Linear estimation of coherent structures in wall-bounded turbulence at $Re_\tau = 2000$

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Abstract.
The estimation problem for a fully-developed turbulent channel flow at $Re_\tau = 2000$ is considered. Specifically, a Kalman filter is designed using a Navier–Stokes-based linear model. The estimator uses time-resolved velocity measurements at a single wall-normal location (provided by DNS) to estimate the time-resolved velocity field at other wall-normal locations. The estimator is able to reproduce the largest scales with reasonable accuracy for a range of wavenumber pairs, measurement locations and estimation locations. Importantly, the linear model is also able to predict with reasonable accuracy the performance that will be achieved by the estimator when applied to the DNS. A more practical estimation scheme using the shear stress at the wall as measurement is also considered. The estimator is still able to estimate the largest scales with reasonable accuracy, although the estimator’s performance is reduced.

1. Introduction
Linear estimation has been used to successfully detect perturbations and coherent structures in wall bounded shear flows at laminar [1–3], transitional [4–6], and turbulent [7, 8] Reynolds numbers. Estimation is useful for obtaining information about quantities that cannot be measured directly, and can provide feedback in a control setup. The design of a linear estimator is a well-understood topic in control theory, and therefore the implementation is relatively straightforward. However, a model of the flow is required. A suitable model for this purpose is the linearized Navier-Stokes equations.

Previous studies have analysed the linearized Navier-Stokes equations to understand the growth and decay of perturbations. The majority of studies have focused on laminar shear flows by considering the temporal development of small perturbations about the laminar velocity profile; and the response to external forcing [9–12]. Of particular significance was the realization that, even for stable flows, perturbations can experience significant transient growth before ultimately decaying [13,14]. Linear modelling for fully-developed turbulent shear flows, although able to claim a relatively long history [15, 16], has received less attention. In recent years, though, there has been a renewed interest in linear models for turbulent shear flows, motivated by increasing evidence that linear mechanisms play an important role here as well.

Existing research on linear mechanisms in fully-developed turbulent shear flows can be divided into two broad camps: those that, following earlier work [17], include an eddy viscosity in the linear operator [18–21]; and those that do not [22–25].
This study investigates model-based estimation of channel flow at $Re_T = 2000$ using a linear, Navier-Stokes-based model (introduced in §2). The estimator uses time-resolved velocity measurements (or shear-stress measurements) at a single wall-normal location to estimate the time-resolved velocity field at other wall-normal locations. Like many previous studies [17–21] the model includes an eddy viscosity in the linear operator; but in contrast to them it does not discard the remaining nonlinear terms, instead absorbing them into an unknown (but non-zero) forcing term following [22].

The work is composed of three parts. The first part of the study serves to consolidate earlier results [8] which were obtained for channel flow at $Re_T = 1000$. The present study builds on that work in three ways. First, applying the techniques to a different dataset at a different Reynolds number provides an important consolidation and confirmation of those earlier results. Second, with access to significantly more time-resolved data at $Re_T = 2000$ we obtain significantly better convergence in the results. This will be of particular importance when comparing the results obtained in DNS with those predicted by the linear model. Third, with access to all wall-normal locations we are able to compare velocity fields not only in single wall-normal planes (as done in [8]), but also across all wall-normal heights. This allows us to qualitatively assess the structures that are estimated by the linear model when it knows only the time-resolved velocity field at a single wall-normal height. In the second part of the study we look at making the measurement scheme more practical. Measuring the velocity in the interior of a flow is difficult to achieve experimentally. For this reason wall-mounted shear stress measurements have been used in a number of previous studies on flow estimation and flow control [1–7, 26–28]. Therefore in the second part of the study we replace the (less practical) velocity measurements used in the first part with (more practical) wall shear stress measurements. Having looked at estimator performance at individual wall-normal heights in the first two parts, the third and final part looks at estimator performance over a range of estimation and measurement planes (for velocity measurements only). Specifically, we quantify the estimator performance for all pairs of measurement and estimation locations over the entire channel half-height. Although less practical (since the sensor could be anywhere), this is interesting because it shows us what is possible with a single sensor.

2. Methods

2.1. Linear model

We consider a fully developed turbulent channel flow at $Re_T = 2000$. Here $Re_T$ is the friction Reynolds number defined as $u_f h/\nu$, where $\nu$ is the kinematic viscosity, $h = 1$ the channel half height, $u_f = \sqrt{\tau_w/\rho}$ the friction velocity, $\tau_w$ the wall shear stress, and $\rho$ the density. Thus, the spatial variables are normalised by $h$, velocities by $u_f$, time by $h/u_f$ and the pressure $p$ by $\rho u_f^2$. The streamwise, spanwise, and wall-normal spatial coordinates of the flow are represented by $[x, y, z]$ and velocities by $u = [u, v, w]$. The channel flow dynamics are represented by a linear model for perturbations $\mathbf{u}$ about a full-developed mean velocity profile $\mathbf{U} = (U(z), 0, 0)$, as done in [18,19]):

$$\frac{\partial \mathbf{u}}{\partial t} = - (\mathbf{U} \cdot \nabla) \mathbf{u} - (\mathbf{u} \cdot \nabla) \mathbf{U} - \nabla p + \nabla \cdot \left[ \frac{\nu_T}{\nu} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) \right] + \mathbf{d}, \quad \mathbf{u} \cdot \nabla = 0, \quad (1)$$

where $\nu_T(z)$ is the eddy viscosity profile and $\mathbf{d} = -(\mathbf{u} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u}$ contains all non-linear terms. An analytical approximation of the eddy viscosity profile, as used in [8,18,19,24] is given by the relation:

$$\nu_T(z) = \frac{\nu}{2} \left( 1 + \frac{\kappa^2 Re^2}{9} (2z - z^2)^2 (3 - 4z + z^2)^2 \left[ 1 - \exp\left( \frac{-Re_T z}{A} \right) \right]^2 \right)^{\frac{1}{2}} + \frac{\nu}{2}. \quad (2)$$
The constants $\kappa = 0.426$ and $A = 25.4$ give the best fit to the mean velocity profile at $Re_x = 2003$ [16, 17, 29]. The mean velocity profile $U$ is obtained by integrating $(1 - z)/\nu_T(z)$ in the wall-normal direction.

2.2. Linear state-space model

We require a linear time-invariant state-space model, which will allow us to design and implement a Kalman filter. We begin with the linear model introduced in 2.1, take the Fourier transforms in the homogeneous directions ($x$ and $y$), and transform it into the Orr-Sommerfeld Squire form. The wall-normal direction is then discretised using Chebyshev collocation of order 151 [30]. Convergence has been checked for the setup. The state-space model at a single wavenumber pair is thus obtained as:

$$\begin{align*}
\dot{x}(t) &= Ax(t) + Bd(t), \\
y(t) &= Cy(t) + n(t), \\
z(t) &= Cz(t),
\end{align*}$$

where $x = [\hat{w}, \hat{v}]^T$ represents the wall-normal velocity and wall-normal vorticity, $u = [\hat{u}, \hat{v}, \hat{w}]^T$ the streamwise, spanwise, and wall-normal velocity, $y$ a sensor reading contaminated by noise $n$, $z$ a quantity of interest, and $d = [\hat{d}_x, \hat{d}_y, \hat{d}_z]^T$ represents the non-linear effects (see §2.1).

For the purpose of the linear estimator design, the disturbances $d$ and $n$ are treated as unknown forcing which is white in space and time. The sensor noise is chosen to be small enough (relative to the size of disturbances) that the estimation results are insensitive to it. The state-space model (3) can be expressed as a transfer function by using the Laplace transform:

$$\begin{bmatrix} y \\ z \end{bmatrix} = P(s) \begin{bmatrix} d \\ n \end{bmatrix}.$$  

See Appendix A for a full definition of the state-space model (3) and the transfer function (4).

2.3. Linear estimator design

The linear time-invariant model, introduced in the previous section, allows us to design an estimator using common tools from control theory. We begin the estimator design process by stating the optimal estimation problem: given sensor measurements $y$, which are contaminated by noise $n$, estimate a quantity of interest $z$. We use the plant model $P(s)$ and the $H_2$ optimal control framework to design the Kalman filter, which is denoted as $F(s)$ in figure 1 (for example, see Seron et al. [31]). The filter generates the estimate $\hat{z}(s)$:

$$\hat{z}(s) = F(s)y(s).$$

Combining $F(s)$ with $P(s)$ forms a new transfer function, $G(s)$, as shown in figure 1. The estimation task can be summarized as: minimize the estimation error $e(s)$ (6) in the presence of the exogenous inputs $[d(s) \ n(s)]^T$:

$$e(s) = z(s) - \hat{z}(s) = G(s) \begin{bmatrix} d(s) \\ n(s) \end{bmatrix}.$$  

Now, the estimation design procedure can now be summarized as: given $P(s)$, design $F(s)$ such that $G(s)$ is small. Thus, we quantify the size of $G(s)$ via a $H_2$-norm, which is defined as:

$$||G||_2 = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Trace}[G^*(j\omega)G(j\omega)]d\omega} = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \sum \sigma_i(j\omega)^2 d\omega},$$

where $\sigma_i(j\omega)$ are the singular values of $G(j\omega)$. This expression is known as the $H_2$-norm of the transfer function $G(s)$.
where $\sigma_i$ are the singular values of $G$ at frequency $\omega$. The singular values represent a transfer function’s gain when there are multiple inputs and multiple outputs, and hence the $H_2$-norm can be considered as an average gain over all frequencies and all directions.

### 2.4. DNS dataset

We require two sets of time-resolved DNS data: first, the velocity or shear stress fields in the measurement plane(s) to provide the input for the estimator, and second the velocity field in the estimation plane(s) to compare the estimates with the truth. These data sets are provided by the Polytechnic University of Madrid (UPM). They were generated by direct numerical simulations (DNS) performed in a channel at $Re_\tau = 2000$. The streamwise and spanwise directions are discretised by Fourier expansion and the wall-normal direction is discretised using a compact difference scheme of 7th order. See table 1 for the details on the spatial dimensions and wavenumbers considered. Only positive spanwise wavenumbers ($k_y$) are considered, because the data is real-valued in physical space, and therefore the coefficients for modes ($+k_x, +k_y$) are those for ($+k_x, -k_y$) [32]. The largest temporal frequency is approximated using Taylor’s hypothesis as: $\omega_{max} = \max(|k_x|)U_c = 195$, where $U_c$ is the velocity at the channel centre, and $\max(|k_x|)$ the largest streamwise wavenumber considered in this study. We therefore have $2\pi/(\omega_{max}\Delta t) = 3$ samples per period for the highest frequency, which fulfils the Nyquist criterion. Refer to table 2 for the temporal dimensions. Full details of the DNS dataset can be found in [32].

### 2.5. Estimator performance

The filter uses the sensor measurements $y(t)$ from the DNS data set at a chosen wall-normal height and provides estimates $\hat{z}(t)$ at different wall-normal heights by simulating the filter in time. The error $e(t) = \hat{z}(t) - z(t)$ is then formed by comparing the true flow fields $z(t)$ at the estimated wall-normal heights with their estimates $\hat{z}(t)$. The $H_2$-norm quantifies the performance, which

![Diagram](image)

**Figure 1.** The combined Plant $P(s)$ and Filter model $F(s)$. The overall transfer function $G(s)$ describes the input-output relationship of the estimation problem.

### Table 1. The $Re_\tau = 2000$ channel flow spatial dimension

| Direction  | Size  | Spacing | Resolved data | Smallest wavelength | Range       |
|------------|-------|---------|---------------|---------------------|-------------|
| Streamwise ($x$): | $8\pi$ | $\Delta k_x = 1/4$ | 65 wavenumbers | $\lambda_x \geq 2\pi h/8 \approx 0.785$ | $|k_x| \leq 8$ |
| Spanwise ($y$):   | $3\pi$ | $\Delta y_z = 2/3$ | 33 wavenumbers | $\lambda_y \geq 6\pi h/64 \approx 0.295$ | $0 \leq k_y \leq 64/3$ |

### Table 2. The $Re_\tau = 2000$ channel flow temporal dimension used for estimation

| $Re_\tau$ | $\Delta t$ | $t_{max}$ | $U_c$ | $U_c\Delta t$ | $t_{max}U_c/(8\pi)$ |
|------------|-------------|-----------|-------|---------------|---------------------|
| 2000       | 0.0111      | 12.72     | 24.37 | 0.272         | 12.33               |
is obtained by integrating the error in time: $||e||_2 = \left[ \int_0^{t_{\text{max}}} e^*(t) e(t) dt \right]^{1/2}$. It is normalised by $||z||_2 = \left[ \int_0^{t_{\text{max}}} z^*(t) z(t) dt \right]^{1/2}$ to form the performance parameter:

$$\gamma = \frac{||e||_2}{||z||_2}$$  \(8\)

We can also generate performance predictions $\gamma_{\text{pred}}$ of the filter by taking $||G||_2$ (7) and normalising it with the $H_2$ norm of $P$. This is equivalent to taking sensor readings $y(t)$ form the linear model itself, obtain estimates $\hat{z}(t)$ at different wall-normal heights with the filter, and then extract the true flow fields $z(t)$ at the estimated wall-normal heights from the linear model.

3. Results

We now provide the linear estimator (described in §2.3) with time-resolved DNS data at a single wall-normal height and use it to estimate the time-resolved velocity at other wall-normal heights. In practice this means forming an estimator for each of the $33 \times 65 = 2145$ wavenumber pairs of interest (see table 1).

3.1. Estimation in Physical space

We begin by looking at the estimator performance in physical space by comparing the estimate of the streamwise velocity with the true field from DNS. The filter is supplied with all three velocity components at a single wall height of $z = 0.2$ ($z^+ = 400$). Figure 2 compares the estimated streamwise velocity with the true field at $z = 0.1$ ($z^+ = 200$) at a single instant in time (parts (a) and (b)); and as a function of time for a single point in the plane (part (c)). Figure 3 repeats the analysis of figure 2, this time for a wall height of $z = 0.3$ ($z^+ = 600$). In both figures we see good agreement between the model-based estimate and the true velocity field.

To show how estimation performance varies with wall-normal distance in physical space we have also included the streamwise velocity perturbations at $y = \pi/4$ in figure 4 and at $x = 2\pi/4$ in figure 5. Good agreement is seen between the model based estimated and the true velocity field, but it reduces with distance from the sensor. The average magnitude of the streamwise velocity component is over-predicted when estimating near the wall. Similarly it is under-predicted when estimating near the channel centre.

3.2. Estimation in wavenumber space

We now provide a more quantitative assessment of the estimator by examining its performance across all wavenumber pairs of interest (see table 1). As in §3.1, the estimator receives all three velocity components at $z = 0.2$ and uses them to estimate all three velocity components at $z = 0.1$ and $z = 0.3$.

Figure 6(a) plots $\gamma$ as a function of wavenumber pair $(k_x, k_y)$. In order to achieve better convergence, we calculate $\gamma$ in each of the two halves of the channel and average over the two. The colour scale is chosen to cover the range [0, 0.7] to focus on the region over which the estimator performs well. (The maximum value of $\gamma$ across all wavenumber pairs is 0.94.) We see that the estimator performs best for wavenumbers satisfying approximately $|k_x| \leq 2$ and $|k_y| \leq 10$.

By assuming that the unknown disturbances $d$ are random forcing which is white in space and white in time we can also calculate the predicted estimator performance, which we label $\gamma_{\text{pred}}$. For this predicted estimator performance, the filter is designed for the linear model, but also applied to the linear model, and thus the DNS data is not needed at all. The predicted estimator performance $\gamma_{\text{pred}}$ is shown in figure 6(b), from which we see good agreement with the performance that is achieved in DNS. This is an encouraging result because it suggests that the physics of the largest scales are well-captured by the linear model.
3.3. Shear stress measurements

Velocity measurements inside the flow might not be available in an experimental setup. Therefore we now consider the performance of the estimator when its input is instead the streamwise wall shear stress, $\tau_x$ (see Appendix A).

Figure 7 is a repeat of figure 6 for streamwise wall shear stress measurements. (The quantity estimated is still all three velocity components at $z = 0.1$ and $z = 0.3$ and the maximum value of $\gamma$ across all wavenumber pairs is 1.) It is clear from figure 7 that the estimator performance is reduced significantly relative to figure 6 (i.e. relative to velocity measurements at $z = 0.2$).

To better compare results for these two measurement types, we now plot $\gamma$ and its prediction $\gamma_{\text{pred}}$ over a range of spanwise wavenumbers $k_y$ for all $|k_x| \leq 0.5$ (averaged across $k_x$ to improve convergence). This comparison is shown in figure 8(a). The predicted performance, $\gamma_{\text{pred}}$, achieves its minimum for $1.4 < k_y < 2.1$, which is consistent with the results of Illingworth et al. [8].

This corresponds well with the range of $k_y$ over which the potential for transient growth is greatest [19]; and over which the largest amplification is seen for both stochastic and harmonic forcing [21]. It appears, however, that the best performance (i.e. smallest $\gamma$) achieved by the estimator in DNS occurs at a slightly higher spanwise wavenumber of $k_y \approx 3.33$. In a similar way, in figure 8(b) we plot $\gamma$ and $\gamma_{\text{pred}}$—this time as a function of the streamwise
Figure 3. Estimation of the streamwise velocity perturbation at $z = 0.3$ ($z^+ = 600$) using the linear filter. Legend same as figure 2.

Figure 4. Estimation of the streamwise velocity perturbation at $y = 3\pi/4$ ($y^+ = 4712$) using the linear filter: (a) DNS data; and (b) linear estimate using measurements of all three velocity components at $z = 0.2$ ($z^+ = 400$). Sixty-five contour levels are shown from $u = -5.5$ to $u = +5.5$ (red).
Figure 5. Estimation of the streamwise velocity perturbation at $x = 2\pi/4 \ (y^+ = 9425)$ using the linear filter: (a) DNS data; and (b) linear estimate using measurements of all three velocity components at $z = 0.2 \ (z^+ = 400)$. Sixty-five contour levels are shown from $u = -5.5$ to $u = +5.5$ (red).

Figure 6. (a) Normalized estimation error $\gamma$ as a function of $(k_x, k_y)$ and (b) the prediction of the estimation error $\gamma_{\text{pred}}$. The estimator receives all three velocity components at $z = 0.2$ and uses them to estimate all three velocity components at $z = 0.1$ and $z = 0.3$. The same colour scale $\gamma \in [0, 0.7]$ is used for both plots.

Figure 7. Normalized estimation error $\gamma$ as a function of $(k_x, k_y)$ and (b) the prediction of the estimation error $\gamma_{\text{pred}}$. The estimator receives the streamwise shear stress component at the wall and uses it to estimate all three velocity components at $z = 0.1$ and $z = 0.3$. The same colour scale $\gamma \in [0, 0.7]$ is used for both plots.
Figure 8. Normalized estimation error $\gamma$ (−□−) and its prediction $\gamma^{\text{pred}}$ (—) : (a) as a function of $k_y$ averaged over all $k_x \leq |0.5|$; and (b) as a function of $k_x$ averaged over all $4 \leq k_y \leq 8$. Setup same as figure 6 for black results and figure 7 for blue results. The DNS results have been averaged across the two halves of the channel to improve convergence.

wavenumber $k_x$—for all $k_y$ satisfying $4 \leq k_y \leq 8$ (again, averaged across these $k_y$ to improve convergence). We see that, for both measurement types, the predicted performance agrees well with the performance actually achieved in DNS. For shear stress measurements, however, we do observe some more significant differences between the predicted and true performance (up to 22%) for the smallest values of $|k_x|$.

3.4. Variation of estimation performance with wall-normal height

So far we have looked at the estimator performance when the measurement location is fixed (at $z = 0.2$ or at the wall) and when estimation is performed at $z = 0.1$ and $z = 0.3$. It is now interesting to investigate the estimator performance as these measurement and estimation locations are varied. To make the parameter space under consideration manageable, we will do this only for streamwise wavenumbers satisfying $|k_x| \leq 0.5$; and for four individual spanwise wavenumbers: $k_y = 2, 4, 6$ and 8.

We begin by fixing the measurement location at $z = 0.2$ as before, and plotting the variation of the estimator performance as the location of the (single) estimation plane is varied. This is shown in figure 9. We also show results when the measurement is instead the streamwise shear stress at the wall. The results are shown separately for the top half and bottom half of the channel to give some indication of the convergence of the results. We can make a number of observations which apply to all four spanwise wavenumbers: i) the best performance (obviously) occurs at the sensor location and increases as we move away from it; ii) the worst performance occurs at the channel centre; iii) within the proximity of the measurement location, the true performance ($\gamma$) is always better than that predicted ($\gamma^{\text{pred}}$); and (iv) a local maximum occurs in $\gamma$ in the log region at approximately $z = 0.025$ or $z^+ = 50$. An extra observation can be made for the $k_y = 2$ and $k_y = 4$ cases: (v) a local minimum occurs in $\gamma$ in the buffer layer at approximately $z = 0.0075$ or $z^+ = 15$. This could be due to the wall signature of the (relatively large) eddies that are measured at certain $z_m$ [32, 33].

Having fixed the measurement location for the velocity sensor at a single plane, we can now also vary it by adding a third dimension. This is shown in figure 10, where the estimator performance $\gamma$ is plotted as a function of both the measurement location $z_m$ and the estimation location $z_e$. As one would expect, the best performance is achieved when the measurement and estimation locations coincide, i.e. when $z_e = z_m$. It is interesting that, for estimation locations in the vicinity of a given measurement location, the true estimator performance ($\gamma$) is in general
Figure 9. Normalized estimation error $\gamma$ for the two halves of the channel ($\cdots$) and its prediction $\gamma_{\text{pred}}$ ($\cdots$) as a function of wall normal location $z$ averaged over all $k_x \leq |0.5|$. This is plotted for (a) $k_y = 2$; (b) $k_y = 4$; (c) $k_y = 6$; (d) $k_y = 8$. The estimator measures all three velocity components for the black results and the streamwise shear stress component for the blue results.

smaller (i.e. better) than that predicted by the linear model ($\gamma_{\text{pred}}$).

4. Discussion

We now give some broader discussion on the results obtained for the two measurement types considered (i.e. velocity measurements and wall shear stress measurements).

Using velocity measurements at a single wall-normal location we are able to estimate the flow at other wall-normal locations with reasonable accuracy, as demonstrated in figures 2 - 6. This is consistent with the results observed in an earlier study [8] at $Re_\tau = 1000$. The present study at $Re_\tau = 2000$, however, benefits from the availability of significantly more time-resolved DNS data, and as a result we see significantly improved convergence in the normalized estimation error ($\gamma$) in figure 6. Importantly, the linear model (1) not only achieves good estimation performance, but is also able to predict the range of wavenumber pairs (figures 6 and 8) and the range of wall heights (figures 9 and 10) over which the estimator should perform well. In this respect the results in figure 6—which perhaps benefit most from the improved convergence described above—are particularly encouraging. Together these results suggest that, for the largest scales considered, linear mechanisms are important, and that they are well-represented by the linear model (1).

By measuring instead the wall shear stress it is still possible to achieve reasonable estimation performance for the largest scales. However, it is clear in figures 6 and 7 that the estimator is significantly less successful than its velocity-based counterpart. This finding is consistent with studies carried out for laminar profiles [2,26]: the leading eigenvector of the linear system
Figure 10. Normalized estimation error $\gamma$ (top row) and its prediction $\gamma_{\text{pred}}$ (bottom row) as function of the wall-normal measurement location $z_m$ and the wall-normal estimation location $z_e$. This is plotted for (a,e) $k_y = 2$; (b,f) $k_y = 4$; (c,g) $k_y = 6$; (d,h) $k_y = 8$. All results are averaged over all $k_x \leq |0.5|$ and the two channel halves.

includes several centred modes, with little support near the wall. (The modeshapes are only weak functions of Reynolds number.)

We also observe in figure 8 that the agreement between the true estimator performance ($\gamma$) and its prediction ($\gamma_{\text{pred}}$) is not as good as that seen for velocity measurements. This is an interesting observation, the cause of which is still under investigation. One possibility is that, for wall-based measurements, the assumption on $d$ in (1) of delta-correlated forcing (in space and in time) is less appropriate—in other words, that second-order statistics become more important. Zare et al. [34] have recently considered coloured forcing in the linearized Navier Stokes equation to reproduce second order dynamics. It would be interesting to see if this modified linear model is able to improve i) the estimator performance and ii) the agreement between the estimator performance ($\gamma$) and its prediction ($\gamma_{\text{pred}}$)—both for velocity measurement and for wall shear stress measurements.

5. Conclusions
A linear Navier–Stokes-based model has been used to design a Kalman filter for fully-developed turbulent channel flow. The Kalman filter is able to estimate the largest scales with reasonable accuracy at $Re_\tau = 2000$. The results build on earlier work [8] which was performed at $Re_\tau = 1000$, and for which significantly less time-resolved data were available. Using velocity measurements as input, the filter is able to estimate the largest scales with reasonable accuracy for a range of wavenumber pairs, measurement planes, and estimation planes. Furthermore there is good agreement between the performance predicted by the model and the performance actually achieved in DNS. Using wall shear stress measurements as input, the filter is still able
to estimate the largest scales with reasonable accuracy, although the filter’s performance is significantly reduced relative to the velocity-based filter. It remains to be seen whether either measurement strategy is good enough for control purposes. The results warrant further research on the linear model and its applications. A further application currently under consideration is the optimal sensor placement problem: If one has access only to a single sensor, where should it be placed, and what quantity should be measured?

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Appendix A. Linear state space model of the flow

The state-space model (3) of the linear model is (1):

\[
\frac{d}{dt} \begin{bmatrix}
\hat{w} \\
\hat{\eta}
\end{bmatrix} = \begin{bmatrix}
\Delta^{-1}L_{OS} & 0 \\
-ik_y U' & L_{SQ}
\end{bmatrix} \begin{bmatrix}
\hat{w} \\
\hat{\eta}
\end{bmatrix} + \begin{bmatrix}
-ik_x \Delta^{-1}D & -ik_y \Delta^{-1}D & -k^2 \Delta^{-1}
\end{bmatrix} \begin{bmatrix}
d_x \\
d_y \\
d_z
\end{bmatrix},
\]

(A.1a)

\[
\begin{bmatrix}
\hat{u} \\
\hat{v} \\
\hat{w}
\end{bmatrix} = \frac{1}{k^2} \begin{bmatrix}
ik_x D & -ik_y \\
iki_y D & ik_x \\
k^2 & 0
\end{bmatrix} \begin{bmatrix}
\hat{w} \\
\hat{\eta}
\end{bmatrix},
\]

(A.1b)

where

\[
L_{OS} = -jk_x U(z)\Delta + ik_x U''(z) + \nu_T \Delta^2 + 2\nu_T' D \Delta + \nu''_T (D^2 + k^2),
\]

(A.2a)

\[
L_{SQ} = -jk_x U(z) + \nu_T \Delta + \nu'_T D,
\]

(A.2b)

\[D = \frac{\partial}{\partial z}, \quad ()' = \frac{\partial}{\partial z}(), \quad k^2 = k_x^2 + k_y^2, \quad \text{and} \quad \Delta = D - k^2. \]

The boundary conditions are:

\[
\hat{w}(t) = \hat{w}'(t) = \hat{\eta}(t) = 0.
\]

To obtain instead the wall shear stress \([2,35]\) we replace (A.1b) with:

\[
\tau_x = \frac{1}{Re} \left. \frac{d\hat{u}}{dz} \right|_{wall} = \frac{i}{Re} \left. \frac{k_x D^2 - k_y D}{k^2} \right|_{wall} \begin{bmatrix}
\hat{w} \\
\hat{\eta}
\end{bmatrix}. 
\]

(A.3)