On a Nonlinear Mixed Problem for a Parabolic Equation with a Nonlocal Condition

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Abstract: The aim of this work is to prove the well-posedness of some linear and nonlinear mixed problems with integral conditions defined only on two parts of the considered boundary. First, we establish for the associated linear problem a priori estimate and prove that the range of the operator generated by the considered problem is dense using a functional analysis method. Then by applying an iterative process based on the obtained results for the linear problem, we establish the existence, uniqueness and continuous dependence of the weak solution of the nonlinear problem.

Keywords: energy inequality; integral boundary conditions; strong solution; weak solution; second order parabolic equation

MSC: 35B45; 35K61

1. Introduction and Statement of the Problem

Some problems related to physical and technical issues can be described in terms of nonlocal problems with integral conditions in partial differential equations. Nonlocal conditions arise mainly when the values of the studied function on the boundary cannot be measured directly, while their average are known. Parabolic equation, describe several physical phenomena belongs to this of problem i.e., nonlocal problems. Therefore, the problem of parabolic equation with integral condition is stated as follows:

We consider in the rectangular domain $\Omega = [0, 1] \times [0, T]$, where the problem is to find a solution $\sigma(x, t)$ of the following non-classical boundary value problem such that

$$\mathcal{L}\sigma = \frac{\partial\sigma}{\partial t} - \frac{\partial}{\partial x} \left( a \frac{\partial\sigma}{\partial x} \right) = g(x, t, \sigma, \frac{\partial\sigma}{\partial x}), \quad \text{for } (x, t) \in [0, 1] \times [0, T],$$

with the initial condition

$$l\sigma = \sigma(x, 0) = \varphi(x), \quad \text{for } x \in [0, 1],$$

and Dirichlet boundary condition

$$\sigma(0, t) = \sigma(1, t), \quad \text{for } t \in [0, T],$$

and the nonlocal condition

$$\int_0^\alpha K_1(x)\sigma(x, t)dx + \int_\alpha^1 K_2(x)\sigma(x, t)dx = 0, \quad 0 < \alpha < 1 \quad \forall t \in [0, T].$$
where the functions \( g(x,t,\sigma, \frac{\partial \sigma}{\partial x}) \), \( \varphi(x) \) are known, and we assume that the following matching conditions are satisfied

\[
\begin{align*}
\begin{cases}
\varphi(0) = 0, \\
\int_0^a K_1(x) \varphi(x) dx + \int_a^1 K_2(x) \varphi(x) dx = 0.
\end{cases}
\end{align*}
\]

We also assume that there exists a positive constant \( d \) such that

\[
\left| g\left(x,t,\sigma_1, \frac{\partial \sigma_1}{\partial x}\right) - g\left(x,t,\sigma_2, \frac{\partial \sigma_2}{\partial x}\right) \right| \leq d\left( |\sigma_1 - \sigma_2| + \left| \frac{\partial \sigma_1}{\partial x} - \frac{\partial \sigma_2}{\partial x}\right| \right),
\]

for all \((x,t) \in \Omega\).

In addition, we assume that the functions \( a(x,t) \), \( K_1(x) \), \( K_2(x) \) are differentiable on \( \Omega \), respectively on \([0,1]\) and their derivatives satisfy the conditions

\[
\begin{align*}
\begin{cases}
0 < a_0 \leq a(x,t) \leq a_1 & \forall (x,t) \in \Omega, \\
c_2 \leq \frac{\partial a}{\partial t}(x,t) \leq c_1, & \forall (x,t) \in \Omega, \\
\left| \frac{\partial a}{\partial x}(x,t) \right| \leq b, & \forall (x,t) \in \Omega, \\
0 < m_0 \leq K_1(x) \leq m_1, 0 < m_2 \leq K_2(x) \leq m_3, & \forall x \in [0,1], \\
\left| \frac{dK_1(x)}{dx} \right| \leq b_1, \left| \frac{dK_2(x)}{dx} \right| \leq b_2, & \forall x \in [0,1],
\end{cases}
\end{align*}
\]

(5)

This type of problem can be found in various problems arising from physics, such as heat conduction [1–4], plasma physics [5], thermoelasticity [6], electrochemistry [7], chemical diffusion [8] and underground water flow [9–11]. Several results published in the literature such as in [1–4,7,12–24] have solved the parabolic equation by combining the integral condition with Dirichlet condition or Neumann condition, or with purely integral conditions, using various methods. For hyperbolic equations, the unicity and existence of the solution have been studied in [13,25–32] and the mixed type equations in [33–37]. The elliptic equations were considered in [38–40].

The linear problem associated with the problem stated in Equations (1)–(4), for \( \alpha = 0 \) and \( K_2(x) = 1 \), has been studied in [20] and for \( \alpha = 1 \) and \( K_1(x) = 1 \) in [17] with Dirichlet condition.

The main purpose of the present paper is to study and found a solution to the posed problem without imposing any conditions on the functions \( K_1(x) \), \( K_2(x) \) and on the constant \( \alpha \) in the interval \([0,1]\). In addition, the nonlinear problem of the parabolic equation with weighted integral condition defined on two parts of the boundary is solved.

The following methodology to solve the posed problem is summarized below

First, an a priori estimate is established for the associated linear problem and the density of the operator range generated by the considered problem is proved using the functional analysis method. Subsequently, by applying an iterative process based on the obtained results for the linear problem, the existence and uniqueness of the weak solution of the nonlinear problems is established.

The rest of the paper is organized as follows: In Section 2, the associated linear problem is stated. Section 3 deals with the proof of the uniqueness of the solution using an a priori estimate, while Section 4 gives the solvability of the considered linear problem. Finally, in Section 5, based on the obtained results in Sections 3 and 4, and on the use of an iterative process, we prove the existence and uniqueness of the solution of the nonlinear problem.
2. Statement of the Associated Linear Problem

Let us in this section give the position of the linear problem and introduce the different function spaces needed to investigate the mixed nonlocal problem given by Equation (6), below

\[ Lu = \frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) = f(x, t), \]  

and supplemented by the conditions Equations (2)–(4).

The given problem Equations (6), (2)–(4) can be considered to be a solving of the operator equation

\[ Lu = (\xi u, lu) = F, \]

where the operator \( L \) has a domain of definition \( \mathcal{D}(L) \) consisting of functions \( u \in L^2(\Omega) \) such that \( \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}(x, t) \in L^2(\Omega) \) and satisfying the conditions Equations (3) and (4).

The operator \( L \) is an operator defined on \( E \) into \( F \), where \( E \) is the Banach space of functions \( u \in L^2(\Omega) \), with the finite norm

\[ ||u||_E = \int_\Omega \Phi(x) \left[ \left| \frac{\partial u}{\partial t} \right|^2 + \left| \frac{\partial^2 u}{\partial x^2} \right|^2 \right] dx + \int_0^1 \Phi(x) \left[ \left| \frac{\partial u}{\partial x} \right|^2 + |u|^2 \right] dx. \]  

\( F \) is the Hilbert space of functions \( F = (f, \varphi), f \in L^2(\Omega), \varphi \in H^1(0,1) \) with the finite norm

\[ ||F||_F^2 = \int_\Omega \Phi(x)|f(x, t)|^2 dx + \int_0^1 \Phi(x) \left[ \frac{d\varphi}{dx} \right]^2 + |\varphi|^2 dx. \]  

where

\[ \Phi(x) = \begin{cases} 
\frac{x^2}{\alpha^2}, & 0 \leq x \leq \alpha, \\
\frac{1-x}{\alpha}^2, & \alpha \leq x \leq 1.
\end{cases} \]

Then, we show that the operator \( L \) has a closure \( \overline{L} \) and establish an energy inequality:

\[ ||u||_E \leq k ||Lu||_F, \quad \forall u \in \mathcal{D}(L). \]  

**Definition 1.** A solution of the operator equation \( \overline{L}u = F = (f, \varphi) \) is called a strong solution of problem Equations (6), (2)–(4).

Since the points of the graph of the operator \( \overline{L} \) are limits of sequences of points of the graph of \( L \), we can extend the a priori estimate Equation (9) to be applied to strong solutions by taking limits, i.e., we have the inequality

\[ ||u||_E \leq k ||L_{\overline{L}}u||_F, \quad \forall u \in \mathcal{D}(\overline{L}). \]  

From this inequality, we deduce the uniqueness of a strong solution, if it exists, and that the range of the operator \( \overline{L} \) coincides with the closure of the range of \( L \).

**Proposition 1.** The operator \( L : E \rightarrow F \) admits a closure \( \overline{L} \).

The following a priori estimate gives the uniqueness of the solution if it exists of the posed linear problem.
3. An Energy Inequality and Its Application

**Theorem 1.** There exists a positive constant $k$, such that for each function $u \in D(L)$ we have

$$\|u\|_E \leq k\|Lu\|_F. \quad (11)$$

**Proof of Theorem 1.** Let

$$Mu = \begin{cases} \frac{x^2K_1}{a(0,t)}e^{\lambda x}\frac{\partial u}{\partial t} + \frac{2x}{a(0,t)}e^{\lambda x} \int_0^x K_1(\mu)\frac{\partial u}{\partial t}d\mu, & 0 \leq x \leq \alpha, \\ \frac{(1-x)^2K_2}{a(1,t)}e^{\beta(1-x)}\frac{\partial u}{\partial t} + \frac{2(1-x)}{a(1,t)}e^{\beta(1-x)} \int_0^x K_2(\mu)\frac{\partial u}{\partial t}d\mu, & \alpha \leq x \leq 1. \end{cases}$$

where

$$\left\{ \begin{array}{l} \frac{a^2}{a(0,t)}e^{\lambda x}K_1(\alpha) = \frac{(1-a)^2}{a(1,t)}e^{\beta(1-x)}K_2(\alpha), \\ \lambda > \max \left\{ 0, \sup_{0 \leq x \leq \alpha} \frac{K_1'(x)}{K_1(x)} \right\}, \\ \beta > \max \left\{ 0, \sup_{\alpha \leq x \leq 1} \frac{K_2'(x)}{K_2(x)} \right\}. \end{array} \right. \quad (12)$$

We consider the quadratic form obtained by multiplying Equation (6) by $e^{-\zeta t}Mu$, with $0 \leq s \leq T$, $c > 0$, integrating over $\Omega^s = [0, 1] \times [0, s]$ and taking the real part, we obtain

$$\Phi(u, u) = \text{Re} \int_{\Omega^s} e^{-\zeta t} \frac{\partial u}{\partial t} Mu dx dt - \text{Re} \int_{\Omega^s} e^{-\zeta t} \frac{\partial}{\partial x} \left( a(x, t) \frac{\partial u}{\partial x} \right) Mu dx dt. \quad (13)$$

Substituting $Mu$ by its expression in the right term in the right-hand side of Equation (13), integrating with respect to $x$, using the boundary condition Equation (3), the integral condition Equations (4) and (12), we obtain

$$\text{Re} \int_{\Omega^s} e^{-\zeta t} \frac{\partial u}{\partial t} Mu dx dt = f_0^s f_0^a \frac{x^2K_1}{a(0,t)}e^{-\zeta t + \lambda x} \left| \frac{\partial u}{\partial t} \right|^2 dx dt$$

$$+ f_0^s \int_0^x \frac{(1-x)^2K_2}{a(1,t)}e^{-\zeta t + \beta(1-x)} \left| \frac{\partial u}{\partial t} \right|^2 dx dt + f_0^s \int_0^a \frac{K_1(\lambda K_1 - K_1')}{a(0,t)K_1^2}e^{-\zeta t + \lambda x} \left| \int_0^x K_1(\mu)\frac{\partial u}{\partial t}d\mu \right|^2 dx dt$$

$$+ f_0^s \int_a^1 \frac{K_2(\beta K_2 + K_2')}{a(1,t)K_2^2}e^{-\zeta t + \beta(1-x)} \left| \int_a^x K_2(\mu)\frac{\partial u}{\partial t}d\mu \right|^2 dx dt. \quad (14)$$
\[-\text{Re} \int_{1}^{\infty} e^{-ct} \frac{\partial}{\partial x} \left( a(x,t) \frac{\partial u}{\partial x} \right) \mathcal{M} du dt = \]
\[\text{Re} \int_{0}^{s} \int_{0}^{t} \frac{K_{2} a e^{-ct+\lambda x} \frac{\partial u}{\partial x}}{(a/t)} dx dt + \text{Re} \int_{0}^{s} \int_{a}^{1} (1-x)^{2} K_{2} e^{-ct+\beta(1-x)} \frac{\partial u}{\partial x} dx dt + \text{Re} \int_{0}^{s} \int_{a}^{1} (1+\beta(x-1)) K_{2} e^{-ct+\beta(1-x)} \frac{\partial u}{\partial t} dx dt \]
\[+ 2 \text{Re} \int_{0}^{s} \int_{0}^{t} (1+\lambda \alpha) K_{1} a \frac{\partial u}{\partial x} dx dt + 2 \text{Re} \int_{0}^{s} \int_{a}^{1} (1+\beta(x-1)) K_{2} e^{-ct+\beta(1-x)} \frac{\partial u}{\partial t} dx dt \]
\[+ \text{Re} \int_{0}^{s} \int_{0}^{t} a K_{2} e^{-ct+\lambda x} \frac{\partial u}{\partial x} dx dt + \text{Re} \int_{0}^{s} \int_{a}^{1} (1+\lambda \alpha) a \frac{\partial u}{\partial x} dx dt \]
\[= 2 \text{Re} \int_{0}^{s} \int_{0}^{t} a \frac{\partial u}{\partial x} dx dt + \text{Re} \int_{0}^{s} \int_{a}^{1} (1+\lambda \alpha) a \frac{\partial u}{\partial x} dx dt \]
\[\text{(15)}\]

Integrating the first four terms with respect to \(t\) in Equation (15) using the condition Equation (2) we have
\[\text{Re} \int_{0}^{s} \int_{0}^{t} a \frac{\partial u}{\partial x} dx dt = \frac{1}{2} \int_{0}^{s} \int_{0} a \frac{\partial u}{\partial x} dx dt \]
\[\int_{0}^{s} \int_{0}^{t} a \frac{\partial u}{\partial x} dx dt = \frac{1}{2} \int_{0}^{s} \int_{0} a \frac{\partial u}{\partial x} dx dt \]
\[\text{Re} \int_{0}^{s} \int_{0}^{t} a \frac{\partial u}{\partial x} dx dt = \]
\[\frac{1}{2} \int_{0}^{s} \int_{0} a \frac{\partial u}{\partial x} dx dt \]
\[\int_{0}^{s} \int_{0}^{t} a \frac{\partial u}{\partial x} dx dt = \frac{1}{2} \int_{0}^{s} \int_{0} a \frac{\partial u}{\partial x} dx dt \]

and
\[\text{Re} \int_{0}^{s} \int_{0}^{t} a \frac{\partial u}{\partial x} dx dt = \]
\[\frac{1}{2} \int_{0}^{s} \int_{0} a \frac{\partial u}{\partial x} dx dt \]
\[\int_{0}^{s} \int_{0}^{t} a \frac{\partial u}{\partial x} dx dt = \]
\[\frac{1}{2} \int_{0}^{s} \int_{0} a \frac{\partial u}{\partial x} dx dt \]
Combining the previous equalities with Equations (14) and (15), Equation (13) becomes

\[
\int_0^s \int_0^a e^{-ct+\lambda x} a \frac{\partial u}{\partial t} \, dx \, dt + \int_0^s \int_0^1 \left( 1-x \right)^2 K_1 e^{-ct+\beta(1-x)} a \frac{\partial u}{\partial t} \, dx \, dt
\]

\[
+ \frac{1}{2} \int_0^s \int_0^a c a \frac{\partial u}{\partial t} \left( a \frac{\partial u}{\partial t} - a \frac{\partial u}{\partial t} \right) x^2 K_1 e^{-ct+\lambda x} + \frac{1}{2} \int_0^s \int_0^a c a \frac{\partial u}{\partial t} \left( a \frac{\partial u}{\partial t} - a \frac{\partial u}{\partial t} \right) \left( 1-x \right)^2 K_2 e^{-ct+\beta(1-x)} a \frac{\partial u}{\partial t} \, dx \, dt
\]

\[
= \int_0^s \int_0^a e^{-ct+\lambda x} a \frac{\partial u}{\partial t} \left( a \frac{\partial u}{\partial t} - a \frac{\partial u}{\partial t} \right) x^2 K_1 e^{-ct+\lambda x} + \frac{1}{2} \int_0^s \int_0^a e^{-ct+\beta(1-x)} a \frac{\partial u}{\partial t} \left( a \frac{\partial u}{\partial t} - a \frac{\partial u}{\partial t} \right) \left( 1-x \right)^2 K_2 e^{-ct+\beta(1-x)} a \frac{\partial u}{\partial t} \, dx \, dt
\]

Using Young inequalities and using the fact that

\[
0 \leq e^{-ct+\lambda x} a \frac{\partial u}{\partial t} \leq 4m_1 a_1 \int_0^s \int_0^a \frac{K_1}{a(0,t)} x^2 e^{-ct+\lambda x} a \frac{\partial u}{\partial t} \, dx \, dt,
\]

\[
0 \leq e^{-ct+\beta(1-x)} a \frac{\partial u}{\partial t} \leq 4m_2 a_1 \int_0^s \int_0^1 \frac{K_2}{a(1,t)} (1-x)^2 e^{-ct+\lambda x} a \frac{\partial u}{\partial t} \, dx \, dt,
\]

we obtain

\[
0 \leq \int_0^s \int_0^a \frac{K_1'}{a(0,t)} e^{-ct+\lambda x} a \frac{\partial u}{\partial t} \, dx \, dt \leq \int_0^s \int_0^a \frac{x^2 a_1}{a(0,t)} e^{-ct+\lambda x} a \frac{\partial u}{\partial t} \, dx \, dt + \frac{1}{4} \int_0^s \int_0^a \frac{a_1^2 (b_1 + \lambda m_1)^2}{a(0,t)} \frac{\partial u}{\partial t} \, dx \, dt
\]

\[
0 \leq \int_0^s \int_0^a \frac{K_2'}{a(1,t)} e^{-ct+\beta(1-x)} a \frac{\partial u}{\partial t} \, dx \, dt \leq \int_0^s \int_0^a \frac{x^2 a_1}{a(1,t)} e^{-ct+\beta (1-x)} a \frac{\partial u}{\partial t} \, dx \, dt,
\]

\[
0 \leq \int_0^s \int_0^a \frac{a_1^2 (b_1 + \lambda m_1)^2}{a(1,t)} \frac{\partial u}{\partial t} \, dx \, dt.
\]
\[
\begin{align*}
\frac{1}{4} \int_0^s \int_0^1 (1-x)^2 K_2 e^{ct+\lambda x} \left| \frac{\partial u}{\partial t} \right|^2 dx dt + \frac{a_1^2(b_2 + \beta m_3)^2}{a_0 m_2^2} \int_0^s \int_0^1 (1-x)^2 K_2 e^{ct+\lambda x} \left| \frac{\partial u}{\partial x} \right|^2 dx dt, \\
2 \text{Re} \int_0^s \int_0^1 \frac{\lambda a + (1 + \lambda x) \left( \lambda a + \frac{\partial a}{\partial t} \right)}{a(x,t)} e^{-ct+\lambda x} u \int_x^1 K_1 \frac{\partial u}{\partial t} d\zeta dx dt \leq \\
16 \left( \frac{8m_3}{a_0} + \frac{32}{m_2 a_0^2} \right) \int_0^s \int_0^1 x^2 e^{-ct+\lambda x} f^2 dx dt + \frac{1}{4} \int_0^s \int_0^1 \frac{K_1}{a(x,t)} x^2 e^{-ct+\lambda x} \left| \frac{\partial u}{\partial t} \right|^2 dx dt, \\
2 \text{Re} \int_0^s \int_0^1 e^{-ct} f M u dx dt \leq \\
\left( \frac{8m_1}{a_0} + \frac{32}{m_2 a_0^2} \right) \int_0^s \int_0^1 (1-x)^2 e^{-ct+\beta(1-x)} f^2 dx dt + \frac{1}{4} \int_0^s \int_0^1 \left( 1-x \right)^2 K_2 e^{-ct+\beta(1-x)} \left| \frac{\partial u}{\partial t} \right|^2 dx dt.
\end{align*}
\]

We choose the constant \( c \) such that

\[
c > \frac{a_1}{a_0} \max \left\{ \left( \frac{M_1}{a_0} + 16 \frac{(\lambda a_1 + (\lambda a_1+b))^2}{a_0} \right) (1 + \lambda a_1), \left( \frac{M_2}{a_0} + 16 \frac{\beta a_1 + (\beta a_1+b)^2}{a_0} \right) (1 + \beta(1-a_1)), \right\}
\]

(17)

where

\[
M_1 = \sup_{(x,t) \in \Omega} \left( a(0,t) \frac{\partial a(0,t)}{\partial t} - a \frac{\partial a(0,t)}{\partial t} \right), \quad M_2 = \sup_{(x,t) \in \Omega} \left( \frac{\partial a}{\partial t} a(1,t) - a \frac{\partial a(1,t)}{\partial t} \right).
\]

Then by combining the previous inequalities with Equation (16), we obtain

\[
\int_\Omega \Phi(x) \left[ \left| \frac{\partial u}{\partial t} \right|^2 + \left| \frac{\partial u}{\partial x} \right|^2 \right] dx dt + \int_0^1 \phi \left( \Phi(x) \left| \frac{\partial u}{\partial x} \right|^2 + |u|^2 \right) dx \leq K^2 \int_\Omega \Phi(x) \left( \frac{\partial \varphi}{\partial x} \right)^2 + |\varphi|^2 \right) dx,
\]

(18)

where

\[
K^2 = \max \left\{ \left( \frac{8m_3}{a_0} + \frac{32}{m_2 a_0^2} \right) a^2 e^{\lambda x_1} \left( \frac{8m_3}{a_0} + \frac{32}{m_2 a_0^2} \right) (1-a)^2 e^{\beta(1-a)} \left( \frac{1+\lambda a_1}{a_0} \right) \left( \frac{1+\beta(1-a)}{a_0} \right) e^{ct} \right\}
\]

\[
\min \left\{ \frac{e^{\lambda x_1}}{2t_1} a^2 e^{\lambda x_1} \left( \frac{e^{\lambda x_1} - M_1}{2t_1} \right)^2 \left( 1-a \right)^2 m_1^2 \left( \frac{e^{\lambda x_1} - M_2}{2t_1} \right)^2 \left( \frac{e^{\lambda x_1} - M_3}{2t_1} \right)^2 \left( 1-a \right)^2 m_2 \left( \frac{e^{\lambda x_1} - M_4}{2t_1} \right)^2 \left( \frac{e^{\lambda x_1} - M_5}{2t_1} \right)^2 \left( \frac{e^{\lambda x_1} - M_6}{2t_1} \right)^2 \right\}
\]

From Equations (6) and (18), we deduce
\[
\int_{\Omega} \Phi(x) \left[ \left| \frac{\partial u}{\partial t} \right|^2 + \left| \frac{\partial^2 u}{\partial x^2} \right|^2 \right] dx dt + \int_{\Omega} \left( \Phi(x) \left| \frac{\partial u}{\partial x} \right|^2 + |u|^2 \right) dx \bigg|_{t=s} \leq \kappa_2^2 \left[ \int_{Q} \left| f \right|^2 dx dt + \int_{0}^{1} \left( \Phi(x) \left| \frac{\partial \varphi}{\partial x} \right|^2 + |\varphi|^2 \right) dx \right],
\]

If we drop the second term in the last inequality and by taking the least upper bound of the left side with respect to \( s \) from 0 to \( T \), we obtain the desired estimate Equation (11) with \( \kappa_2^2 = K_2^2 + \frac{4 + 2k^2 + 4b^2k^2}{d_0^2} \).

Then, we deduce the uniqueness and continuous dependence of the solution on the input data of the problem Equations (6), (2)–(4).

Corollary 1. If a strong solution of Equations (6), (2)–(4) exists, it is unique and depends continuously on \( F = (f, \varphi) \).

Corollary 2. The range \( R(\mathcal{L}) \) of \( \mathcal{L} \) is closed in \( F \) and \( R(\mathcal{L}) = R(\mathcal{L}) \).

Corollary (2) shows that to prove that problem Equations (6), (2)–(4) has a strong solution for arbitrary \( F \), it suffices to prove that the set \( R(\mathcal{L}) \) is dense in \( F \).

4. Solvability of Problem Equations (6), (2)–(4)

To prove the solvability of problem Equations (6), (2)–(4) it is sufficient to show that \( R(\mathcal{L}) \) is dense in \( F \). The proof is based on the following lemma

Lemma 1. Suppose that the function \( a \) and its derivatives are bounded.

Let \( u \in D_0(L) = \{u \in D(L), u(x, 0) = 0\} \). If, for \( u \in D_0(L) \) and some functions \( w \in L^2(\Omega) \), we have

\[
\int_{\Omega} \Phi(x) f \varphi dx dt = 0, \quad (19)
\]

then \( w \) vanish almost everywhere in \( \Omega \).

Proof of Lemma 1. Equation (19), can be written as follows

\[
\int_{\Omega} \frac{\partial u}{\partial t} \varphi dx dt = \int_{\Omega} A(t) u \varphi dx dt, \quad (20)
\]

where

\[ \rho = \Phi(x) w, \]

and

\[ A(t) u = \frac{\partial}{\partial x} \left( a(x, t) \frac{\partial u}{\partial x} \right). \]

We introduce the smoothing operators [17] \( I_{\varepsilon}^{-1} = \left( I - \varepsilon \frac{\partial}{\partial t} \right)^{-1} \) and \( (I_{\varepsilon}^{-1})^n = \left( I + \varepsilon \frac{\partial}{\partial t} \right)^{-1} \) Substituting the function \( u \) in Equation (20) by the smoothing function \( u_{\varepsilon} \) and using the relation

\[ A(t) u_{\varepsilon} = I_{\varepsilon}^{-1} A(t) u - \varepsilon I_{\varepsilon}^{-1} B_{\varepsilon}(t) u_{\varepsilon}, \]

where
\[
B_{c}(t)u_{e} = \frac{\partial A(t)}{\partial t}u_{e} = \frac{\partial}{\partial x} \left( \frac{\partial a}{\partial t} \frac{\partial u_{e}}{\partial x} \right),
\]
we obtain
\[
- \int_{\Omega} u \frac{\partial \rho_{e}^{c}}{\partial t} dx dt = \int_{\Omega} (A(t)u - eB_{c}(t)u_{e}) \rho_{e}^{c} dx dt. \tag{21}
\]

Since the operator \( A(t) \) has a continuous inverse in \( L^{2}(0, 1) \) defined by
\[
A^{-1}(t)g = \int_{0}^{1} \frac{d\zeta}{a} \int_{0}^{\zeta} g(\eta) d\eta + C_{1}(t) \int_{0}^{1} \frac{d\zeta}{a} + C_{2}(t),
\]
the functions \( C_{1}(t) \) and \( C_{2}(t) \) satisfies
\[
\begin{align*}
C_{1}(t) &= - \int_{0}^{1} k(x) dx \int_{0}^{\zeta} g(\eta) d\eta \frac{d\zeta}{a} + \int_{0}^{1} \frac{d\zeta}{a} \int_{0}^{\zeta} \frac{g(\eta) d\eta}{a}, \\
C_{2}(t) &= - \int_{0}^{1} k(x) dx \int_{0}^{\zeta} g(\eta) d\eta \frac{d\zeta}{a} + \int_{0}^{1} \frac{d\zeta}{a} \int_{0}^{\zeta} \frac{g(\eta) d\eta}{a} - \int_{0}^{1} \frac{1}{a} dx + \int_{0}^{1} \frac{g(\eta) d\eta}{a} d\zeta,
\end{align*}
\]
where
\[
K(x) = \begin{cases} K_{1}(x) & 0 \leq x \leq a, \\ K_{2}(x) & a \leq x \leq 1. \end{cases}
\]

Then, we have \( \int_{0}^{a} K_{1}(x) A^{-1}(t)u dx + \int_{a}^{1} K_{2}(x) A^{-1}(t)u dx = 0 \), hence, the function \( J_{e}^{-1}u = u_{e} \) can be represented in the form
\[
u_{e} = J_{e}^{-1}A^{-1}(t)A(t)u,
\]
then
\[
B_{c}(t)g = \frac{\partial^{2} a}{\partial t^{2}} \int_{0}^{1} C_{1}(t) \frac{\partial}{\partial x} \int_{0}^{\zeta} g(\eta) d\eta \frac{d\zeta}{a} + \frac{\partial a}{\partial t} \int_{0}^{1} \frac{\partial}{\partial t} \frac{1}{a} C_{1}(t) + \int_{0}^{1} \frac{g(\eta) d\eta}{a}.
\]

Consequently, Equation (21), becomes
\[
- \int_{\Omega} u \frac{\partial \rho_{e}^{c}}{\partial t} dx dt = \int_{\Omega} A(t)u \rho_{e}^{c} dx dt, \tag{22}
\]
where
\[
h_{e} = \rho_{e}^{c} - eB_{c}(t)\rho_{e}^{c},
\]
and \( B_{c}(t) \) is the adjoint operator of \( B_{c}(t) \).

The left-hand side of Equation (22) is a continuous linear functional of \( u \), hence, the function \( h_{e} \) has the derivatives \( \frac{\partial h_{e}}{\partial x} \), \( \frac{\partial^{2} h_{e}}{\partial x^{2}} \in L^{2}(\Omega) \) and the following condition are satisfied
\[
h_{e}(0, t) = h_{e}(1, t) = 0, \quad a(0, t) \frac{\partial h_{e}}{\partial x}(0, t) = a(1, t) \frac{\partial h_{e}}{\partial x}(1, t).
\]
For a sufficiently small $\epsilon$ and the operator $\frac{(I_\epsilon^{-1})^a}{\partial t}$ is bounded in $L^2(\Omega)$, we have

$$\left\| \frac{(I_\epsilon^{-1})^a}{\partial t} \right\|_{L^2(\Omega)} < 1,$$

hence, the operator $I - \epsilon \frac{(I_\epsilon^{-1})^a}{\partial t}$ has a bounded inverse in $L^2(\Omega)$, we deduce that $\frac{\partial \rho_\epsilon}{\partial x}, \frac{\partial^2 \rho_\epsilon}{\partial x^2} \in L^2(\Omega)$ and the following condition is satisfied

$$\rho_\epsilon^*(0, t) = \rho_\epsilon^*(1, t) = 0, \quad a(0, t) \frac{\partial \rho_\epsilon}{\partial x}(0, t) = a(1, t) \frac{\partial \rho_\epsilon}{\partial x}(1, t). \quad (23)$$

We introduce the function $v$ such that

$$\begin{cases}
K_1v = \frac{x}{\alpha}w + \frac{1}{\alpha} \int_0^x wd\zeta, & x \in (0, \alpha), \\
K_2v = \frac{1-x}{1-\alpha}w - \frac{1}{1-\alpha} \int_0^x wd\zeta, & x \in (\alpha, 1),
\end{cases}$$

then, the function $\rho(x)$ can be expressed as follows

$$\rho(x) = \begin{cases}
\frac{x^2}{\alpha^2}w = \frac{x}{\alpha}K_1v - \frac{1}{\alpha} \int_0^x K_1vd\zeta, & x \in (0, \alpha), \\
(1-x)^2 \frac{1}{(1-\alpha)^2}w = \frac{1-x}{1-\alpha}K_2v + \frac{1}{1-\alpha} \int_0^x K_2vd\zeta, & x \in (\alpha, 1),
\end{cases}$$

and we deduce that

$$\begin{cases}
v(0, t) = v(1, t) = 0, \quad \int_0^a K_1vd\zeta = \int_a^1 K_2vd\zeta = 0, \\
a(0, t)K_1(0) \frac{\partial v}{\partial x}(0, t) = a(1, t)K_2(1) \frac{\partial v}{\partial x}(1, t).
\end{cases}$$

and

$$\frac{\partial \rho}{\partial x} = H(x) \frac{\partial [K(x)v]}{\partial x}, \text{ where } H(x) = \begin{cases}
\frac{x}{\alpha}, & x \in (0, \alpha), \\
1 - \frac{x}{1-\alpha}, & x \in (\alpha, 1).
\end{cases}$$

Putting

$$u = \int_0^t \exp(c\tau)v d\tau,$$

in Equation (20) and integrating with respect to $x$ and $t$, using Equation (23) we obtain

$$\text{Re} \left| \int_{\Omega} A(t)u\rho d\zeta dt \right| = -\int_{\Omega} \frac{K(x)H(x)}{2} \left( \frac{c}{a} - \frac{\partial u}{\partial t} \right) e^{-c\tau} \left| \frac{\partial u}{\partial x} \right|^2 dx dt$$

$$- \int_0^1 \frac{K(x)H(x)}{2} a e^{-c\tau} \left| \frac{\partial u}{\partial x} \right|^2 dx \bigg|_{t=\tau} - \text{Re} \int_{\Omega} K(x)H(x) ae^{-c\tau} \frac{\partial u}{\partial x} d\zeta dt,$$

and

$$\text{Re} \int_0^s \int_0^1 \frac{\partial u}{\partial t} \rho d\zeta dt = \int_0^s \int_0^1 e^{c\tau}H(x) |v|^2 dx dt,$$

we choose
then, we obtain
\[
\int_{Q} \exp(ct)K(x)H(x)|v|^2dxdt = 0,
\]
then \(v = 0\) a.e., which implies \(w = 0\).

**Theorem 2.** The range \(R(L)\) of the operator \(L\) is dense in \(F\).

**Proof of Theorem 2.** Since \(F\) is a Hilbert space, we have \(\overline{R(L)} = F\) if and only if the relation
\[
\int_{Q} \Phi(x)f \overline{g} dx dt + \int_{0}^{1} \Phi(x) \frac{du}{dx} \overline{\phi} dx + \int_{0}^{1} lu \overline{\phi} dx = 0,
\]
(24)
for arbitrary \(u \in D(L)\) and \((g, \phi) \in F\), implies that \(g = 0\) and \(\phi = 0\).

Putting \(u \in D_{0}(L)\) in Equation (24), we conclude from the Lemma 1 that \(g = w = 0\), a.e. then \(g = 0\).

Taking \(u \in D(L)\) in Equation (24) yields
\[
\int_{0}^{1} \Phi(x) \frac{du}{dx} \overline{\phi} dx + \int_{0}^{1} lu \overline{\phi} dx = 0,
\]
(25)
Since the two terms in the previous equality vanish independently and since the range of the trace operator \(l\) is everywhere dense in Hilbert space with the norm
\[
\int_{0}^{1} \Phi(x) \left| \frac{d\phi}{dx} \right|^2 dx + \int_{0}^{1} |\phi|^2 dx,
\]
hence, \(\phi = 0\). Thus, \(\overline{R(A)} = F\). Then the problem Equations (6), (2)–(4) has a strong solution for an arbitrary \(F\).

5. Study of the Nonlinear Problem

This section is devoted to the proof of the existence, uniqueness and continuous dependence of the solution on the data of the problem Equations (1)–(4).

If the solution of problem Equations (1)–(4) exists, it can be expressed in the form \(u = w + U\), where

- \(U\) is a solution of the homogeneous problem

\[
\begin{align*}
\mathcal{L}U &= \frac{\partial U}{\partial t} - a \frac{\partial U}{\partial x} = 0, \quad \forall (x, t) \in \Omega, \\
U_{0} &= U(x, 0) = \varphi(x), \quad \forall x \in [0, 1], \\
U(0, t) &= U(1, t), \quad \forall t \in [0, T], \\
\int_{0}^{\alpha} K_{1}(x) U(x, t) dx + \int_{\alpha}^{1} K_{2}(x) U(x, t) dx &= 0, \quad \forall t \in [0, T].
\end{align*}
\]
(26)
(27)
(28)
(29)
and \(w\) is a solution of the problem...
\[ \mathcal{L}w = \frac{\partial w}{\partial t} - \frac{\partial}{\partial x} \left( a \frac{\partial w}{\partial x} \right) = F \left( x, t, w, \frac{\partial w}{\partial x} \right), \quad \forall (x, t) \in \Omega, \quad (30) \]

where

\[ w(x, 0) = 0, \quad \forall x \in [0, 1], \quad (31) \]

\[ w(0, t) = w(1, t), \quad \forall t \in [0, T], \quad (32) \]

with the scalar product

\[ \int_0^a \partial_1 K_1(x)v(x, t)\,dx + \int_1^a \partial_2 K_2(x)v(x, t)\,dx = 0, \quad \forall t \in [0, T]. \quad (33) \]

Substituting the expression of \( Mv \)

\[ \text{by taking the real part, we obtain} \]

\[ |F(x, t, u_1, v_1) - F(x, t, u_2, v_2)| \leq d(|u_1 - u_2| + |v_1 - v_2|) \quad \forall x, t \in \Omega. \quad (34) \]

According to Theorem 1 and Lemma 1, the problem Equations (26)–(29) has a unique solution that depends continuously on \( U_0 \in V^{1,0}(0, 1) \) where \( V^{1,0}(0, 1) \) is a Hilbert space with the scalar product

\[ (u, v)_{V^{1,0}(0, 1)} = \int_0^1 \Phi(x) \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} \,dx + \int_0^1 \bar{u} \bar{v} \,dx. \]

and with associated norm

\[ ||u||_{V^{1,0}(0, 1)} = \left( \int_0^1 \Phi(x) \left| \frac{\partial u}{\partial x} \right|^2 \,dx + \int_0^1 |u|^2 \,dx \right)^{1/2}. \]

We shall prove that the problem Equations (30)–(33) has a weak solution using an approximation process and passage to the limit.

Assume that \( v \) and \( w \in C^1(\Omega) \), and the following conditions are satisfied

\[ \begin{cases} v(x, T) = 0, \int_0^a K_1(x)v(x, t)\,dx + \int_1^a K_2(x)v(x, t)\,dx = 0, \\ w(x, 0) = w(1, t), w(0, t) = 0. \end{cases} \quad (35) \]

Taking the scalar product in \( L^2(\Omega) \) of Equation (30) and the integrodifferential operator

\[ \Lambda v = \begin{cases} \frac{x}{a(0,t)} \int_x^a K_1(\mu)\,d\mu, & 0 \leq x \leq a, \\ \frac{1-x}{a(1,t)} \int_a^x K_2(\mu)\,d\mu, & a \leq x \leq 1. \end{cases} \]

by taking the real part, we obtain

\[ H(w, v) = \text{Re} \int_\Omega F \left( x, t, w, \frac{\partial w}{\partial x} \right) \bar{v} \,dxdt \]

\[ = \text{Re} \int_\Omega \frac{\partial w}{\partial x} (\frac{\partial }{\partial x}) \bar{v} \,dxdt - \text{Re} \int_\Omega \frac{\partial w}{\partial x} \frac{\partial \bar{v}}{\partial x} \,dxdt. \quad (36) \]

Substituting the expression of \( \Lambda v \) in the first integral of the right hand-side of Equation (36), integrating with respect to \( t \), using the condition Equation (35), we obtain

\[ \text{Re} \int_\Omega \frac{\partial w}{\partial x} \bar{v} \,dxdt = - \text{Re} \int_0^T \int_0^a w \left( \frac{x}{a(t)} \int_x^a K_1(\mu)\,d\mu - \frac{x_0(0,t)}{a(t)} \int_0^a K_1(\mu)\,d\mu \right) dxdt \]

\[ - \text{Re} \int_0^T \int_a^1 w \left( \frac{(1-x)}{a(1,t)} \int_a^x K_2(\mu)\,d\mu - \frac{(1-x)a(1,t)}{a(t)} \int_a^1 K_2(\mu)\,d\mu \right) dxdt. \quad (37) \]

Substituting the expression of \( Mv \) in the second integral of the right hand-side of Equation (36), integrating with respect to \( x \), using the condition Equation (35), we obtain
By a weak solution of problem Equations (30)–(33) we mean a function obtained by integrating the right-hand side of Equation (36) with respect to \( x \). New problem \( w \) has a unique solution for \( n \). From Theorem 1 and Lemma 1, we deduce that for fixed \( w \), starting with \( w_0 \), we solve the problem \( H(\nu, \mu) \) satisfying the identity Equation (39) and the integral condition Equation (33).

We will construct an iteration sequence in the following way. Starting with \( w_0 = 0 \), the sequence \( (w_n)_{n \in \mathbb{N}} \) is defined as follows: given \( w_{n-1} \), then for \( n \geq 1 \), we solve the problem

\[
\mathcal{L}w_n = \frac{\partial w_n}{\partial t} - \frac{\partial}{\partial x} \left( a \frac{\partial w_n}{\partial x} \right) = F \left( x, t, w_{n-1}, \frac{\partial w_{n-1}}{\partial x} \right), \quad \forall (x, t) \in \Omega, \tag{40}
\]

\[
w_n(x, 0) = 0, \quad \forall x \in [0, 1], \tag{41}
\]

\[
w_n(0, t) = w_n(1, t), \quad \forall t \in [0, T]. \tag{42}
\]

\[
\int_0^1 K_1(x)w_n(x, t)dx + \int_0^1 K_2(x)w_n(x, t)dx = 0, \quad \forall t \in [0, T]. \tag{43}
\]

From Theorem 1 and Lemma 1, we deduce that for fixed \( n \), each problem Equations (40)–(43) has a unique solution \( w_n(x, t) \). If we set \( V_n(x, t) = w_{n+1}(x, t) - w_n(x, t) \), we obtain the new problem

\[
\mathcal{L}V_n = \frac{\partial V_n}{\partial t} - \frac{\partial}{\partial x} \left( a \frac{\partial V_n}{\partial x} \right) = \sigma_{n-1}, \quad \forall (x, t) \in \Omega, \tag{44}
\]

\[
V_n(x, 0) = 0, \quad \forall x \in [0, 1], \tag{45}
\]

\[
V_n(0, t) = V_n(1, t), \quad \forall t \in [0, T], \tag{46}
\]

\[
\int_0^1 K_1(x)V_n(x, t)dx + \int_0^1 K_2(x)V_n(x, t)dx = 0, \quad \forall t \in [0, T]. \tag{47}
\]
Assume that the condition Equation (34) holds, for the linearized problem

\[ \| V_n \|_{L^2(0, T; V^1,0(0, 1))} \leq k \| V_{n-1} \|_{L^2(0, T; V^1,0(0, 1))}. \]  

**Theorem 3.** Assume that the condition Equation (34) holds, for the linearized problem Equations (44)-(47), there exists a positive constant \( k \), such that

\[ \| V_n \|_{L^2(0, T; V^1,0(0, 1))} \leq k \| V_{n-1} \|_{L^2(0, T; V^1,0(0, 1))}. \]  

**Proof of Theorem 3.** We denote by

\[ QV_n = \begin{cases} \frac{x^2}{a(0, t)} e^{\lambda t} \frac{\partial V_n}{\partial t} + \frac{2x}{a(0, t)} e^{\lambda t} \int_x^a K_1(\mu) \frac{\partial V_n}{\partial t} d\mu, & 0 \leq x \leq \alpha, \\ \frac{(1-x)^2}{a(1, t)} e^{\beta(1-x)} \frac{\partial V_n}{\partial t} + \frac{2(1-x)}{a(1, t)} e^{\beta(1-x)} \int_x^a K_2(\mu) \frac{\partial V_n}{\partial t} d\mu, & \alpha \leq x \leq 1. \end{cases} \]

where

\[ \frac{\alpha^2}{a(0, t)} e^{\lambda t} K_1(\alpha) = \frac{(1-\alpha)^2}{a(1, t)} e^{\beta(1-\alpha)} K_2(\alpha), \]

\[ \lambda > \max \left( 0, \sup_{0 \leq x \leq \alpha} \frac{\lambda(x)}{\lambda(x)} \right), \]

\[ \beta > \max(0, \sup_{\alpha \leq x \leq 1} \frac{\lambda(x)}{\lambda(x)}). \]

We consider the quadratic form obtained by multiplying Equation (44) by \( e^{-ct}QV_n \), with the constant \( c \) satisfying Equation (17), integrating over \( \Omega = [0, 1] \times [0, s] \), with \( 0 \leq s \leq T \), taking the real part, we obtain

\[ \Phi(V_n, V_n) = \Re \int_{\Omega} e^{-ct} \sigma_{n-1} QV_n dx dt = \Re \int_{\Omega} e^{-ct} x^2 (1-x)^2 \frac{\partial V_n}{\partial t} QV_n dx dt - \Re \int_{\Omega} e^{-ct} \partial_x \left( a \frac{\partial V_n}{\partial x} \right) QV_n dx dt. \]  

(50)

Following the same procedure done in establishing the proof of Theorem 1, using Equation (34), we obtain

\[ \| V_n \|_{L^2(0, T; V^1,0(0, 1))}^2 \leq k^2 \| V_{n-1} \|_{L^2(0, T; V^1,0(0, 1))}^2. \]  

(51)

\[ K^2 = \max \left\{ \frac{8 m_1}{\mu_0} + \frac{32}{\mu_0 \mu_1 \mu_2}, \frac{8 m_3}{\mu_0} + \frac{32}{\mu_0 \mu_1 \mu_2}, \frac{(1-\alpha)^2 \phi(1-\alpha)}{\alpha} \right\} \]

\[ \min \left\{ \frac{\alpha^2}{a(0, t)} e^{\lambda t} K_1(\alpha), \frac{(1-\alpha)^2}{a(1, t)} e^{\beta(1-\alpha)} K_2(\alpha) \right\} \]

Since \( V_n(x, t) = w_{n+1}(x, t) - w_n(x, t) \), then the sequence \( w_n(x, t) \) can be written as follows

\[ w_n(x, t) = \sum_{k=1}^{n-1} V_k + w_0(x, t), \]

the sequence \( w_n(x, t) \) converge to an element \( w \in L^2(0, T; V^{1,0}(0, 1)) \) if
\[
\begin{align*}
\begin{aligned}
&d^2 < 
\min \left\{ \frac{a^2 m_0}{a_0^2}, (1 - a)^2, \frac{C^2}{2 a_1}, \frac{a_0 w_0 (1 - a)^3}{2 a_1}, \frac{a m_0 a^2}{a_0^2} \right\} e^{-\xi T}.
\end{aligned}
\end{align*}
\]

Now to prove that this limit function \( w \) is a solution of the problem under consideration Equations (44)-(47), we should show that \( w \) satisfies Equations (33) and (39).

For problem Equations (40)-(43), we have

\[
H(w_n - w, v) + H(w, v) =
\]

\[
\begin{align*}
\text{Re} \int_0^T \int_0^a K_1 K_0 \xi \left( \mathcal{F} \left( \eta, t, \omega_n - 1, \frac{\partial \omega_n}{\partial \eta} \right) - \mathcal{F} \left( \eta, t, w, \frac{\partial w}{\partial \eta} \right) \right) d\eta dx dt
\end{align*}
\]

\[
\begin{align*}
\text{Re} \int_0^T \int_a^1 K_2 \xi \left( \mathcal{F} \left( \eta, t, \omega_n - 1, \frac{\partial \omega_n}{\partial \eta} \right) - \mathcal{F} \left( \eta, t, w, \frac{\partial w}{\partial \eta} \right) \right) d\eta dx dt
\end{align*}
\]

\[
\begin{align*}
\text{Re} \int_0^T \int_0^a K_3 \xi \left( \frac{\partial (w_n - w)}{\partial x} \right) d\xi dx dt + \int_0^T \int_a^1 K_4 \xi \left( \frac{\partial (w_n - w)}{\partial x} \right) d\xi dx dt.
\end{align*}
\]

From Equation (40), we have

\[
H(w_n - w, v) = \text{Re} \int_0^a \frac{\partial (w_n - w)}{\partial t} M \xi dx dt - \text{Re} \int_0^a \frac{\partial (w_n - w)}{\partial x} M \xi dx dt.
\]

Integrating with respect to \( t \) and \( x \) using the conditions Equation (35), we obtain

\[
\begin{align*}
H(w_n - w, v) &= - \text{Re} \int_0^T \int_0^a \left( \frac{\partial (w_n - w)}{\partial t} \right) \left( \int_x^a K_1 \frac{\partial w_n}{\partial t} d\xi - \int_x^a K_1 \frac{\partial w}{\partial t} d\xi \right) dx dt
\end{align*}
\]

\[
\begin{align*}
- \text{Re} \int_0^T \int_0^a \left( \frac{\partial (w_n - w)}{\partial x} \right) \left( \int_a^x K_2 \frac{\partial w_n}{\partial x} d\xi - \int_a^x K_2 \frac{\partial w}{\partial x} d\xi \right) dx dt
\end{align*}
\]

\[
\begin{align*}
+ \text{Re} \int_0^T \int_0^a \left( \frac{\partial (w_n - w)}{\partial x} \right) \left( \frac{\partial (w_n - w)}{\partial t} \right) dx dt + \int_0^T \int_a^1 \left( \frac{\partial (w_n - w)}{\partial x} \right) d\xi dx dt
\end{align*}
\]

\[
\begin{align*}
+ \text{Re} \int_0^T \int_a^1 \left( \frac{\partial (w_n - w)}{\partial x} \right) \left( \frac{\partial (w_n - w)}{\partial t} \right) d\xi dx dt.
\end{align*}
\]

Each term of the left-hand side of Equation (53) is controlled by

\[
\begin{align*}
- \text{Re} \int_0^T \int_0^a \left( \frac{\partial (w_n - w)}{\partial t} \right) \left( \int_x^a K_1 \frac{\partial w_n}{\partial t} d\xi - \int_x^a K_1 \frac{\partial w}{\partial t} d\xi \right) dx dt \leq 
\end{align*}
\]

\[
\max \left\{ \frac{a^2 m_0 \max(|c_1|, |c_2|)}{a_0^2}, \frac{a^2 m_1}{a_0^2} \right\} \left( \int_x^a |w_n - w|^2 dx dt \right) \frac{1}{2} \left( \int_a^t \left| \frac{\partial w}{\partial t} \right|^2 + |v|^2 dx dt \right) \frac{1}{2},
\]

\[
- \text{Re} \int_0^T \int_0^a \left( \frac{\partial (w_n - w)}{\partial x} \right) \left( \int_a^x K_2 \frac{\partial w_n}{\partial x} d\xi - \int_a^x K_2 \frac{\partial w}{\partial x} d\xi \right) dx dt \leq 
\]

\[
\max \left\{ \frac{(1 - a)^2 m_1 \max(|c_1|, |c_2|)}{a_0^2}, \frac{(1 - a)^2 m_3}{a_0} \right\} \left( \int_x^a |w_n - w|^2 dx dt \right) \frac{1}{2} \left( \int_a^t \left| \frac{\partial w}{\partial t} \right|^2 + |v|^2 dx dt \right) \frac{1}{2},
\]

\[
\text{Re} \int_0^T \int_0^a \frac{K_1 a}{a_0} \left[ - \frac{x^2 (w_n - w)}{a^2} + (w_n - w) \right] dx dt - \int_0^T \int_0^a \frac{1}{a_0} \frac{\partial w_n}{\partial x} (w_n - w) \int_x^a K_1 \frac{\partial w}{\partial t} d\xi dx dt \leq 
\]

\[
\max \left\{ \frac{a m_0 a^2}{a_0^2}, \frac{a m_1 a^2}{a_0^2} \right\} \left( \int_0^T \int_0^a x^2 \left[ a_0^2 \frac{\partial (w_n - w)}{\partial t} \right] + |w_n - w|^2 dx dt \right) \frac{1}{2} \left( \int_0^T \int_0^a \left( \frac{\partial w}{\partial t} \right)^2 + |v|^2 dx dt \right) \frac{1}{2},
\]
\[
\text{Re}\left(\int_0^T \int_0^1 \frac{K_2 a}{a(1,t)} \left[ (1-x) \frac{\partial(w_1 - w_2)}{\partial x} + w \right] dxdt + \int_0^T \int_0^1 \frac{1}{a(1,t)} \frac{\partial}{\partial x}(w_1 - w_2) \int_a^1 K_2 \frac{\partial}{\partial t} \xi \, dxdt \right) \leq \max\left( \frac{(1-a)m_1}{a_0}, \frac{(1-a)m_3}{a_0} \right) \left( \int_0^T \int_0^1 (1-x)^2 \left[ \frac{\partial(w_1 - w_2)}{\partial x} \right]^2 + |w_1 - w_2|^2 \right) dxdt \right) \frac{1}{2},
\]

from the previous inequalities, we deduce that

\[
|H(w_n - w)| \leq C \left( \|w_n - w\|_{L^2(0,T;V^{1,0}(0,1))} \right) \left( \int_\Omega |w|^2 \, dx \right) \frac{1}{2},
\]

(54)

Using the condition Equation (34) and Cauchy-Schwartz inequality in the first two terms in the left-hand side in Equation (52), we obtain

\[
\text{Re} \left( \int_0^T \int_0^1 \frac{K_2 a}{a(1,t)} \left[ (1-x) \frac{\partial(w_1 - w_2)}{\partial x} + w \right] dxdt \right) \leq \max\left( \frac{(1-a)m_1}{a_0}, \frac{(1-a)m_3}{a_0} \right) \left( \int_0^T \int_0^1 (1-x)^2 \left[ \frac{\partial(w_1 - w_2)}{\partial x} \right]^2 + |w_1 - w_2|^2 \right) dxdt \leq \left( \int_\Omega |w|^2 \, dx \right) \frac{1}{2},
\]

(55)

From Equations (54) and (55) and passing to the limit in Equation (52) as \( n \to +\infty \), we deduce that

\[
H(w, v) = \text{Re} \left( \int_0^T \int_0^a \frac{K_1}{a(1,t)} v \int_0^1 T F \left( \zeta, t, w, \frac{\partial w}{\partial \zeta} \right) \, d\zeta \, dxdt \right) + \text{Re} \left( \int_0^T \int_0^1 \frac{K_2}{a(1,t)} v \int_0^1 (1 - \zeta) F \left( \zeta, t, w, \frac{\partial w}{\partial \zeta} \right) \, d\zeta \, dxdt \right).
\]

Now we show that Equation (33) holds. Since \( \lim_{n \to +\infty} \|w_n - w\|_{L^2(0,T;V^{1,0}(0,1))} = 0 \), then

\[
\lim_{n \to +\infty} \left| \int_0^a K_1(w_n - w) \, dx + \int_0^1 K_2(w_n - w) \, dx \right|^2 \leq \left( m_1 \sqrt{\alpha} + m_3 \sqrt{1 - \alpha} \right) \lim_{n \to +\infty} \int_0^1 |w_n - w|^2 \, dx \to 0.
\]

(56)

from Equation (56) we conclude that \( \int_0^a K_1 \omega \, dx + \int_0^1 K_2 \omega \, dx = 0 \). Then the problem Equations (30)–(33) has a weak solution for arbitrary \( F \).

Thus, we have proved the following

**Theorem 4.** If condition Equation (34) is satisfied, then the solution of problem Equations (30)–(33) is unique.

**Proof of Theorem 4.** Suppose that \( w_1, w_2 \in L^2(0, T : V^{1,0}(0, 1)) \) are two solution of Equations (30)–(33), the function \( v = w_1 - w_2 \) is in \( L^2(0, T : V^{1,0}(0, 1)) \) and satisfies

\[
\frac{\partial v}{\partial t} - \frac{\partial}{\partial x} \left( a(x) \frac{\partial v}{\partial x} \right) = G(x, t), \quad (x, t) \in \Omega,
\]

(57)

\[
v(x, 0) = 0, \quad x \in [0, 1],
\]

(58)

\[
v(0, t) = 0, \quad t \in [0, T],
\]

(59)

\[
\int_0^a K_1 \omega \, dx + \int_0^1 K_2 \omega \, dx = 0, \quad t \in [0, T],
\]

(60)

where \( G(x, t) = F \left( x, t, w_1, \frac{\partial w_1}{\partial x} \right) - F \left( x, t, w_2, \frac{\partial w_2}{\partial x} \right) \).
Taking the inner product in $L^2(\Omega)$ of Equation (57) and the integro-differential operator

$$Mu = \begin{cases} \frac{x^2 K_1}{a(0, t)} e^{(1-a)\frac{\partial v}{\partial x}} + \frac{2w}{a(0, t)} e^{\lambda x} \int_0^a K_1(\mu) \frac{\partial v}{\partial \mu} d\mu, & 0 \leq x \leq a, \\ \frac{(1-x)^2 K_2}{a(1, t)} e^{\beta(1-x)} \frac{\partial v}{\partial t} + \frac{2(1-x)}{a(1, t)} e^{\beta(1-x)} \int_0^a K_2(\mu) \frac{\partial v}{\partial \mu} d\mu & a \leq x \leq 1. \end{cases}$$

where

$$\frac{\alpha^2}{a(0, t)} e^{\lambda x} K_1(a) = \frac{(1-a)^2}{a(1, t)} e^{\beta(1-a)} K_2(a),$$

$$\lambda > \max \left( 0, \sup_{0 \leq x \leq a} \frac{K_1(x)}{K_1(x)}, \right)$$

$$\beta > \max \left( 0, \sup_{a \leq x \leq 1} \frac{K_2(x)}{K_2(x)} \right).$$

Following the same procedure done in establishing the proof of Theorem 1, we obtain

$$\|v\|^2_{L^2(0, T; V^{1,0}(0, 1))} \leq K^2 \|v\|^2_{L^2(0, T; V^{1,0}(0, 1))},$$

where

$$K^2 = \max \left\{ \frac{8\eta_{m_1}}{\eta_0} + \frac{32}{m_1^2_2}, \frac{8\eta_{m_2}}{\eta_0} + \frac{32}{m_3^2_2}, 2\left(1-a\right)^2 e^{\beta(1-a)} \right\} \min \left\{ a^2 m_2 \frac{c_2 - M_1}{2\pi}, a^2 m_2 \frac{c_2 - M_2}{2\pi}, \frac{a_0 m_2 (1-a)^2}{2\pi}, \frac{a_0 m_2 e^{\beta(1-a)}}{2\pi} \right\} d^2 e^{\lambda x},$$

Since $k^2 < 1$, then $v = 0$, which implies that $w_1 = w_2 \in L^2(0, T; V^{1,0}(0, 1)).$

Then the uniqueness of the weak solution holds. □

6. Conclusions

In this work we studied the existence, uniqueness and continuous dependence of a weak solution for some classes of mixed nonlinear problems with nonlocal conditions (boundary integral conditions). The used method is one of the most efficient functional analysis methods for solving linear partial differential equations with boundary integral conditions, the so-called energy-integral method or a priori estimates method. We constructed for each problem suitable multiplicators, which provide the a priori estimate, from which it was possible to establish the solvability of the problem. For the nonlocal mixed problems for nonlinear equations, we first established the a priori estimate for an associated linear problem and hence the solvability of this associated linear problem. Then, by applying an iterative process based on the obtained results for the linear problem, we proved the existence, uniqueness and continuous dependence of a weak solution of the considered nonlinear problem.

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