Coefficient Problems in the Subclasses of Close-to-Star Functions

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Abstract. For two subclasses of close-to-star functions we estimate early logarithmic coefficients, coefficients of inverse functions, Hankel determinant $H_{2,2}$ and Zalcman functional $J_{2,3}$. All results are sharp.

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1. Introduction

Given $r > 0$, let $D_r := \{ z \in \mathbb{C} : |z| < r \}$, and let $D := D_1$. Let $\overline{D} := \{ z \in \mathbb{C} : |z| \leq 1 \}$ and $\Gamma := \partial \mathcal{D}$. Let $\mathcal{H}$ be the class of all analytic functions in $\mathcal{D}$ and $\mathcal{A}$ be its subclass of $f$ normalized by $f(0) := 0$ and $f'(0) := 1$, i.e., of the form

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_1 := 1, \quad z \in \mathcal{D}. \quad (1.1)$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ of all univalent functions and $\mathcal{S}^*$ be the subclass of $\mathcal{S}$ of all starlike functions, namely, $f \in \mathcal{S}^*$ if $f \in \mathcal{A}$ and

$$\Re \frac{zf'(z)}{f(z)} > 0, \quad z \in \mathcal{D}.$$ 

A function $f \in \mathcal{A}$ is called close-to-star if there exist $g \in \mathcal{S}^*$ and $\beta \in \mathbb{R}$ such that

$$\Re \frac{e^{i\beta} f(z)}{g(z)} > 0, \quad z \in \mathcal{D}. \quad (1.2)$$
Denote by $\mathcal{CST}$ the class of all close-to-star functions introduced by Reade [30]. Note that $f \in \mathcal{CST}$ if and only if a function

$$F(z) := \int_0^z \frac{f(t)}{t} dt, \quad z \in \mathbb{D}, \quad (1.3)$$

is close-to-convex [15], [12, Vol. II, p. 3]. The class of close-to-star functions and its subclasses were intensively studied by various authors (e.g., MacGregor [25], Sakaguchi [32], Causey and Merkes [4]; for further references, see [12, Vol. II, pp. 97–104]). Given $g \in S^*$ and $\beta \in \mathbb{R}$, let $\mathcal{CST}_\beta(g)$ be the subclass of $\mathcal{CST}$ of all $f$ satisfying (1.2). The classes $\mathcal{CST}_0(g_i), i = 1, 2, 3$, where

$$g_1(z) := \frac{z}{1 - z^2}, \quad g_2(z) := \frac{z}{(1 - z)^2}, \quad g_3(z) := z, \quad z \in \mathbb{D},$$

are particularly interesting and were separately studied by authors. In this paper we deal with the classes $\mathcal{CST}_0(g_1) =: \mathcal{ST}(i)$ and $\mathcal{CST}_0(g_2) =: \mathcal{ST}(1)$ which elements $f$ in view of (1.2) satisfy the condition

$$\text{Re}\left\{(1 - z^2) \frac{f(z)}{z}\right\} > 0, \quad z \in \mathbb{D}, \quad (1.4)$$

and

$$\text{Re}\left\{(1 - z)^2 \frac{f(z)}{z}\right\} > 0, \quad z \in \mathbb{D}, \quad (1.5)$$

respectively. Let us add the inequality (1.4) defines the subclass of the class of functions starlike in the direction of the real axis introduced by Robertson [31]. Moreover, each function $F$ given by (1.3) over the class $\mathcal{ST}(i)$ maps univalently $\mathbb{D}$ onto a domain $F(\mathbb{D})$ convex in the direction of the imaginary axis. The concept of convexity in one direction belongs to Roberston [31] (see e.g., [12, p. 199]). Each function $F$ given by (1.3) over the class $\mathcal{ST}(1)$ maps univalently $\mathbb{D}$ onto a domain $F(\mathbb{D})$ called convex in the positive direction of the real axis, i.e., \{w + it: t \geq 0\} $\subset$ $f(\mathbb{D})$ for every $w \in f(\mathbb{D})$ [2,8,9,11,20,21]. Let us remark that the condition (1.4) was generalized by replacing the expression $1 - z^2$ by the expression $1 - \alpha^2 z^2$ with $\alpha \in [0, 1]$ in [13].

In this paper we find the sharp estimates of early logarithmic coefficients (Sect. 2), of the Hankel determinant $H_{2,2}$ and of Zalcman functional $J_{2,3}$ (Sect. 3) and of the early inverse coefficients (Sect. 4) of functions in the classes $\mathcal{ST}(i)$ and $\mathcal{ST}(1)$. Since both classes $\mathcal{ST}(i)$ and $\mathcal{ST}(1)$ have a representation using the Carathéodory class $\mathcal{P}$, i.e., the class of functions $p \in \mathcal{H}$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D}, \quad (1.6)$$

having a positive real part in $\mathbb{D}$, the coefficients of functions in $\mathcal{ST}(i)$ and $\mathcal{ST}(1)$ have a suitable representation expressed by the coefficients of functions in $\mathcal{P}$. Therefore to get the upper bounds of considered functionals our computing is based on parametric formulas for the second and third coefficients.
in \( \mathcal{P} \). However both classes are rotation non-invariant. Thus to solve discussed problems we will apply a general formula for \( c_3 \) recently found in \([7]\). The formula (1.7) was proved by Carathéodory [3] (see e.g., [10, p. 41]). The formula (1.8) can be found in [28, p. 166]. The formula (1.9) was shown in a recent paper [7], where the extremal functions (1.11) and (1.12) were computed also. For \( c_1 \geq 0 \) the formula (1.9) is due to by Libera and Zlotkiewicz [22,23].

**Lemma 1.1.** If \( p \in \mathcal{P} \) is of the form (1.6), then

\[
\begin{align*}
  c_1 &= 2\zeta_1, \\
  c_2 &= 2\zeta_1^2 + 2(1 - |\zeta_1|^2)\zeta_2
\end{align*}
\]  

(1.7) 

and

\[
\begin{align*}
  c_3 &= 2\zeta_1^3 + 4(1 - |\zeta_1|^2)\zeta_1\zeta_2 \\
  &\quad - 2(1 - |\zeta_1|^2)\zeta_1\zeta_2^2 + 2(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)\zeta_3
\end{align*}
\]  

(1.9) 

for some \( \zeta_i \in \mathbb{D} \), \( i \in \{1,2,3\} \).

For \( \zeta_1 \in \mathbb{T} \), there is a unique function \( p \in \mathcal{P} \) with \( c_1 \) as in (1.7), namely,

\[
p(z) = \frac{1 + \zeta_1 z}{1 - \zeta_1 z}, \quad z \in \mathbb{D}.
\]  

(1.10)

For \( \zeta_1 \in \mathbb{D} \) and \( \zeta_2 \in \mathbb{T} \), there is a unique function \( p \in \mathcal{P} \) with \( c_1 \) and \( c_2 \) as in (1.7)–(1.8), namely,

\[
p(z) = \frac{1 + (\zeta_1\zeta_2 + \zeta_1) z + \zeta_2 z^2}{1 + (\zeta_1\zeta_2 - \zeta_1) z - \zeta_2 z^2}, \quad z \in \mathbb{D}.
\]  

(1.11)

For \( \zeta_1, \zeta_2 \in \mathbb{D} \) and \( \zeta_3 \in \mathbb{T} \), there is a unique function \( p \in \mathcal{P} \) with \( c_1, c_2 \) and \( c_3 \) as in (1.7)–(1.9), namely,

\[
p(z) = \frac{1 + (\zeta_2\zeta_3 + \zeta_1\zeta_2 + \zeta_1) z + (\zeta_1\zeta_3 + \zeta_1\zeta_2\zeta_3 + \zeta_2) z^2 + \zeta_3 z^3}{1 + (\zeta_2\zeta_3 + \zeta_1\zeta_2 - \zeta_1) z + (\zeta_1\zeta_3 - \zeta_1\zeta_2\zeta_3 - \zeta_2) z^2 - \zeta_3 z^3}, \quad z \in \mathbb{D}.
\]  

(1.12)

2. **Logarithmic Coefficients**

Given \( f \in \mathcal{S} \) let

\[
\log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n z^n, \quad z \in \mathbb{D}\setminus\{0\}, \quad \log 1 := 0.
\]  

(2.1)
The numbers $\gamma_n$ are called logarithmic coefficients of $f$. Differentiating (2.1) and using (1.1) we get

$$
\begin{align*}
\gamma_1 &= \frac{1}{2}a_2, \\
\gamma_2 &= \frac{1}{2} \left(a_3 - \frac{1}{2}a_2^2\right), \\
\gamma_3 &= \frac{1}{2} \left(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right).
\end{align*}
$$

(2.2)

As it well known, the logarithmic coefficients play a crucial role in Milin conjecture ([26], see also [10, p. 155]). It is surprising that for the class $\mathcal{S}$ the sharp estimates of single logarithmic coefficients $\mathcal{S}$ are known only for $\gamma_1$ and $\gamma_2$, namely,

$$
|\gamma_1| \leq 1, \quad |\gamma_2| \leq \frac{1}{2} + \frac{1}{e} = 0.635\ldots
$$

and are unknown for $n \geq 3$. Logarithmic coefficients is one of the topic recently being of interest by various authors (e.g., [1,18,33]).

Logarithmic coefficients can be considered for functions $f$ from the class $\mathcal{A}$ however under the assumption that the branch of logarithm $\mathbb{D} \ni z \mapsto \log f(z)/z$ exists. From (1.4) and (1.5) it follows that $g(z) := f(z)/z \neq 0$ in $\mathbb{D} \setminus \{0\}$ for $f \in \mathcal{S}T(i)$ and $f \in \mathcal{S}T(1)$. However $g(\mathbb{D})$ needs not be necessarily a simply connected domain. Therefore, let $\mathcal{S}T_0(i)$ and $\mathcal{S}T_0(1)$ be the subclasses of $\mathcal{S}T(i)$ and $\mathcal{S}T(1)$ respectively, of all functions $f$ for which the branch $\mathbb{D} \ni z \mapsto \log f(z)/z$ with $\log 1 := 0$ exists.

**Theorem 2.1.** If $f \in \mathcal{S}T_0(i)$ is of the form (1.1), then

$$
|\gamma_1| \leq 1, \quad |\gamma_2| \leq \frac{3}{2}, \quad |\gamma_3| \leq 1.
$$

All inequalities are sharp.

**Proof.** By (1.4) there exists $p \in \mathcal{P}$ of the form (1.6) such that

$$
(1 - z^2) \frac{f(z)}{z} = p(z).
$$

(2.3)

Substituting the series (1.1) and (1.6) into (2.3) by equating the coefficients we get

$$
a_2 = c_1, \quad a_3 = c_2 + 1, \quad a_4 = c_1 + c_3.
$$

(2.4)

The inequality $|\gamma_1| \leq 1$ follows directly from (2.2), (2.4) and (1.7) with sharpness for the function $f$ given by (2.3), where $p$ is as in (1.10).

Substituting (1.7) and (1.8) into (2.4) from (2.2) it follows that

$$
|\gamma_2| = \frac{1}{2} \left|a_3 - \frac{1}{2}a_2^2\right| = \frac{1}{2} \left|c_2 - \frac{1}{2}c_2^2 + 1\right| = \frac{1}{2} \left|2(1 - |\zeta_1|^2)\zeta_2 + 1\right| \leq \frac{1}{2} + (1 - |\zeta_1|^2)|\zeta_2| \leq \frac{3}{2}
$$
with sharpness for the function \( f \) given by (2.3), where \( p \) is as in (1.11) with \( \zeta_1 = 0 \) and any \( \zeta_2 \in \mathbb{T} \).

By (2.2) and (2.4) we have
\[
6\gamma_3 = c_1^3 - 3c_1c_2 + 3c_3.
\]
Hence and by (1.7)–(1.9) we get
\[
3\gamma_3 = 2\zeta_1^2 - 3(1 - |\zeta_1|^2)\bar{\zeta}_1\zeta_2^2 + 3(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)\zeta_3,
\]
where \( \zeta_i \in \mathbb{D}, i = 1, 2, 3 \). Thus by setting \( x := |\zeta_1| \in [0,1] \) and \( y := |\zeta_2| \in [0,1] \) we obtain
\[
3|\gamma_3| \leq 2x^3 + 3(1 - x^2)xy^2 + 3(1 - x^2)(1 - y^2) \\
= 2x^3 - 3x^2 + 3 - 3(1 - x^2)(1 - x)y^2 \\
\leq 2x^3 - 3x^2 + 3 \leq 3, \quad (x, y) \in [0,1] \times [0,1].
\]
Thus \( |\gamma_3| \leq 1 \) with sharpness for the function \( f \) given by (2.3), where \( p \) is as in (1.12) with \( \zeta_1 = \zeta_2 = 0 \) and any \( \zeta_3 \in \mathbb{T} \).

\[\square\]

**Theorem 2.2.** If \( f \in S \mathcal{T}_0(1) \) is of the form (1.1), then
\[
|\gamma_1| \leq 2, \quad |\gamma_2| \leq \frac{3}{2}, \quad |\gamma_3| \leq \frac{1}{3}(1 + \sqrt{2}).
\]
All inequalities are sharp.

**Proof.** By (1.5) there exists \( p \in \mathcal{P} \) of the form (1.6) such that
\[
(1 - z)^2 \frac{f(z)}{z} = p(z).
\]
Substituting the series (1.1) and (1.6) into (2.5) by equating the coefficients we get
\[
a_2 = c_1 + 2, \quad a_3 = 3 + 2c_1 + c_2, \quad a_4 = 4 + 3c_1 + 2c_2 + c_3.
\]
(2.6)
The inequality \( |\gamma_1| \leq 2 \) follows directly from (2.2), (2.6) and (1.7) with sharpness for the function \( f \) given by (2.5), where \( p \) is as in (1.10).

Substituting (1.7) and (1.8) into (2.6) from (2.2) it follows that
\[
|\gamma_2| = \frac{1}{2} \left| a_3 - \frac{1}{2}a_2^2 \right| = \frac{1}{2} \left| c_2 - \frac{1}{2}c_1^2 + 1 \right| \\
= \frac{1}{2} \left| 2(1 - |\zeta_1|^2)\zeta_2 + 1 \right| \leq \frac{1}{2} + (1 - |\zeta_1|^2)|\zeta_2| \leq \frac{3}{2}
\]
with sharpness for the function \( f \) given by (2.3), where \( p \) is as in (1.11) with \( \zeta_1 = 0 \) and any \( \zeta_2 \in \mathbb{T} \).

By (2.2) and (2.6) we have
\[
6\gamma_3 = 2 + c_1^3 + 3c_3 - 3c_1c_2.
\]
Hence and by (1.7)–(1.9) we get
\[
3\gamma_3 = 1 + \zeta_1^3 - 3(1 - |\zeta_1|^2)\bar{\zeta}_1\zeta_2^2 + (1 - |\zeta_1|^2)(1 - |\zeta_2|^2)\zeta_3,
\]
where \( \zeta_i \in \overline{D} \), \( i = 1, 2, 3 \). Thus by setting \( x := |\zeta_1| \in [0, 1] \) and \( y := |\zeta_2| \in [0, 1] \) we obtain

\[
3|\gamma_3| \leq 1 + x^3 - 3(1 - x^2)xy^2 + (1 - x^2)(1 - y^2) \\
= 2 - x^2 + x^3 + (1 - x^2)(3x - 1)y^2 =: F(x, y).
\]

(2.7)

We have \( F(1/3, y) = 52/27 \). Moreover for \( x \in (1/3, 1] \) and \( x \in [0, 1/3) \) we get

\[
F(x, y) \leq F(x, 1) \leq 1 + 3x - 2x^3 \leq 1 + \sqrt{2}, \quad y \in [0, 1],
\]

and

\[
F(x, y) \leq F(x, 0) \leq 2 - x^2 + x^3 \leq 2, \quad y \in [0, 1],
\]

respectively. Thus by (2.7), \( |\gamma_3| \leq (1 + \sqrt{2})/3 \) with sharpness for the function \( f \) given by (2.3), where \( p \) is as in (1.12) with \( \zeta_1 = 1/\sqrt{2}, \zeta_2 = i \) and any \( \zeta_3 \in \mathbb{T} \). □

3. Zalcman Functional and Hankel Determinant

Now we compute the sharp upper bound of the Zalcman functional \( J_{2,3}(f) := a_2a_3 - a_4 \) being a special case of the generalized Zalcman functional \( J_{n,m}(f) := a_na_m - a_{n+1,m-1}, n, m \in \mathbb{N} \setminus \{1\} \), which was investigated by Ma [24] for \( f \in S \) (see also [29] for relevant results on this functional). We will find also the sharp bound of the second Hankel determinant \( H_{2,2}(f) = a_2a_4 - a_3^2 \). Both functionals \( J_{2,3} \) and \( H_{2,2} \) have been studied recently by various authors (see e.g., [5,6,14,16,17,19,27]).

Theorem 3.1. If \( f \in ST(i) \) is of the form (1.1), then

\[
|a_2a_3 - a_4| \leq 2.
\]

The inequality is sharp with the extremal function

\[
f(z) = \frac{z}{(1 - z)^2}, \quad z \in \mathbb{D}.
\]

(3.1)

Proof. From (2.4) by using (1.7)–(1.9) it follows that

\[
|a_2a_3 - a_4| = |c_1c_2 - c_3|
\]

\[
= 2 \left| \zeta_1^3 + 2(1 - |\zeta_1|^2)\zeta_1\zeta_2^2 - (1 - |\zeta_1|^2)(1 - |\zeta_2|^2)\zeta_3 \right|
\]

\[
\leq 2 \left[ |\zeta_1|^3 + 2(1 - |\zeta_1|^2)|\zeta_1||\zeta_2|^2 - (1 - |\zeta_1|^2)(1 - |\zeta_2|^2) \right] \quad (3.2)
\]

\[
= 2 \left[ 1 - |\zeta_1|^2 + |\zeta_1|^3 - 2(1 - |\zeta_1|^2)(1 - |\zeta_1||\zeta_2|^2) \right]
\]

\[
\leq 2 \left( 1 - |\zeta_1|^2 + |\zeta_1|^3 \right) \leq 2,
\]

with sharpness for the function (3.1).

To find sharp estimate for \( H_{2,2} \) over \( ST(i) \) we use the following lemma. □
Proposition 3.2.

\[ |4z^2 - 4z - 1| \leq \begin{cases} 
1 + 4|z| - 4|z|^2, & |z| \leq (-1 + \sqrt{2})/2, \\
\sqrt{2}(1 + 4|z|^2), & (-1 + \sqrt{2})/2 \leq |z| \leq 1.
\end{cases} \tag{3.3} \]

Proof. Since the inequality (3.3) clearly holds for \( z = 0 \), assume that \( z = re^{i\theta} \) with \( 0 < r \leq 1 \) and \( 0 \leq \theta < 2\pi \). A simple computation gives

\[ |4z^2 - 4z - 1|^2 = \varphi(\cos \theta), \tag{3.4} \]

where \( \varphi: [-1, 1] \to \mathbb{R} \) is a function defined by

\[ \varphi(x) := -16r^2x^2 - 8r(4r^2 - 1)x + 16r^4 + 24r^2 + 1. \]

Note that \( \varphi'(x) = 0 \) occurs only when \( x = (1 - 4r^2)/(4r) =: x_0 \).

When \( r \leq (-1 + \sqrt{2})/2 \), we have \( x_0 > 1 \) or \( 1 - 4r - 4r^2 > 0 \). Therefore

\[ \varphi'(x) \geq 8r(1 - 4r - 4r^2) > 0, \quad x \in [-1, 1]. \]

Hence we get

\[ \varphi(x) \leq \varphi(1) = (1 + 4r - 4r^2)^2. \tag{3.5} \]

Thus from (3.4) and (3.5) it follows that the inequality (3.3) holds for \( |z| \leq (-1 + \sqrt{2})/2 \).

When \( (-1 + \sqrt{2})/2 \leq r \leq 1 \), we have \( x_0 \in [-1, 1] \). Then

\[ \varphi(x) \leq \varphi(x_0) = 2(1 + 4r^2)^2, \quad x \in [-1, 1]. \tag{3.6} \]

Combining (3.4) and (3.6) we see that the inequality (3.3) holds for \( (-1 + \sqrt{2})/2 \leq |z| \leq 1 \). \( \Box \)

Theorem 3.3. If \( f \in ST(i) \) is of the form (1.1), then

\[ |a_2a_4 - a_3^2| \leq \frac{28}{3}. \tag{3.7} \]

The inequality is sharp with the extremal function

\[ f(z) = \frac{z(3 + z + 3z^2)}{3(1 - z^2)^2}, \quad z \in \mathbb{D}. \tag{3.8} \]

Proof. From (2.4) by using (1.7)–(1.9) we have

\[ a_2a_4 - a_3^2 = c_1^2 + c_1c_3 - c_2^2 - 2c_2 - 1 \]

\[ = 4\zeta_1^2 - 4\zeta_1 - 1 - 4(1 - |\zeta_1|^2)\zeta_2 - 4(1 - |\zeta_1|^2)\zeta_3^2 \]

\[ + 4\zeta_1(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)\zeta_3, \tag{3.9} \]

where \( \zeta_i \in \overline{\mathbb{D}}, \ i = 1, 2, 3. \) Let \( x := |\zeta_1| \in [0, 1] \) and \( y := |\zeta_2| \in [0, 1] \).

Assume first that \( x \in [0, x_0] \), where \( x_0 := (-1 + \sqrt{2})/2 \). Then by (3.9) and Proposition 3.2 for \( y \in [0, 1] \) we get

\[ |a_2a_4 - a_3^2| \leq 1 + 8x - 4x^2 - 4x^3 + 4(1 - x^2)y \]

\[ + 4x(1 - x^2)y^2 =: F(x, y). \]
Clearly, for each $x \in [0, x_0]$, the function $[0, 1] \ni y \mapsto F(\cdot, y)$ is increasing and therefore for $y \in [0, 1]$,

$$F(x, y) \leq F(x, 1) = 9 + 4x - 12x^2 \leq \frac{28}{3} = 9.333\ldots \quad (3.10)$$

Assume now that $x \in [x_0, 1]$. Then by (3.9) and Proposition 3.2 for $y \in [0, 1]$ we get

$$|a_2a_4 - a_3^2| \leq \sqrt{2} + 4x + 4\sqrt{2}x^2 - 4x^3 + 4(1 - x^2)y + 4(1 - x^2)(1 - x)y^2 =: G(x, y).$$

Note first that

$$G(1, y) = 5\sqrt{2} = 7.071\ldots, \quad y \in [0, 1]. \quad (3.11)$$

Clearly, for each $x \in [x_0, 1]$, the function $[0, 1] \ni y \mapsto G(\cdot, y)$ is increasing and therefore for $y \in [0, 1]$,

$$G(x, y) \leq G(x, 1) = 8 + \sqrt{2} - 4(2 - \sqrt{2})x^2 \leq -2 + 8\sqrt{2} = 9.133\ldots.$$

Hence, from (3.10) and (3.11) it follows that the inequality (3.7) is true. Equality in (3.7) holds for the function $f$ given by (2.3), where $p$ is given by (1.12) with $\zeta_1 := 1/6$ and $\zeta_2 = \zeta_3 := 1$, i.e., for the function (3.8). \hfill \Box

**Theorem 3.4.** If $f \in ST(1)$ is of the form (1.1), then

$$|a_2a_3 - a_4| \leq 20.$$

The inequality is sharp with the extremal function

$$f(z) = \frac{z(1 + z)}{(1 - z)^3}, \quad z \in \mathbb{D}. \quad (3.12)$$

**Proof.** From (2.6), by using (1.7) and the inequality $|c_1c_2 - c_3| \leq 2$ which was proved in (3.2), we obtain

$$|a_2a_3 - a_4| = |2 + 4c_1 + 2c_1^2 + c_1c_2 - c_3| \leq 2 + 4|c_1| + 2|c_1|^2 + |c_1c_2 - c_3| \leq 20.$$ 

with sharpness for the function (3.12). \hfill \Box

**Theorem 3.5.** If $f \in ST(1)$ is of the form (1.1), then

$$|a_2a_4 - a_3^2| \leq 17. \quad (3.13)$$

The inequality is sharp with the extremal function (3.12).
Proof. From (2.6) by using (1.7)–(1.9) we have
\begin{align}
a_2a_4 - a_3^2 &= -1 - 2c_1 - c_1^2 - 2c_1c_2 - 2c_2 - c_1^2 + 2c_1c_3 + 2c_3 \\
&= -1 - 4\zeta_1 - 8\zeta_1^2 - 4\zeta_1^3 - 4(1 - |\zeta_1|^2)\zeta_2 \\
&\quad - 4(1 + \zeta_1)(1 - |\zeta_1|^2)\overline{\zeta_1}\zeta_2^2 \\
&\quad + 4(1 + \zeta_1)(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)\zeta_3,
\end{align}
(3.14)
where $\zeta_i \in \overline{D}$, $i \in \{1, 2, 3\}$. Set $x := |\zeta_1| \in [0, 1]$ and $y =: |\zeta_2| \in [0, 1]$. By (3.14) we have
\[|a_2a_4 - a_3^2| \leq 5 + 8x + 4x^2 + 4(1 - x^2)y \\
- 4(1 - x^2)y^2 =: F(x, y), \quad x, y \in [0, 1].\]

Note first that
\[F(1, y) = 17, \quad y \in [0, 1].\] (3.15)

Let now $x \in [0, 1)$. Then for $y \in [0, 1]$ we have
\[\frac{\partial F}{\partial y} = 4(1 - x^2)[1 - 2(1 - x^2)y] = 0\]
iff $y = 1/2(1 - x^2) =: y_0$. Since $y_0 \geq 1$ for each $x \in [1/\sqrt{2}, 1)$, so then the function $[0, 1] \ni y \mapsto F(\cdot, y)$ is increasing and therefore
\[F(x, y) \leq F(x, 1) = 5 + 8x + 8x^2 - 4x^4 \leq 17, \quad y \in [0, 1].\] (3.16)

For $x \in [0, 1/\sqrt{2})$ we have
\[F(x, y) \leq F(x, y_0) = F\left(x, \frac{1}{2(1 - x^2)}\right) \\
= 6 + 8x + 4x^2 \leq 8 + 4\sqrt{2} = 13.656 \ldots, \quad y \in [0, 1].\]
Hence by (3.15) and (3.16) it follows that the inequality (3.13) is true. Equality in (3.13) holds for the function $f$ defined by (3.12). \hfill \Box

4. Inverse Coefficients

Since $ST(i)$ is a compact class and $f'(0) = 1$ for every $f \in ST(i)$, there exists $r_0 \in (0, 1)$ such that every $f \in ST(i)$ is invertible in the disk $D_{r_0}$. Thus there exists $\delta > 0$ such that the inverse function $\hat{f}$ of $f|_{D_{r_0}}$ has a series expansion in the disk $D_{\delta}$ of the form
\[\hat{f}(w) = w + \sum_{n=2}^{\infty} \beta_n w^n, \quad w \in D_\delta.\] (4.1)

Thus for $f \in ST(i)$ of the form (1.1) the following relations hold (see e.g., [12, Vol. I, p. 57])
\[\beta_2 = -a_2, \quad \beta_3 = 2a_2^2 - a_3, \quad \beta_4 = -5a_2^3 + 5a_2a_3 - a_4.\] (4.2)
Similar situation holds for the class $ST(1)$.

**Theorem 4.1.** If $\hat{f}$ is the inverse function of $f \in ST(i)$ of the form (4.1), then

(i) $|\beta_2| \leq 2$;
(ii) $|\beta_3| \leq 7$;
(iii) $|\beta_4| \leq 30$.

All inequalities are sharp with the extremal function

$$f(z) = \frac{z(1 + iz)}{(1 - z^2)(1 - iz)}, \quad z \in \mathbb{D}.$$  \hspace{1cm} (4.3)

**Proof.** Substituting (2.4) into (4.2) we get

$$\beta_2 = -c_1, \quad \beta_3 = 2c_1^2 - c_2 - 1$$  \hspace{1cm} (4.4)

and

$$\beta_4 = -5c_1^3 + 5c_1c_2 + 4c_1 - 3.$$  \hspace{1cm} (4.5)

By (4.4) and (1.7) the inequality (i) follows immediately. From (4.4) with (1.7) and (1.8) we have

$$|\beta_3| = |2c_1^2 - c_2 - 1| = |6\zeta_1^2 - 2(1 - |\zeta_1|^2)\zeta_2 - 1|$$

$$\leq 6|\zeta_1|^2 + 2(1 - |\zeta_1|^2)|\zeta_2| + 1 \leq 4|\zeta_1|^2 + 3 \leq 7.$$

Now we prove (iii). By (4.5) and (1.7)-(1.9) we have

$$|\beta_4| = | -22\zeta_3^2 + 8\zeta_1 + 16(1 - |\zeta_1|^2)\zeta_1\zeta_2$$

$$+ 2(1 - |\zeta_1|^2)\zeta_1\zeta_2 - 2(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)\zeta_3|$$

$$\leq 2 + 8x - 2x^2 + 22x^3 + 16x(1 - x^2)y - 2(1 - x)^2(1 + x)y^2$$

$$=: F(x, y),$$

where $\zeta_i \in \overline{\mathbb{D}}, \ i = 1, 2, 3, \ x := |\zeta_1| \in [0, 1]$ and $y := |\zeta_2| \in [0, 1]$.

Note first that

$$F(1, y) = 30, \quad y \in [0, 1].$$  \hspace{1cm} (4.6)

Let now $x \in [0, 1)$. Then for $y \in [0, 1]$ we have

$$\frac{\partial F}{\partial y} = 4(1 - x^2)[4x - (1 - x)y] = 0$$

iff $y = 4x/(1 - x) =: y_0$. Since $y_0 \geq 1$ for each $x \in [1/5, 1)$, so then the function $[0, 1] \ni y \mapsto F(\cdot, y)$ is increasing and therefore

$$F(x, y) \leq F(x, 1) = 26x + 4x^3 \leq 30, \quad y \in [0, 1].$$  \hspace{1cm} (4.7)

For $x \in [0, 1/5]$ we have

$$F(x, y) \leq F(x, y_0) = F\left( x, \frac{4x}{1 - x} \right)$$

$$= 2 + 72x + 30x^2 - 10x^3 \leq \frac{438}{25} = 15.52, \quad y \in [0, 1].$$  \hspace{1cm} (4.8)
Hence by (4.6)–(4.8) it follows that the inequality in (iii) is true. All inequalities are sharp with the extremal function (4.3).

\[ \square \]

**Theorem 4.2.** If \( \hat{f} \) is the inverse function of \( f \in \mathcal{S} \mathcal{T}(1) \) of the form (4.1), then

(i) \( |\beta_2| \leq 4; \)
(ii) \( |\beta_3| \leq 23; \)
(iii) \( |\beta_4| \leq 156. \)

All inequalities are sharp with the extremal function

\[ f(z) = \frac{z(1 + z)}{(1 - z)^3}, \quad z \in \mathbb{D}. \]  

**(Proof.** Substituting (2.6) into (4.2) we get

\[ \beta_2 = -c_1 - 2, \quad \beta_3 = 2c_1^2 + 6c_1 - c_2 + 5 \]  

and

\[ \beta_4 = -5c_1^3 - 20c_1^2 + 2c_1 + 5c_1c_2 + 8c_2 - c_3 - 14. \]  

By (4.10) and (1.7) the inequality (i) follows immediately. From (4.10) with (1.7) and (1.8) we have

\[
|\beta_3| = |2c_1^2 + 6c_1 - c_2 + 5| = |6\zeta_1^2 + 12\zeta_1 - 2(1 - |\zeta_1|^2)\zeta_2 + 5| \\
\leq 6|\zeta_1|^2 + 12|\zeta_1| + 2(1 - |\zeta_1|^2) + 5 = 4|\zeta|^2 + 12|\zeta| + 7 \leq 23.
\]

Now we prove (iii). By (4.11) and (1.7)–(1.9) we have

\[
|\beta_4| = |-22\zeta_1^3 - 64\zeta_1^2 - 56\zeta_1 - 14 + 16(1 - |\zeta_1|^2)\zeta_2 \\
+ 16(1 - |\zeta_1|^2)\zeta_1\zeta_2 + 2(1 - |\zeta_1|^2)\zeta_1\zeta_2^2 \\
- 2(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)\zeta_3| \\
\leq 22x^3 + 62x^2 + 56x + 16 + 16(1 - x^2)(1 + x)y \\
- 2(1 - x^2)(1 - x)y^2 =: F(x, y),
\]

where \( \zeta_i \in \overline{\mathbb{D}}, \ i = 1, 2, 3, \ x := |\zeta_1| \in [0, 1] \text{ and } y := |\zeta_2| \in [0, 1]. \)

Note first that

\[ F(1, y) = 156, \quad y \in [0, 1]. \]  

Let now \( x \in [0, 1). \) Then for \( y \in [0, 1] \) we have

\[
\frac{\partial F}{\partial y} = 4(1 - x^2)[4(1 + x) - (1 - x)y] = 0
\]

iff \( y = 4(1 + x)/(1 - x) =: y_0. \) Since \( y_0 \geq 1 \) for each \( x \in (0, 1), \) so the function \( [0, 1] \ni y \mapsto F(\cdot, y) \) is increasing and therefore

\[ F(x, y) \leq F(x, 1) = 4x^3 + 48x^2 + 74x + 30 \leq 156, \quad y \in [0, 1]. \]

Hence and from (4.12) it follows that the inequality in (iii) is true.

All inequalities are sharp with the extremal function (4.9). \( \square \)
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