Abstract

In the framework of locally covariant quantum field theory, a theory is described as a functor from a category of spacetimes to a category of $\ast$-algebras. It is proposed that the global gauge group of such a theory can be identified as the group of automorphisms of the defining functor. Consequently, multiplets of fields may be identified at the functorial level. It is shown that locally covariant theories that obey standard assumptions in Minkowski space, including energy compactness, have no proper endomorphisms (i.e., all endomorphisms are automorphisms) and have a compact automorphism group. Further, it is shown how the endomorphisms and automorphisms of a locally covariant theory may, in principle, be classified in any single spacetime. As an example, the endomorphisms and automorphisms of a system of finitely many free scalar fields are completely classified.

1 Introduction

Algebraic quantum field theory (AQFT) has been highly successful in analysing the structural properties of general quantum field theories in Minkowski space [38]. For many years, however, rigorous quantum field theory in curved spacetimes was restricted to particular free models, or to spacetimes of maximal symmetry. This situation has changed, following the introduction, by Brunetti, Fredenhagen and Verch (henceforth abbreviated as BFV) of a framework of locally covariant quantum field theory [8]. This framework, in which a quantum field theory is defined as a functor between a category of spacetimes and a category of $(C)^\ast$-algebras, developed from a formulation given in Verch’s proof of a general spin–statistics connection [51] and has subsequently played an important role in the completion of the perturbative construction of interacting theories in curved spacetime.
a Reeh–Schlieder theorem, and an analysis of superselection sectors. It has also proved possible to begin an analysis of the fundamental question of what it is that makes a theory of physics the same in different spacetimes (see also for a short summary). In addition, more quantitative applications have been made to particular models in the context of the Casimir effect and quantum energy inequalities and to cosmology.

One of the major successes of AQFT in flat spacetime is undoubtedly the Doplicher–Haag–Roberts (DHR) analysis of superselection sectors and the reconstruction of the field algebra and gauge group from the algebra of observables in the vacuum sector. Brunetti and Ruzzi have employed ideas of local covariance in order to develop a parallel analysis in curved spacetime. One could characterize their approach as being ‘bottom-up’, as the analysis is performed in each spacetime and questions of covariance are then addressed. From the functorial viewpoint, it would be more natural to proceed in a ‘top-down’ manner, identifying the relevant structures and definitions as properties of the functor defining the theory. With this eventual aim in mind, the present paper begins by discussing how the global gauge group of a locally covariant theory may be identified in terms of the functor. Our proposal is simply that the global gauge group is the group of automorphisms of the functor defining the theory.

To explain this, we recall, first, that every functor has an associated group of automorphisms , whose elements are the natural isomorphisms from to itself, equipped with the group structure induced by composition. The automorphism group often carries important structural information about the functor and so, even from a purely mathematical perspective, it is important to understand what significance can be assigned to the automorphism group of a functor defining a locally covariant QFT, or, more generally, to the monoid of endomorphisms of a functor.

The physical motivation for our study rests on the interpretation of a natural transformation between functors describing two theories as an embedding of theory as a subtheory of . Thus an endomorphism of is a way of embedding as a subtheory of itself, and an automorphism is a means of doing this isomorphically. It is therefore very natural to interpret the automorphism group as the global gauge group of the theory.

As we will show, this interpretation is supported by a comparison with the Minkowski space DHR analysis. There, the (maximal) global gauge group consists of (all) unitaries on the Hilbert space of the vacuum representation of the field algebra that commute with the action of the proper orthochronous Poincaré transformations, map each local field

\[ a \rightarrow b \], a Reeh–Schlieder theorem, and an analysis of superselection sectors. It has also proved possible to begin an analysis of the fundamental question of what it is that makes a theory of physics the same in different spacetimes (see also for a short summary). In addition, more quantitative applications have been made to particular models in the context of the Casimir effect and quantum energy inequalities and to cosmology.

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\[ a \rightarrow b \].
algebra isomorphically to itself and preserve the vacuum state. As we show in Sec. 3.1, these properties are respected, and generalized, in our approach. To a large extent they hold in a representation-independent sense in arbitrary spacetimes (Sec. 2.2), and they hold in exactly the DHR sense in representations induced by gauge-invariant states, with the Poincaré group replaced by the bijective spacetime isometries preserving (time)-orientation. In particular, (a) the automorphisms representing global symmetries commute with spacetime symmetries—thus, spacetime and internal symmetries are completely independent, in a manner reminiscent of the Coleman–Mandula theorem; (b) in Minkowski space, the unitary implementation of the automorphism group in the Minkowski vacuum representation is a subgroup of the maximal DHR group—a key open question is to understand whether these groups are equal. We also describe how the gauge group acts on an abstract algebra of fields introduced in [26] and allows the definition of multiplets of fields at the functorial level. The fundamental particle–antiparticle symmetry can be seen at this level. In addition, by taking fixed-points under the action of the gauge group, one can define a new locally covariant theory that is a candidate for the description of the observables of the theory. This is not completely satisfactory, because it can happen that there are no nontrivial fixed points (as occurs with the Weyl algebra if the gauge group is continuous, for example); an approach that generates the observables in suitable representations would be preferred, but is not pursued here.

More generally, we will consider the proper endomorphisms of locally covariant theories, i.e., those embeddings of a theory as a subtheory of itself that are not gauge transformations. For example, starting with any nontrivial theory \( F \), the countably infinite tensor product theory \( F^\otimes\infty := \bigotimes_{n=1}^\infty F \) admits a proper endomorphism with components \( \zeta_M A = 1_{F(M)} \otimes A \) (there are many others). The existence of proper endomorphisms seems, in general, to indicate pathological behaviour: if \( \eta : F \to F \) is a proper endomorphism, so are all of its positive integer powers\(^\text{c}\) and we obtain an infinite chain of properly nested subtheories of \( F \), each of which is itself equivalent to \( F \). Any individual physical element of the theory, such as a species of particle, must be replicated in each of these nested theories, suggesting that each such element appears with infinite multiplicity. In Minkowski space QFT, it has long been understood [39, 13] that the latter situation is typically incompatible with a particle interpretation or good thermodynamic properties. Accordingly, it is of interest to understand what conditions on a theory exclude the existence of proper endomorphisms. We address this issue in Sec. 4 for theories described using \( C^* \)-algebras (to be thought of as field algebras) with a suitable state space and which obey a number of standard conditions in Minkowski space, most notably a mild energy compactness assumption, inspired by those of [39, 13]. Under these assumptions, we show that any endomorphism of the theory is unitarily implemented in the Poincaré invariant Minkowski representation by an element of the maximal gauge global group in the DHR sense [22]. Therefore, assuming that there are no ‘accidental’ gauge symmetries in Minkowski space, every endomorphism is an automorphism. We also give a direct proof

\(^c\)As \( \eta \) is monic, it cannot happen that any two powers are equal, given that \( \eta \) is not an isomorphism, so these nested theories are indeed all distinct.
that the automorphism group is compact; see \cite{22, 24} for proofs in Minkowski space under different hypotheses.

We emphasise that we are discussing endomorphisms of the functor, rather than of the algebras corresponding to individual spacetimes, which admit many proper endomorphisms (indeed the DHR superselection theory makes essential use of algebra endomorphisms). In any $M$ spacetime with global topology $\mathbb{R}^4$, for example, the timeslice axiom gives an isomorphism $\mathcal{A}(M) \cong \mathcal{A}(D)$, where $\mathcal{A}(M)$ is the algebra associated to the full spacetime, while $\mathcal{A}(D)$ is the algebra associated to the domain of dependence of a set that is diffeomorphic to an open 3-ball and lies within a Cauchy surface. But as $\mathcal{A}(D)$ is naturally embedded as a proper subalgebra of $\mathcal{A}(M)$, it follows that there is a proper endomorphism of $\mathcal{A}(M)$. Of course, this endomorphism is not at all canonical, because it depends sensitively on the choice of $D$. The force of our result is that there is no way of choosing a proper endomorphism of each spacetime’s algebra in a natural way.

Our results on proper endomorphisms can be applied to discuss some simple comparisons between different theories. If $\mathcal{A} \rightarrow \mathcal{B}$ and $\mathcal{B} \rightarrow \mathcal{A}$ are subtheory embeddings, and at least one of these theories obeys our conditions, then one may easily show that both subtheory embeddings are in fact isomorphisms (this is a strong analogue of the Cantor–Schröder–Bernstein theorem of set theory). In particular, if there is any subtheory embedding $\mathcal{A} \rightarrow \mathcal{B}$ that is not an isomorphism, then there can be no embedding $\mathcal{B} \rightarrow \mathcal{A}$. For example, if $\mathcal{A}$ obeys the conditions then there is no subtheory embedding $\mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{A}$ unless $\mathcal{B}$ is trivial. There is another parallel with set theory: recall that a set is finite precisely when there is no injection from it into itself that is not a bijection; a notion of finiteness ascribed to Dedekind. In a general category, an object is said to be Dedekind finite if it has no monic proper endomorphisms (for example, finite-dimensional vector spaces). Our result may be paraphrased as indicating that energy compactness, together with our other assumptions, implies Dedekind finiteness in this sense for locally covariant QFTs.

In Sec. 5 we show how the gauge group may be computed for a system of finitely many free scalar fields with any given mass spectrum, both as a classical theory and as a quantum field theory. As one might expect, gauge transformations preserve sectors with different mass. Within each given mass sector, the gauge transformations act by orthogonal transformations among the different fields with the same mass; in the massless quantized case this is augmented by the freedom to add multiples of the unit element of the algebra, resulting in a noncompact gauge group. This corresponds to the broken symmetry of the Lagrangian under addition of a constant to the field. In both the classical and quantized cases, there are no proper endomorphisms; in the quantized case there is a side condition that we restrict to endomorphisms preserving the class of states with distributional $n$-point functions. The algebra of observables is also computed in the quantum case, and we describe briefly how the theory of Sec. 4 can be applied to the Weyl algebra quantisation of the theory, in the case where all fields are massive.

Open questions and directions resulting from these results include the ‘top-down’ analysis of superselection sectors mentioned above, and the possible extension of results on classification of subsystems \cite{14, 15} to curved spacetime. A key question is to understand
the conditions under which there are no accidental symmetries in Minkowski space, i.e.,
that the maximal DHR gauge group coincides with the functorial definition. Finally we
mention, as a related work, a paper of Ciolli, Ruzzi and Vasselli \[16\] that constructs a
rather general covariant theory with given symmetry group, from which it is hoped (and
proved in some cases) that theories of interest can be obtained as specific representations
and which may even provide hints towards the inclusion of local gauge transformations.

2 General framework

2.1 Locally covariant theories

We begin with a brief summary of the main ideas in the BFV framework, as refined in \[32\].

**Spacetimes** The category \(\text{Loc}\) has, as objects, quadruples of the form \(M = (\mathcal{M}, g, o, t)\),
where \((\mathcal{M}, g)\) is a nonempty smooth paracompact globally hyperbolic Lorenzian spacetime
of dimension \(n\) with at most finitely many connected components and \(o\) and \(t\) represent
choices of orientation and time-orientation. The spacetime dimension \(n \geq 2\) is fixed and
the signature convention is \(+−⋅⋅⋅−\). Morphisms in \(\text{Loc}\) are smooth isometric embeddings,
preserving orientation and time-orientation, with causally convex image (i.e., containing all
causal curves whose endpoints it contains); our notation will not distinguish the morphism
from its underlying map of manifolds.\(^d\) Note that causal convexity requires, in particular,
that disjoint components of the image are causally disjoint. The connected spacetimes
form a full subcategory of \(\text{Loc}\), denoted \(\text{Loc}_0\).

**Theories as functors** Locally covariant theories can be described as covariant functors
from \(\text{Loc}\) (or \(\text{Loc}_0\)) to a suitable category \(\text{Phys}\) of physical systems. Thus, to each \(M \in \text{Loc}\)
there is an object \(\mathcal{A}(M)\) of \(\text{Phys}\), and to each morphism \(\psi : M \to N\) of \(\text{Loc}\) there is a
morphism \(\mathcal{A}(\psi)\) of \(\text{Phys}\); we require \(\mathcal{A}(\psi \circ \varphi) = \mathcal{A}(\psi) \circ \mathcal{A}(\varphi)\) for all composable \(\psi, \varphi\), and
\(\mathcal{A}(\text{id}_M) = \text{id}_{\mathcal{A}(M)}\) for each \(M\). BFV mainly studied the case where \(\text{Phys}\) is the category
\(\text{Alg}\) of unital \(*\)-algebras\(^e\) with unit-preserving \(*\)-monomorphisms as the morphisms, or its
full subcategory \(\mathcal{C}^*-\text{Alg}\) of \(\mathcal{C}^*\)-algebras. When discussing classical fields, we will employ
categories of (pre)symplectic spaces. It will always be assumed that all morphisms of \(\text{Phys}\)
are monic.

The BFV framework provides a natural description of local physics. Let \(\mathcal{O}(M)\) be the
set of all open causally convex subsets \(O\) of \(M\) with finitely many connected components.
If \(O \in \mathcal{O}(M)\) is nonempty, we define \(M|_O\) to be the set \(O\) equipped with the metric
and (time)-orientation induced from \(M\) and regarded as a spacetime in its own right,
with \(\iota_{M|O} : M|_O \to M\) being the canonical inclusion morphism. Then we may define

\(^d\)Note, however, that the same map of manifolds can induce morphisms between many different pairs
of objects in \(\text{Loc}\), and that these morphisms are to be distinguished. A similar comment applies to various
other categories that will be discussed in this paper.

\(^e\)We exclude the zero algebra, a convention that was also used, albeit unstated, in \[32, 33\].
\( \mathcal{A}^{\text{kin}}(M;O) \) to be the image of the map \( \mathcal{A}(\iota_{M;O}) \)

(There are other ways of defining local physics, for example, the dynamics-based approach introduced in [32] – note that in that reference \( \mathcal{A}^{\text{kin}}(M;O) \) was taken to be the domain of \( \mathcal{A}(\iota_{M;O}) \), rather than its image.)

**Relative Cauchy evolution**  Dynamics is incorporated in a very natural way. A morphism \( \psi : M \rightarrow N \) whose image contains a Cauchy surface for \( N \) is described as *Cauchy*. A theory \( \mathcal{T} : \text{Loc} \rightarrow \text{Phys} \) satisfies the *timeslice axiom* if \( \mathcal{T} \) maps Cauchy morphisms to isomorphisms in \( \text{Phys} \). If \( M = (\mathcal{M}, g, o, t) \), then to every smooth metric perturbation \( h \) of compact support for which \( M[h] = (\mathcal{M}, g + h, o, t_h) \) is also globally hyperbolic (where \( t_h \) is determined by requiring agreement with \( t \) outside \( \text{supp}(h) \)) there is an automorphism \( r_{\mathcal{M}[h]} \) of \( \mathcal{T}(M) \), called the *relative Cauchy evolution*, that compares the dynamics in \( M \) with that in \( M[h] \). The details of the construction can be found in BFV and (slightly reformulated) [32] and will not be repeated here. In circumstances where the relative Cauchy evolution can be functionally differentiated with respect to \( h \), the functional derivative can be interpreted in terms of a stress-energy tensor (see, e.g., Sec. 2.3), an interpretation supported by computations in specific models.

**Examples**  Many standard models of quantum field theory in curved spacetime have been formulated in the locally covariant framework, including the free scalar [8] and the Dirac quantum field [46], and, importantly, the respective extended algebras of Wick products: see [40] for scalar fields (refined in [6 §5.5.3]) and [18] for the Dirac case. Theories with gauge invariance also fit into the framework modulo important caveats relating to global issues, and at the time of writing, a definitive understanding is yet to be reached. Relevant references include [19, 28, 47, 3]. A common theme is that injectivity of the morphisms may be lost for certain observables of global nature, or alternatively, that certain morphisms in \( \text{Loc} \) should be excluded from consideration. In this paper we work with injective morphisms on the basis that too much is lost if injectivity is dropped wholesale, and a clean characterisation of the ‘global’ observables would be required to incorporate such ideas at the axiomatic level. Moreover, it is argued that these pathologies might be removed in a fully interacting theory [47, Remark 4.15].

The quantum field theories mentioned can all be given state spaces (see below), for example, based on Hadamard states. Treatments of classical theories include linear models described in symplectic spaces [33] and, for general field theories [35, 7].

### 2.2 Endomorphisms and automorphisms

**Definition and basic properties**  The functors from \( \text{Loc} \) to \( \text{Phys} \) form the objects of the category of locally covariant theories, \( \text{LCT} \), introduced in [32]. We will use the notation \( \text{LCT}_{\text{Phys}} \) if the target category is not clear from context. The morphisms in this category are natural transformations \( \zeta : \mathcal{A} \rightarrow \mathcal{B} \): that is, to each \( M \) there is a morphism \( \zeta_M : \)}

\( \notag \)Not all categories associate ‘images’ to morphisms, although all those we use do. In more general situations, the local physics is better described as the subobject of \( \mathcal{A}(M) \) determined by \( \mathcal{A}(\iota_{M;O}) \).
\( \mathcal{A}(M) \rightarrow \mathcal{B}(M) \), so that the equality \( \zeta_N \circ \mathcal{A}(\psi) = \mathcal{B}(\psi) \circ \zeta_M \) holds for every morphism \( \psi : M \rightarrow N \). The interpretation is that \( \zeta \) embeds \( \mathcal{A} \) as a subtheory of \( \mathcal{B} \), so an endomorphism \( \zeta : \mathcal{A} \rightarrow \mathcal{A} \) of \( \mathcal{A} \) is an embedding of \( \mathcal{A} \) as a subtheory of itself; specialising further, \( \zeta \) is an automorphism of \( \mathcal{A} \) if each component \( \zeta_M \) is an isomorphism \( \zeta_M : \mathcal{A}(M) \rightarrow \mathcal{A}(M) \).

The following observations are crucial.

**Proposition 2.1** Suppose \( \mathcal{A} : \text{Loc} \rightarrow \text{Phys} \), \( \eta \in \text{End}(\mathcal{A}) \) and let \( M \) be any spacetime. Then, (a) we have

\[
\eta_M \circ \mathcal{A}(\iota_{M,O}) = \mathcal{A}(\iota_{M,O}) \circ \eta_M \quad \text{and} \quad \eta_M \circ \mathcal{A}(\psi) = \mathcal{A}(\psi) \circ \eta_M
\]

(1)

for every nonempty \( O \in \mathcal{O}(M) \) and \( \psi \in \text{End}(M) \); (b) if \( \mathcal{A} \) also satisfies the timeslice axiom, then

\[
\text{rce}_M[h] \circ \eta_M = \eta_M \circ \text{rce}_M[h]
\]

(2)

in every spacetime \( M \) and for all permitted metric perturbations \( h \in H(M) \).

**Proof.** Eq. (1) is simply two instances of the definition of naturality, while Eq. (2) is a special case of \([32, \text{Proposition 3.8}]\).

Although part (a) of the result is completely elementary, it immediately tells us that endomorphisms act locally, and automorphisms act strictly locally: if \( \text{Phys} = \text{Alg} \) or \( \text{C}^*\text{-Alg} \), for instance, \( \mathcal{A}^{\text{kin}}(M;O) \) is the image of \( \mathcal{A}(\iota_{M,O}) \) and we have

\[
\eta_M(\mathcal{A}^{\text{kin}}(M;O)) \subset \mathcal{A}^{\text{kin}}(M;O),
\]

(3)

with equality if \( \eta \) is an automorphism. Moreover, the second part of (a) asserts that endomorphisms of the theory commute with spacetime symmetries (indeed, even with spacetime endomorphisms). Thus two of the defining properties of an internal symmetry in AQFT are met, and generalised, by automorphisms of a locally covariant theory. In Sec. 3.1 we will see how other standard properties are realised in representations.

The fact that endomorphisms commute with relative Cauchy evolution will be important when we come to classify them in particular models. In circumstances where the relative Cauchy evolution may be differentiated with respect to the metric perturbation, Eq. (2) asserts that endomorphisms preserve the stress-energy tensor.

On operational grounds, it is important to understand the extent to which an endomorphism of a locally covariant theory is determined by its behaviour in any single spacetime; put another way, if two endomorphisms have the same action in one spacetime, what can be said about their action in others? A full treatment requires additional assumptions (see below) but we may make some preliminary observations:

**Lemma 2.2** Consider a theory \( \mathcal{A} : \text{Loc} \rightarrow \text{Phys} \) obeying the timeslice condition. Let \( \eta, \eta' \in \text{End}(\mathcal{A}) \) and suppose that \( \eta_M = \eta'_M \) for some spacetime \( M \). Then the following are true: (i) if \( L \xrightarrow{\psi} M \) then \( \eta_L = \eta'_L \); (ii) if \( M \xrightarrow{\varphi} N \) is Cauchy then \( \eta_N = \eta'_N \); (iii) \( \eta_L = \eta'_L \) for any spacetime \( L \) whose Cauchy surfaces are oriented-diffeomorphic to those of \( M|_O \) for some \( O \in \mathcal{O}(M) \).
Proof. (i) Because \( \eta \) and \( \eta' \) are natural, we have \( \mathcal{A}(\psi) \circ \eta_L = \eta_M \circ \mathcal{A}(\psi) = \eta'_M \circ \mathcal{A}(\psi) = \mathcal{A}(\psi) \circ \eta'_L \) and since \( \mathcal{A}(\psi) \) is monic, \( \eta_L = \eta'_L \). (The time slice property is not required for this argument.) (ii) As \( \mathcal{A}(\varphi) \) is an isomorphism, we have \( \eta'_N = \mathcal{A}(\varphi) \circ \eta'_M \circ \mathcal{A}(\varphi)^{-1} = \mathcal{A}(\varphi) \circ \eta_M \circ \mathcal{A}(\varphi)^{-1} = \eta_N \) as required. For (iii), we use “Cauchy wedge connectedness” [32, Proposition 2.4] (a formalisation of spacetime deformation arguments going back to [37]) to obtain a chain of morphisms

\[
L \xleftarrow{c} L' \xrightarrow{c} L'' \xleftarrow{c} L''' \xrightarrow{c} M|_O \xrightarrow{i_{M,O}} M,
\]

in which a ‘c’ above a morphism indicates that it is Cauchy. Starting at the right-hand end of this chain, where \( \eta_M = \eta'_M \), we use parts (i) and (ii) to move leftwards, deducing that the components of \( \eta \) and \( \eta' \) agree in \( M_O \), \( L'' \) (using part (i) twice), \( L'' \) (using part (ii)), \( L' \) (part (i)) and finally \( L \) (part (ii) again).

Additivity and the determination of an endomorphism from a single spacetime

The theories we will study satisfy additivity properties of the type expected of field theories. Namely, in each spacetime \( M \), the object \( \mathcal{A}(M) \) is generated in a suitable sense by its subobjects \( \mathcal{A}^{\text{kin}}(M;O) \) as \( O \) runs over a set of subspacetimes of \( M \). For the latter, we will use the truncated multi-diamonds [32, Definition 2.5], which are sets of the form \( \mathcal{N} \cap D_M(B) \), where \( \mathcal{N} \) is an open globally hyperbolic neighbourhood of a Cauchy surface \( \Sigma \) for \( M \), and \( B \) is a union of finitely many disjoint subsets of \( \Sigma \) each of which is a nonempty open ball in suitable local coordinates. Sets of the above form with \( \mathcal{N} = M \) are called multi-diamonds.

The sense in which the \( \mathcal{A}^{\text{kin}}(M;O) \) generate \( \mathcal{A}(M) \) depends on the category, and can be expressed abstractly using the notion of a categorical union. A category \( C \) is said to have unions [21 §1.9] if, given any family \( (m_i)_{i \in I} \) of monic \( C \)-morphisms \( m_i : M_i \to A \), representing \( C \)-subobjects of \( A \), there exists a monic \( m : M \to A \) such that (1) each \( m_i \) factorises as \( m_i = m \circ \tilde{m}_i \), and (2) given any \( f : A \to B \) and a monic \( n : N \to B \) such that every \( f \circ m_i \) factorises as \( n \circ \tilde{n}_i \), then there is a unique morphism \( \tilde{f} : M \to N \) such that \( n \circ \tilde{f} = f \circ m \) and \( \tilde{f} \circ \tilde{m}_i = \tilde{n}_i \) for all \( i \in I \). In other words, commutativity of the outer portion of the diagram

\[
\begin{array}{ccc}
M_i & \xrightarrow{\tilde{m}_i} & M \\
\downarrow{\tilde{n}_i} & & \downarrow{f} \\
N & \xrightarrow{n} & B \\
\end{array}
\]

for each \( i \in I \) (with the understanding that \( m_i = m \circ \tilde{m}_i \)) entails the existence of a unique \( \tilde{f} \) making the diagram commute in full. The union subobject \( m : M \to A \) is defined up to isomorphism and we write

\[
m \cong \bigvee_{i \in I} m_i : \bigvee_{i \in I} M_i \to A.
\]

(See [21] and [32, Appendix B] for more details.) Among the categories we employ for \( \text{Phys} \), both \( \text{Alg} \) and \( C^*-\text{Alg} \) have unions, corresponding to the \((C)^*\)-subalgebra generated
by a family of $(C)^*$-subalgebras. The same is true of the category of complexified presymplectic spaces appearing in Sec. 5 (linear span of presymplectic subspaces). However, the category of symplectic spaces does not have unions—note that the linear span of symplectic subspaces need not be symplectic. We may now give a precise statement of additivity.

**Definition 2.3** A theory $\mathcal{A} : \text{Loc} \to \text{Phys}$ is said to be additive if $\text{Phys}$ has unions and, for each spacetime $M$,

$$\mathcal{A}(M) = \bigvee_{D \subseteq M} \mathcal{A}^{\text{kin}}(M; D),$$

or, more precisely,

$$\text{id}_{\mathcal{A}(M)} \cong \bigvee_{D \subseteq M} \mathcal{A}(\iota_M; D),$$

where the union runs over the set of all truncated multi-diamond subsets of $M$.

In particular, any dynamically local theory is additive in this sense [32, Theorem 6.3].

It will be convenient to consider categories where the existence of unions would either be tedious to demonstrate or even fails, but where there is a related category that does have unions. In such circumstances, the following generalised definition is useful.

**Definition 2.4** A theory $\mathcal{A} : \text{Loc} \to \text{Phys}$ is said to be $\mathcal{U}$-additive if $\mathcal{U}$ is a faithful functor $\mathcal{U} : \text{Phys} \to \text{Phys}'$, where $\text{Phys}'$ is a category possessing unions and all of whose morphisms are monic, such that $\mathcal{U} \circ \mathcal{A}$ is additive.

$\mathcal{U}$-additivity includes additivity as a special case, if $\text{Phys}$ has unions, by taking $\mathcal{U}$ to be the identity functor on $\text{Phys}$.

We will need a simple technical lemma, applying if $C$ has both unions and equalizers for arbitrary pairs of morphisms. Here, an equalizer of $f, g : B \to C$ in $C$ is a morphism $h : A \to B$ such that $f \circ h = g \circ h$ and satisfying the property that, if $k$ is any morphism with $f \circ k = g \circ k$ then $k$ factorizes uniquely via $h$, i.e., $k = h \circ m$ for a unique morphism $m$. Equalizers are determined up to isomorphism by this definition; we write $h \cong \text{eq}(f, g)$.

The categories $\text{Alg}$ and $(C^*)\text{-Alg}$ have equalizers: morphisms $\alpha, \beta : A \to B$ are equalized by the inclusion map in $A$ of the $(C)^*$-subalgebra of $A$ on which $\alpha$ and $\beta$ agree.

**Lemma 2.5** Suppose $C$ has unions and equalizers. Let $(m_i)_{i \in I}$ be a class-indexed family of subobjects of $A \in C$ with union $m : M \to A$. If morphisms $g$ and $h$ obey $g \circ m_i = h \circ m_i$ for all $i \in I$, then $g \circ m = h \circ m$. If, additionally, $m$ is an isomorphism, then $g = h$.

**Proof.** We have $g \circ m_i = h \circ m_i$ and hence a factorisation $m_i = \text{eq}(g, h) \circ \tilde{n}_i$ for each $i \in I$. Setting $B = A$, $f = \text{id}_A$ and $N$ equal to the domain of $\text{eq}(g, h)$, the outer portion of diagram [1] commutes for all $i \in I$, and there is therefore a morphism $\tilde{f}$ to make the diagram commute in full. In particular, $\text{eq}(g, h) \circ \tilde{f} = m$, so $g \circ m = g \circ \text{eq}(g, h) \circ \tilde{f} = h \circ \text{eq}(g, h) \circ \tilde{f} = m$ as required. The last statement is immediate (it would be enough for $m$ to be epic).

We can now complete the discussion begun in Lemma 2.2.

---

That is, $\mathcal{U}$ is injective as a map of morphisms.
Theorem 2.6 Suppose \( \mathcal{A} \) obeys the timeslice axiom and is \( \mathcal{U} \)-additive with respect to \( \mathcal{U} : \text{Phys} \to \text{Phys}' \), where \( \text{Phys}' \) has unions and equalizers. Then every \( \eta \in \text{End}(\mathcal{A}) \) is uniquely determined by its component \( \eta_M \) in any given spacetime \( M \).

Remark. In particular, the conclusion holds if \( \text{Phys} \) has unions and equalizers and \( \mathcal{A} \) is additive, by taking \( \mathcal{U} \) to be the identity functor.

Proof. Suppose \( \eta' \in \text{End}(\mathcal{A}) \) agrees with \( \eta \) in \( M \), i.e., \( \eta'_M = \eta_M \). If \( N \) is any spacetime and \( D \) is any truncated multi-diamond in \( N \) then \( N|_D \) has Cauchy surfaces oriented-diffeomorphic to any truncated multi-diamond in \( M \) with the same number of connected components as \( D \). Accordingly, \( \eta_N = \eta'_N \) by Lemma 2.2(iii), and the naturality of \( \eta \) and \( \eta' \) gives

\[
\eta_N \circ \mathcal{A}(\iota_{N,D}) = \mathcal{A}(\iota_{N,D}) \circ \eta_N|_D = \mathcal{A}(\iota_{N,D}) \circ \eta'_N|_D = \eta'_N \circ \mathcal{A}(\iota_{N,D}).
\]

Applying \( \mathcal{U} \), we have \( \mathcal{U}(\eta_N) \circ \mathcal{U}(\mathcal{A}(\iota_{N,D})) = \mathcal{U}(\eta'_N) \circ \mathcal{U}(\mathcal{A}(\iota_{N,D})) \). By Lemma 2.5, it follows that \( \eta_N = \eta'_N \) because \( \bigvee_{D \subset N} \mathcal{U}(\mathcal{A}(\iota_{N,D})) \) is an isomorphism and \( \mathcal{U} \) is faithful. As \( N \) was arbitrary, \( \eta = \eta' \).

2.3 States and twisted locality

The discussion of the previous subsections was conducted quite abstractly, in order to emphasise the general applicability of the ideas. In order to make contact with quantum field theory, we now describe more specific categories of physical systems that incorporate not only \(*\)-algebras, but also states, and allow for the Bose/Fermi distinction. Our discussion of state spaces is based almost entirely on that in BFV, but the discussion of twisted locality is new and, in fact, is made possible by the discussion above.

States  By a state space for an algebra \( \mathcal{A} \in \text{Alg} \), we mean a subset \( S \) of normalized positive linear functionals on \( \mathcal{A} \) that is closed under convex linear combinations, and operations induced by \( \mathcal{A} \) [i.e., to each \( \omega \in S \) and \( B \in \mathcal{A} \) with \( \omega(B^*B) > 0 \), the state \( \omega_B(A) := \omega(B^*AB)/\omega(B^*B) \) is also an element of \( S \)]. BFV raised this idea to the functorial level: along with a functor \( \mathcal{A} : \text{Loc} \to \text{Alg} \), they considered a contravariant functor \( \mathcal{I} \) from \( \text{Loc} \) to a suitable category of state spaces, with the property that each \( \mathcal{I}(M) \) is a state space for \( \mathcal{A}(M) \) and that each \( \mathcal{I}(\psi) \) is an appropriate restriction of the dual map \( \mathcal{A}(\psi)^* \). Then \( \mathcal{I} \) is called a state space for \( \mathcal{A} \). The state space may be given various additional attributes \([8]\); in particular, we say that \( \mathcal{I} \) is faithful if

\[
\bigcap_{\omega \in \mathcal{I}(M)} \ker \pi_\omega = \{0\},
\]

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where $\pi_\omega$ is the GNS representation of $\mathcal{A}(M)$ induced by $\omega$. Given the other properties of a state space, faithfulness also implies that
\[ \bigcap_{\omega \in \mathcal{S}(M)} \ker \omega = \{0\}. \]

In the $C^*$-case, the state space is said to be \textit{locally quasi-equivalent} if, for every spacetime $M$, relatively compact $O \subset M$ and states $\omega_i \in \mathcal{S}(M)$ ($i = 1, 2$), the GNS representations $\langle \mathcal{H}_i, \pi_\omega, \Omega_i \rangle$ restrict to quasi-equivalent representations of $\mathcal{A}_{\text{kin}}(M; O)$, i.e., the sets of states on $\mathcal{A}_{\text{kin}}(M; O)$ induced by density matrices on $\mathcal{H}_1$ and $\mathcal{H}_2$ coincide.

The following simple observation will be useful.

\textbf{Lemma 2.7} Suppose $\eta \in \text{End}(\mathcal{A})$, where $\mathcal{A} : \text{Loc} \to \text{Alg} \text{ or } C^*\text{-Alg}$. If $\psi \in \text{Aut}(M)$ and $\omega$ is an $\mathcal{A}(\psi)$-invariant state on $\mathcal{A}(M)$ then $\eta_M^* \omega$ is also invariant.

\textbf{Proof.} $\mathcal{A}(\psi)^*(\eta_M^* \omega) = \omega \circ \eta_M \circ \mathcal{A}(\psi) = \omega \circ \mathcal{A}(\psi) \circ \eta_M = \omega \circ \eta_M = \eta_M^* \omega$. \hfill \Box

\textbf{Graded algebras, states and (twisted) locality} We combine the description of the algebras and their state spaces as a single mathematical object. At the same time we build the possibility of describing Bose and Fermi statistics by considering a category of graded algebras with states, $\text{grAS}$, whose objects are triples $\langle \mathcal{A}, \gamma, S \rangle$ consisting of an algebra $\mathcal{A}$ together with a choice of state space $S$ for $\mathcal{A}$, and an involutive automorphism $\gamma$ of $\mathcal{A}$ obeying $\gamma^* S = S$, which determines a $\mathbb{Z}_2$-grading. A morphism between triples $\langle \mathcal{A}, \gamma, S \rangle$ and $\langle \mathcal{B}, \delta, T \rangle$ is determined by any $\alpha : \mathcal{A} \to \mathcal{B}$ in $\text{Alg}$ with the property that $\alpha \circ \gamma = \delta \circ \alpha$ and $\alpha^* T \subset S$. The association between any $\text{grAS}$ morphism and its underlying $\text{Alg}$ morphism determines a faithful functor $\mathcal{U} : \text{grAS} \to \text{Alg}$ such that $\mathcal{U}(\langle \mathcal{A}, \gamma, S \rangle) = \mathcal{A}$. We also have an obvious analogue $\text{grC}^*\text{AS}$, obtained by replacing $\text{Alg}$ by $C^*\text{-Alg}$ throughout and to which the following remarks apply \textit{mutatis mutandis}.

A theory $\mathcal{X} \in \text{LCT}_{\text{grAS}}$ assigns a triple $\mathcal{X}(M) = \langle \mathcal{A}(M), \gamma_M, \mathcal{S}(M) \rangle \in \text{grAS}$ to each $M \in \text{Loc}$, and to each morphism $\psi : M \to N$ a corresponding morphism in $\text{grAS}$. It follows immediately that $\mathcal{U} \circ \mathcal{X}$ is a theory in $\text{LCT}_{\text{Alg}}$, with $\mathcal{U} \circ \mathcal{X}(M) = \mathcal{A}(M)$; similarly, the $\mathcal{S}(M)$ form a state space for $\mathcal{A}$. Moreover, the $\gamma_M$ form the components of an automorphism $\gamma \in \text{Aut}(\mathcal{A})$ obeying $\gamma^2 = \text{id}_\mathcal{A}$ and under which $\mathcal{S}$ is invariant.

A subtheory embedding between $\mathcal{X} = \langle \mathcal{A}, \gamma, \mathcal{S} \rangle$ and $\mathcal{Y} = \langle \mathcal{B}, \delta, \mathcal{F} \rangle$ in $\text{LCT}_{\text{grAS}}$ is, as usual, a natural transformation $\zeta : \mathcal{X} \to \mathcal{Y}$. The morphisms $\mathcal{U}(\zeta_M)$ form the components of a natural $\mathcal{U}(\zeta) : \mathcal{A} \to \mathcal{B}$, such that $\mathcal{U}(\zeta) \circ \gamma = \delta \circ \mathcal{U}(\zeta)$ and $\mathcal{U}(\zeta)^* \mathcal{F}$ is a subfunctor of $\mathcal{S}$. As $\mathcal{U}$ is faithful, $\zeta \mapsto \mathcal{U}(\zeta)$ determines an isomorphism
\[ \text{Aut}(\langle \mathcal{A}, \gamma, \mathcal{S} \rangle) \cong \{ \eta \in \text{Aut}(\mathcal{A}) : \eta \circ \gamma = \gamma \circ \eta, \eta^* \mathcal{S} = \mathcal{S} \} \] (5)
so the introduction of the grading and state space can break the symmetry group of $\mathcal{A}$ to a subgroup of the centralizer of $\gamma$ in $\text{Aut}(\mathcal{A})$. As $\gamma$ is an element of the right-hand

\footnote{If $\omega(A) = 0$ for all $\omega \in \mathcal{S}(M)$ then also $\omega(B^* AB) = 0$ for all $\omega \in \mathcal{S}(M)$ and $B \in \mathcal{A}(M)$; polarising, $\omega(B^* AC) = 0$ for all $\omega \in \mathcal{S}(M)$ and $B, C \in \mathcal{A}(M)$, so $\pi_\omega(A) = 0$ for every $\omega \in \mathcal{S}(M)$.}
side of \([\mathfrak{H}]\), it follows that there is a (unique) \(\hat{\gamma} \in \text{Aut}(\langle \mathcal{A}, \gamma, \mathcal{F} \rangle)\) such that \(\mathcal{U}(\hat{\gamma}) = \gamma\). Furthermore, \(\hat{\gamma}^2 = \text{id}_{\langle \mathcal{A}, \gamma, \mathcal{F} \rangle}\), and \(\hat{\gamma}\) is evidently central in \(\text{Aut}(\langle \mathcal{A}, \gamma, \mathcal{F} \rangle)\). (In passing, note that if we replace \(\mathcal{F}\) by an extended state space \(\tilde{\mathcal{F}}\) with

\[
\tilde{\mathcal{F}}(M) = \text{co} \bigcup_{\eta \in \text{Aut}(\mathcal{A})} \eta^*_M \mathcal{F}(M),
\]

where \(\text{co}\) denotes closure under (finite) convex linear combinations, the gauge group will coincide with the centralizer of \(\gamma\). If \(\gamma\) is central in \(\text{Aut}(\mathcal{A})\), this would also ensure that \(\text{Aut}(\langle \mathcal{A}, \gamma, \tilde{\mathcal{F}} \rangle) \cong \text{Aut}(\mathcal{A})\).

The automorphism \(\gamma\) may be used to define a graded commutator on \(\mathcal{A}(M)\) by

\[
[A, B] = AB - (-1)^{\sigma'\sigma} BA
\]

for \(A, B \in \mathcal{A}(M)\) such that \(\gamma_M(A) = (-1)^{\sigma} A\), \(\gamma_M(B) = (-1)^{\sigma'} B\) \((\sigma, \sigma' \in \{0, 1\})\), and extended by linearity. As \(\hat{\gamma}\) is central in \(\text{Aut}(\langle \mathcal{A}, \gamma, \mathcal{F} \rangle)\), it follows that the graded commutator is equivariant in the sense that

\[
[\mathcal{U}(\zeta)_M A, \mathcal{U}(\zeta)_M B] = \mathcal{U}(\zeta)_M [A, B]
\]

for all \(\zeta \in \text{Aut}(\langle \mathcal{A}, \gamma, \mathcal{F} \rangle)\), \(M \in \text{Loc}\) and \(A, B \in \mathcal{A}(M)\). The theory \(\langle \mathcal{A}, \gamma, \mathcal{F} \rangle\) can then be said to obey twisted locality if

\[
[[\mathcal{A}^{\text{kin}}(M; O_1), \mathcal{A}^{\text{kin}}(M; O_2)] = \{0\}
\]

whenever \(O_i \in \mathcal{O}(M)\) are causally disjoint, which implements standard commutation relations for a mixture of bosonic and fermionic degrees of freedom and reduces to commutation at spacelike separation if \(\gamma = \text{id}_{\mathcal{A}}\).

We briefly connect these new structures with some of the ideas in the previous subsections. First, if \(\langle \mathcal{A}, \gamma, \mathcal{F} \rangle\) obeys the timeslice axiom and the state space \(\mathcal{F}\) is faithful for \(\mathcal{A}\), the connection between the relative Cauchy evolution and the stress-energy tensor can be made more specific. It is easily seen that \(\mathcal{U}\) maps the relative Cauchy evolution of \(\langle \mathcal{A}, \gamma, \mathcal{F} \rangle\) to that of \(\mathcal{A}\): \(\mathcal{U}(\text{rce}_M^{\langle \mathcal{A}, \gamma, \mathcal{F} \rangle}(h)) = \text{rce}_M^\mathcal{A}(h)\); moreover, \(\text{rce}_M^\mathcal{A}(h) \circ \gamma_M = \gamma_M \circ \text{rce}_M^\mathcal{A}(h)\) and \(\text{rce}_M^\mathcal{A}(h) \ast \mathcal{F}(M) = \mathcal{F}(M)\). The relative Cauchy evolution is said to be weakly differentiable with respect to \(\mathcal{F}(M)\) on all \(A \in \mathcal{A}(M)\), if for each smooth 1-parameter family \(\lambda \mapsto h(\lambda) \in H(M)\), there exists a (unique, due to faithfulness) element, denoted \([T_M(f), A] \in \mathcal{A}(M)\) such that

\[
\omega([T_M(f), A]) = 2i \frac{d}{d\lambda} \omega(\text{rce}_M[h(\lambda)] A) \bigg|_{\lambda=0}
\]

for all \(\omega \in \mathcal{F}(M)\), where \(f = \dot{h}(0)\). This defines a stress-energy tensor as a (possibly outer) symmetric derivation on \(\mathcal{A}(M)\). Under these circumstances, suppose that \(\eta \in \text{End}(\langle \mathcal{A}, \gamma, \mathcal{F} \rangle)\). Then we may differentiate the identity \(\omega(\text{rce}_M[h(\lambda)] \circ \eta_M A) = \omega(\eta_M \circ...

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\[ \text{rce}_M[h(\lambda)]A = (\eta_M^*\omega)(\text{rce}_M[h(\lambda)]A) \quad \text{(using Proposition 2.1)} \]
and again use faithfulness of \( \mathcal{S} \) to obtain

\[ [T_M(f), \eta_M A] = \eta_M[T_M(f), A] \quad \forall A \in \mathcal{A}(M), \]

so the stress-energy derivation commutes with all endomorphisms, and in particular with the grading \( \gamma_M \). This makes precise the sense in which endomorphisms preserve the stress-energy tensor, and shows that the latter is necessarily Bosonic (as one would expect).

Finally, a theory \( \langle \mathcal{A}, \gamma, \mathcal{S} \rangle \in \text{LCT}_{\text{grAS}} \) is \( \mathcal{U} \)-additive if and only if \( \mathcal{A} \) is additive in \( \text{LCT}_{\text{Alg}} \). Subject to that condition and the timeslice property, Theorem 2.6 applies to \( \langle \mathcal{A}, \gamma, \mathcal{S} \rangle \) and permits the determination of its endomorphisms through the action in any individual spacetime.

### 3 The gauge group and algebra of observables

#### 3.1 Gauge group

In this section we study the automorphism group of theories in \( \text{LCT}_{\text{grAS}} \) and \( \text{LCT}_{\text{grC\ast AS}} \) in terms of the GNS representations of the underlying algebras induced by their state spaces. This makes direct contact with, and again generalises, the global gauge group of Minkowski AQFT [22].

Throughout this subsection we consider a fixed theory \( \langle \mathcal{F}, \gamma, \mathcal{S} \rangle \) which obeys the timeslice axiom and \( \mathcal{U} \)-additivity as described above, and write \( G = \text{Aut}(\langle \mathcal{F}, \gamma, \mathcal{S} \rangle) \). We now endow \( G \) with a topology and investigate its properties.

**Definition 3.1** The natural weak topology on \( G \) is the weakest group topology in which \( G \ni \eta \mapsto \omega(\eta_M F) \) is continuous for all \( M \in \text{Loc}, \omega \in \mathcal{S}(M) \) and \( F \in \mathcal{F}(M) \).

**Proposition 3.2** If \( \mathcal{S} \) is faithful then the natural weak topology of \( G \) is Hausdorff (and therefore finer than the indiscrete topology provided \( G \) is nontrivial).

**Proof.** Suppose without loss that \( G \) is nontrivial and let \( \eta, \zeta \in G \) be arbitrary, with \( \eta \neq \zeta \). Theorem 2.6 entails that \( \eta_M \neq \zeta_M \), so there exists \( F \in \mathcal{F}(M) \) such that \( [\eta_M F] \neq [\zeta_M F] \). As \( \mathcal{S} \) is faithful, there is \( \omega \in \mathcal{S}(M) \) such that \( \omega(\eta_M F) \neq \omega(\zeta_M F) \). Thus the topology separates \( \eta \) and \( \zeta \) and is therefore Hausdorff.

**Proposition 3.3** Suppose \( \omega \in \mathcal{S}(M) \) is gauge-invariant, i.e., \( \eta_M^*\omega = \omega \) for all \( \eta \in G \), and induces a faithful GNS representation \( (\mathcal{H}_\omega, \mathcal{D}_\omega, \pi_\omega, \Omega_\omega) \) of \( \mathcal{F}(M) \). Then there is a faithful and strongly continuous representation \( G \ni \eta \mapsto U_\eta \) such that \( \pi_\omega(\eta_M F) = U_\eta \pi_\omega(F)U_\eta^{-1} \) and \( U_\eta \Omega_\omega = \Omega_\omega \) \( \forall \eta \in G, F \in \mathcal{F}(M) \) and which acts strictly locally, that is \( U_\eta \mathcal{F}^\text{kin}(M; O)U_\eta^{-1} = \mathcal{F}^\text{kin}(M; O) \) for all nonempty \( O \in \mathcal{O}(M) \). Moreover, if \( \pi_\omega \) is irreducible, the representation of \( G \) commutes with the unitary representation of any spacetime automorphism under which \( \omega \) is invariant.
Proof. The existence of the unitary representation is immediate from gauge-invariance of $\omega$. If $U_\eta = 1_{H_\omega}$, then $\eta_M F = F$ for all $F \in \mathcal{F}(M)$, because $\pi_\omega$ is faithful. Thus $\eta$ and $id_{\mathcal{F}(\gamma, \gamma)}$ have equal components in $M$ and are therefore equal by Theorem 2.6 so $\eta \mapsto U_\eta$ is faithful. By definition of the natural weak topology of $G$ and closure of the state space under operations, the maps $\eta \mapsto \omega((1 + \lambda A)^*(\eta_M B)(1 + \lambda A))$ are continuous for all $A, B \in \mathcal{F}(M)$ and all $\lambda \in \mathbb{C}$ of sufficiently small modulus. Expanding in $\lambda$ and $\lambda$, we deduce that the maps $\eta \mapsto \omega(A^* \eta_M B)$ and $\eta \mapsto \omega(\eta_M B A)$ [and $\eta \mapsto \omega(A^* (\eta_M B) A)$] are continuous for any $A, B \in \mathcal{F}(M)$, and hence that $\eta \mapsto U_\eta$ is strongly continuous on the dense domain $\pi_\omega(\mathcal{F}(M)) \Omega_\omega$. An $\epsilon/3$ argument completes the proof of strong continuity. The strict locality of the representation follows immediately from Proposition 2.1(a), cf. Eq. (3).

Now let $H_\omega$ be the (possibly trivial) subgroup of $\text{Aut}(M)$ leaving $\omega$ invariant, i.e., $\mathcal{F}(\psi)^* \omega = \omega$ for all $\psi \in H_\omega$. Then there is also a unitary representation $H_\omega \ni \psi \mapsto V_\psi$, with $\pi_\omega(\mathcal{F}(\psi) F) = V_\psi \pi_\omega(F) V_\psi^{-1}$ and $V_\psi \Omega_\omega = \Omega_\omega$ ($\psi \in H_\omega, F \in \mathcal{F}(M)$). The computation

$$U_\eta V_\psi \pi_\omega(F) V_\psi^* U_\eta^* = \pi_\omega(\eta_M \circ \mathcal{F}(\psi) F) = \pi_\omega(\mathcal{F}(\psi) \circ \eta_M F) = V_\psi U_\eta \pi_\omega(F) U_\eta^* V_\psi$$

shows that $U_\eta$ and $V_\psi$ commute up to phase, by irreducibility of $\pi_\omega$; as both operators leave $\Omega_\omega$ invariant, it follow that $U_\eta$ and $V_\psi$ commute.

Under the hypotheses of Proposition 3.3, we may define a new group topology on $G$, namely the weakest in which the representation $\eta \mapsto U_\eta$ is strongly continuous (this is necessarily weaker than the original topology) – we call this the topology induced by $\omega$, or the $\omega$-topology. Theorem B.3 of Appendix B shows that the natural weak topology is in fact equivalent to the topology induced by the Minkowski space vacuum state for theories in $\text{LCT}_{\text{GC+AS}}$ obeying suitable conditions. Furthermore, Proposition 1.2 gives conditions for these topologies to be compact.

### 3.2 Action on fields

The gauge group of a theory acts in a natural way on the associated locally covariant fields. Some relevant definitions are needed from BFV and [26], adapted slightly to our setting. To start, consider a general category of physical systems $\text{Phys}$, equipped with a functor $\mathcal{V} : \text{Phys} \to \text{Set}$, the category of sets and (not necessarily injective) functions. Fix a functor $\mathcal{D} : \text{Loc} \to \text{Set}$. Then any natural transformation $\Phi : \mathcal{D} \to \mathcal{V} \circ \mathcal{I}$ will be described as a field of type $\mathcal{D}$ associated with $\mathcal{I}$. That is, to each $M$ there is a function $\Phi_M : \mathcal{D}(M) \to \mathcal{V}(\mathcal{I}(M))$, (not assumed to be injective) such that

$$\mathcal{V}(\mathcal{I}(\psi)) \Phi_M(f) = \Phi_N(\mathcal{D}(\psi)f)$$

for each $\psi : M \to N$. We use $\text{Fld}(\mathcal{D}, \mathcal{I})$ to denote the set of all such fields (suppressing $\mathcal{V}$ from the notation). For example, with $\mathcal{D}(M) = C^\infty_0(M)$ (as a set) and $\mathcal{D}(M) \xrightarrow{\psi} N$
given by the push-forward $\mathcal{D}(M \xrightarrow{\psi} N) = \psi_*$, where

$$(\psi_*(f))(p) = \begin{cases} f(\psi^{-1}(p)) & p \in \psi(M) \\ 0 & \text{otherwise} \end{cases} \quad (f \in C^\infty_0(M), \ p \in N),$$

(6)

we obtain the scalar fields associated with the theory. Fields indexed by (possibly distributional) sections in more general bundles can also be described in a similar way – see, e.g., [20] – we restrict to the scalar case in this subsection for simplicity. The functor $\mathcal{V}$ is usually obvious. For $\text{Phys} = \text{Alg}$ or $\text{C}^*\text{-Alg}$, we take the forgetful functor sending each algebra to its underlying set and each morphism to the underlying function; for $\text{grAS}$ and $\text{grC}^*\text{AS}$, we use the functor sending $\langle A, \gamma, S \rangle$ to the underlying set of $A$ and morphisms to the underlying functions.

For a theory $\langle \mathcal{F}, \gamma, \mathcal{I} \rangle \in \text{LCT}_{\text{grAS}}$, the set $\text{Fld}(\mathcal{D}, \langle \mathcal{F}, \gamma, \mathcal{I} \rangle)$ (and, similarly, $\text{Fld}(\mathcal{D}, \langle \mathcal{F} \rangle)$) may be given a unital $*$-algebra structure under pointwise operations inherited from the algebras $\mathcal{F}(M)$ [20]. Thus $(\Phi + \lambda \Psi)_M(f) = \Phi_M(f) + \lambda \Psi_M(f)$, $(\Phi \Psi)_M(f) = \Phi_M(f) \Psi_M(f)$, $(\Phi^*)_M(f) = \Phi_M(f)^*$ and the unit field is $1_M(f) = 1_{\mathcal{F}(M)}$, for all $f \in C^\infty_0(M)$, $M \in \text{Loc}$. If $\langle \mathcal{F}, \gamma, \mathcal{I} \rangle \in \text{LCT}_{\text{grC}^*\text{AS}}$ then

$$\|\Phi\| = \sup_{M \in \text{Loc}} \sup_{f \in \mathcal{F}(M)} \|\Phi_M(f)\|_{\mathcal{F}(M)}$$

is a $C^*$-norm on the $*$-subalgebra $\text{Fld}^\infty(\mathcal{D}, \mathcal{F})$ on which it is finite. (Some set-theoretical niceties are glossed over here; see [20].)

These abstract algebras of fields carry an action of the automorphism group $G = \text{Aut}(\langle \mathcal{F}, \gamma, \mathcal{I} \rangle)$ in an obvious way. Given any $\eta \in G$, and $\Phi \in \text{Fld}(\mathcal{D}, \langle \mathcal{F}, \gamma, \mathcal{I} \rangle)$, define $\eta \cdot \Phi$ by

$$(\eta \cdot \Phi)_M(f) = \eta_M \Phi_M(f) \quad (f \in C^\infty_0(M), \ M \in \text{Loc}).$$

This is easily seen to define a field $\eta \cdot \Phi \in \text{Fld}(\mathcal{D}, \langle \mathcal{F}, \gamma, \mathcal{I} \rangle)$ by the calculation

$$\mathcal{F}(\psi)(\eta \cdot \Phi)_M(f) = \mathcal{F}(\psi) \circ \eta_M(\Phi_M(f)) = \eta_N \circ \mathcal{F}(\psi)(\Phi_M(f) = \eta_N \Phi_N(\psi_M(f)) = (\eta \cdot \Phi)_N(\psi_M(f)$$

for any $\psi : M \to N$, $f \in C^\infty_0(M)$. Moreover, the action of $\eta$ is evidently a $*$-automorphism of $\text{Fld}(\mathcal{D}, \langle \mathcal{F}, \gamma, \mathcal{I} \rangle)$ and gives a group homomorphism $G \mapsto \text{Aut}(\text{Fld}(\mathcal{D}, \langle \mathcal{F}, \gamma, \mathcal{I} \rangle))$ [and a $C^*$-automorphism of $\text{Fld}^\infty(\mathcal{D}, \langle \mathcal{F}, \gamma, \mathcal{I} \rangle)$, and corresponding group homomorphism, if relevant]. Endowing $\text{Fld}(\mathcal{D}, \langle \mathcal{F}, \gamma, \mathcal{I} \rangle)$ with the weakest topology in which every function $\Phi \mapsto \omega(\Phi_M(f))$ ($M \in \text{Loc}$, $f \in \mathcal{D}(M)$, $\omega \in \mathcal{I}(M)$) is continuous, this action of $G$ is continuous with respect to the natural weak topology $^2$

In particular, this gives a continuous linear representation of $G$ on $\text{Fld}(\mathcal{D}, \langle \mathcal{F}, \gamma, \mathcal{I} \rangle)$, regarded as a vector space. A multiplet of fields can now be defined as any subspace of

\footnotesize

\begin{itemize}
  \item If $\Phi$ is a linear field, this definition makes $\Phi^*$ conjugate linear.
  \item $\eta \mapsto \eta \cdot \Phi$ is continuous iff the functions $\eta \mapsto \omega((\eta \cdot \Phi)_M(f)) = \omega(\eta_M \Phi_M(f))$ are continuous, which they are by definition.
\end{itemize}

\end{itemize}
Fld(\(\mathcal{D}, \langle \mathcal{F}, \gamma, \mathcal{I} \rangle \)) transforming under an indecomposable representation of \(G\), and every field can be associated with an equivalence class of \(G\)-representation. Let \(\rho, \sigma\) be the equivalence classes corresponding to fields \(\Phi, \Psi\). Then \(\Phi^*\) transforms in the complex conjugate representation \(\bar{\rho}\) to \(\rho\), any linear combination of \(\Phi\) and \(\Psi\) transforms in a subrepresentation of a quotient of \(\rho \oplus \sigma\), and \(\Phi \Psi\) and \(\Psi \Phi\) transform in (possibly different) subrepresentations of quotients of \(\rho \otimes \sigma\). The quotients reflect any algebraic relationships among the fields in the multiplets of \(\Phi, \Psi\) under the linear combination or product. For example, if \(\Phi\) and \(\Psi\) belong to a common multiplet, then their linear combinations belong to the same multiplet.

The fact that both \(\sigma\) and \(\bar{\sigma}\) appear expresses the fundamental particle–antiparticle symmetry of quantum field theory. Algebras of bi-local and multi-local fields can be defined, and similar comments apply to them.

### 3.3 The algebra of observables

In AQFT, the local observables are precisely those elements of the local field algebras that are fixed points under the gauge group. An analogous construction may be carried out for any theory \(\mathcal{F} : \text{Loc}_0 \to \text{Phys}\) provided that \(\text{Phys}\) has equalizers over arbitrary families of morphisms: in each \(M\), let \(\alpha_M\) be an equalizer for all the morphisms \(\eta_M\) where \(\eta \in \text{Aut}(\mathcal{F})\). Thus \(\eta_M \circ \alpha_M = \alpha_M\) for all \(\eta \in \text{Aut}(\mathcal{F})\) and, if some \(\beta\) should have the same property (replacing \(\alpha_M\) by \(\beta\)) then \(\beta = \alpha_M \circ \gamma\) for a uniquely determined \(\gamma\). We write \(\mathcal{A}(M)\) for the domain of \(\alpha_M\). Next, if \(\psi : M \to N\), observe that

\[
\eta_N \circ \mathcal{F}(\psi) \circ \alpha_M = \mathcal{F}(\psi) \circ \eta_M \circ \alpha_M = \mathcal{F}(\psi) \circ \alpha_M \quad (\eta \in \text{Aut}(\mathcal{F})),
\]

so there is a unique morphism \(\mathcal{A}(\psi) : \mathcal{A}(M) \to \mathcal{A}(N)\) such that \(\mathcal{F}(\psi) \circ \alpha_M = \alpha_N \circ \mathcal{A}(\psi)\).

**Proposition 3.4** \(\mathcal{A} : \text{Loc}_0 \to \text{Phys}\) is a functor, and the maps \(\alpha_M\) constitute a subtheory embedding \(\alpha : \mathcal{A} \to \mathcal{F}\). Moreover, if \(\beta : \mathcal{B} \to \mathcal{F}\) is any subtheory embedding such that \(\eta \circ \beta = \beta\) for all \(\eta \in \text{Aut}(\mathcal{F})\), then there is a unique \(\hat{\beta} : \mathcal{B} \to \mathcal{A}\) so that \(\beta = \alpha \circ \hat{\beta}\).

**Proof.** The functorial nature of \(\mathcal{A}\) is justified by the calculations \(\alpha_M \circ \mathcal{A}(\text{id}_M) = \mathcal{F}(\text{id}_M) \circ \alpha_M = \alpha_M\) and

\[
\alpha_N \circ \mathcal{A}(\psi) \circ \mathcal{A}(\psi') = \mathcal{F}(\psi) \circ \mathcal{F}(\psi') \circ \alpha_M = \mathcal{F}(\psi \circ \psi') \circ \alpha_M = \alpha_N \circ \mathcal{A}(\psi \circ \psi')
\]

together with the monic property of \(\alpha_M\). By construction the \(\alpha_M\) constitute a natural \(\alpha : \mathcal{A} \to \mathcal{F}\), with the property \(\eta \circ \alpha = \alpha\) for all \(\eta \in \text{Aut}(\mathcal{F})\). If \(\beta : \mathcal{B} \to \mathcal{F}\) with \(\eta \circ \beta = \beta\) for all \(\eta \in \text{Aut}(\mathcal{F})\), we take components in \(M\) and use the equalizing property of \(\alpha_M\) to deduce that \(\beta_M = \alpha_M \circ \hat{\beta}_M\) for uniquely determined \(\hat{\beta}_M : \mathcal{B}(M) \to \mathcal{A}(M)\). We then calculate

\[
\alpha_N \circ \hat{\beta}_N \circ \mathcal{B}(\psi) = \beta_N \circ \mathcal{B}(\psi) = \mathcal{F}(\psi) \circ \beta_M = \mathcal{F}(\psi) \circ \alpha_M \circ \hat{\beta}_M = \alpha_N \circ \mathcal{A}(\psi) \circ \hat{\beta}_M.
\]

\(^1\)For the moment, we restrict to connected spacetimes; see comments below.
which proves (again, because $\alpha_N$ is monic) that $\hat{\beta} : B \rightarrow \mathcal{A}$ and $\beta = \alpha \circ \hat{\beta}$.

The theory $\mathcal{A}$ is a natural candidate for the theory of observables determined by the field functor $\mathcal{F}$. In the case $\text{Phys} = \text{Alg}$, of course, the algebra $\mathcal{A}(M)$ may be identified concretely with the subalgebra of $\mathcal{F}(M)$ of fixed points under $\eta_M$ ($\eta \in \text{Aut}(\mathcal{F})$).

However, there are various reasons to be cautious regarding this definition. First, there is no guarantee that $\text{Aut}(\mathcal{A})$ is trivial, although this is what one would expect if $\mathcal{F}$ is a ‘reasonable’ field functor, and could be used as a selection criterion for candidate field functors $\mathcal{F}$. As an example of an ‘unreasonable’ field functor, suppose that indeed $\mathcal{F}$ is given so that $\text{Aut}(\mathcal{A})$ is trivial, and adopt $\mathcal{F} \otimes \mathcal{A}$ as the field functor. In the simplest case, $\text{Aut}(\mathcal{F} \otimes \mathcal{A}) = \text{Aut}(\mathcal{F}) \otimes \text{id}_{\mathcal{A}}$, and the corresponding observable functor would be $\mathcal{A} \otimes \mathcal{A}$, which has a nontrivial automorphism corresponding to the interchange of factors. Thus $\text{Aut}(\mathcal{A} \otimes \mathcal{A})$ has a $\mathbb{Z}_2$ subgroup.

Second, if one applies the same construction to theories defined on possibly disconnected spacetimes $\text{Loc}$, it can result in ‘observables’ that are built from ‘unobservable’ elements in different spacetime components, whose operational significance is unclear, to say the least (we will see examples in Sec. 5). In these circumstances, it is tempting to define the ‘true algebra of observables’ to be the subalgebra of $\mathcal{A}(M)$ generated by the images of $\mathcal{A}(\iota_{M,C}(\mathcal{A}(C)))$ as $C$ runs over the connected components of $M$, with canonical inclusions $\iota_{M,C} : C \rightarrow M$.

Third, in some cases it can happen that the theory $\mathcal{A}$ is trivial. For example, in the case of the classical fields discussed in Sec. 5 there are no nonzero elements of the symplectic space that are invariant under the action of all elements of the symmetry group. Similar problems occur if the theory is then quantized using the Weyl algebra, which has no fixed points (other than multiples of the unit) under a faithful continuous group action if there are no fixed-points in the underlying symplectic space.

Nonetheless, the above definition is worthy of further investigation and will turn out to give the expected theory of observables in the scalar field examples, when quantized using the infinitesimal Weyl algebra.

4 Energy compactness excludes proper endomorphisms

One might suspect that theories admitting endomorphisms that are not automorphisms are unphysical in some way. In this section, we confirm such suspicions for locally covariant theories whose Minkowski space versions obey standard assumptions of the Haag–Araki–Kastler framework, of which the most important will be an energy compactness requirement weaker than the nuclearity conditions of [13]. Provided such a theory has no ‘accidental’ gauge symmetries in Minkowski space—internal symmetries of the Minkowski space net that do not arise from automorphisms of the locally covariant theory—then all its endomorphisms are automorphisms. Moreover, we show that the automorphism group can be given the structure of a compact topological group. Our argument here is more direct than standard presentations and uses weaker hypotheses; it is therefore of independent interest.
Energy compactness conditions were first introduced by Haag and Swieca [39] in an attempt to understand the general conditions under which a quantum field theory admits a particle interpretation. A major development in this line of thought occurred with the introduction of nuclearity criteria by Buchholz and Wichmann [13], which gave more stringent criteria closely linked to the split property (itself linked to a rich mathematical theory of standard split inclusions of von Neumann algebras [24]) and good thermodynamic behaviour of the theory [11]. A variety of nuclearity conditions have been proposed subsequently, see [12] for a review and [4] for a more recent variant. The underlying physical idea of all these approaches, arising from the uncertainty principle, is that the number of degrees of freedom available in small phase space volumes should be finite; this cannot be implemented literally, and compactness or its variants often stand in for ‘finiteness’ in the technical conditions imposed.

Given that endomorphisms of a locally covariant theory are injections and preserve both localization and the energy scale, they intuitively map any given volume of phase space onto itself in a volume preserving way. Thus the existence of a proper endomorphism, (i.e., one that is not an automorphism) can be expected to conflict with energy compactness on physical grounds, and this is exactly what we will establish. Note that we are not claiming, nor do we expect, that all models violating energy compactness admit proper endomorphisms.

Our result will be proved for locally covariant theory \( \mathcal{F}, \gamma, \mathcal{S} \) : \( \text{Loc} \to \text{grC}^*\text{AS} \) obeying a number of conditions that will now be introduced and discussed. The first assumption collects the basic conditions required in general spacetimes, while the others relate specifically to Minkowski space and are largely standard assumptions in algebraic QFT.

1. **Twisted locality, time-slice, \( \mathcal{H} \)-additivity and local quasi-equivalence** Here, \( \mathcal{H} \) is the usual faithful functor \( \mathcal{H} : \text{grC}^*\text{AS} \to \text{C}^*\text{-Alg} \), so \( \mathcal{F} \) is additive as a theory in LCT\( _{\text{C}^*\text{-Alg}} \).

2. **Unique Poincaré invariant state** In Minkowski space \( M_0 \), there is a unique state \( \omega_0 \in \mathcal{S}(M_0) \) that is \( \mathcal{F}(\psi) \)-invariant for all proper orthochronous Poincaré transformations \( \psi \).

Note that assumption (2) does not apply in the case of the massless free scalar field, which admits a 1-parameter family of Poincaré invariant vacuum states in Minkowski space. We will see (albeit not in the \( C^* \)-setting) that there are no proper endomorphisms of that theory either, but that the automorphism group is noncompact, in contrast to the situation discussed in this section.

The invariant state \( \omega_0 \) induces a GNS representation \((\mathcal{H}, \pi, \Omega)\) of the theory in Minkowski space, and hence a local net of \( C^* \)-algebras \( \mathfrak{A}(O) := \pi(\mathcal{F}^{\text{kin}}(M_0; O)) \) on \( \mathcal{H} \) indexed by relatively compact, connected and nonempty \( O \in \mathcal{O}(M_0) \). Taking double-commutants we obtain a net of von Neumann algebras \( \mathfrak{M}(O) := \mathfrak{A}(O)^{\prime\prime} \), with the same index set. The GNS representation and local nets are assumed to obey a number of standard conditions:

3. **Faithfulness, irreducibility and separability** The GNS representation \( \pi \) is a faithful and irreducible representation of \( \mathcal{F}(M_0) \) on a separable Hilbert space \( \mathcal{H} \).
4. **Covariance and spectrum condition** (a) The algebra automorphisms of $\mathcal{F}(M_0)$ induced by the proper orthochronous Poincaré group can be unitarily implemented in $(\mathcal{H}, \pi, \Omega)$ by a strongly continuous unitary representation $\Lambda \mapsto U(\Lambda)$ so that $U(\Lambda)\Omega = \Omega$ and

$$U(\Lambda)\mathfrak{F}(O)U(\Lambda)^{-1} = \mathfrak{F}(\Lambda O);$$

(b) the self-adjoint generators $P_\mu$ of the translation subgroup have joint spectrum contained in the forward light-cone.

5. **Reeh–Schlieder** For all nonempty $O \in \mathcal{O}(M_0)$, the subspace $\mathfrak{F}(O)\Omega$ is dense in $\mathcal{H}$.

6. **Energy compactness** For some nonempty $O \in \mathcal{O}(M_0)$ and $\beta > 0$, the set

$$\mathcal{N} = \{e^{-\beta H}W\Omega : W \in \mathfrak{M}(O) \text{ s.t. } W^*W = 1\}$$

is a relatively compact subset of $\mathcal{H}$ (with necessarily dense linear span, by the Reeh–Schlieder condition and because $e^{-\beta H} = (e^{-\beta H})^*$ has trivial kernel), where $H = P_0$ is the Hamiltonian with respect to some system of inertial coordinates.

The last of the standard assumptions, **twisted duality** [22], requires some notation. Let $\Gamma$ be the unitary implementing $\gamma_{M_0}$, so $\Gamma^2 = 1$, $\Gamma\Omega = \Omega$ and $\Gamma\pi(A)\Gamma^{-1} = \pi(\gamma_{M_0}(A))$ for all $A \in \mathcal{F}(M_0)$, and define a unitary

$$Z = \frac{1 - i}{2} + \frac{1 + i}{2}\Gamma.$$

For any subset of bounded operators $\mathfrak{M} \subset \mathcal{B}(\mathcal{H})$, let $\mathfrak{M}^t = Z\mathfrak{M}Z^{-1}$ and write $\mathfrak{M}^t(O) = \mathfrak{M}(O)^t$. If $\gamma_{M_0}(B) = (-1)^{\sigma}B$ ($\sigma = \{0, 1\}$) then

$$[\pi(A), Z\pi(B)Z^{-1}] = \pi([A, B])(-i\Gamma)^{\sigma}$$

for any $A \in \mathcal{F}(M_0)$, from which the expression for general $B$ may be obtained by linearity. Twisted locality of $(\mathcal{F}, \gamma, \mathcal{F})$ implies that $\mathfrak{F}(O)$ and $\mathfrak{F}(\tilde{O})^t$ commute for any spacelike separated $O, \tilde{O} \in \mathcal{O}(M_0)$, and hence $\mathfrak{M}(O) \subset \mathfrak{M}^t(\tilde{O})^t$. Twisted duality is the following more specific statement.

7. **Twisted duality**° There is a subset $\mathcal{K}_\circ \subset \mathcal{O}(M_0)$ such that every $\tilde{O} \subset \mathcal{K}_\circ$ is a diamond (i.e., a multi-diamond with one connected component) and (a) for all nonempty relatively compact and connected $O \in \mathcal{O}(M_0)$,

$$\mathfrak{F}(O) = \bigvee_{\tilde{O} \subset O} \mathfrak{F}(\tilde{O}),$$

°In the literature, proofs of (twisted) Haag duality for particular models in Minkowski space typically apply to Cauchy developments of sufficiently well-behaved subsets of constant-time hypersurfaces, e.g., double-cones. While one might expect the same to be true for general diamonds, we err on the side of caution by allowing the possibility that twisted duality might only be known for a special class of diamonds, which is nonetheless large enough to generate any local algebra.
where the $C^*$-algebraic join is taken over $\hat{O} \in \mathcal{K}$ contained in $O$, and (b) for every $O \in \mathcal{K}_0$, we have

$$\mathcal{M}(O) = \bigcap_{\hat{O} \subseteq O'} \mathcal{M}(\hat{O})',$$

where the intersection runs over all $\hat{O} \in \mathcal{K}_0$ contained in the causal complement $O' = M \setminus \text{cl} J_M(O)$ of $O$.

As already mentioned, assumptions (1)–(7) are standard conditions in Minkowski space AQFT and are satisfied, for example by models of free scalar fields (cf. Sec. 5.4). Indeed, the version of twisted duality stated here is slightly weaker than that in [22] and the crucial assumption on energy compactness is much weaker than the nuclearity condition of [13], which would require $\mathcal{N}$ to be a nuclear subset of $\mathcal{H}$, with nuclearity index obeying prescribed bounds in terms of the size of $O$ and the inverse temperature $\beta$. Here, our condition is not required to hold for all $\beta$ or $O$ and therefore can incorporate some theories with a maximum temperature. The exponential energy damping is not critical. One could work just as well with a spectral projection of $H$, as in the Haag–Swieca criterion [39]; again, our condition would be weaker, because Haag and Swieca also impose conditions on the ‘approximate dimension’ of the sets they consider. The utility of energy compactness arises from the next result, which is proved at the end of this section.

**Lemma 4.1** Suppose $\mathcal{N}$ is a relatively compact subset of a separable Hilbert space $\mathcal{H}$, with dense linear span. (a) If $T \in \mathcal{B}(\mathcal{H})$ is an isometry with $TN \subseteq \mathcal{N}$, then $T$ is unitary, and $T \text{cl} \mathcal{N} = \text{cl} \mathcal{N}$. (b) Let $G$ be the group of unitary operators $U \in \mathcal{B}(\mathcal{H})$ obeying $U \text{cl} \mathcal{N} = \text{cl} \mathcal{N}$. Then $G$ is compact with respect to the strong operator topology.

This result permits us to give an apparently new proof of the compactness of the maximal global gauge group $G_{\text{max}}$, defined by Doplicher, Haag and Roberts [22], consisting of all unitary operators $U$ on $\mathcal{H}$ that commute with the representation of the Poincaré group, preserve the vacuum vector and act strictly locally on the net of local von Neumann algebras, in the sense that $U\mathcal{M}(O)U^{-1} = \mathcal{M}(O)$ for all relatively compact connected nonempty $O \in \mathcal{O}(M_0)$. Compactness of $G_{\text{max}}$ has been proved under various assumptions in the past, e.g., the existence of an asymptotically complete scattering theory with finite particle multiplets [22], or under the assumption of nuclearity, which implies the split property [13] and hence compactness of $G_{\text{max}}$ by (the proof of) [24, Theorem 10.4]. However, we wish to point out that compactness of $G_{\text{max}}$ may be established directly and under the weaker energy compactness condition assumed here, as a consequence of Lemma 4.1(b). (Twisted duality is not needed for this argument.)

**Proposition 4.2** Under assumptions (1)–(7), the group $G_{\text{max}}$ is compact in the strong operator topology, and $\text{Aut}(\langle \mathcal{F}, \gamma, \mathcal{S} \rangle)$ is compact in the natural weak topology.

**Proof.** Any $U \in G_{\text{max}}$ preserves the relatively compact set $\mathcal{N} = e^{-\beta H} \mathcal{M}(O) \Omega$, because $U$ acts strictly locally, commutes with the Hamiltonian and obeys $U\Omega = \Omega$. Combining both parts of Lemma 4.1, $G_{\text{max}}$ is therefore contained in a group of unitaries that is compact.
in the strong operator topology. As \( G_{\text{max}} \) is closed in this topology, because its defining relations are preserved under strong limits \([22]\), it is compact. Hence, the \( \omega_0 \)-topology on \( \text{Aut}(\langle \mathcal{F}, \gamma, \mathcal{I} \rangle) \) [defined at the end of Sec. 3.1] is compact, and is equivalent to the natural weak topology by Theorem 3.3 of Appendix B.

Returning to the main theme of excluding proper endomorphisms, our first task is to show that any endomorphism is \( \langle \mathcal{F}, \gamma, \mathcal{I} \rangle \) is indistinguishable, in Minkowski space, from a gauge transformation in \( G_{\text{max}} \). In what follows, we abuse notation slightly by using the same symbol for both an endomorphism of \( \langle \mathcal{F}, \gamma, \mathcal{I} \rangle \) and the underlying endomorphism of \( \mathcal{F} \). Our first result extends the unitary representation of \( \text{Aut}(\langle \mathcal{F}, \gamma, \mathcal{I} \rangle) \) from Proposition 3.3.

**Theorem 4.3** Under assumptions \( (1) \sim (3) \), there is a faithful homomorphism of monoids \( \rho : \text{End}(\langle \mathcal{F}, \gamma, \mathcal{I} \rangle) \to G_{\text{max}} \), obeying

\[
\rho(\eta)\pi(A)\Omega = \pi(\eta M_0 A)\Omega \quad (\eta \in \text{End}(\langle \mathcal{F}, \gamma, \mathcal{I} \rangle), \ A \in \mathcal{F}(M_0)); \tag{7}
\]

in particular, each \( \rho(\eta) \) unitarily implements \( \eta M_0 \).

**Proof.** Let \( \eta \in \text{End}(\langle \mathcal{F}, \gamma, \mathcal{I} \rangle) \). By definition of the category \( \text{LCT}_{\text{grC-AS}} \), we have \( \eta \circ \gamma = \gamma \circ \eta \) and \( \gamma^* \mathcal{I} \subset \mathcal{I} \). As \( \omega_0 \) is the unique Poincaré invariant state, Lemma 2.7 entails that \( \eta M_0 \omega_0 = \omega_0 \). Then the calculation, for arbitrary \( A \in \mathcal{F}(M_0) \),

\[
\|\pi(\eta M_0 A)\Omega\|^2 = \omega_0((\eta M_0 A)^*\eta M_0(A)) = \omega_0(A^* A) = \|\pi(A)\Omega\|^2
\]

shows that the equation \( T \pi(A)\Omega = \pi(\eta M_0 A)\Omega \) (for all \( A \in \mathcal{F}(M_0) \)) defines \( T \) unambiguously on a dense domain in the GNS Hilbert space, and extends by continuity to define an isometry of \( \mathcal{H} \) into itself.

Next, let \( \tau_t : (x^0, \mathbf{x}) \mapsto (x^0 + t, \mathbf{x}) \) be the time translation automorphism of \( M_0 \) in some system of standard inertial coordinates, unitarily implemented so that \( e^{iHt} \pi(A)\Omega = \pi(\mathcal{F}(\tau_t)A)\Omega \), where \( H \) is the Hamiltonian in these coordinates. Because \( \eta \) is natural, we must have \( \eta M_0 \circ \mathcal{F}(\tau_t) = \mathcal{F}(\tau_t) \circ \eta M_0 \), which gives

\[
Te^{iHt} \pi(A)\Omega = \pi(\eta M_0 \circ \mathcal{F}(\tau_t)A)\Omega = \pi(\mathcal{F}(\tau_t) \circ \eta M_0 A)\Omega = e^{iHt}T \pi(A)\Omega
\]

i.e., the isometry \( T \) commutes with \( e^{iHt} \) on a dense domain and hence all of \( \mathcal{H} \). We deduce that \( T \) also commutes with \( e^{-\beta H} \) for any \( \beta \geq 0 \).

For any unital \( C^* \)-algebra \( A \), we write \( A_{(1)} = \{ A \in A : A^* A = 1 \} \). According to the energy compactness assumption, we may choose nonempty \( O \in \mathcal{O}(M_0) \) and \( \beta > 0 \) so that \( \mathcal{N} = e^{-\beta H} \mathcal{F}_{(1)}(O)\Omega \) is a subset of the relatively compact set \( e^{-\beta H} \mathcal{M}_{(1)}(O)\Omega \). Hence \( \mathcal{N} \) is relatively compact, with dense linear span. Moreover,

\[
TN = e^{-\beta H} T \pi(\mathcal{F}_{(1)}^{\text{kin}}(M_0; O))\Omega = e^{-\beta H} \pi(\eta M_0 \mathcal{F}_{(1)}^{\text{kin}}(M_0; O))\Omega
\]

\[
\subset e^{-\beta H} \pi(\mathcal{F}_{(1)}^{\text{kin}}(M_0; O))\Omega = \mathcal{N}.
\]

\( ^n \) Take any Schwartz test function \( f \) with \( \hat{f}(\lambda) = e^{-\beta \lambda} \) for \( \lambda \geq 0 \); then \( \int f(t) \langle \varphi | e^{-iHt} \psi \rangle dt = \langle \varphi | e^{-\beta H} \psi \rangle \) for any \( \varphi, \psi \in \mathcal{H} \), from which \( Te^{-\beta H} = e^{-\beta H} T \) follows (cf. the proof of Theorem VIII.13 in [24]).

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and Lemma 4.1(a) entails that $T$ is unitary. Consequently, considering the action on the dense domain $T\pi(\mathcal{F}(M_0))\Omega$, we see that $T$ implements $\eta_{M_0}$, i.e., $T\pi(A)T^{-1} = \pi(\eta_{M_0}A)$. Therefore

$$T\mathcal{F}(O)T^{-1} = \pi(\eta_{M_0}(\mathcal{F}^{\text{kin}}(M_0;O))) \subset \pi(\mathcal{F}^{\text{kin}}(M_0;O)) = \mathcal{F}(O)$$

for all nonempty $O \in \mathcal{O}(M_0)$. Were $T^{-1}$ known to implement an endomorphism of $\langle \mathcal{F}, \gamma, \mathcal{I} \rangle$ (as would be the case if $\eta$ was an automorphism), the above inclusion would become an equality. Lacking such knowledge, however, we proceed using twisted duality.

Passing to the local von Neumann algebras $\mathcal{M}(O)$, the argument so far has established that $T\mathcal{M}(O)T^{-1} \subset \mathcal{M}(O)$, for all relatively compact connected nonempty $O \in \mathcal{O}(M_0)$. As $\eta$ commutes with $\gamma$, we have $T\Gamma = \Gamma T$, and therefore $TZ = ZT$. In particular, $T\mathcal{M}(O)T^{-1} = (T\mathcal{M}(O)T^{-1})^t \subset \mathcal{M}(O)$. Thus, if $\hat{O}$ is spacelike to $O$,

$$[T^{-1}\mathcal{M}(O)T, \mathcal{M}(\hat{O})] = T^{-1}[\mathcal{M}(O), T\mathcal{M}(\hat{O})]T^{-1}T \subset T^{-1}[\mathcal{M}(O), \mathcal{M}(\hat{O})]T = \{0\}$$

and by twisted duality, we see that $T^{-1}\mathcal{M}(O)T \subset \mathcal{M}(O)$ for all $O \in \mathcal{K}_c$. Putting this together with our earlier result gives $T\mathcal{M}(O)T^{-1} = \mathcal{M}(O)$ for all $O \in \mathcal{K}_c$. Now if $O \in \mathcal{O}(M_0)$ is nonempty, connected and relatively compact, the additivity assumption (7a) yields

$$\mathcal{F}(O) = \bigvee_{\hat{O} \subset O} \mathcal{F}(\hat{O}) \quad \text{and hence} \quad \mathcal{M}(O) = \bigvee_{\hat{O} \subset O} \mathcal{M}(\hat{O}),$$

where the joins run over $\hat{O} \in \mathcal{K}_c$; the first is taken in the sense of $C^*$-algebras, while the second is taken in the von Neumann sense and follows on taking weak closures. It follows immediately that $T\mathcal{M}(O)T^{-1} = \mathcal{M}(O)$ for all such $O$.

Summarising: $T$ is a unitary operator, commuting with the unitary representation of the proper, orthochronous Poincaré group, acting strictly locally on the net and preserving $\Omega$. Setting $\rho(\eta) = T$, we have a map $\rho : \text{End}(\langle \mathcal{F}, \gamma, \mathcal{I} \rangle) \to G_{\max}$ obeying Eq. (11) and hence $\rho(\eta)\pi(A)\rho(\eta)^{-1} = \pi(\eta_{M_0}A)$. It is evident from these equations that $\rho$ is a homomorphism and obeys $\rho(\text{id}_{\langle \mathcal{F}, \gamma, \mathcal{I} \rangle}) = 1_{\mathcal{K}_c}$. Further, as $\pi$ is faithful, $\rho(\eta) = 1$ if and only if $\eta_{M_0} = \text{id}_{\mathcal{F}(M_0)}$. By Theorem 2.6 we have $\eta = \text{id}_{\langle \mathcal{F}, \gamma, \mathcal{I} \rangle}$, which completes the proof that $\rho$ is a faithful monoid homomorphism.

To say more, we require an additional assumption.

8. **Implementation of rce** The relative Cauchy evolution can be unitarily implemented in the GNS representation of $\omega_0$, i.e., to each $h \in H(M_0)$, there is a unitary $V(h)$ on $\mathcal{H}$, such that

$$V(h)\pi(A)V(h)^{-1} = \pi(\text{rce}_M[h]A) \quad (A \in \mathcal{F}(M)).$$

We assume in addition that $\langle \Omega \mid V(h)\Omega \rangle \neq 0$ for all $h \in H(M_0)$.

\footnote{There are cases where proper endomorphisms of algebras are unitarily implemented, e.g., shrinking scale transformation on a suitable local algebra in a conformally covariant theory.}
Note that $V(h)$ cannot leave the vacuum invariant (unless the relative Cauchy evolution is trivial) by Proposition 4.3 below. The assumption that $\langle \Omega \mid V(h)\Omega \rangle \neq 0$ is motivated by the idea that $V(h)\Omega$ is a ‘squeezed’ vacuum. This can be seen explicitly for free Bose theories, where the relative Cauchy evolution of the QFT is a Bogoliubov transformation induced by the relative Cauchy evolution in the underlying classical theory. (One might suspect that it actually follows from the other axioms, but we do not have a proof of this at present.) In fact, the assumption should hold generally at least for $h$ in a neighbourhood of the zero test tensor, if the relative Cauchy evolution is assumed to be continuous at $h = 0$, which in turn is a precondition for the existence of a stress-energy tensor as a field in the GNS representation of $\omega_0$. The utility of this condition is made clear in the following result.

**Proposition 4.4** Under assumptions (1)-(8), the homomorphism $\rho$ of Theorem 4.3 obeys

$$\text{Im} \rho \subset G_{\text{re}} := G_{\max} \cap \{V(h) : h \in H(M_0)\}' .$$

The group $G_{\text{re}}$ is compact in the strong operator topology of $\mathcal{H}$.

**Proof.** Let $\eta \in \text{End}(\langle \mathcal{F}, \gamma, \mathcal{F} \rangle)$. For any $A \in \mathcal{P}(M_0)$ and $h \in H(M_0)$, the intertwining property (2) entails

$$\rho(\eta)V(h)\pi(A)V(h)^{-1}\rho(\eta)^{-1} = \pi(\eta_{M_0} \circ \text{rce}_{M_0}[h]A) = \pi(\text{rce}_{M_0}[h] \circ \eta_{M_0}A) = V(h)\rho(\eta)\pi(A)\rho(\eta)^{-1}V(h)^{-1} .$$

As $\pi$ is irreducible, this implies that $\rho(\eta)V(h) = \alpha_\eta(h)V(h)\rho(\eta)$, for some constant $\alpha_\eta(h)$, necessarily of unit modulus. But $\rho(\eta)\Omega = \Omega = \rho(\eta)^*\Omega$ and $\langle \Omega \mid V(h)\Omega \rangle \neq 0$, so $\alpha_{\eta}(h) = 1$. Thus $\text{Im} \rho \subset G_{\text{re}}$; as the defining relations of this group are preserved under strong limits, it is a closed subset of $G_{\max}$ and hence compact, by Proposition 4.2.

As a minor digression, we also note the following application of Proposition 4.4 (from which the affiliation of the stress-energy tensor to the local net would follow).

**Proposition 4.5** Subject to assumptions (1)-(8) above, if $h \in H(M_0;O)$ for some $O \in \mathcal{K}_0$, then $V(h) \in \mathcal{M}(O)$. Moreover, $V(h)\Omega \in \mathcal{C}\Omega$ only if $\text{rce}_{M_0}[h]$ is the identity automorphism.

**Proof.** If $\tilde{O} \subseteq O(M_0)$ is any relatively compact and connected subset of the causal complement $O' = M_0 \setminus \text{cl} J_{M_0}(O)$ of $O$, then $\text{rce}_{M_0}[h]A = A$ for all $A \in \mathcal{P}_{\text{kin}}(M;\tilde{O})$ by Proposition 3.5 of [32], and hence $V(h) \in \mathcal{F}(\tilde{O})' = \mathcal{M}(\tilde{O})'$. Applying Proposition 4.4 to $\gamma$, we see that $V(h)$ commutes with $\Gamma$ and hence $Z$, so we also have $V(h) \in \mathcal{M}(\tilde{O})'$. The result $V(h) \in \mathcal{M}(O)$ for $O \in \mathcal{K}_0$ now follows from twisted duality and arbitrariness of $\tilde{O}$. If $V(h)\Omega = \alpha\Omega$ ($\alpha \in \mathbb{C}$), then $V(h) = \alpha \mathbf{1}$ as $\Omega$ is a separating vector for $\mathfrak{M}(O)$ by twisted locality and the Reeh–Schlieder property. As $\pi$ is faithful, this implies $\text{rce}_{M_0}[h]$ is trivial.

Subject to the assumptions made so far, we have shown that any endomorphism of $\langle \mathcal{F}, \gamma, \mathcal{F} \rangle$ acts as an automorphism of the Minkowski space net. Provided that all such
9. **Absence of accidental gauge symmetries** To each \( U \in G_{\text{rec}} \) there is an automorphism \( \zeta(U) \in \text{Aut}(\langle \mathcal{F}, \gamma, \mathcal{S} \rangle) \) such that \( \rho(\zeta(U)) = U \).

**Theorem 4.6** For a theory \( \langle \mathcal{F}, \gamma, \mathcal{S} \rangle \in \text{LCT}_{\text{grC}^*} \) obeying assumptions (1)–(9) above,

\[
\text{End}(\langle \mathcal{F}, \gamma, \mathcal{S} \rangle) = \text{Aut}(\langle \mathcal{F}, \gamma, \mathcal{S} \rangle) \cong G_{\text{rec}}
\]

(an isomorphism of groups). Moreover, \( G_{\text{max}} \) and \( G_{\text{rec}} \) are compact in the strong operator topology, while \( \text{Aut}(\langle \mathcal{F}, \gamma, \mathcal{S} \rangle) \) is compact in the natural weak topology given in Sec. 3.1.

**Proof.** Let \( \eta \in \text{End}(\langle \mathcal{F}, \gamma, \mathcal{S} \rangle) \). Proposition 4.4 and assumption (9) yield \( \rho(\eta) = \rho(\zeta(\rho(\eta))) \); faithfulness of \( \pi \) gives \( \eta_{M_0} = \zeta(\rho(\eta))_{M_0} \) and hence \( \eta = \zeta(\rho(\eta)) \) by Theorem 2.6. Thus \( \text{End}(\langle \mathcal{F}, \gamma, \mathcal{S} \rangle) = \text{Aut}(\langle \mathcal{F}, \gamma, \mathcal{S} \rangle) \) and the monoid homomorphism \( \rho \), already known to be faithful, is now bijective and becomes a group isomorphism. The compactness statements were proved in Propositions 4.2 and 4.4.

Some remarks are in order. First, it would be interesting to derive the content of assumption (9) from the other axioms or another more primitive requirement. We leave this as an open problem. Alternatively, one could drop this assumption and read our results as proving that proper endomorphisms are associated with accidental symmetries of the Minkowski net. Second, in general categories, objects admitting no proper monic endomorphisms are called Dedekind finite: for the category of sets these are indeed the finite sets, while for the category of vector spaces they are the finite-dimensional spaces (see [48] for other examples of Dedekind finiteness in different categories, including a number of salutary counterexamples). Thus theories obeying our assumptions are Dedekind-finite objects in the category of locally covariant theories. An immediate consequence is a strong version of the Schröder–Bernstein property for locally covariant theories.

**Corollary 4.7** Suppose \( \langle \mathcal{A}, \gamma, \mathcal{S} \rangle \) and \( \langle \mathcal{B}, \delta, \mathcal{T} \rangle \) are theories in \( \text{LCT}_{\text{grC}^*} \), at least one of which obeys assumptions (1)–(9). If there are subtheory embeddings \( \alpha : \langle \mathcal{A}, \gamma, \mathcal{S} \rangle \rightarrow \langle \mathcal{B}, \delta, \mathcal{T} \rangle \) and \( \beta : \langle \mathcal{B}, \delta, \mathcal{T} \rangle \rightarrow \langle \mathcal{A}, \gamma, \mathcal{S} \rangle \) then both \( \alpha \) and \( \beta \) are isomorphisms.

**Proof.** Without loss, suppose that \( \langle \mathcal{A}, \gamma, \mathcal{S} \rangle \) obeys the assumptions (1)–(9). Then \( \eta = \beta \circ \alpha \) is an endomorphism of \( \langle \mathcal{A}, \gamma, \mathcal{S} \rangle \), and hence an automorphism by Theorem 4.6. As \( \beta \) is monic, we may deduce that both \( \beta \) and \( \alpha \) are isomorphisms.

Finally, we comment on an alternative (though more restrictive) energy compactness condition for Minkowski space theories, known as the microscopic phase space condition [4]. Theories obeying this condition, along with other standard assumptions, have a definable

\[ \beta \circ (\alpha \circ \eta^{-1}) = \text{id}_{\langle \mathcal{A}, \gamma, \mathcal{S} \rangle}, \quad \text{and} \quad (\alpha \circ \eta^{-1}) \circ \beta = \beta = \beta \circ \text{id}_{\langle \mathcal{B}, \delta, \mathcal{T} \rangle}, \]

entailing that \( (\alpha \circ \eta^{-1}) \circ \beta = \text{id}_{\mathcal{B}} \) using the monic property of \( \beta \). Hence \( \beta \) is invertible, and so is \( \alpha = \beta^{-1} \circ \eta \).

---

\[ p \] Evidently, \( \beta \circ (\alpha \circ \eta^{-1}) = \text{id}_{\langle \mathcal{A}, \gamma, \mathcal{S} \rangle}, \) and \( \beta \circ (\alpha \circ \eta^{-1}) \circ \beta = \beta = \beta \circ \text{id}_{\langle \mathcal{B}, \delta, \mathcal{T} \rangle}, \) entailing that \( (\alpha \circ \eta^{-1}) \circ \beta = \text{id}_{\mathcal{B}} \) using the monic property of \( \beta \). Hence \( \beta \) is invertible, and so is \( \alpha = \beta^{-1} \circ \eta \).
field content forming finite dimensional subspaces that are sufficient to describe the theory at different orders of a short-distance approximation at sufficiently low energies, and reproduce the total field content of [34]. This might offer an alternative approach to the results of this section that avoids the need for twisted locality and twisted duality.

It remains to prove the Hilbert space result used in our argument.

Proof of Lemma 4.1. (a) For any $\psi \in \mathcal{N}$, the sequence $T^k\psi$ is contained in $\mathcal{N}$ and therefore has a subsequence $T^{kr}\psi$ converging in $\mathcal{H}$. In particular, for any $\epsilon > 0$ there is $R > 0$ such that $\|T^{kr} - T^{ks}\| < \epsilon$ for all $s > r > R$. But $T$ is an isometry, so we also have $\|T^{kr}\psi - \psi\| < \epsilon$ and we may deduce the existence of a sequence $j_r \to \infty$ with $T^{jr}\psi \to \psi$. Hence $\mathcal{N} \subset \text{cl}(TN)$, which, together with $TN \subset \mathcal{N}$, implies $\text{cl}(TN) = \text{cl}(\mathcal{N})$. Moreover, as $T^{(j_r-1)}\psi = T^r T^{jr}\psi \to T^r\psi$, we have $\|T^r\psi\| = \lim_r \|T^{j_r-1}\psi\| = \|\psi\|$ for all $\psi \in \mathcal{N}$. Now $1 - TT^*$ is a projection, so the elementary identity $\|(1 - TT^*)\psi\|^2 = \langle \psi | (1 - TT^*)\psi \rangle = \|\psi\|^2 - \|T^r\psi\|^2 = 0$ gives $TT^r\psi = \psi$ for all $\psi \in \mathcal{N}$. As $\mathcal{N}$ has dense linear span, we conclude that $T$ is unitary. Accordingly a sequence in $\mathcal{H}$ is Cauchy if and only if its image under $T$ is, from which we may obtain $T\text{cl}\mathcal{N} = \text{cl}TN = \text{cl}\mathcal{N}$.

(b) As $\mathcal{H}$ is separable, the strong operator topology is metrisable on any bounded subset of $\mathfrak{B}(\mathcal{H})$ [19] Proposition 2.7] and convergence and compactness can be determined sequentially. Now, $G$ is certainly closed: any strong limit of a sequence in $G$ must be an isometry mapping the compact set $\text{cl}\mathcal{N}$ into itself; by Lemma 4.1(a) the limit is therefore contained in $G$. Turning to compactness, choose a sequence $U_n$ in $G$, and fix a countable linearly independent set of vectors $\psi_j \in \mathcal{N}$ with dense linear span. The sequence $U_n\psi_1$ in $\text{cl}\mathcal{N}$ must have convergent subsequences, and we choose a subsequence $U_{n(k)}\psi_1$ so that $U_{n(k)}\psi_1$ converges. Proceeding inductively, we choose successive subsequences $U_{n(k)}\psi_j$ so that $U_{n(k)}\psi_j$ converges for each $1 \leq j \leq k$. The diagonal subsequence $V_k = U_{n(k)}\psi_k$ converges strongly on each $\psi_j$ and hence on any finite linear combination of the $\psi_j$. As span $\{\psi_j\}$ is dense and the $V_k$ are uniformly bounded, it follows that the sequence $V_k$ converges strongly to a limit in $G$. Thus $G$ is sequentially compact and hence compact. 

5 Scalar fields

As a concrete example, we consider theories of finitely many free minimally coupled scalar fields, specified by a finite set $\mathcal{M} \subset [0, \infty)$ representing the mass spectrum and a function $\nu : \mathcal{M} \to \mathbb{N}$ giving the number of field species with each mass (NB $\nu(m) > 0$ for each $m \in \mathcal{M}$). The total number of species is denoted $|\nu| = \sum_{m \in \mathcal{M}} \nu(m)$. We consider both the classical and quantum theories, working in a fixed spacetime dimension $n \geq 2$, though some of our results on the quantum field theory hold only in dimension $n \geq 3$.

5.1 Classical theory

The classical theory is described using the category of complexified symplectic spaces, $\text{Sympl}$, objects of which are triples $(V, \Gamma, \sigma)$, consisting of a complex vector space $V$, an
antilinear conjugation \( \Gamma : V \rightarrow V \) and a weakly nondegenerate antisymmetric bilinear form \( \sigma : V \times V \rightarrow \mathbb{C} \) such that \( \sigma(\Gamma v, \Gamma w) = \overline{\sigma(v, w)} \) for all \( v, w \in V \). In our applications, \( V \) will be a space of complex-valued functions and \( \Gamma \) will be induced by complex conjugation: \( (\Gamma f)(p) = \overline{f(p)} \). A morphism \( S : (V, \Gamma, \sigma) \rightarrow (V', \Gamma', \sigma') \) in \( \text{Sympl} \) is a (necessarily injective) complex linear map \( S : V \rightarrow V' \), such that \( \sigma'(Sv, Sw) = \sigma(v, w) \) and \( \Gamma' Sv = STv \) hold for all \( v, w \in V \). Relaxing the requirement for the bilinear forms to be nondegenerate, we obtain the category of complexified presymplectic spaces \( \text{preSympl} \). There is an obvious forgetful functor \( \mathcal{U} : \text{Sympl} \rightarrow \text{preSympl} \).

The theory of a single minimally coupled field of mass \( m \geq 0 \) is standard and was described in detail from the functorial perspective in [33]. We recall the essential facts only. For any \( M \in \text{Loc} \), let \( \mathcal{L}_m(M) \) be the space of complex-valued solutions \( \phi \) to

\[
(\Box_M + m^2)\phi = 0
\]

that have compact support on Cauchy surfaces in \( M \). Equipping \( \mathcal{L}_m(M) \) with complex conjugation and the antisymmetric bilinear form

\[
\sigma_{m,M}(\phi, \phi') = \int_\Sigma (\phi n^a \nabla_a \phi' - \phi' n^a \nabla_a \phi) \, d\Sigma,
\]

where \( \Sigma \) is a Cauchy surface with future-pointing unit normal \( n^a \), \( \mathcal{L}_m(M) \) becomes a complexified symplectic space (independent of the choice of \( \Sigma \)). The space \( \mathcal{L}_m(M) \) can also be expressed as \( \mathcal{L}_m(M) = E_{m,M}C_0^\infty(M) \), where \( E_{m,M} \) is the solution operator obtained as the difference of the advanced and retarded Green functions for (8). If \( \psi : M \rightarrow N \) in \( \text{Loc} \), the push-forward \( \psi_* : C_0^\infty(M) \rightarrow C_0^\infty(N) \), defined in Eq. (8), induces a unique linear map \( \mathcal{L}_m(\psi) : \mathcal{L}_m(M) \rightarrow \mathcal{L}_m(N) \) such that \( \mathcal{L}_m(\psi)E_{m,M} f = E_{m,N} \psi_* f \); this is in fact a \( \text{Sympl} \)-morphism and makes \( \mathcal{L}_m : \text{Loc} \rightarrow \text{Sympl} \) a functor.

The full theory is described by a functor \( \mathcal{L} : \text{Loc} \rightarrow \text{Sympl} \) with

\[
\mathcal{L}(M) = \bigoplus_{m \in M} \mathcal{L}_m(M) \otimes \mathbb{C}^{\nu(m)},
\]

equipped with complex conjugation and antisymmetric form

\[
\sigma_M((\phi_m \otimes z_m)_{m \in M}, (\phi'_m \otimes z'_m)_{m \in M}) = \sum_{m \in M} \sigma_m(\phi_m, \phi'_m) z'_m T z_m,
\]

where each \( \phi_m \in \mathcal{L}_m(M) \), \( z_m \in \mathbb{C}^{\nu(m)} \); here, we regard \( \mathbb{C}^k \) as a space of column vectors and write \( T \) for transpose.

The theory \( \mathcal{L} \) inherits the timeslice property from the theories \( \mathcal{L}_m \) [33] and therefore has a relative Cauchy evolution, which is differentiable in the weak symplectic topology

\[\text{Our notation differs slightly from that of [33], where the mass was not indicated explicitly; in particular, for } m = 0 \text{ the present } \mathcal{L}_0(M) \text{ does not coincide with the space } \mathcal{L}_0(M) \text{ studied in [33], which permits solutions that are compactly supported on Cauchy surfaces following modification by a locally constant function. See also Sec. 5.3 here.}\]
(see [33]). Let \( \text{Sym}(M) \) denote the space of smooth symmetric second rank covariant tensor fields of compact support on each \( M \in \text{Loc} \). Then, for each \( M \in \text{Loc} \) and \( f \in \text{Sym}(M) \), there exists a linear and symplectically skew-adjoint map \( F_M[f] \) on \( \mathcal{L}(M) \) such that

\[
\sigma_M(F_M[f] \phi, \phi') = \frac{d}{ds} \sigma_M(rce_M[sf] \phi, \phi') \bigg|_{s=0} \quad (\phi \in \mathcal{L}(M)).
\]

(9)

The maps \( F_M[f] \) are related to the classical stress-energy tensor: in fact,

\[
\sigma_M(F_M[f] \phi, \phi) = \int f_{ab} T^b_M[\phi] d\text{vol}_M,
\]

(10)

where \( T_M[\phi] \) is the stress-energy tensor of \( \phi \in \mathcal{L}(M) \). In view of the intertwining property (2), any endomorphism \( \eta \in \mathcal{L} \) obeys \( F_M[f] \eta_M = \eta_M F_M[f] \) and indeed

\[
T_M[\eta_M \phi] = T_M[\phi] \quad (\phi \in \mathcal{L}(M)).
\]

Although \( \text{Sympl} \) does not admit general categorical unions, \( \text{preSympl} \) does, namely the linear span of (conjugation-invariant) subspaces. Moreover, \( \text{preSympl} \) also has equalizers; the equalizer of two morphisms being the inclusion morphism of the subspace on which they agree (i.e., the kernel of their difference) in their common domain. Let \( M \in \text{Loc} \) be any spacetime and \( \varphi \in \mathcal{L}(M) \). We may write \( \varphi = E_M f \) for some \( f \in C^\infty_0(M; \mathbb{C}^{\nu}) \) and then use a partition of unity to write \( f \) as a finite sum \( f = \sum_i f_i \) where each \( f_i \) is supported in a diamond \( D_i \) in \( M \). Then it is evident that \( \varphi \) is contained in the span of \( E_M C^\infty_0(D_i; \mathbb{C}^{\nu}) \), which is the image of \( \mathcal{U}(\mathcal{L}((M,D_i)) \). Hence \( \mathcal{L} \) is \( \mathcal{U} \)-additive.

The endomorphisms of \( \mathcal{L} \) will now be classified in terms of the group

\[
O(\nu) = \prod_{m \in M} O(\nu(m)),
\]

where the factors are the standard groups of real orthogonal matrices.

**Lemma 5.1** Each \( R = (R_m)_{m \in M} \in O(\nu) \) induces an automorphism \( S(R) \) of \( \mathcal{L} \) given by

\[
S(R)_M = \bigoplus_{m \in M} 1_{\mathcal{L}_m(M)} \otimes R_m,
\]

(11)

and the map \( R \mapsto S(R) \) is an injective group homomorphism of \( O(\nu) \) into \( \text{Aut}(\mathcal{L}) \).

**Proof.** It is easily seen that \( S(R)_M \) is an endomorphism of \( \mathcal{L}(M) \) and has inverse \( S(R^{-1})_M \). If \( \psi : M \rightarrow N \) then

\[
S(R)_N \circ \mathcal{L}(\psi) \bigoplus_{m \in M} \phi_m \otimes z_m = S(R)_M \bigoplus_{m \in M} (\mathcal{L}_m(\psi) \phi_m) \otimes z_m
\]

\[
= \bigoplus_{m \in M} \mathcal{L}_m(\psi) \phi_m \otimes R_m z_m = \mathcal{L}(\psi) \bigoplus_{m \in M} \phi_m \otimes z_m
\]

\[
= \mathcal{L}(\psi) \circ S(R)_M \bigoplus_{m \in M} \phi_m \otimes z_m
\]

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for arbitrary $\bigoplus_{m \in M} \phi_m \otimes z_m \in \mathcal{L}(M)$ and (extending by linearity) we see that $S(R)$ is natural. Thus $S(R) \in \text{Aut}(\mathcal{L})$; the homomorphism and injectivity properties are clear.

The main result of this section is that these are the only endomorphisms of $\mathcal{L}$.

**Theorem 5.2** Every endomorphism of $\mathcal{L}$ is an automorphism, and

$$\text{End}(\mathcal{L}) = \text{Aut}(\mathcal{L}) \cong O(\nu),$$

with the isomorphism given by the homomorphism of Lemma 5.1.

**Proof.** Owing to the timeslice property, $\mathcal{U}$-additivity and Theorem 5.1, any $\eta \in \text{End}(\mathcal{L})$ is uniquely determined by its Minkowski space component $\eta_{M_0}$. Using the fact that $\eta_{M_0}$ preserves the stress-energy tensor and commutes with translations of Minkowski space and complex conjugation, Proposition A.1 in Appendix A shows that there are real orthogonal matrices $R_m$ such that $\eta_{M_0} = S(R)_{M_0}$, where $R = (R_m)_{m \in M} \in O(\nu)$. Thus $\eta = S(R) \in \text{Aut}(\mathcal{L})$. In summary, all endomorphisms are automorphisms, and the homomorphism of Lemma 5.1 is surjective and hence an isomorphism.

### 5.2 Quantized theory: Field algebra

Any $(V, \sigma, C) \in \text{Sympl}$ has a quantization given by the unital $\ast$-algebra $\mathcal{Q}(V, \sigma, C)$, whose underlying complex vector space is the symmetric tensor vector space over $V$,

$$\mathcal{Q}(V, \sigma, C) = \Gamma \otimes V^\otimes V = \bigoplus_{n \in \mathbb{N}_0} V^\otimes n,$$

(all tensor products and direct sums being purely algebraic) with a product such that

$$u^\otimes m \ast v^\otimes n = \sum_{r=0}^{\min\{m,n\}} \left( \frac{i \sigma(u, v)}{2} \right)^r \frac{m! n!}{r! (m-r)! (n-r)!} \left( S(u^\otimes (m-r) \otimes v^\otimes (n-r)) \right), \tag{13}$$

where $S$ denotes symmetrisation, and a $\ast$-operation defined by $(u^\otimes)^\ast = (C u)^\otimes$; both operations being extended by (anti-)linearity to general elements of $\Gamma \otimes V$. By convention $u^\otimes 0 = 1 \in V^\otimes 0 = \mathbb{C}$. The nondegeneracy of $\sigma$ ensures that the algebra $\mathcal{Q}(V, \sigma, C)$ is simple. Moreover, the quantization becomes a functor from $\text{Sympl}$ to $\text{Alg}$ if we also assign

$$\mathcal{Q}(f) = \Gamma \circ (f) = \bigoplus_{n \in \mathbb{N}_0} f^\otimes n \tag{14}$$

to any morphism $f : (V, \sigma, C) \rightarrow (V', \sigma', C')$ in $\text{Sympl}$. We refer to [33] for a full exposition.

The quantization of the classical theory $\mathcal{L}$ studied in the previous subsection is given by $\mathcal{F} = \mathcal{Q} \circ \mathcal{L}$, which is additive because $\mathcal{L}$ is $\mathcal{U}$-additive. It will be useful to describe $\mathcal{F}$

\footnote{In [33], the analogous theory was denoted as $\mathcal{A}$, as is traditional in QFT in CST; here, we adopt the AQFT convention, that field algebras are denoted with an ‘$F$’ and observable algebras with an ‘$A$’.}
using ‘symplectically smeared fields’: for each $M \in \text{Loc}$, let $\Phi_M : \mathcal{L}(M) \to \Gamma_\otimes(\mathcal{L}(M))$ be the canonical injection $\Phi_M(\phi) = 0 \oplus \phi \oplus 0 \oplus \cdots$. Then $\Phi$ is a field of type $L$; $\Phi \in \text{Fld}(\mathcal{L}, \mathcal{F})$ [we suppress the forgetful functors from $\text{Symp}$ and $\text{Alg}$ to $\text{Set}$], obeying

$$\Phi_M : \phi \to \Phi_M(\phi) \text{ is } \mathbb{C}\text{-linear}$$

$$\Phi_M(\phi)^* = \Phi(\phi)$$

$$[\Phi_M(\phi), \Phi_M(\phi')] = i\sigma_M(\phi, \phi') \mathbb{1}_F(M)$$

for all $\phi, \phi' \in \mathcal{L}(M)$. Indeed, these relationships characterize the theory: given any map $\Phi_M$ from $\mathcal{L}(M)$ to some $A \in \text{Alg}$ obeying the above relations, there is an $\text{Alg}$-morphism $\alpha : \mathcal{F}(M) \to A$ such that $\alpha \Phi_M(\phi) = \Phi_M(\phi)$, which is an isomorphism if $A$ is generated by the $\Phi_M(\phi)$ and $\mathbb{1}_A$. The relative Cauchy evolution of $\mathcal{F}$ is closely linked to that of $\mathcal{L}$ by $\text{rec}_M^{(\mathcal{F})}[h] = \mathcal{D}(\text{rec}_M^{(\mathcal{L})}[h])$ and hence

$$\text{rec}_M^{(\mathcal{F})}[h] \Phi(\phi) = \Phi(\text{rec}_M^{(\mathcal{L})}[h] \phi).$$

The usual spacetime smeared fields of the theory are obtained by setting $\Phi_M(f) = \Phi_M(EM f)$ for $f \in C_0^\infty(M; \mathbb{C}[\nu]) \cong \bigoplus_{m \in M} C_0^\infty(M; \mathbb{C}[\nu(m)])$, where

$$EM = \bigoplus_{m \in M} E_{m,m} \otimes 1_{\nu(m)}.$$

We introduce a state space as follows. For each $M$, let $\mathcal{I}(M)$ be the set of all states $\omega$ on $\mathcal{F}(M)$ such that all $k$-point functions are distributional, in the sense that

$$C_0^\infty(M; \mathbb{C}[\nu])^\otimes k \ni f_1 \otimes \cdots \otimes f_k \mapsto \omega(\Phi_M(f_1) \cdots \Phi_M(f_k))$$

is continuous in the usual test-function topology for each $k \in \mathbb{N}$. This is a much larger state space than the Hadamard class usually employed; our results would be unaltered by restricting to this class. It is easily seen that $\mathcal{I}(M)$ is closed under convex linear combinations and with respect to operations induced by $\mathcal{F}(M)$; moreover, the pull-back of a distribution by a smooth embedding is also a distribution, so $\mathcal{F}(\psi)^* \mathcal{I}(N) \subset \mathcal{I}(M)$ for any $\psi : M \to N$. Thus, $\mathcal{I}(\psi)$ may be defined uniquely so that the diagram

$$\begin{array}{ccc}
\mathcal{I}(N) & \xrightarrow{\mathcal{I}(\psi)} & \mathcal{I}(M) \\
\mathcal{F}(N)^* & \xrightarrow{\mathcal{F}(\psi)^*} & \mathcal{F}(M)^*
\end{array}$$

commutes, where the unlabelled maps are the obvious inclusions. With this definition, $\mathcal{I}$ becomes a state space functor. Combining $\mathcal{F}$ and $\mathcal{I}$ we obtain a theory $\langle \mathcal{F}, \text{id}, \mathcal{I} \rangle \in \text{LCT}_{\text{grAS}}$ that obeys locality in the sense of commutation at spacelike separation.

*The morphism is necessarily monic because $\mathcal{F}(M)$ is simple (and $A$ is not the zero algebra).
Our task is now to classify \( \text{End}(\langle \mathcal{F}, \text{id}, \mathcal{I} \rangle) \), i.e., natural transformations \( \eta : \mathcal{F} \to \mathcal{F} \) such that \( \eta^*_{\mathcal{M}} \mathcal{I}(\mathcal{M}) \subseteq \mathcal{I}(\mathcal{M}) \) for all \( \mathcal{M} \). As a first observation, note that any \( \zeta \in \text{End}(\mathcal{L}) \) lifts to an endomorphism \( \mathcal{D}[\zeta] \) of \( \mathcal{F} \) with components \( \mathcal{D}[\zeta]_\mathcal{M} = \mathcal{D}(\zeta_\mathcal{M}) \), and that \( \mathcal{D}[\zeta] \) is an automorphism if \( \zeta \in \text{Aut}(\mathcal{L}) \), because functors preserve isomorphisms. Thus Theorem 5.2 shows that \( O(\nu) \ni R \mapsto \mathcal{D}[S(R)] \in \text{Aut}(\mathcal{F}) \) is a group homomorphism. For theories with no massless components (i.e., \( \nu(0) = 0 \)) this will turn out to give the full class of endomorphisms of \( \langle \mathcal{F}, \text{id}, \mathcal{I} \rangle \). However, massless theories admit additional automorphisms and the general result Theorem 5.4 is that \( \text{Aut}(\langle \mathcal{F}, \text{id}, \mathcal{I} \rangle) \) is isomorphic to

\[
G(\nu) = O(\nu) \ltimes \mathbb{R}^{(0)}_k,
\]

where \( \mathbb{R}^{(0)}_k \) is the additive group of real \( k \)-dimensional row vectors (with the convention that this is the trivial group if \( k = 0 \)) and the semidirect product is defined by

\[
((R_m)_{m \in \mathcal{M}}, \ell) \cdot ((R'_m)_{m \in \mathcal{M}}, \ell') = ((R_m R'_m)_{m \in \mathcal{M}}, \ell R'_0 + \ell').
\]

To explain the action of \( G(\nu) \), we require some notation. For any \( \phi \in \mathcal{L}(\mathcal{M}) \), let \( \phi_0 \) be the component of \( \phi \) in \( \mathcal{L}_0(\mathcal{M}) \otimes \mathbb{C}^{\nu(0)} \), regarded as a \( \nu(0) \)-dimensional column vector with entries in \( \mathcal{L}_0(\mathcal{M}) \). Similarly, for \( f \in C^\infty_0(\mathcal{M}; \mathbb{C}^{\nu(0)}) \), \( f_0 \) denotes the component in \( C^\infty_0(\mathcal{M}; \mathbb{C}^{\nu(0)}) \), so that \( \phi_0 = E_{0,\mathcal{M}} f_0 \). With these conventions, we set

\[
\langle \ell, \mathcal{L}(\psi)\phi \rangle_{\mathcal{N}} = \langle \ell, \phi \rangle_{\mathcal{M}}
\]

which is well-defined because all elements of \( \ker E_{0,\mathcal{M}} = \Box_{\mathcal{M}} C^\infty(\mathcal{M}) \) have vanishing space-time integral. Equivalently, we have \( \langle \ell, \phi \rangle_{\mathcal{M}} = \sigma_{\mathcal{M}}(\ell \cdot \phi_0, 1_{\mathcal{M}}) \), where \( 1_{\mathcal{M}} \) is the unit constant on \( \mathcal{M} \) (extending the notation if \( \mathcal{M} \) has noncompact Cauchy surfaces, so \( 1_{\mathcal{M}} \neq \mathcal{L}(\mathcal{M}) \)). It is easily verified that

\[
\langle \ell, \mathcal{L}(\psi)\phi \rangle_{\mathcal{N}} = \langle \ell, \phi \rangle_{\mathcal{M}}
\]

holds for all \( \phi \in \mathcal{L}(\mathcal{M}) \) and \( \psi : \mathcal{M} \to \mathcal{N} \). The group \( G(\nu) \) acts in the following way.

**Proposition 5.3** There is a group monomorphism \( \zeta : G(\nu) \to \text{Aut}(\langle \mathcal{F}, \text{id}, \mathcal{I} \rangle) \) such that

\[
\zeta(R, \ell)_{\mathcal{M}} \Phi(\phi) = \Phi(S(R)_{\mathcal{M}} \phi) + \langle \ell, \phi \rangle_{\mathcal{M}} 1_{\mathcal{F}(\mathcal{M})}
\]

for all \( \mathcal{M} \in \text{Loc}, \phi \in \mathcal{L}(\mathcal{M}) \) and \( (R, \ell) \in G(\nu) \).

**Proof.** Let \( (R, \ell) \in G(\nu) \) be arbitrary, and write the right-hand side of (19) as \( \tilde{\Psi}_{\mathcal{M}}(\phi) \). Then \( \phi \mapsto \tilde{\Psi}_{\mathcal{M}}(\phi) \) is a linear map of \( \mathcal{L}(\mathcal{M}) \) to \( \mathcal{F}(\mathcal{M}) \) obeying \( \tilde{\Psi}_{\mathcal{M}}(\phi)^* = \tilde{\Psi}_{\mathcal{M}}(\phi) \) and \([\tilde{\Psi}_{\mathcal{M}}(\phi), \tilde{\Psi}_{\mathcal{M}}(\phi')] = i \sigma_{\mathcal{M}}(\phi, \phi') 1_{\mathcal{F}(\mathcal{M})} \). Thus there is a unique endomorphism \( \zeta(R, \ell)_{\mathcal{M}} \) of \( \mathcal{F}(\mathcal{M}) \) such that (19) holds. This is invertible, with inverse \( \zeta((R, \ell)^{-1})_{\mathcal{M}}, \) so \( \zeta(R, \ell)_{\mathcal{M}} \in \text{Aut}(\mathcal{F}(\mathcal{M})) \). Moreover, if a state \( \omega \) has distributional \( k \)-point functions, so does \( \zeta(R, \ell)_{\mathcal{M}} \omega; \)

\(^1\)While it would be interesting to obtain a (purely algebraic) classification of \( \text{End}(\mathcal{F}) \), our current proof involves continuity arguments that require the specification of a state space.
here, we use the fact that \( \langle \ell, E_M f \rangle_M \) is evidently continuous in \( f \in C_0^\infty(M; \mathbb{C}^n) \). Thus \( \zeta(R, \ell)_M : \mathcal{F}(M) = \mathcal{F}(M) \), as \( \zeta(R, \ell)_M \) is an isomorphism. Moreover, we have

\[
\mathcal{F}(\psi) \zeta(R, \ell)_M \Phi_M(\phi) = \mathcal{F}(\psi) (\Phi_M(S(R)M\phi) + \langle \ell, \phi \rangle_M 1_{\mathcal{F}(M)}) \\
= \Phi_N(S(R)M\phi) + \langle \ell, \phi \rangle_M 1_{\mathcal{F}(N)} \\
= \Phi_N(S(R)N\mathcal{L}(\psi)\phi) + \langle \ell, \mathcal{L}(\psi)\phi \rangle_N 1_{\mathcal{F}(N)}
\]

for any \( M \xrightarrow{\psi} N \), using \( S(R) \in \text{End}(\mathcal{L}) \) and Eq. (18), so the components \( \zeta(R, \ell)_M \) form a natural transformation \( \zeta(R, \ell) : \mathcal{F} \rightarrow \mathcal{F} \), whereupon \( \zeta(R, \ell) \in \text{Aut}(\langle \mathcal{F}, \text{id}, \mathcal{J} \rangle) \). That \( \zeta : G(\nu) \ni (R, \ell) \mapsto \zeta(R, \ell) \in \text{Aut}(\langle \mathcal{F}, \text{id}, \mathcal{J} \rangle) \) is a homomorphism holds because

\[
\zeta(R, \ell)_M \zeta(R', \ell')_M \Phi_M(\phi) = \zeta(R, \ell)_M \left[ \Phi(S(R')M\phi) + \langle \ell', \phi \rangle_M 1_{\mathcal{F}(M)} \right] \\
= \Phi(S(R)M\phi) + \langle \ell, S(R')M\phi \rangle_M + \langle \ell', \phi \rangle_M 1_{\mathcal{F}(M)} \\
= \zeta(RR', \ell R_0 + \ell') \Phi_M(\phi),
\]

and the proof that \( \zeta \) is a monomorphism is straightforward. \( \square \)

The main result of this section is that every endomorphism of \( \langle \mathcal{F}, \text{id}, \mathcal{J} \rangle \) is one of the automorphisms just constructed. Our proof uses the Fock representation of the Minkowski vacuum state; as there is no such state for massless fields in \( n = 2 \) spacetime dimensions, we must exclude this case from the current treatment.

**Theorem 5.4** For spacetime dimension \( n \geq 3 \), every endomorphism of \( \langle \mathcal{F}, \text{id}, \mathcal{J} \rangle \) is an automorphism. Moreover, the monomorphism \( \zeta : G(\nu) \rightarrow \text{Aut}(\langle \mathcal{F}, \text{id}, \mathcal{J} \rangle) \) of Proposition 5.3 is an isomorphism of groups, so we have

\[
\text{End}(\langle \mathcal{F}, \text{id}, \mathcal{J} \rangle) = \text{Aut}(\langle \mathcal{F}, \text{id}, \mathcal{J} \rangle) \cong G(\nu).
\]

The same result holds in two dimensions if \( \nu(0) = 0 \), whereupon \( \text{Aut}(\langle \mathcal{F}, \text{id}, \mathcal{J} \rangle) \cong O(\nu) \).

**Remark:** In the purely massive case \( (\nu(0) = 0) \), it is well-known that the maximal DHR group for the affiliated local \( C^* \)-algebraic net is precisely \( G_{\text{max}} = O(\nu) \). Thus the conclusion of Theorem 1.6 holds in this case (with \( G_{\text{rec}} = G_{\text{max}} \)). A direct application of the theory of Sec. 4 to the Weyl algebra is described in Sec. 5.4.

**Proof.** Given any \( \eta \in \text{End}(\langle \mathcal{F}, \text{id}, \mathcal{J} \rangle) \), the transformed symplectically smeared field \( \tilde{\Psi} = \eta \cdot \Phi \) obeys many of the same properties as \( \Phi \); namely, its behaviour under adjoints

\[
\tilde{\Psi}_M(\phi)^* = (\eta_M \Phi_M(\phi))^* = \eta_M(\Phi_M(\phi))^* = \eta_M \Phi_M(\overline{\phi}) = \tilde{\Psi}_M(\overline{\phi})
\]

and the commutation relations

\[
[\tilde{\Psi}_M(\phi), \tilde{\Psi}_M(\phi')] = \eta_M[\Phi_M(\phi), \Phi_M(\phi')] = i\sigma_M(\phi, \phi') 1_{\mathcal{F}(M)}.
\]

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Moreover, Eqs. (16) and (2) entail that

\[ \text{rec}_M^{(\mathcal{F})}[h] \Psi_M(\phi) = \Psi_M(\text{rec}_M^{(\mathcal{F})}[h]\phi). \]  

We now focus on Minkowski space \( M_0 \). Using properties of the vacuum representation, we show in Proposition 5.6 below that \( \Psi_{M_0} \) takes the form

\[ \Psi_{M_0}(\phi) = \Phi_{M_0}(S\phi) + (\ell, \phi)_{M_0} 1_{\mathcal{F}(M_0)} \quad (\phi \in \mathcal{L}(M_0)) \]  

for some linear map \( S : \mathcal{L}(M_0) \to \mathcal{L}(M_0) \) and \( \ell \in \mathbb{C}^0(0)^* \) [with \( \ell = 0 \) if \( \nu(0) = 0 \)]. Substituting in Eq. (22) and using Eq. (16), we find

\[ \Phi_{M_0}(\text{rec}_M^{(\mathcal{F})}[h]S\phi) + (\ell, \phi)_{M_0} 1_{\mathcal{F}(M_0)} = \Phi_{M_0}(S\text{rec}_M^{(\mathcal{F})}[h]\phi) + (\ell, \text{rec}_M^{(\mathcal{F})}[h]\phi)_{M_0} 1_{\mathcal{F}(M_0)} \]

and may deduce that \( S \) commutes with the relative Cauchy evolution in \( M_0 \) (the identity \( (\ell, \text{rec}_M^{(\mathcal{F})}[h]\phi)_{M_0} = (\ell, \phi)_{M_0} \) gives no additional constraint). Accordingly, \( \phi \) and \( S\phi \) have identical (classical) stress-energy tensors; moreover \( S \) commutes with the action of space-time translations by naturality of \( \eta \), and obeys \( S\phi = \phi \) owing to Eq. (20). Proposition A.1 implies that \( S = S(R)_{M_0} \) for some \( R = (R_m)_{m \in M} \in O(\nu) \), and Eq. (21) also shows that \( \ell \) is real. Hence \( \eta_{M_0} = \zeta_{M_0}(R, \ell) \), and as \( (\mathcal{F}, \text{id}, \mathcal{F}) \) obeys the timeslice property and is \( \mathcal{W} \)-additive with respect to the forgetful functor \( \mathcal{W} : \text{grAS} \to \text{Alg} \), Theorem 2.6 entails that \( \eta = \zeta(R, \ell) \in \text{Aut}(\langle \mathcal{F}, \text{id}, \mathcal{F} \rangle) \), and the result follows immediately.

The remaining task is to prove Proposition 5.6. Our argument uses particular features of the standard Minkowski vacuum state \( \omega \), which exists for \( n \geq 3 \), or \( n = 2 \) provided \( \nu(0) = 0 \). Let us briefly recall some properties of the induced GNS representation \( (\mathcal{F}, \pi, \mathcal{D}, \Omega) \) of \( \mathcal{F}(M_0) \). First, we will denote the one-particle Hilbert space by \( \mathcal{H} \), so \( \mathcal{F} = \mathcal{F}_0(\mathcal{H}) \). Second, each translation \( T^y(x) = x + y \), there is a unitary \( U(y) = \exp(iy^a P_a) \) so that \( U(y)\pi(A)U(y)^{-1} = \pi(T^yA) \) for all \( A \in \mathcal{F}(M_0) \). The momentum operators \( P_a \) are defined on a domain in \( \mathcal{F} \) including \( \mathcal{D} = \pi(\mathcal{F}(M_0)) \), on which \( y \mapsto U(y) \) is therefore strongly differentiable; the joint spectrum of the \( P_a \) lies in the closed forward lightcone. Third, the discrete spectrum of the mass-squared operator \( P^a P_a \) is \( \sigma_{\text{disc}}(P^a P_a) = \{ 0 \} \cup \{ m^2 : m \in \mathcal{M} \} \). Denoting the eigenspace of eigenvalue \( m^2 \) by \( \mathcal{H}_m \), the subspaces \( \mathcal{H}_m \) for \( m > 0 \) are subspaces of the one-particle space \( \mathcal{H} \), while \( \mathcal{H}_0 \) is spanned by the one-particle states of any massless fields and the vacuum vector, i.e.,

\[ \mathcal{H}_0 = \mathbb{C}\Omega \oplus \left( \bigoplus_{m \in \mathcal{M} \setminus \{ 0 \}} \mathcal{H}_m \right) \perp \subset \mathcal{F}, \]

where the orthogonal complement is taken in \( \mathcal{H} \). Fourth, the vacuum vector \( \Omega \) is separating for the representation \( \pi \), i.e., \( \pi(A)\Omega = 0 \) iff \( A = 0 \). Fifth, we have the following:

**Lemma 5.5** Suppose \( A \in \mathcal{F}(M_0) \) is such that \( \pi(A)\Omega \in \mathcal{H}_m \) for some \( m \geq 0 \). Then \( A = \Phi_{M_0}(\phi) + \alpha 1_{\mathcal{F}(M_0)} \) for unique \( \phi \in \mathcal{L}(M_0) \) and \( \alpha \in \mathbb{C} \), which obey \( M\phi = m\phi \). If \( m > 0 \), \( \alpha \) must vanish.
Theorem. As $\mathcal{F}(M_0) = \Gamma(\mathcal{L}(M_0))$, each nonzero $A \in \mathcal{F}(M_0)$ therefore has a degree $\deg A$, which is the maximum $n \in \mathbb{N}_0$ such that $A^{(n)} \neq 0$, where $A^{(n)}$ is the component of $A$ in $\mathcal{L}(M_0)^{\otimes n}$; we assign a degree $-1$ to the zero element of $\mathcal{F}(M_0)$. Moreover, the vacuum $\Omega$ induces a normal ordering operation on $\mathcal{F}(M_0)$, so that $\pi(\cdot : A : \Omega) = \pi(A)\pi(\cdot)$, where the normal ordering on the right-hand side is that of the Fock space $\mathcal{F}$. A key fact is that $\deg\{A - \cdot : A : \Omega\} = \max\{-1, \deg(A) - 2\}$. Now $\deg A$ is also the maximum $n \in \mathbb{N}_0$ for which $\mathcal{Y} = \pi(A)\Omega$ has vanishing component in the $n$-particle space in Fock space: to see this, note that $\mathcal{Y}$ clearly has vanishing component in all higher $n$-particle spaces, while its projection onto the $\deg A$-particle space coincides with $\pi(A):\Omega = \pi(A)\Omega$; if this should vanish then $A : \Omega = 0$ because $\Omega$ is separating for $\pi$, and hence $\deg A = \max\{\deg(A) - 2, -1\}$, which implies $A = 0$.

Now suppose that $\pi(A)\Omega \in \mathcal{H}_m$, which lies in the 1-particle space if $m \neq 0$, or the span of the 0 and 1-particle spaces if $m = 0$. Accordingly, if $A \neq 0$, we must have $\deg A = 1$ for $m > 0$ and $\deg A \leq 1$ if $m = 0$. The result follows.

Finally, we may state and prove the remaining result.

Proposition 5.6 Let $\eta \in \text{End}(\langle \mathcal{F}, \text{id}, \mathcal{L} \rangle)$ and define $\hat{\Psi} = \eta \cdot \hat{\Phi}$. Then there exists a unique linear map $S : \mathcal{L}(M_0) \rightarrow \mathcal{L}(M_0)$ and $\ell \in \mathbb{C}^{\nu(0)*}$ [with $\ell = 0$ if $\nu(0) = 0$] and

$$
\hat{\Psi}_{M_0}(\phi) = \hat{\Phi}_{M_0}(S\phi) + \langle \ell, \phi \rangle_{M_0} 1_{\mathcal{F}(M_0)},
$$

for all $\phi \in \mathcal{L}(M_0)$.

Proof. In the vacuum representation, consider vectors of the form $\pi(\hat{\Psi}_{M_0}(\phi))\Omega$ for $\phi \in \mathcal{L}(M_0)$. The properties of $U(y)$ mentioned above entail that

$$
U(y)\pi(\hat{\Psi}_{M_0}(\phi))\Omega = \pi(\mathcal{F}(\tau^y)\hat{\Psi}_{M_0}(\phi))\Omega = \pi(\hat{\Psi}_{M_0}(L(\tau^y)\phi))\Omega.
$$

Let $e_a$ ($a = 0, \ldots, n - 1$) be standard inertial basis vectors. Putting $y = se_a$, the left-hand side may be differentiated with respect to $s$ to give $iP_a\pi(\hat{\Psi}_{M_0}(\phi))\Omega$ at $s = 0$. To evaluate the derivative of the right-hand side, we set $\phi = E_Mf$ for $f \in C_0^\infty(M_0; \mathbb{C}^{[\nu]}).$ Then

$$
s^{-1}(\mathcal{L}(\tau^{se_a})\phi - \phi) + \nabla_a\phi = E_{M_0}(s^{-1}[\tau^{se_a}f - f] + \nabla_a f)
$$

and the parenthesis on the right-hand side tends to 0 in $C_0^\infty(M_0; \mathbb{C}^{[\nu]}).$ Now

$$
\|\pi(\hat{\Psi}(E_{M_0}h))\Omega\|^2 = (\eta_{M_0}\omega)(\hat{\Phi}(E_{M_0}\bar{h})\hat{\Phi}(E_{M_0}h)),
$$

which is a 2-point function for the state $\eta_{M_0}\omega \in \mathcal{S}(M_0)$, and therefore a distribution. Thus $\pi(\hat{\Psi}(E_{M_0}h))\Omega \rightarrow 0$ in $\mathcal{F}$ as $h \rightarrow 0$ in $C_0^\infty(M_0; \mathbb{C}^{[\nu]})$ and the left-hand side of (24) has the derivative one would expect, namely $-\pi(\hat{\Psi}_{M_0}(\nabla_a\phi))\Omega$. Accordingly, $P_a\pi(\hat{\Psi}_{M_0}(\phi))\Omega = \pi(\hat{\Psi}_{M_0}(i\nabla_a\phi))\Omega$ for any $\phi \in \mathcal{L}(M_0)$ and hence

$$
P_aP^a\pi(\hat{\Psi}_{M_0}(\phi))\Omega = \pi(\hat{\Psi}_{M_0}(\Box_{M_0}\phi))\Omega = \pi(\hat{\Psi}_{M_0}(M^2\phi))\Omega,
$$

(25)
where we have abused notation by writing $\Box_M$ and $M^2$ as a shorthand for the operators
\[
\bigoplus_{m \in M} \Box_M \otimes 1_{\nu(m)}, \quad \bigoplus_{m \in M} m^2 1_{L(M)} \otimes 1_{\nu(m)}
\]
on $\mathcal{L}(M)$. From Eq. (25) we see that $M\phi = m\phi$ implies $\pi(\Psi_{M,0}(\phi))\Omega \in \mathcal{H}_m$. According to Lemma 5.5, we may therefore deduce that, for general $\phi \in \mathcal{L}(M_0)$,
\[
\Psi_{M,0}(\phi) = \Phi_{M,0}(S\phi) + \alpha(\phi_0)1_{\mathcal{F}(M_0)},
\]
where $\phi_0$ is the component of $\phi$ in $\mathcal{L}_0(M_0) \otimes \mathbb{C}^{\nu(0)}$ and $S : \mathcal{L}(M_0) \to \mathcal{L}(M_0)$ and $\alpha : \mathcal{L}_0(M_0) \otimes \mathbb{C}^{\nu(0)} \to \mathbb{C}$ are uniquely determined and necessarily linear maps (we can also deduce $SM = MS$). It remains to determine $\alpha$. By Eq. (26), it is clear that $\alpha(\phi_0) = \langle \Omega | \Psi_{M,0}(\phi)\Omega \rangle = (\eta^*_M\omega)(\Phi_{M,0}(\phi))$.

As $\eta^*_M\omega$ is a translationally invariant state in $\mathcal{F}(M_0)$ by Lemma 2.7, it follows that $\mathcal{D}(M_0) \otimes \mathbb{C}^{\nu(0)} \ni f \mapsto (\alpha \circ (E_{0,M_0} \otimes 1_{\nu(0)}))(f) \in \mathbb{C}$ may be regarded as a row vector of translationally invariant distributions, each component of which must be constant. Thus there is $\ell \in \mathbb{C}^{\nu(0)*}$ such that $(\alpha \circ (E_{0,M_0} \otimes 1_{\nu(0)}))(f) = \int_{M_0} \ell \cdot f d\text{vol}_{M_0}$ and in fact we must have $\ell \in \mathbb{R}^{\nu(0)*}$ to obtain $\alpha(\overline{\phi_0}) = \overline{\alpha(\phi_0)}$, whereupon $\alpha(\phi_0) = \langle \ell, \phi \rangle_{M_0}$ for all $\phi \in \mathcal{L}(M_0)$, completing the proof. $\Box$

5.3 Quantized theory: Algebra of Observables

In each spacetime $M$, the algebra of observables $\mathcal{A}(M)$ may be concretely constructed as the subalgebra of the field algebra $\mathcal{F}(M)$ of elements invariant under $\zeta(R, \ell)_M$ for all $(R, \ell) \in \mathcal{G}(\nu) = O(\nu) \ltimes \mathbb{R}^{\nu(0)*}$.

We begin with the issue of $\mathbb{R}^{\nu(0)*}$-invariance (assuming $\nu(0) > 0$). Let $\mathcal{L}_0(M)$ be the subspace of $\mathcal{L}_0(M)$ consisting of solutions with vanishing symplectic product with the constant unit solution. We call these ‘charge-zero’ solutions, because this symplectic product is precisely the Noether charge corresponding to the rigid gauge freedom to add a constant solution (the same constant in all connected components of $M$). The subspace $\mathcal{L}_0(M)$ has codimension 1 in $\mathcal{L}_0(M)$; we choose any $\theta \in \mathcal{L}_0(M)$ with $\sigma_{0,M}(\theta, 1_M) = 1$, which then spans a complementary subspace to $\mathcal{L}_0(M)$ in $\mathcal{L}_0(M)$. For notational simplicity, it is also convenient to write $\mathcal{L}_m(M) = \mathcal{L}_m(M)$ for any $m > 0$. With these choices, any $A \in \mathcal{A}(M)$ may be written in the form
\[
A = \sum_{k=0}^{\deg A} \sum_{|\alpha| \leq k} S \left( \bigotimes_{i=1}^{\nu(0)} (\theta \otimes e_i)^{\otimes \alpha_i} \right) \otimes Z_{k,\alpha},
\]
where the sum runs over all multi-indices \( \alpha \) of total order \( |\alpha| \leq k \), the \( e_i \) are a standard real basis for \( \mathbb{C}^{\nu(0)} \), \( S \) is the symmetrisation operator and

\[
Z_{k,\alpha} \in \mathring{\mathcal{L}}(M)^{\otimes(k-|\alpha|)}, \quad \text{where} \quad \mathring{\mathcal{L}}(M) := \bigoplus_{m \in M} \mathring{\mathcal{L}}_m(M) \otimes \mathbb{C}^{\nu(m)}.
\]

Let \( e_i^* \in \mathbb{R}^{\nu(0)*} \) be the dual basis to \( e_i \). Then because the polynomial \( \lambda \mapsto \zeta((1, \lambda e_j^*)_R A \) is constant, the coefficient of \( \lambda \) must vanish, i.e.,

\[
\sum_{k=1}^{\deg A} \sum_{|\alpha| \leq k} \alpha_j S \left( \bigotimes_{i=1}^{\nu(0)} (\theta \otimes e_i)^{\otimes(\alpha_i - \delta_{ij})} \right) \otimes Z_{k,\alpha} = 0,
\]

whereupon every \( Z_{k,\alpha} \) vanishes for which \( \alpha_j > 0 \). (To see this, one works downwards in degree.) As \( 1 \leq j \leq \nu(0) \) is arbitrary, this gives \( Z_{k,\alpha} = 0 \) for all \( |\alpha| > 0 \). Accordingly, all nontrivial generators of \( A \in \mathfrak{A}(M) \) belong to \( \mathring{\mathcal{L}}(M) \), so \( \mathfrak{A}(M) \subset \Gamma_0(\mathring{\mathcal{L}}(M)) \).

Turning to the \( O(\nu) \) invariance, let us now suppose that \( A \in \Gamma_0(\mathring{\mathcal{L}}(M)) \) obeys

\[
\zeta(R,0)_R A = A \quad \text{for all} \quad R \in O(\nu).
\]

Because \( \zeta(R,0)_R = \Gamma_0(S(R)_R \right) \), where \( S(R)_R \) is defined in Eq. (11), the component \( A_k \) of \( A \) in each \( \mathring{\mathcal{L}}(M)^{\otimes k} \) must be invariant under \( S(R)^{\otimes k}_R \). Now we may identify

\[
\mathring{\mathcal{L}}(M)^{\otimes k} = \bigoplus_{m \in M} \mathring{\mathcal{L}}_m(M) \otimes \mathbb{C}^{\nu(m)}, \quad \mathfrak{A}(M) \otimes \mathbb{C}^{\nu(m)},
\]

\[
= \bigoplus_{m \in M \times k} \bigotimes_{m' \in m} \left( \mathring{\mathcal{L}}_{m'}(M)^{\otimes \mu_{m'}(m')} \otimes \mathbb{C}^{\nu(m')} \right) \otimes \mu_{m'}(m'),
\]

where \( m \in M \times k \) is a \( k \)-tuple \( m = \langle m_1, \ldots, m_k \rangle \), and \( \mu_{m'}(m') \) is the multiplicity of \( m' \) as an element of \( m \), and the product in the last expression is indexed over elements of \( m \) disregarding multiplicity. With respect to the last decomposition, we have

\[
S(R)^{\otimes k}_R \equiv \bigoplus_{m \in M \times k} m' \in m \bigotimes \text{id} \otimes R^{\otimes \mu_{m'}(m')}_{m' \in m}.
\]

Owing to the direct sum structure, the element \( A_k \) decomposes into components \( A_{k,m} \), each of which is an eigenvector of unit eigenvalue for every \( \bigotimes_{m' \in m} (\text{id} \otimes R^{\otimes \mu_{m'}(m')} \otimes \nu(m')) \in \bigoplus_{m' \in m} O(\nu(m')) \). Some multi-linear algebra (cf. e.g., \[33\], Appendix A) entails that

\[
A_{k,m} \in \bigotimes_{m' \in m} \mathring{\mathcal{L}}_{m'}(M)^{\otimes \mu_{m'}(m')} \otimes Y_{m'},
\]

where each \( Y_{m'} \subset (\mathbb{C}^{\nu(m'))}^{\otimes \mu_{m'}(m')} \) is an eigenspace of unit eigenvalue for \( R^{\otimes \mu_{m'}(m')} \) for all \( R \in O(\nu(m')) \). Its elements are thus isotropic tensors (under the full orthogonal group) of rank \( \mu_{m'}(m') \) in \( \nu(m') \) dimensions. By classical results \([54], \S 2.9\), these are known to be
scalars at rank 0, products of Kronecker deltas for other even ranks, and vanishing for odd rank. As \( A \) is an element of the symmetric tensor vector space, we have shown that \( \mathcal{A}(M) \) is contained in the \(*\)-subalgebra of \( \mathcal{F}(M) \) generated by all bilinear elements of the form

\[
\sum_{i=1}^{\nu(m)} \Phi(\phi \otimes e_i)\Phi(\phi' \otimes e_i)
\]  

(27)

for \( m \in M, \phi, \phi' \in \hat{L}_m(M) \) and with \( e_i \) the standard real basis of \( \mathbb{C}^{\nu(m)} \), orthonormal with respect to the standard inner product. As this subalgebra is manifestly invariant under the action of \( G(\nu) \), we have proved:

**Theorem 5.7** The algebra of observables \( \mathcal{A}(M) \) is the \(*\)-subalgebra of \( \mathcal{F}(M) \) generated by all bilinear elements of the form Eq. (27) where \( \phi, \phi' \in \hat{L}_m(M) \) and \( m \in M \).

In the case where \( M \) has more than one connected component, we see that the solutions \( \phi, \phi' \) appearing in Eq. (27) may have support in more than one component of the space-time. One might not wish to regard these as observables. Adopting the more restricted ‘true algebra of observables’ described at the end of Sec. 3.3, the generating set would be restricted to bilinears for which \( \phi, \phi' \) have support in a single common component of \( M \).

Finally, let us specialize to the case of a single massless field \( u \) and compare the algebra of observables obtained here with the discussion of the massless current in [33], in which the classical invariance of the action under addition of locally constant solutions (which may take distinct values on different connected components of space-time) is treated as a classical gauge symmetry. This gives a classical phase space of gauge equivalence classes \([\phi]\) of solutions to the massless Klein–Gordon equation whose symplectic products with locally constant functions must be taken to vanish in order to obtain a well-defined (and, in fact, weakly non-degenerate) symplectic product on the quotient. Quantizing, a locally covariant theory \( \mathcal{C} \) is obtained, in which all \( \mathcal{C}(M) \) are simple, with generators \( \hat{J}_M([\phi]) \) obeying relations analogous to those in (15).

This is most directly comparable to the fixed-point subalgebra \( \hat{\mathcal{F}}(M) \) of \( \mathcal{F}(M) \) under the noncompact factor in the gauge group \( \mathbb{Z}^2 \rtimes \mathbb{R}^* \). Here, the generators are labelled by solutions with vanishing symplectic product with the constant solution \( 1_M \), i.e., the space \( \hat{\mathcal{F}}_0(M) \). If \( M \) is connected, and has noncompact Cauchy surfaces, then \( \hat{\Phi}_M(\phi) \mapsto \hat{J}_M([\phi]) \) determines an isomorphism of the two algebras. However, this isomorphism breaks down if \( M \) is disconnected (because not all solutions in \( \hat{\mathcal{F}}_0(M) \) have vanishing symplectic product with every locally constant function) or has compact Cauchy surface (because then \( \hat{\mathcal{F}}_0(M) \) contain nonzero locally constant functions \( \chi \), which correspond to nonzero elements in the centre of \( \hat{\mathcal{F}}(M) \); on the other hand, \([\chi]\) and hence \( \hat{J}_M([\chi]) \) vanish).

\(^a\)Analogous comments apply to any of the theories with \( \nu(0) > 0 \).
\(^b\)These solutions are permitted to have noncompact support, but must be locally constant outside the causal future and past of some compact set.
The first problem may be removed by passing to the ‘true algebra of observables’, namely the $*$-subalgebra $\hat{F}_0(M)$ of $\hat{F}(M)$ generated by the $\hat{J}(\phi)$ with vanishing symplectic product with the characteristic function of each component of $M$, and hence with every locally constant function. However, the second problem remains, whenever $M$ has a component with compact Cauchy surface; in general $\mathcal{C}(M)$ (with the modification mentioned) is isomorphic to the quotient of $\hat{F}_0(M)$ by the ideal generated by its centre. We mention that these central elements are also responsible for the failure of the theory $\hat{F}$ to be dynamically local [32] [33].

5.4 Quantized theory: Weyl algebra

We briefly explain how the symmetries of the Weyl algebra quantization may be classified. Here, one works with the category of real symplectic spaces $\text{Sympl}^\mathbb{R}$ and the real-valued solutions to our system, which form a functor $\mathcal{L}_\mathbb{R} : \text{Loc} \to \text{Sympl}^\mathbb{R}$; composing with the CCR functor $\text{CCR} : \text{Sympl}^\mathbb{R} \to \mathcal{C}^*\text{-Alg}$ gives the theory $\mathcal{W} = \text{CCR} \circ \mathcal{L}_\mathbb{R}$ – details can be found in e.g., [8, 2, 33]. For simplicity, we restrict to the case in which massless fields are absent, $\nu(0) = 0$; we also write $W_M(\phi)$ for the Weyl generator of $\mathcal{W}(M)$ labelled by $\phi \in \mathcal{L}(M)$. Endowing $\mathcal{W}$ with a state space $\mathcal{S}$, consisting of the closure of the set of quasifree Hadamard states with respect to operations and local quasiequivalence, $\langle \mathcal{W}, \text{id}, \mathcal{S} \rangle \in \mathcal{L}CT_{\text{grC-AS}}$ obeys assumptions (1)–(7) of Sec. 4 (see, e.g., [8] [13] [1] – duality holds (at least) for the set of double cone regions [i.e., nonempty sets of the form $I^+(p) \cap I^-(q)$ for timelike separated $p$ and $q$], which suffices for assumption (7)). Hence by Theorem 4.3, there is a faithful embedding of $\text{End}(\langle \mathcal{W}, \text{id}, \mathcal{S} \rangle)$ in $G_{\text{max}}$. The latter is well-known for this theory: $G_{\text{max}}$ is the group of unitaries $U(R)$ ($R \in O(\nu)$) such that $U(R)W_{M_0}(\phi)U(R)^{-1} = W_{M_0}(R\phi)$ and $U(R)\Omega = \Omega$. As each $R \in O(\nu)$ induces an automorphism of $\langle \mathcal{W}, \text{id}, \mathcal{S} \rangle$ by $\zeta(R)M W_M(\phi) = W_M(R\phi)$, assumption (8) holds with the replacement of $G_{\text{rc}}$ by $G_{\text{max}}$. The following is immediate:

**Theorem 5.8** In spacetime dimension $n \geq 3$, and subject to $\nu(0) = 0$, the Weyl algebra theory $\langle \mathcal{W}, \text{id}, \mathcal{S} \rangle$ obeys $\text{End}(\langle \mathcal{W}, \text{id}, \mathcal{S} \rangle) = \text{Aut}(\langle \mathcal{W}, \text{id}, \mathcal{S} \rangle) \cong O(\nu)$.

Although we have circumvented assumption [8], the relative Cauchy evolution is unitarily implemented as a consequence of Wald’s work on the $S$-matrix [32] and the implementors have nonzero vacuum expectation value. Thus assumption [8] also holds and we see that $G_{\text{rc}} = G_{\text{max}}$. We expect that the case $\nu(0) > 0$ can also be treated, resulting in the automorphism group $G(\nu)$ as in Theorem 5.4 but at the expense of more technicality.

*Note, however, that the $S$-matrix is a unitary map between two representations of the Weyl algebra for the perturbed spacetime, while the relative Cauchy evolution is an automorphism of the algebra of the unperturbed spacetime. The implementation of the relative Cauchy evolution is (up to unitaries) the inverse of the $S$-matrix.*
6 Conclusion

We have argued that the global gauge group of a locally covariant quantum field theory can be identified with the automorphism group of its defining functor (or, sometimes, a subgroup thereof – see footnote b), and that this interpretation provides a natural generalization of the standard concepts in Minkowski space AQFT. Furthermore, we have argued that proper endomorphisms of a theory are pathological, and shown that they can be excluded under reasonable general assumptions, which also entail that the gauge group is compact. As mentioned, it is expected that this viewpoint can contribute to the theory of superselection in curved spacetime (see, e.g., remarks in the conclusion of [50]). Moreover, it should have applications in other areas where locally covariance plays a role, for example the Batalin–Vilkovisky formalism developed by Fredenhagen and Rejzner [35, 36] and the general discussion of classical theories in [7]. Elsewhere [29] it will also be used to give a new perspective on twisted quantum fields [42].

Finally, it may be worth commenting on the special role given to Minkowski space in some of our arguments. From the perspective of traditional QFT in curved spacetime, it might seem unsatisfactory that some of our technical hypotheses are made only on the theory as it is formulated in Minkowski space. Indeed, it is presently unknown how to formulate energy compactness in arbitrary spacetimes in an elegant way. However, the assumption of energy compactness in Minkowski space does place constraints on the theory in any other given spacetime – what is really lacking is a convenient technical expression for them – and the physical interpretation is the same as in [39, 13], namely to restrict the number of degrees of freedom available in each given volume of the phase space of the theory. From this perspective, our use of Minkowski space is a matter of technical simplicity. It remains an open and important issue to formulate energy compactness in curved spacetimes, or, in the spirit of our other results, directly at the level of the functorial description of the theory.

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A Maps preserving the stress-energy tensor

Proposition A.1 Suppose \( S : \mathcal{L}(M_0) \rightarrow \mathcal{L}(M_0) \) is a linear map [no continuity is assumed] so that \( S\phi \) and \( \phi \) have identical total stress-energy tensors for all \( \phi \in \mathcal{L}(M_0) \) and such that \( S \) commutes with complex conjugation and the action of spacetime translations on \( \mathcal{L}(M_0) \). Then there are orthogonal matrices \( R_m \in O(\nu(m)) \) such that

\[
S = \bigoplus_{m \in \mathcal{M}} 1_{\mathcal{L}_m(M_0)} \otimes R_m.
\]

Proof. We may regard any \( \phi \in \mathcal{L}(M_0) \) as a smooth function on \( M_0 \) taking values in \( \mathbb{C}[\nu] := \bigoplus_{m \in \mathcal{M}} \mathbb{C}(\nu(m)) \). Putting the standard norm on the latter space, we have \( T_{ab}^c \ell^d \ell^b |_p = \)
\[ \| \ell^a \nabla \phi \|_p^2 \] for every point \( p \in M_0 \), null vector \( \ell^a \) and \( \phi \in \mathcal{L}(M_0) \); it follows that \( S \) must obey \( \| (\ell^a \nabla_a S \phi) |_p^2 = \| \ell^a \nabla_a \phi \|_p^2 \). It is also convenient to work in terms of Cauchy data on, e.g., the surface in standard inertial coordinates on \( M_0 \) in which the interval is \( d\tau^2 = dt^2 - \delta_{ij} dx^i dx^j \). Then each \( \phi \in \mathcal{L}(M_0) \) is uniquely associated with a pair \( (\varphi, \pi) \in C_0^\infty(\mathbb{R}^{n-1}; \mathbb{C}^\nu) \oplus C_0^\infty(\mathbb{R}^{n-1}; \mathbb{C}^\nu) \), where \( \varphi(x) = \phi(0, x) \), \( \pi(x) = (\partial \varphi / \partial t)(0, x) \) and \( S \) induces a linear map \( \tilde{S} \) on \( C_0^\infty(\mathbb{R}^{n-1}; \mathbb{C}^{|\nu|}) \oplus C_0^\infty(\mathbb{R}^{n-1}; \mathbb{C}^{|\nu|}) \) such that \((\varphi', \pi') = \tilde{S}(\varphi, \pi)\) is the Cauchy data of \( S\phi \). Then the identity \( \| (\ell^a \nabla_a S \phi) |_p \| = \| \ell^a \nabla_a \phi \|_p \) implies

\[
\left\| \left( \ell \cdot \nabla \ 1_\nu \right) \tilde{S} \left( \begin{array}{c} \varphi \\ \pi \end{array} \right) (x) \right\| = \left\| \left( \ell \cdot \nabla \ 1_\nu \right) \left( \begin{array}{c} \varphi \\ \pi \end{array} \right) (x) \right\|
\]

for all unit vectors \( \ell \in \mathbb{R}^{n-1} \). In particular, defining the map

\[
U : \left( \begin{array}{c} \varphi \\ \pi \end{array} \right) \mapsto \left( \ell \cdot \nabla \ 1_{|\nu|} \right) \tilde{S} \left( \begin{array}{c} \varphi \\ \pi \end{array} \right) (0) \in \mathbb{C}^{|\nu|}
\]

on \( C_0^\infty(\mathbb{R}^{n-1}; \mathbb{C}^{|\nu|}) \oplus C_0^\infty(\mathbb{R}^{n-1}; \mathbb{C}^{|\nu|}) \), we have the estimate

\[
\left\| U \left( \begin{array}{c} \varphi \\ \pi \end{array} \right) \right\| \leq \| (\ell \cdot \nabla \varphi)(0) \| + \| \pi(0) \|,
\]

which proves that \( U \) is a \((|\nu| \times 2|\nu|)\)-matrix of distributions each of which is supported at the origin. We may therefore conclude that

\[
U \left( \begin{array}{c} \varphi \\ \pi \end{array} \right) = A \varphi(0) + B \pi(0) + C^j(\nabla_j \varphi)(0) + D^j(\nabla_j \pi)(0)
\]

for \((|\nu| \times |\nu|)\)-matrices \( A \), \( B \), \( C^j \) and \( D^j \), where \( 1 \leq j \leq n-1 \). It is easy to show that \( A = D^j = 0 \) for all \( j \), on considering the estimate \((29)\) in cases where \( \pi = \ell \cdot \nabla \varphi = 0 \), and likewise that only derivatives of \( \varphi \) along \( \ell \) can contribute. This gives

\[
U \left( \begin{array}{c} \varphi \\ \pi \end{array} \right) = R (\ell \cdot \nabla \varphi)(0) + B \pi(0)
\]

for matrices \( R \) and \( B \), which in principle may depend on \( \ell \). Moreover, Eq. \((28)\) entails

\[
\| R (\ell \cdot \nabla \varphi)(0) + B \pi(0) \| = \| (\ell \cdot \nabla \varphi)(0) + \pi(0) \|,
\]

from which we may conclude that \( R \) and \( B \) are unitary and in fact equal (again, by considering cases in which \( \pi(0) = 0 \) or \( \ell \cdot \nabla \varphi(0) = 0 \)). In terms of the Cauchy data \((\varphi', \pi')\) of \( S\phi \), the discussion so far has shown that

\[
(\ell \cdot \nabla \varphi')(0) + \pi'(0) = R_\ell (\ell \cdot \nabla \varphi)(0) + R_\ell \pi(0)
\]

for some unitary matrix \( R_\ell \), from which we may deduce

\[
(\ell \cdot \nabla \varphi')(0) = \frac{1}{2} ([R_\ell + R_{-\ell}](\ell \cdot \nabla \varphi)(0) + [R_\ell - R_{-\ell}]\pi(0)) \tag{30}
\]

\[
\pi'(0) = \frac{1}{2} ([R_\ell - R_{-\ell}](\ell \cdot \nabla \varphi)(0) + [R_\ell + R_{-\ell}]\pi(0)). \tag{31}
\]
Considering data with \( \varphi \equiv 0 \), we see from the \( \ell \)-independence of the left-hand side of Eq. (31) that \( \frac{1}{2}(R_\ell + R_{-\ell}) = R \), independent of \( \ell \). Then Eq. (30) becomes

\[
(\ell \cdot \nabla \varphi')(0) = \frac{1}{2}[R_\ell - R_{-\ell}]\pi(0)
\]

and, as \( \pi(0) \in \mathbb{C}^{|\nu|} \) is arbitrary, we deduce by linearity of the left-hand side in \( \ell \) that \( \frac{1}{2}[R_\ell - R_{-\ell}] = \ell^i \Delta_i \) for \((|\nu| \times |\nu|)\)-matrices \( \Delta_i \) (1 \( \leq i \leq n \)). Substituting back into Eqs. (30) and (31), and considering data where \( \pi \equiv 0 \), we have \( \pi'(0) = \ell^i \ell^j \Delta_i (\nabla_j \varphi)(0) \) for all unit vectors \( \ell \). As the right-hand side is \( \ell \)-independent, we have \( \Delta_i (\nabla_j \varphi)(0) \propto \delta_{ij} \) for all \( \varphi \in C^\infty_0(\mathbb{R}^{n-1}; \mathbb{C}^{|\nu|}) \), which is possible only if \( \Delta_i = 0 \) for all \( 1 \leq i \leq n - 1 \). Thus \( R_\ell = R \), independent of \( \ell \), and it follows that \( \pi'(0) = R\pi(0) \) and \( (\nabla \varphi')(0) = R\varphi(0) \) for general data \((\varphi, \pi)\). Further, because \( S \) and hence \( \tilde{S} \) commute with spatial translations, there is a fixed unitary \( R \in \mathcal{U}(|\nu|) \) so that

\[
\tilde{S} \begin{pmatrix} \varphi \\ \pi \end{pmatrix}(x) = \begin{pmatrix} R\varphi(x) \\ R\pi(x) \end{pmatrix}.
\]

Finally, because \( S \) also commutes with time translations we have \((S\phi)(t, x) = R\phi(t, x)\). But both \( \phi \) and \( S\phi \) solve the same field equation \( \Box \varphi + \mathcal{M}^2 \varphi = 0 \), so \( R \) commutes with \( \mathcal{M}^2 \) and decomposes into block diagonal form, \( R = \bigoplus_{m \in \mathcal{M}} R_m \), where each \( R_m \in \mathcal{U}(\nu(m)) \). Since \( S \) commutes with complex conjugation, the \( R_m \) are real and hence orthogonal. \( \square \)

## B  Equivalent topologies on the automorphism group

In this appendix, we study various topologies that can be placed on the automorphism group \( \text{Aut}((\mathcal{F}, \gamma, \mathcal{F})) \) of a theory obeying assumptions (1)–(5) of Sec. 4. The main purpose is to conclude the proof of Theorem 4.6 but some of the arguments may be of independent use. The topologies concerned are:

- the \( \mathcal{M} \)-topology, defined as the weakest in which the function \( \eta \mapsto \omega(\eta_M(A)) \) is continuous on \( G \), for every \( A \in \mathcal{F}(\mathcal{M}) \), \( \omega \in \mathcal{F}(\mathcal{M}) \);
- the \textit{local} \( \mathcal{M} \)-topology, defined as above, but restricting to \( A \) belonging to kinematic local algebras of \( \mathcal{F}^{\text{kin}}(\mathcal{M}; D) \) of truncated multi-diamonds \( D \) in \( \mathcal{M} \);
- the \( \omega \)-topology induced by any gauge-invariant state \( \omega \) inducing a faithful GNS representation (see the discussion following Proposition 3.3);
- the \textit{diamond topology}, namely the join of all \( \mathcal{M} \)-topologies as \( \mathcal{M} \) runs over truncated multi-diamond spacetimes.
- the \textit{natural weak topology} (as defined in Sec. 3.1), namely the join of all \( \mathcal{M} \)-topologies as \( \mathcal{M} \) runs through \( \text{Loc} \).

We need a technical result. Suppose \( G \) is a topological group acting (not necessarily continuously) by automorphisms on a unital C*-algebra \( \mathcal{A} \). For any state \( \omega \) on \( \mathcal{A} \), we say that an element \( B \in \mathcal{A} \) is \((\omega, G)\)-continuous if \( \eta \mapsto F_{A, B}(\eta) := \omega(A^* \eta(B) A) \) is continuous on \( G \) for every \( A \in \mathcal{A} \) with \( \omega(A^* A) \neq 0 \). The set of all \((\omega, G)\)-continuous elements of \( \mathcal{A} \)
will be denoted $\mathcal{C}_{\omega,G}(A)$; any subalgebra of $A$ contained in $\mathcal{C}_{\omega,G}(A)$ will be described as $(\omega,G)$-continuous.

**Proposition B.1** With the preceding definitions and notation,

(i) $\mathcal{C}_{\omega,G}(A)$ is a self-adjoint, norm-closed, linear subspace of $A$;

(ii) if $B$ is $(\omega,G)$-continuous then $\eta \mapsto \omega(P\eta(B)Q)$ is continuous for every $P,Q \in A$;

(iii) if $B$ and $C$, together with at least one of $BB^*$ or $C^*C$, are $(\omega,G)$-continuous, then so is $BC$;

(iv) if $\mathcal{B}_\alpha$ are $(\omega,G)$-continuous subalgebras of $A$, then $\bigvee_\alpha \mathcal{B}_\alpha$ is $(\omega,G)$-continuous;

(v) if $\omega_n \to \omega$ in the uniform topology on $A^*_{+,1}$ then $\bigcap_n \mathcal{C}_{\omega_n,G}(A) \subset \mathcal{C}_{\omega,G}(A)$.

**Proof.** (i). Linearity and self-adjointness are obvious. As $A$ is a $C^*$-algebra, we may estimate $|\omega(A^*\eta(B)A)| \leq \omega(A^*A)\|B\|$ and deduce that $B_n \to B$ in $A$ implies that $F_{A,B_n}$ converges uniformly to $F_{A,B}$, which is therefore continuous; thus $\mathcal{C}_{\omega,G}(A)$ is norm-closed.

(ii). By polarisation, it is enough to show that $B \in \mathcal{C}_{\omega,G}(A)$ implies that $F_{P,B}$ is continuous for every $P \in A$ (regardless of whether $\omega(P^*P) \neq 0$); the latter is seen to hold on noting that $F_{1+\lambda P,B}$ must be continuous for all sufficiently small $\lambda \in \mathbb{C}$, and consequently each term in the expansion of $F_{1+\lambda P,B}$ in $\lambda,\tilde{\lambda}$ must also be continuous. (iii). Let $\eta_0 \in G$ be arbitrary and assume without loss that $BB^* \in \mathcal{C}_{\omega,G}(A)$. The identity $\eta(BC) = \eta_0(B)\eta(C) + (\eta(B) - \eta_0(B))\eta(C)$ entails

$$F_{A,BC}(\eta) = \omega(A^*\eta_0(B)\eta(C)A) + \omega(A^*(\eta(B) - \eta_0(B))\eta(C)A),$$

the first term of which is continuous in $\eta$ by part (ii). It suffices to show that the last term vanishes in the limit $\eta \to \eta_0$. To this end, the Cauchy–Schwarz inequality gives,

$$|\omega(A^*(\eta(B) - \eta_0(B))\eta(C)A)|^2 \leq \omega(A^*(\eta(B) - \eta_0(B))(\eta(B^*) - \eta_0(B^*))A)\omega(A^*\eta(C^*C)A) \leq \|A\|^2\|C\|^2\omega(A^*(\eta(B) - \eta_0(B))(\eta(B^*) - \eta_0(B^*))A).$$

Expanding the right-hand side, we obtain a sum of functions known to be continuous in $\eta$, using $BB^* \in \mathcal{C}_{\omega,G}(A)$ and part (ii); moreover the expression vanishes for $\eta = \eta_0$. Thus $BC \in \mathcal{C}_{\omega,G}(A)$. Part (iv) is now immediate, using (iii) together with norm-closure and linearity of $\mathcal{C}_{\omega,G}(A)$. (v) If $B$ is $(\omega_n,G)$-continuous for each $n$, then the estimate $|\omega(A^*\eta(B)A) - \omega_n(A^*\eta(B)A)| \leq \|\omega - \omega_n\|\|A\|^2\|B\|$ shows that each $F_{A,B}$ is the uniform limit of continuous functions, so $B \in \mathcal{C}_{\omega,G}(A)$; this establishes $\bigcap_n \mathcal{C}_{\omega_n,G}(A) \subset \mathcal{C}_{\omega,G}(A)$ and we take closures to complete the proof.

As a first application of this result, let $G = \text{Aut}((\mathcal{F}, \gamma, \mathcal{I}))$, endowed with the local $M$-topology for some $M$, and let $\omega \in \mathcal{I}(M)$. Then for every truncated truncated multidiamond $D$ in $M$, $\mathcal{F}^{\text{kin}}(M;D)$ is $(\omega,G)$-continuous; as these subalgebras generate $\mathcal{F}(M)$, the whole algebra is $(\omega,G)$-continuous by part (iv). Letting $\omega$ vary in $\mathcal{I}(M)$, we see that every function generating the $M$-topology is continuous in the local $M$-topology; as the latter is trivially weaker than the former, we have proved:

If $C^*C \in \mathcal{C}_{\omega,G}(A)$, we may use the following argument to conclude that $C^*B^* \in \mathcal{C}_{\omega,G}(A)$ and the required result follows on taking adjoints.
Proposition B.2 The $M$-topology coincides with the local $M$-topology.

For a second application, suppose $\omega_0 \in \mathcal{S}(M)$ is a gauge-invariant state inducing a faithful GNS representation of $\mathcal{F}(M)$, and endow $G$ with the $\omega_0$-topology induced by the strong operator topology in this representation. Hence the corresponding unitary representation of $G$ is strongly continuous and it follows that $\mathcal{F}(M)$ is $(\omega,G)$-continuous with respect to any vector state $\omega$ in the GNS representation; one sees by part (v) that this is also true for any $\omega$ in the folium Fol($\omega_0$) of $\omega_0$. Now consider any $\omega \in \mathcal{S}(M)$ and a truncated multi-diamond $D$ in $M$. By local quasi-equivalence, there is a state $\omega' \in$ Fol($\omega_0$) that agrees with $\omega$ on $\mathcal{F}^\text{kin}(M;D)$, which is $(\omega',G)$-continuous, and hence $(\omega,G)$-continuous. Applying part (iv) again, additivity entails that $\mathcal{F}(M)$ is $(\omega,G)$-continuous for every $\omega \in \mathcal{S}(M)$. Summarising:

Proposition B.3 Suppose $\omega_0 \in \mathcal{S}(M)$ is a gauge-invariant state inducing a faithful GNS representation of $\mathcal{F}(M)$. Then the $\omega_0$-topology coincides with the $M$-topology, and hence the $M$-local topology.

We now specialise to Minkowski space, $M_0$, and the Minkowski vacuum state $\omega_0$, with GNS representation $(\mathcal{H}_0, \pi_0, \Omega_0)$. Our aim is to show that the $M_0$- and $\omega_0$-topologies, which coincide by Proposition [B.3] are equivalent to the diamond topology.

Let $D$ be any multi-diamond of $M_0$, and let $\iota : D \rightarrow M_0$ be the canonical inclusion of $D = M_0|_D$ in $M_0$. Then $\mathcal{S}(\iota)\omega_0 \in \mathcal{S}(D)$ is gauge-invariant and induces a GNS representation $(\mathcal{H}_D, \pi_D, \Omega_D)$ of $\mathcal{F}(D)$ on which $G$ is unitarily represented. As $\omega_0$ has the Reeh–Schlieder property, we may take $\mathcal{H}_D = \mathcal{H}_0$, $\Omega_D = \Omega_0$, $\pi_D = \pi_0\mathcal{F}^\text{kin}(M;D)$, whereupon the unitary implementations of $G$ in the two representations also coincide. It follows (a) that the topologies induced on $G$ by $\mathcal{S}(\iota)\omega_0$ and $\omega_0$ are equivalent, and (b) that $\pi_D$ is faithful, so the $\mathcal{S}(\iota)\omega_0$-topology and $D$-topologies coincide by Proposition [B.3]. Thus the $D$-topology coincides with the $\omega_0$-topology and hence the $M_0$-topology. Using the time-slice property, we conclude immediately that:

Proposition B.4 The $M_0$-topology coincides with the $D$-topology for every truncated multi-diamond spacetime $D$, and hence with the diamond topology.

Our final observation is that the local $M$-topology is trivially weaker than the diamond topology for any $M$. Combining this with Props. [B.4] and [B.2] we find that the $M_0$-topology is stronger than every $M$-topology. Using the definition of the natural weak topology, we obtain in conclusion:

Theorem B.5 For a theory $\langle \mathcal{F}, \gamma, \mathcal{S} \rangle \in$ LCT$_{\text{BC-AS}}$ obeying assumptions (1)–(3) of Sec. [4] the natural weak topology on Aut($\langle \mathcal{F}, \gamma, \mathcal{S} \rangle$) coincides with the $\omega_0$-topology (and hence with the diamond- and $M_0$-topologies).

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