Local asymptotic stability of a multi-source power system

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Abstract—We derive a nonlinear multi-source power system model representing synchronous machines or closed-loop DC/AC converters and study its local stability behavior. We reframe the local asymptotic stability problem in terms of small deviation from a well-defined equilibrium set, representing the rotational symmetry of the synchronous steady state manifold. We develop a theoretical framework that builds upon the celebrated Hartman-Grobman theorem to allow for systems with subspace invariance, whose Jacobian matrix, has one zero eigenvalue associated with one-dimensional kernel representing the same invariance. We prove the existence of a homeomorphism that maps linear trajectories approaching the linear subspace into nonlinear trajectories in some neighborhood of the invariance set. Upon projection into the angle and frequency domain, we exploit our key result to show local asymptotic stability of the rotational invariance, determined by the projected level set of a parameterized Lyapunov function, if each generator has sufficient reactive power and resistive damping.

I. INTRODUCTION

In the advent of the high penetration of renewable energy resources in the electrical network [1], power system stability remains at the heart of the understanding of the repercussions of these unprecedented changes affecting the generation, operation and distribution of energy.

Despite their intrinsic differences, related to microgrid systems consisting of renewable generation connected to the network mainly through DC/AC converters, being of smaller size, than that of a conventional large interconnected synchronous machines, with relatively shorter feeders, operating at medium voltage levels and presenting a lower reactance to resistance ratio compared to conventional infrastructure [2], synchronous machines and DC/AC converters have structural similarities, that can be exploited to design efficient control strategies endowing resilience to the grid, and also to draw constructive conclusions improving the stability assessment in the still prevailing conventional power generation.

Power system literature is rich in papers addressing the question of stability from an energy-shaping perspective in the aftermath of a severe failure or small disturbance. From a physical perspective, these approaches aim at minimizing an energy function comprising kinetic energy, reflecting the presence of rotating, or emulating the rotating components, as well as potential energy, induced by the electromechanical interaction or coupling forces depending on the angles, and represented in incremental coordinates, i.e., with respect to a synchronous steady state trajectory.

Based on Kuramoto models, local exponential stability can be proven in [3] with explicit characterization of the region of attraction of a class of these systems, building on the efforts in [4] to exploit the relationship between the power network model and a first-order model of coupled nonuniform Kuramoto oscillators, and under the assumption of overdamped generators. Recently, the authors in [5] introduce a relaxed notion of power system stability in terms of phase cohesiveness and frequency boundedness, where purely algebraic conditions are derived, that depend on a bound on the angle cohesiveness and the parameter of a typical class of energy functions consisting of a potential energy induced by conservative coupling forces across transmission lines, as well as kinetic energy and related cross-terms. The small-signal stability analysis reveals local exponential convergence to the equilibrium subspace, which depends on the smallest nonzero eigenvalue, and the ratio of the largest to the smallest resulting Laplacian.

Relying on parameterized energy functions, goes hand in hand, with exploiting incremental passivity theory [6], whether in multi-machine setup [7]–[9] or a DC/AC converter setup via the so-called Bregman storage functions [10], where the aim is to develop a theoretical framework that comprises the kinetic energy associated with the elements emulating the rotating machine and terms taking into account the reactive power stored in the lines and dissipated on shunt elements, while satisfying suitable dissipation inequalities.

The main contributions of this paper are three-fold. First, we present a detailed nonlinear power system model, covering in its generality classical high-order models of synchronous machines [11], and DC/AC converters with control strategies that render the closed-loop system structurally equivalent to a conventional machine, e.g., matching control [12], droop control [13], virtual synchronous machines [14].

The steady state manifold possesses a rotational symmetry, that is characteristic for synchronous steady state of a power system [15]. For convenience, we transform the system into error coordinates, describing how disturbances affect the deviation from rotational invariance equilibrium, while preserving the original topology.

Second, we build on the stability analysis conducted in [16] of the linearized system at a desired operating point, whose system matrix possesses one-dimensional kernel, descending from the rotational invariance of the nonlinear system. For this, we develop an extension of the Hartman-Grobman theorem to include general nonlinear systems with subspace invariance, whose Jacobian matrix, has a zero eigenvalue of algebraic and geometric multiplicity one, associated with the same invariance. We prove the existence of
a homeomorphism mapping linearized complement trajectories approaching the invariance subspace into its nonlinear counterparts in a well-defined neighborhood of the invariance subspace. Our developed theory connects to contraction analysis and in particular to notions like partial or semi-contraction.

Third, we apply our theoretical result to study the local asymptotic stability behavior of the rotational invariance, whose first two coordinates agree, with that of invariance subspace of its Jacobian matrix. The lemma proving the existence of a homeomorphism is utilized in the stability analysis, where the sub-level sets of the Lyapunov function from [16], serve as an estimate for a suitable neighborhood, as long as the sufficient stability condition from [16] is satisfied.

The rest of the paper is structured as follows. Section II introduces a high-order model for various types of generation within power systems, ranging from synchronous machines to closed-loop DC/AC converters emulating a parallel behavior. Section III proposes an important lemma, concerning Jacobian matrices with one zero eigenvalue and one-dimensional zero subspace, and proving the existence of a homeomorphism, mapping the trajectories of a general linearized system around a linear subspace invariance, into nonlinear trajectories in some neighborhood. Followed by Section IV, under sufficient and fully decentralized condition from [16], we endeavor to show that the rotational symmetry is locally asymptotically stable, and this, in the light of our preliminary results, where we draw conclusion on the nonlinear trajectories of the power system model in a neighborhood identified with level sets of the Lyapunov function in [16], based on asymptotic behavior of linear trajectories around the linear subspace invariance.

Notation: We define an undirected graph \( G = (V, E) \), where \( V \) is the set of nodes with \( |V| = N \) and \( E \subseteq V \times V \) is the set of interconnected edges with \( |E| = m \). We assume that the topology specified by \( E \) is arbitrary and we define the map \( E \rightarrow \mathcal{V} \), which associates each oriented edge \( e_{ij} = (i, j) \in E \) to an element from the subset \( \mathcal{I} = \{-1,0,1\}^{|V|} \), resulting in the incidence matrix \( B \in \mathbb{R}^{N \times m} \). We denote by \( I \in \mathbb{R}^{2 	imes 2} \) the identity matrix \( I \), \( I_p \) the \( p \)-th dimensional identity matrix, \( p \in \mathbb{N} \), and \( J_{2p} = I_p \otimes J \) with \( J = \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] \). We define the rotation matrix \( R(\gamma) = \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) \\ \sin(\gamma) & \cos(\gamma) \end{bmatrix} \), and \( R(\gamma) = I_p \otimes R(\gamma) \). Let diag(\( v \)) denote a diagonal matrix, whose diagonals are \( v \) and \( R(\gamma) = \text{diag}(r(\gamma)) \), \( k = 1 \ldots N \), with \( r(\gamma) = \begin{bmatrix} -\sin(\gamma) \\ \cos(\gamma) \end{bmatrix} \).

Let \( I_N \) be the \( N \)-th dimensional vector with all entries one, and \( T^N = S^1 \times \ldots \times S^1 \), the \( N \)-th dimensional torus. Given a set \( A \), we denote by \( |x|_A = d(x,A) = \inf_{a \in A} d(x,a) \), the common point-to-set distance. Let \( K_{\infty}\)-comparison functions, be defined by all the maps \( \alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \) that are continuous and strictly increasing, where \( \alpha(s) \rightarrow \infty \), as \( s \rightarrow \infty \).

II. MODELING

We depart from the following general model describing the evolution of \( n \)-identical multi-source generators: synchronous machines [17] or DC/AC converters in closed-loop with the matching control [12], droop control [13] or any control strategy that renders the closed-loop system structurally similar to a synchronous machine, interconnected with identical \( m \)-resistive and inductive lines.

A. Multi-source power system dynamics

We model the dynamics of the balanced three-phase \( k \)-th generator by the following first-order differential equations. To eliminate the time-dependence of the sinusoidal trajectories, we transform the model into a frame rotating at the nominal steady state frequency \( \omega_n > 0 \), with angle \( \theta_{dq}(t) = \int_0^t \omega_n \, d\tau \), by so-called Clark transformation [11],

\[
\begin{bmatrix}
\dot{y}_k \\
M \ddot{\alpha}_k \\
\dot{L}_{idq,k} \\
C \dot{V}_{dq,k}
\end{bmatrix} =
\begin{bmatrix}
\eta(\alpha_k - \alpha_0) \\
-D(\alpha_k - \alpha_0) + \frac{\mu}{\alpha_k} (\gamma_k) \dot{I}_{dq,k} \\
-(R_l + L_0 \alpha_2) \dot{I}_{dq,k} + \frac{\mu}{\alpha_k} (\gamma_k) \dot{V}_{dq,k} \\
-(G_l + C_0 \alpha_2) \dot{V}_{dq,k} + J_{dq,k} - i_{out}
\end{bmatrix} +
\begin{bmatrix}
0 \\
P_{m,k} \\
0 \\
0
\end{bmatrix}
\]

where \( \gamma_k \in S^1 \) is the rotor or virtual converter angle, \( \eta \) is a positive gain and \( \alpha_0 \in \mathbb{R} \) is the frequency. The parameter \( M > 0 \) is the rotational inertia constant or DC capacitance and \( \mu > 0 \) is the constant field current or the modulation magnitude. The machine damping, or equivalently DC conductance is represented by \( D > 0 \). Let \( i_{dq,k} \in \mathbb{R}^2 \) be the inductance current, \( v_{dq,k} \in \mathbb{R}^2 \) the output voltage, and \( i_{dc,k} \in \mathbb{R}^2 \) the line current. The filter resistance and inductance are represented by \( R > 0 \) and \( L > 0 \) respectively and the capacitor \( C > 0 \) is set in parallel to a resistive load conductance \( G > 0 \) to ground and connected to the network via line current \( i_{net} \in \mathbb{R}^2 \). We denote by \( u_k = P_{m,k} \in \mathbb{R} \), constant mechanical power or current source representing the input to the \( k \)-th generation unit.

By lumping the states of \( N \)-identical nodes and \( m \)-identical inductive and resistive lines and defining the impedance matrices \( Z_R = R \cdot I_{2N} + L_0 \alpha_n \cdot I_{2N} \), \( Z_C = G \cdot I_{2N} + C_0 \alpha_n \cdot I_{2N} \), \( Z_t = R_l \cdot I_{2N} + L_0 \alpha_n \cdot I_{2N} \), we obtain the following model,

\[
\begin{bmatrix}
\dot{y}_k \\
M \ddot{\alpha}_k \\
\dot{L}_{idq} \\
C \dot{V}_{dq}
\end{bmatrix} =
\begin{bmatrix}
\eta(\alpha - \alpha_0) \\
-D(\alpha - \alpha_0) + \frac{\mu}{\alpha} (\gamma) \dot{I}_{dq} \\
-Z \dot{V}_{dq} + J_{dq} - B_{\ell} i_{dc} \\
-Z \dot{V}_{dq} - B_{\ell} i_{dc}
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

where \( \gamma = [\gamma_1 \ldots \gamma_N]^T \in T^N \), \( \omega = [\omega_1 \ldots \omega_N]^T \in \mathbb{R}^N \). The last equation in (2) describes the line dynamics and in particular the evolution of the line current state \( i_t = [i_{dt} \ldots i_{dt}] \in \mathbb{R}^{2m} \), where \( R > 0 \) is the line resistance, \( L > 0 \) is the line inductance and \( B = B \otimes I \). The converter input is represented by \( u = [P_{m,1} \ldots P_{m,N}] \in \mathbb{R}^N \).

We introduce the state vector \( z = [\gamma \omega x]^T \), where \( x = [i_{dq} \dot{v}_{dq} i_{dc}]^T \in \mathbb{R}^{(4N+2m)} \) and denote the product space \( T^N \times \mathbb{R}^N \times \mathbb{R}^{(4N+2m)} \) by \( X \). By putting it all together, we arrive at the nonlinear power system model described by,

\[
\dot{z} = K^{-1} \begin{bmatrix}
\eta \alpha \\
-D(\alpha - \alpha_0) + \frac{\mu}{\alpha} (\gamma) \dot{I}_{dq} \\
-Z \dot{V}_{dq} + J_{dq} - B_{\ell} i_{dc} \\
-Z \dot{V}_{dq} - B_{\ell} i_{dc}
\end{bmatrix} + K^{-1} u = f(z,u).
\]
with $K = \text{diag}(I_N, M \cdot I_N, L \cdot I_{2M}, C \cdot I_{2N}, L \cdot I_{2M})$, $\omega = \omega_0 I_N$, $u = \begin{bmatrix} 0 & u & 0 & 0 \end{bmatrix}^\top$, and $f(z, u)$ denotes the resulting nonlinear vector field.

### B. Steady state manifold

In the light of Section II-A, we aim to understand for an input vector $u$, the properties of the induced nonlinear steady state manifold,

$$\mathcal{M} = \{z^* \in \mathcal{X} | f(z^*, u) = 0\},$$  \hfill (4)

resulting from setting (3) to zero.

In the sequel, we investigate the properties of the steady state manifold $\mathcal{M}$, and answer questions related to its feasibility, rotational symmetry and angle periodicity.

**Lemma II.1.** Consider the steady state manifold $\mathcal{M}$ as described by (4). Then, it has the following properties:

1. **Frequency synchronization:** The frequencies of all generators have the same nominal value $\omega_0 I_N$, and hence $\omega^* = 0$.
2. **Rotational symmetry:** $\mathcal{M}$ has a rotational symmetry given by the vector
   $$\mathbf{m}(\alpha, z^*) = \left[\begin{array}{c} \alpha 1_N, 0, R_{2N}(\alpha)\tilde{v}_{dq}, R_{2N}(\alpha)\tilde{v}_{dq}, R_{2N}(\alpha)\tilde{v}_{dq} \end{array}\right]^\top$$
   for all $\alpha \in \mathbb{R}$, or equivalently,
   $$\forall z^* \in \mathcal{M}, z^* + \mathbf{m}(\alpha, z^*) \in \mathcal{M}.$$  \hfill (5)
3. **Feasibility:** If $u(z^*) = -\frac{1}{\eta} \mathbf{R}^\top(\gamma^*) \tilde{v}_{dq}$ with $\mathbf{R}(\gamma^*) = \text{diag}(\mathbf{R}(\gamma^*_N))$ for $k = 1 \ldots N$, then $\mathcal{M}$ is non-empty.
4. **Angle periodicity:** The steady state angles $\gamma^* \in \mathbb{T}^N$, are $2\pi$-periodic.

**Proof.**

1. This follows from that setting the angle dynamics to zero (and by definition of a synchronous steady state manifold $\mathcal{M}$ in (4)).
2. The existence of a rotational symmetry results from
   $$\left[\begin{array}{c} \alpha 1_N, 0 \end{array}\right]^\top \in \ker \left(\begin{bmatrix} 0 & \eta I_N \\ 0 & -D I_N \end{bmatrix}\right), \alpha \in \mathbb{R},$$
   and up to re-defining the transformation into a rotating frame [11], with a new angle $\Theta'_{dq} = \Theta_{dq} + \alpha$, where $i'_{dq} = R_{2N}(\alpha) i_{dq}$, $vi'_{dq} = R_{2N}(\alpha) vi_{dq}$ and $i^r_{dq} = R_{2N}(\alpha) i^r_{dq}$. In fact, the matrix $R_{2N}(\alpha) \in \mathbb{R}^{N \times N}$, commutes with the matrices $Z_r, Z_c, Z_d$ and $J_{2p}$, while satisfying the steady state equations [4].
3. The feasibility condition follows from the steady state equations, of the frequency $\omega_k$ at $k$-the generator, after substituting $\omega^* = 0$, and solving for the input $u_k$, $k = 1 \ldots N$, given by $-\frac{1}{\eta} \mathbf{R}^\top(\gamma^*_k) \tilde{v}_{dq,k} + u_k = 0$.
4. Angle periodicity is a direct consequence of the periodicity of the trigonometric functions involved in the rotation vector $r(\gamma^*_k), k = 1 \ldots N$.

**Assumption 1** (Feasibility of the steady states). Assume that the input $u$ is given by $u = u(z^*)$.

### III. Existence of homeomorphism

#### A. Preliminaries

We consider the system

$$\dot{z} = f(z),$$  \hfill (6)

where $f$ a continuously differentiable vector field defined on the state space $\mathcal{X}$. Let $\mathcal{M} \subseteq \mathbb{R}^n$ be a non-empty steady state manifold of (6) and the non-empty, closed and invariant set $S \subseteq \mathbb{R}^n$ be defined by,

$$f(z^* + S) = f(z^*) = 0,$$

for all $z^* \in \mathcal{M}$.

We rewrite the nonlinear vector field (6) in error coordinates $\zeta = z - z^*, z^* \in \mathcal{M}$, as follows

$$\dot{\zeta} = f(z) - f(z^*) = \tilde{f}(\zeta),$$  \hfill (7)

where $\tilde{f}$ is the resulting vector field with its corresponding trajectory $\phi_{\tilde{f}}(t, \zeta_0)$ defined at time $t > 0$, starting from the initial state $\zeta_0 = [\tilde{\theta}_0 \ \tilde{\omega}_0 \ \tilde{\alpha}_0]^\top$. We are interested in the study of the local stability properties of the set $S \subseteq \mathbb{R}^n$. For this, we define asymptotic stability for a general closed and invariant set $S \subseteq \mathbb{R}^n$, and given initial state vector $\zeta_0 = \tilde{z}(0)$.

**Definition III.1** (Asymptotic stability with respect to $S$). A set $S \subseteq \mathbb{R}^n$ is asymptotically stable, if

1. $S$ is stable, if for all $\epsilon > 0$, there exists $\delta(\epsilon, \zeta_0)$, such that,
   $$|\zeta_0| < \delta \implies |\phi_{\tilde{f}}(t, \zeta_0)| < \epsilon,$$

2. $S$ is attractive, if there exists a scalar $\xi > 0$, having the property that, whenever $|\tilde{z}_0| < \xi$,
   $$\lim_{t \to \infty} |\phi_{\tilde{f}}(t, \zeta_0)| \to 0.$$

$S$ is locally asymptotically stable, if it is locally stable and attractive.

Notice that, if $S$ is asymptotically stable, for all $\tilde{z}_0 \in \mathbb{R}^n$, Definition III.1 coincides with the definition given in [18] of global asymptotic stability of non-empty, closed and invariant set. Notice that, if $S$ is asymptotically stable, for all $\tilde{z}_0 \in \mathbb{R}^n$, Definition III.1 coincides with the definition given in [18] of global asymptotic stability of non-empty, closed and invariant set.

#### B. Existence of a homeomorphism

The following Lemma characterizes the local behavior of trajectories, around a linear subspace $S$ based on its linearization.

**Lemma III.2.** (Existence of a homeomorphism) Consider the system given in (7), with the steady state manifold $\mathcal{M} \subseteq \mathbb{R}^n$ of (6). For $z^* \in \mathcal{M}$, let $S \subseteq \mathbb{R}^n$, be a linear subspace, so that $z^* + S \subseteq \mathcal{M}$, lying in some neighborhood $U = S \oplus W$, $W \subseteq S^\perp$. If the Jacobian $J(z^*) = \frac{\partial f}{\partial z} |_{z=z^*}$, has the real part of all its eigenvalues in the open left or in the open right half-plane except for one at zero, with one-dimensional kernel $S \subseteq \mathbb{R}^n$, then there exists a homeomorphism $\psi$ and a neighborhood $U' = S \oplus W'$, contained in $U$, where $\psi : S^\perp \to W'$, with $W' \subseteq W$, which maps linear trajectories approaching $S$, into nonlinear trajectories of $W'$ approaching $S$, and preserves the directions.
Proof. The claim in Lemma III.2 is equivalent to proving the existence of a continuous, bijective and inversely continuous mapping from $S^\perp$ to $\mathcal{W}'$, that preserves the directions.

Define the neighborhood $\mathcal{U} = S \oplus \mathcal{W}, \mathcal{W} \subseteq S^\perp$, and the relation $\sim$, by the following equivalence class:

$$
\{ \tilde{z} \} = \{ z \in S, \tilde{z}, z \in T_z \mathcal{X} \},
$$

with $\tilde{z} = z - z'$ and $z' = z' - z^*, z^* \in \mathcal{M}$ and $T_z \mathcal{X}$ is the tangent manifold of $\mathcal{X}$ at $z^*$. We introduce the quotient (i.e., continuous, surjective and open) map $p$, as shown in the diagram in Figure 1 as well as, $z_P = p(\tilde{z})$ of the corresponding system $z_P = J_P z_P$, with its system matrix $J_P : \mathbb{R}^n / S \rightarrow \mathbb{R}^n / S$ given by,

$$
J_P : \tilde{z} = J_P : z + S := [J(z^*) : \tilde{z}] = J(z^*) : \tilde{z} + S,
$$

where $J_P : [0] = [0]$ and $J_P : [v_i] = \lambda_i [v_i]$, with $v_i$ being the right eigenvector of $J(z^*)$ corresponding to an eigenvalue $\lambda_i < 0$ (or $\lambda_i > 0$), for all $i = 1, \ldots, n - 1$. Let $T = (v_1 \ldots v_n)$, where $S = \text{span}(v_i)$, be the transformation matrix of $J(z^*)$ into its Jordan Normal form $\hat{J}$ and given by $T^{-1} J(z^*) T = \hat{J}$, up to reordering of the Jordan blocks,. Notice that the transformation matrix $T_P : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1} / S, (a_1, \ldots, a_{n-1})^T \mapsto \alpha_1 [v_1] + \alpha_2 [v_2] + \cdots + \alpha_{n-1} [v_{n-1}], fulfilling T_P^{-1} J_P T_P = \hat{J}_P$ and hence obtained by eliminating one row and one column vector of $\hat{J}$ corresponding to the zero subspace $S$. This shows that $J_P$, has all its eigenvalues in the open left or in the open right half-plane, and the origin is a hyperbolic equilibrium on $T_z \mathcal{X} \sim$.

Next, we define a projection of the trajectories onto $\mathcal{W}$, the complement of $S$ resulting into the coordinates $z_s = \Pi_S \cdot \tilde{z}, \tilde{z} \in \mathcal{U},$ with $\Pi_S = I_s - \frac{\partial \tilde{z}}{\partial z} \frac{\partial z}{\partial S}, s \in S$ a projection matrix defining the system $\dot{z}_s = \Pi_S \hat{f}(z) = \Pi_S \tilde{f}(z_s + s) = f_s(z_s)$.

A linearization of the vector field results in the Jacobian $J_s = \frac{\partial f_s(z_s)}{\partial z_s} = \frac{\partial \Pi_S / \Pi_S}{\partial z_s} = \Pi_S \frac{\partial \Pi_S}{\partial z_s} = \Pi_S J_s \Pi_S$. This implies that $J_s = \Pi_S T_s \hat{J} T_s^{-1} = \hat{T} \hat{T}^{-1}$, where the matrix $\hat{T} = \Pi S \cdot \tilde{T} \cdot$, with $\tilde{v}_i = \Pi S \cdot \tilde{v}_i$, $i = 1, \ldots, n - 1$, $T^\top$ is the pseudo-inverse of $T$, i.e., $T^\top T = T T^\top = \Pi_S$, and hence up to a re-ordering of the Jordan blocks in $\hat{J}$, the matrices $J_s$ and $J_P$ are similar.

By Hartman–Groban Theorem [19], there exists a homeomorphism $h$ mapping linear trajectories of $T_z \mathcal{X} \sim$, determined by $z_P = J_P z_P$, into its nonlinear counterpart in some neighborhood of the origin $U_0 \subseteq \mathcal{W}$, whose dynamics are governed by $\dot{z}_s = f_s(z_s)$, and the origin is a hyperbolic equilibrium, for any $z_s(0) \in U_0$.

The diagram in Figure 2 summarizes useful connections to the tangent space $T_z \mathcal{X}$. The universal property of the quotient map implies the existence of a unique homeomorphism $\varphi$ such that $\varphi \circ p = \Pi_s$ between $T_z \mathcal{X} \sim$ and $S^\perp$.

From setting $\mathcal{W}' = \mathcal{U}_h$, it follows that $\psi : S^\perp \rightarrow \mathcal{W}'$, $\tilde{z} \mapsto h \circ \varphi^{-1}(\tilde{z})$ is a homeomorphism, where $\mathcal{W}' \subseteq \mathcal{W}$ from $S^\perp$ to $\mathcal{W}'$. By defining $\mathcal{U}' = S \oplus \mathcal{W}'$, we have proven the existence of $\mathcal{U}' \subseteq \mathcal{U}$.

Finally, to prove the topological conjugacy between trajectories of $\mathcal{W}'$ and $T_z \mathcal{X} \sim$, we show that $\psi^{-1}(\varphi_t(\tilde{z}_0)) = e^{\sum^t \psi^{-1}(\tilde{z}_0)}$ for all $t > 0$ and $\tilde{z}_0 \in U_0$.

By definition of the homeomorphism $h$ that preserves the direction of the trajectories, for all $\tilde{z}_0 \in U_0$ from the Hartman-Grobman theorem, we have that $h^{-1}(\varphi_t(\tilde{z}_0)) = e^{\sum^t \varphi^{-1}(\tilde{z}_0)}$, for all $t > 0$. Hence $\psi(\varphi_t(\tilde{z}_0)) = e^{\sum^t \varphi^{-1}(\tilde{z}_0)}$, following the same spirit as the arguments, in p.124 of [20], form the proof of Hartman-Grobman theorem.

Remark 1. (Link to existing theories) Lemma III.2 is an extension of the Hartman-Grobman theorem, originally formulated for a hyperbolic steady state [20], where the Jacobian is allowed to have one-dimensional kernel, corresponding to a zero eigenvalue with algebraic multiplicity one. Our analysis links closely to well-established concepts in contraction theory, where one evaluates the stability of trajectories of dynamical systems with respect to one another [21]. Our analysis is reminiscent, in particular of partial contraction [22] (or also semi-contraction [23]) theory, allowing to extend the application of standard contraction analysis, to include convergence to behaviors, e.g., convergence to a steady state manifold. This can be interpreted as the contraction in all directions up to some direction (see Example 4.2, [22]) corresponding in our analysis to the linear subspace $S$. It is illuminating to rewrite the error system in (8) as,

$$
\delta \tilde{z} = \frac{\partial \tilde{f}}{\partial \tilde{z}} |_{\tilde{z}=0} \cdot \delta \tilde{z},
$$

where $\delta \tilde{z}$ is a virtual displacement, referring to the distance between infinitesimally separated trajectories. For any trajectory, starting in the neighborhood $U'$, centered about the invariant set $S$, the symmetric part of the Jacobian of the subsystem, evolving on $\mathcal{W}'$, is negative definite, and $\mathcal{W}'$ is thus a contracting region for trajectories of (10), except for directions along $S$.

IV. LOCAL ASYMPTOTIC STABILITY OF THE MULTI-SOURCE POWER SYSTEM

Our analysis of the Jacobian of the nonlinear power system model [3] in [16], [24], takes under the loop the behavior of
Asymptotic stability of the linearized system is guaranteed from (3) to (4) as follows, where $\tilde{z} = z - z^*$, $z^* \in \mathcal{M}$. In essence, the power system model in error-coordinates (13) describes how power disturbances drive the error state vector, to deviate from the rotational invariance set $m(\alpha, z^*)$.

The following theorem characterizes the stability behavior of the rotational invariance $m(\alpha, z^*)$ in a well-defined neighborhood.

**Theorem IV.2 (Local asymptotic stability).** Consider the system in (13) under Assumption [1] and Assumption 4 from [16]. The rotational symmetry $m(\alpha, z^*)$, $\alpha \in \mathbb{R}$ and $z^*$ in $\mathcal{M}$, given by (5) is locally asymptotically stable for any trajectory of (7) in some neighborhood, defined by the region of attraction $\Pi(\Omega_c(z^*))$, $\Omega_c(z^*) = \{ \tilde{z} \in \mathcal{X} | V(\tilde{z}) \leq c, c > 0 \}$, where $\tilde{z} = z - z^*$, $\Pi$ is a projection matrix into $(\tilde{z}, \partial t)$—subspace and $V(\tilde{z}) : \mathcal{T}_r \mathcal{X} \to \mathbb{R}_0$, is a Lyapunov function with respect to $\{v(z^*)\}$, given by (12).

**Proof.** We perform a projection into the coordinate space consisting of angles and frequencies $(\tilde{q}, \tilde{\phi}) \in \Pi(\mathcal{X})$, where $\Pi$ is a projection matrix. By observing that $\Pi(m(\alpha, z^*)) = \Pi(\text{span}\{v(z^*)\}) = \text{span}\{[I_N \quad 0]^{\top}\}$, for $\alpha \in \mathbb{R}$ and $z^*$ in $\mathcal{M}$, our proof follows two main steps: First, we show that $[I_N \quad 0]^{\top}$ is locally asymptotically stable following Definition III.1 by applying Lemma III.2 with $\mathcal{S} = \text{span}\{[I_N \quad 0]\}^{\top}$. Later, we conclude that $m(\alpha, z^*)$ is locally asymptotically stable.

Under Assumptions 4 in [16], the linear subspace $\text{span}\{v(z^*)\}$ is asymptotically stable for any trajectory of (11), from which follows that $\mathcal{S} = \Pi(\text{span}\{v(z^*)\})$ is asymptotically stable on the projected space $(\tilde{q}, \tilde{\phi})$. This implies that for all $M > 0$ and $\tilde{z}_0 \in \mathcal{T}_r \mathcal{X}$, $\|\tilde{z}_0\|_S < M \implies \lim_{t \to \infty} \|\Pi(\Phi(t, \tilde{z}_0))\|_S = 0$. The homoeomorphic $\psi$ deforms continuously the projected linear system trajectories of $\Phi(t, \tilde{z}_0(0))$ into its nonlinear counterpart $\Psi(\Phi(t, \tilde{z}_0(0)))$, in some neighborhood $\mathcal{U}$. From which follows, that for all $M' > 0$ and $\tilde{z}_0 \in \mathcal{U}'$, so that $|\Pi(\tilde{z}_0)|_S < M'$, implies that $\lim_{t \to \infty} |\Pi(\Phi(t, \tilde{z}_0))|_S = 0$.

Second, verifying asymptotic stability of the remaining coordinates of $m(\alpha, z^*)$ is a matter of verifying the dynamics of the subsystem of (7), driven by the input $u = [\tilde{q} \quad \tilde{\phi}]^{\top}$, with output $y = \begin{bmatrix} i_{dq} \\ i_{dq} \end{bmatrix}$. Straightforward calculations show that $\tilde{\xi} = A \cdot \tilde{\xi}$, with $\tilde{\xi} = \gamma$ and $A = \begin{bmatrix} -i_{dN} & 0 \\ -i_{dN} -2c & -B \end{bmatrix}$, is a Hurwitz matrix (since $A + A^{\top} < 0$). Hence, local asymptotic stability of $\mathcal{S} = \Pi(m(\alpha, z^*))$ implies local asymptotic stability of $[R_{2N}(\alpha) i_{dq} \\ R_{2N}(\alpha) i_{dq} R_{2m}(\alpha) i_{dq}]^{\top}$. The rotational invariance $m(\alpha, z^*)$ of (7) is thus locally asymptotically stable.

By resorting to the Lyapunov function $V(\tilde{z})$ in (12) to prove asymptotic stability of $\text{span}\{v(z^*)\}$ for any trajectory of (11), the linear subspace $\mathcal{S}$ lies in a neighborhood $\mathcal{U}$, given by sublevel set $\Pi(\Omega_c(z^*))$, which serves as a region of attraction for $\Pi(m(\alpha, z^*))$, and hence also of $m(\alpha, z^*)$ for any trajectory (7) starting in $\mathcal{U} = \mathcal{U}'$.

The rotational symmetry $m(\alpha, z^*)$ invoked in (5), refers to a characteristic property in a multi-source power system:
once all the generators are at steady state and we shift their angles with the same value, the steady state flow is maintained. This is interpreted by the absence of an absolute reference angle in power system [11]. To deal with the rotational symmetry, it is customary to introduce traditional grounded coordinate, by appointing one of the buses as reference and hence changing the original network topology [26], [27]. For this, one performs a coordinate transformation into the so-called grounded model, expressed in relative angle coordinates of the form $\theta = B^{-1} \gamma \in \mathbb{R}^{m-1}$, mapping the rotational invariance into an isolated point. This transformation implies in some cases, the knowledge of the left-inverse of the incidence $B$, necessary for the explicit characterization of isolated equilibria, leading to restricting the network topology e.g., to a tree as in [28].

Theorem IV.2 alleviates this restriction, to encompass more accurate and faithful representations of the network topology and generation dynamics, (than a Swing equation [11]) by integrating the rotational invariance in our stability analysis. For this, we define the asymptotic stability with respect to an equilibrium subspace as in Definition II.1 adopted lately also in [5]. We provide useful tools to assess the network stability in the presence of the rotational invariance based on the study of the linearized system [16], where we provide a sufficient and fully-decentralized condition for asymptotic stability of the subspace invariance $\overline{\text{span}} \{v(z^*)\}$. Thanks to Theorem IV.2, we can draw direct conclusions on the local behavior of trajectories around $m(\alpha, z^*)$ for the power system model. An estimate for the neighborhood $U'$ is obtained from the sublevel sets of the Lyapunov function in [12], provided there is sufficient reactive power and resistive damping (Assumption 4 in [16]).

V. CONCLUSION

We presented asymptotic stability analysis of a nonlinear power system model from a local perspective in terms of the deviation from an equilibrium set constituting the rotational invariance of the steady state manifold. Building upon our stability analysis of the linearized system, where we derived, following a Lyapunov approach sufficient and fully decentralized stability conditions, we prove local asymptotic stability of the rotational invariance in a well-defined neighborhood, resting on the existence of a homeomorphism between linear and nonlinear angle and frequency trajectories in a neighborhood of the consensus subspace. Based on local stability analysis, our future work aims towards conducting extensive simulations of typical small disturbance scenarios and investigating almost global synchronization of the nonlinear power system model.

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