Large and moderate deviations of weak record numbers in random walks

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Abstract

Record numbers are basic statistics in random walks, whose deviation principles are not very clear so far. In this paper, the asymptotic probabilities of large and moderate deviations for numbers of weak records in right continuous or left continuous random walks are proved.

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1. Introduction

"Record", according the Oxford dictionary, can be referred to an extreme attainment, the best (or worst) performance ever attested in a particular activity. The study of record statistics has become an integral part of diverse fields such as meteorology, hydrology, economics and sports. In mathematics, record statistics in the setting of i.i.d random variables are well understood in many situations. For example, suppose the family of i.i.d random variables \( \{X_n, n \geq 0\} \) is a stochastic model for achievements in a sequence of activities. Let \( M_n = \max_{0 \leq i \leq n} X_i \). Then \( M_n \) is the record at time \( n \) and the statistic

\[
K_n = \sum_{i=1}^{n} 1\{X_i=M_n\}
\]

counts the numbers of the current record. Brands et al. [2] and references therein studied the asymptotic behaviours of \( K_n \), as \( n \to +\infty \), and Khmaladze, et al. [10] discussed the number of the so-called \( \varepsilon \)-repetitions of the current record value.

From the angle of real world, it is more reasonable that the series of \( X_n \) are correlated. In this case, we say that a record event happens at time \( k \), if \( X_k \) is larger than all the

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previous values in the series. Majumdar and Ziff [21] used random walks to model the time series of achievements in the particular activities and discussed the growth of record numbers and surviving ages. For more works using random walks to study the record phenomenons, please see Godrèche et al [11] and the reference therein. Usually, the main goal of the theory of records is to answer these questions: (a) How many records occur up to step $n$, (b) how long does a record survive, and (c) what is the age of the longest surviving record? (See Majumdar [20]). In this paper, we are interested in the asymptotic properties of record numbers in random walks as the step $n \to +\infty$. The aim of this paper is to study the deviations between the record numbers and their asymptotic limits.

In order to present the definition of record number in our setting, let $\{X_k, k \geq 1\}$ be a sequence of i.i.d. random variables. $S = \{S_n, n \geq 0\}$ is a random walk on line, i.e. for $n \geq 1$

$$S_n = \sum_{k=1}^{n} X_k,$$

and $S_0 = 0$. Let

$$M_n = \max_{0 \leq m \leq n} S_m \quad (1.1)$$

for all $n \geq 1$. Let $T_0 = 0$. For each $n \geq 1$, define

$$T_n = \inf\{m > T_{n-1}, S_m \geq M_{m-1}\}, \quad (1.2)$$

and

$$A_n = \sup\{k \geq 1, T_k \leq n\}, \quad (1.3)$$

where by convention $\inf \emptyset = +\infty$ and $\sup \emptyset = 0$. Obviously, $S_{T_n}$ is the maximum value among $S_0, S_1, \cdots, S_{T_n}$ and $\{A_n, n \geq 1\}$ is a counting process which records the numbers that $S$ arrives at its maximum values. In this paper, we call $A_n$ the weak record numbers up to time $n$. Here, we use “weak” to emphasize that we not only consider the time when a new record appears but also keep eyes on the time when the current record is repeated. We remark that our weak record numbers up to time $n$ is also different from the record numbers studied in Katzenbeisser and Panny [15], Kirschenhofer and Prodinger [17] and Pătăneă [22] and the references therein, where they discussed the number of the events $\{S_k = M_n\}$ that occur up to time $n$.

In the field of random walks, $A_n$ is also called a number of weak ladder points which is a footstone in the fluctuation theory of random walks. The fluctuation theory was set forth by Spitzer [24] and Feller [7], and has drawn much attentions since then because of its wide applications and elaborated but fascinating theory. For the basic conceptions and some earlier results, we can also refer to Karlin and Taylor [14, Chapter 17]. Omey and Teugels [12] proved that a normed version of the bivariate ladder process $(T_n, S_{T_n})_n$ converges in law to the bivariate ladder process of a Lévy process $X$, if the normed $(S_n)$ converges in law to $X$. As an immediate result, one can derive that a normed version of $A_n$
(numbers of ladder points) of $S$ converges in law to the local time at the supremum of $X$. Chaumont and Doney [3] in 2010 extended this result to a general case, where it is proved that when a normed sequence of random walks $S^{(n)}$ converges a.s. on the Skorokhod space toward a Lévy process $X$, then a normed version of the counting processes of ladder points of $S^{(n)}$ converges uniformly on compact sets in probability toward the local time at the supremum of $X$. Based on these results, one may further ask how about the deviations between the normed version of $A_n$ and its limit. To our knowledge, there is little literature to investigate such problem.

To precisely state our problem discussed in this paper, let $S$ be the simple symmetric random walk on line, for example. It is easy to see that as $n \to +\infty$, 

$$A_n/\sqrt{n} \to 2 \max_{0 \leq t \leq 1} B(t) := 2B^*(1)$$

in distribution, where $B$ is a standard Brownian motion. This result suggests that if we regard $A_n$ as $2\sqrt{n}B^*(1)$, then we only ignore an insignificant probability. In this paper, we are interested in finding the insignificant probabilities via the asymptotic probabilities of $P(A_n \geq \sqrt{nc_n})$ where $c_n$ tends to $\infty$ besides other constraints. We will get the large deviation principle (LDP) and moderate deviation principle (MDP) for $A_n$, respectively. For the general theory of LDP and MDP, we refer to Dembo and Zeitouni [6].

Let $Y_k = T_k - T_{k-1}$ for $k \geq 1$. The strong Markovian property of random walks implies that $Y_k$ are i.i.d. $A_n$ can be written by

$$A_n = \sup\{k; \sum_{i=1}^{k} Y_i \leq n\}.$$ 

Namely, $\{A_n\}_{n \geq 1}$ is a discrete time renewal process with the inter-occurrence time sequence $\{Y_n\}$. There are many results on the theory of deviations for renewal processes or renewal reward processes. See, for example, Serfozo [23], Glynn and Whitt [10], Jiang [13], Chi [5], Frolov, etal [8], Lefevere etal [18], Borovkov and Mogulskii [1], Tsirelson [26], Logachov and Mogulskii [19] and the references therein. To the author’s best knowledge, little result can include or be directly applied to our cases, because most of them need the assumptions on moments or moment generating functions for inter-occurrence times, which are not fulfilled by $A_n$ in our situations.

Due to the celebrated invariance principle, one will naturally connect $A_n$ with the maximal value process $B^*(n)$ of a Brownian motion when the increments of random walk $S$ have finite variance, and conceive that we can get the LDP and MDP of $A_n$ by extending the asymptotical results of $2B^*(n)$. However, our results (see Theorem 2.1 and Corollary 2.1) show that this method may not be optimal if it is applicable. In this paper we investigate the LDP and MDP for $A_n$ via the deviation theory of occupation time of Markov process.

The remainder of this paper is organized as follows. In Section 2, we summarize the main results of this paper, and make some comments. In Section 3, we provide some
auxiliary conclusions for the main results. In Section 4 we prove the LDP and in the last section i.e. the section 5, we prove the MDP.

2. Main results

Let $S = \{S_n\}_{n \geq 0}$ be a random walk on the integer lattice $\mathbb{Z}$, namely, for every $n \geq 1$,

$$S_n = S_0 + \sum_{i=1}^{n} X_i, \quad S_0 = 0,$$

where $X_1, X_2, \cdots$ are i.i.d. integer-valued random variables. We say $S$ is right-continuous if the probability mass function (p.m.f) of $X_i$ is read as

$$0 < q = \mathbb{P}(X_i = 1), \quad \mathbb{P}(X_i = -n) = p_n, \quad n \geq 0. \quad (2.1)$$

Similarly, we say $S$ is left-continuous if the p.m.f of $X_i$ is read as

$$0 < q = \mathbb{P}(X_i = -1), \quad \mathbb{P}(X_i = n) = p_n, \quad n \geq 0. \quad (2.2)$$

Define

$$\phi(s) = q + \sum_{n=0}^{\infty} p_n s^{n+1}, \quad \text{for all } s \in [0, 1].$$

For convenience, in the sequel, we call the random walk $S$ is right or left continuous with $\phi$ if the p.m.f of its increments has the form of (2.1) or (2.2), respectively.

By some mathematical analyses, we can see that for each $s \in [0, 1]$, the equation $x = s\phi(x)$ has a solution $x \in [0, 1]$. We denote the minimum non-negative solution by $h(s)$.

For every $\lambda \in (-\infty, 0]$ let

$$\Lambda_r(\lambda) = \ln \left(1 + qe^{\lambda} - \frac{qe^{\lambda}}{h(e^{\lambda})}\right),$$

and

$$\Lambda_l(\lambda) = \lambda + \ln \left(\frac{1 - \phi(h(e^{\lambda}))}{1 - h(e^{\lambda})}\right).$$

As what we show in Lemma 3.3 in the next section, $\Lambda_r'(\lambda)$ and $\Lambda_l'(\lambda)$ are continuous monotone functions and $\Lambda_r(0) = \Lambda_l(0) = +\infty$, $\Lambda_r(-\infty) = \Lambda_l(-\infty) = 1$. Then for any $x \in (1, +\infty)$, there exist unique $\lambda_l, \lambda_r \in (0, -\infty]$ such that $x = \Lambda_r'(\lambda_r)$ and $x = \Lambda_l'(\lambda_l)$. Denote $\lambda_l, \lambda_r$ by $G_l(x)$ and $G_r(x)$, respectively. For each $x \geq 0$, define

$$\Lambda_r^*(x) = \sup_{\lambda \leq 0} \{x\lambda - \Lambda_r(\lambda)\} = \begin{cases} xG_r(x) - \Lambda_r(G_r(x)), & x > 1, \\ -\ln(q + p_0), & x = 1, \\ +\infty, & x < 1. \end{cases}$$

and

$$\Lambda_l^*(x) = \sup_{\lambda \leq 0} \{x\lambda - \Lambda_l(\lambda)\} = \begin{cases} xG_l(x) - \Lambda_l(G_l(x)), & x > 1, \\ -\ln(1 - q), & x = 1, \\ +\infty, & x < 1. \end{cases}$$
Let $A_n$ be the weak record number of $S$ up to time $n$, which is defined by \cite{[13]}. We have the following LDP for $A_n$.

**Theorem 2.1** Suppose $\phi'(1) = 1$. For any $x > 0$, if $S$ is right continuous with $\phi$, then
\[
\lim_{n \to +\infty} \frac{1}{n} \ln P(A_n \geq xn) = -x \Lambda^*_r(1/x).
\]
If $S$ is left continuous with $\phi$, then
\[
\lim_{n \to +\infty} \frac{1}{n} \ln P(A_n \geq xn) = -x \Lambda^*_l(1/x).
\]

To facilitate our discussion in MDP, we propose the following assumption.

**Assumption (H):** There exist $1 > \alpha > 0$ and $c > 0$ such that
\[
\lim_{s \to 1^-} \frac{1 - s \phi'(h(s))}{(1 - s) \alpha} = c.
\]

The MDP for $A_n$ is read as follows.

**Theorem 2.2** Suppose $\phi'(1) = 1$ and Assumption (H) holds. Let $\{c_n\}$ be a sequence of positive numbers such that $c_n \to +\infty$ and $c_n = o(n)$.

1. If $S$ is right continuous with $\phi$, then for any $x > 0$,
\[
\lim_{n \to +\infty} \frac{1}{c_n} \ln P(A_n \geq xn^{1-\alpha}c_n^{\alpha}) = -\frac{\alpha}{1-\alpha} \left( \frac{q}{c} \right)^{1/\alpha} x^{1/\alpha}.
\]
2. If $S$ is left continuous with $\phi$, then for any $x > 0$,
\[
\lim_{n \to +\infty} \frac{1}{c_n} \ln P(A_n \geq xn^{\alpha}c_n^{1-\alpha}) = -[c(1-\alpha)^{2-\alpha} \alpha]^\alpha (1-\alpha) x^{1/(1-\alpha)}.
\]

Applying Theorem 2.2 to some special cases we get the following corollaries.

**Corollary 2.1** Suppose $\phi'(1) = 1$ and $\sigma := \sqrt{\phi''(1)} < +\infty$. Let $\{c_n\}$ be a sequence of positive numbers such that $c_n \to +\infty$ and $c_n = o(n)$.

1. If $S$ is right continuous with $\phi$, then for any $x > 0$,
\[
\lim_{n \to +\infty} \frac{1}{c_n} \ln P(A_n \geq xn^{1/2}c_n^{1/2}) = -\frac{q^2 x^2}{2\sigma^2}.
\]
2. If $S$ is left continuous with $\phi$, then for any $x > 0$,
\[
\lim_{n \to +\infty} \frac{1}{c_n} \ln P(A_n \geq xn^{1/2}c_n^{1/2}) = -\frac{\sigma^2 x^2}{8}.
\]

**Corollary 2.2** Suppose $\phi(s) = s + \gamma(1-s)^{1+\beta}$ where $\gamma, \beta \in (0,1)$. Let $\{c_n\}$ be a sequence of positive numbers such that $c_n \to +\infty$ and $c_n = o(n)$. 
(1) If $S$ is right continuous with $\phi$, then for any $x > 0$,
\[
\lim_{n \to +\infty} \frac{1}{c_n} \ln \mathbb{P}(A_n \geq x n^{1/(1+\beta)} c_n^{\beta/(1+\beta)}) = -\frac{\beta \gamma}{(1+\beta)^{2+1/\beta}} x^{1+1/\beta}.
\]

(2) If $S$ is left continuous with $\phi$, then for any $x > 0$,
\[
\lim_{n \to +\infty} \frac{1}{c_n} \ln \mathbb{P}(A_n \geq x n^{\beta/(1+\beta)} c_n^{1/(1+\beta)}) = -\frac{\gamma \beta}{(1+\beta)^{2+1/\beta}} x^{1+\beta}.
\]

**Remark 2.1** When $\phi'(1) = 1$ and $\sigma^2 = \phi''(1) < +\infty$, the expectation and the variance of $X_i$ of the increments of $S$ are 0 and $\sigma^2$, respectively. In this case, by the strong invariance principle, $S$ is approximated by a Brownian motion with variance parameter $\sigma^2$, whether $S$ is right continuous or left continuous. However, as what Theorem 2.1 and Corollary 2.1 indicate, the right or left continuity of random walk $S$ leads to different rate functions for the LDP and MDP of $A_n$. These observations show that it may not be the best route for the problems investigated here by simply extending the asymptotic results of Brownian motions to random walks via the invariance principle.

### 3. Some results for queueing models

Let $p_{-1}, p_0, p_1, \ldots$ be a sequence of non-negative real numbers such that $\sum_{n=-1}^{+\infty} p_n = 1$ and $p_0 < 1$. Let $W = \{W_n; n \geq 0\}$ be a Markov chain with transition probability matrix $(p_{i,j})_{i,j \geq 0}$, where

\[
p_{i,j} = \begin{cases} 
p_k, & j = i + k, k \geq 0, i \geq 1; 
p_k, & j = k, i = 0, k \geq 1; 
p_0 + p_{-1}, & j = i = 0; 
0, & \text{others.}
\end{cases}
\]

or

\[
p_{i,j} = \begin{cases} 
p_{i-j}, & i \geq 1, 0 < j \leq i; 
\sum_{k=1}^{+\infty} p_k, & i \geq 0, j = 0; 
p_{-1}, & j = i + 1, i \geq 0; 
0, & \text{others.}
\end{cases}
\]

From the opinion of applications, when $p_{i,j}$ has the form of (3.2), $W$ is the length of the waiting line when a new customer enters the server system. If $p_{i,j}$ has the form of (3.1), $W$ is also near the length of the waiting line when a customer leaves the server system. The following results are not only the bases of our main results, but are also of independent interest. They seem to be some fundamental conclusions for the process $W$, however, we cannot find a suitable reference. For convenience of reference, we state them and provide the detailed proofs here.
For any \( s \in [0, 1] \), let
\[
\phi(s) = \sum_{n=-1}^{+\infty} p_n s^{n+1}.
\]
We have that

**Lemma 3.1** If \( \phi'(1) = 1 \), then there exists a unique differentiable function \( h(s) \in [0, 1] \) such that \( h(s) = s\phi(h(s)) \) for all \( s \in [0, 1] \). Furthermore,

1. as \( s \to 0^+ \), \( h(s)/s \to \phi(0) = p_{-1} \) and as \( s \to 1^- \), \( h(s) \to 1 \).
2. for all \( s \in (0, 1) \),
\[
h'(s) = \frac{\phi(h(s))}{1 - s\phi'(h(s))} > 0.
\]
3. as \( s \to 0^+ \), \( (h(s) - p_{-1}s)/s^2 \to p_0p_{-1} \).

4. if \( \sqrt{\phi''(1)} = \sigma < +\infty \), then
\[
\lim_{s \to 1^-} \frac{1 - s\phi'(h(s))}{\sqrt{1 - s}} = 2\sigma.
\]

**Proof.** Following the basic mathematical analysis, one can readily find that there exists a unique differentiable function \( h(s) \in [0, 1] \) such that
\[
h(s) = s\phi(h(s))
\]
for all \( s \in [0, 1] \), from which we can directly get items (1) and (2); the proofs are omitted. We only show the details of proof for the items (3) and (4).

To prove (3), we can get by using the L’Hospital’s law and the formula of \( h'(s) \) in the item (2) that
\[
\lim_{s \to 0^+} \frac{h(s) - p_{-1}s}{s^2} = \lim_{s \to 0^+} \frac{h'(s) - p_{-1}}{2s} = \lim_{s \to 0^+} \frac{\phi(h(s)) - p_{-1}(1 - s\phi'(h(s)))}{2(1 - s\phi'(h(s)))s} = \lim_{s \to 0^+} \frac{\phi'(h(s))h'(s) + p_{-1}\phi'(h(s)) + p_{-1}s\phi''(h(s))h'(s)}{2(1 - s\phi'(h(s))) - 2s(\phi'(h(s)) + s\phi''(h(s))h'(s))}.
\]
Note that as \( s \to 0 \), \( h(s) \to 0 \), \( \phi'(s) \to p_0 \) and \( h'(s) \to p_{-1} \). We obtain that
\[
\lim_{s \to 0^+} \frac{h(s) - p_{-1}s}{s^2} = \lim_{s \to 0^+} \frac{\phi'(h(s))h'(s) + p_{-1}\phi'(h(s))}{2} = p_{-1}p_0.
\]
To prove (4), by the L’Hospital’s law again,
\[
\lim_{s \to 1^-} \frac{(1 - s\phi'(h(s)))^2}{1 - s} = \lim_{s \to 1^-} \frac{2(1 - s\phi'(h(s)))(\phi'(h(s)) + s\phi''(h(s))h'(s))}{2(1 - s\phi'(h(s)))\phi''(h(s))h'(s)} = \lim_{s \to 1^-} \frac{2(1 - s\phi'(h(s)))\phi''(h(s))h'(s)}{2(1 - s\phi'(h(s)))\phi''(h(s))h'(s)}.
\]
From the item (2), we have that \( \phi(h(s)) = (1 - s\phi'(h(s)))h'(s) \). Therefore,
\[
\lim_{s \to 1^-} \frac{(1 - s\phi'(h(s)))^2}{1 - s} = \lim_{s \to 1^-} 2\phi(h(s))\phi''(h(s)) = 2\sigma^2,
\]
which implies the desired result. \( \square \)

Let \( \tau = \inf\{n > 0, W_n = 0\} \), and define
\[
f_k(s) = \mathbb{E}(s^\tau | W_0 = k).
\]

**Lemma 3.2** Suppose that for each every \( i, j \), the transition probability \( p_{i,j} \) is given by \((3.1)\). Then
\[
f_0(s) = 1 + p_{-1}s - \frac{p_{-1}s}{h(s)}.
\]

**Proof.** By the theory of Markov chains, we know that the family of functions \( \{f_k(s); k \geq 0\} \) satisfies the following equations,
\[
f_0(s) = s\left(\sum_{k=1}^{+\infty} p_k f_k(s) + p_0 + p_{-1}\right), \tag{3.3}
f_1(s) = s\left(\sum_{k=1}^{+\infty} p_{k-1} f_k(s) + p_{-1}\right). \tag{3.4}
\]

By the Markov property and noting the fact that \( W \) is homogenous in the states \( k \geq 1 \) under the setting of \((3.1)\), we have that
\[
f_k(s) = (f_1(s))^k,
\]
which plus \((3.4)\) implies that \( f_1(s) = h(s) \). Therefore from \((3.3)\) we get that
\[
f_0(s) = s\left(\sum_{k=1}^{+\infty} p_k h^{k+1}(s) + (p_0 + p_{-1})h(s)\right)/h(s)
= s\left(\phi(h(s)) - p_{-1} + p_{-1}h(s)\right)/h(s) = 1 + p_{-1}s - \frac{p_{-1}s}{h(s)},
\]
which is the desired result. \( \square \)

**Lemma 3.3** Suppose that for each every \( i, j \), the transition probability \( p_{i,j} \) is given by \((3.2)\). Then
\[
f_0(s) = \frac{s(1 - \phi(h(s)))}{(1 - h(s))}.
\]

**Proof.** By the theory of Markov chains, we know that the family of functions \( \{f_k(s); k \geq 0\} \) satisfies the following equations
\[
f_k(s) = s\left(\sum_{n=k}^{+\infty} p_n\right) + s \sum_{j=-1}^{k-1} p_j f_{k-j}(s), \quad k \geq 0. \tag{3.5}
\]
For every \((u, s) \in [0, 1] \times [0, 1]\), define
\[
F(u, s) = \sum_{k=0}^{+\infty} u^k f_k(s).
\]
From (3.5) we get that
\[
F(u, s) = \sum_{k=0}^{+\infty} u^k s \left( \sum_{n=k}^{+\infty} p_n \right) + s \sum_{j=-1}^{k-1} p_j f_{k-j}(s)
\]
\[
= s \sum_{n=0}^{+\infty} p_n \sum_{k=0}^{n} u^k + s \sum_{j=-1}^{+\infty} \sum_{k=j+1}^{+\infty} u^k f_{k-j}(s)
\]
\[
= s \frac{1 - \phi(u)}{1 - u} + s \frac{\phi(u)(F(u, s) - f_0(s))}{(1 - u)(u - s\phi(u))},
\]
which yields that
\[
F(u, s) = \frac{su(1 - \phi(u)) - s(1 - u)\phi(u)f_0(s)}{(1 - u)(u - s\phi(u))}.
\]
For each \(s \in (0, 1)\), let
\[
D_+(s) = \{ u \in [0, 1]; u - s\phi(u) > 0 \} \quad \text{and} \quad D_-(s) = \{ u \in [0, 1]; u - s\phi(u) < 0 \}.
\]
Due to that \(t - s\phi(t)\) is non-decreasing in \(t \in [0, 1]\), we know that for each \(u \in D_-(s), v \in D_+(s)\),
\[
0 < h(s) < v.
\]
In addition, due to the fact that \(F(u, s) \geq 0\) for all \((u, s) \in [0, 1] \times [0, 1]\), we have that for all \(v \in D_+(s)\),
\[
sv(1 - \phi(v)) - s(1 - v)\phi(v)f_0(s) \geq 0,
\]
which means that for all \(v \in D_+(s)\),
\[
f_0(s) \leq \frac{v(1 - \phi(v))}{(1 - v)\phi(v)}.
\]
Similarly, for all \(u \in D_-(s)\),
\[
f_0(s) \geq \frac{u(1 - \phi(u))}{(1 - u)\phi(u)}.
\]
Observe that for each \(s \in (0, 1)\),
\[
\left( \frac{s(1 - \phi(s))}{(1 - s)\phi(s)} \right)' = \frac{\phi(s) - \phi^2(s) - s(1 - s)\phi'(s)}{(1 - s)^2\phi^2(s)}.
\]
From the facts \(\phi'(1) = 1, p_0 < 1\) and convexity of \(\phi\), we know that \(p_{-1} = \phi(0) > 0\) and for all \(s \in (0, 1)\), \(\phi(s) > s\) and
\[
\phi(s) - \phi^2(s) - s(1 - s)\phi'(s) > s(1 - \phi(s) - (1 - s)\phi'(s)) > 0.
\]
Therefore the function
\[
\frac{s(1 - \phi(s))}{(1 - s)\phi(s)}
\]
is non-decreasing for \(s \in [0, 1]\), which implies that
\[
\lim_{u \to h(s)-} \frac{u(1 - \phi(u))}{(1 - u)\phi(u)} \leq f_0(s) \leq \lim_{v \to h(s)+} \frac{v(1 - \phi(v))}{(1 - v)\phi(v)},
\]
namely,
\[
f_0(s) = \frac{h(s)(1 - \phi(h(s)))}{(1 - h(s))\phi(h(s))},
\]
which leads to the desired result. \(\square\)

For each \(\lambda < 0\), let
\[
\Lambda(\lambda) = \ln f_0(e^\lambda).
\]

**Lemma 3.4** If (3.1) or (3.2) holds, \(\Lambda'(\lambda)\) is monotone increasing and
\[
\lim_{\lambda \to -\infty} \Lambda'(\lambda) = 1, \quad \lim_{\lambda \to 0^-} \Lambda'(\lambda) = +\infty.
\]

**Proof.** From the fact that
\[
\Lambda(\lambda) = \ln(f_0(e^\lambda)) = \ln(\mathbb{E}(e^{\lambda \tau} | W_0 = 0)),
\]
we have that
\[
\Lambda''(\lambda) = \frac{\mathbb{E}(\tau^2 e^{\lambda \tau} | W_0 = 0)\mathbb{E}(e^{\lambda \tau} | W_0 = 0) - \mathbb{E}(\tau e^{\lambda \tau} | W_0 = 0)^2}{\mathbb{E}(e^{\lambda \tau} | W_0 = 0)^2}.
\]
Therefore, the Hölder inequality implies that \(\Lambda''(\lambda) > 0\) for all \(\lambda < 0\) and hence \(\Lambda'(\lambda)\) is monotone increasing for \(\lambda \in (-\infty, 0)\).

To show the limits, we first consider the case where (3.1) holds. In this case, from Lemma 3.2 it follows that
\[
\Lambda(\lambda) = \ln \left(1 + p_{-1}e^\lambda - \frac{p_{-1}e^\lambda}{h(e^\lambda)}\right).
\]
By some simple computations, we get that
\[
\Lambda'(\lambda) = \frac{p_{-1}e^\lambda \left(h(e^\lambda)(1 - e^\lambda\phi'(h(e^\lambda))) + e^\lambda\phi'(h(e^\lambda))\right)}{(1 - e^\lambda\phi'(h(e^\lambda))) \left(1 + p_{-1}e^\lambda\right) - p_{-1}e^\lambda}.
\]
Using Lemma 3.3 and noting that \(\phi(0) = p_{-1}, \phi'(0) = p_0\) and \(\phi(1) = \phi'(1) = 1\), we have that
\[
\lim_{\lambda \to 0^-} \Lambda'(\lambda) = \lim_{\lambda \to 0^-} \frac{p_{-1}}{1 - e^\lambda\phi'(h(e^\lambda))} = +\infty,
\]
\[
\lim_{\lambda \to -\infty} \Lambda'(\lambda) = \lim_{\lambda \to -\infty} \frac{p_{-1}e^\lambda \left(h(e^\lambda) + e^\lambda\phi'(h(e^\lambda))\right)}{(1 + p_{-1}e^\lambda)h(e^\lambda) - p_{-1}e^\lambda} = \lim_{\lambda \to -\infty} \frac{p_{-1}(p_{-1} + p_0)e^{2\lambda}}{p_{-1}(p_{-1} + p_0)e^{2\lambda}} = 1.
\]
Now, we consider the case where (3.2) holds. From Lemma 3.3 it follows that

\[ \Lambda(\lambda) = \ln \left( \frac{e^{\lambda} h(e^{\lambda})}{1 + p_{-} e^{\lambda} h(e^{\lambda}) - p_{-} e^{\lambda}} \right). \]

It is easy to see that

\[ \Lambda'(\lambda) = 1 + \frac{h'(e^{\lambda}) e^{\lambda}}{1 - h(e^{\lambda})} - \frac{\phi'(h(e^{\lambda})) h'(e^{\lambda}) e^{\lambda}}{1 - \phi(h(e^{\lambda}))}. \]

Using Lemma 3.1 and noting that \( \phi(0) = p_{-}, \phi'(0) = p_{0} \) and \( \phi(1) = \phi'(1) = 1 \), we get that

\[ \lim_{\lambda \to -\infty} \Lambda'(\lambda) = 1 + \lim_{s \to 0} \left[ \frac{h'(s) s}{1 - h(s)} - \frac{\phi'(h(s)) h'(s) s}{1 - \phi(h(s))} \right] = 1, \]

and that

\[ \lim_{\lambda \to 0} \Lambda'(\lambda) = \lim_{s \to 1-} \left[ \frac{1 - \phi(h(s)) - \phi'(h(s))(1 - h(s))}{(1 - h(s))(1 - \phi(h(s)))} h'(s) s + 1 \right]. \]

From Lemma 3.1 we know that as \( s \to 1, h(s) \to 1 \) and \( h'(s) \to +\infty \). It is easy to see that

\[ \lim_{s \to 1} \frac{1 - \phi(h(s)) - \phi'(h(s))(1 - h(s))}{(1 - h(s))(1 - \phi(h(s)))} h'(s) = \infty. \]

Consequently, \( \lim_{\lambda \to 0} \Lambda'(\lambda) = \infty. \)

**Lemma 3.5** If (3.1) holds, then as \( \lambda \to -\infty, \lambda - \Lambda(\lambda) \to -\ln(p_{-} + p_{0}). \) If (3.2) holds, then as \( \lambda \to -\infty, \lambda - \Lambda(\lambda) \to -\ln(1 - p_{-}). \)

**Proof.** When (3.1) holds,

\[ \lambda - \Lambda(\lambda) = \ln \left( \frac{e^{\lambda} h(e^{\lambda})}{1 + p_{-} e^{\lambda} h(e^{\lambda}) - p_{-} e^{\lambda}} \right). \]

Then by the L’Hospital law, we get that

\[ \lim_{\lambda \to -\infty} \lambda - \Lambda(\lambda) = \lim_{\lambda \to -\infty} \ln \left( \frac{e^{\lambda} h(e^{\lambda})}{1 + p_{-} e^{\lambda} h(e^{\lambda}) - p_{-} e^{\lambda}} \right) = \ln \left( \lim_{s \to 0} \frac{sh(s)}{(1 + p_{-}) h(s) - p_{-}} \right). \]

Using the item (3) in Lemma 3.1 we obtain that

\[ \lim_{\lambda \to -\infty} \lambda - \Lambda(\lambda) = \ln \left( \lim_{s \to 0} \frac{p_{-} s^2}{p_{-} s + p_{-} p_{0} s^2 + p_{-} s^2 - p_{-} s} \right) = \ln \left( \frac{1}{p_{-} + p_{0}} \right) = -\ln(p_{-} + p_{0}). \]

When (3.2) holds, we have that

\[ \lim_{\lambda \to -\infty} \lambda - \Lambda(\lambda) = \lim_{\lambda \to -\infty} \ln \left( \frac{1 - h(e^{\lambda})}{1 - \phi(h(e^{\lambda}))} \right) = -\ln(1 - p_{-}). \]

The proofs are completed. \( \square \)
Lemma 3.6 Let \( P(s) = 1/(1 - f_0(s)) \) for \( s \in (0, 1) \). Suppose that there exist \( 1 > \alpha > 0 \) and \( c > 0 \) such that
\[
\lim_{s \to 1^-} \frac{1 - s\phi'(h(s))}{(1 - s)^\alpha} = c.
\]
Then in the case where (3.1) holds,
\[
\lim_{s \to 1^-} P(s)(1 - s)^{1 - \alpha} = \frac{(1 - \alpha)c}{p_{-1}},
\]
and in the case where (3.2) holds,
\[
\lim_{s \to 1^-} P(s)(1 - s)^{\alpha} = \frac{1}{(1 - \alpha)c}.
\]

Proof. If (3.1) holds, then by Lemma 3.1 (2), Lemma 3.2 and Assumption (H),
\[
\lim_{s \to 1^-} P(s)(1 - s)^{1 - \alpha} = \lim_{s \to 1^-} \frac{h(s)(1 - s)^{1 - \alpha}}{p_{-1}s(1 - h(s))}
\]
\[
= \lim_{s \to 1^-} \frac{1 - \alpha}{p_{-1}} \lim_{s \to 1^-} \frac{1 - s\phi'(h(s))}{(1 - s)^\alpha} = \frac{c(1 - \alpha)}{p_{-1}}.
\]
If (3.2) holds, then Lemma 3.3 implies that
\[
P(s) = \frac{1}{1 - f_0(s)} = \frac{1 - h(s)}{1 - h(s) - s + s\phi(h(s))} = \frac{1 - h(s)}{1 - s}.
\]
Therefore, from Assumption (H), we get that
\[
\lim_{s \to 1^-} P(s)(1 - s)^{\alpha} = \lim_{s \to 1^-} \frac{1 - h(s)}{(1 - s)^{1 - \alpha}} = \frac{1}{(1 - \alpha)c}.
\]
The proof is completed. \( \square \)

4. The proof of LDP

In this section, we will provide the proof of LDP. For this end, we define \( \bar{S}_0 = 0 \) and for every \( n \geq 1 \),
\[
\bar{S}_n = M_n - S_n,
\]
where \( \{S_n\} \) is the random walk given in Section 2 and
\[
M_n = \max_{0 \leq k \leq n} S_k.
\]
We can readily see that \( \{\bar{S}_n, n \geq 0\} \) is a nonnegative homogeneous Markov chain with the transition probability matrix \( (p_{ij})_{i,j \geq 0} \), which is given by (3.1) if \( S \) is a right continuous random walk or is given by (3.2) if \( S \) is a left continuous random walk, where \( p_{-1} = q \).
Let $L^0_n(\bar{S})$ be the occupation time of $\bar{S}$ at the site 0 up to time $n$, i.e.

\[ L^0_0(\bar{S}) = 0 \quad \text{and} \quad L^0_n(\bar{S}) = \sum_{k=1}^{n} 1_{\{S_k=0\}}, \quad n \geq 1. \]

Then it is obvious that for every $n \geq 0$,

\[ A_n = L^0_n(\bar{S}). \quad (4.1) \]

Let $\bar{\tau}_1 = \inf\{n > 0, \bar{S}_n = 0\}$ and

\[ \bar{\tau}_{k+1} = \inf\{n > \tau_k, \bar{S}_n = 0\}, \]

for all $k \geq 1$. $(4.1)$ suggests that

\[ A_n = L^0_n(\bar{S}) = \sup\{k \geq 1, \bar{\tau}_k \leq n\}. \quad (4.2) \]

The Markov property indicates that $\bar{\tau}_1$ and $\bar{\tau}_{k+1} - \bar{\tau}_k$, $k \geq 1$, are i.i.d.

The proof of the LDP of $A_n$ is read as follows.

**Proof.** Let $Y_i$, $i \geq 1$ be a sequence of i.i.d. random variables with the same distribution as that of $\bar{\tau}_1$. Then

\[ L^0_n(\bar{S}) = \sup_k\{k \geq 0, \sum_{i=1}^{k} Y_i \leq n\}. \]

Therefore, for any $0 < x \leq 1$,

\[ \mathbb{P}(L^0_n(\bar{S}) \geq \lfloor xn \rfloor) \leq \mathbb{P}(A_n \geq xn) = \mathbb{P}(L^0_n(\bar{S}) \geq xn) \leq \mathbb{P}(L^0_n(\bar{S}) \geq \lceil xn \rceil), \]

which implies that

\[ \mathbb{P}\left(\lfloor \sum_{i=1}^{\lfloor xn \rfloor} Y_i \rfloor \leq n\right) \leq \mathbb{P}(A_n \geq xn) \leq \mathbb{P}\left(\lceil \sum_{i=1}^{\lfloor xn \rfloor} Y_i \rceil \leq n\right), \]

where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ are the minimal integer larger than $\cdot$ and maximal integer less than $\cdot$, respectively.

When $S$ is right continuous, since $\bar{S}_0 = 0$ and $Y \overset{d}{=} \bar{\tau}_1$, from Lemma 3.2 we have that for every $\lambda < 0$,

\[ \mathbb{E}(e^{\lambda Y}) = \Lambda_r(\lambda) \]

and $\mathbb{E}(Y) = +\infty$. Applying Cramér Theorem [6, P.27], Lemma 3.4 and Lemma 3.5, we obtain that

\[ \lim_{n \to +\infty} \frac{1}{n} \ln \mathbb{P}\left(\sum_{i=1}^{n} Y_i \leq xn\right) = -\Lambda^*_r(x). \]

Similarly, when $S$ is left continuous,

\[ \lim_{n \to +\infty} \frac{1}{n} \ln \mathbb{P}\left(\sum_{i=1}^{n} Y_i \leq xn\right) = -\Lambda^*_l(x). \]

The rest is same as the proof of Theorem 2 in Gantert and Zeitouni [2] and is omitted. $\Box$
5. The proof of MDP

In this section, we will first prove the MDP for \( A_n \) under the assumptions (H). Then we provide some sufficient conditions such that the assumption (H) holds. Based on these sufficient condition, we can directly get Corollary 2.1 and Corollary 2.2. The following lemma is nothing but a special presentation of Chen [4, Theorem 2] in our situations.

**Lemma 5.1** Suppose that there is a non-decreasing positive function \( a(t) \) on \([1, \infty)\) such that \( a(t) \uparrow \infty \),

\[
\lim_{n \to +\infty} \frac{1}{a(n)} \sum_{k=1}^{n} \mathbb{P}(\overline{S}_n = 0|\overline{S}_0 = 0) = 1, 
\]

and there is a constant \( p \in [0, 1) \) such that for every \( t > 0 \),

\[
\lim_{\lambda \to +\infty} \frac{a(\lambda t)}{a(\lambda)} = t^p.
\]

Let \( \{b_n\} \) be a positive sequence satisfying that as \( n \to +\infty \)

\[
b_n \to +\infty \text{ and } b_n/n \to 0.
\]

Then

\[
\lim_{n \to +\infty} \frac{1}{b_n} \ln \mathbb{P}\left( \sum_{k=1}^{n} 1_{\{\overline{S}_k = 0\}} > \lambda a\left(\frac{n}{b_n}\right) b_n \right) = -(1 - p) \left( \frac{\lambda p}{\Gamma(p+1)} \right)^{(1-p)^{-1}}.
\]

Below, we provide the proofs of the MDP.

**Proof of Theorem 2.2** (1) By the basic theory of Markov chains, we have that

\[
\sum_{n=0}^{+\infty} \mathbb{P}(\overline{S}_n = 0|\overline{S}_0 = 0)s^n = \frac{1}{1 - f_0(s)},
\]

for any \( s \in [0, 1) \). Therefore, from Lemma 3.6 we know that as \( s \to 1^- \)

\[
\sum_{n=0}^{+\infty} \mathbb{P}(\overline{S}_n = 0|\overline{S}_0 = 0)s^n \sim \frac{(1 - \alpha)c}{q} (1 - s)^{\alpha-1},
\]

where \( q = p_{-1} \) for the setting of Theorem 2.2. By Tauberian Theorem [24], we know that

\[
\lim_{n \to +\infty} \frac{\sum_{k=0}^{n} \mathbb{P}(\overline{S}_n = 0|\overline{S}_0 = 0)}{n^{1-\alpha}} = \frac{(1 - \alpha)c}{q\Gamma(2 - \alpha)}. \tag{5.1}
\]

Note that

\[
A_n = \sum_{k=1}^{n} 1_{\{\overline{S}_k = 0\}}.
\]

From (5.1), we know that Lemma 5.1 is fulfilled as

\[
p = 1 - \alpha, \quad a(t) = \frac{(1 - \alpha)c}{q\Gamma(2 - \alpha)} t^{1-\alpha}, \quad b_n = c_n.
\]
Consequently,
\[
\lim_{n \to +\infty} \frac{1}{c_n} \ln \mathbb{P}(A_n \geq x n^{1-\alpha} c_n) = \lim_{n \to +\infty} \frac{1}{c_n} \ln \mathbb{P}(A_n \geq x \left( \frac{n}{c_n} \right)^{1-\alpha} c_n) = \lim_{n \to +\infty} \frac{1}{c_n} \ln \mathbb{P}(A_n \geq x \Gamma(2-\alpha) q a(n/c_n) c_n) = -\alpha \left( \frac{x q}{(1-\alpha) c} \right)^{1/\alpha} c_n \ln \mathbb{P}(A_n \geq x^{1/(1-\alpha)}).
\]

(2) In this case, by the same discussion we know that Lemma 5.1 is fulfilled for
\[p = \alpha, \quad a(t) = \frac{1}{(1-\alpha) c \Gamma(1+\alpha)} t^{\alpha}, \quad b_n = c_n.\]

Therefore,
\[
\lim_{n \to +\infty} \frac{1}{c_n} \ln \mathbb{P}(A_n \geq x n^{\alpha} c_n^{1-\alpha}) = \lim_{n \to +\infty} \frac{1}{c_n} \ln \mathbb{P}(A_n \geq x \left( \frac{n}{c_n} \right)^{\alpha} c_n) = \lim_{n \to +\infty} \frac{1}{c_n} \ln \mathbb{P}(A_n \geq x \Gamma(1+\alpha) (1-\alpha) c a(n/c_n) c_n) = -\left[ c(1-\alpha) 2^{-\alpha} \alpha^{1/(1-\alpha)} \right] x^{1/(1-\alpha)}.
\]

The proof is complete. \(\square\)

**Remark 5.1** From Lemma 3.1 we know that when \(\sqrt{\varphi''(1)} = \sigma < \infty\), Assumption (H) always holds for \(\alpha = 1/2\) and \(c = \sqrt{2\sigma}\).

For the case of \(\varphi''(1) = \infty\), we have the following special results.

**Lemma 5.2** If \(\phi(s) = s + \frac{\gamma}{1+s} (1-s)^{1+\beta} \) for some \(\beta \in (0,1)\) and \(\gamma \in (0,1)\), then the assumption (H) holds for \(c = \gamma^{1/(1+\beta)} (1+\beta)^{\beta/(1+\beta)}\) and \(\alpha = \beta/(1+\beta)\).

**Proof.** From \(h(s) = s \phi(h(s))\) we get that
\[h(s) = sh(s) + \frac{s \gamma}{1+\beta} (1-h(s))^{1+\beta},\]
which leads to that
\[
\frac{1-s}{1-h(s)} = M(s)(1-h(s))^{\beta}, \tag{5.2}
\]
where \(M(s) = s \gamma /[ (1+\beta) h(s) ] \). Therefore for any \(s \in (0,1)\),
\[
1-s \phi'(h(s)) = 1-s(1-\gamma(1-h(s))^{\beta}) = 1-s + \frac{s \gamma (1-h(s))^{1+\beta}}{1-h(s)} = 1-s + \frac{(1+\beta) h(s) (1-s)}{1-h(s)} = (1+\beta h(s)) \frac{1-s}{1-h(s)}.
\]
Repeatedly using (5.2) we get that for any $k \geq 1$,
\[
1 - s\phi'(h(s)) = (1 + \beta h(s))M(s)(1 - h(s))^{\beta} = (1 + \beta h(s))M(s)\left(\frac{1 - h(s)}{1 - s}\right)^{\beta}(1 - s)^{\beta} \\
= (1 + \beta h(s))M(s)M(s)^{-\beta}(1 - h(s))^{-\beta^2}(1 - s)^{\beta} \\
= (1 + \beta h(s))M(s)(\frac{1 - h(s)}{1 - s})^{-\beta^2}(1 - s)^{\beta}(1 - s)^{-\beta^2} \\
= (1 + \beta h(s))M(s)\beta M(s)^{\beta^2}(1 - h(s))^\beta(1 - s)^{\beta^2}(1 - s)^{-\beta^2} \\
= \cdots \\
= (1 + \beta h(s))M(s)^{1+\cdot\cdot\cdot+(-\beta)^k}(1 - s)^{\beta+\cdot\cdot\cdot+(-1)^{k+1}\beta}(1 - h(s))^{-1)^{k+1},
\]
Since $0 < \beta < 1$ and $0 < 1 - h(s) < 1$ for all $s \in (0, 1)$, letting $k \to +\infty$, we get that
\[
M(s)^{1-\beta+\cdot\cdot\cdot+(-1)^k\beta} \to M(s)^{1/(1+\beta)}, \quad (1 - h(s))^{-1)^{k+1} \to 1
\]
and
\[
(1 - s)^{-\beta^2+\cdot\cdot\cdot+(-1)^{k+1}\beta} \to (1 - s)^{\beta/(1+\beta)}.
\]
Consequently,
\[
1 - s\phi'(h(s)) = (1 + \beta h(s))M(s)^{1/(1+\beta)}(1 - s)^{\beta/(1+\beta)},
\]
which implies that
\[
\lim_{s \to 1} \frac{1 - s\phi'(h(s))}{(1 - s)^{\beta/(1+\beta)}} = \lim_{s \to 1} (1 + \beta h(s))M(s)^{1/(1+\beta)} = \gamma^{1/(1+\beta)}(1 + \beta)^{\beta/(1+\beta)},
\]
where we use the fact that as $s \to 1$, $h(s) \to 1$ and $M(s) \to \gamma$. Therefore, Assumption (H) holds for $c = \gamma^{1/(1+\beta)}(1 + \beta)^{\beta/(1+\beta)}$ and $\alpha = \beta/(1 + \beta)$.

From Theorem 2.2, Remark 5.1 and Lemma 5.2 we can readily get the corollaries 2.1 and 2.2, where the fact $q = \gamma/(1 + \beta)$ is used for the latter. The details are omitted.

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