Abstract. We study finiteness conditions on large tilting modules over arbitrary rings. We then turn to a hereditary artin algebra $R$ and apply our results to the (infinite dimensional) tilting module $L$ that generates all modules without preprojective direct summands. We show that the behaviour of $L$ over its endomorphism ring determines the representation type of $R$. A similar result holds true for the (infinite dimensional) tilting module $W$ that generates the divisible modules. Finally, we extend to the wild case some results on Baer modules and torsion-free modules proven in [5] for tame hereditary algebras.

Introduction

The category mod-$R$ of all finitely generated modules over a hereditary artin algebra $R$ is well understood. Let us briefly recall its main properties. First, every finitely generated $R$-module has an essentially unique indecomposable decomposition. Further, the finitely generated indecomposable modules are depicted in the Auslander-Reiten quiver of $R$. If $R$ is indecomposable and has infinite representation type, this quiver has the following shape

```
✛

✚

... 

... 

✘

✙

... 

... 

ptq
```

where $p$ contains all indecomposable projectives and is called the preprojective component, $q$ contains all indecomposable injectives and is called the preinjective component, and $t$ consists of infinitely many infinite components, called regular components.

Much less is known about the category Mod-$R$ of all $R$-modules, if $R$ is representation infinite. In his seminal paper [25] from 1979, Ringel initiated the study of the infinite dimensional modules by investigating some torsion pairs in Mod-$R$ constructed from the Auslander-Reiten components of $R$.

For example, he considered the torsion pair $(\mathcal{R}, \mathcal{D})$ cogenerated by $t$. It provides a cut of Mod-$R$ into a torsion-free class $\mathcal{R}$ containing $p$ and $t$, and a torsion class $\mathcal{D}$ containing $q$, and in some sense, it is maximal with respect to this property, see [24] and [5.5]. When $R$ is of tame representation type, the torsion pair $(\mathcal{R}, \mathcal{D})$ splits, and in view of the striking analogies with the category of abelian groups, the modules in $\mathcal{D}$ are called divisible, while the modules in $\mathcal{R}$ are called reduced.

Date: November 21, 2018.
We acknowledge support by Università di Padova, Progetto di Ateneo CDPA048343. First author also partially supported by PRIN 2005 "Prospettive in teoria degli anelli, algebre di Hopf e categorie di moduli", by the DGI and the European Regional Development Fund, jointly, through Project MTM2005-00934, and by the Commissionat per Universitats i Recerca of the Generalitat de Catalunya, Project 2005SGR00206. Third author acknowledges support by GAČR 201/06/0510 and MSM 0021620839.
Ringel also considered the torsion pair \((\mathcal{P}, \mathcal{L})\) generated by \(p\). Here the torsion-free class \(\mathcal{P}\) contains \(p\), the torsion class \(\mathcal{L}\) contains \(t\) and \(q\), and again, the torsion pair is maximal with respect to this property in the sense of [5,3]. However, \((\mathcal{P}, \mathcal{L})\) is not a split torsion pair unless \(R\) is of finite representation type.

Finally, there are also the dual constructions: the torsion pair \((\mathcal{F}, \text{Gen}t)\) generated by \(t\), and the torsion pair \((\mathcal{C}, \text{Q})\) generated by \(q\), see [25], or [5,3] and [5,2].

The aim of our paper is to study these torsion pairs from the point of view of infinite dimensional tilting theory. Indeed, there are tilting modules \(W\) and \(L\) such that \(\mathcal{D} = \text{Gen}W\) and \(\mathcal{L} = \text{Gen}L\). If \(R\) is tame, then it is shown in [24] that \(W\) can be chosen as the direct sum of a set of representatives of the Prüfer modules and the generic module \(G\). A construction of \(W\) in the wild case, as well as a construction of \(L\) in case \(R\) has infinite representation type, can be found in the works of Lukas [22, 23]; for more details we refer to the paper [19].

It turns out that \(W\) and \(L\) play a remarkable role both in the tame and in the wild case. Indeed, they control the behaviour of the category \(\text{Mod-}R\): one can read off the representation type of \(R\) from finiteness conditions satisfied by \(W\) or by \(L\). For example, \(R\) is of tame representation type if and only if \(L\) is noetherian when viewed as a module over its endomorphism ring \(\text{End}L\). Moreover, if \(L\) has finite length over \(\text{End}L\), then \(R\) has finite representation type (Theorems [18] and [19]).

These results are applications of more general investigations carried out in the first part of the paper. We consider arbitrary tilting modules over an arbitrary ring \(R\). As explained in Section 1, using results from [4, 12, 13], every tilting class \(T \perp\) in \(\text{Mod-}R\) corresponds bijectively to a resolving subcategory \(\mathcal{S}\) of \(\text{mod-}R\), and also to a cotilting class \(\perp C\) in the category of left \(R\)-modules \(\text{R-Mod}\). This allows us to associate to \(T\) cotorsion pairs in \(\text{Mod-}R\) and \(\text{R-Mod}\). In Section 2, we characterize finiteness conditions on \(T\) in terms of these cotorsion pairs and of the resolving subcategory \(\mathcal{S}\).

In Section 3, we restrict to the case where \(T\) has projective dimension one. Then \(T\) gives rise to a torsion pair in \(\text{Mod-}R\) with torsion class \(\text{Gen}T\). If \(R\) is a hereditary artin algebra, and \(\mathcal{S}\) is a union of Auslander-Reiten-components, also the torsion pair \((\mathcal{F}, \mathcal{G})\) in \(\text{Mod-}R\) with torsion-free class \(\mathcal{F} = \lim_{\to} \mathcal{S}\) is of importance. We prove that \(T\) is product-complete if and only if \((\mathcal{F}, \mathcal{G})\) is a split torsion pair (Corollary [14]).

In fact, the latter result is a consequence of our investigations in Section 4 devoted to the class \(\mathcal{B}\) of all Baer modules for the torsion class \(\mathcal{G}\). Recall that \(\mathcal{B}\) is the class of all modules \(M\) such that \(\text{Ext}_R^1(M, G) = 0\) for all \(G \in \mathcal{G}\). In Theorem [13] we show that a module belongs to \(\mathcal{B}\) if and only if it is \(\mathcal{S}\)-filtered, generalizing a result from [5].

Finally, in Section 5, we apply our results to the case where \(R\) is a hereditary artin algebra, and \(\mathcal{S} = \text{add} p\). This enables us to prove in Section 6 that the tilting modules \(L\) and \(W\) determine the representation type of \(R\). Moreover, we give an alternative proof of Ringel’s result [25] 3.7 - 3.9 stating that \(R\) is tame if and only if the torsion pair \((\mathcal{C}, \mathcal{Q})\) splits. We also extend to the wild case some results on Baer modules and torsion-free modules obtained in [5].

1. Preliminaries

1.1. Notation. Let \(R\) be a ring, and let \(\text{Mod-}R\) and \(\text{R-Mod}\) be the categories of all right and left \(R\)-modules, respectively. We denote by \(\text{mod-}R\) the subcategory of all modules possessing a projective resolution consisting of finitely generated modules, and we define \(\text{R-mod}\) correspondingly.
Similarly, the classes \(C^\omega\) modules generated and cogenerated, respectively, by the modules in \(M\) and \(B\) are denoted by \(S\) morphic to direct summands of (finite) direct sums of modules of \(M\) consisting of all modules isomorphic to direct summands of direct products of modules of \(C\). We will say that a module \(M\) is a tilting module if for every module \(A\) and \(B\), we obtain a cotorsion pair \((A, B)\) for all \(i > 0\).

We denote by \(\text{Add} C\) (respectively, \(\text{add} C\)) the class consisting of all modules isomorphic to direct summands of (finite) direct sums of modules of \(C\). The class consisting of all modules isomorphic to direct summands of direct products of modules of \(C\) is denoted by \(\text{Prod} C\). Finally, \(\text{Gen} C\) and \(\text{Cogen} C\) denote the class of all modules generated and cogenerated, respectively, by the modules in \(C\).

We will say that a module \(M_R\) with the endomorphism ring \(S\) is endonoetherian if \(M\) is noetherian when viewed as a left \(S\)-module. If \(S\) is a module with finite length then \(M\) is called endofinite. Finally, following [20], a module \(M\) with \(\text{Add} M\) closed under direct products will be called product-complete.

### 1.2. Tilting and cotilting cotorsion pairs.

A module \(T\) is said to be a \((n-)\) tilting module if it satisfies

- (T1) \(\text{proj.dim}(T) \leq n\);
- (T2) \(\text{Ext}_R^i(T, T(\mathcal{I})) = 0\) for each set \(\mathcal{I}\) and each \(i > 0\); and
- (T3) there are \(r \in \mathbb{N}\) and an exact sequence \(0 \to R \to T_0 \to T_1 \to \ldots \to T_r \to 0\)

where \(T_i \in \text{Add}(T)\) for all \(i \leq r\).

The class \(T^\perp\) is then called the tilting class induced by \(T\).

Note that \(T\) is a \(1\)-tilting module if and only if the class \(T^\perp\) coincides with \(\text{Gen} T\).

One then has a tilting torsion pair \((T^\perp, \text{Gen} T)\). The inclusion \(T^\perp \subseteq \text{Gen} T\) holds true for any \(n\)-tilting module \(T\), see [2, 2.3].

**Cotilting modules and classes** are defined dually and have the dual properties.

Tilting and cotilting classes arise naturally in cotorsion pairs. A cotorsion pair is a pair of classes of modules \((\mathcal{A}, \mathcal{B})\) such that \(\mathcal{A} = \mathcal{B}^\perp\) and \(\mathcal{B} = \mathcal{A}^\perp\). If \(S\) is a class of right \(R\)-modules, we obtain a cotorsion pair \((\mathcal{A}, \mathcal{B})\) by setting \(\mathcal{A} = \mathcal{S}^\perp\) and \(\mathcal{B} = \mathcal{S}^\perp\). It is called the cotorsion pair generated by \(S\). Dually, if \(S\) is a class of right \(R\)-modules, we obtain a cotorsion pair \((\mathcal{A}, \mathcal{B})\) by setting \(\mathcal{A} = \mathcal{S}^\perp\) and \(\mathcal{B} = \mathcal{S}^\perp\). It is called the cotorsion pair cogenerated by \(S\).

A cotorsion pair \((\mathcal{A}, \mathcal{B})\) is said to be complete if for every module \(X\) there are short exact sequences \(0 \to X \to B \to A \to 0\) and \(0 \to B' \to A' \to X \to 0\) where \(A, A' \in \mathcal{A}\) and \(B, B' \in \mathcal{B}\). Cotorsion pairs generated by a set of modules or cogenerated by a class of pure-injective modules are always complete [15, 3.2.1 and 3.2.9].

---

1We adopt the convention of writing the torsion-free class on the left side of the torsion pair.

2Our terminology follows [15], hence it differs from previous use.
Cotorsion pairs \((A, B)\) with \(B = T^⊥\) for some \(n\)-tilting module \(T\) are called \(n\)-tilting cotorsion pairs, and cotorsion pairs \((A, B)\) with \(A = T^⊥\) for some \(n\)-cotilting module \(C\) are called \(n\)-cotilting cotorsion pairs. We are now going to describe them as cotorsion pairs generated, respectively cogenerated, by certain classes of modules.

Recall that a subcategory \(S\) of \(\text{mod-R}\) is said to be resolving, if it is closed under direct summands, extensions, kernels of epimorphisms, and contains \(R\). If \(S\) is resolving, then \(S^⊥ = S^⊥_1\), and \(S^⊥_1(S^⊥) = S^⊥(S^⊥_1)\), see \([18, 2.2.11]\).

Dually, we denote by \(\mathcal{PI}\) the full subcategory of \(\text{Mod-R}\) consisting of the pure-injective modules, and we say that a subcategory \(S\) of \(\mathcal{PI}\) is coresolving if it is closed under direct summands, extensions, cokernels of monomorphisms, and contains all the injective modules. Moreover, for \(S\) coresolving, \(S^⊥_1S = S^⊥\), and \((S^⊥)^{\perp_1} = (S^⊥)^{\perp_1}\), see \([18, 2.2.11]\).

The following Theorem, relying on work of Bazzoni, Herbera, and Šťovíček, is essential for our investigation.

**Theorem 1.** Let \(R\) be a ring, and let \((A, B)\) be a cotorsion pair. The following statements hold true.

1. \((A, B)\) is an \(n\)-tilting cotorsion pair if and only if it is generated by a resolving subcategory \(S\) of \(\text{mod-R}\) consisting of modules of projective dimension at most \(n\).
2. \((A, B)\) is an \(n\)-cotilting cotorsion pair if and only if it is cogenerated by a coresolving subcategory \(S\) of \(\mathcal{PI}\) consisting of modules of injective dimension at most \(n\) such that \(S^⊥_1\) is closed under direct products.

In particular, tilting and cotilting classes are always definable classes, that is, they are closed under direct products, direct limits, and pure submodules.

**Proof.** For (1) see \([12, 13]\) or \([18, 5.2.23]\).

(2) For the if-part, note that \((A, B)\) is a complete cotorsion pair by \([18, 3.2.9]\). Further, from the assumption on \(S\) it follows that \(A\) is closed under direct products and kernels of epimorphisms \([18, 2.2.11]\) and \(B\) consists of modules of injective dimension at most \(n\), see \([2, 2.2]\). Then \((A, B)\) is an \(n\)-cotilting cotorsion pair by \([2, 4.2]\).

For the converse implication, we use that every cotilting module is pure-injective, see \([11, 28]\), or \([18, 8.1.7]\). So, if \(C\) is a cotilting module such that \(C^{\perp} = A\), and \(S\) consists of the pure-injective modules in \(B\), then \(S\) is a coresolving subcategory of \(\mathcal{PI}\) that contains \(C\) and has the stated properties, see \([18, 8.1.10]\). Hence \((A, B)\) is cogenerated by \(S\). \(\square\)

Notice that a resolving subcategory \(S\) of \(\text{mod-R}\) as above is uniquely determined by the tilting class \(B\). Indeed, we have the following result.

**Theorem 2.** \([4, 2.2 and 2.3]\) There is a one-to-one correspondence between the \(n\)-tilting classes in \(\text{Mod-R}\) and the \(n\)-cotilting classes in \(\text{R-Mod}\) that are cogenerated by a set \(S^*\) of dual modules. The correspondence is given by the assignment

\[S^⊥ \mapsto S^⊥(= S^T)\]

where \(S\) is a resolving subcategory of \(\text{mod-R}\) as in Theorem \(\[1\]\). Moreover, if \(T\) is an \(n\)-tilting right module, the dual module \(T^*\) is an \(n\)-cotilting left module inducing the corresponding cotilting class.

We remark that if \(R\) is left artinian, the assignment in Theorem \(\[2\]\) even yields a one-to-one correspondence between the 1-tilting classes in \(\text{Mod-R}\) and the 1-cotilting classes in \(\text{R-Mod}\); however, this need not be the case for general rings, cf. \([18, 8.2.8 and 8.2.13]\).
2. Finiteness conditions on large tilting modules

Throughout this section, $R$ denotes a ring and $S$ a resolving subcategory of mod-$R$ consisting of modules of projective dimension at most $n$. According to the results discussed in Section 1, the class $S$ gives rise to the following cotorsion pairs:

2.1. **The cotorsion pair $(\mathcal{M}, \mathcal{L})$ generated by $S$ in mod-$R$,** which is a tilting cotorsion pair by Theorem 1. We fix an $n$-tilting module $T_R$ such that $\mathcal{L} = T^\perp (= S^\perp)$.

2.2. **The cotorsion pair $(\mathcal{C}, \mathcal{C}^\perp)$ cogenerated by $S^\ast$ in R-Mod,** which is the cotilting cotorsion pair corresponding to $(\mathcal{M}, \mathcal{L})$ under the bijection of Theorem 2. The dual module $T^\ast$ is an $n$-cotilting left module such that $\mathcal{C} = T^\perp = T^\ast = S^\ast$.

If $R$ is left noetherian and $n = 1$, then obviously $\mathcal{C} = \lim_{\rightarrow} \mathcal{C}^{\leq \omega}$.

2.3. **The cotorsion pair $(\perp \mathcal{D}, \mathcal{D})$ generated by $\mathcal{C}^{\leq \omega}$ in R-Mod.** Of course, we have $\mathcal{C}^\perp \subseteq \mathcal{D}$, and $\mathcal{C}^{\leq \omega}$ is a resolving subcategory in $R$-mod. Moreover, if $\mathcal{C}^{\leq \omega}$ consists of modules of projective dimension at most $n$, then $(\perp \mathcal{D}, \mathcal{D})$ is a tilting cotorsion pair by Theorem 1 and we can fix an $n$-tilting module $rW$ such that $\mathcal{D} = W^\perp (= (\mathcal{C}^{\leq \omega})^\perp)$.

2.4. **The cotorsion pair $(\mathcal{F}, \mathcal{E})$ cogenerated by $\mathcal{C}^\ast$ in Mod-$R$.** This is the closure of the cotorsion pair $(\mathcal{M}, \mathcal{L})$ studied in [8]. Indeed, by the well-known Ext-Tor-relations, we see that $\mathcal{C}^\ast$ coincides with the class of all dual modules in $\mathcal{L}$, cf. [3, 9.4]. So $(\mathcal{F}, \mathcal{E})$ is cogenerated by the class of all dual modules in $\mathcal{L}$, and $\mathcal{F} = \lim_{\leftarrow} \mathcal{M} = \lim_{\leftarrow} S = \mathcal{T}(S^\ast) = \mathcal{T}\mathcal{C}$ by [8, 2.1 and 2.3].

Assume now that $\mathcal{C} = \lim_{\rightarrow} \mathcal{C}^{\leq \omega}$ and that $\mathcal{C}^{\leq \omega}$ consists of modules of projective dimension at most $n$. Then $\mathcal{F} = \mathcal{T}(\mathcal{C}^{\leq \omega}) = (\mathcal{C}^{\leq \omega})^\ast$. Hence $(\mathcal{F}, \mathcal{E})$ is the cotilting cotorsion pair corresponding to $(\perp \mathcal{D}, \mathcal{D})$ under the bijection of Theorem 2 and the dual module $W^\ast$ is an $n$-cotilting right module such that $\mathcal{F} = \mathcal{T}(W^\ast) = \mathcal{T}W$. In particular, note that in this case $\mathcal{F}$ is closed under direct products, so [15, 4.2] implies that $S$ is covariantly finite in mod-$R$.

We now collect some characterizations of the case when $\mathcal{M}$ is closed under direct limits.

**Theorem 3.** Assume that $\mathcal{C} = \lim_{\rightarrow} \mathcal{C}^{\leq \omega}$, and that $\mathcal{C}^{\leq \omega}$ consists of modules of projective dimension at most $n$. The following statements are equivalent.

1. $\mathcal{M}$ is closed under direct limits, that is, it coincides with $\mathcal{F}$.
2. $\mathcal{M}$ is closed under direct products.
3. The cotorsion pair $(\mathcal{F}, \mathcal{E})$ is generated by some subcategory of mod-$R$.
4. $T$ is product-complete.

If $R$ is right noetherian, or $\mathcal{F}$ consists of modules of bounded projective dimension, then (1) - (4) are further equivalent to

5. $W$ is endonoetherian.

**Proof.** Of course, (1) implies (2) since $\mathcal{M}$ is then a cotilting class by [2, 3]. So, the equivalence of the first four conditions follows immediately from [6, 2.3 and 3.1]. Moreover, under the additional assumptions, condition (3) means that the class $\mathcal{E}$ is closed under direct sums, see [6, 4.10], or [15, 5.1.16]. By [6, 3.2], the latter is equivalent to $W^\ast$ being $\Sigma$-pure-injective. Now we use [3, 9.9], where it is shown that a tilting module is endonoetherian if and only if its dual module is $\Sigma$-pure-injective, and we obtain the equivalence of (3) and (5).

Symmetrically, we obtain
Theorem 4. Assume that $\mathcal{C} = \lim_{\to} \mathcal{C}^{<\omega}$, and that $\mathcal{C}^{<\omega}$ consists of modules of projective dimension at most $n$. The following statements are equivalent:

(1) $\perp \mathcal{D}$ is closed under direct limits, that is, it coincides with $\mathcal{C}$.
(2) $\perp \mathcal{D}$ is closed under direct products.
(3) The cotorsion pair $(\mathcal{C}, \mathcal{C}^{\perp})$ is generated by some subcategory of $\mathcal{R}$-mod.
(4) $W$ is product-complete.

If $R$ is left noetherian, or $\mathcal{C}$ consists of modules of bounded projective dimension, then (1) - (5) are further equivalent to

(5) $T$ is endonoetherian.

Proof. Consider the cotorsion pair $(\perp \mathcal{D}, \mathcal{D})$ generated by the resolving subcategory $\mathcal{C}^{<\omega}$ of $\mathcal{R}$-mod consisting of modules of projective dimension at most $n$. Note that $\mathcal{C} = \lim_{\to} \mathcal{C}^{<\omega} = \lim_{\to} \perp \mathcal{D}$ by [8, 2.3]. Moreover, recall that $\mathcal{F} = \lim_{\to} \mathcal{S}$ is the cotilting class corresponding to $\mathcal{D}$, and $\mathcal{F}^{<\omega} = \mathcal{S}$. So, we can apply Theorem 3, keeping in mind that the roles of $T$ and $W$ are now switched. □

Corollary 5. Let $R$ be a noetherian ring (or a right artinian ring, or a right noetherian ring of finite global dimension). Assume that $\mathcal{C} = \lim_{\to} \mathcal{C}^{<\omega}$, and that $\mathcal{C}^{<\omega}$ consists of modules of projective dimension at most $n$. Then the following statements are equivalent.

(1) $T$ is endofinite
(2) $M$ coincides with $\mathcal{F}$, and $\perp \mathcal{D}$ coincides with $\mathcal{C}$.
(3) $W$ is endofinite.

Proof. Combine Theorems 3 and 4, and use that a module is endofinite if and only if it is endonoetherian and product complete (see e.g. [16, p.43]). □

Corollary 6. Assume that $R$ is an Artin algebra, $\mathcal{C} = \lim_{\to} \mathcal{C}^{<\omega}$, and $\mathcal{C}^{<\omega}$ consists of modules of projective dimension at most $n$. Then the following statements are equivalent.

(1) $\mathcal{S}$ is contravariantly finite in $\text{mod-}R$.
(2) $T$ can be chosen finitely generated.
(3) $W$ can be chosen finitely generated.
(4) $\mathcal{C}^{<\omega}$ is contravariantly finite in $\mathcal{R}$-mod.

In particular, in this case, we have cotorsion pairs $(\mathcal{F}, \mathcal{L})$ in $\text{Mod-}R$, and $(\mathcal{C}, \mathcal{D})$ in $\mathcal{R}$-Mod, where $\mathcal{L} = \lim_{\to} \mathcal{L}^{<\omega}$ and $\mathcal{D} = \lim_{\to} \mathcal{D}^{<\omega}$.

Proof. We apply some results from [10]. For the equivalence of (1)$\iff$(2) and (3)$\iff$(4), we refer to [4, 4.1]. Moreover, (1) implies that $D(\mathcal{S})$ is coresolving and covariantly finite, hence (1)$\implies$(4) holds true by [10, p.125]. Now assume that $W$ is finitely generated. Then $\lim_{\to} \mathcal{C}^{<\omega} = \perp \mathcal{D}$ by [21, 2.4]. Moreover, $W$ and $D(W)$ are endofinite, and $T$ is endofinite by Corollary 5. Since $D(T)$ is a cotilting module with $\perp D(T) = \mathcal{C}$, and $W$ is a tilting module with $W^{\perp} = \mathcal{D}$, we obtain $\text{Prod}D(T) = \mathcal{C} \cap D = \text{Add}W$, see [2, 2.4]. It follows that $D^2(T) \in \text{Prod}D(W) = \text{Add}D(W)$, and since the endofinite module $T$ is a direct summand in $D^2(T)$, and $D(W)$ is finitely generated, we deduce that $T$ is equivalent to a finitely generated tilting module. So we have verified (3)$\iff$(2).

The last claim is shown in [21, 2.4], cf. [6, 5.3]. □
3. Torsion pairs and Auslander-Reiten components

Let us now consider the case of $n = 1$, that is, let $S$ be a resolving subcategory of $\text{mod-} R$ consisting of modules of projective dimension at most one. From the classes $L = S^{-}$, $C = S^{1}$, $D = (C^{\leq \omega})^{-}$ and $F = \lim_{\rightarrow} S$ of the previous section we also obtain some interesting torsion pairs.

First of all, we have the tilting torsion pair $(T^o, L)$ in $\text{Mod-} R$ where $L = \text{Gen} T = T^{-}$, and the cotilting torsion pair $(C, o(T^*))$ in $\text{Mod-} R$ where $C = \text{Cogen} T^*$.

Moreover, if $C^{< \omega}$ consists of modules of projective dimension at most one, we also have the tilting torsion pair $(W^o, \mathcal{D})$ in $\text{Mod-} R$ where $\mathcal{D} = (C^{< \omega})^{-} = \text{Gen} W$.

Finally, if $R$ is also left noetherian, then $C = \lim_{\rightarrow} C^{< \omega}$, and we have the cotilting torsion pair $(F, o(W^*))$ in $\text{Mod-} R$.

Let us look at the last torsion pair in more detail. As we are going to see, if we assume the existence of almost split sequences, and take for $S$ a union of Auslander-Reiten-components, then the torsion class coincides with $\text{Gen} L^{\leq \omega}$.

**Definition.** [9] Let $R$ be a right artinian ring. A subcategory $c$ of $\text{mod-} R$ consisting of indecomposable modules is called an *Auslander-Reiten component* in $\text{mod-} R$ if it satisfies the following conditions.

1. For any $X \in c$ there are a left almost split morphism $X \to Z$ and a right almost split morphism $Y \to X$ in $\text{Mod-} R$ with $Z, Y \in \text{mod-} R$.
2. If $X \to Y$ is an irreducible map in $\text{mod-} R$ with one of the modules lying in $c$, then both modules are in $c$.
3. The Auslander-Reiten-quiver of $c$ is connected.

The next result and its dual allow us to replace the Auslander-Reiten-formula [16] when dealing with artinian rings that need not have a self-duality.

**Lemma 7.** [9 3.6] Let $0 \to A \to B \to C \to 0$ be an almost split sequence in $\text{Mod-} R$, where $R$ is an arbitrary ring, and $X_R$ a module. If $\text{Hom}_R (X, A) = 0$, then also $\text{Ext}^1_R (C, X) = 0$. The converse holds if $C$ has projective dimension at most one.

**Corollary 8.** [11 1.4] Let $R$ be a hereditary ring, and let $0 \to A \to B \to C \to 0$ be an almost split sequence in $\text{Mod-} R$. Then $o A = C^{-}$ and $C^o = \frac{1}{o} A$.

**Proposition 9.** Let $S$ be a resolving subcategory of $\text{mod-} R$ consisting of modules of projective dimension at most one, and set $L = S^{-}$.

1. Assume $R$ is right noetherian. Then there is a torsion pair $(\lim(T^o)^{< \omega}, G)$ in $\text{Mod-} R$ with the torsion-free class $\lim(T^o)^{< \omega} = (L^{< \omega})^o$ and the torsion class $G = \text{Gen} L^{< \omega} = \lim_{\rightarrow} L^{< \omega}$.
2. Assume that $R$ is artinian and hereditary, and that every finitely generated indecomposable non-injective right $R$-module is first term of an almost split sequence in $\text{Mod-} R$ consisting of finitely generated modules. Assume further that the indecomposable modules of $S$ are not injective and form a union of Auslander-Reiten-components. Then $\lim(T^o)^{< \omega} = F$, and there is a torsion pair $(F, \text{Gen} L^{< \omega})$ in $\text{Mod-} R$.

**Proof.** (1) Since $R$ is right noetherian, we know from [18 4.5.2] that $(T^o, L)$ induces a torsion pair $((T^o)^{< \omega}, L^{< \omega})$ in $\text{mod-} R$. Now we apply [15 4.4] to obtain a torsion pair $(\lim(T^o)^{< \omega}, \lim L^{< \omega})$ in $\text{Mod-} R$ with the stated properties.
2. First, since $R$ is left noetherian and $C^{< \omega}$ consists of modules of projective dimension at most one, we have the cotilting torsion pair $(F, o(W^*))$ in $\text{Mod-} R$.

Furthermore, the assumptions on $S$ imply by Corollary [9] that $L = S^{-} = \circ S$. Hence $(T^o, L)$ coincides with the torsion pair $(\circ S)^o, \circ S$ generated by $S$. 

LARGE TILTING MODULES AND REPRESENTATION TYPE 7
We claim that $(T^o)^{<\omega} = \mathcal{S}$. The inclusion $\subseteq$ is obvious. For the reverse one, consider $A \in (T^o)^{<\omega} \subseteq \mathcal{L}^o$, and assume w.l.o.g. that $A$ is indecomposable. Since the tilting class $\mathcal{L}$ generates all the injective modules, $A$ is not injective. Thus there is an almost split sequence $0 \to A \to B \to C \to 0$ in $\text{Mod}-R$ consisting of finitely generated modules, and for all $X \in \mathcal{L}$ we have by Corollary 5.2.1 that $X \in oA = C^{<\omega}$, which shows that $C \in \mathcal{M}$. Then $C \in \mathcal{M}^{<\omega} = \mathcal{S}$ by [15, 5.2.1], hence also $A \in \mathcal{S}$.

Finally, by (1), $\mathcal{F} = \lim_{\to} C = \lim_{\to} (T^o)^{<\omega}$, which concludes the proof. □

**Example 10.** Assume that $R$ is an indecomposable, artinian, hereditary, left pure-semisimple ring of infinite representation type. Choose $\mathcal{S} = \text{add} \mathfrak{p}$ where $\mathfrak{p}$ is the preprojective component of $\text{mod}-R$, and consider the corresponding torsion and cotorsion pairs.

It is shown in [1, 4.3] that in this case the module $rW$ from section 2.3 is finitely generated and product-complete, and the classes $+\mathcal{D}$ and $\mathcal{C}$ in 2.3 and 2.2 coincide, cf. Theorem 3. In other words, we have a (co)tilting cotorsion pair $(\mathcal{C}, \mathcal{D})$ in $\text{Mod}-R$ generated by $C^{<\omega}$, and $W$ is a (co)tilting module (co)generating this cotorsion pair.

Moreover, in $R\text{-Mod}$ we have a split cotilting torsion pair $(C, o(T^o))$, and a split tilting torsion pair $(W^o, D)$, where $W^o \subseteq \mathcal{C}$, see [1, 4.6 and 4.7]. In $\text{Mod}-R$ we have a tilting torsion pair $(T^o, \mathcal{L})$, and by Proposition 9(2), we have a cotilting torsion pair $(\lim \text{add} \mathfrak{p}, \text{Gen}\mathcal{L}^{<\omega})$.

Now, since we are assuming that $R$ has infinite representation type, we are also assuming that there are finitely generated indecomposable non-injective left $R$-modules which are not first terms of an almost split sequence in $R\text{-mod}$, see for example [20]. So the assumptions of Proposition 9(2) are not satisfied in $R\text{-Mod}$. Indeed, let us consider the tilting torsion pair $(W^o, D)$ in $R\text{-Mod}$. By Proposition 3(1) we obtain a torsion pair $(\lim \mathcal{W}^{<\omega}(W^o), \mathcal{G})$ in $R\text{-Mod}$. Here the torsion-free class $\lim \mathcal{W}^{<\omega}(W^o) = (D^{<\omega})^o$ is contained in the cotilting torsion-free class $\mathcal{C} = \lim \mathcal{C}^{<\omega}$, but in contrast to Proposition 9(2), this inclusion is proper. In fact, $W$ belongs to $\mathcal{C}$, but it does not belong to $(D^{<\omega})^o$ as $W \in D^{<\omega}$.

4. Relative Baer modules

Let $R$ be an arbitrary ring and $\mathcal{T}$ be a torsion class in $\text{Mod}-R$. A module $M \in \text{Mod}-R$ is a **Baer module** for $\mathcal{T}$ provided that $\text{Ext}^1_R(M, T) = 0$ for all $T \in \mathcal{T}$.

It is shown in [5] that the study of Baer modules can often be reduced to the countably generated case. To recall this, we need further notation.

Let $\sigma$ be an ordinal. An increasing chain of submodules, $\mathcal{J} = (M_\alpha \mid \alpha \leq \sigma)$, of a module $M$ is called a filtration of $M$ provided that $M_0 = 0$, $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$ for all limit ordinals $\alpha \leq \sigma$ and $M_\sigma = M$.

Given a class of modules $\mathcal{C}$ and a module $M$, a filtration $\mathcal{J}$ is a $\mathcal{C}$-filtration of $M$ provided that $M_{\alpha+1}/M_\alpha$ is isomorphic to some element of $\mathcal{C}$ for each $\alpha < \sigma$. In this case we say that $M$ is $\mathcal{C}$-filtered.

**Theorem 11.** [5, Theorem 1] Let $R$ be an $\mathcal{N}_0$-noetherian ring and $\mathcal{T}$ be a torsion class in $\text{Mod}-R$ such that $+\mathcal{T} = +\mathcal{T}$. Assume that either $\mathcal{T}$ consists of modules of finite injective dimension, or $+\mathcal{T}$ consists of modules of finite projective dimension. Then a module $M$ is a Baer module for $\mathcal{T}$ if and only if it has a filtration $M = (M_\alpha \mid \alpha \leq \kappa)$ such that, for each $\alpha < \kappa$, $M_{\alpha+1}/M_\alpha$ is a countably generated Baer module for $\mathcal{T}$.

Assume again that $\mathcal{S}$ is a resolving subcategory of $\text{mod}-R$ consisting of modules of projective dimension at most one, and let $(\mathcal{M}, \mathcal{L})$ be the cotorsion pair generated
by $\mathcal{S}$. Notice that $\mathcal{M} = \mathcal{L}^\perp = \mathcal{L}^\perp$ consists of modules of projective dimension $\leq 1$, see [2] 2.2.

We will now focus on Baer modules for the torsion class $\mathcal{G} = \text{Gen}\mathcal{L}^{<<\omega}$ from Proposition [9]. We will denote this class by $\mathcal{B}$.

**Lemma 12.** Let $R$ be a right noetherian ring. Assume that every module $A \in \mathcal{L}$ can be purely embedded in a direct product of modules from $\mathcal{L}^{<<\omega}$, and that $\mathcal{B}$ consists of modules of projective dimension $\leq 1$. Then $\mathcal{B} = \mathcal{M}$.

**Proof.** Since $\text{Gen}\mathcal{L}^{<<\omega} \subseteq \mathcal{L}$, we have $\mathcal{M} = \mathcal{L}^\perp \subseteq \mathcal{B}$. For the reverse inclusion, let $M \in \mathcal{B}$. By Theorem 11 and by the Eklof Lemma [16, 3.1.2], we can assume w.l.o.g. that $M$ is countably generated. Since $M^\perp$ contains all injective modules, and each indecomposable injective module is finitely generated, $\mathcal{G}$ contains all homomorphic images of injective modules. This implies that $\mathcal{B}$ consists of modules of projective dimension $\leq 1$. Hence Lemma 12 applies and gives $\mathcal{B} = \mathcal{M}$. Moreover, the Eklof Lemma [18, 3.1.2] gives that each $\mathcal{S}$-filtered module is in $\mathcal{M}$.

Conversely, let $M \in \mathcal{M}$. By Theorem 11 $M$ is $\mathcal{S}$-filtered. Then $\mathcal{B}$ is the class of all countably generated modules from $\mathcal{M}$. So it remains to prove that each countably generated module $M \in \mathcal{M}$ is $\mathcal{S}$-filtered.

By [18, 3.2.4], there is a module $N$ which is a union of an $\mathcal{S}$-filtration $(N_i \mid i \leq \sigma)$ such that $M$ is a direct summand in $N$. By the Hill Lemma [18, 4.2.6], we can w.l.o.g. assume that $\sigma = \omega$.

By induction on $i < \omega$, we will construct an $\mathcal{S}$-filtration, $(M_i \mid i \leq \omega)$, of the module $M$. Let $\{g_i \mid i < \omega\}$ be an $R$-generating subset of $M$. Denote by $t$ the torsion radical corresponding to the torsion pair $(\mathcal{F}, ^*\mathcal{F})$ (where $\mathcal{F} = \varinjlim \mathcal{S}$ is the torsion-free class from [2, 4]). Put $M_0 = 0$, and if $M_i$ is defined so that $M_i$ is finitely generated and $t(M/M_i) = 0$, we consider the least index $j < \omega$ such that $(M_i, g_i) \subseteq N_j$, and let $M_{i+1} = M \cap N_j$.

Then $M_{i+1}$ is finitely generated because $N_j$ is such, and moreover $M/M_{i+1} \cong (M + N_j)/N_j \subseteq N/N_j$, so $t(M/M_{i+1}) = 0 = t(N/N_j)$ because $N/N_j$ is $\mathcal{S}$-filtered.

Since $g_i \in M_{i+1}$ for each $i < \omega$, we have $M = M_\omega = \bigcup_{i<\omega} M_i$. Finally $\mathcal{F}$ is a resolving class, so the exact sequence $0 \to M_{i+1}/M_i \to M/M_i \to M/M_{i+1} \to 0$ yields $M_{i+1}/M_i \in \mathcal{F}^{<<\omega} = \mathcal{S}$, proving that $(M_i \mid i \leq \omega)$ is an $\mathcal{S}$-filtration of $M$. □

**Remark.** Theorem 13 was first proved in the particular case when $R$ is a tame hereditary artin algebra and $\mathcal{S}$ is the class of all preprojective modules, see [3] 2.2. There are many more analogies with [5]:

- Let $R$ be a hereditary artin algebra, and $\mathcal{S}$ is a resolving subcategory of $\text{mod-}R$ satisfying the assumptions of Proposition [9] 2). Denote by $t$ the torsion radical corresponding to the torsion pair $(\mathcal{L}^\alpha, \mathcal{L})$. By Proposition [9] 2) we have $\mathcal{L}^\alpha \subseteq \mathcal{F}$.

- Let $\mathcal{B}$ be the class of all Baer modules for $\mathcal{G} = \text{Gen}\mathcal{L}^{<<\omega}$. Two modules $B, B' \in \mathcal{B}$ are called equivalent iff $B/t(B) \cong B'/t(B')$. 

As in [5], one can prove in our general setting that
(1) $F$ is exactly the class of all pure epimorphic images of the modules in $B$.
(2) If $M \in F$ then $\ell(M)$ is a pure submodule of $M$; if moreover $M \in B$ then $\ell(B) \in \text{Add}(T)$.
(3) If $B, B' \in B$, then $B$ is equivalent to $B'$ iff there exist $L, L' \in \text{Add}(T)$ such that $B \oplus L \cong B' \oplus L'$.
(4) Equivalence classes of modules in $B$ correspond bijectively to isomorphism classes of modules in the torsion–free class $L^\circ$.

However, there does not appear to be any general decomposition theorem for countably generated Baer modules extending [5, Prop.13] and [23, 4.3].

As a consequence of our description of the Baer modules, we obtain yet another characterization of when $M$ is closed under direct limits.

**Corollary 14.** Let $R$ be a hereditary artin algebra. Assume further that the indecomposable modules of $S$ are not injective and form a union of Auslander-Reiten-components. Then $M$ coincides with $F$ if and only if $(F, \text{Gen}L^{<\omega})$ is a split torsion pair.

**Proof.** We know from Proposition [9] that $(F, \text{Gen}L^{<\omega})$ is a torsion pair. So, we have to show that $M = F$ if and only if Ext$_R^1(Z, X) = 0$ for all $X \in \text{Gen}L^{<\omega}$ and $Z \in F$.

The only-if part follows immediately from the fact that $\text{Gen}L^{<\omega} \subseteq L = M^\perp$.

For the if-part, observe that if $F \subseteq \perp \text{Gen}L^{<\omega}$, then $F$ consists of Baer modules for $\text{Gen}L^{<\omega}$, hence it is contained in $M$ by Theorem [13]. □

5. Applications to hereditary artin algebras

¿From now on, we will assume that $R$ is an indecomposable representation-infinite hereditary artin algebra with the standard duality $D: \text{mod}-R \to R\text{-mod}$.

Let $\mathfrak{q}$, $\mathfrak{t}$ and $\mathfrak{p}$ denote representative sets of all indecomposable finitely generated preinjective, regular, and preprojective right modules, respectively. The corresponding sets of left modules are denoted by $R\mathfrak{q}$, $R\mathfrak{t}$ and $R\mathfrak{p}$.

We now apply our previous considerations to the resolving category $S = \text{add} \mathfrak{p}$. Then our torsion pairs look as follows.

5.1. **The torsion pair generated by $\mathfrak{p}$ in $\text{Mod}-R$.** By the Auslander-Reiten formula

$$L = \mathfrak{p}^\perp = \circ \mathfrak{p}$$

so $L$ is the class of all right modules having no non-zero homomorphism to $\mathfrak{p}$, or in other words, the class of all modules that have no non-zero finitely generated preprojective direct summands (see [24 Corollary 2.2]). There is a countably infinitely generated tilting right module generating $L$, called the Lukas tilting module, and denoted by $L$, cf. [19].

The torsion–free class corresponding to $L$ will be denoted by $\mathcal{P}$. This is the class of all (possibly infinitely generated) preprojective right modules. We have

$$\mathcal{P} \cap \text{mod}-R = \text{add} \mathfrak{p}$$

(Note: in [22] and [23], preprojective modules are called ‘$\mathcal{P}^\infty$-torsion-free’, and the modules in $L$ are called ‘$\mathcal{P}^\infty$-torsion’).
5.2. **The torsion pair generated by** $Rq$ **in** $R$-**Mod.** By the Auslander-Reiten formula

$$C = \perp (Rq) = Rq^o$$

so $C = Rq^o$ is the class of all left modules having no non-zero homomorphism from $q$, or in other words, the class of all modules that have no non-zero finitely generated preinjective direct summands. By Theorem 2, we know that $D(L)$ is a cotilting module cogenerating $C$.

The corresponding torsion class is denoted by $Q$. This is the class of all (possibly infinitely generated) preinjective left modules, that is, of all (possibly infinite) direct sums of modules from $Rq$, see [25, 3.3].

Note that in the tame case, the torsion pair $(C, Q)$ is a split torsion pair, see [25, 24].

5.3. **The torsion pair cogenerated by** $Rt$ **in** $R$-**Mod.** As $C^{<\omega} = \text{add}(Rp \cup Rt)$, and since from every module in $Rp$ there is a non-zero map to some module in $Rt$, we infer from the Auslander-Reiten formula that

$$D = (Rt)^\perp = o(Rt)$$

is the torsion class of all divisible left modules, see [25]. The corresponding torsion-free class, called the class of all reduced left modules, is denoted by $R$.

We fix a tilting left module $W$ which generates $D$. If $R$ is tame, then it is shown in [24] that $W$ can be chosen as the direct sum of a set of representatives of the Prüfer left $R$-modules and the generic left $R$-module $RG$. This module is called the Ringel tilting module. Moreover, in the tame case, the torsion pair $(R, D)$ is a split torsion pair, see [25, 24].

5.4. **The torsion pair generated by** $t$ **in** $\text{Mod-}R$. Dually we see that $L^{<\omega} = \text{add}(q \cup t)$, hence by the Auslander-Reiten formula

$$F = t^o = \perp t$$

is the class of all torsion-free right modules, see [25]. Moreover, $F = \text{lim add}p$, and $D(W)$ is a cotilting module which cogenerates $F$, see Section 2.4 and Proposition 9. In the tame case, $D(W)$ is the direct product of a set of representatives of the adic right $R$-modules and the generic right $R$-module $GR$.

The corresponding torsion class is $\text{Gen}L^{<\omega} = \text{Gen}t$, called the class of all torsion modules, see [24, 3.5].

Notice that

$$P = L^o = (o(p))^o \subseteq t^o = F, \quad \text{and} \quad Q = o(D(L)) = o(Rq^o) \subseteq o(Rt) = D$$

Moreover, we remark the following properties of the torsion pairs above.

5.5. **Extremal torsion pairs.** The class $Q = o(Rq^o)$ is the smallest possible torsion class in $R$-Mod containing $Rq$, and the class $R = o(Rt)^o$ is the smallest possible torsion–free class in $R$-Mod containing $Rt$. Note that $R = W^o$ also contains $RP$.

So, both torsion pairs $(C, Q)$ and $(R, D)$ have the property that the indecomposable finite length modules in the torsion-free class are precisely the modules in $RP \cup Rt$, and the indecomposable finite length modules in the torsion class are precisely the modules in $Rq$. Moreover, as shown in [24, §3], they are extremal with this property. More precisely, if $(X, Y)$ is a torsion pair in $R$-Mod such that $X$ contains $Rt$ and $Y$ contains $Rq$, then

$$R \subseteq X \subseteq C \quad \text{and} \quad Q \subseteq Y \subseteq D.$$
Similarly, the class $P = \langle p \rangle^\circ$ is the smallest possible torsion–free class in Mod-$R$ that contains $p$, and $\text{Gen}$ is the smallest possible torsion class in Mod-$R$ containing $t$. Moreover, $\text{Gen}$ also contains $q$.

So, both torsion pairs $(P, L)$ and $(F, \text{Gen})$ have the property that the indecomposable finite length modules in the torsion-free class are precisely the modules in $p$, and the indecomposable finite length modules in the torsion class are precisely the modules in $t \cup q$. Furthermore, they are extremal with this property, in the sense that if $(X, Y)$ is a torsion pair in Mod-$R$ such that $X$ contains $p$ and $Y$ contains $t$, then $P \subseteq X \subseteq F$ and $\text{Gen} \subseteq Y \subseteq L$.

5.6. Baer modules. A module $M$ is called Baer provided that $M$ is a Baer module for $\text{Gen}$. As an application of Theorem 13, we obtain the following result (whose particular instance for tame algebras is [5, Theorem 2]).

Corollary 15. Let $R$ be an indecomposable representation-infinite hereditary artin algebra. A right $R$-module $M$ is Baer if and only if it is $p$-filtered.

As in [5] we obtain as consequences

Corollary 16. A module $M$ is Baer if and only if there is an exact sequence $0 \to M \to L_1 \to L_2 \to 0$ where $L_1, L_2 \in \text{Add} L$.

Corollary 17. The following statements are equivalent for a module $M$.

1. $M$ is torsion–free.
2. $M$ is a pure-epimorphic image of direct sum of indecomposable preprojective modules.
3. $M$ occurs as the end term in a pure–exact sequence $0 \to N \to B \to M \to 0$ with a Baer module $B$ and $N \in \text{Add} L$.

6. Representation type

Again we assume that $R$ is an indecomposable hereditary artin algebra. We now consider the cotorsion pairs $(\mathcal{C}, \mathcal{C}^\perp)$ and $(\mathcal{D}, D)$ in $R$-Mod cogenerated by $Rq$ and generated by $Rt$, respectively. Note that they coincide when $R$ is of tame representation type, as shown in [24]. In fact, this characterizes the tame case.

Theorem 18. The following statements are equivalent.

1. $(\mathcal{C}, D)$ is a cotorsion pair.
2. $W$ is product-complete.
3. $L$ is endonoetherian.
4. The torsion pair $(\mathcal{C}, Q)$ splits.
5. $R$ is of tame representation type.

Proof. The equivalence of the first three conditions is just Theorem 4. Moreover, (1) and (4) are equivalent by Corollary 14. Finally, the equivalence of (4) and (5) is shown by Rings [25, 3.7 - 3.9]. Alternatively, one can use Kern’s construction over a wild hereditary algebra of an indecomposable divisible module in $\mathcal{C}$ which does not belong to $\perp D$, see [22, 1.7 and p.416]. This shows that in the wild case $\perp D$ is properly contained in $\mathcal{C}$ and therefore proves (1)$\Rightarrow$(5). The converse implication is proven in [24].

Now let us consider the relationship between the cotorsion pairs $(M, L)$ and $(F, \mathcal{E})$, as defined in 2.1 and 2.4.
Theorem 19. Assume that $R$ is of tame representation type. The following statements are equivalent.

1. $\mathcal{M}$ is closed under direct limits, that is, it coincides with $\mathcal{F}$.
2. $L$ is product-complete.
3. $W$ is endonoetherian.
4. $R$ is of finite representation type.

Proof. The equivalence of the first three conditions is just Theorem 3. Moreover, if $R$ is a ring of finite representation type, then all modules are endofinite. So, (4) trivially implies (3). Conversely, it is known that $\mathcal{M}$ is properly contained in $\mathcal{F}$ when $R$ is tame of infinite representation type. For example, the generic module is torsion-free but not Baer, see [5]. Thus (1) implies (4).

\[ \square \]

Corollary 20. Assume that $R$ is of infinite representation type. Then $(\mathcal{F}, \text{Gen})$ does not split, and $\mathcal{M}$ is properly contained in $\mathcal{F}$.

Proof. By Corollary 14 the torsion pair $(\mathcal{F}, \text{Gen})$ splits if and only if $\mathcal{M} = \mathcal{F}$. But if this is the case, then we know from Theorem 3 that $L$ is product-complete. In particular, $L$ is $\Sigma$-pure-injective, so it has a decomposition $L = \bigoplus L_i$ in indecomposable modules with local endomorphism ring. By [22, 6.1.b(ii)] every $L_i$ has the property that $\text{Add}L_i \subseteq \text{Add}L_i$. By the Theorem of Krull-Remak-Schmidt-Azumaya it follows that all $L_i$ are isomorphic, that is, $L$ is equivalent to an indecomposable product-complete, thus endofinite, tilting module. But by Corollary 3 this implies that $\mathcal{D}$ coincides with $\mathcal{C}$, and $\mathcal{M}$ coincides with $\mathcal{F}$. This means that $R$ is of finite representation type by Theorems 18 and 19.

\[ \square \]

References

[1] L. Angeleri H"ugel, A key module over pure-semisimple hereditary rings, Journal of Algebra 307 (2007), 361-376.
[2] L. Angeleri H"ugel and F. Coelho, Infinitely generated tilting modules of finite projective dimension, Forum Math 13 (2001), 239-250.
[3] L. Angeleri H"ugel, D. Herbera, Mittag-Leffler conditions on modules, Indiana Univ. Math. J., in press.
[4] Angeleri H"ugel, L.; Herbera, D.; Trlifaj, J., Tilting modules and Gorenstein rings, Forum Math. 18 (2006), 217-233.
[5] L. Angeleri H"ugel, D. Herbera, and J. Trlifaj, Baer and Mittag-Leffler modules over tame hereditary algebras, preprint.
[6] L. Angeleri H"ugel, J. ˇSaroch, J. Trlifaj, On the telescope conjecture for module categories, Journal of Pure and Appl. Algebra 212 (2008), 297-310.
[7] L. Angeleri H"ugel; J. Trlifaj, Tilting theory and the finitistic dimension conjectures, Trans. Amer. Math. Soc. 354 (2002), 4345-4358.
[8] L. Angeleri H"ugel, J. Trlifaj, Direct limits of modules of finite projective dimension. In: Rings, Modules, Algebras, and Abelian Groups. LNPAM 236 M. Dekker (2004), 27–44.
[9] L. Angeleri H"ugel, H. Valenta: A duality result for almost split sequences, Colloquium Mathematicum 80 (1999), 267-292.
[10] M. Auslander; I. Reiten, Applications of contravariantly finite subcategories, Adv. Math. 86 (1991), 111-152.
[11] S. Bazzoni, Cotilting modules are pure-injective, Proc. Amer. Math. Soc. 131 (2003), 3665-3672.
[12] S. Bazzoni and D. Herbera, One dimensional tilting modules are of finite type, Algebras and Representation Theory 11 (2008), 45-61.
[13] S. Bazzoni and J. ˇStovíček, All tilting modules are of finite type, Proc. Amer. Math. Soc. 135 (2007), 3771-3781.
[14] W. Crawley-Boevey, Regular modules for tame hereditary algebras, Proc. London Math. Soc. 62 (1991), 490–508.
[15] W. Crawley-Boevey, Locally finitely presented additive categories, Comm. Algebra 22 (1994), 1644-1674.
[16] W. Crawley-Boevey, Infinite dimensional modules in the representation theory of finite dimensional algebras, In: Algebras and modules I (ed. by I. Reiten, S. Smalø and O. Solberg), CMS Conf. Proc. 23 (1998), 29–54.
[17] L. Fuchs and L. Salce, Modules over Non-Noetherian Domains, AMS, Providence 2001.
[18] R. Gorbel and J. Trlifaj, Approximations and Endomorphism Algebras of Modules, de Gruyter, Berlin 2006.
[19] O. Kerner and J. Trlifaj, Tilting classes over wild hereditary algebras, J. Algebra, 290 (2005), 538–556.
[20] H. Krause, M. Saorín, On minimal approximations of modules, In: Trends in the representation theory of finite dimensional algebras (ed. by E. L. Green and B. Huisgen-Zimmermann), Contemp. Math. 229 (1998) 227-236.
[21] H. Krause; O. Solberg, Applications of cotorsion pairs, J. London Math. Soc. 68 (2003), 631–650.
[22] F. Lukas, Infinite-dimensional modules over wild hereditary algebras, J. London Math. Soc. 44 (1991), 401–419.
[23] F. Lukas, A class of infinite-rank modules over tame hereditary algebras, J. Algebra 158 (1993), 18–30.
[24] I. Reiten and C. M. Ringel, Infinite dimensional representations of canonical algebras, Canad. J. Math. 58 (2006), 180–224.
[25] C. M. Ringel, Infinite dimensional representations of finite dimensional hereditary algebras, Symposia Math. 23 (1979), 321–412.
[26] C. M. Ringel, The Ziegler spectrum of a tame hereditary algebra, Coll Math. 76 (1998), 105–115.
[27] J. ˇSaroch and J. ˇŠtovíček, The countable telescope conjecture for module categories, preprint.
[28] J. ˇŠtovíček, All n-cotilting modules are pure-injective, Proc. Amer. Math. Soc. 134 (2006), 1891-1897.
[29] B. Zimmermann-Huisgen, Strong preinjective partitions and representation type of artinian rings, Proc. Amer. Math. Soc. 109 (1990), 309-322.

Dipartimento di Informatica e Comunicazione, Università degli Studi dell’Insubria, Via Mazzini 5, I - 21100 Varese, ITALY
E-mail address: lidia.angeleri@uninsubria.it

Mathematisches Institut, Heinrich-Heine-Universität Düsseldorf, Universitätsstr.1, 40225 Düsseldorf, GERMANY
E-mail address: kerner@math.uni-duesseldorf.de

Charles University, Faculty of Mathematics and Physics, Department of Algebra, Sokolovská 83, 186 75 Prague 8, Czech Republic
E-mail address: trlifaj@karlin.mff.cuni.cz