Weak dependence for a class of local functionals of Markov chains on $\mathbb{Z}^d$

C. Boldrighini$^1$, A. Marchesiello$^2$, and C. Saffirio$^3$

1 Dipartimento di Matematica G. Castelnuovo, Sapienza Università di Roma, Piazzale Aldo Moro 5, 00185 Roma

2 Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University in Prague, Děčín Branch, Pohrancí 1, 40501 Děčín

3 Institut für Mathematik Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich

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Abstract

In some papers on infinite Markov chains in $\mathbb{Z}^d$, and notably in the work of R.A. Minlos and collaborators, one can prove the existence of a spectral gap for a suitable subspace of local functions. We consider functions of the type $f(\hat{\eta})$, where $\hat{\eta} = \{\eta_t\}_{t=0}^\infty$ is the sequence of the states, and $f$ is local. In the case of a simple example of random walk in random environment with mutual interaction we show that there is a natural class of functions $f$, related to the Hölder continuous functions $C^\alpha$ on the torus $T^1$, with $\alpha \in (0,1)$ large enough, depending on the spectral gap, for which the Central Limit Theorem holds for the sequence $f(S^k \hat{\eta})$, $k = 0, 1, \ldots$, where $S$ is the time shift.

1 Introduction

Many problems in mathematical physics lead to consider Markov chains on the $d$-dimensional lattice $\mathbb{Z}^d$, for $d = 1, 2, \ldots$, with local interaction (see [14]). The states of the chain are random fields $\eta_t = \{\eta_t(x) : x \in \mathbb{Z}^d\}, t \in \mathbb{Z}_+ = 0, 1, \ldots$, with $\eta_t(x) \in S$, where $S$ is usually a finite or countable set.

In many models [3,12,14,3,6] one has an invariant measure $\Pi$ on the state space $\Omega = S^{2d}$, and the spectral analysis of the stochastic operator $\mathcal{F}$ allows to establish the existence of an invariant (w.r.t. $\mathcal{F}$) subspace of local functions $\mathcal{H}_M \subset L_2(\Omega, \Pi)$ such that for all $F \in \mathcal{H}_M$ with zero average $\langle F \rangle_\Pi = 0$, we have, for some positive constant $\mu \in (0, 1)$,

$$\|\mathcal{F} F(\xi)\| \leq \mu \|F\|_M, \quad \xi \in \Omega. \quad (1)$$

Here $\| \cdot \|_M$ is a suitable norm on $\mathcal{H}_M$, usually such that $\|F\|_\infty = \sup_\xi |F(\xi)| \leq \|F\|_M$. $\mathcal{H}_M$ is often identified with the help of an expansion in a suitable basis.

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$^*$boldrigh@mat.uniroma1.it

$^1$anto.marchesiello@gmail.com

$^2$chiara.saffirio@math.uzh.ch
The spectral gap condition (1) is usually sufficient to prove a Central Limit Theorem (CLT) for normalized sums of functions of the type $\sum_{t=0}^{T} F(\eta_t)$, where $F$ is “local”, i.e., it depends on a finite number of variables $\eta_t(x)$, or its dependence on the variables $\eta_t(x)$ for large $|x|$ is weak enough.

A natural extension is to consider sums of functions depending on the space-time field $\hat{\eta} = \{\eta_t\}_{t=0}^{\infty} \in \tilde{\Omega} = S^{d} \times \mathbb{Z}^{+}$, of the type $\sum_{t=0}^{T} f(S_t \hat{\eta})$, where $S$ is the time shift, $S\hat{\eta} = \{\eta_t\}_{t=1}^{\infty}$, and $f$ is local in space. One cannot in general obtain weak dependence by relying on properties such as strong mixing and the like [11], which need requirements that may not apply or may be difficult to prove [7].

The aim of the present paper is to establish properties of weak dependence of the terms of the sum $\sum_{t=0}^{T} f(S_t \hat{\eta})$ which hold in the framework described above and are sufficient for the CLT to hold.

The models to which our considerations apply are of different nature, and in particular the space $\mathcal{M}_k$ is based on explicit constructions suggested by the model under consideration. Therefore, in order to keep an intuitive guideline, and to make the text more readable, we work on the example of a simple model of random walk in dynamical random environment with mutual interaction, which was studied in [2, 3]. The Markov chain $\eta_t, t \in \mathbb{Z}^{+}$, describes the “environment from the point of view of the random walk”, a construction which plays an important role in the analysis of random walks in random environment [3].

Moreover, in order to reduce the size of the proof we assume that the local field takes only two values, i.e. $S = \{\pm 1\}$, and that the space dependence of the functions $f$ is limited to one point.

The plan of the paper is as follows. In the next section we describe the model, which, as it is often the case, is a perturbation of an independent model, and present the main features which are relevant to our analysis.

In §3 we prove some key estimates, which are applied in the final section to obtain a weak dependence condition and other results on the basis of which we obtain a proof of the CLT, which should be compared with a result of [11] (Th. 19.3.1) for the independent case.

2 Description of the model

We consider a version of the model studied in [2, 3], which describes a discrete-time random walk $X_t \in \mathbb{Z}^d, d \geq 1, t \in \mathbb{Z}^{+}$, evolving in mutual interaction with a random field $\xi_t = \{\xi_t(x) : x \in \mathbb{Z}^d\}$. The local field takes only two values: $\xi_t(x) \in \{\pm 1\}$, so that the state space is $\Omega = \{\pm 1\}^{2d}$, and the space of the “trajectories” (or “histories”) of the environment $\vec{\xi} = \{\xi_t : t \in \mathbb{Z}^{+}\}$ is $\tilde{\Omega} = \{\pm 1\}^{2d \times \mathbb{Z}^{+}}$. Measurability is understood with respect to the $\sigma$-algebra generated by the cylinder sets.

The pair $(X_t, \vec{\xi}), t \in \mathbb{Z}^{+}$ is a conditionally independent Markov chain [14], i.e., for all measurable sets $A \subset \Omega$, we have

$$P(X_{t+1} = x + u, \vec{\xi}_{t+1} \in A \mid X_t = x, \vec{\xi} = \vec{\xi}) = P(X_{t+1} = x + u \mid X_t = x, \vec{\xi} = \vec{\xi}) P(\vec{\xi}_{t+1} \in A \mid X_t = x, \vec{\xi} = \vec{\xi}).$$  

(2)
If the history of the environment $\xi_t \in \Omega$ is fixed, the first factor on the right of (2) defines the "quenched" random walk, for which we assume the simple form

$$P(X_{t+1} = x + u \mid X_t = x, \xi_t) = P_0(u) + \varepsilon c(u) \xi_t(x), \quad u \in \mathbb{Z}^d, \xi_t \in \Omega.$$  

(3)

Here $\varepsilon > 0$ is a small parameter, $P_0$ is a probability distribution on $\mathbb{Z}^d$ and $c$ a real function on $\mathbb{Z}^d$. They are such that $P_0(u) \pm \varepsilon c(u) \in [0, 1]$, $u \in \mathbb{Z}^d$. We also assume that $P_0$ is even and $c$ odd in $u$, with finite third moments, i.e., $\sum_{u \in \mathbb{Z}^d} |u|^3 (P_0(u) + |c(u)|) < \infty$. Moreover $P_0$ is strongly aperiodic, i.e., its Fourier transform $\hat{P}_0(\lambda) = \sum_{u \in \mathbb{Z}^d} P_0(u) e^{i(\lambda,u)}$ is such that $|\hat{P}_0(\lambda)| = 1$ if and only if $\lambda = 0$.

In order to meet a technical assumption in (3) we also need that the Fourier coefficients of the function $1/\hat{P}_0(\lambda)$ are absolutely summable. By homogeneity in space it is not restrictive to assume $X_0 = 0$.

The evolution of the environment is independent at each site, so that $P(\xi_{t+1} \in \mathcal{A} \mid X_t = x, \xi_t = \xi) = (1 - \delta_{x,y}) Q_0(\xi_t(y), s) + \delta_{x,y} Q_1(\xi_t(y), s)$ where $s = \pm 1$, $Q_0, Q_1$ are symmetric $2 \times 2$ matrices, $Q_0$ has eigenvalues $1, \mu$, $|\mu| < 1$, and $Q_1$ is such that $Q_1 - Q_0 = O(\varepsilon)$. In words, the evolution is given by the transition matrix $Q_0$ everywhere, except at the site where the random walk is located, where the transition matrix is $Q_1$.

A natural probability measure on the state space $\Omega$ is the product $\Pi_0 = \pi_0^{\mathbb{Z}^d}$, with $\pi_0 = (1/2, 1/2)$. If $Q_0 = Q_1$ (no reaction on the environment) $\Pi_0$ is invariant.

The model just described was first considered in (3) both in the annealed and quenched form. For the case $Q_0 = Q_1$ fairly complete results regarding annealed and quenched diffusivity were obtained in a more general setting in (6). A non-perturbative result was obtained in (7).

The "environment from the point of view of the particle" is the process $\{\eta_t : t \in \mathbb{Z}_+\}$, where $\eta_t(x) = \xi_t(X_t + x)$, which is also Markov with state space $\Omega$ (3). It can be shown (6, 9) that it is equivalent to the full process $(X_t, \xi_t)$, i.e., for all $T \in \mathbb{Z}_+, T \geq 1$, given the sequence $\eta_0, \ldots, \eta_T$ one can reconstruct $(X_T, \xi_T), \ldots, (X_0, \xi_0)$, almost-surely.

The stochastic operator $\mathcal{T}$, defined by the relation

$$(\mathcal{T} f)(\eta) = \langle f(\eta_{t+1}) | \eta_t = \eta \rangle, \quad f \in \mathcal{H}$$

(5)

where the average $\langle \cdot \rangle$ is w.r.t. the transition probability (3), can be defined as an operator on the Hilbert space $\mathcal{H} = L_2(\Omega; \Pi_0)$.

Observe that under the previous assumptions the operator $\mathcal{T}$ preserves parity under the exchange $\eta \mapsto -\eta$.

In $\mathcal{H}$ we introduce a convenient basis. As $Q_0$ is symmetric, its eigenvectors are $e_0 = (1, 1)$ and $e_1 = (1, -1)$ with corresponding eigenvalues 1 and $\mu$. We denote their components as $e_j(s)$, so that $e_1(s) = s$, $e_0(s) = 1$, $s = \pm 1$, and set

$$\Phi_\Gamma(\eta) = \prod_{x \in \Gamma} e_1(\eta(x)) = \prod_{x \in \Gamma} \eta(x), \quad \Gamma \in \mathcal{G},$$

(6)

where $\mathcal{G}$ is the collection of the finite subsets of $\mathbb{Z}^d$, with $\Phi_B = 1$. $\{\Phi_\Gamma : \Gamma \in \mathcal{G}\}$ is a discrete orthonormal complete basis in $\mathcal{H}$, so that we can write

$$f(\eta) = \sum_{\Gamma \in \mathcal{G}} f_\Gamma \Phi_\Gamma, \quad f \in \mathcal{H}.$$
For $M > 1$ we define the subspace $\mathcal{H}_M \subset \mathcal{H}$ as

$$\mathcal{H}_M = \{ f = \sum f_\Gamma \Phi_\Gamma : \| f \|_M = \sum f_\Gamma |M^\Gamma | < \infty \}. \quad (7)$$

$\mathcal{H}_M$ equipped with the norm $\| \cdot \|_M$ is a Banach space, and is dense in $\mathcal{H}$. Moreover, as $|\Phi_\Gamma (\eta)| = 1$, for $f \in \mathcal{H}_M$ we have $\| f \|_\infty \leq \| f \|_{\omega} \leq \| f \|_M$.

Another important property is that $\mathcal{H}$ is closed under multiplication. In fact, as it is to see,

$$\Phi_\Gamma \Phi_\Gamma = \Phi_{\Gamma \triangle \Gamma}, \quad \Gamma \triangle \Gamma' = \Gamma \setminus \Gamma' \cup \Gamma' \setminus \Gamma,$$

so that if $f, g \in \mathcal{H}_M$ and $f = \sum f_\Gamma \Phi_\Gamma, g = \sum g_\Lambda \Phi_{\Lambda}$, we have

$$\| fg \|_M = \sum |f_\Gamma g_\Lambda| |M^\Gamma \triangle \Lambda| \leq \| f \|_M \| g \|_M. \quad (8)$$

In the paper [3] an analysis of the expression of the matrix elements of $\mathcal{T}$ and its adjoint $\mathcal{T}^*$, relying on their spectral properties for $\varepsilon = 0$, leads to the following results.

**Theorem 2.1.** If $\varepsilon$ and $|\mu|$ are small enough, the space $\mathcal{H}_M$ is invariant under $\mathcal{T}$, and there is an invariant probability measure $\Pi$ for the chain $\{ \hat{\eta}_t \}$ which is absolutely continuous with respect to $\Pi_0$ with uniformly bounded density $v(\eta)$. Moreover $\mathcal{H}_M$ can be decomposed as

$$\mathcal{H}_M = \mathcal{H}_M^{(0)} + \mathcal{H}_M$$

where $\mathcal{H}_M^{(0)}$ is the space of the constants, and on $\mathcal{H}_M$ the restriction of $\mathcal{T}$ acts as a contraction:

$$\| \mathcal{T} f \|_M \leq \mu \| f \|_M, \quad f \in \mathcal{H}_M,$$

for some $\mu_0 \in (0, 1), \mu_0 = |\mu| + O(\varepsilon)$. Furthermore if $f \in \mathcal{H}_M$, $f = f_0 + \hat{f}, f_0 \in \mathcal{H}_M^{(0)}$, $\hat{f} \in \mathcal{H}_M$, then

$$f_0 = \int f(\eta) d\Pi(\eta) = \int f(\eta) v(\eta) d\Pi_0(\eta).$$

3 Preliminary estimates

We denote by $\mathcal{P}_\Pi$ the probability measure generated by the initial distribution $\Pi$ on $\Omega = \{ \pm 1 \}^{\mathbb{Z} \times \mathbb{Z}_+}$, the space of the trajectories of the Markov field $\hat{\eta} = \{ \eta_t : t \in \mathbb{Z}_+ \}$, and by $\mathcal{M}_0^{t_1}, 0 \leq t_0 \leq t_1$ the sub-$\sigma$-algebras of subsets of $\hat{\Omega}$ generated by $\{ \eta_t \}_{t=0}^{t_1}$. As $\Pi$ is invariant, $\mathcal{P}_\Pi$ is invariant under the time shift.

We consider functions $f$ which depend only on the values of the field at the origin, i.e., on the sequence of random variables $\{ \eta_t (0) \}_{t=0}^{\infty}$. In what follows we set for brevity $\xi = \eta_t (0)$ and $\tilde{\xi} = \{ \xi : t \in \mathbb{Z}_+ \} \subset \Omega_+, \Omega_+ = \{ \pm 1 \}^{\mathbb{Z}_+}$ is the image of $\hat{\Omega}$ under the map $\tilde{\cdot}$. The probability measure induced by $\mathcal{P}_\Pi$ on $\Omega_+$ is denoted by $\varphi$.

$\varphi$ is stationary with respect to the time shift $S$ on $\Omega_+$: $S^k \xi = \{ \xi_1, \xi_2, \ldots \}$, so that if $f$ is a measurable function on $\Omega_+$, the sequence $f(S^k \xi), k = 1, 2, \ldots$ is stationary in distribution. In what follows, by abuse of notation, $f(\xi)$ may denote a function on $\hat{\Omega}$ or on $\Omega_+$, according to the circumstances, and similarly for the shift operator $S$.

We also introduce the $\sigma$-algebras $\mathcal{M}_0^{t_1}, 0 \leq t_0 < t_1$ generated by the variables $(\eta_t (0))_{t=0}^{t_1}$, which are sub-$\sigma$-algebras of the $\sigma$-algebras $\mathcal{M}_0^{t_1}$. By abuse of notation
we will denote the corresponding $\sigma$-algebras of $\Omega_+$ by the same symbol. In what follows we write $\mathcal{M}_t$ and $\mathcal{M}_i$ for $\mathcal{M}_{t_0}$ and $\mathcal{M}_{i_0}$, respectively.

We first consider functions which are measurable with respect to the $\sigma$-algebra $\mathcal{M}_t^n$ for some $n \geq 0$, i.e., functions which depend only on the variables $\xi_0, \ldots, \xi_n$. Any such function can be written as

$$f(\xi) = \sum_{\gamma \subseteq \{0, \ldots, n\}} f_\gamma \Psi_\gamma(\xi)$$

where the sum runs over the subsets of $\{0, \ldots, n\}$ and the functions $\Psi_\gamma$ are defined as

$$\Psi_\gamma(\xi) = \prod_{i \in \gamma} \xi_i, \quad \gamma \neq \emptyset, \quad \Psi_\emptyset(\xi) = 1. \quad (11)$$

The following lemma is a simple consequence of Theorem 2.1.

**Lemma 3.1.** Let $f(\xi)$ be a cylinder function depending only on the variables $\xi_0, \ldots, \xi_{m-1}$, $m \geq 1$. Then $(f(\cdot) | \mathcal{M}_0) \in \mathcal{M}_t$ and

$$\| (f(\cdot) | \mathcal{M}_0) \|_M \leq C \max_{\gamma \subseteq \{0, \ldots, m-1\}} |f_\gamma| (1 + \mu_*)^m, \quad (12)$$

where $\mu_* = M \sqrt{\mu(1 + 2\mu)}$ and $C$ is a positive constant independent of $m$.

**Proof.** As the expansion (10) is finite, the first assertion is equivalent to the assertion that the conditional probabilities

$$G_\gamma(\eta) := \langle \Psi_\gamma | \mathcal{M}_0 \rangle, \quad \eta \in \Omega, \quad (13)$$

where $\gamma$ is any finite nonempty subset of $Z_+$ are in $\mathcal{M}_t$. In fact, if $\gamma = \{l_0, l_1, \ldots, l_k\}$, and $r_j = l_{k-1-j} - l_{k-j}, j = 1, \ldots, k = |\gamma| - 1$, where $|\gamma|$ is the cardinality of $\gamma$, $G_\gamma$ can be written as

$$G_\gamma(\eta) = \Phi_{\{0\}}(\eta) \left[ \mathcal{T}^{l_0} \Phi_{\{0\}} \cdots \mathcal{T}^{l_k} \Phi_{\{0\}} \right](\eta), \quad \eta \in \Omega, \quad (14)$$

i.e., $G_\gamma$ is obtained by successive applications of the operator $\mathcal{T}$ and of the multiplication operator by $\Phi_{\{0\}}$. As both operations leave $\mathcal{M}_t$ invariant, $G_\gamma \in \mathcal{M}_t$.

Moreover it can be shown that

$$\|G_\gamma\|_M \leq M^{|\gamma|} \mu^{\lceil \frac{|\gamma|}{2} \rceil} (1 + 2\mu)^{\lceil \frac{|\gamma| - 1}{2} \rceil} \leq C \mu_*^{\gamma}, \quad (15)$$

where $\lceil \cdot \rceil$ denotes the integer part, and $C > 0$ is a constant which is easily worked out.

The proof of the inequality (15) is deferred to the appendix.

The proof of the lemma follows by observing that the inequality (15) implies

$$\| (f(\cdot) | \mathcal{M}_0) \|_M \leq C \max_{\gamma \subseteq \{0, \ldots, m-1\}} |f_\gamma| \sum_{\gamma \subseteq \{0, \ldots, m-1\}} \mu_*^{\gamma}. \quad (16)$$

The following assertion is a simple consequence of the previous lemma.

**Lemma 3.2.** Under the assumptions of the previous lemma, if $\mu_*$ is so small that $\mu_* < 1$, then the probability measure $\mathcal{M}_t$ on $\Omega_+$ is continuous.
Proof. We need to prove that any point \( \hat{\zeta}^{(0)} = \{ \xi_k \}_{k=0}^\infty \in \Omega_+ \) has zero \( \rho \)-measure.

In fact, consider the cylinders \( Z_n(\hat{\zeta}^{(0)}) = \{ \xi_j : j = 0, 1, \ldots, n-1 \} \), which are decreasing \( Z_{n+1}(\hat{\zeta}^{(0)}) \subset Z_n(\hat{\zeta}^{(0)}) \) and such that \( \cap_n Z_n(\hat{\zeta}^{(0)}) = \{ \hat{\zeta}^{(0)} \} \) and

\[
\rho^\ast \left( Z_n(\hat{\zeta}^{(0)}) \right) = \frac{1}{2^n} \prod_{j=0}^{n-1} (1 + \xi_j) .
\]

(17)

By opening the brackets we get an expansion in terms of the functions \( \Psi_\gamma \)

\[
\prod_{j=0}^{n-1} (1 + \xi_j) = \sum_{\gamma \subset \{0, \ldots, n-1\}} \Psi_\gamma(\hat{\xi}) \Psi_{\hat{\xi}}(\gamma) , \quad \gamma = \{ \xi_j \}_{j=0}^{n-1} .
\]

Recalling that \( |\Psi_{\hat{\xi}}(\gamma)| = 1 \), we have

\[
\left| \sum_{\gamma \subset \{0, \ldots, n-1\}} \Psi_\gamma(\hat{\xi}) \Psi_{\hat{\xi}}(\gamma) \right| \rho^\ast \leq \sum_{\gamma \subset \{0, \ldots, n-1\}} \left| \Psi_\gamma(\hat{\xi}) \right| \rho^\ast = \sum_{\gamma \subset \{0, \ldots, n-1\}} \left| \langle \Psi_\gamma(\gamma) \rangle_\Pi \right| = \sum_{\gamma \subset \{0, \ldots, n-1\}} \left| \langle G_\gamma(\cdot) \rangle_\Pi \right| .
\]

Therefore by the inequality (17) the right side is bounded by

\[
\frac{C}{2^n} \sum_{\gamma \subset \{0, \ldots, n-1\}} \mu_\gamma^n = C \left( \frac{1 + \mu_\gamma}{2} \right)^n .
\]

Hence if \( \mu_\gamma < 1 \), the right side tends to 0 as \( n \to \infty \), which proves the lemma.

From now on we assume that \( \mu_\gamma < 1 \).

We pass to consider more general functions for which the expansion (10) is infinite

\[
f(\hat{\xi}) = \sum_{\gamma \subset \emptyset} f_\gamma \Psi_\gamma(\hat{\xi}) , \quad \gamma \in \mathfrak{g} ,
\]

(18)

where \( \mathfrak{g} \) is the collection of the finite subsets \( \mathbb{Z}_+ \). The functions \( \Psi_\gamma, \gamma \in \mathfrak{g} \), sometimes called “Walsh functions”, are an orthonormal complete basis in the space \( L_2(\Omega_+, \rho_\Omega) \), where \( \rho_\Omega = \pi_{\mathbb{Z}_+} \) is the probability measure on \( \Omega_+ \) corresponding to the random variables \( \zeta_k, k \in \mathbb{Z}_+ \) being i.i.d. with distribution \( \pi(\pm 1) = \frac{1}{2} \). The series (18) is called “Fourier-Walsh expansion” [10].

A map \( \mathcal{F} : \Omega_+ \to T^1 \), where \( T^1 = [0, 1) \mod 1 \) is the one-dimensional torus, is defined by associating to a point \( \hat{\xi} \in \Omega_+ \) the binary expansion \( x = 0, a_0 a_1 \ldots \in [0, 1] \), with \( a_t = \frac{1}{2}, t \in \mathbb{Z}_+ \).

\( \mathcal{F} \) is not invertible because the dyadic points of \( T^1 \) have two binary expansions, but it becomes invertible if we exclude from the domain the sequences such that \( \zeta_t = -1 \) for all \( t \) large enough. Such sequences are a countable set, which has zero \( \rho_\Omega \)-measure, and also, by Lemma 3.2, zero \( \rho \)-measure.

Under the map \( \mathcal{F} \) the basis functions \( \Psi_\gamma \) go into the function

\[
\Psi_\gamma(x) = \prod_{t \in \gamma} \phi_t(x) , \quad \gamma \in \mathfrak{g} .
\]
where \( \phi(x) \) is the image of \( \zeta, t \in \mathbb{Z}_+ \), i.e.,

\[
\phi_0(x) = \begin{cases} 
1, & 0 \leq x < \frac{1}{2} \\
-1, & \frac{1}{2} \leq x < 1 
\end{cases}
\]

and for \( t \geq 0, \phi_t(x) = \phi_0(2^t x) \), where \( 2^t x \) is understood mod 1.

The space \( L_2(\Omega_+, \mathcal{F}_0) \) goes into \( L_2(T^1, dx) \), where \( \{ \psi_\gamma(x) : \gamma \in \mathcal{g} \} \) is a complete orthonormal set of functions. If \( f \in L_2(\Omega_+, \mathcal{F}_0) \) then \( \tilde{f}(x) = f(\mathcal{F}^{-1} x) \) can be expanded in the basis \( \{ \psi_\gamma : \gamma \in \mathcal{g} \} \), with coefficients

\[
f_\gamma = \int_{\Omega_+} f(\hat{\zeta}) d\mathcal{F}_0(\hat{\zeta}) = \int_0^1 \tilde{f}(x) \psi_\gamma(x) dx.
\]

(19)

A natural way of ordering the set \( \mathcal{g} \), which plays an important role in the study of pointwise convergence, is the following. We set \( \gamma_0 = 0 \) and \( \gamma_n = \{ t_1, t_2, \ldots, t_r \} \) where the integers \( r \) and \( 0 \leq t_1 < t_2 < \ldots < t_r \) are uniquely defined by the relation

\[
n = 2^{t_1} + \ldots + 2^{t_r}.
\]

The Walsh series is then written as

\[
f(\hat{\zeta}) = \sum_{n=0}^\infty f_{\gamma_n} \Psi_{\gamma_n}(\hat{\zeta}) \quad \tilde{f}(x) = \sum_{n=0}^\infty f_{\gamma_n} \psi_{\gamma_n}(x).
\]

(20)

A particular role is played by the partial sums

\[
S_{2^k}(\tilde{f}; x) = \sum_{\gamma \subseteq \{0,1,\ldots,k-1\}} f_\gamma \Psi_\gamma(x) = \sum_{n=0}^{2^k-1} f_{\gamma_n} \psi_{\gamma_n}(x)
\]

for which it can be seen [10] that

\[
S_{2^k}(\tilde{f}; x) = 2^k \int_{\alpha_k}^{\beta_k} f(y) dy, \quad \alpha_k = m 2^{-k}, \beta_k = (m+1)2^{-k}
\]

(21)

where the integer \( m \) is such that \( \alpha_k \leq x < \beta_k \).

The following result is proved in [10]. We repeat it here, with a shorter proof based on conditional probabilities.

**Lemma 3.3.** Let \( \tilde{f}(x) \) be a bounded function. Then its Walsh-Fourier coefficients \( f_\gamma \), given by (19), satisfy the following inequality

\[
|f_\gamma| \leq \frac{\omega(f; 2^{-n-1})}{2^{n+2}}, \quad n = \max\{ t : t \in \gamma \},
\]

(22)

where \( \omega(f; \delta) \) is the modulus of continuity of \( \tilde{f} \):

\[
\omega(f; \delta) = \sup_{x, x' \in T^1} \frac{|f(x) - f(x')|}{|x - x'|}.
\]

(23)
Proof. We have
\[ f_T = \left\langle f(\tilde{\xi}) \prod_{t \in T} \xi_t \right\rangle_{\rho_0} = \left\langle \prod_{t \in T} \xi_t \left\langle f(\tilde{\xi}) \zeta_n | \mathscr{H}_0^n \right\rangle \right\rangle_{\rho_0}. \]

Going back to \( T^1 \), and setting \( x_n = \frac{a_0}{2} + \ldots + \frac{a_{n-1}}{2} \), \( a_j = 1 - \xi_j \), we have
\[
\left| \left\langle f(\tilde{\xi}) \zeta_n | \mathscr{H}_0^n \right\rangle \right| = 2^n \int_x^{x_n+2^{-n}} f(x) (1 - 2\phi_n(x)) dx = 2^n \int_x^{x_n+2^{-n-1}} [\tilde{f}(x) - \tilde{f}(x + 2^{-n-1})] dx, \tag{24}
\]
from which, taking into account (23), the inequality (22) follows immediately.

The results above allow us to prove the analogue of Lemma 3.1 for functions \( f \) such that their image is Hölder continuous, \( \tilde{f} \in \mathscr{C}^\alpha(T^1) \), for some \( \alpha \in (0, 1) \) large enough. In what follows if \( g \in \mathscr{C}^\alpha(T^1) \) we denote by \( \| \cdot \|_\alpha \) the semi–norm
\[
\|g\|_\alpha = \sup_{x,y \in T^1} \frac{|g(x) - g(y)|}{|x - y|^\alpha}
\]
and we write sometimes, by abuse of notation, \( f \in \mathscr{C}^\alpha \) for a function on \( \Omega_+ \) such that the corresponding \( \tilde{f} \) is in \( \mathscr{C}^\alpha(T^1) \).

Lemma 3.4. Let \( f \) be a function on \( \Omega_+ \), such that the corresponding function \( \tilde{f} \) on \( T^1 \) is Hölder continuous \( \mathscr{C}^\alpha(T^1) \), with \( \alpha > \log_2(1 + \mu_+) \).

Then \( \langle f | \mathcal{M}_0 \rangle \in \mathscr{H}_M \) and the following inequality
\[
\| \langle f | \mathcal{M}_0 \rangle \|_M \leq C_\alpha \frac{\mu_+ + 1}{2^\alpha - \mu_+ - 1} \tag{25}
\]
holds, where \( C_\alpha > 0 \) and \( \mu_+ \) is defined in Lemma 3.1.

Proof. We write the Walsh series (20) and observe that for \( 2^k \leq n < 2^{k+1} \) we have \( \max \{ i \in \gamma_n \} = k \), so that, as \( \delta \omega(\tilde{f}, \delta) \leq \delta^\alpha \| \tilde{f} \|_\alpha \), the inequality (22) gives
\[
|f_{\gamma_n}| \leq \frac{\|\tilde{f}\|_\alpha}{2^1 + \alpha} 2^{-k\alpha}, \quad 2^k \leq n < 2^{k+1}.
\]
Therefore we have
\[
\left\| \sum_{n=2^k}^{2^{k+1}-1} f_{\gamma_n} \left\langle \Psi_{\gamma_n} | \mathcal{M}_0 \right\rangle \right\|_M \leq \frac{\|\tilde{f}\|_\alpha}{2^1 + \alpha} 2^{-k\alpha} \sum_{n=2^k}^{2^{k+1}-1} \| \left\langle \Psi_{\gamma_n} | \mathcal{M}_0 \right\rangle \|_M. \tag{26}
\]
Observe moreover that the number of elements of \( \gamma_n \) is \( r_n = |\gamma_n| = u_n - 1 \) where \( u_n \) is the number of "1" in the binary expansion of \( n \). Hence, by the inequality (15) we find
\[
\sum_{n=2^k}^{2^{k+1}-1} \| \left\langle \Psi_{\gamma_n} | \mathcal{M}_0 \right\rangle \|_M \leq C \sum_{s=0}^{k-1} \binom{k-1}{s} \mu_+^s = C(1 + \mu_+)^{k-1},
\]
which implies (25).
4 Weak dependence and the Central Limit Theorem

In this section we prove our main results for sequences of the type \( f(S^t \xi_t), t = 0, 1, \ldots \), where \( f \) satisfies the assumptions of Lemma 5.1 and Lemma 5.4. Analogous results were proved in the book of Ibragimov and Linnik [11] for functions of sequences of independent random variables (Chapter 18 and Chapter 19, Section 3).

In the present section we will sometimes denote by \( \langle \cdot \rangle \) an average with respect to \( \rho \) or \( \mathcal{H}_R \), according to the context.

Theorem 4.1. Let \( f \) be a function on \( \Omega_+ \), depending only on \( \xi_0, \ldots, \xi_{m-1}, m \geq 1 \), and such that \( \langle f \rangle \rho = 0 \). Then the dispersion of normalized sums

\[
S_n^{(f)} = \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} f(S^i \xi_i)
\]

(27)

tends, as \( n \to \infty \), to a finite non-negative limit

\[
\sigma_f^2 = \langle f^2(\cdot) \rangle \rho + 2 \sum_{i=1}^{\infty} \langle f(\cdot)f(S^i \cdot) \rangle \rho
\]

(28)

and the series is absolutely convergent. Moreover, if \( \sigma_f^2 > 0 \), the sequence \( S_n^{(f)}(\xi) \) tends weakly to the centered gaussian distribution with dispersion \( \sigma_f^2 \).

Proof. Let \( \langle f \rangle_{\mathcal{M}_0} := G^{(f)} \) and

\[
\Sigma^{(f)}(\xi) = \sum_{i=1}^{m} f(\xi) f(S^i \xi), \quad \tilde{\Sigma}^{(f)}(\xi) = \Sigma^{(f)}(\xi) - \langle \Sigma^{(f)}(\xi) \rangle.
\]

(29)

As \( \langle f \rangle \rho = 0 \), by Lemma 5.1, \( G^{(f)} \in \mathcal{H}_M \). We first prove that \( \Sigma^{(f)} \in \mathcal{H}_M \) and

\[
\left\| \Sigma^{(f)} \right\|_{\mathcal{M}} \leq \sum_{i=1}^{m} \left\| f(\cdot)f(S^i \cdot) \right\|_{\mathcal{M}} \leq C_m \| f \|_{\infty} \| G^{(f)} \|_{\mathcal{M}},
\]

(30)

where the constant \( C_m \) depends on \( m \) and on the parameters of the model.

In fact for \( i \geq m \) we have

\[
\langle f(\cdot)f(S^i \cdot) \rangle_{\mathcal{M}_0} = \langle f(\xi) G^{(f)}(\eta_i) \rangle_{\mathcal{M}_0} = \sum_{\gamma \subset \{1, \ldots, m-1\}} f_{\gamma} \langle \Psi(\xi) G^{(f)}(\eta_i) \rangle_{\mathcal{M}_0}.
\]

(31)

As \( \| f \|_{\infty} \), applying Proposition 5.1, we see that

\[
\left\| f(\cdot)f(S^i \cdot) \right\|_{\mathcal{M}} \leq C_{m} \hat{\mu}_{m+1} \| f \|_{\infty} \| G^{(f)} \|_{\mathcal{M}} (1 + \mu_{i})^m,
\]

(32)

which implies (30), and proves the absolute convergence of the series in (28).

Supposing now that \( \sigma_f^2 > 0 \) we expand the log of the characteristic function

\[
\psi_n(\lambda) = \log \left( e^{i\lambda S_n^{(f)}} \right) = \log \left( \exp \left( \frac{\lambda}{\sqrt{n}} \sum_{i=0}^{n-1} f(S^i \cdot) \right) \right)
\]

(33)
in Taylor series at \( \lambda = 0 \), and, as usual, we only need to show that the third order remainder vanishes as \( n \to \infty \) for any fixed \( \lambda \). For this we need to estimate the third cumulant of the sum, which, as \( \langle f \rangle = 0 \), reduces to the third moment

\[
n \langle f^3 (\cdot) \rangle + 3 \sum_{t_1 \neq t_2} \langle f^2 (S^{t_1}) f (S^{t_2}) \rangle + 6 \sum_{t_1 < t_2 < t_3} \langle f (S^{t_1}) f (S^{t_2}) f (S^{t_3}) \rangle,
\]  

(34)

where all sums go over \( 0, \ldots, n - 1 \). We will show that the sums \( (34) \) are of order \( O(n) \).

As \( f^2 \) is also a cylinder function, the proof that the series \( \sum_{n=1}^{\infty} \langle f^2 (\cdot) f (S^{t}) \rangle |M_0| \) converges in norm \( \| \cdot \|_M \) is done exactly as before.

For the series \( \sum_{n=1}^{\infty} \langle f (\cdot) f^2 (S^{t}) \rangle \), observe that, as \( \langle f \rangle = 0 \), it is equal to

\[
\sum_{n=1}^{\infty} \langle f (\cdot) \hat{f}^{(2)} (S^{t}) \rangle,
\]

and the convergence of the corresponding series in \( \mathcal{H}_M \) also follows as before.

Passing to the third term in \( (34) \), as \( \langle f \rangle = 0 \), we can write

\[
\sum_{t_1, t_2=1}^{\infty} \langle f (\cdot) f (S^{t_1}) f (\cdot) f (S^{t_1+t_2}) \rangle = \sum_{i=1}^{\infty} \langle f (\cdot) \hat{\Sigma}^{(i)} (S^{t}) \rangle = \sum_{i=1}^{\infty} \langle f (\cdot) \hat{\Sigma}^{(i)} (S^{t}) \rangle,
\]

(35)

with \( \hat{\Sigma}^{(i)} \in \mathcal{H}_M \). Proceeding as before, we apply Proposition \( 5.1 \) to the terms \( \langle \hat{f} (\xi) \hat{\Sigma} (S^{t_1}) \rangle \), for \( t \geq m \), obtaining the analogue of inequality \( (30) \) and the absolute convergence of the series in \( (35) \).

In conclusion, all terms in \( (34) \) are \( O(n) \), and the correction to the second order expansion of \( \psi_\mu (\lambda) \) is \( O(n^{-2/3}) \). The theorem is proved.

We finally consider functions satisfying the assumptions of Lemma \( 3.4 \). In what follows we set

\[
\kappa = 2^{-\alpha} (1 + \mu), \quad \kappa_* = 2^{-\alpha} (1 + 2 \mu).
\]

(36)

As a first step we need, as in \( [11] \), that the functions are well approximated by their conditional probabilities on finite \( \sigma \) algebras. This property is provided by the representation \( (21) \) for the partial sums, which gives

\[
\left| f (\xi) - S_2^n (f, \xi) \right| \leq \| f \|_A 2^{-\alpha n},
\]

(37)

\[
\textbf{Lemma 4.2.} \text{ Let } f \text{ be a function on } \Omega_+, \text{ satisfying the assumptions of Lemma } 3.4 \text{ and such that } \langle f \rangle_{\beta_0} = f_0 = 0. \text{ Then, if } \mu \text{ is so small that } \kappa_* < 1, \text{ we have}
\]

\[
\left\| \left\langle \hat{\Sigma} f (\cdot) \right|_{M_0} \right\|_M \leq \sum_{i=1}^{\infty} \left\langle f (\cdot) f (S^{t}) \right|_{M_0} \leq C_2 \| f \|_{A}^2,
\]

(38)

where \( \Sigma^{(i)} \) is as in \( (29) \) and \( C_2 \) is a constant depending on the parameters of the model.

\textbf{Proof.} The proof is deferred to the Appendix.
Remark 1. Under the assumptions of Lemma 4.2 it is easy to see, by a simple inspection of the proof of that lemma that if we consider the partial sums

$$\sum_{i=1}^{n} \hat{S}^{(f)}_{2n}(\hat{\xi}) = \sum_{i=1}^{n} \tilde{S}^{(f)}_{2n}(\hat{\xi}), \quad \tilde{S}^{(f)}_{2n}(\hat{\xi}) = S^{(f)}_{2n}(\hat{\xi}) - \langle \tilde{S}^{(f)}_{2n}(-) \rangle,$$  \hspace{1cm} (39)

with $$\sum_{i=1}^{\infty} \hat{f}(\hat{\xi}) f(S^{h} \hat{\xi})$$, then we have, uniformly in $$m,n$$

$$\left\| \sum_{i=1}^{n} \langle \hat{S}^{(f)}_{m,n}(-) \rangle \right\|_{M} \leq \sum_{i=1}^{n} \left\| \langle \hat{S}^{(f)}_{m,n}(-) \rangle \right\|_{M} \leq C \|f\|_{2}. \hspace{1cm} (40)$$

Theorem 4.3. Assume that $$\zeta_{\ast} < 1$$ and let $$f$$ be a function on $$\Omega_{+}$$, satisfying the conditions of Lemma 3.4 and such that $$\langle f \rangle_{\rho} = 0$$. Then the dispersion of the normalized sums

$$S^{(f)}_{n} = \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} f(S^{h} \hat{\xi}) \hspace{1cm} (41)$$

tends, as $$n \to \infty$$, to a finite non-negative limit

$$\sigma^{2}_{j} = \langle f^{2}(-) \rangle_{\rho} + 2 \sum_{i=1}^{\infty} \langle f(-) f(S^{m}_{j}) \rangle_{\rho} \hspace{1cm} (42)$$

were the series on the right is absolutely convergent. Moreover, if $$\sigma^{2}_{j} > 0$$, the sequence $$S^{(f)}_{n}(\hat{\xi})$$ tends weakly to the centered gaussian distribution with dispersion $$\sigma^{2}_{j}$$.

Proof. The absolute convergence of the series in (42) follows from Lemma 4.2.

For the third moment (34), observe that, as $$f^{2} \in \mathcal{C}_{\alpha}$$, the absolute summability of the series $$\sum_{i=1}^{\infty} \langle f^{2}(-) f(S^{m}_{j}) \rangle_{\rho}$$ and $$\sum_{i=1}^{\infty} \langle f(-) f^{2}(S^{m}_{j}) \rangle$$ are obtained by trivial changes in the proof of Lemma 4.2. Therefore the first and second terms in (34) are $$O(n)$$.

We are left with the third term in (34), which by translation invariance in time can be written as

$$\sum_{i=0}^{n-2} \sum_{i=1}^{n-1} \sum_{i=1}^{u_{1}+1} \sum_{i=1}^{u_{2}+1} \langle f(-) f(S^{m_{1}} (-) f(S^{m_{1}+m_{2}} (-)) \rangle. \hspace{1cm} (43)$$

Let $$m = 4 \left\lfloor \frac{m}{\alpha} \right\rfloor$$, where $$[\cdot]$$ denotes the integer part, and $$\Delta^{(f)}_{m}(-) = f(-) - \hat{S}^{(f)}_{2m}(-)$$. By the inequality (37), as $$2^{-m} = \Theta(n^{-4})$$, we have

$$\sum_{i=0}^{n-2} \sum_{i=1}^{n-1} \sum_{i=1}^{u_{1}+1} \sum_{i=1}^{u_{2}+1} \left| \langle \Delta^{(f)}_{m}(-) f(S^{m_{1}}(-) f(S^{m_{1}+m_{2}}(-)) \rangle \right| \leq 2^{-m} \|f\|_{\rho}^{2} \left( n \right)^{2} = \Theta(n^{-1}).$$

For the remaining term we first evaluate the sum for $$u_{1} > m$$, i.e.

$$\sum_{i=0}^{n-2} \sum_{i=1}^{n-1} \left( \tilde{S}^{(f)}_{2m}(-) \right) \langle S^{(f)}_{\omega_{n-m-t-1}}(S^{m_{1}}) \rangle \hspace{1cm} (44)$$

As $$\langle S^{(f)}_{\omega_{n-m-t-1}}(-) \rangle_{\rho} = 0$$, for the running term we have

$$\left\langle \tilde{S}^{(f)}_{2m}(-) \right\rangle \langle S^{(f)}_{\omega_{n-m-t-1}}(S^{m_{1}}) \rangle \right\rangle = \left\langle \tilde{S}^{(f)}_{2m}(-) \right\rangle \langle S^{(f)}_{\omega_{n-m-t-1}}(S^{m_{1}}) \rangle \right\rangle.$$
Setting $G^{(f)}_r(\eta) = \langle \tilde{S}^{(f)}_n(\cdot) | \mathfrak{M}_0 \rangle \in \hat{\mathcal{H}}_M$ we have for $u > m$

$$\left\langle \tilde{S}^{(f)}_{2m} (\cdot) | \tilde{S}^{(f)}_{m,\gamma}(S^u) | \mathfrak{M}_0 \right\rangle = \sum_{k=0}^m \sum_{\gamma,\gamma' = k} f_{\gamma} \left\langle \Psi_{\gamma} (\xi) G^{(f)}_r(\eta_\gamma) | \mathfrak{M}_0 \right\rangle .$$

By Lemma 4.2 and the previous remark, which implies that the norms $\| G^{(f)}_r \|_M$ are bounded in $r$, we find, setting $C = C_1 C_2$, 

$$\left| \sum_{\gamma,\gamma' = k} f_{\gamma} \left\langle \Psi_{\gamma} (\xi) G^{(f)}_r(\eta_\gamma) | \mathfrak{M}_0 \right\rangle \right| \leq C \| f \|_2 \kappa^k \mu^{u-k} \| G^{(f)}_r \|_M \leq C \| f \|_2 \kappa^k \mu^{u-k} ,$$

which shows that the quantity (44) is of order $O(n)$.

For the sum over $u_1 \leq m$, proceeding as above we see that

$$\sum_{u_2 = 1}^{n-u_1} \left\langle \tilde{S}^{(f)}_{2m} (\cdot) f(S^{u_1+u_2}) \right\rangle = \sum_{u_2 = 1}^{n-u_1} \left\langle \tilde{S}^{(f)}_{2m} (\cdot) \tilde{S}^{(f)}_{m,\gamma}(S^{u_1+u_2}) \right\rangle + O\left( \frac{1}{n^3} \right).$$

The sum over $t$, for $u_1, u_2 \leq m$, gives a contribution of order $O(\mu^{u_1} n m)$.

We are left with the sum

$$\sum_{u_1 = 1}^{m} \sum_{u_2 = m+1}^{n-u_1} \left\langle \tilde{S}^{(f)}_{2m} (\cdot) \tilde{S}^{(f)}_{m,\gamma}(S^{u_1+u_2}) \right\rangle ,$$

(45)

and we need to estimate the term

$$\left\langle \tilde{S}^{(f)}_{2m} (\cdot) \tilde{S}^{(f)}_{m,\gamma}(S^{u_1+u_2}) \right\rangle = \sum_{\gamma,\gamma' \subseteq \{0, \ldots, m \}} f_{\gamma} f_{\gamma'} \left\langle \Psi_{\gamma} (\eta_{u_1+u_2}) G^{(f)}(\eta_{u_1+u_2}) | \mathfrak{M}_0 \right\rangle .$$

The estimate follows the lines of the proof of Lemma 4.2, with the only difference that the sums are finite, as $\sum_{\gamma,\gamma' \subseteq \{0, \ldots, m \}} f_{\gamma} f_{\gamma'} \left\langle \Psi_{\gamma} (\eta_{u_1+u_2}) G^{(f)}(\eta_{u_1+u_2}) | \mathfrak{M}_0 \right\rangle \leq \text{const} \| f \|_2^2 \kappa^m \mu^{u_2-m} \| G^{(f)} \|_M .

(46)

where the constant does not depend on $m$.

Therefore, summing over $u_2$, we see that the double sum (45) is of order $m$.

Summing up the results so far obtained we see that the sum (43) is at most of order $O(\mu^{u_1} n m^2) = O(n \log_2^3 n)$. As in the expansion of the log of the characteristic function (32) in Taylor series (up to the third order) the third moment is divided by $n^2$, we come to the conclusion.

5. Appendix

Proof of inequality (15). By symmetry with respect to the change of sign $\eta(x) \to -\eta(x)$, $x \in \mathbb{Z}_d$, the density $v(\eta)$ is even. Moreover any finite trajectory of the Markov chain has the same probability of the trajectory obtained by a change of sign.

The functions $\Phi_r$ defined by (2) are even (odd) for $|\Gamma|$ even (odd). Therefore for $|\Gamma|$ odd we have $\langle \Phi_r \rangle_{\Gamma} = 0$, and also $\langle \mathcal{F}^r \Phi_r \rangle_{\Gamma} = 0, r > 0$. The functions $\Psi_r$ are also even (odd) for $|\Gamma|$ even (odd), and for $|\Gamma|$ odd $\langle \Psi_r \rangle_{\Gamma} = \langle G_r \rangle_{\Gamma} = 0$. 

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For $|\gamma|$ even we set

$$G_\gamma = \langle G_\gamma \rangle + \hat{G}_\gamma, \quad \hat{G}_\gamma \in \hat{H}_M.$$  \hspace{1cm} (47)

If $|\gamma| > 1, \gamma = \{ t_0, \ldots, t_k \}, |\gamma| = k + 1$, we have, by (13),

$$G_\gamma(\eta) = \Phi_{\{ t_0 \}}(\eta) \left[ \mathcal{F}_t^\gamma (\hat{G}_{\{ t_0 \}}) \right](\eta).$$

Therefore, if $|\gamma| \geq 2$ is even we have

$$\|G_\gamma\|_M \leq M \hat{\mu}^{\nu_0} \|G_{\{ t_0 \}}\|_M,$$

and if $|\gamma| > 1$ is odd

$$\|G_\gamma\|_M \leq M \left( |\langle (G_{\{ t_0 \}})(\hat{\eta}) \rangle| + \hat{\mu}^{\nu_0} \|\hat{G}_{\{ t_0 \}}\|_M \right) \leq M(1 + 2 \hat{\mu}^{\nu_0}) |\langle G_{\{ t_0 \}}\rangle|, \hspace{1cm} (49)$$

where in the second inequality we take into account that $|\langle G_{\{ t_0 \}}\rangle| \leq |\langle G_\gamma\rangle| \leq |\langle G_\gamma\rangle|$. As for $|\gamma| = 1, G_{\{ t_0 \}}(\hat{\eta}) = \Phi_{\{ t_0 \}}(\hat{\eta})$ so that $\|G_{\{ t_0 \}}\|_M = M$, and for $|\gamma| = 2$ we have $\|G_\gamma\|_M \leq M \|\mathcal{F}_t^\gamma \Phi_{\{ t_0 \}}\|_M \leq M \hat{\mu}^{\nu_0}$. Therefore the inequalities (48) and (49) imply that

$$\|G_\gamma\|_M \leq M \hat{\mu}^{\nu_0} \prod_{j \text{ odd}} \hat{\mu}^{\nu_j} \prod_{j \text{ even}} (2 \hat{\mu}^{\nu_j} + 1)$$

which implies (13).

The following proposition is a simple consequence of the previous proof.

**Proposition 5.1.** Under the assumptions of lemma [5.7] if $\gamma = \{ t_0, \ldots, t_k \}, G \in \hat{H}_M$, and $t > t_k$, the following inequality holds, for some positive constant $C_*$.

$$\left| \langle \Psi_\gamma(\zeta) \; G(\eta) \rangle |M_0\rangle \right|_M \leq C_* \|G\|_M \hat{\mu}^{t-h} \hat{\mu}^{\hat{\gamma}}.$$  \hspace{1cm} (51)

**Proof.** Let $|\gamma| > 1$. It is easy to see, proceeding as in the previous proof, that if $G$ is odd, we get, in analogy with (50),

$$\left| \langle \Psi_\gamma(\zeta) \; G(\eta) \rangle |M_0\rangle \right|_M \leq M^{t-h} \|G\|_M \hat{\mu}^{t-h} \prod_{j \text{ even}} \hat{\mu}^{\nu_j} \prod_{j \text{ odd}} (1 + 2 \hat{\mu}^{\nu_j}),$$

and if $G$ is even, by small modification of the previous proof, we get

$$\left| \langle \Psi_\gamma(\zeta) \; G(\eta) \rangle |M_0\rangle \right|_M \leq M^{t-h} \|G\|_M \hat{\mu}^{t-h} \prod_{j \text{ odd}} \hat{\mu}^{\nu_j} \prod_{j \text{ even}} (1 + 2 \hat{\mu}^{\nu_j}).$$

Writing $G(\eta) = G^{(+)}(\eta) + G^{(-)}(\eta)$, where $G^{(+)}(\eta) = \frac{G(\eta) + G(-\eta)}{2} \in \hat{H}_M$, and observing that $\|G\|_M = \|G^{(+)}\|_M + \|G^{(-)}\|_M$, we get the result (51). \hfill \Box

**Proof of Lemma [4.7]** In what follows by “const” we denote different positive constants depending on the parameters of the model.

Observe first that

$$\langle f \rangle_\gamma = f_\theta + \sum_{\gamma} f_\gamma \langle \Psi_\gamma \rangle_\theta = 0,$$
so that $f(\hat{\zeta}) = \sum_{\gamma \neq s} f_{\gamma} \hat{\Psi}_{\gamma} \hat{\Psi}_{\gamma} = \Psi_{\gamma} - \langle \hat{\Psi}_{\gamma} \rangle$. Therefore we can write

$$\langle f() f(\mathcal{S}^t) \rangle = \sum_{\gamma = 0}^{\infty} \sum_{\gamma = 0}^{\infty} f_{\gamma} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} f_{\gamma} \langle \hat{\Psi}_{\gamma} \rangle,$$

where, for $\gamma = \{i_0, \ldots, i_k\}$, we set $\gamma + t = \{i_0 + t, \ldots, i_k + t\}$, $t \geq -i_0$, and $\min \gamma = i_0, \max \gamma = i_k$. Consider the sum for $\max \gamma < \min \gamma + t$. By applying Proposition 5.1 and taking into account that $\|G_{\gamma}\|_M \leq 2\|G_{\gamma}\|_M \leq 2C\mu^{\gamma}$, we get

$$\sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \sum_{\gamma = 0}^{\infty} \sum_{\gamma = 0}^{\infty} |f_{\gamma}| \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} f_{\gamma} \langle \hat{\Psi}_{\gamma} \rangle \langle \Psi_{\gamma} \rangle \leq \|f\|_a^2 \sum_{m=0}^{\infty} 2^{-\frac{t+1}{m}} \sum_{r=0}^{\infty} |K^m_{\gamma}| \mu^{r+m} \|G_{\gamma}\|_M.$$

By Inequality 15 the previous sum is bounded by

$$\|f\|_a^2 \sum_{m=0}^{\infty} 2^{-\frac{t+1}{m}} \sum_{r=0}^{\infty} |K^m_{\gamma}| \mu^{r+m} \|G_{\gamma}\|_M \leq \|f\|_a^2 \|K\|_a \mu^\gamma.$$

Passing to the sum for $\max \gamma \geq \min \gamma + t$, we first consider the terms containing the average values $\langle \Psi_{\gamma + t} \rangle = \langle \Psi_{\gamma} \rangle$, and, proceeding as before, we get

$$\sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \sum_{\gamma = 0}^{\infty} \sum_{\gamma = 0}^{\infty} |f_{\gamma}| \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} f_{\gamma} \langle \hat{\Psi}_{\gamma} \rangle \langle \Psi_{\gamma} \rangle \leq \|f\|_a^2 \|K\|_a \mu^\gamma.$$

As $\Psi_{\gamma} \Psi_{\gamma} = \Psi_{\gamma \Delta \gamma}$, where $\gamma \Delta \gamma = \gamma \setminus \gamma \cup \gamma \setminus \gamma$, we are left with the quantity

$$\sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \sum_{\gamma = 0}^{\infty} \sum_{\gamma = 0}^{\infty} |f_{\gamma}| \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} f_{\gamma} \langle \Psi_{\gamma \Delta (\gamma + t)} \rangle \langle \Psi_{\gamma} \rangle.$$

Let $\Omega^n_{\gamma} = \{t, \ldots, t + n\}$ and $\hat{\gamma} = \gamma \cap \{\gamma + t\}$. We have $\hat{\gamma} \in \Omega^n_{\gamma \setminus \gamma}$, with $n = \min\{s, k\}$, and, setting $\gamma = \gamma \setminus \hat{\gamma} \cup \gamma \setminus \hat{\gamma}$, with $\gamma \cap \gamma \setminus \hat{\gamma} = 0$, the generic term in the sum in (57) is estimated by

$$|f_{\gamma} f_{\gamma}| \langle \Psi_{\hat{\gamma} \cup \gamma} \rangle \leq C\|f\|_a^2 2^{-s} \mu^{r+s+m} \mu^s |\hat{\gamma}| + |\gamma|.$$

We first consider the sum over $s > k, s = k + \ell, \ell = 1, \ldots$ and set $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$ with $\gamma_1 \in \Omega^n_{\gamma \setminus \gamma}$, $\gamma_2 \in \Omega^n_{\gamma \setminus \gamma}$, $\gamma_3 \in \Omega^n_{\gamma \setminus \gamma \setminus \gamma_2}$. As $|\hat{\gamma}| = |\gamma_1| + |\gamma_2| + |\gamma_3|$, summing over $\ell$ and $\gamma_3$, for everything else fixed, taking into account (58), we get

$$\text{const}\|f\|_a^2 2^{-s} \mu^{r+s+m} \mu^s |\hat{\gamma}| + |\gamma_2| + |\gamma_3|.$$

If $\pi = (\hat{\gamma}, \gamma_2, \gamma_3)$ is an ordered partition of the set $\Omega^n_{\gamma \setminus \gamma}$, with $|\Omega^n_{\gamma \setminus \gamma}| = k + 1$, taking into account that $p, |\gamma_2| = j \leq k + 1$ can be chosen in $k+1 \choose j$ ways, and for each such choice the remaining set can be divided into $\gamma_2, \gamma_3$ in $2^{k+j} - j$ ways we find

$$\sum_{\pi} \mu^{\gamma_2} |\gamma_2| = (1 + 2\mu_*)^{k+1}.$$
As $\kappa < 1$, summing over $k$, and then over $m$ and $\gamma_1$, we see that the sum (57) for $s > k$ is estimated by
\[\text{const} \|f\|_2^2 2^{-\alpha t}.\] (60)

For $s \leq k$, we set $\gamma_1 = \gamma_1 \cup \gamma_2$, $\gamma' + t = \gamma_1 \cup \gamma_2$, with $\gamma_2 \subset \Omega_{1+m}$ and $\gamma_2 \subset \Omega^{(k-s)}_{m+s+1}$. After summing over $k$ and $\gamma_1$ for fixed $s$ we get an inequality of the type (59), where $\gamma_2$ is replaced by $\gamma_1$. Proceeding as in the previous case, we get the same estimate (60) holds.

In conclusion we see that
\[\langle f(\cdot) \langle f(S^t \cdot) \mathcal{M}_0 \rangle \rangle_M \leq \text{const} \|f\|_2^2 \kappa',\] (61)
which implies the inequalities (58). \qed

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