Stability and instability of Kelvin waves

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Abstract
The \( m \)-waves of Kelvin are uniformly rotating patch solutions of the 2D Euler equations with \( m \)-fold rotational symmetry for \( m \geq 2 \). For Kelvin waves sufficiently close to the disc, we prove a nonlinear stability result up to an arbitrarily long time in the \( L^1 \) norm of the vorticity, for \( m \)-fold symmetric perturbations. To obtain this result, we first prove that the Kelvin wave is a strict local maximizer of the energy functional in some admissible class of patches, which had been claimed by Wan in 1986. This gives an orbital stability result with a support condition on the evolution of perturbations, but using a Lagrangian bootstrap argument which traces the particle trajectories of the perturbation, we are able to drop the condition on the evolution. Based on this unconditional stability result, we establish that long time filamentation, or formation of long arms, occurs near the Kelvin waves, which have been observed in various numerical simulations. Additionally, we discuss stability of annular patches in the same variational framework.

Mathematics Subject Classification 35Q31 · 37K58

1 Introduction

A few coherent vortices have been found in the two dimensional Euler equations, such as (rotating) disks, (sliding) dipoles, etc. The study of their stability (and instability) has been a classical topic in fluid dynamics it is believed to be relevant for long time behavior of high Reynolds number flows. Among them, we revisit the \( m \)-waves of Kelvin, which are uniformly rotating patch solutions of the two-dimensional incompressible Euler equations on \( \mathbb{R}^2 \) in the vorticity form:
Here, $\omega(t, \cdot) : \mathbb{R}^2 \to \mathbb{R}$ and $u(t, \cdot) : \mathbb{R}^2 \to \mathbb{R}^2$ denote the vorticity and velocity of the fluid at time $t$, respectively. For any integer $m \geq 2$ and real $r_0 > 0$, the Kelvin waves can be parametrized by $\beta > 0$ (see [5]); for a sufficiently small $\beta$, we shall write
\[
\omega^{m,\beta} = 1_{A^{m,\beta}}, \quad A^{m,\beta} = \{(r, \theta) : r < r_0 + g^{m,\beta}(\theta)\}
\]
as the $m$-wave of Kelvin with parameter $\beta$, characterized by the property that
\[
g^{m,\beta}(\theta) = \beta \cos(m \theta) + o(\beta),
\]
where the $o(\beta)$-term consists of expressions $\cos(km \theta)$ with $k > 1$. It turns out that for $\beta$ small, the function $g^{m,\beta}$ can be chosen uniquely in a way that
\[
\omega^{m,\beta}_\Omega(x), \quad t \geq 0, \ x \in \mathbb{R}^2
\]
defines a solution of the Euler equations (1.1) for some $\Omega^{m,\beta} \in \mathbb{R}$, which is the angular velocity of the Kelvin wave $\omega^{m,\beta}$. Here, we are using the notation $\omega_{\alpha} := \omega(\mathbf{R} - \alpha \mathbf{x})$ where $R_\alpha$ is the counterclockwise rotation matrix by angle $\alpha$ with respect to the origin. In the rest of the introduction, we fix the reference length $r_0$ to be 1 for simplicity, and take $B$ to be the open ball centered at the origin with radius 1.

Kirchhoff has discovered that ellipses define uniformly rotating patch solutions for any aspect ratio [38], which correspond to the case $m = 2$ in the above. For $m \geq 3$, the existence of $m$-fold symmetric rotating patches bifurcating from the disc was first hinted by Kelvin in 1880 (see Lamb [39, 231p]), who computed that an infinitesimal perturbation of the disc with period $m$ rotates with the angular speed $1/2 - 1/2m$. Then, an argument of existence was given by Burbea [5] in 1982 (who coined the term “$m$-waves of Kelvin”) and then rigorously by Hmidi, Mateu, and Verdera in [34] only in 2013. We also refer to the work [32] of Hassainia, Masmoudi, and Wheeler where the authors study the behavior of the whole branch of solutions. Thanks to these works, we know that the boundary of these rotating patches are $C^\infty$-smooth and even real analytic.

In this paper, we focus on nonlinear stability and instability of the Kelvin waves close to the disc, and obtain the following results:

**Long time stability** (Theorem 1.1). For any $T > 0$, sufficiently localized $L^1$ small perturbations of the Kelvin wave stay close to the rotating Kelvin wave (1.3) in the time interval $[0, T]$.

**Instability: large perimeter growth** (Theorem 1.5). For any $M > 0$, there exists an $L^1$-small patch perturbation of the Kelvin wave whose perimeter grows to become larger than $M$ in finite time.

The precise statements will be given in Sect. 1.1, but see Fig. 1 which illustrates both stable and unstable behavior of a threefold rotating state. While the “bulk” of the patch seems to converge to a Kelvin wave, long arms are constantly growing in time. This type of filamentation instability can be generically observed for vortex patches [11, 21, 23, 24, 37, 50].

### 1.1 Main results

Let us recall that $\omega^{m,\beta}_\Omega(x)$ in (1.3) is the rotating solution of Euler with $\omega^{m,\beta}$ as the initial data. Our first result shows that localized and $L^1$ small perturbations of the Kelvin wave (for
sufficiently small $\beta$) stay $L^1$ close for an arbitrarily long time. Roughly speaking, the core part of the perturbed patch solutions is equal to that of the rotating Kelvin wave (1.3).

**Theorem 1.1** (Stability) Let $m \geq 2$ be an integer. There are $\beta_1 > 0$ and $r' > 1$ such that for any $\beta \in (0, \beta_1]$, $\epsilon > 0$, and $T > 0$, there exists $\delta = \delta(m, \beta, \epsilon, T) > 0$ such that if $\omega_0 = 1_{A_0}$ for some $m$-fold symmetric open set $A_0 \subset \mathbb{R}^2$ satisfying

$$\|\omega_0 - \omega^{m, \beta}\|_{L^1(\mathbb{R}^2)} \leq \delta \quad \text{and} \quad A_0 \subset B_{r'}$$

then the solution $\omega(t)$ to (1.1) with initial data $\omega_0$ satisfies

$$\sup_{t \in [0, T]} \|\omega(t) - \omega^{m, \beta}\|_{L^1(\mathbb{R}^2)} \leq \epsilon.$$ (1.5)

Here, $B_r$, $r > 0$ is the open ball centered at the origin with radius $r$, and a set $A \subset \mathbb{R}^2$ is $m$-fold symmetric if $R_{2\pi/m}[A] = A$.

**Remark 1.2** The initial support condition [the second condition in (1.4)] means that the perturbations should be localized near the Kelvin set $A^{m, \beta}$. Indeed, due to the property $r' > 1$, if $\beta > 0$ is sufficiently small, then $A^{m, \beta} \subset B_{r'}$. Thus, the condition holds once we simply assume that for some small $\mu = \mu(r') > 0$,

$$A_0 \subset \{x \in \mathbb{R}^2 : \text{dist}(x, A^{m, \beta}) < \mu\}.$$

(i.e. for any $x \in A_0$, there exists a point $x' \in A^{m, \beta}$ such that $|x - x'| < \mu$.) The important point is that $r' > 1$ (so $\mu > 0$) is a constant depending only on $m \geq 2$ and in particular does not need to become smaller as we vary $\epsilon$ and $T$. In that sense, the assumption is not too much restrictive. However, we note that the condition seems inevitable for stability as long as we

Fig. 1 Evolution of the patch initially defined by the region $\{(r, \theta) : r < 2 + \sin(3\theta)\}$ at time moments $t = 0, 3, 6, 9, 15, 20$. Courtesy of Junho Choi.
use a variational idea saying that each Kelvin wave is a local maximizer of the kinetic energy in some class. A counterexample for the case \( m = 2 \) is given in [45] (case of the ellipse), and it can be modified to be a counterexample for any \( m \geq 3 \).

When proving the above long time stability, we use three main ingredients. The first one is the following conditional orbital stability, which appeared already in Wan’s paper [47, Section 5, Theorem 7] in 1986 with only a very rough sketch of the proof. The key observation is that Kelvin waves become non-degenerate local energy maximums when perturbations are assumed to have the same \((m\text{-fold})\) symmetry. This idea was already used for Kirchhoff’s ellipse by Tang [45] in 1987. We would like to point out that by the time of [47], even the existence of Kelvin waves have not been rigorously established; the first rigorous existence proof came out only 27 years later in [34]. We believe that at the time of [47], a complete proof was impossible since the proof of stability requires not only existence but also smoothness of the boundary of Kelvin waves. In this paper, we provide a detailed and complete proof based on the method suggested in [47], using rigorous results from [34].

**Proposition 1.3** (Conditional Orbital Stability) For each integer \( m \geq 2 \), there exist constants \( \bar{r} > 1 \) and \( \bar{\beta} > 0 \) satisfying the following property:

Fix any \( 0 < \beta < \bar{\beta} \). Then, for any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if \( \omega_0 = 1_{A_0} \) where

\[
A_0 \text{ is } m\text{-fold symmetric, } A_0 \subset B_{\bar{r}}, \text{ and } \| \omega_0 - \omega_{m,\beta} \|_{L^1(\mathbb{R}^2)} \leq \delta,
\]

then the solution \( \omega(t) = 1_{A(t)} \) of (1.1) with initial data \( \omega_0 \) is stable up to rotations as long as it is contained in \( B_{\bar{r}} \); more precisely, if for some \( 0 < T \leq \infty \) we have

\[
\bigcup_{t \in [0,T)} A(t) \subset B_{\bar{r}}, \tag{1.6}
\]

then there exists a function \( \Theta(\cdot) : [0, T) \to \mathbb{R} \) satisfying

\[
\sup_{t \in [0,T)} \| \omega(t) - \omega_{m,\beta}^{\Theta(t)} \|_{L^1(\mathbb{R}^2)} \leq \varepsilon. \tag{1.7}
\]

The above result is conditional in the sense that the perturbed solution needs to be assumed to stay in a given ball [see (1.6)] during the evolution, and orbital as it only gives that the perturbation is \( L^1 \) close only to some unspecified rotation \( \Theta(t) \) of the Kelvin wave.\(^1\) Even assuming that the second condition can be removed, the first support condition (1.6) on the evolution is fatal in the sense that it seems to be guaranteed only for time of at most order 1.

Let us now focus on how the latter issue is handled: orbital stability is very typical when one obtains stability by applying variational idea (e.g. see [1, 6, 7, 9, 15, 45, 47]). Intuitively, it is natural to expect that the perturbed solution is close not only to some unknown \( \Theta(t) \)-rotation of the Kelvin wave but also to the actual rotating Kelvin wave solution (1.3). Therefore, to arrive at Theorem 1.1 from Proposition 1.3, we need an intermediate result which shows that if \( \Delta t \ll 1 \), then

\[
\Delta \Theta(t) \text{ is close to } \Omega_{m,\beta}^m \Delta t,
\]

that is, the perturbed solution almost rotates with the angular speed \( \Omega_{m,\beta}^m \) of the Kelvin wave. This is the second ingredient of the proof of Theorem 1.1.

To state the result, we denote \( \mathbb{T}_m = \mathbb{R}/(2\pi / m \mathbb{Z}) \) by the torus of length \( 2\pi / m \), which we identify with the interval \( [-\pi / m, \pi / m] \). Since the Kelvin wave is \( m\text{-fold symmetric} \), it is natural

\(^1\) These two restrictions have been lifted in Theorem 1.1, at the cost of restricting the solution to a finite (but arbitrarily large) time interval.
to assume that the rotation angle $\Omega^{m,\beta} t$ in (1.3) belongs to $T_m$. Similarly, we shall view the function $\Theta$ as a map $\Theta(\cdot) : [0, T) \to T_m$, and denote the projection $T_m : \mathbb{R} \to T_m$ by

$$T_m[\alpha] = \tilde{\alpha} \in T_m \text{ if } \alpha - \tilde{\alpha} = 2k\pi/m \text{ for some integer } k.$$  

**Proposition 1.4** (Estimate on the required rotation) (continued from Proposition 1.3) For $m \geq 2$, there exist constants $C_0, c_0, \beta_0 > 0$ such that if $\beta \in (0, \min(\beta_0, \beta))$ and $\varepsilon \in (0, c_0\beta^2]$, then the function $\Theta(\cdot)$ in Proposition 1.3 satisfies

$$|T_m[\Theta(t')] - (\Theta(t) + \Omega^{m,\beta}(t' - t))| \leq C_0 \cdot \varepsilon^{1/2} \text{ whenever } t, t' \in [0, T) \text{ satisfy } |t - t'| \leq c_0\beta.$$  

(1.8)

For the third and last ingredient, we employ the fact that trajectories of the Kelvin wave solution are closed curves, and they stay close to the rotating Kelvin set $R_{\Omega^{m,\beta}}[A^{m,\beta}]$ if they are close at the initial time. Then, thanks to conditional stability (Propositions 1.3, 1.4), we are able to prove the long time stability (Theorem 1.1).

Now, as an application of Theorem 1.1, we obtain perimeter growth for finite but arbitrary long time for certain perturbations of the Kelvin wave.

**Theorem 1.5** (Instability) For each integer $m \geq 2$, there exists $C' > 0$ such that if $\beta > 0$ is sufficiently small, then for any $M, \delta > 0$, there exists an $m$-fold symmetric data $\omega_0 = 1_{A_0}$ with $C^\infty$ smooth boundary $\partial A_0$ satisfying

$$\|\omega_0 - \omega^{m,\beta}\|_{L^1(\mathbb{R}^2)} \leq \delta, \text{ perim}(A_0) \leq 20$$

such that the corresponding solution $\omega(t) = 1_{A(t)}$ satisfies

$$\sup_{t \in [0, CM]} \text{perim}(A(t)) \geq M.$$  

We remark that the improved stability statement (Theorem 1.1), rather than a conditional orbital stability result (Proposition 1.3), is essential in the proof of instability. While we have proved a similar instability result for perturbations of the Lamb dipole in [18] (see also [16] for the case of the Hill’s spherical vortex), in this previous work the filament is separating from the center of the perturbed dipole linearly in time, which makes the proof much easier.

### 1.2 Previous works on stability of $V$-states

The Kelvin waves, and more generally uniformly rotating patch solutions, commonly referred to as “$V$-states”, to Euler have been intensively studied in the past decades: existence and rigidity of $V$-states ([5, 8, 12, 13, 20–22, 25, 26, 28–30, 32–34, 43, 46]), linear and nonlinear stability ([10, 14, 41, 44, 45, 47, 48]), instability ([17, 19, 31]), numerical computations ([11, 21, 23, 24, 37, 50]).

Let us briefly review the existing results on the stability of $V$-states. The basic idea is to characterize a given $V$-state as the unique extremizer of a conserved quantity in an appropriate admissible class. Arnol’d [2] suggested to use the kinetic energy, which is natural since steady solutions are characterized by critical points of the energy. We also refer to the recent work [27] for discussions. Some serious work was necessary to apply this idea to the patch case, as it is not a smooth solution to Euler. For the case of the disc $\omega = 1_B$, this was achieved by Wan–Pulvirenti [48] and Tang [45]: the circular patch is actually the unique energy maximizer under a mass constraint, which gives nonlinear stability for perturbations in $L^1$. A different approach is to observe that the circular patch is the unique impulse minimizer under a mass
constraint, which again yields nonlinear stability in [44] (also see [14]). Indeed, the energy and impulse are two different coercive conservation laws for the two-dimensional vorticity equation. It turns out that, in the case of the ellipse, one can show that upon fixing both the mass and impulse, each ellipse for the aspect ratio between 1 and 3 is the unique local maximizer of the energy [45]. The threshold 3 is sharp, as suggested by previous linear analysis [41] and nonlinear instability for larger aspect ratios [31].

Under the additional constraint of \( m \)-fold symmetry, Kelvin waves close to the disc (i.e. \( 0 < \beta \ll 1 \)) for each \( m \geq 3 \) can be characterized by the unique local maximum of energy, as stated in Wan [47]. Even though the stability requires \( m \)-fold symmetry of perturbations, this can be used to prove filamentation simply by taking symmetric perturbations (proof of instability requires stability). It is interesting that when \( m = 2 \) (i.e. Kirchhoff’s ellipses), the stability was obtained not only for small \( \beta > 0 \) but also up to the aspect ratio 3. It is mainly due to the fact that the stream function for each ellipse is explicitly known (e.g. see [39]) so that the computation in spectral analysis in [45] is exact while the representation \((1.2)\) of Kelvin waves for \( m \geq 3 \) works only for small \( \beta \).

Lastly, we note that when proving stability of steady solutions of the Euler equations, monotonicity of the profiles is frequently assumed (e.g. see [3, 4, 14, 36, 40, 42, 49, 51]) because this property gives coercivity in a certain sense. In the Appendix, we demonstrate that the same spectral approach can give nonlinear stability for patches supported on an annulus by imposing \( m \)-fold symmetry for large \( m \) to perturbations. It is interesting to study stability of such an annular patch since it is a non-monotone, radial steady solution.

**Organization of the paper**

The rest of the paper is organized as follows. In Sect. 2, we collect a few basic facts about the two-dimensional Euler equations and derive the asymptotic rotation speed of the Kelvin \( m \)-waves. Then, Proposition 1.3 is proved in Sect. 3. Lastly in Sect. 4, we prove Proposition 1.4, Theorems 1.1, and 1.5. Sections 3 and 4 begin with an overview of the proof. In the Appendix, we discuss stability of annular patches.

**2 Preliminaries**

**2.1 Two-dimensional incompressible Euler**

For the two-dimensional Euler equations, the stream function is defined by

\[
G[\omega](x) = (-\Delta)^{-1} \omega(x) := \frac{1}{2\pi} \int \ln \frac{1}{|x - x'|} \omega(x') dx'.
\]

When \( \omega \) is bounded and compactly supported in \( \mathbb{R}^2 \), then we have that \( G[\omega] \in C^{1,\alpha}_{loc}(\mathbb{R}^2) \) with any \( \alpha < 1 \). The energy functional is defined by

\[
E[\omega] = \frac{1}{2} \langle \omega, G\omega \rangle = \frac{1}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \omega(x)\omega(x') \ln \frac{1}{|x - x'|} dxdx'.
\]

For bounded solutions to \((1.1)\) decaying sufficiently fast at infinity, it is not difficult to check that \( E \) is a conserved quantity in time. We just remark that, strictly speaking, \( E \) is not the
kinetic energy of the fluid unless $\omega$ is of mean zero in $\mathbb{R}^2$: in this case, we have

$$E[\omega] = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla G\omega|^2 dx \geq 0.$$  

In general, $E[\omega]$ is not positive and this quantity is sometime referred to as the pseudo-energy in the literature.

### 2.2 Kelvin waves

Let $r_0 > 0$ and denote $B$ by the open ball with radius $r_0$ centered at the origin. Then, it is not difficult to check that in polar coordinates, the corresponding stream function is given by

$$G[1_B](r, \theta) = \begin{cases} \frac{1}{4} (r_0^2 - r^2) - \frac{1}{2} r_0^2 \ln r, & 0 \leq r \leq r_0, \\ -\frac{1}{2} r_0^2 \ln r, & r > r_0. \end{cases}$$

Let us now revisit the computation of Kelvin, and assume that there exists a uniformly rotating patch $\omega^{m, \beta}$ with boundary $r_0 + g(\theta)$ with $g \simeq \beta \cos(m\theta)$. We shall give a sketch of the derivation of the rotation speed $\Omega^{m, \beta}$ in the limit $\beta \to 0$. A rigorous argument is given in [34]. For some $C > 0$, once we define the relative stream function by

$$\psi^{m, \beta} = G\omega^{m, \beta} + \frac{1}{2} \Omega^{m, \beta} r^2 + C,$$

so that for all $\theta \in [0, 2\pi]$, we have

$$\psi^{m, \beta}(r_0 + g(\theta), \theta) \equiv 0.$$  

Introducing for convenience $\zeta := G(\omega^{m, \beta} - 1_B)$, by differentiating the above relation in $\theta$, we obtain

$$0 = \left( -\frac{1}{2} + \Omega^{m, \beta} \right) r_0 \partial_\theta g^{m, \beta}(\theta) + \partial_\theta \zeta \partial_\theta g^{m, \beta}(\theta) + \partial_\theta \zeta. \quad (2.2)$$

Note that

$$\zeta(r, \theta) = \frac{1}{2\pi} \int_{S^1} \int_{r_0}^{r_0 + g(\theta')} \ln \frac{1}{|re^{i\theta'} - r'e^{i\theta'}|} r' dr' d\theta'.$$

Assuming $\int \omega^{m, \beta} = \int 1_B$, we may expand the above in the case $r > r_0 - |g(\theta')|$ as follows: using $g(\theta) = \beta \cos(m\theta) + o(\beta)$ and that the small term is orthogonal to $1, \cos(m\theta)$,

$$\zeta(r, \theta) = \frac{1}{2\pi} \int_{S^1} \int_{r_0}^{r_0 + g(\theta')} \text{Re} \sum_{n \geq 1} \frac{1}{n} \left( \frac{r'}{r} \right)^n e^{in(\theta' - \theta)} r' dr' d\theta'$$

$$= \frac{1}{2m} \frac{r_0^{m+1}}{r^m} \beta \cos(m\theta) + o(\beta).$$

Similarly, one may compute that

$$\partial_\theta \zeta(r, \theta) = -\frac{r_0^{m+1}}{2r^m} \beta \sin(m\theta) + o(\beta), \quad \partial_\theta \zeta(r, \theta) = -\frac{r_0^{m+1}}{2r^m} \beta \cos(m\theta) + o(\beta).$$

Since we know that $\zeta \in C^{1,\alpha}$ for any $\alpha < 1$, these formulas can be justified up to the boundary of the rotating patch. Applying these to (2.2), we obtain that

$$\Omega^{m, \beta} = \frac{1}{2} - \frac{1}{2m} + o(\beta).$$
The remainder term seems to be of order $\beta^2$ (see [5]), but we shall not need this fact in what follows.

3 Stability of Kelvin waves

3.1 Outline of the proof of Proposition 1.3

This section is devoted to the proof of the stability result. To state the key proposition, let us first define the following class of perturbations, fixing some $m \geq 2$ and $\beta > 0$. The value of $\beta$ will be assumed to be sufficiently small whenever it becomes necessary, but in a way depending only on $m$. (Inspecting the proof, one can see that $\beta \lesssim m^{-2}$ is sufficient.) We take

$$\mathcal{M}(\omega_{m,\beta}) := \left\{ \omega = 1_A : \int r^m \sin(m \theta) \omega(x) dx = 0, \int r^2 (\omega(x) - \omega_{m,\beta}) = 0, \right\}.$$

Then, we say $\omega \in \mathcal{M}_m(\omega_{m,\beta})$ if $\omega \in \mathcal{M}(\omega_{m,\beta})$ and furthermore $\omega$ is $m$-fold symmetric.

Next, given an open set $D$, we define the class

$$\mathcal{N}_{\varepsilon,D}(\omega_{m,\beta}) := \left\{ \omega = 1_A : A \subset D, \| \omega - \omega_{m,\beta} \|_{L^1} < \varepsilon \right\}.$$

We are now ready to state the key technical result of this section, which shows that within the class $\mathcal{N}_{\varepsilon,D} \cap \mathcal{M}_m(\omega_{m,\beta})$ for $D = B_{\bar{r}r_0}$ with some $\bar{r} > 1$, $\omega_{m,\beta}$ is the strict maximizer of the energy. As before, we shall fix $r_0 = 1$ for simplicity.

**Proposition 3.1** For any $m \geq 2$, there exist $\beta_0 > 0$ and $\bar{r} > 1$ depending on $m$ such that the following statement holds. For the Kelvin $m$-wave with parameter $0 < \beta < \beta_0$, there exist $C, \varepsilon > 0$ depending on $m, \beta$ such that

$$E[\omega_{m,\beta}] - E[\omega] \geq C \| \omega_{m,\beta} - \omega \|_{L^1}^2$$

(3.1)

for any $\omega \in \mathcal{N}_{\varepsilon,B_{\bar{r}r_0}} \cap \mathcal{M}_m(\omega_{m,\beta})$.

In Sect. 3.2, we show how our main stability (Proposition 1.3) follows from the above proposition, which is rather straightforward. Then, the remainder of this section is devoted to establishing Proposition 3.1. The structure of the argument is parallel to that for the ellipse stability by Tang [45], which corresponds to the case $m = 2$, and mainly consists of two steps: (i) reduction to a graph-type perturbation and (ii) energy comparison for graph type perturbations. To be more precise, given $\omega$ satisfying the assumptions of Proposition 3.1, we shall find $\tilde{\omega}$ such that

$$E[\omega_{m,\beta}] - E[\tilde{\omega}] \geq C \| \omega_{m,\beta} - \tilde{\omega} \|_{L^1}^2$$

(3.2)

and

$$-E[\omega] + E[\tilde{\omega}] \geq C \| \omega - \tilde{\omega} \|_{L^1}^2$$

(3.3)

holds. Combining the above two inequalities, we obtain (3.1). We shall identify such a $\tilde{\omega}$ and prove (3.3) in Sect. 3.3. Then, (3.2) is proved in Sect. 3.4: this part is the heart of the matter and requires a spectral analysis of the linearized operator coming from the Green’s function for the Laplacian.
In what follows, we shall use a simple change of variables \((r, \theta) \rightarrow (\xi, \eta)\) near \(\{r = r_0\}\), so that \(\eta(r, \theta) = \theta\) and \(\xi(r, \theta) = r - g^{m, \beta}(\theta)\). Then, \(\{\xi = r_0\}\) corresponds to \(\partial A^{m, \beta}\). The Jacobian \(J = J^{m, \beta}\) from \(x = (x_1, x_2)\) to \((\xi, \eta)\) is \(\xi + \beta \cos(m \theta) + o(\beta)\).

### 3.2 Proof of stability

In this section, we show how Proposition 1.3 follows readily from Proposition 3.1. This procedure is straightforward, the key point being that the energy difference is controlled by the \(L^1\) difference of vorticities. First, we prove an intermediate result, namely nonlinear \(L^1\) stability under the natural constraints on the initial vorticity.

**Lemma 3.2** Fix some \(m \geq 2\) and assume that \(0 < \beta\) is sufficiently small. Then, for any sufficiently small \(\varepsilon > 0\), there exists \(\delta > 0\) such that for \(\tilde{\omega}_0 \in M_m \cap N_\delta, B_r(\omega^{m, \beta})\), we have

\[
\|\tilde{\omega}(t, \cdot) - \omega^m, \beta\|_{L^1} < \varepsilon, \quad \forall t \geq 0,
\]

for some \(t' = t'(t)\), provided that \(\text{supp}(\tilde{\omega}(s, \cdot)) \subset B_r\) for all \(s \in [0, t]\).

Note that for two vorticities \(\omega\) and \(\tilde{\omega}\) which are compactly supported in \(\mathbb{R}^2\), we have

\[
|E[\omega] - E[\tilde{\omega}]| = \frac{1}{2} |\langle \omega - \tilde{\omega}, G[\tilde{\omega}] \rangle - \langle \omega, G[\tilde{\omega} - \omega] \rangle| \leq C \|\omega - \tilde{\omega}\|_{L^1}
\]

where \(C > 0\) depends on the radius of the support (see [45, Lemma 5.1]).

**Proof of Lemma 3.2 assuming Proposition 3.1** This is nothing but [45, Lemma 5.3], although we provide a simplified argument. Let us suppose that \(\tilde{\omega}_0\) verifies the assumptions in the above. Furthermore, it will be convenient to consider the Euler equations in a rotating frame in which \(\omega^{m, \beta}\) becomes a steady state, and denote \(\tilde{\omega}(t, \cdot)\) to be the solution defined under this frame. Note that the solution \(\tilde{\omega}(t, \cdot)\) belongs to the class \(M_m\), possibly except for the condition

\[
\int r^m \sin(m \theta) \tilde{\omega}(t, \cdot) dx = 0.
\]

**Proof of (3.4) under the assumption of (3.6) and \(\tilde{\omega}(t) \in N_{\varepsilon, B_r}\).** For the moment, assume that (3.6) holds at some time \(t\) and \(\tilde{\omega}(t) \in N_{\varepsilon, B_r}\). Then, from Proposition 3.1 and (3.5), we derive

\[
\frac{1}{C} \|\omega^{m, \beta} - \tilde{\omega}_0\|_{L^1} \geq E[\omega^{m, \beta}] - E[\tilde{\omega}_0] = E[\omega^{m, \beta}] - E[\tilde{\omega}(t)] \geq C \|\omega^{m, \beta} - \tilde{\omega}(t)\|_{L^1}^2
\]

and hence

\[
\|\omega^{m, \beta} - \tilde{\omega}(t)\|_{L^1} \leq C \delta^{\frac{1}{2}} < \frac{\varepsilon}{4},
\]

where the last inequality follows simply by taking \(\delta > 0\) small in a way depending on \(\varepsilon\).

**Removing the additional assumptions.** We observe that the quantity \(\|\omega^{m, \beta} - \tilde{\omega}(t)\|_{L^1}\) is Lipschitz continuous in time, which follows from the fact that the boundary of the support of \(\omega^{m, \beta}\) is smooth and that the velocity of \(\tilde{\omega}(t)\) is uniformly bounded in time. Therefore, from the continuity, one can take some small \(T > 0\) such that on \([0, T]\), we have

\[
\|\omega^{m, \beta} - \tilde{\omega}(t)\|_{L^1} < \frac{\varepsilon}{2}.
\]
Since the condition \( \text{supp}(\tilde{\omega}(s, \cdot)) \subset B_{r} \) is given in the statement of the lemma, we obtain on \([0, T]\) that \( \tilde{\omega}(t) \in \mathcal{N}_{r, B_{r}} \). Then, at \( t = T \), it is not difficult to see that by rotating \( \tilde{\omega}(T) \) with some small angle \( \tau \) (taking \( \varepsilon \) smaller if necessary), we can arrange that

\[
\int r^{m} \sin(m\theta)\tilde{\omega}_{\tau}(T, \cdot)dx = 0.
\]

See the last part of the proof of Lemma 3.3 for the details of this argument. Recalling the argument in the above, this shows that we can actually upgrade the estimate (3.7) to

\[
\|\omega_{-\tau}^{\alpha_{\beta}} - \tilde{\omega}(T)\|_{L^{1}} = \|\omega_{-\tau}^{\alpha_{\beta}} - \tilde{\omega}_{\tau}(T)\|_{L^{1}} < \frac{\varepsilon}{4}.
\]

Since we may choose \( T \) depending only on \( \omega_{-\tau}^{\alpha_{\beta}} \) and \( \varepsilon \) (using Lipschitz continuity in time of the quantity \( \|\omega^{\alpha_{\beta}} - \tilde{\omega}(t)\|_{L^{1}} \)), we may inductively obtain bounds of the \( L^{1} \) difference on time intervals \([T, 2T], [2T, 3T], \) and so on. \( \square \)

**Proof of Proposition 1.3 from Lemma 3.2** We first note that Lemma 3.2 works for general \( r_{0} > 0 \) by rescaling. Then the idea for proof of Proposition 1.3 is to simply “adjust” both the initial data \( \omega_{0} \) and the Kelvin wave \( \omega_{-\tau}^{\alpha_{\beta}} \) in a way that we are reduced to the setup of Lemma 3.2. Given \( \omega_{0} \) satisfying the assumptions of Proposition 1.3, we may find \( \lambda, \beta', \tau \) verifying

\[
|\lambda - 1|, |\beta' - \beta|, |\tau| \ll 1
\]

(from the inverse function theorem) such that the rotated initial data \( (\omega_{0})_{\tau}(x) = \omega_{0}(R_{-\tau}x) \) and the rescaled, \( \beta' \)-reparametrized wave \( \omega^{\alpha_{\beta'}, \lambda}(x) := \omega^{\alpha_{\beta'}, \lambda}(\lambda x) \) satisfy

\[
\int r^{m} \sin(m\theta)(\omega_{0})_{\tau}(x)dx = 0,
\]

and

\[
\int \omega_{0}(x)dx = \int \omega^{\alpha_{\beta'}, \lambda}(x)dx,
\]

respectively, by taking \( \delta > 0 \) smaller if necessary depending on \( m, \beta \). We observe that if we set \( \omega_{0} := (\omega_{0})_{\tau} \), we have

\[
\|\tilde{\omega}_{0} - \omega_{-\tau}^{\alpha_{\beta'}, \lambda}\|_{L^{1}} = \|\omega_{0} - \omega_{-\tau}^{\alpha_{\beta'}, \lambda}\|_{L^{1}}
\]

\[
\leq \|\omega_{0} - \omega_{-\tau}^{\alpha_{\beta'}}\|_{L^{1}} + \|\omega_{-\tau}^{\alpha_{\beta'}} - \omega_{-\tau}^{\alpha_{\beta'}}\|_{L^{1}} + \|\omega_{-\tau}^{\alpha_{\beta'}} - \omega_{-\tau}^{\alpha_{\beta'}, \lambda}\|_{L^{1}}
\]

\[
+ \|\omega_{-\tau}^{\alpha_{\beta'}, \lambda} - \omega_{-\tau}^{\alpha_{\beta'}, \lambda}\|_{L^{1}},
\]

and the right hand side can be made arbitrarily small by assuming \( \delta > 0 \) small again. Thus we get \( \omega_{0} \in \mathcal{M}_{m} \cap \mathcal{N}_{\delta, B_{r}}(\omega^{\alpha_{\beta'}, \lambda}) \) so that we can apply Lemma 3.2 (for general \( r_{0} > 0 \)) to \( \tilde{\omega}_{0} \) with the Kelvin wave \( \omega^{\alpha_{\beta'}, \lambda} \), which gives

\[
\|\tilde{\omega}(t) - \omega_{t'}^{\alpha_{\beta'}, \lambda}\|_{L^{1}} < \frac{\varepsilon}{3}
\]

for some angle \( t' = t'(t) \). (For this, we may take \( \tilde{r} > 0 \) in the statement of Proposition 1.3 slightly smaller than the original \( \tilde{r} > 0 \) given in Proposition 3.1.)
Then by choosing appropriate angle $t_1 = t_1(t')$, we have
\[
\|\omega(t) - \omega_{t_1}(t')\|_{L_1} \leq \|\tilde{\omega}(t) - \omega_{t_1}(t')\|_{L_1} \\
\leq \|\tilde{\omega}(t) - \omega_{t_1}(t')\|_{L_1} + \|\omega_{t_1}(t') - \omega_{t_1}(t')\|_{L_1} + \|\omega_{t_1}(t') - \omega_{t_1}(t')\|_{L_1} \\
\leq \frac{\varepsilon}{3} + \|\omega_{t_1}(t') - \omega_{t_1}(t')\|_{L_1} + \|\omega_{t_1}(t') - \omega_{t_1}(t')\|_{L_1}.
\]

The last two terms on the right hand side can be taken to be less than $\varepsilon/3$ by choosing $\delta$ small. This finishes the proof. \hfill $\square$

### 3.3 Reduction to graph perturbations

We set $\mathcal{E}$ to be a sufficiently small neighborhood of $\psi^{m, \beta}$ in the $C^1$-topology, where $\psi^{m, \beta}$ is the relative stream function of $\omega^{m, \beta}$ defined in (2.1). Before we proceed, observe that the set $\{\psi^{m, \beta} > 0\}$ consists of two components, with the inner one describing the set $A^{m, \beta}$. Since the gradient of $\psi^{m, \beta}$ is non-degenerate on $\partial A^{m, \beta}$ for $\beta$ small, we have that for any $\psi \in \mathcal{E}$, the inner component of $\{\psi > 0\}$ is an open set close to $A^{m, \beta}$. Note that, as we take $\beta \to 0$, the relative stream functions $\psi^{m, \beta}$ converge in $C^{1, \alpha}$ to the limit $\psi^{m, 0}$:

\[
\lim_{\beta \to 0} \psi^{m, \beta} = G[1_{B_m}] + \frac{1}{2} \omega^{m, 0}_r r^2 + C^{m, 0} = -\frac{1}{2} \ln r + \frac{1}{2} \left( \frac{1}{2} - \frac{1}{2m} \right) (r^2 - 1).
\]

The function described on the right hand side is strictly positive on $0 < r < 1$, negative on $1 < r < r^*$, and again positive on $r^* < r$, for a constant $r^* > 1$ depending only on $m$.

We conclude that, once we pick any $1 < \tilde{r} < r^*$, then there exists $\beta > 0$ such that for any $0 < \beta < \tilde{\beta}$, the relative stream function $\psi^{m, \beta}$ is strictly negative in the region $B_\tilde{r} \setminus A^{m, \beta}$.

#### Lemma 3.3 (The reduction lemma)

Given $\omega_1 = 1_{A_1} \in \mathcal{N}_{\mathcal{E}, B_{\tilde{r}}} \cap \mathcal{M}_m(\omega^{m, \beta})$, there exists a $C^1$-smooth $\tilde{\psi}$ close to $\psi^{m, \beta}$ defined in (2.1) such that if we denote the inner component of $\{\tilde{\psi} \geq 0\}$ by $\tilde{A}$.

\[
\tilde{\omega} := 1_{\tilde{A}} \in \mathcal{M}_m(\omega^{m, \beta}) \quad \text{and} \quad \langle \tilde{\omega} - \omega_1, G \omega_1 - \tilde{\psi} \rangle = 0.
\]

#### Proof

Given $\psi \in \mathcal{E}$, we define

\[
\psi_{\mu} := \psi + \frac{1}{2} \mu_1 r^2 + \mu_2.
\]

Given $\mu = (\mu_1, \mu_2)$ which is close to $0 = (0, 0)$, we may set $A_{\psi, \mu}$ to be the inner component of $\{\psi_{\mu} \geq 0\}$ and $\omega_{\psi, \mu} = 1_{A_{\psi, \mu}}$. Define for a small neighborhood $\mathcal{O} \subset \mathbb{R}^2$ of the origin,

\[
F = (F_1, F_2) : \mathcal{E} \times \mathcal{O} \to \mathbb{R}^2
\]

by

\[
F_1(\psi, \mu) := \int r^2(\omega_{\psi, \mu} - \omega^{m, \beta}) dx, \quad F_2(\psi, \mu) := \int (\omega_{\psi, \mu} - \omega^{m, \beta}) dx.
\]

Assuming that $\mathcal{O}$ is smaller if necessary, we have that $\partial A_{\psi, \mu}$ is described by a graph $h = h_{\psi, \mu} : S^1 \to \mathbb{R}$ with

\[
\psi_{\mu}(r_0 + h(\eta), \eta) = 0.
\]
In particular, we have that $h_{\psi^m,\beta,0} = 0$. Based on this, we compute
\[
\partial_t \psi|_{(\psi, \mu) = (\psi^m, \beta, 0)} = \partial_t \psi^m|_{\xi = \pi} = \partial_t \psi^m(r_0 + g^{m,\beta}(\theta), \theta)
\]
\[
= -\frac{1}{2m}(r_0 + \beta \cos(m\theta)) + \partial_t G(\omega^{m,\beta} - 1)(r_0 + g^{m,\beta}(\theta), \theta)
\]
\[
= -\frac{1}{2m}(r_0 + \beta \cos(m\theta)) - \frac{1}{2} \beta \cos(m\theta) + o(\beta).
\]

Then, from
\[
\partial_{\mu_j} h = \frac{\partial_{\mu_j} \psi}{\partial_t \psi}, \quad \partial_{\mu_2} h = \frac{1}{\partial_t \psi},
\]
we obtain that
\[
\partial_{\mu_1} h = \frac{r^2}{2 \partial_t \psi}, \quad \partial_{\mu_2} h = \frac{1}{\partial_t \psi}.
\]

This allows us to compute $\partial_{\mu} F$ at $(\psi^m, \beta, 0)$. For convenience we introduce the notation $f = o^1(A)$ to mean that the function $f$ satisfies $|f| \ll A$ and $f$ is orthogonal in $L^2(S^1)$ with 1 and $\cos(\theta)$. To begin with,
\[
\partial_{\mu_1} F_1 = \int_0^{2\pi} (r^2 |f\rangle|_{\xi = \pi}) \partial_{\mu_1} h \eta = \int_0^{2\pi} \left(\frac{(r_0 + \beta \cos(m\theta))^2(\theta) + o(\theta)}{-\frac{1}{m} r_0 - (1 + \frac{1}{m}) \beta \cos(m\theta) + o^1(\theta)}\right) d\theta
\]
\[
= -m \int_0^{2\pi} \left(r^4 \beta^3 \cos(m\theta) + 6r^2 \beta^2 \cos(2m\theta) + o(\beta^2))(1 + \frac{\beta}{r_0} \cos(m\theta) + o^1(\beta))\right)
\]
\[
\times (1 - (1 + m) \beta \cos(m\theta) + (m + 1)^2 \beta^2 \frac{2\pi}{r_0} \cos(\theta) + o(\beta^2)) d\theta.
\]

Then, this gives
\[
\partial_{\mu_1} F_1 = -mr^4 \int_0^{2\pi} 1 + (4 - 5(1 + m) + 6(1 + m)^2) \frac{\beta^2}{r_0} \cos^2(m\theta) d\theta + o(\beta^2)
\]
\[
= -mr^4 \left(2\pi + (m(m - 3) + 6) \frac{\beta^2}{r_0^2} \pi\right) + o(\beta^2).
\]

Next, one can similarly compute that
\[
\partial_{\mu_2} F_1 = \int_0^{2\pi} (r^2 |f\rangle|_{\xi = \pi}) \partial_{\mu_2} h \eta = -mr^4 \left(2\pi + (1 - m + m^2) \frac{\beta^2}{r_0^2} \pi\right) + o(\beta^2),
\]
\[
\partial_{\mu_1} F_2 = \int_0^{2\pi} J |_{\xi = \pi} \partial_{\mu_1} h \eta = -mr^2 \left(2\pi + (1 - m + m^2) \frac{\beta^2}{r_0^2} \pi\right) + o(\beta^2),
\]
and
\[
\partial_{\mu_2} F_2 = \int_0^{2\pi} J |_{\xi = \pi} \partial_{\mu_2} h \eta = -m \left(2\pi + (1 + m + m^2) \frac{\beta^2}{r_0^2} \pi\right) + o(\beta^2).
\]

Therefore, we conclude that
\[
\det(\nabla F) = (2\pi mr^2)^2 \left(\frac{5 \beta^2}{2r_0^2} + o(\beta^2)\right).
\]
In particular, there exists some \( \beta_0 > 0 \) so that for \( \beta \in (0, \beta_0) \), \( \det(\nabla F) > 0 \). Fixing such a \( \beta \) and applying the inverse function theorem to the map \( F \) at \( (\psi, \mu) = (\psi^{m, \beta}, 0) \), we obtain existence of a unique

\[
\tilde{\psi} := (G\omega_1)_{\mu} = G\omega_1 + \frac{1}{2}\mu_1 r^2 + \mu_2.
\]

close to \( \psi^{m, \beta} \) such that the corresponding vorticity \( \tilde{\omega} \) satisfies \( F(G\omega_1, \mu) = 0 \), namely

\[
\int r^2(\tilde{\omega} - \omega^{m, \beta})dx = 0, \quad \int (\tilde{\omega} - \omega^{m, \beta})dx = 0. \quad (3.8)
\]

It is clear that \( \tilde{\psi} \) (and therefore \( \tilde{\omega} \)) is \( m \)-fold rotationally symmetric. For \( \tilde{\omega} \) to belong to the class \( \mathcal{M}_m \), it still remains to verify the condition

\[
\int_{r_m} \sin(m\theta)\tilde{\omega} dx = 0.
\]

This is done by rotating \( \tilde{\psi} \) around the origin; that is, define \( \tilde{\psi}_{\tau}(r, \theta) := \tilde{\psi}(r, \theta + \tau) \) in polar coordinates and denote the corresponding vorticity (defined as the characteristic set of the inner component of \( \{\tilde{\psi}_{\tau} \geq 0\} \)) by \( \tilde{\omega}_{\tau} \). Observe that

\[
\int r^m \sin(m\theta)\omega^{m, \beta}_{\tau} dx = 0
\]

and since \( \tilde{\psi} \) is close to \( \psi^{m, \beta} \) in the \( C^1 \) topology, we have

\[
\left| \int r^m \sin(m\theta)\tilde{\omega} dx \right| \ll 1, \quad \left| \frac{d}{d\tau} \left( \int r^m \sin(m\theta)\tilde{\omega}_{\tau} dx - \int r^m \sin(m\theta)\omega^{m, \beta}_{\tau} dx \right) \right| \ll 1.
\]

Here, \( \ll 1 \) means that the constant can be arbitrarily small by taking \( \varepsilon \to 0 \) where \( \varepsilon \) is from \( \mathcal{N}_{\varepsilon, B_r} \). Since

\[
\frac{d}{d\tau} \int r^m \sin(m\theta)\omega^{m, \beta}_{\tau} dx = \frac{d}{d\tau} \int r^m \sin(m\theta - m\tau)\omega^{m, \beta} dx = \int r^m m \cos(m\theta - m\tau)\omega^{m, \beta} dx
\]

is strictly positive at \( \tau = 0 \), we can find some \( \tau \) satisfying \( |\tau| \ll 1 \) such that

\[
\int r^m \sin(m\theta)\tilde{\omega}_{\tau} dx = 0.
\]

Observe that rotating around the origin does not alter (3.8). The proof is complete. \( \square \)

**Lemma 3.4** Given \( \omega_1 \) satisfying the assumptions of Lemma 3.3, let \( \tilde{\omega} \) to be the associated graph-type vorticity from Lemma 3.3. Then, we have

\[
E[\tilde{\omega}] - E[\omega_1] \geq C\|\tilde{\omega} - \omega_1\|_{L^1}^2.
\]

**Proof** Using the formula for the energy difference, we proceed as follows:

\[
E[\tilde{\omega}] - E[\omega_1] = \langle \tilde{\omega} - \omega_1, G\omega_1 \rangle + \frac{1}{2} \langle \tilde{\omega} - \omega_1, G(\tilde{\omega} - \omega_1) \rangle \geq \langle \tilde{\omega} - \omega_1, G\omega_1 \rangle = \langle \tilde{\omega} - \omega_1, \tilde{\psi} \rangle = \int_{A \setminus A_1} \tilde{\psi} \geq \int_{A_1 \setminus A} \tilde{\psi} \geq \int_{A \setminus A_1} \tilde{\psi}.
\]
It is important to note that the assumption supp \((\omega_1) \subset B_\varepsilon\) is used to guarantee that \(\bar{\psi} \leq 0\) on \(A_1 \setminus \bar{A}\). From a uniform lower bound for \(\partial_\xi \bar{\psi}\) near \(\partial \bar{A}\), it is not difficult to show that the last expression has a lower bound of the form \(C\|\bar{\omega} - \omega_1\|_{L^1}^2\), since \(\bar{\psi} = 0\) on \(\partial \bar{A}\) and \(|\bar{A} \setminus A_1| \gtrsim \|\bar{\omega} - \omega_1\|_{L^1}\).

\[\square\]

### 3.4 Spectral analysis

In this section, we shall consider graph-type perturbations of \(\omega^{m, \beta}\). For this purpose, it will be convenient to work on the coordinate system \((\xi, \eta)\) adapted to \(\omega^{m, \beta}\), after fixing some \((m, \beta)\) with \(m \geq 2\) and \(\beta > 0\) sufficiently small in a way depending on \(m\). Furthermore, \(S^1\) will denote the set \(\{(\xi, \eta) : \xi = \xi_0, 0 \leq \eta < 2\pi\}\) in \(\mathbb{R}^2\), unless otherwise specified. Let \(h \in C^1(S^1)\) be a function with sufficiently small \(\|\nabla\|_{C^1}\)-norm in the \(\eta\) variable. In this section, let us use the notation

\(\omega_h := 1_{A_h}, \quad A_h := \{(\xi, \eta) : \xi \leq \xi_0 + h(\eta)\}\).

For \(h\) sufficiently small in \(C^1\), the closed set \(A_h\) is well-defined and close to the set \(A^{m, \beta}\). In this notation, note that we have \(\omega^{m, \beta} = \omega_0\) where \(0\) is the zero function on \(S^1\).

Now observe that \(\omega_h\) is \(m\)-fold symmetric in \(\mathbb{R}^2\) if and only if \(h\) is \(m\)-fold symmetric in the sense that \(h(\eta) = h(\eta + \frac{2\pi}{m})\) for any \(\eta \in S^1\). For such a function \(h\), we have the following simple result.

**Lemma 3.5** Let \(h\) be \(m\)-fold symmetric on \(S^1\). Then for any integer \(0 < n < m\), we have that

\[\int_{S^1} e^{in\eta} h(\eta) d\eta = 0.\]

**Proof** With the change of variables \(\eta \rightarrow \eta + \frac{2\pi}{m}\),

\[\int_{S^1} e^{in\eta} h(\eta) d\eta = \int_{S^1} e^{in\eta + 2\pi i m} h(\eta + \frac{2\pi}{m}) d\eta = \int_{S^1} e^{in\eta + 2\pi i m} h(\eta) d\eta.\]

This gives

\[(1 - e^{2\pi i m}) \int_{S^1} e^{in\eta} h(\eta) d\eta = 0.\]

When \(0 < n < m\), we have that \(e^{2\pi i \frac{n}{m}} \neq 1\) and we are done.

\[\square\]

The following result from Tang [45] gives the expansion of the energy for graph-type perturbations. Since it applies to general rotating solutions of Euler, the Lemma is directly applicable in our case.

**Lemma 3.6** ([45, Lemma 4.1]) Let \(\omega^* = 1_{A^*}\) be a rotating patch solution where \(\partial A^*\) is described by a smooth graph \(g^*\). Let \(\psi^*\) be the relative stream of \(\omega^*\). Furthermore, let \((\xi, \eta)\) be a coordinate system defined near \(\partial A^*\) satisfying \(\eta = \theta\) and \(\{\xi = \xi_0\} = \partial A^*\), and \(J_0\) is the Jacobian of \(x \mapsto (\xi, \eta)\) restricted to \(S^1 := \{\xi = \xi_0\}\). Consider \(C^1\) graph-type perturbations of \(\omega^*\), namely \(\omega_h\) satisfying \(\|h\|_{C^1(S^1)} \ll 1\). Furthermore, assume that we have

\[\int (\omega_h - \omega^*) d\mathbf{x} = 0, \quad \int x(\omega_h - \omega^*) d\mathbf{x} = 0, \quad \int |x|^2(\omega_h - \omega^*) d\mathbf{x} = 0.\]
Then, we have that for $q(\eta) := J_0(\eta)h(\eta)$,

$$E[\omega_h] - E[\omega^*] = \frac{1}{2} \langle q, Lq \rangle + o(\|h\|_{L^2}^2),$$

(3.10)

where

$$Lq := I_0q + \int_{S^1} K(\eta, \eta')q(\eta')d\eta'$$

with

$$I_0 := \frac{\partial_\xi \psi^*|_{\xi = r_0}}{J_0}, \quad K(\eta, \eta') := \frac{1}{2\pi} \ln \frac{1}{|x(r_0, \eta) - x(r_0, \eta')|}.$$ 

Here, $\langle , \rangle$ denotes the $L^2$ inner product on $S^1$.

Given the above key lemma, we are in a position to conclude the main result of this section.

**Proposition 3.7** For $\omega_h \in N_{r, B_2}(\omega^{m, \beta}) \cap M_1(\omega^{m, \beta})$ with $h \in C^1(S^1)$, we have

$$E[\omega^{m, \beta}] - E[\omega_h] \geq C \|\omega^{m, \beta} - \omega_h\|_{L^1}^2$$

for some $C > 0$.

**Proof** Note that $\|h\|_{L^2}$ and $\|q\|_{L^2}$ are equivalent up to constants. We apply Lemma 3.6 with $\omega^* = \omega^{m, \beta}$ and proceed in several steps.

**Step 1: Cancellation conditions.** To begin with, we note that the first condition from (3.9) implies

$$0 = \int_0^{2\pi} \int_{r_0}^{r_0 + h} Jd\xi d\eta,$$

(3.11)

which gives after expanding $J(\xi, \eta) = J_0(\eta) + O(|\xi - r_0|)$ and integrating in $\xi$,

$$\int_{S^1} q(\eta)d\eta = O(\|h\|_{L^2}^2).$$

(3.12)

Similarly, the last condition from (3.9) gives

$$0 = \int_0^{2\pi} \int_{r_0}^{r_0 + h} (\xi + g^{m, \beta}(\eta))^2 J(\xi, \eta)d\xi d\eta.$$ 

Writing $\xi = r_0 + (\xi - r_0)$, applying (3.11) and expanding $J$ as above, we obtain that

$$2r_0 \int_{S^1} g^{m, \beta} q d\eta = O(\|h\|_{L^2}^2) + o(\|h\|_{L^2}).$$

That is,

$$\int_{S^1} \cos(m\eta) q(\eta)d\eta = O(\beta^{-1}\|h\|_{L^2}^2) + o(\|h\|_{L^2}).$$

(3.13)

Next, from the condition

$$\int r^m \sin(m\eta) \omega_h dx = 0,$$
we obtain that
\[
0 = \int_0^{2\pi} \int_{r_0}^{r_0 + h} (r_0 + g^{m,\beta}(\eta) + (\xi - r_0))^m \sin(m\eta) J(\xi, \eta) d\xi d\eta.
\]

Then, it follows
\[
\int_{S^1} \sin(m\eta) q(\eta) d\eta = O(\beta \|h\|_{L^2}) + O(\|h\|^2_{L^2}).
\]

Lastly, applying Lemma 3.5 to \(q\) (note that \(J_0\) is \(m\)-fold symmetric and so does \(q\)) gives
\[
\int_{S^1} e^{ijn\eta} q(\eta) d\eta = 0, \quad 0 < n < m.
\]

**Step 2: Computation for \(I_0\).** We compute that
\[
I_0 = -\frac{1}{r_0 + \beta \cos(m\theta) + o(\beta)} \left( \frac{r_0}{2m} + \left( \frac{1}{2} + \frac{1}{2m} \right) \beta \cos(m\theta) + o(\beta) \right) = -\left( \frac{1}{2m} + \left( \frac{1}{2} + \frac{1}{2m} \right) \frac{\beta}{r_0} \cos(m\theta) + o(\beta) \right).
\]

This gives
\[
\|I_0 q + \frac{1}{2m} q\|_{L^2} \leq C \beta \|q\|_{L^2}.
\]

**Step 3: Computation for \(K\).** We shall replace \(K\) with \(K^*\) up to an \(O(\beta)\) error, which is the convolution operator arising in the disc case. The operator \(K^*\) simply corresponds to the case \(\beta = 0\). To this end, we first note that using the condition (3.12), we have that
\[
K[q](\eta) := \int_{S^1} K(\eta, \eta') q(\eta') d\eta' = -\frac{1}{2\pi} \int_{S^1} \ln \left| \frac{r_0 + g^{m,\beta}(\eta)}{r_0 + g^{m,\beta}(\eta')} \right| q(\eta') d\eta' = -\frac{1}{2\pi} \int_{S^1} \ln \left| 1 - \frac{r_0 + g^{m,\beta}(\eta)}{r_0 + g^{m,\beta}(\eta')} \right| e^{i(\eta' - \eta)} q(\eta') d\eta' + O(\|h\|^2_{L^2}).
\]

We define
\[
K^*[q](\eta) := -\frac{1}{2\pi} \int_{S^1} \ln \left| 1 - e^{i(\eta' - \eta)} \right| q(\eta') d\eta'.
\]

Then, with pointwise bounds
\[
\left| 1 - \frac{r_0 + g^{m,\beta}(\eta')}{r_0 + g^{m,\beta}(\eta)} \right| \leq C \beta |\eta' - \eta|, \quad \left| 1 - e^{i(\eta' - \eta)} \right| \geq c |\eta' - \eta|,
\]

we obtain that
\[
|K^*[q] - K[q]|(\eta) \leq C \beta \|q\|_{L^1} \leq C \beta \|q\|_{L^2}.
\]

**Step 4: Coercivity.** From the previous step and (3.16), we have
\[
\langle Lq, q \rangle \leq \langle I_0 q, q \rangle + \langle K^* q, q \rangle + C \beta \|q\|^2_{L^2} \leq -\frac{1}{2m} \|q\|^2_{L^2} + \langle K^* q, q \rangle + C \beta \|q\|^2_{L^2}.
\]
We now expand $q$ in Fourier series

$$q = \sum_{n \in \mathbb{Z}} q_ne^{in\eta}.$$ 

Since $q$ is real, we have that $q_{-n} = \overline{q_n}$. Now, we recall the exact formula

$$\frac{1}{2n} = -\frac{1}{2\pi} \int_{0}^{2\pi} \ln |1 - e^{i\eta'}| e^{in\eta'} d\eta', \quad n > 0$$

so that

$$\langle K^* q, q \rangle = \alpha_0 |q_0|^2 + \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{2n} |q_n|^2 = I + II, \quad \alpha_0 := K^*1,$$

where for some $C_0 > 0$ depending on $m$, we have

$$II := \sum_{|n| > m} \frac{1}{2n} |q_n|^2 < \left( \frac{1}{2m} - C_0 \right) \sum_{|n| > m} |q_n|^2.$$

Next,

$$I := \alpha_0 |q_0|^2 + \sum_{0 < |n| \leq m} \frac{1}{2n} |q_n|^2 \leq C \beta^2 \|q\|_{L^2},$$

using (3.12), (3.13), (3.14) and (3.15), and taking $\|h\|_{L^2} \leq \beta^2$. Then, using the Plancherel theorem, we continue estimating as follows:

$$\langle \mathcal{L} q, q \rangle \leq -\frac{1}{2m} \|q\|_{L^2}^2 + \left( \frac{1}{2m} - C_0 \right) \sum_{|n| > m} |q_n|^2 + C \beta \|q\|_{L^2}^2 \leq -\frac{C_0}{2} \|q\|_{L^2}^2,$$

by taking $\beta > 0$ smaller if necessary in a way depending only on $C_0$. (Recall that $C_0$ depends only on $m$.)

**Step 5: Completion of the proof.** From (3.10) in Lemma 3.6, we have that

$$E[\omega^{m,\beta}] - E[\omega_h] = -\frac{1}{2} \langle \mathcal{L} q, q \rangle + o(\|h\|_{L^2}^2) \geq \frac{C_0}{8} \|q\|_{L^2}^2.$$

However, it is clear that

$$\|\omega_h - \omega^{m,\beta}\|_{L^1} = \int_0^{2\pi} \left| \int_{r_0 + h}^2 Jd\xi \right| d\eta \leq C \|q\|_{L^2}(1 + \|q\|_{L^2}).$$

For $\|q\|_{L^2}$ small, we conclude that

$$E[\omega^{m,\beta}] - E[\omega_h] \geq \|\omega_h - \omega^{m,\beta}\|_{L^1}^2.$$

This finishes the proof. \qed

## 4 Refined stability and filamentation

In this entire section, we fix an integer $m \geq 2$ so that every estimate and constant appeared in this section may depend on the choice of the integer $m$ even though we do not specify the dependency on $m$ for simpler presentation. When considering a Kelvin wave, we always assume $r_0 = 1$ so that $A^{m,\beta} = \{r < 1 + g^{m,\beta}(\theta)\}$. Let us give an outline of the arguments.
4.1 Outline of the proof

**Refined stability (Proposition 1.4)**

To prove the refined estimate

\[ |T_m[\Delta \Theta(t) - \Omega_{m, \beta} \Delta t]| \lesssim \varepsilon^{1/2}, \quad \Delta t = O(\beta), \quad (4.1) \]

we combine a bootstrap argument with the orbital stability result (Proposition 1.3). Indeed, we first show that the degree of adaptive rotation \( \Theta \) cannot change significantly over a small period of time (Lemma 4.3):

\[ |T_m[\Delta \Theta(t)]| \lesssim \varepsilon^{1/2}, \quad \Delta t = O(\varepsilon) \quad \text{if} \quad \varepsilon \lesssim \beta^2. \]

It means that our perturbed solution behaves very similarly to the rotating Kelvin wave at least for a short period of time (of order \( \varepsilon \)). Since the “Kelvin set” \( A_{m, \beta} \) rotates exactly under its own flow map, we can show that if we leave the set \( A_{m, \beta} \) in the perturbed solution for a short time, then the set lies on a small neighborhood of the precisely rotated Kelvin set \( A_{m, \beta}[\Omega_{m, \beta}] \):

\[ \phi(t, A_{m, \beta}) \subset \{ x \in \mathbb{R}^2 : \text{dist}(x, A_{m, \beta}[\Omega_{m, \beta}]) \lesssim \beta \cdot \varepsilon^{1/2} \}. \]

This detailed information leads to the above conclusion (4.1).

**Unconditional stability up to finite time without any adjusting rotation (Theorem 1.1)**

For any fixed time \( T > 0 \), we show that the rotating Kelvin wave is stable in \( L^1 \)-sense without any adjusting rotation and without a condition on the evolution such as (1.6). To do this, we add the above refined estimate (4.1) for small time repeatedly to derive a finite time result:

\[ |T[\Theta(t) - \Omega t]| \lesssim \left( \frac{T}{\beta} + 1 \right) \varepsilon^{1/2} \quad \text{for all} \quad t \in [0, T). \]

However, it requires that the perturbation should remain for the given time duration in a certain small ball containing the Kelvin wave [see the condition (1.6)]. By observing the dynamics of the Kelvin wave and by comparing it with the perturbed one, we derive the initial condition (1.4) that guarantees the hypothesis during the evolution.

**Filamentation (Theorem 1.5)**

To prove perimeter growth of boundary, we recall that the Kelvin waves are close to the unit disk when the parameter \( \beta > 0 \) is sufficiently small. We also note that the angular velocity of the disk has a non-trivial derivative in the radial direction outside the disk. As is well known, the further out of the disk, the slower the angular speed. This idea was already used in [17] when deriving an example of perimeter growth near the disk. Similarly, we take two points from the boundary of a perturbed patch, and trace their trajectories. From the above finite time stability, each trajectory remains arbitrary close to the original orbit from the Kelvin wave for a large desired amount of time. This process is possible by assuming that the perturbation is small enough in \( L^1 \). When considering any curve lying on the initial boundary connecting these points, the curve is transported by the perturbed flow so that its length increases by the difference multiplied by time.
4.2 Notations for Kelvin wave and simple estimates

If $\beta > 0$ is small enough so that the Kelvin wave $\omega^{m,\beta}$ exists, then we simply denote,

$$T = T_m, \quad T[\alpha] = T_m[\alpha] : \mathbb{R} \to T_m, \quad \Omega = \Omega^{m,\beta}, \quad g = g^{m,\beta},$$

$$\bar{A} = A^{m,\beta}, \quad \bar{A}_\alpha = R_\alpha[\bar{A}] \quad \text{for} \quad \alpha \in \mathbb{R},$$

where $R_\alpha$ is the (counter-clockwise) rotation map by the angle $\alpha$, and

$$\bar{\omega}_\alpha := \mathbf{1}_{\bar{A}_\alpha}, \quad \bar{\omega} = \bar{\omega}_0 := \mathbf{1}_{\bar{A}}.$$

We also set

$$\bar{I}_\alpha := \int_{\mathbb{R}^2} e^{im\theta} \omega_\alpha(x) dx = \int_{\bar{A}_\alpha} e^{im\theta} dx \in \mathbb{C}, \quad \bar{I} := \bar{I}_0 \in \mathbb{C}. \tag{4.2}$$

Here we use the polar coordinate $(r, \theta)$ for $x \in \mathbb{R}^2$. Then it is easy to check, for each $\alpha \in \mathbb{R}$,

$$\bar{I}_\alpha = \bar{I} e^{im\alpha} = \bar{I}_{T[\alpha]}.$$

We collect some properties of Kelvin waves.

**Lemma 4.1** There exist constants $c_i > 0$ for $i = 1, \ldots, 5$ such that

$$c_1 \beta \leq |\bar{I}| \leq c_2 \beta, \tag{4.3}$$

$$c_3 \cdot |T[\alpha]| \cdot \beta \leq |\bar{I} - \bar{I}_\alpha|, \quad \forall \alpha \in \mathbb{R}, \tag{4.4}$$

$$\sup_{\theta \in \mathbb{T}} |g'(\theta)| \leq c_4, \tag{4.5}$$

$$|\bar{A} \setminus \bar{A}_\alpha| \leq c_5 T[\alpha], \quad \forall \alpha \in \mathbb{R}, \tag{4.6}$$

hold for any sufficiently small $\beta > 0$.

**Proof** From the representation (1.2) of $\bar{\omega}$, we get (4.3). Then, by (4.2), we have

$$|\bar{I} - \bar{I}_\alpha| = |\bar{I}| |1 - e^{im\alpha}|,$$

which gives (4.4). Lastly, (4.5), (4.6) follow from $g^{m,\beta} \to g^{m,0} \equiv 0$ in $C^1$ as $\beta \to 0$ (e.g. see [34]). \qed

For $\eta \geq 0$ and for $\alpha \in \mathbb{R}$, we denote the $\eta$-neighborhood of $\bar{A}_\alpha$ by

$$\bar{A}^\eta_\alpha := \{x \in \mathbb{R}^2 : dist(x, \bar{A}_\alpha) < \eta\}, \quad \bar{A}^\eta := \bar{A}^\eta_0. \tag{4.7}$$

When $\beta > 0$ is sufficiently small, then we observe that

$$\bar{A}^\eta \subset \{(r, \theta) : r \leq 1 + g(\theta) + C\eta\}$$

for some $C > 0$ thanks to the estimate (4.5) in Lemma 4.1. It implies

**Lemma 4.2** There exists some $C_1 > 0$ such that

$$|\bar{A}^\eta_\alpha \setminus \bar{A}_\alpha| = |\bar{A}^\eta \setminus \bar{A}| \leq C_1 \eta, \quad \forall \alpha \in \mathbb{R} \tag{4.8}$$

holds for any sufficiently small $\beta > 0$. 
We also denote \( \Omega^* = \Omega^*(m) = \frac{m-1}{2m} > 0 \), and observe
\[
\Omega = \Omega^{m, \beta} \rightarrow \Omega^* \text{ as } \beta \rightarrow 0.
\]

From now on, we always assume \( \beta > 0 \) sufficiently small to have Lemmas 4.1, 4.2 and to satisfy \( \frac{1}{2} \Omega^* \leq \Omega \leq 2 \Omega^* \).

For any given \((t_0, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^2\), we denote the trajectory map \( \tilde{\phi}(\cdot, (t_0, x)) \) from the Kelvin wave solution \( \tilde{\omega}(t) = 1_{\tilde{A}_\Omega} \) with \( \tilde{\omega} := K * \omega \) by solving
\[
\begin{align*}
\begin{cases}
\frac{d}{dt} \tilde{\phi}(t, (t_0, x)) = \tilde{u}(t, \tilde{\phi}(t, (t_0, x))), \\
\tilde{\phi}(t_0, (t_0, x)) = x.
\end{cases}
\end{align*}
\]
(4.9)

We remark that \( \tilde{\omega} \) is Lipschitz in space-time from regularity of \( \partial \tilde{A} \), and
\[
\phi(t, (t_0, \tilde{A}_\Omega), (t_0, x)) = \tilde{\phi}(t, (t_0, x)), \quad \forall t, \forall t_0 \geq 0.
\]

Lastly, we take any constant \( \hat{C} > 0 \) such that any function \( f \in L^1 \cap L^\infty(\mathbb{R}^2) \) satisfies
\[
\| \frac{1}{|x|} * f \|_{L^\infty(\mathbb{R}^2)} \leq \hat{C} \left( \| f \|_{L^1} \| f \|_{L^\infty} \right)^{1/2}
\]
(4.10)
e.g. see Lemma 2.1 of [35]).

### 4.3 Only small jumps in \( \Theta(t) \)

When considering an initial data \( \omega_0 = 1_{A_0} \) for some \( m \)-fold symmetric open set \( A_0 \subset \mathbb{R}^2 \), we set
\[
I(t) = \int_{\mathbb{R}^2} e^{im\theta} \omega(t, x) dx \in \mathbb{C},
\]
where \( \omega(t) = 1_{A(t)} \) is the corresponding solution. As in (4.9), for given \((t_0, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^2\), the trajectory map \( \phi(\cdot, (t_0, x)) \) for the solution \( \omega(t) \) is defined by
\[
\begin{align*}
\begin{cases}
\frac{d}{dt} \phi(t, (t_0, x)) = u(t, \phi(t, (t_0, x))), \\
\phi(t_0, (t_0, x)) = x.
\end{cases}
\end{align*}
\]
(4.11)

We observe that the adjusting function \( \Theta \) in Proposition 1.3 satisfying (1.7) may not be continuous. Even, it does not have to be uniquely determined. We first prove that the function \( \Theta \) is allowed to have at most small jumps of order \( \sqrt{\varepsilon} \) (up to \( 2\pi/m \)-additions).

**Lemma 4.3** There exist constants \( \bar{\beta} > 0, \bar{K} > 0, \) and \( \bar{C} > 0 \) satisfying the following statement:
Let \( \beta \in (0, \bar{\beta}] \) and \( \tilde{\omega} = \omega^{m, \beta} \) be the Kelvin wave with \( r_0 = 1 \). If a \( m \)-fold symmetric solution \( \omega(t) = 1_{A(t)} \) with a function \( \Theta : [0, T) \rightarrow \mathbb{R} \) for some \( 0 < T \leq \infty \) satisfies
\[
\sup_{t \in [0, T)} \| \omega(t) - \tilde{\omega}_{\Theta(t)} \|_{L^1(\mathbb{R}^2)} \leq \varepsilon
\]
(4.12)
for some \( \varepsilon \in (0, \beta^2] \), then the function \( \Theta \) satisfies
\[
| \mathcal{T}[\Theta(t) - \Theta(t')] | \leq \bar{K} \cdot \varepsilon^{1/2}
\]
(4.13)
whenever \( t, t' \in [0, T) \) satisfies \( |t - t'| \leq \bar{C} \cdot \varepsilon \).
Proof Let \( \tilde{\beta} \in (0, \beta] \) be sufficiently small to satisfy all the estimates in Sect. 4.2, where \( \beta > 0 \) comes from Proposition 1.3, and consider \( \beta \in (0, \beta] \). For a simple presentation, we denote

\[ \Theta = \Theta(t), \quad \Theta' = \Theta(t'), \quad \omega = \omega(t), \quad \omega' = \omega(t'). \]

1. We first remark that

\[ \tilde{I}_{T[\Theta' - \Theta]} = \tilde{I}_{\Theta - \Theta'} \quad \text{and} \quad |\tilde{I} - \tilde{I}_{\Theta - \Theta'}| = |\tilde{I}_\Theta - \tilde{I}_{\Theta'}| \]

so that the conclusion (4.13) follows once we prove

\[ |\tilde{I}_\Theta - \tilde{I}_{\Theta'}| \leq c_3 \beta \tilde{K} \cdot \varepsilon^{1/2} \]

thanks to (4.4) in Lemma 4.1 (\( c_3 > 0 \) is the constant from the lemma).

2. We begin the estimate

\[ |\tilde{I}_\Theta - \tilde{I}_{\Theta'}| \leq |\tilde{I}_\Theta - I(t)| + |I(t) - I(t')| + |I(t') - \tilde{I}_{\Theta'}|. \tag{4.14} \]

By the stability assumption (4.12), we estimate the first term by

\[ |\tilde{I}_\Theta - I(t)| \leq \int_{\mathbb{R}^2} |\tilde{\omega}_\Theta - \omega| \, dx \leq \varepsilon. \]

Similarly, \( |I(t) - I(t')| \leq \varepsilon \). For the term in the middle, we estimate

\[ |I(t) - I(t')| \leq \|\omega - \omega'\|_{L^1} = |A(t') \Delta A(t)| = 2|A(t') \setminus A(t)|, \]

where \( |\cdot| \) is the Lebesgue measure in \( \mathbb{R}^2 \). Then, for the particle trajectory map \( \phi \) (4.11) from the solution \( \omega(t) \), we note

\[ A(t') = \phi(t', (t, A(t))), \quad A(t) = (\tilde{A}_\Theta \cap A(t)) \cup (A(t) \setminus \tilde{A}_\Theta), \]

and

\[ |\phi(t', (t, A(t) \setminus \tilde{A}_\Theta)))| = |A(t) \setminus \tilde{A}_\Theta| \leq \|\tilde{\omega}_\Theta - \omega(t)\|_{L^1} \leq \varepsilon. \]

Thus, we can estimate

\[ |A(t') \setminus A(t)| \leq |A(t') \setminus \tilde{A}_\Theta| + |(\tilde{A}_\Theta \setminus A(t))| \]

\[ \leq |\phi(t', (t, (\tilde{A}_\Theta \cap A(t))) \setminus \tilde{A}_\Theta)| + |\phi(t', (t, (A(t) \setminus \tilde{A}_\Theta))) + \varepsilon \]

\[ \leq |\phi(t', (t, \tilde{A}_\Theta))) \setminus \tilde{A}_\Theta| + 2\varepsilon. \tag{4.15} \]

3. We recall the flow speed is uniformly bounded for all time:

\[ \sup_{t \geq 0} \|u(t)\|_{L^\infty} \leq C \sup_{t \geq 0} \|\omega(t)\|_{L^1}^{1/2} \|\omega(t)\|_{L^\infty}^{1/2} \leq C_2 < \infty \]

for some \( C_2 > 0 \). Now we take \( \tilde{C} := (2C_2C_1)^{-1} \) and \( \tilde{K} := 7/c_3 \).

As a consequence of the previous step, we get

\[ \phi(t', (t, \tilde{A}_\Theta)) \subset \tilde{A}_\Theta^{C_2|t - t'|}, \]

which gives, from (4.15) and from (4.8),

\[ |A(t') \setminus A(t)| \leq |\tilde{A}_\Theta^{C_2|t - t'|} \setminus \tilde{A}_\Theta| + 2\varepsilon \leq C_1 C_2 |t' - t| + 2\varepsilon. \]
Thus, from (4.14), for $|t - t'| \leq \tilde{C}\varepsilon$, 
\[
|\tilde{I}_\theta - \tilde{I}_{\theta'}| \leq |I(t) - I(t')| + 2\varepsilon \leq 2|A(t') \setminus A(t)| + 2\varepsilon 
\leq 2C_2C_1\tilde{C}\varepsilon + 6\varepsilon = 7\varepsilon \leq 7\sqrt{\varepsilon} \beta \leq c_3\beta \tilde{K}\varepsilon^{1/2}.
\]
This finishes the proof. \qed

4.4 Proof of Proposition 1.4

Now we will prove refined stability (Proposition 1.4) using orbital stability (Proposition 1.3) and Lemma 4.3.

Proof of Proposition 1.4 We prove the result by a bootstrap argument.

1. We set $\beta_0 = \min\{\tilde{\beta}, \bar{\beta}\}$ where $\tilde{\beta}, \bar{\beta} > 0$ are the constants from Lemma 4.3 and Proposition 1.3, respectively. We take $c_0 \in (0, 1]$, which will be chosen sufficiently small during the proof [see (4.28)]. For $\beta \in (0, \beta_0]$, we consider a $m$-fold symmetric solution $\omega(t) = 1_{A(t)}$ with a function $\Theta : [0, T) \to \mathbb{R}$

\[
\sup_{t \in [0, T)} \|\omega(t) - \tilde{\omega}_{\Theta(t)}\|_{L^1(\mathbb{R}^2)} \leq \varepsilon \tag{4.16}
\]

for some $0 < T \leq \infty$ and for some $\varepsilon \in (0, c_0\beta^2]$.  

2. Fix any $t_0 \in [0, T)$. We will find some constants $c_0, C_0 > 0$, which are independent of the choice of $t_0$, satisfying the following property: 

Goal. For all $t \in [t_0, t_0 + c_0\beta] \cap [0, T)$,

\[
|T[\Theta(t) - (\Theta(t_0) + \Omega(t - t_0))]| \leq C_0 \cdot \varepsilon^{1/2}. \tag{4.17}
\]

For the rest of the proof, every time variable is assumed to be on $[0, T)$.

3. First, we prove the following claim:

**Initial claim.** There exists a constant $\eta = \eta(\varepsilon) > 0$ such that, for any $t \in [t_0, t_0 + \eta]$, we have

\[
|T[\Theta(t) - (\Theta(t_0) + \Omega(t - t_0))]| \leq \frac{1}{2} C_0 \cdot \varepsilon^{1/2}. \tag{4.18}
\]

This estimate (4.18) directly follows from Lemma 4.3. Indeed, the lemma implies that

\[
|T[\Theta(t) - (\Theta(t_0) + \Omega(t - t_0))]| \leq |T[\Theta(t) - \Theta(t_0)]| + |T[\Omega(t - t_0)]|
\leq \tilde{K}\varepsilon^{1/2} + 2\Omega^*|t - t_0| \quad \text{whenever} \quad |t - t_0| \leq \tilde{C}\varepsilon,
\]

where $\tilde{C}, \tilde{K}$ come from Lemma 4.3. We just take any constants $C_0 > 0$ large and $\eta = \eta(\varepsilon) > 0$ small to satisfy

\[
\tilde{K} \leq \frac{1}{4} C_0, \quad \eta \leq \tilde{C}\varepsilon, \quad \text{and} \quad 2\Omega^*\eta \leq \frac{1}{4} C_0 \varepsilon^{1/2},
\]

which gives (4.18).

4. From now on, we may assume that (4.17) is valid for $t \in [t_0, t^*]$ with some $t^* > t_0$, i.e.

\[
|T[\Theta(t) - (\Theta(t_0) + \Omega(t - t_0))]| \leq C_0 \cdot \varepsilon^{1/2}, \quad \forall t \in [t_0, t^*]. \tag{4.19}
\]

The existence of such a moment $t^* > t_0$ is guaranteed by **Initial claim** (4.18). We shall prove the following bootstrap claim:
Bootstrap claim. There exists a small constant $c_0 > 0$ such that if (4.19) holds for some $t^* \leq t_0 + c_0 \beta$, then we have for all $t \in [t_0, t^*]$,

$$|\mathcal{T}[\Theta(t) - (\Theta(t_0) + \Omega(t - t_0))]| \leq \frac{1}{2} C_0 \cdot \epsilon^{1/2}. \quad (4.20)$$

We note that the coefficient of $\sqrt{\epsilon}$ in (4.19) is $C_0$ while that in (4.20) is $(1/2)C_0$.

5. Before proving (4.20), we will perform a refined estimate for the trajectory map $\phi(t)$. First, we set the constant $\gamma_0 > 0$ by

$$\gamma_0 := \left(\frac{1}{2} c_1 \frac{1}{2} C_0\right) / (2C_1) > 0, \quad (4.21)$$

where $C_1 > 0$ is the constant from (4.8) of Lemma 4.2. Then we claim, for any $t \in [t_0, t^*] \subset [t_0, t_0 + c_0 \beta]$,

$$\phi(t, (t_0, \tilde{A}_{\theta(t_0)})) \subset \tilde{A}_{\theta(t_0) + \Omega(t - t_0)}^{\gamma_0 \beta \epsilon^{1/2}}. \quad (4.22)$$

Here, the superscript to a set $\tilde{A}_\alpha$ represents the neighborhood of the set [see (4.7)].

6. To prove (4.22), we fix any $x_0 \in \tilde{A}_{\theta(t_0)}$ and consider

$$\psi(t) := \tilde{\phi}((\Theta(t_0) / \Omega) + (t - t_0), (\Theta(t_0) / \Omega), x_0)). \quad (4.23)$$

where $\tilde{\phi}$ is the trajectory map (4.9) from the Kelvin wave the solution $\tilde{\omega}(t) = \mathbb{1}_{\tilde{A}_{\Omega t}}$. Then $\psi$ defined in (4.23) satisfies $\psi(t_0) = x_0$ and

$$\frac{d}{dt} \psi(t) = \tilde{u}((\Theta(t_0) / \Omega) + (t - t_0), \psi(t)). \quad (4.24)$$

As a result, we observe

$$\psi(t) \in \tilde{A}_{\theta(t_0) + \Omega(t - t_0)} \quad \text{for any} \quad t \geq t_0. \quad (4.25)$$

By denoting $\phi(t) := \phi(t, (t_0, x_0))$, we just need to show

$$|\phi(t) - \psi(t)| \leq \gamma_0 \cdot \beta \cdot \epsilon^{1/2}. \quad (4.26)$$

7. First we decompose

$$\frac{d}{dt} (\phi(t) - \psi(t)) = u(t, \phi(t)) - \tilde{u}((\Theta(t_0) / \Omega) + (t - t_0), \psi(t))$$

$$= u(t, \phi(t)) - \tilde{u}((\Theta(t_0) / \Omega) + (t - t_0), \phi(t))$$

$$+ \tilde{u}((\Theta(t_0) / \Omega) + (t - t_0), \phi(t)) - \tilde{u}((\Theta(t_0) / \Omega) + (t - t_0), \psi(t))$$

$$+ \tilde{u}((\Theta(t_0) / \Omega) + (t - t_0), \phi(t)) - \tilde{u}((\Theta(t_0) / \Omega) + (t - t_0), \psi(t))$$

$$=: I(t) + II(t) + III(t). \quad (4.27)$$

From the stability assumption (4.16), we have

$$|I(t)| \leq \tilde{C} \epsilon^{1/2}. \quad (4.28)$$

For $II(t)$, we first find $\tilde{\Theta}(t) \in \mathbb{R}$ satisfying

$$\tilde{\Theta}(t) = \Theta(t) + \frac{2\pi}{m} \cdot k \quad \text{for some integer} \quad k \quad \text{and} \quad \left(\tilde{\Theta}(t) - (\Theta(t_0) + \Omega(t - t_0))\right) \in \mathbb{T}. \quad (4.29)$$
We observe that $\bar{u} = u^{m, \beta}$ is time-periodic of period $\frac{2\pi}{m\Omega}$ and is Lipschitz (in space-time) of $\bar{u} = u^{m, \beta}$ where the Lipschitz norm is uniformly bounded when $\beta > 0$ is sufficiently small. Let’s denote the Lipschitz constant by $C_{Lip} = C_{Lip}(m) > 0$. Then we get

$$|II(t)| = |\bar{u}(\tilde{\Theta}(t)/\Omega, \phi(t)) - \bar{u}(\Theta(t_0)/\Omega + (t - t_0), \phi(t))|$$

$$\leq \frac{C_{Lip}}{\Omega} |\tilde{\Theta}(t) - (\Theta(t_0) + \Omega(t - t_0))|$$

$$= \frac{C_{Lip}}{\Omega} |T[\tilde{\Theta}(t) - (\Theta(t_0) + \Omega(t - t_0))]|$$

$$= \frac{C_{Lip}}{\Omega} |T[\Theta(t) - (\Theta(t_0) + \Omega(t - t_0))]|$$

$$\leq 2 \frac{C_{Lip}}{\Omega^*} \cdot (C_0 \cdot \varepsilon^{1/2}),$$

where we used the assumption (4.19) in the last inequality. For $III(t)$, we simply have

$$|III(t)| \leq C_{Lip} |\phi(t) - \psi(t)|.$$

Thus we have, for $t \in [t_0, t^*],$

$$\frac{d}{dt} |\phi(t) - \psi(t)| \leq C_{Lip} |\phi(t) - \psi(t)| + C_3 \varepsilon^{1/2},$$

where $C_3 > 0$ is some constant (depending only on $m$). With Gronwall’s inequality, we deduce when $t \in [t_0, t^*] \subset [t_0, t_0 + c_0\beta]$ that

$$|\phi(t) - \psi(t)| \leq e^{C_{Lip}c_0\beta} \cdot \int_{t_0}^{t_0 + c_0\beta} C_3 \varepsilon^{1/2} ds \leq \left(e^{C_{Lip}c_0\beta} C_3\right) \cdot c_0 \beta \cdot \varepsilon^{1/2}.$$

We just take a small constant $c_0 > 0$ satisfying

$$\left(e^{C_{Lip}c_0\beta} C_3\right) \cdot c_0 \leq \gamma_0$$

(see (4.21) for $\gamma_0$), which gives

$$|\phi(t) - \psi(t)| \leq \gamma_0 \cdot \beta \cdot \varepsilon^{1/2}.$$

Thanks to (4.24), we have proved (4.22) for any $t \in [t_0, t^*] \subset [t_0, t_0 + c_0\beta]$. Now we are ready to show the Bootstrap claim (4.20) for $t \in [t_0, t^*] \subset [t_0, t_0 + c_0\beta]$.

8. To prove, we simply denote

$$E_t := \phi(t, (t_0, \bar{A}_{\Theta(t_0)})), \quad t \geq t_0,$$

For instance, we observe $E_{t_0} = \phi(t_0, (t_0, \bar{A}_{\Theta(t_0)})) = \bar{A}_{\Theta(t_0)}$, and (4.22) gives

$$E_t \subset \bar{A}_{\Theta(t_0) + \Omega(t - t_0)}, \quad \forall t \in [t_0, t^*] \subset [t_0, t_0 + c_0\beta].$$

Towards a contradiction, suppose that the property (4.20) on the interval $[t_0, t^*] \subset [t_0, t_0 + c_0\beta]$ fails, i.e.

$$\text{there is some } t' \in [t_0, t^*] \text{ satisfying } |T[\Theta(t') - (\Theta(t_0) + \Omega(t' - t_0))]|$$

$$\geq \frac{1}{2} C_0 \cdot \varepsilon^{1/2}.$$  \hspace{1cm} (4.27)

From now one, we will show

$$\|\omega(t') - \omega_{\tilde{\Theta}(t')}\|_{L^1} \geq 2\varepsilon,$$
which gives a contradiction to the stability assumption (4.16). We begin the estimate with
\[
\|\omega(t') - \tilde{\omega}_\Theta(t')\|_{L^1} \geq \|\omega(t')\mathbf{1}_{E_{t'}} - \tilde{\omega}_\Theta(t')\|_{L^1} - \|\omega(t')\mathbf{1}_{(E_{t'})^c}\|_{L^1} =: I(t') - II(t').
\]
For \(II(t')\), we estimate
\[
\|\omega(t')\mathbf{1}_{E_{t'}} - \tilde{\omega}_\Theta(t')\|_{L^1} = \|\omega(t_0)\mathbf{1}_{E_{t_0}} - \tilde{\omega}_\Theta(t_0)\|_{L^1(E_{t_0})} + \|\omega(t_0) - \tilde{\omega}_\Theta(t_0)\|_{L^1(E_{t_0})} \leq \|\omega(t_0) - \tilde{\omega}_\Theta(t_0)\|_{L^1} \leq \varepsilon.
\]
For \(I(t')\), we have
\[
\|\omega(t')\mathbf{1}_{E_{t'}} - \tilde{\omega}_\Theta(t')\|_{L^1} \geq \|\tilde{\omega}_\Theta(t')\|_{L^1((E_{t'})^c)} \geq \|\tilde{\omega}_\Theta(t')\|_{L^1((\tilde{\Theta}(t_0)+\Omega(t'-t_0))^c)} \geq \|\tilde{\omega}_\Theta(t')\|_{L^1(\tilde{\Theta}(t_0)+\Omega(t'-t_0) - \tilde{\Theta}(t_0)+\Omega(t'-t_0)).}
\]

Then, by using (4.8), we continue to estimate
\[
\ldots \geq |\tilde{\Theta}(t') \setminus \tilde{\Theta}(t_0)+\Omega(t'-t_0)| - C_1 \cdot \gamma_0 \cdot \beta \varepsilon^{1/2} = \frac{1}{2} |\tilde{\Theta}(t') \Delta \tilde{\Theta}(t_0)+\Omega(t'-t_0)| - C_1 \cdot \gamma_0 \cdot \beta \varepsilon^{1/2} \geq \frac{1}{2} |\tilde{\Theta}(t') \setminus (\tilde{\Theta}(t_0)+\Omega(t'-t_0)) - \tilde{\Theta}(t') \setminus (\tilde{\Theta}(t_0)+\Omega(t'-t_0))| - C_1 \cdot \gamma_0 \cdot \beta \varepsilon^{1/2}.
\]

Now we can use (4.4) of Lemma 4.1 to get
\[
\ldots \geq \frac{1}{2} c_3 \beta |\mathcal{T}[\Theta(t') - (\Theta(t_0)+\Omega(t'-t_0))]| - C_1 \cdot \gamma_0 \cdot \beta \varepsilon^{1/2} \geq \frac{1}{2} c_3 \beta \frac{1}{2} C_0 \varepsilon^{1/2} - C_1 \cdot \gamma_0 \cdot \beta \varepsilon^{1/2},
\]
where we used the hypothesis (4.27) in the last inequality. Thanks to the definition of \(\gamma_0\) in (4.21), we have obtained
\[
\ldots \geq \frac{1}{4} c_3 \beta \frac{1}{2} C_0 \varepsilon^{1/2}.
\]
which gives
\[
\|\omega(t') - \tilde{\omega}_\Theta(t')\|_{L^1} \geq \frac{1}{4} c_3 \beta \frac{1}{2} C_0 \varepsilon^{1/2} - \varepsilon.
\]
We make \(c_0 > 0\) smaller than before (if necessary) to satisfy
\[
3\sqrt{c_0} \leq \frac{1}{4} c_3 \frac{1}{2} C_0.
\]
(4.28)
By this choice of \(c_0 > 0\), we get, whenever \(0 < \varepsilon \leq c_0 \beta^2\),
\[
\|\omega(t') - \tilde{\omega}_\Theta(t')\|_{L^1} \geq 2\varepsilon,
\]
which is a contradiction to (4.16). Hence, the hypothesis (4.27) cannot be true, which implies that we have proved Bootstrap claim (4.20) for \([t_0, t^*] \subset [t_0, t_0 + c_0 \beta]\).

9. Lastly, we are ready to show Bootstrap claim (4.20) for \([t_0, t^*] \subset [t_0, t_0 + c_0 \beta]\) since we can extend the interval satisfying Goal (4.17) by applying Bootstrap claim (4.20) with Initial claim (4.18). Indeed, we know that there is \(t^* \in (t_0, t_0 + c_0 \beta)\) such that Goal (4.17) on the
interval \([t_0, t^*]\) holds by using Initial claim (4.18). Then by applying Bootstrap claim (4.20) on the interval \([t_0, t^*]\), we get
\[
|T[\Theta(t) - (\Theta(t_0) + \Omega(t - t_0))]| \leq \frac{1}{2} C_0 \cdot \varepsilon^{1/2}, \quad \forall t \in [t_0, t^*].
\]

Then we use Initial claim (4.18) by replacing \(t_0\) with \(t^*\) so that we get, for any \(t \in [t^*, t^* + \eta]\),
\[
|T[\Theta(t) - (\Theta(t^*) + \Omega(t - t^*))]| \leq \frac{1}{2} C_0 \cdot \varepsilon^{1/2}.
\]

By adding the above two estimates, we get, for any \(t \in [t^*, t^* + \eta]\),
\[
|T[\Theta(t) - (\Theta(t_0) + \Omega(t - t_0))]| \leq C_0 \cdot \varepsilon^{1/2}.
\]

In short, we have obtained Goal on the extended interval \([t_0, t^* + \eta]\). By repeating this process, we can get Goal by using (4.7) on the interval \([t_0, t^* + n\eta]\) for each \(n \geq 1\) until the process eventually covers \([t_0, t_0 + c_0\beta]\). \(\square\)

**Remark 4.4** In Proposition 1.3, we can always take \(\Theta(0) = 0\) (by assuming \(\delta \leq \varepsilon\) if necessary). Then summing the estimate (1.8) of Proposition 1.4 gives
\[
|T[\Theta(t) - \Omega t]| \leq C_0 \cdot \varepsilon^{1/2} \left(\frac{t}{c_0\beta} + 1\right) \quad \text{for all} \quad t \in [0, T).
\]

### 4.5 Proof of Theorem 1.1

Here we will prove finite time stability (Theorem 1.1) by using orbital stability (Proposition 1.3) with refined stability (Proposition 1.4). For \(\beta > 0\), we denote
\[
\tilde{m} = \tilde{m}(\beta) := \sup_{\theta \in \Gamma}(1 + g(\theta)) > 0. \tag{4.29}
\]

The next lemma says that when \(\beta > 0\) is small enough, the trajectories induced from the Kelvin wave starting near the wave remain close.

**Lemma 4.5** (I) For each \(\tau > 0\), there exist \(\beta' > 0\) and \(\mu > 0\) such that if \(\beta \in [0, \beta']\), then
\[
\bigcup_{t \geq 0} \tilde{\phi}(t, (0, \tilde{A}^\mu)) \subset B_{\tilde{m} + \tau},
\]

where \(\tilde{\phi}\) is the trajectory of \(\tilde{\phi}(t) = 1_{\tilde{A}^\mu}\) as in (4.9), and \(\tilde{A}^\mu := (\text{dist}(x, \tilde{A}) < \mu)\) as in (4.7).

(II) There exists \(\kappa \in (0, 1)\) such that for any \(\tau > 0\), there exists \(\beta' > 0\) such that if \(\beta \in [0, \beta']\), then for any \(x \in \mathbb{R}^2\) with \(1/2 \leq |x| \leq 1 + \kappa\), we get, for any \(t \geq 0\),
\[
\tilde{\phi}(t, (0, x)) \in \left( B_{|x| + \tau} \setminus B_{|x| - \tau} \right).
\]

**Proof** The first statement simply follows from the second statement. The second statement simply follows the facts that
\[
\psi^{m,\beta} \to \psi^{m,\beta}|_{\beta=0} \quad \text{in} \quad C^0\text{-norm on} \quad \overline{B_2} \quad \text{as} \quad \beta \to 0
\]
(e.g. see [34]), where \(\psi^{m,\beta}\) is the relative stream defined in (2.1), and that all the level sets of \(\psi^{m,\beta}|_{\beta=0}\) are circles. Indeed, we recall
\[
-\partial_r \psi^{m,\beta}|_{\beta=0}(r, \theta) = -\partial_r \left( G1_{B_1} + \frac{1}{\beta} (m - \frac{1}{2} m^2 r^2) \right) = \begin{cases} \frac{1}{2} \frac{m-1}{2m} r, & r \leq 1, \\ \frac{1}{2} \frac{m-1}{2m} r, & r > 1 \end{cases},
\]

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which gives
\[ \inf_{r \in [1/3, 1+2\kappa]} (-\partial_r \psi^{m,\beta}|_{\beta=0}(r, \theta)) \geq c > 0 \]
for some \( c = c(m) > 0 \) and for some small \( \kappa = \kappa(m) > 0 \). We consider any small \( \tau > 0 \) such that \( [(1/2) - \tau, 1 + \kappa + \tau] \subset [1/3, 1 + 2\kappa] \subset [0, 2] \). Denote
\[ \tilde{\psi} := \psi^{m,\beta}, \quad \tilde{\psi} := \psi^{m,\beta}|_{\beta=0}. \]
For any given point \( x \in \mathbb{R}^2 \) satisfying \( 1/2 \leq |x| \leq 1 + \kappa \), we take any points \( y', y'' \in \mathbb{R}^2 \) such that \( |y'| = |x| - \tau \) and \( |y''| = |x| + \tau \). We observe that \( \tilde{\psi} \) is radially symmetric and
\[ \tilde{\psi}(y') - \tilde{\psi}(x) \geq c\tau. \]
Then, by using the uniform convergence \( \tilde{\psi} \to \hat{\psi} \), we can take \( \beta > 0 \) small enough to get
\[ \sup_{|y'| = |x| + \tau} |\tilde{\psi}(y') - \hat{\psi}(y')| \leq \frac{c\tau}{8}, \quad \sup_{|y'| = |x| - \tau} |\tilde{\psi}(y'') - \hat{\psi}(y'')| \leq \frac{c\tau}{8}, \quad |\tilde{\psi}(x) - \hat{\psi}(x)| \leq \frac{c\tau}{8}, \]
Thus we get
\[ \sup_{|y'| = |x| + \tau} \tilde{\psi}(y) < \tilde{\psi}(x) < \inf_{|y'| = |x| - \tau} \tilde{\psi}(y), \]
which implies that the connected component of the level set of \( \tilde{\psi} \) containing the point \( x \) completely lies on the annulus \( B_{|x|+\tau} \setminus \overline{B}_{|x|-\tau} \). Since the trajectory \( \tilde{\psi}(t, (0, x)), t \geq 0 \) should lie on the connected component of the level set of \( \tilde{\psi} \) containing the point \( x \), we are done. \( \square \)

To prove Theorem 1.1, we just need the first statement of the above lemma while the second one will be used in the next subsection when proving Theorem 1.5.

**Proof of Theorem 1.1** 1. We first borrow the constant \( \beta_0 > 0 \) from Proposition 1.4, and consider small \( \beta_2 \in (0, \beta_0] \) satisfying
\[ \tilde{m}(\beta) \leq 1 + 2\beta < \tilde{r} \quad \text{and} \quad \tilde{n}(\beta) \geq 1 - 2\beta, \quad \forall \beta \in (0, \beta_2], \]
where \( \tilde{m} = \tilde{m}(\beta) \in (0, \tilde{r}) \) is defined in (4.29), where \( \tilde{r} > 0 \) is the constant required in (1.6) for orbital stability of Proposition 1.3, and where \( \tilde{n}(\beta) \) is the minimum radius of the Kelvin wave: \( \tilde{n} = \tilde{n}(\beta) := \inf_{\theta \in \mathbb{T}} (1 + g(\theta)) > 0 \). Then we simply set
\[ \tau := \frac{\tilde{r} - (1 + 2\beta_2)}{2} > 0, \]
and take the two constants \( \beta' = \beta'(\tau) > 0 \) and \( \mu = \mu(\tau) > 0 \) from (I) of Lemma 4.5. Let \( \beta_1 > 0 \) small enough to have
\[ \beta_1 \leq \min(\beta_2, \beta') \quad \text{and} \quad 2\beta_1 \leq \frac{\mu}{2}. \]
We also set
\[ r' := 1 + \frac{\mu}{2} > 1. \]
We may assume
\[ r' < \tilde{r} \]
(by redefining \( \mu > 0 \) if necessary). Then, (I) of Lemma 4.5 says that
\[ \bigcup_{t \geq 0} \hat{\phi}(t, (0, \tilde{A}^{\mu})) \subset B_{\tilde{m}+\tau}. \]
2. From now on, we fix any \( \beta \in (0, \beta_1] \). Let \( T, \varepsilon' > 0 \). We define \( C_4 = C_4(T, \beta) > 0 \) by
\[
C_4(T, \beta) := C_0 \cdot \left( \frac{T}{c_0 \beta} + 1 \right),
\]
where \( c_0, C_0 > 0 \) are the constants from Proposition 1.4. We set \( C_5 = C_5(T, \beta) > 0 \) by
\[
C_5(T, \beta) := \hat{C} + \frac{2 \cdot C_{Lip}}{\Omega} \cdot C_4(T, \beta),
\]
(4.36)
where \( \hat{C} > 0 \) comes from (4.10). We consider any small \( \varepsilon \in (0, c_0 \beta^2] \) satisfying
\[
\left( e^{C_{Lip} \cdot C_5} \right) \cdot T \cdot \varepsilon^{1/2} \leq \frac{t}{2} \quad \text{and} \quad \varepsilon + 2c_5 \varepsilon^{1/2} \leq \varepsilon',
\]
(4.37)
where \( c_5 > 0 \) is the constant of (4.6) in Lemma 4.1.
Then, Proposition 1.3 together with summing the estimate (1.8) of Proposition 1.4 says that there is \( \delta' := \delta(\beta, \varepsilon) > 0 \), where \( \delta(\beta, \varepsilon) \) is the constant from Proposition 1.3, such that if a \( m \)-fold symmetric initial data \( \omega_0 = 1_{A_0} \) satisfies
\[
\| \omega_0 - \omega \|_{L^1} \leq \delta'
\]
and if the corresponding solution \( \omega(t) = 1_{A(t)} \) satisfies
\[
\text{Range hypothesis for } t' : \cup_{t \in [0, t')} A(t) \subset B_{\delta'}
\]
(4.38)
for some \( t' \in (0, T] \), then there exists a function \( \Theta : [0, t') \rightarrow \mathbb{T} \) such that
\[
\sup_{t \in [0, t')} \| \omega(t) - \omega \Theta(t) \|_{L^1(\mathbb{R}^2)} \leq \varepsilon
\]
(4.39)
and
\[
\sup_{t \in [0, t')} | T[\Theta(t) - \Omega t] | \leq C_5 \varepsilon^{1/2}.
\]
(4.40)
(e.g. see Remark 4.4).
3. From now on, we consider any \( m \)-fold symmetric initial data \( \omega_0 = 1_{A_0} \) satisfying the initial condition (1.4). We will show that Range hypothesis (4.38) is valid for \( t' = T \). First, due to the initial assumption with (4.34), the hypothesis is true for some \( t' > 0 \) since the flow speed is uniformly bounded. For a contradiction, let’s suppose that the hypothesis fails for \( t' = T \). Then, there exists some moment \( T_0 \in (0, T) \) such that
\[
\text{Range hypothesis holds for } t' = T_0 \text{ while the hypothesis fails for every } t' > T_0.
\]
(4.41)
4. We note that since the hypothesis is true for \( t' = T_0 \), the estimates (4.39) and (4.40) hold for \( t' = T_0 \). For \( x \in \mathbb{R}^2 \), we denote \( \phi(t) := \phi(t, (0, x)) \) from (4.11) and \( \tilde{\phi}(t) := \tilde{\phi}(t, (0, x)) \) from (4.9). In the computations below, we consider \( t \in [0, T_0) \). Similarly in (4.25), we compute
\[
\frac{d}{dt} \left( \phi(t) - \tilde{\phi}(t) \right) = u(t, \phi(t)) - \tilde{u}(t, \tilde{\phi}(t))
\]
\[
= u(t, \phi(t)) - \tilde{u}(\Theta(t) / \Omega, \phi(t)) + \tilde{u}(\Theta(t) / \Omega, \phi(t)) - \tilde{u}(t, \phi(t)) + \tilde{u}(t, \phi(t)) - \tilde{u}(t, \tilde{\phi}(t)) =: I(t) + II(t) + III(t).
\]
(4.42)
Then the estimate (4.39), we have

\[ |I(t)| \leq \hat{C} \varepsilon^{1/2}. \]

For II (t), as in (4.26), we first find \( \tilde{\Theta}(t) \) by

\[ \tilde{\Theta}(t) = \Theta(t) + \frac{2\pi}{m} \cdot k \quad \text{for some integer } k \quad \text{and } \left( \tilde{\Theta}(t) - \Omega t \right) \in \mathbb{T}. \]

Then, by using time-periodicity and (space-time) Lipschitz continuity of \( \bar{u} \), we get

\[ |II(t)| = |\tilde{u}(\tilde{\Theta}(t)/\Omega, \phi(t)) - \tilde{u}(t, \phi(t))| \leq \frac{2 \cdot C_{Lip}}{\Omega^*} |\tilde{\Theta}(t) - \Omega t| \]

\[ = \frac{2 \cdot C_{Lip}}{\Omega^*} |T[\tilde{\Theta}(t) - \Omega t]| = \frac{2 \cdot C_{Lip}}{\Omega^*} |T[\Theta(t) - \Omega t]| \]

\[ \leq \frac{2 \cdot C_{Lip}}{\Omega^*} \cdot C_4 \varepsilon^{1/2}, \]

where we used the estimate (4.40) in the last inequality. For III (t), we get

\[ |III(t)| \leq C_{Lip} |\phi(t) - \tilde{\phi}(t)|, \]

which gives,

\[ \frac{d}{dt} |\phi(t) - \tilde{\phi}(t)| \leq C_{Lip} |\phi(t) - \tilde{\phi}(t)| + C_5 \varepsilon^{1/2}. \]

where \( C_5 > 0 \) was already defined in (4.36). With Gronwall’s inequality and with the smallness assumption (on \( \varepsilon \)) (4.37), we get

\[ |\phi(t) - \tilde{\phi}(t)| \leq \left( e^{C_{Lip} T} C_5 \right) \cdot T \cdot \varepsilon^{1/2} \leq \frac{\tau}{2}. \]

Together with the fact (4.35) and the definition (4.31) of \( \tau \), the argument above implies

\[ \bigcup_{t \in [0, T_0]} \phi(t, (0, \tilde{A}^\mu)) \subset B(\bar{m} + \tau) + (\tau/2). \]

which gives

\[ \bigcup_{t \in [0, T_0]} \phi(t, (0, \tilde{A}^\mu)) \subset B(\bar{m} + \tau) + (\tau/2). \]

On the other hand, the definition (4.33) of \( r' \) together with (4.32) and (4.30) implies

\[ B(\bar{n} + \mu) \subset \tilde{A}^\mu. \]

Thus, the assumption \( A_0 \subset B(\bar{n} + \mu) \) gives

\[ \bigcup_{t \in [0, T_0]} \bar{A}(t) = \bigcup_{t \in [0, T_0]} \phi(t, (0, A_0)) \subset \bigcup_{t \in [0, T_0]} \phi(t, (0, \tilde{A}^\mu)) \subset \overline{B(\bar{m} + \tau) + (\tau/2)}. \]

On the other hand, we observe

\[ (\bar{m} + \tau) + (\tau/2) < \bar{r} \]

thanks to (4.31), (4.30). By recalling that the flow speed is bounded, there should exist some moment \( T_1 > T_0 \) such that Range hypothesis is true for \( t' = T_1 \), which contradicts the assumption (4.41). Hence, Range hypothesis (4.38) for \( t' = T \) is valid. As a result, we obtain the estimates (4.40), (4.39) for \( t' = T \).
5. Lastly, by the estimate (4.6) of Lemma 4.1 and by (4.39), (4.40) for \( t' = T \), we get, for any \( t \in [0, T) \),

\[
\| \omega(t) - \tilde{\omega}\|_{L^1} \leq \| \omega(t) - \tilde{\omega}_{\Theta(t)} \|_{L^1} + \| \tilde{\omega}_{\Theta(t)} - \tilde{\omega}\|_{L^1} \leq \varepsilon + 2|\tilde{\Delta}\Omega(t) - \tilde{\Delta}\Omega| \leq \varepsilon + 2c_5|T[\Theta(t) - \Omega t]| \leq \varepsilon + 2c_5C_4\varepsilon^{1/2}.
\]

By the smallness assumption (4.37) (on \( \varepsilon \)), we get the stability ((1.4) \( \Rightarrow \) (1.5)\( \varepsilon' \)). It finishes the proof of Theorem 1.1. \( \square \)

### 4.6 Proof of Theorem 1.5

Now we are ready to prove *perimeter growth theorem* (Theorem 1.5) by using *finite time stability* (Theorem 1.1) and (11) of Lemma 4.5.

**Proof of Theorem 1.5**

1. We recall that the angular velocity \( \hat{u}_\theta \) of \( \hat{u} := K * 1_{B_1} \) is

\[
\hat{u}_\theta(r) = \begin{cases} 
\frac{1}{2}, & r \leq 1, \\
\frac{1}{2}r^2, & r > 1
\end{cases}
\]

2. We borrow the constants \( \beta_1 > 0, r' > 1 \) from Theorem 1.1 and set \( \mu := r' - 1 > 0 \). We also take the constant \( \kappa > 0 \) from (11) of Lemma 4.5. We set \( r_1 := 1 \) and \( r_2 := 1 + \frac{\min\{\mu, \kappa\}}{2} \).

We note \( r_2 < r' \) and \( r_2 < 1 + \kappa \). Then we consider any constant \( \tau \in (0, 1/4] \) satisfying

\[
\tau \leq \frac{\min\{\mu, \kappa\}}{20},
\]

which will be chosen again to be small during the proof.

3. We denote the intervals

\[
I_i = [r_i - 2\tau, r_i + 2\tau], \quad I'_i = [r_i - \tau, r_i + \tau]
\]

for \( i = 1, 2 \). We set

\[
U^1 := \left( \inf_{r \in I_1} \hat{u}_\theta(r) \right) > 0 \quad \text{and} \quad U^2 := \left( \sup_{r \in I_2} \hat{u}_\theta(r) \right) > 0.
\]

Since \( \hat{u}_\theta(r_1) > \hat{u}_\theta(r_2) \) and \( \hat{u}_\theta \) is continuous, we can assume \( U^1 - U^2 > 0 \) by making \( \tau > 0 \) smaller than before (if necessary). By fixing such a constant \( \tau > 0 \), we take \( \beta' = \beta'(\tau) > 0 \) from (11) of Lemma 4.5. We also denote

\[
\bar{U}^1 := U^1 - \frac{U^1 - U^2}{4} \quad \text{and} \quad \bar{U}^2 := U^1 + \frac{U^1 - U^2}{4},
\]

and note that \( \Delta \bar{U} := \bar{U}^1 - \bar{U}^2 > 0 \).

4. Let \( M > 0 \) and \( \delta > 0 \). We take any large \( T > 0 \) such that

\[
(T \Delta \bar{U} - 2\pi) > 2M.
\]

(4.43)

Let \( \varepsilon > 0 \) be small enough to satisfy

\[
\bar{C}(\varepsilon)^{1/2} \leq \frac{U^1 - U^2}{8},
\]

(4.44)
where \( \hat{C} > 0 \) comes from (4.10), and
\[
\left( e^{C_{\text{lip}} T} \right) \cdot T \cdot \hat{C}(\varepsilon)^{1/2} \leq \tau. \tag{4.45}
\]

5. From now on, we consider a sufficiently small \( \beta \in (0, \min\{\beta_1, \beta'\}] \) satisfying the following:

(a) The perimeter of \( \partial \bar{A} \) is smaller than 10.
(b) \( \partial A \cap \{ r = 1 \} \neq \emptyset \)
(c) The velocity \( \bar{u} = K \ast 1_{\bar{A}} \) for the Kelvin wave with parameter \( \beta > 0 \) is close enough to the velocity \( \hat{u} \) for the circular patch in the sense that
\[
\| \bar{u} - \hat{u} \|_{L^\infty} \leq \frac{U^1 - U^2}{8}. \tag{4.46}
\]

We may assume \( \delta > 0 \) small enough to satisfy \( \delta \leq \delta(m, \beta, \varepsilon, T) \), where \( \delta(m, \beta, \varepsilon, T) > 0 \) is the constant from Theorem 1.1.

6. We take any initial data \( 1_{A_0} \) with the following properties:

(a) \( A_0 \) is an open \( m \)-fold symmetric set with \( C^\infty \)-smooth connected boundary \( \partial A_0 \).
(b) The perimeter of \( \partial A_0 \) is smaller than 20.
(c) \( A_0 \subset B_{r'} \) and \( \| \omega_0 - \tilde{\omega} \|_{L^1(\mathbb{R}^2)} \leq \delta \).
(d) For each \( i = 1, 2 \), \( \exists \) a point \( x^i = (r_i \cos \theta_i, r_i \sin \theta_i) \in \partial A_0 \) satisfying \( |\theta_i| \leq \frac{2\pi}{m} \).

Then, Theorem 1.1 implies that the perturbed solution \( \omega(t) = 1_{A_t} \) satisfies
\[
\sup_{t \in [0,T]} \| \omega(t) - \tilde{\omega}_{\Omega t} \|_{L^1(\mathbb{R}^2)} \leq \varepsilon'. \tag{4.47}
\]

Set \( L_0 : [0, 1] \to \partial A_0 \) be an injective parametrized curve lying on the sector \( \{ (r, \theta) : \theta \in \mathbb{T} \} \) satisfying \( L_0(0) = x^1, L_0(1) = x^2 \), and consider \( L_T := \phi(t, (0, L_0)) \). We will show that the length of the curve \( L_T \) is larger than \( M \), which finishes the proof thanks to \( L_T \subset \partial A_T \).

7. For \( i = 1, 2 \), we denote
\[
\phi^i(t) := \phi(t, (0, x^i)) \quad \text{and} \quad \tilde{\phi}^i(t) := \tilde{\phi}(t, (0, x^i)),
\]
where \( \phi, \tilde{\phi} \) are the trajectories from the perturbed solution \( \omega(t) \) and the Kevin wave solution \( \tilde{\omega}(t) = 1_{\tilde{A}_{\Omega t}} \) as in (4.11), (4.9), respectively. For any interval \( I = [a, b] \subset \mathbb{R}_{>0} \), we denote the annulus
\[
R_I := \overline{B_b} \setminus B_a.
\]

We observe
\[
\tilde{\phi}^i(t) \in R_{I'}, \quad \forall t \geq 0, \quad i = 1, 2. \tag{4.48}
\]
by using (II) of Lemma 4.5.

8. We claim
\[
\phi^i(t) \in R_{I_i}, \quad \forall t \in [0, T] \tag{4.49}
\]
for \( i = 1, 2 \). Thanks to (4.48), it is enough to show that
\[
|\phi^i(t) - \tilde{\phi}^i(t)| \leq \tau, \quad \forall t \in [0, T].
\]
Similarly in (4.25), (4.42), we compute
\[
\frac{d}{dt}(\phi(t) - \tilde{\phi}(t)) = u(t, \phi(t)) - \tilde{u}(t, \tilde{\phi}(t)) = u(t, \phi(t)) - \tilde{u}(t, \tilde{\phi}(t)) \\
+ \tilde{u}(t, \phi(t)) - \tilde{u}(t, \tilde{\phi}(t)) =: I(t) + II(t).
\]

From the stability (4.47), we have
\[
|I(t)| \leq \hat{C}(\varepsilon')^{1/2}.
\]
For II(t), we note Lipschitz continuity of \(\tilde{u}\) to get
\[
|II(t)| \leq CLip |\phi(t) - \tilde{\phi}(t)|.
\]
Thus we get, for \(t \in [0, T]\),
\[
\frac{d}{dt}|\phi(t) - \tilde{\phi}(t)| \leq CLip |\phi(t) - \tilde{\phi}(t)| + \hat{C}(\varepsilon')^{1/2}.
\]

With Gronwall’s inequality, we get
\[
|\phi(t) - \tilde{\phi}(t)| \leq \left(e^{CLip T} \cdot T \cdot \hat{C}(\varepsilon')^{1/2}.\right.
\]

From the smallness assumption (4.45) on \(\varepsilon' > 0\), we obtain the claim (4.49).

9. Lastly, we observe
\[
|u(t, \phi^i(t)) - \tilde{u}(\phi^i(t))| \leq |u(t, \phi^i(t)) - \tilde{u}(t, \phi^i(t))| + |\tilde{u}(t, \phi^i(t)) - \tilde{u}(\phi^i(t))| \leq \frac{U^1 - U^2}{4},
\]
where the last inequality follows from (4.47), (4.44), (4.46). Thus the above claim (4.49) implies that the angular velocity of \(u(t, \phi^1(t))\) is bigger than \(\bar{U}^1\) while that of \(\tilde{u}(t, \phi^2(t))\) is smaller than \(\bar{U}^2\). Thus we simply observe that the difference between the winding number (with respect to the origin) of trajectory \(\phi^1(t)\) on \([0, T]\) starting at \(x^1\) and the winding number of trajectory \(\phi^2(t)\) starting at \(x^2\) is bigger than
\[
\frac{(T \Delta \bar{U} - 2\pi)}{2\pi}.
\]
Since
\[
\phi^i(t) \in R_{L_i} \subset \mathbb{R}^2 \setminus B_{1/2}, \quad i = 1, 2
\]
on \([0, T]\), our choice (4.43) of \(T\) implies that the length of \(L(T)\) should be larger than \(M\).

The proof is complete. \(\Box\)

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Appendix A: Stability of the annulus

In this section, we provide a sketch of the fact that any annulus is a strict local maximum of the energy within a suitable admissible class of patches. Based on this fact, one can derive nonlinear stability and instability results as in the Kelvin wave case. We believe that this is interesting at least for the following reasons:

- While it is known that monotone decreasing and radial vorticities define nonlinear stable steady states (e.g. see [14]), this seems to be a fist instance where nonlinear stability for non-monotone radial solution can be obtained. Moreover, long time filamentation can be proved near any annulus using the nonlinear stability.
- It is likely that under certain mass, impulse, and $m$-fold symmetry constraint, there exist at least two strict local maximum of the energy, one given by an $m$-fold symmetric Kelvin wave and the other being an annulus, especially when both of them are sufficiently close to the disc. (Strictly speaking, we do not know the precise range of existence/stability in $\beta$ for Kelvin waves with large $m$.)

A.1 Admissible class and key proposition for the annulus

For $0 < r_1 < r_2$, we consider the annulus

$$\tilde{\omega}_{r_1, r_2} := 1_{[r_1, r_2]}(r).$$

We shall often omit writing out the subscripts $r_1$ and $r_2$, and define the admissible class of perturbations

$$\mathcal{A}[\tilde{\omega}] := \left\{ \tilde{\omega} = 1_A : \int \tilde{\omega} = \int \tilde{\omega}, \int |x|^2 \tilde{\omega} = \int |x|^2 \tilde{\omega} \right\}$$

and set

$$\mathcal{A}^m = \mathcal{A}[\tilde{\omega}] \cap \{ \tilde{\omega} \text{ is } m\text{-fold symmetric} \}.$$

It is interesting to note that, imposing the mass and impulse constraint simultaneously picks out (at most) one annulus. Next, we set

$$\mathcal{N}_{E,D}[\tilde{\omega}] := \left\{ \tilde{\omega} = 1_A : A \subset D, \|\tilde{\omega} - \tilde{\omega}\|_{L^1} < \varepsilon \right\}.$$

Let us now state our key proposition.

**Proposition A.1** For any $0 < r_1 < r_2$, there exist $m \geq 2$, $\tilde{r} > 1$, $\varepsilon_0 > 0$, and $c_0 > 0$ such that

$$E[\tilde{\omega}_{r_1, r_2}] - E[\omega] \geq c_0 \|\tilde{\omega}_{r_1, r_2} - \omega\|_{L^1}^2$$

for any $\omega \in \mathcal{A}^m \cap \mathcal{N}_{E,D}[\tilde{\omega}_{r_1, r_2}]$ with $0 < \varepsilon < \varepsilon_0$.

All of the constants $m, \tilde{r}, \varepsilon_0$, and $c_0$ depend on $r_1$ and $r_2$ in a rather complicated way. The rest of this section is devoted to the proof of the above proposition. We omit the details as the arguments are parallel to the case of the Kelvin waves.
A.2 Relative stream function

We now modify the stream function of $\tilde{\phi}$ in a way that it vanishes on the boundary of the annulus. We recall that for any radial vorticity $\tilde{\omega}$, $\tilde{G} := G[\tilde{\omega}] = \frac{1}{2\pi} \ln \frac{1}{|x|} \ast \tilde{\omega}$ is given by

$$-\partial_r \tilde{G} = \frac{1}{r} \int_0^r s \tilde{\omega}(s) ds.$$ 

Indeed, using the above formula it is immediate to see that $\Delta \tilde{G} = (\partial_r^2 + \frac{\partial^2}{r^2}) \tilde{G} = -\tilde{\omega}$. We see that

$$-\partial_r \tilde{G}(r) = \begin{cases} 0 & r \leq r_1, \\ r/2 - r_1^2/(2r) & r_1 < r \leq r_2, \\ (r_2^2 - r_1^2)/(2r) & r_2 < r. \end{cases}$$

We have that $\tilde{G}(0) = \int_{r_1}^{r_2} r \ln \frac{1}{r} dr$ and $\tilde{G}(r)$ is monotone decreasing in $r$. We claim that there exists a unique pair of constants $C_0, C_1$ such that the relative stream function defined by

$$\tilde{\psi} = \tilde{G} + C_0 + C_1 r^2$$

(A.1)

satisfies

$$\tilde{\psi}(r_1) = \tilde{\psi}(r_2) = 0.$$ 

The unique choice is given by

$$\tilde{\psi} = \tilde{G} - \tilde{G}(0) - C_1 r_1^2 + C_1 r^2, \quad C_1 := \frac{1}{4} - \frac{r_1^2}{2(r_2^2 - r_1^2)} \ln \frac{r_2}{r_1} > 0.$$ 

We note that

$$\partial_r \tilde{\psi}(r_1) = 2C_1 r_1 > 0, \quad \partial_r \tilde{\psi}(r_2) = 2C_1 r_2 + \tilde{G}'(r_2) = -\frac{r_2^2 r_2}{r_2^2 - r_1^2} \ln \frac{r_2}{r_1} + \frac{r_1^2}{2r_2} < 0$$

for any $0 < r_1 < r_2$. See Fig. 2 for a plot of $\tilde{G}$ and $\tilde{\psi}$ in the case $r_1 = 1/2$ and $r_2 = 1$. Note that there is a critical radius $r^* > r_2$ (depending on $r_1$ and $r_2$) such that $\tilde{\psi}(r^*) = 0$ and $\tilde{\psi}(r) < 0$ for $r_2 < r < r^*$. This determines $r^*$ in Proposition A.1; we need to take $1 < \tilde{r} < r^*/r_2$.

A.3 Graph type perturbation

We fix some $\tilde{\omega} = \tilde{\omega}_{r_1, r_2}$ and consider graph type perturbations, which are described by a pair of functions defined on $S^1$; given $(h_1(\theta), h_2(\theta))$ which are assumed to be sufficiently small (depending on $r_1$ and $r_2 - r_1$) in the $C^1$ norm, we set

$$\tilde{\omega}_{h_1, h_2} := 1_{\tilde{A}_{h_1, h_2}}, \quad \tilde{A}_{h_1, h_2} = \{(r, \theta) : r_1 + h_1(\theta) < r < r_2 + h_2(\theta)\}.$$ 

We easily compute that, with notation $||h_i||^2 := \int_{S^1} |h_i|^2 d\theta$,

- **Mass:**

$$\int \tilde{\omega} dx = \int \tilde{\omega} dx - r_1 \int h_1 d\theta + r_2 \int h_2 d\theta + \frac{1}{2} \int h_2^2 d\theta - \frac{1}{2} \int h_1^2 d\theta.$$
Based on these computations, we immediately see that the requirement \( \tilde{\omega} \in \mathcal{A}[\bar{\omega}] \) forces that
\[
\int h_2 d\theta, \quad \int h_1 d\theta = O(\|h\|^2),
\]
where \( \|h\|^2 := \|h_1\|^2 + \|h_2\|^2 \). This small mass condition will be used frequently in the following.

\section*{A.4 Reduction to graph type perturbation}

Given a fixed annulus \( \bar{\omega} \) and a (general) patch perturbation \( \omega^* \) belonging to the admissible class \( \mathcal{A}^m \cap \mathcal{N}_{r, B_{r/2}} \), we need to find a graph type perturbation \( \tilde{\omega} \) which satisfies
\[
E[\omega^*] - E[\tilde{\omega}] \leq 0, \quad E[\bar{\omega}] - E[\tilde{\omega}] \leq 0
\]
and still belonging to the admissible class. The proof of the second inequality is the goal of the next section. For the first inequality, having \( \omega^* \) close to \( \bar{\omega} \) in \( L^1 \) implies that the function \( G[\omega^*] + C_0 + C_1 r^2 \) is close to \( \bar{\psi} \) in the \( C^{1,\alpha} \) topology with any \( 0 < \alpha < 1 \), where \( C_0 \) and \( C_1 \) are the constants from (A.1). Then, there is a unique way to slightly perturb the constants \( C_0 \) and \( C_1 \) to \( C'_0 \) and \( C'_1 \) respectively, so that if we define \( \tilde{\omega} \) to be the patch supported on the inner component of the set \( \{ \tilde{G} > 0 \} \) where
\[
\tilde{G} := G[\omega^*] + C'_0 + C'_1 r^2,
\]
then \( \tilde{\omega} \) belongs to the admissible class. This can be proved using a determinant computation arising from matching the mass and impulse simultaneously. Then, proving \( E[\omega^*] - E[\tilde{\omega}] \leq 0 \) is straightforward, using that \( \partial_r \tilde{\psi}(r_1) > 0 > \partial_r \tilde{\psi}(r_2) \).
A.5 Energy difference for graph type perturbation

Finally, we may assume that $\tilde{\omega}$ is a graph type perturbation and write

$$E[\tilde{\omega}] - E[\omega] = \langle \tilde{\omega} - \omega, \tilde{G} \rangle + \frac{1}{2} \langle \tilde{\omega} - \omega, G(\tilde{\omega} - \omega) \rangle = I + II.$$  

\textit{Computation for I:} Using that $\tilde{\omega} \in A$, we may write

$$I = \langle \tilde{\omega} - \omega, \tilde{\psi} \rangle = \int \int_{r_2}^{r_2 + h_2} \tilde{\psi}(r) r dr d\theta - \int \int_{r_1}^{r_1 + h_1} \tilde{\psi}(r) r dr d\theta =: I_2 + I_1.$$  

Then, we compute using that $\tilde{\psi}(r_2) = 0$

$$I_2 = \int \int_{r_2}^{r_2 + h_2} (\partial_r \tilde{\psi}(r_2)(r - r_2) + o(r - r_2))(r_2 + r - r_2) r dr d\theta$$

$$= r_2 \partial_r \tilde{\psi}(r_2) \int \int_{r_2}^{r_2 + h_2} (r - r_2) dr d\theta + o(h_2^2) = \frac{r_2 \partial_r \tilde{\psi}(r_2)}{2} \int h_2^2 dr d\theta + o(h_2^2).$$

Similarly, we have that

$$I_1 = -\frac{r_1 \partial_r \tilde{\psi}(r_1)}{2} \int h_1^2 dr d\theta + o(h_1^2).$$

Note the negative sign in the first term of the right hand side. Therefore, we have that

$$I \leq -2c_0 \|h\|^2 + o(\|h\|^2)$$

for some $c_0 > 0$ depending only on $r_1$ and $r_2$.

\textit{Computation for II:} Next, we consider the quadratic expression $II$ in polar coordinates after writing

$$\tilde{\omega} - \omega = (\tilde{\omega}_h - \omega_h) - (\tilde{\omega}_r - \omega_r), \quad G_j := G(\tilde{\omega}_h - \omega_h);$$

$$II = \frac{1}{2} \int \int (\tilde{\omega}_h - \omega_h)(r, \theta)G_2(r, \theta)r dr d\theta - \frac{1}{2} \int \int (\tilde{\omega}_h - \omega_h)(r, \theta)G_2(r, \theta)r dr d\theta$$

$$- \frac{1}{2} \int \int (\tilde{\omega}_h - \omega_h)(r, \theta)G_1(r, \theta)r dr d\theta + \frac{1}{2} \int \int (\tilde{\omega}_h - \omega_h)(r, \theta)G_1(r, \theta)r dr d\theta$$

$$=: II_{22} + II_{21} + II_{12} + II_{11}. $$

As in [45], we see that

$$II_{jj} = (r_j)^2 \langle h_j, Kh_j \rangle + o(\|h_j\|^2)$$

for $j = 1, 2$ and

$$II_{12} = II_{21} = -r_1 r_2 \langle h_1, \tilde{K} h_2 \rangle + o(\|h\|^2).$$

Here, $\tilde{K}$ is the convolution operator defined on $S^1$ by

$$(\tilde{K} h_2)(\theta) = \frac{1}{2\pi} \int \ln \frac{1}{|r_1 e^{i\theta} - r_2 e^{i\theta'}|} h_2(\theta') d\theta'.$$

Therefore, we have that

$$II = r_2^2 \langle h_2, Kh_2 \rangle + r_1^2 \langle h_1, Kh_1 \rangle - 2r_1 r_2 \langle h_1, \tilde{K} h_2 \rangle + o(\|h\|^2).$$
Under the small mass condition (A.2), we may replace $\tilde{K} h_2$ with the convolution

$$\frac{1}{2\pi} \int \frac{1}{|r_1 e^{i(\theta - \theta')} / r_2 - 1|} h_2(\theta')d\theta',$$

whose eigenfunctions are simply $e^{in\theta}$ for $n \in \mathbb{Z}$. The eigenvalues of this operator depend on $r_1$ and $r_2$ but decays to 0 as $|n| \to \infty$, just like those for $K$. Therefore, we deduce that for $\tilde{\omega} \in A_m[\tilde{\omega}]$,

$$|II| \leq o_m(1)\|h\|^2.$$

**Conclusion** We have that

$$I + II \leq -c_0\|h\|^2,$$

for $m$ sufficiently large and $\|h\|$ sufficiently small. This concludes the proof of Proposition A.1.

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