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Multiplication formulas for $q$-Appell polynomials and the multiple $q$-power sums

Abstract. In the first article on $q$-analogues of two Appell polynomials, the generalized Apostol-Bernoulli and Apostol-Euler polynomials, focus was on generalizations, symmetries, and complementary argument theorems. In this second article, we focus on a recent paper by Luo, and one paper on power sums by Wang and Wang. Most of the proofs are made by using generating functions, and the (multiple) $q$-addition plays a fundamental role. The introduction of the $q$-rational numbers in formulas with $q$-additions enables natural $q$-extension of vector forms of Raabes multiplication formulas. As special cases, new formulas for $q$-Bernoulli and $q$-Euler polynomials are obtained.

1. Introduction. In 2006, Luo and Srivastava [8, p. 635-636] found new relationships between Apostol–Bernoulli and Apostol–Euler polynomials. This was followed by the pioneering article by Luo [10], where multiplication formulas for the Apostol–Bernoulli and Apostol–Euler polynomials of higher order, together with $\lambda$-multiple power sums were introduced. Luo also expressed these $\lambda$-multiple power sums as sums of the above polynomials. One year later, Wang and Wang [12] introduced generating functions for $\lambda$-power sums, some of the proofs use a symmetry reasoning, which lead

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to many four-line identities for Apostol–Bernoulli and Apostol–Euler polynomials and $\lambda$-power sums; as special cases, some of the above Luo identities were obtained.

In [5] it was proved that the $q$-Appell polynomials form a commutative ring; in this paper we show what this means in practice. Thus, the aim of the present paper is to find $q$-analogues of most of the above formulas with the aid of the multiple $q$-addition, the $q$-rational numbers, and so on. Many formulas bear a certain resemblance to the $q$-Taylor formula, where $q$-rational numbers appear to the right in the function argument; this means that the alphabet is extended to $\mathbb{Q}_{\oplus q}$. In some proofs, both $q$-binomial coefficients and a vector binomial coefficient occur, this is connected to a vector form of the multinomial theorem, with binomial coefficients, unlike the case in [3, p. 110].

This paper is organized as follows: In this section we give the general definitions. In each section, we then give the specific definitions and special values which we use there.

In Section 2, multiple $q$-Apostol–Bernoulli polynomials and $q$-power sums are introduced and multiplication formulas for $q$-Apostol–Bernoulli polynomials are proved, which are $q$-analogues of Luo [10].

In Section 3, multiplication formulas for $q$-Apostol–Euler polynomials are proved. In Section 4, formulas containing $q$-power sums in one dimension, $q$-analogues of Wang and Wang, [12] are proved. Then in Section 5, mixed formulas of the same kind are proved. Most of the proofs are similar, where different functions, previously used for the case $q = 1$, are used in each proof.

We now start with the definitions. Some of the notation is well-known and can be found in the book [3]. The variables $i, j, k, l, m, n, \nu$ will denote positive integers, and $\lambda$ will denote complex numbers when nothing else is stated.

**Definition 1.** The Gauss $q$-binomial coefficient are defined by

\[(1) \quad \binom{n}{k}_q \equiv \frac{[n]_q!}{[k]_q!(n-k)_q!}, k = 0, 1, \ldots, n.\]

Let $a$ and $b$ be any elements with commutative multiplication. Then the NWA $q$-addition is given by

\[(2) \quad (a \oplus_q b)^n = \sum_{k=0}^{n} \binom{n}{k}_q a^k b^{n-k}, n = 0, 1, 2, \ldots\]

If $0 < |q| < 1$ and $|z| < |1 - q|^{-1}$, the $q$-exponential function is defined by

\[(3) \quad E_q(z) \equiv \sum_{k=0}^{\infty} \frac{1}{[k]_q!} z^k.\]
The following theorem shows how Ward numbers usually appear in applications.

**Theorem 1.1.** Assume that \( n, k \in \mathbb{N} \). Then

\[
(\pi_q)^k = \sum_{m_1 + \ldots + m_n = k} \binom{k}{m_1, \ldots, m_n}_q,
\]

where each partition of \( k \) is multiplied with its number of permutations.

The semiring of Ward numbers, \((\mathbb{N} \oplus q, \oplus_q, \odot_q)\) is defined as follows:

**Definition 2.** Let \((\mathbb{N} \oplus q, \oplus_q, \odot_q)\) denote the Ward numbers \( k q \), \( k \geq 0 \) together with two binary operations: \( \oplus_q \) is the usual Ward \( q \)-addition. The multiplication \( \odot_q \) is defined as follows:

\[
\pi_q \odot_q m_q \sim nm_q,
\]

where \( \sim \) denotes the equivalence in the alphabet.

**Theorem 1.2.** Functional equations for Ward numbers operating on the \( q \)-exponential function. First assume that the letters \( m_q \) and \( n_q \) are independent, i.e. come from two different functions, when operating with the functional. Then we have

\[
E_q(\pi_q n_q t) = E_q(\pi_q t).
\]

Furthermore,

\[
E_q(\sum_j m_q) = E_q(\sum_q)^m = E_q(\pi_q)^j = E_q(\pi_q \odot_q m_q).
\]

**Proof.** Formula (6) is proved as follows:

\[
E_q(\pi_q n_q t) = E_q((1 \oplus_q 1 \oplus_q \cdots \oplus_q 1)\pi_q t),
\]

where the number of 1s to the left is \( m \). But this means exactly \( E_q(\pi_q t)^m \), and the result follows. 

**Definition 3.** The notation \( \sum_{\vec{m}} \) denotes a multiple summation with the indices \( m_1, \ldots, m_n \) running over all non-negative integer values.

Given an integer \( k \), the formula

\[
m_0 + m_1 + \ldots + m_j = k
\]

determines a set \( J_{m_0, \ldots, m_j} \in \mathbb{N}^{j+1} \).

Then if \( f(x) \) is the formal power series \( \sum_{l=0}^{\infty} a_l x^l \), its \( k \)'th NWA-power is given by

\[
(\oplus_{q,l=0}^{\infty} a_l x^l)^k \equiv (a_0 \oplus_q a_1 x \oplus_q \cdots)^k \equiv \sum_{|\vec{m}|=k} \prod_{m_i \in J_{m_0, \ldots, m_j}} (a_i x^l)^m_l \binom{k}{m_l}_q.
\]
We will later use a similar formula when \( q = 1 \) for several proofs.

In order to solve systems of equations with letters as variables and Ward number coefficients, we introduce a division with a Ward number. This is equivalent to \( q \)-rational numbers with Ward numbers instead of integers.

**Definition 4.** Let \( \mathbb{Q}_{\oplus_q} \) denote the set of objects of the following type:

\[
\frac{m_q}{n_q}, \text{ where } \frac{m_q}{n_q} \equiv 1,
\]

together with a linear functional

\[
v, \mathbb{R}[x] \times \mathbb{Q}_{\oplus_q} \to \mathbb{R},
\]

called the evaluation. If \( v(x) = \sum_{k=0}^{\infty} a_k x^k \), then

\[
v\left(\frac{m_q}{n_q}\right) \equiv \sum_{k=0}^{\infty} a_k \left(\frac{m_q}{n_q}\right)^k.
\]

**Definition 5.** For every power series \( f_n(t) \), the \( q \)-Appell polynomials or \( \Phi_q \) polynomials of degree \( \nu \) and order \( n \) have the following generating function:

\[
f_n(t)E_q(xt) = \sum_{\nu=0}^{\infty} t^\nu \{\nu\}_q \Phi^{(n)}_{\nu,q}(x).
\]

For \( x = 0 \) we get the \( \Phi^{(n)}_{\nu,q} \) number of degree \( \nu \) and order \( n \).

**Definition 6.** For \( f_n(t) \) of the form \( h(t)^n \), we call the \( q \)-Appell polynomial \( \Phi_q \) in (14) **multiplicative**.

Examples of multiplicative \( q \)-Appell polynomials are the two \( q \)-Appell polynomials in this article.

### 2. The NWA \( q \)-Apostol–Bernoulli polynomials.

**Definition 7.** The generalized NWA \( q \)-Apostol–Bernoulli polynomials \( B_{\text{NWA,} \lambda, \nu,q}^{(n)}(x) \) are defined by

\[
t^n \frac{E_q(xt)}{(\lambda E_q(t) - 1)^n} = \sum_{\nu=0}^{\infty} \frac{t^\nu B_{\text{NWA,} \lambda, \nu,q}^{(n)}(x)}{\{\nu\}_q \nu!}, \quad |t + \log \lambda| < 2\pi.
\]

Notice that the exponent \( n \) is an integer.

**Definition 8.** A \( q \)-analogue of [10, (20) p. 381], the multiple \( q \)-power sum is defined by

\[
s^{(l)}_{\text{NWA,} \lambda, m,q}(n) \equiv \sum_{|\vec{j}|=l} \left(\frac{t}{j}\right) \lambda^k \left(\overline{k_q}\right)^m,
\]

where \( k \equiv j_1 + 2j_2 + \cdots + (n-1)j_{n-1}, \forall j_i \geq 0. \)
Definition 9. A $q$-analogue of [10, (46) p. 386], the multiple alternating $q$-power sum is defined by

\[ \sigma^{(l)}_{\text{NWA},\lambda,m,q}(n) \equiv (-1)^l \sum_{|\vec{j}|=l} (-\lambda)^k \left( \frac{k_q}{m_q} \right)^m, \]

where $k \equiv j_1 + 2j_2 + \cdots + (n-1)j_{n-1}$, $\forall j_i \geq 0$.

Remark 1. For $l = 1$, formulas (16) and (17) reduce to single sums, as will be seen in section 4.

We now start rather abruptly with the theorems; we note that limits like $\lambda \rightarrow 1$ and $q \rightarrow 1$ can be taken anywhere in the paper, and also in the next one [6]; see the subsequent corollaries. Much care is needed in the proofs, since the Ward numbers need careful handling.

Theorem 2.1. A $q$-analogue of [10, p. 380], multiplication formula for $q$-Apostol–Bernoulli polynomials.

\[ \mathcal{B}^{(n)}_{\text{NWA},\lambda,\nu,q}(m_q x) = \left( \frac{m_q q}{m_q} \right)^{\nu} \sum_{|\vec{j}|=n} \lambda^k \left( \frac{n}{\lambda_q} \right)^n \mathcal{B}^{(n)}_{\text{NWA},\lambda^m,\nu,q}(x \oplus_q \frac{k_q}{m_q}), \]

where $k = j_1 + 2j_2 + \cdots + (m-1)j_{m-1}$, and $\frac{k_q}{m_q} \in \mathbb{Q} \oplus_q$.

Proof. We use the well-known formula for a geometric sum.

\[ \sum_{\nu=0}^{\infty} \mathcal{B}^{(n)}_{\text{NWA},\lambda,\nu,q}(m_q x) \frac{t^\nu}{\nu_q!} = \frac{t^n}{(\lambda q(x)_q - 1)^n} E_q(m_q x t) \]

\[ = \frac{t^n}{(\lambda q(x)_q - 1)^n} \left( \sum_{i=0}^{m-1} \lambda^i E_q(x_q^i) \right)^n E_q(m_q x t) \]

\[ \text{by (7),} \]

\[ = \left( \frac{t}{(\lambda q(x)_q - 1)} \right)^n \sum_{|\vec{j}|=n} \lambda^k E_q \left( x \oplus_q \frac{k_q}{m_q} \right)^n \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu_q!}. \]

The theorem follows by equating the coefficients of $\frac{t^\nu}{\nu_q!}$. \qed

Corollary 2.2. A $q$-analogue of [10, p. 381]:

\[ \mathcal{B}_{\text{NWA},\lambda,\nu,q}(m_q x) = \left( \frac{m_q q}{m_q} \right)^{\nu} \sum_{j=0}^{m-1} \lambda^j \mathcal{B}^{(n)}_{\text{NWA},\lambda^m,\nu,q}(x \oplus_q \frac{k_q}{m_q}). \]
Corollary 2.3. A q-analogue of Carlitz formula [2], [10, p. 381]

\[
B_{\text{NWA},\nu,q}^{(n)}(x) = \frac{(m_q)^\nu}{(m_q)^n} \sum_{|\nu| = n} \binom{n}{j} B_{\text{NWA},\nu,q}^{(n)} \left( x \oplus_q \frac{k}{m_q} \right),
\]

where \( k = j_1 + 2j_2 + \cdots + (m-1)j_{m-1} \), and \( \frac{k}{m_q} \in \mathbb{Q}_q \).

Theorem 2.4. A formula for a multiple q-power sum, a q-analogue of [10, (25) p. 382]:

\[
s_{\text{NWA},\lambda,m,q}^{(l)}(n) = \sum_{j=0}^{l} \binom{l}{j} \frac{(-1)^{l-j}\lambda^{(n-1)j+l}}{(m+1)_{l,q}} \times \sum_{k=0}^{m+l} \binom{m+l}{k} B_{\text{NWA},\lambda,k,q}^{(j)} \left( (n-1)j + l_q \right) B_{\text{NWA},\lambda,m+l-k,q}^{(l-j)}. \tag{22}
\]

Proof. We use the generating function technique. Put \( k = j_1 + 2j_2 + \cdots + (n-1)j_{n-1} \). It is assumed that \( j_i \geq 0, 1 \leq i \leq n-1 \), zeros are neglected.

\[
\sum_{\nu=0}^{\infty} s_{\text{NWA,\nu,q}}^{(l)}(n) \frac{t^\nu}{\nu!} = \sum_{\nu=0}^{\infty} \sum_{|\nu|=l} \binom{l}{j} \lambda^k \left( k_q \right)^\nu \frac{t^\nu}{\nu!} \tag{23}
\]

The theorem follows by equating the coefficients of \( \frac{t^\nu}{\nu!} \). \( \square \)
Corollary 2.5. A $q$-analogue of [10, (26) p. 382]: The generating function for $s_{\text{NWA},\lambda,\nu,q}^{(l)}(n)$ is

$$
\sum_{\nu=0}^{\infty} s_{\text{NWA},\lambda,\nu,q}^{(l)}(n) \frac{t^{\nu}}{\{\nu\}_{q}!} = \left( \frac{\lambda^n E_{q}(\tau q t)}{\lambda E_{q}(t) - 1} - \frac{\lambda E_{q}(t)}{\lambda E_{q}(t) - 1} \right)^l 
$$

$$
= (\lambda E_{q}(t) + \lambda^2 E_{q}(2q t) + \cdots + \lambda^{n-1} E_{q}(n - 1 t))^l. 
$$

Theorem 2.6. A recurrence relation for $q$-Apostol–Bernoulli numbers, a $q$-analogue of [10, (32) p. 384].

$$
(m_q)^l B_{\text{NWA},\lambda,n,q}^{(l)} = \sum_{j=0}^{n} \binom{n}{j} B_{\text{NWA},\lambda,\nu,q}^{(l)} m_q^{n-j} B_{\text{NWA},\lambda,\nu,j,q}^{(l)} s_{\text{NWA},\lambda,\nu,j,q}(m),
$$

where $k = j_1 + 2j_2 + \cdots + (m-1)j_{m-1}$.

Proof. We use the definition of $q$-Appell numbers as $q$-Appell polynomial at $x = 0$.

$$
(m_q)^l B_{\text{NWA},\lambda,n,q}^{(l)} \overset{\text{by (18)}}= \sum_{|\nu|-l} \lambda^k \left( l \frac{1}{\nu} \right) B_{\text{NWA},\lambda,\nu,q}^{(l)} \left( \frac{t_{q}}{m_q} \right) 
$$

$$
= (m_q)^n \sum_{|\nu|-l} \lambda^k \left( l \frac{1}{\nu} \right) \sum_{j=0}^{n} \binom{n}{j} B_{\text{NWA},\lambda,\nu,j,q}^{(l)} \left( \frac{t_{q}}{m_q} \right)^{n-j} 
$$

$$
= \sum_{j=0}^{n} \binom{n}{j} \frac{(m_q)^n}{m_q^{n-j}} B_{\text{NWA},\lambda,\nu,j,q}^{(l)} \sum_{|\nu|-l} \lambda^k \left( l \frac{1}{\nu} \right) \left( \frac{t_{q}}{m_q} \right)^{n-j} \overset{\text{by (16)}}= \text{LHS}. 
$$

3. The NWA $q$-Apostol–Euler polynomials. We start with some repetition from [3]:

Definition 10. The generating function for the first $q$-Euler polynomials of degree $\nu$ and order $n$, $F_{\text{NWA},\nu,q}^{(n)}(x)$, is given by

$$
\frac{2^n E_{q}(xt)}{(E_{q}(t) + 1)^n} = \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\{\nu\}_{q}!} F_{\text{NWA},\nu,q}^{(n)}(x), \ |t| < \pi. 
$$

Definition 11. The generalized NWA $q$-Apostol–Euler polynomials $T_{\text{NWA},\lambda,\nu,q}^{(n)}(x)$ are defined by

$$
\frac{2^n}{(\lambda E_{q}(t) + 1)^n} E_{q}(xt) = \sum_{\nu=0}^{\infty} \frac{t^{\nu} T_{\text{NWA},\lambda,\nu,q}^{(n)}(x)}{\{\nu\}_{q}!}, \ |t + \log \lambda| < \pi. 
$$
Theorem 3.1. A $q$-analogue of [10, (37) p. 385], first multiplication formula for $q$-Apostol–Euler polynomials.

\[
\mathcal{F}_{NWA,\lambda,\nu,q}(\overline{m}_q x)^{(n)} = (\overline{m}_q)^\nu \sum_{|\vec{j}|=n} (-\lambda)^k \binom{n}{\vec{j}} \mathcal{F}_{NWA,\lambda^m,\nu,q}(x \oplus_q \frac{\overline{k}_q}{\overline{m}_q}),
\]

where $k = j_1 + 2j_2 + \cdots + (m - 1)j_{m-1}$, $m$ odd.

Proof.

\[
\sum_{\nu=0}^{\infty} \mathcal{F}_{NWA,\lambda,\nu,q}(\overline{m}_q x)^{(n)} \frac{t^\nu}{\nu!} = \frac{2^n}{(\lambda^m E_q(\overline{m}_q t) + 1)^n} E_q(\overline{m}_q x)
\]

\[
= \left(\frac{2}{(\lambda^m E_q(\overline{m}_q t) + 1)}\right)^n \sum_{|\vec{j}|=n} (-\lambda)^k E_q\left(\left(x \oplus_q \frac{\overline{k}_q}{\overline{m}_q}\right)\overline{m}_q t\right)
\]

\[
= \sum_{\nu=0}^{\infty} \left(\overline{m}_q\right)^\nu \sum_{|\vec{j}|=n} \binom{n}{\vec{j}} (-\lambda)^k \mathcal{F}_{NWA,\lambda^m,\nu,q}(x \oplus_q \frac{\overline{k}_q}{\overline{m}_q}) \frac{t^\nu}{\nu!}.
\]

The theorem follows by equating the coefficients of $\frac{t^\nu}{\nu!}$. \qed

Theorem 3.2. A $q$-analogue of [10, (38) p. 385], second multiplication formula for $q$-Apostol–Euler polynomials.

\[
\mathcal{F}_{NWA,\lambda,\nu,q}(\overline{m}_q x)
\]

\[
= (-2)^n (\overline{m}_q)^{\nu+n} \sum_{|\vec{j}|=n} (-\lambda)^k \binom{n}{\vec{j}} \mathcal{F}_{NWA,\lambda^m,\nu+n,q}(x \oplus_q \frac{\overline{k}_q}{\overline{m}_q}),
\]

where $k = j_1 + 2j_2 + \cdots + (m - 1)j_{m-1}$, $m$ even.

Corollary 3.3. A $q$-analogue of [10, (43) p. 386]:

\[
\mathcal{F}_{NWA,\lambda,\nu,q}(\overline{m}_q x) = \left\{
\begin{array}{ll}
(\overline{m}_q)^\nu \sum_{j=0}^{m-1} (-\lambda)^j \mathcal{F}_{NWA,\lambda^m,\nu,q}(x \oplus_q \frac{\overline{j}_q}{\overline{m}_q}), & m \text{ odd}, \\
-2(\overline{m}_q)^{\nu+1} \sum_{j=0}^{m-1} (-\lambda)^j \mathcal{B}_{NWA,\lambda^m,\nu+1,q}(x \oplus_q \frac{\overline{j}_q}{\overline{m}_q}), & m \text{ even},
\end{array}
\right.
\]

where $\frac{\overline{j}_q}{\overline{m}_q} \in \mathbb{Q}_{\oplus_q}$. 

Theorem 3.4. A formula for a multiple alternating $q$-power sum, a $q$-analogue of \cite{10}, (51) p. 387:
\[
\sigma_{\text{NWA}, \lambda, m, q}^{(l)}(n) = 2^{-l} \sum_{j=0}^{\lfloor m+1 \rfloor t} \binom{l}{j} (-1)^{jn} \lambda^{(n-1)j+l} \\
\times \left( \sum_{k=0}^{m+l} \binom{m+l}{k} \mathcal{F}_{\text{NWA}, \lambda, k, q}^{(l-j)} \right) q^{(n-1)j+l_k} q^{(l-j)}. 
\]

Proof. We use the generating function technique. Put $k_j = j_1 + 2j_2 + \cdots + (n-1)j_{n-1}$. It is assumed that $j_i \geq 0$, $1 \leq i \leq n-1$.
\[
\sum_{\nu=0}^{\infty} \sigma_{\text{NWA}, \lambda, \nu, q}^{(l)}(n) \frac{t^\nu}{\nu!} = \sum_{\nu=0}^{\infty} \sum_{[j]=l} \binom{l}{j} (-1)^{l} (-\lambda)^k (E_\nu^q)^k \frac{t^\nu}{\nu!} = \\
(\lambda E_q(t) - \lambda^2 E_q(\bar{2}q) + \cdots + (-1)^n \lambda^{n-1} E_q(n-1q)) = \\
\left( \frac{(-\lambda)^n E_q(\bar{n}q)}{\lambda E_q(t) + 1} + \frac{\lambda E_q(t)}{\lambda E_q(t) + 1} \right)^l = \\
\sum_{j=0}^{l} \binom{l}{j} (-1)^{l-j} \left( \frac{(-\lambda)^n E_q(\bar{n}q)}{\lambda E_q(t) + 1} \right)^j \left( \frac{\lambda E_q(t)}{\lambda E_q(t) + 1} \right)^{l-j} = \\
\sum_{i=0}^{\infty} \mathcal{F}_{\text{NWA}, \lambda, i, q}^{(l-j)} \frac{t^i}{\{i\}_q!} = \sum_{\nu=0}^{\infty} \left( 2^{-l} \sum_{j=0}^{l} \binom{l}{j} (-1)^{jn} \lambda^{(n-1)j+l} \sum_{k=0}^{m+l} \binom{m+l}{k} \mathcal{F}_{\text{NWA}, \lambda, k, q}^{(l-j)} \right) q^{(n-1)j+l_k} q^{(l-j)} \\
\times \sum_{k=0}^{m+l} \binom{m+l}{k} \mathcal{F}_{\text{NWA}, \lambda, k, q}^{(l-j)} \frac{t^i}{\{i\}_q!}. 
\]

The theorem follows by equating the coefficients of $\frac{t^\nu}{\nu! q}$. 

Corollary 3.5. A $q$-analogue of \cite{10}, (52) p. 387: The generating function for $\sigma_{\text{NWA}, \lambda, \nu, q}^{(l)}(n)$ is
\[
\sum_{\nu=0}^{\infty} \sigma_{\text{NWA}, \lambda, \nu, q}^{(l)}(n) \frac{t^\nu}{\nu! q} = \left( \frac{(-\lambda)^n E_q(\bar{n}q)}{\lambda E_q(t) + 1} + \frac{\lambda E_q(t)}{\lambda E_q(t) + 1} \right)^l = \\
(\lambda E_q(t) - \lambda^2 E_q(\bar{2}q) + \cdots + (-1)^n \lambda^{n-1} E_q(n-1q)) = 
\]
Theorem 3.6. A q-analogue of [10, p. 389]. For m odd, we have the following recurrence relation for q-Apostol–Euler numbers.

\[
F_{NWA,\lambda,m,q}(n) = (-1)^j \sum_{j=0}^{n} \binom{n}{j} \frac{(\overline{m}_q)^n}{q(\overline{m}_q)^{n-j}} F_{NWA,\lambda^m,j,q}\sigma_{NWA,\lambda,n-j,q}(m),
\]

where \(k = j_1 + 2j_2 + \cdots + (m-1)j_{m-1}\).

Proof.

\[
F_{NWA,\lambda,n,q} \text{ by (20)} = \sum_{|\vec{\nu}| = l} (\lambda^l) \frac{k}{q} \sum_{j=0}^{n} \binom{n}{j} q^{n-j} F_{NWA,\lambda,m,j,q}(\overline{k}_q) \sigma_{NWA,\lambda,n-j,q}(m),
\]

\[
= \sum_{j=0}^{n} \binom{n}{j} q^{n-j} F_{NWA,\lambda,m,j,q}(\overline{k}_q) \sigma_{NWA,\lambda,n-j,q}(m)
\]

4. Single formulas for Apostol q-power sums. In order to keep the same notation as in [3], we make a slight change from [12, p. 309]. The following definitions are special cases of the q-power sums in section 2.

Definition 12. Almost a q-analogue of [12, p. 309], the q-power sum and the alternate q-power sum (with respect to \(\lambda\)), are defined by

\[
s_{NWA,\lambda,m,q}(n) = \sum_{k=0}^{n-1} \lambda^k (\overline{k}_q)^m \quad \text{and} \quad \sigma_{NWA,\lambda,m,q}(n) = \sum_{k=0}^{n-1} (-1)^k \lambda^k (\overline{k}_q)^m.
\]

Their respective generating functions are

\[
\sum_{m=0}^{\infty} s_{NWA,\lambda,m,q}(n) \frac{t^m}{\{m\}_{q}!} = \frac{\lambda^n E_q(\overline{\pi}_q t) - 1}{\lambda E_q(t) - 1}
\]

and

\[
\sum_{m=0}^{\infty} \sigma_{NWA,\lambda,m,q}(n) \frac{t^m}{\{m\}_{q}!} = \frac{(-1)^{n+1} \lambda^n E_q(\overline{\pi}_q t) + 1}{\lambda E_q(t) + 1}.
\]

Proof. Let us prove (38). We have

\[
\sum_{m=0}^{\infty} s_{NWA,\lambda,m,q}(n) \frac{t^m}{\{m\}_{q}!} = \sum_{m=0}^{\infty} \sum_{k=0}^{n-1} \lambda^k (\overline{k}_q)^m \text{ by (6)} = \sum_{k=0}^{n-1} \lambda^k (E_q(t))^k = \text{RHS}.
\]
We have the following special cases:

\begin{align}
\sigma_{\text{NWA}, \lambda, m, q}(1) &= \delta_{0, m}, \\
\sigma_{\text{NWA}, \lambda, m, q}(2) &= \delta_{0, m} + \lambda, \quad \sigma_{\text{NWA}, \lambda, m, q}(2) = \delta_{0, m} - \lambda.
\end{align}

**Theorem 4.1.** A \( q \)-anologue of [12, p. 310], and extensions of [3, p. 121, 131]:

\begin{align}
\sigma_{\text{NWA}, \lambda, m, q}(n) &= \frac{\lambda^n B_{\text{NWA}, \lambda, m+1, q}(\tilde{\pi}_q) - B_{\text{NWA}, \lambda, m+1, q}}{(m+1)_q} \\
\sigma_{\text{NWA}, \lambda, m, q}(n) &= \frac{(-1)^{n+1} \lambda^n \sigma_{\text{NWA}, \lambda, m+1, q}(\tilde{\pi}_q) - \sigma_{\text{NWA}, \lambda, m+1, q}}{2}.
\end{align}

**Theorem 4.2.** A \( q \)-anologue of [12, (18), p. 311],

\begin{align}
\sum_{k=0}^{n} \binom{n}{k} \frac{(\tilde{\tau}_q)^k}{q^k} (\tilde{\tau}_q)^{n-k} B_{\text{NWA}, \lambda, m, q} (\tilde{\tau}_q x) s_{\text{NWA}, \lambda, n-k, q}(i) \\
= \sum_{k=0}^{n} \binom{n}{k} \frac{(\tilde{\tau}_q)^k}{q^k} (\tilde{\tau}_q)^{n-k} B_{\text{NWA}, \lambda, m, q} (\tilde{\tau}_q x) s_{\text{NWA}, \lambda, n-k, q}(j)
\end{align}

\begin{align}
= \frac{(\tilde{\tau}_q)^n}{i} \sum_{m=0}^{j-1} \lambda^m B_{\text{NWA}, \lambda, m, q} \left( \tilde{\tau}_q x \otimes q \frac{im_q}{i_q} \right) \\
= \frac{(\tilde{\tau}_q)^n}{j} \sum_{m=0}^{j-1} \lambda^m B_{\text{NWA}, \lambda, m, q} \left( \tilde{\tau}_q x \otimes q \frac{im_q}{j_q} \right).
\end{align}

**Proof.** Define the following function, symmetric in \( i \) and \( j \).

\begin{align}
f_q(t) &\equiv \frac{t E_q(\tilde{\tau}_q x t) (\lambda^j E_q(\tilde{\tau}_q t) - 1)}{\lambda E_q(\tilde{\tau}_q t) - 1}(\lambda^j E_q(\tilde{\tau}_q t) - 1) \\
= \frac{(\tilde{\tau}_q t)^1 E_q(\tilde{\tau}_q x t)}{\lambda E_q(\tilde{\tau}_q t) - 1} \left( \frac{\lambda^j E_q(\tilde{\tau}_q t) - 1}{\lambda^j E_q(\tilde{\tau}_q t) - 1} \right) \frac{1}{i}.
\end{align}

By using the formula for a geometric sequence, we can expand \( f_q(t) \) in two ways:

\begin{align}
f_q(t) &= \sum_{\nu=0}^{\infty} B_{\text{NWA}, \lambda, \nu, q} (\tilde{\tau}_q x \otimes q \nu)_{q^\nu} \sum_{m=0}^{\infty} s_{\text{NWA}, \lambda, m, q}(i) \left( \frac{(\tilde{\tau}_q t)^m}{q_m} \right) \frac{1}{i} \\
= \frac{(\tilde{\tau}_q t)^1 \sum_{m=0}^{i-1} \lambda^m \left( E_q \left( \tilde{\tau}_q x \otimes q \frac{im_q}{i_q} \right) \right) \frac{1}{i}}{\lambda E_q(\tilde{\tau}_q t) - 1} \\
= \sum_{\nu=0}^{\infty} \frac{(\tilde{\tau}_q t)^\nu}{i} \sum_{m=0}^{i-1} \lambda^m B_{\text{NWA}, \lambda, \nu, q} \left( \tilde{\tau}_q x \otimes q \frac{im_q}{i_q} \right) \frac{1}{\nu}.\]

The theorem follows by equating the coefficients of $\frac{t^\nu}{(\nu)_q}$ and using the symmetry in $i$ and $j$ of $f_q(t)$. □

**Corollary 4.3.** A $q$-analogue of [12, (19), p. 311],

$$B_{NWA,\lambda,n,q}(\tilde{t}_q x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(\tilde{t}_q)^k}{k} B_{NWA,\lambda^i,k,q}(x) s_{NWA,\lambda,n-k,q}(i)$$

(47)

$$= \frac{(\tilde{t}_q)^n}{i} \sum_{m=0}^{i-1} \lambda^{m} B_{NWA,\lambda^i,n,q} \left( x \oplus_q \frac{m_q}{\tilde{t}_q} \right).$$

**Proof.** Put $j = 1$ in (44) and use (41). □

**Remark 2.** This proves formula (20) again.

**Corollary 4.4.** A $q$-analogue of [12, (20), p. 311],

$$\sum_{m=0}^{1} \lambda^{m} B_{NWA,\lambda^2,n,q} \left( \tilde{t}_q x \oplus_q \frac{m_q}{\tilde{t}_q} \right)$$

(48)

$$= \frac{2}{(2q)^n} \sum_{k=0}^{n} \binom{n}{k} \frac{(\tilde{t}_q)^k}{k} \left( \frac{2q}{2q} \right)^{n-k} B_{NWA,\lambda^i,k,q}(\tilde{t}_q x) s_{NWA,\lambda^2,n-k,q}(i)$$

$$= \frac{2}{(2q)^n} \frac{(\tilde{t}_q)^n}{i} \sum_{m=0}^{i-1} \lambda^{2m} B_{NWA,\lambda^i,n,q} \left( \tilde{t}_q x \oplus_q \frac{2m_q}{\tilde{t}_q} \right).$$

**Proof.** Put $j = 2$ in (44) and multiply by $\frac{2}{(2q)^n}$. □

Moreover, we have

(49)  

$$B_{NWA,\lambda,n,q}(x) = \frac{(\tilde{t}_q)^n}{2} \sum_{m=0}^{1} \lambda^{m} B_{NWA,\lambda^2,n,q} \left( x \oplus_q \frac{m_q}{\tilde{t}_q} \right).$$

**Proof.** Put $i = 2$ in (47) and replace $x$ by $x \frac{1}{\tilde{t}_q}$. □

For $\lambda = 1$ and $x = 0$, this reduces to

(50)  

$$B_{NWA,n,q} \left( \frac{1}{\tilde{t}_q} \right) = \left( \frac{2}{(2q)^n} - 1 \right) B_{NWA,n,q}.$$
Theorem 4.5. A q-analogue of [12, (22) p. 312]. Assume that i and j are either both odd, or both even, then we have

\[
\sum_{k=0}^{n} \binom{n}{k} q^n (\overline{t}_q)^{n-k} \mathcal{F}_{\text{NWA},\lambda',k,q} (\overline{t}_q x) \sigma_{\text{NWA},\lambda',n-k,q}(i) = \sum_{k=0}^{n} \binom{n}{k} q^n (\overline{j}_q)^{n-k} \mathcal{F}_{\text{NWA},\lambda',k,q} (\overline{j}_q x) \sigma_{\text{NWA},\lambda',n-k,q}(i)
\]

(51)

\[
= (\overline{t}_q)^n \sum_{m=0}^{i-1} \lambda^m (-1)^m NWA_{\lambda',n,q} \left( \overline{j}_q x + q \frac{jmq}{t_q} \right)
\]

\[
= (\overline{j}_q)^n \sum_{m=0}^{j-1} \lambda^m (-1)^m NWA_{\lambda',n,q} \left( \overline{i}_q x + q \frac{imq}{j_q} \right).
\]

Proof. Define the following symmetric function

\[
f_q(t) = \frac{E_q(\overline{t}_q x t)((-1)^{i+1} \lambda^j E_q(\overline{t}_q t) + 1)}{(\lambda E_q(\overline{t}_q t) + 1)(\lambda E_q(\overline{j}_q t) + 1)}
\]

(52)

By using the formula for a geometric sequence, we can expand \( f_q(t) \) in two ways:

\[
f_q(t) = \frac{1}{2} \sum_{\nu=0}^{\infty} \mathcal{F}_{\text{NWA},\lambda',\nu,q} (\overline{t}_q x)^{\nu} \left( \sum_{m=0}^{i} \sigma_{\text{NWA},\lambda',n,q}(i) \frac{(\overline{t}_q t)^m}{m!} \right)
\]

(53)

\[
= \frac{1}{2} \sum_{\nu=0}^{\infty} \mathcal{F}_{\text{NWA},\lambda',\nu,q} \left( \overline{j}_q x + q \frac{jmq}{t_q} \right)^{\nu} \left( \sum_{m=0}^{i-1} \lambda^m E_q \left( \overline{j}_q x + q \frac{jmq}{t_q} \right) \frac{t_q^m}{m!} \right)
\]

The theorem follows by equating the coefficients of \( \frac{t_q^\nu}{\nu!} \) and using the symmetry in \( i \) and \( j \) of \( f_q(t) \).

\[ \Box \]

Theorem 4.6. (A q-analogue of [12, (24) p. 313]) For i odd we have

\[
\mathcal{F}_{\text{NWA},\lambda,n,q}(\overline{t}_q x) = \sum_{k=0}^{n} \binom{n}{k} q^n (\overline{t}_q)^{n-k} \mathcal{F}_{\text{NWA},\lambda',k,q} (x) \sigma_{\text{NWA},\lambda,n-k,q}(i)
\]

(54)

\[
= (\overline{t}_q)^n \sum_{m=0}^{i-1} (-\lambda)^m \mathcal{F}_{\text{NWA},\lambda',n,q} \left( x + q \frac{mq}{t_q} \right).
\]
(A $q$-analogue of [12, (25) p. 313]) For $i$ even,

$$
\sum_{m=0}^{1} \lambda^{im}(-1)^{m} F_{\text{NW,A},q,n,q} \left( i_{q} x \oplus \frac{m_{q}}{q} \right)
$$

(55)

$$
= \frac{1}{(2q)^{n}} \sum_{k=0}^{n} \binom{n}{k} (\bar{q}_{q})^{k} (\sigma_{q})^{n-k} F_{\text{NW,A},\nu,k,q} (\sigma_{q} x) \sigma_{\text{NW,A},\lambda^{2},n-k,q}(i)
$$

$$
= \frac{(i_{q})^{n}}{(2q)^{n}} \sum_{m=0}^{i-1} (-1)^{m} \lambda^{2m} F_{\text{NW,A},q,n,q} \left( \frac{\sigma_{q} x \oplus \frac{m_{q}}{q}}{i_{q}} \right).
$$

Proof. Put $j = 1$ or 2 in (51), and divide by $(2q)^{n}$. □

Remark 3. This proves the first part of formula (32) again.

5. Apostol $q$-power sums, mixed formulas. We now turn to mixed formulas, which contain polynomials of both kinds.

Theorem 5.1. A $q$-analogue of [12, (26) p. 313]. If $i$ is even then

$$
\sum_{k=0}^{n} \binom{n}{k} \left( \frac{\bar{q}_{q}}{i} \right)^{k} (\bar{j}_{q})^{n-k} F_{\text{NW,A},\lambda^{1},n,q} (\bar{j}_{q} x) \sigma_{\text{NW,A},\lambda^{1},n-k,q}(i)
$$

(56)

$$
= -\frac{(i_{q})^{n}}{2} \sum_{k=0}^{n-1} \binom{n-1}{k} (\bar{j}_{q})^{k} (\bar{q}_{q})^{n-k-1} \times F_{\text{NW,A},\lambda^{1},n-k,q}(\bar{j}_{q} x) \sigma_{\text{NW,A},\lambda^{1},n-k,q}(i)
$$

$$
= \frac{(i_{q})^{n}}{i} \sum_{m=0}^{i-1} (-1)^{m} \lambda^{im} F_{\text{NW,A},\lambda^{1},n,q} \left( \frac{\sigma_{q} x \oplus \frac{m_{q}}{q}}{i_{q}} \right)
$$

$$
= -\frac{(i_{q})^{n}}{2} \sum_{m=0}^{i-1} \lambda^{im} F_{\text{NW,A},\lambda^{1},n-1,q} \left( \frac{\sigma_{q} x \oplus \frac{m_{q}}{q}}{i_{q}} \right).
$$

Proof. Define the following function

$$
f_{q}(t) \equiv \frac{t F_{q}(\bar{j}_{q} x t)((-1)^{i+1} \lambda^{i} E_{q}(\bar{j}_{q} t) + 1)}{(\lambda^{i} E_{q}(\bar{j}_{q} t) - 1)(\lambda^{i} E_{q}(\bar{j}_{q} t) + 1)}
$$

(57)

$$
= \frac{(i_{q} t)^{i} E_{q}(\bar{j}_{q} x t)}{(\lambda^{i} E_{q}(\bar{j}_{q} t) - 1)} \frac{(-1)^{i+1} \lambda^{i} E_{q}(\bar{j}_{q} t) + 1}{\lambda^{i} E_{q}(\bar{j}_{q} t) + 1} \frac{1}{i}.
$$
By using the formula for a geometric sequence, we can expand $f_q(t)$ in two ways:

$$f_q(t) = \left( \sum_{\nu=0}^{\infty} B_{NWA, \lambda, \nu, q} (\tilde{t}_q x) \right) \left( \sum_{m=0}^{\infty} \sigma_{NWA, \lambda, m, q}(i) \frac{(\tilde{t}_q t^m)}{(m)_{q!}} \right) \frac{1}{i}$$

\[ (58) = \left( \frac{(\tilde{t}_q)^1}{\lambda^1 E_q(\tilde{t}_q t)} - 1 \right) \sum_{m=0}^{i-1} (-1)^m \lambda^m E_q \left( \frac{\tilde{t}_q x + \overline{jm_q}}{\tilde{t}_q} \right) \frac{(\tilde{t}_q x^i)}{(i)_{q!}} \frac{1}{i} \]

$$= \sum_{\nu=0}^{\infty} \left( \frac{(\tilde{t}_q)^\nu}{\nu!} \right) \sum_{m=0}^{i-1} (-1)^m \lambda^m B_{NWA, \lambda, \nu, q} \left( \frac{\tilde{t}_q x + \overline{jm_q}}{\tilde{t}_q} \right) \frac{(\tilde{t}_q x^i)}{(i)_{q!}} \frac{1}{i}.$$

By equating the coefficients of $\frac{t^\nu}{(\nu)!}$, we obtain rows 1 and 3 of formula (56).

On the other hand, we can rewrite $f_q(t)$ in the following way:

$$f_q(t) = -t \frac{2 \left( \lambda^1 E_q(\tilde{t}_q t) \right)}{\lambda^1 E_q(\tilde{t}_q t) + 1}$$

\[ (59) = -t \frac{2 \left( \lambda^1 E_q(\tilde{t}_q t) \right)}{\lambda^1 E_q(\tilde{t}_q t) + 1} \left( \frac{\lambda^1 E_q(\tilde{t}_q t) - 1}{\lambda^1 E_q(\tilde{t}_q t) - 1} \right). \]

By using the formula for a geometric sequence, we can expand (59) in two ways:

$$f_q(t) = -t \left( \sum_{\nu=0}^{\infty} F_{NWA, \lambda, \nu, q} (\tilde{t}_q x) \right) \left( \sum_{m=0}^{\infty} \sigma_{NWA, \lambda, m, q}(j) \frac{(\tilde{t}_q t^m)}{(m)_{q!}} \right) \frac{1}{i}$$

\[ (60) = -t \frac{2 \lambda^m}{\lambda^1 E_q(\tilde{t}_q t) + 1} \left( \frac{\tilde{t}_q x + \overline{jm_q}}{\tilde{t}_q} \right) \frac{(\tilde{t}_q x^i)}{(i)_{q!}} \frac{1}{i} \]

$$= -t \sum_{\nu=0}^{\infty} \left( \frac{(\tilde{t}_q)^\nu}{\nu!} \right) \sum_{m=0}^{i-1} \lambda^m \tilde{t}_q^\nu F_{NWA, \lambda, \nu, q} \left( \frac{\tilde{t}_q x + \overline{jm_q}}{\tilde{t}_q} \right) \frac{(\tilde{t}_q x^i)}{(i)_{q!}} \frac{1}{i}.$$

By equating the coefficients of $\frac{t^\nu}{(\nu)!}$, we obtain rows 2 and 4 of formula (56).

\[ \square \]

**Corollary 5.2.** A $q$-analogue of [12, (28) p. 313]. If $i$ is even, then

$$F_{NWA, \lambda, n-1, q} (\tilde{t}_q x)$$

\[ (61) = -2 \left( \frac{\tilde{t}_q}{n} \right) \sum_{k=0}^{n} \binom{n}{k} \frac{(\tilde{t}_q)^k}{k} B_{NWA, \lambda, k, q} (x) \sigma_{NWA, \lambda, n-k, q}(i) \]

$$= -2 \frac{(\tilde{t}_q)^n}{i \{n\}_q} \sum_{m=0}^{i-1} (-\lambda)^m B_{NWA, \lambda, m, q} \left( x + \overline{jm_q} \right).$$

**Proof.** Put $j = 1$ in formula (56) and multiply by $-\frac{2}{\{n\}_q}$. \[ \square \]
Corollary 5.3. A $q$-analogue of [12, (29) p. 313].

$$F_{\text{NWA}, \lambda, n-1, q}(x) = -\frac{2}{\{n\}_q} \sum_{k=0}^{n} \binom{n}{k} \frac{(\tau_q^k)}{q} B_{\text{NWA}, \lambda^k, q} \left( \frac{x}{2q} \right) \sigma_{\text{NWA}, \lambda, n-k, q}(2)$$

$$= -\frac{(\tau_q^n)}{\{n\}_q} \sum_{m=0}^{1} (-\lambda)^m B_{\text{NWA}, \lambda^m, n, q} \left( \frac{x}{2q} \oplus q \frac{m q}{2q} \right).$$

Proof. Put $i = 2$ in formula (61), and replace $x$ by $\frac{x}{2q}$. □

Corollary 5.4. A $q$-analogue of [12, (31) p. 314]. If $i$ is even, then

$$1 \sum_{m=0}^{1} \lambda^m F_{\text{NWA}, \lambda^2, n-1, q} \left( \tau_q x \oplus_q \frac{m q}{2q} \right)$$

$$= -\frac{2}{\{n\}_q (\tau_q^n)} \sum_{k=0}^{n-1} \binom{n}{k} \frac{(\tau_q^k)}{q} B_{\text{NWA}, \lambda^k, q} \left( \tau_q x \right) \sigma_{\text{NWA}, \lambda, n-k, q}(i)$$

$$= \frac{1}{(\tau_q^n)} \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{(\tau_q^k)}{q} B_{\text{NWA}, \lambda^k, q} \left( \tau_q x \right) \sigma_{\text{NWA}, \lambda, n-k-1, q}(2)$$

$$= -\frac{2}{\{n\}_q (\tau_q^n)} \binom{\tau_q^n}{i} \sum_{m=0}^{i-1} (-1)^m \lambda^{2m} B_{\text{NWA}, \lambda^m, n, q} \left( \tau_q x \oplus \frac{m q}{\tau_q} \right).$$

Proof. Put $j = 2$ in formula (56) and multiply by $-\frac{2}{\{n\}_q (\tau_q^n) n-1}$. □

Corollary 5.5. A $q$-analogue of [12, (32) p. 314].

$$1 \sum_{m=0}^{1} (-1)^{m+1} \lambda^m B_{\text{NWA}, \lambda, n, q} \left( x \oplus_q \frac{2m q}{2q} \right)$$

$$= \frac{\{n\}_q (\tau_q^n)}{(\tau_q^n)^n} \sum_{m=0}^{1} \lambda^m F_{\text{NWA}, \lambda, n-1, q} \left( x \oplus \frac{2m q}{\tau_q} \right).$$

Proof. Put $i = 2$ in formula (63), replace $x$ and $\lambda^2$ by $\frac{x}{\tau_q}$ and $\lambda$, and multiply by $\frac{\{n\}_q (\tau_q^n)}{(\tau_q^n)^n}$. □
Corollary 5.6. A q-analogue of [12, (33) p. 314].

\[
\sum_{m=0}^{1} (-1)^{m} \lambda^{j} m \mathfrak{B}_{\text{NWA}, \lambda^{2}, n, q} \left( T_{q} x \oplus q \frac{j m}{2 q} \right)
\]

\[
= -\left\{ n \right\}_{q} \sum_{k=0}^{n-1} \binom{n-1}{k} \left( \mathfrak{T}_{\text{NWA}, \lambda^{j}, k, q} \left( \frac{T_{q} x}{2 q} \right) \right)^{n-k-1} \mathfrak{F}_{\text{NWA}, \lambda^{2}, n-k-1, q}(j)
\]

\[
= -\left\{ n \right\}_{q}^{2} \sum_{m=0}^{j-1} \lambda^{2m} \mathfrak{F}_{\text{NWA}, \lambda^{j}, n-1, q} \left( \frac{T_{q} x \oplus q}{2 q} \frac{j m}{2 q} \right).
\]

Proof. Put \( i = 2 \) in formula (56) and multiply by \( \frac{2}{(2q)^n} \). \( \square \)

6. Discussion. As was indicated in [5], we have considered q-analogues of the currently most popular Appell polynomials, together with corresponding power sums. The beautiful symmetry of the formulas comes from the ring structure of the q-Appell polynomials. We have not considered JHC q-Appell polynomials, since we are looking for maximal symmetry in the formulas. The q-Taylor formulas have not been used in the proofs, since the generating functions were mostly used. In a further paper [6], we will find similar expansion formulas for q-Appell polynomials of arbitrary order.

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