POSITIVE GROUND STATE SOLUTIONS FOR A QUASILINEAR ELLIPTIC EQUATION WITH CRITICAL EXPONENT

YINBIN DENG* AND WENTAO HUANG
School of Mathematics and Statistics, Central China Normal University
Wuhan 430079, China

Abstract. In this paper, we study the following quasilinear elliptic equation with critical Sobolev exponent:
\[-\Delta u + V(x)u - \left[ \Delta (1 + u^2)^{1 \over 2} \right] u = |u|^{2^* - 2} u + |u|^p - 2 u, \quad x \in \mathbb{R}^N,\]

which models the self-channeling of a high-power ultra short laser in matter, where \( N \geq 3, \quad 2 < p < 2^* = \frac{2N}{N-2} \) and \( V(x) \) is a given positive potential. Combining the change of variables and an abstract result developed by Jeanjean in [14], we obtain the existence of positive ground state solutions for the given problem.

1. Introduction and main results. In this paper, we study the existence of positive ground state solutions for the following quasilinear elliptic equation:
\[-\Delta u + V(x)u - \left[ \Delta (1 + u^2)^{1 \over 2} \right] u = |u|^{2^* - 2} u + |u|^p - 2 u, \quad x \in \mathbb{R}^N,\]

where \( N \geq 3, \quad 2 < p < 2^* \) with \( 2^* = \frac{2N}{N-2} \) the critical Sobolev exponent, and \( V(x) \) is a given positive potential.

The solutions of this type of quasilinear elliptic equation are related to the solitary wave solutions for quasilinear Schrödinger equations of the form
\[i \partial_t z = -\Delta z + W(x)z - h(z) - \Delta l(|z|^2)l'(|z|^2)z,\]

where \( z : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}, \quad W : \mathbb{R}^N \to \mathbb{R} \) is a given potential and \( h, \quad l : \mathbb{R} \to \mathbb{R} \) are suitable functions. Quasilinear equations (2) have been derived as models of several physical phenomena and have been the subject of extensive study in recent years. The case \( l(s) = s \) models the time evolution of the condensate wave function in super-fluid film ([16, 17]). This equation has been called the superfluid film equation in fluid mechanics by Kurihara [16]. In the case \( l(s) = (1+s)^{1/2} \), problem (2) models the self-channeling of a high-power ultrashort laser in matter, see [4, 6, 8, 27].

Set \( z(t,x) = \exp(-iEt)u(x) \), where \( E \in \mathbb{R} \) and \( u \) is a real function, (2) can be reduced to the corresponding equations of elliptic type:
\[-\Delta u + V(x)u - \Delta l(u^2)l'(u^2)u = h(u), \quad x \in \mathbb{R}^N.\]

2010 Mathematics Subject Classification. Primary: 35J20; Secondary: 35J62, 35B33.
Key words and phrases. Ground state solutions, quasilinear elliptic equation, critical exponent.
* Corresponding author: Yinbin Deng.
If we set \( l(s) = s \), we get the superfluid film equation in plasma physics:

\[
-\Delta u + V(x)u - (u^2)u = h(u), \quad x \in \mathbb{R}^N. \tag{4}
\]

If \( l(s) = (1 + s)^{\frac{1}{2}} \), we get the equation:

\[
-\Delta u + V(x)u - \left[ \Delta (1 + u^2)^{\frac{1}{2}} \right] \frac{u}{2(1 + u^2)^{\frac{1}{2}}} = h(u), \quad x \in \mathbb{R}^N, \tag{5}
\]

which models the self-channeling of a high-power ultrashort laser in matter.

Problem (4) has been studied extensively recently. The existence of a positive ground state solution of problem (4) has been proved by Poppenberg et al. [26] and Liu and Wang [22] by using a constrained minimization argument, which gives a solution to the equation with an unknown Lagrange multiplier \( \lambda \) in front of the nonlinear term. In [21], by utilizing the Nehari method, Liu et al. treated more general quasilinear problems and obtained positive and sign-changing solutions. By a change of variables in [20], the quasilinear problem was transformed to a semilinear one and an Orlicz space framework was used as the working space, and a positive solution of problem (4) was obtained by using the Mountain Pass Lemma [1]. This argument was also used later in [7], but the usual Sobolev space \( H^1(\mathbb{R}^N) \) framework was used as the working space. Following the idea in [7, 20] for subcritical problems, the authors of [3, 29] took this transformation for (4) with critical growth to obtain a positive solution. For more results about the existence of positive solutions for (4) with critical growth, we can refer to [18, 23, 24, 25] and so on.

A natural question is whether there is a unified approach to study (3) with general functions \( l(s) \)? To answer this question, we denote

\[
g^2(u) = 1 + \frac{(l(u^2))'^2}{2},
\]

then problem (3) can be reduced to quasilinear elliptic equations (see [28]):

\[
-\text{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = h(u), \quad x \in \mathbb{R}^N. \tag{6}
\]

To find solitary wave solutions of (2), it is sufficient to find positive solutions of quasilinear elliptic equations (6). By introducing a new variable replacement as follows

\[
v = G(u) = \int_0^u g(t)dt,
\]

Shen and Wang in [28] studied the existence of positive solitary wave solutions for (2) with a general functions \( l(s) \). The existence of positive solutions for problem (6) is obtained under some assumptions on \( g, V \) and \( h \) with \( h \) is subcritical growth. By using the same change of variables and variational argument, the first author, Peng and Yan studied the generalized problem (6) with critical growth, and obtained the existence of positive solutions in [9, 10].

More precisely, the authors in [10] found that the critical exponents for problem (6) with general \( g(u) \) are \( \alpha 2^* \) if \( \lim_{t \to \infty} \frac{g(t)}{t^{\alpha - 1}} = \beta > 0 \) for some \( \alpha \geq 1 \). Using this fact, they proposed the critical problem for given \( g(u) \) as follows:

\[
-\text{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = |u|^{p-2}u + |u|^{\alpha 2^*-2}u, \quad x \in \mathbb{R}^N, \tag{8}
\]

where \( 2^* = \frac{2N}{N-2} \) and \( 2\alpha < p < \alpha 2^* \). As a corollary, in [10], they established the existence of positive solutions for problem (8) if \( g \) satisfies the following assumption:
(g_1) \ g \in C^1(\mathbb{R}) \text{ is an even positive function and } g'(t) \geq 0 \text{ for all } t \geq 0, \ g(0) = 1.

Moreover, there exist some constants \( \alpha \geq 1, \beta > 0 \) and \( \gamma \in (-\infty, \alpha) \) such that
\[
g(t) = \beta t^{\alpha-1} + O(t^{\gamma-1}) \quad \text{as } t \to +\infty \quad \text{and} \quad (\alpha - 1)g(t) \geq g'(t)t, \quad \forall t \geq 0. \tag{9}
\]

It is interesting to note that \( \alpha 2^* = \frac{2\alpha N}{N - \alpha} \) behaves like a critical exponent for equation (8). Indeed, by using a Pohozaev type variational identity they deduced that the equation
\[
-\text{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = |u|^{q-2}u, \quad x \in \mathbb{R}^N, \tag{10}
\]
has no positive solution in \( H^1(\mathbb{R}^N) \) with \( \int_{\mathbb{R}^N} g^2(u)|\nabla u|^2dx < \infty \) under the assumption (g_1) if \( g \geq \alpha 2^* \) and \( \nabla V(x) \cdot x \geq 0 \) in \( \mathbb{R}^N \) (see [10]).

However, assumption (g_1) fails to the typical case when \( g^2(u) = 1 + \frac{u^2}{2(1 + u^2)} \) (i.e., \( l(s) = (1 + s)^\frac{3}{2} \)) which corresponds to problem (5). Since
\[
\lim_{u \to +\infty} g(u) = \lim_{u \to +\infty} \sqrt{1 + \frac{u^2}{2(1 + u^2)}} = \sqrt{\frac{3}{2}},
\]
we find, in this case, that \( \alpha = 1, \beta = \sqrt{\frac{3}{2}} \) and hence the corresponding critical exponent should be \( \alpha 2^* = 2^*. \) As far as we know, it seems that there is little progress on the existence and nonexistence results for problem (5), i.e. the case \( l(s) = (1 + s)^\frac{3}{2} \). By utilizing the change of variables (7) with \( g(t) = \sqrt{1 + \frac{t^2}{2(1 + t^2)}} \), Yang et al. obtained the existence of positive solutions for problem (5) with \( h(u) = |u|^{r-1}u, \ 12 - 4\sqrt{6} < r + 1 < 2^* \) and \( N \geq 3 \), see [31]. Recently, in [11], the first author, Peng and Yan considered the problem (5) with general nonlinearity including the critical exponent. As a direct consequence of their main result, the existence of positive solution for problem (1) was proved by standard variational method if either \( p \in (12 - 4\sqrt{6}, 2^*) \) for \( N \geq 4 \) or \( p \in (4, 6) \) for \( N = 3 \). The method used in [11] fails to deal with the case when \( p \in (2, 12 - 4\sqrt{6}] \) for \( N \geq 4 \).

It is interesting to discuss the existence and nonexistence of positive solutions for problem (1) when \( p \in (2, 12 - 4\sqrt{6}] \) for \( N \geq 4 \) or \( p \in (2, 4] \) for \( N = 3 \).

The main purpose of the present paper is to study the existence of positive ground state solutions for problem (1) by an abstract result developed by Jeanjean in [14] for all \( p \in (2, 2^*) \).

We observe that formally (1) is the Euler-Lagrange equation associated with the nature energy functional
\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left(1 + \frac{u^2}{2(1 + u^2)}\right)|\nabla u|^2dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)u^2dx - \frac{1}{p} \int_{\mathbb{R}^N} |u|^pdx \tag{11}
- \frac{1}{2^*} \int_{\mathbb{R}^N} |u|^{2^*}dx.
\]

It should be pointed out that we can not apply variational methods directly because the lack of an appropriate function space. We can see that the above functional is well defined in \( H^1(\mathbb{R}^N) \) for \( N \geq 3 \), but it doesn’t posses both smooth and compactness properties in this Sobolev space. Making use of the change of variables (7) with
\[
g(t) = \sqrt{1 + \frac{t^2}{2(1 + t^2)}},
\]
Therefore, in order to find nontrivial solutions of (1), it suffices to study the following

\[
I(u) = J(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)|G^{-1}(v)|^2) \, dx - \frac{1}{p} \int_{\mathbb{R}^N} |G^{-1}(v)|^p \, dx
- \frac{1}{2^*} \int_{\mathbb{R}^N} |G^{-1}(v)|^{2^*} \, dx.
\]  

(12)

Since \( g(t) \) is a nondecreasing bounded function, we can deduce that \( \frac{1}{g(\infty)}|s| = \sqrt{\frac{2}{3}}|s| \leq |G^{-1}(s)| \leq \frac{1}{g(0)}|s| = |s| \). From this, it is clear that \( J \) is well defined in \( H^1(\mathbb{R}^N) \) and \( J \in C^1 \) if \( V(x) \) satisfies the following assumption \((V_1)\).

If \( u \) is a nontrivial solution of (1), then it should satisfy

\[
\int_{\mathbb{R}^N} \left[ (1 + \frac{u^2}{2(1 + u^2)}) \nabla u \nabla \phi + \frac{1}{2} |\nabla u|^2 \frac{u}{(1 + u^2)^2} + V(x)u \phi \right.
- |u|^{p-2} u \phi - |u|^{2^*-2} u \phi \left.] \, dx = 0
\]

for all \( \phi \in C_0^\infty(\mathbb{R}^N) \). Let \( \varphi = \frac{1}{g(u)} \psi \), then it can be checked that (13) is equivalent to

\[
\langle J'(v), \psi \rangle = \int_{\mathbb{R}^N} \left[ \nabla v \nabla \psi + V(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} \psi \right.
- \frac{|G^{-1}(v)|^{p-2} G^{-1}(v)}{g(G^{-1}(v))} \psi
\]

\[
\left. - \frac{|G^{-1}(v)|^{2^*-2} G^{-1}(v)}{g(G^{-1}(v))} \psi \right] \, dx = 0, \quad \forall \psi \in C_0^\infty(\mathbb{R}^N).
\]

(14)

Therefore, in order to find nontrivial solutions of (1), it suffices to study the following equation

\[-\Delta v + V(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} - \frac{|G^{-1}(v)|^{p-2} G^{-1}(v)}{g(G^{-1}(v))} - \frac{|G^{-1}(v)|^{2^*-2} G^{-1}(v)}{g(G^{-1}(v))} = 0. \quad (15)\]

We assume that the potential \( V \in C^1(\mathbb{R}^N, \mathbb{R}) \) satisfies the following conditions:

\((V_1)\) \quad 0 < V_0 \leq V(x) \leq V_\infty := \lim_{|x| \to \infty} V(x) \text{ for all } x \in \mathbb{R}^N;

\((V_2)\) \quad there exists a positive constant \( C_0 < \frac{(N-2)^2}{2} \) such that

\[ |(\nabla V(x) \cdot x) | < \frac{C_0}{|x|^2}, \quad \forall x \in \mathbb{R}^N \backslash \{0\}.\]

Our main result can be stated as follows.

**Theorem 1.1.** Assume \((V_1) - (V_2)\) hold. Then problem (1) which equivalent to (15),
exists a positive ground state solution if either \( p \in (2, 2^*) \) for \( N \geq 4 \) or \( p \in (4, 6) \)
for \( N = 3 \).

**Remark 1.**

(i) It was shown in [11] by using a Pohozaev type identity that (5) has no positive solution in \( H^1(\mathbb{R}^N) \) if \( h(u) = |u|^{q-2} u, \quad q \geq q^* \) and \( V \) satisfies \( \nabla V(x) \cdot x \geq 0 \) in \( \mathbb{R}^N \), where \( q^* \in [2^*, \frac{4N}{N-2}] \). It is an interesting issue whether \( q^* \) can be exactly \( 2^* \) or not.

(ii) There are indeed functions which satisfy \((V_1) - (V_2)\). An example is given by
\( V(x) = V_\infty - \frac{C_0}{4(1+|x|^2)} \), where \( 0 < C_0 < \min\{ \frac{(N-2)^2}{2}, 4V_\infty \} \) is a constant.
Remark 2. Since we are going to discuss the existence of positive solutions of problem (15), we rewrite the corresponding variational functional $J(v)$ in the following form:

$$J(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)|G^{-1}(v)|^2)dx - \frac{1}{p} \int_{\mathbb{R}^N} (G^{-1}(v))^pdx - \frac{1}{2^*} \int_{\mathbb{R}^N} (G^{-1}(v^+))^2^* dx,$$

where $v^+ = \max\{v, 0\}$. We claim that all nontrivial critical points of $J$ are the positive solutions of (15). In fact, let $v \in H^1(\mathbb{R}^N)$ be a nontrivial critical point of $J$, then $v$ must be a nontrivial solution of

$$-\Delta v + V(x)v = V(x)\left(v - \frac{G^{-1}(v)}{g(G^{-1}(v))}\right) + \frac{(G^{-1}(v^+))^{p-1}}{g(G^{-1}(v))} + \frac{(G^{-1}(v^+))^{2^*-1}}{g(G^{-1}(v))}.$$  

(16)

Standard regularity argument show that $v \in C^2(\mathbb{R}^N)$. Moreover, it is easy to check that the right hand side of (16) is nonnegative. Therefore, we know from the strong maximum principle that $v$ is positive.

In order to prove Theorem 1.1, we have to solve two difficulties. Firstly, to deal with the difficulty caused by the lack of compactness due to the nonlinearity with the critical growth, we should estimate precisely the mountain pass value. Secondly, as we have seen in [11] and [31], the effect of change of variables (7) with $g(t) = \sqrt{1 + \frac{t^2}{2^*1+t}}$, which prevent us from using the standard way to prove the boundedness of $(PS)$ sequence for $p \in (2, 12 - 4\sqrt{6})$. To overcome these difficulties, we use an abstract result developed by Jeanjean in [14], where the author studied the problem of the form

$$-\Delta u + Ku = f(x, u), \quad x \in \mathbb{R}^N$$  

(17)

when $f(x, t)$ is asymptotically linear in $t$ and periodic in $x_i$, $1 \leq i \leq N$. That is, we introduce a family of $C^1$-functionals defined as

$$J_\lambda(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)|G^{-1}(v)|^2)dx - \lambda \int_{\mathbb{R}^N} F(v)dx, \quad \forall \lambda \in [1, 2],$$

where $F(s) = \int_0^s f(t)dt = \frac{1}{p}|G^{-1}(s)|^p + \frac{1}{2^*}|G^{-1}(s)|^{2^*}$. By (V1) and Lemma 2.4 below, for a.e. $\lambda \in [1, 2]$, there exists a bounded $(PS)_{c_\lambda}$ sequence $\{v_n\} \subset H^1(\mathbb{R}^N)$ for $J_\lambda$, where $c_\lambda$ is given in Lemma 3.1 below. To prove the convergence of bounded $(PS)_{c_\lambda}$ sequence for $J_\lambda$ and obtain a nontrivial critical point $v_\lambda$ of $J_\lambda$ with $J_\lambda(v_\lambda) = c_\lambda$ for a.e. $\lambda \in [1, 2]$, we need to establish a version of global compactness lemma related to the functional $J_\lambda$ and its limiting functional

$$J_{\lambda, \infty}(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V_{\infty}|G^{-1}(v)|^2)dx - \lambda \int_{\mathbb{R}^N} F(v)dx.$$

Finally, choosing a sequence $\{(\lambda_m, v_{\lambda_m})\} \subset [\frac{1}{2}, 1] \times H^1(\mathbb{R}^N)$ with $\lambda_m \to 1$ and $v_{\lambda_m} \neq 0$ satisfying

$$J'_\lambda(v_{\lambda_m}) = 0 \quad \text{and} \quad J_{\lambda_m}(v_{\lambda_m}) = c_{\lambda_m},$$

we can prove that $\{v_{\lambda_m}\}$ is a bounded $(PS)$ sequence for the original functional $J = J_1$ satisfying $\lim_{m \to \infty} J(v_{\lambda_m}) = c_1$ and $\|v_{\lambda_m}\| \neq 0$, which will yields Theorem 1.1.
First, we collect some properties of the change of variables $G$ are crucial to insure the compactness of bounded (PS) sequence. In Section 3, we employ the change of variables and an abstract result developed by Jeanjean in [14] to prove Theorem 1.1.

2. Some preliminary lemmas. In this section, we give some preliminary lemmas. First, we collect some properties of the change of variables $G^{-1}(s)$.

**Lemma 2.1.** (See [31]) The function $G^{-1}(s)$ enjoys the following properties:

1. $\frac{2}{3} s \leq G^{-1}(s) \leq s$, for all $s \geq 0$.
2. $|G^{-1}(s)| \leq 1$, for all $s \in \mathbb{R}$.
3. $\lim_{s \to 0} \frac{s}{G^{-1}(s)} = 1$.
4. $\lim_{s \to \infty} \frac{G^{-1}(s)}{s} = \sqrt{\frac{2}{3}}$.

By a standard argument (see for example [2]), we can obtain the following Pohozaev type identity.

**Lemma 2.2.** (Pohozaev identity) Assume $V(x)$ satisfies $(V_1)$ and $(V_2)$. Let $v$ be a weak solution of problem (15) which equivalent to (1) and $2 < p < 2^*$, then we have the following Pohozaev identity:

$$
\frac{N-2}{2N} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|G^{-1}(v)|^2 dx + \frac{1}{2N} \int_{\mathbb{R}^N} (\nabla V(x) \cdot x)|G^{-1}(v)|^2 dx
= \frac{1}{p} \int_{\mathbb{R}^N} |G^{-1}(v)|^p dx + \frac{1}{2} \int_{\mathbb{R}^N} |G^{-1}(v)|^2 dx.
$$

Using the Brezis-Lieb lemma in [5], we can prove the following lemma.

**Lemma 2.3.** (See Lemma 2.2 in [33]) Suppose that $f \in C(\mathbb{R}^N \times \mathbb{R})$ and there exists a constant $M < +\infty$ such that

$$
\lim_{s \to 0} \left| \frac{f(x,s)}{s} \right| \leq M \quad \text{and} \quad \lim_{s \to \infty} \frac{f(x,s)}{|s|^{2^*-1}} = 0, \text{ uniformly in } x \in \mathbb{R}^N
$$

hold. Let $\{v_n\} \subset H^1(\mathbb{R}^N)$ be a bounded sequence and $v \in H^1(\mathbb{R}^N)$ with $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N)$. Then

$$
\lim_{n \to \infty} \left[ \int_{\mathbb{R}^N} F(x,v_n) dx - \int_{\mathbb{R}^N} F(x,v) dx - \int_{\mathbb{R}^N} F(x,v,v_n-v) dx \right] = 0,
$$

where $F(x,s) = \int_0^s f(x,t) dt$.

On the existence of bounded (PS) sequences, we shall introduce the following abstract result developed by Jeanjean [14].
Lemma 2.4. Let $X$ be a Banach space equipped $\|\cdot\|$, and let $L \subset \mathbb{R}^+$ be an interval. We consider a family $(I_\lambda)_{\lambda \in L}$ of $C^1$-functionals on $X$ of the form

$$I_\lambda(u) = A(u) - \lambda B(u), \quad \forall \lambda \in L,$$

where $B(u) \geq 0$, $\forall u \in X$, and such that either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\| \to \infty$. We assume that there are two points $(v_1, v_2)$ in $X$, such that setting

$$\Gamma = \{ \gamma \in C([0, 1], X) \mid \gamma(0) = v_1, \gamma(1) = v_2 \},$$

there hold, $\forall \lambda \in L$,

$$c_\lambda := \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} I_\lambda(\gamma(t)) > \max\{ I_\lambda(v_1), I_\lambda(v_2) \}.$$

Then, for almost every $\lambda \in L$, there is a bounded $(PS)_{c_\lambda}$ sequence in $X$. Moreover, the map $\lambda \mapsto c_\lambda$ is continuous from the right.

In the following, we introduce some facts about the following nonlinear scalar field equation in $\mathbb{R}^N$ with critical growth:

$$-\Delta u = k(u), \quad u \in H^1(\mathbb{R}^N), \quad (19)$$

where $k : \mathbb{R} \to \mathbb{R}$ is continuous function. The solution $\omega(x)$ is said to be a least energy solution (or ground state) of $(19)$ if $I(\omega) = m$, where

$$m = \inf\{ I(u) : u \in H^1(\mathbb{R}^N) \setminus \{0\} \text{ is a solution of } (19) \},$$

and $I$ is the natural energy functional corresponding to $(19)$:

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} K(u) dx, \quad u \in H^1(\mathbb{R}^N),$$

where $K(s) = \int_0^s k(t) dt$.

The following lemma can be regarded as a form of generalization of Berestycki and Lions [2] about the subcritical case to the critical case for $N \geq 3$, we can find the details in Zhang and Zou [32].

Lemma 2.5. Assume $q > 2$, $N \geq 4$ or $q > 4$, $N = 3$ and $k$ satisfies

1. $k(s) \in C(\mathbb{R}, \mathbb{R})$ is odd.
2. $\lim_{s \to 0} \frac{k(s)}{s} = -a < 0$.
3. $\lim_{s \to +\infty} \frac{k(s)}{s^{2^* - 1}} = \mu > 0$.
4. There exist $C > 0$ and $q < 2^*$ such that $k(s) - \mu s^{2^* - 1} + as \geq Cs^{q - 1}$ for all $s > 0$.

Then (19) admits a positive radial least energy solution $\omega$ and $c = m$, where $c$ is the mountain pass value of $I$. Moreover, there exists a path $\gamma \in C([0, 1], H^1(\mathbb{R}^N))$ such that $\omega(x) \in \gamma([0, 1])$ and $\max_{t \in [0, 1]} I(\gamma(t)) = I(\omega)$.

From now on, we consider the autonomous problem related to (15)

$$-\Delta v + V_\infty \frac{G^{-1}(v)}{g(G^{-1}(v))} = \frac{|G^{-1}(v)|^{p-2}G^{-1}(v)}{g(G^{-1}(v))} + \frac{|G^{-1}(v)|^{2^* - 2}G^{-1}(v)}{g(G^{-1}(v))}, \quad (20)$$
and the corresponding variational functional is
\[
J^\infty(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V_\infty |G^{-1}(v)|^2) \, dx - \frac{1}{p} \int_{\mathbb{R}^N} |G^{-1}(v)|^p \, dx - \frac{1}{2^*} \int_{\mathbb{R}^N} |G^{-1}(v)|^{2^*} \, dx.
\]
(21)

Define
\[
m^\infty = \inf \{ J^\infty(v) : v \in H^1(\mathbb{R}^N) \setminus \{0\}, (J^\infty)'(v) = 0 \}.
\]

As a consequence of Lemma 2.5, we can obtain the following result.

**Lemma 2.6.** Assume \( p > 2, N \geq 4 \) or \( p > 4, N = 3 \). Then \( m^\infty > 0 \) and there exists a positive least energy solution \( \omega \) of (20). Moreover, there exists \( \gamma \in C([0,1],H^1(\mathbb{R}^N)) \) such that \( \gamma(0) = 0, J^\infty(\gamma(1)) < 0, \omega(x) \in \gamma([0,1]) \) and max \( \{ \gamma(t) \} = J^\infty(\omega) \).

**Proof.** We observe that equation (20) is of the form \(-\Delta v = k(v)\) with
\[
k(s) = \frac{|G^{-1}(s)|^p - 2G^{-1}(s)}{g(G^{-1}(s))} + \frac{|G^{-1}(s)|^{2^*} - 2G^{-1}(s)}{g(G^{-1}(s))} - V_\infty \frac{G^{-1}(s)}{g(G^{-1}(s))}. \]
(22)
The result follows after verifying that the function \( k(s) \) satisfies the assumptions \((k_1) - (k_4)\) presented in Lemma 2.5. Indeed, the fact that \((k_1)\) holds is trivial. By Lemma 2.1 (3), we have that \( k'(0) = \lim_{s \to 0} \frac{k(s)}{s} = -V_\infty < 0 \). It follows from Lemma 2.1 (4) that
\[
\lim_{s \to +\infty} \frac{k(s)}{s^{2^*-1}} = \left( \frac{2}{3} \right)^{\frac{2}{p}} > 0.
\]
Finally, from the fact that \( 1 \leq g(t) \leq \sqrt{\frac{3}{2}} \) and Lemma 2.1 (1), we know that
\[
k(s) \geq \left( \frac{2}{3} \right)^{\frac{2}{p}} s^{p-1} + \left( \frac{2}{3} \right)^{\frac{2}{2^*}} s^{2^*-1} - V_\infty s \quad \text{for all } s \geq 0.
\]
Therefore, we can verify that \( k \) satisfies the assumptions \((k_1) - (k_4)\) if we take \( a = V_\infty \) and \( \mu = \left( \frac{2}{3} \right)^{\frac{2}{2^*}} \). Then the proof follows directly from Lemma 2.5. \( \square \)

3. **Proof of Theorem 1.1.** In this section, we devote to the case when the potential \( V(x) \) is not constant. Set \( L = [1,2] \), we consider a family of \( C^1 \)-functionals on \( H^1(\mathbb{R}^N) \)
\[
J_\lambda(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)|G^{-1}(v)|^2) \, dx - \lambda \int_{\mathbb{R}^N} F(v) \, dx, \quad \forall \lambda \in [1,2],
\]
(23)
where \( F(s) = \int_0^s f(t) \, dt = \frac{1}{p}|G^{-1}(s)|^p + \frac{1}{2^*}|G^{-1}(s)|^{2^*} \). Denote
\[
A(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)|G^{-1}(v)|^2) \, dx, \quad B(v) = \int_{\mathbb{R}^N} F(v) \, dx,
\]
then we have \( J_\lambda(v) = A(v) - \lambda B(v) \). From Lemma 2.1 and \((V_1)\) we find that
\[
A(v) \geq \frac{1}{3} \int_{\mathbb{R}^N} (|\nabla v|^2 + V_0 v^2) \, dx \to +\infty \quad \text{as } \|v\| \to +\infty, \quad B(v) \geq 0.
\]

Now, we can verify that the functional \( J_\lambda \) has a Mountain Pass geometry.

**Lemma 3.1.** Assume that \((V_1)\) holds. Then for all fixed \( \lambda \in [1,2] \), the functional \( J_\lambda \) satisfies

(i) there exists a \( w \in H^1(\mathbb{R}^N) \setminus \{0\} \) such that \( J_\lambda(w) \leq 0; \)
(ii) \( c_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} J_\lambda(\gamma(t)) > \max\{J_\lambda(0), J_\lambda(w)\} \), where 
\[ \Gamma = \{ \gamma \in C([0, 1], H^1(\mathbb{R}^N)) \mid \gamma(0) = 0, \gamma(1) = w \}. \]

**Proof.** For fixed \( u \in H^1(\mathbb{R}^N) \setminus \{0\} \) and any \( \lambda \in [1, 2] \), it follows from Lemma 2.1 (1) and (V1) that
\[ J_\lambda(tu) \leq J_\lambda(\epsilon u) \leq \frac{t^2}{2} \int_{\mathbb{R}^N} \left( |\nabla u|^2 + V_\infty u^2 \right) dx - \frac{\epsilon^2}{2^*} \left( \frac{2}{3} \right) \int_{\mathbb{R}^N} |u|^{2^*} dx \to -\infty \]
as \( t \to +\infty \), which gives the result (i) if we take \( w = tu \) with \( t \) large.

On the other hand, from Lemma 2.1, it is easy to check that for any \( \epsilon > 0 \), there exists \( C_\epsilon > 0 \) such that \( F(v) \leq \epsilon v^2 + C_\epsilon |v|^{2^*} \). Then choosing \( \epsilon \) small enough, by Sobolev inequality, we have
\[ J_\lambda(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)|G^{-1}(v)|^2) dx - \lambda \int_{\mathbb{R}^N} F(v) dx \]
\[ \geq \frac{1}{3} \int_{\mathbb{R}^N} (|\nabla v|^2 + V_0 v^2) dx - \epsilon \lambda \int_{\mathbb{R}^N} v^2 dx - C\epsilon \lambda \int_{\mathbb{R}^N} |v|^{2^*} dx \]
\[ \geq C\|v\|^2 - C\|v\|^{2^*}. \]

Then we see that \( J_\lambda \) has a strict local minimum at 0 and hence \( c_\lambda > 0 \). \( \square \)

In the following, we will give an appropriate estimate on the mountain pass value \( c_\lambda \).

**Lemma 3.2.** Assume that (V1) holds and let \( \lambda \in [1, 2] \) be arbitrary but fixed. Then, the level \( c_\lambda < \frac{\lambda^{1-\frac{N}{p}}}{N} (\frac{3}{2} S)^{\frac{N}{p}} \) if either \( p \in (2, 2^*) \) for \( N \geq 4 \) or \( p \in (4, 6) \) for \( N = 3 \). Here, \( S \) is the best constant of embedding \( D^{1,2}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N) \), i.e., \( S = \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{(\int_{\mathbb{R}^N} |u|^{2^*} dx)^{\frac{2}{2^*}}} \).

**Proof.** It suffices to show that there exists \( v \in H^1(\mathbb{R}^N) \setminus \{0\} \) such that
\[ \sup_{t \geq 0} J_\lambda(tv) < \frac{\lambda^{1-\frac{N}{p}}}{N} (\frac{3}{2} S)^{\frac{N}{p}}. \]

One of the possible candidates for \( v \) is \( v_\epsilon = \phi w_\epsilon \), where \( \phi \) is a smooth cut-off function such that \( \phi(x) = 1 \) if \( |x| \leq 1 \), \( \phi(x) = 0 \) if \( |x| \geq 2 \) and \( |\nabla \phi| \leq 2 \), and
\[ w_\epsilon(x) = \frac{(N(N-2)\varepsilon)^{\frac{N-2}{2}}}{(\varepsilon + |x|^2)^{\frac{N-2}{2}}} \]
\[ \varepsilon > 0. \]

It is well-known that \( w_\epsilon \) satisfies the equation \(-\Delta u = |u|^{2^*-2} u \) in \( \mathbb{R}^N \), i.e., \( w_\epsilon \) is the minimizer of \( S = \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{(\int_{\mathbb{R}^N} |u|^{2^*} dx)^{\frac{2}{2^*}}} \), and
\[ \|\nabla w_\epsilon\|_{L^2}^2 = \|w_\epsilon\|_{L^{2^*}}^2 = S^{\frac{2^*}{2}}. \]

Then by a direct computation we get the following estimations:
\[ \int_{\mathbb{R}^N} |\nabla v_\epsilon|^2 dx = S^{\frac{2^*}{2}} + O(\varepsilon^{\frac{N-2}{2}}), \quad \int_{\mathbb{R}^N} |v_\epsilon|^2 dx = S^{\frac{2^*}{2}} + O(\varepsilon^{\frac{N}{2}}), \]
which implies the claim.

Since \( \lim_{t \to +\infty} J_\lambda(tv_\varepsilon) = -\infty \), there exists \( t_\varepsilon > 0 \) such that \( J_\lambda(t_\varepsilon v_\varepsilon) = \sup_{t \geq 0} J_\lambda(tv_\varepsilon) \).

We claim that there exist positive constants \( t_0, t_1 > 0 \) such that \( t_0 \leq t_\varepsilon \leq t_1 \).

First we prove that \( t_\varepsilon \) is bounded from below by a positive constant. Otherwise, we could find a sequence \( \varepsilon_n \to 0 \) such that \( t_{\varepsilon_n} \to 0 \). By the above estimations, up to a subsequence, we have \( t_{\varepsilon_n} v_{\varepsilon_n} \to 0 \) in \( H^1(\mathbb{R}^N) \). Therefore, \( 0 < \sup_{t \geq 0} J_\lambda(t_{\varepsilon_n} v_{\varepsilon_n}) \to J_\lambda(0) = 0 \), which is a contradiction.

On the other hand, we have

\[
0 \leq J_\lambda(t_\varepsilon v_\varepsilon) \leq \frac{1}{2} t_\varepsilon^2 \int_{\mathbb{R}^N} (|\nabla v_\varepsilon|^2 + V_\infty v_\varepsilon^2) dx - \frac{\lambda}{2} t_\varepsilon^2 \int_{\mathbb{R}^N} |v_\varepsilon|^2 dx \leq C t_\varepsilon^2 - C t_\varepsilon^2 ,
\]

which implies the claim.

Therefore

\[
sup_{t \geq 0} J_\lambda(tv_\varepsilon)
= \frac{1}{2} t_\varepsilon^2 \int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|G^{-1}(t_\varepsilon v_\varepsilon)|^2 dx - \frac{\lambda}{p} \int_{\mathbb{R}^N} |G^{-1}(t_\varepsilon v_\varepsilon)|^p dx
- \frac{\lambda}{2} \int_{\mathbb{R}^N} |G^{-1}(t_\varepsilon v_\varepsilon)|^2 dx
\leq \frac{1}{2} t_\varepsilon^2 \int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 dx - \frac{\lambda}{2} \left( \sqrt{\frac{2}{3}} t_\varepsilon \right)^2 \int_{\mathbb{R}^N} |v_\varepsilon|^2 dx + \frac{1}{2} t_\varepsilon^2 \int_{\mathbb{R}^N} V_\infty v_\varepsilon^2 dx
- \frac{\lambda}{p} \left( \sqrt{\frac{2}{3}} t_\varepsilon \right)^p \int_{\mathbb{R}^N} |v_\varepsilon|^p dx
\leq \frac{1}{2} t_\varepsilon^2 \xi^N - \frac{\lambda}{2} \left( \sqrt{\frac{2}{3}} t_\varepsilon \right)^2 S^{\frac{N}{2}} + O(\varepsilon^{\frac{N-p}{2}}) + C t_\varepsilon \int_{\mathbb{R}^N} v_\varepsilon^2 dx - C t_\varepsilon^p \int_{\mathbb{R}^N} |v_\varepsilon|^p dx.
\]

Consider the function \( \xi : [0, \infty) \to \mathbb{R} \) given by

\[
\xi(t) = \frac{1}{2} t^2 - \frac{\lambda}{2} \left( \frac{2}{3} \right)^{\frac{N}{2}} t^2 ,
\]

we have \( t_0 = \lambda^{-\frac{1}{2}} \left( \frac{3}{2} \right)^{-\frac{N}{2}} \) is the maximum point of \( \xi \) and \( \xi(t_0) = \frac{3}{2} \lambda^{-\frac{1}{2}} \left( \frac{3}{2} \right)^{-\frac{N}{2}} = \frac{\lambda^{-\frac{N}{2}}}{N} \left( \frac{3}{2} \right)^{\frac{N}{2}} \). Thus we deduce that

\[
\sup_{t \geq 0} J_\lambda(tv_\varepsilon) \leq \frac{\lambda^{-\frac{N}{2}}}{N} \left( \frac{3}{2} \right)^{\frac{N}{2}} + O(\varepsilon^{\frac{N-p}{2}}) + C \int_{\mathbb{R}^N} v_\varepsilon^2 dx - C \int_{B_1(0)} w_\varepsilon^p dx
\leq \Delta \frac{\lambda^{-\frac{N}{2}}}{N} \left( \frac{3}{2} \right)^{\frac{N}{2}} + I .
\]

Noting that
\[
\int_{B_1(0)} w_\varepsilon^p dx = C \varepsilon^{\frac{N-2p}{2}} \int_0^{1/\varepsilon} \frac{s^{N-1}}{(1 + s^2)^{N-2p}} ds,
\]
and for \( \varepsilon > 0 \) sufficiently small, we have
\[
\int_0^1 s^{N-1} \frac{1}{(1 + s^2)^{N-2p}} ds \geq \int_0^1 \frac{s^{N-1}}{(1 + s^2)^{N-2p}} ds \geq \frac{2^{N-2p}}{N}.
\]
Consequently,
\[
I \leq C \varepsilon^{\frac{N-2}{2}} - C \varepsilon^{\frac{N-2}{2}} p + C \left\{ \begin{array}{ll}
\varepsilon + O(\varepsilon^{N-2}) & \text{if } N \geq 5, \\
\varepsilon |\ln \varepsilon| + O(\varepsilon) & \text{if } N = 4, \\
O(\varepsilon^{1/2}) & \text{if } N = 3.
\end{array} \right.
\]
From (24) and (25), we only need to prove that \( I < 0 \) for small \( \varepsilon \). By a simple computation, we know that this is true if either \( p \in (2, 2^*) \) for \( N \geq 4 \) or \( p \in (4, 6) \) for \( N = 3 \).

**Remark 3.** We consider the limiting functional \( J_{\lambda, \infty} \) associated to the functional \( J_\lambda \). For \( \lambda \in [1, 2] \), define
\[
J_{\lambda, \infty}(v) = \frac{1}{2} \int_{\mathbb{R}^N} \left( |\nabla v|^2 + V_\infty |G^{-1}(v)|^2 \right) dx - \lambda \int_{\mathbb{R}^N} F(v) dx
\]
and
\[
c_{\lambda, \infty} = \inf_{\gamma \in \Gamma^\infty} \max_{t \in [0, 1]} J_{\lambda, \infty}(\gamma(t)),
\]
where \( \Gamma^\infty = \{ \gamma \in C([0, 1], H^1(\mathbb{R}^N)) \mid \gamma(0) = 0, J_{\lambda, \infty}(\gamma(1)) < 0 \} \). Then it is not difficult to check that the conclusions of Lemmas 3.1 and 3.2 hold for \( J_{\lambda, \infty} \).

For \( \lambda \in [1, 2] \), we denote
\[
f^*_\lambda(x, s) = V(x) \left[ s - \frac{G^{-1}(s)}{g(G^{-1}(s))} \right] + \lambda \left[ \frac{|G^{-1}(s)|^{p-2}G^{-1}(s)}{g(G^{-1}(s))} + \frac{|G^{-1}(s)|^{2-2}G^{-1}(s)}{g(G^{-1}(s))} - \left( \frac{2}{3} \right)^2 s |s|^{2^*-2} \right],
\]
\[
F^*_\lambda(x, s) = \int_0^s f^*_\lambda(x, t) dt = \frac{1}{2} V(x) \left[ s^2 - |G^{-1}(s)|^2 \right] + \frac{\lambda}{p} |G^{-1}(s)|^p + \frac{\lambda}{2^*} |G^{-1}(s)|^{2^*}
\]
\[
- \frac{\lambda}{2^*} \left( \frac{2}{3} \right)^{2^*} |s|^{2^*},
\]
\[
\bar{f}_\lambda(s) = V_\infty \left[ s - \frac{G^{-1}(s)}{g(G^{-1}(s))} \right] + \lambda \left[ \frac{|G^{-1}(s)|^{p-2}G^{-1}(s)}{g(G^{-1}(s))} + \frac{|G^{-1}(s)|^{2-2}G^{-1}(s)}{g(G^{-1}(s))} - \left( \frac{2}{3} \right)^2 s |s|^{2^*-2} \right]
\]
and
\[
\bar{F}_\lambda(s) = \int_0^s \bar{f}_\lambda(t) dt = \frac{1}{2} V_\infty \left[ s^2 - |G^{-1}(s)|^2 \right] + \frac{\lambda}{p} |G^{-1}(s)|^p + \frac{\lambda}{2^*} |G^{-1}(s)|^{2^*}
\]
\[
- \frac{\lambda}{2^*} \left( \frac{2}{3} \right)^{2^*} |s|^{2^*}.
\]
Then the functional $J_\lambda$ and $J_{\lambda, \infty}$ can be rewritten by

$$J_\lambda(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)v^2) dx - \int_{\mathbb{R}^N} F_\lambda^*(x, v) dx - \frac{\lambda}{2} \left( \frac{2}{3} \right)^{\frac{2}{3}} \int_{\mathbb{R}^N} |v|^2 dx$$

and

$$J_{\lambda, \infty}(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V_\infty v^2) dx - \int_{\mathbb{R}^N} \tilde{F}_\lambda^*(v) dx - \frac{\lambda}{2} \left( \frac{2}{3} \right)^{\frac{2}{3}} \int_{\mathbb{R}^N} |v|^2 dx,$$

respectively. Moreover, by Lemma 2.1, it is not difficult to check that

$$\lim_{s \to 0^+} \frac{F_\lambda^*(x, s)}{s^2} = 0, \quad \lim_{s \to +\infty} \frac{F_\lambda^*(x, s)}{s^2} = 0, \quad \lim_{s \to 0^+} \frac{f_\lambda^*(x, s)}{s} = 0, \quad \lim_{s \to +\infty} \frac{f_\lambda^*(x, s)}{s^{2/3} - 1} = 0$$

uniformly in $x \in \mathbb{R}^N$ and $\lambda \in [1, 2]$. Motivated by the ideas in [15] or [12, 30], we can establish a version of global compactness lemma related to the functional $J_\lambda$ and its limiting functional $J_{\lambda, \infty}$.

**Lemma 3.3.** Assume that $(V_1) - (V_2)$ holds. Let $\lambda \in [1, 2]$ be fixed and $\{v_n\} \subset H^1(\mathbb{R}^N)$ be a bounded (PS)$_c$ sequence of $J_\lambda$ with $c_\lambda \in (0, \frac{\lambda^2}{N} - (\frac{N}{3}S)^2)$. Then exist a subsequence of $\{v_n\}$, still denoted by $\{v_n\}$, an integer $l \in \mathbb{N} \cup \{0\}$, sequence $\{y_n^k\} \subset \mathbb{R}^N$ and $w^k \in H^1(\mathbb{R}^N)$ for $1 \leq k \leq l$ such that

(i) $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N)$ with $J_\lambda'(v) = 0$,

(ii) $|y_n^k| \to +\infty$ and $|y_n^k - y_n^{k'}| \to +\infty$ for $k \neq k'$,

(iii) $w^k \neq 0$ and $J_{\lambda, \infty}'(w^k) = 0$ for $1 \leq k \leq l$,

(iv) $\|v_n - v - \sum_{k=1}^l w^k (\cdot - y_n^k)\| \to 0$,

(v) $J_{\lambda}(v_n) \to J_{\lambda}(v) + \sum_{k=1}^l J_{\lambda, \infty}(w^k)$,

where we agree that in the case $l = 0$, the above holds without $w^k$ and $\{y_n^k\}$.

**Proof.** Since $\{v_n\} \subset H^1(\mathbb{R}^N)$ is bounded, we may assume, up to a subsequence, $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N)$. Then $J_\lambda'(v_n) \to 0$ implies that for any $\psi \in C^\infty_0(\mathbb{R}^N)$

$$\langle J_\lambda'(v_n), \psi \rangle = \int_{\mathbb{R}^N} \nabla v_n \nabla \psi dx + \int_{\mathbb{R}^N} V(x) G^{-1}(v_n) \psi dx - \lambda \int_{\mathbb{R}^N} f(v_n) \psi dx$$

$$= o_n(1) \|\psi\|$$

as $n \to \infty$. By using Lebesgue Dominated Theorem, we can obtain that $J_\lambda'(v) = 0$. Thus, (i) holds.

By Lemma 2.2, the Pohozaev identity gives that

$$J_\lambda(v) = \frac{1}{N} \int_{\mathbb{R}^N} |\nabla v|^2 dx - \frac{1}{2N} \int_{\mathbb{R}^N} (\nabla V(x) \cdot x) |G^{-1}(v)|^2 dx.$$

From (V2) and Hardy inequality (see [13]):

$$\int_{\mathbb{R}^N} |\nabla v|^2 dx \geq \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{v^2}{|x|^2} dx, \quad \forall v \in H^1(\mathbb{R}^N),$$

where we agree that in the case $l = 0$, the above holds without $w^k$ and $\{y_n^k\}$. Therefore,
we conclude that
\[
\frac{1}{N} \int_{\mathbb{R}^N} |\nabla v|^2 \, dx \geq \frac{(N-2)^2}{4N} \int_{\mathbb{R}^N} v^2 \, dx \geq \frac{(N-2)^2}{4NC_0} \int_{\mathbb{R}^N} (\nabla V(x) \cdot x)v^2 \, dx
\]
\[
> \frac{1}{2N} \int_{\mathbb{R}^N} (\nabla V(x) \cdot x)v^2 \, dx.
\]
Therefore, from Lemma 2.1 (1), we obtain that
\[
J_\lambda(v) \geq C \int_{\mathbb{R}^N} |\nabla v|^2 \, dx \geq 0. \tag{33}
\]

**Step 1.** Set \( u^1_n = v_n - v \), then, if \( n \to \infty \)

(a.1) \( \|v^1_n\|^2 = \|v_n\|^2 - \|v\|^2 + o_n(1) \),

(b.1) \( \|v^1_n\|_{2^*}^2 = \|v_n\|_{2^*}^2 - \|v\|_{2^*}^2 + o_n(1) \),

(c.1) \( J_\lambda(v_n) - J_\lambda(v) = J_{\lambda,\infty}(v^1_n) + o_n(1) \),

(d.1) \( J_{\lambda,\infty}(v^1_n) \to 0 \) in \( H^{-1}(\mathbb{R}^N) \).

The proof of (a.1) and (b.1) are standard, which follow from the Brezis-Lieb lemma. By (30) and Lemma 2.3, we can check that
\[
\int_{\mathbb{R}^N} F_\lambda^*(x, v^1_n) \, dx = \int_{\mathbb{R}^N} F_\lambda^*(x, v_n) \, dx - \int_{\mathbb{R}^N} F_\lambda^*(x, v) \, dx + o_n(1). \tag{34}
\]

Combining (a.1), (b.1), (34) and the fact that \( v^1_n \to 0 \) in \( H^1(\mathbb{R}^N) \), we deduce that
\[
J_\lambda(v_n) - J_\lambda(v) = J_{\lambda,\infty}(v^1_n) + o_n(1) = J_{\lambda,\infty}(v^1_n) + o_n(1),
\]
which gives the item (c.1). Finally, we prove item (d.1). By elliptic estimate, we have \( v \in L^\infty(\mathbb{R}^N) \). Then from Lemma 8.9 in [30], one has \( \forall \psi \in C_0^\infty(\mathbb{R}^N) \)
\[
\left| \int_{\mathbb{R}^N} (|v_n|^{2^*-2}v_n - |v|^{2^*-2}v - |v_n - v|^{2^*-2}(v_n - v)) \psi \, dx \right| = o_n(1)\|\psi\|. \tag{35}
\]

By the similar argument of Lemma 8.1 in [30], we also have \( \forall \psi \in C_0^\infty(\mathbb{R}^N) \)
\[
\left| \int_{\mathbb{R}^N} (f_\lambda^*(x, v_n) - f_\lambda^*(x, v) - f_\lambda^*(x, v^1_n)) \psi \, dx \right| = o_n(1)\|\psi\|. \tag{36}
\]

On the other hand, a direct computation shows that
\[
\langle J_\lambda'(v_n) - J_\lambda'(v), \psi \rangle
\]
\[
= \int_{\mathbb{R}^N} ((\nabla v_n - \nabla v) \nabla \psi + V(x)(v_n - v) \psi) \, dx - \int_{\mathbb{R}^N} (f_\lambda^*(x, v_n) - f_\lambda^*(x, v)) \psi \, dx
\]
\[
- \lambda(\frac{2}{3})^2 \frac{2^*}{2} \int_{\mathbb{R}^N} (|v_n|^{2^*-2}v_n - |v|^{2^*-2}v) \psi \, dx, \quad \forall \psi \in C_0^\infty(\mathbb{R}^N).
\tag{37}
\]

Combining (35)-(37) and the fact that \( v^1_n \to 0 \) in \( H^1(\mathbb{R}^N) \), there holds
\[
\langle J_{\lambda,\infty}'(v^1_n), \psi \rangle = \langle J_{\lambda,\infty}'(v^1_n), \psi \rangle + o_n(1) = \langle J_{\lambda}'(v_n) - J_{\lambda}'(v), \psi \rangle + o_n(1) = o_n(1).
\]
Hence, \( \{v^1_n\} \) is a \((PS)\) sequence of \( J_{\lambda,\infty} \).

Let
\[
\sigma^1 = \lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |v^1_n|^2 \, dx.
\]
Vanishing: If \( \sigma^1 = 0 \), then by Lion’s compactness lemma [19],
\[
v^1_n \to 0 \quad \text{in } L^q(\mathbb{R}^N) \quad \text{for } 2 < q < 2^*.
\tag{38}
Combining (c.1), (d.1) and (38), we deduce that
\[ J_\lambda(v_n) - J_\lambda(v) = \frac{1}{2}\|v_n\|^2 - \frac{\lambda}{2\gamma} \int_{\mathbb{R}^N} |v_n'|^{2\gamma} dx + o_n(1) \]
and
\[ \|v_n'\|^2 = \lambda \left( \frac{2}{3} \right) \frac{\gamma^2}{\gamma - 1} \int_{\mathbb{R}^N} |v_n'|^{2\gamma} dx + o_n(1). \]

Without loss of generality, we assume that \( \int_{\mathbb{R}^N} |v_n'|^{2\gamma} dx = d + o_n(1) \), then \( \|v_n'\|^2 = \lambda \left( \frac{2}{3} \right) \frac{\gamma^2}{\gamma - 1} d + o_n(1) \). By the Sobolev inequality, we have
\[ o_n(1) + S d \frac{\gamma}{\gamma - 1} = S \left( \int_{\mathbb{R}^N} |v_n'|^{2\gamma} dx \right)^{\frac{\gamma}{\gamma - 1}} \leq \|v_n'\|^2 = \lambda \left( \frac{2}{3} \right) \frac{\gamma^2}{\gamma - 1} d + o_n(1). \]

If \( d > 0 \), we can get that
\[ d \geq \left( \frac{1}{\lambda} S \right) \frac{\gamma}{\gamma - 1} \left( \frac{3}{2} \right)^{\frac{N}{4}} n^{N/4}. \]

From (33), we have
\[ c_\lambda \geq J_\lambda(v_n) - J_\lambda(v) + o_n(1) = \left( \frac{1}{2} - \frac{1}{2\gamma} \right) \lambda \left( \frac{2}{3} \right) \frac{\gamma^2}{\gamma - 1} d \geq \frac{\lambda^{1-\frac{\gamma}{N}}}{N} \left( \frac{3}{2} S \right)^{\frac{\gamma}{\gamma - 1}}, \]
a contradiction. Hence, \( d = 0 \), then \( v_n \to v \) in \( H^1(\mathbb{R}^N) \) and Lemma 3.3 holds with \( l = 0 \).

Non-vanishing: If \( \sigma^1 > 0 \), then there exists a sequence \( \{y_n^1\} \subset \mathbb{R}^N \) such that
\[ \int_{B_1(y_n^1)} |v_n'|^2 \, dx \geq \frac{\sigma^1}{2} > 0. \]

Set \( w_n^1 = v_n^1 (\cdot + y_n^1) \). Then \( \{w_n^1\} \) is bounded in \( H^1(\mathbb{R}^N) \) and we may assume that \( w_n^1 \to w^1 \) in \( H^1(\mathbb{R}^N) \). Since
\[ \int_{B_1(0)} |w_n^1|^2 \, dx \geq \frac{\sigma^1}{2}, \]
we see that \( w^1 \neq 0 \). Moreover, \( v_n^1 \to 0 \) in \( H^1(\mathbb{R}^N) \) implies that \( \{y_n^1\} \) is unbounded. Hence, we may assume that \( |y_n^1| \to +\infty \). We see that it is not difficult to verify that \( J_{\lambda, \infty}(w^1) = 0 \). Moreover, we claim that
\[ J_{\lambda, \infty}(w^1) > 0. \tag{39} \]

Indeed, combined with the Pohozaev identity, which similar to (18) with \( V(x) = V_\infty \), it follows that
\[ J_{\lambda, \infty}(w^1) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla w^1|^2 + V_\infty |G^{-1}(w^1)|^2) \, dx - \frac{N - 2}{2N} \int_{\mathbb{R}^N} |\nabla w^1|^2 \, dx \]
\[ - \frac{1}{2} \int_{\mathbb{R}^N} V_\infty |G^{-1}(w^1)|^2 \, dx \]
\[ = \frac{1}{N} \int_{\mathbb{R}^N} |\nabla w^1|^2 \, dx > 0. \]

**Step 2.** Set \( v_n^2 = v_n - v - w^1 (\cdot - y_n^1) \). We can similarly check that
(a.2) \( \|v_n^2\|^2 = \|v_n\|^2 - \|v\|^2 - \|w^1\|^2 + o_n(1) \),
(b.2) \( \|v_n^2\|^2 = \|v_n\|^2 - \|v\|^2 - \|w^1\|^2 + o_n(1) \),
(c.2) \( J_\lambda(v_n) - J_\lambda(v) = J_{\lambda, \infty}(w^1) = J_{\lambda, \infty}(v_n^2) + o_n(1) \),
(d.2) \( J_{\lambda, \infty}(v_n^2) \to 0 \) in \( H^{-1}(\mathbb{R}^N) \).
Similar to the argument in Step 1, let
\[ \sigma^2 = \lim_{n \to \infty} \sup_{\gamma \in \mathbb{R}^N} \int_{B_1(y)} |v_n^2|^2 \, dx. \]
If vanishing occurs, by (39) and the similar argument of vanishing case in Step 1, we know that \( |v_n^2| \to 0 \), i.e. \( v_n - v - w^1 (\cdot - y_n^1) \to 0 \) in \( H^1(\mathbb{R}^N) \). Moreover, by (c.2), we see that \( J_\lambda(v_n) + o_n(1) = J_\lambda(v) + J_{\lambda, \infty}(w^1) \) and Lemma 3.3 holds with \( l = 1 \).

If non-vanishing occurs, then there exists a sequence \( \{ y_n^2 \} \subset \mathbb{R}^N \) and a nontrivial \( w^2 \in H^1(\mathbb{R}^N) \) such that \( w^2 = v_n^2 (\cdot + y_n^2) \to w^2 \) in \( H^1(\mathbb{R}^N) \). Then by (d.2), we have that \( J_{\lambda, \infty}'(w^2) = 0 \). Furthermore, \( v_n^2 \to 0 \) in \( H^1(\mathbb{R}^N) \) implies that \( |y_n^2| \to +\infty \) and \( |y_n^2 - y_n^1| \to +\infty \).

Finally, we proceed by iteration. Similar to (39), if \( w^k \) is a nontrivial critical point of \( J_{\lambda, \infty} \), then \( J_{\lambda, \infty}(w^k) > 0 \). So there exists some finite \( l \in \mathbb{N} \) such that only the vanishing case occurs in Step \( l \). Then the lemma is proved.

On the convergence of bounded \((PS)\) sequence \( \{ v_n \} \) for \( \lambda \), we can establish the following lemma.

**Lemma 3.4.** Assume that \((V_1) - (V_2)\) holds. Let \( \lambda \in [1, 2] \) be fixed and \( \{ v_n \} \subset H^1(\mathbb{R}^N) \) be a bounded \((PS)_{c_{\lambda}}\) sequence of \( J_{\lambda} \) with \( c_{\lambda} \in (0, \frac{\lambda^{\frac{2}{N}}}{\lambda - N} (\frac{2}{N})^\frac{2}{N}) \). Then there exist subsequence of \( \{ v_n \} \), still denoted by \( \{ v_n \} \) and \( 0 \neq v_\lambda \in H^1(\mathbb{R}^N) \) such that
\[ v_n \to v_\lambda \quad \text{in} \quad H^1(\mathbb{R}^N). \]

**Proof.** By Lemma 3.3, for \( \lambda \in [1, 2] \), there exists a \( v_\lambda \in H^1(\mathbb{R}^N) \) such that \( v_n \to v_\lambda \) in \( H^1(\mathbb{R}^N) \) and \( J_{\lambda}'(v_\lambda) = 0 \). Moreover,
\[ \|v_n - v_\lambda - \sum_{k=1}^{l} w^k (\cdot - y_n^k)\| \to 0 \quad \text{and} \quad J_{\lambda}(v_n) \to J_{\lambda}(v_\lambda) + \sum_{k=1}^{l} J_{\lambda, \infty}(w^k), \quad l \geq 0, \]
where \( w^k (1 \leq k \leq l) \) are nontrivial critical points of \( J_{\lambda, \infty} \).

Thus to prove that \( v_n \to v_\lambda \) in \( H^1(\mathbb{R}^N) \) it suffices to show that \( l = 0 \). Indeed, suppose by contradiction that \( l > 0 \). Then by (33), we deduce that
\[ c_{\lambda} = \lim_{n \to \infty} J_{\lambda}(v_n) = J_{\lambda}(v_\lambda) + \sum_{k=1}^{l} J_{\lambda, \infty}(w^k) \geq \lambda m_{\lambda, \infty} \geq m_{\lambda, \infty}, \quad (40) \]
where \( m_{\lambda, \infty} = \inf \{ J_{\lambda, \infty}(v) : J_{\lambda, \infty}'(v) = 0, \ v \neq 0 \} \). On the other hand, we consider the autonomous problem
\[ -\Delta v + V_\infty \frac{G^{-1}(v)}{g(G^{-1}(v))} = \lambda \frac{|G^{-1}(v)|^{p-2} G^{-1}(v)}{g(G^{-1}(v))} + \lambda \frac{|G^{-1}(v)|^{2-2} G^{-1}(v)}{g(G^{-1}(v))}. \quad (41) \]
Arguing as in Lemma 2.6, we can obtain a least energy solution \( \omega_\lambda \) of (41) and there exists \( \gamma \in C([0, 1], H^1(\mathbb{R}^N)) \) such that \( \gamma(0) = 0, \ J_{\lambda, \infty}(\gamma(1)) < 0, \ \omega_\lambda(x) \in \gamma([0,1]) \) and
\[ \max_{t \in [0,1]} J_{\lambda, \infty}(\gamma(t)) = J_{\lambda, \infty}(\omega_\lambda). \]
Since \( V(x) \leq V_\infty \) and \( V(x) \not\equiv V_\infty \), it follows from the definition of \( c_{\lambda} \) that
\[ c_{\lambda} \leq \max_{t \in [0,1]} J_{\lambda}(\gamma(t)) < \max_{t \in [0,1]} J_{\lambda, \infty}(\gamma(t)) = J_{\lambda, \infty}(\omega_\lambda) = m_{\lambda, \infty}, \]
which contradicts with (40). Hence, \( l = 0 \), i.e. \( v_n \to v_\lambda \) in \( H^1(\mathbb{R}^N) \) and then \( v_\lambda \) is a nontrivial critical point for \( J_\lambda \) and \( J_\lambda(v_\lambda) = c_\lambda \).

As a result, we complete the proof. \( \square \)

Combining Lemmas 2.4 and 3.1, we deduce that for a.e. \( \lambda \in [1, 2] \), there exists a bounded \((PS)\) sequence \( \{v_n\} \subset H^1(\mathbb{R}^N) \) such that \( J_\lambda(v_n) \to c_\lambda < \frac{\lambda^2}{2N} \left( \frac{\alpha}{2} \right)^{\frac{2}{N}} \).

Then by Lemma 3.4, we deduce that \( J_\lambda \) has a nontrivial critical point \( v_\lambda \in H^1(\mathbb{R}^N) \) with \( J_\lambda(v_\lambda) = c_\lambda \) for a.e. \( \lambda \in [1, 2] \). As a special case we obtain the existence of a sequence \( \{(\lambda_m, v_{\lambda_m})\} \subset [1, 2] \times H^1(\mathbb{R}^N) \) with \( \lambda_m \to 1 \) as \( m \to \infty \) and \( v_{\lambda_m} \neq 0 \) satisfying

\[ J'_{\lambda_m}(v_{\lambda_m}) = 0 \quad \text{and} \quad J_{\lambda_m}(v_{\lambda_m}) = c_{\lambda_m}, \tag{42} \]

where \( c_{\lambda_m} \in (0, \frac{(\lambda_m^2}{2N} (\frac{\alpha}{2})^{\frac{2}{N}}) \). In order to prove Theorem 1.1, we need to show that the critical point sequence \( \{v_{\lambda_m}\} \) obtained in (42) is bounded and that is a \((PS)\) sequence for \( J = J_1 \) satisfying \( \lim_{m \to \infty} J(v_{\lambda_m}) = c_1 \), where \( J \) is given by (12).

Then applying Lemma 3.4 again, we obtain a nontrivial critical point of \( J \) and the proof is completed.

**Proof of Theorem 1.1.** First, we show that the sequence \( \{v_{\lambda_m}\} \subset H^1(\mathbb{R}^N) \) obtained in (42) is bounded. Since \( J_{\lambda_m}(v_{\lambda_m}) = c_{\lambda_m} \leq c_1 \) and from Lemma 2.2, we deduce that

\[ c_1 \geq J_{\lambda_m}(v_{\lambda_m}) = \frac{1}{N} \int_{\mathbb{R}^N} |\nabla v_{\lambda_m}|^2 dx - \frac{1}{2N} \int_{\mathbb{R}^N} (\nabla V(x) \cdot x)(G^{-1}(v_{\lambda_m}))^2 dx. \]

By the similar argument to prove (33), we can obtain \( \int_{\mathbb{R}^N} |\nabla v_{\lambda_m}|^2 dx \leq C \). Next, we only need to show the boundedness of \( \int_{\mathbb{R}^N} v_{\lambda_m}^2 dx \). In fact, recall that \( J'_{\lambda_m}(v_{\lambda_m}) = 0 \), we have

\[ \int_{\mathbb{R}^N} (|\nabla v_{\lambda_m}|^2 dx + V(x) \frac{G^{-1}(v_{\lambda_m})}{g(G^{-1}(v_{\lambda_m}))} v_{\lambda_m}) dx = \lambda_m \int_{\mathbb{R}^N} f(v_{\lambda_m}) v_{\lambda_m} dx. \]

Therefore, by Lemma 2.1, for any \( \varepsilon > 0 \) there exists \( C_\varepsilon > 0 \) such that

\[
\int_{\mathbb{R}^N} v_{\lambda_m}^2 dx \leq C \int_{\mathbb{R}^N} (|\nabla v_{\lambda_m}|^2 dx + V(x) \frac{G^{-1}(v_{\lambda_m})}{g(G^{-1}(v_{\lambda_m}))} v_{\lambda_m}) dx \\
\leq C \int_{\mathbb{R}^N} f(v_{\lambda_m}) v_{\lambda_m} dx \\
\leq C \varepsilon \int_{\mathbb{R}^N} v_{\lambda_m}^2 dx + C_\varepsilon \left( \int_{\mathbb{R}^N} |\nabla v_{\lambda_m}|^2 dx \right)^{\frac{2}{N}},
\]

by choosing \( \varepsilon > 0 \) small enough, we obtain \( \int_{\mathbb{R}^N} v_{\lambda_m}^2 dx \) is bounded. Therefore, \( \{v_{\lambda_m}\} \) is bounded in \( H^1(\mathbb{R}^N) \). Then we have for any \( \psi \in C_0^\infty(\mathbb{R}^N) \)

\[ |\langle J'_{\lambda_m}(v_{\lambda_m}), \psi \rangle - \langle J'(v_{\lambda_m}), \psi \rangle| = |(\lambda_m - 1) \int_{\mathbb{R}^N} f(v_{\lambda_m}) \psi dx| \to 0 \]

as \( m \to \infty \). And notice that

\[
\lim_{m \to \infty} J(v_{\lambda_m}) = \lim_{m \to \infty} \left[ J_{\lambda_m}(v_{\lambda_m}) + (\lambda_m - 1) \int_{\mathbb{R}^N} F(v_{\lambda_m}) dx \right] = \lim_{m \to \infty} c_{\lambda_m} = c_1,
\]

where we used the fact that the map \( \lambda \to c_\lambda \) is continuous from the right. That is \( \{v_{\lambda_m}\} \) is a bounded \((PS)\) sequence for \( J \) satisfying \( \lim_{m \to \infty} J(v_{\lambda_m}) = c_1 < \frac{1}{N} \left( \frac{\alpha}{2} \right)^{\frac{2}{N}} \).
Then applying Lemma 3.4 again, we obtain a nontrivial critical point $v_0 \in H^1(\mathbb{R}^N)$ for $J$ and $J(v_0) = c_1$.

Finally, we end this proof by showing the existence of a ground state solution for problem (15) which equivalent to (1). Let

$$d = \inf \{ J(v) : J'(v) = 0, \ v \neq 0 \}.$$  

Then $0 < d \leq J(v_0) < J^\infty(v_0) < \frac{1}{N} \frac{2}{3} S^{\frac{3}{2}}$. In fact, for any $v$ satisfying $J'(v) = 0$, by standard argument we see \( \|v\| \geq \rho \) for some positive constant $\rho$. Similar to the argument to prove (33), we infer

$$J(v) \geq C \int_{\mathbb{R}^N} |\nabla v|^2 \, dx. \quad (43)$$  

Therefore, $d \geq 0$. In the following we rule out $d = 0$. Suppose by contradiction that $\{v_n\}$ be a critical point sequence of $J$ satisfying $\lim_{n \to \infty} J(v_n) = 0$. From (43), we have

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx = 0.$$  

This conclusion combined with $\langle J'(v_n), v_n \rangle = 0$, we can verify that $\lim_{n \to \infty} \int_{\mathbb{R}^N} v_n^2 \, dx = 0$. Therefore, we obtain $\lim_{n \to \infty} \|v_n\| = 0$, a contradiction with $\|v_n\| \geq \rho > 0$ for all $n \in \mathbb{N}$.

Then let $\{v_n\} \subset H^1(\mathbb{R}^N)$ be a sequence of nontrivial critical point of $J$ satisfying $J(v_n) \to d < \frac{1}{N} \frac{2}{3} S^{\frac{3}{2}}$. By (V2) and Hardy inequality, we can similarly deduce that $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$, i.e., $\{v_n\}$ is a bounded $(PS)_d$ sequence for $J$. Similar to the arguments in Lemma 3.4, there exists a nontrivial $w \in H^1(\mathbb{R}^N)$ such that $J(w) = d$ and $J'(w) = 0$. Moreover, Remark 2 can show that $w > 0$. \[ \square \]

**Acknowledgments.** The first author was supported by the Natural Science Foundation of China (11371160,11629101). The second author was partially supported by the excellent doctoral dissertation cultivation grant from Central China Normal University (2016YBZZ081).

**REFERENCES**

[1] A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Func. Anal.*, 14 (1973), 349–381.

[2] H. Berestycki and P. L. Lions, Nonlinear scalar field equations. I. Existence of a ground state, *Arch. Rational Mech. Anal.*, 82 (1983), 313–345.

[3] J. M. Bezerra do Ó, O. H. Miyagaki and S. H. M. Soares, Soliton solutions for quasilinear Schrödinger equations with critical growth, *J. Differential Equations*, 248 (2010), 722–744.

[4] H. Brandi, C. Manus, G. Mainfray, T. Lehner and G. Bonnaud, Relativistic and ponderomotive self-focusing of a laser beam in a radially inhomogeneous plasma, *Phys. Fluids B*, 5 (1993), 3539–3550.

[5] H. Brezis and E. Lieb, A relation between pointwise convergence of function and convergence of functional, *Proc. Amer. Math. Soc.*, 88 (1983), 486–490.

[6] Y. L. Chen and R. N. Sudan, Necessary and sufficient conditions for self-focusing of short ultraintense laser pulse in underdense plasma, *Phys. Rev. Lett.*, 70 (1993), 2082–2085.

[7] M. Colin and L. Jeanjean, Solutions for a quasilinear Schrödinger equation: A dual approach, *Nonlinear Anal. TMA.*, 56 (2004), 213–226.

[8] A. De Bouard, N. Hayashi and J. Saut, Global existence of small solutions to a relativistic nonlinear Schrödinger equation, *Commun. Math. Phys.*, 189 (1997), 73–105.

[9] Y. Deng, S. Peng and S. Yan, Positive soliton solutions for generalized quasilinear Schrödinger equations with critical growth, *J. Differential Equations*, 258 (2015), 115–147.

[10] Y. Deng, S. Peng and S. Yan, Critical exponents and solitary wave solutions for generalized quasilinear Schrödinger equations, *J. Differential Equations*, 260 (2016), 1228–1262.

[11] Y. Deng, S. Peng and S. Yan, Solitary wave solutions to a quasilinear Schrödinger equation modeling the self-channeling of a high-power ultrashort laser in matter, submitted.
[12] M. F. Furtado, L. A. Maia and E. S. Medeiros, Positive and nodal solutions for a nonlinear Schrödinger equation with indefinite potential, *Adv. Nonlinear Stud.*, 8 (2008), 353–373.

[13] J. P. García Azorero and I. Peral Alonso, Hardy inequalities and some critical elliptic and parabolic problems, *J. Differential Equations*, 144 (1998), 441–476.

[14] L. Jeanjean, On the existence of bounded Palais-Smale sequences and application to a Landesman-Lazer-type problem set on $\mathbb{R}^N$, *Proc. Roy. Soc. Edinburgh Sect. A*, 129 (1999), 787–809.

[15] L. Jeanjean and K. Tanaka, A positive solution for a nonlinear Schrödinger equation on $\mathbb{R}^N$, *Indiana Univ. Math. J.*, 54 (2005), 443–464.

[16] S. Kurihara, Large-amplitude quasi-solitons in superfluid films, *J. Phys. Soc. Japan*, 50 (1981), 3262–3267.

[17] E. Laedke, K. Spatschek and L. Stenflo, Evolution theorem for a class of perturbed envelope soliton solutions, *J. Math. Phys.*, 24 (1983), 2764–2769.

[18] H. F. Lins and E. A. B. Silva, Quasilinear asymptotically periodic elliptic equations with critical growth, *Nonlinear Anal.*, 71 (2009), 2890–2905.

[19] P. L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 1 (1984), 109–145.

[20] J. Liu, Y. Wang and Z. Wang, Soliton solutions for quasilinear Schrödinger equations, II, *J. Differential Equations*, 187 (2003), 473–493.

[21] J. Liu, Y. Wang and Z. Wang, Solutions for quasilinear Schrödinger equations via the Nehari method, *Comm. Partial Differential Equations*, 29 (2004), 879–901.

[22] J. Liu and Z. Wang, Soliton solutions for quasilinear Schrödinger equations, I, *Proc. Amer. Math. Soc.*, 131 (2003), 441–448.

[23] X. Liu, J. Liu and Z. Wang, Quasilinear elliptic equations with critical growth via perturbation method, *J. Differential Equations*, 254 (2013), 102–124.

[24] X. Liu, J. Liu and Z. Wang, Ground states for quasilinear Schrödinger equations with critical growth, *Calc. Var. Partial Differential Equations*, 46 (2013), 641–669.

[25] A. Moameni, Existence of soliton solutions for a quasilinear Schrödinger equation involving critical exponent in $\mathbb{R}^N$, *J. Differential Equations*, 229 (2006), 570–587.

[26] M. Poppenberg, K. Schmitt and Z. Wang, On the existence of soliton solutions to quasilinear Schrödinger equations, *Calc. Var. Partial Differential Equations*, 14 (2002), 329–344.

[27] B. Ritchie, Relativistic self-focusing and channel formation in laser-plasma interactions, *Phys. Rev. E*, 50 (1994), 687–689.

[28] Y. Shen and Y. Wang, Soliton solutions for generalized quasilinear Schrödinger equations, *Nonlinear Anal. TMA.*, 80 (2013), 194–201.

[29] E. A. B. Silva and G. F. Vieira, Quasilinear asymptotically periodic Schrödinger equations with critical growth, *Calc. Var. Partial Differential Equations*, 39 (2010), 1–33.

[30] M. Willem, *Minimax Theorems*, Birkhäuser, Boston, 1996.

[31] J. Yang, Y. Wang and A. A. Abdelgadir, Soliton solutions for quasilinear Schrödinger equations, *J. Math. Phys.*, 54 (2013), 071502, 19pp.

[32] J. Zhang and W. Zou, A Berestycki-Lions theorem revisited, *Commun. Contemp. Math.*, 14 (2012), 1250033, 14 pp.

[33] X. Zhu and D. Cao, The concentration-compactness principle in nonlinear elliptic equations, *Acta Math. Sci.*, 9 (1989), 307–328.

Received October 2016; revised March 2017.

E-mail address: ybdeng@mail.ccnu.edu.cn
E-mail address: wthuang1014@aliyun.com