Equivalent String Networks and Uniqueness of BPS States

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Abstract

We analyze string networks in 7-brane configurations in IIB string theory. We introduce a complex parameter $\mathcal{M}$ characterizing equivalence classes of networks on a fixed 7-brane background and specifying the BPS mass of the network as $M_{\text{BPS}} = |\mathcal{M}|$. We show that $\mathcal{M}$ can be calculated without knowing the particular representative of the BPS state. Based on detailed examination of backgrounds with three and four 7-branes we argue that equivalent networks may not be simultaneously BPS, an essential requirement of consistency.
1 Introduction

D-branes in superstring theories have recently given new insight into and powerful tools for studying supersymmetric field theories. The worldvolume theory of parallel Dp-branes is a p+1 dimensional U(n) SUSY Yang-Mills, where the Higgs expectation values correspond to the separation of the branes. The massive gauge bosons arise as open strings stretched between the D-branes.

7+1 dimensional SYM theories with gauge groups other than U(n) may also be constructed in IIB string theory [1, 2, 3], due to the existence of mutually nonperturbative 7-branes. These branes are labeled by two coprime integers \([p, q]\) or more precisely by the two-by-two matrix \(
\begin{pmatrix}
1 & pq \\
-q^2 & 1-pq
\end{pmatrix}
\), which transforms as a tensor under the SL(2,Z) duality group. The \([p, q]\)-branes serve as endpoints for \((p, q)\)-strings [4] and are sources of monodromies of the axion-dilaton field \(\tau\). The single-valuedness of \(\tau\) requires the introduction of branch cuts and as a consequence, the charges of a general \((p, q)\)-string change upon crossing the cuts. The configurations with mutually nonlocal 7-branes give rise to enhanced gauge groups \(SO(n), E_6, E_7\) and \(E_8\). By introducing a D3-brane near the 7-branes, an \(N=2\) SUSY Yang-Mills theory is realized in its 3+1 dimensional world volume [5, 6], with the above groups appearing as global (flavor) symmetries.

The gauge group of the worldvolume theory of the 7-branes is not as manifest as it is in pure D-brane configurations. The states are realized as either \((p, q)\)-strings between two branes or as multi-pronged string networks ending on more than two branes [4]. The identification of the gauge group amounts to finding all these BPS string states in a background with nontrivial metric and SL(2,Z)-monodromy. As an alternative, one may lift the setup to F-theory [8] and deduce the enhanced symmetry from the singularity of the K3 as the 7-branes collapse [4].

The string theory picture of the BPS junctions has been subject to intensive study [10, 11, 12, 13, 14, 15, 16, 17] and the emerging picture is quite rich and interesting. It has been found that the states may correspond to different objects – strings or multi-pronged networks – in string theory, and the realization of a given state may change as we move around in the moduli space. The necessity of three-pronged junctions in the BPS spectrum was first anticipated by Gaberdiel and Zwiebach [7] in 7-brane backgrounds; the transitions between open strings and junctions were studied in [18]. Their
existence among the ultrashort multiplets in d=4, N=4 SU(3) SYM as a worldvolume theory of three D3-branes was showed by Bergman \cite{19} (see also \cite{20, 21}), while the role in the spectrum of the d=4, N=2 SU(2) SYM was clarified in \cite{22, 23}. String webs also play an important role in 5-dimensional field theories realized as 5-brane webs \cite{24, 25}.

Having anticipated that the multipronged strings are necessary ingredients of certain IIB configurations the consistency of the picture demands that the field theory states have unique string theory representatives. In \cite{18} this was shown in special 7-brane configurations with constant \(\tau\) background and purely conical metric singularities, by studying the transitions between strings and junctions. Partial results for general \(\tau\) were also reported showing that a string may not be simultaneously BPS with a junction which is created from it by a single 7-brane crossing. The aim of the present notes is to extend the above reasoning to arbitrary 7-brane configurations and string networks, and argue that a given BPS state is realized by a unique object at any point in the moduli space. We believe that our results are useful both in 7-brane theories and in 3-brane theories in 7-brane background.

We will address the above problem in two steps. First, one should identify the strings and junctions which – on the basis of charge assignments – are thought to give rise to the same state in the brane field theory. These are related to each other by a series of crossing transformations, the process when an open brane (in our case a string) is created between two branes as they cross each other. (For an exhaustive classification of the equivalence classes, see \cite{26}.) The determination of the equivalent networks is based on the monodromies of the axion-dilaton field, but is unaffected by the actual positions of the branes. We take advantage of this fact by defining the notion of a graph which refers to the topology and homotopy of the object. As the second step, the graphs are mapped to the concrete spacetime and we may ask the whether this map results in a geodesic object which possesses the BPS property. One of the purposes of the paper is to argue that two graphs in the equivalence class of an open string may not give rise to BPS networks simultaneously. We will define a complex parameter, \(M\) which depends on the equivalence class under consideration but not on the particular representative and show that the mass of the BPS network is \(|M|\). One can also express the slopes of the constituent strings of a network in terms of \(M\) which will be an important ingredient in the argument for the uniqueness of the BPS representative.

The physics we are considering also possesses a description in terms of F-theory and M-theory. The strings are lifted to cycles in K3 or membranes and the BPS condition translates to the requirement that these membranes
be embedded holomorphically, moreover the properties of the states are determined by the geometry \cite{27, 28, 25, 21}. The uniqueness of the BPS representative of a state is then seen from the uniqueness of the holomorphic embedding of the zero-genus curve corresponding to the open string. (The networks are not unique if they correspond to higher genus curves). We would like to offer an alternative viewpoint to this picture; our approach throughout the paper is conservative in that we restrict ourselves to arguments based on IIB string theory only.

The plan of the paper is as follows. In sec. 2 we define the equivalence classes of string networks. In sec. 3 we demonstrate the uniqueness of the BPS object in the simple example of a background of three 7-branes. In sec. 4 we present the generalizations of the statements of sec. 3 which we apply to the next simplest case of four 7-branes in sec. 5. In sec. 6 we show an explicit construction of the BPS network generalizing the reconstruction of a geodesic from one of its points and the slope at that point. In the last section we discuss concrete transitions in constant \(\tau\)-background and present an interesting application showing how the string tensions modify Plateau’s problem on the graphs of least length connecting the vertices of a polygon.

## 2 Equivalent graphs

We consider configurations of parallel 7-branes, whose worldvolume spans the \(x^0 \ldots x^7\) directions and as a consequence of this 8-dimensional translational invariance we will restrict ourselves to the two dimensions of the \(x^8 x^9\) plane. In the F-theory picture these two directions are compactified on a sphere and this fixes the number of 7-branes to be 24, but since we will be interested in transitions in the close vicinity of groups of branes, we do not need to care about such “ultra-global” behavior of the spacetime.

Let us first fix the global properties of the background in the \(x^8 x^9\) plane. The 7-branes act as sources for the axion-dilaton field, \(\tau\) whose monodromies can described by defining the charges \([p_i, q_i]\) together with branch cuts emanating from the branes. The precise position of these cuts is non-physical, but the relative order is relevant. A typical configuration is shown in Fig. 1(a); note that the figure is not meant to indicate any particular position of the branes in spacetime, only the global properties of \(\tau\) are defined (see \(\text{[18]}\) for the precise definition of the covering space). Our conventions are the same as in
the examples will involve $A$- and $C$-branes whose monodromies are:

\[
A = [1, 0]: K_A = T^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},
\]

\[
C = [1, 1]: K_C = T^2S = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix},
\]

here a \((p, q)\) string becomes a $K(p, q)$-string upon crossing the corresponding cut in the anticlockwise direction.

\[\text{(a) } \quad \text{(b) } \quad \text{(c)}\]

Figure 1: (a) We first define the monodromies of $\tau$. The filled circles stand for $A$-branes, the empty squares are $C$-branes. (b) A graph that is allowed by charge conservation. (c) The same graph as in (b).

As far as the homotopy of curves is concerned any string or network can be represented within the above figure. We define a graph to be a collection of curves joining three-pronged junctions and/or branes. An example is presented in Fig. 1(b) where three curves join a three-pronged junction with three 7-branes. We do not allow for four- or more-pronged junctions, these will arise as limiting cases of the objects described by our graphs. The curves in a graph represent \((p, q)\)-strings and thus charge conservation restricts the allowed graphs. In Fig. 1(b) and in most cases that we shall consider, the charges are uniquely determined (up to an overall sign) by the homotopy and we will not indicate them. We stress that the notion of a graph does not refer to the actual paths or angles in spacetime thus there is no distinction between the graphs in Fig. 1(b) and (c): they are said to be equal. Also we consider two graphs equal if one is obtained from the other by pulling a \((p, q)\)-curve through a \([p, q]\)-brane as a string does not feel monodromy around a brane it may end on.
Two graphs will be said to be *crossing-transformed* of each other if one is obtained from the other by pulling one of the strings through a brane and creating an extra prong [29] as for example in Fig. 2. See [6, 18] for a detailed description of the charges and charge conservation. In general the resulting prong may not have coprime charges.

![Crossing-transformation](image)

**Figure 2: Crossing-transformation.**

We will call two graphs *equivalent* if they are related by a series of crossing-transformations and channel-transitions, the latter corresponds to changing a “t-channel subdiagram” to an “s-channel subdiagram” without altering the outgoing prongs, see Fig. 3.

![Channel transition](image)

**Figure 3: Channel transition.**

A *generalized string* or *network* is a continuous map of a graph onto $S^2$ with the obvious condition that the “branes” are mapped to the corresponding branes and the homotopy is preserved. Two junctions are said to be equivalent if their graphs are equivalent. In this paper we shall restrict ourselves to networks which are equivalent to an open string; these do not have moduli parameters and will be uniquely realized.

We will be looking for graphs that are BPS in the background of the 7-branes. We will mostly rely on two necessary conditions [12]: in a BPS string network the strings are smooth geodesics and the tensions balance each other at the junction points [10].
3 BPS state in the vicinity of three 7-branes

As a warm-up we restrict ourselves to the vicinity of three branes. This situation in the background of constant axion-dilaton field was discussed in [18]. We will be interested in networks whose graphs are equivalent to that of a given open string, the one shown in Fig. 4(a). Application of crossing transformations through the various branes results in the equivalent graphs shown in Fig. 4(b), (c), and (d). We shall see that the endpoints of a graph determine the BPS network (independent of the multiplicity of the prongs), thus as far as the possible endings are concerned, the four graphs form an exhaustive list. Notice that it is unnecessary to extend the list with graphs which have both incoming and outgoing prongs on the same brane. The reason is that these pairs of prongs are not fixed to the brane, they can be considered as parts of a string passing by; Fig. 5 shows a network equivalent to the ones in Fig. 4.

Figure 4: Four equivalent graphs in the vicinity of three branes.

Depending on the positions of the three 7-branes a given graph of Fig. 4 may or may not be realized as a BPS object. In the following we will show that there is no point in the moduli space where any two of the above graphs may be BPS.

Let us assume that for any of the above four graphs there is a region in the moduli space where a BPS state homotopic to the graph exists. Based on this assumption we shall determine the BPS mass and the slopes of the geodesics at the branes. We will find that this data will be the same for the four graphs.

We choose $z$ as the complex coordinate on the cut plane and the positions of the branes are denoted by $P_1$, $P_2$ and $P_3$ (see Fig. 6). Let the 7-branes at $P_1$
Figure 5: (a) A network with a pair of incoming and outgoing strings on the middle brane. (b) A deformation of the previous network shows that the middle brane does not play an important role and we obtain an A-A string with a closed loop in the middle. This can never be BPS as will be shown in sec. 5.

and $P_3$ be $[p, q]$-branes and the one at $P_2$ an $[r, s]$-brane and also define $e$ as

$$ e = ps - qr. \quad (3.1) $$

Note that the charges of the 7-branes are now arbitrary, they need not be A- and C-branes.

Figure 6: Direct string corresponding to Fig. 4. When there is a BPS state represented by this graph its mass is given as an integral along $\gamma_{13}$.

The mass of a $\binom{p}{q}$ string can be calculated using the tension-weighed metric:

$$ ds_{\gamma_{13}}^2 = T_{\binom{p}{q}} d^2 s = |h_{\binom{p}{q}}(\tau(z), z) dz|^2 \quad (3.2) $$
where $h_{(p,q)}(z)$ and $\tau$ is given in terms of the position of the branes, $z_i$ as:

$$h_{(p,q)}(\tau(z), z) = (p - q\tau(z))\eta^2(\tau)\prod_i(z - z_i)^{-\frac{1}{12}}$$  \hspace{1cm} (3.3)

$$j(\tau(z)) = \frac{4(24f)^3}{\Delta}; \quad \Delta \equiv 27g^2 + 4f^3 = \prod_{i=1}^{24}(z - z_i).$$  \hspace{1cm} (3.4)

When the direct string – which is represented in Fig. 4(a) and Fig. 6 – runs along a geodesic, its mass is given by:

$$m = |M_{(a)}|$$  \hspace{1cm} (3.5)

$$M_{(a)} = \int_{\gamma_{13}} h_{(p,q)}(\tau(z), z)dz,$$  \hspace{1cm} (3.6)

where the path of the integration, $\gamma_{13}$ goes from $P_1$ to $P_3$ and below $P_2$ as is shown in the figure, but is otherwise arbitrary, in particular it need not run along the geodesic. To simplify most of the formulas we introduced the complex quantity, $M_{(a)}$ whose absolute value is the mass of the BPS state.

Now we determine the slope of this geodesic at its left endpoint, $P_1$. Since the metric is locally flat, in the coordinate

$$w \equiv \int_{P_1}^{z} h(\tau(z), z)dz,$$  \hspace{1cm} (3.7)

the geodesic is a straight line. In the above integral $h(z)$ is the instantaneous metric that the string feels along its path which in this case equals $h_{(p,q)}(z)$. The slope of the string at a generic point, $z$ is given by:

$$\text{Arg}(dz) = \text{Arg} \left( \frac{dw}{dz} \right) = \text{Arg} \left( \frac{M_{(a)}}{h(z)} \right),$$  \hspace{1cm} (3.8)

and thus the initial slope at $P_1$ is equal to:

$$\text{slope}_{P_1} = \text{Arg} \left( \frac{M_{(a)}}{h(P_1)} \right).$$  \hspace{1cm} (3.9)

Note that in this formula $h(P_1)$ is $h_{(p,q)}(P_1)$, which is nonsingular at a $[p,q]$-brane as follows from (3.3).

When the arrangement of the branes gives rise to a BPS state represented by the graph of Fig. 4(b) the mass can be computed similarly but one must note
that for the above arrangement of branch cuts this is a string with nonuniform charges. Nevertheless the mass is still given as length times tension:

\[
\mathcal{M}_\text{(b)} = \int_{P_1}^{C_2} h_{(p,q)}(\tau(z), z)dz + \int_{C_2}^{C_3} h_{(p-er,q-es)}(\tau(z), z)dz + \int_{C_3}^{P_2} h_{(-er-es)}(\tau(z), z)dz.
\]

The integration is performed along the curve \(\gamma_{12}\) which is depicted in Fig. 7(a) and represents a \((p,q)\)-string between \(P_1\) and the \(C_2\) cut followed by a \((p-er,q-es)\)-string between \(C_2\) and \(C_3\) and a \((-er-es)\) which ends on \(P_2\).

Figure 7: (a) The curve \(\gamma_{12}\) which is used to compute the mass of the BPS indirect string. (b) We transform \(\gamma_{12}\) into \(\gamma_{3232} \circ \gamma_{13}\) to benefit from that the integral over \(\gamma_{3232}\) vanishes.

Let us now deform the curve into a special one so that the integral in (3.10) becomes simple. The new path is shown in Fig. 7(b): it first goes along \(\gamma_{13}\) which hits \(P_3\), then loops around \(P_2\) and then \(P_3\) following \(\gamma_{3232}\) which ends on \(P_2\). The integral along \(\gamma_{3232}\) can be evaluated by shrinking the path into a single curve between \(P_3\) and \(P_2\) which is covered by \(\gamma_{3232}\) three times with different charges:

\[
\int_{\gamma_{3232}} h(\tau(z), z)dz = \int_{P_3}^{P_2} \left( h_{(p,q)}(\tau(z), z) - h_{(p-er,q-es)}(\tau(z), z) + h_{(-er-es)}(\tau(z), z) \right)dz = 0,
\]

(3.11)
by the linearity of the metric in the charges. The vanishing of this integral implies that $M(b)$ is given by the integral over $\gamma_{13}$ only and is equal to $M(a)$. We again define $w$ as in (3.7) keeping in mind that the instantaneous metric is not $h(p)$ along the whole path, nevertheless it is analytic because of the SL(2,Z)-invariance at the branch cuts. The slope at $P_1$ is again given by

$$\text{slope}_{P_1} = \text{Arg} \left( \frac{M(b)}{h(P_1)} \right) = \text{Arg} \left( \frac{M(a)}{h(P_1)} \right).$$

(3.12)

Entirely similar argument holds for the indirect string of Fig. 4(c) and we conclude that:

$$M(a) = M(b) = M(c) = \int_{\gamma_{13}} h(p) (\tau(z), z) dz.$$  

(3.13)

Now let us turn to the three-pronged junction of Fig. 4(d), which was discussed previously [18]. The calculation of $M(d)$ is simplified if we use the freedom of relocating the $C_2$ cut which we place right on top of the $P_2$-prong as is shown in Fig 8(a)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig8.png}
\caption{(a) The three-pronged junction of Fig. 4(d) with a convenient placement of the $C_2$-cut. (b) The length of the $P_2$-prong is equal to the integral over the loop around $P_2$.}
\end{figure}

The junction point which is also on the $C_2$-cut, is denoted with $S$. Our claim is that the tension-weighed integral along the $P_2$-prong is actually equal to the integral along the loop which departs and arrives at $S$ and surrounds $P_2$
weighed with the tension of the \((p_q)-\text{string}\) (Fig. 4). Let us pause for a short comment here. This Ansatz is not as ad hoc as it seems at first sight but is motivated by the physical picture which one has in mind when visualizing what really happens in the Hanany-Witten effect considered as a real process. Fig. 9 shows a (non-BPS) string in the vicinity of a [1,1]-brane. (To make the argument more clear it is now useful to place the cut on the string side – as opposed to the junction side in Fig. 7.) The \((0_{-1})\) and \((1_0)\) strings attract each other and are pulled towards each other until they form the \((1_1)\) bound state. If the other ends of the strings are fixed, then the bound state can only be formed on a short segment originating from the [1,1]-brane and we see a junction as a result.

![Figure 9: A junction is formed as a result of the Hanany-Witten effect.](image)

Back to Fig 8(b), the integral is most easily performed after shrinking the loop to a curve between \(S\) and \(P_2\) which is covered twice:

\[
\oint_S h_{(p_q)}(\tau(z),z)dz = \int_{P_2} \left( h_{(p_q)}^{left}(\tau(z),z) - h_{(p_q)}^{right}(\tau(z),z) \right) dz =
\]

\[
= -e \int_{P_2} h_{(r_s)}^{left}(\tau(z),z)dz,
\]

(3.14)

where we used that the limiting values of the metric on the two sides of the cut relate to each other as \(h_{(p_q)}^{right} = h_{(r_s)}^{left} M_{r,s}^{-1}(p_q) = h_{(p+er)}^{left} M_{r,s}^{-1}(q+s)\). If we now define the quantity analogous to \(M_{(a)}\),

\[
M_{(d)} \equiv \int_{\gamma_1} h_{(p_q)}(\tau(z),z)dz + (-e \int_{\gamma_2} h_{(r_s)}^{left}(\tau(z),z)dz) +
\]

\[
+ \int_{\gamma_3} h_{(r_s)}^{left}(\tau(z),z)dz,
\]

(3.15)
then we again learn that $\mathcal{M}_{(d)} = \mathcal{M}_{(a)}$. Let us now show that the mass of the geodesic object is again $m = |\mathcal{M}_{(d)}|$. The absolute value of each of the three complex terms in (3.15) is equal to the mass of the corresponding geodesic prong, therefore the sum of them gives the mass of the junction. For a general placement of the junction point $S$, this mass will be greater than the BPS bound. This bound – which equals the mass of the BPS three-pronged string – is easy to find: the sum of the absolute values of the three complex numbers is greater then the absolute value of the sum,

$$m \geq |\mathcal{M}_{(d)}|,$$

and is minimal, when they are parallel on the complex plane, and exactly in this case the total mass becomes:

$$m_{\text{geod.junction}} = |\mathcal{M}_{(d)}|. \quad (3.17)$$

In [18] we pointed out a geometrical interpretation of this result: in the coordinate $w(q)(z) \equiv \int z h(q)$, the junction can be represented as a broken line segment between $P_1$ and $P_2$. This crosses the image of the $P_3$-cut, which shows up as an excised region on the $w$-plane. Eq. (3.14) shows that the length of the $P_3$-prong equals the line segment inside the excised region and it is clear that the total length is minimal when the three parts of this broken segment are parallel, which is equivalent to the above result. Also, in the vicinity of the junction point $S$ the fact that the three complex numbers in (3.15) have equal argument translates to the balance of forces [12].

Now we calculate the slope at $P_1$, similarly to the previous cases. We find that

$$\text{slope}_{P_1} = \text{Arg} \left( \frac{w(S) - w(P_1)}{h(P_1)} \right) = \text{Arg} \left( \frac{\int_{P_1} h(q) dz}{h(P_1)} \right) = \text{Arg} \left( \frac{\mathcal{M}_{(d)}}{h(P_1)} \right) = \text{Arg} \left( \frac{\mathcal{M}_{(a)}}{h(P_1)} \right). \quad (3.18)$$

The calculation can be repeated for the other two endpoints.

We have learned that the four quantities $\mathcal{M}_{(a)}, \mathcal{M}_{(b)}, \mathcal{M}_{(c)}$ and $\mathcal{M}_{(d)}$ whose role is to define the mass of the BPS state in a certain regime of the moduli space, are equal, independent of the position of the branes. This shows that the mass of a BPS state is defined unambiguously, no matter which of the four integrals is used and is independent of the actual graph which represents
the BPS state. Moreover, whichever graph is realized as a geodesic object, its slopes at the endpoints are given in terms of the quantity \( M(a) \) and \( h(P_i) \) which depend only on the brane configuration.

Now it follows that there is no 7-brane configuration where two of these equivalent graphs would simultaneously be realized as BPS states. To see this recall that a geodesic curve is fully determined by one of its points and the slope at that point. Take for example the two strings of Fig. 4(a) and (b), these have a common endpoint at \( P_1 \). If the two strings were both BPS, their slope at \( P_1 \) would be equal which would imply that the two geodesics are identical. In other words, the slope at \( P_1 \) is given in terms of the metric and the string “decides” which other brane it wants to end on. Let us now turn to Fig. 4(d) and compare it to one of the open strings, say (a). They have two common endpoints on the A-branes, where the initial slopes are determined. One can construct both of the entire geodesics based on the initial data. It is now clear that they either coincide and give rise to the open string or intersect each other at an angle, which leads to the three-pronged network but can not do both; thus there may not be any configuration when both a string and a junction is BPS. This is again clear from the geometrical interpretation: the line segment between two branes on the \( w \)-plane either avoids or crosses the excised region of the cut of the third brane; in the previous case the BPS object is a string while in the latter case it is a junction.

4 General properties of equivalent graphs

In this section we would like to outline those lessons from the case of three 7-branes which generalize to more complicated graphs. We will not present a rigorous general proof for the existence and uniqueness of the BPS representative of equivalence classes, but we believe and our experience shows that these properties are sufficient to analyze concrete configurations.

The main result of the previous section was that we found certain relevant quantities which characterize an equivalence class of the networks and depend only on the details of the background but not on the particular representative. We assigned a complex number, \( M \) to an open string and observed that it was not changed as we performed crossing transformations. We emphasize that this does not mean that \( M \) is invariant as we move the the 7-branes and the BPS object changes its shape, but rather that at a given point in the moduli space one can determine this number without knowing the
representative of the equivalent networks to which it belongs. The central idea – that \( M \) does not change when a crossing transformation is made – is based on the observation that the integral over the prong which was created (removed) equals an integral over the string that surrounds the brane, see eq.(3.14). Note also that the statement is not restricted to the cases when a single prong is created as \( e = ps - qr \) could be any integer. This holds independently of the particular background and the network that this prong is part of; thus we conclude that \( M \) is a parameter of equivalence classes of string networks and the mass of the BPS representative of the given class is:

\[
m_{\text{BPS}} = |M|
\]  

(4.19)

We also discussed the slopes of the strings in a network and found that at any endpoint the slope of the string is independent of the representative. What we showed was that the slope can be calculated in terms of \( M \) and thus is invariant under crossing transformation. According to the previous paragraph, this is again a general result, moreover nothing relied on that the point where we calculate the slope is the endpoint of a string, thus we arrive at the following conclusion. The slope of a \( (p_q) \)-string at point \( z \) on the \( x^8 x^9 \)-plane, in a BPS network which belongs to the equivalence class characterized by \( M \) is:

\[
slope(p_q)(z) = \text{Arg} \left( \frac{M}{h(p_q)(z)} \right)
\]  

(4.20)

In the following section we will show how these results can be applied to concrete situations.

5 An example with four 7-branes

In section 3 we asked what happens when a mutually nonperturbative 7-brane appears in the vicinity of two D-branes, where these two alone would correspond to a spontaneously broken SU(2)-theory. In the 7+1 dimensional worldvolume of the three 7-branes it is still an SU(2) gauge theory, however the W-boson is no longer simply an open string between the two \([1,0]\) branes, rather – depending on where we are in the moduli space – it is represented by one of four equivalent graphs.
If we add more branes, we may get other gauge groups and the interesting SO(8) and exceptional symmetries can be realized. When we consider these theories with more and more 7-branes, both the number of equivalence classes and the number of equivalent graphs in a given class increase dramatically.  

[26] gives a systematic approach to finding the equivalent networks and also provides the classification of equivalence classes in terms of representations of the underlying gauge symmetry. Enumerating the equivalent graphs for many 7-branes is complicated and we will see that the vicinity of four branes already gives a great deal of insight into the problem.

The particular setup that we shall now investigate is that of Fig. 1(a) which corresponds to a resolution of a singularity possessing the SU(3) gauge group. This corresponds to a fiber of type IV in the elliptic fibration. The simplest representatives of the W-bosons corresponding to the positive roots are shown in Fig. 10(a), (b) and (c). The filled circles stand for [1,0]-branes and the empty squares for [1,1]-branes. We shall study the state of Fig. 10(a), for which a family of equivalent junctions is shown in Fig. 11.

As we can see, six equivalent strings are listed together with four different three-pronged junctions and two four-pronged ones, the latter differing from each other by a channel-transition. The charges for any of the open strings can be read off from the branes and are determined for the other objects by noting that they are related by crossing-transformations.

The graphs of Fig. 11 – when realized as BPS networks – correspond to a particular W-boson of the SU(3)-model. The consistency of the low energy theory requires that there be a unique representative of this state. We shall now demonstrate – using the arguments of section 4 – that this is the case. Since we do not have a general proof in hand, we do this by pairwise comparison of the members of this class; that is we show that no two graphs may be
Figure 11: Twelve equivalent graphs in the vicinity of four 7-branes.
realized simultaneously as the BPS state. First, recall that at a given point in the moduli space the complex number $\mathcal{M}$ is a parameter of the whole class and also that whichever network is realized, the slope of a string at a given point $z$ is determined by its charges and $\mathcal{M}$.

First of all, none of the four-pronged networks, (a) and (b) may “coexist” with any of the other graphs: if say (a) and (e) were both BPS, that would be in conflict with the fact that the slopes of the three prongs ending on the three leftmost brane (which are given in terms of $\mathcal{M}$) determine the whole graph of (e) and there is no fourth prong. (This is analogous to the argument in sec. 3 about the graphs of Fig. 4(a) and (d)). Notice also that there are groups of graphs where one of the four branes has no prong on it, like (c)-(g)-(j)-(l), (d)-(h)-(k)-(l), (e)-(g)-(h)-(j) and (f)-(i)-(j)-(k). For these the arguments of sec. 3 regarding the absence of simultaneous BPS junctions can be repeated without modification, because they did not depend on how the nearby 7-branes affect the metric. (One should not worry about their cut either as it can be transformed away by an SL(2,Z)-transformation [18].)

Now consider any two of the six open strings of (g)–(l) and notice that they necessarily have a common point where both have the same charges. At this point on the $z$-plane, the slope of the string is given in terms of $\mathcal{M}$ which can not give rise to two different geodesics. What about the pairs like (e) and (j), a three-pronged network and a string with one common endpoint? The string is certainly determined in terms of its slope on the common brane but it is not clear that the three-pronged junction cannot be simultaneously BPS with one of its prongs on top of the (j)-string as in Fig. 12.

![Figure 12: Comparing the equivalent string of Fig. 11 (j) and three-pronged network (e) on the $z$-plane.](image-url)
Note that at the intersection point $Y$, the slope argument does not work, since the charges of the two objects are different. However, the two networks can not both be BPS between $S$ and $X$ as follows from the following simple argument. Consider the $(\frac{1}{0})$ and any $(\frac{p}{q})$ strings in a similar configuration depicted in Fig. 13. If the $(\frac{1}{0})$ and $(\frac{p}{q})$ geodesics intersect at two points, one of the angles, $\phi_i$ at the intersection points have to fall between $0$ and $\pi$ while the other should be between $\pi$ and $2\pi$. Supersymmetry is maintained at the two intersection points if the angles satisfy [12]:

$$\phi_i = \text{Arg}(p - q\tau(z_i)), \text{ or } \phi_i = \text{Arg}(p - q\bar{\tau}(z_i)),$$

(5.21)

which however would imply that $\text{Im}\tau$ is positive at one of the intersection points and negative at the other, but the latter is not possible. Having ruled out this last possibility we conclude that the BPS representative of the equivalence class in Fig. 11 is unique.

The above analysis can be repeated without difficulty for the other two equivalence classes of Fig. 10(b) and (c).
6 Invariant charges and construction of the BPS network

To find the entire trajectory of a BPS string it is enough to determine a point of a curve with the slope at that point. Starting from that point one can follow the geodesic path until the 7-brane where the string ends is hit. The argument in connection with Fig. 12 suggests a more general feature: it seems that the general network can be constructed similarly from the same data, even when it is more complicated than a string. We did not prove this conjecture in general but will now present the idea of this explicit construction, which has been applicable to every concrete case we studied.

To understand the construction we need to know the concept of invariant charges assigned to the 7-branes. This is explained in detail in [26], we summarize here the main points only. Given a 7-brane background and a graph, we can assign to every brane an integer which is expressed in terms of the number of prongs ending on that brane and the number of prongs crossing its branch cut. The definition of this invariant charge is such that it does not change in crossing transformations and thus corresponds to the equivalence class of the networks and not the particular graph.

Let us return to the simplest case of the configuration with three branes and take the equivalence class of Fig. 4. The invariant charges of the leftmost, middle and rightmost branes are +1, 0 and -1, respectively when the orientation of \( (1,0) \)-string points from the left to the right. Let us assume that our BPS object starts on the leftmost brane and try to construct it. The slope at the initial point is given so we simply have to follow the \( (1,0) \)-geodesic on the \( z \)-plane. Now there are two possibilities: this geodesic either crosses the cut of the middle brane or it does not. In the latter case it simply hits the rightmost brane and we are done. (It cannot miss it as follows from (3.6) and (3.8).)

However, if the geodesic crosses that cut then the invariant charge of the middle brane is temporarily changed which we should compensate by starting a prong (with the slope given in terms of \( \mathcal{M} \)) from the brane. This new prong again runs along its geodesic which we can follow until it meets our first prong. Now there are again two possibilities: they may meet smoothly such that they together form a geodesic and we end up with Fig. 11(b). Note that in this case the charges work out only if the cut of the rightmost brane is crossed, when we meet that cut with either geodesic, we change the invariant
Figure 14: Construction of the BPS indirect string. (a) The geodesic string hits the branch cut of the [1,1]-brane. (b) To restore the invariant charge of the [1,1]-brane a prong is started with the given initial slope. (c) Because the string crossed the branch cut of the rightmost brane, no prong should end on that brane to preserve its invariant charge. The geodesics of the leftmost and middle brane meet smoothly.

Figure 15: Construction of the BPS junction. (a) The geodesic string hits the branch cut of the [1,1]-brane. (b) To restore the invariant charge of the [1,1]-brane a prong is started with the given initial slope which meets the previous geodesic at the junction point. (c) The initial slope of the third prong determines the third geodesic ending on the rightmost brane.
charge of the rightmost brane which is compensated by erasing its prong, see Fig. 13. If the two geodesics meet under an angle at a point $S$, then a third geodesic should be started with the proper charges and slope from the junction point which eventually hits the third brane, see Fig. 15. Note that it is guaranteed by supersymmetry that the two geodesics meet under the proper angle to form the BPS junction.

We used the simplest configuration to illustrate our idea of how an initial point, the slope of the string at that point and the set of invariant charges determine the whole BPS object. We believe that the recipe that we gave above similarly determines BPS networks in more complicated backgrounds.

7 Four-pronged network in constant $\tau$

Looking at some of the networks in Fig. 11 one could say that we might overestimate the complicatedness of the background and though charge conservation permits all, some of the graphs will never be BPS. This question is hard to answer because the explicit form of the metric is indeed very complex and it is unlikely that an analytic determination of the geodesics is possible, though a numerical analysis would be desirable and useful. We will not do that, but will argue that the moduli space of graphs is not less complex than suggested by Fig. 11 using a similar but far more tractable configuration.

The idea is borrowed from [18] where we used configurations of 7-branes with constant axion-dilaton field. This can be achieved by collapsing some of the branes; in particular $\tau = e^{i\pi/3}$ can be maintained if the background consists of coinciding pairs of $[1,0]$- and $[1,1]$-branes. Thus we may get some insight into the moduli space of the equivalence class of Fig. 11 if we put a $[1,0]$-brane on top of each $[1,1]$ and a $[1,1]$-brane on top of each $[1,0]$ as shown in Fig. 16. This is of course no longer the general resolution of an SU(3) singularity, but rather a special resolution of an $E_6$ configuration (see sec. 3. of [18]).

This configuration possesses a much simpler metric than a general one: in the $w$-coordinate (see (3.7)) all strings are straight lines and the four singularities are purely conical, giving rise to an excised region of deficit angle $60^\circ$ emanating from each puncture. Now it is just an exercise in planar geometry to show that every single graph of Fig. 11 is realized at certain regions of the moduli space. We leave it to the reader to confirm that the moduli space is an extended version of Fig. 13 of [18] and we only present an interesting
Figure 16: (a) Constant $\tau = e^{i\pi/3}$ resolution of the $E_6$ singularity. (b) A convenient arrangement of the cuts used later in Fig. 17 and a three-pronged network.

It is reassuring to see that not only the crossing transformations but also the channel-transitions are smooth processes. It is instructive to see this in more detail. We show in Fig. 18 the (b) $\rightarrow$ (d) process of Fig. 17 in a clearer context. Since the branch cuts do not play any role now, we relocate them in such a way that they are not crossed by any of the strings. In Fig. 18(a) the network is again shown on the $w$-plane with its endpoints forming a rectangle whose vertical edges will be decreased in (b) and (c).

Naively, one could expect a jump in the process of moving the branes when the square is reached, since the shortest network seems to become the one with horizontal string in the middle (this is known as Plateau's problem). However, we should not forget about the tensions: although

$$T_{(1)}^{(1)} = T_{(1)}^{(0)} = T_{(1)}^{(1)},$$

(7.22)

the horizontal prong is born with a different value:

$$T_{(1)}^{(1)} = \sqrt{3}T_{(0)}^{(1)},$$

(7.23)

and this implies that the angles at the new junction will be $60^\circ$–$150^\circ$–$150^\circ$, instead of $120^\circ$–$120^\circ$–$120^\circ$. As a consequence, the network on Fig. 18(c) becomes shorter exactly when the vertical $(1)$-string shrinks to zero size and the transition happens smoothly through the four-pronged junction of Fig. 18(b).
Figure 17: Metamorphosis of a four-pronged network
8 Conclusions

The conjecture, that string networks appear on equal footing with conventional open strings in IIB D-brane theories has given rise to a number of new questions. A great deal of evidence has been accumulated supporting the necessity of these nonperturbative objects in various 3- and 7-brane configurations, which calls for a consistent description of the space of these networks. In the present paper we addressed three issues: the equivalence classes of networks, the uniqueness of the representative at a point in the moduli space and transitions between equivalent networks.

In our paper we examined the behavior of \( (p/q) \)-strings and their junction on a nontrivial background created by parallel 7-branes which also served as endpoints for the network. We first defined what we mean by the equivalence of graphs and then turned to the question of which of many equivalent graphs is realized as a BPS network is realized in a particular background. We analyzed in detail the case of four equivalent networks in the vicinity of three 7-branes and concluded that the representative of the equivalence class changes as we move around in the moduli space but never is more than one realized in a supersymmetric fashion simultaneously.

In this exercise we learned two important general features about networks: the calculation of the mass of the network or the slope of a particular string at a given point is independent of the particular representative and depends only...
on the background metric and the equivalence class. We defined a complex parameter, \( \mathcal{M} \) in terms of the positions of the 7-branes through which the above dependence can be simply expressed and found in particular that the BPS mass is \( m = |\mathcal{M}| \). The determination of the slope of the strings can be used to prove the uniqueness of the BPS object. We carried out this argument for the equivalence class of 12 graphs in the vicinity of four 7-branes and although we did not extend this to the general case, we suggested a procedure for the construction of a BPS object in an arbitrary background making use of the invariant charges.

The most appreciated novelty in these 7-brane configurations is the appearance of exceptional gauge groups, which require at least eight 7-branes. While having more 7-branes means more complication in general, transitions on the constant \( \tau \) branches can be studied explicitly. An interesting concrete application was the channel-transition of the four-pronged network which may be viewed as a modification of Plateau’s problem of the graphs of least length between the vertices of a polygon.

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