Scattering of instantons, monopoles and vortices in higher dimensions

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Abstract

We consider Yang-Mills theory on manifolds $\mathbb{R} \times X$ with a $d$-dimensional Riemannian manifold $X$ of special holonomy admitting gauge instanton equations. Instantons are considered as particle-like solutions in $d+1$ dimensions whose static configurations are concentrated on $X$. We study how they evolve in time when considered as solutions of the Yang-Mills equations on $\mathbb{R} \times X$ with moduli depending on time $t \in \mathbb{R}$. It is shown that in the adiabatic limit, when the metric in the $X$ direction is scaled down, the classical dynamics of slowly moving instantons corresponds to a geodesic motion in the moduli space $\mathcal{M}$ of gauge instantons on $X$. Similar results about geodesic motion in the moduli space of monopoles and vortices in higher dimensions are briefly discussed.
1 Introduction and summary

Instantons in four dimensions are nonperturbative Bogomolny-Prasad-Sommerfield (BPS) configurations solving first-order anti-self-duality equations for gauge fields which imply the full Yang-Mills equations [1]. If one makes the assumption that the non-abelian gauge potential is independent of one coordinate on $\mathbb{R}^4$ then the anti-self-duality equations are reduced to Bogomolny equations on $\mathbb{R}^3$ describing non-abelian monopoles [2]. Furthermore, considering anti-self-dual Yang-Mills equations on a four-manifold $\Sigma_2 \times S^2$ and imposing SO(3)-equivariance condition on gauge fields, one obtains vortex equations on a two-dimensional Riemannian manifold $\Sigma_2$ (see e.g. [2, 3] and references therein). Vortices, monopoles and instantons are important objects in modern field theories describing nonperturbative physics [4]-[7].

Non-abelian monopoles are also particle-like static solutions of Yang-Mills-Higgs equations in Minkowski space $\mathbb{R}^{3,1}$ [1]. Vortices can also be obtained as static solutions of Yang-Mills-Higgs equations in $2 + 1$ dimensions [4]-[7]. Similarly instantons can be considered as solitons in $4 + 1$ dimensional Yang-Mills theory. One can ask about the dynamics of all these solitons which can evolve according to the second-order field equations of Yang-Mills-Higgs or Yang-Mills theory. In the seminal paper [8] Manton suggested that in the “slow motion limit” the monopole dynamics can be described in terms of geodesics in the moduli space of static multi-monopole solutions. These geodesics are defined via a metric on the multi-monopole moduli space [8]. This heuristic approach was extended to vortices in 2+1 dimensions [9], domain walls in 3+1 dimensions [10] and instantons in 4+1 dimensions (see e.g. [11]). Higher derivative corrections to the lowest-order (adiabatic) results were considered in [12]. The Manton approach was rigorously justified both for monopoles and vortices by Stuart [13]. However, we are not aware about such a justification for scattering instantons in 4+1 dimensions. Here we provide a derivation of motion of instantons along geodesics in the multi-instanton moduli space by using the adiabatic approach. Generalizing [11], we will consider this approach for instantons not only in four but also in higher dimensions.

Instanton equations on a $d$-dimensional Riemannian manifold $X$ can be introduced as follows [14]-[16]. Suppose there exists a 4-form $Q$ on $X$. Then there exists a $(d-4)$-form $\Sigma := *Q$, where $*$ is the Hodge operator on $X$ defined with the help of a metric $g$ on $X$. Let $A$ be a connection on a rank-$k$ vector bundle $E$ over $X$ with the curvature $F_X = dA + A \wedge A$. For simplicity we choose SU($k$) as our gauge group and therefore both $A$ and $F_X$ take values in the Lie algebra $\text{su}(k)$. The generalized anti-self-duality (instanton) equation on the gauge field $F_X$ is [16]

\[ *F_X + \Sigma \wedge F_X = 0 \, . \tag{1.1} \]

For $d > 4$ these equations can be defined on manifolds $X$ with special holonomy, i.e. such that the holonomy group $G^h$ of the Levi-Civita connection on the tangent bundle $TX$ is a subgroup in SO($d$). Solutions of (1.1) satisfy the Yang-Mills equations

\[ D_\mu F_X^{\mu \nu} := \frac{1}{\sqrt{\det g}} \partial_\mu (\sqrt{\det g} F_X^{\mu \nu}) + [A_\mu, F_X^{\mu \nu}] = 0 \, , \tag{1.2} \]

where the derivatives $\partial_\mu := \partial/\partial x^\mu$ are taken with respect to local coordinates $x^\mu$ on $X$ and $g = g_{\mu \nu} dx^\mu dx^\nu$, $\mu, \nu, \ldots = 1, \ldots, d$. The instanton equations are also well defined on manifolds $X$ with non-integrable $G^h$-structures, i.e. when $d\Sigma \neq 0$. In this case (1.1) imply the Yang-Mills equations with (3-form) torsion $T := *d\Sigma$, as is discussed e.g. in [17]-[20]. Such torsionful Yang-Mills equations naturally appear in heterotic string compactifications with $H$-flux.
We extend the manifold $X$ by the time axis $\mathbb{R}$ and introduce on the Lorentzian manifold $M = \mathbb{R} \times X$ a metric
\[ \hat{g}_\varepsilon = -dt^2 + \varepsilon^2 g , \] (1.3)
where $t = x^0$ is a coordinate on $\mathbb{R}$ and $\varepsilon$ is a real parameter. Denoting by $\{x^\mu\} = \{x^0, x^\mu\}$ local coordinates on $M = \mathbb{R} \times X$, we introduce the Yang-Mills equations on $M$,
\[ \hat{D}_\mu \hat{F}^{\mu\nu} := \frac{1}{\sqrt{\det g}} \partial_\mu (\sqrt{\det g} \hat{F}^{\mu\nu}) + [\hat{A}_\mu, \hat{F}^{\mu\nu}] = 0 , \] (1.4)
where we used the fact that $|\det \hat{g}_\varepsilon| = \varepsilon^{2d} \det g$.

It is not easy to construct non-trivial time-dependent solutions of the Yang-Mills equations (1.4). The adiabatic limit method, based on Manton's idea, provides a useful and powerful tool for describing such solutions. The adiabatic limit refers to the geometric process of shrinking the metric (1.3) in the $X$ direction by taking the limit $\varepsilon \to 0$. We will show that solutions of the Yang-Mills equations (1.4) in the limit $\varepsilon \to 0$ for the metric (1.3) converge to the solutions of one-dimensional sigma-model describing a map from $\mathbb{R}$ into the moduli space of gauge instantons on $X$. For connections $A$ not depending on one coordinate of $X$ we will get geodesics in the moduli space of (generalized) monopoles on a $(d-1)$-dimensional submanifold of $X$. Similar reductions to geodesic in moduli space of (generalized) vortices on $(d-2)$-dimensional submanifolds of $X$ will also be described.

2 Moduli space of instantons in $d \geq 4$

**Moduli space of connections.** Let $X$ be an oriented smooth manifold of dimension $d$, $G$ a semisimple compact Lie group, $\mathfrak{g}$ its Lie algebra, $P$ a principal $G$-bundle over $X$, $A$ a connection 1-form on $P$ and $\hat{F}_X = dA + A \wedge A$ its curvature. We consider also the bundle of groups $\text{Int} P = P \times_G G$ ($G$ acts on itself by internal automorphisms: $h \mapsto ghg^{-1}$, $h, g \in G$) associated with $P$, the bundle of Lie algebras $\text{Ad} P = P \times_G \mathfrak{g}$ and a complex vector bundle $E = P \times_G V$, where $V$ is the space of some irreducible representation of $G$. All these associated bundles inherit their connection $A$ from $P$. For the simplicity one can consider $G = \text{SU}(k)$, $\mathfrak{g} = \text{su}(k)$ and $V = \mathbb{C}^k$.

We denote by $\mathcal{A}'$ the space of connections on $P$ and by $\mathcal{G}'$ the infinite-dimensional group of gauge transformations (automorphisms of $P$ which induce the identity transformation of $X$),
\[ A \mapsto A^g = g^{-1}Ag + g^{-1}dg , \] (2.1)
which can be identified with the space $\Gamma(X, \text{Int} P)$ of global sections of the bundle $\text{Int} P$. Correspondingly, the infinitesimal action of $\mathcal{G}'$ is defined by global sections $\chi$ of the bundle $\text{Ad} P$,
\[ A \mapsto \delta_\chi A = d\chi + [A, \chi] =: D_A\chi \] (2.2)
with $\chi \in \text{Lie} \mathcal{G}' = \Gamma(X, \text{Ad} P)$.

We restrict ourselves to the subspace $\mathcal{A} \subset \mathcal{A}'$ of irreducible connections and to the subgroup $\mathcal{G} = \mathcal{G}' / Z(\mathcal{G}')$ of $\mathcal{G}'$ which acts freely on $\mathcal{A}$. Then the moduli space of irreducible connections on $P$

\[1\text{In this section we follow the discussion of [21].}\]
(and on $E$) is defined as the quotient $\mathbb{A}/G$. Classes of gauge equivalent connections are points $[A]$ in $\mathbb{A}/G$.

Since $\mathbb{A}$ is an affine space, for each $A \in \mathbb{A}$ we have a canonical identification between the tangent space $T_A\mathbb{A}$ and the space $\Lambda^1(X, \text{Ad}P)$ of 1-forms on $X$ with values in the vector bundle $\text{Ad}P$. Our $\mathfrak{g} = \text{su}(k)$ is a matrix Lie algebra, with the metric defined by the trace. The metrics on $X$ and on the Lie algebra $\text{su}(k)$ induce an inner product on $\Lambda^1(X, \text{Ad}P)$.

\[
\langle \xi_1, \xi_2 \rangle = \int_X \text{tr} (\xi_1 \wedge \ast \xi_2) \quad \text{for} \quad \xi_1, \xi_2 \in \Lambda^1(X, \text{Ad}P) .
\]

This inner product is transferred to $T_A\mathbb{A}$ by the canonical identification. It is invariant under the $G$-action on $\mathbb{A}$, whence we get a metric $(2.3)$ on the moduli space $\mathbb{A}/G$.

**Instanton connections.** Suppose there exists a $(d-4)$-form $\Sigma$ on $X$ which allows us to introduce the instanton equation

\[
\ast F_X + \Sigma \wedge F_X = 0
\]
discussed in Section 1. We denote by $\mathcal{N} \subset \mathbb{A}$ the space of irreducible connections subject to $(2.4)$ on the rank-$k$ complex vector bundle $E \rightarrow X$. This space $\mathcal{N}$ of instanton solutions on $X$ is a subspace of the affine space $\mathbb{A}$, and we define the moduli space $\mathcal{M}$ of instantons as the quotient space

\[
\mathcal{M} = \mathcal{N}/G
\]
together with a projection

\[
\pi : \mathcal{N} \overset{G}{\rightarrow} \mathcal{M} .
\]

According to the bundle structure $(2.6)$, at any point $A \in \mathcal{N}$, the tangent bundle $T_A\mathcal{N} \rightarrow \mathcal{N}$ splits into the direct sum

\[
T_A\mathcal{N} = \pi^*T_{[A]}\mathcal{M} \oplus T_A\mathcal{G} .
\]

In other words,

\[
T_A\mathcal{N} \ni \tilde{\xi} = \xi + D_A\chi \quad \text{with} \quad \xi \in \pi^*T_{[A]}\mathcal{M} \quad \text{and} \quad D_A\chi \in T_A\mathcal{G} ,
\]

where $\tilde{\xi}, \xi \in \Lambda^1(X, \text{Ad}P)$ and $\chi \in \Lambda^0(X, \text{Ad}P) = \Gamma(X, \text{Ad}P)$. The choice of $\xi$ corresponds to a local fixing of a gauge. We denote by $\xi_\alpha$ a local basis of vector fields on $\mathcal{M}$ (sections of the tangent bundle $TM$) with $\alpha = 1, \ldots, \dim\mathcal{M}$. Restricting the metric $(2.3)$ on $\mathbb{A}/G$ to the subspace $\mathcal{M}$ provides a metric $G = (G_{\alpha\beta})$ on the instanton moduli space,

\[
G_{\alpha\beta} = \int_X \text{tr} (\xi_\alpha \wedge \ast \xi_\beta) .
\]

Using this metric on $\mathcal{M}$, we can introduce Christoffel symbols

\[
\Gamma^\gamma_{\alpha\beta} = \frac{1}{2} G^{\gamma\kappa} (\partial_\alpha G_{\beta\kappa} + \partial_\beta G_{\alpha\kappa} - \partial_\kappa G_{\alpha\beta}) ,
\]

where the derivatives $\partial_\alpha := \partial/\partial \phi^\alpha$ are taken with respect to local coordinates $\phi^\alpha$ on $\mathcal{M}$ in which $G = G_{\alpha\beta} d\phi^\alpha d\phi^\beta$. One can also introduce Riemannian tensor, Ricci tensor etc.
3 Adiabatic limit for the Yang-Mills equations in $d \geq 4$

**Splitting of the Yang-Mills equations.** So, we consider the manifold

$$M = \mathbb{R} \times X$$

with a metric

$$\hat{g}_\varepsilon = -dt^2 + \varepsilon^2 g = -dt^2 + \varepsilon^2 g_{\mu\nu} dx^\mu dx^\nu,$$

and rank-$k$ complex vector bundle $E \to M$ with an $\text{su}(k)$-valued connection $A$ as well as the curvature 2-form

$$\hat{F} = \frac{1}{2} F_{\hat{\mu}\hat{\nu}} dx^\hat{\mu} \wedge dx^\hat{\nu} = F_{0\mu} dx^0 \wedge dx^\mu + \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu.$$  \hspace{1cm} (3.3)

Recall that we assume that the second part in (3.3),

$$F_X = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu,$$

satisfies the instanton equation (2.4) and for the connection $\hat{A}$ on $E \to \mathbb{R} \times X$ we have

$$\hat{A} = A_\mu dx^\mu = A_0 dx^0 + A_\mu dx^\mu = A_0 dt + A$$  \hspace{1cm} (3.5)

where $A$ has components only along $X$ but depends on all coordinates $(t, x^\mu)$ on $M$.

We assume that $A$ satisfies the instanton equation (2.4) for any $t$ and depend on $t$ only via moduli $\phi^\alpha$ (collective coordinates) described in Section 2. On the other hand, the full Yang-Mills equations (1.4) impose restrictions on dynamics of $\phi^\alpha(t)$. In order to find them we note that for the metric (3.2) we have

$$\hat{F}_\nu^{\mu_0} = \hat{g}^{00} \hat{g}^{\mu\nu} F_{0\nu} = \varepsilon^{-2} F^{0\nu}, \quad \hat{F}^{\mu\nu} = \varepsilon^{-4} F^{\mu\nu},$$  \hspace{1cm} (3.6)

where in $F^{0\mu}$ and $F^{\mu\nu}$ indices are raised by $g^{00}$ and $g^{\mu\nu}$. After substitution of (3.6) into (1.4) we obtain the equations

$$D_\mu F^{0\nu} = g^{\mu\nu} D_\mu F_{0\nu} = 0,$$

$$g^{\mu\nu} D_0 F_{\nu\mu} = 0,$$  \hspace{1cm} (3.7)  \hspace{1cm} (3.8)

where we used that $D_\mu F^{\mu\nu} = 0$ since $A_\mu$ is an instanton on $X$.

**Projection on $M$.** For $t \in \mathbb{R}$ varying, the connection $A = A(\phi^\alpha(t), x^\mu)$ on the bundle $E \to \{t\} \times X$ defines a map

$$\phi : \mathbb{R} \to \mathcal{M} \quad \text{with} \quad \phi(t) = \{\phi^\alpha(t)\},$$  \hspace{1cm} (3.9)

where $\phi^\alpha$ with $\alpha = 1, \ldots, \dim_{\mathbb{R}} \mathcal{M}$ are local coordinates on $\mathcal{M}$. This map is not free - it is constrained by the equations (3.7)-(3.8). Since $A$ belongs to the solution space $\mathcal{N}$ of the instanton equation (2.4), its derivative $\partial_0 A$ is a solution of the linearized form of (2.4) around $A$, i.e. $\partial_0 A$ belongs to the vector space $T_A \mathcal{N}$. Using (2.7), one can decompose $\partial_0 A_\mu$ into two parts,

$$T_A \mathcal{N} = \pi^* T_{[A]} \mathcal{M} \oplus T_A \mathcal{G} \quad \Leftrightarrow \quad \partial_0 A_\mu = (\partial_0 \phi^\alpha) \xi_{\alpha\mu} + D_\mu \epsilon_0,$$  \hspace{1cm} (3.10)
where $\xi_\alpha = \xi_{\alpha\mu} dx^\mu$ is a local basis of vector fields on $\mathcal{M}$ and $\epsilon_0$ is an su($k$)-valued gauge parameter which is determined by the gauge-fixing equations
\begin{equation}
 g^{\mu \nu} D_\mu \xi_{\alpha \nu} = 0 \tag{3.11}
\end{equation}
and therefore from (3.10) and (3.11) we get
\begin{equation}
 g^{\mu \nu} D_\mu \partial_0 A_\nu = g^{\mu \nu} D_\mu \epsilon_0 . \tag{3.12}
\end{equation}
Note that
\begin{equation}
 F_{\nu 0} = D_\nu A_0 - D_0 A_\nu . \tag{3.13}
\end{equation}
Let us fix the gauge of the Yang-Mills fields on $\mathbb{R} \times X$ by choosing
\begin{equation}
 A_0 := \epsilon_0 . \tag{3.14}
\end{equation}
Then from (3.10) we obtain
\begin{equation}
 F_{\nu 0} = - \dot{\phi}^\alpha \xi_{\alpha \nu} , \tag{3.15}
\end{equation}
where we denoted by dot the derivative with respect to time $t$. From (3.11) and (3.15) we see that the equations (3.7) are satisfied. Furthermore, since
\begin{equation}
 \partial_0 A_\mu = \dot{\phi}^\alpha \frac{\partial A_\mu}{\partial \phi^\alpha} , \tag{3.16}
\end{equation}
we get from (3.12) that
\begin{equation}
 A_0 = \epsilon_0 = \dot{\phi}^\alpha \epsilon_\alpha , \tag{3.17}
\end{equation}
where the gauge parameters $\epsilon_\alpha$ can be obtained as solutions of the equations
\begin{equation}
 g^{\mu \nu} D_\mu D_\nu \epsilon_\alpha = g^{\mu \nu} D_\mu \frac{\partial A_\nu}{\partial \phi^\alpha} , \tag{3.18}
\end{equation}
which follow from (3.12),(3.16) and (3.17). Notice that $F_{0 \mu}$, given in (3.15), is the projection of $\partial_0 A_\mu$ from $T_\mathcal{A} N$ to $T_{[\mathcal{A}]} \mathcal{M}$ (cf. [8]):
\begin{equation}
 \pi_\alpha \partial_0 A_\mu = F_{0 \mu} = \dot{\phi}^\alpha \xi_{\alpha \mu} . \tag{3.19}
\end{equation}

Geodesics. Although the evolution of the gauge fields does not exactly follow a trajectory $\phi^\alpha(t)$ in the set of exact static solutions (moduli space $\mathcal{M}$ of instantons on $X$ in our case), it does a good approximation. Following [22], we will show that in the adiabatic limit $\varepsilon \to 0$ the approximation becomes exact and $\phi(t)$ is a geodesic motion on $\mathcal{M}$. To show this, we substitute (3.15) in the remaining unsolved equations (3.8) and obtain
\begin{equation}
 g^{\mu \nu} \frac{d}{dt} \epsilon_\beta \xi_{\beta \nu} = g^{\mu \nu} \dot{\phi}^\beta \xi_{\beta \nu} , \tag{3.20}
\end{equation}
Now let us multiply these equations on $\dot{\phi}^\alpha \xi_{\alpha \mu}$, take trace tr over su($k$) and integrate over $X$. We get the equations\footnote{The right hand side of (3.20) vanishes since $g^{\mu \nu} \dot{\phi}^\alpha \dot{\phi}^\beta \xi_{\alpha \mu}, \xi_{\beta \nu}$ tr due to converting symmetric and antisymmetric in ($\alpha \beta$) parts.}
\begin{equation}
 \frac{d}{dt} \left( G_{\alpha \beta} \dot{\phi}^\alpha \dot{\phi}^\beta \right) = 0 . \tag{3.21}
\end{equation}
on the moduli space $\mathcal{M}$. In deriving (3.21) we identify $t$ with the affine parameter $s$ entering in definition of the metric

$$ds^2 = G_{\alpha\beta} \, d\phi^\alpha \, d\phi^\beta$$

(3.22)
on the moduli space $\mathcal{M}$, where the metric components $G_{\alpha\beta}$ were introduced in (2.3):

$$G_{\alpha\beta} = \int g_{\mu\nu} \, tr(\xi_\alpha^\mu \wedge * \xi_\beta^\nu)$$

(3.23)
with the Hodge operator $*$ on $X$.

Equation (3.21) defines geodesics on $\mathcal{M}$. To see them in more standard form, with Christoffel symbols

$$\Gamma^\alpha_{\beta\gamma} = G^{\alpha\lambda} \left( \frac{\partial}{\partial \phi^\gamma} G_{\beta\lambda} + \frac{\partial}{\partial \phi^\beta} G_{\alpha\lambda} - \frac{\partial}{\partial \phi^\lambda} G_{\alpha\beta} \right),$$

(3.24)
we consider the action functional

$$\tilde{S} = \int dt \, G_{\alpha\beta} \dot{\phi}^\alpha \dot{\phi}^\beta.$$  

(3.25)
The Euler-Lagrange equations for (3.25) are

$$\ddot{\phi}^\alpha + \Gamma^\alpha_{\beta\gamma} \dot{\phi}^\beta \dot{\phi}^\gamma - \dot{\phi}^\alpha \frac{d}{dt} \ln(G_{\beta\gamma} \dot{\phi}^\beta \dot{\phi}^\gamma) = 0 \quad (3.21) \quad \ddot{\phi}^\alpha + \Gamma^\alpha_{\beta\gamma} \dot{\phi}^\beta \dot{\phi}^\gamma = 0$$

(3.26)
In other words, (3.20)-(3.21) yield equations (3.26) of geodesics on the moduli space $\mathcal{M}$ of instantons on $X$. This also reflects the well-known (classical) equivalence of the action functional (3.25) and the functional

$$S = \int dt \, G_{\alpha\beta} \dot{\phi}^\alpha \dot{\phi}^\beta$$

(3.27)
for which (3.26) are the Euler-Lagrange equations. Note that (3.27) is the effective action for the standard Yang-Mills action functional on $\mathbb{R} \times X$ in the limit $\varepsilon \to 0$. It stems from the term

$$\int_M d\text{vol} \, tr(F_{0\mu} F^{0\mu})$$

(3.28)
Finally, note that the pair $(A_0(\phi(t)), A_\mu(\phi(t), x))$ can be understood as a connection on $\mathbb{R} \times X$ which obeys part of the Yang-Mills equations and in the neighbourhood of $(A_0, A_\mu)$ there is a solution of the full Yang-Mills equations with $\varepsilon \neq 0$ at least for $\varepsilon$ sufficiently small (cf. [8, 13, 7]). This follows from the implicit function theorem and means also the bijectivity of moduli space of the time-dependent solutions for $\varepsilon = 0$ and small $\varepsilon \neq 0$ (cf. [13, 7]).

**Monopoles and vortices.** It is well known that instanton equations on $X^d$ can be reduced to monopole equations on a submanifold $X^{d-1}$ in $X^d$ and similarly (generalized) vortex equations can be obtained by a reduction on a submanifold $X^{d-n}$ with $n \geq 2$. That is why we will be brief and mention only some examples.

Canonical example is given by the case $d = 4$. Considering $X^4 = \mathbb{R}^4$ and imposing translation invariance with respect to the fourth coordinate $x^4$ on $\mathbb{R}^4$ one sees that anti-self-dual Yang-Mills equations on $\mathbb{R}^4$ are reduced to the Yang-Mills-Higgs Bogomolny equations on $\mathbb{R}^3$ describing non-abelian monopoles [1, 2, 5, 6]. Then our consideration produces geodesics on the monopole moduli
space reproducing Manton’s result [8]. In principle, the same can be done for $d > 4$. For example, monopoles on $G_2$-holonomy manifolds $X^7$ can be obtained from Spin(7)-instantons on $X^8$ as in [23].

Similarly, as was mentioned in the Introduction, the anti-self-dual Yang-Mills equations on the manifold $X^4 = \Sigma_2 \times S^2$ are reduced by imposing SO(3)-symmetry to vortex equations on a Riemannian 2-manifold $\Sigma_2$ (see e.g. [2, 3] and references therein). In other words, vortices on $\Sigma_2$ can be considered as SO(3)-symmetric instantons on $\Sigma_2 \times S^2$. Then the adiabatic approach to the Yang-Mills equations on $\mathbb{R} \times \Sigma_2 \times S^2$ yields to geodesics on vortex moduli space. The same reduction from instantons to vortices can be done for $d > 4$ (see e.g. [24, 25]) for $X^d = X^{2p} \times X^{2q}$. Then one obtains generalized vortex equations (see e.g. [24, 25]) on $X^{2p}$ and the adiabatic approach will describe slowly moving vortices via geodesics on moduli space of vortices on $X^{2p}$ or symmetric instantons on $X^{2p} \times X^{2q}$.

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