I. INTRODUCTION

Shell model is a recognized tool for microscopic studies of nuclear structure. No-core shell model (NCSM) [1, 2, 3], the version of the shell model where all A nucleons are spectroscopically active, is widely used now (see, e.g., Refs. [4, 5, 6]) for ab initio calculations of light nuclei (up through A = 16) with modern realistic nucleon-nucleon and three-nucleon forces.

NCSM utilizes the basis of Slater determinants of single-particle oscillator states. These basis functions are known to have spurious contributions of center-of-mass (CM) excitations. The wave functions of physically acceptable eigenstates of intrinsic NCSM Hamiltonian

\[ H_A = \frac{1}{A} \sum_{i<j} (p_i - p_j)^2 + \sum_{i<j} V_{NN,ij} + \sum_{i<j<k} V_{NNN,ijk}, \]

where \( m \) is the nucleon mass, \( V_{NN,ij} \) is the two-nucleon interaction (including both strong and electromagnetic components), \( V_{NNN,ijk} \) is the three-nucleon interaction, should be arranged as spurious-free linear combinations of basis Slater determinants that can be used as a new spurious-free basis for direct diagonalization of the intrinsic Hamiltonian (1). The complete spurious-free basis corresponding to the \( 0\hbar\Omega \) CM-excitations, is much smaller than the basis of all Slater determinants including all \( \neq \hbar\Omega \) CM-excitations with \( \kappa \leq N \) where \( N \) is the maximal oscillator quanta of the \( N\hbar\Omega \) NCSM model space used in the calculations. Therefore it is expected that the use of the projection operator \( P_{CM} \) will simplify essentially the NCSM studies of nuclear structure, will make it possible to arrange the calculations in a larger \( N\hbar\Omega \) model spaces with the same computer facilities and hence to improve the accuracy of the NCSM predictions, etc.

II. CM-PROJECTOR \( P_{CM} \)

Let

\[ \Psi = \sum_{\kappa=0}^{N} \alpha_{\kappa} \Psi_{\kappa} \]  

be a vector (wave function) defined in the \( N\hbar\Omega \) model space; \( N \) is the maximal possible CM-excitation quanta in this model space. Equation (7) presents expansion
The function \( \Psi \) is a superposition of the spurious-free CM excitation quanta which project out components with given values of the CM momentum:

\[
H_{CM}\Psi_\kappa = \left( \kappa + \frac{3}{2} \right) \hbar \Omega \Psi_\kappa. \tag{8}
\]

Due to Eq. (8), the operator \( \tilde{Q}_0 \) defined by Eq. (9) acts as ‘anti-projector’: it projects out the spurious-free component \( \Psi_{sf} \equiv \Psi_0 \) of the wave function,

\[
\tilde{Q}_0 \Psi_{sf} = 0. \tag{9}
\]

We can also define anti-projectors

\[
\tilde{Q}_\kappa \equiv H_{CM} - \left( \kappa + \frac{3}{2} \right) \hbar \Omega \tag{10}
\]

which project out components with given values of the CM excitation quanta \( \kappa \):

\[
\tilde{Q}_\kappa \Psi_\kappa = 0. \tag{11}
\]

We can extract the spurious-free content \( \tilde{\Psi}_{sf} \) of \( \Psi \) by the subsequent use of the operators (10):

\[
\begin{align*}
\Psi_1 &= \tilde{Q}_1 \Psi, \\
\Psi_2 &= \tilde{Q}_2 \Psi_1, \\
&\vdots \\
\tilde{\Psi}_{sf} &\equiv \Psi_N = \tilde{Q}_N \Psi_{N-1}. \tag{12c}
\end{align*}
\]

Equations (12) are equivalent to the following equation:

\[
\tilde{\Psi}_{sf} = \bar{P} \Psi, \tag{13}
\]

where the operator

\[
\bar{P} = \prod_{\kappa=1}^{N} \tilde{Q}_\kappa. \tag{14}
\]

Let us call \( \bar{P} \) ‘quasi-projector’. Mathematically \( \bar{P} \) is not a projection operator since it does not fit the standard property of the projection operators,

\[
P^2 = P. \tag{15}
\]

The function \( \Psi \) is a superposition of the spurious-free \( \Psi_{sf} \equiv \Psi_0 \) and spurious components \( \Psi_\kappa \) with \( \kappa \neq 0 \). The standard projection operator property (15) guarantees that

\[
P \Psi = \alpha_0 \Psi_{sf}. \tag{16}
\]

Instead of (16), the quasi-projector \( \bar{P} \) when applied to \( \Psi \) results in

\[
\bar{P} \Psi = \tilde{\Psi}_{sf} = D \alpha_0 \Psi_{sf}. \tag{17}
\]

The constant \( D \) can be easily calculated using Eqs. (8) and (11):

\[
D = (-1)^N N! (\hbar \Omega)^N. \tag{18}
\]

To become a projector, the quasi-projector \( \bar{P} \) should be ‘normalized’:

\[
P = \frac{1}{D} \bar{P}. \tag{19}
\]

In applications, one can use either \( \bar{P} \) or \( P \). Really, it is usually needed to extract from \( \Psi \) its normalized spurious-free component \( \Psi_{sf} \). The multiplier \( \alpha_0 \) is usually unknown. Hence after using either the quasi-projector \( \bar{P} \) or the projector \( P \), one needs to normalize either the function \( D \alpha_0 \Psi_{sf} \) or the function \( \alpha_0 \Psi_{sf} \). Clearly, the same computational efforts are required to normalize the functions \( D \alpha_0 \Psi_{sf} \) and \( \alpha_0 \Psi_{sf} \).

### III. OTHER USEFUL PROJECTORS

The same idea can be utilized for the construction of other useful projectors. As an example, let us construct the projector on the states with a definite value of the orbital momentum.

Let \( \hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 \) be the standard orbital momentum operator. Its eigenvalues are known to be \( L(L+1) \). We define now the operators

\[
\tilde{Q}_L \equiv \hat{L}^2 - L(L+1) \tag{20}
\]

and

\[
\bar{P}^{L\text{max}} = \prod_{\kappa=0}^{L-1} \tilde{Q}_\kappa \prod_{\kappa=L+1}^{L\text{max}} \tilde{Q}_\kappa, \tag{21}
\]

where \( L_{\text{max}} \) is the maximal accessible orbital momentum in the given \( N \hbar \Omega \) shell model space. The non-normalized quasi-projector \( \bar{P} \) can be used like the CM non-normalized quasi-projector \( \tilde{P} \) to extract (non-normalized) component with the definite value of the orbital momentum \( L \) by the algorithm described briefly by Eq. (13) or in more detail by Eqs. (12).

The projector \( \bar{P}^{L\text{max}} \) can be expressed as

\[
\bar{P}^{L\text{max}} = \frac{1}{D^{L\text{max}}} \bar{P}^{L\text{max}} \tag{22}
\]

where

\[
D^{L\text{max}} = \prod_{\kappa=0}^{L-1} [\kappa(\kappa+1) - L(L+1)] \times \prod_{\kappa=L+1}^{L\text{max}} [\kappa(\kappa+1) - L(L+1)]. \tag{23}
\]
The structure of the projectors $P_{J}^{L_{\max}}$, $P_{S}^{S_{\max}}$ and $P_{T}^{T_{\max}}$ on the states with given values of the total angular momentum $J$, total spin $S$ or isospin $T$, is exactly the same. The only difference is that in the case of an odd-$A$ system, one should use half-integer $J$, $T$ or $S$ values and modify respectively the products in Eqs. (21) and (22).

The standard projection operator property (15) is valid for all projectors (but not quasi-projectors) discussed above.

IV. CONCLUSIONS

Expression of the projection operators on the states with definite value of the angular momentum in the form of the expansion in the powers of the SU(2) generators, are known in the literature (see, e. g., Ref. [7]). However, in the general case, this polynomial includes an infinite number of terms and is inconvenient for the use in the nuclear shell model applications. As it was shown above, in the case of the shell model, the projector can be taken in the form of a finite polynomial in generators that is much more useful for the applications. The suggested projectors $P_{L}^{L_{\max}}$, $P_{J}^{J_{\max}}$, $P_{S}^{S_{\max}}$ and $P_{T}^{T_{\max}}$ are of this form.

The CM-projector $P_{CM}$ is also suggested as a finite expansion in the powers of a simple CM harmonic oscillator operator $H_{CM}$. To the best of my knowledge, similar expression for the CM-projector were never discussed in the literature.

The Lanczos iteration approach is utilized in the modern shell model codes, i. e. the basis vectors are obtained successively by acting by the Hamiltonian on the vector obtained on the previous step. The intrinsic Hamiltonian (1) and the NCSM Hamiltonian (2) cannot produce CM-excited states or to change the value of the total angular momentum of the state. Hence it is possible to project only the pivot vector (the initial vector in the Lanczos iteration approach) on the spurious-free subspace with the given definite value of the total angular momentum $J$; all the rest basis vectors will be produced spurious-free and with the same value of $J$ by the Lanczos iterations.

Formally one can use the projected pivot vector and the intrinsic Hamiltonian (1) instead of the auxiliary Hamiltonian (2) in the NCSM applications. However it is well known that the spurious states will be produced in the Lanczos iteration approach due to the computer noise (round-off errors). The term $\beta Q_0$ in Eq. (2) stabilizes the NCSM calculations reducing the computer noise if $\beta$ is sufficiently large. Therefore it looks reasonable to utilize the auxiliary Hamiltonian (1) in the applications; probably it is reasonable to add the term $\gamma |\hat{J}^2 - J(J+1)|$ with sufficiently large $\gamma$ to the Hamiltonian (2) to reduce the computer noise in the calculations of the states with the definite value $J$ of the total angular momentum.

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