FUCHSIAN EQUATION, HERMITE-KRICHEVER ANSATZ AND PAINLEVÉ EQUATION

KOUICHI TAKEMURA

Abstract. Several results on Heun’s equation are generalized to a certain class of Fuchsian differential equations. Namely, we obtain integral representations of solutions and develop Hermite-Krichever Ansatz on them. In particular, we investigate linear differential equations that produce Painlevé equation by monodromy preserving deformation and obtain solutions of the sixth Painlevé equation which include Hitchin’s solution. The relationship with finite-gap potential is also discussed.

1. Introduction

It is well known that a Fuchsian differential equation with three singularities is transformed to a Gauss hypergeometric equation, and plays important roles in substantial fields in mathematics and physics. Several properties of solutions to the hypergeometric equation have been explained in various textbooks. A canonical form of a Fuchsian equation with four singularities is written as

\[
\left( \left( \frac{d}{dw} \right)^2 + \left( \frac{\gamma}{w} + \frac{\delta}{w-1} + \frac{\epsilon}{w-t} \right) \frac{d}{dw} + \frac{\alpha \beta w - q}{w(w-1)(w-t)} \right) \tilde{f}(w) = 0
\]

with the condition

\[
\gamma + \delta + \epsilon = \alpha + \beta + 1,
\]

and is called Heun’s equation. Despite that Heun’s equation was resolved in the 19th century; several results of solutions have only been recently revealed. Namely, integral representations of solutions, global monodromy in terms of hyperelliptic integrals, relationships with the theory of finite-gap potential and the Hermite-Krichever Ansatz for the case \( \gamma, \delta, \epsilon, \alpha - \beta \in \mathbb{Z} + \frac{1}{2} \) are contemporary (see [11, 12, 10, 13, 14, 15, 16, 19] etc.), though they are not written in a textbook on Heun’s equation [8].

In this paper, we consider differential equations which have additional apparent singularities to Heun’s equation. More precisely, we consider the equation

\[
\left\{ \frac{d^2}{dw^2} + \left( \frac{\frac{1}{2} - l_1}{w} + \frac{\frac{1}{2} - l_2}{w-1} + \frac{\frac{1}{2} - l_3}{w-t} + \sum_{i'=1}^{M} \frac{-r_{i'}}{w-b_{i'}} \right) \frac{d}{dw} + \frac{\left( \sum_{i=0}^{3} l_i + \sum_{i'=1}^{M} r_{i'} \right)(-1 - l_0 + \sum_{i=1}^{3} l_i + \sum_{i'=1}^{M} r_{i'} w + \sum_{i'=1}^{M} \frac{\delta_{i'}}{w-b_{i'}})}{4w(w-1)(w-t)} \right\} \tilde{f}(w) = 0,
\]

for the case \( l_i \in \mathbb{Z}_{\geq 0} (0 \leq i \leq 3), r_{i'} \in \mathbb{Z}_{>0} (1 \leq i' \leq M) \) and the regular singular points \( b_{i'} (1 \leq i' \leq M) \) are apparent.

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By a certain transformation, Eq. (1.3) is rewritten in terms of elliptic functions such as
\begin{equation}
\left\{-\frac{d^2}{dx^2} + \sum_{i=0}^{3} l_i(l_i + 1)\varphi(x + \omega_i) \right. \\
+ \sum_{i'=1}^{M} \left( \frac{r_{i'}}{2} \left( \frac{r_{i'}}{2} + 1 \right) \left( \varphi(x - \delta_{i'}) + \varphi(x + \delta_{i'}) \right) + \frac{s_{i'}}{\varphi(x) - \varphi(\delta_{i'})} \right) - E \right\} f(x) = 0,
\end{equation}
with the condition that logarithmic solutions around the singularities \(x = \pm \delta_{i'}\) \((i' = 1, \ldots, M)\) disappear. We then establish that solutions to Eq. (1.4) have an integral representation and they are also written as a form of the Hermite-Krichever Ansatz. For details see Proposition 3.3 and Theorem 3.5. Note that the results on the Hermite-Krichever Ansatz are related to Picard’s theorem on differential equations with coefficients of elliptic functions [5, §15.6]. By the Hermite-Krichever Ansatz, we can obtain information on the monodromy of solutions to differential equations.

Results on integral representation and the Hermite-Krichever Ansatz are applied for particular cases. One example is Painlevé equation. For the case \(M = 1\) and \(r_1 = 1\), it is known that Eq. (1.3) produces the sixth Painlevé equation by monodromy preserving deformation (see [4]). On the other hand, solutions to Eq. (1.4) are expressed as a form of the Hermite-Krichever Ansatz for the case \(l_i \in \mathbb{Z}_{\geq 0} \ (i = 0, 1, 2, 3)\), and we obtain an expression of monodromy. Fixing monodromy corresponds to the monodromy preserving deformation; thus, we obtain solutions to the sixth Painlevé equation by fixing monodromy (see section 4). For the case \(l_0 = l_1 = l_2 = l_3 = 0\), we recover Hitchin’s solution [4].

Results on integral representation and the Hermite-Krichever Ansatz are also applicable to differential equations related with finite-gap potential. The potential of the Schrödinger operator as Eq. (1.4) for the case \(M = 0\) is called Treibich-Verdier potential [19], and is an example of a finite-gap potential. For this case, the differential equation is transformed to Heun’s equation. If \(M = 1, r_1 = 2, s_1 = 0\) and \(b_1\) satisfies a certain algebraic equation (\(s_1\) and \(b_1\) appear in Eq. (1.4)), then it is seen [18, 11] that the potential is a finite-gap and is also Picard’s in the sense of [3]. For this potential, in this paper we provide a viewpoint from a Fuchsian equation with an apparent singularity, and more results are produced in [17].

This paper is organized as follows. In section 2 we introduce Fuchsian differential equations and rewrite them to the form of elliptic functions. The definition of apparent singularity and its property are mentioned. In section 3 we obtain integral representations of solutions to the differential equation of the class mentioned above and rewrite them to the form of the Hermite-Krichever Ansatz. To obtain an integral representation, we introduce doubly-periodic functions that satisfy a differential equation of order three. Some properties related with this doubly-periodic function are investigated, and we obtain another expression of solutions that looks like the form of the Bethe Ansatz (see Proposition 3.12). In section 4 we consider the relationship with the sixth Painlevé equation. We show that solutions of the sixth Painlevé equation are obtained from solutions expressed in the form of the Hermite-Krichever Ansatz of linear differential equations considered in section 3 by fixing monodromy. Some explicit solutions that include Hitchin’s solution are displayed. In section 5 we discuss the relationship with the results on finite-gap potential. In section 6 we
give concluding remarks and present an open problem. In the appendix, we note definitions and formulae for elliptic functions.

2. Fuchsian differential equation

To begin with, we introduce the following differential equation:

\[
\left\{ \frac{d^2}{dz^2} + \left( \sum_{i=1}^{3} \frac{1}{z-e_i} + \sum_{i' = 1}^{M} \frac{-r_{i'}}{z-b_{i'}} \right) \frac{d}{dz} + \frac{N(N - 2l_0 - 1)z + p}{4(z-e_1)(z-e_2)(z-e_3)} \right\} \hat{f}(z) = 0,
\]

where \( N = \sum_{i=0}^{3} l_i + \sum_{i'=1}^{M} r_{i'} \). This equation is Fuchsian, i.e., all singularities \( \{ e_i \}_{i=1,2,3}, \{ b_{i'} \}_{i'=1,...,M} \) and \( \infty \) are regular. The exponents at \( z = e_i \) (\( i = 1,2,3 \)) (resp. \( z = b_{i'} (i' = 1,\ldots,M) \)) are 0 and \( l_i + 1/2 \) (resp. 0 and \( r_{i'} + 1 \)), and the exponents at \( z = \infty \) are \( N/2 \) and \( (N - 2l_0 - 1)/2 \). Conversely, any Fuchsian differential equation that has regular singularities at \( \{ e_i \}_{i=1,2,3}, \{ b_{i'} \}_{i'=1,...,M} \) and \( \infty \) such that one of the exponents at \( e_i \) and \( b_{i'} \) for all \( i \in \{ 1,2,3 \} \) and \( i' \in \{ 1,\ldots,M \} \) are zero is written as Eq.\((2.1)\). By the transformation \( z \to z + \alpha \), we can change to the case \( e_1 + e_2 + e_3 = 0 \). In this paper we restrict discussion to the case \( e_1 + e_2 + e_3 = 0 \). We remark that any Fuchsian equation with \( M + 4 \) singularities is transformed to Eq.\((2.1)\) with the condition \( e_1 + e_2 + e_3 = 0 \).

It is known that, if \( e_1 + e_2 + e_3 = 0 \) and \( e_1 \neq e_2 \neq e_3 \neq e_1 \), then there exists some periods \( (2\omega_1, 2\omega_3) \) such that \( \wp(\omega_1) = e_1 \) and \( \wp(\omega_3) = e_3 \), where \( \wp(x) \) is the Weierstrass \( \wp \)-function with periods \( (2\omega_1, 2\omega_3) \). We set \( \omega_0 = 0 \) and \( \omega_2 = -\omega_1 - \omega_3 \). Then we have \( \wp(\omega_2) = e_2 \).

Now we rewrite Eq.\((2.1)\) in an elliptic form. We set

\[
\Phi(z) = \prod_{i=1}^{3} (z - e_i)^{-l_i/2} \prod_{i'=1}^{M} (z - b_{i'})^{-r_{i'}/2}, \quad z = \wp(x),
\]

and \( \hat{f}(z)\Phi(z) = f(x) \). Then we have

\[
(H - E)f(x) = 0,
\]

where \( H \) is a differential operator defined by

\[
H = -\frac{d^2}{dx^2} + v(x),
\]

\[
v(x) = \sum_{i=0}^{3} l_i(l_i + 1)\wp(x + \omega_i)
+ \sum_{i'=1}^{M} r_{i'} \left( \frac{r_{i'}}{2} + 1 \right) \left( \wp(x - \delta_{i'}) + \wp(x + \delta_{i'}) \right) + \frac{s_{i'}}{\wp(x) - \wp(\delta_{i'})}.
\]
and
\[ \varphi(\delta_{i'}) = b_{i'}, \quad (i' = 1, \ldots, M), \]
\[ \alpha_{i'} = -s_{i'} + r_{i'} \left\{ \frac{1}{8} r_{i'}(12b_{i'}^2 - g_2) + \frac{1}{2} (4b_{i'}^3 - g_2b_{i'} - g_3) \left( \sum_{i'' \neq i'} \frac{r_{i''}}{(b_{i'} - b_{i''})} \right) 
+ 2(l_1(b_{i'} - e_2)(b_{i'} - e_3) + l_2(b_{i'} - e_1)(b_{i'} - e_3) + l_3(b_{i'} - e_1)(b_{i'} - e_2)) \right\}, \]
\[ p = E + (e_1l_1^2 + e_2l_2^2 + e_3l_3^2) - 2(l_1l_2e_3 + l_2l_3e_1 + l_3l_1e_2) - \frac{1}{2} \sum_{i'=1}^M b_{i'}r_{i'}^2 
+ 2 \sum_{i'=1}^M \sum_{i''=1}^3 l_ir_{i'}(e_i + b_{i'}) + 2 \left( \sum_{i'=1}^M b_{i'}r_{i'} \right) \left( \sum_{i'=1}^M r_{i'} \right), \]
\[ g_2 = -4(e_1e_2 + e_2e_3 + e_3e_1), \quad g_3 = 4e_1e_2e_3. \]

Conversely, Eq. (2.11) is obtained from Eq. (2.8) by the transformation above.

We consider another expression. Set
\[ H_g = -\frac{d^2}{dx^2} + \sum_{i'=1}^M \frac{r_{i'}\varphi'(x)}{\varphi(x) - \varphi(\delta_{i'})} \frac{d}{dx} + \left( l_0 + \sum_{i'=1}^M r_{i'} \right) \left( l_0 + 1 - \sum_{i'=1}^M r_{i'} \right) \varphi(x) 
+ \sum_{i=1}^3 l_i(i_i + 1)\varphi(x + \omega_i) + \sum_{i'=1}^M \frac{\tilde{s}_{i'}}{\varphi(x) - \varphi(\delta_{i'})}, \]
\[ f_g(x) = f(x)\Psi_g(x), \quad \Psi_g(x) = \prod_{i'=1}^M (\varphi(x) - \varphi(\delta_{i'}))^{r_{i'}/2}. \]

Then Eq. (2.3) is also equivalent to
\[ (H_g - E - C_g)f_g(x) = 0, \]
where
\[ \tilde{s}_{i'} = s_{i'} - r_{i'} \left\{ \frac{1}{8} r_{i'}(12b_{i'}^2 - g_2) + \frac{1}{2} (4b_{i'}^3 - g_2b_{i'} - g_3) \left( \sum_{i'' \neq i'} \frac{r_{i''}}{(b_{i'} - b_{i''})} \right) \right\}, \]
\[ C_g = -\frac{1}{2} \sum_{i'=1}^M b_{i'}r_{i'}^2 + 2 \left( \sum_{i'=1}^M b_{i'}r_{i'} \right) \left( \sum_{i'=1}^M r_{i'} \right). \]

In this paper, we consider solutions to Eq. (2.1), which is equivalent to Eq. (2.3) or
Eq. (2.12) for the case \( l_i \in \mathbb{Z} \), and the regular singular point \( z = b_{i'} \) is apparent for
all \( i' \). Here, a regular singular point \( x = a \) of a linear differential equation of order
two is said to be apparent, if and only if the differential equation does not have a
logarithmic solution at \( x = a \) and the exponents at \( x = a \) are integers. It is known
that the regular singular point \( x = a \) is apparent, if and only if the monodromy
matrix around \( x = a \) is a unit matrix. Note that Smirnov investigated solutions in
[11] with the assumptions \( s_{i'} = 0 \) and \( r_{i'} \in 2\mathbb{Z} \) for all \( i' \).

We consider the condition that the regular singular point \( x = a \) is apparent. More
precisely, we describe the condition that a differential equation of order two does not
have logarithmic solutions at a regular singular point \( x = a \) for the case \( \alpha_2 - \alpha_1 \in \mathbb{Z} \),
where \( \alpha_1 \) and \( \alpha_2 \) are exponents at \( x = a \). If \( \alpha_1 = \alpha_2 \), then the differential equation
has logarithmic solutions at \( x = a \). We assume that the exponents satisfy \( \alpha_2 - \alpha_1 = n \in \mathbb{Z}_{\geq 1} \). Since the point \( x = a \) is a regular singular, the differential equation is written as

\[
(2.15) \quad \left\{ \frac{d^2}{dx^2} + \sum_{j=0}^{\infty} p_j(x-a)^{j-1} \frac{d}{dx} + \sum_{j=0}^{\infty} q_j(x-a)^{j-2} \right\} f(x) = 0.
\]

Let \( F(t) \) be the characteristic polynomial at the regular point \( x = a \), i.e. \( F(t) = t^2 + (p_0 - 1)t + q_0 \). From the definition of exponents, we have \( F(\alpha_1) = F(\alpha_1 + n) = 0 \). We can now calculate solutions to Eq.(2.15) in the form

\[
(2.16) \quad f(x) = \sum_{j=0}^{\infty} c_j(x-a)^{\alpha_1+j},
\]

where \( f(x) \) is normalized to satisfy \( c_0 = 1 \). By substituting it into Eq.(2.15) and comparing the coefficients of \( (x-a)^{\alpha_1+j-2} \), we obtain the relations

\[
(2.17) \quad F(\alpha_1 + j)c_j + \sum_{j'=0}^{j-1} \{(\alpha_1 + j')p_{j-j'} + q_{j-j'}\}c_{j'} = 0.
\]

If the positive integer \( j \) satisfies \( F(\alpha_1 + j) \neq 0 \) (i.e. \( j \neq 0, n \)), then the coefficient \( c_j \) is determined recursively. For the case \( j = n \), we have \( F(\alpha_1 + n) = 0 \) and

\[
(2.18) \quad \sum_{j'=0}^{n-1} \{(\alpha_1 + j')p_{n-j'} + q_{n-j'}\}c_{j'} = 0.
\]

Eq.(2.18) with recursive relations (2.17) for \( j = 1, \ldots, n - 1 \) is a necessary and sufficient condition that Eq.(2.15) does not have a logarithmic solution for the case \( \alpha_2 - \alpha_1 = n \in \mathbb{Z}_{\geq 1} \). In fact, if \( p_0, q_0, \ldots, p_n, q_n \) satisfy Eq.(2.18), then there exist solutions to Eq.(2.15) that include two parameters \( c_0 \) and \( c_n \). Thus any solutions are not logarithmic at \( x = a \). Conversely, if Eq.(2.18) is not satisfied, there exists a logarithmic solution written as \( f(x) = \sum_{j=0}^{\infty} c_j(x-a)^{\alpha_1+j} + \log(x-a)\sum_{j=n}^{\infty} \tilde{c}_j(x-a)^{\alpha_1+j} \).

It follows from \( \wp(\delta_v^i) = b_v \) and \( \wp'(\delta_v^i) \neq 0 \) that, the monodromy matrix to Eq.(2.11) around a regular singular point \( z = b_v^i \) is a unit matrix, if and only if the monodromy matrix to Eq.(2.12) around a regular singular point \( x = \pm \delta_v^i \) is a unit matrix. It is obvious that, if the monodromy matrix to Eq.(2.11) around a regular singular point \( z = b_v^i \) is a unit matrix, then we have \( r_v^i \in \mathbb{Z}_{\neq 0} \). In this paper we assume that \( r_v^i \in \mathbb{Z}_{\geq 0} \) for all \( i' \).

3. Integral representation and the Hermite-Krichever Ansatz

We introduce doubly-periodic functions to obtain an integral expression of solutions to Eq.(2.3) for the case \( l_i \in \mathbb{Z}_{\geq 0} \) \((i = 0, 1, 2, 3)\), \( r_i \in \mathbb{Z}_{\geq 0} \) \((i' = 1, \ldots, M)\) and the regular singular points \( z = b_v^i \) \((i' = 1, \ldots, M)\) of Eq.(2.1) are apparent.

**Proposition 3.1.** If \( l_i \in \mathbb{Z}_{\geq 0} \) \((i = 0, 1, 2, 3)\), \( r_i \in \mathbb{Z}_{\geq 0} \) \((i' = 1, \ldots, M)\) and regular singular points \( z = b_v^i \) \((i' = 1, \ldots, M)\) of Eq.(2.1) are apparent, then the equation

\[
(3.1) \quad \left\{ \frac{d^3}{dx^3} - 4(v(x) - E) \frac{d}{dx} - 2\frac{dv(x)}{dx} \right\} \Xi(x) = 0,
\]
has an even nonzero doubly-periodic solution that has the expansion

\[(3.2) \quad \Xi(x) = c_0 + \sum_{i=0}^{3} \sum_{j=0}^{l_i-1} b_j^{(i)} \phi(x + \omega_i)^{l_i-j} + \sum_{i'=1}^{M} \sum_{j=0}^{r_{i'}-1} \frac{d_j^{(i')}}{(\phi(x) - \phi(\delta_{i'}))^r_{i'-j}}.\]

**Proof.** First, we show a lemma that is related to the monodromy of solutions to Eq. (2.12).

**Lemma 3.2.** If $l_0, l_1, l_2, l_3 \in \mathbb{Z}_{>0}$, then the monodromy matrix of Eq. (2.12) around a point $x = n_1\omega_1 + n_3\omega_3$ ($n_1, n_3 \in \mathbb{Z}$) is a unit matrix.

**Proof.** Due to periodicity, it is sufficient to consider the case $x = \omega_i$ ($i = 0, 1, 2, 3$). We first deal with the case $i = 1, 2, 3$. The exponents at the singular point $x = \omega_i$ ($i = 1, 2, 3$) are $-l_i$ and $l_i + 1$. Because Eq. (2.12) is invariant under the transformation $x - \omega_i \rightarrow -(x - \omega_i)$ and the gap of the exponents at $x = \omega_i$ (i.e. $l_i + 1 - (-l_i)$) is odd, there exist solutions in the form $f_{i,1}(x) = (x - \omega_i)^{-l_i}(1 + \sum_{j=1}^{\infty} a_j^i (x - \omega_i)^{2j})$ and $f_{i,2}(x) = (x - \omega_i)^{l_i+1}(1 + \sum_{j=1}^{\infty} a_j^i (x - \omega_i)^{2j})$. Since the functions $f_{i,1}(x)$ and $f_{i,2}(x)$ form a basis for solutions to Eq. (2.12) and they are holomorphic around the point $x = \omega_i$, the monodromy matrix around $x = \omega_i$ is a unit matrix. For the case $i = 0$, the exponents at $x = 0$ are $-l_0 - \sum_{i'=1}^{M} r_{i'}$ and $l_0 + 1 - \sum_{i'=1}^{M} r_{i'}$, and similarly it is shown that the monodromy matrix around the point $x = 0$ is a unit matrix. Hence we obtain the lemma. □

We continue the proof of Proposition 3.1. Let $M_i$ ($i = 1, 3$) be the transformations obtained by the analytic continuation $x \rightarrow x + 2\omega_i$. It follows from double-periodicity of Eq. (2.12) that, if $f_g(x)$ is a solutions to Eq. (2.12), then $M_i f_g(x)$ ($i = 1, 3$) is also a solution to Eq. (2.12). From the assumption that regular singular points $z = b_i$ are apparent for all $i'$, the monodromy matrix to Eq. (2.12) around a regular singular point $x = \pm\delta_{i'}$ is a unit matrix for all $i'$. By combining with Lemma 3.2 it follows that all local monodromy matrices around any singular points are units. Hence the transformations $M_i$ do not depend on the choice of paths. From the fact that the fundamental group of the torus is commutative, we have $M_1 M_3 = M_3 M_1$. Recall that the operators $M_i$ act on the space of solutions to Eq. (2.12) for each $E$, which is two dimensional. By the commutativity $M_1 M_3 = M_3 M_1$, there exists a joint eigenvector $\hat{\Lambda}_g(x)$ for the operators $M_1$ and $M_3$. It follows from Proposition 3.2 and the appearance of singular points that the function $\hat{\Lambda}_g(x)$ is single-valued and satisfies equations $(H_g - E - C_g)\hat{\Lambda}_g(x) = 0$, $M_1 \hat{\Lambda}_g(x) = \tilde{m}_1 \hat{\Lambda}_g(x)$ and $M_3 \hat{\Lambda}_g(x) = \tilde{m}_3 \hat{\Lambda}_g(x)$ for some $\tilde{m}_1, \tilde{m}_3 \in \mathbb{C} \setminus \{0\}$. By changing parity $x \leftrightarrow -x$, it follows immediately that $(H_g - E - C_g)\hat{\Lambda}_g(-x) = 0$, $M_1 \hat{\Lambda}_g(-x) = \tilde{m}_1^{-1} \hat{\Lambda}_g(-x)$ and $M_3 \hat{\Lambda}_g(-x) = \tilde{m}_3^{-1} \hat{\Lambda}_g(-x)$. Then the function $\hat{\Lambda}_g(x)\hat{\Lambda}_g(-x)$ is single-valued, even and doubly-periodic. We set $\hat{\Lambda}(x) = \hat{\Lambda}_g(x)/\Psi_g(x)$. Then $\hat{\Lambda}(x)$ and $\Lambda(-x)$ are solutions to Eq. (2.3).

Now consider the function $\Xi(x) = \hat{\Lambda}_g(x)\hat{\Lambda}_g(-x)/\Psi_g(x)^2$. Since the function $\Psi_g(x)^2$ is single-valued, even and doubly-periodic, the function $\Xi(x)$ is single-valued, even (i.e. $\Xi(x) = \Xi(-x)$), doubly-periodic (i.e. $\Xi(x + 2\omega_1) = \Xi(x + 2\omega_3) = \Xi(x)$), and satisfies the equation

\[
\left\{ \frac{d^2}{dx^2} - 4(v(x) - E) \frac{d}{dx} - 2\frac{dv(x)}{dx} \right\} \Xi(x) = 0
\]

that the products of any pair of solutions to Eq. (2.3) satisfy.
Since the function $\Xi(x)$ is an even doubly-periodic function that satisfies the differential equation (3.1) and the exponents of Eq. (3.1) at $x = \omega_i$ ($i = 0, \ldots, 3$) (resp. $x = \pm \delta_i$ ($i' = 1, \ldots, M$)) are $-2l_i, 1, 2l_i + 2$ (resp. $-r_i, 1, r_i + 2$), it is written as a rational function of variable $\wp(x)$, and it admits the expansion as Eq. (3.2) by considering exponents.

The function $\Xi(x)$ is calculated by substituting Eq. (3.2) into the differential equation (3.1) and solving simultaneous equations for the coefficients. We introduce an integral formula for a solution to the differential equation Eq. (2.3) in use of the function $\Xi(x)$. Set

$$Q = \Xi(x)^2 (E - v(x)) + \frac{1}{2} \Xi(x) \frac{d^2 \Xi(x)}{dx^2} - \frac{1}{4} \left( \frac{d \Xi(x)}{dx} \right)^2.$$  (3.3)

It follows from Eq. (3.1) that

$$\frac{dQ}{dx} = \frac{1}{2} \Xi(x) \left( 4 \frac{d \Xi(x)}{dx} (E - v(x)) - 2 \Xi(x) \frac{dv(x)}{dx} + \frac{d^3 \Xi(x)}{dx^3} \right) = 0.$$  (3.4)

Hence the value $Q$ is independent of $x$.

**Proposition 3.3.** Let $\Xi(x)$ be the doubly-periodic function defined in Proposition 3.1 and $Q$ be the value defined in Eq. (3.3). Then the function

$$\Lambda(x) = \sqrt{\Xi(x)} \exp \int \frac{\sqrt{-Q dx}}{\Xi(x)},$$  (3.5)

is a solution to the differential equation (2.3), and the function

$$\Lambda_g(x) = \Psi_g(x) \sqrt{\Xi(x)} \exp \int \frac{\sqrt{-Q dx}}{\Xi(x)},$$  (3.6)

is a solution to the differential equation (2.12).

**Proof.** From Eqs. (3.3) and (3.4), we have

$$\frac{\Lambda'(x)}{\Lambda(x)} = \frac{1}{2} \Xi(x) + \frac{\sqrt{-Q}}{\Xi(x)},$$  (3.7)

$$\frac{\Lambda''(x)}{\Lambda(x)} = \frac{1}{2} \frac{\Xi''(x)}{\Xi(x)} - \frac{1}{4} \left( \frac{\Xi'(x)}{\Xi(x)} \right)^2 - \frac{Q}{\Xi(x)^2} = v(x) - E.$$  (3.8)

Hence we have $- \frac{d^2}{dx^2} \Lambda(x) + v(x) \Lambda(x) = E \Lambda(x)$. It follows from the equivalence of Eq. (2.3) and Eq. (2.12) that the function $\Lambda_g(x)$ is a solution to Eq. (2.12).  \(\square\)

**Proposition 3.4.** If $Q \neq 0$, then the functions $\Lambda(x)$ and $\Lambda(-x)$ are linearly independent and any solution to Eq. (2.3) is written as a linear combination of $\Lambda(x)$ and $\Lambda(-x)$.

**Proof.** It follows from Eq. (3.6) and the evenness of the function $\Xi(x)$ that

$$\frac{\frac{d}{dx} \Lambda(-x)}{\Lambda(-x)} = \frac{1}{2} \frac{\Xi'(x)}{\Xi(x)} - \frac{\sqrt{-Q}}{\Xi(x)}.$$  (3.9)

Hence we have

$$\Lambda(-x) \frac{d}{dx} \Lambda(x) - \Lambda(x) \frac{d}{dx} \Lambda(-x) = \Lambda(x) \Lambda(-x) \frac{2\sqrt{-Q}}{\Xi(x)}.$$  (3.10)
If \( \Lambda(x) \) and \( \Lambda(-x) \) are linearly dependent, then the l.h.s. of Eq. (3.9) must be zero; however, this is impossible because \( Q \neq 0 \). Hence the functions \( \Lambda(x) \) and \( \Lambda(-x) \) are linearly independent. It follows from the invariance of Eq. (2.3) with respect to the transformation \( x \leftrightarrow -x \) that \( \Lambda(-x) \) is also a solution to Eq. (2.3).

Since solutions to Eq. (2.3) form a two-dimensional vector space and the functions \( \Lambda(x) \) and \( \Lambda(-x) \) are linearly independent, the functions \( \Lambda(x) \) and \( \Lambda(-x) \) form a basis of the space of solutions to Eq. (2.3), and any solution to Eq. (2.3) is written as a linear combination of \( \Lambda(x) \) and \( \Lambda(-x) \).

It follows from Proposition 3.4 that, if \( Q \neq 0 \), then the functions \( \Lambda_g(x) \) and \( \Lambda_g(-x) \) are linearly independent, and any solution to Eq. (2.12) is written as a linear combination of \( \Lambda_g(x) \) and \( \Lambda_g(-x) \).

From the formulae \( 3.4 \) \( 3.5 \) and the doubly-periodicity of the functions \( \Xi(x) \) and \( \Psi_g(x)^2 \), we have

\[
\begin{align*}
\Lambda(x + 2\omega_j) &= \pm \Lambda(x) \exp \int_{0+\varepsilon}^{2\omega_j+\varepsilon} \frac{\sqrt{-Qdx}}{\Xi(x)}, \quad (j = 1, 3), \\
\Lambda_g(x + 2\omega_j) &= \pm \Lambda_g(x) \exp \int_{0+\varepsilon}^{2\omega_j+\varepsilon} \frac{\sqrt{-Qdx}}{\Xi(x)}, \quad (j = 1, 3),
\end{align*}
\]

with \( \varepsilon \) a constant determined so as to avoid passing through the poles while integrating. The sign \( \pm \) is determined by the analytic continuation of the function \( \sqrt{\Xi(x)} \), and the integrations in Eqs. (3.10) \( 3.11 \) may depend on the choice of the path. The function \( \Lambda(x) \) may have branching points, although the function \( \Lambda_g(x) \) does not have branching points and is meromorphic on the complex plane, because \( \Lambda_g(x) \) is a solution to Eq. (2.12) and any singularity of Eq. (2.12) is apparent. It follows from Eq. (3.11) that there exists \( m_1, m_3 \in \mathbb{C} \) such that

\[
\Lambda_g(x + 2\omega_j) = \exp(\pi \sqrt{-1} m_j) \Lambda_g(x), \quad (j = 1, 3).
\]

We now show that a solution to Eq. (2.3) can be expressed in the form of the Hermite-Krichever Ansatz. We set

\[
\Phi_i(x, \alpha) = \frac{\sigma(x + \omega_i - \alpha)}{\sigma(x + \omega_i)} \exp(\zeta(\alpha)x), \quad (i = 0, 1, 2, 3),
\]

where \( \sigma(x) \) (resp. \( \zeta(x) \)) is the Weierstrass sigma (resp. zeta) function. Then we have

\[
\left( \frac{d}{dx} \right)^k \Phi_i(x + 2\omega_j, \alpha) = \exp(-2\eta_j \alpha + 2\omega_j \zeta(\alpha)) \left( \frac{d}{dx} \right)^k \Phi_i(x, \alpha)
\]

for \( i = 0, 1, 2, 3, j = 1, 2, 3 \) and \( k \in \mathbb{Z}_{\geq 0} \), where \( \eta_j = \zeta(\omega_j) \) (\( j = 1, 2, 3 \)).

**Theorem 3.5.** Set \( \tilde{l}_0 = l_0 + \sum_{i'=1}^{M} r_{i'} \) and \( \tilde{l}_i = l_i \) \( (i = 1, 2, 3) \). The function \( \Lambda_g(x) \) in Eq. (3.5) is expressed as

\[
\Lambda_g(x) = \exp(\kappa x) \left( \sum_{i=0}^{3} \sum_{j=0}^{\tilde{l}_i-1} b_j^{(i)} \left( \frac{d}{dx} \right)^j \Phi_i(x, \alpha) \right)
\]
for some values $\alpha$, $\kappa$ and $\bar{b}_j^{(i)}$ ($i = 0, \ldots, 3$, $j = 0, \ldots, \bar{l}_i - 1$), or

\begin{equation}
\Lambda_g(x) = \exp(\bar{\kappa}x) \left( \bar{c} + \sum_{i=0}^{3} \sum_{j=0}^{\bar{l}_i-2} \bar{b}_j^{(i)} \frac{d}{dx} \right)^j \varphi(x + \omega_i) + \sum_{i=1}^{3} \bar{c}_i \varphi'(x) - e_i \right)
\end{equation}

for some values $\bar{\kappa}$, $\bar{c}$, $\bar{c}_i$ ($i = 1, 2, 3$) and $\bar{b}_j^{(i)}$ ($i = 0, \ldots, 3$, $j = 0, \ldots, \bar{l}_i - 2$).

If the function $\Lambda_g(x)$ is expressed as Eq.\,(3.15), then

\begin{equation}
\Lambda_g(x + 2\omega_j) = \exp(-2\eta_j \alpha + 2\omega_j \zeta(\alpha) + 2\kappa \omega_j) \Lambda_g(x), \quad (j = 1, 3),
\end{equation}

else

\begin{equation}
\Lambda_g(x + 2\omega_j) = \exp(2\bar{\kappa} \omega_j) \Lambda_g(x), \quad (j = 1, 3).
\end{equation}

**Proof.** Set

\begin{equation}
\alpha = -m_1 \omega_3 + m_3 \omega_1,
\end{equation}

where $m_1$ and $m_3$ are determined in Eq.\,(3.12).

If $\alpha \not\equiv 0$ (mod $2\omega_1 \mathbb{Z} \oplus 2\omega_3 \mathbb{Z}$), then we set

\begin{equation}
\kappa = \zeta(m_1 \omega_3 - m_3 \omega_1) - m_1 \eta_3 + m_3 \eta_1.
\end{equation}

It follows from Legendre’s relation $\eta_1 \omega_3 - \eta_3 \omega_1 = \pi \sqrt{-1}/2$ that

\begin{equation}
\exp(\kappa(x + 2\omega_j)) \left( \frac{d}{dx} \right)^k \Phi_i(x + 2\omega_j, \alpha)
\end{equation}

\begin{align*}
&= \exp(-2\eta_j \alpha + 2\omega_j (\zeta(\alpha) + \kappa)) \exp(\kappa x) \left( \frac{d}{dx} \right)^k \Phi_i(x, \alpha) \\
&= \exp(2m_1(\eta_j \omega_3 - \eta_3 \omega_j) + 2m_3(\eta_1 \omega_j - \eta_j \omega_1)) \exp(\kappa x) \left( \frac{d}{dx} \right)^k \Phi_i(x, \alpha) \\
&= \exp(\pi \sqrt{-1} m_j) \exp(\kappa x) \left( \frac{d}{dx} \right)^k \Phi_i(x, \alpha)
\end{align*}

for $i = 0, 1, 2, 3$, $j = 1, 3$ and $k \in \mathbb{Z}_{\geq 0}$. Hence the function $\Lambda_g(x)$ and the functions $\exp(\kappa x) \left( \frac{d}{dx} \right)^k \Phi_i(x, \alpha)$ have the same periodicity with respect to periods $(2\omega_1, 2\omega_3)$. Since the meromorphic function $\Lambda_g(x)$ satisfies Eq.\,(2.12), it is holomorphic except for $\mathbb{Z} \omega_1 \oplus \mathbb{Z} \omega_3$ and has a pole of degree $\bar{l}_i$ or zero of degree $\bar{l}_i + 1$ at $x = \omega_i$ ($i = 0, 1, 2, 3$). The function $\exp(\kappa x) \left( \frac{d}{dx} \right)^k \Phi_i(x, \alpha)$ has a pole of degree $k + 1$ at $x = \omega_i$.

By subtracting the functions $\exp(\kappa x) \left( \frac{d}{dx} \right)^k \Phi_i(x, \alpha)$ from the function $\Lambda_g(x)$ to erase the poles, we obtain a holomorphic function that has the same periods as $\Phi_0(x, \alpha)$, and must be zero. Hence we obtain the expression \(3.15\). The periodicity (see Eq.\,(3.17)) follows from Eq.\,(3.21).

If $\alpha \equiv 0$ (mod $2\omega_1 \mathbb{Z} \oplus 2\omega_3 \mathbb{Z}$) (i.e. $m_1 \omega_3 \equiv m_3 \omega_1$ (mod $2\omega_1 \mathbb{Z} \oplus 2\omega_3 \mathbb{Z}$)), then we set

\begin{equation}
\bar{\kappa} = -m_1 \eta_3 + m_3 \eta_1.
\end{equation}

The function $\Lambda_g(x)$ and the function $\exp(\bar{\kappa} x)$ have the same periodicity with respect to periods $(2\omega_1, 2\omega_3)$. Hence the function $\Lambda_g(x) \exp(-\bar{\kappa} x)$ is doubly periodic, and we obtain the expression \(3.16\) by considering the poles. Periodicity (see Eq.\,(3.18)) follows immediately. \(\square\)
We investigate the situation that Eq. (2.12) has a non-zero solution of an elliptic function. Let \( \mathcal{F}_{\epsilon_1, \epsilon_3} \) and \( \bar{\mathcal{F}}_{\epsilon_1, \epsilon_3} \) be the spaces defined by
\[
(3.23) \quad \mathcal{F}_{\epsilon_1, \epsilon_3} = \{ f(x) : \text{meromorphic} \mid f(x + 2\omega_1) = \epsilon_1 f(x), \ f(x + 2\omega_3) = \epsilon_3 f(x) \},
\]
and
\[
(3.24) \quad \bar{\mathcal{F}}_{\epsilon_1, \epsilon_3} = \left\{ f(x) \left| \begin{array}{c}
\quad f(x)\Psi_g(x) \in \mathcal{F}_{\epsilon_1, \epsilon_3} \text{ and holomorphic} \\
\quad \text{except for } \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_3, \text{ and the degree of the pole at} \\
\quad x = \omega_i \text{ is no more than} \left\{ \begin{array}{l}
\quad l_i, \quad i = 1, 2, 3, \\
\quad l_0 + \sum_{i=1}^M r_i, \quad i \neq 1, 2, 3,
\end{array} \right. \\
\end{array} \right. \right\},
\]
where \( (2\omega_1, 2\omega_3) \) are basic periods of elliptic functions. Then \( \bar{\mathcal{F}}_{\epsilon_1, \epsilon_3} \) is a finite-dimensional vector space. Note that, if a solution \( f(x) \) to Eq. (2.13) satisfies the condition \( f(x + 2\omega_1)\Psi_g(x + 2\omega_1) = \epsilon_1 f(x)\Psi_g(x) \) and \( f(x + 2\omega_3)\Psi_g(x + 2\omega_3) = \epsilon_3 f(x)\Psi_g(x) \) for some \( \epsilon_1, \epsilon_3 \in \{ \pm 1 \} \), then we have \( f(x) \in \mathcal{F}_{\epsilon_1, \epsilon_3} \), because the position of the poles and their degree are restricted by the differential equation.

**Proposition 3.6.** Assume that Eq. (2.3) has a non-zero solution in the space \( \bar{\mathcal{F}}_{\epsilon_1, \epsilon_3} \) for some \( \epsilon_1, \epsilon_3 \in \{ \pm 1 \} \). Then the signs \( (\epsilon_1, \epsilon_3) \) are determined uniquely for each \( E, \tilde{s}' \) \((i = 1, \ldots, M)\) etc.

**Proof.** Assume that Eq. (2.3) has a non-zero solution in both the spaces \( \bar{\mathcal{F}}_{\epsilon_1, \epsilon_3} \) and \( \bar{\mathcal{F}}_{\epsilon_1, \epsilon_3} \). Let \( f_1(x) \) (resp. \( f_2(x) \)) be the solution to the differential equation (2.3) in the space \( \bar{\mathcal{F}}_{\epsilon_1, \epsilon_3} \) (resp. the space \( \bar{\mathcal{F}}_{\epsilon_1, \epsilon_3} \)). Then periodicity of the function \( f_1(x)\Psi_g(x) \) and \( f_2(x)\Psi_g(x) \) is different, more precisely there exists \( j \in \{ 1, 3 \} \) such that
\[
(3.25) \quad \begin{cases} 
 f_1(x + 2\omega_j)\Psi_g(x + 2\omega_j) = \pm f_1(x)\Psi_g(x), \\
 f_2(x + 2\omega_j)\Psi_g(x + 2\omega_j) = \mp f_2(x)\Psi_g(x).
\end{cases}
\]
Then the functions \( f_1(x) \) and \( f_2(x) \) are linearly independent. Since the functions \( f_1(x) \) and \( f_2(x) \) satisfy Eq. (2.3), we have \( \frac{df}{dx} (f_2(x)f_1'(x) - f_1(x)f_2'(x)) = f_2(x)f_1''(x) - f_1(x)f_2''(x) = 0 \). Therefore \( f_2(x)f_1'(x) - f_1(x)f_2'(x) = C \) for constants \( C \), and \( C \) is non-zero, which follows from linear independence. By Eq. (3.25), the function \( (f_2(x)f_1'(x) - f_1(x)f_2'(x))\Psi_g(x)^2 \) is anti-periodic with respect to the period \( 2\omega_j \), but it contradicts to \( C \neq 0 \). Hence, we proved that Eq. (2.3) does not have a non-zero solution in both the spaces \( \bar{\mathcal{F}}_{\epsilon_1, \epsilon_3} \) and \( \bar{\mathcal{F}}_{\epsilon_1, \epsilon_3} \). \( \square \)

**Proposition 3.7.** If \( Q = 0 \), then we have \( \Lambda(x) \in \bar{\mathcal{F}}_{\epsilon_1, \epsilon_3} \) for some \( \epsilon_1, \epsilon_3 \in \{ \pm 1 \} \).

**Proof.** It follows from Eq. (3.4) and the double-periodicity of the function \( \Xi(x)\Psi_g(x)^2 \) that
\[
(3.26) \quad \Lambda(x + 2\omega_j)\Psi_g(x + 2\omega_j)^2 = \Lambda(x)\Psi_g(x)^2 = \Xi(x)\Psi_g(x)^2,
\]
for \( j = 1, 3 \). Hence \( \Lambda(x + 2\omega_j)\Psi_g(x + 2\omega_j) = \pm \Lambda(x)\Psi_g(x) \) \((j = 1, 3)\) and we have \( \Lambda(x) \in \bar{\mathcal{F}}_{\epsilon_1, \epsilon_3} \) for some \( \epsilon_1, \epsilon_3 \in \{ \pm 1 \} \). \( \square \)

It follows from Proposition 3.1 that the dimension of the space of solutions to Eq. (3.1), which are even doubly-periodic, is no less than one. Since the exponents of Eq. (3.1) at \( x = 0 \) are \(-2l_0, 1 \) and \( 2l_0 + 2 \), the dimension of the space of even solutions to Eq. (3.1) is at most two. Hence, the dimension of the space of solutions to Eq. (3.1), which are even doubly-periodic, is one or two.

**Proposition 3.8.** Assume that the dimension of the space of solutions to Eq. (2.3), which are even doubly-periodic, is two. The all solutions to Eq. (2.3) are contained in the space \( \bar{\mathcal{F}}_{\epsilon_1, \epsilon_3} \) for some \( \epsilon_1, \epsilon_3 \in \{ \pm 1 \} \).
Proof. Since the differential equation (2.3) is invariant under the change of parity $x \leftrightarrow -x$, a basis of the solutions to Eq. (2.3) is taken as $f_x(x)$ and $f_o(x)$ such that $f_x(x)$ (resp. $f_o(x)$) satisfies $f_x(-x) = f_x(x)$ (resp. $f_o(-x) = -f_o(x)$). Then the functions $f_x(x)^2$ and $f_o(x)^2$ are even and they are solutions to Eq. (3.1). Since the dimension of the space of even solutions to Eq. (3.1) is at most two, and the dimension of the space of solutions to Eq. (3.1), which are even doubly-periodic, is two, the even functions $f_x(x)^2$ and $f_o(x)^2$ must be doubly-periodic. Hence $(f_x(x) + 2\omega_j)\Psi_g(x + 2\omega_j))^2 = (f_x(x)\Psi_g(x))^2 (j = 1, 3)$ and it follows that $f_x(x) + 2\omega_j)\Psi_g(x + 2\omega_j) = \pm f_x(x)\Psi_g(x) (j = 1, 3)$. Therefore we have $f_x(x) \in \tilde{F}_{\epsilon_1, \epsilon_3}$ for some $\epsilon_1, \epsilon_3 \in \{\pm 1\}$. Similarly we have $f_o(x) \in \tilde{F}_{\epsilon'_1, \epsilon'_3}$ for some $\epsilon'_1, \epsilon'_3 \in \{\pm 1\}$, and it follows from Proposition 3.9 that $\epsilon'_j = \epsilon_j (j = 1, 3)$. Since $f_x(x)$ and $f_o(x)$ are a basis of solutions to Eq. (2.3), all solutions to Eq. (2.3) are contained in the space $\tilde{F}_{\epsilon_1, \epsilon_3}$.

Proposition 3.9. If $M = 0$ or $(M = 1$ and $r_1 = 1$), then the dimension of the space of solutions to Eq. (3.1), which are even doubly-periodic, is one.

Proof. Assume that the dimension of the space of solutions to Eq. (3.1), which are even doubly-periodic, is two. From Proposition 3.3 all solutions to Eq. (2.3) are contained in the space $\tilde{F}_{\epsilon_1, \epsilon_3}$ for some $\epsilon_1, \epsilon_3 \in \{\pm 1\}$. Since the differential equation (2.12) is invariant under the change of parity $x \leftrightarrow -x$, a basis of the solutions to Eq. (2.12) is taken as $f_1(x)$ and $f_2(x)$ such that $f_1(x)$ (resp. $f_2(x)$) is even (resp. odd) function. From the assumption that $l_i \in \mathbb{Z} (i = 0, 1, 2, 3)$ and that regular singular points $b_i$ are apparent ($i' = 1, \ldots, M$), the functions $f_1(x)$ and $f_2(x)$ are meromorphic. Since the function $f_1(x)$ (resp. $f_2(x)$) satisfies Eq. (2.12), it does not have poles except for $\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_3$. Hence the function $f_1(x)$ admits the expression $f_1(x) = \varphi_1(x)\beta_1\varphi_2(x)\beta_2\varphi_3(x)\beta_3(P^{(1)}(\varphi(x)) + \varphi'(x)P^{(2)}(\varphi(x)))$, where $\varphi_i(x)$ is a co-\varphi functions and $P^{(1)}(z), P^{(2)}(z)$ are polynomials in $z$. Since the function $f_1(x)$ is even, we have $P^{(1)}(z) = 0$ or $P^{(2)}(z) = 0$. By combining with the relation $\varphi'(x) = -2\varphi_1(x)\varphi_2(x)\varphi_3(x)$, the function $f_1(x)$ is expressed as

\begin{equation}
(3.27) \quad f_1(x) = \varphi_1(x)^{\beta_1}\varphi_2(x)^{\beta_2}\varphi_3(x)^{\beta_3}P_1(\varphi(x)),
\end{equation}

where $P_1(z)$ is a polynomial in $z$. Because the exponents of Eq. (2.12) at $x = \omega_i (i = 1, 2, 3)$ are $-l_i$ and $l_i + 1$, we have $\beta_i \in \{-l_i, l_i + 1\} (i = 1, 2, 3)$. Similarly the function $f_2(x)$ is expressed as

\begin{equation}
(3.28) \quad f_2(x) = \varphi_1(x)^{\beta'_1}\varphi_2(x)^{\beta'_2}\varphi_3(x)^{\beta'_3}P_2(\varphi(x)),
\end{equation}

where $P_2(z)$ is a polynomial in $z$ and $\beta'_i \in \{-l_i, l_i + 1\}$.

Since the functions $\varphi_i(x)$ are odd and the parity of functions $f_1(x)$ and $f_2(x)$ is different, we have $\beta_1 + \beta_2 + \beta_3 \equiv \beta'_1 + \beta'_2 + \beta'_3$ (mod 2). Since $f_j(x + 2\omega_1) = (-1)^{\beta_2 + \beta_3}f_j(x), f_j(x + 2\omega_3) = (-1)^{\beta_1 + \beta_2}f_j(x) (j = 1, 2, 3)$, we have $\beta_2 + \beta_3 \equiv \beta'_2 + \beta'_3$ (mod 2) and $\beta_1 + \beta_2 \equiv \beta'_1 + \beta'_2$ (mod 2). Hence we have $\beta_i \equiv \beta'_i (mod 2)$ for $i = 1, 2, 3$. Therefore $\beta_i, \beta'_i = (l_i, l_i + 1)$ or $\beta_i, \beta'_i = (l_i + 1, -l_i)$ for each $i \in \{1, 2, 3\}$. Let $\beta_0$ (resp. $\beta'_0$) be the exponent of the function $f_1(x)$ (resp. $f_2(x)$) at $x = 0$. Since the parity of functions $f_1(x)$ and $f_2(x)$ is different and the exponents of Eq. (2.12) at $x = 0$ are $-l_0 - \sum_{r_1=1}^{M} r_{1v}$ and $0 + 1 - \sum_{r_1=1}^{M} r_{1v}$, we have $\beta_0, \beta'_0 = (-l_0 - \sum_{r_1=1}^{M} r_{1v}, l_0 + 1 - \sum_{r_1=1}^{M} r_{1v})$ or $\beta_0, \beta'_0 = (l_0 + 1 - \sum_{r_1=1}^{M} r_{1v}, -l_0 - \sum_{r_1=1}^{M} r_{1v})$.

Since the function $f_1(x)$ is doubly-periodic with periods $(4\omega_1, 4\omega_3)$, the sum of degrees of zeros of $f_1(x)$ on the basic domain is equal to the sum of degrees of poles of $f_1(x)$. Since the function $f_1(x)$ does not have poles except for $\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_3$, we
have $\sum_{i=0}^{3} \beta_i \leq 0$. Similarly we have $\sum_{i=0}^{3} \beta_i' \leq 0$. Hence $0 \geq \sum_{i=0}^{3} (\beta_i + \beta_i') = 4 - 2 \sum_{i' = 1}^{M} r_{i'}$. Therefore we have $\sum_{i' = 1}^{M} r_{i'} \geq 2$.

Thus we obtain that, if $M = 0$ or $(M = 1$ and $r_1 = 1)$, then the dimension of the space of solutions to Eq. (3.30), which are even doubly-periodic, is one. □

Note that the case $M = 0$ corresponds to Heun’s equation, and the case $M = 1$ and $r_1 = 1$ is related with the sixth Painlevé equation.

**Example 1.** Let us consider the following differential equation:

$$\left\{-\left(\frac{d}{dx}\right)^2 + \left(\frac{\varphi'(x)}{\varphi(x) + \sqrt{\frac{2}{12}}} + \frac{\varphi'(x)}{\varphi(x) - \sqrt{\frac{2}{12}}}\right)\right\} f(x) = 0 \quad (3.29)$$

This equation corresponds to the case $l_0 = 1$, $l_1 = l_2 = l_3 = 0$, $M = 2$ and $r_1 = r_2 = 1$, if $g_2 \neq 0$. From the relation

$$\frac{\varphi'(x)}{\varphi(x) + \sqrt{\frac{2}{12}}} + \frac{\varphi'(x)}{\varphi(x) - \sqrt{\frac{2}{12}}} = \frac{\varphi'''(x)}{\varphi'(x)},$$

a basis of the solutions to Eq. (3.29) is $\{\pm i\}$. The dimension of the solutions to Eq. (3.30), which are even doubly-periodic, is two, and a basis of the solutions to Eq. (3.30) is written as $1/\varphi'(x)$, $\varphi'(x)^2/\varphi''(x)$, $\varphi'(x)/\varphi''(x)$.

**Proposition 3.10.** Assume that the dimension of the space of the solutions to Eq. (3.1), which are even doubly-periodic, is one. Let $c_0$, $b_j^{(i)}$ and $d_j^{(i)}$ be constants defined in Eq. (3.2).

(i) If there exists a non-zero solution to Eq. (2.3) in the space $\tilde{\mathcal{F}}_{\epsilon_1, \epsilon_3}$ for some $\epsilon_1, \epsilon_3 \in \{\pm 1\}$, then we have $Q = 0$.

(ii) If $Q \neq 0$ and $l_i \neq 0$, then $b_0^{(i)} \neq 0$.

(iii) If $Q \neq 0$ and $l_0 = 0$, then $\Xi(0) \neq 0$. In particular, if $Q \neq 0$ and $l_0 = l_1 = l_2 = l_3 = 0$, then $c_0 \neq 0$.

**Proof.** First we prove (i). Suppose that there exists a non-zero solution to Eq. (2.3) in the space $\tilde{\mathcal{F}}_{\epsilon_1, \epsilon_3}$ and $Q \neq 0$. From the condition $Q \neq 0$, the functions $\Lambda(x)$ and $\Lambda(-x)$ form the basis of the space of the solutions to the differential equation (2.3). Since there is a non-zero solution to Eq. (2.3) in the space $\tilde{\mathcal{F}}_{\epsilon_1, \epsilon_3}$, there exist constants $(C_1, C_2) \neq (0, 0)$ such that $C_1 \Lambda(x) + C_2 \Lambda(-x) \in \tilde{\mathcal{F}}_{\epsilon_1, \epsilon_3}$. By shifting $x \to x + 2\omega_i$ ($i = 1, 3$), it follows from Eq. (3.12) that

$$\sum_{i=1}^{3} (C_1 \Lambda(x + 2\omega_i) + C_2 \Lambda(-x + 2\omega_i))\Psi_g(x + 2\omega_i) = C_1 \Lambda(x + 2\omega_i)\Psi_g(x + 2\omega_i) + C_2 \Lambda(-x - 2\omega_i)\Psi_g(-x - 2\omega_i)$$

$$= C_1 \exp(\pi \sqrt{-1} m_1)\Lambda(x)\Psi_g(x) \pm C_2 \exp(-\pi \sqrt{-1} m_1)\Lambda(-x)\Psi_g(-x)$$

$$= (C_1 \exp(\pi \sqrt{-1} m_1)\Lambda(x) + C_2 \exp(-\pi \sqrt{-1} m_1)\Lambda(-x))\Psi_g(x),$$

where the sign ± is determined by the branching of the function $\Psi_g(x)$, and the function $C_1 \exp(\pi \sqrt{-1} m_1)\Lambda(x) + C_2 \exp(-\pi \sqrt{-1} m_1)\Lambda(-x)$ also satisfies Eq. (2.3). On the other hand, it follows from the definition of the space $\tilde{\mathcal{F}}_{\epsilon_1, \epsilon_3}$ that $(C_1 \Lambda(x + 2\omega_i) + C_2 \Lambda(-x + 2\omega_i))\Psi_g(x + 2\omega_i) = \pm (C_1 \Lambda(x) + C_2 \Lambda(-x))\Psi_g(x)$ for signs ±. By comparing two expressions, we have $\exp(\pi \sqrt{-1} m_1) \in \{\pm 1\}$ ($i = 1, 3$) and the periodicities of the functions $\Lambda(x)\Psi_g(x)$ and $(C_1 \Lambda(x) + C_2 \Lambda(-x))\Psi_g(x)$ coincide. Thus $\Lambda(x)$, $\Lambda(-x) \in \tilde{\mathcal{F}}_{\epsilon_1, \epsilon_3}$. The functions $\Lambda(x)^2$ and $\Lambda(-x)^2$ are even doubly-periodic.
function and satisfy Eq. (3.1), because they are the products of a pair of solutions to Eq. (2.3). Hence the dimension of the space of solutions to Eq. (2.3), which are even doubly-periodic, is one.

Next we show (ii). Assume that \( l_i \neq 0 \). Since the exponents of Eq. (2.3) at \( x = \omega_i \) are \( -l_i \) or \( l_i + 1 \), the function \( \Lambda(x) \) has a pole of degree \( l_i \) or a zero of degree \( l_i + 1 \) at \( x = \omega_i \). It follows from the periodicity (see Eq. (3.12)) that, if the function \( \Lambda(x) \) has a zero at \( x = \omega_i \), then \( \Lambda(x) \) has also a zero at \( x = -\omega_i \). Hence the function \( \Lambda(-x) \) has a zero at \( x = \omega_i \). From the assumption \( Q \neq 0 \), any solution to Eq. (2.3) is written as a linear combination of functions \( \Lambda(x) \) and \( \Lambda(-x) \). But it contradicts that one of the exponents at \( x = \omega_i \) is \( -l_i \). Hence the function \( \Lambda(x) \) has a pole of degree \( l_i \) and \( b_0(i) \neq 0 \).

(iii) is proved similarly by showing that the function \( \Lambda(x) \) does not have zero at \( x = 0 \).

By combining Propositions 3.7 and 3.10 (i) we obtain the following proposition:

**Proposition 3.11.** Assume that the dimension of the space of solutions to Eq. (2.3), which are even doubly-periodic, is one. Then the condition \( Q = 0 \) is equivalent to that there exists a non-zero solution to Eq. (2.3) in the space \( \mathcal{F}_{\epsilon_1, \epsilon_3} \) for some \( \epsilon_1, \epsilon_3 \in \{ \pm 1 \} \).

We show that the function \( \Lambda(x) \) admits an expression of the Bethe Ansatz type.

**Proposition 3.12.** Set \( l = \sum_{i=0}^{3} l_i + \sum_{i'=1}^{M} r_{i'}, \ l_i = l_0 + \sum_{i'=1}^{M} r_{i'} \) and \( \tilde{l}_i = l_i \ (i = 1, 2, 3) \). Assume that \( Q \neq 0 \) and the dimension of the space of the solutions to Eq. (2.3), which are even doubly-periodic, is one.

(i) The function \( \Lambda(x) \) in Eq. (3.4) is expressed as

\[
\Lambda(x) = \frac{C_0 \prod_{j=1}^{l} \sigma(x - t_j)}{\Psi_g(x) \sigma(x)^{l_0} \sigma_1(x)^{l_1} \sigma_2(x)^{l_2} \sigma_3(x)^{l_3}} \exp(cx),
\]

for some \( t_1, \ldots, t_l, c \) and \( C_0 \neq 0 \), where \( \sigma_i(x) \ (i = 1, 2, 3) \) are co-sigma functions.

(ii) \( t_j + t_j' \neq 0 \) (mod \( 2\omega_1 \mathbb{Z} \oplus 2\omega_3 \mathbb{Z} \)) for all \( j, j' \).

(iii) If \( t_j \neq \pm \delta_{i'} \) (mod \( 2\omega_1 \mathbb{Z} \oplus 2\omega_3 \mathbb{Z} \)) for all \( i' \in \{ 1, \ldots, M \} \), then we have \( t_j \neq t_{j'} \) (mod \( 2\omega_1 \mathbb{Z} \oplus 2\omega_3 \mathbb{Z} \)) for all \( j' \neq j \).

(iv) If \( t_j \equiv \pm \delta_{i'} \) (mod \( 2\omega_1 \mathbb{Z} \oplus 2\omega_3 \mathbb{Z} \)), then \# \( \{ j' \mid t_j \equiv t_{j'} \ (\text{mod} \ 2\omega_1 \mathbb{Z} \oplus 2\omega_3 \mathbb{Z}) \} = r_{i'} + 1 \).

(v) If \( l_0 \neq 0 \) (resp. \( l_0 = 0 \)), then we have \( c = \sum_{i=1}^{l} \zeta(t_j) \) (resp. \( c = \sum_{i=1}^{l} \zeta(t_j) + \sqrt{-Q/\Xi(0)} \)).

(Note that it follows from Proposition 3.10 that \( \sqrt{-Q/\Xi(0)} \) is finite.)

(vi) Set \( z = \varphi(x) \) and \( z_j = \varphi(t_j) \). Then

\[
\left. \frac{d\Xi(x)}{dz} \right|_{z=z_j} = \frac{2\sqrt{-Q}}{\varphi'(t_j)}.
\]

**Proof.** Let \( \alpha \) be the value defined in Eq. (3.19). First, we consider the case \( \alpha \neq 0 \) (mod \( 2\omega_1 \mathbb{Z} \oplus 2\omega_3 \mathbb{Z} \)). Let \( \kappa \) be the value defined in Eq. (3.20). Then the function \( \Lambda_g(x)/(\exp(\kappa x) \Phi_0(x, \alpha)) \) is meromorphic and doubly-periodic. Hence there exists \( a_1, \ldots, a_{\nu}, b_1, \ldots, b_{\nu} \) such that \( a_1 + \cdots + a_{\nu} = b_1 + \cdots + b_{\nu} \) and

\[
\Lambda_g(x)/(\exp(\kappa x) \Phi_0(x, \alpha)) = \frac{\prod_{i=1}^{\nu} \sigma(x - a_i)}{\prod_{i=1}^{\nu} \sigma(x - b_i)}.
\]
For the case $\alpha \equiv 0 \pmod{2\omega_1 \mathbb{Z} \oplus 2\omega_3 \mathbb{Z}}$ the function $\Lambda_g(x)/\exp(\bar{\kappa}x)$ is similarly expressed as

$$\Lambda_g(x)/\exp(\bar{\kappa}x) = \frac{\prod_{i=1}^l \sigma(x - a_i)}{\prod_{i=1}^{l'} \sigma(x - b_i)}.$$  

Since the function $\Lambda_g(x)$ satisfies Eq. (2.12), it does not have poles except for $\omega_1 \mathbb{Z} \oplus \omega_3 \mathbb{Z}$. From Proposition 3.10 (ii), it has poles at $x = \omega_i$ of degree $\bar{l}_i$. Hence we have the expression

$$\Lambda_g(x) = \frac{C_0 \prod_{j=1}^l \sigma(x - t_j)}{\sigma(x)\sigma_0(x)\sigma_1(x)\sigma_2(x)\sigma_3(x)} \exp(cx),$$

for some $t_1, \ldots, t_l$, $c$ and $C_0 (\neq 0)$ such that $t_j \not\equiv (0 \mod{\omega_1 \mathbb{Z} \oplus \omega_3 \mathbb{Z}})$. Therefore we obtain (i) and that $2t_j \not\equiv 0 \pmod{2\omega_1 \mathbb{Z} \oplus 2\omega_3 \mathbb{Z}}$.

Suppose that $t_j + t_{j'} \equiv 0 \pmod{2\omega_1 \mathbb{Z} \oplus 2\omega_3 \mathbb{Z}}$ for some $j$ and $j'$. From Eq. (3.32) and $-t_j \equiv t_{j'} \pmod{2\omega_1 \mathbb{Z} \oplus 2\omega_3 \mathbb{Z}}$, we have $\Lambda_g(t_j) = \Lambda_g(-t_j) = 0$. Since $Q \neq 0$, all solutions to Eq. (2.12) are written as linear combinations of $\Lambda_g(x)$ and $\Lambda_g(-x)$. Hence $t_j$ is a zero for all solutions to Eq. (2.12), but they contradict that one of the exponents at $x = t_j$ is zero. Therefore we obtain (ii).

If $t_j \not\equiv \pm \delta_{i'v}, \omega_i \pmod{2\omega_1 \mathbb{Z} \oplus 2\omega_3 \mathbb{Z}}$ for all $i$ and $i'$, then the exponents of Eq. (2.12) at $x = t_j$ are 0 and 1. Hence $x = t_j$ is a zero of $\Lambda_g(x)$ of degree one. Incidentally, the exponents of Eq. (2.12) at $x = \pm \delta_{i'v}$ are 0 and $r_{i'v} + 1$. Hence, if $t_j \equiv \pm \delta_{i'v} \pmod{2\omega_1 \mathbb{Z} \oplus 2\omega_3 \mathbb{Z}}$, then $x = t_j$ is a zero of $\Lambda_g(x)$ of degree $r_{i'v} + 1$. Thus we obtain (iii) and (iv).

It follows from Eq. (3.32) that

$$\Lambda'(x)/\Lambda(x) = c - \bar{l}_0 \sigma'(x)/\sigma(x) - \sum_{i=1}^3 \bar{l}_i \sigma'_i(x)/\sigma_i(x) + \sum_{j=1}^l \sigma'(x - t_j)/\sigma(x - t_j) - \sum_{k=1}^{M} r_{k'v} \varphi'(x)/2 \varphi(x) - \varphi(\delta_{i'v}).$$

By expanding Eq. (3.6) at $x = 0$ and observing coefficient of $x^0$, we obtain

$$c - \sum_{j=1}^l \zeta(t_j) = \frac{\sqrt{-Q}}{\Xi(x)} \bigg|_{x=0},$$

because the functions $\sigma'(x)/\sigma(x), \sigma'_i(x)/\sigma_i(x), \varphi'(x)/\varphi(x - \varphi(\delta_{i'v}))$ and $\Xi'(x)/\Xi(x)$ are odd and $\sigma'(-t)/\sigma(-t) = -\zeta(t)$. It follows from $Q \neq 0$ and Proposition 3.10 that, if $l_0 \neq 0$, then $\frac{\sqrt{-Q}}{\Xi(x)} \bigg|_{x=0} = 0$, and if $l_0 = 0$, then $\frac{\sqrt{-Q}}{\Xi(x)} \bigg|_{x=0}$ is finite. Thus we obtain (v).

We show (vi). The function $\Lambda(x)\Lambda(-x)$ is even doubly-periodic and satisfies Eq. (3.11), because it is a product of the solutions to Eq. (2.3). Since the dimension of the space of the solutions to Eq. (2.3), which are even doubly-periodic, is one, we have $\Xi(x) = C\Lambda(x)\Lambda(-x)$ for some non-zero constant $C$. Hence we have $\Xi(t_j) = \Xi(-t_j) = 0$. On the other hand, we have $\Lambda(-t_j) \neq 0$ from (ii). At $x = -t_j$, the l.h.s. of Eq. (3.10) is finite, and the denominator of the r.h.s. is zero. Therefore we have

$$\Xi'(x) \bigg|_{x=-t_j} + 2\sqrt{-Q} = 0.$$

By changing the variable $z = \varphi(x)$ and the oddness of the function $\varphi'(x)$, we obtain (vi).
4. The case $M = 1, r_1 = 1$ and Painlevé equation

We consider Eq. (2.12) for the case $M = 1, r_1 = 1$. For this case, Eq. (2.12) is written as
\begin{equation}
(H_g - \tilde{E}) f_g(x) = 0,
\end{equation}
where
\begin{equation}
H_g = -\frac{d^2}{dx^2} + \frac{\varphi'(x)}{\varphi(x) - \varphi(\delta_1)} \frac{d}{dx} + \frac{\tilde{s}_1}{\varphi(x) - \varphi(\delta_1)} + \sum_{i=0}^{3} l_i (l_i + 1) \varphi(x + \omega_i).
\end{equation}

We set
\begin{equation}
\Psi_g(x) = \sqrt{\varphi(x) - \varphi(\delta_1)}, \quad b_1 = \varphi(\delta_1),
\end{equation}
\begin{equation}
\mu_1 = \frac{-\tilde{s}_1}{4b_1^3 - g_2 b_1 - g_3} + \sum_{i=1}^{3} \frac{l_i}{2(b_1 - e_i)},
\end{equation}
\begin{equation}
p = \tilde{E} - 2(l_1 l_2 e_3 + l_2 l_3 e_1 + l_3 l_1 e_2) + \sum_{i=1}^{3} l_i (l_i e_i + 2(e_i + b_1)).
\end{equation}

The condition that, the regular singular points $x = \pm \delta_1$ is apparent, is written as
\begin{equation}
p = (4b_1^3 - g_2 b_1 - g_3) \left\{-\mu_1^2 + \sum_{i=1}^{3} \frac{l_i + \frac{1}{2}}{b_1 - e_i} \mu_1 \right\}
- b_1 (l_1 + l_2 + l_3 - l_0) (l_1 + l_2 + l_3 + l_0 + 1).
\end{equation}

From now on we assume that $l_0, l_1, l_2, l_3 \in \mathbb{Z}_{\geq 0}$ and the eigenvalue $\tilde{E}$ satisfies Eqs. (4.5, 4.6). Then the assumption in Proposition 3.1 is true, and propositions and theorem in the previous section are valid. The function $\Xi(x)$ in Proposition 3.1 is written as
\begin{equation}
\Xi(x) = c_0 + \frac{d_0}{(\varphi(x) - \varphi(\delta_1))} + \sum_{i=0}^{3} \sum_{j=0}^{l_i - 1} b_j^{(i)} \varphi(x + \omega_i)^{l_i-j}.
\end{equation}

It follows from Proposition 3.9 that the function $\Xi(x)$ is determined uniquely up to multiplicative constant. Ratios of the coefficients $c_0/d_0$ and $b_j^{(i)}/d_0$ ($i = 0, 1, 2, 3, j = 0, \ldots, l_i - 1$) are written as rational functions in variables $b_1$ and $\mu_1$, because the coefficients $b_j^{(i)}$, $c_0$ and $d_0$ satisfy linear equations whose coefficients are rational functions in $b_1$ and $\mu_1$, which are obtained by substituting Eq. (4.7) into Eq. (3.11). The value $Q$ is calculated by Eq. (3.3) and it is expressed as a rational function in $b_1$ and $\mu_1$ multiplied by $d_0^2$. We set
\begin{equation}
\Lambda(x) = \sqrt{\Xi(x)} \exp \int \frac{\sqrt{-Q} dx}{\Xi(x)}, \quad \Lambda_g(x) = \Lambda(x) \Psi_g(x).
\end{equation}

Due to Proposition 3.3, the function $\Lambda_g(x)$ is a solution to the differential equation (4.1). By Theorem 3.5, the eigenfunction $\Lambda_g(x)$ is also expressed in the form of the Hermite-Krichever Ansatz. Namely, it is expressed as
\begin{equation}
\Lambda_g(x) = \exp (\kappa x) \left( \sum_{i=0}^{3} \sum_{j=0}^{l_i - 1} b_j^{(i)} \left( \frac{d}{dx} \right)^j \Phi_i(x, \alpha) \right)
\end{equation}
satisfies Eq. (4.6). Let \( \alpha \) and \( \mu \) be the values determined by the Hermite-Krichever Ansatz (see Eq. (4.9)). Then \( \wp \) for integers

\[
(4.13) \quad \wp = \frac{\varphi(x + \omega_i)}{\varphi(x) - e_i}
\]

Proof. We assume that \( Q \neq 0, \) the proposition is shown by considering a continuation from the case \( Q = 0. \) Now we investigate

\[
(4.11) \quad \Lambda_g(x + 2\omega_j) = \exp(-2\eta_j\alpha + 2\omega_j(\zeta(\alpha) + \kappa\omega_j))\Lambda_g(x), \quad (j = 1, 3).
\]

Proposition 4.1. Assume that \( M = 1, r_1 = 1, l_0, l_1, l_2, l_3 \in \mathbb{Z}_{\geq 0} \) and the value \( p \) satisfies Eq. (4.9). Let \( \alpha \) and \( \kappa \) be the values determined by the Hermite-Krichever Ansatz (see Eq. (4.9)). Then \( \varphi(\alpha) \) is expressed as a rational function in variables \( b_1 \) and \( \mu_1, \varphi'(\alpha) \) is expressed as a product of \( \sqrt{-Q} \) and a rational function in variables \( b_1 \) and \( \mu_1, \) and \( \kappa \) is expressed as a product of \( \sqrt{-Q} \) and a rational function in variables \( b_1 \) and \( \mu_1. \)

\[
(4.12) \quad \Lambda_g(x + 2\omega_j) = \exp \left( 2\eta_j \left( -\sum_{i' = 1}^{l} t_{i'} + \sum_{i = 1}^{3} l_i\omega_i \right) + 2\omega_j \left( c - \sum_{i = 1}^{3} l_i\eta_i \right) \right) \Lambda_g(x)
\]

for \( j = 1, 3. \) By comparing with Eq. (3.17), we have

\[
(4.13) \quad -2\eta_1\alpha + 2\omega_1(\zeta(\alpha) + \kappa) = -2\eta_1 \left( \sum_{i' = 1}^{l} t_{i'} - \sum_{i = 1}^{3} l_i\omega_i \right) + 2\omega_1 \left( c - \sum_{i = 1}^{3} l_i\eta_i \right) + 2\pi\sqrt{-1}n_1,
\]

and

\[
(4.14) \quad -2\eta_3\alpha + 2\omega_3(\zeta(\alpha) + \kappa) = -2\eta_3 \left( \sum_{i' = 1}^{l} t_{i'} - \sum_{i = 1}^{3} l_i\omega_i \right) + 2\omega_3 \left( c - \sum_{i = 1}^{3} l_i\eta_i \right) + 2\pi\sqrt{-1}n_3
\]

for integers \( n_1, n_3. \) It follows that

\[
(4.15) \quad \left( \alpha - \left( \sum_{i' = 1}^{l} t_{i'} - \sum_{i = 1}^{3} l_i\omega_i \right) \right) (-2\eta_1\omega_1 + 2\eta_3\omega_1) = 2\pi\sqrt{-1}(n_1\omega_3 - n_3\omega_1),
\]

and

\[
(4.16) \quad \left( \zeta(\alpha) + \kappa - c + \sum_{i = 1}^{3} l_i\eta_i \right) (2\eta_2\omega_1 - 2\eta_1\omega_3) = 2\pi\sqrt{-1}(n_1\eta_3 - n_3\eta_1).
\]

From Legendre’s relation \( \eta_1\omega_3 - \eta_3\omega_1 = \pi\sqrt{-1}/2, \) we have

\[
(4.17) \quad \alpha \equiv \sum_{i' = 1}^{l} t_{i'} - \sum_{i = 1}^{3} l_i\omega_i \quad (\text{mod } 2\omega_1\mathbb{Z} \oplus 2\omega_3\mathbb{Z}).
\]
Combining Eqs. (4.15, 4.16) with Proposition 3.12 (v) and relations \( \zeta(\alpha + 2\omega_i) = \zeta(\alpha) + 2\eta_i \) \((i = 1, 3)\), we have

\[
(4.18) \quad \kappa = -\zeta \left( \sum_{j=1}^{l} t_j - \sum_{i=1}^{3} l_i \omega_i \right) + \sum_{j=1}^{l} \zeta(t_j) - \sum_{i=1}^{3} l_i \eta_i + \delta_{l_0,0} \sqrt{-Q} \Xi(0)
\]

Next, we investigate values \( \varphi(\alpha) \), \( \varphi'(\alpha) \) and \( \kappa \). The functions \( \varphi(\sum_{j=1}^{l} t_j - \sum_{i=1}^{3} l_i \omega_i) \), \( \varphi'(\sum_{j=1}^{l} t_j - \sum_{i=1}^{3} l_i \omega_i) \) and \( \zeta(\sum_{j=1}^{l} t_j - \sum_{i=1}^{3} l_i \omega_i) \) are doubly-periodic in variables \( t_1, \ldots, t_l \). Hence by applying addition formulae of elliptic functions and considering the parity of functions \( \varphi(x) \), \( \varphi'(x) \) and \( \zeta(x) \), we obtain the expression

\[
(4.19) \quad \varphi \left( \sum_{j=1}^{l} t_j - \sum_{i=1}^{3} l_i \omega_i \right) = \sum_{j_1 < j_2 < \cdots < j_m \text{ m: even}} f_{j_1,\ldots,j_m}^{(1)}(\varphi(t_1), \ldots, \varphi(t_l)) \varphi'(t_{j_1}) \ldots \varphi'(t_{j_m}),
\]

\[
\varphi' \left( \sum_{j=1}^{l} t_j - \sum_{i=1}^{3} l_i \omega_i \right) = \sum_{j_1 < j_2 < \cdots < j_m \text{ m: odd}} f_{j_1,\ldots,j_m}^{(2)}(\varphi(t_1), \ldots, \varphi(t_l)) \varphi'(t_{j_1}) \ldots \varphi'(t_{j_m}),
\]

\[
\zeta \left( \sum_{j=1}^{l} t_j - \sum_{i=1}^{3} l_i \omega_i \right) = \sum_{j_1 < j_2 < \cdots < j_m \text{ m: odd}} f_{j_1,\ldots,j_m}^{(3)}(\varphi(t_1), \ldots, \varphi(t_l)) \varphi'(t_{j_1}) \ldots \varphi'(t_{j_m}),
\]

where \( f_{j_1,\ldots,j_m}^{(i)}(x_1, \ldots, x_l) \) \((i = 1, 2, 3)\) are rational functions in \( x_1, \ldots, x_l \). From Eq. (3.33), the function \( \varphi'(t_j)/\sqrt{-Q} \) is expressed as a rational function in \( b_1, \mu_1 \) and \( \varphi(t_j) \). Hence, \( \varphi(\sum_{j=1}^{l} t_j - \sum_{i=1}^{3} l_i \omega_i), \varphi'(\sum_{j=1}^{l} t_j - \sum_{i=1}^{3} l_i \omega_i)/\sqrt{-Q} \) and \( \zeta(\sum_{j=1}^{l} t_j - \sum_{i=1}^{3} l_i \omega_i) - \sum_{j=1}^{l} \zeta(t_j) + \sum_{i=1}^{3} l_i \eta_i) /\sqrt{-Q} \) are expressed as rational functions in the variable \( \varphi(t_1), \ldots, \varphi(t_l), b_1 \) and \( \mu_1 \), and they are symmetric in \( \varphi(t_1), \ldots, \varphi(t_l) \). Since the dimension of the space of the solutions to Eq. (2.3), which are even doubly-periodic, is one, we have \( \Xi(x) = CA(x)A(-x) \) for some non-zero scalar \( C \). Hence, we have the following expression;

\[
(4.20) \quad \Xi(x) \Psi_g(x)^2 = \frac{D \prod_{j=1}^{l} (\varphi(x) - \varphi(t_j))}{(\varphi(x) - e_1)^{l_1}(\varphi(x) - e_2)^{l_2}(\varphi(x) - e_3)^{l_3}}
\]

for some value \( D(\neq 0) \). Thus

\[
(4.21) \quad \prod_{j=1}^{l} (\varphi(x) - \varphi(t_j)) = \Xi(x) \Psi_g(x)^2 (\varphi(x) - e_1)^{l_1}(\varphi(x) - e_2)^{l_2}(\varphi(x) - e_3)^{l_3}/D.
\]

Hence, the elementary symmetric functions \( \sum_{j_1 < \cdots < j_{l'}} \varphi(t_{j_1}) \cdots \varphi(t_{j_{l'}}) \) \((l' = 1, \ldots, l)\) are expressed as rational functions in \( b_1 \) and \( \mu_1 \). By substituting elementary symmetric functions into the symmetric expressions of \( \varphi(\sum_{j=1}^{l} t_j - \sum_{i=1}^{3} l_i \omega_i), \varphi'(\sum_{j=1}^{l} t_j - \sum_{i=1}^{3} l_i \omega_i) \) and \( \zeta(\sum_{j=1}^{l} t_j - \sum_{i=1}^{3} l_i \omega_i) - \sum_{j=1}^{l} \zeta(t_j) + \sum_{i=1}^{3} l_i \eta_i) /\sqrt{-Q} \), it follows that \( \varphi'(\sum_{j=1}^{l} t_j - \sum_{i=1}^{3} l_i \omega_i), \varphi'(\sum_{j=1}^{l} t_j - \sum_{i=1}^{3} l_i \omega_i)/\sqrt{-Q} \) and \( \zeta(\sum_{j=1}^{l} t_j - \sum_{i=1}^{3} l_i \omega_i) - \sum_{j=1}^{l} \zeta(t_j) + \sum_{i=1}^{3} l_i \eta_i) /\sqrt{-Q} \) are expressed as rational functions in \( b_1 \) and \( \mu_1 \). Hence,
\( \frac{d\lambda}{dt} = \frac{1}{2} \left( \frac{1}{\lambda} + \frac{1}{\lambda - 1} + \frac{1}{\lambda - t} \right) \left( \frac{d\lambda}{dt} \right)^2 - \left( \frac{1}{t} + \frac{1}{t - 1} + \frac{1}{\lambda - t} \right) \frac{d\lambda}{dt} \)

A remarkable property of this differential equation is that its solutions do not have movable singularities other than poles. This equation is also written in terms of a Hamiltonian system by adding the variable \( \mu \), which is called the sixth Painlevé system:

\[
\frac{d\lambda}{dt} = \frac{\partial H_{VI}}{\partial \mu}, \quad \frac{d\mu}{dt} = -\frac{\partial H_{VI}}{\partial \lambda}
\]

with the Hamiltonian

\[
H_{VI} = \frac{1}{t(t - 1)} \left\{ \lambda(\lambda - 1)(\lambda - t) \mu^2 - \{\kappa_0(\lambda - 1)(\lambda - t) + \kappa_1 \lambda(\lambda - t) + (\kappa_t - 1)\lambda(\lambda - 1)\} \mu + \kappa(\lambda - t) \right\},
\]

where \( \kappa = ((\kappa_0 + \kappa_1 + \kappa_t - 1)^2 - \kappa^2_\infty)/4 \). The sixth Painlevé equation for \( \lambda \) is obtained by eliminating \( \mu \) in Eq. \( \text{(4.23)} \). Set \( \omega_1 = 1/2, \omega_3 = \tau/2 \) and write

\[
t = \frac{e_3 - e_1}{e_2 - e_1}, \quad \lambda = \frac{\varphi(\delta) - e_1}{e_2 - e_1}.
\]

Then the sixth Painlevé equation is equivalent to the following equation (see [7,12]):

\[
\frac{d^2\delta}{d\tau^2} = -\frac{1}{4\pi^2} \left\{ \frac{\kappa^2_\infty}{2} \varphi'(\delta) + \frac{\kappa^2_0}{2} \varphi' \left( \delta + \frac{1}{2} \right) + \frac{\kappa^2_1}{2} \varphi' \left( \delta + \frac{\tau + 1}{2} \right) + \frac{\kappa^2_t}{2} \varphi' \left( \delta + \frac{\tau}{2} \right) \right\},
\]

where \( \varphi'(z) = (\partial/\partial z) \varphi(z) \).

It is widely known that the sixth Painlevé equation is obtained by the monodromy preserving deformation of a certain linear differential equation. Let us introduce the following Fuchsian differential equation:

\[
\frac{d^2y}{dw^2} + p_1(w) \frac{dy}{dw} + p_2(w)y = 0,
\]

where

\[
p_1(w) = \frac{1 - \kappa_0}{w} + \frac{1 - \kappa_1}{w - 1} + \frac{1 - \kappa_t}{w - t} - \frac{1}{w - \lambda},
\]

\[
p_2(w) = \frac{\kappa}{w(w - 1)} - \frac{t(t - 1)H_{VI}}{w(w - 1)(w - t)} + \frac{\lambda(\lambda - 1)\mu}{w(w - 1)(w - \lambda)}.
\]

This equation has five regular singular points \( \{0, 1, \tau, \infty, \lambda\} \) and the exponents at \( w = \lambda \) are 0 and 2. It follows from Eq. \( \text{(4.24)} \) that the regular singular point \( w = \lambda \) is
apparent. Then the sixth Painlevé equation is obtained by the monodromy preserving deformation of Eq. (4.23), i.e., the condition that the monodromy of Eq. (4.27) is preserved as deforming the variable $t$ is equivalent to that $\mu$ and $\lambda$ satisfy the Painlevé system (see Eq. (4.23)), provided $\kappa_0, \kappa_1, \kappa_t, \kappa_{\infty} \notin \mathbb{Z}$. For details, see [6].

Now we transform Eq. (4.27) into the form of Eq. (4.2). We set

$$w = \frac{\varphi(x) - e_1}{e_2 - e_1}, \quad y = f_g(x) \prod_{i=1}^{3} (\varphi(x) - e_i)^{l_i/2},$$

(4.30)

$$t = \frac{e_3 - e_1}{e_2 - e_1}, \quad \lambda = \frac{b_1 - e_1}{e_2 - e_1}, \quad \varphi(\delta_1) = b_1.$$ (4.31)

Then we obtain Eq. (4.2) by setting

$$\kappa_0 = l_1 + 1/2, \quad \kappa_1 = l_2 + 1/2, \quad \kappa_t = l_3 + 1/2, \quad \kappa_{\infty} = l_0 + 1/2,$$ (4.32)

$$\mu = (e_2 - e_1)\mu_1, \quad \kappa = (l_1 + l_2 + l_3 + l_0 + 1)(l_1 + l_2 + l_3 - l_0),$$ (4.33)

$$H_{VI} = \frac{1}{t(1-t)} \left\{ \frac{p + \kappa e_3}{e_2 - e_1} + \lambda(1 - \lambda)\mu \right\},$$ (4.34)

(see Eqs. (4.31, 4.32)), and Eq. (4.24) is equivalent to Eq. (4.6), that means that the appearance of regular singularity is inherited. Mapping from the variable $x$ to the variable $w$ (see Eq. (4.30)) is a double covering from the punctured torus $(\mathbb{C}/(2\omega_1\mathbb{Z} + 2\omega_2\mathbb{Z})) \setminus \{0, \omega_1, \omega_2, \omega_3\}$ to the punctured Riemann sphere $\mathbb{P}^1 \setminus \{0, 1, t, \infty\}$. A solution $y(w)$ to Eq. (4.27) corresponds to a solution $f_g(x)$ to Eq. (4.2) by $y(w) = f_g(x) \prod_{i=1}^{3} (\varphi(x) - e_i)^{l_i/2}$. Hence the monodromy preserving deformation of Eq. (4.27) in $t$ corresponds to the monodromy preserving deformation of Eq. (4.2) in $\tau$.

Now we consider monodromy preserving deformation in the variable $\tau$ ($\omega_1 = 1/2, \omega_3 = \tau = 2/\mathcal{Q}$) by applying solutions obtained by the Hermite-Krichever Ansatz for the case $l_i \in \mathbb{Z}_{\geq 0}$ ($i = 0, 1, 2, 3$). Let $\alpha$ and $\kappa$ be values determined by the Hermite-Krichever Ansatz (see Eq. (4.9)). We consider the case $Q \neq 0$. Then a basis for solutions to Eq. (2.12) is given by $\Lambda_g(x)$ and $\Lambda_g(-x)$, and the monodromy matrix with respect to the cycle $x \rightarrow x + 2\omega_j$ ($j = 1, 3$) is diagonal. The elements of the matrix are obtained from Eq. (4.11). Hence, the eigenvalues $\exp(\pm(2\eta_j\alpha + 2\omega_j\zeta(\alpha) + 2\kappa\omega_j))$ ($j = 1, 3$) of the monodromy matrices are preserved by the monodromy preserving deformation. We set

$$-2\eta_1\alpha + 2\omega_1\zeta(\alpha) + 2\kappa\omega_1 = \pi\sqrt{-1}C_1,$$ (4.35)

$$-2\eta_3\alpha + 2\omega_3\zeta(\alpha) + 2\kappa\omega_3 = \pi\sqrt{-1}C_3,$$ (4.36)

for constants $C_1$ and $C_3$. By Legendre’s relation, we have

$$\alpha = C_3\omega_1 - C_1\omega_3,$$ (4.37)

$$\kappa = \zeta(C_1\omega_3 - C_3\omega_1) + C_3\eta_1 - C_1\eta_3,$$ (4.38)

(see Eqs. (3.19, 3.20)). From Proposition 4.1, the value $\varphi(\alpha) = \varphi(C_3\omega_1 - C_1\omega_3))$ is expressed as a rational function in variables $b_1$ and $\mu_1$, the value $\varphi(\alpha) = \varphi(C_3\omega_1 - C_1\omega_3))$ is expressed as a product of $\sqrt{-Q}$ and a rational function in variables $b_1$ and $\mu_1$, and the value $\kappa = \zeta(C_1\omega_3 - C_3\omega_1) + C_3\eta_1 - C_1\eta_3) \neq 0$. For details, see [6].
be the path in the $x$-plane which is obtained by the pullback of the cycle turning
the origin around anti-clockwise in the $w$-plane, where $x$ and $w$ are related with
$w = (\varphi(x) - e_1)/(e_2 - e_1)$. Then the monodromy matrix on $\gamma_0$ with respect to the
basis $(\Lambda_g(x), \Lambda_g(-x))$ is written as

$$
(\Lambda_g(x), \Lambda_g(-x)) \rightarrow (\Lambda_g(-x), \Lambda_g(x)) = (\Lambda_g(x), \Lambda_g(-x)) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
$$

(4.39)

and does not depend on $\tau$. Since the fundamental group on the punctured Riemann
sphere $\mathbb{P}^1 \setminus \{0, 1, t, \infty\}$ is generated by the images of $\gamma_0$ and the cycles $x \rightarrow x + 2\omega_i$
($i = 1, 3$), Eqs. (4.37, 4.38) describe the condition for the monodromy preserving
deformation on the punctured Riemann sphere by rewriting the variable $\tau$ to $t$. Sum-
marizing, we have the following proposition.

**Proposition 4.2.** We set $\omega_1 = 1/2$, $\omega_3 = \tau/2$ and assume that $l_i \in \mathbb{Z}_{\geq 0}$ ($i = 0, 1, 2, 3$) and $Q \neq 0$. By solving the equations in Proposition 4.1 in variable $b_1 = \varphi(\delta_1)$ and $\mu_1$, we express $\varphi(\delta_1)$ and $\mu_1$ in terms of $\varphi(\alpha)$, $\varphi'(\alpha)$ and $\kappa$, and we replace $\varphi(\alpha)$, $\varphi'(\alpha)$
and $\kappa$ with $\varphi(C_3\omega_1 - C_1\omega_3)$, $\varphi'(C_3\omega_1 - C_1\omega_3)$ and $\zeta(C_3\omega_3 - C_3\omega_1) + C_3\eta_1 - C_1\eta_3$.
Then $\delta_1$ satisfies the sixth Painlevé equation in the elliptic form

$$
d^2\delta_1 \over d\tau^2 = -\frac{1}{8\pi^2} \left\{ \sum_{i=0}^{3} (l_i + 1/2)^2 \varphi'(\delta_1 + \omega_i) \right\}.
$$

(4.40)

We observe the expressions of $b_1$ and $\mu_1$ in detail for the cases $l_0 = l_1 = l_2 = l_3 = 0$
and $l_0 = 1$, $l_1 = l_2 = l_3 = 0$.

4.1. **The case** $M = 1$, $r_1 = 1$, $l_0 = l_1 = l_2 = l_3 = 0$. We investigate the case $M = 1$,
$r_1 = 1$, $l_0 = l_1 = l_2 = l_3 = 0$ in detail. The differential equation (4.1) is written as

$$
\left\{ -\frac{d^2}{dx^2} + \frac{\varphi'(x)}{\varphi(x) - b_1} \frac{d}{dx} - \frac{\mu_1(4b_1^3 - g_2b_1 - g_3)}{\varphi(x) - b_1} - p \right\} f_1(x) = 0,
$$

(4.41)

We assume that $b_1 \neq e_1, e_2, e_3$. The condition that the regular singular points $x = \pm \delta_1$
($\varphi(\delta_1) = b_1$) are apparent is written as

$$
p = -(4b_1^3 - g_2b_1 - g_3)\mu_1^2 + (6b_1^2 - g_2/2)\mu_1
$$

(see Eq. (4.6)). The doubly-periodic function $\Xi(x)$ (see Eq. (4.7)) which satisfies
Eq. (3.11) is calculated as

$$
\Xi(x) = 2\mu_1 + \frac{1}{\varphi(x) - b_1}.
$$

(4.43)

The value $Q$ (see Eq. (3.3)) is calculated as

$$
Q = 2\mu_1(2\mu_1(e_1 - b_1) + 1)(2(e_2 - b_1)\mu_1 + 1)(2\mu_1(e_3 - b_1) + 1).
$$

(4.44)

We set

$$
\Lambda_g(x) = \sqrt{\Xi(x)(\varphi(x) - b_1)} \exp \int \frac{\sqrt{-Q} \, dx}{\Xi(x)},
$$

(see Eq. (1.8)). Then a solution to Eq. (4.41) is written as $\Lambda_g(x)$, and is expressed in
the form of the Hermite-Krichever Ansatz as

$$
\Lambda_g(x) = \exp(kx)\Phi_0(x, \alpha).
$$

(4.46)
for generic \((\mu_1, b_1)\). The values \(\alpha\) and \(\kappa\) are determined as

\[
\varphi(\alpha) = b_1 - \frac{1}{2\mu_1}, \quad \varphi'(\alpha) = -\frac{\sqrt{-Q}}{2\mu_1^2}, \quad \kappa = \frac{\sqrt{-Q}}{2\mu_1}.
\]

Hence we have

\[
\mu_1 = -\frac{\kappa}{\varphi'(\alpha)}, \quad b_1 = \varphi(\alpha) - \frac{\varphi'(\alpha)}{2\kappa}.
\]

From Proposition 4.2, the function \(\delta_1\) determined by

\[
\varphi(\delta_1) = b_1 = \varphi(C_3\omega_1 - C_1\omega_3) - \frac{\varphi'(C_3\omega_1 - C_1\omega_3)}{2(\zeta(C_1\omega_3 - C_3\omega_1) - C_1\eta_3 + C_3\eta_1)}
\]

\[
= \varphi(C_1\omega_3 - C_3\omega_1) + \frac{\varphi'(C_1\omega_3 - C_3\omega_1)}{2(\zeta(C_1\omega_3 - C_3\omega_1) - (C_1\eta_3 - C_3\eta_1))}
\]

is a solution to the sixth Painlevé equation in the elliptic form (see Eq. (4.40)). This solution coincides with the one found by Hitchin [4] when he studied Einstein metrics and isomonodromy deformations.

Now we consider the case \(Q = 0\). If \(Q = 0\), then \(\mu_1 = 0\) or \(\mu_1 = 1/(2(b_1 - e_i))\) for some \(i \in \{1, 2, 3\}\).

If \(\mu_1 = 0\), then a solution to Eq. (4.41) is \(1 = \Lambda_g(x)\) and another solution is written as

\[
\zeta(x) + b_1 x = \int - (\varphi(x) - b_1) dx.
\]

We investigate the monodromy preserving deformation on the basis \(s_1(x) = B(\tau)\) and \(s_2(x) = \zeta(x) + b_1 x\), where \(B(\tau)\) is a constant that is independent of \(x\). The monodromy matrix with respect to the path \(\gamma_0\) is written as \(\text{diag}(1, -1)\). Since \(s_2(x + 2\omega_i) = s_2(x) + 2(\eta_i + \omega_i b_1)\) \((i = 1, 3)\), the monodromy matrix with respect to the basis \((s_1(x), s_2(x))\) on the cycle \(x \rightarrow x + 2\omega_1\) \((i = 1, 3)\) is written as

\[
\begin{pmatrix}
1 & \frac{2(\eta_i + \omega_i b_1)}{B(\tau)} \\
0 & \frac{1}{B(\tau)}
\end{pmatrix}.
\]

To preserve monodromy, the matrix elements should be constants of the variable \(\tau = \omega_3/\omega_1\). Hence we obtain

\[
2(\eta_1 + \omega_1 b_1) = D_1 B(\tau),
\]

\[
2(\eta_3 + \omega_3 b_1) = D_3 B(\tau),
\]

for some constants \(D_1\) and \(D_3\). By using Legendre’s relation, we obtain that \(B(\tau) = \pi \sqrt{-1}/(D_1\omega_3 - D_3\omega_1)\) and

\[
\varphi(\delta_1) = b_1 = \frac{D_1\eta_3 - D_3\eta_1}{D_1\omega_3 - D_3\omega_1}.
\]

Since Eq. (4.53) is obtained by monodromy preserving deformation, the function \(\delta_1\) satisfies the sixth Painlevé equation.

If \(\mu_1 = 1/(2(b_1 - e_i))\) for some \(i \in \{1, 2, 3\}\), then \(\phi_i(x) = \Lambda_g(x)\) is a solution to Eq. (4.41), and another solution is written as

\[
\phi_i(x) \left\{ \frac{e_i - b_1}{(e_i - e_{ii})(e_i - e_{ij})} \zeta(x + \omega_i) + (1 - \frac{e_i - b_1}{(e_i - e_{ii})(e_i - e_{ij})}) x \right\} = \phi_i(x) \int \frac{\phi(x) - b}{\phi(x) - e_i} dx.
\]
where \( i' \) and \( i'' \) are elements in \( \{1, 2, 3\} \) such that \( i' \neq i, \, i'' \neq i \) and \( i' < i'' \). By calculating similarly to the case \( \mu = 0 \), we obtain that the function \( \delta_1 \), which is determined by

\[
\varphi(\delta_1) = b_1 = \frac{(g_2/4 - 2e_1^2)(D_1\omega_3 - D_3\omega_1) + e_1(D_1\eta_3 - D_3\eta_1)}{e_1(D_1\omega_3 - D_3\omega_1) + (D_1\eta_3 - D_3\eta_1)},
\]

is a solution to the sixth Painlevé equation for constants \( D_1 \) and \( D_3 \).

We now show that Eqs. (4.53, 4.55) are obtained by suitable limits from Eq. (4.49).

Set \( (C_1, C_3) = (CD_1, CD_3) \) in Eq. (4.49) and consider the limit \( C \to 0 \), then we recover Eq. (4.53). Similarly, set \( (C_1, C_3) = (CD_1, -1 + CD_3) \) (resp. \( (C_1, C_3) = (-1 + CD_1, 1 + CD_3) \)) and consider the limit \( C \to 0 \), then we recover Eq. (4.55) for the case \( i = 1 \) (resp. \( i = 2, i = 3 \)). Hence the space of the parameters of the solutions to the sixth Painlevé equation (i.e. the space of initial conditions) for the case \( l_0 = l_1 = l_2 = l_3 = 0 \) is obtained by blowing up four points on the surface \( \mathbb{C}/(2\pi\sqrt{-1}\mathbb{Z}) \times \mathbb{C}/(2\pi\sqrt{-1}\mathbb{Z}) \), and this reflects the \( A_1 \times A_1 \times A_1 \times A_1 \) structure of Riccati solutions by Saito and Terajima [9].

4.2. The case \( M = 1, \, r_1 = 1, \, l_0 = 1, \, l_1 = l_2 = l_3 = 0 \). The differential equation (4.1) for this case is written as

\[
\begin{align*}
\left\{ -\frac{d^2}{dx^2} + \frac{\varphi'(x)}{\varphi(x)} \frac{d}{dx} - \frac{\mu_1(4b_1^3 - g_2b_1 - g_3)}{\varphi(x)} - b_1 \right\} f(x) = 0,
\end{align*}
\]

We assume that \( b_1 \neq e_1, e_2, e_3 \). The condition that the regular singular points \( x = \pm \delta_1 \) (\( \varphi(\delta_1) = b_1 \)) are apparent is written as

\[
\begin{align*}
p &= -(4b_1^3 - g_2b_1 - g_3)\mu_1^2 + (6b_1^2 - g_2/2)\mu_1 + 2b_1
\end{align*}
\]

(see Eq. (4.6)). The doubly-periodic function \( \Xi(x) \) (see Eq. (4.7)), which satisfies Eq. (3.1), is calculated as

\[
\begin{align*}
\Xi(x) &= \varphi(x) + ((-4b_1^3 + b_1g_2 + g_3)\mu_1^2 + (6b_1^2 - g_2/2)\mu_1) \mu_1
\end{align*}
\]

\[
+ ((-4b_1^3 + b_1g_2 + g_3)\mu_1^2 + 3b_1^2 - g_2/4)/(\varphi(x) - b_1)
\]

The value \( Q \) (see Eq. (3.3)) is calculated as

\[
\begin{align*}
Q &= -((2(4b_1^3 - b_1g_2 - g_3)\mu_1^3 - (12b_1^2 - g_2)\mu_1^2 + 4)(2(b_1^2 + e_1b_1 + e_2e_3)\mu_1 - 2b_1 - e_1)
\end{align*}
\]

\[
+ (2(b_1^2 + e_2b_1 + e_1e_2)\mu_1 - 2b_1 - e_2)(2(b_1^2 + e_3b_1 + e_1e_3)\mu_1 - 2b_1 - e_3).
\]

We set

\[
\Lambda_g(x) = \sqrt{\Xi(x)}(\varphi(x) - b_1) \exp \int \frac{\sqrt{-Q} dx}{\Xi(x)},
\]

(see Eq. (4.8)). Then a solution to Eq. (4.41) is written as \( \Lambda_g(x) \), and it is expressed in the form of the Hermite-Krichever Ansatz as

\[
\Lambda_g(x) = \exp(\kappa x) \left\{ \Phi_0(x, \alpha) + \frac{d}{dx} \Phi_0(x, \alpha) \right\}
\]
for generic \((\mu_1, b_1)\). The values \(\alpha\) and \(\kappa\) are determined as
\[
\varphi(\alpha) = \frac{2(4b_1^3 - b_1g_2 - g_3)b_1\mu_1^3 + (-24b_1^2 + 4g_2b_1 + 3g_3)\mu_1^2 + (24b_1^2 - 2g_2)\mu_1 - 8b_1}{2(4b_1^3 - b_1g_2 - g_3)\mu_1^3 - (12b_1^2 - g_2)\mu_1^2 + 4},
\]
\[
\varphi'(\alpha) = \frac{-4((4b_1^3 - b_1g_2 - g_3)\mu_1^3 - (12b_1^2 - g_2)\mu_1^2 + 12b_1\mu_1 - 4)\sqrt{-Q}}{(2(4b_1^3 - b_1g_2 - g_3)\mu_1^3 - (12b_1^2 - g_2)\mu_1^2 + 4)^2},
\]
\[
\kappa = \frac{2\mu_1}{2(4b_1^3 - b_1g_2 - g_3)\mu_1^3 - (12b_1^2 - g_2)\mu_1^2 + 4\sqrt{-Q}}.
\]
Hence we have
\[
b_1 = \frac{2\varphi(\alpha)\kappa^3 - 3\varphi'(\alpha)\kappa^2 + (6\varphi(\alpha)^2 - g_2)\kappa - \varphi(\alpha)\varphi'(\alpha)}{2(\kappa^3 - 3\varphi(\alpha)\kappa + \varphi'(\alpha))},
\]
\[
\mu_1 = \frac{-2\varphi'(\alpha)\kappa^3 + (12\varphi(\alpha)^2 - g_2)\kappa^2 - 6\varphi(\alpha)\varphi'(\alpha)\kappa + \varphi'(\alpha)^2}{2(\kappa^3 - 3\varphi(\alpha)\kappa + \varphi'(\alpha))\kappa}.
\]
From Proposition 4.2 the function \(\delta_1\) determined by
\[
\varphi(\delta_1) = b_1 = \frac{2\varphi(\omega)(\zeta(\omega) - \eta)^3 + 3\varphi'(\omega)(\zeta(\omega) - \eta)^2 + (6\varphi(\omega)^2 - g_2)(\zeta(\omega) - \eta) + \varphi(\omega)\varphi'(\omega)}{2((\zeta(\omega) - \eta)^3 - 3\varphi(\omega)(\zeta(\omega) - \eta) - \varphi'(\omega))},
\]
\[
(\omega = C_1\omega_3 - C_3\omega_1, \quad \eta = C_1\eta_3 - C_3\eta_1),
\]
is a solution to the sixth Painlevé equation in the elliptic form (see Eq. (4.40)). In the sixth Painlevé equation, it is known that the case \((\kappa_0, \kappa_1, \kappa_i, \kappa_\infty) = (1/2, 1/2, 1/2, 3/2)\) is linked to the case \((\kappa_0, \kappa_1, \kappa_i, \kappa_\infty) = (1/2, 1/2, 1/2, 1/2)\) by Bäcklund transformation. For a table of Bäcklund transformation of the sixth Painlevé equation, see [20]. By transforming the solution in Eq. (4.49) of the case \((\kappa_0, \kappa_1, \kappa_i, \kappa_\infty) = (1/2, 1/2, 1/2, 1/2)\) to the one of the case \((\kappa_0, \kappa_1, \kappa_i, \kappa_\infty) = (1/2, 1/2, 1/2, 3/2)\), we recover the solution in Eq. (4.67).

Now we consider the case \(Q = 0\). If \(Q = 0\), then \(\mu_1\) is a solution to the equation
\[
2(4b_1^3 - b_1g_2 - g_3)\mu_1^3 - (12b_1^2 - g_2)\mu_1^2 + 4 = 0
\]
or
\[
\mu_1 = \frac{(2b_1 + e_i)/(2(b_i^2 + e_ib_1 + e_i^2 - g_2/4))}{\text{for some } i \in \{1, 2, 3\}}
\]
for some \(i \in \{1, 2, 3\}\). We set \(\omega = D_1\omega_3 - D_3\omega_1\) and \(\eta = D_1\eta_3 - D_3\eta_1\), where \(D_1\) and \(D_3\) are constants. For the case that \(\mu_1\) is a solution to the equation
\[
2(4b_1^3 - b_1g_2 - g_3)\mu_1^3 - (12b_1^2 - g_2)\mu_1^2 + 4 = 0,
\]
the corresponding solutions to the sixth Painlevé equation are written as the function \(\delta_1\), where
\[
\varphi(\delta_1) = b_1 = \frac{4\eta^3 + g_2\omega^2\eta - 2g_3\omega^3}{\omega(g_2\omega^2 - 2\eta^2)}.
\]
For the case \(\mu_1 = (2b_1 + e_i)/(2(b_i^2 + e_ib_1 + e_i^2 - g_2/4))\) (\(i \in \{1, 2, 3\}\), we have
\[
\varphi(\delta_1) = b_1 = \frac{-g_2e_i\omega/2 + (6e_i^2 - g_2)\eta}{(6e_i^2 - g_2)\omega - 6e_i\eta}.
\]
Note that these solutions are also obtained by suitable limits from Eq. (4.67), and Eq. (4.68) (resp. Eq. (4.69)) is transformed by Bäcklund transformation from Eq. (4.68) (resp. Eq. (4.69)).
5. Relationship with finite-gap potential

5.1. The case $M = 0$ and Heun’s equation. For the case $M = 0$, Eq. (2.1) is transformed to Heun’s equation, and the potential of the operator $H(= -\frac{d^2}{dx^2} + v(x))$ (see Eq. (2.4)) is written as

\begin{equation}
 v(x) = \sum_{i=0}^{3} l_i (l_i + 1) \wp(x + \omega_i).
\end{equation}

If $l_0, l_1, l_2, l_3 \in \mathbb{Z}_{\geq 0}$, then the function $v(x)$ in Eq. (5.1) is called the Treibich-Verdier potential, and is an example of algebro-geometric finite-gap potential (see [19, 2, 10, 15]). Recall that, if there exists an odd-order differential operator

\begin{equation}
 A = \left( \frac{d}{dx} \right)^{2g + 1} + \sum_{j=0}^{2g-1} b_j(x) \left( \frac{d}{dx} \right)^{2g-1-j}
\end{equation}

such that

\begin{equation}
 \left[ A, -\frac{d^2}{dx^2} + v(x) \right] = 0,
\end{equation}

then $v(x)$ is called an algebro-geometric finite-gap potential. For the case $M = 0$ and $l_0, l_1, l_2, l_3 \in \mathbb{Z}_{\geq 0}$, statements in Propositions 3.1, 3.3, 3.4, 3.10, 3.11, 3.12 and Theorem 3.5 in this paper hold true, because there is no constraint relation for the appearance of additional regular singularity. Moreover, the function $\Xi(x)$ in Proposition 3.1 is written as

\begin{equation}
 \Xi(x) = E^g + \sum_{i=1}^{g} a_{g-i}(x) E^i,
\end{equation}

for some $g \in \mathbb{Z}_{\geq 1}$ and even doubly-periodic functions $a_i(x)$ ($i = 0, \ldots, g-1$), and the constant $Q$ is a monic polynomial in $E$ of degree $2g + 1$ (see [13]). The commuting operator $A$ is described by using functions $a_i(x)$ ($i = 1, \ldots, g$). The eigenfunction $\Lambda(x)$ (see Eq. (3.2)) of the operator $-\frac{d^2}{dx^2} + v(x)$ is expressed in a form of the Hermite-Krichever Ansatz (see Theorem 3.5). It is shown in [16] that the values $\wp(\alpha)/\sqrt{-Q}$ and $\kappa/\sqrt{-Q}$ are expressed as a rational function in $E$, and it follows that the global monodromy of Heun’s equation for the case $l_0, l_1, l_2, l_3 \in \mathbb{Z}_{\geq 0}$ is written as an elliptic integral. On the other hand it is known that global monodromy is also expressed by a hyperelliptic integral (see [15]). By comparing the two expressions, we obtain a hyperelliptic-to-elliptic integral reduction formula (see [16]).

5.2. The case $M = 1$ and $r_1 = 2$. For the case $M = 1$ and $r_1 = 2$, the differential equation in the elliptic form is written as

\begin{equation}
 \left( -\frac{d^2}{dx^2} + v(x) - E \right) f(x) = 0,
\end{equation}

where $v(x)$ is written as

\begin{equation}
 v(x) = 2(\wp(x - \delta_1) + \wp(x + \delta_1)) + \frac{s_1}{\wp(x) - \wp(\delta_1)} + \sum_{i=0}^{3} l_i (l_i + 1) \wp(x + \omega_i).
\end{equation}
If \( x \) is an odd degree in Heun’s equation in [15]. In fact, a commuting operator of an odd degree can be reported in [17].

Thus, if we restrict our discussion to the case \( s_1 = 0 \) and that \( b_1 \) satisfies Eq. (5.10), then the function \( \Xi(x) \) is expressed as Eq. (5.4), and similar properties are valid as in similar arguments written on Heun’s equation in [15]. In fact, a commuting operator of an odd degree is constructed from the function \( \Xi(x) \), and we recover the results by Treibich [18]. Moreover, a solution to Eq. (5.5) is expressed in the form of the Hermite-Krichever Ansatz, monodromy has two integral representations that are elliptic and hyperelliptic, and hyperelliptic-to-elliptic integral reduction formulae are obtained. Details will be reported in [17].

Thus, if we restrict our discussion to the case \( s_1 = 0 \) and that \( b_1 \) satisfies Eq. (5.10), then the potential of the operator \( H \) (see Eq. (5.6)) is a finite-gap, which recover the results by Treibich [18] and Smirnov [11]. In other words, the potential is Picard’s in the sense of Gesztesy and Weikard [3]. Note that Smirnov obtained expressions like Eqs. (5.4) and calculated several examples in [11].

6. Concluding remarks

We have shown in sections 3 and 4 that solutions of the linear differential equation that produces the sixth Painlevé equation have integral representations and that they are expressed in the form of the Hermite-Krichever Ansatz. Furthermore we got a procedure for obtaining solutions of the sixth Painlevé equation (see Eq. (4.26)) for the cases \( \kappa_0, \kappa_1, \kappa_t, \kappa_\infty \in \mathbb{Z} + \frac{1}{2} \) by fixing the monodromy, and we presented explicit solutions for the cases \( (\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \) and \( (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}) \).

By Bäcklund transformation of the sixth Painlevé equation (see [20 etc.], Hitchin’s solution (i.e., solutions for the case \( (\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \)) is transformed to
the solutions for the case \((\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) \in O_1 \cup O_2\), where

\[
O_1 = \left\{ (\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) \mid \kappa_0, \kappa_1, \kappa_t, \kappa_\infty \in \mathbb{Z} + \frac{1}{2} \right\},
\]

(6.1)

\[
O_2 = \left\{ (\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) \mid \begin{array}{l}
\kappa_0, \kappa_1, \kappa_t, \kappa_\infty \in \mathbb{Z} \\
\kappa_0 + \kappa_1 + \kappa_t + \kappa_\infty \in 2\mathbb{Z}
\end{array} \right\}.
\]

(6.2)

Note that solutions for the case \((\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) = (0, 0, 0, 0) \in O_2\) are already known and are called Picard’s solution.

For the case \((\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) \in O_1\), solutions of the linear differential equation are investigated by our method, and solutions of the sixth Painlevé equation follow from them. On the other hand, for the case \((\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) \in O_2\), we cannot obtain results on integral representation and the Hermite-Krichever Ansatz by our method, although solutions of the sixth Painlevé equation are obtained in principle by Bäcklund transformation. Note that the condition \((\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) \in O_1\) corresponds to the condition \(l_0, \ldots, l_3 \in \mathbb{Z} + \frac{1}{2}, l_0 + l_1 + l_2 + l_3 \in 2\mathbb{Z}\).

Now we propose a problem to investigate solutions and their monodromy of the linear differential equation (Eq. (4.1) with the condition (4.6)) for the cases \(l_0, \ldots, l_3 \in \mathbb{Z} + \frac{1}{2}, l_0 + l_1 + l_2 + l_3 \in 2\mathbb{Z}\). In particular, how can we investigate solutions and their monodromy of the linear differential equation for the case \(\kappa_0 = \kappa_1 = \kappa_t = \kappa_\infty = 0\) (i.e. \(l_0 = l_1 = l_2 = l_3 = -1/2\))?

\section*{Appendix A. Elliptic functions}

This appendix presents the definitions of and the formulas for the elliptic functions.

The Weierstrass \(\wp\)-function, the Weierstrass sigma-function and the Weierstrass zeta-function with periods \((2\omega_1, 2\omega_3)\) are defined as follows:

\[
\wp(z) = \frac{1}{z^2} + \sum_{(m,n) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0,0)\}} \left( \frac{1}{(z - 2m\omega_1 - 2n\omega_3)^2} - \frac{1}{(2m\omega_1 + 2n\omega_3)^2} \right),
\]

(A.1)

\[
\sigma(z) = z \prod_{(m,n) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0,0)\}} \left( 1 - \frac{z}{2m\omega_1 + 2n\omega_3} \right) \cdot \exp \left( \frac{z}{2(2m\omega_1 + 2n\omega_3)^2} \right),
\]

\[
\zeta(z) = \frac{\sigma'(z)}{\sigma(z)}.
\]

Setting \(\omega_2 = -\omega_1 - \omega_3\) and

(A.2) \quad \epsilon_i = \wp(\omega_i), \quad \eta_i = \zeta(\omega_i) \quad (i = 1, 2, 3)

yields the relations

(A.3) \quad \epsilon_1 + \epsilon_2 + \epsilon_3 = \eta_1 + \eta_2 + \eta_3 = 0,
\]

\[
\eta_1 \omega_3 - \eta_3 \omega_1 = \eta_3 \omega_2 - \eta_2 \omega_3 = \eta_2 \omega_1 - \eta_1 \omega_2 = \pi \sqrt{-1}/2,
\]

\[
\wp(z) = -\zeta'(z), \quad (\wp'(z))^2 = 4(\wp(z) - \epsilon_1)(\wp(z) - \epsilon_2)(\wp(z) - \epsilon_3),
\]

\[
\wp(z) - \wp(\bar{z}) = -\frac{\sigma(z + \bar{z})\sigma(z - \bar{z})}{\sigma(z)^2\sigma(\bar{z})^2}
\]
The periodicity of functions $\wp(z), \zeta(z)$ and $\sigma(z)$ are as follows:

\begin{equation}
\wp(z + 2\omega_i) = \wp(z), \quad \zeta(z + 2\omega_i) = \zeta(z) + 2\eta_i \quad (i = 1, 2, 3),
\end{equation}

\[ \sigma(z + 2\omega_i) = -\sigma(z) \exp(2\eta_i(z + \omega_i)), \quad \frac{\sigma(z + t + 2\omega_i)}{\sigma(z + 2\omega_i)} = \exp(2\eta_i t) \frac{\sigma(z + t)}{\sigma(z)} \]

The constants $g_2$ and $g_3$ are defined by

\begin{equation}
g_2 = -4(e_1 e_2 + e_2 e_3 + e_3 e_1), \quad g_3 = 4e_1 e_2 e_3.
\end{equation}

The co-sigma functions $\sigma_i(z) \quad (i = 1, 2, 3)$ and co-$\wp$ functions $\wp_i(z) \quad (i = 1, 2, 3)$ are defined by

\begin{equation}
\sigma_i(z) = \exp(-\eta_i z) \frac{\sigma(z + \omega_i)}{\sigma(\omega_i)}, \quad \wp_i(z) = \frac{\sigma_i(z)}{\sigma(z)}.
\end{equation}

and satisfy

\begin{equation}
\wp_i(z) = \wp(z) - e_i, \quad (i, i' = 1, 2, 3)
\end{equation}

\[ \wp_i(z + 2\omega_{i'}) = \exp(2(\eta_i \omega_i - \eta_i' \omega_{i'})) \wp_i(z) = (-1)^{\delta_{i,i'}} \wp_i(z). \]

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DEPARTMENT OF MATHEMATICAL SCIENCES, YOKOHAMA CITY UNIVERSITY, 22-2 SETO, KANAZAWA-KU, YOKOHAMA 236-0027, JAPAN.
E-mail address: takemura@yokohama-cu.ac.jp