Tachyons and (non)vanishing scalar masses in six-dimensional gauge theories with flux compactification

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24 August 2021

Abstract

In this paper, we study the possibility to obtain a massless scalar boson for which quantum corrections to the mass vanish at all loop-order, which has been recently understood to be due to a shift symmetry making the scalar a Goldstone boson. We present the effective four-dimensional Lagrangian of a six-dimensional gauge theory compactified on a torus with magnetic flux. Because of this magnetic field, a symmetry of translation in the extra dimensions is broken which implies the existence of a massless scalar boson. We then explicitly check that a model with two U(1) gauge symmetries contains a scalar boson with finite mass but protected from large quantum corrections. Finally, we study the presence of tachyons in the model with non-abelian gauge symmetry. In particular, we propose a way to eliminate these tachyons and we compute the full mass spectrum of the scalars in this theory. Finally, we show that our method preserve the chirality of fermions in the model.
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1 Introduction

The Standard Model of fundamental interactions has been a successfully tested theory for decades. Nevertheless, it has left some fundamental questions unanswered, as the hierarchy problem which has been the main guideline for the search of physics beyond the standard model. These unanswered questions suggest that the Standard Model cannot be the fundamental theory of fundamental interactions. Thus, at high energy scales, one should find signs of new physics, i.e. deviations from the Standard Model predictions. As new physics has not been observed yet, the scale of possible underlying physics goes higher and higher and the fine-tuning necessary to obtain the 125 GeV of the Higgs gets stronger with time.

This report is mainly focused on the hierarchy problem which consists of explaining the low mass of the Higgs boson while the quantum corrections to its mass are huge in the framework of the standard model. A desirable solution could be that the Higgs mass is zero at tree level and only due to quantum corrections. During this internship, we explored the promising possibility to use flux compactification for this purpose.

Flux compactification plays an important role in string theory and field theories. It has been studied extensively as it has many nice properties. Indeed, it leads to a multiplicity of chiral fermions, explains the number of lepton-quark generations [1], it can break supersymmetry [2], and stabilize the compact dimensions [3]. In the context of compact extra dimensions, it is known that there is a possibility to use zero-modes of higher dimensional gauge fields as Higgs field in four dimensions [4, 5]. It appears that identifying these zero modes with the Higgs scalar gives a finite Higgs mass protected by the size of the extra dimensions $m_{Higgs}^2 \propto L^{-2}$ [6]. Thus this mechanism is interesting if the extra dimensions are large.

However, if the extra dimension is small, i.e. if the UV cutoff is much higher than the electroweak scale, one has to find an alternative. Using magnetic compactification, it has been shown recently that quantum corrections to the mass of the zero modes of the scalar field induced by extra components of the gauge field vanish. This was first shown by [7] in a supersymmetric model. The vanishing of the one-loop correction has been studied more carefully in [8], taking particular care to the regularization of divergent momentum integrals via dimensional regularization. In a second paper [9], the authors of [7] studied the cancellation of one-loop corrections in more detail and showed explicitly how this cancellation is due to an exact shift symmetry in the higher dimensional theory.

This was done without supersymmetry as the cancellation is due to the shift symmetry and not SUSY. The one-loop corrections to the zero-mode mass have then been calculated in a six-dimensional Yang-Mills theory with flux compactification [10]. As expected, the quantum corrections are canceled. Moreover, [11] studied the possibility to produce a nonvanishing finite Wilson line scalar mass. Indeed, the Wilson line must have a finite mass if one wants to identify it with the Higgs field which is massive. They therefore classified the interactions terms breaking the shift symmetry. As for cosmological applications, [12] proposed a new inflation scenario where the massless Wilson line is identified with an inflaton.

This paper is organized as follows. First, we introduce flux compactification by a simple example in abelian gauge theories. Then, we present the supersymmetric model proposed by [7] and compute the 4d effective Lagrangian. The non-supersymmetric model of [9] is then introduced. In the next two subsections, we compute the one-loop corrections to the Wilson line scalar mass and explain its cancellation with a shift symmetry of the 6d action [9]. We then show explicitly how a $U(1) \times U(1)$ gauge symmetry could lead to a finite Higgs mass with vanishing one-loop corrections. This could lead to a finite Higgs mass protected from high corrections due to the possibly high compactification scale.

In section 3, we study a six-dimensional Yang-Mills theory compactified on a torus with magnetic flux. [7] showed the presence of tachyons in the $SU(2)$ model, implying that the effective action corresponds to an expansion around an extremal point which is not the true ground state. Rather than studying tachyon condensation, we try to eliminate the tachyon by adding a scalar boson charged under the gauge symmetry, which gets a vacuum expectation value. Finally, we check if this model could include chiral fermions.
2 Interacting model in 6 dimensions with flux compactification

2.1 Flux compactification on a torus

Let us begin by introducing flux compactification in a simple example. Flux compactification produces a mass spectrum in the spirit of the Landau levels in quantum mechanics. Let us consider a scalar field of charge q under the abelian gauge symmetry of coupling constant g carried by the gauge field $A^M$. The six dimensional action for the scalar is

$$S_6 = -\int d^6x D_M \bar{\chi} D^M \chi$$  \hspace{1cm} (2.1)

with

$$D_M = \partial_M + igqA_M$$  \hspace{1cm} (2.2)

First we must compactify the extra dimensions. A common way to compactify the extra dimensions is to compactify on a torus. For a factorizable torus, the compact dimensions are treated as circles of length $2\pi r$. This correspond to the identification $y \leftrightarrow y + 2\pi r$. In this case, $V_n = (2\pi r)^n$. A magnetic flux in the background correspond to a constant flux density $f$ in the internal dimensions. One can make the following gauge choice:

$$A_5 = -\frac{1}{2} fx_6 \hspace{1cm} A_6 = \frac{1}{2} fx_5 \hspace{1cm} F_{56} = f$$  \hspace{1cm} (2.3)

For a torus of finite volume $L^2$, the flux is quantized

$$\frac{qg}{2\pi} \int_{T^2} F = \frac{qg}{2\pi} L^2 f = N \in \mathbb{Z}$$  \hspace{1cm} (2.4)

We choose without loss of generality $qf < 0$. By splitting the kinetic term into a 4D kinetic term and a kinetic term on $T^2$, one can find :

$$S_6 = -\int \eta^{\mu\nu} D_\mu \bar{\chi} D_\nu \chi - \bar{\chi} H^2 \chi$$  \hspace{1cm} (2.5)

where

$$H^2 = -\frac{2}{qgf} (a^\dagger a + 1/2)$$  \hspace{1cm} (2.6)

is written by defining a creation and an annihilation operator functions of the internal space components.

$$a = \frac{i}{\sqrt{-2qgf}} (\bar{\partial} - qgfz), \hspace{1cm} a^\dagger = \frac{i}{\sqrt{-2qgf}} (\bar{\partial} + qgfz)$$  \hspace{1cm} (2.7)

with

$$z = \frac{1}{2} (x_5 + ix_6), \hspace{1cm} \bar{\partial} = \partial_{\bar{z}} = \partial_5 - i\partial_6$$  \hspace{1cm} (2.8)

The ladder operators $a, a^\dagger$ satisfy the relation $[a, a^\dagger] = 1$. Thus, based on the work of [2], we have showed that the internal Hamiltonian can be written in the form of a harmonic oscillator Hamiltonian, completing the analogy with the Landau levels. We denote the internal fields $\xi_{n,j}$ where $n$ is the index for the Landau levels and $j$ is the degeneracy index going from 0 to $|N| - 1$. Starting from the lowest mass fields,

$$a\xi_{0,j} = 0 \hspace{1cm} a^\dagger \xi_{0,j} = 0,$$  \hspace{1cm} (2.9)

one defines all the field profiles using the ladder operators.

$$\xi_{n,j} = \frac{1}{\sqrt{n!}} (a^\dagger)^n \xi_{0,j} \hspace{1cm} \bar{\xi}_{n,j} = \frac{1}{\sqrt{n!}} (a)^n \bar{\xi}_{0,j}$$  \hspace{1cm} (2.10)

\footnote{This is showed in the appendix of [9]. For another approach, see footnote 2 of [8].}

\footnote{The common notation is to write the nth extra component with index n, even if consistency with the 4-vector notation would ask for index n-1.}
They satisfy the same orthonormality relation as the Kaluza-Klein level. However, the KK levels are not orthonormal to the Landau levels.

\[ \int_{T^2} d^2 x' \xi_{n',j'} \xi_{n,j} = \delta_{n,n'} \delta_{j,j'} \]  

(2.11)

The function \( \xi_{n,j} \) forms a complete set of functions and we can therefore develop the charged fields with respect to the Landau levels.

\[ \chi(x_M) = \sum_{n,j} \chi_{n,j}(x_\mu) \xi_{n,j}(x_m) \quad \bar{\chi}(x_M) = \sum_{n,j} \bar{\chi}_{n,j}(x_\mu) \bar{\xi}_{n,j}(x_m) \]  

(2.12)

Using the harmonic oscillator algebra, the 6D action then becomes:

\[ S_4 = \int d^4 x \sum_{n,j} (-D_\mu \bar{\chi}_{n,j} D^\mu \chi_{n,j} + 2qgf(n + 1/2) \bar{\chi}_{n,j} \chi_{n,j}) \]  

(2.13)

which contains a mass term for the scalars \( \chi_{n,j} \).

\[ m_{n,j} = -2qgf(n + 1/2) = \frac{2\pi|N|}{L^2}(n + 1/2) \]  

(2.14)

2.2 \( N=1 \) supersymmetric gauge theory in 6 dimension with flux compactification

The simplest 6D matter multiplet is the hypermultiplet which decomposes into two 4D chiral multiplets of opposite charge. We decompose these chiral multiplets \( Q, \bar{Q} \) with respect to the Landau levels.

\[ Q(x_M, \theta, \bar{\theta}) = \sum_{n,j} Q_{n,j}(x_\mu, \theta, \bar{\theta}) \xi_{n,j}(x_m) \]

(2.15)

\[ \bar{Q}(x_M, \theta, \bar{\theta}) = \sum_{n,j} \bar{Q}_{n,j}(x_\mu, \theta, \bar{\theta}) \bar{\xi}_{n,j}(x_m) \]

In this section, we will recall the models presented in [7] and compute the effective mass spectrum for a charged matter multiplet with abelian gauge interactions compactified on a torus with magnetic flux.

To bring an abelian gauge interaction into the model, we will need an uncharged 6D vector multiplet. Since it is uncharged, it decomposes in the usual KK modes as in equation (A.11)

The \( N = 2 \) vector multiplet corresponds in \( N = 1 \) SUSY to one vector multiplet and one chiral multiplet, we note them \( V \) and \( \phi \). To see this more explicitly, the 6D vector field is composed of one six components vector field (4 d.o.f.) and a Dirac gaugino (4 d.o.f) which adds up to 8 d.o.f.

In four dimensions, the vector field split into one 4-vector field and one complex scalar field and the Dirac gaugino split into two Weyl gauginos. Thus, in four dimensions, we get one vector field, one complex scalar and two spinors which correspond to a 4D \( N=2 \) vector multiplet or one \( N=1 \) vector multiplet plus one chiral multiplet. Let us remind that \( x_5 \) and \( x_6 \) play the role of indices for the superfields \( V \) and \( \phi \), this dependence will be implicit for the rest of this work. We will only consider the zero modes of the uncharged sector: \( \phi_0 \) and \( V_0 \).

It is \( \phi_0 \) that contains the internal component of the gauge field and therefore the magnetic flux. Indeed, \( \phi|_{\theta=\bar{\theta}=0} = \frac{1}{\sqrt{2}}(A_6 + iA_5) \). Let us decompose explicitly \( \phi_0 \) into its vacuum expectation value (vev) and its perturbation. The perturbation is \( \varphi \), the Wilson line. And the vev is the magnetic flux’s contribution to the background.

\[ \phi_0|_{\theta=\bar{\theta}=0} = \frac{f}{2\sqrt{2}}(x_5 - ix_6) + \varphi \]  

(2.16)
The supersymmetric action for a N=2 vector multiplet is [13]:

\[ S_6 = \int d^6x \left\{ \int d^4\theta \left( \partial V \overline{\partial V} + \partial \overline{\phi} + \sqrt{2} V (\partial \overline{\phi} + \overline{\partial \phi}) \right) + \left( \frac{1}{4} \int d^2\theta W^a W_a + h.c. \right) \right\} \] (2.17)

The action for a hypermultiplet interacting with the vector multiplet \( \{V, \phi\} \) is the following. The hypermultiplet being chiral, one must choose the gaugino field to be of the same chirality.

\[ S_6 = \int d^6x \left\{ \int d^4\theta \left( \overline{Q}_n e^{2gqV_0} Q_n + \overline{\tilde{Q}}_n e^{-2gqV_0} \tilde{Q}_n \right) + \left( \int d^2\theta \overline{\tilde{Q}}_n (\partial + \sqrt{2}gq\phi) Q_n + h.c. \right) \right\} \] (2.18)

By replacing the appropriate parts with ladder operators and the chiral fields by their expansions in KK and Landau modes, one can show that the effective action is [7]:

\[ S_4 = \int d^4x \left[ \int d^4\theta \left( \overline{Q}_n e^{2gqV_0} Q_n + \overline{\tilde{Q}}_n e^{-2gqV_0} \tilde{Q}_n \right) + 2fV_0 \right] \]

\[ + \int d^2\theta \left( \frac{1}{4} W^a W_a,0 + \frac{1}{2} \theta \overline{\theta} \overline{\theta} \overline{\theta} + \frac{1}{2} \sqrt{\frac{2}{gq}} \overline{\tilde{Q}}_n e^{2gqV_0} Q_n + h.c. \right) \] (2.19)

This action contains a superpotential with mass terms for the charged superfields, and an interaction term coupling them to the Wilson lines. The kinetic terms in the action gives kinetic terms for the charged fields, the 4D gauge fields, the Wilson line, and a Fayet-Iliopoulos term \( 2fV_0 \). In order to obtain the physical mass spectrum and interactions terms for the component fields, one has to eliminate the auxiliary fields. We denote the component fields as follows.

\[ Q = Q + \sqrt{2}\theta \chi + \theta^2 F \]

\[ V_0 = -\theta \sigma^\mu \overline{\theta} A_\mu + i\overline{\theta} \theta \lambda + \frac{1}{2} \theta \overline{\theta} \overline{\theta} \overline{\theta} + \frac{1}{2} \theta \overline{\theta} \overline{\theta} \overline{\theta} + \frac{1}{2} \theta \overline{\theta} \overline{\theta} \overline{\theta} \] (2.20)

This leads to the effective Lagrangian for the on-shell component fields given in section 3.2 of [7]. We will not give the full expression which is quite tedious. However, it has the following form:

\[ \mathcal{L}_{eff} = \mathcal{L}_{kin} + \mathcal{L}_M + \mathcal{L}_{int} - \frac{1}{2} f^2 \] (2.21)

and gives the following masses:

\[ m^2_{\overline{Q}_{n,j}} = m^2_{Q_{n,j}} = \frac{2\pi |N|}{L^2} (2n + 1) \]

\[ m^2_{\Psi_{n,j}} = \frac{4\pi |N|}{L^2} (2n + 1) \] (2.22)

where \( \Psi_{n,j} \) is formed by chiral fermions paired up as:

\[ \Psi_{n,j} = \left( \begin{array}{c} \tilde{\chi}_{n+1,j} \\ \chi_{n,j} \end{array} \right) \] (2.23)

and the \( \tilde{\chi}_{0,j} \) are the only charged massless fields. Moreover, the interacting part of the Lagrangian contains interaction terms for the Wilson line.

\[ \mathcal{L}_{int} \supset -i\sqrt{2}gq \sum_{n,j} \sqrt{\alpha(n + 1)} \overline{\phi} \left( \overline{Q}_{n+1,j} \tilde{Q}_{n,j} - \overline{Q}_{n,j} Q_{n+1,j} \right) + h.c. \]

\[ - 2q^2 g^2 \sum_{n,j} |\phi|^2 \left( |Q_{n,j}|^2 + |Q_{n,j}|^2 \right) - \left( \sqrt{2}gq \phi \chi_{n,j} \chi_{n,j} + h.c. \right) \] (2.24)

where we introduced \( \alpha = -2gqf \). These are the interactions that will contribute to the quantum corrections to the Wilson line mass.
2.3 Non supersymmetric fermion with abelian flux background

In the next section, we will compute quantum corrections to the Wilson line scalar mass and show that it vanishes at first order. Before that, we would like to propose a simpler model than the one introduced above. As we will see, the cancellation is independent of supersymmetry. Therefore, we introduce a simple model to work without supersymmetry: A 6D fermion charged under an abelian gauge theory with a magnetic flux in the background. This model was presented in detail in [9].

Let us write the 6D Lagrangian for a single fermion with abelian gauge symmetry.

\[
S_6 = \int d^6x \left( -\frac{1}{4} F^{MN} F_{MN} + i \bar{\Psi} \Gamma^M D_M \Psi \right) \tag{2.25}
\]

where \( D_M = \partial_M + iq A_M \). Before moving on, we introduce some conventions for our six-dimensional fermion. The 6D Weyl spinor is splitted into two two-components Weyl spinors \( \psi \) and \( \chi \) with charge \( q \) and \(-q\), respectively.

\[
\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \quad \psi_L = \begin{pmatrix} \psi_0 \\ 0 \end{pmatrix}, \quad \chi_R = \begin{pmatrix} 0 \\ \chi \end{pmatrix} \tag{2.26}
\]

\[
\gamma_5 \psi_L = -\psi_L \quad \gamma_5 \psi_R = \psi_R
\]

For the gamma matrices \( \Gamma^M \) we choose the following basis.

\[
\Gamma_\mu = \begin{pmatrix} \gamma_\mu & 0 \\ 0 & \gamma_\mu \end{pmatrix}, \quad \Gamma_5 = \begin{pmatrix} 0 & i \gamma_5 \\ i \gamma_5 & 0 \end{pmatrix}, \quad \Gamma_6 = \begin{pmatrix} 0 & -\gamma_5 \\ \gamma_5 & 0 \end{pmatrix} \tag{2.27}
\]

These 6D gamma matrices satisfy the algebra \( \{ \Gamma_M, \Gamma_N \} = -2 \eta_{MN} \). Developing the fermionic part of the action (2.25), one find

\[
S_{6f} = \int d^6x \left( -i \psi \sigma^\mu D_\mu \bar{\psi} - i \chi \sigma^\mu D_\mu \bar{\chi} - \chi (\partial + \sqrt{2q} \phi) \psi - \bar{\chi} (\partial + \sqrt{2q} \phi) \bar{\psi} \right) \tag{2.28}
\]

Similarly, the gauge part of the action is developed as follows.

\[
S_{6g} = \int d^6x \left( -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \partial^\mu \phi \partial_\mu \phi - \frac{1}{4} (\phi + \sqrt{2q} \phi)^2 - \frac{1}{2} \partial_\mu A^\mu \partial_\rho A^\rho - \frac{i}{\sqrt{2q}} \partial_\mu A^\mu (\partial \phi - \sqrt{2q} \phi) \right) \tag{2.29}
\]

Again, we compactify on a torus \( T^2 \) with a quantized magnetic flux in the background (2.3). To obtain the 4D effective action, a method similar to the one used in the preceding sections is used. For simplicity, let us choose here, without loss of generality, \( qf > 0 \). The ladder operators then take the form

\[
a_+^\dagger = \frac{i}{\sqrt{2qf}} (\bar{\phi} - qf \xi) \quad a_+ = \frac{i}{\sqrt{2qf}} (\partial + qf \bar{\xi}) \tag{2.30}
\]

\[
a_-^\dagger = \frac{i}{\sqrt{2qf}} (\partial - qf \bar{\xi}) \quad a_- = \frac{i}{\sqrt{2qf}} (\bar{\partial} + qf \xi)
\]

and satisfy the relations \([a_+, a_-] = 1, \quad [a_+, a_+] = [a_-, a_-] = 0\). One defines the orthonormal set of wave functions as

\[
a_+ \xi_{0,j} = 0 \quad a_- \bar{\xi}_{0,j} = 0
\]

\[
\xi_n,j = \frac{i^n}{\sqrt{n}} (a_+^\dagger)^n \xi_{0,j} \quad \bar{\xi}_{n,j} = \frac{i^n}{\sqrt{n}} (a_-^\dagger)^n \bar{\xi}_{0,j} \tag{2.31}
\]
where \( j \) takes values between 0 and \(|N| - 1\). One can expand the Weyl spinors with respect to these mode functions similarly to (2.12).

\[
\psi = \sum_{n,j} \psi_{n,j} \xi_{n,j} \\
\chi = \sum_{n,j} \chi_{n,j} \xi_{n,j}
\]

(2.32)

As for the gauge fields, they do not feel the magnetic flux and thus expand with respect to KK modes functions. An important remark is needed: The fermionic zero modes are chiral. Indeed, starting from the Lagrangian (2.28), one can write the Euler-Lagrange equations for the Weyl fermions. Let us compute as an example the equation for \( \psi \).

\[
L_f \supset -i\psi \sigma^\mu \partial_\mu \psi - \chi (\partial + qf \tau) \psi \]

(2.33)

where we used \( \phi = \frac{f}{\sqrt{2}} \tau + \varphi \) and partial integration to isolate \( \psi \). By performing a similar calculation for \( \chi \), \( \psi \) and combining the coupled equations, one can find

\[
\Box \psi - M_+^2 \psi = 0
\]

(2.34)

\[
\Box \chi - M_-^2 \chi = 0
\]

(2.35)

where

\[
M_+^2 = 2qf a_+ a_-
\]

\[
M_-^2 = 2qf (a_+ a_- + 1)
\]

(2.36)

where we used the property \( \sigma^\mu \tau^\nu \partial_\mu \partial_\nu = \Box \). Hence, there are \(|N|\) left-handed fermionic zero modes.

Using the decomposition in Landau and KK modes, one can write the 4D effective Lagrangian of the theory

\[
S_4 = \int d^4x \left[ -\partial^\mu \overline{\varphi}_0 \partial_\mu \varphi_0 + \sum_{n,j} \left( i\overline{\psi}_{L,j} \gamma^\mu D_\mu \psi_{L,j} + i\overline{\Psi}_{n,j} \gamma^\mu D_\mu \Psi_{n,j} \right) + \sqrt{2}qf (n + 1) \overline{\Psi}_{n,j} \psi_{L,j} + \sqrt{2}qf (n + 1) \overline{\psi}_{L,j} \psi_{n,j} \right]
\]

(2.37)

where \( \psi_{L,j} \) and \( \Psi_{n,j} \) are \(|N|\) left-handed fermions and \( \Psi_{n,j} \) is an infinite tower of massive Dirac fermions.

Therefore, the model with magnetic flux contains massless chiral fermions and the Wilson line scalar is massless. We will see later that the vanishing of the Wilson line scalar mass is due to a continuous shift symmetry. Moreover, the vacuum expectation value of the WL scalar does not give mass to the chiral fermions, which would be necessary to construct a realistic model.

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\[^3\text{Again, we only include the zero modes of the uncharged fields } A_\mu, \varphi.\]
2.4 Quantum corrections

In this section, we will compute one-loop corrections to the Wilson line scalar mass. We will begin by the quantum corrections without flux as an example. Then, we will show that the quantum corrections in the case with flux vanish in both supersymmetric and non-supersymmetric cases.

2.4.1 Quantum corrections without flux

Let us consider the action without flux found in [7]. Expressed in terms of the component fields, it reads

\[ L_4 \supset -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \partial_\mu \phi \partial^\mu \phi + \sum_{n,m} \left( -D_\mu \bar{Q}_{n,m} D^\mu Q_{n,m} - |M_{n,m} + \sqrt{2} g q \phi|^2 \bar{Q}_{n,m} Q_{n,m} \right. \]

\[ \left. - i\chi_{n,m} \sigma^\mu D_\mu \chi_{n,m} - i\bar{\chi}_{n,m} \sigma^\mu D_\mu \bar{\chi}_{n,m} + \left( M_{n,m} + \sqrt{2} g q \phi \right) \chi_{n,m} \chi_{n,m} + h.c. \right) \]

From this Lagrangian, one obtains the one-loop contributions to the \( \phi \) mass. We will separate the bosonic and fermionic contributions.

\[ \delta m_{b,1}^2 = 2g^2 q^2 \sum_{n,m} \int \frac{d^4k_{eu}}{(2\pi)^4} \left( \frac{1}{k_{eu}^2 + |M_{n,m}|^2} - \frac{|M_{n,m}|^2}{(k_{eu}^2 + |M_{n,m}|^2)^2} \right) \]

\[ \delta m_{f,1}^2 = -4g^2 q^2 \sum_{n,m} \int \frac{d^4k_{eu}}{(2\pi)^4} \frac{k_{eu}^2}{(k_{eu}^2 + |M_{n,m}|^2)^2} \]

\[ \Rightarrow \delta m_1^2 = \delta m_{b,1}^2 + \delta m_{f,1}^2 = 0 \]

where \( k_{eu}^2 = \sum_i k_i^2 \) is the euclidean norm squared of \( k \). Starting from now on, the euclidean notation will be understood in one-loop integrals. The bosonic and fermionic contributions cancel each other, as expected for a supersymmetric theory.

However, these quantum corrections are infinite sums of quadratically divergent terms [14] and a consistent treatment asks for a regularization scheme as well as renormalization conditions. This is discussed in [7]. It is useful to use the Schwinger representation of propagators

\[ \frac{1}{p^2 + m^2} = \int_0^\infty dt e^{-t(p^2 + m)} \]  

(2.40)

This allows us to rewrite the bosonic correction as

\[ \delta m_{b,1}^2 = \frac{g^2 q^2}{\pi^3 L^2} \sum_{r,s} \frac{1}{r^2 + s^2} \]  

(2.41)

in which the full divergence is contained in the term \( r = s = 0 \). A counterterm is needed to eliminate this divergence. For the finite part, the latter result has been compared to other works using different methods, as dimensional regularization or the Wilson line effective potential.

Figure 1: Bosonic (first two diagrams) and fermionic (last diagram) contributions to the Wilson line mass without flux.
2.4.2 Quantum corrections with flux in a supersymmetric model

Let us now compute the one-loop corrections for the Wilson line scalar mass with flux compactification. We thus have to take into account the full tower of Landau levels of bosons and fermions. Using the interaction Lagrangian (2.24), one can compute the one loop correction to the scalar mass.

\[ \delta m_{\text{b,1}}^2 = -4q^2g^2|N| \sum_n \int \frac{d^4k}{(2\pi)^4} \left( \frac{n}{k^2 + \alpha(n+1/2)} - \frac{n+1}{k^2 + \alpha(n+3/2)} \right) \]

where \( \alpha = -2qgf \). Using the Schwinger representation, one finds

\[ \delta m_{\text{b,1}}^2 = 4q^2g^2|N| \sum_n \int \frac{d^4k}{(2\pi)^4} \left( \frac{n}{k^2 + \alpha n} - \frac{n+1}{k^2 + \alpha(n+1)} \right) \]

\[ \delta m_{\text{f,1}}^2 = 4q^2g^2|N| \sum_n \int \frac{d^4k}{(2\pi)^4} \left( \frac{n}{k^2 + \alpha(n+1/2)} - \frac{n+1}{k^2 + \alpha(n+3/2)} \right) \]

where the cancellation is showed by using the Schwinger representation and inverting the summation and integration after having integrated on momentum space. Surprisingly, we have shown that both contributions to the correction vanish individually. Therefore, the quantum correction vanishes as in the case without flux but, in this case, the cancellation is not due to supersymmetry.

In [8], particular care is given to the regularization of the quantum corrections. Indeed, the one-loop integrals are divergent and one needs to perform a UV regularization consistent with the symmetry of the theory. The dimensional regularization used ensures that the series of these integrals are well defined. Furthermore, in the limit of removing the regulator, any divergences of the series give the form of the corresponding counterterms. Furthermore, using dimensional regularization ensures that all symmetries are respected (including gauge symmetries). In the limit of removing the regularization, i.e. \( \epsilon = 4 - d \to 0 \), \( \delta m_1^2(q^2) \) where \( q^2 \) is the external momenta has a pole in \( \epsilon \) dictating the form of the counterterm.

\[ \delta m_1^2(q^2) \propto \alpha^2 \left( -\frac{q^2}{\alpha} \left( \frac{1}{\epsilon} + \frac{C_1}{\alpha} \right) + C_2q^2 + \mathcal{O}(q^2/\alpha^3) \right) \]

where \( C_{1,2} \) are real constants. This counterterm implies the presence of a ghost state of squared mass proportional to \( \alpha = -2qgf \). After having identified the counterterm, they set \( q^2 = 0 \) and find \( \delta m_{\text{b,1}}^2 = \delta m_{\text{f,1}}^2 = 0 \) as expected.

One would like to show that the quantum corrections to the Wilson line mass in a non-supersymmetric model still vanish. As we will see in the next subsection, this is precisely the case. This suggests that another symmetry, due to the magnetic flux, should protect the scalar mass.

In [7], it is also shown that no \( |\phi|^4 \)-term is generated at one-loop order. As predicted by [2], this suggests that the entire one-loop effective potential vanishes.
2.4.3 Quantum corrections with flux in a non-supersymmetric model

Let us compute the quantum corrections in the non-supersymmetric model of section 2.3. We will use the action (2.36) which gives two fermionic contributions to the corrections to the Wilson line (WL) scalar mass. One obtain from these diagrams the following correction.

\[
\delta m_1^2 = -2q^2 |N| \sum_n \int \frac{d^4k}{(2\pi)^4} \frac{2k^2}{(k^2 + 2qfn)(k^2 + 2qf(n + 1))} \\
= 4q^2 |N| \sum_n \int \frac{d^4k}{(2\pi)^4} \left( \frac{n}{k^2 + 2qfn} - \frac{n + 1}{k^2 + 2qf(n + 1)} \right) \\
= \frac{q^2}{4\pi^2} |N| \int_0^\infty \frac{dt}{t^2} \sum_n \left( ne^{-2qft} - (n + 1)e^{-2qf(n + 1)} \right) \\
= \frac{q^2}{4\pi^2} |N| \int_0^\infty \frac{dt}{t^2} \left( \frac{e^{2qft}}{(e^{2qft} - 1)^2} - \frac{e^{2qft}}{(e^{2qft} - 1)^2} \right) \\
= 0
\]

Figure 3: One-loop contributions to the Wilson line mass with flux.

Hence, the cancellation of one-loop correction is indeed independent of supersymmetry. So what is the origin of this cancellation? We will investigate this question in the next section.

Cancellation of the one-loop corrections in a Yang-Mills SU(2) non-supersymmetric model has been verified in [10]. They use a model similar to the SU(2) one in [7]. Quantum corrections to the WL mass also vanish thanks to the shift symmetry under the translation in extra space. In this extension, the cancellation is non-trivial since they take into account the ghost field contribu-

2.4.4 Nonvanishing finite contributions to the scalar mass

The results above cannot be applied as it stands to the hierarchy problem. Indeed, if one wants to identify the WL scalar with the Higgs boson of the standard model, the WL scalar must be a pseudo-Nambu-Goldstone boson with finite mass. Therefore, it is necessary to have some mechanism generating a term breaking the translational symmetry in compactified space. In [11], the possibility to generate finite scalar mass for the Higgs boson has been studied. They generalize one-loop quantum corrections and derive the conditions for the corrections to be nonvanishing and finite. Dimensional regularization is used to treat UV divergences in loop integrals.

The model used contain a scalar \( \phi \) and a fermion field with SU(2) gauge symmetry in six dimensions with flux compactification on a torus. A particular attention is given to the following simple term that generate nonvanishing correction.

\[
\mathcal{L}_{\text{breaking}} = \kappa (\phi + \bar{\phi}) \Phi \Phi
\]

where \( \kappa \) is a dimensionless coupling constant. This term gives the following correction to the scalar mass.

\[
\delta m_1^2 = \frac{|N| \ln 2 \kappa^2}{32\pi^2 \bar{L}^2}
\]

\[\text{Footnote: Faddev-Popov ghosts are introduced into gauge quantum field theories to maintain consistency of the path integral formulation. In Yang-Mills theories, the ghosts are complex scalar fields but they anticommute like fermions. Hence, we talk of non-physical particles.}\]
and \( \delta m^2 = 0 \) is reproduced for \( \kappa \to 0 \). This Wilson line scalar mass could be considered as a Higgs mass. Indeed, even if the compactification scale \( 1/L \) is the Planck scale, a realistic Higgs mass at the TeV scale could be realized by the interaction term by some dynamics at the TeV scale \( \kappa/L \).

### 2.5 Underlying symmetry

In this section we will try to explain the cancellation of one-loop correction to the WL mass. Let us begin by observing that the actions (2.29) and (2.28) without magnetic flux are invariant under translation \( \delta_T = \epsilon \partial + \epsilon \overline{\partial} \) on the torus \( T^2 \), acting on \( \chi, \psi, A^\mu \). By adding a magnetic flux in the background, we explicitly break the translational symmetry because the magnetic field depends on \( x_5, x_6 \). This breaking of the invariance can be compensated by a shift in the scalar fluctuation \( \varphi \).

\[
\delta_T \varphi = (\epsilon \partial + \epsilon \overline{\partial}) \varphi + \frac{\epsilon}{\sqrt{2}} f
\]

The generators of translations do not commute with the mass operators (2.35). However, the mode functions \( \psi_{n,j} \) and \( \chi_{n,j} \) are eigenfunctions of these operators. Therefore, the action of the translation on the mode function cannot be trivial. Instead, one will see that all the tower is reshuffled.

To construct a general transformation of the mode functions one can combine the translation \( \delta_T = \epsilon \partial + \epsilon \overline{\partial} \) with another symmetry of the action. The action is invariant under the following local transformation.

\[
\varphi_\Lambda = -\frac{1}{\sqrt{2}} \overline{\partial} \Lambda \quad \psi_\Lambda = e^{q \Lambda} \psi \quad \chi_\Lambda = e^{-q \Lambda} \chi \quad \Lambda = f(\alpha \overline{\sigma} - \overline{\alpha z})
\]

where \( \alpha \) is a complex parameter. These transformations which first considered in [15]. For the WL, the transformation corresponds therefore to a shift \( \delta \varphi = \frac{\sqrt{2}}{2} f \). Combining the two symmetries of the Lagrangian, one obtains the simple infinitesimal transformation:

\[
\delta \psi = (\delta_T + \delta_\Lambda) \psi = -i \sqrt{2 q f} (\epsilon a_+ + \epsilon a_+^\dagger) \psi
\]

Therefore, the transformation above reshuffle the different modes, connecting neighbouring mode functions. Using equation (2.31), one can find explicitly how the different mode functions are reshuffled.

\[
\delta \psi = \sum_{n,j} \delta \psi_{n,j} \xi_{n,j}
\]

\[
\delta \psi_{n,j} := \sqrt{2 q f} (\epsilon \sqrt{n+1} \psi_{n+1,j} - \epsilon \sqrt{n} \psi_{n-1,j})
\]

Analogously,

\[
\delta \chi = \sum_{n,j} \delta \chi_{n,j} \xi_{n,j}
\]

\[
\delta \chi_{n,j} := \sqrt{2 q f} (\epsilon \sqrt{n+1} \chi_{n+1,j} - \epsilon \sqrt{n} \chi_{n-1,j})
\]

The invariance of (2.36) under these transformation is explicitly verified in [9]. As expected, the invariance is showed under the condition that \( \varphi \) transforms as

\[
\delta \varphi_0 = \sqrt{2 \epsilon} f
\]

Indeed, the shift part of \( \delta_T \varphi \) corresponds to the transformation of \( \varphi_0 \) above while the translation part is \( \epsilon \partial + \epsilon \overline{\partial} \). As explained in [9], this part of the transformation corresponds to the massive part of the scalar KK tower. This massive part of the uncharged field has to be considered if one wants to write the full Lagrangian of the theory which contains new interactions terms. Furthermore, the full Lagrangian has to be taken into account to compute higher loop order corrections. For example, the Kaluza-Klein modes of the vector fields appear in two-loop order correction to the WL scalar mass. The transformation laws of the massive gauge
fields mode functions are computed in \[9\] and will be summarized in section 2.6. It appears that the full 4D Lagrangian also has an exact symmetry under which the scalar zero modes transform under the shift (2.53).

How does this shift symmetry explain the cancellation of the one-loop correction to the scalar mass? Let us summarize what we did here.

We started by identifying two symmetries of the 6D actions, a translation, and gauge invariance. From these, we identified a transformation law of the four-dimensional fields which leaves the 4D effective action invariant. In particular, the complex scalar zero-mode transforms with a shift which prevents the generation of a mass term. Indeed, the mass of a scalar field is generated by the presence of an extremum in the scalar potential. If the Lagrangian is invariant under a shift of this scalar, this implies the absence of a unique ground state and therefore the absence of mass. Therefore, the shift symmetry only allows for derivative terms. Moreover, the fact that the full action containing the full tower of vector and scalar preserves this shift symmetry of $\varphi_0$ suggests that the scalar mass is protected at two-loop order and higher. The calculation of two-loop order corrections has been done in \[16\] and the cancellation has been shown up to this order.

Furthermore, the Wilson line scalar can be identified with the Nambu-Goldstone boson of the translational symmetry on the torus \[7, 16\]. Indeed, the shift symmetry (2.53) is precisely the definition of an NG boson. And one can see that $\varphi_0$ is the NG boson corresponding to the breaking of the translational symmetry spontaneously broken by the vacuum expectation value of the internal components of the gauge field.

2.6 $U(1)xU(1)$ gauge group

As mentioned in 2.4.4, there is a necessity to break the translational symmetry to generate a finite mass of the TeV order for the Higgs boson. Rather than adding ad-hoc interactions terms as \[11\], we would like to investigate the possibility to generate non-vanishing finite corrections to the scalar mass using only the fundamental structure of the theory. It has been mentioned in \[7\] that in the case of a gauge group with several $U(1)$ gauge factors, the WL lines are the pseudo-Nambu-Goldstone bosons of the translational symmetry.

Let us recall that the physical reason for the cancellation of quantum corrections to the WL scalar mass in the $U(1)$ case is that the WL can be seen as the Nambu-Goldstone boson of the translational symmetry on the torus. In a case where the gauge symmetry contains more than one $U(1)$, the situation becomes more subtle. Contrary to the discussion in \[7\], let us discuss this matter in the non-supersymmetric model of \[9\] and section 2.3, and let us study the simplest case of a $U(1)_1 \times U(1)_2$ gauge symmetry. There are two gauge fields and therefore two Wilson Line scalars denoted $\varphi^1$ and $\varphi^2$. The covariant derivative reads

$$D_M = \partial_M + iq_\alpha A_M^\alpha$$

where $q_\alpha$ is the charge of $\Psi$ associated to the gauge symmetry $U(1)_\alpha$ and $A_M^\alpha$ is the gauge vector field associated with $U(1)_\alpha$.

$$A_5^\alpha = -\frac{1}{2} f^\alpha x_6, \quad A_6^\alpha = \frac{1}{2} f^\alpha x_5, \quad F_{56}^\alpha = f^\alpha$$

One could easily include an arbitrary number of fermions families $N_f$. In this case, the charge would take the form of a matrix $q_\alpha$.

Following the same procedure as in section 2.5, one can show (equations (2.62) to (2.65)) that under the condition that the Lagrangian transforms into a total derivative, the Wilson lines transform as

$$q_\alpha \delta \varphi_0^\alpha = \sqrt{2} q_\alpha \tau f^\alpha$$

This indicates that a field charged under two $U(1)$ gauge symmetries should feel an effective flux depending on the two charges and the flux of the gauge groups $U(1)_\alpha$. Conversely, a field may not feel any flux even though there is a flux in the background on the two gauge fields. For example, let us consider two fermions doublets $\{\Psi^i\}_{i=1,2}$ with the following charges.

$$q_{i\alpha} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$
For the flux choice \( f^1 = f^2 = f \), one has

\[
q_{i\alpha} \delta \varphi^\alpha_0 = \sqrt{2} f \begin{pmatrix} 2\pi^1 \\ 0 \end{pmatrix}
\]

and therefore, the second fermion \( \Psi^2 \) does not feel any flux. Indeed, if, \( q_{i\alpha} f^\alpha = 0 \), the ladder operators (2.30) do not satisfy the harmonic oscillator algebra and there are no longer Landau levels in the spectrum of the theory.

Additionally, one can see that in this example, there is one Goldstone boson corresponding to \( \varphi_1^0 + \varphi_2^0 \) while \( \varphi_1^0 - \varphi_2^0 \) does not transform nonlinearly. Hence, there is only one Goldstone boson. This is a general feature of this theory. There are two translational symmetries in the extra space. One along \( x_5 \) and the other along \( x_6 \). When at least one of the gauge fields contains a flux in its background, both of these symmetries are broken and a complex NG boson arises. There is no other symmetry to be broken and therefore any other symmetry would be accidental and potentially broken by interactions terms. Therefore one can have at most one Nambu-Goldstone boson and any other NG boson would be a "pseudo-NG boson" with a non-vanishing mass.

The reason why we are interested in these pseudo-NG bosons is that even if their mass are not zero, they should be small compared to a standard boson mass. Let us verify that. One can do the calculations of section 2.3 with two \( U(1) \) instead of one and verify that the effective action has exactly the same form as (2.36) with \( q\varphi_0 \) replaced by \( q_{i\alpha} \varphi_0^\alpha \). One can even take an arbitrary number of fermion families and check that the total action is simply a sum over \( i = 1, 2, \ldots N_f \) of the action (2.36) for \( \Psi^i \). The interactions terms are:

\[
\mathcal{L}_4 \supset \sum_{i=1}^{N_f} \sum_{n \geq 0} \sum_{j=0}^{N_i-1} \left( \sqrt{2} q_{i\alpha} \varphi_0^\alpha \left( \bar{\Psi}_0^{i,j} \frac{1 - \gamma_5}{2} \psi_L^{i,j} + \bar{\Psi}_{n+1,j}^{i} \gamma_5 \frac{1 - \gamma_5}{2} \psi_{n,j}^{i} \right) + \sqrt{2} q_{i\alpha} \varphi_0^\alpha \left( \psi_L^{i,j} \frac{1 + \gamma_5}{2} \bar{\Psi}_0^{i,j} + \psi_{n,j}^{i} \frac{1 + \gamma_5}{2} \bar{\Psi}_{n+1,j}^{i} \right) \right)
\]

(2.59)

where \( N_i \) is the quantization of the flux associated to the charge \( i \).

\[
\frac{q_{i\alpha}}{2\pi} f^\alpha = N_i \in \mathbb{Z}
\]

(2.60)

With the interaction Lagrangian above, one can compute the one-loop correction to the Wilson lines masses. Here, because of the multitude of Wilson lines, one have to consider possible mixing between the two scalars. Indeed, there are eight one-loop diagrams to consider.

Figure 4: One-loop contributions to the Wilson lines masses for \( N_f = 1 \).
One has for an arbitrary number of families $N_f$:

$$\left(\delta m^2_i\right)_{\alpha\beta} = -2 \sum_{i=1}^{N_f} q_{i\alpha} q_{i\beta} |N_i| \sum_n \int \frac{d^4k}{(2\pi)^4} \frac{2k^2}{(k^2 + 2q_{i\gamma} f \gamma n)(k^2 + 2q_{i\nu} f \nu (n+1))}$$

$$= 4 \sum_{i=1}^{N_f} q_{i\alpha} q_{i\beta} |N_i| \sum_n \int \frac{d^4k}{(2\pi)^4} \left( \frac{n}{k^2 + 2q_{i\gamma} f \gamma n} - \frac{n+1}{k^2 + 2q_{i\nu} f \nu (n+1)} \right)$$

$$(2.61)$$

if $q_{i\alpha} f^\alpha \neq 0$. We used the fact that the series of integrals above vanish as showed in equations (2.45), independently of $q_{i\alpha}$. Thus, one-loop corrections of Wilson lines scalar masses always vanish for non-vanishing effective flux.

Nonetheless, the scalar mass should be finite and non-vanishing in some cases. Therefore, one expects that quantum corrections to the WL masses should be finite in some cases beyond one-loop order. However, this result is interesting because it confirms that even the pseudo-NG bosons masses are protected from one-loop order corrections and one could hope to produce a realistic mass in the TeV range. Further calculations at two-loop order would be needed to compute the precise spectrum of Wilson lines.

If finite corrections are expected, one should explicitly observe symmetry breaking terms in the Lagrangian. The transformations of fermions Landau modes founded in [7] can be generalized to the $U(1)_1 \times U(1)_2$ case with $N_f$ families.

$$\delta \psi^i_{n,j} = \sqrt{2}q_{i\alpha} f^\alpha \left( \epsilon^i \sqrt{n+1} \psi^i_{n+1,j} - \epsilon^i \sqrt{n} \psi^i_{n-1,j} \right)$$

$$\delta \chi^i_{n,j} = \sqrt{2}q_{i\alpha} f^\alpha \left( -\epsilon^i \sqrt{n} \chi^i_{n-1,j} + \epsilon^i \sqrt{n+1} \chi^i_{n+1,j} \right)$$

As explained before, we expect the symmetry to be broken by interaction terms. One can compute the transformation of the interaction terms of the complete Lagrangian given in [7]. The variation of the mass term

$$\delta \left( -\sum_{n,j} \sqrt{2}q_{i\alpha} f^\alpha (n+1) \chi^i_{n,j} \psi^i_{n+1,j} \right) = -2q_{i\alpha} f^\alpha \sum_{n,j} \sqrt{n+1} \left( -\epsilon^i \sqrt{n} \chi^i_{n-1,j} \psi^i_{n+1,j} + \epsilon^i \sqrt{n+1} \chi^i_{n+1,j} \psi^i_{n+1,j} \right.$$  

$$+ \epsilon^i \sqrt{n+2} \chi^i_{n,j} \psi^i_{n+2,j} - \epsilon^i \sqrt{n+1} \chi^i_{n,j} \psi^i_{n,j} \right) = 2\epsilon^i q_{i\alpha} f^\alpha \sum_{n,j} \chi^i_{n,j} \psi^i_{n,j}$$

$$(2.63)$$

which must compensate the variation of the Yukawa term

$$\delta \left( -\sum_{n,j} \sqrt{2}q_{i\alpha} \phi^i_0 \chi^i_{n,j} \psi^i_{n,j} \right) = \sum_{n,j} \left( -\sqrt{2}q_{i\alpha} \phi^i_0 \chi^i_{n,j} \psi^i_{n,j} - \sqrt{2}q_{i\alpha} \phi^i_0 \delta (\chi^i_{n,j} \psi^i_{n,j}) \right)$$

$$(2.64)$$

$$(2.64)$$

These equations imply the transformation law of the Wilson lines.

$$q_{i\alpha} \delta \phi^i_0 = \sqrt{2} \epsilon^i q_{i\alpha} f^\alpha$$

$$(2.65)$$

Thus, we have found the transformation law mentioned at the beginning of this section. As stated then, one can see that this transformation cannot leave the full Lagrangian invariant. However, the fact that the $N_f$ shift symmetries are preserved by the Yukawa interaction explains the fact that the one-loop correction
vanishes. We will now show that the interaction term involved in two-loop corrections will break these $N_f$ shift symmetries to only preserve one particular shift transformation. Following the work [9], one can write the transformation law of massive modes of gauge fields.

$$\delta \phi^a_{l,m} = (\epsilon^a M_{l,m} - \tau^a M_{l,m}) \phi^a_{l,m}$$

$$\delta A^a_{\mu,l,m} = (\epsilon^a M_{l,m} - \tau^a M_{l,m}) A^a_{\mu,l,m}$$

(2.66)

where we have introduced two different parameters $\epsilon^a$ corresponding to the transformation of the two sets of gauge fields. Let us consider as an example an interaction term between massive gauge fields found in [9] and extend it to the $U(1) \times U(1)$ case.

$$\mathcal{L}_4 \supset -\sqrt{2} \sum_{i=1}^{N_f} N_f \sum_{l,m;n,j,n',j'} C^{l,m}_{n,j,n',j'} q^a_{i\alpha} \phi^a_{l,m} \chi^i_{n,j} \psi^i_{n,j}$$

(2.67)

The transformation of this term under (2.66) gives

$$-\sqrt{2} \sum_{i=1}^{N_f} N_f \sum_{l,m;n,j,n',j'} q^a_{i\alpha} \left[ - (\tau^a M_{l,m} - \epsilon^a M_{l,m}) C^{l,m}_{n,j,n',j'} ight.$$

$$+ \sqrt{2} q^a_{i\alpha} f^a \left( \epsilon^i \sqrt{n+1} C^{l,m}_{n,j,n',j'} - \delta^i \sqrt{n} C^{l,m}_{n,j,n',j'} \right)$$

$$+ \sqrt{2} q^a_{i\alpha} f^a \left( \epsilon^i \sqrt{n+1} C^{l,m}_{n,j,n',j'} - \delta^i \sqrt{n} C^{l,m}_{n,j,n',j'} \right) \right] \phi^a_{l,m} \chi^i_{n,j} \psi^i_{n,j}$$

(2.68)

which is not zero in general. Indeed, only one particular case gives a zero variation:

$$\epsilon^i = \epsilon^a \equiv \epsilon \quad \forall i, \alpha$$

(2.70)

Hence, there is only one parameter for the shift symmetry of the Wilson lines: only one shift symmetry. The only linear combination of the Wilson lines transforming with a shift and there being the Goldstone boson is

$$\frac{\phi^0 + \phi^2}{\sqrt{2}}$$

(2.71)

3 Six-dimensional Yang-Mills theories with flux compactification

3.1 Tachyons in 6D Yang-Mills theories with flux compactification

To construct an effective theory interesting from the phenomenological point of view, it is inevitable to extend the model to Yang-Mills gauge theories. It has been shown in [7] that the effective action for non-abelian flux contains tachyons, i.e. fields with an imaginary mass. This was predicted by [2]. Indeed, the mass shift for the component of a multiplet living in six dimensions with flux compactification is:

$$\delta M^2_q = (2n + 1) |qg| + 2qg \Sigma$$

(3.1)
$\Sigma$ is the internal helicity which is equal to $\pm 1/2$ for fermions, 0 for scalars and $A^\mu$ the 4D components of the scalar field, and $\pm 1$ for $A_{5,6}$ the internal components of the vector field. Let us note however that Bachas’s mass equation does not predict all the spectrum found in [7] because some of the fields are absorbed via the Stückelberg mechanism\(^5\). Indeed, the flux background breaks the gauge symmetry and the charged vector gauge fields acquire a mass. Thus they have to absorb some scalar field to complete their longitudinal degree of freedom, and one expects to find a Goldstone modes among the scalars of the theory.

Let us comment briefly on the presence of tachyons. The abelian flux background of section 2.2 was perturbatively stable. Indeed, the perturbation theory is done around the stable ground state. Therefore, all fields have a non-negative mass. For a non-abelian flux background, some of the extra-dimensional gauge fields become tachyonic. Hence, the effective action is due to an expansion around a local maximal point rather than a ground state. Therefore, one would be interested in investigating tachyon condensation to find the ground state of the theory.

In this section, we will try to compute the scalar spectrum of a model with an SU(2) gauge symmetry with flux compactification on a torus. As supersymmetry does not play any role in our model, we will work in a non-supersymmetric model. A priori, the presence of tachyons should not be due to supersymmetry. Indeed, the Bachas formula (3.1) predicts the presence of tachyon without considering supersymmetry. We consider the kinetic Lagrangian for a 6D vector field.

\[ L_6 = -\frac{1}{2} \text{Tr}\{ F_{MN} F^{MN} \} \] (3.2)

where the fields are in the adjoint representation of SU(2) and decomposed along the basis of the SU(2) algebra which are the Pauli matrices \( \{ T_1, T_2, T_3 \} \) normalized such that \( \text{Tr} T_a T_b = \frac{1}{2} \delta_{ab} \). For non abelian gauge theories, 

\[ F_{MN} = \partial_M A_N - \partial_N A_M - ig[A_M, A_N] \] (3.3)

To find the 4D effective action, we split the six dimensions into the four classical dimensions and two extra dimensions compactified on a torus \( T^2 \).

\[ F_{56} = \frac{1}{\sqrt{2}} (\partial \phi + \partial \bar{\phi} - \sqrt{2} g [\phi, \bar{\phi}]) \]

\[ F_{\mu 5}^2 + F_{\mu 6}^2 = (F_{\mu 6} + iF_{\mu 5})(F_{\mu 6} - iF_{\mu 5}) \]

\[ F_{\mu 6} + iF_{\mu 5} = \sqrt{2} \partial_\mu \phi - i \partial A_\mu - i \sqrt{2} g [A_\mu, \phi] \] (3.4)

One can rewrite (3.2) as

\[ \mathcal{L}_6 = -\frac{1}{2} \text{Tr}\{ F_{\mu \nu} F^{\mu \nu} \} \]

- 2 \text{Tr}\left\{ \left( \partial_\mu \bar{\phi} - \frac{i}{\sqrt{2}} \partial A_\mu - i \sqrt{2} g [A_\mu, \bar{\phi}] \right) \left( \partial_\mu \phi - \frac{i}{\sqrt{2}} \partial A_\mu - i g [A_\mu, \phi] \right) \right\} \]

\[ - \frac{1}{2} \text{Tr}\left\{ \left( \partial \bar{\phi} + \partial \phi - \sqrt{2} g [\phi, \bar{\phi}] \right) \left( \partial \bar{\phi} + \partial \phi - \sqrt{2} g [\phi, \bar{\phi}] \right) \right\} \] (3.5)

The mass term of the scalar is contained in the third line of the Lagrangian.

To reformulate the theory in terms of charged components of the gauge fields, one defines a new basis of generators \( T_+, T_-, T_3 \) where

\[ T_\pm = T_1 \pm i T_2. \] (3.6)

\(^5\)The Stückelberg mechanism is a particular case of the Brout-Englert-Higgs mechanism where the Higgs mass has been sent to infinity.
The basis elements satisfy:
\[
\begin{align*}
\text{Tr}\{T_3^2\} &= \frac{1}{2}, & \text{Tr}\{T_+T_-\} &= 1, & \text{Tr}\{T_\pm T_\mp\} &= 0, & \text{Tr}\{T_\pm^2\} &= 0, \\
[T_+, T_-] &= 2T_3, & [T_3, T_\pm] &= \pm T_\pm.
\end{align*}
\]
Eq. (3.7)

An arbitrary field \(\phi\) in the adjoint representation is decomposed as
\[
\phi = \phi_3 T_3 + \phi_+ \frac{T_+}{\sqrt{2}} + \phi_- \frac{T_-}{\sqrt{2}}
\]
Eq. (3.8)
\[
\bar{\phi} = \bar{\phi}_3 T_3 + \bar{\phi}_+ \frac{T_+}{\sqrt{2}} + \bar{\phi}_- \frac{T_-}{\sqrt{2}}
\]
Eq. (3.9)

And the vector fields are decomposed in the same way. The vector field being real, one has \(\bar{A}_\mu 3 = A_{\mu,3}\) and \(\bar{A}_\mu \pm = A_{\mu,\mp}\). We choose the basis such that the flux background is only encode in \(\phi_3\).
\[
< \phi_\pm >= 0 & < \phi_3 >= \frac{f}{2\sqrt{2}} (x_5 - ix_6)
\]
Eq. (3.10)

The relation \([T_3, T_\pm] = \pm T_\pm\) shows that the components \(\phi_+, A_\mu^a\) and \(\phi_-, A_\mu^a\) have opposite charges with respect to \(\phi_3, A_\mu^3\). Therefore, their field profile will be similar to the field profile in the abelian case. Indeed, charge conservation forbids the existence of quadratic terms (harmonic oscillator terms) mixing fields of different charges. Therefore, the difference between the abelian and non-abelian cases will lie only in the interacting part of the Lagrangian.

Let us compute the mass spectrum of the scalar fields.
\[
\mathcal{L}_0 \supset -\frac{1}{4} \left( \overline{\partial \phi_3} + \overline{\partial \phi_3} + \sqrt{2}g(|\phi_+|^2 - |\phi_-|^2)^2 \right)^2 \\
- \frac{1}{2} \left( \overline{\partial \phi_+} + \overline{\partial \phi_-} - \sqrt{2}g(\phi_3 \overline{\phi_+} - \overline{\phi}_3 \phi_-) \right) \left( \overline{\partial \phi_+} + \overline{\partial \phi_-} + \sqrt{2}g(\phi_3 \overline{\phi_+} - \overline{\phi}_3 \phi_-) \right)
\]
\[
= -\frac{1}{4} \left( \sqrt{2}f + \overline{\partial \phi_3} + \overline{\partial \phi_3} + \sqrt{2}g(|\phi_+|^2 - |\phi_-|^2)^2 \right)^2
\]
\[
- \frac{1}{2} \left( -i\sqrt{2q_0 a_+^\dagger} - \sqrt{2g \phi_3} \phi_+ + (\sqrt{2q_0 a_-} + \sqrt{2g \phi_3}) \phi_- \right)
\times \left( -i\sqrt{2q_0 a_+^\dagger} - \sqrt{2g \phi_3} \phi_+ + (\sqrt{2q_0 a_-} + \sqrt{2g \phi_3}) \phi_- \right)
\]
\[
\supset -gf |\phi_+|^2 - |\phi_-|^2 + gf \left( a_+^\dagger \phi_+ + a_-^\dagger \phi_- \right) \left( a_+ \phi_+ + a_- \phi_- \right)
\]
Eq. (3.11)

Integrating over \(T^2\) and using (2.30) (2.31), one can get the four dimensional effective Lagrangian.
\[
\mathcal{L}_4^{\text{gauge mass}} = -gf \sum_{n,j} \left( |\phi_{+;n,j}|^2 - |\phi_{-;n,j}|^2 \right)
\]
\[
- gf \sum_{n,j,n',j'} \int d^2z \left( \sqrt{n+1} \phi_{+;n,j} \xi_{n+1,j} - \sqrt{n} \phi_{-;n,j} \xi_{n-1,j} \right) \times
\]
\[
\times \left( \sqrt{n'+1} \phi_{+;n',j'} \xi_{n'+1,j'} - \sqrt{n'} \phi_{-;n',j'} \xi_{n'-1,j'} \right)
\]
Eq. (3.12)

Using the orthonormility relation (2.11), one can rewrite this Lagrangian as
\[
\mathcal{L}_4^{\text{mass}} = gf|\phi_{-0,j}|^2 - gf \sum_{n,j} \left( \bar{\phi}_{+n,j} \phi_{-n+2,j} \right) \left( \begin{array}{c} n + 2 \\ n + 1 \\ \sqrt{(n + 1)(n + 2)} \\ n + 1 \end{array} \right) \left( \begin{array}{c} \phi_{+n,j} \\ \bar{\phi}_{-n+2,j} \end{array} \right)
\]

which gives the scalar mass spectrum. Let us summarise the different scalar states we found.

1. \( \phi_{-0,j} \) is a tachyon with negative mass squared.

\[
m^2_{\phi_{-0,j}} = -gf
\]

This confirms the result of [7] and the formula (3.1).

2. \( \phi_{-1,j} \) is a massless scalar.

3. One of the eigenstates of the scalar mass matrix is the tower of Goldstone bosons

\[
\phi^G_{n,j} = \sqrt{\frac{n + 1}{2n + 3}} \phi_{+,n,j} + \sqrt{\frac{n + 2}{2n + 3}} \phi_{-,n+2,j}
\]

4. The other eigenstate corresponds to the following massive scalars.

\[
\phi^M_{n,j} = -\sqrt{\frac{n + 2}{2n + 3}} \phi_{+,n,j} + \sqrt{\frac{n + 1}{2n + 3}} \phi_{-,n+2,j}
\]

and its mass is

\[
m^2_{\phi^M_{n,j}} = gf(2n + 3)
\]

### 3.2 Eliminating the tachyons

As an alternative to tachyon condensation, we will propose in this section another approach which is to eliminate the tachyons. Nonetheless, studying tachyon condensations without eliminating them could lead to interesting results, in particular in string theory. How could we eliminate the tachyon? Here, we propose to bring a scalar boson \( \Phi \) in the adjoint representation of the gauge group SU(2) taking a non-zero vacuum expectation value. We expect this vev to give a positive contribution to the masses of all gauge scalar fields. Then, for sufficient values of the vev, we could eliminate the tachyons by increasing their mass up to a positive value.

\[
\Phi = \Phi_3 T_3 + \Phi_+ \frac{T_+}{\sqrt{2}} + \Phi_- \frac{T_-}{\sqrt{2}}
\]

The Lagrangian describing the scalar boson and the gauge fields is

\[
\mathcal{L}_6 = -\frac{1}{2} \text{Tr} F_{MN} F^{MN} - (D_M \Phi)^\dagger D^M \Phi - V(\Phi)
\]

where \( V(\Phi) \) is the scalar potential

\[
V(\Phi) = -m^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2
\]

and \( D_M \) is the covariant derivative.

\[
D_M \Phi = \partial_M - ig [A_M, \Phi]
\]

The minima of this potential is not 0 and give a finite vacuum expectation value to the scalar boson. A rotation of the basis can be performed such that

\[
< \Phi_3 > = v = \sqrt{\frac{m^2}{2\lambda}}
\]
In its matrix form, $\Phi$ reads
\[
\Phi = \frac{1}{2} \begin{pmatrix} \Phi_3 & \sqrt{2}\Phi_- \\ \sqrt{2}\Phi_+ & -\Phi_3 \end{pmatrix}
\] (3.23)

The adjoint representation of $SU(2)$ is real, thus $\Phi^\dagger = \Phi$ and
\[
\text{Tr} \, \Phi \Phi^\dagger = \frac{1}{2}(\Phi_3^2 + 2\Phi_+ \Phi_-)
\] (3.24)

Moreover,
\[
\Phi \cdot \Phi = \Phi_1^2 + \Phi_2^2 + \Phi_3^2 = \Phi_3^2 + 2\Phi_+ \Phi_- = 2 \text{Tr} \, \Phi \Phi^\dagger
\] (3.25)

such that
\[
\mathcal{L}_0 = -\frac{1}{2} \text{Tr} \, F_{MN} F^{MN} - 2 \text{Tr} \, D_M \Phi D^M \Phi \quad \text{with} \quad <\Phi_3> = v \neq 0
\] (3.26)

Let us compute the mass spectrum of the model. One can write the covariant derivative of $\Phi$ in components in the basis $\{T_3, T_+, T_-\}$. First, the commutator reads
\[
[A_M, \Phi] = (A_M^- \Phi_+ - A_M^+ \Phi_-)T_3 + (A_M^3 \Phi_+ - A_M^3 \Phi_-) \frac{T_+}{\sqrt{2}} + (A_M^3 \Phi_3 - A_M^3 \Phi_3) \frac{T_-}{\sqrt{2}}
\] (3.27)

which gives for the covariant derivative:
\[
D_M \Phi = (\partial_M \Phi_3 - ig(A_M^- \Phi_+ - A_M^+ \Phi_-)) T_3 + (\partial_M \Phi_- - ig(A_M^3 \Phi_+ - A_M^3 \Phi_-)) \frac{T_+}{\sqrt{2}}
\]
\[
+ (\partial_M \Phi_+ - ig(A_M^3 \Phi_3 - A_M^3 \Phi_3)) \frac{T_-}{\sqrt{2}}
\] (3.28)

From this decomposition, one can see that the vev of $\Phi_3$ gives indeed a mass to the tachyons. Using the trace properties of $T_{3,+,−}$, one can compute the traces in the Lagrangian (3.26) and find that
\[
\text{Tr} \, D_M \Phi D^M \Phi \supset \frac{1}{2} g^2 (v^2 A_3^+ A_3^M + v^2 A_3^+ A_3^M) = g^2 v^2 A_3^+ A_3^M
\] (3.29)

Moreover,
\[
\phi^i = \frac{f}{\sqrt{2}} (A_6^+ + iA_5^+) \quad \phi^\pm = \frac{1}{\sqrt{2}} (\phi^1 \pm i\phi^2)
\] (3.30)

\[
\Rightarrow \phi_\pm = \frac{1}{\sqrt{2}} (A_6^\pm + iA_5^\pm) \quad \phi_\mp = \frac{1}{\sqrt{2}} (A_6^\mp - iA_5^\mp)
\]

which allows to express the extra components of the vector field decomposed in the basis $T_{3,+,−}$ in terms of the higgs-gauge scalar field $\phi$ decomposed in the same basis.
\[
A_6^+ = \frac{1}{\sqrt{2}} (\phi_+ + \phi_-) \quad A_5^+ = \frac{1}{\sqrt{2}i} (\phi_+ - \phi_-)
\]
\[
A_6^- = \frac{1}{\sqrt{2}i} (\phi_+ + \phi_-) \quad A_5^- = \frac{1}{\sqrt{2}i} (\phi_- - \phi_+)
\] (3.31)

The trace (3.29) then reads
\[
\text{Tr} \, D_M \Phi D^M \Phi \supset g^2 v^2 (\phi_+ + \phi_-)
\] (3.32)

We must be cautious with this result. Indeed, the term above is not the only contribution to the scalar masses. Indeed, mixing terms of the type $\Phi_+ \phi_−, \Phi_- \phi_+$ should appear because of the definition of the covariant derivative. We expect to find a mass matrix mixing the states $\phi_+, \phi_−$, and $\Phi_\pm$. Therefore, one should have
in the Lagrangian a 3 by 3 mass matrix including the 2 by 2 matrix of \([3.13]\). We expect the size of the matrix to be 3 and not 4 because the scalar field \(\Phi\) is real and thus
\[
\Phi_\pm = \Phi_+ 
\]  
(3.33)

Now that we have clarified this issue, let us compute the relevant terms in the Lagrangian by including the mass terms of \(\Phi\) and \(\Phi\).

\[
\text{Tr } D_M \Phi D^M \Phi \supset 1/2 \left( \partial_M \Phi_+ - ig(A^3_M > \Phi_+ - A^-_M < \Phi_3) \right) \left( \partial^M \Phi_+ - ig(A^3_M < \Phi_3 > - < A^3_M > \Phi_+) \right)
\]
\[
+ 1/2 \left( \partial_M \Phi_+ - ig(A^3_M < \Phi_3 > - < A^3_M > \Phi_+) \right) \left( \partial_M \Phi_- - ig(A^3_M > \Phi_- - A^-_M < \Phi_3) \right)
\]
\[
\supset (\partial_5 \Phi_- - ig(A^3 > \Phi_- - A^- v)) \left( \partial_5 \Phi_+ - ig(A^3 v - < A^3 > \Phi_+) \right)
\]
\[
+ (\partial_6 \Phi_- - ig(A^6 > \Phi_- - A^- v)) \left( \partial_6 \Phi_+ - ig(A^6 v - < A^6 > \Phi_+) \right)
\]  
(3.34)

The expression above depends on \(\partial_5\) and \(\partial_6\). One would like to introduce \(\partial\) and \(\overline{\partial}\) to obtain the ladder operators that will act on the Landau levels of the charged fields. From \((2.8)\), one can find
\[
\partial_5 = \frac{\partial + \overline{\partial}}{2}, \quad \partial_6 = \frac{\overline{\partial} - \partial}{2i}
\]  
(3.35)

Additionally, we will use \((2.3,3.10)\) to write the third component of \(A_5\) and \(A_6\) as
\[
A_5 = -\frac{f}{2i}(z - \overline{\tau}) \quad A_6 = \frac{f}{2i}(z + \overline{\tau})
\]  
(3.36)

The mass terms in the Lagrangian \((3.26)\) then read
\[
\mathcal{L}^\text{mass}_6 = -1/2 \left( (\partial + \overline{\partial}) \Phi_- - ig \left( -\frac{f}{i}(z - \overline{\tau}) \Phi_- + \frac{\sqrt{2}}{i}(\Phi_+ - \Phi_-) \right) \right) \times
\]
\[
\times \left( (\partial + \overline{\partial}) \Phi_+ - ig \left( \frac{\sqrt{2}}{i}(\Phi_+ - \overline{\Phi}_-) v + \frac{f}{i}(z - \overline{\tau}) \Phi_+ \right) \right)
\]
\[
- 1/2 \left( -i(\overline{\partial} - \partial) \Phi_- - ig \left( f(z + \overline{\tau}) \Phi_- - \sqrt{2}(\overline{\Phi}_+ + \Phi_-) \right) \right) \times
\]
\[
\times \left( -i(\overline{\partial} - \partial) \Phi_+ - ig \left( \sqrt{2}v(\Phi_+ + \overline{\Phi}_-) - f(z + \overline{\tau}) \Phi_+ \right) \right)
\]  
(3.37)

We can reorganize the terms as follows.
\[
\mathcal{L}^\text{mass}_6 = -1/2 \left( (\partial - gf\tau) \Phi_- + (\overline{\partial} + gf\tau) \Phi_- - \sqrt{2}gv(\overline{\Phi}_+ - \Phi_-) \right) \times
\]
\[
\times \left( (\partial + gf\tau) \Phi_+ + (\overline{\partial} - gf\tau) \Phi_+ - \sqrt{2}gv(\phi_+ - \overline{\phi}_-) \right)
\]
\[
+ 1/2 \left( (\partial - gf\tau) \Phi_- - (\overline{\partial} + gf\tau) \Phi_- + \sqrt{2}gv(\overline{\Phi}_+ + \Phi_-) \right) \times
\]
\[
\times \left( (\partial + gf\tau) \Phi_+ - (\overline{\partial} - gf\tau) \Phi_+ - \sqrt{2}gv(\phi_+ + \overline{\phi}_-) \right)
\]  
(3.38)
We have explicitly highlighted the ladder operators (2.30) in the Lagrangian. This allows us to write the mass terms for \( \phi \) in the Lagrangian in the following compact form.

\[
L_{\text{scalar mass}}^6 = -\frac{1}{2} \left( -i \sqrt{2g f}(a_+ + a_-) - \sqrt{2g v}(\overline{\phi} + \phi) \right) \left( -i \sqrt{2g f}(a_+ - a_-) - \sqrt{2g v}(\phi - \overline{\phi}) \right)
+ \frac{1}{2} \left( -i \sqrt{2g f}(a_+^\dagger - a_-^\dagger) + \sqrt{2g v}(\overline{\phi} - \phi) \right) \left( -i \sqrt{2g f}(a_+^\dagger - a_-^\dagger) + \sqrt{2g v}(\phi - \overline{\phi}) \right)
= 2gf \left( a_+^\dagger \Phi_+ a_+ + a_-^\dagger \Phi_- a_- a_+ \Phi_+ \right) - 2g^2 v^2 (\overline{\phi} \phi + \phi \overline{\phi})
+ 2i \sqrt{gf gv} \left( a_+^\dagger \Phi_+ a_+^\dagger - a_- \Phi_- a_+ + \phi \overline{\phi} \right)
\]
(3.39)

As we expected, the Lagrangian contains mass terms (3.32) for the charged gauge scalar fields, mass terms for the charged scalar fields \( \Phi_+ \) and \( \Phi_- \), and also mass terms mixing the two types of scalar fields. To this contribution, one must add the gauge scalar mass terms (3.11). The complete scalar masses Lagrangian is

\[
L_{\text{mass}}^6 = 2gf \left( a_+^\dagger \Phi_+ a_+^\dagger a_+ a_+ \Phi_+ + a_-^\dagger \Phi_- a_- a_+ \Phi_+ \right) + 2i \sqrt{gf gv} \left( a_+^\dagger \Phi_+ a_+^\dagger - a_- \Phi_- a_+ + \phi \overline{\phi} \right)
- (2g^2 v^2 + gf)|\phi_+|^2 - (2g^2 v^2 - gf)|\phi_-|^2 + gf \left( a_+^\dagger \Phi_+ a_+^\dagger + \phi \overline{\phi} \right)
\]
(3.40)

Starting from this Lagrangian, one can compute the effective 4D Lagrangian by decomposing the charged scalar fields \( \Phi_\pm \) with respects to the Landau mode functions (2.31).

\[
\Phi_+ = \sum_{n,j} \Phi_{+,n,j} \xi_{n,j}
\]
\( (3.41) \)

\[
\Phi_- = \sum_{n,j} \Phi_{-,n,j} \xi_{n,j}
\]

Integrating these definitions into the Lagrangian (3.40) and using the definition of Landau mode functions (2.31) allows to make the ladder operators act on mode function.

\[
L_{\text{mass}}^6 = \sum_{n,j,n',j'} 2gf \left( -i \sqrt{n + 1} \Phi_{-,n,j} \xi_{n+1,j} \times (-i \sqrt{n'} + 1) \Phi_{+,n',j'} \xi_{n'+1,j'} \right)
+ i \sqrt{n} \Phi_{-,n,j} \xi_{n-1,j} \times i \sqrt{n'} \Phi_{+,n',j'} \xi_{n'-1,j'}
+ 2i \sqrt{gf gv} \sum_{n,j,n',j'} \left( -i \sqrt{n + 1} \Phi_{-,n,j} \xi_{n+1,j} \Phi_{+,n',j'} \xi_{n'+1,j} - i \sqrt{n} \Phi_{-,n,j} \xi_{n-1,j} \Phi_{+,n',j'} \xi_{n'-1,j} \right)
- i \sqrt{n + 1} \Phi_{-,n',j'} \xi_{n'+1,j'} \Phi_{+,n,j} \xi_{n+1,j} - i \sqrt{n} \Phi_{-,n',j'} \xi_{n'-1,j'} \Phi_{+,n,j} \xi_{n-1,j}
- \sum_{n,j,n',j'} \left( (2g^2 v^2 + gf) \overline{\phi}_{+,n,j} \xi_{n,j} \phi_{+,n',j'} \xi_{n'+j'} + (2g^2 v^2 - gf) \phi_{-,n,j} \overline{\phi}_{+,n',j'} \xi_{n'+j'} \right)
- gf \sum_{n,j,n',j'} \left( \sqrt{n + 1} \Phi_{+,n,j} \xi_{n+1,j} - \sqrt{n} \Phi_{-,n,j} \xi_{n-1,j} \right)
\times \left( \sqrt{n' + 1} \Phi_{+,n',j'} \xi_{n'+1,j'} - \sqrt{n'} \Phi_{-,n',j'} \xi_{n'-1,j'} \right)
\]

(3.42)
The orthonormality relation (2.11) can be applied and one can get after integration on $T^2$:

\[
\mathcal{L}^{\text{mass}}_4 = -2gf \sum_{n,j} (n\Phi_{-n-1,j}\Phi_{+n+1,j} + (n+1)\Phi_{-n+1,j}\Phi_{+n+1,j})
\]

\[
+ 2\sqrt{gf}gv \sum_{n,j} (\sqrt{n}\Phi_{-n-1,j}\Phi_{+n+1,j} + \sqrt{n+1}\Phi_{-n+1,j}\Phi_{+n-1,j} + \sqrt{n}\Phi_{-n,j}\Phi_{+n+1,j} + \sqrt{n+1}\Phi_{-n,j}\Phi_{+n+1,j})
\]

\[
- (2g^2v^2 + gf)|\phi_{+n,j}|^2 - (2g^2v^2 - gf)|\phi_{-n,j}|^2 - gf \sum_{n,j} (\sqrt{n}\Phi_{+n+1,j} - \sqrt{n+1}\Phi_{+n+1,j}) \times
\]

\[
(\sqrt{n}\Phi_{+n+1,j} - \sqrt{n+1}\Phi_{+n+1,j})
\]

(3.43)

Let us perform some translations in the Landau levels.

\[
\mathcal{L}^{\text{mass}}_4 = -2gf \sum_{n,j} ((2n+3)\Phi_{-n+1,j}\Phi_{+n+1,j} + \Phi_{-n,0,j}\Phi_{+n+1,j}) + 2\sqrt{gf}gv \sum_{n,j} (\Phi_{-n,0,j}\Phi_{-n+1,j} + \Phi_{-1,j}\Phi_{+n+1,j})
\]

\[
+ \sqrt{n+2}\Phi_{-n+1,j}\Phi_{-n+2,j} + \sqrt{n+1}\Phi_{-n+1,j}\Phi_{-n+2,j} + \sqrt{n}\Phi_{-n+2,j}\Phi_{+n+1,j} + \sqrt{n+1}\Phi_{-n+2,j}\Phi_{+n+1,j})
\]

\[
- (2g^2v^2 + gf)|\phi_{+n,j}|^2 - (2g^2v^2 - gf)|\phi_{-n+1,j}|^2 + |\phi_{-n,0,j}|^2 + |\phi_{-1,j}|^2
\]

\[
- gf \sum_{n,j} (\sqrt{n+1}\Phi_{+n+1,j} - \sqrt{n}\Phi_{+n+1,j}) (\sqrt{n+1}\Phi_{+n+1,j} - \sqrt{n}\Phi_{+n+1,j}) - gf \sum_{j} |\phi_{-n,j}|^2
\]

(3.44)

We can now rewrite this Lagrangian in a more practical way. The matrix form of the Lagrangian will allow us to compute the mass spectrum and the expression of the different eigenstates.

\[
\mathcal{L}^{\text{mass}}_4 = - \sum_{n,j} (\Phi_{-n,j} \Phi_{-n,0,j}) \times
\]

\[
\begin{pmatrix}
\frac{gf(n+2) + 2g^2v^2}{2gv\sqrt{gf(n+1)}} & -gf\sqrt{(n+1)(n+2)} & -2gv\sqrt{gf(n+1)} \\
-gf\sqrt{(n+1)(n+2)} & \frac{gf(n+1) + 2g^2v^2}{2gv\sqrt{gf(n+1)}} & -2gv\sqrt{gf(n+2)} \\
-2gv\sqrt{gf(n+1)} & -2gv\sqrt{gf(n+2)} & 2gf(2n+3)
\end{pmatrix}
\]

(3.45)

\[
-2 \sum_{j} (\phi_{-1,j} \Phi_{-n,0,j}) \left( \frac{g^2v^2}{-gv\sqrt{gf}} - \frac{g^2v^2}{gv\sqrt{gf}} \right) \left( \frac{\Phi_{-1,j}}{\Phi_{-n,0,j}} \right) - (2g^2v^2 - gf)|\phi_{-n,j}|^2
\]

At first sight, one can see that some appropriate value of the vev could eliminate the tachyon (3.14). Moreover the Lagrangian contains two matrices that need to be studied in more depth. To simplify the calculation, one can write the 3x3 matrix as follows.

\[
\mathcal{M}^2_{3x3} = \begin{pmatrix}
b^2 + c^2 & -ab & -2ac \\
-ab & a^2 + 2c^2 & -2bc \\
-2ac & -2bc & 2(a^2 + b^2)
\end{pmatrix}
\]

(3.46)

whose determinant is zero, as expected. Here,

\[
a = \sqrt{gf(n+2)} \quad b = \sqrt{gf(n+1)} \quad c = gv
\]

(3.47)

Let us summarize the resulting scalar mass spectrum.
1. The 3x3 matrix and the 2x2 matrices both have a massless eigenstate.

\[ \Phi_{G,1}^{n,j} = \sqrt{\frac{n+1}{2n+3+v/f}} \Phi_{+,n,j} + \sqrt{\frac{n+2}{2n+3+v/f}} \Phi_{-,n+2,j} + \sqrt{\frac{v/f}{2n+3+v/f}} \Phi_{+,n+1,j} \]

\[ \Phi_{G,2}^{n,j} = \sqrt{\frac{1}{1+gv^2/f}} \Phi_{-,1,j} + \sqrt{\frac{1}{1+gv^2/f}} \Phi_{+,0,j} \]

(3.48)

2. The 3x3 matrix have two other eigenstates which are massive and the 2x2 matrix have a second eigenstate which is also massive.

\[ \Phi_{M,1}^{n,j} = -\sqrt{\frac{v}{f(2n+3+v/f)(2n+3)}} \Phi_{+,n,j} \]

\[ -\sqrt{\frac{v}{f(2n+3+v/f)(2n+3)}} \Phi_{-,n+2,j} + \sqrt{\frac{2n+3}{2n+3+v/f}} \Phi_{+,n+1,j} \]

\[ \Phi_{M,2}^{n,j} = -\sqrt{\frac{n+2}{2n+3}} \Phi_{+,n,j} + \sqrt{\frac{n+1}{2n+3}} \Phi_{-,n+2,j} \]

\[ \Phi_{M,3}^{n,j} = -\sqrt{\frac{1}{1+gv^2/f}} \Phi_{-,1,j} + \sqrt{\frac{1}{1+gv^2/f}} \Phi_{+,0,j} \]

and their masses are

\[ (m_{M,1}^{n,j})^2 = 2(gf(2n+3) + g^2v^2) \]

\[ (m_{M,2}^{n,j})^2 = gf(2n+3) + 2g^2v^2 \]

\[ (m_{M,3}^{n,j})^2 = 2(gf + g^2v^2) \]

(3.49)

3. The last scalar is the one that was a tachyon before the addition of the scalar \( \Phi \). As we hoped, the addition of \( \Phi \) cures the problem by adding a positive value to the mass. Indeed, \( \phi_{-,0,j} \) is not a tachyon if

\[ 2gv^2 \geq f \]

(3.50)

We have shown that by adding a scalar in the adjoint representation, the tachyons can be eliminated. Furthermore, for sufficiently high values of the vev of this scalar, the non-abelian flux background is perturbatively stable. In other words, none of the fields is tachyonic and the effective action is due to an expansion around the true ground state. In addition, let us note that the elimination of the tachyon depends on the physical parameters of the model. For fixed magnetic flux and vev, we can see that the presence of tachyon depends on the gauge coupling. In particular, one may wonder what happens when one renormalize the theory to energy scale below the limit \( g = \frac{f}{2v} \). Let us note in parallel that the presence of the scalar field does not change the fact that the Wilson line \( \phi_3 \) is still massless in this model. The Lagrangian terms for \( \Phi \) do not give rise to a \( \phi_3 \) mass term.

3.3 Chiral fermions in the SU(2) model without tachyon

In this last section, we would like to add fermions into the model. Indeed, if we wants to use the Wilson line as a Higgs field, it is important to verify that fermions in this model can be chiral. Let us add a \( SU(2) \) doublet in the model. The covariant derivative for this doublet is

\[ D_M = \partial_M + igqA_M \]

(3.51)
and the complete Lagrangian must contains a covariant derivative term for the gauge field, the fermion
doublet and the scalar triplet, a scalar potential, and an interaction term between the scalar field and the
fermion doublet (Yukawa term).

\[ L_6 = -\frac{1}{2} \operatorname{Tr} F_{MN} F^{MN} + i \bar{\Psi} \Gamma^M (\partial_M + igqT^a A_M^a) \Psi - (D_M \Phi)^\dagger D^M \Phi - V(\Phi) + \bar{\Psi} T^a \Psi \Phi^a \] (3.53)

where

\[ \Psi = \begin{pmatrix} \Psi^1 \\ \Psi^2 \end{pmatrix} \quad \Psi^i = \begin{pmatrix} \psi^i \\ 0 \\ 0 \\ \chi^i \end{pmatrix} \] (3.54)

and

\[ A_M = T^a A_M^a = \frac{1}{2} \begin{pmatrix} A_M^3 & \sqrt{2} A_M^- \\ \sqrt{2} A_M^+ & -A_M^3 \end{pmatrix} \] (3.55)

Let us assign a charge \( q_1 \) and \( q_2 \) to the fermions of the doublet. We have already calculated in section 2.3
that the fermionic part of the Lagrangian reads (2.28) which combine classical 4D fermionic terms and the
following internal components terms.

\[ L_6 \supset -\bar{\chi}(\partial + \sqrt{2} g q^2)\psi - \bar{\chi}(\partial + \sqrt{2} g q^2)\psi \] (3.56)

where \( \phi = T^a \phi^a \). Using the matrix form (3.55), one finds

\[ L_6 \supset -\begin{pmatrix} \chi^1 & \chi^2 \end{pmatrix} \begin{pmatrix} \partial + \sqrt{2} g q\phi & 2 g q \phi_i \\ 2 g q \phi & \partial - \sqrt{2} g q \phi_j \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} - \begin{pmatrix} \chi^1 & \chi^2 \end{pmatrix} \begin{pmatrix} \partial + \sqrt{2} g q\phi & 2 g q \phi_i \\ 2 g q \phi & \partial - \sqrt{2} g q \phi_j \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} \] (3.57)

Let us develop these matrices multiplications in components.

\[ L_6 \supset -\chi^1(\partial + \sqrt{2} g q\phi)^i \psi^i - \chi^2(\partial - \sqrt{2} g q\phi)^j \psi^j - \bar{\chi}^1(\bar{\partial} + \sqrt{2} g q\phi)^i \bar{\psi}^i - \bar{\chi}^2(\bar{\partial} - \sqrt{2} g q\phi)^j \bar{\psi}^j \]

\[ -2 g \begin{pmatrix} q_1 \chi^2 \phi^+ \psi^1 + q_2 \chi^1 \phi^- \psi^2 + q_1 \chi^2 \phi^- \bar{\psi}^i + q_2 \chi^1 \phi^+ \bar{\psi}^j \end{pmatrix} \] (3.58)

This Lagrangian contains interactions terms between the charged fields and mass terms for the fermion
fields. Using the annihilation and creation operators, we obtain the mass-squared operators for fermions. For
\( q_1 = -q_2 = 1/2 \), one has

\[ M^2_{\psi^1} = gf a^+_1 a_+ \]
\[ M^2_{\psi^2} = gf (a^+_1 a_- + 1) \]
\[ M^2_{\chi^1} = gf a^+_1 a_+ \]
\[ M^2_{\chi^2} = gf (a^+_1 a_- + 1) \] (3.59)

One has to take into account the Yukawa terms. We denote the Yukawa coupling constant \( h \). The Yukawa
terms takes the form

\[ L_{Yuk} = h \bar{\Psi} \Phi \Psi = \frac{1}{2} \begin{pmatrix} \Psi^1 & \Psi^2 \end{pmatrix} \begin{pmatrix} \Phi^3 & \sqrt{2} \Phi^- \frac{\sqrt{2} \Phi_+}{-\Phi^3} \end{pmatrix} \begin{pmatrix} \Psi^1 \\ \Psi^2 \end{pmatrix} \] (3.60)

\[ = 0 \]

Indeed, for chiral 6D fermions, the Lorentz invariant \( \bar{\Psi} \Psi \) is zero. Therefore, there are no Yukawa terms in
our simple model containing only one doublet.
Let us compute the spectrum to verify that it satisfy the Bachas formula \[ (3.1) \].

\[
\mathcal{L}_6^\text{fermion-mass} = -\chi^1(\partial + gqfz)\psi^1 - \chi^2(\partial + gqfz)\psi^2 - \chi^1(\bar{\partial} + gfz)\overline{\psi}^1 - \chi^2(\bar{\partial} + gqfz)\overline{\psi}^2
\]

\[
= i\sqrt{2}gqf \sum_{n,j; n', j'} \left[ \chi^1_{n,j} a + \psi^1_{n', j'} + \chi^2_{n,j} a - \psi^2_{n', j'} \right] \xi_{n,j} \xi_{n', j'} + \left[ \chi^2_{n,j} a + \psi^2_{n', j'} + \chi^1_{n,j} a - \psi^1_{n', j'} \right] \xi_{n,j} \xi_{n', j'}
\]

\[
= -\sqrt{2}gqf(n + 1) \sum_{n,j} \left[ \chi^1_{n,j} \psi^1_{n+1,j} + \chi^2_{n,j} \overline{\psi}^2_{n+1,j} + \chi^2_{n,j} \psi^2_{n+1,j} + \chi^1_{n,j} \overline{\psi}^1_{n+1,j} \right]
\]

This confirm the Bachas formula \[ (3.1) \] in the case of fermions which predicted \( \delta M_{\text{fermions}}^2 = g f (n+1/2 \pm 1/2) \).

Indeed, the Lagrangian below implies

\[
m^2_{\psi_{n,j}} = gf n
\]

\[
m^2_{\chi_{n,j}} = gf(n + 1)
\]

And therefore the fermions are chiral, as one may have already see with equations \[ (3.59) \]. In particular, the left-handed zero modes are massless while the right-handed zero modes are massive. This is an important feature that the scalar \( \Phi \) preserve the chirality of fermions since chiral fermions are necessary to phenomenology.

### 4 Conclusion and Outlook

In this work, we aimed to continue the work undertaken by \[ 7, 9 \] to use the gauge vector field of a six-dimensional model with flux compactification on a torus to obtain a massless scalar boson.

First, we have explained that the extra component of the gauge fields with a magnetic flux in the background forms a complex scalar field - the Wilson line - which is massless independently of supersymmetry. Indeed, contrary to the case without flux where the Wilson line receives quantum corrections depending on the volume of the torus, the Wilson line scalar is protected from one-loop corrections in U(1) and SU(2) gauge symmetries. However, the SU(2) model contains a finite number of tachyons which encourages investigating the nature of the true ground state by tachyon condensation or to eliminate these tachyons.

Furthermore, we recalled that it has been shown that the Wilson line scalar mass is expected to vanish at all-loop order. Indeed, the Lagrangian of the model is invariant under a shift symmetry of the Wilson line. Therefore, the massless scalar is the Goldstone boson of a translational symmetry in the internal space.

Since one would like to construct a realistic model of the Higgs boson, we investigated the possibility to obtain a non-vanishing mass protected from large corrections. For this purpose, we added a gauge symmetry to the original one and proved that there can be at most only one Nambu-Goldstone boson. The two gauge fields produce two Wilson lines which have a shift symmetry leaving the Yukawa term of the Lagrangian invariant. However, only one particular combination of these two shift symmetries leaves the other interactions terms invariant. Therefore, when the effective fluxes felt by the fermions do not vanish, both of the Wilson lines are pseudo-Nambu-Goldstone bosons with non-vanishing two-loop mass correction. This opens the possibility to interpret the Pseudo scalars as Higgs bosons with finite mass protected from large quantum corrections. However, we must make an important comment. The Wilson lines we are talking about here are uncharged under the gauge symmetry groups while the Higgs boson of the Standard Model is a charged SU(2) doublet. Thus, one would need a Wilson line produced by the U(1) gauge group that would be a doublet of SU(2) (the one of the SM) while preserving the shift symmetry of U(1).

Our second aim was to investigate the possibility to eliminate the tachyons present in the model with an SU(2) gauge symmetry. We first showed that the presence of tachyons was independent of supersymmetry.
Then, we added a scalar boson in the adjoint representation of SU(2) and computed the complete charged scalar mass spectrum. By diagonalizing the mass matrices obtained, we concluded that the model contains two charged Goldstone bosons that are absorbed via the Stückelberg mechanism by the charged vector fields $A_{\mu, n,j}^\pm$, three massive scalar modes of masses (3.50), and one possible tachyonic mode. This latter tachyon is shown to be eliminated for appropriate values of the vacuum expectation value of the uncharged component of SU(2) scalar triplet. In particular, the vev must be greater than $\sqrt{f_2/g}$. Finally, it was verified that the scalar field $\Phi$ of our model preserve the chirality of the fermions.

The present paper suggests several extensions.

1. As explained above, one could incorporate the SU(2) symmetry of the SM such that the Higgs boson is the Wilson line of the U(1) gauge fields with magnetic flux in the background and is a doublet under SU(2). Verification of the fact that the shift symmetry is preserved would be needed.

2. We have shown that the Wilson lines in a $U(1) \times U(1)$ case have a non-vanishing mass protected from one-loop corrections. One would like to precise this result by computing the two-loop quantum corrections to find the compute the Higgs mass and check that a TeV value is possible. In particular, one can envisage large extra dimensions where $1/R$ would be of the order of $10\text{TeV}$ such that a Higgs boson protected from 1-loop or even 2-loop corrections would have a realistic mass.

3. One could investigate the inequality (3.51) and the possibility that tachyons appear depending on the gauge coupling value.

4. We have mainly focused our attention on the hierarchy problem. However, an easier application of the models introduced in this work would be for inflation models. Indeed, contrary to the Higgs boson, the inflaton is uncharged, as the scalar field of this work. As an example, [12] is a recent model of inflation using the massless Wilson line as an inflaton.

**Acknowledgement**

This work is dedicated to Charles Franken, my grandfather.

Finally, I would like to thank Emilian Dudas for his great guidance and supervision throughout my internship at CPHT.
A Appendix: Conventions

In this work, we follow the conventions of [17]. For example, we work with the Minkowski metric \( g_{mn} = \text{diag}(-1, 1, 1, 1) \). We will use Weyl spinors. For a Dirac spinor \( \Psi \) with four components, we write:

\[
\Psi = \begin{pmatrix} \psi_\alpha \\ \chi^{\dot{\alpha}} \end{pmatrix}
\]

(A.1)

where the two components of \( \Psi \) are Weyl spinors and have two components. The Dirac spinor is a representation \((1/2, 1/2)\) of the Lorentz group and the two Weyl components are the two irreducible representations \((1/2, 0)\) and \((0, 1/2)\) respectively. They are chirality eigenstates. Indeed,

\[
\Psi_L = \frac{1 - \gamma_5}{2} \psi = \begin{pmatrix} \psi_\alpha \\ 0 \end{pmatrix}
\]

(A.2)

\[
\Psi_R = \frac{1 + \gamma_5}{2} \psi = \begin{pmatrix} 0 \\ \chi^{\dot{\alpha}} \end{pmatrix}
\]

(A.3)

where we use the Weyl basis for the gamma matrices and \( \alpha, \dot{\alpha} \in \{1, 2\} \) are the Weyl spinors components.

\[
\gamma^\mu = \begin{pmatrix} 0 & \sigma^m_{\alpha\beta} \\ \sigma^m_{\dot{\alpha}\dot{\beta}} & 0 \end{pmatrix}, \quad \gamma_5 = \gamma^5 = -i \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \begin{pmatrix} -\mathbb{I}_{2\times 2} & 0 \\ 0 & \mathbb{I}_{2\times 2} \end{pmatrix}
\]

(A.4)

where \( \sigma^i \) are the Pauli matrices, \( \sigma^0 = \sigma^0 = \text{diag}(-1, -1) \) and \( \sigma^{1,2,3} = -\sigma^{1,2,3} \). The gamma matrices satisfy the Clifford algebra.

\[
\{ \gamma_m, \gamma_n \} = -2\eta_{mn}
\]

(A.5)

The Dirac adjoint is defined by \( \overline{\Psi} = \Psi^\dagger \gamma_0 \). Therefore,

\[
\overline{\Psi} = \left( \chi^\alpha \quad \psi^\dot{\alpha} \right).
\]

(A.6)

Let us make some precision on spinor notations. We define Weyl spinors as the objects carrying the fundamental representation of \( SL(2, \mathbb{C}) \) (the set of nxn matrices of determinant 1). A Weyl spinor transforms under an element of \( Sl(2, \mathbb{C}) \) as

\[
\psi_\alpha \rightarrow \mathcal{M}_{\alpha\beta}^\beta \psi_\beta
\]

(A.7)

with \( \alpha, \beta = 1, 2 \) labelling the components. The complex conjugate of a representation transforms as

\[
\overline{\psi}_{\dot{\alpha}} \rightarrow \mathcal{M}_{\dot{\alpha}\dot{\beta}}^{\dot{\beta}} \overline{\psi}_{\dot{\beta}}
\]

(A.8)

where we used dotted indices to make the difference explicit. One can see that \( \overline{\psi}_{\alpha} = (\psi_\alpha)^* \). Finally, it is useful to introduce the antisymmetric tensor

\[
-\epsilon_{\alpha\beta} = -\epsilon_{\dot{\alpha}\dot{\beta}} = \epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

(A.9)

with the following properties.

\[
\psi \chi = \chi \psi = \psi^\alpha \chi_\alpha \quad \overline{\psi} \overline{\chi} = \overline{\chi} \overline{\psi} = \overline{\psi}_{\dot{\alpha}} \chi^{\dot{\alpha}}
\]

\[
\psi^\dot{\alpha} = \epsilon^{\alpha\beta} \psi_\beta \quad \chi^\alpha \sigma^m_{\alpha\beta} \psi_\beta = -\psi_\beta \sigma^m_{\alpha\beta} \chi^\alpha
\]

(A.10)

\[
\sigma^m_{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} \epsilon^{\alpha\beta} \sigma^m_{\beta\dot{\beta}}
\]

We refer to [17] or [18] for further explanations on spinors and their notations.
Let us introduce our notations and conventions for Kaluza-Klein expansions of higher dimensional fields. Considering a $\text{N}=1$ vector multiplet $V$ as an example, we develop the superfield in its KK modes.

$$V(x_M, \theta, \overline{\theta}) = \sum_{n,m} V_{n,m}(x_\mu, \theta, \overline{\theta}) \psi_{n,m}(x_m)$$  \hspace{1cm} (A.11)

where $V_{n,m}$ are the KK modes and $\psi_{n,m}$ are a complete set of mode functions. Following the convention of [7], we will choose them to be:

$$\psi_{n,m}(x_m) = \frac{1}{2\pi r} \exp \left\{ \frac{i}{r}(nx_5 + mx_6) \right\}$$  \hspace{1cm} (A.12)

where $L^2 = (2\pi r)^2$ is the area of the torus $T^2$ and $n, m \in \mathbb{Z}$. They satisfy the following relations:

$$\int_{T_2} d^2x \bar{\psi}_{n,m} \psi_{k,l} = \delta_{n,k} \delta_{m,l}$$

$$\int_{T_2} d^2x \bar{\psi}_{n,m} \psi_{k,l} \psi_{r,s} = \frac{1}{L} \delta_{n,k+r} \delta_{m,l+s}$$  \hspace{1cm} (A.13)

$$\int_{T_2} d^2x \bar{\psi}_{n,m} \psi_{k,l} \psi_{r,s} \psi_{u,v} = \frac{1}{L^2} \delta_{n,k+r+u} \delta_{m,l+s+v}$$

The reality of $V$ implies $\overline{V}_{n,m} = V_{-n,-m}$. We often note

$$M_{mn} = \frac{2\pi}{r} (m + ln)$$  \hspace{1cm} (A.14)

which are called KK masses
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