Convergence of the Wick Star Product

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Abstract

We construct a Fréchet space as a subspace of $C^\infty(\mathbb{C}^n)$ where the Wick star product converges and is continuous. The resulting Fréchet algebra $A_\hbar$ is studied in detail including a $^*$-representation of $A_\hbar$ in the Bargmann-Fock space and a discussion of star exponentials and coherent states.

1 Introduction

Deformation quantization usually comes in two flavours: formal and strict deformations:

In formal deformation quantization as introduced by [2], see also [20,25] for recent reviews, one considers the Poisson algebra of smooth complex-valued functions $C^\infty(M)$ on a Poisson manifold as observable algebra in classical mechanics. A formal star product $\star$ is a $\mathbb{C}[[\lambda]]$-bilinear associative product for $C^\infty(M)[[\lambda]]$ such that in zeroth order $f \star g$ is the pointwise product and in first order of $\lambda$ the $\star$-commutator gives $i$ times the Poisson bracket. Usually one requires the higher orders to be given by bidifferential operators. The algebra $C^\infty(M)[[\lambda]]$ then serves as a model for the quantum mechanical observables corresponding to the classical system described by $M$. In particular, the formal parameter $\lambda$ corresponds to Planck’s constant $\hbar$ and should be replaced by $\hbar$ whenever one can establish convergence of the formal series.

On the other hand, in strict deformation quantization as introduced in [44], see also [35], one works on the framework of $C^*$-algebras where the dependence of the deformed product $\star_\hbar$ on the deformation parameter $\hbar$ is now required to be continuous. This is made precise using the notion of continuous fields of $C^*$-algebras.

While in the first approach one has very strong existence [18,22–24,34,41] and classification results [3,17,26,34,38,39,47], the formal character of Planck’s constant is, of course, physically not acceptable. Here the second approach is much more appealing as it directly uses the analytical
framework suitable for quantum mechanics. On the other hand, however, a general construction and reasonable classification of strict quantizations seems still to be missing.

Many examples like the global symbol calculus on cotangent bundles [6–8], Berezin-Toeplitz quantization on Kähler manifolds [5,13–16,33] as well as [4] suggest that the formal star products should be seen as asymptotic expansions for \( \hbar \to 0 \) of their convergent counterparts in strict quantization. On the other hand, many formal star products allow for large subalgebras, where the formal series actually converge, whence in some sense the asymptotics can be used again to recover the strict result, a heuristic statement for which a general theorem unfortunately is still missing. The above examples also suggest that there is a framework in between formal and \( C^\ast \)-algebraic, namely one can try to construct deformations of \( C^\infty(M) \) (or suitable subalgebras of it) in the framework of Fréchet or more generally locally convex algebras. Early results in this direction have been obtained in [27,30,36], see also [40]. Moreover, a general set-up of smooth deformations has been established and exemplified in [21], in [43] holomorphic deformations were studied.

The example we are going to discuss will provide an entire holomorphic deformation of a Fréchet subalgebra of \( C^\omega(M) \). More specifically, we consider the most simple phase space \( M = \mathbb{C}^n \) with its canonical Poisson structure \( \{ z^k, z^\ell \} = 2i\delta^k_\ell \) and the formal Wick star product

\[
f \star_{\text{Wick}} g = \sum_{r=0}^{\infty} \frac{(2\lambda)^r}{r!} \sum_{i_1, \ldots, i_r} \frac{\partial^{\ell} f}{\partial z^{i_1} \cdots \partial z^{i_r}} \frac{\partial^{\ell} g}{\partial z^{i_1} \cdots \partial z^{i_r}},
\]

where \( z^1, \ldots, z^n \) are the canonical, global, holomorphic coordinates on \( \mathbb{C}^n \). Our convergence scheme to construct the ‘convergent’ subalgebra \( A_\hbar \) is then based on the crucial observation that the Wick star product enjoys a very strong positivity property [10–12]: every \( \delta \)-functional is a positive \( \mathcal{C}[[\lambda]] \)-linear functional in the sense of formal power series. After choosing a point \( p \in \mathbb{C}^n \) and \( \hbar > 0 \) this will allow us to construct recursively a system of seminorms for which the Wick star product is continuous, thereby defining our algebra \( A_{p,\hbar} \). Moreover, we shall construct, via the GNS construction corresponding to the positive functional \( \delta_p \), a faithful \( \ast \)-representation of \( A_{p,\hbar} \) on a dense subspace of the Bargmann-Fock space giving an interpretation of the \( \delta \)-functionals as coherent states with respect to the Heisenberg group \( H_\hbar \) acting on \( \mathbb{C}^n \). We treat this example in quite some detail as we believe that it may serve as a good starting point for geometric generalizations to Wick star products on Kähler manifolds [9,31] suitable for a bottom-up approach to [5,13–16,33]. Moreover, in a future project we shall discuss the field-theoretic generalization for infinitely many degrees of freedom.

The paper is organized as follows: In Section 2 we briefly recall some basic properties of \( \star_{\text{Wick}} \), the formal GNS construction for \( \delta \)-functionals and the Bargmann-Fock space. Section 3 is devoted to the construction of the seminorms, depending on \( p \in \mathbb{C}^n \) and \( \hbar > 0 \). This gives the space \( A_{p,\hbar} \) which is shown to be a subspace of \( C^\omega(\mathbb{C}^n) \) with a Fréchet topology. In Section 4 we show that \( A_{p,\hbar} \) is a subalgebra of \( C^\omega(\mathbb{C}) \) such that the pointwise product as well as the Poisson bracket are continuous, i.e. \( A_{p,\hbar} \) becomes a Fréchet-Poisson algebra. Moreover, we show that the formula (1.1) for \( \star_{\text{Wick}} \) actually converges on \( A_{p,\hbar} \) in the Fréchet topology resulting in a continuous product. This way, \( A_{p,\hbar} \) becomes a holomorphic deformation. In Section 5 we discuss the dependence on the a priori chosen point \( p \) and on the value \( \hbar > 0 \) of Planck’s constant. It turns out that the translation group acts on \( A_{p,\hbar} \) by inner \( \ast \)-automorphisms whence \( A_{p,\hbar} = A_\hbar \) does not depend on the choice of \( p \). As a side remark we show that the star exponential, see [2], of linear functions converges in the topology of \( A_\hbar \). Moreover, for all values \( \hbar > 0 \) the algebras \( A_\hbar \) are isomorphic in a canonical way. Finally, in Section 6 we show how the GNS construction yields a \( \ast \)-representation of \( A_\hbar \) in the sense of [45] in the Bargmann-Fock space. The action of the Heisenberg group by inner \( \ast \)-automorphisms gives easily the coherent states.
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2 Preliminary results

In this section we shall collect some well-known results on the Wick star product which we shall use in the sequel, see e.g. [9, 10].

On the classical phase space \( \mathbb{R}^{2n} \cong \mathbb{C}^n \) with standard symplectic form \( \omega = \frac{1}{\hbar} \, dz^k \wedge dz^{\bar{k}} \) one defines the formal Wick star product by

\[
f \star_{\text{Wick}} g = \sum_{r=0}^{\infty} \frac{(2\lambda)^r}{r!} \sum_{i_1, \ldots, i_r} \frac{\partial^r f}{\partial z^{i_1} \partial z^{\bar{i}_1} \cdots \partial z^{i_r} \partial z^{\bar{i}_r}} \frac{\partial^r g}{\partial z^{\bar{i}_1} \cdots \partial z^{\bar{i}_r}},
\]

where \( f, g \in C^\infty(\mathbb{C}^n)[[\lambda]] \), the formal parameter \( \lambda \) corresponds to Planck's constant \( \hbar \) without any further prefactors and \( z, \bar{z} \) denote the usual global holomorphic/anti-holomorphic coordinates on \( \mathbb{C}^n \). Then \( \star_{\text{Wick}} \) is known to be an associative star product quantizing the canonical Poisson bracket corresponding to \( \omega \). It has the separation of variable property in the sense of Karabegov [31, 32] and is in fact the name-giving example of a star product of Wick type in the sense of [9]. Moreover, \( \star_{\text{Wick}} \) is Hermitian

\[
f \star_{\text{Wick}} g = \overline{g} \star_{\text{Wick}} f,
\]

where according to our interpretation of \( \lambda \) the formal parameter is defined to be real \( \lambda = \lambda \). We shall also frequently make use of multiindex notation: Let \( R = (r_1, \ldots, r_n) \in \mathbb{N}^n \) be a multiindex, then one defines \( |R| = r_1 + \cdots + r_n \), \( R! = r_1! \cdots r_n! \) as well as \( z^R = (z^1)^{r_1} \cdots (z^n)^{r_n} \) etc. Moreover, we define \( R \leq L \) if \( r_i \leq l_i \) for all \( i = 1, \ldots, n \). The Wick star product can equivalently be written as

\[
f \star_{\text{Wick}} g = \sum_{R=0}^{\infty} \frac{(2\lambda)|R|}{R!} \frac{\partial^{|R|} f}{\partial z^R} \frac{\partial^{|R|} g}{\partial \bar{z}^R}.
\]

The Wick star product enjoys a very strong positivity property which e.g. the Weyl-Moyal star product does not share: If \( \delta_p : C^\infty(\mathbb{C}^n)[[\lambda]] \rightarrow \mathbb{C}[[\lambda]] \) denotes the evaluation functional at \( p \in \mathbb{C}^n \) then we have

\[
\delta_p(\overline{f} \star_{\text{Wick}} f) = \sum_{R=0}^{\infty} \frac{(2\lambda)|R|}{R!} \left| \frac{\partial^{|R|} f}{\partial \bar{z}^R} (p) \right|^2 \geq 0,
\]

where the positivity is understood in the sense of the canonical ring ordering of \( \mathbb{R}[[\lambda]] \), see [10, 46] for a detailed discussion on the physical relevance of this notion of positivity. This very strong positivity property is not true for general Hermitian star products: instead one has to add 'quantum corrections' to a given classically positive functional (here \( \delta_p \)) in order to obtain a positive functional with respect to the star product. Since positive functionals play the role of states the above simple observation implies that for the Wick star product any classical state defines a quantum state without any quantum correction, see [46] for a detailed discussion on the general situation. In fact, the Wick star product is used in an essential way for proving that for an arbitrary Hermitian star product one can always construct quantum corrections for a classical state, see [11, 12].

Since \( \delta_p \) is a positive functional for \( \star_{\text{Wick}} \) we have a corresponding GNS representation of the Wick star product algebra \( C^\infty(\mathbb{C}^n)[[\lambda]] \). In fact, this example was one of the first examples of
GNS constructions in deformation quantization. Since we need the construction in the following, we briefly review the results from [10]. The Gel’fand ideal \( \mathcal{J}_p \) of \( \delta_p \) is given by

\[
\mathcal{J}_p = \left\{ f \left| \delta_p(f \star_{\text{Wick}} f) = 0 \right. \right\} = \left\{ f \left| \frac{\partial^{\left| R \right|} f}{\partial z^R}(p) = 0 \text{ for all } R \right. \right\}. 
\]

The GNS pre Hilbert space \( \mathcal{H}_p = C^\infty(\mathbb{C}^n)[[\lambda]]/\mathcal{J}_p \) can canonically be identified with

\[
\mathcal{H}_{BF} = \mathbb{C}[[\bar{y}^1, \ldots, \bar{y}^n]][[\lambda]] 
\]

with the \( \mathbb{C}[[\lambda]] \)-valued positive definite inner product

\[
\langle \phi, \psi \rangle = \sum_{R=0}^\infty \frac{(2\lambda)^{|R|}}{R!} \frac{\partial^{\left| R \right|} \phi}{\partial \bar{y}^R}(0) \frac{\partial^{\left| R \right|} \psi}{\partial \bar{y}^R}(0), 
\]

where the identification of a class \( \psi_f \in \mathcal{H}_p \) is given by its formal anti-holomorphic Taylor expansion, i.e.

\[
\mathcal{H}_p \ni \psi_f \mapsto \sum_{R=0}^\infty \frac{1}{R!} \frac{\partial^{\left| R \right|} f}{\partial z^R}(p) \bar{y}^R, 
\]

where \( \psi_f \) denotes the equivalence class of the function \( f \) in \( \mathcal{H}_p \). Then the GNS representation on \( \mathcal{H}_p \) defined by \( \pi_p(f) \psi_g = \psi_{f \star_{\text{Wick}} g} \) is translated into

\[
e_p(f) = \sum_{R,S=0}^\infty \frac{(2\lambda)^{|R|}}{R!S!} \frac{\partial^{\left| R \right|+\left| S \right|} f}{\partial z^R \partial \bar{y}^S}(p) \bar{y}^S \frac{\partial^{\left| R \right|}}{\partial \bar{y}^R},
\]

via the unitary map (2.8). In particular, for \( p = 0 \) we see that this gives the formal analog of the usual Bargmann-Fock space and the Bargmann-Fock representation: indeed, recall that the Bargmann-Fock space is the Hilbert space

\[
\mathcal{H}_{BF} = \left\{ f \in \mathcal{O}(\mathbb{C}^n) \left| \frac{1}{(2\pi \hbar)^n} \int |f(\bar{z})|^2 e^{-\frac{|z|^2}{2\hbar}} dz d\bar{z} < \infty \right. \right\} 
\]

of anti-holomorphic functions which are square-integrable with respect to the Gaussian measure, see [1]. Then it is well-known that \( \mathcal{H}_{BF} \) is actually a closed subspace of the \( L^2 \)-space for this measure and hence a Hilbert space itself. Moreover, the \( L^2 \)-inner product can be evaluated by the same formula (2.7) if one replaces \( \lambda \) by \( \hbar \), where the series now converges absolutely. A Hilbert basis for \( \mathcal{H}_{BF} \) is given by the monomials

\[
e_R(\bar{z}) = \frac{1}{\sqrt{(2\hbar)^{|R|} R!}} \bar{z}^R. 
\]

From (2.7) we see that the Hilbert space \( \mathcal{H}_{BF} \) can be interpreted as the space of those anti-holomorphic functions whose Taylor coefficients at 0 form a sequence in a (weighted) \( \ell^2 \)-space.

### 3 Construction of the Fréchet space \( \mathcal{A}_{p,\hbar} \)

The motivation for our convergence scheme is rather simple: we fix \( \hbar > 0 \) and we fix a point \( p \in \mathbb{C}^n \). Then we are looking for a subalgebra of \( C^\infty(\mathbb{C}^n)[[\lambda]] \) such that \( \delta_p(f) \) converges for \( \lambda = \hbar \). Though this looks rather innocent at the beginning, we obtain a hierarchy of conditions: For \( f, g \) in our algebra we want \( f \star_{\text{Wick}} g \) to be in the algebra as well whence \( \delta_p(f \star_{\text{Wick}} g) \) has to converge for \( \lambda = \hbar \).
as well. The idea is now to estimate the convergence of \( \delta_p(f \ast_{\text{Wick}} g) \mid_{\lambda = h} \) by \( \delta_p(f \ast_{\text{Wick}} f) \mid_{\lambda = h} \) and \( \delta_p(\bar{f} \ast_{\text{Wick}} g) \mid_{\lambda = h} \) using the Cauchy-Schwarz inequality for the positive \( \delta \)-functional. An iteration of this procedure will lead to countably many unary conditions for a function \( f \) to belong to our algebra, which we shall interpret as seminorms determining the algebra. This approach can hence be used as a heuristic motivation for the following definition of the seminorms.

**Definition 3.1** Let \( R, S \in \mathbb{N}^n \) be multiindices, \( m \in \mathbb{N} \) and \( \ell = 0, \ldots, 2^m - 1 \). Then we define recursively for \( f \in C^\infty(\mathbb{C}^n) \)

\[
h^{p,h}_{0,0,R,S}(f) = (2h)^{|R|+|S|} \left( \frac{\partial^{|R|+|S|} f}{\partial z^R \partial \bar{z}^S} \ast_{\text{Wick}} \frac{\partial^{|R|+|S|} f}{\partial z^R \partial \bar{z}^S} \right)(p) \bigg|_{\lambda = h} \tag{3.1}
\]

and

\[
h^{p,h}_{m,\ell,R,S}(f) = \begin{cases} \sum_{N=0}^\infty \frac{1}{N!} \left( \sum_{I=0}^R \sum_{J=0}^S \binom{R}{I} \binom{N+S}{J} h^{p,h}_{m-1,\ell/2,I,I}(f) \right)^2 & \ell \text{ even} \\ \sum_{N=0}^\infty \frac{1}{N!} \left( \sum_{I=0}^R \sum_{J=0}^S \binom{R}{I} \binom{N+S}{J} h^{p,h}_{m-1,(\ell-1)/2,I,I}(f) \right)^2 & \ell \text{ odd}. \end{cases} \tag{3.2}
\]

The precise form of the recursive definition will become either clear by following the above heuristic argument in detail or from the proof of Proposition 3.7: the binomial coefficients arise from the Leibniz rule.

Thanks to the positivity (2.4) of the \( \delta \)-functional it is clear that \( h^{p,h}_{0,0,R,S}(f) \) either converges absolutely or diverges absolutely to \(+\infty\), as it is a series consisting of non-negative terms only. By induction, the same is true for all other \( h^{p,h}_{m,\ell,R,S}(f) \). Hence we have

\[
h^{p,h}_{m,\ell,R,S}(f) \in [0, +\infty], \tag{3.3}
\]

where ‘convergence’ is always absolute and does not depend on the order of summation.

**Definition 3.2** For \( f \in C^\infty(\mathbb{C}^n) \) we define

\[
\|f\|_{m,\ell,R,S}^{p,h} = \sqrt{h^{p,h}_{m,\ell,R,S}(f)} \in [0, +\infty]. \tag{3.4}
\]

Moreover, we define

\[
\widetilde{A}_{p,h} = \left\{ f \in C^\infty(\mathbb{C}^n) \mid \|f\|_{m,\ell,R,S}^{p,h} < \infty \text{ for all } m, \ell, R, S \right\}. \tag{3.5}
\]

The first step is now to show that \( \tilde{\mathcal{A}}_{p,h} \) is a vector space and that the \( \|\cdot\|_{m,\ell,R,S}^{p,h} \) are seminorms on \( \tilde{\mathcal{A}}_{p,h} \). This will be a consequence of the following proposition:

**Proposition 3.3** The maps \( \|\cdot\|_{m,\ell,R,S}^{p,h} : C^\infty(\mathbb{C}^n) \longrightarrow [0, +\infty] \) enjoy the following properties:

i.) \( \|\alpha f\|_{m,\ell,R,S}^{p,h} = |\alpha| \|f\|_{m,\ell,R,S}^{p,h} \) for \( \alpha \in \mathbb{C} \).

ii.) \( \|f + g\|_{m,\ell,R,S}^{p,h} \leq \|f\|_{m,\ell,R,S}^{p,h} + \|g\|_{m,\ell,R,S}^{p,h} \).
A

\text{Theorem 3.6}

Let \( \text{Corollary 3.4} \)

Then we have the following simple corollary:

\( \text{Definition 3.5} \)

\( \text{Proof:} \) The first part is clear by a simple induction. For the second part the case \( m = 0 \) follows directly from Minkowski’s inequality. Then \( m > 0 \) is shown inductively by using again Minkowski’s inequality, for both cases of odd and even \( \ell \). The remaining inequalities between the \( \| f \|_{m,\ell,R,S}^{p,h} \) for different values of the parameters are simply obtained by omitting all but one term for a specific \( N \) in the defining summations of (3.2). For example, in the third part one considers \( N = 0 \) and \( I = R, J = S \) only, while for the fourth and fifth one uses \( N = S = J \) and \( N = R = I \).

Thus the maps \( \| \cdot \|_{m,\ell,R,S}^{p,h} \) restricted to \( \tilde{A}_{p,h} \) give indeed seminorms. Moreover, the labels \( R, S \) play only a minor role thanks to the estimates in the last part of the proposition. This motivates the following definitions. For \( f \in \mathbb{C}^\infty(\mathbb{C}^n) \) we define

\( \| f \|_{m,\ell}^{p,h} = \| f \|_{m,\ell,0,0}^{p,h} \)

\begin{equation}
\| f \|_{m}^{p,h} = \max_{0 \leq \ell \leq 2^m-1} \left\{ \| f \|_{m,\ell}^{p,h} \right\}.
\end{equation}

Then we have the following simple corollary:

\textbf{Corollary 3.4} The set \( \tilde{A}_{p,h} \subseteq \mathbb{C}^\infty(\mathbb{C}^n) \) is a subvector space and the \( \| \cdot \|_{m,\ell,R,S}^{p,h} \) are seminorms on \( \tilde{A}_{p,h} \). Moreover, the seminorms \( \| \cdot \|_{m,\ell}^{p,h} \) as well as the seminorms \( \| \cdot \|_{m}^{p,h} \) determine the same locally convex topology on \( \tilde{A}_{p,h} \).

In the following, we shall equip \( \tilde{A}_{p,h} \) always with this locally convex topology induced by the seminorms \( \| \cdot \|_{m,\ell,R,S}^{p,h} \). However, this topology has one unpleasant feature: it is non-Hausdorff as a function \( f \) whose Taylor expansion at \( p \) vanishes identically has clearly \( \| f \|_{m,\ell,R,S}^{p,h} = 0 \) for all parameters \( m, \ell, R, S \). On the other hand, as one sees already from the seminorm \( \| \cdot \|_1^{p,h} \) the functions with vanishing \( \infty \)-jet \( j_\infty^p f \) at \( p \) are the only functions with this property. Thus we identify them to be zero in order to have a Hausdorff topology:

\textbf{Definition 3.5} We define

\( A_{p,h} = \tilde{A}_{p,h}/ \{ f \in \mathbb{C}^\infty(\mathbb{C}^n) \mid j_\infty^p f = 0 \} \),

and equip \( A_{p,h} \) with the induced locally convex topology determined by the seminorms \( \| \cdot \|_{m,\ell,R,S}^{p,h} \) (or equivalently, by the seminorms \( \| \cdot \|_{m}^{p,h} \)).

Clearly, \( A_{p,h} \) is now a Hausdorff locally convex topological vector space. The following theorem shows that the abstract quotient can be viewed as a certain subspace of the real-analytic functions on \( \mathbb{C}^n \):

\textbf{Theorem 3.6} Let \( h > 0 \) and \( p \in \mathbb{C}^n \).
i.) $A_{p,h}$ is a Hausdorff locally convex topological vector space.

ii.) Every class $[f] \in A_{p,h}$ has a unique real-analytic representative $f \in C^\omega(\mathbb{C}^n)$. Therefore, we identify $A_{p,h}$ with the corresponding subspace of $C^\omega(\mathbb{C}^n)$ from now on.

iii.) Every function $f \in A_{p,h}$ has a unique extension to a function $\hat{f} \in \mathcal{O} \times \overline{\mathcal{O}}(\mathbb{C}^n \times \mathbb{C}^n)$, i.e. holomorphic in the first and anti-holomorphic in the second argument, such that

$$f = \Delta^* \hat{f},$$

where $\Delta : \mathbb{C}^n \ni z \mapsto (z, z) \in \mathbb{C}^n \times \mathbb{C}^n$ is the diagonal.

iv.) Any $f \in C^\omega(\mathbb{C}^n)$ such that there exist constants $a, b, c > 0$ with

$$\left| \frac{\partial^{[R]+|S|} f}{\partial z^R \partial \overline{z}^S} (p) \right| \leq c a^{|R| b^{|S|}}$$

belongs to $A_{p,h}$. In particular $\mathbb{C}[z, \overline{z}] \subseteq A_{p,h}$.

**PROOF:** The first part is clear. For the second, we consider

$$\|f\|^2_{p,h,1,0,0} \geq \frac{1}{R!} (\|f\|^2_{0,0,R,0}) \geq \frac{(2h)^{2|R|+2|S|} R!}{2!(S!)^2} \left| \frac{\partial^{[R]+|S|} f}{\partial z^R \partial \overline{z}^S} (p) \right|^4$$

for all $R, S$. Thus we obtain

$$\left| \frac{\partial^{[R]+|S|} f}{\partial z^R \partial \overline{z}^S} (p) \right| \leq \|f\|^p_{1,1,0,0} \sqrt[4]{R!} \frac{\sqrt[S!]}{(2h)^{|R|+|S|}}$$

But this implies that the series

$$\hat{f}(z, \overline{w}) = \sum_{R,S=0}^{\infty} \frac{1}{R! S!} \frac{\partial^{[R]+|S|} f}{\partial z^R \partial \overline{z}^S} (p)(z - p)^R (\overline{w} - \overline{p})^S$$

converges for all $z, w \in \mathbb{C}^n$. Thus $\hat{f} \in \mathcal{O} \times \overline{\mathcal{O}}(\mathbb{C}^n \times \mathbb{C}^n)$ and clearly $\Delta^* \hat{f}$ is in the same equivalence class as $f$. This shows the second and third part. Now assume $f \in C^\omega(\mathbb{C}^n)$ satisfies (3.10). Then

$$h^p_{0,0,R,S} (f) \leq c^2 e^{2h b^2} (2h a^2)^{|R|} (2h b^2)^{|S|},$$

whence there are constants $c_{0,0} = e^{2h b^2}$, $a_{0,0} = 2h a^2$ and $b_{0,0} = 2h b$ such that

$$h^p_{0,0,R,S} (f) \leq c_{0,0} a_{0,0}^{|R|} b_{0,0}^{|S|}.$$

We claim that for all $m, \ell$ there are constants $a_{m,\ell}$, $b_{m,\ell}$ and $c_{m,\ell}$ such that

$$h^p_{m,\ell,R,S} (f) \leq c_{m,\ell} a_{m,\ell}^{|R|} b_{m,\ell}^{|S|}.$$

Indeed, a recursive argument shows that

$$c_{m,\ell} = c_{m-1,\ell/2}^2 e^{a_{m-1,\ell/2}^2}, \quad a_{m,\ell} = (1 + a_{m-1,\ell/2})^2, \quad b_{m,\ell} = (1 + b_{m-1,\ell/2})^2$$
for even $\ell$ and
\[
c_{m,\ell} = c_{m-1,(\ell-1)/2}^2 e^{n(1 + a_{m-1,(\ell-1)/2})^2}, \quad a_{m,\ell} = (1 + b_{m-1,(\ell-1)/2})^2, \quad b_{m,\ell} = (1 + a_{m-1,(\ell-1)/2})^2
\]
for odd $\ell$ will do the job. But then all seminorms of $f$ are finite. \hfill \blacksquare

Since the polynomials are in $A_{p,h}$ we shall make intense use of them. The next proposition gives a first hint on the continuity of the Wick product. Here and in the following we shall use the notation $\star_{\text{Wick}}^h$ for the Wick product with $\lambda$ being replaced by $h$.

**Proposition 3.7** Let $f \in C^\infty(\mathbb{C}^n)$ and let $g \in \mathbb{C}[z, \overline{z}]$ be a polynomial.

i.) $f \star_{\text{Wick}}^h g$ is a finite sum and thus a well-defined smooth function.

ii.) $\|f \star_{\text{Wick}}^h g\|_{m,\ell,R,S}^{p,h} \leq \|f\|_{m+1,2^m+\ell,R,S}^{p,h} \|g\|_{m+1,\ell,R,S}^{p,h}$.

**Proof:** The first part is clear from the explicit form of $\star_{\text{Wick}}^h$. For the second part we first have by the Cauchy-Schwarz inequality
\[
\left| \sum_{N=0}^{\text{finite}} \frac{(2h)^{|N|}}{N!} \frac{\partial^{|N|} f}{\partial z^N}(p) \frac{\partial^{|N|} g}{\partial \overline{z}^N}(p) \right|^2 \leq \left( \sum_{N=0}^{\infty} \frac{(2h)^{|N|}}{N!} \left| \frac{\partial^{|N|} f}{\partial z^N}(p) \right|^2 \right) \left( \sum_{N=0}^{\infty} \frac{(2h)^{|N|}}{N!} \left| \frac{\partial^{|N|} g}{\partial \overline{z}^N}(p) \right|^2 \right) \quad (\ast)
\]

\[
= h_{0,0,0,0}^{p,h}(f) h_{0,0,0,0}^{p,h}(g).
\]
Now partial derivatives are still *derivations* of $\star_{\text{Wick}}$ and hence of $\star_{\text{Wick}}^h$ if one of the functions is a polynomial. This allows the following computation

$$h_{0,0,R,S}^{p,h}(\mathcal{F} \star_{\text{Wick}}^h g)$$

$$= \sum_{N=0}^{\infty} \frac{(2h)^{|N|+|R|+|S|}}{N!} \frac{\partial^{N+S} \partial_z \partial \mathcal{F} \star_{\text{Wick}}^h g}{(N+1)^2} (p)$$

$$= \sum_{N=0}^{\infty} \frac{(2h)^{|N|+|R|+|S|}}{N!} \sum_{I=0}^{R} \sum_{J=0}^{N+S} (R_I) (N+S-J) \frac{\partial^{N+S-J} \partial_z \partial \mathcal{F} \star_{\text{Wick}}^h g}{(N+1)^2} (p)$$

$$\leq \sum_{N=0}^{\infty} \frac{(2h)^{|N|+|R|+|S|}}{N!} \left( \sum_{I=0}^{R} \sum_{J=0}^{N+S} (R_I) (N+S-J) \frac{\partial^{N+S-J} \partial_z \partial \mathcal{F} \star_{\text{Wick}}^h g}{(N+1)^2} (p) \right)^2$$

$$\leq \sum_{N=0}^{\infty} \frac{1}{N!} \left( \sum_{I=0}^{R} \sum_{J=0}^{N+S} (R_I) (N+S-J) h_{0,0,I,J}^p (f) \right)^2 \left( \sum_{I=0}^{R} \sum_{J=0}^{N+S} (R_I) (N+S-J) h_{0,0,I,J}^p (g) \right)^2$$

$$\leq \sqrt{\sum_{N=0}^{\infty} \frac{1}{N!} \left( \sum_{I=0}^{R} \sum_{J=0}^{N+S} (R_I) (N+S-J) h_{0,0,I,J}^p (f) \right)^2} \sqrt{\sum_{N=0}^{\infty} \frac{1}{N!} \left( \sum_{I=0}^{R} \sum_{J=0}^{N+S} (R_I) (N+S-J) h_{0,0,I,J}^p (g) \right)^2}$$

$$= \sqrt{h_{1,0,R,S}^{p,h} (f)} \sqrt{h_{1,0,R,S}^{p,h} (g)},$$

which proves the second part for the case $m = 0$. Note the necessity that one function (we have chosen $g$) is polynomial since otherwise the ‘function’ $\mathcal{F} \star_{\text{Wick}}^h g$ is a priori not defined as a smooth function. The general case is now obtained by a straightforward induction on $m$ only using the Cauchy-Schwarz inequality. $
$

**Corollary 3.8** The pointwise complex conjugation is a continuous map $A_{p,h} \rightarrow A_{p,h}$. We have

$$\| \mathcal{F} \|^p_{m,t,R,S} \leq \| 1 \|^p_{m+1,t,R,S} \| f \|^p_{m+1,2^m,t,R,S} \cdot$$ (3.12)

Note however that the seminorms themselves are not invariant under complex conjugation $f \leftrightarrow \bar{f}$, though the complex conjugation is continuous.

We also note that the second part of the proposition already shows some nice continuity properties of the Wick star product, at least if one function is a polynomial. Note however, that the above argument will not extend to arbitrary $f, g$ whence we shall need another route.

**Theorem 3.9** The polynomials $\mathbb{C}[z, \bar{z}] \subseteq A_{p,h}$ are dense. More specifically, the Taylor expansion

$$f(z, \bar{z}) = \sum_{I,J=0}^{\infty} \frac{1}{I!J!} \frac{\partial^{I+|J|} f}{\partial z^I \partial \bar{z}^J} (p)(z - p)^I (\bar{z} - \bar{p})^J$$ (3.13)
of $f \in A_{p,h} \subseteq C^\omega(C^n)$ converges unconditionally to $f$ with respect to the topology of $A_{p,h}$. In particular, the truncated Taylor polynomials

$$f^{(N,M)}(z, \overline{z}) = \sum_{I=0}^{N} \sum_{J=0}^{M} \frac{1}{I!J!} \frac{\partial^{I+J} f}{\partial z^I \partial \overline{z}^J} (p)(z - p)^I (\overline{z} - \overline{p})^J \quad (3.14)$$

converge unconditionally to $f$ in the topology of $A_{p,h}$.

**Proof:** Clearly, we only have to show the later statement. First we rewrite the seminorms $\| \cdot \|_{m,\ell,R,S}^{p,h}$ in the following ‘measure-theoretic’ way

$$h_{m,\ell,R,S}^{p,h}(f) = \sum_{I_1,J_1=0}^{\infty} \cdots \sum_{I_s,J_s=0}^{\infty} \mu_{m,\ell,R,S}^{p,h} \left| \frac{\partial^{I_1+J_1} f}{\partial z^{I_1} \partial \overline{z}^{J_1}} (p) \right|^2 \cdots \left| \frac{\partial^{I_s+J_s} f}{\partial z^{I_s} \partial \overline{z}^{J_s}} (p) \right|^2,$$

where $\mu_{m,\ell,R,S}^{p,h}$ are numerical constants not depending on $p$ and $s = 2^m$. This can be seen by induction easily. The concrete form of the coefficients $\mu_{m,\ell,R,S}^{p,h}$ is not important for the following argument. Now we define for $f \in C^\omega(C^n)$ a non-negative function $\phi_p(f) : \mathbb{N}^{2sn} \rightarrow [0, \infty)$ by

$$\phi_p(f)(I_1, \ldots, I_s, J_1, \ldots, J_s) = \left| \frac{\partial^{I_1+J_1} f}{\partial z^{I_1} \partial \overline{z}^{J_1}} (p) \right|^2 \cdots \left| \frac{\partial^{I_s+J_s} f}{\partial z^{I_s} \partial \overline{z}^{J_s}} (p) \right|^2.$$

Then we can interpret $h_{m,\ell,R,S}^{p,h}(f)$ as the ‘integral’ of $\phi_p(f)$ over $\mathbb{N}^{2sn}$ with respect to the weighted counting measure $d \mu_{m,\ell,R,S}^{p,h}$ determined by the coefficients $\mu_{m,\ell,R,S}^{p,h}$, i.e.

$$h_{m,\ell,R,S}^{p,h}(f) = \int_{\mathbb{N}^{2sn}} \phi_p(f) \, d \mu_{m,\ell,R,S}^{p,h}.$$

Now let $K, L \subseteq \mathbb{N}^n$ be finite subsets and define the polynomial

$$f^{(K,L)}(z, \overline{z}) = \sum_{I \in K} \sum_{J \in L} \frac{1}{I!J!} \frac{\partial^{I+J} f}{\partial z^I \partial \overline{z}^J} (p)(z - p)^I (\overline{z} - \overline{p})^J.$$

Then we clearly have

$$\phi_p \left( f - f^{(K,L)} \right)(I_1, \ldots, I_s, J_1, \ldots, J_s) = \begin{cases} 0 & \text{if } I_1, \ldots, I_s \in K, J_1, \ldots, J_s \in L \\ \phi_p(f) & \text{else,} \end{cases} \quad (3.15)$$

Thus when $K, L$ exhaust $\mathbb{N}^n$, the function $\phi_p(f - f^{(K,L)})$ converges pointwise and monotonically to zero, i.e. for $K \subseteq K'$ and $L \subseteq L'$ we have

$$\phi_p \left( f - f^{(K,L)} \right) \geq \phi_p \left( f - f^{(K',L')} \right),$$

and for all $I_1, \ldots, I_s, J_1, \ldots, J_s$

$$\lim_{K,L \rightarrow \mathbb{N}^n} \phi_p \left( f - f^{(K,L)} \right)(I_1, \ldots, I_s, J_1, \ldots, J_s) = 0$$

in these sense of nets convergence for the net of finite subsets of $\mathbb{N}^n$. Now an order of summation in (3.13) corresponds to a strictly increasing sequence $K_i \times L_i \subseteq \mathbb{N}^n \times \mathbb{N}^n$ which exhausts $\mathbb{N}^n \times \mathbb{N}^n$. Then

$$\lim_{i \rightarrow \infty} h_{m,\ell,R,S}^{p,h}(f - f^{(K_i,L_i)}) = \lim_{i \rightarrow \infty} \int \phi_p \left( f - f^{(K_i,L_i)} \right) \, d \mu_{m,\ell,R,S}^{p,h} = 0$$
by dominated convergence. But this is equivalent to the unconditional convergence of (3.13), see also [29, Sect. 14.6, Thm. 1]. ■

We now come to the main result of this section:

**Theorem 3.10** The locally convex topology of \( \mathcal{A}_{p,h} \) is complete, i.e. \( \mathcal{A}_{p,h} \) is a Fréchet space.

**Proof:** Since the topology is determined by countably many seminorms we only have to consider Cauchy sequences and not Cauchy nets. Thus let \( f_i \in \mathcal{A}_{p,h} \) be a Cauchy sequence, i.e. for all \( \epsilon > 0 \) we find a \( K(m, \ell, R, S, \epsilon) \) such that for \( i, j \geq K \) we have

\[
\| f_i - f_j \|_{m, \ell, R, S}^h < \epsilon.
\]

We first evaluate this for \( m = 1, \ell = 1, R, S = 0 \). Let

\[
f_i(z, \bar{z}) = \sum_{I, J = 0}^{\infty} \frac{1}{I! J!} a_{I,J}^{(i)} (z - p)^I (\bar{z} - \bar{p})^J
\]

be the Taylor expansion of \( f_i \) then from (3.11) we see that the Taylor coefficients \( a_{I,J}^{(i)} \) form a Cauchy sequence for each \( I, J \). Denote their limit by

\[
a_{I,J} = \lim_{i \to \infty} a_{I,J}^{(i)}
\]

and define

\[
f(z, \bar{z}) = \sum_{I, J = 0}^{\infty} \frac{1}{I! J!} a_{I,J} (z - p)^I (\bar{z} - \bar{p})^J.
\]

Then we want to show \( f \in \mathcal{A}_{p,h} \) and \( f_i \to f \). To this end we first choose a smooth function \( \tilde{f} \in C^\infty(\mathbb{C}^n) \) with Taylor coefficients at \( p \) given by (\( \ast \)), which is possible thanks to the Borel Lemma. Since \( f_i \) is a Cauchy sequence with respect to \( \| \cdot \|_{m, \ell, R, S}^h \) the sequence of seminorms \( \| f_i \|_{m, \ell, R, S}^h \) stays bounded as \( i \to 0 \). Thus we can again use the measure-theoretic point of view and write with the notation from the previous proof

\[
h_{m, \ell, R, S}^{p,h}(\tilde{f}) = \int_{N_{2n}} \phi(\tilde{f}) \, d\mu_{m, \ell, R, S, h}^{p,h} = \int_{N_{2n}} \lim_{i \to \infty} \phi(f_i) \, d\mu_{m, \ell, R, S, h}^{p,h} = \int_{N_{2n}} \liminf_i \phi(f_i) \, d\mu_{m, \ell, R, S, h}^{p,h} \\
\leq \liminf_i \int_{N_{2n}} \phi(f_i) \, d\mu_{m, \ell, R, S, h}^{p,h} \\
\leq \sup_i (\| f_i \|_{m, \ell, R, S}^{p,h})^{2m+1} < \infty,
\]

by Fatou’s Lemma. Thus \( \tilde{f} \in \bar{\mathcal{A}}_{p,h} \) and hence \( f \) as in (\( \ast \)) is the unique real-analytic representative in \( \mathcal{A}_{p,h} \). Next we compute using (3.14)

\[
\| f - f_i \|_{m, \ell, R, S}^{p,h} \\
\leq \| (f - f_i) - (f - f_i) \|_{m, \ell, R, S}^{(N,M)}^{p,h} + \| (f(N,M) - f_j^{(N,M)}) \|_{m, \ell, R, S}^{p,h} + \| f_j^{(N,M)} - f_i^{(N,M)} \|_{m, \ell, R, S}^{p,h} \\
\leq \| (f - f_i) - (f - f_i) \|_{m, \ell, R, S}^{(N,M)}^{p,h} + \| (f(N,M) - f_j^{(N,M)}) \|_{m, \ell, R, S}^{p,h} + \| f_j - f_i \|_{m, \ell, R, S}^{p,h}.
\]
Now we fix $\epsilon > 0$ and $K$ such that the last term is smaller than $\epsilon/3$ for $i, j > K$. For such an $i$ we fix $N, M$ such that the first term is smaller $\epsilon/3$ thanks to Theorem 3.9. Finally, for this choice of $N, M$ we can find $j$ large enough that the second term is smaller $\epsilon/3$ since we have two polynomials of fixed degree $(N, M)$ whose coefficients converge. This finally proves $f_i \to f$ with respect to $\| \cdot \|_{m, \ell, R, S}$. \hfill \blacksquare

4 The continuity of $\star_{\text{Wick}}$

From Proposition 3.7 we know that on the subspace $\mathbb{C}[z, \overline{z}] \subseteq \mathcal{A}_{p, h}$ the Wick star product $\star_{\text{Wick}}^h$ is well-defined and continuous with respect to the topology of $\mathcal{A}_{p, h}$. Since on the other hand $\mathbb{C}[z, \overline{z}]$ is a dense subspace by Theorem 3.9 the Wick star product extends uniquely to a continuous product on $\mathcal{A}_{p, h}$ which thereby becomes a Fréchet algebra. However, from this abstract extension we cannot yet conclude whether the formula (2.3) with $\lambda$ being replaced by $h$ is still true. Thus we need an additional argument.

Let $\ell \in \{0, \ldots, 2^m - 1\}$ be written as $\ell = \ell_{m-1}2^{m-1} + \cdots + \ell_12 + \ell_0$ with $\ell_{m-1}, \ldots, \ell_0 \in \{0, 1\}$. Then we define $\epsilon_\ell = (-1)^{\ell_{m-1}+\cdots+\ell_0}$. With this notation we can prove the following continuity property of the partial derivatives:

**Proposition 4.1** Let $f \in C^\infty(\mathbb{C}^n)$ then we have

$$\sqrt{(2h)^{|I|+|J|}} \left| \frac{\partial^{|I|+|J|} f}{\partial z^I \partial \overline{z}^J} \right|_{m, \ell, R, S, S}^{p, h} \leq \left\{ \begin{array}{ll} \| f \|_{m, \ell, R+I, S, J}^{p, h} & \text{for } \epsilon_\ell = +1 \\
\| f \|_{m, \ell, R+I, S, J}^{p, h} & \text{for } \epsilon_\ell = -1. \end{array} \right. \quad (4.1)$$

**Proof:** Clearly, for $m = 0$ (and hence $\ell = 0$) we even have the equality

$$\sqrt{(2h)^{|I|+|J|}} \left| \frac{\partial^{|I|+|J|} f}{\partial z^I \partial \overline{z}^J} \right|_{0, 0, R, S}^{p, h} = \| f \|_{0, 0, R+I, S, J}^{p, h}$$

by the very definition of $\| \cdot \|_{0, 0, R, S}^{p, h}$. Thus we prove the claim by induction on $m$. Let first be $\ell$ even and $\epsilon_\ell = +1$. Then $\ell/2 = +1$ as well and we have by induction

$$h_{m, \ell, R, S}^{p, h} \left( \sqrt{(2h)^{|I|+|J|}} \frac{\partial^{|I|+|J|} f}{\partial z^I \partial \overline{z}^J} \right) \leq \sum_{N=0}^\infty \frac{1}{N!} \left( \sum_{K=0}^R \sum_{L=0}^{N+S} \binom{R}{K} \binom{N+S}{L} h_{m-1, \ell/2, K, L}^{p, h} (f) \right)^2 \leq \sum_{N=0}^\infty \frac{1}{N!} \left( \sum_{K=I}^{R+I} \sum_{L=J}^{N+S+J} \binom{R+I}{K} \binom{N+S+J}{L} h_{m-1, \ell/2, K, L}^{p, h} (f) \right)^2 \leq \sum_{N=0}^\infty \frac{1}{N!} \left( \sum_{K=I}^{R+I} \sum_{L=J}^{N+S+J} \binom{R+I}{K} \binom{N+S+J}{L} h_{m-1, \ell/2, K, L}^{p, h} (f) \right)^2 \leq h_{m, \ell, R+I, S, J}^{p, h} (f).$$

For $\epsilon_\ell = \epsilon_{\ell/2} = -1$ we get $h_{m, \ell, R+I, S, J}^{p, h} (f)$ instead and the two cases with odd $\ell$ are analogous. \hfill \blacksquare

From Theorem 3.9 we see that the polynomials

$$\zeta_p^{IJ}(z, \overline{z}) = (z-p)^I(\overline{z}-\overline{p})^J \quad (4.2)$$

form a countable unconditional topological basis for $\mathcal{A}_{p, h}$. The proposition now implies that we even have a Schauder basis:
Corollary 4.2 The polynomials \( \{ \zeta_p^{IJ} \}_{I,J \in \mathbb{N}^n} \) form an unconditional Schauder basis for \( A_{p,h} \).

Proof: By Theorem 3.9 we have the unconditional convergence

\[
  f = \sum_{I,J=0}^{\infty} \frac{1}{I! J!} \delta_p^{IJ}(f) \zeta_p^{IJ},
\]

and the \((I,J)\)-th derivative \( \delta_p^{IJ} \) of the \( \delta \)-functionals are continuous linear functionals on \( A_{p,h} \) by Proposition 4.1. This implies the result, see e.g. [29, Sect. 14.2] for a definition of a Schauder basis.

The next proposition shows that the pointwise product is continuous in the topology of \( A_{p,h} \):

Proposition 4.3 Let \( f, g \in C^\infty(\mathbb{C}^n) \). Then we have

\[
  \|fg\|_{m, R, S}^{p, h} \leq \|f\|_{m+1, \ell, R, S}^{p, h} \|g\|_{m+1, \ell, R, S}^{p, h},
\]

whence \( A_{p,h} \) is a Fréchet *-algebra with respect to the pointwise product.

Proof: First recall that from the explicit form of the Wick star product we obtain

\[
  (2h)^{|I|+|J|} \left( \frac{\partial^{I+J} f}{\partial z^I \partial \bar{z}^J} \right)^2 (p) \leq h_{0,0,I,J}^{p, h}(f).
\]

Now we first consider the case \( m = 0 \). Here we have by the Leibniz rule

\[
  h_{0,0,R,S}^{p, h}(fg) = \sum_{N=0}^{\infty} \frac{(2h)^{|N|+|R|+|S|}}{N!} \left( \sum_{I=0}^{R} \sum_{J=0}^{N+S} \binom{R}{I} \binom{N+S}{J} \frac{\partial^{I+J} f}{\partial z^I \partial \bar{z}^J} \frac{\partial^{R-I+|N+S-J|} g}{\partial z^{R-I} \partial \bar{z}^{N+S-J}} \right)^2 (p)
\]

\[
  \leq \sum_{N=0}^{\infty} \frac{(2h)^{|N|+|R|+|S|}}{N!} \left( \sum_{I=0}^{R} \sum_{J=0}^{N+S} \binom{R}{I} \binom{N+S}{J} \frac{\partial^{I+J} f}{\partial z^I \partial \bar{z}^J} \frac{\partial^{R-I+|N+S-J|} g}{\partial z^{R-I} \partial \bar{z}^{N+S-J}} \right)^2 (p)
\]

\[
  \leq \sum_{N=0}^{\infty} \frac{1}{N!} \left( \sum_{I=0}^{R} \sum_{J=0}^{N+S} \binom{R}{I} \binom{N+S}{J} h_{0,0,I,J}^{p, h}(f) \right) \left( \sum_{I=0}^{R} \sum_{J=0}^{N+S} \binom{R}{I} \binom{N+S}{J} h_{0,0,I,J}^{p, h}(g) \right)
\]

\[
  \leq \sqrt{\sum_{N=0}^{\infty} \frac{1}{N!} \left( \sum_{I=0}^{R} \sum_{J=0}^{N+S} \binom{R}{I} \binom{N+S}{J} h_{0,0,I,J}^{p, h}(f) \right)^2} \times \sqrt{\sum_{N=0}^{\infty} \frac{1}{N!} \left( \sum_{I=0}^{R} \sum_{J=0}^{N+S} \binom{R}{I} \binom{N+S}{J} h_{0,0,I,J}^{p, h}(g) \right)^2}
\]

\[
  = h_{1,0,R,S}^{p, h}(f) \sqrt{h_{1,0,R,S}^{p, h}(g)},
\]

using twice the Cauchy Schwarz inequality in the last steps. For \( m \geq 1 \) we proceed by a straightforward induction for the two cases of \( \ell \) even and odd separately.
Corollary 4.4 $A_{p,h}$ is a Fréchet-Poisson $\ast$-algebra.

PROOF: Clearly, the canonical Poisson bracket is continuous as $\{f, g\}$ is the sum of pointwise products of partial derivatives of $f$ and $g$.

Combining the last two propositions we can finally show the continuity of the Wick star product: we even show that the series (2.3) converges in the topology of $A_{p,h}$ if we replace $\lambda$ by any complex number $\alpha$:

Theorem 4.5 The Wick star product

$$f \ast_{\text{Wick}}^\alpha g = \sum_{N=0}^\infty \frac{(2\alpha)^{|N|}}{N!} \frac{\partial |N| f \partial |N| g}{\partial z^N \partial \bar{z}^N}$$

(4.4)

converges absolutely in the topology of $A_{p,h}$ for all $\alpha \in \mathbb{C}$ and gives a continuous associative product. If $\alpha = h > 0$ then $A_{p,h}$ becomes a Fréchet $\ast$-algebra with respect to $\ast_{\text{Wick}}^h$ as the product and the complex conjugation as the $\ast$-involution.

PROOF: Let $f, g \in A_{p,h}$ then we have for even $\ell$ and $\epsilon_\ell = +1$

$$\left\| \sum_{N=0}^\infty \frac{(2\alpha)^{|N|}}{N!} \frac{\partial |N| f \partial |N| g}{\partial z^N \partial \bar{z}^N} \right\|_{m,\ell,0,0}^{p,h} \leq \sum_{N=0}^\infty \frac{|2\alpha|^{|N|}}{N!} \left\| \frac{\partial |N| f \partial |N| g}{\partial z^N \partial \bar{z}^N} \right\|_{m,\ell,0,0}^{p,h}$$

Prop. 4.3

$$\leq \sum_{N=0}^\infty \frac{\alpha^{|N|}}{N!} \left\| f \right\|_{m+1,\ell,N,0}^{p,h} \left\| g \right\|_{m+1,\ell,0,N}^{p,h}$$

Prop. 4.1

$$\leq \sqrt{\sum_{N=0}^\infty \frac{\alpha^{|N|}}{N!} \left( \left\| f \right\|_{m+1,\ell,N,0}^{p,h} \right)^2} \sqrt{\sum_{N=0}^\infty \frac{\alpha^{|N|}}{N!} \left( \left\| g \right\|_{m+1,\ell,0,N}^{p,h} \right)^2}$$

Prop. 3.3

$$\leq \sqrt{\sum_{N=0}^\infty \frac{\alpha^{|N|}}{N!} \left( \sum_{m=0}^{2m+3} \sqrt{N!} \left\| f \right\|_{m+2,2\ell+1,0,0}^{p,h} \right)^2} \sqrt{\sum_{N=0}^\infty \frac{\alpha^{|N|}}{N!} \left( \sum_{m=0}^{2m+3} \sqrt{N!} \left\| g \right\|_{m+2,2\ell,0,0}^{p,h} \right)^2}$$

$$= \left( \sum_{N=0}^\infty \frac{\alpha^{|N|}}{N!} \sqrt{\sum_{m=0}^{2m+2} \sqrt{N!}} \left\| f \right\|_{m+2,2\ell+1,0,0}^{p,h} \right)^{\epsilon_\ell} \left( \sum_{N=0}^\infty \frac{\alpha^{|N|}}{N!} \sqrt{\sum_{m=0}^{2m+2} \sqrt{N!}} \left\| g \right\|_{m+2,2\ell,0,0}^{p,h} \right)^{\epsilon_\ell}$$

Since $c_m(\alpha)$ converges for all $\alpha \in \mathbb{C}$ we have shown the convergence of (4.4) with respect to $\left\| \cdot \right\|_{m,\ell,0,0}^{p,h}$ for even $\ell$ and $\epsilon_\ell = +1$. The other three cases are shown analogously. As the topology of $A_{p,h}$ is already determined by the seminorms $\left\| \cdot \right\|_{m,\ell,0,0}^{p,h}$ the absolute convergence in the topology of $A_{p,h}$ follows. From the above estimate (and the analogous ones for odd $\ell$ etc.) one also obtains the continuity of $\ast_{\text{Wick}}^\alpha$. If $\alpha = h$ is real, then the complex conjugation is a $\ast$-involution showing the last statement.
Corollary 4.6 The Wick star product $\star_{\text{Wick}}^\alpha$ is a holomorphic deformation of the pointwise product in the sense of [43].

Corollary 4.7 For $f, g \in \mathcal{A}_{p,\hbar}$ we have

$$\left\| \mathcal{T} \star_{\text{Wick}}^\hbar f \right\|_{p,\hbar}^{m,\ell,R,S} \leq \left\| f \right\|_{m+1,2^m+\ell,R,S}^{p,\hbar} \left\| g \right\|_{m+1,\ell,R,S}^{p,\hbar}.$$  (4.5)

**Proof:** This follows from Proposition 3.7, the density of $/BV[z,\bar{z}]$ in $\mathcal{A}_{p,\hbar}$ and the continuity of $\star_{\text{Wick}}^\hbar$. ■

Though $\mathcal{A}_{p,\hbar}$ becomes a Fréchet $^*$-algebra, the topology is not locally $m$-convex in the sense of [37], see also [43, App. A]. Recall that a locally convex algebra is called locally $m$-convex if there exist a set of seminorms $\| \cdot \|_i$ defining the topology such that $\|ab\|_i \leq \|a\|_i \|b\|_i$. Such locally $m$-convex algebras always have a holomorphic calculus which fails for $\mathcal{A}_{p,\hbar}$:

**Example 4.8** We consider the entire function $f \in \mathcal{O}(\mathcal{C})$ defined by

$$f(z) = \sum_{r=0}^{\infty} \frac{z^r}{\sqrt{r!}}.$$  (4.6)

Then it is easy to see that $f \star_{\text{Wick}}^\hbar \mathcal{T}$ evaluated at $z = 0$ converges only for $\hbar = 0$. Since clearly the $\delta$-functional $\delta_p : \mathcal{A}_{p,\hbar} \rightarrow \mathcal{C}$ is continuous we conclude that $f \not\in \mathcal{A}_{0,\hbar}$. This shows that $\mathcal{A}_{0,\hbar}$ does not allow a holomorphic functional calculus as $z \in \mathcal{A}_{0,\hbar}$ and the $\star_{\text{Wick}}^\hbar$-Taylor expansion of $f$ would again coincide with $f$ since $\star_{\text{Wick}}^\hbar$-power of $z$ coincide with the corresponding pointwise powers. Analogous arguments apply also for $p \neq 0$ and higher dimensions $n \geq 1$.

**Corollary 4.9** The topology of $\mathcal{A}_{p,\hbar}$ is not locally $m$-convex with respect to the Wick star product $\star_{\text{Wick}}^\alpha$ for all $\alpha$ and there is no general holomorphic calculus for $\mathcal{A}_{p,\hbar}$.

In the following we shall equip $\mathcal{A}_{p,\hbar}$ always with the Wick star product $\star_{\text{Wick}}^\hbar$.

**Remark 4.10** At this point it would be interesting to compare our algebra to the construction obtained in [40]: Here the authors consider the Weyl-Moyal star product, which on the formal level is known to be equivalent to the Wick star product, and establish a convergence scheme to obtain a certain Fréchet algebra as the completion of the polynomials. However, their construction is rather different from ours whence it seems difficult to investigate whether the usual formal equivalence transformation survives the convergence conditions.

## 5 Translations and rescalings

We shall now discuss the dependence of $\mathcal{A}_{p,\hbar}$ on the point $p \in \mathbb{C}^n$ and on the value $\hbar > 0$. We start with the dependence on the point $p$.

Let $\alpha \in \mathbb{C}^n$ then for $f \in \mathcal{A}_{p,\hbar}$ we define

$$(\tau_\alpha f)(z, \bar{z}) = \hat{f}(z + \alpha, \bar{z})$$  \hspace{1cm} (5.1)

and

$$(\overline{\tau_\alpha f})(z, \bar{z}) = \hat{f}(z, \bar{z} + \bar{\alpha}),$$  \hspace{1cm} (5.2)
which is well-defined according to Theorem 3.6, Part iii.) Moreover, we consider the functions
\[ e^{\alpha,\beta}(z,\bar{z}) = e^{\alpha\beta}e^{\alpha z + \beta\bar{z}}, \tag{5.3} \]
which are elements in \( A_{p,\hbar} \) according to Theorem 3.6, Part iv.) The following lemma is a simple computation, the results of which are well-known in the case of the formal Wick star product:

**Lemma 5.1** Let \( \alpha, \beta, \gamma, \delta \in \mathbb{C}^n \). Then we have for all \( f \in A_{p,\hbar} \):

i.) \( e^{\alpha,\beta} \star_{\text{Wick}} e^{\gamma,\delta} = e^{\alpha\delta - \beta\gamma}e^{\alpha + \gamma,\beta + \delta} \).

ii.) \( e^{\alpha,\beta} \star_{\text{Wick}} f = e^{\alpha,\beta}T_{2\imath\hbar}f \).

iii.) \( f \star_{\text{Wick}} e^{\alpha,\beta} = e^{\alpha,\beta}T_{2\hbar}f \).

iv.) The maps \( \tau_\alpha \) and \( \overline{\tau_\alpha} \) are continuous linear bijections

\[ \tau_\alpha, \overline{\tau_\alpha} : A_{p,\hbar} \to A_{p,\hbar}. \tag{5.4} \]

**Proof:** The only non-trivial point here is that for \( f \in A_{p,\hbar} \) we have

\[ \tau_\alpha f = \sum_{N=0}^{\infty} \frac{\alpha^N}{N!} \frac{\partial \lvert N \rvert f}{\partial z^N} \]

and analogously for \( \overline{\tau_\alpha} \) since \( f \) has a extension to \( \hat{f} \in \mathcal{O} \times \overline{\mathcal{O}}(\mathbb{C}^n \times \mathbb{C}^n) \). Then the computations for the first three parts are folklore. The last part follows from

\[ \tau_\alpha f = e_{0, -\frac{\alpha}{\sqrt{\hbar}}} \left( f \star_{\text{Wick}} e_{0, \frac{\alpha}{\sqrt{\hbar}}} \right) \tag{*} \]

and the continuity of the pointwise multiplication as well as of the continuity of \( \star_{\text{Wick}} \). The same argument applies for \( \overline{\tau_\alpha} \). From (*) one can easily work out explicit estimates for \( \|\tau_\alpha f\|_{p,\hbar}^{m,\ell,R,S} \) and \( \|\overline{\tau_\alpha} f\|_{m,\ell,R,S}^{p,\hbar} \) using Proposition 4.3 and Corollary 4.7. \( \blacksquare \)

**Corollary 5.2** Let \( \alpha, \beta \in \mathbb{C}^n \).

i.) \( e^{\alpha,\beta} \in A_{p,\hbar} \) is invertible with respect to \( \star_{\text{Wick}} \) with inverse given by \( e_{-\overline{\alpha}, -\overline{\beta}} \).

ii.) \( e^{\alpha,\beta} \) is unitary iff \( \alpha = -\beta \) since in general \( e^{\alpha,\beta} = e_{\overline{\alpha}, \overline{\beta}} \).

We introduce now the following notation. For \( w \in \mathbb{C}^n \) we denote the translation by \( w \) by \( T_w(z) = z + w \) whence we have the corresponding pull-back on functions \( (T^*_w f)(z) = f(z + w) \). Moreover, we set

\[ u_w = e^{\frac{1}{2\hbar} w, -\frac{1}{2\hbar} w} \in A_{p,\hbar}, \tag{5.5} \]

which is a unitary element of \( A_{p,\hbar} \) according to Corollary 5.2.

**Proposition 5.3** The translation group \( \mathbb{C}^n \) acts via pull-backs by continuous inner \( \ast \)-automorphisms

\[ T^*_w = \text{Ad}_{\text{Wick}}(u_w) \tag{5.6} \]
on \( A_{p,\hbar} \).
Proof: This is a simple consequence of Lemma 5.1 and the fact that \( u_w \) is unitary.

Remark 5.4 We remark that on a heuristic level (or on polynomial functions only) the statement of this proposition is folklore. Note that it is clear, that for the formal Wick star product the statement is wrong: the translations are only outer automorphisms as the elements \( u_w \) are not well-defined as formal series in \( \lambda \). In fact, it was one of our main motivations to find a reasonably large algebra where the statement of the proposition is still true, extending the polynomials.

The fact that the translations act by inner automorphisms will immediately imply the following result:

Theorem 5.5 Let \( p, p' \in \mathbb{C}^n \). Then
\[
A_{p, h} = A_{p', h}
\] as Fréchet *-algebras.

Proof: Let \( f \in A_{p, h} \) be given and \( w = p' - p \). Then we have on one hand
\[
\| f \|_{m, \ell, R, S}^{p', h} = \| T_w^* f \|_{m, \ell, R, S}^{p, h} < \infty
\]
since
\[
\frac{\partial |I| + |J| f}{\partial z^I \partial \bar{z}^J}(p') = \frac{\partial |I| + |J| (T_w^* f)}{\partial z^I \partial \bar{z}^J}(p),
\]
and in the definition of \( \| \cdot \|_{m, \ell, R, S}^{p, h} \) only the Taylor coefficients of \( f \) at \( p \) are used while the combinatorial coefficients in the construction are the same for all points \( p', p \). This shows \( f \in A_{p', h} \) whence by symmetry the equality (5.7) as vector spaces follows. On the other hand we have
\[
\| f \|_{m, \ell, R, S}^{p', h} = \| u_w \star_{\text{Wick}} f \star_{\text{Wick}} u_{-w} \|_{m, \ell, R, S}^{p, h} \leq \| u_w \|_{m+1, 2m+1, \ell, R, S}^{p, h} \| f \|_{m+2, 2m+2, \ell, R, S}^{p, h} \| u_w \|_{m+2, \ell, R, S}^{p, h}.
\]
This shows that the seminorms \( \| \cdot \|_{m, \ell, R, S}^{p, h} \) can be estimated against the seminorms \( \| \cdot \|_{m, \ell, R, S}^{p, h} \) whence the topology of \( A_{p', h} \) is coarser than the one of \( A_{p, h} \). By symmetry \( p' \leftrightarrow p \) we see that they actually coincide. The algebraic structures are the same anyway whence the theorem is shown.

Thus we see a posteriori that the construction of the Fréchet algebra \( A_{p, h} \) does not depend on our choice \( p \in \mathbb{C}^n \). Hence we can simply write \( A_h = A_{p, h} \) from now on. Note however, that the system of seminorms \( \| \cdot \|_{m, \ell, R, S}^{p, h} \) depends on the choice of \( p \), only the induced topology is independent of \( p \).

As in the formal case, all \( \delta \)-functionals are positive:

Corollary 5.6 All \( \delta \)-functionals
\[
\delta_p : A_h \longrightarrow \mathbb{C}
\] (5.8)
are continuous positive linear functionals.

Proof: The positivity is clear from (4.4) with \( \alpha = h > 0 \) and the continuity follows from the theorem as \( |\delta_p(f)| \leq \|f\|_{0, 0, 0, 0}^{p, h} \).

Corollary 5.7 The Fréchet topology of \( A_h \) is finer than the topology of pointwise convergence.
In the next step we want to analyze the continuity properties of the group representation \( w \mapsto T^*_w \) further. To this end we consider the dependence of the elements \( u_w \) on \( w \). Since
\[
u_w \star^h_{\text{Wick}} u_v = e^{-\frac{i}{2h} \Im(\overline{w}v)} u_{w+v}
\] (5.9)
the map \( w \mapsto u_w \) is not a group morphism. Since \( e^{-\frac{i}{2h} \Im(\overline{w}v)} \) is even a non-trivial group cocycle we need to pass to the central extension of the translation group \( \mathbb{C}^n \) by this cocycle, i.e. to the Heisenberg group \( H_n \). Here we use the convention that \( H_n = \mathbb{C}^n \times \mathbb{R} \) with multiplication law
\[
(w, c) \cdot (w', c') = (w + w', c + c' + \Im(\overline{w}w')).
\] (5.10)
Then it follows that
\[
H_n \ni (w, c) \mapsto u_{(w,c)} = e^{-\frac{i}{h}c} u_w \in U(A_h)
\] (5.11)
is a group morphism from \( H_n \) into the group of unitaries \( U(A_h) \) in \( A_h \). Since \( e^{-\frac{i}{h}c} \) is central, (5.11) factors to the group morphism \( w \mapsto T^*_w \), i.e. we have
\[
\Ad_{\text{Wick}} (u_{(w,c)}) = \Ad_{\text{Wick}}(u_w) = T^*_w
\] (5.12)
for all \((w, c) \in H_n\). The Lie algebra \( h^n \) of \( H_n \) can be identified with \( H_n \) via the exponential map. Then the Lie bracket is given by \([ (w, c), (w', c') ] = (0, 2 \Im(\overline{w}w')) \). The next theorem shows that the group morphism \( H_n \mapsto U(A_h) \) is analytic and induces a Lie algebra morphism \( h^n \mapsto A_h \):

**Theorem 5.8**  

**i.** The map \( h^n \cong H_n \ni (w, c) \mapsto u_{(w,c)} \) is analytic with respect to the topology of \( A_h \).

**ii.** The generator \( J_{(w,c)} \) of the one-parameter group \( t \mapsto u_{(tw, tc)} \) is given by
\[
J_{(w,c)}(z, \overline{z}) = \left. \frac{d}{dt} \right|_{t=0} u_{(tw, tc)} = -\frac{i}{2h}c + \frac{1}{2h} (\overline{w}z - wz),
\] (5.13)
and \( h^n \ni (w, c) \mapsto J_{(w,c)} \in A_h \) is a Lie algebra morphism where \( A_h \) is equipped with the \( \star^h_{\text{Wick}} \)-commutator as Lie bracket.

**iii.** We have for all \( k \)
\[
\left. \frac{d^k}{dt^k} \right|_{t=0} u_{(tw, tc)} = J_{(w,c)} \star^h_{\text{Wick}} \cdots \star^h_{\text{Wick}} J_{(w,c)},
\] (5.14)
and explicitly for \( c = 0 \)
\[
\left. \frac{d^k}{dt^k} \right|_{t=0} u_{(tw, 0)} = \sum_{\ell=0}^{[k/2]} \frac{k!}{\ell! (k - 2\ell)!} \left( \frac{\overline{w}w}{4h} \right)^{\ell} \left( \frac{\overline{w}z - wz}{2h} \right)^{k-2\ell}. \] (5.15)

**Proof:** We have \( u_{(w,c)}(z, \overline{z}) = e^{-\frac{i}{2h}c - \frac{i}{h}(\overline{w}z - wz)} \). The first factors \( e^{-\frac{i}{h}c} \) and \( e^{-\frac{i}{2h}w} \) are clearly analytic as they are analytic functions times a fixed element (the identity) in \( A_h \). For the remaining factor we see that the \((z, \overline{z})\)-Taylor expansion, which converges unconditionally in the topology of \( A_h \) coincides up to numerical factors with the \((w, \overline{w})\)-Taylor expansion of this function. Thus the \((w, \overline{w})\)-Taylor expansion converges also in \( A_h \) unconditionally which shows the first part. The second part is a trivial computation. For the third part, the contribution of \( c \) is not essential whence we discuss the case \( c = 0 \) only. A straightforward computation of the Taylor expansion in
\( t \) gives immediately (5.15). On the other hand, the \( k \)-th power of the linear function \( J_w = J_{(w,0)} \) satisfies the recursion formula

\[
J_w^*k = J_w J_w^{*(k-1)} + (k - 1) \frac{\overline{w}w}{2\hbar} J_w^{*(k-2)},
\]

which follows easily from the fact that partial derivatives are derivations of \( \star_\text{Wick} \) and \( \overline{w} \frac{\partial}{\partial z} J_w = \frac{\overline{w}}{2\hbar} \) is central. In a last step one shows that also the right hand side of (5.15) satisfies this recursion with the same initial conditions for \( k = 0, 1 \).\( \blacksquare \)

**Corollary 5.9** Let \( w \in \mathbb{C}^n \) then the \( \star_\text{Wick}_h \)-exponential function

\[
\exp(tJ_w) = \sum_{k=0}^{\infty} \frac{t^k}{k!} J_w \star_\text{Wick}_h \cdots \star_\text{Wick}_h J_w = u_tw
\]  

(5.16)

converges unconditionally in the topology of \( A_h \) for all \( t \in \mathbb{R} \).

**Remark 5.10** Again, the importance of this corollary is not the explicit computation of the star exponential which is folklore. Instead, we have found a well-defined analytic framework where the formula actually converges inside an algebra of functions. Note also, that in general we cannot expect such a convergence as \( A_h \) does not allow a holomorphic functional calculus in general.

Let us now discuss the dependence on the parameter \( h \). From a simple dimensional analysis we see that with our convention for the Poisson bracket \( \{ z^k, \overline{z}^\ell \} = \frac{2i}{\hbar} \delta^{k\ell} \) the coordinates have to have the physical dimension \([\text{action}]^{1/2}\). Thus a rescaling of \( h \), which physically is of course absurd, has to be reinterpreted as a rescaling of the coordinates \((z, \overline{z})\): we are not changing the value \( h \) but the unit system. On the other hand, from a purely mathematical point of view we cannot distinguish these two interpretations. As we do not have any additional absolute scale in our approach, the corresponding algebras \( A_h \) and \( A_{h'} \) should be isomorphic in order to be physically reasonable. The following theorem will show that this is indeed the case.

We define for \( \alpha > 0 \) the diffeomorphism \( R_\alpha : \mathbb{C}^n \longrightarrow \mathbb{C}^n \)

\[
R_\alpha(z) = \sqrt{\alpha}z,
\]  

(5.17)

whose inverse is \( R_\frac{1}{\alpha} \).

**Theorem 5.11** The pull-back \( R_\alpha^* \) induces an isomorphism of Fréchet *-algebras

\[
R_\alpha^* : A_{\alpha h} \longrightarrow A_h.
\]  

(5.18)

In particular, we have for all \( f \in A_{\alpha h} \)

\[
\|R_\alpha f\|_{m,t,R,S}^{0,h} = \|f\|_{m,t,R,S}^{0,\alpha h}.
\]  

(5.19)

**Proof:** Let \( f \in A_{\alpha h} \) be given. Then we have

\[
\frac{\partial^{\vert I\vert + \vert J\vert}}{\partial z^I \partial \overline{z}^J} (R_\alpha^* f) = \sqrt{\alpha^{\vert I\vert + \vert J\vert}} R_\alpha \left( \frac{\partial^{\vert I\vert + \vert J\vert}}{\partial z^I \partial \overline{z}^J} f \right)
\]
and thus
\[ \| R_\alpha^* f \|_{0,0,R,S}^{0,h} = \sum_{N=0}^{\infty} \frac{(2h)^{|N|+|R|+|S|}}{N!} \left| \frac{\partial^{|N|+|R|+|S|}}{\partial z^N \partial z^{N+S}} (R_\alpha^* f) (0) \right|^2 \]
\[ = \sum_{N=0}^{\infty} \frac{(2\alpha h)^{|N|+|R|+|S|}}{N!} \left| \frac{\partial^{|N|+|R|+|S|}}{\partial z^N \partial z^{N+S}} f (0) \right|^2 \]
\[ = \| f \|_{0,0,R,S}^{0,\alpha h} . \]

Since the higher seminorms are constructed in a purely combinatorial way out of \( \| \cdot \|_{0,0,R,S}^{0,h} \), we can conclude (5.19). This shows that \( R_\alpha^* : A_{\alpha h} \to A_h \) is an isomorphism of Fréchet spaces as we can exchange the role of \( h \) and \( \alpha h \) by passing from \( \alpha \) to \( \frac{1}{\alpha} \). Then it is easy to see that \( R_\alpha^* \) is an algebra morphism: we use the convergence of (4.4) and the continuity of \( R_\alpha^* \) to obtain
\[ R_\alpha^* (f \ast_{\text{Wick}} g) = R_\alpha^* \left( \sum_{N=0}^{\infty} \frac{(2\alpha h)^{|N|}}{N!} \frac{\partial^{|N|}}{\partial z^N} (R_\alpha^* f) \frac{\partial^{|N|}}{\partial z^N} (R_\alpha^* g) \right) \]
\[ = \sum_{N=0}^{\infty} \frac{(2h)^{|N|}}{N!} \frac{\partial^{|N|}}{\partial z^N} (R_\alpha^* f) \frac{\partial^{|N|}}{\partial z^N} (R_\alpha^* g) \]
\[ = R_\alpha^* f \ast_{\text{Wick}} R_\alpha^* g. \]

The compatibility with the complex conjugation \( \overline{R_\alpha^* f} = R_\alpha^* \overline{f} \) is obvious. 

**Remark 5.12** Of course we can also directly compare the seminorms for different values of \( h \). Clearly, one has
\[ \| f \|_{0,0,R,S}^{0,h} \leq \| f \|_{0,0,R,S}^{0,h'} \] for \( h \leq h' \) whence by induction
\[ \| f \|_{m,\ell,R,S}^{0,h} \leq \| f \|_{m,\ell,R,S}^{0,h'} \] as well. For \( h \leq h' \) this gives the inclusion
\[ A_{h'} \subseteq A_h. \] (5.22)

### 6 The GNS construction and coherent states

We shall now discuss the GNS construction corresponding to the positive \( \delta \)-functionals
\[ \delta_p : A_h \to \mathbb{C}, \] (6.1)
now in the convergent situation.

**Proposition 6.1** Let \( p \in \mathbb{C}^n \). Then the Gel’fand ideal of \( \delta_p \) is given by
\[ \mathfrak{J}_p = \left\{ f \in A_h \mid \forall I : \frac{\partial^{|I|}}{\partial z^I} (p) = 0 \right\}, \] (6.2)
and the GNS pre-Hilbert space \( \mathcal{D}_p = A_h / \mathfrak{J}_p \) is a Fréchet space in the natural way, where the topology of \( \mathcal{D}_p \) is determined by the seminorms
\[ \| [f] \|_{m,\ell,R,S}^{p,h} = \inf \left\{ \| f + g \|_{m,\ell,R,S}^{p,h} \mid g \in \mathfrak{J}_p \right\}. \] (6.3)
Proof: The statement (6.2) is obvious. Since \( \delta_p \) is continuous, the Gel’fand ideal is a closed subspace of \( \mathcal{A}_h \). Thus the quotient \( \mathfrak{D}_p \) is again a Fréchet space by general arguments, see e.g. [29, Sect. 4.4, Prop. 1].

Since the translation group acts by inner \(*\)-automorphisms we can safely specialize to the case \( p = 0 \) in a first step. Then we can describe the quotient \( \mathfrak{D}_0 \) more explicitly:

Theorem 6.2 Let \( f, g \in \mathcal{A}_h \).

i.) The \( \overline{z} \)-Taylor expansion \( \Psi : \mathcal{A}_h \rightarrow \mathcal{A}_h \) defined by

\[
\Psi : f \mapsto \left( z \mapsto \Psi_f(z) = \sum_{l=0}^{\infty} \frac{1}{l!} \frac{\partial^{\left| J \right|}}{\partial \overline{z}^J} f(0) \overline{z}^l \right)
\]  

is a continuous projection with

\[
\ker \Psi = \mathfrak{j}_0.
\]  

In fact,

\[
\|\Psi_f\|_{m,\ell,R,S}^{0,h} \leq \|f\|_{m,\ell,R,S}^{0,h},
\]

where equality holds if and only if \( f \) is anti-holomorphic.

ii.) The quotient \( \mathfrak{D}_0 \) is canonically isomorphic as a Fréchet space to the image \( \mathfrak{D} = \text{im } \Psi \) of \( \Psi \) via

\[
\mathfrak{D}_0 \ni [f] \mapsto \Psi_f \in \mathfrak{D}.
\]  

In particular,

\[
\|[f]\|_{m,\ell,R,S}^{0,h} = \|\Psi_f\|_{m,\ell,R,S}^{0,h}.
\]  

iii.) The space \( \mathfrak{D} \) is a dense subspace of the Bargmann-Fock Hilbert space \( \mathfrak{H}_{BF} \). The map (6.7) is an isometry of pre-Hilbert spaces and the Fréchet topology of \( \mathfrak{D} \) is finer than the topology induced from the Hilbert space \( \mathfrak{H}_{BF} \).

iv.) The GNS representation on \( \mathfrak{D}_0 \) induces via (6.7) the Bargmann-Fock representation of \( \mathcal{A}_h \) on \( \mathfrak{D} \), explicitly given by

\[
\pi(f) \Psi_g = \sum_{l=0}^{\infty} \frac{(2h)^{\left| J \right|}}{l!} \left( \sum_{J=0}^{\infty} \frac{1}{J!} \frac{\partial^{\left| J \right|}}{\partial z^J} f(0) \overline{z}^l \right) \frac{\partial^{\left| J \right|} \Psi_g}{\partial \overline{z}^l},
\]

where both series converge in the topology of \( \mathfrak{D} \).

v.) The bilinear map \( \mathcal{A}_h \times \mathfrak{D} \ni (f, \Psi_g) \mapsto \pi(f) \Psi_g \in \mathfrak{D} \) is continuous with respect to the Fréchet topologies of \( \mathcal{A}_h \) and \( \mathfrak{D} \), respectively.

Proof: For the first part we have \( \Psi_f(\overline{z}) = \hat{f}(0, \overline{z}) \) whence \( \Psi_f \) is indeed a well-defined anti-holomorphic function. Moreover, (6.6) is clear from our consideration in the proof of Theorem 3.9 since in the seminorm of \( \Psi_f \) simply less Taylor coefficients contribute compared to the corresponding seminorm of \( f \). Thus \( \Psi_f \in \mathcal{A}_h \) and \( \Psi \) is continuous. From the explicit form of \( \Psi \) the equality (6.5) and \( \Psi^2 = \Psi \) are obvious. For the second part we first notice that (6.7) is well-defined and bijective since \( f - \Psi_f \in \ker \Psi = \mathfrak{j}_0 \) using the fact that \( \Psi \) is a projection. Moreover, (6.8) follows directly from (6.6) since obviously \( \|[f]\|_{m,\ell,R,S}^{0,h} \leq \|\Psi_f\|_{m,\ell,R,S}^{0,h} \). Since the inverse of (6.7) is simply given
by the well-defined and continuous map $\Psi_f \mapsto [f]$, we see that (6.7) is indeed an isomorphism of Fréchet spaces. For the third part we compute explicitly the GNS inner product on $\mathcal{D}_0$

$$\langle [f], [g] \rangle_{\mathcal{D}_0} = \delta_0(\mathcal{T}^h_{\text{Wick}} f, \mathcal{T}^h_{\text{Wick}} g)$$

$$= \sum_{N=0}^{\infty} \frac{(2\hbar)^{|N|}}{N!} \frac{\partial^{N|f|} \Psi_f}{\partial z^{Nf}(0)} \frac{\partial^{N|g|} \Psi_g}{\partial z^{Ng}(0)}$$

$$= \sum_{N=0}^{\infty} \frac{(2\hbar)^{|N|}}{N!} \frac{\partial^{N|f|} \Psi_f}{\partial z^{Ng}(0)} \frac{\partial^{N|g|} \Psi_g}{\partial z^{Ng}(0)}$$

$$= \langle \Psi_f, \Psi_g \rangle_{\mathcal{BF}}$$

whence (6.7) is isometric. In particular $\mathcal{D} \subseteq \mathcal{H}_{\text{BF}}$ follows, as

$$\langle \Psi_f, \Psi_f \rangle_{\mathcal{BF}} = (\mathcal{T}^h_{\text{Wick}} (f))(0) = (\|f\|^0, 0, 0, 0, 0, 0)^2 < \infty.$$  

Since $\mathcal{C}[\bar{z}] \subseteq \mathcal{D} \subseteq \mathcal{H}_{\text{BF}}$, the subspace $\mathcal{D}$ is dense in $\mathcal{H}_{\text{BF}}$. Moreover, since $\|f\|^0, 0, 0, 0, 0, 0 = \|\Psi_f\|^0, 0, 0, 0, 0, 0$ the estimate $(\ast)$ implies that the Fréchet topology of $\mathcal{D}$ is finer than the topology induced from $\mathcal{H}_{\text{BF}}$. This shows the third part. For the fourth part recall that the GNS representation is defined by $\varrho(f)[g] = [f \star_{\text{Wick}}^h g]$ which translates via (6.7) into

$$\pi(f)\Psi_g = \Psi_f^h_{\text{Wick}} g$$

$$= \Psi \sum_{N=0}^{\infty} \frac{(2\hbar)^{|N|}}{N!} \frac{\partial^{N|f|} \Psi_f}{\partial z^{Ng}(0)} \frac{\partial^{N|g|} \Psi_g}{\partial z^{Ng}(0)}$$

$$= \sum_{N=0}^{\infty} \frac{(2\hbar)^{|N|}}{N!} \frac{\partial^{N|f|} \Psi_f}{\partial z^{Ng}(0)} \frac{\partial^{N|g|} \Psi_g}{\partial z^{Ng}(0)}$$

$$= \sum_{N=0}^{\infty} \frac{(2\hbar)^{|N|}}{N!} \left( \sum_{M=0}^{\infty} \frac{1}{M!} \frac{\partial^{N|+|M|} \Psi_f}{\partial z^{Ng}(0)} \frac{\partial^{N|} \Psi_g}{\partial z^{Ng}(0)} \right) \frac{\partial^{N|} \Psi_g}{\partial z^{Ng}(0)}$$

where (a) holds since $\Psi$ is continuous and (b) holds since obviously $\Psi$ is a homomorphism of the pointwise product. Finally, in the last step we have used that $\Psi$ commutes with derivatives in $\bar{z}$-direction. This shows the fourth part and the last part follows immediately from $\pi(f)\Psi_g = \Psi_f^h_{\text{Wick}} g = \Psi_f^h_{\text{Wick}} \Psi_g$ and the continuity of $\Psi$ and $\star_{\text{Wick}}^h$.

**Corollary 6.3** The Bargmann-Fock representation of $\mathcal{A}_h$ is injective.

The following corollary is remarkable in so far as closed subspaces of Fréchet spaces usually do not have complementary closed subspaces:

**Corollary 6.4** The algebra $\mathcal{A}_h$ decomposes into two complementary closed subspaces $\mathcal{A}_h = \mathcal{D} \oplus \mathcal{J}_0$.

**Remark 6.5** Since the Bargmann-Fock representation is injective and since

$$\pi(\bar{z}^i) = 2\hbar \frac{\partial}{\partial \bar{z}^i} = a_i$$  

(6.10)
\[ \pi(\varphi_i) = \varphi'_i = a_i^\dagger \]  

(6.11)

are the annihilation and creation operators we find another interpretation of the algebra \( \mathcal{A}_h \): it is a (rather large) completion of the polynomials in the creation and annihilation operators in a certain Fréchet topology. In particular, this completion contains the usual unitary generators of the Weyl algebra, i.e. the exponential functions of \( a_i \) and \( a_i^\dagger \). The are given by \( \pi(\phi_{\alpha,\beta}) \) for suitable \( \alpha, \beta \in \mathbb{C}^n \).

**Remark 6.6** Since the Bargmann-Fock representation is a *-representation of the *-algebra \( \mathcal{A}_h \) by (in general) unbounded operators with common domain \( \mathcal{D} \), one can investigate the resulting \( O^* \)-algebra of unbounded operators by techniques as developed in e.g. [45]. In particular, it would be interesting to find more concrete characterizations of the Fréchet topologies of \( \mathcal{D} \) and \( \mathcal{A}_h \).

We now discuss the action of the translation group \( \mathbb{C}^n \) and its central extension \( \mathbb{H}_n \). The following statement is obvious, the representation itself being well-known:

**Lemma 6.7** The map

\[ H_n \ni (w, c) \mapsto U_{(w, c)} = \pi(u_{(w, c)}) \in \mathcal{U}(\mathcal{F}_{BF}) \]  

(6.12)

is a strongly continuous unitary representation of the Heisenberg group. Explicitly,

\[ (U_{(w, c)} \psi)(\varphi) = e^{-\frac{i}{2\hbar}c - \overline{ww}} e^{-\frac{i}{2\hbar} \varphi + \overline{\varphi}} \]  

(6.13)

for \( \psi \in \mathcal{F}_{BF} \). It factors to a projective representation \( U_w = U_{(w, 0)} \) of the translation group \( \mathbb{C}^n \).

**Proof:** Since \( \pi \) is a *-representation it follows that \( \pi(u_{(w, c)}) \) is a unitary operator defined on the dense domain \( \mathcal{D} \). Thus it extends to a unitary operator on \( \mathcal{F}_{BF} \). The group representation property is obvious from (5.11). The explicit formula is a simple consequence of (6.9). The fact that (6.12) is strongly continuous is well-known but can also be shown within our approach directly: let \( \phi, \psi \in \mathcal{D} \) then \( g(w, c) = \langle \psi, U_{(w, c)} \phi \rangle_{BF} \) is real-analytic since \( \langle \cdot, \cdot \rangle_{BF} \) and \( \pi \) are continuous with respect to the Fréchet topology and \( (w, c) \mapsto u_{(w, c)} \) is real-analytic according to Theorem 5.8. Since \( g(0, 0) = \langle \psi, \phi \rangle_{BF} \) we see that on the dense domain \( \mathcal{D} \) the representation (6.12) is weakly continuous at the identity. But this implies that it is strongly continuous on the whole group \( H_n \) and on the whole Hilbert space \( \mathcal{F}_{BF} \). Since the contribution of \( c \) is only an overall phase, the representation clearly factors to a projective representation of \( \mathbb{C}^n \).

Since we have a group action of \( H_n \) we can formulate now the following covariance property of the Bargmann-Fock representation which is an obvious consequence of the fact that the *-automorphisms are inner.

**Theorem 6.8** The Bargmann-Fock representation is \( H_n \)-covariant with respect to the action by *-automorphisms on \( \mathcal{A}_h \) and the action by unitaries on \( \mathcal{F}_{BF} \), i.e. we have

\[ \pi(\text{Ad}_{\text{Wick}}(u_{(w, c)}) f) = U_{(w, c)} \pi(f) U^*_{(w, c)} = U_w \pi(f) U_w^* \]  

(6.14)

for all \( (w, c) \in H_n \) and \( f \in \mathcal{A}_h \).

Since the GNS representation is cyclic with cyclic vector \( \Psi_1 = 1 \) we obtain coherent states with respect to the representation of \( H_n \). We define the coherent state vector \( \psi_{(w, c)} \in \mathcal{F}_{BF} \) by

\[ \psi_{(w, c)} = U_{(w, c)}^{-1} \psi_1 \]  

(6.15)
explicitly given by
\[ \psi_{(w,c)}(z) = e^{\frac{1}{2\hbar}c + \frac{w}{4\hbar} e^{w} z}. \] (6.16)

From the covariance property (6.14) we immediately have the following characterization of the \( \delta \)-functionals at arbitrary points in \( \mathbb{C}^n \):

**Corollary 6.9** The coherent state vectors give the \( \delta \)-functionals as expectation value functionals, i.e. we have
\[ \delta_w(f) = \langle \psi_{(w,c)}, \pi(f) \psi_{(w,c)} \rangle_{\mathcal{BF}} \] (6.17)
for all \( f \in \mathcal{A}_h \) and \((w,c) \in H_n\). In particular, the group action of \( H_n \) on the coherent state vectors factors through to a group action of the translation group \( \mathbb{C}^n \) on the coherent states \( \delta_w : \mathcal{A}_h \to \mathbb{C} \).

**Remark 6.10** This corollary gives finally the justification to view the \( \delta \)-functionals of the Wick star product algebra as coherent states with respect to the translation group. Though the explicit formula (6.16) is folklore (it is just the Bergmann kernel) we would like to emphasize that in our approach the coherent states emerge out of properties of the observable algebra instead of more conventional approaches based on group actions on the state vectors in some Hilbert space, see e.g. [42, 48] for more references. In this sense our approach supports the idea that the observable algebra is the more fundamental object in both, quantum and classical mechanics.

**Remark 6.11** We also note that the statement of Theorem 6.8 as well as the Corollary 6.9 are not possible for the formal Bargmann-Fock representation. Again, this was one of our main motivations to consider a suitable convergence scheme for the Wick star product.

The result of Theorem 6.8 and Corollary 6.9 suggest the following general definition of coherent states with respect to some symmetry based on the observable algebra:

**Definition 6.12** Let \( A \) be a \(*\)-algebra with unit \( 1 \) and let \( G \) be a group acting on \( A \) by \(*\)-automorphisms \( \Phi_g : A \to A \). Let \( \omega \) be a state of \( A \) such that the GNS representation is \( G \)-covariant, i.e. there exists a unitary (or more general: projectively unitary) representation \( U \) of \( G \) on the GNS pre-Hilbert space \( \mathcal{H}_\omega \). Then the states \( \omega_g \) with
\[ \omega_g(a) = (\omega \circ \Phi_g)(a) = \langle \psi_g, \pi(a) \psi_g \rangle, \] (6.18)
where \( \psi_g = U_g^* \psi_1 \in \mathcal{H}_\omega \) are called coherent with respect to \( G \).

Clearly, in case of a projective representation only the coherent states \( \omega_g \) are well-defined, while for a unitary representation also the coherent state vectors \( \psi_g \) are well-defined.

With the Heisenberg group acting on \( \mathcal{A}_h \) and the \( \delta \)-functional we are in this situation: if the action is realized by inner \(*\)-automorphisms then the representation is always covariant for a (in general only projective) representation on the GNS pre-Hilbert space. Note also, that for an invariant state \( \omega \) the GNS representation is trivially covariant. In this case \( \psi_g = \psi_1 \) coincides with the vacuum vector for all \( g \in G \). Thus the interesting coherent states arise from non-invariant vacua such that the GNS representation is nevertheless covariant. For a survey on covariant \(*\)-representation theory for \(*\)-algebras over ordered rings we refer to [28].

Let us now come to a further property of the subspace \( \mathcal{D} \subseteq \mathcal{F}_{\mathcal{BF}} \). We have already seen that the action of \( H_n \) leaves \( \mathcal{D} \) invariant.

**Theorem 6.13** The vectors in \( \mathcal{D} \) are analytic with respect to the unitary representation \( U \) of \( H_n \).
**Proof:** Let $\psi \in \mathcal{D}$ be fixed then we have to show that $H_n \ni (w,c) \mapsto U(w,c)\psi$ is analytic with respect to the topology of $\mathcal{H}_{BF}$. But this is simple since $U(w,c)\psi = \pi(u(w,c))\psi$ and $(w,c) \mapsto u(w,c)$ is analytic in the topology of $A_h$. Moreover, the bilinear map $(f,\psi) \mapsto \pi(f)\psi$ is continuous in the topologies of $A_h$ and $\mathcal{D}$. Thus the map $(w,c) \mapsto U(w,c)\psi$ is analytic with respect to the topology of $\mathcal{D}$. Since by Theorem 6.2 this topology is finer than the one of $\mathcal{H}_{BF}$, the proof is complete. ■

Thus it would be interesting to know whether $\mathcal{D}$ coincides with the space of all analytic vectors. A positive answer would help to understand the (still rather complicated) Fréchet topology of $\mathcal{D}$ and hence the one of $A_h$.

Using the convergence of the star exponential (5.16) we even can specify the analyticity of the vectors in $\mathcal{D}$ further. Since $\pi$ is continuous with respect to the topologies of $A_h$ and $\mathcal{D}$, we have

$$U(w,c)\psi = \pi(u(w,c))\psi = \pi \left( \text{Exp}(J(w,c)) \right) \psi = \sum_{r=0}^{\infty} \frac{1}{r!} \pi \left( J(w,c) \right)^r \psi \tag{6.19}$$

with respect to the topology of $\mathcal{D}$. Again, since the topology of $\mathcal{D}$ is finer than the one of $\mathcal{H}_{BF}$, the series converges unconditionally also in the Hilbert space sense. As usual, the contribution of $c$ is not essential.

**Corollary 6.14** Let $\psi \in \mathcal{D}$ and $w \in \mathbb{C}^n$. Then the series

$$U_w\psi = \sum_{r=0}^{\infty} \frac{1}{r!} \left( \frac{1}{2\hbar} \right)^r \left( \sum_{i=1}^{n} w_i a_i - w_i a_i^\dagger \right)^r \psi \tag{6.20}$$

converges unconditionally in the Hilbert space topology.

Of course we can also rewrite this in terms of the position and momentum operators

$$Q_k = \frac{1}{2} (a_k + a_k^\dagger) = \frac{1}{2} \pi(z_k + z_k^\dagger) \tag{6.21}$$

and

$$P_k = \frac{1}{2i} (a_k - a_k^\dagger) = \frac{1}{2i} \pi(z_k - z_k^\dagger) \tag{6.22}$$

defined as unbounded symmetric operators on $\mathcal{D}$. Then (6.19) shows that for $\psi \in \mathcal{D}$ we have the unconditionally convergent series

$$e^{\frac{i\bar{q}\cdot \hat{Q}}{\hbar}}\psi = \sum_{r=0}^{\infty} \frac{1}{r!} \left( \frac{i\bar{q} \cdot \hat{Q}}{\hbar} \right)^r \psi \tag{6.23}$$

and

$$e^{\frac{i\bar{p}\cdot \hat{P}}{\hbar}}\psi = \sum_{r=0}^{\infty} \frac{1}{r!} \left( \frac{i\bar{p} \cdot \hat{P}}{\hbar} \right)^r \psi \tag{6.24}$$

in the Hilbert space topology, for all $\bar{q}, \bar{p} \in \mathbb{R}^n$ substituting $w$ suitably.

Of course, this also follows by ‘Hilbert space techniques’ from the strong continuity of the representation $U$ and Theorem 6.13. Note however, that the above argument using the convergence of the star exponential is independent.
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