Internal DLA in Higher Dimensions

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Abstract

Let \( A(t) \) denote the cluster produced by internal diffusion limited aggregation (internal DLA) with \( t \) particles in dimension \( d \geq 3 \). We show that \( A(t) \) is approximately spherical, up to an \( O(\sqrt{\log t}) \) error.

In the process known as internal diffusion limited aggregation (internal DLA) one constructs for each integer time \( t \geq 0 \) an occupied set \( A(t) \subset \mathbb{Z}^d \) as follows: begin with \( A(0) = \emptyset \) and \( A(1) = \{0\} \). Then, for each integer \( t > 1 \), form \( A(t + 1) \) by adding to \( A(t) \) the first point at which a simple random walk from the origin hits \( \mathbb{Z}^d \setminus A(t) \). Let \( B_r \subset \mathbb{R}^d \) denote the ball of radius \( r \) centered at 0, and write \( B_r := B_r \cap \mathbb{Z}^d \). Let \( \omega_d \) be the volume of the unit ball in \( \mathbb{R}^d \). Our main result is the following.

**Theorem 1.** Fix an integer \( d \geq 3 \). For each \( \gamma \) there exists an \( a = a(\gamma, d) < \infty \) such that for all sufficiently large \( r \),

\[
P\left\{ B_{r-a\sqrt{\log r}} \subset A(\omega_d r^d) \subset B_{r+a\sqrt{\log r}} \right\}^c \leq r^{-\gamma}.
\]

We treated the case \( d = 2 \) in [JLS10] (see also the overview in [JLS09]), where we obtained a similar statement with \( \log r \) in place of \( \sqrt{\log r} \). Together with a Borel-Cantelli argument, this in particular implies the following [JLS10]:

**Corollary 2.** The maximal distance from \( \partial B_r \) to a point in one (but not both) of \( B_r \) and \( A(\omega_d r^d) \) is a.s. \( O(\log r) \) when \( d = 2 \) and \( O(\sqrt{\log r}) \) when \( d > 2 \).

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These results show that internal DLA in dimensions $d \geq 3$ is extremely close to a perfect sphere: when the cluster $A(t)$ has the same size as a ball of radius $r$, its fluctuations around that ball are confined to the $\sqrt{\log r}$ scale (versus $\log r$ in dimension 2).

In [JLS10] we explained that our method for $d = 2$ would also apply in dimensions $d \geq 3$ with the $\log r$ replaced by $\sqrt{\log r}$. We outlined the changes needed in higher dimensions (stating that the full proof would follow in this paper) and included a key step: Lemma A, which bounds the probability of “thin tentacles” in the internal DLA cluster in all dimensions. The purpose of this note is to carry out the adaptation of the $d = 2$ argument of [JLS10] to higher dimensions. We remark that in [JLS10] we used an estimate from [LBG92] to start this iteration, while here we have modified the argument slightly so that this a priori estimate is no longer required.

One way for $A(\omega,d^2,t)$ to deviate from the radius $r$ sphere is for it to have a single “tentacle” extending beyond the sphere. The thin tentacle estimate [JLS10, Lemma A] essentially says that in dimensions $d \geq 3$, the probability that there is a tentacle of length $m$ and volume less than a small constant times $m^d$ (near a given location) is at most $e^{-cm^2}$. By summing over all locations, one may use this to show that the length of the longest “thin tentacle” produced before time $t$ is $O(\sqrt{\log t})$. To complete the proof of Theorem 1, we will have to show that other types of deviations from the radius $r$ sphere are also unlikely.

Lemma A of [JLS10] was also proved for $d = 2$, albeit with $e^{-cm^2}$ replaced by $e^{-cm^2/\log m}$. However, when $d = 2$ there appear to be other more “global” fluctuations that swamp those produced by individual tentacles. (Indeed, we expect, but did not prove, that the $\log r$ fluctuation bound is tight when $d = 2$.) We bound these other fluctuations in higher dimensions via the same scheme introduced in [JLS09, JLS10], which involves constructing and estimating certain martingales related to the growth of $A(t)$. It turns out the quadratic variations of these martingales are, with high probability, of order $\log t$ when $d = 2$ and of constant order when $d \geq 3$, closely paralleling what one obtains for the discrete Gaussian free field (as outlined in more detail in [JLS10]). The connection to the Gaussian free field is made more explicit in [JLS11].

Section 1 proves Theorem 1 by iteratively applying higher dimensional analogues of the two main lemmas of [JLS10]. The lemmas themselves are proved in Section 3, which is the heart of the argument. Section 2 contains preliminary estimates about random walks that are used in Section 3.
A brief history of internal DLA fluctuation bounds

The history of fluctuation bounds such as the one in Corollary 2 is as follows. In 1991, Lawler, Bramson, and Griffeath proved that the limit shape of internal DLA from a point is the ball in all dimensions [LBG92]. In 1995 Lawler gave a more quantitative proof, showing that the fluctuations of \( A(\omega d r^d) \) from the ball of radius \( r \) are at most of order \( O(r^{1/3} \log^4 r) \) [Law95]. In December 2009, the present authors announced the bound \( O(\log r) \) on fluctuations in dimension \( d = 2 \) [JLS09] and gave an overview of the argument, making clear that the details remained to be written. In April 2010, Asselah and Gaudilliére [AG10a] gave a proof, using different methods from [JLS09], of the bound \( O(r^{1/(d+1)}) \) in all dimensions, improving the Lawler bound for all \( d \geq 3 \). In September 2010, Asselah and Gaudilliére improved this to \( O((\log r)^2) \) in all dimensions \( d \geq 2 \) with an \( O(\log r) \) bound on “inner” errors [AG10b]. In October 2010 the present authors proved the \( O(\log r) \) bounds (announced in December 2009) for dimension \( d = 2 \) and outlined the proof of the \( O(\sqrt{\log r}) \) bound for dimensions \( d \geq 3 \) [JLS10]. In November 2010, Asselah and Gaudilliére gave a second proof of the \( O(\sqrt{\log r}) \) bound [AG10c]. Their proof uses methods from [AG10b] along with Lemma A of [JLS10] to bound “outer” errors and a new large deviation bound (in some sense symmetric to Lemma A) to bound “inner” errors.

More references and a more general discussion of internal DLA history appear in [JLS10].

1 Proof of Theorem 1

Let \( m \) and \( \ell \) be positive real numbers. We say that \( x \in \mathbb{Z}^d \) is \( m \)-early if

\[
x \in A(\omega d |x| - m)^d,
\]

where \( \omega d \) is the volume of the unit ball in \( \mathbb{R}^d \). Likewise, we say that \( x \) is \( \ell \)-late if

\[
x \notin A(\omega d |x| + \ell)^d.
\]

Let \( \mathcal{E}_m[T] \) be the event that some point of \( A(T) \) is \( m \)-early. Let \( \mathcal{L}_\ell[T] \) be the event that some point of \( \mathcal{B}_{(T/\omega d)^{1/d-\ell}} \) is \( \ell \)-late. These events correspond to “outer” and “inner” deviations of \( A(T) \) from circularity.

Lemma 3. (Early points imply late points) Fix a dimension \( d \geq 3 \). For each \( \gamma \geq 1 \), there is a constant \( C_0 = C_0(\gamma, d) \), such that for all sufficiently large \( T \), if \( m \geq C_0 \sqrt{\log T} \) and \( \ell \leq m/C_0 \), then

\[
P(\mathcal{E}_m[T] \cap \mathcal{L}_\ell[T]^c) < T^{-10\gamma}.
\]
Lemma 4. (Late points imply early points) Fix a dimension \( d \geq 3 \). For each \( \gamma \geq 1 \), there is a constant \( C_1 = C_1(\gamma, d) \) such that for all sufficiently large \( T \), if \( m \geq \ell \geq C_1 \sqrt{\log T} \) and \( \ell \geq C_1((\log T)m)^{1/3} \), then

\[
P(\mathcal{E}_m[T]^{c} \cap \mathcal{L}_\ell[T]) \leq T^{-10^\gamma}.
\]

Theorem 1 will follow because \( m_T > m_0 = T \) is impossible and the other rectangles on the right are all (by Lemmas 3 and 4) unlikely.

We now proceed to derive Theorem 1 from Lemmas 3 and 4. The lemmas themselves will be proved in Section 3. Let \( C = \max(C_0, C_1) \). We start with

\[
m_0 = T.
\]

Note that \( A(T) \subset B_T \), so \( P(\mathcal{E}_T[T]) = 0 \). Next, for \( j \geq 0 \) we let

\[
\ell_j = \max(C((\log T)m_j)^{1/3}, C\sqrt{\log T})
\]

and

\[
m_{j+1} = C\ell_j.
\]

By induction on \( j \), we find

\[
P(\mathcal{E}_{m_j}[T]) < 2jT^{-10^\gamma}
\]

and

\[
P(\mathcal{L}_{\ell_j}[T]) < (2j + 1)T^{-10^\gamma}.
\]
To estimate the size of $\ell_j$, let $K = C^4 \log T$ and note that $\ell_j \leq \ell_j'$, where

$$\ell_0' = (KT)^{1/3}; \quad \ell_{j+1}' = \max((K\ell_j')^{1/3}, K^{1/2}).$$

Then

$$\ell_j' \leq \max(K^{1/3+1/9+\cdots+1/3^j}T^{1/3^j}, K^{1/2})$$

so choosing $J = \log T$ we have

$$T^{1/3^j} < 2$$

and

$$\ell_J \leq 2K^{1/2} \leq C\sqrt{\log T}.$$

The probability that $A(T)$ has $\ell_J$-late points or $m_J$-early points is at most

$$(4J + 1)T^{-10\gamma} < T^{-9\gamma} < r^{-\gamma}.$$

Setting $T = \omega d^d$, $\ell = \ell_J$ and $m = m_J$, we conclude that if $a$ is sufficiently large, then

$$\mathbb{P}\left\{ B_{r-a\sqrt{\log r}} \subset A(\omega d^d) \subset B_{r+a\sqrt{\log r}} \right\} \leq \mathbb{P}(E_m[T] \cup L_{\ell[T]}) < r^{-\gamma}$$

which completes the proof of Theorem 1.

2 Green function estimates on the grid

This section assembles several Green function estimates that we need to prove Lemmas 3 and 4. The reader who prefers to proceed to the heart of the argument may skip this section on a first read and refer to the lemma statements as necessary. Fix $d \geq 3$ and consider the $d$-dimensional grid

$$\mathcal{G} = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : \text{at most one } x_i \notin \mathbb{Z}\}.$$ 

In many of the estimates below, we will assume that a positive integer $k$ and a $y \in \mathbb{Z}^d$ have been fixed. We write $s = |y|$ and

$$\Omega = \Omega(y, k) := \mathcal{G} \cap B_{s+k}\setminus\{y\}.$$ 

For $x \in \partial\Omega$, let

$$P(x) = P_{y,k}(x)$$

be the probability that a Brownian motion on the grid $\mathcal{G}$ (defined in the obvious way; see [JLS10]) starting at $x$ reaches $y$ before exiting $B_{s+k}$. Note
that $P$ is grid harmonic in $\Omega$ (i.e., $P$ is linear on each segment of $\Omega \setminus \mathbb{Z}^d$, and for each $x \in \Omega \cap \mathbb{Z}^d$, the sum of the slopes of $P$ on the $2d$ directed edge segments starting at $x$ is zero). Boundary conditions are given by $P(y) = 1$ and $P(x) = 0$ for $x \in (\partial \Omega) \setminus \{y\}$. The point $y$ plays the role that $\varsigma$ played in [JLS10], and $P$ plays the role of the discrete harmonic function $H_\varsigma$. One difference from [JLS10] is that we will take $y$ inside the ball (i.e., $k \geq 1$) instead of on the boundary.

To estimate $P$ we use the discrete Green function $g(x)$, defined as the expected number of visits to $x$ by a simple random walk started at the origin in $\mathbb{Z}^d$. The well-known asymptotic estimate for $g$ is [Uch98]

$$\left| g(x) - a_d|x|^{2-d} \right| \leq C|x|^{-d}$$

for dimensional constants $a_d$ and $C$ (i.e., constants depending only on the dimension $d$). We extend $g$ to a function, also denoted $g$, defined on the grid $\mathcal{G}$ by making $g$ linear on each segment between lattice points. Note that $g$ is grid harmonic on $\mathcal{G} \setminus \{0\}$.

Throughout we use $C$ to denote a large positive dimensional constant, and $c$ to denote a small positive dimensional constant, whose values may change from line to line.

**Lemma 5.** There is a dimensional constant $C$ such that

(a) $P(x) \leq C/(1 + |x - y|^{d-2})$.

(b) $P(x) \leq Ck(s + k + 1 - |x|)/|x - y|^d$, for $|x - y| \geq k/2$.

(c) $\max_{x \in B_r} P(x) \leq Ck/(s - r - k)^{d-1}$ for $r < s - 2k$.

**Proof.** The maximum principle (for grid harmonic functions) implies $Cg(x - y) \geq P(x)$ on $\Omega$, which gives part (a).

The maximum principle also implies that for $x \in \Omega$,

$$P(x) \leq C(g(x - y) - g(x - y^*))$$

where $y^*$ is the one of the lattice points nearest to $(s + 2k + C_1)y/s$. Indeed, both sides are grid harmonic on $\Omega$, and the right side is positive on $\partial B_{s+k}$ by (1), so it suffices to take $C = (g(0) - g(y - y^*))^{-1}$.

Combining (1) and (2) yields the bound

$$P(x) \leq \frac{Ck}{|x - y|^{d-1}}, \text{ for } |x - y| \geq 2k.$$
Next, let $z \in \partial B_{s+k}$ be such that $|z - y| = 2L$, with $L \geq 2k$. The bound above implies

$$P(x) \leq Ck \frac{L^d}{d-1}, \quad \text{for } x \in B_L(z)$$

Let $z^*$ be one of the lattice points nearest to $(s + k + L + C_1)z/|z|$. Then

$$F(x) = a_d L^{2-d} - g(x - z^*)$$

is comparable to $L^{2-d}$ on $\partial B_{2L}(z^*)$ and positive outside the ball $B_L(z^*)$ (for a large enough dimensional constant $C_1$ — in fact, we can also do this with $C_1 = 1$ with $L$ large enough). It follows that

$$P(x) \leq C(k/L^d-1)(L^{d-2})F(x)$$

on $\partial (B_{2L}(z^*) \cap \Omega)$ and hence by the maximum principle on $B_{2L}(z^*) \cap \Omega$. Moreover,

$$F(x) \leq C (s + k + 1 - |x|)/L^{d-1}$$

for $x$ a multiple of $z$ and $s + k - L \leq |x| \leq s + k$. Thus for these values of $x$,

$$P(x) \leq C(k/L)F(x) \leq Ck(s + k + 1 - |x|)/L^d$$

We have just confirmed the bound of part (b) for points $x$ collinear with 0 and $z$, but $z$ was essentially arbitrary. To cover the cases $|x - y| \leq 2k$ one has to use exterior tangent balls of radius, say $k/2$, but actually the upper bound in part (a) will suffice for us in the range $|x - y| \leq Ck$.

Part (c) of the lemma follows from part (b).

The mean value property (as typically stated for continuum harmonic functions) holds only approximately for discrete harmonic functions. There are two choices for where to put the approximation: one can show that the average of a discrete harmonic function $u$ over the discrete ball $B_r$ is approximately $u(0)$, or one can find an approximation $w_r$ to the discrete ball $B_r$ such that averaging $u$ with respect to $w_r$ yields exactly $u(0)$. The divisible sandpile model of [LP09] accomplishes the latter. In particular, the following discrete mean value property follows from Theorem 1.3 of [LP09].

**Lemma 6.** (Exact mean value property on an approximate ball) For each real number $r > 0$, there is a function $w_r : \mathbb{Z}^d \to [0,1]$ such that

- $w_r(x) = 1$ for all $x \in B_{r-c}$, for a constant $c$ depending only on $d$.
- $w_r(x) = 0$ for all $x \notin B_r$. 

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• For any function $u$ that is discrete harmonic on $B_r$,
\[ \sum_{x \in \mathbb{Z}^d} w_r(x)(u(x) - u(0)) = 0. \]

The next lemma bounds sums of $P$ over discrete spherical shells and discrete balls. Recall that $s = |y|$.

**Lemma 7.** There is a dimensional constant $C$ such that

(a) $\sum_{x \in B_{r+1} \setminus B_r} P(x) \leq Ck$ for all $r \leq s + k$.

(b) $\left| \sum_{x \in B_r} (P(x) - P(0)) \right| \leq Ck$ for all $r \leq s$.

(c) $\left| \sum_{x \in B_{s+k}} (P(x) - P(0)) \right| \leq Ck^2$.

**Proof.** Part (a) follows from Lemma 5: Take the worst shell, when $r = s$. Then the lattice points with $|x - y| \leq k$, $s \leq |x| \leq s + 1$ are bounded by Lemma 5(a)
\[ \int_0^k s^{2-d} s^{d-2} ds = k \]
(volume element on disk with thickness 1 and radius $k$ in $\mathbb{Z}^{d-1}$ is $s^{d-2}ds$.) For the remaining portion of the shell, Lemma 5(b) has numerator $k(s+k-s) = k^2$, so that
\[ \int_k^\infty k^2 s^{-d} s^{d-2} ds = k \]
Next, for part (b), let $w_r$ be as in Lemma 6. Since $P$ is discrete harmonic in $B_s$, we have for $r \leq s$
\[ \sum_{x \in \mathbb{Z}^d} w_r(x)(P(x) - P(0)) = 0. \]

Since $w_r$ equals the indicator $1_{B_r}$ except on the annulus $B_r \setminus B_{r-c}$, and
$|w_r| \leq 1$, we obtain
\[
\left| \sum_{x \in B_r} (P(x) - P(0)) \right| \leq \sum_{x \in B_r \setminus B_{r-c}} |w_r(x)| |P(x) - P(0)| \\
\leq \sum_{x \in B_r \setminus B_{r-c}} (P(x) + P(0)) \\
\leq Ck.
\]
In the last step we have used part (a) to bound the first term; the second term is bounded by Lemma 5(b), which says that $P(0) \leq Ck/s^{d-1}$.

Part (c) follows by splitting the sum over $B_{s+k}$ into $k$ sums over spherical shells $B_{s+j} \setminus B_{s+j-1}$ for $j = 1, \ldots, k$, each bounded by part (a), plus a sum over the ball $B_s$, bounded by part (b).

Fix $\alpha > 0$, and consider the level set
\[
U = \{x \in \mathcal{G} \mid g(x) > \alpha\}.
\]
For $x \in \partial U$, let $p(x)$ be the probability that a Brownian motion started at the origin in $\mathcal{G}$ first exits $U$ at $x$.

**Lemma 8.** Choose $\alpha$ so that $\partial U$ does not intersect $\mathbb{Z}^d$. For each $x \in \partial U$, the quantity $p(x)$ equals the directional derivative of $g/2d$ along the directed edge in $U$ starting at $x$.

**Proof.** We use a discrete form of the divergence theorem
\[
\int_U \text{div} V = \sum_{\partial U} \nu_U \cdot V. \tag{3}
\]
where $V$ is a vector-valued function on the grid, and the integral on the left is a one-dimensional integral over the grid. The dot product $\nu_U \cdot V$ is defined as $e_j \cdot V(x - 0e_j)$, where $e_j$ is the unit vector pointing toward $x$ along the unique incident edge in $U$. To define the divergence, for $z = x + te_j$, where $0 \leq t < 1$ and $x \in \mathbb{Z}^d$, let
\[
\text{div} V(z) := \frac{\partial}{\partial x_j} e_j \cdot V(z) + \delta_x(z) \sum_{j=1}^d (e_j \cdot V(x + 0e_j) - e_j \cdot V(x - 0e_j)).
\]

If $f$ is a continuous function on $U$ that is $C^1$ on each connected component of $U - \mathbb{Z}^d$, then the gradient of $f$ is the vector-valued function
\[
V = \nabla f = (\partial f/\partial x_1, \partial f/\partial x_2, \ldots, \partial f/\partial x_d)
\]
with the convention that the entry $\partial f/\partial x_j$ is 0 if the segment is not pointing in the direction $x_j$. Note that $\nabla f$ may be discontinuous at points of $\mathbb{Z}^d$.

Let $G = -g/2d$, so that $\text{div} \nabla G = \delta_0$. If $u$ is grid harmonic on $U$, then $\text{div} \nabla u = 0$ and
\[
\text{div} (u \nabla G - G \nabla u) = u(0) \delta_0.
\]
Indeed, on each segment this is the same as $(uG' - u'G)' = u'G' - u'G' + uG'' - u''G = 0$ because $u$ and $G$ are linear on segments. At lattice points $u$ and $G$ are continuous, so the divergence operation commutes with the factors $u$ and $G$ and gives exactly one nonzero delta term, the one indicated.

Let $u(y)$ be the probability that Brownian motion on $U$ started at $y$ first exits $U$ at $x$. Since $u$ is grid-harmonic on $U$, we have $\text{div} \nabla u = 0$ on $U$, hence by the divergence theorem
\[
u_{U} \cdot \nabla G.
\]

Next we establish some lower bounds for $P$.

**Lemma 9.** There is a dimensional constant $c > 0$ such that
(a) $P(0) \geq ck/s^{d-1}$.
(b) Let $k = 1$, and $z = (1 - 2m/s)y$. Then
\[
\min_{x \in B(z,m)} P(x) \geq c/m^{d-1}.
\]

**Proof.** By the maximum principle, there is a dimensional constant $c > 0$ such that
\[
P(x) \geq c(g(x - y) - a_d(k/2)^{2-d})
\]
for $x \in B_{k/2}(y)$. In particular,
\[
P(x) \geq ck^{2-d} \quad \text{for all } |x - y| \leq k/4
\]
Now consider the region
\[
U = \{x \in \mathcal{G} : g(x) > a_d(s')^{2-d}\}
\]
where $s'$ is chosen so that $|s' - (s - k/8)| < 1/2$ and all of the boundary points of $U$ are non-lattice points. (A generic value of $s'$ in the given range will suffice.)
By (1), this set is within unit distance of the ball of radius \( s - k/8 \). Let \( p(z) \) represent the probability that a Brownian motion on the grid starting from the origin first exits \( U \) at \( z \in \partial U \). Thus

\[
u(0) = \sum_{z \in \partial U} u(z)p(z)
\] (4)

for all grid harmonic functions \( u \) in \( U \).

Take any boundary point of \( z \in \partial U \). Take the nearest lattice point \( z^* \). Let \( z_j \) be a coordinate of \( z \) largest in absolute value. Then \( |z_j| \geq |z|/d \). The rate of change of \( |x|^2 - d \) in the \( j \)th direction near \( z \) has size \( \geq 1/d|z|^{d-1} \), which is much larger than the error term \( C|z|^{-d} \) in (1). It follows that on the segment in that direction, where the function \( g(x) - a_d(s - k/8)^{2-d} \) changes sign, its derivative is bounded below by \( 1/2d|z|^{d-1} \). In other words, by Lemma 8, within distance 2 of every boundary point of \( z \in \partial U \) there is a point \( z' \in \partial U \) for which \( p(z') \geq c/s^{d-1} \). There are at least \( ck^{d-1} \) such points in the ball \( B_{k/4}(y) \) where the lower bound for \( P \) was \( ck^2/d - 1 \), so

\[
P(0) \geq ck^{2-d}k^{d-1}/s^{d-1} = ck/s^{d-1}.
\]

Next, the argument for Lemma 9(b) is nearly the same. We are only interested in \( k = 1 \). It is obvious that for points \( x \) within constant distance of \( y \) (and unit distance from the boundary at radius \( s + 1 \), the values of \( P(x) \) are bounded below by a positive constant. We then bound \( P((s - 2m)y/|y|) \) from below using the same argument as above, but with Green’s function for a ball of radius comparable to \( m \). Finally, Harnack’s inequality says that the values of \( P(x) \) for \( x \) in the whole ball of size \( m \) around this point \((s - 2m)y/|y|\) are comparable. \( \square \)

3 Proofs of main lemmas

The proofs in this section make use of the martingale

\[
M(t) = M_{y,k}(t) := \sum_{x \in A_{y,k}(t)} (P(x) - P(0))
\]

where \( A_{y,k}(t) \) is the modified internal DLA cluster in which particles are stopped if they exit \( \Omega \). As in [JLS10], we view \( A_{y,k}(t) \) as a multiset: points on the boundary of \( \Omega \) where many stopped particles accumulate are counted with multiplicity in the sum defining \( M \). In addition to these stopped particles, the set \( A_{y,k}(t) \) contains one more point, the location of the currently active particle performing Brownian motion on the grid \( \mathcal{G} \).
Recall that $P = P_{y,k}$ and $M = M_{y,k}$ depend on $k$, which is the distance from $y$ to the boundary of $\Omega$. We will choose $k = 1$ for the proof of Lemma 3, and $k = a\ell$ for a small constant $a$ in the proof of Lemma 4. Taking $k > 1$ is one of the main differences from the argument in [JLS10].

Proof of Lemma 3. The proof follows the same method as [JLS10, Lemma 12]. We highlight here the changes needed in dimensions $d \geq 3$. We use the discrete harmonic function $P(x)$ with $k = 1$. Fix $z \in \mathbb{Z}^d$, let $r = |z|$ and $y = (r + 2m)z/r$. Let

$$T_1 = \lceil \omega_d(r - m)^d \rceil$$

where $\omega_d$ is the volume of the unit ball in $\mathbb{R}^d$. If $z$ is $m$-early, then $z \in A(T_1)$; in particular, this means that $r \geq m$, so that $r + m, r + 2m$ are all comparable to $r$. Since $k = 1$, we have by Lemmas 5(c) and 9(a)

$$P(0) \approx 1/r^{d-1},$$

where $\approx$ denotes equivalence up to a constant factor depending only on $d$.

First we control the quadratic variation

$$S(t) = \lim_{0=t_0 \leq \ldots \leq t_N = t} \sum_{i=1}^{N} (M(t_i) - M(t_{i-1}))^2$$

on the event $\mathcal{E}_{m+1}[T]^c$ that there are no $(m+1)$-early points by time $T$. As in [JLS10, Lemma 9], there are independent standard Brownian motions $\tilde{B}^0, \tilde{B}^1, \ldots$ such that each increment $(S(n+1) - S(n))1_{\mathcal{E}_{m+1}[T]^c}$ is bounded above by the first exit time of $\tilde{B}^n$ from the interval $[-a_n, b_n]$, where

$$a_n = P(0) \approx \frac{1}{r^{d-1}}$$

$$b_n = \max_{|x| \leq (n/\omega_d)^{1/d} + m + 1} P(x) \leq \frac{1}{[r + 2m - ((n/\omega_d)^{1/d} + m + 1)]^{d-1}}.$$  

Here we have used Lemma 5(b) in the bound on $b_n$.

Unlike in dimension 2, we will use the large deviation bound for Brownian exit times [JLS10, Lemma 5] with $\lambda = cm^2$ instead of $\lambda = 1$. Here $c$ is a constant depending only on $d$. Note that $b_n \leq 1/m^{d-1}$, for all $n \leq T_1$, so this is a valid choice of $\lambda$ in all dimensions $d \geq 3$ (that is, the hypothesis
\[ \sqrt{\lambda}(a_n + b_n) \leq 3 \text{ of [JLS10, Lemma 5] holds}. \]

We obtain

\[ \log \mathbb{E}\left[e^{\lambda S(T_1)}1_{E_{m+1}[T]^c}\right] \leq \sum_{n=1}^{T_1} 10\lambda a_n b_n \]
\[ \leq \int_1^{T_1} \frac{\lambda C}{r^{d-1}} \frac{1}{(r + m - (n/\omega_d)^{1/d} - 1)^{d-1}} dn \]
\[ \leq \int_1^r \frac{\lambda C}{r^{d-1}} \frac{1}{(r + m - j - 1)^{d-1}} dj \]
\[ \leq \int_1^r \frac{C\lambda dj}{(r + m - j - 1)^{d-1}} \leq C\lambda/m^{d-2}. \]

Note that the last step uses \( d \geq 3 \). Taking \( \lambda = cm^2 \) for small enough \( c \) we obtain

\[ \mathbb{E}\left[e^{cm^2 S(T_1)}1_{E_{m+1}[T]^c}\right] \leq e^{m^2/m^{d-2}} \leq e^m. \]

Therefore, by Markov’s inequality,

\[ \mathbb{P}(\{S(T_1) > 1/c\} \cap E_{m+1}[T]^c) \leq e^{m-m^2} < T^{-20\gamma}. \quad (5) \]

Fix \( z \in B_T \) and \( t \in \{1, \ldots, T\} \), and let \( Q_{z,t} \) be the event that \( z \in A(t) \setminus A(t-1) \) and \( z \) is \( m \)-early and no point of \( A(t-1) \) is \( m \)-early. This event is empty unless \( (t/\omega_d)^{1/d} + m \leq |z| \leq (t/\omega_d)^{1/d} + m + 1 \); in particular, the first inequality implies \( t \leq T_1 \). We will bound from below the martingale \( M(t) \) on the event \( Q_{z,t} \cap L_t[T]^c \). With no \( \ell \)-late point, the ball \( B_{r-m-\ell-1} \) is entirely filled by time \( t \). Lemma 7(b) shows that the sites in this ball contribute at most a constant to \( M(t) \) (recall that \( k = 1 \)). The thin tentacle estimate [JLS10, Lemma A] says that except for an event of probability \( e^{-cm^2} \), there are order \( m^d \) sites in \( A(t) \) within the ball \( B(z, m) \). By Lemma 9(b), \( P \) is bounded below by \( c/m^{d-1} \) on this ball, so these sites taken together contribute order \( m \) to \( M(t) \). Each of the remaining terms in the sum defining \( M(t) \) is bounded below by \( -P(0) \), and there are at most \( \ell r^{d-1} \) sites in \( A(t) \setminus B_{r-m-\ell-1} \). So these terms contribute at least

\[ -\ell r^{d-1}/r^{d-1} = -\ell \geq -m/C \]

which cannot overcome the order \( m \) term. Thus

\[ \mathbb{P}(Q_{z,t} \cap \{M(t) < m/C\} \cap L_t[T]^c) < e^{-cm^2}. \quad (6) \]

We conclude that

\[ \mathbb{P}(Q_{z,t} \cap L_t[T]^c) \leq \mathbb{P}(Q_{z,t} \cap \{S(t) > 1/c\}) \]
\[ + \mathbb{P}(Q_{z,t} \cap \{M(t) < m/C\} \cap L_t[T]^c) \]
\[ + \mathbb{P}(\{S(t) \leq 1/c\} \cap \{M(t) \geq m/C\}). \]
The first two terms are bounded by (5) and (6). Since \( M(t) = B(S(t)) \) for a standard Brownian motion \( B \), the final term is bounded by

\[
P \left\{ \sup_{0 \leq s \leq 1/c} B(s) \geq m/C \right\} < e^{-c(m/C)^2/2} < T^{-20\gamma}. \]

Proof of Lemma 4. Fix \( y \in \mathbb{Z}^d \), and let \( L[y] \) be the event that \( y \) is \( \ell \)-late. Let \( s = |y| \), and set \( k = a\ell \) in the definition of \( P \). Here \( a > 0 \) is a small dimensional constant chosen below. Note that the hypotheses on \( m \) and \( \ell \) imply that \( \ell \) is at least of order \( \sqrt{\log T} \); after choosing \( a \), we take the constant \( C_1 \) appearing in the statement of the lemma large enough so that \( k^2 > 1000\gamma \log T \).

Case 1. \( 1 \leq s \leq 2k \). Then \( P(0) \approx 1/s^{d-2} \). Let

\[
T_1 = \lfloor \omega_d (s + \ell)^d \rfloor
\]

With \( a_n = P(0) \) and \( b_n = 1 \), we have \( S(n+1) - S(n) \leq \tau_n \), where \( \tau_n \) is the first exit time of the Brownian motion \( B^n \) from the interval \([-a_n, b_n]\). (Note that because we take \( b_n = 1 \), the indicator \( 1_{\xi_{m+1}[T]^c} \) is not needed here as it was in the proof of Lemma 3.) We obtain

\[
\log \mathbb{E} e^{S(T_1)} \leq \sum_{t=1}^{T_1} \log \mathbb{E} e^{\tau_n} \leq T_1 P(0).
\]

Let \( Q = T_1 P(0) \). By Markov’s inequality, \( \mathbb{P}(S(T_1) > 2Q) \leq e^{-Q} \).

On the event \( L[y] \), the site \( y \) is still not occupied at time \( T_1 \). Accordingly, the largest \( M(T_1) \) can be is if \( A_{y,k}(T_1) \) fills the whole ball \( B_{s+k} \) (except for \( y \)), and then the rest of the particles will have to collect on the boundary where \( P \) is zero. The contribution from \( B_{s+k} \) is at most \( Ck^2 \) by Lemma 7(c). The number of particles stopped on the boundary is at least

\[
T_1 - 2\omega_d (s + k)^d \geq \frac{T_1}{2}.
\]

Therefore, on the event \( L[y] \) we have

\[
M(T_1) \leq Ck^2 - \frac{T_1}{2} P(0). \tag{7}
\]

Note that \( Q := T_1 P(0) \approx (s + \ell)^d/s^{d-2} \geq \ell^d/(k/2)^{d-2} \), so by taking \( a = k/\ell \) sufficiently small, we can ensure that the right side of (7) is at most \( -Q/4 \).
Also, \( Q \geq \ell^2 \geq 1000\gamma \log T \). Since \( M(t) = B(S(t)) \) for a standard Brownian motion \( B \), we conclude that

\[
\mathbb{P}(L[y]) \leq \mathbb{P}(S(T_1) > 2Q) + \mathbb{P}\left\{ \inf_{0\leq s \leq 2Q} B(s) \leq -Q/4 \right\} \\
\leq e^{-Q} + e^{-(Q/4)^2/4Q} \\
< T^{-20\gamma}.
\]

**Case 2.** \( s \geq 2k \). Then by Lemma 5(c) with \( r = 1 \), and Lemma 9(a), we have \( P(0) \approx k/s^{d-1} \). First take

\[
T_0 = \lfloor \omega_d(s + k - 3m)^d \rfloor \quad \text{(or } T_0 = 0 \text{ if } s + k - 3m \leq 0 \text{).}
\]

As in the previous lemma (but taking \( \lambda = 1 \) instead of \( \lambda = cm^2 \)) we have

\[
\log \mathbb{E}_c \left[ e^{S(T_1)} 1_{E_m[T]} \right] \leq C \frac{k}{s^{d-1}} \int_0^{T_0} \frac{dn}{(s + k - (n/\omega_d)^{1/d})^{d-1}} \leq Ck/m^{d-2} \leq C.
\]

The last inequality follows from \( d \geq 3 \) and \( m \geq k/a \). By Markov’s inequality,

\[
\mathbb{P}(\{S(T_0) > C + k^2\} \cap E_m[T]^c) < e^{-k^2} < T^{-20\gamma}.
\]

Now since

\[
(T_1 - T_0)P(0) \approx ms^{d-1}(k/s^{d-1}) = km
\]

we have

\[
\log \mathbb{E}_c e^{S(T_1) - S(T_0)} \leq Ckm.
\]

Thus (since \( km \geq k^2 \))

\[
\mathbb{P}(\{S(T_1) > 2Ckm\} \cap E_m[T]^c) < 2T^{-20\gamma}. \quad (8)
\]

As in case 1, the martingale \( M(T) \) is largest if the ball \( B_{s+k} \) is completely filled, and in that case the total contribution of sites in this ball is at most \( Ck^2 \). On the event \( L[y] \), the number of particles stopped on the boundary of \( \Omega \) at time \( T_1 \) is at least

\[
T_1 - \#B_{s+k} \geq \omega_d((s + \ell)^d - (s + k + C)^d) \approx \ell s^{d-1}.
\]
Each such particle contributes \(-P(0) \approx -k/s^{d-1}\) to \(M(T_1)\), for a total contribution of order \(-k\ell = -k^2/a\). Taking \(a\) sufficiently small we obtain \(M(T_1) \leq Ck^2 - k^2/a \leq -k^2\). We conclude that

\[
P(L[y] \cap E_m[T \cap T_1]) \leq P\left\{ S(T_1) > 2Ckm \right\} \cap E_m[T \cap T_1] + P\left\{ S(T_1) \leq 2Ckm \right\} \cap \{ M(T_1) \leq -k^2 \}.
\]

The first term is bounded above by (8), and the second term is bounded above by

\[
P\left\{ \inf_{s \leq 2Ckm} B(s) \leq -k^2 \right\} \leq e^{-k^4/4Ckm} < T^{-20\gamma}.
\]

Hence \(P(L[y] \cap E_m[T \cap T_1]) < 3T^{-20\gamma}\). Since \(\mathcal{L}_\ell[T]\) is the union of the events \(L[y]\) for \(y \in \mathcal{B} := B_{(T/\omega_d)^{1/d-\ell}}\), summing over \(y \in \mathcal{B}\) completes the proof.

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