Integral Policy Iterations for Reinforcement Learning Problems in Continuous Time and Space *

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Abstract

Policy iteration (PI) is a recursive process of policy evaluation and improvement to solve an optimal decision-making, e.g., reinforcement learning (RL) or optimal control problem and has served as the fundamental to develop RL methods. Motivated by integral PI (IPI) schemes in optimal control and RL methods in continuous time and space (CTS), this paper proposes on-policy IPI to solve the general RL problem in CTS, with its environment modelled by an ordinary differential equation (ODE). In such continuous domain, we also propose four off-policy IPI methods—two are the ideal PI forms that use advantage and Q-functions, respectively, and the other two are natural extensions of the existing off-policy IPI schemes to our general RL framework. Compared to the IPI methods in optimal control, the proposed IPI schemes can be applied to more general situations and do not require an initial stabilizing policy to run; they are also strongly relevant to the RL algorithms in CTS such as advantage updating, Q-learning, and value-gradient based (VGB) greedy policy improvement. Our on-policy IPI is basically model-based but can be made partially model-free; each off-policy method is also either partially or completely model-free. The mathematical properties of the IPI methods—admissibility, monotone improvement, and convergence towards the optimal solution—are all rigorously proven, together with the equivalence of on- and off-policy IPI. Finally, the IPI methods are simulated with an inverted-pendulum model to support the theory and verify the performance.

Key words: policy iteration, reinforcement learning, optimization under uncertainties, continuous time and space, iterative schemes, adaptive systems

1 Introduction

Policy iteration (PI) is a recursive process to solve an optimal decision-making/control problem by alternating between policy evaluation to obtain the value function with respect to the current policy (a.k.a. the current control law in control theory) and policy improvement to improve the policy by optimizing it using the obtained value function (Sutton and Barto, 2017; Lewis and Vrabie, 2009). PI was first proposed by Howard (1960) in the stochastic environment known as Markov decision process (MDP) and is strongly relevant to reinforcement learning (RL) and approximate dynamic programming (ADP). PI has served as a fundamental principle to develop RL and ADP methods especially when the underlying environment is modelled or approximated by an MDP in a discrete space. There are also model-free off-policy PI methods using Q-functions and their extensions to incremental RL algorithms (e.g., Lagoudakis and Parr, 2003; Farahmand, Ghavamzadeh, Mannor, and Szepesvári, 2009; Maei, Szepesvári, Bhatnagar, and Sutton, 2010). Here, off-policy PI is a class of PI methods whose policy evaluation is done while following a policy, termed as a behavior policy, which is possibly different from the target policy to be evaluated; if the behavior and target policies are same, it is called an on-policy method. When the MDP is finite, all the on- or off-policy PI methods converge towards the optimal solution in finite time. Another advantage is that compared to backward-in-time dynamic programming, the forward-in-time computation of PI like the other ADP methods (Powell, 2007) alleviates the problem known as the curse of dimensionality. In continuing tasks, a discount factor γ is normally introduced to PI and RL to suppress the future reward and thereby have a finite return. Sutton and Barto (2017) gives a comprehensive overview of PI, ADP, and RL algorithms with their practical applications and recent success in the RL field.

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1.1 PI and RL in Continuous Time and Space (CTS)

Different from the MDP in discrete space, the dynamics of real physical world is usually modelled by (ordinary) differential equations (ODEs) inevitably in CTS. PI has been also studied in such continuous domain mainly under the framework of deterministic optimal control, where the optimal solution is characterized by the partial differential Hamilton-Jacobi-Bellman equation (HJBE) which is however extremely difficult or hopeless to be solved analytically, except in a very few special cases. PI in this field is often referred to as successive approximation of the HJBE (to recursively solve it!), and the main difference among them lies in their policy evaluation—the earlier versions of PI solve the associated \( \text{infinitesimal} \) Bellman equation (a.k.a. Lyapunov or Hamiltonian equation) to obtain the value function for the current policy (e.g., Leake and Liu, 1967; Kleinman, 1968; Saridis and Lee, 1979; Beard, Saridis, and Wen, 1997; Abu-Khalaf and Lewis, 2005, to name a few); Murray, Cox, Lendaris, and Saeks (2002) proposed a PI algorithm along with the trajectory-based policy evaluation that does not rely on the system model and can be viewed as a deterministic Monte-Carlo policy evaluation (Sutton and Barto, 2017). Motivated by those two approaches above, Vrabie and Lewis (2009) recently proposed a partially model-free PI scheme called integral PI (IPI), which is more relevant to RL/ADP in that the Bellman equation associated with its policy evaluation is of a temporal difference form—see (Lewis and Vrabie, 2009) for a comprehensive overview. By \textit{partially model-free}, it is meant in this paper that the PI can be done without explicit use of some or any input-independent part of the dynamics. IPI is then extended to a series of completely or partially model-free off-policy IPI methods (e.g., Lee, Park, and Choi, 2012, 2015; Luo, Wu, Huang, and Liu, 2014; Modares, Lewis, and Jiang, 2016, to name a few), with a hope to further extend them to incremental off-policy RL methods for adaptive optimal control in CTS—see (Vamvoudakis, Vrabie, and Lewis, 2014) for an incremental extension of the on-policy IPI method. The on/off-policy equivalence and the mathematical properties of stability/admissibility/monotone-improvement of the generated policies and convergence towards the optimal solution were also studied in the literatures above all regarding PI in CTS.

On the other hand, the aforementioned PI methods in CTS were all designed via Lyapunov’s stability theory (Haddad and Chellaboina, 2008) to guarantee that the generated policies are all asymptotically stable and thereby yield finite returns (at least on a bounded region around an equilibrium state), provided that so is the initial policy. There are two main restrictions, however, for these stability-based works to be extended to the general RL framework. One is the fact that except in the LQR case (e.g., Modares et al., 2016), there is no (or less if any) direct connection between stability and discount factor \( \gamma \) in RL. This is why in the nonlinear cases, the aforementioned PI methods in CTS only consider the total case \( \gamma = 1 \), but \textit{not} the discounted case \( 0 < \gamma < 1 \). The other restriction of those stability-based designs is that the initial policy needs to be asymptotically stabilizing to \textit{run the PI methods}. This is quite contradictory for IPI methods since they are partially or completely model-free, but it is hard or even impossible to find a stabilizing policy \textit{without knowing the dynamics}. Besides, compared with the RL frameworks in CTS, e.g., those in (Doya, 2000; Mehta and Meyn, 2009; Frémaux, Sprekeler, and Gerstner, 2013), this stability-based approach rather restricts the class of the cost (or reward) and the dynamics. For instances, the dynamics was assumed to have at least one equilibrium state,\(^2\) and the goal was always to (locally) stabilize the system in an optimal fashion to that equilibrium state although there may also exist multiple isolated equilibrium states to be considered or bifurcation; for such optimal stabilization, the cost was always crafted to be non-negative for all points and zero at the equilibrium.

Independently of the research on PI, several RL methods have come to be proposed also in CTS. Advantage updating was proposed by Baird III (1993) and then reformulated by Doya (2000) under the environment represented by an ODE. Doya (2000) also extended TD(\( \lambda \)) to the CTS domain and then combined it with the two policy improvement methods—the continuous actor with its update rule and the value-gradient based (VGB) greedy policy improvement; see also (Frémaux et al., 2013) for an extension of Doya (2000)’s continuous actor-critic using spiking neural networks. Mehta and Meyn (2009) defined the Hamiltonian function as a Q-function and then proposed a Q-learning method in CTS based on stochastic approximation. Unlike in MDP, however, these RL methods in CTS and the related action-dependent (AD) functions such as advantage and Q-functions are barely relevant to the PI methods in CTS due to the gap between optimal control and RL.

1.2 Contributions and Organizations

The main goal of this paper is to build up a theory on IPI in a general RL framework when the time and the state-action space are all continuous and the environment is modelled by an ODE. As a result, a series of IPI methods are proposed in the general RL framework with mathematical analysis. This also provides the theoretical connection of IPI to the aforementioned RL methods in CTS. The main contributions of this paper can be summarized as follows.

1. Motivated by the work of IPI (Vrabie and Lewis, 2009; Lee et al., 2015) in the optimal control framework, we propose the corresponding on-policy IPI scheme in the general RL framework and then prove its mathematical properties of admissibility/monotone-improvement

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\(^1\) In RL community, the return or the RL problem (in a discrete space) is said to be \textit{discounted} if \( 0 \leq \gamma < 1 \) and \textit{total} if \( \gamma = 1 \).

\(^2\) For an example of a dynamics with no equilibrium state, see (Haddad and Chellaboina, 2008, Example 2.2).
of the generated policies and the convergence towards the optimal solution (see Section 3). To establish them, we rigorously define the general RL problem in CTS with its environment formed by an ODE and then build up a theory regarding policy evaluation and improvement (see Section 2).

(2) Extending on-policy IPI in Section 3, we propose four off-policy IPI methods in CTS—two named integral advantage PI (IAPI) and integral Q-PI (IQPI) are the ideal PI forms of advantage updating (Baird 1993; Doya, 2000) and Q-learning in CTS, and the other two named integral explorized PI (IEPI) and integral C-PI (ICPI) are the natural extensions of the existing off-policy IPI methods (Lee et al., 2015) to our general RL problem. All of the off-policy methods are proven to generate the effectively same policies and value functions to those in on-policy IPI—they all satisfy the above mathematical properties of on-policy IPI. These are all shown in Section 4 with detailed discussions and comparisons.

The proposed on-policy IPI is basically model-based but can be made partially model-free by slightly modifying its policy improvement (see Section 3.2). IEPI is also partially model-free: IAPI and IQPI are even completely model-free; so is ICPI, but only applicable under the special $u$-affine-and-concave $(u$-AC) setting shown in Section 3.3. Here, we emphasize that Doya (2000)’s VGB greedy policy improvement is also developed under this $u$-AC setting, and ICPI provides its model-free version. Finally, to support the theory and verify the performance, simulation results are provided in Section 5 for an inverted-pendulum model. As shown in the simulations and all of the IPI algorithms in this paper, the initial policy is not required to be asymptotically stable to achieve the learning objective. Conclusions follow in Section 6. This theoretical work lies between the fields of optimal control and machine learning and also provides the unified framework of unconstrained and input-constrained formulations in both RL and optimal control (e.g., Doya, 2000; Abu-Khalaf and Lewis, 2005; Mehta and Meyn, 2009; Vrabie and Lewis, 2009; Lee et al., 2015 as the special cases of our framework).

Notations and Terminologies. $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ are the sets of all natural numbers, integers, and real numbers, respectively; $\mathbb{R} = \mathbb{R} \cup \{-\infty, \infty\}$ is the set of all extended real numbers; $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, the set of all nonnegative integers; $\mathbb{R}^{n \times m}$ is the set of all $n$-by-$m$ real matrices; $A^T$ and rank $(A)$ denote the transpose and the rank of a matrix $A \in \mathbb{R}^{n \times m}$, respectively. $\mathbb{R}^n = \mathbb{R}^{n \times 1}$ is the $n$-dimensional Euclidean space; span $\{x_1, \cdots, x_m\}$ is the linear subspace of $\mathbb{R}^n$ spanned by the vectors $x_1, \cdots, x_m \in \mathbb{R}^n$; $\|x\|$ denotes the Euclidean norm of $x \in \mathbb{R}^n$. A subset $\Omega$ of $\mathbb{R}^n$ is said to be compact if it is closed and bounded; for any Lebesque measurable subset $E$ of $\mathbb{R}^n$, we denote the Lebesque measure of $E$ by $|E|$. A function $f : D \to \mathbb{R}^m$ on a domain $D \subseteq \mathbb{R}^n$ is said to be $C^1$ if its first-order partial derivatives exist and are all continuous; $\nabla f : D \to \mathbb{R}^m$ denotes the gradient of $f$; $\text{Im} f \subseteq \mathbb{R}^m$ is the image of $f$. The restriction of $f$ on a subset $\Omega \subseteq D$ is denoted by $f|_{\Omega}$.

2 Preliminaries

In this paper, the state space $\mathcal{X}$ is given by $\mathcal{X} = \mathbb{R}^n$, and the action space $\mathcal{U} \subseteq \mathbb{R}^m$ is an $m$-dimensional manifold in $\mathbb{R}^m$ with (or without) boundary; $t \geq 0$ denotes a given specific time instant. The environment considered in this paper is the deterministic one in CTS described by the following ODE:

$$\dot{X}_\tau = f(X_\tau, U_\tau),$$

where $\tau \in [t, \infty)$ is the time variable; $f : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$ is a continuous function; $X_\tau \in \mathcal{X}$ denotes the state vector at time $\tau$; the action trajectory $U_\tau : [t, \infty) \to \mathcal{U}$ is a right continuous function over $[t, \infty)$.

A (non-stationary) policy $\mu \equiv \mu(\tau, x)$ refers to a function $\mu : [t, \infty) \times \mathcal{X} \to \mathcal{U}$ such that

1. for each fixed $\tau \geq t$, $\mu(\tau, \cdot)$ is continuous;
2. for each fixed $x \in \mathcal{X}$, $\mu(\cdot, x)$ is right continuous;
3. for each $x \in \mathcal{X}$, the state trajectory $X_\tau$ generated under $U_\tau = \mu(\tau, X_\tau)$ $\forall \tau \geq t$ and the initial condition $X_t = x$ is uniquely defined over the whole time interval $[t, \infty)$.

It is said to be stationary if there is a function $\pi : \mathcal{X} \to \mathcal{U}$ such that $\mu(\tau, x) = \pi(x)$ for all $(\tau, x) \in [t, \infty) \times \mathcal{X}$, in which case we call $\pi$ a (stationary) policy. In this paper, we use $\pi$ to indicate a stationary policy, and $\mu$ to denote any non-stationary behavior policy; the latter will be not shown until Section 4.

For notational efficiency and consistency to the classical RL framework, we will use the notation

$$E_\pi[Z | X_t = x]$$

(or its abbreviation $E_\pi^X[Z]$),

which plays no stochastic role but just means the deterministic value $Z$ when $X_t = x$ and $U_\tau = \pi(X_\tau)$ for all $\tau \geq t$. Using this notation, the state vector $X_\tau$ at time $\tau \geq t$ generated under the initial condition $X_t = x$ and the policy $\pi$ is denoted by $E_\pi[X_\tau | X_t = x]$ or simply $E_\pi^X[X_\tau]$. Throughout the paper, we also denote $\Delta t > 0$ the time difference, and for simplicity, $t' = t + \Delta t$, $X_{t'} = X_t + v t$, and $U_{t'} = U_t$. For any differentiable function $v : \mathcal{X} \to \mathbb{R}$, its time derivative $\dot{v} : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ is defined as

$$\dot{v}(X_t, U_t) = \lim_{\Delta t \to 0} \frac{v(X_{t'}) - v(X_t)}{\Delta t},$$

which is explicitly expressed by the chain rule as

$$\dot{v}(X_t, U_t) = \nabla v(X_t) f(X_t, U_t),$$

where $X_t \in \mathcal{X}$ and $U_t \in \mathcal{U}$ are free variables.
2.1 RL Problem in Continuous Time and Space

The RL problem considered in this paper is to find the best policy $\pi_*$ that maximizes the value function $v_{\pi} : X \rightarrow \mathbb{R}$

$$v_{\pi}(x) \doteq E_{\pi}[G_t | X_t = x], \quad (2)$$

where $G_t$ is the discounted or total return defined as

$$G_t \doteq \int_t^\infty \gamma^{t-i} R_{\tau} d\tau,$$

with the immediate reward $R_\tau \doteq R(X_\tau, U_\tau) \in \mathbb{R}$ at time $\tau$ and the discount factor $\gamma \in (0, 1)$. The reward $R \doteq R(x, u)$ here is a continuous upper-bounded function of $x \in X$ and $u \in U$. For any policy $\pi$, we also define the $\pi$-reward $R^\pi$ (and $R^\pi_t$) as $R^\pi(x) \doteq R(x, \pi(x))$ and $R^\pi_t \doteq R^\pi(X_t)$. In this paper, an admissible policy $\pi$ is defined as follows.

**Definition 1** A policy $\pi$ (or its value function $v_{\pi}$) is said to be admissible, denoted by $v_{\pi} \in \mathcal{V}_a$, if

$$|v_\pi(x)| < \infty \text{ for all } x \in X,$$

where $\mathcal{V}_a \doteq \{v_\pi : \pi \text{ is admissible}\}$, the set of all admissible value functions $v_{\pi}$.

To make our RL problem feasible, this paper assumes that every $v_\pi \in \mathcal{V}_a$ is $C^1$ and that there is an optimal admissible policy $\pi_*$ such that $v_{\pi_*} \preceq v_{\pi}$ holds for any policy $\pi$. Here, $v_\pi$ is the optimal value function; the partial ordering $v \preceq w$ for any two functions $v, w : X \rightarrow \mathbb{R}$ means $v(x) \leq w(x)$ for all $x \in X$. There may exist another optimal policy $\pi'_*$ than $\pi_*$, but the optimal value function is always same by $v_{\pi_*} \preceq v_{\pi'_*}$ and $v_{\pi_*} \preceq v_{\pi'_*}$. In this paper, $\pi_*$ indicates any one of the optimal policies, and $v_\pi$ denotes the unique optimal value function for them. Also note that in the discounted case $\gamma \in (0, 1)$, $v_\pi$ is upper-bounded since so is $R$. Hence, for the consistency to the discounted case, we assume only for the total case $\gamma = 1$ that $v_\pi$ is upper-bounded and

$$\forall \text{admissible } \pi : \limsup_{k \rightarrow \infty} l_{\pi}(x, k; v_{\pi}) \leq 0, \quad (3)$$

where $l_{\pi}(x, k; v) \doteq \gamma^k \Delta^k E_{\pi} [v(X_{t+k}\Delta_t) | X_t = x]$. Here, (3) is also true in the discounted case since $\lim_{k \rightarrow \infty} \gamma^k \Delta^k = 0$ and $v_\pi$ is upper-bounded.

**Remark 1** Admissibility of $\pi$ strongly depends not only on the policy $\pi$ itself, but on the choice of $R$ and $\gamma$ in $v_\pi$ and the properties (e.g., boundedness and stability) of the system (1) and the state trajectory $X$, under $\pi$ as exemplified below.

(1) For $\gamma \in (0, 1]$, if there exist $\alpha \in [0, \gamma^{-1})$ and a function $\bar{R} : X \rightarrow [0, \infty)$ such that $R_\tau$ is bounded as

$$\forall x \in X : \quad E_{\pi}[|R_\tau| | X_t = x] \leq \alpha^{\tau-t} R(x), \quad (4)$$

then $v_{\pi} \in \mathcal{V}_a$ since for $k = -\frac{\ln \alpha}{\ln \gamma}$ and each $x \in X$,

$$|v_\pi(x)| \leq k \bar{R}(x) < \infty. \quad (4)$$

For the total case $\gamma = 1$, the condition (4) implies that the immediate reward $R_\tau$ exponentially converges to 0 with the rate $\alpha \in [0, 1)$.

(2) For $\gamma \in (0, 1)$, if the reward $R$ or $R^\pi$ is lower-bounded, or the trajectory $\mathbb{E}^\pi_{x}[X_{\tau}]$ is bounded for each $x \in X$, then $v_{\pi} \in \mathcal{V}_a$. This is because such boundedness gives the bound (4) for $\alpha = 1$ and some constant $R > 0$. In this case, we have $\sup_{x \in X} |v_{\pi}(x)| < \infty$, so $v_{\pi}$ is also bounded. Especially, when $R$ is lower-bounded, such property is independent of the policy $\pi$—in other words, $v_{\pi}$ is bounded (and thus admissible) for any given policy $\pi$ (see the simulation setting in Section 5 for a practical example of this bounded $R$ case).

Consider the Hamiltonian function $h : X \times U \times \mathbb{R}^{1 \times n} \rightarrow \mathbb{R}$

$$h(x, u, p) \doteq R(x, u) + p f(x, u), \quad (5)$$

by which the time derivative of any differentiable function $v : X \rightarrow \mathbb{R}$ can be represented as

$$\dot{v}(x) = h(x, u, \nabla v(x)) - R(x, u), \quad (6)$$

for all $(x, u) \in X \times U$. Substituting $u = \pi(x)$ for any policy $\pi$ into (6), we also have the following expression:

$$\mathbb{E}^\pi_{x}[R_t + \dot{v}(X_t, U_t)] = h(x, \pi(x), \nabla v(x)) \forall x \in X, \quad (7)$$

which converts a time-domain expression (the left-hand side) into the Hamiltonian formula expressed in the state-space (the right-hand side). In addition, essentially required in our theory is the following lemma, which also provides conversions of (in)equalities between time domain and state space.

**Lemma 1** Let $v : X \rightarrow \mathbb{R}$ be any differentiable function. Then, for any policy $\pi$,

$$v(x) \leq E_{\pi} \left[ \int_0^t \gamma^{t-i} R_{\pi} d\tau + \gamma^{t-i} v(X_{t-i}^\tau) | X_t = x \right] \quad (8)$$

for all $x \in X$ and all $\Delta t > 0$ iff

$$-\ln \gamma \cdot v(x) \leq h(x, \pi(x), \nabla v(x)) \forall x \in X. \quad (9)$$

Moreover, the equalities in (8) and (9) are also necessary and sufficient. That is, the equality in (8) holds $\forall x \in X$ and $\forall \Delta t > 0$ if the equality in (9) is true $\forall \pi \in \notin X$.
Proof. By the standard calculus,
\[
\frac{d}{d\tau}(\gamma^{t-\tau} v(X_\tau)) = \gamma^{t-\tau} \left( \ln \gamma \cdot v(X_\tau) + \dot{v}(X_\tau, U_\tau) \right)
\]
for any \(x \in \mathcal{X}\) and any \(\tau \geq t\). Hence, by (6) and (9), we have for any \(\tau \geq t\) that
\[
0 \leq \mathbb{E}_\pi^x \left[ \gamma^{t-\tau} \cdot \left( h(X_\tau, \pi(X_\tau), \nabla v(X_\tau)) + \ln \gamma \cdot v(X_\tau) \right) \right]
\]
\[
= \mathbb{E}_\pi^x \left[ \gamma^{t-\tau} \cdot \left( R_\tau + \dot{v}(X_\tau, U_\tau) + \ln \gamma \cdot v(X_\tau) \right) \right]
\]
\[
= \mathbb{E}_\pi \left[ \gamma^{t-\tau} R_\tau + \frac{d}{d\tau} \left( \gamma^{t-\tau} v(X_\tau) \right) \right] 
\quad \forall x \in \mathcal{X},
\]
and, for any \(x \in \mathcal{X}\) and any \(\Delta t > 0\), integrating it from \(t\) to \(t' (= t + \Delta t)\) yields (8). One can also show that the equality in (9) for all \(x \in \mathcal{X}\) implies the equality in (8) for all \(x \in \mathcal{X}\) and \(\Delta t > 0\) by following the same procedure. Finally, the proof of the opposite direction can be easily done by following the similar procedure to the derivation of (11) from (10) below. □

2.2 Bellman Equation with Boundary Condition

By time-invariance\(^6\) of a stationary policy \(\pi\) and
\[
G_t = \int_t^{t'} \gamma^{t-\tau} R_\tau \, d\tau + \gamma^\Delta t G_{t'},
\]
we can see that \(v_\pi \in \mathcal{V}_\alpha\) satisfies the Bellman equation:
\[
v_\pi(x) = \mathbb{E}_\pi \left[ \int_t^{t'} \gamma^{t-\tau} R_\tau \, d\tau + \gamma^\Delta t v_\pi(X_{t'}) \right] \bigg| X_t = x
\]
for any \(x \in \mathcal{X}\) and any \(\Delta t > 0\). Using (10), we obtain the boundary condition of \(v_\pi \in \mathcal{V}_a\) at \(\tau = \infty\).

Proposition 1 Suppose that \(\pi\) is admissible. Then, \(\forall x \in \mathcal{X}, \lim_{\tau \to \infty} \gamma^{\tau-t} \cdot \mathbb{E}_\pi^x [v_\pi(X_{t+\tau})] = 0\).

Proof. Taking the limit \(\Delta t \to \infty\) of (10) yields
\[
v_\pi(x) = \lim_{\Delta t \to \infty} \mathbb{E}_\pi^x \left[ \int_t^{t+\Delta t} \gamma^{t-\tau} R_\tau \, d\tau + \gamma^\Delta t v_\pi(X_{t'}) \right]
\]
\[
= v_\pi(x) + \lim_{\Delta t \to \infty} \gamma^\Delta t \cdot \mathbb{E}_\pi^x [v_\pi(X_{t+\tau})],
\]
which implies \(\lim_{\tau \to \infty} \gamma^{\tau-t} \cdot \mathbb{E}_\pi^x [v_\pi(X_{t+\tau})] = 0\). □

Corollary 1 Suppose that \(\pi\) is admissible. Then, for any \(x \in \mathcal{X}\) and any \(\Delta t > 0\), \(\lim_{k \to \infty} l_\pi(x, k; v_\pi) = 0\).

Proof. Since \(\pi\) is admissible, \(\mathbb{E}_\pi [v_\pi(X_{t+k})] \big| X_t = x\) is finite for all \(\tau \geq t\) and \(x \in \mathcal{X}\). First, let \(\tilde{v}(x) = v_\pi(x) - v_\pi(x)\) and suppose \(v\) satisfies (13). Then, subtracting (10) from (13) yields \(\tilde{v}(x) = \gamma^\Delta t \cdot \mathbb{E}_\pi^x [\tilde{v}(X_{t+\tau})]\), whose repetitive applications to itself results in
\[
\tilde{v}(x) = \gamma^\Delta t \mathbb{E}_\pi^x [\tilde{v}(X_{t+\Delta t})] = \cdots = \gamma^{k\Delta t} \mathbb{E}_\pi^x [\tilde{v}(X_{t+k\Delta t})].
\]
For an admissible policy \(\pi\), rearranging (10) as
\[
\frac{1 - \gamma^\Delta t}{\Delta t} \cdot v_\pi(x) = \mathbb{E}_\pi \left[ \int_t^{t'} \gamma^{t-\tau} R_\tau \, d\tau + \gamma^\Delta t \cdot \frac{v_\pi(X_{t'}) - v_\pi(X_{t})}{\Delta t} \bigg| X_t = x \right],
\]
limiting \(\Delta t \to 0\), and using (7) yield the infinitesimal form:
\[
- \ln \gamma \cdot v_\pi(x) = h_\pi(x, \pi(x)) = h(x, \pi(x), \nabla v_\pi(x)) \quad \forall x \in \mathcal{X},
\]
where \(h_\pi : \mathcal{X} \times \mathcal{U} \to \mathbb{R}\) is the Hamiltonian function for a given admissible policy \(\pi\) and is defined, with a slight abuse of notation, as
\[
h_\pi(x, u) := h(x, u, \nabla v_\pi(x)),
\]
which is obviously continuous since so are the functions \(f, R, \) and \(\nabla v_\pi\) (see also (5)). Both \(h_\pi(x, u)\) and \(h(x, u, \nabla v_\pi(x))\) will be used interchangeably in this paper for convenience to indicate the same Hamiltonian function for \(\pi\); the Hamiltonian function for the optimal policy \(\pi_\star\) will be also denoted by \(h_\star(x, u)\), or equivalently, \(h_\star(x, u, \nabla v_\star(x))\).

The application of Lemma 1 shows that finding \(v_\pi\) satisfying (10) and (11) are both equivalent. In the following theorem, we state that the boundary condition (15), the counterpart of that in Corollary 1 is actually necessary and sufficient for a solution \(v\) of the Bellman equation (13) or (14) to be equal to the corresponding value function \(v_\pi\).

Theorem 1 Let \(\pi\) be admissible and \(v : \mathcal{X} \to \mathbb{R}\) be a function such that either of the followings holds \(\forall x \in \mathcal{X}\):

1) \(v\) satisfies the Bellman equation for some \(\Delta t > 0\):
\[
v(x) = \mathbb{E}_\pi \left[ \int_t^{t'} \gamma^{t-\tau} R_\tau \, d\tau + \gamma^\Delta t v(X_{t'}) \bigg| X_t = x \right];
\]
Then, \(\lim_{k \to \infty} l_\pi(x, k; v) = v(x) - v_\pi(x)\) for each \(x \in \mathcal{X}\). Moreover, \(v = v_\pi\) over \(\mathcal{X}\) if (and only if)
\[
\forall x \in \mathcal{X} : \lim_{k \to \infty} l_\pi(x, k; v) = 0.
\]

Proof. Since \(\pi\) is admissible, \(\mathbb{E}_\pi [v_\pi(X_{t+k})] \big| X_t = x\) is finite for all \(\tau \geq t\) and \(x \in \mathcal{X}\). First, let \(\tilde{v}(x) = v(x) - v_\pi(x)\) and suppose \(v\) satisfies (13). Then, subtracting (10) from (13) yields \(\tilde{v}(x) = \gamma^\Delta t \mathbb{E}_\pi^x [\tilde{v}(X_{t+\tau})]\), whose repetitive applications to itself results in
\[
\tilde{v}(x) = \gamma^\Delta t \mathbb{E}_\pi^x [\tilde{v}(X_{t+\Delta t})] = \cdots = \gamma^{k\Delta t} \mathbb{E}_\pi^x [\tilde{v}(X_{t+k\Delta t})].
\]
Therefore, by limiting $k \to \infty$ and using Corollary 1, we obtain $\lim_{k \to \infty} l_{\pi}(x, k; v) = \bar{v}(x)$, which also proves $\bar{v} = 0$ under (15). The proof of the other case for $v$ satisfying (14) instead of (13) is direct by Lemma 1. Conversely, if $v = v_\pi$ over $\mathcal{X}$, then (15) is obviously true by Corollary 1. □

**Remark 2** (15) is always true for any $\gamma \in (0, 1)$ and any bounded $v$. Hence, whenever $v_\pi$ is bounded and $0 < \gamma < 1$, e.g., the second case in Remark 1 including the simulation example in Section 5, any bounded function $v$ satisfying the Bellman equation (13) or (14) is equal to $v_\pi$ by Theorem 1.

2.3 Optimality Principle and Policy Improvement

Note that the optimal value function $v_* \in \mathcal{V}_a$ satisfies

$$v_*(x) = \max_{\pi} v_{\pi}(x)$$

and hence, by principle of optimality, the following Bellman optimality equation:

$$v_*(x) = \max_{\pi} \left\{ E_\pi \left[ \int_t^{t+1} \gamma^{t+1-\tau} R_{\pi}(\tau) d\tau + \gamma^{t+1} v_{\pi}(X_{t+1}) \right] \right\}$$

(16)

for all $x \in \mathcal{X}$, where $\max_{\pi}$ denotes the maximization among the all stationary policies. Hence, by the similar procedure to derive (11) from (10), we obtain from (16)

$$- \ln \gamma \cdot v_* = \max_{\pi} h_\pi(x, \pi(x), \nabla v_\pi(x)) \quad \forall x \in \mathcal{X}.$$  

Here, the above maximization formula can be characterized as the following HJBE:

$$- \ln \gamma \cdot v_*(x) = \max_{u \in \mathcal{U}} h(x, u, \nabla v_\pi(x)), \quad \forall x \in \mathcal{X}, \quad (17)$$

and the optimal policy $\pi_*$ as

$$\pi_*(x) \in \arg \max_{u \in \mathcal{U}} h_\pi(x, u), \quad \forall x \in \mathcal{X}$$

under the following assumption. \(^7\)

**Assumption 1** For any admissible policy $\pi$, there exists a policy $\pi'$ such that for all $x \in \mathcal{X}$,

$$\pi'(x) \in \arg \max_{u \in \mathcal{U}} h_\pi(x, u). \quad (18)$$

The following theorems support the argument.

**Theorem 2 (Policy Improvement Theorem)** Suppose $\pi$ is admissible and Assumption 1 holds. Then, the policy $\pi'$ given by (18) is also admissible and satisfies $v_{\pi'} \geq v_\pi$.

**Proof.** Since $\pi$ is admissible, (18) in Assumption 1 and (11) imply that for any $x \in \mathcal{X}$,

$$h_\pi(x, \pi'(x)) \geq h_\pi(x, \pi(x)) = - \ln \gamma \cdot v_\pi(x).$$

By Lemma 1, it is equivalent to

$$v_\pi(x) \leq \mathbb{E}_\pi \left[ \int_t^{t+1} \gamma^{t+1-\tau} R_{\pi}(\tau) d\tau + \gamma^{t+1} v_{\pi}(X_{t+1}) \right],$$

and by the repetitive applications itself,

$$v_\pi(x) \leq \mathbb{E}_\pi \left[ \int_t^{t+k\Delta t} \gamma^{t+k\Delta t-\tau} R_{\pi}(\tau) d\tau + \gamma^{t+k\Delta t} v_{\pi}(X_{t+k\Delta t}) \right].$$

Let $V_* \in \mathbb{R}$ be an upper bound of $v_*$. Then, in the limit $k \to \infty$, we obtain for each $x \in \mathcal{X}$

$$v_\pi(x) \leq v_{\pi'}(x) + \limsup_{k \to \infty} l_{\pi'}(x, k; v_{\pi'}) \leq v_{\pi'}(x) + \limsup_{k \to \infty} \gamma^{k\Delta t} \mathbb{E}_{\pi'}[v_\pi(X_{t+k\Delta t})] \quad (19)$$

from which and $v_\pi \in \mathcal{V}_a$, we can conclude that $v_{\pi'}(x)$ for each $x \in \mathcal{X}$ has a lower bound; since it also has an upper bound as $v_{\pi'}(x) \leq v_\pi(x) \leq V_*$, $\pi'$ is admissible. Finally, (3) with the admissible policy $\pi'$ and (19) imply that for each $x \in \mathcal{X}$,

$$v_\pi(x) \leq v_{\pi'}(x) + \limsup_{k \to \infty} l_{\pi'}(x, k; v_{\pi'}) \leq v_{\pi'}(x) \leq v_{\pi'}(x),$$

which completes the proof. □

**Corollary 2** Under Assumption 1, $v_*$ satisfies the HJBE (17).

**Proof.** Under Assumption 1, let $\pi'_* \in \mathbb{R}$ be a policy such that $\pi'_*(x) \in \arg \max_{u \in \mathcal{U}} h_\pi(x, u)$. Then, $\pi'_*$ is admissible and $v_* \leq v_{\pi'_*}$ holds by Theorem 2; trivially, $v_{\pi'_*} \leq v_*$. Hence, $\pi'_*$ is an optimal policy. Noting that any admissible $\pi$ satisfies (11), we obtain from “(11)” with $\pi = \pi'_*$ and $v_\pi = v_*$:

$$- \ln \gamma \cdot v_\pi(x) = h(x, \pi'_*(x), \nabla v_\pi(x)) = \max_{u \in \mathcal{U}} h(x, u, \nabla v_\pi(x))$$

for all $x \in \mathcal{X}$, which is exactly the HJBE (17). □

For the uniqueness of the solution $v_*$ to the HJBE, we further assume throughout the paper that

\(^7\) If $\mathcal{U}$ is compact, then by continuity of $h_\pi$, $\arg \max_{u \in \mathcal{U}} h_\pi(x, u)$ in (18) is non-empty for any admissible $\pi$ and any $x \in \mathcal{X}$. This guarantees the existence of a function $\pi'$ satisfying (18). For the other cases, where the action space $\mathcal{U}$ is required to be convex, see Section 3.3 (specifically, (30) and (34)).
Assumption 2. There is one and only one element \( w_* \in \mathcal{V}_a \) over \( \mathcal{V}_a \) that satisfies the HJBE:

\[-\ln \gamma \cdot w_*(x) = \max_{u \in \mathcal{U}} h(x, u, \nabla w_*(x)), \quad \forall x \in \mathcal{X}.
\]

Corollary 3. Under Assumptions 1 and 2, \( v_* = w_* \). That is, \( v_* \) is the unique solution to the HJBE (17) over \( \mathcal{V}_a \).

Remark 3. The policy improvement equation (18) in Assumption 1 can be generalized and rewritten as

\[
\pi'(x) \in \arg \max_{u \in \mathcal{U}} \left[ \kappa \cdot h_\pi(x, u) + b_\pi(x) \right]
\]

for any constant \( \kappa > 0 \) and any function \( b_\pi : \mathcal{X} \to \mathbb{R} \).显然，(18) is the special case of (20) with \( \kappa = 1 \) and \( b_\pi(x) = 0 \); policy improvement of our IPI methods in this paper can also be considered to be equal to "(20) with a special choice of \( \kappa \) and \( b_\pi \)" (as long as the associated functions are perfectly estimated in their policy evaluation).

3 On-policy Integral Policy Iteration (IPI)

Now, we are ready to state our basic primary PI scheme, which is named on-policy integral policy iteration (IPI) and estimate the value function \( v_\pi \) only (in policy evaluation) based on the on-policy state trajectory \( \mathcal{X} \) generated under \( \pi \) during some finite time interval \([t, t']\). The value function estimate obtained in policy evaluation is then utilized in the maximization process (policy improvement) yielding the next improved policy.

**Algorithm 1a:** On-policy IPI for the General Case (1)–(2)

1. Initialize: \( \pi_0 : \mathcal{X} \to \mathcal{U} \), the initial admissible policy;
   \( \Delta t > 0 \), the time difference;
2. \( i \leftarrow 0 \);
3. repeat
   4. Policy Evaluation: given policy \( \pi_i \), find the solution \( v_i : \mathcal{X} \to \mathbb{R} \) to the Bellman equation: for any \( x \in \mathcal{X} \),
   \[
   v_i(x) = \mathbb{E}_{\pi_i}^x \left[ \int_t^{t'} \gamma^{t'-t} R_{t'} d\tau + \gamma^{\Delta t} v_i(X_{t'}) \right];
   \] (21)
5. Policy Improvement: find a policy \( \pi_{i+1} \) such that
   \[
   \pi_{i+1}(x) \in \arg \max_{u \in \mathcal{U}} h(x, u, \nabla v_i(x)) \quad \forall x \in \mathcal{X};
   \] (22)
6. \( i \leftarrow i + 1 \);
   until convergence is met.

Algorithm 1a describes the whole procedure of on-policy IPI—it starts with an initial admissible policy \( \pi_0 \) (line 1) and performs policy evaluation and improvement until \( v_i \) and/or \( \pi_i \) converge (lines 4–7). In policy evaluation (line 4), the agent solves the Bellman equation (21) to find the value function \( v_i = v_\pi \) for the current policy \( \pi_i \). Then, in policy improvement (line 5), the next policy \( \pi_{i+1} \) is obtained by maximizing the associated Hamiltonian function.

3.1 Admissibility, Monotone Improvement, & Convergence

As stated in Theorem 3 below, on-policy IPI guarantees the admissibility and monotone improvement of \( \pi_i \) and the perfect value function estimation \( v_i = v_\pi \) at each \( i \)-th iteration under Assumption 1 and the boundary condition:

**Assumption 3a.** For each \( i \in \mathbb{Z}_+ \), if \( \pi_i \) is admissible, then

\[
\lim_{k \to \infty} l_{\pi_i}(x, k; v_i) = 0 \quad \text{for any } x \in \mathcal{X}.
\]

**Theorem 3.** Let \( \{\pi_i\}_{i=0}^\infty \) and \( \{v_i\}_{i=0}^\infty \) be the sequences generated by Algorithm 1a under Assumptions 1 and 3a. Then,

\[\begin{align*}
(P1) & \quad \forall i \in \mathbb{Z}_+ : v_i = v_{\pi_i}; \\
(P2) & \quad \forall i \in \mathbb{Z}_+ : \pi_{i+1} \text{ is admissible and satisfies} \\
& \quad \pi_{i+1}(x) \in \arg \max_{u \in \mathcal{U}} h_{\pi_i}(x, u); \\
(P3) & \quad \text{the policy is monotonically improved, i.e.,} \\
& \quad v_{\pi_0} \preceq v_{\pi_1} \preceq \cdots \preceq v_{\pi_i} \preceq v_{\pi_{i+1}} \preceq \cdots \preceq v_*. 
\end{align*}\]

**Proof.** \( \pi_0 \) is admissible by the first line of Algorithm 1a. For any \( i \in \mathbb{Z}_+ \), suppose \( \pi_i \) is admissible. Then, since \( v_i \) satisfies Assumption 3a, \( v_i = v_{\pi_i} \) by Theorem 1. Moreover, Theorem 2 under Assumption 1 shows that \( \pi_{i+1} \) is admissible and satisfies \( v_{\pi_i} \preceq v_{\pi_{i+1}} \preceq v_* \). Furthermore, (23) holds by (12), (22), and \( v_i = v_{\pi_i} \). Finally, the proof is completed by mathematical induction. □

From Theorem 3, one can directly see that for any \( x \in \mathcal{X} \), the real sequence \( \{v_i(x)\}_{i=0}^\infty \) satisfies

\[
v_0(x) \leq \cdots \leq v_i(x) \leq v_{i+1}(x) \leq \cdots \leq v_*(x) < \infty,
\]

implying pointwise convergence to some function \( \hat{v}_* \). Since \( v_i (v_i = v_{\pi_i}) \) is continuous by the \( C^1 \)-assumption on every \( v_\pi \in \mathcal{V}_a \) (see Section 2.1), the convergence is uniform on any compact subset of \( \mathcal{X} \). By Dini’s theorem (Thomson, Bruckner, and Bruckner, 2001) provided that \( \hat{v}_* \) is continuous. This is summarized and sophisticated in the following theorem.

**Theorem 4.** Under the same conditions to Theorem 3, there is a Lebesque measurable, lower semicontinuous function \( \hat{v}_* \), defined as \( \hat{v}_*(x) = \sup_{i \in \mathbb{Z}_+} v_i(x) \) such that
(1) \( v_i \to \hat{v}_* \) pointwisely on \( \mathcal{X} \);

(2) for any \( \varepsilon > 0 \) and any compact set \( \Omega \) of \( \mathcal{X} \), there exists its compact subset \( E \subseteq \Omega \) such that \( |\Omega \setminus E| < \varepsilon \), \( \hat{v}_*|_E \) is continuous, and \( v_i \to \hat{v}_* \) uniformly on \( E \).

Moreover, if \( \hat{v}_* \) is continuous over \( \mathcal{X} \), then the convergence \( v_i \to \hat{v}_* \) is uniform on any compact subset of \( \mathcal{X} \).

**Proof.** By (24), the sequence \( \{v_i(x) \in \mathbb{R}\}_{i=0}^\infty \) for any fixed \( x \in \mathcal{X} \) is monotonically increasing and upper bounded by \( v_0(x) < \infty \). Hence, \( v_i(x) \) converges to \( \hat{v}_*(x) \) by monotone convergence theorem (Thomson et al., 2001), the pointwise convergence \( v_i \to \hat{v}_* \). Since \( v_i \) is continuous, \( \hat{v}_* \) is Lebesgue measurable and lower semicontinuous by its construction (Folland, 1999, Propositions 2.7 and 7.11c). Next, by Lusin’s theorem (Loeb and Talvila, 2004), for any \( \varepsilon > 0 \) and any compact set \( \Omega \subseteq \mathcal{X} \), there exists a compact subset \( E \subseteq \Omega \) such that \( |\Omega \setminus E| < \varepsilon \) and the restriction \( v_i|_E \) is continuous. Hence, the monotone sequence \( v_i \) converges to \( \hat{v}_* \) uniformly on \( E \) (and on any compact subset of \( \mathcal{X} \)) if \( \hat{v}_* \) is continuous over \( \mathcal{X} \) by Dini’s theorem (Thomson et al., 2001).

Next, we prove the convergence \( v_i \to v_* \) to the optimal solution \( v_* \) using the PI operator \( T : \mathcal{V}_a \to \mathcal{V}_a \) defined on the space \( \mathcal{V}_a \) of admissible value functions as

\[
Tv_\pi \doteq v_\pi^*.
\]

under Assumption 1, where \( \pi^* \) is the next admissible policy that satisfies (18) and is obtained by policy improvement with respect to the given value function \( v_\pi \in \mathcal{V}_a \). Let its \( N \)-th recursion \( T^N \) be defined as \( T^N v_\pi \doteq T^{N-1} [Tv_\pi] \) and \( T^0 v_\pi \doteq v_\pi \). Then, any sequence \( \{v_i \in \mathcal{V}_a\}_{i=0}^\infty \) generated by Algorithm 1a under Assumptions 1 and 3a satisfies

\[
T^N v_0 = v_N \text{ for any } N \in \mathbb{N}.
\]

**Lemma 2** Under Assumptions 1 and 2, the optimal value function \( v_* \) is the unique fixed point of \( T^N \) for all \( N \in \mathbb{N} \).

**Proof.** See Appendix A.

To precisely state our convergence theorem, let \( \Omega \) be any given compact subset and define the uniform pseudometric \( d_\Omega : \mathcal{V}_a \times \mathcal{V}_a \to [0, \infty) \) on \( \mathcal{V}_a \) as

\[
d_\Omega(v, w) \doteq \sup_{x \in \Omega} |v(x) - w(x)| \text{ for } v, w \in \mathcal{V}_a.
\]

**Theorem 5** For the value function sequence \( \{v_i\}_{i=0}^\infty \) generated by Algorithm 1a under Assumptions 1, 2, and 3a,

(C1) there exists a metric \( d : \mathcal{V}_a \times \mathcal{V}_a \to [0, \infty) \) such that \( T \) is a contraction (and thus continuous) under \( d \) and \( v_i \to v_* \) in the metric \( d \), i.e., \( \lim_{i \to \infty} d(v_i, v_*) = 0 \);

(C2) if \( \hat{v}_* \in \mathcal{V}_a \) and for every compact subset \( \Omega \subseteq \mathcal{X} \), \( T \) is continuous under \( d_\Omega \), then \( v_i \to v_* \) pointwisely on \( \mathcal{X} \) and uniformly on any compact subset of \( \mathcal{X} \).

**Proof.** By Lemma 2 and Bessaga (1959)’s converse of the Banach’s fixed point principle, there exists a metric \( d \) on \( \mathcal{V}_a \) such that \( (\mathcal{V}_a, d) \) is a complete metric space and \( T \) is a contraction (and thus continuous) under \( d \). Moreover, by Lemma 2 and Banach’s fixed point principle (e.g., Kirk and Sims, 2013, Theorem 2.2),

\[
\forall \nu \in \mathcal{V}_a : \lim_{N \to \infty} v_N = \lim_{N \to \infty} T^n v_0 = v_* \text{ in the metric } d.
\]

To prove the second part, suppose that \( \hat{v}_* \in \mathcal{V}_a \) and that \( T \) is continuous under \( d_\Omega \) for every compact subset \( \Omega \subseteq \mathcal{X} \). Then, since \( \hat{v}_* \in \mathcal{V}_a \) is \( C^1 \) by assumption and thereby, continuous, \( v_i \) converges to \( \hat{v}_* \) pointwisely on \( \mathcal{X} \) and uniformly on any compact \( \Omega \subseteq \mathcal{X} \) by Theorem 4; the latter implies \( v_i \to v_* \) in \( d_\Omega \). Therefore, in the uniform pseudometric \( d_\Omega \),

\[
\hat{v}_* = \lim_{i \to \infty} v_{i+1} = \lim_{i \to \infty} T v_i = T \left( \lim_{i \to \infty} v_i \right) = T \hat{v}_*.
\]

by continuity of \( T \) under \( d_\Omega \). That is, \( \hat{v}_*|_\Omega = (T \hat{v}_*)|_\Omega \) for every compact \( \Omega \subseteq \mathcal{X} \). This implies \( \hat{v}_* = T \hat{v}_* \) and thus, \( \hat{v}_* = v_* \) by Lemma 2, which completes the proof.

3.2 Partially Model-Free Nature

Policy evaluation (21) can be done without using the explicit knowledge of the system dynamics \( f(x, u) \) in (1)—there is no explicit term of \( f \) shown in (21), and all of the necessary information on \( f \) are captured by the observable state trajectory \( X \) during a finite time interval \([t, t']\). Hence, IPI is model-free as long as so is its policy improvement (22), which is not unfortunately (see the definition (5) of \( h \)). Nevertheless, the policy improvement (22) can be modified to yield the partially model-free IPI. To see this, consider the decomposition (25) of the dynamics \( f \) below:

\[
f(x, u) = f_d(x) + f_c(x, u),
\]

where \( f_d : \mathcal{X} \to \mathcal{X} \) is independent of \( u \) and called a drift dynamics, and \( f_c : \mathcal{X} \times \mathcal{U} \to \mathcal{X} \) is the corresponding input-coupling dynamics. Substituting the definitions (5) and (12) into (20), choosing \( \kappa = 1 \) and \( h(x) = -\nabla v_\pi(x)f_d(x) \) and replacing \( (\pi, v_\pi) \) with \( (\pi_i, v_i) \) then yield

\[
\pi_{i+1}(x) \in \arg \max_{u \in \mathcal{U}} \{ R(x, u) + \nabla v_i(x)f_c(x, u) \},
\]

a partially model-free version of (22). In summary, the whole procedure of Algorithm 1a can be done even when the drift dynamics \( f_d \) is completely unknown.

---

9 There are an infinite number of ways of choosing \( f_d \) and \( f_c \); one typical choice is \( f_d(x) = f(x, 0) \) and \( f_c(x, u) = f(x, u) - f_d(x) \).
3.3 Case Studies

The partially model-free policy improvement (26) can be even more simplified if:

(1) the system dynamics \( f(x, u) \) is affine in \( u \), i.e.,
\[
    f(x, u) = f_0(x) + F_c(x)u, \tag{27}
\]
where \( F_c : \mathcal{X} \to \mathbb{R}^{n \times m} \) is a continuous matrix-valued function; it is (25) with \( f_c(x, u) = F_c(x)u \);
(2) the action space \( \mathcal{U} \subseteq \mathbb{R}^m \) is convex; \( \tag{28} \)
(3) the reward \( R(x, u) \) is strictly concave in \( u \) and given by
\[
    R(x, u) = R_0(x) - S(u), \tag{29}
\]
where \( R_0 : \mathcal{X} \to \mathbb{R} \) is a continuous upper-bounded function called the state reward, and \( S : \mathcal{U} \to \mathbb{R} \), named the action penalty, is a strictly convex \( C^1 \) function whose restriction \( S|_{\mathcal{U}_{\text{int}}} \) on the interior \( \mathcal{U}_{\text{int}} \) of \( \mathcal{U} \) satisfies \( \text{Im}(\nabla S|_{\mathcal{U}_{\text{int}}}) = \mathbb{R}^m \).

In this case, solving the maximization in (26) (or (22)) is equivalent to finding the regular point \( u \in \mathcal{U} \) such that
\[
    -\nabla S(u) + \nabla v(u)F_c(x) = 0,
\]
where the gradient \( \nabla S \) of \( S \) is a strictly monotone mapping that is also bijective when its domain \( \mathcal{U} \) is restricted to its interior \( \mathcal{U}_{\text{int}} \). Rearranging it with respect to \( u \), we obtain the explicit closed-form expression of (26) also known as the VGB greedy policy (Doya, 2000):
\[
    \pi_{i+1}(x) = \sigma \left( F_c^T(x)\nabla v^i_T(x) \right), \tag{30}
\]
where \( \sigma : \mathbb{R}^m \to \mathcal{U}_{\text{int}} \) is defined as \( \sigma = (\nabla S|_{\mathcal{U}_{\text{int}}})^{-1} \), which is also strictly monotone, bijective, and continuous. Therefore, in the \( u \)-affine-and-concave \((u\text{-AC}) \) case (27)–(29), the complicated maximization process in (26) over the continuous action space can be obviated by directly calculating the next policy \( \pi_{i+1} \) by (30). Also note that under the \( u \)-AC setting (27)–(29), the VGB greedy policy \( \pi^* = \sigma \circ (F_c^T \nabla v^*_T) \) for an admissible \( \pi \) is the unique policy satisfying (18).

\[\text{Remark 4} \] Whenever each \( j \)-th component \( U_{\tau,j} \) of \( U_{\tau} \) meets some physical limitation \( |U_{\tau,j}| \leq U_{\text{max},j} \) for some threshold \( U_{\text{max},j} \in (0, \infty) \), one can formulate the action space \( \mathcal{U} \) in (28) as \( \mathcal{U} = \{ u \in \mathbb{R}^m : |U_{\tau,j}| \leq U_{\text{max},j}, 1 \leq j \leq m \} \) and determine the action penalty \( S(u) \) in (29) as
\[
    S(u) = \lim_{v \to u} \int_0^v (s^T)^{-1}(w) \cdot \Gamma dw \tag{31}
\]
for a positive definite matrix \( \Gamma \in \mathbb{R}^{m \times m} \) and a continuous function \( s : \mathbb{R}^m \to \mathcal{U}_{\text{int}} \) such that
\( \tag{32} \)

(1) \( s \) is strictly monotone, odd, and bijective;
(2) \( S \) in (31) is finite at any point on the boundary \( \partial \mathcal{U} \).

This gives the closed-form expression \( \sigma(\xi) = s(\Gamma^{-1}\xi) \) of the function \( \sigma \) in (30) and includes the sigmoidal example in Section 5 as its special case. \( \tag{33} \)
Another well-known example is:
\[
    \mathcal{U} = \mathbb{R}^m \ (i.e., \ U_{\text{max},j} = \infty, \ 1 \leq j \leq m) \text{ and } s(u) = u/2,
\]
in which case (31) becomes \( S(u) = u^T\Gamma u \).

The well-known special case of (27)–(29) is the following linear quadratic regulation (LQR) (31), (32) and
\[
    f_0(x) = Ax, \quad F_c(x) = B, \quad R_0(x) = -||Cx||^2, \quad \tag{33}
\]
where \((A, B, C)\) for \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, \text{ and } C \in \mathbb{R}^{p \times n} \) is stabilizable and detectable. In this LQR case, if the policy \( \pi_i \) is linear, i.e., \( \pi_i(x) = K_i x \ (K_i \in \mathbb{R}^{m \times n}) \), then its value function \( v_{\pi_i} \), if finite, can be represented in a quadratic form \( v_{\pi_i}(x) = x^T P_{\pi_i} x \ (P_{\pi_i} \in \mathbb{R}^{n \times n}) \). Moreover, when \( v_i \) is quadratically represented as \( v_i(x) = x^T P_i x \), (30) becomes
\[
    \pi_{i+1}(x) = K_{i+1} x \text{ and } K_{i+1} = \Gamma^{-1} B^T P_i, \tag{34}
\]
a linear policy again. This observation gives the policy evaluation and improvement in Algorithm 1b below. Moreover, whenever the given policy \( \pi \) is linear and (31)–(33) are all true, the process \( Z_\tau \) generated by
\[
    \dot{Z}_\tau = (A + \frac{\ln \gamma}{\tau} I) Z_\tau + BU_\tau \tag{35}
\]
yields the following value function expression in terms of \( Z_\tau \) without the discount factor \( \gamma \) as
\[
    v_\pi(x) = \mathbb{E}\pi \left[ \int_t^\infty R(Z_\tau, U_\tau) \, dt \mid Z_t = x \right].
\]

11 \( \partial \mathcal{U} = \{ U_\tau \in \mathbb{R}^m : U_{\tau,j} = U_{\text{max},j}, 1 \leq j \leq m \} \)

12 See also (Doya, 2000; Abu-Khalaf and Lewis, 2005).

**Algorithm 1b: On-policy IPI for the LQR Case (31)–(33)**

1. Initialize: \( \pi_0(x) = K_0 x \), the init. admissible policy;
   \( \Delta t > 0 \), the time difference;
2. \( i \leftarrow 0; \)
3. repeat (under the LQR setting (31), (32), and (33))
4. Policy Evaluation: given policy \( \pi_i(x) = K_i x \), find the solution \( v_i(x) = x^T P_i x \) to the Bellman equation (21);
5. Policy Improvement: \( K_{i+1} = \Gamma^{-1} B^T P_i \);
6. \( i \leftarrow i + 1; \)
until convergence is met.
where \( \tilde{E}_\pi[Y|Z_t = x] \) denotes the value of \( Y \) when \( Z_t = x \) and \( U_\tau = \pi(Z_\tau) \ \forall \tau \geq t \). This transforms any discounted LQR problem into the total one (with its state \( Z_\tau \) in place of \( X_n \)). Hence, the application of the standard LQR theory (Anderson and Moore, 1989) shows that

\[
\begin{aligned}
(1) \quad & v_* (x) = x^T P_* x \leq 0 \text{ for some } P_* \in \mathbb{R}^{n \times n}, \\
(2) \quad & \pi_* (x) = K_* x \text{ with } K_* = \Gamma^{-1} B^T P_*; \\
(3) \quad & \pi_* \text{ is the unique solution to the HJBE (17)}. 
\end{aligned}
\]

Furthermore, the application of the analytical result of IPI (Lee et al., 2014, Theorem 5 and Remark 4 with \( h \to \infty \)) gives the following statements.

**Lemma 3 (Policy Improvement Theorem: the LQR Case)**

Let \( \pi \) be linear and admissible. Then, the linear policy \( \pi' \) given by \( \pi'(x) = K'x \) and \( K' = \Gamma^{-1} B^T P_* \) under the LQR setting (31)–(33) is admissible and satisfies \( P_0 \leq P_{i+1} \leq 0 \).

**Theorem 6**

Let \( \{\pi_i\}_{i=0}^\infty \) and \( \{v_i\}_{i=0}^\infty \) be the sequences generated by Algorithm 1b and parameterized as \( \pi_i(x) = K_i x \) and \( v_i(x) = x^T P_i x \). Then,

\[
\begin{aligned}
(1) \quad & \pi_i \text{ is admissible and } P_i = P_{i+1} \text{ for all } i \in \mathbb{Z}_{+}; \\
(2) \quad & \lim_{i \to \infty} P_i = P_e; \\
(3) \quad & \lim_{i \to \infty} P_i = P_e; \\
(4) \quad & \text{the convergence } P_i \to P_e \text{ is quadratic}.
\end{aligned}
\]

**Remark 5**

In the LQR case, Assumptions 1, 2, and 3a are all true—Assumptions 1 and 2 are trivially satisfied as shown above; Assumption 3a is also true by

\[
\begin{aligned}
\lim_{k \to \infty} l_{\pi_i} (x; k; v_i) &= \lim_{k \to \infty} \mathbb{E}_{\pi_i} \left[ X_{t+k}\Gamma_k X_{t+k}\Gamma_k D \right] \\
&= \lim_{k \to \infty} \mathbb{E}_{\pi_i} \left[ Z_{t+k}\Gamma_k P Z_{t+k}\Gamma_k D \right] = 0,
\end{aligned}
\]

where we have used the equality \( \mathbb{E}_{\pi_i} [Z_{t+k}] = \mathbb{E}_{\pi_i} [(\gamma/2)^{t+k} X_{t+k}] \) and the fact that a linear policy \( \pi \) is admissible if it stabilizes the \( Z \)-system (35) and thus satisfies \( \lim_{t \to \infty} \mathbb{E}_{\pi} [Z_t] = 0 \) \( \forall x \in \mathcal{X} \) (Lee et al., 2014, Section 2). To the best authors’ knowledge, however, the corresponding theory does not exist for the nonlinear discounted case \( \gamma \in (0, 1) \). See (Abu-Khalaf and Lewis, 2009; Vrabie and Lewis, 2009; Lee et al., 2015) for the u-AC case (27)–(29) with \( \gamma = 1 \) and \( R \leq 0 \).

4 Extensions to Off-policy IPI Methods

In this section, we propose a series of completely/partially model-free off-policy IPI methods, which are effectively same to on-policy IPI but uses data generated by a behavior policy \( \mu \), rather than the target policy \( \pi_* \). For this, we first introduce the concept of action-dependent (AD) policy.

**Definition 2**

For a non-empty subset \( U_0 \subseteq \mathcal{U} \) of the action space \( \mathcal{U} \), a function \( \mu : \mathcal{X} \times U_0 \to \mathcal{U} \), denoted by \( \mu(x, u) = u \) for \( \tau \geq t \) and \( (x, u) \in \mathcal{X} \times U_0 \), is said to be an AD policy over \( U_0 \) (starting at time \( t \)) if:

\[
\begin{aligned}
(1) \quad & \mu(t, x, u) = u \text{ for all } x \in \mathcal{X} \text{ and all } u \in U_0; \\
(2) \quad & \text{for each fixed } u \in U_0, \mu(\cdot, u) \text{ is a policy (starting at time } t), \text{ which is possibly non-stationary}.
\end{aligned}
\]

An AD policy is actually a policy parameterized by \( u \in U_0 \); the purpose of such a parameterization is to impose the condition \( U_\tau = u \) at the initial time \( t \) through the first property in Definition 2 so as to make it possible to explore the state-action space \( \mathcal{X} \times \mathcal{U} \) (or its subset \( \mathcal{X} \times U_0 \), rather than the state space \( \mathcal{X} \) alone. The simplest form of an AD policy \( \mu \) is

\[
\mu(\tau, x, u) = u \text{ for all } \tau \geq t \text{ and all } (x, u) \in \mathcal{X} \times U_0
\]

used in Section 5; another useful important example is

\[
\mu(\tau, x, u) = \pi(x) + e(\tau, x, u) \text{ (with } \mathcal{U} = \mathbb{R}^m),
\]

for a policy \( \pi \) and a probing signal \( e(\tau, x, u) \) given by

\[
e(\tau, x, u) = (u - \pi(x)) e^{-\sigma (\tau - t)} + \sum_{j=1}^{N} A_j \sin (\omega_j (\tau - t)),
\]

where \( \sigma > 0 \) regulates the vanishing rate of the first term; \( A_j \in \mathbb{R}^m \) and \( \omega_j \in \mathbb{R} \) are the amplitude and the angular frequency of the \( j \)-th sin term in the summation, respectively.

In what follows, for an AD policy \( \mu \) starting at time \( t \geq 0 \), we will interchangeably use

\[
\mathbb{E}_\mu[Z|X_t = x, U_t = u] \text{ and } \mathbb{E}_\mu^{(x, u)}[Z|t],
\]

with a slight abuse of notations, to indicate the deterministic value \( Z \) when \( X_t = x \) and \( U_t = \mu(\tau, X_\tau, u) \) for all \( \tau \geq t \). Using this notation, the state vector \( X_\tau \) at time \( \tau \geq t \) that starts at time \( t \) and is generated under \( \mu \) and the initial condition \( X_0 = x \) and \( U_t = u \) is denoted by \( \mathbb{E}_\mu^{(x, u)}[X_t|t] \).

For any non-AD policy \( \mu \), we also denote \( \mathbb{E}_\mu[Z|X_t = x] \) and \( \mathbb{E}_\mu^{(x, u)}[Z|t] \), instead of (36), to indicate the value \( Z \) when \( X_t = x \) and \( U_t = \mu(\tau, X_\tau) \) \( \forall \tau \geq t \), where the condition \( U_t = u \) is obviated.

Each off-policy IPI method in this paper is designed to generate the policies and value functions satisfying the same properties in Theorems 3 and 4 as those in on-policy IPI (Algorithm 1a), but they are generated using the off-policy trajectories generated by a (AD) behavior policy \( \mu \), rather than the target policy \( \pi_* \). Specifically, each off-policy method estimates \( \pi_i \) and/or a (AD) function in policy evaluation using the off-policy state and action trajectories and then employs the estimated function in policy improvement to find a next improved policy. Each off-policy IPI method will be described only with its policy evaluation and improvement steps since the others are all same to those in Algorithm 1a.

4.1 Integral Advantage Policy Iteration (IAPI)

First, we consider a continuous function \( a_{\pi} : \mathcal{X} \times \mathcal{U} \to \mathbb{R} \) called the advantage function for an admissible policy \( \pi \)
(Baird III, 1993; Doya, 2000), which is defined as

$$a_\pi(x,u) = h_\pi(x,u) + \ln \gamma \cdot v_\pi(x)$$  \hspace{1cm} (37)

and satisfies $a_\pi(x,\pi(x)) = 0$ by (11). Considering (20) in Remark 3 with $h_\pi(x) = \ln \gamma \cdot v_\pi(x)$ and $\kappa = 1$, we can see that the maximization process (18) can be replaced by

$$\pi'(x) \in \arg \max_{u \in U} a_\pi(x,u) \forall x \in \mathcal{X};$$  \hspace{1cm} (38)

the optimal advantage function $a_* = a_{\pi_*}$ also characterizes the HJBE (17) and the optimal policy $\pi_*$ as

$$\max_{u \in U} a_*(x,u) = 0 \text{ and } \pi_*(x) \in \arg \max_{u \in U} a_*(x,u).$$

These ideas give model-free off-policy IPI named integral advantage policy iteration (IAPI), whose policy evaluation and improvement steps are shown in Algorithm 2 while the other parts are, as mentioned above, all same to those in Algorithm 1a and thus omitted. Given $\pi_i$, the agent tries to find/estimate in policy evaluation both $v_i(x)$ and $a_i(x,u)$ satisfying (40) and the off-policy Bellman equation (39) for “an AD policy $\mu$ over the entire action space $\mathcal{U}$.” Here, $v_i$ and $a_i$ corresponds to the value and the advantage functions with respect to the $i$-th admissible policy $\pi_i$ (see Theorem 7 in Section 4.5). Then, the next policy $\pi_{i+1}$ is updated in policy improvement by using the advantage function $a_i(x,u)$ only.

Notice that this IAPI provides the ideal PI form of advantage updating and the associated ideal Bellman equation—see (Baird III, 1993) for advantage updating and the approximate version of the Bellman equation (39).

**Algorithm 2: Integral Advantage Policy Iteration (IAPI)**

**Policy Evaluation:** given $\pi_i$ and an AD policy $\mu$ over $\mathcal{U}$,

find a $C^1$ function $v_i : \mathcal{X} \to \mathbb{R}$

such that

1. for all $(x,u) \in \mathcal{X} \times \mathcal{U}$,

$$v_i(x) = \mathbb{E}_\mu^{(x,u)} \left[ \int_t^{t'} \gamma^{t-t'} Z_\tau d\tau + \gamma^{t'} v_i(X'_\tau) \right] t,$$  \hspace{1cm} (39)

where $Z_\tau = R_\tau - a_i(X_\tau, U_\tau) + a_i(X_\tau, \pi_i(X_\tau));$

2. $a_i(x,\pi_i(x)) = 0$ for all $x \in \mathcal{X};$  \hspace{1cm} (40)

**Policy Improvement:** find a policy $\pi_{i+1}$ such that

$$\pi_{i+1}(x) \in \arg \max_{u \in U} a_i(x,u) \forall x \in \mathcal{X};$$  \hspace{1cm} (41)

4.2 Integral Q-Policy-Iteration (IQPI)

Our next model-free off-policy IPI named integral IQ-policy-iteration (IQPI) estimates and uses a general Q-function $q_{\pi}$:

**Algorithm 3a: Integral Q-Policy-Iteration (IQPI)**

**Policy Evaluation:** given

\{ the current policy $\pi_i$

\{ an weighting factor $\beta > 0$

find a continuous function $q_i : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ such that for all $(x,u) \in \mathcal{X} \times \mathcal{U}$:

$$q_i(x,\pi_i(x)) = \mathbb{E}_\mu^{(x,u)} \left[ \int_t^{t'} \beta^{t-t'} Z_\tau d\tau + \beta^{t'} q_i(X'_\tau, \pi_i(X'_\tau)) \right] t,$$  \hspace{1cm} (42)

where $Z_\tau = \kappa_1 R_\tau - \kappa_2 q_i(X_\tau, U_\tau) + \kappa_3 q_i(X_\tau, \pi_i(X_\tau)),$

$$\kappa_1 \kappa_2 > 0 \text{ and } \kappa_3 = \kappa_2 - \ln(\gamma^{-1}\beta);$$

**Policy Improvement:** find a policy $\pi_{i+1}$ such that

$$\pi_{i+1}(x) \in \arg \max_{u \in U} q_i(x,u) \forall x \in \mathcal{X};$$  \hspace{1cm} (43)

$\mathcal{X} \times \mathcal{U} \to \mathbb{R}$ defined as

$$q_{\pi}(x,u) = \kappa_1 \cdot (v_{\pi}(x) + a_{\pi}(x,u)/\kappa_2)$$  \hspace{1cm} (44)

for an admissible policy $\pi$, where $\kappa_1$, $\kappa_2 \in \mathbb{R}$ are any two nonzero real numbers that have the same sign, so that $\kappa_1/\kappa_2 > 0$ holds.\footnote{Our general Q-function $q_\pi$ includes the previously proposed Q-functions in CTS as special cases—Baird III (1993)’s Q-function ($\kappa_1 = 1$, $\kappa_2 = 1/\Delta t$); $b_\pi$ for $\gamma \in (0.1)$ ($\kappa_1 = \kappa_2 = -\ln \gamma$), and its generalization for $\gamma \in (0,1]$ (any $\kappa_1 = \kappa_2 > 0$) both recognized as Q-functions by Mehta and Meyn (2009).}

By its definition and continuities of $v_{\pi}$ and $a_{\pi}$, the Q-function $q_{\pi}$ is also continuous over its whole domain $\mathcal{X} \times \mathcal{U}$. Here, since both $\kappa_1$ and $\kappa_2$ are nonzero, the Q-function (44) does not lose both information on $v_{\pi}$ and $a_{\pi}$; thereby, $q_\pi$ plays a similar role of the DT Q-function—on one hand, it holds the property

$$\kappa_1 \cdot v_{\pi}(x) = q_{\pi}(x,\pi(x)),$$  \hspace{1cm} (45)

and on the other, it replaces (18) and (38) with

$$\pi'_i(x) \in \arg \max_{u \in U} q_i(x,u) \forall x \in \mathcal{X}$$  \hspace{1cm} (46)

by (20) for $\kappa = \kappa_1/\kappa_2 > 0$ and $b_\pi(x) = \kappa_1 (1 + \ln \kappa_2) \cdot v_{\pi}(x)$; the HJBE (17) and the optimal policy $\pi_*$ are also characterized by the optimal Q-function $q_* = q_{\pi_*}$ as

$$\kappa_1 \cdot v_* = \max_{u \in U} q_*(x,u) \text{ and } \pi_*(x) \in \arg \max_{u \in U} q_*(x,u).$$

Algorithm 3a shows the policy evaluation and improvement of IQPI—the former is derived by substituting (45) and $\kappa_1 a_{\pi}(x,u) = \kappa_2 \cdot (q_{\pi}(x,u) - q_{\pi}(x,\pi(x)))$ (obtained from (44) and (45)) into (39) in IAPI, and the latter directly from (46). At each iteration, while IAPI needs to find/estimate both $v_i$ and $a_i$, IQPI just estimate and use in its loop $q_i$ only. In addition, the constraint on the AD function such as (40).
in IAPI, making the algorithm simpler. As will be shown in Theorem 7 in Section 4.5, \( q_t \) in Algorithm 3a corresponds to the Q-function \( q_\pi \), for the \( i \)-th admissible policy \( \pi_i \), and the policies \( \{ \pi_i \}_{i=0}^\infty \) generated by IQPI satisfy the same properties to those in on-policy IPI.

Notice that simplification of IQPI is possible by setting
\[
\kappa = \ln(\gamma^{-1} \beta) = \kappa_1 = \kappa_2 \text{ and } \gamma \neq \beta, \quad (47)
\]
in which case \( Z_\pi \) in IQPI is dramatically simplified to (49) shown in the Algorithm 3b, the simplified IQPI, and the Q-function \( q_\pi \) in its definition (44) becomes
\[
q_\pi(x,u) = \kappa \cdot v_\pi(x) + a_\pi(x,u). \quad (48)
\]
In this case, the gain \( \kappa \) (\( \neq 0 \)) of the integral is the scaling factor of \( v_\pi \) in the Q-function \( q_\pi \), relative to \( a_\pi \). As mentioned by Baird III (1993), a bad scaling between \( v_\pi \) and \( a_\pi \), e.g., extremely large \( \kappa \), may result in significant performance degradation or extremely slow Q-learning.

Compared with the other off-policy IPI methods, the use of the weighting factor \( \beta \in (0, \infty) \) is one of the major distinguishing feature of IQPI—\( \beta \) plays a similar role to the discount factor \( \gamma \in (0,1) \) in the Bellman equation; it can be arbitrarily set in the algorithm; it can be equal to \( \gamma \) or not. In the special case (47), \( \beta \) should not be equal to \( \gamma \) since the log ratio \( \ln(\beta/\gamma) \) of the two determines the non-zero scaling gain \( \kappa \) in (48) and (49). Since Algorithm 3b is a special case of IQPI (Algorithm 3a), it also has the same mathematical properties shown in Section 4.5.

Algorithm 3b: IQPI with the Simplified Setting (47)

| Policy Evaluation | Policy Improvement |
|-------------------|--------------------|
| given \( \{ \text{ the current policy } \pi_i \} \) \( \beta > 0 \), \( \text{ an AD policy } \mu \) over \( U \) | find a policy \( \pi_{i+1} \) satisfying (43) |

\[
\text{Policy Evaluation:} \quad \text{given } \{ \text{ the current policy } \pi_i \}, \beta > 0, \mu \text{ over } U
\]

find a continuous function \( q_t : \mathcal{X} \times U \to \mathbb{R} \) such that (42) holds for all \( (x,u) \in \mathcal{X} \times U \) and for \( Z_\tau \) given by
\[
Z_\tau = \kappa \cdot (R_\tau - q_t(X_\tau, U_\tau)); \quad (49)
\]

4.3 Integral Explorized Policy Iteration (IEPI)

The on-policy IPI (Algorithm 1a) can be easily generalized and extended to its off-policy version without introducing any AD function such as \( a_t \) in IAPI and \( q_t \) in IQPI. In this paper, we name it integral explorized policy iteration (IEPI) following the perspectives of Lee et al. (2012) and present its policy evaluation and improvement loop in Algorithm 4a. Similarly to on-policy IPI with its policy improvement (22) replaced by (26), IEPI is also partially model-free—the input-coupling dynamics \( f_c \) has to be used in both policy evaluation and improvement while the drift term \( f_d \) is not when the system dynamics \( f \) is decomposed to (25).

Algorithm 4a: IEPI for the General Case (1)–(2)

| Policy Evaluation | Policy Improvement |
|-------------------|--------------------|
| find a \( C^1 \) function \( v_t : \mathcal{X} \to \mathbb{R} \) such that for all \( x \in \mathcal{X} \), \( v_t(x) = \mathbb{E}_\mu_x \left[ \int^\infty_t \gamma^{t-\tau} Z_\tau \, d\tau + \gamma^{\Delta t} v_t(X_\tau) \bigg| X_t = x \right] \), (50) | find a policy \( \pi_{i+1} \) satisfying (26) |

Note that the difference of IEPI from on-policy IPI lies in its Bellman equation (50)—it contains the compensating term \( \nabla v_t(X_\tau) (f_c(\tau) - f_c^\pi(\tau)) \) that naturally emerges due to the difference between the behavior policy \( \mu \) and the target one \( \pi_i \). For \( \mu = \pi_i \), the compensating term becomes identically zero, in which case the Bellman equation (50) becomes (21) in on-policy IPI. For any given policy \( \mu, \) IEPI in fact generates the same result \( \{ \{ v_t, \pi_t \} \}_{t=0}^\infty \) to its on-policy version (Algorithm 1a) under the same initial condition as shown in Theorem 7 (and Remark 7) in Section 4.6.

In what follows, we are particularly interested in IEPI under the \( u\)-AC setting (27)–(29) shown in Algorithm 4b. In this case, the maximization process in the policy improvement is simplified to the update rule (30) also known as the VGB greedy policy (Doya, 2000). On the other hand, the compensation term in \( Z_\tau \) of the Bellman equation (50) is also simplified to \( \nabla v_t(X_\tau) F_c(X_\tau) \xi_\tau^c \) which is linear in the difference \( \xi_\tau^c = U_\tau - \pi_t(X_\tau) \) at time \( \tau \) and contains the function \( \nabla v_t \cdot F_c \) also shown in its policy improvement rule (30) in common. This observation brings our next off-policy IPI method named integral C-policy-iteration (ICPI).

Algorithm 4b: IEPI in the \( u\)-AC Setting (27)–(29)

| Policy Evaluation | Policy Improvement |
|-------------------|--------------------|
| find a \( C^1 \) function \( v_t : \mathcal{X} \to \mathbb{R} \) such that (50) holds for all \( x \in \mathcal{X} \) and for \( Z_\tau \) given by \( Z_\tau = R_\tau^c - \nabla v_t(X_\tau) F_c(X_\tau) \xi_\tau^c \), where \( \xi_\tau^c = U_\tau - \pi_t(X_\tau) \) is the policy difference; | update the next policy \( \pi_{i+1} \) by (30); |

4.4 Integral C-Policy-Iteration (ICPI)

In the \( u\)-AC setting (27)–(29), we now modify IEPI (Algorithm 4b) to make it model-free by employing a function \( c_\pi : \mathcal{X} \to \mathbb{R}^m \) defined for a given admissible policy \( \pi \) as
\[
c_\pi(x) = F^T_c(x) \nabla v_\pi^c(x), \quad (51)
\]
which is continuous by continuity of $F_c$ and $\nabla v_\pi$. Here, the function $c_\pi$ will appear in both policy evaluation and improvement in Common and contains the input-Coupling term $F_c$, so we call it C-function for an admissible policy $\pi$. Indeed, when (27)–(29) are true, the next policy $\pi'$ satisfying (18) for an admissible policy $\pi$ is explicitly given by

$$\pi'(x) = (\sigma \circ c_\pi)(x) = \sigma(c_\pi(x)).$$

In the same way, if $c_i(x) = F_i^T(x) \nabla v_i^T(x)$ is true, then (30) in Algorithm 4b can be replaced by (53), and the compensating term $\nabla v_i(X_\tau) F_c(X_\tau) \xi_{\pi_i}^\tau$ in policy evaluation of IEPI (Algorithm 4b) by $c_i^T(X_\tau) \xi_{\pi_i}^\tau$.

Motivated by the above idea, we propose integral C-policy-iteration (ICPI) whose policy evaluation and improvement are shown in Algorithm 5. In the former, the functions $v_{\pi_i}$ and $c_{\pi_i}$ for the given (admissible) policy $\pi_i$ are estimated by solving the associated off-policy Bellman equation for $v_{\pi_i}$ and $c_i$, and then the next policy $\pi_{i+1}$ is updated using $c_i$ in the latter. In fact, ICPI is a model-free extension of IEPI—where ICPI does not, IEPI obviously needs the knowledge of the input-coupling dynamics $F_c$ to run. A model-free off-policy IPI so-named integral Q-learning by Lee et al. (2012, 2015), which was derived from IEPI under the Lyapunov’s stability framework, also falls into a class of ICPI for the unconstrained total case ($U = \mathbb{R}^m$ and $\gamma = 1$).

Compared with IAPI and IQPI, the advantages of ICPI (at the cost of restricting the RL problem to the $u$-AC one (27)–(29)) are as follows.

(1) As in IEPI, the complicated maximization in the policy improvement of IAPI and IQPI has been replaced by the simple update rule (53), which is a kind of model-free VGB greedy policy (Doya, 2000).

(2) By virtue of the fact that there is no AD function to be estimated in ICPI as in IEPI, the exploration over its smaller space $\mathcal{X} \times \{u_j\}_{j=0}^m$, rather than the entire state-action space $\mathcal{X} \times \mathcal{U}$, is enough to obtain the desired result “$v_i = v_{\pi_i}$ and $c_i = c_{\pi_i}$” in its policy evaluation (see Algorithm 5 and Theorem 7 in the next subsection). Here, $u_j$’s are any vectors in $\mathcal{U}$ such that

$$\text{span}\{u_j - u_{j-1}\}_{j=1}^m = \mathbb{R}^m. \quad (54)$$

Remark 6 One might consider a general version of ICPI by replacing the term $\nabla v_i(x) f_i(x, u)$ in the general IEPI (Algorithm 4a) with an AD function, say $v_i^0(x, u)$. In this case, however, it loses the merits of ICPI over IAPI and IQPI shown above. Furthermore, the solution $(v_i, c_i)$ of the associated Bellman equation is not uniquely determined—a pair of $v_i$ and any $c_i^0(x, u) = c_i^0(x, u) + b(x)$ for a continuous function $b(x)$ is also a solution to (39) for $Z_\tau$ given by

$$Z_\tau = R^\pi_\tau - c_i^T(X_\tau, U_\tau) + c_i^T(X_\tau, \pi_i(X_\tau)). \quad (52)$$

4.5 Mathematical Properties of Off-policy IPI Methods

Now, we show that every off-policy IPI method is effectively same to on-policy IPI in a sense that the sequences $\{v_i\}_{i=0}^\infty$ and $\{\pi_i\}_{i=0}^\infty$ generated satisfy Theorems 3 and 4 under the same assumptions and are equal to those in on-policy IPI under the uniqueness of the next policy $\pi'$ in Assumption 1. In the case of IQPI, we let $v_i = q(\cdot, \pi_i(\cdot))/\kappa_i$ and assume

Assumption 3b For each $i \in \mathbb{Z}_+$, if $\pi_i$ is admissible, then $v_i = q(\cdot, \pi_i(\cdot))/\kappa_i$ is $C^1$ and

$$\lim_{k \to \infty} l_{\pi_i}(x, k; v_i) = 0 \text{ for all } x \in \mathcal{X}.'$$

Theorem 7 Under Assumptions 1 and 3a (or 3b in IQPI), the sequences $\{\pi_i\}_{i=0}^\infty$ and $\{v_i\}_{i=0}^\infty$ generated by any off-policy IPI method (IAPI, IQPI, IEPI, or ICPI) satisfy the properties (P1)–(P3) in Theorem 3. Moreover,

$$a_i = a_{\pi_i} \text{ (IAPI), } q_i = q_{\pi_i} \text{ (IQPI), and } c_i = c_{\pi_i} \text{ (ICPI)}$$

for all $i \in \mathbb{Z}_+$; if Assumption 2 also holds, then $v_i$ converges towards the optimal solution $v_*$ in a sense that $\{v_i\}_{i=0}^\infty$ satisfies the convergence properties (C1)–(C2) in Theorem 5.

Proof. See Appendix B. □

Remark 7 If the policy $\pi'$ satisfying (18) in Assumption 1 is unique for each admissible $\pi$, then Theorem 7 also shows that $\{\pi_i\}_{i=0}^\infty$ and $\{v_i\}_{i=0}^\infty$ generated by any off-policy IPI in this paper are even equivalent to those in on-policy IPI under the same initial $\pi_0$. An example of this is the $u$-AC case (27)–(29), where the next policy $\pi_{i+1}$ is always uniquely given by the VGB greedy policy (30) for given $v_i \in \mathcal{V}_\pi$.  

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14 The name ‘integral Q-learning’ does not imply that it is involved with our Q-function (44). Instead, its derivation was based on the value function with singularly-disturbed actions (Lee et al., 2012).

15 When $\mathcal{U}$ contains the zero vector, any linearly independent subset $\{u_j\}_{j=1}^m$ and $u_0 = 0$ is an example of such $u_j$’s in (54).
Table 1
Summary of the off-policy IPI methods.

| Name  | Model-free | $R_z$ or $R_z^\tau$ | Functions involved | Search Space | Algorithm No. | Constraint(s) |
|-------|------------|----------------------|-------------------|--------------|---------------|--------------|
| IAPI  | O          | $R_z$                | $v_\pi$ and $a_\pi$ | $\mathcal{X} \times \mathcal{U}$ | 2             | (40)         |
| IQPI  | O          | $R_z$                | $q_\pi$           | $\mathcal{X} \times \mathcal{U}$ | 3a            | X            |
|       |            |                      |                   |              | 3b            | (47)         |
| IEPI  | $\triangle$| $R_z^\tau$           | $v_\pi$           | $\mathcal{X}$ | 4a            | X            |
|       |            |                      |                   |              | 4b            | (27)–(29)    |
| ICPI  | O          | $R_z^\tau$           | $v_\pi$ and $c_\pi$ | $\mathcal{X} \times \{u_j\}$ | 5             | (27)–(29)    |

4.6 Summary and Discussions

The off-policy IPI methods presented in this section are compared and summarized in Table 1. As shown in Table 1, all of the off-policy IPI methods are model-free except IEPI which needs the full-knowledge of a input-coupling dynamical $f_\pi$ in (25) to run; here, ICPI is actually a model-free version of the $u$-AC IEPI (Algorithm 4b). While IAPI and IQPI explore the whole state-action space $\mathcal{X} \times \mathcal{U}$ to learn their respective functions $(v_\pi, a_\pi)$ and $q_\pi$, IEPI and ICPI search only the significantly smaller spaces $\mathcal{X}$ and $\mathcal{X} \times \{u_j\}^m_{j=0}$, respectively. This is due to the fact that IEPI and ICPI both learn no AD function such as $a_\pi$ and $q_\pi$ as shown in the fourth column of Table 1. While IAPI and IQPI employ the reward $R_z$, both IEPI and ICPI use the $\pi_i$-reward $R_z^\tau_i$ at each $i$-th iteration.

Table 1 also summarizes the constraint(s) on each algorithm. IAPI has the constraint (40) on $a_\pi$ and $\pi_i$ in the policy evaluation that reflects the equality $a_\pi(x, \pi_i(x)) = 0$ similarly to advantage updating (Baird III, 1993; Doya, 2000). ICPI is designed under the $u$-AC setting (27)–(29), which gives:

1. the uniqueness of the target solution $(v_1, c_1) = (v_{1\pi}, c_{1\pi})$ of the Bellman equation (39) for $Z_\pi$, given by (52);
2. the exploration of a smaller space $\mathcal{X} \times \{u_j\}^m_{j=0}$, rather than the whole state-action space $\mathcal{X} \times \mathcal{U}$;
3. the simple update rule (53) in policy improvement, the model-free version of the VGB greedy policy (Doya, 2000), in place of the complicated maximization over $\mathcal{U}$ for each $x \in \mathcal{X}$ such as (41) and (43) in IAPI and IQPI.

The special IEPI scheme (Algorithm 4b) designed under the $u$-AC setting (27)–(29) also updates the next policy $\pi_{i+1}$ via the simple policy improvement update rule (30) (a.k.a. the VGB greedy policy (Doya, 2000)), rather than performing the maximization (26). IQPI can be also simplified to Algorithm 3b under the different weighting (or discounting) by $\beta (\neq \gamma)$ and the gain setting $k_1 = k_2 = \kappa (\kappa \doteq \ln(\beta/\gamma))$ shown in (47). In this case, $\beta \in (0, \infty)$ determines the gain of the integral in policy evaluation and scales $v_\pi$ with respect to $a_\pi$ in $q_\pi$ (see (48) and (49)).

For any of the model-free methods, if $U_z = \pi_i(X_z)$, rather than $U_z = \mu(\tau, X_z, u)$, then their AD parts summarized in Table 2 and shown in their off-policy Bellman equations (or their $Z_\pi$’s) become all zeros and thus no longer detectable—the need for the behavior policy $\mu$ different from the target policy $\pi_i$ to obtain or estimate the respective functions in such AD terms. In the case of IEPI, if $U_z = \pi_i(X_z)$ for all $\tau \in [t, t']$, then it becomes equal to “on-policy IPI with its policy improvement (22) replaced by (26).”

| Name  | The AD-Parts |
|-------|--------------|
| IAPI  | $a_i(X_z, U_z) - a_i(X_z, \pi_i(X_z))$ |
| IQPI  | $\kappa_2 \cdot (q_i(X_z, U_z) - q_i(X_z, \pi_i(X_z)))$ |
| ICPI  | $c_i(X_z)\xi_i$ |

5 Inverted-Pendulum Simulation Examples

To support the theory and verify the performance, we present the simulation results of the IPI methods applied to the 2nd-order inverted-pendulum model $(n = 2$ and $m = 1)$:

$$\ddot{\theta}_z = -0.01\dot{\theta}_z + 9.8 \sin \theta_z - U_z \cos \theta_z,$$

where $\theta_z, U_z \in \mathbb{R}$ are the angular position of and the external torque input to the pendulum at time $\tau$, respectively, with the torque limit given by $|U_z| \leq U_{\max}$ for $U_{\max} = 5 \text{ Nm}$. Note that this model is exactly same to that used by Doya (2000) except that the action $U_z$, the torque input, is coupled with the term ‘$\cos \theta_z$’ rather than the constant ‘1,’ which makes our problem more realistic and challenging. Letting $X_z \doteq [\theta_z, \dot{\theta}_z]^T$, then the inverted-pendulum model can be expressed as (1) and (27) with

$$f_\delta(x) = \begin{bmatrix} x_2 \\ 9.8 \sin x_1 - 0.01x_2 \end{bmatrix}$$

and

$$F_\xi(x) = \begin{bmatrix} 0 \\ -\cos x_1 \end{bmatrix},$$

where $x = [x_1 \ x_2]^T \in \mathbb{R}^2$. Here, our learning objective is to make the pendulum swing up and eventually settle down at the upright position $\theta_z = 2\pi k$ for some $k \in \mathbb{Z}$. The reward $R_z$ to achieve such a goal under the limited torque was therefore set to (29) and (31) with $U = [-U_{\max}, U_{\max}]$, $\Gamma = 1$, and the functions $R_0(x)$ and $s(\xi)$ given by

$$R_0(x) = 10^2 \cos x_1$$

and

$$s(\xi) = U_{\max} \tanh(\xi/U_{\max});$$
the sigmoid function \( s \) with \( \Gamma = 1 \) then gives the following expressions of the functions \( \sigma(\xi) \) in (30) and \( S(u) \) in (29):

\[
\sigma(\xi) = U_{\text{max}} \tanh(\xi/U_{\text{max}}),
\]

\[
S(u) = (U_{\text{max}}^{2}/2) \cdot \ln \left( u_{+} \cdot u_{-} \right),
\]

where \( u_{\pm} = 1 \pm u/U_{\text{max}} \). Here, note that \( S(u) \) is finite for all \( u \in \mathcal{U} \) and has its maximum at the endpoints \( u = \pm U_{\text{max}} \) as \( S(\pm U_{\text{max}}) = (U_{\text{max}}^{2})/2 \approx 17.3287 \). The initial policy \( \pi_{0} \) was given as \( \pi_{0} = 0 \) and for its admissibility \( v_{x0} \in \mathcal{V}_{a} \), we set the discount factor as \( \gamma = 0.1 \), less than 1. This is a high gain on the state-reward \( R_{0}(x) = 10^{2} \cos x_{1} \) and low discounting (\( \gamma = 0.1 \)) scheme, which made it possible to achieve the learning objective merely after the first iteration.

Under the above \( u \)-AC framework, we simulated the four off-policy methods (Algorithms 2, 3b, 4b, and 5) with their parameters \( \Delta t = 10 \) [ms] and \( \beta = 1 \). On-policy IPI in Section 3 is a special case \( \mu = \pi \) of IEPI and thus omitted. The behavior policy \( \mu \) used in the simulations was \( \mu = 0 \) for IEPI and \( \mu(t,x,u) = u \) for the others; the next target policy \( \pi_{i+1} \) was given by \( \pi_{i+1} = \sigma(y_{i}(x)) \), where \( y_{i}(x) = F_{i}^{T}(x)\nabla v_{i}^{T}(x) \) in IEPI, \( y_{i}(x) = c_{i}(x) \) in ICPI; in IAPI and IQPI, \( y_{i}(x) \) is approximately equal to the output of a radial basis function network (RBFN) to be trained by policy improvement using \( a_{i} \) and \( q_{i} \), respectively. The functions \( v_{i}, a_{i}, q_{i}, \) and \( c_{i} \) were all approximated by RBFNs as well. Instead of the whole spaces \( \mathcal{X} \) and \( \mathcal{X} \times \mathcal{U} \), we considered their compact regions \( \Omega_{x} \cong [-\pi, \pi] \times [-6, 6] \) and \( \Omega_{x} \times \mathcal{U} \) in our whole simulations; since our inverted-pendulum system and the value function are \( 2\pi \)-periodic in the angular position \( x_{1} \), the state value \( x \in \mathcal{X} \) was normalized to \( x \in [-\pi, \pi] \times \mathbb{R} \) whenever input to the RBFNs. The details about the RBFNs and the implementation methods of the policy evaluation and improvement are shown in Appendix D. Every IPI method ran up to the 10th iteration.

Fig. 1 shows the estimated values of \( v_{x}(x) \) for \( x \in \Omega_{x} \) after the learning has been completed (at \( i = 10 \)), where after convergence, \( v_{x} \) may be considered to be an approximation of the optimal value function \( v_{x}^{*} \). Although there are some small ripples in the case of IQPI, the final value function estimates shown in Fig. 1 that are generated by different IPI methods (IEPI, ICPI, IAPI, and IQPI) are all consistent to each other. We also generated the state trajectories \( \mathbf{X} \) shown in Fig. 2 for the initial condition \( \theta_{0} = (1 + \epsilon_{0})\pi \) with \( \epsilon_{0} = 0.1 \) and \( \theta_{0} = 0 \) under the estimated policy \( \pi_{i} \) obtained at the last iteration (\( i = 10 \)) of each IPI method. As shown in Fig. 2, all of the policies \( \pi_{10} \) obtained by different IPI methods generate the state trajectories that are almost consistent with each other—they all achieved the learning objective at around \( t = 4 \) [s], and the whole state trajectories generated are almost same (or very close).
to each other. Also note that the IPI methods achieved our learning objective without using an initial stabilizing policy that is usually required in the optimal control setting under the total discounting \( \gamma = 1 \) (e.g., Abu-Khalaf and Lewis, 2005; Vrabie and Lewis, 2009; Lee et al., 2015).

6 Conclusions

In this paper, we proposed the on-policy IPI scheme and four off-policy IPI methods (IAPI, IQPI, IEPI, and ICPI) which solve the general RL problem formulated in CTS. We proved their mathematical properties of admissibility, monotone improvement, and convergence, together with the equivalence of the on- and off-policy methods. It was shown that on-policy IPI can be made partially model-free by modifying its policy improvement, and the off-policy methods are partially model-free (IEPI), completely model-free (IAPI, IQPI), or model-free but only implementable in the \( \nu\)-AC setting (ICPI). The off-policy methods were discussed and compared with each other as listed in Table 1. Numerical simulations were performed with the 2nd-order inverted-pendulum model to support the theory and verify the performance, and the results with all algorithms were consistent and approximately equal to each other. Unlike the IPI methods in the stability-based framework, an initial stabilizing policy is not required to run any of the proposed IPI methods. This work also provides the ideal PI forms of RL in CTS such as advantage updating (IAPI), Q-learning in CTS (IQPI), VGB greedy policy improvement (on-policy IPI and IEPI under \( \nu\)-AC setting), and the model-free VGB greedy policy improvement (ICPI). Though the proposed IPI methods are not online incremental RL algorithms, we believe that this work provides the theoretical background and intuition to the (online incremental) RL methods to be developed in the future and developed so far in CTS.

References

Abu-Khalaf, M. and Lewis, F. L. Nearly optimal control laws for nonlinear systems with saturating actuators using a neural network HJB approach. *Automatica*, 41(5):779–791, 2005.

Aleksandar, B., Lever, G., and Barber, D. Nesterov’s accelerated gradient and momentum as approximations to regularised up-date descent. *arXiv preprint arXiv:1607.01981v2*, 2016.

Anderson, B. and Moore, J. B. Optimal control: linear quadratic methods. Prentice-Hall, Inc., 1989.

Baird III, L. C. Advantaging update. Technical report, DTIC Document, 1993.

Beard, R. W., Saridis, G. N., and Wen, J. T. Galerkin approximations of the generalized Hamilton-Jacobi-Bellman equation. *Automatica*, 33(12):2159–2177, 1997.

Bessaga, C. On the converse of banach “fixed-point principle”. *Colloquium Mathematicae*, 7(1):41–43, 1959.

Doya, K. Reinforcement learning in continuous time and space. *Neural computation*, 12(1):219–245, 2000.

Farahmand, A. M., Ghavamzadeh, M., Mannor, S., and Szepesvári, C. Regularized policy iteration. In *Advances in Neural Information Processing Systems*, pages 441–448, 2009.

Folland, G. B. *Real analysis: modern techniques and their applications*. John Wiley & Sons, 1999.

Frémaux, N., Sprekeler, H., and Gerstner, W. Reinforcement learning using a continuous time actor-critic framework with spiking neurons. *PLoS Comput. Biol.*, 9(4):e1003024, 2013.

Haddad, W. M. and Chellaboina, V. *Nonlinear dynamical systems and control: a Lyapunov-based approach*. Princeton University Press, 2008.

Howard, R. A. *Dynamic programming and Markov processes*. Prentice Hall and John Wiley & Sons Inc., 1960.

Kirk, W. and Sims, B. *Handbook of metric fixed point theory*. Springer Science & Business Media, 2013.

Kleinman, D. On an iterative technique for Riccati equation computations. *IEEE Trans. Autom. Cont.*, 13(1):114–115, 1968.

Lagoudakis, M. G. and Parr, R. Least-squares policy iteration. *J. Mach. Learn. Res.*, 4(Dec):1107–1149, 2003.

Leake, R. J. and Liu, R.-W. Construction of suboptimal control sequences. *SIAM Journal on Control*, 5(1):54–63, 1967.

Lee, J. Y., Park, J. B., and Choi, Y. H. Integral Q-learning and explorized policy iteration for adaptive optimal control of continuous-time linear systems. *Automatica*, 48(11):2850–2859, 2012.

Lee, J. Y., Park, J. B., and Choi, Y. H. On integral generalized policy iteration for continuous-time linear quadratic regulations. *Automatica*, 50(2):475–489, 2014.

Lee, J. Y., Park, J. B., and Choi, Y. H. Integral reinforcement learning for continuous-time input-affine nonlinear systems with simultaneous invariant explorations. *IEEE Trans. Neural Networks and Learning Systems*, 26(5):916–932, 2015.

Lewis, F. L. and Vrabie, D. Reinforcement learning and adaptive dynamic programming for feedback control. *IEEE Circuits and Systems Magazine*, 9(3):32–50, 2009.

Loeb, P. A. and Talvila, E. Lusin’s Theorem and Bochner integration. *Scientiae Mathematicae Japonicae*, 10:55–62, 2004.

Luo, B., Wu, H.-N., Huang, T., and Liu, D. Data-based approximate policy iteration for affine nonlinear continuous-time optimal control design. *Automatica*, 50(12):3281–3290, 2014.

Maci, H. R., Szepesvári, C., Bhatnagar, S., and Sutton, R. S. Toward off-policy learning control with function approximation. In *Proceedings of the 27th International Conference on Machine Learning (ICML-10)*, pages 719–726, 2010.

Mehta, P. and Meyn, S. Q-learning and pontryagin’s minimum principle. In *Proc. IEEE Int. Conf. Decision and Control, held jointly with the Chinese Control Conference (CDC/CCC)*, pages 3598–3605, 2009.

Modares, H., Lewis, F. L., and Jiang, Z.-P. Optimal output-feedback control of unknown continuous-time linear systems using off-policy reinforcement learning. *IEEE Trans. Cybern.*, 46(11):2401–2410, 2016.

Murray, J. J., Cox, C. J., Lendaris, G. G., and Saeks, R. Adaptive dynamic programming. *IEEE Trans. Syst. Man Cybern. Part C- Appl. Rev.*, 32(2):140–153, 2002.

Powell, W. B. Approximate dynamic programming: solving the curses of dimensionality. Wiley-Interscience, 2007.

Saridis, G. N. and Lee, C. S. G. An approximation theory of optimal control for trainable manipulators. *IEEE Trans. Syst. Man Cybern.*, 9(3):152–159, 1979.

Sutton, R. S. and Barto, A. G. *Reinforcement learning: an introduction*. Second Edition in Progress, MIT Press, Cambridge, MA (available at http://incompleteideas.net/sutton), 2017.

Thomson, B. S., Bruckner, J. B., and Bruckner, A. M. *Elementary real analysis*. Prentice Hall, 2001.

Vamvoudakis, K. G., Vrabie, D., and Lewis, F. L. Online adaptive algorithm for optimal control with integral reinforcement learning. *Int. J. Robust and Nonlinear Control*, 24(17):2686–2710, 2014.
Appendix

A Proof of Lemma 2

In this proof, we first focus on the case \( N = 1 \) and then generalize the result. By Theorem 2, \( T v_* \) is an admissible value function and satisfies \( v_* \leq T v_* \), but \( T v_* \leq v_* \) since \( v_* \) is the optimal value function. Therefore, \( T v_* = v_* \) and \( v_* \) is a fixed point of \( T \).

**Claim A.1** \( v_* \) is the unique fixed point of \( T \).

**Proof.** To show the uniqueness, suppose \( v_{\pi} \in \mathcal{V} \) is another fixed point of \( T \) and let \( \pi' \) be the next policy obtained by policy improvement with respect to the fixed point \( v_* \). Then, \( \pi' \) is admissible by Theorem 2, and it is obvious that

\[
-\ln \gamma \cdot (Tv_{\pi})(x) = h(x, \pi'(x), \nabla (Tv_{\pi})(x)) \quad x \in \mathcal{X},
\]

by \( v_{\pi'} = T v_{\pi} \) and \"(11)\" for the admissible policy \( \pi' \). The substitution of \( Tv_{\pi} = v_* \) into it results in

\[
-\ln \gamma \cdot v_{\pi}(x) = h(x, \pi'(x), \nabla v_{\pi}(x)) = \max_{u \in \mathcal{U}} h(x, u, \nabla v_{\pi}(x))
\]

for all \( x \in \mathcal{X} \), the HJBE. Therefore, \( v_{\pi} = v_* \) by Corollary 3 and Assumptions 1 and 2, a contradiction, implying that \( v_* \) is the unique fixed point of \( T \). \( \square \)

Now, we generalize the result to the case with any \( N \in \mathbb{N} \). Since \( v_* \) is the fixed point of \( T \), we have

\[
T^N v_* = T^{N-1}[Tv_*] = T^{N-1}v_* = \cdots = Tv_* = v_*,
\]

showing that \( v_* \) is also a fixed point of \( T^N \) for any \( N \in \mathbb{N} \). To prove that \( v_* \) is the unique fixed point of \( T^N \) for all \( N \), suppose that there is some \( M \in \mathbb{N} \) and \( v \in \mathcal{V} \) such that \( T^M v = v \). Then, it implies \( T v = v \) since we have

\[
v \leq T v \leq T^2 v \leq \cdots \leq T^M v = v
\]

by the repetitive applications of Theorem 2. Therefore, we obtain \( v = v_* \) by Claim A.1, which completes the proof. \( \square \)

B Proof of Theorem 7

For the proof, we employ the following lemma regarding the conversion from an time-integral to an algebraic equation. Its proof is given in Appendix C.

**Lemma B.1** Let \( v : \mathcal{X} \to \mathbb{R} \) and \( Z : \mathcal{X} \times \mathcal{U} \to \mathbb{R} \) be any \( C^1 \) and continuous functions, respectively. If there exist \( \Delta t > 0 \), a weighting factor \( \beta > 0 \), and \"an AD policy \( \mu \) over a non-empty subset \( \mathcal{U}_0 \subseteq \mathcal{U} \)\" such that for each \( (x, u) \in \mathcal{X} \times \mathcal{U}_0 \),

\[
v(x) = \mathbb{E}_\mu \left[ \int_t^{t+\Delta t} \beta^{r-1} Z_r \, dt + \beta^{\Delta t} v(X'_r) \right] \quad x_t = x, U_t = u,
\]

(B.1)

where \( Z_r = Z(X_r, U_r) \) for \( r \geq t \), then

\[-\ln \beta \cdot v(x) = Z(x, u) + \nabla v(x)f(x, u)\]

holds for all \( (x, u) \in \mathcal{X} \times \mathcal{U}_0 \).

The applications of Lemma B.1 to the Bellman equations of the off-policy IPI methods (IAPI, IQPI, IEPI, and ICPI) provides the following claim.

**Claim B.1** If \( \pi_i \) is admissible, then \( v_i \) and \( \pi_{i+1} \) obtained by the i-th policy evaluation and improvement of any off-policy IPI method satisfy \( v_i = v_{\pi_i} \) and (23). Moreover,

\[
a_i = a_{\pi_i} \quad (IAPI), \quad q_i = q_{\pi_i} \quad (IQPI), \quad \text{and} \quad c_i = c_{\pi_i} \quad (ICPI).
\]

Suppose that \( \pi_i \) is admissible. Then, if Claim B.1 is true, then \( \pi_{i+1} \) in any off-policy IPI method satisfies (23) and hence Theorem 2 with Assumption 1 proves that \( \pi_{i+1} \) is also admissible and satisfies \( v_{\pi_i} \leq v_{\pi_{i+1}} \leq v_{\pi_*} \). Since \( v_0 \) is admissible in the off-policy IPI methods, mathematical induction proves the first part of the theorem. Moreover, now that we have the properties (P1)-(P3), if Assumption 2 additionally holds, then we can easily prove the convergence properties (C1)-(C2) in Theorem 4 by following its proof.

**Proof of Claim B.1.** (IAPI/IQPI) Applying Lemma B.1 with \( \mathcal{U}_0 = \mathcal{U} \) to (39) in IAPI and to (42) in IQPI and then substituting the definition (5) of \( h(x, u, p) \) show that \( (v_i, a_i) \) in IAPI and \( (v_i, q_i) \) in IQPI (with \( v_i = \tilde{q}_i(\cdot, \pi_i(\cdot))/\kappa_1 \)) satisfy

\[
-\ln \gamma \cdot v_i(x) = h(x, u, \nabla v_i(x)) - a_i(x, u) + a_i(x, \pi_i(x))
\]

(B.2)

\[
-\ln \gamma \cdot v_i(x) = h(x, u, \nabla v_i(x)) - \frac{\kappa_2}{\kappa_1} (q_i(x, u) - \kappa_1 v_i(x))
\]

(B.3)

for all \( (x, u) \in \mathcal{X} \times \mathcal{U} \), respectively. Furthermore, the substitutions of \( u = \pi_i(x) \) into (B.2) and (B.3) yield

\[
-\ln \gamma \cdot v_i(x) = h(x, \pi_i(x), \nabla v_i(x)) \quad \forall x \in \mathcal{X},
\]

(B.4)

which implies \( v_i = v_{\pi_i} \) by Theorem 1 and Assumption 3a. Next, substituting \( v_i = v_{\pi_i} \) into (B.2) and (B.3) and then rearranging it with (12) and (40) in the IAPI case result in

\[
a_i(x, u) = h_{\pi_i}(x, u) + \ln \gamma \cdot v_{\pi_i}(x),
\]

\[
q_i(x, u) = \kappa_1 (v_{\pi_i}(x) + a_i(x, u)/\kappa_2),
\]
and hence we obtain \( a_i = \alpha_i \) and \( q_i = \gamma_i \) by the definitions (37) and (44). By this and the respective policy improvement of IAPI and IQPI, it is obvious that the next policy \( \pi_{i+1} \) in each algorithm satisfies

\[
\forall x \in \mathcal{X} : \left\{ \begin{array}{l}
\pi_{i+1}(x) \in \arg \max_{u \in \mathcal{U}} a_{\pi_i}(x, u) \text{ (IAPI)}; \\
\pi_{i+1}(x) \in \arg \max_{u \in \mathcal{U}} q_{\pi_i}(x, u) \text{ (IQPI)}.
\end{array} \right.
\]

Since they are equivalent to (20) with \( \pi = \pi_i \) and some special choices of \( \delta_\pi \) and \( \kappa > 0 \), and (20) is equivalent to (18), \( \pi_{i+1} \) in both IAPI and IQPI satisfy (23) in (P2).

**IEPI** By (C.2) in Appendix C and (25), the Bellman equation (50) can be expressed as

\[
0 = \mathbb{E}_\mu \left[ \int_t^{t'} \gamma^{t-t'} \phi_i(X_\tau) \, d\tau \bigg| X_t = x \right],
\]

where \( \phi_i : \mathcal{X} \to \mathbb{R} \) is given by

\[
\phi_i(x) \doteq R(x, \pi_i(x)) + \ln \gamma \cdot v_i(x) + \nabla v_i(x) f(x, \pi_i(x)),
\]

which is obviously continuous since so are all functions contained in it. Thus, the term “\( \gamma^{t-t'} \phi_i(X_{t'}) \)” is integrable over \([t, t']\), and Claim C.1 in Appendix C with \( w(x, u) = \phi_i(x) \) for all \((x, u) \in \mathcal{X} \times \mathcal{U}\) implies \( \phi_i = 0 \), which results in (B.4) and hence \( v_i = v_{\pi_i} \) by Theorem 1 and Assumption 3a. Since the policy improvement (26) in IEPI is equivalent to solving (22), it is equivalent to (23) by \( v_i = v_{\pi_i} \) and (12).

**ICPI** Applying Lemma B.1 to policy evaluation of ICPI and rearranging it using (5) and (27), we obtain for each \((x, u) \in \mathcal{X} \times \mathcal{U}_0\):

\[
- \ln \gamma \cdot v_i(x) = R(x, \pi_i(x)) - c_i^T(x)(u - \pi_i(x)) + \nabla v_i(x) f(x, u)
= h(x, \pi_i(x), \nabla v_i(x))(u - \pi_i(x))^T \psi(x),
\]

where \( \psi(x) \doteq F^T(x) \nabla v_i^T(x) - c_i(x) \). Next, let \( x \in \mathcal{X} \) be an arbitrary fixed value. Then, for each \( j \in \{1, 2, \cdots, m\} \), subtracting (B.5) for \( u = u_j - 1 \) from the same equation but for \( u = u_j \) yields \( 0 = (u_j - u_{j-1}) \psi(x) \). This can be rewritten in the following matrix-vector form:

\[
(E_{1:m} - E_{0:m-1}) \psi(x) = 0,
\]

where \( E_{k:m} \doteq [u_k \ 0 \ 0 \cdots \ u_{k+m-1}] \) for \( 0 \leq k \leq l \leq m \) is the \( m \times (l-k+1) \)-matrix constructed by the column vectors \( u_k, u_{k+1}, \cdots, u_l \). Since (54) implies that \( \{u_j - u_{j-1}\}_{j=1}^m \) is a basis of \( \mathbb{R}^m \), we have rank \( (E_{1:m} - E_{0:m-1}) = m \) and by (B.6), \( \psi(x) = 0 \). Since \( x \in \mathcal{X} \) is arbitrary, \( \psi(x) = 0 \).

Now that we have \( \psi(x) = 0 \) \( \forall x \in \mathcal{X} \), (B.5) becomes (B.4) and thus \( v_i = v_{\pi_i} \) by Theorem 1 and Assumption 3a. This also implies \( c_i = c_{\pi_i} \) by \( \psi = 0 \) and the definition of \( \psi \).

Moreover, by \( v_i = v_{\pi_i} \), the policy improvement (53) is equal to \( \pi_{i+1}(x) = \sigma(c_{\pi_i}(x)) \) for all \( x \in \mathcal{X} \), which is the closed-form solution of (23) in the \( u\)-AC setting (27)–(29).

**C Proof of Lemma B.1**

The proof is done using the following claim.

**Claim C.1** Let \( \mu \) be a policy starting at \( t \) and \( w : \mathcal{X} \times \mathcal{U} \) be a continuous function. If there exist \( \Delta t > 0 \) and \( \beta > 0 \) such that for all \( x \in \mathcal{X} \),

\[
0 = \mathbb{E}_\mu \left[ \int_t^{t'} \beta^{t-t'} w(X_\tau, U_\tau) \, d\tau \bigg| X_t = x \right],
\]

then, \( w(x, \mu(t, x)) = 0 \) for all \( x \in \mathcal{X} \).

By the standard calculus, for any \( \Delta t > 0 \) and \( \beta > 0 \),

\[
\beta^{t-t'}w(x_\tau)\bigg|_t^{t'} = \int_t^{t'} \frac{d}{d\tau} \left( \beta^{t-t'}w(x_\tau) \right) \, d\tau = \int_t^{t'} \beta^{t-t'} \ln \beta \cdot v(x_\tau) + \nabla v(x_\tau) f(x_\tau, U_\tau) \, d\tau.
\]

Hence, (B.1) can be rewritten for any \((x, u) \in \mathcal{X} \times \mathcal{U}_0 \) as

\[
0 = \mathbb{E}_\mu \left[ \int_t^{t'} \beta^{t-t'} w(X_\tau, U_\tau) \, d\tau \bigg| X_t = x, U_t = u \right],
\]

where \( w(x, u) \doteq Z(x, u) + \ln \beta \cdot v(x) + \nabla v(x) f(x, u) \). Here, \( w \) is continuous since so are \( v, \nabla v, Z, \) and \( f \), and thus the term “\( \beta^{t-t'} w(X_\tau, U_\tau) \)” is integrable over the compact time interval \([t, t']\). Now, fix \( u \in \mathcal{U}_0 \). Then, one can see that

1. \( \mu(\cdot, \cdot, u) \) is obviously a policy;
2. the condition \( U_t = u \) in (C.3) is violated for fixed \( u \);
3. \( \mu(t, x, u) = u \) holds for all \( x \in \mathcal{X} \).

Hence, by Claim C.1, we obtain

\[
0 = w(x, \mu(t, x, u)) \doteq w(x, u) \quad \text{for all} \quad x \in \mathcal{X}.
\]

Since \( u \in \mathcal{U}_0 \) is arbitrary, we finally have \( w(x, u) = 0 \) for all \((x, u) \in \mathcal{X} \times \mathcal{U}_0 \), which completes the proof.

**Proof of Claim C.1.** To prove the claim, let \( x_0 \in \mathcal{X} \) and \( x_k \equiv \mathbb{E}_\mu[x_{k-1} | X_t = x_{k-1}] \) for \( k = 1, 2, 3, \cdots \). Then, (C.1) obviously holds for each \( x = x_k \in \mathcal{X} (k \in \mathbb{Z}_+) \). Denote

\[
t_0 \doteq t \text{ and } t_k \doteq t + k \Delta t \quad \text{for any} \quad k \in \mathbb{N},
\]

and define \( \tilde{\mu}_k : [t_k, \infty) \times \mathcal{X} \to \mathcal{U} \) for each \( k \in \mathbb{Z}_+ \) as

\[
\tilde{\mu}_k(\tau, x) \doteq \mu(\tau - t_k + t, x) \quad \text{for any} \quad \tau \geq t_k \text{ and any} \quad x \in \mathcal{X}.
\]
Then, obviously, $\bar{\mu}_k$ is a policy that starts at time $t_k$. Moreover, since in our framework, the non-stationarity, i.e., the explicit time-dependency, comes only from, if any, that of the applied policy, we obtain by the above process and (C.1) that

$$0 = \mathbb{E}[\bar{\mu}_k \left[ \int_{t_k}^{t_{k+1}} \beta^{t-t} w(X_{\tau}, U_{\tau}) d\tau \bigg| X_{t_k} = x_k \right]] \tag{C.4}$$

for all $k \in \mathbb{Z}_+$. Next, construct $\bar{\mu} : [t, \infty) \times \mathcal{X} \to \mathcal{U}$ by

$$\bar{\mu}(\tau, x) = \bar{\mu}_k(\tau, x) \text{ for } \tau \in [t_k, t_{k+1}) \text{ and } x \in \mathcal{X}.$$ 

Then, for each fixed $x \in \mathcal{X}$, $\bar{\mu}(\cdot, x)$ is right continuous since for all $k \in \mathbb{Z}_+$, so is $\bar{\mu}_k(\cdot, x)$ on each time interval $[t_k, t_{k+1})$. In a similar manner, for each fixed $\tau \in [t, \infty)$, $\bar{\mu}(\tau, \cdot)$ is continuous over $\mathcal{X}$ since for all $k \in \mathbb{Z}_+$, so is $\bar{\mu}_k(\cdot, \tau)$ for each fixed $\tau \in [t_k, t_{k+1})$. Moreover, since $\bar{\mu}_k$ is a policy starting at $t_k$, the state trajectory $\mathbb{E}[X | X_{t_k} = x_k]$ is uniquely defined over $[t_k, t_{k+1})$. Therefore, noting that $x_k$ is represented (by definitions and the recursive relation) as

$$x_k = \mathbb{E}[\bar{\mu}_{k-1} \left[ X_{t_k} | X_{t_k-1} = x_{k-1} \right]] \text{ for any } k \in \mathbb{N},$$

we conclude that the state trajectory $\mathbb{E}_{\bar{\mu}}[X | X_t = x_0]$ is also uniquely defined for each $x_0 \in \mathcal{X}$ over $[t, \infty)$, implying that $\bar{\mu}$ is a policy starting at time $t$. Finally, using the policy $\bar{\mu}$, we obtain from (C.4)

$$0 = \sum_{k=0}^{\infty} \left( \mathbb{E}[\bar{\mu}_k \left[ \int_{t_k}^{t_{k+1}} \beta^{t-t} w(X_{\tau}, U_{\tau}) d\tau \bigg| X_{t_k} = x_k \right]] \right)$$

$$= \mathbb{E}[\bar{\mu} \left[ \int_{t}^{\infty} \beta^{t-t} w(X_{\tau}, U_{\tau}) d\tau \bigg| X_{t} = x_0 \right]],$$

which implies $W(t; x_0) = 0$ for all $t \geq 0$. Since

$$\frac{\partial}{\partial t} \int_{t}^{\infty} \beta^{t} w(X_{\tau}, U_{\tau}) d\tau = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{t}^{t+\Delta t} \beta^{t} w(X_{\tau}, U_{\tau}) d\tau$$

$$= w(X_{t}, U_{t}),$$

by (right) continuity of $w$, $X_{\tau}$, and $U_{\tau}$, we therefore obtain

$$0 = \frac{\partial W(t; x_0)}{\partial t} = - \ln \beta \cdot W(t; x_0) + \beta^{-t} \cdot w(x_0, \bar{\mu}(t, x_0))$$

and thereby, $w(x_0, \mu(t, x_0)) = 0$. Since $x_0 \in \mathcal{X}$ is arbitrary, it implies $w(x, \mu(t, x)) = 0$ for all $x \in \mathcal{X}.$ □

D Inverted-Pendulum Simulation Methods

D.1 Linear Function Approximations by RBFNs

To describe the methods in a unified manner, we denote any network input by $z$ and its corresponding input space by $\mathcal{Z}$.

They correspond to $z = (x, u)$ and $Z = \mathcal{X} \times \mathcal{U}$ when the network is AD, and $z = x$ and $Z = \mathcal{X}$ when it is not. In the simulations in Section 5, the functions $v_i, a_i, q_i$, and $c_i$ are all approximated by RBFNs as shown below:

$$v_i(x) = \tilde{v}_i(z; \theta^v_i) = \phi^v(\tilde{z}; \theta^v_i),$$

$$c_i(x) = \tilde{c}_i(z; \theta^c_i) = \phi^c(\tilde{z}; \theta^c_i),$$

$$a_i(x, u) = \tilde{a}(z; \theta^a_i) = \phi^a(\tilde{z}; \theta^a_i),$$

$$q_i(x, u) = \tilde{q}(z; \theta^q_i) = \phi^q(\tilde{z}; \theta^q_i).$$

(D.1)

where $\tilde{z} \in \mathcal{Z}$ represents the input $z \in \mathcal{Z}$ to each network whose state-component $x$ is normalized to $\bar{x} \in [-\pi, \pi] \times \mathbb{R}$ by adding $\pm 2\pi k$ to its first component $x_1$ for some $k \in \mathbb{Z}_+$: $\theta^v_i, \theta^c_i \in \mathbb{R}^N$, and $\theta^a_i, \theta^q_i \in \mathbb{R}^M$ are the weight vectors of the networks; $N, M \in \mathbb{N}$ are the numbers of hidden neurons; the RBFs $\phi : \mathcal{Z} \to \mathbb{R}^N$ with $\mathcal{Z} = \mathcal{X}$ and $\phi_{AD} : \mathcal{Z} \to \mathcal{R}^M$ with $\mathcal{Z} = \mathcal{X} \times \mathcal{U}$ are defined as

$$\phi_j(x) = e^{-\|x-x_j\|^2_2} \text{ and } \phi_{AD,j}(z) = e^{-\|z-z_j\|^2_2}.$$ 

Here, $\phi_j$ and $\phi_{AD,j}$ are the $j$-th components of $\phi$ and $\phi_{AD}$, respectively; $\|x\|_{\Sigma_1}$ and $\|z\|_{\Sigma_2}$ are weighted Euclidean norms defined as $\|x\|_{\Sigma_1} = (x^T \Sigma_1 x)^{1/2}$ and $\|z\|_{\Sigma_2} = (z^T \Sigma_2 z)^{1/2}$ for the diagonal matrices $\Sigma_1 = \text{diag}(1, 0.5)$ and $\Sigma_2 = \text{diag}(1, 0.5, 1)$; $x_j \in \mathcal{X}$ for $1 \leq j \leq N$ and $z_j \in \mathcal{X} \times \mathcal{U}$ for $1 \leq j \leq M$ are the center points of RBFs that are uniformly distributed within the compact regions $\Omega_x$ and $\Omega_x \times \mathcal{U}$, respectively. In all of the simulations, we choose $N = 13^2$ and $M = 13^3$, so we have $13^2$-RBFs in $\tilde{v}$ and $\tilde{c}$, and $13^3$-RBFs in $\tilde{a}$ and $\tilde{q}$.

D.2 Policy Evaluation Methods

Under the approximation (D.1), the Bellman equations in Algorithms 2, 3b, 4b, and 5 can be expressed, with the approximation error $\varepsilon : \mathcal{Z} \to \mathcal{R}$, as the following uniform form:

$$\psi^v(t) \cdot \theta_v = b^v(z) + \varepsilon(z),$$

(D.2)

where the parameter vector $\theta_v \in \mathbb{R}^L$ is to be estimated, with its dimension $L$, and the associated functions $\psi : \mathcal{Z} \to \mathbb{R}^{1 \times L}$ and $b : \mathcal{Z} \to \mathbb{R}$ are given in Table D.1 for each IPI method. In Table D.1, $\mathcal{I}_v(Z), D_v(u)$, and $\phi_{V}^{\pi_0}$ defined as

$$\mathcal{I}_v(Z) = \int_t^{t+\Delta t} \alpha^{t-t} Z(X_{\tau}, U_{\tau}) d\tau,$$

$$D_v(u) = v(X_{t}) - \alpha \Delta t v(X_{t}),$$

and $\phi_{AD}^{\pi_0}(\cdot, \pi_0) \text{ were used for simplicity, where } X_{t} = \text{ the state value } X_{t} \text{ normalized to } [-\pi, \pi] \times \mathbb{R}.$ \textsuperscript{17} We set $\beta = 1$ in our IQPI simulation.

\textsuperscript{17} $X_{t}$ is normalized to $X_{t}$ whenever input to the RBFN(s). Other than that, the use of $X_{t}$ instead of $X_{t}$ (or vice versa) does not affect the performance (e.g., $R_0(X_{t}) = R_0(X_{t})$ in our setting).
In each \(i\)-th policy evaluation of each method, \(\psi(z)\) and \(b(z)\) in (D.2) were evaluated at the given data points \(z = z_{\text{init,j}}\) \((j = 1, 2, \cdots, L_{\text{init}})\) that are uniformly distributed over the respective compact regions \(\Omega_x \times \mathcal{U}\) (IAPI and IQPI), \(\Omega_x (\text{IEPI})\), and \(\Omega_x \times U_0\) with \(U_0 = \{-U_{\text{max}}, U_{\text{max}}\}\) (ICPI). The trajectory \(x\) over \([t, t']\) with each data point \(z_{\text{init,j}}\) used as its initial condition was generated using the 4th-order Runge-Kutta method with its time step \(\Delta t/10 = 1\) [ms], and the trapezoidal approximation:

\[
\mathcal{I}_0(Z) \approx \left( \frac{Z(\bar{X}_t, U_t) + \alpha \Delta t Z(\bar{X}_{t'}, U_{t'})}{2} \right) \cdot \Delta t
\]

was used in the evaluation of \(\psi\) and \(b\). The number \(L_{\text{init}}\) of the data points \(z_{\text{init,j}}\) used in each IPI algorithm was \(17^3\) (IAPI), \(25 \times 31 \times 25\) (IQPI), \(17^2\) (IEPI), and \(17^2 \times 2\) (ICPI). In the \(i\)-th policy evaluation of IAPI, we also evaluated the vectors \(\rho(x) = \left[ \phi_{\text{Ad}}(x, \pi_i(x)) \right] \in \mathbb{R}^k\) at the grid points \(x_{\text{grid,k}}\) \((k = 1, 2, \cdots, L_{\text{grid}})\) with \(L_{\text{grid}} = 50^2\) uniformly distributed in \(\Omega_x\), in order to take the constraint

\[
\rho^T(x) \theta_i = \varepsilon_{\text{const}}(x) \tag{D.3}
\]

obtained from \(a_i(x, \pi_i(x)) = 0\) into considerations, where \(\varepsilon_{\text{const}} : \mathcal{X} \to \mathbb{R}\) is the residual error. After evaluating \(\psi(\cdot)\) and \(b(\cdot)\) at all points \(z_{\text{init,j}}\) (and in addition to that, \(\rho(\cdot)\) at all points \(x_{\text{grid,j}}\) in the IAPI case), the parameters \(\theta_i\) in (D.2) were estimated using least squares as

\[
\hat{\theta}_i = \left( \sum_{j=1}^{L_{\text{init}}} \psi_j \psi_j^T \right)^{-1} \left( \sum_{j=1}^{L_{\text{init}}} \psi_j b_j \right) \tag{D.4}
\]

in the case of IQPI, IEPI, and ICPI, where \(\psi_j = \psi(z_{\text{init,j}})\) and \(b_j = b(z_{\text{init,j}})\). This \(\hat{\theta}_i\) minimizes the squared error

\[
J(\theta_i) = \varepsilon^2(z_{\text{init,1}}) + \cdots + \varepsilon^2(z_{\text{init,L_{init}}}).
\]

In IAPI, \(\theta_i\) in (D.2) and (D.3) were estimated also in the least-squares sense as

\[
\hat{\theta}_i = \left( \sum_{j=1}^{L_{\text{grid}}} \psi_j \psi_j^T + \sum_{k=1}^{L_{\text{grid}}} \rho_k \rho_k^T \right)^{-1} \left( \sum_{j=1}^{L_{\text{grid}}} \psi_j b_j \right), \tag{D.5}
\]

where \(\rho_k = \rho(x_{\text{grid,k}})\); this \(\hat{\theta}_i\) minimizes the squared error

\[
J_{\text{API}}(\theta_i) = J(\theta_i) + \varepsilon^2_{\text{const}}(x_{\text{grid,1}}) + \cdots + \varepsilon^2_{\text{const}}(x_{\text{grid,L_{grid}}}).
\]

### D.3 Policy Improvement Methods

In each \(i\)-th policy improvement, the next policy \(\pi_{i+1}\) was obtained using the estimates \(\hat{\theta}_i, \hat{\theta}_i', \hat{\theta}_i''\), or \(\hat{\theta}_i^3\) obtained at the \(i\)-th policy evaluation by (D.4) or (D.5) depending on the algorithms. In all of the simulations, the next policy \(\pi_{i+1}\) was parameterized as \(\pi_{i+1}(x) \approx \sigma(\hat{y}_i(x))\), where \(\hat{y}_i(x)\) was directly determined in IPIE and ICPI (Algorithms 4b and 5) as \(\hat{y}_i(x) = F_i^T(x) \nabla \hat{v}_i^T(x; \hat{\theta}_i^3)\) and \(\hat{y}_i(x) = \hat{c}(x; \hat{\theta}_i^3)\), respectively. In IAPI and IQPI, \(\hat{y}_i(x)\) is the output of the RBFN we additionally introduced:

\[
\hat{y}_i(x) = \phi^T(\bar{x}) \hat{\theta}_i^3
\]

to perform the respective maximizations (41) and (43). Here, \(\hat{\theta}_i^3 \in \mathbb{R}^N\) is updated by the mini-batch regularized update descent (RUD) (Alesksandar, Lever, and Barber, 2016) shown in Algorithm D.1, which is a variant of stochastic gradient descent, to perform such maximizations with improved convergence speed. In Algorithm D.1, \(J_{\text{imp}}(x; \theta)\) is given by

\[
J_{\text{imp}}(x; \theta) = \begin{cases} 
\hat{q}(x, u; \hat{\theta}_i)_{\mid u = \sigma(\phi^T(x) \theta)} & \text{in IQPI,} \\
\hat{a}(x, u; \hat{\theta}_i^3)_{\mid u = \sigma(\phi^T(x) \theta)} & \text{in IAPI;}
\end{cases}
\]

the error tolerance \(0 < \delta \ll 1\) was set to \(\delta = 0.01\); the smoothing factor \(\lambda_j \in (0, 1)\) and the learning rate \(\eta_j > 0\) were scheduled as \(\lambda_j = (1 - \epsilon) \cdot (1 - 10^3 \eta_j)\) with \(\epsilon = 10^{-3}\) and

\[
\eta_j = \begin{cases} 
10^{-3} & \text{for } 1 \leq j \leq 30; \\
10^{-3} / (j - 30) & \text{for } j > 30.
\end{cases}
\]

**Algorithm D.1: Mini-Batch RUD for Policy Improvement**

1. **Initialize:**
   \[
   \hat{\theta}_i^u = \mathbf{v} = 0 \in \mathbb{R}^N; \\
   0 < \delta \ll 1 \text{ be a small constant;}
   \]
2. \( j \leftarrow 1;\)
3. **repeat**
4. **Calculate** \(\frac{\partial J_{\text{imp}}(x; \hat{\theta}_i^u)}{\partial \hat{\theta}_i^u} \) at all grid points \(\{x_{\text{grid},k}\}_{k=1}^{L_{\text{grid}}};\)
5. \( \mathbf{v} \leftarrow \lambda_j \mathbf{v} + \eta_j \left( \sum_{k=1}^{L_{\text{grid}}} \frac{\partial J_{\text{imp}}(x_{\text{grid},k}; \hat{\theta}_i^u)}{\partial \hat{\theta}_i^u} \right);\)
6. \( \hat{\theta}_i^u \leftarrow \hat{\theta}_i^u + \mathbf{v};\)
7. \( j \leftarrow j + 1;\)
8. **until** \( \|\mathbf{v}\| < \delta\)
9. **return** \(\hat{\theta}_i^u.\)