APPLICATION OF THE METHOD OF APPROXIMATION OF ITERATED \( \text{Itô} \) STOCHASTIC INTEGRALS BASED ON GENERALIZED MULTIPLE FOURIER SERIES TO THE HIGH-ORDER STRONG NUMERICAL METHODS FOR NON-COMMUTATIVE SEMILINEAR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

DMITRIY F. KUZNETSOV

ABSTRACT. We consider a method for the approximation of iterated stochastic integrals of arbitrary multiplicity \( k \ (k \in \mathbb{N}) \) with respect to the infinite-dimensional \( Q \)-Wiener process using the mean-square approximation method of iterated \( \text{Itô} \) stochastic integrals with respect to the scalar standard Wiener processes based on generalized multiple Fourier series. The case of multiple Fourier–Legendre series is considered in details. The results of the article can be applied to construction of high-order strong numerical methods (with respect to the temporal discretization) for the approximation of mild solution for non-commutative semilinear stochastic partial differential equations with multiplicative trace class noise.

1. INTRODUCTION

There exists a lot of publications on the subject of numerical integration of stochastic partial differential equations (SPDEs) (see, for example, [1]–[25]). One of the perspective approaches to the construction of high-order strong numerical methods (with respect to the temporal discretization) for SPDEs is based on the Taylor formula in Banach spaces and exponential formula for the mild solution of SPDEs [12] (2015), [13] (2016). As shown in [12] (2015) and [17] (2007) the exponential Milstein type approximation method has the strong order of convergence \( 1.0 - \varepsilon \) (where \( \varepsilon \) is an arbitrary small positive real number) [12] or 1.0 [17]. In [13] the exponential Wagner–Platen type numerical method for SPDEs with strong order \( 1.5 - \varepsilon \) (where \( \varepsilon \) is an arbitrary small positive real number) has been considered. An important feature of these numerical methods is a presence in them of the so-called iterated stochastic integrals with respect to the infinite-dimensional \( Q \)-Wiener process [19].

Approximation of these stochastic integrals is a complex problem. This problem can be significantly simplified if special commutativity conditions be fulfilled [12, 13]. In [25] (2019) two methods of the mean-square approximation of simplest iterated (double) stochastic integrals with respect to the infinite-dimensional \( Q \)-Wiener process are considered and theorems on the convergence of these methods are given (the basic idea about Karhunen–Loeve expansion of the Brownian bridge process was taken from monograph [26] (1988, In Russian)). It is important to note that the approximation of iterated stochastic integrals with respect to the infinite-dimensional \( Q \)-Wiener process can be reduced to the approximation of iterated \( \text{Itô} \) stochastic integrals with respect to the scalar standard Wiener processes. In a lot of author’s publications [27]–[65] the effective methods for the mean-square approximation of iterated \( \text{Itô} \) and Stratonovich stochastic integrals with respect to the scalar standard Wiener processes were proposed and developed. One of these methods [30] (also see [31]–[65]) is based on generalized multiple Fourier series, in particular, on multiple Fourier–Legendre series. The purpose of this article is an adaptation of the method [30]–[65] for the mean-square approximation of iterated
stochastic integrals of multiplicity \( k \) (\( k \in \mathbb{N} \)) with respect to the finite-dimensional approximation of the infinite-dimensional \( \mathcal{Q} \)-Wiener process.

Let \( U, H \) be separable \( \mathbb{R} \)-Hilbert spaces and \( L_{HS}(U, H) \) be a space of Hilbert–Schmidt operators mapping from \( U \) to \( H \). Let \( (\Omega, \mathbb{F}, \mathbb{P}) \) be a probability space with a normal filtration \( \{ \mathbb{F}_t, t \in [0, T] \} \) \cite{19}, let \( \mathbf{W}_t \) be an \( U \)-valued \( \mathbb{Q} \)-Wiener process with respect to \( \{ \mathbf{F}_t, t \in [0, T] \} \), which has a covariance trace class operator \( \mathbf{Q} \in L(U) \). Here \( L(U) \) denotes all bounded linear operators mapping from \( U \) to \( U \). Consider the semilinear parabolic SPDE

\[
(1) \quad dX_t = (AX_t + F(X_t)) \, dt + B(X_t) \, d\mathbf{W}_t, \quad X_0 = \xi, \quad t \in [0, T],
\]

where nonlinear operators \( F, B \) (\( F : H \to H, B : H \to L_{HS}(U_0, H) \)), linear operator \( A : D(A) \subset H \to H \) as well as the initial value \( \xi \) are assumed to satisfy the conditions of existence and uniqueness of the SPDE \cite{1} mild solution \cite{22} (see also \cite{12, 13}). Here \( U_0 \) is an \( \mathbb{R} \)-Hilbert space defined by \( U_0 = \mathbb{Q}^{1/2}(U) \). The scalar product in \( U_0 \) is defined as follows \( \langle u, w \rangle_{U_0} = \langle Q^{-1/2}u, Q^{-1/2}w \rangle_U \) for all \( u, w \in U_0 \).

As it is known, strong numerical methods with high-orders of accuracy (with respect to the temporal discretization) for approximating the mild solution of the SPDE \cite{1}, which are based on the Taylor formula in Banach spaces and an exponential formula for the mild solution of SPDEs, contain iterated stochastic integrals with respect to the \( \mathbb{Q} \)-Wiener process \cite{8, 10, 13, 17}.

Note that the exponential Milstein type numerical scheme \cite{12, 17, 24} and exponential Wagner–Platen type numerical scheme \cite{13} contain, for example, the following iterated stochastic integrals

\[
(2) \quad \int_0^T B(Z) \, d\mathbf{W}_t, \int_0^T B'(Z) \left( \int_0^t B(Z) \, d\mathbf{W}_t \right) \, dt,
\]

\[
(3) \quad \int_0^T F'(Z) \left( \int_0^t B(Z) \, d\mathbf{W}_t \right) \, dt, \int_0^T B'(Z) \left( \int_0^t B'(Z) \left( \int_0^t B(Z) \, d\mathbf{W}_t \right) \, dt \right) \, d\mathbf{W}_t,
\]

\[
(4) \quad \int_0^T B'(Z) \left( \int_0^t F(Z) \, dt \right) \, d\mathbf{W}_t, \int_0^T B''(Z) \left( \int_0^t B(Z) \, d\mathbf{W}_t \right) \left( \int_0^t B(Z) \, d\mathbf{W}_t \right) \, dt,
\]

where \( 0 \leq t < T \leq \bar{T}, Z : \Omega \to H \) is an \( \mathbf{F}_t/B(H) \)-measurable mapping and \( F', B', B'' \) denote Fréchet derivatives. At that, the exponential Milstein type scheme \cite{12} contains integrals \( 2 \) while the exponential Wagner–Platen type scheme \cite{13} contains integrals \( 2 \)–\( 4 \). It is easy to notice that the numerical schemes for SPDEs with higher orders of convergence (with respect to the temporal discretization) in contrast with numerical schemes from \cite{12, 13} will include iterated stochastic integrals (with respect to the \( \mathbb{Q} \)-Wiener process) with multiplicities \( k > 3 \). So, this work is partially devoted to the approximation of iterated stochastic integrals of the form

\[
(5) \quad I[\Phi^{(k)}(Z)]_{T, t} = \int_0^T \Phi_k(Z) \left( \ldots \left( \int_0^t \Phi_2(Z) \left( \int_0^t \Phi_1(Z) \, d\mathbf{W}_t \right) \, dt \right) \ldots \right) \, d\mathbf{W}_t,
\]

where \( Z : \Omega \to H \) is an \( \mathbf{F}_t/B(H) \)-measurable mapping, \( \Phi_k(v)(\ldots(\Phi_2(v)(\Phi_1(v))\ldots)) \) is a \( k \)-linear Hilbert–Schmidt operator mapping from \( U_0 \times \ldots \times U_0 \) to \( H \) for all \( v \in H \), and \( 0 \leq t < T \leq \bar{T} \).
In Sect. 5 we consider the approximation of more general iterated \( \text{Itô} \) stochastic integrals than \( \{5\} \). In Sect. 6, 7 some other types of iterated stochastic integrals of multiplicities 2–4 with respect to the \( Q \)-Wiener process will be considered. In this paper, in all the integrals mentioned above, the infinite-dimensional \( Q \)-Wiener process will be replaced by its finite-dimensional approximation. In \([59]-[61]\) we consider the approximation of iterated stochastic integrals \( \{2\}–\{4\} \) with respect to the \( Q \)-Wiener process. In \([44]\), Chapter 7 one can find a continuation of the studies begun in this work. In \([44]\), \([29]-[61]\) we consider the approximation of iterated stochastic integrals \( \{2\}–\{4\} \) with respect to the \( Q \)-Wiener process.

Note that the second stochastic integral in \( \{4\} \) is not a special case of the stochastic integral \( \{5\} \) for \( k = 3 \). Nevertheless, the expanded representation of the approximation of stochastic integral \( \{4\} \) has a close structure to \( \{9\} \) for \( k = 3 \) (see below). Moreover, the mentioned representation of stochastic integral \( \{4\} \) contains the same iterated \( \text{Itô} \) stochastic integrals of third multiplicity as in \( \{9\} \) for \( k = 3 \) (see Sect. 6). These conclusions mean that the main result of this article (Theorem 4, Sect. 5) for \( k = 3 \) can be reformulated naturally for the stochastic integral \( \{4\} \) (see Sect. 6).

It should be noted that by developing an approach from the work \([13]\), which uses the Taylor formula in Banach spaces and a formula for the mild solution of the SPDE \( \{1\} \), we obviously obtain a number of other iterated stochastic integrals with respect to the \( Q \)-Wiener process. For example, the following stochastic integrals

\[
\int_{t}^{T} B''(Z) \left( \int_{t}^{t_2} B(Z) dW_{t_1} \right) \int_{t}^{t_2} B(Z) dW_{t_2} dt_3,
\]

\[
\int_{t}^{T} B'(Z) \left( \int_{t}^{t_2} B(Z) dW_{t_1} \right) \int_{t}^{t_2} B(Z) dW_{t_2} dt_3,
\]

\[
\int_{t}^{T} B'(Z) \left( \int_{t}^{t_2} B(Z) dW_{t_1} \right) \int_{t}^{t_2} B(Z) dW_{t_2} dt_3,
\]

\[
\int_{t}^{T} F'(Z) \left( \int_{t}^{t_2} B(Z) dW_{t_1} \right) dt_2,
\]

\[
\int_{t}^{T} F'(Z) \left( \int_{t}^{t_2} B(Z) dW_{t_1} \right) dt_2,
\]

\[
\int_{t}^{T} B''(Z) \left( \int_{t}^{t_2} F(Z) dt_1 \right) \int_{t}^{t_2} B(Z) dW_{t_2} dt_3
\]

will be considered in Sect. 7. Here \( Z : \Omega \to H \) is an \( F_t/B(H) \)-measurable mapping and \( B', B'', B''', F', F'' \) are Fréchet derivatives.

Consider eigenvalues \( \lambda_i \) and eigenfunctions \( e_i(x) \) of the covariance operator \( Q \), where \( i = (i_1, \ldots, i_d) \in J, x = (x_1, \ldots, x_d) \), and \( J = \{i : i \in \mathbb{N}^d, \text{ and } \lambda_i > 0\} \).

The series representation of the \( Q \)-Wiener process has the following form \([19]\)

\[
W(t,x) = \sum_{i \in J} e_i(x) \sqrt{\lambda_i} w_i(t), \quad t \in [0,T],
\]

or in the shorter notations
\[ W_t = \sum_{i \in J} e_i \sqrt{\lambda_i} w_t^{(i)}, \quad t \in [0, T], \]

where \( w_t^{(i)}, i \in J \) are independent standard Wiener processes. Note that eigenfunctions \( e_i, i \in J \) form an orthonormal basis of \( U \) [19].

Consider the finite-dimensional approximation of \( W_t \) [19]

\[ W_t^M = \sum_{i \in J_M} e_i \sqrt{\lambda_i} w_t^{(i)}, \quad t \in [0, T], \]

where \( J_M = \{ i : 1 \leq i_1, \ldots, i_d \leq M, \text{ and } \lambda_i > 0 \} \).

Using (6) and the relation [19]

\[ w_t^{(i)} = \frac{1}{\sqrt{\lambda_i}} \langle e_i, W_t \rangle_U, \quad i \in J, \]

we obtain

\[ W_t^M = \sum_{i \in J_M} e_i \langle e_i, W_t \rangle_U, \quad t \in [0, T], \]

where \( \langle \cdot, \cdot \rangle_U \) is a scalar product in \( U \).

Taking into account (7), (8), we note that the approximation \( I_{\Phi(Z)}^{(k)}|_{T,t}^M \) of iterated stochastic integral \( I_{\Phi(Z)}^{(k)}|_{T,t} \) (see (5)) can be rewritten with probability 1 (further w. p. 1) in the following form

\[ I_{\Phi(Z)}^{(k)}|_{T,t}^M = \int_t^T \Phi_k(Z) \left( \ldots \left( \int_t^{t_3} \Phi_2(Z) \left( \int_t^{t_2} \Phi_1(Z) dW_t^M \right) dW_{t_2}^M \right) \ldots \right) dW_{t_k}^M = \]

\[ = \sum_{r_1, \ldots, r_k \in J_M} \Phi_k(Z) \left( \ldots \left( \Phi_2(Z) (\Phi_1(Z) e_{r_1}) e_{r_2} \right) \ldots \right) e_{r_k} \times \]

\[ \times \int_t^T \ldots \int_t^{t_3} \int_t^{t_2} d\langle e_{r_1}, W_{t_1} \rangle_U d\langle e_{r_2}, W_{t_2} \rangle_U \ldots d\langle e_{r_k}, W_{t_k} \rangle_U = \]

\[ = \sum_{r_1, \ldots, r_k \in J_M} \Phi_k(Z) \left( \ldots \left( \Phi_2(Z) (\Phi_1(Z) e_{r_1}) e_{r_2} \right) \ldots \right) e_{r_k} \sqrt{\lambda_{r_1} \lambda_{r_2} \ldots \lambda_{r_k}} \times \]

\[ \times \int_t^T \ldots \int_t^{t_3} \int_t^{t_2} dW_{t_1}^{(r_1)} dW_{t_2}^{(r_2)} \ldots dW_{t_k}^{(r_k)}, \]

where \( 0 \leq t < T \leq \bar{T} \).

**Remark 1.** Obviously, without the loss of generality we can write \( J_M = \{ 1, 2, \ldots, M \} \).

When special conditions of commutativity for SPDEs in the form (1) be fulfilled it is proposed to simulate numerically the stochastic integrals (2)–(4) using the simple formulas [12], [13]. In this
case, the numerical simulation of mentioned stochastic integrals requires the use of increments of the Q-Wiener process only. However, if these commutativity conditions are not fulfilled (which often corresponds to SPDEs in numerous applications), the numerical simulation of stochastic integrals becomes much more difficult. In [25] two methods for the mean-square approximation of simplest iterated (double) stochastic integrals with respect to the Q-Wiener process are proposed. In this article, we consider a substantially more general and effective method for the mean-square approximation of iterated stochastic integrals of multiplicity \( k (k \in \mathbb{N}) \) with respect to the Q-Wiener process. The convergence analysis in the transition from \( J_M \) to \( J \), i.e. from \( \mathbf{W}_t^M \) to \( \mathbf{W} \) is carried out in [44] (Sect.7.4.2), [45] (Sect.7.4.2), [46], [59], [60] for stochastic integrals of multiplicity \( k (k = 1, 2, 3) \) and some new results (Sect. 5–7). The monographs [43] (Chapters 5 and 6) and [44] or [45], [46] (Chapters 1, 2, and 5) (also see [40–42, 47–58]) are devoted to constructing of efficient methods of the mean-square approximation of iterated Itô stochastic integrals with respect to the scalar standard Wiener processes. These results are adapted for iterated Stratonovich stochastic integrals [27]–[58]. Below (Sect. 2–4) we consider a very short review of results from monographs [43] (Chapters 5 and 6) and [44] or [45], [46] (Chapters 1, 2, and 5) and some new results (Sect. 5–7).

2. Method of Approximation of Iterated Itô Stochastic Integrals Based on Generalized Multiple Fourier Series

Consider more general iterated Itô stochastic integrals than in [59].

\[
J[\psi^{(k)}]_{T,t}^{(i_1,\ldots,i_k)}(T,t) = \int_t^T \psi_1(t_1) \psi_2(t_2) \cdots \psi_k(t_k) d\mathbf{W}_1^{(i_1)}(t_1) \cdots d\mathbf{W}_k^{(i_k)}(t_k),
\]

where \( 0 \leq t < T \leq T^* \) and every \( \psi_l(\tau) \) \( (l = 1, \ldots, k) \) is a continuous non-random function on \( [t,T] \); \( \mathbf{W}_i^{(j)} \) \( (i = 1, \ldots, m) \) are independent standard Wiener processes (see Sect. 1) and \( \mathbf{W}_\tau^{(0)} = \tau; \).

Suppose that \( \{\phi_j(x)\}_{j=0}^\infty \) is a complete orthonormal system of functions in \( L_2([t,T]) \). Define the following function on the hypercube \( [t,T]^k \)

\[
K(t_1,\ldots,t_k) = \begin{cases} \psi_1(t_1) \cdots \psi_k(t_k), & t_1 < \ldots < t_k \\ 0, & \text{otherwise} \end{cases} = \prod_{l=1}^k \psi_l(t_l) \prod_{l=1}^{k-1} 1_{(t_l < t_{l+1})},
\]

where \( t_1,\ldots,t_k \in [t,T] \) for \( k \geq 2 \) and \( K(t_1) = \psi_1(t_1) \) for \( t_1 \in [t,T] \). Here \( 1_A \) is the indicator of the set \( A \).

The function \( K(t_1,\ldots,t_k) \) is piecewise continuous on the hypercube \( [t,T]^k \). At this situation it is well known that the generalized multiple Fourier series of \( K(t_1,\ldots,t_k) \in L_2([t,T]^k) \) converges to \( K(t_1,\ldots,t_k) \) in the hypercube \( [t,T]^k \) in the mean-square sense, i.e.

\[
\lim_{p_1,\ldots,p_k \to \infty} \left\| K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} C_{j_k}\ldots j_1 \prod_{l=1}^k \phi_{j_l}(t_l) \right\|_{L_2([t,T]^k)} = 0,
\]
where
\[
C_{j_k \ldots j_1} = \int_{[t,T]^k} K(t_1, \ldots, t_k) \prod_{i=1}^k \phi_{j_i}(t_i) dt_1 \ldots dt_k
\]
is the Fourier coefficient and
\[
\|f\|_{L^2([t,T]^k)} = \left( \int_{[t,T]^k} f^2(t_1, \ldots, t_k) dt_1 \ldots dt_k \right)^{1/2}.
\]

Consider the discretization \( \{\tau_j\}_{j=0}^N \) of \([t,T]\) such that
\[
t = \tau_0 < \ldots < \tau_N = T, \quad \Delta_N = \max_{0 \leq j \leq N-1} \Delta \tau_j \to 0 \text{ if } N \to \infty, \quad \Delta \tau_j = \tau_{j+1} - \tau_j.
\]

**Theorem 1** (30) (2006) (also see \([31-60]\)). Suppose that every \( \psi_l(\tau) \) \((l = 1, \ldots, k)\) is a continuous non-random function on \([t,T]\) and \(\{\phi_j(x)\}_{j=0}^\infty\) is a complete orthonormal system of continuous functions in \(L^2([t,T])\). Then
\[
\begin{align}
J_{\psi^{(k)}}(\{t_1 \ldots t_k\})_{[T,T]} &= \lim_{p_1, \ldots, p_k \to \infty} \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \left( \prod_{i=1}^k \xi_{j_i}(i) \right) - \lim_{N \to \infty} \sum_{(l_1, \ldots, l_k) \in G_k} \phi_{j_1}(\tau_{l_1}) \Delta w_{\tau_{l_1}}^{(i_1)} \ldots \phi_{j_k}(\tau_{l_k}) \Delta w_{\tau_{l_k}}^{(i_k)},
\end{align}
\]
where
\[
G_k = H_k \setminus L_k; \quad H_k = \{(l_1, \ldots, l_k): l_1, \ldots, l_k = 0, 1, \ldots, N - 1\},
\]
\[
L_k = \{(l_1, \ldots, l_k): l_1, \ldots, l_k = 0, 1, \ldots, N - 1; l_g \neq l_r \ (g \neq r); \ g, r = 1, \ldots, k\},
\]
\(\text{l.i.m. is a limit in the mean-square sense, } i_1, \ldots, i_k = 0, 1, \ldots, m,\)
\[
\xi_{j_i}^{(i)} = \int_{t_i}^T \phi_j(s) dw_s^{(i)}
\]
are independent standard Gaussian random variables for various \(i\) or \(j\) \((i \neq 0)\), \(C_{j_k \ldots j_1}\) is the Fourier coefficient \((13)\), \(\Delta w_{\tau_{l_i}}^{(i)} = w_{\tau_{l_i+1}}^{(i)} - w_{\tau_{l_i}}^{(i)} \ (i = 0, 1, \ldots, m)\), \(\{\tau_j\}_{j=0}^N\) is the discretization of \([t,T]\), which satisfies the condition \((14)\).

Note that in \([30-67]\) the version of Theorem 1 for systems of Haar and Rademacher–Walsh functions has been considered. Another modifications and generalizations of Theorem 1 can be found in the monographs \([44-46]\) (also see Theorem 2 below).

It is not difficult to see that for the case of pairwise different numbers \(i_1, \ldots, i_k = 1, \ldots, m\) from Theorem 1 we obtain
\[
J_{\psi^{(k)}}(\{t_1 \ldots t_k\})_{[T,T]} = \lim_{p_1, \ldots, p_k \to \infty} \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \xi_{j_1}^{(i_1)} \ldots \xi_{j_k}^{(i_k)}.
\]
In order to evaluate the significance of Theorem 1 for practical purposes, we will demonstrate its transformed particular cases for \( k = 1, \ldots, 6 \) [39–58] (the cases \( k = 7 \) and \( k > 7 \) can be found in [34, 39, 43–46]).

\[
J[\psi(1)]_{T,t} = \text{l.i.m.} \sum_{j_1=0}^{p_1} C_{j_1} \zeta_{j_1},
\]

(17)

\[
J[\psi(2)]_{T,t} = \text{l.i.m.} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2j_1} \left( \zeta_{j_2} \zeta_{j_1} \right) - \mathbf{1}_{\{i_1=i_2=0\}} \mathbf{1}_{\{j_1=j_2\}},
\]

(18)

\[
J[\psi(3)]_{T,t} = \text{l.i.m.} \sum_{j_3=0}^{p_3} \sum_{j_2=0}^{p_2} \sum_{j_1=0}^{p_1} C_{j_3j_2j_1} \left( \zeta_{j_3j_2} \zeta_{j_1} \right) - \mathbf{1}_{\{i_1=i_3=0\}} \mathbf{1}_{\{j_1=j_3\}} \mathbf{1}_{\{j_2=j_2\}} - \mathbf{1}_{\{i_1=i_2=0\}} \mathbf{1}_{\{j_1=j_2\}} \mathbf{1}_{\{j_2=j_2\}},
\]

(19)

\[
J[\psi(4)]_{T,t} = \text{l.i.m.} \sum_{j_4=0}^{p_4} \sum_{j_3=0}^{p_3} \sum_{j_2=0}^{p_2} \sum_{j_1=0}^{p_1} C_{j_4j_3j_2j_1} \left( \prod_{i=1}^{4} \zeta_{j_i} \right) - \mathbf{1}_{\{i_1=i_2=0\}} \mathbf{1}_{\{j_1=j_2\}} \mathbf{1}_{\{j_3=j_3\}} \mathbf{1}_{\{j_4=j_4\}} - \mathbf{1}_{\{i_1=i_4=0\}} \mathbf{1}_{\{j_1=j_4\}} \mathbf{1}_{\{j_2=j_2\}} \mathbf{1}_{\{j_3=j_3\}} - \mathbf{1}_{\{i_2=i_3=0\}} \mathbf{1}_{\{j_2=j_3\}} \mathbf{1}_{\{j_1=j_1\}} \mathbf{1}_{\{j_4=j_4\}} - \mathbf{1}_{\{i_2=i_4=0\}} \mathbf{1}_{\{j_2=j_4\}} \mathbf{1}_{\{j_1=j_1\}} \mathbf{1}_{\{j_3=j_3\}} + \mathbf{1}_{\{i_1=i_3=0\}} \mathbf{1}_{\{j_1=j_3\}} \mathbf{1}_{\{i_2=i_4=0\}} \mathbf{1}_{\{j_2=j_4\}} \mathbf{1}_{\{j_3=j_3\}} + \mathbf{1}_{\{i_1=i_4=0\}} \mathbf{1}_{\{j_1=j_4\}} \mathbf{1}_{\{i_2=i_3=0\}} \mathbf{1}_{\{j_2=j_3\}} \mathbf{1}_{\{j_3=j_3\}} + \mathbf{1}_{\{i_2=i_3=0\}} \mathbf{1}_{\{j_2=j_3\}} \mathbf{1}_{\{i_1=i_4=0\}} \mathbf{1}_{\{j_1=j_4\}} \mathbf{1}_{\{j_3=j_3\}} - \mathbf{1}_{\{i_3=i_4=0\}} \mathbf{1}_{\{j_3=j_3\}} \mathbf{1}_{\{j_2=j_4\}} \mathbf{1}_{\{j_1=j_1\}} \mathbf{1}_{\{j_4=j_4\}} - \mathbf{1}_{\{i_2=i_4=0\}} \mathbf{1}_{\{j_2=j_4\}} \mathbf{1}_{\{j_1=j_1\}} \mathbf{1}_{\{j_3=j_3\}} \mathbf{1}_{\{j_4=j_4\}},
\]

(20)
\[ J_{\ell_0}^{(i_0)}(\ell_{\ell_1}^{(i_{\ell_1})}) = \lim_{\ell \to \infty} \sum_{\ell_1=0}^{\ell_1} \sum_{\ell_2=0}^{\ell_2} C_{\ell_1 \ell_2} \left( \prod_{i=1}^{n} \zeta_{j_i} \right) - \\
\sum_{i=1}^{n} \left( -1 \right) \left( i_1 \neq i_0 \right) \left( j_1 = j_0 \right) \left( \sum_{i=1}^{n} \left( i_2 \right) \xi_{j_3} \right) + \\
\left( i_3 \neq j_0 \right) \left( \sum_{i=1}^{n} \left( i_4 \right) \xi_{j_5} \right) + \\
\left( i_5 \neq j_0 \right) \left( \sum_{i=1}^{n} \left( i_6 \right) \xi_{j_7} \right) - \\
\left( i_2 \neq i_0 \right) \left( j_2 = j_0 \right) \left( \sum_{i=1}^{n} \left( i_3 \right) \xi_{j_4} \right) + \\
\left( i_4 \neq i_0 \right) \left( j_4 = j_0 \right) \left( \sum_{i=1}^{n} \left( i_5 \right) \xi_{j_6} \right) + \\
\left( i_6 \neq i_0 \right) \left( j_6 = j_0 \right) \left( \sum_{i=1}^{n} \left( i_7 \right) \xi_{j_8} \right) \right) \\
(21) \]
case of an arbitrary complete orthonormal system of functions.

The first part consists of

\[ (24) \]

We will say that (23) is a partition and consider the sum with respect to all possible partitions

Below there are several examples of sums in the form (24):

\[ \sum_{(g_1, g_2) = (1, 2)} a_{g_1 g_2} = a_{12}, \]
orthonormal system of functions in the space \(X\), converging in the mean-square sense is valid, where the notations are the same as in Theorem 2. Therefore, we can formulate the following generalization of Theorem 1.

**Theorem 2** \cite{Sect. 1.11, Sect. 15}. Suppose that \(\{\phi_j(x)\}_{j=0}^{\infty}\) is an arbitrary complete orthonormal system of functions in the space \(L_2([t, T])\) and \(\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t, T])\). Then the following expansion

\[
J[\psi^{(k)}(t_1, \ldots, t_k)]_{T, t} = \lim_{p_1, \ldots, p_k \to \infty} \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \left( \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} + \sum_{r=1}^{\lfloor k/2 \rfloor} (-1)^r \prod_{s=1}^{r} \mathbf{1}_{i_{q_{s-1}} = i_{q_s} \neq 0} \mathbf{1}_{j_{q_{s-1}} = j_{q_s}} \prod_{l=1}^{k-r} \zeta_{j_l}^{(i_l)} \right)
\]

converging in the mean-square sense is valid, where \(\lfloor x \rfloor\) is an integer part of a real number \(x\); another notations are the same as in Theorem 1.

In particular, from (25) for \(k = 5\) we obtain

\[
J[\psi^{(5)}(t_1, \ldots, t_5)]_{T, t} = \sum_{j_1, \ldots, j_5=0}^{\infty} C_{j_5 \ldots j_1} \left( \prod_{l=1}^{5} \zeta_{j_l}^{(i_l)} + \sum_{(i_1, i_2, i_3) \neq 0} \mathbf{1}_{i_1 = i_2 \neq 0} \mathbf{1}_{j_1 = j_2} \prod_{l=1}^{3} \zeta_{j_l}^{(i_l)} \right)
\]
The last equality obviously agrees with (21). Note that the correctness of formulas (17)–(22) can be verified by the fact that if \(i_1 = \ldots = i_6 = i = 1, \ldots, m\) and \(\psi_1(s), \ldots, \psi_6(s) = \psi(s)\), then we can derive from (17)–(22) the well known equalities

\[
J[\psi(1)]_{T,t} = \frac{1}{1!} \delta_{T,t}^{(i)}, \\
J[\psi(2)]_{T,t} = \frac{1}{2!} \left( \left( \delta_{T,t}^{(i)} \right)^2 - \Delta_{T,t} \right), \\
J[\psi(3)]_{T,t} = \frac{1}{3!} \left( \left( \delta_{T,t}^{(i)} \right)^3 - 3 \delta_{T,t}^{(i)} \Delta_{T,t} \right), \\
J[\psi(4)]_{T,t} = \frac{1}{4!} \left( \left( \delta_{T,t}^{(i)} \right)^4 - 6 \left( \delta_{T,t}^{(i)} \right)^2 \Delta_{T,t} + 3 \Delta^2_{T,t} \right), \\
J[\psi(5)]_{T,t} = \frac{1}{5!} \left( \left( \delta_{T,t}^{(i)} \right)^5 - 10 \left( \delta_{T,t}^{(i)} \right)^3 \Delta_{T,t} + 15 \delta_{T,t}^{(i)} \Delta^2_{T,t} \right), \\
J[\psi(6)]_{T,t} = \frac{1}{6!} \left( \left( \delta_{T,t}^{(i)} \right)^6 - 15 \left( \delta_{T,t}^{(i)} \right)^4 \Delta_{T,t} + 45 \left( \delta_{T,t}^{(i)} \right)^2 \Delta^2_{T,t} - 15 \Delta^3_{T,t} \right)
\]

w. p. 1 [31]-[43], where

\[
\delta_{T,t}^{(i)} = \int_t^T \psi(s) d\mathcal{W}_s^{(i)}, \quad \Delta_{T,t} = \int_t^T \psi^2(s) ds.
\]

The above equalities can be independently obtained using the Itô formula and Hermite polynomials [66].

3. Calculation of the Mean-Square Approximation Error of Iterated Itô Stochastic Integrals in Theorems 1, 2

Assume that \(J[\psi^{(k)}(i_1, \ldots, i_k)_{T,t}]_{p_1, \ldots, p_k}\) is an approximation of (10), which is the expression on the right-hand side of (25) before passing to the limit \(l.l.m. \rightarrow \infty\). Let us denote

\[
E^{(i_1, \ldots, i_k)p_1, \ldots, p_k} = \mathbb{M} \left\{ \left( J[\psi^{(k)}(i_1, \ldots, i_k)_{T,t}]_{p_1, \ldots, p_k} - J[\psi^{(k)}(i_1, \ldots, i_k)_{T,t}]_{p_1, \ldots, p_k} \right)^2 \right\},
\]

\[
E^{(i_1, \ldots, i_k)p} = E^{(i_1, \ldots, i_k)p_1, \ldots, p_k} \bigg|_{p_1 = \ldots = p_k = p},
\]

\[
I_k = \|K\|^2_{L_2([t, T]^k)} = \int_{[t, T]^k} K^2(t_1, \ldots, t_k) dt_1 \ldots dt_k.
\]

In [39]-[46], [55]-[57] it was shown that

\[
E^{(i_1, \ldots, i_k)p_1, \ldots, p_k} \leq k! \left( I_k - \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} C_{j_k, \ldots, j_1}^2 \right),
\]
where \( i_1, \ldots, i_k = 1, \ldots, m \) for \( 0 < T - t < \infty \) and \( i_1, \ldots, i_k = 0, 1, \ldots, m \) for \( 0 < T - t < 1 \). Note that the estimate \( \text{(27)} \) is valid under the conditions of Theorem 2.

The exact calculation of \( E^{(i_1 \ldots i_k)p} \) is presented in the following theorem.

**Theorem 3** \([44]\) (Sect. 1.12). Suppose that \( \{ \phi_j(x) \}_{j=0}^{\infty} \) is an arbitrary complete orthonormal system of functions in the space \( L_2([t, T]) \) and \( \psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t, T]), i_1, \ldots, i_k = 1, \ldots, m \). Then

\[
E^{(i_1 \ldots i_k)p} = I_k - \sum_{j_1, \ldots, j_k = 0}^{p} C_{j_k \ldots j_1} M \left\{ \int_{t}^{T} \phi_{j_k}(t_k) \ldots \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)} \right\},
\]

where \( J[\psi^{(k)}]|_{T,t}^{(i_1 \ldots i_k)} \) is the expression on the right-hand side of \( \text{(25)} \) before passing to the limit \( \lim_{p_1, \ldots, p_k \to \infty} \) for \( p_1 = \ldots = p_k = p \); the expression

\[
\sum_{(j_1, \ldots, j_k)}
\]

means the sum with respect to all possible permutations \( (j_1, \ldots, j_k) \). At the same time if \( j_r \) swapped with \( j_q \) in the permutation \( (j_1, \ldots, j_k) \), then \( i_r \) swapped with \( i_q \) in the permutation \( (i_1, \ldots, i_k) \); another notations are the same as in Theorems 1, 2.

Note that

\[
M \left\{ \int_{t}^{T} \phi_{j_k}(t_k) \ldots \phi_{j_1}(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)} \right\} = C_{j_k \ldots j_1}
\]

for \( i_1 \ldots i_k = 1, \ldots, m \).

Then from Theorem 3 for \( i_1, \ldots, i_k = 1, \ldots, m \) we obtain \([40], [42]-[46]\)

\[
(29) \quad E^{(i_1 \ldots i_k)p} = I_k - \sum_{j_1, \ldots, j_k = 0}^{p} C_{j_k \ldots j_1}^2 \quad \text{(pairwise different } i_1, \ldots, i_k),
\]

\[
E^{(i_1 i_2)p} = I_2 - \sum_{j_1, j_2 = 0}^{p} C_{j_2 j_1}^2 - \sum_{j_1, j_2 = 0}^{p} C_{j_2 j_1} C_{j_1, j_2} \quad (i_1 = i_2),
\]

\[
E^{(i_1 i_2 i_3)p} = I_3 - \sum_{j_1, j_2, j_3 = 0}^{p} C_{j_3 j_2 j_1}^2 - \sum_{j_1, j_2, j_3 = 0}^{p} C_{j_3 j_1 j_2} C_{j_3 j_2 j_1} \quad (i_1 = i_2 \neq i_3),
\]

\[
E^{(i_1 i_2 i_3 i_4)p} = I_4 - \sum_{j_1, j_2, j_3, j_4 = 0}^{p} C_{j_4 j_3 j_2 j_1} \left( \sum_{(j_3, j_4)} \left( \sum_{(j_1, j_2)} C_{j_4 j_3 j_2 j_1} \right) \right) \quad (i_1 = i_2 \neq i_3 = i_4),
\]
\[
E^{(i_1 i_2 i_3 i_4 \ldots)} = I_5 - \sum_{j_1, j_2, j_3, j_4, j_5 = 0}^{p} C_{j_5 j_4 j_3 j_2 j_1} \left( \sum_{i_3, i_4} \left( \sum_{j_5, j_4, j_3, j_2} C_{j_5 j_4 j_3 j_2 j_1} \right) \right)
\]
where \( i_1 = i_2 = i_5 \neq i_3 = i_4 \).

4. Some Examples of the Mean-Square Approximations of Iterated Itô Stochastic Integrals Using Legendre Polynomials

Denote
\[
I_{(11111)}^{(i_1 i_2 i_3 i_4 i_5)} = \int_t^T dw_{t_1}^{(i_1)} dw_{t_2}^{(i_2)} dw_{t_3}^{(i_3)} dw_{t_4}^{(i_4)} dw_{t_5}^{(i_5)},
\]
where \( i_1, i_2, i_3, i_4, i_5 = 1, \ldots, m. \)

The complete orthonormal system of Legendre polynomials in the space \( L_2([t, T]) \) looks as follows
\[
\phi_j(x) = \sqrt{\frac{2j + 1}{T - t}} P_j \left( \left( x - \frac{T + t}{2} \right) \frac{2}{T - t} \right); \quad j = 0, 1, 2, \ldots,
\]
where \( P_j(x) \) is the Legendre polynomial.

Using the system of functions (30) and Theorems 1, 2 we obtain the following approximations of iterated Itô stochastic integrals [27-65]

\[
I_{(1)}^{(i_1)} = \sqrt{T - t} \zeta_0^{(i_1)},
\]
\[
I_{(01)}^{(i_1)} = \frac{(T - t)^{3/2}}{2} \left( \zeta_0^{(i_1)} + \frac{1}{\sqrt{3}} \zeta_1^{(i_1)} \right),
\]
\[
I_{(10)}^{(i_1)} = \frac{(T - t)^{3/2}}{2} \left( \zeta_0^{(i_1)} - \frac{1}{\sqrt{3}} \zeta_1^{(i_1)} \right),
\]
\[
I^{(i_1 i_2) q_1}_{(11) T, t} = \frac{T - t}{2} \left( \zeta_0 (i_1) \beta_0 (i_2) + \sum_{i=1}^{q_1} \frac{1}{\sqrt{4i^2 - 1}} \left( \zeta_1 (i_1) \beta_1 (i_2) - \zeta_1 (i_1) \beta_1 (i_2) - \zeta_1 (i_1) \beta_1 (i_2) - \zeta_1 (i_1) \beta_1 (i_2) \right) \right),
\]

\[
I^{(i_1 i_2 i_3) q_3}_{(111) T, t} = \sum_{j_1, j_2, j_3=0}^{q_3} C_{j_1 j_2 j_3} \left( \zeta_1 (j_1) \beta_1 (j_2) \zeta_1 (j_3) \beta_1 (j_3) - \zeta_1 (j_1) \beta_1 (j_2) \zeta_1 (j_3) \beta_1 (j_3) - \zeta_1 (j_1) \beta_1 (j_2) \zeta_1 (j_3) \beta_1 (j_3) - \zeta_1 (j_1) \beta_1 (j_2) \zeta_1 (j_3) \beta_1 (j_3) \right),
\]

\[
-1 \{i_2 = i_3\} \{j_2 = j_3\} \zeta_j (i_1) - 1 \{i_1 = i_3\} \{j_1 = j_3\} \beta_j (i_2)
\]

\[
I^{(i_1 i_2 i_3)}_{(111) T, t} = \frac{1}{6} (T - t)^{3/2} \left( \zeta_0 (i_1)^3 - 3 \zeta_0 (i_1) \right),
\]

\[
I^{(i_1 i_2 i_3 i_4) q_4}_{(1111) T, t} = \sum_{j_1, j_2, j_3, j_4=0}^{q_4} C_{j_1 j_2 j_3 j_4} \left( \zeta_1 (j_1) \beta_1 (j_2) \zeta_1 (j_3) \beta_1 (j_4) \zeta_1 (j_4) \beta_1 (j_4) - \zeta_1 (j_1) \beta_1 (j_2) \zeta_1 (j_3) \beta_1 (j_4) \zeta_1 (j_4) \beta_1 (j_4) - \zeta_1 (j_1) \beta_1 (j_2) \zeta_1 (j_3) \beta_1 (j_4) \zeta_1 (j_4) \beta_1 (j_4) - \zeta_1 (j_1) \beta_1 (j_2) \zeta_1 (j_3) \beta_1 (j_4) \zeta_1 (j_4) \beta_1 (j_4) \right),
\]

\[
-1 \{i_1 = i_3\} \{j_1 = j_3\} \zeta_j (i_2) \beta_j (i_2) - 1 \{i_1 = i_3\} \{j_1 = j_3\} \beta_j (i_2) \zeta_j (i_2) - 1 \{i_1 = i_3\} \{j_1 = j_3\} \beta_j (i_2) \zeta_j (i_2) - 1 \{i_1 = i_3\} \{j_1 = j_3\} \beta_j (i_2) \zeta_j (i_2)
\]

\[
I^{(i_1 i_2 i_3 i_4)}_{(1111) T, t} = \frac{1}{24} (T - t)^2 \left( \zeta_0 (i_1)^4 - 6 \zeta_0 (i_1)^2 + 3 \right),
\]
\begin{equation}
+ \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \mathbf{1}_{\{i_4=i_5\}} \mathbf{1}_{\{j_4=j_5\}} \mathbf{1}_{\{i_4\}} \mathbf{1}_{\{j_4\}} \mathbf{1}_{\{i_3\}} \mathbf{1}_{\{j_3\}} \zeta^{(i_1)} + \\
+ \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \mathbf{1}_{\{i_3\}} \mathbf{1}_{\{j_3\}} \mathbf{1}_{\{i_4\}} \mathbf{1}_{\{j_4\}} \mathbf{1}_{\{i_5\}} \mathbf{1}_{\{j_5\}} \zeta^{(i_1)} + \\
I_{\{(11111)_{T,t}\}}^{(i_1i_2i_3)} = \frac{1}{120} (T-t)^{5/2} \left( \left( \zeta^{(i_1)}_0 \right)^5 - 10 \left( \zeta^{(i_1)}_0 \right)^3 + 15 \zeta^{(i_1)}_0 \right),
\end{equation}

\begin{align}
C_{j_3j_2j_1} &= \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}(T-t)^{3/2}}{8} \tilde{C}_{j_3j_2j_1}, \\
C_{j_4j_3j_2j_1} &= \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)(2j_5+1)}(T-t)^2}{16} \tilde{C}_{j_4j_3j_2j_1}, \\
C_{j_5j_4j_3j_2j_1} &= \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)(2j_5+1)(2j_6+1)}(T-t)^{5/2}}{32} \tilde{C}_{j_5j_4j_3j_2j_1},
\end{align}

\begin{align}
\tilde{C}_{j_3j_2j_1} &= \int_{-1}^{1} P_{j_3}(z) \int_{-1}^{1} P_{j_2}(y) \int_{-1}^{1} P_{j_1}(x) dx dy dz, \\
\tilde{C}_{j_4j_3j_2j_1} &= \int_{-1}^{1} P_{j_4}(u) \int_{-1}^{1} P_{j_3}(z) \int_{-1}^{1} P_{j_2}(y) \int_{-1}^{1} P_{j_1}(x) dx dy dz du, \\
\tilde{C}_{j_5j_4j_3j_2j_1} &= \int_{-1}^{1} P_{j_5}(v) \int_{-1}^{1} P_{j_4}(u) \int_{-1}^{1} P_{j_3}(z) \int_{-1}^{1} P_{j_2}(y) \int_{-1}^{1} P_{j_1}(x) dx dy dz du dv.
\end{align}

Random variables \( \zeta_j^{(i)} \) are defined by (16), and

\begin{align}
I_{\{(111)_{T,t}\}}^{(i_1i_2i_3)} &= \text{l.i.m.} \int_{\{(111)_{T,t}\}}^{(i_1i_2i_3)q} \text{l.i.m.} \int_{\{(111)_{T,t}\}}^{(i_1i_2i_3)q_1} \text{l.i.m.} \int_{\{(1111)_{T,t}\}}^{(i_1i_2i_3i_4)q_2} \text{l.i.m.} \int_{\{(11111)_{T,t}\}}^{(i_1i_2i_3i_4i_5)q_3}.
\end{align}

Note that \( T-t < 1 \) (\( T-t \) is an integration step with respect to the temporal variable). Thus \( q_1 << q \) (see Table 1, 30, 32, 12, 16). Moreover, the values \( \tilde{C}_{j_3j_2j_1}, \tilde{C}_{j_4j_3j_2j_1}, \tilde{C}_{j_5j_4j_3j_2j_1} \) do not depend on \( T-t \). This feature is important because we can use a variable integration step \( T-t \). Coefficients \( \tilde{C}_{j_3j_2j_1}, \tilde{C}_{j_4j_3j_2j_1}, \tilde{C}_{j_5j_4j_3j_2j_1} \) are calculated once and before the start of the numerical scheme. Some examples of the exact calculation of coefficients \( \tilde{C}_{j_3j_2j_1}, \tilde{C}_{j_4j_3j_2j_1}, \tilde{C}_{j_5j_4j_3j_2j_1} \) via Python programming language can be found in Tables 2–4 (the database with 270,000 exactly calculated Fourier–Legendre coefficients was described in [62, 63]).
Table 1. Minimal numbers $q, q_1$ such that $E^{(i_{1}i_{2})q}, E^{(i_{1}i_{2}i_{3})q_1} \leq (T-t)^{4}$, $q_1 \ll q$.

| $T-t$ | 0.08222 | 0.05020 | 0.02310 | 0.01956 |
|-------|---------|---------|---------|---------|
| $q$   | 19      | 51      | 235     | 328     |
| $q_1$ | 1       | 2       | 5       | 6       |

Table 2. Coefficients $\tilde{C}_{3jk}$.

| $j$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  |
|-----|----|----|----|----|----|----|----|
| 0   | -1 | 1  | -1 | 1  | -1 | 1  | -1 |
| 1   | -1 | 1  | -1 | 1  | -1 | 1  | -1 |
| 2   | -1 | 1  | -1 | 1  | -1 | 1  | -1 |
| 3   | -1 | 1  | -1 | 1  | -1 | 1  | -1 |
| 4   | -1 | 1  | -1 | 1  | -1 | 1  | -1 |
| 5   | -1 | 1  | -1 | 1  | -1 | 1  | -1 |
| 6   | -1 | 1  | -1 | 1  | -1 | 1  | -1 |

Table 3. Coefficients $\tilde{C}_{21kl}$.

| $k$ | 0  | 1  | 2  |
|-----|----|----|----|
| 0   | 1  | -1 | 1  |
| 1   | 1  | -1 | 1  |
| 2   | 1  | -1 | 1  |

Table 4. Coefficients $\tilde{C}_{101lr}$.

| $l$ | 0  | 1  |
|-----|----|----|
| 0   | 1  | -1 |
| 1   | 1  | -1 |

$$E^{(i_{1}i_{2})q} = M\left\{ \left( I^{(i_{1}i_{2})}_{(11)T,t} - I^{(i_{1}i_{2})}_{(11)T,t} \right)^{2} \right\},$$

$$E^{(i_{1}i_{2}i_{3})q_1} = M\left\{ \left( I^{(i_{1}i_{2}i_{3})}_{(111)T,t} - I^{(i_{1}i_{2}i_{3})}_{(111)T,t} \right)^{2} \right\},$$

$$E^{(i_{1}i_{2}i_{3}i_{4})q_2} = M\left\{ \left( I^{(i_{1}i_{2}i_{3}i_{4})}_{(1111)T,t} - I^{(i_{1}i_{2}i_{3}i_{4})}_{(1111)T,t} \right)^{2} \right\},$$

$$E^{(i_{1}i_{2}i_{3}i_{4}i_{5})q_3} = M\left\{ \left( I^{(i_{1}i_{2}i_{3}i_{4}i_{5})}_{(11111)T,t} - I^{(i_{1}i_{2}i_{3}i_{4}i_{5})}_{(11111)T,t} \right)^{2} \right\}.$$
Then for pairwise different \(i_1, i_2, i_3, i_4, i_5 = 1, \ldots, m\) from Theorem 3 we obtain \([27]-[65]\)

\[
E^{(i_1i_2)}q = \frac{(T-t)^2}{2} \left( \frac{1}{2} - \sum_{i=1}^{q} \frac{1}{4i^2 - 1} \right),
\]

\[
E^{(i_1i_2i_3)}q_1 = \frac{(T-t)^3}{6} - \sum_{j_1,j_2,j_3=0}^{q_1} C^2_{j_3j_2j_1},
\]

\[
E^{(i_1i_2i_3i_4)}q_2 = \frac{(T-t)^4}{24} - \sum_{j_1,j_2,j_3,j_4=0}^{q_2} C^2_{j_4j_3j_2j_1},
\]

\[
E^{(i_1i_2i_3i_4i_5)}q_3 = \frac{(T-t)^5}{120} - \sum_{j_1,j_2,j_3,j_4,j_5=0}^{q_3} C^2_{j_5j_4j_3j_2j_1}.
\]

On the basis of the presented approximations of iterated Itô stochastic integrals we can see that increasing of multiplicities of these integrals leads to increasing of orders of smallness with respect to \(T - t\) \((T - t \ll 1)\) in the mean-square sense for iterated Itô stochastic integrals. This leads to a sharp decrease of member quantities in the approximations of iterated Itô stochastic integrals, which are required for achieving the acceptable accuracy of approximation \((q_1 \ll q)\).

From \([37]-[39]\) we obtain \([30]-[39], [42]-[46]\)

\[
E^{(i_1i_2i_3i_4i_5)}q_1 \bigg|_{q_1=6} \approx 0.01956000(T-t)^3,
\]

\[
E^{(i_1i_2i_3i_4i_5)}q_2 \bigg|_{q_2=2} \approx 0.02360840(T-t)^4,
\]

\[
E^{(i_1i_2i_3i_4i_5)}q_3 \bigg|_{q_3=1} \approx 0.00759105(T-t)^5.
\]

It is not difficult to see that the accuracy in \([41]\) and \([42]\) is significantly better than in \([40]\) \((T - t \ll 1)\) even for \(q_2 = 2\) and \(q_3 = 1\). This means that in such situation in formulas \([34], [35]\) the number of terms can be chosen significantly less than \(3^4 (q_2 = 2)\) and \(2^5 (q_3 = 1)\). So, in practice, we can leave only few terms in these formulas. For more details see \([62]-[65]\).

5. APPROXIMATION OF ITERATED STOCHASTIC INTEGRALS OF MULTIPlicity \(k\) WITH RESPECT TO THE \(Q\)-WIENER PROCESS

Consider the iterated stochastic integral with respect to the \(Q\)-Wiener process in the form

\[
I[\Phi^{(k)}(Z), \psi^{(k)}]_{T,t} = \int_{t}^{T} \Phi_{k}(Z) \left( \cdots \left( \int_{t}^{t_3} \Phi_{2}(Z) \left( \int_{t}^{t_2} \Phi_{1}(Z) \psi_{1}(t_1) dW_{t_1} \right) \psi_{2}(t_2) dW_{t_2} \right) \cdots \right) \psi_{k}(t_k) dW_{t_k},
\]

\[
I[\Phi^{(k)}(Z), \psi^{(k)}]_{T,t} = \int_{t}^{T} \Phi_{k}(Z) \left( \cdots \left( \int_{t}^{t_3} \Phi_{2}(Z) \left( \int_{t}^{t_2} \Phi_{1}(Z) \psi_{1}(t_1) dW_{t_1} \right) \psi_{2}(t_2) dW_{t_2} \right) \cdots \right) \psi_{k}(t_k) dW_{t_k},
\]

\[
I[\Phi^{(k)}(Z), \psi^{(k)}]_{T,t} = \int_{t}^{T} \Phi_{k}(Z) \left( \cdots \left( \int_{t}^{t_3} \Phi_{2}(Z) \left( \int_{t}^{t_2} \Phi_{1}(Z) \psi_{1}(t_1) dW_{t_1} \right) \psi_{2}(t_2) dW_{t_2} \right) \cdots \right) \psi_{k}(t_k) dW_{t_k},
\]

\[
I[\Phi^{(k)}(Z), \psi^{(k)}]_{T,t} = \int_{t}^{T} \Phi_{k}(Z) \left( \cdots \left( \int_{t}^{t_3} \Phi_{2}(Z) \left( \int_{t}^{t_2} \Phi_{1}(Z) \psi_{1}(t_1) dW_{t_1} \right) \psi_{2}(t_2) dW_{t_2} \right) \cdots \right) \psi_{k}(t_k) dW_{t_k},
\]
where \( Z : \Omega \to H \) is an \( \mathbf{f}/\mathcal{B}(H) \)-measurable mapping, \( \Phi_k(v)(\ldots(\Phi_2(v)(\Phi_1(v))\ldots) \) is a \( k \)-linear Hilbert–Schmidt operator mapping from \( U_0 \times \ldots \times U_0 \) to \( H \) for all \( v \in H \), and \( \psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t, T]) \).

Let \( I[\Phi_k(Z), \psi^{(k)}]_{T,t}^M \) be an approximation of the stochastic integral (13).

\[
I[\Phi_k(Z), \psi^{(k)}]_{T,t}^M =
\]

\[
= \int_t^T \Phi_k(Z) \left( \ldots \left( \int_t^{t_1} \Phi_2(Z) \left( \int_t^{t_2} \Phi_1(Z) \psi_1(t_1) dW_{t_1}^M \right) \psi_2(t_2) dW_{t_2}^M \right) \ldots \right) \psi_k(t_k) dW_{t_k}^M =
\]

\[
= \sum_{r_1, r_2, \ldots, r_k \in J_M} \Phi_k(Z) \left( \ldots (\Phi_2(Z)(\Phi_1(Z)e_{r_1} e_{r_2}) \ldots) e_{r_k} \times
\]

\[
\int_t^T \psi_1(t_1) \int_{t_1}^{t_2} \psi_2(t_2) \int_{t_2}^{t_3} \psi_3(t_3) dW_{t_3}^{(r_1)} dW_{t_2}^{(r_2)} \ldots dW_{t_k}^{(r_k)}
\]

(44)

where \( 0 \leq t < T \leq \bar{T} \), and

\[
J[\psi^{(k)}]_{r_1 \ldots r_k} = \int_t^T \psi(t_1) \ldots \int_t^{t_2} \psi(t_2) \ldots \int_t^{t_k} \psi(t_k) dW_{t_1}^{(r_1)} dW_{t_2}^{(r_2)} \ldots dW_{t_k}^{(r_k)}
\]

is the iterated Itô stochastic integral (13), \( r_1, r_2, \ldots, r_k \in J_M \).

Let \( I[\Phi_k(Z), \psi^{(k)}]_{T,t}^{M,p_1 \ldots p_k} \) be an approximation of the stochastic integral (14).

\[
I[\Phi_k(Z), \psi^{(k)}]_{T,t}^{M,p_1 \ldots p_k} =
\]

\[
= \sum_{r_1, r_2, \ldots, r_k \in J_M} \Phi_k(Z) \left( \ldots (\Phi_2(Z)(\Phi_1(Z)e_{r_1} e_{r_2}) \ldots) e_{r_k} \times
\]

\[
\int_t^T \psi(t_1) \ldots \int_t^{t_2} \psi(t_2) \ldots \int_t^{t_k} \psi(t_k) dW_{t_1}^{(r_1)} dW_{t_2}^{(r_2)} \ldots dW_{t_k}^{(r_k)}
\]

(45)

where \( J[\psi^{(k)}]_{T,t}^{(r_1 \ldots r_k)p_1 \ldots p_k} \) is defined as a prelimit expression on the right-hand side of (25).

\[
J[\psi^{(k)}]_{T,t}^{(r_1 \ldots r_k)p_1 \ldots p_k} = \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \left( \prod_{l=1}^{k} \right)^{[k/2]} + \sum_{m=1}^{[k/2]} (-1)^m \times
\]

\[
\prod_{s=1}^{m} 1_{r_{j_{2s-1}} = r_{j_{2s}} \neq 0} 1_{j_{2s-1} = j_{2s}} \prod_{l=1}^{k-2m} \sum_{s=1}^{m} \left( \prod_{l=1}^{k-2m} \right)
\]

(46)
Let \( U, H \) be separable \( \mathbb{R} \)-Hilbert spaces, \( U_0 = Q^{1/2}(U) \), and \( L(U, H) \) be the space of linear and bounded operators mapping from \( U \) to \( H \). Let \( L(U, H)_0 = \{ T | \psi_0 : T \in L(U, H) \} \) (here \( T | \psi_0 \) is the restriction of operator \( T \) to the space \( U_0 \)). It is known \(^{[7]} \) that \( L(U, H)_0 \) is a dense subset of the space of Hilbert–Schmidt operators \( L_{HS}(U_0, H) \).

**Theorem 4** \(^{[44]-[46], [59], [60], [68]} \). Suppose that \( \{ \phi_j(x) \}_{j=0}^{\infty} \) is an arbitrary complete orthonormal system of functions in the space \( L_2([t, T]) \) and \( \psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t, T]) \). Furthermore, let the following conditions be satisfied:

1. \( Q \in L(U) \) is a nonnegative and symmetric trace class operator (\( \lambda_i \) and \( e_i \) (\( i \in J \)) are its eigenvalues and eigenfunctions (which form an orthonormal basis of \( U \)) correspondingly), and \( W_\tau, \tau \in [0, T] \) is an \( U \)-valued \( Q \)-Wiener process.

2. \( Z : \Omega \to \mathcal{F}/\mathcal{B}(H) \)-measurable mapping.

3. \( \Phi_1 \in L(U, H)_0, \Phi_2 \in L(H, L(U, H)_0) \), and \( \Phi_k(v)(\ldots(\Phi_2(v)(\Phi_1(v)))\ldots) \) is a \( k \)-linear Hilbert–Schmidt operator mapping from \( U_0 \times \ldots \times U_0 \) to \( H \) for all \( v \in H \) such that

\[
\| \Phi_k(Z)(\ldots(\Phi_2(Z)(\Phi_1(Z)e_{r_1})e_{r_2})\ldots)e_{r_k} \|^2_H \leq L_k < \infty
\]

w. p. 1 for all \( r_1, r_2, \ldots, r_k \in J_M, M \in \mathbb{N} \).

Then

\[
M \left\{ \left\| [\Phi^{(k)}(Z), \psi^{(k)}]_{T,t}^{M_{\psi_{r1} \ldots r_k}} - I[\Phi^{(k)}(Z), \psi^{(k)}]_{T,t}^{M_{\psi_{r1} \ldots r_k}} \right\|^2_H \right\} \leq L_k(k!)^2 (\text{tr } Q)\left(I_k - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1}^2\right),
\]

(47)

where \( I_k \) is defined by \(^{[26]} \), \( \text{tr } Q = \sum_{i \in J} \lambda_i \), and

\[
C_{j_k \ldots j_1} = \int_{[t,T]^k} K(t_1, \ldots, t_k) \prod_{l=1}^{k} \phi_{j_l}(t_l) dt_1 \ldots dt_k,
\]

\[
K(t_1, \ldots, t_k) = \begin{cases} 
\psi_1(t_1) \ldots \psi_k(t_k), & t_1 < \ldots < t_k \\
0, & \text{otherwise}
\end{cases}
\]

**Remark 2.** It should be noted that the right-hand side of the inequality \(^{[47]} \) is independent of \( M \) and tends to zero if \( p_1, \ldots, p_k \to \infty \) due to the Parseval equality.

**Proof.** Using \(^{[27]} \), we obtain

\[
M \left\{ \left\| [\Phi^{(k)}(Z), \psi^{(k)}]_{T,t}^{M_{\psi_{r1} \ldots r_k}} - I[\Phi^{(k)}(Z), \psi^{(k)}]_{T,t}^{M_{\psi_{r1} \ldots r_k}} \right\|^2_H \right\} = \]

...
\[
= M \left\{ \left\| \sum_{r_1, r_2, \ldots, r_k \in J_M} \Phi_k (Z) \left( \ldots (\Phi_2 (Z) (\Phi_1 (Z) e_{r_1}) e_{r_2}) \ldots \right) e_{r_k} \sqrt{\lambda_{r_1} \lambda_{r_2} \ldots \lambda_{r_k}} \times \right. \right.
\]
\[
\times \left( J[\psi (k)]_{T, t} (r_1 r_2 \ldots r_k) - J[\psi (k)]_{T, t} (r_1 r_2 \ldots r_k) p_1, \ldots, p_k \right) \right\|^2_H \right\}
\]
\[
= \left| M \left\{ \sum_{r_1, r_2, \ldots, r_k \in J_M} \sum_{(r_1, r_2, \ldots, r_k) = (r_1, r_2, \ldots, r_k)} \phi_k (Z) \left( \ldots (\Phi_2 (Z) (\Phi_1 (Z) e_{r_1}) e_{r_2}) \ldots \right) e_{r_k} \times \right. \right.
\]
\[
\times \left( J[\psi (k)]_{T, t} (r_1 r_2 \ldots r_k) - J[\psi (k)]_{T, t} (r_1 r_2 \ldots r_k) p_1, \ldots, p_k \right) \right\} \left| F_t \right| \right\} \right| \leq \sum_{r_1, r_2, \ldots, r_k \in J_M} \sum_{(r_1, r_2, \ldots, r_k) = (r_1, r_2, \ldots, r_k)} \left| M \left\{ \phi_k (Z) \left( \ldots (\Phi_2 (Z) (\Phi_1 (Z) e_{r_1}) e_{r_2}) \ldots \right) e_{r_k} \right. \right.
\]
\[
\times \left( J[\psi (k)]_{T, t} (r_1 r_2 \ldots r_k) - J[\psi (k)]_{T, t} (r_1 r_2 \ldots r_k) p_1, \ldots, p_k \right) \right\} \left| F_t \right| \right\} \right| \leq \sum_{r_1, r_2, \ldots, r_k \in J_M} \sum_{(r_1, r_2, \ldots, r_k) = (r_1, r_2, \ldots, r_k)} \left| M \left\{ \phi_k (Z) \left( \ldots (\Phi_2 (Z) (\Phi_1 (Z) e_{r_1}) e_{r_2}) \ldots \right) e_{r_k} \right. \right.
\]
\[
\leq L_k \sum_{r_1, r_2, \ldots, r_k \in J_M} \sum_{(r_1, r_2, \ldots, r_k): \{r_1, r_2, \ldots, r_k\} = \{r_1, r_2, \ldots, r_k\}} \sqrt{\lambda_{r_1} \lambda_{r_2} \ldots \lambda_{r_k}} \times \\
\times \mathbb{M} \left\{ \left\| \left( J[\psi(k)]_{T,t}^{(r_1 r_2 \ldots r_k)} - J[\psi(k)]_{T,t}^{(r_1 r_2 \ldots r_k) p_1, \ldots, p_k} \right)^2 \right\|^2 \right\}^{1/2} \times \\
\times \mathbb{M} \left\{ \left( J[\psi(k)]_{T,t}^{(r_1 r_2 \ldots r_k)} - J[\psi(k)]_{T,t}^{(r_1 r_2 \ldots r_k) p_1, \ldots, p_k} \right)^2 \right\}^{1/2} \leq \\
\leq L_k \sum_{r_1, r_2, \ldots, r_k \in J_M} \sum_{(r_1, r_2, \ldots, r_k): \{r_1, r_2, \ldots, r_k\} = \{r_1, r_2, \ldots, r_k\}} \sqrt{\lambda_{r_1} \lambda_{r_2} \ldots \lambda_{r_k}} \times \\
\times \left(k! \left( I_k - \sum_{j_1=0}^{p_1} \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \right) \right)^{1/2} \left(k! \left( I_k - \sum_{j_1=0}^{p_1} \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \right) \right)^{1/2} \leq \\
\leq L_k \sum_{r_1, r_2, \ldots, r_k \in J_M} k! \lambda_{r_1} \lambda_{r_2} \ldots \lambda_{r_k} \left(k! \left( I_k - \sum_{j_1=0}^{p_1} \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \right) \right) = \\
= L_k (kl)^2 \sum_{r_1, r_2, \ldots, r_k \in J_M} \lambda_{r_1} \lambda_{r_2} \ldots \lambda_{r_k} \left( I_k - \sum_{j_1=0}^{p_1} \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \right) \leq 
\]
\[ \leq L_k (k!)^2 (\text{tr } Q)^k \left( I_k - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1}^2 \right), \]

where \( \langle \cdot, \cdot \rangle_H \) is a scalar product in \( H \), and

\[ \sum_{(r_1, r_2, \ldots, r_k): \{r_1, r_2, \ldots, r_k\} = \{r_1, r_2, \ldots, r_k\}} \]

means the sum with respect to all possible permutations \( (r_1, r_2, \ldots, r_k) \) such that

\[ \{r_1, r_2, \ldots, r_k\} = \{r_1, r_2, \ldots, r_k\}. \]

The transition from (48) to (49) is based on the following theorem.

**Theorem 5** [44]-[46], [68]. The following equality is true

\[
\mathbb{M}\left\{ \left( J^{(k)}_{\psi(k)}(r_1, \ldots, r_k) \big|_{T,t} - J^{(k)}_{\psi(k)}(r_1, \ldots, r_k)p_1 \ldots p_k \big|_{T,t} \right) \times \right.
\]

\[
\left. \times \left( J^{(k)}_{\psi(k)}(m_1, \ldots, m_k) - J^{(k)}_{\psi(k)}(m_1, \ldots, m_k)p_1 \ldots p_k \big|_{T,t} \right) \right\} F_t = 0
\]

w. p. 1 for all \( r_1, \ldots, r_k, m_1, \ldots, m_k \in J_M (M \in \mathbb{N}) \) such that \( \{r_1, \ldots, r_k\} \neq \{m_1, \ldots, m_k\} \).

**Proof.** Using the standard moment properties of the Itô stochastic integral, we obtain

\[
\mathbb{M}\left\{ J^{(k)}_{\psi(k)}(r_1, \ldots, r_k) J^{(k)}_{\psi(k)}(m_1, \ldots, m_k) \big|_{T,t} \right\} F_t = 0
\]

w. p. 1 for all \( r_1, \ldots, r_k, m_1, \ldots, m_k \in J_M \) such that \( (r_1, \ldots, r_k) \neq (m_1, \ldots, m_k), M \in \mathbb{N} \).

From the proof of Theorem 1.18 in [44] (Sect. 1.12) it follows that

\[
\prod_{l=1}^{k} c_{j_l}^{(r_l)} + \sum_{m=1}^{[k/2]} (-1)^m \times
\]

\[
\sum_{\{(q_1, q_2, \ldots, 1, 2, \ldots, k) \in \mathbb{N}^k \mid (q_1, q_2, q_{2m-1}, q_{2m}) \neq (q_1, q_2, \ldots, 1, 2, \ldots, k) \}} \prod_{s=1}^{m} 1^{(s, 2s-1, r_{2s-1} \neq 0)} 1^{(s, 2s, j_{2s})} \prod_{l=1}^{k-2m} c_{j_l}^{(r_l)} =
\]
(52) \[ \sum_{(j_1, \ldots, j_k)} \int_t^T \phi_{j_k}(t_k) \cdots \phi_{j_1}(t_1)dw_{r_1}(t_1) \cdots dw_{r_k}(t_k) \] w. p. 1,

where

\[ \sum_{(j_1, \ldots, j_k)} \]

means the sum with respect to all possible permutations \((j_1, \ldots, j_k)\). At the same time if \(j_l\) swapped with \(j_q\) in the permutation \((j_1, \ldots, j_k)\), then \(r_l\) swapped with \(r_q\) in the permutation \((r_1, \ldots, r_k)\); another notations are the same as in (52).

Using (25) and (52), we get

(53) \[ J_{(r_1 \cdots r_k)_{T,t}}^{(p_1 \cdots p_k)} = \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} C_{j_k \cdots j_1} \sum_{(j_1, \ldots, j_k)} \int_t^T \phi_{j_k}(t_k) \cdots \phi_{j_1}(t_1)dw_{r_1}(t_1) \cdots dw_{r_k}(t_k), \]

where notations are the same as in (52).

Then w. p. 1

\[ M \left\{ J_{(\psi(k)_{T,t}}^{(m_1 \cdots m_k)} J_{(\psi(k)_{T,t}}^{(r_1 \cdots r_k)_{p_1 \cdots p_k}} \mid F_t \right\} = \]

\[ = \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} C_{j_k \cdots j_1} \times \]

\[ \times M \left\{ J_{(\psi(k)_{T,t}}^{(m_1 \cdots m_k)} \sum_{(j_1, \ldots, j_k)} \int_t^T \phi_{j_k}(t_k) \cdots \phi_{j_1}(t_1)dw_{r_1}(t_1) \cdots dw_{r_k}(t_k) \mid F_t \right\}. \]

From the standard moment properties of the Itô stochastic integral it follows that

\[ M \left\{ J_{(\psi(k)_{T,t}}^{(m_1 \cdots m_k)} \sum_{(j_1, \ldots, j_k)} \int_t^T \phi_{j_k}(t_k) \cdots \phi_{j_1}(t_1)dw_{r_1}(t_1) \cdots dw_{r_k}(t_k) \mid F_t \right\} = 0 \]

w. p. 1 for all \(r_1, \ldots, r_k, m_1, \ldots, m_k \in J_M\) such that \(\{r_1, \ldots, r_k\} \neq \{m_1, \ldots, m_k\}\), \(M \in \mathbb{N}\).

Then

(54) \[ M \left\{ J_{(\psi(k)_{T,t}}^{(m_1 \cdots m_k)} J_{(\psi(k)_{T,t}}^{(r_1 \cdots r_k)_{p_1 \cdots p_k}} \mid F_t \right\} = 0 \]

w. p. 1 for all \(r_1, \ldots, r_k, m_1, \ldots, m_k \in J_M\) \((M \in \mathbb{N})\) such that \(\{r_1, \ldots, r_k\} \neq \{m_1, \ldots, m_k\}\).

From (53) it follows that
This section is devoted to the approximation of iterated stochastic integrals of the following form (see Sect. 1)

\begin{align*}
M\left\{ J_{\{\psi(k)\}}(r_1,...,r_k)p_1,...,p_k J_{\{\psi(k)\}}(m_1,...,m_k)p_1,...,p_k \right| F_t \right\} = \\
= \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k...j_1} \sum_{q_1=0}^{p_1} \ldots \sum_{q_k=0}^{p_k} C_{q_k...q_1} \times \\
\times M\left\{ \left( \sum_{(j_1,...,j_k)} \int_t^{t_1} \phi_{j_k}(t_k) \ldots \int_t^{t_2} \phi_{j_1}(t_1) dW_{t_1}^{(r_1)} \ldots dW_{t_k}^{(r_k)} \right) \times \\
\times \left( \sum_{(q_1,...,q_k)} \int_t^{t_1} \phi_{q_k}(t_k) \ldots \int_t^{t_2} \phi_{q_1}(t_1) dW_{t_1}^{(m_1)} \ldots dW_{t_k}^{(m_k)} \right) \right| F_t \right\} = 0
\end{align*}

w. p. 1 for all $r_1,...,r_k,m_1,...,m_k \in J_M \ (M \in \mathbb{N})$ such that \{r_1,...,r_k\} \neq \{m_1,...,m_k\}.

From (51), (54), and (55) we obtain (50). Theorem 5 is proved.

**Corollary 1** [44-46, 68]. The following equality is true

\begin{align*}
M\left\{ (J_{\{\psi(k)\}}(r_1,...,r_k)p_1,...,p_k) (J_{\{\psi(l)\}}(m_1,...,m_l) - J_{\{\psi(l)\}}(m_1,...,m_l)q_1,...,q_l) \right| F_t \right\} = 0
\end{align*}

w. p. 1 for all $l = 1,2,...,k-1,$ and $r_1,...,r_k,m_1,...,m_l \in J_M, p_1,...,p_k,q_1,...,q_l = 0,1,2,...$

6. **Approximation of Some Iterated Stochastic Integrals of Second and Third Multiplicity with Respect to the Q-Wiener Process**

This section is devoted to the approximation of iterated stochastic integrals of the following form (see Sect. 1)

\begin{align*}
I_0[B(Z), F(Z)]_{T,t}^M = \int_t^T B'(Z) \left( \int_t^{t_1} F(Z) dt_1 \right) dW_{t_2}^M, \\
I_1[B(Z), F(Z)]_{T,t}^M = \int_t^T F'(Z) \left( \int_t^{t_1} B(Z) dW_{t_1}^M \right) dt_2, \\
I_2[B(Z)]_{T,t}^M = \int_t^T B''(Z) \left( \int_t^{t_1} B(Z) dW_{t_1}^M, \int_t^{t_2} B(Z) dW_{t_2}^M \right) dW_{t_2}^M.
\end{align*}
Let conditions 1 and 2 of Theorem 4 be fulfilled. Let $B''(v)(B(v), B(v))$ be a 3-linear Hilbert–Schmidt operator mapping from $U_0 \times U_0 \times U_0$ to $H$ for all $v \in H$.

Then we have w. p. 1 (see (41))

$$I_0[B(Z), F(Z)]_{T,t}^M = \sum_{r_1 \in J_M} B'(Z)F(Z)e_{r_1} \sqrt{\lambda_{r_1}} f^{(0r_1)}_{(01)T,t},$$

(59)

$$I_1[B(Z), F(Z)]_{T,t}^M = \sum_{r_1 \in J_M} F'(Z)(B(Z)e_{r_1}) \sqrt{\lambda_{r_1}} f^{(r_10)}_{(10)T,t},$$

(60)

$$I_2[B(Z)]_{T,t}^M = \sum_{r_1, r_2, r_3 \in J_M} B''(Z) (B(Z)e_{r_1}, B(Z)e_{r_2}) e_{r_3} \sqrt{\lambda_{r_1} \lambda_{r_2} \lambda_{r_3}} \times$$

(61)

$$\times \int_t^T \left( \int_t^s dw^{(r_1)}_t \int_t^s dw^{(r_2)}_t \right) dw^{(r_3)}_s.$$

Using the Itô formula, we obtain

$$\int_t^s dw^{(r_1)}_t \int_t^s dw^{(r_2)}_t = f^{(r_1r_2)}_{(11)s,t} + f^{(r_2r_1)}_{(11)s,t} + 1_{\{r_1 = r_2\}} (s - t) \text{ w. p. 1.}$$

(62)

From (62) we have

$$\int_t^T \left( \int_t^s dw^{(r_1)}_t \int_t^s dw^{(r_2)}_t \right) dw^{(r_3)}_s = f^{(r_1r_2r_3)}_{(111)s,t} + f^{(r_2r_1r_3)}_{(111)s,t} + 1_{\{r_1 = r_2\}} f^{(0r_3)}_{(01)s,t} \text{ w. p. 1.}$$

(63)

Note that in (59), (60), (62), and (63) we use the notations from Sect. 4.

After substituting (63) into (61), we have

$$I_2[B(Z)]_{T,t}^M = \sum_{r_1, r_2, r_3 \in J_M} B''(Z) (B(Z)e_{r_1}, B(Z)e_{r_2}) e_{r_3} \sqrt{\lambda_{r_1} \lambda_{r_2} \lambda_{r_3}} \times$$

(64)

$$\times \left( f^{(r_1r_2r_3)}_{(111)T,t} + f^{(r_2r_1r_3)}_{(111)T,t} + 1_{\{r_1 = r_2\}} f^{(0r_3)}_{(01)T,t} \right) \text{ w. p. 1.}$$

Taking into account (31) and (32), we put for $q = 1$

$$I^{(0r_3)q}_{(01)T,t} = I^{(0r_3)q}_{(01)T,t} = \frac{(T - t)^{3/2}}{2} \left( \zeta_0^{(r_3)} + \frac{1}{\sqrt{3}} \zeta_1^{(r_3)} \right) \text{ (q = 1) w. p. 1,}$$

(65)

$$I^{(r_2r_3)q}_{(10)T,t} = I^{(r_2r_3)q}_{(10)T,t} = \frac{(T - t)^{3/2}}{2} \left( \zeta_0^{(r_3)} - \frac{1}{\sqrt{3}} \zeta_1^{(r_3)} \right) \text{ (q = 1) w. p. 1,}$$

(66)
where $I_{(01)}^{(r_1)q}, I_{(10)}^{(r_0)q}$ denote the approximations of corresponding iterated Itô stochastic integrals.

Denote by $I_0[B(Z), F(Z)]_{T,t}^{M,q}, I_1[B(Z), F(Z)]_{T,t}^{M,q}, I_2[B(Z)]_{T,t}^{M,q}$ the approximations of iterated stochastic integrals \([59], [60], [64]\)

\begin{equation}
I_0[B(Z), F(Z)]_{T,t}^{M,q} = \sum_{r_1 \in J_M} B'(Z) F(Z) e_{r_1} \sqrt{\lambda_{r_1} I_{(01)}^{(r_0)q}},
\end{equation}
\begin{equation}
I_1[B(Z), F(Z)]_{T,t}^{M,q} = \sum_{r_1 \in J_M} F'(Z) (B(Z)e_{r_1}, B(Z)e_{r_2}) e_{r_3} \sqrt{\lambda_{r_1} \lambda_{r_2} \lambda_{r_3}} \times
\end{equation}
\begin{equation}
\times \left( I_{(11)}^{(r_1r_2r_3)q} + I_{(111)}^{(r_1r_2)q} + 1_{\{r_1=r_2\}} I_{(01)q} \right),
\end{equation}

where $q \geq 1$, and the approximations $I_{(11)}^{(r_1r_2r_3)q}, I_{(111)}^{(r_1r_2)q}$ are defined by \([63]\).

From \([59], [60], [64], [67] - [69]\) it follows that

\begin{equation}
I_0[B(Z), F(Z)]_{T,t}^{M,q} - I_0[B(Z), F(Z)]_{T,t}^{M,q} = 0 \quad \text{w. p. 1},
\end{equation}
\begin{equation}
I_1[B(Z), F(Z)]_{T,t}^{M,q} - I_1[B(Z), F(Z)]_{T,t}^{M,q} = 0 \quad \text{w. p. 1},
\end{equation}
\begin{equation}
I_2[B(Z)]_{T,t}^{M,q} - I_2[B(Z)]_{T,t}^{M,q} = \sum_{r_1, r_2, r_3 \in J_M} B''(Z) (B(Z)e_{r_1}, B(Z)e_{r_2}) e_{r_3} \sqrt{\lambda_{r_1} \lambda_{r_2} \lambda_{r_3}} \times
\end{equation}
\begin{equation}
\times \left( I_{(111)}^{(r_1r_2r_3)q} - I_{(111)}^{(r_1r_2r_3)q} \right) + \left( I_{(111)}^{(r_1r_2)q} - I_{(111)}^{(r_1r_2)q} \right) \quad \text{w. p. 1}.
\end{equation}

Repeating with an insignificant modification the proof of Theorem 4 for the case $k = 3$, we obtain

\[ M \left\{ \left\| I_2[B(Z)]_{T,t}^{M,q} - I_2[B(Z)]_{T,t}^{M,q} \right\| \right\} \leq \]
\[ \leq 4C(3!)^2 \left( \sum_{j_1, j_2, j_3=0}^q C_{j_1j_2j_3}^2 \right), \]

where here and further constant $C$ has the same meaning as constant $L_k$ in Theorem 4 ($k$ is the multiplicity of the iterated stochastic integral), and
Consider the stochastic integral

\[ C_{j_3,j_2,j_1} = \frac{\sqrt{(2j_1 + 1)(2j_2 + 1)(2j_3 + 1)}(T - t)^{3/2}}{8} C_{j_3,j_2,j_1}, \]

\[ \bar{C}_{j_3,j_2,j_1} = \int_{-1}^{1} P_{j_3}(z) \int_{-1}^{z} P_{j_2}(y) \int_{-1}^{y} P_{j_1}(x) dx dy dz, \]

where \( P_j(x) \) is the Legendre polynomial.

7. Approximation of Some Iterated Stochastic Integrals of Third and Fourth Multiplicity with Respect to the \( Q \)-Wiener Process

In this section, we consider the approximation of iterated stochastic integrals of the following form (see Sect. 1)

\[ I_3[B(Z)]_{T,t}^{M} = \int_{t}^{T} B'''(Z) \left( \int_{t}^{t_2} B(Z) dW_{t_2}^{M} \int_{t}^{t_2} B(Z) dW_{t_1}^{M} \int_{t}^{t_2} B(Z) dW_{t_1}^{M} \right) dW_{t_2}^{M}, \]

\[ I_4[B(Z)]_{T,t}^{M} = \int_{t}^{T} B''(Z) \left( \int_{t}^{t_3} B(Z) dW_{t_3}^{M} \int_{t}^{t_3} B(Z) dW_{t_1}^{M} \int_{t}^{t_3} B(Z) dW_{t_1}^{M} \right) dW_{t_2}^{M}, \]

\[ I_5[B(Z)]_{T,t}^{M} = \int_{t}^{T} B''(Z) \left( \int_{t}^{t_3} B(Z) dW_{t_3}^{M} \int_{t}^{t_3} B'(Z) \left( \int_{t}^{t_3} B(Z) dW_{t_1}^{M} \right) dW_{t_2}^{M} \right) dW_{t_3}^{M}, \]

\[ I_6[B(Z), F(Z)]_{T,t}^{M} = \int_{t}^{T} F''(Z) \left( \int_{t}^{t_3} B'(Z) \left( \int_{t}^{t_3} B(Z) dW_{t_1}^{M} \right) dW_{t_2}^{M} \right) dt_3, \]

\[ I_7[B(Z), F(Z)]_{T,t}^{M} = \int_{t}^{T} F''(Z) \left( \int_{t}^{t_2} B(Z) dW_{t_2}^{M} \int_{t}^{t_2} B(Z) dW_{t_1}^{M} \right) dt_2, \]

\[ I_8[B(Z), F(Z)]_{T,t}^{M} = \int_{t}^{T} B''(Z) \left( \int_{t}^{t_2} F(Z) dt_1 \int_{t}^{t_2} B(Z) dW_{t_1}^{M} \right) dW_{t_2}^{M}. \]

Consider the stochastic integral \( I_3[B(Z)]_{T,t}^{M} \). Let conditions 1 and 2 of Theorem 4 be fulfilled. Let \( B'''(v)(B(v), B(v), B(v)) \) be a 4-linear Hilbert–Schmidt operator mapping from \( U_0 \times U_0 \times U_0 \times U_0 \) to \( H \) for all \( v \in H \).

We have (see (13))
From \[43\] (pp. A.438–A.439) (also see \[44\]–\[46\]) or using the Itô formula we obtain

\[
I_{(1)s,t}^{(r_1)} I_{(1)s,t}^{(r_2)} I_{(1)s,t}^{(r_3)} = \sum_{(r_1, r_2, r_3)} I^{(r_1 r_2 r_3)}_{(111)s,t} + (s-t) \left( \sum_{(r_1, r_2, r_3)} I^{(r_2 r_3 r_1)}_{(111)s,t} + I^{(r_3 r_1 r_2)}_{(111)s,t} + I^{(r_1 r_3 r_2)}_{(111)s,t} \right)
\]

(71)

where

\[
\sum_{(r_1, r_2, r_3)}
\]

means the sum with respect to all possible permutations \((r_1, r_2, r_3)\) and we use the notations from Sect. 4.

After substituting (71) into (70), we obtain

\[
I_3[B(Z)]_{T,t}^{M} = \sum_{r_1, r_2, r_3 \in I_3} B''(Z) (B(Z)e_{r_1}, B(Z)e_{r_2}, B(Z)e_{r_3}) e_{r_4} \sqrt{\lambda_{r_1} \lambda_{r_2} \lambda_{r_3} \lambda_{r_4}} \times
\]

(72)

\[
\times \left( \sum_{(r_1, r_2, r_3)} I^{(r_1 r_2 r_3)}_{(111)s,t} - 1_{r_1 = r_2} I^{(r_3 r_4)}_{(01)s,t} - 1_{r_1 = r_3} I^{(r_2 r_4)}_{(01)s,t} - 1_{r_2 = r_3} I^{(r_1 r_4)}_{(01)s,t} \right) \text{ w. p. 1,}
\]

where

\[
I^{(r_1 r_2)}_{(01)s,t} = \int_{t}^{T} (t-s) \int_{t}^{s} dw_{x}^{(r_1)} dw_{x}^{(r_2)}.
\]

Denote by \(I_3[B(Z)]_{T,t}^{M,q}\) the approximation of the iterated stochastic integral (72), which has the following form.
where notations are the same as in Theorems 1, 2.

The approximations \( I^{(r_1 r_2 r_3 r_4)}_{(1111)T,t} \) and \( J^{(r_1 r_2)}_{(01)T,t} \) are based on Theorems 1, 2 and Legendre polynomials.

The approximation \( J^{(r_1 r_2)}_{(01)T,T,t} \) of the stochastic integral \( J^{(r_1 r_2)}_{(01)T,t} \) \((r_1, r_2 = 1, \ldots, M)\), which is based on Theorems 1, 2 and Legendre polynomials has the following form (see [43] (formula (6.91), p. A.544) or [39] (formula (5.19), p. A.252-A.253))

\[
J^{(r_1 r_2)}_{(01)T,T,t} = -\frac{T-t}{2} J^{(r_1 r_2)}_{(01)T,t} - \frac{(T-t)^2}{4} \left( \frac{1}{\sqrt{3}} I_0^{(r_1 r_2)} \right)
\]

\[
+ \sum_{i=0}^{q} \left( \frac{(i+2) \zeta_i^{(r_1)} \zeta_i^{(r_2)} - (i+1) \zeta_i^{(r_1)} \zeta_i^{(r_2)}}{\sqrt{(2i+1)(2i+5)(2i+3)}} \right)
\]

\[
J^{(r_1 r_2)}_{(11)T,t} = \frac{T-t}{2} \left( I_0^{(r_1 r_2)} + \sum_{i=1}^{q} \frac{1}{4i^2-1} \left( \zeta_i^{(r_1)} \zeta_i^{(r_2)} - \zeta_i^{(r_1)} \zeta_i^{(r_2)} \right) - 1 \right),
\]

where notations are the same as in Theorems 1, 2.

Moreover (see [43] (formula (6.106), p. A.551) or [39] (formula (5.19), p. A.252-A.253)),

\[
M \left\{ \left( J^{(r_1 r_2)}_{(01)T,t} - J^{(r_1 r_2)}_{(01)T,t} \right)^2 \right\} = \frac{(T-t)^4}{16} \times
\]

\[
\left( \frac{5}{9} - 2 \sum_{i=2}^{q} \frac{1}{4i^2-1} - \sum_{i=1}^{q} \frac{1}{(2i-1)^2(2i+3)^2} - \sum_{i=0}^{q} \frac{(i+2)^2 + (i+1)^2}{(2i+1)(2i+5)(2i+3)^2} \right) (r_1 \neq r_2).
\]

From (27), (29) we obtain

\[
M \left\{ \left( J^{(r_1 r_2)}_{(01)T,t} - J^{(r_1 r_2)}_{(01)T,t} \right)^2 \right\} \leq
\]

\[
\leq \frac{(T-t)^4}{8} \left( \frac{5}{9} - 2 \sum_{i=2}^{q} \frac{1}{4i^2-1} - \sum_{i=1}^{q} \frac{1}{(2i-1)^2(2i+3)^2} - \sum_{i=0}^{q} \frac{(i+2)^2 + (i+1)^2}{(2i+1)(2i+5)(2i+3)^2} \right).
\]

where \( r_1, r_2 = 1, \ldots, M \).

From (20), (21) it follows that
where \( P \times (80) \)

\[ I_3[B(Z)]_{T,t}^M - I_{5}[B(Z)]_{T,t}^{M,q} = \]

\[ = \sum_{r_1, r_2, r_3, r_4 \in J_M} B''(Z)(B(Z)e_{r_1}, B(Z)e_{r_2}, B(Z)e_{r_3}) e_{r_4} \sqrt{\lambda_{r_1} \lambda_{r_2} \lambda_{r_3} \lambda_{r_4}} \times \]

\[ \times \left( \sum_{(r_1, r_2, r_3)} \left( I_{(1111)T,t}^{(r_1, r_2, r_3)} - I_{(1111)T,t}^{(r_1, r_2, r_3)} \right) - 1_{\{r_1 = r_2\}} \left( J_{(01)T,t}^{(r, r_4)} - J_{(01)T,t}^{(r, r_4)} \right) - \right) \]

\[ - 1_{\{r_1 = r_3\}} \left( J_{(01)T,t}^{(r, r_4)} - J_{(01)T,t}^{(r, r_4)} \right) - 1_{\{r_2 = r_3\}} \left( J_{(01)T,t}^{(r_1 r_4)} - J_{(01)T,t}^{(r_1 r_4)} \right) \]

(78) \[ \text{w. p. 1.} \]

Repeating with an insignificant modification the proof of Theorem 4 for the cases \( k = 2, 4 \), we obtain

\[ M\left\{ \left\| I_3[B(Z)]_{T,t}^M - I_{5}[B(Z)]_{T,t}^{M,q} \right\|_H \right\}^2 \leq \]

\[ \leq C (\text{tr } Q)^{1} \left( 6^2 (4!)^2 \left( \frac{(T - t)^4}{24} - \sum_{j_1, j_2, j_3, j_4 = 0}^q C_{j_4, j_3, j_2, j_1}^2 \right) \right) + 3^2 (2!)^2 E_q, \]

where \( E_q \) is the right-hand side of (77), and

\[ C_{j_4, j_3, j_2, j_1} = \frac{\sqrt{(2j_1 + 1)(2j_2 + 1)(2j_3 + 1)(2j_4 + 1)(T - t)^2}}{16} \bar{C}_{j_4, j_3, j_2, j_1}, \]

\[ \bar{C}_{j_4, j_3, j_2, j_1} = \int_{-1}^1 P_{j_4}(u) \int_{-1}^u P_{j_3}(z) \int_{-1}^z P_{j_2}(y) \int_{-1}^y P_{j_1}(x) dx dy dz du, \]

where \( P_j(x) \) is the Legendre polynomial.

Consider the stochastic integral \( I_4[B(Z)]_{T,t}^M \). Let conditions 1 and 2 of Theorem 4 be fulfilled. Let \( B''(v)(B''(v)(B(v), B(v))) \) be a 4-linear Hilbert–Schmidt operator mapping from \( U_0 \times U_0 \times U_0 \times U_0 \) to \( H \) for all \( v \in H \).

We have (see (14))

\[ I_4[B(Z)]_{T,t}^M = \sum_{r_1, r_2, r_3, r_4 \in J_M} B''(Z)(B''(Z)(B(Z)e_{r_1}, B(Z)e_{r_2}) e_{r_3}) e_{r_4} \sqrt{\lambda_{r_1} \lambda_{r_2} \lambda_{r_3} \lambda_{r_4}} \times \]

\[ \times \int_{t}^{T} \int_{t}^{s} \left( \int_{t}^{\tau} d\mathbf{w}_{u}^{(r_1)} \int_{t}^{\tau} d\mathbf{w}_{u}^{(r_2)} \right) d\mathbf{w}_{s}^{(r_3)} d\mathbf{w}_{s}^{(r_4)} \]

w. p. 1.
From (80) and (81) we obtain

\[
I_4[B(Z)]_{T,t}^M = \sum_{r_1, r_2, r_3, r_4 \in J_M} B'(Z) (B''(Z) (B(Z)e_{r_1}, B(Z)e_{r_2}) e_{r_3}) e_{r_4} \sqrt{\lambda_{r_1} \lambda_{r_2} \lambda_{r_3} \lambda_{r_4}} \times
\]

\[
I_{(1111)T,t}^{(r_1 r_2 r_3 r_4)} + I_{(1111)T,t}^{(r_2 r_3 r_4)} - 1_{\{r_1 = r_2\}} J_{(10)T,t}^{(r_3 r_4)} \] w. p. 1,
\]

where

\[
J_{(10)T,t}^{(r_3 r_4)} = \int_t^T \int_t^s (t - \tau) d\nu_{\tau}^{(r_3)} d\nu_{\tau}^{(r_4)}.
\]

Denote by \( I_4[B(Z)]_{T,t}^{M,q} \) the approximation of the iterated stochastic integral (81), which has the following form

\[
I_4[B(Z)]_{T,t}^{M,q} = \sum_{r_1, r_2, r_3, r_4 \in J_M} B'(Z) (B''(Z) (B(Z)e_{r_1}, B(Z)e_{r_2}) e_{r_3}) e_{r_4} \sqrt{\lambda_{r_1} \lambda_{r_2} \lambda_{r_3} \lambda_{r_4}} \times
\]

\[
I_{(1111)T,t}^{(r_1 r_2 r_3 r_4)} + I_{(1111)T,t}^{(r_2 r_3 r_4)} - 1_{\{r_1 = r_2\}} J_{(10)T,t}^{(r_3 r_4)} \] w. p. 1,
\]

where the approximations \( I_{(1111)T,t}^{(r_1 r_2 r_3 r_4)} \), \( J_{(10)T,t}^{(r_3 r_4)} \) are based on Theorems 1, 2 and Legendre polynomials.

The approximation \( J_{(10)T,t}^{(r_3 r_4)} \) of the stochastic integral \( J_{(10)T,t}^{(r_1 r_2)} \) \((r_1, r_2 = 1, \ldots, M)\), which is based on Theorems 1, 2 and Legendre polynomials has the following form (see [43] (formula (6.92), p. A.544) or [39] (formula (5.8), p. A.249))

\[
J_{(10)T,t}^{(r_3 r_4)} = -\frac{T - t}{2} J_{(1111)T,t}^{(r_1 r_2)} - \frac{(T - t)^2}{4} \left( \frac{1}{\sqrt{3}} \zeta_{(r_2)} \zeta_{(r_1)} + \sum_{i=0}^q \left( \frac{(i + 1) \zeta_{(r_2)} \zeta_{(r_1)}}{\sqrt{(2i + 1)(2i + 3)}} - \frac{(i + 2) \zeta_{(r_2)} \zeta_{(r_1)}}{\sqrt{(2i + 1)(2i + 3)} + \zeta_{(r_2)} \zeta_{(r_1)}} \right) \right),
\]

where the approximation \( I_{(11)T,t}^{(r_1 r_2)} \) is defined by (76).

Moreover,

\[
M \left\{ \left( J_{(10)T,t}^{(r_1 r_2)} - J_{(10)T,t}^{(r_1 r_2)} \right)^2 \right\} = E_q \ (r_1 \neq r_2),
\]
where $E_q$ is the right-hand side of (77) (see [43] (formula (6.106), p. A.551) or [39] (formula (5.19), p. A.252–A.253)).

From (31), (33) it follows that

$$I_4[B(Z)]_{T,t}^M - I_4[B(Z)]_{T,t}^{M,q} =$$

$$= \sum_{r_1,r_2,r_3,r_4 \in J_M} B'(Z) \left( B''(Z) (B(Z)e_{r_1}, B(Z)e_{r_2}) e_{r_3} \sqrt{\lambda_1 \lambda_2 \lambda_3 \lambda_4} \timesight.$$  

$$\times \left( I_{(1111)T,t}^{(r_1r_2r_3r_4q)} - I_{(1111)^T,t}^{(r_1r_2r_3r_4q)} \right) + \left( I_{(1111)T,t}^{(r_2r_1r_3r_4q)} - I_{(1111)^T,t}^{(r_1r_2r_3r_4q)} \right) -$$

$$- 1_{\{r_1=r_2\}} \left( J_{(10)^T,t}^{(r_2r_3r_4q)} - J_{(10)^T,t}^{(r_3r_4q)} \right) \text{ w. p. 1.}$$

Repeating with an insignificant modification the proof of Theorem 4 for the cases $k = 2, 4,$ we obtain

$$\mathcal{M} \left\{ \left\| I_4[B(Z)]_{T,t}^M - I_4[B(Z)]_{T,t}^{M,q} \right\|_H^2 \right\} \leq$$

$$\leq C \left( \text{tr } Q \right)^4 \left( 2^2(4!)^2 \frac{(T-t)^4}{24} - \sum_{j_1,j_2,j_3,j_4=0}^{q} C_{j_1j_2j_3j_4}^2 \right) + (2!)^2 E_q,$$

where $E_q$ is the right-hand side of (77) and $C_{j_1j_2j_3j_4}$ is defined by (79).

Consider the stochastic integral $I_3[B(Z)]_{T,t}^M$. Let conditions 1 and 2 of Theorem 4 be fulfilled. Let $B''(v)(B(v), B'(v)(B(v)))$ be a 4-linear Hilbert–Schmidt operator mapping from $U_0 \times U_0 \times U_0 \times U_0$ to $H$ for all $v \in H$.

We have (see [43])

$$I_3[B(Z)]_{T,t}^M = \sum_{r_1,r_2,r_3,r_4 \in J_M} B''(Z)(B(Z)e_{r_3}, B'(Z)(B(Z)e_{r_2})e_{r_1}) e_{r_4} \sqrt{\lambda_1 \lambda_2 \lambda_3 \lambda_4} \times$$

$$\times \int_t^T \left( \int_t^s d\mathbf{w}^{(r_3)}_{\tau} \int_t^s d\mathbf{w}^{(r_1)}_{\tau} \int_t^s d\mathbf{w}^{(r_2)}_{\tau} d\mathbf{w}^{(r_4)}_{\tau} \right) \text{ w. p. 1.}$$

Using the theorem on the integration order replacement in iterated Itô stochastic integrals (see [43] (p. A.150, p. A.163), [44–46], [67]) or the Itô formula, we obtain

$$\int_t^T \left( \int_t^s d\mathbf{w}^{(r_3)}_{\tau} \int_t^s d\mathbf{w}^{(r_1)}_{\tau} \int_t^s d\mathbf{w}^{(r_2)}_{\tau} d\mathbf{w}^{(r_4)}_{\tau} \right) d\mathbf{w}^{(r_4)}_{\tau} =$$
where we use the notations from Sect. 4, and $J^{(r_1 r_2)}_{(10) , (01) T,t}$ are defined by (87), (88).

After substituting (87) into (86), we obtain

$$I_5 [B(Z)]_{T,t}^{M} = \sum_{r_1,r_2,r_3,r_4 \in J_M} B''(Z)(B(Z)e_{r_3}, B'(Z)(B(Z)e_{r_2})e_{r_4})e_{r_4} \sqrt{\lambda_{r_1} \lambda_{r_2} \lambda_{r_3} \lambda_{r_4}} \times$$

$$\times \left( J^{(r_2 r_3 r_4)}_{(01) T,t} + J^{(r_2 r_3 r_4)}_{(1111) T,t} + J^{(r_2 r_3 r_4)}_{(1111) T,t} + \right. +$$

$$\left. + 1_{\{r_1=r_3\}} \left( J^{(r_2 r_4)}_{(10) T,t} - J^{(r_2 r_4)}_{(01) T,t} \right) - 1_{\{r_2=r_3\}} J^{(r_1 r_4)}_{(10) T,t} \right) \text{ w. p. 1.} \tag{88}$$

Denote by $I_5 [B(Z)]_{T,t}^{M,q}$ the approximation of the iterated stochastic integral (88), which has the following form

$$I_5 [B(Z)]_{T,t}^{M,q} = \sum_{r_1,r_2,r_3,r_4 \in J_M} B''(Z)(B(Z)e_{r_3}, B'(Z)(B(Z)e_{r_2})e_{r_4})e_{r_4} \sqrt{\lambda_{r_1} \lambda_{r_2} \lambda_{r_3} \lambda_{r_4}} \times$$

$$\times \left( J^{(r_2 r_3 r_4)q}_{(1111) T,t} + J^{(r_2 r_3 r_4)q}_{(1111) T,t} + J^{(r_2 r_3 r_4)q}_{(1111) T,t} + \right. +$$

$$\left. + 1_{\{r_1=r_3\}} \left( J^{(r_2 r_4)q}_{(10) T,t} - J^{(r_2 r_4)q}_{(01) T,t} \right) - 1_{\{r_2=r_3\}} J^{(r_1 r_4)q}_{(10) T,t} \right) \text{ w. p. 1.} \tag{89}$$

where the approximations $J^{(r_1 r_2 r_3 r_4)q}_{(1111) T,t}$, $J^{(r_1 r_2)q}_{(01) T,t}$, and $J^{(r_1 r_2)q}_{(10) T,t}$ are based on Theorems 1, 2 and Legendre polynomials.

From (88), (89) it follows that

$$I_5 [B(Z)]_{T,t}^{M} - I_5 [B(Z)]_{T,t}^{M,q} = \sum_{r_1,r_2,r_3,r_4 \in J_M} B''(Z)(B(Z)e_{r_3}, B'(Z)(B(Z)e_{r_2})e_{r_4})e_{r_4} \sqrt{\lambda_{r_1} \lambda_{r_2} \lambda_{r_3} \lambda_{r_4}} \times$$

$$\times \left( I^{(r_2 r_3 r_4)q}_{(1111) T,t} - I^{(r_2 r_3 r_4)q}_{(1111) T,t} - (r_2 r_3 r_4)q \right) +$$

$$+ 1_{\{r_1=r_3\}} \left( J^{(r_2 r_4)q}_{(10) T,t} - J^{(r_2 r_4)q}_{(01) T,t} \right) - 1_{\{r_2=r_3\}} J^{(r_1 r_4)q}_{(10) T,t} \right) \text{ w. p. 1.} \tag{90}$$
Repeating with an insignificant modification the proof of Theorem 4 for the cases \( k = 2, 4 \) and taking into account (85), we obtain

\[
M \left\{ \left\| I_5[B(Z)]_{T,t}^M - I_5[B(Z)]_{T,t}^q \right\|_H^2 \right\} \leq C (\text{tr } Q)^4 \left( \frac{3^2 (4!)^2}{24} - \sum_{j_1, j_2, j_3, j_4 = 0}^q C_{j_1 j_2 j_3 j_4}^2 \right) + 3^2 (2!)^2 E_q \right),
\]

where \( E_q \) is the right-hand side of (77), and \( C_{j_1 j_2 j_3 j_4} \) is defined by (79).

Consider the stochastic integral

\[
I_6[B(Z), F(Z)]_{T,t}^M = \sum_{r_1, r_2 \in J_M} F'(Z)(B'(Z)(B(Z)e_{r_1})e_{r_2}) \sqrt{\lambda_{r_1} \lambda_{r_2}} \times
\]

\[
\times \left( T - t \right) I_{(11)T,t}^{(r_1 r_2)} + J_{(01)T,t}^{(r_1 r_2)} \right) \text{ w. p. } 1.
\]

Using the theorem on the integration order replacement in iterated Itô stochastic integrals (see [43] (p. A.150, p. A.163), [44]–[46], [67]) or the Itô formula, we obtain

\[
\int_t^T \int_t^{s} \int_t^{\tau} d\mathbf{w}_u^{(r_1)} d\mathbf{w}_\tau^{(r_2)} d s = (T - t) I_{(11)T,t}^{(r_1 r_2)} + J_{(01)T,t}^{(r_1 r_2)} \right) \text{ w. p. } 1.
\]

After substituting (91) into (90) we have

\[
I_6[B(Z), F(Z)]_{T,t}^M = \sum_{r_1, r_2 \in J_M} F'(Z)(B'(Z)(B(Z)e_{r_1})e_{r_2}) \sqrt{\lambda_{r_1} \lambda_{r_2}} \times
\]

\[
\times \left( (T - t) I_{(11)T,t}^{(r_1 r_2)} + J_{(01)T,t}^{(r_1 r_2)} \right) \text{ w. p. } 1.
\]

Denote by \( I_6[B(Z), F(Z)]_{T,t}^{M,q} \) the approximation of the iterated stochastic integral (92), which has the following form

\[
I_6[B(Z), F(Z)]_{T,t}^{M,q} = \sum_{r_1, r_2 \in J_M} F'(Z)(B'(Z)(B(Z)e_{r_1})e_{r_2}) \sqrt{\lambda_{r_1} \lambda_{r_2}} \times
\]

\[
\times \left( (T - t) I_{(11)T,t}^{(r_1 r_2)} + J_{(01)T,t}^{(r_1 r_2)} \right),
\]
where the approximations \( J^{(r_1,r_2)q}_{11,T,t}, I^{(r_1r_2)q}_{11,T,t} \) are defined by (75), (76).

From (92), (93) it follows that

\[
\begin{align*}
I_6 [B(Z), F(Z)]_{T,t}^{M} - I_6 [B(Z), F(Z)]_{T,t}^{M,q} &= \sum_{r_1,r_2 \in J_M} F'(Z)(B'(Z)e_{r_1})e_{r_2} \sqrt{\lambda_{r_1} \lambda_{r_2}} \\
&\quad \times \left( (T-t) \left( I^{(r_1,r_2)q}_{11,T,t} - I^{(r_1r_2)q}_{11,T,t} \right) + \left( J^{(r_1,r_2)q}_{(01),T,t} - J^{(r_1r_2)q}_{(01),T,t} \right) \right) \quad \text{w. p. 1}.
\end{align*}
\]

Repeating with an insignificant modification the proof of Theorem 4 for the case \( k = 2 \), we obtain

\[
\begin{align*}
M \left\{ \left\| I_6 [B(Z), F(Z)]_{T,t}^{M} - I_6 [B(Z), F(Z)]_{T,t}^{M,q} \right\|_H^2 \right\} &\leq \leq 2C(2!)^2 (\text{tr } Q)^2 (T-t)^2 G_q + E_q, \\
\end{align*}
\]

where \( G_q \) and \( E_q \) are the right-hand sides of (36) and (77) correspondingly.

Consider the stochastic integral \( I_7 [B(Z), F(Z)]_{T,t}^{M} \). Let conditions 1 and 2 of Theorem 4 be fulfilled. Then we have (see (44)) w. p. 1

\[
\begin{align*}
I_7 [B(Z), F(Z)]_{T,t}^{M} &= \sum_{r_1,r_2 \in J_M} F''(Z) (B(Z)e_{r_1}, B(Z)e_{r_2}) \sqrt{\lambda_{r_1} \lambda_{r_2}} \\
&\quad \times \int_t^T \left( \int_t^s d\mathbf{w}_t^{(r_1)} \int_t^s d\mathbf{w}_t^{(r_2)} \right) ds.
\end{align*}
\]

Using the Itô formula, we obtain

\[
\begin{align*}
\int_t^s d\mathbf{w}_t^{(r_1)} \int_t^s d\mathbf{w}_t^{(r_2)} = I^{(r_1r_2)}_{(11),s,t} + I^{(r_2r_1)}_{(11),s,t} + 1_{\{r_1 = r_2\}} (s-t) \quad \text{w. p. 1},
\end{align*}
\]

where we use the notations from Sect. 4.

From (99) and (91) we have

\[
\begin{align*}
\int_t^T \left( \int_t^s d\mathbf{w}_t^{(r_1)} \int_t^s d\mathbf{w}_t^{(r_2)} \right) ds &= \int_t^T I^{(r_1r_2)}_{(11),s,t} ds + \int_t^T I^{(r_2r_1)}_{(11),s,t} ds + 1_{\{r_1 = r_2\}} \frac{(T-t)^2}{2}.
\end{align*}
\]
\[(T - t) \left( I^{(r_1 r_2)}_{(11)T,t} + I^{(r_2 r_1)}_{(11)T,t} \right) + J^{(r_1 r_2)}_{(01)T,t} + J^{(r_2 r_1)}_{(01)T,t} + 1_{\{r_1 = r_2\}} \frac{(T - t)^2}{2} = \]

\[(T - t) \left( I^{(r_1)}_{(1)T,t} I^{(r_2)}_{(1)T,t} - 1_{\{r_1 = r_2\}} (T - t) \right) + J^{(r_1 r_2)}_{(01)T,t} + J^{(r_2 r_1)}_{(01)T,t} + 1_{\{r_1 = r_2\}} \frac{(T - t)^2}{2} = \]

\[(T - t) I^{(r_1)}_{(1)T,t} I^{(r_2)}_{(1)T,t} + J^{(r_1 r_2)}_{(01)T,t} + J^{(r_2 r_1)}_{(01)T,t} - 1_{\{r_1 = r_2\}} \frac{(T - t)^2}{2} \text{ w. p. 1.} \]

After substituting (96) into (94) we obtain

\[I_T[B(Z), F(Z)]_{T,t}^{M} = \sum_{r_1, r_2 \in J_M} F''(Z) (B(Z)e_{r_1}, B(Z)e_{r_2}) \sqrt{\lambda_{r_1} \lambda_{r_2}} \times \]

\[\times \left( (T - t) I^{(r_1)}_{(1)T,t} I^{(r_2)}_{(1)T,t} + J^{(r_1 r_2)}_{(01)T,t} + J^{(r_2 r_1)}_{(01)T,t} - 1_{\{r_1 = r_2\}} \frac{(T - t)^2}{2} \right) \text{ w. p. 1.} \]

Denote by \(I_T[B(Z), F(Z)]_{T,t}^{M,q}\) the approximation of the iterated stochastic integral (97), which has the following form

\[I_T[B(Z), F(Z)]_{T,t}^{M,q} = \sum_{r_1, r_2 \in J_M} F''(Z) (B(Z)e_{r_1}, B(Z)e_{r_2}) \sqrt{\lambda_{r_1} \lambda_{r_2}} \times \]

\[\times \left( (T - t) I^{(r_1)}_{(1)T,t} I^{(r_2)}_{(1)T,t} + J^{(r_1 r_2)}_{(01)T,t} + J^{(r_2 r_1)}_{(01)T,t} - 1_{\{r_1 = r_2\}} \frac{(T - t)^2}{2} \right) , \]

where the approximation \(J^{(r_1 r_2)q}_{(01)T,t}\) is defined by (75).

From (97), (98) it follows that

\[I_T[B(Z), F(Z)]_{T,t}^{M} - I_T[B(Z), F(Z)]_{T,t}^{M,q} = \sum_{r_1, r_2 \in J_M} F''(Z) (B(Z)e_{r_1}, B(Z)e_{r_2}) \sqrt{\lambda_{r_1} \lambda_{r_2}} \times \]

\[\times \left( J^{(r_1 r_2)}_{(01)T,t} - J^{(r_1 r_2)q}_{(01)T,t} \right) + \left( J^{(r_2 r_1)}_{(01)T,t} - J^{(r_2 r_1)q}_{(01)T,t} \right) \text{ w. p. 1.} \]

Repeating with an insignificant modification the proof of Theorem 4 for the case \(k = 2\), we obtain
APPLICATION OF THE METHOD OF APPROXIMATION OF ITERATED IT\'O STOCHASTIC INTEGRALS

\[ M \left\{ \left\| I_7[B(Z), F(Z)]_{T,t}^M - I_7[B(Z), F(Z)]_{T,t}^M, q \right\|^2_H \right\} \leq \]
\[ \leq 2^2C(2!)^2 (\text{tr } Q)^2 E_q, \]

where \( E_q \) is the right-hand side of (77).

Consider the stochastic integral \( I_8[B(Z), F(Z)]_{T,t}^M \). Let conditions 1 and 2 of Theorem 4 be fulfilled. Then we have (see (44)) w. p. 1

\[ I_8[B(Z), F(Z)]_{T,t}^M = - \sum_{r_1, r_2 \in J_M} B''(Z)(F(Z), B(Z)e_{r_1}) e_{r_2} \sqrt{\lambda_{r_1} \lambda_{r_2}} J_{(01)}^{(r_1, r_2)}_{T,t}. \]

Denote by \( I_8[B(Z), F(Z)]_{T,t}^M, q \) the approximation of the iterated stochastic integral (99), which has the following form

\[ I_8[B(Z), F(Z)]_{T,t}^M, q = - \sum_{r_1, r_2 \in J_M} B''(Z)(F(Z), B(Z)e_{r_1}) e_{r_2} \sqrt{\lambda_{r_1} \lambda_{r_2}} J_{(01)}^{(r_1, r_2)q}_{T,t}, \]

where the approximation \( J_{(01)}^{(r_1, r_2)q} \) is defined by (75).

From (99), (100) it follows that

\[ I_8[B(Z), F(Z)]_{T,t}^M - I_8[B(Z), F(Z)]_{T,t}^M, q = \]
\[ = - \sum_{r_1, r_2 \in J_M} B''(Z)(F(Z), B(Z)e_{r_1}) e_{r_2} \sqrt{\lambda_{r_1} \lambda_{r_2}} \times \]
\[ \times \left( J_{(01)}^{(r_1, r_2)}_{T,t} - J_{(01)}^{(r_1, r_2)q}_{T,t} \right) \text{ w. p. 1.} \]

Repeating with an insignificant modification the proof of Theorem 4 for the case \( k = 2 \), we obtain

\[ M \left\{ \left\| I_8[B(Z), F(Z)]_{T,t}^M - I_8[B(Z), F(Z)]_{T,t}^M, q \right\|^2_H \right\} \leq C(2!)^2 (\text{tr } Q)^2 E_q, \]

where \( E_q \) is the right-hand side of (77).

Using computational experiments it was shown in [64], [65] (also see [44], Sect. 5.4) that we can neglect the multiplier factor \( k! \) in the estimate (27). As a result, the computational costs for the approximation of iterated Itô stochastic integrals are significantly reduced. For the same reason, we can replace the multiplier factor \( (k!)^2 \) by \( k! \) in the formula (47) in practical calculations.
Acknowledgement. I would like to thank Leonid Makarovsky for his help in translation this article into English and Konstantin Rybakov for useful discussion of some presented results.

REFERENCES

[1] Gyöngy I. Lattice approximations for stochastic quasi-linear parabolic partial differential equations driven by space-time white noise. I. Potential Anal. 9, 1 (1998), 1-25.
[2] Gyöngy I. Lattice approximations for stochastic quasi-linear parabolic partial differential equations driven by space-time white noise. II. Potential Anal. 11, 1 (1999), 1-37.
[3] Gyöngy I. and Krylov N. An accelerated splitting-up method for parabolic equations. SIAM J. Math. Anal. 37, 4 (2005), 1070-1097.
[4] Hausenblas E. Numerical analysis of semilinear stochastic evolution equations in Banach spaces. J. Comp. Appl. Math. 147, 2 (2002), 485-516.
[5] Hutzenthaler M. and Jentzen A. Non-globally Lipschitz counterexamples for the stochastic Euler scheme. arXiv:0905.0273 [math.NA] (2009), 22 pp.
[6] Jentzen A. Pathwise numerical approximations of SPDEs with additive noise under non-global Lipschitz coefficients. Potential Anal. 31, 4 (2009), 375-404.
[7] Jentzen A. Taylor expansions of solutions of stochastic partial differential equations. arXiv:0904.2232 [math.NA] (2009), 32 pp.
[8] Jentzen A. and Kloeden P.E. Overcoming the order barrier in the numerical approximation of stochastic partial differential equations with additive space-time noise. Proc. R. Soc. Lond. Ser. Math. Phys. Eng. Sci. 465, 2102 (2009), 649-667.
[9] Jentzen A. and Kloeden P.E. Taylor expansions of solutions of stochastic partial differential equations with additive noise. Ann. Prob. 38, 2 (2010), 532-569.
[10] Jentzen A. and Kloeden P.E. Taylor approximations for stochastic partial differential equations. SIAM, Philadelphia, 2011, 224 pp.
[11] Jentzen A. and Röckner M. A Milstein scheme for SPDEs. Foundations Comp. Math. 15, 2 (2015), 313-362.
[12] Kruse R. Optimal Error Estimates of Galerkin Finite Element Methods for Stochastic Partial Differential Equations with Multiplicative Noise. IMA J. Numer. Anal. 34, 1 (2014), 217-251.
[13] Kruse R. Consistency and stability of a Milstein-Galerkin finite element scheme for semilinear SPDE. Stoch. PDE: Anal. Comp. 2, 4 (2014), 471-516.
[14] Milstein G.N. Numerical integration of stochastic differential equations. Ural Univ. Press, Sverdlovsk, 1988, 225 pp.
Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MATLAB Programs, 6th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2018), A.1-A.1073. Available at: http://diffjournal.spbu.ru/EN/numbers/2018.4/article.2.1.html

Kuznetsov, D.F. Strong approximation of iterated Ito and Stratonovich stochastic integrals based on generalized multiple Fourier series. Application to numerical solution of Ito SDEs and semilinear SPDEs. arXiv:2003.14184 [math.PR], 2020, 859 pp. [In English].

Kuznetsov D.F. Strong approximation of iterated Ito and Stratonovich stochastic integrals based on generalized multiple Fourier series. Application to numerical solution of Ito SDEs and semilinear SPDEs. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2020), A.1-A.606. Available at: http://diffjournal.spbu.ru/EN/numbers/2020.4/article.1.8.html

Kuznetsov, D.F. Mean-Square Approximation of Iterated Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Ito SDEs and Semilinear SPDEs. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2021), A.1-A.798. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.4/article.1.9.html

Kuznetsov D.F. Expansion of iterated Ito stochastic integrals of arbitrary multiplicity based on generalized multiple Fourier series converging in the mean. [In English]. arXiv:1712.09736 [math.PR], 2017, 107 pp.

Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series: multiplicities 1 to 5 and beyond. [In English]. arXiv:1712.09510 [math.PR], 2022, 173 pp.

Kuznetsov D.F. Mean-square approximation of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 6 from the Taylor-Ito and Taylor-Stratonovich expansions using Legendre polynomials. [In English]. arXiv:1801.00231 [math.PR], 2017, 100 pp.

Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity based on generalized iterated Fourier series converging pointwise. [In English]. arXiv:1801.00784 [math.PR], 2018, 74 pp.

Kuznetsov D.F. Strong numerical methods of orders 2.0, 2.5, and 3.0 for Ito stochastic differential equations based on the unified stochastic Taylor expansions and multiple Fourier–Legendre series. [In English]. arXiv:1807.02190 [math.PR], 2018, 39 pp.

Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of fifth multiplicity based on generalized multiple Fourier series. [In English]. arXiv:1802.00643 [math.PR], 2018, 91 pp.

Kuznetsov D.F. Exact calculation of the mean-square error in the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series. [In English]. arXiv:1801.01079 [math.PR], 2018, 57 pp.

Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series. [In English]. Ufa Mathematical Journal, 11, 4 (2019), 49-77. DOI: http://doi.org/10.13108/2019-11-4-49 Available at: http://matem.anrb.ru/en/article?art_id=604

Kuznetsov D.F. On Numerical Modeling of the Multidimensional Dynamic Systems Under Random Perturbations With the 2.5 Order of Strong Convergence. [In English]. Automation and Remote Control, 80, 5 (2019), 867-881. DOI: http://doi.org/10.1134/S0005117919050060

Kuznetsov D.F. A comparative analysis of efficiency of using the Legendre polynomials and trigonometric functions for the numerical solution of Ito stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 59, 8 (2019), 1236-1250. DOI: http://doi.org/10.1134/S0965542519080116

Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Ito stochastic differential equations. [In English]. arXiv:1901.02349 [math.GM], 2019, 34 pp.

Kuznetsov D.F. Explicit one-step numerical method with the strong convergence order of 2.5 for Ito stochastic differential equations with a multi-dimensional nonadditive noise based on the Taylor-Stratonovich expansion. [In English]. Computational Mathematics and Mathematical Physics, 60, 3 (2020), 379-389. DOI: http://doi.org/10.1134/S0965542520030100

Kuznetsov D.F. Application of multiple Fourier-Legendre series to strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear stochastic partial differential equations. Differential Equations and Control Processes, 3 (2020), 129-162. Available at: http://diffjournal.spbu.ru/EN/numbers/2020.3/article.1.6.html

Kuznetsov D.F. Application of multiple Fourier-Legendre series to implementation of strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear stochastic partial differential equations. [In English]. arXiv:1912.02012 [math.PR], 2019, 32 pp.

Kuznetsov D.F., Kuznetsov M.D. Mean-square approximation of iterated stochastic integrals from strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear SPDEs based on multiple Fourier–Legendre series. Recent Developments in Stochastic Methods and Applications. ICSM-5 2020. Springer Proceedings in Mathematics & Statistics, vol 371, Eds. Shiryaev, A.N., Samouylov, K.E., Kozyrev, D.V. Springer, Cham, 2021, pp. 17-32. DOI: http://doi.org/10.1007/978-3-030-83266-7_2

Kuznetsov M.D., Kuznetsov D.F. SDE-MATH: A software package for the implementation of strong high-order numerical methods for Ito SDEs with multidimensional non-commutative noise based on multiple Fourier–Legendre
series. [In English]. Differential Equations and Control Processes, 1 (2021), 93-422. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.1/article.1.5.html

[63] Kuznetsov M.D., Kuznetsov D.F. Implementation of strong numerical methods of orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Ito SDEs with non-commutative noise based on the unified Taylor-Ito and Taylor-Stratonovich expansions and multiple Fourier-Legendre series. arXiv:2009.14011 [math.PR], 2020, 336 pp. [In English].

[64] Kuznetsov M.D., Kuznetsov D.F. Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals of multiplicities 1 to 5 from the unified Taylor-Ito expansion based on multiple Fourier-Legendre series arXiv:2010.13564 [math.PR], 2020, 59 pp. [In English].

[65] Kuznetsov M.D., Kuznetsov D.F. Optimization of the mean-square approximation procedures for iterated Ito stochastic integrals based on multiple Fourier–Legendre series. arXiv:1801.04634 [math.PR], 2018, 27 pp.

[66] Kloeden P.E. and Platen E. Numerical solution of stochastic differential equations. Springer-Verlag, Berlin, 1992, 632 pp.

[67] Kuznetsov D.F. Integration order replacement technique for iterated Ito stochastic integrals and iterated stochastic integrals with respect to martingales. [In English]. arXiv:1801.04634 [math.PR], 2018, 27 pp.

[68] Kuznetsov D.F. Application of the method of approximation of iterated stochastic Ito integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. Differential Equations and Control Processes. 3 (2019), 18-62. Available at: http://diffjournal.spbu.ru/EN/numbers/2019.3/article.1.2.html

Dmitriy Feliksovich Kuznetsov
Peter the Great Saint-Petersburg Polytechnic University, Institute of Applied Mathematics and Mechanics, Department of Mathematics, Polytechnicheskaya ul., 29, 195251, Saint-Petersburg, Russia
Email address: sde_kuznetsov@inbox.ru