On the corner elements of the CKM and PMNS matrices

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Abstract – Recent experiments show that the top-right corner element \( U_{e3} \) of the PMNS matrix is small but nonzero, and suggest further via unitarity that it is smaller than the bottom-left corner element \( U_{\tau 1} \). Here, it is shown that if to the assumption of a universal rank-one mass matrix, long favoured by phenomenologists, one adds that this matrix rotates with scale, then it follows that:

A) by inputting the mass ratios \( m_c/m_t, m_s/m_b, m_\mu/m_\tau, \) and \( m_2/m_3 \), i) the corner elements are small but nonzero, ii) \( V_{ub} < V_{td} \), \( U_{e3} < U_{\tau 1} \), iii) estimates result for the ratios \( V_{ub}/V_{td} \) and \( U_{e3}/U_{\tau 1} \), and
B) by inputting further the experimental values of \( V_{us}, V_{tb}, U_{e2}, U_{\mu3} \), iv) estimates result for the values of the corner elements themselves. All the inequalities and estimates obtained are consistent with present data within expectation for the approximations made.

The purpose of this letter is to try to understand the smallness of the corner elements of the quark [1,2] and lepton [3,4] mixing matrices and their asymmetry about the diagonals. It is shown that a scheme with a rank-one rotating mass matrix (R2M2) devised to explain the hierarchical masses and mixing of fermions [5] will automatically also give the asymmetry.

To theoreticians, the mixing matrices of quarks and leptons are a bit of an embarrassment. While their experimental colleagues have improved the measurements of the CKM matrix elements and gained an increasingly clear picture even of the PMNS matrix, no commonly accepted theoretical understanding has been achieved even of the qualitative features.

The latest injection from experiment is a batch of new results [6–10] which give a nonzero value for the upper corner element \( U_{e3} \) of the PMNS matrix. A combined fit [11] gives a value of

\[
\sin^2(2\theta_{13}) = 0.096 \pm 0.013. \tag{1}
\]

And, with the CP phase \( \delta = 0 \), the PMNS matrix is [12]

\[
U_{PMNS} = \begin{pmatrix}
0.8205 & 0.549 & 0.152 \\
0.374 & 0.573 & 0.6935 \\
0.3815 & 0.567 & 0.6835
\end{pmatrix}, \tag{2}
\]

as compared to the measured CKM matrix given in [13]:

\[
V_{CKM} = \begin{pmatrix}
0.97427 & 0.22534 & 0.00351 \\
0.22520 & 0.97344 & 0.0412 \\
0.00867 & 0.0404 & 0.999146
\end{pmatrix}. \tag{3}
\]

We quote here only the absolute values, as we shall deal mainly with these, but we shall return later to the important question of the Kobayashi-Maskawa CP-violation phase [4].

One notes that the two matrices (2) and (3) share some qualitative features. In both, the corner elements are rather small and the bottom-left corner element is larger than the top-right corner element by about a factor of 2. This asymmetry, together with the small values of the corner elements, are what particularly interests us here, for the prediction of both would be a delicate test no ad hoc model is likely to reproduce.

The similarity between the two matrices suggests, to us at least, that they be treated similarly and understood together as two facets of the same phenomenon, and the R2M2 scheme is an attempt to do so.

The R2M2 scheme was suggested some years ago [14–16] as a possible explanation for the hierarchical mass spectrum and mixing pattern of quarks and leptons observed in experiment and incorporated per se into the standard model. We note that any rank-one mass
matrix can be written [17] in terms of a unit vector $\alpha$ in generation space:

$$m = m_T \alpha \alpha^\dagger.$$  

(4)

This means that there is only one massive generation. With $\alpha$ “universal” in the sense of being independent of the fermion types $T$ (i.e., whether quarks or lepton), it further implies that there is no mixing between up and down states. Now such a form of the fermion mass matrix has long been suggested by phenomenologists [18,19] as a good starting point for understanding mass hierarchy and mixing. What R2M2 adds to this is that the vector $\alpha$ rotates with changing scales under renormalization.

How R2M2 can lead to mixing and mass hierarchy can be seen most simply by considering just the two heaviest generations, and assuming for further simplification that $\alpha$ is real and that $m_T$ is scale independent. The eigenvector $\alpha$ at scale $\mu = m_t$ is the state vector $t$ of $t$, and the orthogonal vector $c$ is that of $c$. As the scale decreases, $\alpha$ rotates through an angle, say $\theta_c$, when it reaches the scale $\mu = m_c$, where it becomes the state vector $b$ of $b$. The vector orthogonal to $\alpha$ is then the state vector $s$ of $s$.

We have therefore two dyads, $\{t, c\}$ and $\{b, s\}$, linked by the mixing matrix

$$
\begin{pmatrix}
V_{ts} & V_{tb} \\
V_{cs} & V_{cb}
\end{pmatrix} =
\begin{pmatrix}
\cos \theta_b & -\sin \theta_b \\
\sin \theta_b & \cos \theta_b
\end{pmatrix}.

(5)

As to hierarchical masses, we have $m_t = m_U$ and $m_b = m_D$ for $U$- and $D$-type quarks. At $\mu = m_t$, the eigenvector $c$ has zero eigenvalue, but this is not the mass of the $c$ state, which should be evaluated at $\mu = m_c$. Indeed, $m_c$ is to be taken as the solution to the equation

$$\mu = \langle c | m(\mu) | c \rangle = m_U |\langle c | \alpha(\mu) \rangle|^2.$$  

(6)

A nonzero solution exists since the scale on the LHS decreases from $\mu = m_t$ while the RHS increases from zero at that scale. At $\mu < m_t$, the vector will have rotated from $t$ to a different direction so that it will have acquired a nonzero component, $(\sin \theta_c)$ in the direction of $c$ giving

$$m_c = m_t \sin^2 \theta_c.$$  

(7)

Further $m_c$ will be small if the rotation is not too fast. Similarly for $m_s$.

Despite these attractive features, R2M2 invites immediately, of course, some fundamental questions, e.g.: i) How are the masses appearing in (6), and the rotating mass matrix itself, formally related to the mass defined by the Green’s functions of the quarks and leptons? ii) What is really meant by the scale here in relation to the renormalisation theory? iii) Why, from the theoretical point of view, should the rotating vector $\alpha$ appearing in (4) be universal? It seems to us that all these questions can be properly answered only when one has actually a consistent theory which produces R2M2 as a consequence. We have worked and are still working strenuously towards such ends, producing already some of the answers needed [20]. Our aim in this paper, however, is just to expand on R2M2’s attractive features so as to engage the community’s attention on and help in resolving the questions raised.

In the realistic case when all three generations are taken into account basically the same arguments apply as when considering only the 2 heaviest. As the scale changes, the unit vector $\alpha$ now traces out a curve on the unit sphere and applying the same physical arguments as before, one deduces, say for the $U$-type quarks, the following formulae:

$$
t = \alpha(m_t), \quad c = u \times t, \quad u = \left(\frac{\alpha(m_c) \times \alpha(m_t)}{|\alpha(m_c) \times \alpha(m_t)|}\right),$$  

(8)

and

$$
m_t = m_U, \quad m_c = m_U |\langle c | \alpha(m_c) \rangle|^2, \quad m_u = m_U |\langle u | \alpha(m_u) \rangle|^2.$$  

(9)

Together, these 2 sets of coupled equations allow us to evaluate both the state vectors and the masses. Similar equations and remarks apply also to $D$-type quarks as well as to the leptons. With the state vectors so determined, the mixing matrices can then be evaluated, e.g., for quarks:

$$V_{\text{CKM}} \sim \begin{pmatrix} u \cdot d & u \cdot s & u \cdot b \\ c \cdot d & c \cdot s & c \cdot b \\ t \cdot d & t \cdot s & t \cdot b \end{pmatrix}.$$  

(10)

The expression for the lepton mixing matrix $U_{\text{PMNS}}$ would be similar.

An interesting feature of R2M2 is that the mass matrix remains of rank one and chiral invariant throughout. Yet, simply because the mass matrix rotates, the lower generations all acquire nonzero masses as if by “leakage” from the heaviest states. For a more detailed discussion of this point, the reader is referred to [5], especially 1.4 therein.

That the mass spectra and mixing matrices so obtained from a rank-one rotating mass matrix (R2M2) actually do resemble those observed in experiment can be checked in two ways: i) invent a model trajectory, then evaluate the mass spectra and mixing matrices of quarks and leptons to compare with experiment, or conversely, ii) fit a trajectory through the experimental data on these quantities. Both have been tried with encouraging success [5,16,21]. We quote here in fig. 1 a particularly simple example from an early work [22] which will be of use to us later. As seen in this figure, the rotation angle $\theta$ obtained via (5) and (7) from the experimental masses and mixing angles in the 2-generation simplification all fall on a smooth curve as a function of $\mu$ as expected.

However, both the above methods i) and ii) for testing the R2M2 hypothesis involve some manipulations of the data or assumptions about the shape of the rotation
trajectory, and may in the process obscure somewhat the basic simplicity of the result. What we wish to do here simplify the argument as to make the correlations between the various bits of data immediately obvious, in particular those in relation to the corner elements of the mixing matrices. To do so, we rely on the fact that most of the rotation angles involved are relatively small so that use can be made of the differential Serret-Frenet-Darboux formulae for curves lying on a surface.

At every point of the trajectory for \( \alpha \), we set up a Darboux triad, consisting of first the normal to the surface \( n \) (which for the special case here is the radial vector \( \alpha \)), the tangent vector \( \tau \) to the trajectory, and the normal \( \nu \) to both the above. The Serret-Frenet-Darboux formulae then reduce to

\[
\begin{align*}
\alpha' &= \alpha(s + \delta s) = \alpha(s) - \tau(s) \delta s, \\
\tau' &= \tau(s + \delta s) = \tau(s) + \alpha(s) \delta s + \kappa_g \nu(s) \delta s, \\
\nu' &= \nu(s + \delta s) = \nu(s) - \kappa_g \tau(s) \delta s.
\end{align*}
\]

\( s \) being the arc length and \( \kappa_g \) the geodesic curvature.

At \( \mu = m_t \), we recall that \( \alpha \) coincides with the state vector \( t \) for the \( t \) quark. As the three quarks, \( t, c \) and \( u \) are by definition independent quantum states, the state vectors \( c \) and \( u \) must be orthogonal to \( \alpha \) and to each other. They must therefore be related to \( \tau \) and \( \nu \) by a rotation about \( \alpha \) by an angle, say \( \omega_U \).

\[
(t, c, u) = (\alpha, \cos \omega_U \tau + \sin \omega_U \nu, \cos \omega_U \nu - \sin \omega_U \tau).
\]

Similarly, for the \( D \)-type quarks we can write

\[
(b, s, d) = (\alpha', \cos \omega_D \tau' + \sin \omega_D \nu', \cos \omega_D \nu' - \sin \omega_D \tau'),
\]

where \( \{\alpha', \tau', \nu'\} \) is the Darboux triad taken at \( \mu = m_b \), and \( \omega_D \) is the corresponding rotation angle about \( \alpha' \).

Now the angle \( \theta_{tb} \) between \( t \) and \( b \), which on the unit sphere is also the arc-length between the two, is envisaged in the rotation scheme to be rather small. As seen in fig. 1, the rotation angle deduced from data seems to approach an asymptote at \( \mu = \infty \), the best fit to the data there being in fact an exponential. The rotation will speed up as \( \mu \) decreases, but for \( \mu \) between \( m_t \) and \( m_b \), the rotation remains rather small. Indeed, according to (5) it is given by the CKM matrix element \( V_{tb} = \cos \theta_{tb} \). As seen above in (3), this is measured to have the value

\[
\theta_{tb} = \delta s \approx 0.04331 \pm 0.0005.
\]

To this order of smallness then, we can take the expressions in (11) as the Darboux triad at \( \mu = m_b \) and hence (13) as the \( D \)-triad. This gives immediately the CKM matrix, according to (10), as

\[
\text{see eq. (15) above}
\]

with \( \omega = \omega_D - \omega_U \).

Although correct only to first order in \( \theta_{tb} \), (15) exhibits succinctly some of the special properties arising from rotation with clear correspondence with experiment, which we shall now examine.

First, by virtue of the fact that \( \theta_{tb} \) is small, we note already that: i) the off-diagonal elements in the last row and the last column are all of order \( \theta_{tb} \) and therefore small compared to the others; ii) the three diagonal elements are markedly different, the first two differing from unity by an amount of order \( \theta_{tb} \) while the last stands alone, differing from unity by only order \( \theta_{tb}^2 \); iii) the elements \( V_{us} \) and \( V_{cd} \) are equal also to first order in \( \theta_{tb} \). A glance at (3) shows that these are all in agreement with what is experimentally observed.

Next, focussing now on the corner elements

\[
\begin{align*}
V_{ub} &= u \cdot b = \sin(\omega_U) \theta_{tb}, \\
V_{td} &= t \cdot d = -\sin(\omega_D) \theta_{tb}.
\end{align*}
\]

we recall that \( \omega_U \) arises as a consequence of the rotation of the vector \( \alpha(\mu) \) as \( \mu \) changes from \( m_t \) to \( m_c \), and is thus generically of the same order of magnitude as \( \theta_{tb} \), which, according to (7) or (9) above, gives rise to the mass ratio

\[
\frac{\mu}{\mu_2} 
\]
The same conclusion applies to \( \omega_D \), namely that it should be of the same order as \( \theta_{bs} \) which gives rise to the mass ratio \( m_s/m_b \). Hence it follows that the two corner elements, since they are respectively of order \( \theta_{bs} \) and \( \theta_{tc} \) must both be particularly small compared with the others as a result of the rotation scenario. These corner elements are small basically because they are given by the twist of the trajectory, and the geodesic torsion \( \tau_g \) being zero on a sphere, the twist can only arise as a second order effect of the rotation. We have thereby a ready explanation in the rotation scheme for why the corner elements in the CKM matrix are so particularly small, as experimentally observed.

As a corollary of both \( \omega_U \) and \( \omega_D \) being small, it follows from (15) that the elements \( V_{cb} \) and \( V_{ts} \) will have about the same value as \( \theta_{bs} \), i.e., by (14) \( \sim 0.041 \), again as experimentally observed in (3). Estimates on \( V_{cb} \) and \( V_{ts} \) can be made immediately because they are proportional in (13) to the normal curvature \( \kappa_n = 1 \), but not on the two similarly placed elements \( V_{us}, V_{cd} \), since these are proportional to the geodesic curvature \( \kappa_g \), which depends on the rotation trajectory.

Going further, we notice in fig. 1 that rotation seems to be speeding up from \( \mu \sim m_t \) to \( \mu \sim m_b \). More concretely, taking the mass values in GeV,

\[
\begin{align*}
m_t &= 173.5 \pm 0.6 \pm 0.8, \\
m_b &= 4.18 \pm 0.03, \\
m_s &= 1.275 \pm 0.025, \\
m_c &= 0.955 \pm 0.005
\end{align*}
\]  
(17)

(cited by PDG [13], one obtains

\[
\begin{align*}
\theta_{tc} &\sim \sqrt{m_c/m_t} \sim 0.085 \pm 001, \\
\theta_{bs} &\sim \sqrt{m_s/m_b} \sim 0.151 \pm 0.004.
\end{align*}
\]  
(18)

Now in the rotation picture, as explained above, \( \omega_U \) is closely related to \( \theta_{tc} \) and \( \omega_D \) to \( \theta_{bs} \), so that in as much as \( \theta_{tc} < \theta_{bs} \), so will

\[
V_{ub} < V_{td},
\]  
(19)

which is the correct asymmetry in the corner elements observed in experiment, again readily explained here by rotation. As a corollary of (19), one can deduce the following inequalities:

\[
\frac{V_{ub}}{V_{tb}} < \frac{V_{ts}}{V_{tb}} < \frac{V_{cd}}{V_{us}}.
\]  
(20)

implied by (19) and (15) which are equally satisfied by experiment.

One can go further still to make a semi-quantitative estimate for the size of the asymmetry between the two corner elements as follows. The angle we are after is \( \omega_U \), which is the angle between \( c \), the state vector of the \( c \) quark, and the original tangent vector \( \tau \). This is easily seen in fig. 2, as can be checked also by an explicit calculation using elementary differential geometry, to have half the value of the change in direction of the tangent vector itself, in the limit when the latter value is small. Hence, we have the results

\[
\omega_U = \frac{1}{2} \kappa_g \theta_{tc}, \quad \omega_D = \frac{1}{2} \kappa_g \theta_{bs},
\]  
(21)

although in the “\( D \)” case \( \kappa_g \) should in principle refer to the geodesic curvature taken at \( \mu = m_t \), not at \( \mu = m_s \), as in the “\( U \)” case. However, if we ignore this difference, which is of order \( \theta_{tb} \) compared with \( \kappa_g \) and therefore negligible to the order we are working, we obtain

\[
\frac{V_{ub}}{V_{tb}} \sim \sin \omega_U \sim \sin \theta_{tc}, \quad \frac{V_{us}}{V_{tu}} \sim \sin \omega_D \sim \sin \theta_{bs}.
\]  
(22)

Taking the estimates obtained before in (18) for these angles one then obtains the following estimate compared to experiment:

\[
\frac{V_{ub}}{V_{tb}} \sim 0.56; \quad \frac{V_{ub}}{V_{tb}} = 0.41 \pm 0.03.
\]  
(23)

This is as close an agreement as one can expect, since the starting formulae (11) are correct only to order \( \delta S \sim \theta_{tb} \) and from (14) one would expect an error in the matrix elements of the order of \( \theta_{tb}^2 \sim 0.0017 \), which is not much smaller than the actual values of the matrix elements themselves.
So far one has input from experiment only the values of the mass ratios (18) and the fact that $\theta_{ub}$ is small. One can go even further still to estimate the actual values of the corner elements by inputting in addition $\theta_{ub}$ from (14) to set the scale and, say, the Cabibbo angle $V_{us}$ from (3) to estimate the value of the geodesic curvature $\kappa_0$. Indeed, using the formulae for $V_{us}$ in (15), and for $\omega_U, \omega_D$ in (21), one easily obtains

$$\kappa_0 \sim 3.0, \quad \omega_U \sim 0.128, \quad \omega_D \sim 0.23. \quad (24)$$

This then gives the following estimates for the actual values of the corner elements compared with experiment:

$$V_{ub}^{\text{est}} \sim 0.005, \quad V_{ub}^{\text{exp}} = 0.00351^{+0.00015}_{-0.00014},$$

$$V_{td}^{\text{est}} \sim 0.009, \quad V_{td}^{\text{exp}} = 0.00867^{+0.00022}_{-0.00031}. \quad (25)$$

Again the agreement is about as good as can be expected, given the intrinsic errors in the starting Serret-Frenet-Darboux formulae of first order.

Now that one has reproduced by the above means the corner elements, one can proceed to give values to all elements of the unitary CKM matrix:

$$V_{\text{CKM}}^{\text{est}} = \begin{pmatrix} 0.97427 & 0.2254 & 0.005 \\ 0.2252 & 0.97346 & 0.0408 \\ 0.009 & 0.0401 & 0.999146 \end{pmatrix}, \quad (26)$$

where the entries in italics are inputs, the rest being evaluated. This compares very well with (3) from experiment.

One important feature missed out in the above treatment of the CKM matrix is, of course, the K-M CP-violating phase to which, as promised, we now return. Interestingly, R2M2 automatically provides an explanation for this phase, and in a particularly novel way by linking it to the theta-angle in the QCD action of topological origin, thereby offering even a new solution to the old strong CP problem. This comes about as follows. The mass matrix (4), being of rank one throughout, has two zero eigenvalues, remaining thus chiral invariant at every scale. Hence, a chiral transformation can be performed to eliminate the theta-angle from the QCD action without making the mass matrix (4) complex. This chiral transformation has, however, to be performed on the fermion state in the direction of the Darboux vector $\nu$, which changes in direction with scale. Hence, its effect will get transmitted on to the state vectors of the various quarks by rotation, and further on to the CKM matrix, appearing there as a CP-violating phase. In other words, at one stroke, R2M2 can eliminate the strong CP angle theta, solving thereby the strong CP problem and, by transforming theta via rotation, to give the CKM matrix a (weak) CP-violating phase even with $\alpha$ real in (4). Amazingly, it even follows that for a theta-angle of order unity, the numbers in (3) or (26) will automatically give a Jarlskog invariant $J$ of order $10^{-5}$ as is observed in experiment. The details have been given already in [24]. The only new addition from the present Darboux approach is an explicit, though only approximate, expression for the Jarlskog invariant [25]:

$$J = -\cos \omega_U \cos \omega_D \sin(\omega_U + \omega_D) \kappa_0 \theta_{ub} \sin(\theta/2). \quad (27)$$

The next exciting question is whether the same arguments can apply to the PMNS matrix for leptons. Qualitatively, this seems attractive. Given that the rotation rate in fig. 1 is seen to increase for decreasing scale, it would follow immediately that the leptons at lower scales have larger mixings than the quarks at higher scales, just as is seen in experiment. In the details, however, there are a couple of serious reservations. First, the expansion parameter $\delta s$ is now given, in parallel to $\theta_{ub}$, above, by $U_{e3}$ in (2) as $\sin \theta_{e3} \sim 0.73$, which will take a lot of imagination to consider as a small expansion parameter. Secondly, on the physical masses of the neutrinos as given in experiment, the question arises whether they should be regarded as the Dirac masses which figure in the rotation formulae of (9) above, or as the masses obtained from these via some seesaw mechanism. This question matters in the estimate for the rotation angles $\theta_{\nu_{ei}}$ between $\nu_i$ and the heaviest neutrino, and the next heaviest $\nu_2$, which is related to the Dirac masses. However, let us be cavalier here and boldly ignore the first reservation and repeat the arguments for the cases suggested.

Starting then from the experimental values of $m_\tau = 1.777, m_\mu = 0.106$ in GeV [12], we first obtain

$$\sin \theta_{\tau\mu} = \sqrt{m_\mu/m_\tau} = 0.244, \quad (28)$$

the experimental errors here being negligible. Then taking the experimental neutrino mass differences [13]

$$\Delta m_{32}^2 = (2.427^{+0.042}_{-0.065}) \times 10^{-3} \text{ eV}^2,$$

$$\Delta m_{21}^2 = (7.59^{+0.18}_{-0.19}) \times 10^{-5} \text{ eV}^2, \quad (29)$$

and assuming normal hierarchy, we obtain

$$m_1 = 0.0493 \pm 0.0005 \text{ eV}, \quad m_2 = 0.0087 \pm 0.0001 \text{ eV}. \quad (30)$$

These experimental error are much too small to be carried meaningfully for our further analysis, and will henceforth be dropped. Hence for the Dirac case (D), we have straightforwardly:

$$\sin \theta_{\tau\nu_\mu}^D = \sqrt{m_\tau/m_3} \sim 0.42. \quad (31)$$

For the see-saw case (ss), we take for the see-saw mechanism [26] the simplest model, i.e., Type-I quadratic see-saw. This then gives the physical masses of the 3 neutrino states $\nu_i$ as respectively $m_i = (m_i^D)^2/m_R$, with $m_i^D$ being their Dirac masses and $m_R$ the right-handed neutrino mass. Then

$$\sin \theta_{\tau\nu_\mu}^{\text{ss}} = \sqrt{m_\tau/m_D} = \sin \theta_{\tau\nu_\mu}^D \sim 0.65. \quad (32)$$

We note first that for both cases (D and ss), $\sin \theta_{\tau\mu} < \sin \theta_{\tau\nu_\mu}$, it follows that the corner elements of the PMNS
matrix will have the same asymmetry as in the CKM matrix, i.e., $U_{e3} < U_{\tau 1}$, as seems to be implied by experiment in (2). In fact, for the values adopted there for $\theta_{12}$ and $\theta_{23}$, this asymmetry will persist up to $\sin^2 2\theta_{13} = 0.23$ or to $\theta_{13} = 0.25$, i.e., way outside the experimental errors.

Secondly, accepting the result (21) obtained above we have for charged leptons ($L$) and neutrinos ($\nu$):

$$\omega_L = \frac{1}{2} \kappa_g \theta_{13}, \quad \omega_\nu = \frac{1}{2} \kappa_g \theta_{23} \nu_2,$$

and assuming again that $\kappa_g$ is constant we obtain

$$\frac{U_{e3}}{U_{\tau 1}} \sim \frac{\theta_{13}}{\theta_{23} \nu_2} \sim 0.6 \, (D), \quad 0.4 \, (ss),$$

as compared with the value 0.40 obtained for the matrix in (2) above and agreeing with the noted asymmetry. Lastly, pushing it all the way, if we repeat the previous arguments to estimate the values of the corner elements for the CKM matrix, we obtain the estimates

$$U_{e3}^{D} \sim 0.06; \quad U_{e3}^{ss} \sim 0.05.$$

These estimates maintain the above observation that the corner elements will be small and be asymmetric about the diagonal to roughly the order indicated by experiment.

In summary, we conclude that the R2M2 (rotation) hypothesis gives automatically small but nonzero corner elements to both the CKM and PMNS matrices with the right asymmetry and roughly the right magnitudes as observed in experiment. This is, we believe, a nontrivial test for the hypothesis. The merit of the small-angle approximation used here is the utter simplicity and transparency of its derivations, with the minimum of assumptions on the rotation trajectory and without resorting to numerical methods. Thus, though not giving as extensive and as accurate results, it is a valuable complement to the approach by numerical integration of the Serret-Frenet-Darboux formulæ based on an explicit parametrization of the rotation trajectory [27].

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