JACQUET MODULES AND LOCAL LANGLANDS CORRESPONDENCE

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Abstract. In this paper, we explicitly compute the semisimplifications of all Jacquet modules of irreducible representations with generic $L$-parameters of $p$-adic split odd special orthogonal groups or symplectic groups. Our computation represents them in terms of linear combinations of standard modules with rational coefficients. The main ingredient of this computation is to apply Mœglin’s explicit construction of local $A$-packets to tempered $L$-packets.

1. Introduction

When $G$ is a $p$-adic reductive group and $P = MN$ is a parabolic subgroup, there is the normalized induction functor

$$\text{Ind}_G^P : \text{Rep}(M) \to \text{Rep}(G).$$

The (normalized) Jacquet functor

$$\text{Jac}_P : \text{Rep}(G) \to \text{Rep}(M)$$

is the left adjoint functor of $\text{Ind}_G^P$. For $\pi \in \text{Rep}(G)$, the object $\text{Jac}_P(\pi) \in \text{Rep}(M)$ is called the Jacquet module of $\pi$ with respect to $P$. In the representation theory of $p$-adic reductive groups, the induction functors and the Jacquet functors are ones of the most basic and important terminologies. One of the reasons why they are so important is that they are both exact functors.

The Jacquet modules have many applications. For example:

- Looking at the Jacquet modules of irreducible representation $\pi$ of $G$, one can take a parabolic subgroup $P = MN$ and an irreducible supercuspidal representation $\rho_M$ of $M$ such that $\pi \hookrightarrow \text{Ind}_G^P(\rho_M)$. Such a $\rho_M$ is called the cuspidal support of $\pi$.
- Casselman’s criterion says that the growth of matrix coefficients of an irreducible representation $\pi$ is determined by exponents of the Jacquet modules of $\pi$.
- Mœglin explicitly constructed the local $A$-packets, which are the “local factors of Arthur’s global classification”, by taking Jacquet functors intelligently.

In this paper, we shall give an explicit description of the semisimplifications of Jacquet modules of tempered representations of split odd special orthogonal groups $\text{SO}_{2n+1}(F)$ or symplectic groups $\text{Sp}_{2n}(F)$, where $F$ is a non-archimedean local field of characteristic zero. To do this, it is necessary to have some sort of classification of irreducible representations of these groups. We use the local Langlands correspondence established by Arthur [Ar13] for such a classification.

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The local Langlands correspondence attaches each irreducible representation $\pi$ of $G(F) = SO_{2n+1}(F)$ or $G(F) = Sp_{2n}(F)$ to its $L$-parameter $(\phi, \eta)$, where
\[
\phi : W_F \times SL_2(\mathbb{C}) \to GL_N(\mathbb{C})
\]
is a self-dual representation of the Weil–Deligne group $W_F \times SL_2(\mathbb{C})$ with a suitable structure, and
\[
\eta \in \text{Irr}(A_\phi)
\]
is an irreducible character of the component group $A_\phi$ associated to $\phi$ (which is trivial on the central element).

The Jacquet modules will be computed by two main theorems (Theorems 4.2 and 4.3) and Tadić’s formula (Theorem 2.7) together with Lemma 2.6. Fix an irreducible unitary supercuspidal representation $\rho$ of $GL_d(F)$. By abuse of notation, we denote by the same notation $\rho$ the irreducible representation of $W_F$ corresponding to $\rho$ by the local Langlands correspondence. Let $P_d = M_dN_d$ be the standard parabolic subgroup of $G(F)$ with Levi subgroup $M_d \cong GL_d(F) \times G_0(F)$ for some classical group $G_0$ of the same type as $G$. For an irreducible representation $\pi$, if the semisimplification of $\text{Jac}_{P_d}(\pi)$ is of the form
\[
\bigoplus_{i \in I} \tau_i \boxtimes \pi_i,
\]
we set
\[
\text{Jac}_{\rho|\pi}(\pi) = \bigoplus_{i \in I} \pi_i
\]
for $x \in \mathbb{R}$. The first main theorem is the description of $\text{Jac}_{\rho|\pi}(\pi)$ for tempered $\pi$ (Theorem 4.2). To state this theorem clearly, we introduce an enhanced component group $A_\phi$ attached to $\phi$ with a canonical surjection $A_\phi \twoheadrightarrow A_\phi$ in §3.1.

For discrete series $\pi$, Theorem 4.2 has been proven by Xu [X17a, Lemma 7.3] to describe the cuspidal support of $\pi$ in terms of its $L$-parameter. As related works, Aubert–Moussaoui–Solleveld [AMS18, AMSa, AMSb] defined the “cuspality” of $L$-parameters $(\phi, \eta)$ by a geometric way, and compared this notion with the cuspidal supports or the Bernstein components of corresponding $\pi$. Theorem 4.2 gives us more information for $\pi$ than its cuspidal support. The main ingredient for the proof of Theorem 4.2 is Mœglin’s explicit construction of tempered $L$-packets (Theorem 3.3).

The second main theorem (Theorem 4.3) is a reduction of the computation of the Jacquet module $\text{s.s.}\text{Jac}_{P_k}(\pi)$ with respect to any maximal parabolic subgroup $P_k$ to the one of $\text{Jac}_{\rho|\pi}(\pi)$. Using Theorems 4.2 and 4.3 (together with Lemma 2.6), we can explicitly compute the semisimplifications of all Jacquet modules of irreducible tempered representations $\pi$. In fact, using a generalization of the standard module conjecture by Mœglin–Waldspurger (Theorem 3.2) and Tadić’s formula (Theorem 2.7), we can apply this explicit computation to any irreducible representation $\pi$ with generic $L$-parameter $(\phi, \eta)$.

This paper is organized as follows. In §2 we review some basic results on induced representations and Jacquet modules for classical groups. In particular, Tadić’s formula, which computes the Jacquet modules of induced representations, is stated in §2.2. In §3 we explain the local Langlands correspondence and Mœglin’s explicit construction of tempered $L$-packets.
In [4] we state the main theorems (Theorems 4.2 and 4.3) and give some examples. Finally, we prove the main theorems in [5].

**Notation.** Let $F$ be a non-archimedean local field of characteristic zero. We denote by $W_F$ the Weil group of $F$. The norm map $| \cdot | : W_F \to \mathbb{R}^\times$ is normalized so that $|\text{Frob}| = q^{-1}$, where Frob $\in W_F$ is a fixed (geometric) Frobenius element, and $q = q_F$ is the cardinality of the residual field of $F$.

Each irreducible supercuspidal representation $\rho$ of $GL_d(F)$ is identified with the irreducible bounded representation of $W_F$ of dimension $d$ via the local Langlands correspondence for $GL_d$. Through this paper, we fix such a $\rho$. For each positive integer $a$, the unique irreducible algebraic representation of $SL_2(\mathbb{C})$ of dimension $a$ is denoted by $S_a$.

For a $p$-adic group $G$, we denote by $\text{Rep}(G)$ (resp. $\text{Irr}(G)$) the set of equivalence classes of smooth admissible (resp. irreducible) representations of $G$. For $\Pi \in \text{Rep}(G)$, we write $\text{s.s.}(\Pi)$ for the semisimplification of $\Pi$.

## 2. Induced Representations and Jacquet Modules

In this section, we recall some results on induced representations and Jacquet modules.

### 2.1. Representations of $GL_k(F)$

Let $P = MN$ be a standard parabolic subgroup of $GL_k(F)$, i.e., $P$ contains the Borel subgroup consisting of upper half triangular matrices. Then the Levi subgroup $M$ is isomorphic to $GL_{k_1}(F) \times \cdots \times GL_{k_r}(F)$ with $k_1 + \cdots + k_r = k$. For smooth representations $\tau_1, \ldots, \tau_r$ of $GL_{k_1}(F), \ldots, GL_{k_r}(F)$, respectively, we denote the normalized induced representation by

$$\tau_1 \times \cdots \times \tau_r := \text{Ind}_{P}^{GL_k(F)}(\tau_1 \boxtimes \cdots \boxtimes \tau_r).$$

A **segment** is a symbol $[x, y]$, where $x, y \in \mathbb{R}$ with $x - y \in \mathbb{Z}$ and $x \geq y$. We identify $[x, y]$ with the set $\{x, x - 1, \ldots, y\}$ so that $\#[x, y] = x - y + 1$. Then the normalized induced representation

$$\rho| \cdot |^x \times \cdots \times \rho| \cdot |^y$$

of $GL_{d(x-y+1)}(F)$ has a unique irreducible subrepresentation, which is denoted by

$$\langle \rho; x, \ldots, y \rangle.$$

If $y = -x \leq 0$, this is called a **Steinberg representation** and is denoted by

$$\text{St}(\rho, 2x + 1) = \langle \rho; x, \ldots, -x \rangle,$$

which is a discrete series representation of $GL_{d(2x+1)}(F)$. In general, $\langle \rho; x, \ldots, y \rangle$ is the twist $| \cdot |^{\frac{x+y}{2}} \text{St}(\rho, x - y + 1)$. We say that two segments $[x, y]$ and $[x', y']$ are **linked** if $[x, y] \not\subset [x', y']$, $[x', y'] \not\subset [x, y]$ as sets, and $[x, y] \cup [x, y']$ is also a segment. The linked-ness gives an irreducibility criterion for induced representations.

**Theorem 2.1** (Zelevinsky [Z80 Theorem 9.7]). Let $[x, y]$ and $[x', y']$ be segments, and let $\rho$ and $\rho'$ be irreducible unitary supercuspidal representations of $GL_d(F)$ and $GL_{d'}(F)$, respectively. Then the induced representation

$$\langle \rho; x, \ldots, y \rangle \times \langle \rho'; x', \ldots, y' \rangle$$

is irreducible unless $[x, y]$ are $[x', y']$ are linked, and $\rho \cong \rho'$. 
Let $\text{Irr}_\rho(\text{GL}_{dm}(F))$ be the subset of $\text{Irr}(\text{GL}_{dm}(F))$ consisting of $\tau$ with cuspidal support of the form $\rho|^{x_1} \times \cdots \times \rho|^{x_m}$, i.e.,

$$\tau \mapsto \rho|^{x_1} \times \cdots \times \rho|^{x_m}$$

for some $x_1, \ldots, x_m \in \mathbb{R}$. We understand that $1 := 1_{\text{GL}_0} \in \text{Irr}_\rho(\text{GL}_0(F))$. It is easy to see that

- for pairwise distinct irreducible unitary supercuspidal representations $\rho_1, \ldots, \rho_r$, if $\tau_i \in \text{Irr}_{\rho_i}(\text{GL}_{d,m_i}(F))$ for $i = 1, \ldots, r$, then the induced representation $\tau_1 \times \cdots \times \tau_r$ is irreducible;

- any irreducible representation of $\text{GL}_k(F)$ is of the above form for some $\tau_i \in \text{Irr}_{\rho_i}(\text{GL}_{d,m_i}(F))$.

**Lemma 2.2.** Let $\Omega_m$ be the subset of $\mathbb{R}^m$ consisting of elements

$$\underline{x} = (x_1, x_1 - 1, \ldots, y_1, x_2, x_2 - 1, \ldots, y_2, \ldots, x_t, x_t - 1, \ldots, y_t)$$

such that $x_{i-1} \leq x_i$ for $1 < i \leq t$, and $y_{i-1} \leq y_i$ if $x_{i-1} = x_i$. Let $\Delta_i = \langle \rho; x_i, x_i - 1, \ldots, y_i \rangle$ be the discrete series representation of $\text{GL}_{d(x_i - y_i + 1)}(F)$ corresponding to the segment $[x_i, y_i]$. Then the induced representation $\Delta_{\underline{x}} := \Delta_1 \times \cdots \times \Delta_t$ has a unique irreducible subrepresentation $\tau_{\underline{x}}$. The map $\underline{x} \mapsto \tau_{\underline{x}}$ gives a bijection

$$\Omega_m \rightarrow \text{Irr}_\rho(\text{GL}_{dm}(F)).$$

**Proof.** This follows from the Langlands classification and Theorem 2.1. See also [Z80, Proposition 9.6].

For a partition $(k_1, \ldots, k_r)$ of $k$, we denote by $\text{Jac}_{(k_1, \ldots, k_r)}$ the normalized Jacquet functor on $\text{Rep}(\text{GL}_k(F))$ with respect to the standard maximal parabolic subgroup $P = MN$ with $M \cong \text{GL}_{k_1}(F) \times \cdots \times \text{GL}_{k_r}(F)$. The Jacquet module of $\langle \rho; x, \ldots, y \rangle$ with respect to a maximal parabolic subgroup is computed by Zelevinsky.

**Proposition 2.3 ([Z80 Proposition 9.5]).** Suppose that $x \not= y$ and set $k = d(x - y + 1)$. Then $\text{Jac}_{(k_1, k_2)}(\langle \rho; x, \ldots, y \rangle) = 0$ unless $k_1 \equiv 0 \pmod{d}$. If $k_1 = dm$ with $1 \leq m \leq x - y$, we have

$$\text{Jac}_{(k_1, k_2)}(\langle \rho; x, \ldots, y \rangle) = \langle \rho; x, \ldots, x - (m - 1) \rangle \boxtimes \langle \rho; x - m, \ldots, y \rangle.$$  

If

$$\text{s.s.Jac}_{(d, k-d)}(\tau) = \bigoplus_{i \in I} \tau_i \boxtimes \tau'_i,$$

for $x \in \mathbb{R}$, we set

$$\text{Jac}_{\rho|^{x}}(\tau) = \bigoplus_{\tau_i \cong \rho|^{x_i}} \tau'_i.$$  

For $\underline{x} = (x_1, \ldots, x_r) \in \mathbb{R}^r$, we also define

$$\text{Jac}_{\rho|^{\underline{x}}} = \text{Jac}_{\rho|^{x_r}} \circ \cdots \circ \text{Jac}_{\rho|^{x_1}}.$$  

This is a functor

$$\text{Jac}_{\rho|^{\underline{x}}}: \text{Rep}(\text{GL}_k(F)) \rightarrow \text{Rep}(\text{GL}_{k-d\underline{r}}(F)).$$

In particular, when $\tau \in \text{Rep}(\text{GL}_{dm}(F))$ is of finite length, for $\underline{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$, the Jacquet module $\text{Jac}_{\rho|^{\underline{x}}}(\tau)$ is a representation of the trivial group $\text{GL}_0(F)$ of finite length so that it is a finite dimensional $\mathbb{C}$-vector space.
Lemma 2.4. Let \( \underline{x} = (x_1, \ldots, y_1, \ldots, x_t, \ldots, y_t) \in \Omega_m \) such that \( x_{i-1} \leq x_i \) for \( 1 < i \leq t \), and \( y_{i-1} \leq y_i \) if \( x_{i-1} = x_i \), as in Lemma 2.2. For \( (x, y) \in \{(x_i, y_i)\}_i \), if we set \( m_{(x,y)} = \#\{i \mid (x_i, y_i) = (x, y)\} \), then for \( y \in \Omega_m \), we have

\[
\dim \mathcal{C} \text{Jac}_{\rho|\mathcal{U}(\tau_x)} = \begin{cases} 
\prod_{(x, y) \in \{(x_i, y_i)\}_i} m_{(x,y)}! & \text{if } \underline{x}' = \underline{x}, \\
0 & \text{if } y < \underline{x}.
\end{cases}
\]

Here, we regard \( \mathbb{R}^m \) as a totally ordered set with respect to the lexicographical order.

Proof. Fix \( x \in \mathbb{R} \). We note that \( \text{Jac}_{\rho|\mathcal{U}}(\Delta_{\underline{x}}) \neq 0 \) if and only if \( x \in \{x_1, \ldots, x_t\} \). Let \( \underline{x}_1, \ldots, \underline{x}_t \in \Omega_{m-1} \) be the elements obtained by removing \( x \) from a component of \( \underline{x} \), and rearranging it (so that \( l = \#\{i \mid x_i = x\} \)). Then \( \text{Jac}_{\rho|\mathcal{U}}(\Delta_{\underline{x}}) = \sum_{i=1}^{l} \Delta_{\underline{x}_i} \). Using this, we obtain the lemma by induction on \( m \).

When \( y > \underline{x} \), one can also compute \( \dim \mathcal{C} \text{Jac}_{\rho|\mathcal{U}(\tau_x)} \) inductively.

Let \( \mathcal{R}_k \) be the Grothendieck group of the category of smooth representations of \( \text{GL}_k(F) \) of finite length. By the semisimplification, we identify the objects in this category with elements in \( \mathcal{R}_k \). Elements in \( \text{Irr}(\text{GL}_k(F)) \) form a \( \mathbb{Z} \)-basis of \( \mathcal{R}_k \). Set \( \mathcal{R} = \oplus_{k \geq 0} \mathcal{R}_k \). The induction functor gives a product

\[
m: \mathcal{R} \otimes \mathcal{R} \to \mathcal{R}, \quad \tau_1 \otimes \tau_2 \mapsto \text{s.s.}(\tau_1 \times \tau_2).
\]

This product makes \( \mathcal{R} \) an associative commutative ring. On the other hand, the Jacquet functor gives a coproduct

\[
m^*: \mathcal{R} \to \mathcal{R} \otimes \mathcal{R}
\]

which is defined by the \( \mathbb{Z} \)-linear extension of

\[
\text{Irr}(\text{GL}_k(F)) \ni \tau \mapsto \sum_{k_1 = 0}^{k} \text{s.s.} \text{Jac}_{(k_1, k, k_1)}(\tau).
\]

Then \( m \) and \( m^* \) make \( \mathcal{R} \) a graded Hopf algebra, i.e., \( m^*: \mathcal{R} \to \mathcal{R} \otimes \mathcal{R} \) is a ring homomorphism.

2.2. Representations of \( \text{SO}_{2n+1} \) and \( \text{Sp}_{2n} \). We set \( G \) to be split \( \text{SO}_{2n+1} \) or \( \text{Sp}_{2n} \), i.e., \( G \) is the split algebraic group of type \( B_n \) or \( C_n \). Fix a Borel subgroup of \( G(F) \), and let \( P = MN \) be a standard parabolic subgroup of \( G(F) \). Then the Levi part \( M \) is of the form \( \text{GL}_{k_1}(F) \times \cdots \times \text{GL}_{k_r}(F) \times G_0(F) \) with \( G_0 = \text{SO}_{2n+1} \) or \( G_0 = \text{Sp}_{2n_0} \) such that \( k_1 + \cdots + k_r + n_0 = n \). For a smooth representation \( \tau_1 \boxtimes \cdots \boxtimes \tau_r \boxtimes \pi_0 \) of \( M \), we denote the normalized induced representation by

\[
\tau_1 \times \cdots \times \tau_r \times \pi_0 = \text{Ind}_P^G(F)(\tau_1 \boxtimes \cdots \boxtimes \tau_r \boxtimes \pi_0)
\]

The functor \( \text{Ind}_P^G(F): \text{Rep}(M) \to \text{Rep}(G(F)) \) is exact.

On the other hand, for a smooth representation \( \pi \) of \( G(F) \), we denote the normalized Jacquet module with respect to \( P \) by

\[
\text{Jac}_P(\pi),
\]

and its semisimplification by \( \text{s.s.} \text{Jac}_P(\pi) \). The functor \( \text{Jac}_P: \text{Rep}(G(F)) \to \text{Rep}(M) \) is exact. The Frobenius reciprocity asserts that

\[
\text{Hom}_{G(F)}(\pi, \text{Ind}_P^G(F)(\sigma)) \cong \text{Hom}_M(\text{Jac}_P(\pi), \sigma)
\]
for \( \pi \in \text{Rep}(G(F)) \) and \( \sigma \in \text{Rep}(M) \).

The maximal parabolic subgroup with Levi \( \text{GL}_k(F) \times G_0(F) \) is denoted by \( P_k = M_kN_k \). If

\[
\text{s.s.} \text{Jac}_{P_k}(\pi) = \bigoplus_{i \in I} \tau_i \boxtimes \pi_i,
\]

for a real number \( x \), we set

\[
\text{Jac}_{\rho|.|^x}(\pi) = \bigoplus_{i \in I} \tau_i.
\]

This is a representation of \( G_0(F) \), which is \( \text{SO}_{2(n-d)+1}(F) \) or \( \text{Sp}_{2(n-d)}(F) \). Also, for \( x = (x_1, \ldots, x_r) \in \mathbb{R}^r \), we set

\[
\text{Jac}_{\rho|.|^x}(\pi) = \text{Jac}_{\rho|.|^{x_1}} \circ \cdots \circ \text{Jac}_{\rho|.|^{x_r}}(\pi).
\]

We recall some properties of \( \text{Jac}_{\rho|.|^x} \).

**Lemma 2.5** ([X17a, Lemmas 5.3, 5.6]). Let \( \pi \) be an irreducible representation of \( G(F) \).

1. Suppose that \( \text{Jac}_{\rho|.|^x}(\pi) \) is nonzero. Then there exists an irreducible constituent \( \sigma \) of \( \text{Jac}_{\rho|.|^x}(\pi) \) such that we have an inclusion

\[
\pi \hookrightarrow \rho|.|^{x_1} \times \cdots \times \rho|.|^{x_r} \times \sigma.
\]

2. If \( |x - y| \neq 1 \), then \( \text{Jac}_{\rho|.|^x,\rho|.|^y}(\pi) = \text{Jac}_{\rho|.|^y \circ \cdots \circ \rho|.|^x}(\pi) \).

Let \( \mathcal{R}_n(G) \) be the Grothendieck group of the category of smooth representations of \( G(F) \) of finite length, where \( n = \text{rank}(G) \), i.e., \( G = \text{SO}_{2n+1} \) or \( G = \text{Sp}_{2n} \). Set \( \mathcal{R}(G) = \oplus_{n \geq 0} \mathcal{R}_n(G) \).

The parabolic induction defines a module structure

\[
\times : \mathcal{R} \otimes \mathcal{R}(G) \to \mathcal{R}(G), \quad \tau \otimes \pi \mapsto \text{s.s.}(\tau \times \pi),
\]

and the Jacquet functor defines a comodule structure

\[
\mu^* : \mathcal{R}(G) \to \mathcal{R} \otimes \mathcal{R}(G)
\]

by

\[
\text{Irr}(G(F)) \ni \pi \mapsto \sum_{k=0}^{\text{rank}(G)} \text{s.s.} \text{Jac}_{P_k}(\pi).
\]

When

\[
\mu^*(\pi) = \bigoplus_{i \in I} \tau_i \otimes \pi_i,
\]

we define \( \mu^*_\rho(\pi) \) by

\[
\mu^*_\rho(\pi) = \bigoplus_{\tau_i \in \text{Irr}_\rho(\text{GL}_{dm}(F))} \tau_i \otimes \pi_i.
\]

**Lemma 2.6.** If we define \( \iota : \mathcal{R}(G) \to \mathcal{R} \otimes \mathcal{R}(G) \) by \( \pi \mapsto 1_{\text{GL}_d(F)} \otimes \pi \), we have

\[
\mu^* = \circ \rho \left((m \otimes \text{id}) \circ (\text{id} \otimes \mu^*_\rho)\right) \circ \iota,
\]

where \( \rho \) runs over all irreducible unitary supercuspidal representations of \( \text{GL}_d(F) \) for \( d > 0 \).
Proof. Fix an irreducible representation $\pi$ of $G(F)$. First, we note that there are only finitely many $\rho$ such that $\mu^\ast(\pi) \neq 0$. Second, we claim that

$$(m \otimes \text{id}) \circ (\text{id} \otimes \mu^\ast_\rho) \circ \mu^\ast_\rho(\pi) = (m \otimes \text{id}) \circ (\text{id} \otimes \mu^\ast_\rho) \circ \mu^\ast_\rho(\pi)$$

for distinct $\rho$ and $\rho'$. In fact, this is the sum of subrepresentations appearing $\mu^\ast(\pi)$ of the form

$$(\tau \times \tau') \otimes \pi_0 = (\tau' \times \tau) \otimes \pi_0,$$

where $\tau$ is of type $\rho$ and $\tau'$ is of type $\rho'$. By the same argument, we have

$$\mu^\ast = \circ \rho((m \otimes \text{id}) \circ (\text{id} \otimes \mu^\ast_\rho)) \circ \iota(\pi),$$

as desired. \(\square\)

Tadić established a formula to compute $\mu^\ast$ for induced representations. The contragredient functor $\tau \mapsto \tau^\vee$ defines an automorphism $\vee: \mathcal{R} \to \mathcal{R}$ in a natural way. Let $s: \mathcal{R} \otimes \mathcal{R} \to \mathcal{R} \otimes \mathcal{R}$ be the homomorphism defined by $\sum_i \tau_i \otimes \tau'_i \mapsto \sum_i \tau'_i \otimes \tau_i$.

**Theorem 2.7** (Tadić [T95]). Consider the composition

$$M^\ast = (m \otimes \text{id}) \circ (\vee \otimes m^\ast) \circ s \circ m^\ast: \mathcal{R} \to \mathcal{R} \otimes \mathcal{R}.$$

Then for the maximal parabolic subgroup $P_k = M_k N_k$ of $G(F)$ and for an admissible representation $\tau \boxtimes \pi$ of $M_k$, we have

$$\mu^\ast(\tau \boxtimes \pi) = M^\ast(\pi) \times \mu^\ast(\pi).$$

In particular, we have the following.

**Corollary 2.8.** For a segment $[x, y]$, we have

$$\mu^\ast_\rho((\rho; x, \ldots, y) \times \pi) = \sum_{k \geq 0 \atop k + l \leq x - y + 1} \left(\left(\rho; -y, \ldots, -y - k + 1\right) \times \left(\rho; x, \ldots, x - l + 1\right)\right) \otimes \left(\rho; x - l, \ldots, y + k\right) \times \mu^\ast_\rho(\pi).$$

3. Local Langlands correspondence

In this section, we review the local Langlands correspondence for split $\text{SO}_{2n+1}$ or $\text{Sp}_{2n}$ over $F$.

3.1. $L$-parameters. A homomorphism

$$\phi: W_F \times \text{SL}_2(\mathbb{C}) \to \text{GL}_k(\mathbb{C})$$

is called a representation of the Weil–Deligne group $W_F \times \text{SL}_2(\mathbb{C})$ if

- $\phi(\text{Frob}) \in \text{GL}_k(\mathbb{C})$ is semisimple;
- $\phi|W_F$ is smooth, i.e., has an open kernel;
- $\phi|\text{SL}_2(\mathbb{C})$ is algebraic.
We say that a representation $\phi$ of $W_F \times \text{SL}_2(\mathbb{C})$ is \textbf{tempered} if $\phi(W_F)$ is bounded. The local Langlands correspondence for $\text{GL}_k$ asserts that there is a canonical bijection

$$\text{Irr}(\text{GL}_k(F)) \leftrightarrow \{\phi: W_F \times \text{SL}_2(\mathbb{C}) \to \text{GL}_k(\mathbb{C})\}/\sim,$$

which preserves the tempered-ness.

An \textbf{L-parameter} for $\text{SO}_{2n+1}$ is a symplectic representation $\phi: W_F \times \text{SL}_2(\mathbb{C}) \to \text{Sp}_{2n}(\mathbb{C})$.

Similarly, an \textbf{L-parameter} for $\text{Sp}_{2n}$ is an orthogonal representation $\phi: W_F \times \text{SL}_2(\mathbb{C}) \to \text{SO}_{2n+1}(\mathbb{C})$.

For $G = \text{SO}_{2n+1}$ or $G = \text{Sp}_{2n}$, we let $\Phi(G)$ be the set of equivalence classes of L-parameters for $G$. We say that

- $\phi \in \Phi(G)$ is \textbf{discrete} if $\phi$ is a multiplicity-free sum of irreducible self-dual representations of the same type as $\phi$;
- $\phi \in \Phi(G)$ is \textbf{of good parity} if $\phi$ is a sum of irreducible self-dual representations of the same type as $\phi$;
- $\phi \in \Phi(G)$ is \textbf{tempered} if $\phi(W_F)$ is bounded;
- $\phi \in \Phi(G)$ is \textbf{generic} if the adjoint $L$-function $L(s, \phi, \text{Ad})$ is regular at $s = 1$.

We denote by $\Phi_{\text{disc}}(G)$ (resp. $\Phi_{\text{gp}}(G)$, $\Phi_{\text{temp}}(G)$, and $\Phi_{\text{gen}}(G)$) the subset of $\Phi(G)$ consisting of discrete L-parameters (resp. L-parameters of good parity, tempered L-parameters, and generic L-parameters). Then we have inclusions

$$\Phi_{\text{disc}}(G) \subset \Phi_{\text{gp}}(G) \subset \Phi_{\text{temp}}(G) \subset \Phi_{\text{gen}}(G) \subset \Phi(G).$$

For $\phi \in \Phi(G)$, we can decompose

$$\phi = m_1\phi_1 \oplus \cdots \oplus m_r\phi_r \oplus (\phi' \oplus \phi''),$$

where $\phi_1, \ldots, \phi_r$ are distinct irreducible self-dual representations of the same type as $\phi$, $m_i \geq 1$ is the multiplicity of $\phi_i$ in $\phi$, and $\phi'$ is a sum of irreducible representations which are not self-dual or self-dual of the opposite type to $\phi$. We define the \textbf{component group} $A_\phi$ of $\phi$ by

$$A_\phi = \bigoplus_{i=1}^r (\mathbb{Z}/2\mathbb{Z})\alpha_{\phi_i} \cong (\mathbb{Z}/2\mathbb{Z})^r.$$

Namely, $A_\phi$ is a free $\mathbb{Z}/2\mathbb{Z}$-module of rank $r$ and $\{\alpha_{\phi_1}, \ldots, \alpha_{\phi_r}\}$ is a basis of $A_\phi$ with $\alpha_{\phi_i}$ associated to $\phi_i$. We set

$$z_\phi = \sum_{i=1}^r m_i\alpha_{\phi_i} \in A_\phi$$

and we call $z_\phi$ the \textbf{central element} in $A_\phi$.

We shall introduce an \textbf{enhanced component group} $A_\phi$ associated to $\phi \in \Phi(G)$. Write $\phi = \phi_{\text{gp}} \oplus (\phi' \oplus \phi''')$, where $\phi_{\text{gp}}$ is the sum of irreducible self-dual representations of the same type as $\phi$, and $\phi'$ is a sum of irreducible representations which are not of the same type as $\phi$. We decompose

$$\phi_{\text{gp}} = \bigoplus_{i \in I} \phi_i.$$
into the sum of irreducible representations. Then we define the enhanced component group $A_{\phi}$ associated to $\phi$ by

$$A_{\phi} = \bigoplus_{i \in I} (\mathbb{Z}/2\mathbb{Z}) \alpha_i.$$  

Namely, $A_{\phi}$ is a free $\mathbb{Z}/2\mathbb{Z}$-module whose rank is equal to the length of $\phi_{\text{gp}}$. By abuse of notation, we put $z_\phi = \sum_{i \in I} \alpha_i \in A_{\phi}$ and call it the central element of $A_{\phi}$. There is a canonical surjection

$$A_{\phi} \to A_\phi, \; \alpha_i \mapsto \alpha_{\phi_i}.$$  

The kernel of this map is generated by $\alpha_i + \alpha_j$ for $\phi_i \cong \phi_j$. Moreover, this map preserves the central elements.

3.2. Local Langlands correspondence. We denote by $\text{Irr}_{\text{disc}}(G(F))$ (resp. $\text{Irr}_{\text{temp}}(G(F))$) the set of equivalence classes of irreducible discrete series (resp. tempered) representations of $G(F)$. The local Langlands correspondence established by Arthur is as follows:

**Theorem 3.1** ([Ar13, Theorem 2.2.1]). Let $G$ be a split $\text{SO}_{2n+1}$ or $\text{Sp}_{2n}$.

1. There exists a canonical surjection

$$\text{Irr}(G(F)) \to \Phi(G).$$

For $\phi \in \Phi(G)$, we denote by $\Pi_\phi$ the inverse image of $\phi$ under this map, and call $\Pi_\phi$ the $L$-packet associated to $\phi$.

2. There exists an injection

$$\Pi_\phi \to \widehat{A}_{\phi},$$

which satisfies certain endoscopic character identities. Here, $\widehat{A}_{\phi}$ is the Pontryagin dual of $A_{\phi}$. The image of this map is

$$\{ \eta \in \widehat{A}_{\phi} \mid \eta(z_\phi) = 1 \}.$$  

When $\pi \in \Pi_\phi$ corresponds to $\eta \in \widehat{A}_{\phi}$, we write $\pi = \pi(\phi, \eta)$.

3. For $* \in \{ \text{disc, temp} \}$,

$$\text{Irr}^*_*(G(F)) = \bigcup_{\phi \in \Phi^*_*(G)} \Pi_\phi.$$  

4. Assume that $\phi = \phi_\tau \oplus \phi_0 \oplus \phi_0^\vee \in \Phi_{\text{temp}}(G)$, where

- $\phi_0 \in \Phi_{\text{temp}}(G_0)$ with a classical group $G_0$ of the same type as $G$;
- $\phi_\tau$ is a tempered representation of $W_F \times \text{SL}_2(\mathbb{C})$ of dimension $k$.

Let $\tau$ be the irreducible tempered representation of $GL_k(F)$ corresponding to $\phi_\tau$. Then for $\pi_0 \in \Pi_{\phi_0}$, the induced representation $\tau \times \pi_0$ decomposes into a direct sum of irreducible tempered representations of $G(F)$. The $L$-packet $\Pi_\phi$ is given by

$$\Pi_\phi = \{ \pi \mid \pi \subset \tau \times \pi_0, \; \pi_0 \in \Pi_{\phi_0} \}.$$  

Moreover there is a canonical inclusion $A_{\phi_0} \hookrightarrow A_{\phi}$. If $\pi(\phi, \eta)$ is a direct summand of $\tau \times \pi_0$ with $\pi_0 = \pi(\phi_0, \eta_0)$, then $\eta_0 = \eta|_{A_{\phi_0}}$.

5. Assume that

$$\phi = \phi_1 \cdot |^{s_1} \oplus \cdots \oplus \phi_\tau \cdot |^{s_\tau} \oplus \phi_0 \oplus \phi_0^\vee \cdot |^{-s_\tau} \oplus \cdots \oplus \phi_0^\vee \cdot |^{-s_1},$$

where

- $\phi_0 \in \Phi_{\text{temp}}(G_0)$ with a classical group $G_0$ of the same type as $G$;
• \( \phi_i \) is a tempered representation of \( W_F \times \text{SL}_2(\mathbb{C}) \) of dimension \( k_i \) for \( 1 \leq i \leq r \);

• \( s_i \) is a real number such that \( s_1 \geq \cdots \geq s_r > 0 \).

Let \( \tau_i \) be the irreducible tempered representation of \( \text{GL}_{k_i}(F) \) corresponding to \( \phi_i \). Then the \( L \)-packet \( \Pi_{\phi} \) consists of the unique irreducible quotients \( \pi \) of the standard modules

\[
\tau_1 \cdot \vert \cdot \vert^{s_1} \times \cdots \times \tau_r \cdot \vert \cdot \vert^{s_r} \cdot \pi_0,
\]

where \( \pi_0 \) runs over \( \Pi_{\phi_0} \). Moreover there is a canonical inclusion \( A_{\phi_0} \hookrightarrow A_{\phi} \), which is in fact bijective. If \( \pi(\phi, \eta) \) is the unique irreducible quotient of the above standard module with \( \pi_0 = \pi(\phi_0, \eta_0) \), then \( \eta_0 = \eta|_{A_{\phi_0}} \). In this case, we denote this standard module by \( I(\phi, \eta) \).

The injection \( \Pi_{\phi} \hookrightarrow \hat{A}_{\phi} \) is not canonical when \( G = \text{Sp}_{2n} \). To specify this, we implicitly fix an \( F^{\times 2} \)-orbit of non-trivial additive characters of \( F \) through this paper.

We have the following irreducibility criterion for standard modules.

**Theorem 3.2** (Generalized standard module conjecture). For \( \phi \in \Phi(G) \), the standard module \( I(\phi, 1) \) attached to \( \pi(\phi, 1) \) is irreducible if and only if \( \phi \) is generic. Moreover, if \( \phi \) is generic, then all standard modules \( I(\phi, \eta) \), where \( \eta \in \hat{A}_{\phi} \) with \( \eta(z_\phi) = 1 \), are irreducible.

**Proof.** The first assertion is the usual standard module conjecture proven in [CS98, Mu01, HM07, HO13]. The second assertion was proven by Mœglin–Waldspurger [MW12, Corollaire 2.14] for special orthogonal groups and symplectic groups. Heiermann [H16] also proved the second assertion in a more general setting. Note that their definition of generic \( L \)-parameters might look different from ours. The equivalence of two definitions is called a conjecture of Gross–Prasad and Rallis, which was proven by Gan–Ichino [GI16, Proposition B.1]. \( \square \)

However, even if \( \phi \) is not generic, there might exist an irreducible standard module \( I(\phi, \eta) \). An example of such standard modules will be given by Corollary 5.6 and Example 5.7 below.

### 3.3. Extension to enhanced component groups

To describe Jacquet modules of \( \pi(\phi, \eta) \) for \( \phi \in \Phi_{\text{sp}}(G) \), it is useful to extend \( \pi(\phi, \eta) \) to the case where \( \eta \) is a character of the enhanced component group \( \mathcal{A}_\phi \) as follows. Recall that there exists a canonical surjection \( \mathcal{A}_\phi \twoheadrightarrow \hat{A}_\phi \) so that we have an injection \( \hat{A}_\phi \hookrightarrow \hat{\mathcal{A}_\phi} \). For \( \eta \in \hat{\mathcal{A}_\phi} \), set

\[
\pi(\phi, \eta) = \begin{cases} 
\pi(\phi, \eta) & \text{if } \eta \in \hat{A}_\phi, \ \eta(z_\phi) = 1, \\
0 & \text{otherwise}.
\end{cases}
\]

In particular, \( \pi(\phi, \eta) \) is irreducible or zero for any \( \eta \in \hat{\mathcal{A}_\phi} \).

### 3.4. Mœglin’s construction of tempered \( L \)-packets

The \( L \)-packets are used for local classification. On the other hand, Arthur [Ar13, Theorem 2.2.1] introduced the notion of \( A \)-packets for global classification. Mœglin constructed the local \( A \)-packets in her consecutive works (e.g., [M06, M09], etc.). For a detailed why Mœglin’s local \( A \)-packets agree with Arthur’s, one can see Xu’s paper [X17b] in addition to the original papers of Mœglin. Since the tempered \( A \)-packets are the same notion as the tempered \( L \)-packets, Mœglin’s construction can be applied to the tempered \( L \)-packets.
We explain Mœglin’s construction of $\Pi_{\phi}$ for $\phi \in \Phi_{gp}(G)$. Write

$$\phi = \left( \bigoplus_{i=1}^{t} \rho \boxtimes S_{a_i} \right) \oplus \phi_e$$

with $a_1 \leq \cdots \leq a_t$ and $\rho \boxtimes S_a \not\subset \phi_e$ for any $a > 0$. Take a new $L$-parameter

$$\phi_{\gg} = \left( \bigoplus_{i=1}^{t} \rho \boxtimes S_{a'_i} \right) \oplus \phi_e$$

for a bigger group $G'$ of the same type as $G$ such that

- $a'_1 < \cdots < a'_t$;
- $a'_i \geq a_i$ and $a'_i \equiv a_i \mod 2$ for any $i$;

Then we can identify $A_{\phi_{\gg}}$ with $A_\phi$ canonically, i.e., $A_{\phi_{\gg}} = A_\phi$ and if $A_{\phi_{\gg}} \ni \alpha \mapsto \alpha_{\phi_{\gg}} S_{a'_i} \in A_{\phi_{\gg}}$, then $A_\phi \ni \alpha \mapsto \alpha \boxtimes S_{a_i} \in A_\phi$. Let $\eta_{\gg} \in \widehat{A_{\phi_{\gg}}}$ be the character corresponding to $\eta \in \widehat{A_\phi}$, i.e., $\eta_{\gg} = \eta$ via $A_{\phi_{\gg}} = A_\phi$.

**Theorem 3.3** (Mœglin). With the notation above, we have

$$\pi(\phi, \eta) = \Jac_{\rho | \left| \frac{a'_1}{2}, \ldots, \frac{a'_t-1}{2}, \frac{a'_t+1}{2}, \ldots, \frac{a_t-1}{2}, \ldots, \frac{\rho}{2} \right|} \circ \cdots \circ \Jac_{\rho | \left| \frac{a_t}{2}, \ldots, \frac{\rho}{2} \right|} \left( \left( \pi(\phi_{\gg}, \eta_{\gg}) \right) \right).$$

Using this theorem repeatedly, we can construct $L$-packets $\Pi_{\phi}$ for $\phi \in \Phi_{gp}(G)$ from $L$-packets associated to discrete $L$-parameters for bigger groups.

**Example 3.4.** We construct $\Pi_{\phi}$ for $\phi = S_2 \oplus S_4 \oplus S_4 \oplus S_6 \oplus S_6 \in \Phi_{gp}(SO_{23})$. Note that $A_\phi = (\mathbb{Z}/2\mathbb{Z})\alpha_{S_2} \oplus (\mathbb{Z}/2\mathbb{Z})\alpha_{S_4} \oplus (\mathbb{Z}/2\mathbb{Z})\alpha_{S_6}$ with $z_{\phi} = \alpha_{S_2}$. Let $\eta \in \widehat{A_\phi}$. We write $\eta(\alpha_{S_2}) = \eta_{\phi} \in \{ \pm 1 \}$. If $\eta(z_{\phi}) = 1$, then $\eta_1 = +1$.

Now we take new $L$-parameters

$$\phi_{\gg} = S_2 \oplus S_4 \oplus S_6 \oplus S_8 \oplus S_{10} \in \Phi_{disc}(SO_{31}),$$
$$\phi' = S_2 \oplus S_4 \oplus S_4 \oplus S_8 \oplus S_{10} \in \Phi_{disc}(SO_{29}),$$
$$\phi'' = S_2 \oplus S_4 \oplus S_4 \oplus S_6 \oplus S_{10} \in \Phi_{disc}(SO_{27}),$$

and we consider $\eta_{\gg} \in \widehat{A_{\phi_{\gg}}}$, $\eta' \in \widehat{A_{\phi'}}$ and $\eta'' \in \widehat{A_{\phi''}}$ given by

- $\eta_{\gg}(\alpha_{S_2}) = \eta'(\alpha_{S_2}) = \eta''(\alpha_{S_2}) = \eta_1 = +1$;
- $\eta_{\gg}(\alpha_{S_4}) = \eta_{\gg}(\alpha_{S_6}) = \eta'(\alpha_{S_4}) = \eta'(\alpha_{S_6}) = \eta''(\alpha_{S_6}) = \eta''(\alpha_{S_6}) = \eta_3$.

Then Theorem 3.3 says that

$$\Jac_{\left| \frac{\rho}{2} \right|} \left( \left( \pi(\phi_{\gg}, \eta_{\gg}) \right) \right) = \pi(\phi', \eta'),$$
$$\Jac_{\left| \frac{\rho}{2} \right|} \left( \pi(\phi', \eta') \right) = \pi(\phi'', \eta''),$$
$$\Jac_{\left| \frac{\rho}{2} \right|} \left( \left( \pi(\phi'', \eta'') \right) \right) = \pi(\phi, \eta).$$

**4. Description of Jacquet modules**

In this section, we state the main theorems, which compute the semisimplifications of the Jacquet modules of $\pi(\phi, \eta)$ for $\phi \in \Phi_{gp}(G)$. 

4.1. Statements. Note that:

**Lemma 4.1.** For \( \phi \in \Phi_{\text{gp}}(G) \) and \( x \in \mathbb{R} \), if \( \text{Jac}_{\rho|\pi}^{x}(\pi) \neq 0 \) for some \( \pi \in \Pi_{\phi} \), then \( x \) is a non-negative half-integer and \( \rho \otimes S_{2x+1} \subset \phi \).

**Proof.** When \( \phi \in \Phi_{\text{disc}}(G) \), it follows from [14, Lemma 7.3]. We may assume that \( \phi \in \Phi_{\text{gp}}(G) \setminus \Phi_{\text{disc}}(G) \). Then there exists an irreducible representation \( \rho' \otimes S_{a} \) which \( \phi \) contains at least multiplicity two. By Theorem 3.1 (4), we have

\[
\text{Lemma 4.1.} \quad \text{For} \quad \phi \in \Phi_{\text{gp}}(G) \quad \text{and} \quad \pi \in \Pi_{\phi}, \quad \text{then} \quad \phi \text{ is irreducible or zero.}
\]

\[
\text{Proof.} \quad \text{When} \quad \phi \in \Phi_{\text{disc}}(G), \quad \text{it follows from [14, Lemma 7.3]. We may assume that} \quad \phi \in \Phi_{\text{gp}}(G) \setminus \Phi_{\text{disc}}(G). \quad \text{Then there exists an irreducible representation} \quad \rho' \otimes S_{a} \quad \text{which} \quad \phi \quad \text{contains at least multiplicity two. By Theorem 3.1 (4), we have}
\]

\[
\phi \in \Phi_{\text{gp}}(G) \quad \text{and} \quad \rho \otimes S_{2x+1} \subset \phi.
\]

Hence if \( \text{Jac}_{\rho|\pi}^{x}(\pi) \neq 0 \), then \( \text{Jac}_{\rho|\pi}^{x}(\pi) \neq 0 \) or \( \rho \otimes S_{2x+1} \cong \rho' \otimes S_{a} \subset \phi \). By induction, we conclude that \( \rho \otimes S_{2x+1} \subset \phi \) as desired. \( \square \)

The following is the first main theorem, which is a description of \( \text{Jac}_{\rho|\pi}^{x}(\pi(\phi, \eta)) \).

**Theorem 4.2.** Let \( \phi \in \Phi_{\text{gp}}(G) \) and \( \eta \in \widehat{A}_{\phi} \) such that \( \pi(\phi, \eta) \neq 0 \). Fix a non-negative half-integer \( x \in (1/2)\mathbb{Z} \). Write

\[
\phi = \phi_{0} \oplus (\rho \otimes S_{2x+1})^{\oplus m}
\]

with \( \rho \otimes S_{2x+1} \nsubseteq \phi_{0} \) and \( m > 0 \).

1. Assume that \( m \geq 3 \). Take \( \delta \in \{1, 2\} \) such that \( \delta \equiv m \mod 2 \). Then

\[
\text{Jac}_{\rho|\pi}^{x}(\pi(\phi, \eta)) = (m - \delta) \cdot \left( \rho; x, x - 1, \ldots, -(x - 1) \right) \times \pi(\phi - (\rho \otimes S_{2x+1})^{\oplus 2}, \eta) + \text{St}(\rho, 2x + 1) \times \cdots \times \text{St}(\rho, 2x + 1) \times \text{Jac}_{\rho|\pi}^{x}(\pi(\phi_{0} \oplus (\rho \otimes S_{2x+1})^{\oplus \delta}, \eta)).
\]

Here, we canonically identify the (usual) component groups of \( \phi - (\rho \otimes S_{2x+1})^{\oplus 2} \) and \( \phi_{0} \oplus (\rho \otimes S_{2x+1})^{\oplus \delta} \) with \( A_{\phi} \), so that we regard \( \eta \) as a character of these groups.

2. Assume that \( x > 0 \) and \( m = 1 \). Set

\[
\phi' = \phi - (\rho \otimes S_{2x+1}) \oplus (\rho \otimes S_{2x-1}).
\]

There is a canonical inclusion \( A_{\phi'} \hookrightarrow A_{\phi} \), which is in fact bijective if \( x > 1/2 \). Let \( \eta' \in \widehat{A}_{\phi'} \) be the character corresponding to \( \eta \in \widehat{A}_{\phi} \), i.e., \( \eta' = \eta|A_{\phi'} \). Then

\[
\text{Jac}_{\rho|\pi}^{x}(\pi(\phi, \eta)) = \pi(\phi', \eta').
\]

In particular, \( \text{Jac}_{\rho|\pi}^{x}(\pi(\phi, \eta)) \) is irreducible or zero.
(3) Assume that $x > 0$ and $m = 2$. Set $\eta_+ = \eta$, and take the unique character $\eta_- \in \hat{A}_\phi$ so that $\eta_-|A_{\phi_0} = \eta_+|A_{\phi_0}$, $\eta_-(z_\phi) = \eta_+(z_\phi) = 1$ but $\eta_- \neq \eta_+$. For $\phi'$ as in (3) and for $e \in \{\pm 1\}$, let $\eta'_e \in \hat{A}_{\phi'}$ be the character corresponding to $\eta_e \in \hat{A}_\phi$ via the canonical inclusion $A_{\phi'} \hookrightarrow A_\phi$. Then

$$\text{Jac}_{\rho|_{x}}(\pi(\phi, \eta)) = (\rho; x, x - 1, \ldots, -(x - 1)) \times \pi(\phi_0, \eta|A_{\phi_0}) + \pi(\phi', \eta'_+) - \pi(\phi', \eta'_-) .$$

(4) Assume that $x = 0$. If $m = 1$, then $\text{Jac}_\rho(\pi(\phi, \eta)) = 0$. If $m = 2$, then $\text{Jac}_\rho(\pi(\phi, \eta)) = \pi(\phi_0, \eta|A_{\phi_0})$.

When $\phi \in \Phi_{\text{disc}}(G)$, Theorem 4.2 has been already proven by Xu (X17a, Lemma 7.3). In (2) (resp. (3)), we note that $\pi(\phi', \eta')$ (resp. $\pi(\phi', \eta'_e)$) can be zero even if $\pi(\phi, \eta) \neq 0$. In (3), the character $\eta_-$ is characterized so that

$$\pi(\phi, \eta_+) + \pi(\phi, \eta_-) = \text{St}(\rho, 2x + 1) \times \pi(\phi_0, \eta|A_{\phi_0}) .$$

The second main theorem concerns $\mu_\rho^*(\pi)$.

**Theorem 4.3.** Let $\phi \in \Phi_{\text{sp}}(G)$, and write

$$\phi = \left( \bigoplus_{i=1}^{t} \rho \boxtimes S_{a_i} \right) \oplus \phi_e$$

with $a_1 \leq \cdots \leq a_t$ and $\rho \boxtimes S_a \not\cong \phi_e$ for any $a > 0$, and $a_i = 2x_i + 1$. For $0 \leq m \leq (2d)^{-1} \cdot \dim(\phi)$, we denote by $K_\phi^{(m)}$ the set of tuples of integers $k = (k_1, \ldots, k_t)$ such that

- $0 \leq k_i \leq a_i$ for any $i$;
- $k_{i-1} \geq k_i$ if $a_{i-1} = a_i$;
- $k_1 + \cdots + k_t = m$.

For $k \in K_\phi^{(m)}$, set

$$x(k) = (x_1, \ldots, x_1 - k_1 + 1, \ldots, x_t, \ldots, x_t - k_t + 1) \in \Omega_m .$$

For $k, l \in K_\phi$, we set

$$m_{k,l} = \dim \text{Jac}_{\rho|_{x(k)}}(\Delta_{x(k)}) ,$$

and define $(m_{k,l})_{k,l \in K_\phi^{(m)}}$ to be the inverse matrix of $(m_{k,l})_{k,l \in K_\phi^{(m)}}$, i.e.,

$$\sum_{k' \in K_\phi^{(m)}} m_{k',k} \cdot m_{k',l} = \begin{cases} 1 & \text{if } k = l, \\
0 & \text{if } k \neq l . \end{cases}$$

Then for $\pi \in \Pi_\phi$, we have

$$\mu_\rho^*(\pi) = \left( (2d)^{-1} \dim(\phi) \right) \sum_{k,l \in K_\phi^{(m)}} m_{k,l}^t \cdot \Delta_{x(k)} \otimes \text{Jac}_{\rho|_{x(k)}}(\pi) .$$

When we formally regard $(\Delta_{x(k)})_{k \in K_\phi^{(m)}}$ and $(\otimes \text{Jac}_{\rho|_{x(k)}}(\pi))_{k \in K_\phi^{(m)}}$ as column vectors, we have

$$\sum_{k,l \in K_\phi^{(m)}} m_{k,l}^t \cdot \Delta_{x(k)} \otimes \text{Jac}_{\rho|_{x(k)}}(\pi) = t(\Delta_{x(k)}) \cdot (m_{k,l})^{-1} \cdot (\otimes \text{Jac}_{\rho|_{x(k)}}(\pi)) .$$
By Lemma 2.4, \( (m_{k,l})_{k,l \in K^{(m)}_\phi} \) is a “triangular matrix”, which can be computed inductively. Here, we regard \( K^{(m)}_\phi \) as a totally ordered set with respect to the lexicographical order. The diagonal entries \( m_{k,k} \) are given in Lemma 2.4 explicitly.

By Tadić’s formula (Theorem 2.7, Lemma 2.6 and Theorems 4.2, 4.3, we can deduce the following corollary.

**Corollary 4.4.** We can compute \( \mu^*(\pi) \) explicitly for any \( \pi \in \Pi_\phi \) with \( \phi \in \Phi_{gou}(G) \).

### 4.2. Examples

We shall give some examples.

**Example 4.5.** Fix two positive integers \( a, b \) such that \( a \equiv b \mod 2 \) and \( a < b \), and consider

\[ \phi = \rho \otimes (S_a \oplus S_b) \in \Phi_{\text{disc}}(\text{SO}_{d(a+b)+1}). \]

Then \( \Pi_\phi = \{ \pi_+(a,b), \pi_-(a,b) \} \) with generic \( \pi_+(a,b) \) and non-generic \( \pi_-(a,b) \). Note that both \( \pi_+(a,b) \) and \( \pi_-(a,b) \) are discrete series. We compute \( \mu^*(\pi_\epsilon(a,b)) \) for \( \epsilon \in \{ \pm 1 \} \).

Note that \( K^{(m)}_\phi = \{(k_1,k_2) \in \mathbb{Z}^2 \mid 0 \leq k_1 \leq a, 0 \leq k_2 \leq b, k_1 + k_2 = m \} \). For \( (k_1,k_2) \in K^{(m)}_\phi \),

\[ \Delta_{\pi(k_1,k_2)} = \left\langle \rho; \frac{a-1}{2}, \ldots, \frac{a+1}{2} - k_1 \right\rangle \times \left\langle \rho; \frac{b-1}{2}, \ldots, \frac{b+1}{2} - k_2 \right\rangle. \]

This induced representation is irreducible unless \( (a + 3)/2 - k_1 \leq (b + 1)/2 - k_2 \leq (a + 1)/2 \), i.e., \( (b - a)/2 \leq k_2 \leq (b - a)/2 + k_1 - 1 \). Moreover, one can easy to see that for \( (l_1,l_2) \), the virtual representation

\[ \sum_{(k_1,k_2) \in K^{(m)}_\phi} m_{(k_1,k_2),(l_1,l_2)} \cdot \Delta_{\pi(k_1,k_2)} \]

is the unique irreducible subrepresentation \( \tau_{\pi(l_1,l_2)} \) of \( \Delta_{\pi(l_1,l_2)} \) (cf. see [ZS0] Proposition 4.6). Note that

\[ \text{Jac}_{\rho|_{\mathbb{Z}^2_{(k_1,k_2)}}} (\pi_\epsilon(a,b)) = \begin{cases} \pi_\epsilon(a - 2k_1, b) & \text{if } k_1 \leq a/2, \\ 0 & \text{otherwise.} \end{cases} \]

Here, when \( k_1 = a/2 \), we understand that \( \pi_+(0,b) \) is the unique element in \( \Pi_{\rho \otimes S_b} \), and \( \pi_-(0,b) = 0 \). Moreover, for \( (k_1,k_2) \in K^{(m)}_\phi \), when \( k_1 \leq a/2 \) and \( k_2 \leq (b - a)/2 + k_1 \), we have

\[ \text{Jac}_{\rho|_{\mathbb{Z}^2_{(k_1,k_2)}}} (\pi_\epsilon(a,b)) = \pi_\epsilon(a - 2k_1, b - 2k_2). \]

When \( k_1 \leq a/2 \) and \( (b - a)/2 + k_1 + 1 \leq k_2 \leq b/2 \), we have

\[ \text{Jac}_{\rho|_{\mathbb{Z}^2_{(k_1,k_2)}}} (\pi_\epsilon(a,b)) = \left\langle \rho; \frac{a-1}{2} - k_1, \ldots, \frac{b-1}{2} + k_2 \right\rangle \times 1_{\text{SO}_1(F)}. \]

When \( k_1 \leq a/2 \) and \( b/2 \leq k_2 \leq (a + b)/2 - k_1 \), we have

\[ \text{Jac}_{\rho|_{\mathbb{Z}^2_{(k_1,k_2)}}} (\pi_\epsilon(a,b)) = \left\langle \rho; \frac{a-1}{2} - k_1, \ldots, \frac{b-1}{2} + k_2 \right\rangle \times 1_{\text{SO}_1(F)}. \]
In particular, if $k_1+k_2 = (a+b)/2$, then $\text{Jac}_{\rho_{1[\Delta(k_1,k_2)}}(\pi_\epsilon(a,b)) = 1_{SO_4(F)}$. Hence s.s.~$\text{Jac}_{P(d(a+b)/2)}(\pi_\epsilon(a,b))$ contains the irreducible representation
\[
\left\langle \rho; \frac{a-1}{2}, \ldots, \frac{a+1}{2} - k_1 \right\rangle \times \left\langle \rho; \frac{b-1}{2}, \ldots, \left( \frac{a-1}{2} - k_1 \right) \right\rangle \otimes 1_{SO_4(F)}
\]
with multiplicity one if $k_1 < a/2$, or if $k_1 = a/2$ and $\epsilon = +1$.

**Example 4.6.** Consider the $L$-parameter $\phi = S_2 \oplus S_4 \oplus S_4 \in \Phi_{\text{gp}}(SO_{1})$. Then $\Pi_\phi$ has two elements $\pi_+(2,4,4)$ and $\pi_-(2,4,4)$ with generic $\pi_+(2,4,4)$ and non-generic $\pi_-(2,4,4)$. Then
\[
K^{(m)} = \{(k_1,k_2,k_3) \in \mathbb{Z}^3 \mid 0 \leq k_1 \leq 2, 0 \leq k_3 \leq k_2 \leq 4, k_1 + k_2 + k_3 = m\}
\]
for $0 \leq m \leq 5$. Write $\Pi_{\Delta(k)} = \text{Jac}_{\rho_{1[\Delta(k)}}(\pi_\epsilon(2,4,4))$ for $\epsilon \in \{\pm 1\}$, and $\text{St}_a = \text{St}(1_{GL_4(F)},a)$. We denote by $\text{det}_a$ the determinant character of $GL_4(F)$.
1. When $m = 1$, we have $K^{(1)}_\phi = \{(1,0,0) > (0,1,0)\}$. Since
\[
(\Delta_{\Delta(1,0,0)} \quad \Delta_{\Delta(0,1,0)}) = \left(\begin{array}{cc} | \cdot |^{\frac{1}{2}} & | \cdot |^{\frac{1}{2}} \\ | \cdot |^{\frac{1}{2}} & | \cdot |^{\frac{1}{2}} \end{array} \right),
\]
we have
\[
(m_{k,k'})^{-1} = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)
\]
Moreover
\[
\left(\begin{array}{c} \Pi_{\Delta(1,0,0)} \\ \Pi_{\Delta(0,1,0)} \end{array} \right) = \left(\begin{array}{c} \pi_+(4,4) \\ | \cdot |^{\frac{1}{2}} \text{St}_3 \times \pi_+(2) + \epsilon \cdot \pi_+(2,2,4) \end{array} \right).
\]
Hence
\[
\text{s.s.~} \text{Jac}_{P_1}(\pi_\epsilon(2,4,4)) = \left| \cdot \right|^{\frac{1}{2}} \otimes \pi_+(4,4)
\]
\[
+ \left| \cdot \right|^{\frac{1}{2}} \otimes \left(\left| \cdot \right|^{\frac{1}{2}} \text{St}_3 \times \pi_+(2) + \epsilon \cdot \pi_+(2,2,4) \right).
\]
2. When $m = 2$, we have $K^{(2)}_\phi = \{(2,0,0) > (1,1,0) > (0,2,0) > (0,1,1)\}$. Since
\[
(\Delta_{\Delta(2,0,0)} \quad \Delta_{\Delta(1,1,0)} \quad \Delta_{\Delta(0,2,0)} \quad \Delta_{\Delta(0,1,1)})
\]
\[
= \left(\begin{array}{cccc} \text{St}_2 & | \cdot |^{\frac{1}{2}} \times | \cdot |^{\frac{1}{2}} & | \cdot |^{\frac{1}{2}} \times | \cdot |^{\frac{1}{2}} & | \cdot |^{\frac{1}{2}} \times | \cdot |^{\frac{1}{2}} \end{array} \right),
\]
we have
\[
(m_{k,k'})^{-1} = \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)
\]
Moreover
\[
\left(\begin{array}{c} \Pi_{\Delta(2,0,0)} \\ \Pi_{\Delta(1,1,0)} \end{array} \right) = \left(\begin{array}{c} 0 \\ | \cdot |^{\frac{1}{2}} \text{St}_3 \times 1_{SO_4(F)} + \pi_+(2,4) - \pi_-(2,4) \\ | \cdot |^{\frac{1}{2}} \text{St}_2 \times \pi_+(2) + | \cdot |^{\frac{1}{2}} \text{St}_3 \times 1_{SO_4(F)} + \epsilon \left(\left| \cdot \right|^{\frac{1}{2}} \times \pi_+(4) + \pi_+(2,2,2) \right) \\
(1 + \epsilon) \cdot \pi_+(2,2,2) \end{array} \right).
\]
Hence

\[
\text{s.s.}\text{Jac}_{P_2}(\pi_\epsilon(2, 4, 4)) = |\det_2|^{\frac{1}{2}} \otimes \left( | \cdot |^{\frac{1}{2}}St_3 \times \mathbb{1}_{\text{SO}_1(F)} + \pi_\epsilon(2, 4) - \pi_- (2, 4) \right) \\
+ | \cdot |^{\frac{1}{2}}St_2 \otimes \left( | \cdot |^{\frac{1}{2}}St_2 \times \pi_\epsilon(2) \right) + \left( | \cdot |^{\frac{1}{2}} \times \pi_\epsilon(4) + \pi_\epsilon(2, 4) \right) + \epsilon \left( | \cdot |^{\frac{1}{2}} \times \pi_\epsilon(4) + \pi_\epsilon(2, 4) \right) \\
+ \left( | \cdot |^{\frac{1}{2}} \times | \cdot |^{\frac{1}{2}} \right) \otimes \frac{1 + \epsilon}{2} \cdot \pi_\epsilon(2, 2, 2).
\]

(3) When \( m = 3 \), we have \( K^{(3)}_\rho = \{ (2, 1, 0) > (1, 2, 0) > (1, 1, 1) > (0, 3, 0) > (0, 2, 1) \} \).

Since

\[
\left( \Delta_{\mathbb{R}}(2, 1, 0) \quad \Delta_{\mathbb{R}}(1, 2, 0) \quad \Delta_{\mathbb{R}}(1, 1, 1) \quad \Delta_{\mathbb{R}}(0, 3, 0) \quad \Delta_{\mathbb{R}}(0, 2, 1) \right) \\
= \left( \text{St}_2 \times | \cdot |^{\frac{3}{2}} \mid | \cdot |^{\frac{1}{2}} \times \mid | \cdot |^{\frac{1}{2}} \text{St}_2 \mid | \cdot |^{\frac{1}{2}} \times | \cdot |^{\frac{3}{2}} \times | \cdot |^{\frac{3}{2}} \mid | \cdot |^{\frac{3}{2}} \text{St}_3 \mid | \cdot |^{\frac{3}{2}} \text{St}_2 \times | \cdot |^{\frac{3}{2}} \right),
\]

we have

\[
(m_{k, k'})^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
0 & 0 & 2 & 0 & 1
\end{pmatrix}^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 1
\end{pmatrix}.
\]

Moreover

\[
\begin{pmatrix}
\Pi'_{\mathbb{R}}(2, 1, 0) \\
\Pi'_{\mathbb{R}}(1, 2, 0) \\
\Pi'_{\mathbb{R}}(1, 1, 1) \\
\Pi'_{\mathbb{R}}(0, 3, 0) \\
\Pi'_{\mathbb{R}}(0, 2, 1)
\end{pmatrix} = \begin{pmatrix}
0 \\
| \cdot |^{\frac{1}{2}} \text{St}_2 \times \mathbb{1}_{\text{SO}_1(F)} + \epsilon \cdot \pi_\epsilon(4) \\
2 \cdot \pi_\epsilon(2, 2) \\
| \cdot |^{\frac{3}{2}} \times \pi_\epsilon(2) + \epsilon \cdot \pi_\epsilon(4) \\
0 \\
(1 + \epsilon) \cdot \pi_\epsilon(2, 2, 2) + \pi_- (2, 2)
\end{pmatrix}.
\]

Hence

\[
\text{s.s.}\text{Jac}_{P_3}(\pi_\epsilon(2, 4, 4)) = \left( | \cdot |^{\frac{1}{2}} \times | \cdot |^{\frac{1}{2}} \text{St}_2 \right) \otimes \left( | \cdot |^{\frac{1}{2}} \text{St}_2 \times \mathbb{1}_{\text{SO}_1(F)} + \epsilon \cdot \pi_\epsilon(4) \right) \\
+ \left( | \cdot |^{\frac{1}{2}} \times | \cdot |^{\frac{3}{2}} \right) \otimes \pi_\epsilon(2, 2) \\
+ \left( | \cdot |^{\frac{3}{2}} \text{St}_3 \right) \otimes \left( | \cdot |^{\frac{3}{2}} \times \pi_\epsilon(2) + \epsilon \cdot \pi_\epsilon(4) \right) \\
+ \left( | \cdot |^{\frac{3}{2}} \right) \otimes \left( (1 + \epsilon) \cdot | \cdot |^{\frac{3}{2}} \times \pi_\epsilon(2) + \pi_\epsilon(2, 2) + \epsilon \cdot \pi_- (2, 2) \right).
\]
Moreover

\[
\begin{pmatrix}
\Pi_{z(2,0)}^+ \\
\Pi_{z(2,1)}^+ \\
\Pi_{z(1,3,0)}^+ \\
\Pi_{z(1,2,1)}^+ \\
\Pi_{z(0,4,0)}^+ \\
\Pi_{z(0,3,1)}^+ \\
\Pi_{z(0,2,2)}^+
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
| \cdot | \frac{\pi}{2} \times 1_{SO_1(F)} \\
| \cdot | \frac{\pi}{2} \times 1_{SO_1(F)} + \epsilon \cdot \pi_+(2) \\
\pi_+(2) \\
(3 + 2\epsilon)| \cdot | \frac{\pi}{2} \times 1_{SO_1(F)} + (1 + 2\epsilon) \cdot \pi_+(2)
\end{pmatrix}.
\]

Hence

\[
s.s.Jac_4(\pi_+(2, 4, 4)) = 
\left( | \cdot | \frac{\pi}{2} \times | \cdot | \frac{\pi}{2} \right) \otimes | \cdot | \frac{\pi}{2} \times 1_{SO_1(F)}
+ (| \cdot | 1_{St_2} \times |det_2|) \otimes \left( | \cdot | \frac{\pi}{2} \times 1_{SO_1(F)} + \epsilon \cdot \pi_+(2) \right)
+ St_4 \otimes \pi_+(2)
+ \left( | \cdot | \frac{\pi}{2} \times | \cdot | \frac{\pi}{2} \right) \otimes (1 + \epsilon) \cdot \pi_+(2)
+ \left( | \cdot | 1_{St_2} \times | \cdot | 1_{St_2} \right) \otimes \left( (1 + \epsilon)| \cdot | \frac{\pi}{2} \times 1_{SO_1(F)} + \frac{1 + \epsilon}{2} \cdot \pi_+(2) \right).
\]
Remark 4.7. In Theorem 4.3, one can replace \( \Delta_{\mathbf{k}} \) with its unique irreducible subrepresentation \( \tau_{\mathbf{k}} \). Then one should consider the matrix \( (M_{\mathbf{k}}) = (\dim_{\mathbb{C}} \text{Jac}_{\mathbf{r}}(\Delta_{\mathbf{k}})) \). One might seem that \( (M_{\mathbf{k}}) \) easier than \( (m_{\mathbf{k}}) \). For instance, if \( \phi \) is in Example 4.5, all \( (M_{\mathbf{k}}) \) are the identity matrix, but not so is some \( (m_{\mathbf{k}}) \). However, in general, \( (M_{\mathbf{k}}) \) is not always diagonal. In Example 4.6, such a non-diagonal \( (M_{\mathbf{k}}) \) appears.

5. Proof of main theorems

In this section, we prove main theorems (Theorems 4.2 and 4.3).
5.1. The case of higher multiplicity. We prove Theorem 4.2 (1) in this subsection. It immediately follows from the case of $\phi \in \Phi_{\text{disc}}(G)$ together with Tadić’s formula (Corollary 2.8).

**Proof of Theorem 4.2 (1).** We prove the assertion by induction on $m$. Since

$$\pi(\phi, \eta) = \text{St}(\rho, 2x + 1) \times \pi(\phi - (\rho \boxtimes S_{2x+1})^\oplus 2, \eta),$$

by Corollary 2.8, we have

$$\text{Jac}_{\rho|\pi} (\pi(\phi, \eta)) = 2 \cdot (\rho; x, \ldots, -(x - 1)) \times \pi (\phi - (\rho \boxtimes S_{2x+1})^\oplus 2, \eta) + \text{St}(\rho, 2x + 1) \times \text{Jac}_{\rho|\pi} (\pi (\phi - (\rho \boxtimes S_{2x+1})^\oplus 2, \eta)).$$

This proves the assertion when $m = 3$ or $m = 4$. When $m \geq 5$, since

$$\text{St}(\rho, 2x + 1) \times (\rho; x, \ldots, -(x - 1)) \times \pi (\phi - (\rho \boxtimes S_{2x+1})^\oplus 2, \eta)$$

$$\cong (\rho; x, \ldots, -(x - 1)) \times \text{St}(\rho, 2x + 1) \times \pi (\phi_0 \oplus (\phi - (\rho \boxtimes S_{2x+1})^\oplus 4), \eta)$$

$$\cong (\rho; x, \ldots, -(x - 1)) \times \pi (\phi - (\rho \boxtimes S_{2x+1})^\oplus 2, \eta),$$

we obtain the assertion by the induction hypothesis. \hfill \Box

5.2. The case of multiplicity one. Next, we prove Theorem 4.2 (2). Let $\phi = \phi_0 \oplus (\rho \boxtimes S_{2x+1})$ with $\rho \boxtimes S_{2x+1} \not\subset \phi_0$, and $\eta \in \hat{A}_0$. Set

$$\phi' = \phi - (\rho \boxtimes S_{2x+1}) \oplus (\rho \boxtimes S_{2x-1}).$$

**Proof of Theorem 4.2 (2).** First, we assume that $x > 0$ and $\pi(\phi', \eta') \neq 0$. We apply Mœglin’s construction to $\Pi_{\phi'}$. Write

$$\phi' = \left( \bigoplus_{i=1}^{t} \rho \boxtimes S_{a_i} \right) \oplus \phi'_e$$

with $a_1 \leq \cdots \leq a_t$ and $\rho \boxtimes S_{a} \not\subset \phi'_e$ for any $a > 0$. Set

$$t_0 = \max\{ i \in \{ 1, \ldots, t \} \mid t_i = 2x - 1 \}.$$

Take a new $L$-parameter

$$\phi''_e = \left( \bigoplus_{i=1}^{t} \rho \boxtimes S_{a'_i} \right) \oplus \phi'_e$$

such that

- $a'_1 < \cdots < a'_t$;
- $a'_i \geq a_i$ and $a'_i \equiv a_i \mod 2$ for any $i$;
- $a'_{t_0} \geq 2x + 1$.

We can identify $\mathcal{A}_{\phi''_e}$ with $\mathcal{A}_{\phi'}$ canonically. Let $\eta''_e \in \hat{A}_{\phi''_e}$ be the character corresponding to $\eta' \in \hat{A}_{\phi'}$. Then Theorem 3.3 says that

$$\pi(\phi', \eta'') = \text{Jac}_{\rho|\pi} \left( \bigoplus_{i=1}^{a_{t_0}+1} (\pi (\phi''_e, \eta''_e)) \right).$$

When $i = t_0$, we note that

$$\text{Jac}_{\rho|\pi} \left( \bigoplus_{i=1}^{a'_{t_0}+1} \right) = \text{Jac}_{\rho|\pi} \circ \text{Jac}_{\rho|\pi} \left( \bigoplus_{i=1}^{a'_{t_0}+1} \right).$$
Since $\phi'$ does not contain $\rho \boxtimes S_{2x+1}$ for $i > k_0$ and $a_i < 2x' + 1 \leq a_i'$ with $2x' + 1 \equiv a_i \mod 2$, we have $x' - x > 1$. By Lemma 2.5 (2), we see that $\pi(\phi', \eta')$ is the image of

$$\text{Jac}_{\rho|\frac{\pi_1}{\tau}, \ldots, \rho|\frac{\pi_n}{\tau}} \circ \cdots \circ \text{Jac}_{\rho|\frac{\pi_1}{\tau}, \ldots, \rho|\frac{\pi_n}{\tau}} \circ \cdots \circ \text{Jac}_{\rho|\frac{\pi_1}{\tau}, \ldots, \rho|\frac{\pi_n}{\tau}}(\pi(\phi', \eta'))$$

under $\text{Jac}_{\rho|\pi}$. However, by applying Theorem 3.3 again, we see that this representation is isomorphic to $\pi(\phi, \eta)$. Therefore $\pi(\phi', \eta') = \text{Jac}_{\rho|\pi}(\pi(\phi, \eta))$, as desired.

Next, we assume that $\pi(\phi', \eta') = 0$. We claim that $\pi(\phi', \eta) = 0$. When $\phi \in \Phi_{\text{disc}}(G)$, this was proven in [X17a, Lemma 7.3]. When $\phi \in \Phi_{\text{gp}}(G) \setminus \Phi_{\text{disc}}(G)$, there exists an irreducible representation $\phi_1$ which is contained in $\phi$ with multiplicity at least two. Then $\pi(\phi, \eta)$ is a subrepresentation of $\tau_1 \times \pi(\phi - \phi_1^{\otimes 2}, \eta)$, where $\tau_1$ is the irreducible tempered representation of $GL_k(F)$ corresponding to $\phi_1$. Since $\rho \boxtimes S_{2x+1}$ is contained in $\phi$ with multiplicity one, we have $\phi_1 \not\sim \rho \boxtimes S_{2x+1}$. This implies that

$$\text{Jac}_{\rho|\pi}(\tau_1 \times \pi(\phi - \phi_1^{\otimes 2}, \eta)) = \tau_1 \times \text{Jac}_{\rho|\pi}(\pi(\phi - \phi_1^{\otimes 2}, \eta)).$$

By the induction hypothesis, $\text{Jac}_{\rho|\pi}(\pi(\phi - \phi_1^{\otimes 2}, \eta)) = 0$ unless $\phi_1 = \rho \boxtimes S_{2x-1}$ and $\phi$ contains it with multiplicity exactly two. In this case, one can take $\eta_- \in \widehat{A}_\phi$ such that $\pi(\phi, \eta_-) \neq 0$ and

$$\pi(\phi, \eta) \oplus \pi(\phi, \eta_-) = \text{St}((\rho, 2x - 1) \times \pi(\phi - \phi_1^{\otimes 2}, \eta)).$$

Then by the first case, we see that $\text{Jac}_{\rho|\pi}(\pi(\phi, \eta_-)) \neq 0$ and $\text{St}((\rho, 2x - 1) \times \text{Jac}_{\rho|\pi}(\pi(\phi - \phi_1^{\otimes 2}, \eta)))$ is irreducible. Hence $\text{Jac}_{\rho|\pi}(\pi(\phi, \eta))$ must be zero. This completes the proof of Theorem 4.2 (2).

By the same argument as the last part, one can prove that $\text{Jac}_{\rho}(\pi(\phi, \eta)) = 0$ when $x = 0$ and $m = 1$.

5.3. Description of small standard modules. Before proving Theorem 4.2 (3), we describe the structures of some standard modules.

Lemma 5.1. Let $\phi \in \Phi_{\text{disc}}(G)$. Suppose that $x > 0$, and $\phi \supset \rho \boxtimes S_{2x-1}$ but $\phi \not\equiv \rho \boxtimes S_{2x+1}$. Let $\eta \in \widehat{A}_\phi$ such that $\pi(\phi, \eta) \neq 0$. We set

- $\Pi = \rho|\cdot^x \rtimes \pi(\phi, \eta)$ to be a standard module;
- $\sigma$ to be the unique irreducible quotient of $\Pi$;
- $\phi' = \phi - (\rho \boxtimes S_{2x-1}) \oplus (\rho \boxtimes S_{2x+1})$;
- $\eta' \in \widehat{A}_{\phi'}$ to be the character corresponding to $\eta \in \widehat{A}_\phi$ via the canonical identification $\widehat{A}_{\phi'} = \widehat{A}_\phi$.

Then there exists an exact sequence

$$0 \longrightarrow \pi(\phi', \eta') \longrightarrow \Pi \longrightarrow \sigma \longrightarrow 0.$$

In particular, $\Pi$ has length two.

Proof. We note that $\pi(\phi', \eta')$ is an irreducible subrepresentation of $\Pi$ by Theorem 4.2 (2) and Lemma 2.5 (1).

If $\phi'$ is an irreducible subquotient of $\Pi$ which is non-tempered, by Tadić’s formula and Casselman’s criterion, there exists a maximal parabolic subgroup $P_k$ of $G(F)$ such that
s.s.$\text{Jac}_{P_k}(\sigma')$ contains an irreducible representation of the form $(\rho|\cdot|^{-x} \times \tau) \boxtimes \sigma_0$. In particular, we have $\text{Jac}_{\rho|\cdot|^{-x}}(\sigma') \neq 0$. However, since $\text{Jac}_{\rho|\cdot|^{-x}}(\Pi) = \pi(\phi, \eta)$ is irreducible, we see that $\sigma' = \sigma$, i.e., $\Pi$ has only one irreducible non-tempered subquotient.

Let $\Pi^{\text{sub}}$ be the maximal proper subrepresentation of $\Pi$, i.e., $\Pi/\Pi^{\text{sub}} \cong \sigma$. By the above argument, all irreducible subquotients of $\Pi^{\text{sub}}$ must be tempered. Since they have the same cuspidal support, they share the same Plancherel measure. This implies that all irreducible subquotients of $\Pi^{\text{sub}}$ belong to the same $L$-packet $\Pi_{\sigma'}$ (see [GI16] Lemma A.6), so that they are all discrete series. Hence $\Pi^{\text{sub}}$ is semisimple. In particular, any irreducible subquotient $\pi'$ of $\Pi^{\text{sub}}$ is a subrepresentation of $\Pi$, so that $\text{Jac}_{\rho|\cdot|^{-x}}(\pi') \neq 0$. However, since $\text{Jac}_{\rho|\cdot|^{-x}}(\Pi) = \pi(\phi, \eta)$ is irreducible, $\Pi$ has only one irreducible subrepresentation. Therefore $\Pi^{\text{sub}} = \pi(\phi', \eta')$.

This completes the proof. \qed

We describe the standard module appearing in Theorem 1.2 (3). When $x = 1/2$, the standard module was described in Lemma 5.1. Hence we assume $x > 1/2$.

**Proposition 5.2.** Let $\phi \in \Phi_{\text{gp}}(G)$. Suppose that $x > 1/2$ and $\phi \not\supset \rho \boxtimes S_{2x+1}$. Let $\eta \in \hat{A}_\phi$ such that $\pi(\phi, \eta) \neq 0$. We set

- $\Pi = (\rho; x, x-1, \ldots, -(x-1)) \times \pi(\phi, \eta)$ to be a standard module;
- $\sigma$ to be the unique irreducible quotient of $\Pi$;
- $\phi' = \phi \oplus (\rho \boxtimes S_{2x-1}) \oplus (\rho \boxtimes S_{2x+1})$;
- $\eta_+^\prime$ and $\eta_-^\prime$ to be the two distinct characters of $\hat{A}_{\phi'}$ such that $\eta^\prime_{\pm}|A_\phi = \eta$ and $\eta^\prime_{\pm}(z_{\phi'}) = 1$.

Then there exists an exact sequence

$0 \longrightarrow \pi(\phi', \eta_+^\prime) \oplus \pi(\phi', \eta_-^\prime) \longrightarrow \Pi \longrightarrow \sigma \longrightarrow 0.$

In particular, $\Pi$ has length 2 or 3 according to $\phi \supset \rho \boxtimes S_{2x-1}$ or not.

**Proof.** First, we show that there is an inclusion $\pi(\phi', \eta_+^\prime) \hookrightarrow \Pi$ for each $\epsilon \in \{\pm 1\}$. To do this, we may assume that $\pi(\phi', \eta_+^\prime) \neq 0$. Note that $\phi'$ contains $\rho \boxtimes S_{2x+1}$ with multiplicity one. By Theorem 4.2 (2), we see that $\text{Jac}_{\rho|\cdot|^{-x}}(\pi(\phi', \eta_+^\prime))$ is nonzero and is an irreducible subrepresentation of $\text{St}(\rho, 2x-1) \times \pi(\phi, \eta)$. By Lemma 2.5 (1), we have an inclusion

$$\pi(\phi', \eta_+^\prime) \hookrightarrow \rho|\cdot|^{-x} \times \text{St}(\rho, 2x-1) \times \pi(\phi, \eta).$$

Since $\Pi$ is a subrepresentation of $\rho|\cdot|^{-x} \times \text{St}(\rho, 2x-1) \times \pi(\phi, \eta)$ such that

$$\text{Jac}_{\rho|\cdot|^{-x}}(\rho|\cdot|^{-x} \times \text{St}(\rho, 2x-1) \times \pi(\phi, \eta)) = \text{Jac}_{\rho|\cdot|^{-x}}(\Pi),$$

the above inclusion factors through $\pi(\phi', \eta_+^\prime) \hookrightarrow \Pi$.

If s.s.$\text{Jac}_{P_k}(\Pi)$ contains an irreducible representation $\tau \boxtimes \pi_0$ such that the central character of $\tau$ is of the form $x^s$ with $x$ unitary and $s < 0$, by Tadić’s formula (Theorem 2.7) and Casselman’s criterion, $\tau = |\cdot|^{-\frac{1}{2}}\text{St}(\rho, 2x) \times (\times_{i=1}^r \pi_i)$, where $\pi_i$ is a discrete series representation of $\text{GL}_{k_i}(F)$ such that the corresponding irreducible representation $\phi_i$ of $W_F \times \text{SL}_2(\mathbb{C})$ is contained in $\phi$ with multiplicity at least two, and $\pi_0 = \pi(\phi_0, \eta_0)$ with $\phi_0 = \phi - (\oplus_{i=1}^r \phi_i)^{\boxtimes 2}$ and $\eta_0 = \eta|A_{\phi_0}$. Since such an irreducible representation $\tau \boxtimes \pi_0$ is also contained in s.s.$\text{Jac}_{P_k}(\sigma)$, we see that $\sigma$ is the unique irreducible non-tempered subquotient of $\Pi$. Namely, if we let $\Pi^{\text{sub}}$ be the maximal proper subrepresentation of $\Pi$, i.e., $\Pi/\Pi^{\text{sub}} \cong \sigma$, then all irreducible subquotients of $\Pi^{\text{sub}}$ must be tempered. Moreover since these irreducible subquotients have
the same cuspidal support so that they share the same Plancherel measure, they belong to the same $L$-packet $\Pi_{\phi'}$ (see [GIT], Lemma A.6)).

Now we show that $\Pi^\text{sub}$ is isomorphic to $\pi(\phi', \eta'_+) \oplus \pi(\phi', \eta'_-)$. We separate the cases as follows:

- $\phi$ is discrete and $\rho \boxtimes S_{2x-1} \not\subset \phi$;
- $\phi$ is discrete and $\rho \boxtimes S_{2x-1} \subset \phi$;
- $\phi$ is general.

When $\phi$ is discrete and $\rho \boxtimes S_{2x-1} \not\subset \phi$, we note that $\phi' = \phi \oplus (\rho \boxtimes S_{2x-1}) \oplus (\rho \boxtimes S_{2x+1})$ is also discrete. Then since all irreducible subquotients of $\Pi^\text{sub}$ are discrete series, $\Pi^\text{sub}$ is semisimple. In particular, any irreducible subquotient $\pi'$ of $\Pi^\text{sub}$ is a subrepresentation of $\Pi$ so that $\text{Jac}_{\rho|\pi'}(\pi') \neq 0$. However, since $\text{Jac}_{\rho|\pi'}(\Pi) = \text{Jac}_{\rho|\pi'}(\pi(\phi', \eta'_+) \oplus \pi(\phi', \eta'_-))$, we have $\Pi^\text{sub} \cong \pi(\phi', \eta'_+) \oplus \pi(\phi', \eta'_-)$. When $\phi$ is discrete and $\rho \boxtimes S_{2x-1} \subset \phi$, any irreducible subquotient $\pi'$ of $\Pi^\text{sub}$ belongs to $\Pi_{\phi'}$ with $\phi' = \phi_0' \oplus (\rho \boxtimes S_{2x-1}) \boxtimes \mathbb{R}$, where $\phi_0' = \phi - (\rho \boxtimes S_{2x-1}) \boxtimes (\rho \boxtimes S_{2x+1})$ is discrete such that $\rho \boxtimes S_{2x-1} \not\subset \phi_0'$. Hence the Jacquet module $s.s.\text{Jac}_{\rho_1|\pi'}(\pi')$ contains an irreducible representation of the form $\text{St}(\rho, 2x - 1) \otimes \pi_0'$. By Tadić’s formula (Corollary 2.8), the sum of irreducible representations of this form appearing in $s.s.\text{Jac}_{\rho_1|\pi'}(\Pi)$ is

$$\text{St}(\rho, 2x - 1) \otimes s.s.(\rho \cdot | \sigma \times \pi(\phi, \eta)).$$

By Lemma 5.1 we have an exact sequence

$$0 \longrightarrow \pi(\phi'_0, \eta'_0) \longrightarrow \rho| \cdot | \sigma \times \pi(\phi, \eta) \longrightarrow \sigma' \longrightarrow 0,$$

where $\sigma'$ is the unique irreducible quotient of $\rho| \cdot | \sigma \times \pi(\phi, \eta)$, and $\eta'_0 \in \widehat{A}_\phi$ is the character corresponding to $\eta \in \widehat{A}_\phi$. Now there exists $\epsilon \in \{\pm 1\}$ such that $\pi(\phi', \eta'_\epsilon) = 0$. Moreover, $s.s.\text{Jac}_{\rho_1|\pi'}(\Pi_{\phi', \eta'_\epsilon}) \subset \text{St}(\rho, 2x - 1) \otimes \pi(\phi', \eta'_0)$ since

$$\eta'_0(\alpha \boxtimes S_{2x+1}) = \eta'_0(\alpha \boxtimes S_{2x-1}) = \eta(\alpha \boxtimes S_{2x-1}) = \eta_0(\alpha \boxtimes S_{2x+1}).$$

On the other hand, since $\sigma \hookrightarrow \text{St}(\rho, 2x - 1) \times \rho| \cdot | \sigma \times \pi(\phi, \eta)$, we see that $\text{Jac}_{\rho|\pi'}(\sigma)$ is nonzero and contains $\text{St}(\rho, 2x - 1) \otimes \sigma'$. Hence

$$s.s.\text{Jac}_{\rho_1|\pi'}(\Pi) - s.s.\text{Jac}_{\rho_1|\pi'}(\Pi_{\phi', \eta'_\epsilon}) - s.s.\text{Jac}_{\rho_1|\pi'}(\sigma)$$

has no irreducible representation of the form $\text{St}(\rho, 2x - 1) \otimes \pi'_0$. This shows that $\Pi^\text{sub} = \pi(\phi', \eta'_0)$.

In general, we prove the claim by induction on the dimension of $\phi$. When $\phi$ is not discrete, there exists an irreducible representation $\phi_1$ of $W_F \times \text{SL}_2(\mathbb{C})$ which $\phi$ contains with multiplicity at least two. Note that $\phi_1 \not\cong \rho \boxtimes S_{2x+1}$. Set $\phi_0' = \phi - \phi_1 \boxtimes \mathbb{R}$, and $\eta_0' = \eta|\mathbb{A}_{\phi_0'}$. Take $\Pi_0$, $\sigma_0$, $\phi'_0$ and $\eta'_0, \epsilon \in \mathbb{A}_{\phi'_0}$ as in the statement of the proposition. By induction hypothesis, we have an exact sequence

$$0 \longrightarrow \pi(\phi'_0, \eta'_{0,+}) \oplus \pi(\phi'_0, \eta'_{0,-}) \longrightarrow \Pi_0 \longrightarrow \sigma_0 \longrightarrow 0.$$

Let $\tau$ be the irreducible discrete series representation of $\text{GL}_k(F)$ corresponding to $\phi_1$. The above exact sequence remains exact after taking the parabolic induction functor $\pi_0 \mapsto \tau \times \pi_0$. Note that $\tau \times \langle \rho; x, x - 1, \ldots, -(x - 1) \rangle \cong \langle \rho; x, x - 1, \ldots, -(x - 1) \rangle \times \tau$ by Theorem 2.1. Since $\sigma_0$ is unitary, the parabolic induction $\tau \times \sigma_0$ is semisimple. In particular, any irreducible subquotient of $\tau \times \sigma_0$ is non-tempered. Considering the cases where
Therefore, it is enough to show that Jac
ρ
we have Jac
Since
\[ A \]
Proof. Let
\[ \text{Lemma 5.4.} \]
5.4. The case of multiplicity two. Finally, we prove Theorem 4.2 (3).

Lemma 5.3. Let \( \phi \in \Phi_{\text{sp}}(G) \), \( \eta \in \hat{A}_\phi \) and \( x > 0 \). Suppose that \( \phi \) contains both \( \rho \boxtimes S_{2x+1} \) and \( \rho \boxtimes S_{2x+3} \) with multiplicity one. Then we have
\[
\text{Jac}_{\rho|^{x+1},\rho|^{x},\rho|^{x}}(\pi(\phi, \eta)) \subset 2 \cdot \text{Jac}_{\rho|^{x},\rho|^{2},\rho|^{2}}(\pi(\phi, \eta)).
\]

Proof. We may assume that \( \text{Jac}_{\rho|^{x+1},\rho|^{x},\rho|^{x}}(\pi(\phi, \eta)) \neq 0 \). By Lemma 2.5 (1), there exists an irreducible subquotient \( \sigma \) of this Jacquet module such that
\[
\pi(\phi, \eta) \hookrightarrow \rho|^{x+1} \times \rho|^{x} \times \rho|^{x} \times \sigma.
\]
Since there exists an exact sequence
\[
0 \longrightarrow \langle \rho|^{x+1}, x, x \rangle \longrightarrow \rho|^{x+1} \times \rho|^{x} \longrightarrow \langle \rho|^{x}, x, x \rangle \longrightarrow 0,
\]
where \( \langle \rho|^{x}, x, x \rangle \) is the unique irreducible subrepresentation of \( \rho|^{x} \times \rho|^{x} \), we see that \( \pi(\phi, \eta) \) is a subrepresentation of \( \langle \rho|^{x}, x, x \rangle \times \rho|^{x} \times \sigma \) or \( \langle \rho|^{x}, x, x \rangle \times \rho|^{x} \times \sigma \). Since \( \langle \rho|^{x}, x, x \rangle \times \rho|^{x} \times \sigma \), we have \( \text{Jac}_{\rho|^{x},\rho|^{x}}(\pi(\phi, \eta)) \neq 0 \).

By Theorem 4.2 (2), \( \text{Jac}_{\rho|^{x+1}}(\pi(\phi, \eta)) \neq 0 \) and \( \text{Jac}_{\rho|^{x}}(\pi(\phi, \eta)) \neq 0 \) imply that \( \sigma' = \text{Jac}_{\rho|^{x+1},\rho|^{x},\rho|^{x}}(\pi(\phi, \eta)) \) is nonzero and irreducible. Moreover, we have \( \text{Jac}_{\rho|^{x}}(\sigma') = 0 \) and \( \text{Jac}_{\rho|^{x+1}}(\sigma') = 0 \). By Lemma 2.5 (1), we have an inclusion
\[
\pi(\phi, \eta) \hookrightarrow \rho|^{x} \times \rho|^{x+1} \times \rho|^{x} \times \sigma'.
\]
Since
\[
\text{Jac}_{\rho|^{x+1},\rho|^{x},\rho|^{x}}(\rho|^{x} \times \rho|^{x} \times \rho|^{x+1} \times \rho|^{x} \times \sigma') = 2 \cdot \sigma',
\]
we have \( \text{Jac}_{\rho|^{x+1},\rho|^{x},\rho|^{x}}(\pi(\phi, \eta)) \subset 2 \cdot \sigma' \), as desired.

Suppose that \( x > 0 \). Let \( \phi = \phi_0 \oplus (\rho \boxtimes S_{2x+1})^{\otimes 2} \) with \( \rho \boxtimes S_{2x+1} \not\subset \phi_0 \), and \( \eta \in \hat{A}_\phi \).

Lemma 5.4. Set \( \phi_1 = \phi - (\rho \boxtimes S_{2x+1})^{\otimes 2} \oplus (\rho \boxtimes S_{2x-1})^{\otimes 2} \). We canonically identify \( A_{\phi_1} \) with \( A_\phi \), and let \( \eta_1 \in \hat{A}_{\phi_1} \) be the character corresponding to \( \eta \in \hat{A}_\phi \). Then we have
\[
\text{Jac}_{\rho|^{x},\rho|^{x}}(\pi(\phi, \eta)) = 2 \cdot \pi(\phi_1, \eta_1).
\]

Proof. Let \( \eta_+ = \eta \) and \( \eta_- \in \hat{A}_\phi \) be as in the statement of Theorem 4.2 (3). We also take \( \eta_{1, \pm} \in \hat{A}_{\phi_1} \) corresponding to \( \eta_{\pm} \). Then we have
\[
\begin{align*}
\pi(\phi, \eta) & \oplus J(\phi, \eta) = \text{St}(\rho, 2x + 1) \times \pi(\phi_0, \eta|A_{\phi_0}); \\
\pi(\phi_1, \eta) & \oplus J(\phi_1, \eta_{1, -}) = \text{St}(\rho, 2x - 1) \times \pi(\phi_0, \eta|A_{\phi_0}); \\
\text{Jac}_{\rho|^{x},\rho|^{x}}(\text{St}(\rho, 2x + 1) \times \pi(\phi_0, \eta|A_{\phi_0})) & \simeq 2 \cdot \text{St}(\rho, 2x - 1) \times \pi(\phi_0, \eta|A_{\phi_0}).
\end{align*}
\]
Therefore, it is enough to show that \( \text{Jac}_{\rho|^{x},\rho|^{x}}(\pi(\phi, \eta)) \subset 2 \cdot \pi(\phi_1, \eta_1) \).

We apply Mœglin’s construction to \( \Pi_\phi \). Write
\[
\phi = \bigoplus_{i=1}^{t} \rho \boxtimes S_{U_i} \oplus \phi_e
\]
with \(a_1 \leq \cdots \leq a_t\) and \(\rho \otimes S_0 \not\subset \phi_e\) for any \(a > 0\). There exists \(t_0 > 1\) such that \(a_{t_0-1} = a_{t_0} = 2x + 1\). Take a new \(L\)-parameter

\[
\phi_\gg = \left( \bigoplus_{i=1}^t \rho \otimes S_{a_i'} \right) \oplus \phi_e
\]

such that

- \(a_i' < \cdots < a_t'\);
- \(a_i' \geq a_i\) and \(a_i' \equiv a_i \mod 2\) for any \(i\).

In particular, \(a_{t_0}' \geq 2x + 3\). We can identify \(\mathcal{A}_{\phi_{\gg}}\) with \(\mathcal{A}_\phi\) canonically. Let \(\eta_\gg \in \widehat{\mathcal{A}_{\phi_{\gg}}}\) be the character corresponding to \(\eta \in \widehat{\mathcal{A}_\phi}\). Then Theorem 3.3 says that

\[
\pi(\phi, \eta) = \text{Jac}_{\rho| \frac{a_i'-1}{2}, \ldots, \rho| \frac{a_t'+1}{2}} \circ \cdots \circ \text{Jac}_{\rho| \frac{a_i'-1}{2}, \ldots, \rho| \frac{a_1'+1}{2}}(\pi(\phi_{\gg}, \eta_{\gg})).
\]

Note that \((a_{t_0} + 1)/2 = x + 1\). By Lemma 2.5 (2), we see that

\[
\text{Jac}_{\rho| \frac{x+1}{2}, \rho| \frac{x}{2}}(\pi(\phi, \eta)) = \text{Jac}_{\rho| \frac{a_i'-1}{2}, \ldots, \rho| \frac{a_t'+1}{2}} \circ \cdots \circ \text{Jac}_{\rho| \frac{a_i'-1}{2}, \ldots, \rho| \frac{a_1'+1}{2}}(\pi(\phi_{\gg}, \eta_{\gg})).
\]

By Lemma 5.3, we have

\[
\text{Jac}_{\rho| \frac{x+1}{2}, \rho| \frac{x}{2}}(\pi(\phi, \eta)) \subset 2 \cdot \text{Jac}_{\rho| \frac{x}{2}, \rho| \frac{x+1}{2}}(\pi(\phi, \eta)).
\]

Since

\[
\text{Jac}_{\rho| \frac{a_i'-1}{2}, \ldots, \rho| \frac{a_t'-1}{2}} \circ \cdots \circ \text{Jac}_{\rho| \frac{a_i'-1}{2}, \ldots, \rho| \frac{a_1'-1}{2}}(\pi(\phi_{\gg}, \eta_{\gg})) = \pi(\phi_1, \eta_1)
\]

by Theorem 3.3, we have \(\text{Jac}_{\rho| \frac{x}{2}, \rho| \frac{x+1}{2}}(\pi(\phi, \eta)) \subset 2 \cdot \pi(\phi_1, \eta_1)\), as desired. \(\square\)

Now we can prove Theorem 4.2 (3).

**Proof of Theorem 4.2** (3). By Corollary 2.8, we have

\[
\text{Jac}_{\rho| \frac{x}{2}}(\pi(\phi, \eta_+) \oplus \pi(\phi, \eta_-)) = 2 \cdot (\rho; x, x - 1, \ldots, -(x - 1)) \times \pi(\phi_0, \eta|\mathcal{A}_0).
\]

By Proposition 5.2, we have an exact sequence

\[
0 \longrightarrow \pi(\phi', \eta'_+) \oplus \pi(\phi', \eta'_-) \longrightarrow \Pi \longrightarrow \sigma \longrightarrow 0,
\]
where Π = \langle ρ; x, x - 1, \ldots, -(x - 1) \rangle × \pi(φ_0, η|A_{φ_0}), and σ is the unique irreducible quotient of Π. Fix ε ∈ {±1}. By Lemma 5.4, we see that Jac_{ρ}|_{x}(π(φ, η_{ε})) ⊃ 2 · π(φ', η_{ε}). On the other hand, since s.s.Jac_{P_{2x}}(π(φ, η_{ε})) ⊃ St(ρ, 2x + 1) ⊗ π(φ_0, η|A_{φ_0}), we have

\text{s.s.Jac}_{P_{2x}}(Jac_{ρ}|_{x}(π(φ, η_{ε}))) ⊃ \left| \cdot - \frac{1}{2} \right| St(ρ, 2x) ⊗ π(φ_0, η|A_{φ_0}).

This implies that Jac_{ρ}|_{x}(π(φ, η_{ε})) contains an irreducible non-tempered representation, which must be σ. Hence

Jac_{ρ}|_{x}(π(φ, η_{ε})) ⊃ 2 · π(φ', η_{ε}) + σ = Π + π(φ', η_{ε}) - π(φ', η_{ε}).

Considering Jac_{ρ}|_{x}(π(φ, η_{+}) ⊕ π(φ, η_{-})), we see that this inclusion must be an equality. □

If x = 0 and m = 2, we see that Jac_{ρ}(π(φ, η)) ⊃ π(φ_0, η|A_{φ_0}). By the same argument, this inclusion must be an equality. This completes the proof of Theorem 4.2 (4), so that the ones of all statements of Theorem 4.2.

5.5. Description of \( \mu^{*}_{π} \). We prove Theorem 4.3 in this subsection. To do this, we need the following specious lemma.

Lemma 5.5. Let \( φ ∈ \Phi_{gp}(G) \) and \( \underline{\pi} = (x_1, \ldots, x_m) ∈ R^m \). Suppose that Jac_{ρ}|_{x}(π) ≠ 0 for some \( π ∈ Π_{φ} \).

(1) If \( x_m < 0 \), then \( x_i = -x_m \) for some \( i \).

(2) Suppose that \( \underline{θ} \) is of the form

\[ \underline{θ} = (x^{(1)}_1, \ldots, x^{(1)}_{m_1}, \ldots, x^{(k)}_1, \ldots, x^{(k)}_{m_k}) \]

with \( x^{(j)}_{i-1} > x^{(j)}_{i} \) for \( 1 ≤ j ≤ k, 1 < i ≤ m_j \), and \( x^{(1)}_1 ≤ \cdots ≤ x^{(k)}_1 \). Then \( x^{(j)}_1 ≥ 0 \) for \( j = 1, \ldots, k, and \)

\[ Φ ⊃ \bigoplus_{j=1}^{k} ρ ⊗ S_{2x^{(j)}_1+1}. \]

Proof. We prove the lemma by induction on \( m \). By Lemma 4.1, we see that \( 2x_1 + 1 \) is a positive integer, and \( φ \) contains \( ρ ⊗ S_{2x_1+1} \). In particular, we obtain the lemma for \( m = 1 \).

Suppose that \( m ≥ 2 \) and put \( \underline{θ}' = (x_2, \ldots, x_m) ∈ R^{m-1} \). By Theorem 4.2, one of the following holds.

- Jac_{ρ}|_{θ'}(π') ≠ 0 for some \( π' ∈ Π_{φ'} \) with \( φ' = φ - (ρ ⊗ S_{2x_1+1}) ⊕ (ρ ⊗ S_{2x_1-1}) \);
- Jac_{ρ}|_{θ'}((ρ; x_1, x_1 - 1, \ldots, -(x_1 - 1)) × π_0) ≠ 0 for some \( π_0 ∈ Π_{φ_0} \) with \( φ_0 = φ - (ρ ⊗ S_{2x_1+1}) ⊕ 2 \).

The former case can occur only if \( x_1 > 0 \), and the latter case can occur only if \( φ ⊃ (ρ ⊗ S_{2x_1+1}) ⊕ 2 \).

We consider the former case. Assume that \( x_1 > 0 \) and Jac_{ρ}|_{θ'}(π') ≠ 0 for some \( π' ∈ Π_{φ'} \) with \( φ' = φ - (ρ ⊗ S_{2x_1+1}) ⊕ (ρ ⊗ S_{2x_1-1}) \). By the induction hypothesis, we have \( x_i = -x_m \) for some \( i ≥ 2 \) when \( x_m < 0 \), and

\[ φ' ⊃ (ρ ⊗ S_{2x^{(i)}_1 - 1}) ⊕ \bigoplus_{j=2}^{k} ρ ⊗ S_{2x^{(j)}_1+1}. \]
when $\underline{x}$ is of the form in (2) since $\underline{x}'$ is also of the form. This implies the assertion for $\phi$.

We consider the latter case. Assume that $\phi \supset (\rho \boxtimes S_{2x+1})^{\otimes 2}$, and that

$$\text{Jac}_{\rho|\underline{x}'}((\rho; x_1, x_1 - 1, \ldots, -(x_1 - 1)) \times \pi_0) \neq 0$$

for some $\pi_0 \in \Pi_{\phi_0}$ with $\phi_0 = \phi - (\rho \boxtimes S_{2x+1})^{\otimes 2}$. By Corollary 2.8 we can divide

$$(2, \ldots, m) = \{i_1, \ldots, i_{m_1}\} \sqcup \{j_1, \ldots, j_{m_2}\} \sqcup \{k_1, \ldots, k_{m_3}\}$$

with $i_1 < \cdots < i_{m_1}$, $j_1 < \cdots < j_{m_2}$, $k_1 < \cdots < k_{m_3}$ and $m_2 + m_3 \leq 2x_1$ such that

- $\text{Jac}_{\rho|\underline{x}'}[(\rho; y_1, \ldots, \rho; y_{m_1})(\pi_0)] \neq 0$ with $y_t = x_{i_t}$;
- $x_{j_t} = x_1 + 1 - t$ for $t = 1, \ldots, m_2$;
- $x_{k_t} = x_1 - t$ for $t = 1, \ldots, m_3$.

Considering the following four cases, we can prove the existence $x_i$ satisfying $x_i = -x_m$ when $x_m < 0$.

- When $m = i_{m_1}$, by the induction hypothesis, we have $x_{i_t} = -x_m$ for some $t$.
- When $m = j_{m_2}$, we have $x_m = x_1 + 1 - m_2 < 0$ so that $x_{j_t} = -x_m$ with $t = 2x_1 + 2 - m_2$.
- When $m = k_{m_3}$ and $m_3 < 2x_1$, we have $x_m = x_1 - m_3 < 0$ so that $x_{k_t} = -x_m$ with $t = 2x_1 - m_3$.
- When $m = k_{m_3}$ and $m_3 = 2x_1$, we have $x_1 = -x_m$.

On the other hand, when $\underline{x}$ is of the form in (2), since $x_1^{(j_0)} \geq x_1^{(1)} = x_1$, there is at most one $j_0 \geq 2$ such that

$$x_1^{(j_0)} \in \{x_{j_t} \mid t = 1, \ldots, m_2\} \cup \{x_{k_t} \mid t = 1, \ldots, m_3\},$$

in which case, $x_1^{(j_0)} = x_1$. By the induction hypothesis, we have

$$\phi_0 \supset \bigoplus_{2 \leq j < k \neq j_0} (\rho \boxtimes S_{2x_1^{(j)}+1}).$$

This implies the assertion for $\phi$. This completes the proof.

Now we can prove Theorem 4.3.

**Proof of Theorem 4.3.** Since the subgroup of $\mathcal{R}_m$ spanned by $\text{Irr}_\rho(\text{GL}_{dm}(F)) = \{\tau_{\underline{x}} \mid \underline{x} \in \Omega_m\}$ has another basis $\{\Delta_{\underline{x}} \mid \underline{x} \in \Omega_m\}$, we can write

$$\mu_\rho^*(\pi) = \sum_{m \geq 0} \sum_{\underline{x} \in \Omega_m} \Delta_{\underline{x}} \otimes \Pi_{\underline{x}}(\pi)$$

for some virtual representation $\Pi_{\underline{x}}(\pi)$. For $\underline{y} \in \Omega_m$, applying $\text{Jac}_{\rho|\underline{x}}$ to $s.s.\text{Jac}_{P_{dm}}(\pi)$, we have

$$\text{Jac}_{\rho|\underline{y}}(\pi) = \sum_{\underline{x} \in \Omega_m} \text{Jac}_{\rho|\underline{y}}(\Delta_{\underline{x}}) \otimes \Pi_{\underline{x}}(\pi) = \sum_{\underline{x} \in \Omega_m} m(\underline{y},\underline{x}) \cdot \Pi_{\underline{x}}(\pi),$$

where $m(\underline{y},\underline{x}) = \dim \text{Jac}_{\rho|\underline{y}}(\Delta_{\underline{x}})$. If $m'(x',y) \in \mathbb{Q}$ satisfies that

$$\sum_{y \in \Omega_m} m'(x',y)m(\underline{y},\underline{x}) = \delta_{\underline{x},\underline{y}},$$

we have

$$\Pi_{\underline{y}}(\pi) = \sum_{y \in \Omega_m} m'(x,y) \cdot \text{Jac}_{\rho|\underline{y}}(\pi).$$
Hence we have
\[ \mu^*_\rho(\pi) = \sum_{m \geq 0} \sum_{x, k \in \Omega_m} m'\langle x, x(k) \rangle \cdot \Delta x \otimes \text{Jac}_{\rho|\pi}(\pi). \]

One can easily prove by induction that \( m'\langle x, x(k) \rangle = 0 \) unless \( x = x(k') \) for some \( k' \in K_\phi \) (see also the proof of Lemma 2.4). Therefore, by Lemma 5.5 we have
\[ \mu^*_\rho(\pi) = \sum_{m \geq 0} \sum_{k, k' \in K_\phi} m'\langle x(k'), x(k) \rangle \cdot \Delta x \otimes \text{Jac}_{\rho|\pi}(\pi). \]

This completes the proof of Theorem 4.3. \( \Box \)

### 5.6. A remark on standard modules.

As a consequence of Theorem 4.2, we can prove the irreducibility of certain standard modules.

**Corollary 5.6.** Let \( \phi \in \Phi_{\text{gp}}(G) \) and \( \eta \in \hat{A}_\phi \) such that \( \pi(\phi, \eta) \neq 0 \). Suppose that \( \phi \supset \rho \boxtimes S_{2x+1} \) for \( x > 0 \) but \( \text{Jac}_{\rho|\pi}(\pi(\phi, \eta)) = 0 \). Then the standard module
\[ \Pi = \langle \rho; x, x-1, \ldots, -(x-1) \rangle \times \pi(\phi, \eta) \]

is irreducible.

**Proof.** Let \( \sigma \) be the unique irreducible quotient of \( \Pi \), which is non-tempered. By the same argument as the proof of Proposition 5.2, we see that \( \sigma \) is the unique irreducible non-tempered subquotient of \( \Pi \). Suppose that \( \Pi \) is reducible. If \( \pi' \) is another irreducible subquotient of \( \Pi \), by considering its cuspidal support or its Plancherel measure, we see that \( \pi' \in \Pi_{\phi'} \) with \( \phi' = \phi \oplus (\rho \boxtimes S_{2x-1}) \oplus (\rho \boxtimes S_{2x+1}) \). Since \( \phi \supset \rho \boxtimes S_{2x+1} \), we see that \( \phi' \supset (\rho \boxtimes S_{2x+1}) \oplus 2 \). By Theorem 4.2, \( \text{Jac}_{\rho|\pi'(\Pi)} = \text{St}(\rho, 2x-1) \times \pi(\phi, \eta) \) consists of tempered representations. This is a contradiction. \( \Box \)

**Example 5.7.** Consider \( \phi = S_2 \oplus S_4 \oplus S_6 \in \Phi_{\text{disc}}(\text{SO}_{13}) \) and \( \eta \in \hat{A}_\phi \) given by \( \eta(\alpha_{S_{2a}}) = (-1)^a \) for \( a = 1, 2, 3 \). Then \( \pi(\phi, \eta) \) is an irreducible supercuspidal representation. Moreover, the standard module
\[ \Pi = \langle 1_{\text{GL}_1(F)}; \frac{5}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{3}{2} \rangle \times \pi(\phi, \eta) \]

of \( 1_{\text{SO}}(F) \) is irreducible. Note that the \( \tilde{L} \)-parameter \( \phi' = \phi \oplus | \cdot |^{\frac{5}{2}} S_5 \oplus | \cdot |^{-\frac{3}{2}} S_5 \) of \( \Pi \) is non-generic since
\[ L(s, \phi', \text{Ad}) = \zeta_F(s-1)\zeta_F(s)^3\zeta_F(s+1)^{13}\zeta_F(s+2)^{10}\zeta_F(s+3)^{12}\zeta_F(s+4)^5\zeta_F(s+5)^3 \]

has a pole at \( s = 1 \), where \( \zeta_F \) is the local zeta function associated to \( F \). In particular, the standard module
\[ \Pi_0 = \langle 1_{\text{GL}_1(F)}; \frac{5}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{3}{2} \rangle \times \pi(\phi, 1) \]

is reducible by Theorem 4.2.
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