The Differential Geometry and Physical Basis for the Applications of Feynman Diagrams

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Abstract

This paper recalls the development of gauge theory culminating in Yang-Mills theory, and the application of differential geometry including connections on fiber bundles to field theory. Finally, we see how the preceding is used to explain the Feynman diagrams appearing on the Feynman postage stamp released in May 2005. Version 2 included the Feynman diagrams, Version 3 corrected typos and Version 4 included an appendix for the derivation of the Yang-Mills transformation and field strength. Version 5 indicates that the article has been published in the Notices of the AMS and in July 2009 appears in Chinese translation in a journal of the Chinese Academy of Sciences. It also corrects some typos and adds to the appendix a heuristic derivation of the Yang-Mills field strength.

On May 11, the late Richard Feynman’s birthday, a stamp was dedicated to Feynman at the Post Office in Far Rockaway, New York City, Feynman’s boyhood home. (At the same time, the United States Postal Service issued three other stamps honoring the scientists Josiah Willard Gibbs and Barbara McClintock, and the mathematician John von Neumann.)

The design of the stamp tells a wonderful story. The Feynman diagrams on it show how Feynman’s work originally applicable to QED, for which he won the Nobel prize, was then later used to elucidate the electroweak force. The design is meaningful to both mathematicians and physicists. For mathematicians, it demonstrates the application of differential geometry; for physicists, it depicts the verification of QED, the application of the Yang-Mills equations and the establishment and experimental verification of the electroweak force, the first step in the creation of the standard model. The physicists used gauge theory to achieve this and were for the most part unaware of the developments in differential geometry. Similarly mathematicians developed fiber-bundle theory without knowing that it could be applied to physics. We should, however, remember that in general relativity, Einstein introduced geometry into physics. And as we will relate below, Weyl did so for electromagnetism. General relativity sparked mathematicians interest in parallel transport, eventually leading to the development of fiber-bundles in differential geometry. After physicists achieved success using gauge theory, mathematicians applied it to differential geometry. The story begins with Maxwell’s equations. In this story the vector potential $\mathbf{A}$ goes from being a mathematical

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construct used to facilitate problem solution in electromagnetism to taking center stage by causing the shift in the interference pattern in the Aharonov-Bohm solenoid effect. As the generalized four-vector $A_\mu$, it becomes the gauge field that mediates the electromagnetic interaction, and the electroweak and strong interactions in the standard model of physics – $A_\mu$ is understood as the connection on fiber-bundles in differential geometry. The modern reader would be unaccustomed to the form in which Maxwell equations first appeared. They are easily recognizable when expressed using vector analysis in the Heaviside-Gibbs formulation.

**Maxwell’s Equations**

The equations used to establish Maxwell’s equations in vacuo expressed in Heaviside-Lorentz rationalized units are:

1. $\nabla \cdot \vec{E} = \rho$ (Gauss’s law)
2. $\nabla \cdot \vec{B} = 0$ (No magnetic monopoles)
3. $\nabla \times \vec{B} = \vec{J}$ (Ampere’s law)
4. $\nabla \times \vec{E} = -\partial \vec{B}/\partial t$ (Faraday’s and Lenz’s law)

where $\vec{E}$ and $\vec{B}$ are respectively the electric and magnetic fields; $\rho$ and $\vec{J}$ are the charge density and electric current. The continuity equation which dictates the conservation of charge:

$\nabla \cdot \vec{J} + \partial \rho/\partial t = 0$

indicates that Maxwell’s equations describe a local theory since you cannot destroy a charge locally and recreate it at a distant point instantaneously. The concept that the theory should be local is the corner-stone of the gauge theory used in quantum field theory, resulting in the Yang-Mills theory, the basis of the standard model.

Maxwell realized that since:

$\nabla \cdot \nabla \times \vec{B} = 0$ (6)

equation (3) is inconsistent with (5), he altered (3) to read

$(3’) \quad \nabla \times \vec{B} = \vec{J} + \partial \vec{E}/\partial t$

Thus a local conservation law mandated the addition of the $\partial \vec{E}/\partial t$ term. Although equations (1), (2), (3’) and (4) are collectively known as Maxwell’s equations, Maxwell himself was only responsible for (3’).

Maxwell calculated the speed of a wave propagated by the final set of equations, and found its velocity very close to the speed of light. He thus hypothesized that light was an electromagnetic wave. Since the curl of a vector cannot be calculated in two-dimensions, Maxwell’s equations indicate that light, as we know it, cannot exist in a two-dimensional world. This is the first clue that
electromagnetism, is bound up with geometry. In fact equation (6) is the vector analysis equivalent of the differential geometry result stating that if $\beta$ is a p-form, and $d\beta$ is its exterior derivative, then $d(d\beta)$, or $d^2\beta = 0$.

Unlike the laws of newtonian mechanics, Maxwell’s equations carry over to relativistic frames. The non-homogeneous equations, (1) and (3’), become

$$(7) \partial_\mu F^{\mu\nu} = J^\nu$$

while the homogeneous equations, (2) and (4) become

$$(8) \epsilon^{\alpha\beta\gamma\delta} \partial_\beta F^\gamma_\delta = 0$$

where $\epsilon^{\alpha\beta\gamma\delta}$ is the Levi-Civita symbol, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, and $A^0$ is the scalar potential and $A^i$s ($i = 1, 2, 3$) the components of the vector potential $\vec{A}$. Note that both equations (7) and (8) are manifestly covariant. Yang remarked that equation (8) is related to the geometrical theorem that the boundary of a region has no boundary. In a later section, we will show that equation (8) is due to the principle $d^2\omega = 0$, where $\omega$ is a p-form. Yang’s geometrical explanation can be understood in differential geometry terms using the generalized Stoke’s theorem: $$\int_M d\omega = \int_{\partial M} \omega,$$ where $\omega$ is an n-form and $M$ is an $n + 1$ dimension oriented manifold with boundary $\partial M$. For the purposes of this article a manifold is simply a surface that is locally Euclidean. Because $d^2\omega = 0$, this leads to $$\int_M d^2\omega = \int_{\partial M} d\omega = \int_{\partial^2 M} \omega = 0,$$ where $\partial^2 M$ indicates the boundary of a boundary of a region. If we assume that $\omega$ is non-vanishing, then $\partial^2 M$ is $\emptyset$. The Möbius strip can be used as another example of the theorem Yang cites.

**Gauge Invariance**

In a 1918 article, Hermann Weyl tried to combine electromagnetism and gravity by requiring the theory to be invariant under a local scale change of the metric $g_{\mu\nu} \rightarrow g_{\mu\nu} e^{\alpha(x)}$, where $x$ is a 4-vector. This attempt was unsuccessful and was criticized by Einstein for being inconsistent with observed physical results. It predicted that a vector parallel transported from point $p$ to $q$ would have a length that was path dependent. Similarly, the time interval between ticks of a clock would also depend on the path on which the clock was transported. The article did, however, introduce

- The term “gauge invariance”, his term was *Eichinvarianz*. It refers to invariance under his scale change. The first use of “gauge invariance” in English was in Weyl’s 1929 English version of his famous 1929 paper.
- The geometric interpretation of electromagnetism,
- The beginnings of non-abelian gauge theory. The similarity of Weyl’s theory to non-abelian gauge theory is more striking in his 1929 paper.

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2 Yang, C.N., (1980), *Physics Today*, 6 42
3 Weyl, Hermann, (1919), *Sitzungsber. Preuss. Akad.*, Berlin, 465
4 See Jackson, J. D. and Okun, L. B., (2001). *Rev. Mod. Physics*, 73, 663.
5 Weyl, H. (1929). *Proc. Natl. Acad. Sci.*, 15, 32.
By 1929 Maxwell’s equations had been combined with quantum mechanics to produce the start of quantum electrodynamics. Weyl in his 1929 article turned from trying to unify electromagnetism and gravity to following a suggestion originally thought to have been made by Fritz London in his 1927 article and introduced as a phase factor an exponential in which the phase \( \alpha \) is preceded by the imaginary unit \( i \), e.g., \( e^{+iq\alpha(x)} \), in the wave function for the wave equations (for instance, the Dirac equation is \( (i\gamma^\mu \partial_\mu - m)\psi = 0 \)). It is here that Weyl correctly formulated gauge theory as a symmetry principle from which electromagnetism could be derived. It was to become the driving force in the development of quantum field theory. In their 2001 Rev. Mod. Phys. paper Jackson and Okun point out that in a 1926 paper pre-dating London’s, Fock showed that for a quantum theory of charged particles interacting with the electromagnetic field, invariance under a gauge transformation of the potentials required multiplication of the wave function by the now well-know phase factor. Many subsequent authors incorrectly cited the date of Fock’s paper as 1927. Weyl’s 1929 article along with his 1918 one and Fock’s and London’s, and other key articles appear in translation in a work by O’Raifeartaigh with his comments. Yang discusses Weyl’s gauge theory results as reported by Pauli, as a source for Yang-Mills gauge theory (although Yang didn’t find out until much later that these were Weyl’s results):

I was very much impressed with the idea that charge conservation was related to the invariance of the theory under phase changes, an idea, I later found out, due originally to H. Weyl. I was even more impressed with the fact the gauge-invariance determined all the electromagnetic interactions.

For the wave equations to be gauge invariant, i.e., have the same form after the gauge transformation as before, the local phase transformation \( \psi(x) \rightarrow \psi(x)e^{+iq\alpha(x)} \) has to be accompanied by the local gauge transformation

\[
(9) \quad A_\mu \rightarrow A_\mu - \partial_\mu \alpha(x)
\]

(The phase and gauge transformations are local because \( \alpha(x) \) is a function of \( x \).) This dictates that the \( \partial_\mu \) in the wave equations be replaced by \( \partial_\mu + iqA_\mu \) in order for the \( \partial_\mu \alpha(x) \) terms to cancel each other. Thus gauge invariance determines the type of interaction – here, the inclusion of the vector potential. This is called the gauge principle and \( A_\mu \) is called the gauge field or gauge potential. Gauge invariance is also called gauge symmetry. In electromagnetism, \( A \) is the space-time vector potential representing the photon field, while in electroweak theory, \( A \) represents the intermediate vector bosons \( W^\pm \) and \( Z^0 \) fields and in the strong interaction, \( A \) represents the colored gluon fields. The fact that the \( q \) in \( \psi(x)e^{+iq\alpha(x)} \) must be the same as the \( q \) in \( \partial_\mu + iqA_\mu \) to insure gauge invariance, means that the charge \( q \) must be conserved. Thus gauge invariance dictates charge conservation. By Noether’s theorem, a conserved current is associated with a symmetry. Here the symmetry is the non-physical rotation invariance in an internal space called a fiber. In electromagnetism the

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\(^6\) Weyl, Hermann, (1929). Zeit. f. Physic, 330 56.
\(^7\) London, Fritz, (1927). Zeit. f. Physic, 42 375.
\(^8\) Fock, V., (1926). Z. Phys, 39 226.
\(^9\) O’Raifeartaigh L., (1997) *The Dawning of Gauge Theory*, Princeton University.
\(^10\) Yang, C.N., (2005) *Selected Papers (1945-1980) With Commentary*, p19, World Scientific.
\(^11\) Pauli, W., (1941). Rev. Mod. Phys., 13, 203.
\(^12\) See section 4.6 of Aitchison, I.J.R., and Hey, A.J.G., (1989) *Gauge Theories in Particle Physics*, Adam Hilger.
rotations form the group U(1), the group of unitary 1-dimensional matrices. U(1) is an example of a **structure group** and the fiber is $S^1$, the circle.

A **fiber bundle** is determined by two manifolds and the structure group $G$ which acts on the fiber: the first manifold, called the total space $E$ consists of many copies of the fiber $F$, one for each point in the second manifold, the base manifold $M$ which for our discussion is the space-time manifold. The fibers are said to project down to the base manifold. A **principal fiber bundle** is a fiber bundle in which the structure group, $G$, is a Lie group that acts on the total space $E$ in such a way that each fiber is mapped onto itself and the action of an individual fiber looks like the action of the structure group on itself by left-translation. In particular, the fiber $F$ is diffeomorphic to the structure group $G$.

The gauge principle shows how electromagnetism can be introduced into quantum mechanics. The transformation $\partial_{\mu} \to \partial_{\mu} + iqA_{\mu}$ is also called the **minimal principle** and the operation $\partial_{\mu} + iqA_{\mu}$ is the covariant derivative of differential geometry, $D = d + iqA$, where $A$ is the connection on a fiber bundle. A connection on a fiber bundle allows one to identify fibers over points $b_i \in M$ via parallel transport along a path $\gamma$ from $b_1$ to $b_2$. In general, the particular identification is path dependent. It turns out that the parallel transport depends only on the homotopy class of the path if and only if the curvature of the connection vanishes identically. Recall that two paths are homotopic if one can be deformed continuously onto the other keeping the end points fixed.

Weyl in his 1929 paper also includes an expression for the curvature $\Omega$ of the connection $A$, namely Cartan’s second structural equation which in modern differential geometry notation is $\Omega = dA + A \wedge A$. It is the same form as the equation used by Yang and Mills which in modern notation is $\Omega = dA + [A, A]$, where $[,]$ is the Lie bracket. Since the transformations in (9) form an abelian group U(1), the space-time vector potential $A$ commutes with itself. Thus in electromagnetism the curvature of the connection $A$ is just

$$\Omega = dA.$$ (10)

which, as we will see in the next section, is the field strength $F$ defined as $F = dA$.

**Differential Geometry**

Differential geometry principally developed by Levi-Civita, Cartan, Poincaré, de Rham, Whitney, Hodge, Chern, Steenrod and Ehresmann led to the development of fiber-bundle theory which is used in explaining the geometric content of Maxwell’s equations. It was later used to explain Yang-Mills theory and to develop string theory. The successes of gauge theory in physics sparked mathematicians interest in it. In the 1970’s Sir Michael Atiyah initiated the study of the mathematics of the Yang-Mills equations and in 1983 his student Simon Donaldson using Yang-Mills theory discovered a unique property of smooth manifolds in $\mathbb{R}^4$. Michael Freedman went on to prove that there exists multiple exotic differential structures only on $\mathbb{R}^4$. It is known that in other dimensions, the standard differential structure on $\mathbb{R}^n$ is unique.

In 1959 Aharonov and Bohm established the primacy of the vector potential by proposing an electron diffraction experiment to demonstrate a quantum mechanical effect: A long solenoid lies

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13 In giving these definitions, we restrict attention to the smooth manifolds which is adequate for our discussion.

14 Donaldson, S. K. (1983), *Bull. Amer. Math. Soc.* 8, 81.

15 Y. Aharonov and D. Bohm, (1959) *Phys. Rev.* 115, 485.
behind a wall with two slits and is positioned between the slits and parallel to them. An electron source in front of the wall emits electrons that follow two paths. One path through the upper slit and the other path through the lower slit. The first electron path flows above the solenoid and the other path flows below it. The solenoid is small enough so that when no current flows through it, the solenoid doesn’t interfere with the electrons’ flow. The two paths converge and form a diffraction pattern on a screen behind the solenoid. When the current is turned on, there is no magnetic or electric field outside the solenoid so the electrons cannot be effected by these fields; however there is a vector potential $\mathbf{A}$ and it effects the interference pattern on the screen. Thus Einstein’s objection to Weyl’s 1918 paper can be understood as saying that there is no Aharonov–Bohm effect for gravity. Because of the necessary presence of the solenoid, the upper path cannot be continuously deformed into the lower one. Therefore, the two-paths are not homotopically equivalent.

The solution of $(1/2m)(-i\hbar \nabla - q\mathbf{A}/c)^2 \psi + qV \psi = E\psi$, the time-independent Schrödinger’s equation for a charged particle, is $\psi_0(x)e^{i(q/c) \int A(x) dx}$ where $\psi_0(x)$ is the solution of the equation for $A$ equals zero and $s(x)$ represents each of the two paths. Here $c$ is the speed of light and $\hbar$ is Plank’s constant divided by $2\pi$. The interference term in the superposition of the solution for the upper path and that for the lower path produces a difference in the phase of the electron’s wave function called a phase shift. Here the phase shift is $(q/ch) \oint A(x) dx$. By Stoke’s theorem, the phase shift is $(q/ch) \phi$ where $\phi$ is the magnetic flux in the solenoid, $\int \mathbf{B} \cdot d\mathbf{S}$. Mathematically, their proposal corresponds to the fact that even if the curvature [the electromagnetic field strength] of the connection vanishes [as it does outside of the solenoid] parallel transport along non-homotopic paths can still be path-dependent [producing a shift in the diffraction pattern].

Chambers\textsuperscript{16} performed an experiment to test the Aharonov and Bohm (AB) effect. The experiment, however, was criticized because of leakage from a tapered magnetic needle. Tonomura\textsuperscript{17} et. al. performed beautiful experiments that indeed verified the AB prediction. Wu and Yang\textsuperscript{18} analyzed the prediction of Aharonov and Bohm and comment that different phase shifts $(q/ch)\phi$ may describe the same interference pattern, whereas the phase factor $e^{i(q/c)\phi}$ provides a unique description. The equation $e^{2\pi Ni} = 1$, where $N$ is an integer means that $e^{i(q/c)(\phi + 2\pi N c h/q)} = e^{i(q/c)\phi} e^{2\pi Ni} = e^{i(q/c)\phi}$. Thus flux of $\phi$, $\phi + 2\pi ch/q$, $\phi + 4\pi ch/q$... all describe the same interference pattern. Moreover, they introduced a dictionary relating gauge theory terminology to bundle terminology. For instance, the gauge theory phase factor corresponds to the bundle parallel transport; and as we shall see, the Yang-Mills gauge potential corresponds to a connection on a principal fiber bundle.

Let’s see how using the primacy of the four-vector potential $\mathbf{A}$, we can derive the homogeneous Maxwell’s equations from differential geometry simply by using the gauge transformation. Then we’ll get the non-homogeneous Maxwell’s equations for source-free ($\mathbf{J} = 0$) electromagnetism using the fact that our world is a four-dimensional (space-time) world.

We will also show that Maxwell’s equations are invariant under the transformations $\mathbf{A}_\mu \rightarrow \mathbf{A}_\mu + \partial_\mu \alpha(x)$, or expressed in differential geometry terms, $\mathbf{A} \rightarrow \mathbf{A} + d\alpha(x)$. We want $\alpha(x)$ to vanish when a function of $\mathbf{A}$ is assigned to the $\mathbf{E}$ and $\mathbf{B}$ fields. Taking the exterior derivative of $\mathbf{A}$ will do this since $d^2 \alpha(x) = 0$. Set $\mathbf{A}$ to the 1-form $\mathbf{A} = -A_0 dt + A_x dx + A_y dy + A_z dz$. Evaluating $d\mathbf{A}$ and realizing that the wedge product $dx^0 \wedge dx^1 = -dx^1 \wedge dx^0$ and therefore $dx^1 \wedge dx^3 = 0$ where $dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 = 0$ produces the 2-form $d\mathbf{A}$ consisting of terms like

\textsuperscript{16}Chambers, R. G., (1960) \textit{Phys. Rev. Lett.} 5, 3
\textsuperscript{17}Tonomura, Akira, et. al., (1982) \textit{Phys. Rev. Lett.} 48, 1443, and (1986) \textit{Phys. Rev. Lett.} 56, 792
\textsuperscript{18}Wu, T. T. and Yang, C. N., (1975) \textit{Phys. Rev. D}, 12, 3845
\[(\partial_x A_0 + \partial_t A_x)dt\,dx\) and \[(\partial_x A_y - \partial_y A_x)dx\,dy\]. When all the components are evaluated, these terms become respectively \(\nabla A_0 + \partial_t \bar{A}/\partial_t\) and \(\nabla \times \bar{A}\). The analysis up to now has been purely mathematical. To give it physical significance we associate these terms with the field strengths \(\vec{B}\) and \(\vec{E}\). In electromagnetic theory, two fundamental principles are \(\nabla \cdot \vec{B} = 0\) (no magnetic monopoles) and for time-independent fields \(\vec{E} = -\nabla A_0\) (the electromagnetic field is the gradient of the scalar potential), so consistency dictates that in the time-dependent case, we assign the two terms to \(\vec{B}\) and \(\vec{E}\) respectively:

\[(11) \quad \vec{B} = \nabla \times \bar{A} \quad \text{and} \quad \vec{E} = -\nabla A_0 - \partial_t \bar{A}\]

The gradient, curl and divergence are spatial operators – they involve the differentials \(dx, dy\) and \(dz\). The exterior derivative of a scalar is the gradient, the exterior derivative of a spatial 1-form is the curl, and the exterior derivative of a spatial two-form is the divergence. In the 1-form \(A\), the \(-A_0 dt\) is a spatial scalar and when the exterior derivative is applied gives rise to \(\nabla A_0\). The remaining terms in \(A\) are the coefficients of \(dx^i\) constituting a spatial 1-form and thus produce \(\nabla \times \bar{A}\).

We define the field strength, \(F\) as \(F = dA\) and from equation (10), we see that the field strength is the curvature of the connection \(A\). Using the equations in (11) and the 2-form \(dA\) we get

\[(12) \quad F = E_x dx\,dt + E_y dy\,dt + E_z dz\,dt + B_x dy\,dz + B_y dz\,dx + B_z dx\,dy\]

where for example \(dx^i dt\) is the wedge product \(dx \wedge dt\). Since \(d^2 A = 0\)

\[(13) \quad dF = 0\]

Evaluating \(dF\) gives the homogeneous Maxwell’s equations. In equation (12) since the \(E\) part is a spatial 1-form, when the exterior derivative is applied, it produces the \(\nabla \times \bar{E}\) part of Maxwell’s homogenous equations. Since the \(B\) part of equation (12) is a spatial 2-form, it results in the \(\nabla \cdot \vec{B}\) part. Since \(dF = 0\), \(F\) is said to be a closed 2-form.

To get the expression for the non-homogeneous Maxwell’s equations, i.e., the equivalent of equation (7), we use

\[(14) \quad J = \rho dt + J_x dx + J_y dy + J_z dz\]

and calculate the Hodge dual using the Hodge star operator. The Hodge Duals are defined\(^{19}\) by \(\ast F_{\alpha\beta} = 1/2\epsilon_{\alpha\beta\gamma\delta} F^{\gamma\delta}\) and \(\ast J_{\alpha\beta\gamma} = \epsilon_{\alpha\beta\gamma\delta} J^{\delta}\). The Hodge star\(^{20}\) operates on the differentials in equation (12) and (14) using \(\ast(dx^i dt) = dx^j dx^k\) and \(\ast(dx^j dx^k) = -dx^i dt\) where \(i, j, k\) refer to \(x, y\) and \(z\), and are taken in cyclic order. The metric used is \((+++\). Thus the Hodge star takes a spatial 1-form \(dx^i dt\) into a spatial 2-form and vice versa with a sign change.

The non-homogeneous Maxwell’s equations are then expressed by

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\(^{19}\)Misner, C. W., Thorne, K. S., and Wheeler, J. A., (1973) Gravitation. Freeman, San Francisco.

\(^{20}\)Flanders, H., (1963). Differential Forms. Academic Press.
\[(15) \quad d*F = 0 \quad \text{(source-free)}\]

\[(15') \quad d*F = *J \quad \text{(non-source-free)}\]

where the 2-form \(*F\) and the 3-form \(*J\) are respectively the Hodge duals of \(F\) and \(J\). \(*F\) and \(*J\) are defined as

\[(16) \quad *F = -B_x dx dt - B_y dy dt - B_z dz dt + E_x dy dz + E_y dz dx + E_z dx dy\]

\[(17) \quad *J = \rho dx dy dz - J_x dt dy dz - J_y dt dz dx - J_z dt dx dy\]

Thus the Hodge star reverses the rolls of \(\vec{E}\) and \(\vec{B}\) from what they were in \(F\). In \(*F\) the coefficient of the spatial 1-form is now \(-\vec{B}\) which will produce the curl in the non-homogeneous Maxwell’s equations, and the coefficient of the spatial 2-form is \(\vec{E}\) which will produce the divergence. In \(\mathbb{R}^n\), the Hodge star operation on a p-form produces an (n-p)-form. Thus the form of Maxwell’s equations is dictated by the fact that we live in a four-dimensional world. When the 1-form \(A\) undergoes the local gauge transformation \(A \rightarrow A + d\alpha(x)\), \(dA\) remains the same since \(d^2\alpha = 0\). Since \(\vec{B}\) and \(\vec{E}\) are unchanged, Maxwell’s theory is gauge invariant.

**The Dirac and Electromagnetism Lagrangians**

To prepare for the discussion of the Yang-Mills equations, let’s investigate the Dirac and Electromagnetism Lagrangians. The Dirac equation is

\[(20) \quad (i\gamma^\mu \partial_\mu - m)\psi = 0\]

where the speed of light, \(c\), and Plank’s constant \(h\) are set to one. Its Lagrangian density is

\[(21) \quad \mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi\]

The Euler-Lagrange equations minimize the action \(S\) where \(S = \int \mathcal{L} dx\). Using the Euler-Lagrange equation where the differentiation is with respect to \(\bar{\psi}\), i.e.,

\[(22) \quad \partial_\mu (\bar{\psi} \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)}) - \frac{\partial \mathcal{L}}{\partial \psi} = 0\]

yields equation (20).

The same gauge invariant argument used in the Gauge Invariance section applies here. In order for the Lagrangian to be invariant under the phase transformation \(\psi(x) \rightarrow \psi(x)e^{+i\alpha(x)}\), this transformation has to be accompanied by the local gauge transformation \(A_\mu \rightarrow A_\mu - e^{-1}\partial_\mu \alpha(x)\) and \(\partial_\mu\) has to be replaced by \(\partial_\mu + ieA_\mu\). The Lagrangian density becomes
\begin{equation}
(23) \quad \mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - e\bar{\psi}\gamma^\mu \psi A_\mu
\end{equation}

The last term is the equivalent of the interaction energy with the electromagnetic field, \( j^\mu A_\mu \). In order for the Euler-Lagrangian equation differentiated with respect to \( A_\mu \) to yield the inhomogeneous Maxwell equation (7) we must add \(-\left(\frac{1}{4}\right)(F_{\mu\nu})^2\) getting

\begin{equation}
(24) \quad \mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - e\bar{\psi}\gamma^\mu \psi A_\mu - \left(\frac{1}{4}\right)(F_{\mu\nu})^2
\end{equation}

The Euler-Lagrange equation yields

\begin{equation}
(25) \quad \partial_\mu F^{\mu\nu} = e\bar{\psi}\gamma^\nu \psi
\end{equation}

which equals \( J^\nu \). Note that the gauge field \( A_\mu \) does not carry a charge and there is no gauge field self-coupling which would be indicated by an \([A_\mu, A_\nu]\) term in (25). The Lagrangian density does not yield the homogeneous Maxwell equations. They are satisfied trivially because the definition of \( F^{\mu\nu} \) satisfies the homogeneous equations automatically \(^{21}\).

From this it is apparent that the Lagrangian density for the electromagnetic field alone

\begin{equation}
(26) \quad \mathcal{L} = -J^\mu A_\mu - \left(\frac{1}{4}\right)(F_{\mu\nu})^2
\end{equation}

yields all of Maxwell’s equations.

In differential geometry, if \( j = 0 \), this Lagrangian density becomes

\begin{equation}
(27) \quad \mathcal{L} = \left(\frac{1}{2} F \wedge * F\right)
\end{equation}

**The Yang-Mills Theory**

The Yang-Mills theory incorporates isotopic spin symmetry introduced in 1932 by Heisenberg who observed that the proton and neutron masses are almost the same (938.272 MeV versus 939.566 MeV respectively). He hypothesized that if the electromagnetic field was turned off, the masses would be equal and the proton and neutron would react identically to the strong force, the force that binds the nucleus together and is responsible for the formation of new particles and the rapid (typically their lifetimes are about \(10^{-20}\) seconds) decay of others. In a non-physical space (also known as an internal space) called isospin space, the proton would have isospin up, for instance, and the neutron, isospin down; but other than that, they would be identical. The wave function for each particle could be transformed to that for the other by a rotation using the spin matrices of the non-abelian group \( SU(2) \). Because of charge independence, the strong interactions are invariant under rotations in isospin space. Since the ratio of the electromagnetic to strong force is approximately \( \alpha \), where \( \alpha = e^2/4\pi\hbar c = 1/137 \), to a good approximation we can neglect the fact that the electromagnetic forces break this symmetry. By Noether’s theorem, if there is a rotational symmetry in isospin

\(^{21}\) Jackson, J. D., (1998). *Classical Electrodynamics, 3rd Ed*, p600. John Wiley and Sons.
space, the total isotopic spin is conserved. This hypothesis enables us to estimate relative rates of the strong interactions in which the final state has a given isospin. The spin matrices turn out to be the Pauli matrices $\sigma_i$. The theory just described is a global one, i.e., the isotopic spin is independent of the space-time coordinate and thus no connection is used. We will see that Yang and Mills elevated this global theory to a local one. In 1954 they proposed applying the isospin matrices to electromagnetic theory in order to describe the strong interactions. Ultimately their theory was used to describe the interaction of quarks in the electroweak theory and the gluons fields of the strong force. In the next section we will give an example using the up quark $u$ which has a charge of $\frac{2}{3}e$ and down quark $d$ which has a charge of $\frac{1}{3}e$.

We have seen that the field strength (which is also the curvature of the connection on the fiber) is given by $F = dA + A \wedge A$. In electromagnetism $A$ is a 1-form with scalar coefficients for $dx^i$ so $A \wedge A$ vanishes. If, however, the coefficients are non-commuting matrices $A \wedge A$ does not vanish and provides for gauge field self-coupling. Yang and Mills formulated the field strength, using the letter $B$ instead of $A$, so we will follow suit. The field is

$$F_{\mu\nu} = (\partial_\nu B_\mu - \partial_\mu B_\nu) + i\epsilon(B_\mu B_\nu - B_\nu B_\mu)$$

or equivalently $F_{\mu\nu} = (\partial_\nu B_\mu - \partial_\mu B_\nu) + i\epsilon[B_\mu, B_\nu]$, where $B$ is the connection on a principal fiber bundle, i.e., the gauge potential. So as opposed to the electromagnetic field strength which is linear, their field strength is non-linear. They proposed using a local phase. For instance, one could let

$$\psi(x) \rightarrow \psi(x) e^{i\alpha_j(x)\sigma^j}$$

where $\sigma^j$ are the Pauli matrices and $j$ goes from 1 to 3. Thus the exponent includes the dot product (or inner product) in $\mathbb{R}^3$. The Pauli matrices do not commute, $[\sigma_i, \sigma_j] = i\epsilon^{ijk}\sigma_k$. Since $B_\mu = \frac{1}{2}b^i_\mu \sigma_i$ or $B_\mu = \frac{1}{2}\vec{b} \cdot \vec{\sigma}$ (where $b^i_\mu$ is called the isotopic spin vector gauge field) the four-vectors $B_\mu$ and $B_\nu$ in (28) do not commute. The purpose of the Pauli spin matrices in the connection $B$ is to rotate the particles in isospin space so that they retain their identities at different points in $\mathbb{R}^4$. Equation (28) can be rewritten so that the curvature is defined as $F = dB + i\epsilon[B, B]$. As opposed to the Maxwell’s equations case, the exterior derivative of the curvature $dF$, does not equal zero because of the commutator in the expression for the curvature. Thus the exterior derivative for the 2-form $F$ has to be altered to include the connection $B$.

The Lagrangian

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu(D_\mu - m)\psi - (\frac{1}{4})Tr(F_{\mu\nu}F^{\mu\nu})$$

is invariant under the gauge transformation for the covariant derivative given as

$$D_\mu = \partial_\mu - i\epsilon B_\mu$$

where $\epsilon$ is the coupling constant analogous to $q$ in (9). The connection $B_\mu$ transforms as

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22 Yang, C. N. and Mills, R. L., (1954). Phys. Rev. 96, p91
23 The weak and electromagnetic forces are the two manifestations of the electroweak force
(33) $B_\mu \rightarrow B_\mu + \epsilon^{-1} \partial_\mu \alpha + [\alpha, B_\mu]$.

the fiber is the sphere, $S^2$ and the structure group is SU(2).

Since there are three components of the vector gauge field $b^i_\mu$, there are three vector gauge fields representing three gauge particles having spin one. They were later identified as the intermediate vector bosons $W^\pm$ and $Z^0$ which mediate the electroweak interactions. The fact that there are three gauge particles is dictated by the fact that the gauge field is coupled with the three Pauli spin matrices. Also, since the charges of the up quark and down quark differ by one, the gauge field particles that are absorbed and emitted by them in quark-quark interactions can have charges of $\pm 1$ or zero. It’s astonishing that Yang and Mills in their 1954 paper predicted the existence of the three intermediate vector bosons.

The gauge particles predicted by the Lagrangian (30) have zero mass since any mass term added to (30) would make the Lagrangian non-invariant under a local gauge transformation. So the force associated with the particles would have infinite range as the photons of the electromagnetic interaction do. Of course the weak force (the force responsible for particle decaying slowly, typically their lifetimes are about $10^{-10}$ seconds or much less) and strong nuclear force are short range. This discrepancy was corrected some years later by the introduction of spontaneous symmetry breaking in the electroweak $SU(2) \times U(1)$ theory of Weinberg, Salam and Glashow (WSG) using the Higgs mechanism. The WSG theory, which explains the electromagnetic and weak forces, predicts the existence of four gauge bosons: the three massive ones, $W^\pm$ and $Z^0$, and the photon. Moreover, it predicts the mass of the $W^\pm$ ($80.37 \pm 0.03$ GeV) and $Z^0$ ($92 \pm 2$ GeV), where GeV represents a billion electron volts. The $W^\pm$ was discovered in 1983 (its mass is now reported at 80.425 GeV $\pm$ 0.033 GeV) and later that year the $Z^0$ was discovered (its mass is now reported at a mass of 91.187 $\pm$ 0.002 GeV).

The Euler-Lagrange equations for equation (30) give the Dirac equation

\[ (34) \ \gamma^\mu (\partial_\mu - ieB_\mu)\psi + m\psi = 0 \]

and also the vector equation for the vector field $F$, namely

\[ (35) \ \partial_\mu F_{\mu\nu} - i\epsilon[B^\mu, F_{\mu\nu}] = -\frac{1}{2}i\bar{\psi}\gamma_\nu\sigma\psi = -J_\nu \]

which, if it weren’t for the commutator, is the same form as the non-homogeneous four-vector Maxwell equation. The commutator causes the gauge particles to interact with themselves.

The effect of these equation is explained by ’t Hooft who with Veltman proved the renormalizability of Yang-Mills theories.

\[ ^{24} \text{Arnison, G. et. al., (1983).} \text{ Phys. Lett.} 122B, 103 \]
\[ ^{25} \text{Arnison, G. et. al., (1983).} \text{ Phys. Lett.} 126B, 398 \]
\[ ^{26} \text{’t Hooft, Gerardus, editor, (2005) 50 Years of Yang-Mills Theory, World Scientific.} \]
also a force that rotates these particles in isospin space, which means that elementary reactions involving the transmutation of particles into their isospin partners will result. A novelty in the Yang-Mills theory was that the $B$ quanta are predicted to interact directly with one another. These interactions originate from the commutator term in the $F_{\mu\nu}$ field [equation (35)], but one can understand physically why such interactions have to occur: in contrast with ordinary photons, the Yang-Mills quanta also carry isospin, so they will undergo isospin transitions themselves, and furthermore, some of them are charged, so the neutral components of the Yang-Mills fields cause Coulomb-like interactions between these charged particles.

So the Yang-Mills equations indicate that for instance for the up quark down quark doublet, the $W^-$ generates a force that rotates the $d$ into the $u$ in isospin space exhibited by the transition $d \rightarrow u + W^-$. The commutator in equation (35) is responsible for interactions like $W \rightarrow W + Z$ occurring\textsuperscript{27} and the $W$ can radiate producing a photon in $W \rightarrow W + \gamma$.

The Yang-Mills equations can be derived from the differential geometry Lagrangian density, where $k$ is a constant

\begin{equation}
L = -k Tr(F \wedge *F)\text{.}
\end{equation}

The Euler-Lagrange equations produce $d_B F = 0$ (the Bianchi identity) and in the absence of currents, $d_B * F = 0$ where $d_B$ is the exterior covariant derivative. These are the Yang-Mills equations in compact form.

**The Feynman Stamp**

In QED after Schwinger, Tomonaga and Feynman addressed the singularites produced by the self-energy of the electron by renormalizing the theory, they were then exceedingly successful in predicting phenomena such as the Lamb shift and anomalous magnetic moment of the electron.

Feynman introduced\textsuperscript{28} schematic diagrams, today called *Feynman diagrams*, to facilitate calculations of particle interaction parameters. External particles, represented by lines (edges) connected to only one vertex are real, i.e., observable. They are said to be on the mass shell, meaning their four-momentum squared equals their actual mass, i.e., $m^2 = E^2 - p^2$. Internal particles are represented by lines that connect vertices and are therefore intermediate states – that’s why they are said to *mediate* the interaction. They are virtual and are considered to be off the mass shell. This means their four-momentum squared differs from the value of their actual mass. This is done so that four-momentum is conserved at each vertex. The rationale for this difference is the application of the uncertainty principle $\Delta E \cdot \Delta t = \hbar$. Since $\Delta t$, the time spent between external states is very small, for that short time period, $\Delta E$ and thus the difference between the actual and calculated mass can be large. In the following Feynman diagrams, the time axis is vertical upwards.

The diagram on the upper-left of the stamp (Figure 1) is a vertex diagram, and as such represents a component of a Feynman diagram. It illustrates the creation of an electron-positron pair from a photon, $\gamma$; it’s called *pair production*. The $\gamma$ is represented by a wavy line. The Feynman-Stuckelberg interpretation of negative-energy solutions indicates that here the positron,
the electron’s antiparticle, which is propagating forward in time is in all ways equivalent to an electron going backwards in time. If all the particles here were external, the process would not conserve energy and momentum. To see this you must first remember that since the photon has zero mass – due to the gauge invariance of electromagnetic theory – its energy and momentum are equal. Thus $\beta$ which equals $\frac{p}{E}$ has the value 1; but $\beta = \frac{v}{c}$ so that the photon’s velocity is always $c$, the speed of light. In the electron-positron center of mass frame (more aptly called the center of momentum frame, since the net momentum of all the particles is zero there), the electron and positron momenta are equal and are in opposite directions. The photon travels at the speed of light and therefore its momentum cannot be zero; but there is no particle to cancel its momentum, so the interaction cannot occur (for it to occur requires a Coloumb field from a nearby nucleus to provide a virtual photon that transfers momentum producing a nuclear recoil). Therefore the $\gamma$ in the diagram is internal. Its mass is off the mass shell and cannot equal its normal value, i.e., zero.

The diagram on the lower-left of the stamp (Figure 2) is also a vertex diagram and represents an electron-positron pair annihilation producing a $\gamma$. Again, if all the particles are external, conservation energy and momentum prohibits the reaction from occurring, So the $\gamma$ must be virtual.

The diagram (Figure 3) on the bottom to the right of Feynman was meant to represent an electron-electron scattering with a single photon exchange. This is called Møller scattering. (It can, however, represent any number of interactions exchanging a photon.) The diagram represents the $t$-channel of Møller scattering; there is another diagram not shown here representing the $u$-channel contribution where $u$, $t$ and another variable $s$ are called the Mandelstam variables. They are used in general to describe 2-body $\rightarrow$ 2-body interactions. If you rotate the diagram in Figure 3 by 90° you have the $s$-channel diagram for electron-positron scattering called Bhabha scattering shown in Figure 4 but not on the stamp. Here an electron and positron annihilate producing a virtual photon which in turn produces an electron-positron pair. There is also a $t$-channel contribution to Bhabha scattering. The cross section for Bhabha scattering can be easily obtained from the one for Møller scattering by interchanging the $s$ and $u$ in the cross section expression in a process called crossing. Small angle Bhabha scattering is used to test the luminosity in $e^+e^-$ colliding beam accelerators.

To the right of the Møller scattering diagram is a vertex correction to electron scattering shown in Figure 5 where the extra photon forms a loop. It is used to calculate both the anomalous magnetic moment of the electron and muon, also the anomalous magnetic moment contribution to the Lamb shift. The other two contributions to the Lamb shift are the vacuum polarization and the electron mass renormalization. The Lamb shift explains the splitting in the spectrum of the $2S_{\frac{1}{2}}$ and $2P_{\frac{1}{2}}$ levels of hydrogen; whereas Dirac theory alone incorrectly predicted that these two levels should be degenerate.

The low-order solution of the Dirac equation predicts a value of 2 for the $g$-factor used in the expression for the magnetic moment of the electron. The vertex correction shown in Figure 5, however, alters the $g$-factor producing an anomalous magnetic moment contribution written as $\frac{g - 2}{2}$. When this and higher order contributions are included, the calculated value of $\frac{g - 2}{2}$ for the electron is $1159\,652\,460(127)(75) \times 10^{-12}$ and the experimental value is $1159\,652\,193(10) \times 10^{-12}$ where the numbers in parenthesis are the errors. This seven-significant figure agreement is a spectacular triumph for QED. We need not emphasise that the calculations for all these diagrams use the gauge principal for quantum electrodynamics.

The other diagrams on the stamp are all vertex diagrams and show how Feynman’s work originally

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29 See for instance p156, Griffith, David, (1987) Introduction to Elementary Particles John Wiley and Sons.
applicable to QED was then later used to elucidate the electroweak force. This is exemplified on the stamp by flavor changing transitions, e.g., $d \rightarrow W^- + u$ shown in Figure 6 and flavor conserving transitions, e.g., $d \rightarrow Z^0 + d$ of the electroweak force – the $u$ and $d$ quarks have different values of flavor. The process in Figure 6 occurs for instance in $\beta$ decay where a neutron (udd) decays into a proton (udu) and electron and an anti-neutrino. What happens is that the transition $d \rightarrow u + W^-$ corresponds to a rotation in isospin space. This rotation is caused by the virtual $W^-$ which mediates the decay. It in turn decays into an electron and an anti-neutrino. The calculations for these transitions all use the Yang-Mills equations. Although the quarks are confined in the hadrons – particles that undergo strong interactions like the proton and neutron – they are free to interact with the intermediate vector bosons.

Who Designed The Stamp?

Feynman’s daughter Michelle was sent a provisional version of the stamp by the United States Postal Service and advised on the design of the stamp by among others, Ralph Leighton, coauthor with Richard Feynman of two popular books; and Cal Tech’s Steven Frautschi and Kip Thorne. Frautschi and Leighton edited the Feynman diagrams, and Frautschi rearranged them and composed the final design. The person who chose the original Feynman diagrams that form the basis for the stamp remains a mystery.

Acknowledgements

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Appendix A: Yang-Mills Derivation
We begin by performing a phase transformation

(A1) $\psi' = S\psi$

where $S = e^{i\alpha(x) \cdot \sigma}$ and use the covariant derivative $D_\mu = \partial_\mu - i\epsilon B_\mu$ which transforms in the same way as indicated in equation (A2)

(A2) $D'^\mu \psi' = SD\psi$. Then

(A3) $(\partial_\mu - i\epsilon B'_\mu)S\psi = (\partial_\mu S)\psi + S\partial_\mu \psi - i\epsilon B'_\mu \psi$

But (A3) equals $S\partial_\mu \psi - i\epsilon SB_\mu \psi$.

Cancelling $S\partial_\mu \psi$ on both sides we get,

(A4) $(\partial_\mu S)\psi - i\epsilon B'_\mu S\psi = -i\epsilon SB_\mu \psi$, or

(A5) $-i\epsilon B'_\mu S = -i\epsilon SB_\mu - (\partial_\mu S)$ or $B'_\mu S = SB_\mu + (\partial_\mu S)/(i\epsilon)$, thus
(A6) \( B'_\mu S = S B_\mu - i(\partial_\mu S)/\epsilon \) or

(A7) \( B'_\mu = SB_\mu S^{-1} - i(\partial_\mu S)S^{-1}/\epsilon \)

For \( \alpha \) infinitesimal, \( S = 1 + i\alpha \cdot \sigma \), so

(A8) \( B'_\mu = (1 + i\alpha \cdot \sigma)B_\mu (1 - i\alpha \cdot \sigma) - i(1/\epsilon)\partial_\mu (1 + i\alpha \cdot \sigma)(1 - i\alpha \cdot \sigma) \) and

Remembering that \((a \cdot \sigma)(b \cdot \sigma) = a \cdot b + i\sigma \cdot (a \times b)\), setting \( B_\mu = \sigma \cdot b_\mu \), and since \( \alpha \) is infinitesimal, we drop terms of order \( \alpha^2 \) getting

(A9) \( b'_\mu \cdot \sigma = b_\mu \cdot \sigma + i[(\alpha \cdot \sigma)(b_\mu \cdot \sigma), (b_\mu \cdot \sigma)(\alpha \cdot \sigma)] + (1/\epsilon)\partial_\mu (\alpha \cdot \sigma) \) and finally

(A10) \( b'_\mu = b_\mu + 2(b_\mu \times \alpha) + (1/\epsilon)\partial_\mu \alpha \), which is equation (10) in the Yang-Mills paper.

Pauli, in equation (22a) of Part I of his 1941 Rev. Mod. Phys. article gives the electromagnetic field strength as \( [D_\mu, D_\nu] = -i\epsilon F_{\mu\nu} \) which apart from the minus sign agrees with our conventions and where \( D_\mu = \partial_\mu - i\epsilon A_\mu \). So by following suit, the field strength for the Yang-Mills strength can be obtained from the commutator

(A11) \([D_\mu, D_\nu] = (\partial_\mu - i\epsilon B_\mu)(\partial_\nu - i\epsilon B_\nu) - (\partial_\nu - i\epsilon B_\nu)(\partial_\mu - i\epsilon B_\mu)\)

operating on the wave function \( \psi \). Note that \(-\partial_\mu(B_\nu \psi) = -(\partial_\mu B_\nu)\psi - B_\nu \partial_\mu \psi \). So we get an apparently extra \(-B_\nu \partial_\mu \) and a \( B_\mu \partial_\nu \) term. Thus expanding (A11) we get

(A12) \( \partial_\mu \partial_\nu - i\epsilon \partial_\mu B_\nu - i\epsilon B_\mu \partial_\nu - i\epsilon B_\nu \partial_\mu - \epsilon^2 B_\mu B_\nu - \partial_\nu \partial_\mu + i\epsilon \partial_\nu B_\mu + i\epsilon B_\nu \partial_\mu + i\epsilon B_\mu \partial_\nu + \epsilon^2 B_\nu B_\mu \)

which reduces to

(A13) \( i\epsilon(\partial_\nu B_\mu - \partial_\mu B_\nu) - \epsilon^2 [B_\mu, B_\nu] \) or

(A14) \([D_\mu, D_\nu] = i\epsilon F_{\mu\nu} \) where \( F_{\mu\nu} \) is given by equation (28).

If we let \( B_\mu = \sigma \cdot b_\mu \) we can rewrite the equation \( F_{\mu\nu} = (\partial_\nu B_\mu - \partial_\mu B_\nu) + i\epsilon(B_\mu B_\nu - B_\nu B_\mu) \) as

(A15) \( F_{\mu\nu} = (\partial_\nu B_\mu - \partial_\mu B_\nu) + i\epsilon(2i\sigma \cdot b_\mu \times b_\nu) \)

If we further let \( F_{\mu\nu} = f_{\mu\nu} \cdot \sigma \), we get

(A16) \( f_{\mu\nu} = (\partial_\nu b_\mu - \partial_\mu b_\nu) - 2i\epsilon b_\mu \times b_\nu \)
which is equation (9) in the Yang-Mills paper.

Appendix B, Finding the Field Strength

We reconstruct how one can go about determining the field strength. Since

\[ F'_{\mu\nu} = S^{-1} F_{\mu\nu} S \]  \hspace{1cm} (1)

under an isotopic gauge transformation, let’s start off with the electromagnetic-like field strength in the primed system

\[ F'_{\mu\nu} = \partial_{\nu} B'_{\mu} - \partial_{\mu} B'_{\nu} \]  \hspace{1cm} (2)

and express it in terms of the non-primed system fields. We calculate \( \partial_{\nu} B'_{\mu} \) from \( B'_{\mu} = S^{-1} B_{\mu} S + i S^{-1} (\partial_{\mu} S) / \epsilon \), equation (A7), obtaining

\[ \partial_{\nu} B'_{\mu} = -S^{-1} (\partial_{\nu} S) S^{-1} B_{\mu} S + S^{-1} (\partial_{\nu} B_{\mu}) S + S^{-1} B_{\mu} \partial_{\nu} S + \]

\[ i/\epsilon [-S^{-1} (\partial_{\nu} S) S^{-1} \partial_{\mu} S + S^{-1} \partial_{\nu} \partial_{\mu} S] \]  \hspace{1cm} (3)

So

\[ \partial_{\nu} B'_{\mu} - \partial_{\mu} B'_{\nu} = -S^{-1} [(\partial_{\nu} S) S^{-1} B_{\mu} S - (\partial_{\mu} S) S^{-1} B_{\nu}] S 
+ S^{-1}[\partial_{\nu} B_{\mu} - \partial_{\mu} B_{\nu}] S + S^{-1}[B_{\mu} \partial_{\nu} - B_{\nu} \partial_{\mu}] S + 
\]

\[ i/\epsilon [-S^{-1} (\partial_{\nu} S) S^{-1} \partial_{\mu} S + S^{-1} (\partial_{\mu} S) S^{-1} \partial_{\nu} S] \]  \hspace{1cm} (4)

We see that the \(+ S^{-1} [(\partial_{\nu} S) S^{-1} B_{\mu} S - (\partial_{\mu} S) S^{-1} B_{\nu}] S\) term satisfies equation (1) if the field strength only had the electromagnetic-like contribution. The other terms must either represent the transformed non-electromagnetic-like part of \( F_{\mu\nu} \) or be cancelled by adding the non-electromagnetic terms to equation (2). Since \( S \) is only used for the transformation, it should not appear in the expression for \( F_{\mu\nu} \).

The \( i/\epsilon \) term in equations (4) dictates that a term multiplied by \( i \epsilon \) be added to equation (2). Since \( S^{-1} (\partial_{\mu} S) \) and \( S^{-1} \partial_{\mu} S \) appear in the expressions for \( B'_{\mu} \) and \( B'_{\nu} \) respectively, the product of \( S^{-1} (\partial_{\mu} S) \) and \( S^{-1} \partial_{\nu} S \) that appears in the last term of equation (4) suggests that we should start our quest to eliminate extra terms in equation (4) by adding \( i \epsilon B'_{\mu} B'_{\nu} \) to that equation. This product gives

\[ i \epsilon B'_{\mu} B'_{\nu} = i \epsilon [S^{-1} B_{\mu} S + i S^{-1} (\partial_{\mu} S) / \epsilon] \quad [S^{-1} B_{\nu} S + i S^{-1} (\partial_{\nu} S) / \epsilon] \]

which equals
\[ i\epsilon S^{-1}B_\mu B_\nu S - i/\epsilon S^{-1}(\partial_\mu S)S^{-1}\partial_\nu S - \\
S^{-1}B_\mu \partial_\nu S - S^{-1}(\partial_\mu S)S^{-1}B_\nu S \]  \hspace{1cm} (6)

All but the first term (which represents the transformation of \(i\epsilon B_\mu B_\nu\)) cancel components of the extraneous terms in equation (4). And \(i\epsilon(B'_\mu B'_\nu - B'_\nu B'_\mu)\) cancels all of the extraneous terms except the transformation of \(i\epsilon(B_\mu B_\nu - B_\nu B_\mu)\).

After performing the cancellation, we get

\[ \partial_\nu B'_\mu - \partial_\mu B'_\nu + i\epsilon(B'_\mu B'_\nu - B'_\nu B'_\mu) = \\
S^{-1}[\partial_\nu B_\mu - \partial_\mu B_\nu + i\epsilon(B_\mu B_\nu - B_\nu B_\mu)]S \]  \hspace{1cm} (7)

which satisfies equation (1).
Figure 1. A pair production vertex.

Figure 2. A pair annihilation vertex.

Figure 3. Electron-electron (Møller) scattering.
Figure 4. Electron-positron (Bhabha) scattering.

Figure 5. Radiative Correction.

Figure 6. A flavor non-conserving transition vertex.