FINITE GROUPS CONTAIN LARGE CENTRALIZERS

BY

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ABSTRACT

Every finite non-abelian group of order $n$ has a non-central element whose centralizer has order exceeding $n^{1/3}$. The proof does not rely on the classification of finite simple groups, yet it uses the Feit–Thompson theorem.

1. Introduction

A classical theorem of Brauer and Fowler [2] states that a finite non-abelian group $G$ of even order with a center of odd order has a non-central element $x$ such that

$$|G| < |C_G(x)|^3.$$ 

For finite non-abelian solvable groups, Bertram [1] proved the same inequality and asked whether the exponent 3 could be improved to 2. This question was answered affirmatively by Isaacs [5], who showed that every finite non-abelian solvable group contains a non-central element whose centralizer has order exceeding its index.

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In [4], Guralnick and Robinson considered some variants of the Brauer–Fowler theorem. Among other results, they prove in [4, Theorem 5] that any finite non-abelian group $G$ has a non-central element $x$ of $G$ such that

$$|G| < \frac{6}{5}|C_G(x)|^3.$$ 

Their proof does not rely on the classification of finite simple groups but uses the Feit–Thompson odd order theorem as well as a degenerate case of a result of Griess [3]. In fact, using the classification they slightly improved this result showing the following:

**Theorem 1.1** ([4]): Let $G$ be a finite non-abelian group. Then, there exists a non-central element $x$ of $G$ such that

$$|G| < |C_G(x)|^3.$$ 

The purpose of this short note is to give a proof of this result without using the classification, but still using the Feit–Thompson theorem.

Note that as a consequence of the aforementioned result of Bertram [1] (see also [4, Lemma 5.1]), to prove Theorem 1.1 it suffices to consider finite nonsolvable groups. Hence, since all finite groups of odd order are solvable by the Feit–Thompson odd order theorem, we are reduced to considering a finite non-abelian group $G$ such that $G/Z(G)$ has even order. Therefore, Theorem 1.1 follows from the following statement:

**Theorem 1.2:** Let $G$ be a finite non-abelian group of even order and let $t$ be a non-central element of $G$ such that $t^2$ is central. Then, there exists a non-central element $x$ of $G$ such that

$$|G| \leq |C_G(t)|^2\left(|C_G(x)| - \frac{1}{2}\right).$$

We remark that in general the exponents in Theorem 1.1 and Theorem 1.2 cannot be improved as occurs in $\text{SL}(2,2^n)$, where the centralizer of an involution has order $2^n$ and the maximum order of a centralizer of a non-identity element is $2^n + 1$.

**2. Proof of Theorem 1.2**

Let $G$ be a finite non-abelian group and let $t$ be a non-central element of $G$ such that $t^2$ is central. Write $Z = Z(G)$ and $k(G)$ for the number of conjugacy classes of $G$. Also, we let $i(Z)$ denote the number of involutions of $Z$. 

CLAIM: The following equation holds:
\[ |G| \leq (1 + i(Z))|C_G(t)| + (k(G) - |Z|)|C_G(t)|^2. \]

Proof of Claim. Let \( W \) be the set of pairs \( (x, y) \) in \( G \times G \) such that \( x \) is a conjugate of \( t \) which inverts \( y \), that is
\[ W = \{(x, y) \in t^G \times G : y^x = y^{-1}\}, \]
and set
\[ W_y = \{x \in t^G : (x, y) \in W\} \]
for an element \( y \) of \( G \).

It is clear that a central element \( y \) of \( G \) with \( W_y \neq \emptyset \) must be an involution or the identity element and in any case \( W_y = t^G \). For an arbitrary involution \( y \) of \( G \), the set \( W_y \) equals \( t^G \cap C_G(y) \) and for any other element \( y \) of \( G \) we have that either \( W_y \) is empty or equals \( t^G \cap C_G(y)x \) for any \( x \in W_y \). In particular, we have that \( |W_y| \leq |C_G(y)| \) for every \( y \in G \setminus Z \). Therefore, this yields
\[ |W| = \sum_{y \in G} |W_y| \leq (1 + i(Z))|t^G| + \sum_{y \in G \setminus Z} |C_G(y)| \]
\[ = (1 + i(Z))|t^G| + \sum_{i=1}^r |y_i^G||C_G(y_i)|, \]
where \( y_1, \ldots, y_r \) are the representatives of the non-central conjugacy classes of \( G \). Thus \( r = k(G) - |Z| \) and so
\[ (1) \quad |W| \leq (1 + i(Z))\frac{|G|}{|C_G(t)|} + (k(G) - |Z|)|G|. \]

On the other hand, observe that every element \( x \in t^G \) inverts all elements of \([x, G] \), since \( x^{-2} \) is central and so
\[ [x, g]^x = x^{-2}g^{-1}xgx = g^{-1}x^{-1}gx = [x, g]^{-1} \]
for any \( g \) in \( G \). As
\[ |[x, G]| = |x^{-1}x^G| = |t^G| \]
for \( x \in t^G \), we then have that
\[ (2) \quad |W| \geq \sum_{x \in t^G} |[x, G]| = |t^G|^2 = \frac{|G|^2}{|C_G(t)|^2}. \]

Hence, comparing (1) and (2) we get the desired equation. 

Now, let $x$ be an element of $G \setminus Z$ such that the order $|C_G(x)|$ is the maximum of all orders of centralizers for non-central elements, i.e.,

$$|C_G(x)| = \max\{|C_G(y)| : y \in G \setminus Z\}.$$

Then, the class equation yields that

$$|G| \geq |Z| + (k(G) - |Z|)\frac{|G|}{|C_G(x)|}$$

and so $k(G) - |Z| < |C_G(x)|$, since certainly $|G| < |Z| + |G|$. Thus, we get that $k(G) - |Z| \leq |C_G(x)| - 1$. Combining this with the equation given by the Claim, it follows that

$$|G| \leq (1 + i(Z))|C_G(t)| + (|C_G(x)| - 1)|C_G(t)|^2$$

$$\leq |C_G(t)|^2 \left(|C_G(x)| - 1 + \frac{|Z|}{|C_G(t)|}\right),$$

since $1 + i(Z) \leq |Z|$. This yields the desired inequality.

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