SMALL DATA SCATTERING OF SEMIRELATIVISTIC HARTREE EQUATION

CHANGHUN YANG

Abstract. In this paper we study the small data scattering of Hartree type semirelativistic equation in space dimension 3. The Hartree type nonlinearity is $|V * |u|^2|u$ and the potential $V$ which generalizes the Yukawa has some growth condition. We show that the solution scatters to linear solution if an initial data given in $H^{s,1}$ is sufficiently small and $s > \frac{1}{4}$. Here, $H^{s,1}$ is Sobolev type space taking in angular regularity with norm defined by $\|\varphi\|_{H^{s,1}} = \|\varphi\|_{H^s} + \|\nabla_S \varphi\|_{H^s}$. To establish the results we employ the recently developed Strichartz estimate which is $L^2$-averaged on the unit sphere $S^2$ and construct the resolution space based on $U^p$-space.

1. Introduction

In this paper we consider the following Cauchy problem:

\begin{equation}
\begin{aligned}
&i\partial_t u = \Lambda_m u + F(u) \text{ in } \mathbb{R}^{1+3}, \\
&u(x,0) = \varphi(x) \text{ in } \mathbb{R}^3,
\end{aligned}
\end{equation}

where $\Lambda_m$ is the fourier multiplier defined by $\Lambda_m = (m - \Delta)^{\frac{1}{2}}$ and $F(u)$ is nonlinear term of Hartree type such that $F(u) = |V * |u|^2|u$ with a smooth $V$ in $\mathbb{R}^3 \setminus \{0\}$. Here $m > 0$ is mass and * denotes the convolution in $\mathbb{R}^3$. The concerned Hartree potential is defined as follows:

Definition 1.1. For $0 \leq \gamma_1, \gamma_2 < 3$ the potential $V$ is said to be of type $(\gamma_1, \gamma_2)$ if it satisfies the growth condition such that $\tilde{V} \in C^4(\mathbb{R}^3 \setminus \{0\})$ and for $0 \leq k \leq 4$

\begin{equation}
|\nabla^k \tilde{V}(\xi)| \lesssim |\xi|^{-\gamma_1-k} \text{ for } |\xi| \leq 1, \quad |\nabla^k \tilde{V}(\xi)| \lesssim |\xi|^{-\gamma_2-k} \text{ for } |\xi| > 1.
\end{equation}

The Coulomb potential $V(x) = |x|^{-1}$ is of such type corresponding to $\gamma_1 = \gamma_2 = 2$ and the Yukawa potential $V(x) = e^{-\mu_0|x|} |x|^{-1}$, $\mu_0 > 0$ is corresponding to $\gamma_1 = 0, \gamma_2 = 2$. The equation (1.1) with these two potentials, which is called semirelativistic Hartree equation, arises in the mean-field limit of large systems of bosons, see, e.g., [10, 11, 18]. In this paper we study (1.1) with the above generalized potentials.

By Duhamel’s formula, (1.1) is written as an integral equation

\begin{equation}
u = e^{-it\Lambda_m} \varphi - i \int_0^t e^{-i(t-s)\Lambda_m} F(u)(s) \, ds.
\end{equation}

Here we define the linear propagator $e^{-it\Lambda_m}$ given by the solution to the linear problem $i\partial_t v = \Lambda_m v$ with initial datum $v(0) = \varphi$. It is formally written by

\begin{equation}
e^{-it\Lambda_m} \varphi = \mathcal{F}^{-1} \left( e^{-i\sqrt{m+|\xi|^2}} \mathcal{F}(\varphi) \right) = (2\pi)^{-3} \int_{\mathbb{R}^3} e^{i(x - \xi - t\sqrt{m+|\xi|^2})} \varphi(\xi) \, d\xi.
\end{equation}

2010 Mathematics Subject Classification. 35Q55, 35Q53.

Key words and phrases. semirelativistic Hartree equation, Yukawa type potential, small data scattering, angularly averaged Strichartz estimate, $U^p$ and $V^p$ spaces.
The purpose of this research is to study the existence and uniqueness of solutions and observe the behaviour of solutions as time goes to infinity, in particular comparing them with the linear solutions, which is well-known in area of dispersive PDE as the well-posedness and scattering problem respectively. The following is formal definition of scattering.

**Definition 1.2.** We say that a solution $u$ to \((1.1)\) scatters (to $u_{\pm}$) in a Hilbert space $\mathcal{H}$ if there exist $\varphi_{\pm} \in \mathcal{H}$ (with $u_{\pm}(t) = e^{-it\Lambda_{\infty}} \varphi_{\pm}$) such that $\lim_{t \to \pm \infty} \|u(t) - u_{\pm}\|_{\mathcal{H}} = 0$.

One of candidates for Hilbert space $\mathcal{H}$ is the Sobolev spaces $H^s(\mathbb{R}^3)$. That is, we will show the well-posedness and scattering results when the initial data is given in $H^s(\mathbb{R}^3)$. Especially we want to find the minimum value of $s$ that ensures the scattering states of corresponding solutions, which is called low regularity problem.

There have been a lot of results on this subject. Firstly, Lenzmann [17] established the global existence of solutions for Yukawa type potential using energy methods provided the initial data given in $H^{1/2}(\mathbb{R}^3)$ is sufficiently small. Herr and Lenzmann [13] showed that for Coulomb type potential the almost optimal local well-posedness holds for initial data with $s > 1/4$ (and $s > 0$ if the data is radially symmetric) using localized Strichartz estimates and the Bourgain spaces. In [3, 4, 5, 6], they considered the generalized potential from Coulomb type, namely, $V(x) = |x|^{-\gamma}$, for $0 < \gamma < 3$ (corresponding to $\gamma_1 = \gamma_2 = 3 - \gamma$ in our definition) and investigated well-posedness and scattering of equations. The most recent results on the Yukawa potential were obtained by Herr and Tesfahun [15] where they showed the small data scattering result for initial data with $s > 1/2$ (and $s > 0$ if the data is radially symmetric) using $U^p - V^p$ spaces method which has proved effective to derive scattering result.

In this paper we consider the range $0 < s \leq 1/2$ where the scattering result has been proved only when radial assumption is given to initial data [15] and aim to obtain the similar result with a weaker assumption. We prove the scattering result when $s > 1/4$ by imposing additional one angular regularity to the initial data. Let us introduce angular derivative and angularly regular Sobolev space. The spherical gradient $\nabla_{\mathbb{S}}$ is restriction of the gradient on the unit sphere which is well-defined, that is, independent of coordinates of $\mathbb{S}^2$. It satisfies a following relation

$$\nabla = \frac{1}{r} \nabla_{\mathbb{S}} + \theta \frac{\partial}{\partial r}, \quad x = r\theta, \quad \theta \in \mathbb{S}^2,$$

and also has a concrete formula $\nabla_{\mathbb{S}} = x \times \nabla$. A function space $H^{s,1}$ is the set of all $H^s$ functions whose angular derivative is also in $H^s$. The norm is defined by $\|f\|_{H^{s,1}} := \|f\|_{H^s} + \|\nabla_{\mathbb{S}} f\|_{H^s}$. It contains all radially symmetric functions.

Our main result is the following.

**Theorem 1.3.** Let $s > 1/4$. Suppose the potential $V$ in \((1.1)\) is radially symmetric and of type $(\gamma_1, \gamma_2)$ with $0 \leq \gamma_1 < 1$ and $1/2 < \gamma_2 < 3$. Then there exists $\delta > 0$ such that for any $\varphi \in H^{s,1}$ with $\|\varphi\|_{H^{s,1}} \leq \delta$, \((1.1)\) has a unique solution $u \in (C \cap L^\infty)(\mathbb{R};H^{s,1})$ which scatters in $H^{s,1}$.

**Remark 1.** The potential in Theorem 1.3 includes the Yukawa. Concerning the Coulomb potential, non-existence scattering results [1] and modified scattering results [14] have been established.

Our proof is fundamentally based on fixed point argument and Littlewood-Paley decomposition. In order to occur a contraction, we use frequency-localized spherical Strichartz estimates and
construct a resolution space using $U^p, V^p$ spaces where linear estimates for free solutions could be transferred.

We have an application to the following Hartree Dirac equations:

\[
\begin{align*}
\left\{ 
\begin{array}{l}
\alpha \cdot D + m\beta \psi &= [V * |\psi|^2]\psi, \\
\psi(x,0) &= \psi_0(x), 
\end{array}
\right. 
\quad \text{in } \mathbb{R}^{1+3},
\end{align*}
\]

(1.5)

where $D = -i\nabla, \psi : \mathbb{R}^{1+3} \to \mathbb{C}^4$ is the Dirac spinor, $m > 0$ is mass and $\beta$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ are the Dirac matrices. If $V$ is Coulomb potential, (1.5) appears when Maxwell-Dirac system with zero magnetic field is uncoupled [2]. And in the same paper [2] it is conjectured that (1.5) with Yukawa also might be obtained by uncoupling Dirac-Klein-Gordon system as Maxwell-Dirac case. For more information about Dirac equation, see [2] and references therein.

Following [3] (see also [4]) we introduce the projection operators $\Pi^m_{\pm}(D)$ with symbol

\[
\Pi^m_{\pm}(\xi) = \frac{1}{2}[I \pm \frac{1}{(\xi \cdot \alpha + m\beta)}],
\]

We then define $\psi_{\pm} := \Pi^m_{\pm}(D)\psi$ and split $\psi = \psi_{+} + \psi_{-}$. By applying the operators $\Pi^m_{\pm}(D)$ to the equation (1.5), and using the identity

\[
\alpha \cdot D + m\beta = \Lambda_m(\Pi^m_{\pm}(D) - \Pi^m(D))
\]

we obtain the following system of equations

\[
\begin{align*}
\left\{ 
\begin{array}{l}
(-i\partial_t + \Lambda_m)\psi_+ &= \Pi^m_{\pm}(D)(V * |\psi|^2)\psi, \\
(-i\partial_t - \Lambda_m)\psi_- &= \Pi^m_{\pm}(D)(V * |\psi|^2)\psi,
\end{array}
\right. 
\end{align*}
\]

(1.6)

with initial data $\psi_{\pm}^0 = \Pi^m_{\pm}(D)\psi_0$. Observe that the linear propagators for this system have same formula as (1.4) except for sign and the nonlinear term is also same except for projection operator. Note that Strichartz estimates holds regardless of sign and the Sobolev norm has an equivalence under this operator, i.e., $\|\Pi^m_{\pm}(D)f\|_{H^s} + \|\Pi^m_{\pm}(D)f\|_{H^s} \sim \|f\|_{H^s}$. Since our proof for Theorem 1.3 does not require any structure of equation but relies on Strichartz estimates, function spaces and Littlewood-Paley decomposition, one can easily check the following Corollary:

**Corollary 1.4.** Let $s > \frac{1}{4}$. Suppose the potential $V$ in (1.1) is radially symmetric and of type $(\gamma_1, \gamma_2)$ with $0 \leq \gamma_1 < 1$ and $\frac{2}{3} < \gamma_2 < 3$. Then there exists $\delta > 0$ such that for any $\psi_0 \in H^{s,1}$ with $\|\psi_0\|_{H^{s,1}} \leq \delta$, (1.5) has a unique solution $\psi \in (C \cap L^\infty)(\mathbb{R}; H^{s,1})$ which scatters in $H^{s,1}$.

2. **Notations and Preliminaries**

2.1. **Notations.** The Fourier transform of $f$ is denoted by $\hat{f} = \mathcal{F}(f)$ and the inverse Fourier transform is by $\mathcal{F}^{-1}$ such that

\[
\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^3} e^{-ix \cdot \xi} f(x) \, dx, \quad \mathcal{F}^{-1}(g)(x) = (2\pi)^{-3} \int_{\mathbb{R}^3} e^{ix \cdot \xi} g(\xi) \, d\xi.
\]

We denote the frequency variables by capital letters $M, N > 0$ which is assumed dyadic number, that is of the form $2^m, 2^n$ with $m, n \in \mathbb{Z}$. Let $\rho \in C^\infty_{0,rad}(-2,2)$ be such that $\rho(s) = 1$ if $|s| < 1$ and $\chi_M$ be defined by $\chi_M(s) = \rho(\frac{s}{M}) - \rho(\frac{M}{s})$ for $M > 0$. Then supp $\chi_M = \{ s \in \mathbb{R} : \frac{M}{2} < |s| < 2M \}$. Fix $N_0 > 1$ and let $\beta_{N_0} := \sum_{M \leq N_0} \chi_M$ and $\beta_N := \chi_N$ for $N > N_0$. Then supp $\beta_{N_0} = \{ s \in \mathbb{R} : |s| < 2N_0 \}$ and supp $\beta_N = \text{supp } \chi_N$ for $N > N_0$. Denote $\bar{\chi}_M(s) := \chi_M(s/2) + \chi_M(s) + \chi_M(2s)$
Lemma 2.3. Let all functions \( \tilde{u} \) and \( \tilde{v} \) with the convention of Lemma 2.1. The properties in Lemma 2.1 also hold for the spaces \( U_{p} \) and \( V_{p} \).

2.2. Function spaces. In this subsection we introduce the \( U_{p}, V_{p} \) function spaces. For the general theory, see e.g. [14], [16].

Let \( 1 \leq p < \infty \). We call a finite set \( \{t_{0}, \ldots, t_{J}\} \) a partition if \( -\infty < t_{0} < t_{1} < \cdots < t_{J} \leq \infty \), and denote the set of all partitions by \( \mathcal{T} \). A corresponding step-function \( a : \mathbb{R} \rightarrow L^{2}(\mathbb{R}^{3}) \) is called \( U_{p} \)-atom if

\[
a(t) = \sum_{j=1}^{J} 1_{(t_{j-1}, t_{j})}(t) f_{j}, \quad \sum_{j=1}^{J} \|f_{j}\|_{L^{2}(\mathbb{R}^{3})}^{p} = 1, \quad \{t_{0}, \ldots, t_{J}\} \in \mathcal{T},
\]

and \( U_{p} \) is the atomic space. The norm is defined by

\[
\|u\|_{U_{p}} := \inf \left\{ \sum_{k=1}^{\infty} |\lambda_{k}| : u = \sum_{k=1}^{\infty} \lambda_{k} a_{k}, \text{ where } a_{k} \text{ are } U_{p} \text{-atoms and } \lambda_{k} \in \mathbb{C} \right\}.
\]

Further, let \( V_{p} \) be the space of all right-continuous \( v : \mathbb{R} \rightarrow L^{2}(\mathbb{R}^{3}) \) satisfying

\[
\|v\|_{V_{p}} := \sup_{\{t_{0}, \ldots, t_{J}\} \in \mathcal{T}} \left( \sum_{j=1}^{J} \|v(t_{j}) - v(t_{j-1})\|_{L^{2}(\mathbb{R}^{3})}^{p} \right)^{\frac{1}{p}}.
\]

with the convention \( v(t_{J}) = 0 \) if \( t_{J} = \infty \). Likewise, let \( V_{p}^{rc} \) denote the spaces of all functions \( v : \mathbb{R} \rightarrow L^{2}(\mathbb{R}^{3}) \) satisfying \( v(-\infty) = 0 \) and \( \|v\|_{V_{p}} < \infty \), equipped with the norm (2.1). We define \( V_{p, rc} \) by the closed subspace of all right continuous \( V_{p} \) functions.

Now we list some useful Lemmas on \( U_{p}, V_{p} \) spaces.

Lemma 2.1. Let \( 1 \leq p < q < \infty \).

1. \( U_{p}, V_{p}, V_{p}^{rc} \) is Banach spaces.
2. The embeddings \( U_{p} \hookrightarrow V_{p, rc} \hookrightarrow U_{q} \hookrightarrow L^{\infty}(\mathbb{R}; L^{2}) \) are continuous.
3. The embeddings \( V_{p} \hookrightarrow V_{q} \) and \( V_{p}^{rc} \hookrightarrow V_{q}^{rc} \) are continuous.
4. (Duality) For \( 1 < p < \infty \), \( \|u\|_{U_{p}} = \sup_{\{v \in V_{p}^{rc} : \|v\|_{V_{p}^{rc}} \leq 1\}} \int_{-\infty}^{\infty} (u'(t), v(t))_{L^{2}} dt. \)

Definition 2.2 (Adapted function spaces). We define \( U_{m}^{p} \) (and \( V_{m}^{p} \) respectively) by the spaces of all functions \( u \) such that \( e^{it\Lambda_{m}} u \in U_{p} \) (\( e^{it\Lambda_{m}} v \in V_{p} \) respectively) with the norm

\[
\|u\|_{U_{m}^{p}} := \|e^{it\Lambda_{m}} u\|_{U_{p}}, \quad \|v\|_{V_{m}^{p}} := \|e^{it\Lambda_{m}} v\|_{V_{p}}.
\]

The properties in Lemma 2.1 also hold for the spaces \( U_{m}^{p} \) and \( V_{m}^{p} \).

Lemma 2.3 (Transfer principle). Let \( T : L^{2} \rightarrow L^{1}_{loc}(\mathbb{R}^{3}; \mathbb{C}) \) be a linear operator satisfying that

\[
\|T(e^{-it\Lambda_{m}} f)\|_{L^{1}_{X}} \lesssim \|f\|_{L^{2}}.
\]
for some $1 \leq q < \infty$ and a Banach space $X \subset L^1_{loc}(\mathbb{R}^3; \mathbb{C})$. Then
\[ \|T(u)\|_{L^q_X} \lesssim \|u\|_{L^q_{m}}. \]

2.3. Strichartz estimates. Let the pair $(q, r)$ satisfy that $2 \leq q, r \leq \infty$, $\frac{2}{q} + \frac{3}{r} = \frac{3}{2}$. Then it holds from \[7\]
\[ \langle M \rangle^\frac{3}{2r} \|e^{-it\Delta_m} \hat{P}_M \varphi\|_{L^q_r L^r_{\varphi}} \lesssim \|\hat{P}_M \varphi\|_{L^r_r}. \]

The case $q = 2$ is sufficient in our discussion. For the $N_0 \gg 1$ we have
\[ \|e^{-it\Delta_m} P_{N_0} \varphi\|_{L^2_r L^2_{\varphi}} \lesssim N_0 \|P_{N_0} \varphi\|_{L^2}, \]
which gives by transfer principle in Lemma \[2.3\]
\[ \|P_{N_0} u\|_{L^2_r L^2_{\varphi}} \lesssim N_0 \|P_{N_0} \varphi\|_{L^2}. \]

This endpoint estimate can be extended to a wider range with weaker angular integrability in the left term. That is, we consider the following $L^1_{r} L^q_{\varphi} L^r_{\varphi}$ norm with $r_* \leq r < \infty$ defined by
\[ \|u\|_{L^1_{r} L^q_{\varphi} L^r_{\varphi}} = \left( \int_{\mathbb{R}} \left( \int_{S^2} |u(t, \rho \theta)|^r d\theta \right)^\frac{q}{r} \right)^\frac{1}{q}. \]

If $r = \infty$, then we define $L^\infty_{\varphi} = L^\infty_{R^3}$. Then for $\frac{10}{3} < r < 6$, there holds
\[ \|e^{-it\Delta_m} \hat{P}_M \varphi\|_{L^1_{r} L^q_{\varphi} L^r_{\varphi}} \lesssim \|\hat{P}_M \varphi\|_{L^r_r} \times \begin{cases} M^{\frac{3}{2r} - \frac{3}{4}} \langle M \rangle^{\frac{3}{4} - \frac{3}{4}}, & \text{if } \frac{10}{3} < r < 4 \\ M^{-\frac{1}{4}} \langle M \rangle^{\frac{3}{4} + \epsilon}, & \text{if } r = 4 \text{ and } \epsilon > 0 \\ M^{1 - \frac{3}{2}}, & \text{if } 4 < r < 6. \end{cases} \]

For this see the Klein-Gordon case of Theorem 3.3 in \[12\]. Especially if $\frac{10}{3} < r < 4$ and $N > N_0$ we have for $u \in U^2_m$ by transfer principle into $U^2_m$ spaces
\[ \|P_N u\|_{L^1_{r} L^q_{\varphi} L^r_{\varphi}} \lesssim N^{\frac{1}{2}} \|P_N u\|_{L^2_m}, \]
which we will intensively use in following argument.

Remark 2. Note that the minimum loss of regularity occurs when $r$ is close to 4. And this is essentially related with the regularity condition on initial data $s > \frac{1}{4}$. It can be easily checked that in the range $4 < r < 6$ the bound is sharp if we consider the homogeneous case and scaling argument, but in the other range the sharpness is not known yet. If we can improve the bound in this range we might obtain better regularity result, i.e., threshold of well-posedness could be lowered.

2.4. Properties of angular derivative. In this section we introduce a series of lemmas concerning angular derivative.

Lemma 2.4. Let $\psi, f$ be smooth and let $\psi$ be radially symmetric. Then
\[ \nabla_S (\psi * f) = \psi * \nabla_S f. \]

From this we check the order of the projection operator and angular derivative can be reversed:
\[ \nabla_S \hat{P}_M f = \hat{P}_M \nabla_S f \text{ for } M > 0. \]

The next one is on the Sobolev inequality on the unit sphere \[8\].
Lemma 2.5. For any $2 < \tilde{r} < \infty$
\[
\|f\|_{L^\tilde{r}_m(\mathbb{S}^2)} \lesssim \|f\|_{L^2_m(\mathbb{S}^2)} + \|
abla_s f\|_{L^2_m(\mathbb{S}^2)}, \quad \|f\|_{L^\tilde{r}_m(\mathbb{S}^2)} \lesssim \|f\|_{L^2_m(\mathbb{S}^2)} + \|
abla_s f\|_{L^2_m(\mathbb{S}^2)}.
\]

The final one is extended Young’s convolution estimates.

Lemma 2.6 (Lemma 7.1 of [3]). If $\psi$ is radially symmetric, then
\[
\|\psi * f\|_{L^p_\delta L^q_\delta} \leq \|\psi\|_{L^p_\delta} \|f\|_{L^q_\delta L^1_\delta},
\]
for all $p_1, p_2, p, q_1 \in [1, \infty]$ satisfying
\[
\frac{1}{p_1} + \frac{1}{p_2} - 1 = \frac{1}{p}, \quad \frac{1}{q_1} + \frac{1}{p_2} - 1 \leq \frac{1}{q}.
\]

2.5. Norm of Potential. We calculate the $L^p$ norm of $\hat{P}_M V$ and $F^{-1} \chi_M$ for $1 < p < \infty$. We simply denote $\hat{P}_M V$ by $V_M$.
\[
\int |V_M(x)|^p dx = \int_{|x| \leq M^{-1}} |V_M(x)|^p dx + \int_{|x| > M^{-1}} |x|^{-4p} |x|^4 |V_M(x)|^p dx
\]
\[
\lesssim M^{-3} \|V_M\|_{L^\infty}^p + M^{4p-3} \|x^4 |V_M|^p\|_{L^\infty}
\]
\[
\lesssim M^{-3} \|\chi_M(\xi) \hat{V}(\xi)\|_{L^1}^p + M^{4p-3} \|\nabla^2_\xi (\chi_M(\xi) \hat{V}(\xi))\|_{L^1}^p.
\]

Using the assumption (1.2) of $V$ we estimate $\|\nabla^k (\chi_M(\xi) \hat{V}(\xi))\|_{L^1} \lesssim M^{-k-\gamma+3}$ for $0 \leq k \leq 4$, where $\gamma = \gamma_1$ if $0 < M \leq 1$, or $\gamma = \gamma_2$ if $M > 1$. Thus we have
\[
(2.5) \quad \|V_M\|_{L^p_\delta} \lesssim \begin{cases} M^{3-\frac{2}{p}-\gamma_1}, & \text{if } 0 < M \leq 1 \\ M^{3-\frac{2}{p}-\gamma_2}, & \text{if } M > 1. \end{cases}
\]

Also we can check by simple calculation
\[
\|F^{-1} \chi_M\|_{L^p_\delta} \lesssim M^{3-\frac{2}{p}}.
\]

3. PROOF OF MAIN THEOREM

Let us define the Banach space $X^s$ by
\[
X^s := \left\{ u : \mathbb{R} \to H^s \bigg| P_N u, \nabla_s P_N u \in L^2_m(\mathbb{R}; L^2_s) \forall N \geq N_0 \right\}
\]
with the norm
\[
\|u\|_{X^s} = \left( \sum_{N \geq N_0} N^{2s} \|P_N u\|_{L^2_m}^2 \right)^{\frac{1}{2}} + \|u\|_{L^2_m} + \|\nabla_s u\|_{L^2_m}.
\]

Let $X^s_+$ be the restricted space defined by
\[
X^s_+ := \left\{ u \in C([0, \infty); H^s) \bigg| u(0) \in X^s \right\}
\]
with norm $\|u\|_{X^s_+} := \|u\|_{X^s}$.

Let $D^s_\delta(\delta)$ be a complete metric space $\{ u \in X^s_+ \big| \|u\|_{X^s_+} \leq \delta \}$ equipped with the metric $d(u, v) := \|u - v\|_{X^s_+}$. Then we will show that the nonlinear functional $\Psi(u) = e^{-i t \Lambda_m \varphi} + N_m(u, u, u)$ is a contraction on $D^s_\delta(\delta)$, where
\[
N_m(u_1, u_2, u_3) = -i \int_0^t e^{-i(t-t') \Lambda_m} [V * (u_1 \tilde{u}_2) u_3] dt'.
\]
Clearly, $\|e^{-it\Lambda_m}\varphi\|_{X^s_+} \lesssim \|\varphi\|_{H^{s,1}}$ so it suffices to show that

$$\|N_m(u_1, u_2, u_3)\|_{X^s_+} \lesssim \prod_{j=1}^3 \|u_j\|_{X^s_+}^3$$

This readily implies estimates for difference

$$\|N_m(u, u, u) - N_m(v, v, v)\|_{X^s_+} \lesssim (\|u\|_{X^s_+} + \|v\|_{X^s_+})^2 \|u - v\|_{X^s_+}, \text{ for } u, v \in X^s$$

and thus we can find $\delta$ small enough for $\Psi$ to be a contraction mapping on $D^s_+(\delta)$. From now, we simply denote $N_m(u, u, u)$ by $N_m(u)$.

Since $e^{it\Lambda_m}P_N N_m(u)$ and $e^{it\Lambda_m}P_N \nabla \delta N_m(u)$ are in $V^2_{-\infty, \infty}(\mathbb{R}; L^2)$, and

$$\sum_{N \geq N_0} N^{2s} (\|e^{it\Lambda_m}P_N N_m(u)\|_{V^2} + \|e^{it\Lambda_m}P_N \nabla \delta N_m(u)\|_{V^2})^2 < \infty$$

from (3.1), $\lim_{t \to +\infty} e^{it\Lambda_m}N_m(u)$ exists in $H^{s,1}$. Define a scattering state $u_+$ with

$$\varphi_+ := \varphi + \lim_{t \to +\infty} e^{it\Lambda_m}N_m(u).$$

By time symmetry we can argue in a similar way for the negative time. Thus we get the desired result.

We start to show (3.1). We may assume that $u(t) = 0$ for $-\infty < t < 0$. From the duality in Lemma 2.1,

$$\|P_N N_m(u_1, u_2, u_3)\|_{L^{2\epsilon}} \lesssim \sup_{\|v\|_{L^{2\epsilon}} \leq 1} \left| \int_{\mathbb{R}} \int_{\mathbb{R}^3} [V * (u_1 \bar{u}_2)](t) P_N v(t) \overline{v} dx dt \right|.$$ Using Littlewood-Paley decomposition and applying Lemma 2.4 and Leibniz rule, we have

$$\|N_m(u)\|_{X^s_+}^2 \lesssim \sum_{N \geq N_0} N^{2s} \sup_{\|v\|_{L^{2\epsilon}} \leq 1} \left( \sum_{N_1, N_2, N_3 \geq N_0} \sum_{k=0}^3 I_k(N, N_1, N_2, N_3) \right)^2,$$

where

\begin{align*}
I_0(N, N_2, N_3, N) &= \left| \int_{\mathbb{R}} \int_{\mathbb{R}^3} V * (P_{N_1} u_1 P_{N_2} \bar{u}_2) P_{N_3} u_3 P_N \bar{v} \overline{v} dx dt \right|, \\
I_1(N, N_2, N_3, N) &= \left| \int_{\mathbb{R}} \int_{\mathbb{R}^3} V * (\nabla \delta P_{N_1} u_1 P_{N_2} \bar{u}_2) P_{N_3} u_3 P_N \overline{v} dx dt \right|, \\
I_2(N, N_2, N_3, N) &= \left| \int_{\mathbb{R}} \int_{\mathbb{R}^3} V * (P_{N_1} u_1 \nabla \delta P_{N_2} \bar{u}_2) P_{N_3} u_3 P_N \overline{v} dx dt \right|, \\
I_3(N, N_2, N_3, N) &= \left| \int_{\mathbb{R}} \int_{\mathbb{R}^3} V * (P_{N_1} u_1 P_{N_2} \bar{u}_2) V \delta P_{N_3} u_3 P_N \overline{v} dx dt \right|.
\end{align*}

Since the argument will not be affected by complex conjugation, we drop the conjugate symbol. By Lemma 2.3 we can change the order of derivative operator $\nabla \delta$ and projection $P$. Thus to show (3.1), we suffice to prove the following:

$$\sum_{N \geq N_0} N^{2s} \sup_{\|v\|_{L^{2\epsilon}} \leq 1} \left( \sum_{N_1, N_2, N_3 \geq N_0} I(N, N_1, N_2, N_3) \right)^2 \lesssim \prod_{i=1,2,3} \|u_i\|_{X^s_+}^2,$$

where

$$I(N, N_2, N_3, N) := \left| \int_{\mathbb{R}} \int_{\mathbb{R}^3} V * (P_{N_1} u_1 P_{N_2} u_2) P_{N_3} u_3 P_N v \overline{v} dx dt \right|.$$
and at most one of \( u_i \) could take an angular derivative \( \nabla_\delta \), i.e. \( u_i = \nabla_\delta u_i \). To prove this inequality we introduce the following proposition.

**Proposition 3.1.** Let \( s > \frac{1}{4} \). Suppose \( P_N u_i \in U_{m}^{2,1} \), \( P_N v \in V_{m}^{2} \) for \( i = 1, 2, 3 \) and at most one of \( u_i \) could take an angular derivative \( \nabla_\delta \). Then for \( \frac{1}{4} < \frac{1}{r} < \min(s, \frac{3}{10}) \) it holds

\[
I(N_1, N_2, N_3, N) \lesssim C(N, N_1, N_2, N_3) \| P_N u_1 \|_{U_{m}^{2,1}} \| P_N u_2 \|_{U_{m}^{2,1}} \| P_N u_3 \|_{U_{m}^{2,1}} \| P_N v \|_{V_{m}^{2}},
\]

(3.4)

\[
C(N, N_1, N_2, N_3) = \begin{cases} 
N_1^\frac{1}{2} N_2^\frac{1}{2} & \text{for } N_3 \gtrsim N, \\
\min(N_1, N_2)^\frac{1}{2} N_3^\frac{1}{2} & \text{for } N_3 \ll N.
\end{cases}
\]

Here the implicit constant only depends on \( r, N_0 \).

Now, we postpone the proof of Proposition 3.1 in a moment and explain how this result implies (3.3). Let us split the summation of LHS in (3.3) into two parts as follows:

\[
\text{LHS of (3.3)} = \sum_{N_3 \gtrsim N} + \sum_{N_3 \ll N} \implies S_1 + S_2.
\]

Fix \( r \) as in Proposition 3.1. Apply the first case of (3.3) to \( S_1 \)

\[
S_1 \lesssim \sum_{N \geq N_0} N^{2s} \left( \sum_{N_1, N_2 \geq N_0} N_1^{\frac{1}{2}} \| P_N u_1 \|_{U_{m}^{2,1}} N_2^{\frac{1}{2}} \| P_N u_2 \|_{U_{m}^{2,1}} \sum_{N_3 \gtrsim N} \| P_N u_3 \|_{U_{m}^{2,1}} \right)^2
\]

\[
\lesssim \sum_{N \geq N_0} \left( \sum_{N_1, N_2 \geq N_0} N_1^{\frac{1}{2} - s} N_2^{\frac{1}{2} - s} N_3^{\frac{1}{4}} \| P_N u_1 \|_{U_{m}^{2,1}} N_2^{\frac{1}{2}} \| P_N u_2 \|_{U_{m}^{2,1}} \sum_{N_3 \gtrsim N} \left( \frac{N}{N_3} \right)^s N_3^{\frac{1}{4}} \| P_N u_3 \|_{U_{m}^{2,1}} \right)^2
\]

\[
\lesssim \| u_1 \|_{X_+^1}^2 \| u_2 \|_{X_+^1}^2 \sum_{N \geq N_0} \left( \sum_{N_3 \gtrsim N} \frac{N}{N_3}^s N_3^{\frac{1}{4}} \| P_N u_3 \|_{U_{m}^{2,1}} \right)^2
\]

\[
\lesssim \prod_{i=1,2,3} \| u_i \|_{X_+^1}^2.
\]

\( S_2 \) is estimated using the second case of (3.3). By symmetry we may assume \( N_1 \leq N_2 \).

\[
S_2 \lesssim \sum_{N \geq N_0} N^{2s} \left( \sum_{N_0 \leq N_1 \leq N_2} N_1^{\frac{1}{2} - s} N_2^{\frac{1}{2} - s} N_3^{\frac{1}{4}} \| P_N u_1 \|_{U_{m}^{2,1}} \| P_N u_2 \|_{U_{m}^{2,1}} N_3^{\frac{1}{4}} \| P_N u_3 \|_{U_{m}^{2,1}} \right)^2
\]

\[
\lesssim \| u_1 \|_{X_+^1}^2 \| u_3 \|_{X_+^1}^2 \sum_{N \geq N_0} \left( \sum_{N_2 \gtrsim N} \frac{N}{N_2}^s N_2^{\frac{1}{4}} \| P_N u_2 \|_{U_{m}^{2,1}} \right)^2
\]

\[
\lesssim \prod_{i=1,2,3} \| u_i \|_{X_+^1}^2.
\]

So it remains to prove the Proposition 3.1. To simplify the notations, we assume all the functions are localized one, i.e., \( P_N u_i = u_i \) for \( i = 1, 2, 3 \) and \( P_N v = v \). And we use the bold notation \( u_i \) when it could take an angular derivative or not. But be cautious that at most one of bold \( u_i \) could take. In other words, the estimates hold true even if at most one of \( u_i \) take an angular derivative.
4. Proof of Proposition 3.1

We perform an additional decomposition for $I$:

\[ I(N_1, N_2, N_3, N) \lesssim \sum_{M > 0} \left| \int \int \hat{P}_M V * (u_1 u_2) \hat{P}_M (u_3 v) \, dx \, dt \right| \]

\[ = \sum_{M > 0} \left| \int \int V_M * (u_1 u_2) \hat{P}_M (u_3 v) \, dx \, dt \right|, \]

where at most one of bold $u_i$ could take the angular derivative.

4.1. Case 1: $N_3 \gtrsim N$. In this subsection we prove that

\[ I \lesssim N_1^4 N_2^4 \| u_1 \|_{L^{2,1}_m} \| u_3 \|_{L^{2,1}_m} \| u_3 \|_{L^{2,1}_m} \| v \|_{V_2^2}. \]

Since the localized Strichartz estimates we apply have different admissible pairs whether the support in frequency side is low part or not, that is, (2.3) or (2.4), we proceed to prove dividing the case whether $N_i$ is equal to $N_0$ or not for $i = 1, 2, 3$.

Note that the support properties from Littlewood-Paley decomposition would restrict the range of summation over $M$.

4.1.1. $N_0 = \min(N_1, N_2) \sim \max(N_1, N_2)$. In this case the support condition gives $M \lesssim N_0$. We estimate using Hölder and Young’s inequality

\[ I \lesssim \sum_{M \lesssim N_0} \| V_M * (u_1 u_2) \|_{L^2 \rightarrow L^\infty} \| \hat{P}_M (u_3 v) \|_{L^\infty \rightarrow L^2} \]

\[ \lesssim \sum_{M \lesssim N_0} \| V_M \|_{L^2 \rightarrow L^\infty} \| u_1 \|_{L^2_m} \| u_2 \|_{L^2_m} \| u_3 \|_{L^\infty \rightarrow L^2} \| v \|_{L^\infty \rightarrow L^2}. \]

By (2.3) and the embeddings $U^2_m, V^2_m \hookrightarrow L^\infty \rightarrow L^2$ in Lemma 2.1 we obtain

\[ I \lesssim N_0 \sum_{M \lesssim N_0} \| V_M \|_{L^2 \rightarrow L^{2,1}} \| u_1 \|_{L^{2,1}_m} \| u_2 \|_{L^{2,1}_m} \| u_3 \|_{L^\infty \rightarrow L^2} \| v \|_{L^\infty \rightarrow L^2}. \]

Here, bold $u_3$ means that the estimates hold for both $u_3$ and $\nabla_x u_3$ cases. From (2.3) we estimate

\[ \sum_{M \lesssim N_0} \| V_M \|_{L^2 \rightarrow L^{2,1}} \leq \sum_{0 < M \leq 1} M^{1 - \gamma_1} + \sum_{1 < M \lesssim N_0} M^{1 - \gamma_2} \lesssim C(N_0), \]

where the assumption $\gamma_1$ be less than 1 is essential. Thus we have

\[ I \lesssim N_0 \| u_1 \|_{L^{2,1}_m} \| u_2 \|_{L^{2,1}_m} \| u_3 \|_{U^{2,1}_m} \| v \|_{V_2^2}. \]

4.1.2. $N_0 = \min(N_1, N_2) \ll \max(N_1, N_2)$. In this case $M$ should be comparable to $\max(N_1, N_2)$. We divide the case according to whether $u_3$ takes the angular derivative or not.

(1) $u_3$ case: In this case at most one of $u_1$, $u_2$ could take the angular derivative. We denote this by bold $u_1$, $u_2$. We have by Hölder inequality

\[ I \lesssim \sum_{M > \max(N_1, N_2)} \| V_M * (u_1 u_2) \|_{L^1_{t,x} W^{6,6}_\rho} \| \hat{P}_M (u_3 v) \|_{L^\infty \rightarrow L^1_{t,x} W^{6,6}_\rho}. \]

We compute the first norm. We assume $N_0 = N_1 < N_2$. We apply Lemma 2.6

\[ (4.1) \quad \| V_M * (u_1 u_2) \|_{L^1_t L^{\infty}_x W^{6,6}_\rho} \lesssim \| V_M \|_{L^\infty \rightarrow L^1_{t,x} W^{6,6}_\rho} \| u_1 u_2 \|_{L^1_{t,x} W^{6,6}_\rho}. \]
We estimate using Lemma 2.5
\[
\|u_1 u_2\|_{L_t^1 \mathcal{L}^{3, \infty}_x L^\delta_x} \lesssim \|u_1\|_{L^7_t \mathcal{L}^6_x L^\delta_x} \|u_2\|_{L^7_t \mathcal{L}^6_x L^\delta_x} \lesssim \left( \|u_1\|_{L^7_t \mathcal{L}^6_x L^\delta_x} + \|\nabla_S u_1\|_{L^7_t \mathcal{L}^6_x L^\delta_x}\right) \|u_2\|_{L^7_t \mathcal{L}^6_x L^\delta_x}
\]
(4.2)
where in the last inequality we used Strichartz estimates \(2.3\) and \(2.4\). Similarly we estimate
\[
\|u_1 u_2\|_{L_t^1 \mathcal{L}^{3, \infty}_x L^\delta_x} \lesssim \|u_1\|_{L^7_t \mathcal{L}^6_x L^\delta_x} \|u_2\|_{L^7_t \mathcal{L}^6_x L^\delta_x} \lesssim \|u_1\|_{L^7_t \mathcal{L}^6_x L^\delta_x} \left( \|u_2\|_{L^7_t \mathcal{L}^6_x L^\delta_x} + \|\nabla_S u_2\|_{L^7_t \mathcal{L}^6_x L^\delta_x}\right)
\]
(4.3)
The case \(N_0 = N_2 < N_1\) can be bounded similarly. Thus from (4.1), (4.2), and (4.3) we obtain
\[
\|V_M \ast (u_1 u_2)\|_{L_t^1 \mathcal{L}^{3, \infty}_x L^\delta_x} \lesssim \|V_M\|_{L_t^\infty \mathcal{L}^\infty_x L^\delta_x} \max(N_1, N_2) \|u_1\|_{U_m^{2,1}} \|u_2\|_{U_m^{2,1}}.
\]
Next we estimate the second norm using Lemma 2.6
\[
\|\tilde{P}_M (u_3 v)\|_{L_t^\infty \mathcal{L}^{3, \infty}_x L^\delta_x} \lesssim \|F^{-1} \chi_M \|_{L_t^\infty \mathcal{L}^\infty_x L^\delta_x} \|u_3\|_{L_t^\infty \mathcal{L}^\infty_x L^\delta_x} \lesssim \left( \|u_3\|_{L_t^\infty \mathcal{L}^\infty_x L^\delta_x} + \|\nabla_S u_3\|_{L_t^\infty \mathcal{L}^\infty_x L^\delta_x}\right) \|v\|_{L_t^\infty \mathcal{L}^\infty_x L^\delta_x},
\]
where we applied Lemma 2.3 with \(r > \frac{40}{11}\). In conclusion, we have
\[
I \lesssim N_0 \sum_{M \sim \max(N_1, N_2)} \|V_M\|_{L_t^\infty \mathcal{L}^{3, \infty}_x L^\delta_x} \|F^{-1} \chi_M\|_{L_t^\infty \mathcal{L}^\infty_x L^\delta_x} \|u_1\|_{U_m^{2,1}} \|u_2\|_{U_m^{2,1}} \|u_3\|_{U_m^{2,1}} \|v\|_{V_m^{2,1}}
\]
where we used \(\sum_{M \sim \max(N_1, N_2)} \|V_M\|_{L_t^\infty \mathcal{L}^{3, \infty}_x L^\delta_x} \|F^{-1} \chi_M\|_{L_t^\infty \mathcal{L}^\infty_x L^\delta_x} \lesssim \sum_{M \sim \max(N_1, N_2)} M^{3/2 + \gamma_2} < C\) for \(r\) we consider by 2.4.

(2) \(\nabla_S u_3\) case: In this case neither \(u_1\) nor \(u_2\) takes the angular derivative. We have by Hölder inequality
\[
I \lesssim \sum_{M \sim \max(N_1, N_2)} \|V_M \ast (u_1 u_2)\|_{L_t^1 \mathcal{L}^{3, \infty}_x L^\delta_x} \|\tilde{P}_M (\nabla_S u_3 v)\|_{L_t^\infty \mathcal{L}^\infty_x L^\delta_x}.
\]
We consider the former. By symmetry we may assume \(N_0 = N_1 < N_2\). We apply Lemma 2.6
\[
\|V_M \ast (u_1 u_2)\|_{L_t^1 \mathcal{L}^{3, \infty}_x L^\delta_x} \lesssim \|V_M \ast (u_1 u_2)\|_{L_t^1 \mathcal{L}^{3, \infty}_x L^\delta_x} + \|\nabla_S V_M \ast (u_1 u_2)\|_{L_t^1 \mathcal{L}^{3, \infty}_x L^\delta_x}.
\]
By applying Lemma 2.6 and Hölder inequality we estimate
\[
\|V_M \ast (u_1 u_2)\|_{L_t^1 \mathcal{L}^{3, \infty}_x L^\delta_x} \lesssim \|V_M\|_{L_t^\infty \mathcal{L}^{3, \infty}_x L^\delta_x} \|u_1 u_2\|_{L_t^7 \mathcal{L}^6_x L^\delta_x} \lesssim \|V_M\|_{L_t^\infty \mathcal{L}^{3, \infty}_x L^\delta_x} \|u_1\|_{L_t^7 \mathcal{L}^6_x L^\delta_x} \|u_2\|_{L_t^7 \mathcal{L}^6_x L^\delta_x} \lesssim N_0 \|V_M\|_{L_t^\infty \mathcal{L}^{3, \infty}_x L^\delta_x} \|u_1 u_2\|_{U_m^{2,1}} \|u_2\|_{U_m^{2,1}}.
\]
The derivative term can be estimated by the same argument as above because by Lemma 2.4 and Leibniz rule, we have
\[
\|\nabla_S V_M \ast (u_1 u_2)\|_{L_t^1 \mathcal{L}^{3, \infty}_x L^\delta_x} \leq \|V_M \ast (\nabla_S u_1 u_2)\|_{L_t^1 \mathcal{L}^{3, \infty}_x L^\delta_x} + \|V_M \ast (u_1 \nabla_S u_2)\|_{L_t^1 \mathcal{L}^{3, \infty}_x L^\delta_x}.
\]
Then we finally obtain
\[
\|V_M \ast (u_1 u_2)\|_{L_t^1 \mathcal{L}^{3, \infty}_x L^\delta_x} \lesssim N_0 \|V_M\|_{L_t^\infty \mathcal{L}^{3, \infty}_x L^\delta_x} \max(N_1, N_2) \|u_1\|_{U_m^{2,1}} \|u_2\|_{U_m^{2,1}}.
\]
For the latter in (4.4) we only use Hölder inequality and the embedding
\[ \| \tilde{P}_M (\nabla_S u_3 v) \|_{L^\infty_x L^1_t} \lesssim \| \nabla_S u_3 \|_{L^2_m} \| v \|_{V^2_m}. \]

In conclusion we get as in the previous case
\[ I \lesssim N_0 \sum_{M \sim \max(N_1, N_2)} M^{\frac{3}{4} + \frac{3}{2} - \gamma_2} \max(N_1, N_2)^{\frac{3}{2}} \| u_1 \|_{U^{2,1}_m} \| u_2 \|_{U^{2,1}_m} \| u_3 \|_{U^{2,1}_m} \| v \|_{V^2_m}. \]

4.1.3. \( N_0 < N_1, N_2 \). We apply Hölder inequality
\[ I \lesssim \sum_{M > 0} \| V_M \ast (u_1 u_2) \|_{L^1_t L^\infty_x} \| \tilde{P}_M (u_3 v) \|_{L^\infty_t L^1_x}. \]

We claim that \( \| u_1 u_2 \|_{L^1_t L^\infty_x} \lesssim N_1^{\frac{3}{2}} N_2^{\frac{3}{2}} \| u_1 \|_{U^{2,1}_m} \| u_2 \|_{U^{2,1}_m} \| u_3 \|_{U^{2,1}_m} \| v \|_{V^2_m}. \) Indeed, we estimate applying Lemma 2.26 and spherical Strichartz estimate (2.24)
\[ \| u_1 u_2 \|_{L^1_t L^\infty_x} \lesssim \| u_1 \|_{L^\infty_t L^2_x} \| u_2 \|_{L^\infty_t L^2_x} \lesssim \| u_1 \|_{L^\infty_t L^2_x} \| u_2 \|_{L^\infty_t L^2_x} (\| u_2 \|_{L^\infty_t L^2_x} + \| \nabla_S u_2 \|_{L^\infty_t L^2_x}) \lesssim N_1^{\frac{3}{2}} N_2^{\frac{3}{2}} \| u_1 \|_{U^{2,1}_m} \| u_2 \|_{U^{2,1}_m} \| u_3 \|_{U^{2,1}_m} \| v \|_{V^2_m}. \]

Also, we can change the role of \( u_1 \) and \( u_2 \), which implies the claim. Thus we have
\[ I \lesssim \sum_{M > 0} \| V_M \|_{L^\infty_x} \| u_1 \|_{U^{2,1}_m} \| u_2 \|_{U^{2,1}_m} \| u_3 \|_{U^{2,1}_m} \| v \|_{V^2_m}. \]

We compute the summation over \( M \) using (2.26)
\[ \sum_{M > 0} \| V_M \|_{L^\infty_x} = \sum_{0 < M \leq 1} M^{-\gamma_1} + \sum_{M > 1} M^{-\gamma_2} < C, \]
which is finite if we choose \( r \) so that \( r > 6/\gamma_2 \).

4.2. Case2: \( N_3 \ll N \). In this subsection we prove
\[ I \lesssim \min(N_1, N_2)^{\frac{3}{2}} N_3^{\frac{3}{2}} \| u_1 \|_{U^{2,1}_m} \| u_2 \|_{U^{2,1}_m} \| u_3 \|_{U^{2,1}_m} \| v \|_{V^2_m}. \]

In this case we should further divide the case whether \( N_3 \) is \( N_0 \) or not. Among them the case \( N_0 = \min(N_1, N_2) \sim \max(N_1, N_3) \) is already considered in section 4.1.1.

Note that in this range we have \( M \sim N \lesssim \max(N_1, N_2) \).

4.2.1. \( N_0 = N_3 \ll N \). Suppose \( N_0 = \min(N_1, N_2) \ll \max(N_1, N_2) \). We estimate
\[ I \lesssim \sum_{M \sim N} \| V_M \ast (u_1 u_2) \|_{L^2_t L^\infty_x} \| \tilde{P}_M (u_3 v) \|_{L^2_t L^\infty_x} \]
\[ \lesssim \sum_{M \sim N} \| V_M \|_{L^\infty_x} \| u_1 \|_{L^2_t L^\infty_x} \| u_2 \|_{L^2_t L^\infty_x} \| u_3 \|_{L^2_t L^\infty_x} \| v \|_{L^\infty_t L^2_x} \]
\[ \lesssim N_0 \sum_{M \sim N} M^{-\gamma_2} \| u_1 \|_{U^{2,1}_m} \| u_2 \|_{U^{2,1}_m} \| u_3 \|_{U^{2,1}_m} \| v \|_{V^2_m}, \]
which is complete since \( \gamma_2 > \frac{3}{2} \).
Suppose \( N_1, N_2 > N_0 \). We have

\[
I \lesssim \sum_{M \sim N} \left\| V_M * (u_1 u_2) \right\|_{L^2_t L^\infty_x} \left\| \hat{P}_M (u_3 v) \right\|_{L^2_t L^\infty_x}.
\]

We bound the first term. We assume \( \min(N_1, N_2) = N_1 \). By Lemma 2.6 we have

\[
\left\| V_M * (u_1 u_2) \right\|_{L^2_t L^\infty_x} \lesssim \left\| V_M \right\|_{L^\infty_t L^\infty_x} \left\| u_1 u_2 \right\|_{L^2_t L^\infty_x}.
\]

We estimate

\[
\left\| u_1 u_2 \right\|_{L^2_t L^\infty_x} \lesssim \left\| u_1 \right\|_{L^2_t L^\infty_x} \left\| u_2 \right\|_{L^\infty_t L^\infty_x}
\]

\[
\lesssim \left( \left\| u_1 \right\|_{L^2_t L^\infty_x} + \left\| \nabla u_1 \right\|_{L^t L^\infty_x} \right) \left\| u_2 \right\|_{L^\infty_t L^2_x}
\]

\[
\lesssim N_1^\frac{1}{2} \left( \left\| u_1 \right\|_{U_2^1} + \left\| \nabla u_1 \right\|_{U_2^1} \right) \left\| u_2 \right\|_{U_2^1},
\]

where we used Lemma 2.5 since \( \frac{2\nu}{4-r} > 2 \). Or, exchanging a spherical pair for Hölder inequality we estimate

\[
\left\| u_1 u_2 \right\|_{L^2_t L^\infty_x} \lesssim \left\| u_1 \right\|_{L^2_t L^\infty_x} \left\| u_2 \right\|_{L^\infty_t L^\infty_x}
\]

\[
\lesssim \left( \left\| u_1 \right\|_{L^2_t L^\infty_x} + \left\| \nabla u_1 \right\|_{L^t L^\infty_x} \right) \left\| u_2 \right\|_{L^\infty_t L^2_x}
\]

\[
\lesssim N_1^\frac{1}{2} \left( \left\| u_1 \right\|_{U_2^1} + \left\| \nabla u_1 \right\|_{U_2^1} \right) \left\| u_2 \right\|_{U_2^1}.
\]

Since we can change the role of \( u_1 \) and \( u_2 \), (4.10), (4.7) and (4.8) imply

\[
\left\| V_M * (u_1 u_2) \right\|_{L^2_t L^\infty_x} \lesssim \left\| V_M \right\|_{L^\infty_t L^\infty_x} \min(N_1, N_2)^\frac{1}{2} \left\| u_1 \right\|_{U_2^1} \left\| u_2 \right\|_{U_2^1}.
\]

Next we bound the second term in (4.5) by applying Lemma 2.6

\[
\left\| \hat{P}_M (u_3 v) \right\|_{L^2_t L^\infty_x} \lesssim \left\| \mathcal{F}^{-1} \chi_M \right\|_{L^\infty_t L^\infty_x} \left\| u_3 \right\|_{L^2_t L^\infty_x} \left\| v \right\|_{L^\infty_t L^2_x}
\]

\[
\lesssim N_0 \left\| \mathcal{F}^{-1} \chi_M \right\|_{L^\infty_t L^\infty_x} \left\| u_3 \right\|_{U_2^1} \left\| v \right\|_{V_2^1}.
\]

In conclusion we obtain

\[
I \lesssim \sum_{M \sim N} \left\| V_M \right\|_{L^\infty_t L^\infty_x} \left\| \mathcal{F}^{-1} \chi_M \right\|_{L^\infty_t L^\infty_x} \left\| u_1 \right\|_{U_2^1} \left\| u_2 \right\|_{U_2^1} \left\| u_3 \right\|_{U_2^1} \left\| v \right\|_{V_2^1}.
\]

which implies the desired result since we have from

\[
\sum_{M \sim N} \left\| V_M \right\|_{L^\infty_t L^\infty_x} \left\| \mathcal{F}^{-1} \chi_M \right\|_{L^\infty_t L^\infty_x} \lesssim \sum_{M \sim N} M^{\frac{3}{4} + \frac{1}{4} - 2} < C.
\]

4.2.2. \( N_0 < N_3 \ll N \) and \( N_0 = \min(N_1, N_2) \ll \max(N_1, N_2) \). We divide the case according to whether \( u_3 \) takes the angular derivative or not.

(1) \( u_3 \) case: We have

\[
I \lesssim \sum_{M \sim N} \left\| V_M * (u_1 u_2) \right\|_{L^2_t L^\infty_x} \left\| \hat{P}_M (u_3 v) \right\|_{L^2_t L^\infty_x}.
\]

We compute the first norm. By symmetry we may assume \( \min(N_1, N_2) = N_1 \).

\[
\left\| V_M * (u_1 u_2) \right\|_{L^2_t L^\infty_x} \lesssim \left\| V_M \right\|_{L^\infty_t L^\infty_x} \left\| u_1 u_2 \right\|_{L^2_t L^\infty_x} \lesssim \left\| V_M \right\|_{L^\infty_t L^\infty_x} \left\| u_1 \right\|_{L^2_t L^2_x} \left\| u_2 \right\|_{L^\infty_t L^2_x}
\]

\[
\lesssim N_0 \left\| V_M \right\|_{L^\infty_t L^\infty_x} \left\| u_1 \right\|_{U_2^1} \left\| u_2 \right\|_{U_2^1}.
\]
And we estimate the second term using Lemma 2.6
\[
\| \tilde{P}_M(u_3 v) \|_{L^2_t L^6_x} \lesssim \| F^{-1} \chi_M \|_{L^6_t} \| u_3 \|_{L^2_t L^6_x} \| v \|_{L^6_t L^6_x} \lesssim (\| u_3 \|_{L^2_t L^6_x}^2 + \| \nabla u_3 \|_{L^2_t L^6_x}) \| v \|_{L^6_t L^6_x} \\
\lesssim N_3 \left( \| u_3 \|_{L^2_t} + \| \nabla u_3 \|_{L^2_t} \right) \| v \|_{V^2_m}.
\]

Thus we have
\[
I \lesssim \sum_{M < N} M^{(\frac{1}{6} + \frac{1}{2}) - \gamma_3} N_3^2 \| u_1 \|_{U^2_m} \| u_2 \|_{U^1_m} \| u_3 \|_{U^2_m} \| v \|_{V^2_m} \\
\lesssim N_3^2 \| u_1 \|_{U^2_m} \| u_2 \|_{U^1_m} \| u_3 \|_{U^2_m} \| v \|_{V^2_m}.
\]

(2) $\nabla u_3$ case: We have
\[
I \lesssim \sum_{M \sim \max(N_1, N_2)} \| V_M * (u_1 u_2) \|_{L^2_t L^6_x} \| \tilde{P}_M (\nabla u_3 v) \|_{L^2_t L^6_x}.
\]

We estimate the first norm. Applying Lemma 2.3 we obtain
\[
\| V_M * (u_1 u_2) \|_{L^2_t L^6_x} \lesssim \| V_M * (u_1 u_2) \|_{L^2_t L^6_x} + \| \nabla V_M * (u_1 u_2) \|_{L^2_t L^6_x}.
\]

By Young's and Hölder inequality we have
\[
\| V_M * (u_1 u_2) \|_{L^2_t L^6_x} \lesssim \| V_M \|_{L^\infty_t} \| u_1 \|_{L^2_t L^2_x} \| u_2 \|_{L^\infty_t L^2_x} \lesssim N_0 \| V_M \|_{L^\infty_t} \| u_1 \|_{U^2_m} \| u_2 \|_{U^2_m}.
\]

Then by Leibniz rule we can bound the derivative term similarly and finally get
\[
\| V_M * (u_1 u_2) \|_{L^2_t L^6_x} \lesssim \| V_M \|_{L^\infty_t} \| u_1 \|_{U^2_m} \| u_2 \|_{U^2_m}.
\]

We apply the Lemma 2.6 to the second term
\[
\| \tilde{P}_M (\nabla u_3 v) \|_{L^2_t L^6_x} \lesssim \| F^{-1} \chi_M \|_{L^6_t} \| \nabla u_3 \|_{L^2_t L^6_x} \| v \|_{L^6_t L^6_x} \lesssim N_3 \| \nabla u_3 \|_{U^2_m} \| v \|_{V^2_m}.
\]

By (4.10) and (4.11) we obtain
\[
I \lesssim N_0 \sum_{M \sim N} \| V_M \|_{L^\infty_t} \| u_1 \|_{U^2_m} \| u_2 \|_{U^2_m} \| u_3 \|_{U^2_m} \| v \|_{V^2_m} \\
\lesssim N_3 \| u_1 \|_{U^2_m} \| u_2 \|_{U^2_m} \| u_3 \|_{U^2_m} \| v \|_{V^2_m}.
\]

4.2.3. $N_0 < N_1, N_2, N_3$ and $N_3 \ll N$.

\[
I \lesssim \sum_{M \sim N} \| V_M * (u_1 u_2) \|_{L^2_t L^6_x} \| \tilde{P}_M (u_3 v) \|_{L^2_t L^6_x}.
\]

The first term is bounded as in (4.10). For the second one we apply Lemma 2.6
\[
\| \tilde{P}_M (u_3 v) \|_{L^2_t L^6_x} \lesssim \| F^{-1} \chi_M \|_{L^6_t} \| u_3 v \|_{L^2_t L^6_x} \| v \|_{L^6_t L^6_x} \lesssim \| F^{-1} \chi_M \|_{L^6_t} \| u_3 \|_{L^2_t L^6_x} \| v \|_{L^6_t L^6_x} \lesssim \| F^{-1} \chi_M \|_{L^6_t} \| u_3 \|_{U^2_m} \| v \|_{V^2_m}.
\]

Then the claim follows since $\sum_{M \sim N} \| V_M \|_{L^\infty_t} \| F^{-1} \chi_M \|_{L^6_t} \lesssim \sum_{M \sim N} M^{\frac{3}{2} - \gamma_3} < C$ by (2.5).
Acknowledgements

The author would like to thank Prof. Yonggeun Cho for his encouragement and advice on the paper. And the author is grateful to the referee for careful reading of the paper and valuable comments. This work was supported by NRF (NRF-2015R1D1A1A09057795).

References

[1] I. Bejenaru and S. Herr, On global well-posedness and scattering for the massive Dirac-Klein-Gordon system, J. Eur. Math. Soc. 19 (2017), no. 8, 2445–2467.

[2] Chadam, J. M. and Glassey, R. T., On the Maxwell-Dirac equations with zero magnetic field and their solution in two space dimensions, J. Math. Anal. Appl. 53 (1976), no. 3, 495–507.

[3] Y. Cho and K. Nakanishi, On the global existence of semirelativistic Hartree equations, RIMS Kokyuroku Bessatsu, B22 (2010), 145-166.

[4] Y. Cho and T. Ozawa, On the semi-relativistic Hartree type equation, SIAM J. Math. Anal., 38 (2006), no. 4, 1060–1074.

[5] Y. Cho and T. Ozawa, Global solutions of semirelativistic Hartree type equations, J. Korean Math. Soc., 44 (2007), no.5, 1065–1078.

[6] Y. Cho, T. Ozawa, H. Sasaki and Y. Shim, Remarks on the semirelativistic Hartree equations, Discrete Contin. Dyn. Syst. 23 (2009), no. 4, 1277-1294.

[7] Y. Cho, T. Ozawa and S. Xia, Remarks on some dispersive estimates, Commun. Pure Appl. Anal., 10 (2011), no. 4, 1121-1128.

[8] T. Coulhon, E. Russ and V. Tardivel-Nachef, Sobolev Algebras on Lie Groups and Riemannian Manifolds, American Journal of Mathematics 123, no. 2 (2001), 283–342.

[9] P. D’Ancona, D. Foschi and S. Selberg, Null structure and almost optimal local regularity for the Dirac-Klein-Gordon system, J. Eur. Math. Soc. (JEMS) 9 (2007), no.4, 877–899.

[10] A. Elgart, Mean field dynamics of boson stars, Comm. Pure Appl. Math. 60, No. 4 (2007) 500–545.

[11] J. Fröhlich and E. Lenzmann, Mean-field limit of quantum Bose gases and nonlinear Hartree equation, Séminaire: Équations aux Dérivées Partielles. 2003–2004 Sémin. Équ. Dériv. Partielles (2004), Exp. no. XIX, 26.

[12] Z. Guo, Z. Hani and K. Nakanishi, Scattering for the 3D Gross-Pitaevskii equation, Comm. Math. Phys. 259 (2018), no.1, 265-295.

[13] S. Herr and E. Lenzmann, The Boson star equation with initial data of low regularity, Nonlinear Anal. 97 (2014), 125–137.

[14] M. Hadac, S. Herr and H. Koch, Well-posedness and scattering for the KP-II equation in a critical space, Ann. Inst. H. Poincare Anal. Non Lineaire 26 (2009), no. 3, 917–941.

[15] S. Herr and T. Tesfahun, Small data scattering for semi-relativistic equations with Hartree type nonlinearity, J. Differential Equations 259, (2015), no.10, 5510–5532.

[16] H. Koch, D. Tataru and M. Visan, Dispersive Equations and Nonlinear Waves, Oberwolfach Seminars 45 (2014).

[17] E. Lenzmann, Well-posedness for semi-relativistic Hartree equations of critical type, Math. Phys. Anal. Geom. 10 (2007), no. 1, 43–64.

[18] E, Lieb and H. Yau, The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics, Comm. Math. Phys. 112 (1987), no.1, 147–174.

[19] F. Pusateri, Modified scattering for the boson star equation, Comm. Math. Phys. 332 (2014), no.3, 1203–1234.

Department of Mathematical Sciences, Seoul National University, Seoul 151-747, Republic of Korea
E-mail address: maticionych@snu.ac.kr