1. INTRODUCTION

The current picture of terrestrial planet formation relies heavily on our understanding of the dynamical evolution of planetesimals—asteroid-like bodies thought to be planetary building blocks. In this study we investigate the growth of eccentricities and inclinations of planetesimals in spatially homogeneous protoplanetary disks using methods of kinetic theory. Emphasis is put on clarifying the effect of gravitational scattering between planetesimals on the evolution of their random velocities. We explore disks with a realistic mass spectrum of planetesimals evolving in time, similar to that obtained in self-consistent simulations of planetesimal coagulation: the distribution scales as a power law of mass for small planetesimals and is supplemented by an extended tail of bodies at large masses representing the ongoing runaway growth in the system. We calculate the behavior of planetesimal random velocities as a function of the planetesimal mass spectrum both analytically and numerically; results obtained by the two approaches agree quite well. Scaling of random velocity with mass can always be represented as a combination of power laws corresponding to different velocity regimes (shear- or dispersion-dominated) of planetesimal gravitational interactions. For different mass spectra we calculate analytically the exponents and time-dependent normalizations of these power laws, as well as the positions of the transition regions between different regimes. It is shown that random energy equipartition between different planetesimals can only be achieved in disks with very steep mass distributions (the differential surface number density of planetesimals falling off steeper than $m^{-4}$) or in the runaway tails. In systems with shallow mass spectra (shallower than $m^{-3}$) the random velocities of small planetesimals turn out to be independent of their masses. We also discuss the damping effects of inelastic collisions between planetesimals and of gas drag, and their importance in modifying planetesimal random velocities.

Key words: Kuiper belt — planetary systems: formation — solar system: formation

DYNAMICAL EVOLUTION OF PLANETESIMALS IN PROTOPLANETARY DISks

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ABSTRACT

The current picture of terrestrial planet formation relies heavily on our understanding of the dynamical evolution of planetesimals—a asteroid-like bodies thought to be planetary building blocks. In this study we investigate the growth of eccentricities and inclinations of planetesimals in spatially homogeneous protoplanetary disks using methods of kinetic theory. Emphasis is put on clarifying the effect of gravitational scattering between planetesimals on the evolution of their random velocities. We explore disks with a realistic mass spectrum of planetesimals evolving in time, similar to that obtained in self-consistent simulations of planetesimal coagulation: the distribution scales as a power law of mass for small planetesimals and is supplemented by an extended tail of bodies at large masses representing the ongoing runaway growth in the system. We calculate the behavior of planetesimal random velocities as a function of the planetesimal mass spectrum both analytically and numerically; results obtained by the two approaches agree quite well. Scaling of random velocity with mass can always be represented as a combination of power laws corresponding to different velocity regimes (shear- or dispersion-dominated) of planetesimal gravitational interactions. For different mass spectra we calculate analytically the exponents and time-dependent normalizations of these power laws, as well as the positions of the transition regions between different regimes. It is shown that random energy equipartition between different planetesimals can only be achieved in disks with very steep mass distributions (the differential surface number density of planetesimals falling off steeper than $m^{-4}$) or in the runaway tails. In systems with shallow mass spectra (shallower than $m^{-3}$) the random velocities of small planetesimals turn out to be independent of their masses. We also discuss the damping effects of inelastic collisions between planetesimals and of gas drag, and their importance in modifying planetesimal random velocities.

Key words: Kuiper belt — planetary systems: formation — solar system: formation
system with a quasi-stationary planetesimal mass spectrum. Understanding this problem is a logical step necessary to provide a clearer perspective on how to build a fully self-consistent theory of planet formation.

We describe the statistical approach to the problem of the evolution of planetesimal random velocities in § 2. Our model of the planetesimal mass spectrum is introduced § 3. In § 4 we outline the results for the distribution of planetesimal random velocities versus planetesimal masses as functions of the input mass distribution and time; we also compare these analytical predictions with numerical results. Table 1 summarizes our major findings in a concise form. In § 5 we comment on how gas drag and inelastic collisions between planetesimals can affect planetesimal velocity spectra. We conclude in § 6 with a discussion of our results and a comparison with previous studies.

### Table 1

Summary of Analytical Results for the Velocity Scaling with Mass

| Range of α | Mass Interval | \( \eta = d \ln s / d \ln x \) | \( \eta_s = d \ln s_s / d \ln x \) | Reference |
|------------|---------------|-------------------------------|-------------------------------|-----------|
| 0 < α < 2  | \( x \leq x_0 \) | 0                             | 0                             | Eq. (29)  |
|            | \( x_0 \leq x \leq x_{\text{shear}} \) | \(-1/2\)                      | \(-1/2\)                      | Eq. (32)  |
|            | \( x \geq x_{\text{shear}} \) | \(-1/6\)                      | \(-1/2\)                      | Eq. (37)  |
| 2 < α < 3  | \( x \leq x_0 \) | 0                             | 0                             | Eq. (40)  |
|            | \( x_0 \leq x \leq x_{\text{s}} \) | \(-1/4\)                      | \(-1/4\)                      | Eq. (42)  |
|            | \( x_\text{s} \leq x \leq x_{\text{shear}} \) | \(-1/2\)                      | \(-1/2\)                      | Eq. (43)  |
|            | \( x \geq x_{\text{shear}} \) | \(-5/6\)                      | \(-7/6 \alpha < 5/2\)        | Eqs. (46) and (47) |
| 3 < α < 4  | \( x \leq x_0 \) | \(-\beta\)                     | \(-\beta\)                    | Eq. (C2)  |
|            | \( x_0 \leq x \leq x_{\text{s}} \) | \(-1/2\)                      | \(-1/2\)                      | Eq. (C4)  |
|            | \( x_{\text{s}} \leq x \leq x_{\text{shear}} \) | \(-1/6\)                      | \(-1/6\)                      | Eq. (C5)  |
|            | \( x \geq x_{\text{shear}} \) | \(-1/6\)                      | \(-1/2\)                      | Eq. (C6)  |
| α > 4      | \( x \leq x_{\text{shear}} \) | \(-1/2\)                      | \(-1/2\)                      | Eq. (C3), Appendix C.2 |
|            | \( x \geq x_{\text{shear}} \) | \(-1/6\)                      | \(-1/2\)                      | Appendix C.2 |

Two important consequences immediately follow from this set of assumptions. First, the gravitational scattering of planetesimals during their encounter can be studied in the framework of a Hill approximation (Hénon & Petit 1986; Ida 1990; Rafikov 2003a). In this approach the gravitational interaction between two bodies with masses \( m \) and \( m^* \) introduces a natural length scale—the Hill radius:

\[
R_H = a \left( \frac{m + m^*}{M_c} \right)^{1/3} = 10^{-4} \text{ AU} \quad a_{\text{AU}} \left( \frac{m + m^*}{2 \times 10^{21} \text{ g}} \right)^{1/3} ,
\]

where \( a \) is the distance from the central object. The numerical estimate made in (1) assumes that \( M_c = M_\odot \) and \( a_{\text{AU}} \equiv a/(1 \text{ AU}) \) and is intended to illustrate that typically \( R_H \ll a \). The Hill approximation yields two significant simplifications:

1. The outcome of the interactions between two bodies depends only on their relative velocities and distances.
2. If all relative distances are scaled by \( R_H \) and relative velocities of interacting bodies are scaled by \( \Omega R_H \), then the outcome of the gravitational scattering depends only on the initial values of the scaled relative quantities.

Second, numerical studies (Greenzweig & Lissauer 1992; Ida & Makino 1992) have demonstrated that, in homogeneous planetesimal disks, the distribution function \( \psi(e, i) \) of absolute values of planetesimal eccentricities \( e \) and inclinations \( i \) is well represented by the Rayleigh distribution:

\[
\psi(e, i) \; de \; di = \frac{e \; de \; di}{\sigma_e \sigma_i} \exp \left( -\frac{e^2}{2 \sigma_e^2} - \frac{i^2}{2 \sigma_i^2} \right) ,
\]

where \( \sigma_e \) and \( \sigma_i \) are the aforementioned dispersions of eccentricity and inclination. It also follows from azimuthal symmetry that the horizontal and vertical epicyclic phases \( \tau \) and \( \omega \) are distributed uniformly in the interval \((0, 2\pi)\). These facts have important ramifications, as we demonstrate below.

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1 This definition follows the original work of Hénon & Petit (1986) and differs from the \( R_H \equiv d(m + m^*)/3M_\odot \)\(^1\) often used in the literature (e.g., Ida 1990; Stewart & Ida 2000).
Let us consider the gravitational interaction of two planetesimal populations, one with mass $m$ and eccentricity and inclination dispersions $\sigma_e$ and $\sigma_i$, and the other with mass $m^*$ and dispersions $\sigma_e^*$ and $\sigma_i^*$. When the velocity distribution function of the planetesimals has the form (2) it can be shown (Rafikov 2003a) that the evolution of $\sigma_e$ and $\sigma_i$ due to the gravitational interaction with a population of mass $m^*$ proceeds according to

$$
\frac{\partial \sigma_e^2}{\partial t} = \left| A N^* a^4 \left( \frac{m + m^*}{M_c} \right) \right|^4 \left( \frac{m^*}{m + m^*} \right)^{4/3} \frac{m^*}{m + m^*} \times \left( \frac{m^*}{m + m^*} H_1 + 2 \frac{\sigma_e^2}{\sigma_e + \sigma_e^2} H_2 \right),
$$

(3a)

$$
\frac{\partial \sigma_i^2}{\partial t} = \left| A N^* a^4 \left( \frac{m + m^*}{M_c} \right) \right|^4 \left( \frac{m^*}{m + m^*} \right)^{4/3} \frac{m^*}{m + m^*} \times \left( \frac{m^*}{m + m^*} K_1 + 2 \frac{\sigma_i^2}{\sigma_i + \sigma_i^2} K_2 \right),
$$

(3b)

where $N^*$ is the surface number density of planetesimals of mass $m^*$, and $A = -(r/2) d\Omega/d\tau$ is a measure of shear in the disk [in Keplerian disks $A = (3/4)\Omega$]. Scattering coefficients $H_{1,2}$ are defined as

$$
H_1 = H_1(\sigma_e, \sigma_i) = \int d\varepsilon d\dot{\varepsilon} \psi_1(\varepsilon, \dot{\varepsilon}) \times \int \frac{\partial \dot{\varepsilon}}{\partial \dot{\varepsilon}} \left( \Delta \varepsilon_{\text{sc}} \right)^2, \tag{4a}
$$

$$
H_2 = H_2(\sigma_e, \sigma_i) = \int d\varepsilon d\dot{\varepsilon} \psi_2(\varepsilon, \dot{\varepsilon}) \times \int \frac{\partial \dot{\varepsilon}}{\partial \dot{\varepsilon}} \left( \Delta \dot{\varepsilon}_{\text{sc}} \right), \tag{4b}
$$

where $\Delta \varepsilon_{\text{sc}}$ and $\Delta \dot{\varepsilon}_{\text{sc}}$ represent a change of $\varepsilon$ produced in the course of scattering, which is a function of not only absolute values of $\varepsilon$ and $\dot{\varepsilon}$, but also of the relative epicyclic phases (for which reason we use vector eccentricities and inclinations here). The distribution function of relative eccentricities and inclinations of planetesimals $\psi_1(\varepsilon, \dot{\varepsilon})$ is analogous in its functional form to equation (2) but with $\sigma_e$ replaced by relative velocity dispersions $\sigma_{e,ir}$. Expressions analogous to (4a) and (4b) can be written down also for the inclination scattering coefficients $K_1$ and $K_2$. Note that the only assumption used in deriving equations (3a) and (3b) is that of the specific form (2) of the distribution function of planetesimal eccentricities and inclinations (Rafikov 2003a).

Equations (3a) and (3b) describe dynamical evolution driven only by a gravitational scattering of planetesimals, implicitly assuming that they are point masses. The physical size of the planetesimal $r_p \approx 3 \times 10^{-7}$ AU $[m/(10^{21} \text{ g})]^{1/3}$ (for physical density $3 \text{ g cm}^{-3}$) is very small compared with the corresponding Hill radius in equation (1), which often justifies the neglect of physical collisions between planetesimals. However, at high relative velocities—higher than the escape speed from planetesimal surfaces—one can no longer disregard highly inelastic physical collisions, which strongly damp planetesimal random motions. In this study we proceed by assuming, first, that velocities of planetesimals are below their escape velocities and then discussing in § 5 how abandoning this assumption affects our conclusions.

Equations (3a) and (3b) and definitions (4a) and (4b) have been previously derived by other authors in a slightly different form (Ida 1990; Tanaka & Ida 1996; Ohtsuki 1999; Stewart & Ida 2000; Ohtsuki, Stewart, & Ida 2002):

$$
\frac{\partial \sigma_e^2}{\partial t} = \left| A N^* a^4 \left( \frac{m + m^*}{M_c} \right) \right|^4 \left( \frac{m^*}{m + m^*} \right)^{4/3} \frac{m^*}{m + m^*} \times \left[ \frac{m^*}{m + m^*}(H_1 + 2H_2) + 2 \frac{m}{m + m^*} \sigma_e^2 + \sigma_e^2 \frac{H_1}{H_2} \right. \\
\left. - 2 \frac{m^*}{m + m^*} \sigma_e^2 \frac{H_1}{H_2} \right].
$$

(5)

In this form the first term in brackets on the right-hand side describes the so-called viscous stirring, which is proportional to the phase space average of $\Delta (e_i^2)$ (see the definitions of $H_1$ and $H_2$). Depending on $\sigma_e$ and $\sigma_i$ it can be either positive or negative. The second term represents the phenomenon of dynamical friction well known from galactic dynamics. In the limit $m \gg m^*$ its contribution is proportional to the mass of the particle under consideration and to the surface mass density of field particles, but it is independent of individual masses of field particles (see Binney & Tremaine 1987). This term is negative since it represents the gravitational interaction of a moving body with the wake of field particles formed behind it as a result of gravitational focusing; thus, $H_2 < 0$ and $K_2 < 0$. Finally, the third term describes the increase of random velocities of a particular body at the expense of the random motion of the field planetesimals it interacts with. This effect is analogous to the first-order Fermi mechanism of the cosmic-ray acceleration via the scattering of energetic particles by randomly moving magnetic field inhomogeneities (Fermi 1949). Note that, in previous studies of planetesimal scattering, it is the combination of the second and third terms on the right-hand side of (5) that is called the dynamical friction (Ida 1990; Stewart & Ida 2000). The combined effect of these two terms is to drive the planetesimal system to an equipartition of random energy between planetesimals of different mass (which would be realized in the absence of viscous stirring).

In this work we analyze planetesimal velocity evolution with the aid of equations (3a) and (3b). We call the first (positive) term on the right-hand side gravitational stirring or heating (different from viscous stirring), while second (negative) term is called the gravitational friction or cooling (different from dynamical friction). In this sense the terms “stirring” and “friction” are only used to describe positive and negative contributions to the growth rate of the planetesimals’ random motion. The use of equations (3a) and (3b), rather than (5), has two obvious advantages: (1) the gravitational stirring and friction have different dependences on $m^*/(m + m^*)$, which considerably simplifies the

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1 Throughout the paper we refer to the eccentricities and inclinations of planetesimals as their random velocities.

2 I am grateful to Peter Goldreich and Re’em Sari for pointing this out to me.
analysis of the velocity evolution equations, and (2) the stirring and friction terms have definite signs (unlike the viscous stirring and dynamical friction used in previous studies of planetesimal dynamics).

The gravitational scattering of planetesimals can proceed in two rather different velocity regimes, shear-dominated and dispersion-dominated. The former one is realized when \( \sigma^2_x + \sigma^2_z < (R_H/a)^2 \), while the latter holds when \( \sigma^2_x + \sigma^2_z > (R_H/a)^2 \). Analytical arguments and numerical calculations demonstrate that, in the dispersion-dominated regime, \( \sigma_x \) and \( \sigma_y \) are of the same order and evolve in a similar fashion (e.g., Ida 1990; Stewart & Ida 2000). This happens because scattering in this regime has a three-dimensional character, making the different components of velocity ellipsoid comparable to each other. Thus, a rough idea of the dynamical evolution in this case can be obtained by studying only one of the equations (3a) or (3b). Moreover, the behavior of the scattering coefficients in the dispersion-dominated regime can be calculated analytically as a function of \( \sigma_x \) and \( \sigma_y \) in a two-body approximation (Stewart & Wetherill 1988; Tanaka & Ida 1996), and one finds that

\[
\begin{align*}
H_1 & = A_1(\sigma_x/\sigma_y) \ln \Lambda, \\
K_1 & = C_1(\sigma_x/\sigma_y), \\
H_2 & = -A_2(\sigma_x/\sigma_y) \ln \Lambda, \\
K_2 & = C_2(\sigma_x/\sigma_y),
\end{align*}
\]

where \( A_{1,2}, C_{1,2} \) are positive functions of the inclination-to-eccentricity ratio \( \sigma_x/\sigma_y \), and \( \ln \Lambda \) is a Coulomb logarithm (Binney & Tremaine 1987), which in our case can be represented as (Rafikov 2003a)

\[
\Lambda \approx a_1 \sigma_x/\sigma_y + a_2 \sigma_y,
\]

with \( a_{1,2} \) being some constants. Explicit analytical expressions for the coefficients \( A_{1,2}, C_{1,2} \) have been derived by Stewart & Ida (2000). One can deduce a specific useful property of these functions by considering a single-mass planetesimal population: in a dispersion-dominated regime the distribution of random energy between the vertical and horizontal motions tends to reach a quasi-equilibrium state (see, e.g., Ida & Makino 1992). Then the ratio \( \sigma_y/\sigma_x \) is almost constant, and both \( \sigma_y \) and \( \sigma_x \) grow with time, meaning that (see eqs. [3] and [6])

\[
A_1 > 2A_2, \quad C_1 > 2C_2.
\]

This property will be used later in \( \S 4 \) and Appendix C.

In the shear-dominated regime \( \sigma_x \) and \( \sigma_y \) evolve along different routes: eccentricity is excited much more strongly than inclination because in this regime the disk is geometrically thin and forcing in the plane of the disk is stronger than in the perpendicular direction. The eccentricity evolution is also quite rapid in this case because of the vigorous scattering (the relative horizontal velocity increases by \( \sim \Omega R_H \) in each synodic passage of two bodies initially separated by \( \sim R_H \) in the semimajor axes). Simple reasoning confirmed by numerical experiments suggests the following behavior of scattering coefficients in the shear-dominated regime:

\[
\begin{align*}
H_1(\sigma_x, \sigma_y) & \approx B_1, \\
K_1(\sigma_x, \sigma_y) & \approx D_1 \sigma_x^2, \\
H_2(\sigma_x, \sigma_y) & \approx -B_2 \sigma_x^2, \\
K_2(\sigma_x, \sigma_y) & \approx -D_2 \sigma_x^2,
\end{align*}
\]

where \( B_{1,2}, D_{1,2} \) are some positive constants, which can be fixed using numerical orbit integrations; see Appendix A. A simple qualitative derivation of equations (6) and (9) can be found in Ida & Makino (1993) and Rafikov (2003c) (see also Ohtsuuki et al. 2002).

For further convenience, we now reduce the evolution equations to a dimensionless form. We relate planetesimal masses to the smallest planetesimal mass \( m_0 \) by using a dimensionless quantity \( x = m/m_0 \). Instead of \( N(m) \) we introduce the dimensionless mass distribution function \( f(x) \), such that

\[
N(m) = \frac{\Sigma_p}{m_0^2} f(x), \quad N(m)dm = \frac{\Sigma_p}{m_0} f(x) dx,
\]

where \( \Sigma_p \) is the total mass surface density of planetesimals in the disk, which is a conserved quantity. We also recscale eccentricity and inclination dispersions as follows:

\[
s = \sigma_x \left( \frac{m_0}{M_c} \right)^{-1/3}, \quad s_z = \sigma_y \left( \frac{m_0}{M_c} \right)^{-1/3}.
\]

This is equivalent to rescaling all distances by the Hill radius of the smallest planetesimals. In this notation the boundary between the shear- and dispersion-dominated regimes is given by the conditions

\[
s^2 + s_z^2 \approx (x + x^*)^{2/3}, \quad s^2 + s_z^2 \approx (x + x^*)^{2/3}.
\]

We also introduce the dimensionless time \( \tau \) by

\[
\tau = \frac{t}{t_0}, \quad t_0 = |A|^{-1} \left( \frac{m_0}{\Sigma_p a^3} \right)^{2/3} \left( \frac{m_0}{10^{21} \text{ g}} \right)^{1/3},
\]

where we assumed that \( \Sigma_p (a) = \Sigma_p (a_0) a^{-3/2} \) (Hayashi 1981). The dynamical timescale is rather short when \( s, s_z \sim 1 \), but it grows very rapidly as epicyclic velocities of planetesimals increase \( (t \sim t_0 a^4) \) in the dispersion-dominated regime.

The random velocities of planetesimals of mass \( x \) evolve as a result of the interaction with all other bodies spanning the whole mass spectrum. Using our dimensionless notation, we can rewrite equations (3a), (3b), (6), and (9) in the following general form:

\[
\frac{\partial x^2}{\partial \tau} = \int_1^\infty dx' f(x') x' (x + x^*)^{1/3} \left( \frac{x'}{x + x^*} \right)^{2} H_1 + 2 \frac{s^2}{s^2 + s_z^2} H_2,
\]

(15)

- Protoplanetary disks should contain planetesimals of different sizes, but we assume that bodies lighter than \( m_0 \) are not important for the disk evolution. In the case considered here the choice of \( m_0 \) is dictated by the mass at which the distribution would depart from the given power-law form toward shallower scaling with mass (see also \( \S 4.1.1 \)).
\[
\frac{\partial s^2}{\partial \tau} = \int_1^\infty dx^* f(x^*) x^*(x + x^*)^{1/3} \\
\times \left( \frac{x^*}{x + x^*} K_1 + 2 \frac{s^2}{s^2 + s_{x}^2} K_2 \right),
\]
where \( s \equiv s(x) \), \( s^* \equiv s(x^*) \) and similarly for \( s_c, s_{x}^* \); in the shear-dominated regime \([s^2 + s_{x}^2, s^2 + s_{x}^2 \ll (x + x^*)^{2/3}]\)

\[
H_1 \approx C_1, \quad H_2 \approx -C_2 \frac{s^2 + s_{x}^2}{(x + x^*)^{2/3}}, \quad K_1 \approx D_1 \frac{s^2 + s_{x}^2}{(x + x^*)^{2/3}}, \quad K_2 \approx -D_2 \frac{s^2 + s_{x}^2}{(x + x^*)^{2/3}},
\]
and in the dispersion-dominated regime \([s^2 + s^2, s^2 + s^2 \gg (x + x^*)^{2/3}]\)

\[
\begin{align*}
\left( \frac{H_1}{K_1} \right) & = \frac{A_1}{B_1} \frac{(x + x^*)^{2/3}}{s^2 + s_{x}^2} \ln \Lambda, \\
\left( \frac{H_2}{K_2} \right) & = -\frac{A_2}{B_2} \frac{(x + x^*)^{2/3}}{s^2 + s_{x}^2} \ln \Lambda.
\end{align*}
\]
We explore this system in § 4 using two approaches: asymptotic analysis utilizing analytical methods and the direct numerical calculation of velocity evolution.

3. MASS SPECTRUM

Starting from equations (15)-(18), one would like to obtain the behavior of \( s \) and \( s_c \) as functions of \( x \) and \( \tau \), given some planetesimal mass spectrum \( f(x) \) as an input. In general, to do this one has to evolve equations (15)-(18) numerically. However, it is usually true that the planetesimal size distribution spans many orders of magnitude in mass; thus, one would expect that some general predictions can be made analytically about the asymptotic properties of the planetesimal velocity spectrum.

In this study we assume that the planetesimal mass distribution has a “self-similar” form; specifically, we take

\[
f(x, \tau) = \psi(\tau) \varphi \left( \frac{x}{x_c(\tau)} \right),
\]
where \( x_c(\tau) \gg 1 \) is some fiducial planetesimal mass, which steadily grows in time as a result of coagulation. \( \psi(\tau) \) is a temporal modulation, and the function \( \varphi \) represents a self-similar shape of the mass spectrum. We also assume in this study that asymptotically

\[
\varphi(y) \sim \begin{cases} y^{-\alpha}, & y \ll 1, \\ \exp(-y), & y \gg 1, \end{cases}
\]
where \( \alpha > 0 \) (see Fig. 1). Such scaling behavior is often found in coagulation simulations. We will see in § 4 that, for clarifying the asymptotic properties of the planetesimal velocity spectrum, it is enough to know only the asymptotic behavior of \( \varphi(y) \) and not its exact shape. Moreover, in § 6 we demonstrate how our results for power-law size distributions can be generalized for other mass spectra.

The normalization \( \psi(\tau) \) is not an independent function. It is related to \( x_c(\tau) \) because of the conservation of the total (dimensionless) planetesimal surface density

\[
M_1 = \int_1^\infty x f(x) dx. \tag{21}
\]

Taking this constraint into account, one finds that

\[
\psi(\tau) = \begin{cases} x_{c}^{-2}(\tau), & \alpha < 2, \\ x_{c}^{-4}(\tau), & \alpha > 2. \end{cases} \tag{22}
\]

In the case \( \alpha > 2 \) the mass spectrum behaves as \( f(x, \tau) = x^{-\alpha} \) for \( x \ll x_c(\tau) \); only the position of the high-mass cutoff \( x_c(\tau) \) shifts toward higher and higher masses with time.

We define \( M_{\psi}(\tau) \) to be the \( \psi \)-th-order moment of the mass distribution:

\[
M_{\psi}(\tau) = \int_1^\infty x^{\psi} f(x, \tau) dx. \tag{23}
\]

The integral in (23) is dominated by the upper end of the power-law part of the mass spectrum (i.e., by \( x^* \sim x_c \)), if \( \nu > \alpha - 1 \); in this case we can write (using eqs. [19], [22])

\[
M_{\psi}(\tau) = M_{\psi}[x_c(\tau)]^{\psi+1} \psi(\tau) = M_{\psi} \left\{ x_c^{\psi-1}, \alpha < 2, x_c^{\psi+1-\alpha}, \alpha > 2 \right\},
\]

\[
\tilde{M}_{\psi} = \int_0^\infty y^{\psi} \varphi(y) dy \tag{24}
\]

where \( \tilde{M}_\psi \)—the reduced moment of order \( \nu \)—is a time-independent constant for a given mass distribution.

One of the interesting features exhibited by self-consistent coagulation simulations is the development of the tail of high-mass bodies beyond the cutoff of the bulk distribution of planetesimals (Wetherill & Stewart 1989; Inaba et al.
This is interpreted as the manifestation of the runaway phenomenon taking place in a coagulating system. To explore the possibility of the runaway scenario, we add to the mass spectrum in equation (20) a tail of high-mass planetesimals extending beyond the exponential cutoff $x_c$ (see Appendix A for details). In doing this we always make sure that the runaway tail contains a negligible part of the system’s mass and does not affect its dynamical state (i.e., the stirring and friction caused by these high-mass planetesimals are small), which is true if the biggest bodies are not too massive (Rafikov 2003c). Under these assumptions the explicit functional form of the runaway tail is completely unimportant. The evolutionary sequence of the typical mass distribution studied here is shown schematically in Figure 1. Throughout this study we use the term “planetesimals” for bodies belonging to the power-law part of the spectrum ($x \leq x_c$) and “massive” or “runaway” bodies to denote the constituents of the tail ($x \gg x_c$).

We do not restrict the variety of possible functional dependences of $x_c$ on time $\tau$. It will turn out that all our results can be expressed as some function of $x_c$, which leads to the time dependence of planetesimal velocities in a very general form. In some cases it will be important that $x_c(\tau)$ does not grow too fast, which, however, we believe is a fairly weak constraint (see the discussion in § 4.1.2).

4. VELOCITY SCALING LAWS

To facilitate our treatment of planetesimal velocities, we introduce a set of simplifications into our consideration as follows:

1. We split the mass spectrum into several regions, such that in each of them planetesimal interactions can be considered as occurring in a single dynamical regime (shear-dominated or dispersion-dominated). Transitions between such regions are not considered, but in principle might be treated by interpolation.

2. When studying the dispersion-dominated regime, we neglect the difference between the vertical and horizontal velocity dispersions and treat them by a single equation. We also set the Coulomb logarithm equal to a constant, because of its rather weak dependence on $s$, $s_r$, or $x$.

The validity and impact of these assumptions on the velocity spectrum are further checked using numerical techniques. Since we are mostly interested in the qualitative behavior of the mass spectrum, $\alpha < 2$, intermediate mass spectrum, $2 < \alpha < 3$, and steep spectrum, $\alpha > 3$ (this regime splits into two more important subcases; see § 4.3). Note that, for the sake of avoiding additional complications, we do not consider the borderline cases $\alpha = 2, 3$; they can be easily studied in the framework of our approach if the need arises.

In our analytical work planetesimals are started with large enough $s$ and $s_r$ so that they interact with each other in the dispersion-dominated regime. We then also assume that planetesimals with masses $x \leq x_c(\tau)$ (containing most of the mass) stay in the dispersion-dominated regime with respect to each other at a later time as well; i.e.,

$$ s(x, \tau), s_r(x, \tau) \gg x^{1/3} \text{ for } x \leq x_c(\tau) . \quad (25) $$

This is a reasonable assumption for all mass spectra at the beginning of the evolution, although, for steep size distributions it may break down when the maximum planetesimal mass becomes very large (see § 4.3).

4.1. Shallow Mass Spectrum

Planetesimal mass distributions shallower than $m^{-2}$ result from the coagulation of high-velocity planetesimals when gravitational focusing is unimportant and the collision rate is determined by the geometrical cross-section of the colliding bodies (Wetherill & Stewart 1993; Kenyon & Luu 1998). It is worth remembering, however, that in highly dynamically excited disks (1) the energy dissipation in inelastic collisions must be important (see § 5) and (2) planetesimal fragmentation cannot be ignored. Despite that, the case of a shallow mass spectrum is interesting because it facilitates the understanding of disks with other planetesimal size distributions.

4.1.1. Velocities of Planetesimals

We start by considering planetesimals $x \leq x_c(\tau)$—the part of the size distribution containing most of the mass. Regarding them as interacting in the dispersion-dominated regime (i.e., condition [25] is fulfilled) we may write, using equations (15) and (18), that

$$ \frac{\partial s^2}{\partial \tau} = \int_1^\infty dx f(x) \frac{x^*(x + x^*)}{s^2 + s^{*2}} \times \left( \frac{x^*}{x + x^*} A_1 - 2 \frac{s^2}{s^2 + s^{*2}} A_2 \right) \quad (26) $$

for $x \ll x_c(\tau)$ (the Coulomb logarithm is set to constant and absorbed into coefficients $A_{1,2}$). Since we use a single equation to describe the evolution of both $s$ and $s_r$, the values of constants $A_{1,2}$ are not well defined, but this is not important for deriving general properties of the velocity spectrum.

We can represent (26) asymptotically in the following form:

$$ \frac{\partial s^2}{\partial \tau} \simeq A_1 \int_1^{\infty} dx \frac{x^2 f(x^*)}{s^2 + s^{*2}} - 2 A_2 s^2 \int_1^{\infty} dx \frac{x^2 f(x^*)}{(s^2 + s^{*2})^2} + \int_1^{\infty} dx \frac{x^2 f(x^*)}{s^2 + s^{*2}} \left( A_1 - 2 \frac{s^2}{s^2 + s^{*2}} A_2 \right) . \quad (27) $$

The first two terms on the right-hand side represent, correspondingly, the gravitational stirring and friction by bodies smaller than $x$. The last term is responsible for the combined effect of the stirring and friction produced by bodies bigger than $x$. From this equation it is easy to see that $s(\tau)$ independent of $x$ is a legitimate solution of (27) in the case $\alpha < 2$. Indeed, assuming $s$ to be independent of $x$, we find that all integrals over $x^*$ in (27) are dominated by their upper limits when the mass spectrum is shallow. Using
equations (19)–(24), we can estimate that
\[
\frac{\partial^2 \mathbf{x}}{\partial \tau^2} \approx A_1 \frac{\mathbf{x}_c}{2\mathbf{x}^3} \left( \frac{\mathbf{x}}{\mathbf{x}_c} \right)^3 - 2A_2 \frac{\mathbf{x}_c}{4\mathbf{x}^3} \left( \frac{\mathbf{x}}{\mathbf{x}_c} \right)^3 + (A_1 - A_2) \frac{M_2(\tau)}{2\mathbf{x}^2}. \tag{28}
\]

Note that, in deriving this expression, we have extended the integration range of the last term in equation (27) from 1 to infinity (not from \(\sim \mathbf{x}\)). This is legitimate because this integral is dominated by the contribution coming from \(\sim \mathbf{x}_c \gg \mathbf{x}\) and, thus, adding the interval from 1 to \(\sim \mathbf{x}\) to the integration range would introduce only a subdominant contribution to the final answer. Clearly, the first two terms in (28) are negligible compared with the third one for \(\mathbf{x} \ll \mathbf{x}_c\), (see eq. [24]) and the third term is positive (see eq. [8]) and independent of \(\mathbf{x}\), meaning that \(s(\tau)\) is also independent of mass, in accord with our assumption. One can then easily find that\(^5\) (dropping the constant factors \(A_1, A_2\) but bearing in mind condition [8])
\[
s(\mathbf{x}, \tau) \approx s_0 \equiv \left[ \int_0^\tau M_2(\tau') d\tau' \right]^{1/4} = \left[ \mathcal{M}_2 \int_0^\tau \mathbf{x}_c(\tau') d\tau' \right]^{1/4}, \quad \alpha < 2, \quad \mathbf{x} \ll \mathbf{x}_c \tag{29}
\]

[if \(s(\mathbf{x}, \tau) \gg s(\mathbf{x}, 0)\)]. The second equality holds only for \(\alpha < 2\), according to equations (22) and (24). Thus, in the case of a mass spectrum shallower than \(\mathbf{x}^{-2}\), random velocities are independent of planetesimal masses and uniformly grow with time. Both gravitational stirring and friction are dominated by the biggest planetesimals \((\mathbf{x} \sim \mathbf{x}_c)\), with stirring being more important; friction by small bodies is completely negligible, and energy equipartition between planetesimals of different mass is not reached. For this reason, the planetesimal velocity expressed in physical units using equations (11), (13), and (29) is independent of the choice of \(m_0\). A schematic representation of the velocity spectrum for the shallow mass distribution is displayed in Figure 2.

4.1.2. Velocities of Runaway Bodies

Now we turn our attention to the runaway bodies \(\mathbf{x} \gtrsim \mathbf{x}_c\). The presence of the runaway tail in the case of a shallow mass spectrum might seem rather artificial, since such mass distributions usually originate in the course of orderly planetesimal coagulation when gravitational focusing is weak and runaway bodies do not form (Safro nov 1962; Wetherill & Stewart 1989; Kenyon & Luu 1998). However, the shallow part of the mass spectrum can appear after the runaway tail formation in cases with \(\alpha > 2\) (see §§ 4.2 and 5) as a result of vigorous stirring by the largest planetesimals, which raises the velocities of the small planetesimals above their escape velocities and reduces the role of gravitational focusing, validating the following discussion. By assumption, the runaway tail contains so little mass that it cannot affect the velocities not only of the planetesimals but also of the runaway bodies themselves.

Because of our assumption of the dispersion-dominated interaction between all planetesimals with \(\mathbf{x} \ll \mathbf{x}_c\), it is natural to expect that at least those runaway bodies that are not too heavy still experience dispersion-dominated scattering by planetesimals with masses \(\lesssim \mathbf{x}_c\). Assuming that the velocity dispersion of these runaway bodies is much smaller than that of the bulk of planetesimals—a natural consequence of the tendency to equipartition of random epicyclic energy between bodies of different mass—we can rewrite equation (26) as
\[
\frac{\partial \mathbf{s}^2}{\partial \tau^2} \approx A_1 \int_1^\infty d\mathbf{x}^* \frac{\mathbf{x}^{*2} f(\mathbf{x}^*)}{s^{*2}} - 2A_2 s^2 \mathbf{x} \int_1^\infty d\mathbf{x}^* \frac{\mathbf{x}^{*2} f(\mathbf{x}^*)}{s^{*4}}. \tag{30}
\]

Although the integration range of all terms in the right-hand side is extended to infinity, the exponential cutoff of \(f(\mathbf{x})\) at \(\sim \mathbf{x}_c\) effectively restricts the integration range to be from 1 to \(\sim \mathbf{x}_c\).

One can easily see that \(s(\mathbf{x}, \tau) = \text{const}(\mathbf{x})\) cannot be a solution of (30) because the dynamical friction is too strong. Indeed, if we were to assume the opposite, we would find that \(s(\mathbf{x}, \tau)\) is still given by equation (29). However, the direct substitution of (29) into (30) shows that the gravitational friction term exceeds other contributions for \(\mathbf{x} \gtrsim \mathbf{x}_c\). This might tempt one to suggest that the heating (first) term in the right-hand side of (30) should be neglected compared with the friction (second) term. In this case one would find that \(s(\mathbf{x}, \tau)\) exponentially decays in time, and very soon the heating term

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\(^5\) Throughout the paper we imply by \(s_0\) the velocity dispersion of the smallest planetesimals, and \(s_0\) is different for different mass spectra; see eqs. (40) and (53).
becomes the dominant one, contrary to the initial assumption. These results suggest the only remaining possibility: that the heating of massive bodies by planetesimals almost exactly balances the friction by planetesimals, and the minute difference between them is accounted for by the time dependence of $s(x, \tau)$ embodied in the left-hand side of equation (30). Given that this assumption is correct, we immediately find that

$$s(x, \tau) \approx \frac{1}{\sqrt{x}} \left[ \int_{1}^{\infty} dx^* \frac{x^{s_0} f(x^*)}{s^{s_0^2}} \right]^{1/2} \times \left[ \int_{1}^{\infty} dx^* \frac{x^{s_0} f(x^*)}{s^{s_0^2}} \right]^{-1/2}.$$

(31)

It is easy to see that scaling $s \propto x^{-1/2}$ follows directly from assuming that (1) the interaction is in the dispersion-dominated regime and (2) the heating of runaway bodies by planetesimals is balanced by friction also due to planetesimals. This specific result is completely independent of the mass or velocity spectra of small bodies. We will highlight this again when we obtain similar results for intermediate and steep mass spectra in § 4.2.2 and Appendix C.2. For the specific case $\alpha < 2$ considered in this section, $s^* = s_0(\tau)$, and one finds that

$$s(x, \tau) \approx \frac{s_0(\tau)}{\sqrt{x}} \left[ \frac{M_2(\tau)}{M_1} \right]^{1/2} = s_0 \left( \frac{x_c}{x} \right)^{1/2} \left( \frac{M_2}{M_1} \right)^{1/2},$$

$$\alpha < 2, \quad x_c \lesssim x \lesssim x_{\text{shear}}$$

(32)

($x_{\text{shear}}$ is defined below in eq. [33]). Clearly, $s(x, \tau) \sim s_0(\tau)$, in agreement with equation (29).

The time dependence enters (32) eventually through $x_c(\tau)$. Thus, it is the behavior of $x_c(\tau)$ that determines $\partial s^2/\partial \tau$ in equation (30). The timescale on which $x_c$ varies is usually much longer than the dynamical relaxation time of the system; thus, one would expect the left-hand side of equation (30) to be small, in accordance with the assumptions that led to equation (31). The requirement of the negligible effect of time derivatives in situations like the one described here is the only constraint that must be imposed on the possible behavior of $x_c(\tau)$.

As we consider even bigger $x$, we find that, at some mass, runaway bodies start interacting with bodies of similar mass in the shear-dominated regime; this happens when $x^{1/3} \sim s_0(x_c/x)^{1/2}$, or $x \sim x_c(s_0/x_c^{1/3})^{6/5}$. This, however, does not affect the dynamics of these bodies, because by our assumption the runaway tail contains too little mass to dynamically affect its own constituents. It is only after the runaway bodies start interacting in the shear-dominated regime with planetesimals (i.e., bodies lighter than $x_c$) that the velocity evolution of the tail gets affected. In our case this clearly happens when $x^{1/3} \sim s_0(\tau)$. Thus, runaway bodies with masses

$$x \gtrsim x_{\text{shear}} \equiv s_0^3$$

(33)

interact with all planetesimals in the shear-dominated regime.

As mentioned in § 2, in the shear-dominated case eccentricities and inclinations may no longer evolve along similar routes and we have to consider them separately. Using equations (15)–(17), we write

$$\frac{\partial s^2}{\partial \tau} = \int_{1}^{\infty} dx^* f(x^*) \left( x + x^* \right)^{1/3} \times \frac{C_1}{C_2} \left[ \frac{s^2}{x + x^*} - 2C_2 \frac{s^2}{(x + x^*)^{2/3}} \right],$$

$$\frac{\partial x^2}{\partial \tau} = \int_{1}^{\infty} dx^* f(x^*) x^* \left( s^2 + s^* \right)^{2/3},$$

$$\frac{\partial s_2}{\partial \tau} = \int_{1}^{\infty} dx^* f(x^*) x^* \left( s_2^2 + s^* \right)^{1/3} \times \frac{D_1}{D_2} \left[ \frac{s^2}{x + x^*} - 2D_2 \frac{s^2}{s_2^2 + s^*} \right].$$

(34)

(35)

Assuming that $s, s_c \ll s^* \approx s_0^3$ (because of the gravitational friction) and $x \gg x^*$ (both heating and cooling are dominated by planetesimals, not runaway bodies), and neglecting the left-hand side of both equations (similar to the previously discussed situation for $x_c \lesssim x \lesssim x_{\text{shear}}$), we obtain the following result:

$$s(x, \tau) \approx x^{-1/6} \left( \frac{M_2(\tau)}{M_1} \right)^{1/2},$$

$$s_c(x, \tau) \approx \frac{1}{\sqrt{x}} \left[ \frac{M_2}{M_1} \int_{1}^{\infty} dx^* x^{s_0^2} f(x^*) \right]^{1/2}.$$

(36)

Finally, setting $s^* \approx s_0(\tau)$ as appropriate for $\alpha < 2$, we find that

$$s(x, \tau) \approx x^{-1/6} \left( \frac{M_2}{M_1} \right)^{1/2} x_c^2,$$

$$s_c(x, \tau) \approx s_0 \left( \frac{x_c}{x} \right)^{1/2} \left( \frac{M_2}{M_1} \right)^{1/2}, \quad \alpha < 2, \quad x \gtrsim x_{\text{shear}}.$$ 

(37)

These expressions demonstrate that the transition to the shear-dominated regime is accompanied by a change of the power-law index of eccentricity scaling with mass, while the power-law slope of the inclination scaling stays the same. This happens because in the case of inclination evolution the heating term is functionally similar to the friction term (exactly as in the dispersion-dominated case), which is not the case for eccentricity evolution (see eq. [17]). As a result, for $x \gtrsim x_{\text{shear}}$ one finds that $s \gg s_c$—the planetesimal velocity ellipsoid flattens considerably in the vertical direction, which is very unlike the dispersion-dominated case (see Fig. 2). This conclusion is very general and should be found whenever runaway bodies interact with all planetesimals in the shear-dominated regime.

4.1.3. Comparison with Numerical Results

To check the prediction of our asymptotic analysis, we have performed numerical calculations of planetesimal velocity evolution for a distribution of planetesimal masses that was artificially evolved in time according to the prescriptions outlined in § 3. The details of our numerical procedure can be found in Appendix A.

In Figure 3 we show the evolution of planetesimal eccentricity and inclination dispersions for a shallow mass spectrum with $\alpha = 1.5$. For each curve displaying an $s$ or $s_c$, run with $x$ at a specific moment of time, we also computed a power-law spectral index $\eta \equiv d \ln s/d \ln x$. We then compare the power-law indexes $\eta_1$ and $\eta_2$ with the analytical
Different curves correspond to increasing values of maximum planetesimal mass $x_c$. The velocity spectrum is plotted as a function of dimensionless mass $x$, while in the top panels the power-law index $\eta = d \ln \sigma / d \ln x$ of the spectrum is displayed. Different curves correspond to increasing values of maximum planetesimal mass $x_c$ (and, consequently, time) indicated by arrows of corresponding line type in the top panels. The dotted line in the bottom panels is $x = x_c^{1/3}$ (the boundary between different velocity regimes). The dotted curve in the top panels is the theoretical prediction for the run of the spectral index $\eta$ corresponding to the largest $x_c$ displayed (for which the numerical result is always shown by a thick solid line). These results are to be compared with the theoretical predictions shown in Fig. 2.

The dotted lines in the upper panels of Figure 3 represent the analytical predictions for $\eta$ and $\eta_z$ at the moment when the snapshot of the velocity spectrum displayed by the solid line (corresponding to the highest $x_c$ shown) was taken. One can see that the agreement between two approaches is quite good. The power-law slopes of both eccentricity dispersion $\eta$ and inclination $\eta_z$ exhibit a well-defined plateau at masses considerably smaller than $x_c$, which is predicted by equation (29). Above the upper mass cutoff $\eta$ and $\eta_z$ plunge toward $-1/2$, which is in agreement with equation (32), although they do not follow this value very closely. The reason is most likely the lack of dynamical range—transitions between different regimes typically occupy substantial intervals in mass. Finally, at $x$ above $x_{\text{shear}}$, where runaway bodies interact with all planetesimals in the shear-dominated regime, $\eta$ goes up to $-1/6$, while $\eta_z$ stays at a value of $-1/2$, exactly as equation (37) predicts. Glitches in the curves of $\eta$ and $\eta_z$ in the vicinity of $x_{\text{shear}}$ are artifacts of the interpolation of scattering coefficients between the shear- and dispersion-dominated regimes.

In Figure 4a we display the time evolution of $s_0$—the eccentricity dispersion of the smallest planetesimals together with the analytical prediction for $s_0(\tau)$ given equation (29). One can see that the agreement between the two approaches is excellent—the analytical and numerical results closely follow each other as time increases. The vertical offset between them amounts to a constant factor by which the numerically determined $s_0(\tau)$ differs from the analytical $s_0(\tau)$. This, of course, is expected because we completely ignored constant coefficients in our asymptotic analysis. The numerical results shown in Figure 4a allow us to fix a constant coefficient in equation (29), and we do this when we calculate $x_{\text{shear}}$ in Figure 3 using equation (33). Note that the rapid growth of $s_0$ with time should certainly increase the velocities of planetesimals beyond their escape speeds (if it were not the case initially). The dynamical evolution in this case is discussed in more detail in § 5.

An apparent break in the behavior of $s_0(\tau)$ at $\tau \approx 10^5$ can be seen in Figure 4a. It occurs because this is the time at which the mass scale $x_c$ starts to grow: the prescription for $x_c(\tau)$ that we use (described in Appendix A) is such that $x_c$ is almost constant for $\tau \lesssim 10^5$; as a result, $s_0 \propto \tau^{1/4}$ according to equation (29). At $\tau \gtrsim 10^5$ the scale factor starts to increase rapidly and this causes $s_0$ to grow faster. To conclude, the results of this section show that the overall agreement between the numerical calculations and the analytical theory of planetesimal velocity evolution is rather good.

4.2. Intermediate Mass Spectrum

Self-consistent coagulation calculations (Kenyon & Luu 1998) and N-body simulations of planet formation (Kokubo & Ida 1996, 2000) often yield intermediate mass distributions of planetesimals ($2 < \alpha < 3$). Kokubo & Ida (1996) have found, for example, $\alpha \approx 2.5 \pm 0.4$ in their N-body calculation of the collisional evolution of several thousand massive ($m = 10^{23}$ g) planetesimals. Such an outcome seems to be ubiquitous for planetesimals that agglomerate under the action of strong gravitational focusing (planetesimal velocities are well below the escape velocities from their surfaces) in the dispersion-dominated regime, which makes the case of intermediate mass spectra very important.

4.2.1. Velocities of Planetesimals

As in § 4.1 we start our consideration of mass spectra with $2 < \alpha < 3$ by concentrating on planetesimals with masses
true provided that
\[ x \leq x_k(\tau) = \frac{M_2(\tau)}{M_1} = x_c \frac{\dot{M}_2}{\dot{M}_1} x^{2-\alpha} \leq x_c . \]  
(39)

The last equality follows from the fact that \( M_2 = \dot{M}_2 x^{2-\alpha} \) when \( 2 < \alpha < 3 \). Thus, for small planetesimals (\( x \leq x_k \)) friction by even smaller ones is again unimportant; their velocities are excited by big planetesimals \( x \sim x_c \), and as a result (cf. eq. [29])

\[ s(x, \tau) \approx s_0(\tau) = \left( \frac{\dot{M}_2}{\dot{M}_1} \int_{x_k}^{\tau} [x(\tau')]^{3-\alpha} d\tau' \right)^{1/4} = \text{const}(x) , \]
\[ 2 < \alpha < 3, \quad x \leq x_k . \]  
(40)

However, for \( x \gtrsim x_k \), gravitational friction by small planetesimals is no longer negligible compared with the velocity excitation by big ones.

One would then expect the friction to lower the velocities of big bodies below those of small planetesimals (because of the tendency to energy equipartition associated with the friction term). Starting with equation (27) and using (40), we obtain the following equation for \( x \gtrsim x_k \):

\[ \frac{\partial s^2}{\partial \tau} = \frac{A_1}{s_0^2} \int_{x_k}^{x} dx^* (x^*)^2 f(x^*) + A_1 \int_{x_k}^{\infty} dx^* (x^*)^2 f(x^*) \]
\[ - 2A_2 x^{2-\alpha} \int_{x_k}^{x} dx^* x^2 f(x^*) \]
\[ - 2A_2 x^{2-\alpha} \int_{x_k}^{\infty} dx^* x^2 f(x^*) \]
\[ + \frac{A_1}{s_0^2} \int_{x_k}^{\infty} dx^* (x^*)^2 f(x^*) . \]  
(41)

In this equation different terms represent stirring or friction by different parts of the planetesimal mass spectrum, below and above \( x_k \). We now take an educated guess that the solution for \( s \) can be obtained by equating the third and fifth terms in the right-hand side of (41). Physically, this means that the velocity spectrum results from the balance of velocity stirring by the biggest planetesimals (masses \( \sim x_c \gg x \), similar to the case of the shallow mass spectrum) and friction which is mainly contributed by the smallest planetesimals. One then finds that (see eq. [39])

\[ s(x, \tau) \approx s_0(\tau) \left( \frac{M_2(\tau)}{M_1} \right)^{1/4} x^{-1/4} \approx s_0(\tau) \left( \frac{x_k(\tau)}{x} \right)^{1/4} , \]
\[ 2 < \alpha < 3, \quad x_k \leq x \leq x_c \].  
(42)

Substituting this solution into (41) we easily verify that our neglect of all other terms in the right-hand side of (41) is indeed justifiable. The left-hand side of (41) can be neglected for \( x_k \leq x \leq x_c \) on the basis of arguments identical to those advanced in \( \S \) 4.1.2. Thus, \( s \), given by (42), is indeed a legitimate solution in this mass range.

Equations (40) and (42) highlight an interesting feature of the case \( 2 < \alpha < 3 \): although the mass spectrum has a power-law form with a continuous slope all the way up to \( x_c \), the velocity spectrum exhibits a knee at some intermediate mass \( x_k \) (see Fig. 5). The resemblance of the velocity spectrum for \( x \leq x_k \) to the spectrum for \( \alpha < 2 \) is due to the fact that, in both cases, stirring is done by the biggest planetesimals and

\[ x \leq x_k(\tau) = \frac{M_2(\tau)}{M_1} = x_c \frac{\dot{M}_2}{\dot{M}_1} x^{2-\alpha} \leq x_c . \]  
(39)

The last equality follows from the fact that \( M_2 = \dot{M}_2 x^{2-\alpha} \) when \( 2 < \alpha < 3 \). Thus, for small planetesimals (\( x \leq x_k \)) friction by even smaller ones is again unimportant; their velocities are excited by big planetesimals \( x \sim x_c \), and as a result (cf. eq. [29])

\[ s(x, \tau) \approx s_0(\tau) = \left( \frac{\dot{M}_2}{\dot{M}_1} \int_{x_k}^{\tau} [x(\tau')]^{3-\alpha} d\tau' \right)^{1/4} = \text{const}(x) , \]
\[ 2 < \alpha < 3, \quad x \leq x_k . \]  
(40)

However, for \( x \gtrsim x_k \), gravitational friction by small planetesimals is no longer negligible compared with the velocity excitation by big ones.

One would then expect the friction to lower the velocities of big bodies below those of small planetesimals (because of the tendency to energy equipartition associated with the friction term). Starting with equation (27) and using (40), we obtain the following equation for \( x \gtrsim x_k \):

\[ \frac{\partial s^2}{\partial \tau} = \frac{A_1}{s_0^2} \int_{x_k}^{x} dx^* (x^*)^2 f(x^*) + A_1 \int_{x_k}^{\infty} dx^* (x^*)^2 f(x^*) \]
\[ - 2A_2 x^{2-\alpha} \int_{x_k}^{x} dx^* x^2 f(x^*) \]
\[ - 2A_2 x^{2-\alpha} \int_{x_k}^{\infty} dx^* x^2 f(x^*) \]
\[ + \frac{A_1}{s_0^2} \int_{x_k}^{\infty} dx^* (x^*)^2 f(x^*) . \]  
(41)

In this equation different terms represent stirring or friction by different parts of the planetesimal mass spectrum, below and above \( x_k \). We now take an educated guess that the solution for \( s \) can be obtained by equating the third and fifth terms in the right-hand side of (41). Physically, this means that the velocity spectrum results from the balance of velocity stirring by the biggest planetesimals (masses \( \sim x_c \gg x \), similar to the case of the shallow mass spectrum) and friction which is mainly contributed by the smallest planetesimals. One then finds that (see eq. [39])

\[ s(x, \tau) \approx s_0(\tau) \left( \frac{M_2(\tau)}{M_1} \right)^{1/4} x^{-1/4} \approx s_0(\tau) \left( \frac{x_k(\tau)}{x} \right)^{1/4} , \]
\[ 2 < \alpha < 3, \quad x_k \leq x \leq x_c \]  
(42)

Substituting this solution into (41) we easily verify that our neglect of all other terms in the right-hand side of (41) is indeed justifiable. The left-hand side of (41) can be neglected for \( x_k \leq x \leq x_c \) on the basis of arguments identical to those advanced in \( \S \) 4.1.2. Thus, \( s \), given by (42), is indeed a legitimate solution in this mass range.

Equations (40) and (42) highlight an interesting feature of the case \( 2 < \alpha < 3 \): although the mass spectrum has a power-law form with a continuous slope all the way up to \( x_c \), the velocity spectrum exhibits a knee at some intermediate mass \( x_k \) (see Fig. 5). The resemblance of the velocity spectrum for \( x \leq x_k \) to the spectrum for \( \alpha < 2 \) is due to the fact that, in both cases, stirring is done by the biggest planetesimals and
Fig. 5.—Same as Fig. 2, but for the intermediate mass spectrum. The specific case of $2 < \alpha < 5/2$ is displayed. The notation is explained in the text. The analogous plot for $5/2 \leq \alpha < 3$ would exhibit different behavior of $s_c$ in the high-mass tail and can be easily constructed using the results of § 4.2.2.

friction has a small effect. However, for $x \gtrsim x_h$ the effect of friction becomes important, and now it is dominated by a part of mass spectrum (the smallest planetesimals) different from that responsible for stirring (the biggest planetesimals). This causes the velocity spectrum to change its slope from 0 to $-1/4$ in this mass range (see Fig. 5).

### 4.2.2. Velocities of Runaway Bodies

Similar to the case of the shallow mass spectrum, some (the lightest) runaway bodies interact with the bulk of planetesimals in the dispersion-dominated regime. In this case all the considerations of § 4.1.2 hold; i.e., the evolution equation (30) stays unchanged, and following the same line of argument we arrive at expression (31) for the planetesimal velocity dispersion. Integrals entering (31) must be reevaluated anew using equations (40) and (42) appropriate for $2 < \alpha < 3$. Performing this procedure and substituting the resulting expressions into (31), we find that

$$s(x, \tau) \approx \frac{s_0}{\sqrt{x}} \left[ \frac{\bar{M}_{\text{s}}(\tau)}{M_1 x_k^{1/2} \bar{M}_{\text{s}}(\tau)} \right]^{1/2} = s_0 \left( \frac{x_c}{x} \right)^{1/2} \left( \frac{\bar{M}_{\text{s}}}{M_1} \sqrt{\frac{x_k}{x_c}} \right)^{1/2},$$

$$2 < \alpha < 3, \quad x_c \lesssim x \lesssim x_h.$$  

(43)

This velocity distribution is valid only for runaway bodies lighter than

$$x_k \equiv \left[ s(x_c) \right]^{3/4} \approx s_0^{3} \left( \frac{x_k}{x_c} \right)^{3/4},$$

(44)

because only such bodies interact with all small-mass planetesimals ($x \lesssim x_c$) in the dispersion-dominated regime. In agreement with what we found in § 4.1.2, the velocity dispersion decreases with mass as $x^{-1/2}$. At $x \sim x_c$ this expression matches the low-mass result in equation (42).

Runaway bodies heavier than $x_h$ but still lighter than $x_{\text{shear}}$ are in a mixed state; they interact with the most massive planetesimals in the shear-dominated regime and with lighter ones in the dispersion-dominated regime. This, of course, is a direct consequence of the fact that the velocity spectrum of planetesimals has a knee at $x_k$; see § 4.2.1. Introducing

$$x_k(x) \equiv x_k \left( \frac{s_0}{x^{1/3}} \right)^4,$$  

(45)

we find that runaway bodies with masses $x$ between $x_k$ and $x_{\text{shear}}$ interact with planetesimals less massive than $x_k(x)$ in the dispersion-dominated regime, and with planetesimals heavier than $x_k(x)$ in the shear-dominated regime. Self-consistent analysis of the planetesimal velocity spectrum for $x_h \lesssim x \lesssim x_{\text{shear}}$ is described in detail in Appendix B, and only the final results are shown here. It turns out that the eccentricity scales as

$$s(x, \tau) \approx \frac{s_0}{x^{5/6}} \left[ \frac{2 M_2(\tau)}{M_1} \right]^{1/2} = \frac{s_0}{x^{5/6}} x_k^{1/2},$$

$$2 < \alpha < 3, \quad x_h \lesssim x \lesssim x_{\text{shear}}.$$  

(46)

while the inclination behaves in a different manner:

$$s_c(x, \tau) \approx \frac{s_0}{x^{7/6}} \left[ \frac{4 M_3(\tau)}{M_1} \right]^{1/2} \frac{x_k^{1/2} M_3}{M_2}, \quad \alpha < \frac{5}{2},$$  

(47)

$$s_z(x, \tau) \approx \frac{s_0}{x^{7/6}} \left[ \frac{4 M_3(\tau)}{M_1} \right]^{1/2} x^{(4\alpha-17)/6}, \quad \alpha > \frac{5}{2}.$$  

(48)

In the case of $\alpha = 5/2$ a more general expression for $s_z$ following directly from equation (B4) should be used.

Finally, bodies heavier than $x_{\text{shear}}$ interact with all planetesimals in the shear-dominated regime. The expressions in equation (36) adequately describe velocity behavior in this case. Manipulating them with the aid of equations (40) and (42) we find that

$$s(x, \tau) \approx \frac{x_k^{1/2}}{x^{1/6}}, \quad 2 < \alpha < 3, \quad x \gtrsim x_{\text{shear}},$$  

(49)

while

$$s_c(x, \tau) \approx \frac{x_k^{1/2}}{x} \left[ \frac{x_k}{x_c} \right]^{1/2} \left( \frac{M_3}{M_2} \right)^{1/2}, \quad \alpha < \frac{5}{2},$$  

(50)

$$s_z(x, \tau) \approx s_0 \left( \frac{x_k}{x} \right)^{1/2} \left( \frac{x_k}{x_c} \right)^{(3-\alpha)/2}, \quad \alpha > \frac{5}{2}.$$  

(51)

Scalings of velocity dispersions with mass $x$ in (49)–(51) are the same as in the case of the shallow mass spectrum, but the time dependences are different. The velocity behavior in the case of the intermediate mass spectrum is displayed in Figure 5.

### 4.2.3. Comparison with Numerical Results

We calculated numerically the evolution of $s$ and $s_c$ for two intermediate mass spectra: $\alpha = 2.3$ and $\alpha = 2.7$. In
Figures 4b and 4c we compare numerically calculated \( s_0(C^2) \)—the eccentricity dispersion of the smallest planetesimals—with predictions of our asymptotic theory. As in the case of the shallow mass spectrum discussed in § 4.1.3, the agreement between the two approaches is very nice. This allows us to fix the coefficient in equation (40), the absence of which in our asymptotic analysis causes a constant offset between numerical and analytical curves in Figures 4b and 4c. The results for the velocity dependences on mass at different moments of time and their comparison with the analytical theory of intermediate mass spectra outlined above are shown in Figures 6 and 7.

In the case of \( \alpha = 2.3 \) one can clearly see the appearance of all features we described in §§ 4.2.1 and 4.2.2. Indeed, in Figure 6 one can observe both the zero-slope part of the spectrum at \( x \lesssim x_k \) and the trend of reaching the slope \(-1/4\) for \( x_k \lesssim x \lesssim x_c \); the last feature is not fully developed when we stop our calculation, but it is almost certain that \( \eta \)
and $\eta_k$ would reach slope $-1/4$ given enough time and mass range. The calculation has to be stopped when $x$ reaches $\sim 10^8$ because later on the biggest planetesimals start interacting with bodies of similar size in the shear-dominated regime. Such a possibility was not explicitly considered in the present study (but it can be easily dealt with using the analytical apparatus developed in this work).

The results for the biggest runaway bodies ($x \gtrsim x_{\text{shear}}$) are in nice concordance with the analytical predictions given by equation (51). In the range $x_c \lesssim x \lesssim 2_x \text{shear}$ velocities of the runaway bodies clearly have not yet converged to a steady state but are evolving in the right direction. The range of masses where the velocity spectrum should exhibit a slope $-1/2$ ($x_c \lesssim x \lesssim x_k$, see eq. [43]) is so narrow\(^6\) for the last (solid) curve (because planetesimals with mass $\sim x_c$ are already very close to shear-dominated regime) that we do not see this regime realized. At the same time, it is conceivable that runaway bodies that interact with planetesimals in the mixed regime ($x_k \lesssim x \lesssim x_{\text{shear}}$; see Appendix B) will finally reach their asymptotic velocity state with $\eta = -5/6$ and $\eta_k = -7/6$ (see eq. [48]), although it will take a very long time for them to get there.

The comparison for the case $\alpha = 2.7$ (Fig. 7) is very similar. All major features corresponding to the intermediate mass spectrum can be traced in this case as well. Unfortunately, the comparison with analytical results is complicated by the fact that, in this case, we have to stop the numerical calculation even earlier than in the $\alpha = 2.3$ case to avoid bringing the most massive planetesimals into the shear-dominated regime (for this reason the mass scale in Fig. 7 does not go as far as in Fig. 6). As a result, even a velocity plateau at small masses does not have time to fully develop, not speaking of runaway bodies in “mixed” interaction state. Despite this, the overall agreement between the numerical results and asymptotic theory of the velocity evolution of intermediate mass distributions is rather good, especially if we make allowance for the short duration of our calculation and the small mass range of the planetesimals.

### 4.3. Steep Mass Spectrum

Planetesimal spectra decreasing with mass steeper than $m^{-3}$ are usually not found in calculations of planetesimal agglomeration. Still, this case is quite interesting, and we briefly discuss it here.

As demonstrated in §§ 4.1 and 4.2, the stirring of planetesimals is determined solely by the biggest bodies with masses $\sim x_c$ when $\alpha < 3$. This is no longer true when the mass spectrum becomes steeper than $x^{-3}$. Indeed, let us consider the evolution of $s_0$—the velocity dispersion of the smallest planetesimals having a dimensionless mass $x = 1$. Using equation (26) we find that

$$
\frac{\partial s_0^2}{\partial \tau} \approx \frac{1}{s_0^2} \int_1^\infty dx^* f(x^*) \frac{x^*(x + x^*)}{1 + s^{2*/s_0^2}} \times \left( \frac{x^*}{x + x^*} A_1 - \frac{1}{1 + s^{2*/s_0^2}} A_2 \right). \quad (52)
$$

Assuming that $s^*/s_0 \to 0$ when $x^*/x \to \infty$ as a result of gravitational friction we find that the integral in (52) is dominated by its lower integration limit for $\alpha > 3$. This means that the velocity of the smallest planetesimals is now mediated by the smallest planetesimals themselves, unlike the case of $\alpha < 3$. We also find that the temporal behavior of $s_0$ is given simply by

$$
s_0(\tau) \approx \tau^{1/4}, \quad \alpha > 3. \quad (53)
$$

We now turn our attention to studying the planetesimal velocities in the mass range $1 \lesssim x \lesssim x_c$. We make an a priori assumption that the planetesimal velocity spectrum has the form of a simple power law (with a constant power-law index),

$$
s(x, \tau) \approx s_0(\tau) x^{-\beta}, \quad x \lesssim x_c \quad \text{(54)}
$$

and verify later whether it is correct. In Appendix C we show that, whenever $3 < \alpha < 4$, both the heating and friction of planetesimals of a particular mass $x$ are determined by bodies of similar mass, $x^* \sim x$. This is somewhat unusual in view of our previous results, for which only the extrema of the power-law mass spectrum were important (see §§ 4.1 and 4.2). In this case the power-law index $\beta$ can only be found by numerically solving equation (C2); there is no simple expression for $\beta$, but its value has to satisfy the condition

$$
\frac{\alpha - 2}{4} < \beta < \frac{1}{2}. \quad (55)
$$

For even steeper mass distributions, $\alpha > 4$, we demonstrate in Appendix C that the smallest planetesimals dominate both the stirring and the fraction, not only of themselves but also of bigger bodies. This occurs because the mass spectrum is very steep and high-mass planetesimals contain too little mass to be of any dynamical importance. This is very similar to the velocity evolution of the runaway tail in the dispersion-dominated regime described in § 4.1.2, and, not surprisingly, we find that $\beta = 1/2$ when $\alpha > 4$. Thus, a solution leading to a complete energy equipartition between planetesimals ($s \propto x^{-1/2}$) is only possible for very steep mass spectra. The velocity behavior of runaway bodies in the case of a steep mass spectrum is described in detail in Appendix C. The theoretical spectra of planetesimal velocities are schematically illustrated in Figure 8a (for $3 < \alpha < 4$) and Figure 8b (for $\alpha > 4$).

We verify the accuracy of these predictions by numerically calculating the velocity evolution of planetesimal disks with $\alpha = 3.2$ and 4.3. In the first case, shown in Figure 9, the slope $\beta$ of planetesimal velocity spectrum for $x \leq x_c$ is evidently shallower than $-1/2$ but steeper than $-(\alpha - 2)/4 = -0.3$, in agreement with the constraint in equation (55). The analytical prediction for the behavior of $\eta(x)$ in the top panels of Figure 9 is computed using the numerically determined value of $\beta$ and equations (C4), (C5), and (C6). It agrees with the numerical result (solid curves, corresponding to the largest $x_c$ displayed) rather well, especially for $x \gtrsim x_{\text{shear}}$.

In the case of $\alpha = 4.3$ (Fig. 10) the slope of the planetesimal velocity scaling with mass is essentially indistinguishable from $-1/2$ for $x \leq x_c$, in accordance with our asymptotic prediction. The velocity evolution of the runaway tail also agrees with the discussion in Appendix C.2

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\(^6\)The position of $x_k$ in Fig. 6 is determined by eq. (44), which assumes that between $x_k$ and $x_c$, the velocity is $\sim x^{-1/4}$. In reality, as we discussed above, the velocity slope in that mass region is somewhat steeper, causing eq. (44) to overestimate $x_k$. 

quite well. In both the $\alpha = 3.2$ and $\alpha = 4.3$ cases the final velocity curves displayed are at the limit where we can still apply our theory: $s(x_c) \sim x_c^{1/3}$; this washes out some details of the velocity spectra of runaway tails predicted in Appendix C.2. Differences of $\eta_1$ and $\eta_z$ from their predicted values at the smallest planetesimal masses are caused by boundary effects. Note (see Fig. 10) that the eccentricity dispersion $s$ is virtually independent of time above $x_{\text{shear}}$ for both $\alpha = 3.2$ and 4.3 (while the inclination dispersion $s_z$ slowly grows as $\tau$ increases), which is in complete agreement with equation (C6). Finally, the numerically computed $s_0(\tau)$ follows reasonably well the analytical result represented by formula (53), which is demonstrated in Figure 4d.

5. VELOCITY SCALING IN THE PRESENCE OF DISSIPATIVE EFFECTS

Until now, we have been assuming that gravity is the only force acting on the planetesimals. In reality there should be other factors, such as gas drag and inelastic collisions...
between planetesimals, which might influence their eccentricities and inclinations. Here we briefly comment on the possible changes these effects can give rise to.

In the presence of gas drag the eccentricity and inclination of a particular planetesimal with mass $m$ and physical size $r_p$ tend to decrease with time. Adachi, Hayashi, & Nakazawa (1976) investigated the damping of planetesimal eccentricities and inclinations assuming gas drag force proportional to the square of planetesimal velocity relative to the gas flow. They came out with an orbit-averaged prescription for the evolution of the random velocity of planetesimals, which can be written at the level of accuracy we are pursuing in this study (dropping all possible constant factors and assuming that eccentricity is of the same order as inclination) as

$$\frac{\partial e^2}{\partial t} = -e^2 (e + \eta \frac{\rho_g \Omega a}{\rho_p v_p}),$$

where $\rho_g$ is the mass density of nebular gas, $\rho_p$ is the physical density of the planetesimal, and $\eta \equiv (c_s/\Omega a)^2 \ll 1$, with $c_s$ being the sound speed of nebular gas. In our notation (see §2) this equation translates into

$$\frac{\partial s^2}{\partial \tau} \approx -\zeta s^{1/3} \left[ s + \eta \left( \frac{M_s}{m_0} \right)^{1/3} \right],$$

where $\Sigma_g$ is the surface mass density of the gas disk, and a numerical estimate is made for $\rho_g = 3$ g cm$^{-3}$ and $M_s = M_\odot$. For protoplanetary nebula consisting of solar-metallicity gas one would expect $\Sigma_g/\Sigma_p \approx 50–100$, but during the late stages of nebula evolution this ratio should be greatly reduced by gas removal.

We use the results of §4 to discuss the effect of the gas drag on the planetesimal random velocities for mass distributions with $\alpha < 3$. We know from §§4.1 and 4.2 that stirring in this case is dominated by the biggest planetesimals and the stirring rate is $\approx M_s(\tau)/s^2$ (see equations [28], [38], and [41]). Comparing this heating rate with the damping due to gas drag (57), we immediately see that gas drag can become important for small planetesimals if planetesimal velocities are large ($s \gg 1$). In this case dynamical excitation by the biggest planetesimals is balanced by the gas drag. Very interestingly, it turns out that such a balance leads to $s \propto x^{1/12}$ or $x^{1/15}$ [depending on whether $s$ is bigger or smaller than $\eta (M_s/m_0)^{1/3}$]; i.e., smaller planetesimals have smaller random velocities. Such behavior has been previously seen in coagulation simulations, including the effects of gas drag on small planetesimals (e.g., Wetherill & Stewart 1993). Because of the very weak dependence of the velocity on mass in this case, only very small bodies will be able to enter the shear-dominated regime with respect to the biggest planetesimals, which justifies our use of the dispersion-dominated formulae for reasonably large planetesimal sizes. The time dependence of the velocities of small planetesimals affected by the gas drag should also be different from that given by equations (29) or (40). In Figure 11 we schematically display the scaling of $s$ as a function of $x$ in the presence of gas drag for $2 < \alpha < 3$ and different values of $\zeta$.

Inelastic collisions between planetesimals become important for their dynamical evolution roughly when the relative

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7 Note that, for planet formation in some exotic environments such as the early universe prior to metal enrichment or metal-poor globular clusters, this ratio can be much larger than 100.
velocity at infinity with which two planetesimals approach each other becomes comparable to the escape speed from the biggest of them; in this case gravitational focusing is inefficient. The escape speed \( v_{\text{esc}} \approx 1 \text{ m s}^{-1} \frac{r_p}{(1 \text{ km})} \) (for physical density \( \rho = 3 \text{ g cm}^{-3} \)) is much larger than the Hill velocity for the same mass \( \Omega R_H \approx 5 \text{ cm s}^{-1} a_{\text{AU}} \frac{r_p}{(1 \text{ km})} \) (for \( M_\ast = M_{\odot} \)), which justifies initially our previous neglect of inelastic collisions (especially far from the Sun, where \( \Omega R_H \) is small). Later on, however, continuing dynamical heating in the disk would certainly bring the velocities of small planetesimals above their escape speeds. When this happens, collisions with bodies of similar size lead to strong velocity dissipation. This damping should produce an effect on planetesimal random velocities similar to the one caused by gas drag. Indeed, kinetic energy losses in physical collisions between high-velocity planetesimals are roughly proportional to their initial kinetic energies, which is similar to the behavior of the gas drag losses.

At the same time, the velocities of light bodies should still be smaller than the escape velocity from the biggest planetesimals, since, otherwise, these planetesimals would not be able to dynamically heat the disk and it would “cool” below their escape velocity. Thus, planetesimal velocities cannot exceed the escape speed from the surfaces of bodies doing most of the stirring, a result dating back to a classical study by Safronov (1972). This implies that gravitational stirring must still be done by the highest-mass planetesimals (if \( \alpha < 3 \)), in accordance with what we assumed in the case of gas drag. As a result, one would again expect \( s \) to exhibit a power-law dependence on mass with a positive (but small) slope. The exact value of the power-law index of this dependence is determined by the scaling of energy losses with the mass of planetesimals involved in the collision. The effect of the collisions should be most pronounced for small planetesimals, for which relative encounter velocities can be much higher than their \( v_{\text{esc}} \). The threshold mass at which highly inelastic collisions become important must constantly increase in time as the planetesimal disk is heated up by massive planetesimals. One should also remember that planetesimal fragmentation in high-velocity encounters may become important in this regime, which would significantly complicate the simple picture we described here.

In summary, dissipative effects are unlikely to affect the dynamics of massive planetesimals. However, they become important for small-mass planetesimals and lead to a positive correlation between the random velocities of planetesimals and their masses (unlike the case of pure gravity studied in § 4), which has been previously observed in self-consistent coagulation simulations (Wetherill & Stewart 1993). It is thus fairly easy to modify our simple picture developed in § 4 to incorporate the effects of gas drag and inelastic collisions.

6. DISCUSSION

Although our study has concentrated on a particular case of power-law size distributions (most of the mass is locked up in the power-law part of the planetesimal mass spectrum), its results have a wider range of applicability. The reason for this is that the velocity scalings derived in § 4 depend only on what part of the planetesimal spectrum dominates the stirring and friction and not on the exact shape of the mass distribution. For example, suppose that \( f(x, \tau) \) does not have a self-similar power-law form of the kind studied in § 4.2 (intermediate mass spectrum), but that its first moment \( M_1 \) is still dominated by the smallest masses (most of the mass is locked up in the smallest planetesimals, which dominate friction), while the second moment \( M_2 \) is mainly determined by the highest masses (heating is determined by the biggest planetesimals \( \sim x_\ast \)). Then it is easy to see from our discussion in § 4.2.1 that the velocity distribution has the form of a broken power law in mass, changing its slope from 0 at small masses to \(-1/4\) at high masses (see eqs. [40] and [42]). If, on the contrary, most of the mass is not in the smallest bodies but in largest ones \((M_1 \text{ and } M_2 \text{ converge near the cutoff of the planetesimal spectrum, } \sim x_\ast)\), then both friction and stirring are dominated by the biggest planetesimals, and the velocity spectrum must have zero slope for the entire range of planetesimal masses, in accord with our consideration of shallow mass distributions (§ 4.1.1). Whenever both the stirring and cooling of planetesimals of mass \( m \) are dominated by planetesimals of similar mass, we go back to case 1 of § 4.3. Finally, in all cases when stirring and friction are produced by planetesimals much smaller than those under consideration, the velocity profile scales as \( m^{-1/2} \) (energy equipartition; see case 2 in § 4.3). Similar rules can be derived on the basis of the results of § 4 for more complicated mass spectra, and this makes our approach very versatile.

One of the interesting features found in our study of purely gravitational scattering is the development of a plateau at the low-mass end of the planetesimal velocity distribution for shallow and intermediate mass spectra (\( \alpha < 3 \)). These mass spectra are among the most often realized in the coagulation simulations, which makes this prediction quite important for the interpretation of numerical results. One
should be cautious enough not to confuse these plateaus with the manifestations of dissipative processes, such as gas drag or inelastic collisions, which tend to endow the velocity distribution with only a very weak (positive) dependence on mass (see § 5).

Another observation that can be made based on our results is that the bulk of planetesimals (the power-law part where most of the mass is locked up) rarely reaches a complete equipartition in energy. Equipartition is realized when \( \sigma_{e,i} \propto m^{-1/2} \), or in our notation \( s_i \propto x^{-1/2} \). Our calculations (§ 4.3) show that, when \( x \lesssim x_\alpha \), such a velocity spectrum is only possible for very steep mass distributions, \( \alpha > 4 \), when both the dispersion-dominated stirring and friction are due to the smallest planetesimals with masses \( m \sim m_0 (x \sim 1) \). Mass distributions that steep are not often encountered in self-consistent coagulation or \( N \)-body simulations (Wetherill & Stewart 1993; Kenyon 2002; Kokubo & Ida 1996, 2000), and yet the equipartition argument is very often used to draw important conclusions about the bulk of the planetesimal population (see below).

At the same time, random energy equipartition is ubiquitous for runaway bodies in the high-mass tail (\( x \gtrsim x_i \)) if they still interact with the rest of planetesimals in the dispersion-dominated regime. Since, by our assumption, the runaway tail contains very little mass, this equipartition is very similar to the case of \( \alpha > 4 \), because in both situations the stirring as well as the friction are driven by planetesimals much lighter than the bodies under consideration. It is also worth mentioning that this conclusion changes when runaway bodies start interacting with small planetesimals in the shear-dominated regime (see, e.g., § 4.1.2).

Although we did not follow the evolution of the planetesimal mass spectrum self-consistently, we can already constrain some of the theories explaining mass distributions in coagulation cascades. For example, \( N \)-body simulations of gravitational agglomeration of \( 10^3-10^4 \) massive planetesimals by Kokubo & Ida (1996, 2000) produce a roughly power-law mass spectrum with an index of \( \approx -2.5 \) within some range of masses. To explain this result, Makino et al. (1998) explored planetesimal growth in the dispersion-dominated regime with strong gravitational focusing. They postulated epicyclic energy to be in equipartition and found that the planetesimal mass distribution has a power-law form with a slope of \(-8/3 \approx -2.7\).

However, using the results of § 4.2.1, one can immediately pinpoint the contradiction with the basic assumptions that lead to this conclusion. Indeed, a slope of \(-8/3\) corresponds to the intermediate mass spectrum (see § 4.2), which can never reach energy equipartition. Moreover, this mass spectrum cannot even have the power-law velocity distribution with a continuous slope necessary for the theory of Makino et al. (1998) to work. As a result, the explanation provided by these authors for the mass spectrum found in \( N \)-body simulations cannot be self-consistent. More work in this direction has to be done.

Although the theory developed in § 4 is in good agreement with our numerical calculations, one should bear in mind that our analytical results are asymptotic by construction; i.e., they are most accurate when the planetesimal disk has evolved for a long time and when the planetesimal size distribution spans a wide range in mass (this can be easily seen by comparing Figs. 6 and 7). Real protoplanetary disks during their evolution might not enjoy the luxury of such conditions having been fully realized; this can complicate quantitative applications of our results to real systems. We believe, however, that this does not devalue our analytical results, because they provide a basic understanding of processes involved in planet formation, allow one to clarify important trends seen in simulations, and enable the quick and efficient classification of possible evolutionary outcomes for a wide range of parameters of protoplanetary disks.

Several previous studies have concentrated on exploring velocity equilibria in systems with nonevolving mass spectra in which inelastic collisions between planetesimals acted as damping mechanism necessary to balance gravitational stirring (Kaula 1979; Hornung, Pellat, & Barge 1985; Stewart & Wetherill 1988). We have outlined a qualitative picture of the effects of dissipative processes in § 5, and it agrees rather well with these previous investigations. \( N \)-body calculations and “particle-in-a-box” coagulation simulations represent another class of studies in which velocity distributions have been routinely computed. As we already commented in § 1, these calculations usually absorb a lot of diverse physical phenomena, which makes them difficult targets for comparisons. Still, there are several generic features almost all simulations agree upon, such as (1) a roughly power-law mass spectrum, typically with slope \( \alpha < 3 \); (2) constant or slowly increasing with mass velocity dispersion of small bodies; and (3) velocity dispersion decreasing with mass for large bodies. This general picture is in good agreement with our analytical predictions.

7. Conclusions

We have carried out an exhaustive census of the dynamical properties of planetesimal disks characterized by a variety of mass distributions. Here we briefly summarize our results for the case of velocity evolution driven by the gravitational scattering of planetesimals.

Whenever the planetesimal mass distribution in a disk has a power-law slope shallower than \(-2\), we find that planetesimal velocity is constant up to the upper mass cutoff. For slopes between \(-2 \) and \(-3\), velocity dispersions are constant at small masses but switch to an \( m^{-1/4} \) dependence above some intermediate mass. As a result, the mass spectrum exhibits a pronounced “knee” at this mass. Distributions with slopes between \(-3 \) and \(-4\) have purely power-law velocity spectra with slope \(-\beta\) satisfying \( 1/4 < \beta < 1/2 \), which can be determined numerically using equation (C2). Finally, planetesimals in disks characterized by mass spectra steeper than \( m^{-4} \) are in energy equipartition; i.e., velocity dispersions scale as \( m^{-1/2} \). Such differences in behavior are caused by the fact that the gravitational stirring and dynamical friction receive major contributions from different parts of the planetesimal mass spectra in different cases.

We also consider the possibility that the mass distribution has a tail of massive “runaway” bodies sticking out beyond the exponential cutoff of the power-law mass spectrum. We have found that the low-mass end of the tail experiencing dispersion-dominated scattering by all planetesimals exhibits complete energy equipartition: \( \sigma_{e,i} \propto m^{-1/2} \). The most massive runaway bodies interact with all planetesimals in the shear-dominated regime, which leads to highly anisotropic velocity distributions of these bodies: \( \sigma_e \propto m^{-1/6} \), while \( \sigma_x \propto m^{-1/2} \).

We have found, as a general rule, that the complete equipartition of random energy (\( \sigma_{e,i} \propto m^{-1/2} \)) of planetesimals
(as well as runaway bodies) is possible only if both their gravitational stirring and friction are mainly controlled by the dispersion-dominated interaction with bodies of much smaller mass. The time dependence of planetesimal velocities is determined primarily by the evolution of the cutoff of the power-law mass spectrum $m_{\nu}(\nu)$. Our asymptotic results derived using analytical means are in good agreement with the numerical calculations of planetesimal dynamical evolution presented in §4. They can be easily generalized to cover more complicated planetesimal mass distributions.

Finally, we investigate qualitatively the impact of damping processes such as gas drag or inelastic collisions on the planetesimal disk dynamics. We find that manifestations of these effects are only important for small-mass planetesimals; they exhibit themselves in the form of a gentle increase of planetesimal random velocities with mass, which has been previously observed in particle-in-a-box coagulation simulations (Wetherill & Stewart 1993).

Future work in this direction should target the self-consistent coupling of the evolution of planetesimal velocities to the evolution of the mass spectrum of planetesimals due to their coagulation. The successful solution of this problem would greatly contribute to our understanding of how terrestrial planets grow out of swarms of planetesimals in protoplanetary disks.

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APPENDIX A
NUMERICAL PROCEDURE

The evolution of planetesimal eccentricities and inclinations is studied through numerical integration of equations (3a) and (3b) with scattering coefficients smoothly interpolated between shear- and dispersion-dominated regimes. Planetesimals of different masses are distributed in logarithmically spaced mass bins, with bodies in each bin being more massive by 1.2 times than the planetesimals in previous bin. Using numerical orbit integrations, we have determined the values of constant coefficients in equation (9) relevant for the shear-dominated scattering to be

$$B_1 \approx 11.4, \quad B_2 \approx 4.0, \quad D_1 \approx 11.4, \quad D_2 \approx 4.2. \quad (A1)$$

The scattering coefficients in the dispersion-dominated regime are taken from Stewart & Ida (2000), who derived analytical expressions for viscous stirring and dynamical friction rates in the two-body approximation. We translate their results for our gravitational stirring and friction functions $H_{1,2}$ and $K_{1,2}$. This provides us with the dependence of coefficients $A_{1,2}$ and $C_{1,2}$ in equation (6) on the inclination-to-eccentricity ratio $\sigma_i/\sigma_e$. The constants $a_{1,2}$ in equation (7) are fixed to be $a_1 = 1.7$, $a_2 = 1$.

We assume that the mass scale evolves as $x_s(\nu) = F + (\nu \nu)\nu$, where $F \approx 40$, $\epsilon = 10^{-4}$, and $\chi = 4$ in the case of a shallow mass spectrum, where $\alpha = 1.5$ and $\chi = 1.5$ in all other cases. The small parameter $\epsilon$ is introduced to mimic the difference between the timescale of dynamical evolution of the planetesimal disk and the timescale of planetesimal growth (which is usually much longer). Besides this, such a form of $x_s(\nu)$ is chosen arbitrarily. The runaway tail is assumed to have a power-law form with the same slope $\alpha$ but much smaller normalization than the planetesimal mass distribution. The tail always extends 8 orders of magnitudes beyond $x_s$ to allow interesting velocity regimes to fully develop. The initial distribution of planetesimal eccentricities and inclinations is set to $s(x, 0) = s_s(x, 0) = 3x^{-1/2}$.

APPENDIX B
VELOCITY EVOLUTION IN A “MIXED” STATE FOR AN INTERMEDIATE MASS SPECTRUM

In the case of an intermediate mass spectrum runaway bodies with masses $x$ between $x_b$ (defined in eq. [44]) and $x_{\text{shear}}$ interact with small planetesimals [less massive than $x_s(x)$ defined by eq. (45)] in the dispersion-dominated regime, but with large ones [heavier than $x_s(x)$] in the shear-dominated regime. As a result, the equations for $s$ and $s_s$ are now hybrid versions of equations (30) and (35):

$$\frac{\partial s^2}{\partial \tau} = \int_1^{x_s(x)} dx^* \frac{x^2 s f(x^*)}{s^4} - 2A_3 s^2 x \int_1^{x_s(x)} dx^* \frac{x^2 s f(x^*)}{s^6} + C_1 \int_1^{x_s(x)} dx^* x^2 s f(x^*) - 2C_2 \int_1^{x_s(x)} dx^* x^4 s^3 f(x^*), \quad (B1)$$

$$\frac{\partial s_s^2}{\partial \tau} = \int_1^{x_s(x)} dx^* \frac{x^2 s f(x^*)}{s^4} - 2B_3 s_s^2 x \int_1^{x_s(x)} dx^* \frac{x^2 s f(x^*)}{s^6} + D_1 \int_1^{x_s(x)} dx^* x^2 s^2 f(x^*) - 2D_2 \int_1^{x_s(x)} dx^* x^4 s^3 f(x^*). \quad (B2)$$

Following the approach taken in §4.1.2 we neglect the left-hand side of both equations and use equations (40) and (42) to eval-
ulate the integrals in the evolution equations. After careful comparison of different contributions in the right-hand side we find that

$$C_1 \frac{M_2}{x_{c,1}^3} \approx 2A_2 M_1 \frac{s^2 x}{s_0^2} ,$$  \hspace{1cm} (B3)$$

$$B_1 \frac{x^{7/2-\alpha}}{x_0^2 x^{1/2}} + D_1 \frac{s_0^{1/2}}{x_{s_0}^{2/3}} \int_{x_{c}}^{\infty} dx^k (x^k)^{3/2} f(x^k) \approx 2B_2 M_1 \frac{s^2 x}{s_0^2} .$$ \hspace{1cm} (B4)

Equation (B3) implies that the eccentricity stirring of runaway bodies in the mass range $x_b \ll x \ll x_{\text{shear}}$ is done mainly by planetesimals with masses $\sim x_c$, for which the interaction proceeds in the shear-dominated regime (left-hand side of [B3]). This heating is balanced by dynamical friction due to the smallest planetesimals, lighter than $x_b$, which interact with these massive bodies in the dispersion-dominated regime.

Vertical heating is somewhat different. Gravitational friction is again dominated by the smallest planetesimals, which are in the dispersion-dominated mode (right-hand side of [B4]). Stirring, however, depends on the shape of the mass distribution. If $\alpha < 5/2$, the second term in the left-hand side of (B4)—shear-dominated heating by planetesimals with masses between $\sim x_c$ and $\sim x_b$—dominates over the first term, which represents the effect of dispersion-dominated heating by planetesimals with masses $\sim x_c$. In the opposite case, when $\alpha > 5/2$, both terms in the left-hand side of (B4) produce roughly equal contributions. Using (B3) and (B4), one can easily derive expressions (46)–(48).

APPENDIX C

DETAILS OF VELOCITY EVOLUTION FOR THE CASE OF A STEEP MASS SPECTRUM

Substituting our guess in equation (54) into equation (26) and going through all possibilities appropriate for $\alpha > 3$, we find that self-consistent solutions of a power-law type can exist in only two cases:

Case 1: $3-\alpha + 2\beta > 0$ and $2-\alpha + 4\beta > 0$.

Case 2: $3-\alpha + 2\beta < 0$ and $2-\alpha + 4\beta < 0$.

We now consider separately these two possibilities.

C1. PLANETESIMAL VELOCITIES

Case 1: In this case one can easily see that the integrals in the right-hand side of equation (26) are mostly contributed by $x^k \sim x$. This allows us to rewrite (26) in the following form:

$$x^{-2\beta} \frac{\partial x^2}{\partial t} = \frac{x^{3-\alpha} + 2\beta}{s_0^2} \int_0^\infty dt \frac{\Gamma(1+\alpha) t^\alpha}{(1+t^{-2\beta})} \left( \frac{t}{1+t} \right) .$$  \hspace{1cm} (C1)

In arriving at this expression, we have used the fact that $f(x, \tau) = x^{-\alpha}$ for $\alpha > 3$ and $1 \leq x \leq x_c$, and extended the integration range over $t \equiv x^k/x$ from 0 to $\infty$ (because only the local region $t \sim 1$ matters). From this equation we can readily see that the left-hand side of (C1) can be neglected for $x \gg 1$. As a result, a self-consistent power-law solution for $s$ exists only if the integral in (C1) is equal to zero. This leads to the following constraint on the required value of $\beta$:

$$I_1(\alpha, \beta) = 2A_2 , \quad I_1(\alpha, \beta) = \int_0^\infty dt \frac{\Gamma(1+\alpha) t^\alpha}{(1+t^{-2\beta})} , \quad I_2(\alpha, \beta) = \int_0^\infty dt \frac{t^{2-\alpha}}{1+t^{-2\beta}} .$$  \hspace{1cm} (C2)

Solving this equation for given $\alpha$ and $A_1/A_2$, one can find the slope of the mass spectrum $\beta(\alpha, A_1/A_2)$. At first sight it is not at all clear that (C2) should in general possess a solution for $\beta$. However, a closer look at the problem reveals some interesting patterns.

First of all, it follows from equation (8) that the right-hand side of (C2) has to be bigger than 1. Second, in the case of energy equipartition $I_1(\alpha, 1/2)/I_2(\alpha, 1/2) = 1$. Third, $I_1(\alpha, \beta) \to \infty$ as $\beta \to (\alpha - 2)/4$, while $I_2$ stays finite. These observations prove that (C2) always has a solution for $\beta$ satisfying the constraint posed by equation (55). Combining restriction (55) with the initial constraints of case 1, we find that case 1 is only possible for mass spectra with a power-law slope $3 < \alpha < 4$.

In fact, one can do things even better by considering separately the eccentricity and inclination scalings with mass, i.e., using both equations (15) and (16). Assuming that the ratio of inclination to eccentricity $x_e/x$ is constant for $x \leq x_c$, one would obtain, in addition to (C2), another equation dictated by the inclination evolution. It would be identical to (C2), but with $A_{1,2}$ replaced by $B_{1,2}$. Including also the dependence of $A_{1,2}$ and $B_{1,2}$ on $x_e/x$ (see eq. [6]), one would obtain two equations for two unknowns: $\beta$ and $x_e/x$. Solving them, one can uniquely fix both the power-law index of the planetesimal velocity dependence on mass and the ratio of inclination to eccentricity of planetesimals (which is left undetermined in our simplified analysis of § 4).

\* For $\alpha < 5/2$ integral in eq. (B4) converges at the upper end of its range, near $x_c$. When $\alpha = 5/2$ this integral is contributed to roughly equally by equal logarithmic intervals in mass between $\sim x_c$ and $\sim x_b$; then the second term in the left-hand side of eq. (B4) dominates over the first one by $\sim \ln (x_b/x_c)$. 


C2. VELOCITIES OF RUNAWAY BODIES

Case 2: Restrictions imposed on $\alpha$ and $\beta$ by the conditions of case 2 imply that all integrals in the right-hand side of equation (26) are dominated by the lower end of their integration range, i.e., by masses $x \sim 1$. Neglecting the time derivative in (26) and balancing contributions in the right-hand side, we find that $\beta = 1/2$ and

$$s(x, \tau) \approx s_0(\tau)x^{-1/2}.$$  

(C3)

This solution demonstrates that, for $x \gg 1$, our omission of the left-hand side of (26) is justified for arbitrary behavior of $x(\tau)$. It also imposes an important restriction on the mass spectra for which this scaling can be realized: $\alpha$ must be bigger than 4. This means that we have found solutions for all possible positive values of $\alpha$, and our study is complete in this sense.

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Case 1 ($3 < \alpha < 4$): The velocity evolution of runaway bodies in case 1 is similar to the situation with the intermediate mass spectrum. One finds that, when the mass $x$ satisfies $x_c \lesssim x \lesssim x_h \equiv (s_0x^{3/4})$, the velocity profile is shaped by the dispersion-dominated interaction with all small-mass planetesimals ($x \leq x_c$):

$$s(x, \tau) \approx s_c(x, \tau) \approx \frac{s_0(\tau)}{x^{1/2}} \left( x_c^{-1/2} \frac{M_2^{2/3} + 1}{M_1^{1/3}} \right)^{1/2}, \quad 3 < \alpha < 4, \quad x_c \lesssim x \lesssim x_h.$$  

(C4)

The biggest contribution to both stirring and dynamical friction comes from the most massive planetesimals, with masses $\sim x_c$. Going to heavier bodies, $x_h \leq x \leq x_{\text{shear}} \equiv s_0^{1/2}$, one finds scattering to be in mixed state: some planetesimals [those heavier than $x_h(\equiv (s_0/\sqrt{x^{1/3}})^{1/2})$] interact with big bodies in the shear-dominated regime, while the others (those lighter than $x_c$) are in the dispersion-dominated regime relative to runaway bodies. It turns out that, when $3 < \alpha < 4$, the biggest contributors to both the heating and friction of big bodies are planetesimals of mass $\sim x_c$. Since this is just at the boundary between the shear- and dispersion-dominated regimes, one can expect $s_c$ to behave in the same way as $s$ (similar to [C4]). Indeed, we find this to be the case (Fig. 8a):

$$s(x, \tau) \approx s_c(x, \tau) \approx \frac{s_0(\tau)}{x^{1/6}} \left( \frac{x_c^{1/2}}{x^{1+1/3}/6} \right)^{1/2}, \quad 3 < \alpha < 4, \quad x_h \lesssim x \lesssim x_{\text{shear}}.$$  

(C5)

Runaway bodies with $x \gtrsim x_{\text{shear}} \equiv s_0^{1/2} \approx \tau^{3/4}$ experience shear-dominated scattering by the smallest planetesimals; using equation (36), one finds that

$$s(x, \tau) \approx x^{-1/6} \left( \frac{M_2}{M_1} \right)^{1/2}, \quad s_c(x, \tau) \approx s_0(\tau)x^{-1/2}, \quad 3 < \alpha > 4, \quad x \approx x_{\text{shear}}.$$  

(C6)

Case 2 ($\alpha > 4$): We saw in Appendix C.1 that in case 2 planetesimal velocity evolution is determined purely by the smallest bodies. As a result, it does not matter whether one studies the dynamics of planetesimals or runaway bodies—all the conclusions of Appendix C.1 stay unchanged as long as the interaction with the smallest planetesimals occurs in the dispersion-dominated regime. Thus, we predict that, for $x \lesssim x_{\text{shear}}$, the velocity profile is still given by equation (C3). Runaway bodies of larger mass, $x \gtrsim x_{\text{shear}}$, feel shear-dominated scattering by all planetesimals, and their velocity dispersions are given by (C6) (exactly as in case 1).