The Heyde theorem on a group $\mathbb{R}^n \times D$, where $D$ is a discrete Abelian group

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Abstract

Heyde proved that a Gaussian distribution on the real line is characterized by the symmetry of the conditional distribution of one linear statistic given another. The present article is devoted to a group analogue of the Heyde theorem. We describe distributions of independent random variables $\xi_1, \xi_2$ with values in a group $X = \mathbb{R}^n \times D$, where $D$ is a discrete Abelian group, which are characterized by the symmetry of the conditional distribution of the linear statistic $L_2 = \xi_1 + \delta \xi_2$ given $L_1 = \xi_1 + \xi_2$, where $\delta$ is a topological automorphism of $X$ such that $\text{Ker}(I + \delta) = \{0\}$.

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1 Introduction

Characterization problems in mathematical statistics are statements in which the description of possible distributions of random variables follows from properties of some functions in these variables. A large number of studies have been devoted to theorems characterizing the Gaussian distribution on the real line. In particular, in 1970 C.C. Heyde proved the following theorem.

The Heyde theorem ( [14], [15, § 13.4.1]). Let $\xi_j$, $j = 1, 2, \ldots, n$, $n \geq 2$, be independent random variables. Consider linear statistics $L_1 = \alpha_1 \xi_1 + \cdots + \alpha_n \xi_n$ and $L_2 = \beta_1 \xi_1 + \cdots + \beta_n \xi_n$, where the coefficients $\alpha_j$, $\beta_j$ are nonzero real numbers such that $\beta_i \alpha_i^{-1} \pm \beta_j \alpha_j^{-1} \neq 0$ for all $i \neq j$. If the conditional distribution of $L_2$ given $L_1$ is symmetric then all random variables $\xi_j$ are Gaussian.

The theory of characterization theorems is being actively developed in a situation when random variables take values in a locally compact Abelian group (see e.g. [5]). Group analogues of the Heyde theorem were studied in the case when independent random variables take values in locally compact Abelian groups and coefficients of the linear statistics are either topological automorphisms of the group ( [2–4, 7, 8, 11, 16, 17, 19]) or integers ( [18]). Basically, the following problem was studied.

Let $X$ be a second countable locally compact Abelian group. Denote by Aut($X$) the group of topological automorphisms of the group $X$ and denote by $I$ the identity automorphism. Let $\xi_1$ and $\xi_2$ be independent random variables with values in $X$ and distributions $\mu_1$ and $\mu_2$. Let $\delta \in \text{Aut}(X)$ and

$$I \pm \delta \in \text{Aut}(X).$$

(1)

What can be said about the distributions $\mu_1$ and $\mu_2$, if the conditional distribution of the linear statistic $L_2 = \xi_1 + \delta \xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric?
This problem was studied on discrete groups, the real $m$-dimensional space, products of these groups, $a$-adic solenoids. The proofs of the obtained results essentially used condition (1).

It turns out (see [3][11]) that in some cases theorems, which were proved under condition (1), remain true if condition (1) is replaced by the weaker condition

$$\text{Ker}(I + \delta) = \{0\}. \quad (2)$$

In this article we generalize results of [10] and describe distributions on a group of the form

$$X = \mathbb{R}^n \times D, \quad (3)$$

where $D$ is a discrete Abelian group, which are characterized by the symmetry of the conditional distribution of one linear statistic given another. Note that our result also generalizes the result of the article [16], where the group analogue of the Heyde theorem was proved on the group of form (3), where $n = 1$ and condition (1) holds.

2 Notation and definitions

In the article we use standard results on the structure theory of locally compact Abelian groups and abstract harmonic analysis (see e.g. [13]). Let $X$ be a second countable locally compact Abelian group, $Y = X^*$ be its character group, and $(x,y)$ be the value of a character $y \in Y$ at an element $x \in X$. Let $K$ be a subgroup of $Y$. Denote by $A(X,K) = \{x \in X : (x,y) = 1 \quad \forall \ y \in K\}$ the annihilator of $K$. Denote by $b_X$ the subgroup of all compact elements of $X$, and denote by $c_X$ the connected component of zero of $X$. If $\delta$ is a continuous endomorphism of the group $X$ then the adjoint endomorphism $\delta$ of the group $X$ is defined by the formula $\langle x, \tilde{\delta}y \rangle = (\delta x, y)$ for all $x \in X$, $y \in Y$. Let $p$ be a prime number. The $p$-component of an Abelian group is a subgroup consisting of elements whose order is a power of $p$. Denote by $X_p$ the $p$-component of the group $X$. A torsion group $X$ is called $p$-primary, if $X = X_p$. For each integer $n$, $n \neq 0$, let $f_n : X \mapsto X$ be the endomorphism $f_n x = nx$. Set $X^{(n)} = f_n(X)$, $X_{(n)} = \text{Ker} f_n$. A subgroup $G$ of a group $X$ is said to be characteristic if $G$ is invariant under each topological automorphism of a group $X$. The subgroups $b_X$, $c_X$, $X^{(n)}$, $X_{(n)}$ are characteristic.

Let $f(y)$ be a function on $Y$, and $h \in Y$. Denote by $\Delta_h$ the finite difference operator

$$\Delta_h f(y) = f(y + h) - f(y), \quad y \in Y.$$ 

A function $f(y)$ on $Y$ is called a polynomial if

$$\Delta_h^{n+1} f(y) = 0$$

for some $n$ and for all $y, h \in Y$.

Let $M^1(X)$ be the convolution semigroup of probability distributions on $X$. Let $\hat{\mu}(y) = \int_X(x,y)d\mu(x)$ be the characteristic function of a distribution $\mu \in M^1(X)$, and $\sigma(\mu)$ be the support of $\mu$. Define $\hat{\mu} \in M^1(X)$ by the formula $\hat{\mu}(B) = \mu(-B)$ for all Borel sets $B$ in $X$. Then $\hat{\mu}(y) = \hat{\mu}(y)$. Denote by $m_K$ the Haar distribution of a compact subgroup $K$ of a group $X$ and denote by $E_x$ the degenerate distribution concentrated at the point $x \in X$. Note that the characteristic function of the distribution $m_K$ has the form

$$\hat{m}_K(y) = \begin{cases} 1, & \text{if } y \in A(Y,K), \\ 0, & \text{if } y \notin A(Y,K). \end{cases} \quad (4)$$

Denote by $\Gamma(\mathbb{R}^n)$ the set of Gaussian distributions on $\mathbb{R}^n$. 


3 Main theorem

The main result of the article is the following theorem.

**Theorem 1** Let \( X = \mathbb{R}^n \times D \), where \( D \) is a countable discrete Abelian group. Let \( \delta \in \text{Aut}(X) \) such that condition \((2)\) is fulfilled. Let \( \xi_1 \) and \( \xi_2 \) be independent random variables with values in \( X \) and distributions \( \mu_1 \) and \( \mu_2 \). If the conditional distribution of the linear statistic \( L_2 = \xi_1 + \delta \xi_2 \) given \( L_1 = \xi_1 + \xi_2 \) is symmetric, then \( \mu_j = \gamma_j * \rho_j * m_E * E_g \), where \( \gamma_j \in \Gamma(\mathbb{R}^n) \), \( \sigma(\rho_j) \subset D_2 \), \( F \) is a finite subgroup of \( D \) without elements of order 2, \( g_j \in D \), \( j = 1, 2 \). Moreover, \( \delta(F) = F \).

As appears from the paper \([19]\), on a \( 2 \)-primary finite group, even under condition \((1)\), one can hardly expect to obtain a reasonable description of distributions characterized by the symmetry of the conditional distribution of one linear statistic given another.

We need some lemmas to prove Theorem \((1)\).

**Lemma 1** \(([3], \text{Lemma 16.1})\) Let \( X \) be a second countable locally compact Abelian group, \( Y = X^* \). Let \( \delta \in \text{Aut}(X) \). Let \( \xi_1 \) and \( \xi_2 \) be independent random variables with values in \( X \) and distributions \( \mu_1 \) and \( \mu_2 \). The conditional distribution of the linear statistic \( L_2 = \xi_1 + \delta \xi_2 \) given \( L_1 = \xi_1 + \xi_2 \) is symmetric if and only if the characteristic functions \( \hat{\mu}_j(y) \) satisfy the equation

\[
\hat{\mu}_1(u+v)\hat{\mu}_2(u+\varepsilon v) = \hat{\mu}_1(u-v)\hat{\mu}_2(u-\varepsilon v), \quad u, v \in Y,
\]

where \( \varepsilon = \delta \).

**Lemma 2** \(([3])\) Let \( Y \) be a locally compact Abelian group, \( f(y) \) be a continuous polynomial on \( Y \). Then \( f(y) = \text{const} \) for \( y \in b_Y \).

For convenience, we formulate the following well-known statements.

**Lemma 3** Let \( X \) be a second countable locally compact Abelian group, \( Y = X^* \). Let \( \mu \in M^1(X) \). Then the set \( E = \{ y \in Y : \hat{\mu}(y) = 1 \} \) is a closed subgroup of \( Y \) and \( \sigma(\mu) \subset A(X, E) \).

**Lemma 4** \(([3], \text{Lemma 2.13})\) Let \( X \) be a topological group, \( G \) be a Borel subgroup of \( X \), \( \mu \in M^1(G) \), \( \mu = \mu_1 * \mu_2 \), where \( \mu_j \in M^1(X) \). Then the distributions \( \mu_j \) can be replaced by their shifts \( \mu'_j \) in such a manner that \( \mu = \mu'_1 * \mu'_2 \) and \( \mu'_j \in M^1(G) \).

The following lemma is crucial for the proof of Theorem \((1)\).

**Lemma 5** Let \( X = \mathbb{R}^n \times G \), where \( G \) is a countable discrete \( 2 \)-primary Abelian group. Let \( \delta \in \text{Aut}(X) \) such that condition \((2)\) is fulfilled. Let \( \xi_1 \) and \( \xi_2 \) be independent random variables with values in \( X \) and distributions \( \mu_1 \) and \( \mu_2 \). If the conditional distribution of the linear statistic \( L_2 = \xi_1 + \delta \xi_2 \) given \( L_1 = \xi_1 + \xi_2 \) is symmetric, then \( \mu_j = \gamma_j * \rho_j \), where \( \gamma_j \in \Gamma(\mathbb{R}^n) \), \( \sigma(\rho_j) \subset G \), \( j = 1, 2 \).

**Proof.** The group \( Y = X^* \) is topologically isomorphic to the group \( \mathbb{R}^n \times H \), where \( H = G^* \). To avoid introducing new notation we will suppose that \( Y = \mathbb{R}^n \times H \). Denote by \( x = (t,g) \), where \( t \in \mathbb{R}^n \), \( g \in G \), elements of the group \( X \). Denote by \( y = (s,h) \), where \( s \in \mathbb{R}^n \), \( h \in H \), elements of the group \( Y \). Put \( \varepsilon = \delta \).

1. First we shall show that the restriction of the endomorphism \( I - \delta \) on \( G \) has a zero kernel. If \( x \in G_2 \), \( x \neq 0 \), then \((I - \delta)x = (I + \delta)x \). It follows from the condition \((2)\) that \( x \not\in \text{Ker}(I - \delta) \).
Suppose that there exists an element \( x_0 \in G \setminus G(2) \) such that \( (I - \delta)x_0 = 0 \). Since the subgroup \( G \) is 2-primary, the element \( x_0 \) has an order \( 2^k \) for some \( k > 1 \), i.e. \( 2^kx_0 = 0 \) and \( 2^{k-1}x_0 \neq 0 \). Since \( (I - \delta)x_0 = 0 \), we have \( (I - \delta)2^{k-1}x_0 = 0 \). Since \( 2^{k-1}x_0 \in G(2) \setminus \{0\} \), as has been shown above, \( 2^{k-1}x_0 \notin \text{Ker}(I - \delta) \). We obtain the contradiction. Thus, we get

\[
(I - \delta)x \neq 0, \quad x \in G \setminus \{0\}. \tag{6}
\]

Since the subgroups \( G \) and \( H \) are characteristic, conditions (2), (3) and the compactness of \( H \) implies that

\[
(I \pm \varepsilon)H = H. \tag{7}
\]

2. We reduce the proof of the lemma to the case when the subgroup \( G \) is bounded. It follows from Lemma 1 that the characteristic functions \( \hat{\mu}_j(y) \) satisfy equation (5). Put \( \nu_j = \mu_j * \tilde{\mu}_j \). Then \( \hat{\nu}_j(y) = |\hat{\mu}_j(y)|^2 \geq 0 \), \( y \in Y \). It is obvious that the characteristic functions \( \hat{\nu}_j(y) \) satisfy equation (5) too. Since the subgroup \( H \) is characteristic, we can consider the restriction of equation (5) on the subgroup \( H \).

Let \( U \) be a neighborhood of zero of \( H \) such that all characteristic functions \( \nu_j(y) > 0 \) for \( y \in U \). Put \( \varphi_j(y) = -\ln \hat{\nu}_j(y) \), \( y \in U \).

Let \( V \) be a neighborhood of zero of \( H \) such that

\[
\sum_{j=1}^{8} \lambda_j(V) \subset U
\]

for all endomorphisms \( \lambda_j \in \{I, \varepsilon\} \).

Since \( G \) is a discrete torsion group, the group \( H \) is compact totally disconnected. Then each neighborhood of zero of \( H \) contains an open compact subgroup (\[13\], 7.7)). Let \( W \subset V \), where \( W \) is an open subgroup of \( H \). We prove that the functions \( \varphi_j(y) \) are polynomials on a subgroup.

It follows from (5) that the functions \( \varphi_j(y) \) satisfy equation

\[
\varphi_1(u + v) + \varphi_2(u + \varepsilon v) - \varphi_1(u - v) - \varphi_2(u - \varepsilon v) = 0, \quad u, \ v \in W. \tag{8}
\]

We use the finite differences method. Let \( k_1 \) be an arbitrary element of \( W \). Substitute in (8) \( u + \varepsilon k_1 \) for \( u \) and \( v + k_1 \) for \( v \). Subtracting equation (8) from the resulting equation we obtain

\[
\Delta_{l_{11}} \varphi_1(u + v) + \Delta_{l_{12}} \varphi_2(u + \varepsilon v) - \Delta_{l_{13}} \varphi_1(u - v) = 0, \quad u, v \in W, \tag{9}
\]

where \( l_{11} = (I + \varepsilon)k_1, l_{12} = 2\varepsilon k_1, l_{13} = (\varepsilon - I)k_1 \). Let \( k_2 \) be an arbitrary element of \( W \). Substitute in (9) \( u + k_2 \) for \( u \) and \( v + k_2 \) for \( v \). Subtracting equation (9) from the resulting equation we obtain

\[
\Delta_{l_{21}} \Delta_{l_{11}} \varphi_1(u + v) + \Delta_{l_{22}} \Delta_{l_{12}} \varphi_2(u + \varepsilon v) = 0, \quad u, v \in W, \tag{10}
\]

where \( l_{21} = 2k_2, l_{22} = (I + \varepsilon)k_2 \). Let \( k_3 \) be an arbitrary element of \( W \). Substitute in (10) \( u - \varepsilon k_3 \) for \( u \) and \( v + k_3 \) for \( v \). Subtracting equation (10) from the resulting equation we obtain

\[
\Delta_{l_{31}} \Delta_{l_{21}} \Delta_{l_{11}} \varphi_1(u + v) = 0, \quad u, v \in W, \tag{11}
\]

where \( l_{31} = (I - \varepsilon)k_3 \). Putting \( v = 0 \) in (11), we get

\[
\Delta_{l_{31}} \Delta_{l_{21}} \Delta_{l_{11}} \varphi_1(u) = 0, \quad u \in W. \tag{12}
\]
Since elements \( k_j \) are arbitrary, it follows from (7), (12) and the expressions of \( l_{11}, l_{21}, l_{31} \) that the function \( \varphi_1(y) \) satisfy the equation

\[
\Delta^3_{xy} \varphi_1(y) = 0, \quad h, y \in B,
\]

where the subgroup \( B = (I + \varepsilon)W \cap (I - \varepsilon)W \cap W^{(2)} \). Thus the function \( \varphi_1(y) \) is a polynomial on the subgroup \( B \).

Similarly, we get that the function \( \varphi_2(y) \) satisfy equation (13) too.

Since the subgroup \( W \) is open, it is closed and therefore compact. Hence the subgroup \( B \) is also compact. It follows from this, Lemma 2 and the condition \( \hat{\nu}_j(0) = 1 \) that \( \varphi_j(y) = 0 \) for \( y \in B \). Hence, \( \hat{\nu}_j(y) = 1 \) for \( y \in B \). Lemma 3 implies that \( \sigma(\nu_j) \subset A(X, B) \). Put \( D = A(X, B) \). Note that the subgroup \( D \) is generated by the annihilators \( A(X, (I + \varepsilon)W \cap (I - \varepsilon)W) \) and \( A(X, W^{(2)}) \).

Put \( W' = (I + \varepsilon)W \cap (I - \varepsilon)W \). It follows from condition (7) that the endomorphisms \( I + \varepsilon, I - \varepsilon \) are open. Then the subgroup \( \text{Ann}(W) \) is also open in \( H \). Note that \( (Y/W')^* \approx A(X, W') \). Since the factor-group \( H/W' \) is finite, the annihilator \( A(X, W') = A(\mathbb{R}^n \times G, W') = \mathbb{R}^n \times A(G, W') = \mathbb{R}^n \times F', \) where \( F' \) is a finite subgroup of \( G \).

We have

\[
A(X, W^{(2)}) = \{ x \in X : (x, 2w) = 1 \ \forall w \in W \} = \{ x \in X : (2x, w) = 1 \ \forall w \in W \} = \{ x \in X : 2x \in A(X, W) \} = f_2^{-1}(A(X, W)).
\]

Note that \( \text{Kerf}_2 = X(2) = G(2) \). Since \( (Y/W')^* \approx A(X, W) \) and \( W \) is an open subgroup of \( H \), it follows from the finiteness of the factor group \( H/W \) that the annihilator \( A(X, W) = \mathbb{R}^n \times F'' \), where \( F'' \) is a finite subgroup of \( G \). We get that the subgroup \( A(X, W^{(2)}) \) is generated by \( G(2) \) and the subgroup \( \mathbb{R}^n \times F'' \).

Thus, the subgroup \( D \) is generated by the subgroups \( \mathbb{R}^n \times F', \mathbb{R}^n \times F'' \) and \( G(2) \). Hence \( D \subset \mathbb{R}^n \times G(k) \) for some \( k \). So we get that \( \hat{\nu}_j(y) = 1 \) for \( y \in B \). Lemma 3 implies that \( \sigma(\nu_j) \subset D \subset \mathbb{R}^n \times G(k) \) for some \( k \).

Lemma 4 implies that the distributions \( \mu_j \) are concentrated on the sets \( x_j + (\mathbb{R}^n \times G(k)) \) for some \( x_j \in X \). Since \( x_j = (t_j, g_j) \), where \( 2^i j = 0 \) for some nonnegative integers \( l_j \), there exists a nonnegative integer \( l \) such that the supports \( \sigma(\mu_j) \subset \mathbb{R}^n \times G(l) \). Since the subgroup \( \mathbb{R}^n \times G(l) \) is characteristic, we can prove the lemma in the case when \( X = \mathbb{R}^n \times G(l) \).

3. So, let \( X = \mathbb{R}^n \times G \), where \( G \) is a discrete bounded 2-primary group. Then \( Y = \mathbb{R}^n \times H \), where \( H \) is a compact bounded 2-primary group. It is obvious that conditions (2), (6) and (7) are fulfilled. Further reasoning is similar to the reasoning of the article 19.

Write equation (14) in the form

\[
\hat{\mu}_1(s + s', h + h')\hat{\mu}_2(s + \varepsilon s', h + \varepsilon h') = \hat{\mu}_1(s - s', h - h')\hat{\mu}_2(s - \varepsilon s', h - \varepsilon h'), \quad (s, h), (s', h') \in Y.
\]

Put \( h = h' = 0 \) in (3). We get

\[
\hat{\mu}_1(s + s', 0)\hat{\mu}_2(s + \varepsilon s', 0) = \hat{\mu}_1(s - s', 0)\hat{\mu}_2(s - \varepsilon s', 0), \quad s, s' \in \mathbb{R}^n.
\]

It was proved in 19 that all solutions of this equation are the characteristic functions of Gaussian distributions, i.e.

\[
\hat{\mu}_1(s, 0) = \exp\{-\langle A_1 s, s \rangle + i \langle t_1, s \rangle\}, \quad \hat{\mu}_2(s, 0) = \exp\{-\langle A_2 s, s \rangle + i \langle t_2, s \rangle\}, \quad s \in \mathbb{R}^n,
\]
where \( A_j \geq 0 \) are positive semidefinite matrices, \( t_j \in \mathbb{R}^n \), \( j = 1, 2 \).

We will prove by induction by \( k \), where \( 2^k \) is the order of an element \( h \), that

\[
\hat{\mu}_1(s, h) = \phi_1(s) \psi_1(h), \quad \hat{\mu}_2(s, h) = \phi_2(s) \psi_2(h), \quad s \in \mathbb{R}^n, h \in H, \tag{16}
\]

where \( \phi_j(0) = \psi_j(0) = 1, j = 1, 2 \).

Substituting \( s = -\varepsilon s', h' = h \) into (13), we get the equation

\[
\hat{\mu}_1((I - \varepsilon)s', 2h)\hat{\mu}_2(0, (I + \varepsilon)h) = \hat{\mu}_1(-(I + \varepsilon)s', 0)\hat{\mu}_2(-2\varepsilon s', (I - \varepsilon)h), \quad (s, h), (s', h') \in Y. \tag{17}
\]

If \( k = 1 \), i.e. \( 2h = 0 \) then equation (17) is of the form

\[
\hat{\mu}_1((I - \varepsilon)s', 0)\hat{\mu}_2(0, (I + \varepsilon)h) = \hat{\mu}_1(-(I + \varepsilon)s', 0)\hat{\mu}_2(-2\varepsilon s', (I - \varepsilon)h), \quad (s, h), (s', h') \in Y. \tag{18}
\]

It follows from (15) that \( \hat{\mu}_1(-(I + \varepsilon)s', 0) \neq 0 \). Since the condition (7) the equality \(-2\varepsilon(\mathbb{R}^n) = \mathbb{R}^n\) are fulfilled, we obtain from (18) representation (16) for \( \hat{\mu}_2(s, h) \).

Substituting \( s' = -s, h = \varepsilon h' \) into (13), we get the equation

\[
\hat{\mu}_1(0, (I + \varepsilon)h)\hat{\mu}_2((I - \varepsilon)s, 2\varepsilon h') = \hat{\mu}_1(2s, -(I - \varepsilon)h')\hat{\mu}_2((I + \varepsilon)s, 0), \quad (s, h), (s', h') \in Y. \tag{19}
\]

It follows from (19) that for \( k = 1 \), i.e. \( 2h' = 0 \), the function \( \hat{\mu}_1(s, h) \) is of the form (16). Thus, the statement is proved for \( k = 1 \).

Assume that (16) holds if \( h \) has order \( 2^k \). Let \( h \) have order \( 2^{k+1} \). Then \( 2h \) has order \( 2^k \), and we have in (17) \( \hat{\mu}_1((I - \varepsilon)s', 2h) = \phi_1((I - \varepsilon)s')\psi_1(2h) \) by induction hypothesis. Arguing similarly to the case of \( k = 1 \), we obtain from (17) representation (16) for \( \hat{\mu}_2(s, h) \). Similarly we obtain from (19) the representation (16) for \( \hat{\mu}_1(s, h) \). The function \( \phi_1(s) \) is a characteristic function of a distribution \( \gamma_1 \in \Gamma(\mathbb{R}^n) \), and the function \( \psi_1(h) \) is a characteristic function of a distribution \( \rho_1 \) such that \( \sigma(\rho_1) \subset G \). Thus, \( \mu_1 = \gamma_1 \ast \rho_1 \). Similarly we obtain that \( \mu_2 = \gamma_2 \ast \rho_2 \).

The following corollary follows from the proof of Lemma 5.

**Corollary 1** Let \( Y = \mathbb{R}^n \times H \), where \( H \) is a compact 2-primary Abelian group. All solutions of equation (5) on \( Y \) have form (16), where the functions \( \phi_j(s) \) have form (15).

**Lemma 6** (12) Let \( Y \) be a connected compact Abelian group, \( X = Y^* \). Let \( \varepsilon \in \text{Aut}(Y) \) and \( (I + \varepsilon)Y = Y \). Let \( \mu_1 \) and \( \mu_2 \) be distributions on the group \( X \) such that the characteristic functions \( \hat{\mu}_j(y) \) satisfy equation (5). Then \( \hat{\mu}_j(y) = (x_j, y) \), where \( x_j \in X, j = 1, 2 \).

**Lemma 7** Let \( X = \mathbb{R}^n \times D \), where \( D \) is a countable discrete Abelian group. Let \( \delta \in \text{Aut}(X) \) such that condition (2) is fulfilled. Let \( \xi_1 \) and \( \xi_2 \) be independent random variables with values in \( X \) and distributions \( \mu_1 \) and \( \mu_2 \). If the conditional distribution of the linear statistic \( L_2 = \xi_1 + \delta \xi_2 \) given \( L_1 = \xi_1 + \xi_2 \) is symmetric, then the random variables \( \xi_j \) can be replaced by their shifts \( \xi_j' \) with distributions \( \mu_j' \) in such a manner that \( \sigma(\mu_j') \subset R^n \times b_D \) and the conditional distribution of the linear statistic \( L_2 = \xi_1 + \delta \xi_2 \) given \( L_1 = \xi_1 + \xi_2 \) is symmetric.
Lemma 7 was actually proved in the article [10], but was not formulated as a separate statement. For the sake of exposition, we present its proof in the article.

Proof of Lemma 7. We shall reduce the proof of the lemma to the case when the subgroup $D$ is torsion. The group $Y = X^*$ is topologically isomorphic to the group $\mathbb{R}^n \times K$, where $K = D^*$. To avoid introducing new notation we will suppose that $Y = \mathbb{R}^n \times K$. Denote by $x = (t, d)$, where $t \in \mathbb{R}^n$, $d \in D$, elements of the group $X$. Denote by $y = (s, k)$, where $s \in \mathbb{R}^n$, $k \in K$, elements of the group $Y$.

It follows from Lemma 1 that the characteristic functions $\hat{\mu}_j(y)$ satisfy equation (5). Since the subgroup $K$ consists of all compact elements of the group $Y$, it is characteristic. Hence, the subgroup $c_K$ is also characteristic, i.e. $\varepsilon(c_K) = c_K$. Consider the restriction of equation (5) on the subgroup $c_K$.

We shall show that

$$(I + \varepsilon)(c_K) = c_K. \quad (20)$$

Since $A(X, c_K) = \mathbb{R}^n \times b_D$, we get that $(c_K)^* \approx X/\mathbb{R}^n \times b_D$. Since the subgroup $\mathbb{R}^n \times b_D$ is characteristic, the automorphism $\delta$ induces an automorphism $\bar{\delta}$ on the factor group $X/\mathbb{R}^n \times b_D$.

The condition (20) is equivalent to the condition

$$Ker(I + \bar{\delta}) = \{0\}. \quad (21)$$

If $(t_0, d_0) \in X$ and $[(t_0, d_0)] \in Ker(I + \bar{\delta})$ then $[(I + \delta)(t_0, d_0)] = [0]$. Hence, $(I + \delta)(t_0, d_0) \in \mathbb{R}^n \times b_D$. Since the subgroup $b_D$ is torsion, we get that $k(I + \delta)(t_0, d_0) \in \mathbb{R}^n$ for some natural $k$.

It follows from this that $(I + \delta)k(t_0, d_0) \in \mathbb{R}^n$, i.e.

$$(I + \delta)(kt_0, kd_0) = (t_1, 0) \quad (22)$$

for some $t_1 \in \mathbb{R}^n$. It is obvious that $(I + \delta)(\mathbb{R}^n) \subset \mathbb{R}^n$. It follows from (2) that the restriction of the continuous endomorphism $I + \delta$ of the group $X$ into the subgroup $\mathbb{R}^n$ is a topological automorphism of the group $\mathbb{R}^n$. Therefore

$$(t_1, 0) = (I + \delta)(t_2, 0) \quad (23)$$

for some $t_2 \in \mathbb{R}^n$. In view of (2), it follows from (22) and (24) that $kd_0 = 0$. Hence $d_0 \in b_D$ and $(t_0, d_0) \in \mathbb{R}^n \times b_D$. It follows from this that $[(t_0, d_0)] = 0$. Thus, (21) is fulfilled. So, (20) is fulfilled too.

Lemma 4 implies that $\hat{\mu}_j(y) = (x_j, y), y \in c_K, j = 1, 2$. By the theorem on the extension of a character from the closed subgroup to the group, we can suppose that $x_j \in X, j = 1, 2$.

Substituting these expressions of $\hat{\mu}_j(y)$ into equation (5) and taking into account the equality $A(X, c_K) = \mathbb{R}^n \times b_D$, we get

$$2(x_1 + \delta x_2) \in \mathbb{R}^n \times b_D. \quad (24)$$

Since $b_D$ consists of elements of finite order of the group $X$, it follows from (24) that

$$x_1 + \delta x_2 \in \mathbb{R}^n \times b_D. \quad (25)$$

Consider new random variables $\xi'_1 = \xi_1 + \delta x_2$ and $\xi'_2 = \xi_2 - x_2$ with values in the group $X$. Denote by $\mu'_j$ distributions of the random variables $\xi'_j$. Then $\mu'_1 = \mu_1 * E_{\delta x_2}$, $\mu'_2 = \mu_2 * E_{-x_2}$.

It is easy to see that the characteristic functions $\hat{\mu}'_j(y)$ satisfy equation (5). Then Lemma 1 implies that the conditional distribution of the linear statistic $L'_2 = \xi'_1 + \delta \xi'_2$ given $L'_1 = \xi'_1 + \xi'_2$ is symmetric. We have that $\mu'_j(y) = 1, y \in c_K$. In view of (25), we get that $\mu'_1(y) = 1, y \in c_K$.

It follows from Lemma 4 that $\sigma(\mu'_j) \subset A(X, c_K) = \mathbb{R}^n \times b_D, j = 1, 2$. ■
**Lemma 8** (10) Let $X$ be a countable discrete Abelian group, $G$ be a subgroup generated by all elements of odd order of the group $X$. Let $\delta \in \text{Aut}(X)$ such that condition (2) is fulfilled. Let $\xi_1$ and $\xi_2$ be independent random variables with values in $X$ and distributions $\mu_1$ and $\mu_2$. If the conditional distribution of the linear statistic $L_2 = \xi_1 + \delta \xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric, then $\mu_j = \rho_j * m_K * E_{x_j}$, where $\sigma(\rho_j) \subset X_2$, $K$ is a finite subgroup of $G$, $x_j \in X$, $j = 1, 2$.

**Lemma 9** (10) Let $X$ be a countable discrete Abelian group without elements of order 2. Let $\delta \in \text{Aut}(X)$ such that condition (2) is fulfilled. Let $\xi_1$ and $\xi_2$ be independent random variables with values in $X$ and distributions $\mu_1$ and $\mu_2$. If the conditional distribution of the linear statistic $L_2 = \xi_1 + \delta \xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric, then $\mu_j = m_F * E_{x_j}$, where $F$ is a finite subgroup of $X$, $x_j \in X$, $j = 1, 2$. Moreover, $\delta(F) = F$.

**Proof of Theorem 1** Since the subgroup $\mathbb{R}^n \times b_D$ of the group $X$ is characteristic, Lemma 7 implies that it suffices to prove the theorem in the case where $D$ is a torsion subgroup.

A discrete torsion group can be decomposed into a weak direct product of its $p$-primary components: $D = P \times D_p$, where $P$ is the set of prime numbers (see [12, Theorem 8.4]). Put $G = D_2$, $L = \sum_{p \in P} D_p$. Then $X = \mathbb{R}^n \times G \times L$ and $Y \approx \mathbb{R}^n \times H \times M$, where $H = G^*, M = L^*$. To avoid introducing new notation we will assume that $Y = \mathbb{R}^n \times H \times M$. Denote by $(s, h, m)$, $s \in \mathbb{R}^n$, $h \in H$, $m \in M$, elements of the group $Y$. Since $\mathbb{R}^n, H, M$ are characteristic subgroups, each automorphism $\varepsilon \in \text{Aut}(Y)$ can be written in the form $\varepsilon(s, h, m) = (\varepsilon s, \varepsilon h, \varepsilon m)$, $(s, h, m) \in Y$.

It follows from Lemma 1 that the characteristic functions $\hat{\mu}_j(y)$ satisfy equation (5). Consider the restriction of equation (5) to the subgroup $M$. Lemma 9 and (1) imply that

$$\hat{\mu}_1(0, 0, m) = \begin{cases} (l_1, m), & m \in A(M, F); \\ 0, & m \notin A(M, F); \end{cases} \quad \hat{\mu}_2(0, 0, m) = \begin{cases} (l_2, m), & m \in A(M, F); \\ 0, & m \notin A(M, F); \end{cases}$$

(26)

where $l_j \in L$, $F$ is a finite subgroup of $L$ such that $\delta(F) = F$.

Substituting (26) into equation (5) and considering the restriction of equation (5) to the subgroup $M$, we get

$$2(l_1 + \delta l_2) \in F.$$  

(27)

Since $L$ does not contain elements of order 2, it follows from (27) that

$$l_1 + \delta l_2 \in F.$$  

(28)

Considering new random variables $\zeta_1 = \xi_1 + \delta l_2$ and $\zeta_2 = \xi_2 - l_2$ and reasoning as in the end of the proof of Lemma 7 we obtain that we can suppose from the beginning that

$$\hat{\mu}_1(0, 0, m) = \begin{cases} 1, & m \in A(M, F); \\ 0, & m \notin A(M, F); \end{cases} \quad \hat{\mu}_2(0, 0, m) = \begin{cases} 1, & m \in A(M, F); \\ 0, & m \notin A(M, F). \end{cases}$$

(29)

It follows from this that $\sigma(\mu_j) \subset A(X, A(M, F)) = \mathbb{R}^n \times G \times F$, $j = 1, 2$.

Since the subgroups $\mathbb{R}^n$ and $G$ are characteristic and $\delta(F) = F$, it suffices to prove the theorem in the case when $X = \mathbb{R}^n \times G \times L$, where the subgroup $L$ is finite and $\delta(L) = L$.

Moreover, the following representations are valid

$$\hat{\mu}_1(0, 0, m) = \begin{cases} 1, & m = 0; \\ 0, & m \neq 0; \end{cases} \quad \hat{\mu}_2(0, 0, m) = \begin{cases} 1, & m = 0; \\ 0, & m \neq 0. \end{cases}$$

(30)

Putting $u = v = (0, 0, m)$ into (5) $u = v = (0, 0, m)$, we get
\[
\hat{\mu}_1(0, 0, 2m)\hat{\mu}_2(0, 0, (I + \varepsilon)m) = \hat{\mu}_2(0, 0, (I - \varepsilon)m), \quad m \in M. \quad (31)
\]

Since the subgroup \(M\) does not contain elements of order 2, \(2m = 0\) if and only if \(m = 0\). Then \(\hat{\mu}_1(0, 0, 2m) = 0\) for \(m \neq 0\), and it follows from \((31)\) that \(\hat{\mu}_2(0, 0, (I - \varepsilon)m) = 0\) if and only if \(m \neq 0\). It follows from this and the representations \((30)\) that the restriction of the endomorphism \(I - \varepsilon\) into \(M\) has a zero kernel. Since the subgroup \(M\) is finite, the restriction of the endomorphism \(I - \varepsilon\) into \(M\) is an automorphism of \(M\), i.e.

\[(I - \varepsilon)M = M. \quad (32)\]

Consider the restriction of equation \((5)\) on the subgroup \(H \times M\). Lemma \(8\) implies that

\[
\hat{\mu}_1(0, h, m) = \begin{cases} 
\psi_1(h), & m = 0; \\
0, & m \neq 0;
\end{cases} \quad \hat{\mu}_2(0, h, m) = \begin{cases} 
\psi_2(h), & m = 0; \\
0, & m \neq 0.
\end{cases} \quad (33)
\]

where \(\psi_j(h)\) are characteristic functions on \(H\).

Rewrite equation \((5)\) in the form

\[
\hat{\mu}_1(s + s', h + h', l + l')\hat{\mu}_2(s + \varepsilon s', h + \varepsilon h', m + \varepsilon m') = 
\hat{\mu}_1(s - s', h - h', l - l')\hat{\mu}_2(s - \varepsilon s', h - \varepsilon h', m - \varepsilon m'), \quad (s, h, m), (s', h', m') \in Y. \quad (34)
\]

Putting \(s' = s, h' = -h, m' = -m\) into \((34)\), we get

\[
\hat{\mu}_1(2s, 0, 2m)\hat{\mu}_2((I + \varepsilon)s, (I - \varepsilon)h, (I - \varepsilon)m) = 
\hat{\mu}_1(0, 2h, 2m)\hat{\mu}_1((I - \varepsilon)s, (I + \varepsilon)h, (I + \varepsilon)m), \quad (s, h, m) \in Y. \quad (35)
\]

It follows from \((33)\) that \(\hat{\mu}_1(0, 2h, 2m) = 0\) for \(m \neq 0\). Hence, \(\hat{\mu}_1(2s, 0, 0)\hat{\mu}_2((I + \varepsilon)s, (I - \varepsilon)h, (I - \varepsilon)m) = 0\) for \(m \neq 0\). Consider the restriction of equation \((5)\) on the subgroup \(\mathbb{R}^n\). It was proved in \([9]\) that all solutions of this equation are the characteristic functions of Gaussian distributions, i.e. have form \((15)\). It follows from \((15)\) that the function \(\hat{\mu}_1(2s, 0, 0)\) do not vanish. Then we get from \((33)\) that

\[
\hat{\mu}_2((I + \varepsilon)s, (I - \varepsilon)h, (I - \varepsilon)m) = 0, \quad s \in \mathbb{R}^n, h \in H, \quad (36)
\]

for \(m \neq 0\). Note that it follows from \((2)\) that \((I + \varepsilon)\mathbb{R}^n = \mathbb{R}^n\). As in the proof of the part 1 of Lemma \(5\) we obtain \((7)\). Taking into account \((32)\), it follows from \((36)\) that

\[
\hat{\mu}_2(s, h, m) = 0, \quad s \in \mathbb{R}^n, h \in H, m \neq 0. \quad (37)
\]

Similarly we get that

\[
\hat{\mu}_1(s, h, m) = 0, \quad s \in \mathbb{R}^n, h \in H, m \neq 0. \quad (38)
\]

Put \(m = m' = 0\) in \((3)\). Corollary \(1\) implies that

\[
\hat{\mu}_1(s, h, 0) = \phi_1(s)\psi_1(h), \quad \hat{\mu}_2(s, h, 0) = \phi_2(s)\psi_2(h), \quad (39)
\]

where the functions \(\phi_1(s)\), \(\phi_2(s)\) are of the form \((15)\), and the functions \(\psi_1(h), \psi_2(h)\) are the characteristic functions of distributions \(\rho_j\) such that \(\sigma(\rho_j) \subset G\).

We deduce from \((37)-(39)\) that
\[
\hat{\mu}_1(s, h, m) = \begin{cases} 
\phi_1(s)\psi_1(h), & m = 0; \\
0, & m \neq 0;
\end{cases}
\]
\[
\hat{\mu}_2(s, h, m) = \begin{cases} 
\phi_2(s)\psi_2(h), & m = 0; \\
0, & m \neq 0.
\end{cases}
\] (40)

It is easy to see that the assertion of the theorem follows from (40).

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