Illusory Quantum Hair

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Abstract

We analyze the quantum hair model proposed recently by Coleman, Preskill and Wilczek. We give arguments suggesting that the potential hair is expected to be destroyed by the instability of the black-hole. We also discuss the general implications of such arguments on the prospects for formulating a quantum extension of the classical “no hair” theorems.

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1 Introduction

The classical “no hair” theorems state that a stationary black hole is characterized by a very limited set of parameters, essentially its mass, angular momentum and charges corresponding to massless gauge fields\(^2\). This means that an external observer can get very little information about the matter configuration that formed the black hole. This situation is directly connected to the thermal nature of the Hawking radiation and if the black hole evaporates completely this leads to a violent contradiction with the unitary evolution of a system, according to quantum theory.

One way out would be to find additional degrees of freedom of a black hole that are observable only by quantum effects – “quantum hair”. Coleman, Preskill and Wilczek (CPW) \(^2\) proposed recently a model in which such a quantum hair seems to exist. Using semiclassical thermodynamics, neglecting the black-hole evaporation through the Hawking radiation, they calculated the effects of such a hair that can be measured by an external observer.

The effect found is very small – non-perturbative in \(\hbar\) – and therefore a natural question that arises is whether it is reasonable to assume that this effect is not influenced by the Hawking radiation (which is of first order in \(\hbar\)). Our analysis indicates that it is not. Indeed we show that the characteristic time of the effect is much longer than the whole (semiclassical) lifetime of the black hole and for such a long time the thermodynamic equilibrium hypothesis is certainly not justified. This means that the results indicating the existence of the quantum hair are not reliable. Furthermore, an analysis of an analogous toy model suggests that the instability of the black hole – which causes the energy levels to have finite width – “hides” the potential hair.

We start, in section 2 by reviewing CPW’s model, in section 3 we show its inconsistency and in section 4 we analyze a toy model which demonstrates what we expect really happens. We conclude by discussing the general question of the existence of a quantum hair in view of the above results.

2 The Proposed Hair

In this section we will review briefly the construction of the hair proposed by CPW \(^2\). The proposed model is an Abelian Higgs model coupled to gravity, where the charge of the scalar field \(\varphi\) that “condenses” (acquires a non-trivial vacuum expectation value \(|<\varphi>| = \frac{1}{\sqrt{2}} v\)) is a multiple of the elementary charge quantum: \(Q_\varphi = N\hbar e\). The action is (c=1)

\[
S = S_{grav} + S_{em} + S_{Higgs} + S_{mat}\]

\(^2\)for details, see e.g. \([4, p. 322 – 324]\)
where
\[ S_{\text{grav}} = -\frac{1}{16\pi G} \left[ \int d^4x \sqrt{g}R + 2 \oint d^3x \sqrt{h}K \right] \] (2)

\( h \) is the boundary metric and \( K \) is the extrinsic curvature on the boundary, both induced by \( g \)

\[ S_{\text{em}} = \frac{1}{16\pi} \int d^4x \sqrt{g}F_{\mu\nu}F^{\mu\nu}; \quad F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} \] (3)

\[ S_{\text{higgs}} = \frac{1}{4\pi} \int d^4x \sqrt{g} \left[ g^{\mu\nu}(\partial_{\mu} + ieNA_{\mu})\varphi(\partial_{\nu} - ieNA_{\nu})\varphi^* + \frac{\lambda}{2} \left(|\varphi|^2 - \frac{v^2}{2}\right)^2 \right] \] (4)

and \( S_{\text{mat}} \) is the action of a matter field \( \xi \) of charge \( Q_\xi = \hbar e \) that does not condense. Consequently, although the \( U(1) \) symmetry is spontaneously broken, there is a residual symmetry which is respected by the vacuum:

\[ \Omega = e^{i2\pi \frac{\Omega}{\hbar}} \in \mathbb{Z}_N \subset U(1) \] (5)

under which \( \xi \rightarrow \Omega \xi \) and \( \varphi \) and \( A_\mu \) are invariant.

The existence of a \( \mathbb{Z}_N \) (primary) hair would mean that the black-hole states are labeled by a \( \mathbb{Z}_N \) charge which can be measured by an external observer. CPW propose to identify the charge dependence through the canonical partition function in the black-hole sector. The partition function for a gauge theory can be expressed by a Euclidean path integral [3]:

\[ Z(\beta) \equiv \text{tr}(e^{-\beta H}) = \int_{\beta \hbar} \mathcal{D}A_\mu \mathcal{D}\phi_i e^{-S_E/\hbar} \] (6)

where \( \phi_i \) represent the fields of the model, \( S_E \) is the Euclidean action and the integration \( \int_{\beta \hbar} \) is over the configurations which are periodic in the Euclidean time \( \tau \) with period \( \beta \hbar \) and satisfy

\[ A_\tau(\tau,\bar{x}) \rightarrow 0 \quad \text{as} \quad |\bar{x}| \rightarrow \infty. \] (7)

As suggested by Gibbons and Hawking [4], to treat gravitation quantum mechanically, one integrates over (Euclidean) geometries as well. For the black-hole sector the geometries are restricted to a \( \mathbb{R}^2 \times S^2 \) topology.

Since the charge \( Q \) commutes with the Hamiltonian \( H \), it is meaningful to consider the partition function \( Z(\beta, Q) \), restricted to states of a given charge \( Q \). For \( U(1) \) gauge group these are, by definition, states \( |Q> \) which satisfy

\[ U(\Omega)|Q> = e^{i\omega Q/\hbar e}|Q> \] (8)

where \( \Omega(\bar{x}) \) is a gauge transformation with the asymptotic form

\[ \Omega(\bar{x}) \rightarrow e^{i\omega} = \text{const.} \quad \text{as} \quad |\bar{x}| \rightarrow \infty \] (9)
and $U(\Omega)$ is the unitary representation of the gauge group in the space of states. Inserting the appropriate projection operator $P_Q$ into the trace one obtains

$$Z(\beta, Q) \equiv \text{tr}(P_Q e^{-\beta H}) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega \frac{Q}{\beta}} \tilde{Z}(\beta, \omega)$$

where

$$\tilde{Z}(\beta, \omega) = \int_{\beta h, \omega} Dg_{\mu\nu} DA_{\mu} D\phi e^{-S_E/\hbar}$$

and the integration $\int_{\beta h, \omega}$ is over the configurations which are periodic in $\tau$ with period $\beta h$ and satisfy

$$\int_0^{\beta h} eA_\tau d\tau \to \omega \quad \text{as} \quad |\vec{x}| \to \infty.$$ (12)

The functional integral in $\tilde{Z}(\beta, \omega)$ is calculated semiclassically therefore only configurations with finite action contribute. This means that the geometry must be asymptotically flat. In the Higgs model this also means $|D\phi| \to 0$ and $|\varphi| \to \frac{1}{\sqrt{2}} v$ which implies

$$\omega = 2\pi \frac{k}{N}, \quad k \in \mathbb{Z}$$

($k$ is called “vorticity”), so the expression for the partition function is, finally

$$Z(\beta, Q) = \frac{1}{N} \sum_{k=-\infty}^{\infty} e^{-i2\pi \frac{k}{N} \frac{Q}{\beta}} \tilde{Z}(\beta, k)$$

and using standard thermodynamics one obtains a $Q$-dependent correction to the relation between the mass of the black-hole and its temperature:

$$M(\beta, Q) - M(\beta, Q = 0) = -\frac{\partial}{\partial \beta} \ln \frac{Z(\beta, Q)}{Z(\beta, Q = 0)}.$$ (16)

The same method provides expectation values for local observables such as the components of the electric field $E_j = iF_{j\tau}$, in a given $Q$-sector:

$$<E_j>_{\beta,Q} = \frac{1}{Z(\beta, Q)} \frac{1}{N} \sum_{k=-\infty}^{\infty} e^{-i2\pi \frac{k}{N} \frac{Q}{\beta}} \int_{\beta h, k} iF_{j\tau} e^{-S_E/\hbar}$$

and a non trivial $Q$-dependence provides (at least conceptually) a method to measure the $\mathbb{Z}_N$-charge.
In the semiclassical limit $\hbar \to 0$, the functional integral is dominated by the configuration which minimizes the Euclidean action and therefore is a solution of the classical (Euclidean) equations of motion – an instanton:

$$\tilde{Z}(\beta, k) \sim e^{-S_{\text{clas}}/\hbar}. \hspace{1cm} (18)$$

There is an instanton solution for each vorticity $k$. The corresponding action $S_k$ is even in $k$ and increases with $|k|$, therefore the leading $Q$-dependence is obtained by retaining only $k = -1, 0, 1$. The result is

$$\frac{Z(\beta, Q)}{Z(\beta, Q = 0)} \approx 1 - 2 \left[ 1 - \cos \left( \frac{2\pi Q}{N\hbar} \right) \right] \frac{\tilde{Z}(\beta, k = 1)}{\tilde{Z}(\beta, k = 0)} \hspace{1cm} (19)$$

with

$$\frac{\tilde{Z}(\beta, k = 1)}{\tilde{Z}(\beta, k = 0)} \approx \frac{1}{2} Ce^{-\Delta S/\hbar} \hspace{1cm} (20)$$

where $\Delta S = S_1 - S_0$ and $\frac{1}{2} C(\beta \hbar)$ is a ratio of functional determinants (which can be shown to be positive and $O(1)$ in the limit $\hbar \to 0$, $\beta \hbar = \text{const.}$). This leads (neglecting terms suppressed by additional powers of $\hbar$) to:

$$M(\beta, Q) - M(\beta, Q = 0) \approx - \left[ 1 - \cos \left( \frac{2\pi Q}{N\hbar} \right) \right] C \frac{\partial \Delta S}{\partial \beta \hbar} e^{-\Delta S/\hbar}. \hspace{1cm} (21)$$

The final step is the calculation of $C$ and $\Delta S$. $S_0$ is the action of the (Euclidean) Schwarzschild black-hole, which is (after substructing the (infinite) flat action)

$$S_0 = \frac{(\beta \hbar)^2}{16\pi G} \hspace{1cm} (22)$$

For $k = 1$ one seeks a rotationally invariant solution that obeys the boundary conditions at spatial infinity $r \to \infty$. The corresponding action is evaluated for two limiting values of the Compton wavelength of the massive vector meson $\mu^{-1} = (N ev)^{-1}$, called “string thickness”:

- The thick string limit $\mu^{-1} \gg r_+$:
  (where $r_+ \approx \frac{\beta \hbar}{4\pi}$ is the radius of the horizon)

In this limit the action can be expanded in the small parameter $(\mu r_+)^2 = (r_+ Nev)^2$ and the leading term is obtained by setting $v^2 = 0$ which corresponds to the unbroken $U(1)$ gauge theory. Consequently the solution is the Euclidean Reissner-Nordström black-hole, so $\Delta S$ is given by

$$\Delta S \approx \frac{\pi}{2(Ne)^2} \left[ 1 + \frac{1}{2}(\beta \hbar)^2 \left( \frac{2\pi}{Ne} \right)^2 \right]. \hspace{1cm} (23)$$
• The thin string limit $\mu^{-1} \ll r_+$:

In this limit it is argued that the solution in the $r - \tau$ plane looks just like a cross section of a cosmic string at the origin $r = 0$ (recall that $\tau$ here is an angular coordinate) and using an effective action one obtains

$$\Delta S \approx \frac{1}{4\pi} (\beta \hbar)^2 T_{st} (1 - 4 GT_{st})^{-1}$$

where $T_{st}$ is the string tension. Small back reaction corresponds to $GT_{st} \ll 1$ and in this limit it can be shown that $T_{st}$ has the form

$$T_{st} \approx \frac{1}{4} v^2 f(\lambda/e^2)$$

where $f$ is a slowly varying function such that $f(1) = 1$ therefore for small back reaction one has

$$\Delta S \approx \frac{(\beta \hbar)^2}{16\pi} v^2 f(\lambda/e^2).$$

To summarize, by calculating the canonical partition function in different $Q$-sectors, one discovers observable quantities (e.g. mass, electric field) which depend on the $Z_N$ charge $Q$ of the black-hole and consequently enable a measurement of this charge.

3 The Elusiveness of the Hair

In the process of getting the results of the previous section, a few simplifying assumptions were made – an inevitable step in the current state of knowledge of quantum gravity. A necessary condition (though not sufficient) for the validity of the results is that they be consistent with the assumptions. The assumption that underlies the whole method (as stated by the authors themselves) is that to calculate the effects of the $Z_N$ charge, one can neglect the Hawking radiation. Indeed, the use of a canonical partition function applies to a stationary system in thermal equilibrium with a heat reservoir (in our case this translates to a black-hole in equilibrium with a surrounding radiation bath). It is not clear if the very existence of such a system is allowed by the laws of physics (Since it does not satisfy Einstein’s equations). So in writing down the canonical partition function, one actually refers to an evaporating black-hole with the hope that it can be considered in an approximate equilibrium (a notion that seems to make sense semiclassically since the evaporation process is a quantum effect – of order $\hbar$).

Aposteriori, the problem with this assumption is that the effect that was discovered is non-perturbative in $\hbar$ and therefore much weaker then the neglected Hawking radiation. Let us illustrate this. We are concerned with the limit $\hbar \rightarrow 0$, $\beta \hbar =$const., in which $\beta$ is very large, and therefore

$$Z(\beta, Q) \approx < Q | e^{-\beta H} | Q >$$

(27)
where $|Q >$ is the lowest-energy state with charge $Q$. We define the “topological vacua”

$$|n > \equiv \sum_{q=0}^{N-1} e^{-i2\pi \frac{q}{N}} |Q = qhe >$$  \hspace{1cm} (28)

and obtain

$$< n + \hat{k}|e^{-\beta H}|n > = \hat{Z}(\beta, \hat{k}) \equiv \sum_{N|k-\hat{k}} \tilde{Z}(\beta, k).$$  \hspace{1cm} (29)

(Note that $n$ and $\hat{k}$ are defined modulo $N$.) This means that $\hat{Z}(\beta = \frac{i}{\hbar}t, \hat{k})$ is the transition amplitude between topological vacua with $\Delta n = \hat{k}$ mod $N$, during the time period $t$:

$$\hat{Z}(\beta = \frac{i}{\hbar}t, \hat{k}) = < n + \hat{k}|e^{-\frac{i}{\hbar}tH}|n >$$  \hspace{1cm} (30)

(and the leading contribution $\tilde{Z}(\beta, k)$ is the one with the smallest $|k|$.) This is the exact situation as in gauge field theories (or for a particle in a one dimensional periodic potential). The existence of Euclidean configurations labeled by a “topological charge” (in the present case it is the vorticity $k$) which contribute to the vacuum transition amplitude, implies the existence of a set of states, called “topological vacua” between which the Euclidean configurations interpolate. (This follows from the usual interpretation of Euclidean configurations as contributions to tunneling amplitudes.)

The statement that the states $\{|Q >\}$ are non degenerate is equivalent to the statement that the topological vacua are not stationary states, or in other words, the transition amplitudes between different vacua do not vanish. In the context of a semiclassical calculation such a statement can be checked only for times which are not longer then the semiclassical life-time of a black-hole which is

$$t \approx \frac{10\hbar^2}{\pi^2 G \beta^3}. \hspace{1cm} (31)$$

Let us calculate the transition amplitude:

$$A_{\Delta n}(t) \equiv < n + \Delta n|e^{-\frac{i}{\hbar}tH}|n > =$$  \hspace{1cm} (32)

3 The transition (tunneling) between neighboring topological vacua $|k| = 1$ was interpreted in [2] as a process of nucleation and annihilation of a virtual cosmic string which envelopes the black-hole, however this interpretation is not required for our results.

4 Actually, since the Hawking radiation was neglected, the time should be shorter then the mean time between the emission of Hawking-radiation quanta, which is (for photons with energy $\sim \frac{1}{\beta}$)

$$t \sim \frac{240}{\pi} \beta \hbar.$$
\[ \sum_{q=0}^{N-1} e^{i2\pi \frac{q \Delta n}{N}} < e^{-\frac{i}{\hbar} t H} >_{Q=q e} \]

Using eq. (21) we have

\[ M(q) \equiv < H >_{Q=q e} \approx M_0 + 2\Delta M \cos \left(2\pi \frac{q}{N}\right) \] (33)

where

\[ 2\Delta M = C \frac{\partial \Delta S}{\partial \beta} e^{-\Delta S / \hbar} \] (34)

therefore

\[ A_{\Delta n}(t) = e^{-\frac{i t M_0}{\hbar}} \sum_{q=0}^{N-1} e^{i2\pi \frac{q \Delta n}{N}} \sum_{l=0}^{\infty} \frac{1}{l!} \left[ -\frac{i}{\hbar} t \Delta M \left( e^{i2\pi \frac{q}{N}} + e^{-i2\pi \frac{q}{N}} \right) \right]^l \]

\[ = N e^{-\frac{i t M_0}{\hbar}} \sum_{l_+, l_-} \frac{1}{l_+! l_-!} \left( -\frac{i}{\hbar} t \Delta M \right)^{l_+ + l_-} \] (35)

where the sum is over all non-negative integers \( l_+, l_- \) which satisfy

\[ l_+ - l_- = \Delta n \mod N. \] (36)

Without loss of generality, we choose \(|\Delta n| \leq \frac{N}{2}\) so for \(|t \Delta M| \ll \hbar\) we have\(^5\)

\[ |A_{\Delta n}(t)| = \frac{N}{\Delta n!} \frac{|t \Delta M| |\Delta n|}{\hbar} \left( 1 + \mathcal{O} \left( \frac{t \Delta M}{\hbar} \right) \right) \ll 1. \] (37)

Finally we evaluate \(t \Delta M / \hbar\) taking \(\Delta S\) from eqs. (23 – 26) and \(t\) from (31):

- for the thick string

\[ \left| \frac{t \Delta M}{\hbar} \right| \lesssim \frac{40}{\pi} C \left( \frac{\Delta S}{\hbar} \right)^2 e^{-\Delta S / \hbar} \] (38)

with

\[ \frac{\Delta S}{\hbar} \gtrsim \frac{\pi}{2\alpha} \] (39)

where

\[ \alpha \equiv \hbar (Ne)^2 \] (40)

is the dimensionless gauge field coupling constant (analogous to the fine structure constant in electromagnetism).

\(^5\) Actually for \(|\Delta n| = \frac{N}{2}\) the leading term is twice bigger but this does not spoil our argument
• for the thin string (neglecting back reaction)
\[
\frac{|t \Delta M|}{\hbar} \approx \frac{40}{\pi} C \frac{1}{G \nu_{st}} \left( \frac{\Delta S}{\hbar} \right)^2 e^{-\Delta S/\hbar}
\] (41)

with
\[
\Delta S \approx \frac{(\beta \hbar)^2}{16\pi \hbar} v^2 f \left( \frac{\lambda}{e^2} \right) \gg \frac{1}{\alpha}
\]

(42)

(the inequality follows from \(\frac{d\hbar}{4\pi} \approx r_+ \gg (N_{ev})^{-1}\)).

Both expressions vanish exponentially with \(1/\alpha\) and since \(\alpha\) must be small, it follows that indeed \(|t \Delta M|/\hbar\) is negligibly small.

To summarize, we have shown that the semiclassical approximation does not give any reliable indication for the instability of the topological vacua – the transition between such states during the semiclassical life of the black-hole is extremely improbable. This means that there is no indication for the non-degeneracy of the \(|Q>\) states – no \(\mathbb{Z}_N\)-hair. This, of course, does not prove that the quantum hair does not exist and to discover what really happens, one must go beyond the semiclassical approximation. Since we do not know how to do that, it would be useful to get intuition about what can be expected, by analyzing a simpler model which shares with the real problem the relevant properties. We will analyze such a toy model in the next section, concluding that the \(\mathbb{Z}_N\) hair is indeed expected to be illusory.

4 A Toy Model

We will consider a particle in a one dimensional space, subjected to a potential as described in figure 1. There are three high potential-barriers (labeled \(-2, 0, 2\)) which divide the space into four regions (labeled \(-3, -1, 1, 3\)). The two central regions represent the black-hole sector and the barrier between them is what causes the splitting of the spectrum of this sector – the analogue of the quantum hair. The outer regions represent the “vacuum” into which the black-hole evaporates and the Hawking radiation is modeled by the penetration through the side barriers. This is of course a very crude model and it is only meant to demonstrate how the instability of the internal states influences the ability to measure the splitting. The details of the analysis are given in the appendix and here we only summarize the results.

The stationary state of energy \(E\) can be written in each region, in the WKB approximation (assumed to be valid there, except near the sides), in the form:

• Between the barriers \((V(x) < E)\):
\[
\Psi_t = \frac{1}{\sqrt{k}} \left[ A^+ e^{i \int_{x_l} x_k} kdx' \right] + A^- e^{-i \int_{x_l} x_k} kdx'
\]

(43)
Figure 1: The Potential

- "Under" the barriers ($V(x) > E$):

$$\Psi_l = e^{-i\frac{\pi}{4}} \frac{1}{\sqrt{2k}} \left[ 2B_l^+ e^{\int_{a_l}^{x} kdx'} + B_l^- e^{-\int_{a_l}^{x} kdx'} \right]$$  \hspace{1cm} (44)

- where $\hbar k(x) = \sqrt{2m[E - V(x)]}$.

The relations between the parameters in adjacent regions are given by the "connection formula" for each "turning point" ($V(x) = E$). All the characteristics of the system can be described by the following parameters (functions of the energy $E$):

A barrier is represented by the "penetrability"

$$\gamma = \frac{1}{2} e^{-\int kdx}$$  \hspace{1cm} (45)

and an allowed region (between barriers) is represented by the "total phase"

$$\varphi = \int kdx$$  \hspace{1cm} (46)

Note that each region is described by two expressions, one for each side of the region. This and the unusual normalization of the $B$-coefficients are meant to make these relations as simple as possible (see the appendix).
(where the integration is over the relevant region)\[7\]

We will compare probabilities corresponding to different wave-functions (see below) and for this they have to be normalized. Therefore we will perform the quantization in a box, very large compared to any relevant length parameter in the problem.

For $\gamma_{0,\pm 1} \to 0$ (infinite barriers) there are four disconnected regions and each region has its own states. The inner ("black-hole") energy spectrum is determined by the quantization conditions

$$\cot \varphi_{\pm 1} = 0.$$ \hspace{1cm} (47)

We will be interested in a degenerate level, i.e. we will consider a neighborhood of an energy $E^0$ which is in the spectrum of both left and right inner regions:

$$\cot \varphi_1^0 = \cot \varphi_{-1}^0 = 0.$$ \hspace{1cm}

For this level we will check when it is possible to recognize a splitting caused by finiteness of the central barrier ($\gamma_0 > 0$).

### 4.1 The Symmetric Case

In analogy to the $\mathbb{Z}_N$ symmetry of the black-hole we first assume a $\mathbb{Z}_2$ symmetry of the potential:

$$V(-x) = V(x).$$ \hspace{1cm} (48)

Consequently, the solutions are either symmetric or antisymmetric. Defining

$$\beta_1 \equiv B_1^+/B_1^-$$ \hspace{1cm} (49)

it can be shown that a symmetric solution corresponds to $\beta_1 = +\gamma_0$ and an antisymmetric one corresponds to $\beta_1 = -\gamma_0$ therefore $\beta_1/\gamma_0 = \pm 1$ should be identified as the $\mathbb{Z}_2$-charge of a state.

We want to explore the conditions in which the charge and the energy of the black-hole states are correlated in an observable way so that the charge would be a genuine (i.e. measurable) quantum hair. In the limit $\gamma_{\pm 2} \to 0$, which corresponds to

\[7\] For the validity of the connection formula, we must have

$$\gamma \ll 1 \text{ and } \varphi \gtrsim 10.$$

The first restriction is unimportant since anyway we are interested only in barriers very high above the energy level. But the second restriction means that the analysis is invalid for the lowest levels. This restriction does not appear in the alternative model in which the turning points are replaced by vertical walls. At first look the results relevant to our purposes seem to be identical in both models.
to a completely stable black-hole, the correlation is clear. The black-hole sector is isolated and every degenerate level splits to two levels

\[ dE \equiv E - E^0 = \pm \frac{\gamma_0}{\pi D_1} \]  

(50)

where

\[ D \equiv \frac{dn}{dE} = \frac{1}{\pi} \frac{d\varphi}{dE} \]  

(51)

is the (local) density of states. This splitting corresponds to

\[ d\varphi_1 \equiv \varphi_1 - \varphi_0^1 = \pm \gamma_0 = -\beta_1 \]  

(52)

and this means that we have a separation between states of different charge. Note also that the separation is proportional to the “penetrability” \( \gamma_0 \) of the barrier.

When the side barriers are finite (i.e. the black hole is not stable), the states are not restricted to a single region. In particular, there are no exclusively-black-hole states (each state has “tails” in the outer regions) so we must investigate the whole spectrum of states. This is a very dense spectrum (tends to the continuum in the infinite box limit) and both charges are equally distributed in it, so correlations can emerge only from differences in the probabilities to find (in the black-hole region) states of different charges in different regions of energy. In the appendix we calculate the probability \( P_1 \) to find the particle in the right black-hole region as a function of charge (represented by \( \beta_1 \)) and energy (represented by \( d\varphi \)). The results are shown in figure 2: for each charge...

1. the maximal \( P_1 \) is obtained at

\[ d\varphi_1 = -\beta_1 = \pm \gamma_0; \]  

(53)
2. the “width” $2\Delta\varphi_1$ at half height is

$$\Delta\varphi_1 \approx \gamma_2^2,$$  \hspace{1cm} (54)

so each level in the $\gamma_2 \to 0$ limit is replaced by a band of levels with width $\gamma_2^2$. This is actually a recovery of a very well-known phenomenon – the width of a resonance.

Now the situation should be clear. When the instability $\gamma_2^2$ of the black hole is small compared to the penetrability $\gamma_0$ through the central barrier we have a sharp separation between the two bands of opposite charge and the charge is therefore measurable (this is the situation, in particular, for $\gamma_2 = 0$). As the instability increases, the bands get wider and when $\gamma_2^2 \gg \gamma_0$ they overlap completely and there is no practical way to determine the charge knowing the energy: the instability of the internal states cancels the splitting, as anticipated.

### 4.2 The General Case

Strictly speaking, the symmetric potential is the suitable model for the real situation, so it should be explained why it is interesting to consider a more general case. One motivation is to see if the results change by a small perturbation (i.e. in the case of an approximate symmetry). Also, since this model is very naive, it will be reassuring to realize that the desired effect is more general and exists also when the degeneracy is accidental. We indeed expect it to be universal because the phenomenon of a width of a resonance is commonly seen as a consequence of the uncertainty principle

$$\Delta E \Delta t \approx \hbar.$$  \hspace{1cm} (55)

As in the symmetric case, our task is to identify, in the neighborhood of a degenerate energy, two bands of states and to determine their width and the distance between them. One might expect that the qualitative situation is the same as in the symmetric case (especially when there is an approximate symmetry), namely that there are two (approximate) values of $\beta_1$ and to each of them corresponds a band in a different location. This expectation was found to be wrong, but the conclusions nevertheless remains unchanged. As shown in the appendix, assuming a small instability $\gamma_{2 \pm 2}$, the quantization condition leads this time to pairs of states, one with $\beta_1 \approx \cot \varphi_1$ and the other with $\beta_1 \approx \gamma_0^2 / \cot \varphi_1$. As illustrated in figure (3) for each of these cases, $P_1$ is an even function of $d\varphi_1$ with two local maxima around $d\varphi_1 = \pm \gamma_0 / \lambda_0$, where

$$\lambda_0^2 \equiv \left. \frac{d\varphi_{-1}}{d\varphi_1} \right|_{E_0},$$  \hspace{1cm} (55)

($\lambda_0^2$ can be recognized as the ratio of the (local) densities of states in the two black-hole regions). The only difference between these cases is the width of the minima but in both cases it is proportional to the instability $\gamma_{2 \pm 2}$ of the black hole.
Now we see that the results of the symmetric case are in fact general: a degenerate level splits into two bands which are well separated for a relatively stable black hole $\gamma_{\pm 2}^2 \ll \gamma_0$. The width is proportional to the instability $\gamma_{\pm 2}^2$ so as $\gamma_{\pm 2}^2$ increases the band gets wider and for $\gamma_{\pm 2}^2 \gg \gamma_0$ one should expect that there will be no trace of the splitting.

Finally, to see in what way the thermodynamical analysis of the black hole fails, let us apply the same method to this quantum mechanical model. For the sake of simplicity, we assume $V_{\text{min}} = 0$ and the minimum is reached only once in each region (at $m_{-3}, m_{-1}, m_1, m_3$ respectively). The path-integral expression for the partition function is

$$Z(\beta) = \int_{Eh} D[x(\tau)] e^{SE/Eh} \quad (56)$$

where the integration is over paths $x(\tau)$ periodic in the Euclidean time $\tau$ with period $\beta \hbar$. In the semiclassical limit $\hbar \to 0$ the integral is dominated by classical paths of a particle in an inverted potential $U = -V$ (see figure 4) and in the limit $\tau \to \infty$ only paths between minima contribute. These limits correspond to $\beta \to \infty$ (low temperature) which means that only the lowest-energy states contribute to the partition function.

When the barriers are infinite ($\gamma_{0,\pm 2} \to 0$), the only finite action paths are constant paths and these give the degenerate ground-state energy at the two black-hole regions. When $\gamma_0$ is finite there are additional paths going between the minima $m_{-1}$ and $m_1$ and their contribution is what removes the degeneracy. These are the instantons, analogous to those used in the CPW model. But when $\gamma_{\pm 2}$ are also finite, there are still additional paths going to and from the vacuum areas. As we saw, these paths change the spectrum significantly (unless $\gamma_{\pm 2}$ are sufficiently small). This suggests that to correct the calculation of the black hole energy spectrum, one
should include configurations analogous to this last type of paths. These might be configurations that interpolate between (Euclidean) black holes of different mass (since we know that these are connected through the process of Hawking radiation).

5 General Consequences

In this work we have shown that the quantum hair suggested by CPW is most probably illusory. Essentially, this follows directly from the fact that the effect was much smaller than the Hawking radiation therefore the same arguments should apply to a larger class of would-be quantum hair, in particular those that correspond to non-perturbative effects, as the one analyzed here. This puts the search for quantum hair in a rather delicate situation. On the one hand, the classical theory implies that these hair must be weak – vanishing in the $\hbar \to 0$ limit. On the other hand, the instability of the black hole implies that they cannot be too weak otherwise they will be shadowed by the evaporation process. Another aspect of this problem (not unrelated) is the fact that so far we only know how to make semiclassical calculations and these are considered reliable only for large black holes (small curvature) and weak effects – typically too weak in view of the above arguments. All this leads to a conclusion that there is little hope to find genuine quantum hair using semiclassical methods. Perhaps these arguments can be made more precise to form a quantum extension of the classical “no hair” theorems.

Beyond the semiclassical approximation the situation may be quite different. In particular, it is possible that the black hole does not evaporate completely and instead stabilizes as a Planck-scale remnant [5]. Our arguments do not apply to this case and such a remnant might have arbitrarily weak quantum hair. Thus
until we understand Planck-scale physics, quantum hair cannot be excluded. This however should not make a difference for large black holes which seem to be bald also quantum-mechanically.

### Appendix: Calculational details

In this appendix we derive the results of section 4. The quantization in a box implies – by conservation of probability – that the probability current vanishes and this implies (With the notation defined in section 4):

$$ |A_i^+| = |A_i^-| $$

(57)

and

$$ \beta_i \equiv \frac{B_i^+}{B_i^-} \text{ is real.} $$

(58)

The first of these relations means that $|A_i| \equiv |A_i^+| = |A_i^-|$ is a suitable measure for the probability density. More precisely, the probability for finding the particle in a certain allowed region with a side $a_i$ is

$$ P_i \approx \pi \frac{\hbar^2}{2m} D_i |A_i|^2. $$

(59)

where $D_i$ is the (local) density of states:

$$ D \equiv \frac{dn}{dE} = \frac{1}{\pi} \frac{d\varphi}{dE}. $$

(60)

The relations among the amplitudes are determined by the connection formula, which obtains in the present notation the simple form

$$ \begin{pmatrix} B^+ \\ B^- \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} A^+ \\ A^- \end{pmatrix}. $$

(61)

---

This follows from direct differentiation

$$ D \approx \frac{m}{\pi \hbar^2} \frac{\Delta x}{\bar{k}} $$

where $\bar{k}$ is the average wave number:

$$ \frac{1}{\bar{k}} \equiv \frac{1}{\Delta x} \int \frac{dx}{k}. $$

Note also that $D$ is proportional to the size $\Delta x$ of the region so in the external regions the density is very large (this is extensively used in the following).
The relations between the $B$’s across a barrier are
\[
\begin{pmatrix}
B^-_B \\
B^+_B
\end{pmatrix} = \begin{pmatrix}
\gamma^{-1} & 0 \\
0 & \gamma
\end{pmatrix}
\begin{pmatrix}
B^-_B \\
B^+_B
\end{pmatrix}
\] (62)
and the relations across an allowed region are
\[
\begin{pmatrix}
A^+_q \\
A^-_q
\end{pmatrix} = \begin{pmatrix}
e^{-i\varphi} & 0 \\
0 & e^{i\varphi}
\end{pmatrix}
\begin{pmatrix}
A^-_p \\
A^+_p
\end{pmatrix}
\] (63)
which implies
\[
\begin{pmatrix}
B^+_q \\
B^-_q
\end{pmatrix} = \begin{pmatrix}
\cos \varphi & -\sin \varphi \\
\sin \varphi & \cos \varphi
\end{pmatrix}
\begin{pmatrix}
B^-_p \\
B^+_p
\end{pmatrix}
\] (64)
($a_p, a_q$ denote the sides of the relevant region). From these relations follow some simple and important properties of the real parameter $\beta_l$ defined in (58) which make it a central parameter in the analysis:

- it is directly connected to the ratio of amplitudes on both sides ($a_p, a_q$) of a barrier:
  \[
  \left| \frac{A_q}{A_p} \right|^2 = \frac{\gamma^{-2} \beta_p^2 + \gamma^2}{\beta_p^2 + 1}
  \] (65)
  (this is a monotonically increasing function of $\beta_p$, going from $\gamma^2$ to $\gamma^{-2}$);
- there are simple relations among the $\beta$’s:
  over a barrier: $\beta_p \beta_q = \gamma^2$ (66)
  over an allowed region: $\frac{\beta_p + \beta_q}{1 - \beta_p \beta_q} = \kappa \equiv \cot \varphi$ (67)
- the boundary conditions $\psi(a_{\pm4}) = 0$ imply
  $\beta_{\pm3} = \kappa_{\pm3}$ (68)
  where $\kappa_{\pm3} \equiv \cot(\varphi_{\pm3} - \frac{\pi}{4})$. (69)

Combining all these relations, the quantization condition can be expressed as
\[
\beta_{-1} \beta_1 = \gamma_0^2
\] (70)
where
\[
\beta_{\pm1} = \frac{\kappa_{\pm1} - \gamma_{\pm2}^2}{\kappa_{\pm3}}.
\] (71)
The symmetric case: The $Z_2$-symmetry \((48)\) of the potential implies
\[
\begin{align*}
\kappa_{-1} &= \kappa_1, \\
\gamma_{-2} &= \gamma_2, \\
\kappa_{-3} &= \kappa_3
\end{align*}
\] (72)
and therefore
\[
\beta_{-1} = \beta_1 = \pm \gamma_0.
\] (73)
We assume throughout $\gamma \ll 1$. Among other things this allows us to neglect the probability to find the particle under a barrier, so we have
\[
2P_1 + 2P_3 \approx 1.
\] (74)
Substituting (59) and using (65 – 67) we obtain
\[
\frac{1}{2P_1} \approx 1 + \frac{D_3}{D_1} \left| \frac{A_3}{A_1} \right|^2 \approx 1 + \lambda_2^2 \left[ \gamma_2^2 + \gamma_2^{-2} \sin^2(d\varphi_1 + \beta_1) \right]
\] (75)
where $\lambda_2 = \sqrt{\frac{D_3}{D_1}} \gg 1$. We recall that for each charge $\beta_1$ is constant and from this follows the shape of $P_1$ as described in section 4.

The general case: Proceeding as before, we have
\[
P_{-3} + P_{-1} + P_1 + P_3 \approx 1
\] (76)
which leads to
\[
\frac{1}{P_1} \approx 1 + \frac{D_3}{D_1} \left| \frac{A_3}{A_1} \right|^2 + \frac{D_{-1}}{D_1} \left| \frac{A_{-1}}{A_1} \right|^2 \left( 1 + \frac{D_{-3}}{D_{-1}} \left| \frac{A_{-3}}{A_{-1}} \right|^2 \right)
\]
\[
\approx 1 + \lambda_2^2 \left[ \gamma_2^2 + \gamma_2^{-2} \cos^2(\varphi_1 + \delta\varphi_1) \right] +
\]
\[
+ \left[ \frac{1 + \beta_{-1}^2}{1 + \beta_1^2} \left( \frac{\lambda_0}{\gamma_0 \beta_1} \right)^2 \right] \left[ 1 + \lambda_2^2 \left[ \gamma_2^{-2} + \gamma_2^{-2} \cos^2(\varphi_{-1} + \delta\varphi_{-1}) \right] \right]
\] (77)
where we define:
\[
\lambda_0 \equiv \sqrt{\frac{D_{-1}}{D_1}}, \quad \lambda_{\pm 2} \equiv \sqrt{\frac{D_{\pm 3}}{D_{\pm 1}}}
\] (78)
and
\[
\delta\varphi_{\pm 1} \equiv \arctan \beta_{\pm 1}.
\] (79)
In the following we assume
\[
\lambda_{\pm 2}^2 \ll \gamma_{\pm 2}^2 \ll \lambda_0 \gamma_0, \quad \frac{\gamma_0}{\lambda_0} \ll 1
\] (80)

\footnote{Note that the following analysis does not use directly the symmetry \((48)\) of the potential and it is only needed that \((72)\) will be satisfied in a neighborhood of $E_0$.}
• $\gamma_0 \ll \lambda_0 \ll \gamma_0^{-1}$ to insure $|d\varphi_{\pm 1}| \ll 1$ in the relevant area;
• $\gamma_{\pm 2}^2 \ll \lambda_0 \gamma_0, \gamma_0/\lambda_0$ to obtain a sharp splitting;
• $\lambda_{\pm 2}^{-1} \ll \gamma_{\pm 2}$ to insure sufficient density of states.

It is reasonable to take for the $\gamma$’s and $\lambda$’s their values at $E^0$ so, recalling that

$$d\varphi_{-1} = \lambda_0^2 d\varphi_1, \quad \beta_{-1} = \frac{\gamma_0^2}{\beta_1}$$

we see that $P_1$ can be considered as a function of the two variables $d\varphi_1 \equiv \varphi_1 - \varphi_0^0$ and $\beta_1$. Therefore to obtain the function $P_1(d\varphi_1)$, we have to find the dependence of $\beta_1$ on $d\varphi_1$. This is obtained from the quantization condition (70) and (71). The quantization points are determined by the intersection of

$$\beta_1 = \frac{\kappa_1 - \frac{\gamma_0^2}{\kappa_3}}{1 + \kappa_1 \frac{\gamma_0^2}{\kappa_3}}$$

with

$$\beta_1 = \frac{\kappa_1 - \frac{\gamma_0^2}{\kappa_3}}{1 + \kappa_1 \frac{\gamma_0^2}{\kappa_3}} = \frac{\gamma_0^2}{\beta_{-1}} = \gamma_0^2 \left[ \frac{1 + \kappa_1 - \frac{\gamma_0^2}{\kappa_3}}{\kappa_1 - \frac{\gamma_0^2}{\kappa_3}} \right]$$

(see figure [F]). It is clear from the graph that generically we have pairs of states, one with $\beta_1 \approx \kappa_1$ (which corresponds roughly to $\kappa_3 \approx 0$) and the other with $\beta_1 \approx \frac{\gamma_0^2}{\kappa_3}$ (which corresponds roughly to $\kappa_3 \approx 0$)\(^{10}\). We check first the $\beta_1 \approx \kappa_1$ states. For $|d\varphi_1| \ll 1$ we have

$$\delta \varphi_1 \approx \beta_1 \approx \kappa_1 \approx -d\varphi_1.$$ 

Substituting all this to (71) we obtain

$$P_1^{-1} \approx (\lambda_2 \gamma_2)^2 + \left( \frac{\lambda_0}{\gamma_0} \lambda_{-2} \right)^2 \left[ (1 + \beta_{-1}^2) \beta_{-1}^2 \gamma_{-2}^0 + \gamma_{-2}^0 \cos^2(\varphi_{-1} + \delta \varphi_{-1}) \right].$$

This is an even function of $d\varphi_1$ (recall (81)), with two local maxima around $d\varphi_1 = \pm \gamma_0/\lambda_0$ (see figure [F]). In the regions of the the maxima $|d\varphi_1| \gg \gamma_0^2$ therefore $|\beta_{-1}| \ll 1$ and we finally obtain, for the $\beta_1 \approx \kappa_1$ states:

\(^{10}\) Other values occur only when $\kappa_3$ and $\kappa_{-3}$ vanish (almost) simultaneously and this is very rare if $\varphi_3$ and $\varphi_{-3}$ are not correlated. In this respect, the symmetric case is exceptional (recall that we get there $\beta_1$ independent of $\kappa_1$) and an approximate symmetry is closer to the general case then to the symmetric one.
\( \beta = -\gamma_0 \pm 2 \ll 1 \) \( \beta = \kappa_1 \pm 1 \)

**Figure 5:** The Quantization condition for \( \gamma_{\pm 2} \ll 1 \)

- **the location of the maxima is:**
  \( d\varphi_1 \approx \pm \frac{\gamma_0}{\lambda_0} \); (86)

- **the maximal value is:**
  \( P_{1_{\text{max}}} \approx \left[ (\lambda_2 \gamma_2)^2 + (\lambda_{-2} \gamma_{-2})^2 \right]^{-1} \); (87)

- **the width** \( 2\Delta\varphi_1 \) **at half height is:**
  \[ \Delta\varphi_1 \approx \frac{1}{2} \left( \frac{\gamma_{-2}}{\lambda_0} \right)^2 \sqrt{1 + \left( \frac{\lambda_2 \gamma_2}{\lambda_{-2} \gamma_{-2}} \right)^2} \]. (88)

The other set of states \( \beta_1 \approx \frac{\gamma_2^2}{\kappa_{-1}} \) corresponds to \( \beta_{-1} \approx \kappa_{-1} \) which is the same as the first set but for the left black-hole region. For these states \( P_1 \) looks the same: it has the same maximal values and at the same locations. Only the width is different:

\[ \Delta\varphi_1 \approx \frac{1}{2} (\lambda_0 \gamma_2)^2 \left( \frac{\lambda_{-2}}{\lambda_2} \right)^2 \sqrt{1 + \left( \frac{\lambda_2 \gamma_2}{\lambda_{-2} \gamma_{-2}} \right)^2} \] (89)

but it is still proportional to \( \gamma_{\pm 2}^2 \).

\footnote{Actually the exact location differs slightly from \( \frac{\gamma_2^2}{\lambda_0} \) but the difference is negligible with respect to the width of the band.}
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