Nonsingular rotating black hole solutions

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We present regular (nonsingular) rotating black hole solutions that are identified asymptotically \((r \gg k, k > 0)\) constant exactly as Kerr-Newman black holes, and as Kerr black holes when \(k = 0\). The radiating counterpart renders a nonsingular generalization of Carmeli solution as well as Vaidya solution in the appropriate limits.

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I. INTRODUCTION

The celebrated theorems of Penrose and Hawking \cite{1}, state under some circumstances singularities are inevitable in general relativity. For the Kerr solution these singularities have the shape of a ring, and it is timelike. The Kerr metric \cite{2} is undoubtedly the most remarkable exact solution in the Einstein theory of general relativity, which represents the prototypical black hole that can arise from gravitational collapse, which contains an event horizon \cite{3}. Thanks to the no-hair theorem, that the vacuum region outside a stationary black hole has a Kerr geometry. It is believed that spacetime singularities do not exist in Nature; they are creation of the general relativity. It turns out that what amount to a singularity in general relativity could be adequately explained by some other theory, say, quantum gravity. However, we are yet affair from a specific theory of quantum gravity. So a suitable course of action is to understand the inside of a black hole and resolve its singularity by carrying out research of classical black holes, with regular (nonsingular) properties, where spacetime singularities can be avoided in presence of horizons. This can be motivated by quantum arguments of Sakharov \cite{4} and Gliner \cite{5} who proposed that spacetime in the highly dense central region of a black hole should be de Sitter-like for \(r \simeq 0\), which was later explored and refined by Mukhanov and his collaborators \cite{6}. This indicates that an unlimited increase of spacetime curvature during a collapse process, which may halt it, if quantum fluctuations dominate the process. This puts an upper bound on the value of curvature and compels the formation of a central core.

Bardeen \cite{7} realized the idea of a central matter core, by proposing the first regular black hole, replacing the singularity by a regular de Sitter core, which is solution of the Einstein equations coupled an electromagnetic field, yielding an alteration of the Reissner-Nordström metric. One of the exact self-consistent solutions for the regular black hole for the dynamics of gravity coupled to nonlinear electrodynamics have been searched later \cite{8}, which also shares most properties of Bardeen’s black hole. Subsequently, there has been intense activities in the investigation of regular black holes \cite{9–11}, and more recently \cite{12–14}, but most of these solutions are more or less based on Bardeen’s proposal. However, these non-rotating black holes cannot be tested by astrophysical observations, as the black hole spin plays a critical and key role in any astrophysical process. This prompted generalization of these regular black holes to the axially symmetric case or the Kerr-like black hole, via Newman-Janis algorithm \cite{17}, was addressed recently \cite{14-15}. However, these solutions go over to Kerr solution in the appropriate limits, but not to Kerr-Newman solution.

This Letter searches for a new class of three parameter stationary, axisymmetric solutions that describe regular (nonsingular) rotating black holes. The solution depends on the mass \((M)\) and spin \((a)\) of the black hole as well a free parameter \((k)\) that measure potential deviations from the Kerr solution \cite{17} and also generalizes Kerr-Newman solution \cite{17}, in Boyer-Lindquist coordinates reads

\[
ds^2 = -\left(1 - \frac{2MrM}{\Sigma}\right)dr^2 + \frac{\Sigma}{\Delta}d\theta^2 + \frac{\Sigma}{\Delta}d\phi^2 + \left[r^2 + a^2 + \frac{2MrM}{\Sigma}\sin^2 \theta\right]d\phi^2, \quad (1)
\]

with \(r^2 = a^2 \cos^2 \theta, \Delta = r^2 + a^2 - 2Me^{-k/r}, \) and \(M, a, k\) are three parameters, which will be assumed to be positive. This class of solutions includes the Kerr solution as the special case if deviation parameter, \(k = 0\), and the Schwarzschild solution for \(a = k = 0\). Note that the solution \cite{10} asymptotically \((r \gg k)\) behaves as rotating counterpart of Reissner-Nordström solution or Kerr-Newman solution \cite{17}, i.e.,

\[
g_{tt} = 1 - \frac{(2Mr - q^2)}{\Sigma} + \mathcal{O}(k^2/r^2),
\Delta = r^2 + a^2 - 2M + q^2 + \mathcal{O}(k^2/r^2).
\]

This happens when the charge \(q\) and mass \(M\) are related to the parameter \(k\) via \(q^2 = 2Mk\). Thus, the solution \cite{10}
is stationary, axisymmetric with killing field \((\frac{\partial}{\partial \theta})^a\) and \((\frac{\partial}{\partial t})^a\), and all known stationary black holes are encompassed by the three parameter family solutions, and it also generalizes Kerr-Newman solution \([1]\). It is not difficult to find numerically a range of \(M\) and \(k\) for which the solution \([1]\) is a black hole, in addition to regular everywhere. Henceforth, for definiteness we shall address the solution \([1]\) as the regular (nonsingular) rotating black hole.

We approach regularity problem of solution by studying the behavior of invariant \(R = R_{ab}R^{ab} (R_{ab} \text{ is the Ricci tensor})\) and the Kretschmann invariant \(K = R_{abcd}R^{abcd} (R_{abcd} \text{ is the Riemann tensor})\). For the solution \([1]\), they read

\[
\begin{align*}
R &= \frac{2k^2M^2e^{-\frac{2k}{r}}}{r^2\Sigma^4} (\Sigma^2k^2 - 4r^3\Sigma k + 8r^6), \\
K &= \frac{4M^2e^{-\frac{2k}{r}}}{r^6\Sigma^6} (\Sigma^4k^4 - 8r^3\Sigma^3k^3 + Ak^2 + Bk + C)(2)
\end{align*}
\]

where \(A, B\) and \(C\) are functions of \(r\) and \(\theta\) given by

\[
\begin{align*}
A &= -24r^4\Sigma(-r^4 + a^4\cos^4 \theta), \\
B &= -24r^5\left(r^6 + a^6\cos^6 \theta - 5r^2a^2\cos^2 \theta \Sigma\right), \\
C &= 12r^6\left(r^6 - a^6\cos^6 \theta\right) - 180r^8a^2\cos^2 \theta (r^2 - a^2\cos^2 \theta).
\end{align*}
\]

These invariants, for \(M \neq 0\), are regular everywhere, including at the origin \(r = 0\), where noteworthy they vanish (cf. Fig.1).

In addition, the solution \([1]\) is singular at the points where \(\Sigma \neq 0\), and \(\Delta = 0\), is a coordinate singularity this surface is called event horizon (EH). The numerical analysis of the transcendental equation \(\Delta = 0\) reveals that it is possible to find non-vanishing values of parameters \(a\) and \(k\) for which \(\Delta\) has a minimum, and it admits two positive roots \(r_+\). It turns out that \(r = r_+\) are coordinate singularities of the same nature as the singularity at \(r = 2M\) in the Schwarzschild spacetime, the metric can be smoothly extended across \(r = r_+\), with \(r = r_+\) being a smooth null hypersurface, and the simplest possible extension could be rewriting solution \([1]\) in Eddington-Finkelstein coordinates as shown below (cf. Eq. \([7]\)). It turns out that, for a given \(a\), there exists a critical value of \(k, k_+^{EH}\), and one of \(r, r_c^{EH}\), such that \(\Delta = 0\) has a double root which corresponds to a regular extremal black hole with degenerate horizons \((r^{EH} = r^{+}_{EH} = r^{−}_{EH})\). When \(k < k_c^{EH}\), \(\Delta = 0\) has two simple zeros, and has no zeros for \(k > k_c^{EH}\) (cf. Fig.2). These two cases corresponds, respectively, a regular non-extremal black hole with a Cauchy horizon and an EH, and a regular spacetime. It is worthwhile to mention that the critical values of \(k_c^{EH}\) and \(r_c^{EH}\) are a dependent, e.g., for \(a = 0.3, 0.5\), respectively \(k_c^{EH} = 0.63, 0.48\) and \(r_c^{EH} = 0.82, 0.89\) (cf. Fig.2). Indeed, the \(k_c^{EH}\) decreases with the increase in \(a\), on the other hand, the radius \(r_c^{EH}\) increases with an increase in \(a\). The timelike killing vector \(\xi^a = (\frac{a}{m})^b\) of the solution has norm

\[
\xi^a \xi_a = g_{tt} = -\left(1 - \frac{2Mr^2k/r}{\Sigma}\right),
\]

becomes positive in the region where \(r^2 + a^2\cos^2 \theta - 2Me^{-k/r} < 0\). This killing vector is null at the stationary limit surface (SLS), whose locations are, for different \(k\), depicted in Fig.3. The analysis of the zeros of the \(g_{tt} = 0\), for a given value of \(a\) and \(\theta\), disseminate a critical parameter \(k_{cSLS}\) such that \(g_{tt} = 0\) (e.g., \(k_{cSLS} = 0.72\) and 0.66, respectively for \(a = 0.3\) and 0.5) has no roots if \(k > k_{cSLS}\).
they do not (cf. Table I) as in the usual Kerr/Kerr-EH coincides. On other hand, outside this symmetry, the solution decreases when compared to analogous Kerr case ($k = 0$). Notice, that for $\theta = 0$, $\pi$, the SLS and EH coincides. On other hand, outside this symmetry, they do not (cf. Table I) as in the usual Kerr/Kerr-Newman. The region between $r^E < r < r^SLS$ is called ergosphere, where the asymptotic time translation Killing field $\xi^a = (\frac{\Sigma}{kr})^a$ becomes spacelike and an observer follow an orbit of $\xi^a$. The shape of the ergosphere, therefore, depends on the spin $a$, and parameter $k$. It came as a great surprise when Penrose [20] suggested that energy can be extracted from a black hole with an ergosphere. On the other hand, the Penrose process [20] relies on the presence of an ergosphere, which for the solution ($I$) grows with the increase of parameter $k$ as well with spin $a$ as demonstrated in Table II. This in turn is likely to have impact on energy extraction, which being investigated separately.

The vacuum state is obtained by letting horizons size go to zero or by making black hole disappear this amount to $r \to \infty$. One thus conclude that the solution is asymptotically flat as the metric components approaches those of Minkowski spacetime in spheroidal coordinates.

If the Einstein equations, $G_{ab} = kT_{ab}$, are used for the solution ($I$), it is supported by stresses

$$
T_t^t = \frac{Mke^{-k/r} \left\{ \Sigma ka^2 \sin^2 \theta - 2a^2 r^3 (\cos^2 \theta - 2) \right\}}{r^3 \Sigma^3},
$$

$$
T_r^t = -\frac{2Mke^{-k/r}}{\Sigma^2}, \quad T_\theta^{\theta} = \frac{Mke^{-k/r} \left( 2r^3 - k \Sigma \right)}{r^3 \Sigma^3},
$$

$$
T_\phi^\phi = \frac{Mke^{-k/r} \left( 2r^3 - 2a^2 r^3 (\cos^2 \theta - 2) - (a^2 + r^2) \Sigma k \right)}{r^3 \Sigma^3},
$$

$$
T_\phi^t = \frac{Mke^{-k/r} \left( k \Sigma - 4r^3 \right)}{r^3 \Sigma^3}, \quad T_\phi^t = \frac{Mke^{-k/r} \sin^2 \theta (a^2 + r^2)(4r^3 - k \Sigma)}{r^3 \Sigma^3}. \quad (4)
$$

These stresses vanish when $k = 0$, also for $M = 0$, fall off rapidly at large $r$ for $M, k \neq 0$, and for $r \gg k$ and $a = 0$, they are, to $O(k^2/r^2)$, exactly stress energy tensor of Maxwell charge given by $q^2/r^4 \text{diag}[-1, -1, 1, 1]$. In this limit, the solution exactly takes form of Kerr-Newman
solution. Further, the causal (horizon) structure of the solution (1) is similar to that of Kerr/Kerr-Newman solutions, except that the scalar polynomial singularity of the Kerr/Kerr-Newman solutions, at center \((r = 0)\), is no more exists with regular behaviors of the scalars at center as shown in Fig. 1. Thus, the solution (1), which asymptotically behaves as Kerr-Newman, can be understood as a rotating regular black hole of general relativity coupled to a suitable nonlinear electrodynamics governed by the action

\[
\frac{1}{2} \int \sqrt{-g} dt dx(R - \mathcal{L}(F)),
\]

where \(R\) is ricci scalar, \(\mathcal{L}(F)\) is function of \(F = 1 + F_{ab} F^{ab}, F_{ab}\) is Maxwell tensor.

Next, we add radiation by rewriting the static solution (1) in terms of the Eddington-Finkelstein coordinates \((v, r, \theta, \phi)\) [18]:

\[v = t + \frac{\int r^2 + a^2}{\Delta}, \quad \phi = \phi + \frac{\int a}{\Delta},\]

and allow mass \(M\) and parameter \(k\) to be function time \(v\), and dropping bar, we get

\[ds^2 = -\left(1 - \frac{2M(v) r e^{-k(v)/r}}{\Sigma} \right) dv^2 + 2dvdr + \Sigma d\theta^2 - 4aM(v) r e^{-k(v)/r} \sin^2 \theta d\theta d\phi - 2a \sin^2 \theta dr d\phi + \left[r^2 + a^2 + \frac{2M(v) r^2 e^{-k(v)/r}}{\Sigma} \sin^2 \theta \right] \sin^2 \theta d\phi d\Omega^2\]

Again relating \(M(v), q(v)\) with \(k(v)\) as done earlier, then all stresses of the solution (7) have same form as that of the solution (1), but (7) has some additional stresses corresponding to the energy-momentum tensor of ingoing null radiation [19]. The solution (7) describe exterior of radiating objects, recovering Carmeli solution (or rotating Vaidya solutions) [19] for \(k = 0\), and Vaidya solution [20] when \(k = 0\). The radiating rotating solution (7) is a natural generalization of the stationary rotating solution (1), but it is Petrov type-II with a twisting, shear free, null congruence the same as for stationary rotating solution, which is of Petrov type \(D\). Thus, the radiating solution (7) bear the same relation to stationary solution (1) as does the Vaidya solution to the Schwarzschild solution.

According to the no-hair theorem, all astrophysical black holes are expected to be like Kerr black holes. The regular solution (1) contains the Kerr metric as the special case when the deviation parameter, \(k = 0\), also for \(r \gg k\) behaves as Kerr-Newman and it stationary, ax-symmetric, asymptotically flat, and makes this solution a well-suited framework for astrophysical applications. This warrants to further study geometrical properties, causal structures and thermodynamics of this black hole solution, which is being investigated.

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