QUANTUM ALGEBRAS IN NUCLEAR STRUCTURE

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ABSTRACT
Quantum algebras are a mathematical tool which provides us with a class of symmetries wider than that of Lie algebras, which are contained in the former as a special case. After a self-contained introduction to the necessary mathematical tools ($q$-numbers, $q$-analysis, $q$-oscillators, $q$-algebras), the $su_q(2)$ rotator model and its extensions, the construction of deformed exactly soluble models (Interacting Boson Model, Moszkowski model), the use of deformed bosons in the description of pairing correlations, and the symmetries of the anisotropic quantum harmonic oscillator with rational ratios of frequencies, which underly the structure of superdeformed and hyperdeformed nuclei, are discussed in some detail. A brief description of similar applications to molecular structure and an outlook are also given.

1. Introduction

Quantum algebras (also called quantum groups) are deformed versions of the usual Lie algebras, to which they reduce when the deformation parameter $q$ is set equal to unity. From the mathematical point of view they are Hopf algebras. Their use in physics became popular with the introduction of the $q$-deformed harmonic oscillator (sec. 10) as a tool for providing a boson realization of the quantum algebra $su_q(2)$ (sec. 14), although similar mathematical structures had already been known (sec. 11). Initially used for solving the quantum Yang–Baxter equation, quantum algebras have subsequently found applications in several branches of physics, as, for example, in the description of spin chains, squeezed states, rotational and vibrational nuclear and molecular spectra, and in conformal field theories. By now several kinds of generalized deformed oscillators (sec. 12) and generalized deformed $su(2)$ algebras (sec. 17) have been introduced.

It is clear that quantum algebras provide us with a class of symmetries which is richer than the class of Lie symmetries, which are contained in the former as a special case. It is therefore conceivable that quantum algebras can turn out to be appropriate for describing symmetries of physical systems which are outside the realm of Lie algebras.
Here we shall confine ourselves to applications of quantum algebras in nuclear structure physics. The structure of this review is as follows: In order to make this review self-contained, we are going first to give a brief account of the necessary tools: $q$-numbers and $q$-analysis (secs 2–9), $q$-deformed oscillators (secs 10–13), $q$-deformed $su(2)$ algebras (secs 14–18). The remainder will be devoted to specific applications in nuclear structure problems, starting with phenomenology and advancing towards more microscopic subjects. The $su_q(2)$ rotator model (secs 19–23) and its extensions (secs 24–26), as well as the formulation of deformed exactly soluble models (Interacting Boson Model (secs 27–29), Moszkowski model (sec. 30)) will be covered in some detail. Subsequently, the use of quantum algebraic techniques for the description of pairing correlations in nuclei (secs 31–33), as well as the symmetries of the anisotropic quantum harmonic oscillators with rational ratios of frequencies (sec. 34) will also be considered in some detail. The latter are of current interest in connection with the symmetries underlying superdeformed and hyperdeformed nuclear bands (sec. 34). Finally, a brief account of applications of the same techniques to molecular structure (sec. 35) and an outlook (sec. 36) will be given.

2. $q$-numbers

The $q$-number corresponding to the ordinary number $x$ is defined as

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad (2.1)$$

where $q$ is a parameter. The same definition holds if $x$ is an operator. We remark that $q$-numbers remain invariant under the substitution $q \to q^{-1}$.

If $q$ is real, $q$-numbers can easily be put in the form

$$[x] = \frac{\sinh(\tau x)}{\sinh(\tau)}, \quad (2.2)$$

where $q = e^{\tau}$ and $\tau$ is real.

If $q$ is a phase factor, $q$-numbers can be written as

$$[x] = \frac{\sin(\tau x)}{\sin(\tau)}, \quad (2.3)$$

where $q = e^{i\tau}$ and $\tau$ is real.

In both cases it is clear that in the limit $q \to 1$ (or, equivalently, $\tau \to 0$) $q$-numbers (or operators) tend to the ordinary numbers (or operators):

$$\lim_{q \to 1} [x] = x. \quad (2.4)$$

A few examples of $q$-numbers are given here:

$$[0] = 0, \quad [1] = 1, \quad [2] = q + q^{-1}, \quad [3] = q^2 + 1 + q^{-2}. \quad (2.5)$$
Identities between $q$-numbers exist. They are, however, different from the familiar identities between usual numbers. As an exercise one can show (using the definition of $q$-numbers) that

$$[a][b + 1] - [b][a + 1] = [a - b]. \quad (2.6)$$

The $q$-factorial of an integer $n$ is defined as

$$[n]! = [n][n-1] \ldots [2][1]. \quad (2.7)$$

The $q$-binomial coefficients are defined as

$$\binom{m}{n} = \frac{[m]!}{[m-n]![n]!}, \quad (2.8)$$

while the $q$-binomial expansion is given by

$$[a \pm b]^m = \sum_{k=0}^{m} \binom{m}{k} a^{m-k} (\pm b)^k. \quad (2.9)$$

In the limit $q \to 1$ we obviously have

$$[n]! \to n! \quad \text{and} \quad \binom{m}{n} \to \binom{m}{n}, \quad (2.10)$$

where $n!$ and $\binom{m}{n}$ are the standard factorial and binomial coefficients respectively.

It should be noticed that two-parametric deformed numbers have also been introduced

$$[x]_{p,q} = \frac{q^x - p^{-x}}{q - p^{-1}}. \quad (2.11)$$

In the special case $p = q$ they reduce to the usual $q$-numbers.

**3. $q$-deformed elementary functions**

In addition to $q$-deformed numbers and operators, $q$-deformed elementary functions can be introduced. The $q$-exponential function is defined as

$$e_q(ax) = \sum_{n=0}^{\infty} \frac{a^n}{[n]!} x^n, \quad (3.1)$$

while the $q$-trigonometric functions are defined as

$$\sin_q(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{[2n+1]!}, \quad (3.2)$$

$$\cos_q(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{[2n]!}. \quad (3.3)$$
It should also be noticed that $q$-deformed polynomials, such as $q$-deformed Hermite polynomials \(7, 8, 9\) and $q$-deformed Laguerre polynomials \(7\) also exist (see also subsec. 34.3).

4. *q*-derivatives

Proceeding along this path one can build a new differential calculus, based on $q$-deformed quantities (see \(10, 11\) for concise expositions). For this purpose the $q$-derivative is defined as

$$D_q^x f(x) = \frac{f(qx) - f(q^{-1}x)}{(q - q^{-1})x}.$$  \hspace{1cm} (4.1)

The similarity between the present definition and the one of $q$-numbers (eq.(2.1)) is clear.

Using the definition of the $q$-derivative one can easily see that

$$D_q^x(ax^n) = a[n]x^{n-1},$$ \hspace{1cm} (4.2)

$$D_q^x e_q(ax) = ae_q(ax).$$ \hspace{1cm} (4.3)

From the definition of the $q$-derivative one can further derive the sum rule

$$D_q^x(f(x) + g(x)) = D_q^x f(x) + D_q^x g(x),$$ \hspace{1cm} (4.4)

as well as the rule

$$D_q^x[x_1 \pm bx_2]^m = \pm [m]b[x_1 \pm bx_2]^{m-1},$$ \hspace{1cm} (4.5)

where $a$ and $b$ are constants and $[ax_1 \pm bx_2]^m$ is given by the $q$-binomial expansion (eq. (2.9)). One can also prove the $q$-integration by parts formula

$$D_q^x(f(x)g(x)) = \frac{f(qx)g(qx) - f(q^{-1}x)g(q^{-1}x)}{(q - q^{-1})x}.$$ \hspace{1cm} (4.6)

From this, the following two forms of the Leibnitz rule can be derived

$$D_q^x(f(x)g(x)) = (D_q^x f(x))g(q^{-1}x) + f(qx)(D_q^x g(x)),$$ \hspace{1cm} (4.7)

$$D_q^x(f(x)g(x)) = (D_q^x g(x))f(q^{-1}x) + g(qx)(D_q^x f(x)).$$ \hspace{1cm} (4.8)

In addition one can show the property

$$D_q^x f(qx) = qD_q^x f(x)|_{x=qx},$$ \hspace{1cm} (4.9)

and the chain rules

$$D_q^{ax} f(x) = \frac{1}{a}D_q^x f(x),$$ \hspace{1cm} (4.10)

$$D_q^x f(x^n) = [n]x^{n-1}D_q^n f(x^n),$$ \hspace{1cm} (4.11)
where $a$ is a constant. Another useful result is

$$D^q_x f(x) = \frac{1}{[n]} \sum_{k=0}^{n-1} D^q_x f(q^{2k-(n-1)} x).$$  \hspace{1cm} (4.12)

5. $q$-integration

The $q$-integration (see [2] for concise expositions) in the interval $[0, a]$ is defined by

$$\int_0^a f(x) d_q x = a(q^{-1} - q) \sum_{n=0}^\infty q^{2n+1} f(q^{2n+1} a),$$  \hspace{1cm} (5.1)

while for the interval $[0, \infty)$ one has

$$\int_0^\infty f(x) d_q x = (q^{-1} - q) \sum_{n=-\infty}^\infty q^{2n+1} f(q^{2n+1}).$$  \hspace{1cm} (5.2)

The indefinite $q$-integral is defined as

$$\int f(x) d_q x = (q^{-1} - q) \sum_{n=0}^\infty q^{2n+1} x f(q^{2n+1} x) + \text{constant},$$  \hspace{1cm} (5.3)

where $0 < q < 1$. For entire functions $f(x)$ one can easily see that this $q$-integral approaches the Riemann integral as $q \to 1$, and also that the operators of $q$-differentiation and $q$-integration are inverse to each other

$$D^q_x \int f(x) d_q x = f(x) = \int D^q_x f(x) d_q x.$$  \hspace{1cm} (5.4)

One can also easily see that

$$\int a x^{n-1} d_q x = \frac{1}{[n]} a x^n + \text{constant},$$  \hspace{1cm} (5.5)

$$\int e_q(ax) d_q x = \frac{1}{a} e_q(ax) + \text{constant}.$$  \hspace{1cm} (5.6)

From (4.6) one can also prove the following formulae of integration by parts

$$\int_0^a f(qx)(D^q_x g(x)) d_q x = f(x)g(x)|_{x=a}^{x=0} - \int_0^a (D^q_x f(x)) g(q^{-1} x) d_q x,$$  \hspace{1cm} (5.7)

$$\int_0^a f(q^{-1} x)(D^q_x g(x)) d_q x = f(x)g(x)|_{x=0}^{x=a} - \int_0^a (D^q_x f(x)) g(qx) d_q x.$$  \hspace{1cm} (5.8)

The following formulae can also be proved

$$\int f(x) d_{aq} x = a \int f(x) d_q x.$$  \hspace{1cm} (5.9)
\[
\int f(x^n) d_q x^n = [n] \int x^{n-1} f(x^n) d_q x, \tag{5.10}
\]

\[
\int f(x) d_q x = \frac{1}{[n]} \sum_{k=0}^{n-1} q^{2k-(n-1)} \int f(q^{2k-(n-1)} x) d_q x. \tag{5.11}
\]

The \(q\)-analogue for Euler's formula for the function \(\Gamma(x)\) is

\[
\int_0^x \kappa x^n d_q x = [n][n-1][n-2] \ldots [1] = [n]!. \tag{5.12}
\]

A proof of this formula can be found in [10].

6. \(Q\)-numbers

The definition of \(q\)-numbers given in sec. 2 is not the only possible one. Much literature exists using the definition of \(Q\)-numbers

\[
[x]_Q = \frac{Q^x - 1}{Q - 1}, \tag{6.1}
\]

where \(x\) can be a number or an operator and \(Q\) is a deformation parameter. \(Q\) is a real number \((Q \neq 0, 1)\). The notation \(Q = e^T\), where \(T\) a real number, will be often used. The subscript \(Q\) will be used in this review in order to distinguish deformed numbers defined as in eq. (6.1) from these defined by eq. (2.1). It is clear that in the limit \(Q \to 1\) (or, equivalently, \(T \to 0\)) \(Q\)-numbers become ordinary numbers, i.e. \([x]_Q \to x\).

A few examples of \(Q\)-numbers are given here:

\[
[0]_Q = 0, \quad [1]_Q = 1, \quad [2]_Q = Q + 1, \quad [3]_Q = Q^2 + Q + 1. \tag{6.2}
\]

\(Q\)-numbers clearly do not remain invariant under the substitution \(Q \to Q^{-1}\). One can easily prove that

\[
[x]_Q = Q^{x-1} [x]_{1/Q}. \tag{6.3}
\]

\(Q\)-numbers are connected to \(q\)-numbers through the relation

\[
[x] = q^{1-x} [x]_Q, \quad \text{with} \quad Q = q^2. \tag{6.4}
\]

The definitions of \(Q\)-factorials and \(Q\)-binomial coefficients still look like the ones given in eqs. (2.7)–(2.8):

\[
[n]_Q! = [n]_Q [n-1]_Q \ldots [1]_Q, \tag{6.5}
\]

\[
\binom{m}{n}_Q = \frac{[m]_Q!}{[m-n]_Q! [n]_Q!}. \tag{6.6}
\]
As it can be easily seen from eq. (6.1) under the substitution $Q \to Q^{-1}$ one obtains

$$[n]_Q! = Q^{n(n-1)/2} [n]_{1/Q},$$  

(6.7)

and

$$\left[ \begin{array}{c} n \\ k \end{array} \right]_Q = Q^{k(n-k)} \left[ \begin{array}{c} n \\ k \end{array} \right]_{1/Q}. \quad (6.8)$$

$Q$-factorials are connected to $q$-factorials by

$$[n]! = q^{-n(n-1)/2}[n]_Q!, \quad \text{with} \quad Q = q^2. \quad (6.9)$$

7. $Q$-deformed elementary functions

The definitions of $Q$-deformed elementary functions look similar to those given in sec. 3. The $Q$-deformed exponential function is defined as

$$e_Q(ax) = \sum_{n=0}^{\infty} \frac{a^n}{[n]_Q!} x^n, \quad (7.1)$$

and satisfies the property

$$e_Q(x)e_{1/Q}(-x) = 1. \quad (7.2)$$

(Notice that $e_Q(x)e_Q(-x) \neq 1$.)

The $Q$-deformed trigonometric functions are defined as

$$\sin_Q(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{[2n+1]_Q!}, \quad (7.3)$$

$$\cos_Q(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{[2n]_Q!}. \quad (7.4)$$

One can easily show that

$$\sin_Q(x) = \frac{1}{2i}(e_Q(ix) - e_Q(-ix)), \quad (7.5)$$

$$\cos_Q(x) = \frac{1}{2}(e_Q(ix) + e_Q(-ix)). \quad (7.6)$$

Instead of the familiar identity $\sin^2(x) + \cos^2(x) = 1$ one has

$$\sin_Q(x) \sin_{1/Q}(x) + \cos_Q(x) \cos_{1/Q}(x) = 1. \quad (7.7)$$

$Q$-deformed polynomials, such as $Q$-deformed Hermite polynomials and $Q$-deformed Laguerre polynomials also exist.
8. **Q-derivative**

Given the function $f(x)$ one defines its $Q$-derivative $D_x^Q$ by the expression

$$D_x^Q f(x) = \frac{f(Qx) - f(x)}{(Q - 1)x}.$$  

(8.1)

The similarity between this definition and the one of $Q$-numbers (eq. (6.1)) is clear.

One can easily prove that

$$D_x^Q x^n = \frac{Q^n x^n - x^n}{(Q - 1)x} = [n]_Q x^{n-1},$$  

(8.2)

which looks exactly like eq. (4.2). In addition one has

$$D_x^Q e_Q(ax) = ae_Q(ax),$$  

(8.3)

$$D_x^Q e_{1/Q}(ax) = ae_{1/Q}(aQx),$$  

(8.4)

$$D_x^Q \sin_Q(ax) = a \cos_Q(ax),$$  

(8.5)

$$D_x^Q \cos_Q(ax) = -a \sin_Q(ax).$$  

(8.6)

One can also easily see that $\sin_Q(ax)$ and $\cos_Q(ax)$ are the linearly independent solutions of the $Q$-differential equation

$$(D_x^Q)^2 u(x) + a^2 u(x) = 0,$$  

(8.7)

while the functions $\sin_{1/Q}(ax)$ and $\cos_{1/Q}(ax)$ satisfy the equation

$$(D_x^Q)^2 u(x) + a^2 u(Q^2 x) = 0.$$  

(8.8)

The following *Leibnitz rules* can also be shown:

$$D_x^Q (f_1(x) f_2(x)) = (D_x^Q f_1(x)) f_2(Qx) + f_1(x) \left(D_x^Q f_2(x)\right),$$  

(8.9)

$$D_x^Q (f_1(x) f_2(x)) = (D_x^Q f_1(x)) f_2(x) + f_1(Qx) \left(D_x^Q f_2(x)\right).$$  

(8.10)

One can further obtain

$$D_x^Q \frac{f_1(x)}{f_2(x)} = \frac{(D_x^Q f_1(x)) f_2(x) - f_1(x) \left(D_x^Q f_2(x)\right)}{f_2(Qx)f_2(x)}.$$  

(8.11)

For the second derivative of $f(x)$ one has

$$(D_x^Q)^2 f(x) = (Q - 1)^{-2}Q^{-1}x^{-2} \left\{ f(Q^2x) - (Q + 1)f(Qx) + Qf(x) \right\},$$  

(8.12)
and by mathematical induction we obtain the general formula

\[(D^Q)^n f(x) = (Q - 1)^{-n} Q^{-n(n-1)/2} x^{-n} \sum_{k=0}^{n} \binom{n}{k} (Q^{-1})^k Q^{k(n-1)/2} f(Q^{n-k}x). \quad (8.13)\]

9. **Q-integration**

In a way analogous to that of sec. 5 the definite Q-integral of the function \( f(x) \) in the interval \([0, 1]\) is defined as follows

\[\int_0^1 f(x) d_Qx = (1 - Q) \sum_{s=0}^{\infty} f(Q^s) Q^s, \quad (9.1)\]

assuming that \( Q \) is real and \(|Q| < 1\), while for the definite integral of \( f(x) \) in the interval \([0, \infty]\), we have

\[\int_0^\infty f(x) d_Qx = (1 - Q) \sum_{s=-\infty}^{\infty} Q^s f(Q^s). \quad (9.2)\]

For the indefinite Q-integral of \( f(x) \) one has

\[\int f(x) d_Qx = (1 - Q)x \sum_{s=-\infty}^{\infty} Q^s f(Q^s x) + \text{const.} \quad (9.3)\]

One can easily check that Q-differentiation and Q-integration are operations inverse to each other

\[D^Q \int f(x) d_Qx = f(x). \quad (9.4)\]

The formula for Q-integration by parts reads

\[\int (D^Q f_1(x)) f_2(x) d_Qx = f_1(x) f_2(x) - \int f_1(Qx) \left(D^Q f_2(x)\right) d_Qx. \quad (9.5)\]

10. **The q-deformed harmonic oscillator**

The interest for possible applications of quantum algebras in physics has been triggered in 1989 by the introduction of the q-deformed harmonic oscillator, of which earlier equivalent versions existed.

The q-deformed harmonic oscillator is defined in terms of the creation and annihilation operators \(a^\dagger\) and \(a\) and the number operator \(N\), which satisfy the commutation relations

\[[N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \quad (10.1)\]

\[aa^\dagger - q^\pm 1 a^\dagger a = q^\pm N. \quad (10.2)\]
In addition the following conditions of hermitian conjugation hold

\[(a^\dagger)^\dagger = a, \quad N^\dagger = N.\] (10.3)

Eq. (10.1) is the same as in ordinary quantum mechanics, while eq. (10.2) is modified by the presence of the deformation parameter $q$. For $q \to 1$ it is clear that eq. (10.2) goes to the usual boson commutation relation $[a, a^\dagger] = 1$. An immediate consequence of (10.2) is that

\[a^\dagger a = [N], \quad aa^\dagger = [N + 1].\] (10.4)

Thus the number operator $N$ is not equal to $a^\dagger a$, as in the ordinary case. The operators $a^\dagger$ and $a$ are referred to as $q$-deformed boson creation and annihilation operators respectively.

The basis of the Fock space is defined by repeated action of the creation operator $a^\dagger$ on the vacuum state, which is annihilated by $a$:

\[a|0\rangle = 0, \quad |n\rangle = \frac{(a^\dagger)^n|0\rangle}{\sqrt{n!}}.\] (10.5)

The action of the operators on the basis is given by

\[N|n\rangle = n|n\rangle,\] (10.6)

\[a^\dagger|n\rangle = \sqrt{|n + 1|}|n\rangle,\] (10.7)

\[a|n\rangle = \sqrt{|n|}|n - 1\rangle.\] (10.8)

We remark that these equations look very similar to the ones of the ordinary case, the only difference being that $q$-numbers appear under the square roots instead of usual numbers.

The Hamiltonian of the $q$-deformed harmonic oscillator is

\[H = \frac{\hbar \omega}{2}(aa^\dagger + a^\dagger a),\] (10.9)

and its eigenvalues in the basis given above are

\[E(n) = \frac{\hbar \omega}{2}([n] + [n + 1]).\] (10.10)

One can easily see that for $q$ real the energy eigenvalues increase more rapidly than the ordinary case, in which the spectrum is equidistant, i.e. the spectrum gets “expanded”. In contrast, for $q$ being a phase factor ($q = e^{i\tau}$ with $\tau$ real) the eigenvalues of the energy increase less rapidly than the ordinary (equidistant) case, i.e. the spectrum is “compressed”. In particular, for $q$ real ($q = e^{i\tau}$) the eigenvalues can be written as

\[E(n) = \frac{\hbar \omega}{2} \sinh \left( \tau \left( n + \frac{1}{2} \right) \right) \frac{\sinh \frac{\tau}{2}}{\sinh \frac{\tau}{2}},\] (10.11)
while for $q$ being a phase factor ($q = e^{i\tau}$) one has

$$E(n) = \frac{\hbar\omega}{2} \sin \left( \frac{\tau (n + \frac{1}{2})}{\sin \frac{\tau}{2}} \right). \quad (10.12)$$

In both cases in the limit $q \to 1$ ($\tau \to 0$) the ordinary expression

$$E(n) = \hbar\omega \left( n + \frac{1}{2} \right) \quad (10.13)$$

is recovered.

In addition, the following commutation relation holds

$$[a, a^\dagger] = [N + 1] - [N]. \quad (10.14)$$

For $q$ being a phase factor, this commutation relation takes the form

$$[a, a^\dagger] = \cos \left( \frac{(2N+1)\tau}{2} \right). \cos \frac{\tau}{2}. \quad (10.15)$$

It is useful to notice that the $q$-deformed boson operators $a^\dagger$ and $a$ can be expressed in terms of usual boson operators $\alpha^\dagger$ and $\alpha$ (satisfying $[\alpha, \alpha^\dagger] = 1$ and $N = \alpha^\dagger\alpha$) through the relations \cite{19, 22, 23}

$$a = \sqrt{\frac{N+1}{N+1}} \alpha = \alpha \sqrt{\frac{N}{N}}, \quad a^\dagger = \alpha^\dagger \sqrt{\frac{N+1}{N+1}} = \sqrt{\frac{N}{N}} \alpha^\dagger. \quad (10.16)$$

The square root factors in the last equation have been called $q$-deforming functionals.

For $q$ being a primitive root of unity, i.e. $q = e^{2\pi i/k}$ ($k = 2, 3, \ldots$), it is clear the the representation of eqs (10.5)–(10.8) becomes finite-dimensional and has dimension $k$, since only the vectors $|0>, |1>, \ldots, |k-1>$ can be present. This case has been related to the system of two anyons \cite{23}. In what follows we are going to assume that $q$ is not a primitive root of unity.

### 11. The $Q$-deformed harmonic oscillator

A different version of the deformed harmonic oscillator can be obtained by defining \cite{24, 25, 26} the operators $b$, $b^\dagger$ through the equations

$$a = q^{1/2}bq^{-N/2}, \quad a^\dagger = q^{1/2}q^{-N/2}b^\dagger. \quad (11.1)$$

Eqs. (10.1) and (10.2) then give

$$[N, b^\dagger] = b^\dagger, \quad [N, b] = -b, \quad (11.2)$$

$$bb^\dagger - q^2b^\dagger b = 1. \quad (11.3)$$
This oscillator has been first introduced by Arik and Coon \(^1\) and later considered also by Kuryshkin \(^2\). One then easily finds that

\[
\begin{align*}
  b^\dagger b &= [N]_Q, \\
  bb^\dagger &= [N+1]_Q,
\end{align*}
\]

(11.4)

where \(Q = q^2\) and \(Q\)-numbers are defined in (6.1). The basis is defined by

\[
  b|0\rangle = 0, \quad |n\rangle = \frac{(b^\dagger)^n}{\sqrt{|n|_Q!}}|0\rangle,
\]

(11.5)

while the action of the operators on the basis is given by

\[
  N|n\rangle = n|n\rangle, \quad b^\dagger |n\rangle = \sqrt{|n+1|_Q}|n+1\rangle, \quad b|n\rangle = \sqrt{|n|_Q}|n-1\rangle.
\]

(11.6, 11.7, 11.8)

The Hamiltonian of the corresponding deformed harmonic oscillator has the form

\[
  H = \frac{\hbar \omega}{2} (bb^\dagger + b^\dagger b),
\]

(11.9)

the eigenvalues of which are

\[
  E(n) = \frac{\hbar \omega}{2} ([n]_Q + [n+1]_Q).
\]

(11.10)

One can easily see that for \(Q = e^T\), where \(T > 0\) and real, the spectrum increases more rapidly than the ordinary (equidistant) spectrum, while for \(Q = e^T\), with \(T < 0\) and real, the spectrum is increasing less rapidly than the ordinary (equidistant) case.

From the above relations, it is clear that the following commutation relation holds

\[
  [b, b^\dagger] = Q^N.
\]

(11.11)

12. The generalized deformed oscillator

In addition to the oscillators described in the last two sections, many kinds of deformed oscillators have been introduced in the literature (see \(^2\) for a list). All of them can be accommodated within the common mathematical framework of the generalized deformed oscillator \(^2\), which is defined as the algebra generated by the operators \(\{1, a, a^\dagger, N\}\) and the structure function \(\Phi(x)\), satisfying the relations

\[
  [a, N] = a, \quad [a^\dagger, N] = -a^\dagger, \quad a^\dagger a = \Phi(N) = [N], \quad aa^\dagger = \Phi(N+1) = [N+1],
\]

(12.1, 12.2)
where $\Phi(x)$ is a positive analytic function with $\Phi(0) = 0$ and $N$ is the number operator. From eq. (12.2) we conclude that

$$N = \Phi^{-1}(a^\dagger a),$$  \hspace{1cm} (12.3)

and that the following commutation and anticommutation relations are obviously satisfied:

$$[a, a^\dagger] = [N + 1] - [N], \quad \{a, a^\dagger\} = [N + 1] + [N].$$  \hspace{1cm} (12.4)

The structure function $\Phi(x)$ is characteristic to the deformation scheme. In Table 1 the structure functions corresponding to different deformed oscillators are given. They will be further discussed at the end of this section.

It can be proved that the generalized deformed algebras possess a Fock space of eigenvectors $|0>, |1>, \ldots, |n>, \ldots$ of the number operator $N$

$$N|n> = n|n>, \quad <n|m> = \delta_{nm},$$  \hspace{1cm} (12.5)

if the vacuum state $|0>\,$ satisfies the following relation:

$$a|0> = 0.$$  \hspace{1cm} (12.6)

These eigenvectors are generated by the formula:

$$|n> = \frac{1}{\sqrt{[n]!}} (a^\dagger)^n |0>,$$  \hspace{1cm} (12.7)

where

$$[n]! = \prod_{k=1}^{n} [k] = \prod_{k=1}^{n} \Phi(k).$$  \hspace{1cm} (12.8)

The generators $a^\dagger$ and $a$ are the creation and annihilation operators of this deformed oscillator algebra:

$$a|n> = \sqrt{[n]} a|n - 1>, \quad a^\dagger|n> = \sqrt{[n + 1]} a|n + 1>.$$  \hspace{1cm} (12.9)

These eigenvectors are also eigenvectors of the energy operator

$$H = \frac{\hbar \omega}{2} (aa^\dagger + a^\dagger a),$$  \hspace{1cm} (12.10)

corresponding to the eigenvalues

$$E(n) = \frac{\hbar \omega}{2} (\Phi(n) + \Phi(n + 1)) = \frac{\hbar \omega}{2} ([n] + [n + 1]).$$  \hspace{1cm} (12.11)

For

$$\Phi(n) = n$$  \hspace{1cm} (12.12)
Table 1: Structure functions of special deformation schemes

| Structure function | Reference |
|--------------------|-----------|
| $\Phi(x)$ | harmonic oscillator, bosonic algebra |
| $x$ | $q$-deformed harmonic oscillator |
| $\frac{q^x - q^{-x}}{q - q^{-1}}$ | $q$-deformed harmonic oscillator |
| $\frac{q^x - 1}{q - 1}$ | Arik–Coon, Kuryshkin, or $Q$-deformed oscillator |
| $\frac{q^x - p^x}{q - p}$ | 2-parameter deformed oscillator |
| $x(p + 1 - x)$ | parafermionic oscillator |
| $\frac{\sinh(\tau x) \sinh(\tau(x+1-x))}{\sinh^2(\tau)}$ | $q$-deformed parafermionic oscillator |
| $x \cos^2(\pi x/2) + (x + p - 1) \sin^2(\pi x/2)$ | parabosonic oscillator |
| $\frac{\sinh(\tau x) \cosh(\tau(x+2N_0-1))}{\sinh(\tau(x+2N_0-1)) \cosh(\tau)} \frac{\cos^2(\pi x/2)}{\cos(\tau)} + \frac{\sin^2(\tau x)}{\sinh(\tau) \cosh(\tau)} \frac{\sin^2(\pi x/2)}{\cosh(\tau)}$ | $q$-deformed parabosonic oscillator |
| $\sin^2(\pi x/2)$ | fermionic algebra |
| $q^x - 1 \sin^2(\pi x/2)$ | $q$-deformed fermionic algebra |
| $\frac{1 - (-q)^x}{1 + q}$ | generalized $q$-deformed fermionic algebra |
| $x^n$ | |
| $\frac{\sin(\tau x)}{\sin(\tau)}$ | |

one obtains the results for the ordinary harmonic oscillator. For

$$\Phi(n) = \frac{q^n - q^{-n}}{q - q^{-1}} = [n]$$  \hspace{1cm} (12.13)

one has the results for the $q$-deformed harmonic oscillator, while the choice

$$\Phi(n) = \frac{Q^n - 1}{Q - 1} = [n]_Q$$  \hspace{1cm} (12.14)

leads to the results of the $Q$-deformed harmonic oscillator. Many more cases are shown in Table 1, on which the following comments apply:

i) Two-parameter deformed oscillators have been introduced, in analogy to the one-parameter deformed oscillators.
ii) Parafermionic oscillators of order $p$ represent particles of which the maximum number which can occupy the same state is $p$. Parabosonic oscillators can also be introduced.

iii) $q$-deformed versions of the parafermionic and parabosonic oscillators have also been introduced.

iv) $q$-deformed versions of the fermionic algebra have also been introduced, as well as $q$-deformed versions of generalized $q$-deformed fermionic algebras. It has been proved, however, that $q$-deformed fermions are fully equivalent to the ordinary fermions.

13. The physical content of deformed harmonic oscillators

In order to get a feeling about the physical content of the various deformed harmonic oscillators it is instructive to construct potentials giving spectra similar to these of the oscillators.

13.1. Classical potentials equivalent to the $q$-oscillator

Let us consider the $q$-deformed harmonic oscillator first. For small values of $\tau$ one can take Taylor expansions of the functions appearing there and thus find an expansion of the $q$-number $[n]$ of eq. (2.1) in terms of powers of $\tau^2$. The final result is

$$[n] = n \pm \frac{\tau^2}{6}(n - n^3) + \frac{\tau^4}{360}(7n - 10n^3 + 3n^5) \pm \frac{\tau^6}{15120}(31n - 49n^3 + 21n^5 - 3n^7) + \ldots,$$

(13.1)

where the upper (lower) sign corresponds to $q$ being a phase (real). Using this expansion the energy of the $q$-deformed harmonic oscillator of eq. (10.10) can be rewritten as

$$E(n)/\hbar = (n + \frac{1}{2})(1 \pm \frac{\tau^2}{24}) \mp \frac{\tau^2}{6}(n + \frac{1}{2})^3 + \ldots$$

(13.2)

On the other hand, one can consider the potential

$$V(x) = V_0 + kx^2 + \lambda x^4 + \mu x^6 + \xi x^8 + \ldots$$

(13.3)

If $\lambda, \mu, \xi$ are much smaller than $k$, one can consider this potential as a harmonic oscillator potential plus some perturbations and calculate the corresponding spectrum through the use of perturbation theory (see also subsec. 32.2). In order to keep the subsequent formulae simple, we measure $x$ in units of $(\hbar/(2m\omega))^{1/2}$. Using standard first order perturbation theory one finds that the corresponding spectrum up to the order considered is

$$E(n) = E_0 + (2\kappa + 25\mu)(n + \frac{1}{2}) + (6\lambda + 245\xi)(n + \frac{1}{2})^2 + 20\mu(n + \frac{1}{2})^3 + 70\xi(n + \frac{1}{2})^4.$$  (13.4)
By equating coefficients of the same powers of \((n + \frac{1}{2})\) in eqs. (13.2) and (13.4), we can determine the coefficients appearing in the expansion of the potential given in eq. (13.3). The final result for the potential, up to the order considered, is

\[
V(x) = \left(\frac{1}{2} \pm \frac{\tau^2}{8}\right)x^2 \mp \frac{\tau^2}{120}x^6. \tag{13.5}
\]

We see therefore that to lowest order one can think of the \(q\)-oscillator with a small value of the parameter \(\tau\) as a harmonic oscillator perturbed by a \(x^6\) term. It is clear that if we go to higher order, the next term to appear will be proportional to \(\tau^4 x^{10}\).

The results of this subsection are corroborated by an independent study of the relation between the \(q\)-deformed harmonic oscillator and the ordinary anharmonic oscillator with \(x^6\) anharmonicities.

13.2. Classical potentials equivalent to the \(Q\)-deformed oscillator

Similar considerations can be made for the \(Q\)-oscillator. Defining \(Q = e^T\) it is instructive to construct the expansion of the \(Q\)-number of eq. (6.1) in powers of \(T\). Assuming that \(T\) is small and taking Taylor expansions in eq. (6.1) one finally has

\[
[n]_Q = n + \frac{T}{2}(n^2 - n) + \frac{T^2}{12}(2n^3 - 3n^2 + 1) + \frac{T^3}{24}(n^4 - 2n^3 + n^2) + \ldots \tag{13.6}
\]

Then the corresponding expansion of the energy levels of the oscillator of eq. (11.10) is

\[
E(n)/\hbar \omega = E'_0 + (1 - \frac{T}{2} + \frac{T^2}{8} - \frac{T^3}{16} + \ldots)(n + \frac{1}{2})^2 + (\frac{T}{2} - \frac{T^2}{4} + \frac{5T^3}{48} - \ldots)(n + \frac{1}{2})^3 + (\frac{T^2}{6} - \frac{T^3}{12} + \ldots)(n + \frac{1}{2})^4. \tag{13.7}
\]

Comparing this expansion to eq. (13.4) and equating equal powers of \((n + \frac{1}{2})\) we arrive at the following expression for the potential

\[
V(x) = \frac{T^2}{12} + \left(\frac{1}{2} - \frac{T}{4} - \frac{T^2}{24} + \frac{T^3}{48}\right)x^2 + \left(\frac{T}{12} - \frac{T^2}{24} - \frac{T^3}{144}\right)x^4 + \left(\frac{T^2}{120} - \frac{T^3}{240}\right)x^6 + \frac{T^3}{1680}x^8. \tag{13.8}
\]

Thus to lowest order one can think of the \(Q\)-oscillator with small values of the parameter \(T\) as a harmonic oscillator perturbed by a \(x^4\) term. A similar expression is found, for example, by Taylor expanding the modified Pöschl–Teller potential \([13]\), which, among other applications, has been recently used in the description of hypernuclei \([13][14][15]\). The modified Pöschl–Teller potential (see also subsec. 32.2) has the form

\[
V(x)_{\text{PT}} = -\frac{A}{\cosh(ax)^2}. \tag{13.9}
\]
Its Taylor expansion is

$$V(x)_{PT} = A(-1 + a^2 x^2 - \frac{2}{3} a^4 x^4 + \frac{17}{45} a^6 x^6 - \ldots).$$  \hspace{1cm} (13.10)$$

We remark that this expansion contains the same powers of $x$ as the expansion (13.8). Furthermore, the signs of the coefficients of the same powers of $x$ in the two expansions are the same for $T < 0$.

### 13.3. WKB-EPs for the $q$-deformed oscillator

The potentials obtained above are only rough lowest order estimates. More accurate methods exist for constructing WKB equivalent potentials (WKB-EPs) giving (within the limits of WKB approximation) the same spectrum as the above mentioned oscillators. A method by which this can be achieved has been given by Wheeler and is described by Chadan and Chabatier. Applying this method to the $q$-deformed oscillator (sec. 10) with $q$ being a phase factor one finds the potential

$$V(x) = \left(\frac{\tau}{2 \sin(\tau/2)}\right)^2 m\omega^2 x^2 \left[1 - \frac{8}{15} \left(\frac{x}{2R_e}\right)^4 + \frac{4448}{1575} \left(\frac{x}{2R_e}\right)^8 - \frac{345344}{675675} \left(\frac{x}{2R_e}\right)^{12} + \ldots\right],$$ \hspace{1cm} (13.11)

where

$$R_e = \frac{1}{\tau} \left(\frac{\hbar^2}{2m}\right)^{1/2} \left(\frac{2 \sin(\tau/2)}{\hbar \omega}\right)^{1/2},$$ \hspace{1cm} (13.12)

while for the $q$-deformed oscillator with $q$ real one has

$$V(x) = \left(\frac{\tau}{2 \sinh(\tau/2)}\right)^2 m\omega^2 x^2 \left[1 + \frac{8}{15} \left(\frac{x}{2R_h}\right)^4 + \frac{4448}{1575} \left(\frac{x}{2R_h}\right)^8 + \frac{345344}{675675} \left(\frac{x}{2R_h}\right)^{12} + \ldots\right],$$ \hspace{1cm} (13.13)

where

$$R_h = \frac{1}{\tau} \left(\frac{\hbar^2}{2m}\right)^{1/2} \left(\frac{2 \sinh(\tau/2)}{\hbar \omega}\right)^{1/2}.$$ \hspace{1cm} (13.14)

The results of this subsection are corroborated by an independent study of the relation between the $q$-deformed harmonic oscillator and the ordinary anharmonic oscillator with $x^6$ anharmonicities.

### 13.4. WKB-EPs for the $Q$-deformed oscillator
Using the same technique one finds that the WKB equivalent potential for the $Q$-deformed harmonic oscillator (sec. 11) takes the form

$$V(x) = V_{\text{min}} + \frac{(\ln Q)^2}{Q} \left( \frac{Q+1}{Q-1} \right)^2 \frac{1}{2} m\omega^2 x^2$$

$$\left[ 1 - \frac{2}{3} \left( \frac{x}{R'} \right)^2 + \frac{23}{45} \left( \frac{x}{R'} \right)^4 - \frac{134}{315} \left( \frac{x}{R'} \right)^6 + \frac{5297}{14172} \left( \frac{x}{R'} \right)^8 - \ldots \right],$$

(13.15)

where

$$V_{\text{min}} = \frac{\hbar \omega (\sqrt{Q} - 1)}{2\sqrt{Q}(\sqrt{Q} + 1)},$$

(13.16)

and

$$R' = \left( \frac{\hbar \sqrt{Q}(Q-1)}{\omega m(Q+1)} \right)^{1/2} \frac{\ln Q}{\sqrt{2}}.$$

(13.17)

We remark that this WKB-EP contains all even powers of $x$, in contrast to the WKB-EPs for the $q$-oscillator (eqs (13.11), (13.13)), which contains only the powers $x^2, x^6, x^{10}, \ldots$. This is in agreement with the lowest order results obtained in subsecs 13.1 and 13.2.

14. The quantum algebra $su_q(2)$

Quantum algebras are generalized versions of the usual Lie algebras, to which they reduce when the deformation parameter $q$ is set equal to unity. A simple example of a quantum algebra is provided by $su_q(2)$, which is generated by the operators $J_+, J_0, J_-$, satisfying the commutation relations

$$[J_0, J_\pm] = \pm J_\pm,$$

(14.1)

$$[J_+, J_-] = [2J_0],$$

(14.2)

with $J_0^\dagger = J_0, (J_+)^\dagger = J_-$. We remark that eq. (14.1) is the same as in the case of the ordinary $su(2)$ algebra, while eq. (14.2) is different, since in the usual $su(2)$ case it reads

$$[J_+, J_-] = 2J_0.$$ 

(14.3)

In the rhs of eq. (14.3) one has the first power of the $J_0$ operator, while in the rhs of eq. (14.2) one has the $q$-operator $[2J_0]$, defined in sec. 2. Because of eqs. (2.2) and (2.3) it is clear that if one writes the rhs of eq. (14.2) in expanded form all odd powers of $J_0$ will appear:

$$[J_+, J_-] = \frac{1}{\sinh(\tau)} \left( \frac{2\tau J_0}{1!} + \frac{(2\tau J_0)^3}{3!} + \frac{(2\tau J_0)^5}{5!} + \ldots \right)$$

for $q = e^\tau$, (14.4)

$$[J_+, J_-] = \frac{1}{\sin(\tau)} \left( \frac{2\tau J_0}{1!} - \frac{(2\tau J_0)^3}{3!} + \frac{(2\tau J_0)^5}{5!} - \ldots \right)$$

for $q = e^{i\tau}$. (14.5)
Thus $\text{su}_q(2)$ can be loosely described as a nonlinear generalization of the usual $\text{su}(2)$: While in usual Lie algebras the commutator of two generators is always producing a linear combination of generators, in the case of quantum algebras the commutator of two generators can contain higher powers of the generators as well.

The irreducible representations (irreps) $D^J$ of $\text{su}_q(2)$ (which have dimensionality $2J + 1$) are determined by highest weight states with $J = 0, \frac{1}{2}, 1, \ldots$. The basic states $|J, M >$ (with $-J \leq M \leq J$) are connected with highest weight states $|J, J >$ as follows

$$|J, M > = \sqrt{\frac{(J + M)!}{(2J)![(J - M)!]} (J_-)^{J-M} |J, J >}, \quad (14.6)$$

with $J_- |J, J > = 0$ and $< J, J |J, J > = 1$. The action of the generators of the algebra on these basic vectors is given by

$$J_0 |J, M > = M |J, M >, \quad (14.7)$$

$$J_\pm |J, M > = \sqrt{[J \mp M][(J \mp M) + 1]} |J, M \pm 1 >. \quad (14.8)$$

These expressions look similar to the ones of the usual $\text{su}(2)$ algebra, the only difference being that $q$-numbers appear under the square root instead of ordinary numbers.

The second order Casimir operator of $\text{su}_q(2)$ is determined from the condition that it should commute with all of the generators of the algebra. The resulting operator is

$$C^q_2 = J_- J_+ + [J_0][J_0 + 1] = J_+ J_- + [J_0][J_0 - 1]. \quad (14.9)$$

Its eigenvalues in the above mentioned basis are given by

$$C^q_2 |J, M > = [J][J + 1] |J, M >, \quad (14.10)$$

while for the usual $\text{su}(2)$ the eigenvalues of the Casimir operator are $J(J + 1)$. One can easily check that for real $q$ ($q = e^\tau$ with $\tau$ real) the eigenvalues $[J][J + 1]$ produce a spectrum increasing more rapidly than $J(J + 1)$ (an expanded spectrum), while for $q$ being a phase factor ($q = e^{i\tau}$ with $\tau$ real) the eigenvalues $[J][J + 1]$ correspond to a spectrum increasing less rapidly than $J(J + 1)$ (a compressed spectrum).

It should be noticed that the generators $J_+, J_0, J_-$ of $\text{su}_q(2)$ are connected to the generators $j_+, j_0, j_-$ of the usual $\text{su}(2)$, which satisfy the commutation relations

$$[j_0, j_\pm] = \pm j_\pm, \quad [j_+, j_-] = 2j_0, \quad (14.11)$$

through the relations

$$J_0 = j_0, \quad J_+ = \sqrt{\frac{[j_0 + j][j_0 + 1 - j]}{(j_0 + j)(j_0 - 1 - j)}} j_+, \quad J_- = j_- \sqrt{\frac{[j_0 + j][j_0 - 1 - j]}{[j_0 + j][j_0 - 1 - j]}}, \quad (14.12)$$

where $j$ is determined by the relation for the second order Casimir operator of $\text{su}(2)$

$$C = j_- j_+ + j_0(j_0 + 1) = j_+ j_- + j_0(j_0 - 1) = j(j + 1). \quad (14.13)$$
15. Realization of su_q(2) in terms of q-deformed bosons

Realizations of Lie algebras in terms of (ordinary) bosons are useful not only as a convenient mathematical tool, but also because of their applications in physics. In the case of quantum algebras it turns out that boson realizations are possible in terms of the q-deformed boson operators already introduced in sec. 10.

In the case of su_q(2) the generators can be mapped onto q-deformed bosons as follows:

\[ J_+ = a_1^\dagger a_2, \quad J_- = a_2^\dagger a_1, \quad J_0 = \frac{1}{2}(N_1 - N_2), \]

where \(a_1^\dagger\), \(a_i\) and \(N_i\) are q-deformed boson creation, annihilation and number operators as these introduced in sec. 10. One can easily prove that the boson images satisfy the commutation relations (14.1) and (14.2). For example, one has

\[ [J_+, J_-] = [a_1^\dagger a_2 a_2^\dagger a_1 - a_2^\dagger a_1 a_1^\dagger a_2] = [N_1][N_2 + 1] - [N_1 + 1][N_2] = [N_1 - N_2] = [2J_0], \]

where use of the identity (2.6) has been made.

In the q-boson picture the normalized highest weight vector is

\[ |JJ> = \frac{(a_1^\dagger)^{2J}}{\sqrt{[2J]!}}|0>, \]

while the general vector \(|JM>\) is given by

\[ |JM> = \frac{(a_1^\dagger)^{J+M}(a_2^\dagger)^{J-M}}{\sqrt{[J+M]!}\sqrt{[J-M]!}}|0>. \]

It should be noticed that it was the search for a boson realization of the su_q(2) algebra that led to the introduction of the q-deformed harmonic oscillator in 1989.

Starting from su_q(2) one can formulate a q-deformed version of angular momentum theory. Some references are listed here:
- i) Clebsch-Gordan coefficients for su_q(2) can be found in [61,62,63,64,65,66,67,68].
- ii) 6-j symbols for su_q(2) can be found in [69,70,71,72].
- iii) 9-j symbols for su_q(2) can be found in [73].
- iv) The q-deformed version of the Wigner–Eckart theorem can be found in [74,75].

In addition, it should be noticed that a two-parameter deformation of su(2), labelled as su_{p,q}(2) has been introduced [76,77,78,79,80]. Clebsch-Gordan coefficients for su_{p,q}(2) have been discussed in [81,82,83,84].

The way in which the algebra su_q(2) can be realized in terms of the Q-deformed bosons of sec. 11 is given in subsec. 33.1 (see eqs (33.13)–(33.16)).

16. The quantum algebra su_q(1,1)
In this section we shall give a brief account of the algebra $\mathfrak{su}_q(1,1)$. In the classical case the $\mathfrak{so}(2,1)$ generators satisfy the commutation relations

$$[K_1, K_2] = -iK_3, \quad [K_2, K_3] = iK_1, \quad [K_3, K_1] = iK_2,$$

which differ from the classical $\mathfrak{so}(3)$ commutation relations in the sign of the r.h.s. of the first commutator. Defining

$$K_+ = K_1 + iK_2, \quad K_- = K_1 - iK_2, \quad K_3 = K_z,$$

one obtains the $\mathfrak{su}(1,1)$ commutation relations

$$[K_z, K_\pm] = \pm K_\mp, \quad [K_+, K_-] = -2K_z,$$

which differ from the familiar $\mathfrak{su}(2)$ commutation relations in the sign of the r.h.s. of the last commutator. The generators of $\mathfrak{su}(1,1)$ accept the following boson representation

$$K_+ = a_1^\dagger a_2^\dagger, \quad K_- = a_1a_2, \quad K_z = \frac{1}{2}(a_1^\dagger a_1 + a_2^\dagger a_2 + 1),$$

where $a_1^\dagger, a_1, a_2^\dagger, a_2$ satisfy usual boson commutation relations.

The second order Casimir operator of $\mathfrak{so}(2,1)$ is

$$C_2[\mathfrak{so}(2,1)] = -(K_1^2 + K_2^2 - K_3^2),$$

while for $\mathfrak{su}(1,1)$ one has

$$C_2[\mathfrak{su}(1,1)] = [K_0][K_0 - 1] - K_+K_- = [K_0][K_0 + 1] - K_-K_+.$$
since the basis has the form $|\kappa \mu \rangle = |n_1 \rangle > |n_2 \rangle$, with

$$|n_i \rangle = \frac{1}{\sqrt{|n_i|!}} (a_i^\dagger)^{|n_i|} |0 \rangle.$$  \hfill (16.12)

In this basis the possible values of $\mu$ are given by

$$\mu = \kappa, \kappa + 1, \kappa + 2, \ldots,$$  \hfill (16.13)

up to infinity, while $\kappa$ may be any positive real number. The action of the generators on this basis is given by

$$K_0 |\kappa \mu \rangle = \mu |\kappa \mu \rangle,$$  \hfill (16.14)

$$K_\pm |\kappa \mu \rangle = \sqrt{\mu \pm \kappa} |\mu \mp \kappa \pm 1 \rangle |\kappa \mu \pm 1 \rangle.$$  \hfill (16.15)

Clebsch-Gordan and Racah coefficients for $su_q(1,1)$ can be found in [1, 2, 3, 4]. Furthermore, a two-parameter deformed version of $su_q(1,1)$, labelled as $su_{p,q}(1,1)$, has been introduced [5, 6, 7].

\section*{17. Generalized deformed $su(2)$ algebras}

In the same way that in addition to the $q$-deformed oscillators one can have generalized deformed oscillators, it turns out that generalized deformed $su(2)$ algebras, containing $su_q(2)$ as a special case and having representation theory similar to that of the usual $su(2)$, can be constructed [8, 9]. It has been proved that it is possible to construct an algebra

$$[J_0, J_\pm] = \pm J_\pm,$$  \hfill (17.1)

$$[J_+, J_-] = \Phi(J_0(J_0) + 1)) - \Phi(J_0(J_0) - 1)),$$

where $J_0, J_+, J_-$ are the generators of the algebra and $\Phi(x)$ is any increasing entire function defined for $x \geq -1/4$. Since this algebra is characterized by the function $\Phi$, we use for it the symbol $su_\Phi(2)$. The appropriate basis $|l, m \rangle$ has the properties

$$J_0 |L, M \rangle = M |L, M \rangle,$$  \hfill (17.2)

$$J_+ |L, M \rangle = \sqrt{\Phi(L(L + 1)) - \Phi(M(M + 1))} |L, M + 1 \rangle,$$  \hfill (17.3)

$$J_- |L, M \rangle = \sqrt{\Phi(L(L + 1)) - \Phi(M(M - 1))} |L, M - 1 \rangle,$$  \hfill (17.4)

where

$$L = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \ldots,$$  \hfill (17.5)

and

$$M = -L, -L + 1, -L + 2, \ldots, L - 2, L - 1, L.$$  \hfill (17.6)

The Casimir operator is

$$C = J_- J_+ + \Phi(J_0(J_0 + 1)) = J_+ J_- + \Phi(J_0(J_0 - 1)),$$  \hfill (17.7)
its eigenvalues indicated by
\[ C|L, M >= \Phi(L(L + 1))|L, M >. \] (17.8)
The usual su(2) algebra is recovered for
\[ \Phi(x(x + 1)) = x(x + 1), \] (17.9)
while the quantum algebra su_q(2)
\[ [J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = [2J_0]_q, \] (17.10)
occurs for
\[ \Phi(x(x + 1)) = [x][x + 1]. \] (17.11)

The su_\Phi(2) algebra occurs in several cases, in which the rhs of the last equation in (17.1) is an odd function of \( J_0 \). It can be seen that other algebraic structures, like the quadratic Hahn algebra QH(3) and the finite W algebra W_0 can be brought into the su_\Phi(2) form, the advantage being that the representation theory of su_\Phi(2) is already known. It can also be proved that several physical systems, like the isotropic oscillator in a 2-dim curved space with constant curvature, the Kepler system in a 2-dim curved space with constant curvature, and the system of two identical particles in two dimensions can also be put into an su_\Phi(2) form. More details can be found in [18].

18. Generalized deformed parafermionic oscillators

It turns out that the generalized deformed su_\Phi(2) algebras mentioned in the last section are related to generalized deformed parafermionic oscillators, which we will therefore describe here.

It has been proved that any generalized deformed parafermionic algebra of order \( p \) can be written as a generalized oscillator (sec. 12) with structure function
\[ F(x) = x(p + 1 - x)(\lambda + \mu x + \nu x^2 + \rho x^3 + \sigma x^4 + \ldots), \] (18.1)
where \( \lambda, \mu, \nu, \rho, \sigma, \ldots \) are real constants satisfying the conditions
\[ \lambda + \mu x + \nu x^2 + \rho x^3 + \sigma x^4 + \ldots > 0, \quad x \in \{1, 2, \ldots, p\}. \] (18.2)

Considering an su_\Phi(2) algebra with structure function
\[ \Phi(J_0(J_0 + 1)) = AJ_0(J_0 + 1) + BJ_0(J_0 + 1))^2 + CJ_0(J_0 + 1)^3, \] (18.3)
and making the correspondence
\[ J_+ \rightarrow A^\dagger, \quad J_- \rightarrow A, \quad J_0 \rightarrow N, \] (18.4)
one finds that the $\text{su}_q(2)$ algebra is equivalent to a generalized deformed parafermionic oscillator of the form

$$F(N) = N(p + 1 - N)$$

$$[-(p^2(p+1)C + pB) + (p^3 + (p-1)B)N + ((p^2 - p+1)C + B)N^2 + (p-2)CN^2 + CN^4],$$

if the condition

$$A + p(p+1)B + p^2(p+1)^2C = 0$$

holds. The condition of eq. (18.2) is always satisfied for $B > 0$ and $C > 0$.

In the special case of $C = 0$ one finds that the $\text{su}_q(2)$ algebra with structure function

$$\Phi(J_0(J_0 + 1)) = AJ_0(J_0 + 1) + B(J_0(J_0 + 1))^2$$

is equivalent to a generalized deformed parafermionic oscillator characterized by

$$F(N) = BN(p + 1 - N)(-p + (p-1)N + N^2),$$

if the condition

$$A + p(p+1)B = 0$$

is satisfied. The condition of eq. (18.2) is satisfied for $B > 0$.

Including higher powers of $J_0(J_0 + 1)$ in eq. (18.3) results in higher powers of $N$ in eq. (18.5) and higher powers of $p(p+1)$ in eq. (18.6). If, however, one sets $B = 0$ in eq. (18.7), then eq. (18.8) vanishes, indicating that no parafermionic oscillator equivalent to the usual $\text{su}(2)$ rotator can be constructed.

It turns out that several other mathematical structures, like the finite W algebras $\overline{W}_0$ and $W^{(2)}_3$ (see subsec. 34.5) can be put into the generalized deformed parafermionic oscillator form. The same is true for several physical systems, such as the isotropic oscillator and the Kepler problem in a 2-dim curved space with constant curvature and the Fokas–Lagerstrom, Smorodinsky–Winternitz, and Holt potentials. Further details can be found in [97].

19. The $\text{su}_q(2)$ rotator model

It has been suggested by Raychev, Roussev and Smirnov, and independently by Iwao and others, that rotational spectra of deformed nuclei can be described by the $q$-deformed rotator, which corresponds to the 2nd order Casimir operator of the quantum algebra $\text{su}_q(2)$, already studied in sec. 14. We shall show here that this assumption works and discuss the reasons behind this success, as well as the relation between the $\text{su}_q(2)$ model and the Variable Moment of Inertia (VMI) model (see sec. 20).

The $q$-deformed rotator corresponds to the Hamiltonian

$$H = \frac{1}{2I} C_2(\text{su}_q(2)) + E_0,$$
where $I$ is the moment of inertia and $E_0$ is the bandhead energy (for ground state bands $E_0 = 0$). For $q$ real, i.e. with $q = e^\tau$ with $\tau$ real, the energy levels of the $q$-rotator are

$$E(J) = \frac{1}{2I} [J][J + 1] + E_0 = \frac{1}{2I} \sinh(\tau J) \sinh(\tau (J + 1)) \sinh(\tau) + E_0. \quad (19.2)$$

For $q$ being a phase, i.e. $q = e^{i\tau}$ with $\tau$ real, one obtains

$$E(J) = \frac{1}{2I} [J][J + 1] + E_0 = \frac{1}{2I} \sin(\tau J) \sin(\tau (J + 1)) \sin(\tau) + E_0. \quad (19.3)$$

Raychev et al. have found that good fits of rotational spectra of even–even rare earths and actinides are obtained with eq. (19.3). It is easy to check that eq. (19.2) fails in describing such spectra. In order to understand this difference, it is useful to make Taylor expansions of the quantities in the numerator of eq. (19.2) (eq. (19.3)) and collect together the terms containing the same powers of $J(J + 1)$ (all other terms cancel out), finally summing up the coefficients of each power. In the first case the final result is

$$E(J) = E_0 + \frac{1}{2I} \left(\frac{1}{\sqrt{\pi} I_{1/2}(\tau)}\right)^2 (\sqrt{\frac{\pi}{2\tau}} I_{1/2}(\tau) J(J + 1) + \tau \sqrt{\frac{\pi}{2\tau}} I_{3/2}(\tau) (J(J + 1))^2$$

$$+ \frac{2\tau^2}{3} \sqrt{\frac{\pi}{2\tau}} I_{5/2}(\tau) (J(J + 1))^3 + \frac{\tau^3}{3} \sqrt{\frac{\pi}{2\tau}} I_{7/2}(\tau) (J(J + 1))^4 + \ldots) \quad (19.4)$$

where $\sqrt{\frac{\pi}{2\tau}} I_{n+\frac{1}{2}}(\tau)$ are the modified spherical Bessel functions of the first kind.

In the second case (eq. (19.3)) following the same procedure one obtains

$$E(J) = E_0 + \frac{1}{2I} \left(\frac{1}{j_0(\tau)}\right)^2 (j_0(\tau) J(J + 1) - \tau j_1(\tau) (J(J + 1))^2$$

$$+ \frac{2}{3} \tau^2 j_2(\tau) (J(J + 1))^3 - \frac{1}{3} \tau^3 j_3(\tau) (J(J + 1))^4 + \frac{2}{15} \tau^4 j_4(\tau) (j(j + 1))^5 - \ldots), \quad (19.5)$$

where $j_n(\tau)$ are the spherical Bessel functions of the first kind.

Both results are of the form

$$E(J) = A J(J + 1) + B (J(J + 1))^2 + C (J(J + 1))^3 + D (J(J + 1))^4 + \ldots. \quad (19.6)$$

Empirically it is known that nuclear rotational spectra do show such a behaviour, the coefficients $A$, $B$, $C$, $D$, ... having alternating signs (starting with $A$ positive) and magnitudes dropping by about 3 orders of magnitude each time one moves to the next higher power of $J(J + 1)$.

It is interesting to check if the empirical characteristics of the coefficients $A$, $B$, $C$, $D$ are present in the case of the expansions of eqs. (19.2), (19.3), especially for small values of $\tau$. (Since we deal with rotational spectra, which are in first order...
approximation described by the usual algebra $su(2)$, we expect $\tau$ to be relatively small, i.e. the deviation of $su_q(2)$ from $su(2)$ to be small. This is in agreement to the findings of [107], where $\tau$ is found to be around 0.03.)

One can easily see that in eq. (19.2) it is impossible to get alternating signs, while in eq. (19.3) the condition of alternating signs is readily fulfilled. This fact has as a result that the energy levels given by eq. (19.2) increase more rapidly than the levels given by the $J(J+1)$ rule, while the levels given by eq. (19.3) increase less rapidly than $J(J+1)$. In order to check the order of magnitude of the coefficients for small values of $\tau$, it is useful to expand the spherical Bessel functions appearing in eq. (19.3) and keep only the lowest order term in each expansion. The result is

$$E(J) = E_0 + \frac{1}{2I} (J(J+1) - \frac{\tau^2}{3} (J(J+1))^2 + \frac{2\tau^4}{45} (J(J+1))^3$$

$$- \frac{\tau^6}{315} (J(J+1))^4 + \frac{2\tau^8}{14175} (J(J+1))^5 - \ldots).$$

(19.7)

We remark that each term contains a factor $\tau^2$ more than the previous one. For $\tau$ in the area of 0.03, $\tau^2$ is of the order of $10^{-3}$, as it should. We conclude therefore that eq. (19.6) is suitable for fitting rotational spectra, since its coefficients have the same characteristics as the empirical coefficients of eq. (19.6).

Extended comparisons of the $su_q(2)$ predictions to experimental data for ground state bands of rare earth and actinide nuclei can be found in [107,110,114,115]. More recently, the $su_q(2)$ formalism has been used for the description of $\beta$- and $\gamma$-bands of deformed rare earths and actinides, with satisfactory results [116].

It is necessary for $E(J)$ to be an increasing function of $J$. In order to guarantee this in eq. (19.3) one must have

$$\tau(J + 1) \leq \frac{\pi}{2}. \quad (19.8)$$

In the case of $\tau = 0.036$ (as in $^{232}$U in [110]), one finds $J \leq 42$, this limiting value being larger than the highest observed $J$ in ground state bands in the actinide region [117]. Similarly, for $\tau = 0.046$ (as in $^{178}$Hf in [110]), one finds $J \leq 32$, this limiting value being again higher than the highest observed $J$ in ground state bands in the rare earth region [117].

20. Comparison of the $su_q(2)$ model to other models

20.1. The Variable Moment of Inertia (VMI) model

In lowest order approximation rotational nuclear spectra can be described by the formula

$$E(J) = \frac{J(J + 1)}{2\Theta}, \quad (20.1)$$
where Θ is the moment of inertia of the nucleus, which is assumed to be constant. However, in order to get closer agreement to experimental data, one finds that he has to include higher order terms in this formula, as shown in eq. (19.6).

Another way to improve agreement with experiment is to let the moment of inertia Θ to vary as a function of the angular momentum \( J \). One thus obtains the Variable Moment of Inertia (VMI) model \(^{118}\). In this model the levels of the ground state band are given by

\[
E(J) = \frac{J(J+1)}{2\Theta(J)} + \frac{1}{2}C(\Theta(J) - \Theta_0)^2, \tag{20.2}
\]

where \( \Theta(J) \) is the moment of inertia of the nucleus at the state with angular momentum \( J \), while \( C \) and \( \Theta_0 \) are the two free parameters of the model, fitted to the data. The parameter \( \Theta_0 \) corresponds to the ground state moment of inertia, while instead of the parameter \( C \) it has been found meaningful to use the parameter combination

\[
\sigma = \frac{1}{2C\Theta_0^3}, \tag{20.3}
\]

which is related to the softness of the nucleus. The moment of inertia at given \( J \) is determined through the variational condition

\[
\frac{\partial E(J)}{\partial \Theta(J)}|_{J} = 0, \tag{20.4}
\]

which is equivalent to the cubic equation

\[
\Theta(J)^3 - \Theta(J)^2\Theta_0 - \frac{J(J+1)}{2C} = 0. \tag{20.5}
\]

This equation has only one real root, which can be written as

\[
\Theta(J) = \sqrt[3]{\frac{J(J+1)}{4C}} + \frac{\Theta_0^3}{27} + \sqrt[3]{\frac{(J(J+1))^2}{16C^2}} + \frac{\Theta_0^3J(J+1)}{54C} + \Theta_0 \tag{20.6}
\]

Expanding the roots in this expression one obtains

\[
\Theta(J) = \Theta_0(1 + \sigma J(J+1) - 2\sigma^2(J(J+1))^2 \\
+7\sigma^3(J(J+1))^3 - 30\sigma^4(J(J+1))^4 + \ldots). \tag{20.7}
\]

Using eq. (20.7) in eq. (20.2) one obtains the following expansion for the energy

\[
E(J) = \frac{1}{2\Theta_0}(J(J+1) - \frac{1}{2}\sigma(J(J+1))^2)
\]
\[ + \sigma^2(J(J+1))^3 - 3\sigma^3(J(J+1))^4 + \ldots. \] 

(20.8)

Empirically it is known \[18\] that for rotational nuclei the softness parameter \(\sigma\) is of the order of \(10^{-3}\). Therefore the expansion of eq. (20.8) has the same characteristics as the expansion of eq. (19.6) (alternating signs, successive coefficients falling by about 3 orders of magnitude).

20.2. Comparison of the \(\text{su}_q(2)\) model to the VMI and related models

We now turn to the comparison of the expansion of eq. (19.5) to the Variable Moment of Inertia (VMI) model, discussed in the previous subsection. Comparing eqs (19.5) and (20.8) we see that both expansions have the same form. The moment of inertia parameter \(I\) of (19.5) corresponds to the ground state moment of inertia \(\Theta_0\) of (20.8). The small parameter of the expansion is \(\tau^2\) in the first case, while it is the softness parameter \(1/(2C\Theta_0^3)\) in the second. However, the numerical coefficients in front of each power of \(J(J+1)\) are not the same.

In \[110\] a comparison is made between the parameters obtained by fitting the same spectra by the \(\text{su}_q(2)\) and VMI formulae. The agreement between \(1/(2I)\) and \(1/(2\Theta_0)\) is very good, as it is the agreement between \(\tau^2\) and \(\sigma\) as well. Therefore the known \[118\] smooth variation of \(\Theta_0\) and \(\sigma\) with the ratio \(R_4 = E(4)/E(2)\) is expected to hold for the parameters \(I\) and \(\tau^2\) as well. This is indeed seen in \[110\].

The difference between the expansions of eqs (19.5) (or (19.7)) and (20.8) is also demonstrated by forming the dimensionless ratios \(AC/(4B^2)\) and \(A^2D/(24B^3)\) in eq. (19.6). In the case of eq. (20.8) both quantities are equal to 1, as expected, since it is known that the VMI is equivalent \[110,128\] to the Harris expansion \[121\], in which both quantities are known to be equal to 1. In the case of eq. (19.7) the corresponding values are \(AC/(4B^2) = 1/10\) and \(A^2D/(24B^3) = 1/280\). According to the empirical values of these ratios given in \[113,122\] the ratios given from eq. (19.7) are better than the ratios given by eq. (20.8), especially the second one.

20.3. The hybrid model

The hybrid model of nuclear collective motion \[124,126\] has been introduced in order to provide a link between the two successful ways of describing low-lying nuclear excitations: the extended form of the Bohr-Mottelson model (BMM) \[127,128\] and the Interacting Boson Model (IBM) \[24\] (see secs 27, 29 for more details). The hybrid model combines the advantages of both models, i.e. the geometrical significance of the collective coordinates inherent in the extended BMM, and the use of group theoretical concepts characterizing IBM. In the framework of the rotational limit of the hybrid model, associated with the chain \(u(6)\supseteq\text{su}(3)\), Partensky and Quesne \[130,131\], starting from the fact that in the geometrical description the square of the deformation is proportional to the moment of inertia of the ground state band, proved that the
energy levels of the ground state band are given by
\[ E(J) = \frac{A}{J(J+1)} + \frac{B}{J(J+1)}, \quad (20.9) \]
with \( A \) being a free parameter and \( B \) given by
\[ B = 8N^2 + 22N - 15, \quad (20.10) \]
where \( N \) is the sum of the number of valence proton pairs (or proton-hole pairs, when more than half of the proton valence shell is filled) \( N_\pi \) and the number of the valence neutron pairs (or neutron-hole pairs, when more than half of the neutron valence shell is filled) \( N_\nu \). We remark that in the framework of this model the moment of inertia
\[ \Theta(J,N) = \frac{J(J+1) + \frac{B}{2A}}{2A} \quad (20.11) \]
depends on both the angular momentum \( J \) and the valence pair number \( N \).

It is instructive to expand \( E(J) \) in powers of \( J(J+1) \)
\[ E(J) = A \sum_{k=0}^{\infty} (-1)^k \left( \frac{J(J+1)}{B} \right)^{k+1} = \frac{A}{B}(J(J+1)) \]
\[ -\frac{1}{B}(J(J+1))^2 + \frac{1}{B^2}(J(J+1))^3 - \frac{1}{B^3}(J(J+1))^4 + \ldots. \quad (20.12) \]
Comparing the present expansion to the one of eq. (19.7) for the su\(_2\) model one has
\[ \tau = \sqrt{\frac{3}{B}} = \sqrt{\frac{3}{8N^2 + 22N - 15}}, \quad (20.13) \]
 obtaining thus a connection between the \( \tau \) parameter of the su\(_2\) model and a microscopic quantity, the valence nucleon pair number. From this equation it is clear that \( \tau \) decreases for increasing \( N \). Thus the minimum values of \( \tau \) are expected near the midshell regions, where the best rotators are known to be located. This is a reasonable result, since \( \tau \) describes the deviations from the su\(_2\) (rigid rotator) limit. It is clear that the minimum deviations should occur in the case of the best rigid rotators.

From eq. (20.13) one can obtain for each nucleus the value of \( \tau \) from the number of valence nucleon pairs present in it. These predictions for \( \tau \) have been compared in \[32\] to the values of \( \tau \) found empirically by fitting the corresponding spectra, with good results in both the rare earth and the actinide regions. In addition eq. (20.13) indicates that in a given shell nuclei characterized by the same valence nucleon pair number \( N \) will correspond to the same value of \( \tau \). Such multiplets have been studied in the framework of the hybrid model by \[33\].

It is worth noticing that fits of \( \gamma_1 \)-bands in the rare earth region (Er, Yb isotopes) \[116\] give \( \tau \) parameters very similar to the ones coming from fitting the corresponding
ground state bands, in addition exhibiting the same as the one mentioned above
behaviour of $\tau$ as a function of $N$.

Taking further advantage of the above connection between the $su_q(2)$ model and
the hybrid model, the parameter $\tau$ has been connected to the nuclear deformation
parameter $\beta$, as well as to the electromagnetic transition probabilities $B(E2:2_1^+ \rightarrow 0_1^+)$. Since both $\beta$ and $B(E2:2_1^+ \rightarrow 0_1^+)$ are known to increase with increasing collectivity,
$\tau$ is expected to decrease with increasing $\beta$ or increasing $B(E2:2_1^+ \rightarrow 0_1^+)$. This
expectation is corroborated by the results reported in 134.

20.4. Other models

It should be noticed that an empirical formula very similar to that of eq. (19.3)
has been recently proposed by Amal’skii on completely different physical grounds.
The formula reads

\[ E(J) = A \sin^2 \left( \frac{\pi J}{B} \right), \]

(20.14)

where $A$ and $B$ are free parameters. This formula should be compared to eq. (19.3),
with which it is almost identical.

A different formula, also giving very good fits of rotational spectra, has been
introduced by Celeghini, Giachetti, Sorace and Tarlini, based on the $q$-Poincaré
rotator.

21. Electromagnetic transitions in the $su_q(2)$ model

We have already seen that the $su_q(2)$ formalism provides an alternative to the
VMI model, the deformation parameter $q$ being connected to the softness parameter
of the VMI model.

The stretching effect present in rotational energy levels, which can equally well be
described in terms of the VMI model and the $su_q(2)$ symmetry, should also manifest
itself in the $B(E2)$ transition probabilities among these levels. If deviations from
the $su(2)$ symmetry are observed in the energy levels of a band, relevant deviations
should also appear in the $B(E2)$ transitions connecting them. In the case of the
VMI model no way has been found for making predictions for the $B(E2)$ transition
probabilities connecting the levels of a collective band. The $su_q(2)$ symmetry naturally
provides such a link. Before studying the $su_q(2)$ case, though, it is useful to recall the
predictions of other models on this matter.

21.1. The collective model of Bohr and Mottelson

In rotational bands one has

\[ B(E2: J + 2 \rightarrow J) = \frac{5}{16\pi} Q_0^2 |C_{K,0,-K}^{t+2,2,J}|^2, \]

(21.1)
i.e. the B(E2) transition probability depends on the relevant Clebsch-Gordan coefficient of su(2), while $Q_0^2$ is the intrinsic electric quadrupole moment and $K$ is the projection of the angular momentum $J$ on the symmetry axis of the nucleus in the body-fixed frame. For $K = 0$ bands, as the ground state bands, one then has

$$B(E2 : J + 2 \to J) = \frac{5}{16\pi} Q_0^2 \frac{3}{2} \frac{(J + 1)(J + 2)}{(2J + 3)(2J + 5)}. \quad (21.2)$$

It is clear that the B(E2) values should saturate with increasing $J$.

### 21.2. The Interacting Boson Model (IBM)

It is also instructive to mention what happens in the case of the Interacting Boson Model (IBM)\(^{129}\), the successful algebraic model of nuclear structure with which we are going to deal in more detail later (see sections 27, 29). In the case of the su(3) limit of the IBM, which is the limit applicable to deformed nuclei, the corresponding expression is\(^{133}\)

$$B(E2 : J + 2 \to J) = \frac{5}{16\pi} Q_0^2 \frac{3}{2} \frac{(2N - J)(2N + J + 3)}{(2N + 3/2)^2}, \quad (21.3)$$

where $N$ is the total number of bosons. Instead of saturation one then gets a decrease of the B(E2) values at high $J$, which finally reach zero at $J = 2N$. This is a well known disadvantage of the simplest version of the model (IBM-1) due to the small number of collective bosons ($s (J = 0)$ and $d (J = 2)$) taken into account. It can be corrected by the inclusion of higher bosons ($g (J = 4)$, $i (J = 6)$, etc), which approximately restore saturation (see\(^{129,138}\) for full list of references).

Another way to avoid the problem of decreasing B(E2) values in the su(3) limit of IBM at high $J$ is the recently proposed\(^{139}\) transition from the compact su(3) algebra to the noncompact sl(3,R) algebra. The angular momentum at which this transition takes place is fitted to experiment. In this way an increase of the B(E2) values at high $J$ is predicted, which agrees well with the experimental data for $^{236}$U.

### 21.3. The $su_q(2)$ model

In order to derive a formula similar to (21.2) in the $su_q(2)$ case, one needs to develop an $su_q(2)$ angular momentum theory. As mentioned in sec. 15, this has already been achieved. It turns out that an equation similar to (21.1) holds in the $q$-deformed case, the only difference being that the Clebsch-Gordan coefficient of the $su_q(2)$ algebra must be used instead. These coefficients have the form\(^{4}\)

$$q^{C_{K,0,-K}^{J+2,2,J}} = q^{2K} \sqrt{\frac{[3][4][J - K + 2][J - K + 1][J + K + 1][J + K + 2]}{[2][2J + 2][2J + 3][2J + 4][2J + 5]}}. \quad (21.4)$$
For $K = 0$ bands one then has

$$B_q(E2; J + 2 \to J) = \frac{5}{16\pi} Q_0^2 \frac{[3][4][J+1]^2[J+2]^2}{[2][2J+2][2J+3][2J+4][2J+5]}.$$  \hspace{1cm} (21.5)

For $q = e^{i\tau}$ this equation takes the form

$$B_q(E2; J + 2 \to J) = \frac{5}{16\pi} Q_0^2 \frac{\sin(3\tau) \sin(4\tau)}{\sin(2\tau) \sin(\tau)} \frac{(\sin(\tau(J + 1)))^2(\sin(\tau(J + 2)))^2}{\sin(\tau(2J + 2)) \sin(\tau(2J + 3)) \sin(\tau(2J + 4)) \sin(\tau(2J + 5))}. \hspace{1cm} (21.6)$$

It is useful to get an idea of the behaviour of this expression as a function of $J$, especially for the small values of $\tau$ found appropriate for the description of ground state spectra. Expanding all functions and keeping corrections of the leading order in $\tau$ only, one has

$$B_q(E2, J + 2 \to J) = \frac{5}{16\pi} Q_0^2 \frac{3(J + 1)(J + 2)}{2(2J + 3)(2J + 5)} (1 + \frac{\tau^2}{3}(6J^2 + 22J + 12)). \hspace{1cm} (21.7)$$

We see that the extra factor, which depends on $\tau^2$, contributes an extra increase with $J$, while the usual su(2) expression reaches saturation at high $J$ and IBM even predicts a decrease.

21.4. Comparison to experiment

Is there any experimental evidence for such an increase? In order to answer this question one should discover cases in which the data will be consistent with the su$_q$(2) expression but inconsistent with the classical su(2) expression. (The opposite cannot happen, since the classical expression is obtained from the quantum expression for the special parameter value $\tau = 0$.) Since error bars of $B(E2)$ values are usually large, in most cases both symmetries are consistent with the data. One should expect the differences to show up more clearly in two cases:

i) In rare earth nuclei not very much deformed (i.e. with an $R_4 = E(4)/E(2)$ ratio around 3.0). These should be deformed enough so that the su$_q$(2) symmetry will be able to describe them, having, however, at the same time values of $\tau$ not very small. Since in several of these nuclei backbending (or upbending) occurs at $J = 14$ or 16, one can expect only 5 or 6 experimental points to compare the theoretical predictions with.

ii) In the actinide region no backbending occurs up to around $J = 30$, so that this is a better test ground for the two symmetries. However, most nuclei in this region are well deformed, so that small values of $\tau$ should be expected, making the distinction between the two theoretical predictions difficult.

A few characteristic examples (4 rare earths and an actinide) are given in [14]. In all cases it is clear that the su$_q$(2) curve follows the experimental points, while the
su(2) curve has a different shape which cannot be forced to go through all the error bars. Several comments are now in place:

i) For a given nucleus the value of the parameter $\tau$ obtained from fitting the B(E2) values among the levels of the ground state band should be equal to the value obtained from fitting the energy levels of the ground state band. In $^{140}$ it is clear that both values are similar, although in most cases the value obtained from the B(E2)s is smaller than the value obtained from the spectra. It should be taken into account, however, that in most cases the number $n'$ of levels fitted is different (larger) than the number $n$ of the B(E2) values fitted. In the single case ($^{184}$W) in which $n = n'$, the two $\tau$ values are almost identical, as they should.

ii) One can certainly try different fitting procedures. Using the value of $\tau$ obtained from the B(E2) values for fitting the spectrum one gets a reasonably good description of it, although the squeezing of the spectrum is not as much as it should have been (with the exception of $^{184}$W). Using the value of $\tau$ obtained from the spectrum for fitting the B(E2) values one obtains an increase more rapid than the one shown by the data (again with the exception of $^{184}$W). One can also try to make an overall fit of spectra and B(E2)s using a common value of $\tau$. Then both the squeezing of the spectrum and the rise of the B(E2)s can be accounted for reasonably well although not exactly. One should notice, however, that the experimental uncertainties of the B(E2)s are much higher than the uncertainties of the energy levels.

iii) Concerning energy levels, the rigid rotator model and the su(3) limit of the IBM predict a $J(J + 1)$ increase, while the su$_q$(2) model and the VMI model predict squeezing, which is seen experimentally.

iv) Concerning the B(E2) values, the VMI makes no prediction, the rigid rotator predicts saturation at high $J$, the su(3) limit of the IBM predicts decrease, while the su$_q$(2) model predicts an increase. The evidence presented in $^{140}$ supports the su$_q$(2) prediction, but clearly much more work, both experimental and analytical, is needed before final conclusions can be drawn. The modified su(3) limit of IBM described in $^{18}$ also supports the increase of the B(E2) values at high $J$. Increasing BE(2) values are also predicted in the framework of the Fermion Dynamical Symmetry Model $^{141}$. There is also empirical evidence for increasing B(E2) values in the recent systematics by Zamfir and Casten $^{142}$.

v) It is clear that much further work is needed as far as comparisons of the su$_q$(2) predictions to experimental BE(2) values are concerned for safe conclusions to be reached.

vi) Since the quadrupole operator is not a member of the symmetry algebra su$_q$(2) under consideration, it is clear that the B(E2) values studied here do not contain any dynamical deformation effects, but only the kinematical ones (through the use of the $q$-deformed Clebsch-Gordan coefficients). A more complete approach to the problem will be the construction of a larger algebra, of which the quadrupole operator will be a member and it will also be an irreducible tensor under su$_q$(2) or so$_q$(3). Work in this direction is in progress.
22. Superdeformed bands

One of the most impressive experimental discoveries in nuclear physics during the last decade was that of superdeformation \textsuperscript{143} (see \textsuperscript{144,145,146} for relevant reviews). The energy levels of superdeformed bands follow the $J(J+1)$ rule much more closely than the usual rotational bands. Levels with $J$ larger than 60 have been observed. A compilation of superdeformed bands has been given in \textsuperscript{147}. The best examples have been found in the $A \approx 150$ mass region, while additional examples have been found in the $A \approx 80$, $A \approx 130$ and $A \approx 190$ regions. It is understood that the superdeformed bands in the $A \approx 150$ region correspond to elongated ellipsoidal shapes with an axis ratio close to 2:1, while in the $A \approx 130$ and $A \approx 190$ regions the ratios 3:2 and 1.65:1 respectively appear closer to reality.

Since the $su_q(2)$ model has been found suitable for describing normal deformed bands, it is plausible that it will also be successful in describing superdeformed bands as well. A test has been performed in \textsuperscript{148}. The $su_q(2)$ model has been found to give good results in all mass regions, the deformation parameter $\tau$ being smaller than in the case of normal deformed bands, thus indicating smaller deviations from the $su_q(2)$ symmetry. In particular, $\tau$ has been found to obtain values about 0.01 in the $A \approx 130$ and $A \approx 190$ regions, while it obtains even smaller values, around 0.004, in the $A \approx 150$ region, which contains the best examples of superdeformed bands observed so far. These results should be compared to the values of $\tau$ around 0.03 obtained in the case of normal deformations.

Concerning the corresponding B(E2) values, the experimental information is still quite poor for allowing a meaningful comparison of the $su_q(2)$ predictions to experiment.

23. The physical content of the $su_q(2)$ model

From the above it is clear that the $su_q(2)$ model offers a way of describing nuclear stretching, i.e. the departure of deformed nuclei from the $su(2)$ symmetry of the rigid rotator, similar to the one of the VMI model. The parameter $\tau$ describes this departure quantitatively, vanishing in the rigid rotator limit. Therefore the deformation parameter $\tau$ should not be confused with nuclear deformation; it is in fact related to nuclear softness, as already discussed in sec. 20.

On the other hand, the increase of the moment of inertia with increasing $J$ in the framework of the VMI model means that collectivity gets increased \textsuperscript{136}. The $su_q(2)$ model is an alternative way for describing this increase in collectivity. But increased collectivity implies increased B(E2) transitions. Therefore it is not surprising that the $su_q(2)$ model predicts B(E2) values increasing with $J$.

Given the $su_q(2)$ generators $J_+, J_-, J_0$, it is instructive to define as usual the operators $J_x, J_y, J_z$ by

\begin{equation}
J_+ = J_x + iJ_y, \quad J_- = J_x - iJ_y, \quad J_0 = J_z.
\end{equation}

\textsuperscript{23.1}
The $su_q(2)$ commutation relations can then be rewritten in the form

$$[J_x, J_y] = \frac{i}{2}[2J_z], \quad [J_y, J_z] = iJ_x, \quad [J_z, J_x] = iJ_y, \quad (23.2)$$

which is a generalization of the $so(3)$ commutation relations, obtained in the limit $q \to 1$. We remark that while in the classical $so(3)$ case the three commutation relations have exactly the same form, in the quantum case the first commutation relation differs (in the right hand side) from the other two, thus indicating that in the framework of the problem under study the z-direction is not any more equivalent to the x- and y- directions. This is of course a phenomenological way to describe the softness of deformed nuclei by adding the appropriate perturbations to the pure $su(2)$ Hamiltonian and has nothing to do with the isotropy of space, as implied in 149.

It is worth remarking at this point that the Casimir operator of $su_q(2)$ is also invariant under the usual $su(2)$. Therefore the quantum number $J$ characterizing the irreps of $su_q(2)$, and as a result the nuclear levels, is exactly the same as the quantum number $J$ used in the case of the usual $su(2)$. Therefore there is no reason for a “total reformulation of quantum mechanics”, as implied in 149, the $su_q(2)$ generators being connected to their $su(2)$ counterparts by the $q$-deforming functionals of sec. 14. In other words, one continues to believe in usual angular momentum theory and usual quantum mechanics. All what is done in the framework of the $su_q(2)$ model is to add to the usual $su(2)$ Hamiltonian several perturbations which have a special form making them suitable to be summed up, including the original $su(2)$ term, into the form of the $su_q(2)$ Hamiltonian.

24. The $u_{p,q}(2)$ rotator model

An extension of the $su_q(2)$ model is the $u_{p,q}(2)$ model, which is based on a two-parameter deformed algebra (see sec. 15 for a list of references). For $p = q$ (using the definition of $(p, q)$-numbers of eq. (2.11)) this model reduces to the $su_q(2)$ one. This model has been successfully applied to superdeformed nuclear bands. When Taylor expanded, it becomes clear that the eigenvalues of the Hamiltonian of this model (which is the second order Casimir operator of $u_{p,q}(2)$) contain terms of the form $J(J(J + 1))^n$, in addition to the $(J(J + 1))^n$ ones. It is therefore closer to the modification of $su_q(2)$ which will be discussed in subsec. 26.3.

25. Generalized deformed $su(2)$ models

Another formula giving very good results for rotational spectra has been introduced by Holmberg and Lipas, and rediscovered in 153. In this case the energy levels are given by

$$E(J) = a \left[ \sqrt{1 + bJ(J + 1)} - 1 \right]. \quad (25.1)$$

Taylor expansion of the square root immediately shows that the present formula is a
special case of eq. (19.6). This formula can be derived from the collective model of Bohr and Mottelson [36].

It has been argued in [122] that the Hamiltonian of eq. (25.1) gives better agreement to rotational nuclear spectra than the one coming from the $su_q(2)$ symmetry. Using the techniques described in detail in sec. 17 one can construct a generalized deformed algebra $su\Phi(2)$, characterized by a function $\Phi(J(J+1))$, giving the spectrum of eq. (25.1) exactly. In this particular case the algebra is characterized by the structure function

$$\Phi(J(J+1)) = a \left[ \sqrt{1 + bJ(J+1)} - 1 \right].$$

(25.2)

It is of interest to check if this choice of structure function also improves the agreement between theory and experiment in the case of the electromagnetic transition probabilities connecting these energy levels. In order to study this problem, one has to construct the relevant generalized Clebsch-Gordan coefficients. This problem is still open.

26. Quantum algebraic description of vibrational and transitional nuclear spectra

We have already seen that the $su_q(2)$ model describes successfully deformed and superdeformed bands. It is not surprising that the applicability of the $su_q(2)$ formalism is limited to the rotational region (where the ratio $R_4 = E(4)/E(2)$ obtains values between 3.0 and 3.33), since it is based on a deformation of the rotation algebra. For describing nuclear spectra in the vibrational ($2.0 \leq R_4 \leq 2.4$) and transitional ($2.4 \leq R_4 \leq 3.0$) regions it is clear that an extension of the model is needed. In order to be guided towards such an extension, we briefly review the existing experience of other successful models.

26.1. The Interacting Boson Model

In the rotational ($su(3)$) limit [35] of the Interacting Boson Model (IBM) (see secs 27, 29 for more details) the spectrum is described by a $J(J+1)$ expression, while in the vibrational ($u(5)$) [52] and transitional ($o(6)$) [55] limits expressions of the form $J(J+c)$ with $c > 1$ appear. In the $u(5)$ limit, in particular, the energy levels are given by

$$E(N, n_d, v, n_\Delta, J, M_J) = E_0 + \epsilon n_d + \alpha n_d(n_d + 4) + \beta 2v(v + 3) + \gamma 2J(J + 1),$$

(26.1)

where $N$ is the total number of bosons, $n_d$ is the number of d-bosons, $v$ is the seniority, $n_\Delta$ is the “missing” quantum number in the reduction from $o(5)$ to $o(3)$, $M_J$ is the third component of the angular momentum $J$, while $E_0, \epsilon, \alpha, \beta, \gamma$ are free parameters. The ground state band, in particular, is characterized by quantum numbers $n_d = 0, 1, 2, \ldots, v = n_d, n_\Delta = 0, J = 2n_d$, so that the energy expression for it reads

$$E(J) = E_0 + \frac{\epsilon}{2} J + \frac{\alpha}{4} J(J + 8) + \frac{\beta}{2} J(J + 6) + 2\gamma J(J + 1).$$

(26.2)
In the o(6) limit the energy is given by

\[ E(N, \sigma, \tau, \nu_\Delta, J, M_J) = E_0 + \beta 2\tau(\tau + 3) + \gamma 2J(J + 1) + \eta 2\sigma(\sigma + 4), \]  

(26.3)

where \( \sigma \) is the quantum number characterizing the irreducible representations (irreps) of \( o(6) \), \( \tau \) is the quantum number characterizing the irreps of \( o(5) \), \( \nu_\Delta \) is the missing quantum number in the reduction from \( o(5) \) to \( o(3) \), while \( E_0, \beta, \gamma, \eta \) are free parameters. The ground state band is characterized by the quantum numbers \( \sigma = N, \tau = 0, 1, 2, \ldots, \nu_\Delta = 0, J = 2\tau \), so that the relevant energy expression takes the form

\[ E(J) = E_0 + \frac{\beta}{2} J(J + 6) + \gamma 2J(J + 1) + \eta 2N(N + 4). \]  

(26.4)

The message from eqs (26.2) and (26.4) is that nuclear anharmonicities are described by expressions in which \( J \) and \( J^2 \) appear with different coefficients, and not with the same coefficient as in \( J(J + 1) \). The earliest introduction of this idea is in fact the Ejiri formula \[158\]

\[ E(J) = kJ(J + 1) + aJ, \]  

(26.5)

which has been subsequently justified microscopically in \[159\].

### 26.2. Generalized VMI

The two-parameter VMI model is known to continue giving good fits in the transitional and even in the vibrational region. In these regions, however, the accuracy of the model is substantially improved by adding a third parameter, which essentially allows for treating \( J \) and \( J^2 \) with a different coefficient \[160, 161, 162\].

The usual VMI model has been briefly reviewed in subsec. 20.1. One of the (essentially equivalent) three-parameter extensions of the model, which give improved fits of vibrational and transitional spectra, is the generalized VMI (GVMI) model \[160, 161\] in which the energy levels are described by

\[ E(J) = \frac{J + xJ(J - 2)}{\Phi(J)} + \frac{1}{2}k(\Phi(J) - \Phi_0)^2, \]  

(26.6)

which can be easily rewritten in the form

\[ E(J) = \frac{J(J + x')}{2\Phi'(J)} + \frac{1}{2}k'(\Phi'(J) - \Phi'_0)^2, \]  

(26.7)

where \( x' = x^{-1} - 2 \). It is clear that for \( x = 1/3 \) the GVMI reduces to the usual VMI, while for transitional and vibrational nuclei \( x \) obtains lower values, so that \( x' \) becomes greater than 1. The variational condition determining the moment of inertia still has the form of eq. (20.4), while the expansion of the energy turns out to be

\[ E(J) = \frac{1}{2\Phi'_0}(J(J + x') - \frac{\sigma'}{2}(J(J + x'))^2 \]
\[ (+\sigma')^2 (J(J + x'))^3 - 3(\sigma')^2 (J(J + x'))^4 + \ldots), \] (26.8)

where

\[ \sigma' = \frac{1}{2k'(\Phi'_0)^3}. \] (26.9)

We remark that an expansion in terms of \( J(J + x') \) is obtained, as compared to an expansion in terms of \( J(J+1) \) in the case of the usual VMI. The physical content of the parameters is clear: the centrifugal stretching effect is accounted for by the softness parameter \( \sigma' \), as in the case of the usual VMI, while anharmonicities, important in the vibrational region, are introduced by \( x' > 1 \). Since centrifugal stretching and anharmonicities are two effects of different origins, it is reasonable to describe them by two different parameters.

### 26.3. Modification of the **su_q(2)** model

The evidence coming from the IBM and the generalized VMI model described above, suggests a model in which the spectrum is given by

\[ E(J) = \frac{1}{2I}[J]_q[J + c]_q, \] (26.10)

which contains 3 parameters: the moment of inertia \( I \), the deformation parameter \( q \) and the new parameter \( c \), which is expected to be 1 in the rotational limit and larger than 1 in the vibrational and transitional regions. This energy expression can be expanded as

\[ E(J) = \frac{1}{2I} \frac{1}{(j_0(\tau))^2} (j_0(\tau)J(J + c) - \tau j_1(\tau)(J(J + c))^2 \]
\[ + \frac{2}{3} \tau^2 j_2(\tau)(J(J + c))^3 - \frac{1}{3} \tau^3 j_3(\tau)(J(J + c))^4 + \frac{2}{15} \tau^4 j_4(\tau)(J(J + c))^5 - \ldots), \] (26.11)

which is similar to eq. (19.5) with \( J(J+1) \) replaced by \( J(J+c) \).

It is expected that the deformation parameter \( \tau \), which plays the role of the small parameter in the expansion, as the softness parameter does in the case of the VMI, will describe the centrifugal stretching effect, while the parameter \( c \) will correspond to the anharmonicity effects. These expectations are corroborated from fits of the experimental data reported in [163]. The following comments can be made:

i) The anharmonicity parameter \( c \) is clearly decreasing with increasing \( R_4 \), i.e. with increasing collectivity. It obtains high values (8-18) in the vibrational region, while in the rotational region it stays close to 1. (It should be noted that by fixing \( c = 1 \) in the rotational region the fits are only very slightly changed, as expected.) In the transitional region its values are close to 3.

ii) The deformation parameter \( \tau \), which corresponds to the centrifugal stretching, is known from the **su_q(2)** model to obtain values close to 0.3-0.4 in the rotational region, a fact also seen here. The same range of values appears in the vibrational
region as well, while in the transitional region $\tau$ reaches values as high as 0.6. It is not unreasonable for this parameter, which is connected to the softness of the nucleus, to obtain its highest values in the region of $\gamma$-soft nuclei.

iii) It is worth remarking that eq. (26.10) coincides for $q = 1$ and $c=\text{integer}=N$ with the eigenvalues of the Casimir operator of the algebra $o(N+2)$ in completely symmetric states $^16_4, ^16_5$. In the rotational region the fits gave $N=1$, which corresponds to $o(3)$, as expected, while in the transitional region the fits gave approximately $N=3$, which corresponds to $o(5)$, which is a subalgebra contained in both the $u(5)$ and $o(6)$ limits of the IBM.

iv) It is also worth remarking that a special case of the expression of eq. (26.10) occurs in the $q$-deformed version of the $o(6)$ limit of the Interacting Boson Model, which will be reported below (see eq. (29.7)).

v) The $su_q(2)$ symmetry is known to make specific predictions for the deviation of the behaviour of the $B(E2)$ values from the rigid rotator model (subsec. 21.3). It will be interesting to connect the spectrum of eq. (26.10) to some deformed symmetry, at least for special values of $c$, and examine the implications of such a symmetry for the $B(E2)$ values. Such a study in the framework of the $q$-deformed version of the $o(6)$ limit of IBM, mentioned in iv), is also of interest.

vi) It is worth noticing that an expansion in terms of $J(J+c)$ can also be obtained from a generalized oscillator (sec. 12) with a structure function
\[ F(J) = [J(J+c)]_Q, \]
where $[x]_Q$ stands for the $Q$-numbers introduced in sec. 6 and $Q = e^T$, with $T$ real. This is similar to an oscillator successfully used for the description of vibrational spectra of diatomic molecules (see sec. 35). It can also be considered as a deformation of the oscillator corresponding to the Morse potential (see sec. 35).

We have therefore introduced an extension of the $su_q(2)$ model of rotational nuclear spectra, which is applicable in the vibrational and transitional regions as well. This extension is in agreement with the Interacting Boson Model and the Generalized Variable Moment of Inertia model. In addition to the overall scale parameter, the model contains two parameters, one related to the centrifugal stretching and another related to nuclear anharmonicities. In the rotational region the model coincides with the usual $su_q(2)$ model, while in the transitional region an approximate $o(5)$ symmetry is seen. These results give additional motivation in pursuing the construction of a deformed version of the Interacting Boson Model. This problem will be discussed in the next sections.

27. A toy Interacting Boson Model with $su_q(3)$ symmetry

The Interacting Boson Model (IBM) is a very popular algebraic model of nuclear structure. In the simplest version of IBM low lying collective nuclear spectra are described in terms of $s$ ($J = 0$) and $d$ ($J = 2$) bosons, which are supposed to be correlated fermion pairs. The symmetry of the simplest version of the model
is $u(6)$, which contains $u(5)$ (vibrational), $su(3)$ (rotational) and $o(6)$ ($\gamma$-unstable) chains of subalgebras (see also sec. 29). A simplified version of the model, having the $su(3)$ symmetry with $su(2)$ and $so(3)$ chains of subalgebras also exists\cite{166}. It can be considered as a toy model for two-dimensional nuclei, but it is very useful in demonstrating the basic techniques used in the full IBM.

In the present section we will construct the $q$-deformed version of this toy model. Since this project requires the construction of a realization of $su_q(3)$ in terms of $q$-deformed bosons, we will also use this opportunity to study $su_q(3)$ in some detail.

27.1. The $su_q(3)$ algebra

In the classical version of the toy IBM\cite{166} one introduces bosons with angular momentum $m = 0, \pm 2$, represented by the creation (annihilation) operators $a_0^+, a_0^-$, $a_+^+$, $a_-^-$ (together). They satisfy usual boson commutation relations

$$[a_i, a_j^\dagger] = \delta_{ij}, \quad [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0. \quad (27.1)$$

The 9 bilinear operators

$$\Lambda_{ij} = a_i^\dagger a_j \quad (27.2)$$

satisfy then the commutation relations

$$[\Lambda_{ij}, \Lambda_{kl}] = \delta_{jk} \Lambda_{il} - \delta_{il} \Lambda_{kj}, \quad (27.3)$$

which are the standard $u(3)$ commutation relations. The total number of bosons

$$N = \Sigma_i \Lambda_{ii} = a_0^+ a_0^\dagger + a_+^+ a_+^\dagger + a_-^+ a_-^\dagger \quad (27.4)$$

is kept constant. Since we are dealing with a system of bosons, only the totally symmetric irreps $\{N, 0, 0\}$ of $u(3)$ occur.

In the quantum case one has the $u_q(3)$ commutation relations given in Table 2\cite{163}, where $A_{ij}$ are the generators of $u_q(3)$ and the $q$-commutator is defined as

$$[A, B]_q = AB - qBA. \quad (27.5)$$

In order to obtain a realization of $u_q(3)$ in terms of the $q$-bosons described in sec. 10, one starts with

$$A_{12} = a_1^\dagger a_2, \quad A_{21} = a_2^\dagger a_1, \quad A_{23} = a_2^\dagger a_3, \quad A_{32} = a_3^\dagger a_2. \quad (27.6)$$

One can easily verify that the $u_q(3)$ commutation relations involving these generators are satisfied. For example, one has

$$[A_{12}, A_{21}] = [N_1 - N_2], \quad [A_{23}, A_{32}] = [N_2 - N_3], \quad (27.7)$$

using the identity of eq. (2.6) and the identifications

$$N_1 = A_{11}, \quad N_2 = A_{22}, \quad N_3 = A_{33}. \quad (27.8)$$
Table 2: $u_q(3)$ commutation relations\textsuperscript{167}, given in the form $[A, B]_a = C$. $A$ is given in the first column, $B$ in the first row. $C$ is given at the intersection of the row containing $A$ with the column containing $B$. $a$, when different from 1, follows $C$, enclosed in parentheses.

|     | $A_{11}$ | $A_{22}$ | $A_{33}$ | $A_{12}$ | $A_{23}$ | $A_{13}$ |
|-----|----------|----------|----------|----------|----------|----------|
| $A_{11}$ | 0        | 0        | 0        | $A_{12}$ | 0        | $A_{13}$ |
| $A_{22}$ | 0        | 0        | 0        | $-A_{12}$ | $A_{23}$ | 0        |
| $A_{33}$ | 0        | 0        | 0        | 0        | $-A_{23}$ | $-A_{13}$ |
| $A_{12}$ | $-A_{12}$ | $A_{12}$ | 0        | 0        | $A_{13}$ | $(q)$ |
| $A_{23}$ | 0        | $-A_{23}$ | $A_{23}$ | $-q^{-1}A_{13}(q^{-1})$ | 0        | $(q)$ |
| $A_{13}$ | $-A_{13}$ | 0        | $A_{13}$ | 0$(q)$  | 0$(q^{-1})$ | 0        |
| $A_{21}$ | $A_{21}$ | $-A_{21}$ | 0        | $-[A_{11} - A_{22}]$ | 0        | $A_{23}q^{A_{11} - A_{22}}$ |
| $A_{32}$ | 0        | $A_{32}$ | $-A_{32}$ | 0        | $-[A_{22} - A_{33}]$ | $-q^{-A_{22} + A_{33}}A_{12}$ |
| $A_{31}$ | $A_{31}$ | 0        | $-A_{31}$ | $-q^{-A_{11} + A_{22}}A_{32}$ | $-A_{21}q^{A_{22} - A_{33}}$ | $-[A_{11} - A_{33}]$ |

|     | $A_{21}$ | $A_{32}$ | $A_{31}$ |
|-----|----------|----------|----------|
| $A_{11}$ | $-A_{21}$ | 0        | $-A_{31}$ |
| $A_{22}$ | $A_{21}$ | $-A_{32}$ | 0        |
| $A_{33}$ | 0        | $A_{32}$ | $A_{31}$ |
| $A_{12}$ | $[A_{11} - A_{22}]$ | 0        | $-q^{-A_{11} + A_{22}}A_{32}$ |
| $A_{23}$ | 0        | $[A_{22} - A_{33}]$ | $A_{21}q^{A_{22} - A_{33}}$ |
| $A_{13}$ | $-A_{23}q^{A_{11} - A_{22}}$ | $q^{-A_{22} + A_{33}}A_{12}$ | $[A_{11} - A_{33}]$ |
| $A_{21}$ | 0        | $-qA_{31}(q)$ | 0$(q^{-1})$ |
| $A_{32}$ | $A_{31}(q^{-1})$ | 0        | $(q)$ |
| $A_{31}$ | 0$(q)$  | 0$(q^{-1})$ | 0        |

One can now determine the boson realizations of $A_{13}$ and $A_{31}$ from other commutation relations, as follows

$$A_{13} = [A_{12}, A_{23}]_q = a_1^\dagger a_3 q^{-N_2}, \quad (27.9)$$

$$A_{31} = [A_{32}, A_{21}]_{q^{-1}} = a_3^\dagger a_1 q^{N_2}. \quad (27.10)$$

Using eq. (2.6) once more one can verify that the relation

$$[A_{13}, A_{31}] = [N_1 - N_3] \quad (27.11)$$

is fulfilled by the boson images of (27.9), (27.10). It is by now a straightforward task to verify that all commutation relations of Table 2 are fulfilled by the boson images obtained above.

Before turning to the study of the two limits of the model, we give for completeness some additional information on $su_q(3)$:

i) The irreps of $su_q(3)$ have been studied in\textsuperscript{166,167,170,171,172,173,174,175,176,177}.
ii) Clebsch-Gordan coefficients for $\text{su}_q(3)$ have been given in $^{65,70,78,79}$. 

iii) The Casimir operators of $\text{su}_q(3)$ and their eigenvalues have been given explicitly in $^{180}$. Using the Elliott quantum numbers $^{181,182,183}$

\[
\lambda = f_1 - f_2, \quad \mu = f_2,
\]

where $f_i$ represents the number of boxes in the $i$-th line of the corresponding Young diagram, the irreps of $\text{su}_q(3)$ are labelled as $(\lambda, \mu)$. The eigenvalues of the second order Casimir operator then read

\[
C_2 = \left[ \frac{\lambda}{3} - \frac{\mu}{3} \right]^2 + \left[ \frac{2\lambda}{3} + \frac{\mu}{3} + 1 \right]^2 + \left[ \frac{\lambda}{3} + \frac{2\mu}{3} + 1 \right]^2 - 2,
\]

while the eigenvalues of the third order Casimir operator are

\[
C_3 = 2 \left[ \frac{\lambda}{3} - \frac{\mu}{3} \right] \left[ \frac{2\lambda}{3} + \frac{\mu}{3} + 1 \right] \left[ \frac{\lambda}{3} + \frac{2\mu}{3} + 1 \right].
\]

In the limit $q \to 1$ these reproduce the ordinary results for $\text{su}(3)$:

\[
C_2 = \frac{2}{3} (\lambda^2 + \mu^2 + \lambda\mu + 3\lambda + 3\mu),
\]

\[
C_3 = \frac{2}{27} (\lambda - \mu)(2\lambda + \mu + 3)(\lambda + 2\mu + 3).
\]

iv) A different deformation of $\text{sl}(3)$ has been studied in $^{184}$.

27.2. The $\text{su}_q(2)$ limit

We shall study the $\text{su}_q(2)$ limit of the model first, since it is technically less demanding.

So far we have managed to write a boson realization of $\text{u}_q(3)$ in terms of 3 $q$-bosons, namely $a_1$, $a_2$, $a_3$. Omitting the generators involving one of the bosons, one is left with an $\text{su}_q(2)$ subalgebra. Omitting the generators involving $a_3$, for example, one is left with $A_{12}$, $A_{21}$, $N_1$, $N_2$, which satisfy usual $\text{su}_q(2)$ commutation relations if the identifications

\[
J_+ = A_{12}, \quad J_- = A_{21}, \quad J_0 = \frac{1}{2}(N_1 - N_2)
\]

are made. $J_0$ alone forms an $\text{so}_q(2)$ subalgebra. Therefore the relevant chain of subalgebras is

\[
\text{su}_q(3) \supset \text{su}_q(2) \supset \text{so}_q(2).
\]

The second order Casimir operator of $\text{su}_q(2)$ has been given in eq. (14.9). Substituting the above expressions for the generators one finds

\[
C_2(\text{su}_q(2)) = \left[ \frac{N_1 + N_2}{2} \right] \left[ \frac{N_1 + N_2}{2} + 1 \right].
\]
All of the above equations go to their classical counterparts by allowing \( q \to 1 \), for which \( [x] \to x \), i.e. \( q \)-numbers become usual numbers. In the classical case one chooses to leave out the three bosons \( (a_0, a_+, a_-) \) forming \( \text{su}(3) \), one chooses to leave out \( a_0 \), the boson with zero angular momentum, in order to be left with the \( \text{su}(2) \) subalgebra formed by \( a_+ \) and \( a_- \), the two bosons of angular momentum two. The choice of the \( \text{su}_q(2) \) subalgebra made above is then consistent with the following correspondence between classical bosons and \( q \)-bosons

\[
a_+ \to a_1, \quad a_- \to a_2, \quad a_0 \to a_3.
\]  
(We have opted in using different indices for usual bosons and \( q \)-bosons in order to avoid confusion.)

In the classical case the states of the system are characterized by the quantum numbers characterizing the irreducible representations (irreps) of the algebras appearing in the classical counterpart of the chain of eq. (27.18). For \( \text{su}(3) \) the total number of bosons \( N \) is used. For \( \text{su}(2) \) and \( \text{so}(2) \) one can use the eigenvalues of \( J^2 \) and \( J_0 \), or, equivalently, the eigenvalues of \( a_+^\dagger a_+ + a_-^\dagger a_- \) and \( L_3 = 4J_0 \), for which we use the symbols \( n_d \) (the number of bosons with angular momentum 2) and \( M \). Then the basis in the classical case can be written as

\[
|N, n_d, M > = \frac{(a_0^\dagger)^{N-n_d} (a_+^\dagger)^{n_d/2-M/4} (a_-^\dagger)^{n_d/2+M/4}}{(N-n_d)! (n_d/2 + M/4)! (n_d/2 - M/4)!} |0 > .
\]  
(27.21)

In the quantum case for each oscillator one defines the basis as in sec. 10. Then the full basis in the \( q \)-deformed case is

\[
|N, n_d, M >_q = \frac{(a_3^\dagger)^{N-n_d} (a_1^\dagger)^{n_d/2+M/4} (a_2^\dagger)^{n_d/2-M/4}}{[N-n_d]! [n_d/2 + M/4]! [n_d/2 - M/4]!} |0 > ,
\]  
(27.22)

where \( N = N_1 + N_2 + N_3 \) is the total number of bosons, \( n_d = N_1 + N_2 \) is the number of bosons with angular momentum 2, and \( M \) is the eigenvalue of \( L = 4J_0 \). \( n_d \) takes values from 0 up to \( N \), while for a given value of \( n_d \), \( M \) takes the values \( \pm 2n_d, \pm 2(n_d - 2), \ldots, \pm 2 \) or 0, depending on whether \( n_d \) is odd or even. In this basis the eigenvalues of the second order Casimir operator of \( \text{su}_q(2) \) are then

\[
C_2(\text{su}_q(2))|N, n_d, M >_q = \left[ \frac{n_d}{2} \right] \left[ \frac{n_d}{2} + 1 \right] |N, n_d, M >_q .
\]  
(27.23)

In the case of \( N = 5 \) one can easily see that the spectrum will be composed by the ground state band, consisting of states with \( M = 0, 2, 4, 6, 8, 10 \) and \( n_d = M/2 \), the first excited band with states characterized by \( M = 0, 2, 4, 6 \) and \( n_d = M/2 + 2 \), and the second excited band, containing states with \( M = 0, 2 \) and \( n_d = M/2 + 4 \).

In the case that the Hamiltonian has the \( \text{su}_q(2) \) dynamical symmetry, it can be written in terms of the Casimir operators of the chain (27.18). Then one has

\[
H = E_0 + AC_2(\text{su}_q(2)) + BC_2(\text{so}_q(2)) ,
\]  
(27.24)
where $E_0$, $A$, $B$ are constants. Its eigenvalues are

$$E = E_0 + A \left[ \frac{n_d}{2} \right] \left[ \frac{n_d}{2} + 1 \right] + BM^2. \quad (27.25)$$

Realistic nuclear spectra are characterized by strong electric quadrupole transitions among the levels of the same band, as well as by interband transitions. In the framework of the present toy model one can define, by analogy to the classical case [166], quadrupole transition operators

$$Q_+ = a_1^\dagger a_3 + a_3^\dagger a_2, \quad Q_- = a_2^\dagger a_3 + a_3^\dagger a_1. \quad (27.26)$$

In order to calculate transition matrix elements of these operators one only needs eqs. (10.7), (10.8), i.e. the action of the $q$-boson operators on the $q$-deformed basis. The selection rules, as in the classical case, are $\Delta M = \pm 2$, $\Delta n_d = \pm 1$, while the corresponding matrix elements are

$$q < N, n_d + 1, M \pm 2|Q_\pm |N, n_d, M >_q = \sqrt{[N - n_d]\left[ \frac{n_d}{2} \pm \frac{M}{4} + 1 \right]}, \quad (27.27)$$

$$q < N, n_d - 1, M \pm 2|Q_\pm |N, n_d, M >_q = \sqrt{[N - n_d + 1]\left[ \frac{n_d}{2} \mp \frac{M}{4} \right]}. \quad (27.28)$$

From these equations it is clear that both intraband and interband transitions are possible.

In order to get a feeling of the qualitative changes in the spectrum and the transition matrix elements resulting from the $q$-deformation of the model, a simple calculation for a system of 20 bosons ($N = 20$) has been performed in [183]. Two cases are distinguished: i) $q$ real ($q = e^\tau$, with $\tau$ real), ii) $q$ a phase factor ($q = e^{i\tau}$, with $\tau$ real). The main conclusions are:

i) When $q$ is real the spectrum is increasing more rapidly than in the classical case, while when $q$ is a phase the spectrum increases more slowly than in the classical case, in agreement with the findings of the $q$-rotator model.

ii) The transition matrix elements in the case that $q$ is real increase more rapidly than in the classical case, while they increase less rapidly than in the classical case when $q$ is a phase.

iii) Transition matrix elements are much more sensitive to $q$-deformation than energy spectra. This is an interesting feature, showing that $q$-deformed algebraic models can be much more flexible in the description of transition probabilities than their classical counterparts.

27.3. The $so_q(3)$ limit

The classical $su(3)$ toy model has, in addition to the above mentioned $su(2)$ chain of subalgebras, an $so(3)$ chain. However, the problem of constructing the $su_q(3) \supset so_q(3)$ decomposition is a very difficult one. Since this decomposition is needed
in constructing the $q$-deformed versions of several collective models, including the Elliott model and the su(3) limit of the IBM, we report here the state of the art in this problem:

i) As far as the completely symmetric irreps of $su_q(3)$ are concerned, the problem has been solved by Van der Jeugt. This suffices for our needs in the framework of the toy IBM, since only completely symmetric $su_q(3)$ irreps occur in it.

ii) Sciarrino started from $so_q(3)$ and obtained a deformed $gl(3)$ containing $so_q(3)$ as a subalgebra. However, it was not clear how to impose the Hopf structure on this larger algebra. Trying the other way around, he found that by starting from a $gl_q(3)$ algebra, which already possesses the Hopf structure, one loses the Hopf structure of the principal 3-dim subalgebra, which should have been $so_q(3)$.

iii) Pan and Del Sol Mesa et al attacked the problem through the use of $q$-deforming functionals of secs 10, 14.

iv) Quesne started with $q$-bosonic operators transforming as vectors under $so_q(3)$ and constructed a $q$-deformed $u(3)$ by tensor coupling.

v) $q$-deformed subalgebras of several $q$-deformed algebras have recently been studied by Sciarrino.

vi) A simplified version of the $so_q(3)$ subalgebra of $u_q(3)$ has been constructed. Furthermore, explicit expressions for the irreducible vector and quadrupole tensor operators under $so_q(3)$ have been given and the matrix elements of the latter have been calculated.

In what follows, it suffices to use the solution given in 187, since in the model under study only completely symmetric irreps of $u_q(3)$ enter. Using the notation

$$a_+ \rightarrow a_1, \quad a_- \rightarrow a_2, \quad a_0 \rightarrow a_3,$$  \hspace{1cm} (27.29)

the basis states are of the form

$$|n_+n_0n_- > = \frac{(a_+^\dagger)^{n_+}(a_0^\dagger)^{n_0}(a_-^\dagger)^{n_-}}{\sqrt{n_+!n_0!n_-!}}|0 >, \hspace{1cm} (27.30)$$

with $a_i|0 > = 0$ and $N_i |n_+n_0n_- > = n_i |n_+n_0n_- >$, where $i = +, 0, -$. The principal subalgebra $so_q(3)$ is then generated by

$$L_0 = N_+ - N_-,$$  \hspace{1cm} (27.31)

$$L_+ = q^{N_- - \frac{1}{2}N_0} \sqrt{q^{N_+} + q^{-N_+}} \quad a_+^\dagger a_0 + a_0^\dagger a_+ q^{N_+ - \frac{1}{2}N_0} \sqrt{q^{N_+} + q^{-N_+}},$$  \hspace{1cm} (27.32)

$$L_- = a_0^\dagger a_+ q^{N_- - \frac{1}{2}N_0} \sqrt{q^{N_+} + q^{-N_+}} + q^{N_+ - \frac{1}{2}N_0} \sqrt{q^{N_+} + q^{-N_+}} a_-^\dagger a_0,$$  \hspace{1cm} (27.33)

satisfying the commutation relations

$$[L_0, L_\pm] = \pm L_\pm, \quad [L_+, L_-] = [2L_0].$$  \hspace{1cm} (27.34)

$L_0$ alone generates then the $so_q(2)$ subalgebra. Therefore the relevant chain of subalgebras is

$$su_q(3) \supset so_q(3) \supset so_q(2).$$  \hspace{1cm} (27.35)
The so\(_q(3)\) basis vectors can be written in terms of the vectors of eq. (27.30) as

\[
|v(N, L, M)\rangle = q^{-[(L+M)(L+M-1)]/4} \sqrt{\frac{[N + L]!![2L + 1][L + M]![L - M]!}{[N - L]!![N + L + 1]!}} \\
\sum_x q^{(2L-1)x/2} s^{(N-L)/2} \frac{|x, L + M - 2x, x - M\rangle}{\sqrt{[2x]!![L + M - 2x]![2x - 2M]!!}},
\]

where

\[
s = (a^\dagger_0)^2 q^{N_+ + N_- + 1} - \sqrt{\frac{[2N_+][2N_-]}{[N_+][N_-]}} a^\dagger_+ a^- q^{-N_0 - \frac{1}{2}},
\]

\(x\) takes values from \(\text{max}(0,M)\) to \([(L + M)/2]\) in steps of 1, \(L = N, N - 2, \ldots, 1\) or 0, \(M = -L, -L + 1, \ldots, +L\), and \([2x]!! = [2x][2x - 2] \ldots [2]\). The action of the generators of so\(_q(3)\) on these states is given by

\[
L_0 |v(N, L, M)\rangle = M |v(N, L, M)\rangle,
\]

\[
L_\pm |v(N, L, M)\rangle = \sqrt{[L + M][L \pm M + 1]} |v(N, L, M \pm 1)\rangle.
\]

The second order Casimir operator of so\(_q(3)\) has the form

\[
C_2(\text{so}_q(3)) = L^2 = L_- L_+ + [L_0][L_0 + 1].
\]

Its eigenvalues in the above basis are given by

\[
C_2(\text{so}_q(3)) |v(N, L, M)\rangle = [L][L + 1] |v(N, L, M)\rangle.
\]

All of the above equations go to their classical counterparts by allowing \(q \to 1\), for which \([x] \to x\), i.e. \(q\)-numbers become usual numbers. In the classical case, the states of the system are characterized by the quantum numbers characterizing the irreducible representations (irreps) of the algebras appearing in the classical counterpart of the chain of eq. (27.35). For su(3) the total number of bosons \(N\) is used. For so(3) and so(2) one can use \(L\) and \(M\), respectively. In the classical case, however, the eigenvalue of \(L_0' = 2L_0\) is used, which is \(M' = 2M\).

Since the rules for the decomposition of the totally symmetric \(u_q(3)\) irreps into so\(_q(3)\) irreps are the same as in the classical case, it is easy to verify that for a system with \(N = 6\) the spectrum will be composed by the ground state band, consisting of states with \(M' = 0, 2, 4, 6, 8, 10, 12\) and \(L = N\), the first excited band with states characterized by \(M' = 0, 2, 4, 6, 8\) and \(L = N - 2\), the second excited band, containing states with \(M = 0, 2, 4\) and \(L = N - 4\), and the third excited band, containing a state with \(M' = 0\) and \(L = N - 6\).

In the case that the Hamiltonian has the so\(_q(3)\) dynamical symmetry, it can be written in terms of the Casimir operators of the chain (27.35). Then one has

\[
H = E_0 + AC_2(\text{so}_q(3)) + BC_2(\text{so}_q(2)),
\]
where $E_0$, $A$, $B$ are constants. Its eigenvalues are

$$E = E_0 + A[L][L + 1] + BM^2.$$  \hfill (27.43)

It is then clear that in this simple model the internal structure of the rotational bands is not influenced by $q$-deformation. What is changed is the position of the bandheads.

We turn now to the study of electromagnetic transitions. In the present limit one can define, by analogy to the classical case, quadrupole transition operators $Q_\pm$ proportional to the $so_3(3)$ generators $L_\pm$

$$Q_\pm = L_\pm.$$  \hfill (27.44)

In order to calculate transition matrix elements of these operators one only needs eq. (27.39). The selection rules, as in the classical case, are $\Delta M' = \pm 2$, $\Delta L = 0$, i.e. only intraband transitions are allowed. The relevant matrix elements are

$$< v(N, L, M' + 2)|Q_+|v(N, L, M')> = \sqrt{\left[L - \frac{M'}{2}\right]\left[L + \frac{M'}{2} + 1\right]},$$  \hfill (27.45)

$$< v(N, L, M' - 2)|Q_-|v(N, L, M')> = \sqrt{\left[L + \frac{M'}{2}\right]\left[L - \frac{M'}{2} + 1\right]}.$$  \hfill (27.46)

In order to get a feeling of the qualitative changes in the spectrum and the transition matrix elements resulting from the $q$-deformation of this limit of the model, a simple calculation has been done in 28. Again the cases of $q$ being real or $q$ being a phase factor have been considered. The main conclusions are:

i) $q$-deformation influences only the position of bandheads, while it leaves the internal structure of the bands intact.

ii) When $q$ is real the bandheads are increasing more rapidly than in the classical case, while when $q$ is a phase the bandheads increase more slowly than in the classical case. This result is in qualitative agreement with the findings of the $su_2(2)$ model.

iii) Transition matrix elements in the case that $q$ is real have values higher than in the classical case, while they have values lower than in the classical case when $q$ is a phase.

28. The question of complete breaking of symmetries and some applications

In the cases examined so far the $q$-deformed symmetries considered were close to their classical counterparts, to which they reduce for $\tau = 0$ $(q = 1)$, since the values of $\tau$ were relatively small. One can then argue that results similar to the ones provided by the quantum symmetries can also be obtained from the usual Lie symmetries through the addition of suitable perturbations. What can be very useful is to start with one limiting symmetry and, through large deformations, reach another limiting symmetry. We shall refer to this as the complete breaking of the symmetry.
In this way one could hope to “bridge” different Lie symmetries through the use of \( q \)-deformations, providing in addition new symmetries in the regions intermediate between the existing Lie ones.

The question of complete breaking of the symmetry in the framework of the toy IBM under study has been studied by Cseh \(^{199}\), Gupta \(^{200}\), and Del Sol Mesa et al. \(^{194}\).

Cseh \(^{199}\) started with the \( su_q(2) \) (vibrational) limit (in a form different from the one used above) and tried to reach the \( so_q(3) \) (rotational) limit. He noted that for \( q \) being a phase factor this is not possible, while for real \( q \) some rotational features are obtained, but without all of the requirements for rotational behaviour being satisfied simultaneously.

Gupta \(^{200}\) started with the \( su_q(2) \) limit, in the form given above. He noted that for \( q \) being a phase factor a recovery of the \( su(3) \) symmetry occurs (see also the next paragraph), while for real \( q \), and even better for \( q \) complex (\( q = e^s \) with \( s = a + ib \)), the \( so_q(3) \) limit is indeed reached.

Del Sol Mesa et al. \(^{194}\) considered the \( o_q(3) \) limit of the model, since it corresponds to the symmetry of a \( q \)-deformed version of the spherical Nilsson Hamiltonian with spin-orbit coupling term. They found that for \( q \) being a phase factor (\( q = e^{i\tau} \)) and for \( \tau \) obtaining values in the region \( 0.5 \leq \tau \leq 2 \) the \( u(3) \) symmetry, which is broken in the initial model because of the presence of the spin-orbit term, is recovered. This offers a way of recovering the \( u(3) \) symmetry alternative to the one developed for the spherical Nilsson model \(^{201, 202}\) and the deformed Nilsson model \(^{203, 204}\) through the use of appropriate unitary operators.

Complex deformations have also been used in \(^{205}\) in the framework of a deformed \( u(2) \) model, possessing the \( u(2) \supset u(1) \) and \( u(2) \supset o(2) \) chains. Again it has been found that complex deformations can bridge the two limiting symmetries.

Possible complete breaking of the symmetry has also been studied in the framework of the \( q \)-deformed version of the full Interacting Boson Model (see sec. 29).

A problem associated with complex deformations as the ones considered above is that the energy eigenvalues become complex as well. A way to avoid this problem has been introduced recently by Januissis and collaborators \(^{206, 207}\).

Finally, the \( o_q(3) \) limit of the model has been used for describing the \( ^{16}O + \alpha \) cluster states in \(^{20}Ne \). It turns out that an improved description of the energy spectrum and the \( \alpha \)-particle spectroscopic factors occurs for \( q = e^{0.124} \).

29. \( q \)-deformation of the Interacting Boson Model (IBM)

The Interacting Boson Model \(^{137, 156, 157}\) (see \(^{129, 138}\) for recent overviews) is the most popular algebraic model of nuclear structure. It describes the collective properties of medium-mass and heavy nuclei away from closed shells in terms of bosons, which correspond to correlated valence fermion pairs. In its simplest form, called IBM-1, only \( s \) (\( J = 0 \)) and \( d \) (\( J = 2 \)) bosons are used. The overall symmetry of the model is \( u(6) \), possessing three limiting symmetries: the \( u(5) \) (vibrational) limit, corresponding
to the chain of subalgebras
\[ u(6) \supset u(5) \supset o(5) \supset o(3), \] (29.1)
the $su(3)$ (rotational) limit, characterized by the chain
\[ u(6) \supset su(3) \supset o(3), \] (29.2)
and the $o(6)$ ($\gamma$-unstable) limit, for which the relevant chain is
\[ u(6) \supset o(6) \supset o(5) \supset o(3). \] (29.3)

If one of these dynamical symmetries is present, the Hamiltonian can be written in terms of the Casimir operators of the algebras appearing in the relevant chain. Thus the Hamiltonian can be analytically diagonalized in the corresponding basis. This is a great advantage of IBM and its numerous generalizations: they provide us with a large number of exactly soluble models, the predictions of which can be directly compared to experiment, without any need for lengthy numerical calculations.

From what we have already seen in sec. 27, it is worth examining if a $q$-deformed version of the IBM has any advantages in comparison to the standard version. In order to accomplish this, one has to construct the $q$-analogues of the three chains mentioned above. The difficulties associated with the $su(3)$ chain have already been discussed in subsec. 27.3. In what follows we are going to focus attention on the $o(6)$ chain, for which the relevant construction has been carried out \[209\]. The technique used is based on the notion of complementary subalgebras, which is explained in detail in \[210\], while here only final results will be reported. We only mention here that the notion of complementary subalgebras was introduced by Moshinsky and Quesne \[211\], \[212\], \[213\]. Two subalgebras $A_1$ and $A_2$ of a larger algebra $A$ are complementary within a definite irrep of $A$, if there is an one-to-one correspondence between all the irreps of $A_1$ and $A_2$ contained in this irrep of $A$.

In the $o(6)$ limit of IBM the Hamiltonian is
\[ H = E_0 + \beta C_2(o(5)) + \gamma C_2(o(3)) + \eta C_2(o(6)). \] (29.4)
The eigenvalues of the energy in the relevant basis have already been given in subsec. 26.1.

Using the notion of complementarity it turns out that, instead of using the $o(6)$ chain mentioned above, it suffices to study the chain
\[ su^{sd}(1,1) \otimes so(6) \supset su^d(1,1) \otimes so(5) \supset so(3), \] (29.5)
where $su^{sd}(1,1)$ is the algebra closed by the pair operators formed out of the $s$ and $d$ bosons, while $su^d(1,1)$ is the algebra closed by the pair operators formed out of $d$ bosons alone. (Details on the basis for symmetric irreps of $su(1,1)\otimes o(5)$ can be found in \[214\].) The irreps of $su^{sd}(1,1)$ are characterized by the same quantum numbers as the irreps of $o(6)$ in the $o(6)$ chain of the IBM, while the irreps of $su^d(1,1)$ are
characterized by the same quantum numbers as the irreps of o(5) in the o(6) limit of IBM. Therefore in the Hamiltonian one can use the Casimir operators of the su\(^{\text{ad}}(1,1),\) su\(^{\text{ad}}(1,1)\) and su(2) subalgebras (the deformed versions of which are well known, as seen in secs 14–16) instead of the Casimir operators of o(6), o(5), o(3) respectively. Keeping the same notation as in eq. (26.3) the final result reads

\[ E(N, \sigma, \tau, \nu_\Delta, J, M_J) = E_0 + \beta 8 \left[ \frac{\tau}{2} \right]_q \frac{[\tau + 3]}{2}_q \]

\[ + \gamma 2[J]_q [J + 1]_q + \eta 8 \left[ \frac{\sigma}{2} \right]_q \frac{[\sigma + 4]}{2}_q, \]

(29.6)

where the free parameters have been chosen so that the present equation reduces to its classical counterpart for \(q \to 1\). For the ground state band then the analog of eq. (26.4) is

\[ E(J) = E_0 + \beta' 8[J]_{q'/4} [J + 6]_{q'/4} + \gamma 2[J]_q [J + 1]_q + \eta' 8[N]_{q'/2} [N + 4]_{q'/2}, \]

(29.7)

where the identities

\[ \left[ \frac{x}{2} \right]_q = [x]_{q^{1/2}} (q^{1/2} + q^{-1/2})^{-1}, \]

(29.8)

\[ \left[ \frac{x}{4} \right]_q = [x]_{q^{1/4}} (q^{1/4} + q^{-1/4})^{-1} (q^{1/4} + q^{-1/4})^{-1}, \]

(29.9)

have been used and \(\beta', \eta'\) are related to \(\beta, \eta\) and \(q\) in an obvious way. We remark that the Casimir operator of su\(^{\text{ad}}(1,1)\), which is complementary to o(5) in the undeformed case, leads to a term of the form \([J]_{q'} [J + 6]_{q'}\) with \(q' = q^{1/4}\).

In refs 214, 215 the question has been studied if large values of the deformation parameter can lead us from the o(6) limit to the su(3) (rotational) or u(5) (vibrational) limits, so that complete breaking of the symmetry, in the sense of sec. 28, could be obtained. It turns out that for \(q\) real the spectrum of the ground state band goes towards the rotational limit, while \(q\) being a phase factor leads towards the vibrational limit. Many more detailed studies, of both spectra and electromagnetic transition probabilities, are required before such a claim can be made.

A different method for constructing the \(q\)-deformed versions of the u(5) and o(6) limits of the IBM has been used by Pan 217. The method is based on the use of \(q\)-deforming functionals (see secs 10, 14). The same method has been used in 193 for studying the \(q\)-deformed version of the su(3) \(\supseteq so(3)\) decomposition. The final result for the energy eigenvalues in the o(6) case is similar to the one reported in eq. (29.6), the main difference being that different deformation parameters are allowed in each of the three deformed terms in the rhs. Some comparisons of the model predictions for spectra and B(E2) values to the experimental data have been performed 217.

It is clear that several deformed versions of the IBM can be constructed, providing us with a large number of exactly soluble models. In order to demonstrate their usefulness, one has to show that by deforming the model one gets some advantages over the classical (non-deformed) version. One way to achieve this is the use of
parameter-independent tests based on systematics of the data, like the ones used in 218, 219, 220 for the usual IBM. It is also desirable for the deformation parameter to be associated with some physical quantity, as in the case of the $su_q(2)$ model. Much work is still required in these directions. Some mathematical results which can be useful in these efforts are reported below:

i) Casimir operators for $su_q(n)$ have been given in 221, while the quadratic Casimir of $so_q(5)$ can be found in 222.

ii) Raising and lowering operators for $u_q(n)$ have been given by Quesne 223.

iii) Irreps of $u_q(m+n)$ in the $u_q(m) \oplus u_q(n)$ basis have been constructed in 224, while generalized $q$-bosonic operators acting in a tensor product of $m$ Fock spaces have been constructed as double irreducible tensors with respect to $u_q(m) \oplus u_q(n)$ in 225, 226.

30. Deformed versions of other collective models

The Moszkowski model 227 is a schematic two-level model which provides a description of the phase transition from the vibrational regime to the rotational one. A $q$-deformed version of the model has been developed 228 and the RPA modes in it have been discussed 229. Furthermore, the $q$-deformed Moszkowski model with cranking has been studied in the mean field approximation and the relation between $q$-deformation and temperature has been discussed 230. It should be noticed here that quantum algebraic techniques have also been found useful in describing thermal effects in the framework of the $q$-deformed Thouless model for superconductivity 231.

The Lipkin–Meshkov–Glick (LMG) model 232 is an exactly soluble schematic shell model. $q$-deformed versions of the 2-level LMG model (in terms of an $su_q(2)$ algebra) 233, 234, 235, 236, 237, and of the 3-level LMG model (in terms of an $su_q(3)$ algebra) 238 have been developed.

31. Fermion pairs as deformed bosons: approximate mapping

We have seen so far that several quantum algebraic phenomenological models have been proposed for the description of nuclear collective properties. These models make use of $q$-deformed bosons, which satisfy commutation relations differing from the standard boson commutation relations, to which they reduce in the limit $q \to 1$.

On the other hand, it is known that vibrational nuclear spectra, which are described in the simplest way by a pairing Hamiltonian, show anharmonicities (see also sec. 26), described, for example, by the Anharmonic Vibrator Model (AVM) 159

$$E(J) = aJ + bJ(J - 2). \quad (31.1)$$

In the framework of the single-j shell model 239, 240, 241, 242, 243, which can be extended to several non-degenerate j-shells 244, 245, 246, 247, these anharmonicities are related to the fact that correlated fermion pairs satisfy commutation relations which resemble boson commutation relations but in addition include corrections due to the presence of the Pauli principle. This fact has been the cause for the development of boson mapping
techniques (see the recent reviews by Klein and Marshalek \[2\] and Hecht \[24\] and references therein), by which the description of systems of fermions in terms of bosons is achieved. In recent years boson mappings have attracted additional attention in nuclear physics as a necessary tool in providing a theoretical justification for the success of the phenomenological Interacting Boson Model \[25\] and its various extensions, in which low lying collective states of medium and heavy mass nuclei are described in terms of bosons.

From the above observations it is clear that both q-bosons and correlated fermion pairs satisfy commutation relations which resemble the standard boson commutation relations but they deviate from them, due to the q-deformation in the former case and to the Pauli principle in the latter. A question is thus created: Are q-bosons suitable for the approximate description of correlated fermion pairs? In particular, is it possible to construct a boson mapping in which correlated fermion pairs are mapped onto q-bosons, in a way that the q-boson operators approximately satisfy the same commutation relations as the correlated fermion pair operators? In this section we show for the simple case of su(2) that such a mapping is indeed possible.

31.1. The single-j shell model

Let us consider the single-j shell model \[239, 240, 241, 242, 243\]. One can define fermion pair and multipole operators as

\[
A_{JM}^\dagger = \frac{1}{\sqrt{2}} \sum_{mm'} (jmjm'|JM) a_{jm}^\dagger a_{jm'}^\dagger,
\]

\[
B_{JM} = \frac{1}{\sqrt{2J+1}} \sum_{mm'} (jmj - m'|JM)(-1)^{j-m'} a_{jm}^\dagger a_{jm'},
\]

with the following definitions

\[
A_{JM} = [A_{JM}]^\dagger, \quad B_{JM} = [B_{JM}]^\dagger.
\]

In the above $a_{jm}^\dagger$ ($a_{jm}$) are fermion creation (annihilation) operators and $(jmjm'|JM)$ are the usual Clebsch–Gordan coefficients.

The pair and multipole operators given above satisfy the following commutation relations:

\[
[A_{JM}^\dagger, A_{J'M'}^\dagger] = 0,
\]

\[
[A_{JM}, A_{J'M'}^\dagger] = \delta_{JJ'} \delta_{MM'} - 2 \sum_{J''} (-1)^{2j+M} \sqrt{(2J+1)(2J'+1)(2J''+1)} (J - M, J'M'|J''M' - M) \left\{ \begin{array}{ccc} J & J' & J'' \\ j & j & j \end{array} \right\} B_{J''M' - M},
\]

\[
[B_{JM}^\dagger, A_{J'M'}^\dagger] = \sum_{J''} 2 \sqrt{2J'+1} (-1)^{2j-M} (J - M, J'M'|J''M' - M)
\]
\[
\begin{align*}
\{ J_j J'_j J''_j \} A_{j',M'-M} & \frac{1 + (-1)^{j''}}{2}, \\
[B_{jM}, B_{j'M'}] & = \sum_{j''} (-1)^{2j-j''} [1 - (-1)^{j+j'+j''}] \sqrt{2j'' + 1}
\end{align*}
\]

(31.7)

\[
\begin{align*}
\{ J_j J'_j J''_j \} A_{j',M'-M} & \frac{1 + (-1)^{j''}}{2}, \\
[B_{jM}, B_{j'M'}] & = \sum_{j''} (-1)^{2j-j''} [1 - (-1)^{j+j'+j''}] \sqrt{2j'' + 1}
\end{align*}
\]

(31.8)

in which the curly brackets are the usual 6-j symbols. These are the commutation relations of the so(2(2j+1)) algebra.

31.2. Fermion pairs of zero angular momentum

In the present subsection we will restrict ourselves to fermion pairs coupled to angular momentum zero. The relevant commutation relations take the form

\[
\begin{align*}
[A_0, A_0^\dagger] & = 1 - \frac{N_F}{\Omega}, \quad \frac{N_F}{2}, A_0^\dagger = A_0^\dagger, \quad \frac{N_F}{2}, A_0 = -A_0,
\end{align*}
\]

(31.9)

where \(N_F\) is the number of fermions, \(2\Omega = 2j + 1\) is the size of the shell, and

\[B_0 = \frac{N_F}{\sqrt{2\Omega}}.\]  

(31.10)

With the identifications

\[
\begin{align*}
J_+ & = \sqrt{\Omega} A_0^\dagger, \quad J_- = \sqrt{\Omega} A_0, \quad J_0 = \frac{N_F - \Omega}{2},
\end{align*}
\]

(31.11)

eqs (31.9) take the form of the usual su(2) commutation relations

\[
\begin{align*}
[J_+, J_-] & = 2J_0, \quad [J_0, J_+] = J_+, \quad [J_0, J_-] = -J_-.
\end{align*}
\]

(31.12)

An exact boson mapping of the su(2) algebra is given in [243, 60]

\[
\begin{align*}
A_0^\dagger & = a_0^\dagger \sqrt{1 - \frac{n_0}{\Omega}}, \quad A_0 = \sqrt{1 - \frac{n_0}{\Omega}} a_0, \quad N_F = 2n_0,
\end{align*}
\]

(31.13)

where \(a_0^\dagger (a_0)\) are boson creation (annihilation) operators carrying angular momentum zero and \(n_0\) is the number of these bosons.

The simplest pairing Hamiltonian one can consider has the form

\[
H = -G\Omega A_0^\dagger A_0.
\]

(31.14)

The Casimir operator of su(2) can be written as

\[
\{ A_0^\dagger, A_0 \} + \frac{\Omega}{2} \left(1 - \frac{N_F}{\Omega} \right)^2 = \frac{\Omega}{2} + 1,
\]

(31.15)
while the pairing energy takes the form
\[
\frac{E}{(-G\Omega)} = \frac{N_F}{2} - \frac{N_F^2}{4\Omega} + \frac{N_F}{2\Omega}.
\] (31.16)

Our aim is to check if there is a boson mapping for the operators \(A_0^\dagger, A_0\) and \(N_F\) in terms of \(q\)-deformed bosons, having the following properties:

i) The mapping is simpler than the one of eq. (31.13), i.e. to each fermion pair operator \(A_0^\dagger, A_0\) corresponds a bare \(q\)-boson operator and not a boson operator accompanied by a square root (the Pauli reduction factor).

ii) The commutation relations (31.9) are satisfied up to a certain order.

iii) The pairing energies of eq. (31.16) are reproduced up to the same order.

31.3. The \(q\)-deformed oscillator: A story of failure

In the case of the \(q\)-deformed harmonic oscillator (sec. 10), the commutation relation
\[
[a, a^\dagger] = [N + 1] - [N]
\] (31.17)
for \(q\) being a phase can be written as
\[
[a, a^\dagger] = \frac{\cos\left(\frac{(2N+1)\tau}{2}\right)}{\cos\frac{\tau}{2}}.
\] (31.18)

In physical situations \(\tau\) is expected to be small (i.e. of the order of 0.01). Therefore in eq. (31.18) one can take Taylor expansions of the functions appearing there and thus find an expansion of the form
\[
[a, a^\dagger] = 1 - \frac{\tau^2}{2} (N^2 + N) + \frac{\tau^4}{24} (N^4 + 2N^3 - N) - \ldots.
\] (31.19)

We remark that the first order corrections contain not only a term proportional to \(N\), but in addition a term proportional to \(N^2\), which is larger than \(N\). Thus one cannot make the simple mapping
\[
A_0 \rightarrow a, \quad A_0^\dagger \rightarrow a^\dagger, \quad N_F \rightarrow 2N,
\] (31.20)
because then one cannot get the first of the commutation relations (31.9) correctly up to the first order of the corrections. The same problem appears in the case that \(q\) is real as well. In addition, by making the simple mapping of eq. (31.20) the pairing Hamiltonian can be written as
\[
\frac{H}{-G\Omega} = a^\dagger a = [N].
\] (31.21)
In the case of small $\tau$, one can take Taylor expansions of the functions appearing in the definition of the $q$-numbers (eq. (2.2) or (2.3)) and thus obtain the following expansion

$$[N] = N \pm \frac{\tau^2}{6} (N - N^3) + \frac{\tau^4}{360} (7N - 10N^3 + 3N^5)$$

$$\pm \frac{\tau^6}{15120} (31N - 49N^3 + 21N^5 - 3N^7) + \ldots ,$$

where the upper (lower) sign corresponds to $q$ being a phase factor (real). We remark that while the first order corrections in eq. (31.16) are proportional to $N_F^2$ and $N_F$, here the first order corrections are proportional to $N$ and $N^3$. Thus neither the pairing energies can be reproduced correctly by this mapping.

### 31.4. The $Q$-deformed oscillator: A story of success

In the case of the $Q$-oscillator of sec. 11, however, the commutation relation among the bosons is

$$[b, b^\dagger] = Q^n.$$  \hfill (31.23)

Defining $Q = e^T$ this can be written as

$$[b, b^\dagger] = 1 + TN + \frac{T^2N^2}{2} + \frac{T^3N^3}{6} + \ldots.$$  \hfill (31.24)

We remark that the first order correction is proportional to $N$. Thus, by making the boson mapping

$$A_0^\dagger \rightarrow b^\dagger, \quad A_0 \rightarrow b, \quad N_F \rightarrow 2N,$$  \hfill (31.25)

one can satisfy the first commutation relation of eq. (31.9) up to the first order of the corrections by determining $T = -2/\Omega$.

We should now check if the pairing energies (eq. (31.16)) can be found correctly up to the same order of approximation when this mapping is employed. The pairing Hamiltonian in this case takes the form

$$\frac{H}{-G\Omega} = b^\dagger b = [N]_Q.$$  \hfill (31.26)

Defining $Q = e^T$ it is instructive to construct the expansion of the $Q$-number of eq. (6.1) in powers of $T$. Assuming that $T$ is small and taking Taylor expansions in eq. (6.1) one finally has

$$[N]_Q = N + \frac{T}{2} (N^2 - N) + \frac{T^2}{12} (2N^3 - 3N^2 + 1) + \frac{T^3}{24} (N^4 - 2N^3 + N^2) + \ldots.$$  \hfill (31.27)

Using the value of the deformation parameter $T = -2/\Omega$, determined above from the requirement that the commutation relations are satisfied up to first order corrections, the pairing energies become

$$\frac{E}{-G\Omega} = N - \frac{N^2 - N}{\Omega} + \frac{2N^3 - 3N^2 + 1}{3\Omega^2} - \frac{N^4 - 2N^3 + N^2}{3\Omega^3} + \ldots.$$  \hfill (31.28)
The first two terms in the rhs of eq. (31.28), which correspond to the leading term plus the first order corrections, are exactly equal to the pairing energies of eq. (31.16), since \( N_F \rightarrow 2N \). We therefore conclude that through the boson mapping of eq. (31.25) one can both satisfy the fermion pair commutation relations of eq. (31.9) and reproduce the pairing energies of eq. (31.16) up to the first order corrections.

The following comments are also in place:

i) By studying the spectra of the two versions of the \( q \)-deformed harmonic oscillator, given in eqs. (10.10) and (11.10), one can easily draw the following conclusions: when compared to the usual oscillator spectrum, which is equidistant, the spectrum of the \( q \)-oscillator is getting shrunk for \( q \) being a phase, while the spectrum of the \( Q \)-oscillator (where \( Q = e^{T} \)) gets shrunk when \( T < 0 \). In a similar way, the spectrum of the \( q \)-oscillator gets expanded for \( q \) real, while the spectrum of the \( Q \)-oscillator gets expanded for \( T > 0 \). In physical situations (secs 19–23) it has been found that the physically interesting results are gotten with \( q \) being a phase. Thus in the case of the \( Q \)-oscillator it is the case \( T < 0 \) the one which corresponds to the physically interesting case. As we have already seen, it is exactly for \( T = -2/\Omega < 0 \) that the present mapping gives the fermion pair results.

ii) It should be recalled that the pairing model under discussion is studied under the assumptions that the degeneracy of the shell is large (\( \Omega >> 1 \)), that the number of particles is large (\( N >> 1 \)), and that one stays away from the center of the shell (\( \Omega - N = O(N) \)). The accuracy of the present mapping in reproducing the pairing energies has been checked in \(^{248}\), where results for \( \Omega = 11 \) (the size of the nuclear fp major shell), \( \Omega = 16 \) (the size of the nuclear sdg major shell) and \( \Omega = 22 \) (the size of the nuclear pfh major shell) are reported, along with results for the case \( \Omega = 50 \) (as an example of a large shell). In all cases good agreement between the classical pairing model results and the \( Q \)-Hamiltonian of eq. (31.26) is obtained up to the point at which about 1/4 of the shell is filled. The deviations observed near the middle of the shell are expected, since there the expansion used breaks down.

We have thus shown that an approximate mapping of the fermion pairs coupled to angular momentum zero in a single-\( j \) shell onto suitably defined \( q \)-bosons (the \( Q \)-bosons) is possible. The \( \text{su}(2) \) commutation relations are satisfied up to the first order corrections, while at the same time the eigenvalues of a simple pairing Hamiltonian are correctly reproduced up to the same order. The small parameter of the expansion, which is \( T \) (where \( Q = e^{T} \)), turns out to be negative and inversely proportional to the size of the shell.

The present results are an indication that suitably defined \( q \)-bosons could be used for approximately describing systems of correlated fermions under certain conditions in a simplified way. The construction of \( q \)-bosons which would exactly satisfy the fermion pair \( \text{su}(2) \) commutation relations will be undertaken in the following section.

### 32. Fermion pairs as deformed bosons: exact mapping

From the contents of the previous section, the following question is created: Is it
possible to construct a generalized deformed oscillator (as in sec. 12) using deformed bosons in such a way that the spectrum of the oscillator will exactly correspond to the pairing energy in the single-j shell model, while the commutation relations of the deformed bosons will exactly correspond to the commutation relations of the correlated fermion pairs in the single-j shell under discussion? In this section we show that such an oscillator can indeed be constructed by using the method of sec. 12.

32.1. An appropriate generalized deformed oscillator

We apply the procedure of sec. 12 in the case of the pairing in a single-j shell mentioned before. The boson number is half the fermion number, i.e. \( N = N_F/2 \). Then eq. (31.16) can be written as

\[
\frac{E}{-G\Omega} = N - \frac{N^2}{\Omega} + \frac{N}{\Omega}. \tag{32.1}
\]

One can then use a generalized deformed oscillator with structure function

\[
F(N) = a^\dagger a = N - \frac{N^2}{\Omega} + \frac{N}{\Omega}. \tag{32.2}
\]

In addition one has

\[
F(N + 1) = aa^\dagger = N + 1 - \frac{(N + 1)^2}{\Omega} + \frac{N + 1}{\Omega}. \tag{32.3}
\]

What we have constructed is a boson mapping for the operators \( A_0, A_0^\dagger, N_F \):

\[
A_0 \rightarrow a, \quad A_0^\dagger \rightarrow a^\dagger, \quad N_F \rightarrow 2N. \tag{32.4}
\]

From eq. (32.2) it is clear that this mapping gives the correct pairing energy. In addition one has

\[
[a, a^\dagger] = F(N + 1) - F(N) = 1 - \frac{2N}{\Omega}, \tag{32.5}
\]

in agreement to the first commutation relation of eq. (31.9). Thus the correct commutation relations are also obeyed. (The last two commutation relations of eq. (31.9) are satisfied because of (12.1).)

As we have already seen in the previous section, an exact hermitian boson mapping for the su(2) algebra is known to have the form of eq. (31.13). In this mapping the Pauli principle effects are carried by the square roots accompanying the ordinary boson operators, while in the mapping of eq. (32.4) the Pauli principle effects are “built in” the deformed bosons.

The generalized oscillator obtained here has energy spectrum

\[
E_N = \frac{1}{2}(F(N) + F(N + 1)) = N + \frac{1}{2} - \frac{N^2}{\Omega}, \tag{32.6}
\]

which is the spectrum of an anharmonic oscillator.
32.2. Related potentials

The classical potential giving the same spectrum, up to first order perturbation theory, can be easily determined (see also subsecs 13.1, 13.2). The potential

\[ V(x) = \kappa x^2 + \lambda x^4, \]  

is known to give in first order perturbation theory the spectrum

\[ E_n = \kappa(2n + 1) + \lambda(6n^2 + 6n + 3) = (2\kappa + 6\lambda)(n + \frac{1}{2}) + 6\lambda n^2. \]  

(32.8)

Comparing eqs. (32.6) and (32.8) one finds

\[ \kappa = \frac{1}{2}(1 + \frac{1}{\Omega}), \quad \lambda = -\frac{1}{6\Omega}. \]  

(32.9)

Then the classical potential giving the same spectrum, up to first order perturbation theory, as the generalized oscillator determined here, is

\[ V(x) = \frac{1}{2}(1 + \frac{1}{\Omega})x^2 - \frac{1}{6\Omega}x^4. \]  

(32.10)

It is therefore demonstrated that the Pauli principle effects in a single-j shell with pairing interaction are equivalent to an \( x^4 \) anharmonicity.

The generalization of the results obtained in this section for the pairing hamiltonian to any anharmonic oscillator is straightforward. For example, the potential

\[ V(x) = \kappa x^2 + \lambda x^4 + \mu x^6 + \xi x^8, \]  

(32.11)

is known to give up to first order perturbation theory the spectrum

\[ E_n = \kappa(2n + 1) + \lambda(6n^2 + 6n + 3) + \mu(20n^3 + 30n^2 + 40n + 15) + \xi(70n^4 + 140n^3 + 350n^2 + 280n + 105), \]  

(32.12)

which can be rewritten in the form

\[ E_n = (n + (n + 1))(\kappa + 5\mu) + (n^2 + (n + 1)^2)(3\lambda + 70\xi) + (n^3 + (n + 1)^3)(10\mu) + (n^4 + (n + 1)^4)(35\xi). \]  

(32.13)

Taking into account eq. (12.11), from eq. (32.13) one gets

\[ \frac{F(N)}{2} = (\kappa + 5\mu)n + (3\lambda + 70\xi)n^2 + (10\mu)n^3 + (35\xi)n^4. \]  

(32.14)

For \( \mu = \xi = 0 \) and \( \kappa, \lambda \) given from eq. (32.9), the results for the pairing problem are regained.
It is worth mentioning at this point that the energy spectrum of the generalized oscillator corresponding to the pairing correlations (eq. (32.6)) can be rewritten as

\[ E_N = \frac{2}{\Omega} \left( -\frac{1}{8} + \frac{\Omega + 1}{2} \left( N + \frac{1}{2} \right) - \frac{1}{2} \left( N + \frac{1}{2} \right)^2 \right), \quad (32.15) \]

On the other hand, it is known that for the modified Pöschl–Teller potential (see also subsec. 13.2)

\[ V(x) = D \tanh^2(x/R), \quad (32.16) \]

the energy spectrum is given by

\[ E_N = \frac{\hbar^2}{mR^2} \left( -\frac{1}{8} + \frac{1}{2} \sqrt{8mDR^2/\hbar^2} + 1 \left( N + \frac{1}{2} \right) - \frac{1}{2} \left( N + \frac{1}{2} \right)^2 \right). \quad (32.17) \]

It is thus clear that the energy spectrum of the generalized oscillator studied here can be obtained from the modified Pöschl–Teller potential for special values of the potential depth \( D \).

It is also worth remarking that the “structure function” \( F(N) \) of the generalized oscillator obtained here (eq. (32.2)) can be written as

\[ F(N) = \frac{N}{\Omega} (\Omega + 1 - N), \quad (32.18) \]

which is similar to the one of the para-fermionic oscillator of Ohnuki and Kamefuchi (see also secs 12, 18).

In summary, we have constructed a generalized deformed oscillator which satisfies the same commutation relations as fermion pair and multipole operators of zero angular momentum in a single-j shell, and, in addition, reproduces the pairing energy of this shell exactly. We have thus demonstrated that an exact hermitian boson mapping of a system of angular-momentum-zero fermion pairs in terms of bare deformed bosons can be constructed, while in the usual case the ordinary bosons are accompanied by square roots due to the Pauli principle effects. The oscillator corresponding to the pairing problem has a spectrum which can be reproduced up to first order perturbation theory by a harmonic oscillator with an \( x^4 \) anharmonicity. The construction of a generalized deformed oscillator corresponding to any anharmonic oscillator has also been achieved.

The results obtained in this section indicate that deformed bosons might be a convenient tool for describing systems of fermion pairs under certain conditions. The generalisation of the results obtained here to fermion pairs of nonzero angular momentum, which will allow for a fuller treatment of the single-j shell in terms of deformed bosons, is a very interesting problem.
33. The seniority scheme

In the previous two sections we have seen how correlated fermion pairs of zero angular momentum can be mapped onto deformed bosons. It is however known that pairs of non-zero angular momentum play an important role in the formation of nuclear properties. In the present section a first step in the direction of describing the \( J \neq 0 \) pairs in terms of deformed bosons is taken.

33.1. Uncovering a dynamical symmetry

In the usual formulation of the theory of pairing in a single-j shell, fermion pairs of angular momentum \( J = 0 \) are created by the pair creation operators

\[
S^\dagger = \frac{1}{\sqrt{\Omega}} \sum_{m>0} (-1)^{j+m} a^\dagger_{jm} a^\dagger_{j-m},
\]

(33.1)

where \( a^\dagger_{jm} \) are fermion creation operators and \( 2\Omega = 2j + 1 \) is the degeneracy of the shell. In addition, pairs of nonzero angular momentum are created by the \( \Omega - 1 \) operators

\[
B^\dagger_J = \sum_{m>0} (-1)^{j+m}(jmj - m|J0)a^\dagger_{jm} a^\dagger_{j-m},
\]

(33.2)

where \( (jmj - m|J0) \) are the usual Clebsch Gordan coefficients. The fermion number operator is defined as

\[
N_F = \sum_m a^\dagger_{jm} a_{jm} = \sum_{m>0} (a^\dagger_{jm} a_{jm} + a^\dagger_{j-m} a_{j-m}).
\]

(33.3)

As we have already seen, the \( J = 0 \) pair creation and annihilation operators satisfy the commutation relation

\[
[S, S^\dagger] = 1 - \frac{N_F}{\Omega},
\]

(33.4)

while the pairing Hamiltonian is

\[
H = -G\Omega S^\dagger S.
\]

(33.5)

The seniority \( V_F \) is defined as the number of fermions not coupled to \( J = 0 \). If only pairs of \( J = 0 \) are present (i.e. \( V_F = 0 \)), the eigenvalues of the Hamiltonian are (as already seen in eq. (31.16))

\[
E(N_F, V_F = 0) = -G\Omega \left( \frac{N_F}{2} + \frac{N_F}{2\Omega} - \frac{N_F^2}{4\Omega} \right).
\]

(33.6)

For non-zero seniority the eigenvalues of the Hamiltonian are

\[
E(N_F, V_F) = -\frac{G}{4}(N_F - V_F)(2\Omega - N_F - V_F + 2).
\]

(33.7)
We denote the operators $N_F$, $V_F$ and their eigenvalues by the same symbol for simplicity.

In subsec. 31.4 it has been proved that the behaviour of the $J = 0$ pairs can be described, up to first order corrections, in terms of $Q$-bosons. In particular, making the mapping

$$S^\dagger \to b^\dagger, \quad S \to b, \quad N_F \to 2N,$$

(33.8)

the relevant pairing Hamiltonian of eq. (33.5) becomes

$$H(N, V = 0) = -G\Omega b^\dagger b = -G\Omega [N]_Q,$$

(33.9)

which coincides with eq. (33.6) up to first order corrections in the small parameter, which is identified as $T = -2/\Omega$. Furthermore, the $Q$-bosons satisfy the commutation relation of eq. (31.24), which coincides with eq. (33.4) up to first order corrections in the small parameter, which is, consistently with the above finding, identified as $T = -2/\Omega$. Therefore the fermion pairs of $J = 0$ can be approximately described as $Q$-bosons, which correctly reproduce both the pairing energies and the commutation relations up to first order corrections in the small parameter.

For the case of nonzero seniority, one observes that eq. (33.7) can be written as

$$E(N_F, V_F) = G\Omega \left( \frac{V_F}{2} + \frac{V_F^2}{4\Omega} \right) - G\Omega \left( \frac{N_F}{2} + \frac{N_F^2}{4\Omega} \right),$$

(33.10)

i.e. it can be separated into two parts, formally identical to each other. Since the second part (which corresponds to the $J = 0$ pairs) can be adequately described by the $Q$-bosons $b$, $b^\dagger$, and their number operator $N$, as we have already seen, it is reasonable to assume that the first part can also be described in terms of some $Q$-bosons $d$, $d^\dagger$, and their number operator $V$ (with $V_F \to 2V$), satisfying commutation relations similar to eqs (11.2) and (11.3):

$$[V, d^\dagger] = d^\dagger, \quad [V, d] = -d, \quad dd^\dagger - Qd^\dagger d = 1.$$

(33.11)

From the physical point of view this description means that a set of $Q$-bosons is used for the $J = 0$ pairs and another set for the $J \neq 0$ pairs. The latter is reasonable, since in the context of this theory the angular momentum value of the $J \neq 0$ pairs is not used explicitly. The $J \neq 0$ pairs are just counted separately from the $J = 0$ pairs. A Hamiltonian giving the same spectrum as in eq. (33.10), up to first order corrections in the small parameter, can then be written as

$$H(N, V) = G\Omega ([V]_Q - [N]_Q).$$

(33.12)

Using eq. (31.27) it is easy to see that this expression agrees to eq. (33.10) up to first order corrections in the small parameter $T = -2/\Omega$.

Two comments concerning eq. (33.12) are in place:

i) In the classical theory states of maximum seniority (i.e. states with $N = V$) have zero energy. This is also holding for the Hamiltonian of eq. (33.12) to all orders in the deformation parameter.
ii) A landmark of the classical theory is that $E(N, V) - E(N, V = 0)$ is independent of $N$. This also holds for eq. (33.12) to all orders in the deformation parameter.

Knowing the Schwinger realization of the $su_q(2)$ algebra in terms of $q$-bosons (sec. 15), one may wonder if the operators used here close an algebra. It is easy to see that the operators $b^\dagger d, d^\dagger b$ and $N - V$ do not close an algebra. Considering, however, the operators (see 31 with $p = 1$)

$$J_+ = b^\dagger Q^{-V/2}d, \quad J_- = d^\dagger Q^{-V/2}b, \quad J_0 = \frac{1}{2}(N - V),$$ \hfill (33.13)

one can easily see that they satisfy the commutation relations

$$[J_0, J_\pm] = \pm J_\pm, \quad J_+J_- - Q^{-1}J_-J_+ = [2J_0]_Q. \hfill (33.14)$$

Using the transformation

$$J_0 = \tilde{J}_0, \quad J_+ = Q^{(1/2)(J_0-1/2)}\tilde{J}_+, \quad J_- = \tilde{J}_-Q^{(1/2)(J_0-1/2)},$$ \hfill (33.15)

one goes to the usual $su_q(2)$ commutation relations

$$[\tilde{J}_0, \tilde{J}_\pm] = \pm \tilde{J}_\pm, \quad [\tilde{J}_+, \tilde{J}_-] = [2\tilde{J}_0], \hfill (33.16)$$

where $q^2 = Q$. One can thus consider eq. (33.14) as a rewriting of the algebra $su_q(2)$, suitable for boson realization in terms of $Q$-bosons.

It is clear that $N + V$ is the first order Casimir operator of the $u_Q(2)$ algebra formed above (since it commutes with all the generators given in eq. (33.13)), while $N - V$ is the first order Casimir operator of its $u_Q(1)$ subalgebra, which is generated by $J_0$ alone. Therefore the Hamiltonian of eq. (33.12) can be expressed in terms of the Casimir operators of the algebras appearing in the chain $u_Q(2) \supset u_Q(1)$ as

$$E(N, V) = G\Omega\left(\frac{C_1(u_Q(2)) - C_1(u_Q(1))}{2}Q - \frac{C_1(u_Q(2)) + C_1(u_Q(1))}{2}\right), \hfill (33.17)$$

i.e. the Hamiltonian has a $u_Q(2) \supset u_Q(1)$ dynamical symmetry.

33.2. Comparison to experiment

In the construction given above we have shown that $Q$-bosons can be used for the approximate description of correlated fermion pairs in a single-$j$ shell. The results obtained in the $Q$-formalism agree to the classical (non-deformed) results up to first order corrections in the small parameter. However, the $Q$-formalism contains in addition higher order terms. The question is then born if these additional terms are useful or not. For answering this question, the simplest comparison with experimental data which can be made concerns the classic example of the neutron pair separation energies of the Sn isotopes, used by Talmi 252,253.
In Talmi’s formulation of the pairing theory, the energy of the states with zero seniority is given by

$$E(N)_{cl} = NV_0 + \frac{N(N - 1)}{2} \Delta,$$

(33.18)

where \(N\) is the number of fermion pairs and \(V_0, \Delta\) are constants. We remark that this expression is the same as the one in eq. (33.6), with the identifications

$$\Delta / (2V_0) = -1/\Omega, \quad \Delta = 2G, \quad N_F = 2N.$$

(33.19)

The neutron pair separation energies are given by

$$\Delta E(N + 1)_{cl} = E(N + 1)_{cl} - E(N)_{cl} = V_0 \left( 1 + \frac{\Delta}{V_0} N \right).$$

(33.20)

Thus the neutron pair separation energies are expected to decrease linearly with increasing \(N\). (Notice from eq. (33.19) that \(\Delta/V_0 < 0\), since \(\Omega > 0\).) A similar linear decrease is predicted also by the Interacting Boson Model.

In our formalism the neutron pair separation energies are given by

$$\Delta E(N + 1)_Q = -G\Omega([N + 1]_Q - [N]_Q) = -G\Omega Q^N = -G\Omega e^{TN}.$$

(33.21)

Since, as we have seen, \(T\) is expected to be \(-2/\Omega\), i.e. negative and small, the neutron pair separation energies are expected to fall exponentially with increasing \(N\), but the small value of \(T\) can bring this exponential fall very close to a linear one.

The neutron pair separation energies of the even Sn isotopes from \(^{104}\text{Sn}\) to \(^{130}\text{Sn}\) (i.e. across the whole sdg neutron shell) have been fitted in eq. (33.21) using both theories. Furthermore, in a fit of the logarithms of the energies has been performed, since eq. (33.21) predicts a linear decrease of the logarithm of the energies with increasing \(N\). Both fits give almost identical results. Eq. (33.21) (in which the free parameters are \(G\Omega\) and \(T\)), gives a better result than eq. (33.20) (in which the free parameters are \(V_0\) and \(\Delta/V_0\)) for every single isotope, without introducing any additional parameter, indicating that the higher order terms can be useful.

One should, however, remark that \(^{116}\text{Sn}\) lies in the middle of the sdg neutron shell. Fitting the isotopes in the lower half of the shell (\(^{104}\text{Sn}\) to \(^{116}\text{Sn}\)) and the isotopes in the upper half of the shell (\(^{118}\text{Sn}\) to \(^{130}\text{Sn}\)) separately, one finds that both theories give indistinguishably good results in both regions. Therefore \(Q\)-deformation can be understood as expressing higher order correlations which manifest themselves in the form of particle-hole asymmetry. It is also known that a strong subshell closure exists at \(N=64\) (which corresponds to \(^{114}\text{Sn}\)). The presence of this subshell closure can also affect the neutron pair separation energies in a way similar to the one shown by the data.

In a fit of the neutron pair separation energies of the Pb isotopes from \(^{186}\text{Pb}\) to \(^{202}\text{Pb}\) has also been attempted. In this case both theories give indistinguishably good fits. This result is in agreement with the Sn findings, since all of these Pb isotopes lie in the upper half of the pfh neutron shell. Unfortunately, no neutron pair separation energy data exist for Pb isotopes in the lower part of the pfh neutron shell.
Concerning the values of $T$ obtained in the case of the Sn isotopes ($T = -0.0454, = -0.0447$), one observes that they are slightly smaller than the value ($T = -0.0488$) which would have been obtained by considering the neutrons up to the end of the sdg shell as lying in a single-j shell. This is, of course, a very gross approximation which should not be taken too seriously, since it ignores the fact that most properties of nuclei can be well accounted for by the valence nucleons alone, without being affected by the closed core. In the case of the Pb isotopes mentioned above, however, the best fit was obtained with $T = -0.0276$, which is again slightly smaller than the value of $T = -0.0317$ which corresponds to considering all the neutrons up to the end of the pfh shell as lying in a single-j shell.

In summary, we have shown that pairing in a single-j shell can be described, up to first order corrections, by two $Q$-oscillators, one describing the $J = 0$ pairs and the other corresponding to the $J \neq 0$ pairs, the deformation parameter $T = \ln Q$ being related to the inverse of the size of the shell. These two oscillators can be used for forming an $su_Q(2)$ algebra. A Hamiltonian giving the correct pairing energies up to first order corrections in the small parameter can be written in terms of the Casimir operators of the algebras appearing in the $u_Q(2) \supset u_Q(1)$ chain, thus exhibiting a quantum algebraic dynamical symmetry. The additional terms introduced by the $Q$-oscillators serve in improving the description of the neutron pair separation energies of the Sn isotopes, with no extra parameter introduced.

In the previous section a generalized deformed oscillator describing the correlated fermion pairs of $J = 0$ exactly has been introduced. This generalized deformed oscillator is the same as the one giving the same spectrum as the Morse potential (sec. 35), up to a shift in the energy spectrum. The use of two generalized deformed oscillators for the description of $J = 0$ pairs and $J \neq 0$ pairs in a way similar to the one of the present section is a straightforward task, while the construction out of them of a closed algebra analogous to the $su_Q(2)$ obtained here is an open problem. The extension of the ideas presented here to the case of the BCS theory is an interesting open problem.

### 33.3. Other approaches

A $q$-deformed version of the pairing theory was assumed by Petrova and Shelly Sharma, with satisfactory results when compared to experimental data. The present construction offers some justification for this assumption, since in both cases the basic ingredient is the modification of eq. (33.4). It should be noticed, however, that the deformed version of eq. (33.4) considered in is different from the one obtained here (eq. (31.24)). A basic difference is that the deformed theory of reduces to the classical theory for $q \to 1$, so that $q$-deformation is introduced in order to describe additional correlations, while in the present formalism the $Q$-oscillators involved for $Q \to 1$ reduce to usual harmonic oscillators, so that $Q$-deformation is introduced in order to attach to the oscillators the anharmonicity needed by the energy expression (eq. (33.6)).
Continueing along the same line Shelly Sharma and Sharma derived Random Phase Approximation (RPA) equations for the pairing vibrations of nuclei differing by two nucleons in comparison to the initial one and applied their method to the study of the $0^+$ states of the Pb isotopes, which offer a good example of pairing vibrations in nonsuperconducting nuclei. Furthermore, using deformed quasi-particle pairs coupled to zero angular momentum they developed a deformed version of the quasi-boson approximation for $0^+$ states in superconducting nuclei and tested it against a schematic two-level shell model. Another deformed two-level shell model has been developed by Avancini and Menezes.

34. Anisotropic quantum harmonic oscillators with rational ratios of frequencies

3-dim anisotropic harmonic oscillators with rational ratios of frequencies (RHOs) are of current interest because of their relevance as possible underlying symmetries of superdeformed and hyperdeformed nuclei. In particular, it is thought that superdeformed nuclei correspond to a ratio of frequencies of 2:1, while hyperdeformed nuclei correspond to a 3:1 ratio. In addition they have been recently connected to the underlying geometrical structure in the Bloch–Brink $\alpha$-cluster model, and possibly to the interpretation of the observed shell structure in atomic clusters, especially after the realization that large deformations can occur in such systems. The 2-dim RHO is also of interest, since its single particle level spectrum characterizes the underlying symmetry of “pancake” nuclei.

RHOs are examples of maximally superintegrable systems in N dimensions. Superintegrable systems in N dimensions have more than N independent integrals (constants of motion). Maximally superintegrable systems in N dimensions have $2N-1$ independent integrals.

The two-dim and three-dim anisotropic harmonic oscillators have been the subject of several investigations, both at the classical and the quantum mechanical level. The special cases with frequency ratios 1:2 and 1:3 have also been considered. While at the classical level it is clear that the su(N) or sp(2N,R) algebras can be used for the description of the N-dimensional anisotropic oscillator, the situation at the quantum level, even in the two-dimensional case, is not as simple.

In this section we are going to prove that a generalized deformed $u(2)$ algebra is the symmetry algebra of the two-dimensional anisotropic quantum harmonic oscillator, which is the oscillator describing the single-particle level spectrum of “pancake” nuclei, i.e. of triaxially deformed nuclei with $\omega_x >> \omega_y, \omega_z$. The method can be extended to the 3-dim RHO in a rather straightforward way.
34.1. A deformed $u(2)$ algebra

Let us consider the system described by the Hamiltonian:

$$H = \frac{1}{2} \left( p_x^2 + p_y^2 + \frac{x^2}{m^2} + \frac{y^2}{n^2} \right), \quad (34.1)$$

where $m$ and $n$ are two natural numbers mutually prime ones, i.e. their great common divisor is $\gcd(m, n) = 1$.

We define the creation and annihilation operators:

$$a^\dagger = \frac{x/m - ip_x}{\sqrt{2}}, \quad a = \frac{x/m + ip_x}{\sqrt{2}}, \quad (34.2)$$

$$b^\dagger = \frac{y/n - ip_y}{\sqrt{2}}, \quad b = \frac{y/n + ip_y}{\sqrt{2}}. \quad (34.3)$$

These operators satisfy the commutation relations:

$$[a, a^\dagger] = \frac{1}{m}, \quad [b, b^\dagger] = \frac{1}{n}, \quad \text{other commutators} = 0. \quad (34.4)$$

Further defining

$$U = \frac{1}{2} \left\{ a, a^\dagger \right\}, \quad W = \frac{1}{2} \left\{ b, b^\dagger \right\}, \quad (34.5)$$

one can consider the enveloping algebra generated by the operators:

$$S_+ = (a^\dagger)^m (b)'^n, \quad S_- = (a)^m (b^\dagger)^n, \quad (34.6)$$

$$S_0 = \frac{1}{2} \left( U - W \right), \quad H = U + W. \quad (34.7)$$

These generators satisfy the following relations:

$$[S_0, S_i] = \pm S_i, \quad [H, S_i] = 0, \quad \text{for } i = 0, \pm, \quad (34.8)$$

and

$$S_+ S_- = \prod_{k=1}^m \left( U - 2k - 1 \right) \prod_{\ell=1}^n \left( W + \frac{2\ell - 1}{2n} \right) \quad (34.9)$$

$$S_- S_+ = \prod_{k=1}^m \left( U + 2k - 1 \right) \prod_{\ell=1}^n \left( W - \frac{2\ell - 1}{2n} \right). \quad (34.10)$$

The fact that the operators $S_i, \ i = 0, \pm$ are integrals of motion has been already realized in [268].

The above relations mean that the harmonic oscillator of eq. (34.1) is described by the enveloping algebra of the generalization of the $u(2)$ algebra formed by the generators $S_0, S_+, S_-$ and $H$, satisfying the commutation relations of eq. (34.8) and

$$[S_-, S_+] = F_{m,n}(H, S_0 + 1) - F_{m,n}(H, S_0), \quad (34.11)$$

where $F_{m,n}$ are some functions.
where

\[ F_{m,n}(H, S_0) = \prod_{k=1}^{m} \left( \frac{H}{2} + S_0 - \frac{2k-1}{2m} \right) \prod_{\ell=1}^{n} \left( \frac{H}{2} - S_0 + \frac{2\ell-1}{2n} \right). \]  

(34.12)

In the case of \( m = 1, n = 1 \) this algebra is the usual \( u(2) \) algebra, and the operators \( S_0, S_\pm \) satisfy the commutation relations of the ordinary \( u(2) \) algebra, since in this case one easily finds that

\[ [S_-, S_+] = -2S_0. \]  

(34.13)

In the rest of the cases, the algebra is a deformed version of \( u(2) \), in which the commutator \([S_-, S_+]\) is a polynomial of \( S_0 \) of order \( m + n - 1 \). In the case with \( m = 1, n = 2 \) one has

\[ [S_-, S_+] = 3S_0^2 - HS_0 - \frac{H^2}{4} + \frac{3}{16}, \]  

(34.14)

i.e. a polynomial quadratic in \( S_0 \) occurs, while in the case of \( m = 1, n = 3 \) one finds

\[ [S_-, S_+] = -4S_0^3 + 3HS_0^2 - \frac{7}{9}S_0 - \frac{H^3}{4} + \frac{H}{4}, \]  

(34.15)

i.e. a polynomial cubic in \( S_0 \) is obtained.

34.2. The representations

The finite dimensional representation modules of this algebra can be found using the concept of the generalized deformed oscillator (sec. 12), in a method similar to the one used in for the study of quantum superintegrable systems. The operators:

\[ \mathcal{A}^\dagger = S_+, \quad \mathcal{A} = S_-, \quad \mathcal{N} = S_0 - u, \quad u = \text{constant}, \]  

(34.16)

where \( u \) is a constant to be determined, are the generators of a deformed oscillator algebra:

\[ [\mathcal{N}, \mathcal{A}^\dagger] = \mathcal{A}^\dagger, \quad [\mathcal{N}, \mathcal{A}] = -\mathcal{A}, \quad \mathcal{A}^\dagger \mathcal{A} = \Phi(H, \mathcal{N}), \quad \mathcal{A} \mathcal{A}^\dagger = \Phi(H, \mathcal{N} + 1). \]  

(34.17)

The structure function \( \Phi \) of this algebra is determined by the function \( F_{m,n} \) in eq. (34.12):

\[ \Phi(H, \mathcal{N}) = F_{m,n}(H, \mathcal{N} + u) = \prod_{k=1}^{m} \left( \frac{H}{2} + \mathcal{N} + u - \frac{2k-1}{2m} \right) \prod_{\ell=1}^{n} \left( \frac{H}{2} - \mathcal{N} - u + \frac{2\ell-1}{2n} \right). \]  

(34.18)

The deformed oscillator corresponding to the structure function of eq. (34.18) has an energy dependent Fock space of dimension \( N + 1 \) if

\[ \Phi(E, 0) = 0, \quad \Phi(E, N + 1) = 0, \quad \Phi(E, k) > 0, \quad \text{for } k = 1, 2, \ldots, N. \]  

(34.19)
The Fock space is defined by:
\[ H |E, k > = E |E, k >, \quad \mathcal{N} |E, k > = k |E, k >, \quad a |E, 0 > = 0, \quad (34.20) \]
\[ \mathcal{A}^\dagger |E, k > = \sqrt{\Phi(E, k + 1)} |E, k + 1 >, \quad \mathcal{A} |E, k > = \sqrt{\Phi(E, k)} |E, k - 1 >. \quad (34.21) \]
The basis of the Fock space is given by:
\[ |E, k > = \frac{1}{\sqrt{[k]!}} (\mathcal{A}^\dagger)^k |E, 0 >, \quad k = 0, 1, \ldots N, \quad (34.22) \]
where the “factorial” \([k]!\) is defined by the recurrence relation:
\[ [0]! = 1, \quad [k]! = \Phi(E, k) [k - 1]!. \quad (34.23) \]

Using the Fock basis we can find the matrix representation of the deformed oscillator and then the matrix representation of the algebra of eqs (34.8), (34.12). The solution of eqs (34.19) implies the following pairs of permitted values for the energy eigenvalue \(E\) and the constant \(u\):
\[ E = N + \frac{2p - 1}{2m} + \frac{2q - 1}{2n}, \quad (34.24) \]
where \(p = 1, 2, \ldots, m, \quad q = 1, 2, \ldots, n, \) and
\[ u = \frac{1}{2} \left( \frac{2p - 1}{2m} - \frac{2q - 1}{2n} - N \right), \quad (34.25) \]
the corresponding structure function being given by:
\[ \Phi(E, x) = \Phi^N_{(p,q)}(x) = \prod_{k=1}^m \left( x + \frac{2p - 1}{2m} - \frac{2k - 1}{2m} \right) \prod_{\ell=1}^n \left( N - x + \frac{2q - 1}{2n} + \frac{2\ell - 1}{2n} \right) \]
\[ = \frac{1}{m^{m} n^{n}} \frac{\Gamma(mx + p)}{\Gamma(mx + p - m)} \frac{\Gamma((N - x)n + q + n)}{\Gamma((N - x)n + q)}, \quad (34.26) \]
where \(\Gamma(x)\) denotes the usual Gamma-function. In all these equations one has \(N = 0, 1, 2, \ldots, \) while the dimensionality of the representation is given by \(N + 1\). Eq. (34.24) means that there are \(m \cdot n\) energy eigenvalues corresponding to each \(N\) value, each eigenvalue having degeneracy \(N + 1\). (Later we shall see that the degenerate states corresponding to the same eigenvalue can be labelled by an “angular momentum”.)

It is useful to show at this point that a few special cases are in agreement with results already existing in the literature.

i) In the case \(m = 1, n = 1\) eq. (34.26) gives
\[ \Phi(E, x) = x(N + 1 - x), \quad (34.27) \]
while eq. (34.24) gives
\[ E = N + 1, \quad (34.28) \]
in agreement with Sec. IV.A of [28].

ii) In the case $m = 1, n = 2$ one obtains for $q = 2$

$$
\Phi(E, x) = x(N + 1 - x) \left( N + \frac{3}{2} - x \right), \quad E = N + \frac{5}{4}, \quad (34.29)
$$

while for $q = 1$ one has

$$
\Phi(E, x) = x(N + 1 - x) \left( N + \frac{1}{2} - x \right), \quad E = N + \frac{3}{4}. \quad (34.30)
$$

These are in agreement with the results obtained in Sec. IV.F of [28] for the Holt potential (for $\delta = 0$).

iii) In the case $m = 1, n = 3$ one has for $q = 1$

$$
\Phi(E, x) = x(N + 1 - x) \left( N + \frac{1}{3} - x \right) \left( N + \frac{2}{3} - x \right), \quad E = N + \frac{2}{3}, \quad (34.31)
$$

while for $q = 2$ one obtains

$$
\Phi(E, x) = x(N + 1 - x) \left( N + \frac{2}{3} - x \right) \left( N + \frac{4}{3} - x \right), \quad E = N + 1, \quad (34.32)
$$

and for $q = 3$ one gets

$$
\Phi(E, x) = x(N + 1 - x) \left( N + \frac{4}{3} - x \right) \left( N + \frac{5}{3} - x \right), \quad E = N + \frac{4}{3}. \quad (34.33)
$$

These are in agreement with the results obtained in Sec. IV.D of [28] for the Fokas–Lagerstrom potential.

In all of the above cases we remark that the structure function has forms corresponding to various versions of the generalized deformed parafermionic algebra of eq. (18.1), the relevant conditions of eq. (18.2) being satisfied in all cases. It is easy to see that the obtained algebra corresponds to this of the generalized parafermionic oscillator in all cases with frequency ratios $1 : n$.

The energy formula can be corroborated by using the corresponding Schrödinger equation. For the Hamiltonian of eq. (34.1) the eigenvalues of the Schrödinger equation are given by:

$$
E = \frac{1}{m} \left( n_x + \frac{1}{2} \right) + \frac{1}{n} \left( n_y + \frac{1}{2} \right), \quad (34.34)
$$

where $n_x = 0, 1, \ldots$ and $n_y = 0, 1, \ldots$. Comparing eqs (34.24) and (34.34) one concludes that:

$$
N = [n_x/m] + [n_y/n], \quad (34.35)
$$

where $[x]$ is the integer part of the number $x$, and

$$
p = \text{mod}(n_x, m) + 1, \quad q = \text{mod}(n_y, n) + 1. \quad (34.36)
$$
The eigenvectors of the Hamiltonian can be parametrized by the dimensionality of the representation $N$, the numbers $p, q$, and the number $k = 0, 1, \ldots, N$. $k$ can be identified as $[n_x/m]$. One then has:

\[ H \left| \frac{N}{(p, q)}, k \right\rangle = \left( N + \frac{2p - 1}{2m} + \frac{2q - 1}{2n} \right) \left| \frac{N}{(p, q)}, k \right\rangle, \]

\[ S_0 \left| \frac{N}{(p, q)}, k \right\rangle = \left( k + \frac{1}{2} \left( \frac{2p - 1}{2m} - \frac{2q - 1}{2n} - N \right) \right) \left| \frac{N}{(p, q)}, k \right\rangle, \]

\[ S_+ \left| \frac{N}{(p, q)}, k \right\rangle = \sqrt{\Phi^N_{(p,q)}(k+1)} \left| \frac{N}{(p, q)}, k+1 \right\rangle, \]

\[ S_- \left| \frac{N}{(p, q)}, k \right\rangle = \sqrt{\Phi^N_{(p,q)}(k)} \left| \frac{N}{(p, q)}, k-1 \right\rangle. \]

34.3. The “angular momentum” quantum number

It is worth noticing that the operators $S_0, S_\pm$ do not correspond to a generalization of the angular momentum, $S_0$ being the operator corresponding to the Fradkin operator $S_{xx} - S_{yy}$. The corresponding “angular momentum” is defined by:

\[ L_0 = -i (S_+ - S_-). \]

The “angular momentum” operator commutes with the Hamiltonian:

\[ [H, L_0] = 0. \]

Let $|\ell\rangle$ be the eigenvector of the operator $L_0$ corresponding to the eigenvalue $\ell$. The general form of this eigenvector can be given by:

\[ |\ell\rangle = \sum_{k=0}^{N} \frac{i^k c_k}{\sqrt{|k|!}} \left| \frac{N}{(p, q)}, k \right\rangle. \]

In order to find the eigenvalues of $L_0$ and the coefficients $c_k$ we use the Lanczos algorithm, as formulated in \[284\]. From eqs (34.39) and (34.40) we find

\[ L_0|\ell\rangle = \ell|\ell\rangle = \ell \sum_{k=0}^{N} \frac{i^k c_k}{\sqrt{|k|!}} \left| \frac{N}{(p, q)}, k \right\rangle = \]

\[ = \frac{1}{i} \sum_{k=0}^{N-1} \frac{i^k c_k \sqrt{\Phi^N_{(p,q)}(k+1)}}{\sqrt{|k|!}} \left| \frac{N}{(p, q)}, k+1 \right\rangle - \frac{1}{i} \sum_{k=1}^{N} \frac{i^k c_k \sqrt{\Phi^N_{(p,q)}(k)}}{\sqrt{|k|!}} \left| \frac{N}{(p, q)}, k-1 \right\rangle. \]

(34.44)
From this equation we find that:

\[
c_k = (-1)^k 2^{-k/2} H_k(\ell/\sqrt{2})/N, \quad N^2 = \sum_{n=0}^{N} 2^{-n} H_n^2(\ell/\sqrt{2})/n!
\]  

(34.45)

where the function \(H_k(x)\) is a generalization of the “Hermite” polynomials (see also \(258\)), satisfying the recurrence relations:

\[
H_{-1}(x) = 0, \quad H_0(x) = 1,\]

(34.46)

\[
H_{k+1}(x) = 2xH_k(x) - 2\Phi^N_{(p,q)}(k)H_{k-1}(x),
\]

(34.47)

and the “angular momentum” eigenvalues \(\ell\) are the roots of the polynomial equation:

\[
H_{N+1}(\ell/\sqrt{2}) = 0.
\]

(34.48)

Therefore for a given value of \(N\) there are \(N + 1\) “angular momentum” eigenvalues \(\ell\), symmetric around zero (i.e. if \(\ell\) is an “angular momentum” eigenvalue, then \(-\ell\) is also an “angular momentum” eigenvalue). In the case of the symmetric harmonic oscillator \((m/n = 1/1)\) these eigenvalues are uniformly distributed and differ by 2. In the general case the “angular momentum” eigenvalues are non-uniformly distributed. For small values of \(N\) analytical formulae for the “angular momentum” eigenvalues can be found \(287\). Remember that to each value of \(N\) correspond \(m \cdot n\) energy levels, each with degeneracy \(N + 1\).

In order to have a formalism corresponding to the one of the isotropic oscillator, let us introduce for every \(N\) and \((m, n, p, q)\) an ordering of the “angular momentum” eigenvalues

\[
\ell_{\mu}^{L,m,n,p,q}, \quad \text{where} \quad L = N \quad \text{and} \quad \mu = -L, -L + 2, \ldots, L - 2, L,
\]

(34.49)

by assuming that:

\[
\ell_{\mu}^{L,m,n,p,q} \leq \ell_{\nu}^{L,m,n,p,q} \quad \text{if} \quad \mu < \nu,
\]

(34.50)

the corresponding eigenstates being given by:

\[
|L, \mu; m, n, p, q\rangle = \sum_{k=0}^{L} \frac{(-i)^k H_k(\ell^{L,m,n,p,q}_\mu/\sqrt{2})}{N^{2^{k/2}[k]!}} N^{(p,q)}_2 k \left( \frac{N}{(p,q)}, k \right) = \sum_{k=0}^{L} d_{k+1} \left( \frac{N}{(p,q)}, k \right)
\]

(34.51)

The above vector elements constitute the analogue corresponding to the basis of “spherical harmonic” functions of the usual oscillator. The calculation of the “angular momentum” eigenvalues of eq. (34.49) and the coefficients \(d_1, d_2, \ldots, d_{L+1}\) in the expansion of eq. (34.51) is a quite difficult task. The existence of general analytic expressions for these quantities is not obvious. The first few “angular momentum” eigenvalues are given by:

\[
\ell_{\pm 1}^{L,m,n,p,q} = \pm \sqrt{\frac{1}{m^m n^n} \frac{\Gamma(m+p)}{\Gamma(p)} \frac{\Gamma(n+q)}{\Gamma(q)}},
\]

(34.52)
and

\[ \ell_{0, m,n,p,q}^2 = 0, \]

(34.53)

\[ \ell_{\pm 2}^{m,n,p,q} = \pm \sqrt{\frac{1}{m^{m+n}} \left( \frac{\Gamma(m+p) \Gamma(2n+q)}{\Gamma(p) \Gamma(n+q)} + \frac{\Gamma(2m+p) \Gamma(n+q)}{\Gamma(m+p) \Gamma(q)} \right)} \]  

(34.54)

For \( L > 2 \) the analytic expressions of the angular momentum eigenvalues and the coefficients \( d_k \) are longer, but their calculation is a straightforward task. Numerical results for these quantities in the cases of frequency ratios 1:2 and 1:3 are given in 289.

After working out a few examples (see 289 for details) one finds out the following points:

i) In the basis described by eqs. (34.16)-(34.19) it is a trivial matter to distinguish the states belonging to the same irrep for any \( m:n \) ratio, while in the Cartesian basis this is true only in the 1:1 case.

ii) In the 1:2 case the irreps have degeneracies 1, 1, 2, 2, 3, 3, 4, 4, \ldots, i.e. “two copies” of the \( u(2) \) degeneracies 1, 2, 3, 4, \ldots are obtained.

iii) In the 1:3 cases the degeneracies are 1, 1, 1, 2, 2, 2, 3, 3, \ldots, i.e. “three copies” of the \( u(2) \) degeneracies are obtained.

iv) It can be easily seen that the 1:n case corresponds to “n copies” of the \( u(2) \) degeneracies.

v) Cases with both \( m, n \) different from unity show more complicated degeneracy patterns, also correctly reproduced by the above formalism. In the 2:3 case, for example, the degeneracy pattern is 1, 1, 1, 1, 1, 2, 1, 2, 2, 2, 3, 2, 3, 3, \ldots.

vi) The only requirement for each energy eigenvalue to correspond to one irrep of the algebra is that \( m \) and \( n \) have to be mutually prime numbers. If \( m \) and \( n \) possess a common divisor other than 1, then some energy eigenvalues will correspond to sums of irreps, i.e. to reducible representations.

vii) The difference between the formalism used in 274, 275, 276, 281 and the one used here is that in the former case for given \( m \) and \( n \) appropriate operators have to be introduced separately for each set of \( (p,q) \) values, while in the present case only one set of operators is introduced.

34.4. Multisections of the isotropic oscillator

In 289 the concept of bisection of an isotropic harmonic oscillator has been introduced. One can easily see that multisections (trisections, tetrasections, \ldots) can be introduced in a similar way. The degeneracies of the various anisotropic oscillators can then be obtained from these of the isotropic oscillator by using appropriate multisections.

Using the Cartesian notation \( (n_x, n_y) \) for the states of the isotropic harmonic oscillator we have the following list:

\[ N=0: \quad (00) \]
\[ N=1: \quad (10) \quad (01) \]
N=2: (20) (02) (11)
N=3: (30) (03) (21) (12)
N=4: (40) (04) (31) (13) (22)
N=5: (50) (05) (41) (14) (32) (23),
where \( N = n_x + n_y \). The corresponding degeneracies are 1, 2, 3, 4, 5, 6, \ldots, i.e. these of \( u(2) \).

A bisection can be made by choosing only the states with \( n_y \)=even. Then the following list is obtained:

| N=0 | 00 |
| N=1 | 10 |
| N=2 | 20, 02 |
| N=3 | 30, 12 |
| N=4 | 40, 04, 22 |
| N=5 | 50, 14, 32 |

The degeneracies are 1, 1, 2, 2, 3, 3, \ldots, i.e. these of the anisotropic oscillator with ratio of frequencies 1:2. The same degeneracies are obtained by choosing the states with \( n_y \)=odd. Therefore a **bisection** of the isotropic oscillator, distinguishing states with \( \text{mod}(n_y,2) = 0 \) and states with \( \text{mod}(n_y,2) = 1 \), results in two interleaving sets of levels of the 1:2 oscillator.

By analogy, a **trisection** can be made by distinguishing states with \( \text{mod}(n_y,3) = 0 \), or \( \text{mod}(n_y,3) = 1 \), or \( \text{mod}(n_y,3) = 2 \). One can easily see that in this case three interleaving sets of states of the 1:3 oscillator, having degeneracies 1, 1, 1, 2, 2, 2, 3, 3, \ldots, occur.

Similarly a **tetrasection** can be made by distinguishing states with \( \text{mod}(n_y,4) = 0 \), or \( \text{mod}(n_y,4) = 1 \), or \( \text{mod}(n_y,4) = 2 \), or \( \text{mod}(n_y,4) = 3 \). The result is four interleaving sets of states of the 1:4 oscillator, having degeneracies 1, 1, 1, 1, 2, 2, 2, 2, 2, 3, 3, 3, 3, \ldots, i.e. degeneracies of the 2:3 oscillator.

By bisecting \( n_x \) and trisecting \( n_y \) one is left with six interleaving sets of states with degeneracies 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, \ldots, i.e. degeneracies of the 2:3 oscillator.

By bisecting (or trisecting, tetresecting, etc) both \( n_x \) and \( n_y \) one is obtaining the original \( u(2) \) degeneracies of the isotropic oscillator.

It is therefore clear that the degeneracies of all \( m:n \) oscillators can be obtained from these of the isotropic oscillator by appropriate multisections. In particular:

i) The degeneracies of the 1 : \( n \) oscillator can be obtained from these of the 1:1 (isotropic) oscillator by \( n \)-secting \( n_y \) or \( n_x \).

ii) The degeneracies of the \( m:n \) oscillator can be obtained from these of the 1:1 oscillator by \( m \)-secting \( n_x \) and \( n \)-secting \( n_y \).

### 34.5. Connection to \( W_3^{(2)} \)

For the special case \( m = 1, n = 2 \) it should be noticed that the deformed algebra received here coincides with the finite \( W \) algebra \( W_3^{(2)} \). The commutation
relations of the $W^{(2)}_3$ algebra are

$$[H_W, E_W] = 2E_W, \quad [H_W, F_W] = -2F_W, \quad [E_W, F_W] = H_W^2 + C_W,$$

$$[C_W, E_W] = [C_W, F_W] = [C_W, H_W] = 0,$$

while in the $m = 1, n = 2$ case one has the relations

$$[\mathcal{N}, \mathcal{A}^\dagger] = \mathcal{A}^\dagger, \quad [\mathcal{N}, \mathcal{A}] = -\mathcal{A}, \quad [\mathcal{A}, \mathcal{A}^\dagger] = 3S_0^2 - \frac{H^2}{4} - HS_0 + \frac{3}{16},$$

$$[H, \mathcal{A}^\dagger] = [H, \mathcal{A}] = [H, S_0] = 0,$$

with $S_0 = \mathcal{N} + u$ (where $u$ a constant). It is easy to see that the two sets of commutation relations are equivalent by making the identifications

$$F_W = \sigma \mathcal{A}^\dagger, \quad E_W = \rho \mathcal{A}, \quad H_W = -2S_0 + kH, \quad C_W = f(H),$$

with

$$\rho \sigma = \frac{4}{3}, \quad k = \frac{1}{3}, \quad f(H) = -\frac{4}{9}H^2 + \frac{1}{4}.$$

34.6. Discussion

In conclusion, the two-dimensional anisotropic quantum harmonic oscillator with rational ratio of frequencies equal to $m/n$, is described dynamically by a deformed version of the $u(2)$ Lie algebra, the order of this algebra being $m + n - 1$. The representation modules of this algebra can be generated by using the deformed oscillator algebra. The energy eigenvalues are calculated by the requirement of the existence of finite dimensional representation modules. An “angular momentum” operator useful for labelling degenerate states has also been constructed. The algebras obtained in the special cases with $1 : n$ ratios are shown to correspond to generalized parafermionic oscillators. In the special case of $m : n = 1 : 2$ the resulting algebra has been identified as the finite W algebra $W^{(2)}_3$. Finally, it is demonstrated how the degeneracies of the various $m : n$ oscillators can be obtained from these of the isotropic oscillator by appropriate multisections.

The extension of the present method to the three-dimensional anisotropic quantum harmonic oscillator is already receiving attention, since it is of clear interest in the study of the symmetries underlying the structure of superdeformed and hyperdeformed nuclei [253, 260].

35. The use of quantum algebras in molecular structure

The techniques developed in this article can be applied in very similar ways in describing properties of diatomic and polytomic molecules. A brief list will be given here.
1) Rotational spectra of diatomic molecules have been described in terms of the $\text{su}(2)$ model \cite{108,293,296,297,298}. As in the case of nuclei, $q$ is a phase factor ($q = e^{i\tau}$). In molecules $\tau$ is of the order of 0.01. The use of the $\text{su}(2)$ symmetry leads to a partial summation of the Dunham expansion describing the rotational–vibrational spectra of diatomic molecules \cite{294}. Molecular backbending (bandcrossing) has also been described in this framework \cite{292}. Rotational spectra of symmetric top molecules have also been considered \cite{300,301} in the framework of the $\text{su}(2)$ symmetry.

2) Vibrational spectra of diatomic molecules have been described in terms of $q$-deformed anharmonic oscillators having the $\text{su}_q(1,1)$ \cite{302} or the $u_q(2) \supset o_q(2)$ \cite{303} symmetry, as well as in terms of generalized deformed oscillators similar to the ones used in sec. 26 \cite{26,299,300,301}. These results, combined with 1), lead to the full summation of the Dunham expansion \cite{303}. A two-parameter deformed anharmonic oscillator with $u_{qp}(2) \supset o_{qp}(2)$ symmetry has also been considered \cite{307}.

3) The physical content of the anharmonic oscillators mentioned in 2) has been clarified by constructing WKB equivalent potentials (WKB-EPs) \cite{309,310} and classical equivalent potentials \cite{4}, similar to the ones of sec. 13. The results have been corroborated by the study of the relation between $\text{su}_q(1,1)$ and the anharmonic oscillator with $x^4$ anharmonicities \cite{308}. The WKB-EP corresponding to the $\text{su}_q(1,1)$ anharmonic oscillator has been connected to a class of Quasi-Exactly Soluble Potentials (QESPs) \cite{310}.

4) Generalized deformed oscillators giving the same spectrum as the Morse potential \cite{311} and the modified Pöschl–Teller potential \cite{312}, as well as a deformed oscillator containing them as special cases \cite{313,314} have also been constructed. In addition, $q$-deformed versions of the Morse potential have been given, either by using the $\text{so}_q(2,1)$ symmetry \cite{315} or by solving a $q$-deformed Schrödinger equation for the usual Morse potential \cite{316}.

5) A $q$-deformed version of the vibron model for diatomic molecules has been constructed \cite{317,318,319,320} in a way similar to that described in sec. 29.

6) For vibrational spectra of polyatomic molecules a model of $n$ coupled generalized deformed oscillators has been built \cite{224}, containing the approach of Iachello and Oss \cite{221,222} as a special case.

7) Quasi-molecular resonances in the systems $^{12}\text{C} + ^{12}\text{C}$ and $^{12}\text{C} + ^{16}\text{O}$ have been described in terms of a $q$-deformed oscillator plus a rigid rotator \cite{323}.

A review of several of the above topics, accompanied by a detailed and self-contained introduction to quantum algebras, has been given by Raychev \cite{324}.

36. Outlook

Nobody likes binding himself by statements concerning the future. However, we attempt to give here a partial list of open problems, roughly following the order of the material in this review:

1) The list of physical systems which can be classified under a generalized deformed $\text{su}(2)$ symmetry (sec. 17) or under a generalized deformed parafermionic oscillator...
scheme (sec. 18) can be enlarged. Self-similar potentials and isospectral oscillator Hamiltonian systems could probably be related to these symmetries.

2) The description of B(E2) values in terms of the su_q(2) model attempted so far (sec. 21) takes into account only the kinematical deformation effects. In order to take into account dynamical deformation effects, one has to build a larger algebra, of which the quadrupole operators will be members. (This will then probably be a deformed version of an su(3) algebra.) These quadrupole operators should behave as irreducible tensors of rank 2 under su_q(2).

3) The su_q(2) prediction about B(E2) values increasing with increasing angular momentum J, supported by the predictions of other models as well (see sec. 21), requires further testing against detailed experimental data.

4) The construction of Clebsch-Gordan coefficients for the subclass of generalized deformed su(2) algebras for which this could be possible (see secs 17, 25) is an open problem.

5) The su_q(3) ⊃ so_q(3) decomposition for su_q(3) irreps other than the completely symmetric ones (see subsec. 27.3 for the current state of the art) remains an open problem, the solution of which is necessary for developing a deformed version of the su(3) limit of the Interacting Boson Model.

6) Realizations of multi-level shells models in terms of deformed bosons (see secs 31–33 for some one- and two-level cases) should be further pursued.

7) The symmetry algebras of the various 3-dim anisotropic harmonic oscillators with rational ratios of frequencies should be worked out, since they are of interest in relation to superdeformed and hyperdeformed nuclei, and possibly to deformed atomic clusters (see sec. 34 for references). A deformed u(3) algebra should occur in this case, which could serve as the basis for building a deformed analog of the Elliott model suitable for superdeformed nuclei.

8) In molecular physics (sec. 35) the study of vibrations of highly symmetric polyatomic molecules (including fullerenes) by these techniques is of interest.

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