A mollifier method for the inverse conductivity problem

P Maass¹, M K Pidcock ² and C Sebu ²

¹ Zentrum für Technomathematik, Universität Bremen, Germany
² Department of Mathematical Sciences, Oxford Brookes University, United Kingdom
E-mail: caebu@brookes.ac.uk

Abstract. In this paper we present an integral equation formulation of the inverse conductivity problem that is regularised using mollifier methods.

1. Introduction

The inverse conductivity problem has attracted much attention because although it is relatively easy to state, it turns out to be both nonlinear and, above all, extremely ill-posed. Furthermore it has a practical realisation which is known as Electrical Impedance Tomography (EIT). Substantial progress has been made in designing practical reconstruction algorithms applicable to noisy measurement data. While the iterative methods based on formulating the inverse problem in the framework of nonlinear optimisation techniques are promising for obtaining accurate reconstructed conductivity values, they may be slow to converge and are quite demanding computationally particularly when addressing the three dimensional problem. This concern has encouraged the search for reconstruction algorithms which reduce the computational demands. Some use a priori information to reconstruct piecewise constant conductivity distributions e.g. [1–3] while others are based on reformulating the inverse problems in terms of integral equations [4–9].

In this paper we describe a reconstruction method suitable for continuous conductivity distributions. It uses a simple transformation to establish a connection between the equations defining the inverse conductivity problem and those used in Inverse Scattering [10]. By combining this process with the powerful concept of mollifiers [11–14] we are able to obtain conductivity reconstructions for arbitrary geometry, extremely rapidly and without relying on accurate a priori information. This is achieved by reformulating the inverse problem in terms of a pair of coupled integral equations, one of which is of the first kind which we solve using mollifier methods. An interesting feature of this method is that the kernel of this integral equation is not given, but can be modified according to the choice of mollifier.

The paper is organized as follows: Section 2 presents the integral equation formulation of our inverse conductivity problem. In Section 3 we describe the reconstruction algorithm, introduce the concept of mollifiers and show how the technique described in Section 2 can be regularized using mollifiers. In Section 4 we present explicit analytical developments in two dimensions. Section 5 contains the results of some numerical experiments which indicate the possibilities and limitations of the current approach.
2. Integral Equation Formulation

Suppose that we have a region $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, with a boundary $\partial \Omega$ that contains Ohmic isotropic material of electric conductivity $\sigma$. It is well known that if an electric current $j = \sigma \frac{\partial \Phi}{\partial n} \in H^{-\frac{1}{2}}(\partial \Omega)$ is applied on the boundary of $\Omega$ then the induced electric potential $\Phi \in H^{\frac{1}{2}}(\Omega)$ satisfies the equation

$$
\nabla \cdot (\sigma(x) \nabla \Phi(x)) = 0, \quad x \in \Omega.
$$

Using the well-known change of variables $\tau = \sqrt{\sigma}$ we can rewrite equation (1) in the form

$$
\Delta \Psi(x) = -V(x)\Psi(x), \quad x \in \Omega
$$

where $V(x) \equiv -\frac{\Delta \tau(x)}{\tau(x)}$, $\Psi(x) \equiv \tau(x)(\Phi(x) + C)$ and $C$ is an arbitrary constant.

In this work we will assume that from data measurements on the boundary we have a knowledge of both $\sigma$ and $\Phi$ and their normal derivatives on $\partial \Omega$. The inverse conductivity problem is to use this information to identify $\sigma$ in $\Omega$. Although it might seem that our inverse problem is overdetermined, we do not use all this information but only certain linear combinations. In terms of the new variables, the reconstruction problem becomes that of finding $V$ (and consequently $\sigma$) from a knowledge of $\Psi$ and $\frac{\partial \Psi}{\partial n}$ on $\partial \Omega$ which is similar to the data normally used. Furthermore, although it is possible to use multiple data measurements we will show that even a single pair of data is sufficient for the reconstruction of a conductivity distribution which fits the available data.

**Definition 1** Let $\mathcal{H}$ be the set of all functions of two variables $H(x, y)$ which satisfy the Helmholtz equation with respect to the second variable, i.e.

$$
\Delta_y H(x, y) + \lambda H(x, y) = 0, \quad x, y \in \Omega,
$$

where $\lambda$ is a positive real constant.

Using Green’s second identity we can derive two integral representations of problem (2):

$$
0 = \zeta(x) - \int_{\Omega} dy \ H(x, y)X(y), \quad x \in \Omega,
$$

$$
\Psi(x) = \zeta_0(x) + \int_{\Omega} dy \ \mathcal{G}_0(x, y)X(y), \quad x \in \Omega,
$$

where $X(y) = (V(y) - \lambda)\Psi(y)$.

$$
\zeta(x) = \int_{\partial \Omega} dy \left( \Psi(y) \frac{\partial H}{\partial n}(x, y) - H(x, y) \frac{\partial \Psi(y)}{\partial n} \right),
$$

$$
\zeta_0(x) = \int_{\partial \Omega} dy \left( \mathcal{G}_0(x, y) \frac{\partial \Psi(y)}{\partial n} - \Psi(y) \frac{\partial \mathcal{G}_0}{\partial n}(x, y) \right).
$$

$\mathcal{G}_0(x, y)$ is the free space Green’s function for the Helmholtz equation (3), i.e.

$$
\mathcal{G}_0(x, y) = \begin{cases} 
-\frac{1}{4} Y_0(\sqrt{\lambda} \lVert x - y \rVert) & \text{ in } 2D, \\
-\frac{1}{4\pi} \cos(\sqrt{\lambda} \lVert x - y \rVert) & \text{ in } 3D,
\end{cases}
$$

where $Y_0$ is the Bessel function of the second kind. This integral equation formulation will allow us to develop a procedure for determining $\sigma$. It is important to note that since equation (4) is a Fredholm integral equation of the first kind, it is severely ill-posed and in this paper we will regularize it using mollifiers. There is some freedom in the choice of the function $H(x, y)$ which can be exploited in a number of ways.
3. A mollifier method for impedance tomography

The reconstruction of $\sigma$ in $\Omega$ using the formulation described in Section 2 consists of four steps:

1. Solve the linear problem $AX = \zeta$ (given by equation (4)) where
   \[ A : L^2(\Omega) \rightarrow L^2(\Omega) \quad \text{and} \quad X \rightarrow \int_{\Omega} dy H(x, y)X(y). \] (8)

   The reconstructed $X(x)$ will be an approximation to $(V(x) - \lambda)\Psi(x)$, $x \in \Omega$.

2. Compute $\Psi(x)$ for all $x \in \Omega$ by means of a simple quadrature using equation (5).

3. Compute $V(x)$ for all $x \in \Omega$ using
   \[ V(x) = \frac{X(x)}{\Psi(x)} + \lambda. \] (9)

   Any computational difficulties related to very small or zero values of $\Psi(x)$ in $\Omega$ can be avoided by an appropriate choice of the constant $C$ defined in equation (2).

4. Compute $\tau(x) = \sqrt{\sigma(x)}$, $x \in \Omega$ by solving
   \[ \Delta \tau(x) = -V(x)\tau(x), \] (10)

   with given boundary values $\tau(x)$, $x \in \partial\Omega$.

Thus, the non-linear ill-posed inverse conductivity problem has been transformed into a linear ill-posed problem (step 1) followed by a non-linear process (step 3) leading to a unique stable solution of the differential equation (10) (step 4).

The ill-posed linear system in step 1 can be stabilized using any regularization method for integral equations of the first kind. However, mollifier methods offer certain advantages:

(i) Locally adapted resolution can be easily incorporated in a mollifier approach.
(ii) The inversion of $A$ is an ill-posed problem and data errors of $\zeta$ are amplified in the reconstruction. This problem is overcome by the mollifier approach, since it requires us to solve an operator equation for every reconstruction point $x$ with an analytically given exact right hand side which can be done with arbitrary precision.
(iii) All the pointwise reconstruction vectors can be precomputed and the final data-dependent reconstruction step simply requires the computation of one scalar product per reconstruction point.
(iv) The mollifier method allows us to optimize the choice of the function $H$.

The mollifier method is a pointwise reconstruction technique which is based on a Dirac-sequence \( \{ e_\gamma(\tilde{y}, y) \} \). A standard choice of mollifier is given by
\[
e_\gamma(\tilde{y}, y) = \frac{1}{|B(\tilde{y}, \gamma)|} \cdot \chi_{B(\tilde{y}, \gamma)}(y) = \frac{1}{|B(\tilde{y}, \gamma)|} \cdot \begin{cases} 1 & : y \in B(\tilde{y}, \gamma) \\ 0 & : \text{otherwise} \end{cases} \] (11)

and this is the one we use in this paper. The parameter $\gamma$ is called the resolution or the regularization parameter at the point $\tilde{y}$ and $B(\tilde{y}, \gamma) = \{ y : ||\tilde{y} - y|| \leq \gamma \}$.

For a given reconstruction point $\tilde{y}$ and a desired resolution $\gamma$ instead of reconstructing $X$ itself, we attempt a reconstruction of
\[ X_\gamma(\tilde{y}) = <e_\gamma(\tilde{y}, \cdot), X >_{L^2(\Omega)}. \] (12)
Theorem 1

The mollified approximation to the solution (3) in $y$

Hence we can aim at best at an effective mollifier $\tilde{\eta}$ of the data $\tilde{\gamma}$.

Consequently, the reconstruction at the point $\tilde{y}$ is achieved by simply computing a scalar product of the data $\zeta$ with the precomputed $w_\gamma(\tilde{y}, \cdot)$.

Let $\tilde{e}_\gamma = A^* w_\gamma$, where $w_\gamma$ is the minimizer of equation (13). This implies that we will actually reconstruct

$$X_\gamma(x) = \int_\Omega d\gamma(x,y)\tilde{e}_\gamma(x,y).$$

In order to determine $\tilde{e}_\gamma$ we first need to determine the range of $A^*$. This range depends on the choice of the kernel $H$.

**Lemma 1** Range $(A^*) \subset \mathcal{H}$.

**Proof**: Let $H$ denote the kernel of $A^*$ and consider $g \in L^2(\Omega)$, then it is straightforward to show that

$$\Delta_y A^* g(y) = -\lambda A^* g(y).$$

Hence we can aim at best at an effective mollifier $\tilde{e}_\gamma(\tilde{y}, y)$ which satisfies the Helmholtz equation (3) in $y$.

**Theorem 1** The mollified approximation to the solution $X$ of equation (4) at $\tilde{y}$ is given by

$$X_\gamma(\tilde{y}) = \left<\tilde{e}_\gamma(\tilde{y}, \cdot), X \right>_L^2(\Omega) = \int_{\partial \Omega} \left( \frac{\partial \tilde{e}_\gamma}{\partial n} (\tilde{y}, x) \Psi(x) - \tilde{e}_\gamma (\tilde{y}, x) \frac{\partial \Psi}{\partial n} (x) \right) dx.$$
4. Analytic development in two dimensions

If we restrict ourselves to problems in disks explicit calculations are possible.

**Theorem 2** Let \( \Omega \) be a two dimensional disk of radius \( R, \Omega = \{ y = (r \cos \theta, r \sin \theta) \in \mathbb{R}^2 : \| y \| \leq R \} \), the orthonormal solutions \( \{ H_l^j : j = 1, 2 \} \) of equation (3) for \( \lambda > 0 \) are given by

\[
\begin{align*}
H_1^1(r, \theta) &= \rho_1 J_1(\sqrt{\lambda}r) \cos(\theta) = \rho_1 \Re \left[ J_1(\sqrt{\lambda}r)e^{i\theta} \right], \\
H_1^2(r, \theta) &= \rho_1 J_1(\sqrt{\lambda}r) \sin(\theta) = \rho_1 \Im \left[ J_1(\sqrt{\lambda}r)e^{i\theta} \right],
\end{align*}
\]

where \( \rho_l \) are the normalization constants:

\[
\rho_l = \frac{1}{R} \sqrt{\frac{(2 - \delta_{l0})}{\pi \left( J_l^2(\sqrt{\lambda}R) - J_{l-1}(\sqrt{\lambda}R)J_{l+1}(\sqrt{\lambda}R) \right)}}.
\]

Hence, the effective mollifier at point \( \tilde{y} \) is given by

\[
\tilde{e}_\gamma(\alpha, \psi; r, \theta) = \frac{2}{\sqrt{\lambda}} J_1(\sqrt{\lambda}r) \sum_{l=0}^{\infty} \rho_l^2 J_1(\sqrt{\lambda}\alpha) J_l(\sqrt{\lambda}r) \cos(l(\psi - \theta)),
\]

where \( \alpha, \psi \) are the polar coordinates of \( \tilde{y} \), the centre of the mollifier disk. Furthermore,

\[
X_\gamma(\alpha, \psi) = \frac{2\pi R J_1(\sqrt{\lambda}r)}{\sqrt{\lambda}} \sum_{l=0}^{\infty} \rho_l^2 J_l(\sqrt{\lambda}\alpha) \left[ (a_l \cos(l\psi) + b_l \sin(l\psi)) J_l(\sqrt{\lambda}r) \right. \\
- \left. (c_l \cos(l\psi) + d_l \sin(l\psi)) J_l(\sqrt{\lambda}r) \right],
\]

where \( a_l, b_l \) and \( c_l, d_l \) are respectively the Fourier coefficients of \( \Psi \) and \( \frac{\partial \Psi}{\partial n} \) on \( \partial \Omega \).

**Proof:** It is straightforward to show that \( J_l(\sqrt{\lambda}r)e^{i\theta} \) are orthogonal solutions to equation (3). In order to guarantee an orthonormal set of solutions \( \{ H_l^j \} \) we require that

\[
\int_{\Omega} dy \left( H_l^j(y) \right)^2 = \rho_l^2 \pi \int_0^R dr r J_l(\sqrt{\lambda}r)^2 = 1
\]

and the result given in equation (19) follows.

For a fixed point \( \tilde{y} \in \Omega, \tilde{e}_\gamma(\tilde{y}, y) \) is the projection of the mollifier \( e_\gamma(\tilde{y}, \cdot) \in L^2(\Omega) \) given in (11) on \( H \), i.e.

\[
\tilde{e}_\gamma(\tilde{y}, y) = \sum_{l=-\infty}^{\infty} \sum_{j=1}^{2} < e_\gamma(\tilde{y}, \cdot), H_l^j >_{L^2(\Omega)} H_l^j(y).
\]

The inner product \( < e_\gamma(\tilde{y}, \cdot), H_l^j >_{L^2(\Omega)} \) is given by

\[
< e_\gamma(\tilde{y}, \cdot), H_l^j >_{L^2(\Omega)} = \int_{\Omega} \frac{dy}{\Omega} e_\gamma(\tilde{y}, y) H_l^j(y) = \frac{1}{\pi \gamma^2} \int_{B(\tilde{y}, \gamma)} dy \left( \frac{2\pi \gamma}{\sqrt{\lambda}} \right) J_1(\sqrt{\lambda}r) H_l^j(\tilde{y}).
\]

Using the Mean Value Theorem for the Helmholtz equation [15] we can show that

\[
\int_{B(\tilde{y}, \gamma)} dy H_l^j(y) = \left( \frac{2\pi \gamma}{\sqrt{\lambda}} \right) J_1(\sqrt{\lambda}r) H_l^j(\tilde{y}).
\]

The result in equation (20) follows immediately. It is now straightforward to derive the result for \( X_\gamma \) by substituting in equation (16) the explicit formula for \( \tilde{e}_\gamma \) given in equation (20). \( \square \)
Lemma 2 If $\lambda \to 0$ then equation (3) reduces to Laplace’s equation and the scalar product (22) is independent of the regularization parameter $\gamma$. Hence, the effective mollifier $\tilde{e}_\gamma(\tilde{y}, y)$ (20) and, consequently, the reconstructed $X_\gamma$ (21) are independent of the resolution $\gamma$.

Proof: If $\lambda \to 0$ then
\[
\int_{B(\tilde{y}, \gamma)} d y H^\gamma_I(y) = \left( \frac{2\pi \gamma}{\sqrt{\lambda}} \right) J_1(\sqrt{\lambda} \gamma) H^\gamma_I(\tilde{y}) \to \pi \gamma^2 H^\gamma_I(\tilde{y}).
\] (24)
Consequently, the scalar product (22) becomes
\[
< e_\gamma(\tilde{y}, \cdot), H^\gamma_I >_{L^2(\Omega)} = H^\gamma_I(\tilde{y}).
\] (25)
Hence, the effective mollifier $\tilde{e}_\gamma(\tilde{y}, y)$ (20) and the reconstructed $X_\gamma$ (21) are independent of the resolution $\gamma$. \(\square\)

5. Numerical examples
Following our approach we should be able to determine an acceptable reconstruction of the conductivity from one measurement only. In principle, this requires an optimal choice of the induced current but in this paper we will investigate what can be reconstructed from a single measurement resulting from a standard cosine pattern for the applied current.

The method described in this paper is not restricted to a two-dimensional circular geometry. However, for simplicity as in the previous section, we consider the geometry of the unit disk
\[
\Omega = \{ y = (r \cos \theta, r \sin \theta) \in \mathbb{R}^2 : \| y \| \leq 1 \}.
\]
We shall consider the input current $j(\theta) = \cos \theta$. Our general procedure for performing the numerical test is as follows. We start by choosing a conductivity $\sigma$ that we attempt to reconstruct. To simulate the measured values of the potential on the boundary we have to solve first the direct problem. In order to avoid inverse crimes we use a forward solver which has no connection with the method under consideration. Thus, we obtain $\Phi$ and $\Psi$ as well as their normal derivatives on the boundary.

The inversion algorithm is to use our mollifiers method to compute $X$ by means of equation (21) and then follow the step by step procedure described in Section 3. Equation (10) can be solved for $\tau$ in a stable way, for instance, by means of a Fredholm equation of the second kind. Since the reconstruction is extremely fast there is no need to use an adaptive resolution. Therefore for the following numerical examples we use the same regularization parameter $\gamma = 0.1$ for all reconstruction points. We present the reconstruction results only for data with 1% random errors and $\lambda = 1$. This noise level of 1% is reasonable in many circumstances but in some medical applications greater accuracy can be achieved [16]. In order to avoid any damping effects in the determination of $V$ by equation (9) the value of $\lambda$ cannot be large. Numerical experiments showed that a value of $\lambda = 1$ is an appropriate choice.

Example 1 As a first example we attempt to reconstruct a conductivity distribution consisting of an off-centered high conductivity region within a constant background, see Figure 1(a)
\[
\sigma_1(x, y) = 1 + \frac{1}{(x - 0.6)^2 + (y - 0.3)^2 + 0.1}.
\]
We present in Figure 1(b) the reconstructed conductivity for data with 1% random errors.

Example 2 The second example is a test for the achievable resolution of our method. The conductivity distribution consists of two high conductivity regions, see Figure 2(a),
\[
\sigma_2(x, y) = 1 + \frac{1}{(x - 0.7)^2 + (y - 0.1)^2 + 0.1} + \frac{2}{(x + 0.6)^2 + (y - 0.2)^2 + 0.2}.
\]
Figure 1. (a) The conductivity distribution $\sigma_1(x, y)$; (b) The reconstructed conductivity for data with 1% errors.

Figure 2. (a) The conductivity distribution $\sigma_2(x, y)$; (b) The reconstructed conductivity for data with 1% errors.

6. Conclusion
In this paper we have investigated the use of mollifiers to the inverse conductivity problem. We assume that the conductivity $\sigma$, the potential $\Phi$ and their normal derivatives are known on the boundary. The inverse problem is reformulated as a linear Fredholm integral equation of the first kind, where the kernel satisfies the Helmholtz equation. The linear problem is solved by applying the method of mollifiers. Since we used the free space Green’s function the approach is not geometrically constrained and can be applied to any simply connected domain. Both the theoretical investigations and the two dimensional numerical experiments indicate that the algorithm reconstructs continuous distributions of conductivity and that it is quite stable with respect to the noise level in the data. Further developments of the our approach are currently under consideration.

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