The Pion Mass Formula

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Abstract

The often used Gell-Mann-Oakes-Renner (GMOR) mass formula for Nambu-Goldstone bosons in QCD, such as the pions, involves the condensate \( \langle \bar{q} q \rangle \), \( f_\pi \) and the quark current masses. Within the context of the Global Colour Model (GCM) for QCD a manifestly different formula was recently found by Cahill and Gunner. Remarkably Langfeld and Kettner have shown the two formulae to be equivalent. Here we explain that the above recent analyses refer to the GCM constituent pion and not the exact GCM pion. Further, we suggest that the GMOR formula is generic. We generalise the Langfeld-Kettner identity to include the full response of the constituent quark propagators to the presence of a non-zero (and running) quark current mass.

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1. Introduction

The properties of the pion continue to be the subject of considerable theoretical and experimental interest in QCD studies. The pion is an (almost) massless Nambu-Goldstone (NG) boson and its properties are directly associated with dynamical chiral symmetry breaking and the underlying quark-gluon dynamics. Recently there has been renewed interest in the mass formula for the pion \[2, 3, 4\] and the relationship with the well known Gell-Mann-Oakes-Renner (GMOR) \[1\] mass formula, as in (1) and (2). Here we extend the study of these relationships and show how one must carefully appreciate the different quantum field theoretic approaches that are actually being employed, often without explicit exposition.

One expects that there should be some perturbative expression for the almost NG boson pion mass in terms of the small quark current masses which is built upon the underlying non-perturbative chiral-limit quark-gluon dynamics. While the relation of the low pion mass to the breaking of chiral symmetry dates back to the current algebra era and PCAC \[1\], the often used implementation in QCD has the form,

\[ m_\pi^2 = \frac{(m_u + m_d)\rho}{f_\pi^2} \] (1)

where the integral \( \rho = \langle \not{q} q \rangle \) is the so-called condensate parameter. For \( N_c = 3 \)
\[ \rho = N_c\text{tr}(G(x = 0)) = 12 \int \frac{d^4q}{(2\pi)^4} \sigma_s(q^2), \] (2)
and \( f_\pi \) is the usual pion decay constant. (Note: our definition for \( \rho \) has an unconventional sign). In (2) \( \sigma_s(s;0) \equiv \sigma_s(s;0) \) is the chiral limit scalar part of the quark propagator which, utilizing only the Lorentz structure (and for later reference \( \sigma_v(s) \equiv \sigma_v(s;0) \)), we can write in full generality as

\[ G(q;m) = (iA(s;m)q.\gamma + B(s;m) + m(s))^{-1} = -iq.\gamma\sigma_v(s;m) + \sigma_s(s;m), \] (3)

from which we easily deduce that

\[ \sigma_s(s;m) = \frac{B(s;m) + m(s)}{sA(s;m)^2 + (B(s;m) + m(s))^2}, \] (4)

\[ \sigma_v(s;m) = \frac{A(s;m)}{sA(s;m)^2 + (B(s;m) + m(s))^2}. \] (5)

Here \( m(s) \) is the running quark current mass, but only the combination \( B(s;m) + m(s) \) appears. We note that the expression for \( \rho \) in (2) is divergent in QCD, because for
\[ s \rightarrow \infty \] $B(s)$ decreases like \(1/(s\ln[s/\Lambda^2]^{1-\lambda})\) where \(\lambda = 12/(33 - 2N_f)\) and \(\Lambda\) is the QCD scale parameter. Some integration cutoff is usually introduced, and the values of \(m\) and \(<\overline{nq}>\) are quoted as being relative to this cutoff momentum, often \(1\text{GeV}\). The GMOR relation has been considered in various approaches, such as operator product expansions (OPE) \(\Box\), QCD sum rules \(\Box\ Box\) and recently, finite energy sum rules and Laplace sum rules \(\Box\).

In \(\Box\) a new mass formula for the pion mass was derived. The analysis in \(\Box\) exploited the intricate interplay between the constituent pion Bethe-Salpeter equation (BSE) and the non-linear Dyson-Schwinger equation (DSE) for the constituent quarks, resulting in the new expression

\[
m_{\pi}^2 = \frac{24m}{f_{\pi}^2} \int \frac{d^4q}{(2\pi)^4} (\epsilon_{s}(s)\sigma_{s}(s) + s\epsilon_{v}(s)\sigma_{v}(s))c(s) + O(m^2), \tag{6}
\]

where \(c(s)\) is a naturally arising cutoff function

\[
c(s) = \frac{B(s;0)^2}{sA(s;0)^2 + B(s;0)^2}. \tag{7}
\]

Here \(\epsilon_{s}(s)\) and \(\epsilon_{v}(s)\) are functions which specify the response of the constituent quark propagator to the turning on of the quark current mass; see (32) and (33). Note that the GMOR mass formula \(\Box\) and \(\Box\) appear to be manifestly different from the new expression in \(\Box\). However Langfeld and Kettner \(\Box\) have shown, by further analysis of the DSE for the constituent quark propagator, and ignoring for simplicity the \(\epsilon_{v}(s)\) vector response term, that the two mass formulae are equivalent, even though the integrands are indeed different.

Here we first demonstrate that in the quantum field theoretic analyses different concepts are often being used and confused in the literature. In this respect we carefully distinguish between the constituent pion and the full or exact pion, and their relevant mass expressions. Little detailed progress has been made in the exact analysis of QCD, and so we use the Global Colour Model (GCM) to illustrate these differences. Further we extend the Langfeld-Kettner identity \(\Box\) to include the vector response function \(\epsilon_{v}(s)\) and a quark running current mass function \(m(s)\). We show that the new mass formula can indeed be written in the GMOR form, with both \(\Box\) and \(\Box\) each now generalised to include a running current mass.

To be clear we note that this report contains no analysis of the mass formula for the full pion in QCD, or even in the GCM. However if the GMOR relation is also the
correct QCD result up to $O(m)$, then that would indicate that the GMOR relation is in fact a generic form that arises whether we are dealing with the full pion or with the constituent pion, and whether we are analysing QCD or some approximation scheme to QCD, such as the GCM, provided we carefully preserve the dynamical consequences of the dynamical breaking of chiral symmetry and its activation by the underlying quark-gluon dynamics. We also note that the GMOR formula has been ‘derived’ in the past, but such analyses in general appear to have brushed over the various subtleties presented herein. Ref.[2], in its appendix, illustrated this by using one example of an incorrect derivation leading to the GMOR relation.

We exploit the GCM [9] of QCD which has proven to be remarkably successful in modelling low energy QCD, as discussed in sec.2. It should be noted that the GCM generates constituent hadrons which necessarily have the form of ladder states. The non-ladder diagrams then arise from hadronic functional integrations over constituent ladder hadrons. In sec.3 we show the difference between the full or exact and the constituent pion. This involves the use of effective actions and the fact that these effective actions refer to constituent hadrons. In sec.4 the effective action for the chiral limit constituent pion is discussed. In sec.5 the constituent quark propagators are given. They arise as the Euler Lagrange equations of the hadronised effective action for the GCM. They define the constituent quarks. Fluctuations about the minimum action configuration introduce constituent mesons, and these are described by ladder BSEs. Ad hoc alterations to these equations can introduce double counting problems. The full (observable) states are produced by dressing each of the constituent states by other states, as is made clear by the functional integral formalism in sec.2. In sec.6 the constituent pion mass formula, (6), is derived, but here generalised to include the quark running mass. In sec.7 the Langfeld-Kettner identity is generalised to include the vector response function and a quark running current mass. This identity leads from the new formula in (6) to the GMOR formula in (1).

2. The Global Colour Model of QCD

An overview and an insight into the nature of the non-perturbative low energy hadronic regime of QCD is provided by the functional integral hadronisation of QCD
In the functional integral approach correlation functions of QCD are defined by
\[ G(\ldots, x, \ldots) = \int D\pi Dq DA \ldots q(x) \ldots \exp(-S_{QCD}[A, \pi, q]) \] (8)
the kernel of which includes (not shown) gluon string structures that render \( G(\ldots, x, \ldots) \) gauge invariant. One example of (8) would be the pion correlation function, which may be defined by the connected part of
\[ G_\pi(x, y, z, w) = \int D\pi Dq DA \ldots \pi(x)i\gamma_5\tau_i q(y)\ldots \pi(z)i\gamma_5\tau_i q(w) \ldots \exp(-S_{QCD}[A, \pi, q]). \] (9)
The \( \pi\pi \) scattering amplitude, for example, is also defined by such a functional integral. The pion mass is defined by the position of the pole, wrt the centre-of-mass (cm) momentum, of the Fourier transform of the translation invariant amplitude \( G_\pi \).

Apart from lattice computations a direct computation of these functional integrals is not attempted. Amplitudes, such as (9), when the on-mass-shell conditions are imposed, define the observables of QCD, such as the pion. Theoretical analysis of these amplitudes proceeds by more circumspect techniques, some of which we clarify here.

The correlation functions, as in (9), may be extracted from the generating functional of QCD, \( Z_{QCD}[\pi, \eta, \ldots] \), defined in (10). However the interactions of low energy hadronic physics, such as \( \pi\pi \) scattering, are known to be well described by effective actions which refer only to hadronic states, although the various parameters in these effective actions could only be determined by fitting to experimental data. Hence we expect that the functional integrals, such as (9), should also be extractable from a hadronic functional integral, as is indeed possible
\[ Z_{QCD}[\pi, \eta, \ldots] = \int D\pi Dq DA \exp(-S_{QCD}[A, \pi, q] + \pi q + \bar{\pi} \bar{q}) \] (10)
\[ \approx \int D\pi DN \ldots \exp(-S_{\text{had}}[\pi, \ldots, N, \ldots] + J_\pi [\pi, \eta] \pi + \ldots) \] (11)
\[ = Z_{\text{had}}[J_\pi [\pi, \eta], \ldots], \] (12)
which produces a hadronic generating functional, \( Z_{\text{had}}[J_\pi [\pi, \eta], \ldots], \) in which source terms for the various hadrons are naturally induced. A partial derivation [3] of this functional transformation proceeds as follows. First, and not showing source terms for convenience, the gluon integrations are formally performed (ghosts also not shown)
\[ \int D\pi Dq DA \exp(-S_{QCD}[A, \pi, q]) \]
\[ Z = \int \mathcal{D}q \mathcal{D}q' \mathcal{D}A \exp(-S_{QCD}[A, q, q'] + \eta q + q' \eta) \]
\[ \approx \int \mathcal{D}q \mathcal{D}A \exp(-S_{GCM}[A, \bar{q}, q] + \bar{q}q + \bar{\eta}) \quad \text{(GCM truncation)} \]

\[ = \int \mathcal{D}B \mathcal{D}D^* \exp(-S_{bd}[B, D, D^*]) \quad \text{(bilocal fields)} \]  

\[ = \int \mathcal{D}\pi...\mathcal{D}N \mathcal{D}N... \exp(-S_{\text{had}}[\pi, ..., N, ...]) \quad \text{(local fields)} \]  

The derived hadronic action that finally emerges from this action sequencing, to low order in fields and derivatives, has the form

\[ S_{\text{had}}[\pi, ..., N, ...] = \]

\[ = \int d^4x \text{tr}\{\mathcal{N}(\gamma.\partial + m_N + \Delta m_N - m_N \sqrt{2i\gamma_5\pi^aT^a} + ..)N\} + \]

\[ + \int d^4x \left[ \frac{f_2^2}{2}(\partial_\mu \pi)^2 + m_\pi^2 \pi^2 \right] + \frac{f_2^2}{2}[-\rho_{\mu} \Box \rho_{\mu} + (\partial_{\mu} \rho_{\mu})^2 + m_\rho^2 \rho_\mu^2] + \]

\[ + \frac{f_2^2}{2}[\rho \rightarrow \omega] - f_\rho f_\pi^2 g_{\rho\pi\pi} \rho_{\mu} \pi \times \partial_\mu \pi - i f_\omega f_\pi^3 \epsilon_{\mu\nu\sigma\tau} \omega_{\mu} \partial_\nu \pi \partial_\sigma \pi \times \partial_\tau \pi + \]

\[ - i f_\omega f_\rho f_\pi G_{\omega\rho\pi} \epsilon_{\mu\nu\sigma\tau} \omega_{\mu} \partial_\nu \rho_\sigma \partial_\tau \pi + \]

\[ + \frac{\lambda_i}{80\pi^2} \epsilon_{\mu\nu\sigma\tau} \text{tr}(\pi.F \partial_\mu \pi.F \partial_\nu \pi.F \partial_\sigma \pi.F \partial_\tau \pi.F) + ....... \]  

(18)

The bilocal fields in (16) arise naturally and correspond to the fact that, for instance, mesons are extended states. In (9) we can see that the pion arises as a correlation function for two bilinear quark structures. This bosonisation/hadronisation arises by functional integral calculus changes of variables that are induced by generalised Fierz transformations that emerge from the colour, spin and flavour structure of QCD.

The final functional integration in (17) over the hadrons give the hadronic observables, and amounts to dressing each hadron by, mainly, lighter mesons. The basic insight is that the quark-gluon dynamics, in (9), is fluctuation dominated, whereas the hadronic functional integrations in (17) are not, and for example the meson dressing of bare hadrons is known to be almost perturbative. In performing the change of variables essentially normal mode techniques are used [9]. In practice this requires detailed numerical computation of the gluon propagator, quark propagators, and meson and baryon propagators. The mass-shell states of the latter are determined by covariant Bethe-Salpeter and Faddeev equations. The Faddeev computations are made feasible by using the diquark correlation propagators; the diquarks being quark-quark correlations within baryons.
3. Constituent Hadrons

We now come to one of the main points. Using the functional hadronisation we can write $G_\pi$ in the form (1) or, from (17), in the form:

$$G_\pi(X, Y) = \int D\pi..D\pi D..\pi(X)\pi(Y) exp(-S_{\text{had}}[\pi, ..., \pi, N, ..]),$$

(19)
in which $X = \frac{x+y}{2}$ and $Y = \frac{z+w}{2}$ are cm coordinates for the pion. We note that now the pion field appears in $S_{\text{had}}[\pi, ..., \pi, N, ..]$ which is in the exponent of (19), and it appears with an effective-action mass parameter $m_\pi$. As we now discuss, it is important to clearly distinguish between this mass (and the equations which define its value) and the pion mass that would emerge from (9) or (19). Eqns. (9) or (19) define the observable pion mass. Whereas the mass in (18) defines the constituent pion mass. There is no reason for these to be equal in magnitude, though they may well both be given by the generic GMOR form.

How do the constituent hadrons arise in (17)? In going from (16) to (17) an expansion about the minimum of $S_{bl}[B, D, D^*]$ is performed. First the minimum is defined by Euler-Lagrange equations (ELE)

$$\frac{\delta S_{bl}}{\delta B} = 0 \quad \text{and} \quad \frac{\delta S_{bl}}{\delta D} = 0.$$

(20)
These equations have solutions $B \neq 0$ and $D = 0$. Eqns. (20) with $D = 0$ is seen, after some analysis, to be nothing more than the DSE for the constituent quark propagator in the rainbow approximation (see (27) and (28) in sec.5). The occurrence of the rainbow form of these equations is not an approximation within the GCM. The non-rainbow diagrams, corresponding to various more complicated gluon dressing of the quarks, are generated by the additional functional integrals in (17). Ad hoc alterations to these DSE constituent quark equations will lead to double counting of certain classes of diagrams. The generation of a minimum with $B \neq 0$ is called the formation of a condensate, here a $\overline{q}q$ condensate. That $D = 0$ means that in the GCM no diquark or anti-diquark type condensates are formed.

Next in going from (16) to (17) we must consider the fluctuations or curvatures of the action for the bilocal fields. One finds that the curvature $\frac{\delta^2 S_{bl}}{\delta B_0 \delta B}$ when inverted gives the meson propagators, but with only ladder gluon exchanges. Again non-ladder diagrams are generated by the functional hadronic integrations in (17).
Similarly inversion of the curvatures in the diquark sector $\delta^2 S_{bl}/\delta D \delta D^*$ leads to diquark propagators, but with only ladder gluon exchanges between the constituent quarks.

We note that the generalised bosonisation with meson and diquark fields leads to some additional complications that we shall consider elsewhere, but which do not impinge on the basic point being made here. This meson-baryon hadronisation is based upon a generalised Fierz transformation \cite{9} that induces the appropriate colour singlet anti-quark - quark correlations, and colour anti-triplet quark-quark correlations that are in the correct colour state for quark-quark correlations within a colour singlet baryon.

An earlier bosonisation \cite{13} used a Fierz transformation that lead to only the meson sector of the GCM. In this bosonisation the meson effective action involves constituent states that are generated by naturally arising and exclusively rainbow or ladder diagrams. Then all of the other diagrams contributing to the observable states are generated by the functional integrations in \cite{17}. Hence we see that in the exponent in \cite{17} there arise particular propagators for quarks, mesons, diquarks and even baryons. These propagators and their associated fields will be defined to be the constituent states. They could also be described as core states. The observables are generated by the hadronic functional integrals in \cite{17}, and correspond to dressing each constituent or core state with other such states. Hence the hadronic effective action in \cite{18} contains a variety of parameters that refer to the constituent states.

Nevertheless one often compares these parameters with the parameter values for the fully dressed constituent states, that is the observable hadronic states. This appears to be valid because in general the dressing produces only a small shift in the parameter values. However one known exception is the nucleon where pion dressing of the constituent nucleon state reduces its mass by some $200 - 300\text{MeV}$. This mass shift emerges from consideration of the functional integral

$$
G_N(X,Y) = \int \mathcal{D}\pi..\mathcal{N}(X)N(Y)exp(-S_{had}[\pi,..,\mathcal{N},N,..]),
$$

where mainly the pions, but as well other mesons are used to dress the nucleon. Of course one usually casts this into the form of a non-linear integral equation for the meson dressed nucleon correlation function.

4. Chiral Limit Constituent Pions
When the quark current masses $M \to 0$ $S_{QCD}[\bar{q},q,A^\mu_\mu]$ has an additional global $U_L(N_F) \otimes U_R(N_F)$ chiral symmetry: writing $\bar{q}\gamma_\mu q = \bar{q}_R\gamma_\mu q_R + \bar{q}_L\gamma_\mu q_L$ where $q_{R,L} = P_{R,L}q$ and $\bar{q}_{R,L} = \bar{q}P_{L,R}$ we see that these two parts are separately invariant under $q_R \to U_R q_R, q_R \to q_R U_R^\dagger$ and $q_L \to U_L q_L, q_L \to q_L U_L^\dagger$. Its consequences may be explicitly traced through the GCM hadronisation. First the ELEs $\delta S_{sl}/\delta B = 0$ have degenerate solutions. In terms of the constituent quark propagator this degeneracy manifests itself in the form

$$G(q;V) = [iA(q)q,\gamma + VB(q)]^{-1} = \zeta^\dagger G(q;1)\zeta$$

where

$$\zeta = \sqrt{V}, \quad V = exp(i\sqrt{2}\gamma_5 \pi^a F^a)$$

in which the $\{\pi^a\}$ are arbitrary real constants. The degeneracy of the minimum implies that some fluctuations in $\delta^2 S_{sl}/\delta B \delta B$ have zero mass; these are the NG BSE states, and this indicates the realisation of the Goldstone theorem.

In the hadronisation, in going from (16) to (17), new variables are forced upon us to describe the degenerate minima (vacuum manifold). This is accomplished by a coordinatisation of the angle variables $\{\pi^a\}$:

$$U(x) = exp(i\sqrt{2}\pi^a(x)F^a)$$

$$V(x) = P_L U(x)^\dagger + P_R U(x) = exp(i\sqrt{2}\gamma_5 \pi^a(x)F^a)$$

The NG part of the hadronisation then gives rise to the constituent pion effective action

$$\int d^4x \left( \frac{f^2}{4} tr(\partial_\mu U \partial_\mu U^\dagger) + \kappa_1 tr(\partial^2 U \partial^2 U^\dagger) + \frac{\rho}{2} tr([1 - \frac{U + U^\dagger}{2}]M) + \kappa_2 tr(\partial_\mu U \partial_\mu U^\dagger)^2 + \kappa_3 tr(\partial_\mu U \partial_\nu U^\dagger \partial_\mu U \partial_\nu U^\dagger) + .... \right)$$

This is the ChPT effective action [14], but with the added insight that all coefficients are given by explicit and convergent integrals in terms of $A$ and $B$, which are in turn determined by $D_{\mu\nu}$. The higher order terms contribute to $\pi\pi$ scattering. The dependence of the ChPT coefficients upon $D_{\mu\nu}$ has been studied in [15, 12, 16] in which the GCM constituent pion expressions for the various parameters were used. However in
view of the apparent generic role of the GMOR relation one should keep in mind the possibility that the functional form of the dependences of the parameters $\kappa_1, \ldots$ on the quark correlation functions $A$ and $B$ might also be generic. At present the final functional integral dressing to obtain the pion observables has not been carried out. This amounts to the assumption that the constituent pion forms are sufficiently accurate. The hadronisation procedure also gives a full account of NG-meson - nucleon coupling.

The GCM is in turn easily related to a number of the more phenomenological models of QCD, as indicated in fig.1. They include the Nambu-Jona-Lasinio Model (NJL) [17], ChPT [14], MIT and Cloudy Bag Model (CBM) [18], Soliton Models [13], Quantum Hadrodynamics (QHD) [20] and Quantum Meson Coupling model (QMC) [21]. We also indicate that the pure gluon correlation function in (14) may be obtained from lattice computations and used in the GCM. The relationships indicated in fig.1 are discussed in [22].

\[ A(p^2; m) = \frac{8}{3} \int \frac{d^4 q}{(2\pi)^4} q.p D(p - q) \frac{A(q^2; m)}{q^2 A(q^2; m)^2 + (B(q^2; m) + m(q^2))^2}, \]

\[ B(p^2; m) = \frac{16}{3} \int \frac{d^4 q}{(2\pi)^4} D(p - q) \frac{B(q^2; m) + m(q^2)}{q^2 A(q^2; m)^2 + (B(q^2; m) + m(q^2))^2}, \]

5. Action Minimum and Pionic Fluctuations

The GCM involves the solution of various integral equations for the constituent correlation functions. As we saw in sec.3, the first equation involves the determination of the minimum of the bilocal effective action, and this reduces to solving the DSE for the constituent quark propagator necessarily in the rainbow form (the so-called vacuum equation of the GCM [3, 13])

\[ B(p^2; m) = \frac{16}{3} \int \frac{d^4 q}{(2\pi)^4} D(p - q) \frac{B(q^2; m) + m(q^2)}{q^2 A(q^2; m)^2 + (B(q^2; m) + m(q^2))^2}, \]

\[ [A(p^2; m) - 1]p^2 = \frac{8}{3} \int \frac{d^4 q}{(2\pi)^4} q.p D(p - q) \frac{A(q^2; m)}{q^2 A(q^2; m)^2 + (B(q^2; m) + m(q^2))^2}. \]
For simplicity we use a Feynman-like gauge in which $D_{\mu\nu}(p) = \delta_{\mu\nu}D(p)$ (the quark-gluon coupling is incorporated into $D$). The formal results of the analysis here are not gauge dependent. Even in numerical studies the Landau gauge can also be used; see [12]. We have also included, for generality, a running current mass for the quarks.

Using Fourier transforms (27) may be written in the form, here for $m = 0$,

$$D(x) = \frac{3}{16} B(x) \frac{B(x)}{\sigma_s(x)},$$

which implies that knowledge of the quark propagator determines the effective GCM gluon propagator. Multiplying (29) by $B(x)/D(x)$, and using Parseval’s identity for the RHS, we obtain the identity

$$\int \frac{d^4x}{(2\pi)^4} B(x) \frac{B(x)}{D(x)} = \frac{16}{3} \int \frac{d^4q}{(2\pi)^4} B(q) \sigma_s(q).$$

The second basic equation is the ladder form BSE for the constituent pion mass-shell state, which arises from the mesonic fluctuations about the minimum determined by (27) and (28). Again this ladder form cannot be generalised without causing double counting of some classes of diagrams at a later stage, and without also damaging the intricate interplay between (27), (28) and the BSE

$$\Gamma_f(p, P) = \frac{8}{3} \int \frac{d^4q}{(2\pi)^4} D(p - q) tr_{SF}(G_+T^9G_-T^9) \Gamma^g(q, P)$$

where $G_\pm = G(q \pm \frac{P}{2})$. This BSE is for isovector NG bosons, and only the dominant $\Gamma = \Gamma^f T^f i\gamma_5$ amplitude is retained (see [23] for discussion); the spin trace arises from projecting onto this dominant amplitude. Here $\{T^b, b = 1, \ldots, N_F^2 - 1\}$ are the generators of $SU(N_F)$, with $tr(T^f T^g) = \frac{1}{2} \delta_{fg}$.

The BSE (31) is an implicit equation for the mass shell $P^2 = -M^2$. It has solutions only in the time-like region $P^2 \leq 0$. Fundamentally this is ensured by (27) and (28) being the specification of an absolute minimum of an effective action after a bosonisation [9]. Nevertheless the loop momentum is kept in the space-like region $q^2 \geq 0$; this mixed metric device ensures that the quark and gluon propagators remain close to the real space-like region where they have been most thoroughly studied. Very little is known about these propagators in the time-like region $q^2 < 0$.

The non-perturbative quark-gluon dynamics is expressed here in (27) and (28). Even when $m = 0$ (27) can have non-perturbative solutions with $B \neq 0$. This is the dynamical breaking of chiral symmetry.
When $m = 0$ (31) has a solution for $P^2 = 0$; the Goldstone theorem effect. For the zero linear momentum state $\{P_0 = 0, \vec{P} = \vec{0}\}$ it is easily seen that (31) reduces to (34) with $\Gamma^f(q,0) = B(q^2)$. When $\vec{P} \neq \vec{0}$ then $\Gamma^f(q,P) \neq B(q^2)$, and (31) must be solved for $\Gamma^f(q,P)$.

6. Constituent Pion Mass

We shall now determine an accurate expression for the mass of the constituent pion when $m(s)$ is small but non-zero. This amounts to finding an analytic solution to the BSE (31), when the constituent quark propagators are determined by (27) and (28). The result will be accurate to order $m$.

For small $m \neq 0$ we can introduce the Taylor expansions in $m(s)$

\begin{align}
B(s;m) + m(s) &= B(s) + m(s)\epsilon_s(s) + O(m^2), \\
A(s;m) &= A(s) + m(s)\epsilon_v(s) + O(m^2).
\end{align}

(32) (33)

For large space-like $s$ we find that $\epsilon_s \rightarrow 1$, but for small $s$ we find that $\epsilon_s(s)$ can be significantly larger than 1. This is an infrared region dynamical enhancement of the quark current mass by gluon dressing, and indicates the strong response of the chiral limit constituent quark propagator to the turning on of the current mass. A plot of $\epsilon_s(s)$ is shown in [2]. Higashijima [24] and Elias [25] have also reported similar enhancements of the current quark masses in the infrared region.

Even in the chiral limit the constituent quark running mass $M(s) = B(s)/A(s)$ is essential for understanding any non-perturbative QCD quark effects. The integrand of a BSE contains the gluon correlation function, constituent quark correlation functions and the form factor for the state (see for example (31)). This integrand shows strong peaking at typically $s \approx 0.3GeV^2$. At this value we find [12] that $M(s) \approx 270MeV$.

This is a property of the constituent hadrons. It does not include any effects from the dressing of these hadrons via (17). This mass is called the constituent quark mass. Because of the infrared region enhancement of the quark current mass we find that this constituent mass rises quickly with quark current mass; see [12].

Because the pion mass $m_\pi$ is small when $m$ is small, we can perform an expansion of the $P_\mu$ dependence in the kernel of (31). Since the analysis is Lorentz covariant we
can, without loss of validity, choose to work in the rest frame with $P = (im_\pi, \vec{0})$ giving, for equal mass quarks for simplicity

$$\Gamma(p) = \frac{2}{9}m_\pi^2 \int \frac{d^4q}{(2\pi)^4} D(p-q)I(s)\Gamma(q) +$$

$$\frac{16}{3} \int \frac{d^4q}{(2\pi)^4} D(p-q) \frac{1}{s(A(s) + m(s)e_v(s))^2 + (B(s) + m(s)e_s(s))^2} \Gamma(q) + ... . \quad (34)$$

where

$$I(s) = 6 \left( \sigma_v^2 - 2(\sigma_s\sigma_v + s\sigma_v\sigma_v') - s(\sigma_s\sigma_s'' - (\sigma_v')^2) - s^2(\sigma_v\sigma_v' - (\sigma_v')^2) \right). \quad (35)$$

By using Fourier transforms the integral equation (34), now with explicit dependence on $m_\pi$, can be expressed in the form of a variational mass functional,

$$m_\pi[\Gamma]^2 = -\frac{24}{f_\pi[\Gamma]^2} \int \frac{d^4q}{(2\pi)^4} \frac{\Gamma(q)^2}{s(A(s) + m(s)e_v(s))^2 + (B(s) + m(s)e_s(s))^2} +$$

$$+ \frac{9}{2f_\pi[\Gamma]^2} \int d^4x \frac{\Gamma(x)^2}{D(x)} \quad (36)$$

in which

$$f_\pi[\Gamma]^2 = \int \frac{d^4q}{(2\pi)^4} I(s)\Gamma(q)^2. \quad (37)$$

The functional derivative $\delta m_\pi[\Gamma]^2/\delta \Gamma(q) = 0$ reproduces (34). The mass functional (36) and its minimisation is equivalent to the constituent pion BSE in the near chiral limit. To find an estimate for the minimum we need only note that the change in $m_\pi^2$ from its chiral limit value of zero will be of 1st order in $m$, while the change in the zero linear momentum frame $\Gamma(q)$ from its chiral limit value $B(q^2)$ will be of 2nd order in $m$.

Hence to obtain $m_\pi^2$ to lowest order in $m$, we may replace $\Gamma(q)$ by $B(q^2)$ in (36), and we have that the constituent pion mass is given by

$$m_\pi^2 = \frac{24}{f_\pi[B]^2} \int \frac{d^4q}{(2\pi)^4} m(s)\frac{\epsilon_s(s)B(s) + s\epsilon_v(s)A(s)}{sA(s)^2 + B(s)^2} \frac{B(s)^2}{sA(s)^2 + B(s)^2}$$

$$- \frac{24}{f_\pi[B]^2} \int \frac{d^4q}{(2\pi)^4} \frac{B(s)^2}{sA(s)^2 + B(s)^2} + \frac{9}{2f_\pi[B]^2} \int d^4x \frac{B(x)^2}{D(x)} + O(m^2) \quad (38)$$

However the pion mass has been shown to be zero in the chiral limit. This is confirmed as the two $O(m^0)$ terms in (38) cancel because of the identity (30). Note that it might appear that $f_\pi$ would contribute an extra $m$ dependence from its kernel in (35).
However because the numerator in (30) is already of order \( m \), this extra contribution must be of higher order in \( m \).

Hence we finally arrive at the analytic expression, to \( O(m) \), for the constituent NG boson (mass)\(^2\) from the solution of the BSE in (31)

\[
m_{\pi}^2 = \frac{24}{f_{\pi}B^2} \int \frac{d^4q}{(2\pi)^4} m(s)\epsilon_s(s)B(s) + s\epsilon_v(s)A(s) \frac{B(s)^2}{sA(s)^2 + B(s)^2} + O(m^2).
\]

Eqn.(3) or (39) is the new form of the NG mass formula derived in [2]. It would appear that expression (6) is manifestly different to the conventional GMOR form in (1) and (2). However in the next section we generalise an identity found by Langfeld and Kettner [4] which shows these forms to be equivalent.

7. Relating the Mass Formulae

Inserting (22) and (33) into (27), and expanding in powers of \( m(s) \), we obtain up to terms linear in \( m \), and after using (27) with \( m = 0 \) to eliminate the \( O(m^0) \) terms,

\[
m(p^2)\epsilon_s(p^2) = m(p^2) + \frac{16}{3} \int \frac{d^4q}{(2\pi)^4} D(p - q) \frac{m(q^2)\epsilon_s(q^2)}{q^2A(q^2)^2 + B(q^2)^2} \]
\[
- \frac{16}{3} \int \frac{d^4q}{(2\pi)^4} D(p - q) \frac{B(q^2)^22m(q^2)\epsilon_s(q^2)}{(q^2A(q^2)^2 + B(q^2)^2)^2} \]
\[
- \frac{16}{3} \int \frac{d^4q}{(2\pi)^4} D(p - q) \frac{B(q^2)A(q^2)2m(q^2)q^2\epsilon_v(q^2)}{(q^2A(q^2)^2 + B(q^2)^2)^2}. \quad (40)
\]

We now multiply (10) throughout by \( B(p^2)/(p^2A(p^2)^2 + B(p^2)^2) \), and integrate wrt \( p \). Using again the chiral limit of (27) there is some cancellation of terms, and we are left with a generalised Langfeld-Kettner identity

\[
2 \int d^4p \frac{B(p^2)^2}{p^2A(p^2)^2 + B(p^2)^2} \left( \frac{B(p^2)m(p^2)\epsilon_s(p^2)}{p^2A(p^2)^2 + B(p^2)^2} + \frac{p^2A(p^2)^2m(p^2)\epsilon_v(p^2)}{p^2A(p^2)^2 + B(p^2)^2} \right) =
\]
\[
\int d^4p \frac{m(p^2)B(p^2)}{p^2A(p^2)^2 + B(p^2)^2} \quad (41)
\]

Remarkably, noting (4) and (5), we see that using this identity in (1) or (39) finally completes the derivation of the GMOR expression for the mass of the constituent pion.

We thus see that despite its apparently simple form the GMOR expression actually depends on two identities that follow from the non-linear constituent quark DSE, as
well as on the subtle interplay between this constituent quark equation and the BSE for the constituent pion. These in turn arise from the careful self-consistency rendered by the functional integral prescription which ensures that the fluctuation spectrum for the bilocal action is precisely related to the Euler-Lagrange equations. *Ad hoc* alterations will invalidate this connection and therefore the derivation of the GMOR expression.

8. Conclusion

We have indicated the careful considerations that must be given to modelling QCD via the GCM and the manner in which this leads to hadronic effective actions. We have defined constituent quarks, meson, diquarks and baryons as those states that appear in the effective action, i.e. in the exponent, as in (17). These constituent states are then further dressed by the functional integrations in (17). Remarkably this GCM structuring of the quantum field theoretic analysis implies, at least in the simplest version of the GCM, that the constituent states are described by sums of rainbow or ladder diagrams, and that the functional integrations then build up all the remaining diagrams, amounting to the vast array of crossed diagrams and vertex dressings etc. Because in most cases these extra dressings do not cause large changes in the values of the constituent masses, coupling constants,.. the effect of the inclusion of these extra diagrams is not manifestly large. Of course this is not surprising because the GCM hadronisation allows us to assess the significance of a constituent state through its mass; low mass states should be more important than very massive states. This implies that the pion dressing is the largest such effect. However inclusion of this dressing for the constituent nucleon is known to be significant, and is a result of the inclusion of various non-ladder diagrams in the observable nucleon.

We have also carefully indicated that it is the mass of the *constituent* pion that is analysed here and, by using various identities that follow from the non-linear equation for the constituent quark, one can show that the mass of this constituent pion is indeed given by the GMOR formula, with the scalar part of the constituent quark correlation function appearing. This does not preclude the fact that presumably the observable pion also has its mass obeying a GMOR formula, but one in which the full quark scalar correlation function appears. That is, the GMOR relation is generic.
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