Collective excitations of trapped Fermi or Bose gases

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A new method is developed to calculate all excitations of trapped gases using hydrodynamics at zero temperature for any equation of state $\mu = \mu(n)$ and for any trapping potential. It is shown that a natural scalar product can be defined for the mode functions, by which the wave operator is hermitian and the mode functions are orthogonal. It is also shown that the Kohn-modes are exact for harmonic trapping in hydrodynamic theory. Excitations for fermions are calculated in the BCS-BEC transition region using the equation of state of the mean-field Leggett-model for isotropic harmonic trap potential.

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Several experiments on trapped ultracold gases probed in the past decade the collective excitations of atomic gases. Earlier measurements on bosons [1, 2] and more recent measurements on fermions [3, 4] near Feshbach resonances can be explained rather satisfactorily using hydrodynamics at zero temperature. In his seminal paper [5] Stringari applied first hydrodynamics for trapped bosons undergoing Bose-Einstein condensation. His predictions were confirmed by experiments [1, 2]. Later, using the same approach he predicted [5] also the qualitative behavior of low lying modes for fermions in the whole crossover region from a BCS type superfluid Fermi gas to a molecular Bose-Einstein condensation (BEC) [7, 8, 9].

Now, several recent theoretical papers appeared in the literature [10, 11, 12, 13, 14, 15] using hydrodynamic theory to better explain the measurements on the BCS-BEC transition. In general, no exact solution to the hydrodynamic equations are known, except when the equation of state has the polytropic form $\mu(n) \propto n^\gamma$ [11].

The hydrodynamic approach leads to a wave equation for the density oscillations. In principle, this wave equation can be solved for a single oscillating mode, if the boundary conditions for the density oscillations are known. Bulgac et al. [13] has written the eigenvalue equation for a single mode in such a way that the two sides were hermitian, but did not address the question of the function space to which all the excitations should belong. Here we use a different approach. For a general equation of state $\mu = \mu(n)$ it is usually very difficult to prescribe appropriate boundary conditions at the surface of the gas. There are a few examples where this problem is circumvented using some ansatz on the spatial forms of the excitations [12, 13]. Here we shall introduce a natural scalar product, by which the wave operator itself is hermitian and the boundary conditions can be treated as in quantum-mechanical problems: the mode functions are square integrable functions. The scalar product we shall use automatically ensures particle conservation. By this way finding excitation frequencies are relatively easy: the task is to calculate matrix-elements of the wave operator with the natural scalar product, and then calculate the eigenvalues of the resulting matrix. We shall demonstrate the whole procedure for the mean-field model of the BCS-BEC transition for isotropic trap potential and compare our results with that of the scaling ansatz approach for the lowest $l = 0$ monopole mode.

In hydrodynamic theory for trapped gases at zero temperature density oscillations are given by the continuity equation

$$\frac{\partial n}{\partial t} + \nabla (nu) = 0, \quad (1)$$

and the Euler-equation

$$\frac{\partial u}{\partial t} + u \nabla u = - \frac{1}{mn} (\nabla P) - \frac{\nabla V}{m}, \quad (2)$$

where $u$ is the velocity field, $n$ is the density, $t$ is the time, $P$ is the pressure, $m$ is the particle mass, and $V$ is the external trapping potential.

Knowing the equation of state $\mu = \mu(n)$ at zero temperature in the corresponding homogeneous system the equilibrium density in the trapped case can be determined from the local chemical potential

$$\mu = \mu(r) \equiv \mu(n_0(r)) = \mu_0 - V(r), \quad (3)$$

where $\mu_0$ is the overall constant chemical potential. For confining potentials the solution of this equation for positive $n_0(r)$ supplies an equilibrium density, which has a finite support with a well defined boundary. Typically $n_0$ decreases to zero by approaching the boundary. Using the thermodynamic identity

$$\frac{\partial \mu}{\partial n} = \frac{1}{n} \left( \frac{\partial P}{\partial n} \right) \quad (4)$$
valid also at $T = 0$ the gradient of Eq. 8 gives the important vector identity
\[-A_0(r) \nabla n_0(r) = \nabla V(r), \quad A_0(r) \equiv \left( \frac{\partial P}{\partial n} \right)_{n=n_0(r)}. \tag{5}\]

Eq. 8 plays a central role in the following in simplifying the linearized hydrodynamics. In mechanical equilibrium the equilibrium pressure $P_0$ must satisfy
\[\nabla P_0(r) = -n_0(r) \nabla V(r), \tag{6}\]
otherwise the right hand side of Eq. (2) will not vanish for $u = 0$ (this is the local form of the Archimedes-law for a general external potential). Close to equilibrium $n$ and $\delta n$ are small, and $P$ can be expanded to first order in $\delta n$ as
\[P(r,t) = P_0(r) + \left( \frac{\partial P}{\partial n} \right)_{n=n_0(r)} \delta n(r,t). \tag{7}\]

An important restriction for $\delta n$ is particle conservation $\int d^3r \delta n(r,t) = 0$. Linearizing the continuity equation 11 and the Euler-equation 12 in $\delta n$ and $u$ using the local form of the Archimedes-law 10 and the identity 15 the linearized hydrodynamic equations can be written as
\[\frac{\partial \delta n}{\partial t} + \nabla \langle n_0 u \rangle = 0, \tag{8}\]
\[\frac{\partial n}{\partial t} = -\nabla \left[ \frac{A_0}{n_0 m} \delta n \right]. \tag{9}\]

Let us introduce a new field by
\[\Psi(r,t) = \sqrt{\frac{A_0(r)}{n_0(r)}} \delta n(r,t) \tag{10}\]
where $\Psi$ has the same support as $n_0$, $A_0$ and $\delta n$. From now on, we allow complex fields $\Psi$ (which are more convenient for problems, where angular momentum is conserved). Eliminating $u$ from Eqs. 5 and 9 a wave equation
\[\frac{\partial^2 \Psi}{\partial t^2} + \hat{G}_\Psi \Psi = 0, \tag{11}\]
can be derived for $\Psi$, where $\hat{G}_\Psi$ is given by
\[\hat{G}_\Psi = -\sqrt{\frac{A_0(r)}{n_0(r)}} \cdot \nabla \cdot \frac{n_0(r)}{m} \cdot \nabla \cdot \sqrt{\frac{A_0(r)}{n_0(r)}}. \tag{12}\]

The main advantage of the field $\Psi$ is that its wave operator $\hat{G}_\Psi$ is manifestly hermitian 16 with respect to the scalar product
\[\langle \Psi_1 | \Psi_2 \rangle = \int_{n_0(r)>0} d^3r \Psi_1^*(r) \Psi_2(r). \tag{13}\]

The scalar product 13 is trivially the correct one for a homogeneous system with periodic boundary conditions. It was used for a weakly interacting trapped Bose-gas 17, where $\mu(n)/n$. The same idea of finding a proper scalar product for eigenmodes of a trapped, noninteracting Bose or Fermi gas at finite temperature using the hydrodynamic approach was applied in 18. A single eigenmode
\[\Psi_i(r,t) = \sin(\omega_i t + \phi_0) \Psi_i(r), \tag{14}\]
fulfills the eigenvalue equation
\[\omega_i^2 \Psi_i(r) = \hat{G}_\Psi \Psi_i(r). \tag{15}\]

Solutions of 15 are (or for degenerate eigenvalues can be) orthonormalized with the scalar product 16
\[\delta_{ij} = \int_{n_0(r)>0} d^3r \Psi_i^*(r) \Psi_j(r). \tag{16}\]

$\Psi_0(r) = \text{Const} \cdot \sqrt{n_0(r)}/A_0(r)$ is always a solution to 15 with $\omega_0 = 0$. Eq. 10 implies that $\delta n_i = \Psi_0 \Psi_i$, thus the orthogonality relation 16 shows that all the modes with $i \neq 0$ are automatically particle conserving, and the mode $\Psi_0$ should be cancelled from the solutions.

Taking a complete orthonormal basis, i.e., $\delta_{ij} = \langle \varphi_i | \varphi_j \rangle$ the squared excitation frequencies $\omega_i^2$ can be obtained from the eigenvalues of the matrix
\[G_{i,j} = \langle \varphi_i | \hat{G}_\Psi | \varphi_j \rangle. \tag{17}\]
The matrix elements in 17 require the knowledge or the numerical evaluation of spatial derivatives of the basis functions. Usually this causes big numerical errors because the high lying modes are rapidly oscillating functions. In practice, it is much better to apply the spatial derivatives to the (spatially varying) coefficients of the wave equation, which are usually not oscillating too much.

The wave operator 12 has the structure $\hat{G}_\Psi = -R \nabla Q \nabla R$. If there exists a similar system for which the boundary is the same and the wave operator has also the structure $\hat{G}_{0} = -R_0 \nabla Q_0 \nabla R_0$ but with known spectra and eigenfunctions
\[\hat{G}_{0} | \varphi_i \rangle = \epsilon_i^{(0)} | \varphi_i \rangle \tag{18}\]
then one can eliminate the unwanted spatial derivatives of the basis functions in the matrix elements if the basis is given by $\varphi_i$, ($i = 0, 1, \ldots$). Let us introduce $\alpha$ and $\beta$ by
\[\alpha = \alpha(r) = Q/Q_0, \quad \beta = \beta(r) = R/R_0, \tag{19}\]
then the matrix elements can be written as
\[G_{i,j} = \int d^3r \varphi_i^*(r) \varphi_j(r) G_{i,j}(r). \tag{20}\]
where

\[ G_{i,j}(r) = \frac{\epsilon^{(0)}_i + \epsilon^{(0)}_j}{2} + \alpha^2 \beta + R_0^2 Q_0 \alpha (\nabla \beta) (\nabla \beta) + \frac{1}{2} R_0^2 \nabla \left[ Q_0 \beta^2 (\nabla \alpha) \right]. \] (21)

For an isotropic harmonic trapping potential \( V(r) = m\omega_0^2 r^2 / 2 \) and for any equation of state \( \mu = \mu(n) \) a whole series of exact solutions of the wave equation can be given. If \( \Psi(r) \) is chosen to be

\[ \Psi(r) = \text{Const} \sqrt{\frac{n_0(r)}{A_0(r)}} Y^m_0(\theta, \phi), \quad l > 0 \] (22)

then this mode function fulfills the wave equation with eigenvalue \( \omega^2 = \omega_0^2 l \). The three \( l = 1 \) modes are the Kohn-modes (see Ref. [12]) for isotropic trapping.

As a specific, nontrivial model let us consider the mean-field model of Leggett \[7\] for the BCS-BEC transition. The Leggett model is fixed in homogeneous systems by the gap equation

\[ \sum_k \frac{1}{2} \left( \frac{1}{E_k} - \frac{1}{\varepsilon_k} \right) = -\frac{m}{4\pi \hbar^2 a}, \] (23)

and by the number equation

\[ N = \sum_k \left( 1 - \frac{\varepsilon_k - \mu}{E_k} \right), \] (24)

where \( E_k = \sqrt{(\varepsilon_k - \mu)^2 + \Delta^2} \), \( \varepsilon_k = \hbar^2 k^2 / (2m) \), \( \Delta \) is the pairing gap and \( a \) is the \( s \)-wave scattering length.

The equation of state \( \mu = \mu(n) \) is implicitly given by the model. This model captures the essential features of the BCS-BEC transition. However, recent Monte-Carlo data on the equation of state show \[15\] that there are corrections to the mean-field results of the Leggett model which should be taken into account for the equation of state, especially close to unitarity (i.e., around the \( a = \infty \) point). Here we study the above model for simplicity. In the trapped case we use the \( \mu(n) \) function taken from the model and solve \[3\] for the density profile keeping \( N = \int d^3 r n(r) \) to be fixed. The equilibrium pressure for any trap potential \( V(r) \) can be calculated from the local form of the Archimedean law \[9\]. Knowing the pressure and the density the calculation of \( A_0(r) \) with help of \[6\] is straightforward. The details of the full calculation for the Leggett model will be published elsewhere \[19\].

For isotropic harmonic trapping there is a dimensionless coupling parameter: \( \kappa = d / (a N^{1/6}) \), where \( d = \sqrt{\hbar / m \omega_0} \) is the oscillator length. The spectra depends only on \( \kappa \). In three cases the spectra is exactly known \[2, 10, 11\] because the equation of state has a polytropic form: \( \mu \propto n^\gamma \). These particular values are \( \kappa = -\infty \) (BCS limit), \( \kappa = 0 \) (unitarity limit) and \( \kappa = \infty \) (BEC limit). In these cases all the mode functions can be constructed exactly \[10\], even in the nonisotropic case (The methods of Refs. \[17, 20\] can be easily employed to the polytropic equation of state). We used the \( \kappa = -\infty \) mode functions \[21\] on the BCS-side and the \( \kappa = \infty \) mode functions \[11\] on the BEC-side as basis functions. Our numerical results for different angular momentum \( l \) can be seen on Fig. \[1\]. Arrows on both sides show the limiting well-known collective oscillation frequencies \[4\]. In Fig. \[2\] the behavior of the lowest \( l = 0 \) quadrupole mode can be seen as a function of \( \kappa \). This mode is the lowest \( \kappa \) dependent mode on Fig. \[1\]. The scaling ansatz approach \[12, 13\] gives quite a good result for this particular mode. On the scale of Fig. \[2\] the two curves would be practically
the scaling ansatz is exact at \( \kappa \). We compared the numerically exact excitation frequencies for the experimentally relevant axial resolution. We have preliminary data [19] for the case is so small that is much less than the experimental resolution. We have preliminary data [10] for the excitation frequencies for the experimentally relevant axially symmetric harmonic trapping as well. Once again the scaling ansatz differs a little for the radial and axial quadrupole modes for a general intermediate coupling \( \kappa \). For such large anisotropies as in [2, 11] however, the deviation \( \delta \Omega^2 \) is much bigger than in the isotropic case.

Finally, let us turn to the conclusions. We gave a straightforward method how to solve the hydrodynamic equations in the trapped case if the equation of state is known. We introduced a natural scalar product for the transformed wave operator by which the operator is hermitian. Collective excitations by our method can be found by simply diagonalizing the matrix of the wave operator on some basis. The power of our method lies in the fact that we calculate the whole spectra. We can predict the behavior of the excitations also for those modes for which no scaling ansatz is known. The method is not limited to a particular trap potential (isotropic or not), nor a given mode. There is no additional approximation, the method calculates the (numerically) exact modes given by the hydrodynamic theory. Here we wanted to present the basic formalism and applied to the simplest isotropic harmonic case. We compared the numerically exact excitation frequencies with that of the scaling ansatz for the monopole mode and showed that the difference between the two frequencies are extremely small in the whole region of the BCS-BEC transition for the mean-field Leggett model.

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![Comparison of the exact frequencies and those of the scaling ansatz for harmonic case.](http://example.com/image.png)