An explicit formula of continuant polynomials for periodic parameters

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MSC classes: 05E05, 11B39, 11J70, 30B70, 33C45

Abstract

We give an explicit formula of continuant polynomials by Chebyshev polynomials of the second kind under the periodicity condition for parameters.

1 Introduction

Let \( a := (a_m)_{m \in \mathbb{Z}} \), \( b := (b_m)_{m \in \mathbb{Z}} \) and \( c := (c_m)_{m \in \mathbb{Z}} \) be infinite complex sequences. For an integer \( p \), we put

\[ \alpha_p := (a_p, b_p, c_p). \]

We define the (extended) continuant polynomials \( K_n(\alpha_p) \) by

\[ K_{-1}(\alpha_p) := 0, \quad K_0(\alpha_p) := 1, \quad K_1(\alpha_p) := a_p, \quad K_n(\alpha_p) := \det A_n(\alpha_p), \]

where \( A_n(\alpha_p) \) is the following \( n \times n \) tridiagonal matrix:

\[
A_n(\alpha_p) = \begin{pmatrix}
a_p & b_p & 0 & \cdots & 0 & 0 \\
c_p & a_{p+1} & b_{p+1} & \cdots & 0 & 0 \\
0 & c_{p+1} & a_{p+2} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & a_{p+n-2} & b_{p+n-2} \\
0 & 0 & 0 & \cdots & c_{p+n-2} & a_{p+n-1}
\end{pmatrix}.
\]

There has been many research on continuant polynomials in relation to continued fraction and orthogonal polynomials. For continuant polynomials, several properties have been well known since Euler, especially regarding some explicit formulas [2]. In this article, for a positive integer \( l \), under the following \( l \)-periodicity condition for the sequences \( a, b \) and \( c \)

\[ a_{p+l} = a_p, \quad b_{p+l} = b_p, \quad c_{p+l} = c_p, \]

we give an explicit formula of \( K_n(\alpha_p) \) by Chebyshev polynomials of the second kind.
2 Preliminaries

Throughout the paper, we denote the ring of rational integers by $\mathbb{Z}$. We set the Gauss hypergeometric function

$$2F_1\left(\frac{a,b}{c}; x\right) := \sum_{m \geq 0} \frac{(a)_m (b)_m}{m! (c)_m} x^m, \quad (a)_m := \begin{cases} a(a+1) \cdots (a+m-1) & (m \neq 0) \\ 1 & (m = 0) \end{cases}.$$ 

Chebyshev polynomial of the second kind is defined by

$$U_n(x) := (n + 1)2F_1\left(\frac{-n, n + 2}{\frac{3}{2}}; \frac{1-x}{2}\right) = \sum_{k=0}^{n} \binom{n}{k} \frac{(n+1)_{k+1}}{(\frac{3}{2})_k} \left(\frac{1-x}{2}\right)^k,$$

$$\binom{n}{k} := \begin{cases} \frac{n(n-1) \cdots (n-k+1)}{k!} & (k \neq 0) \\ 1 & (k = 0) \end{cases}.$$ 

It should be remarked that by the definition of $U_n(x)$ we have

$$U_0(x) = 1, \quad U_{-1}(x) = 0, \quad U_{-2}(x) = -1.$$ 

The generating function for the $U_n(x)$ is

$$\frac{1}{1-2xu+u^2} = \sum_{n \geq 0} U_n(x) u^n. \quad (2.1)$$ 

Let $h_n(x, y)$ denote the bivariate complete homogeneous symmetric polynomials of degree $n$

$$h_n(x, y) := \sum_{i+j=n} x^i y^j = \frac{x^{n+1} - y^{n+1}}{x-y}.$$ 

The generating function for the $h_n(x, y)$ is

$$\frac{1}{(1-xu)(1-yu)} = \sum_{n \geq 0} h_n(x, y) u^n. \quad (2.2)$$ 

By (2.1) and (2.2) we have

$$h_n(x, y) = \begin{cases} (xy)^{\frac{n}{2}} U_n \left(\frac{x+y}{\sqrt{xy}}\right) & (xy \neq 0) \\ (x+y)^n & (xy = 0) \end{cases}. \quad (2.3)$$ 

**Lemma 1.** Let $A$ be a complex matrix

$$A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
and $E_2$ be the $2 \times 2$ identity matrix. For any nonnegative integer $m$, we have
\[
A^m = h_{m-1}(\rho_+, \rho_-) A - h_{m-2}(\rho_+, \rho_-)(\det A) E_2
\]
\[
= \begin{cases} 
\left( \det A \right)^{m-1} U_{m-1} \left( \frac{\text{tr} A}{2 \sqrt{\det A}} \right) A - \left( \det A \right)^m U_{m-2} \left( \frac{\text{tr} A}{2 \sqrt{\det A}} \right) E_2 & (\det A \neq 0) \\
(\text{tr} A)^{m-1} A & (\det A = 0) 
\end{cases}.
\] (2.4) 
(2.5)

Here $\rho_+$ and $\rho_-$ are the roots of the characteristic polynomial $\det (\lambda E_2 - A)$.

**Proof.** We consider Euclidean division for $\lambda^m$ and $\det (\lambda E_2 - A) = (\lambda - \rho_+) (\lambda - \rho_-)$. By the Euclidean theorem for division of polynomials, there exist unique polynomial $q(\lambda)$ and two constants $c_1, c_0$ such that
\[
\lambda^m = q(\lambda)(\lambda - \rho_+) (\lambda - \rho_-) + c_1 \lambda + c_0.
\] (2.6)

By substituting $\rho_\pm$ for $\lambda$ in (2.6), we have
\[
\rho_\pm^m = c_1 \rho_\pm + c_0.
\]

Hence we obtain
\[
c_1 = \frac{\rho_+^m - \rho_-^m}{\rho_+ - \rho_-} = h_{m-1}(\rho_+, \rho_-),
\]
\[
c_0 = -\frac{\rho_+^m \rho_- - \rho_+ \rho_-^m}{\rho_+ - \rho_-} = -h_{m-2}(\rho_+, \rho_-) \det A.
\]

We remark that these expressions hold even if the case of $\rho_+ = \rho_-$. □

**Lemma 2** (Fundamental properties of continuant polynomials). (1)
\[
K_{-1}(\alpha_p) := 0, \quad K_0(\alpha_p) := 1, \quad K_n(\alpha_p) = a_p K_{n-1}(\alpha_{p+1}) - b_p c_p K_{n-2}(\alpha_{p+2}).
\] (2.7)

(2) Let
\[
L(\alpha, \beta) := \begin{pmatrix} \alpha & \beta \\ 1 & 0 \end{pmatrix},
\]
\[
A_n(\alpha_p) := L(a_p, -b_p c_p) L(a_{p+1}, -b_{p+1} c_{p+1}) \cdots L(a_{p+n-1}, -b_{p+n-1} c_{p+n-1}).
\] (2.8)

We have
\[
A_n(\alpha_p) = \begin{pmatrix} K_n(\alpha_p) & -b_{p+n-1} c_{p+n-1} K_{n-1}(\alpha_p) \\ K_{n-1}(\alpha_{p+1}) & -b_{p+n-1} c_{p+n-1} K_{n-2}(\alpha_{p+1}) \end{pmatrix}.
\] (2.9)

Especially
\[
\text{tr} A_n(\alpha_p) = K_n(\alpha_p) - b_{p+n-1} c_{p+n-1} K_{n-2}(\alpha_{p+1}), \quad \det A_n(\alpha_p) = \prod_{j=1}^n b_{p+j-1} c_{p+j-1}.
\] (2.10)
(3) Put
\[ k_{n+1}(\alpha_p) := \begin{pmatrix} K_{n+1}(\alpha_p) \\ K_n(\alpha_{p+1}) \end{pmatrix}. \]
For any integer \( m \) such that \( n \geq m \), we have
\[ k_{n+1}(\alpha_p) = A_m(\alpha_p)k_{n+1-m}(\alpha_{p+m}). \]

(4) If for any integer \( n \ c_n = -1 \), then we have
\[ a_p + \frac{n-1}{i=1} b_{p+i-1} \frac{K_i}{a_{p+i}} = \frac{K_n(\alpha_p)}{K_{n-1}(\alpha_{p+1})}, \]
where
\[ a_p + \frac{n-1}{i=1} b_{p+i-1} \frac{K_i}{a_{p+i}} := a_p + \frac{b_p}{a_{p+1} + \frac{b_{p+1}}{a_{p+2} + \frac{b_{p+2}}{a_{p+3} + \cdots + \frac{b_{p+n-2}}{a_{p+n-1}}}}}. \]

Proof. (1) It follows from the definition of \( K_n(\alpha_p) \).
(2) When \( n = 1 \), (2.9) holds. Assume the result true for \( n \). From induction on \( n \) and (2.7), we have
\[ A_{n+1}(\alpha_p) = L(a_p, -b_p c_p)A_n(\alpha_{p+1}) \]
\[ = \begin{pmatrix} a_p & -b_p c_p \\ 1 & 0 \end{pmatrix} \begin{pmatrix} K_n(\alpha_{p+1}) & -b_p c_p K_{n-1}(\alpha_{p+1}) \\ K_{n-1}(\alpha_{p+2}) & -b_p c_p K_{n-2}(\alpha_{p+2}) \end{pmatrix} \]
\[ = \begin{pmatrix} a_p K_n(\alpha_{p+1}) - b_p c_p K_{n-1}(\alpha_{p+2}) & -b_p c_p K_{n-1}(\alpha_{p+1}) \\ K_n(\alpha_{p+1}) & -b_p c_p K_{n-1}(\alpha_{p+1}) \end{pmatrix}. \]
(3) By the definition of \( K_n(\alpha_p) \) and (2.7),
\[ k_{n+1}(\alpha_p) = \begin{pmatrix} K_{n+1}(\alpha_p) \\ K_n(\alpha_{p+1}) \end{pmatrix} = \begin{pmatrix} a_p K_n(\alpha_{p+1}) - b_p c_p K_{n-1}(\alpha_{p+2}) \\ K_n(\alpha_{p+1}) \end{pmatrix} = L(a_p, -b_p c_p)k_n(\alpha_{p+1}). \]
Hence
\[ k_{n+1}(\alpha_p) = L(a_p, -b_p c_p)L(a_{p+1}, -b_{p+1} c_{p+1}) \cdots L(a_{p+m-1}, -b_{p+m-1} c_{p+m-1})k_{n+1-m}(\alpha_{p+m}) \]
\[ = A_m(\alpha_p)k_{n+1-m}(\alpha_{p+m}). \]
(4) It follows from (2.7) and induction on \( n \).
3 Main results

Under the following we assume

\[ a_{p+t} = a_p, \quad b_{p+t} = b_p, \quad c_{p+t} = c_p \quad (p \in \mathbb{Z}), \]

that is to say

\[ \alpha_{p+l} = \alpha_p. \quad (3.1) \]

**Theorem 3.** For any nonnegative integer \( m \), we obtain

\[
K_{lm+1}(\alpha_p) = \begin{cases} 
\left( \det A_l(\alpha_p) \right)^{m-1} U_{m-1} \left( \frac{\text{tr} A_l(\alpha_p)}{2\sqrt{\det A_l(\alpha_p)}} \right) K_l(\alpha_p) \\
- \left( \det A_l(\alpha_p) \right)^{m-2} U_{m-2} \left( \frac{\text{tr} A_l(\alpha_p)}{2\sqrt{\det A_l(\alpha_p)}} \right) K_{l-1}(\alpha_{p+1}) & (\det A_l(\alpha_p) \neq 0)
\end{cases},
\]

\[
K_{lm}(\alpha_{p+1}) = \begin{cases} 
\left( \det A_l(\alpha_p) \right)^{m-1} U_{m-1} \left( \frac{\text{tr} A_l(\alpha_p)}{2\sqrt{\det A_l(\alpha_p)}} \right) K_{l-1}(\alpha_{p+1}) & (\det A_l(\alpha_p) \neq 0)
\end{cases}.
\]

**Proof.** By (2.11) and periodicity (3.1)

\[ k_{lm+1}(\alpha_p) = A_l(\alpha_p) k_{l(m-1)+1}(\alpha_{p+l}) = A_l(\alpha_p) k_{l(m-1)+1}(\alpha_p). \]

Then we have

\[ k_{lm+1}(\alpha_p) = A_l(\alpha_p)^m k_1(\alpha_p). \]

When

\[ \det A_l(\alpha_p) = \prod_{j=1}^l b_{p+j-1}c_{p+j-1} \neq 0, \]

from (2.4) we have

\[ A_l(\alpha_p)^m = \left( \det A_l(\alpha_p) \right)^{m-1} U_{m-1} \left( \frac{\text{tr} A_l(\alpha_p)}{2\sqrt{\det A_l(\alpha_p)}} \right) A_l(\alpha_p) \]

\[ - \left( \det A_l(\alpha_p) \right)^{m-2} U_{m-2} \left( \frac{\text{tr} A_l(\alpha_p)}{2\sqrt{\det A_l(\alpha_p)}} \right) E_2. \]

If \( \det A_l(\alpha_p) = 0 \), then

\[ A_l(\alpha_p)^m = (\text{tr} A_l(\alpha_p))^{m-1} A_l(\alpha_p). \]
Finally, by (2.9)

\[
A_l(\alpha_p) = \begin{pmatrix}
K_l(\alpha_p) & -b_{p+l-1}c_{p+l-1}K_{l-1}(\alpha_p) \\
K_{l-1}(\alpha_{p+1}) & -b_{p+l-1}c_{p+l-1}K_{l-2}(\alpha_{p+1})
\end{pmatrix}
\]

By comparing the entries of the vector \( k_{lm+1}(\alpha_p) \), we obtain the conclusion.

Our main result follows from this theorem immediately.

**Corollary 4.** For \( j = 0, 1, \ldots, l - 1 \), we have

\[
K_{lm+j}(\alpha_{p+1-j}) = K_{j-1}(\alpha_{p-j+1})K_{lm+1}(\alpha_p) - b_{p-1}c_{p-1}K_{j-2}(\alpha_{p-j+1})K_{lm}(\alpha_{p+1})
\]

\[
= \begin{cases}
(det A_l(\alpha_p))\frac{m-1}{2}U_{m-1} \left( \frac{tr A_l(\alpha_p)}{2\sqrt{det A_l(\alpha_p)}} \right) \\
\cdot (K_{j-1}(\alpha_{p-j+1})K_l(\alpha_p) - b_{p-1}c_{p-1}K_{j-2}(\alpha_{p-j+1})K_{l-1}(\alpha_{p+1})) \\
- (det A_l(\alpha_p))\frac{m}{2}U_{m-2} \left( \frac{tr A_l(\alpha_p)}{2\sqrt{det A_l(\alpha_p)}} \right) K_{j-1}(\alpha_{p-j+1}) \\
(tr A_l(\alpha_p))^{m-1} \\
\cdot (K_{j-1}(\alpha_{p-j+1})K_l(\alpha_p) - b_{p-1}c_{p-1}K_{j-2}(\alpha_{p-j+1})K_{l-1}(\alpha_{p+1})) \quad (det A_l(\alpha_p) \neq 0) \\
\cdot (K_{j-1}(\alpha_{p-j+1})K_l(\alpha_p) - b_{p-1}c_{p-1}K_{j-2}(\alpha_{p-j+1})K_{l-1}(\alpha_{p+1})) \quad (det A_l(\alpha_p) = 0)
\end{cases}
\]

(3.3)

Here we define \( K_{-2}(\alpha_{p+1}) \) by

\[-b_{p-1}c_{p-1}K_{-2}(\alpha_{p+1}) := K_0(\alpha_{p-1}) = 1.\]

**Proof.** From (2.11) and (2.9), we have

\[
k_{lm+j}(\alpha_{p-j}) = A_j(\alpha_{p-j})k_{lm+1}(\alpha_p)
\]

\[
= \begin{pmatrix}
K_j(\alpha_{p-j}) & -b_{p-1}c_{p-1}K_{j-1}(\alpha_{p-j}) \\
K_{j-1}(\alpha_{p-j+1}) & -b_{p-1}c_{p-1}K_{j-2}(\alpha_{p-j+1})
\end{pmatrix} \begin{pmatrix} K_{lm+1}(\alpha_p) \\ K_{lm}(\alpha_{p+1}) \end{pmatrix}
\]

(3.4)

By (3.2) and comparing the entries of (3.4), we obtain our main result (3.3).

**4 Examples**

In this section, we give the examples of (3.2) for \( l = 1, 2, 3 \) explicitly.
4.1 \( l = 1 \)

In this subsection, we put
\[
a := a_p = a_{p+1}, \quad b := b_p = b_{p+1}, \quad c := c_p = c_{p+1}.
\]

In the case of \( l = 1 \), since
\[
\text{tr } A_1(\alpha_p) = a, \quad \text{det } A_1(\alpha_p) = bc,
\]
we have the following well-known result:
\[
K_{m}(\alpha_{p+1}) = \begin{cases} 
(bc)^{\frac{m-1}{2}} U_{m-1} \left( \frac{a}{2\sqrt{bc}} \right) & (bc \neq 0) \\
\frac{1}{a^{m-1}} & (bc = 0)
\end{cases}.
\] (4.1)

4.2 \( l = 2 \)

In this subsection, we put
\[
a_1 := a_{2m+1} = a_{2m+3}, \quad b_1 := b_{2m+1} = b_{2m+3}, \quad c_1 := c_{2m+1} = c_{2m+3},
\]
\[
a_2 := a_{2m} = a_{2m+2}, \quad b_2 := b_{2m} = b_{2m+2}, \quad c_2 := c_{2m} = c_{2m+2}.
\]

Since
\[
\text{tr } A_2(\alpha_p) = a_p a_{p+1} - b_p c_p - b_{p+1} c_{p+1}, \quad \text{det } A_1(\alpha_p) = b_1 c_1 b_2 c_2,
\]
our main result (3.3) is
\[
K_{2m+1}(\alpha_p) = \begin{cases} 
(b_1 c_1 b_2 c_2)^{\frac{m-1}{2}} U_{m-1} \left( \frac{a_p a_{p+1} - b_p c_p - b_{p+1} c_{p+1}}{2\sqrt{b_1 c_1 b_2 c_2}} \right) (a_p a_{p+1} - b_p c_p) & (b_1 c_1 b_2 c_2 \neq 0), \\
-(b_1 c_1 b_2 c_2)^{\frac{m-1}{2}} U_{m-2} \left( \frac{a_p a_{p+1} - b_p c_p - b_{p+1} c_{p+1}}{2\sqrt{b_1 c_1 b_2 c_2}} \right) (a_p a_{p+1} - b_p c_p) & (b_1 c_1 b_2 c_2 = 0)
\end{cases}.
\]
\[
K_{2m}(\alpha_{p+1}) = \begin{cases} 
(b_1 c_1 b_2 c_2)^{\frac{m-1}{2}} U_{m-1} \left( \frac{a_p a_{p+1} - b_p c_p - b_{p+1} c_{p+1}}{2\sqrt{b_1 c_1 b_2 c_2}} \right) a_{p+1} & (b_1 c_1 b_2 c_2 \neq 0), \\
(a_p a_{p+1} - b_p c_p - b_{p+1} c_{p+1})^{m-1} a_{p+1} & (b_1 c_1 b_2 c_2 = 0)
\end{cases}.
\] (4.2)

Example 5 (A \( q \)-analogue of Fibonacci numbers). Morier-Genoud and Ovsienko \([1]\) introduced the following notion of \( q \)-deformed rational numbers and continued fractions, motivated by Jones polynomials of rational knots or \( F \)-polynomials of a cluster algebra. For a positive rational number \( \frac{r}{s} \) and its (regular) continued fraction
\[
\frac{r}{s} = a_1 + \frac{2n-1}{K_{i=1} \frac{1}{a_{i+1}}}, \quad a_1, \ldots, a_{2n} > 0,
\]
their \( q \)-analogue are defined by
\[
\left[ \frac{r}{s} \right]_q := [a_1]_q + \sum_{i=1}^{2n-1} q^{(-1)^i a_i} [a_{i+1}]_q,
\]
where \( q \neq 0 \) is a complex parameter and

\[
[a]_q := \frac{1 - q^a}{1 - q}.
\]

Our formulas (3.3) or (4.2) are useful to write down these \( q \)-analogue explicitly. Here we consider Fibonacci numbers defined by

\[
F_1 = 1, \quad F_2 = 1, \quad F_{n+2} = F_{n+1} + F_n
\]
as an example. As is well known, the continued fraction of \( \frac{F_{2n+1}}{F_{2n}} \) is given by

\[
\frac{F_{2n+1}}{F_{2n}} = 1 + \frac{2n-1}{1 + \frac{1}{K_{i=1}^{n-1}}},
\]

Thus, a \( q \)-analogue of this rational number and continued fraction expansion are equal to

\[
\left[ \frac{F_{2n+1}}{F_{2n}} \right]_q = 1 + \frac{2n-1}{1 + \frac{1}{K_{i=1}^{n-1} q^{(-1)^{i-1}}}}.
\]

We put

\[
a_1 = a_2 = 1, \quad b_i = q^{(-1)^{i-1}}, \quad c_1 = c_2 = -1
\]

and

\[
F_{2m+2}(q) := K_{2m+1}(\alpha_2), \quad F_{2m+1}(q) := K_{2m}(\alpha_1).
\]

From Lemma 2 (4), we have

\[
\left[ \frac{F_{2n+1}}{F_{2n}} \right]_q = \frac{F_{2n+1}(q)}{F_{2n}(q)}.
\]

This sequence \( \{F_n(q)\} \) satisfies

\[
F_1(q) = 1, \quad F_2(q) = 1, \quad F_3(q) = 1 + q
\]

and

\[
F_{2m}(q) = F_{2m-1}(q) + q^{-1}F_{2m-2}(q), \quad F_{2m+1}(q) = F_{2m}(q) + qF_{2m-1}(q).
\]

From (4.2), \( \{F_n(q)\} \) has the following explicit expression:

\[
F_{2m+2}(q) = U_{m-1} \left( \frac{1 + q + q^{-1}}{2} \right) (1 + q^{-1}) - U_{m-2} \left( \frac{1 + q + q^{-1}}{2} \right)
\]

\[
F_{2m+2}(q) = U_{m-1} \left( \frac{1 + q + q^{-1}}{2} \right).
\]
4.3 \( l = 3 \)

Put

\[
a_1 := a_{3m+1} = a_{3m+4}, \quad b_1 := b_{3m+1} = b_{3m+4}, \quad c_1 := c_{3m+1} = c_{3m+4},
\]

\[
a_2 := a_{3m+2} = a_{3m+5}, \quad b_2 := b_{3m+2} = b_{3m+5}, \quad c_2 := c_{3m+2} = c_{3m+5},
\]

\[
a_3 := a_{3m} = a_{3m+3}, \quad b_3 := b_{3m} = b_{3m+3}, \quad c_3 := c_{3m} = c_{3m+3}.
\]

From (2.10), we have

\[
\operatorname{tr} A_3(\alpha_p) = a_p a_{p+1} a_{p+2} - a_p b_{p+1} c_{p+1} - a_p b_{p+2} c_{p+2} - a_{p+2} b_p c_p, \quad \det A_1(\alpha_p) = \prod_{j=1}^3 b_j c_j.
\]

Then (3.3) can be written as

\[
K_{3m+2}(\alpha_{p-1}) = a_{p-1} K_{3m+1}(\alpha_p) - b_{p-1} c_{p-1} K_{3m}(\alpha_{p+1})
\]

\[
= \left\{ \begin{array}{l}
\prod_{j=1}^3 (b_j c_j)^{m-1} U_{m-1} \left( \frac{a_p a_{p+1} a_{p+2} - a_p b_{p+1} c_{p+1} - a_p b_{p+2} c_{p+2} - a_{p+2} b_p c_p}{2 \sqrt{b_1 c_1 b_2 c_2 b_3 c_3}} \right) \\
\cdot (a_{p-1} a_p a_{p+1} a_{p+2} - a_{p-1} a_p b_{p+1} c_{p+1} - a_{p-1} a_p b_{p+2} c_{p+2} - a_{p+2} b_p c_p)^{m-1} \\
- \prod_{j=1}^3 (b_j c_j)^{m-1} U_{m-2} \left( \frac{a_p a_{p+1} a_{p+2} - a_p b_{p+1} c_{p+1} - a_p b_{p+2} c_{p+2} - a_{p+2} b_p c_p}{2 \sqrt{b_1 c_1 b_2 c_2 b_3 c_3}} \right) a_{p-1} \\
\end{array} \right. \tag{3.3}
\]

\[
K_{3m+1}(\alpha_p)
\]

\[
= \left\{ \begin{array}{l}
\prod_{j=1}^3 (b_j c_j)^{m-1} U_{m-1} \left( \frac{a_p a_{p+1} a_{p+2} - a_p b_{p+1} c_{p+1} - a_p b_{p+2} c_{p+2} - a_{p+2} b_p c_p}{2 \sqrt{b_1 c_1 b_2 c_2 b_3 c_3}} \right) \\
\cdot (a_p a_{p+1} a_{p+2} - a_{p+1} a_p b_{p+1} c_{p+1} - a_{p+1} a_p b_{p+2} c_{p+2} - a_{p+2} b_p c_p)^{m-1} \\
- \prod_{j=1}^3 (b_j c_j)^{m-1} U_{m-2} \left( \frac{a_p a_{p+1} a_{p+2} - a_p b_{p+1} c_{p+1} - a_p b_{p+2} c_{p+2} - a_{p+2} b_p c_p}{2 \sqrt{b_1 c_1 b_2 c_2 b_3 c_3}} \right) a_{p+1} \\
\end{array} \right. \tag{3.3}
\]

\[
K_{3m}(\alpha_{p+1})
\]

\[
= \left\{ \begin{array}{l}
\prod_{j=1}^3 (b_j c_j)^{m-1} U_{m-1} \left( \frac{a_p a_{p+1} a_{p+2} - a_p b_{p+1} c_{p+1} - a_p b_{p+2} c_{p+2} - a_{p+2} b_p c_p}{2 \sqrt{b_1 c_1 b_2 c_2 b_3 c_3}} \right) a_{p+1} \\
\cdot (a_{p+1} a_p a_{p+2} - a_p b_{p+1} c_{p+1} - a_p b_{p+2} c_{p+2} - a_{p+2} b_p c_p)^{m-1} \\
- \prod_{j=1}^3 (b_j c_j)^{m-1} U_{m-2} \left( \frac{a_p a_{p+1} a_{p+2} - a_p b_{p+1} c_{p+1} - a_p b_{p+2} c_{p+2} - a_{p+2} b_p c_p}{2 \sqrt{b_1 c_1 b_2 c_2 b_3 c_3}} \right) a_{p+1} \\
\end{array} \right. \tag{3.3}
\]

**References**

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