New Common Fixed Points for Total Asymptotically Nonexpansive Mapping in CAT(0) Space

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Abstract:
Strong and Δ-convergence for a two-step iteration process utilizing asymptotically nonexpansive and total asymptotically nonexpansive nonself mappings in the CAT(0) spaces have been studied. As well, several strong convergence theorems under semi-compact and condition (M) have been proved. Our results improve and extend numerous familiar results from the existing literature.

Keywords: Asymptotically nonexpansive mapping, CAT(0) space, Common fixed point, Condition (M), Δ-convergence.

Introduction:
A metric space $G$ is a CAT(0) space, if it is geodesically connected and if each geodesic triangle in $G$ is at least as thin as its comparison triangle in the Euclidean plane. Some typical examples of CAT(0) spaces are R-trees, Pre-Hilbert space and Euclidean buildings (1).

Fixed point theory in CAT(0) spaces was foremost initialized through Kirk (1). He proved that each nonexpansive (single valued) mapping defined on a bounded closed convex subset of a complete CAT(0) spaces permanently has fixed point. Therefore, the fixed point theory for single valued as well multivalued mappings in CAT(0) spaces has intensively been evolved by numerous authors. The convergence for nonexpansive mappings in CAT(0) spaces was studied by Dhompongsa-Panyanak (2). Thereafter, Khan and Abbas (3) studied the strong and Δ-convergence in CAT(0) space for an iteration process that is independent of the Ishikawa iteration process. As well, several of these results obtained for two nonexpansive mappings. It is important to remember that fixed point theorems in CAT(0) space can be stratified to graph theory, computer science and biology (1).

Let $(G, d)$ be a metric space and $u, v \in G$ with $d(u, v) = x$. A geodesic path from $u$ to $v$, this means an isometry $c: [0, x] \rightarrow c([0, 1]) \subseteq G$ such as $c(0) = u$ and $c(x) = v$. The image of every geodesic path between $u$ and $v$ is called geodesic segment. Each point $y$ in the segment is appeared by $\omega u \oplus (1 - \omega)v$, where $\omega \in [0, 1]$ that is $[u, v] = \{\omega u \oplus (1 - \omega)v: \omega \in [0, 1]\}$. The space $(G, d)$ is called a geodesic if each two points of $G$ are joined through a geodesic segment, and $G$ is uniquely geodesic if there exists properly one geodesic joining $u$ and $v$ for every $u, v \in G$. A subset $H$ of $G$ is called convex if $H$ has each geodesic segment joining any two points in $H(4-6)$.

A geodesic triangle $\Delta(u_1, u_2, u_3)$ is a geodesic metric space $(G, d)$ that consists of three points $u_1, u_2, u_3$ in $G$ (the vertices $\Delta$) and a geodesic segment between every pair of vertices (the edges of $\Delta$). A comparison triangle $\tilde{\Delta}(\bar{u}_1, \bar{u}_2, \bar{u}_3)$ in $W^2$ for $\Delta(u_1, u_2, u_3)$ is a triangle in 2-dimensional Euclidean plane $W^2$ with $\bar{u}_1, \bar{u}_2, \bar{u}_3 \in W^2$ such as $d(u_1, u_2) = |\bar{u}_1 - \bar{u}_2|_{W^2}$, $d(u_2, u_3) = |\bar{u}_2 - \bar{u}_3|_{W^2}$, $d(u_3, u_1) = |\bar{u}_3 - \bar{u}_1|_{W^2}$, where $\|\cdot\|_{W^2}$ is the Euclidean norm on $W^2$ (7).

CAT(0): A geodesic space is called CAT(0) space if whole geodesic triangles achieve the following comparison axiom.

Let $\Delta$ be a geodesic triangle in $G$ and $\Delta \sqsubset W^2$ be a comparison triangle for $\Delta$. Therefore, $\Delta$ is called to achieve the CAT(0) inequality if $\forall u, v \in \Delta$ & $\forall \bar{u}, \bar{v} \in \Delta$, $d(u, v) \leq d_{W^2}(\bar{u}, \bar{v})$. 
If \( u, v_1, v_2 \) are the points in CAT(0) and if \( v_0 = \frac{1}{2}(v_1 \oplus v_2) \), therefore the CAT(0) inequality leads to 
\[
d(u, v_0)^2 \leq \frac{1}{2}d(u, v_1)^2 + \frac{1}{2}d(u, v_2)^2 - \frac{1}{4}d(v_1, v_2)^2
\]
Which is the (CN) inequality of Bruhat and Tits. In verity, a geodesic space is a CAT(0) space if it accomplishes (CN).(3).

**Lemma (1)(8):** Let \((G, d)\) be a CAT(0) space. Therefore, 
\[
d((1 - k)a \oplus kb, c)^2 \leq (1 - k)d(a, c)^2 + kd(b, c)^2 - k(1 - k)d(a, b)^2
\]
for all \( k \in [0,1] \) and \( a, b, c \in G \).

Let \( \{u_n\} \) be a bounded sequence in a CAT(0) space \( G \). For \( u \in G \), setting 
\[
r(u, \{u_n\}) = \lim \sup_{n \to \infty} d(u, u_n).
\]
The asymptotic radius \( r(\{u_n\}) \) of \( \{u_n\} \) is given through 
\[
r(\{u_n\}) = \inf\{r(u, \{u_n\}) : u \in G\},
\]
and the asymptotic center \( A(\{u_n\}) \) of \( \{u_n\} \) is defined as 
\[
A(\{u_n\}) = \{ u \in G : r(u, \{u_n\}) = r(\{u_n\}) \}
\]
It is familiar that in CAT(0) space, \( A(\{u_n\}) \) has punctually one point.

Numerous iteration processes have been structured and suggested in order to approximate fixed points. The Picard iteration for a mapping \( T : E \to E \) is defined by
\[
u_1 = u \in E
\]
\[
u_{n+1} = Tu_n
\]
(1)
The modified Mann iteration is considered by Schu (5), as below
\[
u_1 = u \in E
\]
\[
u_{n+1} = (1 - \delta_n)u_n + \delta_n Tu_n
\]
(2)
Where \( \{\delta_n\} \in (0,1) \).

The modified Ishikawa iteration is studied by Tan and Xu (5), as below
\[
u_1 = u \in E
\]
\[
u_{n+1} = (1 - \delta_n)u_n + \delta_n Tu_n + \beta_n T^\nu u_n
\]
\[
u_n = (1 - \beta_n)u_n + \beta_n T^nu_n, \forall n \geq 1
\]
(3)
Where \( \{\delta_n\} \) and \( \{\beta_n\} \in (0,1) \). The iteration decreases to the modified Mann iteration when \( \beta_n = 0, \forall n \geq 1 \).

Lately, the modified S-iteration in a Banach space is introduced by Agarwal et al. (5), as below
\[
u_1 = u \in E
\]
\[
u_{n+1} = (1 - \delta_n)T^n u_n + \delta_n T^nu_n
\]
\[
u_n = (1 - \beta_n)u_n + \beta_n T^n u_n, \forall n \geq 1
\]
(4)
where \( \{\delta_n\} \) and \( \{\beta_n\} \in (0,1) \). Notice that this iteration is independent of Ishikawa and Mann iterations.

Recently, Sahin and Basarir (5) modified the above iteration in a CAT(0) space, as follows
\[
u_1 = u \in E
\]
\[
u_{n+1} = (1 - \delta_n)T^n u_n + \delta_n T^nu_n
\]
\[
u_n = (1 - \beta_n)u_n + \beta_n T^n u_n, \forall n \geq 1
\]
(5)
The following iteration has been studied by M. R. Yadava (9) for common fixed points of two self mappings \( S \) and \( T \),
\[
u_1 = u \in E
\]
\[
u_{n+1} = \delta_n u_n + \gamma_n Tu_n + \beta_n Sv_n
\]
\[
u_n = (1 - \gamma_n)u_n + \gamma_n Tu_n, \quad n \in N
\]
(6)
Where \( \{\delta_n\}, \{\gamma_n\} \) and \( \{\beta_n\} \) are real sequences in \( [0,1] \) with \( \delta_n + \gamma_n + \beta_n = 1 \). This iteration as well decreases to Mann iteration when \( T = I \) or \( \gamma_n = 0 \).

Inspired and motivated by the work of M. R. Yadava (6), the iteration (6) for common fixed points of two mapping asymptotically nonexpansive and total asymptotically nonexpansive nonself mappings in a CAT(0) space is modified, as follows.

Deem \( E \) to be a nonempty closed convex subset of a complete CAT(0) space \( G \). For \( T : E \to E \) to be an asymptotically nonexpansive and \( S : E \to E \) to be a total asymptotically nonexpansive mappings. Presume that \( \{u_n\} \) is a sequence produced by
\[
u_1 = u \in E
\]
\[
u_{n+1} = P(\delta_n u_n \oplus \gamma_n T(PT)^{n+1} u_n \oplus \beta_n S(PS)^{n+1} u_n), \quad n \in N
\]
(7)
where \( \{\delta_n\}, \{\gamma_n\} \) and \( \{\beta_n\} \) are real sequences in \( [0,1] \) with \( \delta_n + \gamma_n + \beta_n = 1 \) and \( P \) is a nonexpansive retraction of \( G \) onto \( E \).

In this paper, a new iteration for approximating a common fixed point of asymptotically nonexpansive and total asymptotically nonexpansive nonself mappings is constructed. Some strong convergence theorems and \( \Delta \)-convergence theorem under appropriate conditions like semi-compact and condition (M) in CAT(0) spaces are proved. As well, numerical example to elucidate our work is provided.

**Preliminaries**

Let \((G, d)\) be a metric space & \( E \) be a nonempty subset of \( G \). Deem \( T : E \to E \) to be a mapping. A point \( a \in E \) is called a fixed point of \( T \) if \( Ta = a \). As well the set of common fixed points of \( T \) and \( S \) denote by \( F \) which is
\[
F = \{ a \in E : Ta = Sa = a \}.
\]
Call that \( E \) is called retract of \( G \) if there is a continuous mapping \( P : G \to E \) such as \( Pa = a, \forall a \in E \). A mapping \( P : G \to E \) is called a retraction if \( P^2 = P \). If \( P \) is a retraction, then \( Pb = b, \forall b \) in the range of \( P \).
• A mapping \( T: E \rightarrow E \) is called nonexpansive (10) if \( d(Ta, Tb) \leq d(a, b) \), \( \forall a, b \in E \).

• A mapping \( T: E \rightarrow E \) is called asymptotically nonexpansive (11) if \( \exists \) a sequence \( \{e_n\} \subset [1, \infty) \) with \( e_n \rightarrow 1 \) such as \( d(T^n a, T^n b) \leq e_n d(a, b) \), \( \forall n \geq 1, \forall a, b \in E \), where \( P \) is a nonexpansive retraction of \( G \) onto \( E \).

• A mapping \( T: E \rightarrow G \) is called total asymptotically nonexpansive (11) if \( \exists \) positive sequences \( \{e_n\}, \{\sigma_n\} \) with \( e_n \rightarrow 0, \sigma_n \rightarrow 0 \) and a strictly nondecreasing continuous function \( \theta: [0, \infty) \rightarrow [0, \infty) \) such as \( d(T^n a, T^n b) \leq d(a, b) + e_n \theta d(a, b) + \sigma_n, \forall n \geq 1, \forall a, b \in E \).

• A mapping \( T: E \rightarrow E \) is called uniformly L-lipschitzain (11) if \( \exists \) a constant \( L > 0 \) such as \( d(T^n a, T^n b) \leq Ld(a, b), \forall n \geq 1, \forall a, b \in E \).

The notion of asymptotically nonexpansive mapping was foremost introduced by Gloeble and Kirk. Therefore Alber et al. introduced the class of total asymptotically nonexpansive, which generalizes some classes of mappings that are spans of asymptotically nonexpansive. Several authors have been extensively studied these classes of mappings (6).

Definition (2)(13): A sequence \( \{u_n\} \) in a CAT(0) space \( G \) is called \( \Delta \)-convergence to \( u \in G \) if \( u \) is the unique asymptotic center of \( \{v_n\} \) \( \forall \) subsequence \( \{v_n\} \) of \( \{u_n\} \). Here, note down \( \Delta - \lim_{n \to \infty} u_n = u \) and \( u \) is the \( \Delta \)-limit of \( \{u_n\} \).

Note that given \( \{u_n\} \subseteq G, \{u_n\} \Delta \)-convergence to \( u \) and \( v \in G \) with \( v \neq u \) through the uniqueness of the asymptotic center that gives

\[
\lim_{n \to \infty} \sup d(u_n, u) \leq \lim_{n \to \infty} \sup d(u_n, v)
\]

Therefore, each CAT(0) space achieves the Opial property.

Lemma (3)(8): Let \( G \) be a CAT(0) space and \( a \in G \). Suppose \( \{s_n\} \) is a sequence in \([z, c]\) for several \( z, c \in (0, 1) \) and \( \{a_n\}, \{b_n\} \) be sequences in \( G \) such as \( \lim_{n \to \infty} \sup d(a_n, h^*) \leq t \), \( \lim_{n \to \infty} \sup d(b_n, h^*) \leq t \) and \( \lim_{n \to \infty} d((1 - \{s_n\})a_n \oplus \{s_n\}b_n) = t \) for several \( t \geq 0 \). Thus \( \lim_{n \to \infty} d(a_n, b_n) = 0 \).

Lemma (4)(6): Let \( \{\zeta_n\}, \{\alpha_n\} \) and \( \{\lambda_n\} \) be the sequences of positive numbers such as \( \zeta_{n+1} \leq (1 + \alpha_n) \zeta_n + \lambda_n, \forall n \geq 1 \).

• If \( \sum_{n=1}^{\infty} \alpha_n < \infty \) and \( \sum_{n=1}^{\infty} \lambda_n < \infty \), therefore \( \lim_{n \to \infty} \zeta_n \) exists.

• If there is a subsequence \( \{\zeta_{n_k}\} \subset \{\zeta_n\} \) such as \( \zeta_{n_k} \rightarrow 0 \) then \( \lim_{n \to \infty} \zeta_n = 0 \).

Lemma (5)(14): Each bounded sequence in a complete CAT(0) space \( G \) holds a \( \Delta \)-convergence subsequence.

Lemma (6)(15): If \( G \) is closed convex subset of a complete CAT(0) space \( G \) and if \( \{u_n\} \) is bounded sequence in \( E \), thus the asymptotic center of \( \{u_n\} \) is in \( E \).

Theorem (7)(11): Let \( E \) be a closed convex subset of a complete CAT(0) space \( G \). Let \( T \) be a mapping accomplishing one of the following conditions:

• \( T: E \rightarrow E \) is an asymptotically nonexpansive mapping with a sequence \( \{e_n\} \subset [1, \infty) \) & \( e_n \rightarrow 1 \).

• \( T: E \rightarrow G \) is an asymptotically nonexpansive nonself mapping.

• \( T: E \rightarrow E \) is a total asymptotically nonexpansive mapping.

Let \( \{u_n\} \) be a bounded sequence in \( E \) such as \( \lim_{n \to \infty} d(u_n, Tu_n) = 0 \) and \( \Delta - \lim_{n \to \infty} u_n = h^* \). Thus, \( Th^* = h^* \), \( h^* \in F \).

The convergence results

In this part, \( \Delta \)-convergence and some strong convergence theorems by using iteration (7) for asymptotically nonexpansive and total asymptotically nonexpansive nonself mappings in CAT(0) spaces are proved.

Theorem (8): Let \( E \) be a nonempty closed convex subset of a complete CAT(0) space \( G \). Let \( T: E \rightarrow E \) be a uniformly \( L \)-lipschitzain and asymptotically nonexpansive and \( S: E \rightarrow G \) be a uniformly \( L \)-lipschitzain total asymptotically nonexpansive nonself mappings with \( F(T, S) \neq \emptyset \). Presume that \( u_n \) is defined by (7). If \( F = F(T) \cap F(S) \) and the following conditions are accomplished:
i) $\sum_{n=1}^{\infty} e_n < \infty$ and $\sum_{n=1}^{\infty} \sigma_n < \infty$.

ii) There is a constant $B^* > 0$ such that $\vartheta(\mu) < B^* \mu$, $\mu \geq 0$.

Thus, the sequence $\{u_n\}$ is $\Lambda$-convergence to a several points $h^* \in F \,(F = F(T) \cap F(S))$.

Proof: Step 1: Firstly, proving that $\lim_{n \to \infty} d(u_n, h^*)$ for each $h^* \in F$ and $\lim_{n \to \infty} d(u_n, F)$ exist.

Since $h^* \in F$, $P h^* = h^*$. Now,

$$d(v_n, h^*) = d(P(1 - \gamma_n)u_n + \gamma_n T(PT)^{n-1} u_n, P h^*)$$

$$\leq d((1 - \gamma_n)u_n + \gamma_n T(PT)^{n-1} u_n, P h^*)$$

$$\leq (1 - \gamma_n)d(u_n, h^*) + \gamma_n d(T(PT)^{n-1} u_n, h^*)$$

$$\leq (1 - \gamma_n)d(u_n, h^*) + \gamma_n d(T(PT)^{n-1} u_n, h^*)$$

$$= (1 - \gamma_n + \gamma_n d)(u_n, h^*)$$

Thus,

$$d(u_{n+1}, h^*) = d(P(\delta_n u_n \oplus \gamma_n T(PT)^{n-1} u_n \oplus \beta_n (SPS)^{n-1} v_n), P h^*)$$

$$\leq d(\delta_n u_n \oplus \gamma_n T(PT)^{n-1} u_n \oplus \beta_n (SPS)^{n-1} v_n, h^*)$$

$$\leq \delta_n d(u_n, h^*) + \gamma_n d(T(PT)^{n-1} u_n, h^*) + \beta_n d((SPS)^{n-1} v_n, h^*)$$

$$\leq \delta_n d(u_n, h^*) + \gamma_n d(T(PT)^{n-1} u_n, h^*) + \beta_n d((SPS)^{n-1} v_n, h^*)$$

$$\leq \delta_n d(u_n, h^*) + \gamma_n d(T(PT)^{n-1} u_n, h^*) + \beta_n [1 + e_n B^*] d(v_n, h^*) + \beta_n \sigma_n$$

$$\leq [\delta_n + \gamma_n \sigma_n] d(u_n, h^*) + \beta_n [1 + e_n B^*] d(v_n, h^*) + \beta_n \sigma_n$$

$$\leq [\delta_n + \gamma_n \sigma_n] d(u_n, h^*) + \beta_n [1 + e_n B^*] d(v_n, h^*) + \beta_n \sigma_n$$

$$\leq [\delta_n + \gamma_n \sigma_n] + \beta_n [1 + e_n B^*] d(v_n, h^*) + \beta_n \sigma_n$$

$$= (1 + \rho_n) d(u_n, h^*) + \theta_n$$

where

$$\rho_n := \beta_n e_n B^* - \beta_n \gamma_n - \beta_n e_n B^* \gamma_n + \beta_n \gamma_n \sigma_n + \beta_n e_n B^* \gamma_n \sigma_n + \theta_n := \beta_n \sigma_n$$

Whereas $\sum_{n=1}^{\infty} e_n < \infty$ and $\sum_{n=1}^{\infty} \sigma_n < \infty$.

Therefore through Lemma (4),

$$\lim_{n \to \infty} d(u_n, h^*), \forall \, h^* \in F$$

and

$$\lim_{n \to \infty} d(u_n, F)$$

exist.

Step 2: Next, proving that

$$\lim_{n \to \infty} d(u_n, Tu_n) = 0$$

and

$$\lim_{n \to \infty} d(u_n, Su_n) = 0.$$
Again,
\[
\lim_{n \to \infty} d(v_n, h^*) = r
\]
\[
\begin{align*}
    r &= \lim_{n \to \infty} d(v_n, h^*) \\
    &= d(P(1 - \gamma_n)u_n \oplus \gamma_n T(PT)^{n-1} u_n), h^*)
\end{align*}
\]

Through Lemma (3), getting
\[
\lim_{n \to \infty} d(T(PT)^{n-1} u_n, u_n) = 0
\]

Notice that
\[
\lim_{n \to \infty} d(v_n, h^*) = \gamma_n d(T(PT)^{n-1} u_n, h^*)
\]

Therefore,
\[
\lim_{n \to \infty} d(v_n, h^*) = 0
\]

Now,
\[
\begin{align*}
d(u_{n+1}, v_n) &= d(P((\delta_n u_n \oplus \gamma_n T(PT)^{n-1} u_n) \\
    &\quad \oplus \beta_n S(PS)^{n-1} v_n), u_n) \\
    &\leq d((\delta_n + \gamma_n T(PT)^{n-1} u_n) \\
    &\quad + \beta_n S(PS)^{n-1} v_n, u_n) \\
    &\leq (1 - \beta_n) d(T(PT)^{n-1} u_n, u_n) \\
    &\quad + \beta_n d(S(PS)^{n-1} v_n, u_n)
\end{align*}
\]

This gives
\[
\lim_{n \to \infty} d(u_{n+1}, u_n) = 0
\]

Thus,
\[
d(u_{n+1}, v_n) \leq d(u_{n+1}, u_n) + d(v_n, u_n) \\
&\rightarrow 0 \text{ as } n \to \infty
\]

Which gives
\[
\lim_{n \to \infty} d(u_{n+1}, v_n) = 0
\]

Moreover, from
\[
d(u_{n+1}, S(PS)^{n-1} v_n) \\
\leq d(u_{n+1}, u_n) + d(u_n, T(PT)^{n-1} u_n) \\
&\quad + d(T(PT)^{n-1} u_n, S(PS)^{n-1} v_n)
\]

That gives
\[
\lim_{n \to \infty} d(u_{n+1}, S(PS)^{n-1} v_n) = 0
\]

\[
\begin{align*}
d(u_n, T(PT)^{n-1} u_n) \\
&\leq d(u_n, T(PT)^{n-1} u_n) \\
&\quad + d(T(PT)^{n-1} u_n, S(PS)^{n-1} v_n) \\
&\quad + d(S(PS)^{n-1} v_n, T(PT)^{n-1} u_n)
\end{align*}
\]

Gives that
\[
\lim_{n \to \infty} d(u_n, T(PT)^{n-1} u_n) = 0
\]

And
\[
d(u_n, Tu_n) \leq d(u_n, u_{n+1}) \\
&\quad + d(u_{n+1}, T(PT)^{n-1} u_n) \\
&\quad + d(T(PT)^{n-1} u_n, Tu_n)
\]

By uniformly L-lipschitzain, getting
\[
\begin{align*}
(1 + L)d(u_n, u_{n+1}) &\leq d(u_n, T(PT)^{n-1} u_n) \\
&\quad + L d(P(PT)^{n-1} u_n, u_n) \\
&\leq (1 + L)d(u_n, u_{n+1}) + d(u_{n+1}, T(PT)^{n-1} u_n) \\
&\quad + L d(T(PT)^{n-1} u_n, u_n)
\end{align*}
\]

Therefore,
\[
\lim_{n \to \infty} d(u_n, Tu_n) = 0
\]

This means
\[
\lim_{n \to \infty} d(u_n, Su_n) = d(u_n, Su_n)
\]

\[
\text{Step 3: Now, proving that}
\]
\[
\exists_{\Delta}(u_n) := \left\{ (z_n) \mid \sup_n A(z_n) \subseteq F(T,S) \right\}
\]

And \(\exists_{\Delta}(u_n)\) has punctually one point. Since
\[
\lim_{n \to \infty} d(u_n, Tu_n) = 0 \text{ and } \lim_{n \to \infty} d(u_n, Su_n) = 0
\]

Let \(\exists_{\Delta}(u_n) := \bigcup (z_n) \subseteq (u_n) A(z_n) \subseteq F(T,S)\), where the union is taken over all subsequence \(\{u_n\}\) over \(\{z_n\}\). To belay that \(\Delta\)-convergence of \(\{z_n\}\) to a common fixed point of \(T\) and \(S\), first, elucidating that \(\exists_{\Delta}(u_n) \subseteq F(T,S) \& \exists_{\Delta}(u_n)\) is a singleton set. To show that \(\exists_{\Delta}(u_n) \subseteq F(T,S)\), presume that \(\exists_{\Delta}(u_n) \subseteq \{z_n\}\). Therefore, there is a subsequence \(\{z_n\}\) of \(\{u_n\}\). Such that \(A(z_n) = \{z\}\). Through Lemma (5) & (6), \(\forall u_n \subseteq \{z_n\}\) of \(\{u_n\}\) such as \(\Delta \lim_{n \to \infty} y_n = y\) and \(y \in E\).

Since
\[
\lim_{n \to \infty} d(y_n, Ty_n) = 0 \text{ and } \lim_{n \to \infty} d(y_n, Sy_n) = 0.
\]

It follows up from Theorem (7) that \(y \in E\). By the Opial property
\[
\lim_{n \to \infty} \sup d(y_n, y) \leq \lim_{n \to \infty} \sup d(y_n, Ty_n) + \lim_{n \to \infty} \sup d(y_n, Sy_n).
\]

Thus, \(Ty = y\) and \(Sy = y\) i.e. \(y \in F\). Now, claiming that \(z = y\). If not, by step (1), \(\lim_{n \to \infty} d(u_n, y)\) exists and holding to the uniqueness of the asymptotic centers,
\[
\lim \sup d(y_n, y) < \lim \sup d(y_n, z)
\]

\[
\leq \lim \sup d(z_n, z)
\]

\[
< \lim \sup d(z_n, y)
\]

\[
\leq \lim \sup d(u_n, y)
\]

\[
= \lim \sup d(y_n, y)
\]

Which is a contradiction. Thus, \(z = y\). To confirm that and \(\exists_{\Delta}(u_n)\) is a singleton, let \(\{z_n\}\) be a subsequence of \(\{u_n\}\).

By Lemma (5) & (6), there exists a subsequence \(\{y_n\}\) of \(\{z_n\}\) such as \(\Delta \lim_{n \to \infty} y_n = y\) and \(y \in E\). Let \(A(\{z_n\}) = \{z\}\) and \(A(\{u_n\}) = \{u\}\). Previously, showing that \(z = y\). Therefore, it is sufficient to show \(y = u\), thus by step (1) \(\lim_{n \to \infty} d(u_n, y)\) converges. By uniqueness
\[
\lim \sup d(y_n, y) < \lim \sup d(y_n, u)
\]

\[
\leq \lim \sup d(u_n, u)
\]

\[
= \lim \sup d(y_n, y)
\]

Whish is a contradiction. Thus, \(y = u\) the conclusion is delayed.

Lastly, proving \(\{u_n\}\) \(\Delta\)-convergence to a common fixed point of \(S\) and \(T\). Of step (1) \(d(u_n, h^*)\), \(\forall h^* \in F\), and from step (2) \(\lim_{n \to \infty} d(u_n, Tu_n) = 0 \& \lim_{n \to \infty} d(u_n, Su_n) = 0, \exists_{\Delta}(u_n)\) has
punctually one point. Hence, \{u_n\} is \(\Delta\)-convergence to a common fixed point of \(F\).

**Theorem (9):** Under the presumption of Theorem (8). Therefore the sequence \(\{u_n\}\) is defined by:

\[
\begin{align*}
  u_{n+1} & = \delta_n u_n + \gamma_n T^n u_n + \beta_n S^n v_n \\
  v_n & = (1 - \gamma_n) u_n + \gamma_n T^n u_n
\end{align*}
\] (12)

\(\Delta\)-convergence to a common fixed point of \(S\) and \(T\).

**Proof:** \(T\) and \(S\) are self-mappings from \(E\) to \(E\), take \(P = I\) (the identity mapping on \(E\)).

Therefore, \((TP)^{n+1} = T^n \cdot U_n\). The consequence of this Theorem is got of Theorem (8).

**Theorem (10):** Under the presumption of Theorem (8). Presume that \(\{u_n\}\) is defined by (7). If

\[
\lim_{n \to \infty} d(u_n, F) = 0 \quad \text{or} \quad \lim_{n \to \infty} \sup d(u_n, F) = 0,
\]

then the sequence \(\{u_n\}\) converges strongly to a point in \(F\).

**Proof:** Through Theorem (8).

\[
\begin{align*}
  d(u_{n+1}, h^*) & \leq (1 + \rho_n) d(u_n, F) + \theta_n \\
  & \leq (1 + \rho_n) d(u_{n-1}, h^*) + \theta_n \quad \forall n \geq 1
\end{align*}
\] (13)

Since \(\forall u \geq 0, 1 + u \leq e^u\), gives that

\[
\begin{align*}
  \lim_{n \to \infty} d(u_n, h^*) & \leq e^{\rho_n + 1} d(u_0, h^*) + \theta_n \\
  & \leq e^{\rho_n + 1} d(u_0, h^*) + \theta_n
\end{align*}
\]

Next, \(\{u_n\}\) is a cauchy sequence in \(E\). In verity, from (13) \(\forall u \in F\),

\[
\lim_{n \to \infty} d(u_n, h) = 0.
\]

Thus, the sequence \(\{u_n\}\) converges strongly to a point in \(F\).

**Proof:** Through Theorem (8),

\[
\lim_{n \to \infty} d(u_n, Tu_n) = 0 \quad \text{and} \quad \lim_{n \to \infty} d(u_n, Su_n) = 0
\]

Theorem (12): Under the presumption of Theorem (8). If either \(S\) or \(T\) is semi-compact, hence the sequence \(\{u_n\}\) converges strongly to a point of \(F\).

**Proof:** Presume that \(S\) is semi-compact, By Theorem (8), getting \(\lim_{n \to \infty} d(u_n, Su_n) = 0\).

Hence, \(H^* \subseteq E\), where

\[
\begin{align*}
  H & = e^{\sum_{n=1}^{\infty} \rho_n} < \infty \quad \text{and} \\
  \sum_{n=1}^{\infty} \rho_n & < \infty
\end{align*}
\]

A mapping \(T: E \to E\) is called semi-compact (16) if for a sequence \(\{u_n\}\) in \(E\) with \(\lim_{n \to \infty} d(u_n, Tu_n) = 0\), there is a subsequence \(\{u_{n_k}\}\) of \(\{u_n\}\) such as \(u_{n_k} \to h^* \in E\).

**Theorem (12):** Under the presumption of Theorem (8). If either \(S\) or \(T\) is semi-compact, hence the sequence \(\{u_n\}\) converges strongly to a point of \(F\).

**Proof:** Presume that \(S\) is semi-compact. By Theorem (8), getting \(\lim_{n \to \infty} d(u_n, Su_n) = 0\).

Thus, \(\{u_n\}\) of \(\{u_n\}\) such as \(H^* \to h^*\).

Now, by Theorem (8) encloses that

\[
\lim_{n \to \infty} d(u_{n+1}, h^*) \leq (1 + \rho_n) d(u_n, F) + \theta_n
\]

where \(\sum_{n=1}^{\infty} \rho_n < \infty\) and \(\sum_{n=1}^{\infty} \theta_n < \infty\), by Lemma (4) \(\lim_{n \to \infty} d(u_n, h^*)\) exists and \(u_n \to h^* \in F\) gives that \(u_n \to h^*\). This proves that \(\{u_n\}\) converges strongly.
The consequence is

\[ u_n \text{ converges to zero.} \]

### Table 1. Numerical results for 40 steps

| n   | \( u_1 = 0.76, k \) | \( u_1 = 0.88, k \) | \( u_1 = 0.9, k \) |
|-----|---------------------|---------------------|---------------------|
| 1   | 0.4902              | 0.6380              | 0.7200              |
| 2   | 0.3162              | 0.4626              | 0.5760              |
| 3   | 0.2039              | 0.3353              | 0.4608              |
| 4   | 0.1315              | 0.2431              | 0.3686              |
| 5   | 0.0848              | 0.1763              | 0.2949              |
| 6   | 0.0547              | 0.1278              | 0.2359              |
| 7   | 0.0353              | 0.0927              | 0.1887              |
| 8   | 0.0228              | 0.0672              | 0.1510              |
| 9   | 0.0147              | 0.0487              | 0.1208              |
| 10  | 0.0095              | 0.0353              | 0.0966              |
| 11  | 0.0061              | 0.0256              | 0.0773              |
| 12  | 0.0039              | 0.0186              | 0.0618              |
| 13  | 0.0025              | 0.0135              | 0.0495              |
| 14  | 0.0016              | 0.0098              | 0.0396              |
| 15  | 0.0011              | 0.0071              | 0.0317              |
| 16  | 0.0007              | 0.0051              | 0.0253              |
| 17  | 0.0004              | 0.0037              | 0.0203              |
| 18  | 0.0003              | 0.0027              | 0.0162              |
| 19  | 0.0002              | 0.0020              | 0.0130              |
| 20  | 0.0001              | 0.0014              | 0.0104              |
| 21  | 0.0001              | 0.0010              | 0.0083              |
| 22  | 0.0000              | 0.0007              | 0.0066              |
| 23  | 0.0000              | 0.0005              | 0.0053              |
| 24  | 0.0000              | 0.0004              | 0.0043              |
| 25  | 0.0000              | 0.0003              | 0.0034              |
| 26  | 0.0000              | 0.0002              | 0.0027              |
| 27  | 0.0000              | 0.0001              | 0.0022              |
| 28  | 0.0000              | 0.0001              | 0.0017              |
| 29  | 0.0000              | 0.0001              | 0.0014              |
| 30  | 0.0000              | 0.0001              | 0.0011              |
| 31  | 0.0000              | 0.0000              | 0.0009              |
| 32  | 0.0000              | 0.0000              | 0.0007              |
| 33  | 0.0000              | 0.0000              | 0.0006              |
| 34  | 0.0000              | 0.0000              | 0.0005              |
| 35  | 0.0000              | 0.0000              | 0.0004              |
| 36  | 0.0000              | 0.0000              | 0.0003              |
| 37  | 0.0000              | 0.0000              | 0.0002              |
| 38  | 0.0000              | 0.0000              | 0.0002              |
| 39  | 0.0000              | 0.0000              | 0.0002              |
| 40  | 0.0000              | 0.0000              | 0.0001              |

**Figure 1. Convergence behaviors for different initial points for 40 steps.**

**Authors' declaration:**

- Conflicts of Interest: None.
We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for re-publication attached with the manuscript.

Ethical Clearance: The project was approved by the local ethical committee in University of Baghdad.

References:
1. Abbas M, Ibrahim Y, Khan AR, Sen M. Split variational inclusion problem and fixed point problem for a class of multivalued mappings in CAT(0) spaces. Mathematics. 2019;7(749).
2. Dhompongsa S, Panyanak B. On \( \Delta \)-convergence theorems in CAT(0) spaces, Comput. Math. Appl. 2008; 56: 2572-2579.
3. Khan SH, Abbas M. Strong and \( \Delta \)-convergence of some iteration schemes in CAT(0) spaces. Comput. Math. Appl. 2011; 61: 109-116.
4. Saluja GS. On the convergence of modified S-iteration process for generalized asymptotically quasi-nonexpansive mappings CAT(0) spaces. published by faculty of Sci. Math. 2014; 6(2): 29-38.
5. Saluja GS, Postolache M. Strong and \( \Delta \)-convergence theorems for two asymptotically nonexpansive mappings in the intermediate sense in CAT(0) spaces. Fixed Point Theory Appl. 2015; 12.
6. Abkar A, Shekarbaigi M. A novel iteration algorithm applied tp totally asymptotically nonexpansive mappings in CAT(0) spaces. Math. 2017; 5(14).

7. Kimura Y. Convergence of a sequence of sets in a Hadamard space and the shrinking projection method for a real hilbert ball, Hindawi. Abstract and Applied Analysis, 2010.
8. Ullah K, Iqbal K, Ashed M. Some convergence results using K iteration process in CAT(0) spaces. Fixed Point Theory Appl. 2018; 11.
9. Yodav MR. Common fixed point for contractive operators by a faster iteration process in real Banach spaces. J. Comp. Math. Sci. 2014; 5(2): 251-257.
10. Abed SS, Hasan ZMM. Common fixed point of a finite-step iteration algorithm under total asymptotically quasi-nonexpansive maps. Baghdad Sci. J. 2019; 16(3).
11. Chang S, Wang L, Jee HW, Chan C. Strong and \( \Delta \)-convergence for mixed type total asymptotically nonexpansive mappings in CAT(0) spaces. Fixed Point Theory Appl. 2013; 122.
12. Abed SS, Hasan ZMM. Multi-step iteration algorithm via two finite families of total asymptotically quasi-nonexpansive maps. ASTESJ. 2019; 4(2).
13. Uddin I, Dala S, Imdad M. Approximating fixed points for generalized nonexpansive mapping in CAT(0) spaces. J. Inequalities Appl. 2014; 155.
14. Kazemi OA, Chinedu IZ, Godwin CU, Oluwatosin TM. On the proximal point algorithm and demimetric mappings in CAT(0) spaces. Demonstr. Math. 2018; 51: 277-294.
15. Godwin CU. Approximating a common fixed point for finite family of demimetric mappings in CAT(0) space. Fixed point theory, 2019; 9(1): 45-60.
16. Husain S, Singh N. \( \Delta \)-convergence for proximal point algorithm and fixed point problem in CAT(0) spaces. Fixed Point Theory Appl. 2019; 8.

CAT(0) space is a complete, simply connected, and locally compact geodesic space. The CAT(0) space is characterized by the triangle inequality, which is a weaker version of the triangle inequality in Euclidean space. In a CAT(0) space, any geodesic triangle is smaller than or equal to the corresponding triangle in the Euclidean plane. The CAT(0) space is a real Hilbert ball. In this paper, we study the convergence and \( \Delta \)-convergence for proximal point algorithm and fixed point problem in CAT(0) spaces. The results improve and extend many results in the literature.