Abstract

We study the Hamiltonian structure of the general parity-invariant model of three-dimensional gravity with propagating torsion, with eight parameters in the Lagrangian. In the scalar sector, containing scalar or pseudoscalar modes with respect to maximally symmetric background, the phenomenon of constraint bifurcation is observed and analyzed. The stability of the Hamiltonian structure under linearization is used to identify dynamically acceptable values of parameters.

1 Introduction

Models of three-dimensional (3D) gravity were introduced to help us in clarifying highly complex dynamical behavior of the realistic four-dimensional general relativity (GR). In the last three decades, they led to a number of outstanding results [1]. However, in the early 1990s, Mielke and Baekler [2] proposed a new, non-Riemannian approach to 3D gravity, based on the Poincaré gauge theory (PGT) [3, 4, 5, 6]. In contrast to the traditional GR with an underlying Riemannian geometry of spacetime, the PGT approach is characterized by a Riemann–Cartan geometry, with both the curvature and the torsion of spacetime as carriers of the gravitational dynamics. Thus, PGT allows exploring the interplay between gravity and geometry in a more general setting.

Three-dimensional GR with or without a cosmological constant, as well as the Mielke–Baekler (MB) model, are topological theories without propagating modes. From the physical point of view, such a degenerate situation is certainly not quite realistic. In the context of Riemannian geometry, this limitation is surmounted by two well-known models: topologically massive gravity [7] and the Bergshoeff–Hohm–Townsend massive gravity [8]. On the other hand, including propagating modes in PGT is much more natural: it is achieved simply by using Lagrangians quadratic in the field strengths [9, 10, 11, 12].

Since the general parity-invariant PGT Lagrangian in 3D is defined by eight arbitrary parameters [11], it is a theoretical challenge to find out which values of the parameters are...
allowed in a viable theory. Following the approach of Sezgin and Niewenhuisen [13], Helayél-Neto et al. [10] used the weak-field approximation around the Minkowski background to analyze this issue in a parity-violating version of PGT, and found a number of interesting restrictions on the parameters. However, one should be very careful with the interpretation of these results, since (i) it is not clear how the transition from Minkowski to (anti-)de Sitter [(A)dS] background might influence the perturbative analysis, and (ii) the weak-field approximation does not always lead to a correct identification of the physical degrees of freedom. Regarding (ii), we note that the constrained Hamiltonian method [14, 4] is best suited for analyzing dynamical content of gauge field theories, respecting fully their nonlinear structure. As noticed by Chen et al. [15] and Yo and Nester [16], it may happen, for some ranges of parameters, that the canonical structure of a theory (the number and/or type of constraints) is changed after linearization in a way that affects its physical content, such as the number of physical degrees of freedom. Based on the canonical stability under linearization as a criterion for an acceptable choice of parameters, Shie et al. [17] were able to define a PGT cosmological model that offers a convincing explanation of dark energy as an effect induced by torsion. Recently, the Bergshoeff–Hohm–Townsend massive gravity is found to be canonically unstable under linearization [18, 19].

In this paper, we use the constrained Hamiltonian formalism to study (a) the phenomenon of “constraint bifurcation” and (b) the stability under linearization of the general parity-invariant PGT in 3D [11], in order to find out the parameter values that define consistent models of 3D gravity with propagating torsion. Because of the complexity of the Hamiltonian structure, we restrict our attention to the scalar sector, with $J^P = 0^+$ or $0^-$ modes, defined with respect to the (A)dS background. Investigation of higher spin modes is left for a future study.

The paper is organized as follows. In Section 2, we review basic Lagrangian aspects of the parity-invariant PGT in 3D. In Section 3, we give a brief account of the weak-field approximation around the (A)dS background, restricting our attention to the scalar sector, with $J^P = 0^+$ or $0^-$. In Section 4, we analyze general aspects of the canonical dynamics of PGT; in particular, we examine how, depending on certain critical values of parameters, some extra primary constraints may appear (if-constraints), leading to a significant effect on the Hamiltonian structure. In Section 5, we analyze the canonical structure of the spin-$0^+$ sector, including the “constraint bifurcation” effects. Then, the test of canonical stability under linearization is used to reveal dynamically acceptable values of parameters. In Section 6, the same type of analysis is carried out for the spin-$0^-$ sector. Section 7 is devoted to concluding remarks, and appendices contain technical details.

Our conventions are as follows: the Latin indices ($i, j, k, ...$) refer to the local Lorentz frame, the Greek indices ($\mu, \nu, \lambda, ...$) refer to the coordinate frame, and both run over 0,1,2; the metric components in the local Lorentz frame are $\eta_{ij} = (+, -, -)$; totally antisymmetric tensor $\varepsilon^{ijk}$ is normalized to $\varepsilon^{012} = 1$.

## 2 Lagrangian formalism

We begin our considerations by a short account of the Lagrangian formalism for PGT. Assuming parity invariance, the dynamics of 3D gravity with propagating torsion is determined
by the gravitational Lagrangian (density) \( \mathcal{L}_G = b \mathcal{L}_G, \)

\[
\mathcal{L}_G = -aR - 2\Lambda_0 + \mathcal{L}_{T^2} + \mathcal{L}_{R^2} ,
\]

where \( \Lambda_0 \) is a bare cosmological constant, \( a = 1/16\pi G, \) and the pieces quadratic in the field strengths read:

\[
\mathcal{L}_{T^2} := \frac{1}{2} T^{ijk} (a_1^{(1)} T_{ijk} + a_2^{(2)} T_{ij} + a_3^{(3)} T_{ijk}) ,
\]

\[
\mathcal{L}_{R^2} := \frac{1}{4} R^{ijkl} (b_4^{(4)} R_{ijkl} + b_5^{(5)} R_{ijkl} + b_6^{(6)} R_{ijkl}) ,
\]

where \( a^{(n)} T_{ijk} \) and \( a^{(n)} R_{ijkl} \) are irreducible components of the torsion and the Riemann–Cartan curvature [11]. Since the Weyl curvature vanishes in 3D, one can rewrite these expressions in the form that is more practical for the canonical analysis:

\[
L_{T^2} = T^{ijk} (\alpha_1 T_{ijk} + \alpha_2 T_{ij} + \alpha_3 \eta_{ij} V_k) ,
\]

\[
L_{R^2} = R^{ij} (\beta_1 R_{ij} + \beta_2 R_{ij} + \beta_3 \eta_{ij} R) =: R^{ij} \mathcal{H}_{ij} ,
\]

Here, \( V_k := T^{m}_{mk} , \) \( R_{ij} := R^m_{imn} \) is the Ricci tensor, \( R \) is the scalar curvature, and

\[
\alpha_1 = \frac{1}{6} (2a_1 + a_3) , \quad \alpha_2 = \frac{1}{3} (a_1 - a_3) , \quad \alpha_3 = \frac{1}{2} (a_2 - a_1).
\]

\[
\beta_1 = \frac{1}{2} (b_4 + b_5) , \quad \beta_2 = \frac{1}{2} (b_4 - b_5) , \quad \beta_3 = \frac{1}{12} (b_6 - 4b_4).
\]

We also introduce the covariant momenta \( \mathcal{H}_{ijk} = \partial \mathcal{L}_G / \partial T^{ijk} \) and \( \mathcal{H}_{ijkl} = \partial \mathcal{L}_G / \partial R^{ijkl} ; \)

\[
\mathcal{H}_{ijk} = 2 (a_1^{(1)} T_{ijk} + a_2^{(2)} T_{ij} + a_3^{(3)} T_{ijk})
\]

\[
-4 (\alpha_1 T_{ijk} + \alpha_2 T_{[kji]} + a_3 \eta_{ij} V_k) ,
\]

\[
\mathcal{H}_{ijkl} = -2a (\eta_{ik} \eta_{jl} - \eta_{ij} \eta_{kl}) + \mathcal{H}'_{ijkl},
\]

\[
\mathcal{H}'_{ijkl} = 2 (b_4^{(4)} R_{ijkl} + b_5^{(5)} R_{ijkl} + b_6^{(6)} R_{ijkl})
\]

\[
-2 (\eta_{ik} \mathcal{H}_{jl} - \eta_{ij} \mathcal{H}_{kl}) - (k \leftrightarrow l) .
\]

General field equations for the PGT theory (2.1) are given in [11]. Without matter contribution, these equations, transformed to the local Lorentz basis, take the form:

\[
\nabla^m \mathcal{H}_{imj} + \frac{1}{2} \mathcal{H}_{imn} (-T_{jmn} + 2\eta_{jm} V_n) - t_{ij} = 0 , \quad (2.2a)
\]

\[
2a T_{kij} + 2T^m_{ij} (\mathcal{H}_{mk} - \eta_{mk} \mathcal{H}) + 4 \nabla_i (\mathcal{H}_{jk} - \eta_{ij} \mathcal{H}) + \varepsilon_{ijm} \varepsilon^{mn} k \mathcal{H}_{mr} = 0 , \quad (2.2b)
\]

where \( \mathcal{H} = \mathcal{H}^{k}_k , \) and \( t_{ij} \) is the energy-momentum tensor of gravity:

\[
t_{ij} := \eta_{ij} \mathcal{L}_G - T^m_{mn} \mathcal{H}_{mnj} + 2a \mathcal{R}_{ji} - 2(\mathcal{R}^n_i \mathcal{H}_{nj} - \mathcal{R}^m_{ij} \mathcal{H}_{nm}) .
\]

Relying again on the vanishing of the Weyl curvature, one can express Bianchi identities in terms of the Ricci tensor. In the local Lorentz basis, these identities take the form:

\[
\varepsilon^{mnv} \nabla_m T_n^r + \varepsilon^{rsm} T_m^i T_{n}^s + 2\varepsilon_r^{mn} R_{mn} = 0 ,
\]

\[
\nabla_k G^{ki} - V_k G^{ki} = 0 ,
\]

\[
\text{where } G_{ki} := R_{ki} - \frac{1}{2} \eta_{ik} R ,
\]
3 Scalar excitations around (A)dS background

Particle spectrum of 3D gravity with torsion (2.1) around the Minkowski background $M_3$ is already known \[10, 11\]. Here, we wish to examine the modification of this spectrum induced by transition to the (A)dS background. This will help us to clarify the relation between the canonical stability of the theory under linearization and its $M_3$ or (A)dS particle spectrum. Our attention is restricted to the scalar sector, with $J^P = 0^+, 0^-$ modes.

Maximally symmetric configuration of 3D gravity with torsion is defined by the set of fields $\bar{\phi} = (\bar{b}_i{}^\mu, \bar{A}^{ij}{}^\mu)$, such that

$$T_{ijk} = p\varepsilon_{ijk}, \quad \bar{R}^{ij}_{mn} = -q (\delta^i_m \delta^j_n - \delta^i_n \delta^j_m),$$

where the parameters $p$ and $q$ define an effective cosmological constant,

$$\Lambda_{\text{eff}} := q - \frac{p^2}{4}.$$  \hspace{1cm} (3.1)

In order for this configuration to be a solution of the field equations in vacuum, the parameters $p$ and $q$ have to satisfy the following conditions \[11\]:

$$p(a + qb_6 + 2a_3) = 0, \quad aq - \Lambda_0 + \frac{1}{2}p^2a_3 - \frac{1}{2}q^2b_6 = 0. \hspace{1cm} (3.2a)$$

In the weak-field approximation around $\bar{\phi}$, the gravitational variables $\phi = (b^\mu_\nu, A^{ij}_\nu)$ take the form $\phi = \bar{\phi} + \tilde{\phi}$. We use the convention that indices of the linear excitations $\tilde{\phi}$ are changed by the background triad and/or metric.

The analysis of the particle spectrum is based on the linearized field equations. In the same approximation, the Bianchi identities read:

$$\varepsilon^{kmn} \nabla_k \bar{T}^{i}_{mn} - 2p\bar{V}^i + 2\varepsilon^{inn} \bar{R}_{mn} = 0, \hspace{1cm} (3.3a)$$

$$\nabla_k \bar{G}^{ki} - q\bar{V}^i = 0. \hspace{1cm} (3.3b)$$

3.1 Spin-0$^+$ mode

Looking at the particle spectrum of the theory (2.1) on the $M_3$ background, see section 3 in \[11\], one finds that the spin-0$^+$ mode has a finite mass (and propagates) if

$$a_2(b_4 + 2b_6) \neq 0.$$  \hspace{1cm} (3.4a)

In order to study the spin-0$^+$ mode, we adopt the following, somewhat simplified conditions:

$$a_2, b_6 \neq 0, \quad a_1 = a_3 = b_4 = b_5 = 0. \hspace{1cm} (3.4a)$$

In fact, this choice is not unique since the existence of a spin-0$^+$ mode can be realized, for instance, without requiring $b_1 = 0$. However, our “minimal” choice (3.4a) greatly simplifies the calculations, and moreover, one does not expect that any essential dynamical feature of the spin-0$^+$ mode will be thereby lost, see \[15, 16\]. The corresponding Lagrangian reads:

$$\mathcal{L}_G^+ = -aR - 2\Lambda_0 + \frac{1}{2}a_2V^kV_k + \frac{1}{12}b_6R^2,$$  \hspace{1cm} (3.4b)
and the conditions (3.2) reduce to

\[ p(a + qb_6) = 0, \quad aq - \Lambda_0 - \frac{1}{2}q^2b_6 = 0. \]  

(3.4c)

Now, we are going to show that the Minkowskian conditions (3.4a) equally well define the spin-0\(^+\) mode with respect to the (A)dS background (3.1). We start by noting that, under the conditions (3.4a), the linearized field equations (2.2) read:

\[ (a + qb_6) \tilde{G}^i_j + a_2 \eta^i_j \tilde{\nabla}^k \tilde{V}_k + \frac{b_6q}{3} \eta^i_j \tilde{R} = 0, \]  

(3.5a)

\[ (a + qb_6) \tilde{T}^i_jk - \frac{pb_6}{6} \varepsilon^i_jk \tilde{R} + a_2 \eta^i_j \tilde{V}_k + \frac{b_6}{3} \eta^i_j \tilde{\nabla}^k \tilde{V}_k = 0, \]  

(3.5b)

and their traces are

\[ -2a_2 \tilde{\nabla}^i \tilde{V}_i + (a - qb_6) \tilde{R} = 0, \]  

(3.6a)

\[ (a + qb_6 + a_2) \tilde{V}_k + \frac{b_6}{3} \tilde{\nabla}^k \tilde{R} = 0. \]  

(3.6b)

In the generic case, by combining \( \tilde{\nabla}^k \tilde{V}^k \) of (3.6a) with \( \tilde{\nabla}^k \tilde{V}^k \) of (3.6b), one obtains

\[ (\tilde{\nabla}^i \tilde{V}^i + m_{0+}^2) \tilde{\sigma} = 0, \quad m_{0+}^2 = \frac{3(a - qb_6)(a + qb_6 + a_2)}{2a_2b_6} \]  

(3.7)

where \( \tilde{\sigma} := \tilde{\nabla}^i \tilde{V}^i \). Thus, the field \( \tilde{\sigma} \) can be identified as the spin-0\(^+\) excitation with respect to the (A)dS background, the mass of which is finite. In the limit of vanishing \( q \), \( m_{0+}^2 \) reduces to the corresponding Minkowskian expression.

### 3.2 Spin-0\(^-\) mode

Similar analysis can be applied to the spin-0\(^-\) excitation. We start from the Minkowskian condition that the spin-0\(^-\) mode has a finite mass (and propagates) [11],

\[ (a_1 + 2a_3)b_5 \neq 0. \]  

(3.8a)

We describe dynamics of the spin-0\(^-\) sector by the simplified conditions:

\[ a_3, b_5 \neq 0, \quad a_1 = a_2 = b_4 = b_6 = 0. \]  

(3.8a)

The related Lagrangian has the form

\[ \mathcal{L}_{G} = -aR - 2\Lambda_0 + 3a_3 \mathcal{A}^2 + b_5 R_{[ij]} R^{[ij]}, \]  

(3.8b)

with \( \mathcal{A} = \varepsilon^{ijk} T_{ijk}/6 \), and the conditions (3.2) reduce to

\[ p(a + 2a_3) = 0, \quad aq - \Lambda_0 - \frac{1}{2}p^2a_3 = 0. \]  

(3.8c)
Starting from the linearized field equations,
\[ a_3 \varepsilon_{ijk} \nabla^k \tilde{A} + a_3 p \eta_{ij} \tilde{A} + \frac{4a_3}{3} p \varepsilon_{(imn} \tilde{f}_{j)mn} - a_3 p \varepsilon_{ijk} \tilde{V}^k + a \tilde{G}_{ji} + b_5 q \tilde{R}_{[ij]} = 0, \]  
(3.9a)
\[ a \tilde{T}_{ijk} + pb_5 \varepsilon^{nk} j_k \tilde{R}_{ni} + b_5 \nabla_{[i} ( \tilde{R}_{k]} - \tilde{R}_{ik}) + 2a_3 \varepsilon_{ijk} \tilde{A} = 0. \]  
(3.9b)
the axial irreducible components of these equations read:
\[ a_3 \nabla^i \tilde{A} - a_3 p \tilde{V}^i - \frac{1}{2} (a - qb_5) \varepsilon^{ijk} \tilde{R}_{jk} = 0, \]
\[ (a + 2a_3) \tilde{A} + \frac{1}{3} b_5 \varepsilon^{ijk} \nabla_i \tilde{R}_{jk} = 0. \]

Then, the divergence of the first equation combined with the second one yields
\[ a_3 \nabla^i \nabla_i \tilde{A} - pa_3 \nabla_i \tilde{V}^i + \frac{1}{2} (a - qb_5) \frac{3(a + 2a_3)}{b_5} \tilde{A} = 0. \]  
(3.11)
Now, using the divergence of the first Bianchi identity (3.3a) and the commutator identity
\[ [\nabla_m, \nabla_n] \tilde{X}_i = -p \varepsilon_{mnk} \nabla^k \tilde{X}_i - 2q \eta_{[m} \tilde{X}_{n]}, \]
we find
\[ \sigma \equiv \nabla_k \tilde{V}^k = -\frac{3}{2} p (a + 2a_3) \tilde{A} = 0, \]
as a consequence of (3.8c). Hence, (3.11) implies
\[ (\nabla_k \nabla^k + m^2_{0-}) \tilde{A} = 0, \quad m^2_{0-} = \frac{3(a - qb_5)(a + 2a_3)}{2a_3 b_5}. \]  
(3.12)
Thus, generically, \( \tilde{A} \) can be identified as the spin-0 excitations with respect to the (A)dS background. For \( q = 0 \), \( m^2_{0-} \) takes the Minkowskian form.

4 Hamiltonian structure

In this section, we analyze general features of the Hamiltonian structure of 3D gravity with propagating torsion, defined by the Lagrangian (2.1); see [4, 20].

4.1 Primary constraints

We begin our study by analyzing the primary constraints. The canonical momenta corresponding to basic dynamical variables \((b^i, A^{ij})\) are \((\pi_i, \Pi_{ij})\); they are given by
\[ \pi_i := \frac{\partial \mathcal{L}}{\partial (\partial_0 b^i)} = b \mathcal{H}_i 0^i, \quad \Pi_{ij} := \frac{\partial \mathcal{L}}{\partial (\partial_0 A^{ij})} = b \mathcal{H}_{ij} 0^i. \]
Since the torsion and the curvature do not involve the velocities \(\partial_0 b^i_0\) and \(\partial_0 A^{ij}_0\), one obtains the so-called "sure" primary constraints
\[ \pi_i 0 \approx 0, \quad \Pi_{ij} 0 \approx 0, \]  
(4.1)
which are always present, independently of the values of coupling constants. If the Lagrangian (2.1) is singular with respect to some of the remaining velocities \( \partial_b b^i_\alpha \) and \( \partial_b A^{ij}_\alpha \), one obtains further primary constraints. The existence of these primary “if-constraints” (ICs) is determined by the critical values of the coupling constants.

**The torsion sector.** The gravitational Lagrangian (2.1) depends on the time derivative \( \partial_0 b^i_\alpha \) only through the torsion tensor, appearing in \( L_{T^2} \). It is convenient to decompose \( T_{ijk} \) into the parallel and orthogonal components with respect to the spatial hypersurface \( \Sigma \) (see Appendix A),

\[
T_{ijk} = T_{ijk} + 2T_i[j\perp n_k] = T_{ijk} + \mathcal{T}_{ijk} ,
\]

where \( T_{ijk} := T_{ijk} \) does not depend on velocities and the unphysical variables \( (b^i_0, A^{ij}_0) \), and \( n_k \) is the normal to \( \Sigma \). Now, by introducing the parallel gravitational momentum

\[
\hat{\pi}^k_i = \pi^a b^k_\alpha \ (\hat{\pi}^k_i n_k = 0),
\]

one obtains

\[
\hat{\pi}^k_i = J\mathcal{H}_{i\perp k}(T) ,
\]

where \( J := \det(b^i_\alpha) \), and

\[
\mathcal{H}_{i\perp k} = 2[2\alpha_1 T_{i\perp k} + \alpha_2 (T_{k\perp i} - T_{i\perp k}) + \alpha_3 (n_i V_k - n_k V_i)] .
\]

The linearity of \( \mathcal{H}_{i\perp k}(T) \) in the torsion tensor allows us to rewrite (4.2a) in the form

\[
\phi_{i\perp k} := \frac{\hat{\pi}^k_i}{J} - \mathcal{H}_{i\perp k}(T) = \mathcal{H}_{i\perp k}(T) ,
\]

where the “velocities” \( T_{i\perp k} \) appear only on the right-hand side. This system of equations can be decomposed into irreducible parts with respect to the group of two-dimensional rotations in \( \Sigma \). Going over to the parameters \( a_1, a_2, a_3 \), one obtains:

\[
\hat{\phi}_{\perp k} \equiv \frac{\hat{\pi}^k_i}{J} - (a_2 - a_1)T^m_{\perp m_k} = (a_1 + a_2)T_{\perp k} ,
\]

\[
S^\phi \equiv \frac{S\hat{\pi}^k_i}{J} = -2a_2 T^m_{m\perp} ,
\]

\[
A^\phi_{i\perp k} \equiv \frac{A\hat{\pi}^k_i}{J} - \frac{2}{3}(a_1 - a_3)T_{\perp i\perp k} = -\frac{2}{3}(a_1 + 2a_3)T_{[i\perp k]\perp} ,
\]

\[
T^\phi_{i\perp k} \equiv \frac{T\hat{\pi}^k_i}{J} = -2a_1 T_{k\perp} ,
\]

where \( S^\phi, A^\phi_{i\perp k}, T^\phi_{i\perp k} \) are the trace (scalar), antisymmetric and traceless-symmetric parts of \( \phi_{i\perp k} \) (Appendix A).

If the critical parameter combinations appearing on the right-hand sides of Eqs. (4.3) vanish, the corresponding expressions \( \phi_K \) become additional primary constraints, the primary ICs. After a suitable reordering, the result of the analysis is summarized as follows:

- For \( a_2 = 0, a_1 + 2a_3 = 0, a_1 + a_2 = 0 \) and/or \( a_1 = 0 \), the expressions \( S^\phi, A^\phi_{i\perp k}, \phi_{i\perp k} \) and/or \( T^\phi_{i\perp k} \) become primary ICs (see Table 1 below).
The curvature sector. In order to examine how the gravitational Lagrangian depends on the velocities $\partial_0 A^{ij}_\alpha$, we start with the following decomposition of the curvature tensor:

$$R_{ijmn} = R_{ij\tilde{m}\tilde{n}} + 2R_{ij[n\perp m_n]} = R_{ijmn} + \mathcal{R}_{ijmn},$$

where $R_{ijmn} := R_{ij\tilde{m}\tilde{n}}$ does not depend on the “velocities” $R_{ij\perp \tilde{k}}$ and the unphysical variables. The parallel gravitational momentum $\dot{\Pi}_{ij\tilde{k}} := \Pi_{ij\alpha} b^\alpha \ (\Pi_{ij\perp \tilde{k}} n_k = 0)$ is given as

$$\dot{\Pi}_{ij\tilde{k}} = J\mathcal{H}_{ij\perp \tilde{k}}(R), \quad (4.4a)$$

where

$$\mathcal{H}_{ij\perp \tilde{k}} = -4an[i\eta_{j\tilde{k}} + 4n[i\mathcal{H}_{j\tilde{k}} - 4\eta_{[i\tilde{k}} \mathcal{H}_{j]\perp} \]
= 4n[i\eta_{j\tilde{k}} (-a + 2\beta_3 R)
+ 4\beta_1 (n[i\eta_{j\tilde{k}} - \eta_{[i\tilde{k}} R_{j]\perp}) + 4\beta_2 (n[i\eta_{j\tilde{k}} - \eta_{[i\tilde{k}} R_{j}\perp]) \].$$

Since the “velocities” $R_{ij\perp \tilde{k}}$ are contained only in $\mathcal{R}$, we rewrite this equation as

$$\Phi_{ij\tilde{k}} := \frac{\dot{\Pi}_{ij\tilde{k}}}{J} + 4an[i\eta_{j\tilde{k}} - \mathcal{H}_{ij\perp \tilde{k}}(R) = \mathcal{H}_{ij\perp \tilde{k}}(\mathcal{R}). \quad (4.4b)$$

The components of a tensor $X_{\perp ij}$ can be decomposed into the trace, antisymmetric and symmetric-traceless piece (Appendix A). Such a decomposition of (4.4b) yields:

$$s\Phi_{\perp} \equiv \frac{s\dot{\Pi}_{\perp}}{J} + 4a - \frac{2}{3}(b_6 - b_4) R^{k\perp \tilde{k} \perp} k \equiv \frac{2}{3}(b_4 + 2b_6) R^{k\perp \tilde{k} \perp}, \quad (4.5a)$$

$$\lambda\Phi_{\perp ij} \equiv \frac{\lambda\dot{\Pi}_{\perp ij}}{J} + 2b_7 R^{ij}_{\perp \tilde{j} \tilde{k}} = 2b_7 R_{ij\perp \tilde{j} \tilde{k}} \perp, \quad (4.5b)$$

$$t\Phi_{\perp ij} \equiv \frac{t\dot{\Pi}_{\perp ij}}{J} - b_4 \left(2R_{(i\tilde{k}j)} \perp - \eta_{i\tilde{k}} R^{\tilde{m} \tilde{n} \tilde{m} \tilde{n}} \right) = b_4 \left(2R_{(i\tilde{j}j)} \perp - \eta_{i\tilde{j}} R^{\tilde{k} \tilde{k} \perp \tilde{k} \perp} \right). \quad (4.5c)$$

For a tensor $X_{ij\tilde{k}} = -X_{j\tilde{k}i}$, the pseudoscalar ($\varepsilon^{ij}_{\perp k} X_{ij\tilde{k}}$) and the symmetric-traceless piece ($X_{i(jk)} - \text{traces}$) identically vanish. Hence, Eq. (4.4b) implies one more relation:

$$V\Phi_{i} \equiv \frac{V\dot{\Pi}_{i}}{J} - (b_4 - b_5) R_{i\perp \tilde{k}} \perp \tilde{k} = (b_4 + b_5) R^{\tilde{k} \tilde{k} \perp \tilde{k} \perp}, \quad (4.5d)$$

where $VX_{ij} = X^\perp_{ij}$ (Appendix A).

Thus, when the parameters appearing on the right-hand sides of (4.5b) vanish, we have the additional primary constraints $\Phi_K$. Combining these relations with those obtained in the torsion sector, one find the complete set of primary ICs, including their spin-parity characteristics ($J^P$), as shown in Table 1.
Table 1. Primary if-constraints

| Critical conditions | Primary constraints | \( J^P \) |
|---------------------|---------------------|------------|
| \( a_2 = 0 \)       | \( S\phi \approx 0 \) | 0⁺        |
| \( b_4 + 2b_6 = 0 \) | \( S\Phi \perp \approx 0 \) | 0⁻        |
| \( a_1 + 2a_3 = 0 \) | \( A\phi_{\perp k} \approx 0 \) | 1          |
| \( b_5 = 0 \)       | \( A\Phi_{\perp \perp k} \approx 0 \) | 2          |
| \( a_1 + a_2 = 0 \) | \( T\phi_{\perp k} \approx 0 \) |            |
| \( b_4 + b_5 = 0 \) | \( T\Phi_{\perp \perp k} \approx 0 \) |            |

This classification has a noteworthy interpretation: whenever a pair of the ICs with specific \( J^P \) is absent, the corresponding dynamical mode is liberated and becomes a *physical degree of freedom* (DoF). Thus, for \( a_2(b_4 + 2b_6) \neq 0 \), the spin-0⁺ ICs are absent, and the related DoF becomes physical. Similarly, \( (a_1 + 2a_3)b_5 \neq 0 \) implies that the spin-0⁻ DoF becomes physical. The results obtained here refer to the full nonlinear theory; possible differences with respect to the perturbative analysis (Section 3) will be discuss in Sections 5 and 6.

### 4.2 General form of the Hamiltonian

Once we know the complete set of the primary ICs, we can construct first the canonical and then the total Hamiltonian. Being interested only in the gravitational degrees of freedom, we disregard the matter contribution.

**Canonical Hamiltonian.** In the absence of matter, the canonical Hamiltonian (density) is defined by

\[
\mathcal{H}_c = \pi^i_\alpha b^j_\alpha + \frac{1}{2} \Pi_{ij}^\alpha A^{ij}_\alpha - bL_G.
\]

Using the lapse and shift functions \( N \) and \( N^\alpha \), defined in Appendix A, one can rewrite \( \mathcal{H}_c \) in the Dirac–ADM form [4, 20]:

\[
\mathcal{H}_c = N\mathcal{H}_\perp + N^\alpha \mathcal{H}_\alpha - \frac{1}{2} A^{ij}_0 \mathcal{H}_{ij} + \partial_\alpha D^\alpha,
\]

where

\[
\mathcal{H}_\perp = \hat{\pi}^i_\beta T^i_{\perp j} + \hat{\pi}^i_\beta R^{ij}_{\perp} - J\mathcal{L}_G - n^i \nabla_\alpha \pi^i_\alpha,
\]

\[
\mathcal{H}_\alpha = \pi^i_\beta T^i_{\alpha \beta} + \frac{1}{2} \Pi_{ij}^\alpha A^{ij}_{\alpha \beta} - b^i_\alpha \nabla_\beta \pi^i_\alpha,
\]

\[
\mathcal{H}_{ij} = 2\pi^i_\alpha b^j_\alpha + \nabla_\alpha \Pi_{ij} \alpha,
\]

\[
D^\alpha = b^i_\alpha \pi^i_\alpha + \frac{1}{2} \Pi_{ij}^\alpha A^{ij}_\alpha.
\]

The canonical Hamiltonian is linear in the unphysical variables \( (b^i_0, A^{ij}_0) \), and \( \mathcal{H}_\perp \) is the only dynamical part of \( \mathcal{H}_c \). The “velocities” \( T^i_{\perp k}, R^{ij}_{\perp k} \) appearing in \( \mathcal{H}_\perp \) can be expressed
in terms of the phase-space variables, using Eqs. (1.3) and (1.5). Explicit calculation is
simplified by separating the torsion and the curvature contributions in \( \mathcal{H}_\perp \):

\[
\mathcal{H}_\perp = 2A_0 J + \mathcal{H}^T_\perp + \mathcal{H}^R_\perp,
\]

\[
\mathcal{H}^T_\perp := \dot{n}^{ij} T_{i\perp j} - J \mathcal{L}_{T \perp} - n^i \nabla_a \pi^i_a,
\]

\[
\mathcal{H}^R_\perp := \frac{1}{2} \dot{\Pi}^{ijk} R_{i\perp j\perp k} - J \mathcal{L}_{R \perp} + aJ \mathbf{R}.
\]  \( \text{(4.7)} \)

The torsion piece of \( \mathcal{H}_\perp \) turns out to have the form (Appendix A)

\[
\mathcal{H}^T_\perp = \frac{1}{2} J \phi^2 - J \mathcal{L}_{T \perp} (T) - n^i \nabla_a \pi^i_a,
\]  \( \text{(4.8a)} \)

\[
\phi^2 := \frac{\lambda(a_1 + a_2)}{a_1 + a_2} (\phi_{\perp i})^2 + \frac{\lambda(a_2)}{2a_2} (\Phi^T)^2
\]

\[
+ \frac{3}{2} \frac{\lambda(a_1 + 2a_3)}{(a_1 + 2a_3)} (\phi_{ij})^2 + \frac{\lambda(a_1)}{2a_1} (\phi_{ij})^2.
\]  \( \text{(4.8b)} \)

where \( \lambda(x) \) is the singular function

\[
\frac{\lambda(x)}{x} = \begin{cases} 
\frac{1}{x}, & x \neq 0 \\
0, & x = 0
\end{cases}
\]

which takes care of the conditions under which ICs become true constraints. Similar calculations for the curvature part yields

\[
\mathcal{H}^R_\perp = \frac{1}{4} J \Phi^2 - J \mathcal{L}_{R \perp} (\mathbf{R}) + aJ \mathbf{R},
\]  \( \text{(4.9a)} \)

\[
\Phi^2 := \frac{\lambda(b_5)}{b_5} (A_{\perp jk})^2 + \frac{\lambda(b_4)}{b_4} (\Phi_{\perp jk})^2
\]

\[
+ \frac{3}{2} \frac{\lambda(b_4 + 2b_6)}{b_4 + 2b_6} (\Phi_{\perp})^2 + 2\frac{\lambda(b_4 + b_5)}{b_4 + b_5} (\Phi_{\perp}^T)^2.
\]  \( \text{(4.9b)} \)

**Total Hamiltonian.** The total Hamiltonian is defined by the expression

\[
\mathcal{H}_{\text{tot}} = \mathcal{H}_c + u^k_0 \phi^0_k + \frac{1}{2} u^{ij} \Phi^0_{ij} + (u \cdot \phi) + (v \cdot \Phi),
\]  \( \text{(4.10a)} \)

where \( u \)'s and \( v \)'s are arbitrary multipliers and \( (u \cdot \phi) + (v \cdot \Phi) \) denotes the contribution of all the primary ICs. Formally, the existence of ICs is regulated by the form of the related multipliers; for instance, \( u_{\perp} \) is given as \( u_{\perp} := [1 - \lambda(a_1 + a_2)]u_{\perp}, \) and so on. Using the irreducible decomposition technique, we find:

\[
(u \cdot \phi) := u^{\perp} \phi_{\perp} + T u^{\perp} T \phi_{\perp} + A u^{\perp} A \phi_{\perp} + \frac{1}{2} S u S \phi,
\]

\[
(v \cdot \Phi) := T v^{\perp} T \Phi_{\perp} + A v^{\perp} A \Phi_{\perp} + \frac{1}{2} S v S \Phi + V v^{\perp} V \Phi.
\]  \( \text{(4.10b)} \)

**Consistency conditions.** Having found the form of the total Hamiltonian, we can now apply Dirac’s consistency algorithm to the primary constraints, \( \dot{\phi}_K = \{ \phi_K, \mathcal{H}_{\text{tot}} \} \approx 0, \) where \( \mathcal{H}_{\text{tot}} = \int d^3x \mathcal{H}_{\text{tot}} \) and \( \{ X, Y \} \) is the Poisson bracket (PB) between \( X \) and \( Y \); then, the procedure continues with the secondary constraints, and so on [20]. In what follows, our attention will be focused on the scalar sector, with \( J^P = 0^+ \) or \( 0^- \) modes.
5 Spin-0+ sector

As one can see from Table 1, the absence of two spin-0+ constraints, $S\phi$ and $S\Phi_\perp$, is ensured by the condition $a_2(b_4 + 2b_6) \neq 0$, whereby the spin-0+ degree of freedom becomes physical. To study the dynamical content of this sector, we adopt the relaxed conditions

$$a_2, b_6 \neq 0, \quad a_1 = a_3 = b_4 = b_5 = 0,$$

which define the Lagrangian $\mathcal{L}^+_G$ as in (3.4b).

5.1 Hamiltonian and constraints

Primary constraints. In the spin-0+ sector (5.1), general considerations of the previous section lead to the following conclusions: the set of primary constraints is given by

$$\pi^0_i \approx 0, \quad \Pi^0_{ij} \approx 0,$$

$$A\phi_{ij} := \frac{A^{\pi_{ij}}}{J} \approx 0, \quad T\phi_{ij} := \frac{T^{\pi_{ij}}}{J} \approx 0,$$

$$A\Phi_{\perp ij} := \frac{A^{\Pi_{\perp ij}}}{J} \approx 0, \quad T\Phi_{\perp ij} := \frac{T^{\Pi_{\perp ij}}}{J} \approx 0,$$

$$V\Phi_i := \frac{V^{\Pi_i}}{J} \approx 0,$$

the dynamical part of the canonical Hamiltonian has the form

$$\mathcal{H}_{\perp} = J \left[ \frac{1}{2a_2}(\phi_{\perp k})^2 + \frac{1}{4a_2}(S\phi)^2 + \frac{3}{16b_6}(S\Phi_\perp)^2 \right]$$

$$- J\mathcal{L}^+_G(T, R) - n_i \nabla^i \pi^{i\alpha},$$

where $\phi_{\perp k}, S\phi$, and $S\Phi_\perp$ are the “generalized” momentum variables defined in (4.3) and (4.5), and the total Hamiltonian reads

$$\mathcal{H}_{\text{tot}} = \mathcal{H}_c + A_{ij}A\phi_{ij} + T_{ij}T\phi_{ij} + A_{ij}A\Phi_{\perp ij} + T_{ij}T\Phi_{\perp ij} + V_i V\Phi_i.$$  (5.4)

Secondary constraints. The consistency conditions of the sure primary constraints $\pi^0_i$ and $\pi^0_{ij}$ produce the secondary constraints

$$\mathcal{H}_{\perp} \approx 0, \quad \mathcal{H}_\alpha \approx 0, \quad \mathcal{H}_{ij} \approx 0,$$

where

$$\mathcal{H}_\alpha \approx \hat{\pi}^i_{\perp} T_{\perp\alpha i} - \frac{1}{2} S\hat{\pi} V_{\alpha} + \frac{1}{2} S\hat{\Pi} R_{\perp\alpha} - b^{i}_{\alpha} \nabla_{\beta} \pi^{i\beta},$$

$$\mathcal{H}_{\perp k} \approx \frac{A^{\pi_{\perp k}}}{J} + \frac{S\hat{\Pi}_{\perp}}{2J} T_{\perp k},$$

$$\mathcal{H}_{\perp k} \approx \frac{A^{\Pi_{\perp k}}}{J} + \frac{S\hat{\Pi}}{2J} V_{k} + \nabla_{k} \frac{S\hat{\Pi}}{2J}. $$

(5.5b)
Going over to the (eight) primary ICs, $X_M = (A\phi, T\phi, A\Phi, T\Phi, V\Phi)$, we note that the only nonvanishing PBs among them are

$$\{A\Phi_{\perp ij}, A\phi_{\parallel mk}\} \approx -\frac{S_{\perp ij}}{2J^2} \delta_i^{[\parallel m} \delta_j^{\parallel n]} \delta, \quad \{T\Phi_{\perp ij}, T\phi_{\parallel mk}\} \approx \frac{S_{\perp ij}}{2J^2} \delta_i^{(\parallel m} \delta_j^{\parallel n]} \delta.$$  \hfill (5.6)

As long as $S_{\perp} \neq 0$, the constraints $(A\phi, T\phi, A\Phi, T\Phi)$ are second class (SC) \[4, 20\], and their consistency conditions fix the values of the corresponding multipliers $(A_u, T_u, A_v, T_v)$ in $\mathcal{H}_\text{tot}$. On the other hand, $V\Phi$ commutes with all the other primary constraints, but not with its own secondary pair $\chi_i = \{V\Phi_i, H_\text{tot}\}$, see \[20\]. Using $\chi_i \approx J^{-1}\{V\Phi_i, H_\text{tot}\}$ and

$$\{V\Phi_i, \mathcal{H}_{mn}\} \approx 0, \quad \{V\Phi_i, H_\alpha\} \approx 0,$$

$$\{V\Phi_i, \mathcal{H}_\perp\} \approx J \left[ \frac{\phi_{\perp i}}{a_2} \left( \frac{S_{\perp i}}{4J} - \frac{a_2}{2} \right) + \frac{a_2}{2} V_i + \nabla_i \frac{S_{\perp i}}{4J} \right],$$

one ends up with

$$\chi_i := \frac{\phi_{\perp i}}{a_2} \left( \frac{S_{\perp i}}{4J} - \frac{a_2}{2} \right) - \frac{a_2}{2} V_i - \nabla_i \frac{S_{\perp i}}{4J}.$$  \hfill (5.7)

The only nonvanishing PB involving $\chi_i$ is

$$\{\chi_i, V\Phi_k\} = \frac{2}{a_2 J h k} \frac{S_{\perp i}}{4J} \left( \frac{S_{\perp i}}{4J} - a_2 \right) \delta.$$  \hfill (5.8)

Thus, for $S_{\perp}(S_{\perp} - 4Ja_2) \neq 0$, both $\chi_i$ and $V\Phi_k$ are SC. Consequently, the consistency condition of $\chi_i$ determines the multiplier $V\phi_i$, which completes the consistency algorithm.

If the kinetic energy density in the Hamiltonian \[5.3\] is to be positive definite (“no ghosts”), the coefficients of $(S\phi)^2$ and $(S\Phi_{\perp})^2$ should be positive:

$$a_2 > 0, \quad b_6 > 0.$$  \hfill (5.9)

On the other hand, $(\phi_{\perp k})^2$ gives a negative definite contribution, but it is an interaction term, as can be seen from \[4.3a\] and \[5.5b\].

### 5.2 Constraint bifurcation

In the previous discussion, we identified the conditions for which all the ICs, $X'_M = (X_M, \chi)$, are SC. To calculate the determinant of the $10 \times 10$ matrix $\Delta^+_M = \{X'_M, X'_N\}$,

$$\Delta^+_M \approx \begin{vmatrix}
0 & 0 & \{A\phi, A\Phi\} & 0 & 0 & 0 \\
0 & 0 & 0 & \{T\phi, T\Phi\} & 0 & 0 \\
-\{A\phi, A\Phi\} & 0 & 0 & 0 & 0 & 0 \\
0 & -\{T\phi, T\Phi\} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\{V\phi, \chi\} \\
0 & 0 & 0 & 0 & -\{V\phi, \chi\} & 0
\end{vmatrix}$$
we use (5.6) and (5.8), which leads to
\[ \Delta^+ \sim \left( \frac{\Pi_{\perp}}{4J} \right)^{10} \left( \frac{\Pi_{\perp}}{4J} - a_2 \right)^4. \] (5.10)

Introducing a convenient notation
\[ W := \frac{\Pi_{\perp}}{4J}, \] (5.11)
we see that \( \Delta^+ \) can vanish only on a set (of spacetime points) of measure zero, defined by \( W = 0 \) or \( W - a_2 = 0 \). In other words, the condition
\[ W(W - a_2) \neq 0 \] (5.12)
is fulfilled \textit{almost everywhere} (everywhere except on a set of measure zero). Thus, our previous discussion can be summarized by saying that all of the ICs are SC almost everywhere; the related (generic) classification of constraints is shown in Table 2.

| Table 2. Generic constraints in the \( 0^+ \) sector |
|---------------------------------|---------------------------------|
| First class                    | Second class                    |
| Primary                        | \( \pi_i^0, \Pi_{ij}^0 \)       | \( \chi_M \) |
| Secondary                      | \( \mathcal{H}_{\perp}, \mathcal{H}_\alpha, \mathcal{H}_{ij} \) | \( \chi_i \) |

The Hamiltonian constraints \( \mathcal{H}_{\perp}, \mathcal{H}_\alpha \) and \( \mathcal{H}_{ij} \) are first class (FC) \([4, 20]\); they are obtained from (5.3) and (5.5b) by adding the contributions containing the determined multipliers. With \( N = 18, N_1 = 12 \) and \( N_2 = 10 \), the dimension of the phase space is given as \( N^* = 2N - 2N_1 - N_2 = 2 \). Thus, the theory exhibits a single Lagrangian DoF almost everywhere.

However, the determinant \( \Delta^+ \), being a field-dependent object, may vanish in some regions of spacetime, changing thereby the number and/or type of constraints and the number of physical degrees of freedom, as compared to the generic situation described in Table 2. This effect, known as the phenomenon of \textit{constraint bifurcation}, can be fully understood by analyzing the dynamical behavior of the two factors in (5.12). Although the complete analysis can be carried out in the canonical formalism, we base our arguments on the Lagrangian formalism, in order to simplify the exposition (see Appendix B).

1. Starting with the second factor,
\[ \Omega := W - a_2 \approx -\left( a - \frac{1}{6} b_i R + a_2 \right), \] (5.13)
where we used (4.5a) to clarify the geometric interpretation, one can prove the relation
\[ -\Omega V_k + 2\partial_k \Omega \approx 0, \] (5.14)
which implies that the behavior of \( \Omega \) is limited to the following two options (Appendix B):
(a) either \( \Omega(x) \) vanishes globally, on the whole spacetime manifold,
(b) or it does not vanish anywhere.
Which of these two options is realized depends upon the initial conditions for $\Omega$; choosing them in accordance with (b) extends the generic behavior of $\Omega$, $\Omega \neq 0$ almost everywhere, to the whole spacetime. This mechanism is the same as the one observed in the spin-$0^+$ sector of the 4-dimensional PGT; compare (5.14) with equation (4.20) in [16].

2. We now focus our attention to the first factor in (5.12),

$$W \approx -\left( a - \frac{1}{6} b_6 R \right). \quad (5.15)$$

It is interesting that a solution for the $W$-bifurcation ($W = 0$) can be found by relying on the solution for the $\Omega$-bifurcation, which is based on choosing $\Omega \neq 0$ on the initial spatial surface $\Sigma$. Indeed, the choice $\Omega > 0$ on $\Sigma$ implies

$$\Omega > 0 \quad \text{globally.} \quad (5.16a)$$

Then, since $\Omega = W - a_2$ ($a_2$ is positive), we find

$$W > a_2 \quad \text{globally.} \quad (5.16b)$$

Thus, with $\Omega > 0$ and $W > a_2$, the problem of constraint bifurcation simply disappears. Note that geometrically, the condition $W > a_2$ represents a restriction on the Cartan scalar curvature, $b_6 R > 6(a + a_2)$. An equivalent form of this relation is obtained by using the identity $R = \tilde{R} - 2\sigma$, where $\tilde{R}$ is Riemannian scalar curvature.

Thus, with a suitable choice of the initial conditions, one can ensure the generic condition $\Delta^+ \neq 0$ to hold globally, so that the constraint structure of the spin-$0^+$ sector is described exactly as in Table 2. Any other situation, with $W = 0$ or $W - a_2 = 0$, would not be acceptable—it would have a variable constraint structure over the spacetime, the property that could not survive the process of linearization.

5.3 Stability under linearization

Now, we are going to compare the canonical structure of the full nonlinear theory with its linear approximation around maximally symmetric background.

In the linear approximation, the condition of canonical stability (5.12) is to be taken in the lowest order (zeroth) approximation. Using $\tilde{R} = -6q$, it reduces to

$$(a + qb_6)(a + qb_6 + a_2) \neq 0. \quad (5.17)$$

The three cases displayed in Table 3 define characteristic sectors of the linear regime (see Appendix C).

| Table 3. Canonical stability in the $0^+$ sector | $a + qb_6$ | $a + qb_6 + a_2$ | DoF | stability |
|-------------------------------------------------|-----------|-----------------|-----|----------|
| (a)                                             | $\neq 0$  | $\neq 0$        | 1   | stable   |
| (b)                                             | $= 0$     | $\neq 0$        | 0   | unstable |
| (c)                                             | $\neq 0$  | $= 0$           | 1   | stable*  |
(a) When the condition (5.17) is satisfied, the nature of the constraints remains the same as in Table 2, and we have a single Lagrangian DoF, the massive spin-0$^+$ mode.

(b) Here, all ICs become FC, but only six of them are independent. Thus, $N_1 = 12 + 6 = 18$, and with $N_2 = 0$, the number of DoF’s is zero: $N^* = 36 - 2 \times 18 = 0$.

(c) In this case, $\tilde{\chi}_k$ is not an independent constraint, and $V\tilde{\Phi}_k$ is FC. As compared to (a), the number and type of constraints is changed according to $N_1 \to N_1 + 2$, $N_2 \to N_2 - 4$, but the number of DoF’s remains one ($N^* = 2$), corresponding to the massless spin-0$^+$ mode.

The case when both $a + qb_6$ and $a + qb_6 + a_2$ vanish is not possible, since $a_2 \neq 0$.

To clarify the case (c), we need a more detailed analysis. Consider first the case (a), in which the constraint $\tilde{\chi}_i$, defined in (C.3), is replaced by an equivalent expression, $\tilde{\chi}_i' = \tilde{\pi}_{\perp i}/J$. Then, the pair of SC constraints $(V\tilde{\Phi}_k, \tilde{\chi}_k')$, with the related Dirac brackets, defines the reduced phase space $\tilde{R}(a)$. Next, consider the case (c), where $\tilde{\chi}_i$ does not exist and $V\tilde{\Phi}_k$ is FC. Here, we can introduce a suitable gauge condition associated to $V\tilde{\Phi}_k$, given by $\tilde{\chi}_k'' = \tilde{\pi}_{\perp i}/J$.

The pair $(V\tilde{\Phi}_k, \tilde{\chi}_k'')$ defines the reduced phase space $\tilde{R}(c)$, which coincides with the reduced phase space $\tilde{R}(a)$, subject to the additional condition $a + qb_6 + a_2 = 0$. Thus, the “massless” nonlinear theory, defined by $a + qb_6 + a_2 = 0$, is essentially (up to a gauge fixing) stable under the linearization. The star symbol in Table 3 (stable*) is used to remind us of this gauge fixing condition.

For the $M_3$ background ($p = q = 0$ and $a \neq 0$), the case (b) is not possible.

6 Spin-0$^-$ sector

For $(a_1 + 2a_3)b_5 \neq 0$, the constraints $A\phi_k, A\Phi_{\perp k}$ in Table 1 are absent, and the spin-0$^-$ mode becomes a physical degree of freedom. Here, we study canonical features of the spin-0$^-$ sector by using the specific conditions

$$a_3, b_5 \neq 0, \quad a_1 = a_2 = b_4 = b_6 = 0, \quad \text{(6.1)}$$

which define the Lagrangian $L_G$ as in (3.8b).

6.1 Hamiltonian and constraints

**Primary constraints.** Applying the conditions (6.1) to the general considerations of Section 4, we find the following set of the primary (sure and if-) constraints:

$$\pi^0_i \approx 0, \quad \Pi^0_{ij} \approx 0,$$

$$S_\phi := \frac{S\tilde{\pi}}{J} \approx 0, \quad T_{\phi ij} := \frac{T\tilde{\pi}_{ij}}{J} \approx 0,$$

$$\phi_{\perp i} := \frac{\tilde{\pi}_{\perp i}}{J} \approx 0, \quad \phi_{\perp ij} := \frac{\tilde{\pi}_{\perp ij}}{J} \approx 0,$$

$$S\Phi_{\perp} := \frac{S\tilde{\Pi}_{\perp}}{J} + 4a \approx 0, \quad T\Phi_{\perp ij} := \frac{T\tilde{\Pi}_{\perp ij}}{J} \approx 0. \quad \text{(6.2)}$$
The dynamical part of the canonical Hamiltonian has the form

\[ \mathcal{H}_\perp = J \left[ \frac{3}{8a_3} (A\phi_{ij})^2 + \frac{1}{4b_5} (A\Phi_{\perp ij})^2 + \frac{1}{2b_5} (V\Phi_i)^2 \right] - J L_G^-(T, R) - n_i \nabla \alpha \pi^{i\alpha}, \]  

(6.3)

where \( A\phi_{ij}, A\Phi_{\perp ij} \) and \( V\Phi_i \) are the “generalized” momentum variables defined in (4.3) and (4.5), and the total Hamiltonian reads:

\[ \mathcal{H}_T = \mathcal{H}_c + \frac{1}{2} s^i \dot{\phi} + T u^{ij} \dot{\phi}_{ij} + \frac{1}{2} S v^{1\perp} S \Phi + T v^{1\perp} T \Phi_{\perp ij}. \]  

(6.4)

**Secondary constraints.** The consistency conditions of the primary constraints \( \pi_i^0 \) and \( \Pi_{ij}^0 \) produce the usual secondary constraints:

\[ \mathcal{H}_\perp \approx 0, \quad \mathcal{H}_a \approx 0, \quad \mathcal{H}_{ij} \approx 0, \]  

(6.5a)

where

\[ \mathcal{H}_a \approx A_{\perp ij}^i T^{i\alpha} + A_{\perp ij} \Pi_{\perp ij}^i R_{\perp \alpha}^i \]  

\[ \mathcal{H}_{ij} \approx a T_{\perp ij}^i + \frac{V\Pi_{\perp ij}^i}{2J} T_{\perp ij}^i + \nabla \beta \frac{V\Pi_{\perp ij}^i}{J}, \]  

\[ \mathcal{H}_{\perp i} \approx a \Pi_{\perp i}^{\perp m} T_{\perp i}^{\perp m} + \frac{V\Pi_{\perp i}^{\perp m}}{2J} T_{\perp i}^{\perp m} + \frac{1}{2} \nabla \beta \frac{A_{\perp i}^{\perp m}}{J}. \]  

(6.5b)

Using the PB algebra between the primary ICs \( Y_M = (S\phi, T\phi, \phi_{\perp k}, S\Phi, T\Phi) \) (Appendix D), one finds that generically, for \( A_{\perp ik} \neq 0 \), they are SC; their consistency conditions result in the determination of the corresponding multipliers \( (S_\pi, T_\pi, u_{\perp k}, S_v, T_v) \). Moreover, the secondary constraints (6.5a), corrected by the contributions of the determined multipliers, are FC, so that their consistency conditions are trivially satisfied. Thus, in the generic case, the consistency algorithm is completed at the level of secondary constraints.

The first two terms in \( \mathcal{H}_\perp \), proportional to the squares of \( A\phi_{ik} \) and \( A\Phi_{\perp ik} \), describe the contribution of the spin-0\(^{-}\) mode to the kinetic energy density, see Table 1. This contribution is positive definite for

\[ a_3 > 0, \quad b_5 > 0. \]  

(6.6)

At the same time, the contribution of the third term, the square of \( V\Phi_k \), becomes negative definite (“ghost”), which is a serious problem for the physical interpretation. As we shall see, this is not the only problem.

### 6.2 Constraint bifurcation

Based on the PB algebra of the (eight) primary ICs \( Y_M \), we can now calculate the determinant of the \( 8 \times 8 \) matrix \( \Delta_{MN}^- = \{ Y_M, Y_N \} \) (Appendix D); the result takes the form

\[ \Delta^- \sim A_{\perp ij} A_{\perp \alpha \beta} \left( \frac{4a_2^2}{J^2} + \frac{1}{8J} A_{\perp \alpha \beta} A_{\perp \gamma \delta} \right)^2. \]  

(6.7)
Since the second factor is always positive definite, $\Delta^-$ remains different from zero only if
\[ A_{\bar{m}\bar{n}} \neq 0. \]
(6.8)
This condition holds everywhere except on a set of measure zero, so that $\Delta^- \neq 0$ almost everywhere. Thus, generically, the eight primary ICs are SC, as shown in Table 4; the primes in $\mathcal{H}'_A$, $\mathcal{H}'_\alpha$, and $\mathcal{H}'_{ij}$ denote the presence of corrections induced by the determined multipliers.

Table 4. Generic constraints in the $0^-$ sector

| First class | Second class |
|-------------|--------------|
| Primary $\pi^0_i, \Pi^0_{ij}$ | $Y_M$ |
| Secondary $\mathcal{H}'_A, \mathcal{H}'_\alpha, \mathcal{H}'_{ij}$ |

Using $N = 18$, $N_1 = 12$ and $N_2 = 8$, we find $N^* = 2N - 2N_1 - N_2 = 4$. Surprisingly, the theory exhibits two Lagrangian DoF: one is the expected spin-$0^-$ mode, and the other is the spin-$1$ “ghost” mode, represented canonically by $V\Phi_k$.

In Appendix E, we analyze the nature of the critical condition $A_{\bar{m}\bar{n}} = 0$. In the region of spacetime where it holds, we find the phenomenon of constraint bifurcation: the number of DoF is changed to zero. Although such a situation is canonically unstable under linearization, it is interesting to examine basic aspects of the linearized theory.

6.3 Linearization

In the linearized theory, the term $A_{\bar{k}j\bar{k}}$ in the determinant $\Delta^-$ takes the form
\[ A_{\bar{k}j\bar{k}} = -2a_3 \bar{\varepsilon}_{\bar{i}jk} p. \]
(6.9)
Hence, the canonical structure of the linearized theory crucially depends on the value of the background parameter $p$, as shown in Table 5.

Table 5. Canonical instability in the $0^-$ sector

| DoF | stability |
|-----|-----------|
| (a) $p \neq 0$ | 2 | stable almost everywhere |
| (b) $p = 0$ | 1 | unstable |

(a) For $p \neq 0$ (Riemann–Cartan background, massless spin-$0^-$ mode), the determinant $\bar{\Delta}^-$ is positive definite, all the primary ICs are SC, as in the generic sector of the full nonlinear theory, and consequently, $N^* = 4$. However, this is not true in the critical region $A_{\bar{i}j} = 0$, where $N^* = 0$ and the theory is canonically unstable.

(b) For $p = 0$ (Riemannian background, massive or massless spin-$0^-$ mode), the situation is changed (Appendix F). First, the determinant $\bar{\Delta}^-$ vanishes, since the primary IC $\tilde{\phi}_{\perp L}$ commutes with itself, see (D.11). By calculating its consistency condition (which was not needed for $p \neq 0$), one finds its secondary pair $\tilde{\chi}_i$. Now, the PB of $\tilde{\phi}_{\perp L}$ with the modified secondary pair $\tilde{\chi}'_i = \tilde{\chi}_i - \bar{H}_i$ does not vanish. Thus, there are two SC constraints more than in the case (a) so that $N^* = 2$, and we have the canonical instability under linearization.

Thus, in both cases (a) and (b), the theory canonically unstable.
7 Concluding remarks

In this paper, we studied the Hamiltonian structure of the general parity-invariant model of 3D gravity with propagating torsion, described by the eight-parameter PGT Lagrangian (2.1). Because of the complexity of the problem, we focused our attention on the scalar sector, containing $J^P = 0^+$ or $0^-$ modes with respect to maximally symmetric background. By investigating fully nonlinear “constraint bifurcation” effects as well as the canonical stability under linearization, we were able to identify the set of dynamically acceptable values of parameters for the spin-0$^+$ sector, as shown in Table 3. On the other hand, the spin-0$^-$ sector is found to be canonically unstable for any choice of parameters, see Table 5. Transition from an (A)dS to Minkowski background simplifies the results.

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A The 1+2 decomposition of spacetime

To derive the Dirac-ADM form of the Hamiltonian, it is convenient to pass from the tetrad basis $h_i = h_i^\mu \partial_\mu$ to the ADM basis $(n, h_\alpha)$, where $n$ is the unit vector with $n_k = h_k^0 / \sqrt{g^{00}}$, orthogonal to the vectors $h_\alpha = \partial_\alpha$ lying in the $x^0 =$ const. hypersurface $\Sigma$, see [4, 20].

(1) Introducing the projectors on $n$ and $\Sigma$, $(P_\perp)^i_k = n^i n_k$ and $(P_\parallel)^i_k = \delta^i_k - n^i n_k$, any vector $V_k$ can be decomposed in terms of its normal and parallel projections:

\[ V_k = n_k V_\perp + V_\parallel, \quad V_{\perp} := n^k V_k, \quad V_{\parallel} := (P_\parallel)^i_k V_i = h_\alpha^k V_\alpha. \]  

The decomposition of $V_0 = b_0^k V_k$ in the ADM basis yields $V_0 = N V_\perp + N^\alpha V_\alpha$, where the lapse and shift functions $N$ and $N^\alpha$, respectively, are linear in $b_0^k$:

\[ N := n_k b_0^k, \quad N^\alpha := h^\alpha_\beta b_0^\beta. \]  

The decomposition (A.1) can be extended to any tensor field. Thus, a second rank antisymmetric tensor $X_{ik} = -X_{ki}$ can be decomposed as

\[ X_{ik} = X_{\perp k} + (X_{\perp i} n_k - X_{\perp k} n_i). \]  

The parallel tensors, like $X_{\perp k}$, lie in $\Sigma$, and can be further decomposed into the irreducible parts with respect to the spatial rotations:

\[ X_{ik} = T X_{\perp k} + A X_{\perp k} + \frac{1}{2} \eta_{ik} S X, \]  

(A.4a)
As a consequence, the product $X_{ik}^i Y_{ik}$ is given by

$$X_{ik}^i Y_{ik} = T X_{ik}^i Y_{ik} + A X_{ik}^i A Y_{ik} + \frac{1}{2} S X S Y.$$  \hspace{1cm} (A.4b)

For a tensor $\Phi_{ijk} = -\Phi_{jki}$, the pseudoscalar $(\varepsilon^{ijk} \Phi_{ijk})$ and traceless-symmetric piece $(\Phi_{i(jk)} - \text{traces})$ identically vanish, so that the only nontrivial piece is the vector $V \Phi_i := \Phi_{ik}$.

$$\Phi_{ijk} = 2 \eta_{ijk} V \Phi_i, \quad \Phi^{ijk} Q_{ijk} = 2 V^i V Q_i.$$  \hspace{1cm} (A.5)

(2) These results can be now used to find the Dirac-ADM form of the Hamiltonian. Starting with the torsion sector, we use the formula $T = T + \mathcal{T}$ to rewrite $\mathcal{L}_T$ in the form

$$\mathcal{L}_T = \frac{1}{4} \mathcal{H}^{ijk}(T) T_{ijk} + \frac{1}{4} \mathcal{H}^{ijk}(\mathcal{T}) T_{ijk} + \frac{1}{4} \mathcal{H}^{ijk}(T) T_{ijk}$$

which yields

$$\mathcal{H}^T_T = \frac{1}{2} J \phi^{i j} T_{i j} - J \mathcal{L}_T(\mathcal{T}) - n^i \nabla_\alpha \pi_\alpha^i.$$  \hspace{1cm} (A.6a)

Then, the irreducible decomposition

$$\phi^{i j} T_{i j} = \phi^{i j} T_{i j} + A \phi^{i j} T_{i j} + T \phi^{i j} T_{i j} + \frac{1}{2} S \phi T_{i j}^q$$  \hspace{1cm} (A.6b)

in conjunction with (4.3), leads to (4.5).

Similar calculations for the curvature part yield

$$\mathcal{H}^R_T = \frac{1}{4} J \Phi^{ijk} R_{ijk} + J \mathcal{L}_T(\Phi) + a J R.$$  \hspace{1cm} (A.7a)

Then, the irreducible decompositions

$$\Phi^{ijk} R_{ijk} = 2 \Phi^{ijk} R_{ijk} + 2 V \Phi^{ijk} R_{ijk},$$

$$2 \Phi^{ijk} R_{ijk} = 2 \left( A \Phi^{ijk} A R_{ijk} + T \Phi^{ijk} T R_{ijk} + \frac{1}{2} S \Phi^{ijk} R_{ijk} \right),$$  \hspace{1cm} (A.7b)

combined with (4.5), lead directly to (4.9).
B Constraint bifurcation in the spin-0$^+$ sector

In this appendix, we study the phenomenon of constraint bifurcation in the spin-0$^+$ sector, determined by the critical condition $\Omega = 0$.

1. We start our discussion by writing the field equations for the spin-0$^+$ sector:

\begin{align}
2a_2\eta_{ij}[\nabla^k V_k] + 2 \left( a - \frac{b_6}{6} R \right) G_{ji} - \eta_{ij} \left( \frac{a_2}{2} V^2 + \frac{b_6}{12} R^2 + 2\Lambda_0 \right) &= 0, \\
\left( a - \frac{b_6}{6} R \right) T_{ijk} + a_2 \eta_{ij} V_k + \frac{b_6}{3} \eta_{ij} \nabla_k R &= 0,
\end{align}

where $V^2 = V_k V^k$. The content of these equations can be expressed in terms of their irreducible components. For the first equation, we find

\begin{align}
-a_2 \nabla_{[i} V_{j]} + 2 \left( a - \frac{b_6}{6} R \right) R_{[ij]} &= 0, \\
-a_2 \left( \nabla_{(i} V_{j)} - \frac{1}{3} \eta_{ij} \sigma \right) + 2 \left( a - \frac{b_6}{6} R \right) \left( R_{(ij)} - \frac{1}{3} \eta_{ij} R \right) &= 0, \\
-2a_2 \sigma + \frac{3}{2} a_2 V^2 + a R + \frac{b_6}{12} R^2 + 6\Lambda_0 &= 0,
\end{align}

where $\sigma := \nabla_i V^i$. The irreducible components of the second equation are:

\begin{align}
\left( a - \frac{b_6}{6} R \right) A &= 0, \\
\left( a - \frac{b_6}{6} R \right) T_{ijk} &= 0, \\
\left( a - \frac{b_6}{6} R + a_2 \right) V_k + \frac{b_6}{3} \nabla_k R &= 0.
\end{align}

2. Now, we focus our attention on the factor $\Omega = W - a_2$ in $\Delta^+$. Its dynamical evolution is determined by Eq. (B.4c), which can be written in the form

\begin{align}
-\Omega V_k + 2\partial_k \Omega &\approx 0,
\end{align}

Note that this equation is an extension of Eq. (5.13) from $\Sigma$ to the whole spacetime $\mathcal{M}$.

The spacetime continuum $\mathcal{M}$ on which 3D PGT lives is a differentiable manifold with topology $\mathcal{M} = R \times \Sigma$, where $R$ corresponds to time, and $\Sigma$ to the spatial section of $\mathcal{M}$. Let us now assume that: (i) $\Omega$ vanishes at some point $x = a$ in $\mathcal{M}$, (ii) $\Omega$ is an infinitely differentiable function on $\mathcal{M}$, and (iii) $V_k$ and all its derivatives are finite at $x = a$. Then, one can notice that $\Omega = 0$ at $x = a$. In the next step, we apply the differential operator $\partial_k$ to $\Omega$ and conclude that $\partial_k \Omega = 0$ at $x = a$. Continuing this procedure, we eventually conclude that for every $n, \partial_{k_n} \cdots \partial_{k_1} \partial_k \Omega = 0$ at $x = a$. In general, the behavior of $\Omega$ on the whole $\mathcal{M}$ is not determined by its properties at a single point. However, if (iv) $\Omega$ is an analytic function on $\mathcal{M}$, its Taylor expansion around $x = a$ implies that $\Omega = 0$ on the whole $\mathcal{M}$.
The result obtained can be formulated in a more useful form: if there is at least one point in $\mathcal{M}$ at which $\Omega \neq 0$, then $\Omega \neq 0$ on the whole $\mathcal{M}$. Thus, by choosing the initial data so that $\Omega \neq 0$ at $x^0 = 0$, it follows that $\Omega$ stays nonvanishing for any $x^0 > 0$. In other words, for a suitable choice of initial data, the configuration $\Omega = 0$ is kind of a barrier that the system cannot cross during its dynamical evolution. Moreover, since $\Omega$ is a continuous function, it has a definite sign for any $x^0 > 0$.

C The linearized spin-0$^+$ sector

In the weak-field approximation, the primary ICs of the spin-0$^+$ sector take the form

$$A_{\phi ij} := \frac{A_{\phi ij}}{J} \approx 0, \quad T_{\phi ij} := \frac{T_{\phi ij}}{J} \approx 0,$$

$$A_{\Phi,ij} := \frac{A_{\Phi,ij}}{J} - 2(a + qb_6)\tilde{b}_{[ij]} \approx 0,$$

$$T_{\Phi,ij} := \frac{T_{\Phi,ij}}{J} - 2(a + qb_6)\tilde{b}_{ij} \approx 0,$$

$$V_{\Phi,i} := \frac{V_{\Phi,i}}{J} - 2(a + qb_6)\tilde{b}_{i,i} \approx 0,$$  \hspace{1cm} (C.1)

and the secondary Hamiltonian constraints are given by

$$\hat{\mathcal{H}}_i = J \left( 2\frac{s_{\Pi,i}}{J} - (a + qb_6)(\tilde{R}_{ij} - 4\tilde{b}_i^j) \right) - \bar{n}_i \tilde{\nabla}^\alpha \tilde{n}^{i\alpha}, \hspace{1cm} (C.2a)$$

$$\hat{\mathcal{H}}_\alpha = p\varepsilon_{ijk}\tilde{b}_j^i \tilde{\zeta}_k^i - \tilde{b}_\alpha \tilde{\nabla}^\beta \tilde{n}^{i\beta} - 2(a + qb_6)(\tilde{R}_\beta - 2\tilde{b}_\beta + 4\tilde{b}_\beta^j \tilde{b}_{ij}), \hspace{1cm} (C.2b)$$

$$\hat{\mathcal{H}}_{ij} \approx \frac{A_{\phi ij}}{J} - (a + qb_6)\tilde{T}_{i,j} + p\varepsilon_{ij} \left( \frac{s_{\Pi,j}}{4J} + \frac{1}{2}(a + qb_6)\tilde{b}_k^j \right) \approx 0, \hspace{1cm} (C.2c)$$

$$\hat{\mathcal{H}}_{i,i} \approx \tilde{n}_{i,i} + 2(a + qb_6)\tilde{T}_{i,i} + \tilde{\nabla}_i \left( \frac{s_{\Pi,i}}{2J} + (a + qb_6)\tilde{b}_k^i \right) \approx 0. \hspace{1cm} (C.2d)$$

The consistency condition of $V_{\Phi,i}$ can be expressed in the form

$$\tilde{\chi}_i = \frac{1}{2} \hat{\mathcal{H}}_{i,i} - \frac{a + qb_6 + a_2 \tilde{\zeta}_{i,i} + \tilde{n}_i}{a_2} \approx \frac{a + qb_6 + a_2 \tilde{\zeta}_{i,i}}{a_2} \approx \frac{a + qb_6 + a_2 \tilde{\zeta}_{i,i}}{J}. \hspace{1cm} (C.3)$$

For $a + qb_6 \neq 0$ and $a + qb_6 + a_2 \neq 0$, the type and the number of constraints remains the same as in the full non-linear theory, and we have the canonical stability under linearization.

The case $a + qb_6 = 0$. In this case, the analysis depends on the value of $p$.

(i) for $p \neq 0$, the six secondary constraints $\mathcal{H}_M = (\hat{\mathcal{H}}, \tilde{\mathcal{H}}_i, \tilde{\mathcal{H}}_{ij}, \tilde{\mathcal{H}}_{i,i})$, in conjunction with $V_{\Phi,i} \approx 0$ take, respectively, the following form:

$$S_\tilde{\zeta} \approx 0, \quad 0 \approx 0, \quad s_{\Pi,i} \approx 0, \quad \tilde{\zeta}_{i,i} \approx 0.$$

\hspace{1cm} (C.4)
Thus, $\tilde{\chi}_i$ and $\tilde{H}_a$ are identically satisfied, and there are no SC constraints, $N_2 = 0$. Hence, the number of FC constraints is $N_1 = 6 + 6 - 2 + 8 = 18$ and consequently, there are no propagating modes: $N^* = 2 \times 18 - 2 \times 18 = 0$.

(ii) For $p = 0$, the constraints $\mathcal{H}_M$, in conjunction with $V\Phi_k \approx 0$, read:

$$2^S \tilde{\Pi}_{\perp} - \tilde{n}_i \nabla_\alpha \tilde{\pi}_{i_\alpha} \approx 0, \quad \tilde{b}_{\alpha} \nabla_\beta \tilde{\pi}_{i_\beta} \approx 0,$$

$$0 \approx 0, \quad \frac{\tilde{\pi}_{i_{\perp \perp}}}{J} + \nabla_\chi \left( \frac{S\tilde{\Pi}_{\perp}}{2J} \right) \approx 0. \quad \text{(C.5)}$$

Taking into account the form of $\tilde{\chi}_i$, the set $\mathcal{H}_M$ reduces to:

$$\tilde{\pi}_{i_{\perp \perp}} \approx 0, \quad S\tilde{\pi} \approx 0, \quad S\tilde{\Pi}_{\perp} \approx 0.$$

Thus, we again have $N_1 = 18$, $N_2 = 0$, and $N^* = 0$.

The case $a + qb_0 + a_2 = 0$. Compared to the generic case, this condition induces the following change: Eq. (C.3) implies that $\tilde{\chi}_i$ is identically satisfied, whereas $V\Phi_k$ becomes FC. Thus, $N_1 = 6 + 6 + 2 = 14$, $N_2 = 10 - 4 = 6$, and consequently, $N^* = 2$.

D The algebra of ICs in the spin-0$^-$ sector

The non-trivial PBs between the primary ICs $Y_M = (S\phi, T\phi, \phi_{\perp, k}, S\Phi, T\Phi)$ in the spin-0$^-$ sector read:

$$\{S\phi, S\Phi_{\perp}\} \approx -\frac{4a}{J} \delta,$$

$$\{T\phi_{ij}, T\Phi_{\perp, \tilde{m}\tilde{n}}\} \approx -\frac{1}{J^2} \left[ \delta_{i\langle \tilde{m}} A\tilde{\Pi}_{\perp, j\rangle} \tilde{n} \right] \delta,$$

$$\{\phi_{\perp, i}, \phi_{\perp, j}\} \approx \frac{2}{J^2} A\tilde{\Pi}_{ij} \delta,$$

$$\{\phi_{\perp, i}, S\Phi_{\perp}\} \approx \frac{1}{J^2} V\tilde{\Pi}_{i} \delta,$$

$$\{\phi_{\perp, i}, T\Phi_{\perp, \tilde{m}\tilde{n}}\} \approx \frac{1}{J^2} \left( \frac{1}{2} \eta_{\tilde{m}\tilde{n}} V\tilde{\Pi}_{i} - \eta_{\tilde{m}} (V\tilde{\Pi}_{i}) - 4a J n_{(m} \eta_{n)} \right) \delta. \quad \text{(D.1)}$$

Calculating the determinant of the $8 \times 8$ matrix $\Delta^-_{MN} = \{Y_M, Y_N\}$,

$$\Delta^- = \begin{vmatrix}
0 & 0 & -\frac{4a}{J} & 0 & 0 \\
0 & \{T\phi_{ij}, T\phi_{\tilde{m}\tilde{n}}\} & 0 & 0 & \{T\phi_{ij}, T\Phi_{\perp, \tilde{m}\tilde{n}}\} \\
0 & 0 & \{\phi_{\perp, i}, \phi_{\perp, j}\} & \{\phi_{\perp, i}, S\Phi_{\perp}\} & \{\phi_{\perp, i}, T\Phi_{\perp, \tilde{m}\tilde{n}}\} \\
4a & 0 & -\{\phi_{\perp, i}, S\Phi_{\perp}\} & 0 & 0 \\
J & 0 & -\{T\phi_{ij}, T\Phi_{\perp, \tilde{m}\tilde{n}}\} & -\{\phi_{\perp, i}, T\Phi_{\perp, \tilde{m}\tilde{n}}\} & 0 & 0
\end{vmatrix}$$

one obtains the result displayed in Eq. (6.7).
E On the condition $\hat{A}_{\bar{i}\bar{j}} = 0$

In this appendix, we wish to clarify the phenomenon of constraint bifurcation in the spin-0 sector, where the field equations take the form:

$$2a_3\varepsilon_{ijk}(\nabla^k - V^k)A + \eta_{ij}(a_3A^2 - b_5R_{[ij]}R^{[ij]} - 2\Lambda_0) + \frac{8a_3}{3}A\varepsilon_{(imn}t_{j)m}A + 2aG_{ji} + 2b_5R_{[in]}G^i_j = 0,$$

$$T^i_{mn}(a\eta_{ik} + b_5R_{[ik]}) + b_5\nabla_{[m}(R_{n]k} - R_{kn}) + 2a_3\varepsilon_{mn}A = 0.$$  

(E.1)  

Consider now the solutions for which $\hat{A}_{\bar{i}\bar{j}} = 0$, or equivalently, $A = 0$. Although the condition $A = 0$ is imposed from outside, it is, essentially, an additional constraint as compared to the generic case. Hence, using not only $A = 0$ but also $\nabla^0A = 0$, which ensures dynamical consistency of $A = 0$, the field equations reduce to

$$2aG_{ji} + 2b_5R_{[in]}G^i_j = 0,$$

$$T^i_{mn}(a\eta_{ik} + b_5R_{[ik]}) + b_5\nabla_{[m}(R_{n]k} - R_{kn}) = 0.$$  

(E.3a)  

(E.3b)

In what follows, we focus our attention on the first first equation (E.3a). We begin by noting that its components $i, j = \perp \perp$ and $\bar{i}, \bar{j}$, rewritten in the form

$$-aR^{\bar{i}\bar{j}} + 2\Lambda_0 - \frac{\hat{\Pi}_{\bar{i}}^i}{J}R^{\perp \perp} - \frac{1}{4b_5J^2} \left(2\hat{\Pi}_{\bar{i}}^i\hat{\Pi}_{\bar{j}}^j + \hat{A}_{\perp \bar{i}}^\perp\hat{A}_{\perp \bar{j}}^\perp\right) = 0,$$

$$2aR^{\perp \perp} + \frac{\hat{V}_{\bar{i}}^i}{J}G_{\perp \perp} + \frac{\hat{A}_{\bar{i}}^{\perp \perp}}{J}R^{\perp \perp} = 0,$$

represent the secondary FC constraints $\mathcal{H}_{\perp}$ and $\mathcal{H}_{\bar{i}} := h^{\alpha}\mathcal{H}_{\alpha}$. Next, consider the components $[\bar{i}, \bar{j}]$ and $[\perp, \bar{i}]$:

$$(2a + b_5R^{\perp \perp})\hat{A}_{\bar{i} \bar{j}} + 2b_5\hat{V}_{\bar{i}}^iR^{\bar{j}} = 0,$$

$$2a\hat{V}_{\bar{i}}^i - b_5\left(\hat{V}_{\bar{i}}^iG_{\perp \perp} + \hat{A}_{\bar{i}}^{\perp \perp}R^{\perp \perp} + \hat{V}_{\bar{j}}^jG^{\bar{i}}\right) = 0.$$  

(E.6)  

(E.7)

Taking into account the relations

$$aR = \frac{1}{4b_5J^2} \left(2\hat{V}_{\bar{i}}^i\hat{V}_{\bar{j}}^j + \hat{A}_{\bar{i}}^{\perp \perp}\hat{A}_{\bar{j}}^{\perp \perp}\right) - 6\Lambda_0,$$

$$G_{\perp \perp} = -\frac{1}{2}R^{\bar{i}\bar{j}},$$

it follows that (E.6) is a new constraint. Finally, the $\bar{i}\bar{j}$ components of (E.3a) read:

$$\left(2a\eta_{\bar{ik}} + \frac{\hat{A}_{\bar{i}}^{\perp \perp}}{J}\right)G^{\bar{j} \bar{k}} - \eta_{\bar{i} \bar{j}} \left[\frac{1}{4b_5J^2} \left(2\hat{V}_{\bar{m}}^m\hat{V}_{\bar{m}}^n + \hat{A}_{\bar{m}}^{\perp \perp}\hat{A}_{\perp \bar{m}}\hat{A}_{\perp \bar{m}}\right) + 2\Lambda_0\right] = 0.$$

This equation can be solved for $G_{\bar{i}\bar{j}}$ since

$$\det \left[2a\eta_{\bar{ik}} + \frac{\hat{A}_{\bar{i}}^{\perp \perp}}{J}\right] = 4a^2 + \frac{\hat{A}_{\perp \bar{m}}\hat{A}_{\perp \bar{m}}}{2J^2} > 0.$$  

Thus, $G_{\bar{i}\bar{j}}$ can be expressed in terms of the phase-space variables, and consequently, (E.7) is also a constraint. Hence, equations (E.6), (E.7) and $\hat{A}_{\bar{i}\bar{j}} = 0$ describe four additional SC constraints (if any of these were FC, the number of DoF would be negative), which eliminate the two propagating modes of the generic case, so that $N^* = 0.$
F The linearized spin-0\(^{-}\) sector

In this Appendix, we present the canonical structure of the linearized spin-0\(^{-}\) sector around the maximally symmetric background. We start by noting that

\[
\frac{A_{\tilde{\alpha}ij}}{J} = -2a_3 \varepsilon_{\perp ij} \tilde{A},
\]

where \(\tilde{A} = p\). Then, for \(p \neq 0\), Eq. (6.7) implies that the determinant \(\tilde{\Delta}^{-}\) is positive definite, so that the canonical structure remains the same as before linearization, see Section 6. Moreover, in that case the spin-0\(^{-}\) mode is massless, see (3.8c).

To see what happens in the complementary case \(p = 0\) (the spin-0\(^{-}\) mode is either massive or massless), we start with

\[
\tilde{\pi}_i^\alpha = 2a_3 \varepsilon^{\alpha \beta \gamma} b_{i \beta} \tilde{A},
\]

\[
\tilde{\Pi}_{ij}^\alpha = -2 \varepsilon_{ij \kappa} \left[ a \varepsilon^{\kappa \lambda \beta} \tilde{b}_{\lambda} + b_{\lambda} \varepsilon^{\kappa \lambda \beta} \varepsilon^{\lambda \mu \nu} \left( \tilde{R}_{\kappa} - \tilde{R}^{\kappa}_{\mu} \right) \right],
\]

and find the following primary ICs:

\[
S_{\phi} := \frac{\tilde{S}_{\pi}}{J} \approx 0, \quad T_{\phi} := \frac{\tilde{T}_{\pi}}{J} \approx 0, \quad \tilde{\phi}_{\perp i} := \frac{\tilde{\pi}_{\perp i}}{J} \approx 0,
\]

\[
S_{\tilde{\Phi}_\perp} := \frac{\tilde{S}_{\Pi}_{\perp}}{J} + 2 a \tilde{b}_i \approx 0, \quad T_{\tilde{\Phi}_\perp} := \frac{\tilde{T}_{\Pi}_{\perp}}{J} - 2 a \tilde{b}_i \approx 0. \tag{F.1}
\]

The only nontrivial PBs between the primary ICs are:

\[
\{S_{\tilde{\Phi}_\perp}, S_{\phi} \} \approx \frac{4a}{J} \delta, \quad \{T_{\tilde{\Phi}_\perp}, T_{\phi} \} \approx -\frac{2a}{J} \delta(i \delta j) \delta. \tag{F.2}
\]

The secondary constraints \(\tilde{\mathcal{H}}_i\) and \(\tilde{\mathcal{H}}_{ij}\) read:

\[
\tilde{\mathcal{H}}_i = \tilde{J} \left[ 2 \frac{\tilde{S}_{\Pi}_{\perp}}{J} - a(\tilde{R}^{\kappa}_{\mu} - 4 \tilde{b}_i) \right],
\]

\[
\tilde{\mathcal{H}}_\alpha = \tilde{b}_\alpha \left[ -q \left( \frac{\tilde{V}}{J} - 2 a \tilde{b}_{i \perp} \right) - \tilde{\nabla}_\beta \tilde{\pi}_i^\beta - 2 a \tilde{J} \tilde{R}_{\perp i} \right],
\]

\[
\tilde{\mathcal{H}}_{ij} \approx a \tilde{T}_{\perp i j} + \frac{\tilde{A}_{\tilde{\alpha}ij}}{2 \tilde{J}} + \tilde{\nabla}_l \left( \frac{\tilde{V}_{\tilde{\kappa}}}{J} - 2 a \tilde{b}_{\perp l} \right),
\]

\[
\tilde{\mathcal{H}}_{\perp i} \approx a \tilde{V}_i + \frac{1}{2} \tilde{\nabla}_j \left( \frac{\tilde{A}_{\tilde{\alpha}ij}}{J} - 2 a \tilde{b}_{ij} \right). \tag{F.3}
\]

Moreover, \(\tilde{\mathcal{H}}_\alpha\) can be used to find \(\tilde{\mathcal{H}}_i = \tilde{h}_i^\alpha \tilde{\mathcal{H}}_\alpha / \tilde{J}\):

\[
\tilde{\mathcal{H}}_i \approx -q \left( \frac{\tilde{V}_{\tilde{\kappa}}}{J} - 2 a \tilde{b}_{\perp l} \right) + \tilde{\nabla}_j \frac{\tilde{A}_{\tilde{\alpha}ij}}{J} - 2 a \tilde{R}_{\perp i}.
\]
According to (F.2), the consistency of the primary ICs \((S\phi, T\phi; S\Phi, T\Phi)\) results in the determination of the multipliers \((S_u, T_u; S_v, T_v)\), whereas the consistency of \(\tilde{\phi}_\perp\) yields a new, secondary IC:

\[
\tilde{\chi}_i = \nabla^j \tilde{n}_{ij}^\alpha + \frac{1}{b_5} \left( a - q b_5 \right) \left( \frac{v\tilde{\Pi}_i}{J} - 2 a \tilde{\phi}_{\perp i} \right) \approx 0 .
\] (F.4)

The PB of \(\tilde{\phi}_{\perp} \) with its own (modified) secondary pair \(\tilde{\chi}_j' := \tilde{\chi}_j - \tilde{H}_j\) reads:

\[
\{ \tilde{\phi}_{\perp i}, \tilde{\chi}_j' \} = \frac{2a^2}{b_5 J} \eta_{ij} \delta .
\] (F.5)

Thus, the consistency condition of \(\tilde{\chi}_j'\) leads to the determination of the multiplier \(u_{\perp j}\).

According to Eqs. (F.2) and (F.5), the ten ICs \(X_A = \{ S\tilde{\phi}, S\tilde{\Phi}, T\tilde{\phi}_{\perp}, T\tilde{\Phi}_{\perp}, \tilde{\phi}_{\perp}, \tilde{\chi}_j' \} \) are SC. Hence, \(N = 18, N_1 = 12, N_2 = 10\), so that \(N^* = 2\) (one Lagrangian DoF).

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