AN IMPROVED PURE SOURCE TRANSFER DOMAIN DECOMPOSITION METHOD FOR HELMHOLTZ EQUATIONS IN UNBOUNDED DOMAIN

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Abstract. We propose an improved pure source transfer domain decomposition method (PST-DDM) for solving the truncated perfectly matched layer (PML) approximation in bounded domain of Helmholtz scattering problem. The method is based on the source transfer domain decomposition method (STDDM) proposed by Chen and Xiang. The idea is decomposing the domain into non-overlapping layers and transferring the sources equivalently layer by layer so that the solution in each layer can be solved using a PML method defined locally outside its two adjacent layers. Furthermore, we divide the domain into non-overlapping blocks and solve the solution in each block by using a PML method defined locally outside its adjacent blocks. The convergence analysis of the method is provided for the case of constant wave number. Finally, numerical examples are provided where the method is used as both a direct solver and an efficient preconditioner in the GMRES method for solving the Helmholtz equation.

Key words. Helmholtz equation, large wave number, perfectly matched layer, source transfer, domain decomposition method

AMS subject classifications. 65N12, 65N15, 65N30, 78A40

1. Introduction. This paper is devoted to a domain decomposition method based on the STDDM method (cf. [15]) for the Helmholtz problem in the full space $\mathbb{R}^2$ with Sommerfeld radiation condition:

\begin{align}
\Delta u + k^2u &= f \quad \text{in } \mathbb{R}^2, \\
\left| \frac{\partial u}{\partial r} - iku \right| &= o(r^{-1/2}) \quad \text{as } r = |x| \to \infty.
\end{align}

where the wave number $k$ is positive and $f \in H^1(\mathbb{R}^2)'$ having compact support, where $H^1(\mathbb{R}^2)'$ is the dual space of $H^1(\mathbb{R}^2)$. The problem is satisfied in a weak sense (cf. [33]).

Helmholtz boundary value problems appear in various applications, for example, in the context of inverse and scattering problems. Since the huge number of degrees of freedom is required resulting from the pollution error and the highly indefinite nature of Helmholtz problem with large number wave [1], [2], [3], [10], [18], [21], [22], [24], [26], [27], [28], [31], it is challenging to solve the algebraic linear equations resulting from the finite difference or finite element method with large wave number. Considerable efforts in the literature have been made. One way is to find efficient and cheap methods [2], [10], [17], [21], [22], [24], such as the continuous interior penalty finite element method [9], [55], [36], which use less degrees of freedom as the same relative error reached. Another way is to find efficient algorithms for solving discrete Helmholtz equations, e.g. Benamou and Desprès [4], Gander et al. [23] for domain decomposition techniques and Brandt and Livshit [8], Elman et al. [19] for multigrid methods. Recently Engquist and Ying constructed a new sweeping preconditioner for the interior solution [20]. Then Chen and Xiang proposed the source transfer domain decomposition method (STDDM)
in which only some local PML problems defined locally outside the union of two layers are solved. Thus the complexity of STDDM is the sum of the complexity of the algorithms for solving those local problems which reduce the complexity to solve the whole linear system. We are inspired by the key idea of STDDM, and the main lemmas and idea of proofs also come from their work.

In this paper we show the improved pure source transfer domain decomposition method (PSTDDM).

Let $B_l = \{x = (x_1, x_2)^T \in \mathbb{R}^2 : |x_1| < l_1, |x_2| < l_2\}$. Assume that $f$ is supported in $B_l$. We divide the interval $(-l_2, l_2)$ into $N$ segments with the points $\zeta_i = -l_2 + (i-1)\Delta \zeta$ where $\Delta \zeta = 2l_2/N$. Then we denote the layers by

$$
\Omega_0 = \{x = (x_1, x_2)^T \in \mathbb{R}^2 : x_2 < \zeta_1\},
\Omega_i = \{x = (x_1, x_2)^T \in \mathbb{R}^2 : \zeta_i < x_2 < \zeta_{i+1}\}, \ i = 1, \ldots, N,
\Omega_{N+1} = \{x = (x_1, x_2)^T \in \mathbb{R}^2 : \zeta_{N+1} < x_2\}.
$$

Clearly, $\text{supp} \ f \subset \cup_{i=1}^{N} \Omega_i$. Let $f_i = f$ in $\Omega_i$ and $f_i = 0$ elsewhere. Let $f_i^+ = f_i$ and $f_i^- = f_N$. The key idea is that by defining the source transfer function $\Psi_i^\pm$ in the sense that

$$
\begin{align*}
\int_{\Omega_i} f_i^+(y) G(x, y) dy &= \int_{\Omega_{i+1}} \Psi_i^+(f_i^+)(y) G(x, y) dy \quad \forall x \in \cup_{j=i+2}^{N+1} \Omega_j, \\
\int_{\Omega_i} f_i^-(y) G(x, y) dy &= \int_{\Omega_{i-1}} \Psi_i^-(f_i^-)(y) G(x, y) dy \quad \forall x \in \cup_{j=0}^{i-2} \Omega_j,
\end{align*}
$$

and letting $f_i^{\pm} = f_{i+1}^+ + \Psi_i^{\pm}(f_i)$, then we have for any $x \in \Omega_i$

$$
(1.3) \quad u(x) = \left( -\int_{\Omega_i} f_i(y) G(x, y) dy - \int_{\Omega_{i-1}} f_i^-(y) G(x, y) dy \right) + \\
\left( -\int_{\Omega_{i+1}} f_i^+(y) G(x, y) dy \right).
$$

Here, $G(x, y)$ is the Green’s function of the problem (1.1)–(1.2). Observing (2.1), we know that $u(x)$ in $\Omega_i$ consists of two independent parts. The first part only involves the sources in $\Omega_i$ and $\Omega_{i-1}$ and the second one only involves the source in $\Omega_{i+1}$. Thus they could be solved independently by using the PML method outside only $\Omega_{i-1} \cup \Omega_i$ and $\Omega_i \cup \Omega_{i+1}$ respectively. Similar to STDDM, our PSTDDM also consists of two steps which could run in parallel and the complexity of every step is the same as that of STDDM. The error of PSTDDM are not larger than that of STDDM if the same numerical algorithm, such as the finite element or difference method, was used. Besides, since every step of PSTDDM just consists of local PML problems defined in the whole space, not in the half-space with Dirichlet boundary, these local PML problems also can be solved by using our PSTDDM recursively. As a result, the computational domain is divided into blocks and the local PML problems needed to be solved are defined outside the union of four adjacent blocks.

The perfectly matched layer (PML) is a mesh termination technique of effectiveness, simplicity and flexibility in computational wave propagation. After the pioneering work of Bérenger [5, 6], various constructions of PML absorbing layers have been proposed and many theoretical results about Helmholtz problem, such as those about the convergence and stability, have been studied [7, 13, 14, 16, 29, 30, 31]. In this paper, the uniaxial PML methods will be used.
The remainder of this paper is organized as follows. In section 2, pure source transfer domain decomposition method in $\mathbb{R}^2$ and some important lemmas and theorems, which are fundamental and illuminating for the PSTDDM in truncated domain, are introduced. Section 2.3 shows the algorithm in the truncated bounded domain and the main result, that is, the exponentially convergence of the solution of PSTDDM in the truncated domain to the solution in $\mathbb{R}^2$. In section 3, we show that the local PML problems can be solved by out PSTDDM recursively and proof the convergence of the method. At last, we both use the method as a direct solver and an efficient preconditioner in the GMRES method to solve the Helmholtz equation.

2. Source transfer layer by layer. In this section, we introduce the PSTDDM for the PML method in the whole space. First, we recall the progress of deriving the PML method and set the medium properties of perfect matched lays which are a bit different from traditional medium and would be used in the following lemmas and theorems [15]. In subsection 2.1, we also recall some basic lemmas. Then we introduce the two steps of the pore source transfer domain decomposition method in $\mathbb{R}^2$.

2.1. The PML method. In this subsection, we recall some knowledge about the PML method.

The exact solution of equation (1.1) with the radiation condition 1.2 can be written as the acoustic volume potential. Let $G(x, y)$ be the fundamental solution of the Helmholtz problem

$$
\Delta G(x, y) + k^2 G(x, y) = -\delta_y(x) \text{ in } \mathbb{R}^2.
$$

We know $G(x, y) = \frac{i}{4} H_0^{(1)}(k|x-y|)$ where $H_0^{(1)}(z)$, for $z \in \mathbb{C}$, is the first kind Hankel function of order zero.

Then, the solution of (1.1) is given by

$$
u(x) = -\int_{\mathbb{R}^2} f(y) G(x, y) dy \quad \forall x \in \mathbb{R}^2. \quad (2.1)
$$

In this paper we used the uniaxial PML method [7, 15, 12, 29]. the model medium properties are defined by

$$
\alpha_1(x_1) = 1 + i\sigma_1(x_1), \quad \alpha_2(x_2) = 1 + i\sigma_2(x_2)
$$

$$
\sigma_j(t) = \sigma_j(-t) \text{ for } t \in \mathbb{R}^2, \quad \sigma_j = 0 \text{ for } |t| \leq l_j, \quad \sigma_j = \gamma_0 > 0 \text{ for } |t| \geq \bar{l}_j.
$$

where $\sigma_j(x_j) \in C^1(\mathbb{R}^2)$ are piecewise smooth functions, $\bar{l}_j > l_j$ is fixed and $\gamma_0$ is a constant.

For $x = (x_1, x_2)^T$, we define the complex coordinate as $\tilde{x}(x) = (\tilde{x}_1(x_1), \tilde{x}_2(x_1))$, where

$$
\tilde{x}_j(x_j) = \int_0^{x_j} \alpha_j(t) dt = x_j + i \int_0^{x_j} \sigma_j(t) dt, \quad j = 1, 2. \quad (2.2)
$$

We remark that this kind of definition has been proposed in [15] and recall that the requirement, $\sigma_j = \gamma_0$ for $|t| \geq \bar{l}_j$, is very important because of the use of proving the local inf-sup condition (2.33) (cf. [15]) for the truncated PML problem by using the reflection argument of [7, 15] and estimating the dependence of the inf-sup constants on the wave number $k$. 

The complex distance is defined as
\[(2.3) \quad \rho(\tilde{x}, \tilde{y}) = \left[(\tilde{x}_1(x_1) - \tilde{y}_1(y_1))^2 + (\tilde{x}_2(x_2) - \tilde{y}_2(y_2))^2\right]^{1/2}.\]

Here, \(z^{1/2}\) denote the analytic branch of \(\sqrt{z}\) such that \(\text{Re}(z^{1/2}) > 0\) for \(z \in \mathbb{C}\setminus[0, +\infty)\).

The solution to the PML problem is
\[(2.4) \quad \tilde{u}(x) = u(\tilde{x}) = -\int_{\mathbb{R}^2} f(y)G(\tilde{x}, \tilde{y})dy \quad \forall x \in \mathbb{R}^2,\]

Since \(f\) is supported inside \(B_t\), we know that \(\tilde{y} = y\) and \(\tilde{u} = u\) in \(B_t\).

The solution \((2.4)\) satisfies the PML equation
\[(2.5) \quad J^{-1} \nabla \cdot (A\nabla \tilde{u}) + k^2 \tilde{u} = f \quad \text{in} \ \mathbb{R}^2,\]

which could be obtained by the fact that \(\tilde{\Delta} \tilde{u} + k^2 \tilde{u} = f\) in \(\mathbb{R}^2\) and using the chain rule, where \(A(x) = \text{diag}\left(\frac{\alpha_{11}(x)}{\alpha_{11}(x_1)}, \frac{\alpha_{22}(x)}{\alpha_{22}(x_2)}\right)\) and \(J(x) = \alpha_1(x_1)\alpha_2(x_2)\).

The weak formulation of \((2.5)\) is: Find \(\tilde{u} \in H^1(\mathbb{R}^2)\) such that
\[(2.6) \quad (A\nabla \tilde{u}, \nabla v) - k^2 (J \tilde{u}, v) = -\langle Jf, v \rangle \quad \forall v \in H^1(\mathbb{R}^2),\]

where \((\cdot, \cdot)\) is the inner product in \(L^2(\mathbb{R}^2)\) and \(\langle \cdot, \cdot \rangle\) is the duality pairing between \(H^1(\mathbb{R}^2)\) and \(H^1(\mathbb{R}^2)\).

We have the following inf-sup condition for the sesquilinear form associated with the PML problem in \(\mathbb{R}^2\) which has been proved (cf. [15], Lemma 3.3):
\[(2.7) \quad \sup_{\psi \in H^1(\mathbb{R}^2)} \frac{|(A\nabla \phi, \nabla \psi) - k^2 \langle J \phi, \psi \rangle|}{\|\psi\|_{H^1(\mathbb{R}^2)}} \geq \mu_0 \|\phi\|_{H^1(\mathbb{R}^2)} \quad \forall \phi \in H^1(\mathbb{R}^2),\]

where the inf-sup condition \(\mu_0^{-1} \leq Ck^{3/2}\) which is fundamental to our estimates.

The fundamental solution of the PML equation \((2.5)\) is (cf. [7, 34]):
\[(2.8) \quad \tilde{G}(x, y) = J(y)G(\tilde{x}, \tilde{y}) = \frac{1}{4} J(y)H_0^1(k\rho(\tilde{x}, \tilde{y})).\]

**2.2. The PSTDDM for the PML equation in \(\mathbb{R}^2\).** In this subsection, we introduce our PSTDDM for the PML equation in the whole space and give the fundamental theorems.

We recall that the domain \(\{x = (x_1, x_2) : |x_2| \leq l_2\}\) is divided into \(N\) layers (??) and that \(f_i(x) = f(x)|_{\Omega_i}\) for any \(x \in \Omega_i\) and \(f_i(x) = 0\) for any \(x \in \mathbb{R}^2\setminus\Omega_i\). We define smooth functions \(\beta_i^+(x_2)\) and \(\beta_i^-(x_2)\) by
\[(2.9) \quad \beta_i^+ = 1, \beta_i^- = 0, \quad \beta_i^+ = \beta_i^- = 0 \quad \text{as} \quad x_2 \leq \zeta_i, \quad \beta_i^+ = 0, \beta_i^- = 1, \quad \beta_i^+ = \beta_i^- = 0 \quad \text{as} \quad x_2 \geq \zeta_{i+1}, \quad |\beta_i^+| \leq C(\nabla \zeta)^{-1}, \quad |\beta_i^-| \leq C(\nabla \zeta)^{-1},\]

where \(C\) is a constant independent of \(\zeta_i, \zeta_{i+1}\) and the subscript \(i\). Our PSTDDM consists of two steps [1] and [2]. Clearly, the two steps [1] and [2] are independent of each other and can be computed in parallel.
Algorithm 1 Source Transfer I for PML problem in $\mathbb{R}^2$

1. Let $\bar{f}_i^+ = f_i$.
2. While $i = 1, \cdots, N - 2$ do
   - Find $u_i^+ \in H^1(\mathbb{R}^2)$ such that
     \begin{equation}
     J^{-1} \nabla \cdot (A \nabla u_i^+) + k^2 u_i^+ = -\tilde{f}_i^+ - f_{i+1} \quad \text{in } \mathbb{R}^2
     \end{equation}
   - Compute
     \begin{equation}
     \Psi_{i+1}^+(\tilde{f}_i^+) = J^{-1} \nabla \cdot (A \nabla (\beta_{i+1}^+ u_i^+)) + k^2 (\beta_{i+1}^+ u_i^+).
     \end{equation}
   - Set $\bar{f}_{i+1} = f_{i+1} + \Psi_{i+1}^+(\tilde{f}_i^+)$ in $\Omega_{i+1}$ and $\tilde{f}_{i+1} = 0$ elsewhere.
End while
3. For $i = N - 1$, find $u_{N-1}^+ \in H^1(\mathbb{R}^2)$ such that
   \begin{equation}
   J^{-1} \nabla \cdot (A \nabla u_{N-1}^+) + k^2 u_{N-1}^+ = -\tilde{f}_{N-1}^+ - f_N \quad \text{in } \mathbb{R}^2
   \end{equation}

Algorithm 2 Source Transfer II for PML problem in $\mathbb{R}^2$

1. Let $\bar{f}_N^- = f_N$;
2. While $i = N, \cdots, 3$,
   - Find $u_i^- \in H^1(\mathbb{R}^2)$ such that
     \begin{equation}
     J^{-1} \nabla \cdot (A \nabla u_i^-) + k^2 u_i^- = -\tilde{f}_i^- \quad \text{in } \mathbb{R}^2
     \end{equation}
   - Compute
     \begin{equation}
     \Psi_{i-1}^-(\tilde{f}_i^-) = J^{-1} \nabla \cdot (A \nabla (\beta_{i-1}^- u_i^-)) + k^2 (\beta_{i-1}^- u_i^-).
     \end{equation}
   - Set $\bar{f}_{i-1} = f_{i-1} + \Psi_{i-1}^- (\tilde{f}_i^-)$ in $\Omega_{i-1}$ and $\tilde{f}_{i-1}^+ = 0$ elsewhere.
End while
3. For $i = 2$, find $u_2^- \in H^1(\mathbb{R}^2)$ such that
   \begin{equation}
   J^{-1} \nabla \cdot (A \nabla u_2^-) + k^2 u_2^- = -f_1 - \tilde{f}_2^- \quad \text{in } \mathbb{R}^2
   \end{equation}

By (2.10), (2.14) and (2.15), we know that $u_i^+$ is given by
\begin{equation}
   u_i^+(x) = \int_{\Omega_i \cup \Omega_{i+1}} (\tilde{f}_i^+ + f_{i+1}) J(y) G(\tilde{x}, y) dy \quad \forall x \in \mathbb{R}^2, \quad i = 1, \cdots, N - 1,
\end{equation}
and $u_i^-$ is given by
\begin{align}
   u_i^-(x) &= \int_{\Omega_i} \tilde{f}_i^- (y) J(y) G(\tilde{x}, y) dy \quad \forall x \in \mathbb{R}^2, \quad i = N, \cdots, 3, \\
   u_2^-(x) &= \int_{\Omega_1 \cup \Omega_2} (\tilde{f}_2^- (y) + f_1(y)) J(y) G(\tilde{x}, y) dy.
\end{align}
By simple calculation, we have the equivalent form of the source transfer operator \( \Psi_{i+1}^+ \):
\[
(2.19) \quad \Psi_{i+1}^+(f_i^+) = J^{-1} \nabla (A \nabla \beta_{i+1}^+ u_i^+) + J^{-1} \nabla \beta_{i+1}^+ \cdot (A \nabla u_i^+) - \beta_{i+1}^+ f_{i+1}.
\]
and it’s easily obtained that \( \Psi_{i+1}^+(f_i^+) + \beta_{i+1}^+ f_{i+1} \) is in \( L^2(\Omega_{i+1}) \) and supported in \( \Omega_{i+1} \). Similarly, we can get the equivalent form for \( \Psi_{i-1}^- \):
\[
(2.20) \quad \Psi_{i-1}^-(f_i^-) = J^{-1} \nabla (A \nabla \beta_{i-1}^- u_i^-) + J^{-1} \nabla \beta_{i-1}^- \cdot (A \nabla u_i^-).
\]

The proof of the following two lemmas is quit similar to Lemma 2.6 in [15]. We omit the details.

**Lemma 2.1.** For \( i = 1, \cdots, N - 2 \), we have \( u_i^+ \in H^1(\mathbb{R}^2) \) and \( \|u_i^+\|_{H^1(\mathbb{R}^2)} \leq C \|f\|_{H^1(\mathbb{R}^2)^\prime} \). Let \( M_0 = l, M_i = \sqrt{2}M + (1 + \sqrt{2})M_{i-1} \), where \( l \) is the diameter of \( B_i \) and \( M = \max(l_1, l_2) \). Then there exists a constant \( C > 0 \) such that
\[
\|u_i^+(x)\| + \|\Delta u_i^+(x)\| \leq Ce^{-\frac{k}{2}k^{m_0}|x|} \|f\|_{H^1(\mathbb{R}^2)^\prime} \forall |x| \geq M_i,
\]
where \( H^1(\mathbb{R}^2)^\prime \) is the dual space of \( H^1(\mathbb{R}^2) \).

**Lemma 2.2.** For \( i = N, \cdots, 3 \), we have \( u_i^- \in H^1(\mathbb{R}^2) \) and \( \|u_i^-\|_{H^1(\mathbb{R}^2)} \leq C \|f\|_{H^1(\mathbb{R}^2)^\prime} \). Let \( M_0 = l, M_i = \sqrt{2}M + (1 + \sqrt{2})M_{i-1} \), where \( l \) is the diameter of \( B_i \) and \( M = \max(l_1, l_2) \). Then there exists a constant \( C > 0 \) such that
\[
\|u_i^-(x)\| + \|\Delta u_i^-(x)\| \leq Ce^{-\frac{k}{2}k^{m_0}|x|} \|f\|_{H^1(\mathbb{R}^2)^\prime} \forall |x| \geq M_i,
\]
where \( H^1(\mathbb{R}^2)^\prime \) is the dual space of \( H^1(\mathbb{R}^2) \).

Lemma 2.1 and Lemma 2.2 show that \( u_i^+ \) and \( u_i^- \) decay exponentially at infinity, which will be used in the following theorems.

**Theorem 2.3.** The following assertions hold:

(i) For \( i = 1, \cdots, N - 2 \), we have, for any \( x \in \Omega(\zeta_{i+2}, +\infty) \),
\[
(2.21) \quad \int_{\Omega_i} f_i^+(y) \tilde{G}(x, y) dy = \int_{\Omega_{i+1}} \Psi_{i+1}^+(f_i^+) \tilde{G}(x, y) dy.
\]

(ii) For the solution \( u_i^+ \) in (2.10), we have, for any \( x \in \Omega_{i+1}, i = 1, \cdots, N - 1 \),
\[
(2.22) \quad u_i^+(x) = \int_{\Omega(-\infty, \zeta_{i+2})} f(y) \tilde{G}(x, y) dy.
\]

**Proof.** We first prove (2.21). By the property of (2.8) (cf. e.g. [15], 2.11-2.13, [17], Theorem 2.8 and [31], Theorem 4.1), we know that for any \( x \in \Omega(\zeta_{i+2}, +\infty) \) and \( y \in \Omega(\zeta_i, \zeta_{i+2}) \)
\[
\nabla_y \cdot (A \nabla_y (J^{-1} \tilde{G}(x, y))) + k^2 J (J^{-1} \tilde{G}(x, y)) = 0.
\]

For \( x \in \Omega(\zeta_{i+2}, +\infty), y \in \Omega_j, j = 1, \cdots, i+1 \), \( \tilde{G}(x, y) \) decays exponentially as \( |y| \to 0 \) (cf. [15], Lemma 2.5). By Lemma 2.1, we know that \( u_i^+(y) \) decays also exponentially at infinity. By integrating by parts, we have
\[
\int_{\Omega_i} f_i^+ \tilde{G}(x, y) dy = - \int_{\Omega(-\infty, \zeta_{i+1})} J^{-1} \nabla_y \cdot (A \nabla_y u_i^+(y)) + k^2 J u_i^+(y) \tilde{G}(x, y) dy
\]
\[
= - \int_{\Gamma_{i+1}} [(A \nabla_y u_i^+(y) \cdot e_2) J^{-1} \tilde{G}(x, y) - (A \nabla_y (J^{-1} \tilde{G}(x, y)) \cdot e_2) u_i^+(y)] ds(y),
\]
where \( e_2 \) is the unit vector in the \( x_2 \) axis. By using (2.9), we can do integration by parts to have

\[
\int_{\Omega_i} \tilde{f}_i^+ \tilde{G}(x,y) dy = \int_{\partial\Omega_{i+1}} \left[ (A \nabla_y (\beta^+_{i+1} u^+_i(y)) \cdot n) J^{-1} \tilde{G}(x,y) - (A \nabla_y (J^{-1} \tilde{G}(x,y)) \cdot n) \beta^+_{i+1} u^+_i(y) \right] ds(y)
\]

\[
= \int_{\Omega_{i+1}} J^{-1} [\nabla_y \cdot (A \nabla_y (\beta^+_{i+1} u^+_i(y))) + k^2 J \beta^+_{i+1} u^+_i(y)] \tilde{G}(x,y) dy
\]

\[
= \int_{\Omega_{i+1}} \Psi^+_{i+1}(\tilde{f}_i^+)(y) \tilde{G}(x,y) dy,
\]

where \( n \) is the unit outer normal to \( \partial\Omega_{i+1} \).

Since \( \tilde{y}(y) = y \) and \( J(y) = 1 \) for any \( y \in B_i \), By using (2.16) and (2.21) we could prove (2.22). For any \( x \in \Omega_{i+1} \)

\[
u_i^+(x) = \int_{\Omega_i \cup \Omega_{i+1}} (\tilde{f}_i^++f_{i+1})J(y)G(\tilde{x},\tilde{y})dy
\]

\[
= \int_{\Omega_{i+1}} f_{i+1}(y)G(\tilde{x},\tilde{y})dy + \int_{\Omega_i} f_i(y)G(\tilde{x},\tilde{y})dy + \int_{\Omega_i} \Psi^+_i(\tilde{f}_{i-1}^+)(y)J(y)G(\tilde{x},\tilde{y})dy
\]

\[
= \int_{\Omega_i \cup \Omega_{i+1}} f(y)G(\tilde{x},\tilde{y})dy + \int_{\Omega_{i-1}} \tilde{f}_{i-1}^+(y)J(y)G(\tilde{x},\tilde{y})dy
\]

\[
= \cdots = \int_{\Omega_{i-1} \cup \Omega_{i+1}} f(y)G(\tilde{x},\tilde{y})dy.
\]

This completes the proof. \( \square \)

The second step \( \square \) is similar to the first one \( \square \) of the PSTDDM for the PML equation in the whole space. So by argument similar to the proof above, we can easily obtain the following results.

**Theorem 2.4.** The following assertions hold:

(i) For \( i = N, \cdots, 3 \), we have, for any \( x \in \Omega(-\infty,\zeta_{i-1}) \),

\[
f_i^- \tilde{G}(x,y)dy = \int_{\Omega_{i-1}} \Psi_{i-1}^- (\tilde{f}_i^-)(y) \tilde{G}(x,y)dy.
\]

(ii) For the solution \( u^-_i \), \( i = N, \cdots, 3 \), in (2.14), we have, for any \( x \in \Omega_{i-1} \),

\[
u^-_i(x) = \int_{\Omega(\zeta_{i-1}^+)} f(y)G(\tilde{x},\tilde{y})dy.
\]

(iii) For the solution \( u^-_2 \) in (2.15), we have, for any \( x \in \Omega_1 \),

\[
u^-_2(x) = \int_{\Omega_1} f(y)G(\tilde{x},\tilde{y})dy.
\]

Combining Theorem 2.3 and Theorem 2.4 we could obtain the main result in this section.

**Theorem 2.5.** We define \( u^+_0(x) \) and \( u^-_{N+1}(x) \), for any \( x \in \Omega^2 \). For any \( x \in \Omega_i, i = 1, \cdots, N \), we have

\[
u(x) = -(u^-_{i-1}(x) + u^+_i(x))
\]
Proof. From (2.25), it’s easy to see that the lemma holds for \( i = 1 \). Using the definition of \( \tilde{u}(x) \) (2.4) and (2.22), (2.24), we have, for any \( x \in \Omega_i, i = 2, \ldots, N, \)

\[
\tilde{u}(x) = -\int_{\mathbb{R}^2} f(y)G(\bar{x}, \bar{y})dy \\
= -\left( \int_{\Omega(-\infty, \zeta_{i+1})} f(y)G(\bar{x}, \bar{y})dy + \int_{\Omega(\zeta_{i+1}, +\infty)} f(y)G(\bar{x}, \bar{y})dy \right) \\
= -(u_i^{L-1}(x) + u_i^{L+1}(x)),
\]

where we have used \( \tilde{y}(y) = y \) in \( B_l \). \( \Box \)

2.3. The PSTDDM for the PML equation in the truncated bounded domain. The PSTDDM for PML equation in the truncated bounded domain and the most important results in this paper are introduced in this section. First we introduce some notation. Let \( U \) be a bounded domain in \( \mathbb{R}^2 \) and \( \partial U = \Gamma \). Then the weighted norms are written as

\[
\|u\|_{H^1(U)} = \left( \|\nabla u\|_{L^2(U)}^2 + \|k u\|_{L^2(U)}^2 \right)^{1/2}, \|v\|_{H^{1/2}(\Gamma)} = \left( d_U^{-1} \|v\|_{L^2(\Gamma)}^2 + |v|_{L^1,\Gamma}^2 \right)^{1/2},
\]

where \( d_U = \text{diam}(U) \) and

\[
|v|_{L^1,\Gamma}^2 = \int_{\Gamma} \int_{\Gamma} \frac{|v(x) - v(x')|^2}{|x - x'|^2} ds(x)ds(x').
\]

The following inequality are given (cf. [15], 3.1),

\[
\|v\|_{H^{1/2}(\Gamma)} \leq (|\Gamma| d_U^{-1})^{1/2} \|v\|_{L^\infty(\Gamma)} + |\Gamma| \|\nabla v\|_{L^\infty(\Gamma)}, \forall v \in W^{1,\infty}(\Gamma),
\]

The inequality (2.26) is easily derived from the definition of weighted norms.

For simplicity, the following assumption about the medium property is adopted:

\[
H1 \quad \int_{l_1}^{l_1 + d_1} \sigma_1(t) dt = \int_{l_2}^{l_2 + d_2} \sigma_2(t) dt =: \bar{\sigma}, \quad \int_{l_1}^{l_1 + d_1} \sigma_1(t) dt \geq \bar{\sigma}
\]

and \( l_1 \leq l_2, \quad d_1 = 2d_2. \)

This assumption is not essential. Those lemmas and theorems are also valid with a bit modification of the proof if the assumption is changed.

Denote by \( B_L = (-l_1 - d_1, l_1 + d_1) \times (-l_2 - d_2, l_2 + d_2) \) where \( l_1 + d_1 > l_1 \) and \( l_2 + d_2 > l_2 \). Clearly, \( B_L \) contains \( B_l \). We will show how to get a approximation of the solution to the PML equation in the truncated domain \( B_L \).

We introduce local PML problems by using the PML complex coordinate stretching outside the domain \((-l_1, l_1) \times (\zeta_1, \zeta_{i+2})\). The PML stretching is \( \tilde{x}_i(x) = (\tilde{x}_{i,1}(x_1), \tilde{x}_{i,2}(x_2))^T \), which has been proposed in [15], where \( \tilde{x}_{i,1}(x_1) = \tilde{x}_1(x_1) \) and

\[
\tilde{x}_{i,2}(x_2) = \begin{cases} 
 x_2 + i \int_{\zeta_{i+2}}^{x_2} \sigma_2(t + \zeta_{N+1} - \zeta_{i+2})dt & \text{if } x_2 > \zeta_{i+2}, \\
 x_2 + i \int_{\zeta_{i}}^{x_2} \sigma_2(t - \zeta_i + \zeta_1)dt & \text{if } \zeta_i \leq x_2 \leq \zeta_{i+2}, \\
 x_2 + i \int_{\zeta_{i+2}}^{x_2} \sigma_2(t - \zeta_i + \zeta_1)dt & \text{if } x_2 < \zeta_i.
\end{cases}
\]

We define

\[
A_i(x) = \text{diag} \left( \begin{array}{c} \tilde{x}_{i,2}(x_2) \\ \tilde{x}_{i,1}(x_1) \\ \tilde{x}_{i,2}(x_2) \end{array} \right), \quad J_i(x) = \tilde{x}_{i,1}(x_1) \tilde{x}_{i,2}(x_2).
\]
Denote by $\Omega_{i}^{\text{PML}} = \{x = (x_1, x_2) \in B_L : \zeta_i - d_2 \leq x_2 \leq \zeta_{i+2} + d_2\}$. The local PML problems in truncated domains can be defined for some wave source $F \in H^1(\Omega_{i}^{\text{PML}})$ as: find $\phi \in H^1(\Omega_{i}^{\text{PML}})$ such that

\begin{equation}
(A_{i} \nabla \phi, \nabla \psi) - k^2(J_{i} \phi, \psi) = - (J F, \psi) \quad \forall \psi \in H^1(\Omega_{i}^{\text{PML}}).
\end{equation}

Then our PSTDDDM for the PML equation in a truncated bounded domain consist of two steps\cite{3} and \cite{4}.

Before proving the convergence of the method, we introduce some functions and an important result which would be used often. The functions are $\bar{u}_i^+, i = 1, \cdots, N - 1$, and $\bar{u}_i^-, i = N, \cdots, 2$ which are defined as:

\begin{align}
\bar{u}_i^+(x) &= \int_{\Omega_i \cup \Omega_{i+1}} ((f_i^+(y) + f_{i+1}(y))J_i(y)G(\bar{x}_i, \bar{y}_i)dy, \quad i = 1, \cdots, N - 1, \\
\bar{u}_i^-(x) &= \int_{\Omega_i} f_i^-(y)J_{i-1}(y)G(\bar{x}_{i-1}, \bar{y}_{i-1})dy, \quad i = N, N - 1, \cdots, 3, \\
\bar{u}_2^+(x) &= \int_{\Omega_1 \cup \Omega_2} (f_2^+ + f_1)J_1(y)G(\bar{x}_1, \bar{y}_1)dy.
\end{align}

The result is about the inf − sup condition of the PML equations in truncated domains.

**Theorem 2.6.** Let $\sigma_0 d_2$ be sufficiently large. There’s some constant $\alpha < 1$ such that

\begin{equation}
\sup_{\phi \in H^1(\Omega_{i}^{\text{PML}})} \frac{(A_{i} \nabla \phi, \nabla \psi) - k^2(J_{i} \phi, \psi)}{\|\psi\|_{H^1(\Omega_{i}^{\text{PML}})}} \geq \mu \|\phi\|_{H^1(\Omega_{i}^{\text{PML}})} \quad \forall \phi \in H^1(\Omega_{i}^{\text{PML}}),
\end{equation}

where $\mu^{-1} \leq C k^{1+\alpha}$. $C$ is independent of $k$.

We remark that the recent work (cf. [5], 3.16) of Chen and Xiang shows that the inequality in the theorem above holds for $\alpha = 1/2$. Besides, we know that the inf-sup condition number is about $k^{-1}$ (cf. [22], [11]) for the Helmholtz problem (1.1) with Sommerfeld radiation condition [12] or Robin boundary condition.

The proof of the following lemma is omitted. The reader can complete it easily by arguments similar to Lemma 3.5--Lemma 3.7 in [15].

**Lemma 2.7.** Let $\sigma_0 d_2 \geq 1$ be sufficiently large, we have

(i) For $i = 1, \cdots, N - 1$, 
\[\|\bar{u}_i^+\|_{H^{1/2}(\partial \Omega_{i+1}^{\text{PML}})} \leq C k(1 + kL)e^{-\frac{1}{2}k\gamma \bar{\sigma}} \|f\|_{H^1(B_i)^{\prime}}.\]

(ii) For $i = N, N - 1, \cdots, 2$, 
\[\|\bar{u}_i^-\|_{H^{1/2}(\partial \Omega_{i-1}^{\text{PML}})} \leq C k(1 + kL)e^{-\frac{1}{2}k\gamma \bar{\sigma}} \|f\|_{H^1(B_i)^{\prime}}.\]

**Theorem 2.8.** Let $\sigma_0 d_2 \geq 1$ be sufficiently large, we have

(i) For $i = 2, \cdots, N - 1$, 
\[\|\bar{f}_i^+ - \hat{f}_i^+\|_{H^{-1}(\Omega_{i}^{\text{PML}})} \leq C k^{\alpha(i-1)} k(1 + kL)^2 e^{-\frac{1}{2}k\gamma \bar{\sigma}} \|f\|_{H^1(B_i)^{\prime}}.\]

(ii) For $i = N - 1, \cdots, 2$, 
\[\|\bar{f}_i^- - \hat{f}_i^-\|_{H^{-1}(\Omega_{i}^{\text{PML}})} \leq C k^{\alpha(N-i)} k(1 + kL)^2 e^{-\frac{1}{2}k\gamma \bar{\sigma}} \|f\|_{H^1(B_i)^{\prime}}.\]
Algorithm 3 Source Transfer I for Truncated PML problem

1. Let $f_1^+ = f_1$;
2. While $i = 1, \ldots, N-2$, do
   \begin{itemize}
   \item Find $\hat{u}_i^+ \in H^1_0(\Omega_i^{PML})$, where $\Omega_i^{PML} = (-l_i - d_i, l_i + d_i) \times (\zeta_i - d_i, \zeta_i + d_i)$, such that
   \end{itemize}
   \begin{equation}
   (A_i \nabla \hat{u}_i^+, \nabla \psi) - k^2 (J_i \hat{u}_i^+, \psi) = \left\langle J_i (f_i^+ + f_{i+1}), \psi \right\rangle \quad \forall \psi \in H^1_0(\Omega_i^{PML}),
   \end{equation}
   \begin{itemize}
   \item Compute $\hat{\Psi}_{i+1}^+(\hat{f}_i^+) \in H^{-1}(\Omega_i^{PML})$ such that
   \begin{equation}
   \hat{\Psi}_{i+1}^+(\hat{f}_i^+) = J_i^{-1} (A_i \nabla (\beta_{i+1}^+ \hat{u}_i^+)) + k^2 (\beta_{i+1}^+ \hat{u}_i^+).
   \end{equation}
   \item Set $\hat{f}_{i+1}^+ = f_{i+1} + \hat{\Psi}_{i+1}^+(\hat{f}_i^+)$ in $\Omega_{i+1} \cap B_L$ and $\hat{f}_{i+1}^+ = 0$ elsewhere.
   \end{itemize}
End while
3. For $i = N-1$, find $\hat{u}_{N-1}^+ \in H^1_0(\Omega_{N-1}^{PML})$ where $\Omega_{N-1}^{PML} = (-l_1 - d_1, l_1 + d_1) \times (\zeta_{N-1} - d_1, \zeta_{N-1} + d_1)$, such that $\forall \psi \in H^1_0(\Omega_{N-1}^{PML})$
   \begin{equation}
   (A_{N-1} \nabla \hat{u}_{N-1}^+, \nabla \psi) - k^2 (J_{N-1} \hat{u}_{N-1}^+, \psi) = \left\langle J_i (\hat{f}_{N-1}^+ + f_N), \psi \right\rangle.
   \end{equation}

Algorithm 4 Source Transfer II for Truncated PML problem

1. Let $f_N^- = f_N$;
2. While $i = N, \ldots, 3$
   \begin{itemize}
   \item Find $\hat{u}_i^- \in H^1_0(\Omega_i^{PML})$ such that
   \end{itemize}
   \begin{equation}
   (A_{i-1} \nabla \hat{u}_i^-, \nabla \psi) - k^2 (J_{i-1} \hat{u}_i^-, \psi) = \left\langle J_{i-1} \hat{f}_i^-, \psi \right\rangle \quad \forall \psi \in H^1_0(\Omega_{i-1}^{PML}),
   \end{equation}
   \begin{itemize}
   \item Compute $\hat{\Psi}_{i-1}^-(\hat{f}_i^-) \in H^{-1}(\Omega_i^{PML})$ such that
   \begin{equation}
   \hat{\Psi}_{i-1}^-(\hat{f}_i^-) = J_{i-1}^{-1} (A_{i-1} \nabla (\beta_{i-1}^- \hat{u}_i^-)) + k^2 (\beta_{i-1}^- \hat{u}_i^-).
   \end{equation}
   \item Set $\hat{f}_{i-1}^- = f_{i-1} + \hat{\Psi}_{i-1}^-(\hat{f}_i^-)$ in $\Omega_{i-1}$ and $\hat{f}_{i-1}^- = 0$ elsewhere.
   \end{itemize}
End while
3. For $i=2$, find $\hat{u}_2^- \in H^1_0(\Omega_1^{PML})$ such that $\forall \psi \in H^1_0(\Omega_1^{PML})$
   \begin{equation}
   (A_1 \nabla \hat{u}_2^-, \nabla \psi) - k^2 (J_1 \hat{u}_2^-, \psi) = \left\langle J_1 (\hat{f}_2^- + f_1), \psi \right\rangle.
   \end{equation}

Proof. From the expressions (2.16) and (2.30), we have that $u_i^+(x) = \tilde{u}_i^+(x)$ for $x \in \Omega_i \cap \Omega_{i+1}$, which implies

$$
\hat{\Psi}_{i+1}(\tilde{f}_i^+) = J_i^{-1} \nabla \cdot (A_i \nabla (\beta_{i+1}^+ \tilde{u}_i^+)) + k^2 (\beta_{i+1}^+ \tilde{u}_i^+).
$$
By simple calculation, we have for any $\psi \in H^1_0(\Omega^\text{PML}_{i+1})$,
\[
\langle J_i \Psi^+_{i+1}(\bar{f}^+_i), \psi \rangle = -\langle A_i \nabla \beta^+_{i+1} \bar{u}^+_i, \nabla \psi \rangle_{\Omega_{i+1}} + \langle A_i \nabla \bar{u}^+_i, \nabla \beta^+_{i+1} \psi \rangle_{\Omega_{i+1}} - \langle f^+_{i+1}, \beta^+_{i+1} \psi \rangle,
\]
and
\[
\langle J_i \hat{\Psi}^+_{i+1}(\bar{f}^+_i), \psi \rangle = -\langle A_i \nabla \beta^+_{i+1} \bar{u}^+_i, \nabla \psi \rangle_{\Omega_{i+1}} + \langle A_i \nabla \hat{u}^+_i, \nabla \beta^+_{i+1} \psi \rangle_{\Omega_{i+1}} - \langle f^+_{i+1}, \beta^+_{i+1} \psi \rangle.
\]

Therefore,
\[
\langle J_i \Psi^+_{i+1}(\bar{f}^+_i) - \hat{f}^+_i), v \rangle = \langle J_i \Psi^+_{i+1}(\bar{f}^+_i) - \hat{\Psi}^+_{i+1}(\bar{f}^+_i), v \rangle_{\Omega_{i} \cap B_L} + (A_i \nabla \bar{u}^+_i, \nabla v)_{\Omega_{i} \cap B_L} + \langle A_i \nabla (\bar{u}^+_i - \hat{u}^+_i), \nabla \beta^+_i v \rangle_{\Omega_{i} \cap B_L} \leq C_{i, 1} \| \bar{u}^+_i - \hat{u}^+_i \|_{H^1(\Omega_{i} \cap B_L)} \| v \|_{H^1(\Omega_{i} \cap B_L)}.
\]

On the other hand, $\bar{u}^+_i - \hat{u}^+_i = \bar{u}^+_i - \hat{u}^+_i$ on $\partial \Omega^\text{PML}_{i-1}$ and for any $\psi \in H^1_0(\Omega^\text{PML}_{i-1})$
\[
\langle A_i \nabla (\bar{u}^+_i - \hat{u}^+_i), \nabla \psi \rangle - k^2 (J_i (\bar{u}^+_i - \hat{u}^+_i), \psi) = \langle J_i (\bar{f}^+_i - \hat{f}^+_i), \psi \rangle.
\]

By the inf-sup condition 2.33 and Lemma 2.7 we have
\[
\| \bar{u}^+_i - \hat{u}^+_i \|_{H^1(\Omega^\text{PML}_{i-1})} \leq Ck^{1+\alpha} \| \bar{f}^+_i - \hat{f}^+_i \|_{H^{-1}(\Omega^\text{PML}_{i-1})}
\]
\[+ Ck^{1+\alpha} (1 + kL) \| \bar{u}^+_i - \hat{u}^+_i \|_{H^{1/2}(\partial \Omega^\text{PML}_{i-1})} \leq Ck^{1+\alpha} \| \bar{f}^+_i - \hat{f}^+_i \|_{H^{-1}(\Omega^\text{PML}_{i-1})} + Ck^{2+\alpha} (1 + kL)^2 e^{-\frac{1}{2} k \gamma \delta} \| f \|_{H^1(B)}.
\]

Therefore,
\[
\| \bar{f}^+_i - \hat{f}^+_i \|_{H^{-1}(\Omega^\text{PML}_{i-1})} \leq Ck^{\alpha} \| \bar{f}^+_i - \hat{f}^+_i \|_{H^{-1}(\Omega^\text{PML}_{i-1})} + Ck^{1+\alpha} (1 + kL)^2 e^{-\frac{1}{2} k \gamma \delta} \| f \|_{H^1(B)}.
\]

(i) follows from the induction argument and the fact that $\bar{f}^+_1 - \hat{f}^+_1 = 0$. Finally, we could prove (ii) by an argument similar to that of (i). This completes the proof. ☐

**Lemma 2.9.** Let $\sigma_0 d_2 \geq 1$ be sufficiently large.

(i) For $i = 1, 2, \cdots, N - 1$,
\[
\| \bar{u}^+_i - \hat{u}^+_i \|_{H^1(\Omega^\text{PML}_{i-1})} \leq Ck^{\alpha i+2} (1 + kL)^2 e^{-\frac{1}{2} k \gamma \delta} \| f \|_{H^1(B)}.
\]

(ii) For $i = N, \cdots, 2$,
\[
\| \bar{u}^+_i - \hat{u}^+_i \|_{H^1(\Omega^\text{PML}_{i-1})} \leq Ck^{\alpha (N-i+1)+2} (1 + kL)^2 e^{-\frac{1}{2} k \gamma \delta} \| f \|_{H^1(B)}.
\]

**Proof.** By using the fact that $\bar{u}^+_i - \hat{u}^+_i = \bar{u}^+_i$ on $\partial \Omega^\text{PML}_i$ and for any $\psi \in H^1_0(\Omega^\text{PML}_i)$
\[
\langle A_i \nabla (\bar{u}^+_i - \hat{u}^+_i), \nabla \psi \rangle - k^2 (J_i (\bar{u}^+_i - \hat{u}^+_i), \psi) = \langle J_i (\bar{f}^+_i - \hat{f}^+_i), \psi \rangle,
\]
Lemma 2.9 and the inf–sup condition (2.33), the first result can be obtained. The second result can be proved similarly. □

**Theorem 2.10.** We define \( \tilde{u}_0^+ = \tilde{u}_{N+1} = 0 \) in \( \mathbb{R}^2 \). Let \( \tilde{v} = -(\tilde{u}_{i-1}^- + \tilde{u}_{i+1}^-) \) in \( \Omega_i \cap B_L \) for all \( i = 1, 2, \cdots, N \). Then for sufficiently large \( \sigma_0 d_2 \geq 1 \), we have

\[
\| \tilde{u} - \tilde{v} \|_{H^1(B_L)} \leq Ck^{\alpha(N-1)}k^2(1 + kL)^2e^{-\frac{1}{2}k^\beta} \| f \|_{H^1(B_L)},
\]

where \( \tilde{u} \) is the solution to the PML problem (2.6) in the whole space.

**Proof.** From their definitions, we know that \( u_i^+ = \tilde{u}_i^+ \) and \( u_i^- = \tilde{u}_i^- \) for \( i = 1, \cdots, N \). We also define \( \tilde{u}_0^+ = \tilde{u}_{N+1} = 0 \) in \( \mathbb{R}^2 \). Combining with Theorem 2.8, we have \( \tilde{u}(x) = -(\tilde{u}_{i-1}^- (x) + \tilde{u}_{i+1}^- (x)) \) for any \( x \in \Omega_i, i = 1, \cdots, N \). Then, by using Lemma 2.9, we complete the proof. □

Theorem 2.10 shows that the solution \( \tilde{v} \) obtained by our PSTDDM is a fine approximation of the exact solution to the PML problem in the whole space. We remark that the constant ‘C’ in (2.38) generally depends on \( \Omega^i_{PML} \) and the number \( N \) of layers due to the inf–sup condition and the induction argument used in the proofs which are omitted (cf. [15], Theorem 3.7).

3. **Source transfer block by block.** Since every local PML problem in our PSTDDM (cf. 1 and 2) is defined on the whole space \( \mathbb{R}^2 \), we can use the PSTDDM to solve the local PML problems (cf. 2.10 and 2.14) recursively. As a consequence, the domain \( B_L \) is divided into some small squares and we only need to solve local PML problems defined outside the union of four squares.

3.1. **The PSTDDM for the PML problem in \( \mathbb{R}^2 \).** We only show the algorithms (cf. 5 and 6) to solve the local PML equation (2.10). In order to show the details of our method, Some notations are introduced. For simplicity, we set \( l_1 = l_2 \), \( d_1 = d_2 \). In order to use the results obtained in the previous sections, the following assumption about the medium property is adopted:

\[
\text{H2} \quad \int_{l}^{l+d/2} \sigma_1(t)dt \geq \bar{\sigma}, \quad \int_{l}^{l+d/2} \sigma_2(t)dt \geq \bar{\sigma},
\]

and

\[
\int_{l+d/2}^{l+d} \sigma_1(t)dt \geq \bar{\sigma}, \quad \int_{l+d/2}^{l+d} \sigma_2(t)dt \geq \bar{\sigma}.
\]

which is a direct consequence of the assumption H1 (2.27).

We divide the whole space into layers:

\[
\Omega^0 = \{ x = (x_1, x_2)^T \in \mathbb{R}^2 : x_1 < \zeta_1 \},
\]

\[
\Omega^i = \{ x = (x_1, x_2)^T \in \mathbb{R}^2 : \zeta_i < x_1 < \zeta_{i+1} \}, \quad i = 1, \cdots, N,
\]

\[
\Omega^{N+1} = \{ x = (x_1, x_2)^T \in \mathbb{R}^2 : \zeta_{N+1} < x_1 \},
\]

and denote by \( \Omega_{i,j} = \Omega_i \cap \Omega^j \).

We define the PML complex coordinate stretching \( \tilde{x}^{i,j} = (\tilde{x}_1^{i,j}(x_1), \tilde{x}_2^{i,j}(x_2)) \) outside the domain \( (\zeta_i, \zeta_{i+1}) \) by \( \tilde{x}^{i,j}(x_2) = \tilde{x}_{i,2}(x_2) \) and

\[
\tilde{x}^{i,j}(x_1) = \begin{cases} 
  x_1 + i \int_{\zeta_i}^{\zeta_{i+1}} \sigma_1(t + \zeta_{i+1} - \zeta_j)dt & \text{if } x_1 > \zeta_{j+2}, \\
  x_1 + i \int_{\zeta_i}^{\zeta_j} \sigma_1(t - \zeta_j + \zeta_i)dt & \text{if } \zeta_j \leq x_1 \leq \zeta_{j+2}, \\
  x_1 & \text{if } x_1 < \zeta_{j+1}.
\end{cases}
\]
Then the PML equation’s coefficients are defined as

\[ A_{i,j}(x) = \text{diag}\left( \frac{x_{1j}^j(x_2)}{x_{1j}^j(x_2)}, \frac{x_{1i}^j(x_1)}{x_{2i}^j(x_2)}, \frac{x_{1j}^j(x_1)}{x_{2j}^j(x_2)} \right), \quad J_{i,j}(x) = \frac{x_{1j}^j(x_1) x_{2j}^j(x_2)}{x_{1j}^j(x_2)}. \]

Let \( \tilde{f}_{i,j}^+ = \tilde{f}_i^+ + f_{i+1} \) in \( \Omega^1 \) and \( \tilde{f}_{i,j}^+ = 0 \) elsewhere. We denote by \( \gamma_1^+(x_1) \) and \( \gamma_1^-(x_1) \) smooth functions such that \( \gamma_1^+(t) = \beta_1^+(t) \) and \( \gamma_1^-(t) = \beta_1^-(t) \) for any \( t \in \mathbb{R} \).

**Algorithm 5** Source Transfer I\(^+\) for the \( i^{th} \) local PML problem

1. Let \( \tilde{f}_{i,1}^+ = \tilde{f}_{i+1}^+ \) in \( \Omega^1 \) and \( \tilde{f}_{i,1}^+ = 0 \) elsewhere.
   while \( j = 1, \cdots, N-2 \), do
     \begin{itemize}
     \item Find \( \bar{u}_{i,j}^+ \in H^1(\mathbb{R}^2) \), such that
       \begin{equation}
       -\nabla (A_{i,j} \nabla \bar{u}_{i,j}^+) - k^2 J_{i,j} \bar{u}_{i,j}^+ = J_{i,j}(\tilde{f}_{i,j}^+ + \tilde{f}_{i,j+1}^+),
       \end{equation}
     \item Compute \( \bar{\Psi}_{i,j+1}^+(\tilde{f}_{i,j}^+) \in H^{-1}(\mathbb{R}^2) \) such that
       \begin{equation}
       \bar{\Psi}_{i,j+1}^+(\tilde{f}_{i,j}^+) = J_{i,j}^{-1} \nabla (A_{i,j} \nabla (\gamma_{i,j+1}^+ \bar{u}_{i,j}^+)) + k^2 (\gamma_{i,j+1}^+ \bar{u}_{i,j}^+).
       \end{equation}
     \item Set \( \bar{f}_{i,j+1}^+ = \bar{f}_{i,j+1}^+ + \bar{\Psi}_{i,j+1}^+ \) in \( \Omega^{j+1} \) and \( \bar{f}_{i,j+1}^+ = 0 \) elsewhere.
   \end{itemize}
   End while
2. Find \( \bar{u}_{i,N-1}^+ \in H^1(\mathbb{R}^2) \), such that
   \begin{equation}
   -\nabla (A_{i,N-1} \nabla \bar{u}_{i,N-1}^+) - k^2 J_{i,N-1} \bar{u}_{i,N-1}^+ = J_{i,N-1}(\tilde{f}_{i,N-1}^+ + \tilde{f}_{i,N}^+).
   \end{equation}

**Algorithm 6** Source Transfer I\(^-\) for the \( i^{th} \) local PML problem

1. Let \( \bar{f}_{i,N} = \tilde{f}_{i,N}^+ \).
   While \( j = N, \cdots, 3 \), do
     \begin{itemize}
     \item Find \( \bar{u}_{i,j}^- \in H^1(\mathbb{R}^2) \), such that
       \begin{equation}
       -\nabla (A_{i,j-1} \nabla \bar{u}_{i,j}^-) - k^2 J_{i,j-1} \bar{u}_{i,j}^- = J_{i,j-1} \bar{f}_{i,j}^-,
       \end{equation}
     \item Compute \( \bar{\Psi}_{i,j-1}^-(\bar{f}_{i,j}^-) \in H^{-1}(\mathbb{R}^2) \) such that
       \begin{equation}
       \bar{\Psi}_{i,j-1}^-(\bar{f}_{i,j}^-) = J_{i,j-1}^{-1} \nabla (A_{i,j-1} \nabla (\gamma_{j-1}^- \bar{u}_{i,j}^-)) + k^2 (\gamma_{j-1}^- \bar{u}_{i,j}^-).
       \end{equation}
     \item Set \( \bar{f}_{i,j-1}^- = \bar{f}_{i,j-1}^- + \bar{\Psi}_{i,j-1}^- \) in \( \Omega^{j-1} \) and \( \bar{f}_{i,j-1}^- = 0 \) elsewhere.
   \end{itemize}
   End while
2. Find \( \bar{u}_{i,2}^+ \in H^1(\mathbb{R}^2) \) such that
   \begin{equation}
   -\nabla (A_{i,1} \nabla \bar{u}_{i,2}^+) - k^2 J_{i,1} \bar{u}_{i,2}^+ = J_{i,1}(\bar{f}_{i,2}^- + \bar{f}_{i,1}^+).
   \end{equation}

Algorithm 5 and Algorithm 6 show the details of our PSTDDM solving the \( i \)-th PML problem in Algorithm 1. We omit the details about Algorithm 2 to save the
space. Then we can obtain some results similar to those in section 2.2, but only state briefly them when needed.

The following lemma can be proved by their definitions. We omit the details.

**Lemma 3.1.** Let \( \bar{u}_{i,0}^+ \equiv 0 \) and \( \bar{u}_{i,N+1}^- \equiv 0 \). For any \( x \in \Omega_{i,j} \),

\[
\bar{u}_i^+(x) = \bar{u}_{i,j-1}^+(x) + \bar{u}_{i,j+1}^-(x).
\]

**Lemma 3.2.** Let \( \sigma_0 d > 1 \) be sufficiently large. For \( i = 1, \ldots, N-1 \)

(i) For \( j = 1, \ldots, N-2 \), we have for any \( x \in \Omega_{i,j+1}^+ := \{ x = (x_1, x_2)^T : \zeta_{j+1} < x_1 < \zeta_{j+2} \text{ and } |x_2 - \zeta_{j+1}| > \Delta \zeta + d/2 \} \),

\[
|\bar{u}_{i,j}^+(x)| \leq Ck e^{-\frac{1}{2}k\gamma^2}(\|f\|_{H^1(\Omega_i)} + \|\bar{f}\|_{H^1(\Omega_i)})\]

\[
|\nabla \bar{u}_{i,j}^+(x)| \leq Ck^2 e^{-\frac{1}{2}k\gamma^2}(\|f\|_{H^1(\Omega_i)} + \|\bar{f}\|_{H^1(\Omega_i)})\]

(ii) For \( j = N, \ldots, 3 \), we have for any \( x \in \Omega_{i,j-1}^+ \),

\[
|\bar{u}_{i,j}^-(x)| \leq Ck e^{-\frac{1}{2}k\gamma^2}(\|f\|_{H^1(\Omega_i)} + \|\bar{f}\|_{H^1(\Omega_i)})\]

\[
|\nabla \bar{u}_{i,j}^-(x)| \leq Ck^2 e^{-\frac{1}{2}k\gamma^2}(\|f\|_{H^1(\Omega_i)} + \|\bar{f}\|_{H^1(\Omega_i)})\]

Here \( \gamma = \frac{d}{\sqrt{d^2 + (2\ell + 2d)^2}} \).

**Proof.** We give the proof of the first assertion and the second one could be proved by the same argument. For \( j = 1, \ldots, N-2 \), \( \bar{u}_{i,j}^+(x) \) satisfies

\[
\bar{u}_{i,j}^+(x) = \int_{x_1 < \zeta_{j+2}} f(y) J(y) G(\bar{x}, \bar{y}) dy + \int_{x_1 < \zeta_{j+1}} \bar{f}_i(y) J(y) G(\bar{x}, \bar{y}) dy
\]

\[
:= \bar{u}_{i,j}^+(x) + \bar{u}_{i,j}^{+\Pi}(x) \quad \forall x \in \Omega_{i,j+1}^+.
\]

It is clear that

\[
|\bar{u}_{i,j}^{+\Pi}(x)| \leq C \int_{\Omega_{i,j}} |\bar{f}_i(y)| |G(\bar{x}, \bar{y})| dy + C \int_{\Omega_{i,j}^\text{out}} |\bar{f}_i(y)| |G(\bar{x}, \bar{y})| dy,
\]

where \( \Omega_{i,j} = \{ x = (x_1, x_2)^T \in \Omega_i : -l - d/2 < x_1 < \zeta_{j+1} \} \) and \( \Omega_{i,j}^\text{out} = \{ x = (x_1, x_2)^T \in \Omega_i : x_1 < -l - d/2 \} \). By the standard argument (cf. [15], 3.11), we can get

\[
\int_{\Omega_{i,j}} |\bar{f}_i(y)| |G(\bar{x}, \bar{y})| dy \leq Ck^{1/2} e^{-\frac{1}{2}k\gamma^2} \|\bar{f}_i\|_{H^1(\Omega_i)}.
\]

We recall that \( \bar{u}_i(y) \) and \( \nabla \bar{u}_i(y) \) decay exponentially as \( |y| \to \infty \), that is

\[
|\bar{u}_i(y)| \leq Ck^{1/2} e^{-\frac{1}{2}k\gamma^2} \|f\|_{H^1(\Omega_i)} \quad \text{and} \quad |\nabla \bar{u}_i(y)| \leq Ck^{3/2} e^{-\frac{1}{2}k\gamma^2} \|f\|_{H^1(\Omega_i)}
\]

for \( |y| > l_d/2 \) when \( \sigma_0 d \) is large enough, which implies

\[
\int_{\Omega_{i,j}^\text{out}} |\bar{f}_i(y)| |G(\bar{x}, \bar{y})| dy \leq C \sup_{y \in \Omega_{i,j}^\text{out}} (|\bar{u}_{i+1}^-| + |\nabla \bar{u}_{i+1}^-|) \int_{\Omega_{i,j}^\text{out}} |G(\bar{x}, \bar{y})| dy
\]

\[
\leq Ck e^{-\frac{1}{2}k\gamma^2} \|f\|_{H^1(\Omega_i)}.
\]
Thus for any $x \in \Omega^+_{i,j+1}$,
\[ |\tilde{u}^+_i j(x)| \leq Cke^{-\frac{1}{2}k\gamma \delta}(\|f\|_{H^1(B_i')} + \|\tilde{f}_i\|_{H^1(\Omega_i')}).
\]

It’s so easy to get the estimates,
\[ |\tilde{u}^+_i j(x)| \leq Ck^{1/2}e^{-\frac{1}{2}k\gamma \delta}\|f\|_{H^1(B_i')}.
\]

Therefore, we have $|\tilde{u}^+_i j(x)| \leq Ck^{1/2}e^{-\frac{1}{2}k\gamma \delta}(\|f\|_{H^1(B_i')} + \|\tilde{f}_i\|_{H^1(\Omega_i')})$ for $x \in \Omega^+_{i,j+1}$. A similar argument implies that
\[ |\nabla \tilde{u}^+_i j(x)| \leq Ck^2e^{-\frac{1}{2}k\gamma \delta}(\|f\|_{H^1(B_i')} + \|\tilde{f}_i\|_{H^1(\Omega_i')}) \quad \forall x \in \Omega^+_{i,j+1}.
\]

This completes the proof. □

Define $l_N := 2l/N + 2d$ and $\Omega_{i,j}^{PML} := \{x = (x_1,x_2) : \zeta_i - d \leq x_1 \leq \zeta_{i+2} + d, \zeta_j - d \leq x_2 \leq \zeta_{j+2} + d\}$. We have the following lemma.

**Lemma 3.3.** Assume that $\sigma_0 d > 1$ be sufficiently large. There exists a constant $C_b$ independent of $l_N, k$ and $N$ such that

(i) For $j = 1,2,\cdots,N - 1$,
\[ \|\tilde{u}^+_i j\|_{H^{1/2}(\partial \Omega_{i,j}^{PML})} \leq C_b k^3(1 + kl_N)\|f\|_{H^1(B_i')};
\]

(ii) For $j = N, N - 1, \cdots, 2$,
\[ (3.7) \quad \|\tilde{u}^-_{i,j}\|_{H^{1/2}(\partial \Omega_{i,j}^{PML})} \leq C_b k^3(1 + kl_N)\|f\|_{H^1(B_i')}.
\]

**Proof.** The two assertions can be obtained by using arguments similar to those in Lemma 3.6 in [15]. We omit the details and show the result
\[ \|\tilde{u}^+_i j\|_{H^{1/2}(\partial \Omega_{i,j}^{PML})} \leq Ck^{3/2}(1 + kl_N)e^{-\frac{1}{2}k\gamma \delta}(\|f\|_{H^1(B_i')} + \|\tilde{f}_i\|_{H^1(\Omega_i')}).
\]

By the definition of source transfer operator (2.19), we have
\[ \|\tilde{f}_i\|_{H^1(\Omega_i')} \leq C \|\tilde{u}_{i-1}\|_{H^1(\Omega_{i-1}')} \leq Ck^{3/2}\|f\|_{H^1(B_i')}.
\]

Combining the two inequalities above, we have
\[ \|\tilde{u}^+_i j\|_{H^{1/2}(\partial \Omega_{i,j}^{PML})} \leq C_b k^3(1 + kl_N)e^{-\frac{1}{2}k\gamma \delta}\|f\|_{H^1(B_i')}.
\]

Clearly, the $C$’s used here are independent of $l_N, k$ and $N$. Thus we complete the proof of the first assertion and the second one can be proved by the same way. □

**3.2. The PSTDDM for the truncated PML problem.** For the ease of presentation, we denote
\[ \Omega_{i,1}^{tr} = (\zeta_i - d, \zeta_2) \times (\zeta_i - d, \zeta_{i+2} + d),
\]
\[ \Omega_{i,j}^{tr} = (\zeta_j, \zeta_{j+1}) \times (\zeta_i - d, \zeta_{i+2} + d), \quad j = 2, \cdots, N - 1,
\]
\[ \Omega_{i,N}^{tr} = (\zeta_N, \zeta_{N+1} + d) \times (\zeta_i - d, \zeta_{i+2} + d).
\]
We can get the approximation $\tilde{u}_i^+(x)$ of $\bar{u}_i^+(x)$, $i = 1, \ldots, N - 1$, in $\Omega_{i}^{\text{PML}}$ by using Algorithm 7 and Algorithm 8, where $\tilde{u}_i^+(x)$ are defined by

$$
(3.8) \quad \tilde{u}_i^+ = \tilde{u}_{i,1}^+ \text{ in } \Omega_{i,1}^{\text{tr}}, \quad \tilde{u}_i^+ = \tilde{u}_{i,N}^- \text{ in } \Omega_{i,N}^{\text{tr}} \\
\tilde{u}_i^+ = \tilde{u}_{i,j-1}^+ + \tilde{u}_{i,j+1}^+ \text{ in } \Omega_{i,j}^{\text{tr}}, \text{ for } j = 2, \ldots, N - 1.
$$

In Algorithm 7 and Algorithm 8 we have defined $\bar{f}_{i,j}^+ = \bar{f}_{i,j}^+ + f_{i+1}$ in $\Omega_{i,j}^{\text{tr}}$ and $\bar{f}_{i,j}^+$ are defined by

1. Let $\bar{f}_{i,1}^+ = f_1^+$ in $\Omega_{i,1}^{\text{PML}}$.

2. Compute $\hat{\Psi}_{i,j+1}^+ \in H^{-1}(\Omega_{i,j}^{\text{PML}})$ such that

$$
\hat{\Psi}_{i,j+1}^+ = J_{i}^{-1} \nabla \left( A_{i,j} \nabla (\beta_{i,j}^+ \tilde{u}_{i,j}^+) \right) + k^2 (\beta_{i,j}^+ \tilde{u}_{i,j}^+) .
$$

3. Set $\bar{f}_{i,j+1}^+ = f_{i+1} + \hat{\Psi}_{i,j+1}^+$ in $\Omega_{i,j+1} \cap B_L$ and $\bar{f}_{i,j+1}^+ = 0$ elsewhere.

We also could obtain the approximation $\tilde{u}_i^- (x)$ of $\bar{u}_i^- (x)$, $i = N, \ldots, 2$, in $\Omega_{i-1}^{\text{PML}}$. The details are omitted in order to save space.

**Algorithm 7** Source Transfer $I^+$ for local Truncated PML problem

1. Let $\bar{f}_{i,1}^+ = \bar{f}_{i,1}^+^$. While $j = 1, \ldots, N - 2$, do

   - Find $\tilde{u}_{i,j}^+ \in H_0^1(\Omega_{i,j}^{\text{PML}})$, where $\Omega_{i,j}^{\text{PML}} = (\zeta_i - d, \zeta_i + d) \times (\zeta_j - d, \zeta_j + d)$, such that $\forall \psi \in H_0^1(\Omega_{i,j}^{\text{PML}})$

   $$
   (3.9) \quad (A_{i,j} \nabla \tilde{u}_{i,j}^+, \nabla \psi) - k^2 (J_{i,j} \tilde{u}_{i,j}^+, \psi) = \left( J_i (\bar{f}_{i,j}^+ + \bar{f}_{i,j+1}^+), \psi \right),
   $$

   - Compute $\hat{\Psi}_{i,j+1}^+ (\bar{f}_{i,j}^+)$ $\in H^{-1}(\Omega_{i,j}^{\text{PML}})$ such that

   $$
   \hat{\Psi}_{i,j+1}^+ (\bar{f}_{i,j}^+) = J_{i,j} \nabla \left( A_{i,j} \nabla (\gamma_{i,j}^+ \tilde{u}_{i,j}^+) \right) + k^2 (\gamma_{i,j}^+ \tilde{u}_{i,j}^+) .
   $$

   - Set $\bar{f}_{i,j+1}^+ = \bar{f}_{i,j+1}^+ + \hat{\Psi}_{i,j+1}^+ (\bar{f}_{i,j}^+)$ in $D_{j+1} \cap \Omega_{i,j}^{\text{PML}}$ and $\bar{f}_{i,j+1}^+ = 0$ elsewhere.

End while

2. Find $\tilde{u}_{i,N-1}^- \in H_0^1(\Omega_{i,N-1}^{\text{PML}})$ where $\Omega_{i,N-1}^{\text{PML}} = (\zeta_i - d, \zeta_i + d) \times (\zeta_{N-1} - d, \zeta_{N+1} + d)$, such that $\forall \psi \in H_0^1(\Omega_{i,N-1}^{\text{PML}})$

$$
(3.10) \quad (A_{i,N-1} \nabla \tilde{u}_{i,N-1}^+, \nabla \psi) - k^2 (J_{i,N-1} \tilde{u}_{i,N-1}^+, \psi) = \left( J_{i,N-1} (\bar{f}_{i,N-1}^+ + \bar{f}_{i,N}^+), \psi \right).
$$

We can improve the local inf-sup condition (2.33).

**Lemma 3.4.** For sufficiently large $\sigma_0 d > 1$, we have the inf-sup condition for any $\phi \in H_0^1(\Omega_{i,j}^{\text{PML}})$

$$
(3.13) \quad \sup_{\psi \in H_0^1(\Omega_{i,j}^{\text{PML}})} \frac{(A_{i,j} \nabla \phi, \nabla \psi) - k^2 (J_{i,j} \phi, \psi)}{\|\psi\|_{H^1(\Omega_{i,j}^{\text{PML}})}} \geq \mu \|\phi\|_{H^1(\Omega_{i,j}^{\text{PML}})},
$$

where $\mu^{-1} \leq C_{\text{is}} (l_N k)^{3/2}$ if $l_N k$ large enough, and $\mu^{-1} \leq C_{\text{is}}$ if $l_N k \approx 1$. $C_{\text{is}}$ independent of $l_N, k$ and $N$. 

Algorithm 8 Source Transfer I− for local Truncated PML problem

1. Let \( f^{-}_{i,M} = f^{+}_{i,M}. \)
   While \( j = N, \cdots, 3, \)
   - Find \( u_{i,j}^{-} \in H^{1}_{0}(Ω^{PML}_{i,j-1}) \) such that \( \forall ψ \in H^{1}_{0}(Ω^{PML}_{i,j-1}) \)
     \[
     (A_{i,j-1} \nabla u_{i,j}^{-}, \nabla ψ) - k^2(J_{i,j-1} u_{i,j}^{-}, ψ) = \left\langle J_{i,j-1} f^{-}_{i,j}, ψ \right\rangle,
     \]
   - Compute \( Ψ^{-}_{i,j-1}(f^{-}_{i,j}) \in H^{-1}(Ω^{PML}_{i,j-1}) \) such that
     \[
     Ψ^{-}_{i,j-1}(f^{-}_{i,j}) = J^{-1}_{i,j-1} \nabla \left(A_{i,j-1} \nabla (γ^{-}_{i,j-1} u_{i,j}^{-}) \right) + k^2 (γ^{-}_{i,j-1} u_{i,j}^{-}).
     \]
   - Set \( f^{-}_{i,j-1} = f^{+}_{i,j-1} + Ψ^{-}_{i,j-1}(f^{-}_{i,j}) \) in \( Ω_{i,j-1} \) and \( f^{-}_{i,j-1} = 0 \) elsewhere.
   End while
2. Find \( u_{i,2}^{-} \in H^{1}_{0}(Ω^{PML}_{i,1}) \) such that \( ∀ ψ \in H^{1}_{0}(Ω^{PML}_{i,1}) \)
   \[
   (A_{i,1} \nabla u_{i,2}^{-}, \nabla ψ) - k^2(J_{i,1} u_{i,2}^{-}, ψ) = \left\langle J_{i,1} f^{-}_{i,2} + f^{+}_{i,1}, ψ \right\rangle.
   \]

Proof. The inequality can be proved easily by using scaling argument. We know that there is a unique solution \( φ \in H^{1}_{0}(Ω^{PML}_{i,j}) \) to the problem

\[
- \nabla (A_{i,j} \nabla φ) - k^2 J_{i,j} φ = F,
\]
for some \( F \in H^{1}_{0}(Ω^{PML}_{i,j})' \). We define a mapping \( m : I := [0, 1] \times [0, 1] → Ω^{PML}_{i,j} \) as \( m(z) = l_N z + (ζ_i - d, ζ_j - d) \) and denote by \( φ(z) := φ(m(z)) \) and \( F(z) := F(m(z)) \). The equation above implies \( \hat{φ}(z) ∈ H^{1}_{0}(I) \) satisfying

\[
- \nabla z (A_{i,j} (m(z)) \nabla z φ(z)) - (l_N k)^2 J_{i,j} (m(z)) \hat{φ}(z) = l_N^2 \hat{F}(z).
\]
If \( l_N k \) large enough, by the local inf-sup condition (2.33), we get

\[
(l_N k)^2 \left\| \hat{φ}(z) \right\|_{L^2(I)}^2 + \left| \hat{φ}(z) \right|_{H^1(I)}^2 \right)^{1/2} \leq C_{i_α}(l_N k)^{3/2} \left\| l_N^2 \hat{F}(z) \right\|_{H^1(I)',}
\]
where \( \left\| . \right\|_{H^1(I)'} \) is defined as

\[
\sup_{ψ ∈ H^1(I)} \frac{(ψ, \psi)}{(l_N k)^2 \left\| ψ \right\|_{L^2(I)}^2 + \left| ψ \right|_{H^1(I)}^2)^{1/2},
\]
from the definition of weighted norm \( \left\| u \right\|_{H^1(I)} \) at the beginning of section 2.3. However, if \( l_N k \approx 1 \) it’s known that the problem (3.14) is elliptic, then we have

\[
\left( \left| \hat{φ}(z) \right|_{L^2(I)}^2 + \left| \hat{φ}(z) \right|_{H^1(I)}^2 \right)^{1/2} \leq C_{i_α} \left\| l_N^2 \hat{F}(z) \right\|_{H^1(I)'},
\]
Clearly, \( C_{i_α} \) is independent of \( l_N, k \) and \( N \). Finally, The consequence is obtained by combining the inequalities (3.15), (3.16) and the fact that

\[
(l_N k)^2 \left\| \hat{φ}(z) \right\|_{L^2(I)}^2 = k^2 \left\| φ(x) \right\|_{L^2(Ω^{PML}_{i,j})}^2, \left| \hat{φ}(z) \right|_{H^1(I)}^2 = \left| φ(x) \right|_{H^1(Ω^{PML}_{i,j})}^2,
\]
\[
\left\| l_N^2 \hat{F}(z) \right\|_{H^1(I)'} = \left\| F(x) \right\|_{H^1(Ω^{PML}_{i,j})'}.
\]
In general, we can expect that \( l_N k \) is less than \( k \). If \( N \) is large enough such that \( l_N k \approx 1 \), the local truncated PML problems (cf. (3.9)-(3.12)) needed to be solved are about elliptic.

**Lemma 3.5.** Let \( \sigma_0d > 1 \) be sufficiently large. There are constants \( C_1 \) and \( C_{b2} \) independent of \( l_N, k \) and \( N \) such that

(i) For \( i = 1, 2, \cdots, N - 1 \),

\[
\| \tilde{u}_i^+ - \tilde{u}_i^- \|_{H^1(\Omega_{PML}^1)} \leq C_{b2} C_{k,N,i} k^3 (1 + k l_N)^2 e^{-\frac{1}{2} k \gamma \sigma} \| f \|_{H^1(B_i')}.
\]

(ii) For \( i = N, N - 1, \cdots, 2 \),

\[
\| \tilde{u}_i^+ - \tilde{u}_i^- \|_{H^1(\Omega_{PML}^1)} \leq C_{b2} C_{k,N,N+1-i} k^3 (1 + k l_N)^2 e^{-\frac{1}{2} k \gamma \sigma} \| f \|_{H^1(B_i')}.
\]

Here \( C_{k,N,j}, j \in \mathbb{N} \) are defined as

\[
C_{k,N,j} = \sum_{q=1}^{j} \left( \sum_{p=1}^{N-1} (C_i \mu^{-1})^p \right)^q.
\]

**Proof.** We only show the details of the proof for the first assertion and the second one could be proved similarly. At the beginning, we recall the property (cf. [15], Theorem 3.7) of source transfer operators that there’s a constant \( C_1 \) independent of \( l_N, k \) and \( N \), such that

\[
\| \tilde{f}_{i,j}^+ - \tilde{f}_{i,j}^- \|_{H^1(\Omega_{PML}^1)} \leq C_1 \| \tilde{u}_{i,j-1}^+ - \tilde{u}_{i,j-1}^- \|_{H^1(\Omega_{PML}^1)},
\]

\[
\| \tilde{f}_{i,j}^+ - \tilde{f}_{i,j}^- \|_{H^1(\Omega_{PML}^1)} \leq C_1 \| \tilde{u}_{i-1}^+ - \tilde{u}_{i-1}^- \|_{H^1(\Omega_{PML}^1)},
\]

for \( i, j = 2, \cdots, N - 1 \) from their definitions and calculations similar to (2.19)-(2.20). Using the argument in Lemma 2.9 and Lemma 3.3, it’s easy to get

\[
\| \tilde{u}_{i,j}^+ - \tilde{u}_{i,j}^- \|_{H^1(\Omega_{PML}^1)} \leq \mu^{-1} \left\| \left( \tilde{f}_{i,j}^+ + \tilde{f}_{i,j+1}^- \right) - \left( \tilde{f}_{i,j}^- + \tilde{f}_{i,j+1}^+ \right) \right\|_{H^1(\Omega_{PML}^1)},
\]

\[
+ \mu^{-1} (1 + k l_N) \| \tilde{u}_{i,j}^+ \|_{H^{1/2}(\partial_{PML}^1)} \leq C_1 \mu^{-1} \| \tilde{u}_{i,j-1}^+ - \tilde{u}_{i,j-1}^- \|_{H^1(\Omega_{PML}^1)} + \mu^{-1} \| \tilde{f}_{i,j}^- - \tilde{f}_{i,j}^+ \|_{H^1(\Omega_{PML}^1)} + C_{b2} k^3 (1 + k l_N)^2 e^{-\frac{1}{2} k \gamma \sigma} \| f \|_{H^1(B_i')}.
\]

By the induction argument and the fact that

\[
\| \tilde{u}_{i,1}^+ - \tilde{u}_{i,1}^- \|_{H^1(\Omega_{PML}^1)} \leq \mu^{-1} \| \tilde{f}_{i,1}^- - \tilde{f}_{i,1}^+ \|_{H^1(\Omega_{PML}^1)} + C_{b2} k^3 (1 + k l_N)^2 e^{-\frac{1}{2} k \gamma \sigma} \| f \|_{H^1(B_i')}.
\]

(3.20) implies for \( j = 1, \cdots, N - 1 \)

\[
\| \tilde{u}_{i,j}^+ - \tilde{u}_{i,j}^- \|_{H^1(\Omega_{PML}^1)} \leq \sum_{p=0}^{j-1} (C_i \mu^{-1})^p \cdot \mu^{-1} \cdot \left[ \| \tilde{f}_{i}^+ - \tilde{f}_{i}^- \|_{H^1(\Omega_{PML}^1)} + C_{b2} k^3 (1 + k l_N)^2 e^{-\frac{1}{2} k \gamma \sigma} \| f \|_{H^1(B_i')} \right].
\]
Similarly, we have for $j = N, \cdots, 2$

\begin{equation}
(3.22) \quad \| \bar{u}_{i,j} - \bar{\bar{u}}_{i,j} \|_{H^1(\Omega_{i,j}^{\text{pML}})} \leq \sum_{p=0}^{N-j} (C_t \mu^{-1})^p \cdot \mu^{-1}.
\end{equation}

\[ \left[ \| \bar{f}^+ - \bar{f}^+ \|_{H^1(\Omega_{i,j}^{\text{pML}})} \right] \leq \sum_{p=1}^{N-1} \left( C_t \mu^{-1} \right)^p \left[ \| \bar{u}_{i-1} - \bar{\bar{u}}_{i-1} \|_{H^1(\Omega_i \cap B_L)} \right] + C_b k^3 (1 + k l_i N)^2 e^{-\frac{1}{2} k \gamma \sigma} \| f \|_{H^1(B_i')}. \]

From (3.21), (3.22), Lemma 3.1 and the definition 3.8 we obtain

\begin{equation}
(3.23) \quad \| \bar{u}^+ - \bar{\bar{u}}^+ \|_{H^1(\Omega_{i}^{\text{pML}})} \leq \sum_{p=0}^{N-2} (C_t \mu^{-1})^p \cdot \mu^{-1}.
\end{equation}

\[ \left[ \| \bar{f}^+ - \bar{f}^+ \|_{H^1(\Omega_{i}^{\text{pML}})} \right] \leq \sum_{p=1}^{N-1} \left( C_t \mu^{-1} \right)^p \left[ \| \bar{u}_{i-1} - \bar{\bar{u}}_{i-1} \|_{H^1(\Omega_i \cap B_L)} \right] + C_b k^3 (1 + k l_i N)^2 e^{-\frac{1}{2} k \gamma \sigma} \| f \|_{H^1(B_i')}. \]

Since $C_t$ and $C_b$ don’t depend on $l_i, k$ and $N$, we can denote $C_{bt} = \frac{C_b}{C_t}$. Then we complete the proof for the first assertion (3.17) by the induction argument and the fact

\[ \| \bar{u}^+ - \bar{\bar{u}}^+ \|_{H^1(\Omega_{i}^{\text{pML}})} \leq \sum_{p=1}^{N-1} \left( C_t \mu^{-1} \right)^p C_{bt} k^3 (1 + k l_i N)^2 e^{-\frac{1}{2} k \gamma \sigma} \| f \|_{H^1(B_i')}. \]

The following theorem is a direct consequence of Theorem 2.5, Lemma 3.5 and the fact that $\bar{\bar{u}}^+ = \hat{\bar{u}}^+$ in $\Omega_i \cup \Omega_{i+1}$.

**Theorem 3.6.** Let $u_i = 0$ in $B_L$ and $\hat{u}(x) = - (\bar{\bar{u}}_{i-1} + \bar{\bar{u}}_{i+1})$ in $\Omega_i \cap B_L$ for all $i = 1, \cdots, N$. Denote $C_{k,N} = C_{k,N,N-1}$. Then for sufficiently large $\sigma_0 d \geq 1$, we have

\begin{equation}
(3.24) \quad \| \bar{u} - \hat{\bar{u}} \|_{H^1(B_L)} \leq C_{bt} C_{k,N} k^3 (1 + k l_i N)^2 e^{-\frac{1}{2} k \gamma \sigma} \| f \|_{H^1(B_i')}. \]

We remark that from the theorem above, we can know that the larger number $N$ doesn’t mean the solution $\bar{u}$ performing better although the local problem solved may be elliptic. However, our numerical examples in the following section show that the relative errors between $\bar{u}$ and the discrete solutions don’t increase significantly when $N$ becomes larger.

**4. Numerical examples.** In this section, we simulate the problem [1.1] and [1.2] for constant and heterogeneous wave number by FEM and STDDM, where $f$ is given so that the exact solution is

\[ u = \begin{cases} - r^3 (r^3 + 3 r^2 - 12 r + 9) H_0^{(1)}(kr), & r < 1, \\ - H_0^{(1)}(kr), & r \geq 1. \end{cases} \]

Let $d_1 = 0.2$, $d_2 = 0.1$ and $l_1 = l_2 = 1.1$. So the computational domain $B_L$ is $(-1.3, 1.3) \times (-1.2, 1.2)$ and the perfect matched lay is $B_L \setminus B_l$ where $B_l = (-1.1, 1.1)^2$. It is easy to see that $u \in C^2(\mathbb{R}^2)$ and supp $f \subset B_l$. 


We define the medium property \( \tilde{\sigma}_j \) by setting \( l_1 = l_2 = 1.18 \) and \( \sigma_j(t) = \tilde{\sigma}_j(t) + (t - l_j)\tilde{\sigma}_j'(t) \) for \( l_j < t < \tilde{l}_j \), where

\[
\tilde{\sigma}_j(t) = \gamma_0 \left( \int_{l_j}^{t} (s - l_j)^2 (\tilde{l}_j - s)^2 ds \right) \left( \int_{l_j}^{\tilde{l}_j} (s - l_j)^2 (\tilde{l}_j - s)^2 ds \right)^{-1}.
\]

The functions \( \beta_i^+(x_2) \), \( x_2 \in \Omega_l \), \( i = 2, \cdots, N - 1 \), used in the source transfer algorithm are defined as

\[
\beta_i^+(x_2) = \begin{cases} 
1, & \zeta_i \leq x_2 < \zeta_i + \Delta\zeta/4, \\
\eta_i(x_2), & \zeta_i + \Delta\zeta/4 \leq x_2 < \zeta_i + 3\Delta\zeta/4, \\
0, & \zeta_i + 3\Delta\zeta/4 \leq x_2 \leq \zeta_{i+1}, 
\end{cases}
\]

and \( \beta_i^- = 1 - \beta_i^+ \), where

\[
\eta_i(x_2) = 1 + \left( \frac{x_2 - (\zeta_i + \Delta\zeta/4)}{\Delta\zeta/2} \right)^4 - 2 \left( \frac{x_2 - (\zeta_i + \Delta\zeta/4)}{\Delta\zeta/2} \right)^2.
\]

Clearly, \( \beta_i^+(x_2) \), \( i = 2, \cdots, N - 1 \), are in \( C^1(\Omega_l) \) and this fact avoids the discontinuity of \( \beta_i^+(x_2) \)' which may make \( f_i^+ \) oscillate heavily.

We use the finite element method to solve truncated PML problems. The number of nodes in the \( x_j \)-direction is \( n_j = q \cdot 2L_j/\lambda \), \( j = 1, 2 \), where \( q \) is the mesh density which is the number of nodes in each wavelength \( \lambda = 2\pi/k \). Then the number of degree freedom DOF is \( n_1n_2 \). Let \( N \) be the division number in the \( x_2 \)-direction. \( e_i, e_f \) and \( e_s \) denote the relative error in \( H^1 \)-seminorm of the interpolation, the FEM solution and the PSTDDM solution bounded in \( B_l \) respectively.

We first test the algorithms \( 3 \) and \( 4 \) for the wave number \( k/(2\pi) = 25 \) and \( k/(2\pi) = 50 \). The left graph of Figure 4.1 plots the relative error decay of the interpolation, FE solution and PSTDDM solution with a fixed number of lays \( N_2 = 10 \) in terms of DOF for \( k/(2\pi) = 25, 50 \) respectively. We could find that the relative errors of PSTDDM solution is the same to that of FE solution when DOF is equal. This is best result about comparison between the PSTDDM and FEM which we could expect, since the details of the algorithms \( 3 \) and \( 4 \) show that the errors of PSTDDM solutions can not be less than those of FE solutions under the condition that the mesh is same. In the right graph of Figure 4.1 we set DOF = \( 624 \times 10^4 \) and give the relative errors in \( H^1 \)-seminorm of the PSTDDM \( 3 \) \( 4 \) solutions in terms of the number of lays in \( x_2 \)-direction \( N = 1, 5, 10, 20, 25, 50, 100 \), for \( k/(2\pi) = 25, 50 \) respectively, where \( N = 1 \) means that this solution is the FE solution. It is shown that the error of PSTDDM solution remains unchanged even if the number of lays in the \( x_2 \)-direction becomes larger. So we could choose a relatively large number of lays to reduce the computational complexity.

Next we test our further consideration (cf. \( 3 \) \( 8 \) \( 7 \) \( 8 \) about the PSTDDM for \( k/(2\pi) = 25 \) and \( k/(2\pi) = 50 \). The parameters about PML layers are still those provided at the beginning of this section, since they’re not essential from the previous proofs.

In the left graph of Figure 4.2 we set \( N = 10 \), and show the error decay of the FE solution and further PSTDDM solution when mesh density \( q \) increases. The graph is very quite similar to that of Figure 4.1 what we could like to obtain. In the right graph, we show the relative errors of the further PSTDDM (cf. \( 3 \) \( 8 \) \( 7 \) \( 8 \) when
$N = 5, 10, 20, 25$. Thus the number of the squares, which the domain $B_l$ is divided into, is $N^2 = 25, 100, 400, 625$.

We remark that it’s not necessary to set the number of layers $N$ too small because of the fixed width of PML layer resulting in low computational efficiency in practical application.

![Graph 1](image1.png)

**Fig. 4.1.** Left graph: The relative errors $e_i$, $e_f$, $e_s$ for the interpolations, FE solutions and PSTDDM solutions with a fixed number of layers in $x_2$-direction $N = 10$ in terms of the number of degree freedom DOF = $n_1 n_2$ for $k/(2\pi) = 25$ and $k/(2\pi) = 50$ respectively. Right graph: The relative errors for the PSTDDM solutions in term of the number of layers in the $x_2$-direction for $k/(2\pi) = 25$ and $k/(2\pi) = 50$ respectively, and setting DOF $= 624 \times 10^4$.

![Graph 2](image2.png)

**Fig. 4.2.** Left graph: The relative errors $e_f$, $e_s$ for the FE solutions and further PSTDDM solutions (cf. 7) with a fixed number of layers $N = 10$ in terms of the number of degree freedom DOF $= n_1 n_2$ for $k/(2\pi) = 25$ and $k/(2\pi) = 50$ respectively. Right graph: The relative errors for the further PSTDDM solutions (cf. 3.8, 7, 8) in term of the number of squares for $k/(2\pi) = 25$ and $k/(2\pi) = 50$ respectively, and setting DOF $= 624 \times 10^4$.

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