Electric Field Analysis using Schwarz-Christoffel Mapping

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Abstract. Electrical techniques based on AC electrokinetics and impedance spectroscopy are widely used to manipulate and characterize biological particles in the microfluidic systems. This paper presents the application of the Schwarz-Christoffel mapping method to analytically solve the electric field distributions in different microfluidic systems, which are composed of different microelectrode patterns and boundary conditions. The derived results can be further utilized to analyze the movement and electrical response of the biological particles in each system.

1. Introduction
In the field of Micro Total Analysis Systems (µTAS) or the Lab-on-a-chip (LOAC), the manipulation and characterization of biological particles at the micro and nano scale are widely performed by using electrical techniques based on AC electrokinetics and impedance spectroscopy. A number of different microfluidic devices with different microchannel geometries and microelectrode patterns have been fabricated to perform the analysis. The most commonly used system for combing DEP and twDEP techniques is the interdigitated bar electrode arrays. On the other hand, the microfluidic cytometers are highly adopted to perform high throughput analysis for single particles. The most commonly fabricated electrodes configurations are parallel facing electrodes and coplanar electrodes.

Since both AC electrokinetic techniques and impedance spectroscopy are dependent on the electric field distribution in the system, the knowledge of the accurate electric field map within the system is required. In this paper, we present the analytical solutions of the electric field distributions in the interdigitated bar electrode arrays for DEP and twDEP applications and the parallel facing and coplanar electrodes cytometers for impedance measurement.

2. Schwarz-Christoffel Mapping
The Schwarz-Christoffel mapping (SCM) method is a kind of conformal mapping. It is based on the angular change, which is encountered at corners of polygonal configurations as one traces the boundary in a given direction. It maps the inside region of a polygon in the Z-plane (physical plane), into the upper half plane of the T-plane (auxiliary plane). For the details of SCM, see ref [1]. For the electric field calculation, the basic idea is that a non-uniform two-dimensional electric field polygonal region can be transformed into an equivalent rectangle region, as a parallel plate capacitor, where the electric field distribution is uniform.
3. Interdigitated Electrode Arrays

In the DEP case, two AC voltage signals with phases $0^0$ and $180^0$ are connected to the electrodes, alternatively. In the twDEP case, four signals with phase shift of $90^0$ are applied to consecutive electrodes.

3.1 The dielectrophoretic array

The values for the real and imaginary parts of the potential phasor at every electrode in the dielectrophoretic array and the relevant boundary conditions are shown in figure 1 (a). Since the normal component of the total current passing through the electrolyte/lid and electrolyte/substrate interfaces must be continuous and the lid and the substrate are made from glass, which has a much smaller permittivity and conductivity than the electrolyte (water) in the channel, the normal component of the electric field in the electrolyte at the interface is negligible compared to that of the glass. Therefore, we assume that Neumann boundary condition (insulating) holds for the potential at the electrolyte/lid and electrolyte/substrate interfaces: \( \frac{\partial \phi}{\partial n} = 0 \), where \( n \) is the normal to the boundary. For the detail description, see ref [2]. Due to the symmetry, the basic cell (ABCDE) is chosen for analysing the electric field distribution by using SCM. Figure 1(b) shows the planes used in the SCM for the DEP array case.

![Figure 1 (a)](image1.png)

![Figure 1 (b)](image2.png)

Figure 1 (a) Diagram showing the dielectrophoretic electrode array with boundary conditions. (b) Three complex planes of basic cell in the DEP array for performing SCM method.

The chosen cell ABCDE is set in the Z-plane with the complete boundary conditions for the real potential \( \phi_R \). In the auxiliary plane (T-plane), the whole polygon ABCDE in Z-plane is opened at point \( F \) and mapped into the upper half of the T-plane and all the boundaries are mapped on to the real axis. The corresponding points are: \( T_A = (t_A, 0) \), \( T_B = (t_B, 0) \), \( T_C = (t_C, 0) \), \( T_D = (t_D, 0) \) and \( T_E = (t_E, 0) \). The point \( F \) is mapped to the positive infinity, \( T_{F(+)} \) and negative infinity, \( T_{F(-)} \). The SCM integral from the T-plane to the Z-plane is given by:

\[
Z = C_1 \left[ (t-t_E)^{1/2} (t-t_B)^{-1/2} (t-t_C)^{-1/2} (t-t_D)^{-1/2} dt + C_2 \right] \tag{1}
\]

For \( t > t_E > t_B > t_C > t_D \), the solution of equation (1) is an elliptic integral [6]:

\[
Z = \frac{h}{K(k_{hi})} F[\sigma_{hi}, k_{hi}] = \frac{h}{K(k_{hi})} F[\arcsin \left( \frac{(t_B-t_D)(t-t_E)}{(t_B-t_D)(t-E)} \right), \frac{(t_E-t_D)(t_B-t_D)}{(t_B-t_D)(t_E-t_D)}] \tag{2}
\]
with \[ \frac{K(k_{d1})}{K(k_{d1})} = \frac{w + g}{2h}, \quad t_a = \frac{t_k t_e \text{cn}^2 \left( \frac{g K(k_{d1})}{w + g}, k_{d1} \right)}{t_b - t_k \text{sn}^2 \left( \frac{g K(k_{d1})}{w + g}, t_k \right)} \]

where \( F[\sigma_{d1}, k_{d1}] \) is the elliptical integral of the first kind and \( k_{d1} \) is the modulus of the elliptic function. \( K(k_{d1}) \) and \( K(k'_{d1}) \) are the complete elliptic integral of the first kind with \( k'_{d1} = \sqrt{1 - k_{d1}^2} \). \( \text{sn}, \text{cn} \) are the Jacobian elliptic functions. \( w \) is defined as the width of electrode. \( g \) is defined as the distance of the gap between the two adjacent electrodes and \( h \) is defined as the height of the channel.

In the model plane \((W\text{-plane})\), the upper half of the \( T\text{-plane} \) is transformed into a rectangle, where the electric field distribution is uniform. The corresponding points are: \( W_A = (0, i Y_{W_d}), W_B = (X_{W_d}, i Y_{W_d}), W_D = (X_{W_d}, 0) \) and \( W_E = (0, 0) \) where \( X_{W_d} \) and \( Y_{W_d} \) are the size of the region along the real and imaginary axis, respectively. The transformation from the \( T\text{-plane} \) to the \( W\text{-plane} \) is given by:

\[
W = D_1 \int (t - t_E)^{-1/2} (t - t_A)^{-1/2} (t - t_B)^{-1/2} (t - t_D)^{-1/2} \, dt + D_2
\]

(3)

Similarly, the solution of equation (3) becomes:

\[
W = \frac{X_{W_d}}{K(k_{d1})} F[\sigma_{d2}, k_{d2}] = \frac{X_{W_d}}{K(k_{d2})} F \left[ \arcsin \left( \frac{(t_A - t_B)(t - t_E)}{(t_E - t_A)(t - t_B)} \right), \sqrt{\frac{(t_E - t_D)(t_A - t_B)}{(t_A - t_D)(t_E - t_B)}} \right]
\]

(4)

As described above, the polygonal area for the basic cell in DEP array has been mapped into a rectangle in the \( W\text{-plane} \), where the electric field distribution is uniform and restricted by the transformed boundaries. Then, the non-uniform electric field distribution in the \( Z\text{-plane} \) can be calculated as:

\[
\mathbf{E}_{zd} = -\nabla \phi_{zd} = -\nabla \phi_{zd} \left( \frac{dW}{dZ} \right) = \frac{V}{h} \frac{K(k_{d1})}{K(k_{d2})} \left[ \frac{t_A}{t_b} \left( t_E - t_B \right) (t - 1) \right]^{1/2}
\]

\[ \left( \frac{t_A}{t_b} \left( t_E - t_B \right) (t - 1) \right]^{1/2}
\]

(5)

where \( \mathbf{E}_{zd} \) is the electric field distribution in the \( Z\text{-plane} \) for the DEP array case. \( \phi_{zd} \) is the potential distribution in the \( Z\text{-plane} \) and \( W\text{-plane} \) respectively. \( dW/dZ \) represents the complex conjugate of the derivative from \( W \) to \( Z \)

Equation (5) gives the analytical solution of the electric field distribution in the basic cell of DEP array case. The electric field magnitude gets to zero at point \( C \) (when \( t = 1 \)) and infinity at point \( A \) (when \( t = t_A \)), where is the edge of the electrode, as shown in figure 1(c). The numerical evaluation is performed in MATLAB 7.0.

Figure 1 (c) shows the magnitude (log10 scale) of the electric field distribution in the basic cell of DEP array.
3.2 The travelling wave array
The values for the real and imaginary parts of the potential phasor at every electrode in the traveling wave array and the relevant boundary conditions are shown in figure 2 (a). For the detail explanation, see ref [2]. The basic cell (A’B’C’D’E’F’) is chosen for analyzing the electric field distribution. The boundary conditions for imaginary part of the potential phasor are the mirror image of those for the real part, about the centre of the gap. This indicates the field analysis is only needed to be solved once. Figure 2 (b) shows the planes used in the SCM for solving the real part electric field in the twDEP array case.

\[ E_{ZB'} = \left\{ \frac{V}{l} \frac{K(k_1)}{K(k_{1z})} \left[ \frac{t_{\phi}(t_{w'} - t_{w})(t-1)(t-t_{E'})}{\sqrt{t_{w'}(t_{w'} - 1)}} \right]^{1/2} \right\} \] (8)

with \( \frac{K(k_{1z})}{K(k_1)} = \frac{w + g}{h} \), \( k_1 = \frac{t_{\phi}(t_{w'} - 1)}{\sqrt{t_{w'}(t_{w'} - 1)}} \), \( k_{1z} = \frac{t_{\phi}(t_{w'} - t_{w})}{\sqrt{t_{w'}(t_{w'} - t_{w})}} \)

where \( E_{ZB'} \) is the real part of the electric field distribution in the Z-plane for the twDEP array case. Equation (8) gives the analytical solution of the real part of the electric field distribution in the basic cell of twDEP array case. The electric field magnitude gets to zero at point C’ and E’ (when \( t = 1 \) and \( t = t_{E'} \)) and infinity at point A’ and F’ (when \( t = t_{A'} \) and \( t = t_{F'} \)), where are the edge of the electrodes, as shown in figure 2(c).
4. Microfluidic Cytometers

4.1 Parallel Facing Electrodes

Owing to symmetry only one quarter of the geometry is solved, as indicated in the Z-plane and the thickness of the electrode is assumed to be zero. Figure 3(a) shows the planes used in the SCM for the parallel facing electrode design and relevant geometrical parameters.

From Collin [3], the transformation from the T-plane to the Z-plane is given by

\[ Z = \frac{-h_f}{\pi} \ln \left( \sqrt{t} + \sqrt{t+1} \right) + \frac{ih_f}{2} \]  

(9)

The transformation from the T-plane to the W-plane is given by

\[ W = \frac{1}{iK(k_f')} \text{sn}^{-1} \left[ (-t)^{\frac{1}{2}}, k_f' \right] + \frac{K(k_f'K(k_f'))}{K(k_f')}, \]  

(10)

where \( \text{sn}^{-1}[\ldots, \ldots] \) is the inverse Jacobian elliptic function.

The electric field distribution in the Z-plane can be derived as:

\[ E_{z,t} = \left\{ -l \pi V \frac{1}{h_f'K(k_f')} \right\} \sqrt{\frac{a_f}{t + a_f}} \]  

(11)

with \( k_f = a_f^{\frac{1}{2}} = \cosh^{-1} \left( \frac{\pi w_f}{2h_f} \right) \)

According to equation (11), the electric field magnitude gets to infinity at point \( A_i \) (when \( t = -a_f \)), where are the edge of the electrodes. Figure 3(b) shows the electric field distribution in the parallel facing electrode cytometer.

![Figure 3(a)](image1)

![Figure 3(b)](image2)

Figure 3 (a) Three complex planes of basic cell in the parallel facing electrodes cytometer for performing SCM method. (b) The magnitude (log10 scale) of the real electric field distribution in the parallel facing electrodes cytometer.

4.2 Coplanar Electrodes

Figure 4 (a) shows the SCM transformation planes of the coplanar electrodes cytometer. Owing to the symmetry, half of the channel is chosen to analyze. Applying SCM method and following the procedure of Gevorgian [4], the transformation from T-plane to Z-plane is:

\[ Z = \frac{2h_f}{\pi} \ln \left( \sqrt{t} + \sqrt{t-1} \right) \]  

(12)
The transformation from the $T$-plane to the $W$-plane is given by:

$$W = \frac{X_w}{K(k_c)} F \left[ \arcsin \left( \frac{b_c (t_a - a_c)}{a_c (t_c - b_c)} \right) \right]$$  \hspace{1cm} (13)

with

$$k_c = \frac{\tanh \left( \frac{\pi g_c}{(4h_c)} \right)}{\tanh \left( \frac{\pi (g_c + 2w_c)}{(4h_c)} \right)}$$

$$a_c = \cosh^2 \left( \frac{\pi (g_c + 2w_c)}{4h_c} \right)$$

$$b_c = \cosh^2 \left( \frac{\pi g_c}{4h_c} \right)$$

The analytical electric field solution is given by:

$$E_{zc} = \frac{-\pi V}{2h_c K(k_c)} \sqrt{\frac{b_c (a_c - 1)}{(t - a_c)(t - b_c)}}$$ \hspace{1cm} (14)

According to equation (14), the electric field magnitude gets to infinity at point $A_c$ and $B_c$ (when $t = a_c$ and $t = b_c$), where are the edge of the electrodes. Figure 3(b) shows the electric field distribution in the coplanar electrode cytometer.

Figure 4 (a) Three complex planes of basic cell in the coplanar electrodes cytometer for performing SCM method. (b) The magnitude (log10 scale) of the real electric field distribution in the coplanar electrodes cytometer.

5. Conclusion

In this paper, we systematically showed that how to use Schwarz-Christoffel mapping method to analytically solve the electric field distribution problem. The analyzed four different cases contain either different electrode patterns or boundary conditions. SCM method has been demonstrated as a powerful tool to perform electrostatic field analysis.

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