Chaotic, staggered and polarized dynamics in opinion forming: the contrarian effect

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Abstract

We revisit the no tie breaking 2-state Galam contrarian model of opinion dynamics for update groups of size 3. While the initial model assumes a constant density of contrarians $a$ for both opinions, it now depends for each opinion on its global support. Proportionate contrarians are thus found to indeed preserve the former case main results. However, restricting the contrarian behavior to only the current collective majority, makes the dynamics more complex with novel features. For a density $a < a_c = 1/9$ of one-sided contrarians, a chaotic basin is found in the fifty-fifty region separated from two majority-minority point attractors, one on each side. For $1/9 < a \lesssim 0.301$ only the chaotic basin survives. In the range $a > 0.301$ the chaotic basin disappears and the majority starts to alternate between the two opinions with a staggered flow towards two point attractors. We then study the effect of both, decoupling the local update time sequence from the contrarian behavior activation, and a smoothing of the majority rule. A status quo driven bias for contrarian activation is also considered. Introduction of unsettled agents driven in the debate on a contrarian basis is shown to only shrink the chaotic basin.

The model may shed light to recent apparent contradictory elections with on the one hand very tied results like in US in 2000 and in Germany in 2002 and 2005, and on the other hand, a huge majority like in France in 2002.

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1 Introduction

In the last years the study of opinion dynamics has attracted a growing number of works\textsuperscript{11 2 3 4 5 6 7 8 9 10 11 12} making it a major current trend

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of sociophysics \cite{13,14}. Most models consider 2-state opinion agents combined with some local opinion update rule which implements the dynamics. They are found to lead to an opinion polarization of the whole population along one of the two competing opinions. A unifying frame was proposed to incorporate all these models \cite{15}. Continuous opinion models yield similar tendency \cite{16,17}.

More recently, the concept of contrarian was introduced to account for some peculiar behavior of agents \cite{18}. A contrarian is undistinguishable to others, i.e. its opinion evolves also by local rule updates. However, once it leaves the update group, it changes individually its opinion to the other one. The shift is independent of the opinion itself. A contrarian is not a permanent individual state. Each agent has a probability $a$ to behave like a contrarian and $(1-a)$ to stick to its opinion. After each cycle of local updates, on average a proportion $a$ of agents shifts spontaneously their opinion to the other one, while a proportion $(1-a)$ sticks to its current opinion. Given a fixed density of contrarians $a$, the associated opinion dynamics is then studied \cite{18}.

As intuitively expected, the existence of contrarians was shown to avoid total opinion polarization with the creation of stable attractors characterized by a stable coexistence between a large majority and a small minority. However, above some low density, they were found to produce an unexpected reversal of the dynamics, with the merger of the two attractors at the former separator, turning it to the unique stable attractor \cite{18,19}. Accordingly, for whatever initial conditions the dynamics leads to an exact global balance between the two competing opinions. It thus offers a possible explanation to recently observed hung election scenario \cite{18}. A hung election being a two candidate run for which the result is very tied around fifty percent. Chaotic regime was also found in the description of investors in stock markets \cite{20}.

But in today campaigns, polls are regularly publicized making agents aware of which opinion is currently leading at the global level. It is therefore more natural to link the propensity to a contrarian behavior to the current level of global support for a given opinion. Accordingly, in this paper we relax above constraints of fixed independent contrarian density $a$ in two successive steps. First, we study the effect on the dynamics making the density $a$ proportional for each opinion to its current global support. Contrarians become proportionate contrarians. Second, we push the asymmetry further by restricting the contrarian behavior to only the current majority opinion, contrarians being one-sided. While proportionate contrarians are found to preserve the mean features of the fixed density contrarian dynamics, a more complex situation including a chaotic regime is discovered for one-sided contrarians.

For a density $a < a_c = 1/9$, one-sided contrarians produce a chaotic basin located in the fifty-fifty percent region. The associated Lyapunov exponent is calculated. However, there still exist majority-minority coexistence point attractors located on each side of the chaotic basin. Initial conditions determines which regime will dominate, either chaotic outcome around fifty percent, or point attractor with a well defined majority. For $1/9 < a < 0.301$ only the chaotic basin survives. Further, in the range $a > 0.301$, the chaotic basin disappears and the majority starts to alternate between the two opinions with a staggered flow towards two majority-minority point attractors.

On this basis the effect of additional social parameters are studied. A constant shift for contrarian activation is considered. Decoupling the local update time sequence from the contrarian behavior, and a delay in accounting for a
change of majority side are also introduced. Last, unsettled agents are included.
They are found to only shrink the chaotic basin making the outcome very tied.

Given above framework, the results may shed light to recent very unusual hung elections like in the 2000 US presidential vote and in Germany for the 2005 elections. In both cases the outcomes were very tied with almost identical support. At the same time, the model can also provides an explanation for the 2002 French presidential election where the winner obtained a huge majority around 80%.

At this stage it is worth to stress that we are able to embody these contradictory voting outcomes within a single frame. But at the same time, we are not able to decide before hand which one will prevail. To overpass this difficulty would require the collaboration with social scientists to estimate the actual type of contrarians involved in a given election.

The rest of the paper is organized as follows. In the next Section we review the original Galam model of contrarians. Throughout the paper, the size of update local groups is kept equal to three agents. The contrarians are then made proportionate in Section 3. Section 4 considers one-sided contrarians and the corresponding complex dynamics topology. Novel features are obtained. In Section 5 we study the effect of decoupling the local update time sequence from the contrarian behavior activation. A smoothing of the majority rule is also included. The effect of a status quo driven bias for contrarian activation is investigated in Section 6. Section 7 deals with the case of unsettled agents. Last section contains some discussion.

2 The original Galam model: individual contrarians

We start recalling the Galam model of 2-state opinion dynamics model extended to the presence of contrarians [18]. It considers a population of $N$ agents where at a time $t$, $N_A(t)$ persons support one opinion $A$ and $\{N - N_A(t)\}$ persons support another competing opinion $B$. In terms of global proportions among the whole population, it yields $p_t = \frac{N_A(t)}{N}$ for $A$ and $\{1 - p_t\}$ for $B$. These values can be evaluated at any time using polls.

From an initial value $p_t$ at time $t$, a dynamics is implemented in two steps. First, a neighborhood step where agents are distributed randomly among various size groups in which they update their respective individual opinion following the local initial majority [9]. In case of a tie in even size groups, either one opinion is adopted according to some probabilities [21]. The step is accounted by a discrete time increment of +1 leading to a new proportion $p_{t+1}$ of agents supporting opinion $A$. The second step is contrarian, each agent individually either shifts its respective opinion to the other one with a probability $a$, or preserves its current opinion with probability $(1 - a)$. This second step yields an additional increment of time +1 and modifies $p_{t+1}$ to another value $p_{t+2}$.

Throughout this paper we explicit the calculations for the case of local groups with the unique value 3. Above rules thus yield respectively for the first step $p_t \to p_{t+1} = F_m(p_t)$ with

$$p_{t+1} = F_m(p_t) = p_t^3 + 3p_t^2(1 - p_t),$$  \hspace{1cm} (1)
and for the contrarian second step $p_{t+1} \to p_{t+2} = P_c(p_{t+1})$ where

$$p_{t+2} = P_c(p_{t+1}) \equiv (1 - a)p_{t+1} + a[1 - p_{t+1}]. \tag{2}$$

It should be stressed that before performing another cycle of opinion updates, agents are reshuffled \[22\]. Applying above 2-step cycle repeatedly $n$ times results in a proportion $p_{t+2n}$ of agents supporting $A$. All possible variations and extensions of the model can be included within a unifying frame \[15\].

At this stage we make a change of variable from $p$ to $d$ with $p = d + \frac{1}{2}$. It will appear to be more convenient for our investigation. A positive $d$ makes $A$ the majority opinion while a negative value grounds it as minority with a deficit $|d|$ of support with respect to $B$. In terms of the new variable $d$, $P_m(p_t)$ and $P_c(p_{t+1})$ become respectively

$$d_{t+1} = D_m(d_t) \equiv -2d_t^3 + \frac{3}{2}d_t, \tag{3}$$

and

$$d_{t+2} = D_c(d_{t+1}) \equiv (1 - 2a)d_{t+1}, \tag{4}$$

which combine for one full cycle into the single Equation $d_{t+2} = D_c[D_m(d_t)]$, which we denote by

$$d_{t+2} = D_2(d_t) = (1 - 2a)[-2d_t^3 + \frac{3}{2}d_t], \tag{5}$$

where index 2 of $D_2$ means one local rule followed by one contrarian step with a time interval of 2 for the dynamics. At this stage such a 2-step split could appear artificial but it will become instrumental latter on to extend the dynamics to cases where one cycle is built out of $(k-1)$ consecutive local updates followed by one contrarian effect. These cases will be denoted by $d_{t+k} = D_k(d_t)$ with $k$ being the appropriate time interval to study the associated properties of the dynamics. In the original Galam work $k = 2$.

For this last case the main results obtained from Eqs. (1,2) or (3,4) are twofold. At low concentration $a$ of contrarian behavior, total polarization is prevented with two mixed attractors at which a majority and a small minority coexist. The threshold for $A$ victory is at $p_v = \frac{1}{2}$ or $d_v = 0$ which defines the separator of the dynamics. Furthermore, increasing $a$ provokes a continuous phase transition at $a_c = \frac{1}{8}$ turning $p_v = \frac{1}{2}$ or $d_v = 0$ into the unique and stable attractor of the dynamics. The final state is a perfect equality of both opinions at the collective level with ongoing individual opinion shifts \[18\].

### 3 Making contrarian behavior opinion current status dependent: proportionate contrarians

While the original model considers a constant and fixed proportion $a$ of contrarians, it seems more realistic to make it depend on the current state of the system, in particular due to the existence of published polls. We thus suppose that if at some specific time $t$ all agents are informed of the actual value of $d_t$, they react accordingly as contrarians with respect to $d_t$ at time $(t+1)$ leading to $d_{t+1}$. But it thus becomes natural to make the contrarian behavior opinion
dependent. Not to the opinion itself but to its current level of support in the population.

To distinguish the associated contrarians from previous ones, we call them proportionate contrarians. Therefore, agents sharing opinion A react to \( p_t \), i.e., \( d_t \) while agents sharing opinion B react to \((1 - p_t)\), i.e., \( -d_t \). We denote these rates respectively \( a(d) \) and \( b(d) \). From symmetry we have \( b(d) = a(-d) \) since opinions are time reversal.

For the time being, we keep our 2-step cycle, which implies a regular and periodic publication of polls. Such a constraint will be relaxed in a latter Section. On this basis the proportionate contrarian density \( a \) becomes a function of time through the variable \( d_t \). If no agent shares the opinion A, no one will react against it as a proportionate contrarian yielding the constraint \( a(d = -1/2) = 0 \). On the other extreme at \( d = 1/2, a(d = 1/2) = a_0 \) where \( a_0 \) is the proportionate contrarian maximal value which satisfies \( 0 \leq a_0 \leq 1 \).

Keeping in mind that now the proportionate contrarian density is not the same at a given time among agents sharing respectively opinion A and B, Eq. (4) becomes

\[
d_{t+2} = D_2(d_t) = (1 - a_{t+1} - b_{t+1})d_{t+1} + \frac{b_{t+1} - a_{t+1}}{2},
\]

where \( a_{t+1} \) and \( b_{t+1} \) respectively mean \( a(d_{t+1}) \) and \( b(d_{t+1}) \), and \( d_{t+1} \) is given by Eq. (3).

From Eq.(6) we can extract several properties of the associated dynamics. First we note that \( d = 0 \) is a fixed point if and only if \( a(d = 0) = b(d = 0) \). It means no agent is contrarian at perfect equality of opinions. Second we can evaluate its stability by studying the value of the associated eigenvalue \( \lambda \) with respect to one. When \( d = 0 \) is a fixed point, it is an attractor when \( \lambda < 1 \), which in turn implies the condition

\[
6a(d = 0) + a'(d = 0) - b'(d = 0) > 1,
\]

where the prime means a derivative with respect to \( d \). Otherwise when \( 6a(d = 0) + a'(d = 0) - b'(d = 0) < 1 (\lambda > 1) \), the fixed point \( d = 0 \) is a separator. A separator implies the existence of two attractors located respectively on both side at \( d > 0 \) and \( d < 0 \) with a stable coexistence of a majority and a minority.

Using above results we can review few specific functional forms for the \( a d \)-dependence. We start with the linear dependence \( a = a_0 p = a_0(1/2 + d) \). For \( a_0 < 1/5 \), \( d_v = 0 \) is a separator and the associated attractors are located at \( d = \pm \frac{2}{3\sqrt{1 + 5a_0}} \). When \( a_0 \geq 1/5 \), \( d = 0 \) becomes the unique attractor of the dynamics. In other words, the former Galam result is recovered with proportionate contrarians stabilizing a perfect equality between both opinions once their density is larger than a critical value, here \( 1/5 \). Results are shown in Fig. (1).

Considering a power law form \( a = a_0 p^\gamma = a_0(1/2 + d)^\gamma \) we get the condition \( a_0 \geq \frac{2^{-\gamma}}{3^{\gamma - 1}} \) to make \( d = 0 \) the unique attractor. It is a separator with two mixed phases attractors when \( a_0 < \frac{2^{-\gamma}}{3^{\gamma - 1}} \). Taking \( \gamma = 2 \) yields a square dependence with \( \frac{2}{5} \) for the critical value of \( a_0 \). More proportionate contrarians are needed \( (a_0 \geq \frac{2}{5} = 0.286) \) with respect to the linear case \( (a_0 > \frac{1}{5} = 0.20) \) to produce the perfect balance of opinions. When \( d = 0 \) is a separator, the two attractors are
Figure 1: Application $D_2$ given by Eq. \((6)\) for the linear and symmetric case where $a(d) = a_0 p = a_0(1/2 + d)$ and $b(d) = a(-d)$. On the left side $a_0 = 0.1 < 1/5$ with two attractors at coexistence of a majority and a minority. On the right side $a_0 = 0.3 > 1/5$ with one attractor at a perfect balance among both opinions.

located at $d = \pm \frac{1}{2} \sqrt{1 + \frac{1}{a_0}(1 + \sqrt{1 + 8a_0^2})}$. On the opposite, taking a square root dependence with $\gamma = 1/2$, yields $\frac{1}{4\sqrt{2}} \simeq 0.177$ for the critical value of $a_0$ making less proportionate contrarians needed to get the opinion balance as the unique attractor of the dynamics. The former Galam model considers $a = cst$, giving a critical value of $\frac{1}{8} \simeq 0.125$ which is the lowest value we can get.

At this stage we can conclude that proportionate contrarians do not modify qualitatively the former constant contrarian density dynamics opinion.

4 Restricting contrarian behavior to the current majority: one-sided contrarians

On the basis of above results, we go back to the case of a constant density of contrarians, but now restricting the activation of the contrarian behavior to only the current majority opinion. It thus yields for $d > 0$ the conditions $a(d > 0) = a = cst$ (with $0 \leq a \leq 1$) and $b(d > 0) = 0$, while for $d < 0$ the conditions are $a(d < 0) = 0$ and $b(d < 0) = a = cst$, where we have assumed symmetric conditions for both opinions. It is worth to note that this symmetry induces at $d = 0$ the condition $a(d = 0) = b(d = 0) = \frac{a}{2}$ both preserving the same total density $a$ of contrarians and recovering here the former original individual contrarian case. We call these contrarians one-sided contrarians. The associated rule update writes:

$$d_{t+2} = D_{o \rightarrow s}(d_{t+1}) \equiv (1 - a)d_{t+1} - \frac{a}{2} sign[d_{t+1}], \quad (8)$$

where $sign[x] = 1$ if $x > 0$, $-1$ if $x < 0$ and $0$ if $x = 0$.

Using Eq. \((8)\) for $d_{t+1}$ and noting that $sign[d_{t+1}] = sign[d_t]$, Eq. \((6)\) writes:

$$d_{t+2} = D_2(d_t) = (1 - a)(\frac{3}{2} d_t - 2d_t^3) - \frac{a}{2} sign[d_t]. \quad (9)$$
At second order in $d_t$ it yields:

$$D_2(d_t) \simeq \lambda_m d_t - \frac{a}{2} \text{sign}[d_t],$$

(10)

where $\lambda_m = \frac{3}{2}(1 - a)$ is the maximal slope of the application $D_2$.

We can now study the associated fixed points and their stability. We first note that the point $d = 0$ is a singular fixed point. Then, the application being symmetric we restrict the study to $d > 0$. According to Eq. (9), fixed points exist if $a \leq a_c = \frac{1}{9} \simeq 0.11$ and their values are:

$$d_{F\pm} = \frac{1}{4}(1 \pm \sqrt{\frac{1 - 9a}{1 - a}}).$$

(11)

When they exist, $d_{F-}$ is unstable and $d_{F+}$ stable (see Fig. 2). From Eq. (10) it gives:

$$d_{F-} \simeq \frac{a}{2(\lambda_m - 1)},$$

(12)

with the eigenvalue $\lambda \simeq \lambda_m > 1$ ($\lambda_m > 1$ for $a \leq a_c$).

Eq. (10) exhibits clearly condition for expansion when $\lambda_m > 1$, and folding up by the discontinuity at the origin $d = 0$. Thus, for being chaotic, the application $D_2$ has to satisfy two conditions; first, the expansion condition and second, the perpetuity of the chaotic basin [23, 24]. If the application $D_2$ is chaotic, the interval of successive iterated points after transients is called $\Omega_l$. Here $\Omega_l = [-a/2; a/2]$ (see Fig. 2).

1st condition: expansion.

Inside the interval $\Omega_l$ the application $D_2$ possesses a slope $\lambda$, such as $\lambda_m > \lambda > D'_2(a/2)$, where $D'_2$ denotes the derivative of $D_2$ relating to $d$. This implies that necessary $\lambda_m > 1$, i.e. $a < 1/3$. Furthermore, $D'_2(a/2) = 1$ for $a \simeq 0.27$. According to this unique condition, $D_2$ loses its chaotic nature for $a$ including the two last values.

2nd condition: perpetuity of the chaotic basin, i.e. $D_2(\Omega_l) \subset \Omega_l$.

If there are not fixed points, this condition is automatically satisfied. If fixed points exist, i.e. for $a \leq a_c$, they must be outside the interval $[-a/2; a/2]$, i.e. $d_{F-} > a/2$. According to Eq. (12) it implies $\lambda_m < 2$. This condition is always
Figure 3: Successive iterated points by the application $D_2$ given by Eq. (9). Notation here: $d(n + 1) = D_2[d(n)]$. Left side: $a = 0.05$ with the initial value $d(0) = 0.05$. Right side: $a = 0.2$ with the initial value $d(0) = 0.1$.

Figure 4: Error growth and Lyapunov exponent of the application $D_2$ given by Eq. (9) in semi-log plot. Notation here: $d(n + 1) = D_2[d(n)]$. The initial error is $\delta(0) = 10^{-10}$, taken after transients. Left side: $a = 0.05$. From the graph the Lyapunov exponent is $\lambda_{lyap} \approx 0.353$ while $\ln(\lambda_m) \approx 0.354$. Right side: $a = 0.2$. From the graph the Lyapunov exponent is $\lambda_{lyap} \approx 0.166$ while $\ln(\lambda_m) \approx 0.182$.

satisfied here for group size 3 while it is not the case for larger update groups from size 5. Nevertheless, to observe a chaotic behavior initial condition must satisfy $|d(t = 0)| < d_{F-}$. Otherwise, the successive iterated points go to a stable fixed point at $\pm d_{F+}$.

A chaotic behavior is numerically observed until $a \approx 0.301$. Illustrations are shown in Fig. (3) for $a = 0.2$ and $a = 0.05$ (with an appropriate initial condition in the latter case).

Fig. (4) shows sensitivity to initial conditions and permits the evaluation of the Lyapunov exponent $\lambda_{lyap}$. The initial difference $\delta(0)$ grows exponentially, $\delta(n) \simeq \delta(0) e^{n \lambda_{lyap}}$, until saturation at the typical size of the interval $\Omega_t$. The Lyapunov exponent is positive with $\lambda_{lyap} \simeq \ln(\lambda_m)$ as seen from Eq. (10).

The notation adopted in these figures is to consider the $n^{th}$ iteration of the application $D_2$ since the beginning of the electoral campaign as the equivalent time $n$. Thus $d_{t_0+2n} \equiv d(n)$, where $t_0$ is the time at the beginning of the electoral campaign. In other words, $d(n + 1) = D_2[d(n)]$.

At this stage we can make the following comments:
The discontinuity of the application $D_2$ at $d = 0$ is not the unique origin of its chaotical nature. Indeed, we can transform $D_2$ to be continuous and derivable, e.g. via $a \rightarrow a(m(1 - e^{-|d|\alpha}))$. If $a_m < 0.301$ and $\alpha \ll 1$ then this application exhibits a chaotic behavior. Let us recall that sudden variation is the source of the chaotic nature of the application because it permits the folding up of an expansive application (if $\lambda_m > 1$).

- When $a > 0.311$ the application has stable fixed points of doubling period. They are obtained from $D_2(d) = -d$.

- Increasing $a$, the first separation for $d > 0$ inside the interval $\Omega_l$ in two intervals occurs at $a \approx 0.056$. This result can be retrieved considering the doubling iterated application $D_2^{(2)} = D_2 \circ D_2$, i.e. $D_2^{(2)}(d) = D_2(D_2(d))$. Indeed, this occurs when $\lim_{d \rightarrow 0^-} D_2^{(2)}(d) > d^{(2)*}$, i.e. $D_2(a/2) > d^{(2)*}$, where $d^{(2)*}$ is a fixed point of doubling period where $D_2(d^{(2)*}) = -d^{(2)*}$. From Eq. (10) we retrieve $a \approx 1 - \frac{\sqrt{2}}{3} \approx 0.057$. The extension of successive iterated points after transients, $\Omega_l$, is then, for positive values: $[0; D_2(a/2) \cup |D_2^{(2)}(a/2)]]; a/2]$. 

- Starting for instance from opinion A being initially the majority, i.e $d > 0$, we evaluate $d_{ch}$, such as $D_2(d_{ch})=0$. Thus, if $d < d_{ch}$ then $D_2(d) < 0$ and the majority side will change to the opposite side; and reciprocally if $d > d_{ch}$ then $D_2(d) > 0$ and the majority side will keep the same side (see Fig. 2). From Eq. (10):

$$d_{ch} \approx \frac{a}{2\lambda_m}.$$  

(13)

Since successive iterated points are contained into the interval $\Omega_l = [-a/2; a/2]$ and $D_2(a/2) < d_{ch}$ with $\lambda_m \leq 1.5$, we deduce that $d > 0$ cannot remain positive more than twice before turning negative. Note that at second order on $d$, $D_2(a/2) > d_{ch}$ yields a second order equation whose solution is $\lambda_m > \frac{1+\sqrt{5}}{2} \approx 1.62$, the golden number.

To summarize, for $a < a_c = \frac{1}{9}$ this model provides the coexistence of three radically different basins. A chaotic one located around $d = 0$ delimited by
local majority rules writes as $w = 0$. convinced by majority rule. The new intention vote dynamics of the $k$-step process, generated by the one-sided contrarian step occurred after $a$ at the origin at $(3\cdot 2^k)$. For $k = 0$ such that $d \in [-a/2; a/2]$. If $|d(0)| > d_F$, the issue is certain but with a result at the extremes. This could account for contradictory electoral outcomes whose results are either around $p = 50\%$ or with a huge majority at $p \approx 80\%$ like the 2002 French presidential election.

5 Varying the time scales of collective information and local updates

After having considered 2-step processes, we now study the opinion dynamics driven by a $k$-step process where the individual activation of the one-sided contrarian step occurs after $k - 1$ repeated steps of local majority rule update. It accounts for the fact that polls are not made public every single day during a campaign while on the contrary people keep on discussing all the time.

Now $d_{t+k} = D_{a-s}(d_{t+k-1})$ with $d_{t+k-1} = D_m^{(k-1)}(d_t)$ where $D_m^{(k-1)} = D_m \circ D_m \circ \cdots \circ D_m$, i.e. $k - 1$ iterations of $D_m$. In previous sections, $k$ was equal to 2. Accordingly one-sided contrarians consider the collective information with a delay acting at time $t + k - 1$ while considering information at time $t$. However they could as well consider the collective information without delay, just after the last update inside groups at time $t + k - 1$. Indeed, they act according to $\text{sign}[d]$ (see Section 4) and $\text{sign}[d_t] = \text{sign}[d_{t+1}] = \cdots = \text{sign}[d_{t+k-1}]$.

Moreover $d_{t+k-1}$ increases very quickly from the origin $d = 0$, making a contrarian behavior without much effect. For instance $k = 10$ makes the slope at the origin at $(\frac{a}{2})^9 \approx 38$. So, we have to slow down the dynamics driven by the update rule inside groups in order to study the varying time scale of collective information. It is done quite naturally assuming that not every agent eventually changes its opinion to follow the local majority within each cycle of local updates turning Eq. (1) to:

$$p_{t+1} = P_{m,w}(p_t) \equiv w[3p_t^2 - 2p_t^3] + (1 - w)p_t,$$  (14)

where $w$ denotes the propensity of an agent to be convinced by majority rule with $0 \leq w \leq 1$. Using the $d$ variable gives:

$$d_{t+1} = D_{m,w}(d_t) \equiv (1 + w/2)d_t - 2w d_t^3.$$  (15)

Now $k - 1$ iterations of $D_{m,w}$ yields a slower slope at the origin, e.g. for $w = 0.1$ and $k = 10$ it is $(1 + w/2)^9 \approx 1.5$.

Let $a$ the density of contrarians and $w$ the propensity of an agent to be convinced by majority rule. The new intention vote dynamics of the $k$-step process, generated by the one-sided contrarian step occurred after $k - 1$ repeated local majority rules writes as

$$d_{t+k} = D_k(d_t) = D_{a-s}[D_{m,w}^{(k-1)}(d_t)],$$  (16)
Figure 6: Successive iterated points by the application $D_k$ given by Eq. (16).
Notation here: $d(n + 1) = D_k[d(n)]$. Left side: $a = w = 0.1, k = 3$ and an initial value $d(0) = 0.1$. The application is not chaotic. $\lambda_m \simeq 0.99 < 1$. Right side: $a = w = 0.1, k = 18$ and an initial value $d(0) = -0.04$. The application is not chaotic by escaping from the previous chaotic basin. $\lambda_m \simeq 2.06 > 2$.

where $D_{a,w}$ and $D_{m,w}$ are respectively given by Eqs. (8, 15). This yields at second order on $d_t$ the Eq. (10), but now with the slope $\lambda_m = (1-a)(1+w/2)^{k-1}$.

To exhibit a chaotic behavior the condition for expansion $\lambda_m > 1$ gives now:

$$k - 1 > \frac{-\ln(1-a)}{\ln(1+w/2)}, \quad (17)$$

If $a, w \ll 1$ and are of the same order, then $k - 1 > \frac{2a}{w}$, e.g. for $a = w \ll 1, k > 3$. Numerically this can be satisfied until $a = w \leq 0.4$ (see Fig. 6).

With respect of the perpetuity of the chaotic basin, the unstable fixed points (Eq. (12)) are $\pm d_{F-} \simeq \pm a$, if they exist. It implies $\lambda_m > 1$ and $\lambda_m \leq 1+a$ to have $d_{F-} \leq 1/2$, so, $k - 1 > \frac{\ln(1+a)}{\ln(1+w/2)}$. If $a, w \ll 1$ and are on the same order, then $k - 1 > \frac{2a}{w}$, e.g. for $a = w \ll 1, k > 5$. Nevertheless, the second order approximation is not proper as soon as the inequality $d_{F-} \leq 1/2$ doesn’t satisfy $|d| \ll 1$.

To check if successive iterated points cannot escape the chaotic basin, i.e. if $d_{F-} > a/2$, from Eq. (12) we need to have $\lambda_m < 2$, i.e.

$$k - 1 < \frac{\ln(2) - \ln(1-a)}{\ln(1+w/2)} \quad (18)$$

If $a, b \ll 1$ and are on the same order, then $k - 1 < \frac{2\ln(2)+2a}{w}$, e.g. for $a = w = 0.1, k < 16.9$ while numerically, $k < 18$ (see Fig. 6). As in Section 4, initial value has to satisfy $|d(t = 0)| < d_{F-}$. However, contrarily to the precedent section, here successive iterated points can escape the previous chaotic basin.

In addition, for $k > 2$, this $k$-process with $k - 1$ repeated steps of local majority rule updates increase naturally the majority side persistency before changing compared to the previous model.
6 Status quo driven bias

Up to now both opinions were perfectly symmetric. However while dealing with political opinion dynamics in view of an election a difference should be made between the opinion supporting the current political party in power and the one supporting the challenging party. The contrarian behavior should be a little more active against the former winner opinion thus creating a bias in favor of the challenging opinion \[9\].

Assuming B is the former opinion winner and denoting \(s\) the bias in favor of the challenging opinion, Eqs. (9, 10) should be rewritten as:

\[
d_{t+2} = D_2(d_t) = (1 - a)(\frac{3}{2} d_t - 2d_t^3) - \frac{a}{2} \text{sign}[d_t - s],
\]

and 

\[D_2(d_t) \simeq \lambda_m d_t - \frac{a}{2} \text{sign}[d_t - s]\]

where \(\lambda_m = \frac{3}{2} (1 - a)\).

In the case \(a \leq a_c = 1/9\) for which fixed points exist, Fig. (7) \((a = 0.05\) and \(s = 0.02\)) shows that for \(|s| < d_{F-}\), the bias doesn’t modify the position of the fixed points. But now, depending on the ration \(s/a\), the successive iterated points could escape from the previous chaotic basin, and thus reach a point attractor. The values \(s\), at a fixed density \(a\), for which this new phenomenon occurs, is obtained when \(\text{Sup}\{\lim_{d_t \to s} D_2(d_t) ; \lim_{d_t \to s} |D_2(d_t)|\} > d_{F-}\).

Thus, at first order on \(s\), \((a/2 + \lambda_m |s|) > d_{F-}\) and at second order on \(d_t\), Eq. (12) yields:

\[|s| > s_c \simeq a \frac{2 - \lambda_m}{2\lambda_m(\lambda_m - 1)},\]

where here \(1 < \lambda_m < \frac{3}{2}\). For instance \(a = 0.05\) gives \(s_c \simeq 2.37\%\) for the exact value \(s_c \simeq 2.44\%\).

Evaluating the values \(s\), at a fixed density \(a\), for which successive iterated points \(d_t\) have the same sign we find \(|s| > d_{ch}\), where \(d_{ch}\) is defined as \(D_2(d_{ch}) = 0\) for the application without bias (see Fig. 7). From Eq. (13):

\[|s| > a \frac{1}{2\lambda_m}.
\]

For instance \(a = 0.05\) yields \(|s| > 1.75\%\) for the exact value \(|s| > 1.76\%. It is worth noting that for \(s\) sufficiently large the application is no more chaotic. For instance, with \(a = s = 0.2\), the application has a periodic attractor of period 13.

Including a bias in favor of one opinion provides two main effects. First one, after transients, a majority side (in the favored opinion) before changing more persistent than without bias. Second one, for given density \(a\) of one-sided contrarians, it is possible to escape the chaotic basin without bias to reach the point attractor of the favored opinion.

7 Unsettled people and contrarian attraction

We now go back to a population where agents sharing either one opinion evolve by local majority rule updates only without contrarians \[9\]. But we also consider another population of agents who do not take part in the public debate. They are unsettled and hold no opinion. However they are gradually driven in the
public debate on a contrarian basis. At a constant rate \( u \) with \( 0 ≤ u ≤ 1 \) they move to the opinion holding population starting with an opinion opposite to the current majority. Once they adopt an opinion they become identical to other sharing opinion agents, i.e., they evolve by local majority rules.

At time \( t \) the number of persons sharing opinion A, opinion B and no opinion are denoted respectively \( N_A(t) \), \( N_B(t) \) and \( N_U(t) \) with \( N_O(t) = N_A(t) + N_B(t) \) and \( N_O(t) + N_U(t) = N \) where \( N \) is the total number of agents of both populations. Associated probabilities are:

\[
p_t = \frac{N_A(t)}{N_O(t)} \quad \text{and} \quad \frac{N_B(t)}{N_O(t)} = 1 - p_t.
\]  

(22)

We still have a two-step process. The first one is unchanged with \( p_{t+1} = P_m(p_t) \) while the second one is produced by the contrarian unsettled agents partial joining the debate. Note that now the application \( p_t \rightarrow p_{t+2} \) is no longer stationary. To account for the shrinking dynamics of unsettled agents we note \( n \) the time which corresponds to the \( n^{th} \) iteration i.e. \( d_{t_0+2n} = d(n) \) where \( t_0 \) is the time at the beginning of the campaign. Thus, writing \( d(n+1) = D_{2,n}[d(n)] \), it gives:

\[
d(n+1) = D_{2,n}[d(n)] = \frac{\frac{3}{2}d(n) - 2d(n)^3}{1 + a(n)} - \frac{a(n)}{2(1 + a(n))} \text{sign}[d(n)],
\]  

(23)

with

\[
a(n) = u \frac{N_U(n)}{N_O(n)} = u \frac{(1-u)^n}{R - (1-u)^n},
\]  

(24)

where \( R = \frac{N}{N_U(0)} ≥ 1 \). For \( u ≪ 1 \), on the first order on \( u \frac{N_U(n)}{N_O(n)} \) Eqs. (9, 10) are unchanged with now instead of \( a \) an effective time dependent contrarian density: \( a \rightarrow a(n) \).

In the limit \( u ≪ 1 \) from Eq. (12) we can write, \( d_{n,F} ≃ a(n) \) with \( \Omega_{t,n} \) defined for \( 0 < |d| < \frac{a(n)}{2} \). With \( \lim_{n→∞} a(n) = 0 \) the interval \( \Omega_{t,n} \) shrinks to 0.

However this model prohibits \( n \rightarrow ∞ \) since the number of unsettled persons mobilized at each iteration is an integer and not a real one, although the \( n^{th} \) iteration, the number \( uN_U(0)(1-u)^n \) is assumed to be greater than 1.
Figure 8: Applications $D_{2,n}$ given by Eq. (23) with $u = 0.02$ and $R = 2$. $n = 0$ in plain line and $n = 1$ in dashed line.

In the case $(R - 1) \gg u$, $a(n + 1) \simeq (1 - u \frac{R}{R-1}) a(n)$ from Eq. (24). Accordingly points escape the basin bordered by $\pm d_{n,F}^-$ from time $n$ to $n + 1$, if $|d(n + 1)| > d_{n+1,F}^- \simeq a(n + 1)$ implying $|d(n)| > (1 - u \frac{3R}{2(R-1)}) a(n)$ (see Fig. 8). Thus, if $d$ belongs to the interval $\Omega_{l,n}$ at time $n$, successive iterated points will be contained into the successive intervals $\Omega_{l,n}$. (see Fig. 9).

Even if successive iterated points do not escape the basin bordered by $\pm d_{n,F}^-$, the dynamics is no longer chaotic. Indeed the point $d = 0$ is now asymptotically stable. This is due to the non-stationary dynamics effects. Nevertheless, the shrinking dynamics doesn’t affect the sign of iterated $d$. So, the expected winner at the issue of an electoral campaign remains unpredictable.

Including unsettled people driven gradually in the public debate on a contrarian basis provides a shrinking of the chaotic basin of the one-sided contrarians model. Thus, the results may shed light to recent very unusual elections like the hung 2000 USA presidential and German 2005 elections. Indeed, the winners of these long electoral campaigns were very unpredictable and the outcomes very tied.

8 Conclusion

We have presented a simple model giving a deterministic opinion dynamics (see Eq. 9), which can be chaotic. Furthermore, chaotic basin can coexist with two point attractors at the extremes. This model contains two main effects. First effect, amplification one given by the local majority rule inside groups. Second effect, retroaction one given by the action of the one-sided contrarians. The one-sided contrarians act by comparison and opposition to a collective information, the majority. Their action introduces to dynamics a discontinuity.

Afterwards, rooted in this simple model, some others features of electoral campaigns are added, like the fact that polls are not made public every day, or a bias in favor of one opinion, or the influence of unsettled people gradually driven in the public debate on a contrarian basis.

To sum up our various results we come back to the one-sided contrarian
model deterministic equation used throughout this paper:

\[ f(x) = (1-a)\left(\frac{3}{2} x - 2x^3\right) - \frac{a}{2} \text{sign}[x], \quad (25) \]

where \( x \equiv d \in [-1/2; 1/2] \) and \( a \) is the density of one-sided contrarians, \( 0 \leq a \leq 1 \). At the second order on \( x \) approximation, with \( \lambda_m = \frac{3}{2}(1-a) \), \( f(x) \simeq \lambda_m x - \frac{a}{2} \text{sign}[x] \). \( \lambda_m \) is the expansion effect. The term \(-\frac{a}{2} \text{sign}[x]\) is related to the folding up, i.e. the retroaction effect. The term \(-x^3\) can be seen as a non-linear saturation effect and gives the point attractors. It is symmetric with respect to \( x = 0 \).

Afterwards, accounting for some others social parameters enrich the model and modify a little bit its applications. By decoupling the local update time sequence from the one-sided contrarian behavior activation, the two effects, amplification and folding up effects, can act separately. Next, a bias in favor of one opinion introduced as a simple parameter \( s \) fixes the discontinuity position at \( x = s \), i.e., \( \text{sign}[x] \rightarrow \text{sign}[x-s] \). The opinion dynamics is no more symmetric.

Last, unsettled people driven gradually to public debate on a contrarian basis roughly plays with the folding up effect uniquely; \( \lambda_m \approx \frac{3}{2} \) and \( a \rightarrow a(n) \) with \( a(n) \rightarrow 0 \) when \( n \rightarrow \infty \). The opinion forming dynamics is no more stationary.

At this stage it is worth to stress that contrary to what could be expected, our treatment using probabilities, local updates and reshuffling between updates does not define a mean field like frame. This result was demonstrate recently using Monte Carlo simulations of a nearest neighbor ferromagnetic Ising system on a square lattice \[22\]. Indeed it creates a new class of universality in addition to both 2-d Ising and mean field ones. For our current model, in the case of group size four with no contrarians, using a cellular automata was shown to recover our analytical result \[12\].

To conclude it is worth to notice that in terms of real life situations, due to the finite number of iterations imposed by the fact that any public campaign is finite in time, the intention vote dynamics will not exhibit a chaotic behavior although in principle it could. In addition, the non zero fuzziness of poll measurements of the initial intention vote distribution, result automatically in

Figure 9: Successive iterated points with \( D_{2,n} \) given by Eq. (23) where \( u = 0.02 \) and \( R = 2 \). Borders \( \pm \frac{a(n)}{2} \) are included in thin points. Notation here: \( d(n+1) = D_{2,n}[d(n)] \).
Figure 10: Combination of a k-step process and unsettled agents with $k = 10$, $w = 0.05$, $u = 0.05$ and $R = 1.5$. Initial values are 0.1 in plain line and $(0.1 - 0.001)$ in dashed line.

a growing error making difficult to predict the sign of successive iterations (see Fig. 10).

Although this simple model does not pretend to account for an exhaustive explanation of opinion forming during electoral campaigns it exhibits some features which could shed new light on recent surprising voting outcomes, like for instance, on the one hand, an unpredictable issue with a very tied outcome like the German 2005 and the USA 2000 elections, and on the other hand, a well predicted outcome with a huge majority like in the 2002 French presidential elections with a majority around 80%.

However it is worth to stress that at this stage if we are able to embody above contradictory voting outcomes within a single frame, we are not in a position to select the prevailing one. To overpass this difficulty would require the collaboration with social scientists to estimate the actual type of contrarians involved in a given election. It is open for future work.

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References
[1] F. Caruso and P. Castorina arXiv:physics/0503199
[2] F. Wu and B. A. Huberman, arXiv: cond-mat/0407252
[3] C.J. Tessone, R. Toral, P. Amengual, H.S. Wio, and M. San Miguel Eur. Phys. J. B 39, 535 (2004)
[4] M.C. Gonzalez, A.O. Sousa and H.J. Herrmann, To appear in Int. J. Mod. Phys. C 15 (2004)
[5] D. Stauffer and H. Meyer-Ortmanns, Int. J. Mod. Phys. C 15 (2), 241 (2004)
[6] F. Slanina and H. Lavicka, Eur. Phys. J. B 35, 279 (2003)
[7] S. Galam, Physica A 320, 571 (2003)  
[8] M. Mobilia, S. Redner, Phys. Rev. E 68, 046106 (2003)  
[9] S. Galam, Eur. Phys. J.B 25 Rapid Note, 403 (2002)  
[10] F. Schweitzer, J. Zimmermann and H. Mühlenbein, Physica A 303, 189 (2002)  
[11] K. Sznajd-Weron and J. Sznajd-Weron, Int. J. Mod. Phys. C 11, 1157 (2002)  
[12] S. Galam, B. Chopard, A. Masselot and M. Droz, Eur. Phys. J. B 4, 529 (1998)  
[13] S. Galam, Y. Gefen and Y. Shapir, Math. J. Socio. 9, 1 (1982)  
[14] S. Galam, Physica A 336, 49 (2004)  
[15] S. Galam, Europhys. Lett., 70 (6), 705 (2005)  
[16] G. Weisbuch, G. Deffuant, F. Amblard, and J. P. Nadal, Complexity 7, No. 2, 55 (2002)  
[17] R. Hegselmann and U. Krause, Journal of Artificial Societies and Social Simulations 5, 3 (2002)  
[18] S. Galam, Physica A 333, 453 (2004)  
[19] D. Stauffer, J.S. Sa Martins, Physica A 334, 558 (2004)  
[20] A. Corcos, J-P. Eckmann, A. Malaspinas, Y. Malevergne, D. Sornette, Quant. Finance 2, 264 (2002)  
[21] S. Galam, Phys. Rev. 71, 046123 (2005)  
[22] A. O. Sousa, K. Malarz and S. Galam, Int. J. Mod. Phys. C 16, 1507 (2005)  
[23] P. Bergé, Y. Pomeau, Ch. Vidal, L’Ordre dans le chaos, Hermann (1988)  
[24] E. Ott, Chaos in dynamical systems, Cambridge U. P., (2002)