Collapsed Riemannian Manifolds with Bounded Sectional Curvature*

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Abstract

In the last two decades, one of the most important developments in Riemannian geometry is the collapsing theory of Cheeger-Fukaya-Gromov. A Riemannian manifold is called (sufficiently) collapsed if its dimension looks smaller than its actual dimension while its sectional curvature remains bounded (say a very thin flat torus looks like a circle in a bared eyes).

We will survey the development of collapsing theory and its applications to Riemannian geometry since 1990. The common starting point for all of these is the existence of a singular fibration structure on collapsed manifolds. However, new techniques have been introduced and tools from related fields have been brought in. As a consequence, light has been shed on some classical problems and conjectures whose statements do not involve collapsing. Specifically, substantial progress has been made on manifolds with nonpositive curvature, on positively pinched manifolds, collapsed manifolds with an a priori diameter bound, and subclasses of manifolds whose members satisfy additional topological conditions e.g. 2-connectedness.

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One of the most important developments in Riemannian geometry over the last two decades is the structure theory of Cheeger-Fukaya-Gromov for manifolds $M^n$ of bounded sectional curvature, say $|\text{sec}_{M^n}| \leq 1$, which are sufficiently collapsed. Roughly, $M^n$ is called $\epsilon$-collapsed, if it appears to have dimension less than $n$, unless the metric is rescaled by a factor $\geq \epsilon^{-1}$.

For scaling reasons, collapsing and boundedness of tend to oppose one another. Nevertheless, very collapsed manifolds with bounded curvature do in fact exist. For example, a very thin cylinder is very collapsed, although its curvature vanishes identically.

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If one fixes $\epsilon$ and in addition, a bound, $d$, on the diameter, then in each dimension, there only finitely many manifolds, which are not $\epsilon$-collapsed; see [Ch]. The basic result of collapsing theory states the existence of a constant $\epsilon(n) > 0$, such that a manifold which is $\epsilon$-collapsed, for $\epsilon \leq \epsilon(n)$, has a particular kind of singular fibration structure with flat (or “almost flat”) fibers. The fibers lie in the $\epsilon$-collapsed directions; see [CG1,2], [CFG], [Fu1-3].

The first nontrivial example of a collapsing sequence with bounded curvature (described in more detail below) was constructed by M. Berger in 1962; see [CFG]. The first major result on the collapsed manifolds (still a cornerstone of the theory) was M. Gromov’s characterization of “almost flat manifolds” i.e. manifolds admitting a sequence of metrics with curvature and diameter going to zero. Gromov showed that such manifolds are infranil. Later in [Ru], they were shown to actually be nilmanifolds; compare [GMR].

We will survey the development of collapsing theory and its applications to Riemannian geometry since 1990; compare [Fu4]. The common starting point for all of these is the above mentioned singular fibration structure. However, new techniques have been introduced and tools from related fields have been brought in. As a consequence, light has been shed on some classical problems and conjectures whose statements do not involve collapsing. Specifically, substantial progress has been made on manifolds with nonpositive curvature, on positively pinched manifolds, collapsed manifolds with an a priori diameter bound, and subclasses of manifolds whose members satisfy additional topological conditions e.g. 2-connectedness.

1. Collapsed manifolds of bounded sectional curvature

Convention: unless otherwise specified, “collapsing” refers to a sequence of Riemannian manifolds with sectional curvature bounded in absolute value by one and injectivity radii uniformly converge to zero, while “convergence” means “convergence with respect to the Gromov-Hausdorff distance.

Recall that a map from a metric space $(X, d_X)$ to a metric space $(Y, d_Y)$ is called an $\epsilon$-Gromov-Hausdorff approximation, if of $f(X)$ is $\epsilon$-dense in $Y$ and if $|d_X(x, x') - d_Y(f(x), f(x'))| < \epsilon$. The Gromov-Hausdorff distance between two (compact) metric spaces is the infimum of $\epsilon$ as above, for all possible $\epsilon$-Gromov-Hausdorff approximations from $X$ to $Y$ and vice versa. (To be more precise, one should say “pseudo-distance”, since isometric metric spaces have distance zero.) The collection of all compact metric spaces is complete with respect to the Gromov-Hausdorff distance.

a. Flat manifolds, collapsing by scaling and torus actions

For fixed $(M, g)$, the family, $\{ (M, \epsilon^2 g) \}$ converges to a point as $\epsilon \to 0$. However, if the curvature is not identically zero, it blows up. On the other hand, for any compact flat manifold, $(M, g)$, the the manifolds, $(M, \epsilon^2 g)$ continue to be flat. More generally, if $(M, g)$ is a (possibly nonflat) manifold with an isometric torus $T^k$-action for which all $T^k$-orbits have the same dimension, then one obtains a collapsing sequence by rescaling $g$ along the orbits i.e. by putting $g_\epsilon = \epsilon^2 g_0 \oplus g_0^\perp$, 324 Xiaochun Rong
where $g_\epsilon$ is the restriction of $g$ to the tangent space of a $T^k$-orbit and $g_\epsilon^\perp$ is the orthogonal complement. A computation shows that $g_\epsilon$ has bounded sectional curvature independent of $\epsilon$. The collapse constructed by Berger in 1962 was of this type. In his example, $M^3$ is the unit 3-sphere and the $S^1$ action is by rotation in the fibers of the Hopf fibration $S^1 \to S^3 \to S^2$. The limit of this collapse is the 2-sphere with a metric of constant curvature $\equiv 4$; see [Pet].

More generally, a collapsing construction has been given by Cheeger-Gromov for manifolds which admit certain mutually compatible local torus actions (possibly by tori of different dimensions) for which all orbits have positive dimension; see the notion of $F$-structure given below and (1.2.1). As above, for each individual local torus action, one obtains locally defined collapsing sequence. The problem is to patch together these local collapses. If the orbits are not all of the same dimension, the patching requires a suitable scaling of the metric (by a large constant) in the transition regions between orbits of different dimensions; see [CG1]. Hence, in contrast to the Berger example, in general the diameters of such nontrivially patched collapsings necessarily go to infinity.

b. Almost flat manifolds and collapsing by inhomogeneous scaling

Although a compact nilmanifold (based on a nonabelian nilpotent Lie group) admits no flat metric, a sequence metrics on such a manifold which collapses to a point can be constructed by a suitable inhomogeneous scaling process; see [Gr1]. As an example, regard a compact nilmanifold $M^3$ as the total space of a principle circle bundle over a torus. A canonical metric $g$ on $M^3$ splits into horizontal and vertical complements, $g = g_h \oplus g_\perp^h$. Then $g_\epsilon = (\epsilon g_h) \oplus (\epsilon^2 g_\perp^h)$ has bounded sectional curvature independent of $\epsilon$, while $(M^3, g_\epsilon)$ converges to a point. The inhomogeneity of the scaling is essential in order for the curvature to remain bounded; compare Theorem 3.4.

c. Positive rank F-structure and collapsed manifolds

The notion of an $F$-structure may be viewed as a generalization of that of a torus action. An $F$-structure $F$ on a manifold is defined by an atlas $F = \{(V_i, U_i, T^k_i)\}$, satisfying the following conditions:

(1.1.1) $\{U_i\}$ is a locally finite open cover for $M$.
(1.1.2) $\pi_i : V_i \to U_i$ is a finite normal covering and $V_i$ admits an effective torus $T^k_i$-action such that it extends to a $\pi_i^{-1}(U_i) \ltimes T^{k_i}$-action.
(1.1.3) If $U_i \cap U_j \neq \emptyset$, then $\pi_i^{-1}(U_i \cap U_j)$ and $\pi_j^{-1}(U_i \cap U_j)$ have a common finite covering on which the lifting $T^{k_i}$ and $T^{k_j}$-actions commute.

If $k_i = k$, for all $i$, then $F$ is called pure. Otherwise, $F$ is called mixed. The compatibility condition, (1.1.3), implies that $M$ decomposes into orbits. (an orbit at a point is the smallest set containing all the projections of the $T^{k_i}$-orbits at the point.) The minimal dimension of all such orbits is called the rank of $F$. An orbit is called regular, if it has a tubular neighborhood in which the orbits form a fibration. Otherwise, it is called singular. An $F$-structure is called polarized if all $T^{k_i}$-actions are almost free. An $F$-structure is called injective (resp. semi-injective) if the inclusion of any orbit to $M$ induces an injective (resp. nontrivial) map on the
fundamental groups.

A Cr-structure is an injective F-structure with an atlas that satisfies two additional properties: i) \( V_i = D_i \times T^{k_i} \) and \( T^{k_i} \) acts on \( V_i \) by the multiplication. ii) If \( U_i \cap U_j \neq \emptyset \), then \( k_i < k_j \) or vice versa; see [Bu1]. This notion arises in the context of nonpositive curvature.

A metric is called an \( F \)-invariant (or simply invariant), if the local \( T^{k_i} \)-actions are isometric. For any F-structure, there exists an invariant metric.

A manifold may not admit any nontrivial F-structure; compare Corollary 2.5. In fact, a simple necessary condition for a closed manifold \( M^{2n} \) to admit a positive rank F-structure is the vanishing of its Euler characteristic; see [CG1].

A necessary and sufficient condition for the existence of a collapsing sequence of metrics is the existence of an F-structure of positive rank; see [CG1], [CG2].

**Theorem 1.2 (Collapsing and F-structure of positive rank).** ([CG1,2]) Let \( M \) be a manifold without boundary.

(1.2.1) If \( M \) admits a positive rank (resp. polarized) F-structure, then \( M \) admits a continuous one-parameter family of invariant metrics \( g_\epsilon \) such that \( |\text{sec}_{g_\epsilon}| \leq 1 \) and the injectivity radius (resp. volume) of \( g_\epsilon \) converges uniformly to zero as \( \epsilon \to 0 \).

(1.2.2) There exists a constant \( \epsilon(n) \) (the critical injectivity radius) such that if \( M^n \) admits a metric \( g \) with \( |\text{sec}_g| \leq 1 \) and the injectivity radius is less than \( \epsilon(n) \) everywhere, then \( M \) admits a positive rank F-structure almost compatible with the metric.

The F-structure in (1.2.2) is actually a substructure of a so called nilpotent Killing structure on \( M \) whose orbits are infra-nilmanifolds; see [CFG] and compare to Theorem 3.5. Such an infra-nilmanifold orbit at a point contains all sufficiently collapsed directions of the metric; the orbit of its sub F-structure, which is defined by the ‘center’ of the infra-nilmanifold, only contains the most collapsed directions comparable to the injectivity radius at a point. A unsolved problem pertaining to nilpotent structures is whether a collapse as in (1.2.1) can be constructed for which the diameters of the nil-orbits converge uniformly to zero (as holds for F-structures).

The construction of the F-structure in (1.2.2) relies only on the local geometry. Hence, (1.2.2) can be applied to a collapsed region in a complete manifold of bounded sectional curvature. In this way, for such a manifold, one obtains a thick-thin decomposition, in which the thin part carries an F-structure of positive rank; see [CFG].

Theorem 1.2 has been the starting point for many subsequent investigations of collapsing in various situations. The guiding principle is that additional geometrical properties of a collapsing should be mirrored in properties of its associated F-structure, which in turn, puts constraints on the topology. For instance, if a collapsing satisfies additional geometrical conditions such as: i) volume small, ii) uniformly bounded diameter, iii) nonpositive curvature, iv) positive pinched curvature, v) bounded covering geometry i.e. the injectivity radii of the Riemannian universal covering has a uniform positive lower bound, then one may expect corresponding topological properties of the F-structure such as: i) existence of a polarization, ii) pureness, iii) existence of a Cr-structure, iv) the existence of a circle orbit, v)
injective F-structure. Results on such correspondences and their applications will occupy the rest of this paper.

d. Topological invariants associated to a volume collapse

The existence of a sufficiently (injectivity radius) collapsed metric as in (1.2.2) imposes constraints on the underlying topology. For instance, the simplicial volume of $M$ vanishes; see [Gr3]. As mentioned earlier, for a closed $M^{2n}$, the Euler characteristic of $M^{2n}$ also vanishes; see [CFG].

In this subsection, we focus on some topological invariants associated to certain (partially) volume collapsed metrics: the minimal volume, the $L^2$-signature and the limiting $\eta$-invariant; see below.

The minimal volume, $\text{MinVol}(M)$, of $M$, is the infimum of the volumes over all complete metrics with $|\text{sec}_M| \leq 1$. Clearly, $\text{MinVol}(M)$ is a topological invariant. Gromov conjectured that there exists a constant $\epsilon(n) > 0$ such that $\text{MinVol}(M^n) < \epsilon(n)$ implies that $\text{MinVol}(M^n) = 0$ (the gap conjecture for minimal volume). By Theorem 1.2, it would suffice to show that a sufficiently volume collapsed manifold admits a polarized F-structure. On a 3-manifold, any positive rank F-structure has a polarized substructure and thus Theorem 1.2 implies Gromov’s gap conjecture in dimension 3. However, for $n \geq 4$, there are $n$-manifolds which admit a positive rank F-structure but which admit no polarized F-structure; see [CG1].

Theorem 1.3 (Volume collapse and Polarized F-structure). ([Ro2]) There is a constant $\epsilon > 0$ such that if $\text{MinVol}(M^4) < \epsilon$, then $M^4$ admits a polarized F-structure and thus $\text{MinVol}(M^4) = 0$.

For a complete open manifold with bounded sectional curvature and finite volume (necessarily volume collapsed near infinity), the integral of an invariant polynomial of the curvature form may depend on the particular metric; see [CG3]. It is of interest to find a class of metrics for which integral of characteristic forms have a topological interpretation. Cheeger-Gromov showed that for any open complete manifold $M^{4k}$ of finite volume and bounded covering geometry outside some compact subset, the integral of the Hirzebruch signature form over $M^{4k}$ is independent of the metric; see [CG3] and the references therein. Cheeger-Gromov showed that this integral is equal to the so called $L_2$-signature and conjectured that it can take only rational values. (The notion of $L_2$-signature, whose definition involves the concept of Von Neumann dimension, was first introduced by Atiyah and Singer in the context of coverings of compact manifolds.)

Theorem 1.4 (Rationality of geometric signature). ([Ro3]) If an open complete manifold, $M^4$, of finite volume has bounded covering geometry outside a compact subset, then the integral of the Hirzebruch signature form over $M^4$ is a rational number.

The main idea is to show that $M^4$ admits a polarized F-structure $\mathcal{F}$ outside some compact subset and an exhaustion by compact submanifolds, $M^n_i$, such that the restriction of $\mathcal{F}$ to the boundary of $M^n_i$ is injective. The integral over $M^4$ is the limit of the integrals over $M^n_i$, to which we apply the Atiyah-Patodi-Singer formula.
to reduce to showing the rationality of the limit of the $\eta$-invariant terms. By making use of the special property of $\mathcal{F}$ and Theorem 1.5 below, we are able to conclude that the limit of the $\eta$-invariant term is rational.

Cheeger-Gromov showed that if a sequence of volume collapsed metrics on a closed manifold $N^{4n-1}$ have bounded covering geometry, then the sequence of the associated $\eta$-invariants converges and the limit is independent of the particular sequence of such metrics. They conjectured that the limit is rational.

**Theorem 1.5 (Rationality of limiting $\eta$-invariants).** ([Ro1]) If a closed manifold $N^3$ admits a sequence of volume collapsed metrics with bounded covering geometry, then $N^3$ admits an injective $F$-structure and the limit of the $\eta$-invariants is rational.

The idea is to show that $N^3$ admits an injective $F$-structure $\mathcal{F}$. For an injective $F$-structure, the collapsing constructed in (1.2.1) has bounded covering geometry and may be used to compute the limit. Results from 3-manifold topology play a role in the proof of the existence of the injective $F$-structure.

2. Collapsed manifolds with nonpositive sectional curvature

A classical result of Preissmann says that for a closed manifold $M^n$ with negative sectional curvature, any abelian subgroup of the fundamental group is cyclic. By bringing in the discrete group technique, Margulis showed that if the metric is normalized such that $-1 \leq \sec M^n \leq 0$, then there exists at least one point at which the injectivity radius is bounded below by a constant $\epsilon(n) > 0$.

The study of the subsequent study of collapsed manifolds with $-1 \leq \sec \leq 0$ may be viewed as an attempt to describe the special circumstances under which the conclusions of the Preissmann and Margulis theorem can fail, if the hypothesis is weakened to nonpositive curvature; see [Bu1-3], [CCR1,2], [Eb], [GW], [LY], [Sc].

A collapsed metric with nonpositive curvature tends to be rigid in a precise sense; see (2.2.1) and (2.2.2). Namely, there exists a canonical $Cr$-structure whose orbits are flat totally geodesic submanifolds. Of necessity, the construction of this $Cr$-structure is global. By contrast, the construction of less precise (but more generally existing) $F$-structure is local; see [CG2].

Let $M^n = \tilde{M}^n / \Gamma$, where $\tilde{M}^n$ denotes the universal covering space of $M^n$ with the pull-back metric. A local splitting structure on a Riemannian manifold is a $\Gamma$-equivariant assignment to each point (of an open dense subset of $\tilde{M}^n$) a specified neighborhood and a specified isometric splitting of this neighborhood, with a non-trivial Euclidean factor. Hence, a necessary condition for a local splitting structure is the existence of a plane of zero curvature, at every point of $M^n$. A local splitting structure is abelian if the projection to $M^n$ of every nontrivial Euclidean factor as above is a closed embedded flat submanifold, and in addition, if two projected leaves intersect, then one of them is contained in the other.

**Theorem 2.1 (Abelian local splitting structure and $Cr$-structure).** ([CCR1]) Let $M^n$ be a closed manifold of $-1 \leq \sec M^n \leq 0$. 

(2.1.1) If the injectivity radius is smaller than $\epsilon(n) > 0$ everywhere, then $M^n$ admits an abelian local splitting structure.

(2.1.2) If $M^n$ admits an abelian local splitting structure, then it admits a compatible $C^r$-structure, whose orbits are the flat submanifolds (projected leaves) of the abelian local splitting structure. In particular, $\text{MinVol}(M^n) = 0$.

Theorem 2.1 was conjectured by Buyalo, who proved the cases $n = 3, 4$; see [Bu1–3], [Sc].

Let $\hat{x} \in M^n$. Let $\Gamma_\epsilon(\hat{x}) \neq 1$ denote the subgroup of $\Gamma$ generated by those $\gamma$ whose displacement function, $\delta_\epsilon(\hat{x}) = d(x, \gamma(\hat{x}))$, satisfies $d(x, \gamma(\hat{x})) < \epsilon$. (In the application, $\epsilon$ is small.) If all $\Gamma_\epsilon(\hat{x})$ are abelian, then the minimal sets, $\{\text{Min}(\Gamma_\epsilon(\hat{x}))\}$, of the $\Gamma_\epsilon(\hat{x})$ give the desired abelian local splitting structure in (2.1.1). In general, $\Gamma_\epsilon(\hat{x})$ is only Bieberbach. Then, a crucial ingredient in (2.1.1) is the existence of a ‘canonical’ abelian subgroup of $\Gamma_\epsilon(\hat{x})$ of finite index consisting of those elements which are stable in the sense of [BGS]. In spirit, the proof of (2.1.2) is similar to the construction in [CG2], but the techniques used are quite different.

The following are some specific questions pertaining to abelian local splitting structures:

(2.2.1) If some metric $g$ on $M$ of nonpositive sectional curvature has an abelian local splitting structure, does every nonpositively curved metric also have such a structure?

(2.2.2) If $M$ has a $C^r$-structure, does every any nonpositively curved metric on $M$ have a compatible local splitting structure?

Note that an affirmative answer to (2.2.1) and (2.2.2) would imply a kind of semirigidity. It would imply that all nonpositively curved metrics on $M$ are alike in a precise sense.

**Theorem 2.3 (F-structure and local splitting structure).** ([CCR2]) Let $X^n$, $M^n$ be closed manifolds such that $X^n$ admits a nontrivial F-structure. Let $f : X^n \to M^n$ have nonzero degree. Then every metric of nonpositive sectional curvature on $M^n$ has a local splitting structure.

We conjecture that if an F-structure has positive rank, then the local splitting structure is abelian. This conjecture, whose proof would provide an affirmative answer to (2.2.2), has been verified in dimension 3 and in some additional special cases; see [CCR2].

We conclude this section with two consequences of Theorem 2.3.

**Corollary 2.4 (Generalized Margulis Lemma).** ([CCR2]) Let $M^n$ be a closed manifold of nonpositive sectional curvature. If the Ricci curvature is negative at some point, then for every metric with $|\text{sec}| \leq 1$, there is a point with injectivity radius $\geq \delta(n) > 0$.

Another consequence is a geometric obstruction for a nontrivial F-structure.

**Corollary 2.5 (Nonexistence of F-structure).** ([CCR2]) If a closed manifold $M$ admits a metric of nonpositive sectional curvature such that the Ricci curvature is negative at some point, then $M$ does not admit a nontrivial F-structure.
3. Collapsed manifolds with bounded sectional curvature and diameter

In this section, we discuss the class of collapsed manifolds of bounded sectional curvature whose diameters are also bounded. By the Gromov’s compactness theorem, any sequence of such collapsed manifolds contains a convergent subsequence; see [GLP]. Hence, without loss of the generality, we only need to consider convergent collapsing sequences.

(3.1) Let $M^n_i > d_{GH} >> X$ denote a sequence of closed manifolds converging to a compact metric space $X$ such that $|\text{sec}_{M^n_i}| \leq 1$ and $\dim(X) < n$.

Main Problem 3.2. For $i$ large, investigate relations between geometry and topology of $M^n_i$ and that of $X$. The following are some specific problems and questions.

(3.2.1) Find topological obstructions for the existence of $M^n_i$ as in (3.1).

(3.2.2) To what extent is the topology of the $M^n_i$ in (3.1) stable when $i$ is sufficiently large?

(3.2.3) Under what additional conditions is it true that $\{M^n_i\}$ as in (3.1) contains a subsequence of constant diffeomorphism type? If all $M^n_i$ are diffeomorphic, then to what extent do the metrics converge?

Note that by the Cheeger-Gromov convergence theorem, the above problems are well understood in the noncollapsed situation $\dim(X) = n$.

**Theorem 3.3 (Convergence).** ([Ch], [GLP]) Let $M^n_i > d_{GH} >> X$ be as in (3.1) except $\dim(X) = n$. Then for $i$ large, $M^n_i$ is diffeomorphic to some fixed $M^n$ which is homeomorphic to $X$ and there are diffeomorphisms, $f_i : M^n \to M^n_i$, such that the pulled back metrics, $f_i^*(g_i)$, converge to a metric, $g_\infty$, in the $C^{1,\alpha}$-topology ($0 < \alpha < 1$).

Note that as a consequence of Theorem 3.3, topological stability of a sequence as in (3.1) will immediately yield a corresponding finiteness result in terms of the dimension and bounds on curvature and diameter.

e. Structure of collapsed manifolds with bounded diameter

As described in Section 1, any closed nilmanifold $M^n$ admits metrics collapsing to a point.

**Theorem 3.4 (Almost flat manifolds).** ([Gr1]) Let $M^n_i > d_{GH} >> X$ be as in (3.1). If $X$ is a point, then a finite normal covering space of $M^n_i$ of order at most $c(n)$ is diffeomorphic to a nilmanifold $N^n/\Gamma_i$ ($i$ large), where $N^n$ is the simply connected nilpotent group.

Theorem 3.4 can be promoted to a description of convergent collapsing sequence, of manifolds, $M^n_i$, as in (3.1). As mentioned following Theorem 1.2, any sufficiently collapsed manifold admits a nilpotent Killing structure; see [CFG]. Here a bound on diameter forces the nilpotent Killing structure to be pure.

For a closed Riemannian manifold $M^n$, its frame bundle $F(M^n)$ admits a canonical metric determined by the Riemannian connection up to a choice of a bi-invariant metric on $O(n)$. A fibration, $N/\Gamma \to F(M^n) \to Y$, is called $O(n)$-invariant.
if the $O(n)$-action on $F(M^n)$ preserves both the fiber $N/\Gamma$ (a nilmanifold) and the structural group. By the $O(n)$-invariance, $O(n)$ also acts on the base space $Y$. A canonical metric is invariant if its restriction on each $N/\Gamma$ is left-invariant. A pure nilpotent Killing structure on $M$ is an $O(n)$-invariant fibration on $F(M^n)$ for which the canonical metric is also invariant.

**Theorem 3.5 (Fibration).** ([CFG]) Let $M^n_i@d > d_{GH} >> X$ be as in (3.1). Then $F(M^n_i)$ equipped with canonical metrics contains a convergent subsequence, $F(M^n_i)_i@d > d_{GH} >> Y$, and $F(M^n_i)$ admits an $O(n)$-invariant fibration $N/\Gamma_i \rightarrow F(M^n_i) \rightarrow Y$ for which the canonical metric is $\epsilon_i$-close in the $C^1$ sense to some invariant metric, where $\epsilon_i \rightarrow 0$.

The following properties are crucial for the study of particular instances of collapsing as in (3.1).

**Proposition 3.6.** Let $M^n_i@d > d_{GH} >> X$ be as in (3.1).

(3.6.1) (Regularity) ([Ro5]) For any $\epsilon > 0$, $M^n_i$ admits an invariant metric $g_i$ such that $\min(se_{M^n_i}) - \epsilon \leq se_{(M^n_i, g_i)} \leq \max(se_{M^n_i}) + \epsilon$ for $i$ large.

(3.6.2) (Equivariance) ([PT], [GK]) The induced $O(n)$-actions on $Y$ from the $O(n)$-action on $F(M^n_i)$ are $C^1$-close and therefore are all $O(n)$-equivariant for $i$ large.

**f. Obstructions to collapsing with bounded diameter**

**Theorem 3.7 (Polarized F-structure and vanishing minimal volume).** ([CR2]) Let $M^n_i@d > d_{GH} >> X$ be as in (3.1). Then the F-substructure associated to the pure nilpotent Killing structure on $M^n_i$ contains a (mixed) polarized F-structure. In particular, $\text{MinVol}(M^n_i) = 0$.

Theorem 3.7 may be viewed as a weak version of the Gromov’s gap conjecture. Note that the associated F-structure on $M^n_i$ may not be polarized. The existence of a polarized substructure puts constraints on the singularities of the structure.

**Theorem 3.8 (Absence of symplectic structure).** ([FR3]) Let $M^n_i@d > d_{GH} >> X$ be as in (3.1). If $\pi_1(M^n_i)$ is finite, then $M^n_i$ does not support any symplectic structure.

The proof of Theorem 3.8 includes a nontrivial extension of the well known fact that any $S^1$-action on a closed simply connected symplectic manifold which preserves the symplectic structure has a nonempty fixed point set.

A geometric obstruction to the existence of a collapsing sequence in (3.1) is provided by:

**Theorem 3.9 (Geometric collapsing obstruction).** ([Ro7]) Let $M^n_i@d > d_{GH} >> X$ be as in (3.1). Then $\lim \sup(\max_{M^n_i} Riem_{M^n_i}) \geq 0$.

A key ingredient in the proof is a generalization of a theorem of Bochner asserting that a closed manifold of negative Ricci curvature admits no nontrivial invariant pure F-structure (Bochner’s original theorem only guarantees the nonexistence of a nontrivial isometric torus action.)
Theorem 3.10 (Pure injective F-structure). ([CR1]) Let \( M^n_i > d_{GH} >> X \) be as in (3.1). If \( M^n_i \) has bounded covering geometry and \( \pi_1(M^n_i) \) is torsion free, then for \( i \) large \( M^n_i \) admits a pure injective F-structure.

g. The topological and geometric stability

In this subsection, we address Problems (3.2.2) and (3.2.3). Observe that by the Gromov’s Betti number estimate, [Gr2], the sequence in (3.1) contains a subsequence whose cohomology groups, \( H^*(M^n_i, \mathbb{Q}) \), are all isomorphic. On the other hand, examples have been found showing that \( \{H^*(M^n_i, \mathbb{Q})\} \) can contain infinitely many distinct ring structures; see [FR2].

Theorem 3.11 (\( \pi_q \)-Stability). ([FR2]; compare [Ro4], [Tu]) Let \( M^n_i > d_{GH} >> X \) be as in (3.1). Then for \( q \geq 2 \) and after passing to a subsequence, the \( q \)-th homotopy group \( \pi_q(M^n_i) \) are all isomorphic, provided that \( \pi_q(M^n_i) \) are finitely generated (e.g. \( \text{sec} M^n_i \geq 0 \) or \( \pi_1(M^n_i) \) is finite).

Note that in contrast to the Betti number bound, Theorem 3.11 does not hold if upper bound on the sectional curvature is removed; see [GZ].

We now discuss sufficient topological conditions for diffeomorphism stability. Consider the sequence of fibrations, \( N/\Gamma_i \to F(M^n_i) \to Y \), associated to (3.1). One would like to know when all \( N/\Gamma_i \) are diffeomorphic.

Proposition 3.12. ([FR4]) Let \( M^n_i > d_{GH} >> X \) be as in (3.1). If \( \pi_1(M^n_i) \) contains no free abelian group of rank two, then \( N/\Gamma_i \) is diffeomorphic to a torus.

In low dimensions, we have:

Theorem 3.13 (Diffeomorphism stability—low dimensions). ([FR3], [Tu]) For \( n \leq 6 \), let \( M^n_i > d_{GH} >> X \) be as in (3.1). If \( \pi_1(M^n_i) = 1 \), then there is a subsequence all whose members are diffeomorphic.

Note that for \( n \geq 7 \), one cannot expect Theorem 3.13; see [AW]. Hence, additional restrictions are required in higher dimensions. Observe that if \( M^n_i \) are 2-connected, then all \( T^k \to F(M^n_i) \to Y \) are equivalent as principle \( T^k \)-bundles. In particular all \( F(M^n_i) \) are diffeomorphic.

Using (3.6.2), Petrunin-Tuschmann showed that the equivalence can be chosen that is also \( O(n) \)-equivariant, and concluded the diffeomorphism stability for two-connected manifolds; see [PT]. For the special case in which the \( M^n_i \) are positively pinched, the same conclusion was obtained independently in [FR1] via a different approach.

We introduce a topological condition which when \( M^n_i \) is simply connected, reduces to the assumption that \( \pi_2(M^n_i) \) is finite. In the nonsimply connected case however, there are manifolds with \( \pi_2(M) \) infinite, which satisfy our condition.

Let \( \tilde{M} \) denote the universal covering of \( M \). For a homomorphism, \( \rho : \pi_1(M) \to \text{Aut}(\mathbb{Z}^k) \), the semi-direct product, \( \tilde{M} \times_{\pi_1(M)} \mathbb{Z}^k \), is a bundle of \( \rho(\pi_1(M)) \)-modules which can be viewed as a local coefficient system over \( M \). We denote it by \( \mathbb{Z}_\rho^k \).

Let \( b_q(M, \mathbb{Z}_\rho^k) \) denote the rank of the cohomology group, \( H^q(M, \mathbb{Z}_\rho^k) \), with the local
coefficient system \( \mathbb{Z}_p^k \). We refer to the integer

\[
\tilde{b}_q(M, \mathbb{Z}_p^k) = \max_{\rho : \pi_1(M) \to \text{Aut}(\mathbb{Z}_p^k)} \{ \hat{b}_q(M, \mathbb{Z}_p^k) \}
\]

as the \( q \)-th twisted Betti number of \( M \). Clearly, \( \tilde{b}_q(M, \mathbb{Z}_k^p) \) is a topological invariant of \( M \). Moreover, \( k \cdot b_2(M, \mathbb{Z}) \leq \tilde{b}_q(M, \mathbb{Z}_k^p) \), with equality if \( \pi_1(M) = 1 \).

**Theorem 3.14 (Diffeomorphism stability and geometric stability).** ([FR4])

Let \( M^n \mathbb{Q} @ d_{GH} \gg X \) be as in (3.1) such that \( k = n - \dim(X) \). Assume that \( M^n \mathbb{Q} \) satisfies:

1. \( \pi_1(M^n) \) is a torsion group with torsion exponents uniformly bounded from above.
2. The second twisted Betti number \( \tilde{b}_2(M^n, \mathbb{Z}_k^p) = 0 \).

Then there are diffeomorphisms, \( f_i \), from \( M^n \mathbb{Q} \) to (a subsequence of) \( \{ M^n \mathbb{Q} \} \), such that the distance functions of pullback metrics, \( f_i^*(g_i) \), on \( M^n \), converge to a pseudo-metric \( d_\infty \) in \( C_0 \)-norm. Moreover, \( M^n \mathbb{Q} \) admits a foliation with leaves diffeomorphic to flat manifolds (that are not necessarily compact) and a vector \( V \) tangent to a leaf if and only if \( \| V \|_{g_i} \to 0 \).

The proof of Theorem 3.14 is quite involved.

Finally, we mention that J. Lott has systematically investigated the analytic aspects for a collapsing in (3.1); for details, see [Lo1-3].

## 4. Positively pinched manifolds

In this section, we further investigate a subclass of the class of collapsed manifolds with bounded diameter: collapsed manifolds with pinched positive sectional curvature; see [AW], [Ba], [Es], [Pü] for examples.

In the spirit of Theorem 3.4, we first give the following classification result.

**Theorem 4.1 (Maximal collapse with pinched positive curvature).** ([Ro8])

Let \( M^n \mathbb{Q} @ d_{GH} \gg X \) be as in (3.1) such that \( \sec_{M^n} \geq \delta > 0 \). Then \( \dim(X) \geq \frac{n-1}{2} \) and equality implies that \( \tilde{M}_i^n \) isomorphic to \( S^n/\mathbb{Z}_{q_i} \) (a lens space), where \( \tilde{M}_i^n \to M_i^n \) is a covering space of order \( \leq \frac{n+1}{2} \).

By Theorem 3.5, (3.6.1) and Proposition 3.12, the proof of Theorem 4.1 reduces to the classification of positively curved manifolds which admit invariant pure F-structures of maximal rank; see [GS].

**Theorem 4.2 (Positive pinching and almost cyclicity of \( \pi_1 \)).** ([Ro6])

Let \( M^n \mathbb{Q} @ d_{GH} \gg X \) be as in (3.1) such that \( \sec_{M^n} \geq \delta > 0 \). Then for \( i \) sufficiently large, \( \pi_1(M^n) \) has a cyclic subgroup whose index is less than \( w(n) \).

By Theorem 3.5 and (3.6.1), the following result easily implies Theorem 4.2.
Theorem 4.3 (Symmetry and almost cyclicity of $\pi_1$). ([Ro6]) Let $M^n$ be a closed manifold of positive sectional curvature. If $M^n$ admits an invariant pure $F$-structure, then $\pi_1(M^n)$ has a cyclic subgroup whose index is less than a constant $w(n)$.

In the special case of a free isometric action, from the homotopy exact sequence associated to the fibration, $S^1 \to M^n \to M^n/S^1$, together with the Synge theorem, one sees that $\pi_1(M^n)$ is cyclic. The proof of the general case is by induction on $n$ and is rather complicated.

We now consider the injectivity radius estimate. Klingenberg-Sakai and Yau conjectured that the infimum of the injectivity radii of all $\delta$-pinched metrics on $M^n$ is a positive number which depends only on $\delta$ and the homotopy type of the manifold. By a result of Klingenberg, this conjecture is easy in even dimensions. In odd dimensions it is open.

Theorem 4.4 (Noncollapsing). ([FR4]; compare [FR1], [PT]) For $n$ odd, let $M^n$ be a closed manifold satisfying $0 < \delta \leq \sec M^n \leq 1$ and $|\pi_1(M^n)| \leq c$. If $\hat{b}(M^n, \mathbb{Z}^{\oplus 2}) = 0$, then the injectivity radius of $M^n$ is at least $\epsilon(n, \delta, c) > 0$.

If Theorem 4.4 were false, then by Theorem 3.14 and (3.6.1) one could assume the existence of a sequence, $(M, g_i)_{i \geq d \GH} >> X$, with $\delta/2 \leq \sec g_i \leq 1$, such that the distance functions of the metrics $g_i$ also converge. In view of the following theorem this would lead to a contradiction.

Theorem 4.5 (Gluing). ([PRT]) Let $(M, g_i)_{i \geq d \GH} >> X$ as in (1.3). If the distance functions of $g_i$ converge to a pseudo-metric, then $\liminf(\min sec g_i) \leq 0$.

Let $f_i : (M, g_i)_{i \geq d \GH} >> X$ denote an $\epsilon_i$ Gromov-Hausdorff approximation, where $\epsilon_i \to 0$. For an open cover $\{B_j\}$ for $X$ by small (contractible) balls, the assumption on the distance functions implies (roughly) that the tube, $C_{ij} = f_i^{-1}(B_j)$, is a subset of $M$ independent of $i$. Clearly, the universal covering $\tilde{C}_{ij}$ of $C_{ij}$ is noncompact. The idea is to glue together the limits of the $\tilde{C}_{ij}$ (modulo some suitable group of isometries with respect to the pullback metrics) to form a noncompact metric space with curvature bounded below by $\liminf(\min sec g_i)$ in the comparison sense; see [BGP], [Pe]. On the other hand, the positivity of the curvature implies that the space so obtained would have to be compact.

The above results on $\delta$-pinched manifolds may shed a light on the topology of positively curved manifolds. It is tempting to make the following conjecture (which seems very difficult).

Conjecture 4.6. Let $M^n$ denote a closed manifold of positive sectional curvature.

(4.6.1) (Almost cyclicity) $\pi_1(M^n)$ has a cyclic subgroup with index bounded by a constant depending only on $n$.

(4.6.2) (Homotopy group finiteness) For $q \geq 2$, $\pi_q(M^n)$ has only finitely many possible isomorphism classes depending only on $n$ and $q$.

(4.6.3) (Diffeomorphism finiteness) If $\pi_q(M^n) = 0$ ($q = 1, 2$), then $M^n$ can have only finitely many possible diffeomorphism types depending only on $n$. 

Note that (4.6.1)–(4.6.3) are false for nonnegatively curved spaces. By the results in this section, Conjecture 4.6 would follow from an affirmative answer to the following:

**Problem 4.7 (Universal pinching constant).** ([Be], [Ro5]) Is there a constant $0 < \delta(n) \ll 1$ such that any closed $n$-manifold of positive sectional curvature admits a $\delta(n)$-pinched metric?

A partial verification of (4.6.2) is obtained by [FR2].

**Theorem 4.8.** ([FR2]) Let $M^n$ denote a closed manifold of positive sectional curvature. For $q \geq 2$, the minimal number of generators for $\pi_q(M^n)$ is less than $c(q,n)$.

Previously, by Gromov the minimal number of generators of $\pi_1(M^n)$ is bounded above by a constant depending only on $n$.

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