CENTRAL AFFINE CURVE FLOW ON THE PLANE

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Dedicated to Prof. Choquet-Bruhat

Abstract. We give the following results for Pinkall’s central affine curve flow on the plane: (i) a systematic and simple way to construct the known higher commuting curve flows, conservation laws, and a bi-Hamiltonian structure, (ii) Bäcklund transformations and a permutability formula, (iii) infinitely many families of explicit solutions. We also solve the Cauchy problem for periodic initial data.

1. Introduction

The group $SL(2, \mathbb{R})$ acts transitively on $\mathbb{R}^2 \setminus \{0\}$ by $A \cdot y = Ay$. It is noted in [15] that given a smooth curve $\gamma$ on $\mathbb{R}^2 \setminus \{0\}$, if $\det(\gamma, \gamma_s)$ never vanishes, then there is a unique parameter $x$ such that

$$\det(\gamma, \gamma_x) = 1.$$ (In fact, $\frac{dx}{ds} = \det(\gamma, \gamma_s)^{-1}$). Taking $x$-derivative of $\det(\gamma, \gamma_x) = 1$ gives $\det(\gamma, \gamma_{xx}) = 0$. Hence there is a unique smooth function $q$ such that

$$\gamma_{xx} = q \gamma.$$ This parameter $x$ is called the central affine arc-length parameter and $q$ is called the central affine curvature of $\gamma$. Note that $q = \det(\gamma_{xx}, \gamma_x)$.

Let

$$\mathcal{M}_2(I) = \{ \gamma : I \to \mathbb{R}^2 \setminus \{0\} \mid \gamma \text{ smooth curve, } \det(\gamma, \gamma_x) = 1 \},$$ (1.1)

where $I$ is $\mathbb{R}$ or $S^1$.

Note that $X$ lies in the tangent space $T(\mathcal{M}_2(I))_\gamma$ of $\mathcal{M}_2(I)$ at $\gamma$ if and only if $\det(X, \gamma_x) + \det(\gamma, X_x) = 0$. So $X = y_1 \gamma + y_2 \gamma_x$ lies in $T(\mathcal{M}_2(I))_\gamma$ if and only if

$$y_1 = -\frac{(y_2)_x}{2}.$$ This identifies $T(\mathcal{M}_2(I))_\gamma$ as $C^\infty(I, \mathbb{R})$. Henceforth we will use the following notation: given $\xi \in C^\infty(I, \mathbb{R})$, let $\tilde{\xi}$ denote the tangent vector field on $\mathcal{M}_2(I)$ defined by

$$\tilde{\xi}(\gamma) = -\frac{\xi_x}{2} \gamma + \xi \gamma_x.$$ (1.2)

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We call the flow on $\mathcal{M}_2(I)$ defined by a tangent vector field $\tilde{\xi}(\gamma)$ on $\mathcal{M}_2(I)$,
\[\gamma_t = \tilde{\xi}(\gamma) = \frac{\xi_x}{2} \gamma + \xi \gamma_x,\]
a central affine curve flow on $\mathbb{R}^2 \setminus \{0\}$ if $\xi$ is a differential polynomial of the central affine curvature $q$ for $\gamma$ (i.e., $\xi$ is a polynomial of $q$ and the $x$-derivatives of $q$). Note that a central affine curve flow is invariant under the action of $SL(2, \mathbb{R})$ and translations, i.e., if $\gamma$ is a solution of (1.3), $c \in SL(2, \mathbb{R})$, and $x_0, t_0$ are real constants, then $c\gamma$ and $\gamma_1(x, t) = \gamma(x + x_0, t + t_0)$ are again solutions of (1.3).

Pinkall considered the following in [15] third order central affine curve flow on $\mathcal{M}_2(I)$:
\[\gamma_t = \frac{1}{4} q_x \gamma - \frac{1}{2} q \gamma_x,\] (1.3)
i.e., the flow defined by the vector field $\tilde{\xi}$ with $\xi = -\frac{q}{2}$. He proved in [15] that if $\gamma$ is a solution of the curve flow (1.3), then the central affine curvature $q(\cdot, t)$ of $\gamma(\cdot, t)$ satisfies the KdV equation:
\[q_t = \frac{1}{4} (q_{xxx} - 6qq_x).\] (1.4)

He also proved in [15] that
\[w_\gamma(\tilde{\xi}, \tilde{\eta}) = \oint \det(\tilde{\xi}, \tilde{\eta}) = \frac{1}{2} \oint (\eta_x \xi - \xi_x \eta) \, dx = -\oint \xi_x \eta \, dx,\] (1.5)
is a symplectic form on the orbit space $\mathcal{M}_2(S^1)/S^1$ and (1.3) is the Hamiltonian flow for
\[H(q) = \oint \frac{1}{2} q \, dx.\]

Here $S^1$ acts on $\mathcal{M}_2(S^1)$ by
\[(e^{i\theta} \cdot \gamma)(x) = \gamma(x + \theta).\]

Chou and Qu wrote down the traveling wave solutions for the central affine curve flow (1.3) in [5]. Calini, Ivey and Mari-Beffa studied periodic (in $x$) solutions of the curve flow (1.3) whose central affine curvatures are finite-gap solutions of the KdV equation in [3]. Higher order central affine curve flows and conservation laws for (1.3) were given in [5], [3], [8]. A bi-Hamiltonian structure for (1.3) was discussed in [9].

In this paper,
(1) We explain how to use soliton theory of the KdV hierarchy to give a systematic and simple way to obtain higher commuting Hamiltonian flows, conservation laws, and a bi-Hamiltonian structure for the central affine curve flow (1.3).
(2) We construct Bäcklund transformations and a Permutability formula for the central affine curve flow (1.3). Then we use these to construct recursively, infinitely many families of explicit solutions of
whose central affine curvatures are pure n-soliton solutions or rational solutions of the KdV equation.

(3) We solve the Cauchy problem for (1.3) with initial data $\gamma_0$ with rapidly decaying central affine curvature. We also solve the periodic Cauchy problem for (1.3).

A natural generalization of the KdV hierarchy to $(n-1)$ component functions is the $A_{n-1}^{(1)}$-KdV hierarchy, a KdV type hierarchy constructed by Drinfel’d and Sokolov from the affine Kac-Moody algebra $A_{n-1}^{(1)}$ in [7]. This inspired the study of $n$-dimensional central affine curve flows: It follows from the change of variable formula that if $\gamma$ is a smooth curve in $\mathbb{R}^n \setminus \{0\}$ such that

$$\det(\gamma, \gamma_s, \ldots, \gamma^{(n-1)}_s) > 0,$$

then there is a unique orientation preserving parameter $x$ such that

$$\det(\gamma, \gamma_x, \ldots, \gamma^{(n-1)}_x) \equiv 1.$$

Taking the $x$-derivative of the above equation gives

$$\det(\gamma, \gamma_x, \ldots, \gamma^{(n-2)}_x, \gamma^{(n)}_x) \equiv 0.$$

So there exist unique smooth functions $u_1, \ldots, u_{n-1}$ such that

$$\gamma^{(n)}_x = u_1 \gamma + u_2 \gamma_x + \cdots + u_{n-1} \gamma^{(n-1)}_x.$$

This parameter $x$ is called the central affine arc-length parameter and $u_i$ is called the $i$-th central affine curvature of $\gamma$ for $1 \leq i \leq n-1$. It follows from the existence and uniqueness for ordinary differential equations that these $u_i$ form a complete set of local invariants for curves in $\mathbb{R}^n \setminus \{0\}$ under the group $SL(n, \mathbb{R})$. In a forthcoming paper [19], we consider the following central affine curve flow

$$\gamma_t = -\frac{2}{n} u_{n-1} \gamma + \gamma_{xx},$$

on

$$\mathcal{M}_n(I) = \{ \gamma : I \to \mathbb{R}^n \setminus \{0\} \mid \det(\gamma, \gamma_x, \ldots, \gamma^{(n-1)}_x) = 1 \},$$

where $u_{n-1}(\cdot, t)$ is the $(n-1)$-th central affine curvature for $\gamma(\cdot, t)$. When $n = 3$, this central affine curve flow was studied in [12] and [4]. In the forthcoming paper [19], we (i) prove that if $\gamma(x,t)$ is a solution of (1.6) on $\mathcal{M}_n(I)$ then its central affine curvatures $u_1(\cdot, t), \ldots, u_{n-1}(\cdot, t)$ satisfy the second flow in the $A_{n-1}^{(1)}$-KdV hierarchy, (ii) construct for the curve flow (1.6) higher order commuting central affine curve flows, conservation laws, a bi-Hamiltonian structure, Bäcklund transformations, and (iii) obtain recursively, infinitely many families of explicit solutions of (1.6).

This paper is organized as follow: In section 2 we construct a sequence of commuting higher order central affine curve flows on $\mathbb{R}^2 \setminus \{0\}$ and solve the Cauchy problem. In section 3 we construct Bäcklund transformations and a Permutability formula for the central affine curve flow (1.3) and then apply these to the stationary solution of (1.3) to obtain infinitely many
families of explicit solutions of (1.3) whose affine curvatures are pure \(n\)-soliton solutions or rational solutions of the KdV equation. We discuss the Hamiltonian aspect of (1.3) in the final section.

2. Higher order commuting curve flows and Cauchy problem

The outline of this section is as follows:

1. We review the construction of the KdV hierarchy and its Lax pair (for detail see [1], [6], [18]).

2. We use the Lax pairs of the KdV hierarchy to write down the known sequence of commuting higher order central affine curve flows on \(M_2(I)\) constructed in [5], [3], [8], whose central affine curvatures satisfy the higher flows in the KdV hierarchy.

3. We use the solution of the Cauchy problem for the KdV equation to solve the Cauchy problem for (1.3) with initial data \(\gamma_0 \in M_2(\mathbb{R})\) having rapidly decaying central affine curvatures. We also solve the Cauchy problem for (1.3) with periodic initial data \(\gamma_0 \in M_2(S^1)\).

Let \(B : sl(2, \mathbb{R}) \to \mathbb{R}e_{12}\) denote the linear map defined by

\[
B \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}.
\]  

(2.1)

Given a smooth function \(q : \mathbb{R} \to \mathbb{R}\), it is known (cf. [18]) that there exists a unique

\[
Q(q, \lambda) = e_{12}\lambda + \sum_{i \geq 0} Q^{-i}(q)\lambda^{-i}
\]

satisfying

\[
\begin{cases}
\left[ d_x + \begin{pmatrix} 0 & \lambda + q \\ 1 & 0 \end{pmatrix}, Q(q, \lambda) \right] = 0, \\
Q(q, \lambda)^2 = \lambda I_2.
\end{cases}
\]  

(2.2)

Compare coefficients of \(\lambda^{-j}\) of the above equation to get the following recursive formulas:

\[
\begin{cases}
(Q_{-j}(q))_x + [e_{21} + q e_{12}, Q_{-j}(q)] = [Q_{-(j+1)}(q), e_{12}], \\
e_{12}Q_{-(j+1)}(q) + Q_{-(j+1)}(q)e_{12} + \sum_{i=0}^{j} Q^{-i}(q)Q_{-(j-i)}(q) = 0,
\end{cases}
\]  

(2.3)

for all \(j \geq 0\). Write

\[
Q_{-j}(q) = \begin{pmatrix} A_j(q) & B_j(q) \\ C_j(q) & -A_j(q) \end{pmatrix}.
\]
Compare entries of (2.3) to get

\[ C_{j+1}(q) = -((A_j(q))_x + qC_j(q) - B_j(q)), \]  

(2.4)

\[ A_{j+1}(q) = \frac{1}{2}((B_j(q))_x - qA_j(q)), \]  

(2.5)

\[ A_j(q) = -\frac{1}{2}(C_j(q))_x. \]  

(2.6)

It follows that these \( Q_i(q) \)'s can be obtained recursively and they are differential polynomials of \( q \) in \( x \)-variable. For example,

\[ Q_0(q) = \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{pmatrix}, \]

\( Q_{-1}(q) = \frac{1}{4} \begin{pmatrix} q_x & q_{xx} - 2q^2 \\ -2q & -q_x \end{pmatrix}, \]

\[ Q_{-2}(q) = \frac{1}{8} \begin{pmatrix} \frac{1}{2}q_{xxx} - 3qq_x & \frac{1}{4}(q^{(4)} - 6qq_{xx} - 7q_x^2 + 2q^3) \\ 3q^2 - q_{xx} & -\frac{1}{2}q_{xxx} + 3qq_x \end{pmatrix}, \]

\[ Q_{-3}(q) = \begin{pmatrix} -\frac{1}{32}(q^{(4)} + 10q^3 - 5q_x^2 - 10qq_{xx}) & * \\ * & * \end{pmatrix}. \]

The \((2j + 1)\)-th flow in the KdV hierarchy is

\[ q_{t_{2j+1}} = (B_j(q) - C_{j+1}(q))_x - 2qA_j(q), \]  

(2.7)

In particular, the first, third, and fifth flows are:

\[ q_{t_1} = q_x, \]  

(2.8)

\[ q_{t_3} = \frac{1}{4}(q_{xxx} - 6qq_x), \]  

(2.9)

\[ q_{t_5} = \frac{1}{16}(\partial_x^5 q - 10q\partial_x^3 q - 20(\partial_x q)\partial_{xx} q + 30q^2 \partial_x q). \]  

(2.10)

Note that the third flow (2.9) is the KdV equation.

The following two Theorems are well-known and the proofs can be found in many places (cf. [20], [1], [18]).

**Theorem 2.1.** The flows in the KdV hierarchy commute.

**Theorem 2.2.** [Lax pair for the KdV hierarchy]

The following statements are equivalent for \( q \in C^\infty(\mathbb{R}^2, \mathbb{R}) \):

(i) \( q \) is a solution of the \((2j + 1)\)-th flow (2.7).

(ii) The following family of connections on the \((x, t_{2j+1})\)-plane defined by \( q \) is flat for all parameter \( \lambda \in \mathbb{C} \),

\[ \begin{bmatrix} \partial_x + \begin{pmatrix} 0 & q + \lambda \\ 1 & 0 \end{pmatrix}, & \partial_{t_{2j+1}} + (Q(q, \lambda)\lambda^j) - \mathcal{B}(Q_{-(j+1)}(q)) \end{bmatrix} = 0, \]  

(2.11)

where \( \mathcal{B} \) is linear map defined by (2.1). We call (2.11) the Lax pair of the solution \( q \) of the \((2j + 1)\)-th flow in the KdV hierarchy.

(iii) Equation (2.11) holds for \( \lambda = 0 \), i.e.,

\[ \begin{bmatrix} \partial_x + \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix}, & \partial_{t_{2j+1}} + Q_{-j}(q) - \mathcal{B}(Q_{-(j+1)}(q)) \end{bmatrix} = 0. \]  

(2.12)
For example, (2.11) for the third flow (KdV) is
\[
\begin{bmatrix}
\partial_x + \begin{pmatrix} 0 & \lambda + q \\ 1 & 0 \end{pmatrix}, \\
\partial_t + \begin{pmatrix} \frac{1}{4}q_x & \lambda^2 + \frac{1}{2}q \lambda + \frac{1}{4}(q_{xx} - 2q^2) \\ \lambda - \frac{1}{2} & -\frac{1}{4}q_x \end{pmatrix}
\end{bmatrix} = 0. \tag{2.13}
\]

Let \(q\) be a solution of (2.11), and \(c(\lambda) \in SL(2, \mathbb{C})\) holomorphic for \(\lambda \in \mathbb{C}\) satisfying \(c(\bar{\lambda}) = c(\lambda)\). Then there exists a unique \(E(x, t, \lambda) \in SL(2, \mathbb{C})\) satisfying \(E(x, t, \bar{\lambda}) = E(x, t, \lambda)\) and

\[
\begin{align*}
E_x &= E \begin{pmatrix} 0 & \lambda + q \\ 1 & 0 \end{pmatrix}, \\
E_t &= E((Q(q, \lambda) \lambda^j - \mathcal{B}(Q_{-(j+1)}(q))), \\
E(0, 0, \lambda) &= c(\lambda).
\end{align*}
\]

Moreover, the solution \(E(x, t, \lambda)\) is holomorphic for \(\lambda \in \mathbb{C}\). We call \(E\) an extended frame of the solution \(q\) of the \((2j + 1)\)-th flow (2.7).

Next we discuss Pinkall’s result that the central affine curvature \(q(\cdot, t)\) of a solution of (1.3) is a solution of the KdV equation. Let \(\gamma\) be a solution of (1.3) on \(\mathcal{M}_2(I)\), and \(g = (\gamma, \gamma_x)\). Then we have \(\gamma_{xx} = q\gamma\) and \(g_x = g \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix}\).

Use \(\gamma_t = \frac{1}{4}q_x\gamma - \frac{1}{2}q\gamma_x\) to get
\[
(\gamma_x)_t = (\gamma_t)_x = \begin{bmatrix} \frac{1}{4}q_x\gamma - \frac{1}{2}q\gamma_x \end{bmatrix}_x = \begin{bmatrix} \frac{1}{4}q_{xx} - \frac{1}{2}q^2 \gamma - \frac{1}{4}q_x\gamma_x \end{bmatrix}.
\]

So \(g = (\gamma, \gamma_x) \in SL(2, \mathbb{R})\) satisfies
\[
\begin{align*}
g_x &= g \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix}, \\
g_t &= g \begin{pmatrix} \frac{1}{4}q_x & \frac{1}{4}(q_{xx} - 2q^2) \\ -\frac{1}{2} & -\frac{1}{4}q_x \end{pmatrix}. \tag{2.14}
\end{align*}
\]

This implies that
\[
\begin{bmatrix}
\partial_x + \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix}, \\
\partial_t + \begin{pmatrix} \frac{1}{4}q_x & \frac{1}{4}(q_{xx} - 2q^2) \\ -\frac{1}{2} & -\frac{1}{4}q_x \end{pmatrix}
\end{bmatrix} = 0.
\]

By Theorem 2.2, \(q\) is a solution of the KdV equation. This gives Pinkall’s result:

**Proposition 2.3.** (15) If \(\gamma\) is a solution of (1.3) on \(\mathcal{M}_2(I)\), then its central affine curvature \(q(\cdot, t)\) of \(\gamma(\cdot, t)\) is a solution of the KdV equation (1.4).

The converse is also true when \(I = \mathbb{R}\):

**Proposition 2.4.** If \(q : \mathbb{R}^2 \to \mathbb{R}\) is a solution of the KdV equation and \(c_0 \in SL(2, \mathbb{R})\), then

(i) there is a unique \(g : \mathbb{R}^2 \to SL(2, \mathbb{R})\) satisfies (2.14) with \(g(0, 0) = c_0\),
(ii) \( \gamma(x,t) = g(x,t)p_0 \) is a solution of the curve flow \([1.3]\) with central affine curvature \( q \), where \( p_0 = (1,0)^t \).

**Proof.** Statement (i) follows from properties of the Lax pair. Let \( \gamma \) and \( v \) denote the first and the second column of \( g \) respectively. Then the first equation of \([2.14]\) implies that \( \gamma_x = v \) and \( v_x = q \gamma \). So \( \gamma_{xx} = q \gamma \). Since \( g = (\gamma, \gamma_x) \in SL(2, \mathbb{R}) \), \( x \) is the central affine arc-length for the curve \( \gamma(\cdot,t) \) and \( q \) is its central affine curvature. The second equation of \([2.14]\) implies that \( \gamma_t = \frac{1}{2}q_x \gamma - \frac{1}{2}q \gamma_x \). □

If \( g_0 \) is the solution of \([2.14]\) with \( g_0(0,0) = I_2 \), then given \( c \in SL(2, \mathbb{R}) \) the solution \( g \) of \([2.14]\) with \( g(0,0) = c \) is \( cg_0 \). So it follows from Proposition \([2.4]\) that we have

**Corollary 2.5.** Let \( \Psi : M_2(I) \to C^{\infty}(I, \mathbb{R}) \) be the map defined by \( \Psi(\gamma) = \) the central affine curvature of \( \gamma \). Then \( \Psi(\gamma_1) = \Psi(\gamma_2) \) if and only if there is a constant \( c \in SL(2, \mathbb{R}) \) such that \( \gamma_2 = c \gamma_1 \).

We define the holonomy map next:

**Definition 2.6.** Given \( q \in C^{\infty}(S^1, \mathbb{R}) \), The holonomy map \( \Pi \) is the map from \( C^{\infty}(S^1, \mathbb{R}) \) to \( SL(2, \mathbb{R}) \) defined by \( \Pi(q) = \) the holonomy of the connection \( \frac{d}{dx} + \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix} \), i.e., \( \Pi(q) = g(2\pi) \), where \( g \) is the solution of

\[
\begin{cases}
  g^{-1}g_x = \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix}, \\
  g(0) = I_2.
\end{cases}
\]

**Corollary 2.7.** Let \( \Psi : M_2(S^1) \to C^{\infty}(S^1, \mathbb{R}) \) be as in Corollary \([2.4]\) and \( \Pi \) the holonomy map defined above. Then \( \Psi \) induces a bijection from the orbit space \( M_2(S^1)/SL(2, \mathbb{R}) \) onto

\[ C^{\infty}_r(S^1, \mathbb{R}) = \{ q \in C^{\infty}(S^1, \mathbb{R}) | \Pi(q) = I_2 \}. \]

Next we write down Commuting higher order central affine curve flows for \([1.3]\). Recall that \( Y(\gamma) = y_1 \gamma + y_2 \gamma_x \) is a tangent vector field on \( M_2(I) \) if and only if \( y_1 = -\frac{1}{2}(y_2)_x \), and we use \( \tilde{y}_1 \) to denote this vector field. So it follows from \([2.6]\) that \( A_j(q) \gamma + C_j(q) \gamma_x \) is tangent to \( M_2(I) \) at \( \gamma \) and

\[
\gamma_{t_{2j+1}} = A_j(q) \gamma + C_j(q) \gamma_x = -\frac{1}{2}(C_j(q))_x \gamma + C_j(q) \gamma_x \tag{2.15}
\]

is a central affine curve flow on \( M_2(I) \) of order \( 2j + 1 \), where \( Q_{-j}(q) = \begin{pmatrix} A_j(q) & B_j(q) \\ C_j(q) & -A_j(q) \end{pmatrix} \) is the coefficient of \( \lambda^{-j} \) of the solution \( Q(q, \lambda) \) of \([2.2]\).

We call this the \((2j + 1)\)-th central affine curve flow on \( M_2(I) \). For example, the first, third, and fifth (i.e., \( j = 0, 1, 2 \)) central affine curve flow on \( M_2(I) \)
Note that the third central affine curve flow is the curve flow \((1.3)\).

The same proof of Proposition 2.3 implies that if \(\gamma\) is a solution of the central affine curve flow (2.15) then its affine curvature \(q(\cdot, t)\) is a solution of the \((2j + 1)\)-th flow (2.7). Analogous result as Proposition 2.4 for these higher order central affine curve flow can be proved in a similar manner. Since the flows in the KdV hierarchy commute, these central affine curve flows commute. Hence we get the following result proved in \([5], [3],\) and \([8]\):

**Proposition 2.8.** Let \(\Psi\) denote the map defined in Corollary 2.5. Then \(\Psi\) maps the central affine curve flow (2.15) to the \((2j + 1)\)-th flow (2.7) in the KdV hierarchy for all \(j \geq 0\). Moreover,

1. if \(q\) is a solution of the \((2j + 1)\)-th flow of the KdV equation and \(g : \mathbb{R}^2 \to SL(2, \mathbb{R})\) a solution of \(g^{-1}g_x = \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix}\), then \(\gamma = g\begin{pmatrix} 1 \\ 0 \end{pmatrix}\) is a solution of (2.15),
2. these flows (2.15) for all \(j \geq 0\) commute.

We will discuss Cauchy problems for the central affine curve flow (1.3) in the rest of this section. First recall that the Cauchy problem for the KdV equation is solved for two classes of initial data \(q_0\) (\([11], [13]\)):

1. \(q_0 \in \mathcal{S}(\mathbb{R}, \mathbb{R})\), i.e., \(q_0\) is smooth and rapidly decaying,
2. \(q_0 \in C^\infty(S^1, \mathbb{R})\).

Use Proposition 2.4 and the solutions to the Cauchy problem for the KdV equation with initial data \(q_0 \in \mathcal{S}(\mathbb{R}, \mathbb{R})\) to get

**Theorem 2.9.** [Cauchy problem on the line]

Suppose \(\gamma_0 \in \mathcal{M}_2(\mathbb{R})\) has rapidly decaying central affine curvature \(q_0\), and \(q(x, t)\) the solution of the KdV equation with initial data \(q(x, 0) = q_0(x)\). Let \(g : \mathbb{R}^2 \to SL(2, \mathbb{R})\) denote the solution of (2.14) with \(g(0, 0) = (\gamma_0(0), (\gamma_0)_x(0))\). Then \(\gamma(x, t) = g(x, t)(1, 0)^t\) is the solution of (1.3) with \(\gamma(x, 0) = \gamma_0(x)\). Moreover, the central affine curvatures of \(\gamma(\cdot, t)\) are also rapidly decaying.

If \(q(x, t)\) is a solution of the KdV equation such that \(q\) is periodic in \(x\) with period \(2\pi\) and \(g\) is a solution of (2.14), then by Proposition 2.4, \(g(1, 0)^t\) is a solution of (1.3). Although \(q\) is periodic in \(x\), the solution \(g\) of the linear system (2.14) may not be periodic in \(x\). We prove below that if \(g(\cdot, 0)\) is periodic then \(g(\cdot, t)\) is periodic.
Theorem 2.10. [Cauchy Problem with periodic initial data]
Suppose \( q_0 \) is the central affine curvature of \( \gamma_0 \in \mathcal{M}_2(S^1) \) and \( q(x,t) \) is the solution of the KdV equation periodic in \( x \) such that \( q(x,0) = q_0(x) \).
Let \( g : \mathbb{R}^2 \to SL(2, \mathbb{R}) \) be the solution of (2.13) with initial data \( g(0,0) = (\gamma(0)(0), (\gamma_0)_x(0)) \). Then \( \gamma(x,t) = g(x,t)(1,0)^t \) is a solution of (1.3) with initial data \( \gamma(x,0) = \gamma_0(x) \) and \( \gamma(x,t) \) is periodic in \( x \) with period \( 2\pi \).

Proof. Both \( g(x,0) \) and \( (\gamma_0,(\gamma_0)_x) \) satisfy \( g_x = g \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix} \) with the same initial data. So by the uniqueness for ordinary differential equations we have \( g(\cdot,0) = (\gamma_0,(\gamma_0)_x) \). It follows from Proposition [2.4] that \( \gamma(x,t) = g(x,t)(1,0)^t \) is a solution of the curve flow (1.3) with initial data \( \gamma(x,0) = \gamma_0(x) \). It remains to prove that \( \gamma \) is periodic in \( x \).
Since \( \gamma_0 \) is periodic with period \( 2\pi \), \( g(x,0) = (\gamma_0(x),(\gamma_0)_x(x)) \) is periodic in \( x \). Hence \( g(2\pi,0) - g(0,0) = 0 \). We claim that
\[
y(t) = g(2\pi,t) - g(0,t)
\]
is identically zero. To see this, first recall that
\[
Q_{-1}(q) = \frac{1}{4} \begin{pmatrix} q_x & q_{xx} - 2q^2 \\ -2q & -q_x \end{pmatrix}.
\]
Since \( q \) is periodic in \( x \), so is \( Q_{-1}(q) \). Compute directly to get
\[
y_t = g_t(2\pi,t) - g_t(0,t)
= (gQ_{-1}(q))(2\pi,t) - (gQ_{-1}(q))(0,t) = (g(2\pi,t) - g(0,t))Q_{-1}(q)((0,t)
= yQ_{-1}(q)(0,t).
\]
Since \( y \equiv 0 \) is a solution of \( y_t = yQ_{-1}(q)(0,t) \) with \( y(0) = 0 \). But \( y(0) = g(2\pi,0) - g(0,0) = 0 \). So it follows from the uniqueness for ordinary differential equations that \( y \) is identically zero.

\[\square\]

3. Bäcklund Transformations

We construct Bäcklund transformations and a Permutability formula for the central affine curve flow (1.3). Then we apply these transformations to the stationary solutions of (1.3) to construct recursively, infinitely many families of explicit solutions of (1.3).

Bäcklund transformations for the KdV equation were given in several places (cf. [2], [16], [17]). The one we will use to construct Bäcklund transformations for the central affine curve flow (1.3) is given in [17]. These transformations were constructed using the Lax pair for the KdV equation constructed in [1]:
\[
\begin{bmatrix}
\partial_x + \begin{pmatrix} z & q \\ 1 & -z \end{pmatrix}, \\
\partial_{t_3} + \begin{pmatrix} z^3 - \frac{qz}{2} + \frac{q^3}{4} & qz^2 - \frac{q^2}{2} + \frac{2q_xz - 2q^2}{4} \\ \frac{qz}{2} - \frac{q^2}{2} + \frac{2q_x - 2q^2}{4} & 0 \end{pmatrix}
\end{bmatrix} = 0. \quad (3.1)
\]
This Lax pair is gauge equivalent to the Lax pair \([2.13]\) with \(\lambda = z^2\) by
\[
\phi(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix},
\]
i.e.,
\[
\phi(z) \left[ \partial_x + \begin{pmatrix} 0 & z^2 + q \\ 1 & 0 \end{pmatrix}, \partial_t + \begin{pmatrix} \frac{q_4}{4} & \frac{q_z}{2} + \frac{q_x - 2q_2}{4} \\ z^2 - \frac{q}{2} \end{pmatrix} \right] \phi(z)^{-1}
\]
is equal to the left hand side of \((3.1)\). Hence \(E(x, t, \lambda)\) is an extended frame for the Lax pair \((2.13)\) if and only if
\[
F(x, t, z) = \phi(z) E(x, t, z^2) \phi(z)^{-1}
\]
is an extended frame for the Lax pair \((3.1)\) in \(z\)-parameter.

Next we use this conjugation to state results for Bäcklund transformations in \([17]\). Given real constants \(k, \xi\), we call
\[
r_{\xi, k}(\lambda) = \begin{pmatrix} \xi & \xi^2 - k^2 + \lambda \\ 1 & \xi \end{pmatrix}
\]
a simple factor. A direct computation implies that
\[
r_{\xi, k}^{-1}(\lambda) = \frac{r_{-\xi, k}(\lambda)}{\lambda - k^2}.
\]
Note that \(\det(r_{\xi, k}(\lambda)) = k^2 - \lambda\), hence \(r_{\xi, k}(\lambda)\) has a zero at \(\lambda = k^2\).

**Theorem 3.1.** (\([17]\)) Let \(q\) be a solution of the KdV equation, \(E(x, t, \lambda)\) an extended frame of the Lax pair \((2.13)\), and \(k, \xi\) real constants. Set
\[
\begin{pmatrix} y_1(x, t) \\ y_2(x, t) \end{pmatrix} = E(x, t, k^2)^{-1} \begin{pmatrix} -\xi \\ 1 \end{pmatrix},
\]
\[
\tilde{\xi}(x, t) = -\frac{y_1(x, t)}{y_2(x, t)}.
\]
If \(y_2\) does not vanish in an open subset \(O \subset R^2\), then
\[
r_{\xi, k} \ast q := -q + 2(\tilde{\xi}^2(x, t) - k^2)
\]
is a solution of the KdV equation defined on \(O\). Moreover,
\[
\tilde{E}(x, t, \lambda) = \frac{r_{\xi, k}(\lambda) E(x, t, \lambda) r_{-\tilde{\xi}(x, t), k}(\lambda)}{\lambda - k^2}
\]
is holomorphic for \(z \in \mathbb{C}\) and is an extended frame of \(r_{\xi, k} \ast q\).

**Remark 3.2.** For \(k \neq 0\), to prove \(\tilde{E}(x, t, z)\) defined by \((3.3)\) is holomorphic for all \(\lambda \in \mathbb{C}\), it suffices to show that the residues of \(\tilde{E}\) at \(\lambda = k^2\) are zero. For \(k = 0\), we check that the constant coefficient of \(r_{\xi, 0}(\lambda) E(x, t, \lambda) r_{-\tilde{\xi}, 0}(\lambda)\) as a power series in \(\lambda\) is zero. Hence
\[
\tilde{E}(x, t, \lambda) = \lambda^{-1} r_{\xi, 0}(\lambda) E(x, t, \lambda) r_{-\tilde{\xi}, 0}(\lambda)
\]
is holomorphic for \(z \in \mathbb{C}\).
Let \( E \) be an extended frame of a solution \( q \) of the KdV equation. Assume that \( \tilde{E} \) is of the form (3.3) and require that \( \tilde{E}^{-1} d\tilde{E} \) equals to the Lax pair of some solution \( \tilde{q} \). This gives a system of compatible non-linear ODEs for \( \xi \):

**Theorem 3.3.** (17) Let \( k \in \mathbb{R} \) be a constant, and \( q : \mathbb{R}^2 \to \mathbb{R} \) a smooth function. Then the following first order system for \( A \) is solvable if and only if \( q \) is a solution of the KdV equation:

\[
(BT)_{q,k} = \begin{cases} A_x = q - A^2 + k^2, \\ A_t = \frac{q_{xx} - 2q^2}{4} - \frac{q}{2} A + \frac{q(A^2 + k^2)}{2} - k^2 (A^2 - k^2) \end{cases}
\]

Moreover, if \( q \) is a solution of the KdV equation and \( A \) the solution of \((BT)_{q,k} \) with \( A(0,0) = \xi \), then \( A = \xi \) and

\[
r_{\xi,k} * q := -q + 2(\xi^2 - k^2)
\]

is a solution of the KdV equation, where \( \tilde{\xi} \) is defined in Theorem 3.1.

**Remark 3.4.** To construct Bäcklund transformations for a given solution \( q \) of the KdV equation, we can either solve \( E \) from the following linear system

\[
\begin{cases} E_x = E \begin{pmatrix} 0 & k^2 + q \\ 1 & 0 \end{pmatrix}, \\ E_t = E \begin{pmatrix} \frac{1}{4} q_x & k^4 + \frac{1}{2} k^2 q + \frac{1}{4} (q_{xx} - 2q^2) \\ k^2 - \frac{1}{2} q & -\frac{1}{4} q_x \end{pmatrix}, \end{cases}
\]

for constant parameter \( k^2 \) or solve the non-linear system \((BT)_{q,k} \).

Permutability Theorem for the KdV hierarchy follows from a relation among simple factors \( p_{\xi,k} \)’s:

**Proposition 3.5.** (17) Let \( \xi_1 \neq \xi_2, k_1 \neq k_2 \) be real constants, \( \eta_1 = -\xi_2 + \frac{k_1^2 - k_2^2}{\xi_1 - \xi_2}, \ \eta_2 = -\xi_1 + \frac{k_1^2 - k_2^2}{\xi_1 - \xi_2}, \) and \( r_{\xi,k} \) be defined as in (3.2). Then \( r_{\eta_2,k_2} r_{\xi_1,k_1} = r_{\eta_1,k_1} r_{\xi_2,k_2} \).

**Theorem 3.6.** [Permutability] (17)

Let \( \xi_1, \xi_2, k_1, k_2, \eta_1, \eta_2 \) be real constants as in Proposition 3.5. Let \( E \) be an extended frame of the solution \( q \) of the KdV equation, \( (y_{11}, y_{21})^t = E(x, t, k_1^2) - 1(-\xi_1, 1)^t, \)

\[
\tilde{\xi}_i = -(y_{1i}/y_{2i}),
\]

and

\[
q_i = r_{\xi_i,k_i} * q = -q + 2(\xi_i^2 - k_i^2),
\]

for \( i = 1, 2 \). Set

\[
\tilde{\xi}_{12} = -\tilde{\xi}_1 + \frac{k_1^2 - k_2^2}{\xi_1 - \xi_2}, \quad (3.4)
\]

\[
E_{12}(x, t, \lambda) = \frac{r_{\eta_2,k_2}(\lambda) r_{\xi_1,k_1}(\lambda) E(x, t, \lambda) r_{-\tilde{\xi}_1,k_1}(\lambda) r_{-\tilde{\xi}_{12},k_2}(\lambda)}{(\lambda - k_1^2)(\lambda - k_2^2)}, \quad (3.5)
\]

\[
q_{12} = -q_1 + 2(\xi_{12}^2 - k_2^2). \quad (3.6)
\]
Then
\[(1) \quad q_{12} = r_{\eta_2,k_2} \ast (r_{\xi_1,k_1} \ast q) = r_{\eta_1,k_1} \ast (r_{\xi_2,k_2} \ast q) \] is a solution of the KdV equation,
\[(2) \quad E_{12} \text{ is an extended frame for } q_{12}, \]
\[(3) \quad \tilde{\xi}_{12} \text{ is the solution of } (BT)_{q_1,k_2} \text{ with initial data } \eta_2 \text{ and is also the solution of } (BT)_{q_2,k_1} \text{ with initial data } \eta_1. \]

As a consequence of Proposition 2.3, Proposition 2.4, and Theorem 3.1 we obtain:

**Theorem 3.7.** [BT for central affine curve flow (1.3) with \(k \neq 0\)]

Let \(\gamma\) be a solution for (1.3) with central affine curvature \(q\), and \(E(x,t,\lambda)\) the extended frame for \(q\) for the Lax pair (2.13) with \(E(0,0,0) = (\gamma, \gamma_x)\) at \((0,0)\). Given \(k, \xi \in \mathbb{R}\) with \(k \neq 0\), let
\[
\begin{pmatrix}
y_1(x,t) \\
y_2(x,t)
\end{pmatrix} := E(x, t, k^2)^{-1} \begin{pmatrix} -\xi \\ 1 \end{pmatrix}.
\]

Suppose \(y_2(x,t) \neq 0\). Set \(\tilde{\gamma} = -\frac{w}{y_2}\). Then
\[
\tilde{\gamma} = \frac{1}{k} (\xi \gamma - \gamma_x)
\]
is a solution of (1.3) with central affine curvature \(\tilde{q} = r_{\xi,k} \ast q = -q + 2(\xi^2 - k^2)\). Moreover,
\[
\tilde{E}(x,t,\lambda) = \frac{r_{\xi,k}(\lambda) E(x,t,\lambda) r_{-\tilde{\xi}(x,t),k}(\lambda)}{\lambda - k^2}
\]
is an extended frame for \(\tilde{q}\), where \(r_{k,\xi}(\lambda) = \begin{pmatrix} \xi & \xi^2 - k^2 + \lambda \\ 1 & \xi \end{pmatrix}\).

**Proof.** Since both \(E(x,t,0)\) and \((\gamma,\gamma_x)\) are solutions of (2.14) with the same initial data at \((0,0)\), we have \(E(\cdot,\cdot,0) = (\gamma,\gamma_x)\). By Theorem 3.1
\[
\tilde{E}(x,t,\lambda) = \frac{r_{\xi,k}(\lambda) E(x,t,\lambda) r_{-\tilde{\xi}(x,t),k}(\lambda)}{\lambda - k^2}
\]
is an extended frame for the new solution \(\tilde{q}\). By Proposition 2.4, the first column of \(\tilde{E}(x,t,0)\) is a solution of the curve flow (1.3) with \(\tilde{q}\) as its central affine curvature. Hence
\[
\tilde{E}(x,t,0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\frac{1}{k^2} r_{\xi,k}(0) E(x,t,0) r_{-\tilde{\xi},k}(0) \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]
is a solution of (1.3). But \(\frac{1}{k^2} r_{\xi,k}(0) = \frac{1}{k} \begin{pmatrix} \xi & \xi^2 - k^2 \\ 1 & \xi \end{pmatrix}\) is a constant in \(SL(2,\mathbb{R})\). So
\[
\tilde{\gamma} = -\frac{1}{k} E(x,t,0) r_{-\tilde{\xi},k}(0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\frac{1}{k} (\gamma,\gamma_x) \begin{pmatrix} -\xi \\ 1 \end{pmatrix}
\]
is a solution of (1.3). \(\square\)
Theorem 3.8. [BT for (1.3) with \( k = 0 \)]

Let \( \gamma \) be a solution of the flow (1.3), \( q \) its central affine curvature, and \( E \) an extended frame for \( q \) such that \( E(0,0,0) = (\gamma, \gamma_x) \) at \((0,0)\). Write

\[
E(x,t,\lambda) = E_0(x,t) + E_1(x,t)\lambda + E_2(x,t)\lambda^2 + \cdots .
\]

Let \( \xi \in \mathbb{R} \) be a constant, and

\[
\bar{\xi} := \frac{(1,\xi)\gamma_x}{(1,\xi)\gamma}.
\]

Here \( \gamma, \gamma_x \) are column vectors. Then \( E_0(x,t) = E(x,t,0) = (\gamma, \gamma_x) \) and

\[
\tilde{\gamma} = \epsilon_{12}(-\bar{\xi}\gamma + \gamma_x) + \begin{pmatrix} -\bar{\xi} & \bar{\xi}^2 - k^2 \\ 1 & -\bar{\xi} \end{pmatrix} E_1(x,t) \begin{pmatrix} -\bar{\xi} \\ 1 \end{pmatrix}
\]

is a solution of (1.3) with central affine curvature \( \tilde{q} = r_{\xi,0} * q = -q + 2\bar{\xi}^2 \).

Moreover, \( \tilde{E}(x,t,\lambda) = \lambda^{-1}r_{\xi,0}(\lambda)E(x,t,\lambda)r_{-\bar{\xi}(x,t),0}(\lambda) \)

is an extended frame for \( \tilde{q} \).

Proof. Since \( E(\cdot,0) \) and \( g = (\gamma, \gamma_x) \) are solutions of the linear system (2.14) with the same initial condition, we have \( E(x,t,0) = (\gamma, \gamma_x) \). A simple computation implies that

\[
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = E(x,t,0)^{-1} \begin{pmatrix} -\bar{\xi} \\ 1 \end{pmatrix} = (\gamma, \gamma_x)^{-1} \begin{pmatrix} -\bar{\xi} \\ 1 \end{pmatrix} = \begin{pmatrix} -(1,\xi)\gamma_x \\ (1,\xi)\gamma \end{pmatrix}.
\]

Hence \( \bar{\xi} \) defined in Theorem 3.4 is given by (3.7).

By Theorem 3.1 \( \tilde{E}(x,t,\lambda) \) is an extended frame for the new solution \( \tilde{q} \). To compute \( \tilde{E}(x,t,0) \) we only need to compute the coefficient of \( \lambda \) of \( r_{\xi,k}(\lambda)E(x,t,\lambda)r_{-\bar{\xi}(x,t),k}(\lambda) \). So we get

\[
\tilde{E}(x,t,0) = \epsilon_{12}E_0r_{-\bar{\xi},0}(0) + r_{\xi,0}(0)E_0\epsilon_{12} + r_{\xi,0}(0)E_1r_{-\bar{\xi},0}(0).
\]

By Proposition 2.4, the first column of \( \tilde{E}(x,t,0) \) is a solution of (1.3) with \( \tilde{q} \) as its central affine curvature. \( \square \)

As a consequence of Permutability Theorem 3.6 we have

Theorem 3.9. [Permutability for (1.3)]

Let \( \gamma \) be a solution of (1.3) with central affine curvature \( q \), and \( E \) the extended frame for \( q \) with \( E(x,t,0) = (\gamma, \gamma_x) \). Let \( k_i, \xi_i, \bar{\xi}_i \) be as in Theorem 3.6, \( k_1k_2 \neq 0 \), and \( \gamma_i = \frac{1}{k_i}(\bar{\xi}_i\gamma - \gamma_x) \) for \( i = 1,2 \). Then

\[
\gamma_{12} = \frac{1}{k_2}(\bar{\xi}_{12}\gamma_1 - (\gamma_1)_x)
\]

is a solution of (1.3) with central affine curvature \( q_{12} = q - 2(\bar{\xi}_1^2 - k_1^2) + 2(\bar{\xi}_{12}^2 - k_2^2) \), where \( \bar{\xi}_{12} = -\bar{\xi}_1 + (k_1^2 - k_2^2)(\xi_1 - \xi_2)^{-1} \).
Explicit solutions

We apply Bäcklund transformation (BT) (Theorem 3.7) to the stationary solution \( \gamma(x, t) = (1, x)^t \) to get explicit solutions of (1.3), whose central affine curvatures are pure 1-soliton solutions. Then we apply BT again to obtain explicit solutions of (1.3) whose central affine curvature are pure 2-soliton solutions of the KdV equation.

First note that
\[
E(x, t, \lambda) = \begin{pmatrix} \cosh(z x + z^3 t) & z \sinh(z x + z^3 t) \\ z^{-1} \sinh(z x + z^3 t) & \cosh(z x + z^3 t) \end{pmatrix}
\]
(3.11)
is an extended frame for the trivial solution \( q = 0 \) of the KdV equation with \( \lambda = z^2 \) and
\[
\gamma(x, t) = E(x, t, 0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ x \end{pmatrix}.
\]
We apply BT to the stationary solution with \( k \neq 0 \) and \( \xi = 0 \) to \( \gamma \). Then we get
\[
\tilde{\gamma} = \begin{pmatrix} \tanh(k x + k^3 t) \\ x \tanh(k x + k^3 t) - \frac{1}{k} \end{pmatrix}
\]
is a solution of (1.3) with central affine curvature
\[
\tilde{q} = -2k^2 \sech^2(k x + k^3 t).
\]

First we use Permutability formula to write down more explicit solutions of (1.3). Apply BT to the stationary solution with \( \xi = 0 \) and real \( k_1, k_2 \) to get two 1-soliton solutions:
\[
\tilde{\xi}_i = k_i \tanh(k_i x + k_i^3 t),
\]
\[
\tilde{\gamma}_i = \begin{pmatrix} \tanh(k_i x + k_i^3 t) \\ x \tanh(k_i x + k_i^3 t) - \frac{1}{k_i} \end{pmatrix},
\]
with central affine curvature
\[
q_i = -2k_i^2 \sech^2(k_i x + k_i t)
\]
for \( i = 1, 2 \). Then we apply the Permutability formula (3.10) to get the solution
\[
\gamma_{12} = \frac{1}{k_2} (\tilde{\xi}_{12} \gamma_1 - (\gamma_1)_x)
\]
of (1.3) with central affine curvature
\[
q_{12} = 2k_1^2 \sech^2(k_1 x + k_1^3 t) + 2(\tilde{\xi}_{12}^2 - k_2^2),
\]
where
\[
\tilde{\xi}_{12} = -k_1 \tanh(m_1) + \frac{(k_1^2 - k_2^2)(\cosh(m_1 + m_1) + \cosh(m_1 - m_2))}{(k_1 - k_2) \sinh(m_1 + m_2) + (k_1 + k_2) \sinh(m_1 - m_2)}.
\]
Note that in general, \( q_{12} \) and \( \gamma_{12} \) have singularities.

Next we apply BT to \( \gamma_1 \) to get new smooth solutions. Note that
\[
E_1(x, t, \lambda) = \frac{1}{\lambda^2 - k_1^2} \begin{pmatrix} 0 & \lambda - k_1^2 \\ 1 & 0 \end{pmatrix} E(x, t, \lambda) \begin{pmatrix} -\frac{1}{\xi} & \lambda - k_1^2 + \frac{\xi^2}{\lambda} \\ 1 & -\frac{1}{\xi} \end{pmatrix}
\]
(3.12)
is an extended frame for $q_1$. Apply BT to $\gamma_1$ with $k_2 \neq 0$ and $\xi_2 = 0$ to get

$$\begin{pmatrix} y_{11} \\ y_{12} \end{pmatrix} = E(x, t, k_2)^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \tilde{\xi}_1 \cosh(k_2 x + k_2^3 t) - k_2^{-1} (\tilde{\xi}_1^2 + k_2^3) \sinh(k_2 x + k_2^3 t) \\ \cosh(k_2 x + k_2^3 t) - k_2^{-1} \tilde{\xi}_1 \sinh(k_2 x + k_2^3 t) \end{pmatrix},$$

where $\tilde{\xi}_1 = k_1 \tanh(k_1 x + k_1^3 t)$. Let

$$m_1 = k_1 x + k_1^3 t, \quad m_2 = k_2 x + k_2^3 t.$$ 

Then we have

$$\tilde{\gamma}_{12} = -\frac{\tilde{\xi}_1 \tilde{\xi}_{12} + \tilde{\xi}_1^{2} - k_2^{2}}{k_2} \frac{\tilde{\xi}_{12} - \tilde{\xi}_1}{k_2^{2}}$$

is a solution of (1.3) with central affine curvature

$$q_{12} = -q_1 + 2(\tilde{\xi}_1^{2} - k_2^{2}),$$

where

$$\tilde{\xi}_{12} = -2k_1k_2 \sinh m_1 \cosh m_2 - k_2 \sinh m_1 \cosh m_2 \cosh^2 m_1 + k_1 \sinh m_2 \cosh m_1((k_2 - k_1) \cosh(m_1 + m_2) + (k_1 + k_2) \cosh(m_1 - m_2)).$$

Note that if $k_2 > k_1 > 0$, then $\tilde{\xi}_{12}$ is smooth on $\mathbb{R}^2$. Hence $\gamma_{12}$ is smooth on $\mathbb{R}^2$ with smooth 2-soliton solution $q_{12}$ as central affine curvature.

4. Bi-Hamiltonian structure

In this section, we first give a brief review of the bi-Hamiltonian structure of the KdV hierarchy (cf. [14], [10], and [6]) including the sequence of Poisson structures $\{,\}_{2j+1}$ of order $2j + 1$. Then we use the pull back of $\{,\}_{2j+1}$ to get a sequence of compatible Poisson brackets of order $2j - 5$ on $\mathcal{M}(S^1)$ via the map $\Psi : \mathcal{M}_2(S^1) \to C^\infty(S^1, \mathbb{R})$ defined in Corollary 2.5. We note that the pull back of $\{,\}_5$ and $\{,\}_3$ on $\mathcal{M}(S^1)$ give rise to the pre-symplectic forms $\hat{w}_5$ and $\hat{w}_3$ given by Pinkall in [15] and Fujioka and Kurose in [9] respectively. Moreover, $\hat{w}_3$ is a symplectic form on $\mathcal{M}(S^1)/SL(2, \mathbb{R})$, and $\hat{w}_5$ is a symplectic form on $\mathcal{M}(S^1)/S^1$.

The gradient of a functional $H : C^\infty(S^1, \mathbb{R}) \to \mathbb{R}$ with respect to the $L^2$ inner product is defined by

$$dH_q(v) = \langle \nabla H(q), v \rangle = \oint (\nabla H(q)) v \, dx.$$ 

A Poisson operator on $C^\infty(S^1, \mathbb{R})$ is a collection of linear skew-adjoint operators $L_q$ on $C^\infty(S^1, \mathbb{R})$ for $q \in C^\infty(S^1, \mathbb{R})$ such that

$$\{F_1, F_2\}(q) = -\oint L_q(\nabla F_1(q)) F_2(q) \, dx.$$
defines a Poisson structure on \( C^\infty(S^1, \mathbb{R}) \). The Hamiltonian equation for \( H : C^\infty(S^1, \mathbb{R}) \to \mathbb{R} \) with respect to \( \{,\} \) is 
\[
q_t = L_q(\nabla H(q)).
\]

The following results concerning the bi-Hamiltonian structure of the KdV hierarchy are known:

(i) For \( q \in C^\infty(S^1, \mathbb{R}) \), let
\[
(L_1)_q(v) = v_x,
\]
\[
(L_3)_q(v) = \frac{1}{4}(v_{xxx} - 4qv_x - 2q_x v). \tag{4.1}
\]

Then \( L_1, L_3 \) are Poisson operators on \( C^\infty(S^1, \mathbb{R}) \). We let \( \{,\}_1 \) and \( \{,\}_3 \) denote the Poisson structures defined by \( L_1 \) and \( L_3 \) respectively.

(ii) Let \( H_{2j+1} : C^\infty(S^1, \mathbb{R}) \to \mathbb{R} \) denote the functional defined by
\[
H_{2j+1}(q) = \frac{4}{2j+1} \oint C_{-(j+1)}(q) \, dx, \tag{4.3}
\]
where \( Q_i(q) = \begin{pmatrix} A_i(q) & B_i(q) \\ C_i(q) & -A_i(q) \end{pmatrix} \) is defined in section 2. For example,
\[
H_1(q) = -2 \oint q \, dx, \\
H_3(q) = \frac{1}{2} \oint q^2 \, dx, \\
H_5(q) = -\frac{1}{8} \oint (q_x)^2 + 2q^3 \, dx.
\]

Then the \((2j + 1)\)-th flow \((2.7)\) is the Hamiltonian flow for \( H_{2j+1} \) (\( H_{2j+3} \) resp.) with respect to \( \{,\}_3 \) (\( \{,\}_1 \) resp.). Let
\[
P = L_1^{-1} L_3,
\]
where \((L_1)^{-1}_q\) is defined on \( \{ v \in C^\infty(S^1, \mathbb{R}) \mid \oint v \, dx = 0 \} \) and
\[
(L_1)^{-1}_q(p(q)_x) = p(q),
\]
where \( p(q) \) is a polynomial differential of \( q \) without constant term. We have
\[
\nabla H_{2j+1}(q) = -2P_q^j(1) = P_q^{j-1}(q), \tag{4.4}
\]
and the \((2j + 1)\)-th flow in the KdV hierarchy is
\[
q_{t_{2j+1}} = (L_3)_q(\nabla H_{2j+1}(q)) = (L_1)_q(\nabla H_{2j+3}(q)).
\]

Since the flows in the KdV-hierarchy commute, we have
\[
\{H_{2i+1}, H_{2k+1}\}_1 = \{H_{2i+1}, H_{2k+1}\}_3 = 0, \quad [X_{2i+1}, X_{2k+1}] = 0
\]
for all \( i, k \geq 0 \), where
\[
X_{2i+1}(q) = -2(L_3 P^i)(1) = (L_3 P^{i-1})_q(q). \tag{4.5}
\]
(iii) The Poisson structures \{ , \} and \{ , \} are compatible, i.e., \( c_1 \{ , \} + c_3 \{ , \} \) is a Poisson structure for all real constants \( c_1, c_3 \). This implies that

\[
L_{2j+1} = L_3(L^{-1}_1 L_3)^{j-1}, \quad j \geq 0,
\]

is a Poisson operator. We give a heuristic argument how this sequence \( \{L_{2j+1}\} \) arises. Since \( L_1 - \mu L_3 \) is a Poisson structure for all \( \mu \in \mathbb{R} \), the 2-form defined by

\[
w(\mu)_q(v_1, v_2) = \langle (L_3 - \mu L_1)^{-1}_q(v_1), v_2 \rangle
\]
is closed. But

\[
(L_3 - \mu L_1)^{-1} = (L_3(I - \mu L_3^{-1} L_1))^{-1} = (1 - \mu L_3^{-1} L_1)^{-1} L_3^{-1}
\]
\[= \sum_{i \geq 0} \mu^i (L_3^{-1} L_1)^i L_3^{-1}.
\]

So we can write \( w(\mu) = \sum_{i \geq 0} \mu^i w_{2i+3} \), where

\[
(w_{2i+3})_q(v_1, v_2) = \oint ((L_3^{-1} L_1)^i L_3^{-1} q(v_1)) v_2 \, dx
\]
for \( i \geq 0 \). Since \( w(\mu) \) is a closed 2-form for all parameter \( \mu \), \( w_{2i+3} \) is closed for all \( i \geq 0 \). Therefore \( L_{2j+3} = L_3 P_j \) is an order \((2j + 3)\) Poisson operator on the domain of \( P_j \) for \( j \geq 0 \). Note that we can use induction to see that given \( v \in C^\infty(S^1 \mathbb{R}) \), if \( v \) is perpendicular to \( X_1(q), \ldots, X_{2j+1}(q) \) then \( P_j(v) \) is defined (i.e., it is periodic).

The \((2j + 1)\)-th flow in the KdV hierarchy is Hamiltonian flow for \( H_{2i+1} \) with respect to \( \{ , \}_{2(j-i)+1} \), i.e.,

\[
q_{t_{2j+1}} = L_{2(j-i)+1}(\nabla H_{2i+3}) = (L_3 P_j^{-1})_q(q).
\]

Since \( L_{2j+1} = L_3(L_1^{-1} L_3)^{j-1} = (L_3 L_1^{-1})^j L_1 \), the flows in the KdV hierarchy can be obtained by applying the recursive operator \( L_3 L_1^{-1} \) as follows:

\[
q_{t_{2j+1}} = (L_3 L_1^{-1})_q(q_x).
\]

In the rest of the section, we discuss properties of the pull back of the Poisson structure \( \{ , \}_{2j+1} \) on the subring

\[
\Psi^*(C^\infty(S^1 \mathbb{R})) = \{ F \circ \Psi \mid F : C^\infty(S^1 \mathbb{R}) \to \mathbb{R} \}
\]
defined by

\[
\{ F_1 \circ \Psi, F_2 \circ \Psi \}_{2j+1} = \{ F_1, F_2 \}_{2j+1} \circ \Psi.
\]

In order to write down the corresponding Poisson operator, we need the following Propositions:

**Proposition 4.1.** Let \( \Psi : \mathcal{M}_2(S^1) \to C^\infty(S^1 \mathbb{R}) \) be the map defined in Corollary 2.5. Then

\[
d\Psi_\gamma(\tilde{\xi}) = -2(L_3)_q(\xi),
\]

where \( q = \Psi(\gamma), \xi \in C^\infty(S^1 \mathbb{R}), \tilde{\xi} = -\frac{1}{2} \xi_x \gamma + \xi_\gamma, \) and \( (L_3)_q \) is the Poisson operator defined by (4.2).
Proof. Recall that we identify $C^\infty(S^1, \mathbb{R})$ as $T(M_2(S^1))_\gamma$ via

$$\xi \mapsto \tilde{\xi} = -\frac{1}{2}\xi x \gamma + \xi x.$$ 

Write $\delta \gamma = \tilde{\xi}$. Take variation of the equation $\gamma_{xx} = q \gamma$ to get

$$(\delta \gamma)_{xx} = (\delta q) \gamma + q \delta \gamma = (\delta q) \gamma + q(\frac{1}{2} \xi x \gamma + \xi x) = (\delta q - \frac{1}{2} \xi x q) \gamma + \xi x. \quad (4.7)$$

Let $g = (\gamma, \gamma_x)$. Then we have

$$g_x = g \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix}. \quad (4.8)$$

Take $x$-derivative of

$$\delta \gamma = -\frac{1}{2} \xi x \gamma + \xi x = g \begin{pmatrix} -\frac{\xi x}{2} \\ \xi \end{pmatrix}$$

and use (4.8) to get

$$(\delta \gamma)_{xx} = (-\frac{1}{2} \xi x x + 3 \frac{3}{2} q \xi x + q x \xi) \gamma + (\xi x) \gamma x. \quad (4.9)$$

Compare coefficients of $\gamma$ of (4.7) and (4.9) to get $\delta q = -\frac{1}{2} \xi x x + 2q \xi x + q x \xi$, which is equal to $-2(L_3)_q(\xi)$. □

Proposition 4.2. Let $C^\infty_I(S^1, \mathbb{R})$ and $\Psi$ be as in Corollary 2.7, $\gamma \in M_2(S^1)$, and $q = \Psi(\gamma)$.

1. Let $A \in sl(2, \mathbb{R})$, and $\hat{A}(\gamma) = A\gamma$ the infinitesimal vector field for the action of $SL(2, \mathbb{R})$ on $M_2(S^1)$. Then there exists $\xi(\gamma) \in C^\infty(S^1, \mathbb{R})$ such that $\hat{A}(\gamma) = \xi(\gamma)$. In fact, $\xi(\gamma)$ can be computed from $g^{-1}A\gamma = (-\frac{1}{2}G\xi(\gamma)x, \xi(\gamma)) \Gamma$, where $g = (\gamma, \gamma_x)$.

2. $\text{Ker}((L_3)_q)$ is the space of all $\xi \in C^\infty(S^1, \mathbb{R})$ such that $\hat{\xi}(\gamma) = A\gamma$ for some $A \in sl(2, \mathbb{R})$.

3. The tangent space $T(C^\infty_I(S^1, \mathbb{R}))_q$ at $q$ is equal to $\text{Im}((L_3)_q)$, and $\text{Ker}(d\Psi_\gamma) = \{\xi \mid \xi \in \text{Ker}((L_3)_q)\} = T(SL(2, \mathbb{R}) \cdot \gamma)_\gamma$.

Proof. It follows from Proposition 4.1 and Corollary 2.5 that the space of infinitesimal vector fields at $\gamma$ generated by the $SL(2, \mathbb{R})$-action on $M_2(I)$ is equal to $\text{Ker}((L_3)_q)$. This proves (3). Since $\hat{A}(\gamma)$ is tangent to $M_2(S^1)$ at $\gamma$, there exists $\xi \in C^\infty(S^1, \mathbb{R})$ such that $\hat{A}(\gamma) = \xi(\gamma)$. But

$$\hat{A}(\gamma) = A\gamma = -\frac{\xi x}{2} \gamma + \xi x = (\gamma, \gamma_x) \begin{pmatrix} -\frac{\xi x}{2} \\ \xi \end{pmatrix} = g \begin{pmatrix} -\frac{\xi x}{2} \\ \xi \end{pmatrix}$$

implies statements (1) and (2). □

Proposition 4.3. Let $H : C^\infty(S^1, \mathbb{R}) \to \mathbb{R}$ be a functional, and $\hat{H} = H \circ \Psi$. Then $(\nabla H)(\gamma) = y(\gamma)$, where $y(\gamma) = 2(L_3)_q(\nabla H(q))$ and $q = \Psi(\gamma)$.
Proof. Use Proposition 4.1 to get
\[
\delta \hat{H} = \oint (\nabla H(q)) \delta q \, dx = \oint -2 \nabla H(q)(L_3)q(\xi) \, dx = \oint 2(L_3q(\nabla H(q))\xi \, dx,
\]
where \( \delta \gamma = \tilde{\xi} \).

Next we write down the formula for the Poisson operators induced from \( L_{2j+1} \) via the map \( \Psi \):

**Proposition 4.4.** The Poisson operator induced from \( L_{2j+1} \) via the map \( \Psi \) is
\[
(\hat{L}_{2j+1})_\gamma(\tilde{\xi}) = (J_{2j+1})_\gamma(\xi), \quad J_{2j+1} = -\frac{1}{4}(L^{-1}L_3)^{j-1}L_3^{-1},
\]
where \( q = \Psi(\gamma) \).

Proof. Let \( H \) be a functional on \( C^\infty(S^1, \mathbb{R}) \), \( \hat{H} = H \circ \Psi \), and \( \nabla \hat{H}(\gamma) = \tilde{y}(\gamma) \). The Hamiltonian vector field of \( \hat{H} \) with respect to \( \{ \cdot, \cdot \}_{2j+1} \) is \( L_{2j+1}(\nabla H) \). By Proposition 4.3, \( y(\gamma) = 2(L_3q(\nabla H(q)) \xi \equiv (J_2)_{\gamma}(\xi) \). Suppose \( \xi \) is the Hamiltonian vector field of \( \hat{H} \) with respect to the Poisson structure induced from \( L_{2j+1} \) via \( \Psi \). Then \( d\Psi(\tilde{\xi}(\gamma)) = L_{2j+1}(\nabla H) \). By Proposition 4.1 we have
\[
\xi(\gamma) = -\frac{1}{4}L^{-1}L_{2j+1}L_3^{-1}(y(\gamma)) = -\frac{1}{4}J_{2j+1}(y(\gamma)).
\]

Remark 4.5.
(a) The Poisson operator \( \hat{L}_{2j+1} \) has order \( (2j - 5) \). So the induced Poisson structure loses six derivatives.
(b) \( J_3 = -\frac{1}{4}L_3^{-1} \) and \( J_5 = -\frac{1}{4}L_1^{-1} \).
(c) Since \( \hat{L}_3, \hat{L}_5 \) define Poisson brackets, the two forms \( \hat{w}_3, \hat{w}_5 \) defined as below are closed:
\[
(\hat{w}_3)_\gamma(\tilde{\xi}, \tilde{\eta}) = \oint ((J_3)_q^{-1}(\xi))\eta \, dx = -4 \oint ((L_3)_q(\xi))\eta \, dx,
\]
\[
(\hat{w}_5)_\gamma(\tilde{\xi}, \tilde{\eta}) = \oint ((J_5)_q^{-1}(\xi))\eta \, dx = -4 \oint ((L_5)_q(\xi))\eta \, dx,
\]
where \( q = \Psi(\gamma) \). They are degenerate because \( \text{Ker}(L_1)_q = \mathbb{R} \) and \( \text{Ker}(L_3)_q \) is of dimension 3. The infinitesimal vector field defined by the \( S^1 \)-action on \( \mathcal{M}_2(S^1) = \mathcal{I}(\gamma) = \gamma_x \). So \( \hat{w}_5 \) is a symplectic form on \( \mathcal{M}(S^1)/S^1 \). It follows from Proposition 4.2 that \( \hat{w}_3 \) defines a symplectic form on \( \mathcal{M}(S^1)/SL(2, \mathbb{R}) \).
(d) Given \( X = \tilde{\xi} = -\frac{\xi_x}{2} + \xi \gamma_x \) and \( Y = \tilde{\eta} = -\frac{\eta_x}{2} + \eta \gamma_x \) in \( T\mathcal{M}_2(S^1) \), we use \( \gamma_x = q^2 \gamma \) to get
\[
X_x = -\frac{\xi_x}{2} + q\xi \gamma_x + \frac{\xi_x}{2} \gamma_x, \quad Y_x = -\frac{\eta_x}{2} + q\eta \gamma + \frac{\eta_x}{2} \gamma_x.
\]
A direct computation implies that

\[
(\hat{w}_3)_\gamma(X, Y) = -2 \oint \det(X_x, Y_x) + q \det(X, Y) \, dx,
\]

(4.10)

\[
(\hat{w}_5)_\gamma(X, Y) = -4 \oint \det(X, Y) \, dx.
\]

(4.11)

Note that \(\frac{1}{4} \hat{w}_5\) is the symplectic form \([15]\) defined by Pinkall in [15], and \(\hat{w}_3\) is the symplectic form defined by Fujioka and Kurose in [9].

(e) The order \((2j+1)\) central affine curve flow \((2.15)\) is the Hamiltonian flow for \(\hat{H}_{2(j-i)+1}\) with respect to the Poisson structure \(\hat{J}_{2i+1}\) for all \(0 \leq i \leq j\). In fact, \((2.15)\) is

\[
\gamma_{t_{2j+1}} = -\frac{1}{2} P_{q,j-1}(q) = \hat{L}_{2i+1}(\nabla \hat{H}_{2(j-i)+3}), \quad 0 \leq i \leq j,
\]

where \(P = L^{-1}_1 L_3\).

References

[1] Ablowitz, M.J., Kaup, D.J., Newell, A.C., Segur, H., The inverse scattering transform - Fourier analysis for nonlinear problems, Stud. Appl. Math. 53 (1974), 249–315.

[2] Adler, M., On the Bäcklund transformation for the Gel’fand-Dickey equations, Comm. Math. Phys. 80(4) 1981, 517–527.

[3] Calini, A., Ivey, T., Mari Beffa, G., Remarks on KdV-type flows on star-shaped curves, Phys. D 238(8) (2009), 788–797.

[4] Calini, A., Ivey, T., Mari Beffa, G., Integrable flows for starlike curves in centroaffine space, SIGMA Symmetry Integrability Geom. Methods Appl. 9 (2013).

[5] Chou, K.S., Qu, C.Z., The KdV equation and motion of plane curves, J. Phys. Soc. Japan 70(7) (2001), 1912–1916.

[6] Dickey, L.A., Soliton equations and Hamiltonian systems, second edition, Advanced Series in Mathematical Physics 26 (2003), World Scientific Publishing Co. Inc., River Edge, NJ.

[7] Drinfel’d, V.G., Sokolov, V.V., Lie algebras and equations of Korteweg-de Vries type, (Russian) Current problems in mathematics, 24 (1984), 81–180, Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow.

[8] Fujioka, A., Kurose, T., Hamiltonian formalism for the higher KdV flows on the space of closed complex equicentroaffine curves, Int. J. Geom. Methods Mod. Phys. 7(1) (2010), 165-175.

[9] Fujioka, A., Kurose, T., Multi-Hamiltonian structures on space of closed equicentroaffine plane curves associated to higher KdV flows, preprint, Arxiv: math.dg 1310.1688

[10] Gel’fand, I.M., Dikii, L.A., A family of Hamiltonian structures connected with integrable nonlinear differential equations, Akad. Nauk SSSR Inst. Prikl. Mat. Preprint 136 (1978), 41.

[11] Gardner, C.S., Greene, J.M., Kruskal, M.D., Miura, R.M., Korteweg-de Vries equation and generalization. VI. Methods for exact solution, Comm. Pure Appl. Math. 27 (1974), 97–133.

[12] Huang, R.P., Singer, D.A., A new flow on starlike curves in \(\mathbb{R}^3\), Proc. Amer. Math. Soc. 130(9) (2002), 2725-2735.

[13] Lax, Peter D., Periodic solutions of the KdV equation, Comm. Pure Appl. Math. 28 (1975), 141-188.
[14] Magri, F., *A simple model of the integrable Hamiltonian equation*, J. Math. Physics 19 (1978), 1156-1162.
[15] Pinkall, U., *Hamiltonian flows on the space of star-shaped curves*, Results Math. 27(3-4) (1995), 328–332.
[16] Sattinger, D.H., Szmigielski, J.S., *Factorization and the dressing method for the Gel’fand-Dikii hierarchy*, Phys. D 64(1-3) (1993), 1–34.
[17] Terng, C.L., Uhlenbeck, K., *Bäcklund transformations and loop group actions*, Comm. Pure Appl. Math. 53 (2000), 1–75.
[18] Terng, C.L., Uhlenbeck, K., *The n × n KdV hierarchy*, JFPTA 10 (2011), 37–61.
[19] Terng, C.L., Wu, Z., *Central affine curve flows on \( \mathbb{R}^n \setminus \{0\} \)*, preprint
[20] Zaharov, V.E., Faddeev, L.D., *The Korteweg-de Vries equation is a fully integrable Hamiltonian system*, Funkcional. Anal. Priloz 5(4) (1971), 18–27.

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