Reduction of multipartite qubit density matrixes to bipartite qubit density matrixes and criteria of partial separability of multipartite qubit density matrixes

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Abstract

The partial separability of multipartite qubit density matrixes is strictly defined. We give a reduction way from N-partite qubit density matrixes to bipartite qubit density matrixes, and prove a necessary condition that a N-partite qubit density matrix to be partially separable is its reduced density matrix to satisfy PPT condition.

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Recently, an important task in modern quantum mechanics and quantum information is to find the criteria of separability of density matrixes. The first important result is the well-known positive partial transposition (PPT, Peres-Horodecki) criteria[1,2] for $2 \times 2$ and $2 \times 3$ systems. There are many studies about the criteria of separability for the multipartite systems, see [3-8].

Generally, the common so-called ‘separability’, in fact, is the full-separability. For multipartite systems the problems are more complex, there yet is other concept of separability weaker than full-separability, i.e. the ‘partial separability’, e.g. the A-BC-separability, B-AC-separability for a tripartite qubit pure-state $\rho_{ABC}$[8], etc.. Related to Bell-type inequalities and some criteria of
partial separability of multipartite systems, etc., see [9-12]. However, we yet need to stricter define the concept of partial separability and find the simpler criteria. In this paper, first we discuss how to define strictly the concept of the partial separability corresponding to a partition. Next, we give a new way that an arbitrary \( N \) -partite \( (N \geq 3) \) qubit density matrix always can be reduced in one step through to a bipartite qubit density matrix. Thus, we prove an effective criterion: A necessary condition of a \( N \)-partite qubit density matrix to be partially separable with respect to a partition is that the corresponding reduced bipartite qubit density matrix is separable, i.e. it satisfies the PPT condition. Some examples are given.

Suppose that \( \rho_{i_1i_2\cdots i_N} \) is a density matrix for \( N \)-partite qubit Hilbert space \( H = \bigotimes_{s=1}^{N} H_s \), of which the standard basis is \( \{ \bigotimes_{s=1}^{N} | i_s > \} \) \( (i_s = 0, 1) \). Let \( Z_N \) be the integer set \( \{ 1, 2, \cdots, N \} \). If two subsets \( (r)_P \equiv \{ r_1, \cdots, r_P \} \) and \( (s)_{N-P} \equiv \{ s_1, \cdots, s_{N-P} \} \) in \( Z_N \) obey

\[
1 \leq r_1 < \cdots < r_P < N, 1 < s_1 < \cdots < s_{N-P} \leq N
\]

\[
(r)_P \cup (s)_{N-P} = Z_N, (r)_P \cap (s)_{N-P} = \emptyset (1 \leq P < N)
\]

where \( P \) is an integer, \( 1 \leq P \leq N-1 \), the set \( \{(r)_P, (s)_{N-P}\} \) forms a partition of \( Z_N \), in the following we simply call it a ‘partition’, and for the sake of stress we denote it by symbol \( (r)_P \parallel (s)_{N-P} \). A partition \( (r)_P \parallel (s)_{N-P} \) corresponds to a permutation \( S_{(r)_P \parallel (s)_{N-P}} \equiv \begin{pmatrix} 1, \cdots, P, P+1, \cdots, N \\ r_1, \cdots, r_P, s_1, \cdots, s_{N-P} \end{pmatrix} \), by which a new matrix \( \rho_{(r)_P \parallel (s)_{N-P}} \) from \( \rho_{i_1i_2\cdots i_N} \) is defined now, whose entries are

\[
[\rho_{(r)_P \parallel (s)_{N-P}}]_{j_1\cdots j_N, k_1\cdots k_N} = [\rho]_{j_1\cdots j_P j_{s_1}\cdots j_{s_{N-P}} k_1\cdots k_P k_1\cdots k_{N-P}} \quad (2)
\]

For instance, \( \rho_{AB||CD} = \rho_{AB||CD} = \rho_{ABC||D} = \rho_{ABCD} \), and \( [\rho_{ABCD}]_{ijkl,rs} = [\rho_{ABCD}]_{ijkl,rs} = [\rho_{ABCD}]_{ijkl,rs} \), etc.. Generally, if \( \rho_{(r)_P \parallel (s)_{N-P}} \neq \rho_{i_1i_2\cdots i_N} \), unless \( (r)_P \parallel (s)_{N-P} \) just maintains the natural order of \( Z_N \) (i.e. \( (r)_P = (1, \cdots, P), (s)_{N-P} = (P+1, \cdots, N) \)), then \( \rho_{(r)_P \parallel (s)_{N-P}} = \rho_{i_1i_2\cdots i_N} \).

**Lemma.** For any partition \( (r)_P \parallel (s)_{N-P} \), \( \rho_{(r)_P \parallel (s)_{N-P}} \) is still a \( N \)-partite qubit density matrix.

**Proof.** We only consider the case of tripartite qubit, the general cases are completely similar (also see [11]). Notice the permutation \( S_{B||AC} \), then
we have

\[
\rho_{B||AC} = S \rho_{ABC} S^\dagger, \quad S = \begin{bmatrix}
1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
& 0 & 0 &
\end{bmatrix}
\]

(3)

\(S\) is an unitary matrix, therefore \(\rho_{B||AC}\) is still a tripartite qubit density matrix. \(\square\)

Now, we consider how to more strictly define the partial separability. Obviously, if a partition \(\rho(r)_P \parallel \rho(s)_{N-P}\) maintains the natural order of \(Z_N\) (i.e. \(\rho(r)_P = (1, 2, \ldots, P), \rho(s)_{N-P} = (P+1, P+2, \ldots, N)\)), then \(\rho(r)_P \parallel \rho(s)_{N-P}\) under the standard basis \(\{ \otimes_{s=1}^N | i_s \rangle \}\), now the \((r)_P - (s)_{N-P}\) separability can naturally be defined as that if \(\rho_{i_1i_2\cdots i_N}\) can be decomposed as

\[
\rho_{(r)_P \parallel (s)_{N-P}} = \rho_{i_1i_2\cdots i_N} = \sum_{\alpha} p_{\alpha} \rho_{\alpha, (r)_P} \otimes \rho_{\alpha, (s)_{N-P}}
\]

where \(\rho_{\alpha, (r)_P}\) and \(\rho_{\alpha, (s)_{N-P}}\), respectively, are a \(P\)-partite and a \((N-P)\)-partite qubit density matrices acting upon \(\otimes_{m=1}^P H_m\) and \(\otimes_{n=1}^{N-P} H_n\) for all \(\alpha\), then we call \(\rho_{i_1i_2\cdots i_N}\) to be \((r)_P - (s)_{N-P}\)-separable. However, if the natural order of \(Z_N\) has been broken in \((r)_P \parallel (s)_{N-P}\) (i.e. \(s_1 < r_P\)), then generally \(\rho_{(r)_P \parallel (s)_{N-P}} \neq \rho_{i_1i_2\cdots i_N}\), the case is different from the above. For instance, we consider a normalized pure-state \(\rho_{ABCD} = | \Psi_{ABCD} \rangle < \Psi_{ABCD} |\),

\[
| \Psi_{ABCD} \rangle \in H_A \otimes H_B \otimes H_C \otimes H_D
\]

of four spin-\(\frac{1}{2}\) particles A, B, C and D. Now, assume that \(| \Psi_{ABCD} \rangle\) has a special form as

\[
\sum_{i,j,k,l=0,1} c_{ik} c_{jl} | i_A > \otimes | j_B > \otimes | k_C > \otimes | l_D >, \quad \text{where} \quad c_{ik}, c_{jl} \in \mathbb{C}^1.
\]

If we keep up to use the original standard basis, then we cannot directly see the partial separability, because this choice of basis is unsuitable. If we choose other nature basis \(\{ | i_A > \otimes | k_C > \otimes | j_B > \otimes | l_D > \}\) (this, in fact, means that we are using \(\rho_{AC\parallel BD}\), under which we can consider the state

\[
| \Psi_{ACBD} \rangle = | \Psi_{AC} \rangle < \Psi_{AC} | \Psi_{BD} \rangle, \quad \text{where} \quad | \Psi_{AC} \rangle = \sum_{i,k=0,1} c_{ik} | i_A > \otimes | k_C >,
\]

\[
| \Psi_{BD} \rangle = \sum_{j,l=0,1} c_{jl} | j_B > \otimes | l_D >. \quad \text{Now,} \quad \rho_{AC\parallel BD} = \rho_{AC} \otimes \rho_{BD}, \quad \text{where},
\]

\[
\rho_{AC} = | \Psi_{AC} \rangle < \Psi_{AC} |, \quad \rho_{BD} = | \Psi_{BD} \rangle < \Psi_{BD} |. \quad | \Psi_{ABCD} \rangle \quad \text{and} \quad | \Psi_{ABCD} \rangle,
\]

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in fact, are the same in physics, therefore to call $\rho_{ABCD}$ AC-BD-separable is completely reasonable. Similarly, for the rest. Generalize to the cases of mixed-states, thus we can generally define the concept of partial separability as follows.

**Definition.** For the partition $(r)_P \parallel (s)_{N-P}$, a $N$-partite qubit density matrix $\rho_{i_1i_2...i_N}$ acting upon $H = \otimes_{s=1}^{N} H_s$ is called to be $(r)_P - (s)_{N-P}$-separable if the corresponding density matrix $\rho((r)_P || (s)_{N-P})$ can be decomposed as

$$\rho((r)_P || (s)_{N-P}) = \sum_{\alpha} \rho_{\alpha}(r)_P \otimes \rho_{\alpha}(s)_{N-P}$$  \hspace{1cm} (4)

where $\rho_{\alpha}(r)_P$ and $\rho_{\alpha}(s)_{N-P}$, respectively, are a $P$-partite and a $(N - P)$-partite qubit density matrices acting upon $\otimes_{m=1}^{P} H_{r_m}$ and $\otimes_{n=1}^{N-P} H_{s_n}$ for all $\alpha$, and $0 < p_{\alpha} \leq 1$, $\sum_{\alpha} p_{\alpha} = 1$. If $\rho_{i_1i_2...i_N}$ is not $(r)_P - (s)_{N-P}$-separable, then we call it $(r)_P - (s)_{N-P}$-inseparable.

For the distinct partitions $\rho_{i_1i_2...i_N}$ can have distinct separability. Of course, if a $\rho_{i_1i_2...i_N}$ is partially inseparable for some partition, then it must be entangled. Here, in passing, we point out that how to find the general relations between the partial separability and the ordinary separability (full-separability), generally, is not a simple problem. For instance, we can make such a multipartite qubit density matrix $\tilde{\rho}$ (similar to the theorem 1 in [13,14]), and by using of the technique in this paper, we can prove that $\tilde{\rho}$ always is partially separable for all possible partitions $(r)_P \parallel (s)_{N-P} (1 \leq P \leq N - 1)$, but $\tilde{\rho}$ is entangled (not full-separability).

In order to find the criteria of partial separability, first we discuss how to reduce a multipartite qubit density matrix in one step through to a bipartite qubit density matrix. For a given partition $(r)_P \parallel (s)_{N-P}$, let two sets $(r)_P$ and $(s)_{N-P}$, respectively be separated again as follows,

$$\begin{align*}
(r')_{P'} &= \{r'_{1}, \ldots, r'_{P'}\}, (r'')_{P''} = \{r''_{1}, \ldots, r''_{P''}\}, \text{ one of them can be the null set} \\
(s')_{Q'} &= \{s'_{1}, \ldots, s'_{Q'}\}, (s'')_{Q''} = \{s''_{1}, \ldots, s''_{Q''}\}, \text{ one of them can be the null set} \\
r'_{1} < r'_{2} < \cdots < r'_{P'}, & \quad r''_{1} < r''_{2} < \cdots < r''_{P''} \\
s'_{1} < s'_{2} < \cdots < s'_{Q'}, & \quad s''_{1} < s''_{2} < \cdots < s''_{Q''} \\
(r)_P &= (r')_{P'} \cup (r'')_{P''}, (r')_{P'} \cap (r'')_{P''} = \emptyset (0 \leq P', P'' \leq P \text{ and } P' + P'' = P) \\
(s)_{N-P} &= (s')_{Q'} \cup (s'')_{Q''}, (s')_{Q'} \cap (s'')_{Q''} = \emptyset (0 \leq Q', Q'' \leq N - P \text{ and } Q' + Q' = N - P)
\end{align*}$$  \hspace{1cm} (5)
now we rewrite the partition added these partitions as \([ (r^r)_{pr}, (r^m)_{pm} ] \parallel [ (s^s)_{Qs}, (s^m)_{Qm} ] \).

Now we define the matrix \(\rho[(r^r)_{pr}, (r^m)_{pm}] \parallel [(s^s)_{m-pr}, (s^m)_{m-pm}] \) by

\[
\rho[(r^r)_{pr}, (r^m)_{pm}] \parallel [(s^s)_{m-pr}, (s^m)_{m-pm}] = \text{the submatrix in } \rho_{i_1 \cdots i_N} \text{ consisting of all entries with form as } [\rho]_{x_1x_2 \cdots x_N, y_1y_2 \cdots y_N}
\]

which must be a \(4 \times 4\) matrix, where the values of \(x_k\) and \(y_k\) \((k = 1, \cdots, N)\), respectively, are determined by

\[
\begin{align*}
    x_k &= i \text{ for } k \in (r^r)_{pr}, \ x_k = 1 - i \text{ for } k \in (r^m)_{pm} \\
    x_k &= j \text{ for } k \in (s^s)_{Qs}, \ x_k = 1 - j \text{ for } k \in (s^m)_{Qm} \\
    y_k &= u \text{ for } k \in (r^r)_{pr}, \ y_k = 1 - u \text{ for } k \in (r^m)_{pm} \\
    y_k &= v \text{ for } k \in (s^s)_{Qs}, \ y_k = 1 - v \text{ for } k \in (s^m)_{Qm}
\end{align*}
\]

where \(i, j, u, v = 0, 1\). E.g.

\[
\rho[(AC)_{ij}][(B),(D)] = \text{the submatrix in } \rho_{ABCD} \text{ consisting of all entries with form as } [\rho]_{iji(1-j), uvu(1-v)} =
\]

\[
\begin{bmatrix}
    [\rho]_{0001,0001} & [\rho]_{0001,0100} & [\rho]_{0001,1011} & [\rho]_{0001,1110} \\
    [\rho]_{0100,0001} & [\rho]_{0100,0100} & [\rho]_{0100,1011} & [\rho]_{0100,1110} \\
    [\rho]_{1011,0001} & [\rho]_{1011,0100} & [\rho]_{1011,1011} & [\rho]_{1011,1110} \\
    [\rho]_{1110,0001} & [\rho]_{1110,0100} & [\rho]_{1110,1011} & [\rho]_{1110,1110}
\end{bmatrix}
\]

etc.. Now we define the \(4 \times 4\) matrix \(\rho[(r^r)_{pr}, (s^s)_{Qs}]\) by

\[
\rho[(r^r)_{pr}, (s^s)_{Qs}] = \sum \rho[(r^r)_{pr}, (r^m)_{pm}] \parallel [(s^s)_{Qs}, (s^m)_{Qm}] \\
\text{for all possible } [(r^r)_{pr}, (r^m)_{pm}] \parallel [(s^s)_{Qs}, (s^m)_{Qm}], \\
\text{and } \rho[(r^r)_{pr}, (r^m)_{pm}] \parallel [(s^s)_{Qs}, (s^m)_{Qm}] \text{ are not repeated}
\]

where we notice that there are indeed repeated \(\rho[(r^r)_{pr}, (r^m)_{pm}] \parallel [(s^s)_{m-pr}, (s^m)_{m-pm}]\),

in fact, \(\rho[(r^r)_{pr}, (r^m)_{pm}] \parallel [(s^s)_{m-pr}, (s^m)_{m-pm}] = \rho[(r^r)_{pr}, (r^m)_{pm}] \parallel [(s^s)_{m-pr}, (s^m)_{m-pr}]\).

For instance, we have

\[
\rho(A \cup BC) = \rho(A, B, C) + \rho(A, B, \emptyset) + \rho(A, \emptyset, B)
\]
\[
\rho_{(B-ACD)} = \rho [(B),\emptyset][[(ACD),\emptyset] + \rho [(B),\emptyset][[(AC),\emptyset] + \rho [(B),\emptyset][[(AD),\emptyset] + \rho [(B),\emptyset][[(A),\emptyset]]] \\
\rho_{(AC-BD)} = \rho [(AC),\emptyset][[(BD),\emptyset] + \rho [(AC),\emptyset][[(B),\emptyset] + \rho [(A),\emptyset][[(BD),\emptyset] + \rho [(A),\emptyset][[(B),\emptyset]]] \\
\rho_{(AC-BDE)} = \rho [(AC),\emptyset][[(BDE),\emptyset] + \rho [(AC),\emptyset][[(BD),\emptyset] + \rho [(AC),\emptyset][[(BE),\emptyset] + \rho [(A),\emptyset][[(BD),\emptyset]])] \\
+ \rho [(A),\emptyset][[(BE),\emptyset] + \rho [(A),\emptyset][[(BD),\emptyset]]] [10] \\
\] 

etc.

As an example, the above reduction procedures from \(\rho_{ABCD}\) to \(\rho_{(AC-BD)}\) can be described as \(\rho_{ABCD} \rightarrow \rho_{(AC-BD)} = \rho_{[(AC),\emptyset][[(BD),\emptyset] + \rho_{[(AC),\emptyset][[(B),\emptyset]]] + \rho_{[(A),\emptyset][[(BD),\emptyset] + \rho_{[(A),\emptyset][[(B),\emptyset]]]}}\)

\[\equiv \sigma_\Delta + \sigma_\times + \sigma_\circ + \sigma_\wedge,\] where the submatrixes \(\sigma_\Delta, \sigma_\times, \sigma_\circ\) and \(\sigma_\wedge\), respectively, consist of the entries ‘\(\Delta\)’, ‘\(\times\)’, ‘\(\circ\)’ and ‘\(\wedge\)’ in \(\rho_{ABCD}\) as in the following figure (\(\sigma_\times\) is just the matrix in Eq.(8))

\[
\begin{array}{cccccccccccc}
0000 & 0001 & 0010 & 0011 & 0100 & 0101 & 0110 & 0111 & 1000 & 1001 & 1010 & 1011 & 1100 & 1101 & 1110 & 1111 \\
\hline
0000 & \Delta & \Delta & \Delta & \Delta & \Delta & \Delta & \Delta & \Delta & \Delta & \Delta & \Delta & \Delta & \Delta & \Delta & \Delta \\
0001 & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\
0010 & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\
0011 & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\
0100 & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\
0101 & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\
0110 & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\
0111 & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\
1000 & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\
1001 & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\
1010 & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\
1011 & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\
1100 & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\
1101 & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\
1110 & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\
1111 & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\
\end{array}
\] [11]

Similarly, we can consider higher dimensional cases. As for the ordinary bipartite qubit density matrix \(\rho_{AB}\), we can take \(\rho_{(A-B)} \equiv \rho_{AB}\).

Sum up, generally we can define the \(4 \times 4\) matrix \(\rho_{(r),P} | (s),_{N-P}\) for a given \((r),_P | (s),_{N-P}\). In addition, it is easily verified that for any partition
$(r)_P \parallel (s)_{N-P}$, $\rho((s)_{N-P}-(r)_P)$ is the transposition of $\rho((s)_{N-P}-(r)_P)$, therefore from viewpoint of partial separability, we don’t have to distinguish between the partitions $(r)_P \parallel (s)_{N-P}$ and $(s)_{N-P} \parallel (r)_P$.

**Theorem 1.** For any partition $(r)_P \parallel (s)_{N-P}$, $\rho((r)_P-(s)_{N-P})$ is a bipartite qubit density matrix, therefore $\rho((r)_P-(s)_{N-P})$, in fact, is a reduction of the $N$-partite qubit density matrix $\rho_{i_1i_2\cdots i_N}$.

**Proof.** The fact must proved only is that $\rho((r)_P-(s)_{N-P})$ is surely a bipartite qubit density matrix. Here we only discuss in detail the cases of quadripartite qubit states, since the generalization is completely straightforward. In the first place, we prove that the theorem holds for a pure-state $\rho_{ABCD}$. Suppose that $\rho_{ABCD} = |\Psi_{ABCD}\rangle < \psi_{ABCD}$ is a normalized pure-state, where $|\Psi_{ABCD}\rangle = \sum_{i,j,k,l=0,1} c_{ijkl} |i_A \rangle \otimes |j_B \rangle \otimes |k_C \rangle \otimes |l_D \rangle$, $\sum_{i,j,k,l=0,1} |c_{ijkl}|^2 = 1$. Let

$$|\Phi_\Delta\rangle = \sum_{i,j=0,1} c_{ijij} |i_x \rangle \otimes |j_y \rangle, \quad |\Phi_x\rangle = \sum_{i,j=0,1} c_{ij(i-j)} |i_x \rangle \otimes |j_y \rangle \quad (12)$$

where $x$ and $y$ are form particles. Make normalization, we obtain $\rho_\Delta = |\varphi_\Delta\rangle < \varphi_\Delta |$, $|\varphi_\Delta\rangle = \eta_\Delta^{-1} |\Phi_\Delta\rangle$, $\rho_x = |\varphi_x\rangle < \varphi_x |$, $|\varphi_x\rangle = \eta_x^{-1} |\Phi_x\rangle$, $\rho_\omega = |\varphi_\omega\rangle < \varphi_\omega |$, $|\varphi_\omega\rangle = \eta_\omega^{-1} |\Phi_\omega\rangle$, where the normalization factors are

$$\eta_\Delta = \sqrt{\sum_{i,j=0,1} \left|c_{ijij}\right|^2}, \quad \eta_x = \sqrt{\sum_{i,j=0,1} \left|c_{ij(i-j)}\right|^2}$$

$$\eta_\omega = \sqrt{\sum_{i,j=0,1} \left|c_{ij(i-j)}\right|^2}, \quad \eta_\Lambda = \sqrt{\sum_{i,j=0,1} \left|c_{ij(1-i)(1-j)}\right|^2} \quad (13)$$

It can be directly verified that from Eq.(10) we have

$$\rho_{(AC-BD)} = \eta_\Delta^2 \rho_\Delta + \eta_x^2 \rho_x + \eta_\omega^2 \rho_\omega + \eta_\Lambda^2 \rho_\Lambda$$

where $\rho_\Delta, \rho_x, \rho_\omega, \rho_\Lambda$ all are bipartite qubit pure-states. It is easily seen that since $|\Psi_{ABCD}\rangle >$ is normalized, $\eta_\Delta^2 + \eta_x^2 + \eta_\omega^2 + \eta_\Lambda^2 = \sum_{i,j,k,l=0,1} |c_{ijkl}|^2 = 1$. This
means that $\rho_{(AC-BD)}$ is a bipartite qubit density matrix (a mixed state) for this pure-state $\rho_{ABCD}$.

Secondly, if $\rho_{ABCD} = \sum_{\alpha} p_\alpha \rho_{\alpha(ABCD)}$ is a mixed-state, where every $\rho_{\alpha(ABCD)}$ is a quardipartite qubit pure-state with probabilities $p_\alpha$, then from Eq.(10) we have $\rho_{(AC-BD)} = \sum_{\alpha} p_\alpha \rho_{\alpha(AC-BD)}$. Since every $\rho_{\alpha(AC-BD)}$ is a bipartite qubit density matrix, $\rho_{(AC-BD)}$ is a density matrix (a mixed-state).

A similar way can be extended to higher dimensional case, the key is that when $\rho_{i_1,\ldots,i_N}$ is a pure-state, then $\rho_{[(r')_{pr},(r'')_{pr}][[(s')_{m-pr},(s'')_{m-pr}]}$

$$= |\Psi^[(r')_{pr},(r'')_{pr}][[(s')_{m-pr},(s'')_{m-pr}]]><\Psi^[(r')_{pr},(r'')_{pr}][[(s')_{m-pr},(s'')_{m-pr}]]|,$$

where the pure-state

$$|\Psi^[(r')_{pr},(r'')_{pr}][[(s')_{m-pr},(s'')_{m-pr}]] = \sum_{i,j=0,1} c_{x_1x_2\ldots x_N} \mid x_1 > \otimes \cdots \otimes \mid x_N >$$

$(x_1, x_2, \ldots, x_N$ are determined by Eq.(7))

therefore we just have

$$|\Psi_{i_1,\ldots,i_N} >$$

$$= \sum_{\text{for all possible } [(r')_{pr},(r'')_{pr}][[(s')_{m-pr},(s'')_{m-pr}]}, (\text{and } |\Psi^[(r')_{pr},(r'')_{pr}][[(s')_{m-pr},(s'')_{m-pr}]] > \text{ are not repeated})$$

By using of this relation, make the similar states as in Eq.(12), and make generalization to mixes-states, we can prove that generally, a mixed-state density matrix $\rho_{i_1,\ldots,i_N}$ can be reduced through to the bipartite qubit density matrix $\rho_{(r)-(s)_{N-P}}$. □

The following theorem is the main result in this paper, it is an application of PPT condition for multipartite qubit systems.

**Theorem 2 (Criterion).** For a given partition $(r)_P \parallel (s)_{N-P}$, a necessary condition of a N-partite ($N \geq 3$) qubit density matrix $\rho_{i_1i_2\ldots i_N}$ to be $(r)_P \parallel (s)_{N-P}$-separable is that the reduced bipartite qubit density matrix is separable, i.e. $\rho_{(r)-(s)_{N-P}}$ satisfies the PPT condition.

**Proof.** We only discuss in detail the case of quadipartite qubit, it can be straightforwardly generalized to the case of arbitrary N-partite qubit. In the first place, we prove that this theorem holds for a quadipartite qubit pure-state. Suppose that the pure-state $\rho_{ABCD}$ is AC-BD-separable. This means
that if we choose the natural basis \{ | i_A > \otimes | j_C > \otimes | r_B > \otimes | s_D > \}, then 
\rho_{AC\|BD} = \rho_{AC} \otimes \rho_{BD}, \text{ where } \rho_{AC} = | \Psi_{AC} > < \Psi_{AC} |, \text{ and } \rho_{BD} = | \Psi_{BD} > < \Psi_{BD} |, \text{ and } | \Psi_{AC} > = \sum_{i,j=0,1} c_{ij} | i_A > \otimes | j_C >, \text{ and } \rho_{BD} = | \Psi_{BD} > < \Psi_{BD} |, \text{ and } | \Psi_{BD} > = 
\sum_{r,s=0,1} d_{rs} | r_B > \otimes | s_D >, \text{ and } \sum_{r,s=0,1} |d_{rs}|^2 = 1. \text{ From the above ways, it easily checked that the bipartite qubit density matrix } \rho_{(AC-\cdot BD)}; \text{ in fact, can be rewritten as }
\rho_{(AC-\cdot BD)} = \sigma_\Delta + \sigma_\chi + \sigma_\eta + \sigma_\lambda = \sigma_{(AC)} \otimes \sigma_{(BD)} + \sigma_{(AC)} \otimes \sigma_{(\cdot BD)} \tag{17}
\text{ where } \sigma_{(AC)} = | \Phi_{(AC)} > < \Phi_{(AC)} |, \text{ we already write } | \Phi_{(AC)} > = \sum_{i=0,1} e_i, \text{ and } | i_A > \otimes | i_C > \longrightarrow | i_x >. \text{ Similarly, } \sigma_{(AC)} = | \Phi_{(AC)} > < \Phi_{(AC)} |, | \Phi_{(AC)} > = \sum_{j=0,1} f_j, \text{ and } | j_B > \otimes | j_D > \longrightarrow | j_x >, \text{ and similarly for } \sigma_{(BD)}, \sigma_{(\cdot BD)}, \text{ etc..}
\text{ Now, } \rho_{(AC-\cdot BD)} \text{ can be written as }
\rho_{(AC-\cdot BD)} = \eta_{(AC)}^2 \rho_{(AC)} \otimes \rho_{(BD)} + \eta_{(AC)}^2 \rho_{(BD)} \otimes \rho_{(\cdot BD)} \tag{18}
+ \eta_{(AC)}^2 \rho_{(AC)} \otimes \rho_{(BD)} + \eta_{(AC)}^2 \rho_{(\cdot BD)} \otimes \rho_{(\cdot BD)}
\text{ where } \rho_{(AC)} = (\eta_{(AC)})^{-1} | \Phi_{(AC)} > < \Phi_{(AC)} |, \eta_{(AC)} = \sqrt{\sum_{i=0,1} |c_{ii}|^2}. \text{ Now, } \rho_{(AC)} \text{ is a density matrix of a single particle. Similarly, for } \rho_{(AC)}, \rho_{(BD)}, \rho_{(\cdot BD)}. \text{ Since }
\eta_{(AC)}^2 \rho_{(BD)} + \eta_{(AC)}^2 \rho_{(BD)} + \eta_{(AC)}^2 \rho_{(BD)} + \eta_{(AC)}^2 \rho_{(BD)}
= \left( \eta_{(AC)}^2 + \eta_{(AC)}^2 \right) \left( \eta_{(BD)}^2 + \eta_{(BD)}^2 \right) = 1 \tag{19}
\text{ therefore } \rho_{(AC-\cdot BD)} \text{ is a separable bipartite qubit mixed-state. The PPT condition for separability of } 2 \times 2 \text{ systems is sufficient and necessary[2], thus } \rho_{(AC-\cdot BD)} \text{ satisfies the PPT condition. Similarly, for other partial separability.}
Secondly, we prove that this theorem holds yet for partially separable mixed-states. Suppose that $\rho_{ABCD}$ is a AC-BD-separable mixed-state, then under the same natural basis there is a decomposition as $\rho_{AB\parallel CD} = \sum_{\alpha} p_{\alpha} \rho_{\alpha (AC)} \otimes \rho_{\alpha (BD)}$, where $\rho_{\alpha (AC)}$ and $\rho_{\alpha (BD)}$ both are bipartite qubit pure-states as in the above for all $\alpha$, $0 < p_{\alpha} \leq 1$, $\sum_{\alpha} p_{\alpha} = 1$. From the above reduction operation, obviously we have

$$\rho_{(AC-BD)} = \sum_{\alpha} p_{\alpha} \left[ \rho_{\alpha (AC)} \otimes \rho_{\alpha (BD)} \right]_{(AC-BD)}$$

(20)

According to the above mention, every $\left[ \rho_{\alpha (AC)} \otimes \rho_{\alpha (BD)} \right]_{(AC-BD)}$ is a separable bipartite qubit mixed-state, this leads to that the convex sum $\rho_{(AC-BD)}$ in Eq.(20) still is a separable bipartite qubit mixed-state, and it must satisfy the PPT condition.

Similarly, we can prove higher dimensional cases. □

**Corollary.** If the reduced bipartite qubit density matrix $(\rho_{1i_2 \cdots i_N})_{(r)_P-(s)_{N-P}}$ violates the PPT condition for a partition $(r)_P \parallel (s)_{N-P}$, then $\rho_{1i_2 \cdots i_N}$ is $(r)_P - (s)_{N-P}$ inseparable and entangled.

It, in fact, is the inverse-negative proposition of Theorem 2.

**Examples.** Consider two tripartite qubit density matrixes

$$\rho'_{ABC} = \left[ \begin{array}{cccccc}
0 & \frac{1-x}{4} & \frac{1-x}{4} & 0 & \frac{x}{2} & \frac{-x}{2} \\
\frac{1-x}{4} & 0 & 0 & \frac{x}{2} & \frac{-x}{2} & \frac{1-x}{4} \\
0 & 0 & 0 & 0 & \frac{x}{2} & \frac{1-x}{4} \\
\frac{x}{2} & \frac{-x}{2} & \frac{1-x}{4} & 0 & 0 & \frac{x}{2} \\
\frac{-x}{2} & \frac{1-x}{4} & 0 & 0 & 0 & \frac{x}{2} \\
\frac{1-x}{4} & \frac{1-x}{4} & 0 & 0 & 0 & 0
\end{array} \right]$$

$$\rho''_{ABC} = \left[ \begin{array}{cccccc}
0 & \frac{1-x}{4} & 0 & 0 & -\frac{x}{2} \\
\frac{1-x}{4} & 0 & \frac{1-x}{4} & 0 & 0 \\
0 & \frac{1-x}{4} & 0 & 0 & \frac{x}{2} \\
0 & 0 & \frac{x}{2} & 0 & 0 \\
-\frac{x}{2} & 0 & 0 & \frac{x}{2} & 0 \\
\frac{1-x}{4} & 0 & 0 & 0 & 0
\end{array} \right]$$

(21)
then we have

$$(\rho'_{ABC})_{(A-BC)} = (\rho''_{ABC})_{(B-AC)} = \rho_W$$

(22)

where $\rho_W$ is the Werner state[1,15] which consists of a singlet fraction $x$ and a random fraction $(1-x)$,

$$[\rho_W]_{ij,rs} = x S_{ij,rs} + \frac{1}{4} (1-x) \delta_{ir} \delta_{js}$$

$$S_{01,01} = S_{10,10} = -S_{01,10} = -S_{10,01} = \frac{1}{2}$$

(23)

and all the other components of $S$ vanish.

It is known[1] that when $\frac{1}{3} < x \leq 1$ $\rho_W$ violates the PPT condition, it leads to that $\rho'_{ABC}$ is A-BC-inseparable and $\rho''_{ABC}$ is B-AC-inseparable.

By using of the above theorems and corollary, in some special cases we can make a $N$-partite qubit from $2^{N-2}$ bipartite qubit density matrixes, which is partially inseparable for a given partition. As in the above, for the case of tripartite qubit we take two bipartite qubit density matrixes $\sigma_{(1)}$, $\sigma_{(2)}$ and real numbers $p_1$, $p_2$, $0 < p_1, p_2 \leq 1$ such that $\sigma = p_1 \sigma_{(1)} + p_2 \sigma_{(2)}$ is a bipartite qubit entangled state (then it violates the PPT condition). If we want to construct a tripartite qubit entangled state $\rho_{ABC}$ which is B-AC-inseparable, then we can take the entries of $\rho_{ABC}$ by

$$[\rho_{ABC}]_{ijk, rst} = p_1 [\sigma_{(1)}]_{ji, sr}, \text{ for } k = i \text{ and } t = r$$

$$[\rho_{ABC}]_{ijk, rst} = p_2 [\sigma_{(2)}]_{ji, sr}, \text{ for } k = 1 - i \text{ and } t = 1 - r$$

(24)

$$[\rho_{ABC}]_{ijk, rst} = 0, \text{ for the rest } (i, j, k, r, s, t = 0, 1)$$

It can be verified that $\rho_{ABC}$ is a tripartite qubit density matrix, and is B-AC-inseparable. In fact, $\langle \rho_{ABC} \rangle_{(B-AC)} = \tau$ which violates the PPT condition. Similarly, for A-BC and C-AB. The above way can be generalized to obtain a $(r)_p - (s)_{N-p}$-inseparable density matrix from a bipartite qubit entangled state in form as $\tau = \sum_{i=1}^{2^{N-2}} p_i \sigma_{(i)}$, where all $\sigma_{(i)}$ are some bipartite qubit density matrixes.

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