Representations of the compact quantum group $SU_q(N)$ and geometrical quantization

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Abstract

The method of geometrical quantization of symplectic manifolds is applied to constructing infinite dimensional irreducible unitary representations of the algebra of functions on the compact quantum group $SU_q(2)$. A formulation of the method for the general case $SU_q(n)$ is suggested.

1 Introduction

The discovery of new symmetry structures called quantum groups [1]-[5] has leaded to the development of the representation theory of function algebras on quantum groups. In this article we study the connection of the geometrical quantization with the representation theory of the function algebra on the compact quantum group $SU_q(2)$. By geometrical quantization we mean the construction of the Hilbert space $H$ of quantum states and the observable algebra starting with the geometry $(M,\omega)$ of a symplectic manifold giving a model of classical mechanics [6]-[10].

The relation of the geometry of the Poisson Lie group $SU(n)$ [11]-[13] with the theory of infinite dimensional irreducible unitary representations of the function algebra $Fun(SU_q(n))$ was systematically studied by the authors of [14]-[16]. It turned out that these representations are the highest weight representations and can be parametrized by the elements of Weyl group and points of maximal torus of

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SU(n). The aim of this work is to show how infinite dimensional irreducible unitary representations of $\text{Fun}(SU_q(n))$ arise as the result of geometrical quantization in the sense of Kostant applied to the symplectic leaves of some Poisson structure defined on the $SU(n)$ group by a classical $r$-matrix. As a simple example, we consider the case of $SU_q(2)$ and obtain the well-known representations of the algebra $\text{Fun}(SU_q(n))$ having in mind that the found constructions keep their value for an arbitrary $n$. Namely, it is due to the fact that the structure of symplectic leaves in the Poisson Lie group $SU(n)$ is known $[15, 16]$.

2 Geometrical quantization

For constructing representations of the compact quantum groups we need an analogue of so-called ”exact” quantization in which the Dirac axiom takes place.

Let us denote $\mathcal{P}(M, \omega)$ the Poisson algebra of functions on a symplectic manifold $(M, \omega)$. A linear mapping $F \to \hat{F}$ of the algebra $\mathcal{P}(M, \omega)$ to the set $B(H)$ of operators, acting in a Hilbert space $H$ is called Dirac quantization, if it has the following properties

1. $\hat{1} = 1$, i.e. the unity on $M$ correspond the unity operator in $H$.

2. $\{\hat{F}, \hat{G}\} = \frac{2\pi i}{\hbar}[\hat{F}, \hat{G}]$ (Dirac axiom). This means that the mapping $F \to \hat{F}$ is a homomorphism of Lie algebras $\mathcal{P}(M, \omega) \to B(H)$.

3. $\hat{F}^\ast = \hat{F}^\ast$, i.e. real functions from $\mathcal{P}(M, \omega)$ correspond to the self-adjoint operators in $B(H)$.

4. For some full set $\hat{\mathcal{P}}$ of functions $F_1, F_2, \ldots$ from $\mathcal{P}(M, \omega)$ the corresponding operators $\hat{F}_1, \hat{F}_2, \ldots$ form an irreducible representation $\hat{\mathcal{P}}$. The set $\hat{\mathcal{P}}$ is called a full, if any function on $M$, commuting with all $F_i \in \hat{\mathcal{P}}$ with respect to the Poisson brackets is a constant.

A linear mapping $F \to \hat{F}$, obeying only (1)-(3) is said to be prequantization. For a given symplectic manifold $(M, \omega)$ prequantization can be constructed as follows. $[3]$. Let us cover the manifold $M$ by open covers $U_\alpha$ in such a way as to obtain in every cover $U_\alpha$ the identity $\omega = d\theta_\alpha$ for a suitable 1-form $\theta_\alpha$ in $U_\alpha$. Define in $U_\alpha$ the operator:

$$\hat{\tilde{F}}_\alpha = F_\alpha + \frac{\hbar}{2\pi i} \xi_F - \theta_\alpha(\xi_F),$$

that acts on the space $C^\infty(U_\alpha)$. Here $F_\alpha \in C^\infty(U_\alpha)$ and $\xi_F$-is a hamiltonian vector field with the generating function $F$, considering as an operator in $C^\infty(M)$.

If the form $\omega$ defines an integer cohomology class in $H^2(X, R)$, then according to fundamental Kostant’s theorem, the introduced local operators $\hat{\tilde{F}}_\alpha$ can be composed in a one global operator $\hat{\tilde{F}}$, acting on the space of smooth sections of a linear bundle $L$ over $M$. The form $\theta_\alpha$ can be realized as a local expression for some connection $\nabla$ of the bundle $L$ in the trivialization $U_\alpha$. Let us note that in the case of real $\omega$
there exists on $L$ a hermitian structure compatible with the connection $\nabla$, and the prequantization operator $\tilde{F}$ has the following form:

$$\tilde{F} = F + \frac{\hbar}{2\pi i} \nabla \xi_F ,$$

GDE $\nabla \xi_F = \xi_F - \frac{2\pi i}{\hbar} \theta(\xi_F)$.

The prequantization space, i.e. the space of sections $\Gamma(L, M)$ of $L$ over $M$, is too large to fulfill the condition (4). The subspace $\Gamma(L, M, P)$, on which an irreducible representation of the observable algebra acts are distinguished from $\Gamma(L, M)$ by means of a real or complex polarization $P$ [8],[10]. Namely, the elements of $\Gamma(L, M, P)$ are the smooth functions $s \in \Gamma(L, M)$, obeying the equation

$$\nabla_\eta s = 0$$

for any vector field $\eta \in P$.

It should be noted that for the case in question prequantization can be thought of as the assignment of homomorphism of the Poisson algebra $\mathcal{P}(M, \omega)$ in some abstract associative algebra $\mathcal{A}$ with involution (any associative algebra with respect to product can be turned in to the Lie algebra by introducing an ordinary commutator). The quantization procedure is realized then as the construction of irreducible representations of the algebra $\mathcal{A}$.

Let us consider now the algebra of regular functions on the compact quantum group $SU_q(2)$. This is the associative algebra $Fun(SU_q(2))$ with unity and involution $*$, generated by two generators $t_{11}, t_{21}$ modulo the relations:

$$t_{11}^* t_{21} = qt_{21}^* t_{11} , \quad t_{21}^* t_{21} = t_{21}^* t_{21} ,$$

$$t_{11} t_{11}^* - t_{11}^* t_{11} = (1 - q^2) t_{21} t_{21}^* ,$$

$$t_{11}^* t_{11} + t_{21} t_{21}^* = 1 , \quad t_{11} t_{21} = q t_{21} t_{11} .$$

where $q$ is the deformation parameter. The algebra $Fun(SU_q(2))$ has the Hopf algebra structure. It becomes commutative in the limit $q = 1$ and can be identified with the algebra of commutative polynomials generated by matrix elements of the fundamental representation of $SU(2)$, i.e. as an algebra of regular functions on $SU(2)$. Introduce a new parameter $\hbar$ (Plank’s constant) connecting with $q$ as $q = e^{-\hbar}$. Let $\hbar$ belongs to the interval $E = (0, a]$ where $a$ is some positive constant. Then the relations (1) give the family $A_\hbar$ of associative algebras supplied with involution:

$$t_{11} t_{21} = e^{-\hbar} t_{21} t_{11} , \quad t_{21} t_{21}^* = t_{21}^* t_{21} ,$$

$$t_{11} t_{11}^* - t_{11}^* t_{11} = (1 - e^{-2\hbar}) t_{21} t_{21}^* ,$$

$$t_{11} t_{21} = e^{-\hbar} t_{21} t_{11} , \quad t_{11}^* t_{11} + t_{21} t_{21}^* = 1 .$$

It is well known that $SU(2)$ is a Poisson Lie group. [3]. In other words, the function algebra $A_0 = Fun(SU(2))$ is a Poisson Hopf algebra (i.e. $A_0$ has the Hopf as well as the Poisson algebra structure with one and the same product with coproduct $A_0 \rightarrow A_0 \otimes A_0$ being homomorphism of Poisson algebras).
The family (3) of the algebras $A_h$ give a deformation, or in other words, quantization of the Poisson algebra $A_0$ in the sense of associative Hopf algebras [2], i.e. for any $h \in E A_h$ has the Hopf algebra which goes to the structure of $A_0$ when $h \to 0$. In this case the correspondence principle claims that
\[
\{F, G\} = \lim_{h \to 0} \frac{1}{h} [\hat{F}, \hat{G}],
\]
where we have the Poisson brackets of any functions $F$ and $G$ from $A_0$ on the left and the commutator of their images in $A_h$ on the right. The Poisson brackets on $A_0$ arising in such a way are quadratic ones. Thus the construction of the family $A_h$ can be regarded as quantization of the quadratic brackets on the Poisson Lie group $SU(2)$.

Let us note that the defining relations (2) of the noncommutative algebra $A_h$ for the fixed $h$ are specified by a quantum $R$-matrix being a matrix solution of the quantum Yang-Baxter equation (QYBE) and quadratic brackets on $SU(2)$ compatible with the Hopf algebra structure are constructed by using the canonical $r$-matrix:
\[
r = \frac{1}{2} \sum_{\alpha > 0} X_\alpha \otimes X_{-\alpha} - X_{-\alpha} \otimes X_\alpha
\]
giving a solution of the classical Yang-Baxter equation (CYBE) [2, 11]. Here $X_\alpha$ is a basis of positive roots of $su(2)$. Supposing $R$-matrix to be decomposed in the $h$-series:
\[
R = 1 + hr + O(h^2)
\]
in the second order of the QYBE we obtain for $r$ the CYBE. It is the connection between the QYBE and the CYBE that governs by the correspondence principle (3).

Let us consider now $*$-representations $\pi$ of the algebra (2) when $h \in E$ in separable Hilbert space. As it is known [14], every irreducible $*$-representation $\pi$ of the algebra (2) in Hilbert space $H$ is unitary equivalent to the one of the following two series:

1. One dimensional representations $\xi_\psi$ given by the formulae
   \[
   \xi_\psi(t_{11}) = e^{i\psi}, \quad \xi_\psi(t_{21}) = 0, \quad \psi \in \mathbb{R}/2\pi \mathbb{Z}.
   \]

2. Infinite dimensional representations $\rho_\psi$ having in an orthonormal bases $\{e_k\}_{k=0}^\infty$ the form:
   \[
   \rho_\psi(t_{11})e_0 = 0, \quad \rho_\psi(t_{11})e_k = \sqrt{1 - e^{-2kh}} e_{k-1}, \quad \rho_\psi(t_{21})e_k = e^{i\psi - kh} e_k
   \]

Let us describe now the procedure of finding the connection between infinite dimensional representations (3) of the algebra (2) and the geometry of $SU(2)$. In the limit $h \to 0$ the defining relations of the algebra (2) give a degenerate Poisson structure on $SU(2)$. The reduction of this structure on it’s symplectic leave of a maximal
dimension generates a symplectic form $\omega$. We will regard now the symplectic leave with the form $\omega$ as a model of classical mechanics. To construct a Hilbert space $H$ and a set of operators on it giving a quantum analogue of the system in question one needs to show the absence of cohomology obstacles. Namely, the cohomology class of the form $\omega$ must be integer\[6\]. Further we will construct Hilbert space $H$ and the representation of the maximal commutative subalgebra in the Poisson algebra $Fun(SU(2))$ and point out it’s relation with the representation (4) of the algebra (2).

3 Geometrical quantization of symplectic leaves in $SU(2)$

Consider the limit $\hbar \to 0$. The defining relations of the algebra (2) produce the following quadratic Poisson brackets on $Fun(SU(2))$:

\[
\begin{align*}
\{t_{11}, t_{21}\} &= t_{11}t_{21} , \quad \{t_{11}, \bar{t}_{11}\} = t_{21}\bar{t}_{21} , \\
\{t_{11}, \bar{t}_{21}\} &= -t_{11}\bar{t}_{21} , \quad \{t_{11}, t_{21}\} = 0 , \\
\{t_{21}, \bar{t}_{11}\} &= -t_{21}\bar{t}_{11} , \quad \{t_{11}, \bar{t}_{11}\} = -t_{21}\bar{t}_{11} ,
\end{align*}
\]

(1)

where the element $g \in SU(2)$ is parametrized by $t_{11}, t_{21}$:

\[
g = \begin{pmatrix}
t_{11} & -t_{21} \\
t_{21} & t_{11}
\end{pmatrix}
\]

obeying the condition $|t_{11}|^2 + |t_{21}|^2 = 1$, and the bar means the complex conjugation. Brackets (1) generate corresponding to the coordinate functions strictly hamiltonian vector fields on $SU(2)$:

\[
\begin{align*}
\xi_{t_{11}} &= -t_{11} \left( t_{21} \frac{\partial}{\partial t_{21}} + \bar{t}_{21} \frac{\partial}{\partial \bar{t}_{21}} \right) + 2t_{21}\bar{t}_{21} \frac{\partial}{\partial \bar{t}_{11}} , \\
\xi_{t_{21}} &= t_{21} \left( -\bar{t}_{11} \frac{\partial}{\partial \bar{t}_{11}} + t_{11} \frac{\partial}{\partial t_{11}} \right) .
\end{align*}
\]

Moreover, we have $\xi_{\bar{t}_{11}} = -\xi_{t_{11}}$ and $\xi_{\bar{t}_{21}} = -\xi_{t_{21}}$.

The algebra $Fun(SU(2))$ is supplied with the involution that is the complex conjugation. The quadratic brackets (1) are antiinvolutive in opposite to the ordinary brackets on $T^*M$. For example,

\[
\{t_{11}, \bar{t}_{21}\} = -\{\bar{t}_{11}, t_{21}\} .
\]

Thus the reduction of the brackets (1) on a symplectic leave gives a symplectic form $\omega$ having the property $\omega = -\bar{\omega}$, that means the absence of its real part. Therefore, for the real function $F$ ($F = \bar{F}$) the strictly hamiltonian vector field $\xi_F$, defined by equation

\[
2\omega(\xi_F, \ldots) + dF = 0
\]
obeys the condition: $\xi_F = -\bar{\xi}_F$.

Such behavior of brackets (\[4\]) with respect to the unvolution gives a hint that in the geometrical quantization procedure we have to require this time instead of

$$\frac{2\pi i}{\hbar} [\tilde{F}, \tilde{G}] = \{F, G\}$$

then fulfillment of the following relation:

$$\frac{2\pi}{\hbar} [\tilde{F}, \tilde{G}] = \{F, G\} . \tag{2}$$

Then the choice of prequantization operators in the form:

$$\tilde{F} = F + \frac{\hbar}{2\pi} \xi_F - \theta(\xi_F) \tag{3}$$

obeys automatically (\[2\]). Here $\theta$ is such 1-form that $\omega = d\theta$. Since $\omega$ is imaginary and for real functions $F$: $\xi_F = -\bar{\xi}_F$, the form $\theta$ can be chosen to be imaginary. Then

$$\theta(\bar{\xi}_F) = \bar{\theta}(\xi_F) = -\theta(-\xi_F) = \theta(\xi_F),$$

i.e. the function $\theta(\xi_F)$ is real for the real $F$. Thus we have the possibility to realize the prequantization operators in the space of finite functions on a symplectic leave $M$. By introduction of the scalar product:

$$(\varphi_1, \varphi_2) = \int_M \varphi_1 \bar{\varphi}_2 \omega , \tag{4}$$

where $\omega$ is the Liouville measure on the leave $M$, this space is turned to be Hilbert. (For $SU(2)$ the maximal dimension of symplectic leaves is equal to 2). The prequantization operators corresponding to real finite functions are symmetrical and essentially self-adjoint.

Let us find explicitly the symplectic form $\omega$ on the symplectic leave $M$ of the quadratic brackets (\[4\]). Hamiltonian vector fields of the coordinate functions annihilate the following two functions:

$$c_1 = |t_{11}|^2 + |t_{21}|^2 , \quad c_2 = (t_{21}/\overline{t_{21}})^{1/2} , \quad |t_{21}| \neq 0 .$$

Therefore if $|t_{21}| \neq 0$ the symplectic leave consists of the matrices:

$$M_\psi = \left[ g : g = \begin{pmatrix} t_{11} & -t_{21} \\ t_{21} & \bar{t}_{11} \end{pmatrix} \in SU(2), \quad \arg t_{21} = \psi, \quad |t_{21}| \neq 0 \right],$$

where $0 < \psi \leq 2\pi$. $M_\psi$ is described by the one complex coordinate $z = t_{11}$ obeying the condition $z \leq 1$, since $|t_{11}|^2 + |t_{21}|^2 = 1$ and $|t_{21}| > 0$. So we see that $M_\psi$ is an open circle of the radius one on a complex plane, i.e. a noncompact complex manifold. The range of brackets is constant for any point inside the round. Any differential 2-form $\omega$ on $M_\psi$ is:

$$\omega_\psi = a(t_{11}, \bar{t}_{11}, \psi) dt_{11} \wedge d\overline{t}_{11} .$$
The condition $\bar{\omega} = -\omega$ gives that $a$ is real, i.e. $a(t_{11}, \bar{t}_{11}, \psi) \in R$. The reduction of the Poisson brackets on the leave $M_{\psi}$ should coincide with $\omega_{\psi}$:

$$\{F, G\} = 2\omega_{\psi}(\xi_F, \xi_G).$$

From here we find:

$$\omega = \frac{1}{2(1 - |z|^2)}dz \wedge d\bar{z}.$$  \hfill (5)

Let $X = M_{\psi}/\{0\}$ be the open circle on the complex plane without zero. Then there exists on $X$ the 1-form $\theta$ defined by:

$$\theta = \frac{1}{4} \ln(1 - |z|^2)(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}}) = \frac{i}{2} \ln(1 - \rho^2)d\varphi,$$  \hfill (6)

for which $\omega = d\theta$. Here $z = \rho e^{i\varphi}$, $\varphi \in S^1$, $0 < \rho < 1$.

Denote $L(X) \approx X \times C$ a trivial linear bundle over $X$. On $C$ one can choose the usual hermitian structure. Since $L(X)$ is trivial it has the unit section $s$ globally determined on $X$. Let us think of $\theta$ to be a connection form $\tilde{\theta}$ of the bundle $L(X)$, corresponding to the section $s$, i.e.

$$s^* \theta = \tilde{\theta}.$$  

The curvature $\Omega$ of this connection is also defined on the base $X$ by means of $s$. It coincides with $\frac{1}{\hbar} \omega$. Note, that the connection introduced in such a way is not flat but the cohomology class corresponding to $\omega$ in $H^2(X, R)$ is equal to zero. Thus the cohomology obstacles are absent.

Let us construct the prequantization operator for the coordinate function $t_{21}$, using the formula (3):

$$\tilde{t}_{21} = t_{21} + \hbar \frac{i}{2\pi} \xi_{t_{21}} - \theta(\xi_{t_{21}}).$$  \hfill (7)

In the $(\rho, \varphi)$ coordinates we have:

$$\tilde{t}_{21} = t_{21} + t_{21} \left( \frac{\hbar}{2\pi i} \frac{\partial}{\partial \varphi} - \frac{1}{2} \ln(1 - \rho^2) \right).$$  \hfill (8)

Consider the polarization generated by the vector field $\eta = \frac{t_{21}}{i} \frac{\partial}{\partial \varphi}$. The polarization is real ($\eta = -\bar{\eta}$) and it’s leaves on $X$ are circles. The quantization space $H$ associated with the polarization $\eta$ consists of sections $s(\rho, \varphi)$ of the linear bundle $L(X)$ over $X$ that are horizontal ones with respect to the direction defined by the vector field $\eta$. Thus $H$ is modeled by solutions of the equation $\nabla_\eta s = 0$. Being rewritten in the variables $(\rho, \varphi)$ it takes the form:

$$\frac{\hbar}{i\pi} \frac{\partial s}{\partial \varphi} - \ln(1 - \rho^2)s = 0.$$  \hfill (9)

Trying to solve this equation we run into the well-known problem of geometrical quantization [7, 10]. As was mentioned above the leaves of the polarization are
circles, i.e. non simply connected manifolds. Parallel transport of a section along
the closed way $\gamma$ laying in the leave leads to the multiplication of $\gamma$ on
\[ Q(\gamma) = e^{\frac{2\pi}{\hbar} \int \gamma \theta}. \]
Since, generally speaking, an arbitrary way $\gamma$ can not be squeezed to a point
the number $Q(\gamma)$ is not equal to one. Clearly, in this case any solution of equation (9)
equals zero on the leave considered. Denote $\hat{X}$ the set of polarization leaves for
which $Q(\gamma) = 1$. The set $\hat{X}$ is known as Bohr-Zommerfeld submanifold [8].
The way out of the problem consists in considering instead of $\Gamma(L, X, P)$
the set of distributions being solutions of equations (9). All solutions of such a type
(up to a multiplication constant ) have the form:
\[ s_n(\rho, \varphi) = e^{in\varphi} \delta \left( \rho - \sqrt{1 - e^{\frac{\nu \hbar}{\pi}}} \right) \]
and their support coincides with $\hat{X}$ (the unity of circles). In the variables parametriz-
ing points of the symplectic leave $M_{\phi}$, where $\phi \in S^1$, the coordinate function $t_{21}$
is written as follows:
\[ t_{21} = | t_{21} | e^{i\varphi} = \sqrt{1 - \rho^2 e^{i\varphi}}. \] (11)
Let us introduce the set of elements $s_n^\psi(\rho, \varphi, \phi)$ numerating by the parameter $\psi \in S^1$:
\[ s_n^\psi(\rho, \varphi, \phi) = e^{in\varphi} \delta \left( \rho - \sqrt{1 - e^{\frac{\nu \hbar}{\pi}}} \right) \otimes \delta (\phi - \psi). \] (12)
Then the operator $t_{21}$ acts on the elements of $s_n^\psi$ as the multiplication on the function $t_{21}$:
\[ t_{21} s_n^\psi(\rho, \varphi, \phi) = e^{i\psi + \frac{\nu \hbar}{2\pi}} s_n^\psi(\rho, \varphi, \phi). \] (13)
The natural requirement $1 - e^{\frac{\nu \hbar}{\pi}} \geq 0$ defines an allowed region of $n$ to take it’s value:
$n \in N = \{0, -1, -2, \ldots\}$. Thus in the representation $s_n^\psi$ the operator $t_{21}$ has
the spectrum: $e^{i\psi + \frac{\nu \hbar}{2\pi}}$, $n \in N$ that is identical to the one of the operator $\rho(\psi(t_{21}))$ in the
representation (4) when $q = e^{-\frac{\hbar}{\pi}}$.
By analogy we obtain for the operator $t_{21}$:
\[ t_{21} s_n^\psi(\rho, \varphi, \phi) = e^{-i\psi + \frac{\nu \hbar}{2\pi}} s_n^\psi(\rho, \varphi, \phi). \] (14)
The straightforward calculations of the operators corresponding to the coordinate
functions $t_{11}$ and $t_{11}$ show that they do not preserve the quantization space $H_\psi$.
However, let us note that the coordinate functions $t_{21}, t_{21}$ and 1 generate the maximal
commutative subalgebra with respect to the Poisson brackets. Then the operators \( \tilde{t}_{21} \), \( \tilde{t}_{21} \) and 1 can be considered as the representation of this algebra in the space of horizontal sections \( H_{\psi} \) over Bohr-Zommerfeld submanifold. We will show that this representation can be prolonged to representations of the algebra (4) in \( H_{\psi} \) that unitary equivalent to (4).

Introduce in the space \( H_{\psi} \) a scalar product \((,\) in such a way as to obtain \((s_n, s_m) = \delta_{nm}\). Let the operator \( \tilde{t}_{11} \) has the matrix \( A_{nm} = \left\| a_{nm} \right\| \) in the bases \( s_{\psi}^n \), i.E.:

\[
t_{11} s_{\psi}^n = \sum_m a_{nm} s_{\psi}^m.
\]

Then

\[
t_{11} s_{\psi}^n = \sum_m a_{nm}^* s_{\psi}^m.
\]

The relations (2) combined with (13) and (14) give us:

\[
\tilde{t}_{11} \tilde{t}_{11} s_{\psi}^n = (1 - e^{i\frac{nh}{\pi}}) s_{\psi}^n
\]

and

\[
\sum_m e^{i\frac{nh}{\pi}} a_{nm} s_{\psi}^m = \sum_m e^{(m-1)i\frac{nh}{\pi}} a_{nm} s_{\psi}^m.
\]

From the last equation one has:

\[
t_{11} s_{\psi}^n = b_n s_{\psi}^{n+1},
\]

\[
t_{11} s_{\psi}^n = b_{n-1}^* s_{\psi}^{n-1},
\]

where \( b_n \) are coefficients that should be defined. Substituting (17) in (13) we find:

\[
b_n = e^{i\phi_n} \sqrt{1 - e^{i\frac{nh}{\pi}}} , \quad \phi_n \in S^1, \quad n < 0.
\]

Let us show now that the representation \( \hat{T} \) characterized by the infinite dimensional set \( \{\phi_1, \phi_2, \ldots\} \) is unitary equivalent to the representation \( T \) denoted by \( T \). Consider the operator \( U \) acting on \( s_{\psi}^n \) in the following manner:

\[
Us_{\psi}^n = e^{i\sum_{k=1}^{n-1} \phi_k} s_{\psi}^n , \quad Us_0 = 0
\]

Clearly \( U \) is unitary and \( \hat{T}U = T \), i.E. \( T_\phi \) and \( T \) are equivalent.
4 Geometrical quantization of the Poisson Lie group $SU_q(n)$

Here we present the general formulation of the found on the $SU_q(2)$ example the connection of geometrical quantization with the theory of unitary infinite dimensional representations an arbitrary compact quantum group.

Let $Fun(G_q)$ be the function algebra on a compact quantum group $G_q$. When the deformation parameter $q$ goes to one $G_q$ turns into the Poisson Lie group $G$. The Poisson structure arising in such a way is degenerate. It’s symplectic leaves are parametrized by the elements of Weyl group and by points of maximal torus $H$ of the group $G$. Let us reduce the Poisson brackets on the leave $M$ and consider in the Poisson algebra the maximal commutative subalgebra $B$ with respect to the Poisson brackets. Geometrical quantization gives the realization $\pi$ of this subalgebra by differential operators of the first order acting in the space $\Gamma(L, M)$ of sections of a linear bundle $L$ over $M$. $\Gamma(L, M)$ can be identified with the space of prequantization. Let us chose such a polarization $P$ that all operators $\pi(B)$ are diagonal in the corresponding space $\Gamma(L, M, P)$. Since the algebra $B$ is commutative then the polarization with the desired property exists. The space $\Gamma(L, M, P)$ are defined to be the set of distributions horizontal in the direction of the polarization. In $\Gamma(L, M, P)$ the irreducible representation of the function algebra $Fun(G_q)$ acts. It’s reduction on the image of $B$ in $Fun(G_q)$ coincides with $\pi(B)$. Thus geometrical quantization of leaves of Poisson structure gives an explicit realization of the space of the corresponding representation of the quantum group and the realization of $\pi(B)$ the algebra $B$ on this space. Since irreducible infinite dimensional unitary representations of $Fun(G_q)$ are representations with highest weight they are defined up to the unitary equivalence by restrictions on the maximal commutative subalgebra $B$ (an analogue of Cartan subalgebra).

5 Concluding remarks

Note that Dirac axiom plays a crucial role in the considered example. The reason for which the standard geometrical quantization of the Poisson algebra $P(M, \omega)$ does not immediately give representations of a quantum group as follows. The deformation procedure of Hopf algebras is not ”exact” quantization. In general, for any two observables one has $P(M, \omega)$ the equality:

$$[\hat{F}, \hat{G}] = \frac{h}{2\pi} \{\widehat{F}, \widehat{G}\} + O(h^2).$$

Existence of the hole series in Plank’s constant on the left hand side is the well known problem arising in quantization of nonlinear (in particular quadratic) Poisson brackets. Namely, one can not substitute Poisson brackets by an operator analog because it is unknown how to order their right hand side. The choice of some ordering leads to the appearance of highest powers of $h$ in the equality under consideration. Thus, in the case of nonlinear brackets we have only some peculiar subalgebras (in our
example it is a commutative algebra) for which the series on the right vanishes, the quantization becomes "exact" and can be formulated purely in terms of geometry. Hence the natural question arises of whether it is possible to extend somehow the class of subalgebras in $\mathcal{P}(M, \omega)$ being the subject of quantization. Clearly, existence of such an extension is connected with the possibility to generalize the usual procedure of geometrical quantization. This should be done in a way as to realize a quantum group as the Hopf algebra of invariant differential operators (but not the first order). The positive solution of this problem can be applied then to the construction of dynamical systems on quantum groups \cite{17}, \cite{18}.

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