Exponential and strong ergodicity for one-dimensional symmetric stable jump diffusions

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Abstract
We obtain explicit criteria for both exponential ergodicity and strong ergodicity for one-dimensional time-changed symmetric stable processes with $\alpha \in (1, 2)$. Explicit lower bounds for ergodic convergence rates are given.

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1 Introduction and main results
Let $X := (X_t)_{t \geq 0}$ be a symmetric $\alpha$-stable process on $\mathbb{R}$ with infinitesimal generator $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$, $\alpha \in (0, 2)$, where $-(-\Delta)^{\alpha/2}$ is the fractional Laplacian operator. It is well known that $X$ is pointwise recurrent (i.e. it hits single points almost surely) if and only if $\alpha \in (1, 2)$ (see [19, Remark 43.12]), but it is not ergodic since the invariant measure is Lebesgue measure which is infinite.

Consider the following stochastic differential equation:
\begin{equation}
    dY_t = \sigma (Y_{t-}) dX_t,
\end{equation}
where $\sigma$ is a strictly positive continuous function on $\mathbb{R}$. By [11, Proposition 2.1], there is a unique weak solution $Y = (Y_t)_{t \geq 0}$ to the SDE (1), and $Y$ can also be expressed as a time change process $Y_t := X_{\zeta_t}$, where
\[\zeta_t := \inf \left\{ s > 0 : \int_0^s \sigma(X_u)^{-\alpha} \, du > t \right\}.\]
By [8, Section 1.2], the generator of $Y$ is $L = \sigma^\alpha \Delta^{\alpha/2}$ which is symmetric with respect to its invariant measure $\mu(dx) = \sigma(x)^{-\alpha} dx$.

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Note that a time change does not change the recurrence (cf. [8, Theorem 5.2.5]). When \( \alpha \in (1, 2) \), \( Y \) is pointwise recurrent, so that it is Lebesgue irreducible (see [18, Page 42] for the definition). Thus by [18, Proposition 4.1.1 and Theorem 4.2.1], \( Y \) is ergodic whenever \( \mu(\mathbb{R}) < \infty \).

Throughout this paper, we study explicit criteria for both exponential ergodicity and strong ergodicity for this process \( Y \). Furthermore, we obtain explicit estimates for ergodic convergence rates.

Now we assume that \( \mu(\mathbb{R}) < \infty \). Let \( \pi(dx) := \mu(dx)/\mu(\mathbb{R}) \). By [8, Section 1], the associated regular Dirichlet form \( (\mathcal{E}, \mathcal{F}) \) of \( Y \) on \( L^2(\pi) \) is given by

\[
\mathcal{E}(f, g) = \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} (f(x) - f(y))(g(x) - g(y)) \frac{C_{\alpha} dx dy}{|x - y|^{1+\alpha}}, \quad f, g \in \mathcal{F},
\]

where

\[
\mathcal{F} = \{ u \in L^2(\pi) : \mathcal{E}(u, u) < \infty \},
\]

and \( C_{\alpha} = \frac{\alpha}{\sqrt{\pi} \Gamma(1/2) \Gamma((1+\alpha)/2)} \).

Denote by \( (P_t)_{t \geq 0} \) the semigroup of \( Y \), \( \pi(f) := \int_{\mathbb{R}} f d\pi \) and \( ||f||_{L^2(\pi)} := (\pi(f^2))^{1/2} \). We say that the process \( Y \) is exponentially ergodic (or \( L^2 \)-exponentially convergent), if there exists \( \lambda_1 > 0 \), such that for any \( f \in L^2(\pi) \),

\[
||P_t f - \pi(f)||_{L^2(\pi)} \leq e^{-\lambda_1 t} ||f - \pi(f)||_{L^2(\pi)},
\]

see [4, (1.2)] for the definition and [4, Page 158–160,(h) and (i)] for the equivalence between exponential ergodicity and \( L^2 \)-exponential convergence. The optimal constant \( \lambda_1 \) in (4) is equal to the spectral gap

\[
\lambda_1 = \text{gap}(\mathcal{E}) := \inf \{ \mathcal{E}(f, f) : f \in \mathcal{F}, \pi(f^2) = 1, \pi(f) = 0 \},
\]

see [4] for more details.

Our first result is the explicit criterion for exponential ergodicity and the explicit estimate for \( \lambda_1 \).

**Theorem 1** (Exponential ergodicity). \( Y \) is exponentially ergodic if and only if

\[
\delta := \sup_x |x|^\alpha - 1 \int_{\mathbb{R}\setminus(|x|, |x|)} \sigma(y)^{-\alpha} dy < \infty.
\]

Furthermore,

\[
\lambda_1 \geq \frac{1}{4\omega_{\alpha} \delta},
\]

where

\[
\omega_{\alpha} := \frac{1}{\cos(\pi \alpha/2) \Gamma(\alpha)} > 0.
\]

For any open set \( B \subset \mathbb{R} \), define the local Dirichlet eigenvalue by

\[
\lambda_0(B) = \inf \{ \mathcal{E}(f, f) : f \in \mathcal{F}, \pi(f^2) = 1 \text{ and } f|_{B^c} = 0 \}.
\]

In particular, denote by \( \lambda_0 := \lambda_0(\{0\}^c) \).
\( \lambda_0(B) \) is the bottom of spectrum for the part Dirichlet form \((\mathcal{E}, \mathbb{F}^B)\) (see Section 2 for more detail). The probabilistic meaning of \( \lambda_0(B) \) is the \( L^2 \)-decay rate for the killed semigroup \( P^B_t \) (see Section 2 for the definition), i.e.

\[
\| P^B_t g \|_{L^2(\pi)} \leq e^{-\lambda_0(B)t} \| g \|_{L^2(\pi)}, \quad \text{for any } g \in L^2(\pi).
\]

\( \lambda_0(B) \) and \( \lambda_1 \) are closely related. By [7, Theorem 1.4] or [21, Theorem 1.1], we know that if \( \pi(B^c) > 0 \), then \( \lambda_1 \leq \lambda_0(B)/\pi(B^c) \). According to [3, Proposition 3.2], \( \lambda_1 \geq \lambda_0 \). Indeed, the sufficiency of Theorem 1 is based on the following result.

**Theorem 2** (Dirichlet eigenvalues).

1. If

\[
\delta_+ := \sup_{x > 0} x^{-1} \int_x^{\infty} \sigma(z)^{-\alpha} dz < \infty,
\]

then

\[
\lambda_0((0, \infty)) \geq \frac{(\alpha - 1) \Gamma(\alpha/2)^2}{4 \delta_+}.
\]

2. If

\[
\delta := \sup_{x} |x|^{-1} \int_{\mathbb{R}\setminus(-|x|,|x|)} \sigma(y)^{-\alpha} dy < \infty,
\]

then

\[
\frac{2}{\omega_\alpha} \left( \frac{1}{\delta_+} + \frac{1}{\delta_-} \right) \geq \lambda_0 \geq \frac{1}{4 \omega_\alpha \delta},
\]

where \( \omega_\alpha \) is given by (5) and

\[
\delta_- := \sup_{x > 0} x^{-1} \int_{-\infty}^{-x} \sigma(z)^{-\alpha} dz.
\]

Next, we study the strong ergodicity for \( Y \). For this, let \( \| \nu \|_{\text{Var}} := \sup_{|f| \leq 1} |\nu(f)| \) be the total variation of a signed measure \( \nu \), and \( P_t(x, \cdot) \) be the transition function. We say that \( Y \) is **strongly ergodic**, if there exist constants \( 1 < C < \infty \) and \( \kappa > 0 \), such that

\[
\sup_{x \in \mathbb{R}} \| P_t(x, \cdot) - \pi \|_{\text{Var}} \leq Ce^{-\kappa t}.
\]

The optimal convergence rate

\[
\kappa = \lim_{t \to \infty} \frac{1}{t} \log \sup_{x \in \mathbb{R}} \| P_t(x, \cdot) - \pi \|_{\text{Var}},
\]

see [15] for more details.

**Theorem 3** (Strong ergodicity). \( Y \) is strongly ergodic if and only if

\[
I := \int_{\mathbb{R}} \sigma(x)^{-\alpha} |x|^{\alpha-1} dx < \infty.
\]

Moreover, the optional convergence rate \( \kappa \) in the strong ergodicity satisfies

\[
\kappa \geq \frac{1}{\omega_\alpha I} > 0.
\]
Remark 4. (1) Note that when \( \alpha = 1 \), the process \( Y \) is neighborhood recurrent but not pointwise recurrent (see [11, Section 3.3]; also see [13, Theorem I.1.5] for the criteria of general Lévy processes). Therefore, in this case, the Green operator \( U^{(0)} \) killed on hitting the origin, which is important for the proofs of Theorem 1 and Theorem 3, is not valid, for example, \( U^{(0)}1 = \infty \).

(2) Our criteria are somehow comparable with those in the special case \( \alpha = 2 \), the time-changed Brownian motion on \( \mathbb{R} \). In the latter, the process is exponentially ergodic if and only if

\[
\delta_1 := \sup_{x \geq 0} x \int_{\mathbb{R}\setminus[-x,x]} \sigma(z)^{-2}dz < \infty,
\]

and \( \lambda_1 \geq \lambda_0 \geq (4\delta_1)^{-1} \), while it is strongly ergodic if and only if

\[
\int_{\mathbb{R}} \sigma(x)^{-2} |x|^{\alpha-1}dx < \infty.
\]

See [6, Table 5.1 and Theorem 5.8].

(3) By [11, Table 2], for \( \alpha > 1 \), \( \pm \infty \) is entrance boundary if and only if (7) holds. Therefore, Theorem 3 indicates that for a pointwise recurrent time-changed symmetric stable process, the strong ergodicity is equivalent to entrance at \( \pm \infty \).

By using Lyapunov functions, [8, Theorem 1.7] obtained some sufficient conditions for exponential ergodicity and strong ergodicity, which now can be derived by our Theorems 1 and 3.

Corollary 5. \( Y \) is exponentially ergodic if

\[
A_1 := \liminf_{|x| \to \infty} \frac{\sigma(x)}{|x|} > 0,
\]

and \( Y \) is strongly ergodic if

\[
A_2 := \liminf_{|x| \to \infty} \frac{\sigma(x)}{|x|^\gamma} > 0
\]

for some constant \( \gamma > 1 \).

Proof. From [8], there exists \( N_1 > 0 \) such that for any \( x > N_1 \), \( \sigma(x) \geq A_1|x|/2 \), so that

\[
\delta \approx \sup_{|x| > N_1} |x|^{\alpha-1} \int_{\mathbb{R}\setminus[-|x|,|x|]} \sigma(y)^{-\alpha}dy \leq \frac{2^\alpha}{(\alpha - 1)A_1^{\alpha}} < \infty,
\]

where the symbol “\( A \approx B \)” means that there exists \( 0 < c_1, c_2 < \infty \), such that \( c_1B \leq A \leq c_2B \). Then \( Y \) is exponentially ergodic by Theorem 1.

From [10], there exists \( N_2 > 0 \) such that for any \( |x| > N_2 \), \( \sigma(x) \geq A_2|x|^\gamma/2 \), so that

\[
I \approx \int_{|x| > N_2, |x|^\gamma} \sigma(x)^{-\alpha} |x|^{\alpha-1}dx \leq \frac{2^\alpha N_2^{\alpha(1-\gamma)}}{\alpha(\gamma - 1)A_2^{\alpha}} < \infty.
\]

Then \( Y \) is strongly ergodic by Theorem 3.

Actually, if \( \sigma \) is a polynomial, then we have explicit results for the ergodicity and the convergence rates of \( Y \).
Corollary 6. Consider the polynomial case: \( \sigma(x) = (1 + |x|)^\gamma \).

(1) \( Y \) is ergodic if and only if \( \gamma > 1/\alpha \).

(2) For \( \gamma > 1 \),
\[
\frac{2(\alpha \gamma - 1)^{\alpha \gamma} (\alpha (\gamma - 1))^{\alpha(1-\gamma)}}{\omega_\alpha (\alpha - 1)^{\alpha-1}} \geq \lambda_0 \geq \frac{(\alpha \gamma - 1)^{\alpha \gamma} (\alpha (\gamma - 1))^{\alpha(1-\gamma)}}{8 \omega_\alpha (\alpha - 1)^{\alpha-1}}.
\]
and for \( \gamma = 1 \),
\[
\frac{2(\alpha - 1)}{\omega_\alpha} \geq \lambda_0 \geq \frac{\alpha - 1}{8 \omega_\alpha}.
\]

(3) \( Y \) is exponentially ergodic if and only if \( \gamma \geq 1 \). Moreover,
\[
\lambda_1 \geq \frac{(\alpha \gamma - 1)^{\alpha \gamma} (\alpha (\gamma - 1))^{\alpha(1-\gamma)}}{8 \omega_\alpha (\alpha - 1)^{\alpha-1}}
\]
for \( \gamma > 1 \) and \( \lambda_1 \geq (\alpha - 1)/8 \omega_\alpha \) for \( \gamma = 1 \).

(4) \( Y \) is strongly ergodic if and only if \( \gamma > 1 \). Furthermore, \( \kappa \geq (\gamma - 1)/2 \omega_\alpha \).

Proof. (1) Note that \( \mu(\mathbb{R}) = \int_\mathbb{R} (1 + |x|)^{-\alpha \gamma} dx < \infty \) if and only if \( \gamma > 1/\alpha \).

(2) For \( \gamma > 1 \), a direct calculation shows that \( \delta = 2 \delta_+ = 2 \delta_- \) and
\[
\delta_+ = \sup_x |x|^{\alpha-1} \int_{|x|}^{\infty} (y + 1)^{-\alpha \gamma} dy = \frac{(\alpha - 1)^{\alpha-1}}{(\alpha \gamma - 1)^{\alpha \gamma} (\alpha (\gamma - 1))^{\alpha(1-\gamma)}} > 0. \tag{10}
\]
Hence by Theorem 2
\[
\frac{2(\alpha \gamma - 1)^{\alpha \gamma} (\alpha (\gamma - 1))^{\alpha(1-\gamma)}}{\omega_\alpha (\alpha - 1)^{\alpha-1}} \geq \lambda_0 \geq \frac{(\alpha \gamma - 1)^{\alpha \gamma} (\alpha (\gamma - 1))^{\alpha(1-\gamma)}}{8 \omega_\alpha (\alpha - 1)^{\alpha-1}}.
\]
For \( \gamma = 1 \), we have \( \delta = 2 \delta_+ = 2 \delta_- \) and
\[
\delta_+ = \sup_x |x|^{\alpha-1} \int_{|x|}^{\infty} (y + 1)^{-\alpha} dy = \lim_{|x| \to \infty} \frac{|x| + 1}{\alpha - 1} \frac{|x|^{\alpha-1}}{\alpha - 1} = \frac{1}{\alpha - 1} > 0. \tag{11}
\]
So by Theorem 2
\[
\frac{2(\alpha - 1)}{\omega_\alpha} \geq \lambda_0 \geq \frac{\alpha - 1}{8 \omega_\alpha}.
\]

(3) By \((10), (11)\) and Theorem 1 we obtain the lower bound for \( \lambda_1 \), and see that if \( \gamma \geq 1 \), \( Y \) is exponentially ergodic.

If \( 1/\alpha < \gamma < 1 \), then
\[
\delta = \sup_x 2|x|^{\alpha-1} \int_{|x|}^{\infty} (y + 1)^{-\alpha \gamma} dy = \lim_{|x| \to \infty} \frac{2}{\alpha \gamma - 1} |x|^{\alpha-1} (|x| + 1)^{1-\alpha \gamma} = \infty,
\]
hence by Theorem 1 \( Y \) is exponentially ergodic if and only if \( \gamma \geq 1 \).

(4) Note that for \( 1/\alpha < \gamma \leq 1 \),
\[
I \geq 2 \int_0^1 \frac{x^{\alpha-1}}{(1 + x)^{\alpha \gamma}} dx + 2 \int_1^\infty \frac{x^{\alpha-1}}{(1 + x)^{\alpha \gamma}} dx
\]
\[
\geq \frac{1}{\alpha 2^{\alpha \gamma - 1}} + \frac{1}{2^{\alpha \gamma - 1}} \int_1^\infty x^{\alpha - 1 - \alpha \gamma} dx = \infty,
\]
while for \( \gamma > 1 \), \( I \leq 2 \alpha^{-1}(\gamma - 1)^{-1} < \infty \). Thus Theorem 3 gives that \( Y \) is strongly ergodic if and only if \( \gamma > 1 \), and \( \kappa \geq (\omega_\alpha I)^{-1} \geq \alpha (\gamma - 1)/2 \omega_\alpha \).

\[\square\]
2 Killed process, Green function and time change

We first recall some definitions and properties. Given an open set $B \subset \mathbb{R}$, denote by
\[ \tau_B := \inf \{ t > 0 : Y_t \notin B \} \]
the first exit time from $B$ of the time-changed symmetric stable process $Y$. Let $Y^B$ be the sub-process of $Y$ killed upon leaving $B$, whose transition function is
\[ P^B_t(x, A) := \mathbb{P}_x [Y_t \in A, t < \tau_B], \]
for any $x \in B$, and Borel set $A \subset \mathbb{R}$.

The Green potential measure of the killed process $Y^B$ starting from $x$ is a Borel measure defined by
\[ U^B(x, dy) := \int_0^\infty P^B_t(x, dy) dt. \]
The Green operator $U^B$ is given by
\[ U^B f(x) := \int_B f(y)U^B(x, dy), \quad x \in \mathbb{R}, \]
for $f \in \mathcal{B}(\mathbb{R})$ with $U^B|f| < \infty$.

Recall that $(\mathcal{E}, \mathcal{F})$ is the Dirichlet form of $Y$ given by (2) and (3). Denote by $(\mathcal{E}, \mathcal{F}^B)$ the part Dirichlet form, where
\[ \mathcal{F}^B := \{ f \in \mathcal{F}, \tilde{f} = 0, \text{ q.e. on } B^c \}, \]
q.e. stands for quasi-everywhere, and $\tilde{f}$ is a quasi-continuous modification of $f$ (cf. [17, Section 2.2]).

By [17, Theorem 3.5.7], $(\mathcal{E}, \mathcal{F}^B)$ is a symmetric regular Dirichlet form, and $Y^B$ is the process associated with $(\mathcal{E}, \mathcal{F}^B)$.

Note that for any nonempty set $B$, $(\mathcal{E}, \mathcal{F}^B)$ is a transient Dirichlet form. By [17, Theorem 1.3.9], for any $f$ with $\int_{\mathbb{R}} |f(x)|U^B|f|(x)\pi(dx) < \infty$, we have $U^B f \in \mathcal{F}^B$, and for any $u \in \mathcal{F}^B$,
\[ \mathcal{E}(U^B f, u) = \int fud\pi. \quad (12) \]

Let $\lambda_0(B)$ be the bottom of spectrum for $(\mathcal{E}, \mathcal{F}^B)$:
\[ \lambda_0(B) := \inf \{ \mathcal{E}(f, f) : f \in \mathcal{F}^B, \pi(f^2) = 1 \}. \]
Denote by $C_0(B)$ the space of continuous functions with compact support on $B$. Since $\mathcal{F}^B$ is the closure of $C_0(B) \cap \mathcal{F}$ in $\mathcal{F}$, we have
\[ \lambda_0(B) = \inf \{ \mathcal{E}(f, f) : f \in C_0(B) \cap \mathcal{F}, \pi(f^2) = 1 \} \]
\[ = \inf \{ \mathcal{E}(f, f) : f \in \mathcal{F}, \pi(f^2) = 1 \text{ and } f|_{B^c} = 0 \}. \quad (13) \]
We call $\lambda_0(B)$ the local Dirichlet eigenvalue on $B$.

Let $U^B_X$ be the Green operator of $X$ on $B$ and $G^B_X(\cdot, \cdot)$ be the Green (density) function of $X$, i.e. for any $x, y \in \mathbb{R}$,
\[ U^B_X(x, dy) = G^B_X(x, y)dy. \]
Denote by $\tau^B_X$ the first exit time from
and define the additive functional $A_t := \int_0^t \sigma(X_s)^{-\alpha} \, ds$. Then by the basic transform formula for time-change (see [2, Lemma A.3.7]), we have for any $K \in \mathcal{B}(\mathbb{R})$,

$$U_B(x, K) = \mathbb{E}_x \left[ \int_0^{\tau^B} 1_K(Y_t) \, dt \right] = \mathbb{E}_x \left[ \int_0^\infty 1_{\{Y_t \in K, t < \tau^B\}} \, dt \right] = U_B^X(\sigma^{-\alpha} 1_K)(x).$$

Hence for any $f \in \mathcal{B}(\mathbb{R})$ with $U_B|f| < \infty$,

$$U_B f(x) = U_B^X(\sigma^{-\alpha} f)(x) = \int_K f(y) G_B^X(x, y) \sigma(y)^{-\alpha} \, dy.$$

Since $G_B(\cdot, \cdot)$ can be represented by the Green function of $X$, the estimate of $G_B(\cdot, \cdot)$ is obtained from $G_B^X(\cdot, \cdot)$.

For the one-dimensional symmetric $\alpha$-stable process $X$ with $\alpha \in (1, 2)$, its Green function $G_B^X(\cdot, \cdot)$ for an open set $B$ can be expressed explicitly. For example:

1. ([1, Lemma 4]) $B = \mathbb{R}\setminus\{0\}$:
   $$G_X^{[0,\infty)}(x, y) = \frac{\omega_\alpha}{2} \left( |y|^{\alpha-1} + |x|^{\alpha-1} - |y-x|^{\alpha-1} \right),$$

2. ([10, (11)]) $B = [-1, 1]^c$:
   $$G_X^{[-1,1]^c}(x, y) = c_\alpha \left( |x-y|^{\alpha-1} h \left( \frac{|xy| - 1}{|x-y|} \right) - (\alpha - 1) h(x) h(y) \right),$$

3. ([2, Page 388]) $B = (0, \infty)$:
   $$G_X^{(0,\infty)}(x, y) = \frac{1}{\Gamma(\alpha/2)^2} |x - y|^{\alpha-1} J_\alpha \left( \frac{x \wedge y}{|x-y|} \right),$$

where $J_\alpha(t) := \int_0^t [s(s+1)]^{\alpha/2-1} \, ds$.

3 Exponential ergodicity

It is well known that the exponential ergodicity for a reversible Markov process is equivalent to the existence of the spectral gap $\lambda_1$, and we can turn this problem to the estimate of the local Dirichlet eigenvalue $\lambda_0(B)$ for some open set $B$. 
By \[7\] Theorem 1.4 or \[21\] Theorem 1.1, we have an upper bound for \(\lambda_1\) by using \(\lambda_0(B)\):
\[
\lambda_1 \leq \frac{\lambda_0(B)}{\pi(B)} , \text{ for any open set } B \text{ with } \pi(B^c) > 0, \tag{19}
\]
and by \[3\] Proposition 3.2, \(\lambda_1 \geq \lambda_0(\{0\}^c)\). Therefore, to prove exponential ergodicity, our strategy is to estimate the local Dirichlet eigenvalues.

To estimate the upper bound and lower bound of \(\lambda_0(B)\), according to the definition \([6]\) and \([20]\) Theorem 3.2], we have the following variational formula for \(\lambda_0(B)\).

**Lemma 7** (Variational formula for the local Dirichlet eigenvalue). Assume that \(B\) is a nonempty open subset of \(\mathbb{R}\). Then
\[
\inf_{f \in C_b(B)}\sup_{x \in B} \frac{f(x)}{U^B f(x)} \geq \lambda_0(B) \geq \sup_{f \in C_b(B)}\inf_{x \in B} \frac{f(x)}{U^B f(x)},
\]
where \(C_b(B)\) is the space of all bounded continuous functions on \(B\).

**Proof.** First we consider the upper bound. Note that for \(f \in C_b(\mathbb{R})\), \(\int_{\mathbb{R}} |f(x)|U^B f(x)\pi(dx) < \infty\). By \([17]\) Theorem 1.3.9], we see that \(U^B f \in \mathcal{P}^B\), and \(\mathcal{P}^B\) holds. Thus by the definition \([6]\),
\[
\lambda_0 \leq \frac{\mathcal{E}(U^B f, U^B f)}{\pi((U^B f)^2)} = \frac{\int U^B f \pi(dx)}{\pi((U^B f)^2)} \leq \sup_{x \in B} \frac{f(x)}{U^B f(x)}.
\]
So we get the upper bound by the arbitrariness of \(f \in C_b(\mathbb{R})\).

For the lower bound, by \([17]\) Theorem 3.5.7(ii)], \(Y^B\) is also a Hunt process, thus it is a right continuous Markov process. Hence by the proof of \([20]\) Lemma 2.2], for any \(f \in C_b(B)\),
\[
\tilde{L}^B U^B f(x) = \beta U^B f(x) - f(x), \tag{20}
\]
where \(U^B f := \int_0^\infty e^{-\beta t} P^B f(x) dt\) and \(\tilde{L}^B\) is the weak generator for \(Y^B\) (see \([20]\) Definition 2.1]). Therefore, by \([20]\) Theorem 3.2] and \(20\),
\[
\lambda_0(B) \geq \inf_{x \in B} \left( -\frac{\tilde{L}^B U^B f}{U^B f} \right) (x) = \inf_{x \in B} \frac{f(x)}{U^B f(x)} - \beta \geq \inf_{x \in B} \frac{f(x)}{U^B f(x)} - \beta.
\]
Now we obtain the lower bound by letting \(\beta \to 0\). \(\square\)

### 3.1 Estimates for the local Dirichlet eigenvalues

In this section, our main aim is to estimate the bounds of the local Dirichlet eigenvalues on \(\mathbb{R} \setminus \{0\}\) and \((0, \infty)\) by using \([6]\) and Lemma 7.

Before stating the main results, we recall the so-called II-operator
\[
\Pi(f)(x) := \frac{1}{f(x)} U_0 f(x), \quad f \in \{g : g \in C([0, \infty]), g(0) = 0, g > 0\} \tag{21}
\]
constructed by M.F. Chen (cf. \([6]\) Section 6.2]) for diffusion operator
\[
A = a(x) \frac{d^2}{dx^2} + b(x) \frac{d}{dx}
\]
on half line, where \( a > 0 \) and \( b \) are continuous on \([0, \infty)\),

\[
U_0 f(x) := \int_0^x e^{-C(y)} \left( \int_y^\infty f(z)\nu(dz) \right) dy,
\]

\( C(x) = \int_1^x \frac{b(t)}{a(t)}dt \) and \( \nu(dx) = \frac{e^{C(x)}}{a(x)}dx \) is the reversible measure. By choosing

\[ f = \sqrt{\varphi} := \int_0^x e^{-C(t)}dt \]

in \((21), \) \([6, \) Theorem 6.1\) gives the explicit estimate for the Dirichlet eigenvalue

\[ \lambda_0 \geq \inf_x (\sqrt{\varphi}(x))^{-1} \geq (4\eta)^{-1}, \]

where \( \eta := \sup_{x > 0} \varphi(x)\pi([x, \infty)). \)

We interpret the II-operator from a new viewpoint. It is well known that \( U_0 f(x) \) is the classical solution of ordinary differential equation \(-A(U_0 f) = f\) with Dirichlet boundary condition \( U_0 f(0) = 0\), and \( \varphi(x) = \int_0^x e^{-C(t)}dt \) is the harmonic function of \( A \) with \( \varphi(0) = 0\).

By choosing the square root of harmonic function in the II-operator, we obtain the lower bound for \( \lambda_0 \) of diffusion operator.

Now we construct the II-operator for time-changed symmetric stable process \( Y \). Let

\[ U^{(0)} := U^{(0)c} \]

be the Green operator of \( Y \) killed upon \( \{0\}^c \), by \((12), \) \( v := U^{(0)} f \) is a weak solution of \(-Lv = f\) on \( L^2(\pi) \) with \( v(0) = 0\).

Naturally, we define the II-operator by

\[ \Pi(f)(x) := \frac{1}{f(x)}U^{(0)} f(x), \quad \text{for } f \in \mathcal{G} := \{g : g \in C(\mathbb{R}), U^{(0)}|g| < \infty, g(0) = 0, g > 0\}. \]

**Lemma 8.** For any \( x \in \mathbb{R} \) and \( f \in \mathcal{G} \),

\[ \Pi(f)(x) \leq \frac{(\alpha - 1)\omega_\alpha}{f(x)} \int_0^{[x]} z^{\alpha-2} \left( \int_{\mathbb{R}\setminus(-z,z)} f(y)\sigma(y)^{-\alpha} dy \right) dz. \]

**Proof.** By the property of time change \((14)\), for any \( f \) with \( U^{(0)}|f| < \infty\),

\[ U^{(0)} f(x) = \int_\mathbb{R} G^{(0)c}_x (x, y) f(y)\sigma(y)^{-\alpha} dy. \]

By an elementary inequality (see \((14, \) Lemma 4.2)):

\[ |y|^{\alpha-1} + |x|^{\alpha-1} - |y - x|^{\alpha-1} \leq 2(|x| \wedge |y|)^{\alpha-1}, \quad 1 < \alpha < 2, \quad (22) \]

we obtain that for any \( x, y \neq 0\),

\[ G^{(0)c}_x (x, y) = \frac{\omega_\alpha}{2} \left( |y|^{\alpha-1} + |x|^{\alpha-1} - |y - x|^{\alpha-1} \right) \leq \omega_\alpha (|y| \wedge |x|)^{\alpha-1}. \]

Then for any \( f \) with \( U^{(0)}|f| < \infty\),

\[ U^{(0)} f(x) \leq \omega_\alpha \int_\mathbb{R} (|y| \wedge |x|)^{\alpha-1} f(y)\sigma(y)^{-\alpha} dy \]

\[ = \omega_\alpha \left( \int_{\mathbb{R}\setminus(-|x|,|x|)} |x|^{\alpha-1} f(y)\sigma(y)^{-\alpha} dy + \int_{-|x|}^{[x]} |y|^{\alpha-1} f(y)\sigma(y)^{-\alpha} dy \right) \]

\[ = (\alpha - 1)\omega_\alpha \int_0^{[x]} z^{\alpha-2} \left( \int_{\mathbb{R}\setminus(-z,z)} f(y)\sigma(y)^{-\alpha} dy \right) dz. \]
Let \( h_0(x) = (\omega_n/2)|x|^{\alpha-1} \) be the harmonic function for \( P^{(0)c}_t \) on \( \mathbb{R} \setminus \{0\} \) (see Example 1.1). By choosing \( f(x) = \sqrt{h_0(x)} \), we have the following upper estimate:

**Lemma 9.** If

\[
\delta := \sup_x |x|^{\alpha-1} \int_{\mathbb{R} \setminus (-|x|,|x|)} \sigma(z)^{-\alpha} \, dz < \infty,
\]

then for any \( x \in \mathbb{R} \),

\[
\Pi(\sqrt{h_0})(x) \leq 4\omega_\alpha \delta.
\]

**Proof.** Note for any \( y > 0 \),

\[
\int_{\mathbb{R} \setminus (-y,y)} |z|^{(\alpha-1)/2} \sigma(z)^{-\alpha} \, dz = \int_y^\infty z^{(\alpha-1)/2} \sigma(z)^{-\alpha} \, dz + \int_{-\infty}^{-y} (-z)^{\alpha-1/2} \sigma(z)^{-\alpha} \, dz.
\]

By using integration by parts, for any \( y > 0 \), we have

\[
\int_y^\infty z^{(\alpha-1)/2} \sigma(z)^{-\alpha} \, dz = - \int_y^\infty z^{(\alpha-1)/2} \, d\mu((z, \infty))
\]

\[
\leq y^{(\alpha-1)/2} \mu((y, \infty)) + \frac{\alpha - 1}{2} \int_y^\infty z^{(\alpha-3)/2} \, d\mu((z, \infty)) \, dz,
\]

while

\[
\int_{-\infty}^{-y} (-z)^{(\alpha-1)/2} \, d\mu(\cdot) = \int_y^\infty z^{(\alpha-1)/2} \sigma(-z)^{-\alpha} \, dz = - \int_y^\infty z^{(\alpha-1)/2} \, d\mu((\infty, -z))
\]

\[
\leq y^{(\alpha-1)/2} \mu((\infty, -y)) + \frac{\alpha - 1}{2} \int_y^\infty z^{(\alpha-3)/2} \, d\mu((\infty, -z)) \, dz.
\]

Therefore,

\[
\int_{\mathbb{R} \setminus (-y,y)} |z|^{(\alpha-1)/2} \, d\mu(\cdot) \leq y^{(\alpha-1)/2} \mu((-y, y)) + \frac{\alpha - 1}{2} \int_y^\infty z^{(\alpha-3)/2} \, d\mu((-z, z)^c) \, dz.
\]

Note the definition of \( \delta \) gives

\[
\int_{\mathbb{R} \setminus (-y,y)} |z|^{(\alpha-1)/2} \, d\mu(\cdot) \leq \frac{\delta}{y^{(\alpha-1)/2}} + \frac{\delta(\alpha - 1)}{2} \int_y^\infty z^{-(\alpha+1)/2} \, dz = \frac{2\delta}{y^{(\alpha-1)/2}}.
\]

Finally we have

\[
\Pi(\sqrt{h})(x) \leq \frac{(\alpha - 1)\omega_\alpha}{|x|^{(\alpha-1)/2}} \int_0^{|x|} z^{\alpha-2} \frac{2\delta}{z^{(\alpha-1)/2}} \, dz = 4\omega_\alpha \delta.
\]

Next, we consider the local Dirichlet eigenvalue on half-line. According to [2, Lemma 5],

\[
\frac{(\alpha - 1)\Gamma(\alpha/2)^2 G^{(0,\infty)}_X(x, y)}{(x \land y)^{\alpha-1}} \leq 1.
\]
For any $f$ with $U^{(0,\infty)}|f| < \infty$, it follows from \textbf{(14)} that
\[
U^{(0,\infty)}f(x) = U_X^{(0,\infty)}(f\sigma^{-\alpha})(x) = \int_0^\infty C_X^{(0,\infty)}(x, y)f(y)\sigma(y)^{-\alpha}dy
\leq \frac{1}{(\alpha - 1)\Gamma(\alpha/2)^2}\int_0^\infty (x \wedge y)^{\alpha - 1}f(y)\sigma(y)^{-\alpha}dy
\leq \frac{1}{(\alpha - 1)\Gamma(\alpha/2)^2}(\int_0^x y^{\alpha - 1}f(y)\sigma(y)^{-\alpha}dy + \int_x^\infty x^{\alpha - 1}f(y)\sigma(y)^{-\alpha}dy)
\leq \frac{1}{\Gamma(\alpha/2)^2}\int_0^x z^{\alpha - 2}\left(\int_z^\infty f(y)\sigma(y)^{-\alpha}dy\right)dz.
\]
By defining
\[
\Pi^+(f)(x) := \frac{1}{f(x)}U^{(0,\infty)}f(x) = \frac{1}{\Gamma(\alpha/2)^2f(x)}\int_0^x z^{\alpha - 2}\left(\int_z^\infty f(y)\sigma(y)^{-\alpha}dy\right)dz,
\]
a similar argument to Lemma \textbf{9} gives the following estimate.

**Lemma 10.** If
\[
\delta_+ := \sup_{x > 0}x^{\alpha - 1}\int_x^\infty \sigma(z)^{-\alpha}dz < \infty,
\]
then
\[
\Pi^+(\varphi)(x) \leq \frac{4}{\Gamma(\alpha/2)^2(\alpha - 1)}\delta_+,
\]
where $\varphi(x) = x^{(\alpha - 1)/2}$.

To obtain the lower bounds of the local Dirichlet eigenvalues, we use the following approximation, which is modified from \textbf{[3], Lemma 3.4} for jump process.

**Lemma 11.** Let $A$ be an open subset of $\mathbb{R}$, and $\{A_m\}_{m=1}^\infty$ be a sequence of bounded open subsets such that $A_m \uparrow A$. Then we have
\[
\lambda_0(A) = \lim_{m \to \infty} \lambda_0(A_m).
\]

**Proof.** Since $A_m \subset A$, by the definition of the local Dirichlet eigenvalue (\textbf{3}), we have $\lambda_0(A) \leq \lambda_0(A_m)$. So we only need to prove $\lambda_0(A) \geq \lim_{m \to \infty} \lambda_0(A_m)$.

By \textbf{(13)}, for every $m \geq 1$, there is $f_m \in C_0(A) \cap \mathcal{F}$, $\pi(f_m^2) = 1$, such that
\[
\lambda_0(A) \geq \mathcal{E}(f_m, f_m) - \frac{1}{m} \geq \lambda_0(K_m) - \frac{1}{m},
\]
where $K_m := \text{supp} f_m \subset A$ is the compact support of $f_m$.

Since $A_m \uparrow A$, for each $m$, there exists $k_m$ which satisfies that $k_m \uparrow \infty$ as $m \to \infty$, such that $K_m \subset A_{k_m}$, so then $\lambda_0(A_{k_m}) \leq \lambda_0(K_m)$. By combining it with (\textbf{23}), we have
\[
\lambda_0(A) \geq \lambda_0(A_{k_m}) - \frac{1}{m}.
\]
Due to the monotonicity of $\{A_m\}$, the required assertion follows by letting $m \to \infty$. □
Now we establish the estimates for the local Dirichlet eigenvalues.

**Proof of Theorem 2.** Let \( A = (0, \infty) \) (or \( \mathbb{R} \setminus \{0\} \)) and \( A_n = (-n, n) \cap A \), for \( n \geq 1 \). Denote by \( Y_i^n \) the killed process on \( A_n \).

Since the continuous function is bounded on \( A_n \), by Lemma 7, we have for any \( f \in C(A_n) \),

\[
\lambda_0(A_n) \geq \inf_{x \in A_n} \frac{f(x)}{U^{A_n} f(x)}.
\]

It is clear that \( U^{A_n} f \leq U^A f \) by noting \( \tau_{A_n} \leq \tau_A \). By Lemma 11, \( \lambda_0(A) = \lim_{n \to \infty} \lambda_0(A_n) \), so

\[
\lambda_0(A) = \lim_{n \to \infty} \lambda_0(A_n) \geq \inf_{n \to \infty} \inf_{x \in A_n} \frac{f(x)}{U^A f(x)}.
\]

In the case of \( A = (0, \infty) \) and \( A_n = (0, n) \), by letting \( f(x) = x^{(\alpha-1)/2} \), we have

\[
\lambda_0((0, \infty)) \geq \lim_{n \to \infty} \inf_{x \in (0, n)} \frac{f(x)}{U^{(0, \infty)} f(x)} \geq \inf (\Pi^+(f)(x))^{-1} \geq \left[ \frac{4}{\Gamma(\alpha/2)^2(\alpha - 1)} \right]^{-1} = \frac{(\alpha - 1)\Gamma(\alpha/2)^2}{4\delta};
\]

in the case of \( A = \mathbb{R} \setminus \{0\} \), and \( A_n = (-n, n) \setminus \{0\} \), by letting \( f(x) = |x|^{(\alpha-1)/2} \), we have

\[
\lambda_0 \geq \lim_{n \to \infty} \inf_{x \in (0, n)} \frac{f(x)}{U^{(0)} f(x)} \geq \inf (\Pi(f)(x))^{-1} \geq \frac{1}{4\omega_2 \delta}.
\]

Next we estimate the upper bound. For this, we assume that \( \lambda_0 > 0 \), otherwise, the conclusion is trivial.

Since

\[
\| P_t^{(0)c} g \|_{L^2(\pi)} \leq e^{-\lambda_0 t} \| g \|_{L^2(\pi)},
\]

we have that

\[
\| U^{(0)} g \|_{L^2(\pi)} \leq \lambda_0^{-1} \| g \|_{L^2(\pi)},
\]

which implies that \( (\mathcal{E}^{(0)c}, \mathcal{F}^{(0)c}) \) is transient, and

\[
\int g U^{(0)} g \, d\pi \leq \| g \|_{L^2(\pi)} \| U^{(0)} g \|_{L^2(\pi)} \leq \frac{1}{\lambda_0} \| g \|_{L^2(\pi)}^2 < \infty.
\]

By [17, Theorem 1.3.9], we have \( U^{(0)} g \in \mathcal{F} \), and (12) holds. Hence

\[
\mathcal{E}(U^{(0)} g, U^{(0)} g) = \int g U^{(0)} g \, d\pi.
\]

According to the definition of \( \lambda_0 \),

\[
\lambda_0 \leq \frac{\mathcal{E}(U^{(0)} g, U^{(0)} g)}{\| U^{(0)} g \|_{L^2(\pi)}^2} = \frac{\int g U^{(0)} g \, d\pi}{\| U^{(0)} g \|_{L^2(\pi)}^2} \leq \sup_{x > 0} \frac{g(x)}{U^{(0)} g(x)} + \sup_{x < 0} \frac{g(x)}{U^{(0)} g(x)}.
\]

(24)

Note that for \( xy > 0 \),

\[
|y|^{\alpha-1} + |x|^{\alpha-1} - |y-x|^{\alpha-1} \geq (|x| \wedge |y|)^{\alpha-1}, \ 1 < \alpha < 2,
\]

12
so for $x > 0$, 
\[
U^{(0)} g(x) \geq \frac{\omega_\alpha}{2} \int_{0}^{\infty} (x \land y)^{\alpha - 1} g(y) \sigma(y)^{-\alpha} dy, \tag{25}
\]
while for $x < 0$, 
\[
U^{(0)} g(x) \geq \frac{\omega_\alpha}{2} \int_{-\infty}^{0} (- (x \lor y))^{\alpha - 1} g(y) \sigma(y)^{-\alpha} dy. \tag{26}
\]
Fix $x_0 > 0$ and choose 
\[
g(x) = \begin{cases} 
(x \land x_0)^{\alpha - 1}, & x \geq 0, \\
|x \lor (-x_0)|^{\alpha - 1}, & x \leq 0.
\end{cases}
\]
Then $g \in L^2(\pi)$ and $g(0) = 0$. For $x > 0$, by (25), 
\[
\frac{U^{(0)} g(x)}{g(x)} \geq \frac{\omega_\alpha}{2} \frac{1}{(x \land x_0)^{\alpha - 1}} \int_{x \land x_0}^{\infty} (x \land y)^{\alpha - 1} (y \land x_0)^{\alpha - 1} \sigma(y)^{-\alpha} dy \\
\geq \frac{\omega_\alpha}{2} \int_{x \land x_0}^{\infty} (y \land x_0)^{\alpha - 1} \sigma(y)^{-\alpha} dy \geq \frac{\omega_\alpha}{2} \int_{x_0}^{\infty} (y \land x_0)^{\alpha - 1} \sigma(y)^{-\alpha} dy \\
\geq \frac{\omega_\alpha}{2} x_0^{\alpha - 1} \int_{x_0}^{\infty} \sigma(y)^{-\alpha} dy.
\]
For $x < 0$, by (26), 
\[
\frac{U^{(0)} g(x)}{g(x)} \geq \frac{\omega_\alpha}{2} \frac{1}{|x \lor (-x_0)|^{\alpha - 1}} \int_{-\infty}^{x \lor (-x_0)} (- (x \lor y))^{\alpha - 1} |y \lor (-x_0)|^{\alpha - 1} \sigma(y)^{-\alpha} dy \\
\geq \frac{\omega_\alpha}{2} \int_{-\infty}^{x \lor (-x_0)} |y \lor (-x_0)|^{\alpha - 1} \sigma(y)^{-\alpha} dy \geq \frac{\omega_\alpha}{2} \int_{-\infty}^{-x_0} |y \lor (-x_0)|^{\alpha - 1} \sigma(y)^{-\alpha} dy \\
\geq \frac{\omega_\alpha}{2} x_0^{\alpha - 1} \int_{-\infty}^{-x_0} \sigma(y)^{-\alpha} dy.
\]
Now combining (24), (27), and (28), we obtain the desired result.

\[
\square
\]

3.2 Proof of Theorem 1

By combining Theorem 2 with (5), we prove the criterion for exponential ergodicity, and bounds for spectral gap $\lambda_1$.

Proof of Theorem 1. The sufficiency and the estimate for lower bound of $\lambda_1$ follow from $\lambda_1 \geq \lambda_0$ and Theorem 2, so it remains to show the necessity and the upper bound.

First, without loss of generality, assume that 
\[
d_+ := \sup_{x > 0} x^{\alpha - 1} \int_{x}^{\infty} \sigma(y)^{-\alpha} dy = \infty.
\]
Let $h$ be the harmonic function for $P_l^{([-1,1]^c)}$ given by (17). Note that for any $x \in \mathbb{R}$, 
\[
x^{\alpha - 1} \int_{x}^{\infty} \sigma(y)^{-\alpha} dy < \infty, \quad \text{and} \quad \lim_{x \to \infty} \frac{h(x)}{x^{\alpha - 1}} = \frac{1}{\alpha - 1}.
\]

13
So $\delta_+ = \infty$ means that

$$\delta_+(x) := h(x) \int_x^\infty \sigma(y)^{-\alpha} \, dy \to \infty, \text{ as } x \to \infty. \quad (29)$$

By [10, Lemma 3.3], for any $x \notin [-1,1]$,

$$\lim_{y \to \infty} G_{x}^{[-1,1]^c}(x,y) = K_\alpha h(x), \quad (30)$$

where $K_\alpha$ are given by

$$K_\alpha = \frac{2c_\alpha (1 - \frac{\alpha}{2}) \Gamma \left( \frac{\alpha}{2} \right)}{\Gamma \left( 1 - \frac{\alpha}{2} \right)} \int_1^\infty \frac{h'(v)}{1 + v} \, dv < \infty.$$ 

Thus there exists some constant $N_1 > 1$, such that for any $y > N_1$,

$$G_{x}^{[-1,1]^c}(x,y) \geq \frac{1}{2} K_\alpha h(x), \quad x \notin [-1,1].$$

For any fixed $x_0 > N_1$, let $h^{x_0}(x) := h(x \wedge x_0)$, and $u^{x_0}(x) := (U^{[-1,1]^c} h^{x_0})(x)$. Note that

$$u^{x_0}(x) = U_{X}^{[-1,1]^c} (h^{x_0} \sigma^{-\alpha}) (x) \geq \int_{x_0}^\infty G_{x}^{[-1,1]^c}(x,y) h^{x_0}(y) \mu(\,dy) \geq \frac{K_\alpha}{2} h(x_0) \int_{x_0}^\infty \mu(\,dy) \frac{h(x_0)}{2} \delta_+ (x_0) \geq \frac{K_\alpha}{2} h^{x_0}(x_0) \delta_+ (x_0).$$

Hence

$$\langle u^{x_0}, h^{x_0} \rangle = \left( \langle u^{x_0} \rangle^2, \frac{h^{x_0}}{u^{x_0}} \right)_\pi \leq \|u^{x_0}\|_2^2 \pi \left( \frac{K_\alpha}{2} \delta_+ (x_0) \right)^{-1} < \infty,$$

where $\langle \cdot, \cdot \rangle_\pi$ is the inner product on $L^2(\pi)$. By [17, Theorem 1.3.9], we have $u^{x_0} \in \mathcal{F}$ and $\mathcal{E}(u^{x_0}, u^{x_0}) = \langle h^{x_0}, u^{x_0} \rangle_\pi$. According to the definition of the local Dirichlet eigenvalue,

$$\lambda_0([-1,1]^c) \leq \frac{\mathcal{E}(u^{x_0}, u^{x_0})}{\|u^{x_0}\|_{L^2(\pi)}^2} = \frac{\langle u^{x_0}, h^{x_0} \rangle_\pi}{\|u^{x_0}\|_{L^2(\pi)}^2} \leq \left( \frac{K_\alpha}{2} \delta_+ (x_0) \right)^{-1}. \quad (31)$$

Therefore, by letting $x_0 \to \infty$, from (29), we obtain that $\lambda_0([-1,1]^c) = 0$ which implies that $Y$ is non-exponentially ergodic.

### 4 Strong ergodicity

To prove the strong ergodicity of $Y$, we need to estimate the uniform bounds for the first moment of hitting time.

**Proof of Theorem 3**

By a similar argument to [14, Lemma 2.1], we know that strong ergodicity implies that for any closed set $B \subset \mathbb{R}$ with $\pi(B) > 0$, $\sup_x \mathbb{E}_x \tau_B < \infty$. So for the necessity, we only need to prove $\sup_x \mathbb{E}_x \tau_{[-1,1]^c} = \infty$ under the assumption $I = \infty$. According to (30) and the symmetry of $G_{x}^{[-1,1]^c}(x,y)$, we have

$$\lim_{x \to \infty} G_{x}^{[-1,1]^c}(x,y) = K_\alpha h(y), \quad (32)$$
where $K\alpha$ is defined by

$$K\alpha = \frac{2c_\alpha \left(1 - \frac{\alpha}{2}\right) \Gamma \left(\frac{\alpha}{2}\right)}{\Gamma \left(1 - \frac{\alpha}{2}\right)} \int_1^\infty \frac{h'(v)}{1 + v} \, dv < \infty.$$ 

Therefore, by Fatou’s lemma and (32),

$$\sup_x E_x \tau_{[-1,1]} = \sup_x \int_{[0,1]} \liminf_{x \to \infty} G_{[-1,1]}^{(x, y)}(x, y) \sigma(y)^{-\alpha} dy \geq \int_1^\infty \liminf_{x \to \infty} G_{[-1,1]}^{(x, y)}(x, y) \sigma(y)^{-\alpha} dy = \int_1^\infty K\alpha h(y) \sigma(y)^{-\alpha} dy \geq \frac{K\alpha}{(\alpha - 1)} \int_1^\infty (y^{\alpha - 1} - 1) \sigma(y)^{-\alpha} dy = \infty.$$ 

The sufficiency and the lower bound for convergence rate are proved by [16, Theorem 4.3]. We survey the proof below for the readers’ convenience. By (14), we have

$$M_0 := \sup_x E_x \tau_{(0)} = \sup_x \int_{[0,1]} U_{(0)}(x, y) \, dy = \sup_x \int_{[0,1]} G_{(0)}^{(x, y)}(x, y) \sigma(y)^{-\alpha} dy.$$ 

According to (22), we have

$$M_0 \leq \omega_\alpha \int_{[0,1]} |y|^{\alpha - 1} \sigma(y)^{-\alpha} dy = \omega_\alpha I. \quad (33)$$

Note that $\lambda_1 \geq \lambda_0$, and $\lambda_0^{-1} \leq M_0$ (see [12, Lemma 3.2]). Therefore, by [16, Theorem 1.2 (R2)], $\kappa \geq \min\{\lambda_1, M_0^{-1}\} = M_0^{-1} \geq (\omega_\alpha I)^{-1}$, so $Y$ is strongly ergodic.

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