The Operator Valued Autoregressive Filter Problem and the Suboptimal Nehari Problem in Two Variables

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Abstract

Necessary and sufficient conditions are given for the solvability of the operator valued two-variable autoregressive filter problem. In addition, in the two variable suboptimal Nehari problem sufficient conditions are given for when a strictly contractive little Hankel has a strictly contractive symbol.

1 Introduction

The classical autoregressive filter problem asks for the construction of an autoregressive filter based on a finite set of prescribed correlation coefficients \( c_0, \ldots, c_n \). There is a solution to this problem if and only if the Hermitian Toeplitz matrix \( C = (c_{i-j})_{i,j=0}^n \)

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is positive definite, and in that case the filter coefficients can be read off from the first column of $C^{-1}$. While the above problem dates back to the 1950’s other aspects of the theory of positive semidefinite Toeplitz matrices had already been studied in detail in the early 1900’s with the works of Carathéodory, Fejér, Kolomogorov, Riesz, Schur, Szegö, and Toeplitz (see e.g. [13] for a full account). Multivariable versions were considered about halfway through the 20th century. Several questions lead to extensive multivariable results (e.g., [24, 25], [5, 6, 8]), while others lead to counterexamples ([3], [31], [17], [9], [27], [26]). The specific two variable autoregressive filter problem was not completely solved until recently in [18]. The authors found that in addition to an expected positive definiteness requirement of a doubly Toeplitz matrix i.e. a block Toeplitz matrix whose blocks are themselves Toeplitz matrices, a low rank condition on a submatrix is necessary for the existence of a two-variable autoregressive filter with a finite set of prescribed correlation coefficients. As it turns out, this low rank condition may be reformulated as a commutativity condition on matrices built from the correlation coefficients. While this was indirectly present in the results in [18] (see Theorem 2.2.1), it was not fully recognized as essential until now. This commutativity condition allows for a generalization to the operator case which we will present in this paper.

The autoregressive filter result yields sufficient conditions on a partially defined doubly Toeplitz matrix to have a positive definite completion, as follows. The notations row$((c_k)_{k \in K})$ and col$((c_k)_{k \in K})$ stand for a row and column vector containing the entries $c_k, k \in K$, respectively. Note that in the statement below matrices appear that have rows and columns indexed by pairs of integers.

**Theorem 1.1** Let $c_k, k \in \Lambda := \{-n, \ldots, n\} \times \{-m, \ldots, m\} \subset \mathbb{Z} \times \mathbb{Z}$ be given so that

$$(c_k - l)_{k,l \in \{0, \ldots, n\} \times \{0, \ldots, m\}}$$

is positive definite. Put

$$\Phi = (c_k - l)_{k,l \in \{0, \ldots, n-1\} \times \{0, \ldots, m-1\}},$$

$$\Phi_1 = (c_k - l)_{k,l \in \{0, \ldots, n-1\} \times \{0, \ldots, m-1\}, l \in \{1, \ldots, n\} \times \{0, \ldots, m-1\}},$$

$$\Phi_2 = (c_k - l)_{k,l \in \{0, \ldots, n-1\} \times \{0, \ldots, m-1\}, l \in \{0, \ldots, n-1\} \times \{1, \ldots, m\}}.$$

Suppose that $\Phi_1 \Phi^{-1} \Phi_2^* = \Phi_2^* \Phi^{-1} \Phi_1$ and

$$c_{-n,m} = K_{n,m} \Phi^{-1} \tilde{K}_{n,m},$$

where

$$K_{n,m} = \text{row}((c_k - l)_{k=(0,m-1), l \in \{1, \ldots, n\} \times \{0, \ldots, m-1\}}),$$

and

$$\tilde{K}_{n,m} = \text{col}((c_k^* - l)_{k \in \{0, \ldots, n-1\} \times \{0, \ldots, m-1\}, l = (n-1,1)}.$$

Then there exist $c_k, k \notin \Lambda$, so that $(c_k - l)_{k,l \in \mathbb{Z} \times \mathbb{Z}}$ is positive definite (as an operator on $l^2(\mathbb{Z} \times \mathbb{Z}))$.  

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Using the connection between positive definite and contractive completion problems as it was used in the band method (see, e.g., [12, 20, 32]) one may take the ideas that go in to the proof of Theorem 1.1 and apply them to the two-variable Nehari problem. The classical Nehari problem states that a bounded Hankel operator \( H \) has an essentially bounded symbol \( \psi \), and in fact one can choose \( \psi \) so that \( \|\psi\|_\infty = \|H\| \) (see, e.g., [20]). In two or more variables the situation is quite different. First of all, there are several types of Hankels to consider. In two variables the most prominent types are the so-called big Hankel and the little Hankel. In [3] it was shown that there exist bounded big Hankel operators that do not have an essentially bounded symbol. Recently, in [13] it was shown that every bounded small Hankel operator has an essentially bounded symbol. The proof in [13] relies on the dual formulation of the problem, due to [14]. In general, though, one cannot find a symbol \( \psi \) of a small Hankel \( h \), so that \( \|h\| = \|\psi\|_\infty \). We will give sufficient conditions under which this equality can be established in a suboptimal sense. To be more precise, we give sufficient conditions under which \( \|h\| < 1 \) implies the existence of a symbol \( \psi \) so that \( \|\psi\|_\infty < 1 \).

The paper is organized as follows. In Section 2 we treat the autoregressive filter problem and as a corollary obtain Theorem 1.1. In Section 3 we treat the two-variable Nehari problem.

### 2 Operator valued autoregressive filters

A two-variable polynomial \( p(z, w) \) is called stable if \( p(z, w) \) is invertible for \( (z, w) \in \mathbb{D}^2 \), where \( \mathbb{D} \) stands for the closure of \( \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \). Also, we denote \( T = \{z \in \mathbb{C} : |z| = 1\} \). The notation \( B(\mathcal{H}, \mathcal{K}) \) stands for the Banach space of bounded linear Hilbert space operators acting \( \mathcal{H} \to \mathcal{K} \). We abbreviate \( B(\mathcal{H}, \mathcal{H}) \) as \( B(\mathcal{H}) \).

#### Theorem 2.1

Given are bounded linear operators \( c_{ij} \in B(\mathcal{H}), (i, j) \in \Lambda := \{-n, \ldots, n\} \times \{-m, \ldots, m\} \setminus \{(n, m), (-n, m), (n, -m), (-n, -m)\} \). There exists stable polynomials

\[
p(z, w) = \sum_{i,j \in \{0, \ldots, n\}} p_{ij} z^i w^j \in B(\mathcal{H}), r(z, w) = \sum_{i,j \in \{0, \ldots, n\}} r_{ij} z^i w^j \in B(\mathcal{H})
\]

(2.1)

with \( p_{00} > 0 \) and \( r_{00} > 0 \) so that

\[
p(z, w)^{-1} p(z, w)^{-1} = \sum_{(i,j) \in \mathbb{Z}^2} c_{ij} z^i w^j = r(z, w)^{-1} r(z, w)^{-1}, \quad (z, w) \in T^2,
\]

(2.2)

for some \( c_{ij} \in B(\mathcal{H}), (i, j) \not\in \Lambda \), if and only if

(i) \( \Phi_1 \Phi^{-1} \Phi^*_2 = \Phi^*_2 \Phi^{-1} \Phi_1 \),

(ii) when we put

\[
c_{-n,m} = \text{row}(c_{k-l})_{k \in \{0, m-1\}, l \in \{1, \ldots, n\} \times \{0, \ldots, m-1\}} \Phi^{-1} \text{col}(c^*_{k-l})_{k \in \{0, \ldots, n-1\} \times \{0, \ldots, m-1\}, l = (n-1, 1)},
\]

then the matrices

\[
(c_{k-l})_{k \in \{0, \ldots, n\} \times \{0, \ldots, m\} \setminus \{(n,m)\}} \quad \text{and} \quad (c^*_{k-l})_{k \in \{0, \ldots, n\} \times \{0, \ldots, m\} \setminus \{(0,0)\}}
\]

are positive definite.
Here
\[ \Phi = (c_{k-l})_{k,l \in \{0,\ldots,n-1\} \times \{0,\ldots,m-1\}}, \]
\[ \Phi_1 = (c_{k-l})_{k,l \in \{0,\ldots,n-1\} \times \{0,\ldots,m-1\}}, \]
\[ \Phi_2 = (c_{k-l})_{k,l \in \{0,\ldots,n-1\} \times \{0,\ldots,m-1\}}. \]

There is a unique choice for \( c_{n,m} \) that results in \( p_{n,m} = 0 \), namely
\[ c_{n,m} = (c_{k-l})_{k,l \in \{0,\ldots,n\} \times \{0,\ldots,m\}} \]
\[ \times (c_{k-l})_{k,l \in \{0,\ldots,n\} \times \{0,\ldots,m\}}^{-1} \times \]
\[ \times (c_{k-l})_{k,l \in \{0,\ldots,n\} \times \{0,\ldots,m\}} \]
\[ \times (c_{k-l})_{k,l \in \{0,\ldots,n\} \times \{0,\ldots,m\}}^{-1} \times \]
\[ \times (c_{k-l})_{k,l \in \{0,\ldots,n\} \times \{0,\ldots,m\}} \]
\[ \times (c_{k-l})_{k,l \in \{0,\ldots,n\} \times \{0,\ldots,m\}}^{-1} \times \]
\[ \times (c_{k-l})_{k,l \in \{0,\ldots,n\} \times \{0,\ldots,m\}} \]
\[ \times (c_{k-l})_{k,l \in \{0,\ldots,n\} \times \{0,\ldots,m\}}^{-1} \times \]
\[ \times (c_{k-l})_{k,l \in \{0,\ldots,n\} \times \{0,\ldots,m\}} \]
\[ \times (c_{k-l})_{k,l \in \{0,\ldots,n\} \times \{0,\ldots,m\}}^{-1} \times \]
\[ \times (c_{k-l})_{k,l \in \{0,\ldots,n\} \times \{0,\ldots,m\}} \]
\[ \times (c_{k-l})_{k,l \in \{0,\ldots,n\} \times \{0,\ldots,m\}}^{-1} \times \]
\[ \times (c_{k-l})_{k,l \in \{0,\ldots,n\} \times \{0,\ldots,m\}} \]
\[ \times (c_{k-l})_{k,l \in \{0,\ldots,n\} \times \{0,\ldots,m\}}^{-1} \times \]
\[ \times (c_{k-l})_{k,l \in \{0,\ldots,n\} \times \{0,\ldots,m\}} \]
\[ \times (c_{k-l})_{k,l \in \{0,\ldots,n\} \times \{0,\ldots,m\}}^{-1} \times \]
\[ \times (c_{k-l})_{k,l \in \{0,\ldots,n\} \times \{0,\ldots,m\}} \]
\[ \times (c_{k-l})_{k,l \in \{0,\ldots,n\} \times \{0,\ldots,m\}}^{-1} \times \]
\[ \times (c_{k-l})_{k,l \in \{0,\ldots,n\} \times \{0,\ldots,m\}} \]
\[ \times (c_{k-l})_{k,l \in \{0,\ldots,n\} \times \{0,\ldots,m\}}^{-1} \times \]
\[ \times (c_{k-l})_{k,l \in \{0,\ldots,n\} \times \{0,\ldots,m\}} \]
\[ \times (c_{k-l})_{k,l \in \{0,\ldots,n\} \times \{0,\ldots,m\}}^{-1} \times \]
\[ \times (c_{k-l})_{k,l \in \{0,\ldots,n\} \times \{0,\ldots,m\}} \]
\[ \times (c_{k-l})_{k,l \in \{0,\ldots,n\} \times \{0,\ldots,m\}}^{-1} \times \]
\[ \times (c_{k-l})_{k,l \in \{0,\ldots,n\} \times \{0,\ldots,m\}} \]
\[ \times (c_{k-l})_{k,l \in \{0,\ldots,n\} \times \{0,\ldots,m\}}^{-1} \times \]
\[ \times (c_{k-l})_{k,l \in \{0,\ldots,n\} \times \{0,\ldots,m\}} \]
\[ \times (c_{k-l})_{k,l \in \{0,\ldots,n\} \times \{0,\ldots,m\}}^{-1} \times \]
\[ \times (c_{k-l})_{k,l \in \{0,\ldots,n\} \times \{0,\ldots,m\}} \]
\[ \times (c_{k-l})_{k,l \in \{0,\ldots,n\} \times \{0,\ldots,m\}}^{-1} \times \]
\[ \times (c_{k-l})_{k,l \in \{0,\ldots,n\} \times \{0,\ldots,m\}} \]
\[ \times (c_{k-l})_{k,l \in \{0,\ldots,n\} \times \{0,\ldots,m\}}^{-1} \times \]
\[ \times (c_{k-l})_{k,l \in \{0,\ldots,n\} \times \{0,\ldots,m\}} \]

Notice that (i) is equivalent to the statement that \( \Phi_1 \Phi_2 \) and \( \Phi_2 \Phi_1 \) commute. These operators correspond exactly to the operators appearing in Theorem 2.2.1 in [18]. When conditions (i) and (ii) are met, the polynomial \( p \) may be constructed by a Yule-Walker type of equation. Alternatively, the Fourier coefficients \( c_{ij} \) may be constructed by an iterative process.

In the proof of Theorem 2.1 we shall make use of some well-known results, including the 3 \( \times \) 3 positive definite operator matrix completion problem and the one-variable operator valued autoregressive filter problem. We now recall these results.

**Proposition 2.2** Let
\[ \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} C & D \\ D^* & E \end{pmatrix} \]
be positive definite Hilbert space operator matrices. Then there exist operators \( X \) so that
\[ M(X) = \begin{pmatrix} A & B & X \\ B^* & C & D \\ X^* & D^* & E \end{pmatrix} \]
is positive definite. E.g., one may choose \( X = BD^{-1}C ) =: X_0 \). In fact, \( X_0 \) is the unique choice for \( X \) so that \( [M(X)^{-1}]_{13} = 0 \).

It is not hard to prove this result directly. The result also appears in the literature in several places, e.g., in [11], [2], [16, Section XVI.3].

We will need operator valued generalizations of Theorem 2.1.5 in [18] (see also Delsarte et al. [7]) and Lemma 2.3.4 in [18].

**Theorem 2.3** Let
\[ p(z,w) = \sum_{i \in \{0,\ldots,n\}} \sum_{j \in \{0,\ldots,m\}} p_{ij} z^i w^j \in B(H). \]

Then \( p(z,w) \) is stable if and only if \( p(z,w) \) is invertible for all \( z \in \overline{\mathbb{D}} \) and \( w \in \mathbb{T} \) and for all \( z \in \mathbb{T} \) and \( w \in \overline{\mathbb{D}} \).
Proof. Since \( p(z, w) \) is invertible for all \( z \in \mathbb{D} \) and \( w \in T \) we can write
\[
p(z, w)^{-1} = \sum_{k=-\infty}^{\infty} g_k(z)w^k, \quad z \in T, w \in T,
\]
where \( g_k(z) \) is analytic for \( z \in \mathbb{D} \). The second condition implies that \( g_k(z) = 0 \) for \( k < 0 \). Thus \( p(z, w)^{-1} \) is analytic for all \( (z, w) \in \mathbb{D}^2 \). Thus \( p(z, w) \) is invertible for all \( (z, w) \in \mathbb{D}^2 \), and hence \( p(z, w) \) is stable. \( \square \)

**Lemma 2.4** Let \( A \) be a positive definite \( r \times r \) operator matrix with entries \( A_{i,j} \in B(\mathcal{H}) \). Suppose that for some \( 1 \leq j < k \leq r \) we have that \( (A^{-1})_{kl} = 0, \ l = 1, \ldots, j \). Write \( A^{-1} = L^*L \) where \( L \) is a lower triangular operator matrix with positive definite diagonal entries. Then \( L \) satisfies \( L_{kl} = 0, \ l = 1, \ldots, j \). Moreover, if \( \tilde{A} \) is the \( (r-1) \times (r-1) \) matrix obtained from \( A \) by removing the \( k \)th row and column, and \( \tilde{L} \) is the lower triangular factor of \( \tilde{A}^{-1} \) with positive diagonal entries, then
\[
L_{il} = \tilde{L}_{il}, \quad i = 1, \ldots, k-1; l = 1, \ldots, j,
\]
and
\[
L_{i+1,l} = \tilde{L}_{il}, \quad i = k, \ldots, r-1; l = 1, \ldots, j.
\]
In other words, the first \( j \) columns of \( L \) and \( \tilde{L} \) coincide after the \( k \)th row (which contains zeroes in columns \( 1, \ldots, j \)) in \( L \) has been removed.

**Proof.** Analog to the proof of Lemma 2.3.4 in [13]. \( \square \)

A polynomial \( A(z) = A_0 + \ldots + A_nz^n \) is called stable if \( A(z) \) is invertible for \( z \in \mathbb{D} \). We say that \( B(z) = B_0 + \ldots + B_nz^n \) is antistable if \( B(1/z)^* \) is stable.

**Theorem 2.5** (The one variable autoregressive filter problem) Let \( A_j, j = -n, \ldots, n, \) be given Hilbert space operators, so that the Toeplitz matrix \( (A_{i-j})_{i,j=0}^n \) is positive definite. Let \( P_0, \ldots, P_n \) and \( Q_{-n}, \ldots, Q_0 \) be defined via
\[
\begin{pmatrix}
A_0 & \cdots & A_{-n} \\
\vdots & \ddots & \vdots \\
A_n & \cdots & A_0
\end{pmatrix}
\begin{pmatrix}
P_0 \\
\vdots \\
P_n
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 1 & 0
\end{pmatrix}
\begin{pmatrix}
A_0 & \cdots & A_{-n} \\
\vdots & \ddots & \vdots \\
A_n & \cdots & A_0
\end{pmatrix}
\begin{pmatrix}
Q_{-n} & \cdots & 0 \\
0 & \cdots & 0 \\
\vdots & \ddots & \ddots
\end{pmatrix}
= \begin{pmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1
\end{pmatrix}.
\]
Write \( P_0 = BB^* \) and \( Q_0 = CC^* \) with \( B \) and \( C \) invertible, and put \( R_i = P_iB_{i-1}^* \), \( S_i = Q_iC_{i-1}^* \), \( R(z) = \sum_{i=0}^{n} R_iz^i \), and \( S(z) = \sum_{i=-n}^{0} S_iz^i \). Then \( R(z) \) is stable and \( S(z) \) is anti-stable. Moreover,
\[
R(z)^{*-1}R(z)^{-1} = S(z)^{*-1}S(z)^{-1} = \sum_{j=-\infty}^{\infty} A_jz^j, \quad z \in T,
\]
for some \( A_j = A^*_j, j > n \). In fact, \( A_j, |j| > n, \) is given inductively via
\[
A_r^* = A_{-r} = (A_{-1} \cdots A_{-n}) [(A_{i-j})_{i,j=0}^{n-1}]^{-1} \begin{pmatrix} A_{-r+1} \\ \vdots \\ A_{-r+n}\end{pmatrix}, \quad r \geq n + 1.
\]
The matrix version of this result goes back to [10]. The operator valued case appeared first in [21]. One may also consult [32, Section III.3] or [22, Chapter XXXIV].

We will need the notions of left and right stable factorizations of operator valued trigonometric polynomials. Let \( A(z) = \sum_{i=-n}^{n} A_i z^i \) be a matrix-valued trigonometric polynomial that is positive definite on \( \mathbb{T} \), i.e., \( A(z) > 0 \) for \( |z| = 1 \). In particular, since the values of \( A(z) \) on the unit circle are Hermitian, we have \( A_i = A_i^* \), \( i = 0, \ldots, n \).

The positive matrix function \( A(z) \) allows a left stable factorization, that is, we may write

\[
A(z) = M(z)M(1/\overline{z})^*, \quad z \in \mathbb{C} \setminus \{0\},
\]

with \( M(z) \) a stable matrix polynomial of degree \( n \). In the scalar case, this is the well-known Fejér-Riesz factorization and goes back to the early 1900's. For the matrix case the result goes back to [30] and [23]. When we require that \( M(0) \) is lower triangular with positive definite diagonal entries, the stable factorization is unique. We shall refer to this unique factor \( M(z) \) as the left stable factor of \( A(z) \). Similarly, we define right variations of the above notions. In particular, if \( N(z) \) is so that \( A(z) = N(1/\overline{z})^*N(z) \), \( z \in \mathbb{C} \setminus \{0\} \), \( N(z) \) is stable and \( N(0) \) is lower triangular with positive definite diagonal elements, then \( N(z) \) is called the right stable factor of \( A(z) \).

**Proof of Theorem 2.1.** Observe that \( \Phi_1 \Phi_1^{-1} \) and \( \Phi_2^{-1} \Phi_2 \), \( i = 1, 2 \), have the following companion type forms:

\[
\Phi_1 \Phi_1^{-1} = \begin{pmatrix} * & \cdots & * & * \\ I & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ I & 0 & \cdots & I \end{pmatrix}, \quad \Phi_2^{-1} \Phi_1 = \begin{pmatrix} 0 & * \\ I & \cdots & * \\ \vdots & \ddots & \vdots \\ I & * \end{pmatrix}, \quad (2.5)
\]

\[
\Phi_2^* \Phi_2^{-1} = (Q_{ij})_{i,j=0}^{n-1}, \quad Q_{ij} = \begin{pmatrix} * & \cdots & * & * \\ \delta_{ij} I & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \delta_{ij} I & 0 & \cdots & \delta_{ij} I \end{pmatrix}, \quad (2.6)
\]

\[
\Phi^{-1} \Phi_2^* = (R_{ij})_{i,j=0}^{n-1}, \quad R_{ij} = \begin{pmatrix} 0 & * \\ \delta_{ij} I & \cdots & * \\ \vdots & \ddots & \vdots \\ \delta_{ij} I & * \end{pmatrix}, \quad (2.7)
\]

where \( \delta_{ij} = 1 \) when \( i = j \) and \( \delta_{ij} = 0 \) otherwise. Consequently, if \( S = (S_{ij})_{i,j=0}^{n-1} \) satisfies

\[
\Phi_1 \Phi_1^{-1} S = S \Phi_1 \Phi_1^{-1}, \quad (2.8)
\]

then \( S \) is block Toeplitz (i.e., \( S_{ij} = S_{i+1,j+1} \) for all \( 0 \leq i, j \leq n - 2 \)). Next, if \( S = (S_{ij})_{i,j=0}^{n-1} \) satisfies

\[
\Phi_2^* \Phi_2^{-1} S = S \Phi_2 \Phi_2^{-1}, \quad (2.9)
\]

then each \( S_{ij} \) is Toeplitz. It follows from (i) that all expressions of the form

\[
S = \Psi_{i_1} \Phi_1^{-1} \Psi_{i_2} \Phi_1^{-1} \cdots \Phi_1^{-1} \Psi_{i_k}, \quad (2.10)
\]
Due to (2.12) it follows from Lemma 2.4 that $P$ and we factor then

Equivalently, if we let $\Phi \Phi_1^{-1} \Phi_2^*$ be as under (ii). Notice that due to (2.11) we have that

where $c_{-n,m}$ is defined by this equation to be as under (ii). Notice that due to (2.11) we have that

with $e_0 = (1 \ 0 \ \cdots \ 0)^T$. Due to (ii) and the $3 \times 3$ positive definite matrix completion problem, we can choose $c_{n,m} = c_{-n,-m}$ so that $\Gamma := (c_{k,r})_{k,l \in \{0, \ldots, n\} \times \{0, \ldots, m\}} > 0$. View $\Gamma = (C_{i,j})_{i,j=0}^n$ where $C_k = (c_{k,r-s})_{r,s=0}^m$, and extend $\Gamma$ following the one variable theory to $(C_{i,j})_{i,j=0}^\infty$, where

Equivalently, if we let

and we factor $Q_0 = LL^*$ with $L$ lower triangular, and put $P_j = Q_j L^{*-1}$, $j = 0, \ldots, n$, then $P(z) := P_0 + \ldots + z^n P_n$ is stable and

Due to (2.12) it follows from Lemma 2.4 that $P_j$ is of the form

But then it follows that $\tilde{P}(z) := \tilde{P}_0 + \ldots + z^n \tilde{P}_n$ is stable, and that

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where \( \tilde{C}_j \) is obtained from \( C_j \) by leaving out the first row and column.

Similarly, if we let

\[
\begin{pmatrix}
(R_n) \\
\vdots \\
(R_0)
\end{pmatrix} = \Gamma^{-1} \begin{pmatrix} 0 \\
\vdots \\
0 \end{pmatrix},
\]

and we factor \( R_0 = U U^* \) with \( U \) upper triangular, and put \( S_j = R_j U^{*-1}, j = -n, \ldots, 0 \), then \( S(z) := S_0 + \ldots + z^{-n} S_{-n} \) is anti-stable and

\[
\sum_{j=-\infty}^{\infty} C_j z^j = S(z)^{-1} S(z)^{-1}, z \in T.
\]

Due to (2.12) it follows from Lemma 2.4 that \( S_j \) is of the form

\[
S_j = \begin{pmatrix}
\tilde{S}_j & (\tilde{P}_{j, -m}) \\
\vdots & \ddots \\
\tilde{P}_{j, -1} & \ddots \\
0 & \ddots & \tilde{P}_{0j}\n\end{pmatrix}, j = -n, \ldots, 0.
\tag{2.14}
\]

But then it follows that \( \tilde{S}(z) := \tilde{S}_0 + \ldots + z^{-n} \tilde{S}_{-n} \) is anti-stable, and that

\[
\sum_{j=-\infty}^{\infty} \tilde{C}_j z^j = \tilde{S}(z)^{-1} \tilde{S}(z)^{-1}, z \in T,
\]

where \( \tilde{C}_j \) is obtained from \( C_j \) by leaving out the last row and column.

Due to the block Toeplitzness of \( C_j, j = -n, \ldots, n \), we have that \( \tilde{C}_j = \tilde{C}_j, j = -n, \ldots, n \). As \( \tilde{S}(z) \) and \( \tilde{P}(z) \) follow the one variable construction with \( (\tilde{C}_i-j)_{i,j=0}^{n} = (\tilde{C}_i-j)_{i,j=0}^{n} \), we have by the one variable theory that

\[
\tilde{P}(z)^{-1} \tilde{P}(z)^{-1} = \tilde{S}(z)^{-1} \tilde{S}(z)^{-1}, z \in T,
\]

and thus \( \tilde{C}_j = \tilde{C}_j, j \in \mathbb{Z} \). Thus \( C_j \) is Toeplitz for all \( j \). Denote \( C_j = (c_{r,s})_{r,s=0}^{m} \). As \( \sum_{j=-\infty}^{\infty} C_j z^j > 0, j \in \mathbb{T} \), we have that the infinite block Toeplitz \( (C_{i,j})_{i,j=-\infty}^{\infty} \) is positive definite. We may regroup this infinite block Toeplitz matrix with Toeplitz entries as \( (T_{i-j})_{i,j=0}^{m} \) where

\[
T_j = (c_{r-s,j})_{r,s=-\infty}^{\infty}.
\]

Taking equality (2.13), and performing a regrouping and extracting the first column from \( P_i \), one arrives at

\[
\begin{pmatrix}
T_0 & \cdots & T_{-m} \\
\vdots & \ddots & \vdots \\
T_m & \cdots & T_0
\end{pmatrix}
\begin{pmatrix}
\Pi_0 \\
\vdots \\
\Pi_m
\end{pmatrix} =
\begin{pmatrix}
Q_0 \\
\vdots \\
o
\end{pmatrix},
\]

8
where \( \Pi_j = (p_{r-s,j})_{r,s=-\infty}^{\infty}, \) \( Q_0 = \Pi_0^{-1} = (q_{r-s,0})_{r,s=-\infty}^{\infty}, \) \( p_{ij} = 0 \) for \( i < 0 \) or \( i > n, j < 0 \) or \( j > m \) and
\[
q(z) = \sum_{i=-\infty}^{0} q_{00} z^i := (\sum_{i=0}^{n} p_{00} z^i)^{*-1}.
\]

Note that \( q(z) \) is indeed anti-analytic as \( \sum_{i=0}^{n} p_{00} z^i \) is stable. The one variable theory now yields that
\[
\Pi(w) := \Pi_0 + \ldots + \Pi_m w^m
\]
is invertible for all \( w \in \mathbb{D} \). As \( \Pi(w) \) is Toeplitz, its symbol is invertible on \( \mathbb{T} \), and thus \( p(z, w) = \sum_{i=0}^{n} \sum_{j=0}^{m} p_{ij} z^i w^j \) is invertible for all \( |w| \leq 1 \) and \( |z| = 1 \). By reversing the roles of \( z \) and \( w \) one can prove in a similar way that \( p(z, w) \) is invertible for all \( |z| \leq 1 \) and \( |w| = 1 \). Combining these two statements yields by Theorem 2.3 that \( p(z, w) \) is stable. In addition, we obtain that
\[
\Pi(w)^{*-1}\Pi(w)^{-1}
\]
has Fourier coefficients \( T_{-m}, \ldots, T_m \). But then it follows that
\[
p(z, w)^{*-1}p(z, w)^{-1}
\]
has Fourier coefficients \( c_{ij} \). Similarly, one proves that \( r(z, w) := \sum_{i \in \{0, \ldots, n\}} \tilde{p}_{-i-j} z^i w^j \) is stable and \( r(z, w)^{-1} r(z, w)^{*-1} \) has Fourier coefficients \( c_{ij} \). This proves one direction of the theorem.

For the converse, let \( p \) and \( r \) as in 2.1 be stable and suppose that 2.2 holds. Denote \( f(z, w) = \sum_{(i,j) \in \mathbb{Z}^2} c_{ij} z^i w^j \). Write \( f(z, w) = \sum_{i=-\infty}^{\infty} f_i(z) w^i \). Then \( T_k(z) := (f_{i-j}(z))_{i,j=0}^{k} > 0 \) for all \( k \in \mathbb{N}_0 \) and all \( z \in \mathbb{T} \). Next, write
\[
p(z, w) = \sum_{i=0}^{m} p_i(z) w^i, r(z, w) = \sum_{i=0}^{m} r_i(z) w^i,
\]
and put \( p_i(z) = r_i(z) \equiv 0 \) for \( i > m \). By the inverse formula for block Toeplitz matrices 19 we have that for \( k \geq m - 1 \) and \( z \in \mathbb{T} \)
\[
T_k(z)^{-1} = \begin{pmatrix}
   p_0(z) & \cdots & 0 \\
   \vdots & \ddots & \vdots \\
   r_{k+1}(1/\bar{z})^* & \cdots & 0 \\
   r_1(1/\bar{z})^* & \cdots & r_{k+1}(1/\bar{z})^*
\end{pmatrix}
\begin{pmatrix}
   p_0(1/\bar{z})^* & \cdots & p_k(1/\bar{z})^* \\
   \vdots & \ddots & \vdots \\
   \tilde{p}_{-i-j} z^i w^j & \cdots & \tilde{p}_{-i-j} z^i w^j \\
   \tilde{p}_{-i-j} z^i w^j & \cdots & \tilde{p}_{-i-j} z^i w^j
\end{pmatrix}
\begin{pmatrix}
   r_k(z) & \cdots & 0 \\
   \vdots & \ddots & \vdots \\
   r_{k+1}(1/\bar{z})^* & \cdots & 0 \\
   r_1(1/\bar{z})^* & \cdots & r_{k+1}(1/\bar{z})^*
\end{pmatrix} =: E_k(z)
\]
(2.15)

As was proven in 18 Proposition 2.1.2 for the scalar case, we have that for \( k \geq m - 1 \), the left stable factors \( M_k(z) \) and \( M_{k+1}(z) \) of \( E_k(z) \) and \( E_{k+1}(z) \), respectively, satisfy
\[
M_{k+1}(z) = \begin{pmatrix}
   p_0(z) & \cdots & 0 \\
   \text{col}(p_i(z))_{i=1}^{k+1} & M_k(z)
\end{pmatrix}.
\]
(2.16)
Indeed, if we define $M_{k+1}(z)$ by this equality, then writing out the product $M_{k+1}(z)M_{k+1}(1/z)^*$ and comparing it to $E_{k+1}(z)$, it is straightforward to see that $M_{k+1}(z)M_{k+1}(1/z)^* = E_{k+1}(z)$. Since both $p_0(z)$ and $M_k(z)$ are stable, $M_{k+1}(z)$ is stable as well. Moreover, since $p_0(0) > 0$ and $M_k(0)$ is lower triangular with positive diagonal entries, the same holds for $M_{k+1}(0)$. Thus $M_{k+1}(z)$ must be the stable factor of $E_{k+1}(z)$. Let $C_k = (c_{k,r,s})_{r,s=0}^m$ as before. Then we have that $T_m(z) = \sum_{k=-\infty}^{\infty} C_k z^k = M_m(1/z)^{-1} M_m(z)$. Writing $M_m(z) = P_0 + \ldots + z^n P_n$ it follows from the one-variable result that

\[
\begin{pmatrix}
C_0 & \cdots & C_{-n} \\
\vdots & \ddots & \vdots \\
C_n & \cdots & C_0
\end{pmatrix}
\begin{pmatrix}
P_0 \\
\vdots \\
P_n
\end{pmatrix}
= \begin{pmatrix}
P_0^{-1} \\
\vdots \\
0
\end{pmatrix}.
\]

Due to the zeros in $P_1, \ldots, P_n$ (see (2.10)) it follows from Proposition 2.2 that (2.14) holds. By a similar argument, reversing the roles of $z$ and $w$, we obtain that

\[
\begin{pmatrix}
\tilde{C}_0 & \cdots & \tilde{C}_{-n} \\
\vdots & \ddots & \vdots \\
\tilde{C}_n & \cdots & \tilde{C}_0
\end{pmatrix}
\begin{pmatrix}
S_{-n} \\
\vdots \\
S_0
\end{pmatrix}
= \begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix},
\]

where $\tilde{C}_k = (c_{r,s-k})_{r,s=0}^n$ and $S_j$ has the form as in (2.11). Using the zero structure of $S_{-1}, \ldots, S_{-n}$ one obtains equality (2.12) with $\Phi_1 \Phi^{-1} \Phi_2^*$ replaced by $\Phi_2^* \Phi^{-1} \Phi_1$. But then it follows that

\[
(I \ 0 \ \cdots \ 0) \Phi_1 \Phi^{-1} \Phi_2^* \begin{pmatrix}
e_0 & 0 & \cdots & 0 \\
0 & e_0 & \cdots & 0
\end{pmatrix} = (I \ 0 \ \cdots \ 0) \Phi_2^* \Phi^{-1} \Phi_1 \begin{pmatrix}
e_0 & 0 \\
0 & e_0
\end{pmatrix}.
\]

Due to (2.5)-(2.7) it is easily seen that $\Phi_2^* \Phi^{-1} \Phi_1$ and $\Phi_1 \Phi^{-1} \Phi_2^*$ have the same block entries anywhere else, so combining this with (2.17) gives that $\Phi_2^* \Phi^{-1} \Phi_1 = \Phi_1 \Phi^{-1} \Phi_2^*$. This yields (i) and the equality for $c_{n,m}$ in (ii). The positive definiteness of the matrices in (ii) follows as they are restriction of the multiplication operator with symbol $f$, which takes on positive definite values on $\mathbb{T}^2$.

**Proof of Theorem 1.1** Follows directly from Theorem 2.1. □

## 3 Nehari’s problem in two variables

We start by stating a version of the operator valued one-variable Nehari result that will be useful in our two-variable result. The operator valued Nehari result is due to Page [28] who proved it using its connection to the commutant lifting theorem, and independently to Adamjan, Arov and Krein [1] who had a matricial approach. The latter approach is close to the one we employ here.

We let $l_2^2(K)$ denote the Hilbert space of sequences $\eta = (\eta_j)_{j \in K}$ satisfying $\|\eta\| := \sqrt{\sum_{j \in K} \|\eta_j\|^2_{\mathcal{H}}} < \infty$. We shall typically write Hankels in a Toeplitz like format by
reversing the order of the columns of our Hankel matrices. E.g., in the one-variable case our Hankels shall typically act \( l^2(-\mathbb{N}_0) \to l^2(\mathbb{N}_0) \) as opposed to the usual convention of acting \( l^2(\mathbb{N}_0) \to l^2(\mathbb{N}_0) \).

**Theorem 3.1** Let \( \Gamma_i \in L(\mathcal{H}, \mathcal{K}), i \geq 0 \), be bounded linear Hilbert space operators so that the Hankel

\[
H := \begin{pmatrix} \Gamma_1 & \Gamma_0 \\ & \ddots & \Gamma_1 \\ & & \ddots & \ddots \\ & & & \ddots & \ddots \\ & & & & \ddots & \ddots \\ & & & & & \ddots & \ddots \\ & & & & & & \ddots & \ddots \\ & & & & & & & \ddots & \ddots \\ & & & & & & & & \ddots & \ddots \\ & & & & & & & & & \ddots \end{pmatrix} : l^2_K(-\mathbb{N}_0) \to l^2_K(\mathbb{N}_0),
\]

(3.1)

is a strict contraction. Solve for operators \( \Delta_0, D_1, D_2, \ldots, B_0, B_1, \ldots \), satisfying the Yule-Walker type equation

\[
\begin{pmatrix} I & H \\ H^* & I \end{pmatrix} \begin{pmatrix} B \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ \Delta \end{pmatrix},
\]

(3.2)

where

\[
B = \begin{pmatrix} B_0 \\ B_1 \\ B_2 \\ \vdots \end{pmatrix} : \mathcal{K} \to l^2_K(\mathbb{N}_0), D = \begin{pmatrix} \vdots \\ D_{-2} \\ D_{-1} \\ I_K \end{pmatrix} : \mathcal{K} \to l^2_K(-\mathbb{N}_0), \Delta = \begin{pmatrix} \vdots \\ 0 \\ 0 \\ \Delta_0 \end{pmatrix} : \mathcal{K} \to l^2_K(-\mathbb{N}_0).
\]

For \( j = -1, -2, \ldots \), put

\[
\Gamma_j = -\Gamma_{j+1}D_{-1} - \Gamma_{j+2}D_{-2} - \ldots = -\sum_{k=1}^{\infty} \Gamma_{j+k}D_{-k}.
\]

(3.3)

Then \( f = \sum_{j=-\infty}^{\infty} \Gamma_j z^j \) belongs to \( L^\infty_K(\mathbb{T}) \) and \( \| f \|_\infty < 1 \).

Alternatively, the Fourier coefficients \( \Gamma_j \) of \( f \) may be constructed as follows. Solve for operators \( \alpha_0, A_1, A_2, \ldots, C_0, C_{-1}, \ldots \), satisfying the Yule-Walker type equation

\[
\begin{pmatrix} I & H \\ H^* & I \end{pmatrix} \begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix},
\]

(3.4)

where

\[
A = \begin{pmatrix} I_K \\ A_1 \\ A_2 \\ \vdots \end{pmatrix} : \mathcal{H} \to l^2_K(\mathbb{N}_0), C = \begin{pmatrix} C_{-2} \\ C_{-1} \\ C_0 \end{pmatrix} : \mathcal{H} \to l^2_K(-\mathbb{N}_0), \alpha = \begin{pmatrix} \alpha_0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} : \mathcal{H} \to l^2_K(\mathbb{N}_0).
\]

For \( j = -1, -2, \ldots \), \( \Gamma_j^* \) may be calculated from

\[
\Gamma_j^* = -\Gamma_{j+1}^*A_1 - \Gamma_{j+2}^*A_2 - \ldots = -\sum_{k=1}^{\infty} \Gamma_{j+k}^*A_k.
\]

(3.5)
Proof. Let
\[ \tilde{H} = \begin{pmatrix} \Gamma_2 & \Gamma_1 \\ \vdots & \vdots \\ \Gamma_2 & \Gamma_1 \end{pmatrix}. \]

Then it follows from (3.2) that
\[ \left( \begin{array}{cc} I & \tilde{H} \\ \tilde{H}^* & I \end{array} \right) \left( \begin{array}{cc} B \\ \tilde{D} \end{array} \right) = - \left( \begin{array}{c} \Gamma \\ 0 \end{array} \right), \]
where
\[ \Gamma = \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \\ \vdots \end{pmatrix}, \quad \tilde{D} = \begin{pmatrix} \vdots \\ D_{-2} \\ D_{-1} \end{pmatrix}. \]

But then it follows that (3.3) is equivalent to the equation
\[ \Gamma_j = (0, Z_{j+1}) \left( \begin{array}{cc} I & \tilde{H} \\ \tilde{H}^* & I \end{array} \right)^{-1} \left( \begin{array}{c} \Gamma \\ 0 \end{array} \right), \quad j \leq -1, \] (3.6)
where
\[ Z_k = (\cdots, \Gamma_{k+1}, \Gamma_k). \]

But this coincides exactly with the iterative process described in [1] (see also [29, Section 2.2]), and thus the conclusion follows from there.

For the alternative construction of \( \Gamma_j \), use that (3.4) implies that
\[ \left( \begin{array}{cc} I & \tilde{H} \\ \tilde{H}^* & I \end{array} \right) \left( \begin{array}{cc} \tilde{A} \\ \tilde{C} \end{array} \right) = - \left( \begin{array}{c} 0 \\ \tilde{\Gamma} \end{array} \right), \]
where
\[ \tilde{\Gamma} = \begin{pmatrix} \vdots \\ \Gamma_1^* \\ \Gamma_0^* \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} A_1 \\ \vdots \\ A_2 \end{pmatrix}. \]

But then (3.5) is equivalent to the equality
\[ \Gamma_j^* = (\tilde{Z}_{j+1} \quad 0) \left( \begin{array}{cc} I & \tilde{H} \\ \tilde{H}^* & I \end{array} \right)^{-1} \left( \begin{array}{c} 0 \\ \tilde{\Gamma} \end{array} \right), \]
with
\[ \tilde{Z}_k = (\Gamma_k \quad \Gamma_{k+1} \quad \cdots), \]
which yields the same sequence of operators \( \Gamma_k, k \leq -1 \), as in (3.6). \( \square \)

We now come to the main result in this section.
Theorem 3.2 Let $\gamma_{ij} \in L(\mathcal{H}, \mathcal{K})$, $i, j \geq 0$, be given so that the little Hankel operator $h_\gamma : l^2_K(-\mathbb{N}_0 \times \mathbb{N}_0) \to l^2_{\mathcal{H}}(\mathbb{N}_0 \times \mathbb{N}_0)$ defined via

$$h_\gamma = \begin{pmatrix} \Gamma_1 & \Gamma_0 \\ \vdots & \Gamma_1 \end{pmatrix}, \quad \Gamma_j = \begin{pmatrix} \gamma_{j1} & \gamma_{j0} \\ \vdots & \gamma_{j1} \end{pmatrix},$$

is a strict contraction. Put

$$\Phi = P_{\mathbb{N}_0 \times \mathbb{N}_0} \oplus P_{\mathbb{N} \times \mathbb{N}_0} \left( \frac{I}{h_\gamma^*} \right) P_{\mathbb{N}_0 \times \mathbb{N}_0} \oplus P_{\mathbb{N} \times \mathbb{N}_0} \left( \frac{I}{h_\gamma} \right) P_{\mathbb{N}_0 \times \mathbb{N}_0} \oplus P_{\mathbb{N} \times \mathbb{N}_0},$$

$$\Phi_1 = P_{\mathbb{N}_0 \times \mathbb{N}_0} \oplus P_{\mathbb{N} \times \mathbb{N}_0} \left( \frac{I}{h_\gamma^*} \right) P_{\mathbb{N}_0 \times \mathbb{N}_0} \oplus P_{\mathbb{N} \times \mathbb{N}_0} \left( \frac{I}{h_\gamma} \right) P_{\mathbb{N}_0 \times \mathbb{N}_0} \oplus P_{\mathbb{N} \times \mathbb{N}_0},$$

$$\Phi_2 = P_{\mathbb{N}_0 \times \mathbb{N}_0} \oplus P_{\mathbb{N} \times \mathbb{N}_0} \left( \frac{I}{h_\gamma^*} \right) P_{\mathbb{N}_0 \times \mathbb{N}_0} \oplus P_{\mathbb{N} \times \mathbb{N}_0} \left( \frac{I}{h_\gamma} \right) P_{\mathbb{N}_0 \times \mathbb{N}_0} \oplus P_{\mathbb{N} \times \mathbb{N}_0},$$

where the projection $P_K : l^2(M) \to l^2(K)$, $K \subseteq M$, is defined by $P_k((\eta_j)_{j \in K}) = (\eta_j)_{j \in K}$. Suppose that

$$\Phi_1 \Phi_2^{-1} \Phi_2 = \Phi_2 \Phi_2^{-1} \Phi_1. \quad (3.7)$$

Then there exist $\gamma_{ij} \in L(\mathcal{H})$, $(i, j) \in (\mathbb{Z} \times \mathbb{N}) \cup (-\mathbb{N} \times \mathbb{Z})$, so that the operator matrix

$$(\gamma_{i-j,k-l})_{i,j,k,l \in \mathbb{Z}} : l^2_{\mathcal{H}}(\mathbb{Z} \times \mathbb{Z}) \to l^2(\mathbb{Z} \times \mathbb{Z})$$

is a strict contraction. Equivalently, the essentially bounded function $f \sim \sum_{i,j \in \mathbb{Z}} \gamma_{ij} z^i w^j$ satisfies $\|f\|_\infty < 1$.

**Proof.** We start by applying Theorem 3.1 to construct $\Gamma_j, j \leq -1$, via (3.3) or, equivalently, (3.5), yielding the strict contraction

$$(\Gamma_{i-j})_{i,j \in \mathbb{Z}} : l^2_{\mathcal{H}}(-\mathbb{N}_0 \times \mathbb{Z}) \to l^2_{\mathcal{H}}(\mathbb{Z} \times \mathbb{Z}).$$

The main step in the proof is to show that (3.7) implies that $\Gamma_j, j \leq -1$, are also Hankel; that is, they are of the form

$$\Gamma_j = \begin{pmatrix} \gamma_{j1} & \gamma_{j0} \\ \vdots & \gamma_{j1} \end{pmatrix}, j \leq -1,$$

for some operators $\gamma_{ij}, j \geq 0, i \leq -1$. To show this we need to prove the following claim.

**Claim.** Equation (3.7) implies that the operators $D_j, j \leq -1$, in (3.2) are of the form

$$D_j = \begin{pmatrix} \vdots & \vdots & \vdots \\ \cdots & * & * \\ \cdots & * & * \\ \cdots & 0 & * \end{pmatrix} : l^2_{\mathcal{H}}(-\mathbb{N}_0) \to l^2_{\mathcal{H}}(-\mathbb{N}_0).$$
Similarly, (3.7) implies that $A_j$ in (3.4) is of the form

$$A_j = \begin{pmatrix} * & 0 & 0 & \cdots \\ * & * & * & \cdots \\ * & * & * & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}: l^2_N(N_0) \to l^2_N(N_0).$$

**Proof of Claim.** It is not hard to see that $\Phi_1 \Phi^{-1}$ and $\Phi^{-1} \Phi_1$, $i = 1, 2$, have a certain companion type form (variations of the ones in the proof of Theorem 2.1). For instance,

$$\Phi_1 \Phi^{-1} = \begin{pmatrix} \hat{S} & Q \\ 0 & \hat{S} \end{pmatrix}, \Phi^{-1} \Phi_1 = \begin{pmatrix} Z & \hat{Q} \\ 0 & Z \end{pmatrix},$$

where $\hat{S}$ and $\hat{Z}$ have an infinite companion form

$$\hat{S} = \begin{pmatrix} I & * & \cdots \\ I & I & \vdots \\ \vdots & \ddots & \ddots \end{pmatrix}, \hat{Z} = \begin{pmatrix} \cdots & I & * \\ \cdots & I & I \\ \vdots & \ddots & \ddots \end{pmatrix},$$

the operators $S$ and $Z$ are shifts

$$S = \begin{pmatrix} \cdots & I & 0 \\ \cdots & I & 0 \\ 0 & 0 & \ddots \end{pmatrix}, Z = \begin{pmatrix} 0 & 0 & \cdots \\ I & I & \vdots \\ \vdots & \ddots & \ddots \end{pmatrix},$$

and $Q$ and $\hat{Q}$ are zero except for the first block row and last block column, respectively:

$$Q = \begin{pmatrix} * & * & \cdots \\ 0 & 0 & \cdots \\ \vdots & \ddots & \ddots \end{pmatrix}, \hat{Q} = \begin{pmatrix} \cdots & 0 & * \\ \cdots & 0 & * \\ \vdots & \ddots & \ddots \end{pmatrix}.$$

But then, viewing $R := \Phi_2^* \Phi^{-1} \Phi_1 = \Phi_1 \Phi^{-1} \Phi_2^*$ in the four possible ways ($\Phi_2^* \Phi^{-1} \Phi_1$, $\Phi_2^*(\Phi^{-1} \Phi_1)$, $(\Phi_1 \Phi^{-1}) \Phi_2^*$, $\Phi_1 (\Phi^{-1} \Phi_2^*)$) one easily deduces that

$$\Phi_1 \Phi^{-1} \Phi_2^* = \Phi_2^* \Phi^{-1} \Phi_1 = P_{N_0 \times N} \oplus P_{-N \times N} \begin{pmatrix} I & h_{\gamma} \\ h_{\gamma}^* & I \end{pmatrix} P_{N \times N}^* \oplus P_{-N \times -N}^*.$$

Multiplying the above equation on the left with $0 \oplus P_{-N \times \{0\}}$ and on the right with $0 \oplus P_{\{0\} \times -N}^*$ gives that $YW^{-1}U = X$, where $U, W, X$ and $Y$ are defined via

$$Y = 0 \oplus P_{-N \times \{0\}} \begin{pmatrix} I & h_{\gamma} \\ h_{\gamma}^* & I \end{pmatrix} P_{N_0 \times N_0}^* \oplus P_{-N \times -N}^*, \quad W = P_{N_0 \times N_0} \oplus P_{-N \times -N} \begin{pmatrix} I & h_{\gamma} \\ h_{\gamma}^* & I \end{pmatrix} P_{N_0 \times N_0}^* \oplus P_{-N \times -N}^*,$$
\[ U = P_{N_0 \times N_0} \oplus P_{-N \times -N} \left( I \begin{pmatrix} h_\gamma & \gamma \end{pmatrix} \right) 0 \oplus P_{\{0\} \times -N}, \]

and

\[ X = 0 \oplus P_{-N \times \{0\}} \left( I \begin{pmatrix} h_\gamma & \gamma \end{pmatrix} \right) 0 \oplus P_{\{0\} \times -N}. \]

View the operator

\[ M = \left( P_{(-N_0 \times -N_0) \setminus \{(0,0)\}} \begin{pmatrix} I \gamma \gamma \ast \end{pmatrix} \right) \]

after permutation as the operator matrix

\[
\begin{pmatrix}
* & Y & X \\
* & W & U \\
* & * & *
\end{pmatrix}
\]

acting on

\[ [0 \oplus l^2(-N \times \{0\})] \oplus [l^2(N_0 \times N_0) \oplus l^2(-N \times -N)] \oplus [0 \oplus l^2(\{0\} \times -N)]. \]

Then the equality \( YW^{-1}U = X \) together with Proposition 2.2 gives that

\[ (0 \oplus P_{-N \times \{0\}})M^{-1}(0 \oplus P_{\{0\} \times -N}) = 0. \]

This exactly yields the required zeros in \( D_j, j \leq -1. \)

The proof of the zeros in \( A_j, j \geq 1, \) is similar. This proves the claim. \( \square \)

Following the claim, we may now write \( D_j \) and \( A_j \) as

\[ D_j = \begin{pmatrix} \tilde{D}_j & q_j \\ 0 & \delta_j \end{pmatrix}, j \leq -1; \quad A_j = \begin{pmatrix} \alpha_j & 0 \\ r_j & \tilde{A}_j \end{pmatrix}, j \geq 1. \]

Write

\[ \Gamma_j = \begin{pmatrix} \tilde{\Gamma}_j & \gamma_j0 \\ \gamma_j1 & \tilde{\Gamma}_j \end{pmatrix}, \quad \Gamma_j = \begin{pmatrix} \ldots & \gamma_j1 & \gamma_j0 \\ \ldots & \tilde{\Gamma}_j & \ldots \end{pmatrix}. \]

Note that \( \tilde{\Gamma}_j = \tilde{\Gamma}_j, j \geq 0. \) Observe that due to (3.8), equation (3.2) implies

\[ \begin{pmatrix} I & \tilde{h}_\gamma \\ \tilde{h}_\gamma^* & I \end{pmatrix} \begin{pmatrix} B \\ \tilde{D} \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{\Delta} \end{pmatrix}, \]

with

\[ \tilde{h}_\gamma = (\tilde{\Gamma}_{i-j})_{i \in N_0, j \in -N_0}, \quad \tilde{D} = \begin{pmatrix} \vdots \\ \tilde{D}_{-2} \\ \tilde{D}_{-1} \\ I \end{pmatrix}, \quad \tilde{\Delta} = \begin{pmatrix} \vdots \\ 0 \\ 0 \\ \tilde{\Delta}_0 \end{pmatrix}, \]
where \( \tilde{\Delta}_0 \) is obtained from \( \Delta_0 \) by removing the last row and column; that is \( \Delta_0 = (\tilde{\Delta}_0 \ast \ast) \). Moreover, if we define

\[
\tilde{\Gamma}_j = -\sum_{k=1}^{\infty} \tilde{\Gamma}_{j+k}\tilde{D}_{-k}, \ j \leq -1,
\]

then we have that \( \tilde{\Gamma}_j \) corresponds to \( \Gamma_j \) without the last column for \( j \leq -1 \) as well. In other words,

\[
\Gamma_j = (\tilde{\Gamma}_j \ast), \ j \leq -1.
\]

Likewise, due to the form of \( A_j \), we have that \( \hat{A}_j \) may be constructed from (3.4) with \( \Gamma_j \) replaced by \( \hat{\Gamma}_j \). Moreover, if we define

\[
\hat{\Gamma}_j = -\sum_{k=1}^{\infty} \hat{\Gamma}_{j+k}\hat{A}_{j}, \ j \leq -1,
\]

then we have that \( \Gamma_j = (\hat{\Gamma}_j \ast), \ j \leq -1 \). But since \( \hat{\Gamma}_j = \tilde{\Gamma}_j, \ j \geq 0 \), we obtain from Theorem 3.1 that

\[
\tilde{\Gamma}_j = -\sum_{k=1}^{\infty} \tilde{\Gamma}_{j+k}\tilde{D}_{-k} = -\sum_{k=1}^{\infty} \hat{\Gamma}_{j+k}\tilde{D}_{-k} = -\sum_{k=1}^{\infty} \hat{A}_j^*\hat{\Gamma}_{j+k} = \hat{\Gamma}_j, \ j \leq -1.
\]

Since

\[
\Gamma_j = (\hat{\Gamma}_j \ast) = (\tilde{\Gamma}_j \ast), \ j \leq -1,
\]

it now follows that \( \Gamma_j, \ j \leq -1, \) is Hankel.

The last step in the proof is to recognize that

\[
\| (\Gamma_{i-j})_{i,j \in \mathbb{Z}} \| < 1
\]

implies that the Hankel \( (H_{i-j})_{i \in \mathbb{N}_0, j \in -\mathbb{N}_0} \) is a strict contraction, where

\[
H_i = (\gamma_{p-q,i})_{p,q \in \mathbb{Z}, i \geq 0}.
\]

But now it follows that \( H_i = (\gamma_{p-q,i})_{p,q \in \mathbb{Z}, i \leq -1} \), exist so that \( (H_{i-j})_{i,j=-\infty}^{\infty} \) is a strict contraction. \( \Box \)

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