NEW-FROM-OLD FULL DUALITIES VIA AXIOMATISATION

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ABSTRACT. We clarify what it means for two full dualities based on the same algebra to be different. Our main theorem gives conditions on two different alter egos of a finite algebra under which, if one yields a full duality, then the other does too. We use this theorem to obtain a better understanding of several important examples from the theory of natural dualities. Throughout the paper, a fundamental role is played by the universal Horn theory of the dual classes.

There have been several rich interactions between the theory of natural dualities and first-order logic: for example, in the study of algebraically and existentially closed algebras [29, 8], in the theory of optimal dualities [19, 20, 14] and in the axiomatisation of dual classes [7, 4, 3, 5]. In this paper, we further investigate this last connection, by showing how to obtain new full dualities from an existing full duality using a universal Horn axiomatisation of the dual class.

The theory of natural dualities [21, 2] provides a general framework that encompasses several well-known dualities between algebras and topological structures, including:

- Stone duality [30] between Boolean algebras and Boolean spaces (that is, compact totally disconnected spaces);
- Pontryagin duality [27] between abelian groups and compact Hausdorff topological abelian groups;
- Priestley duality [28] between bounded distributive lattices and Priestley spaces (that is, compact totally order-disconnected spaces).

In general, a full duality (in the sense of natural duality theory) is based on an algebra $M$ and a topological structure $M$ (called an alter ego of $M$) that is compatible with the algebra $M$, and provides a dual equivalence between the prevariety $\mathcal{A} := ISP(M)$ of algebras and a category $\mathcal{X}$ of topological structures generated by $M$. For example:

- Stone duality is based on the Boolean algebra $B = \langle\{0, 1\}; \lor, \land, ', 0, 1\rangle$ and the discrete topological space $B = \langle\{0, 1\}; \mathcal{T}\rangle$.
- Pontryagin duality is based on the circle group $S = \langle S^1; -1, 1\rangle$ and the topological circle group $S = \langle S^1; -1, 1, \mathcal{T}\rangle$.
- Priestley duality is based on the bounded lattice $2 = \langle\{0, 1\}; \lor, \land, 0, 1\rangle$ and the discretely topologised chain $2 = \langle\{0, 1\}; \leq, \mathcal{T}\rangle$.

Most of the general theory considers the case where the generating algebra $M$ is finite [2], in which case $\mathcal{A}$ is the quasivariety generated by $M$ (essentially, the universal Horn class). We shall make this assumption throughout the paper.

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Much recent work in the theory of natural dualities has focussed on obtaining a better understanding of full dualities and, in particular, on the extent to which there can be different full dualities based on the same algebra [16, 11, 25, 6, 26, 12, 18]. It is tempting to suppose that a full duality, based on an algebra \( M \) and alter ego \( M' \), would be stable under enriching the structure of the alter ego \( M' \). A powerful counterexample to this has been constructed based on the three-element bounded lattice \( 3 \). There is an infinite ascending chain of alter egos \( 3_0, 3_1, 3_2, \ldots \) of \( 3 \), each one a reduct of the next, such that these alter egos alternately do and do not yield a finite-level full duality on the variety of bounded distributive lattices [12].

This counterexample was constructed using the answer to the following general enrichment question [18, 5.3] (see also Theorem 7.7):

Assume that a pair \( M, M' \) yields a full duality. How can the structure on the alter ego \( M' \) be enriched while retaining a full duality?

In this paper, we consider the corresponding reduction question:

Assume that a pair \( M, M' \) yields a full duality. How can the structure on the alter ego \( M' \) be reduced while retaining a full duality?

Our main theorem gives conditions on a pair of alter egos \( M_1 \) and \( M_2 \) of \( M \) under which, if \( M_1 \) yields a full duality on \( A \), then \( M_2 \) does too (New-from-old Theorem 7.1). The conditions are based on having an axiomatisation of the universal Horn theory of the structure \( M_1 \).

While the statement of the main theorem is rather technical, we use the theorem to obtain a series of self-contained corollaries.

One corollary gives a new and very natural condition under which every finite-level full duality lifts to the infinite level (Theorem 7.3):

Assume that \( M \) is fully dualised by a standard alter ego (that is, the dual class is described by universal Horn sentences). If an alter ego \( M' \) fully dualises \( M \) at the finite level, then \( M' \) is standard and fully dualises \( M \).

It follows from this corollary that, for any quasi-primal algebra \( M \), every finite-level full duality lifts to a full duality (Example 7.4).

Other corollaries include a new characterisation of the alter egos that yield a finite-level full duality (Theorem 7.5) and a new constructive description of the smallest alter ego that yields a finite-level full duality (Theorem 7.6).

We are also able to use our main theorem to elucidate two important counterexamples in the theory of natural dualities:

- The first example of a finite-level duality that is full but not strong [16] was based on the three-element bounded lattice \( 3 = \langle \{0, a, 1\}; \lor, \land, 0, 1 \rangle \) and the alter ego \( 3_h = \langle \{0, a, 1\}; f, g, h, T \rangle \); see Section 8.
- The first example of a full but not strong duality [6] was based on a four-element quasi-primal algebra \( Q = \langle \{0, a, b, 1\}; t, \lor, \land, 0, 1 \rangle \) and the alter ego \( Q_r = \langle \{0, a, b, 1\}; r, T \rangle \); see Example 7.8.

We give a general algorithm for obtaining full dualities (Algorithm 7.9) that can be used to construct the two alter egos \( 3_h \) and \( Q_r \), each based on a known strong duality and an axiomatisation of the universal Horn theory of the corresponding alter ego (see Example 7.10).

In the final section of the paper, we clarify what it really means for two full dualities based on a finite algebra \( M \) to be different. We shall show that the concrete concept of ‘substructure’ in dual classes is not categorical. While the
two dual classes coming from two full dualities must be isomorphic as categories, they can be different in a concrete sense, since the category isomorphism need not preserve substructures.

1. Preliminaries: Full dualities

This section outlines the basic set-up for a full duality. For a comprehensive introduction to the theory of natural dualities, see the Clark–Davey monograph [2].

Fix a finite algebra \( M = (M; F) \) and consider the quasivariety \( \mathcal{A} := ISP(M) \), that is, the class of all isomorphic copies of subalgebras of arbitrary powers of \( M \). Our conventions are that \( \mathcal{A} \) never contains the empty algebra and that \( \mathcal{A} \) always contains the one-element algebras (via the zero power).

- Let \( r \) be an \( n \)-ary relation on \( M \), for some \( n \geq 0 \). Then \( r \) is said to be compatible with \( M \) if it forms a subalgebra \( r \) of \( M^n \).
- Let \( h \) be an \( n \)-ary partial operation on \( M \), for some \( n \geq 0 \). Then \( h \) is said to be compatible with \( M \) if the \((n+1)\)-ary relation

\[
\text{graph}(h) := \{(\bar{a}, h(\bar{a})) \mid \bar{a} \in \text{dom}(h)\}
\]

is compatible with \( M \), or equivalently, if the \( n \)-ary relation \( r := \text{dom}(h) \) is compatible with \( M \) and \( h : r \to M \) is a homomorphism.

An alter ego of \( M \) is a topological structure \( \bar{M} = (M; H, R, \mathcal{T}) \) with the same underlying set as \( M \), where

- \( H \) is a set of partial operations that are compatible with \( M \),
- \( R \) is a set of relations that are compatible with \( M \), and
- \( \mathcal{T} \) is the discrete topology on \( M \).

(It is common to add a set \( G \) of total operations to the signature of \( M \), but to simplify the notation, we include total operations in \( H \).)

The alter ego \( \bar{M} \) is the starting point for creating a potential dual category for the quasivariety \( \mathcal{A} = ISP(M) \). We form the topological prevariety \( \mathcal{X} := IS, P^*(\bar{M}) \) consisting of all isomorphic copies of topologically closed substructures of non-zero powers of \( \bar{M} \). Our conventions are that \( \mathcal{X} \) does not automatically contain any one-element structures and that \( \mathcal{X} \) contains the empty structure if and only if \( H \) contains no nullary operations.

The concept of substructure here is the natural concrete one: we say that \( \mathcal{X} \) is a substructure of \( \mathcal{Y} \) if \( \mathcal{X} \subseteq \mathcal{Y} \) and the relations in \( R^X \cup \text{dom}(H^X) \) are the restrictions of those in \( R^X \cup \text{dom}(H^Y) \) and the partial operations in \( H^X \) are the restrictions of those in \( H^Y \).

The choice of an alter ego \( \bar{M} \) of \( M \) induces a pair of contravariant hom-functors \( D : \mathcal{A} \to \mathcal{X} \) and \( E : \mathcal{X} \to \mathcal{A} \) and a pair of natural transformations \( \varepsilon : \text{id}_\mathcal{A} \to ED \) and \( \varepsilon : \text{id}_\mathcal{X} \to DE \). The hom-functors \( D \) and \( E \) are given on objects by

\[
D(A) := \mathcal{A}(A, M) \leq M^A \quad \text{and} \quad E(X) := \mathcal{X}(X, \bar{M}) \leq M^X,
\]

for all \( A \in \mathcal{A} \) and \( X \in \mathcal{X} \). The compatibility between \( M \) and \( \bar{M} \) guarantees that these hom-functors are well defined. The natural transformations \( \varepsilon \) and \( \varepsilon \) are given by evaluation: for all \( A \in \mathcal{A} \), the embedding \( \varepsilon_A : A \to ED(A) \) is defined by

\[
e_A(a)(x) := x(a), \quad \text{for all } a \in A \text{ and } x \in \mathcal{A}(A, M),
\]

and for all \( X \in \mathcal{X} \), the embedding \( \varepsilon_X : X \to DE(X) \) is defined by

\[
\varepsilon_X(u)(x) := u(x), \quad \text{for all } x \in X \text{ and } u \in \mathcal{X}(X, \bar{M}).
\]
The basic concepts are defined as follows:

1. \( \mathcal{M} \) fully dualises \( \mathcal{M} \) [at the finite level] if the embedding \( e_A : A \rightarrow ED(A) \) is an isomorphism, for each [finite] algebra \( A \in \mathcal{A} \).

2. \( \mathcal{M} \) fully dualises \( \mathcal{M} \) [at the finite level] if, in addition to (1), the embedding \( e_X : X \rightarrow DE(X) \) is an isomorphism, for each [finite] structure \( X \in \mathcal{X} \).

3. \( \mathcal{M} \) strongly dualises \( \mathcal{M} \) [at the finite level] if, in addition to (1) and (2), the alter ego \( \mathcal{M} \) is injective among the [finite] structures in \( \mathcal{X} \).

The following basic lemma will allow us to create new full dualities from old ones. There are two versions of this lemma: the phrases in square brackets can be either included or deleted.

**New-from-old Lemma 1.1.** Let \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) be alter egos of a finite algebra \( \mathcal{M} \). For \( i \in \{1, 2\} \), define \( \mathcal{X}_i := IS_2P^+(\mathcal{M}_i) \). Assume that \( \mathcal{M}_1 \) fully dualises \( \mathcal{M} \) [at the finite level]. Then \( \mathcal{M}_2 \) also fully dualises \( \mathcal{M} \) [at the finite level] provided the following two conditions hold:

1. \( \mathcal{M}_2 \) dualises \( \mathcal{M} \) [at the finite level];
2. for each [finite] structure \( X \) in \( \mathcal{X}_2 \), there is a structure \( X' \) in \( \mathcal{X}_1 \) on the same underlying set as \( X \) such that \( \mathcal{X}_2(X, \mathcal{M}_2) = \mathcal{X}_1(X', \mathcal{M}_1) \).

**Proof.** Define \( \mathcal{A} := ISP(\mathcal{M}) \) and, for \( i \in \{1, 2\} \), let \( D_i : \mathcal{A} \rightarrow \mathcal{X}_i \) and \( E_i : \mathcal{X}_i \rightarrow \mathcal{A} \) be the hom-functors induced by \( \mathcal{M} \) and \( \mathcal{M}_i \). Assume that (1) and (2) hold. Let \( X \) be a [finite] structure in \( \mathcal{X}_2 \). We just need to show that \( e_X : X \rightarrow D_2E_2(X) \) is surjective, that is, we need to show that every homomorphism \( u : E_2(X) \rightarrow \mathcal{M} \) is given by evaluation.

By (2), we have \( \mathcal{X}_2(X, \mathcal{M}_2) = \mathcal{X}_1(X', \mathcal{M}_1) \). Thus \( E_2(X) = E_1(X') \) in \( \mathcal{A} \). As \( \mathcal{M}_1 \) fully dualises \( \mathcal{M} \) [at the finite level], each homomorphism \( u : E_1(X') \rightarrow \mathcal{M} \) is given by evaluation. It follows immediately that each homomorphism \( u : E_2(X) \rightarrow \mathcal{M} \) is given by evaluation. \( \square \)

We close this section with a brief discussion of universal Horn sentences and their role in attempts to axiomatise dual classes.

Fix a signature \((H, R)\) of finitary partial operation and relation symbols. We define a universal Horn sentence (uH-sentence, for short) in the language of \((H, R)\) to be a first-order sentence of the form

\[ \forall \nu \left[ \left( \bigwedge_{i=1}^k \alpha_i(\nu) \right) \rightarrow \gamma(\nu) \right], \]

for some \( \nu \geq 0 \), where each \( \alpha_i(\nu) \) is an atomic formula and \( \gamma(\nu) \) is either an atomic formula or \( \bot \).

**Definition 1.2.** Let \( \mathcal{M} = \langle M; H, R, T \rangle \) be an alter ego of a finite algebra \( \mathcal{M} \), and define \( \mathcal{X} = IS_2P^+(\mathcal{M}) \). The potential dual class \( \mathcal{X} \) is always contained in the class \( \mathcal{Y} \) consisting of all Boolean models of the uH-theory of \( \mathcal{M} \):

\[ \mathcal{X} \subseteq \mathcal{Y} := \{ Y \in \mathcal{Z} \mid Y = Th_{uH}(\mathcal{M}) \}, \]

where \( \mathcal{Z} \) denotes the class of all Boolean structures of signature \((H, R)\). (That is, each member of \( \mathcal{Z} \) is a topological structure with a Boolean topology and with continuous partial operations on closed domains and closed relations.)

If the two classes \( \mathcal{X} \) and \( \mathcal{Y} \) are equal, then we say that the alter ego \( \mathcal{M} \) is standard. For example, the discrete semilattice \( \mathcal{S} = \langle \{0, 1\}; \lor, \mathcal{T} \rangle \) is standard \[22\], but the discrete chain \( \mathcal{C} = \langle \{0, 1\}; \leq, \mathcal{T} \rangle \) is not standard \[31\].
Note that we always have $X_{\text{fin}} = Y_{\text{fin}}$. That is, the finite structures in $X$ are precisely the finite Boolean models of the uH-theory of $M$.

The study of standardness has been a very rich offshoot of the theory of natural dualities; see, for example, [3, 4, 5, 13, 17, 23, 24, 32].

2. Preliminaries: Comparing alter egos

This section gives the more specific background theory that we require. We start by defining the ‘structural reduct’ quasi-order on the alter egos of a finite algebra $M$; see [15, 18]. This is the natural generalisation from algebras to structures of the ‘term reduct’ quasi-order.

Definition 2.1. Given any alter ego $M = \langle M; H, R, T \rangle$ of $M$, we define $\text{Clo}_{\text{ep}}(M)$ to be the enriched partial clone on $M$ generated by $H$, that is, the smallest set of partial operations on $M$ that contains $H$ and the projections, $\pi_i: M^n \to M$ for all $n \geq 1$ and $i \leq n$, and is closed under composition (with maximum, non-empty domain). This corresponds to the usual definition of partial clone, except that we exclude empty domains and we enrich the partial clone by allowing nullary operations.

Definition 2.2. Let $M = \langle M; H, R, T \rangle$ be an alter ego of $M$ and let $k, n \geq 0$.

- We shall call a conjunction of atomic formulæ $\Psi(\vec{v}) = [\psi_1(\vec{v}) \& \cdots \& \psi_k(\vec{v})]$ a conjunct-atomic formula.
- We say that a non-empty $n$-ary relation $r$ on $M$ is conjunct-atomic definable from $M$ if it is described in $M$ by an $n$-variable conjunct-atomic formula $\Psi(\vec{v})$ in the language of $M$, that is, if
  \[ r = \{ (a_1, \ldots, a_n) \in M^n \mid \Psi(a_1, \ldots, a_n) \text{ is true in } M \}. \]
- We define $\text{Rel}_{\text{ca}}(M)$ to be the set of all relations on $M$ that are conjunct-atomic definable from $M$.

Definition 2.3. Let $M_1 = \langle M; H_1, R_1, T \rangle$ and $M_2 = \langle M; H_2, R_2, T \rangle$ be alter egos of $M$. Then we say that $M_1$ is a structural reduct of $M_2$ if

(a) each partial operation in $H_1$ has an extension in $\text{Clo}_{\text{ep}}(M_2)$, and
(b) each relation in $R_1 \cup \text{dom}(H_1)$ belongs to $\text{Rel}_{\text{ca}}(M_2)$.

We say that $M_1$ and $M_2$ are structurally equivalent if each is a structural reduct of the other.

Under the ‘structural reduct’ quasi-order, the alter egos of $M$ form a doubly algebraic lattice $A_M$; see [18, 2.6]. With the help of the following definitions and lemmas, we will be able to describe how the various flavours of duality occur within this lattice.

Definition 2.4. Let $r$ be an $n$-ary relation compatible with $M$, for some $n \geq 0$, and let $r$ be the subalgebra of $M^n$ with $r$ as underlying set.

- We say that the relation $r$ is hom-minimal if every homomorphism from $r$ to $M$ is a projection.
- We say that $M$ is operationally rich at $r$ if every compatible partial operation on $M$ with domain $r$ has an extension in $\text{Clo}_{\text{ep}}(M)$.
Duality Lemma 2.5 ([18, 4.1]). Let $\mathcal{M}$ be an alter ego of a finite algebra $\mathcal{M}$. Then $\mathcal{M}$ dualises $\mathcal{M}$ at the finite level if and only if every hom-minimal relation on $\mathcal{M}$ belongs to $\text{Rel}_{ca}(\mathcal{M})$.

**Remark 2.6.** If a finite algebra $\mathcal{M}$ has an alter ego that yields a duality, then every finite-level duality based on $\mathcal{M}$ lifts to the infinite level; see [18, p. 19]. Note that the same is not true in general for full duality [16]; see Lemma 3.4 and Remark 3.5.

We shall use the description of finite-level full duality provided by the following lemma. In fact, our main theorem will allow us to give a more refined version of this lemma (see Theorem 7.5).

**Full Duality Lemma 2.7 ([18, 4.3]).** Let $\mathcal{M}$ be an alter ego of a finite algebra $\mathcal{M}$. Then $\mathcal{M}$ fully dualises $\mathcal{M}$ at the finite level if and only if

(a) every hom-minimal relation on $\mathcal{M}$ belongs to $\text{Rel}_{ca}(\mathcal{M})$, and

(b) $\mathcal{M}$ is operationally rich at each relation in $\text{Rel}_{ca}(\mathcal{M})$.

**Remark 2.8.** Since $M^n \in \text{Rel}_{ca}(\mathcal{M})$, for all $n \geq 0$, it follows from (b) above that every compatible total operation (that is, every homomorphism $g: M^n \to \mathcal{M}$) belongs to $\text{Clo}_{ep}(\mathcal{M})$. In particular, every element of $\mathcal{M}$ that forms a one-element subalgebra of $\mathcal{M}$ must be the value of a nullary operation in $\text{Clo}_{ep}(\mathcal{M})$.

**Facts 2.9.** The following facts about the lattice of alter egos $\mathcal{A}_\mathcal{M}$ are proved in [18]; see Figure II

1. By the Duality Lemma 2.5, the alter egos that dualise $\mathcal{M}$ at the finite level form a principal filter of $\mathcal{A}_\mathcal{M}$.
2. It follows from the Full Duality Lemma 2.7, that, under the ‘structural reduct’ quasi-order, there is a smallest alter ego $\mathcal{M}_\alpha$ that fully dualises $\mathcal{M}$ at the finite level; see [18, 4.4].
3. The alter egos that fully dualise $\mathcal{M}$ at the finite level form a complete sublattice $\mathcal{F}_\mathcal{M}$ of $\mathcal{A}_\mathcal{M}$ and those that fully dualise $\mathcal{M}$ form an up-set of $\mathcal{F}_\mathcal{M}$; see [18, 5.5].
4. An alter ego strongly dualises $\mathcal{M}$ at the finite level if and only if it is structurally equivalent to the top alter ego $\mathcal{M}_\Omega$, and so there is essentially only one candidate for a strong duality; see [18, 4.6].
Figure 2. The compatible partial operations \( f, g, \sigma \) and \( h \) on 3

3. Motivating example

In this section, we illustrate the general idea behind the proof of our New-from-old Theorem 7.1, using the three-element bounded lattice

\[ 3 = \{0, a, 1\}; \lor, \land, 0, 1, \]

which has played a seminal role as an example in the theory of natural dualities.

We use the four compatible partial operations on 3 shown in Figure 2, the two total operations \( f \) and \( g \), and the two proper partial operations \( \sigma \) and \( h \). The alter ego \( 3 : = \langle \{0, a, 1\}; f, g, \sigma, \tau \rangle \) dualises 3, and can be obtained from Priestley duality using general ‘duality transfer’ techniques [9]. The alter ego

\[ 3_\sigma : = \langle \{0, a, 1\}; f, g, \sigma, \tau \rangle \]

strongly dualises 3, and can be obtained from Priestley duality using general ‘strong duality transfer’ techniques [10]. Note that, since \( 3_\sigma \) strongly dualises 3 at the finite level, it must be equivalent to the top alter ego of 3; see Facts 2.9(4).

The first example of a finite-level full but not strong duality (given by Davey, Haviar and Willard [16]) was based on the alter ego

\[ 3_h : = \langle \{0, a, 1\}; f, g, h, \tau \rangle \]

This alter ego was not found using general techniques. Later in this paper, we shall give a general ‘full duality transfer’ technique that will allow us to obtain this alter ego in a natural way from \( 3_\sigma \); see Example 7.10.

In this section, we give a new proof that \( 3_h \) fully dualises 3 at the finite level. We will show how to transfer the finite-level full duality down from \( 3_\sigma \) to \( 3_h \) by using a basis for the universal Horn theory of \( 3_\sigma \).

We want to apply the New-from-old Lemma 1.1. So we need a way to enrich each finite structure \( X \) in \( X_h := IS_p^\lor(3_h) \) into a structure \( X^3 \) in \( X_\sigma := IS_p^\land(3_\sigma) \). We will check membership of \( X_\sigma \) syntactically: we know that at the finite level \( X_\sigma \) is axiomatised by the universal Horn theory of \( 3_\sigma \).

Definition 3.1. Let \( X = \langle X; f^X, g^X, h^X, \tau^X \rangle \) be a finite structure in \( X_h \). We want to define a structure \( X^3 \) of the same signature as \( 3_\sigma \). The binary partial operation \( \sigma \) is described in \( 3_\sigma \) by the sentence

\[ \forall u v w [\sigma(u, v) = w \leftrightarrow (f(w) = u \land g(w) = v) \}, \]

which is logically equivalent to a conjunction of uH-sentences. So we would like to define the partial operation \( \sigma^X \) on \( X \) by

\[ \text{graph}(\sigma^X) := \{ (x, y, z) \in X^3 \mid f^X(z) = x \land g^X(z) = y \}. \]
As the endomorphisms $f$ and $g$ separate the elements of $3$, the uH-sentence
\[ \forall uv \left[ (f(u) = f(v) \& g(u) = g(v)) \to u = v \right] \]
holds in $3_h$ and therefore in $\mathcal{X}$. This tells us that $\text{graph}(\sigma^{X^3})$ really is the graph of a binary partial operation on $X$ (possibly an empty operation). So we can define
\[ X^3 := \langle X; f^X, g^X, \sigma^{X^3}, \mathcal{I}^X \rangle, \]
and $X^3$ is a (discrete) Boolean structure of the same signature as $3_\sigma$.

**Remark 3.2.** The operation $\sigma^{X^3}$ defined above has a natural interpretation in the case that $\mathcal{X}$ is a concrete structure in $\mathcal{X}_h$. Assume that $\mathcal{X} \leq (3_h)^k$, for some $k > 0$. Then we can impose the ternary relation $\text{graph}(\sigma)$ coordinate-wise on the set $X$. The operation $\sigma^{X^3}$ is defined so that $\text{graph}(\sigma^{X^3}) = \text{graph}(\sigma)^X$. Thus $\sigma^{X^3}$ is the maximum coordinate-wise extension of $\sigma$ to $X$.

**Lemma 3.3.** Let $\mathcal{X}$ be a finite structure in $\mathcal{X}_h := 1S^P(3_h)$. Then the structure $X^3$ defined in (3.2) belongs to $\mathcal{X}_\sigma := 1S^P(3_\sigma)$.

**Proof.** As the structure $X^3$ is finite, we just have to check it is a model of the universal Horn theory of $3_\sigma$. The basis for $\text{Th}_{uH}(3_\sigma)$ given by Clark, Davey, Haviar, Pitkethly and Talukder [4, 3.6] can be reduced to the following set of sentences:

(1) $\forall v \left[ f(v) = f(f(v)) = g(f(v)) \& g(v) = f(g(v)) = g(g(v)) \right]$;
(2) $\forall uvw \left[ (f(u) = v) \leftrightarrow \sigma(u, v) = w \right]$;
(3) $\forall uv \left[ (\sigma(u, v) = \sigma(u, v) \& \sigma(v, u) = \sigma(v, u)) \to u = v \right]$;
(4) $\forall uvw \left[ (\sigma(u, v) = \sigma(u, v) \& \sigma(v, w) = \sigma(v, w)) \to \sigma(u, w) = \sigma(u, w) \right]$.

Since sentence (1) is in the language of $f$ and $g$, it is also part of the uH-theory of $3_h$. So $X^3$ satisfies (1), as $X \in \mathcal{X}_h$. Sentence (2) holds in $X^3$ by construction. Sentence (3) can be transformed into a uH-sentence in the language of $f$ and $g$, using sentence (2):

$$\forall uvwz \left[ (f(x) = u \& g(x) = v \& f(y) = v \& g(y) = u) \to u = v \right].$$

So $X^3$ satisfies (3), again as $X \in \mathcal{X}_h$.

When sentence (4) is translated into the language of $f$ and $g$, it becomes

$$\forall uvwz \left[ (f(x) = u \& g(x) = v \& f(y) = v \& g(y) = w) \to \exists z (f(z) = u \& g(z) = w) \right],$$

which is not a uH-sentence. But we can overcome this problem using the partial operation $h$. It is easy to check that $3_h$ satisfies the sentences

(5) $\forall xy \left[ g(x) = f(y) \to h(x, y) = h(x, y) \right]$,
(6) $\forall xy \left[ h(x, y) = h(x, y) \to f(h(x, y)) = f(x) \right]$,
(7) $\forall xy \left[ h(x, y) = h(x, y) \to g(h(x, y)) = g(y) \right]$.

It follows that $3_h$ satisfies

$$\psi := \forall uvwz \left[ (f(x) = u \& g(x) = v \& f(y) = v \& g(y) = w) \to (f(h(x, y)) = u \& g(h(x, y)) = w) \right],$$

which is logically equivalent to a conjunction of uH-sentences. Since $\psi \vdash \varphi$, it follows that $\mathcal{X} \models \varphi$ and therefore that $X^3$ satisfies (4). \qed
The original proof that $3_h$ fully dualises $3$ at the finite level piggybacked on Priestley duality. We obtain a more ‘generalisable’ proof by piggybacking on the strong duality given by $3_\sigma$.

**Lemma 3.4** ([16]). The alter ego $3_h := \langle \{0, a, 1\}; f, g, h, \mathcal{F} \rangle$ fully dualises the bounded lattice $3$ at the finite level.

**Proof.** We use the fact that $3$ is dualised by $3 := \langle \{0, a, 1\}; f, g, \mathcal{F} \rangle$ and strongly dualised by $3_\sigma := \langle \{0, a, 1\}; f, g, \sigma, \mathcal{F} \rangle$. We shall establish conditions (1) and (2) of the New-from-old Lemma [1] with $M_1 = 3_\sigma$ and $M_2 = 3_h$.

Since $3$ dualises $3$, so does $3_h$. Now let $\mathcal{X}$ be a finite structure in $\mathcal{X}_h$, and construct the structure $\mathcal{X}^f$ as in Definition 3.1. We know that $\mathcal{X}^f \in \mathcal{X}_\sigma$, by the previous lemma. It remains to check that $\mathcal{X}_h(\mathcal{X}, 3_h) = \mathcal{X}_{\sigma}(\mathcal{X}^f, 3_\sigma)$.

Define $\mathcal{X} := IS_{2P}^*(3)$. Let $\mathcal{X}_\sigma$ denote the common reduct of $\mathcal{X}$ and $\mathcal{X}^f$ to the language of $3$; thus $\mathcal{X}_\sigma \in \mathcal{X}$. Consider a morphism $\mu: \mathcal{X} \to 3$. The construction of $\mathcal{X}^f$ ensures that $\mu: \mathcal{X}^f \to 3_\sigma$ is a morphism. Since $\text{graph}(h) \in \text{Rel}_{ca}(3)$, via the sentence

$$\forall uvw [h(u, v) = w \leftrightarrow (f(u) = f(w) \& g(v) = g(w) \& g(u) = f(v))],$$

and since $\mathcal{X} \models \text{Th}_{uH}(3_h)$, we also know that $\mu: \mathcal{X} \to 3_h$ is a morphism. Thus $\mathcal{X}_h(\mathcal{X}, 3_h) = \mathcal{X}(\mathcal{X}_\sigma, 3) = \mathcal{X}_{\sigma}(\mathcal{X}^f, 3_\sigma)$, as required.

**Remark 3.5.** We know that $3_h$ does not fully dualise $3$ [16]. So this proof must break down somewhere for infinite structures in $\mathcal{X}_h$. For any structure $\mathcal{X} \in \mathcal{X}_h$, we can construct $\mathcal{X}^f$ as in Definition 3.1 and we can show that $\mathcal{X}^f$ is a Boolean model of the $uH$-theory of $3_\sigma$. However, this does not imply that the structure $\mathcal{X}^f$ belongs to $\mathcal{X}_\sigma$, because the alter ego $3_\sigma$ is not standard [4, 3.5]. The connection between full dualities and standardness is explored in Section 6.

### 4. Background uH-logic for the general case

The proof of our New-from-old Theorem 3.1 generalises the proof in the previous section: we transfer a [finite-level] full duality from one alter ego $M_1$ to another alter ego $M_2$ by using a basis for $\text{Th}_{uH}(M_1)$. In this section, we present the required background $uH$-logic, all of which is well known and elementary, except perhaps for the ‘partial operations’ twist.

Consider a $uH$-sentence $\forall \overline{v} \left[ (\&_{i=1}^{\nu} \alpha_i(\overline{v})) \rightarrow \gamma(\overline{v}) \right]$ in the language of $(H, R)$. We call $\&_{i=1}^{\nu} \alpha_i(\overline{v})$ the premise or hypothesis of the sentence and $\gamma(\overline{v})$ the conclusion. We identify a particularly simple form of $uH$-sentence.

**Definition 4.1.**

1. A formula $\alpha$ is **hypothetically pure** if it has one of the following forms:
   - (a) $r(v_{i_1}, \ldots, v_{i_n})$, for some $r \in R$, or
   - (b) $h(v_{i_1}, \ldots, v_{i_n}) = v_{i_0}$, for some $h \in H$,
   where $v_{i_0}, v_{i_1}, \ldots, v_{i_n}$ are variables (not necessarily distinct).
2. A formula $\gamma$ is **conclusively pure** if it has one of the following forms:
   - (a) $r(v_{i_1}, \ldots, v_{i_n})$, for some $r \in R$,
   - (b) $h(v_{i_1}, \ldots, v_{i_n}) = h(v_{i_1}, \ldots, v_{i_n})$, for some $h \in H$,
   - (c) $u = v$, or
   - (d) $\perp$,
   where $u, v, v_{i_1}, \ldots, v_{i_n}$ are variables (not necessarily distinct).
(3) A uH-sentence $\forall \vec{v} \left[ (\&_{i=1}^{n} \alpha_i(\vec{v})) \rightarrow \gamma(\vec{v}) \right]$ is pure if
(a) each $\alpha_i$ in the premise is hypothetically pure, and
(b) the conclusion $\gamma$ is conclusively pure.

Lemma 4.2. Every uH-sentence in a language allowing partial operation symbols is logically equivalent to a conjunction of pure uH-sentences.

**Proof.** We show how to transform an impure uH-sentence $\sigma$ into a finite number of new uH-sentences, each of which is nearer to being pure than $\sigma$ (according to some appropriate well-founded measure). By recursively applying this process to each new sentence obtained, we ultimately obtain a finite set of pure uH-sentences whose conjunction is logically equivalent to $\sigma$.

For each transformation in the following list (other than the first transformation), we (a) state the relevant logical equivalence, and (b) display the new uH-sentence $\sigma$ whose conjunction is logically equivalent to some appropriate well-founded measure. By recursively applying this process to statements of the transformations are assumed not to occur in $\sigma$.

Given an impure uH-sentence $\sigma = \forall \vec{v} \left[ (\&_{i=1}^{n} \alpha_i(\vec{v})) \rightarrow \gamma(\vec{v}) \right]$, do the following.

(0) If some $\alpha_k$ is of the form $u = v$, for variables $u$ and $v$, then remove $\alpha_k$ from $\sigma$.

(1) Else if some $\alpha_k$ is of the form $r(t_1, \ldots, t_n)$, with some $t_\ell$ not a variable:
(a) use $\alpha_k \equiv \exists \vec{w} \left[ r(\vec{w}) & (\&_{j=1}^{n} t_j = w_j) \right]$;
(b) replace $\sigma$ by $\forall \vec{v} \vec{w} \left[ (\&_{i \neq k} \alpha_i(\vec{v})) & r(\vec{w}) \& (\&_{j=1}^{n} t_j = w_j) \right] \rightarrow \gamma(\vec{v})$.

(2) Else if some $\alpha_k$ is of the form $s = t$, with $t$ not a variable:
(a) use $\alpha_k \equiv \exists \vec{w} \left[ s = w & t = w \right]$;
(b) replace $\sigma$ by $\forall \vec{v} \vec{w} \left[ (\&_{i \neq k} \alpha_i(\vec{v})) \& s = w \& t = w \right] \rightarrow \gamma(\vec{v})$.

(3) Else if some $\alpha_k$ is of the form $h(t_1, \ldots, t_n) = v_m$, with some $t_\ell$ not a variable:
(a) use $\alpha_k \equiv \exists \vec{w} \left[ h(\vec{w}) = v_m \& (\&_{j=1}^{n} t_j = w_j) \right]$;
(b) replace $\sigma$ by the sentence
$$\forall \vec{v} \vec{w} \left[ (\&_{i \neq k} \alpha_i(\vec{v})) \& h(\vec{w}) = v_m \& (\&_{j=1}^{n} t_j = w_j) \right] \rightarrow \gamma(\vec{v}).$$

(4) Else if $\gamma$ is of the form $r(t_1, \ldots, t_n)$, with some $t_\ell$ not a variable:
(a) use $\gamma \equiv (\&_{j=1}^{n} t_j = t_j) \& \forall \vec{w} \left[ (\&_{j=1}^{n} t_j = w_j) \rightarrow r(\vec{w}) \right]$;
(b) replace $\sigma$ by the $n + 1$ sentences
$$\forall \vec{v} \left[ (\&_{i \neq k} \alpha_i(\vec{v})) \rightarrow t_j = t_j \right], \text{ for } j \in \{1, \ldots, n\},$$
and
$$\forall \vec{v} \vec{w} \left[ (\&_{i \neq k} \alpha_i(\vec{v})) \& (\&_{j=1}^{n} t_j = w_j) \right] \rightarrow r(\vec{w}).$$

(5) Else if $\gamma$ is of the form $s = t$, where $s$ and $t$ are distinct terms, at least one of which is not a variable:
(a) use $\gamma \equiv (s = s \& t = t) \& \forall \vec{w} \left[ (s = w \& t = w') \rightarrow w = w' \right]$;
(b) replace $\sigma$ by the three sentences
$$\forall \vec{v} \left[ (\&_{i \neq k} \alpha_i(\vec{v})) \rightarrow s = s \right],$$
$$\forall \vec{v} \left[ (\&_{i \neq k} \alpha_i(\vec{v})) \rightarrow t = t \right],$$
and
$$\forall \vec{v} \vec{w} \left[ (\&_{i \neq k} \alpha_i(\vec{v})) \& s = w \& t = w' \rightarrow w = w' \right].$$
(6) Else \( \gamma \) is of the form \( h(t_1, \ldots, t_n) = h(t_1, \ldots, t_n) \), with some \( t_\ell \) not a variable:

(a) use \( \gamma = \left( \&_{i=1}^n t_j = t_j \right) \& \forall \vec{w} \left[ \left( \&_{j=1}^n t_j = w_j \right) \rightarrow h(\vec{w}) = h(\vec{w}) \right] \);

(b) replace \( \sigma \) by the \( n+1 \) sentences

\[
\forall \vec{v} \left[ \left( \&_{i=1}^n \alpha_i(\vec{v}) \right) \rightarrow t_j = t_j \right], \quad \text{for } j \in \{1, \ldots, n\}, \text{ and }
\forall \vec{v} \vec{w} \left[ \left( \left( \&_{i=1}^n \alpha_i(\vec{v}) \right) \& \left( \&_{j=1}^n t_j = w_j \right) \right) \rightarrow h(\vec{w}) = h(\vec{w}) \right].
\]

\[\square\]

**Notation 4.3.** Given a structure \( X \) and an \( n \)-variable sentence \( \sigma \) of the form \( \forall \vec{v} [\psi(\vec{v}) \rightarrow \gamma(\vec{v})] \) in the language of \( X \), we use \( \text{pr}_X(\sigma) \) to denote the \( n \)-ary relation on \( X \) defined by the premise of \( \sigma \), that is,

\[
\text{pr}_X(\sigma) := \left\{ (x_1, \ldots, x_n) \in X^n \mid X \models \psi(x_1, \ldots, x_n) \right\}.
\]

**Definition 4.4.** Let \( r \) be a \( k \)-ary relation on \( M \) and let \( s \) be an \( \ell \)-ary relation on \( M \). We say that \( r \) is a *bijective projection* of \( s \) if there is a bijection \( \rho: s \rightarrow r \) of the form \( \rho(a_1, \ldots, a_k) = (\theta(a_1), \ldots, \theta(a_k)) \), for some map \( \theta: \{1, \ldots, k\} \rightarrow \{1, \ldots, \ell\} \).

**Remark 4.5.** Let \( \sigma \) be a \( uH \)-sentence and let \( \Phi \) be the logically equivalent set of pure \( uH \)-sentences obtained via the proof of the previous lemma. In Section 6 we will use the following two facts.

1. For all \( \varphi \in \Phi \), the conclusion of \( \varphi \) is in the same language as the conclusion of the original \( uH \)-sentence \( \sigma \). That is, any partial operation or relation symbol occurring in the conclusion of \( \varphi \) also occurs in the conclusion of \( \sigma \).

2. For all \( \varphi \in \Phi \), the premise of the original \( uH \)-sentence \( \sigma \) is a ‘bijective projection’ of the premise of \( \varphi \). That is, for each structure \( X \) such that \( X \models \sigma \), the relation \( \text{pr}_X(\sigma) \) is a bijective projection of the relation \( \text{pr}_X(\varphi) \).

5. **Proof of the New-from-old Theorem: The sharp functor**

The New-from-old Theorem [7,7] will give conditions under which we can deduce that an alter ego \( \mathcal{M}_2 \) fully dualises \( M \) [at the finite level] if we know that another alter ego \( \mathcal{M}_1 \) fully dualises \( M \) [at the finite level]. In this section and the next, we set up and prove the theorem.

**Assumptions 5.1.** Fix a finite algebra \( M \) and define \( \mathcal{A} := \text{ISP}(M) \). Let

\[
\mathcal{M}_1 = \langle M; H_1, R_1, \mathcal{J} \rangle \quad \text{and} \quad \mathcal{M}_2 = \langle M; H_2, R_2, \mathcal{J} \rangle
\]

be two alter egos of \( M \), and assume that

(hm) every hom-minimal relation on \( M \) belongs to \( \text{Rel}_{ca}(\mathcal{M}_2) \), and

(op) \( \mathcal{M}_2 \) is operationally rich at each relation in \( R_2 \cup \text{dom}(H_2) \).

To mimic the set-up for our motivating example from Section 5, take \( M \) to be the bounded lattice 3 and choose \( \mathcal{M}_1 = 3_\sigma \) and \( \mathcal{M}_2 = 3_h \).

Note that the two conditions (hm) and (op) are necessary for \( \mathcal{M}_2 \) to yield a finite-level full duality, by the Full Duality Lemma [2,7]. Using the following easy lemma, the assumption (op) also ensures that \( \mathcal{M}_2 \) is operationally rich at each relation in \( \text{graph}(H_2) \).

**Lemma 5.2.** Let \( \mathcal{M} \) be an alter ego of a finite algebra \( M \). Let \( r \) and \( s \) be relations compatible with \( M \), and assume that there is a bijective projection \( \rho: s \rightarrow r \). If \( \mathcal{M} \) is operationally rich at \( r \), then \( \mathcal{M} \) is also operationally rich at \( s \).
Proof. Say that \( r \) is \( m \)-ary and \( s \) is \( n \)-ary. The projection \( \rho: s \to r \) is given by \( \rho(a_1, \ldots, a_n) = (a_{\theta(1)}, \ldots, a_{\theta(m)}) \), for some map \( \theta: \{1, \ldots, m\} \to \{1, \ldots, n\} \).

Assume that \( \mathcal{M} \) is operationally rich at \( r \). Let \( h: s \to \mathcal{M} \) be a partial operation compatible with \( \mathcal{M} \). Then \( h \circ \rho^{-1}: r \to \mathcal{M} \) is also a partial operation compatible with \( \mathcal{M} \). So there is an \( m \)-ary term \( t(v_1, \ldots, v_m) \) in the language of \( \mathcal{M} \) such that 

\[
h(a_1, \ldots, a_n) = h \circ \rho^{-1}(\rho(a_1, \ldots, a_n)) = t^M(a_{\theta(1)}, \ldots, a_{\theta(m)}).
\]

Then, for any \( (a_1, \ldots, a_n) \in s \), we have

\[
h(a_1, \ldots, a_n) = h \circ \rho^{-1}(\rho(a_1, \ldots, a_n)) = t^M(a_{\theta(1)}, \ldots, a_{\theta(m)}) = t^M_1(a_1, \ldots, a_n).
\]

Thus \( t^M_1 \) is an extension of \( h \) in \( \text{Clo}_{\text{ep}}(\mathcal{M}) \).

\[\square\]

**Notation 5.3.** Let \( \mathcal{M}_\Omega = \langle \mathcal{M}; H_\Omega, R_\Omega, T \rangle \) denote the top alter ego of \( \mathcal{M} \), where

- \( H_\Omega \) is the set of all partial operations compatible with \( \mathcal{M} \), and
- \( R_\Omega \) is the set of all relations compatible with \( \mathcal{M} \).

For each \( k \in \{1, 2, \Omega\} \), let \( \mathcal{Z}_k \) denote the category of all Boolean structures of signature \( (H_k, R_k) \), and define the two full subcategories

\[
\mathcal{X}_k := \{ \mathcal{X} \in \mathcal{P}^* (\mathcal{M}_k) \} \quad \text{and} \quad \mathcal{Y}_k := \{ \mathcal{Y} \in \mathcal{Z}_k \mid \mathcal{Y} \models \text{Th}_{\text{df}}(\mathcal{M}_k) \}
\]

within \( \mathcal{Z}_k \); note that \( \mathcal{X}_k \subseteq \mathcal{Y}_k \). For each \( k \in \{1, 2\} \), let \( F_k: \mathcal{Z}_\Omega \to \mathcal{Z}_k \) be the natural forgetful functor.

Our aim in this section is to set up a ‘sharp’ functor \( S_2: \mathcal{Y}_2 \to \mathcal{Z}_\Omega \) that enriches each Boolean model of \( \text{Th}_{\text{df}}(\mathcal{M}_2) \) into a Boolean structure of signature \( (H_\Omega, R_\Omega) \). This mimics our motivating example in Section 3, where we enriched each finite structure \( \mathcal{X} \in \mathcal{X}_k \) into a structure \( \mathcal{X}^\Omega \in \mathcal{Z}_\Omega \) by defining the graph of the partial operation \( \sigma^{\mathcal{X}} \) conjunct-atomically in the language of \( \mathcal{X}^\Omega \). In the general situation, not every compatible relation on \( \mathcal{M} \) is conjunct-atomic definable from \( \mathcal{M}_2 \). But we now show that assumption 5.4(hm) ensures that every compatible relation on \( \mathcal{M} \) is primitive-positively definable from \( \mathcal{M}_2 \).

**Definition 5.4.** For each \( n \)-ary compatible relation \( r \) on \( \mathcal{M} \), where \( n \geq 0 \), fix an enumeration \( f_1, \ldots, f_m \) of the hom-set \( \mathcal{A}(r, \mathcal{M}) \) and define the \( (n + m) \)-ary compatible relation

\[
\hat{r} := \{ (\vec{a}, f_1(\vec{a}), \ldots, f_m(\vec{a})) \mid \vec{a} \in r \}
\]

on \( \mathcal{M} \).

In the definition above, the algebra \( r \subseteq \mathcal{M}^n \) is the isomorphic projection of the algebra \( \hat{r} \subseteq \mathcal{M}^{n+m} \) onto its first \( n \) coordinates. By construction, the relation \( \hat{r} \) is hom-minimal on \( \mathcal{M} \). Therefore \( \hat{r} \) is conjunct-atomic definable from \( \mathcal{M}_2 \), by assumption 5.4(hm). This justifies the next definition.

**Definition 5.5.** For each \( n \)-ary compatible relation \( r \) on \( \mathcal{M} \), with \( \hat{r} \) the associated \((n + m)\)-ary hom-minimal relation on \( \mathcal{M} \),

(a) fix an \((n + m)\)-variable conjunct-atomic formula \( \beta_r(\vec{v}, \vec{w}) \) in the language of \( \mathcal{M}_2 \) that defines \( \hat{r} \) in \( \mathcal{M}_2 \), and

(b) define the primitive-positive formula \( \beta_r(\vec{v}) := \exists \vec{w} \beta_r(\vec{v}, \vec{w}) \).

**Lemma 5.6.** Let \( r \) be an \( n \)-ary compatible relation on \( \mathcal{M} \), for some \( n \geq 0 \). Then the formula \( \beta_r(\vec{v}) \) defines the relation \( r \) in \( \mathcal{M}_2 \).
Proof. For all \( \bar{a} \in M^n \), we have the sequence of equivalences
\[
\mathbb{M}_2 \models \beta_{r}(\bar{a}) \iff \exists \bar{c} \in M^n (\bar{a}, \bar{c}) \in \hat{r} \iff \bar{a} \in r,
\]
as required. \( \square \)

Lemma 5.7. Let \( r \) be an \( n \)-ary compatible relation on \( M \), for some \( n \geq 0 \). Let \( X \in \mathcal{Y}_2 \) and let \( r_X \) denote the \( n \)-ary relation defined in \( X \) by the formula \( \beta_r(\vec{v}) \).

1. The relation \( r_X \) is topologically closed in \( X^n \).
2. If \( r \) is the graph of a partial operation on \( M \), then \( r_X \) is the graph of a continuous partial operation on \( X \) with a topologically closed domain.

Proof. (1): Let \( \hat{r}_X \) denote the \( (n + m) \)-ary relation defined in \( X \) by the conjunct-atomic formula \( \beta_r(\vec{v}, \vec{w}) \). Then the relation \( \hat{r}_X \) is topologically closed in \( X^{n+m} \), since \( X \) is a Boolean structure. But \( r_X \) is just the projection of \( \hat{r}_X \) onto its first \( n \) coordinates. Since \( X^{n+m} \) is compact and \( X^n \) is Hausdorff, it follows that \( r_X \) is also topologically closed.

(2): Let \( r \) be the graph of an \( n \)-ary compatible partial operation on \( M \), with corresponding \((n + 1)\)-variable primitive-positive formula \( \beta_r(\vec{v}, u) \). The sentence
\[
\forall \vec{v} u' [ (\beta_r(\vec{v}, u) \land \beta_r(\vec{v}, u')) \rightarrow u = u']
\]
is logically equivalent to a \( \mathcal{U} \)-sentence in the language of \( \mathbb{M}_2 \). Since \( \beta_r(\vec{v}, u) \) defines \( r \) in \( \mathbb{M}_2 \) (by Lemma 5.6), the sentence (1) is true in \( \mathbb{M}_2 \) and therefore true in \( \mathcal{Y}_2 \). Thus \( r_X \) is the graph of an \( n \)-ary partial operation \( h \) on \( X \). It follows from part (1) that \( r_X \) is closed. Since the codomain of \( h \) is compact and Hausdorff and the graph of \( h \) is closed, it follows that \( h \) is continuous. The domain of \( h \) is closed as it is a projection of \( r_X \) from the compact space \( \mathbb{X}^{n+1} \) to the Hausdorff space \( \mathbb{X}^n \). \( \square \)

Definition 5.8. Define the sharp functor \( S_2 : \mathcal{Y}_2 \to \mathcal{Z}_\Omega \) as follows.

1. For each structure \( X \in \mathcal{Y}_2 \), define \( S_2(X) \) to be the Boolean structure of signature \( (H_\Omega, R_\Omega) \) such that:
   - \( S_2(X) \) has the same underlying set and topology as \( X \);
   - for all \( r \in R_\Omega \), the relation \( r^{S_2(X)} \) is the relation \( r_X \) defined in \( X \) by the formula \( \beta_r(\vec{v}) \);
   - for all \( h \in H_\Omega \), the graph of the partial operation \( h^{S_2(X)} \) is the relation \( \text{graph}(h)_X \) defined in \( X \) by the formula \( \beta_{\text{graph}(h)}(\vec{v}, u) \).
   (Note that \( S_2(\mathbb{X}) \in \mathcal{Z}_\Omega \), by the previous lemma.)
2. For each morphism \( \mu : X \to Y \) in \( \mathcal{Y}_2 \), the morphism \( S_2(\mu) : S_2(X) \to S_2(Y) \) has the same underlying set-map as \( \mu \). (This works because morphisms are compatible with primitive-positive formulæ.)

Note 5.9. It follows at once from Lemma 5.6 that \( S_2(\mathbb{M}_2) = \mathbb{M}_\Omega \).

Lemma 5.10. Let \( h \) be a compatible partial operation on \( M \), and let \( X \in \mathcal{Y}_2 \). Then \( \text{dom}(h^{S_2(X)}) = \text{dom}(h)^{S_2(X)} \).

Proof. Consider the compatible relations \( r := \text{dom}(h) \) and \( s := \text{graph}(h) \) on \( M \).
Let the fixed enumerations used in Definition 5.4 be \( f_1, \ldots, f_m \) for \( \mathcal{A}(r, M) \) and \( g_1, \ldots, g_m \) for \( \mathcal{A}(s, M) \). Note that the two hom-sets have the same size, since there is an isomorphism \( \rho : s \to r \), given by \( \rho(\bar{a}, h(\bar{a})) := \bar{a} \). Indeed, there is a permutation \( \theta \) of \( \{1, \ldots, m\} \) such that \( g_i = f_{\theta(i)} \circ \rho \), for all \( i \in \{1, \ldots, m\} \).
We now have
\[ \hat{r} = \{ (\vec{a}, f_1(\vec{a}), \ldots, f_m(\vec{a})) \mid \vec{a} \in r \} \quad \text{and} \quad \hat{s} = \{ (\vec{a}, h(\vec{a}), f_{\theta(1)}(\vec{a}), \ldots, f_{\theta(m)}(\vec{a})) \mid \vec{a} \in r \} \]

We can choose \( j \in \{1, \ldots, m\} \) such that \( h = f_j \). The sentence
\[ \forall \vec{v} w_1 \ldots w_m \left( f_j(\vec{v}, w_1, \ldots, w_m) \land u = w_j \leftrightarrow \beta_j(\vec{v}, u, w_{\theta(1)}, \ldots, w_{\theta(m)}) \right) \]

is equivalent to a conjunction of \( \mathsf{uH} \)-sentences in the language of \( M \). Hence \( \text{dom}(h) \models \text{S}_2(\vec{x}) \) in \( Y_2 \).

Since the implication \( \forall \vec{v} [\beta_{\theta}(\vec{v}) \to \alpha(\vec{v})] \) is logically equivalent to a \( \mathsf{uH} \)-sentence and is true in \( M_2 \), it is true in \( Y_2 \). So it remains to consider the converse implication.

By assumption \( \text{S}_2(\vec{x}) \), the alter ego \( M_2 \) is operationally rich at each relation in \( H_2 \). By Lemma \( \text{S}_2(\vec{x}) \), it follows that \( M_2 \) is operationally rich at each relation in \( \Gamma_h \). So \( M_2 \) is operationally rich at \( s \). Let \( f_1, \ldots, f_m \) be the fixed enumeration of \( \mathcal{A}(s, M) \) used in Definition \( \text{S}_2(\vec{x}) \). Then \( f_1, \ldots, f_m \) have extensions \( g_1, \ldots, g_m \) in \( \text{Clo}_\mathsf{op}(M_2) \) by operational richness. Thus
\[ \hat{s} = \{ (\vec{a}, f_1(\vec{a}), \ldots, f_m(\vec{a})) \mid \vec{a} \in s \} = \{ (\vec{a}, g_1(\vec{a}), \ldots, g_m(\vec{a})) \mid \vec{a} \in s \} \]

Let \( t_1, \ldots, t_m \) be terms in the language of \( M_2 \) that yield the partial operations \( g_1, \ldots, g_m \). Then the sentence
\[ \forall \vec{v} [\alpha(\vec{v}) \to \beta_j(\vec{v}, t_1(\vec{v}), \ldots, t_m(\vec{v}))] \quad (\dagger) \]

is equivalent to a conjunction of \( \mathsf{uH} \)-sentences in the language of \( M_2 \). The sentence \( (\dagger) \) holds in \( M_2 \) and thus in \( Y_2 \). But \( (\dagger) \) logically implies \( \forall \vec{v} [\alpha(\vec{v}) \to \beta_j(\vec{v})) \), as required.

Lemma 5.12.

(1) For each \( \vec{x} \in Y_2 \), we have \( \vec{x} = F_2 \circ S_2(\vec{x}) \).

(2) For each \( \vec{x} \in Y_2 \), we have \( \vec{x} = S_2 \circ F_2(\vec{x}) \).

Proof. Part (1) follows directly from the previous lemma. To prove part (2), first note that the top alter ego \( M_3 \) satisfies the assumptions \( \text{S}_2(\vec{x}) \) and \( \text{S}_2(\vec{x}) \) on \( M_2 \). Consider the functor \( S_2 : Y_3 \to Y_2 \) obtained by applying Definition \( \text{S}_2(\vec{x}) \) with \( M_2 \) as \( M_2 \), but using the same formulæ \( \beta_j \) as for \( M_2 \). Then \( S_2 = S_2 \circ F_2 \). The previous lemma with \( M_2 \) as \( M_2 \) yields \( S_2 = \text{id}_{Y_2} \), as required. 

□
6. Proof of the New-from-old Theorem: The transfer functor

Throughout this section, the assumptions 5.1(hm) and 5.1(op) remain in force. In the previous section, we defined the sharp functor $S_2 : Y_2 \to Z_\Omega$. In this section, we aim to show that the transfer functor

$$T_{21} := F_1 \circ S_2 : Y_2 \to Y_1$$

is well defined. As in the motivating example from Section 3 we will use an axiomatisation $\Sigma_1$ for the uH-theory of $M_1$. By Lemma 4.2 we can assume that all the sentences in $\Sigma_1$ are pure. We will need to strengthen our assumptions on $M_2$, but to do this we require some definitions.

**Definition 6.1.** Let $\varphi = \forall \overline{v} \left[ (\bigwedge_{i=1}^\nu \alpha_i(\overline{v})) \rightarrow \gamma(\overline{v}) \right]$ be a pure uH-sentence in the language of $M_\Omega$. Define $\varphi^2$ to be the sentence in the language of $M_2$ constructed from $\varphi$ as follows.

1. First, simultaneously make the following replacements:
   a. replace each $\gamma(v_{i_1}, \ldots, v_{i_n})$ in $\varphi$ with $\beta_r(v_{i_1}, \ldots, v_{i_n})$;
   b. replace each $h(v_{i_1}, \ldots, v_{i_n}) = v_{i_0}$ in $\varphi$ with $\beta_{\text{graph}(h)}(v_{i_1}, \ldots, v_{i_n}, v_{i_0})$;
   c. if the conclusion $\gamma(\overline{v})$ is $h(v_{i_1}, \ldots, v_{i_n}) = h(v_{i_1}, \ldots, v_{i_n})$, replace it with $\beta_{\text{dom}(h)}(v_{i_1}, \ldots, v_{i_n})$.
2. Then convert the new existential quantifiers in the premise into universal quantifiers out the front.

Let the new sentence so constructed be

$$\varphi^2 = \forall \overline{v} \overline{w}_1 \ldots \overline{w}_\nu \left[ (\bigwedge_{i=1}^\nu \alpha_i^2(\overline{v}, \overline{w}_i)) \rightarrow \gamma^2(\overline{v}) \right].$$

Note that each $\alpha_i^2$ in the premise is of the form $\beta_r(v_{i_1}, \ldots, v_{i_n}, \overline{w}_i)$, for some $r \in R_\Omega$, and therefore is a conjunct-atomic formula in the language of $M_2$. The conclusion $\gamma^2$ is either a primitive-positive formula in the language of $M_2$ or else $\bot$.

**Lemma 6.2.** Let $X \in Y_2$ and let $\varphi$ be a pure uH-sentence in the language of $M_\Omega$. Then $S_2(X) \models \varphi$ if and only if $X \models \varphi^2$.

**Proof.** This follows from the definitions of $S_2(X)$ and $\varphi^2$. The only complication is replacement (6.11c). But Lemma 6.1 tells us that, for all $h \in H_\Omega$, we have

$$S_2(X) \models h(a) = h(\tilde{a}) \iff \tilde{a} \in \text{dom}(h^{S_2(X)}) \iff \tilde{a} \in \text{dom}(h)^{S_2(X)} \iff X \models \beta_{\text{dom}(h)}(\tilde{a}).$$

So the result holds.

Recall that the notation $pr_X(\sigma)$ was introduced in 4.3.

**Lemma 6.3.** Let $X \in Y_2$ and let $\varphi$ be a pure uH-sentence such that $M_\Omega \models \varphi$. Then $S_2(X) \models pr_X(\varphi)$.

**Proof.** Assume that (1) or (2) holds. We want to show that $S_2(X) \models \varphi$. By Lemma 6.2 it suffices to show that $X \models \varphi^2$. Using Note 5.9 we have $S_2(M_2) = M_\Omega \models \varphi$. So it follows by Lemma 6.2 that $M_2 \models \varphi^2$. Say that $\varphi = \forall \overline{v} \left[ (\bigwedge_{i=1}^\nu \alpha_i(\overline{v})) \rightarrow \gamma(\overline{v}) \right]$ and let $\gamma^2(\overline{v})$ be the conclusion of $\varphi^2$; see Definition 6.1. Since the uH-sentence $\varphi$ is pure, we know that its conclusion $\gamma(\overline{v})$ must take one of the following four forms:
we can now assume that \( \gamma(v) \) is a uH-sentence true in \( \mathcal{M}_2 \) and thus in \( X \). So we can now assume that \( \gamma(v) \) is of type (a) or (b).

**Case (1): the conclusion of \( \varphi \) is in the language of \( \mathcal{M}_2 \).** We can construct a uH-sentence \( \psi \) in the language of \( \mathcal{M}_2 \) from \( \varphi^2 \) by changing the conclusion \( \gamma(v) \) back to \( \gamma(v) \). The conclusion \( \gamma(v) \) is \( \beta_r(v_1,\ldots,v_n) \), for some \( r \in R_2 \cup \text{dom}(H_2) \). We know that \( \beta_r \) defines the interpretation of \( r \) in \( \mathcal{M}_2 \) (by Lemma 5.6) and also in \( X \) (by Lemmas 5.10 and 5.11). Thus \( \varphi^2 \leftrightarrow \psi \) is true in both \( \mathcal{M}_2 \) and \( X \). Since \( \mathcal{M}_2 \models \varphi^2 \) and \( \psi \) is a uH-sentence, it follows that \( X \models \varphi^2 \).

**Case (2): \( \mathcal{M}_2 \) is operationally rich at the relation \( \text{pr}_{\mathcal{M}_2}(\varphi^2) \).** To show that \( X \models \varphi^2 \), it is enough to find a set \( \Sigma \) of uH-sentences in the language of \( \mathcal{M}_2 \) such that \( \mathcal{M}_2 \models \Sigma \) and \( \Sigma \vdash \varphi^2 \).

The conclusion \( \gamma(v) \) is \( \beta_r(v_1,\ldots,v_n) \), for some \( r \in R_2 \cup \text{dom}(H_2) \). Define the compatible relation \( p := \text{pr}_{\mathcal{M}_2}(\varphi^2) \) on \( \mathcal{M} \). Since \( \mathcal{M}_2 \models \varphi^2 \) and since \( \beta_r \) defines \( r \) in \( \mathcal{M}_2 \) (by Lemma 5.6), we have

\[
(a, c_1, \ldots, c_r) \in p \iff \mathcal{M}_2 \models \alpha^\nu_{i=1} \alpha^2_j(a, c_i) \implies \mathcal{M}_2 \models \gamma^2(a) \\
\implies \mathcal{M}_2 \models \beta_r(a_1,\ldots,a_n) \iff (a_1,\ldots,a_n) \in r.
\]

Let \( f_1,\ldots,f_m \) be the fixed enumeration of \( \mathcal{A}(r,M) \) used in Definition 6.4. Then, for all \( j \in \{1,\ldots,m\} \), we can define \( g_j : p \to \mathcal{M} \) by

\[
g_j(a_1,\ldots,a_n) := f_j(a_1,\ldots,a_n).
\]

Each \( g_j \) is a compatible partial operation on \( \mathcal{M} \) with domain \( p \). We are assuming that \( \mathcal{M}_2 \) is operationally rich at \( \text{pr}_{\mathcal{M}_2}(\varphi^2) = p \). Thus there are terms \( t_1,\ldots,t_m \) in the language of \( \mathcal{M}_2 \) that define extensions of \( g_1,\ldots,g_m \) in \( \mathcal{M}_2 \). Define the sentence

\[
\psi := \forall \hat{\nu} \hat{\nu}_1 \ldots \hat{\nu}_m \left( \exists \hat{\nu}_1,\ldots,\hat{\nu}_m (\mathcal{M}_2 \models \gamma^2(\hat{\nu} \hat{\nu}_1,\ldots,\hat{\nu}_m)) \right).
\]

Then \( \psi \) is equivalent to a conjunction of uH-sentences in the language of \( \mathcal{M}_2 \), with \( \mathcal{M}_2 \models \psi \) and \( \psi \vdash \varphi^2 \). Hence it follows that \( X \models \varphi^2 \), as required. \( \square \)

The next lemma will be used later to simplify the checking of condition 6.3 (2).

**Lemma 6.4.** Let \( \varphi \) be a pure uH-sentence in the language of \( \mathcal{M}_\Omega \), and define the sentence \( \varphi^3 \) as in 6.4. If \( \mathcal{M}_2 \) is operationally rich at the relation \( \text{pr}_{\mathcal{M}_2}(\varphi) \), then \( \mathcal{M}_2 \) is also operationally rich at \( \text{pr}_{\mathcal{M}_2}(\varphi^2) \).

**Proof.** By Lemma 5.4, it is enough to show that \( \text{pr}_{\mathcal{M}_2}(\varphi) \) is a bijective projection of \( \text{pr}_{\mathcal{M}_2}(\varphi^2) \). Referring to the notation of Definition 6.1 first note that each \( \alpha^2_j(\hat{v},\hat{w}_j) \) in the premise of \( \varphi^2 \) is of the form \( \beta_r(\hat{v}_j,\hat{w}_j) \), for some compatible relation \( r_j \) on \( \mathcal{M} \), some tuple \( \hat{v}_j = (v_{ij_1},\ldots,v_{ij_{n_j}}) \) of variables from \( \hat{v} \), and some tuple of new variables \( \hat{w}_j \) of length \( |\mathcal{A}(r_j,M)| \). Let \( f_j \) be the fixed enumeration of \( \mathcal{A}(r_j,M) \). Then...
used in Definition 5.5. Then

\[(\bar{a}, \bar{c}_1, \ldots, \bar{c}_\nu) \in \text{pr}_{M_2}(\varphi^{\sharp}) \iff M_2 \models \bigwedge_{j=1}^{\nu} \alpha_j^{\sharp}(\bar{a}, \bar{c}_j)\]

\[\iff M_2 \models \bigwedge_{j=1}^{\nu} \beta_{r_j}(\bar{a}_j, \bar{c}_j) \]

\[\iff (\forall j \in \{1, \ldots, \nu\}) (\bar{a}_j \in r_j \& \bar{c}_j = f_j(\bar{a}_j)) \]

\[\iff \bar{a} \in \text{pr}_{M_1}(\varphi) \& (\forall j \in \{1, \ldots, \nu\}) \bar{c}_j = f_j(\bar{a}_j).\]

It now follows that \(\rho: \text{pr}_{M_2}(\varphi^{\sharp}) \to \text{pr}_{M_1}(\varphi)\), given by \((\bar{a}, \bar{c}_1, \ldots, \bar{c}_\nu) \mapsto \bar{a}\), is a bijective projection.

We now add to our initial assumptions, 6.5, in order to ensure that the transfer functor \(T_{21} := F_1 \circ S_2: Y_2 \to Y_1\) is well defined.

**Assumptions 6.5.** Choose a basis \(\Sigma_1\) for the universal Horn theory of \(M_1\) such that each \(u\text{H}-\)sentence in \(\Sigma_1\) is pure. Assume that

- \((ax)\) for each \(\varphi \in \Sigma_1\), if the conclusion of \(\varphi\) is not in the language of \(M_2\), then \(M_2\) is operationally rich at the relation \(\text{pr}_{M_2}(\varphi^{\sharp})\).

**Note 6.6.** Each relation \(\text{pr}_{M_2}(\varphi^{\sharp})\) is conjunct-atomic definable from hom-minimal relations on \(M_1\); see Definition 6.1. So assumption 6.5(ax) is necessary for \(M_2\) to yield a finite-level full duality, by the Full Duality Lemma 2.7.

**Lemma 6.7.** The transfer functor \(T_{21} := F_1 \circ S_2: Y_2 \to Y_1\) is well defined. That is, if \(X \in Y_2\), then \(F_1 \circ S_2(X) \in Y_1\).

**Proof.** Let \(X \in Y_2\) and \(\varphi \in \Sigma_1\). Then \(S_2(X) \models \varphi\), using 6.5(ax) and Lemma 6.3. So \(F_1S_2(X) \models \varphi\). It follows that \(F_1S_2(X) \models \Sigma_1\) and therefore \(F_1S_2(X) \in Y_1\). \(\square\)

**Remark 6.8.** If we assume that \(M_1\) fully dualises \(M\) at the finite level, then \(M_1\) satisfies the first two conditions 5.1(hm) and 5.1(op) that we placed on \(M_2\), by the Full Duality Lemma 2.7. This means that we can use the method of Section 5 to define a sharp functor \(S_1: Y_1 \to Z_Q\) based on \(M_1\). As in Definition 5.3, for each \(r \in R_Q\), we will need to choose some conjunct-atomic formula \(\delta_r(\bar{v}, \bar{w})\) in the language of \(M_1\) that defines \(\tilde{r}\) in \(M_1\), and this formula may well be different from the one \(\beta_r(\bar{v}, \bar{w})\) chosen for \(M_2\). The alter ego \(M_1\) also satisfies condition 6.5(ax) for any pure basis \(\Sigma_2\) for \(\text{Th}_{\text{ah}}(M_2)\), by Note 6.6. So we can follow the method of this section to establish that the transfer functor \(T_{12} := F_2 \circ S_1: Y_1 \to Y_2\) is well defined; see Lemma 6.7.

**Lemma 6.9.** Assume that \(M_1\) fully dualises \(M\) at the finite level. Then the two transfer functors \(T_{12} := F_2 \circ S_1: Y_1 \to Y_2\) and \(T_{21} := F_1 \circ S_2: Y_2 \to Y_1\) are mutually inverse category isomorphisms.

**Proof.** Using Lemma 6.7 and Remark 6.8, the two transfer functors are well defined. We just need to show that they are mutually inverse.

By symmetry, it is enough to show that \(X = T_{12}T_{21}(X)\), for some arbitrary \(X \in Y_2\). Let \(r \in R_2 \cup \text{graph}(H_2)\). We use \(\hat{r}\) to denote the associated hom-minimal relation; see Definition 6.4. We have a conjunct-atomic formula \(\hat{\beta}_r(\bar{v}, \bar{w})\) in the language of \(M_2\) that defines \(\hat{r}\) in \(M_2\), and a conjunct-atomic formula \(\hat{\delta}_r(\bar{v}, \bar{w})\) in the language of \(M_1\) that defines \(\hat{r}\) in \(M_1\); see Definition 5.3 and Remark 6.8. Thus \(M_1\) satisfies the sentence \(\sigma := \forall \bar{v}\bar{w}\ [\hat{\beta}_r(\bar{v}, \bar{w}) \leftrightarrow \hat{\delta}_r(\bar{v}, \bar{w})]\).
By Lemma \(5.11\), the relation \(r^X\) is defined by the formula \(\exists \bar{w} \beta_r(\bar{v}, \bar{w})\) in \(S_2(X)\).
The relation \(r^{T_2T_2(X)}\) is equal to the relation \(r^{S_1F_1S_2(X)}\), which is described by the
formula \(\exists \bar{w} \delta_r(\bar{v}, \bar{w})\) in \(S_2(X)\). So we can show that \(r^X = r^{T_2T_2(X)}\) by checking that
\(S_2(X)\) satisfies the sentence \(\sigma\).

First consider the backwards implication \(\sigma_b := \forall \bar{v} \bar{w} [\delta_r(\bar{v}, \bar{w}) \rightarrow \beta_r(\bar{v}, \bar{w})]\). This
is logically equivalent to a set \(\Sigma_b\) of uH-sentences, each of which holds in \(M\).
Using Lemma 6.10, we can convert \(\Sigma_b\) into a logically equivalent set \(\Phi_b\) of pure
uH-sentences. The conclusion of each sentence in \(\Phi_b\) is in the language of \(M\); see
Remark 4.3. So \(S_2(X) \models \sigma_b\), by Lemma 6.3.

Now consider the forwards implication \(\sigma_f := \forall \bar{v} \bar{w} [\delta_r(\bar{v}, \bar{w}) \rightarrow \beta_r(\bar{v}, \bar{w})]\). This
is logically equivalent to a set \(\Sigma_f\) of uH-sentences, each of which holds in \(M\).
Using Lemma 6.2 again, we can convert \(\Sigma_f\) into a logically equivalent set \(\Phi_f\) of
pure uH-sentences. Let \(\varphi \in \Phi_f\). By Remark 4.3, since \(M_\Omega \models \sigma_f\), there is a
bijective projection \(\rho: \text{pr}_{M_\Omega}(\varphi) \rightarrow \text{pr}_{M_\Omega}(\sigma_f)\). The alter ego \(M_2\) is operationally
rich at the relation \(r = \text{pr}_{M_\Omega}(\sigma_f)\), since it is hom-minimal, and therefore \(M_2\) is
also operationally rich at \(\text{pr}_{M_\Omega}(\varphi)\), by Lemma 5.2. So \(S_2(X) \models \varphi\), by Lemmas 6.3 and 6.4.
It follows that \(S_2(X) \models \sigma_f\), as required.

We wrap up this section with the following result.

**Lemma 6.10.** Assume that \(M_2\) satisfies \((5.1)\) and \((6.5)\). If \(M_1\) fully
dualises \(M\) [at the finite level], then the following are equivalent:

1. \(M_2\) fully dualises \(M\) [at the finite level];
2. the transfer functor \(T_{21} := F_1 \circ S_2: Y_2 \rightarrow Y_1\) sends each [finite] structure
   in \(X_2\) into \(X_1\).

**Proof.** (2) \(\Rightarrow\) (1): Assume that (2) holds. The alter ego \(M_2\) dualises \(M\) at the finite
level, by \((5.1)\) and the Duality Lemma 2.5. If \(M_1\) dualises \(M\), then so does \(M_2\); see
Remark 4.4. By the previous lemma, the transfer functor \(T_{21}: Y_2 \rightarrow Y_1\) is a
category isomorphism that preserves underlying sets and set-maps, and by Note 5.3
we have \(T_{21}(M_2) = M_1\). It follows that \(M_2\) fully dualises \(M\) [at the finite level],
using condition (2) and the New-from-old Lemma 4.1.

(1) \(\Rightarrow\) (2): Assume \(M_2\) fully dualises \(M\) [at the finite level]. For \(i \in \{1, 2, \Omega\}\),
let \(D_i: A \rightarrow X_i\) and \(E_i: X_i \rightarrow A\) denote the hom-functors induced by \(M\) and \(M_i\).
Let \(X\) be a [finite] structure in \(X_2\), and define the algebra \(A := E_2(X) \in A\). Then
\(X \cong D_2E_2(X) = D_2(A)\). Since \(D_i(A) = F_iD_\Omega(A)\), for each \(i \in \{1, 2\}\), using
Lemma 4.12(2) yields
\[
D_1(A) = F_1D_\Omega(A) = F_1S_2F_2D_\Omega(A) = F_1S_2D_2(A).
\]
So we have \(F_1S_2(X) \cong F_1S_2D_2(A) = D_1(A) \in X_1\), as required.

### 7. The New-from-old Theorem and its Applications

We now have all the ingredients necessary to state and prove our main theorem.
Recall that an alter ego \(M\) is **standard** if the potential dual class \(X = IS_\Omega^*(M)\)
consists precisely of all Boolean models of Th_{uH}(M); see Definition 1.2.

**New-from-old Theorem 7.1.** Let \(M\) be a finite algebra, and let \(M_1\) and \(M_2\) be
alter egos of \(M\). Assume that \(M_2\) satisfies \((5.1)\) and \((6.5)\). Then
1. If \(M_1\) fully dualises \(M\) at the finite level, then so does \(M_2\).
2. If \(M_1\) is standard and fully dualises \(M\), then the same is true of \(M_2\).
Proof. Part (1) follows directly from Lemma 6.10 because we automatically have $(X_1)_{fin} = (Y_1)_{fin}$; see Notation 5.3 and Definition 1.2.
To prove part (2), assume that $M_1$ is standard and fully dualises $M$. Since $M_1$ is standard, we have $X_1 = Y_1$. It follows by Lemma 6.10 that $M_2$ fully dualises $M$. To see that $M_2$ is also standard, let $X \subseteq Y_2$. Then $T_{21}(X) \subseteq Y_1 = X_1$. As we have shown that $M_2$ fully dualises $M$, we can use Lemma 6.10 (with the subscripts 1 and 2 swapped) to deduce that $T_{12}T_{21}(X) \subseteq X_2$. Therefore Lemma 6.9 gives $X = T_{12}T_{21}(X) \subseteq X_2$. Thus $M_2$ is standard. □

Warning 7.2. In the signature of the alter ego $M_1 = \langle M; H_1, R_1, T \rangle$, we are not distinguishing between total operations and proper partial operations. To apply the New-from-old Theorem 7.1, the $uH$-basis chosen for $M_1$ must imply all $uH$-sentences true in $M_1$, including those of the form $\forall v_1 \ldots v_n \left[ f(v_1, \ldots, v_n) = f(v_1, \ldots, v_n) \right]$, where $f$ is an $n$-ary total operation on $M$ for some $n \geq 0$.

We now use this rather technical theorem to obtain a series of self-contained corollaries. First we use the theorem to give a new and very natural condition under which every finite-level full duality lifts to the infinite level.

Theorem 7.3. Let $M$ be a finite algebra. Assume that $M$ is fully dualised by a standard alter ego. If an alter ego $\bar{M}$ fully dualises $M$ at the finite level, then $\bar{M}$ is standard and fully dualises $M$.

Proof. Let $M_1$ be a standard alter ego that fully dualises $M$. Assume that $M$ fully dualises $M$ at the finite level. Then we can take $M_2 := \bar{M}$ and the assumptions of the New-from-old Theorem 7.1 are satisfied, by the Full Duality Lemma 2.7 and Note 6.6. Thus $\bar{M}$ is standard and fully dualises $M$. □

Example 7.4. The previous theorem can be applied to quasi-primal algebras, that is, to finite algebras $M$ such that the ternary discriminator $t: M^3 \to M$ is a term function of $M$, where
$$t(x, y, z) = \begin{cases} x & \text{if } x \neq y, \\ z & \text{if } x = y. \end{cases}$$
Davey and Werner [21, 2.7] have shown that every quasi-primal algebra has a standard, strongly dualising alter ego. So, for any quasi-primal algebra $M$, the finite-level full dualities always lift to full dualities.

We can also use the New-from-old Theorem to refine the intrinsic description of finite-level full dualities given by the Full Duality Lemma 2.7.

Theorem 7.5. Let $\bar{M} = \langle M; H, R, T \rangle$ be an alter ego of a finite algebra $M$. Then the following are equivalent:

1. $\bar{M}$ fully dualises $M$ at the finite level;
2. (a) every hom-minimal relation on $M$ belongs to $\text{Rel}_{ca}(M)$, and
   (b) $\bar{M}$ is operationally rich at each relation in $\text{Rel}_{ca}(M)$;
3. (a) every hom-minimal relation on $M$ belongs to $\text{Rel}_{ca}(M)$, 
   (b) $\bar{M}$ is operationally rich at each relation in $R \cup \text{dom}(H)$, and 
   (c) $\bar{M}$ is operationally rich at each relation that is conjunct-atomic definable from hom-minimal relations.

Proof. Using the Full Duality Lemma 2.7 we only need to prove that (3) $\Rightarrow$ (1). So assume that (3) holds. We check that we can apply the New-from-old Theorem 7.1.
with $M_1 = M_\Omega$ and $M_2 = M$. First note that the top alter ego $M_\Omega$ must fully dualise $M$ at the finite level, by the Full Duality Lemma 2.7. Conditions (5.1(hm)) and (5.1(op)) correspond to assumptions (3a) and (3b). Condition (5.1(ax)) holds by assumption (3c), because each relation $pr_M(\varphi^2)$ is conjunct-atomic definable from hom-minimal relations; see Note 6.6.

From the previous result, we easily obtain a ‘constructive’ description of the smallest full-at-the-finite-level alter ego $M_\alpha$; see Facts 2.9(2).

**Theorem 7.6.** Let $M$ be a finite algebra. Define the sets

- $R_\alpha$ of all compatible relations on $M$ that are conjunct-atomic definable from the hom-minimal relations on $M$, and
- $H_\alpha$ of all compatible partial operations on $M$ with domain in $R_\alpha$.

Then $M_\alpha = \langle M; H_\alpha, R_\alpha, T \rangle$ is the smallest alter ego that fully dualises $M$ at the finite level.

**Proof.** This follows from Theorem 7.3 (1)$\iff$(3).

Using the transfer set-up from Section 6 we can give a new proof of the known characterisation of when a full duality is preserved under enriching the alter ego.

**Theorem 7.7 (18, 5.3).** Let $M_1 = \langle M; H_1, R_1, T \rangle$ and $M_2 = \langle M; H_2, R_2, T \rangle$ be alter egos of a finite algebra $M$, with $M_1$ a structural reduct of $M_2$. Assume that $M_1$ fully dualises $M$ [at the finite level]. Then the following are equivalent:

1. $M_2$ fully dualises $M$ [at the finite level];
2. $M_2$ is operationally rich at each relation in $(R_2 \setminus R_1) \cup \text{dom}(H_2 \setminus H_1)$.

**Proof.** By the Full Duality Lemma 2.7, it suffices to prove (2)$\Rightarrow$(1). Assume that (2) holds. Without loss of generality, we can assume $M_1$ is a reduct of $M_2$.

Every hom-minimal relation on $M$ belongs to $\text{Rel}_{ca}(M_1) \subseteq \text{Rel}_{ca}(M_2)$, by the Duality Lemma 2.6. Thus (5.1(hm)) holds. Now let $r \in R_2 \setminus \text{dom}(H_2)$. Using the Full Duality Lemma 2.7, if $r \in R_1 \setminus \text{dom}(H_1)$, then $M_1$ is operationally rich at $r$, and so $M_2$ is too. Otherwise, condition (2) ensures that $M_2$ is operationally rich at $r$. Thus (5.1(op)) holds. Since the language of $M_1$ is contained in that of $M_2$, it follows immediately that (5.1(ax)) holds.

We now apply Lemma 6.10 to show that $M_2$ fully dualises $M$ [at the finite level]. Since $M_1$ is a reduct of $M_2$, the transfer functor $T_{21} := F_1 \circ S_2 : Y_2 \to Y_1$ is the forgetful functor, by Lemma 6.12(1). It follows that $T_{21}$ sends each structure in $X_2$ into $X_1$, as required.

We will now illustrate the general New-from-old Theorem using an important example from natural duality theory: the first known full-but-not-strong duality.

**Example 7.8.** Define the four-element lattice-based algebra

$$Q := \langle \{0, a, b, 1\}; t, \lor, \land, 0, 1 \rangle,$$

where $0 < a < b < 1$ and the operation $t$ is the ternary discriminator. Define two alter egos of $Q$:

$$Q_0 := \langle \{0, a, b, 1\}; \text{graph}(f), T \rangle$$

and

$$Q_1 := \langle \{0, a, b, 1\}; f, g, T \rangle,$$

where the partial automorphisms $f$ and $g$ of $Q$ are shown in Figure 3.

By the Quasi-primal Strong Duality Theorem [23, 3.3.13], the alter ego $Q_1$ strongly dualises $Q$. Since $Q_0$ and $Q_1$ are clearly not structurally equivalent, the alter ego
Q₀ cannot strongly dualise Q. Nevertheless, the alter ego Q₀ fully dualises Q: Clark, Davey and Willard [6] gave three different proofs to celebrate this discovery; we will now give yet another proof.

Since we know that Q₁ dualises Q, it follows easily that Q₀ dualises Q. Thus Q₀ satisfies 5.1(hm), by the Duality Lemma 2.5. Since \( \text{graph}(f) \) is hom-minimal on Q, it is trivial that Q₀ satisfies 5.1(op).

As mentioned in Example 7.4, every quasi-primal algebra is strongly dualised by a standard alter ego. So Q₁ is standard, by Theorem 7.3. It is easy to check that the following three uH-sentences form a basis for Th\( _{uH}(Q₁) \):

\[
\begin{align*}
(1) & \forall uv \ [f(u) = v \rightarrow g(v) = u]; \\
(2) & \forall uv \ [g(u) = v \rightarrow f(v) = u]; \\
(3) & \forall uvw \ [(f(u) = v \land f(v) = w) \rightarrow u = v].
\end{align*}
\]

Sentence (3) is pure, but sentences (1) and (2) are not. Sentence (1) converts into two pure uH-sentences:

\[
\begin{align*}
(1a) & \forall uv \ [f(u) = v \rightarrow g(v) = g(v)]; \\
(1b) & \forall uvw \ [(f(u) = v \land g(v) = w) \rightarrow w = u].
\end{align*}
\]

The purification of (2) is the pair of sentences (2a) and (2b) obtained from (1a) and (1b) by interchanging \( f \) and \( g \). Of the five sentences (1a), (1b), (2a), (2b) and (3), only (1a) and (2a) have conclusions not in the language of Q₀. Since \( \text{pr}_{M₀}(1a) = \text{graph}(f) \) and \( \text{pr}_{M₀}(2a) = \text{graph}(g) \), both of which are hom-minimal, it follows from Lemma 6.4 that Q₀ satisfies 6.5(ax) with respect to these five sentences. Thus Q₀ is standard and fully dualises Q, by the New-from-old Theorem 7.1.

To use the New-from-old Theorem directly, we need first to have come up with a candidate alter ego \( M₂ \) that is going to fully dualise M [at the finite level]. But we can easily adapt the New-from-old Theorem into an algorithm that can help you to find, for your favourite finite algebra M, an alter ego of M that is equivalent to the smallest full-at-the-finite-level alter ego \( Mₐ \).

**Algorithm 7.9.** Let M be a finite algebra. You need the following:

(i) An alter ego \( M₀ = \langle M; H₀, R₀, T \rangle \) of M such that
   (a) \( M₀ \) dualises M at the finite level,
   (b) \( M₀ \) is operationally rich at each relation in \( R₀ \cup \text{dom}(H₀) \), and
   (c) \( M₀ \) is a reduct of \( Mₐ \).
   (The easiest way to guarantee that (c) holds is to ensure that the signature of \( M₀ \) includes only total operations and hom-minimal relations.)

(ii) An alter ego \( M₁ \) that fully dualises M at the finite level.

(iii) A finite basis \( Σ₁ \) for the uH-theory of \( M₁ \).
Start with $M_2 := M_0$. Then an alter ego equivalent to $M_0$ can be obtained by adding partial operations to the signature of $M_2$ as follows.

For each uH-sentence $\psi \in \Sigma_1$ whose conclusion is not in the language of $M_0$, complete the following steps:

1. Convert $\psi$ into a set of pure uH-sentences $\varphi_1, \ldots, \varphi_n$.
2. For each $i \in \{1, \ldots, n\}$ such that the conclusion of $\varphi_i$ is not in the language of $M_0$, calculate the relation $r_i$ on $M$ as follows:
   a. if the premise of $\varphi_i$ is in the language of $M_0$, then $r_i := \text{pr}_{M_0}(\varphi_i)$;
   b. if the premise of $\varphi_i$ is not in the language of $M_0$, then $r_i := \text{pr}_{M_0}(\varphi_i')$.
(See 6.1 and 6.4)
3. For each relation $r_i$ calculated in step (2), add all the compatible partial operations on $M$ with domain $r_i$ to the signature of $M_2$.

At the end of this process, you will have $M_2 \equiv M_0$.

We finish this section by demonstrating this algorithm on the bounded lattice 3, whereby we shall ‘rediscover’ the partial operation $h$ used in Section 3.

**Example 7.10.** Consider the bounded lattice $3 = \{(0, a, 1); \lor, \land, 0, 1\}$, and define the two alter egos

$$3_0 := \{(0, a, 1); f, g, T\} \quad \text{and} \quad 3_1 := \{(0, a, 1); f, g, \sigma, T\},$$

as in Section 3. Then $3_0$ and $3_1$ dualise and strongly dualise 3, respectively.

A uH-axiomatisation for $3_1$ is given in the proof of Lemma 6.3.

1. $\forall v [f(v) = f(f(v)) \land g(v) = g(g(v))];$
2. $\forall uvw [(f(w) = u \land g(w) = v) \leftrightarrow \sigma(u, v) = w];$
3. $\forall uv [(\sigma(u, v) = \sigma(u, v) \land \sigma(v, u) = \sigma(v, u)) \rightarrow u = v];$
4. $\forall uvw [(\sigma(u, v) = \sigma(u, v) \land \sigma(v, w) = \sigma(v, w)) \rightarrow \sigma(u, w) = \sigma(u, w)].$

We only need to consider (4) and the forward direction of (2).

The forward direction of (2) converts into a pair of pure uH-sentences, of which we need only consider (2a):

(2a) $\forall uvw [(f(w) = u \land g(w) = v) \rightarrow \sigma(u, v) = \sigma(u, v)];$
(2b) $\forall uvw [(f(w) = u \land g(w) = v) \land \sigma(u, v) = x] \rightarrow x = w].$

The premise of (2a) defines the ternary relation $\text{graph}(\sigma) = \{000, 01a, 111\}$. This relation is hom-minimal on 3, so every compatible partial operation with domain $\text{graph}(\sigma)$ already has an extension in $\text{Clo}_{cp}(3_0)$.

Using (2), we can rewrite (4) as

$$\forall uvwxy [(f(x) = u \land g(x) = v) \land f(y) = v \land g(y) = w) \rightarrow \sigma(u, w) = \sigma(u, w)]. \quad (4')$$

Note that the premise of (4') is in the language of $3_0$, so step (2a) of Algorithm 7.9 applies. The premise of (4') defines the 5-ary relation

$$r := \text{pr}_{M_0}(4') = \{00000, 0010a, 011a1, 11111\}.$$

This relation forms a four-element chain, and so there are six homomorphisms from $r$ to 3. Thus there is only one compatible partial operation on 3 with domain $r$ that is not the restriction of a projection. We could just add this 5-ary partial operation to the signature of $3_0$, and we would be done.
But instead, we note from the premise of (4)' that the 5-ary relation $r$ is isomorphic (via a projection) to the binary relation defined by $g(x) = f(y)$, which is $\text{dom}(h) = \{00, 0a, a1, 11\}$. The missing partial operation with domain $r$ is a restriction of $h(\pi_4, \pi_5)$. Since $\text{dom}(h)$ is in $\text{Rel}_{\text{ca}}(3_0)$ and since every compatible partial operation with domain $\text{dom}(h)$ is generated from the projections by $f, g, h$, we can add $h$ to $3_0$ to obtain the familiar alter ego $3_2 := \langle\{0, a, 1\}; f, g, h, \mathcal{F}\rangle = 3_a$.

8. Distinguishing full dualities

In this final section, we clarify the precise sense in which there can be two ‘different’ full dualities based on the same algebra $M$. We first recall the categorical description of structural equivalence; see [15, p. 404].

**Lemma 8.1.** Let $M_1$ and $M_2$ be alter egos of a finite algebra $M$. For $i \in \{1, 2\}$, define $X_i := IS_P(M_i)$. Then the following are equivalent:

1. $M_1$ and $M_2$ are structurally equivalent;
2. there is a concrete category isomorphism $F : X_2 \rightarrow X_1$ such that
   
   (a) $F(M_2) = M_1$, and
   
   (b) both $F$ and $F^{-1}$ preserve embeddings.

Moreover, we can take $F$ to be the natural ‘forgetful’ functor.

Using our transfer set-up from Section 6, we obtain the following similar result.

**Lemma 8.2.** Let $M_1$ and $M_2$ be alter egos of a finite algebra $M$. For $i \in \{1, 2\}$, define $X_i := IS_P(M_i)$. Assume that $M_1$ fully dualises $M$. Then the following are equivalent:

1. $M_2$ fully dualises $M$;
2. there is a concrete category isomorphism $F : X_2 \rightarrow X_1$ such that
   
   (a) $F(M_2) = M_1$, and
   
   (b) $F$ preserves embeddings of the form $X \rightarrow (M_2)^S$, where $X$ is closed under all compatible partial operations on $M$.

Moreover, if $M_1$ is a structural reduct of $M_2$, then we can take $F$ to be the natural ‘forgetful’ functor, and if $M_2$ is a structural reduct of $M_1$, then we can take $F^{-1}$ to be the natural ‘forgetful’ functor.

**Proof.** (2) $\Rightarrow$ (1): Assume that we have $F : X_2 \rightarrow X_1$ as in (2). Let $A := IS_P(M)$ and, for $i \in \{1, 2\}$, let $D_i : A \rightarrow X_i$ and $E_i : X_i \rightarrow A$ denote the hom-functors induced by $M$ and $M_i$.

To show that $M_2$ dualises $M$, let $A \in A$. Then $F$ preserves the embedding $D_2(A)^{\text{incl}} = (M_2)^A$, by (2b), and so $FD_2(A)$ is a substructure of $F((M_2)^A)$. We have $F(M_2) = M_1$, by (2a). Therefore $F((M_2)^A) = (F(M_2))^A = (M_1)^A$, as the concrete category isomorphism $F : X_2 \rightarrow X_1$ must preserve concrete products. Since $F$ preserves underlying sets, we have $|FD_2(A)| = |D_2(A)| = A(A, M)$. We can now conclude that $FD_2(A) = D_1(A)$. Therefore

$$X_2(D_2(A), M_2) = X_1(FD_2(A), F(M_2)) = X_1(D_1(A), M_1),$$

and so $E_2D_2(A) = E_1D_1(A)$. Since $M_1$ dualises $M$, it follows that $M_2$ does too.

For each structure $X \in X_2$, we have $X_2(X, M_2) = X_1(F(X), M_1)$. Hence $M_2$ fully dualises $M$, by the New-from-old Lemma 1.

(1) $\Rightarrow$ (2): Assume that $M_2$ fully dualises $M$. Since $M_1$ also fully dualises $M$, the transfer functors $T_{21} : X_2 \rightarrow X_1$ and $T_{12} : X_1 \rightarrow X_2$ are well defined, using
Lemma 6.10 twice. Thus $T_{21}: \mathcal{X}_2 \to \mathcal{X}_1$ is a concrete category isomorphism, by Lemma 6.9. We have $T_{21}(\mathcal{M}_2) = \mathcal{M}_1$, by Note 5.9, and so $T_{21}$ satisfies (2a).

Now let $\mathcal{X} \subseteq (\mathcal{M}_2)^S$ such that $\mathcal{X}$ is closed under all compatible partial operations on $\mathcal{M}$. Then $\mathcal{X} = F_2(\mathcal{X}^2)$, where $\mathcal{X}^2 \subseteq (\mathcal{M}_1)^S$. Using Lemma 5.12(2), we have

$$T_{21}(\mathcal{X}) = T_{21}F_2(\mathcal{X}^2) = F_1S_2F_3(\mathcal{X}^2) = F_1(\mathcal{X}^2) \subseteq (\mathcal{M}_1)^S.$$ 

Note that $T_{21}(\mathcal{M}_2)^S = (T_{21}(\mathcal{M}_2))^S = (\mathcal{M}_1)^S$. Hence $T_{21}$ satisfies (2b).

If $\mathcal{M}_1$ is a structural reduct of $\mathcal{M}_2$, then the transfer functor $T_{21}$ is the natural ‘forgetful’ functor, by Lemma 5.12(1). Similarly, if $\mathcal{M}_2$ is a structural reduct of $\mathcal{M}_1$, then the inverse transfer functor $T_{12}$ is the natural ‘forgetful’ functor. \qed

Remark 8.3. Within the setting of Boolean structures, the natural definition of ‘embedding’ is ‘isomorphism onto a topologically closed substructure’. We now demonstrate that this is not always categorically expressible.

Our example is based on the quasi-primal algebra $\mathcal{Q}$ from Example 7.8. We know that the two alter egos

$$Q_0 = \langle \{0, a, b, 1\}; \text{graph}(f), J \rangle \quad \text{and} \quad Q_1 = \langle \{0, a, b, 1\}; f, g, J \rangle$$

fully dualise $\mathcal{Q}$. For each $i \in \{0, 1\}$, define the dual category $\mathcal{X}_i := ISP^+(\mathcal{Q}_i)$. By Lemma 5.2, the natural ‘forgetful’ functor $F_i: \mathcal{X}_i \to \mathcal{X}_0$ is a category isomorphism. The inverse category isomorphism $F_i^{-1}: \mathcal{X}_0 \to \mathcal{X}_1$ preserves underlying sets and set-maps, but does not preserve embeddings.

For example, consider the substructure $\mathcal{X}$ of $\mathcal{Q}_0$ with $X := \{a\}$. The inclusion $i: \mathcal{X} \to Q_0$ is an embedding. But the one-to-one morphism $F_i^{-1}(i): F_i^{-1}(\mathcal{X}) \to Q_1$ is not an embedding in the natural sense: its image $\{a\}$ does not form a substructure of $Q_1$, as it is not closed under $f$. (The morphism $F_i^{-1}(i)$ is an embedding in $\mathcal{X}_1$ in the category-theoretic sense; see [1] Definition 8.6.)

In the strong dual category $\mathcal{X}_1$, the embeddings correspond exactly to surjections in the quasivariety $\mathcal{A} = ISP(\mathcal{Q})$. The dual category $\mathcal{X}_0$ has more embeddings.

Comparing Lemmas 8.1 and 8.2, we see that it is the non-categorical nature of embeddings that allows a finite algebra to have truly different full dualities, and thus gives the theory of full dualities its richness.

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