CONFORMALLY FLAT PENCILS OF METRICS, FROBENIUS STRUCTURES AND A MODIFIED SAITO CONSTRUCTION

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Abstract. The structure of a Frobenius manifold encodes the geometry associated with a flat pencil of metrics. However, as shown in the authors' earlier work [1], much of the structure comes from the compatibility property of the pencil rather than from the flatness of the pencil itself. In this paper conformally flat pencils of metrics are studied and examples, based on a modification of the Saito construction, are developed.

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1. INTRODUCTION

The Saito construction [10] of a flat structure on the orbit space \( \mathbb{C}^n/W \), where \( W \) is a Coxeter group, has played a foundational role in many areas of mathematics. It is a central construction in singularity theory and contains the kernel of the definition of a Frobenius manifold, this having been done many years before the introduction of a Frobenius manifold by Dubrovin [2].

The initial motivation for this paper was the observation that one may repeat the Saito construction starting with a metric of constant non-zero sectional curvature \( s \). One easily obtains a pencil of metrics \((h, \tilde{h})\) on the orbit space \( \mathbb{C}^n/W \) (if \( s > 0 \)) or \( \mathbb{H}^n \otimes \mathbb{C}/W \) (if \( s < 0 \)) of a Coxeter group \( W \). The pencil \((h, \tilde{h})\) has interesting geometric properties: it is conformally related to the flat pencil provided by the classical Saito construction, the metric \( h \) has constant sectional curvature \( s \) and, as it turns out, the metric \( \tilde{h} \) is flat. This modified Saito construction is developed in Section 2. The construction may be applied, locally, to any Frobenius manifold and this is also illustrated in Section 2.

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Another consequence of Saito’s work is that it provides a construction of so called flat pencils of metrics. This then leads, via the results of Dubrovin and Novikov and Magri, to bi-Hamiltonian structures and the theory of integrable systems. The flatness of such pencils is required for the locality of the bi-Hamiltonian structures; however one may introduce curvature - resulting in non-local Hamiltonian operators - in such a way as to preserve the bi-Hamiltonian property. Geometrically one requires a compatible pencil of metrics rather than a flat pencil.

In the authors’ earlier work the geometry of compatible metrics was studied in detail - this generalizing the results of Dubrovin from flat pencils of metrics to (and curved) pencils of metrics. In Section 3 we continue this study. One way to construct examples is to scale a known flat pencil of metrics by a conformal factor. This introduces curvature but the new metrics remain compatible. The geometry of such conformally scaled compatible pencils is studied in Section 4 and provides a general scheme into which the modified Saito construction of Section 2 falls.

The rest of this Section outlines some standard notations and earlier results.

1.1. Compatible metrics on manifolds. Let $M$ be a smooth manifold. We shall use the following notations: $\mathcal{X}(M)$ for the space of smooth vector fields on $M$; $\mathcal{E}^1(M)$ for the space of smooth 1-forms on $M$. For a pseudo-Riemannian metric $g$ on $M$, $\nabla^g$ will denote its Levi-Civita connection and $R^g$ its curvature. The metric $g$ induces a metric $g^*$ on the cotangent bundle $T^*M$ of $M$. The 1-form corresponding to $X \in \mathcal{X}(M)$ via the pseudo-Riemannian duality defined by $g$ will be denoted $g(X)$. Conversely, if $\alpha \in \mathcal{E}^1(M)$, the corresponding vector field will be denoted $g^*\alpha$.

Following we recall the basic theory of compatible metrics on manifolds; the flat case has been treated in [4]. Let $(g, \tilde{g})$ be an arbitrary pair of metrics on $M$. Recall that the pair $(g, \tilde{g})$ defines a multiplication on $T^*M$ (or on $TM$, by identifying $TM$ with $T^*M$ using the metric $\tilde{g}$). For every constant $\lambda$ we define the inverse metric $g^* := g^* + \lambda \tilde{g}^*$, which, we will assume, will always be non-degenerate and whose Levi-Civita connection and curvature tensor will be denoted $\nabla^\lambda$ and $R^\lambda$ respectively. The metrics $g$ and $\tilde{g}$ are almost compatible if, by definition, the relation

$$g^*_\lambda \nabla^\lambda_X \alpha = g^* \nabla^\gamma_X \alpha + \lambda \tilde{g}^* \nabla^\tilde{g} \alpha$$

holds, for every $X \in \mathcal{X}(M)$, $\alpha \in \mathcal{E}^1(M)$ and constant $\lambda$. The almost compatibility condition is equivalent with the vanishing of the integrability tensor $N_K$ of $K := g^* \tilde{g} \in \text{End}(TM)$, defined by the formula:

$$N_K(X, Y) = -[KX, KY] + K[KX, Y] + K[X, KY] - K^2[X, Y], \quad \forall X, Y \in \mathcal{X}(M)$$

and implies the following two relations:

$$g^*(\nabla^\tilde{g}_g \gamma \alpha - \nabla^\gamma_g \alpha) = \tilde{g}^*(\nabla^\tilde{g}_g \gamma \alpha - \nabla^\gamma_g \alpha), \quad \forall \alpha, \gamma \in \mathcal{E}^1(M)$$

(3) and

$$\tilde{g}^*(\alpha \circ \beta, \gamma) = g^*(\alpha, \gamma \circ \beta), \quad \forall \alpha, \beta, \gamma \in \mathcal{E}^1(M).$$

(4)
Recall now that two almost compatible metrics \((g, \tilde{g})\) are compatible if, by definition, the relation
\[
g_\lambda^*(R_{X,Y} \alpha) = g^*(R_{X,Y} \alpha) + \lambda \tilde{g}^*(R_{X,Y} \alpha)
\]
holds, for every \(\alpha \in \mathcal{E}(M)\), \(X, Y \in \mathcal{X}(M)\) and constant \(\lambda\). The compatibility condition has several alternative formulations: if the metrics \((g, \tilde{g})\) are almost compatible, then they are compatible if and only if the relation
\[
g^*(\nabla^\tilde{g}_X \alpha - \nabla^g_X \alpha, \nabla^\tilde{g}_Y \beta - \nabla^g_Y \beta) = g^*(\nabla^\tilde{g}_Y \alpha - \nabla^g_Y \alpha, \nabla^\tilde{g}_X \beta - \nabla^g_X \beta)
\]
holds for every \(X, Y \in \mathcal{X}(M)\) and \(\alpha, \beta \in \mathcal{E}(M)\), or, in terms of the multiplication “\(\circ\)” associated to the pair \((g, \tilde{g})\),
\[
(\alpha \circ \beta) \circ \gamma = (\alpha \circ \gamma) \circ \beta, \quad \forall \alpha, \beta, \gamma \in \mathcal{E}(M).
\]
If the metrics \((g, \tilde{g})\) are compatible and \(R^\lambda = 0\) for all \(\lambda\) then \((g, \tilde{g})\) are said to form a flat pencil of metrics.

1.2. The Dubrovin correspondence. We end the Introduction by recalling the Dubrovin correspondence between flat pencils of metrics and Frobenius manifolds and its generalizations. We first recall the definition of a Frobenius manifold.

**Definition 1.** \([4]\) A manifold \(M\) is a Frobenius manifold if a structure of a Frobenius algebra (i.e. a commutative, associative algebra with multiplication denoted by “\(\cdot\)”, an identity element “\(e\)” and an inner product “\(<, >\)”, satisfying the invariance condition \(<a \cdot b, c> = <a, b \cdot c>\) is specified on the tangent space \(T_pM\) at any point \(p \in M\) smoothly depending on the point \(p\) such that:

(i) The invariant metric \(\tilde{g} = <, >\) is a flat metric on \(M\);

(ii) The identity vector field \(e\) is covariantly constant with respect to the Levi-Civita connection \(\nabla^\tilde{g}\) of the metric \(\tilde{g}\):
\[
\nabla^\tilde{g} e = 0;
\]

(iii) The \((4,0)\)-tensor \(\nabla^\tilde{g}(\bullet)\) defined by the formula
\[
\nabla^\tilde{g}(\bullet)(X,Y,Z,V) := \tilde{g} \left( \nabla^\tilde{g}_X(\bullet)(Y,Z), V \right), \quad \forall X,Y,Z \in \mathcal{X}(M)
\]
is symmetric in all arguments.

(iv) A vector field \(E\) - the Euler vector field - must be determined on \(M\) such that
\[
\nabla^\tilde{g}(\nabla^\tilde{g} E) = 0
\]
and that the corresponding one-parameter group of diffeomorphisms acts by conformal transformations of the metric and by rescalings of the multiplication “\(\cdot\)”.

Using the flat coordinates \(\{t^i\}\) of the metric \(<, >\) one may express the multiplication in terms of the derivatives of a scalar prepotential \(F\),
\[
< \frac{\partial}{\partial t^1}, \frac{\partial}{\partial \nu} \cdot \frac{\partial}{\partial \nu} > = \frac{\partial^3 F}{\partial t^1 \partial \nu \partial \nu}
\]
where the \(t^1\)-dependence of \(F\) is fixed by the condition
\[
< \frac{\partial}{\partial t^1}, \frac{\partial}{\partial \nu} > = \frac{\partial^3 F}{\partial t^1 \partial \nu \partial \nu}.
\]
The associativity condition then becomes an overdetermined partial differential equation for the prepotential $F$ known as the Witten-Dijkgraaf-Verlinde-Verlinde equation.

Recall now that if $(M, \cdot, \tilde{g}, E)$ is a Frobenius manifold then we can define an inverse metric $g^*$ by the relation $g^* \tilde{g} = E \cdot$. The metrics $(g, \tilde{g})$ form a flat pencil on the open subset of $M$ where $E\cdot$ is an automorphism, satisfying some additional conditions (the quasi-homogeneity conditions). Conversely, a (regular) quasi-homogeneous flat pencil of metrics on a manifold determine a Frobenius structure on that manifold. This construction is known in the literature as the Dubrovin correspondence \cite{Dubrovin}.

It turns out that the key role in the Dubrovin correspondence is played not by the flatness property of the metrics but rather by their compatibility. Weaker versions of the Dubrovin correspondence have been developed in \cite{Dubrovin}. Following \cite{Dubrovin} we recall now a weak version of the Dubrovin correspondence. In general, a pair of metrics $(g, \tilde{g})$ together with a vector field $E$ on a manifold $M$ such that the endomorphism $T(u) := g(E)\cdot u$ of $T^*M$ is an automorphism (the regularity condition) determines a multiplication $u \ast v := u \circ T^{-1}(v)$ on $T^*M$ (or on $TM$, by identifying $TM$ with $T^*M$ using the metric $\tilde{g}$). If the metrics $(g, \tilde{g})$ are compatible, then the multiplication “$\ast$” is associative, commutative, with identity $g(E)$ on $T^*M$, the metrics $g, \tilde{g}$ are “$\ast$”-invariant and $g^* \tilde{g} = E \cdot$. Moreover, if $E$ satisfies the relations

$$L_E(\tilde{g}) = D\tilde{g}, \quad \nabla^g_X(E) = \frac{1-d}{2}X, \quad \forall X \in \mathcal{X}(M),$$

for some constants $D$ and $d$, then $(M, \cdot, \tilde{g}, E)$ is a weak $\mathcal{F}$-manifold, i.e. the following conditions are satisfied:

1. The metric $\tilde{g}$ and the multiplication “$\ast$” define a Frobenius algebra at every tangent space of $M$.
2. The vector field $E$ - the Euler vector field - rescales the metric $\tilde{g}$ and the multiplication “$\ast$” by constants and has an inverse $E^{-1}$ with respect to the multiplication “$\ast$”, which is a smooth vector field on $M$.
3. The $(4,0)$-tensor field $\nabla^g(\cdot)$ of $M$ satisfies the symmetries:

$$\nabla^g(\cdot)(E, Y, Z, V) = \nabla^g(\cdot)(Y, E, Z, V), \quad \forall Y, Z, V \in \mathcal{X}(M).$$

Conversely, a weak $\mathcal{F}$-manifold $(M, \cdot, \tilde{g}, E)$ determines a pair $(g, \tilde{g})$ of compatible metrics, with $g$ defined by the formula $g^* \tilde{g} = E \cdot$, and the Euler vector field $E$ satisfies relations (\cite{Dubrovin}). Therefore, there is a one to one correspondence between (regular) compatible pencils of metrics $(g, \tilde{g})$ with a vector field $E$ satisfying relations (\\cite{Dubrovin}) and weak $\mathcal{F}$-manifolds.

Under a certain curvature condition on the metrics $(g, \tilde{g})$ - see Theorem 23 of \cite{Dubrovin} - the tensor $\nabla^g(\cdot)$ is symmetric in all arguments and then $(M, \cdot, \tilde{g}, E)$ is called an $\mathcal{F}$-manifold. Note that in this case $(M, \cdot)$ is an $\mathcal{F}$-manifold, i.e. the relation

$$L_X\cdot_Y(\cdot) = X \cdot L_Y(\cdot) + Y \cdot L_X(\cdot), \quad \forall X, Y \in \mathcal{X}(M)$$

holds. In fact, Hertling noticed - see Theorem 2.15 of \cite{Dubrovin} - that if $(M, \cdot, \tilde{g})$ satisfies the first of the three conditions mentioned above and “$\cdot$” is the identity vector field of the multiplication “$\ast$”, then relation (\\cite{Dubrovin}) together with the closeness of the coidentity $\tilde{g}(e)$ is equivalent to the total symmetry of the tensor $\nabla^g(\cdot)$.
Remark: The definition of (weak) $\mathcal{F}$-manifolds and all the properties proved about these manifolds in \cite{1} assumed that the Euler vector field rescaled the metric and the multiplication by constants. From now on, when we refer to (weak) $\mathcal{F}$-manifolds we allow the Euler vector field to rescale the metric and the multiplication by not necessarily constant functions. In Section 3 we extend the results of \cite{1} to this more general class of weak $\mathcal{F}$-manifolds and in Section 4 we apply our theory to the metrics obtained by non-constant conformal rescalings of the flat metrics of a Frobenius manifold.

2. A modified Saito construction

The motivation for considering such non-constant conformal rescalings comes from the following Theorem.

Theorem 2. Let $(g, \tilde{g})$ be the flat pencil of the Saito construction on the space of orbits $\mathbb{C}^n/W$ of a Coxeter group $W$. There is a metric $\tilde{h}$ with the following properties:

1. The metric $\tilde{h}$ is flat.
2. The metric $\tilde{h}$ is conformally related to the metric $\tilde{g}$: $\tilde{h} = \Omega^2 \tilde{g}$, for a smooth non-vanishing function $\Omega$.
3. The metric $\tilde{h} := \Omega^2 \tilde{g}$ has constant non-zero sectional curvature $s$. If $s > 0$ then $\tilde{h}$ is defined on $\mathbb{C}^n/W$. If $s < 0$ then $\tilde{h}$ is defined on $\mathbb{H}^n \otimes \mathbb{C}/W$.

Proof. We begin with a review of the salient features of the Saito construction. Details can be found in \cite{2}. Recall that a Coxeter group of a real $n$-dimensional vector space $V = \mathbb{R}^n$ is a finite group of linear transformations of $V$ generated by reflections. Let $\{t_i\}$ be a basis of $W$-invariant polynomials on $V$ with degrees $\deg(t_i) = d_i$, ordered so that

$$h = d_1 > d_2 \geq \ldots \geq d_{n-1} > d_n = 2,$$

where $h$ is the Coxeter number of the group. The action of $W$ extends to the complexified space $V \otimes \mathbb{C} = \mathbb{C}^n$. In the Saito construction of interest is the orbit space

$$M = \mathbb{C}^n/W.$$

Starting with a $W$-invariant metric

$$g := \sum_{i=1}^n (dx^i)^2$$

on $V$ one obtains a flat metric $g$ on the orbit space $M \setminus \text{Discr}(W)$, where $\text{Discr}(W)$ is the discriminant locus of irregular orbits. What Saito showed was that there is another metric

$$\tilde{g} := \text{Lie}_e(g^*)$$

defined on the whole of $M$ which is also flat. Here $e$ is the vector field which, in terms of the basis $\{t^i\}$ of invariant polynomials, is $\frac{\partial}{\partial t^1}$. The basis $\{t^i\}$ of invariant polynomials can be chosen such that the metric $\tilde{g}$ is anti-diagonal with constant entries:

$$\tilde{g}_{ij} = \delta_{i+j,n+1}$$

and is referred in this case as a Saito’s basis of invariant polynomials. Unlike $g$ which is defined only on $M \setminus \text{Discr}(W)$, the metric $\tilde{g}$ is defined on the whole of $M$. 

An important fact of the Saito construction is that the two metrics \((g, \tilde{g})\) are the regular flat pencil of a Frobenius structure on \(M\), with Euler vector field
\[
E = d_1t^1 \frac{\partial}{\partial t^1} + \cdots + d_n t^n \frac{\partial}{\partial t^n}.
\] (10)

Suppose now that one repeats the Saito construction starting with a metric of constant sectional curvature, i.e. let
\[
\tilde{h} := \frac{1}{\{c \sum_{i=1}^n (x_i)^2 + d\}^2} \sum_{i=1}^n (dx_i)^2.
\]
This has constant sectional curvature \(4(cd)\).

Since one can take, without loss of generality, the invariant \(t^n\) to be
\[
t^n = \sum_{i=1}^n (x_i)^2,
\]
the conformal factor is a function of \(t^n\) alone. Hence one obtains a new metric
\[
\tilde{h} := \text{Lie}_e(h^*),
\]
\[
= \frac{1}{\{c \sum_{i=1}^n (x_i)^2 + d\}^2} \sum_{i=1}^n (dx_i)^2 \tilde{g}^*.
\]
defined on \(\mathbb{H}^n \otimes \mathbb{C}/W\) for \((cd) < 0\) and on \(\mathbb{C}^n/W\) for \((cd) > 0\). In terms of the flat coordinates \(\{t^i\}\) for the metric \(\tilde{g}\),
\[
\tilde{h}_{ij} = \frac{1}{(ct^n + d)^2} \delta_{i+j,n+1}.
\]
Notice that the metric
\[
h := \frac{1}{(ct^n + d)^2} \tilde{g}
\]
has constant sectional curvature \(4(cd)\).

It remains to show that \(\tilde{h}\) is flat. This may be proved using the standard formulae for transformation of the curvature tensor under a conformal change. Moreover, the flat coordinates \(\{\tilde{t}^i\}\) for \(\tilde{h}\) can be written down explicitly
\[
\tilde{t}^1 = t^1 - \frac{c}{ct^n + d} \sum_{i=2}^{n-1} t^i t^{n+1-i},
\]
\[
\tilde{t}^i = \frac{t^i}{ct^n + d}, \quad i = 2, \ldots, n - 1,
\]
\[
\tilde{t}^n = \frac{at^n + b}{ct^n + d}, \quad ad - be = 1,
\]
(note that this \(SL(2, \mathbb{C})\)-transformation appears also in [2] in a slightly different context) giving
\[
\tilde{h} = \sum_{i=1}^n d\tilde{t}^i d\tilde{t}^{n+1-i}.
\]
The conclusion follows. □

The construction turns out to be quite general:
Proposition 3. Suppose one has a Frobenius manifold with metrics \( \tilde{g} \) and \( g \) (or \( \eta \) and \( g \) respectively in Dubrovin’s notation). Consider the conformally scaled metrics
\[
\tilde{h} = \Omega^2(t_1) \tilde{g} \\
h = \Omega^2(t_1) g.
\]
(Here \((t^1, \ldots, t^n)\) are \( \tilde{g} \)-flat coordinates with \( \frac{\partial}{\partial t^i} \) being the identity vector field \( e \) and \( t_1 \) is dual to the identity coordinate \( t^1 \), i.e. \( t_1 = \tilde{g}_i^1 t^i \).) Suppose that \( \tilde{h} \) is flat. Then \( h \) has constant sectional curvature.

Proof. The curvature conditions on \( \tilde{h} \) translate to a simple differential equation for the conformal factor. Solving this gives \( \Omega^{-1} = ct_1 + d \) for constants \( c \) and \( d \). This then fixes the metric \( h \). Calculating its curvature (using again the standard formulae for change in the curvature tensor under a conformal change and various properties of the Christoffel symbols of \( g \) in [D]) yields the result. \( \square \)

Note that this conformal factor satisfies the condition \( d\Omega \wedge g(E) = 0 \), where \( E \) is the Euler vector field of the Frobenius manifold. This follows from the following easy computation: \( g(E) = \tilde{g}(e) = \tilde{g}_i^1 dt^i = dt_1 \), the functions \( \tilde{g}_i^1 \) being constant. It turns out, as Section 4 will show, that conformally scaled metrics with this condition have particularly attractive properties.

3. \( \mathcal{F} \)-manifolds and compatible pencils of metrics

In this Section we study the geometry of a pair of compatible metrics together with a vector field satisfying conditions (1), when \( D \) and \( d \) are not necessarily constant.

Proposition 4. Let \((h, \tilde{h})\) be a regular pair of compatible metrics together with a vector field \( E \) on a connected manifold \( M \) of dimension at least three. Suppose that \( L_E(h) = Dh \), \( L_E(\tilde{h}) = D\tilde{h} \), for \( D, \tilde{D} \in C^\infty(M) \). Then \( E \) rescales the multiplication “\( \circ \)” associated to the pair \((h, \tilde{h})\) and vector field \( E \) if and only if \( \tilde{D} - D \) is constant. In this case \( L_E(\circ) = (\tilde{D} - D)\circ \) on \( TM \).

Proof. Recall that if \( g \) is an arbitrary pseudo-Riemannian metric on a manifold \( M \) and \( Z \) is a conformal vector field with \( L_Z(g) = pg \) for a function \( p \in C^\infty(M) \), then
\[
L_Z(\nabla g)_X(\alpha) = \frac{1}{2}[-dp(X)\alpha - \alpha(X)dp + g^*(\alpha, dp)g(X)],
\]
for every \( \alpha \in \mathcal{E}^1(M) \) and \( X \in \mathcal{X}(M) \). Applying this formula for the metrics \( h \) and \( \tilde{h} \) we easily get
\[
L_E(\circ)(u, v) = \frac{1}{2}[\eta^*(u, d\lambda)v + \eta^*(u, v)d\lambda + h^*(v, dD)u - \tilde{h}^*(v, d\tilde{D})\tilde{h}^*(u)] - Du \circ v,
\]
where “\( \circ \)” is the multiplication (11) determined by the pair of metrics \((h, \tilde{h})\), \( u, v \in \mathcal{E}^1(M) \) and \( \lambda := \tilde{D} - D \). Let \( T \) be the automorphism of \( T^*M \) defined by the formula \( T(u) = h(E) \circ u \). From the above relation we easily see that
\[
L_E(T)(u) = \frac{1}{2}[E(\lambda)u + u(E)d\lambda + h^*(u, dD)h(E) - \tilde{h}^*(u, d\tilde{D})\tilde{h}(E)].
\]
Recall now that the multiplications “$o$” and “$\cdot$” on $T^*M$ are related by the formula $u \cdot T(v) = u \circ v$. Taking the derivative with respect to $E$ of this formula we easily see that

$$L_E(\bullet)(u, T(v)) + \frac{1}{2} u \bullet [E(\lambda)v + v(E)d\lambda + h^*(v, dD)h(E)]$$

$$= \frac{1}{2} [h^*(u, d\lambda)v + h^*(u, v)d\lambda + h^*(v, dD)u - h^*(v, d\tilde{D})\tilde{h}(E)]$$

$$- D u \bullet T(v),$$

for every $u, v \in \mathcal{E}^1(M)$. Since $h^*\tilde{h} = E \bullet$ (the metrics $(h, \tilde{h})$ being compatible), an easy argument shows that $u \bullet h(E) = \tilde{h}h^*(u)$. The compatibility of $(h, \tilde{h})$ also implies, as mentioned in Section 1.1, that $\tilde{h}(E)$ is the identity of the multiplication “$\cdot$” on $T^*M$. It follows that

$$L_E(\bullet)(u, T(v)) = \frac{1}{2} [h^*(u, d\lambda)v + h^*(u, v)d\lambda]$$

$$- \frac{1}{2} u \bullet [E(\lambda)v + v(E)d\lambda]$$

$$- D u \bullet T(v),$$

for every $u, v \in \mathcal{E}^1(M)$. From this relation it is easy to see that $E$ rescales the multiplication “$\cdot$” if and only if for every $u, v \in \mathcal{E}^1(M)$, the equality

$$u \bullet [E(\lambda)v + v(E)d\lambda] = h^*(u, d\lambda)v + h^*(u, v)d\lambda$$

(11)

holds and in this case $L_E(\bullet) = -D\bullet$ on $T^*M$, or $L_E(\bullet) = \lambda\bullet$ on $TM$. We will show that relation (11) holds only if $\lambda$ is constant. Indeed, if in relation (11) we take $u$ and $v$ annihilating $E$, then we get

$$h^*(u, d\lambda)v = h^*(v, d\lambda)u$$

which can hold only if $d\lambda = \mu h(E)$ for a function $\mu \in C^\infty(M)$, since the dimension of $M$ is at least three (and hence the annihilator of $E$ in $T^*M$ is of dimension at least two). Relation (11) then becomes

$$\mu[h(E, E)v + v(E)h(E)] \bullet u = \mu[u(E)v + h^*(u, v)h(E)]$$

in which turn implies that $\mu[v(E)u - u(E)v]$ is symmetric in $u$ and $v$, for every $u, v \in \mathcal{E}^1(M)$. This can happen only when $\mu$ is identically zero or $\lambda = \tilde{D} - D$ is constant ($M$ being connected).

**Remark:** Note that relation (11) does not imply that $\lambda$ is constant in dimension two. An easy argument shows that relation (11) imposes that in two dimensions the multiplication “$\bullet$” is of the form

$$d\lambda \bullet d\lambda = 0; \quad d\lambda \bullet h(E) = d\lambda; \quad h(E) \bullet h(E) = h(E)$$

when $\lambda$ is non-constant. Proposition 1 does not hold in dimension two: consider for example the inverse metrics

$$\tilde{h}^* = f \left( \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} + \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial y} \right)$$

$$h^* = x \left( \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} + \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial y} \right)$$
together with the vector field
\[ E = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \]
with \( f \) smooth, non-vanishing, depending only on \( x \), such that \( \frac{xf'(x)}{f(x)} \) is non-constant on a connected open subset \( M \) of
\[ \{(x, y) \in \mathbb{R}^2 : \frac{xf'(x)}{f(x)} \neq \frac{1}{2}, \ x \neq 0\}. \]
The hypothesis of Proposition 4 is satisfied on \( M \), with \( LE(h) = h; \ LE(\tilde{h}) = \left( 2 - \frac{xf'(x)}{f(x)} \right) \tilde{h}. \)
The associated multiplication “\( \bullet \)” on \( T^*M \) has the expression:
\[ dx \bullet dx = 0; \ dx \bullet dy = dx; \ dy \bullet dy = dy \]
and is preserved by \( E: \ LE(\bullet) = \left( 1 - \frac{xf'(x)}{f(x)} \right) \bullet \) on \( TM. \)

We return now to higher dimensions and we note the following consequence of Proposition 4.

**Corollary 5.** Let \((M, \bullet, \tilde{h}, E)\) be a connected weak \( \mathcal{F} \)-manifold of dimension at least three. Then the Euler vector field \( E \) rescales the multiplication “\( \bullet \)” by a constant.

**Proof.** Let \( h \) be the metric on \( M \) defined by the relation \( h^*\tilde{h} = E \bullet \). Suppose that \( LE(h) = \tilde{D}h \) and \( LE(\bullet) = k \bullet \), for \( \tilde{D}, k \in C^\infty(M) \). Then \( LE(h) = (\tilde{D} - k)h \). The same argument used in the proof of Theorem 17 and Proposition 19 of [1] shows that \((h, \tilde{h})\) are compatible and that the multiplication “\( \circ \)” on \( T^*M \) (identified with \( TM \)) is related to the multiplication “\( \circ \)” determined by the pair \((h, \tilde{h})\) by the relation: \( u \circ T(v) = u \circ v \), where \( T \) is the endomorphism of \( T^*M \) defined by \( T(u) := h(E) \circ u \). Even if \( T \) is not necessarily an automorphism, the argument used in the proof of Proposition 4 still holds and implies our conclusion.

In order to simplify terminology, we introduce the following definition (which generalizes Definition 14 of [1]).

**Definition 6.** Let \((h, \tilde{h})\) be a compatible pair of metrics and \( E \) a vector field on a manifold \( M \). The pair \((h, \tilde{h})\) is a weak quasi-homogeneous pencil with Euler vector field \( E \) if the following conditions are satisfied:

1. \( LE(\tilde{h}) = \tilde{D}h; \ \nabla^h(E) = \frac{\tilde{D}}{2} \text{Id} \), where “\( \text{Id} \)” is the identity endomorphism of \( TM \) and \( D, \tilde{D} \in C^\infty(M) \).
2. The difference \( \tilde{D} - D \) is constant.

The weak quasi-homogeneous pair \((h, \tilde{h})\) is regular if the endomorphism \( T(u) := h(E) \circ u \) of \( T^*M \) is an automorphism. (Here “\( \circ \)” is the multiplication [4] associated to the pair of metrics \((h, \tilde{h})\)).

The correspondence between weak quasi-homogeneous pencils of metrics and weak \( \mathcal{F} \)-manifolds can be stated as follows:
Theorem 7.  
(1) Let \((h, \tilde{h})\) be a regular weak quasi-homogeneous pencil of metrics with Euler vector field \(E\) on a manifold \(M\). Let \(\ast\) be the multiplication on \(TM\) associated to the pair of metrics \((h, \tilde{h})\) and vector field \(E\). Then \((M, \ast, \tilde{h}, E)\) is a weak \(\mathcal{F}\)-manifold.

(2) Conversely, let \((M, \ast, \tilde{h}, E)\) be a connected weak \(\mathcal{F}\)-manifold of dimension at least three. Define the metric \(h\) on \(M\) by the formula \(h \ast \tilde{h} = E \ast\). Then \((h, \tilde{h})\) is a weak quasi-homogeneous pencil with Euler vector field \(E\).

Proof. The proof follows the same steps and is totally similar to the proofs of Theorem 17 and Theorem 20 of [1]. Note that in the second statement of the Theorem we have restricted to the case when the manifold \(M\) is connected and of dimension at least three. These additional conditions insure that the pair \((h, \tilde{h})\) satisfies the second condition of Definition [5].

The following Theorem and its Corollary generalize the results from Section 6 of [1].

Theorem 8. Let \((M, \ast, \tilde{h}, E)\) be a weak \(\mathcal{F}\)-manifold with \(L_E(\tilde{h}) = \tilde{D}\tilde{h}, \; L_E(\ast) = k\ast\), where \(k\) is constant and \(\tilde{D} \in C^\infty(M)\). Let \(h\) be the metric on \(M\) defined by the relation \(h \ast \tilde{h} = E \ast\). Consider the multiplication \(\ast\) also on \(T^*M\), by identifying \(TM\) and \(T^*M\) using the metric \(h\). Then \((M, \ast, \tilde{h}, E)\) is an \(\mathcal{F}\)-manifold if and only if the equality

\[
R^h_{X,Y}(\alpha) = R^\tilde{h}_{X,Y}(\alpha) + \left( -R^h_{X,E}(\alpha) + \frac{1}{2} \alpha(X) \tilde{D}\tilde{D} \right) \ast \tilde{h}(E^{-1} \ast Y) + \left( -R^\tilde{h}_{Y,E}(\alpha) + \frac{1}{2} \alpha(Y) \tilde{D}\tilde{D} \right) \ast \tilde{h}(E^{-1} \ast X)
\]

holds, for every \(X, Y \in \mathcal{X}(M)\) and \(\alpha \in \mathcal{E}^1(M)\).

Proof. The argument is similar to the one employed in the proof of Theorem 23 of [1]. The only difference from the case studied in [1] is that \(\tilde{D}\) can be non-constant and then

\[
\nabla^\tilde{h}_{X,Y}(E)_X = R^\tilde{h}_{Y,E}(X) + \frac{1}{2} [-\tilde{h}(X, Y) \tilde{D}\tilde{D} + Y(\tilde{D})\tilde{h}(X) + X(\tilde{D})\tilde{h}(Y)]
\]

for every \(X, Y \in \mathcal{X}(M)\), which is the analogue of Lemma 22 of [1] and can be proved in the same way in this more general context.

Corollary 9. Consider the set-up of Proposition [5] and suppose that \(\tilde{h}\) is flat. Then \((M, \ast, \tilde{h}, E)\) is an \(\mathcal{F}\)-manifold if and only if \(h\) has constant sectional curvature \(s\) and \(d\tilde{D} = -2sh(E)\).

Proof. From Theorem [5] and the flatness of \(\tilde{h}\) we know that \((M, \ast, \tilde{h}, E)\) is an \(\mathcal{F}\)-manifold if and only if the curvature \(R^h\) of \(h\) has the following expression:

\[
R^h_{X,Y}(\alpha) = \frac{1}{2} \tilde{D}\tilde{D} \left( \alpha(X) \tilde{h}(E^{-1} \ast Y) - \alpha(Y) \tilde{h}(E^{-1} \ast X) \right), \tag{12}
\]

for every \(X, Y \in \mathcal{X}(M)\) and \(\alpha \in \mathcal{E}^1(M)\). It is clear now that if \(h\) has constant sectional curvature \(s\) and \(d\tilde{D} = -2sh(E)\), then relation \(\tag{12}\) is satisfied. Conversely,
suppose that \((M, \bullet, \bar{h}, E)\) is an \(\mathcal{F}\)-manifold, so that relation \([12]\) is satisfied. Then
\[
h \left(R^h_{X,Y} Z, V\right) = \frac{1}{2} \left(h(X, Z)d\bar{D}(V \bullet Y \bullet E^{-1}) - h(Y, Z)d\bar{D}(V \bullet X \bullet E^{-1})\right), \quad (13)
\]
for every \(X, Y, Z, V \in \mathcal{X}(M)\). On the other hand, since
\[
h \left(R^h_{X,Y} X, X\right) = -h \left(R^h_{Y,X} X, Y\right), \quad \forall X, Y \in \mathcal{X}(M)
\]
we easily get
\[
h(X, X)(d\bar{D})(Y^2 \bullet E^{-1}) = h(Y, Y)(d\bar{D})(X^2 \bullet E^{-1})
\]
or
\[
h(X, T)(d\bar{D})(Y \bullet S \bullet E^{-1}) = h(Y, S)(d\bar{D})(X \bullet T \bullet E^{-1}), \quad \forall X, Y, S, T \in \mathcal{X}(M).
\]
It follows that \(h(E) \wedge d\bar{D} = 0\) (let \(S = T := E\) in the above relation) or \(d\bar{D} = -2sh(E)\), for a function \(s \in C^\infty(M)\). From relation \([13]\) we deduce that \(s\) is constant and \(\bar{h}\) has constant sectional curvature \(s\).

\[\square\]

4. THE GEOMETRY OF CONFORMALLY SCALED COMPATIBLE PENCILS

In this Section we fix a pair of metrics \((g, \bar{g})\) on a manifold \(M\). The following Lemma will be relevant in our calculations.

**Lemma 10.** Suppose that the metrics \((g, \bar{g})\) are almost compatible. Then, for every \(X, Y \in \mathcal{X}(M)\) and \(\alpha \in \mathcal{E}^1(M)\) the relation
\[
g^* \left(\nabla^g_X \alpha - \nabla^g_X \alpha, \bar{g}(Y)\right) = g^* \left(\nabla^\bar{g}_X \alpha - \nabla^\bar{g}_X \alpha, \bar{g}(X)\right)
\]
holds.

**Proof.** Let \(X := \bar{g}^*(\gamma)\) and \(Y := \bar{g}^*(\delta)\), for \(\gamma, \delta \in \mathcal{E}^1(M)\). Then
\[
g^* \left(\nabla^g_X \alpha - \nabla^g_X \alpha, \bar{g}(Y)\right) = \delta \left(g^*\left(\nabla^\bar{g}_{g^{-1}\gamma} \alpha - \nabla^\bar{g}_{g^{-1}\gamma} \alpha\right)\right)
\]
\[= \delta \left(\bar{g}^*\left(\nabla^\bar{g}_{g^(-1)\gamma} \alpha - \nabla^\bar{g}_{g^{-1}\gamma} \alpha\right)\right)\]
\[= \bar{g}^*(\gamma \circ \alpha, \delta) = \bar{g}^*(\gamma, \delta \circ \alpha)\]
\[= g^* \left(\nabla^\bar{g}_X \alpha - \nabla^\bar{g}_X \alpha, \bar{g}(X)\right),\]
where “\(\circ\)” is the multiplication \([11]\) associated to the pair \((h, \bar{h})\) and we have used relations \([3]\) and \([4]\).

\[\square\]

As a consequence of Lemma \([10]\) we deduce that the compatibility property of two metrics is conformal invariant:

**Proposition 11.** Suppose that the metrics \((g, \bar{g})\) are compatible and let \(\Omega \in C^\infty(M)\), non-vanishing. Then the metrics \((h := \Omega^2 g, \bar{h} := \bar{\Omega}^2 \bar{g})\) are also compatible.

**Proof.** It is obvious that the metrics \(h\) and \(\bar{h}\) are almost compatible, since \(h^* \bar{h} = g^* \bar{g}\) (and hence the integrability tensor of \(h^* \bar{h}\) is identically zero). In order to show the
compatibility of \((h, \tilde{h})\), we first notice that
\[
\nabla^h_X \alpha = \nabla^\tilde{h}_X \alpha - \frac{d\Omega}{\Omega}(X)\alpha - \alpha(X)\frac{d\Omega}{\Omega} + g^* \left( \alpha, \frac{d\Omega}{\Omega} \right) g(X)
\]
\[
\nabla^\tilde{h}_X \alpha = \nabla^\tilde{h}_X \alpha - \frac{d\Omega}{\Omega}(X)\alpha - \alpha(X)\frac{d\Omega}{\Omega} + \tilde{g}^* \left( \alpha, \frac{d\Omega}{\Omega} \right) \tilde{g}(X),
\]
for every \(X \in \mathcal{X}(M)\) and \(\alpha \in \mathcal{E}^1(M)\), from where we deduce that
\[
\nabla^h \alpha - \nabla^\tilde{h} \alpha = \nabla^\tilde{h} \alpha - \nabla^\tilde{h} \alpha + \tilde{g}^* \left( \alpha, \frac{d\Omega}{\Omega} \right) \tilde{g}(X) - g^* \left( \alpha, \frac{d\Omega}{\Omega} \right) g(X).
\] (14)

To prove the compatibility of the metrics \((h, \tilde{h})\) we shall verify relation (15). Notice that, since \(h^* = \Omega^{-2}g^*\), we need to show that the relation
\[
g^* (\nabla^h_X \alpha - \nabla^\tilde{h}_X \alpha, \nabla^h_Y \beta - \nabla^\tilde{h}_Y \beta) = g^* (\nabla^h_Y \alpha - \nabla^\tilde{h}_Y \alpha, \nabla^h_X \beta - \nabla^\tilde{h}_X \beta)
\] (15)
holds, for every \(X, Y \in \mathcal{X}(M)\) and \(\alpha, \beta \in \mathcal{E}^1(M)\). Using the compatibility of the metrics \((g, \tilde{g})\) and relation (14), we easily see that relation (15) is equivalent with
\[
\tilde{g}^* \left( \beta, \frac{d\Omega}{\Omega} \right) [g^* \left( \nabla^\tilde{h}_X \alpha - \nabla^h_X \alpha, \tilde{g}(Y) \right) - g^* \left( \nabla^h_Y \alpha - \nabla^\tilde{h}_Y \alpha, \tilde{g}(X) \right)] +
\tilde{g}^* \left( \alpha, \frac{d\Omega}{\Omega} \right) [g^* \left( \nabla^h_Y \beta - \nabla^\tilde{h}_Y \beta, \tilde{g}(X) \right) - g^* \left( \nabla^\tilde{h}_X \beta - \nabla^h_X \beta, \tilde{g}(Y) \right)] = 0,
\]
which is obviously true from Lemma (10).
\]

For the rest of this Section we suppose that the metrics \((g, \tilde{g})\) are the regular flat metrics of a Frobenius manifold \((M, \bullet, \tilde{g}, E)\). We study the geometry of the pair of scaled metrics \((h := \Omega^2g, \tilde{h} := \Omega^2\tilde{g})\) together with the vector field \(E\). We restrict to the case when the scaled pair is regular and we denote by \(\bullet_h\) the associated multiplication on \(TM\) or \(T^*M\). Recall that the multiplication \(\bullet_g\) on \(TM\) associated to the pair of metrics \((g, \tilde{g})\) together with \(E\) coincides with the multiplication \(\bullet\) of the Frobenius manifold \((M, \bullet, \tilde{g}, E)\).

**Proposition 12.** The multiplications \(\bullet_h\) and \(\bullet_g\) coincide on \(TM\).

**Proof.** From relation (14) we easily see that the multiplications \(\circ_h\) and \(\circ_g\) associated to the pair of metrics \((h, \tilde{h})\) and \((g, \tilde{g})\) respectively are related by the formula
\[
\alpha \circ_h \beta = \Omega^{-2} [\alpha \circ_g \beta + g^* \left( \beta, \frac{d\Omega}{\Omega} \right) \alpha - g^* \left( \alpha, \frac{d\Omega}{\Omega} \right) g \alpha],
\] (16)
for every \(\alpha, \beta \in \mathcal{E}^1(M)\). Define the automorphisms \(T(\alpha) := g(E) \circ_h \alpha\) and \(\tilde{T}(\alpha) := h(E) \circ_h \alpha\) of \(T^*M\). Relation (14) also implies that
\[
\tilde{T}(\alpha) = T(\alpha) + g^* \left( \alpha, \frac{d\Omega}{\Omega} \right) g(E) - \tilde{g}^* \left( \alpha, \frac{d\Omega}{\Omega} \right) \tilde{g}(E).
\] (17)
Since \( \alpha \circ h \beta = \alpha \circ_h T(\beta) \) and similarly \( \alpha \circ g \beta = \alpha \circ_g T(\beta) \) we deduce from (16) and (17) that the relation
\[
\alpha \circ_h [T(\beta) + g^* \left( \beta \cdot \frac{d\Omega}{\Omega} \right) g(E) - \tilde{g}^* \left( \beta \cdot \frac{d\Omega}{\Omega} \right) \tilde{g}(E)]
= \Omega^{-2} [\alpha \circ_g T(\beta) + g^* \left( \beta \cdot \frac{d\Omega}{\Omega} \right) \alpha - \tilde{g}^* \left( \beta \cdot \frac{d\Omega}{\Omega} \right) \tilde{g}g^*(\alpha)]
\]
holds, for every \( \alpha, \beta \in \mathcal{E}^1(M) \). As in the proof of Proposition 4, \( \alpha \circ_h \tilde{g}(E) = \Omega^{-2} \partial h^*(\alpha) \) and \( \alpha \circ_h g(E) = \Omega^{-2} \alpha \) (the metrics \((h, \tilde{h})\) being compatible). It follows that \( \circ_h = \Omega^{-2} \circ_g \) on \( T^*M \), or \( \circ_h = \circ_g \) on \( TM \).

\[\square\]

**Proposition 13.** The following statements are equivalent:

1. The pair \((h, \tilde{h})\) is weak quasi-homogeneous with Euler vector field \(E\).
2. \(g(E) \wedge d\Omega = 0\).
3. \((M, \bullet, \tilde{h}, E)\) is an \(\mathcal{F}\)-manifold.
4. \((M, \bullet, h, E)\) is a weak \(\mathcal{F}\)-manifold.

**Proof.** Before proving the equivalence of the statements, we make some preliminary remarks. Since \((g, \tilde{g})\) are the flat metrics of a Frobenius manifold, \(L_E(g) = (1 - d)g\) and \(L_E(\tilde{g}) = Dg\) for some constants \(D\) and \(d\). It follows that
\[
L_E(h) = \left( 1 - d + \frac{2E(\Omega)}{\Omega} \right) h, \quad L_E(\tilde{h}) = \left( D + \frac{2E(\Omega)}{\Omega} \right) \tilde{h}.
\]
Also,
\[
\nabla^h_X(E) = \nabla^g_X(E) + \frac{d\Omega}{\Omega}(X)E + \frac{E(\Omega)}{\Omega} X - g(X, E)g^* \left( \frac{d\Omega}{\Omega} \right)
= \left( 1 - d + \frac{E(\Omega)}{\Omega} \right) X - \left( E \wedge g^* \left( \frac{d\Omega}{\Omega} \right) \right) (g(X)).
\]
Moreover, Proposition 11 implies that the metrics \((h, \tilde{h})\) are compatible. The equivalence 1 \(\iff\) 2 clearly follows from these facts. The equivalence 2 \(\iff\) 3 follows from Hertling’s observation mentioned at the end of Section 11. Indeed, condition (2) means that the coidentity \(\tilde{h}(e) = \Omega^2 g(E)\) is closed (note that the 1-form \(g(E)\), being equal to \(\tilde{g}(e)\), is closed because \(e\) is \(\nabla^g\)-parallel). To prove the equivalence 3 \(\iff\) 4 we notice that, since \((M, \bullet, \tilde{g}, E)\) is an \(\mathcal{F}\)-manifold, the \((3, 1)\)-tensor field \(\nabla^h(\bullet)\) satisfies the relation
\[
\nabla^h_X(\bullet)(Y, Z) - \nabla^h_Y(\bullet)(X, Z) = \left( \tilde{g}^* \left( \frac{d\Omega}{\Omega} \right) \wedge e \right) (\tilde{g}(Y \bullet Z) \bullet X
- \left( \tilde{g}^* \left( \frac{d\Omega}{\Omega} \right) \wedge e \right) (\tilde{g}(X \bullet Z) \bullet Y,
\]
for every \(X, Y, Z \in \chi(M)\). Suppose now that \((M, \bullet, \tilde{h}, E)\) is a weak \(\mathcal{F}\)-manifold. The symmetry
\[
\nabla^h_X(\bullet)(Y, Z) = \nabla^h_Y(\bullet)(E, Z), \quad \forall Y, Z \in \chi(M)
\]
of the \((3, 1)\)-tensor field \(\nabla^{\tilde{h}}(\bullet)\) becomes, after replacing \(Z\) with \(E^{-1}\bullet Z\), the relation:
\[
(\tilde{g}^* \frac{d\Omega}{\Omega} \wedge e)(\tilde{g}(Z)) \bullet Y = (\tilde{g}^* \frac{d\Omega}{\Omega} \wedge e)(\tilde{g}(Y \bullet Z \bullet E^{-1})) \bullet E.
\]

It is clear now that if \((M, \bullet, \tilde{h}, E)\) is a weak \(\mathcal{F}\)-manifold, then
\[
\nabla^{\tilde{h}}(\bullet)(Y, Z) = \nabla^{\tilde{h}}(\bullet)(X, Z), \quad \forall X, Y, Z \in \mathcal{X}(M)
\]
which implies that \((M, \bullet, \tilde{h}, E)\) is an \(\mathcal{F}\)-manifold (the symmetry of the \((4, 0)\)-tensor field \(\nabla^{\tilde{h}}(\bullet)\) in the last three arguments is a consequence of the fact that \(\tilde{h}\) is "\(\bullet\)"-invariant and of the commutativity of "\(\bullet\")). The equivalence \(3 \iff 4\) follows.

\[\square\]

Note that Hertling’s observation used in the proof of Proposition 13 together with Corollary \(\text{K}\) provide a different viewpoint of Proposition 13.

**Corollary 14.** Let \((g, \tilde{g})\) be the flat metrics of a Frobenius manifold \((M, \bullet, \tilde{g}, E)\). Let \(\Omega \in C^\infty(M)\) non-vanishing which satisfies \(d\Omega \wedge g(E) = 0\). Consider the scaled metrics \((h := \Omega^2 g, \tilde{h} := \Omega^2 \tilde{g})\). If \(h\) is flat, then \(\tilde{h}\) has constant sectional curvature.

**Proof.** The condition \(d\Omega \wedge g(E) = 0\) implies, using Hertling’s observation, that \((M, \bullet, \tilde{h}, E)\) is an \(\mathcal{F}\)-manifold. The conclusion follows from Corollary \(\text{K}\) (since \(h^* \tilde{h} = E\bullet\) and \(\tilde{h}\) is flat).

\[\square\]

5. **The modified Saito construction revisited**

We return now to the modified Saito construction described in Section 2, summarizing the various results in the following Theorem.

**Theorem 15.** Let \((g, \tilde{g})\) be the flat metrics of the Frobenius structure \((M, \bullet, \tilde{g}, E)\) on the space of orbits \(M = \mathbb{C}^n / W\) of a Coexter group \(W\). Let \(\{t^i\}\) be a Saito basis of \(W\)-invariant polynomials with \(\deg(t^i) = d_i\), in which the metric \(\tilde{g}\) is anti-diagonal, the identity vector field \(e\) is \(\frac{\partial}{\partial t^n}\) and the Euler vector field \(E\) has the expression \(\text{[10]}\). Consider the pair of scaled metrics \((h := \Omega^2 g, \tilde{h} := \Omega^2 \tilde{g})\), where \(\Omega \in C^\infty(M_0)\) is non-vanishing on an open subset \(M_0\) of \(M\). The following facts hold:

1. The metrics \((h, \tilde{h})\) are compatible on \(M_0\).
2. The metrics \((h, \tilde{h})\) together with the Euler vector field \(E\) is a weak quasi-homogeneous pencil on \(M_0\) if and only if \(\Omega\) depends only on the last coordinate \(t^n\). If \(\Omega = \Omega(t^n)\) and the weak quasi-homogeneous pair \((h, \tilde{h})\) is also regular, then the associated weak \(\mathcal{F}\)-manifold is \((M_0, \bullet, \tilde{h}, E)\) and is an \(\mathcal{F}\)-manifold.
3. Let \(\Omega(t) = (ct^n + d)^{-1}\), for \(c, d\) constants. The pair \((h, \tilde{h})\) together with \(E\) is weak quasi-homogeneous on \(\mathbb{H}^n \otimes \mathbb{C}/W\) (when \((cd) < 0\)) and on \(\mathbb{C}^n / W\) (when \((cd) > 0\)). It is regular on the open subset where
\[
t^n \neq \frac{d}{c}, \quad t^n \neq \frac{(1 - d_1)d}{(1 + d_1)c}.
\]
Moreover, \(\tilde{h}\) is flat and \(h\) has constant sectional curvature \(4(cd)\).
Proof. The first statement follows from Proposition 11. The second statement is a consequence of Proposition 13: note that \( g(E) = \tilde{g} \left( \frac{\partial}{\partial t_1} \right) = dt^n \). The third statement uses the proof of Proposition 3. Note that the endomorphism \( T \) from Definition 6 has the following expression:

\[
T(u) = \sum_{i=1}^{n} \left( (d_i - 1)u_i + \frac{cu_1}{ct^n + d} d_i n - i + 1 \right) dt^i - \frac{cu(E)}{ct^n + d} dt^n,
\]

for every \( u = \sum_{i=1}^{n} u_i dt^i \). The regularity condition can be easily checked. \( \square \)

From the symmetries of the tensors and the flatness of the metric, one may integrate the equations, via the Poincaré lemma, and express the tensors as derivatives, with respect to the flat coordinates, of a scalar prepotential. The differential equation satisfied by the prepotential being the celebrated Witten-Dijkgraaf-Verlinde-Verlinde (or WDVV) equation.

Thus given a Frobenius manifold with prepotential \( F \) one may conformally rescale the metrics, derive new flat coordinates and multiplication, and calculate the new prepotential \( \tilde{F} \).

\[
\begin{array}{c}
F \\
\downarrow \\
\tilde{F}
\end{array} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \frac{\partial}{\partial t_1} = t^i \quad (i = 1, 2, \ldots, n).
\end{array}
\]

\( \tilde{F} \) is the new prepotential obtained from \( F \) by conformal rescaling. This gives rise to an \( SL(2, \mathbb{C}) \)-symmetry on solution space of the WDVV equation.

Example:

Starting with the prepotential

\[
F = \frac{1}{2} t_1^2 t_3 + \frac{1}{2} t_1 t_2^2 + f(t_2, t_3)
\]

where \( f \) satisfies the differential equation

\[
f_{333} = f_{222}^2 - f_{233} f_{222}
\]

one obtains the new solution

\[
\tilde{F} = \frac{1}{2} \tilde{t}_1^2 \tilde{t}_3 + \frac{1}{2} \tilde{t}_1 \tilde{t}_2^2 + \left\{ \frac{c \tilde{t}_2^4}{8(c \tilde{t}_3 + d)} + (c \tilde{t}_3 + d)^2 f \left( \frac{\tilde{t}_2}{c \tilde{t}_3 + d} \right) \left( \frac{a \tilde{t}_3 + b}{c \tilde{t}_3 + d} \right) \right\}
\]

where \( ad - bc = 1 \). Note that this is a transformation on solutions on the WDVV equation: the transformation breaks the linearity condition on the Euler vector field (except in the very special case identified in [2]) and so does not generate new examples of Frobenius manifolds.

As mentioned in the introduction, these conformally flat pencils will automatically generate bi-Hamiltonian structures and hence certain integrable hierarchies of evolution equations. The properties of these hierarchies will be considered elsewhere.

\(^1\text{For notational convenience indices are dropped in these example only, so } t_i = t^i.\)
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