ON ASYMPTOTIC BOUNDS FOR THE NUMBER OF IRREDUCIBLE COMPONENTS OF THE MODULI SPACE OF SURFACES OF GENERAL TYPE II

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Abstract. In this paper we investigate the asymptotic growth of the number of irreducible and connected components of the moduli space of surfaces of general type corresponding to certain families of surfaces isogenous to a higher product with group \((\mathbb{Z}/2\mathbb{Z})^k\). We obtain a significantly higher growth than the one in our previous paper [LP14].

1. Introduction

It is well known (see [Gie77]) that once two positive integers \(x, y\) are fixed there exists a quasiprojective coarse moduli space \(M_{y,x}\) of canonical models of surfaces of general type with \(x = \chi(S)\) and \(y = K_S^2\). The number \(\iota(x, y)\), resp. \(\gamma(x, y)\), of irreducible, resp. connected, components of \(M_{y,x}\) is bounded from above by a function of \(y\). In fact, Catanese proved that the number \(\iota^0(y, x)\) of components containing regular surfaces, i.e., \(q(S) = 0\), has an exponential upper bound in \(K^2\). More precisely [Cat92, p.592] gives the following inequality

\[
\iota^0(x, y) \leq y^{77/2}y^2.
\]

This result is not known to be sharp and in recent papers [M97, Ch95, GP14, LP14] inequalities are proved which tell how close one can get to this bound from below. In particular, in the last two papers the authors considered families of surfaces isogenous to a product in order to construct many irreducible components of the moduli space of surfaces of general type. The reason why one works with these surfaces, is the fact that the number of families of these surfaces can be easily computed using group theoretical and combinatorial methods.

In our previous work [LP14] we constructed many such families with many different 2-groups. There, we exploited the fact that the number of 2-groups with given order grows very fast in function of the order. In this paper we obtain a significantly better lower bound for \(\iota^0(x, y)\) using only the groups \((\mathbb{Z}/2\mathbb{Z})^k\) and again some properties of the moduli space of surfaces isogenous to a product. Our main result is the following theorem.

**Theorem 1.1.** Let \(h\) be number of connected components of the moduli space of surfaces of general type containing regular surfaces isogenous to a product of curves, admitting \((\mathbb{Z}/2\mathbb{Z})^k\) as group and ramification structure of type \((2^{k(k-1)/2}, 2^{k^2-k-1})\). Then for \(k \to \infty\) we have

\[
h \geq 2^{2\nu + \sqrt{k^2}}.
\]

with \(\nu\) a positive real number. In particular, we obtain sequences \(y_k\) with

\[
\iota^0(x_k, y_k) \geq C y_k^{\sqrt{k}}.
\]

Let us explain now the way in which this paper is organized.

In the next section **Preliminaries** we recall the definition and the properties of surfaces isogenous to a higher product and the its associated group theoretical data. Moreover, we recall a result of Bauer–Catanese [BC] which allows us to count the number of connected components of the moduli space of surfaces isogenous to a product with given group and type of ramification structure.

In the last section we give the proof of the **Theorem 1.1**

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Notation and conventions. We work over the field \( \mathbb{C} \) of complex numbers. By surface we mean a projective, non-singular surface \( S \). For such a surface \( \omega_S = O_S(K_S) \) denotes the canonical bundle, \( p_g(S) = h^0(S, \omega_S) \) is the geometric genus, \( q(S) = h^1(S, \omega_S) \) is the irregularity, \( \chi(O_S) = \chi(S) = 1 - q(S) + p_g(S) \) is the Euler-Poincaré characteristic and \( e(S) \) is the topological Euler number of \( S \).

Definition 2.1. A surface \( S \) is said to be isogenous to a higher product of curves if and only if, \( S \) is a quotient \( (C_1 \times C_2)/G \), where \( C_1 \) and \( C_2 \) are curves of genus at least two, and \( G \) is a finite group acting freely on \( C_1 \times C_2 \).

Using the same notation as in Definition 2.1 let \( S \) be a surface isogenous to a product, and \( G^\circ := G \cap (\text{Aut}(C_1) \times \text{Aut}(C_2)) \). Then \( G^\circ \) acts on the two factors \( C_1, C_2 \) and diagonally on the product \( C_1 \times C_2 \). If \( G^\circ \) acts faithfully on both curves, we say that \( S = (C_1 \times C_2)/G \) is a minimal realization. In [Cat00] it is also proven that any surface isogenous to a product admits a unique minimal realization.

Assumptions. In the following we always assume:

1. Any surface \( S \) isogenous to a product is given by its unique minimal realization;
2. \( G^\circ = G \), this case is also known as unmixed type, see [Cat00].

Under these assumption we have.

Proposition 2.2. [Cat00] Let \( S = (C_1 \times C_2)/G \) be a surface isogenous to a higher product of curves, then \( S \) is a minimal surface of general type with the following invariants:

\[
\chi(S) = \frac{(g(C_1) - 1)(g(C_2) - 1)}{|G|}, \quad e(S) = 4\chi(S), \quad K_S^2 = 8\chi(S).
\]

The irregularity of these surfaces is computed by

\[
q(S) = g(C_1/G) + g(C_2/G).
\]

Among the nice features of surfaces isogenous to a product, one is that their deformation class can be obtained in a purely algebraic way. Let us briefly recall this in the particular case when \( S \) is regular, i.e., \( q(S) = 0 \), \( C_i/G \cong \mathbb{P}^1 \).

Definition 2.3. Let \( G \) be a finite group and \( r \in \mathbb{N} \) with \( r \geq 2 \).

- An \( r \)-tuple \( T = (v_1, \ldots, v_r) \) of elements of \( G \) is called a spherical system of generators of \( G \) if \( \langle v_1, \ldots, v_r \rangle = G \) and \( v_1 \cdot \ldots \cdot v_r = 1 \).
- We say that \( T \) has an unordered type \( \tau := (m_1, \ldots, m_r) \) if the orders of \( (v_1, \ldots, v_r) \) are \( (m_1, \ldots, m_r) \) up to a permutation, namely, if there is a permutation \( \pi \in \mathfrak{S}_r \) such that \( \text{ord}(v_1) = m_{\pi(1)}, \ldots, \text{ord}(v_r) = m_{\pi(r)} \).
- Moreover, two spherical systems \( T_1 = (v_1,1, \ldots, v_{1,r_1}) \) and \( T_2 = (v_2,1, \ldots, v_{2,r_2}) \) are said to be disjoint, if:

\[
\Sigma(T_1) \cap \Sigma(T_2) = \{1\},
\]

where

\[
\Sigma(T_i) := \bigcup_{g \in G} \bigcup_{j=0}^{r_i} g \cdot v_{i,k}^j \cdot g^{-1}.
\]
Definition 2.4. Let $2 < r_i \in \mathbb{N}$ for $i = 1, 2$ and $\tau_i = (m_{i,1}, \ldots, m_{i,r_i})$ be two sequences of natural numbers such that $m_{k,i} \geq 2$. A (spherical-) ramification structure of type $(\tau_1, \tau_2)$ and size $(r_1, r_2)$ for a finite group $G$, is a pair $(T_1, T_2)$ of disjoint spherical systems of generators of $G$, whose types are $\tau_i$, such that:

$$Z \geq \frac{|G|(-2 + \sum_{i=1}^{r_1}(1 - \frac{1}{m_{1,i}}))}{2} + 1 \geq 2, \quad \text{for } i = 1, 2.$$  

Remark 2.5. Following e.g., the discussion in [LP14, Section 2] we obtain that the datum of the deformation class of a regular surface $S$ isogenous to a higher product of unmixed type together with its minimal realization $S = (C_1 \times C_2)/G$ is determined by the datum of a finite group $G$ together with two disjoint spherical systems of generators $T_1$ and $T_2$ (for more details see also [BCG06]).

Remark 2.6. Recall that from Riemann Existence Theorem a finite group $G$ acts as a group of automorphisms of some curve $C$ of genus $g$ such that $C/G \cong \mathbb{P}^1$ if and only if there exist integers $m_r \geq m_{r-1} \geq \cdots \geq m_1 \geq 2$ such that $G$ has a spherical system of generators of type $(m_1, \ldots, m_r)$ and the following Riemann-Hurwitz relation holds:

$$2g - 2 = |G|(-2 + \sum_{i=1}^{r}(1 - \frac{1}{m_i})).$$

Remark 2.7. Note that a group $G$ and a ramification structure determine the main numerical invariants of the surface $S$. Indeed, by (1) and (2) we obtain:

$$4\chi(S) = |G| \cdot \left(-2 + \sum_{k=1}^{r_1}(1 - \frac{1}{m_{1,k}})\right) \cdot \left(-2 + \sum_{k=1}^{r_2}(1 - \frac{1}{m_{2,k}})\right) =: 4\chi(|G|, (\tau_1, \tau_2)).$$

Let $S$ be a surface isogenous to a product of unmixed type with group $G$ and a pair of two disjoint spherical systems of generators of types $(\tau_1, \tau_2)$. By (3) we have $\chi(S) = \chi(G, (\tau_1, \tau_2))$, and consequently, by (1), $K_S^2 = K^2(G, (\tau_1, \tau_2)) = 8\chi(S)$.

Let us fix a group $G$ and a pair of unmixed ramification types $(\tau_1, \tau_2)$, and denote by $\mathcal{M}_{(G, (\tau_1, \tau_2))}$ the moduli space of isomorphism classes of surfaces isogenous to a product admitting these data, by [Cat00, Cat03] the space $\mathcal{M}_{(G, (\tau_1, \tau_2))}$ consists of a finite number of connected components. Indeed, there is a group theoretical procedure to count these components. In case $G$ is abelian it is described in [BC].

Theorem 2.8. [BC, Theorem 1.3] . Let $S$ be a surface isogenous to a higher product of unmixed type and with $q = 0$. Then to $S$ we attach its finite group $G$ (up to isomorphism) and the equivalence classes of an unordered pair of disjoint spherical systems of generators $(T_1, T_2)$ of $G$, under the equivalence relation generated by:

(i) Hurwitz equivalence for $T_1$;

(ii) Hurwitz equivalence for $T_2$;

(iii) Simultaneous conjugation for $T_1$ and $T_2$, i.e., for $\phi \in \text{Aut}(G)$ we let $(T_1 := (x_{1,1}, \ldots, x_{r_1,1}), \quad T_2 := (x_{1,2}, \ldots, x_{r_2,2}))$ be equivalent to

$$\left(\phi(T_1) := (\phi(x_{1,1}), \ldots, \phi(x_{r_1,1})), \quad \phi(T_2) := (\phi(x_{1,2}), \ldots, \phi(x_{r_2,2}))\right).$$

Then two surfaces $S, S'$ are deformation equivalent if and only if the corresponding equivalence classes of spherical generating systems of $G$ are the same.

The Hurwitz equivalence is defined precisely in e.g., [P13]. In the cases that we will treat the Hurwitz equivalence is given only by the braid group action on $T_i$ defined as follows. Recall the Artin presentation of the Braid group of $r_1$ strands

$$B_{r_1} := \langle \gamma_1, \ldots, \gamma_{r_1-1}| \gamma_i \gamma_j = \gamma_j \gamma_i \text{ for } |i - j| \geq 2, \gamma_i \gamma_{i+1} \gamma_i = \gamma_{i+1} \gamma_i \gamma_{i+1} \rangle.$$  

For $\gamma_i \in B_{r_1}$ then:

$$\gamma_i(T_1) = \gamma_i(v_1, \ldots, v_{r_1}) = (v_{i,1}, \ldots, v_{i+1}, v_{i+1,1}^{-1} v_{i+1} v_{i+1,1}, \ldots, v_{r_1}).$$
Moreover, notice that, since we deal here with abelian groups only, the braid group action is indeed only by permutation of the elements on the spherical system of generators.

Once we fix a finite abelian group $G$ and a pair of types $(\tau_1, \tau_2)$ (of size $(r_1, r_2)$) of an unmixed ramification structure for $G$, counting the number of connected components of $\mathcal{M}(G, (\tau_1, \tau_2))$ is then equivalent to the group theoretical problem of counting the number of classes of pairs of spherical systems of generators of $G$ of type $(\tau_1, \tau_2)$ under the equivalence relation given by the action of $B_{r_1} \times B_{r_2} \times \text{Aut}(G)$, given by:

\[(\gamma_1, \gamma_2, \phi) \cdot (T_1, T_2) := (\phi(\gamma_1(T_1)), \phi(\gamma_2(T_2))),\]

where $\gamma_1 \in B_{r_1}$, $\gamma_2 \in B_{r_1}$ and $\phi \in \text{Aut}(G)$, see for more details e.g., [P13].

\section{Proof of Theorem 1.1}

Let us consider the group $G := ((\mathbb{Z}/2\mathbb{Z})^k$, with $k >> 0$ and an integer $l$. We want to give to $G$ many classes of ramification structures of size $(r_1, r_2) = (k(k+1), 2^{l-k+1} + 4)$. Since the elements of $G$ have only order two we will produce in the end ramification structure of type $((2^1), (2^{2^i}))$.

First let us consider the following elements of $G$

\[v_1 = (1,0,\ldots,0)\]
\[v_2 = (1,0,\ldots,0)\]
\[v_3 = (0,1,0,\ldots,0)\]
\[v_4 = (0,1,0,\ldots,0)\]
\[v_5 = (0,1,0,\ldots,0)\]
\[v_6 = (0,1,0,\ldots,0)\]
\[\vdots\]
\[v_{k(k+1)} = (0,\ldots,0,1)\]

and let $T_1 := (v_1, v_2, \ldots, v_{k(k+1)})$. One can see that $< T_1 \cong G$ and by construction the product of the elements in $T_1$ is $1_G$. Define the set $M := G \setminus \{0, v_1, \ldots, v_{k(k+1)}\}$. We have a bijection

\[M \overset{1:1}{\longleftrightarrow} \{n \in \mathbb{N} | n \leq 2^k - k - 1\} =: B.\]

Call $\varphi: B \longrightarrow M$ the bijection map. Consider $(n_1, \ldots, n_{2^k - k - 2})$ a $(2^k - k - 1)$-tuple of elements of $B$ whose sum is $2^{l-k} + 2$. We define a map

\[(n_1, \ldots, n_{2^k - k - 1}) \mapsto T_2 = (\underbrace{\varphi(1), \ldots, \varphi(1)}_{2n_1}, \underbrace{\varphi(2^k - k - 1), \ldots, \varphi(2^k - k - 1)}_{2n_{2^k - k - 1}}).\]

It holds that $< T_2 \cong G$. Moreover, the product of the elements in $T_2$ is $1_G$, hence $T_2$ is a spherical system of generators for $G$ of size $2^{l-k+1}$.

By construction $G$ is abelian and all its elements are of order two, therefore the pair $(T_1, T_2)$ is a ramification structure for $G$ of the desired type.

Now we count how many inequivalent ramification structures of this kind we have under the action of the group defined in Theorem 2.8 and Equation (7). First notice that by construction to any tuple $(n_1, \ldots, n_{2^k - k - 2})$ its associated generating vector $T_2$ is in a different braid orbit. Moreover, the choice of $T_1$ implies that any pair $(T_1, T_2')$ and $(T_1, T_2'')$ are in the same Aut($G$)-orbit if and only if $T_2' = T_2''$.

Hence the number of inequivalent ramification structures is equal to the number of $(2^k - k - 1)$-tuple of positive integers whose sum is $2^{l-k} + 2$.

This condition maybe relaxed to the point that only for the elements of a basis the entry must be strictly positive and maybe non-negative else.
This number is known to be
\[
\frac{\frac{\nu}{2} - 1}{2^k - k - 2} = \frac{2^{l-k} + 1}{2^k - k - 2},
\]
see e.g., [F50, Section II.5]. Let \( \nu > 0 \) be a rational number and let us suppose that \( l = (\nu+2) \cdot k \), then using Stirling’s approximation of the binomial coefficient - more exactly a corresponding lower bound - we obtain
\[
\left( \frac{2^{l-k} + 1}{2^k - k - 2} \right) \frac{\nu}{2} > \frac{1}{e\sqrt{2^k - k - 2}} \cdot \frac{2^k - k - 2}{\nu} > 2^{\nu k (2^k)}.
\]
Since \( e_k = |G|(-2 + \frac{1}{2}r_1)(-2 + \frac{1}{2}r_2) \) implies \( 2e_k = 2^k \cdot 2^{l-k} \cdot (k^2 + k - 4) = 2^{\nu+2} k (k^2 + k - 4) \) we have
\[
(2e_k)^\frac{1}{e\sqrt{2^k - k - 2}} \cdot \frac{k}{(k^2 + k - 4)^{\nu+2}} = k 2^k
\]
Using this, we obtain for \( k \) large enough in the second inequality
\[
\nu(2e_k)^\frac{1}{e\sqrt{2^k - k - 2}} > 2^{\nu k (2^k) + \nu}
\]
We can bound further for \( k \) large enough
\[
2^{\nu k (2^k) + \nu} > 2^{(e_k)^\nu}
\]
We use the identity \( x^f(x) = e^{f(x)\ln x} = 2f(x)\pi^2\ln x \) to get for all \( \alpha < \frac{1}{2} \)
\[
h > e^{\nu k (2^k) + \nu}
\]
if \( k \) is large enough, depending on \( \alpha \). This concludes the proof since \( e_k \) is proportional to \( y_k \). \( \square \)

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