ON SPLITTING OF EXACT DIFFERENTIAL FORMS

V.N. Dumachev  
Voronezh institute of the MVD of Russia  
e-mail: dumv@comch.ru

Abstract. In work the internal structure of de Rham cohomology is considered. As examples the phase flows in $\mathbb{R}^3$ admitting the Nambu poisson structure are studied.

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1 Introduction

Let $\Lambda^k(\mathcal{M})$ - be the exterior graded algebra of differential forms with de Rham complex

$$0 \to \Lambda^0(\mathcal{M}) \to \Lambda^1(\mathcal{M}) \to \ldots \to \Lambda^{n-1}(\mathcal{M}) \to \Lambda^n(\mathcal{M}) \to 0.$$ 

Remember [1], that differential form $\omega \in \Lambda^k(\mathcal{M})$ is called closed if $d\omega = 0$, and exact if $\omega = d\nu$ for some $\nu \in \Lambda^{k-1}(\mathcal{M})$. The quotient of closed $k$-forms on manifolds $\mathcal{M}$ by exact $k$-forms is the $k$'th de Rham cohomology group

$$H^k(\mathcal{M}) = \frac{\ker \left( d : \Lambda^k(\mathcal{M}) \to \Lambda^{k+1}(\mathcal{M}) \right)}{\text{im} \left( d : \Lambda^{k-1}(\mathcal{M}) \to \Lambda^k(\mathcal{M}) \right)}.$$ 

It is known, that cohomology $\mathcal{M} = \mathbb{R}^n$ is vanish. It means, that for anyone $\omega \in \Lambda^k(\mathbb{R}^n)$ such that $d\omega = 0$ there exists a $\nu \in \Lambda^{k-1}(\mathbb{R}^n)$ such that $\omega = d\nu$. In any case the dimension of cohomology group is determined by Betti numbers:

$$b^k = \frac{\dim \left( \omega : d\omega = 0 \right)}{\dim \left( d\nu \right)}.$$ 

2 Cogomology in $\Lambda^2(\mathbb{R}^n)$

Let’s connect with de Ram complex the differential module $\{C, d\}$, then

$$Z(C) = \ker d = \{ x \in C \mid dx = 0 \}$$
are called cocycles of module \( \{C, d\} \) (space of closed forms),
\[
B(C) = \text{im} d = dC = \{x = dy | y \in C\}
\]
are called coboundary of module \( \{C, d\} \) (space of exact forms). In given designations
the group \( i \)-cohomology \( H^i \) be the quotient \( i \)-cocycles by \( i \)-coboundary
\[
H^i = Z^i / B^i.
\]
Assume that \( \omega \in B^2 \subset \Lambda^2(\mathbb{R}^n) \). This means that
\[
d\omega = 0, \quad \omega = d\nu, \quad \text{where} \quad \nu \in \Lambda^1(\mathbb{R}^n).
\]
The received quotient we can write as
\[
H^2_2(\mathbb{R}^n) = \frac{\ker (d : \Lambda^2(\mathbb{R}^n) \to \Lambda^3(\mathbb{R}^n))}{\text{im} (d : \Lambda^1(\mathbb{R}^n) \to \Lambda^2(\mathbb{R}^n))} = Z^2 / B^2.
\]
Note however that in \( \Lambda^2(\mathbb{R}^n) \) there exists a form
\[
\omega = \lambda_1 \wedge \lambda_2, \quad \text{where} \quad \lambda_1, \lambda_2 \in \Lambda^1(\mathbb{R}^n) \quad \text{such that} \quad d\lambda_i = 0.
\]
Let \( \lambda_i \notin H^1(\mathbb{R}^n) \) (i.e. \( \lambda_i = d\mu_i \) ), then
\[
d\omega = 0, \quad \omega = d\mu_1 \wedge d\mu_2.
\]
In other words, exact \( \omega \in \Lambda^2(\mathbb{R}^n) \) be wedge product of the other exact forms. We
can write this space as
\[
B^{1,1} = \{x = dy_1 \wedge dy_2 | y_i \in C\}
\]
then
\[
H^2_{1,1}(\mathbb{R}^n) = \frac{\ker (d : \Lambda^2(\mathbb{R}^n) \to \Lambda^3(\mathbb{R}^n))}{\text{im} (d : \Lambda^0(\mathbb{R}^n) \to \Lambda^1(\mathbb{R}^n)) \oplus \text{im} (d : \Lambda^0(\mathbb{R}^n) \to \Lambda^1(\mathbb{R}^n))} = Z^2 / B^{1,1},
\]
or
\[
b^2_{1,1}(\mathbb{R}^n) = \frac{\dim (\omega : d\omega = 0)}{\dim (d\mu \wedge d\mu)}.
\]
It is obvious that from \( B^2 = \{d\mu_1 \wedge d\mu_2, d\nu\} \), and \( B^{1,1} = \{d\mu_1 \wedge d\mu_2\} \) it follows that
\( B^{1,1} \subset B^2 \). This means that quotient
\[
B^2 / B^{1,1} \simeq H^2_{1,1} / H^2_2
\]
should characterize presence of obstacles (topological defects) for existence of the exact forms in $\Lambda^2(\mathbb{R}^n)$, which are wedge product of exact form from $\Lambda^1(\mathbb{R}^n)$.

**Example 1.** Consider the dynamical systems in $\mathbb{R}^3$

\[
\begin{align*}
  \dot{x} &= -xz; \\
  \dot{y} &= yz; \\
  \dot{z} &= x^2 - y^2.
\end{align*}
\]

According [2], this phase flow has one vectorial

\[
h = \frac{1}{4} \left( (-x^2y + y^3 + yz^2)dx + (x^3 - y^2x + xz^2)dy - 2xyzdz \right)
\]

and two scalar Hamiltonians

\[
H = \frac{1}{2} (x^2 + y^2 + z^2), \quad F = xy.
\]

These Hamiltonians are connected by expressions

\[
dh = dH \wedge dF.
\]

This means that our system admit Poisson structure with vectorial Hamiltonian

\[
\dot{x}_i = \{ h, x_i \} = X_h ] dx_i,
\]

where

\[
X_h = -xz \frac{\partial}{\partial x} + yz \frac{\partial}{\partial y} + (x^2 - y^2) \frac{\partial}{\partial z},
\]

and Poisson structure (Nambu [3])

\[
\dot{x}_i = \{ H, F, x_i \} = X_H ] dF \wedge dx_i = -X_F ] dH \wedge dx_i,
\]

where

\[
\begin{align*}
  X_H &= z \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + x \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + y \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x}, \\
  X_F &= y \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + x \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x}.
\end{align*}
\]

**Example 2.** Divergence-free Lorenz set

\[
\begin{align*}
  \dot{x} &= y - z; \\
  \dot{y} &= -x + xz; \\
  \dot{z} &= x - xy
\end{align*}
\]
has one vectorial
\[ h = \left( \frac{x}{4} (z^2 + y^2) - \frac{x}{3} (z + y) \right) dx + \left( \frac{1}{3} (x^2 - yz + z^2) - \frac{1}{4} x^2 y \right) dy + \left( \frac{1}{3} (y^2 - zy + x^2) - \frac{1}{4} x^2 z \right) dz \]
and two scalar Hamiltonians
\[ H = \frac{1}{2} (x^2 + y^2 + z^2), \quad F = \left( y - \frac{y^2}{2} \right) + \left( z - \frac{z^2}{2} \right) \]
connected by expressions
\[ dh = dH \wedge dF. \]
This means that our system admit Poisson structure with vectorial Hamiltonian
\[ \dot{x}_i = \{ h, x_i \} = X_h \, dx_i \]
where
\[ X_h = (y-z) \frac{\partial}{\partial x} + \left( -x + xz \right) \frac{\partial}{\partial y} + \left( x - xy \right) \frac{\partial}{\partial z}, \]
and Poisson structure in two forms
\[ \dot{x}_i = \{ H, F, x_i \} = X_H \, dF \wedge dx_i = -X_F \, dH \wedge dx_i, \]
where
\[ X_H = \frac{z}{\partial x} \wedge \frac{\partial}{\partial y} + x \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + y \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x}; \]
\[ X_F = \left( 1 - z \right) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + \left( 1 - y \right) \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x}. \]

**Example 3.** Phase flow
\[ \dot{x} = xy; \quad \dot{y} = x - z; \quad \dot{z} = -zy \]
has one vectorial
\[ h = \frac{1}{12} \begin{pmatrix} z(3y^2 + 4x - 4z) \\ -6xyz \\ x(3y + 4z - 4x) \end{pmatrix}, \]
one scalar Hamiltonian
\[ H = x - \frac{y^2}{2} + z \]
and prehamiltonian form
\[ \Theta = -z dx + x dz \]
connected by expressions
\[ dh = dH \wedge \Theta. \]
This means that our system admit Poisson structure with vectorial Hamiltonian
\[ \dot{x}_i = \{ h, x_i \} = X_h dx_i \]
where
\[ X_h = xy \frac{\partial}{\partial x} + (x - z) \frac{\partial}{\partial y} - yz \frac{\partial}{\partial z}, \]
and scalar Poisson structure in the form
\[ \dot{x}_i = X_H \wedge \Theta dx_i, \]
where
\[ X_H = \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} - y \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x}. \]
Therefore,
\[ dh \in B^2 \setminus B^{1,1}. \]
For completeness of a statement we shall notice that as \( d\Theta \neq 0 \), but \( d\Theta \wedge \Theta = 0 \), then Pfaff equation on prehamiltonians form \( \Theta \) has solved with integrating factor
\[ dF = \frac{\Theta}{x^2 + z^2}, \quad \Rightarrow \quad F = \arctan \frac{z}{x}. \]
The caused of global non-integrability of the given system is the presence holes \((x = 0, z = 0)\) in \( x0z \) planes. It is obvious that vanishing of second Hamiltonians has not admitted to enter of Nambu structure with a bracket \( \{ H, F, G \} \).
3 Cogomology in $\Lambda^3(\mathbb{R}^n)$

Further, the top index of the any form will denote its degree, i.e. $\omega^k \in \Lambda^k(\mathbb{R}^n)$. For standard de Rham complexes in $\mathbb{R}^n$ we get $\omega^3 \in B^3 \subset \Lambda^3(\mathbb{R}^n)$. This means that

$$d\omega^3 = 0, \quad \omega^3 = d\nu^2, \quad \text{where} \quad \nu^2 \in \Lambda^2(\mathbb{R}^n).$$

Thus

$$H^3_3(\mathbb{R}^n) = \frac{\ker (d : \Lambda^3(\mathbb{R}^n) \to \Lambda^4(\mathbb{R}^n))}{\im (d : \Lambda^2(\mathbb{R}^n) \to \Lambda^3(\mathbb{R}^n))} = \frac{Z^3}{B^3},$$

$$b^3_3 = \frac{\dim (\omega^3 : d\omega^3 = 0)}{\dim (d\nu^2)}.$$

But in $\Lambda^3(\mathbb{R}^n)$ there exists are forms

$$\omega^3_1 = \lambda^1_1 \land \lambda^1_2 \land \lambda^1_3, \quad \text{where} \quad \lambda^1_i \in \Lambda^1(\mathbb{R}^n), \quad \text{such that} \quad d\lambda^1_i = 0.$$

Let $\lambda^1_i \notin H^1(\mathbb{R}^n)$ (i.e. $\lambda^1_i = d\mu^0_i$), then

$$d\omega^3_1 = 0, \quad \omega^3_1 = d\mu^0_1 \land d\mu^0_2 \land d\mu^0_3.$$

In other words, exact $\omega^3_1 \in \Lambda^3(\mathbb{R}^n)$ be wedge product of the other exact forms. We can write this quotient as

$$H^3_{1,1,1}(\mathbb{R}^n) = \frac{\ker (d : \Lambda^3(\mathbb{R}^n) \to \Lambda^4(\mathbb{R}^n))}{\bigoplus_{k=1}^3 \im_k (d : \Lambda^0(\mathbb{R}^n) \to \Lambda^1(\mathbb{R}^n))} = \frac{Z^3}{B^{1,1,1}},$$

$$b^3_{1,1,1} = \frac{\dim (\omega^3 : d\omega^3 = 0)}{\dim (d\mu^0 \land d\mu^0 \land d\mu^0)}.$$

At the same time in $\Lambda^3(\mathbb{R}^n)$ it is possible also to construct the forms

$$\omega^3_2 = \lambda^1 \land \lambda^2, \quad \text{where} \quad \lambda^k \in \Lambda^k(\mathbb{R}^n)$$

such that $d\lambda^k = 0$. Let $\lambda^k \notin H^k(\mathbb{R}^n)$ (i.e. $\lambda^k_i = d\mu^k_i$), then

$$d\omega^3_2 = 0, \quad \omega^3_2 = d\mu^0 \land d\mu^1.$$
In other words, exact $\omega^3$ be wedge product of the other exact forms. We can write this quotient as

$$H^3_{2,1}(\mathbb{R}^n) = \frac{\ker(d: \Lambda^3(\mathbb{R}^n) \to \Lambda^4(\mathbb{R}^n))}{\im(d: \Lambda^0(\mathbb{R}^n) \to \Lambda^1(\mathbb{R}^n)) \oplus \im(d: \Lambda^1(\mathbb{R}^n) \to \Lambda^2(\mathbb{R}^n))} = Z^3/B^2_1,$$

$$b^3_{1,2} = \frac{\dim(\omega^3 \cdot d\omega^3 = 0)}{\dim(d\mu^0 \wedge d\mu^1)}.$$

Evidently that

$$H^3_{1,1,1}(\mathbb{R}^n) \supset H^3_{1,2}(\mathbb{R}^n) \supset H^3_3(\mathbb{R}^n),$$

such that quotients $H^3_{1,1,1}(\mathbb{R}^n)/H^3_3(\mathbb{R}^n)$ and $H^3_{1,1,1}(\mathbb{R}^n)/H^3_{2,1}(\mathbb{R}^n)$ should characterize presence of obstacles (topological defects) for existence of the exact forms in $\Lambda^3(\mathbb{R}^n)$, which are wedge product of exact form from $\Lambda^1(\mathbb{R}^n)$ or from $\Lambda^2(\mathbb{R}^n)$.

### 4 Cogomology in $\Lambda^k(\mathbb{R}^n)$

Generalizing the previous calculations we consider de Rham complex in $\mathbb{R}^n$ and get $\omega^k \in \Lambda^k(\mathbb{R}^n) \notin H^k(\mathbb{R}^n)$. Then

$$d\omega^k = 0, \quad \omega^k = d\nu^{k-1}, \quad \text{where} \quad \nu^{k-1} \in \Lambda^{k-1}(\mathbb{R}^n),$$

and

$$H^k(\mathbb{R}^n) = \frac{\ker(d: \Lambda^k(\mathbb{R}^n) \to \Lambda^{k+1}(\mathbb{R}^n))}{\im(d: \Lambda^{k-1}(\mathbb{R}^n) \to \Lambda^k(\mathbb{R}^n))} = Z^k/B^k.$$

However, into $\Lambda^k(\mathbb{R}^n)$ there exists are forms

$$\omega^k = \bigwedge_{i=1}^k \lambda^1, \quad \text{where} \quad \lambda^1 \in \Lambda^1(\mathbb{R}^n)$$

such that $d\lambda^1 = 0$. Suppose that $\lambda^1 \notin H^1(\mathbb{R}^n)$ (i.e. $\lambda^1 = d\mu^0$), then

$$d\omega^k = 0, \quad \omega^k = \bigwedge_{i=1}^k d\mu^0.$$

In other words, exact $\omega^k \in \Lambda^k(\mathbb{R}^n)$ be wedge product of the other exact forms. We can write this quotient as

$$H^k_{1,1,...,1}(\mathbb{R}^n) = \frac{\ker(d: \Lambda^k(\mathbb{R}^n) \to \Lambda^{k+1}(\mathbb{R}^n))}{\bigoplus_{i=1}^k \im_i(d: \Lambda^0(\mathbb{R}^n) \to \Lambda^1(\mathbb{R}^n))} = Z^k/B^{1,1,...,1},$$

7
Continuing it is similarly we receive:

for \( \omega^k = d\mu^1 \wedge \bigwedge_{i=1}^{k-2} d\mu_i^0 \)

\[
H^k_{2,1,1,...,1}(\mathbb{R}^n) = \frac{\ker (d : \Lambda^k(\mathbb{R}^n) \to \Lambda^{k+1}(\mathbb{R}^n))}{\text{im} (d : \Lambda^1(\mathbb{R}^n) \to \Lambda^2(\mathbb{R}^n)) \bigoplus_i \text{im}_i (d : \Lambda^0(\mathbb{R}^n) \to \Lambda^1(\mathbb{R}^n))},
\]

\[
b^k_{2,1,1,...,1} = \frac{\dim (\omega^k : d\omega^k = 0)}{\dim \left( d\mu^1 \wedge \bigwedge_{i=1}^{k-2} d\mu_i^0 \right)},
\]

for \( \omega^k = d\mu^2 \wedge \bigwedge_{i=1}^{k-3} d\mu_i^0 \)

\[
H^k_{3,1,1,...,1}(\mathbb{R}^n) = \frac{\ker (d : \Lambda^k(\mathbb{R}^n) \to \Lambda^{k+1}(\mathbb{R}^n))}{\text{im} (d : \Lambda^2(\mathbb{R}^n) \to \Lambda^3(\mathbb{R}^n)) \bigoplus_i \text{im}_i (d : \Lambda^0(\mathbb{R}^n) \to \Lambda^1(\mathbb{R}^n))},
\]

where

\[
b^k_{3,1,1,...,1} = \frac{\dim (\omega^k : d\omega^k = 0)}{\dim \left( d\mu^2 \wedge \bigwedge_{i=1}^{k-3} d\mu_i^0 \right)};
\]

for \( \omega^k = d\mu^3 \wedge \bigwedge_{i=1}^{k-4} d\mu_i^0 \)

\[
H^k_{4,1,1,...,1}(\mathbb{R}^n) = \frac{\ker (d : \Lambda^k(\mathbb{R}^n) \to \Lambda^{k+1}(\mathbb{R}^n))}{\text{im} (d : \Lambda^3(\mathbb{R}^n) \to \Lambda^4(\mathbb{R}^n)) \bigoplus_i \text{im}_i (d : \Lambda^0(\mathbb{R}^n) \to \Lambda^1(\mathbb{R}^n))},
\]

where

\[
b^k_{4,1,1,...,1} = \frac{\dim (\omega^k : d\omega^k = 0)}{\dim \left( d\mu^3 \wedge \bigwedge_{i=1}^{k-4} d\mu_i^0 \right)};
\]

... ... ...

for \( \omega^k = d\mu^1 \wedge d\mu^1 \wedge \bigwedge_{i=1}^{k-4} d\mu_i^0 \)

\[
H^k_{2,2,1,...,1}(\mathbb{R}^n) = \frac{\ker (d : \Lambda^k(\mathbb{R}^n) \to \Lambda^{k+1}(\mathbb{R}^n))}{\bigoplus_i \text{im}_i (d : \Lambda^1(\mathbb{R}^n) \to \Lambda^2(\mathbb{R}^n)) \bigoplus_i \text{im}_i (d : \Lambda^0(\mathbb{R}^n) \to \Lambda^1(\mathbb{R}^n))},
\]
where

\[ b_{2,2,1,1,...,1}^k = \frac{\dim (\omega^k : d\omega^k = 0)}{\dim (d\mu^1 \wedge d\mu^1 \wedge \bigwedge_{i=1}^{k-4} d\mu_i^0)} \]

etc.

For simplification of record we shall enter a multiindex

\[ \#m = \{m_1, m_2, ..., m_i\}, \quad m_1 + m_2 + ... + m_i = k, \quad m_1 \geq m_2 \geq ... \geq m_i \geq 0, \]

which is formed by a rule of construction of the Young diagrams. Then

\[ H_{\#m}^k(\mathbb{R}^n) = \frac{\ker (d : \Lambda^k(\mathbb{R}^n) \to \Lambda^{k+1}(\mathbb{R}^n))}{\bigoplus_{\#m} \text{im}_m (d : \Lambda^{\#m}(\mathbb{R}^n) \to \Lambda^{\#m+1}(\mathbb{R}^n))} = \mathbb{Z}^k / B_{\#m}, \]

\[ b_{\#m}^k = \frac{\dim (\omega^k : d\omega^k = 0)}{\dim (\bigwedge_{\#m} d\mu_i^0)}. \]

So, for \( k = 3 \) we shall receive

\[
\begin{array}{ccc}
\begin{array}{c}
1 \\
1 \\
1
\end{array} & 2 & 3 \\
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array} & \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} & \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}
\end{array}
\]

i.e.

\[ \#m = \{1, 1, 1\} \quad \text{or} \quad H_{\#m}^3(\mathbb{R}^n) = H_{1,1,1}^3(\mathbb{R}^n), \]

\[ \#m = \{2, 1\} \quad \text{or} \quad H_{\#m}^3(\mathbb{R}^n) = H_{2,1}^3(\mathbb{R}^n), \]

\[ \#m = \{3\} \quad \text{or} \quad H_{\#m}^3(\mathbb{R}^n) = H_{3}^3(\mathbb{R}^n). \]

Then the filtered complex of cohomology can be represented as follows

\[ H_{1,1,1}^3 \to H_{2,1}^3 \to H_{3}^3. \]

For \( k = 4 \) we get

\[
\begin{array}{cccccc}
\begin{array}{c}
1 \\
1 \\
1 \\
1
\end{array} & 2 & 3 & 4 \\
\begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array} & \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} & \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} & \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} & \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}
\end{array}
\]
i.e. $\#m = \{1, 1, 1\}$, $\#m = \{2, 1, 1\}$, $\#m = \{2, 2\}$, $\#m = \{3, 1\}$ or $\#m = \{4\}$.
Let’s notice, that at $k \geq 4$ the structure of cohomology $H^k_{\#m}(\mathbb{R}^n)$ is not linear and for some small $k = 4, 5, 6$ is shown in figures:

```
\begin{array}{c}
\begin{tikzpicture}
\node (1111) at (0,0) {$H^4_{1,1,1,1}$};
\node (2111) at (1,0) {$H^4_{2,1,1}$};
\node (311) at (1,1) {$H^4_{3,1}$};
\node (4) at (2,1) {$H^4_4$};
\node (22) at (1,2) {$H^4_{2,2}$};
\path[->] (1111) edge (2111);
\path[->] (2111) edge (311);
\path[->] (311) edge (4);
\path[->] (4) edge (22);
\end{tikzpicture}
\end{array}
```

```
\begin{array}{c}
\begin{tikzpicture}
\node (11111) at (0,0) {$H^5_{1,1,1,1,1}$};
\node (21111) at (1,0) {$H^5_{2,1,1,1}$};
\node (3111) at (1,1) {$H^5_{3,1,1}$};
\node (411) at (2,1) {$H^5_{4,1}$};
\node (51) at (3,1) {$H^5_{5,1}$};
\node (5) at (2,2) {$H^5_5$};
\node (32) at (1,3) {$H^5_{3,2}$};
\path[->] (11111) edge (21111);
\path[->] (21111) edge (3111);
\path[->] (3111) edge (411);
\path[->] (411) edge (51);
\path[->] (51) edge (5);
\path[->] (5) edge (32);
\end{tikzpicture}
\end{array}
```

```
\begin{array}{c}
\begin{tikzpicture}
\node (111111) at (0,0) {$H^6_{1,\ldots,1}$};
\node (211111) at (1,0) {$H^6_{2,1,\ldots,1}$};
\node (31111) at (1,1) {$H^6_{3,1,1,1}$};
\node (4111) at (2,1) {$H^6_{4,1,1}$};
\node (511) at (3,1) {$H^6_{5,1,1}$};
\node (61) at (4,1) {$H^6_{6,1}$};
\node (42) at (2,2) {$H^6_{4,2}$};
\node (52) at (3,2) {$H^6_{5,2}$};
\node (62) at (4,2) {$H^6_{6,2}$};
\node (63) at (5,2) {$H^6_{6,3}$};
\node (64) at (6,2) {$H^6_{6,4}$};
\node (65) at (7,2) {$H^6_{6,5}$};
\node (66) at (8,2) {$H^6_{6,6}$};
\path[->] (111111) edge (211111);
\path[->] (211111) edge (311111);
\path[->] (311111) edge (41111);
\path[->] (41111) edge (5111);
\path[->] (5111) edge (611);
\path[->] (611) edge (61);
\path[->] (61) edge (42);
\path[->] (62) edge (52);
\path[->] (63) edge (52);
\path[->] (64) edge (52);
\path[->] (65) edge (52);
\path[->] (66) edge (52);
\end{tikzpicture}
\end{array}
```

We can see that in the general case the cohomology sequences are not filtered. This means to define cohomology of cogomology is obviously impossible.

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