TRANSPORT IN THE ONE-DIMENSIONAL SCHRÖDINGER EQUATION

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Abstract. We prove a dispersive estimate for the Schrödinger equation on the real line, mapping between weighted $L^p$ spaces with stronger time-decay ($|t|^{-\frac{1}{2}}$ versus $|t|^{-\frac{3}{2}}$) than is possible on unweighted spaces. To satisfy this bound, the long-term behavior of solutions must include transport away from the origin. Our primary requirements are that $\langle x \rangle^3 V$ be integrable and $-\Delta + V$ not have a resonance at zero energy. If a resonance is present (for example, in the free case), similar estimates are valid after projecting away from a rank-one subspace corresponding to the resonance.

In one dimension, the linear propagator of the free Schrödinger equation is given by the explicit convolution
\[
e^{-it\Delta} \psi(x) = \frac{1}{\sqrt{-4\pi it}} \int_{\mathbb{R}} e^{-i|x-y|^2 \frac{4}{it}} \psi(y) \, dy.
\]
This gives rise immediately to the dispersive estimate
\[
|e^{-it\Delta} \psi|_\infty \leq (4\pi|t|)^{-\frac{1}{2}} \|\psi\|_1.
\] (1)

Such an estimate cannot be true in general for the perturbed operator $H = -\Delta + V(x)$. Even small perturbations of the Laplacian may lead to the formation of bound states, i.e. functions $f_j \in L^2$ satisfying $H f_j = -E_j f_j$. Bound states with strictly negative energy are known to possess exponential decay, hence they belong to the entire range of $L^p(\mathbb{R}), 1 \leq p \leq \infty$. For each of these bound states $f_j$, the associated evolution $e^{-itE_j} f_j = e^{-itE_j} f_j$ clearly violates (1).

It is well known [3, 10] that if $V \in L^1(\mathbb{R})$ then the pure-point spectrum of $H$ consists of at most countably many eigenvalues $-E_j < 0$. The absolutely continuous spectrum of $H$ is the entire positive half-line, and there is no singular continuous spectrum. Bound states can therefore be removed easily via a spectral projection, suggesting that one should look instead for dispersive estimates of the form
\[
|e^{itH} P_{ac}(H) \psi|_\infty \lesssim |t|^{-\frac{1}{2}} \|\psi\|_1.
\] (2)

The condition $V \in L^1$ does not always guarantee regularity at the endpoint of the continuous spectrum. We say that zero is a resonance of $H$ if there exists a bounded solution to the equation $H f = 0$. Since resonances are not removed by the spectral projection $P_{ac}(H)$, the validity of dispersive estimates invariably depends on whether zero is a resonance of $H$. Weder [12] and Goldberg-Schlag [5] have shown that (2) holds for all potentials with $(1 + |x|)^2 V \in L^1$, and that $(1 + |x|)^2 V \in L^1$ suffices provided zero is not a resonance.

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The relatively slow time-decay of these estimates (the tail of the function $t^{-\frac{3}{2}}$ is not integrable) makes them unsuitable for many applications. We are therefore interested proving a dispersive estimate which improves the rate of decay by mapping between favorably weighted spaces. Statements of this type appear in the work of Murata [9] and Buslaev-Perelman [2], with weighted $L^2(\mathbb{R})$ as the underlying space. A weighted $L^1 \to L^\infty$ bound was proven recently by Schlag [11]. Our first theorem is a refinement of Schlag’s result.

**Theorem 1.** Suppose $(1 + |x|)^3V \in L^1(\mathbb{R})$ and zero is not a resonance of $H$. The continuous part of the Schrödinger evolution satisfies the bound

$$
\|(1 + |x|)^{-1}e^{itH}P_{ac}(H)\psi\|_\infty \lesssim |t|^{-\frac{3}{2}}\|1 + |x|\psi\|_1.
$$

Recall that $|e^{itH}P_{ac}(H)\psi(\cdot)|$ is always dominated by $|t|^{-\frac{3}{2}}$, by (2). The additional estimate (3) reduces the bound even further for all $|x| \ll |t|$. This suggests that solutions experience transport away from the origin with nonzero velocity.

The assumption that zero energy is not a resonance is a necessary part of Theorem 1. To give an explicit example, consider the case $V = 0$ with initial data $\psi(x) = e^{-\frac{|x|^2}{2}}$. For each $t$, the solution $e^{-it\Delta}\psi(x) = (4\pi(1 - it))^{-\frac{1}{2}}e^{\frac{|x|^2}{4(1 - it)}}$ satisfies (2) but clearly violates (3). There is a significant degree of structure to a resonance at $\lambda = 0$, as is seen in the power-series resolvent expansion of Jensen-Nenciu [7]:

$$(H - (\lambda + i0))^{-1} = \lambda^{-\frac{1}{2}}C_{-1} + C_{0} + \lambda^{\frac{3}{2}}C_{1} + O(\lambda).$$

Here $C_{-1}$ is a projection onto the subspace spanned by the bounded solution of $Hf = 0$, or is vacuous if zero is not a resonance. One consequence is that the worst time-decay must be confined to a rank-one subspace of functions. More precisely, in the one-dimensional setting we prove the following:

**Theorem 2.** Suppose $(1 + |x|)^4V \in L^1(\mathbb{R})$ and there is a nontrivial bounded function $f_{0}$ for which $Hf_{0} = 0$, normalized so that $\lim_{z \to \infty} (|f_{0}(z)|^2 + |f_{0}(-z)|^2) = 2$. Denote by $P_{0}$ the projection onto the span of $f_{0}$ given formally by $P_{0}\psi = \langle \psi, f_{0}\rangle f_{0}$.

The continuous part of the Schrödinger evolution satisfies the bound

$$
\|(1 + |x|)^{-2}(e^{itH}P_{ac}(H) - (-4\pi i t)^{-\frac{1}{2}}P_{0})\psi\|_\infty \lesssim |t|^{-\frac{3}{2}}\|1 + |x|^2\psi\|_1.
$$

The proof of each theorem relies on a decomposition of the propagator $e^{itH}P_{ac}(H)$ according the spectral measure of $H$. Written this way,

$$
e^{itH}P_{ac}(H)\psi = \int_{0}^{\infty} e^{it\lambda}E_{ac}(d\lambda)\psi \ d\lambda$$

where $E_{ac}(d\lambda)$ denotes the absolutely continuous part of the spectral measure of $H$. Since $V$ is assumed to be integrable, it is correct to assume that the absolutely continuous spectrum is supported on the interval $[0, \infty)$. The Stone formula provides additional information about the nature of $E_{ac}(d\lambda)$, namely

$$
\langle E_{ac}(d\lambda)f, g \rangle = \frac{1}{2\pi i}([R_{V}^{+}(\lambda) - R_{V}^{-}(\lambda)]f, g)
$$

where $R_{V}^{\pm}(\lambda) := (-\Delta + V - (\lambda \pm i0))^{-1}$ is the continuation of the resolvent onto the positive real half-line. Substituting this into the previous equation yields

$$
\langle e^{itH}P_{ac}(H)\psi, \varphi \rangle = \frac{1}{2\pi i} \int_{0}^{\infty} e^{it\lambda}([R_{V}^{+}(\lambda) - R_{V}^{-}(\lambda)]\psi, \varphi) \ d\lambda.
$$
It is convenient to make the change of variables \( \lambda \mapsto \lambda^2 \). For the purpose of changing variables inside the resolvent, recall that \( R_+^+(\lambda^2) \) is an analytic continuation of the operator-valued function \((H - z)^{-1}\) from the upper half-plane. The continuation of \((H - z^2)^{-1}\) is therefore \((H - (\lambda + i0)^2)^{-1}\), which is identical to \( R_+^+(\lambda^2) \) along the positive half-line and \( R_+^-(\lambda^2) \) along the negative half-line. This allows us to open up the domain of integration to the entire real line:

\[
\langle e^{itH}P_{ac}(H)\psi, \varphi \rangle = \frac{1}{\pi i} \int_{-\infty}^{\infty} e^{it\lambda^2} \lambda (R_+^+(\lambda^2)\psi, \varphi) \, d\lambda.
\]

For large values of \( \lambda \) we will regard \( R_+^+(\lambda^2) \) as a perturbation of the free resolvent \( R_+^0(\lambda^2) \), which can be expressed explicitly as a convolution. This part of the argument has appeared previously in [11] and requires no further modification.

For small \( \lambda \) we will characterize the resolvent in terms of the Jost solutions of \( H \). The desired estimates will follow from scattering-theory results of Deift and Trubowitz [4], using similar arguments to those in Goldberg-Schlag [5].

To separate the cases of low and high energy, let \( \chi \) be a smooth even cutoff function that is equal to one when \( |x| \leq \lambda_0 \) and is supported on the interval where \( |x| \leq 2\lambda_0 \). The value of \( \lambda_0 \) will be determined later, and depends primarily on the size of \( V \). We will adopt the following notation for discussing polynomially weighted \( L^p \) spaces.

\[
\langle x \rangle := (1 + |x|^2)^{\frac{1}{2}}
\]

\[
\|f\|_{L^p_\sigma} := \|\langle x \rangle^\sigma f\|_{L^p}
\]

1. **HIGH ENERGY ESTIMATES**

Both Theorem 1 and Theorem 2 rely on the same estimate for the high-energy part of the evolution. This result can be found in [11] but we include it here for the sake of completeness.

**Proposition 3.** Assume that \( V \in L^1_1(\mathbb{R}) \) and choose \( \lambda_0 \geq ||V||_1 \). The following estimate is valid for all functions \( \psi, \varphi \in L^1_1(\mathbb{R}) \).

\[
\langle e^{itH} (1 - \chi(\sqrt{H}))\psi, \varphi \rangle \lesssim |t|^{-\frac{1}{2}} \|\langle x \rangle^\sigma \psi\|_1 \|\langle x \rangle^\sigma \varphi\|_1.
\]

**Proof.** By the same spectral argument that led to (5), we are estimating here the integral

\[
\frac{1}{\pi i} \int_{-\infty}^{\infty} e^{it\lambda^2} \lambda(1 - \chi(\lambda))\langle R_+^+(\lambda^2)\psi, \varphi \rangle \, d\lambda.
\]

Integrate by parts once to obtain

\[
\frac{-1}{2\pi i} \int_{-\infty}^{\infty} e^{it\lambda^2} d\lambda \left( (1 - \chi(\lambda))\langle R_+^+(\lambda^2)\psi, \varphi \rangle \right) \, d\lambda.
\]

The perturbed resolvent \( R_+^+(\lambda^2) \) can be linked to the free resolvent \( R_+^0(\lambda^2) \) via the identity \( R_+^+(\lambda^2) = R_+^0(\lambda^2)(I + VR_0^+(\lambda^2))^{-1} \), leading to the Born series expansion

\[
R_+^+(\lambda^2) = \sum_{k=0}^{\infty} R_+^0(\lambda^2)(VR_0^+(\lambda^2))^k.
\]
The free resolvent $R_0^+(\lambda^2)$ has an explicit representation as an integral operator with kernel $K(x, y) = (2\lambda)^{-1} e^{i\lambda|x-y|}$. Substituting this into the identity above leads to the expression

$$\langle R_0^+(\lambda^2)\psi, \varphi \rangle = \sum_{k=0}^{\infty} (2\lambda)^{-k} \int_{\mathbb{R}^{k+2}} e^{i\lambda\sum_{j=0}^{k} |x_{j+1} - x_j|} \psi(x_0)V(x_1)\ldots V(x_k)\varphi(x_{k+1}) \, dx_0 \ldots dx_{k+1}$$

which is a convergent series provided $2|\lambda| \geq \|V\|_1$. The absence of a boundary term in the integration by parts (7) is justified by a similar argument.

When this is substituted back into the integral (7) the differentiation in $\lambda$ leads to two distinct terms. Up to a constant factor, we have

$$(e^{itH}(1 - \chi(\sqrt{H}))\psi, \varphi) =$$

$$(8) \quad \frac{1}{t} \int_{\mathbb{R}^{k+3}} \sum_{k=0}^{\infty} \sum_{m=0}^{k} e^{it\lambda^2} e^{i\lambda\sum_{j=0}^{k} |x_{j+1} - x_j|} \frac{(k+1)(1 - \chi(\lambda)) - \frac{\chi'(\lambda)}{\lambda}}{(2i\lambda)^{k+1}} |x_{m+1} - x_m| \times \psi(x_0)V(x_1)\ldots V(x_k)\varphi(x_{k+1}) \, dx_0 \ldots dx_{k+1} \, d\lambda$$

and

$$(8') \quad -\frac{1}{t} \int_{\mathbb{R}^{k+3}} \sum_{k=0}^{\infty} e^{it\lambda^2} e^{i\lambda\sum_{j=0}^{k} |x_{j+1} - x_j|} \frac{k(1 - \chi(\lambda)) + \frac{\chi'(\lambda)}{\lambda}}{\lambda(2i\lambda)^{k+1}} \times \psi(x_0)V(x_1)\ldots V(x_k)\varphi(x_{k+1}) \, dx_0 \ldots dx_{k+1} \, d\lambda.$$

In each of the terms we may rearrange the order of integration to handle the $d\lambda$ integral first (for the $k=0$ term in (8) this requires restricting to compact support in $\lambda$ and taking limits; otherwise it is permitted by Fubini’s theorem). Evaluate this integral using Plancherel’s identity: The Fourier transform of the oscillatory component $e^{it\lambda^2} e^{i\lambda\sum |x_{j+1} - x_j|}$ is bounded above by $|t|^{-\frac{3}{2}}$ uniformly in the choice of all $x_j$. The Fourier transform of each expression involving the cutoff function (e.g. $\frac{1 - \chi(\lambda)}{\lambda(2i\lambda)^{k+1}}$) is integrable with $L^1(\mathbb{R})$ norm bounded by $k(2\lambda_0)^{-k-1}$. This implies that

$$|\langle e^{itH}(1 - \chi(\sqrt{H}))\psi, \varphi \rangle| \lesssim |t|^{-\frac{3}{2}} \sum_{k=0}^{\infty} k(2\lambda_0)^{-k-1} \times \sum_{m=0}^{k} \langle x_{m+1} - x_m \rangle \langle \psi(x_0)V(x_1)\ldots V(x_k)\varphi(x_{k+1}) \rangle \, dx_0 \ldots dx_{k+1}.$$

The sum of differences $\langle x_{m+1} - x_m \rangle$ can be controlled by $2 \sum_{m=0}^{k} \langle x_m \rangle$ using the triangle inequality. The inner integral is then separable, with the eventual bound

$$|\langle e^{itH}(1 - \chi(\sqrt{H}))\psi, \varphi \rangle| \lesssim |t|^{-\frac{3}{2}} \sum_{k=0}^{\infty} k^2(2\lambda_0)^{-k-1} \|V\|_1^{k-1} \|\langle x \rangle V\|_1 \|\langle x \rangle \psi\|_1 \|\langle x \rangle \varphi\|_1$$

$$\lesssim \lambda_0^{-1} |t|^{-\frac{3}{2}} \|\langle x \rangle \psi\|_1 \|\langle x \rangle \varphi\|_1$$

provided $\lambda_0 \geq \|V\|_1$. \( \square \)

2. Low Energy Estimates

It remains to control the behavior of $e^{itH}\chi(\sqrt{H})\psi$, with the result depending on whether or not $H$ has a resonance at zero. The Born series used previously cannot be made to converge, so we rely instead on a characterization of the resolvent in
terms of Jost solutions. For each $\lambda \in \mathbb{R}$, let $f_{\pm}(x, \lambda)$ be the unique functions which satisfy

$$-f''_{\pm}(x, \lambda) + (V(x) - \lambda^2)f_{\pm}(x, \lambda) = 0, \quad f_{\pm}(x, \lambda) = e^{\pm i \lambda x} \text{ as } x \to \pm \infty$$

and $W(\lambda) := W[f_+(\cdot, \lambda), f_-(\cdot, \lambda)]$ be their Wronskian. Define also the Wronskian $\tilde{W}(\lambda) = W[f_-(\cdot, \lambda), f_+(\cdot, -\lambda)]$. The perturbed resolvent $R_V(\lambda^2)$ is an integral operator whose kernel is given by

$$(9) \quad R_V^+(\lambda^2)(x, y) = \frac{f_+(x, \lambda)f_-(y, \lambda)}{W(\lambda)}$$

for all $x \geq y$, and is symmetric for $x < y$.

Note that $f_{\pm}(\cdot, -\lambda)$ solve the same second-order differential equation as $f_{\pm}(\cdot, \lambda)$, hence they must be linearly dependent. The coefficients in the relation

$$(10) \quad f_-(x, \lambda) = \alpha(\lambda)f_+(x, \lambda) + \beta(\lambda)f_+(x, -\lambda)$$

are given by $\alpha(\lambda) = \frac{W(\lambda)}{2i\lambda}$ and $\beta(\lambda) = \frac{W(\lambda)}{2i\lambda}$. These in turn are closely linked to the reflection and transmission coefficients, namely: $\alpha(\lambda) = \frac{R(\lambda)}{T(\lambda)}$ and $\beta(\lambda) = \frac{1}{T(\lambda)}$.

Conjugate symmetry requires that $\beta(-\lambda) = \overline{\beta(\lambda)}$ and $\alpha(-\lambda) = \overline{\alpha(\lambda)}$. Conservation of energy additionally requires that $|\alpha(\lambda)|^2 + 1 = |\beta(\lambda)|^2$ for every value of $\lambda$.

Since $\beta(\lambda)$ is always positive, $W(\lambda)$ cannot vanish except possibly when $\lambda = 0$. The condition $W(0) = 0$ is satisfied precisely if zero is a resonance; in the generic (non-resonant) case the values of $W(\lambda)$ are everywhere nonzero.

It is common to rewrite the Jost solutions as $f_{\pm}(x, \lambda) = e^{\pm i \lambda x}m_{\pm}(x, \lambda)$, where $m_{\pm}(x, \lambda) \to 1$ as $x \to \pm \infty$. The relevant properties of the functions $m_{\pm}(x, \lambda)$ are summarized below. See [4], Lemma 3 for details.

**Lemma 4.** Suppose $V \in L^1_{\sigma}, \sigma \geq 1$. For each $x$ the functions $m_{\pm}(x, \cdot) - 1$ belong to the Hardy space $H^{2+}$ of analytic functions on the upper half-plane. Consequently, their Fourier transform in the second variable, denoted by $m_{\pm}(x, \hat{\cdot})$, is supported on the halfline $\rho \geq 0$.

Define $I(\rho) := \int_{|t| > \rho} |V(t)| \, dt$. The following pointwise estimates for $m_{\pm}(x, \hat{\cdot})$ are valid over the specified ranges of $x$ and all $\rho > 0$.

$$(11) \quad \text{If } x \geq 0, \text{ then } \begin{cases} |m_+(x, \hat{\cdot}) - \delta_0(\rho)| \lesssim I(\rho) \\ |\frac{\partial}{\partial \rho} m_+(x, \hat{\cdot})| \lesssim I(\rho) + |V(x + \rho)| \\ |\frac{\partial^2}{\partial \rho^2} m_+(x, \hat{\cdot})| \lesssim I(\rho) + |V(x + \rho)| \end{cases}$$

$$\text{If } x \leq 0, \text{ then } \begin{cases} |m_-(x, \hat{\cdot}) - \delta_0(\rho)| \lesssim I(\rho) \\ |\frac{\partial}{\partial \rho} m_-(x, \hat{\cdot})| \lesssim I(\rho) + |V(x - \rho)| \\ |\frac{\partial^2}{\partial \rho^2} m_-(x, \hat{\cdot})| \lesssim I(\rho) + |V(x - \rho)| \end{cases}$$

It follows that each of the above functions involving $m_{\pm}(x, \hat{\cdot})$ belongs to $L^1_{\sigma-1}(\mathbb{R})$, uniformly over all $x$ in the appropriate halfline. Furthermore, the Fourier transform of $\partial_\lambda m_{\pm}(x, \cdot)$ belongs to $L^1_{\sigma-2}(\mathbb{R})$.

**Corollary 5.** Suppose $V \in L^1_{\sigma}(\mathbb{R}), \sigma \geq 1$ and let $\tilde{\chi}(\lambda) = \chi(\frac{\lambda}{\hat{\lambda}})$. The functions $\tilde{\chi}(\lambda)W(\lambda)$ and $\tilde{W}(\lambda)$ both have Fourier transform (with respect to $\lambda$) in the space $L^1_{\sigma-1}(\mathbb{R})$. 

Proof. Recall that \( f_\pm(x,\lambda) = e^{\pm i \lambda x} m_\pm(x,\lambda) \). By this definition,
\[
\hat{\chi}(\lambda)W(\lambda) = \hat{\chi}(\lambda)(m_+(0,\lambda)\partial_x m_-(0,\lambda) - \partial_x m_+(0,\lambda)m_-(0,\lambda)) - 2i\lambda \hat{\chi}(\lambda)m_+(0,\lambda)m_-(0,\lambda)
\]
and
\[
\hat{W}(\lambda) = m_-(0,\lambda)\partial_x m_+(0,\lambda) - \partial_x m_-(0,\lambda)m_+(0,\lambda) - \lambda
\]
According to the pointwise bounds in (11), each individual function \( m_\pm(0,\pm\lambda) \) has Fourier transform in \( L^1_{\sigma-1} \), which is an algebra with respect to convolutions.

Proof of Theorem [11]. The desired bounds have already been established in the high energy case by Proposition [13]. The remaining task is to evaluate the part of the integral not considered in (11), namely
\[
\frac{-1}{2\pi i} \int_{-\infty}^{\infty} e^{it\lambda^2} \frac{d}{d\lambda} [\chi(\lambda)\langle R^+_\lambda (\lambda^2)^2,\varphi \rangle] d\lambda.
\]
After applying the formula (11) for the integral kernel of \( R^+\lambda^2 \) and Plancherel’s identity, it suffices to show that the Fourier transform (in \( \lambda \)) of
\[
\frac{d}{d\lambda} \left[ \chi(\lambda) \frac{f_-(x,\lambda)f_+(y,\lambda)}{\hat{\chi}(\lambda)W(\lambda)} \right]
\]
belongs to \( L^1(\mathbb{R}) \) with norm bounded by \( \langle x \rangle \langle y \rangle \) for all choices of \( x \leq y \). The correct estimate will also hold for \( x > y \) by symmetry of the resolvent.

First consider the case \( x \leq 0 \leq y \). We are interested in the Fourier transform of the function
\[
i(y-x)\frac{e^{i(y-x)(\cdot)}\chi m_-(x,\cdot)m_+(y,\cdot)}{\hat{\chi}W} + e^{i(y-x)(\cdot)}\partial_\lambda \left[ \chi m_-(x,\cdot)m_+(y,\cdot) \right] \frac{\hat{W}}{\hat{\chi}W} - \frac{e^{i(y-x)(\cdot)}\chi m_-(x,\cdot)m_+(y,\cdot)\partial_\lambda[\hat{\chi}W]}{(\hat{\chi}W)^2}.
\]
Lemma [14] ensures that the Fourier transform of each numerator has \( L^1 \) norm bounded uniformly in \( x \leq 0 \leq y \). If zero is not a resonance, then the Wronskian \( W(\lambda) \) is everywhere nonzero. The Wiener Lemma (see, for example, [8], Chapter VIII) then implies that \( \chi(\cdot)\langle \hat{\chi}W \rangle^{-1} \) also has integrable Fourier transform, making the division possible as well. Collectively, the \( L^1 \) norm of the Fourier transform will be bounded by \( |y-x| + 1 \) a constant, which in turn is bounded by \( \langle x \rangle \langle y \rangle \).

In the case \( 0 < x < y \), there is no uniform control over quantities derived from \( m_-(x,\lambda) \). To avoid this problem, use the intertwining coefficients to write
\[
f_-(x,\lambda) \equiv \alpha(\lambda)f_+(x,\lambda) + \beta(\lambda)f_+(x,-\lambda)
\]
\[
= -\frac{1}{4i} \left[ \frac{\hat{W}(\lambda) + W(\lambda)}{\lambda} (f_+(x,\lambda) + f_+(x,-\lambda)) + (\hat{W}(\lambda) - W(\lambda)) \left( e^{i\lambda x} \frac{m_+(x,\lambda) - m_+(x,0)}{\lambda} \right. \right.
\]
\[
\left. + \left. e^{-i\lambda x} \frac{m_+(x,0) - m_+(x,-\lambda)}{\lambda} \right) + 2i \sin(\lambda x) \frac{m_+(x,0)}{\lambda} \right].
\]
The only functions here of any concern are the expressions with \( \lambda \) in the denominator. Observe that not only is \( m_+(x,\hat{\rho}) - m_+(x,0)\delta_0(\hat{\rho}) \in L^1_{\sigma-1}(\mathbb{R}) \), (if one accepts
a delta-function at the origin as integrable), but its integral over the real line is exactly zero. Because of this, the Fourier transform of \( \frac{m_+(x,\lambda) - m_+(x,0)}{\lambda} \) is given by

\[
(13) \quad \left[ m_+(x,\lambda) - m_+(x,0) \right] \hat{\lambda} (s) = -i \int_s^\infty m_+(x,\hat{\lambda}) \, d\hat{\lambda}
\]

which belongs to \( L^1_{\sigma-2}(\mathbb{R}) \) uniformly in \( x \geq 0 \). The term \( (m_+(x,0) - m_+(x,-\lambda)/\lambda \) is treated the same way.

An identical argument holds for the fraction \( \frac{W(\lambda) + W(\lambda)}{\lambda} \) by expanding out each Wronskian according to its definition. One can recognize this as a restatement of the well-known fact about reflection coefficient at zero energy: \( R_1(\lambda) = -1 \).

To complete the calculations for these terms as in the previous case, one may need to deal with derivatives such as \( \frac{d}{d\lambda} [W(\lambda) + W(\lambda)] \). The Fourier transform of such a function is in \( L^1_{\sigma-2}(\mathbb{R}) \), which is still integrable provided \( \sigma \geq 3 \). Depending on where else the derivative in \( (12) \) may fall, one obtains norm bounds of size \( \langle x \rangle + \langle y \rangle + 1 \), which is again bounded by \( \langle x \rangle \langle y \rangle \).

For the term with \( \sin(\lambda x)/\lambda \), it is best to go back to the original integral \( (10) \). Apply Plancherel’s identity to the expression

\[
\int_{-\infty}^{\infty} e^{it\lambda^2} \sin(\lambda x) \chi(\lambda) \frac{(\hat{\chi}(\lambda)-W(\lambda))f_+(y,\lambda)}{\hat{\chi}(\lambda)W(\lambda)} \, m_+(x,0) \, d\lambda
\]

and observe that the Fourier transform of \( e^{it\lambda^2} \sin(\lambda x) \) is a multiple of

\[
t^{-1/2} \left( e^{-i(x-y)^2/4t} - e^{-i(x+y)^2/4t} \right) \leq t^{-3/2} ||\rho|| |x|.
\]

The previous estimation of \( (12) \) is sufficient to show that the Fourier transform of \( \frac{\chi(\lambda)f_+(y,\lambda)}{\hat{\chi}(\lambda)W(\lambda)} (\hat{W}(\lambda) - W(\lambda)) \) belongs to \( L^1_{\sigma-2}(\mathbb{R}) \) with norm controlled by \( \langle y \rangle \). Thus the size of this term is not more than \( t^{-3/2} |x| |y| \), as desired.

The case \( x < y < 0 \) is handled in an identical by using the intertwining relation \( f_+(y,\lambda) = -\alpha(\lambda)f_-(y,\lambda) + \beta(\lambda)f_-(y,-\lambda) \) instead of \( (10) \).

\[\Box\] Proof of Theorem \[2\]. All of the estimates in Lemma \[1\] are still valid in the resonant case. The one fundamental difference is that \( W(\lambda) \) vanishes when \( \lambda = 0 \) (and at no other \( \lambda \in \mathbb{R} \)). Consequently, the functions \( \alpha(\lambda) \) and \( \beta(\lambda) = \frac{W(\lambda)}{2\pi} \) are both continuous and real-valued at the origin. The Fourier transform of \( \alpha \) and \( \beta \) lie in the space \( L^1_{\sigma-2}(\mathbb{R}) \), and moreover \( \beta(\lambda) \neq 0 \) over the entire real line.

Thanks to the resonance, the integral

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\lambda^2} \chi(\lambda) \frac{f_-(x,\lambda)f_+(y,\lambda)}{\hat{\chi}(\lambda)\beta(\lambda)} \, d\lambda
\]

must have a stationary phase contribution on the order of \( |t|^{-1/2} \). The integrand is sufficiently regular that one can isolate the leading term

\[
(14) \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\lambda^2} \frac{f_-(x,0)f_+(y,0)}{\beta(0)} \, d\lambda = (-4\pi i t)^{-\frac{1}{2}} \frac{f_-(x,0)f_+(y,0)}{\beta(0)}
\]

leaving a remainder of order \( |t|^{-2} \). It is clear that \( f_-(\cdot,0) \) and \( f_+(\cdot,0) \) are both scalar multiples of \( f_0 \). The limiting values of \( f_-(x,0) \) as \( x \rightarrow \pm\infty \) are \( \beta(0) + \alpha(0) \).
and 1, respectively. This makes
\[ f_-(x, 0) = \sqrt{1 + (\beta(0) + \alpha(0))^2} f_0(x). \]
A similar argument shows that \( f_+(y, 0) = \sqrt{1 + (\beta(0) - \alpha(0))^2} f_0(y) \). The two square roots have signs in common if \( \beta(0) > 0 \) and are of opposite sign if \( \beta(0) < 0 \). The last line of [14] is obtained from this fact and the identity \( \alpha^2(0) + 1 = \beta^2(0) \).

The remainder term is given explicitly by
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\lambda^2} (G_{x,y}(\lambda) - G_{x,y}(0)) \, d\lambda = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} e^{it\lambda^2} \frac{d}{d\lambda} \left[ \frac{G_{x,y}(\lambda) - G_{x,y}(0)}{\lambda} \right] \, d\lambda
\]
where
\[
G_{x,y}(\lambda) = e^{i\lambda(y-x)} \chi(\lambda) \frac{m_-(x,\lambda)m_+(y,\lambda)}{\chi(\lambda)\beta(\lambda)}
\]
\[
= e^{i\lambda(y+x)} \chi(\lambda)\alpha(\lambda)m_+(x,\lambda)m_+(y,\lambda) + e^{i\lambda(y-x)} \chi(\lambda)m_+(x,-\lambda)m_+(y,\lambda).
\]
One uses the first formula for \( G_{x,y}(\lambda) \) in the case \( x \leq y \) and the second formula when \( 0 < x < y \). There is a third formula, quite similar to the second, which is useful when \( x \leq y < 0 \).

In order to complete the proof it suffices to bound the \( L^1 \) norm of the Fourier transform of \( \frac{d}{d\lambda} \left[ \frac{G_{x,y}(\lambda) - G_{x,y}(0)}{\lambda} \right] \) by the quantity \( \langle x \rangle^2 \langle y \rangle^2 \). If we are using the second formula for \( G_{x,y} \), it is permissible to bound each term separately. All of these estimates are consequences of the general rule stated below.

**Proposition 6.** Suppose the Fourier transform of \( F(\lambda) \) belongs to \( L^1_2(\mathbb{R}) \). Define
\[
G(\lambda) = \frac{d}{d\lambda} \left[ \frac{e^{ik\lambda}F(\lambda) - F(0)}{\lambda} \right].
\]
The Fourier transform of \( G \) is integrable, with the bound \( \|G\|_1 \lesssim \langle k \rangle^2 \| \hat{F} \|_{L^1_2} \).

Write out \( G(\lambda) = \frac{d}{d\lambda} \left[ \frac{e^{ik\lambda} - 1}{\lambda} \right] F(\lambda) + \frac{e^{ik\lambda} - 1}{\lambda} F'(\lambda) + \frac{d}{d\lambda} \left[ \frac{F(\lambda) - F(0)}{\lambda} \right] \).

The Fourier transforms of \( \frac{e^{ik\lambda} - 1}{\lambda} \) and its derivative are integrable, with norms proportional to \( k \) and \( k^2 \) respectively. The Fourier transform of \( \frac{F(\lambda) - F(0)}{\lambda} \) belongs to \( L^1_2(\mathbb{R}) \) (compare to [13] to see that this is controlled by \( \| \hat{F} \|_{L^1_2} \)), and that of its derivative is integrable. By convolution in \( L^1 \), each of the three terms above will yield a bound no greater than \( \langle k \rangle^2 \| \hat{F} \|_{L^1_2} \), as desired. \( \square \)

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