Cantor sets with high-dimensional projections

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Abstract

In 1994, J. Cobb constructed a tame Cantor set in $\mathbb{R}^3$ each of whose projections into 2-planes is one-dimensional. We show that an Antoine’s necklace can serve as an example of a Cantor set all of whose projections are one-dimensional and connected. We prove that each Cantor set in $\mathbb{R}^n$, $n \geq 3$, can be moved by a small ambient isotopy so that the projection of the resulting Cantor set into each $(n-1)$-plane is $(n-2)$-dimensional. We show that if $X \subset \mathbb{R}^n$, $n \geq 2$, is a zero-dimensional compactum whose projection into some plane $\Pi \subset \mathbb{R}^n$ with $\dim \Pi \in \{1, 2, n-2, n-1\}$ is zero-dimensional, then $X$ is tame; this extends some particular cases of the results of D.R. McMillan, Jr. (1964) and D.G. Wright, J.J. Walsh (1982).

We use the technique of defining sequences which comes back to Louis Antoine.

Keywords: Euclidean space, projection, Cantor set, embedding, isotopy, tame Cantor set, wild Cantor set, flat cell, dimension.

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1 Introduction

Any topological space homeomorphic to the standard middle-thirds Cantor set $C \subset I = [0, 1]$ is called a Cantor set. Let $f : C \to I$ be a continuous surjection; its graph $\Gamma(f) \subset \mathbb{R}^2$ is a Cantor set whose projection to $Oy$-axis coincides with $I$. The first description of a continuous surjection $f : C \to I$ was given in 1884 by G. Cantor [14, p. 255–256]; it maps a point $t = \frac{x_1}{3} + \frac{x_2}{3^2} + \ldots \in C$ (where each $x_i \in \{0; 2\}$) to the point $f(t) = \frac{x_1}{2} + \frac{x_2}{2^2} + \ldots$ of the unit segment. The remark concerning the projection of $\Gamma(f)$ into the vertical line can be found in L. Zoretti’s note [39] who described this as a known fact (“ce resultat bien connu etant acquis...”). The aim of Zoretti’s note was to disprove a statement made by F. Riesz who wrote in [31, p. 651] that a projection of a closed hereditarily disconnected subset of plane to a straight line is again closed and hereditarily disconnected (F. Riesz attributed this proposition to R. Baire; this is incorrect since the work of Baire contained a different statement). Zoretti also noticed that there exists a Cantor set such that its projections to a countable set of lines contain segments [39, p. 763]. Moreover, he claimed to have described a set all of whose projections contain segments [39, p. 763] (see also [40]); unfortunately Zoretti did not give enough details even to prove that his set does not contain arcs.

L. Antoine constructed a Cantor set in $\mathbb{R}^2$ whose projections coincide with those of a regular hexagon [4, 9, p. 272; and fig. 2 on p. 273]. Other examples of Cantor sets in plane all of whose projections are segments can be found in [15, p. 124, Example], [18, Prop. 1]. Using Antoine’s idea, we show that in $\mathbb{R}^2$, each polygon contains a Cantor set with exactly the same projections (Statement 5.1).

By K. Borsuk [13], for $n \geq 2$ there exists a Cantor set in $\mathbb{R}^n$ such that its projection to every hyperplane contains a $(n - 1)$-dimensional ball, equivalently, has dimension $(n - 1)$. (For $n = 3$, see an alternative description in [11, Thm. 6.2]; another proof of Borsuk’s result, using universal surjectivity property of $C$, can be found in [19, Prop. 3.1].) As a corollary, the projection of the Borsuk set to each $m$-plane, $m \leq n - 1$, has dimension $m$. We remark that a set with this property can be obtained from any given Cantor set, using a small ambient isotopy (Statement 5.2). For $n \geq 3$, in $\mathbb{R}^n$ there is no Cantor set all of whose projections to $(n - 1)$-planes are convex bodies [15, Thm. 3]; generalizations are obtained in [19, Thm. 4.7], [8, Thm. 1]. In $\mathbb{R}^3$, we construct a convex body $K$ and a Cantor set $A \subset K$ such that the
projections of $K$ and $A$ into each line coincide (Corollary 2.7).

J. Cobb [15, Thm. 1] constructed a Cantor set in $\mathbb{R}^3$ such that its projection to every 2-plane is 1-dimensional. Cobb asked [15, p. 126]: “Could there be Cantor sets all of whose projections are connected, or even cells? ...given $n > m > k > 0$, does there exist a Cantor set in $\mathbb{R}^n$ such that each of its projections into $m$-planes is exactly $k$-dimensional?” (Following [9], we call these sets $(n, m, k)$-sets.) Examples of $(n, m, m-1)$- and $(n, n-1, k)$-Cantor sets are given in [22, Thm. 1] and [9, Thm. 1], respectively. The known $(n, m, k)$-Cantor sets [13], [11, Thm. 6.2], [15, Thm. 1], [22, Thm. 1], [9, Thm. 1] are tame by constructions and by [25, Thm. I.4.2] (see Statement 3.2; for the notions of tame and wild, see Definition 2.1).

Consideration of wild Cantor sets provides another approach to Cobb’s question. We show that there is an easily described (wild) Cantor set in $\mathbb{R}^3$ — a well-known Antoine’s Necklace — all of whose plane projections are connected, one-dimensional (Theorem 2.8); and no of its projections can be homeomorphic to a graph (Corollary 7.3). This answers the first part of Cobb’s question for the case of 3-dimensional space.

Generalizing Antoine’s construction, W.A. Blankinship [12] and A.A. Ivanov [24] described, for $n \geq 3$, (wild) Cantor sets in $\mathbb{R}^n$ with non-simply connected complement. In $\mathbb{R}^n$, $n \geq 3$, there exist wild Cantor sets with simply connected complements [26], [17], [35]. We show that each Cantor set in $\mathbb{R}^n$ can be slightly moved so that the resulting set is an $(n, n-1, n-2)$-set (Theorem 4.1). Projections of a wild Cantor set have rather complicated structure (Statement 7.2, Corollary 7.3).

In terms of projections, several tameness conditions are known. W.A. Blankinship proved that if a compactum $K \subset \mathbb{R}^n$, $n \geq 3$, has a zero-dimensional projection into at least one hyperplane, then $\pi_1(\mathbb{R}^n - K) = 0$ [12, Thm. 3E]. D.G. Wright and J.J. Walsh proved the following: let $p : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the projection onto a fixed hyperplane; if $K \subset \mathbb{R}^n$, $n \geq 4$, is a compactum with $\dim K \leq n - 3$, $\dim p(K) \leq n - 3$, and $\dim (K \cap p^{-1}(z)) \leq 0$ for each $z \in \mathbb{R}^{n-1}$, then $K$ is 1-LCC in $\mathbb{R}^n$; see [37, Thm. 5.3, 5.4] for $n \geq 5$ and [38] for $n = 4$. (By definition, $K \subset \mathbb{R}^n$ is 1-LCC embedded iff for each $x \in K$ and each neighborhood $U$ of $x$ in $\mathbb{R}^n$, there exists a smaller neighborhood $V$ of $x$ in $\mathbb{R}^n$ such that any map $\gamma : S^1 \to V - X$ is null-homotopic in $U - X$). For a zero-dimensional compactum $K \subset \mathbb{R}^n$ the 1-LCC property is equivalent to its tameness in the sense of Definition 2.1; see [23] or [11] for $n = 3$, [28] for $n \geq 5$, and [36], [10] for $n = 4$; for further details, refer to [20] or [16, 3.4]. D.R. McMillan, Jr. showed that if a zero-dimensional compactum
$X \subset \mathbb{R}^{n+m} = \mathbb{R}^n \oplus \mathbb{R}^m$, $n, m \geq 1$, satisfies $\dim p_1(X) = 0 = \dim p_2(X)$, where $p_1 : \mathbb{R}^n \oplus \mathbb{R}^m \to \mathbb{R}^n$ and $p_2 : \mathbb{R}^n \oplus \mathbb{R}^m \to \mathbb{R}^m$ are the projections, then $X$ is tame \cite[Cor. 2]{28}. We show that if $X \subset \mathbb{R}^n$, $n \geq 2$, is a zero-dimensional compactum, $\Pi \subset \mathbb{R}^n$ is a subspace such that $\dim p_\Pi(X) = 0$ and $\dim \Pi \in \{1, 2, n-2, n-1\}$, then $X$ is tame (Theorem 6.1).

J. Cobb remarked that Cantor sets that raise dimension under all projections and those that do not are both dense in the Cantor sets in $\mathbb{R}^n$ \cite[p. 128]{15}. In a forthcoming paper, we show that all projections of a typical (in the sense of Baire category) Cantor set are Cantor sets, partially answering another question of Cobb.

**Notation and Conventions**

An $m$-plane in $\mathbb{R}^n$ is any $m$-dimensional affine subspace. An $(n-1)$-plane is also called a hyperplane. We consider only orthogonal projections. For a plane $\Pi \subset \mathbb{R}^n$, the projection map $\mathbb{R}^n \to \Pi$ is denoted by $p_\Pi$.

The usual Euclidean distance between points $x$ and $y$ of $\mathbb{R}^n$ is denoted by $d(x, y)$. A ball in $\mathbb{R}^n$ (sometimes we refer to it as an $n$-ball) is a set of the form $\{X \in \mathbb{R}^n \mid d(X, A) \leq r\}$, where $A \in \mathbb{R}^n$ and $r > 0$. An $n$-cell is any set homeomorphic to a ball in $\mathbb{R}^n$.

A circle in $\mathbb{R}^3$ is a set obtained by rotating a point around a straight line.

For a topological manifold-with-boundary $M$, denote by $\text{Int} M$ and $\partial M$ the interior and the boundary of $M$, correspondingly.

Any topological space homeomorphic to the usual middle-thirds Cantor set is called a Cantor set. (These are exactly non-empty metric zero-dimensional perfect compacta \cite[Thm. 12.8]{29}.)

All maps are continuous unless otherwise specified.

Let $I = [0, 1]$.

An isotopy of $\mathbb{R}^n$ (also called ambient isotopy) is a level preserving homeomorphism $H : \mathbb{R}^n \times I \cong \mathbb{R}^n \times I$ such that $h_0 = \text{id}$, where $h_t : \mathbb{R}^n \cong \mathbb{R}^n$ is defined by $H(x, t) = (h_t(x), t)$. If, moreover, $H$ is a PL homeomorphism, then we call it a PL isotopy. (For PL topology, see e.g. \cite[1.6]{32}; an overview of PL notions and results can be found in \cite[1.6]{33}). We will write $\{h_t, t \in I\}$ or briefly $\{h_t\}$ instead of $H$. An $\varepsilon$-isotopy of $\mathbb{R}^n$ is an isotopy $\{h_t\} : \mathbb{R}^n \to \mathbb{R}^n$ such that $d(h_t(x), x) \leq \varepsilon$ for each $x \in \mathbb{R}^n$ and each $t \in I$.

A subset $P \subset \mathbb{R}^n$ is called a polyhedron (or polyhedral) if it is the union of a finite collection of simplices.
2 Basic example: Antoine’s Necklace

The main result of this section is Theorem 2.8. I decided to discuss the basic example in a separate section; the description and proof are rather elementary, easier than in the general case (Section 4) where results from general theory of zero-dimensional compacta are needed.

Definition 2.1. A zero-dimensional compact set \( K \subset \mathbb{R}^n \) is called tame if there exists a homeomorphism \( h \) of \( \mathbb{R}^n \) onto itself such that \( h(K) \) is a subset of a straight line in \( \mathbb{R}^n \); otherwise, \( K \) is called wild.

By Statement 3.3 we may replace “a homeomorphism \( h \) of \( \mathbb{R}^n \)” by “an isotopy \( \{h_t\} \) of \( \mathbb{R}^n \)” in Definition 2.1; the new definition is equivalent to the original one.

In \( \mathbb{R}^2 \) each zero-dimensional compactum is tame [3, 75, p. 87–89] (one may also refer to [25, Cor. II.3.2, Cor. II.3.3] or [29, Chap. 13]).

L. Antoine in [1] sketched and in [3, 78, p. 91–92] explicitly constructed a Cantor set in \( \mathbb{R}^3 \) which is now widely known as an Antoine’s necklace; Antoine proved its wildness in [3] (see Statement 2.5). Let us describe Antoine’s construction.

Definition 2.2. Let \( \Pi \) be a 2-plane in \( \mathbb{R}^3 \). Let \( D \subset \Pi \) be a disk of radius \( r > 0 \) with center \( Q \), and \( \ell \subset \Pi \) a straight line such that \( d(Q, \ell) = R > r \). A standard solid torus \( T \) is the solid torus of revolution generated by revolving \( D \) in \( \mathbb{R}^3 \) about \( \ell \). The central circle of \( T \) is the circle generated by rotating the point \( Q \). The center of \( T \) is the center of its central circle.

Definition 2.3. A simple chain in a standard solid torus \( T \subset \mathbb{R}^3 \) is a finite family \( T_1, \ldots, T_q, q \geq 3 \), of pairwise disjoint congruent standard solid tori such that

1) \( T_1 \cup \ldots \cup T_q \subset \text{Int } T \);
2) centers of \( T_1, \ldots, T_q \) are subsequent vertices of a regular convex \( q \)-gon inscribed in the central circle of \( T \);
3) \( T_i \) and \( T_j \) are linked for \( |i - j| \equiv 1 \mod q \), and are not linked otherwise;
4) for each \( i \), the central circle of \( T_i \) is zero-homotopic in \( T \).

The chain \( T_1, \ldots, T_q \) looks like a usual “necklace” which winds once around the hole of \( T \); no one of the tori \( T_i \) embraces the hole of \( T \).

Antoine takes a standard solid torus \( T \) in \( \mathbb{R}^3 \); he assumes that the ratio \( \frac{r}{R} \) is small enough (where \( r \) and \( R \) are the numbers described in Definition 2.2).
There exists an integer $k$ (sufficiently large in comparison with $\frac{r}{R}$) such that a simple chain of cyclically linked $k$ tori, each geometrically similar to $T$, can be placed inside $T$, so that their centers lie on the central circle of $T$ and form a regular convex $k$-gon. Then, he applies a similarity transformation to place a chain of $k$ tori in the interior of each torus of the previous level, and so on. Since diameters of the tori tend to zero, the intersection of sets constructed on all levels is a Cantor set.

To generalize this construction, one allows the tori to be non-standard solid tori, that is, not necessarily tori of revolution; also, one varies the number of tori on each stage. The positions of the tori may also change. To be precise, let us give a definition.

**Definition 2.4.** *An Antoine’s Necklace* is a Cantor set in $\mathbb{R}^3$ which can be obtained as an intersection $A = \bigcap_{i=1}^{\infty} M_i$, where each $M_i$ is the union of a finite number of pairwise disjoint standard solid tori such that:

1) $M_1 \subset \mathbb{R}^3$ is a standard solid torus;
2) $M_{i+1} \subset \text{Int} M_i$ for each $i \geq 1$;
3) for each $i \geq 1$ and each component $T$ of $M_i$, the intersection $M_{i+1} \cap T$ is the union of solid tori, which form a simple chain in $T$.

One thus obtains uncountably many inequivalently embedded Antoine’s necklaces [31 Thm. 2]. Each of them is wild. L. Antoine derived the wildness property for his particular family of necklaces from the next statement proved in [3, 80-82, p. 93-96]; with minor changes, Antoine’s reasoning holds for each set that satisfies Definition 2.3:

**Statement 2.5.** Let $A \subset \mathbb{R}^3$ be an Antoine’s necklace and $\Sigma \subset \mathbb{R}^3$ a set homeomorphic to a 2-sphere. Suppose that $A$ intersects each of the (two) connected components of $\mathbb{R}^3 - \Sigma$. Then $\Sigma \cap A \neq \emptyset$.

This implies

**Statement 2.6.** Each orthogonal projection of each Antoine’s necklace is connected.

Proof. Let $A \subset \mathbb{R}^3$ be an Antoine’s necklace and $\Pi \subset \mathbb{R}^3$ a 2-plane. Recall that $A = \bigcap_{i=1}^{\infty} M_i$, where each $M_i$ is the union of standard solid tori (Definition 2.3).
Suppose that for some integer \( N \), the projection \( p_\Pi(M_N) \) is not connected. One may find a simple closed curve \( L \subset \Pi \) such that \( L \cap p_\Pi(M_N) = \emptyset \), while \( p_\Pi(M_N) \) intersects both connected components \( L_{\text{Int}} \) and \( L_{\text{Ext}} \) of \( \mathbb{R}^2 - L \) (recall that each component of \( M_N \) is a solid torus of revolution). Take a large number \( h \in \mathbb{R} \) such that \( A \cap \{ x_3 = \pm h \} = \emptyset \). The union \( L \times [-h, h] \cup L_{\text{Int}} \times \{ \pm h \} \subset \mathbb{R}^3 \) is homeomorphic to a 2-sphere; this contradicts Statement 2.5.

Consequently, each \( p_\Pi(M_i) \) is connected. The intersection of a nested sequence of connected, compact sets is connected. Hence \( p_\Pi(A) = p_\Pi(\cap_{i=1}^{\infty} M_i) = \cap_{i=1}^{\infty} p_\Pi(M_i) \) is also connected.

We have proved that the projection into each 2-plane is connected; this easily implies that the projection into any straight line is connected.

We get an immediate corollary (compare [19, Thm. 4.7]):

**Corollary 2.7.** Let \( A \subset \mathbb{R}^3 \) be an Antoine’s necklace. There exists a convex body \( K_A \subset \mathbb{R}^3 \) such that \( p_\ell(A) = p_\ell(K_A) \) for each line \( \ell \subset \mathbb{R}^3 \).

**Proof.** Let \( K_A \) be the convex hull of \( A \). We only need to show that \( K_A \) is a body (that is, has a non-empty interior as a subset of \( \mathbb{R}^3 \)). Suppose not; then \( K_A \) lies in a 2-plane. Recall that each zero-dimensional compactum in \( \mathbb{R}^2 \) is tame \([3, 75, \text{p. 87–89}]\); thus \( A \) is tame, a contradiction.

**Theorem 2.8.** There exists an Antoine’s Necklace \( A \) in \( \mathbb{R}^3 \) such that for each 2-plane \( \Pi \subset \mathbb{R}^3 \), the set \( p_\Pi(A) \) is connected and one-dimensional.

This fact, once stated, is easily believed. But it is not as evident as it may seem. The number of tori increases quickly; it may happen that they are not sufficiently thin, and the shadows of the whole family are “fat” (compare Statement 5.2). Therefore we need to take care of the tori widths at each stage.

**Definition 2.9.** Let \( \ell \subset \mathbb{R}^3 \) be a straight line and \( r > 0 \). A tube of radius \( r \) with axis \( \ell \) is the set \( t(\ell, r) = \{ P \in \mathbb{R}^3 \mid d(P, \ell) \leq r \} \).

**Definition 2.10.** A strip in \( \mathbb{R}^3 \) is a (closed) set between two parallel 2-planes.

**Definition 2.11.** The width \( w(X) \) of a set \( X \subset \mathbb{R}^n \) is the infimum of the distances between two parallel hyperplanes such that the part of space between them contains \( X \).

For example, \( w(t(\ell, r)) = 2r \).
**Statement 2.12.** Let \( t_1, \ldots, t_s \) be a collection of tubes in \( \mathbb{R}^3 \) such that \( w(t_1) + \ldots + w(t_s) < 2\varepsilon \). Then, for each 2-plane \( \Pi \subset \mathbb{R}^3 \), the projection \( p_{\Pi}(t_1 \cup \ldots \cup t_s) \) contains no 2-ball of radius \( \varepsilon \).

Proof. Suppose that there exists a 2-plane \( \Pi \) and a 2-ball \( B \subset \Pi \) of radius \( \varepsilon \) such that \( B \subset p_{\Pi}(t_1 \cup \ldots \cup t_s) = p_{\Pi}(t_1) \cup \ldots \cup p_{\Pi}(t_s) \).

For any tube \( t(\ell, r) \), the projection \( p_{\Pi}(t(\ell, r)) \) is either a 2-ball of radius \( r \), or a strip of width \( 2r \). In either case, \( p_{\Pi}(t(\ell, r)) \) can be covered by a strip of width \( 2r \). Hence the ball \( B \) of radius \( \varepsilon \) can be covered by strips of total width less than \( 2\varepsilon \); this contradicts the Plank Theorem (for the planar case, see English translation of A. Tarski’s and H. Moese’s papers (1931–32) in [27, Chapter 7] where the history of the problem is also discussed; the high-dimensional case is treated in [7]).

**Statement 2.13.** Let \( L \subset \mathbb{R}^3 \) be a circle. For each \( \varepsilon > 0 \), there exists a finite family of tubes \( t_1, \ldots, t_s \) such that \( L \subset \text{Int} t_1 \cup \ldots \cup \text{Int} t_s \) and \( w(t_1) + \ldots + w(t_s) < \varepsilon \).

Proof. Denote the radius of \( L \) by \( r \). Let \( L_N \) be a regular convex \( N \)-gon inscribed in \( L \). Around each straight line containing a side of \( L_N \), construct a tube of width \( 4 \left( r - r \cos \frac{\pi}{N} \right) \). It can be easily seen that the interiors of these tubes cover \( L \). Since the total width \( N \times 4 \left( r - r \cos \frac{\pi}{N} \right) \) tends to zero as \( N \to \infty \), the result follows.

Proof of Theorem 2.8. Recall that a subset of \( \mathbb{R}^n \) is \( n \)-dimensional iff it contains an \( n \)-ball [21, Theorems 1.8.10, 4.1.5]. Equivalently, a subset of \( \mathbb{R}^n \) is \( n \)-dimensional iff for some integer \( i \) it contains an \( n \)-ball of radius \( \frac{1}{i} \). The desired Antoine’s necklace is constructed in countable number of steps. On each step \( i \in \mathbb{N} \), we first construct arbitrary “preliminary” simple chains \( M_i \) as described in Definition 2.3, and then, we replace them (without changing their central circles) by thinner ones so that no projection of this new \( M_i \) contains a 2-ball of radius \( \frac{1}{i} \). (This is achieved with the help of Statements 2.12 and 2.13.) Only after that, we pass on to the step \( i + 1 \). We also require that the diameters of the tori which constitute \( M_i \) tend to zero as \( i \to \infty \).

Then \( A := \bigcap_{i=1}^{\infty} M_i \) is an Antoine’s necklace in the sense of Definition 2.3. The projection \( p_{\Pi}(A) = p_{\Pi}(\bigcap_{i=1}^{\infty} M_i) = \bigcap_{i=1}^{\infty} p_{\Pi}(M_i) \) contains no 2-ball, thus \( \dim p_{\Pi}(A) \leq 1 \). To finish the proof, apply Statement 2.6.
3 Additional known facts about Cantor sets in $\mathbb{R}^n$

Before we pass on to the higher-dimensional case, let us discuss some results which will be of use below.

**Definition 3.1.** [25, Def. I.3.2] A set $X$ in an $n$-dimensional topological manifold $M$ is called **cellularly separated** (in $M$) if for each open neighborhood $V$ of $X$ there exists a family $\{u_\alpha\}$ of open subsets of $M$ such that $X \subset \bigcup u_\alpha \subset \bigcup \overline{u}_\alpha \subset V$; each $u_\alpha$ is homeomorphic to $\mathbb{R}^n$; each $\overline{u}_\alpha$ is an $n$-cell; and $\overline{u}_\alpha \cap \overline{u}_\beta = \emptyset$ for $\alpha \neq \beta$.

The next proposition was proved in [25, Thm. I.4.2] (for $n = 3$, see [11, Thm. 1.1]; for arbitrary $n$, one may also refer to [30, Thm. 1]).

**Statement 3.2.** For each $n \geq 1$ and each zero-dimensional compact set $K \subset \mathbb{R}^n$ the following conditions are equivalent:

(a) $K$ is cellularly separated in $\mathbb{R}^n$;

(b) $K$ is tame in $\mathbb{R}^n$.

The proof of [25, Thm. I.4.2] implies:

**Statement 3.3.** Let $U \subset \mathbb{R}^n$, $n \geq 1$, be an open connected set, and $K_1, K_2 \subset U$ be two Cantor sets tame in $\mathbb{R}^n$. There exists an isotopy $\{h_t\}$ of $\mathbb{R}^n$ such that $h_t = \text{id}$ on $\mathbb{R}^n - U$ for each $t \in I$, and $h_1(K_1) = K_2$.

The following definition is essentially the one given in [1, p. 662] (where the term “variétés de définition de l’ensemble” is used) and [3, pp. 79, 82] (“surfaces de définition”).

**Definition 3.4.** Let $K$ be a zero-dimensional compact subset of $\mathbb{R}^n$. A sequence $\{M_i, i \in \mathbb{N}\}$ of subsets of $\mathbb{R}^n$ is called a defining sequence for $K$ if all $M_i$'s are compact polyhedral $n$-manifolds-with-boundary, $M_{i+1} \subset \text{Int } M_i$, and $K = \bigcap_{i=1}^\infty M_i$. (The equality $\dim K = 0$ implies that the maximal diameter of the components of $M_i$ necessarily tends to zero as $i \to \infty$.)

L. Antoine showed that each zero-dimensional compactum in $\mathbb{R}^n$, $n \geq 1$, has a defining sequence [1 p. 662], [3, 69, p.78–80]. Using this result, he proved that each zero-dimensional compactum in $\mathbb{R}^n$, $n \geq 2$, can be extended to a simple arc [3 72, p. 82–84] (this was first stated in [31]). In particular,
there exists a Jordan arc in $\mathbb{R}^3$ which contains an Antoine’s necklace; this is a first example of a wild arc [3, 83, p. 97]. In [2], Antoine announced and in [4] described in detail an embedded 2-sphere in $\mathbb{R}^3$ which contains an Antoine’s necklace, the description can be found also in [29, Thm. 18.7]; this is the first example of a wild surface.

We will need a stronger property of defining sequences. By [5, Lemma 4] each zero-dimensional compact set $K \subset \mathbb{R}^3$ has a defining sequence $\{M_i\}$ such that each $M_i$ is a disjoint union of polyhedral cells-with-handles. In general, Shtan’ko–Bryant dimension theory [20, Prop. 1.2, Thm. 1.4], [16, Thm. 3.4.11, 3.4.12] implies

**Statement 3.5.** Each zero-dimensional compact set $K \subset \mathbb{R}^n$, $n \geq 2$, has a defining sequence $\{M_i\}$ such that each $M_i$ is a regular neighborhood of an $(n-2)$-dimensional polyhedron; in particular, $M_i$ is a PL manifold-with-boundary.

Finally, let us state one more fact which we will make use of (see [28, Cor. 1], [30, Theorem 3]; a very short argument which deduces this from the Klee flattening theorem [33, Thm. 2.5.1], [16, Cor. 2.5.3] can be found in [11, Thm. 2]). This proposition is covered by [37, Thm. 5.3, 5.4], [38]. We obtain another generalization of Statement 3.6 below (see Corollary 6.2).

**Statement 3.6.** If a zero-dimensional compactum $K \subset \mathbb{R}^n$ lies in a hyperplane, then $K$ is tame.

### 4 Getting rid of $(n-1)$-dimensional projections

**Theorem 4.1.** Let $K \subset \mathbb{R}^n$, $n \geq 2$, be any Cantor set. For each $\varepsilon > 0$ there exists an $\varepsilon$-isotopy $\{h_t\} : \mathbb{R}^n \cong \mathbb{R}^n$ such that $\dim p_\Pi(h_1(K)) = n-2$ for each $(n-1)$-plane $\Pi \subset \mathbb{R}^n$.

If $K$ is tame, the result follows from known constructions. To prove Theorem 4.1 for the case of a wild set $K$, we proceed essentially as we did above, with Antoine’s necklace. We choose a defining sequence all of whose elements are regular neighborhoods of $(n-2)$-dimensional polyhedra; “compressing” the elements of the defining sequence to thin neighborhoods of these subpolyhedra, we will eliminate $(n-1)$-dimensional projections. This process resem-
bles that in [6, Thm. 8]. This gives the inequality \( \dim p_{\Pi}(h_1(K)) \leq n - 2 \); the equality will then follow from [37, 38].

Let us extend Definitions 2.9 and 2.10.

**Definition 4.2.** Let \( L \subset \mathbb{R}^n \) be an \((n - 2)\)-dimensional plane and \( r > 0 \). A *tube* of radius \( r \) with base \( L \) is the set \( t(L, r) = \{ P \in \mathbb{R}^n \mid d(P, L) \leq r \} \).

**Definition 4.3.** An \( n \)-*strip* is a (closed) set between two parallel hyperplanes in \( \mathbb{R}^n \).

**Statement 4.4.** Let \( t(L, r) \) be a tube of radius \( r \) with base \( L \) in \( \mathbb{R}^n \), and let \( \Pi \subset \mathbb{R}^n \) be an \((n - 1)\)-plane. The set \( p_{\Pi}(t(L, r)) \) can be covered by an \((n - 1)\)-strip in \( \Pi \) of width \( 2r \).

**Proof.** For any point \( P \in t(L, r) \) we have \( d(P, L) \leq r \). This implies \( d(p_{\Pi}(P), p_{\Pi}(L)) \leq r \), and \( p_{\Pi}(t(L, r)) \subset \{ Q \in \Pi \mid d(Q, p_{\Pi}(L)) \leq r \} \). It can be easily seen that the last set is contained in an \((n - 1)\)-strip in \( \Pi \) of width \( 2r \) (for this, consider two cases: \( \dim p_{\Pi}(L) \) equals \( n - 2 \) or \( n - 3 \)).

**Statement 4.5.** Let \( U \subset \mathbb{R}^n, n \geq 2 \), be an open set; let \( M \subset U \) be a regular neighborhood of an \((n - 2)\)-dimensional polyhedron \( S \subset M \). Then for each \( r > 0 \) there exists a PL isotopy \( \{ h_t \} : \mathbb{R}^n \cong \mathbb{R}^n \) such that \( h_t|_{\mathbb{R}^n-U} = \text{id} \) for each \( t \in I \), \( h_1(M) \subset \text{Int} M \), and \( h_1(M) \) can be covered by the interiors of finitely many tubes \( t(L_1, \delta), \ldots, t(L_s, \delta) \) with \( s \cdot \delta \leq r \). In particular, no projection of \( h_1(M) \) into an \((n - 1)\)-dimensional plane contains an \((n - 1)\)-ball of radius \( r \).

**Proof.** Cover \( S \) by a finite number of \((n - 2)\)-planes \( L_1, \ldots, L_s \). Take a positive number \( \delta < \frac{\varepsilon}{4} \) such that the open \( \delta \)-neighborhood \( O(S, \delta) \) of \( S \) lies in \( \text{Int} M \). Note that \( O(S, \delta) \) is covered by the interiors of the tubes \( t(L_1, \delta), \ldots, t(L_s, \delta) \) and \( s \cdot \delta < r \). Let \( N \subset O(S, \delta) \) be a regular neighborhood of \( S \). There exists a PL isotopy \( \{ h_t \} : \mathbb{R}^n \cong \mathbb{R}^n \) such that \( h_t|_{\mathbb{R}^n-U} = \text{id} \) for each \( t \in I \), and \( h_1(M) = N \) [33, Thm. 1.6.4]. This is the desired isotopy. By Statement 4.4 together with Plank Theorem 7, no projection of \( O(S, \delta) \) (hence also of \( h_1(M) \)) contains an \((n - 1)\)-ball of radius \( r \).

Now we are ready to prove the main result of this section.

**Proof of Theorem 4.1.** Case 1. \( K \) is tame. This case reduces to known results as follows. There are a finite number of pairwise disjoint open connected sets \( U_1, \ldots, U_s \subset U \) such that \( K \subset U_1 \cup \ldots \cup U_s \) and \( \text{diam} U_j < \varepsilon \) for each \( j = 1, \ldots, s \). We may assume that each \( K \cap U_j \) is non-empty; thus
\( K \cap U_j \) is a tame Cantor set. For each \( j \), take a tame \((n, n - 1, n - 2)\)-Cantor set \( K_j \subset U_j \) from [22, Thm. 1] or [9, Thm. 1]. By Statement 3.3 there exists an isotopy \( \{h_t\} : \mathbb{R}^n \cong \mathbb{R}^n \) such that \( h_t|_{\mathbb{R}^n - U_1 \cup \ldots \cup U_s} = \text{id} \) for each \( t \in I \), and \( h_1(K \cap U_j) = K_j \) for each \( j = 1, \ldots, s \). This is the desired map.

**Case 2.** \( K \) is wild. By [37, Thm. 5.3, 5.4] and [38] it suffices to obtain inequalities \( \dim \pi_1(h_1(K)) \leq n - 2 \) for each \((n - 1)\)-plane \( \Pi \subset \mathbb{R}^n \).

For this, we apply Statement 4.3 infinitely many times.

Let \( \{M_i\} \) be a defining sequence for \( K \) such that each \( M_i \) is a regular neighborhood of an \((n - 2)\)-dimensional polyhedron \( K_i \).

Let \( \varepsilon_1 = \varepsilon, \varepsilon_2, \varepsilon_3, \ldots \) be a sequence sufficiently fast decreasing to zero. (The exact meaning of this will be clarified below.)

There exists an integer \( i_1 \) such that the diameters of the components of \( M_{i_1} \) do not exceed \( \varepsilon_1 = \varepsilon \). Apply Statement 4.3 taking \( U := \text{Int} M_{i_1} \), \( M := M_{i_1+1} \) and \( S := S_{i_1+1} \); there exists a PL isotopy \( \{h_t^{(1)}\} \) of \( \mathbb{R}^n \) such that \( h_t^{(1)}|_{\mathbb{R}^n - M_{i_1}} = \text{id} \) for each \( t \in I \), \( h_1^{(1)}(M_{i_1+1}) \subset M_{i_1+1} \), and no projection of \( h_1^{(1)}(M_{i_1+1}) \) contains a unit \((n - 1)\)-ball. Note that \( \{h_t^{(1)}\} \) is an \( \varepsilon_1 \)-isotopy.

There exists an integer \( i_2 > i_1 \) such that the diameters of the components of \( h_1^{(1)}(M_{i_2}) \) do not exceed \( \varepsilon_2 \). There exists a PL isotopy \( \{h_t^{(2)}\} \) of \( \mathbb{R}^n \) such that \( h_t^{(2)}|_{\mathbb{R}^n - h_1^{(1)}(M_{i_2})} = \text{id} \) for each \( t \in I \), \( h_1^{(2)}(h_1^{(1)}(M_{i_2+1})) \subset h_1^{(1)}(M_{i_2+1}) \), and no projection of \( h_1^{(2)}(h_1^{(1)}(M_{i_2+1})) \) contains an \((n - 1)\)-ball of radius \( \frac{1}{2} \). Note that \( \{h_t^{(2)}\} \) is an \( \varepsilon_2 \)-isotopy.

Continuing in this way, for each integer \( k \) we find an integer \( i_k > i_{k-1} \) such that the diameters of the components of \( h_1^{(k-1)} \circ h_1^{(k-2)} \circ \ldots \circ h_1^{(1)}(M_{i_k}) \) do not exceed \( \varepsilon_k \). There exists a PL isotopy \( \{h_t^{(k)}\} \) of \( \mathbb{R}^n \) such that

\[
H_t^{(k)}|_{\mathbb{R}^n - h_1^{(k-1)} \circ \ldots \circ h_1^{(1)}(M_{i_k})} = \text{id} \quad \text{for each} \quad t \in I,
\]

\[
h_1^{(k)} \circ \ldots \circ h_1^{(2)} \circ h_1^{(1)}(M_{i_k+1}) \subset h_1^{(k-1)} \circ \ldots \circ h_1^{(2)} \circ h_1^{(1)}(M_{i_k+1}),
\]

and no projection of \( h_1^{(k)} \circ \ldots \circ h_1^{(2)} \circ h_1^{(1)}(M_{i_k+1}) \) contains an \((n - 1)\)-ball of radius \( \frac{1}{k} \). Note that \( \{h_t^{(k)}\} \) is an \( \varepsilon_k \)-isotopy.

Note that on each step we are free to choose \( \varepsilon_k \) as small as we wish; we choose them so that the sequence \( \{h_t^{(k)} \circ \ldots \circ h_t^{(2)} \circ h_t^{(1)}\} \) converges to an isotopy \( \{h_t\} \) of \( \mathbb{R}^n \), see e.g. [25, Lemma I.4.1].

Let us show that \( \{h_t\} \) is the required isotopy. By construction, \( h_t|_{\mathbb{R}^n - M_{i_1}} = \text{id} \) for each \( t \in I \); thus \( d(h_t(x), x) \leq \varepsilon_1 = \varepsilon \) for each \( x \in \mathbb{R}^n \). No projection of
\(h_1(K)\) contains an \((n - 1)\)-ball. In fact, for each \((n - 1)\)-plane \(\Pi \subset \mathbb{R}^n\) and each \(k \in \mathbb{N}\) we have by construction

\[h_1(K) \subset h_1(M_{ik+1}) \subset h_1^{(k)} \circ \ldots \circ h_1^{(1)}(M_{ik+1});\]

hence

\[p_\Pi(h_1(K)) \subset p_\Pi(h_1^{(k)} \circ \ldots \circ h_1^{(1)}(M_{ik+1}));\]

the last set does not contain an \((n - 1)\)-ball of radius \(\frac{1}{k}\). Therefore \(p_\Pi(h_1(K))\) does not contain \((n - 1)\)-balls, consequently \(\dim p_\Pi(h_1(K)) \leq n - 2\) by [21, Theorems 1.8.10, 4.1.5].

5 Making all projections have maximal possible dimension

In this section, we extend results of L. Antoine and K. Borsuk.

**Statement 5.1.** Let \(L \subset \mathbb{R}^2\) be the union of a finite number of 2-simplices. There exists a Cantor set \(K \subset \mathbb{R}^2\) such that \(p_\ell(K) = p_\ell(L)\) for each line \(\ell \subset \mathbb{R}^2\).

Proof. The union of a finite number of Cantor sets in \(\mathbb{R}^n\) is again a Cantor set [21, Thm. 1.3.1]. Hence it suffices to consider the case of a 2-simplex \(L\). For each \(x \in \partial L\) take a hexagon \(H_x\) (which is meant to be taken together with its interior domain) affinely equivalent to a regular hexagon such that \(x \in H_x \subset L\) and \(x\) not a vertex of \(H_x\).

The triangle \(\partial L\) is compact; there are finitely many sets \(H_{x_1}, \ldots, H_{x_s}\) which cover \(\partial L\). The union \(H_{x_1} \cup \ldots \cup H_{x_s}\) form “an interior collar” for \(\partial L\) in \(L\). For each \(i = 1, \ldots, s\), there exists a Cantor set \(K_i \subset H_{x_i}\) such that \(p_\ell(K_i) = p_\ell(H_{x_i})\) for each straight line \(\ell\) (\(K_i\) is an affine image of the set constructed in [1, 9, p.272; and fig.2 on p.273]). The union \(K = K_1 \cup \ldots \cup K_s\) is the desired set.

**Statement 5.2.** Let \(K \subset \mathbb{R}^n\) be a Cantor set, \(n \geq 2\). For each \(\varepsilon > 0\) there exists an \(\varepsilon\)-isotopy \(\{h_t\} : \mathbb{R}^n \cong \mathbb{R}^n\) such that \(\dim p_\Pi(h_1(K)) = \dim \Pi\) for each plane \(\Pi \subset \mathbb{R}^n\).

Proof. By [21] Theorems 1.8.10, 4.1.5], a subset of \(\mathbb{R}^d\) is \(d\)-dimensional iff it contains a \(d\)-ball; hence it suffices to get the equality \(\dim p_\Pi(h_1(K)) = n - 1\) for each \((n - 1)\)-plane \(\Pi \subset \mathbb{R}^n\).
Let \( \{M_i\} \) be a defining sequence for \( K \). There exists an integer \( N \) such that the diameter of each component of \( M_N \) is less than \( \varepsilon \). Enumerate all components of \( M_N \) by \( M_N^{(1)}, \ldots, M_N^{(s)} \). We may assume that for every \( i \) the set \( K \cap M_N^{(i)} \) is non-empty, hence is a Cantor set. Let \( K_i \subset K \cap M_N^{(i)} \) be a Cantor set which is tame in \( \mathbb{R}^n \) (it can be constructed with the help of Statement 3.2), and let \( B_i \subset \text{Int } M_N^{(i)} \) be a tame Cantor set all of whose projections onto hyperplanes are \((n-1)\)-dimensional [13]. By Statement 3.3 there exists an isotopy \( \{h_t\} : \mathbb{R}^n \cong \mathbb{R}^n \) such that \( h_t|_{\mathbb{R}^n - \bigcup_{i=1}^s M_N^{(i)}} = 1 \) for each \( t \in I \), and \( h_1(K_i) = B_i \) for each \( i = 1, \ldots, s \). This is the desired map.

6 Tameness of a Cantor set with a zero-dimensional projection

The main result of this section extends particular cases of [28, Cor. 2]. On the other side, the case \( \text{dim } \Pi = n-1 \) is covered by [37, Thm. 5.3, 5.4] and [38]; we give a different independent proof. We provide unified arguments for three cases \( \text{dim } \Pi \in \{1, n-2, n-1\} \); the case \( \text{dim } \Pi = 2 \) is easy.

**Theorem 6.1.** Let \( X \subset \mathbb{R}^n \) be a zero-dimensional compact set, \( n \geq 2 \). Suppose that \( \text{dim } p_{\Pi}(X) = 0 \) for some plane \( \Pi \subset \mathbb{R}^n \) whose dimension equals 1, 2, \( n-2 \) or \( n-1 \). Then \( X \) is tame.

**Corollary 6.2.** Let \( X \) be a zero-dimensional compactum such that \( X \subset \mathcal{C} \times \mathbb{R}^{n-1} \subset \mathcal{C} \oplus \mathbb{R}^{n-1} = \mathbb{R}^n, n \geq 2 \). Then \( X \) is tame in \( \mathbb{R}^n \).

In our proof of Theorem 6.1, we use the following lemma inspired by the Klee trick [33, Thm. 2.5.1], [16, Thm. 2.5.1].

**Lemma 6.3.** Let \( Q, \hat{Q}, \tilde{Q} \) be \( k \)-cells in \( \mathbb{R}^k \), \( k \geq 1 \), such that \( \tilde{Q} \subset \text{Int } \hat{Q} \subset \hat{Q} \subset \text{Int } Q \). Let \( L \subset \mathbb{R}^\ell, \ell \geq 1 \), be a compact \( PL \ell \)-manifold-with-boundary and \( O_L \) its open neighborhood. Then there exists an isotopy \( \{F_t\} \) of \( \mathbb{R}^k \times \mathbb{R}^\ell \) such that each \( F_t \) is of the form \( (x,y) \mapsto (f_t(x), y) \) for \( x \in \mathbb{R}^k, y \in \mathbb{R}^\ell \), \( F_t|_{\mathbb{R}^k \times \mathbb{R}^\ell - Q \times O_L} = 1 \) for each \( t \in I \), and \( F_1(\hat{Q} \times L) \subset \tilde{Q} \times L \).

Proof of Lemma 6.3. Let \( N \) be a regular neighborhood of \( L \) in \( \mathbb{R}^\ell \) such that \( N \subset O_L \). The closure \( \overline{N - L} \) is a PL \( \ell \)-manifold-with-boundary; its boundary contains \( \partial L \). By the Collar Neighborhood Theorem, \( \partial L \) has a collar in \( \overline{N - L} \). We may therefore assume that \( \partial L \times [0,1] \) is PL embedded in
of the original inclusion $L \subset O_L$. Take an isotopy $\{g_t\}$ of $Q$ such that $g_0 = \text{id}$, $g_t|_{\partial Q} = \text{id}$ for each $t \in I$, and $g_1(\tilde{Q}) \subset \tilde{Q}$. The desired isotopy $\{F_t\}$ is defined by the formula

$$(x, y) \mapsto \begin{cases} 
(g_t(x), y) \text{ for } (x, y) \in Q \times L; \\
(g_t(1-s)(x), y) \text{ for } (x, y) \in Q \times (\partial L \times \{s\}); \\
(x, y) \text{ otherwise.} 
\end{cases}$$

Proof of Theorem 6.1. We may assume that $\Pi$ is the coordinate subspace $\mathbb{R}^m \times \{0\}^{n-m}$, $m = \dim \Pi$. For brevity, we write $p = p_{\Pi}$ and $Y = p(X)$.

The case of a 2-dimensional plane $\Pi$ is easy. In fact, any zero-dimensional compactum in $\mathbb{R}^2$ is tame [3, 75, p. 87–89], [25, Cor. II.3.2, Cor. II.3.3], [29, Chap. 13]. Take a homeomorphism $h: \mathbb{R}^2 \cong \mathbb{R}^2$ such that $h(p(X)) \subset \{0\} \times \mathbb{R}^1$. Define a self-homeomorphism $H$ of $\mathbb{R}^n = \mathbb{R}^2 \times \mathbb{R}^{n-2}$ by $H = h \times \text{id}_{\mathbb{R}^{n-2}}$; we have $H(X) \subset \{0\} \times \mathbb{R}^{n-1}$. By Statement 3.16 $X$ is tame.

For the other cases, we assume that $X \subset \text{Int } I^n$. We will show that $X$ satisfies condition (a) of Statement 3.2. Fix any $\varepsilon > 0$.

Step 1. For each $y \in p(X)$, the preimage $X \cap p^{-1}(y)$ is a zero-dimensional compact subset of $\{y\} \times \text{Int } I^{n-m}$. There exists a compact $(n-m)$-manifold-with-boundary $M_y \subset \text{Int } I^{n-m}$ such that $X \cap p^{-1}(y) \subset \{y\} \times M_y$, and the diameter of each connected component of $M_y$ is less than $\varepsilon$. Moreover, we will assume that: if $m = 1$ then each $M_y$ is a PL manifold-with-boundary; if $m = n - 1$ then each $M_y$ is a closed segment; and if $m = n - 2$ then each $M_y$ is a flat 2-cell in $\text{Int } I^2$ (see Definition 7.1 and Statement 3.2).

Step 2. We have $X \cap \{(y) \times (I^{n-m} - \text{Int } M_y)\} = \emptyset$ for each $y \in Y$. Both $X$ and $\{y\} \times (I^{n-m} - \text{Int } M_y)$ are compact sets; hence there exists a PL $m$-cell $L_y \subset \text{Int } I^n$ such that $y \in \text{Int } L_y$, $X \cap (L_y \times (I^{n-m} - \text{Int } M_y)) = \emptyset$, and $\text{diam } L_y < \varepsilon$. We have $X \cap p^{-1}(L_y) \subset L_y \times \text{Int } M_y$. Note that for each subset $L \subset L_y$ we have $X \cap p^{-1}(L) \subset L \times \text{Int } M_y$.

Step 3. The collection $\{\text{Int } L_y, y \in Y\}$ is a cover of $Y$ by open subsets of $\mathbb{R}^m$; take its finite subcover $\{\text{Int } L_{y_1}, \ldots, \text{Int } L_{y_q}\}$. Using the equality $\dim Y = 0$, we will replace $L_{y_1}, \ldots, L_{y_q}$ by smaller subsets which are pairwise disjoint (but not necessarily $m$-cells anymore). In fact, by compactness of $Y$, one can find a $\delta > 0$ with the property: for each set $D \subset \mathbb{R}^m$ with $\text{diam } D < \delta$ and $D \cap Y \neq \emptyset$, there is an $i \in \{1, \ldots, q\}$ such that $D \subset \text{Int } L_{y_i}$. Take pairwise disjoint compact PL $m$-manifolds-with-boundary $N_1, \ldots, N_t \subset \text{Int } I^m$ such that $Y \subset \text{Int } N_1 \cup \ldots \cup \text{Int } N_t$, $\text{diam } N_j < \delta$ and $Y \cap N_j \neq \emptyset$ for each
$j \in \{1, \ldots, t\}$ (Statement 3.3). Each $N_j$ is a subset of some $\text{Int } L_{y_j}$ with $i \in \{1, \ldots, q\}$.

Replace the family $\{L_{y_1}, \ldots, L_{y_q}\}$ by $\{N_1, \ldots, N_t\}$. Saving our old notation, we will assume that $L_{y_1}, \ldots, L_{y_q}$ are themselves pairwise disjoint compact PL $m$-manifolds-with-boundary (not necessarily $m$-cells). Take pairwise disjoint open neighborhoods $O(L_{y_1}), \ldots, O(L_{y_q})$ of $L_{y_1}, \ldots, L_{y_q}$ in $\mathbb{R}^m$ with $\text{diam } O(L_{y_i}) < \varepsilon$ for each $i$.

**Step 4.** For each $i \in \{1, \ldots, q\}$, denote the connected components of $M_{y_i}$ by $M_{y_i,1}, \ldots, M_{y_i,\alpha(i)}$; we get

$$X = \bigcup_{i=1}^{q} (X \cap p^{-1}(L_{y_i})) \subset \bigcup_{i=1}^{q} (\text{Int } L_{y_i} \times \text{Int } M_{y_i}) = \bigcup_{i=1}^{q} \bigcup_{j=1}^{\alpha(i)} (\text{Int } L_{y_i} \times \text{Int } M_{y_i,j}).$$

**Step 5.** We are now ready to accomplish the proof.

**Case 1:** $m = 1$. We may assume that $L_{y_1}, \ldots, L_{y_q}$ are pairwise disjoint closed segments. For each $i \in \{1, \ldots, q\}$ take closed segments $\tilde{L}_{y_i}$ and $\hat{L}_{y_i}$ such that $\tilde{L}_{y_i} \subset \text{Int } \hat{L}_{y_i} \subset \text{Int } L_{y_i}$ and $X \subset \bigcup_{i=1}^{q} \bigcup_{j=1}^{\alpha(i)} (\text{Int } \hat{L}_{y_i} \times \text{Int } M_{y_i,j})$.

For each $j \in \{1, \ldots, \alpha(i)\}$ let $O(M_{y_i,1}), \ldots, O(M_{y_i,\alpha(i)})$ be pairwise disjoint open neighborhoods of $M_{y_i,1}, \ldots, M_{y_i,\alpha(i)}$ in $\mathbb{R}^{n-m}$, each of diameter less than $\varepsilon$. For each $M_{y_i,j}$ take an $(n-m)$-ball $B_{y_i,j} \subset \mathbb{R}^{n-m}$ of radius $\varepsilon \sqrt{n-m}$ such that $M_{y_i,j} \subset \text{Int } B_{y_i,j}$ (balls $B_{y_i,j}$ may intersect each other). Using Lemma 6.3, for each $i \in \{1, \ldots, q\}$ and each $j \in \{1, \ldots, \alpha(i)\}$ construct an isotopy $\{F_{(i,j),i}\}$ of $\mathbb{R}^n$ identical outside $L_{y_i} \times O(M_{y_i,j})$ such that $F_{(i,j),1}(\tilde{L}_{y_i} \times M_{y_i,j}) \subset L_{y_i} \times M_{y_i,j} \subset \hat{L}_{y_i} \times \text{Int } B_{y_i,j}$. All isotopies $\{F_{(i,j),i}\}$ “glued together” give an isotopy $\{F_i\}$. The set $\tilde{L}_{y_i} \times B_{y_i,j}$ is an $n$-cell, and $\text{diam}(\tilde{L}_{y_i} \times B_{y_i,j}) < \sqrt{\varepsilon^2 + 4\varepsilon^2(n-m)} = \varepsilon \sqrt{1 + 4(n-m)}$. Note that $\text{diam}(L_{y_i} \times O(M_{y_i,j})) < \varepsilon \sqrt{2}$, hence $\{F_i\}$ is an $\varepsilon \sqrt{2}$-isotopy. Thus $\{F_i\}^{-1}$ is also an $\varepsilon \sqrt{2}$-isotopy, and $(F_i)^{-1}(\tilde{L}_{y_i} \times B_{y_i,j})$ is an $n$-cell of diameter less than $2\sqrt{2\varepsilon} + \varepsilon \sqrt{1 + 4(n-m)}$.

The cells $(F_i)^{-1}(\tilde{L}_{y_i} \times B_{y_i,j})$ are pairwise disjoint, and $X \subset \bigcup_{i=1}^{q} \bigcup_{j=1}^{\alpha(i)} (\text{Int } \hat{L}_{y_i} \times \text{Int } B_{y_i,j})$. Statement 3.2 now implies that $X$ is tame.

**Case 2:** $m = n-1$. For each $i \in \{1, \ldots, q\}$ and each $j \in \{1, \ldots, \alpha(i)\}$ take closed segments $\tilde{M}_{y_i,j}$ and $\hat{M}_{y_i,j}$ such that $\tilde{M}_{y_i,j} \subset \text{Int } \hat{M}_{y_i,j} \subset \hat{M}_{y_i,j} \subset \text{Int } M_{y_i,j}$, $X \subset \bigcup_{i=1}^{q} \bigcup_{j=1}^{\alpha(i)} (\text{Int } L_{y_i} \times \text{Int } \hat{M}_{y_i,j})$, and all $\hat{M}_{y_i,j}$ are pairwise disjoint.
For each \( i \in \{1, \ldots, q\} \) take an open neighborhood \( O(L_{y_i}) \) of \( L_{y_i} \) in \( \mathbb{R}^m \) such that \( \text{diam} \ O(L_{y_i}) < \varepsilon \) and all \( O(L_{y_i}) \) are pairwise disjoint; take an \( m \)-ball \( B_{y_i} \) of radius \( \varepsilon \sqrt{m} \) such that \( L_{y_i} \subset \text{Int} \ B_{y_i} \) (these balls may intersect each other). Using Lemma 6.3, for each \( i \in \{1, \ldots, q\} \) and \( j \in \{1, \ldots, \alpha(i)\} \) construct an isotopy \( \{F_{(i,j),t}\} \) of \( \mathbb{R}^n \) identical outside \( O(L_{y_i}) \times \tilde{M}_{y_{i,j}} \) such that \( F_{(i,j),1}(L_{y_i} \times \tilde{M}_{y_{i,j}}) \subset L_{y_i} \times \tilde{M}_{y_{i,j}} \subset (\text{Int} \ B_{y_i}) \times \tilde{M}_{y_{i,j}} \). As in Case 1, all isotopies \( \{F_{(i,j),t}\} \) glue together to an isotopy \( \{F_t\} \); the sets \( (F_t)^{-1}(B_{y_i} \times \tilde{M}_{y_{i,j}}) \) are pairwise disjoint \( n \)-cells of diameter \( < k\varepsilon \) which cover \( X \) \((k \text{ depends on } n \text{ and } m, \text{ and not on } \varepsilon)\), and Statement 3.2 applies.

Case 3: \( m = n - 2 \). We use the same argument as in Case 2. The only difference is that \( \tilde{M}_{y_{i,j}} \) and \( \tilde{M}_{y_{i,j}} \) are \( 2 \)-cells (since any zero-dimensional compactum in \( \mathbb{R}^2 \) is tame); we also assume that each \( \tilde{M}_{y_{i,j}} \) is flat in \( \mathbb{R}^2 \) (see Definition 7.1). The details are omitted.

7 Tameness of a Cantor set with “simple” projection

In this last section, we make an attempt to investigate relationship between wildness of a Cantor set and “complexity” of its projections.

Definition 7.1. A \( k \)-cell \( X \subset \mathbb{R}^n \) is called flat if there exists a homeomorphism \( h \) of \( \mathbb{R}^n \) onto itself such that \( h(X) \) is a \( k \)-simplex.

For more information on flat and tame embeddings, in particular for examples of cells which are not flat, refer to the books [29], [25], [33], [16].

Statement 7.2. Let \( K \subset \mathbb{R}^n \), \( n \geq 3 \) be a zero-dimensional compact set. Suppose that for some \( m \)-plane \( \Pi \), where \( 1 \leq m \leq n \), there exists a countable family of subsets \( X_1, X_2, \ldots \subset \Pi \) such that each \( X_i \) is a flat cell in \( \Pi \), \( \dim X_i \leq m - 1 \), and \( p_{\Pi}(K) \subset X_1 \cup X_2 \cup \ldots \). Then \( K \) is tame in \( \mathbb{R}^n \).

Proof. Let \( \Pi^\perp \) be the orthogonal complement of \( \Pi \) in \( \mathbb{R}^n \). Since \( K \) is compact, there exists an \((n-m)\)-ball \( B \subset \Pi^\perp \) such that \( K \subset \Pi \times B \). We have \( K \subset (X_1 \times B) \cup (X_2 \times B) \cup \ldots \). We may assume that for each \( i \) the set \( K \cap (X_i \times B) \) is non-empty, hence it is a zero-dimensional compactum. It is easy to see that each \( X_i \times B \) is a flat cell in \( \mathbb{R}^n \), and \( \dim(X_i \times B) \leq n - 1 \). Therefore \( K_i \) is tame in \( \mathbb{R}^n \) (Statement 3.6). Now \( K \) is tame in \( \mathbb{R}^n \) by [30, Thm. 8] (for \( n = 3 \) see also [11, Thm. 6.1]).
Corollary 7.3. Let $K \subset \mathbb{R}^n$ be a zero-dimensional compact set, $n \geq 3$. Suppose that for some 2-plane $\Pi$, there exists a subset $X \subset \Pi$ homeomorphic to a one-dimensional polyhedron such that $p_\Pi(K) \subset X$. Then $K$ is tame in $\mathbb{R}^n$.

Proof. A set homeomorphic to a one-dimensional polyhedron is the union of a finite family of simple arcs. Each simple arc in plane is flat by Antoine’s theorem [3, p.21, 17], [25, Thm. II.4.3], [29, Thm. 10.8]. Statement 7.2 now applies.

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