ON THE CONTINUITY OF THE CONTINUOUS STEINER SYMMETRIZATION

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Dedicated to Roger Wets for his 85th birthday

Abstract. Starting from the Brock’s construction of Continuous Steiner Symmetrization of sets, the problem of modifying continuously a given domain up to obtain a ball, preserving its measure and with decreasing first eigenvalue of the Laplace operator, is considered. For a large class of cases it is shown this is possible, while the general question remains still open.

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1. Introduction

The problem of rounding more and more a given set $\Omega \subset \mathbb{R}^d$, keeping fixed its measure and asymptotically reaching a ball of the same measure, enters in a number of problems and has been widely considered in the literature. More precisely, given a bounded open set $\Omega \subset \mathbb{R}^d$, the goal is to construct a family of domains $(\Omega_t)$, with $t \in [0,1]$, such that $\Omega_0 = \Omega$, $\Omega_1 = \Omega^*$ where $\Omega^*$ is a ball of the same measure as $\Omega$, and $|\Omega_t| = |\Omega|$ for all $t \in [0,1]$, where by $| \cdot |$ we denote the Lebesgue measure.

In addition, we require that the mapping $t \mapsto \Omega_t$ be continuous with respect to some suitable topology, and that the family $(\Omega_t)$ satisfy some monotonicity property that will be specified later.

We notice that, without the last monotonicity requirement, a very simple construction would provide a solution. Take indeed a set $\Omega$ and a point $x_0$ far enough from $\Omega$; denoting by $B(x_0, r)$ the ball of center $x_0$ and radius $r$ and by $\omega_d$ the Lebesgue measure of the unit ball in $\mathbb{R}^d$, the family

$$\Omega_t = (1-t)^{1/d} \Omega \cup B(x_0, r_t) \quad \text{with} \quad r_t = \left( \frac{t |\Omega|}{\omega_d} \right)^{1/d}$$

satisfies the measure constraint $|\Omega_t| = |\Omega|$, is such that $\Omega_0 = \Omega$ and $\Omega_1 = \Omega^*$, and is continuous in several useful topologies. An example of such a family $(\Omega_t)$ is illustrated in Figure 1.

The additional monotonicity conditions that we impose consists in the requirement that a suitable shape functional $F$ is monotone. For instance we could consider:

- $F(\Omega) = P(\Omega)$, the perimeter in the sense of De Giorgi, and we require $P(\Omega)$ is nonincreasing;
- $F(\Omega) = \mathcal{H}^{d-1}(\Omega)$, the Hausdorff $d-1$ dimensional measure, and we require $\mathcal{H}^{d-1}(\Omega)$ is nonincreasing;
Figure 1. The sets $\Omega_0$, $\Omega_{1/2}$, $\Omega_1$ when $\Omega$ is the rectangle $]0, 2[\times]0, 1[.$

- $F(\Omega) = T(\Omega)$, the torsional rigidity defined below, and we require $T(\Omega)$ is nondecreasing;
- $F(\Omega) = \lambda(\Omega)$, the first eigenvalue of the Dirichlet Laplacian defined below, and we require $\lambda(\Omega)$ is nonincreasing;
- $F(\Omega) = h(\Omega)$, the Cheeger constant, and we require $h(\Omega)$ is nonincreasing.

In this paper we focus the attention mostly on the first eigenvalue $\lambda(\Omega)$ and on the torsional rigidity $T(\Omega)$.

More precisely, $\lambda(\Omega)$ is the first eigenvalue of the Laplace operator $-\Delta$ with Dirichlet conditions on $\partial\Omega$, that is the minimal value $\lambda$ such that the PDE
\[
\begin{cases}
-\Delta u = \lambda u & \text{in } \Omega, \\
u \in H_0^1(\Omega),
\end{cases}
\]
has a nonzero solution. Equivalently, by the min-max principle (see for instance [11]) $\lambda(\Omega)$ can be defined through the minimization of the Rayleigh quotient, as
\[
\lambda(\Omega) = \min \left\{ \left[ \int_\Omega |\nabla u|^2 \, dx \right] \left[ \int_\Omega u^2 \, dx \right]^{-1} : u \in H_0^1(\Omega), \ u \neq 0 \right\}.
\]
An important bound for $\lambda(\Omega)$ is the Faber-Krahn inequality (see for instance [11], [12])
\[
\lambda(\Omega^*) \leq \lambda(\Omega),
\]
which can be stated in a scaling free form as
\[
|\Omega|^{2/d} \lambda(\Omega) \geq |B|^{2/d} \lambda(B),
\]
where $B$ is any ball in $\mathbb{R}^d$.

The torsional rigidity $T(\Omega)$ is defined as $\int_\Omega u_\Omega \, dx$, where $u_\Omega$ is the unique solution of the PDE
\[
\begin{cases}
-\Delta u = 1 & \text{in } \Omega, \\
u \in H_0^1(\Omega),
\end{cases}
\]
or equivalently through the maximization problem
\[
T(\Omega) = \max \left\{ \left[ \int_\Omega u \, dx \right]^2 \left[ \int_\Omega |\nabla u|^2 \, dx \right]^{-1} : u \in H_0^1(\Omega), \ u \neq 0 \right\},
\]
where the maximum is reached by $u_\Omega$ itself. Also for $T(\Omega)$ an important inequality holds, the Saint-Venant inequality
\[
T(\Omega) \leq T(\Omega^*),
\]
which can be stated in a scaling free form as
\[ |\Omega|^{-(d+2)/d}T(\Omega) \leq |B|^{-(d+2)/d}T(B) \]
where \( B \) is any ball in \( \mathbb{R}^d \).

The monotonicity properties we require to the family \( (\Omega_t) \) are then:
- the mapping \( t \mapsto \lambda(\Omega_t) \) is nonincreasing;
- the mapping \( t \mapsto T(\Omega_t) \) is nondecreasing.

Concerning the continuity of the map \( t \mapsto \Omega_t \) our requirement is that the solutions \( u_t \) of the PDEs
\[
\begin{cases}
-\Delta u_t = f & \text{in } \Omega_t, \\
u_t \in H^1_0(\Omega_t),
\end{cases}
\]
vary continuously in \( t \) with respect to the strong \( H^1(\mathbb{R}^d) \) convergence, for every right-hand side \( f \in L^2(\mathbb{R}^d) \). This is the \( \gamma \)-convergence, that we describe more precisely in Section 2.

When instead of a continuous family \( (\Omega_t) \) we consider the discrete case of a sequence \( (\Omega_n) \) such that
\begin{enumerate}
  \item \( \Omega_0 = \Omega, \ |\Omega_n| = |\Omega| \) for every \( n, \Omega_n \to \Omega^* \) in the \( \gamma \)-convergence,
  \item \( \lambda(\Omega_{n+1}) \leq \lambda(\Omega_n) \) and \( T(\Omega_{n+1}) \geq T(\Omega_n) \) for every \( n \),
\end{enumerate}
we have the problem that was first considered by Steiner, who proposed to use successive symmetrizations through different hyperplanes. More precisely, given a domain \( \Omega \subset \mathbb{R}^d \) and a direction \( \nu \), the **Steiner symmetrization** of \( \Omega \) with respect to \( \nu \) is defined as
\[ \Omega^*_\nu = \left\{ x \in \mathbb{R}^d : |x \cdot \nu| < \frac{\varphi(\pi(x))}{2} \right\}. \]

Here \( \pi(x) = x - \nu(x \cdot \nu) \) is the projection of a point \( x \in \mathbb{R}^d \) on the hyperplane orthogonal to \( \nu \) and, for each \( y \) in this hyperplane,
\[ \varphi(y) = \mathcal{H}^1(\Omega \cap \pi^{-1}(y)) \]
is the length of the \( y \)-section of \( \Omega \), where by \( \mathcal{H}^1 \) we denote the 1-dimensional Hausdorff measure.

Note that the set \( \Omega^*_\nu \) has the same volume of \( \Omega \) and is symmetric with respect to the hyperplane orthogonal to \( \nu \). In addition, it is well-known (see for instance [1]) that the Steiner symmetrization decreases the first eigenvalue and increases the torsional rigidity, that is
\[ \lambda(\Omega^*_\nu) \leq \lambda(\Omega) \quad \text{and} \quad T(\Omega^*_\nu) \geq T(\Omega). \]

By repeating this symmetrization procedure for a dense sequence of directions \( \nu \), one obtains a sequence \( \Omega_n \) of sets, all with the same measure, which \( \gamma \)-converge as \( n \to \infty \) to the ball \( \Omega^* \).

The question is now to pass from the discrete Steiner symmetrization to a continuous one. Since successive Steiner symmetrizations allow to pass from a generic set to a ball, it is enough to construct a continuous family \( \Omega_t \) of sets which transforms a set \( \Omega \) into its Steiner symmetrization \( \Omega^*_\nu \) for a fixed direction \( \nu \). An explicit construction of a family \( \Omega_t \) was proposed by Brock in [4] (see also [5]) and was called **Continuous Steiner Symmetrization**. We shortly recall the Brock’s construction in Section 3.
Unfortunately, the Brock’s construction provides the $\gamma$-continuity of the family $\Omega_t$ only in very particular situations, as for instance when the initial domain $\Omega$ is convex, while in general discontinuities may occur, due to irregularities of the domains $\Omega_t$. On the other hand, the $\gamma$-continuity would be very useful in several situations, as for instance in the study of some Blaschke-Santaló diagrams, as illustrated in [8].

In the present paper we show that a modification of Brock’s construction could be enough to provide the required $\gamma$-continuity of the family $\Omega_t$, at least for a larger class of domains $\Omega$. In [8] a similar construction was made for polyhedral domains $\Omega$. Even if the arguments are not complete, we believe it could help to better understand the difficulties behind the Continuous Steiner Symmetrization.

In the last section we consider a possible alternative approach based on the De Giorgi theory of minimizing movements.

2. The $\gamma$-Convergence

In this section we recall the definition and the main properties of $\gamma$-convergence; for all details, proofs, and generalization to the class of capacitary measures, we refer the interested reader to [6]. For simplicity, we make the assumption that all the domains we consider are included in a given bounded open subset $D$ of $\mathbb{R}^d$, which is satisfied for the domains we consider later. In the following, for every domain $\Omega$, a function in $H^1_0(\Omega)$ is considered extended by zero on $\mathbb{R}^d \setminus \Omega$.

**Definition 2.1.** A sequence $(\Omega_n)$ of domains is said to $\gamma$-converge to a domain $\Omega$ if for every $f \in L^2(\mathbb{R}^d)$ the solutions $u_{n,f}$ of the PDEs

\[
\begin{cases}
-\Delta u = f & \text{in } \Omega_n \\
u \in H^1_0(\Omega_n)
\end{cases}
\]

converge weakly in $H^1(\mathbb{R}^d)$ to the solution $u_f$ of the PDE

\[
\begin{cases}
-\Delta u = f & \text{in } \Omega \\
u \in H^1_0(\Omega)
\end{cases}
\]

We summarize here below the main properties of the $\gamma$-convergence. We refer to [6] for all the details, properties, and proofs.

- The weak $H^1(\mathbb{R}^d)$ convergence of $u_{n,f}$ to $u_f$ is equivalent to the strong $H^1(\mathbb{R}^d)$ convergence. Indeed, integrating by parts we obtain

\[
\int |\nabla u_{n,f}|^2 \, dx = \int u_{n,f} f \, dx \rightarrow \int u_f f \, dx = \int |\nabla u_f|^2 \, dx.
\]

- In the definition above it is not difficult to show that it is equivalent to require the weak $H^1(\mathbb{R}^d)$ convergence of $u_{n,f}$ to $u_f$ for every $f \in L^2(\mathbb{R}^d)$ or for every $f \in H^{-1}(\mathbb{R}^d)$. Indeed, if $f \in H^{-1}(\mathbb{R}^d)$ it is enough to approximate $f$ by a sequence $f_k \in L^2(\mathbb{R}^d)$, in the $H^{-1}$ norm, to obtain for every test
function $\phi$
\[
\left| \int \nabla u_n \nabla \phi \, dx - \int \nabla u_f \nabla \phi \, dx \right| = \left| \langle f, \phi \rangle_{H^1_0(\Omega_n)} - \langle f, \phi \rangle_{H^1_0(\Omega)} \right| \\
\leq \left| \langle f_k, \phi \rangle_{H^1_0(\Omega_n)} - \langle f_k, \phi \rangle_{H^1_0(\Omega)} \right| + \varepsilon_k \|\phi\| \\
= \left| \int \nabla u_{n,k} \nabla \phi \, dx - \int \nabla u_k \nabla \phi \, dx \right| + \varepsilon_k \|\phi\|.
\]
where $\varepsilon_k \to 0$. Passing to the limit first as $n \to \infty$ and then as $k \to \infty$ gives what claimed.

- The $\gamma$-convergence can be defined in a similar way for quasi-open sets $\Omega \subset D$ or more generally for capacitary measures $\mu$ confined into $D$ (that is $\mu = +\infty$ outside $D$). Quasi-open sets are sets of positivity $\{u > 0\}$ of functions $u \in H^1(\mathbb{R}^d)$, while capacitary measures are regular nonnegative Borel measures $\mu$ on $D$, possibly $+\infty$ valued, such that $\mu(E) = 0$ for every Borel set $E \subset D$ with $\text{cap}(E) = 0$. For all details on quasi-open sets and capacitary measures we refer the interested reader to the book [6]. Here we only recall that for a capacitary measure $\mu$ the corresponding PDE is formally written as
\[
\begin{cases}
-\Delta u + \mu u = f & \text{in } D \\
u \in H^1_0(D) \cap L^2_\mu(D)
\end{cases}
\]
and has to be intended in the weak sense, that is, $u \in H^1_0(D) \cap L^2_\mu(D)$ and
\[
\int_D \nabla u \nabla \phi \, dx + \int_D u \phi \, d\mu = \langle f, \phi \rangle
\]
for all $\phi \in H^1_0(D) \cap L^2_\mu(D)$. We notice that open sets or more generally quasi-open sets can be seen as capacitary measures: for a given domain $\Omega$ the capacitary measure representing it is the measure $\infty_{\Omega^c}$ defined as
\[
\infty_{\Omega^c}(E) = \begin{cases}
0 & \text{if } \text{cap}(E \cap \Omega) = 0 \\
+\infty & \text{otherwise.}
\end{cases}
\]
- In Definition 2.1 it is possible to show (see Remark 4.3.10 of [6]) that requiring the convergence of the solutions $u_n$ to $u$ for every right-hand side $f$ is equivalent to require the convergence $u_n \to u$ only for $f \equiv 1$ and in the $L^2(D)$ sense. In particular, calling $u_\mu$ the unique solution of the PDE $-\Delta u + \mu u = 1$ in $H^1_0(D) \cap L^2_\mu(D)$, the quantity
\[
d_\gamma(\mu_1, \mu_2) = \|u_{\mu_1} - u_{\mu_2}\|_{L^2(D)}
\] (2.1)
is a distance on the space $\mathcal{M}$ of capacitary measures, which is equivalent to $\gamma$-convergence, and so $\mathcal{M}$ endowed with the distance $d_\gamma$ above is a compact metric space. Since the solutions $u_\mu$ are all equi-bounded (for instance they are all below by the solution $w$ of the Dirichlet problem $-\Delta w = 1$ on $H^1_0(D)$, which is a bounded function) the $L^2$ norm in (2.1) can be replaced by any $L^p$ norm, with $1 \leq p < +\infty$. In particular, if $p = 1$ and $\Omega_1 \subset \Omega_2$ we have
\[
\|u_{\Omega_1} - u_{\Omega_2}\|_{L^1} = \int u_{\Omega_2} \, dx - \int u_{\Omega_1} \, dx = T(\Omega_2) - T(\Omega_1),
\]
and the $\gamma$-convergence is then reduced to the convergence of the corresponding torsional rigidities.

- The first eigenvalue $\lambda(\Omega)$ (as well as all the other eigenvalues $\lambda_k(\Omega)$) and the torsional rigidity $T(\Omega)$ are continuous with respect to the $\gamma$-convergence.

- The Lebesgue measure $|\Omega|$, or more generally integral functionals as $\int_{\Omega} f(x) \, dx$ with $f \geq 0$ and measurable, are lower semicontinuous with respect to the $\gamma$-convergence on the domains $\Omega$.

- As stated above, the space $\mathcal{M}$ of capacitary measures, endowed with the $\gamma$-convergence, is a compact metric space. On the contrary, the family of open sets (or also quasi-open sets) is not compact in $\mathcal{M}$; it is actually a dense subset of $\mathcal{M}$. The first example of a sequence of open sets $\Omega_n$ which $\gamma$-converges to a capacitory measure which is not a domain (actually to the Lebesgue measure) was given in [9].

- Several subclasses of $\mathcal{M}$ are dense with respect to the $\gamma$-convergence (see Proposition 4.3.7 and Remark 4.3.8 of [6]). For instance:
  - the class of measures $a(x) \, dx$ with $a \geq 0$ and smooth;
  - the class of smooth domains $\Omega \subset D$;
  - the class of polyhedral domains $\Omega \subset D$;
  - the class of measures of the form $a(x) \, d\mathcal{H}^{d-1}$ with $a \geq 0$ and smooth, where $\mathcal{H}^{d-1}$ is the $d-1$ dimensional Hausdorff measure;
  - the class of measures of the form $\mathcal{H}^{d-1}[S]$, where $S \subset D$ is a smooth $d-1$ manifold.

3. The Brock’s construction

We summarize rapidly here the construction by Brock (see [4], [5]) of the continuous Steiner symmetrization, together with the properties important for our purpose. The first construction is for the unidimensional case; here taking the variable $t$ in $[0, +\infty]$ or in $[0, 1]$ does not make any real difference.

- If $I$ is the interval $]a, b[$, then the continuous Steiner symmetrization $I^t$ is the interval $]a^t, b^t[$, where
  \[ a^t = \frac{(a - b + e^{-t}(a + b))}{2}, \quad b^t = \frac{(b - a + e^{-t}(a + b))}{2}. \]

- If $A$ is an open subset of $\mathbb{R}$ we consider the properties:
  (i) $A(0) = A$;
  (ii) if $I$ is an interval with $I \subset A(s)$, then $I^t \subset A(s + t)$ for every $t \geq 0$.

We define then the continuous Steiner symmetrization $A^t$ as

\[ A^t = \bigcap \{ A(t) : A(t) \text{ satisfies (i) and (ii)} \}. \]

In [4] Brock proves that if $A$ is open then $A^t$ are open sets; in addition the monotonicity property

\[ A \subset B \implies A^t \subset B^t \quad \text{for every } t \]

holds.

- Finally, if $A \subset \mathbb{R}$ is only measurable, we have
  \[ A = \bigcap_n A_n \setminus N \]
with $A_n$ open sets and $N$ Lebesgue negligible. We then define the continuous Steiner symmetrization $A^t$ of $A$ as

$$A^t = \bigcap_n A^t_n.$$  

This definition is unique up to a nullset, and we still call continuous Steiner symmetrization a family $A^t$ such that $|A^t \Delta (\bigcap_n A^t_n)| = 0$.

We can now pass to define the continuous Steiner symmetrization for subsets of $\mathbb{R}^d$, with respect to a hyperplane that, with no loss of generality, we can suppose to be $\mathbb{R}^{d-1}$. For a general set $A$ we define the projection of $A$ on $\mathbb{R}^{d-1}$ as

$$A' = \{ x' \in \mathbb{R}^{d-1} : (x', y) \in A \text{ for some } y \in \mathbb{R} \},$$

and for $x' \in A'$ the intersection of $A$ with $(x', \mathbb{R})$ as

$$A(x') = \{ y \in \mathbb{R} : (x', y) \in A \}.$$ 

Note that $A(x')$ is a one-dimensional set. When $A$ is an open subset of $\mathbb{R}^d$ we define its continuous Steiner symmetrization $A^t$ by

$$A^t = \{ x = (x', y) : x' \in A', \ y \in (A(x'))^t \}. \tag{3.1}$$

If $A \subset \mathbb{R}^d$ is only measurable, we define its continuous Steiner symmetrization by the same formula as (3.1), but up to Lebesgue negligible sets.

We stress that, for a bounded quasi-open set $A$, the previous construction only provides a measurable set defined up to a set of zero Lebesgue measure. In order to obtain that the symmetrized sets be still quasi-open and defined quasi-everywhere, it is convenient, for a bounded quasi-open set $A$, to define (by an abuse of notation) the symmetrized set $A^t$ in the following way: consider a decreasing sequence of bounded open sets $(A_n)$ with $\text{cap}(A_n \setminus A) \to 0$ and $A \subset A_n$. For any $t \in [0, 1]$ the set $A^t_n$ is well defined, and by monotonicity we may define $A^t_n \supset A^t_{n+1}$. Then $(A^t_n)$ is $\gamma$-convergent and we define

$$A^t = \gamma - \lim_{n \to \infty} A^t_n.$$ 

In this way, the set $A^t$ is quasi-open. More details on this issue can be found in [6]; in particular, the proofs that the construction above is independent of the sequence $A_n$ and that the Lebesgue measure is preserved, are still missing.

The continuous Steiner symmetrization can be defined for any positive measurable function $u$ by symmetrizing its level sets:

$$\forall s > 0 \quad \{ u^t > s \} := \{ u > s \}^t.$$ 

The main properties of the Brock’s construction are summarized here below, where $\lambda_k(\Omega)$ denotes the $k$-th eigenvalue of the Dirichlet Laplacian in $\Omega$.

**Proposition 3.1.** For every bounded quasi-open set $\Omega \subset \mathbb{R}^d$ and every positive integer $k$ the mapping $t \mapsto \lambda_k(\Omega^t)$, is lower semicontinuous on the left and upper semicontinuous on the right.

When the starting set $\Omega$ is convex, or more generally when the one-dimensional sections $\Omega(x')$ above are intervals, the $\gamma$-continuity actually occurs. However, this is not always the case, as the example of Figure 2 shows. Up to the moment when the internal fracture appears the $\gamma$-continuity is verified; on the other hand, the
Brock’s construction removes the fracture instantaneously, and the $\gamma$-continuity is lost.

![Figure 2](image_url)

Figure 2. A set $\Omega$ such that $t \mapsto \lambda(\Omega_t)$ is discontinuous.

Since the torsional rigidity $T(\Omega_t)$ is increasing along the family $(\Omega_t)$, it has only countably many discontinuity points. Let $t_0$ be one of these points and assume that at $t_0$ we have two domains $\Omega^-, \Omega^+$ such that $\Omega^- \subset \Omega^+$ and

$$
\begin{cases}
T(\Omega_t) \to T(\Omega^-) \quad &\text{as } t \to t_0 \text{ from the left} \\
T(\Omega_t) \to T(\Omega^+) \quad &\text{as } t \to t_0 \text{ from the right}
\end{cases}
\tag{3.2}
$$

In other words $\Omega^-$ is the domain with fractures, while $\Omega^+$ is the domain where the fractures have been removed.

**Remark 3.2.** In the one-dimensional case the existence of a $\gamma$-continuous family $(\Omega_t)$ cannot be obtained in general, since starting by $\Omega_0$ made of two segments and ending by $\Omega_1$ made of a single segment will necessarily produce a discontinuity of $T(\Omega_t)$ at some point $t_0$, independently of the construction of the family $(\Omega_t)$.

In the case $d \geq 2$ on the contrary, we can fill the discontinuity between $\Omega^-$ and $\Omega^+$ by constructing a $\gamma$-continuous family $(\Omega_t)$, with $\Omega_t$ increasing with respect to the set inclusion, and $\Omega_0 = \Omega^-$, $\Omega_1 = \Omega^+$.

**Theorem 3.3.** Let $d \geq 2$ and let $\Omega_0 \subset \Omega_1$ be two bounded open sets. Then there exists a $\gamma$-continuous family $\Omega_t$ of open sets ($t \in [0, 1]$) such that

$$
\Omega_s \subset \Omega_t \quad \text{for every } s < t.
\tag{3.3}
$$

**Proof.** Let us denote by $C$ a large cube containing $\Omega_1$ and by $\Gamma(t)$ a Peano curve from $[0, 1]$ onto $C$, that is a continuous mapping $\Gamma : [0, 1] \to \mathbb{R}^d$ such that $\Gamma([0, 1]) = C$; we also choose $\Gamma(0) \in \Omega_0$. We define

$$
\Omega_t = (\Omega_1 \setminus \Gamma([0, 1 - t])) \cup \Omega_0 \quad \text{for every } t \in [0, 1].
$$

Note that $\Omega_t$ are open subsets of $\mathbb{R}^d$ and that for $t = 0$ we obtain $\Omega_0$, while for $t = 1$ we obtain $\Omega_1$. The family $\Omega_t$ above clearly satisfies the monotonicity property

In order to show that the family $\Omega_t$ is $\gamma$-continuous, it is enough to prove that

$$
\text{cap}(\Omega_n \Delta \Omega_t) \to 0 \quad \text{whenever } t_n \to t.
$$

This comes from the fact that the mapping $\Gamma(t)$ is uniformly continuous, so that

$$
|\Gamma(t) - \Gamma(t_n)| \leq \omega(|t - t_n|)
$$
for a suitable modulus of continuity \( \omega \). Therefore \( \Omega_t \) and \( \Omega_{t_n} \) differ by a set which has a diameter less than \( 2\omega(|t-t_n|) \), hence of capacity which vanishes as \( t_n \to t \). □

**Remark 3.4.** Since the proof of Theorem 3.3 is only based on capacitary arguments, the same statement is valid in the more general case when \( \Omega_0 \) and \( \Omega_1 \) are quasi-open sets.

**Remark 3.5.** When working with polyhedral domains (i.e. whose boundary is made of a finite number of subsets of hyperplanes) we are in the situation above. In fact, if \( \Omega \) is a polyhedral domain, the Brock’s construction provides a family \( \Omega_t \) made of polyhedral domains, and we have a finite number of discontinuity points \( t_1, t_2, \ldots, t_N \). In addition, for every discontinuity point \( t_k \), the fracture \( S \) is a \( d-1 \) dimensional polyhedral set, \( \Omega^- = \Omega_{t_k} \) while \( \Omega^+ = \Omega_{t_k} \setminus S \), and then Theorem 3.3 applies.

In several situations (see for instance [8]), thanks to the \( \gamma \)-density of polyhedral domains in the class of all domains, Remark 3.5 is sufficient to achieve the required goals. However, the question of existence of \( \gamma \)-continuous paths \((\Omega_t)\), with monotone \( \lambda(\Omega_t) \) and \( T(\Omega_t) \), between a general domain \( \Omega_0 \) and the ball \( B \) with the same Lebesgue measure, remains.

Similar questions arise if, instead of the quantities \( \lambda(\Omega_t) \) and \( T(\Omega_t) \), one considers for instance the perimeter \( P(\Omega_t) \), requiring the continuity of the map \( t \mapsto P(\Omega_t) \) and its decreasing monotonicity.

The procedure of removing fractures mentioned after (3.2) needs to be more rigorous. This can be made through the following result.

**Proposition 3.6.** Let \( \Omega_0 \) be a given quasi open set and let \( m \geq |\Omega_0| \). Then there exists a quasi open set \( \hat{\Omega} \) solving the shape optimization problem

\[
\min \left\{ \lambda(\Omega) : \Omega_0 \subset \Omega, |\Omega| \leq m \right\}.
\]

*Proof.* The proof can be obtained directly by applying the existence result of [7]. □

In an analogous way we can obtain a solution for the shape optimization problem

\[
\max \left\{ T(\Omega) : \Omega_0 \subset \Omega, |\Omega| \leq m \right\}.
\]

In particular, the case \( m = |\Omega_0| \) is interesting; this allows to obtain, for every given \( \Omega_0 \), an optimal domain \( \hat{\Omega} \) containing \( \Omega_0 \) and with the same measure as \( \Omega_0 \), which solves simultaneously the two shape optimization problems

\[
\begin{align*}
\min \left\{ \lambda(\Omega) : \Omega_0 \subset \Omega, |\Omega| = |\Omega_0| \right\}, \\
\max \left\{ T(\Omega) : \Omega_0 \subset \Omega, |\Omega| = |\Omega_0| \right\}.
\end{align*}
\]

Indeed, if \( \Omega_1 \) is an optimal domain for the eigenvalue optimization problem and \( \Omega_2 \) an optimal domain for the torsion optimization problem, it is enough to take \( \hat{\Omega} = \Omega_1 \cup \Omega_2 \).

In other words, if \( \Omega_0 \) is a Lipschitz domain, we have \( \hat{\Omega} = \Omega_0 \) while, in the case the set \( \Omega_0 \) presents some internal fractures, the set \( \hat{\Omega} \) removes them.
4. The minimizing movement approach

An alternative approach to the Brock’s construction of the family $\Omega_t$ through the Continuous Steiner Symmetrization could be the use of the De Giorgi minimizing movement theory, introduced in [10] (see for instance [2], [3] for a detailed presentation and further developments).

In our framework of shape functionals, the metric space $X$ could be the one of all measurable subsets $\Omega$ of the Euclidean space $\mathbb{R}^d$ with a prescribed Lebesgue measure, say $|\Omega| = 1$, endowed with the $L^1$ distance

$$d(\Omega_1, \Omega_2) = |\Omega_1 \Delta \Omega_2|.$$  

Given a shape functional $F$ defined on $X$ one can consider the so-called implicit Euler scheme of time step $\varepsilon$ and initial condition $\Omega_0$, which provides a discrete family $\Omega_{n,\varepsilon}$ constructed recursively in the following way:

$$\Omega_{0,\varepsilon} = \Omega_0, \quad \Omega_{n+1,\varepsilon} \in \text{argmin}_{\Omega \in X} \left\{ F(\Omega) + \frac{1}{2\varepsilon} |\Omega \Delta \Omega_{n,\varepsilon}|^2 \right\}.$$  

We may then set $\Omega_{t,\varepsilon} = \Omega_{[t/\varepsilon],\varepsilon}$, where $[\cdot]$ stands for the integer part function, and say that $\Omega_t$ is a family of sets constructed by the minimizing movement procedure associated to the shape functional $F$ if for every $t \in [0, T]$ we have

$$|\Omega_t \Delta \Omega_{[t/\varepsilon],\varepsilon}| \to 0 \quad \text{as } \varepsilon \to 0.$$  

If the limit above occurs only for a sequence $(\varepsilon_n)$ (independent of $t$), we say that $\Omega_t$ is a generalized minimizing movement.

It is easy to see that the discrete sequence $\Omega_{n,\varepsilon}$ is such that $F(\Omega_{n,\varepsilon})$ decreases. It would be interesting to show, at least in the particular cases when the shape functional $F(\Omega)$ is the first eigenvalue $\lambda(\Omega)$, the opposite $-T(\Omega)$ of the torsional rigidity, or the perimeter $P(\Omega)$, or some convex combination of them, that the map $t \mapsto F(\Omega_t)$ is continuous and decreasing.

We do not know if the map $t \mapsto F(\Omega_t)$ above is continuous and decreasing, and the cases in which, as $t \to \infty$, the limit domain is a ball. Some results in this direction, in the case $F(\Omega) = P(\Omega)$ can be found in [14], while some partial results in the case of spectral functionals can be found in [13].

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