The Weinberg - Faddeev solution to the problems of Quantum Field Theory and Quantum Gravity: Quantum Spacetime

G.H. Gadiyar,
Department of Mathematics, Indian Institute of Technology,
Madras 600 036, INDIA.

Abstract: In this paper a fundamental length is introduced into physics. This is done in a way which respects special relativity and quantum field theory. The theory has formal similarity to quantum field theory though its properties are far better: divergences are got rid of. The problem of quantizing gravity is straightforward in the approach.
Trying to unify the General Theory of Relativity with quantum field theory has led to great difficulties. In this paper a new and simple approach to the problem of quantum gravity is sketched.

This work is inspired by two streams of thought: one due to Weinberg [2] which is rich in physical ideas and the other due to Faddeev [1] which has deep mathematical thoughts. The merging of these two approaches yields a simple solution to two theoretical problems: the problem of divergences in quantum field theory and the issues of quantum gravity.

Weinberg [2] follows an unorthodox approach to the General Theory of Relativity. Traditionally General Relativity is based on the equivalence principle and the geometry of curved spacetime. Thus the approach is heavily based on geometry. According to Weinberg this approach drives a wedge between General Relativity and particle physics. He takes a different point of view in two papers written in the mid-sixties. He bases his arguments on the axioms of quantum field theory. He analyses first the case of massless spin one particles and derives the Maxwell equations. Next he analyses the case of massless spin two particles. Surprisingly he is able to derive the equivalence principle and the Einstein equations from the properties of Lorentz invariance and quantum field theory. However the problems of renormalization remain. Weinberg’s analysis seems to indicate that we are victims of history. If quantum field theory had been developed before Einstein’s equations were discovered curved spacetime would not dominate the thoughts of theoretical
physicists. If one can address the problem of renormalization then gravity can be considered the Lorentz invariant theory of massless spin two particles coupled to the energy momentum tensor.

Faddeev [1] recently made an interesting analysis of relativity and quantum mechanics. He says that both these developments were mathematically what are called deformations. This means that the dimensionful parameters $c$ the velocity of light and $\hbar$ the Planck’s constant were introduced into the older structures of physics in a consistent way. He then says that there is one more constant which is dimensionful, namely, the Planck length $L$. He forcefully argues that the general structure of physics should be modified to include this constant as well. As $L^2 = \frac{G\hbar}{c^3}$ where $G$ is the Newton’s gravitation constant, one suspects that the structure of spacetime will need modification to address the problems of quantum gravity.

Thus the logic of the approach of this paper is as follows: first modify the structure of quantum field theory (which respects special relativity and quantum physics) in a consistent way so as to introduce a fundamental length. To do this requires heavy use of the correspondence principle: we guess that the new structure will be formally similar to the older structure. This is like the fact the both classical and quantum mechanics have underlying Lie algebra structures, one the Poisson bracket and the other the commutator. To quantize gravity the approach pioneered by Weinberg becomes feasible as it turns out that introducing $L$ cures the divergence difficulties. Thus the
solution of one theoretical problem can be used to resolve another.

To introduce a fundamental length in a consistent way requires a couple of assumptions. The definition of interval is modified to

\[ \hat{s}^2 = \hat{x}_\mu \eta_{\mu\nu} \hat{x}_\nu \]  

(1)

with

\[ [\hat{x}_\mu, \hat{x}_\nu^\dagger] = L^2 \eta_{\mu\nu} . \]  

(2)

Here \( x_\mu \) and \( x_\mu^\dagger \) are annihilation and creation operators. Thus the interval is quantized in units of \( L^2 \). However one has to choose the state on which these operators act. Here we argue from the correspondence principle that it is possibly a coherent state

\[ \hat{x}_\mu | x_\mu > = x_\mu | x_\mu > , \]  

(3)

\[ < x_\mu | x'_\mu > = e^{-\frac{(x_\mu - x'_\mu)^2}{2L^2}} . \]  

(4)

We consider only real coherent states which have only real \( x_\mu \). This is possible as the set can be used to construct a partition of unit and still constitutes an overcomplete basis.

In one dimension,

\[ \hat{x} | 0 > = 0 \]

\[ \hat{x} | n > = \sqrt{n} | n - 1 > \]
\[\hat{\mathcal{g}} | n > = \sqrt{n+1} | n+1 >\]
\[\hat{x} | x > = x | x >\]
\[| x > = e^{-\frac{x^2}{2L^2}} \sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n!}} | n >\]

where \(| n >\) has eigen value \(nL^2\). Further we demand

\[\int dx | x >< x | = \sum_{n} | n >< n | = 1.\]

But for notational changes this is the same as the harmonic oscillator.

To pinpoint what is the effect of this modification let us consider a scalar field. Mathematically one has to replace \(x_\mu\) in the usual approach by \(\hat{x}_\mu | x_\mu >\) and hence \(\phi(x_\mu)\) by \(\phi(\hat{x}_\mu) | x_\mu >\) and \(\frac{\partial \phi(x_\mu)}{\partial x_\mu}\) by \(\frac{\partial \phi(x_\mu)}{\partial \hat{x}_\mu} | x_\mu >\). Thus the action is now

\[S = \int < x | \partial_\mu \phi(x) \partial^{\mu} \phi(x) | x > dx .\]

Varying \(\phi(x)\) then yields the equation of motion

\[\delta S = \int < x | \delta \phi(x) \Box \phi(x) | x > dx = 0 .\]

Now \(| x | \delta \phi(x)\) is an arbitrary vector in the Hilbert space. Hence if \(\delta S\) is zero it follows that

\[\Box \phi(x) | x > = 0 .\]

This is akin to the Schrödinger equation. This equation replaces the usual \(\Box \phi(x) = 0\). Further one has the analog of \(e^{ikx}\) which in Dirac’s notation is
< k | x >. Here it can be seen that

\[ \int dk < k | x > < k | x' > = e^{-\frac{(x - x')^2}{2L^2}}. \] (10)

\[ \int dx < k | x > < x | k' > = e^{-\frac{(k - k')^2L^2}{2}}. \] (11)

Notice that the sharp delta function does not appear.

Let us find the Green’s function for the scalar field

\[ (\square + m^2) \phi(x) | x > = J(x) | x >. \] (12)

By going to | k > basis and back to | x > basis one can check that

\[ \phi(x) | x > = \int dk dx' J(x') | x' > < x' | k > < k | x >. \] (13)

So the usual Green’s function \( \int dk \frac{e^{ik(x-x')}}{k^2 + m^2} \) is replaced by

\[ \int dk \frac{x'k > x k >}{k^2 + m^2}. \] This has much better properties as can be seen from the identities (10) and (11) given earlier. Also note that \( \int dk | k > < k | \) is tracing over discrete states \( \sum |n > < n| \) of the interval. Hence one has introduced some type of lattice like structure for spacetime. The sharp delta functions which are the origin of the divergence problem are now softened.

It is now possible to write the path integral by replacing \( \phi(x) \) by \( \phi(x) | x >. \) Thus formally one has (for a very simple case)

\[ \int D\phi(x)|x > \exp(-i)\left( \int dx < x | \phi(x) \square \phi(x)|x > \right) \]
\[- \int dx \langle x | J(x) \phi(x) | x > \rangle \]
\[= \exp \left( -\frac{i}{2} \int dx \; dx' \; dk \; J(x) \langle x > \frac{< x | k > < k | x' >}{k^2} < x' | J(x') \rangle \right). (14)\]

Thus Feynman rules can be derived.

It is also possible to do the quantization in the canonical way. There is no problem when it comes to introducing spinors.

To recapitulate, quantum field theory is modified by redefining the notion of interval. This is also a modification of classical field theory in a consistent way. The coherent state is used because of a correspondence principle argument. The light cone properties are improved. The new theory has far better properties than the older theory. Further there is great formal similarity and one can derive Feynman rules from the Path Integral approach.

To quantize gravity is now possible. As the Green’s functions have much better properties all the integrals occuring in the Feynman diagram calculations will now have cutoffs (of an exponential type). Now one has to follow Weinberg’s approach and use Lorentz Invariant theory of spin two gravitons coupling to the energy momentum tensor.

Thus the idea of Faddeev to deform the laws of physics is implemented. The idea of Weinberg now can be used to quantize gravity. This could be called the Weinberg - Faddeev approach to the problems of quantum field theory and quantum gravity.

The advantage of this approach is that experimental facts can be accommodated. Model building is not affected. What has been made is a general
reformulation of physics and experimental details can be incorporated in the model. This is unlike string theory which has a lot of difficulty in agreement with experiment, which is consistent in 10 or 26 (not 4) space time dimensions and requires Kaluza-Klein theory, Calabi Yau manifolds and several exotic ad hoc assumptions and yet makes no contact with experimental facts enshrined in the Weinberg-Salam Model.

It must be stressed that the approach in this paper is extremely conservative both conceptually and mathematically.

**Acknowledgements:** Thanks to Dr. H.S.Sharatchandra for his encouragement and warm friendship. Thanks to L.Kannan of PPST for his help in analysing some of the problems. Thanks to Dr.Shantha Kamath and Mr.M.S.Kamath for giving the computer on which this paper was typed. The work is funded by CSIR, India.
References:

[1] *On the Relationship between Mathematics and Physics*, L.D.Faddeev,
Asia Pacific Physics News Vol.3, June-July, 1988 pp.21-22.

[2] *Photons and Gravitations in S-Matrix theory: Derivation of Charge Con-
servation and Equality of Gravitational and Inertial Mass*, S.Weinberg,
Phys. Rev. B1049-1056, Vol.135, 4B, 1964.

*Photons and Gravitons in Perturbation Theory: Derivation of Maxwell’s
and Einstein’s Equations*, S.Weinberg, Phys. Rev. B988-1002, Vol.138,
4B, 1965.