TOPOLOGICAL GENERATION OF LINEAR ALGEBRAIC GROUPS I

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Abstract. Let $C_1, \ldots, C_e$ be noncentral conjugacy classes of the algebraic group $G = \text{SL}_n(k)$ defined over a sufficiently large field $k$, and let $\Omega := C_1 \times \ldots \times C_e$. This paper determines necessary and sufficient conditions for the existence of a tuple $(x_1, \ldots, x_e) \in \Omega$ such that $\langle x_1, \ldots, x_e \rangle$ is Zariski dense in $G$. As a consequence, a new result concerning generic stabilizers in linear representations of algebraic groups is proved, and existing results on random $(r, s)$-generation of finite groups of Lie type are strengthened.

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1. Introduction and statement of results

Let $C_1, \ldots, C_e$ be conjugacy classes of a group $G$, and $\Omega := C_1 \times \ldots \times C_e$. A familiar problem is to specify conditions for tuples $\omega \in \Omega$ to generate $G$, or to generate a subgroup of $G$ with special properties. For instance, let $C_1, \ldots, C_e$ be conjugacy classes of $G = \text{SL}_n(\mathbb{C})$. The Deligne-Simpson problem asks to provide necessary and sufficient conditions for the existence of a tuple $(x_1, \ldots, x_e) \in \Omega$ such that $\langle x_1, \ldots, x_e \rangle$ acts irreducibly on the natural module, and $x_1 \ldots x_e = 1$. A solution to this question yields information concerning monodromy groups of regular systems of differential equations on $\mathbb{C}P^1$ (see [28] for further details). Posed by Deligne, this problem has been studied in numerous settings and generalizations [11, 23, 24, 25, 26, 27, 29, 28, 35].

Now assume $C_1, \ldots, C_e$ are noncentral conjugacy classes of the algebraic group $G = \text{SL}_n(k)$ where $k$ is an uncountable algebraically closed field of any

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characteristic. Say \((x_1, \ldots, x_e) \in \Omega\) generates \(G\) topologically if \(\langle x_1, \ldots, x_e \rangle\) is Zariski dense in \(G\). Since \(G(k)\) is infinitely generated, the relevant finitary notion of generation is topological. This paper determines necessary and sufficient conditions for the existence of a tuple \(\omega \in \Omega\) which topologically generates \(G\). Since \(G(k)\) is infinitely generated, the relevant finitary notion of generation is topological. The existence of such an \(\omega \in \Omega\) implies that generic tuples in \(\Omega\) generate \(G\) topologically, with generic taken in the appropriate sense. The solution to this question has applications to generic stabilizers in linear representations of algebraic groups, and random \((r, s)\)-generation of finite groups of Lie type. In future work \([7\] \[15\], conditions are determined for the existence of tuples \(\omega \in \Omega\) which topologically generate classical and exceptional algebraic groups, and related applications are discussed.

Our main result is as follows.

**Theorem 1.1.** Let \(C_1, \ldots, C_e\) be noncentral conjugacy classes of the algebraic group \(G = SL_n(k)\), where \(k\) is an uncountable algebraically closed field, and \(n \geq 3\). Let \(\gamma_i\) be the dimension of the largest eigenspace of \(C_i\). Then there is a tuple \(\omega \in \Omega\) topologically generating \(G\) if and only if the following conditions hold:

1. \(\sum_{i=1}^e \gamma_i \leq n(e - 1)\);
2. it is not the case that \(e = 2\) and \(C_1, C_2\) have degree two minimal polynomials.

Let us make a few comments about the theorem. First of all, the assumption that \(k\) is an uncountable algebraically closed field is made for the sake of convenience. It is evident some condition on \(k\) is required, as \(G(k)\) is locally finite when \(k = \mathbb{F}_p\). However once the main theorem has been established over uncountable fields, analogous results can be recovered under much weaker hypotheses on \(k\). In characteristic \(p = 0\), the result holds over any field (see Corollary 3.8 below). In positive characteristic, the result holds over any algebraically closed field of positive transcendence degree (see Corollary 3.9), or any field of infinite transcendence degree over the prime field.

Now let us describe the proof strategy for Theorem 1.1. The notion of a generic property underlies much of the reasoning given below, so let us fix its meaning in the present context. Assume \(X(k)\) is an irreducible variety defined over an uncountable algebraically closed field. A generic subset of \(X(k)\) contains the complement of a countable union of proper closed subvarieties of \(X(k)\). The complement of a generic subset is a *meagre* set.

To begin, choose noncentral conjugacy classes \(C_1, \ldots, C_e\) of \(G = SL_n(k)\). Let \(M\) be a closed subgroup of \(G\), and

\[ Y_M := \{(x_1, \ldots, x_e) \in \Omega \mid \langle x_1, \ldots, x_e \rangle \subset M^g, \text{ for some } g \in G\} \]

Up to conjugacy, there exist finitely many maximal closed subgroups of \(G\) which are positive-dimensional. Call these subgroups \(M_1, \ldots, M_t\). Note if generic tuples in \(\Omega\) generate an infinite group not contained in \(\bigcup_{i=1}^t Y_{M_i}\), then generically tuples in \(\Omega\) generate a positive-dimensional subgroup of \(G\).
contained in no conjugate of $M_i$, $1 \leq i \leq t$, and hence must generate $G$ topologically.

The argument proceeds by induction on the dimension of the natural module. In the base case, the maximal subgroup structure of $G$ is completely understood. For any closed subgroup $M$ of $G$, Lemma 2.1 provides the useful bounds $\dim Y_M \leq \dim \prod_{j=1}^e (C_j \cap M) + \dim G/M$. By computing dimensions of varieties, it is shown

$$\dim Y_M \leq \dim \prod_{j=1}^e (C_j \cap M) + \dim G/M < \dim \Omega$$

for each maximal closed subgroup, and each subfield subgroup $M$ of $G$. It follows generic tuples in $\Omega$ generate an infinite group not contained in $\bigcup_{i=1}^t Y_{M_i}$.

For the inductive argument, slightly more indirect reasoning is given. Using tools from algebraic geometry and representation theory, it is shown generic tuples in $\Omega$ generate a group acting irreducibly and primitively on the natural module, which contains “strongly regular” elements of infinite order. It is then shown no proper maximal closed subgroup of $G$ can share all these properties simultaneously.

Let us now turn to applications of the main theorem. The first concerns linear representations of algebraic groups. Let $V$ be a linear $G$-variety, and recall for any $x \in V$ the point stabilizer of $x$ in $G$ is written $G_x := \{g \in G \mid gx = x\}$. A subgroup $H$ is said to be a generic stabilizer of $G$ on $V$ if there exists a nonempty open subset $V_0 \subset V$ such that $G_x$ is a conjugate of $H$ for every $x \in V_0$.

In characteristic $p = 0$, the cardinality of a generic stabilizer, and in particular the question of whether a generic stabilizer is finite or trivial, arises frequently in the context of invariant theory. A well-known result states that a generic stabilizer of a connected simple irreducible algebraic group $G$ is nontrivial if and only if its algebra of invariant polynomials $k[V]^G$ is free (see Theorem 8.8 of [42]). In [11] and [40], Elašvili and Popov classify when a generic stabilizer of a simple algebraic group $G$ is finite or trivial.

For semisimple algebraic groups, the question of when a generic stabilizer is trivial is studied in [11], [12] and [42]. More recently, the question of when a generic stabilizer is finite but nontrivial appears in works of M. Bhargava, B.H. Gross, X. Wang and W.Ho ([2], [3], [4], [5]).

In arbitrary characteristic, work of Burness-Guralnick-Liebeck-Testerman [20] shows if $\dim V > \dim G$, then generically a point stabilizer is finite. Garibaldi-Guralnick [13] shows if $\dim V > \dim G$, then in most cases a generic stabilizer is trivial as a group scheme. Furthermore the cardinality of a generic stabilizer is closely related to base sizes for algebraic groups, an area of investigation recently initiated in Burness-Guralnick-Saxl [8].
In Section 4, we use topological generation to establish a new bound \( \alpha \) on the dimension \( V \) such that a generic stabilizer is trivial whenever \( \dim V > \alpha \). This bound holds in arbitrary characteristic.

**Corollary 1.2** (Bounds on generic stabilizers of linear groups). Let \( G = SL_n(k) \) be an algebraic group with \( k \) an algebraically closed field. Assume \( V \) is a linear \( G \)-variety such that \( V^G = 0 \). If \( \dim V > \frac{\alpha}{2}n^2 \), then a generic stabilizer is trivial.

A second application of topological generation concerns a classical group-theoretic problem: the random generation of finite simple groups. The following questions date back to the 19th century:

(i) does the probability that two random elements of \( \text{Alt}_n \) generate \( \text{Alt}_n \) tend to 1 as \( n \to \infty \)?

(ii) which finite simple groups appear as quotients of the modular group \( PSL_2(\mathbb{Z}) \)?

The first question was posed by Netto in 1892 and answered affirmatively by Dixon [10] in 1969, where he conjectured the same property holds of all finite simple groups. In other words, for any finite simple group \( H \) (of some fixed type \( A_n, B_n, \) etc.), the probability that two random elements of \( H \) generate the entire group tends to 1 as \( \vert H \vert \to \infty \). This conjecture was proved by Kantor-Lubotsky [22] and Liebeck-Shalev [31] in the 1990s using probabilistic methods.

The solution of Dixon’s conjecture led to many more fine-grained questions concerning random generation of finite simple groups. A natural refinement of the above question asks whether two random elements of prime orders \( r \) and \( s \) generate \( H \) as \( \vert H \vert \to \infty \). This property is called random \((r,s)\)-generation. Perhaps surprisingly given the strength of the statement, this property has also been found to hold in many cases [32], although counterexamples exist [33], and some questions remain. In particular, by work of Liebeck-Shalev [32] it is known that finite simple classical groups have random \((r,s)\)-generation when the rank of the group is large enough (depending on the orders of \( r, s \)).

Although not immediately obvious, question (ii) stated above concerns \((r,s)\)-generation of finite simple groups. \( PSL_2(\mathbb{Z}) \) is isomorphic to the free product \( C_2 * C_3 \), and hence the question may be restated in the following form: which finite simple groups are \((2,3)\)-generated? Liebeck-Shalev [33] discusses the geometric background of this problem, and its long history in the literature. In [33], it is shown all finite simple classical groups have random \((2,3)\)-generation except \( PSp_4 \). In particular, aside from \( PSp_4 \) all but finitely many finite simple classical groups appear as quotients of \( PSL_2(\mathbb{Z}) \). This result is extended to the remaining finite simple groups (except Suzuki groups) in Guralnick-Liebeck-Lübeck-Shalev [18].

Using random \((2,3)\)-generation as a guide, one might expect random \((r,s)\)-generation to hold with orders \( r, s \) independent of the rank of the
Before proving Theorem 1.1, it will be helpful to establish some general facts about algebraic groups required for our arguments. This background material will be stated in slightly greater generality than necessary so that it may be applied in future work. In [15] conditions are determined for the existence of tuples $\omega \in \Omega$ which topologically generate a classical algebraic group $G$. Applications to generic stabilizers and random $(r, s)$-generation are then discussed. In Burness-Gerhardt-Guralnick [7], bounds on fixed point spaces are used to establish topological generation results for exceptional algebraic groups, and additional applications are considered.

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**2. Preliminary lemmas for simple algebraic groups**

In this section (unless otherwise stated) let $G$ be a simple algebraic group defined over an algebraically closed field. When $G$ is a classical group, let $V$ be the natural module for $G$. Finally, let $C_1, \ldots, C_e$ be noncentral conjugacy classes of $G$, and $\Omega := C_1 \times \cdots \times C_e$. Say a tuple $(x_1, \ldots, x_e) \in G^e$ has a given group property if the closure of $\langle x_1, \ldots, x_e \rangle$ has that property.

For any closed subgroup $M$ of $G$, let:

- $\Delta := \prod_{j=1}^e (C_j \cap M)$
- $X := \bigcup_{g \in G} (M^g \times \cdots \times M^g) \subset G^e$
- $\varphi : G \times \Delta \to X$ be the map $\langle g, (x_1, \ldots, x_e) \rangle \mapsto (x_1^g, \ldots, x_e^g)$
- $Y_M := \{ (x_1, \ldots, x_e) \in \Omega \mid \langle x_1, \ldots, x_e \rangle \subset M^g, \text{ for some } g \in G \}$

Note for any such $M$, $(x_1^g, \ldots, x_e^g) \in \text{im}(\varphi)$ if and only if $(x_1, \ldots, x_e) \in \Omega$ and $\langle x_1, \ldots, x_e \rangle \subset M^g$ for some $g \in G$. In particular, $\text{im}(\varphi) = Y_M$. Up to conjugacy, there exist finitely many maximal positive-dimensional closed subgroups of $G$ (see Corollary 3 in Liebeck-Seitz [37]). Call these subgroups $M_1, \ldots, M_i$. As noted earlier, the overall strategy is to prove generic tuples in $\Omega$ topologically generate an infinite group not contained in $\bigcup_{i=1}^i Y_{M_i}$. In the base case, the following lemma allows us to establish
this fact by performing explicit calculations on the dimensions of relevant varieties.

**Lemma 2.1.** Let $M$ be a closed subgroup of a simple algebraic group $G$ defined over an algebraically closed field, and let $C_1, \ldots, C_e$ be conjugacy classes of $G$. Then $\dim Y_M \leq \dim \Delta + \dim G/M$.

**Proof.** To begin, note any $(x_1, \ldots, x_e)^g \in \text{im}(\varphi)$ will be in the same orbit as $(x_1, \ldots, x_e) \in \Delta$, and hence their fibers will have the same dimension. So without loss of generality, assume $(x_1, \ldots, x_e) \in \Delta$ and consider the map $\varphi' : M \times \Delta \to \Delta, (g, (x_1, \ldots, x_e)) \mapsto (x_1^g, \ldots, x_e^g)$. Clearly $\text{im}(\varphi') = \Delta$, and hence every fiber of $\varphi$ has dimension at least $\dim M$. By the fiber theorem (see Corollary 4 of EGA III [16]), $\dim Y_M \leq \dim \Delta + \dim G/M$.

Over an uncountable field, proving generic tuples in $\Omega$ topologically generate $G$ reduces to showing $\dim \Omega > \dim \Delta + \dim G/M$ for each maximal closed subgroup and each subfield subgroup $M$ of $G$.

For the inductive argument, it is shown generic tuples in $\Omega$ generate a group having a list of properties that no subfield subgroup nor positive-dimensional maximal subgroup can share. Typically this argument proceeds in two stages. The first is to prove the existence of a tuple $\omega \in \Omega$ having some desirable property, and the second is to show that this property is either an open or generic condition.

The following is a version of Burnside’s Lemma for algebraic groups. It allows us to recast Lemma 2.1 in a particularly geometric light, and relates condition (i) of Theorem 1.1 to the existence of an open subset of tuples $\Omega_0 \subset \Omega$ which fix no 1-dimensional subspace of $V$.

**Lemma 2.2.** Let $M$ be a closed subgroup of an algebraic group $G$, with $G$ acting on $G/M$. Then for $x \in M$,

$$\dim G/M - \dim (G/M)^x = \dim C - \dim (C \cap M)$$

where $(G/M)^x$ is the set of fixed points of $x$, and $C$ is the conjugacy class of $x$ in $G$.

**Proof.** This is Proposition 1.14 in [30].

This immediately yields a useful bounds on fixed point spaces.

**Lemma 2.3.** Let $M$ be a closed subgroup of a simple algebraic group $G$, and $x_1, \ldots, x_e$ be elements of the conjugacy classes $C_1, \ldots, C_e$. If $\Omega \subset Y_M$, then $\sum_{j=1}^e \dim (G/M)^{x_j} \geq (e - 1) \cdot \dim G/M$.

**Proof.** Assume $\Omega \subset Y_M$. Lemma 2.1 shows:

$$\dim \Omega \leq \sum_{j=1}^e \dim (C_j \cap M) + \dim G/M$$
Summing from 1 to \(e\), Lemma 2.2 yields:

\[
\dim \Omega = \sum_{j=1}^{e} \dim G/M - \sum_{j=1}^{e} \dim (G/M)^{x_j} + \sum_{j=1}^{e} \dim (C_j \cap M)
\]

Combining these facts:

\[
\sum_{j=1}^{e} \dim (G/M)^{x_j} \geq \left( \sum_{j=1}^{e} \dim G/M \right) - \dim G/M
\]

Now assume \(G\) is a classical algebraic group, and \(M\) is a maximal parabolic subgroup of \(G\) (so that \(M\) is the stabilizer of a totally singular subspace \(W \subset V\)). For linear and symplectic groups, all 1-dimensional subspaces of \(V\) are totally singular, and hence \(G/M\) is isomorphic to projective space \(P_1(V)\). To prove there exists a tuple \(\omega \in \Omega\) fixing no 1-dimensional subspace of \(V\), it suffices to show \(\Omega \not\subset X\). (For orthogonal groups, 1-dimensional non-degenerate subspaces must also be considered).

**Lemma 2.4.** Let \(G\) be a linear or symplectic algebraic group defined over an algebraically closed field, and let \(C_1, ..., C_e\) be conjugacy classes of \(G\). Then \(\sum_{j=1}^{e} \gamma_j \leq n(e - 1) \iff\) there is a tuple \(\omega \in \Omega\) fixing no 1-dimensional subspace of \(V\).

**Proof.** The “only if” direction follows from Lemma 3.6 below.

For the converse direction, assume \(\Omega \subset X\) where \(M\) is the stabilizer of a 1-dimensional subspace of \(V\). Let \(\varphi : G^e \times G/M \to G^e\) be the natural projection map. Since \(M\) is parabolic, \(G/M\) is a complete variety and \(\varphi\) is a closed map. However, \(\varphi(Z) = X\) where

\[
Z = \{(x_1, ..., x_e, v) \mid x_1v = ... = x_ev = v\}
\]

So \(X\) is closed, and in particular \(\Omega \subset Y_M\). Now applying Lemma 2.3,

\[
\sum_{j=1}^{e} \dim (G/M)^{x_j} \geq (e - 1) \cdot \dim G/M
\]

Note the irreducible components of \((G/M)^x\) are projective spaces associated with each eigenspace of \(x\). In particular the dimension of \((G/M)^x\) is one less than the dimension of the largest eigenspace of \(x\) acting on the natural module. So \(\dim (G/M)^{x_j} = \gamma_j - 1\), and \(\dim G/M = n - 1\). Applying this information to the above inequality yields \(\sum_{j=1}^{e} \gamma_j - e \geq (e - 1)(n - 1)\). This implies \(\sum_{j=1}^{e} \gamma_j > n(e - 1)\). \(\Box\)

**Remark.** No restrictions were placed on the conjugacy classes \(C_1, ..., C_e\) of \(G\) since only 1-dimensional subspaces of \(V\) were being considered. When moving to the stabilizers of higher dimensional subspaces, different arguments are required to compute \((G/M)^x\) depending on whether \(x \in C_j\) is
semisimple, unipotent, or mixed. This issue arises when giving inductive arguments to establish topological generation for symplectic and orthogonal algebraic groups.

It can now be shown under the stated hypothesis that an open subset of tuples $\Omega_0 \subset \Omega$ fix no 1-dimensional subspace of $V$. Let $G$ be any simple algebraic group $G$, and $V$ be a rational $kG$-module. Furthermore let $R_{d,e}(V)$ be the set of tuples in $G^e$ fixing a common $d$-dimensional subspace of $V$, $I_{d,e}(V)$ be the set of tuples in $G^e$ that fix no $d$-dimensional subspace of $V$, and $I_e(V)$ be the set of tuples in $G^e$ that act irreducibly on $V$. The following lemma is well-known.

**Lemma 2.5.** Let $G$ be a simple algebraic group defined over an algebraically closed field $k$, and let $V$ be a rational $kG$-module. Then

(i) $R_{d,e}(V)$ is a closed subvariety of $G^e$.

(ii) $I_{d,e}(V)$ is an open subvariety of $G^e$.

(iii) $I_e(V)$ is an open subvariety of $G^e$.

**Proof.** Parts (i) and (ii) follow from Lemma 11.1 in Guralnick-Tiep [19]. For part (iii), assume $V$ is an $n$-dimensional module. Then $\cup_{1 \leq d \leq n} I_{d,e}(V) = I_e(V)$.

Since $\Omega$ is an irreducible variety, Lemmas 2.4 and 2.5 imply an open subset of tuples $\Omega_0 \subset \Omega$ fix no 1-dimensional subspace of $V$ under the hypotheses stated in Theorem 1.1.

An extension of Lemma 2.5 proved in [19] will also be quite useful for establishing properties of topological generation. Let $U = \{W_1, ..., W_t\}$ be a finite collection of irreducible rational $kG$-modules, and $\cap_{W \in U} I_e(W)$ be the set of tuples in $G^e$ that act irreducibly on each $W_i \in U$. Finally let $S_e(G)$ be the set of tuples in $G^e$ that topologically generate a group containing a conjugate of some particular subfield subgroup $G(q_0)$ of $G$ (for $q_0$ appropriately large).

**Lemma 2.6.** Let $G$ be a simply connected classical algebraic group defined over an algebraically closed field $k$. There exists a collection of rational irreducible $kG$-modules $U$ such that $S_e(G) = \cap_{W \in U} I_e(W)$.

**Proof.** This is Theorem 11.6 in [19].

The above lemma implies there is a collection $U$ such that if $H$ is a proper closed subgroup of $G$ acting irreducibly on each $W_i$, then $H$ contains a conjugate of $G(q)$ for some finite field $\mathbb{F}_q$ with $p|q$. Furthermore, since acting irreducibly on a single (and hence any finite collection) of rational $kG$-modules is an open condition, the set of tuples in $G^e$ generating a group containing a conjugate of some fixed subfield subgroup is open in $G^e$. Over an uncountable field, this shows a tuple $\omega \in \Omega$ topologically generates $G$ if and only if generic tuples in $\Omega$ share this property.
Lemma 2.7. Assume $G$ is a simple algebraic group defined over an uncountable algebraically closed field. Let $C_1, \ldots, C_e$ be non-central conjugacy classes of $G$. If there is a tuple $\omega \in \Omega$ topologically generating $G$, then generic tuples in $\Omega$ generate $G$ topologically.

Proof. In characteristic $p = 0$ subfield subgroups do not exist. Hence over any algebraically closed field, Lemma 2.6 shows the existence of a single generating tuple $\omega \in \Omega$ implies an open dense subset of tuples generate $G$ topologically.

In positive characteristic, Lemma 2.6 shows there is a proper closed subvariety of $G^e$ outside of which a tuple $\omega \in G^e$ generates a dense subgroup of $G$ if and only if it generates an infinite subgroup of $G$. Over an uncountable field, the existence of a tuple $\omega \in \Omega$ generating an infinite subgroup implies generic tuples in $\Omega$ share this property (see Lemma 2.10 below).

Now recall a group acts primitively on a vector space $V$ if it acts irreducibly and preserves no additive decomposition of $V$. The following lemmas prove the set of tuples in $G^e$ generating a primitive subgroup is open.

Lemma 2.8. Let $G$ be a subgroup of $GL(V)$, with $V$ an $n$-dimensional vector space defined over an algebraically closed field. Then $G$ acts primitively on $V$ if and only if

(i) $G$ acts irreducibly on $V$

(ii) for each divisor $d < n$, no subgroup $H$ with $[G : H] = n/d$ fixes a $d$-dimensional subspace of $V$

Proof. To start we may assume that $G$ acts irreducibly on $V$ or else there is nothing to prove. If $G$ acts imprimitively and irreducibly, then $V = V_1 \oplus \ldots \oplus V_{n/d}$ with $n/d > 1$, and $G$ permutes the $V_i$ transitively. If $H$ is the stabilizer of $V_1$, then $[G : H] = n/d$.

Conversely, suppose there is a subgroup $H$ of $G$ such that $[G : H] = n/d$, $d < n$, and $H$ fixes a $d$-dimensional subspace $V_i$ of $V$. For $1 \leq i \leq n/d$, let $V_i$ be the conjugates of $V_i$ under $G$. Since $G$ acts irreducibly, $V = V_1 \oplus \ldots \oplus V_{n/d}$, and $G$ is imprimitive.

Now let $F_e$ be the free group on $e$ letters, and fix a divisor $d$ of $n$. Since $F_e$ is finitely generated, it has only finitely many subgroups of index $d$. Label these subgroups $H_1(d), \ldots, H_t(d)$. For each such $H_i(d)$, $1 \leq i \leq t$, choose a collection of generators $w_{i1}, \ldots, w_{is}$.

Theorem 2.9. Let $G = GL(V)$, with $V$ a positive dimensional vector space defined over an algebraically closed field. The set of tuples $\omega \in \Omega$ acting primitively on $V$ forms a Zariski open subset of $G^e$.

Proof. By Lemma 2.5, the set of tuples in $G^e$ generating a subgroup acting reducibly on $V$ is closed. Now suppose $(g_1, \ldots, g_e) \in G^e$ generates a subgroup $J$ of $G$ that acts irreducibly and imprimitively on $V$. 
Then for some \(d|n\) (with \(d < n\)) there is a subgroup \(J_1 \subset J\) such that \([J : J_1] = n/d\), and \(J_1\) fixes a \(d\)-dimensional subspace of \(V\). Choose a surjection \(\varphi : F_e \to J\) so that \(\varphi^{-1}(J_1) = H_i(d)\) for some \(1 \leq i \leq t\). Note \(\langle w_{i1}(g_1,\ldots,g_e),\ldots,w_{is}(g_1,\ldots,g_e)\rangle\) fixes a common \(d\)-dimensional subspace of \(V\).

Repeating the argument given in Lemma 2.4, the set of tuples \((x_1,\ldots,x_e) \in G^e\) such that \(\langle w_{i1}(x_1,\ldots,x_e),\ldots,w_{is}(x_1,\ldots,x_e)\rangle\) fixes a common \(d\)-dimensional subspace of \(V\) is closed in \(G^e\). Considering each subgroup \(H_1(d),\ldots,H_t(d)\) for each divisor \(d\) of \(n\), we find the set of tuples \((x_1,\ldots,x_e) \in G^e\) generating an imprimitive subgroup is closed.

Remark: When \(k\) is not the algebraic closure of a finite field there are tuples \((x_1,\ldots,x_e) \in G^e\) topologically generating \(G = GL_n(V)\), so the open subset described above is nonempty. Of course this set is empty when \(e = 1\).

To establish the remaining generic properties of \(\Omega\) it will be helpful to move between properties holding generically on a simple algebraic group \(G\) and properties holding generically on \(G^e\).

**Lemma 2.10.** Let \(G\) be a semisimple algebraic group defined over an algebraically closed field \(k\). For any nontrivial word \(w \in F_e\), the word map \(\varphi_w : G^e \to G\) which sends \((x_1,\ldots,x_e) \mapsto w(x_1,\ldots,x_e)\) is dominant. In particular, \(Z \subset G\) is a generic subset of \(G\) if any only if \(\{(x_1,\ldots,x_e) \in G^e \mid w(x_1,\ldots,x_e) \in Z\}\) is a generic subset of \(G^e\).

**Proof.** This is a result of Borel, and appears as Proposition 2.5 in Breuillard-Guralnick-Tao [6].

Say a tuple \((x_1,\ldots,x_e) \in G^e\) has infinite order if there is a word \(w \in F_e\) such that \(w(x_1,x_2,\ldots,x_e)\) has that property. Although having infinite order is not an open condition in \(G^e\), the following lemma shows over an uncountable field it is a generic one.

**Lemma 2.11.** Let \(C_1,\ldots,C_e\) be conjugacy classes of a simple algebraic group \(G\) defined over an uncountable algebraically closed field. Then generic tuples in \(G^e\) have infinite order. Furthermore if some \(\omega \in \Omega\) has infinite order, generic tuples in \(\Omega\) share this property.

**Proof.** The set \(\Theta_n = \{x \in G \mid x^n = 1\}\) of elements of order at most \(n\) is closed in \(G\). Over an uncountable field, \((\bigcup_{n \geq 1} \Theta_n)^c\) is a generic subset of \(G\). Applying Lemma 2.10, generic tuples in \(G^e\) have infinite order. If some \(\omega \in \Omega\) has infinite order, then \(\Omega \cap \Theta_n\) is a proper closed subvariety of \(\Omega\) for each \(n\).

Now pick a maximal torus \(T\) of a classical algebraic group \(G\), and let \(Ad : G \to GL_n(Lie(G))\) be the adjoint representation. Note \(Lie(G) \cong V \otimes V^*\), where \(V\) is the natural module for \(G\). Recall an element \(x \in G\) is regular if the dimension of its centralizer is as small as possible, and regular semisimple if all its eigenvalues are distinct on the natural module. Note if an element
$t \in T$ has eigenvalues $\gamma_1, ..., \gamma_n$ on $V$, then $t$ has eigenvalues $\gamma_i \cdot \gamma_j^{-1}$ on $V \otimes V^*$, for $1 \leq i \leq n, 1 \leq j \leq n$. Call a regular semisimple element of $G$ strongly regular if for every $1 \leq i, j, k, l \leq n$, we have that $\gamma_i \cdot \gamma_j^{-1} = \gamma_k \cdot \gamma_l^{-1}$ implies $i = j$ and $k = l$. In other words, $t$ is strongly regular if for any two distinct roots $\alpha_i, \alpha_j$ of $G$, $\alpha_i(t) \neq \alpha_j(t)$.

It is a well-known fact (see Theorem 2.5, [21]) that regular semisimple elements are open and dense in $G$. It follows that generically elements in $G$ are strongly regular: if $\gamma_1, ..., \gamma_n$ are generically distinct, then so are $\gamma_i \cdot \gamma_j^{-1}$ and $\gamma_k \cdot \gamma_l^{-1}$ for $i \neq j, k \neq l$. Say a tuple $(x_1, ..., x_e) \in \Omega$ is strongly regular if there is a word $w \in F_e$ such that $w(x_1, x_2, ..., x_e)$ has that property. Applying Lemma 2.10 the existence of a strongly regular tuple $\omega \in \Omega$ implies that generic tuples in $\Omega$ share this property.

The following lemma provides conditions sufficient to ensure the existence of a tuple $\omega \in \Omega$ topologically generating $G = SL_n(k)$. In the next section it is shown these conditions may in fact be satisfied.

**Lemma 2.12.** Let $C_1, ..., C_e$ be non-central conjugacy classes of $G = SL_n(k)$, where $k$ is an uncountable algebraically closed field, and $n \geq 4$. Assume there is a tuple $\omega \in \Omega$ topologically generating a group that acts irreducibly and primitively on the natural module, and which contains strongly regular elements of infinite order. Then there is a tuple $\omega \in \Omega$ topologically generating $G$.

**Proof.** Let $\omega \in \Omega$ be a tuple topologically generating a group $H$ with the above listed properties, and let $t \in H$ be a strongly regular element of infinite order. Since the non-trivial eigenvalues of $t$ on $V \otimes V^*$ are distinct, it follows that $\text{Lie}(G)$ is $H$-invariant, and in particular that $\text{Lie}(H)$ is generated by root subspaces of $\text{Lie}(G)$. Furthermore, since $t$ has infinite order, $\text{Lie}(H)$ is non-trivial. In other words, $\text{Lie}(H) = T_0 \oplus N_{\alpha_1} \oplus ... \oplus N_{\alpha_j}$ where the $\alpha_i$ are roots of $G$, $T_0 \subset T$ is a torus of $G$ (where $T$ is a maximal torus of $G$), and the $N_{\alpha_i}$ are one-dimensional root subspaces of $\text{Lie}(G)$.

If $\text{Lie}(H) = \text{Lie}(G)$ we are done, so assume $\text{Lie}(H) \subsetneq \text{Lie}(G)$. Then there is a one dimensional root subspace $N_\alpha \subset \text{Lie}(G)$ such that $N_\alpha \subset \text{Lie}(H)$. In particular, we have

$$\text{Lie}(H) \subset T \oplus N_{\alpha_1} \oplus ... \oplus N_{\alpha_j} \subset \text{Lie}(G)$$

However this implies $H$ is contained in a maximal rank subgroup of $G$. From Section 6 of [38] and Table 18.2 of [39], we see there are no proper maximal rank irreducible primitive subgroups of $G$ for $n \geq 4$.

3. **Topological generation of special linear algebraic groups**

We are now in a position to prove the main theorem. In this section, let $C_1, ..., C_e$ be non-central conjugacy classes of $G = SL_n(k)$, where $k$ is an uncountable algebraically closed field, and $V$ is the natural module for $G$. 
The “only if” direction of Theorem 1.1 follows without much difficulty, and is checked at the end of this section.

For the converse direction, the argument proceeds by induction on the dimension of the natural module. For both the base case and the induction, it is helpful to record some information about the maximal subgroup structure of $SL_n(k)$. This follows from more general work on the maximal subgroups of classical groups carried out by Aschbacher [1], Liebeck-Seitz [36] and others. In outline, the results state that the positive-dimensional maximal closed subgroups of classical algebraic groups arise in four naturally occurring geometric families, and one family of groups that act irreducibly and tensor indecomposably on the natural module. The geometric families can be described in a uniform fashion, and are often labelled $C_1$ to $C_6$. Members of the latter family are less easily described in a systematic fashion, but have the nice property that their connected component is an almost simple groups modulo scalars. The classical subgroups are sometimes labelled $C_6$. The following table collects the information that will be needed below. The listed structures should be read as intersecting with $SL_n(k)$ in order to obtain the appropriate maximal closed subgroups.

**Theorem 3.1.** The maximal positive-dimensional closed subgroups of $SL_n(k)$ contained in $C_1 \cup \ldots \cup C_4 \cup C_6$ have one of the following forms.

| Class | structure | conditions | rank |
|-------|-----------|------------|------|
| $C_1$ | $P_m$ | $1 \leq m \leq n - 1$ | $n - 2$ |
| $C_2$ | $GL_m \wr S_t$ | $n = mt, t \geq 2$ | $t(m - 1)$ |
| $C_4$ | $GL_{n_1} \otimes GL_{n_2}$ | $n = n_1 n_2, 2 \leq n_1 < n_2$ | $n_1 + n_2 - 2$ |
| | $(\bigotimes_{i=1}^t GL_m).S_t$ | $n = m^t, m \geq 3, t \geq 2$ | $t(m - 1)$ |
| $C_6$ | $Sp_n$ | $p \neq 2$ | $\frac{n}{2}$ |
| | $SO_n$ | | |

**Proof.**

This is Proposition 18.13 in [39].

**Remark.** The groups in $C_1$ are maximal parabolic subgroups of $SL_n(k)$. They stabilize an $m$-dimensional subspace of $V$, with $1 \leq m \leq n - 1$. The groups in $C_2$ are irreducible and imprimitive: they fix no nontrivial proper subspace, but permute $t$ subspaces of dimension $m$, where $n = mt$. The groups in class $C_4$ preserve or permute a tensor decomposition.

**Corollary 3.2.** Assume $M$ is a maximal positive-dimensional closed subgroup of $SL_3(k)$. Up to conjugacy $M$ has one of the following forms:

(i) $M$ is irreducible and primitive. In this case $M \cong SO_3(k)$.

(ii) $M$ is irreducible and imprimitive. In this case $M \cong (GL_1(k) \wr S_3) \cap SL_3(k)$ is the normalizer of a torus.
(iii) \( M \) is reducible. In this case \( M \) is the stabilizer of a line or a hyperplane.

For the base case of our argument, it will be convenient to record the dimensions of non-central conjugacy classes of \( SL_3(k) \). Recall a class \( C \) is quadratic if it has a degree two minimal polynomial.

**Lemma 3.3.** Let \( C \) be a non-central conjugacy class of \( SL_3(k) \). If \( C \) is quadratic then \( \dim C = 4 \), and \( \dim C = 6 \) otherwise.

**Proof.** Regular elements in \( SL_3(k) \) have two dimensional centralizers, and hence if \( C \) is a conjugacy class of regular elements, \( \dim C = 6 \). If not, \( C \) is quadratic and trivially \( \dim C = 4 \).

The following theorem proves the base case.

**Theorem 3.4.** Let \( C_1, ..., C_e \) be non-central conjugacy classes of \( SL_3(k) \), where \( k \) is an uncountable algebraically closed field. Assume \( \sum_{j=1}^{e} \gamma_j \leq n(e - 1) \), and it is not the case that \( e = 2 \) and \( C_1, C_2 \) are quadratic. Then there is a tuple \( \omega \in \Omega \) topologically generating \( SL_3(k) \).

**Proof.** We first show generic tuples in \( \Omega \) generate an infinite subgroup. The conditions given in Theorem 1.1 together with Lemma 3.3 imply that \( \dim \Omega > 8 \). Let \( M \) be a subfield subgroup of \( G \), and pick a class \( C \) from \( C_1, ..., C_e \). Clearly \( \dim C \cap M = 0 \) and \( \dim G/M = 8 \). Hence applying Lemma 2.1, \( \dim \Omega > \dim Y_M \). Now let \( M_1, M_2, ... \) be a list of the subfield subgroups of \( G \) up to conjugacy. As \( (\bigcup_{i \geq 1} Y_{M_i})^c \) is a countable union of proper closed subvarieties of \( \Omega \), generic tuples in \( \Omega \) generate an infinite subgroup of \( G \).

It remains to be shown \( \dim \Omega > \dim \Delta + \dim G/M \), for each maximal closed subgroup \( M \) of \( SL_3(k) \). Applying Corollary 3.2, it suffices to treat the following cases.

(i) \( M = SO_3(k) \). Pick a class \( C_j \) from \( C_1, ..., C_e \) and assume \( x \in C_j \cap M \). Consider the Jordan form of \( x \) on the natural module for \( M \). Note even sized Jordan blocks come in pairs, and any eigenvalue \( \alpha \neq \pm 1 \) of \( x \) comes paired with another eigenvalue \( \alpha^{-1} \). Hence if \( \dim (C_j \cap SO_3(k)) \neq 0 \), then either \( C_j \) is a unipotent class with Jordan form \( J_3 \), or \( C_j \) is a semisimple class. In either of these cases, an element \( x \in C_j \cap SO_3(k) \) centralizes a 1-dimensional subgroup of \( SO_3(k) \), so \( \dim (C_j \cap SO_3(k)) \leq 2 \). As \( \dim G/M = 5 \), it follows \( \dim \Omega > \dim \Delta + \dim G/M \) when the conditions placed on \( C_1, ..., C_e \) are satisfied.

(ii) \( M \) is the normalizer of a maximal torus \( T \). Pick a class \( C_j \) from \( C_1, ..., C_e \). Since \( N_G(T)/C_G(T) \) is finite and \( \dim C_G(T) = \dim T = 2 \), we have \( \dim G/M = 6 \) and \( \dim (M \cap C_j) \leq 2 \). Applying Lemma 3.3, \( \dim \Omega > 2e + 6 \) unless (a) \( e = 2 \) and at most one of \( C_1, C_2 \) are non-quadratic, or (b) \( e = 3 \) and each of \( C_1 - C_3 \) are quadratic.
When $C_j$ is quadratic, $N_G(T) \cap C_j$ is finite, except when $C_j$ is a class of involutions.

So assume $C_j$ is a class of involutions such that $N_G(T) \cap C_j$ is positive-dimensional. Then $N_G(T) \cap C_j$ is a union of classes of outer involutions in $N_G(T)$ (there are two such classes in characteristic $p \neq 2$, and one in characteristic $p = 2$). These outer involutions have a one dimensional centralizer in $N_G(T)$. Hence if $C_j$ is quadratic, $\dim M \cap C_j \leq 1$. It follows that $\dim \Omega > \dim \Delta + \dim G/M$ under the conditions placed on $C_1, \ldots, C_e$.

(iii) $M$ is reducible. Applying Lemmas 2.4 and 2.5 generic tuples in $\Omega$ do not fix a 1-space or hyperplane (by considering the dual space).

In order to make use of the inductive hypothesis, it is necessary to restrict the conjugacy classes $C_1, \ldots, C_e$ of $SL_n(k)$ to classes $C'_1, \ldots, C'_e$ of $SL_{n-1}(k)$ and check the inequality $\sum_{j=1}^{e} \gamma'_j \leq (n - 1)(e - 1)$ still holds, where $\gamma'_j$ is the dimension of the largest eigenspace of the conjugacy class $C'_j$.

Pick a class $C_j$ in $C_1, \ldots, C_e$ and some element $x \in C_j$. Assume $x$ has Jordan form $\oplus_{i=1}^{l} \lambda_i J_i^{r_i}$ on the natural module. Let $\lambda$ be an eigenvalue corresponding to a maximal dimensional eigenspace $E_\lambda$ of $x$, and let $J_k$ be a Jordan block of minimal dimension such that $\lambda J_k$ occurs in the Jordan decomposition of $x$. Replace the block $\lambda J_k$ in the Jordan form of $x$ with $\lambda J_{k-1}$ (if $k = 1$, remove the block $\lambda J_1$). Note an element $x'$ with this new Jordan form (say viewed in $GL_{n-1}(k)$) has determinant $\lambda^{-1}$. However, by scaling and restricting conjugation, we get a new conjugacy class $C'_e$ of $SL_{n-1}(k)$. Every class $C'_e$ can be extended to the class $C_i$ in the obvious way, and rescaling does not effect generation.

**Lemma 3.5.** Assume $C_1, \ldots, C_e$ are non-central conjugacy classes of $SL_n(k)$, where $k$ is an uncountable algebraically closed field and

(i) $\sum_{i=1}^{e} \gamma_i \leq n(e - 1)$

(ii) it is not the case that $e = 2$, and $C_1, C_2$ are quadratic

Finally, assume $C'_1, \ldots, C'_e$ are the classes of $SL_{n-1}(k)$ described above. Then $\sum_{i=1}^{e} \gamma'_i \leq (n - 1)(e - 1)$.

**Proof.** Pick a class $C_i$ in $C_1, \ldots, C_e$ and $x \in C_i$. Assume $x$ has Jordan form $\oplus_{i=1}^{l} \lambda_i J_i^{r_i}$, and $\lambda_k J_k$ is the Jordan block removed from $x$ in order to obtain the class $C'_i$. If $k = 1$, then $\gamma'_i = \gamma_i - 1$. If $k > 1$, then $\gamma'_i = \gamma_i$. In the latter case, all Jordan blocks of $x$ must have size greater than one. In particular if $\gamma'_i = \gamma_i$ we have $\gamma'_i \leq \frac{n}{2}$ with equality if and only if $C_i$ is quadratic.

We argue by induction on the number of classes $e$. For the base case, assume $e = 2$. Note by condition (i) of Theorem 1.1, $\gamma_1 + \gamma_2 \leq n$. If either $C_1$ or $C_2$ contain a Jordan block of size one, then $\gamma'_1 + \gamma'_2 \leq \gamma_1 + \gamma_2 - 1 \leq n - 1$. So assume $C_1$ and $C_2$ contain no Jordan blocks of size one. As it is not the
case that both $C_1$ and $C_2$ are quadratic, $\gamma'_1 + \gamma'_2 \leq n - 1$. Now assume 
$\sum_{i=1}^{e-1} \gamma'_i \leq (e-2)(n-1)$. Since $C_e$ is a non-central class, $\gamma'_e \leq n - 1$. In particular, 
$\sum_{i=1}^{e} \gamma'_i \leq (e-1)(n-1)$.

Applying the inductive hypothesis, there exists a tuple $\omega' \in \Omega'$ topologically generating $SL_{n-1}(k)$. Extending $\omega'$ to a tuple $\omega \in \Omega$, there is an $\omega \in \Omega$ topologically generating a group which contains an isomorphic copy of $SL_{n-1}(k)$. This implies there is a tuple $\omega \in \Omega$ topologically generating a group which has a composition factor of dimension at least $n - 1$ on the natural module. Applying Lemma 2.5 this is an open condition, so generic tuples in $\Omega$ share this property. Furthermore by Lemma 2.4 generic tuples do not fix a 1-space or a hyperplane (by considering the dual space), and hence generic tuples in $\Omega$ act irreducibly on the natural module.

Second, the existence of a tuple $\omega \in \Omega$ topologically generating a group containing a copy of $SL_{n-1}(k)$ implies there is a tuple $\omega \in \Omega$ generating a group $H$ such that any finite index subgroup of $H$ acts irreducibly on a subspace of dimension greater than $n/2$. This is an open condition, so there is a tuple $\omega \in \Omega$ generating a group with properties (i) and (ii) listed in Lemma 2.8. Applying Theorem 2.9, generic tuples in $\Omega$ act primitively on the natural module.

Similarly using the inductive hypothesis there is a tuple $(x_1, ..., x_e) \in \Omega$ and word $w \in F_e$ such that $w(x_1, ..., x_e)$ is strongly regular. By Lemma 2.10, generic tuples in $\Omega$ share this property. The finite intersection of generic subsets is a generic set, and over an uncountable field generic sets are non-empty (see Lemma 2.4 in [6]). Applying Lemma 2.12, there is a tuple $\omega \in \Omega$ topologically generating $G$.

To complete the proof of Theorem 1.1, we check conditions (i) and (ii) listed in the statement of the theorem are in fact necessary.

**Lemma 3.6.** Let $C_1, ..., C_e$ be conjugacy classes of a classical algebraic group $G$ defined over an algebraically closed field $k$. Assume the natural $G$-module $V$ is $n$-dimensional. If $\sum_{i=1}^{e} \gamma_i > n(e-1)$, then no tuple $\omega \in \Omega$ topologically generates $G$.

**Proof.** Pick a class $C_i$, in $C_1, ..., C_e$. For any element $x \in C_i$, let $E_{\alpha_i}$ be a $\gamma_i$-dimensional eigenspace of $x$ on $V$. We would like to show

$$
\dim \left( \bigcap_{i=1}^{e} E_{\alpha_i} \right) = \sum_{i=1}^{e} \gamma_i - n(e-1)
$$

This can easily be proved by induction on the number of classes $e$. If $e = 2$ and $\gamma_1 + \gamma_2 > n$, then clearly $\dim (E_{\alpha_1} \cap E_{\alpha_2}) = \gamma_1 + \gamma_2 - n$. So assume $\sum_{i=1}^{k} \gamma_i > n(k-1)$, and

$$
\dim \left( \bigcap_{i=1}^{k} E_{\alpha_i} \right) = \sum_{i=1}^{k} \gamma_i - n(k-1)
$$
Then
\[ \dim \left( \bigcap_{i=1}^{k} E_{\alpha_i} \cap E_{\alpha_{k+1}} \right) = \left( \sum_{i=1}^{k} \gamma_i - n(k - 1) \right) + \gamma_{k+1} - n \]

Hence if \( \sum_{i=1}^{e} \gamma_i > n(e - 1) \), then every tuple \( \omega \in \Omega \) generates a reducible subgroup on the natural module. \( \square \)

The second statement is a well-known linear algebra exercise.

Lemma 3.7. Let \( C_1, C_2 \) be quadratic conjugacy classes of \( G = SL_n(k) \), and \( V \) be the natural module for \( G \). If \( \dim V > 2 \), then every tuple \( \omega \in \Omega \) generates a reducible subgroup of \( V \).

Proof. See for instance Lemma 11.1 in [46] \( \square \)

3.1. Variations on the main theorem. In the statement of Theorem 1.1 it was assumed that \( SL_n(k) \) was defined over an uncountable algebraically closed field. However once topological generation has been proved over uncountable fields, analogous results can be recovered under much weaker hypothesis on \( k \). For instance in characteristic \( p = 0 \), the result holds over any field.

Corollary 3.8. Let \( C_1, \ldots, C_e \) be noncentral conjugacy classes of the algebraic group \( G = SL_n(k) \), with \( k \) a field of characteristic zero, and \( n \geq 3 \). Assume

(i) \( \sum_{i=1}^{e} \gamma_i \leq n(e - 1) \)

(ii) it is not the case that \( e = 2 \) and \( C_1, C_2 \) are quadratic.

Then there is a tuple \( \omega \in \Omega \) topologically generating \( G \).

Proof. Let \( \overline{k} \) be the algebraic closure of \( k \). It follows from the remark given after Lemma 2.6 that the set of tuples in \( \Omega(\overline{k}) \) that generate a Zariski dense subset of \( G(\overline{k}) \) contains an open subset of \( \Omega(\overline{k}) \). By Theorem 1.1 this open set is non-empty. However \( \Omega(k) \) is dense in \( \Omega(\overline{k}) \), so there is a tuple in \( \Omega(k) \) topologically generating \( G(k) \). \( \square \)

The result also holds over any algebraically closed field not algebraic over a finite field.

Corollary 3.9. Let \( C_1, \ldots, C_e \) be a collection of noncentral conjugacy classes of \( G = SL_n(k) \), where \( k \) is an algebraically closed field not algebraic over a finite field, and \( n \geq 3 \). Assume

(i) \( \sum_{i=1}^{e} \gamma_i \leq n(e - 1) \)

(ii) it is not the case that \( e = 2 \) and \( C_1, C_2 \) are quadratic.

Then there is a tuple \( \omega \in \Omega \) topologically generating \( G \).

Proof. This follows from Theorem 1.1 taken in combination with Theorem 1.5 of Burness-Gerhardt-Guralnick [7]. \( \square \)
Modifying Theorem 4.5 from [6], one can also show the result holds over any field of infinite transcendence degree over the prime field.

Finally, the choice of starting from $SL_3(k)$ in the statement of Theorem 1.1 was made for purposes of uniformity. Using argumentation given above, it is straightforward to prove an analogous topological generation result for $SL_2(k)$, although the statement is slightly different. Note all noncentral conjugacy classes in $SL_2(k)$ have a maximal one-dimensional eigenspace, and are quadratic. Hence condition (i) of Theorem 1.1 may be omitted, condition (ii) becomes a statement about classes of involutions modulo the center, and it suffices to treat the case where $\Omega = C_1 \times C_2$.

Lemma 3.10. Let $C_1, C_2$ be non-central conjugacy classes $G = SL_2(k)$, where $k$ is an uncountable algebraically closed field. There exists a tuple $\omega \in \Omega$ topologically generating $G$ if and only if it is not the case that $C_1$ and $C_2$ are classes of involutions modulo the center.

Proof. Again it suffices to show $\dim \Omega > \dim \Delta + \dim G/M$, where $M$ is any subfield subgroup, or closed maximal positive-dimensional subgroup of $G$. First assume $M$ is a subfield subgroup $G(q)$ of $G$. Any non-central conjugacy class $C$ of $SL_2(k)$ is two-dimensional. As $\dim C \cap M_i = 0$ and $\dim G/M_i = 3$, it follows $\dim \Omega > \dim \Delta + \dim G/M$.

According to Theorem 3.1, the conjugacy classes of maximal closed positive-dimensional subgroups of $SL_2(k)$ are Borel subgroups and the normalizer of a maximal torus. Assume $M$ is a Borel subgroup. If $C_1, C_2$ are semisimple, then elements in $C_i \cap M$ have a one dimensional centralizer, and $\dim \Omega > \dim \Delta + \dim G/M$.

So without loss of generality assume $C_1$ is unipotent (up to scalars) and pick $x_i \in C_i$. Since $C_1$ is a regular unipotent class, $x_1$ lives in a unique Borel subgroup. Picking some $x_2 \in C_2$ which does not live in this Borel, we have that $\langle x_1, x_2 \rangle$ topologically generates $G$. Furthermore since $M$ is parabolic, $X = \bigcup_{g \in G} M^g \times M_g$ is closed subgroup of $G \times G$, so generically tuples in $\Omega$ topologically generate a group living in no conjugate of $M$.

Finally assume $M$ is the normalizer of a torus. If $C$ is not a class of involutions in $M$, then $\dim C \cap M = 0$. Hence $\dim \Omega > \dim \Delta + \dim G/M$, unless $C_1, C_2$ are involutions in $M$. In the latter case, $C_i$ is a class of involutions or a class of involutions modulo the center in $G$, and $C_i \subset M^g$ for some $g \in G$. So $\Omega \subset X$, and no tuple in $\Omega$ will generate $G$ topologically.

4. Applications to generic stabilizers of special linear algebraic groups

We now turn our attention to applications of topological generation, the first of which concerns linear representations of algebraic groups. In this section, let $G$ be a simple algebraic group defined over an algebraically closed field $k$, and let $V$ be a linear $G$-variety. For any $x \in G$, let $V^x = \{ v \in $
Let $G$ be a simple algebraic group defined over an algebraically closed field $k$, and let $C$ be a conjugacy class in $G$. Assume $d$ elements of $C$ generate $G$ topologically. Then $\dim V(C) \leq \frac{d}{d-1} \dim V + \dim C$. In particular, $\dim V(C) < \dim V$, when $\dim V > d \cdot \dim C$.

**Proof.** Pick $x \in C$, and let $\varphi : G \times V^x \to V$ be the map $(g, v) \mapsto gv$. We claim $\text{im}(\varphi) = V(C)$. If $gv \in \text{im}(\varphi)$ then $gxv = gv$, and in particular $(gxg^{-1})gv = gxv = gv$. So $gv \in V(C)$. Conversely if $v \in V(C)$ there exists a $y \in C$ such that $y = gxg^{-1}$ and $yv = v$. Hence $g^{-1}v \in V^x$ and $\varphi(g, g^{-1}v) = v$. To achieve the desired inequality, we compute a bound on the dimension of a fiber.

Pick $gv \in V(C)$. Note for any $h \in C_G(x)$, we have $(gh^{-1})v = ghh^{-1}v = gv$. Hence every fiber has dimension at least $\dim C_G(x)$. In particular,

$$\dim V(C) \leq \dim G + \dim V^x - \dim C_G(x)$$

Now let $\gamma$ be the dimension of the largest eigenspace of $C$. Since $d$ elements of $C$ generate $G$ topologically, it follows from Lemma 3.6 that $d\gamma \leq (d-1)n$, and in particular that $\dim V^x \leq \gamma \leq \frac{d}{d-1} \cdot \dim V$. Since $\dim C = \dim G - \dim C_G(x)$, we have $\dim V(C) \leq \frac{d}{d-1} \dim V + \dim C$.

Finally, $\dim V > d \cdot \dim C \iff \dim V > \frac{d}{d-1} \dim V + \dim C$, so $\dim V(C) < \dim V$ whenever $\dim V > d \cdot \dim C$.

\[\square\]

Now let $\mathcal{C}$ be the set of conjugacy classes of $G$ containing elements of prime order (and classes of arbitrary unipotent elements in characteristic $p = 0$). The following lemma from the literature allows us to conclude a generic stabilizer is trivial when $\dim V(C) < \dim V$ for all $C \in \mathcal{C}$.

**Lemma 4.2.** Let $G$ be an algebraic group with $G^0$ reductive, and $V$ be a linear $G$-variety such that $V^G = 0$. Let $V(C)$ and $\mathcal{C}$ be as above. If $\dim V(C) < \dim V$ for all $C \in \mathcal{C}$, then a generic stabilizer is trivial.

**Proof.** This follows from Proposition 2.10 in [8] and Lemma 10.2 in [14]. \[\square\]

Finally, let:

$$C^d := \{ C \in \mathcal{C} | \text{d elements of } C \text{ are required to generate } G \text{ topologically} \}$$
\[ \alpha_d := \max \{ \dim C \mid C \in C^d \} \]
\[ \alpha := \max \{ \alpha_d \cdot d \mid 2 \leq d \leq n \} \]

It follows from Theorem 1.1 that for any noncentral class \( C \) in \( G = SL_n(k) \) (with \( k \) an uncountable algebraically closed field), it is possible to generate \( G \) topologically by choosing \( n \) elements from \( C \).

Now assume \( \dim V > \alpha \). Then \( \dim V > d \cdot C \), for every prime order class in \( G \) (and unipotent class in characteristic \( p = 0 \)). Applying Lemma 4.1, \( \dim V(C) < \dim V \), for every \( C \in \mathcal{C} \), and hence by Lemma 4.2 a generic stabilizer is trivial. So to find an upper bound on the dimension of a linear \( G \)-variety with a nontrivial generic stabilizer, it suffices to compute an upper bound for \( \alpha \).

**Theorem 4.3.** Let \( G = SL_n(k) \) where \( k \) is an algebraically closed field, and \( V \) is a linear \( G \)-variety such that \( V^G = 0 \). If \( \dim V > \frac{9}{4} n^2 \), then a generic stabilizer is trivial.

**Proof.** Let \( k' \) be an uncountable algebraically closed field extension of \( k \). Note if a generic stabilizer of \( V(k) \) were nontrivial, the set of points in \( V(k) \) having nontrivial stabilizers would be dense. \( V(k) \) is dense in \( V(k') \), so a generic stabilizer of \( V(k') \) would also be nontrivial. Hence it suffices to assume \( G = SL_n(k') \).

By the reasoning given above, it is enough to show \( \frac{9}{4} n^2 \geq \alpha \). To begin, assume \( d = 2 \) elements generate \( G \) topologically. In this case, the largest dimensional conjugacy class \( C \in \mathcal{C}^2 \) will be regular, and it follows that \( \alpha_2 = n^2 - n \).

Next assume \( 2 < d \leq n \), and pick \( C \in \mathcal{C}^d \). Then no \( d - 1 \) elements from \( C \) will generate \( G \) topologically, and it follows from Theorem 1.1 that \( \gamma \geq \frac{n^2}{d - 1} \cdot n \), where \( \gamma \) is the dimension of the largest eigenspace of \( C \).

First assume \( C \) is a semisimple class in \( \mathcal{C}^d \), and pick \( x \in C \). Since \( \gamma \geq \frac{n^2}{d - 1} \cdot n \), it follows \( C_G(x) \) must contain a copy of \( GL_\beta(k) \), where \( \beta = \lceil \frac{n(d - 2)}{d - 1} \rceil \). Hence the minimal dimension of \( C_G(x) \) is \( \beta^2 + (n - \beta) - 1 \), and the largest possible dimension of a semisimple class \( C \in \mathcal{C}^d \) is \( n^2 - \beta^2 - (n - \beta) \).

Next assume \( C \in \mathcal{C}^d \) is a unipotent class, and elements \( x \in C \) have Jordan forms involving Jordan blocks of size \( \lambda_1 \geq \lambda_2 \geq ... \lambda_k \). Let \( A \) be the Young diagram of \( \lfloor \lambda_1, ..., \lambda_k \rfloor \), and \( [\mu_1, ..., \mu_j] \) be the Young diagram of \( A^T \). For \( x \in C \), \( \dim C_G(x) = (\sum_{j=1}^{j} \mu_j^2) - 1 \) (see Section 1.3 of [21]). Since elements in \( C \) have a largest eigenspace of dimension at least \( \beta \), the Jordan form of these elements must contain at least \( \beta \) Jordan blocks. In particular,

\[ \dim C_G(x) \geq \beta^2 + \sum_{i=2}^{j} \mu_i^2 \geq \beta^2 + (n - \beta) - 1 \]

Combined with the previous paragraph, this implies \( \alpha_d \leq n^2 - \beta^2 - (n - \beta) \).
Finally, we compute an upper bound for $\alpha = \max \{ \alpha_d \cdot d \mid 2 \leq d \leq n \}$. Note $\alpha_2 = n^2 - n$, and for $3 \leq d \leq n$:

$$\alpha_d \leq n^2 - \beta^2 - (n - \beta) < n^2 - \beta^2 \leq n^2 - \frac{n^2(d - 2)^2}{(d - 1)^2}$$

Furthermore

$$d(n^2 - \frac{n^2(d - 2)^2}{(d - 1)^2}) = n^2 \frac{d(2d - 3)}{(d - 1)^2}$$

and $\frac{d(2d - 3)}{(d - 1)^2} \leq \frac{9}{4}$, for $3 \leq d \leq n$. So

$$\alpha \leq \max (2(n^2 - n), n^2 \frac{d(2d - 3)}{(d - 1)^2}) \leq \frac{9}{4} n^2$$

$$\square$$

5. Applications to random $(r, s)$-generation of finite groups of Lie type

We now turn our attention to a second application of topological generation, which concerns the random generation of finite simple groups. Let $I_t(H)$ be the elements of order $t$ in a finite simple group $H$. Assume $H$ contains elements of prime orders $r$ and $s$, and let

$$P_{r,s}(H) := \frac{| \{(x, y) \in I_r(H) \times I_s(H) : [x, y] = H\} |}{| I_r(H) \times I_s(H) |}$$

be the probability that two random elements of orders $r$ and $s$ generate $H$. Furthermore, for nonempty subsets $J$ and $K$ of $H$, let $P(J, K)$ be the probability that a random pair in $J \times K$ generates $H$. If $P_{r,s}(H) \to 1$ as $|H| \to \infty$, then groups of type $H$ are said to have random $(r, s)$-generation. The following result shows that finite simple classical groups have random $(r, s)$-generation when the rank of the group is large enough (depending on the orders of $r, s$).

**Theorem 5.1.** Let $(r, s)$ be primes with $(r, s) \neq (2, 2)$. There exists a positive integer $f(r, s)$ such that if $H$ is a finite simple classical group of rank at least $f(r, s)$, then

$$P_{r,s}(H) \to 1 \text{ as } |H| \to \infty$$

**Proof.** This is the main theorem of Liebeck-Shalev [32]. $\square$

We will use topological generation to prove random $(r, s)$-generation for fixed rank linear and unitary groups with orders $r, s$ independent of the rank, hence strengthening Theorem 5.1. In [15] we use topological generation to establish the relevant random $(r, s)$-generation results for the remaining finite simple classical groups.
Let $G$ be a simple algebraic group of simply-connected type, and $F : G \to G$ be a Steinberg endomorphism. Let $G^F$ or $G(q)$ be the fixed points of $F$ on $G$, and $E(q) := E \cap G(q)$ for any subset $E$ of $G$. Finally, say conjugacy classes $C', D'$ of elements of orders $r, s$ in $G$ are bad if

(i) $\Omega'(q) := C'(q) \times D'(q) \neq \emptyset$;

(ii) there is no tuple $\omega' \in \Omega'$ topologically generating $G$.

Similarly, say classes $C, D$ of elements of orders $r, s$ in $G$ are good if

(i) $\Omega(q) \neq \emptyset$;

(ii) there is a tuple $\omega \in \Omega$ topologically generating $G$;

(iii) for any bad classes $C', D'$ in $G$, $\dim \Omega' < \dim \Omega$.

**Theorem 5.2.** Let $G$ be a simple algebraic group defined over an algebraically closed field of positive transcendence degree. Assume $C$ and $D$ are conjugacy classes of elements of finite order in $G$, and $F : G \to G$ is a sequence of Steinberg endomorphisms such that $C(q) \times D(q) \neq \emptyset$ for each $G^F$. The following are equivalent:

(i) There is a tuple $(x, y) \in C \times D$ such that $\langle x, y \rangle$ is Zariski in $G$.

(ii) $P(C(q), D(q)) \to 1$ as $|G(q)| \to \infty$.

**Proof.** This is part of Theorem 2.6 in Guralnick-Liebeck-Lübeck-Shalev [13]. □

Now let $F : G \to G$ be any sequence of Steinberg endomorphisms. There exist only finitely many conjugacy classes of elements of orders $r, s$ in $G$. If for each $G^F$ there are good classes $C, D$ in $G$, then for any bad classes $C', D'$ we have $\dim \Omega' < \dim \Omega$. Recall $|\Omega(q)|$ is on the order of $q^s$, where $s = \dim \Omega$. Hence if good classes exist for each $G^F$, then bad classes will not contribute asymptotically to the proportion of generating pairs, and $P_{r,s}(G(q)) \to 1$ as $|G(q)| \to \infty$. So to establish random $(r, s)$-generation for the relevant finite groups of Lie type, it suffices to prove for each Steinberg endomorphism $F : G \to G$, there exist good classes $C$ and $D$ in $G$.

**Remark.** Note while the results of Liebeck-Shalev are stated for finite simple classical groups, $G^F$ itself is not typically simple. This small discrepancy is easily resolved. If $G$ is a simply connected simple algebraic group, then $S = G^F/Z(G)^F$ is (almost always) a finite simple group. More generally, $[G^F, G^F]/Z([G^F, G^F])$ is simple except for a very small number of cases. Let $\varphi : G \to S$ be the natural projection map. The center of $G^F$ is contained in the Frattini subgroup of $G$, and hence tuples $(x_1, x_2) \in \Omega(q)$ generate $G^F$ if and only the corresponding pair $(\overline{x_1}, \overline{x_2})$ generates $S$ (of course, the orders of the elements may be different in $S$). Hence if we are concerned with proving random $(r, s)$-generation for elements of prime orders $r$ and $s$ in the simple group $S$, it suffices to prove random $(r, s)$-generation for all prime power order elements $r, s$ in the corresponding groups of Lie type $G^F$. 
\textbf{Lemma 5.3.} Let $G = SL_n(k)$ where $k$ is an uncountable algebraically closed field, and $(r, s) \neq (2, 2)$ are prime powers. Let $F : G \to G$ be a Steinberg endomorphism such that $G^F$ contains elements of orders $r$ and $s$. Then good classes $C$ and $D$ exist in $G$.

\textit{Proof.} Let $F : G \to G$ be a Steinberg endomorphism such that $G^F = SL_n(q)$ and

$$C_t(q) = \{C \mid C \text{ is a class of order } t \text{ elements in } G \text{ and } C(q) \neq \emptyset\}$$

To start, assume both $r, s \neq 2$. According to Theorem 1.1, it suffices to show for any maximal dimensional classes $C$ and $D$ in $C_r(q)$ and $C_s(q)$ that largest eigenspaces of $C$ and $D$ have dimension $< \frac{n}{2}$.

Without loss of generality, consider classes of order $r$ elements in $G$. Recall $G^F = SL_n(q)$ has order $q^{n(n-1)/2}(q^2 - 1)(q^3 - 1)\ldots(q^n - 1)$. If $r \mid q$ (so that the classes under consideration are unipotent) or $r \mid q - 1$, then all classes of order $r$ elements are defined in $G^F$, and a maximal dimensional class in $C_r(q)$ has largest eigenspace of dimension $\leq \frac{n}{2}$. For convenience, exclude these cases in what follows.

Now pick a maximal dimensional class $C$ in $C_r(q)$, and choose $x \in C(q)$. Let $x$ act on $V = \mathbb{F}_q$, and

$$l = \min \{j : r \mid q^j - 1, \text{ for } 1 \leq j \leq n\}$$

$$m = \max \{j : r \mid q^j - 1, \text{ for } 1 \leq j \leq n\}$$

Note $x$ is semisimple, acts reducibly on a space of dimension at least $n - m$ with $n - m < \frac{n}{2}$ (if $r$ is prime, any element of order $r$ in $G(q)$ has a fixed space of dimension at least $n - m$), and has at most $m/l$ irreducible composition factors of dimension $l$, with $1 < l \leq n$. If $l > 2$, then $\gamma \leq \max(n - m, \lceil \frac{n}{2} \rceil) < \frac{n}{2}$, and if $l = 2$, $\gamma \leq \frac{n}{2}$. However, if $C'' \in C_r(q)$ is a class with $\frac{n}{2}$ irreducible composition factors of dimension two, there is a class $C'' \in C_r(q)$ whose elements act trivially on a two dimensional subspace of $V$, dim $C'' >$ dim $C'$, and $\gamma'' < \frac{n}{2}$. Hence $\gamma < \frac{n}{2}$. Without loss of generality, if $r$ and $s$ are both not equal to two, there are good classes $C, D$ in $G$.

Now assume $r = 2$. Then any maximal dimensional class in $C_r(q)$ has a largest eigenspace of dimension $\leq \lceil \frac{n}{2} \rceil$. We may assume $s > 2$, and by reasoning given above maximal dimensional classes in $C_s(q)$ have a largest eigenspace of dimension $< \frac{n}{2}$. Hence good classes $C, D$ exist for all prime powers $(r, s) \neq (2, 2)$.

Finally, assume $F : G \to G$ is a sequence of Steinberg endomorphisms such that $G^F = SU_n(q)$. In this case, $|G^F| = q^{n(n-1)/2}(q^2 - 1)(q^3 + 1)\ldots(q^n - (-1)^n)$. As above, if $r \mid q^k + 1$ and $r \nmid q^j + 1$ for $k < j \leq n$, then $k < \frac{n}{2} - 1$. We may repeat the above reasoning to show good classes $C, D$ exist for prime powers $(r, s) \neq (2, 2)$ in $G$.

\hfill \square
Theorem 5.2 and Lemma 5.3 imply Corollary 1.3, the desired random 
\((r, s)\)-generation result for finite simple linear and unitary groups.

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