Generalized massive gravity and Galilean conformal algebra in two dimensions

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Abstract – Galilean conformal algebra (GCA) in two dimensions arises as contraction of two copies of the centrally extended Virasoro algebra \((t \rightarrow t, x \rightarrow \epsilon x)\) with \(\epsilon \rightarrow 0\). The central charges of GCA can be expressed in terms of Virasoro central charges. For finite and non-zero GCA central charges, the Virasoro central charges must behave as the asymmetric form \(O(1) \pm O(\frac{1}{2})\). We propose that the bulk description for 2d GCA with asymmetric central charges is given by general massive gravity (MG) in three dimensions. It can be seen that, if the gravitational Chern-Simons coupling \(\frac{1}{4}\) behaves as if it were of order \((O(\varphi^2))\) or \((\mu \rightarrow \epsilon \mu)\), the central charges of MG have the above \(\epsilon\)-dependence. So, in the non-relativistic scaling limit \(\mu \rightarrow \epsilon \mu\), we calculated GCA parameters and finite entropy in terms of gravity parameters mass and angular momentum of MG.

Introduction. – The AdS/CFT correspondence [1] has been considered during the past decade. Recently, the non-relativistic version of the duality between gravity theory and boundary quantum field theory has attracted a lot of attention. One of the non-relativistic conformal symmetries is the Schrödinger symmetry group [2]. This group is the largest symmetry of the Schrödinger equation (without potential term), and it can be seen in cold-atoms systems [3]. The gravity dual for field theory with this symmetry was proposed in [4] and [5]. In this paper, we will consider another version of non-relativistic conformal group, named Galilean conformal (GC) group, which arises as parametric contraction of a conformal group \((t \rightarrow t, x_i \rightarrow \epsilon x_i, \text{with} \epsilon \rightarrow 0)\). Unlike the Schrödinger group, the GC group can be given an infinite extension in any space-time dimensions. The generators of this group are \(L^n = -(n + 1) t^n x_i \partial_i - t^{n+1} \partial_n\), \(M^n_i = t^{n+1} \partial_i\) and \(J^n_{ij} = -t^n (x_i \partial_j - x_j \partial_i)\) for an arbitrary integer \(n\), where \(i\) and \(j\) are specified by the spatial directions [6] (see also [7]).

The galilean conformal algebra (GCA) was explored in two dimensions by two copies of Virasoro algebra in the non-relativistic limit [8]. At the quantum level, 2d CFT was constructed by two copies of centrally extended Virasoro algebra, so in the non-relativistic limit, we can also consider the quantum aspects of the centrally extended GCA. The central charges of GCA \((C_1, C_2)\) can be expressed in terms of CFT central charges \((c_L, c_R)\), and it is known that GCA central charges are asymmetric [8]. For finite and non-zero \((C_1, C_2)\) the Virasoro central charges must behave as \(c_L \sim O(1) + O(\frac{1}{2})\) and \(c_R \sim O(1) - O(\frac{1}{2})\).

In this paper we propose the gravity dual for 2d GCA. The gravity dual of the GCA was given in terms of Newton-Cartan-like AdS\(_2\times R^d\) [7]. One option for the bulk description of 2d GCA, with asymmetric central charges would be general massive gravity (MG) in three dimensions [9,10]. Massive gravity in 3d was initiated by topological massive gravity (TMG) [11], where the action of this theory is realized by the usual Einstein-Hilbert (EH) term, which includes the cosmological constant and parity-violating Chern-Simons term, with coupling \(\frac{1}{4}\). In TMG only a single mode of helicity 2 and mass \(\mu\) propagates. At \(\mu \neq 1\) the perturbative massive modes around the AdS background have negative energy and the theory is unstable; however, Strominger et al. claimed [12] that at \(\mu = 1\) there are no negative energy modes and the theory is stable. In fact in the paper [13] it was shown that TMG at \(\mu = 1\) is dual to a logarithmic CFT and so it is not unitary. Massive gravity in three dimensions was extended by a new kind of three-dimensional massive gravity (NMG) [9,10]. NMG is defined by adding a higher-derivative term to the EH term in action, with

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cghting \( \frac{1}{m^2} \). So this model is described by the parity-

invariant action, which yields fourth-order field equations for the metric. In NGM, as in the TMG, the linearized excitation about the Minkowski vacuum describe a propagating massive graviton [14]. These gravitons have two polarization states of helicity ±2 and mass m [15]. Finally, in general massive gravity (GMG) the action contains, the EH term, the gravitational CS term and higher-
derivative terms of NGM. GMG violate parity and the ±2 helicity modes propagate with different masses. The general massive gravity central charges are asymmetric \( (c_{L,R} = \frac{m}{2}(1 - \frac{1}{m^2} \pm \frac{1}{m^3}) \). From these central charges, in the non-relativistic limit \( (\mu \rightarrow \epsilon \mu) \), the GCA central charges \( C_1 \) and \( C_2 \) are finite. In the following we shall see that the central charges of GCA, the scaling dimension \( \Delta \) and rapidity \( \xi \) (which are the eigenvalues of \( L_0 \) and \( M_0 \) in the non-relativistic limit \( (\mu \rightarrow \epsilon \mu, J \rightarrow \epsilon J) \) are finite and expressed by gravity parameters \( (M, J, \mu, m^2) \). Also, the entropy in the non-relativistic limit \( (M \rightarrow M, J \rightarrow \epsilon J, \mu \rightarrow \epsilon \mu) \) is finite and can be expressed in terms of GCA parameters \( (C_1, C_2, \Delta, \xi) \). The rest of the paper is organized as follows: in the second section we give a brief review of 2d CFT and its contraction, the GCA parameters were realized in terms of CFT parameters. In the third section we propose that the general massive gravity is the gravity dual of 2d GCA in the non-relativistic limit \( (\mu \rightarrow \epsilon \mu) \); in this section GCA parameters were constructed in terms of gravity parameters and finally we obtained finite entropy in the non-relativistic limit. The last section is devoted to the conclusions.

**GCA in 2d.** – Galilean conformal algebra in 2d can be described by the action of centrally extended Virasoro algebra. In two dimensions space-time \( (z = x + t, \tau = x - t) \), the CFT generators

\[
\mathcal{L}_n = -z^{n+1}\partial_z, \quad \mathcal{Z}_n = -z^{n+1}\partial_\tau,
\]

obey the centrally extended Virasoro algebra,

\[
[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n}, \quad [\mathcal{L}_m, \mathcal{Z}_n] = (m-n)\mathcal{Z}_{m+n} + \frac{c_G}{12} m(m^2 - 1)\delta_{m+n,0}, \quad [\mathcal{Z}_m, \mathcal{Z}_n] = (m-n)\mathcal{Z}_{m+n} + \frac{c_L}{12} m(m^2 - 1)\delta_{m+n,0}.
\]

By taking the non-relativistic limit \( (t \rightarrow t, x \rightarrow \epsilon x \) with \( \epsilon \rightarrow 0) \), the GCA generators \( L_n \) and \( M_n \) are constructed from Virasoro generators by

\[
L_n = \lim_{\epsilon \rightarrow 0} (\mathcal{L}_n + \mathcal{Z}_n) = -(n+1)\epsilon z^{n+1}\partial_z - \epsilon z^{n+1}\partial_\tau,
\]

\[
M_n = \lim_{\epsilon \rightarrow 0} \epsilon (\mathcal{L}_n - \mathcal{Z}_n) = (n+1)\epsilon z^{n+1}\partial_z.
\]

From eqs. (2) and (3), one obtains centrally extended 2d GCA,

\[
[L_m, L_n] = (m-n)\mathcal{L}_{m+n} + C_1 m(m^2 - 1)\delta_{m+n,0}, \quad [L_m, M_n] = (m-n)\mathcal{M}_{m+n} + C_2 m(m^2 - 1)\delta_{m+n,0}, \quad [M_n, M_m] = 0.
\]

The GCA central charges \( (C_1, C_2) \) are related to CFT central charges \( (c_L, c_R) \) as

\[
C_1 = \lim_{\epsilon \rightarrow 0} \frac{c_L + c_R}{12}, \quad C_2 = \lim_{\epsilon \rightarrow 0} \left( \epsilon \frac{c_L - c_R}{12} \right).
\]

From the above equations, for a non-zero and finite \( (C_2, C_1) \) in the limit \( \epsilon \rightarrow 0 \), it can be seen that we need \( c_L - c_R \approx O(\frac{1}{\epsilon}) \) and \( c_L + c_R \approx O(1) \). Similarly, rapidity \( \xi \) and scaling dimensions \( \Delta \), which are the eigenvalues of \( M_0 \) and \( L_0 \), respectively, are given by

\[
\Delta = \lim_{\epsilon \rightarrow 0} (h + \bar{h}), \quad \xi = \lim_{\epsilon \rightarrow 0} (h - \bar{h}),
\]

where \( h \) and \( \bar{h} \) are eigenvalues of \( L_0 \) and \( Z_0 \), respectively. Equation (6) tells us that \( h + \bar{h} \) is of order \( O(1) \), while \( h - \bar{h} \) must be of order \( O(\frac{1}{\epsilon}) \), for the finite \( \Delta \), \( \xi \).

**GCA realization of general massive gravity.** – In this section we would like to propose that the contracted BTZ black-hole solution of three-dimensional general massive gravity (GMG) is the gravity dual of 2d GCA in the context of AdS/CFT correspondence. It is notable that the GMG (as a gravity dual) has to yield finite parameters \( (\Delta, \xi, C_1, C_2 \) and entropy \( S_{GCA} \) for GCA. The action of the cosmological general massive gravity in three dimensions is [9]

\[
S[g_{\mu\nu}] = \frac{1}{16\pi G} \int \sqrt{-g} \left( R - 2\lambda m^2 + \frac{1}{m^2} \mathcal{L}_{NGM} + \frac{1}{2} \mathcal{L}_{CS} \right) dx^3,
\]

where the NGM term is

\[
\mathcal{L}_{NGM} = R_{\mu\nu} R^{\mu\nu} - \frac{3}{8} R^2,
\]

and the gravitational Chern-Simons (CS) term is

\[
\mathcal{L}_{CS} = \frac{1}{2} \epsilon^{\mu\nu\rho}(\Gamma^\alpha_{\mu\beta} \partial_\nu \Gamma^\beta_{\rho\alpha} + \frac{2}{3} \Gamma^\alpha_{\mu\beta} \Gamma^\beta_{\rho\alpha} \Gamma^\gamma_{\mu\rho}).
\]

The Einstein equation of motion of this action is

\[
G_{\mu\nu} + \lambda m^2 g_{\mu\nu} + \frac{1}{2m^2} K_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} = 0,
\]

where \( G_{\mu\nu} \) is the Einstein tensor, the tensor \( C_{\mu\nu} \) (Cotton tensor), due to the gravitational CS term, reads

\[
C_{\mu\nu} = \epsilon^{0\alpha\rho} D_\alpha (R_{\beta\nu} - \frac{1}{4} R g_{\beta\nu}), \quad \epsilon^{012} = 1,
\]

and the tensor \( K_{\mu\nu} \), coming from NGM term, is

\[
K_{\mu\nu} = 2D^2 R_{\mu\nu} - \frac{1}{2} (D_\rho D_\nu R + g_{\mu\nu} D^2 R) - 8R_{\rho\alpha} R^\alpha_{\mu\nu} + g_{\mu\nu} (3R^2 R - 13R_{\rho\beta} R_{\rho\beta} + 18R_{\rho\mu\beta\gamma} R^\beta_{\rho\mu\gamma}).
\]
The reason of this choice of $L_{NMG}$ is that $g^{\mu \nu}K_{\mu \nu} = L_{NMG}$. The parameter $\lambda$ is dimensionless and characterizes the cosmological constant term, while $m$ has the dimension of the mass and provides the coupling to the NMG term, also CS coupling $\mu$ has the dimension of the mass. The solution of BTZ black hole is given by

$$ds^2 = \left(-f(r) + \frac{16G^2J^2}{r^2}\right)dt^2 + \frac{dr^2}{f(r)} + r^2d\phi^2 + 8GJdtd\phi,$$

where

$$f(r) = \left(\frac{r^2}{l^2} - 8GM + \frac{16G^2J^2}{r^2}\right).$$

The parameters $M$ and $J$ correspond to the mass and angular momentum in the case without the terms $L_{NMG}$ and $L_{CS}$, but their definitions in the case with these terms are [16]

$$M_1 = \left(1 - \frac{1}{2m^2l^2}\right)\left(M + \frac{1}{\mu l^2}\right),$$

$$J_1 = \left(1 - \frac{1}{2m^2l^2}\right)\left(J + \frac{1}{\mu l^2}\right),$$

where $\mu' = \alpha \mu$ and $\alpha = (1 - \frac{1}{2m^2l^2})$. Due to the NMG and CS terms, the Bekenstein-Hawking entropy is given by

$$S_{BH} = S_{EH} + S_{NMG} + S_{CS} = \frac{\pi r_+^2}{2G} \left(1 - \frac{1}{2m^2l^2}\right) + \frac{\pi r_-^2}{2G},$$

namely the sum of contributions from the three terms in the action (7) [17–19]. The contributions to the entropy that is due to the gravitational Chern-Simons term (last term in eq. (16)) was first obtained by Solodukhin [18]. It is curiously proportional to the area of the inner horizon rather than of the outer horizon. Here horizons $r_{\pm}$ are defined by

$$r_{\pm} = \sqrt{2G(lM + J)} \pm \sqrt{2G(lM - J)}.$$

In the GMG case the parity-violating Chern-Simons terms leads to different central charges of the Virasoro algebra [20],

$$c_L = \frac{3\alpha}{2G} \left(1 + \frac{1}{\mu l}\right) = \frac{3\alpha}{2G} \left(1 - \frac{1}{2m^2l^2} + \frac{1}{\mu l}\right),$$

$$c_R = \frac{3\alpha}{2G} \left(1 - \frac{1}{\mu l}\right) = \frac{3\alpha}{2G} \left(1 - \frac{1}{2m^2l^2} - \frac{1}{\mu l}\right).$$

For the BTZ black-hole solution in GMG, $h$ and $\overline{h}$ are calculated as

$$h = \frac{1}{2}(lM_1 + J_1) + \frac{c_L}{24},$$

$$\overline{h} = \frac{1}{2}(lM_1 - J_1) + \frac{c_R}{24}.$$

Then, the microscopic entropy is expressed by the Cardy formula (in the $1 \leq \mu l$ case),

$$S_{CFT} = 2\pi \left(\sqrt{\frac{c_Lh}{6}} + \sqrt{\frac{c_R\overline{h}}{6}}\right) = \pi \frac{r_+^2}{2G} \left(1 - \frac{1}{2m^2l^2}\right) + \frac{\pi r_-^2}{2G},$$

which is in good agreement with Bekenstein-Hawking entropy. Now we consider the non-relativistic limit in three-dimensional general massive gravity,

$$t \to t, \quad r \to r, \quad \varphi \to \epsilon \varphi.$$

The parameters $M$ and $J$ in the BTZ solution should scale like

$$M \to M, \quad J \to \epsilon J.$$

It is seen in (5), for non-zero and finite GCA central charges ($C_1$, $C_2$), that the Virasoro central charges must behave as $c_L + c_R \sim O(1)$ and $c_L - c_R \sim O(\frac{1}{\epsilon}).$ In the GMG case, using eq. (8), we have $c_L - c_R = \frac{3\alpha}{2G}.$ So in the non-relativistic limit we demand that $\mu$ should scale as

$$\mu \rightarrow \epsilon \mu.$$

From eqs. (5), (18), and (23) the GCA central charges $C_1$ and $C_2$ are finite

$$C_1 = \frac{l}{4G} \left(1 - \frac{1}{2m^2l^2}\right), \quad C_2 = \frac{1}{4G\mu}.$$

Similarly, from eqs. (6), (19) and (23), scaling dimensions $\Delta$ and rapidity $\xi$, which are the eigenvalue of $L_0$ and $M_0$ are given by

$$\Delta = \lim_{\epsilon \to 0} \left(lM_1 + \frac{c_R - c_L}{24}\right) \left(1 - \frac{1}{2m^2l^2}\right) \left(lM_1 + \frac{c_R - c_L}{24}\right) = \frac{1}{2} \left(1 - \frac{1}{m^2l^2}\right) \left(lM_1 + \frac{c_R - c_L}{24}\right) = \frac{1}{2} \left(lM_1 + \frac{c_R - c_L}{24}\right) + \frac{C_1}{2},$$

$$\xi = \lim_{\epsilon \to 0} \left(J_1 + \frac{c_R - c_L}{24}\right) = \left(1 - \frac{1}{m^2l^2}\right) \left(\frac{1}{\mu l}M_1 + \frac{C_2}{2}\right).$$

In the above equations, when $M$ and $J$ are large enough, the terms $\frac{c_R}{24}$ and $\frac{c_L}{24}$ can be neglected. The dual theory (GMG) of GCA in the non-relativistic limit (23) has to yield the finite ($C_1$, $C_2$) (24) and finite ($\Delta$, $\xi$) (25). In the following, we would like to obtain the finite entropy of the GCA. The scaling limit ($M \to M, J \to \epsilon J$) requires that the event horizons of the BTZ black hole should scale as

$$r_+ \to 2l\sqrt{2GM}, \quad r_- \to \epsilon \frac{\sqrt{2G}}{M},$$

so the black-hole entropy (16) is given by

$$\lim_{\epsilon \to 0} S_{BH} = \lim_{\epsilon \to 0} S_{CFT} = \pi \left(1 - \frac{1}{2m^2l^2}\right) \sqrt{\frac{2l^2M}{G}} + \frac{J}{\mu l} \sqrt{\frac{1}{2GM}}.$$

From the expressions of the central charges $C_1$ and $C_2$ (24), scaling dimension $\Delta$ and rapidity $\xi$ (25), we can rewrite the finite entropy as

$$S_{GCA} = \pi \left(C_1 \frac{2c}{C_2} + \Delta \frac{2c}{\xi}\right).$$
This expression is the entropy for the GCA in two dimensions, which agrees with the result of [21]. Our results show that the Cardy formula (28), already addressed in paper [21], in general works in our non-relativistic limit.

**Conclusion.** Recently the authors of [21] (see also [22]) have shown that the GCA2 is the asymptotic symmetry of cosmologically topologically massive gravity (CTMG) in the non-relativistic limit. They have obtained the central charges of GCA2, and also a non-relativistic generalization of the Cardy formula. Following this work, in this paper we proposed the general massive gravity (GMG) as a gravity dual of 2d GCA in the context of the non-relativistic AdS$_3$/CFT$_2$ correspondence. At the quantum level the centrally extended GCA arises precisely from a non-relativistic contraction of the two copies of the Virasoro algebra. The relations between GCA and CFT central charges (5), tell us that, for non-zero and finite GCA central charges ($C_1$, $C_2$) and $(\Delta, \xi)$ in terms of gravity parameters are given by eqs. (24) and (25), respectively. Finally, we calculated the finite entropy of the GCA2, which agrees with the result of [21]. An interesting question is this: what is the particle content of the non-relativistic version of theory (7)? As we have mentioned in the introduction, in GMG the $\pm 2$ helicity modes propagate with different masses. How do these particles behave in the non-relativistic limit? We leave these questions for future articles.

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