Spontaneous symmetry breaking in pure 2D Yang-Mills theory

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Abstract

We consider purely topological 2d Yang-Mills theory on a torus with the second Stiefel–Whitney class added to the Lagrangian in the form of a $\theta$-term. It will be shown, that at $\theta = \pi$ there exists a class of $SU(2N)/\mathbb{Z}_2$ ($N > 1$) gauge theories with a two-fold degenerate vacuum, which spontaneously breaks the time reversal and charge conjugation symmetries. The corresponding order parameter is given by the generator $O$ of the $\mathbb{Z}_N$ one-form symmetry.

1 Introduction

The possibility of having a number of degenerate vacua called $\theta$-vacua in two dimensional gauge theories was studied in the 70’s by a number of authors [1, 2, 3, 4, 5, 6]. Both abelian and non-abelian theories were considered and the existence of the multiple vacua was shown to be independent of the spontaneous symmetry breaking of the gauge symmetry. Instead, the presence of some matter fields, either fermionic or scalar, was required.

In this note we consider purely topological 2d Yang-Mills theory on a torus with the second Stiefel–Whitney class added to the Lagrangian in the form of a $\theta$-term. It will be shown, that at $\theta = \pi$ there exists a class of $SU(2N)/\mathbb{Z}_2$ ($N > 1$) gauge theories with a two-fold degenerate vacuum. These two vacuum states are related by the time reversal or the charge conjugation and thus indicate the spontaneous symmetry breaking. The corresponding order parameter is given by the generator $O$ of the $\mathbb{Z}_N$ one-form symmetry with the following action of the charge conjugation on it:

$$COC^{-1} = O^{-1}.$$ (1.1)

The motivation to consider such theories comes from the recent developments in generalized global symmetries and ‘t Hooft anomalies [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26]. In particular, authors of [16] considered $SU(N)$ gauge theory in 4 dimensions and showed that at $\theta = \pi$ there is the discrete ‘t Hooft anomaly involving time reversal and the center symmetry. As a consequence of this anomaly, the vacuum at $\theta = \pi$ cannot be a trivial non-degenerate gapped state. Another example of ‘t Hooft anomaly constraining the vacuum of the theory is related to the 2d $\mathbb{C}P^{n-1}$ model [15], where for $n > 2$ the mixed anomaly between time reversal symmetry and the global $PSU(n)$ symmetry at $\theta = \pi$ leads to the spontaneous breaking of time reversal symmetry with a two-fold degeneracy of the vacuum [26]. The list of the examples could be made longer, but we will conclude by mentioning the works [12, 13], where the ‘t Hooft anomalies for discrete global symmetries in bosonic theories were studied in 2, 3 and 4 dimensions.

Although in this note we are not going to discuss possible relation of the spontaneous symmetry breaking to the anomaly, but one could hypothesize the existence of the mixed anomaly between $(-1)$-form symmetry and the charge conjugation in the theories under consideration

While the note was in preparation, we became aware of the paper by D. Kapec, R. Mahajan and D. Stanford [25], which has partial overlap with our results. In [24] the higher genus partition functions were computed and utilized in the context of random matrix ensembles. Also, the paper by E. Sharpe [27] discussing 1-form symmetries in the various 2d theories appeared soon after the first draft of this note. This paper studies the connection with the cluster decomposition and is based on a number of previous results (to name a few [28] [29] [30]).

The note is organized as follows. In section 2 we review the Hamiltonian approach for computing the partition functions of the pure gauge theories in two dimensions. This method originates from the work of

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\textsuperscript{1}This possibility was pointed out to the author by Z. Komargodski
A. Migdal [31] and was extensively developed in the 80’s and 90’s alongside other approaches for studying the 2d Yang-Mills theories [32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44]. We would also like to mention the path integral approach by M. Blau and G. Thompson [45, 46, 47], which leads to the same results, but requires more involved mathematical structures. In section 3 we use G. ’t Hooft’s twisted boundary conditions [48] to compute the partition function of the SU(2)/Z_2 gauge theory. This computation is equivalent to the approach used by E. Witten [38] to compute the SO(3) partition function starting from the SU(2) gauge theory. We conclude section 3 by introducing the θ-term to the Lagrangian and computing the partition function at θ = π, which repeats one of the results of [49] and [30]. In section 4 we extend all the previous arguments to the case of SU(N)/Z_N theory. However, since there is no spontaneous symmetry breaking in PSU(N) theory for any N, we switch in section 5 to the more general case of SU(N)/Γ, where Γ is the subgroup of the center of SU(N). Indeed, we find out that there exists a class of SU(2N)/Z_2, N > 1 theories with two vacuum states given by the fundamental and antifundamental representations of SU(2N). Additionally, we argue that there exists a broader class of SU(2Nm)/Z_2m theories with degenerate vacuum. Finally, in section 6 we relate the two-fold degeneracy of the vacuum to the spontaneous breaking of C and T symmetries.

2 Review: SU(2) gauge theory

To derive the answer for the partition function on the torus we consider the canonical quantization of the theory on a cylinder:

![Diagram of a cylinder with a U_1 and U_2 labels]

The corresponding propagator [31, 33, 43] is given by

\[ Z(a, U_1, U_2) = \sum_R \chi_R(U_1) \chi_R(U_2) e^{-aC_2(R)}, \]

(2.1)

where \( a = e^2 LT/2 \) is proportional to the surface area of the cylinder. The final answer for the partition function on the torus comes from gluing together the opposite sides of the cylinder:

\[ Z = \int dU_1 Z(a, U_1, U_1^{-1}) = \sum_R e^{-aC_2(R)} \int dU_1 \chi_R(U_1) \chi_R(U_1^{-1}). \]

(2.2)

Using the identity

\[ \int dU \chi_R(U) \chi_R(U^{-1}) W = \chi_R(W) \dim R, \]

(2.3)

we get

\[ Z = \sum_R e^{-aC_2(R)}. \]

(2.4)

For SU(2) we have \( C_2(R) = j(j+1) \) with half-integer \( j \) and hence

\[ Z = \sum_{m=0}^{\infty} e^{-a m(m+2)/4}. \]

(2.5)
\section{SU(2)/Z_2 gauge theory}

Now, we consider the cylinder as a rectangular plaquette with one pair of opposite sides being glued together. According to [48] we can introduce the following boundary conditions for the vector potential $A_\mu (x, t)$:

$$\begin{align*}
A_\mu (L, t) &= \tilde{\Omega}_1 (t) A_\mu (0, t), \\
A_\mu (x, T) &= \tilde{\Omega}_2 (x) A_\mu (x, 0),
\end{align*} \tag{3.1}$$

with the notation $\Omega A_\mu = \Omega A_\mu \Omega^{-1} + \frac{i}{2} \Omega \partial_\mu \Omega^{-1}$. However, since we are using the $A_0 = 0$ gauge, we are left with time-independent gauge transformations:

$$\tilde{\Omega}_1 (t) = \tilde{\Omega}_1 (0). \tag{3.2}$$

Now, making a constant gauge transformation $A_\mu \rightarrow \tilde{\Omega} A_\mu$ with

$$\tilde{\Omega} \tilde{\Omega}_1 \tilde{\Omega}_2^{-1} = \text{Id}, \quad \Omega (x) \equiv \tilde{\Omega} \tilde{\Omega}_2 (x) \tilde{\Omega}_1^{-1} \tag{3.3}$$

we arrive at

$$\begin{align*}
A_1 (L, t) &= A_1 (0, t), \\
A_1 (x, T) &= \Omega (x) A_1 (x, 0),
\end{align*} \tag{3.4}$$

and the consistency condition for $\Omega$ is

$$\Omega (0) = \Omega (L) z, \quad z \in \mathbb{Z}_2. \tag{3.5}$$

Now we should be more accurate with the definition of the holonomy around the boundary:

For each $U_i$ we have

$$U_1 = \text{Pexp} \left( \int_0^L A_1 (x, 0) \, dx \right), \tag{3.6}$$

$$U = \text{Pexp} \left( \int_0^T A_1 (L, t) \, dt \right), \tag{3.7}$$

$$U_2 = \text{Pexp} \left( \int_0^1 A_1 (x, T) \frac{dx}{d\sigma_2} \, d\sigma_2 \right) \quad \text{with} \quad x (\sigma_2 = 0) = L, \ x (\sigma_2 = 1) = 0. \tag{3.8}$$

Since we have two kinds of vector potentials defined by the boundary conditions with $\Omega_0 (0) = \Omega_0 (L)$ and $\Omega_1 (0) = \Omega_1 (L) z_1, \ z_1 \neq \text{Id}$, the total partition function can be represented as the following sum:

$$Z = \frac{1}{2} (Z_0 + Z_1). \tag{3.9}$$

where the factor 1/2 comes from the normalization of the Haar measure to give volume one. Now, $Z_0$ corresponds to the periodic boundary conditions as in the case of pure SU(2) and we already know the answer:

$$Z_0 = \sum_{m=0}^\infty e^{-a m (m + 2)/4}. \tag{3.10}$$

To compute $Z_1$ we use the boundary conditions to derive

$$U_2 = \Omega_1 (0) \text{Pexp} \left( \int_0^1 A_1 (x, 0) \frac{dx}{d\sigma_2} \, d\sigma_2 \right) \Omega_1^{-1} (L) = \Omega_1 (0) U_1^{-1} \Omega_1^{-1} (L). \tag{3.11}$$

Then the partition function for the cylinder is

$$Z_1 (a, U_1, U_2) = \sum_R e^{-a C_2 (R)} \chi_R (U_1) \chi_R (U_2). \tag{3.12}$$
Applying the gluing procedure and integrating over $U_1$ we arrive at

$$Z_1 = \sum_R e^{-aC_2(R)} \int dU_1 \chi_R(U_1) \chi_R(z_1U_1^{-1}) = \sum_R e^{-aC_2(R)} \frac{\chi_R(z_1)}{\dim R}. \quad (3.13)$$

Using the Weyl character formula for the $SU(2)$ case

$$\chi_R\left( \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \right) = \frac{\sin(n\phi)}{\sin(\phi)}, \quad n = \dim R, \quad (3.14)$$

we get $\chi_R(z_1) = n(-1)^{n+1}$ and

$$Z_1 = \sum_{m=0}^{\infty} (-1)^m e^{-a(m+2)/4}. \quad (3.15)$$

Thus, the answer for the total partition function is

$$Z = \sum_{k=0}^{\infty} e^{-a(k+1)}, \quad (3.16)$$

which coincides with the general answer (2.4) for the group $SO(3)$.

### 3.1 Adding $w_2$ to $SU(2)/\mathbb{Z}_2$

Following [38, 49] we are going to use topological approach to the calculation of the partition function with the second Stiefel–Whitney class $w_2$ added to the Lagrangian:

$$Z^{SW} = \int DA e^{iS_{YM} + i\theta w_2}, \quad (3.17)$$

where the dependence on $\theta$ is $2\pi$-periodic and different possible values of theta in the $SU(2)$ case are $\theta = 0, \pi$. Since $w_2$ only depends on the topological type of the bundle, the path integral splits into two parts, corresponding to the trivial and nontrivial $SO(3)$-bundles over the torus. The trivial bundle is defined by the boundary conditions $3.3$ with $\Omega_0(0) = \Omega_0(L)$ and the value of $w_2$ is 0. The nontrivial bundle is defined by the boundary conditions $3.3$ with $\Omega_1(0) = \Omega_1(L)z_1$ and the value of $w_2$ is 1. In this way we get the following answer:

$$Z^{SW}(\theta = \pi) = \frac{1}{2} (Z_0 + e^{i\pi} Z_1) = \sum_{k=0}^{\infty} e^{-a(2k+1)(2k+3)/4}, \quad (3.18)$$

where $SW$ stands for Stiefel–Whitney.

### 4 $SU(N)/\mathbb{Z}_N$ gauge theory

For the group $SU(N)/\mathbb{Z}_N$ we have $N$ non-equivalent periodic boundary conditions $3.4$ with

$$\Omega_k(0) = \Omega_k(L)z_k, \quad z_k \in \mathbb{Z}_N, \quad k = 0, 1, \ldots, N-1. \quad (4.1)$$

Repeating the steps from the previous section we write the partition function as

$$Z = \frac{1}{N} \sum_{k=0}^{N-1} Z_k, \quad (4.2)$$

where

$$Z_k = \sum_R e^{-aC_2(R)} \frac{\chi_R(z_k)}{\dim R}. \quad (4.3)$$

Adding the $\theta$ term to the Lagrangian as in (3.17) will affect the partition function in the similar way as before, but in the $SU(N)$ case theta can take more values inside the $[0, 2\pi)$ interval. Labeling these possible values by $\kappa$, we get

$$\theta_k = \frac{2\pi\kappa}{N}, \quad \kappa = 0, \ldots, N-1. \quad (4.4)$$

Thus, each $Z_k$ acquires the factor of $e^{i\theta_k \kappa}$ and the corresponding partition function is

$$Z^{SW}_\kappa = Z^{SW}(\theta = \theta_\kappa) = \frac{1}{N} \sum_{k=0}^{N-1} e^{i\theta_\kappa \kappa \kappa} Z_k. \quad (4.5)$$
4.1 Example: $SU(3)/\mathbb{Z}_3$

Irreducible representations of $SU(3)$ can be labeled by the Dynkin coefficients $(n,m)$. The two fundamental weights of $SU(3)$ are

$$\mu^1 = \left( \frac{1}{2}, \frac{1}{2\sqrt{3}} \right), \quad \mu^2 = \left( 0, \frac{1}{\sqrt{3}} \right). \quad (4.6)$$

This gives for the characters of $z_k$ in the representation $(n,m)$

$$\chi_{(n,m)}(z_k) = \dim R_{(n,m)} e^{2\pi i k (n+2m)/3}, \quad k = 0, 1, 2. \quad (4.7)$$

Since

$$\sum_{k=0}^{2} e^{2\pi i k (n+2m)/3} = 3 \delta \left( [n+2m] \mod 3 \right), \quad \delta (n) \equiv \delta_{n,0} \quad (4.8)$$

and

$$C_2 \left( R_{(n,m)} \right) = (n^2 + m^2 + n m + 3 n + 3 m) / 3, \quad (4.9)$$

we derive for the partition function

$$Z = \sum_{n,m=0}^{\infty} e^{-a (n^2 + m^2 + n m + 3 n + 3 m) / 3} \delta \left( [n+2m] \mod 3 \right), \quad (4.10)$$

where due to the Kronecker delta function the only non-zero terms are those that have $n+2m \equiv 0 \mod 3$.

Adding $w_2$ with $\theta = \theta_n$ changes the argument of the delta function by $\kappa$ and we get

$$Z_n^{SW} = \sum_{n,m=0}^{\infty} e^{-a (n^2 + m^2 + n m + 3 n + 3 m) / 3} \delta \left( [\kappa+n+2m] \mod 3 \right). \quad (4.11)$$

4.2 General case: $SU(N)/\mathbb{Z}_N$

Labeling representations of $SU(N)$ by the Dynkin coefficients $(q_1, \ldots, q_{N-1}) \equiv \alpha$ and using the fundamental weights, we derive for the characters of $z_k$ in the representation $(q_1, \ldots, q_{N-1})$:

$$\chi_{\alpha} (z_k) = \dim R_{\alpha} e^{2\pi i k (q_1+2 q_2 + \cdots + (N-1) q_{N-1}) / N}, \quad k = 0, 1, \ldots, N-1. \quad (4.12)$$

Then with the help of the simple identity

$$\sum_{k=0}^{N-1} e^{2\pi i k n / N} = N \delta (n \mod N), \quad (4.13)$$

we get for the partition function

$$Z = \sum_{q_1, \ldots, q_{N-1}=0}^{\infty} e^{-a C_2(R_{\alpha})} \delta \left( \left[ \sum_{j=1}^{N-1} j q_j \right] \mod N \right), \quad (4.14)$$

where the only non-zero terms are those that have $\sum_{j=1}^{N-1} j q_j \equiv 0 \mod N$. The eigenvalues of the quadratic Casimir operator in $SU(N)\mathcal{U}$ are given by [50]

$$C_2 \left( R_{\alpha} \right) = \sum_{j,k=1}^{N-1} q_j (q_k + 2) G^{kj}, \quad (4.15)$$

where $G^{ij}$ is the inverse of the symmetrized Cartan matrix $G_{ij}$ [51]:

$$G_{ij} \equiv \frac{8 (\alpha_i, \alpha_j)}{(\alpha_i,\alpha_i)(\alpha_j,\alpha_j)} \quad (4.16)$$

and we are using the normalization, which provides the Killing metric of the form $g_{ab} = \frac{1}{2} \delta_{ab}$ and $(\alpha_i,\alpha_i) = 2$.

Adding the usual $\theta$ term with $\theta = \theta_n$, we obtain

$$Z_n^{SW} = \sum_{q_1, \ldots, q_{N-1}=0}^{\infty} e^{-a C_2(R_{\alpha})} \delta \left( \kappa + \sum_{j=1}^{N-1} j q_j \mod N \right), \quad \kappa = 1, \ldots, N-1. \quad (4.17)$$
5 Looking for two vacua in $SU(N)/\Gamma$, $\Gamma \subset \mathbb{Z}_N$ gauge theory

In this section we will consider the more general case, when the factor group is taken with respect to the subgroup $\Gamma$ of the center of $SU(N)$. By now we went through several derivations of the partition functions and it is clear what is the generalization of $\textbf{(4.17)}$ for $\Gamma \neq \mathbb{Z}_N$. If the order of $\Gamma$ is $n$, then we have $n$ non-equivalent periodic boundary conditions $\textbf{(3.4)}$ and the corresponding partition function is

$$Z_n^{\text{SW}} = \sum_{q_1, \ldots, q_{n-1}=0}^{\infty} e^{-a C_2(R_{q_0},R_{q_1},R_{q_2},R_{q_3})} \delta\left(\left[\kappa + \sum_{j=1}^{n-1} j q_j\right] \mod n\right), \quad \kappa = 1, \ldots, n - 1. \quad \textbf{(5.1)}$$

5.1 $N = 4$

We start with the explicit answer for the case of $SU(4)/\mathbb{Z}_4$:

$$Z_4^{\text{SW}}|_{\Gamma = \mathbb{Z}_4} = \sum_{q_1, q_2, q_3=0}^{\infty} e^{-a C_2(R_{q_1},R_{q_2},R_{q_3})} \delta\left(\left[\kappa + \sum_{j=1}^{3} j q_j\right] \mod 4\right), \quad \kappa = 1, \ldots, 3, \quad \textbf{(5.2)}$$

where

$$C_2(R_{q_1},R_{q_2},R_{q_3}) = \frac{1}{8} (3 q_1^2 + 4 q_2^2 + 3 q_3^2 + 4 q_1 q_2 + 2 q_1 q_3 + 4 q_2 q_3 + 12 q_1 + 16 q_2 + 12 q_3). \quad \textbf{(5.3)}$$

As it can be checked directly, there is no such value of $\kappa$ that would produce two vacua. However, we can also consider the case of $SU(4)/\mathbb{Z}_2$ with $\kappa = 1$ and the following partition function:

$$Z_2^{\text{SW}}|_{\Gamma = \mathbb{Z}_2} = \sum_{q_1, q_2, q_3=0}^{\infty} e^{-a C_2(R_{q_1},R_{q_2},R_{q_3})} \delta\left(\left[1 + \sum_{j=1}^{3} j q_j\right] \mod 2\right). \quad \textbf{(5.4)}$$

In this case the two vacua contributions are given by $q = (1,0,0)$ and $q = (0,0,1)$.

5.2 General case of $SU(2N)/\mathbb{Z}_2$ with $N > 1$

It is easy to show, that the first non-trivial example of 2d theory with two vacua discussed earlier is just one of the infinite series of $SU(2N)/\mathbb{Z}_2$ theories with $N > 1$. We again consider the partition function $Z_2^{\text{SW}}|_{\Gamma = \mathbb{Z}_2}$ with $\kappa = 1$ or, equivalently, $\theta = \pi$:

$$Z_2^{\text{SW}}|_{\Gamma = \mathbb{Z}_2} = \sum_{q_1, \ldots, q_{2N-1}=0}^{\infty} e^{-a C_2(R_{q_0},R_{q_1},R_{q_2},R_{q_3})} \delta\left(1 + \sum_{j=1}^{2N-1} j q_j\right) \mod 2). \quad \textbf{(5.5)}$$

To show that these theories have two vacua we need the following facts about the inverse Cartan matrix $G^{ij}$. The first fact is that all elements of this matrix are strictly positive:

$$\forall i, j : G^{ij} > 0. \quad \textbf{(5.6)}$$

Second, the diagonal elements $G^{ii}$ are given by $\textbf{(52-51)}$:

$$G^{ii} = \frac{i (2N - i)}{4N}, \quad i \leq 2N - 1. \quad \textbf{(5.7)}$$

And finally, the following relations hold:

$$\forall j = 2, \ldots, N : \sum_{i=1}^{2N-1} G^{ij} < \sum_{i=1}^{2N-1} G^{ij}, \quad \textbf{(5.8)}$$

$$\forall j = 1, \ldots, N - 1 : \sum_{i=1}^{2N-1} G^{i,N-j} = \sum_{i=1}^{2N-1} G^{i,N+j}. \quad \textbf{(5.9)}$$

Thus, the two vacua contributions are coming from

$$q = (1,0,\ldots,0) \quad \text{and} \quad q = (0,\ldots,0,1). \quad \textbf{(5.10)}$$

Notice that due to the superselection rules

$$\langle R_2|R_1\rangle = \int dU \chi_{R_1}(U) \chi_{R_2}(U^{-1}) = \delta_{R_1,R_2} \quad \textbf{(5.11)}$$

we indeed have two different ground states that indicate spontaneous symmetry breaking.
5.3 $\Gamma \neq \mathbb{Z}_2$

By studying some particular examples with low enough values of $N$, one can check that the following theories with $\Gamma \neq \mathbb{Z}_2$ have two vacuum states: $SU(8)/\mathbb{Z}_4$ with $\kappa = 2$, $SU(12)/\mathbb{Z}_4$ with $\kappa = 2$, $SU(12)/\mathbb{Z}_6$ with $\kappa = 3$, $SU(16)/\mathbb{Z}_4$ with $\kappa = 2$, $SU(16)/\mathbb{Z}_8$ with $\kappa = 4$. Basically, any $SU(2N m)/\mathbb{Z}_{2m}$ theory with $\kappa = m$ and $N > 1$, $m > 1$ is a candidate for having two-fold degenerate vacuum. The only obstacle to making this statement true in general, is that in principle for higher values of $N$ and $m$ there could be states with energies lower than the energy of the following two states:

$$q_m = 1, \forall j \neq m : q_j = 0 \quad \text{and} \quad q_{(2N-1)m} = 1, \forall j \neq (2N - 1) m : q_j = 0. \quad (5.12)$$

However, explicit computations for a number of different values of $N$ and $m$ suggest that the above states are always the lowest energy states of the theory with $\kappa = m$. If we assume that there are states with even lower energies, then they will also come in pairs. This allows us to conclude that the vacuum of the $SU(2N m)/\mathbb{Z}_{2m}$ theory with $\kappa = m$ is at least two-fold degenerate. Moreover, the lack of discrete symmetries with order higher than 2 hints that the two-fold degeneracy is the only option.

6 Spontaneous symmetry breaking in $SU(2N)/\mathbb{Z}_2$ theories with $\theta = \pi$, $N > 1$

If we look at the Dynkin coefficients corresponding to the two vacuum states, we see that these states are given by the fundamental and antifundamental representations of $SU(2N)$. Hence the question is what transformation brings us from one representation to its complex conjugate. Since the wave functions in the propagator $\mathbb{C}$ are given by

$$\langle U | R \rangle = \chi_R(U) = Tr_R(U), \quad U = Pexp\left(\int_0^L A_1(x, t) \, dx\right), \quad (6.1)$$

the transformation $A_1 \rightarrow -A_1^T$ yields $U \rightarrow (U^{-1})^T = U^*$ and $\chi_R(U) \rightarrow \chi_R^*(U)$. From the Gauss Law constraint one could derive the following transformation rules for the $C$, $P$ and $T$ operators:

$$C: \quad e \rightarrow -e, \quad A_1 \rightarrow -A_1^T, \quad (6.2)$$

$$P: \quad x \rightarrow -x, \quad A_1 \rightarrow -A_1, \quad (6.3)$$

$$T: \quad t \rightarrow -t, \quad A_1 \rightarrow -A_1^T. \quad (6.4)$$

Thus, the $C$-symmetry (as well as $T$) is spontaneously broken and the overall $\text{CPT}$-symmetry is conserved, since both $CT$ and $P$ act trivially on the wave functions $\chi_R(U)$.

Spontaneously broken $C$- and $T$-symmetries lead to the domain wall between the two vacuum states $\chi_f(U)$ and $\chi_F(U)$. In the theories under consideration there is a discrete one-form symmetry $\mathbb{Z}_N$, generated by a local unitary operator $\mathcal{O}$. This local operator picks up a phase when crossing the domain wall. To figure out the phase, we consider the $\mathbb{Z}_2N$ subgroup before factoring out $\mathbb{Z}_2$. As before, the corresponding characters are given by

$$\chi_N(z_k) = \dim R_k e^{2\pi i k (q_1 + q_2 + \cdots + (2N - 1)q_{2N - 1})/(2N)}, \quad k = 0, 1, \ldots, 2N - 1. \quad (6.5)$$

After factoring out $\mathbb{Z}_2$ the generator of $\mathbb{Z}_N$ corresponds to $z_1$ and its action on the wave functions is simply

$$\mathcal{O}|F\rangle = e^{\pi i /N} |F\rangle, \quad \mathcal{O}|\bar{F}\rangle = e^{-\pi i /N} |\bar{F}\rangle, \quad (6.6)$$

where $\mathcal{O}^N = 1$ due to the fact that $(-1) \in \mathbb{Z}_2$ in fundamental and antifundamental representations. Here we also assume that adding second Stiefel–Whitney class only affects the $\mathbb{Z}_2$-charges of the states and $\mathbb{Z}_N$-charges remain the same. In this way the relation between the two expectation values reads

$$\langle \mathcal{O} \rangle_F = e^{2\pi i /N} \langle \mathcal{O} \rangle_{\bar{F}}, \quad (6.7)$$

which fixes the phase factor picked by $\mathcal{O}$ upon crossing the domain wall to be $e^{2\pi i /N}$. Then the action of charge conjugation on $\mathcal{O}$ can be inferred from

$$C \mathcal{O} C^{-1} |F\rangle = e^{-\pi i /N} |F\rangle, \quad C \mathcal{O} C^{-1} |\bar{F}\rangle = e^{\pi i /N} |\bar{F}\rangle. \quad (6.8)$$

The latter implies

$$C \mathcal{O} C^{-1} = \mathcal{O}^{-1}. \quad (6.9)$$
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