SOLITON DYNAMICS FOR A NON-HAMILTONIAN PERTURBATION OF MKDV

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ABSTRACT. We study the dynamics of soliton solutions to the perturbed mKdV equation \( \partial_t u = \partial_x (-\partial_x^2 u - 2u^3) + \epsilon Vu \), where \( V \in C^1_b(\mathbb{R}) \), \( 0 < \epsilon \ll 1 \). This type of perturbation is non-Hamiltonian. Nevertheless, via symplectic considerations, we show that solutions remain \( O(\epsilon t^{1/2}) \) close to a soliton on an \( O(\epsilon^{-1}) \) time scale. Furthermore, we show that the soliton parameters can be chosen to evolve according to specific exact ODEs on the shorter, but still dynamically relevant, time scale \( O(\epsilon^{-1/2}) \). Over this time scale, the perturbation can impart an \( O(1) \) influence on the soliton position.

1. Introduction

We consider the modified Korteweg-de Vries (mKdV) equation with a small external potential

\[
\partial_t u = \partial_x (-\partial_x^2 u - 2u^3) + \epsilon Vu .
\]

where \( 0 < \epsilon \ll 1 \), \( V \in C^1_b(\mathbb{R}) \), i.e. \( V \) and \( V' \) are continuous and bounded.

The unperturbed case of (1.1),

\[
\partial_t u = \partial_x (-\partial_x^2 u - 2u^3)
\]

is globally well-posed in \( H^k \) for \( k \geq 1 \) (see Kenig-Ponce-Vega [19]), and possesses single soliton solutions \( u(x, t) = \eta(x, a + c^2 t, c) \), for \( a \in \mathbb{R} \) and \( c \in \mathbb{R} \setminus \{0\} \), where \( \eta(x, a, c) = cQ(c(x - a)) \) with \( Q(x) = \text{sech}(x) \) (so that \( -Q + Q'' + 2Q^3 = 0 \)). The solitons are orbitally stable as solutions to the unperturbed mKdV (1.2) (see [3, 4, 28, 7]), i.e. the solutions stay close to the soliton manifold

\[
M = \{ \eta(x, a, c) | a \in \mathbb{R}, c > 0 \}
\]

if they are initially close.

Our first main result, Theorem 1.1, shows that this type of orbital stability remains true for the structurally perturbed mKdV (1.1), in the following sense: solutions which start an \( H^1_x \) distance \( \omega \) from the soliton manifold \( M \) remain within an \( H^1_x \) distance \( (\omega + \epsilon t^{1/2})e^{Ct} \) up to time \( \epsilon^{-1} \log \epsilon^{-1} \). Our second main result result, Theorem 1.2, shows that on the shorter time scale \( \epsilon^{-1/2} \log \epsilon^{-1} \), we can predict the location on the soliton manifold by solving a system of two ODE for the position parameter \( a \) and scale parameter \( c \). Strong agreement between this prediction and the numerical
solution of (1.1) is illustrated in Fig. 1.1 and Fig. 1.2. We prove the global well-posedness of (1.1) in $H^1$, by adapting the argument of Kenig-Ponce-Vega [19], in Apx. A.

The forced KdV equation

$$\partial_t u = \partial_x (-u_{xx} - 3u^2) + \epsilon f$$

is a model for free-surface shallow water flow [20] with contributions to $f$ arising from surface pressure and bottom topography. Numerics and experiments discussed in [20] show that this type of perturbation can effect the evolution of a single soliton by generating a procession of small solitons ahead of, and dispersive waves behind, the primary soliton.

Both (1.1) and (1.3) are specific instances of a family of gKdV equations with general perturbation

$$\partial_t u = \partial_x (-u_{xx} - u^p) + \epsilon f$$

for $p \in \mathbb{N}$, $p \geq 2$, and $f = f(x, t, u)$. The case $p = 3$ (mKdV) is the unique member of the gKdV family that avoids a certain anomaly with the symplectic structure. Specifically, for $p = 3$, one has $\partial_x^{-1}\partial_x \eta \in L^2$ but this fails for $p \neq 3$. For $p = 3$, one can symplectically project onto the tangent space of the soliton manifold $M$ rather than a skew space. The difference between $p = 3$ and $p \neq 3$ is illustrated in the fact that the local virial estimate of Martel-Merle [21] simplifies for $p = 3$. Nevertheless, we believe that the analysis of the paper carries over in some form to $p \neq 3$ and more general $f$ of the form $f(x, t, u)$. We chose (1.1) as the mathematically simplest case in which to illustrate our method.

1.1. Statements of main results.

**Theorem 1.1** (orbital stability). Let $\delta > 0$ and $a_0, c_0 \in \mathbb{R}$ such that $2\delta \leq c_0 \leq (2\delta)^{-1}$. Suppose $u(x, t)$ solves (1.1) with initial data $u(x, 0)$ such that

$$\omega \overset{\text{def}}{=} \|u(x, 0) - \eta(x, a_0, c_0)\|_{H^1_x} \lesssim \epsilon^{1/2}$$

Then there exist trajectories $a(t)$ and $c(t)$ so that the following hold, where $T$ is the maximum time such that $\delta \leq c(t) \leq \delta^{-1}$ for all $0 \leq t \leq T$ and $w(x, t) \overset{\text{def}}{=} u(x, t) - \eta(x, a(t), c(t))$. First, we have the following bounds on the deviation $w$:

$$\|w\|_{L^1_{[0, t]}H^1_x} + \|e^{-\alpha|x-a|}w\|_{L^2_{[0, t]}H^1_x} \leq C(\omega + \epsilon t^{1/2})e^{C\epsilon t}$$

Second, we have $T \geq C^{-1}\epsilon^{-1}$ and the following estimates for the trajectories $a(t)$ and $c(t)$:

$$\|\dot{a} - c^2 - \epsilon c^{-1}\{V\eta, (x - a)\eta\}\|_{L^1_{[0, t]} \cap L^\infty_{[0, t]}} + \|\dot{c} - \epsilon \{V\eta, \eta\}\|_{L^1_{[0, t]} \cap L^\infty_{[0, t]}} \leq C(\omega + \epsilon t^{1/2})^2 e^{C\epsilon t}$$

The constants $C$ in (1.4), (1.5) depend on $\|V\|_{C^1}$ and $\delta$. 

We remark that the same result holds for $c_0 < 0$, since $\eta(x, a, -c) = -\eta(x, a, c)$.

**Theorem 1.2** (exact predictive dynamics). Suppose $u(x, t)$ solves (1.1) with initial data $u(x, 0)$ satisfying

$$\omega \overset{\text{def}}{=} \|u(x, 0) - \eta(x, a_0, c_0)\|_{H^1_x} \lesssim \epsilon^{1/2}$$

where $c_0 > 0$. Let $(a(t), c(t))$ evolve according to the ODE system

$$\begin{align*}
\dot{a} &= c^2 + \epsilon c^{-1}\langle V\eta, (x-a)\eta \rangle \\
\dot{c} &= \epsilon \langle V\eta, \eta \rangle
\end{align*}$$

(1.6)

with initial data $a(0) = a_0$, $c(0) = c_0$. Then for

$$0 \leq t \leq T = \sigma \epsilon^{-1/2} \log \epsilon^{-1}, \quad \sigma = \sigma(c_0, \|V\|_{C^1_b}) > 0,$$

we have the following estimates with $w(x, t) = u(x, t) - \eta(x, a(t), c(t))$:

$$\|w\|_{L^\infty_{[0,t]}H^1_x} + \|e^{-\alpha|x-a|}w\|_{L^1_{[0,t]}H^1_x} \leq C(\omega + \epsilon t^{1/2})e^{C\epsilon^{1/2}t}.$$  

(1.7)

where $C = C(c_0, \|V\|_{C^1_b})$.

We remark that if one selects initial data so that $\omega \lesssim \epsilon^{3/4}$, then the two terms on the right-side of the estimate (1.7) balance on the $\epsilon^{-1/2}$ time scale. In this case the bound becomes $\epsilon^{3/4}e^{C\epsilon^{1/2}t}$.

1.2. **Relation to recent work.** The energy-Lyapunov based methods for proving orbital stability of solitons subject to perturbations (of the data, as opposed to the structural perturbations considered here) were developed by Benjamin [3], Bona [4], Weinstein [28], Grillakis-Shatah-Strauss [11, 12]. In the last decade several results have emerged using the same basic framework to address the dynamics of solitons for equations subject to structural perturbations [6, 9, 10, 13, 14, 16, 17, 18, 8, 1, 2, 23, 24]. The nonlinear Schrödinger equation (NLS) with slowly varying potential was considered by Fröhlich-Gustafson-Jonsson-Sigal [9] and a result of “orbital stability” type was obtained, however the estimates were not strong enough to obtain “exact predictive dynamics”. Holmer-Zworski [18] obtained exact predictive dynamics plus refined accuracy by adopting the conceptual perspective of symplectic projection, but also, at the technical level, finding an appropriate distortion of the soliton manifold that enabled refined Lyapunov estimates. This “symplectic projection plus correction term method” has been subsequently pursued in different contexts in Datchev-Ventura [8], Holmer-Lin [14], Holmer-Perelman-Zworski [16], and Pocovnicu [25]. To treat a problem in which the perturbation gives rise to significant dispersive radiation, a different approach was employed by Holmer [13]. He treated the KdV equation with a slowly varying potential, and used the Martel-Merle local virial estimate [23, 24] to supplement the energy Lyapunov estimate. In this paper, we follow this approach as well. We show the method is sufficiently robust to handle small non-Hamiltonian
Figure 1.1. With external potential given by $V_1$ as in (1.8), the top plot gives the rescaled evolution $U(X, T)$, the bottom two plots give the comparison between the evolution of the parameters obtained solving the ODE system and exact PDE evolution, i.e. we fit the solution to $\eta(X, \tilde{A}, \tilde{C})$, and plot $T$ versus $\tilde{A}$ and $\tilde{C}$ respectively.

perturbations, which had not been considered in any of the above papers. A stochastic variant of the problem we consider has been addressed by de Bouard–Debussche [5] without the use of the local virial estimate. Work in progress by Holmer-Setayeshgar [15] will adapt the present paper to the stochastic setting and obtain a refinement of the results of [5].

1.3. Numerics. To solve (1.1) numerically we adapt the method in [26] which is based on the fast fourier transform in $x$, then fourth-order Runge-Kutta for the resulting ODE in $t$. We use the rescaled coordinate frame $X = \epsilon^{-1/3}x$, $T = \epsilon^{-1}t$, and consider the equation on $[-\pi, \pi)$. If $U(X, T)$ solves

$$
\partial_T U = \partial_X (-\partial_X^2 U - 2U^3) + V(X)U,
$$
Figure 1.2. These plots are analogs of Fig. 1.1, the external potential is given by \( V_2 \) as in (1.9).

with initial data

\[
U(0, X) = \eta(X, A_0, C_0) = \eta(X, \epsilon^{1/3}a_0, \epsilon^{-1/3}c_0),
\]

then \( u(x, t) = \epsilon^{1/3}U(\epsilon^{1/3}x, \epsilon t) \) gives a solution of (1.1) on \([-\pi/\epsilon^{1/3}, \pi/\epsilon^{1/3}]\) with initial data \( u(0, x) = \eta(x, a_0, c_0) \), and periodic boundary conditions. Fig. 1.1 and Fig 1.2 plot the evolution of the soliton initial data (after rescaling) in the following external potential respectively

(1.8) \[
V_1 = -10 \cos^2(6X) + 6 \sin(10X),
\]

(1.9) \[
V_2 = 8 \cos^2(4X) - 4 \sin(2X).
\]

Note that to examine the solution \( u(x, t) \) on time interval \( 0 \leq t \leq C\epsilon^{-1/2}(or C\epsilon^{-1}) \), we should let \( U(X, T) \) evolve for time \( C\epsilon^{1/2}(or C) \).
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2. **Background on Hamiltonian structure**

Let $J = \partial_x$, and consider $L^2(\mathbb{R} \mapsto \mathbb{R})$ as a manifold with metric $\langle v_1, v_2 \rangle = \int v_1 v_2 \, dx$, we can define the symplectic form as

\[
\omega(v_1, v_2) = \langle v_1 J^{-1} v_2 \rangle = \langle v_1, \partial_x^{-1} v_2 \rangle,
\]

where $J^{-1}$ is given by

\[
J^{-1} f(x) = \partial_x^{-1} f(x) \overset{\text{def}}{=} \frac{1}{2} \left( \int_{-\infty}^{x} - \int_{x}^{+\infty} \right) f(y) \, dy.
\]

The mKdV equation (1.2) is the Hamiltonian flow associated with

\[
H_0(u) = \frac{1}{2} \int (u_x^2 - u^4),
\]

i.e. we can write (1.2) as

\[
\partial_t u = J H'_0(u).
\]

Solutions to mKdV also satisfy conservation of mass $M(u)$ and momentum $P(u)$, where

\[
M(u) = \int u \, dx, \quad P(u) = \frac{1}{2} \int u^2 \, dx.
\]

We define 2-dimensional manifold of solitons $M$ as

\[
M = \{ \eta(\cdot, a, c) \mid a \in \mathbb{R}, c \in \mathbb{R} \setminus \{0\} \}.
\]

The symplectic form (2.1) restricted to $M$ is given by $\omega|_M = da \wedge dc$. We denote $\eta = \eta(\cdot, a, c)$, the dependence of $(a, c)$ on $\eta$ is always meant implicitly. The tangent space at $\eta$ is given by

\[
T_\eta M = \text{span}\{ \partial_a \eta, \partial_c \eta \}.
\]

Note that $J H'_0(\eta) \in T_\eta M$, thus the flow associated to (1.2) will remain on $M$ if it is initially. Specifically, direct computation shows

\[
J H'_0(\eta) = c^2 \partial_a \eta.
\]

which, together with (2.2), explains the form of the expression for single solitons. This is equivalent to saying that the flow (2.2) restricted to $M$ (and thus stays on $M$) is given by

\[
\begin{cases}
\dot{a} = c^2 \\
\dot{c} = 0
\end{cases}
\]
One can also get (2.4) by first restricting \( H_0 \) to \( M \) to obtain
\[
H_0(\eta) = -\frac{1}{3}c^3,
\]
and then noticing that (2.4) is just the solution to the Hamilton equations of motion for \( H_0(\eta) \) with respect to \( \omega|_M \):
\[
\begin{aligned}
\dot{a} &= -\frac{\partial H_0}{\partial c} = c^2 \\
\dot{c} &= \frac{\partial H_0}{\partial a} = 0
\end{aligned}
\]
(2.5)

Note that we can write (2.3) as
\[
JH_0'(\eta) + c^2JP'(\eta) = 0.
\]
(2.6)

From this, we learned that \( L'(\eta) = 0 \), where
\[
L(u) \overset{\text{def}}{=} H_0(u) + c^2P(u).
\]
(2.7)

which is the Lyapunov functional used in the classical orbital stability theory, see [28].

Next, we define the symplectic orthogonal projection operator at \((a,c)\):
\[
\Pi_{a,c} : L^2 \cong T_\eta L^2 \to T_\eta M,
\]
by requiring that
\[
\langle \Pi_{a,c}^\perp f, J^{-1}\partial_a\eta \rangle = \langle \Pi_{a,c}^\perp f, J^{-1}\partial_c\eta \rangle = 0,
\]
where \( \Pi_{a,c}^\perp = I - \Pi_{a,c} \), equivalently,
\[
\Pi_{a,c}f = \langle f, J^{-1}\partial_a\eta \rangle \partial_a\eta - \langle f, J^{-1}\partial_c\eta \rangle \partial_c\eta.
\]

Note that for mKdV,
\[
J^{-1}\partial_a\eta = -\eta \quad \text{and} \quad J^{-1}\partial_c\eta = c^{-1}(x-a)\eta.
\]

3. DECOMPOSITION OF THE FLOW

We can arrange the modulation parameters \( a(t) \) and \( c(t) \) so that
\[
\Pi_{a(t),c(t)}[u(x,t) - \eta(x,a(t),c(t))] = 0.
\]
This is a standard fact and we recall it in the following

**Lemma 3.1.** Given \( \tilde{a}, \tilde{c} \), there exist \( \delta_1 > 0, C > 0 \), such that if \( u = \eta(\cdot, \tilde{a}, \tilde{c}) + \tilde{w} \) with \( \|\tilde{w}\|_{H^1_x} \leq \delta_1 \), then there exist unique \( a, c \) such that
\[
w(x,t) \overset{\text{def}}{=} u(x,t) - \eta(x,a(t),c(t))
\]
satisfies the symplectic orthogonality conditions
\[
\langle w, J^{-1}\partial_a\eta \rangle = \langle w, J^{-1}\partial_c\eta \rangle = 0.
\]
Moreover,
\[ |a - \tilde{a}| \leq C\|\tilde{w}\|_{H^1} \quad |c - \tilde{c}| \leq C\|\tilde{w}\|_{H^1} \]

Proof. Define \( \phi : H_x^1 \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^2 \) by
\[
\phi(v, a, c) = \begin{bmatrix} \langle v - \eta, \eta \rangle \\ \langle v - \eta, (x - a)\eta \rangle \end{bmatrix}
\]
Using \( \omega|_M = da \wedge dc \), we can get the Jacobian matrix of \( \phi \) with respect to \((a, c)\) at \((\eta(\cdot, \tilde{a}, \tilde{c}), \tilde{a}, \tilde{c})\)
\[
(D_{a,c} \phi)(\eta(\cdot, \tilde{a}, \tilde{c}), \tilde{a}, \tilde{c}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\]
which implies, by the implicit function theorem, that the equation \( \phi(u, a, c) = 0 \) can be solved for \((a, c)\) in terms of \( u \) in a neighborhood of \((\cdot, \tilde{a}, \tilde{c})\).

Now since \( u = w + \eta \) and \( u \) solves \((1.1)\), we compute
\[
\partial_t w = \partial_x(-\partial_x^2 w - 6\eta^2 w - 6\eta w^2 - 2w^3) + \epsilon V w - F_0
\]
\[
= \partial_x(L w - c^2 w - 6\eta w^2 - 2w^3) + \epsilon V w - F_0,
\]
where
\[
L = -\partial_x^2 - 6\eta^2 + c^2,
\]
and \( F_0 \) results from the perturbation and \( \partial_t \) landing on the parameters:
\[
F_0 = (\dot{a} - c^2)\partial_a \eta + \dot{c} \partial_c \eta - \epsilon V \eta.
\]
Next, decompose \( F_0 \) into the symplectically parallel part \( \Pi_{a,c} F_0 \) and symplectically orthogonal part \( \Pi_{a,c}^\perp F_0 \), explicitly,
\[
\Pi_{a,c} F_0 = (\dot{a} - c^2 - \epsilon \langle V \eta, J^{-1} \partial_a \eta \rangle)\partial_a \eta + (\dot{c} + \epsilon \langle V \eta, J^{-1} \partial_a \eta \rangle)\partial_c \eta,
\]
\[
\Pi_{a,c}^\perp F_0 = -\epsilon V \eta + \epsilon \langle V \eta, J^{-1} \partial_a \eta \rangle\partial_a \eta - \epsilon \langle V \eta, J^{-1} \partial_a \eta \rangle\partial_c \eta.
\]
We now obtain the equations for the parameters:

**Lemma 3.2** (effective dynamics). Given \( V \in C^1_b \), suppose that \( w \) defined by \((3.1)\) satisfies the orthogonality conditions \((3.2)\). Then there exists \( \alpha > 0 \) such that
\[
\| \partial_t \eta - c^2 \partial_a \eta - \epsilon \Pi_{a,c}(V \eta)\|_{T_{a,c} M} \lesssim \| e^{-\alpha|x-a|} w \|_{H^1}^2 + \epsilon \| e^{-\alpha|x-a|} w \|_{H^1}.
\]
Explicitly,
\[
|\dot{a} - c^2 - \epsilon \langle V \eta, J^{-1} \partial_a \eta \rangle| \lesssim \| e^{-\alpha|x-a|} w \|_{H^1}^2 + \epsilon \| e^{-\alpha|x-a|} w \|_{H^1},
\]
\[
|\dot{c} + \epsilon \langle V \eta, J^{-1} \partial_a \eta \rangle| \lesssim \| e^{-\alpha|x-a|} w \|_{H^1}^2 + \epsilon \| e^{-\alpha|x-a|} w \|_{H^1}.
\]
As all norms on a finite dimensional space are equivalent, we can take
\[
\| \alpha \partial_a \eta + \beta \partial_c \eta \|_{T_{a,c} M} = |\alpha| + |\beta|
\]
Proof. Recall that
\[ \partial w = JH''(\eta)w - J(6\eta w^2 - 2w^3) + \epsilon V w - F_0. \]
Write \( R \) for the error terms of the same order as the right hand side of \eqref{eq:3.6}, take derivative with respect to \( t \) for \( \langle w, J^{-1}\partial_a \eta \rangle \), we have
\begin{align}
0 &= \langle \partial_t w, J^{-1}\partial_a \eta \rangle + \langle w, J^{-1}\partial_a \partial_t \eta \rangle \\
&= -\langle F_0, J^{-1}\partial_a \eta \rangle + \langle JH''(\eta)w, J^{-1}\partial_a \eta \rangle + \langle w, J^{-1}\partial_a \partial_t \eta \rangle + R \\
&= -\langle F_0, J^{-1}\partial_a \eta \rangle + \langle w, J^{-1}\partial_a (\partial_t \eta - JH'(\eta)) \rangle + R \\
&= -\langle F_0, J^{-1}\partial_a \eta \rangle + \langle w, J^{-1}\partial_a (\Pi_{a,c} F_0) \rangle + R,
\end{align}
where for the penultimate equality we have used \( J^*J^{-1} = -I \) and the self-adjointness of \( H'' \), and for the last that
\[ \partial \eta - JH'(\eta) = (\dot{a} - c^2)\partial_a \eta + \dot{c} \partial_c \eta = \Pi_{a,c} F_0 + O(\epsilon)\partial_a \eta + O(\epsilon)\partial_c \eta. \]

Taking derivative for \( \langle w, J^{-1}\partial_c \eta \rangle \), similar computation gives
\[ 0 = -\langle F_0, J^{-1}\partial_c \eta \rangle + \langle w, J^{-1}\partial_c (\Pi_{a,c} F_0) \rangle + R. \]
Combining with \eqref{eq:3.8}, and applying the orthogonality conditions for the second terms when \( \partial_a \) and \( \partial_c \) land on the coefficients of \( \Pi_{a,c} F_0 \), the lemma follows from Cauchy-Schwarz and the smallness of \( w \).

\[ \Box \]

4. Local virial estimate

In this section we review, and then apply, part of the local virial estimates due to Martel-Merle. Let \( \Phi \in C(\mathbb{R}), \Phi(x) = \Phi(-x), \Phi' \leq 0 \) on \((0, \infty)\), such that
\[ \Phi(x) = 1 \text{ on } [0, 1], \quad \Phi(x) = e^{-x} \text{ on } [2, \infty), \quad e^{-x} \leq \Phi(x) \leq 3e^{-x} \text{ on } [0, \infty). \]
Let \( \Psi(x) = \int_0^x \Phi(y) \, dy \), and for \( A \gg 0 \), set \( \Psi_A(x) = A \Psi(x/A) \), we have following

**Lemma 4.1** (Martel-Merle \cite{21, 22} local virial spectral estimate). There exists \( A \) sufficiently large and \( \lambda_0 \) sufficiently small, such that if \( w \) satisfies the orthogonal condition \eqref{eq:3.2}, then
\[ -\langle \Psi_A(x-a)w, \partial_x \mathcal{L}w \rangle \geq \lambda_0 \int (w_x^2 + w^2)e^{-|x-a|/A} \, dx. \]

Denoting \( \psi(\cdot) \) for \( \Psi_A(\cdot - a) \), we now proceed as in \cite{21}:

**Lemma 4.2** (local virial estimate). Suppose \( V \) is bounded, then there exist \( \alpha > 0 \) and \( \kappa_j > 0, j = 1, 2 \), such that if \( w \) solves \eqref{eq:3.3} and satisfies the orthogonality conditions \eqref{eq:3.2}, then
\begin{equation}
\| e^{-\alpha|x-a|}w \|_{H^1}^2 \leq -\kappa_1 \partial_t \int \psi w^2 \, dx + \kappa_2 \epsilon^2 + \kappa_3 \| w \|_{H^1}^2.
\end{equation}
Proof. From the equation for $\partial_t w$, we have
\[
\partial_t \int \psi w^2 = - \dot{a} \int \psi' w^2 + 2 \int \psi w \partial_t w = - \dot{a} \int \psi' w^2 + 2 \int \psi w \partial_x (\mathcal{L}w) \leftarrow \text{I + II}
\]
\[
- 2c^2 \int \psi w \partial_x w - 12 \int \psi w \partial_x (\eta w^2) \leftarrow \text{III + IV}
\]
\[
- 4 \int \psi w \partial_x (w^3) + 2\epsilon \int \psi V w^2 - 2 \int \psi w F_0 \leftarrow \text{V + VI + VII}
\]
Using integration by parts,
\[
\text{III} = c^2 \int \psi' w^2,
\]
hence
\[
|\text{I + III}| = | - (\dot{a} - c^2) \int \psi' w^2| \lesssim \epsilon \|w\|_{H^1}^2 + \|e^{-\alpha|x-a|} w\|_{H^1}^2 \|w\|_{H^1}^2
\]
by (3.7). Following from the boundedness of $\psi$ and $V$, and the estimate $\|w\|_{L^\infty} \lesssim \|w\|_{H^1}$, we obtain
\[
|\text{IV}| \lesssim \|e^{-\alpha|x-a|} w\|_{H^1}^2 \|w\|_{H^1},
\]
\[
|\text{V}| = |3 \int \psi' w^4| \lesssim \|w\|_{H^1}^2 \|e^{-|x-a|/(2\alpha)} w\|_{L^2}^2,
\]
\[
|\text{VI}| \lesssim \|w\|_{H^1}^2,
\]
where for the second estimate we have used $\psi' = \Phi((x-a)/\alpha)$ and the definition of $\Phi$. Decomposing VII term as
\[
\text{VII} = -2 \int \psi w \Pi F_0 - 2 \int \psi w \Pi^\bot F_0 = \text{VIIA + VIIB},
\]
we have by Lemma 3.2 that
\[
\text{VIIA} \lesssim \epsilon \|w\|_{H^1}^2 + \|e^{-\alpha|x-a|} w\|_{H^1}^2 \|w\|_{H^1},
\]
and by $\Pi^\bot F_0 \sim \epsilon \eta$ (see (3.5)) that for any $\mu > 0$,
\[
\text{VIIB} \lesssim \epsilon \|e^{-\alpha|x-a|} w\|_{H^1} \lesssim \mu^{-1} \epsilon^2 + \mu \|e^{-\alpha|x-a|} w\|_{H^1}^2
\]
Note in above estimates the value of $\alpha$ may change from one line to the next, but we can choose one single small enough $\alpha$ that works for all.

By Lemma 4.1 we have
\[
\text{II} = 2\langle \psi w, \partial_x (\mathcal{L}w) \rangle \leq - \lambda_0 \int (w_x^2 + w^2) e^{-|x-a|/4 A} dx.
\]
Combining with (4.2), (4.3), (4.4) and (4.5), the estimate (4.1) follows by the smallness of $\|w\|_{H^1}$, taking $A$ large enough so that $1/(2A) < \alpha$, and $\mu > 0$ suitably small. □
5. Energy estimate

In this section we formulate the energy estimate necessary for the estimation of the error term \( w \). Recall \( \mathcal{L} = -\partial_x^2 - 6\eta^2 + c^2 \). Let

\[
\mathcal{E} = \frac{1}{2} \langle \mathcal{L}w, w \rangle - \frac{1}{2} \int \eta w^3 dx - \frac{1}{2} \int w^4 dx,
\]

Note that \( \mathcal{L} = H_0''(\eta) + c^2 = L''(\eta) \), see (2.6) and (2.7). We have classical coercivity properties for \( \mathcal{L} \) (for a proof, see e.g. [27, Prop 2.9] or [17, Prop 4.1] for a more direct proof—note that \( \mathcal{L} \) is the operator \( L_+ \) considered there):

**Lemma 5.1** (energy spectral estimate). Suppose that \( w \) satisfies the orthogonality condition (3.2). Then

\[
\langle \mathcal{L}w, w \rangle \gtrsim \|w_x\|_{L^2}^2 + c^2 \|w\|_{L^2}^2,
\]

Since we impose a lower bound on \( c \) in Theorems 1.1, it follows from (5.1) that if \( \|w\|_{H^1_x} \) is smaller than some (\( \epsilon \) independent) constant, then

\[
\|w(t)\|_{H^1_x}^2 \sim \mathcal{E}(t)
\]

**Lemma 5.2** (energy estimate). Suppose we are given \( V \in C_0^1 \), \( \delta_0 > 0 \) and \( w(x, t) \), such that \( \delta_0 < c(t) < \delta_0^{-1} \), \( w \) solves (3.3) and satisfies the orthogonality conditions (3.2), then

\[
|\partial_t \mathcal{E}| \lesssim \epsilon \|w\|_{H^1_x}^2 + \epsilon \|e^{-\alpha|x-a|}w\|_{H^1_x}^2 + \|w\|_{H^1_x}^2 \|e^{-\alpha|x-a|}w\|_{H^1_x}^2 + \|w\|_{H^1_x}^2.
\]

where the implicit constant depends on \( \delta_0, \sigma_0 \) and the bounds on \( V \) and \( V' \).

**Proof.** We compute

\[
\partial_t \mathcal{E} = \langle \mathcal{L}w, \partial_t w \rangle + \epsilon \|w\|_{H^1_x}^2 - 6\langle (\partial_x \eta + \partial_t \eta) \eta w, w \rangle - \langle \partial_t w, 6\eta w^2 + 2w^3 \rangle - 2\langle (\partial_x \eta + \partial_t \eta), w^3 \rangle
\]

Substitute (3.3) into I:

\[
I = \langle \mathcal{L}w, \partial_x (\mathcal{L}w) \rangle - c^2 \langle \mathcal{L}w, \partial_x w \rangle - \langle \mathcal{L}w, \partial_x (6\eta w^2 + 2w^3) \rangle + \langle \mathcal{L}w, \epsilon V w \rangle - \langle \mathcal{L}w, F_0 \rangle
\]

\[= IA + IB + IC + ID + IE\]

First, \( IA = 0 \). Integration by parts yields \( IB = -6c^2 \langle \eta \partial_x w, w^2 \rangle \). By the boundedness of \( V \) and \( V' \),

\[
ID \lesssim \epsilon \|w\|_{H^1_x}^2,
\]

and since \( \mathcal{L}(TM) \subset TM \) (by direct computation), we have

\[
IE = -\langle \mathcal{L}w, \Pi F_0 \rangle - \langle \mathcal{L}w, \Pi^\perp F_0 \rangle = -\langle \mathcal{L}w, \Pi^\perp F_0 \rangle,
\]

but by (3.5)

\[
|\langle \mathcal{L}w, \Pi^\perp F_0 \rangle| \lesssim \epsilon \|e^{-\alpha|x-a|}w\|_{H^1_x}^2.
\]
Combining, we obtain
\begin{equation}
(5.3) \quad I = IB + IC + O \left( \epsilon \|w\|_{H^2_x}^2 + \epsilon \|e^{-\alpha|x-a|}w\|_{H^2_x} \right)
= -6c^2 \langle \eta \eta_x, w^2 \rangle - \langle \mathcal{L}w, \partial_x (6\eta w^2 + 2w^3) \rangle + O \left( \epsilon \|w\|_{H^2_x}^2 + \epsilon \|e^{-\alpha|x-a|}w\|_{H^2_x} \right).
\end{equation}
Substituting (3.3) into IV, we have
\begin{equation}
(5.4) \quad IC + IV = -\langle \partial_x (-c^2w - 6\eta w^2 - 2w^3) + \epsilon Vw - F_0, 6\eta w^2 + 2w^3 \rangle.
\end{equation}
By (3.7), we have
\begin{equation}
(5.5) \quad |\dot{a} - c^2| \lesssim \epsilon + \|e^{-\alpha|x-a|}w\|_{H^2_x}^2, \quad |\dot{c}| \lesssim \epsilon + \|e^{-\alpha|x-a|}w\|_{H^2_x}^2,
\end{equation}
hence
\begin{equation}
|\langle F_0, 6\eta w^2 + 2w^3 \rangle| \lesssim \epsilon \|w\|_{H^1_x}^2 + \|w\|_{H^2_x}^2 \|e^{-\alpha|x-a|}w\|_{H^2_x}^2.
\end{equation}
Note
\begin{equation}
-\langle \partial_x (-c^2w), 6\eta w^2 + 2w^3 \rangle = -2c^2 \int \eta' w^3 \, dx.
\end{equation}
Estimating the rest of the terms in (5.4) using Cauchy-Schwarz and that \( \|w\|_{L^\infty} \lesssim \|w\|_{H^1_x} \), we obtain
\begin{equation}
(5.6) \quad IC + IV = -2c^2 \langle \eta', w^3 \rangle + O(\epsilon \|w\|_{H^2_x}^2 + \|e^{-\alpha|x-a|}w\|_{H^2_x}^2 \|w\|_{H^2_x}^2 + \|w\|_{H^2_x}^6).
\end{equation}
By (5.5) again, and that \( \partial_x \eta = -\partial_a \eta \), we have
\begin{equation}
(5.7) \quad II + V = 2\dot{a} \langle \eta', w^3 \rangle + O(\epsilon \|w\|_{H^2_x}^2 + \|e^{-\alpha|x-a|}w\|_{H^2_x}^2 \|w\|_{H^2_x}^2),
\end{equation}
and
\begin{equation}
(5.8) \quad IB + III = 6(\dot{a} - c^2) \langle \eta \eta_x, w^2 \rangle - 6\langle \dot{c} (\partial_a \eta) \eta, w^2 \rangle
\lesssim \epsilon \|w\|_{H^2_x}^2 + \|e^{-\alpha|x-a|}w\|_{H^2_x}^2 \|w\|_{H^2_x}^2.
\end{equation}
Apply (5.5) again to the sum of (5.6) and (5.7), then combine with (5.3) and (5.8), we can obtain (5.2).

6. Proof of the main theorems

First, we give the proof of Theorem 1.1.
Let \([0, T']\) be the maximal time interval so that
\begin{equation}
(6.1) \quad \|w\|_{L^\infty_{[0, T']}H^1_x} \leq \mu(t)^{-1/4}
\end{equation}
for \( \mu > 0 \) chosen small enough to ensure the validity of Lemmas 3.1, 3.2, 4.2 and 5.2 and also small enough to beat some constants in the estimates that follow (as explained below).
Let
\[ \mathcal{V}(t) \overset{\text{def}}{=} \int_0^t \| e^{-\alpha|x-a(s)|} w(s) \|_{H^1_x}^2 \, ds, \quad \mathcal{F}(t) \overset{\text{def}}{=} \sup_{0 \leq s \leq t} \| w(s) \|_{H^1_x}^2. \]
Integrating the local virial estimate (4.1) gives
\[ (6.2) \quad \mathcal{V}(t) \lesssim \mathcal{F}(t) + \epsilon^2 t + \epsilon \int_0^t \mathcal{F}(s) \, ds. \]
Integrating (5.2) over \( 0 \leq t \leq \tau \) yields
\[ (6.3) \quad \mathcal{E}(\tau) \leq \mathcal{E}(0) + \epsilon \int_0^\tau \mathcal{F}(s) \, ds + \epsilon \tau^{1/2} \mathcal{V}(\tau) + \mathcal{F}(\tau) \mathcal{V}(\tau) + \tau \mathcal{F}(\tau)^3. \]
Using that \( \mathcal{E}(\tau) \sim \| w(\tau) \|_{H^1_x}^2 \), and then taking the sup of the above estimate over \( 0 \leq \tau \leq t \), we obtain
\[ (6.4) \quad \mathcal{F}(t) \lesssim \mathcal{F}(0) + \epsilon \int_0^t \mathcal{F}(s) \, ds + \mathcal{F}(0) + \mu^{-1} \epsilon t + \mu \mathcal{V}(t) \]
Substituting (6.2) into here, taking \( \mu \) (introduced in (6.1) above) small enough to beat the implicit constants,
\[ (6.5) \quad \mathcal{F}(t) \lesssim \epsilon \int_0^t \mathcal{F}(s) \, ds + \mathcal{F}(0) + \epsilon^2 t. \]
Integrating yields
\[ (6.6) \quad \int_0^t \mathcal{F}(s) \, ds \lesssim (e^{\epsilon \kappa t} - 1)(\epsilon^{-1} \mathcal{F}(0) + 1) \]
Substituting this back into (6.3),
\[ (6.7) \quad \mathcal{F}(t) \lesssim e^{\epsilon \kappa t} \mathcal{F}(0) + \epsilon ((e^{\epsilon \kappa t} - 1) + \kappa t) \]
For the second term, we might as well bound \( (e^{\epsilon \kappa t} - 1) + \kappa t \sim e^{\epsilon t}, \) so
\[ (6.8) \quad \mathcal{F}(t) \lesssim e^{\epsilon \kappa t}(\mathcal{F}(0) + t^2) \]
This enables us to reach time \( \sigma \epsilon^{-1} \log \epsilon^{-1}, \) for \( \sigma > 0 \) small, while still reinforcing the bootstrap assumption (6.1). Returning to (6.2), we obtain the bound for \( \mathcal{V}(t), \) thus completing the proof of (1.4). The \( L^1_{[0,T]} \) estimates (1.5) follow from integrating (3.7) in time and applying (1.4). The \( L^\infty_{[0,T]} \) estimates also follow from (3.7) by dropping the spatial localization in the terms on the right-hand side of (3.7) and applying the bound on \( \| w \|_{L^\infty_{[0,T]} H^1_x} \) given by (1.4).
Now we discuss the proof of Theorem 1.2. Let \( \tilde{a}, \tilde{c} \) solve the ODE system

\[
\begin{align*}
\dot{\tilde{a}} - \tilde{c}^2 - \epsilon \tilde{c}^{-1} \langle V \tilde{\eta}, (x - \tilde{a}) \tilde{\eta} \rangle &= 0 \\
\dot{\tilde{c}} - \epsilon \langle V \tilde{\eta}, \tilde{\eta} \rangle &= 0
\end{align*}
\]

with initial data \( \tilde{a}(0) = a_0, \tilde{c}(0) = c_0 \), where \( \tilde{\eta} = \tilde{c} Q(\tilde{c}(x - \tilde{a})) \). Since \( |\dot{c}|, |\ddot{c}| \lesssim \epsilon \), we can assume \( \delta_0 < c, \tilde{c} < \delta_0^{-1} \) on \( [0, T] \). Define

\[
\bar{a} = a - \tilde{a}, \quad \bar{c} = c - \tilde{c},
\]

we have

\[
\langle V \eta, (x - \tilde{a}) \eta \rangle - \langle V \tilde{\eta}, (x - \tilde{a}) \tilde{\eta} \rangle = \beta_1(a - \tilde{a}) + \beta_2(c - \tilde{c}),
\]

where we have defined

\[
\beta_1 = \frac{1}{a - \tilde{a}} \int \left( V \left( \frac{x}{c} + a \right) - V \left( \frac{x}{\tilde{c}} + \tilde{a} \right) \right) x \eta^2 \, dx,
\]

\[
\beta_2 = \frac{1}{c - \tilde{c}} \int \left( V \left( \frac{x}{c} + a \right) - V \left( \frac{x}{\tilde{c}} + \tilde{a} \right) \right) x \eta^2 \, dx,
\]

similarly,

\[
\frac{1}{c} \langle V \eta, \eta \rangle - \frac{1}{\tilde{c}} \langle V \tilde{\eta}, \tilde{\eta} \rangle = \gamma_1(a - \tilde{a}) + \gamma_2(c - \tilde{c}),
\]

where

\[
\gamma_1 = \frac{1}{a - \tilde{a}} \int \left( V \left( \frac{x}{c} + a \right) - V \left( \frac{x}{\tilde{c}} + \tilde{a} \right) \right) \eta^2 \, dx,
\]

\[
\gamma_2 = \frac{1}{c - \tilde{c}} \int \left( V \left( \frac{x}{c} + a \right) - V \left( \frac{x}{\tilde{c}} + \tilde{a} \right) \right) \eta^2 \, dx,
\]

Denote \( R_1, R_2 \) for the error terms in Lemma 3.2, i.e.

\[
\begin{align*}
\dot{a} - c^2 - \epsilon c^{-1} \langle V \eta, (x - a) \eta \rangle - R_1 &= 0 \\
\dot{c} - \epsilon \langle V \eta, \eta \rangle - R_2 &= 0,
\end{align*}
\]

Apply (1.5) to (3.7), we obtain

\[
\| R_j \|_{L^1[0, t]} \leq C(\omega + \epsilon t^{1/2})^2 e^{C_\epsilon t^{1/2}}, \quad j = 1, 2.
\]

Note

\[
\frac{\dot{c}}{c} - \frac{\ddot{c}}{c} = \frac{\dot{c}}{c} - \frac{\ddot{c}}{cc},
\]

and

\[
c\dot{a} - \dot{c} \dot{a} = c \dot{a} + (c - \tilde{c}) \dot{\tilde{a}},
\]

denoting

\[
\theta_1 = \frac{1}{c} \left[ (c^2 + \tilde{c}^2 + cc) - (\tilde{c}^2 + \epsilon \tilde{c}^{-1} \langle V \tilde{\eta}, (x - \tilde{a}) \tilde{\eta} \rangle) + \epsilon \beta_2 \right],
\]

and

\[
\theta_2 = \frac{1}{\epsilon \tilde{c}} = \frac{1}{c} \langle V \tilde{\eta}, \tilde{\eta} \rangle,
\]
we can obtain the equation for \((\bar{a}, \bar{c})\),

\[
(6.5) \quad \begin{bmatrix} \bar{a}' \\ \bar{c}' \end{bmatrix} = \begin{bmatrix} \epsilon \beta_1 c^{-1} & \theta_1 \\ \epsilon c \gamma_1 & \epsilon (\theta_2 + c \gamma_2) \end{bmatrix} \begin{bmatrix} \bar{a} \\ \bar{c} \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}.
\]

Writing
\[
A(t) = \begin{bmatrix} \epsilon \beta_1 c^{-1} & \theta_1 \\ \epsilon c \gamma_1 & \epsilon (\theta_2 + c \gamma_2) \end{bmatrix}.
\]

From the boundedness of \(\beta_j, \gamma_j, \theta_j, j = 1, 2\), which is a result of the boundedness of \(V, V', c\) and \(\tilde{c}\), we have the estimate

\[
(6.6) \quad |A(t)| \lesssim \begin{bmatrix} \epsilon^{-1/2} & 1 \\ \epsilon & \epsilon \end{bmatrix}.
\]

Writing \(p(s) = (\epsilon \bar{a}^2 + \bar{c}^2)^{1/2}\), then by above estimate

\[
|\dot{p}| \lesssim \frac{1}{p} \left[ |\epsilon | |\bar{a}| + |\bar{c}| + |R_1| + |\bar{c}|(\epsilon |\bar{a}| + \epsilon |\bar{c}| + |R_2|) \right]
\]

\[
\lesssim \frac{1}{p} \left[ \epsilon (\epsilon \bar{a}^2 + \bar{c}^2) + \epsilon^{1/2} (\epsilon \bar{a}^2 + \bar{c}^2) + \epsilon |\bar{a}| |R_1| + |\bar{c}| |R_2| \right]
\]

\[
\lesssim \epsilon^{1/2} \frac{1}{p} + \epsilon^{1/2} |R_1| + |R_2|.
\]

By Gronwall and \(p(0) = 0\), we obtain

\[
p(t) \leq C e^{C t^{1/2}} \int_0^t (\epsilon^{1/2} |R_1| + |R_2|) (s) \, ds.
\]

Applying \((6.4)\), we obtain

\[
p(t) \leq C e^{C t^{1/2}} (\omega + \epsilon t^{1/2})^2,
\]

recalling the bounds on \(t\) and \(\omega\) in Theorem \(1.2\), this gives

\[
p(t) \leq C e^{1/2} (\omega + \epsilon t^{1/2}) e^{C t^{1/2}}.
\]

The bounds on \(\bar{a}\) and \(\bar{c}\) now follow from the definition of \(p\):

\[
|\bar{a}| \leq C (\omega + \epsilon t^{1/2}) e^{C t^{1/2}}
\]

\[
|\bar{c}| \leq C (\omega + \epsilon t^{1/2}) e^{C t^{1/2}}.
\]

Compare the above two estimates with \((1.7)\), we can conclude the proof of Theorem \(1.2\).

**Remark 6.1.** The \(\epsilon^{-1/2}\) constraint on the time scale stems from the fact that the eigenvalues of \(\begin{bmatrix} \epsilon & 1 \\ \epsilon & \epsilon \end{bmatrix}\) are only of order \(\epsilon^{1/2}\).
Appendix A. Local and global well-posedness

The global well-posedness for gKdV in energy space was obtained by Kenig-Ponce-Vega in [19], where they introduced new and powerful local smoothing and maximal function estimates, especially, they proved the local well-posedness for (1.2) in $H^s(\mathbb{R})$ for $s \geq 1/4$. To prove well-posedness for (1.1) at $H^1$ level of regularity, the full strength of these estimates is not needed, we here follow the presentation of [16] Apx. A and make necessary modifications.

Let $Q_n = [n - \frac{1}{2}, n + \frac{1}{2}]$, and $\tilde{Q}_n = [n - 1, n + 1]$. An example of notation is:

$$\|u\|_{\ell^\infty_n L^2_T L^2_x Q_n} = \sup_n \|u\|_{L^2_{[0,T]} L^2_x Q_n}.$$  

Note that due to the finite incidence of overlap, we have

$$\|u\|_{\ell^\infty_n L^2_T L^2_x Q_n} \sim \|u\|_{\ell^\infty_n L^2_T \tilde{L}^2_x Q_n}.$$  

We omit the $\epsilon$ in (1.1), and consider

$$(A.1) \quad \partial_t u = \partial_x (-\partial_x^2 u - 2u^3) + Vu, \quad V \in C^1_b.$$  

As in [16], we first prove a local smoothing estimate and a maximal function estimate (weak versions), by an integrating factor method:

**Lemma A.1.** Suppose that

$$(A.2) \quad v_t + v_{xxx} - Vv = f,$$  

then there exists $C > 0$, such that if

$$T \leq C(1 + \|V\|_{L^\infty_x})^{-1},$$  

we have the energy and local smoothing estimates

$$(A.3) \quad \|v\|_{L^\infty_x L^2_t L^2_x} + \|v_x\|_{\ell^\infty_n L^2_T L^2_{Q_n}} \lesssim \|v_0\|_{L^2_x} + \left\{ \begin{array}{c} \|\partial_x^{-1} f\|_{\ell^1_n L^2_T L^2_{Q_n}} \\ \|f\|_{L^1_x L^2} \end{array} \right\}$$  

and the maximal function estimate

$$(A.4) \quad \|v\|_{\ell^\infty_n L^\infty_x L^2_{Q_n}} \lesssim \|v_0\|_{L^2_x} + T^{1/2}\|v\|_{L^2_x H^1_x} + T^{1/2}\|f\|_{L^2_x L^2_x}.$$  

The implicit constants are independent of $V$.

**Proof.** Let $\phi(x) = -\tan^{-1}(x-n)$, and set $w(x,t) = e^{\phi(x)} v(x,t)$. By (A.2),

$$\partial_t w + w_{xxx} - 3\phi' w_{xx} + 3(-\phi'' + (\phi')^2)w_x + (-\phi''' + 3\phi'' \phi' - (\phi')^3)w - Vw = e^{\phi} f,$$  

integrating its product with $\frac{1}{2} w$ over $x$,  

$$\partial_t \|w\|_{L^2_x}^2 = -6\langle \phi', w_x^2 \rangle + \langle -\phi'' + 2(\phi')^3, w^2 \rangle + 2\langle V, w^2 \rangle + 2\langle e^{\phi} f, w \rangle,$$  

where $\langle \cdot, \cdot \rangle$ denotes the $L^2$ product.
integrating this identity over $[0, T]$, and using $\phi'(x) = -(x-n)^{-2}$, we obtain

$$\|w(T)\|_{L^2_x}^2 + 6\|\langle x - n \rangle^{-1} w_x\|_{L^2_x}^2 \leq \|w_0\|_{L^2_x}^2 + C_1 T (1 + \|V\|_{L^\infty_x}) \|w\|_{L^2_x}^2 + C_1 \int_0^T \left| \int e^{\phi} f w \, dx \right| \, dt,$$

for some constant $C_1 > 0$, replace $T$ by $t$, and take supremum over $t \in [0, T]$, we obtain, for $T \leq \frac{1}{2} C_1^{-1} (1 + \|v\|_{L^\infty_x})$, the estimate

$$\|w\|_{L^2_x}^2 + \|\langle x - n \rangle^{-1} w_x\|_{L^2_x}^2 \leq \|w_0\|_{L^2_x}^2 + \int_0^T \left| \int e^{\phi} f w \, dx \right| \, dt,$$

note that $0 < e^{-\pi/2} \leq e^{\phi(x)} \leq e^{\pi/2} < \infty$, we can convert the above estimate back to an estimate for $v$:

$$\|v\|_{L^2_x}^2 + \|v_x\|_{L^2_x}^2 \leq \|v_0\|_{L^2_x}^2 + \int_0^T \left| \int e^{2\phi} f v \, dx \right| \, dt.$$

Estimating as

$$\int_0^T \left| \int e^{2\phi} f v \, dx \right| \, dt \lesssim \|f\|_{L^1_t L^2_x} \|v\|_{L^2_x},$$

and then taking the supremum in $n$ yields the second estimate in \((A.3)\). Estimating instead as

$$\int_0^T \left| \int e^{2\phi} f v \, dx \right| \, dt = \int_0^T \left| \int (\partial_x^{-1} f \partial_x(e^{2\phi} v)) \, dx \right| \, dt \leq \sum_m \|\partial_x^{-1} f\|_{L^1_t L^2_{Q_m}} \|\langle \partial_x \rangle v\|_{L^2_x},$$

and then taking the supremum in $n$ yields the second estimate in \((A.3)\).

For the proof of estimate \((A.4)\), take $\phi(x) = 1$ on $[n - \frac{1}{2}, n + \frac{1}{2}]$ and 0 outside $[n - 1, n + 1]$, set $w = \phi v$, and compute similarly as the above.

Using estimates in the above lemma, we can prove:

**Theorem A.2** (local well-posedness in $H^1_x$). *Suppose that*

\[(A.5)\]

$$M \overset{\text{def}}{=} \|V\|_{L^\infty_x} + \|V'\|_{L^\infty_x} < \infty.$$

*For any $R \geq 1$, take*

$$T \lesssim \min(M^{-1}, R^{-2}),$$

*we have*

(1) *If $\|u_0\|_{H^1_x} \leq R$, there exists a solution $u(t) \in C([0, T]; H^1_x)$ to \((A.1)\) on $[0, T]$ with initial data $u_0(x)$ satisfying*

$$\|u\|_{L^\infty_t H^1_x} + \|u_{xx}\|_{L^2_t L^2_{Q_n}} \lesssim R.$$
(2) This solution $u(t)$ is unique among all solutions in $C([0, T]; H^1_x)$.
(3) The data-to-solution map $u_0 \mapsto u(t)$ is continuous as a mapping $H^1 \to C([0, T]; H^1_x)$.

Proof. We prove the existence by contraction in the space $X$, where

$$X = \left\{ u \mid \|u\|_{C([0,T];H^1_x)} + \|u_{xx}\|_{L^\infty_t L^2_{\Omega_n}} + \|u\|_{L^\infty_t L^2_{\Omega_n}} \leq CR \right\},$$

where the constant $C$ is chosen large enough to (10 times, say) exceed the implicit constants in Lemma A.1. Given $u \in X$, let $\varphi(u)$ denote the solution to

$$\partial_t \varphi(u) + \partial_x^3 \varphi(u) - V \varphi(u) = -2\partial_x (u^3),$$

with initial condition $\varphi(u)(0) = 0$. A fixed point $\varphi(u) = u$ in $X$ will solve (A.1).

The local smoothing estimate (A.3) applied to $v = \varphi(u)$ and the estimate

$$\|v^3_x\|_{L^1_t L^2_x} \lesssim T\|v\|^3_{L^\infty_t H^1_x}$$

give the estimate

$$\|\varphi(u)\|_{L^\infty_t L^2_x} \lesssim \|u_0\|_{H^1_x} + T\|u\|^3_{L^\infty_t H^1_x},$$

The maximal function estimate (A.4) applied to $v = \varphi(u)$ and the estimate

$$\|v^3_x\|_{L^1_t L^2_x} \lesssim T^{1/2}\|v\|^3_{L^\infty_t H^1_x}$$

imply the estimate

$$\|\varphi(u)\|_{L^\infty_t L^2_x} \lesssim \|u_0\|_{L^2_x} + T\|\varphi(u)\|_{L^\infty_t H^1_x} + T\|u\|^3_{L^\infty_t H^1_x}.$$  

Now applying $\partial_x$ to (A.6), and denoting $v = \varphi(u)_x$ instead:

$$v_t + v_{xxx} - Vv = -2(u^3)_{xx} + V' \varphi(u).$$

By (A.3) again,

$$\|\varphi(u)_x\|_{L^\infty_t L^2_x} + \|\varphi(u)_{xx}\|_{L^\infty_t L^2_{\Omega_n}} \lesssim \|u_0\|_{H^1_x} + \|(u^3)_x\|_{L^\infty_t L^2_{\Omega_n}} + \|V' \varphi(u)\|_{L^1_t L^2_x}.$$  

Applying Gagliado-Nirenberg inequality to $\phi(x)u$, where $\phi(x) = 1$ on $[n - \frac{1}{2}, n + \frac{1}{2}]$ and 0 outside $[n - 1, n + 1]$, we obtain (writing $Q$ for $Q_n$ and $\tilde{Q}$ for $\tilde{Q}_n$ for the following):

$$\|u\|^2_{L^2_Q} \lesssim \left( \|u\|_{L^2_{\tilde{Q}}} + \|u_x\|_{L^2_{\tilde{Q}}} \right)^2 \|u\|_{L^2_{\tilde{Q}}};$$

hence

$$\|(u^3)_x\|_{L^2_{\tilde{Q}}} \lesssim \|u_x\|_{L^2_{\tilde{Q}}} \|u\|_{L^2_{\tilde{Q}}} \|u\|_{L^2_{\tilde{Q}}} \left( \|u\|_{L^2_{\tilde{Q}}} + \|u_x\|_{L^2_{\tilde{Q}}} \right).$$

Taking $L^2_T$ norm and applying the Hölder inequality, we obtain

$$\|(u^3)_x\|_{L^2_T L^2_{\tilde{Q}}} \lesssim \|u_x\|_{L^2_T L^2_{\tilde{Q}}} \|u\|_{L^2_T L^2_{\tilde{Q}}} \left( \|u\|_{L^2_T L^2_{\tilde{Q}}} + \|u_x\|_{L^2_T L^2_{\tilde{Q}}} \right).$$

Taking $L^1_t$ norm and applying the Hölder inequality again yields

$$\|(u^3)_x\|_{L^2_T L^2_{\tilde{Q}}} \lesssim \|u_x\|_{L^\infty_T L^2_{\tilde{Q}}}, \|u\|_{L^\infty_T L^2_{\tilde{Q}}} \left( \|u\|_{L^2_T L^2_{\tilde{Q}}} + \|u_x\|_{L^2_T L^2_{\tilde{Q}}} \right).$$
Using the bounds \( \|u_x\|_{L^\infty_T L^2_x} \lesssim \|u\|_{L^\infty_T L^2_x} \),
\[
\|u\|_{L^2_T L^2_x} \lesssim \|u\|_{L^2_T L^2_x} \lesssim T^{1/2}\|u\|_{L^\infty_T L^2_x}
\]
and
\[
\|u_x\|_{L^2_T L^2_x} \lesssim \|u_x\|_{L^1_T L^2_x} \lesssim T^{1/2}\|u_x\|_{L^\infty_T L^2_x},
\]
we obtain
\[
\| (u^3)_x \|_{L^\infty_T L^2_x} \lesssim T^{1/2}\|u\|_{L^\infty_T H^1_x}^2 \|u\|_{L^\infty_T L^2_x},
\]
inserting into (A.9),
\[
\| \varphi(u)_x \|_{L^\infty_T L^2_x} + \| \varphi(u)_x \|_{L^\infty_T L^2_x} \lesssim \|u_0\|_{H^1} + T^{1/2}\|u\|_{L^\infty_T H^1_x}^2 \|u\|_{L^\infty_T L^2_x} + T\|V\|_{L^\infty_T} \|\varphi(u)\|_{L^\infty_T L^2_x}.
\]
Summing (A.7), (A.8) and (A.10), we obtain that \( \|\varphi(u)\|_X \leq CR \) if \( \|u\|_X \leq CR \) provided \( T \leq C_0 \min(M^{-1}, R^{-2}) \), with \( C_0 \) small enough. Thus \( \varphi : X \to X \). A similar argument establishes that \( \varphi \) is a contraction on \( X \).
Now suppose \( u, v \in C([0, T]; H^1_x) \) solve (A.1). By (A.4),
\[
\|u\|_{L^\infty_T L^2_x} \lesssim \|u_0\|_{L^2_x} + T\|u\|_{L^\infty_T H^1_x} + T\|u\|_{L^\infty_T H^1_x}^2, \tag{A.11}
\]
\[
\|v\|_{L^\infty_T L^2_x} \lesssim \|v_0\|_{L^2_x} + T\|v\|_{L^\infty_T H^1_x} + T\|v\|_{L^\infty_T H^1_x}^2, \tag{A.11}
\]
Set \( w = u - v \). Then, with \( g = (u^3 - v^3)/(u - v) = u^2 + uv + v^2 \), we have
\[
w_t + w_{xxx} + 2(gw)_x - V w = 0. \tag{A.12}
\]
Apply (A.3) to \( v = w_x \), we obtain
\[
\|w_x\|_{L^\infty_T L^2_x} \lesssim \|(gw)_x\|_{L^\infty_T L^2_x} + \|V'w\|_{L^1_T L^2_x}. \tag{A.13}
\]
The terms of \( \|(gw)_x\|_{L^\infty_T L^2_x} \) can be bounded in the following manner:
\[
\|u_x v w\|_{L^\infty_T L^2_x} \lesssim \|u_x\|_{L^\infty_T L^2_x} \|v w\|_{L^\infty_T L^2_x} \tag{A.13}
\]
\[
\lesssim \|u_x\|_{L^\infty_T L^2_x} (\|v w\|_{L^\infty_T L^2_x} + \|(v w)_x\|_{L^1_T L^2_x}).
\]
The term in the parentheses is bounded by
\[
\|v w\|_{L^\infty_T L^2_x} \|w\|_{L^\infty_T L^2_x} \|w\|_{L^\infty_T L^2_x} \|w\|_{L^\infty_T L^2_x} \|w\|_{L^\infty_T L^2_x} \|w_x\|_{L^\infty_T L^2_x}
\]
which by (A.11) and
\[
\|u_x\|_{L^\infty_T H^1_x} \lesssim \|u\|_{L^\infty_T H^1_x}, \quad \|v\|_{L^\infty_T L^2_x} \lesssim T^{1/2}\|v\|_{L^\infty_T L^2_x}
\]
implies
\[
\|u_x v w\|_{L^\infty_T L^2_x} \lesssim \|u\|_{L^\infty_T H^1_x} \|v\|_{L^\infty_T H^1_x} T^{1/2}(\|w\|_{L^\infty_T L^2_x} + \|w\|_{L^\infty_T H^1_x}).
\]
Same bounds follow for other terms in $\| (gw)_x \|_{L_T^\infty L_x^2 L_{\tilde{q}_n}}$, combined with $\| V'w \|_{L_T^1 L_x^2} \lesssim T \| V' \|_{L_x^\infty} \| w \|_{L_x^2 H_1^1}$, this establishes the estimate

$$\| w_x \|_{L_T^1 L_x^2} \lesssim T^{1/2} (\| w \|_{L_T^\infty L_x^2 L_{\tilde{q}_n}} + \| w \|_{L_T^\infty H_1^1}),$$

where the implicit constant depends on $\| u \|_{L_T^\infty H_1^1}$ and $\| v \|_{L_T^\infty H_1^1}$. Same estimate follows for $\| w \|_{L_T^\infty L_x^2}$ by applying (A.13) to $v = w$. Hence

(A.14) \[ \| w \|_{L_T^\infty H_1^1} \lesssim T^{1/2} (\| w \|_{L_T^\infty L_x^2 L_{\tilde{q}_n}} + \| w \|_{L_T^\infty H_1^1}), \]

but applying (A.4) to $v = w$ yields

(A.15) \[ \| w \|_{L_T^\infty L_x^2 L_{\tilde{q}_n}} \lesssim T \| w \|_{L_T^\infty H_1^1} \]

since e.g.

$$\| w w_x \|_{L_T^1 L_x^2} \lesssim T^{1/2} \| w \|_{L_T^\infty H_1^1} \| u \|_{L_T^\infty H_1^1} \| v \|_{L_T^\infty H_1^1},$$

which can be proved by the same method as in (A.13), and thus $\| (gw)_x \|_{L_T^1 L_x^2} \lesssim T^{1/2} \| w \|_{L_T^\infty H_1^1}$. Substituting (A.15) into (A.14) implies $w \equiv 0$ for $T$ sufficiently small, which then establishes the uniqueness of solutions in $C([0, T]; H_1^1)$. The continuity of the data-to-solution map can be proved by a similar argument.

We now prove the global well-posedness in $H^1$ by (almost) conservation laws.

**Theorem A.3** (global well-posedness). Suppose $M < \infty$, where $M$ is defined in (A.5), for $u_0 \in H^1$, there is a unique global solution $u \in C_{loc}([0, \infty); H_1^1)$ to (A.1) with $\| u \|_{L_T^\infty H_x^1}$ controlled by $\| u_0 \|_{H^1}$, $T$ and $M$.

**Proof.** First, note from Gagliado-Nirenberg inequality, $\| u \|_{L_1^4} \lesssim \| u \|_{L_2^3}^2 \| u_x \|_{L_2^2}$, we have

$$\| u_x \|_{L_2^2}^2 - \| u \|_{L_2^3}^3 \| u_x \|_{L_2^2} \leq H_0(u) \leq \| u_x \|_{L_2^2}^2.$$

Applying Peter-Paul inequality to the $\| u \|_{L_2^3}^3 \| u_x \|_{L_2^2}$ term gives us

$$\| u_x \|_{L_2^2}^2 + \| u \|_{L_2^2}^6 \sim H_0(u) + \| u \|_{L_2^2}^2.$$

Suppose $u$ solves (A.1), then

$$\left| \frac{d}{dt} H_0(u) \right| = |\langle H_0'(u), JH_0'(u) + Vu \rangle| = |\langle H_0'(u), Vu \rangle|$$

(A.16) \[ \lesssim M (\| u_x \|_{L_2^2}^2 + \| u \|_{L_2^3}^6) \lesssim M (\| u_x \|_{L_2^2}^2 + \| u \|_{L_2^3}^6 \| u_x \|_{L_2^2}) \]

$$\lesssim M (\| u_x \|_{L_2^2}^2 + \| u \|_{L_2^3}^6) \lesssim M (H_0(u) + \| u \|_{L_2^2}^2 + \| u \|_{L_2^3}^6),$$

on the other hand, by

$$\left| \frac{d}{dt} P(u) \right| = |\langle u, Vu \rangle| \lesssim MP(u),$$

and Gronwall inequality, we obtain a bound on $\| u \|_{L_T^\infty L_x^2}$ in terms of $\| u_0 \|_{L^2}$ and $M$, combine this with (A.16), and apply Gronwall again, we obtain a bound on $H_0(u)$ and hence $\| u \|_{H_1^1}$. \qed
Remark A.4. A global well-posedness in $H^k_x$ for $k \geq 1$ can in fact be proved, provided $V \in C^k_b$, by similar arguments.

References

[1] W.K. Abou Salem and C. Sulem, *Stochastic acceleration of solitons for the nonlinear Schrödinger equation*, SIAM J. Math. Anal. 41 (2009), no. 1, pp. 117–152.

[2] W.K. Abou Salem, *Effective dynamics of solitons in the presence of rough nonlinear perturbations*, Nonlinearity 22 (2009), no. 4, pp. 747–763.

[3] T. Benjamin, *The stability of solitary waves*, Proc. Roy. Soc. London Ser. A 328 (1972) pp. 153–183.

[4] J. Bona, *On the stability theory of solitary waves*, Proc. Roy. Soc. London Ser. A 344 (1975) pp. 363–374.

[5] A. de Bouard, A. Debussche, *Soliton dynamics for the Korteweg-de Vries equation with multiplicative homogeneous noise*, Electron. J. Probab. 14 (2009), no. 58, pp. 1727–1744.

[6] J.C. Bronski and R.L. Jerrard, *Soliton dynamics in a potential*, Math. Res. Lett. 7 (2000), no. 2-3, pp. 329–342.

[7] J.L. Bona, P.E. Souganidis, W.A. Strauss, *Stability and instability of solitary waves of Korteweg de Vries type*, Proc. Roy. Soc. London Ser. A 411 (1987), no. 1841, pp. 395–412.

[8] K. Datchev and I. Ventura, *Solitary waves for the Hartree equation with a slowly varying potential*, Pacific J. Math. 248 (2010), no. 1, pp. 63–90.

[9] J. Fröhlich, S. Gustafson, B.L.G. Jonsson, I.M. Sigal, *Solitary wave dynamics in an external potential*, Comm. Math. Physics, 250 (2004), pp. 613–642.

[10] J. Fröhlich, T.-P. Tsai, H.-T. Yau, *On the point-particle (Newtonian) limit of the non-linear Hartree equation*, Comm. Math. Phys. 225 (2002), no. 2, pp. 223–274.

[11] Manoussos Grillakis, Jalal Shatah, and Walter Strauss, *Stability theory of solitary waves in the presence of symmetry. I*, J. Funct. Anal. 74 (1987), no. 1, pp. 160–197.

[12] Manoussos Grillakis, Jalal Shatah, and Walter Strauss, *Stability theory of solitary waves in the presence of symmetry. II*, J. Funct. Anal. 94 (1990), no. 2, pp. 308–348.

[13] J. Holmer, *Dynamics of KdV solitons in the presence of a slowly varying potential*, to appear in IMRN Internat. Math. Notices.

[14] J. Holmer and Q. Lin, *Phase-driven interaction of widely separated nonlinear Schrödinger solitons*, arxiv.org preprint arXiv:1108.4859 [math.AP].

[15] J. Holmer and L. Setayeshgar, *Exact dynamics of solitons for mKdV with multiplicative white noise*, in preparation.

[16] J. Holmer, G. Perelman, M. Zworski, *Effective dynamics of double solitons for perturbed mKdV*, Comm. Math. Phys. 305 (2011) pp. 363–425

[17] J. Holmer and M. Zworski, *Slow soliton interaction with delta impurities*, J. Mod. Dyn. 1 (2007), no. 4, pp. 689–718.

[18] J. Holmer and M. Zworski, *Soliton interaction with slowly varying potentials*, Int. Math. Res. Not. IMRN (2008), no. 10, Art. ID rnm026, 36 pp.

[19] C.E. Kenig, G. Ponce, L. Vega, *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle* Comm. Pure Appl. Math. 46 (1993) pp.527–620

[20] S.J. Lee, G.T. Yates, T.Y. Wu, *Experiments and analyses of upstream-advancing solitary waves generated by moving disturbances*, J. Fluid Mech. 199 (1989) pp. 569–593.
[21] Y. Martel and F. Merle, *Asymptotic stability of solitons for subcritical gKdV equations revisited*, Nonlinearity (2005), no. 18, pp. 55–80.

[22] Y. Martel and F. Merle, *Asymptotic stability of solitons for subcritical generalized KdV equations*, Arch. Ration. Mech. Anal. 157 (2001) pp. 219–254.

[23] C. Muñoz, *On the soliton dynamics under slowly varying medium for generalized KdV equations*, to appear in Analysis and PDE.

[24] C. Muñoz, *Dynamics of soliton-like solutions for slowly varying, generalized KdV equations: refraction vs. reflection*, preprint, arxiv.org [arXiv:0912.4725] [math.AP]

[25] O. Pocovnicu, *Soliton interaction with small Toeplitz potentials for the Szego equation on the real line*, arxiv.org preprint [arXiv:1110.5071] [math.AP].

[26] L.N. Trefethen, *Spectral methods in MATLAB*, 10, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000. xviii+165 pp. ISBN: 0-89871-465-6.

[27] M.I. Weinstein, *Modulational stability of ground states of nonlinear Schrödinger equations*, SIAM J. Math Anal. 16 (1985), no. 3, pp. 472–491.

[28] M.I. Weinstein, *Lyapunov stability of ground states of nonlinear dispersive evolution equations*, Comm. Pure. Appl. Math. 29 (1986) pp. 51–68.