Conservation of the noetherianity by perfect transcendental field extensions

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March, 2001

Abstract

Let $k$ be a perfect field of characteristic $p > 0$, $k(t)_{\text{per}}$ the perfect closure of $k(t)$ and $A$ a $k$-algebra. We characterize whether the ring

$$A \otimes_k k(t)_{\text{per}} = \bigcup_{m \geq 0} (A \otimes_k k(t^{1/p^m}))$$

is noetherian or not. As a consequence, we prove that the ring $A \otimes_k k(t)_{\text{per}}$ is noetherian when $A$ is the ring of formal power series in $n$ indeterminates over $k$.

Keywords: perfect−power series ring−noetherian ring− perfect extension−complete local ring.

Introduction

Motivated by the generalization of the results in [10] (for the case of a perfect base field $k$ of characteristic $p > 0$) in this paper we study the conservation of noetherianity by the base field extension $k \rightarrow k(t)_{\text{per}}$, where $k(t)_{\text{per}}$ is the

*Both authors are partially supported by DGESIC, PB97-0723.
perfect closure of $k(t)$. Since this extension is not finitely generated, the conservation of noetherianity is not clear \textit{a priori} for $k$-algebras which are not finitely generated.

Our main result states that $k(t)_{\text{per}} \otimes_k A$ is noetherian if and only if $A$ is noetherian and for every prime ideal $p \subset A$ the field $\bigcap_{m \geq 0} Qt(A/p)^{p^m}$ is algebraic over $k$ (see theorem 3.6). In particular, we are able to apply this result to the case where $A$ is the ring of formal power series in $n$ indeterminates over $k$.

We are indebted to J. M. Giral for giving us the proof of proposition 2.6 and for other helpful comments.

1 Preliminaries and notations

All rings and algebras considered in this paper are assumed to be commutative with unit element. If $B$ is a ring, we shall denote by $\dim(B)$ its Krull dimension and by $\Omega(B)$ the set of its maximal ideals. We shall use the letters $K, L, k$ to denote fields and $\mathbb{F}_p$ to denote the finite field of $p$ elements, for $p$ a prime number. If $p \in \text{Spec}(B)$, we shall denote by $\text{ht}(p)$ the height of $p$.

Remember that a ring $B$ is said to be \textit{equicodimensional} if all its maximal ideals have the same height. Also, $B$ is said to be \textit{biequicodimensional} if all its saturated chains of prime ideals have the same length.

If $B$ is an integral domain, we shall denote by $Qt(B)$ its quotient field.

For any $\mathbb{F}_p$-algebra $B$, we denote $B^\sharp = \bigcap_{m \geq 0} B^{p^m}$.

We shall first study the contraction-extension process for prime ideals relative to the ring extension $K[t] \subset K[t^{1/p}]$, $K$ being a field of characteristic $p > 0$. Let us recall the following well known result (cf. for example [I], th. 10.8):

\textbf{Proposition 1.1} Let $K$ be a field of characteristic $p > 0$. Let $g(X)$ be a monic polynomial of $K[X]$. Then, the polynomial $f(X) = g(X^p)$ is irreducible in $K[X]$ if and only if $g(X)$ is irreducible in $K[X]$ and not all its coefficients are in $K^p$.

From the above result, we deduce the following corollary.
Corollary. 1.2 Let $K$ be a field of characteristic $p > 0$. Let $P$ be a non zero prime ideal in $K[t^{\frac{1}{p}}]$ and let $F(t) \in K[t]$ be the monic irreducible generator of the contraction $P^e = P \cap K[t]$. Then the following conditions hold:

1. If $F(t) = a_0^p + a_1^p t + \cdots + t^d \in K^p[t]$, then $P = (a_0 + a_1 t^\frac{1}{p} + \cdots + t^d)$.  
2. The equality $P = P^{ce}$ holds if and only if $F(t) \not\in K^p[t]$.

Proof:

1. Consider the polynomial $G(\tau) = a_0 + a_1 \tau + \cdots + \tau^d \in K[\tau](\tau = t^\frac{1}{p})$ and the ring homomorphism $\mu : K[\tau] \rightarrow K[t]$ defined by

$$\mu(\sum a_i \tau^i) = \sum a_i^p t^i.$$ 

From the identity $\mu(G) = F$ we deduce that $G(\tau)$ is irreducible. Since $G(t^\frac{1}{p})^p = F(t) \in P$, we deduce that $G(t^\frac{1}{p}) \in P$ and then $P = (G(t^\frac{1}{p}))$.

2. The equality $P = P^{ce}$ means that $F(t) = F(\tau^p) \in K[\tau]$ generates the ideal $P$, but that is equivalent to saying that $F(\tau^p)$ is irreducible in $K[\tau]$. To conclude, we apply proposition 1.1.

For each $k$-algebra $A$, we define $A(t) := k(t) \otimes_k A$. We also consider the field extension

$$k(\infty) = \bigcup_{m \geq 1} k(t^\frac{1}{p^m}).$$

If $k$ is perfect, $k(\infty)$ coincides with the perfect closure of $k(t)$, $k(t)_{per}$.

For the sake of brevity, we will write $t_m = t^\frac{1}{p^m}$. We also define

$$A_{(m)} := A(t_m) := A \otimes_k k(t_m) = A(t) \otimes_{k(t)} k(t_m), \quad A_{[m]} := A[t_m]$$

and

$$A_{(\infty)} := A \otimes_k k(\infty) = \bigcup_{m \geq 0} A_{(m)}, \quad A_{[\infty]} := \bigcup_{m \geq 0} A_{[m]}.$$ 

Each $A_{(m)}$ (resp. $A_{[m]}$) is a free module over $A(t)$ (resp. over $A[t]$) of rank $p^m$ (because $(t_m)^{p^m} - t = 0$).

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For each prime ideal \( P \) of \( A_\infty \) we denote \( P_\infty := P \cap A_\infty \), \( P_m := P \cap A_m \in \text{Spec}(A_m) \) and \( P_m := P \cap A_m \in \text{Spec}(A_m) \).

In a similar way, if \( Q \) is a prime ideal of \( A_\infty \) we denote \( Q_m := Q \cap A_m \in \text{Spec}(A_m) \).

We have:

- \( P = \bigcup_{m \geq 0} P_m, \ P_\infty = \bigcup_{m \geq 0} P_m \), (resp. \( Q = \bigcup_{m \geq 0} Q_m \)).
- \( P_n \cap A_m = P_m \) and \( P_n \cap A_m = P_m \) for all \( n \geq m \) (resp. \( Q_n \cap A_m = Q_m \) for all \( n \geq m \)).

The following properties are straightforward:

1. The \( k \)-algebras \( A_m \) (respectively \( A_m \)) are isomorphic to each other.
2. If \( S_m = k[t_m] \setminus \{0\} \), then \( A_m = S_m^{-1} A_m \).
3. Since \( (S_m)^n \subset S_0 \subset S_m \), we have \( A_m = S_m^{-1} A_m \) for \( m \geq 0 \). Consequently \( A_\infty = S_0^{-1} A_\infty \).
4. If \( A \) is a domain (integrally closed), then \( A_m \) and \( A_m \) are domains (integrally closed) for all \( m \geq 0 \) or \( m = \infty \).
5. If \( A \) is a noetherian \( k \)-algebra, then \( A_m \) and \( A_m \) are noetherian rings, for every \( m \geq 0 \).
6. If \( A = k[X] = k[X_1, \ldots, X_n] \), then \( A_\infty \) is not noetherian (the ideal generated by the \( t_m, m \geq 0 \), is not finitely generated).
7. If \( I \subset A \) is an ideal, then \((A/I)_\infty = A_\infty/I \).
8. If \( T \subset A \) is a multiplicative subset, then \((T^{-1}A)_\infty = T^{-1}A_\infty \).
9. If \( A = k[X] \), then \( A_\infty = k_\infty[X] \), hence \( A_\infty \) is noetherian. Moreover, \( A_\infty \) is noetherian for every finitely generated \( k \)-algebra \( A \).

The main goal of this paper is to characterize whether the ring \( A_\infty \) is noetherian (see th. 3.6 and corollary 3.8).

**Proposition. 1.3** With the above notations, the following properties hold:
1. The extensions $A_{[m-1]} \subset A_m$ and $A_{(m-1)} \subset A_m$ are finite free, and therefore integral and faithfully flat.

2. The corresponding extensions to their quotient fields are purely inseparable.

Proof: Straightforward.

Corollary 1.4 $A_{[\infty]}$ (resp. $A_{(\infty)}$) is integral and faithfully flat over each $A_{[m]}$ (resp. over each $A_{(m)}$).

From the properties above, we obtain the following lemmas:

Lemma 1.5 Let $P' \subseteq P$ be prime ideals of $A_{(\infty)}$ (resp. of $A_{[\infty]}$). The following conditions are equivalent:

(a) $P' \subsetneq P$

(b) There exists an $m \geq 0$ such that $P'_{(m)} \subsetneq P_{(m)}$ (resp. $P'_{[m]} \subsetneq P_{[m]}$).

(c) For every $m \geq 0$, $P'_{(m)} \subsetneq P_{(m)}$ (resp. $P'_{[m]} \subsetneq P_{[m]}$).

Lemma 1.6 Let $P$ prime ideal of $A_{(\infty)}$ (resp. of $A_{[\infty]}$). The following conditions are equivalent:

(a) $P$ is maximal.

(b) $P_{(m)}$ (resp. $P_{[m]}$) is maximal for some $m \geq 0$.

(c) $P_{(m)}$ (resp. $P_{[m]}$) is maximal for every $m \geq 0$.

Corollary 1.7 With the notations above, for every prime ideal $P$ of $A_{(\infty)}$ we have $\text{ht}(P) = \text{ht}(P_{(m)}) = \text{ht}(P_{[m]})$ for all $m \geq 0$. Moreover, $\dim(A_{(\infty)}) = \dim(A_{(m)})$.

Proof: Since flat ring extensions satisfy the “going down” property, corollary [1.4] implies that $\text{ht}(P \cap A_{(m)}) \leq \text{ht}(P)$. By corollary [1.4] again, $A_{(\infty)}$ is integral over $A_{(m)}$, then $\text{ht}(P) \leq \text{ht}(P \cap A_{(m)})$.

The equality $\text{ht}(P_{(m)}) = \text{ht}(P_{[m]})$ comes from the fact that $A_{(m)}$ is a localization of $A_{[m]}$.

The last relation is a standard consequence of the “going up” property.
Remark. 1.8 Corollary 1.7 remains true if we replace \( A(m) \subset A(\infty) \) by \( A[m] \subset A[\infty] \).

Corollary. 1.9 With the notations above, for every \( Q \in \text{Spec}(A(m)) \) there is a unique \( \tilde{Q} \in \text{Spec}(A(m+1)) \) such that \( \tilde{Q}^c = Q \). Moreover, the ideal \( \tilde{Q} \) is given by \( \tilde{Q} = \{ y \in A(m+1) \mid y^p \in Q \} \).

Proof: This is an easy consequence of the fact that \((A(m+1))^p \subset A(m)\). ■

Corollary. 1.10 Let us assume that \( A \) is noetherian and for every maximal ideal \( m \) of \( A \), the residue field \( A/m \) is algebraic over \( k \). Then for every \( m \geq 0 \) we have:

1. \( \dim(A[\infty]) = \dim(A[m]) = \dim(A[t]) = n + 1 \).
2. \( \dim(A(\infty)) = \dim(A(m)) = \dim(A(t)) = n \).

Proof: The first relation comes from remark 1.8 and the noetherianity hypothesis.

The second relation comes from corollary 1.7 and proposition (1.4) of [10]. ■

The following result is a consequence of theorem (1.6) of [10], lemma 1.6 and corollary 1.10.

Corollary. 1.11 Let \( A \) be a noetherian, biequidimensional, universally catenarian \( k \)-algebra of Krull dimension \( n \), and that for any maximal ideal \( m \) of \( A \), the residue field \( A/m \) is algebraic over \( k \). Then every maximal ideal of \( A(\infty) \) has height \( n \).

2 The biggest perfect subfield of a formal functions field

Throughout this section, \( k \) will be a perfect field of characteristic \( p > 0 \), \( A = k[[X]] \), \( p \subset A \) a prime ideal, \( R = A/p \) and \( K = Qt(R) \).
The aim of this section is to prove that the biggest perfect subfield of $K$, $K^\sharp = \bigcap_{e \geq 0} K^{p^e}$, is an algebraic extension of the field of constants, $k$. This result is proved in prop. 2.6 and it is one of the ingredients in the proof of corollary 3.8.

**Proposition. 2.1** Under the above hypothesis, it follows that $k = R^\sharp$.

**Proof:** Let $m$ be the maximal ideal of $R$. It suffices to prove that $R^\sharp \subseteq k$. If $f \in R^\sharp$, then for every $e > 0$ there exists an $f_e \in R$ such that $f = f^{p^e}_e$.

- Suppose at first that $f$ is not a unit, then $f_e$ is not a unit for any $e > 0$, and $f_e \in m$ for every $e > 0$. Thus, $f \in m^{p^e}$ for every $e > 0$ and by Krull’s intersection theorem,

$$f \in \bigcap_{e \geq 0} m^{p^e} = \bigcap_{r \geq 0} m^r = (0).$$

- If $f$ is unit, then $f = f_0 + \tilde{f}$, with $f_0 \in k \subset R^\sharp$ and $\tilde{f} \in R^\sharp$ and $f_0$ is unit. By the above case $\tilde{f} = 0$, hence $f \in k$.

**Proposition. 2.2** If $p = (0)$, that is $R = k[[X]]$, $K = k((X))$, then $k = K^\sharp$.

**Proof:** It is a consequence of prop. 2.1 and the fact that $R$ is a unique factorization domain.

In order to treat the general case, let us look at some general lemmas.

**Lemma. 2.3** (cf. [3] Chap. 5, § 15, ex. 8) If $L$ is a separable algebraic extension of a field $K$ of characteristic $p > 0$, then $L^\sharp$ is an algebraic extension of $K^\sharp$.

**Proof:** If $x \in L^\sharp$, then $x = y^p_e$ with $y_e \in L$ for all $e \geq 0$. Since $y_e$ is separable over $K$, $K(y_e) = K(y^p_e) = K(x)$, it follows that $y_e = x^{p^{-e}} \in K(x)$ and then $x \in K^{p^e}(x^{p^e})$. Therefore

$$[K^{p^e}(x) : K^{p^e}] = [K^{p^e}(x^{p^e}) : K^{p^e}] = [K(x) : K].$$

Thus $x$ satisfies the same minimal polynomial over $K^{p^e}$ and over $K$ for all $e \geq 0$, and the coefficients of this minimal polynomial must be in $K^\sharp$. So $x$ is algebraic over $K^\sharp$.
Lemma. 2.4 Every algebraic extension of a perfect field is perfect.

Proof: This is obvious because this is true for the finite algebraic extensions.

Lemma. 2.5 Let $C$ be a subring of a domain $D$ and let $\overline{C}$ be the integral closure of $C$ in $D$. If $f(X), g(X)$ are monic polynomials in $D[X]$ such that $f(X)g(X) \in \overline{C}[X]$, then $f(X), g(X) \in \overline{C}[X]$.

Proof: We consider a field $L$ containing $D$ such that the polynomials $f(X), g(X)$ are a product of linear factors: $f(X) = \prod(x - \alpha_i), g(X) = \prod(x - \beta_j), \alpha_i, \beta_j \in L$. Each $\alpha_i$ and $\beta_j$ are roots of $f(X)g(X)$, hence they are integral over $\overline{C}$. Thus the coefficients of $f(X)$ and $g(X)$ are integral over $\overline{C}$ and therefore they are in $\overline{C}$.

Proposition. 2.6 Let $k$ be a perfect field of characteristic $p > 0$, $A = k[[X]] = k[[X_1, \ldots, X_n]], p \subset A$ a prime ideal, $R = A/p$ and $K = \mathbb{Q}_p(R)$. Then $K^\sharp$ is an algebraic extension of $k$.

Proof: Let $r = \dim(A/p) \leq n$. By the normalization lemma for power series rings (cf. [1], 24.5 and 23.7) there is a new system of formal coordinates $Y_1, \ldots, Y_r$ of $A$, such that

- $p \cap k[[Y_1, \ldots, Y_r]] = \{0\}$,
- $k[[Y_1, \ldots, Y_r]] \hookrightarrow \frac{A}{p} = R$ is a finite extension, and
- $k((Y_1, \ldots, Y_r)) \hookrightarrow K$ is a separable finite extension.

The proposition is then a consequence of proposition 2.2 and lemma 2.3.

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1Due to J. M. Giral.
2The proof of the normalization lemma for power series rings in [1] uses generic linear changes of coordinates and needs the field $k$ to be infinite. This proof can be adapted for an arbitrary perfect coefficient field (infinite or not) by using non linear changes of the form $Y_i = X_i + F_i(X_{i+1}^p, \ldots, X_n^p)$, where the $F_i$ are polynomials with coefficients in $\mathbb{F}_p$.
3In particular, if $k$ is algebraically closed, we would have $K^\sharp = k.$
Remark. 2.7 Actually, under the hypothesis of proposition 2.6, J.M. Gir al and the authors have proved that the following stronger properties hold:

(1) If $R$ is integrally closed in $K$, then $K^\sharp = k$.
(2) In the general case, $K^\sharp$ is a finite extension of $k$.

3 Noetherianity of $A \otimes_k k(t)_{per}$

Throughout this section, $k$ will be a perfect field of characteristic $p > 0$, keeping the notations of section 1.

Proposition. 3.1 Let $K$ be a field extension of $k$ and suppose that $K^\sharp$ is algebraic over $k$. For every prime ideal $\mathcal{P} \in \text{Spec}(K_{\infty})$ such that $\mathcal{P} \cap k[t] = 0$ there exists an $m_0 \geq 0$ such that $\mathcal{P}_{[m]}$ is the extended ideal of $\mathcal{P}_{[m_0]}$ for all $m \geq m_0$.

Proof: The extension $k[t] \subset K^\sharp[t]$ is integral and then $\mathcal{P} \cap K^\sharp[t] = 0$.
We can suppose $\mathcal{P} \neq (0)$. From Remark 1.8, we have $ht(\mathcal{P}[i]) = ht(\mathcal{P}) = 1$ for every $i \geq 0$. Let $F_i(t_i) \in K[t_i]$ be the monic irreducible generator of $\mathcal{P}[i]$. From 1.2 for each $i \geq 0$ there are two possibilities:

(1) $F_i \in K^p[t_i]$, then $F_{i+1}(t_{i+1}) = F_i(t_i)^{1/p}$.
(2) $F_i \notin K^p[t_i]$, then $\mathcal{P}[i+1] = \mathcal{P}[i]^e$ and $F_{i+1}(t_{i+1}) = F_i(t_i) = F_i(t_i^p)$.

Since $\mathcal{P} \cap K^\sharp[t] = (0)$, $F_0(t_0) \notin (\bigcap_{m \geq 0} K^{p^m})[t_0] = \bigcup_{m \geq 0} K^{p^m}[t_0]$ and there exists an $m_0 \geq 0$ such that $F_0(t_0) \in K^{p^{m_0}}[t_0]$ and $F_0(t_0) \notin K^{p^{m_0+1}}[t_0]$.

From (1) we have $F_i(t_i) = F_0(t_0)^{1/p^i} \in K^{p^{m_0-i}}[t_i]$ for $i = 0, \ldots, m_0-1$ and $F_{m_0}(t_{m_0}) \notin K^p[t_{m_0}]$. Hence, applying (2) repeatedly we find $F_{j+m_0}(t_{j+m_0}) = F_{m_0}(t_{m_0}) = F_{m_0}(t_{j+m_0})$ and $\mathcal{P}_{[j+m_0]}$ is the extended ideal of $\mathcal{P}_{[m_0]}$ for all $j \geq 1$.

Corollary. 3.2 Under the same hypothesis of proposition 3.1, $\mathcal{P}$ is the extended ideal of some $\mathcal{P}_{m_0}$. 

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Therefore \( f \) is not algebraic over \( P \).

To prove the other inclusion, take an \( s \in \text{Spec}(R) \) and let \( \mathcal{P}_1 \) be a prime ideal in \( B \) such that \( \mathcal{P}_1 \cap S = \emptyset \). Let \( \mathcal{P}_0 = \mathcal{P}_1^c \), \( \mathcal{P}_1 = \mathcal{P}_1^e \) and \( \mathcal{P}_0 = \mathcal{P}_1^c \). If \( \mathcal{P}_1 = \mathcal{P}_0^e \), then \( \mathcal{P}_1 = \mathcal{P}_0^c \).

**Proposition 3.3** With the notations above, let \( \mathcal{P}_1 \) be a prime ideal in \( B \) such that \( \mathcal{P}_1 \cap S = \emptyset \). Let \( \mathcal{P}_0 = \mathcal{P}_1^c \), \( \mathcal{P}_1 = \mathcal{P}_1^e \) and \( \mathcal{P}_0 = \mathcal{P}_1^c \). If \( \mathcal{P}_1 = \mathcal{P}_0^e \), then \( \mathcal{P}_1 = \mathcal{P}_0^c \).

**Proof:** Let \( \{e_i\} \) be a \( A \)-basis of \( B \). Since \( \mathcal{P}_1 \cap S = \emptyset \), it is clear that \( \mathcal{P}_1 = \mathcal{P}_0^c \) and \( \mathcal{P}_0 = \mathcal{P}_0^e \). If \( \mathcal{P}_1 = \mathcal{P}_0^e \), we have

\[
\mathcal{P}_1 = \mathcal{P}_0^e = \mathcal{P}_0^c = (\mathcal{P}_0^e)^c = (\mathcal{P}_0^c)^c = (\mathcal{P}_0^e)^c = \sum_{s \in S} (\mathcal{P}_0^e : s)B \supset \mathcal{P}_0^e.
\]

To prove the other inclusion, take an \( s \in S \) and let \( f = \sum a_i e_i \) be an element of \((\mathcal{P}_0^e : s)B\) with \( a_i \in A \). Then, \( sf = \sum (sa_i)e_i \in \mathcal{P}_0^e \) and from the equality \( \mathcal{P}_0^e = \{\sum b_i e_i \mid b_i \in \mathcal{P}_0\} \) we deduce that \( sa_i \in \mathcal{P}_0 \) and \( a_i \in (\mathcal{P}_0^e : s)_A = \mathcal{P}_0 \). Therefore \( f \in \mathcal{P}_0^e \).

**Proposition 3.4** Let \( R \) be an integral \( k \)-algebra, \( K = \text{Qt}(R) \), and suppose that \( K \) is algebraic over \( k \). Then any prime ideal \( \mathcal{P} \in \text{Spec}(R_{(\infty)}) \) with \( \mathcal{P} \cap k[t] = 0 \) and \( \mathcal{P} \cap R = 0 \) is the extended ideal of some \( \mathcal{P}_{[m_0]} \), \( m_0 \geq 0 \).

**Proof:** Let us write \( T = R - \{0\} \). We have \( K = T^{-1}R \) and \( K_{[m]} = T^{-1}R_{[m]} \) for all \( m \geq 0 \) or \( m = \infty \). We define \( \mathcal{P} = T^{-1}\mathcal{P} \). We easily deduce that \( \mathcal{P}_{[m]} = T^{-1}\mathcal{P}_{[m]} \) for all \( m \geq 0 \).

From proposition 3.3, there exists an \( m_0 \geq 0 \) such that \( \mathcal{P}_{[m]} \) is the extended ideal of \( \mathcal{P}_{[m_0]} \) for every \( m \geq m_0 \). Then, proposition 3.3 tells us that \( \mathcal{P}_{[m]} \) is the extended ideal of \( \mathcal{P}_{[m_0]} \) for every \( m \geq m_0 \), so \( \mathcal{P} = \bigcup \mathcal{P}_{[m]} \) is the extended ideal of \( \mathcal{P}_{[m_0]} \).

**Proposition 3.5** Let \( K \) be a field extension of \( k \) and suppose that \( K \) is not algebraic over \( k \). Then \( K_{(\infty)} \) is not noetherian.
Proof: Let $s \in K^\#$ be a transcendental element over $k$.

For each $m \geq 0$, let $s_m = s^{1/p^m} \in K$ and $\alpha_m = t_m - s_m$. Let $P$ be the ideal in $K(\infty)$ generated by the $\alpha_m, m \geq 0$. We have $\alpha_m = \alpha_{m+1}^p$ and $P_{(m)} = K_{(m)}\alpha_m$ for all $m \geq 0$.

Suppose that $P$ is finitely generated. Then, there exists an $m_0 \geq 0$ such that $P = K(\infty)\alpha_{m_0}$. By faithful flatness, we deduce that $\alpha_{m_0+1} \in K_{(m_0+1)}\alpha_{m_0}$. Let us write $\tau = t_{m_0+1}, \sigma = s_{m_0+1}$. Then, $\alpha_{m_0+1} = \tau - \sigma$ and there exist $\psi(\tau) \in K[\tau] = K_{[m_0+1]}, \varphi(\tau) \in k[\tau] \setminus \{0\}$ such that

$$\varphi(\tau)(\tau - \sigma) = \psi(\tau)(\tau - \sigma)^p.$$

Simplifying and making $\tau = \sigma$ we obtain

$$\varphi(\sigma) = \psi(\sigma)(\sigma - \sigma)^{p-1} = 0$$

contradicting the fact that $s$ is transcendental over $k$.

We conclude that $P$ is not finitely generated and $K(\infty)$ is not noetherian.

\[\square\]

**Theorem 3.6** Let $k$ be a perfect field of characteristic $p > 0$ and let $A$ be a $k$-algebra. The following properties are equivalent:

(a) The ring $A$ is noetherian and for any $p \in \text{Spec}(A)$, the field $Q_t(A/p)^\#$ is algebraic over $k$.

(b) The ring $A(\infty)$ is noetherian.

Proof: Let first prove $(a) \Rightarrow (b)$. By Cohen’s theorem (cf. \[8\], (3.4)), it is enough to prove that any $P \in \text{Spec}(A(\infty)) - \{(0)\}$ is finitely generated.

From corollaries \[1.7\] and \[1.11\], we have

$$\text{ht}(P_{[m]}) = \text{ht}(P_{(m)}) = \text{ht}(P_{[\infty]}) = \text{ht}(P) = r \leq n.$$

Consider the prime ideal of $A$:

$$p := A \cap P = A \cap P_{[\infty]} = A \cap P_{[m]} = A \cap P_{(m)}.$$

There are two possibilities (cf. \[8\], prop. (5.5.3)):

(i) $\text{ht}(p) = r = \text{ht}(P_{[m]})$ and $P_{[m]} = p[1/p_m]$, for every $m \geq 0$. 

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(ii) $\text{ht}(p) = r - 1 = \text{ht}(P_{[m]}) - 1$, $p[t_m] \not\subseteq P_{[m]}$ and $A/p \not\subseteq A[t_m]/P_{[m]}$ is algebraic generated by $t_m$ mod $P_{[m]}$, for every $m \geq 0$.

In case (i), $P_{[\infty]}$ and $P$ are the extended ideals of $p$ and they are finitely generated.

Suppose we are in case (ii). We denote $R = A/p$, $K = Qt(R)$.

Then:

$$R_{[m]} = A_{[m]}/p[t_m], \quad R_{[\infty]} = A_{[\infty]}/A_{[\infty]}p = A_{[\infty]}/\bigcup_{m \geq 0} p[t_m].$$

Define $\mathcal{P} := R_{[\infty]}P_{[\infty]} = P_{[\infty]}/\bigcup_{m \geq 0} p[t_m] \in \text{Spec}(R_{[\infty]})$. We have $\mathcal{P}_{[m]} = \mathcal{P} \cap R_{[m]} = P_{[m]}/p[t_m]$, $\mathcal{P} \cap R = \mathcal{P} \cap k[t] = 0$ and

$$\text{ht}(\mathcal{P}_{[m]}) = \text{ht}(P_{[m]}/p[t_m]) = 1, \quad \text{ht}(\mathcal{P}) = \text{ht} \left( P_{[\infty]}/\bigcup_{m \geq 0} p[t_m] \right) = 1.$$

We conclude by applying proposition 3.4: there exists an $m_0 \geq 0$ such that $\mathcal{P}$ is the extended ideal of $P_{[m_0]}$. Then, $P_{[\infty]}$ is the extended ideal of $P_{[m_0]}$ and $P = A_{(\infty)}P_{[\infty]} = A_{(\infty)}P_{[m_0]}$ is finitely generated.

Let us prove now (b) $\Rightarrow$ (a). Since $A_{(\infty)}$ is faithfully flat over $A$, we deduce that $A$ is noetherian.

Let $p \in \text{Spec}(A)$ and let $R = A/p$, $K = Qt(R)$. Noetherianity of $A_{(\infty)}$ implies, first, noetherianity of $R_{(\infty)}$, and second, noetherianity of $K_{(\infty)}$. To conclude we apply proposition 3.5. \(\blacksquare\)

**Corollary. 3.7** Let $k$ be a perfect field of characteristic $p > 0$ and let $A$ be a noetherian $k$-algebra. The following properties are equivalent:

(a) The ring $A_{(\infty)}$ is noetherian.

(b) The ring $(A_m)_{(\infty)}$ is noetherian for any maximal ideal $m \in \Omega(A)$.

*Proof:* For (a) $\Rightarrow$ (b) we use the fact that $(A_m)_{(\infty)} = A_m \otimes_A A_{(\infty)}$.

For (b) $\Rightarrow$ (a), let $p \subset A$ be a prime ideal and let $m$ be a maximal ideal containing $p$. From hypothesis (b), the ring $(A_m)_{(\infty)}$ is noetherian. Then, from theorem 3.6 we deduce that the field $Qt(A/p)^{\sharp} = Qt(A_m/A_mp)^{\sharp}$ is algebraic over $k$. From theorem 3.6 again we obtain (a). \(\blacksquare\)
Corollary. 3.8 Let $k$ be a perfect field of characteristic $p > 0$, $k'$ an algebraic extension of $k$ and $A = k'[\{X_1, \ldots, X_n\}]$. Then, the ring $A_\infty = k(t)_{\text{per}} \otimes_k A$ is noetherian.

Proof: It is a consequence of lemma 2.4, proposition 2.6 and theorem 3.6.

Corollary. 3.9 Let $k$ be a perfect field of characteristic $p > 0$. If $(B, \mathfrak{m})$ is a local noetherian $k$-algebra such that $B/\mathfrak{m}$ is algebraic over $k$, then $B_\infty = k(t)_{\text{per}} \otimes_k B$ is noetherian. In particular, the field $Qt(B/p)^\#$ is algebraic over $k$ for every prime ideal $p \subset B$.

Proof: Let $k' = B/\mathfrak{m}$. By Cohen structure theorem (cf. [6], Chap. 0, Th. (19.8.8)), the completion $\hat{B}$ of $B$ is a quotient of a power-series ring $A$ with coefficients in $k'$. Since $\hat{B}_\infty$ is also a quotient of $A_\infty$, we deduce from corollary 3.8 that $B_\infty$ is noetherian. Since $\hat{B}$ is faithfully flat over $B$, the ring $\hat{B}_\infty$ is also faithfully flat over $B_\infty$. So, $B_\infty$ is noetherian.

The last assertion is a consequence of theorem 3.6.

Corollary. 3.10 Let $k$ be a perfect field of characteristic $p > 0$. For any noetherian $k$-algebra $A$ such that the residue field $A/\mathfrak{m}$ of every maximal ideal $\mathfrak{m} \in \Omega(A)$ is algebraic over $k$, the ring $A_\infty$ is noetherian. Furthermore, if $A$ is regular and equicodimensional then $A_\infty$ is also regular and equicodimensional of the same dimension as $A$.

Proof: The first part is a consequence of corollaries 3.7 and 3.9. For the last part, we use corollary 1.11, the fact that all $A_{(m)}$, $m \geq 0$ are regular and of the same (global homological = Krull) dimension ([10], th. (1.6)) and [2].

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