Energy Ratios in Rheological Models

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Abstract. The article considers the energy aspect of compaction process of an asphalt concrete mixture with a smooth drum. The ratios between the compaction work and the material stress-strain state were obtained using the simplest rheological models example. To study the asphalt concrete mixture compaction process, a complex rheological model was adopted. Taking into account the obtained ratios for the simplest rheological bodies, an expression was obtained to determine the compaction specific work of an asphalt concrete mixture with a smooth drum. In the course of the compaction energy analysis, it was found that the compaction work is equal to the sum of irreversible losses for viscous and plastic deformation and the energy required to perform an elastic aftereffect.

1. Introduction
Consider the process of cylindrical body uniaxial deformation (figure 1) under the force \( F(t) \) action applied vertically [1]. The area \( S \) of the body surface to which the force is applied will be considered so small that the force on this surface is uniformly distributed. Let us direct the axis of absolute deformations \( h(t) \) as shown in figure 1, aligning the origin with the surface of the undeformed body.

Let us find the work of the external force applied to the body, which is necessary to deform this body. For this, we fix the achieved deformation \( h \) at some moment of time \( t \). This deformation corresponds to a certain force \( F \). Let us continue the deformation and give \( h \) an infinitesimal increment \( dh \). Since the change in deformation is infinitely small, the applied force can be considered constant.

Then the elementary work of this force on displacement \( dh \) will be [2]

\[
dA = F(h)dh.
\]
The total work on the final displacement $\Delta h = h_2 - h_1$ will be equal to

$$A = \int_{h_1}^{h_2} F(h) \, dh.$$  \hspace{1cm} (2)

Changing the integration variable $h$ to $t$ in equation (2), we obtain

$$A = \int_{h_1}^{h_2} F(t) \, h'(t) \, dt.$$  \hspace{1cm} (3)

Expression (3) is general for calculating the work that must be done by an external force during deformation of any body, in the case of uniaxial loading.

Dividing both sides of equation (3) by the area $S$, we obtain expression (4) for the specific work $A_{sp}$ required to deform a unit area of material identical to the material of the body under consideration

$$A_{sp} = \int_{t_1}^{t_2} \sigma(t) \, h'(t) \, dt.$$  \hspace{1cm} (4)

2. Deformation work of ideal bodies

Consider finding the work to be done by an external force $F(t)$ to deform ideal elementary bodies that are part of various rheological models (figures 2,3,4).

2.1. Absolutely elastic body (Hooke's element)

The body behavior (figure 2) is described by the equation

$$F(t) = kh(t),$$  \hspace{1cm} (5)

where $k$ is the element rigidity.

Let us calculate the work required to deform the body using the equation (3)

$$A = \frac{k}{2} \left[ \dot{h}^2 (t_2) - \dot{h}^2 (t_1) \right].$$  \hspace{1cm} (6)

Replacing in (6) the deformation $h(t)$ by its value according to (5), we obtain

$$A = \frac{1}{2k} \left[ F^2 (t_2) - F^2 (t_1) \right].$$  \hspace{1cm} (7)

The potential nature of elastic forces is visible, namely: the work depends only on the initial and final position, it is not affected by the nature of the transition from one position to another.

2.2. Ideally viscous body (Newton's element)

This model (figure 3) is described by the following differential equation

$$F(t) = \mu \dot{h}(t),$$  \hspace{1cm} (8)

where $\mu$ is the element viscosity.

Let us calculate the work required to deform the body using the equation (3)
\[ A = \mu \int_{h_i}^{h_f} [h'(t)]^2 \, dt. \] 
\[ (9) \]

Replacing in (9) the deformation \( h(t) \) by its value according to (8), we obtain
\[ A = \frac{1}{\mu} \int_{h_i}^{h_f} [F(t)]^2 \, dt. \] 
\[ (10) \]

2.3. Ideal plastic body (Saint-Venant element)

Body behavior (figure 4) obeys the following algorithm
\[ F(t) < F_y \text{ - body is motionless;} \]
\[ F(t) = F_y = \text{const} \text{ - plastic deformation.} \]

Here \( F_y \) is the dry friction force corresponding to the material yield strength.

Let us calculate the work required for plastic deformation of the body using the equation (3)
\[ A = F_y \left[ h(t_f) - h(t_i) \right]. \] 
\[ (11) \]

3. Application to complex rheological models

3.1. Asphalt concrete mix compaction work

In applied research, quite often, for a mathematical description of the physical bodies behavior as a result of various force factors action on them, a real body is represented in the form of a certain rheological model [3, 4, 5, 6].

Differential equation
\[ \Phi(\sigma^\prime, \ldots, \sigma^\prime, \sigma, h^\prime, \ldots, h^\prime, h, t) = 0 \] 
\[ (12) \]
describing the model behavior, as a rule, is a function of two unknown variables, namely: the pressure \( \sigma(t) \) acting on the body and the absolute deformation \( h(t) \) caused by this pressure. Thus, the differential equation cannot be unambiguously resolved with respect to these variables. However, in some cases, based on considerations of a physical, geometric or other nature, arising from the real physical process analysis, it is possible to obtain the law of deformations changes \( h(t) \).

Then, substituting the mathematical expression \( h(t) \) into the differential equation (12) and setting the initial conditions, it is possible to obtain the dependence of the pressure change on the time of action \( \sigma(t) \).

Let’s take a model of the asphalt concrete mixture compacting process using a smooth drum roller (figure 5). We’ll take into account the following assumptions:
- loading is considered uniaxial, vertical;
- pressure in the contact zone \( \sigma(t) \) depends on material deformation \( h(t) \);
- the pressure along the generatrices of a drum cylindrical surface is the same;
- with some deformation, the pressure reaches the material yield point \( \sigma_y \) and the latter goes into a plastic state.

![Figure 5. The rheological model under study.](image-url)
Let us find the specific work of the external force from the drum side necessary for mixture compacting. Let’s select from the massif of uncompacted coating a certain volume, which is a vertically located cylindrical body \( M \) with such a small base area that the pressure applied from the side of the drum to the body can be considered constant within this area.

Consider the deformation of the body \( M \). The loading scheme is shown in figure 6, where \( V \) is the speed of drum translational motion, \( R \) is the drum radius, \( h_{res} \) is irreversible permanent deformation.

![Figure 6. The loading scheme.](image)

The moment of drum contact with undeformed body is taken as the time starting point.

From the analysis of this loading scheme, an expression that describes the change in the absolute deformation during the deformation of the body \( M \) was obtained

\[
h(t) = \frac{V^2}{2R} \left( 2t - t^2 \right).
\]  

Recall that the asphalt concrete mixture is presented in the form of a rheological model (figure 5) with dynamic parameters \( b, c, \mu \) and \( \sigma_y \), depending on temperature and compaction coefficient. In this case, it is considered that during one drum pass the model parameters are unchanged. This assumption is justified, since deformation occurs in a short period of time (0.5-0.8 s) [7], during which the temperature can be taken constant. Changing the model parameters due to the increase in the compaction coefficient after one drum pass can be neglected, since this increase is insignificant [8]. The next drum pass along one track will occur after a rather long period of time, after the entire section has been compacted. During this time, the mixture temperature, and, consequently, the model parameters will change significantly. Thus, for the next drum pass, new parameters are adopted, corresponding to the changed compaction conditions, and these parameters are again considered unchanged throughout the entire pass [9, 10].

The behavior of the rheological model shown in figure 5 before the plastic deformations onset is generally described by the differential equation

\[
s' + \frac{b + c}{\mu} s = bh' + \frac{bc}{\mu} h.
\]  

Substituting expression (13) and its time derivative into equation (14), we obtain the model's behavior differential equation under the selected loading conditions

\[
s' + \frac{b + c}{\mu} s = \frac{bV^2}{R} \left( t_2 - t \right) + \frac{bcV^2}{2\mu R} \left( 2t - t^2 \right).
\]  

Solving equation (15) taking into account the initial and boundary conditions, we obtain the dependence \( \sigma(t) \).

The entire deformation process can be conditionally divided into two stages: the loading stage, which corresponds to the time interval from \( t_0 \) to \( t_2 \), and the unloading stage, from \( t_2 \) to \( t_1 \).

At the first stage, from the loading beginning to moment \( t_3 \), the work of the external force is spent on overcoming the material elastic-viscous resistance. In this case, part of the work is spent on elastic deformation, and the other part is irrevocably spent on viscous friction in Newton's element.

At moment \( t_3 \), the material yield point \( \sigma_y \) is reached, and up to moment \( t_2 \), plastic deformation occurs, for which some part of the external force work is spent. This part of the work is spent on
overcoming dry friction in the Saint-Venant element and is irreversibly released in the form of heat. In the process of plastic deformation, the body remains in a stressed state achieved during elastic-viscous loading.

Further, in the interval \( t_2 - t_1 \), the body is unloaded. During this process, the energy stored in elastic elements at the loading stage is spent by the body to restore its original shape. The unloading process can be viewed as the work of a compressed material against an external force. Therefore, the work of the external force in the interval \( t_2 - t_1 \) is negative.

Considering the above, we will find the specific work of material compaction. To do this, we use the general equation (4) and equation (11)

\[
A_y = \int_0^t \sigma_t(t) h'(t) dt + \sigma_e \left[ h(t_2) - h(t_1) \right] + \int_{t_1}^t \sigma_2(t) h'(t) dt.
\]

(16)

The work that must be done to compact a coating having an area \( S \) can be found as

\[
A = A_y S,
\]

(17)

where \( A_y \) is the work performed by one drum when compacting a coating having an area \( S \).

The compaction work of a coating having an area \( S \) with a two-drum road roller in \( n \) passes is

\[
A = S \sum_{i=1}^n \left( A_{y, \text{front } i} + A_{y, \text{rear } i} \right).
\]

(18)

where \( i \) is the road roller current pass number.

3.2. Energy ratios detailed analysis

Consider the energy transformations in the process of accepted rheological model deformation.

We rewrite expression (16) as

\[
A_y = A_{y,2} + A_{y,1},
\]

(19)

where \( A_{y,2} \) is the work done during loading (interval \( t_0-t_2 \)), \( A_{y,1} \) is the unloading work (interval \( t_2-t_1 \)).

Work \( A_{y,2} \), in turn, can be represented as the sum of two components

\[
A_{y,2} = A_{y,2}^{e-v} + A_{y,2}^{pl},
\]

(20)

where \( A_{y,2}^{e-v} \) is the work spent to overcome the material elastic-viscous resistance under loading, \( A_{y,2}^{pl} \) is work of plastic deformation after yield point is reached.

And finally, in work \( A_{y,2}^{e-v} \), two more components can be distinguished

\[
A_{y,2}^{e-v} = A_{y,2}^{e} + A_{y,2}^{v},
\]

(21)

where \( A_{y,2}^{e} \) corresponds to the work spent on deforming all elastic elements, \( A_{y,2}^{v} \) is the work spent to overcome viscous friction.

Thus

\[
A_{y,2} = A_{y,2}^{e} + A_{y,2}^{v} + A_{y,2}^{pl}.
\]

(22)

At the unloading stage, the energy \( A_{y,2}^{e} \) stored during loading will be spent on unloading the body not entirely, but minus the energy required for the elastic aftereffect after the end of the drum contact with the surface of the deformable body. Let's explain what has been said.

By the time \( t_1 \) the coating is lifted off the drum surface, all parts of the rheological model with extremely ideal elasticity, will completely restore their original shape. In the remaining parts of the model, which combine elastic properties with viscous ones, full unloading has not yet been achieved as a result of the viscous elements resistance, i.e. there is some deformation of the elastic elements. Consequently, there is still some reserve of potential energy, which is then spent on elastic aftereffect, after which irreversible permanent deformation is achieved.

In addition, some of the energy stored during loading will be spent in the form of work to overcome viscous friction during the unloading stage.

Thus, the remaining part of the stored potential energy of elastic deformation, which will be spent on performing the work necessary to unload the body in the interval from \( t_2 \) to \( t_1 \), will be equal to
\[
\overline{A_{21}} = A_{21}^c - E_p - A_{21}^v,
\]

where \(E_p\) is the energy reserve required for elastic aftereffect, \(A_{21}^v\) is the work to overcome viscous friction during the unloading stage.

Work \(\overline{A_{21}}\) is performed by the body against an external force. Therefore

\[
A_{21} = -\overline{A_{21}}.
\]

Taking into account (19), (22), (23) and (24), we finally obtain

\[
A_p = A_{21}^c + A_{21}^{pl} + A_{21}^v + E_p.
\]

The first three terms in equation (25) represent the frictional losses in the corresponding elements of the rheological model during deformation.

4. Results
The asphalt concrete mixture compaction process with a smooth drum is considered. The compaction work is equal to the sum of irreversible losses for viscous and plastic deformation and the energy required to perform an elastic aftereffect. The specific compaction work characterizes the energy consumption of the compaction process. Thus, it becomes possible to increase the compaction process efficiency by analyzing energy costs and their redistribution. The compaction process efficiency go up with the increase in work expended on the material plastic deformation.

5. References
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