Gaussian Filtering for Simultaneously Occurring Delayed and Missing Measurements

AMIT KUMAR NAIK\textsuperscript{1}, GUDDU KUMAR\textsuperscript{1}, PRABHAT KUMAR UPADHYAY\textsuperscript{1}, (Senior Member, IEEE), PARESH DATE\textsuperscript{2}, AND ABHINOY KUMAR SINGH\textsuperscript{1}, (Member, IEEE)

\textsuperscript{1}Department of Electrical Engineering, Indian Institute of Technology Indore, Indore 453552, India
\textsuperscript{2}Department of Mathematics, College of Engineering, Design and Physical Sciences, Brunel University London, UB8 3PH Uxbridge, U.K.

Corresponding author: Paresh Date (paresh.date@brunel.ac.uk)

This work was supported by the Department of Science and Technology (DST), Government of India, through Innovation in Science Pursuit for Inspired Research (INSPIRE) Faculty Award, under Grant DST/INSPIRE/04/2018/000089.

\begin{abstract}
Approximate filtering algorithms in nonlinear systems assume Gaussian prior and predictive density and remain popular due to ease of implementation as well as acceptable performance. However, these algorithms are restricted by two major assumptions: they assume no missing or delayed measurements. However, practical measurements are frequently delayed and intermittently missing. In this paper, we introduce a new extension of the Gaussian filtering to handle the simultaneous occurrence of the delay in measurements and intermittently missing measurements. Our proposed algorithm uses a novel modified measurement model to incorporate the possibility of the delayed and intermittently missing measurements. Subsequently, it redesigns the traditional Gaussian filtering for the modified measurement model. Our algorithm is a generalized extension of the Gaussian filtering, which applies to any of the traditional Gaussian filters, such as the extended Kalman filter (EKF), unscented Kalman filter (UKF), and cubature Kalman filter (CKF). A further contribution of this paper is that we study the stochastic stability of the proposed method for its EKF-based formulation. We compared the performance of the proposed filtering method with the traditional Gaussian filtering (particularly the CKF) and three extensions of the traditional Gaussian filtering that are designed to handle the delayed and missing measurements individually or simultaneously.
\end{abstract}

\begin{IEEEkeywords}
Delayed measurements, Gaussian filtering, missing measurements, nonlinear Bayesian filtering.
\end{IEEEkeywords}

\section{Introduction}
Modern scientific tools and technologies often involve sensors that give noisy data \cite{1,2}. In non-engineering applications, survey and experimental data are often noisy \cite{3}. Very often, noisy data are used for determining the hidden states of a system (or process). A popular mathematical tool to handle this problem is estimation, and a recursive process of estimation is known as filtering \cite{1,2}. Some crucial scientific and engineering domains involving the applications of estimation and filtering are target tracking \cite{4}, biomedical modeling \cite{5}, industrial diagnosis \cite{6}, weather forecasting \cite{3}, and forecasting prices of financial derivatives \cite{7}.

A commonly accepted filtering framework is the Bayesian framework that gives a probabilistic solution in terms of the prior and posterior probability density functions (PDF). The popularly known Kalman filter \cite{8} is an optimal linear Bayesian filter. However, no optimal nonlinear Bayesian filter in closed form is reported in the literature till today, and the practitioners rely on approximated methods. The literature witnesses two popular approximated nonlinear filtering methodologies, known as Gaussian filtering \cite{9,10} and particle filtering (PF) \cite{9,11}. Particle filters require specialized hardware in real time applications due to their high computational complexity. We focus on Gaussian filters, i.e., filters which assume Gaussian prior and posterior densities in this paper, which are far easier to implement, are very common in engineering
applications and can also work as a proposal stage for particle filters.

Gaussian filtering [9], [10] approximates the prior and posterior PDFs as Gaussian and characterizes them with the corresponding means and covariances. However, the computation of mean and covariance involves integrals which are unavailable in close form and need to be numerically approximated during filtering. Some of the popular Gaussian filters are the extended Kalman filter (EKF) [1], the unscented Kalman filter (UKF) [12], the cubature Kalman filter (CKF) [13], the cubature quadrature Kalman filter (CQKF) [14], the Gauss-Hermite filter (GHF) [15], and the exponentially-fitted CKF (ECFK) [16]. The Gaussian filters are sub-optimal, mainly due to the Gaussian approximation of arbitrary prior and posterior PDFs. However, they are often accurate enough for practical applications under lenient environments. Here, the term ‘lenient environment’ imposes two major restrictions: i) the measurements are not delayed in time and ii) definite availability of measurements is ensured at every sampling instant. This presents difficulties if practical measurements are often time-delayed and intermittently missing.

It is worth mentioning that a marginal delay in measurements is inherent due to the response time of the measuring devices. Moreover, the delay may further increase and become large due to various reasons, such as network delay [17] and propagation delay [18]. In such cases, ignoring the measurement time delay may have a serious impact on the state estimation accuracy. The early literature on filtering with delayed measurements [19], [20], [21] extended the traditional Gaussian filters for delays up to one or two sampling intervals. Later, [22] extended the traditional Gaussian filtering for larger delays, considering that the delay statistics are known. This method is further extended in [23] for unknown delay statistics.

Similar to the delay, there are various practical reasons for intermittently missing measurements. A predominant reason is the inefficient network and communication channels used for measurement data transmission [24]. Some other common reasons, such as the use of time-sharing sensors and the temporal sensor failures, are reported in [25], [26], and [27]. Besides these possible reasons, the measurements may become unidentifiable in heavy clutter environment, which is also addressed as a missing measurement [28].

In an early development on filtering with missing measurements, [26] modified the traditional EKF for partial missing measurements, assuming that a fraction of each measurement is available. Later, [29], [30] investigated the UKF for nonlinear stochastic systems with missing measurements. In another development, [31] introduced a distributed filtering method for saturated systems. However, its design aspect is limited to linear dynamical models with certain modifications to handle the nonlinearity involved due to the saturation.

The above discussed filters [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29], [30], [31] are designed to handle either the delayed or the missing measurements.

The literature [18], [32], [33], [34] on filtering with simultaneously occurring delayed and missing measurements witnesses some developments for different classes of systems. For example, [32] considered this problem in coupled neural networks and developed an estimation method. Similarly, [18] develops a filtering algorithm for the delayed and missing measurements. In this regard, it introduces separate stochastic models for incorporating the delay and missing measurements possibilities. Subsequently, it introduces a Ricatti-like equation for designing Kalman filtering method. However, [18] restricts the delay up to one sampling interval, wherein the practical delays can often be larger. Furthermore, [33] introduced unbiased finite impulse response-based filtering approach for finite-horizon case, considering the presence of delayed and missing measurements. However, it assumes that the delay is time-stamped, while delays without time-stamping are observed in many practical systems [35], [36]. Finally, [34] introduces three new designs of robust linear Kalman filtering for handling the simultaneously occurring delayed and missing measurements. However, in every design, it restricts the delay as one sampling interval, which can lead to poor accuracy if the real measurement delay is higher. Moreover, [18], [34] rely on the augmented state-space approaches, which may increase the computational complexity significantly.

Although the above-discussed filtering methods [18], [33], [34] can handle the simultaneously occurring delayed and missing measurements, they have two broad limitations: i) they are designed for linear dynamical systems, ii) they fail to handle delays of more than one sampling interval without time-stamping. For the nonlinear dynamical systems, [37] is probably the only existing method to handle the simultaneously occurring delayed and missing measurements. It integrates a likelihood-based technique with the nonlinear Gaussian filtering framework for handling the simultaneously occurring delayed and missing measurements. However, [37] uses an augmented state transition equation, with the size of the system growing by a factor of $d$, where $d$ the maximum number of delays. The authors also use Gaussian mixture with $d$ components as a likelihood function, and mention computational issues with large number of components in the likelihood function. Indeed, the increased size of the covariance matrices itself can present significant computational issues in multidimensional integration involved.

In this paper, we develop a new extension of Gaussian filtering to handle the simultaneously occurring delayed and missing measurements for nonlinear dynamical systems. In this regard, we reformulate the measurement model using a set of Bernoulli random variables to incorporate the possibilities of delayed and missing measurements. Subsequently, we re-derive the Gaussian filtering method for the modified measurement model. It should be mentioned that we modify only the measurement model. Consequently, in the redesigned Gaussian filtering, only the filtering parameters related to the measurements, such as
the measurement estimate, the measurement covariance, and the state-measurement cross-covariance, are re-derived. It is worth mentioning that our contribution is on developing a generalized Gaussian filtering methodology for the problem of delayed as well as missing measurements. Thus, it can be used for extending any of the existing Gaussian filters, such as the EKF, the UKF, and the CKF, for handling the simultaneous occurrence of delayed and missing measurements. We study the stability of the proposed method by formulating it under the EKF-based design strategy. We test the performance of the proposed method under CKF-based formulation due to its popularity for high accuracy at a low computational cost. The performance analysis reveals an improved accuracy for the proposed method compared to the traditional Gaussian filtering method and its extensions in presence of delayed and missing measurements.

In view of the above discussion, we highlight the main contributions of the manuscript as follows:

- We introduce a stochastically formulated measurement model that incorporates the possibility of simultaneously occurring delayed and missing measurements.
- We redesign the traditional Gaussian filtering for the modified measurement model to handle the simultaneous occurrences of the delayed and missing measurements.
- We consider arbitrarily large delays without time-stamping for nonlinear systems, whereas the existing filters such as those reported in [34] and [18] (without time-stamping) and [33] (with time-stamping) address the delayed measurements only for linear systems. Moreover, our algorithm, in contrast to [37], avoids computationally expensive state augmentation and instead relies on analytical expressions for the necessary conditional moments (which are additive in the number of maximum delays).
- We study the stochastic stability of the proposed Gaussian filtering structure for the EKF-based formulation.
- We validate the performance of the proposed Gaussian filtering methodology by two comprehensive simulation examples.

The remaining part of the paper is organized as follows. In Section II, we mathematically formulate the problem of simultaneously occurring delayed and missing measurements, which is followed by the explanation of the proposed methodology in Section III. In Section IV, the stochastic stability of the modified Gaussian filter under the EKF-based formulation is performed. The simulation results are presented in Section V, and finally the discussion and conclusion are highlighted in Section VI.

II. PROBLEM FORMULATION

Our problem is to develop an advanced Gaussian filtering methodology to handle the simultaneous occurrence of delayed and missing measurements. The standard representation of the state-space model in a lenient environment (defined in the previous section) is as follows

\[
x_k = f_k(x_{k-1}, k - 1) + \eta_k
\]

\[
z_k = h_k(x_k, k) + v_k,
\]

where \( x_k \in \mathbb{R}^n \) and \( z_k \in \mathbb{R}^m \) are state and measurement variables, respectively, at \( k^{th} \) sampling instant, \( k \in \{1, 2, \ldots, T_s\} \) with \( T_s \) representing the number of sampling intervals. Moreover, \( f_k : x_{k-1} \rightarrow x_k \) and \( h_k : x_k \rightarrow z_k \) are general nonlinear functions. Finally, \( \eta_k \) and \( v_k \) are zero-mean Gaussian noises representing the process and measurement noises, respectively. The covariances of \( \eta_k \) and \( v_k \) are denoted as \( Q_k \) and \( R_k \), respectively.

Following our problem statement, we need to reformulate the measurement model (Eq. (2)) to address the simultaneous occurrence of randomly delayed and randomly missing measurements. Our reformulation of the measurement model is based on two sets of Bernoulli random variables, denoted by \( \alpha \) and \( \Theta \): \( \alpha \) corresponds to the missing measurements and \( \Theta \) corresponds to the delayed measurements.

The measurements are generally received from multiple sources, and they all may not be missing at the same time. Thus, we consider that the measurement at any sampling instant may be partly missing, i.e., specific elements of the measurement may be missing at any particular instant. Thus, we define a matrix of Bernoulli random variables, \( \lambda_k = \text{diag}(\alpha_1^k, \alpha_2^k, \ldots, \alpha_q^k) \) with \( \alpha_i^k \in \{1, 2, \ldots, q\} \) being \( q \) equiprobable Bernoulli random variables and \( E[\lambda_k] = \mu_k = \text{diag}(\mu_1^k, \mu_2^k, \ldots, \mu_q^k) \). It should be mentioned that \( \alpha_i^k \) is either 0 or 1, with \( \alpha_i^k = 0 \) denoting that the \( i^{th} \) element of the received measurement \( y_k \), denoted as \( y_k(i) \), is missing.

For modeling the delay portion, we restrict the maximum delay to \( N_{\text{max}} \). \( N_{\text{max}} \) is the practitioner’s choice and it can be assigned with a fairly large value if the expected delay is large. Therefore, our model and the proposed filtering technique should not be deemed to be restricted to small delays. We define \( N_{\text{max}} + 1 \) equiprobable Bernoulli random variables: one for each of the current and the \( N_{\text{max}} \) possible delayed instants. At \( k^{th} \) instant, we denote them as \( \Theta_k \) \( \forall j \in \{1, 2, \ldots, N_{\text{max}} + 1\} \) with \( P(\Theta_k^j = 1) = E[\Theta_k^j] = \delta_d \).

Note that \( \Theta_k^{j+1} \) corresponds to \( j^{th} \) delayed instant. We assign \( \Theta_k^0 = 0 \), and model the actual measurement as

\[
y_k = \lambda_k \left( (1 - \Theta_k^0)\Theta_k^1 z_k + (1 - \Theta_k^0)(1 - \Theta_k^1)\Theta_k^2 z_{k-1} + \ldots + (1 - \Theta_k^0)(1 - \Theta_k^1)\ldots(1 - \Theta_k^{N_{\text{max}}}) z_{k-N_{\text{max}}} \right). \tag{3}
\]

The coefficients of \( z_{k-m} \) \( \forall m \in \{1, 2, \ldots, N_{\text{max}}\} \) govern the delay extent. For example, if the measurement is one time-step delayed, i.e., \( y_k = z_{k-1} \), then coefficient of \( z_{k-1} \), i.e., \( (1 - \Theta_k^0)(1 - \Theta_k^1)\Theta_k^2 \) takes the value one, while the random variables associated with \( z_{k-m} \) \( \forall m \neq 1 \) remain zero. At the same time, \( \lambda_k \) regulates the missing measurement possibility. The diagonal elements of
$\lambda_k$ are Bernoulli random variables, which take the values zero or one. The value one ensures that the measurement is received, while the value zero indicates that the measurement is lost.

To this end, let us simplify the notation for the coefficients of $z_{k-m}$ as

$$\Lambda_k(m, j) = \left( \prod_{j=0}^{m} (1 - \Theta_k^j) \right) \Theta_k^{(m+1)}.$$  

(4)

so that the received measurement is $m$-step delayed if $\Lambda_k(m, j) = 1$, which means $\Theta_k^{m+1} = 1$ and $\Theta_k^j = 0$ $\forall j \leq m$.

Subsequently, Eq. (3) can be represented as

$$y_k = \lambda_k \left[ \Lambda_k(0, j)z_k + \Lambda_k(1, j)z_{k-1} + \cdots + \Lambda_k(N_{\max}, j)z_{k-N_{\max}} \right].$$  

(5)

Thus, the measurement model can be finally given as

$$y_k = \lambda_k \sum_{m=0}^{N_{\max}} \Lambda_k(m, j)z_{k-m},$$  

(6)

where $y_k$ is the actual received measurement due to delay and missing possibilities. At this end, $z_k$ may be considered as an ideal measurement that might have been received in the lenient environment. A pictorial diagram representing the sequence to be followed to get $y_k$ from $z_1, z_2, \ldots, z_k$ is shown in Fig. 1.

It is assumed that $\alpha_k^i\Theta_k^j$ and $\alpha_k^j\Theta_k^i$ are independent random variables $\forall (k, i, j)$. Furthermore, $\alpha_k^i\Theta_k^j$ and $\alpha_k^j\Theta_k^i$ are independent of $\alpha_k^j\Theta_k^i$ and $\alpha_k^i\Theta_k^j$, respectively for $j \neq i$. Our objective in the next section is to redesign the Gaussian filtering method for the state-space model represented by Eqs. (1) and (6) so that the possibilities of delayed and missing measurements are incorporated.

The above discussions emphasize the importance of the measurement model (6) for developing the proposed filtering algorithm. As mentioned in the previous section, some of the existing filters, such as [19], [20], [21], [22], and [26], also formulated similar measurement models. However, the models in [19], [20], [21], and [22] characterize the delay possibilities only, while the same in [26] characterizes only the missing measurement possibility. Moreover, [19], [20], [21] characterize only limited and small delays, while [26] characterizes fractionally available measurements instead of being completely missing. Thus, they fail to characterize the general practical scenarios of the simultaneously occurring delayed and missing measurements. Our measurement model in Eq. (6) efficiently characterizes the simultaneously occurring delayed and missing measurement possibilities. It considers any large delays and completely missing measurements unlike [19], [20], [21], and [26], respectively.

Remark 1: Considering the above-discussed competency of our measurement model, the Gaussian filtering algorithm designed for this model should accomplish an improved accuracy for simultaneously occurring delayed and missing measurements.

### III. MODIFIED GAUSSIAN FILTERING FOR DELAYED AND MISSING MEASUREMENTS

The traditional Gaussian filtering is designed with respect to the measurement $z$, modeled in Eq. (2). In this section, we derive the necessary modifications to the algorithm to deal with the modified measurement $y$, modeled in Eq. (6). As the measurement model is changed, we re-derive all the related expressions in the traditional Gaussian filtering to propose the advanced Gaussian filtering for $y$. The traditional Gaussian filtering uses only three such expressions, namely the measurement estimate $\hat{z}$, measurement error covariance $P_{zz}$, and the cross covariance $P_{zx}$, derived for $z$. We re-derive all the measurement related expressions in the Gaussian filtering algorithm for the modified measurement model above. On a different note, it should be mentioned that the state dynamics remains unaffected from the simultaneous occurrence of the delayed and missing measurements. Therefore, the time update step of the proposed filtering technique remains the same as the traditional Gaussian filtering [9], [10].

**A. MODIFIED GAUSSIAN FILTER**

In this part, we re-derive the measurement parameters, such as $\hat{y}$, $P_{yy}$, and $P_{zy}$. Before proceeding to the derivation, it should
be mentioned that only one of $\Lambda_k(m, j) \forall m$ is one and the others are zero at any instant $t_k$ to ensure that only one measurement is received. Although this consideration violates the independence of $\Lambda_k(m, j)$ for different $m$ (i.e., different delay), they will be assumed to be statistically independent in our derivation. We now derive the expressions of $\hat{y}_k$, $P_{y}^{\hat{y}}$, and $P_y^{\hat{y} \hat{y}}$.

1) MEASUREMENT ESTIMATE FOR $y_k$

For $y_k$ given in Eq. (6), the measurement estimate is

$$\hat{y}_{k|k-1} = E[y_k] = E \left[ \left( \sum_{m=0}^{N_{\text{max}}} \lambda_k \Lambda_k(m, j) z_{k-m} \right) \right]. \quad (7)$$

As the missing and delay occurrences are mutually independent events, $\lambda_k$ and $\Lambda_k(m, j)$ are statistically independent. Moreover, $\lambda_k$ and $\Lambda_k(m, j)$ are independent of the measurement value $z_k$ also. Thus, we simplify the above equation as

$$\hat{y}_{k|k-1} = \sum_{m=0}^{N_{\text{max}}} E[\lambda_k] E[\Lambda_k(m, j)] E[z_{k-m}] \quad (8)$$

Following our previous notations, $E[z_{k-m}] = \hat{z}_{k-m|k-1}$. Recalling the previous discussion, we get

$$E[\Lambda_k(m, j)] = \left[ \left( \prod_{j=0}^{m} (1 - \Theta_k^j) \right) \Theta_k^{(m+1)} \right] = (1 - \delta_d)^m \delta_d. \quad (9)$$

Substituting $E[z_{k-m}]$, $E[\lambda_k]$, and $E[\Lambda_k(m, j)]$ in Eq. (8), we get

$$\hat{y}_{k|k-1} = \sum_{m=0}^{N_{\text{max}}} \mu_k (1 - \delta_d)^m \delta_d \hat{z}_{k-m|k-1} \quad (10)$$

2) MEASUREMENT ERROR COVARIANCE FOR $y_k$

The measurement error covariance is

$$P_{y}^{\hat{y}} = E \left[ (y_k - \hat{y}_{k|k-1})(y_k - \hat{y}_{k|k-1})^T \right]. \quad (11)$$

From Eqs. (6) and (10), we get

$$y_k - \hat{y}_{k|k-1} = \sum_{m=0}^{N_{\text{max}}} \lambda_k \Lambda_k(m, j) z_{k-m}$$

$$- \sum_{m=0}^{N_{\text{max}}} \mu_k (1 - \delta_d)^m \delta_d \hat{z}_{k-m|k-1}. \quad (12)$$

We can rewrite this expression as

$$y_k - \hat{y}_{k|k-1} = \sum_{m=0}^{N_{\text{max}}} \lambda_k \Lambda_k(m, j) (z_{k-m} - \hat{z}_{k-m|k-1})$$

$$+ \sum_{m=0}^{N_{\text{max}}} \left( \lambda_k \Lambda_k(m, j) - \mu_k (1 - \delta_d)^m \delta_d \right) \hat{z}_{k-m|k-1}. \quad (13)$$

From Eqs. (11) and (13), we can write

$$E[A_1 A_1^T] = E[A_1 A_1^T] + E[A_2 A_2^T] + E[A_2 A_2^T]. \quad (14)$$

We can now compute every expectation term individually for $A_1$ and $A_2$ defined in Eq. (13) and add them to obtain $P_{y}^{\hat{y} \hat{y}}$. In this regard, for $A_1$ given in Eq. (13), we get

$$E[A_1 A_1^T] = \sum_{m=0}^{N_{\text{max}}} E[\lambda_k^2] E[\Lambda_k^2(m, j)] \times E \left[ (z_{k-m} - \hat{z}_{k-m|k-1})(z_{k-m} - \hat{z}_{k-m|k-1})^T \right]. \quad (15)$$

Following previous discussions, $E[\lambda_k^2] = E[\lambda_k] = \mu_k$, $E[\Lambda_k^2(m, j)] = E[(\Lambda_k(m, j))] = (1 - \delta_d)^m \delta_d$, and $E[\Lambda_k(m, j)] = E[\Lambda_k^2(m, j)] = \mu_k$. Thus, Eq. (15) can be simplified as

$$E[A_1 A_1^T] = \sum_{m=0}^{N_{\text{max}}} \mu_k (1 - \delta_d)^m \delta_d \hat{z}_{k-m|k-1}. \quad (16)$$

Similarly, for $A_2$ given in Eq. (13), we can write

$$E[A_2 A_2^T] = \sum_{m=0}^{N_{\text{max}}} \sum_{t=0}^{N_{\text{max}}} E[\lambda_k \Lambda_k(t, j)] (\lambda_k \Lambda_k(t, j) - \mu_k (1 - \delta_d)^m \delta_d \hat{z}_{k-t|k-1}$$

$$\times (\lambda_k \Lambda_k(t, j) - \mu_k (1 - \delta_d)^m \delta_d \hat{z}_{k-t|k-1})^T]. \quad (17)$$

After further simplification, we get

$$E[A_2 A_2^T] = \sum_{m=0}^{N_{\text{max}}} \sum_{t=0}^{N_{\text{max}}} \left( \lambda_k \Lambda_k(t, j) - \mu_k (1 - \delta_d)^m \delta_d \hat{z}_{k-t|k-1} \hat{z}_{k-t|k-1} \right). \quad (18)$$

This also leads to

$$E[A_2 A_2^T] = \sum_{m=0}^{N_{\text{max}}} \sum_{t=0}^{N_{\text{max}}} \left( \lambda_k \Lambda_k(t, j) - \mu_k (1 - \delta_d)^m \delta_d \hat{z}_{k-t|k-1} \right). \quad (19)$$

Finally, for $A_2$ given in Eq. (13), we have

$$E[A_2 A_2^T] = \sum_{m=0}^{N_{\text{max}}} \left( \lambda_k \Lambda_k(m, j) - \mu_k (1 - \delta_d)^m \delta_d \right) \times \delta_d \hat{z}_{k-m|k-1} \hat{z}_{k-m|k-1}. \quad (20)$$
Applying binomial expansion and simplifying further, we get
\[
E[A_2A_k^T] = \sum_{m=0}^{N_{\text{max}}} \left( E[\lambda_k^2]E[A_k^2(m,j)] \right) + (\mu_k \delta_d (1-\delta_d)^m)^2 \\
\quad - 2E[\lambda_k]E[A_k(m,j)] \mu_k (1-\delta_d)^m \delta_d \\
\quad \times \hat{z}_{k-m|k-1}^T \hat{z}_{k-m|k-1}^T. \tag{21}
\]

Substituting \( E[\lambda_k^2] = \mu_k \) and \( E[A_k^2(m,j)] = (1-\delta_d)^m \delta_d \), we obtain
\[
E[A_2A_k^T] = \sum_{m=0}^{N_{\text{max}}} \left( \mu_k (1-\delta_d)^m \delta_d - (\mu_k (1-\delta_d)^m \delta_d)^2 \right) \\
\quad \times \hat{z}_{k-m|k-1}^T \hat{z}_{k-m|k-1}^T. \tag{22}
\]

We now substitute \( E[A_1A_1^T], E[A_1A_k^T], E[A_2A_k^T], \) and \( E[A_2A_k^T] \) from Eqs. (16), (18), (19), and (22), respectively, in Eq. (14) to obtain \( P_{k|k-1}^{xy} \) as
\[
P_{k|k-1}^{xy} = \sum_{m=0}^{N_{\text{max}}} \mu_k (1-\delta_d)^m \delta_d P_{k-m|k-1}^{zz} + \sum_{m=0}^{N_{\text{max}}} (\mu_k \\
\quad \times (1-\delta_d)^m \delta_d - (\mu_k (1-\delta_d)^m \delta_d)^2) \\
\quad \times P_{k-m|k-1}^{zz} \tag{23}
\]

3) CROSS-COVARIANCE FOR \( y_k \)

The cross-covariance between the state and measurement is
\[
P_{k|k-1}^{xy} = E\left[ (x_k - \hat{x}_{k|k-1})(y_k - \hat{y}_{k|k-1})^T \right]. \tag{24}
\]

Substituting \( y_k - \hat{y}_{k|k-1} \) from Eq. (13), we get
\[
P_{k|k-1}^{xy} = E\left[ (x_k - \hat{x}_{k|k-1})A_1^T \right] + E\left[ (x_k - \hat{x}_{k|k-1})A_2^T \right]. \tag{25}
\]

For \( A_1 \) given in Eq. (13), we get
\[
E\left[ (x_k - \hat{x}_{k|k-1})A_1^T \right] = \sum_{m=0}^{N_{\text{max}}} \left( E[\lambda_k]E[A_k(m,j)] \right) \\
\quad \times E\left[ (x_k - \hat{x}_{k|k-1})(x_k - \hat{x}_{k|k-1})^T \right]. \tag{26}
\]

Substituting \( E[\lambda_k] \) and \( E[A_k(m,j)] \), we obtain
\[
E\left[ (x_k - \hat{x}_{k|k-1})A_1^T \right] = \sum_{m=0}^{N_{\text{max}}} (1-\delta_d)^m \delta_d P_{k-m|k-1}^{xx} \mu_k. \tag{27}
\]

Similarly, for \( A_2 \) given in Eq. (13), we get
\[
E\left[ (x_k - \hat{x}_{k|k-1})A_2^T \right] = \sum_{m=0}^{N_{\text{max}}} E\left[ (\lambda_k A_k(m,j) - \mu_k \\
\quad \times (1-\delta_d)^m \delta_d \right] E\left[ (x_k - \hat{x}_{k|k-1})\hat{z}_{k-m|k-1}^T \right]. \tag{28}
\]

As \( \lambda_k \) and \( \Lambda_k(m,j) \) are independent, \( E[\lambda_k \Lambda_k(m,j)] = E[\lambda_k] E[\Lambda_k(m,j)] = \mu_k (1-\delta_d)^m \delta_d \). Thus, we can write
\[
E\left[ (\lambda_k A_k(m,j) - \mu_k (1-\delta_d)^m \delta_d) \right] = 0. \tag{29}
\]

Substituting this into Eq. (28), we get
\[
E\left[ (x_k - \hat{x}_{k|k-1})A_2^T \right] = 0. \tag{30}
\]

Substituting \( E\left[ (x_k - \hat{x}_{k|k-1})A_1^T \right] \) and \( E\left[ (x_k - \hat{x}_{k|k-1})A_2^T \right] \) from Eqs. (27) and (30), respectively, into Eq. (25), we get
\[
P_{k|k-1}^{xy} = \sum_{m=0}^{N_{\text{max}}} \delta_d (1-\delta_d)^m P_{k-m|k-1}^{xx} \mu_k. \tag{31}
\]

As discussed at the beginning of this section, the proposed filtering method modifies the traditional Gaussian filtering by re-deriving the expressions of measurement estimate, measurement covariance, and state-measurement cross-covariance (Eqs. (10), (23), and (31), respectively). Please follow [9], [10] for a detailed discussion on the traditional Gaussian filtering. The proposed filtering algorithm also follows the same filtering strategy by replacing the expressions of \( \mathbf{z} \), \( P_{zz} \), and \( P_{xz} \) with the re-derived expressions of \( \tilde{y} \), \( P_{yy} \), and \( P_{sy} \), respectively. We provide the pseudo-code for implementing the proposed filtering method in Algorithm 1.

In advancing the traditional Gaussian filtering for handling various measurement irregularities, such as the delayed and missing measurements, the major difficulty appears in incorporating those irregularities through mathematical models. The problem becomes yet more challenging if the irregularities are uncertain to appear at any particular sampling instant, as considered in this paper. We handled this problem by mathematically characterizing such irregularities, particularly the delayed and missing measurements, by formulating a stochastic model, as in Eq. (6).

Remark 2: Our measurement model utilizes a sequence of Bernoulli random variables to characterize the possibility of a measurement coming from various possible past instants. A future research problem may be to introduce a more convenient model by reducing the required number of random variables.

Remark 3: Our filter design strategy concludes that handling the measurement irregularities becomes convenient if an efficient mathematical model for characterizing the concerned irregularities is formulated.

Remark 4: The proposed method fetches some information, such as the measurement estimate, measurement covariance, and state-measurement cross-covariance, from past instants, which causes additional storage capacity requirement. Similar additional storage requirements also occur in existing delay filters, e.g., see [9].

Remark 5: The proposed filtering methodology simplifies to the traditional Gaussian filtering methodology for zero probabilities of delay and missing measurements (\( \mu^d = \delta = 1 \)) and \( N_{\text{max}} = 0 \), if we use the convention \( 0^0 = 1 \).
IV. STOCHASTIC STABILITY OF MODIFIED GAUSSIAN FILTERING

In this section, we analyze the stability of the proposed method for the EKF-based formulation. It should be mentioned that the proposed filtering method is generic and applicable to any of the existing Gaussian filters, such as the EKF, UKF, and CKF. The EKF propagates the locally linearized systems models during the filtering, which makes the stability analysis conveniently realizable. However, other Gaussian filters, such as the UKF and CKF, directly propagate the nonlinear systems models, which makes the stability analysis partially unrealizable with the existing theories of nonlinear dynamics. Thus, our stability analysis is limited to the EKF-based formulation of the proposed method only. In the non-linear filtering literature, it is a common practice to analyze the stability for the EKF-based formulations only [38].

In our stability analysis, we formulate a stochastic model for the estimation error of the EKF-based formulation of the proposed method. Subsequently, we show that the estimation error of the EKF-based formulation of the proposed method is exponentially bounded in a mean square if the filter, noise, and system parameters satisfy a set of presumed conditions. In the remaining part of this manuscript, the EKF-based formulation of the proposed method is abbreviated as MDEKF.

A. STOCHASTIC MODELING OF THE ESTIMATION ERROR

Here, our objective is to formulate the stochastic model representing the dynamics of the MDEKF’s estimation error. Before proceeding further, we would like to introduce the time update and measurement update steps of the ordinary EKF that is designed for \( z \). The EKF determines the time update parameters as

\[
\begin{align*}
\hat{x}_{k|k-1} &= f(\hat{x}_{k-1|k-1}) \\
P_{k|k-1} &= F_{k-1}P_{k-1|k-1}F_{k-1}^T + Q_k,
\end{align*}
\]

(32)

where \( \hat{x}_{k|k-1} \) and \( P_{k|k-1} \) are the prior estimate and error covariance, respectively, at \( k^{th} \) sampling instant. Moreover, \( F_{k-1} = \frac{\partial f(x_{k-1})}{\partial x_{k-1}} \bigg|_{x_{k-1} = \hat{x}_{k-1|k-1}} \) represents the Jacobian matrix of \( f(x_{k-1}) \) computed at \( \hat{x}_{k-1|k-1} \).

Furthermore, the EKF determines the measurement update parameters as

\[
\begin{align*}
\hat{z}_{k|k-1} &= h(\hat{x}_{k|k-1}) \\
P_{ee,k|k-1} &= H_k P_{k|k-1} H_k^T + R_k \\
P_{ey,k|k-1} &= P_{k|k-1} H_k^T \\
\hat{x}_{k|k} &= \hat{x}_{k|k-1} + K (z_k - \hat{z}_{k|k-1}),
\end{align*}
\]

(33)

where \( H_k = \frac{\partial h(x_k)}{\partial x_k} \bigg|_{x_k = \hat{x}_{k|k-1}} \) denotes the Jacobian of \( h(x_k) \) computed at \( \hat{x}_{k|k-1} \).

Assuming that \( f(\cdot) \) is a sufficiently smooth function, let us expand \( f(x_k) \), using the Taylor series expansion around \( \hat{x}_{k|k-1} \), as

\[
f(x_k) = f(\hat{x}_{k|k-1}) + F_k e_{k|k-1} + \Psi_f(x_k, \hat{x}_{k|k-1}),
\]

(34)

where \( e_{k|k-1} = x_k - \hat{x}_{k|k-1} \) represents the estimation error and \( \Psi_f(x_k, \hat{x}_{k|k-1}) \) is the remainder term. A similar expansion of \( h(x_k) \) around \( \hat{x}_{k|k-1} \) gives

\[
h(x_k) = h(\hat{x}_{k|k-1}) + H_k e_{k|k-1} + \Psi_h(x_k, \hat{x}_{k|k-1}),
\]

(35)

where \( e_{k|k-1} = x_k - \hat{x}_{k|k-1} \) is the prediction error and \( \Psi_h(x_k, \hat{x}_{k|k-1}) \) has an explanation similar to \( \Psi_f(x_k, \hat{x}_{k|k-1}) \). Substituting \( x_k \) and \( \hat{x}_{k|k-1} \) from Eqs. (1) and (32), respectively, we obtain

\[
e_{k|k-1} = f(x_{k-1}) - f(\hat{x}_{k-1|k-1}) + \eta_k,
\]

(36)

Using Eq. (34), the above equation can be written as

\[
e_{k|k-1} = F_{k-1} e_{k-1|k-1} - \Psi_f(x_{k-1}, \hat{x}_{k-1|k-1}) + \eta_k.
\]

(37)

Similarly, substituting \( \hat{x}_{k|k-1} \) from Eq. (33), but for \( y_k \), in \( e_{k|k} = x_k - \hat{x}_{k|k} \), we get \( e_{k|k} = x_k - \hat{x}_{k|k} - K (y_k - \hat{y}_{k|k-1}) \). Then, substituting \( x_k, y_k, \hat{y}_{k|k-1}, \) and \( \hat{x}_{k|k-1} \) from Eqs. (1), (6), (10), and (32), respectively, we get

\[
e_{k|k} = f(x_{k-1}) - f(\hat{x}_{k-1|k-1}) + \eta_k - K \left( \lambda_k \sum_{m=0}^{N_{\text{max}}} \Lambda_k(m, j) \right)
\times z_{k-m} - \mu_k \sum_{m=0}^{N_{\text{max}}} \delta_d (1 - \delta_d)^m \hat{z}_{k-m|k-1}.
\]
Let us do the following substitutions: i) $z_{k-m} = h(x_{k-m}) + v_{k-m}$, ii) $\hat{z}_{k-m|k-1} = h(\hat{x}_{k-m|k-1})$, and iii) $f(x_k)$ from Eq. (34). In the resulting expression, let us substitute $h(x_{k-m})$ expanded by Eq. (35). Subsequently, we obtain

$$e_{k|k} = F_{k-1}e_{k-1|k-1} + \Psi f(x_{k-1}, \hat{x}_{k-1|k-1}) + \eta_k - K\lambda_k$$

$$\times \sum_{m=0}^{N_{\text{max}}} \lambda_k m (m, j) v_{k-m}$$

$$- K \sum_{m=0}^{N_{\text{max}}} (\lambda_k \lambda_k m (m, j) - \mu_k \delta_d (1 - \delta_d)^m)$$

$$\times (h(\hat{x}_{k-m|k-1}) + H_{k-m} e_{k-m|k-1}) + \lambda_k \sum_{m=0}^{N_{\text{max}}} \lambda_k m (m, j)$$

$$\times \Psi f(x_{k-m}, \hat{x}_{k-m|k-1})$$

$$+ \sum_{m=0}^{N_{\text{max}}} \mu_k \delta_d (1 - \delta_d)^m H_{k-m} e_{k-m|k-1}$$.

We expand $e_{k-m|k-1}$ using Eq. (37) and substitute the expanded $e_{k-m|k-1}$ in the above equation. Subsequently, after some rearrangements, we obtain the desired stochastic model of the error dynamics in the form of Eq. (38), as shown at the bottom of the page, where $I$ represents an identity matrix of the appropriate dimension.

It is worth mentioning that Eq. (38) represents the error dynamics in terms of stochastic difference equation. Interestingly, the state of this difference equation is the error of the MDEKF. Thus, we can consider Eq. (38) as a time-series representation of a hypothetical system with the state being the error of the MDEKF. Subsequently, the stability of this hypothetical system ensures the stability of the MDEKF.

To this end, it should be mentioned that the literature contains several notions of stability for nonlinear systems. The readers may please refer to [39] for a detailed discussion. In this paper, we particularly use exponential stability, where the stability is ensured if the system’s convergence is bounded with an exponential envelope.

### B. EXPONENTIAL STABILITY

Before proceeding forward to prove that the error dynamics presented in Eq. (38) is exponentially bounded, we mathematically define the exponentially bounded process, as follows.

**Definition 1:** Let us consider that $e_k$ denotes a stochastic process and $\kappa > 0$, $\xi > 0$, and $0 < \beta < 1$ are real numbers. Then, $e_k$ is said to be exponentially bounded in mean square if it satisfies

$$E \left[ \|e_k\|^2 \right] \leq \beta^k \kappa^k \xi E \left[ \|e_0\|^2 \right] + \xi$$

where $\|\cdot\|$ represents the spectral norm for matrices and Euclidean norm for vectors.

It should be mentioned that the above definition is general and does not limit $e_k$ to be the estimation error only. However, our notations and discussions will be focused on the estimation error.

In our stability analysis, we approach Eq. (39) in a different way. In this regard, please refer to the subsequent discussion.

**Remark 6:** Let us consider that $\tau_1 > 0$, $\tau_2 > 0$, $\gamma' > 0$, and $0 < \phi < 1$ denote real numbers, and $V(e_k)$ represents a scalar-valued stochastic process, which satisfies

$$\tau_1 \|e_k\|^2 \leq V(e_k) \leq \tau_2 \|e_k\|^2$$

and

$$E [V(e_k)|e_{k-1}] - V(e_{k-1}) \leq \gamma' - \phi V(e_{k-1}) \leq 0.$$  \hspace{1cm} (41)

Then, the stochastic process $e_k$ satisfies

$$E \left[ \|e_k\|^2 \right] \leq \frac{\tau_2}{\tau_1} E \left[ \|e_0\|^2 \right] (1 - \phi)^k + \frac{\gamma'}{\tau_1} \sum_{i=0}^{k-1} (1 - \phi)^i.$$  \hspace{1cm} (42)

Please refer to [38] and [40], for a detailed discussion.

In the subsequent discussion, we conclude that this remark is another way of defining the exponential bound in mean square.

Since $\sum_{i=0}^{k-1} (1 - \phi)^i \leq \sum_{i=0}^{\infty} (1 - \phi)^i = 1/\phi$, Eq. (42) can be written as

$$E \left[ \|e_k\|^2 \right] \leq \frac{\tau_2}{\tau_1} E \left[ \|e_0\|^2 \right] (1 - \phi)^k + \gamma' \frac{1}{\tau_1 \phi}.$$  \hspace{1cm} (43)
It should be mentioned that with $\tau_2/\tau_1 = k'$, $1 - \phi = \beta$, and $\gamma'/(\tau_1\phi) = \xi$, Eq. (43) is the same as Eq. (39).

**Remark 7:** In conclusion to the above discussion, we state that satisfying Eq. (43) for a stochastic process $e_k$ ensures that $e_k$ is exponentially bounded.

**Remark 8:** Since Eq. (43) is concluded from Eqs. (40) and (41), we can further state that if the stochastic process $e_k$ satisfies Eqs. (40) and (41) for any scalar-valued stochastic process $V(e_k)$, then the stochastic process $e_k$ is exponentially bounded.

We will use Remark 8 as our stability criterion. Alternatively, we will conclude the stability of the MDEKF by inferring that the estimation error of the MDEKF satisfies Eqs. (40) and (41). The proof is based on several assumptions, as follows.

- $F_k$ is a non-singular matrix.
- The system, noise, and filter parameters satisfy the following bounds:

$$\|\eta_k\| \leq \omega \quad \text{and} \quad \|\nu_k\| \leq \nu$$

$$\|\Psi_f(x_k, \hat{x}_{k-1})\| \leq c_1 \|x_k - \hat{x}_{k-1}\|^2$$

$$\|\Psi_h(x_k, \hat{x}_{k-1})\| \leq c_2 \|x_k - \hat{x}_{k-1}\|^2$$

$$\|\Phi_k\| \leq f \quad \text{and} \quad \|H_k\| \leq h$$

$$\rho_1 I \leq P_k(x_k) \leq \rho_2 I$$

$$x_k - \hat{x}_{k-1} = e_k$$

$$\|x_k - \hat{x}_{k-1}\| \leq e'$$

$$\|h(\hat{x}_{k-1})\| \leq h_m$$

$$q I \leq Q_k \leq q' I \quad \text{and} \quad r I \leq R_k \leq r' I$$

with $\omega, \nu, c_1, c_2, f, h, \rho_1, \rho_2, e', h_m, q, q', r, \text{ and } r'$ representing positive real numbers.

Before proving that the MDEKF’s estimation error satisfies Eqs. (40) and (41), we derive a series of inequalities through the subsequent lemmas. These inequalities will be used in the final proof.

**Lemma 1:** For the inequalities presented in Eqs. (47), (48), and (52), the Kalman gain $K$ satisfies

$$\|K\| \leq \frac{\rho_2 h}{r}.$$  

**Proof:** Let us expand $P_{k|k-1}^{yy}$ and $P_{k|k-1}^{yy}$ as follows:

i) Substitute $z_{k-m|k-1}$ and $P_{z_{k-m|k-1}}^{yy}$ obtained using Eqs. (32) and (33), respectively, in $P_{k|k-1}^{yy}$ given in Eq. (23), and

ii) substitute $P_{k|k-1}^{yy}$ from Eq. (33) into $P_{k|k-1}^{yy}$ given in Eq. (31). Substituting the expanded $P_{k|k-1}^{yy}$ and $P_{k|k-1}^{yy}$ in Kalman gain expression $K = P_{k|k-1}^{yy}(P_{k|k-1}^{yy})^{-1}$, we obtain

$$K = \sum_{m=0}^{N_{max}} \delta_d(1 - \delta_d)^m \mu_k \left[ \sum_{m=0}^{N_{max}} \delta_d(1 - \delta_d)^m \right]^{-1} \mu_k \left( H_k P_{k|k-1} H_k^T + R_k \right) + \left( \mu_k \delta_d(1 - \delta_d)^m \right) \left( h(\hat{x}_{k-m|k-1}) h(\hat{x}_{k-m|k-1})^T \right)^{-1}.$$

It is worth mentioning that $H_k P_{k-m|k} H_k^T \geq 0$ because $P_{k-m|k-1}$ is a positive definite matrix. Subsequently, applying the norm property, we can write

$$\|K\| \leq \delta_d \|\mu_k\| \left[ \sum_{m=0}^{N_{max}} (1 - \delta_d)^m P_{k-m|k} H_k \right] \left[ \sum_{m=0}^{N_{max}} (1 - \delta_d)^m \right]^{-1} \mu_k \left( R_k + (1 - \mu_k \delta_d(1 - \delta_d)^m) h(\hat{x}_{k-m|k-1}) \right) \left( h(\hat{x}_{k-m|k-1}) \right)^{-1}.$$

For any invertible matrix $M$, please note that $\sigma^+(M^{-1}) = (\sigma^-(M))^{-1}$, where $\sigma^+(\cdot)$ and $\sigma^-(\cdot)$ represent the largest and smallest singular values, respectively. Thus, substituting $\|\mu_k\| = \mu$ and using the inequalities presented in Eqs. (47) and (48), we get

$$\|K\| \leq \mu \rho_2 h \left( 1 - (1 - \delta_d)^{N_{max}+1} \right) \left[ - \sum_{m=0}^{N_{max}} (1 - \delta_d)^m \right]^{-1} \mu_k \left( R_k + (1 - \mu_k \delta_d(1 - \delta_d)^m) h(\hat{x}_{k-m|k-1}) \right) \left( h(\hat{x}_{k-m|k-1}) \right)^{-1}.$$

The bound presented in Eq. (52) assumes that $R_k$ is a positive definite matrix. Moreover, $h(\hat{x}_{k-m|k-1}) h(\hat{x}_{k-m|k-1})^T$ is a positive semidefinite matrix. Thus, the matrix in the second factor of the above equation is also positive definite. Subsequently, it follows that $\sigma^-(\cdot) = \lambda^-(\cdot)$, with $\lambda^-(\cdot)$ representing the smallest eigenvalue. We now calculate the smallest eigenvalue of the second factor by using the Rayleigh-Ritz characterization [41] in Eq. (55), as shown at the bottom of the page.
As \( h(\hat{x}_k-m_k|k-1)h(\hat{x}_k-m_k|k-1)^T \) is positive semidefinite, the second term on the right side of Eq. (55) is zero. Subsequently, using the bound given in Eq. (52), we obtain

\[
\lambda^2 \left( \sum_{m=0}^{\text{Nmax}} \delta_d(1 - \delta_d)^m \mu_k (R_k + (I - \delta_d(1 - \delta_d)^m)\mu_k) \right) \\
\times h(\hat{x}_k-m_k|k-1)h(\hat{x}_k-m_k|k-1)^T \\
\geq \mu \left( 1 - (1 - \delta_d)^{\text{Nmax}+1} \right) r. 
\]

(56)

As discussed previously, \( \sigma^-(\cdot) = \lambda^2(\cdot) \) for positive definite matrix. Thus, substituting Eq. (56) into Eq. (54) gives the bound of \( K \) as given in Eq. (53).

**Lemma 2:** For non-singular matrix \( F_k \) and \( 0 < \phi < 1 \), the following inequality holds

\[
F_{k-1} \left( I - \delta_d K \mu_k H_k \right)^T P_{k|k}^{-1} \left( I - \delta_d K \mu_k H_k \right) F_{k-1} \\
\leq (1 - \phi) P_{k-1|k-1}. 
\]

(57)

**Proof:** Please note that \( P_{k|k} = E[(x_k - \hat{x}_k|k)(x_k - \hat{x}_k|k)^T] \). Substituting \( x_k \) from Eq. (1) and \( \hat{x}_k = \hat{x}_k|k-1 + K(y_k - \hat{y}_k|k-1) \), we get

\[
P_{k|k} = P_{k|k-1} - P_{k|k-1} P_{k|k}^{yy} K^T - K P_{k|k}^{yy} P_{k|k-1}^T + K P_{k|k}^{yy} K^T.
\]

Substituting \( P_{k|k}^{yy} \) and \( P_{k|k}^{yy} \) from Eqs. (31) and (32), respectively, we obtain

\[
P_{k|k} = P_{k|k-1} - \sum_{m=0}^{\text{Nmax}} \delta_d(1 - \delta_d)^m P_{k|k-m|k-1} \mu_k K^T - K \sum_{m=0}^{\text{Nmax}} \left( \delta_d - (1 - \delta_d)^m \mu_k (P_{k|k-m|k-1} \mu_k K^T - K \sum_{m=0}^{\text{Nmax}} \left( \delta_d - (1 - \delta_d)^m \mu_k \right) \\
\times \left( 1 - \delta_d(1 - \delta_d)^m \mu_k (P_{k|k-m|k-1} \mu_k K^T - K \sum_{m=0}^{\text{Nmax}} \left( \delta_d - (1 - \delta_d)^m \mu_k \right) \right) \right) \\
+ K \left( \sum_{m=0}^{\text{Nmax}} \delta_d(1 - \delta_d)^m P_{k|k-m|k-1} \mu_k \right) \\
+ \sum_{m=0}^{\text{Nmax}} \left( \delta_d - (1 - \delta_d)^m \mu_k \right) \\
\times \left( \hat{z}_{k-m|k-1} \hat{z}_{k-m|k-1}^T \right) \right].
\]

It is worth mentioning that \( \hat{z}_{k-m|k-1} \), \( P_{k|k-m|k-1} \), and \( P_{k|k-m|k-1} \) can be determined using Eq. (33). Substituting these parameters in the above expression and simplifying further, we get \( P_{k|k} \) in the form of Eq. (58), as shown at the bottom of the page.

Please note that \( R_k \) is a positive definite matrix in Eq. (58). Similarly, the positive definiteness of \( P_{k-m|k-1} \) ensures that \( \delta_d(1 - \delta_d)^m \mu_k H_k P_{k-m|k-1} H_k^T (I - \delta_d(1 - \delta_d)^m \mu_k) \) is positive semidefinite. Moreover, \( h(\hat{x}_k-m_k|k-1)h(\hat{x}_k-m_k|k-1)^T \) is a positive semidefinite matrix. Since the sum of the positive definite and positive semidefinite matrices is positive definite, we further conclude

\[
P_{k|k} \geq (I - \delta_d K \mu_k H_k) F_{k-1} \left( I - \delta_d K \mu_k H_k \right)^T - \sum_{m=1}^{\text{Nmax}} P_{k-m|k-1}. 
\]

(59)

Consequently, the following inequality can be deduced from Eq. (58):

\[
P_{k|k} \geq (I - \delta_d K \mu_k H_k) F_{k-1} \left( I - \delta_d K \mu_k H_k \right)^T \\
\times \left( I + \frac{1}{\rho_2^2} \left( q - \zeta^2 \text{Nmax} \rho_2 \right)^{-1} P_{k-1|k-1} \right) \\
\times \left( I + \frac{1}{\rho_2^2} \left( q - \zeta^2 \text{Nmax} \rho_2 \right)^{-1} P_{k-1|k-1} \right)^T.
\]

Substituting Eq. (32), we can rearrange the above inequality as

\[
P_{k|k} \geq \left( I - \delta_d K \mu_k H_k \right) F_{k-1} P_{k-1|k-1} \left[ 1 + P_{k-1|k-1}^{-1} F_{k-1}^{-1} (Q_k \\
\times \left( I - \delta_d K \mu_k H_k \right)^{-1} \right) \\
\times \left( I - \delta_d K \mu_k H_k \right)^{-1} \right] \\
\times F_{k-1} \left( I - \delta_d K \mu_k H_k \right)^{T}.
\]

(59)

Applying the bounds of Eqs. (47), (48), and (52), and taking the inverse, the above inequality can be rearranged as

\[
F_{k-1} \left( I - \delta_d K \mu_k H_k \right) \times P_{k|k} \left( I - \delta_d K \mu_k H_k \right)^{T} \\
\leq \left( 1 + \frac{1}{\rho_2^2} \left( q - \zeta^2 \text{Nmax} \rho_2 \right)^{-1} P_{k-1|k-1} \right)^{-1}.
\]

(58)
satisfied. Subsequently, the above inequality is simplified to Eq. (57).

**Theorem I:** Let us consider the stochastic error model given in Eq. (38) and EKF parameters through Eqs. (32) and (33). Furthermore, let us assume that \( P_k \) is non-singular and the conditions given through (45) to (52) hold. Then, the stochastic process \( e_{k|k} \) is exponentially bounded in mean square, i.e.,

\[
E \left[ \| e_{k|k} \|^2 \right] \leq \frac{T_k}{T_j} E \left[ \| e_{0|0} \|^2 \right] (1 - \phi)^j + \frac{\gamma'}{T_j} \phi.
\]

(60)

**Proof:** Throughout the theorem, we adopt the following simplified notations: \( \hat{x}_{k-m} \), \( \hat{x}_{k-1-m} \), \( e_{k-m} \), and \( e_{k-1-m} \) for \( \hat{x}_{k-m} \), \( \hat{x}_{k-1-m} \), \( e_{k-m} \), and \( e_{k-1-m} \), respectively. Then, Eq. (38) can be expressed as

\[
e_{k|k} = A_k e_{k-1|k-1} + B_k + C_k + D_k.
\]

(61)

We now define a scalar-valued stochastic process \( V : \mathbb{R}^n \rightarrow \mathbb{R} \) as \( e_{k|k}^{-1} \sum P_{k|k}^{-1} e_{k|k} \). It should be mentioned that \( V \) is a positive definite function. Substituting \( e_{k|k} \) from Eq. (61) in the expression of \( V \) and simplifying further, we obtain

\[
V(e_{k|k}) = e_{k-1|k-1}^T A_k^T P_{k|k}^{-1} A_k e_{k-1|k-1} + B_k^T P_{k|k}^{-1} (2A_k
\times e_{k-1|k-1} + B_k) + 2C_k^T P_{k|k}^{-1} (A_k e_{k-1|k-1} + B_k)
\times (A_k e_{k-1|k-1} + B_k + D_k) + D_k^T P_{k|k}^{-1} D_k.
\]

(62)

It should be mentioned that the above equation is scalar. Thus, applying norm property to the second expression on the right side, we get

\[
\left\| B_k^T P_{k|k}^{-1} (2A_k e_{k-1|k-1} + B_k) \right\|
\leq \| B_k \| \left\| P_{k|k}^{-1} \right\| (2 \| A_k \| \| e_{k-1|k-1} \| + \| B_k \|).
\]

(63)

We will now calculate the bound of each term in the right side individually. In this regard, let us recall \( B_k \) defined in Eq. (38) and use norm property to get

\[
\| B_k \|
\leq \| \Psi_f(x_{k-1}, \hat{x}_{k-1}) \|
+ \| K \| \sum_{m=0}^{N_{max}} \lambda_k A_k(m, j) \Psi_f(x_{k-m}, \hat{x}_{k-m})
+ \| K \| \sum_{m=0}^{N_{max}} \delta_d (1 - \delta_d)^m \mu_k H_{k-m} \Psi_f(x_{k-1-m}, \hat{x}_{k-1-m}).
\]

It should be mentioned that the elements of \( \lambda_k \) are Bernoulli random variables, which take on values zero or one. Thus, \( \| \lambda_k \| \leq 1 \). Furthermore, for only a particular value of \( m \), we get \( \lambda_k(m, j) = 1 \), otherwise \( \lambda_k(m, j) = 0 \). Subsequently, applying the bounds presented in Eqs. (45), (46), and (47), and from Lemma 1, we obtain

\[
\| B_k \| \leq c_1 e^{\beta} \left( 1 + \frac{\rho_2 h^2 \mu(1 - (1 - \delta_d)^{N_{max}+1})}{r} \right)
+ \frac{\rho_2 h c_2 e^{\beta}}{r}.
\]

(64)

Similarly, for \( A_k \) given in Eq. (38), we get

\[
\| A_k \| \leq \| I + \delta_d K \mu_k H_k \| \| F_{k-1} \| = 1 + \frac{\rho_2 h^2 \mu \delta_d}{r}.
\]

(65)

Applying the bound given in Eq. (48), we have \( \| P_{k|k}^{-1} \| \leq 1/\rho_1 \). Thus, substituting Eqs. (49), (64), and (65) into Eq. (63), it is simplified as

\[
\left\| B_k^T P_{k|k}^{-1} (2A_k e_{k-1|k-1} + B_k) \right\| \leq \chi_1 e^{\beta},
\]

(66)

where

\[
\chi_1 = \frac{1}{\rho_1} \left[ c_1 \left( 1 + \frac{\rho_2 h^2 \mu(1 - (1 - \delta_d)^{N_{max}+1})}{r} \right) \times \left( 2 + \frac{2\rho_2 h^2 \mu \delta_d}{r} \right)
+ c_1 \epsilon' \left( 1 + \frac{\rho_2 h^2 \mu(1 - (1 - \delta_d)^{N_{max}+1})}{r} \right) \right].
\]

Let us now consider the fourth expression on the right side of Eq. (62). For \( C_k \) defined in Eq. (38), applying \( \| C_k \| = \| C_k \| \) gives

\[
\left\| C_k^T P_{k|k}^{-1} C_k \right\|
\leq \left( \| \eta_k \| + \| K \| \sum_{m=0}^{N_{max}} \mu_k (1 - \delta_d)^m \Psi_f(x_{k-m}, \hat{x}_{k-m}) \right) \left\| P_{k|k}^{-1} \right\|.
\]

(67)

We now substitute \( \| K \| = \mu_k \), \( \| \lambda_k \| \leq 1 \), \( \sum_{m=0}^{N_{max}} \Lambda_k(m, j) = 1 \), and \( \| P_{k|k}^{-1} \| \leq 1/\rho_1 \). Then, using the inequalities presented in Eqs. (44) and (47), and substituting the bound of \( K \) from Lemma 1, we finally get

\[
\left\| C_k^T P_{k|k}^{-1} C_k \right\| \leq \chi_2,
\]

(67)

with \( \chi_2 \) be a constant, given as

\[
\chi_2 = \frac{1}{\rho_1} \left( \omega + \frac{\rho_2 h}{r} \left( \mu_0 h(1 - (1 - \delta_d)^{N_{max}+1}) + \nu \right) \right)^2.
\]

(68)

For \( D_k \) defined in Eq. (38), applying the matrix norm property, the fifth expression on the right side of Eq. (62)
can be expressed as
\[
\|D_k\| \leq \|K\| \left[ \sum_{m=0}^{N_{\text{max}}} \lambda_k \Lambda_k(m, j)(h(\hat{x}_k-m) + H_{k-m}e_{k-m}) \right] + \sum_{m=0}^{N_{\text{max}}} \delta_d(1-\delta_d)^m \mu_k \Lambda_k(m, j)(h(\hat{x}_k-m) + H_{k-m}e_{k-m}) + \sum_{m=1}^{N_{\text{max}}} \delta_d(1-\delta_d)^m \mu_k H_{k-m}F_{k-m}e_{k-1-m} \right].
\]

We now substitute \(\|\mu_k\| = \mu\), \(\|\lambda_k\| \leq 1\), and \(\sum_{m=0}^{N_{\text{max}}} \Lambda_k(m, j) = 1\). Furthermore, using Eqs. (47), (50), and (51), and applying the bound of \(K\) from Lemma 1, the above equation can be expressed as
\[
\|D_k\| \leq \frac{\rho_2 h}{\rho_1} \left( (h_m + h\epsilon') \left( 1 + \mu(1 - (1 - \delta_d)^{N_{\text{max}}+1}) \right) + hf\epsilon' \mu(1 - \delta_d)(1 - (1 - \delta_d)^{N_{\text{max}}}) \right) \cdot (70)
\]

Please note that \(\|2D_k^T P_k^{-1} (A_k e_{k-1|k-1} + B_k)\| \leq \|2D_k^T P_k^{-1}\| [\|A_k e_{k-1|k-1}\| + \|B_k\|]\). Substituting the bounds of \(\|B_k\|\), \(\|A_k\|\), and \(\|D_k\|\) from Eqs. (64), (65), and (69), respectively, into this expression, we get
\[
\|2D_k^T P_k^{-1} (A_k e_{k-1|k-1} + B_k)\| \leq \chi_3 \epsilon',
\]

Let us now consider the sixth term in the summation on the right side of Eq. (62). Applying the norm property, we get \(\|D_k^T P_k^{-1} D_k\| \leq \|D_k\|^2 \|P_k^{-1}\|\). Subsequently, substituting \(\|D_k\|\) from Eq. (69), we obtain
\[
\|D_k^T P_k^{-1} D_k\| \leq \chi_4,
\]

where \(\chi_4\) is a positive real number given as
\[
\chi_4 = \frac{1}{\rho_1} \left( \frac{\rho_2 h}{\rho_1} \left( (h_m + h\epsilon') \left( 1 + \mu(1 - (1 - \delta_d)^{N_{\text{max}}+1}) \right) + hf\epsilon' \mu(1 - \delta_d)(1 - (1 - \delta_d)^{N_{\text{max}}}) \right) \right)^2.
\]

Please note that \((1 - \delta_d) K \mu_k H_k\) \(F_{k-1} = A_k\). We now substitute Eqs. (66), (67), (70), and (72) into Eq. (62), and apply Lemma 2. Then, taking conditional expectation, we obtain
\[
E \left[ V(\mathbf{e}_k|\bar{k}) | e_{k-1|k-1} \right] \leq (1 - \phi) e_{k-1|k-1}^T P_{k-1|k-1}^{-1} e_{k-1|k-1} + E \left[ 2C_k^T P_{k-1|k}^{-1} (A_k e_{k-1|k-1} + B_k + D_k) | e_{k-1|k-1} \right] + \left( \chi_1 \epsilon'^3 + \chi_2 + \chi_3 \epsilon' + \chi_4 \right).
\]

For \(C_k\) defined in Eq. (38), we conclude that \(E[2C_k^T P_{k-1|k}^{-1} (A_k e_{k-1|k-1} + B_k + D_k) | e_{k-1|k-1} = 0\). Then, using the definition of \(V(\mathbf{e}_k)\) and substituting \(\chi_1 \epsilon'^3 + \chi_2 + \chi_3 \epsilon' + \chi_4 = \gamma'\), Eq. (74) can be expressed as
\[
E \left[ V(\mathbf{e}_k|\bar{k}) | e_{k-1|k-1} \right] - V(\mathbf{e}_{k-1|k-1} \leq \gamma' - \phi V(\mathbf{e}_{k-1|k-1} - \right).
\]

We select the parameters \(\gamma'\) and \(\phi\) such that \(V(\mathbf{e}_{k-1|k-1}) \geq \gamma' / \phi\) satisfies. Then, the above equation is the same as Eq. (41).

We now consider the inequality presented in Eq. (48). Taking inverse, then multiplying \(e_{k|k}^T\) and \(e_{k|k}\) from left and right side, respectively, we get
\[
\frac{1}{\rho_2^2} \| e_{k|k} \|^2 \leq V(\mathbf{e}_{k|k}) \leq \frac{1}{\rho_1 \rho_2} \| e_{k|k} \|^2.
\]

Please note that with \(1 / \rho_2 = \tau_1\) and \(1 / \rho_1 = \tau_2\), the above equation is same as Eq. (40). Therefore, we conclude that Eqs. (75) and (76) satisfy Eqs. (41) and (40), respectively. It further infers that, for chosen \(V(\mathbf{e}_k)\), \(\epsilon_1\) satisfies Eq. (42) and hence Eq. (60). Thus, we conclude that MDEKF’s estimation error remains exponentially bounded in mean square.

In this section, we proved the stochastic stability of the proposed method for its EKF-based formulation. The essential requirements of the proof involve a set of bounds on the system, noise, and filter parameters. Moreover, the stability analysis requires the initial estimation error to be bounded. Note that boundedness of noise and system parameters does not automatically imply exponential stability. The stability analysis of the proposed MDEKF reduces to the stability analysis of the traditional EKF for the delay and missing measurements probabilities being zero (\(\mu' = \delta = 1\) and \(N_{\text{max}} = 0\), considering the convention \(0^0 = 1\).

V. SIMULATION RESULTS

In real-life problems, the measurement systems (including the measuring devices and the supplementary units) are usually designed to efficiently capture the measurements. Therefore, they may not be expected to miss many measurements. Subsequently, the missing measurement probability is usually small. Thus, we consider the missing measurement probability up to 0.2 for the simulation, which means around 20% of the measurements are missing. On the other hand, the delay inherently appears in the measurements. Therefore, we consider a sufficiently large range of the delay probability (0.1 \leq 1 - \delta_d \leq 0.9). Moreover, it should be mentioned that the practical measurement systems are designed for small delays. Hence, we restrict the maximum possible delay to one or two time-steps (denoted as 1-delay and 2-delay scenarios), such as the delay up to one or two sampling intervals.
For the performance analysis, we considered three popular and advanced Gaussian filters, namely, the CKF [13], CQKF [14], and GHF [15]. With their extensions under the modified filtering method, which are abbreviated as MDCKF, MDCQKF, and MDGHF, respectively. We use the root mean square error (RMSE) as our performance metrics. Please note that we will frequently use the notation \( \mu_m = 1 - \mu \) to denote the missing measurements probability.

We compare the MDCKF with the following filters:

i) traditional CKF [13], ii) the CKF-based formulation of [22], which extends the Gaussian filtering technique for arbitrary delays, iii) [26], wherein the EKF is modified for missing measurements, and iv) the CKF-based formulation of [37], which considers simultaneously occurring delay and missing measurements. We abbreviate the CKF-based formulations of [22], and [37] as CKF_RD, and MLCKF, respectively, while the EKF-based formulation of [26] is abbreviated as MEKF. Please note that we use the EKF-based design of [26] unlike the CKF-based designs for other filters, as [26] is particularly designed for the EKF and becomes inapplicable to other filters.

A. PROBLEM 1

In the first problem, we consider a two-dimensional nonlinear dynamical system with the state-space model given as

\[
x_k = 2 \cos(x_{k-1}) + \eta_k \quad \text{and} \quad z_k = \sqrt{1 + x_k^2} x_k + \nu_k \quad [22].
\]

The true data of the state and measurement are generated by considering the initial state as \( x_0 = [0.1 \ 0.1]^T \).

The filter is initialized with the initial estimate \( \hat{x}_{0|0} = 0.9x_0 \) and \( P_{0|0} = 7I_2 \). The noise covariances are assigned as \( Q_k = 0.1I_2 \) and \( R_k = 0.1 \). The simulation is performed for 200 time-steps and the RMSEs are obtained by implementing 500 Monte-Carlo simulations.

Figs. 2 and 3 show the mean RMSE plots for varying delay probability under different scenarios formed by changing the maximum delay possibility and the missing measurements probability. It should be mentioned that the mean of the RMSEs is obtained over the 200 time-steps. The mean RMSE plots show a reduced RMSE for the MDCKF compared to the ordinary CKF, CKF_RD, MEKF, and MLCKF, which concludes that the proposed filtering method outperforms the ordinary Gaussian filtering as well as the existing filters for handling the delay and missing measurements. The relative computational times of the CKF, MEKF, CKF_RD, MLCKF, and MDCKF are obtained as 1, 2.86, 3.01, 5.78, and 3.04, respectively. It concludes that the computational time of the proposed method is marginally increased in comparison to some of the existing filters, while it remains marginally lower than others.

B. PROBLEM 2

In the second simulation problem, we consider an identification problem of individual sinusoids from the measurements of multiple superimposed sinusoids [9], [22]. We consider that the superimposed signal consists of three sinusoids. Identification of individual sinusoids is equivalent to estimating...
their amplitudes and frequencies from the measurements of the superimposed signal.

In conclusion to the above discussion, the state-dynamics is formed as $x_k = Ix_{k-1} + \eta_k$, where $x_k = [f_{1,k} \ f_{2,k} \ f_{3,k} \ a_{1,k} \ a_{2,k} \ a_{3,k}]^T$, with $a_n$ and $f_n$ representing the amplitude and frequency, respectively, for the $n^{th}$ sinusoid. Moreover, the measurement is a two-dimensional vector representing the superposition of the real and imaginary parts of the superimposed sinusoids. Please refer to [22] and [9] for a detailed discussion on the two models. The initial true and estimated states are taken as $x_0 = [200 \ 800 \ 1000 \ 2 \ 3]^T$ and $\hat{x}_{0|0} = [205 \ 785 \ 990 \ 4 \ 2 \ 3]^T$, respectively, while the initial covariance is taken as $P_{0|0} = \text{diag}([25 \ 50 \ 4 \ 1 \ 4])$. The noise covariances are taken as $Q_k = \text{diag}([0.01 \ 0.01 \ 0.04 \ 0.25 \ 0.25 \ 0.25])$ and $R_k = \text{diag}([0.9 \ 0.9])$.

The simulation is performed for 800 time-steps and 200 Monte-Carlo runs with a sampling interval of 0.25 milliseconds. Please note that the number of Monte-Carlo runs is reduced to 200 (in comparison to 500 runs used in the first problem), as the existing MEKF and MLCKF failed for higher number of Monte-Carlo runs. We obtain the RMSEs for the amplitude and frequency by taking the square root of the average of the mean square errors of the three amplitudes and frequencies, respectively.

We plot the mean RMSE (obtained over the time-steps) for varying delay probabilities in Figs. 4 and 5 for 1-delay and 2-delay, respectively. The mean RMSE plots show a reduced RMSE for the proposed MDCKF compared to the ordinary CKF and considered delay and missing filters. It concludes that the proposed filtering method has improved accuracy compared to these filters. The relative computational times for the MEKF, CKF, CKF_RD, MLCKF, and MDCKF are observed as 1, 1.56, 1.69, 3.24, and 1.72, respectively. It gives a similar conclusion as discussed in the previous problem.

### C. PERFORMANCE VALIDATION FOR OTHER GAUSSIAN FILTERS

To compare the estimation accuracy of the proposed method and the ordinary Gaussian filtering method, we present the mean RMSEs of CKF-based plots only in Figs. 2-5.
TABLE 2. Problem 2, one-delay scenario: Average RMSEs obtained by MDCQKF, MDGHF, and their counterparts for different delay probabilities.

| States | Filters     | \( \mu_m = 0.1 \) | \( \mu_m = 0.2 \) |
|--------|-------------|---------------------|---------------------|
|        |             | 0.2 | 0.5 | 0.8 | 0.2 | 0.5 | 0.8 |
| Amplitude | CQKF     | 8.78 | 10.24 | 10.54 | 9.52 | 10.33 | 10.51 |
|          | MDCQKF    | 5.55 | 7.36 | 8.70 | 6.10 | 7.61 | 9.10 |
|          | GHF       | 7.25 | 8.57 | 9.57 | 7.36 | 8.82 | 9.96 |
|          | MDGHF     | 5.01 | 5.48 | 6.79 | 4.15 | 5.06 | 6.62 |
| Frequency | CQKF    | 11.70 | 17.10 | 18.55 | 13.89 | 18.34 | 18.09 |
|          | MDCQKF   | 3.78 | 5.92 | 8.71 | 4.40 | 6.61 | 9.02 |
|          | GHF      | 9.27 | 13.60 | 18.66 | 10.60 | 12.86 | 17.57 |
|          | MDGHF    | 3.46 | 5.89 | 8.20 | 4.49 | 6.69 | 8.41 |

TABLE 3. Problem 1, one-delay scenario: Relative computational time comparison of MDCQKF, MDGHF, and their counterparts for 0.3 delay probability.

| \( \mu_m \) | Filters     | CQKF | MDCQKF | GHF | MDGHF |
|------------|-------------|------|--------|-----|-------|
| 0.1        |             | 1.091 | 0.812 | 1.083 |
| 0.2        |             | 1.1  | 0.813  | 0.908 |

However, we further extend the comparative analysis for the other advanced and popular Gaussian filters, such as the CQKF [14] and GHF [15], in Tables 1 and 2. The tables present the mean RMSEs obtained using the CQKF, GHF, MDCQKF, and MDGHF for various delay and missing measurement probabilities. From the tables, we conclude that RMSE is reduced for the MDCQKF and MDGHF compared with their traditional counterparts CKF and GHF, respectively. It is worth mentioning that the results for 2-delay scenarios are qualitatively very similar and are omitted for brevity. The computational times remain similar for the proposed and the existing Gaussian filtering methods (Table 3).

VI. DISCUSSION AND CONCLUSION

The manuscript introduces a new extension of Gaussian filtering to efficiently handle the simultaneously occurring delayed and missing measurements. The proposed method reformulates the measurement model stochastically to introduce the possibility of simultaneously occurring delayed and missing measurements. Subsequently, the proposed filtering method is designed by re-deriving the traditional Gaussian filtering method for the modified measurement model. We compare the proposed filter with the CKF and three well-known filters which handle the delay and missing measurements individually or simultaneously. The performance of the proposed method is validated for two simulation problems. We also studied the exponential stability of the proposed method for its EKF-based design. It is worth mentioning that the computational time of the proposed method remains similar to traditional Gaussian filtering.

DATA ACCESS STATEMENT

This research did not use any experimentally generated data or data from any publicly available dataset. Model definitions (including the specified probability distributions) and parameter values (including the initialization parameters) provided in the paper are adequate for reproducing the exact qualitative behavior of the algorithms illustrated in the paper.

REFERENCES

[1] Y. Bar-Shalom, X. R. Li, and T. Kirubarajan, *Estimation With Applications to Tracking and Navigation: Theory Algorithms and Software*. Hoboken, NJ, USA: Wiley, 2004.
[2] R. G. Brown and P. Hwang, *Introduction to Random Signals and Applied Kalman Filtering*, vol. 3. New York, NY, USA: Wiley, 1992.
[3] M. Dhanya and A. Chandrasekar, “Improved rainfall simulation by assimilating Oceansat-2 surface winds using ensemble Kalman filter for a heavy rainfall event over South India,” *IEEE Trans. Geosci. Remote Sens.*, vol. 52, no. 12, pp. 7721–7726, Dec. 2014.
[4] E. J. Lefferts, F. L. Markley, and M. D. Shuster, “Kalman filtering for spacecraft attitude estimation,” *J. Guid., Control, Dyn.*, vol. 5, no. 5, pp. 417–429, 1982.
[5] E. J. Knobbe and B. Buckingham, “The extended Kalman filter for continuous glucose monitoring,” *Diabetes Technol. Therapeutics*, vol. 7, no. 1, pp. 15–27, Feb. 2005.
[6] S. Huang, K. K. Tan, and T. H. Lee, “Fault diagnosis and fault-tolerant control in linear drives using the Kalman filter,” *IEEE/CAA J. Autom. Sinica*, vol. 6, no. 2, pp. 193–206, Apr. 2019.
[7] F. Gustafsson, “Particle filter theory and practice with positioning applications,” *IEEE Aerosp. Electron. Syst. Mag.*, vol. 25, no. 7, pp. 53–82, Jul. 2010.
[8] A. K. Singh, “Major development under Gaussian filtering since unscented Kalman filter,” 2019.
[9] A. K. Naik et al.: Gaussian Filtering for Simultaneously Occurring Delayed and Missing Measurements
[20] A. K. Singh, S. Bhaumik, and P. Date, “Quadrature filters for one-step randomly delayed measurements,” Appl. Math. Model., vol. 40, nos. 19–20, pp. 8296–8308, Oct. 2016.

[21] A. Hermoso-Carazo and J. Linares-Pérez, “Extended and unscented filtering algorithms using one-step randomly delayed observations,” Appl. Math. Comput., vol. 190, no. 2, pp. 1375–1393, Jul. 2007.

[22] A. K. Singh, P. Date, and S. Bhaumik, “A modified Bayesian filter for randomly delayed measurements,” IEEE Trans. Autom. Control, vol. 62, no. 1, pp. 419–424, Jan. 2016.

[23] A. Kumar Singh, P. Date, and S. Bhaumik, “New algorithm for continuous-discrete filtering with randomly delayed measurements,” IET Control Theory Appl., vol. 10, no. 17, pp. 2298–2305, Nov. 2016.

[24] L. Xu, K. Ma, and H. Fan, “Unscented Kalman filtering for non-linear state estimation with correlated noises and missing measurements,” IEEE Trans. Control Syst. Technol., vol. 24, no. 2, pp. 703–710, Mar. 2015.

[25] H. M. Faridani, “Performance of Kalman filter with missing measurements,” Automatica, vol. 22, no. 1, pp. 117–120, 1986.

[26] J. Hu, Z. Wang, H. Gao, and L. K. Stergioulas, “Extended Kalman filtering with stochastic nonlinearities and multiple missing measurements,” Automatica, vol. 48, no. 9, pp. 2007–2015, Sep. 2012.

[27] R. Razavi-Far, M. Farajzadeh-Zanjani, M. Saif, and S. Chakrabarti, “Correlation clustering imputation for diagnosing attacks and faults with missing power grid data,” IEEE Trans. Smart Grid, vol. 11, no. 2, pp. 1453–1464, Mar. 2019.

[28] J. Chen, Q. Zhu, and Y. Liu, “Modified Kalman filtering based multi-step-length gradient iterative algorithm for ARX models with random missing outputs,” Automatica, vol. 118, Aug. 2020, Art. no. 109034.

[29] L. Xu, K. Ma, and H. Fan, “Unscented Kalman filtering for non-linear state estimation with correlated noises and missing measurements,” Int. J. Control, Autom. Syst., vol. 16, no. 3, pp. 1011–1020, Jun. 2018.

[30] X. Luan, Y. Shi, and F. Liu, “Unscented Kalman filtering for greenhouse climate control systems with missing measurement,” Int. J. INNOV COMPUT, INFOR, AND CONTROL, vol. 8, no. 3, pp. 2173–2180, 2012.

[31] J. Hu, J. Li, Y. Kao, and D. Chen, “Optimal distributed filtering for nonlinear saturated systems with random access protocol and missing measurements: The uncertain probabilities case,” Appl. Math. Comput., vol. 418, Apr. 2022, Art. no. 126844.

[32] J. Liang, Z. Wang, and X. Liu, “State estimation for coupled uncertain stochastic networks with missing measurements and time-varying delays: The discrete-time case,” IEEE Trans. Neural Netw. Learn. Syst., vol. 20, no. 5, pp. 781–793, May 2009.

[33] K. J. Uribe-Murcia, Y. S. Shmaliy, C. Ki Ahn, and S. Zhao, “Unbiased FIR filtering for time-stamped discretely delayed and missing data,” IEEE Trans. Autom. Control, vol. 65, no. 5, pp. 2155–2162, May 2019.

[34] C. Ran and Z. Deng, “Robust fusion Kalman estimators for networked mixed uncertain systems with random one-step measurement delays, missing measurements, multiplicative noises and uncertain noise variances,” Inf. Sci., vol. 534, pp. 27–52, Sep. 2020.

[35] N. Modalalvalasa, G. S. B. Rao, K. S. Prasad, G. Laveti, and M. Kumar, “A new method of target tracking by EKF using bearing and elevation measurements for underwater environment,” Robot. Auto. Syst., vol. 74, pp. 221–228, Dec. 2015.

[36] R. Ramaswami, K. Sivarajan, and G. Sasaki, Optical Networks: A Practical Perspective. Burlington, MA, USA: Morgan Kaufmann, 2009.

[37] R. Esmzad and R. M. Esfanjani, “Bayesian filter for nonlinear systems with randomly delayed and lost measurements,” Automatica, vol. 107, pp. 36–42, Sep. 2019.

[38] K. Reif, S. Günther, E. Yaz, and R. Unbehauen, “Stochastic stability of the discrete-time extended Kalman filter,” IEEE Trans. Autom. Control, vol. 44, no. 4, pp. 714–728, Apr. 1999.

[39] T. Morozan, “On the stability of stochastic discrete systems,” in Control Theory and Topics in Functional Analysis. Vienna, Austria: International Atomic Energy Agency, 1976.

[40] T. Tarn and Y. Rasis, “Observers for nonlinear stochastic systems,” IEEE Trans. Autom. Control, vol. AC-21, no. 4, pp. 441–448, Aug. 1976.

[41] R. A. Horn and C. R. Johnson, Matrix Analysis. Cambridge, U.K.: Cambridge University Press, 2012.

AMIT KUMAR NAIK received the B.E. degree in electrical and electronics engineering from the Bhilai Institute of Technology Raipur, Raipur, India, in 2016, and the M.Tech. degree in power systems from the Maulana Azad National Institute of Technology Bhopal, Bhopal, India, in 2019. He is currently pursuing the Ph.D. degree with the Department of Electrical Engineering, Indian Institute of Technology Indore, Indore, India. His current research interests include nonlinear filtering and stochastic stability analysis.

GUDDU KUMAR received the B.Tech. degree in electronics and communication engineering from the West Bengal University of Technology, West Bengal, India, in 2014, and the M.Tech. degree from the Department of Electronics and Communication Engineering, NIT Patna, India, in 2019. He is currently pursuing the Ph.D. degree with the Electrical Engineering Department, IIT Indore, Madhya Pradesh, India. His research interests include the development of filtering algorithms for delay measurements and circular filtering.

PRABHAT KUMAR UPADHYAY (Senior Member, IEEE) received the Ph.D. degree in electrical engineering from the Indian Institute of Technology (IIT) Delhi, New Delhi, India, in 2011. He joined IIT Indore, in 2012, as an Assistant Professor at the Department of Electrical Engineering, where he has become an Associate Professor, in 2017 and a full Professor, in 2022. He is leading a Wireless Communication (WiCom) Research Group at IIT Indore, which is actively involved in cutting-edge research and development to cater to the emerging needs of the next generation wireless communication systems. He has more than 100 publications in peer-reviewed journals and conferences and has authored a book and six book chapters. His research interests include wireless and mobile communications, the Internet of Things (IoT) networks, signal processing, molecular communications, and nano-networks. He is a fellow of the Institution of Electronics and Telecommunication Engineers (IETE), a fellow of the Institution of Engineers (India), and a member of the IEEE Communications Society and the IEEE Vehicular Technology Society. He has been awarded the Nokia Foundation Visiting Professor Scholarship, in 2022. He has received the Sir Visvesvaraya Young Faculty Research Fellowship under the Ministry of Electronics and Information Technology, Government of India, and the IETE-ProF SVC IAI Memorial Award 2018. He was a co-recipient of the Best Paper Award at the International Conference on Advanced Communication Technologies and Networking, Marrakech, Morocco, in 2018. He received the Exemplary Editor Award of the IEEE COMMUNICATIONS LETTERS, in 2021. He has served as a Guest Editor for the Special Issue on Energy-Harvesting Cognitive Radio Networks in the IEEE TRANSACTIONS ON COGNITIVE COMMUNICATIONS AND NETWORKING. He serves as the Editor for IEEE COMMUNICATIONS LETTERS. He has also been involved in the Technical Program Committee of several IEEE premier conferences.

VOLUME 10, 2022
PARESH DATE received the master’s degree in electrical engineering from the Indian Institute of Technology Bombay, Mumbai, in 1995, and the Ph.D. degree from the Cambridge University Engineering Department (CUED), U.K. From 1995 to 1996, he has worked as an Engineering Executive with the Control and Automation Division, Larsen and Toubro Ltd. In July 2002, he has worked with CUED, as a Research Associate, before joining Brunel University London. He is currently a Reader and the Director of Research with the Department of Mathematics. His research has been funded by Grants from the Engineering and Physical Sciences Research Council, U.K., from charitable bodies, such as the London Mathematical Society, the Royal Society, and the industry. He has held several short term visiting appointments with the Indian Institute of Technology, Mumbai, from 2006 to 2007; the Indian Institute of Management Kolkata, in 2010; the Indian Institute of Technology Patna, in 2013, 2014, 2016, and 2018; and a six month Visiting Professorship with the Indian Institute of Technology Gandhinagar, in 2019. He has published more than 50 refereed papers and supervised ten Ph.D. students to completion, as the Principal Supervisor. His research interests include altering and its applications, especially in financial mathematics. He is a fellow of the Institute of Mathematics and its Applications. He is an Associate Editor for the IMA Journal of Management Mathematics.

ABHINOY KUMAR SINGH (Member, IEEE) received the B.Tech. degree in EEE from the Cochin University of Science and Technology (CUSAT), India, in 2012, and the Ph.D. degree in electrical engineering from the Indian Institute of Technology (IIT) Patna, India, in 2016. Previously, he worked at Shiv Nadar University, India, and McGill University, Canada. He is currently an Inspire Faculty at the EE Department, IIT Indore, India. He works on nonlinear filtering and continuous glucose monitoring.

* * *