CRANK-NICOLSON FINITE ELEMENT APPROXIMATIONS
FOR A LINEAR STOCHASTIC FOURTH ORDER EQUATION
WITH ADDITIVE SPACE-TIME WHITE NOISE

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Abstract. We consider a model initial- and Dirichlet boundary-value problem for a fourth-order linear stochastic parabolic equation, in one space dimension, forced by an additive space-time white noise. First, we approximate its solution by the solution of an auxiliary fourth-order stochastic parabolic problem with additive, finite dimensional, spectral-type stochastic load. Then, fully-discrete approximations of the solution to the approximate problem are constructed by using, for the discretization in space, a standard Galerkin finite element method based on $H^2$-piecewise polynomials, and, for time-stepping, the Crank-Nicolson method. Analyzing the convergence of the proposed discretization approach, we derive strong error estimates which show that the order of strong convergence of the Crank-Nicolson finite element method is equal to that reported in [13] for the Backward Euler finite element method.

1. Introduction

Let $T > 0$, $D = (0,1)$, $(\Omega, \mathcal{F}, P)$ be a complete probability space, and consider a model initial- and Dirichlet boundary-value problem for a fourth-order linear stochastic parabolic equation formulated as follows: seek a stochastic function $v : [0, T] \times \Omega \to \mathbb{R}$ such that

\begin{align*}
v_t + v_{xxxx} &= \dot{W} \quad \text{in } (0, T] \times D, \\
v(t, \cdot)|_{\partial D} &= v_{xx}(t, \cdot)|_{\partial D} = 0 \quad \forall \, t \in (0, T], \\
v(0, x) &= w_0 \quad \forall \, x \in D,
\end{align*}

a.s. in $\Omega$, where $\dot{W}$ denotes a space-time white noise on $[0, T] \times D$ (see, e.g., [24], [11]) and $w_0 : \partial D \to \mathbb{R}$ is a deterministic initial condition. The mild solution of the problem above (cf. [12], [4]) has the form

$v = w + \tilde{u}$

where:

- $w : [0, T] \times \Omega \to \mathbb{R}$ is the solution to the deterministic problem:

\begin{align*}
w_t + w_{xxxx} &= 0 \quad \text{in } (0, T] \times D, \\
w(t, \cdot)|_{\partial D} &= w_{xx}(t, \cdot)|_{\partial D} = 0 \quad \forall \, t \in (0, T], \\
w(0, x) &= w_0(x) \quad \forall \, x \in D,
\end{align*}

which is written as

\begin{equation}
w(t, x) = \int_{\partial D} G(t; x, y) w_0(y) \, dy \quad \forall \, (t, x) \in [0, T] \times \partial D
\end{equation}
with

\begin{equation}
G(t; x, y) = \sum_{k=1}^{\infty} e^{-\lambda_k t} \varepsilon_k(x) \varepsilon_k(y) \quad \forall \ (t, x) \in (0, T] \times \mathbb{D},
\end{equation}

and \( \lambda_k := k \pi \) for \( k \in \mathbb{N} \), and \( \varepsilon_k(z) := \sqrt{2} \sin(\lambda_k z) \) for \( z \in \mathbb{D} \) and \( k \in \mathbb{N} \), and

- \( u : [0, T] \times \mathbb{D} \rightarrow \mathbb{R} \) is a stochastic function (known also as ‘stochastic convolution’) given by

\begin{equation}
\hat{u}(t, x) = \int_0^t \int_D G(t - s; x, y) \, dW(s, y),
\end{equation}

which is, also, the mild solution to the problem \((1.1)\) when the initial condition \( u_0 \) vanishes.

Thus, we can approximate numerically the mild solution \( u \) by approximating separately the functions \( w \) and \( u \). In the work at hand, we focus on the development of a numerical method to approximate the stochastic part \( u \) of the mild solution \( v \) to the problem \((1.1)\). In particular, we will formulate and analyze a numerical method which combines a Crank-Nicolson time-stepping with a finite element method for space discretization.

### 1.1. An approximate problem for \( u \)

To construct computable approximations of \( u \) we formulate an auxiliary approximate stochastic fourth-order parabolic problem with a finite dimensional additive noise inspired by the approach of [1] for the stochastic heat equation with additive space-time white noise (cf. [2, 13, 14, 15]).

Let \( M_* \in \mathbb{N} \) and \( S_{M_*} := \text{span}(\varepsilon_k)_{k=0}^{M_*} \). Also, let \( N_* \in \mathbb{N} \), \( \Delta t := \frac{T}{N_*} \), \( t_n := n \Delta t \) for \( n = 0, \ldots, N_* \) be the nodes of a uniform partition of the interval \([0, T]\) and \( T_n := (t_{n-1}, t_n) \) for \( n = 1, \ldots, N_* \). Then, we consider the fourth-order linear stochastic parabolic problem:

\begin{equation}
\begin{aligned}
\hat{u}_t + \hat{u}_{xxxx} &= \hat{W} \quad \text{in} \ (0, T] \times D, \\
\hat{u}(t, \cdot)|_{\partial D} &= \hat{u}_{xx}(t, \cdot)|_{\partial D} = 0 \quad \forall t \in (0, T], \\
\hat{u}(0, x) &= 0 \quad \forall x \in D,
\end{aligned}
\end{equation}

a.e. in \( \Omega \), where:

\begin{equation}
\hat{W}(\cdot, x)|_{T_n} := \frac{1}{\Delta x} \sum_{i=1}^{M_*} R^n_i \varepsilon_i(x) \quad \forall x \in D
\end{equation}

and

\( R^n_i := \int_{T_n} \int_D \varepsilon_i(x) \, dW(t, x), \quad i = 1, \ldots, M_* \),

for \( n = 1, \ldots, N_* \).

The solution of the problem \((1.6)\), according to the standard theory for parabolic problems (see, e.g., [17]), has the integral representation

\begin{equation}
\hat{u}(t, x) = \int_0^t \int_D G(t - s; x, y) \hat{W}(s, y) \, ds \, dy \quad \forall (t, x) \in [0, T] \times \mathbb{D}.
\end{equation}

**Remark 1.1.** Let \( B^i(t) := \int_0^t \int_D \varepsilon_i(x) \, dW(s, x) \) for \( t \geq 0 \) and \( i \in \mathbb{N} \). According to [24], \( (B^i)_{i \in \mathbb{N}} \) is a family of independent Brownian motions. Thus, the random variables \((R^n_i)_{i \in \mathbb{N}}\) are independent and \( R^n_i \sim N(0, \Delta t) \) for \( i \in \mathbb{N} \) and \( n = 1, \ldots, N_* \).

**Remark 1.2.** The stochastic load \( \hat{W} \) in the right hand side of \((1.6)\) corresponds to a spectral-type representation of the space-time white noise. We can, also, build up a numerical method for \( u \) by using the approximate problem proposed in Section 1.2 of [13], where the stochastic load \( \hat{W} \) is piecewise constant with respect to the time variable but it is a discontinuous piecewise linear function with respect to the space variable.
1.2. Crank-Nicolson fully discrete approximations. Let $M \in \mathbb{N}$, $\Delta \tau := \frac{T}{M}$, $(\tau_m)_{m=0}^{M}$ be the nodes of a uniform partition of $[0, T]$ with width $\Delta \tau$, i.e. $\tau_m := m \Delta \tau$ for $m = 0, \ldots, M$, and corresponding intervals $\Delta_m := (\tau_{m-1}, \tau_m)$ for $m = 1, \ldots, M$. For $p = 2 \text{ or } 3$, let $S^p_h \subset H^2(D) \cap H^1_0(D)$ be a finite element space consisting of functions which are piecewise polynomials of degree at most $p$ over a partition of $D$ in intervals with maximum mesh-length $h$.

The Crank-Nicolson finite element method to approximate the solution $\hat{u}$ to the problem (1.6) is as follows:

Step CN1: Set

(1.9) \[ U^0_h := 0. \]

Step CN2: For $m = 1, \ldots, M$, find $U^m_h \in S^p_h$ such that

(1.10) \[ (U^m_h - U^{m-1}_h, \chi)_{0,D} + \frac{\Delta \tau}{2} B(U^m_h + U^{m-1}_h, \chi) = \int_{\Delta_m} \left( \hat{W}(s, \cdot), \chi \right)_{0,D} ds \quad \forall \chi \in S^p_h, \]

where $(\cdot, \cdot)_{0,D}$ is the usual $L^2(D)$--inner product and $B : H^2(D) \times H^2(D) \to \mathbb{R}$ is a bilinear form given by

\[ B(v_1, v_2) := (\partial_x^2 v_1, \partial_x^2 v_2) \quad \forall v_1, v_2 \in H^2(D). \]

1.3. Motivation, results and references. The recent research activity (see, e.g., [21], [2], [23], [25]) indicates that the Backward Euler finite element method, applied to the stochastic heat equation with additive space-time white noise, has strong order of convergence equal to $\frac{1}{2} - \epsilon$ with respect to the time step $\Delta \tau$ and $\frac{1}{2} - \epsilon$ with respect to maximum length $h$ of the subintervals of the partition used to construct the finite element spaces. Both orders of convergence are optimal since they are consistent to the exponent of the Hölder continuity property of the mild solution to the problem. The lack of smoothness for the mild solution is the reason that the strong order of convergence of a numerical method that combines a high order time stepping with a finite element space discretization, is expected to be equal to the strong order of convergence of the Backward Euler finite element method. However, the convergence analysis in [25] provides a pessimistic strong error estimate for the Crank-Nicolson finite element method of the form $O(\Delta \tau h^{-\frac{3}{2}} + h^2)$, which introduces an uncertainty about the convergence of the method when both $h$ and $\Delta \tau$ freely tend to zero. In addition, a bibliographical quest shows that the Crank-Nicolson method has been analyzed in [9] and [10] under the assumption that the additive space-time noise is smooth in space, while it is not among the time-discretization methods analyzed in [19] (see (3.10) in [19]). This unclear convergence behavior of the Crank-Nicolson method, under the presence of an additive space-time white noise, suggests a direction for further research.

In the work at hand, we consider a different but similar problem, the fourth order stochastic parabolic problem formulated in (1.6), motivated by the fact that its mild solution is one of the components of the mild solution to the nonlinear stochastic Cahn-Hilliard equation (see, e.g., [6], [4]). We approximate the stochastic part $u$ of its mild solution $v$ by the Crank-Nicolson finite element method formulated in Section 1.2 for which we derive strong error estimates. As a first step, we confirm that the solution $\hat{u}$ to the approximate problem (1.6) is really an approximation of $u$ by estimating, in Theorem 3.4, and in terms of $\Delta t$ and $M_*$, the difference $u - \hat{u}$ in the $L^\infty_t (L^2_v(L^2_x))$ norm, arriving at the following modeling error bound:

\[ \max_{t \in [0,T]} \left[ \int_D \left( u(t,x) - \hat{u}(t,x) \right)^2 dx \right] dP \frac{1}{2} \leq C \left( \delta^{-\frac{3}{2}} M_*^{-\frac{3}{2} + \delta} + \Delta t^\delta \right) \quad \forall \delta \in (0, \frac{3}{2}]. \]

Then, for the Crank-Nicolson finite element approximations of $\hat{u}$, we derive (see Theorem 5.4) a discrete in time $L^\infty_t (L^p_v(L^2_x))$ error estimate of the form:

\[ \max_{0 \leq m \leq M} \left[ \int_D \left( U^m_h(x) - \hat{u}(\tau_m,x) \right)^2 dx \right] dP \frac{1}{2} \leq C \left( \epsilon_1^{-\frac{3}{2}} \Delta t^\delta - \epsilon_1 + \epsilon_2^{-\frac{3}{2}} h^\delta - \epsilon_2 \right) \quad \forall \epsilon_1 \in \left(0, \frac{3}{2} \right], \quad \forall \epsilon_2 \in \left(0, \frac{3}{2} \right]. \]

The error estimate above follows by estimating separately the time discretization error in Theorem 4.3 and the space discretization error in Theorem 5.3. The definition of the aforementioned type of errors is made possible by using the Crank-Nicolson time-discrete approximations of $\hat{u}$ introduced in Section 1.2. In particular, the time discretization error is the approximation error of the Crank-Nicolson time-discrete
approximations and the space discretization error is the error between the Crank-Nicolson fully discrete approximations and the Crank-Nicolson time discrete approximations. In both cases, we use the Duhamel principle for the representation of the error along with a low regularity nodal error estimate in a discrete in time $L^2_t(L^2_x)$ norm for modified Crank-Nicolson time discrete approximations of $w$ when the time discretization error is estimated (see Section 4.1) and for modified Crank-Nicolson fully discrete approximations of $w$ when the space discretization error is estimated (see Section 5.1). Roughly speaking, the error analysis for the Crank-Nicolson method differs to that of the Backward Euler method, at the following points:

- the numerical method that one has to analyze for the deterministic problem is a modification of the numerical method applied to the stochastic one
- and the derivation of a low regularity $L^2_t(L^2_x)$ nodal error estimate for the numerical method approximating the solution to the deterministic problem is not a natural outcome of the stability properties of the method.

The main outcome of the present work is that the strong order of convergence of the Crank-Nicolson finite element method is equal to the strong order of convergence of the Backward Euler finite element method, which is due to the low regularity of $u$ (see, e.g., [13, 16]). Adapting properly the convergence analysis developed, we can improve the Crank-Nicolson error estimate in [25], showing that the strong order of convergence of the Crank-Nicolson finite element method applied to the stochastic heat equation with additive space-time white noise is equal to the strong order of convergence of the Backward Euler finite element method obtained in [25] and [23]. Analogous result can be obtained for the linear fourth order problem (1.1) with additive derivative of a space-time white noise (cf. [15]), and the two or three space dimension case of the linear fourth order problem (1.1) (cf. [14]).

We close the section by a brief overview of the paper. Section 2 sets notation, recalls some known results often used in the paper and introduce a usefull projection operator. Section 3 is dedicated to the estimation of the modeling error $u - \tilde{u}$. Section 4 defines the Crank-Nicolson time-discrete approximations of $\tilde{u}$ and analyzes its convergence via the convergence analysis of modified Crank-Nicolson time-discrete approximations of $w$. Finally, Section 5 contains the error analysis for the Crank-Nicolson fully-discrete approximations of $\tilde{u}$.

2. Preliminaries

We denote by $L^2(D)$ the space of the Lebesgue measurable functions which are square integrable on $D$ with respect to Lebesgue’s measure $dx$, provided with the standard norm $\|g\|_{0,D} := (\int_D |g(x)|^2 \, dx)^{1/2}$ for $g \in L^2(D)$. The standard inner product in $L^2(D)$ that produces the norm $\|\cdot\|_{0,D}$ is written as $\langle \cdot, \cdot \rangle_{0,D}$, i.e., $\langle g_1, g_2 \rangle_{0,D} := \int_D g_1(x) \overline{g_2(x)} \, dx$ for $g_1, g_2 \in L^2(D)$. Let $\mathbb{N}_0$ be the set of the non negative integers. Then, for $s \in \mathbb{N}_0$, $H^s(D)$ will be the Sobolev space of functions having generalized derivatives up to order $s$ in the space $L^2(D)$, and by $\|\cdot\|_{s,D}$ its usual norm, i.e. $\|g\|_{s,D} := (\sum_{k=0}^{\infty} \|\partial^k_x g\|^2_{0,D})^{1/2}$ for $g \in H^s(D)$. Also, by $H^s_0(D)$ we denote the subspace of $H^s(D)$ consisting of functions which vanish at the endpoints of $D$ in the sense of trace.

The sequence of pairs $\{(\lambda_k^s, \varepsilon_k)\}_{k=1}^{\infty}$ is a solution to the eigenvalue/eigenfunction problem: find nonzero $\varphi \in H^2(D) \cap H^1_0(D)$ and $\sigma \in \mathbb{R}$ such that $-\varphi'' = \sigma \varphi$ in $D$. Since $\{(\lambda_k^s)_{k=1}^{\infty}\}$ is a complete $\langle \cdot, \cdot \rangle_0$-orthonormal system in $L^2(D)$, we define, for $s \in \mathbb{R}$, a subspace $V^s(D)$ of $L^2(D)$ by

$$V^s(D) := \left\{ g \in L^2(D) : \sum_{k=1}^{\infty} \lambda_k^{2s} (g, \varepsilon_k)^2 < \infty \right\}$$

provided with the norm $\|g\|_{V^s} := (\sum_{k=1}^{\infty} \lambda_k^{2s} (g, \varepsilon_k)^2)^{1/2}$ for $g \in V^s(D)$. For $s \geq 0$, the pair $(V^s(D), \|\cdot\|_{V^s})$ is a complete subspace of $L^2(D)$ and we set $(\hat{H}^s(D), \|\cdot\|_{\hat{H}^s}) := (V^s(D), \|\cdot\|_{V^s})$. For $s < 0$, we define $(\hat{H}^s(D), \|\cdot\|_{\hat{H}^s})$ as the completion of $(V^s(D), \|\cdot\|_{V^s})$, or, equivalently, as the dual of $(\hat{H}^{-s}(D), \|\cdot\|_{\hat{H}^{-s}})$.

Let $m \in \mathbb{N}_0$. It is well-known (see [20]) that

$$\hat{H}^m(D) = \left\{ g \in H^m(D) : \partial_x^i g |_{\partial D} = 0 \text{ if } 0 \leq i < m \right\}$$

$$\hat{H}^m(D) = \left\{ g \in H^m(D) : \partial_x^i g |_{\partial D} = 0 \text{ if } 0 \leq i < m \right\}$$
and there exist constants $C_{m,A}$ and $C_{m,B}$ such that
\begin{equation}
C_{m,A} \|g\|_{m,D} \leq \|g\|_{H^m} \leq C_{m,B} \|g\|_{m,D} \quad \forall g \in \dot{H}^m(D).
\end{equation}
Also, we define on $L^2(D)$ the negative norm $\| \cdot \|_{-m,D}$ by
\begin{equation}
\|g\|_{-m,D} := \sup \left\{ \frac{(g, \varphi)_{0,D}}{\|\varphi\|_{m,D}} : \varphi \in \dot{H}^m(D) \text{ and } \varphi \neq 0 \right\} \quad \forall g \in L^2(D),
\end{equation}
for which, using (2.2), it is easy to conclude that there exists a constant $C_{-m} > 0$ such that
\begin{equation}
\|g\|_{-m,D} \leq C_{-m} \|g\|_{\dot{H}^{-m}} \quad \forall g \in L^2(D).
\end{equation}
Let $\mathbb{L}_2 = (L^2(D), (\cdot, \cdot)_{0,D})$ and $\mathcal{L}(\mathbb{L}_2)$ be the space of linear, bounded operators from $L_2$ to $\mathbb{L}_2$. We say that, an operator $\Gamma \in \mathcal{L}(\mathbb{L}_2)$ is Hilbert-Schmidt, when $\|\Gamma\|_{HS} := \left( \sum_{k=1}^{\infty} \|\Gamma \varepsilon_k\|_{2, D}^2 \right)^{\frac{1}{2}} < +\infty$, where $\|\cdot\|_{HS}$ is the so called Hilbert-Schmidt norm of $\Gamma$. We note that the quantity $\|\Gamma\|_{HS}$ does not change when we replace $(\varepsilon_k)^{\infty}_{k=1}$ by another complete orthonormal system of $L_2$. It is well known (see, e.g., [4]) that an operator $\Gamma \in \mathcal{L}(\mathbb{L}_2)$ is Hilbert-Schmidt if there exists a measurable function $\gamma_\ast : D \times D \rightarrow \mathbb{R}$ such that $\Gamma[v](\cdot) = \int_D \gamma_\ast(\cdot, y) v(y) \, dy$ for $v \in L^2(D)$, and then, it holds that
\begin{equation}
\|\Gamma\|_{HS} = \left( \int_D \int_D \gamma_\ast^2(x, y) \, dx \, dy \right)^{\frac{1}{2}}.
\end{equation}
Let $\mathcal{L}_{HS}(\mathbb{L}_2)$ be the set of Hilbert Schmidt operators of $\mathcal{L}(\mathbb{L}^2)$ and $\Phi : [0, T] \rightarrow \mathcal{L}_{HS}(\mathbb{L}_2)$. Also, for a random variable $X$, let $\mathbb{E}[X]$ be its expected value, i.e., $\mathbb{E}[X] := \int_X X \, dP$. Then, the Itô isometry property for stochastic integrals, which we will use often in the paper, reads
\begin{equation}
\mathbb{E} \left[ \left\| \int_0^T \Phi(t) \, dW(t) \right\|^2_{0,D} \right] = \int_0^T \|\Phi(t)\|^2_{HS} \, dt.
\end{equation}
We recall that: if $c_\ast > 0$, then
\begin{equation}
\sum_{k=1}^{\infty} A_k^{-1(1+c_\ast \delta)} \leq \left( \frac{1+2c_\ast \delta}{2c_\ast \delta} \right)^{\frac{1}{\delta}} \quad \forall \delta \in (0, 2],
\end{equation}
and if $(\mathcal{H}, (\cdot, \cdot)_\mathcal{H})$ is a real inner product space, then
\begin{equation}
(g - v, g)_\mathcal{H} \geq \frac{1}{2} \left[ (g, g)_\mathcal{H} - (v, v)_\mathcal{H} \right] \quad \forall g, v \in \mathcal{H}.
\end{equation}
For a nonempty set $A \subset [0, T]$, we will denote by $X_A : [0, T] \rightarrow \{0, 1\}$ the indicator function of $A$. Also, for any $L \in \mathbb{N}$ and functions $(v^L)_0^L \subset L^2(D)$ we define $v^L := \frac{1}{L} (v^L + v^{L-1})$ for $\ell = 1, \ldots, L$. Finally, for $\alpha \in [0, 1]$ and for $n = 0, \ldots, M - 1$, we define $\tau_{n+\alpha} := \tau_n + \alpha \Delta \tau$.

### 2.1. A projection operator
Let $\mathcal{D} := (0, T) \times D$, $\mathcal{L}$ be a finite dimensional subspace of $L^2(\mathcal{D})$ defined by
\begin{equation}
\mathcal{L} := \left\{ \psi \in L^2(\mathcal{D}) : \exists (\alpha^0)^{N_*}_{n=1} \subset \mathbb{R}^{d_t} \text{ s.t. } \psi(t, x) = \sum_{i=1}^{N_*} \alpha_i^0 \varepsilon_i(x) \quad \forall (t, x) \in T_n \times D, \quad n = 1, \ldots, N_* \right\}
\end{equation}
and $\Pi : L^2(\mathcal{D}) \rightarrow \mathcal{L}$ be the $L^2(\mathcal{D})$—projection operator onto $\mathcal{L}$ which is defined by requiring
\begin{equation}
\int_0^T \int_D \Pi(g; t, x) \varphi(t, x) \, dt \, dx = \int_0^T \int_D g(t, x) \varphi(t, x) \, dt \, dx \quad \forall \varphi \in \mathcal{H}, \quad \forall g \in L^2(\mathcal{D}).
\end{equation}
Then, we have
\begin{equation}
\int_0^T \int_D (\Pi(g; t, x))^2 \, dt \, dx \leq \int_0^T \int_D (g(t, x))^2 \, dt \, dx \quad \forall g \in L^2(\mathcal{D})
\end{equation}
and, after using a typical set of basis function for $\mathcal{L}$, we, easily, conclude that
\begin{equation}
\Pi(g; t, x) = \sum_{i=1}^{N_*} \left( \int_{T_n} (g(s, \cdot), \varepsilon_i)_{0,D} \, ds \right) \varepsilon_i(x) \quad \forall (t, x) \in T_n \times D, \quad n = 1, \ldots, N_*, \quad \forall g \in L^2(\mathcal{D}).
\end{equation}
In the lemma below, we show a representation of the stochastic integral of the projection $\Pi$ of a deterministic function $g \in L^2(\Omega)$ as an $L^2(\Omega)$—inner product of $g$ with the random function $\hat{W}$ defined in Section 1.1.

Lemma 2.1. Let $\hat{W}$ be the random function defined in (1.7). Then, it holds that

$$\int_0^T \int_D \Pi(g; s, y) \, dW(s, y) = \int_0^T \int_D \hat{W}(\tau, x) g(\tau, x) \, d\tau \, dx \quad \forall g \in L^2(\Omega).$$

Proof. Using (2.11) and (1.7), we have

$$\int_0^T \int_D \Pi(g; s, y) \, dW(s, y) = \frac{1}{\Delta T} \sum_{n=1}^{N_s} \int_{T_n} \left[ \sum_{i=1}^{M_s} \left( \int_{T_n} g(\tau, \cdot), \varepsilon_i \right) \, d\tau \right] \varepsilon_i(y) \, dW(s, y)$$

$$= \frac{1}{\Delta T} \sum_{n=1}^{N_s} \sum_{i=1}^{M_s} \left( \int_{T_n} g(\tau, \cdot) \varepsilon_i \, d\tau \right) R^n_i$$

$$= \frac{1}{\Delta T} \sum_{n=1}^{N_s} \sum_{i=1}^{M_s} \left( \int_{T_n} \int_D g(\tau, x) R^n_i \varepsilon_i(x) \, dx \, d\tau \right)$$

$$= \frac{1}{\Delta T} \sum_{n=1}^{N_s} \left( \int_{T_n} \int_D g(\tau, x) \left( \sum_{i=1}^{M_s} R^n_i \varepsilon_i(x) \right) \, dx \, d\tau \right)$$

$$= \sum_{n=1}^{N_s} \int_{T_n} \int_D g(\tau, x) \hat{W}(\tau, x) \, dx \, d\tau$$

$$= \int_0^T \int_D g(\tau, x) \hat{W}(\tau, x) \, dx \, d\tau.$$  

\[\square\]

2.2. Linear elliptic and parabolic operators. We denote by $T_E : L^2(D) \rightarrow \dot{H}^2(D)$ the solution operator of the Dirichlet two-point boundary value problem: for given $f \in L^2(D)$ find $v_E \in \dot{H}^2(D)$ such that

$$v''_E = f \quad \text{in} \quad D,$$

i.e. $T_E \in v_E$. Also, by $T_b : L^2(D) \rightarrow \dot{H}^4(D)$ we denote the solution operator of the Dirichlet biharmonic two-point boundary value problem: for given $f \in L^2(D)$ find $v_b \in \dot{H}^4(D)$ such that

$$v''''_b = f \quad \text{in} \quad D,$$

i.e. $T_b \in v_b$. Due to the type of boundary conditions of (2.12), we conclude that

$$T_b f = T_E f \quad \forall f \in L^2(D),$$

which, easily, yields

$$(T_b v_1, v_2)_{0,D} = (T_b v_1, T_b v_2)_{0,D} = (v_1, T_b v_2)_{0,D} \quad \forall v_1, v_2 \in L^2(D).$$

It is well-known that the inverse elliptic operators $T_E$ and $T_b$ satisfy the following inequalities:

$$\|T_E f\|_{m,D} \leq C_E \|f\|_{m-2,D} \quad \forall f \in H^{m-2}\{0,m-2\}(D), \quad \forall m \in \mathbb{N}_0$$

and

$$\|T_b f\|_{m,D} \leq C_b \|f\|_{m-4,D} \quad \forall f \in H^{m-4}\{0,m-4\}(D), \quad \forall m \in \mathbb{N}_0,$$

where the nonnegative constants $C_E$ and $C_b$ depend only on $D$.

Let $(S(t)w)_{t \in [0,T]}$ be the standard semigroup notation for the solution $w$ of (1.2). For $\ell \in \mathbb{N}_0$, $\beta \geq 0$, $r \geq 0$ and $q \in [0, r+4\ell]$ there exists a constant $C_{r,q,\ell} > 0$ (see, e.g., Appendix A in [12, 20, 18]) such that

$$\|\partial_t^\ell S(t)w\|_{H^q} \leq C_{r,q,\ell} t^{-\frac{q-4\ell}{2}} \|w_0\|_{H^q} \quad \forall t > 0, \quad \forall w_0 \in \dot{H}^q(D),$$

$$\|w_0\|_{H^q} = \left( \sum_{\alpha \leq q} \|D^\alpha w_0\|^2 \right)^{\frac{1}{2}} \quad \forall w_0 \in H^q(D).$$
and a constant $C_\beta > 0$ such that

\begin{equation}
(2.18) \quad \int_{t_a}^{t_b} (\tau - t_a)^2 \| \partial^2_{\tau} S(\tau) w_0 \|_{H^0(D)}^2 d\tau \leq C\beta \| w_0 \|_{H^{4\ell+2\beta-2}}^2 \quad \forall t_b > t_a \geq 0, \quad \forall w_0 \in \dot{H}^{4\ell+2\beta-2}(D).
\end{equation}

2.3. Discrete operators. Let $p = 2$ or $3$, and $\mathbf{S}_h^p \subset H^2(D) \cap H^1_0(D)$ be a finite element space consisting of functions which are piecewise polynomials of degree at most $p$ over a partition of $D$ in intervals with maximum mesh-length $h$. It is well-known (see, e.g., [5], [3]) that the following approximation property holds: there exists a constant $C_{p,m,p} > 0$ such that

\begin{equation}
(2.19) \quad \inf_{\chi \in \mathbf{S}_h^p} \| v - \chi \|_{L^2(D)} \leq C_{p,m,p} h^{l-2} \| v \|_{L^2(D)} \quad \forall v \in H^l(D) \cap H^1_0(D), \quad l = 3, \ldots, p+1.
\end{equation}

Then, we define the discrete biharmonic operator $B_h : \mathbf{S}_h^p \to \mathbf{S}_h^p$ by $(B_h \varphi, \chi)_{H^2(D)} = (\partial^2_x \varphi, \partial^2_x \chi)_{H^2(D)}$ for $\varphi, \chi \in \mathbf{S}_h^p$, the $L^2(D)$–projection operator $P_h : L^2(D) \to \mathbf{S}_h^p$ by $(P_h f, \chi)_{H^2(D)} = (f, \chi)_{H^2(D)}$ for $\chi \in \mathbf{S}_h^p$ and $f \in L^2(D)$, and the standard Galerkin finite element approximation $v_{B,h} \in \mathbf{S}_h^p$ of the solution $v_B$ of (2.12) by requiring

\begin{equation}
(2.20) \quad B_h v_{B,h} = P_h f.
\end{equation}

Letting $T_{B,h} : L^2(D) \to \mathbf{S}_h^p$ be the solution operator of the finite element method (2.20), i.e.,

\begin{equation}
T_{B,h} f := v_{B,h} = B_h^{-1} P_h f \quad \forall f \in L^2(D),
\end{equation}

we can easily conclude that

\begin{equation}
(2.21) \quad (T_{B,h} f, g)_{H^2(D)} = (\partial^2_x (T_{B,h} f), \partial^2_x (T_{B,h} g))_{H^2(D)} \quad \forall f, g \in L^2(D).
\end{equation}

Also, using the approximation property (2.19) of the finite element space $\mathbf{S}_h^p$, we can prove (see, e.g., Proposition 2.2 in [13]) the following $L^2(D)$–error estimate for the finite element method (2.20):

\begin{equation}
(2.22) \quad \| T_{B,h} f - T_h f \|_{H^{-2,D}} \leq C h^p \| f \|_{H^{-2,D}} \quad \forall f \in L^2(D).
\end{equation}

Observing that the Galerkin orthogonality property reads

\begin{equation}
E \left[ (T_{B,h} f - T_h f, \chi)_{H^2(D)} \right] = 0 \quad \forall \chi \in \mathbf{S}_h^p, \quad \forall f \in L^2(D),
\end{equation}

after setting $\chi = T_{B,h} f$ and using the Cauchy-Schwarz inequality along with (2.16), we get

\begin{equation}
(2.23) \quad \| \partial^2_x (T_{B,h} f) \|_{H^2(D)} \leq \| \partial^2_x (T_h f) \|_{H^2(D)} \quad \forall f \in L^2(D).
\end{equation}

3. An estimate of the error $u - \tilde{u}$

In the theorem below, we derive an $L^\infty_t(L^2_x)(\Omega_+)$ bound for the difference $u - \tilde{u}$ in terms of $\Delta t$ and $M_\alpha$.

**Theorem 3.1.** Let $u$ be the stochastic function defined by (1.5) and $\tilde{u}$ be the solution of (1.6). Then, there exists a constant $C > 0$, independent of $\Delta t$ and $M_\alpha$, such that

\begin{equation}
(3.1) \quad \max_{t \in [0,T]} E \left[ \| u(t, \cdot) - \tilde{u}(t, \cdot) \|_{L^2(D)}^2 \right] \leq C \left( \Delta t^{\frac{\hat{\beta}}{2}} + \delta^{-\frac{\hat{\beta}}{2}} M_\alpha^{\frac{\hat{\beta}}{2}} \right) \quad \forall \delta \in (0, \frac{\alpha}{2}].
\end{equation}

**Proof.** Let $Z(t) := E \left[ \| u(t, \cdot) - \tilde{u}(t, \cdot) \|_{L^2(D)}^2 \right]$ for $t \in [0, T]$. We will get (3.1) working with the representations (1.5) and (1.8). In the sequel, we will use the symbol $C$ to denote a generic constant that is independent of $\Delta t$ and $M_\alpha$ and may changes value from one line to the other.

Using (1.5), (1.8), (2.10) and (2.9), we conclude that

\begin{equation}
(3.2) \quad \begin{aligned}
& u(t, x) - \tilde{u}(t, x) = \int_0^T \int_D [X_{(0,t)}(s) G(t - s; x,y) - \mathcal{G}(t,x,s,y)] dW(s,y) \quad \forall (t, x) \in (0, T] \times D,
\end{aligned}
\end{equation}

where $\mathcal{G} : (0, T] \times D \to L^2(D)$ given by

\begin{equation}
(3.3) \quad \begin{aligned}
& \mathcal{G}(t, x, s, y) := \frac{1}{\Delta t} \sum_{i=1}^M \left[ \int_D \varepsilon_i \left( \int_D G(t - s'; x, y') \varepsilon_i(y') dy' \right) ds' \right] \varepsilon_i(y) \quad \forall (s, y) \in T_n \times D
\end{aligned}
\end{equation}
for \((t, x) \in (0, T] \times D\) and \(n = 1, \ldots, N_*\). Thus, using (3.2) and (2.5), we obtain
\[
Z(t) = \left( \int_0^T \left( \int_D \int_D \left[ x_{(0,t)}(s) G(t - s; x, y) - \Theta(t, x; s, y) \right]^2 dxdy \right) ds \right)^{\frac{1}{2}} \quad \forall t \in (0, T].
\]
Now, we introduce the splitting
\[
(3.4) \quad Z(t) \leq Z_A(t) + Z_B(t) \quad \forall t \in (0, T],
\]
where
\[
(3.5) \quad Z_A(t) := \left\{ \sum_{n=1}^{N_*} \int_D \int_D \int_{T_n} \left[ x_{(0,t)}(s) G(t - s; x, y) - \frac{1}{\sqrt{T}} \int_{T_n} x_{(0,t)}(s') G(t - s'; x, y) ds' \right]^2 dxdyds \right\}^{\frac{1}{2}}
\]
and
\[
(3.6) \quad Z_B(t) := \left\{ \sum_{n=1}^{N_*} \int_D \int_D \int_{T_n} \left[ \frac{1}{\sqrt{T}} \int_{T_n} x_{(0,t)}(s') G(t - s'; x, y) ds' - \Theta(t, x; s, y) \right]^2 dxdyds \right\}^{\frac{1}{2}}.
\]
Using (3.3) and the \(L^2(D)\)–orthogonality of \((\varepsilon_k)_{k=1}^\infty\) we obtain
\[
(3.7) \quad \Theta(t, x; s, y) = \frac{1}{\sqrt{T}} \int_{T_n} x_{(0,t)}(s') \left[ \sum_{i=1}^{M_*} e^{-\lambda_i^*(t-s') \varepsilon_i(x) \varepsilon_i(y)} \right] ds' \quad \forall (s, y) \in T_n \times D
\]
for \((t, x) \in (0, T] \times D\) and \(n = 1, \ldots, N_*\). Next, we combine (3.6) and (3.7) and use, again, the \(L^2(D)\)–orthogonality of \((\varepsilon_k)_{k=1}^\infty\) to get
\[
Z_B(t) = \left\{ \frac{1}{\sqrt{T}} \sum_{n=1}^{N_*} \int_D \int_D \int_{T_n} \left[ \int_{T_n} x_{(0,t)}(s') \left( G(t - s'; x, y) - \sum_{i=1}^{M_*} e^{-\lambda_i^*(t-s') \varepsilon_i(x) \varepsilon_i(y)} \right) ds' \right]^2 dxdyds \right\}^{\frac{1}{2}}
\]
\[
= \left\{ \frac{1}{\sqrt{T}} \sum_{n=1}^{N_*} \int_D \int_D \int_{T_n} \left[ \int_{T_n} x_{(0,t)}(s') \left( \sum_{i=1}^{M_*} e^{-\lambda_i^*(t-s') \varepsilon_i(x) \varepsilon_i(y)} \right) ds' \right]^2 dxdyds \right\}^{\frac{1}{2}}
\]
\[
= \left\{ \frac{1}{\sqrt{T}} \sum_{n=1}^{N_*} \int_D \int_D \left[ \int_{T_n} x_{(0,t)}(s') e^{-\lambda_i^*(t-s') \varepsilon_i(x) \varepsilon_i(y)} ds' \right] \varepsilon_i(x) \varepsilon_i(y) \right]^2 dxdy\right\}^{\frac{1}{2}}
\]
\[
= \left\{ \frac{1}{\sqrt{T}} \sum_{n=1}^{N_*} \int_D \left[ \int_{T_n} x_{(0,t)}(s') e^{-\lambda_i^*(t-s') \varepsilon_i(x) \varepsilon_i(y)} ds' \right]^2 \varepsilon_i^2(x) \right\}^{\frac{1}{2}} \quad \forall t \in (0, T].
\]
Then, using the Cauchy-Schwarz inequality and (2.3), we obtain
\[
Z_B(t) \leq \left\{ \sum_{i=M_*+1}^{\infty} \left( \int_0^T e^{-2 \lambda_i^*(t-s')} ds' \right)^2 \right\}^{\frac{1}{2}}
\]
\[
\leq \left( \sum_{i=M_*}^{\infty} \frac{1}{2 \pi^2 \lambda_i^*} \right)^{\frac{1}{2}}
\]
\[
\leq \frac{1}{\sqrt{2 \pi}} \frac{1}{M_*^{2-\delta}} \left( \sum_{i=M_*}^{\infty} \frac{1}{i^{1+\delta}} \right)^{\frac{1}{2}}
\]
\[
\leq C \delta^{-\frac{1}{2}} M_*^{-\frac{1}{2}+\delta} \quad \forall t \in (0, T], \quad \forall \delta \in (0, \frac{1}{2}].
\]
Finally, combining \(3.5\) along with the \(L^2(D)\)–orthogonality of \((\varepsilon_k)_{k=1}^\infty\), we conclude that
\[
Z_A(t) = \left\{ \left( \frac{1}{N^2} \sum_{n=1}^N \int \int \int_{T_n} \left[ \int_{T_n} x_{(0,t)}(s) G(t-s;x,y) - x_{(0,t)}(s') G(t-s';x,y) \right] ds' \right)^2 ds \right\}^{\frac{1}{2}}
\]
\[
= \left\{ \sum_{s=1}^\infty \frac{1}{(\Delta t)^2} \sum_{n=1}^N \left( \int_{T_n} x_{(0,t)}(s) e^{-\Delta t^2(t-s)} - x_{(0,t)}(s') e^{-\Delta t^2(t-s')} \right) ds' \right\}^{\frac{1}{2}} \quad \forall t \in (0,T].
\]

Then, we proceed as in the proof of Theorem 3.1 in [13] to get
\[
Z_A(t) \leq C \Delta t^\frac{3}{2} \quad \forall t \in (0,T].
\]

The error bound \(3.1\) follows by observing that \(Z(0) = 0\) and combining the bounds \(3.4\), \(3.9\) and \(3.8\).

4. Time-Discrete Approximations

4.1. The deterministic problem. In this section we introduce and analyze modified Crank-Nicolson time-discrete approximations, \((W^m)_{m=0}^M\), of the solution \(w\) to the deterministic problem \(1.2\).

We begin by setting
\[
W^0 := w_0
\]
and then by finding \(W^1 \in \dot{H}^4(D)\) such that
\[
W^1 - W^0 + \frac{\Delta t}{2} \partial_x^2 W^1 = 0.
\]

Finally, for \(m = 2, \ldots, M\), we specify \(W^m \in \dot{H}^4(D)\) such that
\[
W^m - W^{m-1} + \Delta t \partial_x^4 W^{m-\frac{1}{2}} = 0.
\]

First, we provide a discrete in time \(L^2(L^2)\) a priori estimate of time averages of the nodal error for the modified Crank-Nicolson time-discrete approximations defined above.

**Proposition 4.1.** Let \((W^m)_{m=0}^M\) be the modified Crank-Nicolson time-discrete approximations of the solution \(w\) to the problem \(1.2\) defined by \(1.1\), \(1.2\) and \(1.3\). Then, there exists a constant \(C > 0\), independent of \(\Delta t\), such that
\[
(4.1) \quad \left( \Delta t \sum_{m=1}^M \|W^{m-\frac{1}{2}} - w^{m-\frac{1}{2}}\|^2_{0,D} \right)^{\frac{1}{2}} \leq C \Delta t^{\frac{1}{2}} \|w_0\|_{\dot{H}^{4\theta-2}} \quad \forall \theta \in [0,1], \quad \forall w_0 \in \dot{H}^2(D),
\]
where \(w^\ell(\cdot) := w(\tau_{\ell}, \cdot)\) for \(\ell = 0, \ldots, M\).

**Proof.** The error bound \(4.1\) follows by interpolation after proving it for \(\theta = 1\) and \(\theta = 0\) (cf. [2], [13], [21]). In the sequel, we will use the symbol \(C\) to denote a generic constant that is independent of \(\Delta t\) and may change values from one line to the other.

**Case** \(\theta = 1\): Let \(E^* := w(\tau_{\frac{1}{2}}, \cdot) - W^1\) and \(E^m := w^{m} - W^m\) for \(m = 0, \ldots, M\). Using \(1.2\) and \(1.3\), we arrive at
\[
(4.2) \quad T_B(E^m - E^{m-1}) + \Delta t E^{m-\frac{1}{2}} = \sigma_m, \quad m = 2, \ldots, M,
\]
where
\[
\sigma_\ell(\cdot) := - \int_{\Delta \ell} \left[ w(\tau, \cdot) - w^{\ell-\frac{1}{2}}(\cdot) \right] d\tau, \quad \ell = 2, \ldots, M.
\]

Taking the \(L^2(D)\)–inner product of both sides of \(4.2\) with \(E^{m-\frac{1}{2}}\), and then using \(2.14\) and summing with respect to \(m\), from 2 up to \(M\), we obtain
\[
(4.3) \quad \|T_B E^m\|^2_{0,D} - \|T_B E^1\|^2_{0,D} + 2 \Delta t \sum_{m=2}^M \|E^{m-\frac{1}{2}}\|^2_{0,D} = 2 \sum_{m=2}^M (\sigma_m, E^{m-\frac{1}{2}})_{0,D}.
\]
Applying the Cauchy-Schwarz inequality and the geometric mean inequality, we have

\[
2 \sum_{m=2}^{M} (\sigma_m, E^{m-\frac{1}{2}})_{0,D} \leq 2 \sum_{m=2}^{M} \|\sigma_m\|_{0,D} \|E^{m-\frac{1}{2}}\|_{0,D}
\]

\[
\leq \sum_{m=2}^{M} \left( \Delta \tau^{-1} \|\sigma_m\|_{0,D}^2 + \Delta \tau \|E^{m-\frac{1}{2}}\|_{0,D}^2 \right),
\]

which, along with (4.7), yields

\[
\Delta \tau \sum_{m=2}^{M} \|E^{m-\frac{1}{2}}\|_{0,D}^2 \leq \|T_k E^1\|_{0,D}^2 + \Delta \tau^{-1} \sum_{m=2}^{M} \|\sigma_m\|_{0,D}^2.
\]

Using (4.6), we bound the quantities (\(\sigma_m\))_{m=2} as follows:

\[
\|\sigma_m\|_{0,D}^2 = \frac{1}{\Delta m} \left( - \int_{\Delta m}^{\tau_m} \partial_x w(s,x) \, ds \, d\tau + \int_{\Delta m}^{\tau} \partial_x w(s,x) \, ds \, d\tau \right)^2 \, dx
\]

\[
\leq \int_D \left( \int_{\Delta m}^{\tau_m} |\partial_x w(s,x)| \, ds \, d\tau \right)^2 \, dx
\]

\[
\leq \Delta \tau \int_{\Delta m}^{\tau} \|\partial_x w(s,\cdot)\|_{0,D}^2 \, ds, \quad m = 2, \ldots, M.
\]

Using that \(E^0 = 0\) and combining (4.8), (4.9) and (2.18) (with \(\beta = 0, \ell = 1, r = 0\)), we obtain

\[
\Delta \tau \sum_{m=1}^{M} \|E^{m-\frac{1}{2}}\|_{0,D}^2 \leq \frac{\Delta \tau}{\Delta m} \|E^1\|_{0,D}^2 + \|T_k E^1\|_{0,D}^2 + \Delta \tau^2 \int_0^\tau \|\partial_x w(s,\cdot)\|_{0,D}^2 \, ds
\]

\[
\leq \frac{\Delta \tau}{\Delta m} \|E^1\|_{0,D}^2 + \|T_k E^1\|_{0,D}^2 + C \Delta \tau^2 \|w_0\|_{\dot{H}^2}^2.
\]

In order to bound the first two terms in the right hand side of (4.10), we introduce the following splittings

\[
\|T_k E^1\|_{0,D}^2 \leq 2 \left( \|T_k (w(\tau_1,\cdot) - w(\tau_{\frac{1}{2}},\cdot))\|_{0,D}^2 + \|T_k E^\ast\|_{0,D}^2 \right)
\]

and

\[
\Delta \tau \|E^1\|_{0,D}^2 \leq 2 \Delta \tau \left( \|w(\tau_1,\cdot) - w(\tau_{\frac{1}{2}},\cdot)\|_{0,D}^2 + \|E^\ast\|_{0,D}^2 \right).
\]

We continue by estimating the terms in the right hand side of (4.11) and (4.12). First, we observe that

\[
\|w(\tau_1,\cdot) - w(\tau_{\frac{1}{2}},\cdot)\|_{0,D}^2 = \int_D \left( \int_{\tau_{\frac{1}{2}}}^{\tau_1} \partial_x w(\tau,\cdot) \, d\tau \right)^2 \, dx
\]

\[
\leq \frac{\Delta \tau}{\Delta} \int_{\tau_{\frac{1}{2}}}^{\tau_1} \|\partial_x w(\tau,\cdot)\|_{0,D}^2 \, d\tau,
\]

which, along with (2.18) (with \(\ell = 1, r = 0, \beta = 0\)), yields

\[
\|w(\tau_1,\cdot) - w(\tau_{\frac{1}{2}},\cdot)\|_{0,D}^2 \leq C \Delta \tau \|w_0\|_{\dot{H}^2}^2.
\]
Next, we use (2.15) and (2.3), to get
\[
\|T_E(w(\tau_1, \cdot) - w(\tau_{1 \downarrow}, \cdot))\|_{0,D}^2 = \int_D \left( \int_{\tau_{1 \downarrow}}^{\tau_1} T_E(\partial_\tau w(\tau, x)) \, d\tau \right)^2 \, dx \\
\leq \Delta^2 \int_{\tau_{1 \downarrow}}^{\tau_1} \|T_E(\partial_\tau w(\tau, \cdot))\|_{0,D}^2 \, d\tau \\
\leq C \Delta \int_{\tau_{1 \downarrow}}^{\tau_1} \|\partial_\tau w(\tau, \cdot)\|_{-2,D}^2 \, d\tau \\
\leq C \Delta \int_{\tau_{1 \downarrow}}^{\tau_1} \|\partial_\tau w(\tau, \cdot)\|_{H^{-2}}^2 \, d\tau.
\]
Observing that
\[
w(\tau, \cdot) = \sum_{k=1}^\infty \lambda_k e^\lambda_k \tau (w_0, \varepsilon_k)_{0,D} \varepsilon_k(\cdot) \quad \forall \tau \in [0, T],
\]
we have
\[
\|\partial_\tau w(\tau, \cdot)\|_{H^{-2}}^2 = \sum_{k=1}^\infty \lambda_k^{-4} \|\partial_\tau w(\tau, \cdot), \varepsilon_k\|_{0,D}^2 \\
= \sum_{k=1}^\infty \lambda_k^4 e^{-2\lambda_k \tau} \|w_0, \varepsilon_k\|_{0,D}^2 \\
\leq \|w_0\|_{H^2}^2 \quad \forall \tau \in [0, T].
\]
Thus, we arrive at
\[
\|T_E(w(\tau_1, \cdot) - w(\tau_{1 \downarrow}, \cdot))\|_{0,D}^2 \leq C \Delta^2 \|w_0\|_{H^2}^2.
\]
Finally, using (1.2) and (4.2) we have
\[
T_E(E^* - E^0) + \Delta^2 E^* = \sigma_*
\]
with
\[
\sigma_*(\cdot) := -\int_0^{\tau_{1 \downarrow}} \left[ w(s, \cdot) - w(\tau_{1 \downarrow}, \cdot) \right] \, ds.
\]
Since \(E^0 = 0\), after taking the \(L^2(D)\)-inner product of both sides of (4.16) with \(E^*\) and using (2.14) and the Cauchy-Schwarz inequality along with the arithmetic mean inequality, we obtain
\[
\|T_E E^*\|_{0,D}^2 + \frac{\Delta^2}{2} \|E^*\|_{0,D}^2 = (\sigma_*, E^*)_{0,D} \\
\leq \frac{1}{\Delta^2} \|\sigma_*\|_{0,D}^2 + \frac{\Delta^2}{2} \|E^*\|_{0,D}^2.
\]
Now, using (4.17) and (2.18) (with \(\beta = 0, \ell = 1, r = 0\) we obtain
\[
\|\sigma_*\|_{0,D}^2 = \int_D \left[ \int_{\tau_{1 \downarrow}}^{\tau_1} \left( \int_0^{\frac{\tau_1}{2}} \partial_\tau w(\tau, x) \, d\tau \right) \right] \, dx \\
\leq \Delta \int_0^{\tau_1} \left( \int_0^{\tau_1} \|\partial_\tau w(\tau, x)\|_{H^{-2}}^2 \, d\tau \right) \, dx \\
\leq \Delta \int_0^{\tau_1} \|\partial_\tau w(\tau, \cdot)\|_{0,D}^2 \, d\tau \\
\leq C \Delta^3 \|w_0\|_{H^2}^2,
\]
which, along with (4.18), yields
\[
\|T_E E^*\|_{0,D}^2 + \frac{\Delta^2}{2} \|E^*\|_{0,D}^2 \leq C \Delta^2 \|w_0\|_{H^2}^2.
\]
Thus, from (4.10), (4.11), (4.12), (4.13), (4.15) and (4.19), we conclude that
\[
\Delta \tau \sum_{m=1}^{M} \| E^{m - \frac{1}{2}} \|_{0,D}^2 \leq C \Delta \tau \| w_0 \|_{H^2}^2.
\]

\textbf{Case} \( \theta = 0 \): First, we observe that (4.2) and (4.3) are equivalent to
\[
T_\beta (W^1 - W^0) + \frac{\Delta \tau}{2} W^1 = 0
\]
and
\[
T_\beta (W^m - W^{m-1}) + \Delta \tau W^{m-\frac{1}{2}} = 0, \quad m = 2, \ldots, M.
\]
Next, we take the \( L^2(D) \)--inner product of both sides of (4.21) with \( W^{m-\frac{1}{2}} \), use (2.14) and sum with respect to \( m \) from 2 up to \( M \), to obtain
\[
\| T_E W^m \|_{0,D}^2 - \| T_E W^1 \|_{0,D}^2 + 2 \Delta \tau \sum_{m=2}^{M} \| W^{m-\frac{1}{2}} \|_{0,D}^2 = 0,
\]
which yields that
\[
\Delta \tau \| W^1 \|_{0,D}^2 + \Delta \tau \sum_{m=2}^{M} \| W^{m-\frac{1}{2}} \|_{0,D}^2 \leq \Delta \tau \| W^1 \|_{0,D}^2 + \| T_E W^1 \|_{0,D}^2.
\]
Now, we take the \( L^2(D) \)--inner product of both sides of (4.20) with \( W^1 \), use (2.14) along with (2.7) to get
\[
\| T_E W^1 \|_{0,D}^2 + \Delta \tau \| W^1 \|_{0,D}^2 \leq \| T_E W^0 \|_{0,D}^2.
\]
Combining (4.22) and (4.23) and then using (2.15) and (2.3), we obtain
\[
\Delta \tau \| W^1 \|_{0,D}^2 + \Delta \tau \sum_{m=2}^{M} \| W^{m-\frac{1}{2}} \|_{0,D}^2 \leq \| T_E w_0 \|_{0,D}^2
\]
\[\leq C \| w_0 \|_{H^2}^2 \]
\[\leq C \| w_0 \|_{H^{-2}}^2.
\]
In addition, we have
\[
\Delta \tau \| w^1 \|_{0,D}^2 + \Delta \tau \sum_{m=2}^{M} \| w^{m-\frac{1}{2}} \|_{0,D}^2 \leq 2 \Delta \tau \sum_{m=1}^{M} \| w^m \|_{0,D}^2
\]
and
\[
2 \Delta \tau \sum_{m=1}^{M} \| w^m \|_{0,D}^2 \leq 2 \Delta \tau - 1 \sum_{m=1}^{M} \int_{D} \left( \int_{\tau_{m-1}}^{\tau_m} \partial_{\tau} \left[ (\tau - \tau_{m-1}) w(\tau,x) \right] d\tau \right)^2 dx
\leq 2 \Delta \tau - 1 \sum_{m=1}^{M} \int_{D} \left( \int_{\tau_{m-1}}^{\tau_m} [w(\tau,x) + (\tau - \tau_{m-1}) w_\tau(\tau,x)] d\tau \right)^2 dx
\leq 4 \sum_{m=1}^{M} \int_{D} \left[ \| w(\tau,\cdot) \|_{0,D}^2 + (\tau - \tau_{m-1})^2 \| w_\tau(\tau,\cdot) \|_{0,D}^2 \right] d\tau
\leq 4 \int_0^T \left[ \| w(\tau,\cdot) \|_{0,D}^2 + \tau^2 \| w_\tau(\tau,\cdot) \|_{0,D}^2 \right] d\tau,
\]
which, along with (2.13) (taking \((\beta,\ell,r) = (0,0,0)\) and \((\beta,\ell,r) = (2,1,0)\)), yields
\[
2 \Delta \tau \sum_{m=1}^{M} \| w^m \|_{0,D}^2 \leq C \| w_0 \|_{H^{-2}}^2.
\]
integrating by parts, we easily arrive at

\[
\Delta \tau \sum_{m=1}^{M} \| E^{m-\frac{1}{2}} \|_{0,D}^2 \leq 2 \left( \Delta \tau \| W^1 \|_{0,D}^2 + \Delta \tau \sum_{m=2}^{M} \| W^{m-\frac{1}{2}} \|_{0,D}^2 \right) + 2 \left( \Delta \tau \| w^1 \|_{0,D}^2 + \Delta \tau \sum_{m=2}^{M} \| w^{m-\frac{1}{2}} \|_{0,D}^2 \right),
\]

which, after using (4.24), (4.25) and (4.26), yields

\[
\sum_{m=1}^{M} \Delta \tau \| E^{m-\frac{1}{2}} \|_{0,D}^2 \leq C \| w_0 \|_{H^{-2}}^2.
\]

Next, we show a discrete in time \(L^2(D)\) a priori estimate of the nodal error for the modified Crank-Nicolson time-discrete approximations.

**Proposition 4.2.** Let \((W^m)_{m=0}^{\infty}\) be the modified Crank-Nicolson time-discrete approximations of the solution \(w\) to the problem (1.2) defined by (1.1), (1.2) and (1.3). Then, there exists a constant \(C > 0\), independent of \(\Delta \tau\), such that

\[
(\Delta \tau \sum_{m=1}^{M} \| W^m - w^m \|_{0,D}^2)^{\frac{1}{2}} \leq C \Delta \tau^{\frac{\delta}{2}} \| w_0 \|_{H^{2(\delta-1)}}, \quad \forall \delta \in [0,1], \quad \forall w_0 \in \dot{H}^2(D),
\]

where \(w^\ell := w(\tau^\ell, \cdot)\) for \(\ell = 0, \ldots, M\).

**Proof.** We will arrive at the error bound (4.27) by interpolation after proving it for \(\delta = 1\) and \(\delta = 0\) (cf. Proposition 4.1). In both cases, the error estimation is based on the following bound

\[
(\Delta \tau \sum_{m=1}^{M} \| W^m - w^m \|_{0,D}^2)^{\frac{1}{2}} \leq S_B + S_C + S_D
\]

where

\[
S_B := \left( \Delta \tau \sum_{m=2}^{M} \| W^m - W^{m-\frac{1}{2}} \|_{0,D}^2 \right)^{\frac{1}{2}},
\]

\[
S_C := \left( \Delta \tau \| W^1 - w^1 \|_{0,D}^2 + \Delta \tau \sum_{m=2}^{M} \| W^{m-\frac{1}{2}} - w^{m-\frac{1}{2}} \|_{0,D}^2 \right)^{\frac{1}{2}},
\]

\[
S_D := \left( \Delta \tau \sum_{m=2}^{M} \| w^{m-\frac{1}{2}} - w^m \|_{0,D}^2 \right)^{\frac{1}{2}}.
\]

In the sequel, we will use the symbol \(C\) to denote a generic constant that is independent of \(\Delta \tau\) and may change value from one line to the other.

- **Case** \(\delta = 1\): Taking the \(L^2(D)\) inner product of both sides of (1.3) with \((W^m - W^{m-1})\) and then integrating by parts, we easily arrive at

\[
\| W^m - W^{m-1} \|_{0,D}^2 + \frac{\Delta \tau}{2} \left( \| \partial_x^2 W^m \|_{0,D}^2 - \| \partial_x^2 W^{m-1} \|_{0,D}^2 \right) = 0, \quad m = 2, \ldots, M.
\]

After summing both sides of (4.29) with respect to \(m\) from 2 up to \(M\), we obtain

\[
\Delta \tau \sum_{m=2}^{M} \| W^m - W^{m-1} \|_{0,D}^2 + \frac{\Delta \tau}{2} \left( \| \partial_x^2 W^M \|_{0,D}^2 - \| \partial_x^2 W^1 \|_{0,D}^2 \right) = 0,
\]
which yields

\[
(4.30) \quad \Delta \tau \sum_{m=2}^{M} \| W^m - W^{m-1} \|^2_{\alpha, D} \leq \frac{\Delta \tau^2}{2} \| \partial_x^2 W^1 \|^2_{\alpha, D}.
\]

Taking the $L^2(D)$-inner product of both sides of (4.2) with $W^1$, and then integrating by parts and using (2.7), we have

\[
\| W^1 \|^2_{\alpha, D} - \| W^0 \|^2_{\alpha, D} + \Delta \tau \| \partial_x^2 W^1 \|^2_{\alpha, D} \leq 0
\]

from which we conclude that

\[
(4.31) \quad \Delta \tau \| \partial_x^2 W^1 \|^2_{\alpha, D} \leq \| w_0 \|^2_{\alpha, D}.
\]

Thus, combining (4.30) and (4.31), we get

\[
S_B = \frac{1}{2} \left( \Delta \tau \sum_{m=2}^{M} \| W^m - W^{m-1} \|^2_{\alpha, D} \right) \leq \frac{1}{2} \Delta \tau \| \partial_x^2 W^1 \|^2_{\alpha, D}
\]

(4.32)

\[
\leq \frac{1}{\sqrt{2}} \Delta \tau \| w_0 \|_{\alpha, D}
\]

Also, we observe that the estimate (4.4), for $\theta = \frac{1}{2}$, yields

\[
(4.33) \quad S_C \leq C \Delta \tau \frac{1}{2} \| w_0 \|_{\alpha, D}.
\]

Finally, using (2.17) (with $\ell = 1$, $r = 0$, $q = 0$), we obtain

\[
(4.34) \quad S_D \leq \left( \frac{\Delta \tau^2}{\Delta \tau} \int_{\Delta \tau}^{T} \| \partial_x w(\tau, \cdot) \|_{\alpha, D}^2 d\tau \right)^{\frac{1}{2}}
\]

\[
\leq C \left( \frac{\Delta \tau^2}{\Delta \tau} \int_{\Delta \tau}^{T} \| w_0 \|_{\alpha, D}^2 d\tau \right)^{\frac{1}{2}}
\]

\[
\leq C \Delta \tau \| w_0 \|_{\alpha, D} \left( \frac{1}{\Delta \tau} - \frac{1}{T} \right)^{\frac{1}{2}}
\]

\[
\leq C \Delta \tau \frac{1}{2} \| w_0 \|_{\alpha, D}.
\]

Thus, from (4.28), (4.32), (4.33) and (4.34) we conclude (4.27) for $\delta = 1$.

- **Case $\delta = 0$:** Taking again the $L^2(D)$-inner product of both sides of (4.2) with $W^1$ and then integrating by parts and using (2.8) and (2.2) along with the arithmetic mean inequality, we obtain

\[
(4.35) \quad \| W^1 \|^2_{\alpha, D} + \frac{\Delta \tau^2}{2} \| \partial_x^2 W^1 \|^2_{\alpha, D} = (w_0, W^1)_{\alpha, D}
\]

\[
\leq \| w_0 \|_{-2, D} \| W^1 \|_{1, D}
\]

\[
\leq C \| w_0 \|_{\alpha-2} \| W^1 \|_{\alpha^2}
\]

\[
\leq C \Delta \tau^{-1} \| w_0 \|^2_{\alpha-2} + \frac{\Delta \tau^2}{2} \| W^1 \|^2_{\alpha^2}.
\]
Now, integrating by parts we have

\[
\| \partial_x^2 W^1 \|_{0,D} = \left( \sum_{k=1}^{\infty} |(\varepsilon_k, \partial_x^2 W^1)_{0,D}|^2 \right)^{\frac{1}{2}}
\]

(4.36)

\[
= \left( \sum_{k=1}^{\infty} |(\partial_x^2 \varepsilon_k, W^1)_{0,D}|^2 \right)^{\frac{1}{2}}
\]

\[
= \left( \sum_{k=1}^{\infty} \lambda_k^4 |(\varepsilon_k, W^1)_{0,D}|^2 \right)^{\frac{1}{2}}
\]

\[
= \| W^1 \|_{H^2},
\]

which, along with (4.35), yields that

\[
\Delta \tau^2 \| \partial_x^2 W^1 \|_{0,D}^2 \leq C \| w_0 \|_{\dot{H}^{-2}}.
\]

(4.37)

Thus, combining (4.30) and (4.37), we conclude that

\[
S_B = \frac{1}{2 \Delta \tau} \left( \Delta \tau \sum_{m=2}^{M} \| W^m - W^{m-1} \|_{0,D}^2 \right)^{\frac{1}{2}}
\]

(4.38)

\[
\leq \frac{1}{2 \Delta \tau} \Delta \tau \| \partial_x^2 W^1 \|_{0,D}
\]

\[
\leq C \| w_0 \|_{\dot{H}^{-2}}.
\]

(4.39)

Also, the estimate (4.4), for \( \theta = 0 \), yields

\[
S_C \leq C \| w_0 \|_{\dot{H}^{-2}}.
\]

Using the Cauchy-Schwarz inequality and (4.26), we have

\[
S_D = \frac{1}{2 \Delta \tau} \left( \Delta \tau \sum_{m=2}^{M} \| W^m - W^{m-1} \|_{0,D}^2 \right)^{\frac{1}{2}}
\]

(4.40)

\[
\leq \frac{\sqrt{2}}{2} \Delta \tau \left( \sum_{m=1}^{M} \| w^m \|_{0,D}^2 \right)^{\frac{1}{2}}
\]

\[
\leq C \| w_0 \|_{\dot{H}^{-2}}.
\]

(4.41)

Thus, from (4.28), (4.38), (4.39) and (4.40) we conclude (4.27) for \( \delta = 0 \). \( \square \)

4.2. The stochastic problem. The Crank-Nicolson time-stepping method for the approximate problem (1.6) constructs, for \( m = 0, \ldots, M \), an approximation \( U^m \) of \( \hat{u}(\tau_m, \cdot) \), first by setting

\[
U^0 := 0,
\]

(4.41)

and then, for \( m = 1, \ldots, M \), by specifying \( U^m \in \dot{H}^4(D) \) such that

\[
U^m - U^{m-1} + \Delta \tau \partial_t^2 U^m = \int_{\Sigma_m} \hat{W} d\sigma \quad \text{a.s}.
\]

(4.42)

In the theorem that follows, using the result of Proposition 4.2 we show a discrete in time \( L_{t}^{\infty}(L^2_{x})(L^2_{v}) \) convergence estimate for the Crank-Nicolson time discrete approximations of \( \hat{u} \) defined above.

**Theorem 4.3.** Let \( \hat{u} \) be the solution of (1.6) and \( (U^m)_{m=0}^{M} \) be the Crank-Nicolson time-discrete approximations of \( \hat{u} \) specified in (4.41) and (4.42). Then, there exists constant \( C > 0 \), independent of \( \Delta t, M, \) and \( \Delta \tau \), such that

\[
\max_{1 \leq m \leq M} \left( E \left[ \| U^m - \hat{u}^m \|_{0,D}^2 \right] \right)^{\frac{1}{2}} \leq C \epsilon^{-\frac{1}{2}} \Delta \tau^{\frac{3}{2} - \epsilon} \quad \forall \epsilon \in (0, \frac{3}{8}]
\]

(4.43)

where \( \hat{u}^\ell := \hat{u}(\tau_\ell, \cdot) \) for \( \ell = 0, \ldots, M \).
Proof. Let \( I : L^2(D) \to L^2(D) \) be the identity operator, \( Y : H^4(D) \to L^2(D) \) be defined by \( Y := 1 - \frac{\Delta}{2} \partial^2_x \) and \( \Lambda : L^2(D) \to \dot{H}^4(D) \) be the inverse elliptic operator \( \Lambda := (I + \frac{\Delta}{2} \partial^2_x)^{-1} \). Finally, for \( m = 1, \ldots, M \), we define an operator \( Q_m : L^2(D) \to \dot{H}^4(D) \) by \( Q_m := (AY)^{m-1} \Lambda \). The operator \( \Lambda \) has Green function \( G(x,y) = \sum_{k=1}^{\infty} \frac{\delta_k(x) \delta_k(y)}{1 + \frac{\Delta}{2} \lambda_k} \), i.e. \( \Lambda f(x) = \int_D G(x,y)f(y)dy \) for \( x \in \overline{D} \) and \( f \in L^2(D) \). Also, \( Y \) has Green function \( G_m(x,y) = \sum_{k=1}^{\infty} (1 - \frac{\Delta}{2} \lambda_k) \delta_k(x) \delta_k(y) \), i.e. \( Yz(x) = \int_D G_m(x,y)z(y)dy \) for \( x \in \overline{D} \) and \( z \in H^4(D) \). Finally, for \( m = 1, \ldots, M \), \( Q_m \) has Green function \( G_{Q_m} \) given by

\[
G_{Q_m} = \sum_{k=1}^{\infty} \frac{(1 - \frac{\Delta}{2} \lambda_k)^{m-1}}{(1 + \frac{\Delta}{2} \lambda_k)^m} \delta_k(x) \delta_k(y),
\]

i.e. \( Q_m f(x) = \int_D G_{Q_m}(x,y)f(y)dy \) for \( x \in \overline{D} \) and \( f \in L^2(D) \).

For a given \( w_0 \in \dot{H}^2(D) \), let \((W^m)_m=0\) be the modified Crank-Nicolson time-discrete approximations defined by \( (4.1) \), \( (4.2) \) and \( (4.3) \). Then, using a simple induction argument, we conclude that

\[
(4.44) \quad W^m = Q_m w_0, \quad m = 1, \ldots, M.
\]

Also, to simplify the notation, we set \( G_m(\tau;x,y) := \chi_{(0,\tau_m)}(\tau) G(\tau_m - \tau;x,y) \) for \( m = 1, \ldots, M \).

In the sequel, we will use the symbol \( C \) to denote a generic constant that is independent of \( \Delta t \), \( M \), and \( \Delta \tau \), and may change value from one line to the other.

Using \( (4.12) \) and an induction argument, we conclude that

\[
U^m = \sum_{\ell=1}^{m} \int_{\Delta \ell} Q_{m-\ell+1} \left( \dot{W}(\tau, \cdot) \right) d\tau, \quad m = 1, \ldots, M,
\]

which yields

\[
U^m(x) = \int_0^x \int_D \mathcal{K}_m(\tau;x,y) \dot{W}(\tau,y) dy d\tau \quad \forall x \in \overline{D}, \quad m = 1, \ldots, M,
\]

with

\[
\mathcal{K}_m(\tau;x,y) := \sum_{\ell=1}^{m} \chi_{\Delta \ell}(\tau) G_{Q_{m-\ell+1}}(x,y) \quad \forall \tau \in [0,T], \ \forall x, y \in \overline{D}.
\]

Let \( m \in \{1, \ldots, M\} \) and \( \mathcal{E}_m := \left( \mathbb{E} \left[ \|U^m - \hat{u}^m\|_{0,D}^2 \right] \right)^{1/2} \). Now, we use \( (4.45) \), \( (1.8) \), \( (2.10) \), \( (2.3) \), \( (2.4) \) and \( (2.8) \), to obtain

\[
(4.46) \quad \mathcal{E}_m = \left( \mathbb{E} \left[ \int_D \left( \int_0^{T_m} \left[ \mathcal{K}_m(\tau;x,y) - G_m(\tau;x,y) \right] \dot{W}(\tau,y) dy d\tau \right)^2 dx \right] \right)^{1/2}
\]

\[
\leq \left( \int_0^{T_m} \left( \int_D \left[ \mathcal{K}_m(\tau;x,y) - G_m(\tau;x,y) \right]^2 dy dx \right) d\tau \right)^{1/2}
\]

\[
\leq \left( \sum_{\ell=1}^{m} \int_{\Delta \ell} \|Q_{m-\ell+1} - S(\tau_m - \tau) \|^2_{HS} d\tau \right)^{1/2}
\]

\[
\leq A_m + B_m,
\]

with

\[
A_m := \left( \sum_{\ell=1}^{m} \int_{\Delta \ell} \|Q_{m-\ell+1} - S(\tau_m - \tau_{\ell-1}) \|^2_{HS} d\tau \right)^{1/2},
\]

\[
B_m := \left( \sum_{\ell=1}^{m} \int_{\Delta \ell} \|S(\tau_m - \tau_{\ell-1}) - S(\tau_m - \tau) \|^2_{HS} d\tau \right)^{1/2}.
\]
Let $\delta \in [0, \frac{3}{4})$. Then, using the definition of the Hilbert-Schmidt norm, the deterministic estimate (4.27) and (2.6), we have

$$A_m = \left[ \sum_{k=1}^{\infty} \left( \Delta \tau \sum_{\ell=1}^{m} \|Q_{m-\ell+1} \varepsilon_k - S(\tau_{m-\ell+1}) \varepsilon_k \|_{0,D}^2 \right) \right]^{\frac{1}{2}} \leq C \Delta \tau^{\frac{1}{2}} \left( \sum_{k=1}^{\infty} \| \varepsilon_k \|_{2}^2 \right)^{\frac{1}{2}} \leq C \Delta \tau^{\frac{1}{2}} \left( \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \right)^{\frac{1}{2}} \leq C \Delta \tau^{\frac{1}{2}} \left( \sum_{k=1}^{\infty} \frac{1}{\lambda_k^{1+\frac{\delta}{2}}} \right)^{\frac{1}{2}} \leq C \left( \frac{3}{8} - \frac{\delta}{2} \right)^{-\frac{1}{2}} \Delta \tau^{\frac{1}{2}},$$

which, after setting $\epsilon = \frac{3}{8} - \frac{\delta}{2} \in (0, \frac{3}{8}]$, yields

$$(4.47) \quad A_m \leq C \epsilon^{-\frac{1}{2}} \Delta \tau^{\frac{1}{2}}.$$ 

Now, using the definition of the Hilbert-Schmidt norm and (4.14), we bound $B_m$ as follows:

$$B_m = \left[ \sum_{k=1}^{\infty} \left( \Delta \tau \sum_{\ell=1}^{m} \int_{\Delta \tau} S(\tau_{m+1-\ell}) \varepsilon_k - S(\tau_{m-\tau}) \varepsilon_k \|_{0,D}^2 \right) d\tau \right]^{\frac{1}{2}} \leq \left[ \sum_{k=1}^{\infty} \left( \Delta \tau \sum_{\ell=1}^{m} \int_{\Delta \tau} \left[ e^{-\lambda_k \Delta \tau} - e^{-\lambda_k (\tau_{m-\tau})} \right]^2 \varepsilon_k^2(x) dx \right) d\tau \right]^{\frac{1}{2}} \leq \left[ \sum_{k=1}^{\infty} \left( \Delta \tau \sum_{\ell=1}^{m} \int e^{-2\lambda_k (\tau_{m-\tau})} \left( 1 - e^{-\lambda_k (\tau_{m-\tau})} \right)^2 \varepsilon_k^2(x) dx \right) d\tau \right]^{\frac{1}{2}} \leq \left[ \sum_{\ell=1}^{m} \left( 1 - e^{-\lambda_k \Delta \tau} \right)^2 \left( \int_{\tau_m}^0 e^{-2\lambda_k (\tau_{m-\tau})} d\tau \right) \right]^{\frac{1}{2}} \leq \frac{1}{\sqrt{2}} \left( \sum_{k=1}^{\infty} \frac{1-e^{-2\lambda_k \Delta \tau}}{\lambda_k^2} \right)^{\frac{1}{2}},$$

from which, applying (3.13) in [13], we obtain

$$(4.48) \quad B_m \leq C \Delta \tau^{\frac{1}{2}}.$$ 

Finally, the estimate (4.43) follows easily combining (4.46), (4.47) and (4.48).  

5. Fully-Discrete Approximations

5.1. The deterministic problem. In this section we construct and analyze finite element approximations, $(W_h^m)_{m=0}^M$, of the modified Crank-Nicolson time-discrete approximations defined in Section 4.1.  

Let $p = 2$ or 3. We begin, by setting

$$(5.1) \quad W_h^0 := P_h w_0$$
and then by finding $W^1_h \in S_h^0$ such that

$$W^1_h - W^0_h + \Delta \tau^2 B_h W^1_h = 0.$$  

(5.2)

Finally, for $m = 2, \ldots, M$, we specify $W^m_h \in S_h^0$ such that

$$W^m_h - W^{m-1}_h + \Delta \tau B_h W^{m-\frac{1}{2}}_h = 0.$$  

(5.3)

First, we show a discrete in time $L^2_t(L^2_x)$ a priori estimate of time averages of the nodal error between the modified Crank-Nicolson time-discrete approximations and the modified Crank-Nicolson fully-discrete approximations defined above.

**Proposition 5.1.** Let $p = 2$ or 3, $w$ be the solution of the problem (1.2). $(W^m_h)^M_{m=0}$ be the Crank-Nicolson time-discrete approximations of $w$ defined by (4.1) – (4.3), and $(W^m_h)^M_{m=0}$ be the modified Crank-Nicolson fully-discrete approximations of $w$ specified by (5.1) – (5.3). Then, there exists a constant $C > 0$, independent of $h$ and $\Delta \tau$, such that

$$\Delta \tau \|W^1 - W^1_h\|_{0,D}^2 + \Delta \tau \sum_{m=2}^M \|W^{m-\frac{1}{2}} - W^{m-\frac{1}{2}}_h\|_{0,D}^2 \leq C h^p \|w_0\|_{H^{3p-2}}$$

for all $\theta \in [0, 1]$ and $w_0 \in \tilde{H}^2(D)$.

**Proof.** We will get the error estimate (5.4) by interpolation after proving it for $\theta = 1$ and $\theta = 0$ (cf. [13]). In the sequel, we will use the symbol $C$ to denote a generic constant that is independent of $\Delta \tau$ and may change values from one line to the other.

- **Case $\theta = 1$:** Letting $\Theta^\ell := W^\ell - W^\ell_h$ for $\ell = 0, \ldots, M$, we use (4.2), (5.2), (4.3) and (5.3), to arrive at the following error equations:

$$T_{B,h}(\Theta^1 - \Theta^0) + \frac{\Delta \tau}{2} \Theta^1 = \frac{\Delta \tau}{2} \xi_1$$

(5.5)

and

$$T_{B,h}(\Theta^m - \Theta^{m-1}) + \Delta \tau \Theta^{m-\frac{1}{2}} = \Delta \tau \xi_m, \quad m = 2, \ldots, M,$$

(5.6)

where

$$\xi_1 := (T_B - T_{B,h})\partial_x^4 W^1$$

(5.7)

and

$$\xi_\ell := (T_B - T_{B,h})\partial_x^4 W^{\ell-\frac{1}{2}}, \quad \ell = 2, \ldots, M.$$  

(5.8)

Taking the $L^2(D)$–inner product of both sides of (5.6) with $\Theta^{m-\frac{1}{2}}$ and then using (2.21), the Cauchy-Schwarz inequality along with the arithmetic mean inequality, we obtain

$$\|\partial_x^2(T_{B,h}(\Theta^m))\|_{0,D}^2 - \|\partial_x^2(T_{B,h}(\Theta^{m-1}))\|_{0,D}^2 + \Delta \tau \|\Theta^{m-\frac{1}{2}}\|_{0,D}^2 \leq \Delta \tau \|\xi_m\|_{0,D}^2, \quad m = 2, \ldots, M.$$  

After summing with respect to $m$ from 2 up to $M$, the relation above yields

$$\Delta \tau \sum_{m=2}^M \|\Theta^{m-\frac{1}{2}}\|_{0,D}^2 \leq \Delta \tau \sum_{m=2}^M \|\xi_m\|_{0,D}^2 + \|\partial_x^2(T_{B,h}(\Theta^1))\|_{0,D}^2,$$

which, easily, yields that

$$\Delta \tau \|\Theta^1\|_{0,D}^2 + \Delta \tau \sum_{m=2}^M \|\Theta^{m-\frac{1}{2}}\|_{0,D}^2 \leq \|\partial_x^2(T_{B,h}(\Theta^1))\|_{0,D}^2 + \Delta \tau \|\Theta^1\|_{0,D}^2 + \Delta \tau \sum_{m=2}^M \|\xi_m\|_{0,D}^2.$$  

(5.9)

Observing that $T_{B,h}(\Theta^0) = 0$, we take the $L^2(D)$–inner product of both sides of (5.5) with $\Theta^1$ and then use the Cauchy-Schwarz inequality along with the arithmetic mean inequality, to get

$$\|\partial_x^2(T_{B,h}(\Theta^1))\|_{0,D}^2 + \Delta \tau \|\Theta^1\|_{0,D}^2 \leq \frac{\Delta \tau}{18} \|\Theta^1\|_{0,D}^2.$$  

(5.10)
Thus, using (5.9), (5.10), (5.7), (5.8) and (2.22), we easily conclude that
\begin{equation}
\Delta \tau \| \Theta^1 \|^2_{0,D} + \Delta \tau \sum_{m=2}^{M} \| \Theta^{m-\frac{1}{2}} \|^2_{0,D} \leq \Delta \tau \| \xi^1 \|^2_{0,D} + \Delta \tau \sum_{m=2}^{M} \| \xi^m \|^2_{0,D}
\end{equation}
(5.11)
\[ \leq C h^{2\rho} \left( \Delta \tau \| \partial_x^3 W^1 \|^2_{0,D} + \Delta \tau \sum_{m=2}^{M} \| \partial_x^3 W^{m-\frac{1}{2}} \|^2_{0,D} \right). \]
Taking the $L^2(D)$–inner product of (4.3) with $\partial_x^3 W^{m-\frac{1}{2}}$, and then integrating by parts and summing with respect to $m$, from 2 up to $M$, it follows that
\[ \| \partial_x W^m \|^2_{0,D} - \| \partial_x W^1 \|^2_{0,D} + 2 \Delta \tau \sum_{m=2}^{M} \| \partial_x^3 W^{m-\frac{1}{2}} \|^2_{0,D} = 0 \]
which yields
\begin{equation}
\Delta \tau \| \partial_x^3 W^1 \|^2_{0,D} + \sum_{m=2}^{M} \Delta \tau \| \partial_x^3 W^{m-\frac{1}{2}} \|^2_{0,D} \leq \frac{1}{\tau} \| \partial_x W^1 \|^2_{0,D} + \Delta \tau \| \partial_x^3 W^1 \|^2_{0,D}.
\end{equation}
(5.12)
Now, take the $L^2(D)$–inner product of (4.2) with $\partial_x^3 W^1$, and then integrate by parts and use (2.7) to get
\[ \| \partial_x W^1 \|^2_{0,D} - \| \partial_x W^0 \|^2_{0,D} + \Delta \tau \| \partial_x^3 W^1 \|^2_{0,D} \leq 0, \]
which, along with (2.2), yields
\begin{equation}
\| \partial_x W^1 \|^2_{0,D} + \Delta \tau \| \partial_x^3 W^1 \|^2_{0,D} \leq \| w_0 \|^2_{\tilde{H}^1}. \end{equation}
(5.13)
Thus, combining (5.11), (5.12) and (5.13), we obtain (5.4) for $\theta = 1$.

- **Case $\theta = 0$:** From (5.2) and (5.3), it follows that
\begin{equation}
T_{b,h}(W^1_h - W^0_h) + \Delta \tau W^1_h = 0
\end{equation}
(5.14)
and
\begin{equation}
T_{b,h}(W^m_h - W^{m-1}_h) + \Delta \tau W^{m-\frac{1}{2}}_h = 0, \quad m = 2, \ldots, M.
\end{equation}
(5.15)
Taking the $L^2(D)$–inner product of (5.14) with $W^{\frac{1}{2}}_h$ and using (2.21), we have
\[ \| \partial_x^2(T_{b,h} W^1_h) \|^2_{0,D} - \| \partial_x^2(T_{b,h} W^{m-1}_h) \|^2_{0,D} + 2 \Delta \tau \| W^{m-\frac{1}{2}}_h \|^2_{0,D} = 0, \quad m = 2, \ldots, M, \]
which, after summing with respect to $m$ from 2 up to $M$, yields
\begin{equation}
\Delta \tau \| W^1_h \|^2_{0,D} + \sum_{m=2}^{M} \Delta \tau \| W^{m-\frac{1}{2}}_h \|^2_{0,D} \leq \frac{1}{\tau} \| \partial_x^2(T_{b,h} W^1_h) \|^2_{0,D} + \Delta \tau \| W^1_h \|^2_{0,D}.
\end{equation}
(5.16)
Now, take the $L^2(D)$–inner product of (5.14) with $W^1_h$ and use (2.7) and (5.1), to have
\[ \| \partial_x^2(T_{b,h} W^1_h) \|^2_{0,D} + \Delta \tau \| W^1_h \|^2_{0,D} \leq \| \partial_x^2(T_{b,h} P_h w_0) \|^2_{0,D}
\]
\[ \leq \| \partial_x^2(T_{b,h} w_0) \|^2_{0,D}. \]
(5.17)
Combining (5.16), (5.17), (2.23) and (2.3), we obtain
\begin{equation}
\left( \Delta \tau \| W^1_h \|^2_{0,D} + \sum_{m=2}^{M} \Delta \tau \| W^{m-\frac{1}{2}}_h \|^2_{0,D} \right)^{\frac{1}{2}} \leq C \| w_0 \|_{\tilde{H}^2}.
\end{equation}
(5.18)
Finally, combine (5.18) with (4.24) to get (5.4) for $\theta = 0$. \qed

Next, we derive a discrete in time $L^2(L^2)$ a priori estimate of the nodal error between the modified Crank-Nicolson time-discrete approximations and the modified Crank-Nicolson fully-discrete approximations.
Proposition 5.2. Let $p = 2$ or $3$, $w$ be the solution of the problem (1.2), $(W^m)^m_{m=0}$ be the modified Crank-Nicolson time-discrete approximations of $w$ defined by (4.1)-(4.3), and $(W^m_m)^m_{m=0}$ be the modified Crank-Nicolson finite element approximations of $w$ specified by (5.1)-(5.3). Then, there exists a constant $C > 0$, independent of $h$ and $\Delta t$, such that

$$
(5.19) \quad \left(\Delta t \sum_{m=1}^M \|W^m - W^m_h\|^2_{0,D}\right)^\frac{1}{2} \leq C \left[\Delta t^\frac{p}{2} \|w_0\|_{R^{2(p-1)}} + h^p \|w_0\|_{R^{2p-2}}\right]
$$

for all $\delta, \theta \in [0, 1]$ and $w_0 \in \tilde{H}^2(D)$.

Proof. The proof is based on the estimation of the terms in the right hand side of the following triangle inequality:

$$
(5.20) \quad \left(\Delta t \sum_{m=1}^M \|W^m - W^m_h\|^2_{0,D}\right)^\frac{1}{2} \leq S_B + S_C + S_D
$$

where

$$
S_B := \left(\Delta t \sum_{m=2}^M \|W^m - W^{m-\frac{1}{2}}\|^2_{0,D}\right)^\frac{1}{2},
$$

$$
S_C := \left(\Delta t \|W^1 - W^1_h\|^2_{0,D} + \Delta t \sum_{m=2}^M \|W^{m-\frac{1}{2}} - W^{m-\frac{1}{2}}_h\|^2_{0,D}\right)^\frac{1}{2},
$$

$$
S_D := \left(\Delta t \sum_{m=2}^M \|W^{m-\frac{1}{2}}_h - W^{m-\frac{1}{2}}_h\|^2_{0,D}\right)^\frac{1}{2}.
$$

In the sequel, we will use the symbol $C$ to denote a generic constant that is independent of $\Delta t$ and may change values from one line to the other.

Taking the $L^2(D)$–inner product of both sides of (5.3) with $(W^m - W^{m-1})$, we have

$$
(5.21) \quad \|W^m - W^{m-1}_h\|^2_{0,D} + \Delta t \left(\|\partial_x^2 W^m_h\|^2_{0,D} - \|\partial_x^2 W^{m-1}_h\|^2_{0,D}\right) = 0, \quad m = 2, \ldots, M.
$$

After summing both sides of (5.21) with respect to $m$ from 2 up to $M$, we obtain

$$
\Delta t \sum_{m=2}^M \|W^m_h - W^{m-1}_h\|^2_{0,D} + \Delta \frac{t}{2} \left(\|\partial_x^2 W^m_h\|^2_{0,D} - \|\partial_x^2 W^{m-1}_h\|^2_{0,D}\right) = 0,
$$

which yields

$$
(5.22) \quad \Delta t \sum_{m=2}^M \|W^m_h - W^{m-1}_h\|^2_{0,D} \leq \Delta \frac{t}{2} \|\partial_x^2 W^1_h\|^2_{0,D}.
$$

Taking the $L^2(D)$–inner product of both sides of (5.2) with $W^1_h$ and then using (2.7), we obtain

$$
\|W^1_h\|^2_{0,D} - \|W^0_h\|^2_{0,D} + \Delta t \|\partial_x^2 W^1_h\|^2_{0,D} \leq 0
$$

from which we conclude that

$$
(5.23) \quad \Delta t \|\partial_x^2 W^1_h\|^2_{0,D} \leq \|w_0\|^2_{0,D}.
$$

Thus, combining (5.22) and (5.23) we have

$$
S_D = \frac{1}{2} \left(\Delta t \sum_{m=2}^M \|W^m_h - W^{m-1}_h\|^2_{0,D}\right)^\frac{1}{2}
$$

$$
(5.24) \quad \leq \frac{1}{2\sqrt{2}} \Delta t \|\partial_x^2 W^1_h\|_{0,D}
$$

$$
\leq \Delta t^\frac{p}{2} \|w_0\|_{0,D}.
$$
Taking again the $L^2(D)$–inner product of both sides of (5.2) with $W_h^1$ and then using (2.3) and (2.2) along with the arithmetic mean inequality, we obtain

$$\|W_h^1\|_{0,D}^2 + \frac{\Delta \tau}{2} \|\partial_x^2 W_h^1\|_{0,D}^2 = (w_0, W_h^1)_{0,D}$$

(5.25)

$$\leq \|w_0\|_{-2,D} \|W_h^1\|_{2,D}$$

$$\leq C \|w_0\|_{H^{-2}} \|W_h^1\|_{H^2}$$

$$\leq C \Delta \tau^{-1} \|w_0\|_{H^{-2}}^2 + \frac{\Delta \tau}{2} \|W_h^1\|_{H^2}^2.$$  

Observing that $\|\partial_x^2 W_h^1\|_{0,D} = \|W_h^1\|_{\dot{H}^2}$ (cf. (4.36)), (5.25) yields that

$$\Delta \tau^2 \|\partial_x^2 W_h^1\|_{0,D}^2 \leq C \|w_0\|_{H^{-2}}^2.$$  

(5.26)

Thus, combining (5.22) and (5.26), we conclude that

$$S_D = \frac{1}{2} \left( \Delta \tau \sum_{m=2}^M \|W_h^m - W_h^{m-1}\|_{0,D}^2 \right)^{\frac{1}{2}}$$

(5.27)

$$\leq \frac{1}{\sqrt{2}} \Delta \tau \|\partial_x^2 W_h^1\|_{0,D}$$

$$\leq C \|w_0\|_{H^{-2}}.$$  

Also, from (4.32) and (4.38), we have

$$S_B \leq C \Delta \tau^\delta \|w_0\|_{0,D}$$

(5.28)

and

$$S_B \leq C \|w_0\|_{H^{-2}}.$$  

(5.29)

By interpolation, from (5.24), (5.28), (5.27) and (5.29), we conclude that

$$S_B + S_D \leq C \Delta \tau^\delta \|w_0\|_{H^{2(\delta-1)}} \forall \delta \in [0,1].$$

(5.30)

Finally, the estimate (5.4) reads

$$S_C \leq C h^p \theta \|w_0\|_{H^{3p-2}} \forall \theta \in [0,1].$$

(5.31)

Thus, (5.19) follows as a simple consequence of (5.20), (5.30) and (5.31). \hfill \Box

5.2. The stochastic problem. The following lemma ensures the existence of a continuous Green function for some discrete operators (cf. Lemma 5.2 in [13], Lemma 5.1 in [15]).

**Lemma 5.1.** Let $p = 2$ or $3$, $f \in L^2(D)$ and $g_{h}, \psi_h, z_{h} \in S_h^p$ such that

$$\psi_h + \frac{\Delta \tau}{2} B_h \psi_h = P_h f$$

(5.32)

and

$$z_{h} = g_{h} - \frac{\Delta \tau}{2} B_h g_{h}.$$  

(5.33)

Then there exist functions $G_{h}, \bar{G}_h \in C(D \times D)$ such that

$$\psi_h(x) = \int_D G_h(x,y) f(y) \, dy \quad \forall x \in \overline{D}$$

(5.34)

and

$$z_h(x) = \int_D \bar{G}_h(x,y) g_h(y) \, dy \quad \forall x \in \overline{D}.$$  

(5.35)
Proof. Let \( \nu_h := \dim(S_h^p) \) and \( \gamma : S_h^p \times S_h^p \to \mathbb{R} \) be an inner product on \( S_h^p \) defined by \( \gamma(\chi, \varphi) := B(\chi, \varphi)_{\alpha, \beta} \) for \( \chi, \varphi \in S_h^p \). Then, we can construct a basis \( (\varphi_j)^{\nu_h}_{j=1} \) of \( S_h^p \) which is \( L^2(D) \)–orthonormal, i.e., \( (\varphi_i, \varphi_j)_{\alpha, \beta} = \delta_{ij} \) for \( i, j = 1, \ldots, \nu_h \), and \( \gamma \)-orthogonal, i.e. there exist positive \( (\varepsilon_{h,j})^{\nu_h}_{j=1} \) such that \( \gamma(\varphi_i, \varphi_j) = \varepsilon_{h,i} \delta_{ij} \) for \( i, j = 1, \ldots, \nu_h \) (see Sect. 8.7 in [8]). Thus, there exist real numbers \( (\mu_j)^{\nu_h}_{j=1} \) and \( (\bar{\mu}_j)^{\nu_h}_{j=1} \) such that \( \psi_h = \sum_{j=1}^{\nu_h} \mu_j \varphi_j \) and \( \bar{\psi}_h = \sum_{j=1}^{\nu_h} \bar{\mu}_j \varphi_j \). Then, (5.32) and (5.33) yield

\[
\mu_\ell = \frac{1}{1+\varepsilon_{h,\ell}} (f, \varphi_\ell)_{\alpha, \beta}, \quad \bar{\mu}_\ell = (1 - 2\varepsilon_{h,\ell}) (g_h, \varphi_\ell)_{\alpha, \beta}
\]
for \( \ell = 1, \ldots, \nu_h \). Using (5.30) we conclude (5.34) and (5.35) with \( G_h(x, y) = \sum_{j=1}^{\nu_h} \frac{1}{1+\varepsilon_{h,j}} \varphi_j(x) \varphi_j(y) \) and \( \bar{G}_h(x, y) = \sum_{j=1}^{\nu_h} (1 - 2\varepsilon_{h,j}) \varphi_j(x) \varphi_j(y) \) for \( x, y \in \overline{D} \). \( \square \)

Now, we are ready to derive a discrete in time \( L^\infty(T^h(L^2_h(L^2_h))) \) a priori estimate of the nodal error between the Crank-Nicolson time-discrete approximations of \( \bar{u} \) and the Crank-Nicolson fully-discrete approximations of \( \bar{u} \).

**Theorem 5.3.** Let \( p = 2 \) or \( 3 \), \( \bar{u} \) be the solution of the problem (1.6), \( (U^m_T)^{m=0} \) be the Crank-Nicolson fully-discrete approximations of \( \bar{u} \) defined by (1.10) and (1.11), and \( (U^m)_{m=0} \) be the Crank-Nicolson time-discrete approximations of \( \bar{u} \) defined by (4.11) and (4.12). Then, there exists a constant \( C > 0 \), independent of \( M, \Delta t, h, \) or \( \Delta \tau \), such that

\[
\max_{1 \leq m \leq M} \left( \mathbb{E} \left[ \|U^m_T - U^m\|^2_{0, D} \right] \right)^{\frac{1}{2}} \leq C \left( \epsilon_1 h^{-\epsilon_1} + \epsilon_2 \Delta \tau^{\frac{\epsilon_2}{2}} \right) \quad \forall \epsilon_1 \in (0, \frac{1}{2}], \quad \forall \epsilon_2 \in (0, \frac{3}{8}] .
\]

**Proof.** Let \( L : L^2(D) \to L^2(D) \) be the identity operator, \( Y_h : S_h^p \to S_h^p \) be defined by \( Y_h := 1 - \frac{\Delta \tau}{2} B_h \) and \( \Lambda_h : L^2(D) \to S_h^p \) be the inverse discrete elliptic operator given by \( \Lambda_h := (1 + \frac{\Delta \tau}{2} B_h)^{-1} P_h \). Also, for \( m = 1, \ldots, M \), we define a discrete operator \( Q_{h,m} : L^2(D) \to S_h^p \) by \( Q_{h,m} := (\Lambda_h Y_h)^{m-1} \Lambda_h \), which has a Green function \( G_{Q_{h,m}} \) (cf. Lemma 5.1). Using, now, an induction argument, from (1.10) we conclude that

\[
U^m_T = \sum_{\ell=1}^{m} \int_{\Delta \tau/2}^{T} Q_{h,m-\ell+1}(\bar{W}(\tau, \cdot)) d\tau, \quad m = 1, \ldots, M,
\]
which is written, equivalently, as follows:

\[
U^m_T(x) = \int_0^T \int_D K_{h,m}(\tau; x, y) \bar{W}(\tau, y) dyd\tau \quad \forall x \in \overline{D}, \quad m = 1, \ldots, M,
\]
where

\[
K_{h,m}(\tau; x, y) := \sum_{\ell=1}^{m} \chi_{\Delta \tau/2}^{(\tau)} G_{Q_{h,m-\ell+1}}(x, y) \quad \forall \tau \in [0, T], \quad \forall x, y \in \overline{D}.
\]

Using (5.30), (1.10), (2.10), (2.24), (2.24) and (2.8), we get

\[
\left( \mathbb{E} \left[ \|U^m - U^m\|^2_{0, D} \right] \right)^{\frac{1}{2}} \leq \left( \int_0^T \int_D \left[ K_m(\tau; x, y) - K_{h,m}(\tau; x, y) \right] dyd\tau \right)^{\frac{1}{2}} \leq \left( \Delta \tau \sum_{\ell=1}^{m} \|Q_{h,m-\ell+1} - Q_{h,m-\ell+1}\|^2_{H^2} \right)^{\frac{1}{2}}, \quad m = 1, \ldots, M.
\]
Let $\delta \in [0, \frac{3}{2})$ and $\theta \in [0, \frac{1}{2})$. Using the definition of the Hilbert-Schmidt norm and the deterministic error estimate (5.19), we obtain

\[
(\mathbb{E} \left[ \|U^m - U^m_h\|_{0,D}^2 \right])^{\frac{1}{2}} \leq \left( \sum_{k=1}^{\infty} \left[ \Delta \tau \sum_{\ell=1}^{m} \|Q_{m-\ell+1} \varepsilon_k - Q_{h,m-\ell+1} \varepsilon_k\|_{0,D}^2 \right] \right)^{\frac{1}{2}} 
\]

\[
\leq \left( \sum_{k=1}^{\infty} \left[ \Delta \tau \sum_{\ell=1}^{m} \|Q_k \varepsilon_k - Q_{h,k} \varepsilon_k\|_{0,D}^2 \right] \right)^{\frac{1}{2}} 
\]

\[
\leq C \left[ \frac{h^{2p\theta}}{\sum_{k=1}^{\infty} \frac{1}{\lambda_k^2(2-\theta)}} \right]^{\frac{1}{2}} + \Delta \tau^\frac{1}{2} \left( \sum_{k=1}^{\infty} \frac{1}{\lambda_k^{4(\delta-\theta)}} \right)^{\frac{1}{2}} 
\]

\[
\leq C \left[ h^{2p\theta} \left( \sum_{k=1}^{\infty} \frac{1}{\lambda_k^{2(2-\theta)}} \right) + \Delta \tau^\frac{1}{2} \left( \sum_{k=1}^{\infty} \frac{1}{\lambda_k^{4(\delta-\theta)}} \right) \right]^{\frac{1}{2}}, \quad m = 1, \ldots, M, 
\]

which, along with (5.39), yields

\[
(5.39) \max_{1 \leq m \leq M} \left( \mathbb{E} \left[ \|U^m - U^m_h\|_{0,D}^2 \right] \right)^{\frac{1}{2}} \leq C \left[ h^{2p\theta} \left( \frac{1}{2} - \theta \right) - \frac{1}{2} + \Delta \tau^\frac{1}{2} \left( \frac{3}{8} - \frac{3}{2} \right) \right]. 
\]

Setting $\epsilon_1 = p \left( \frac{1}{2} - \theta \right) \in (0, \frac{2}{3})$ and $\epsilon_2 = \frac{3}{8} - \frac{3}{2} \in (0, \frac{1}{8}]$, we, easily, arrive at (5.37). \hfill \Box

Now, we are able to formulate a discrete in time $L^\infty_t(L^2_x(L^2_x))$ error estimate for the Crank-Nicolson fully-discrete approximations of $u$.

**Theorem 5.4.** Let $p = 2$ or $3$, $\tilde{u}$ be the solution of problem (1.6), and $(U^m_h)_{m=0}^M$ be the Crank-Nicolson fully-discrete approximations of $u$ constructed by (1.9), (1.10). Then, there exists a constant $C > 0$, independent of $M_*$, $\Delta t$, $h$ and $\Delta \tau$, such that

\[
(5.40) \max_{0 \leq m \leq M} \left( \mathbb{E} \left[ \|U^m_h - \tilde{u}(\tau_m, \cdot)\|_{0,D}^2 \right] \right)^{\frac{1}{2}} \leq C \left( \epsilon_1^\frac{1}{2} h^{\frac{3}{2} - \epsilon_1} + \epsilon_2^\frac{1}{2} \Delta \tau^{\frac{1}{2} - \epsilon_2} \right), \quad \forall \epsilon_1 \in (0, \frac{2}{3}) \quad \forall \epsilon_2 \in (0, \frac{1}{8}].
\]

**Proof.** The estimate is a simple consequence of the error bounds (5.37) and (4.43). \hfill \Box

**Remark 5.1.** For an optimal, logarithmic-type choice of the parameter $\delta$ in (5.31) and of the parameters $\epsilon_1$ and $\epsilon_2$ in (5.40), we refer the reader to the discussion in Remark 3 of [14].

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