Making cobordisms symplectic

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Abstract

We establish an existence \( h \)-principle for symplectic cobordisms of dimension \( 2n > 4 \) with concave overtwisted contact boundary.

1 Introduction

We say that \((W, \omega, \xi_-, \xi_+)\) is a symplectic cobordism between contact manifolds \((\partial W_\pm, \xi_\pm)\), if \(W\) is a smooth cobordism between \(\partial_- W\) and \(\partial_+ W\), and \(\omega\) is a symplectic form which admits a Liouville vector field \(Z\) near \(\partial W\), i.e. a vector field which satisfies \(d(\iota(Z)\omega) = \omega\), which is inward transverse to \(\partial_- W\), outward transverse to \(\partial_+ W\) and such that the contact forms \(\lambda_\pm = \iota(Z)\omega|_{\partial W_\pm}\) define the contact structures \(\xi_\pm\). By a Liouville cobordism \((W, \lambda, \xi_-, \xi_+)\) between contact manifolds \((\partial W_\pm, \xi_\pm)\) we mean a symplectic cobordism \((W, \omega)\) between \((\partial W_\pm, \xi_\pm)\) with a fixed primitive \(\lambda\), \(d\lambda = \omega\), and such that the \(\omega\)-dual to \(\lambda\) Liouville vector field \(Z\) is inward transverse to \(\partial_- W\) and outward transverse to \(\partial_+ W\).

An obvious necessary condition for finding on a fixed smooth cobordism \(W\) a symplectic cobordism structure between contact manifolds \((\partial_- W, \xi_-)\) and \((\partial_+ W, \xi_+)\) is existence of an almost symplectic cobordism structure, that is a non-degenerate but not necessarily closed 2-form \(\eta\) on \(W\) which coincides near \(\partial_\pm W\) with \(d\lambda_\pm\). We call such \(\eta\) an almost symplectic cobordism structure. The homotopy class of a formal symplectic structure is determined by a homotopy class of an almost complex structure \(J\) tamed by \(\eta\) for which \(\xi_\pm\) are \(J\)-complex subbundles.

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The following theorem is the main result of the paper:

**Theorem 1.1.** \( W \) be a smooth cobordism of dimension \( 2n > 2 \) between contact manifolds \((\partial_\pm W, \xi_\pm)\), which are both non-empty. Let \( \eta \) be an almost symplectic cobordism between \((\partial_- W, \xi_-)\) and \((\partial_+ W, \xi_+)\). Suppose that

- the contact structure \( \xi_- \) is overtwisted;
- if \( n = 2 \) the contact structure \( \xi_+ \) is overtwisted as well.

Then there exists a Liouville cobordism structure \((W, \lambda, \xi_-, \xi_+)\) such that \( d\lambda \) and \( \eta \) are in the same relative to the boundary homotopy class of almost symplectic structures.

In particular, any smooth cobordism \( W \) which admits an almost complex structure also admits a structure of a symplectic cobordism between two contact manifolds.

The notion of an overtwisted contact manifold was introduced in [4] in \( \dim = 3 \) and extended to the general case in [1]. Without giving precise definitions, which will be not important for the purposes of this paper, we summarize below the main necessary for us results about overtwisted contact structures. Part 1 of the following theorem is proven in [1] for the case \( n > 1 \) and in [4] for \( n = 1 \). Part 2 is proven in [2], see also Theorem 10.2 in [1].

**Theorem 1.2.** 1. Let \( \eta \) be an almost contact structure on a \((2n + 1)\)-dimensional manifold \( M \) which is genuinely contact on a neighborhood \( O p A \) of a closed subset \( A \subset M \). Then there exists an overtwisted contact structure \( \xi \) on \( M \) which coincides with \( \eta \) on a neighborhood of \( A \) and which belongs to the same relative to \( A \) homotopy class of almost contact structures as \( \eta \). Moreover, any two contact structures which coincide on \( O p A \), overtwisted on every connected component of \( M \setminus A \) and homotopic rel. \( A \) as almost contact structures are isotopic.

2. Let \((M, \{\alpha = 0\})\) be an overtwisted contact manifold and \((D_R, \lambda)\) the disc of radius \( R \) in \( \mathbb{R}^2 \) endowed with a Liouville form \( \mu = ydx - ydx \). Then for a sufficiently large \( R \) the product \((M \times D_R, \{\alpha \oplus \mu = 0\})\) is also overtwisted.

We note that any contact manifold of dimension \( > 1 \) can be made overtwisted by a modification in a neighborhood of one of its points, and that any two definitions of overtwistedness for which Theorem 1.2.1 holds are equivalent.

With any closed form \( \omega \) on \( W \) which is equal to \( d\lambda_\pm \) on \( \partial_\pm W \) one can canonically associate a cohomology class \([\omega] \in H^2(W, \partial W)\). Indeed, take any 1-form \( \lambda \) on \( W \) extending \( \lambda_\pm \). Then the cohomology class of \([\omega - d\lambda] \in H^2(W, \partial W)\) is independent of a choice of the extension \( \lambda \).
Corollary 1.3. Under the assumptions of Theorem 1.1 we can find a symplectic form \( \omega \) on \( W \), equal to \( d\lambda \pm \) near \( \partial W \), which realizes any a priori given cohomology class \( a \in H^2(W, \partial W) \).

Indeed, let \( \lambda \) be the Liouville from provided by Theorem 1.1 and let \( \sigma \) be any closed form with compact support representing \( a \in H^2(W, \partial W) \). Then for a sufficiently large constant \( C \) the form \( \omega = Cd\lambda + \sigma \) is symplectic and has the required properties.

If \( n > 2 \) then for any contact structure \( \xi \) on \( \partial_+ W \) there is a Liouville concordance (i.e. a Liouville cobordism diffeomorphic \( \partial_+ W \times [0,1] \)) between \( \xi \) on the positive end and an overtwisted contact structure \( \xi_{ot} \) on the negative one, see [2] and Corollary 2.5 below. Hence, it is sufficient to prove Theorem 1.1 for the case when both contact structures \( \xi_\pm \) are overtwisted.

Given a \( 2n \)-dimensional manifold \( X \) with boundary \( \partial X \), a symplectic form \( \omega \) on a \( X \setminus p, p \in M \), and a contact structure \( \xi \) on \( \partial X \), we say that \((X,\omega,\xi)\) is a symplectic domain with a conical singularity at \( p \) and a contact boundary \((\partial X,\xi)\) if \((\partial X,\xi)\) is a positive contact boundary in the above sense, and near \( p \) the radial vector field centered at \( p \) is Liouville for \( \omega \). In other words, on a ball centered at \( p \) the form \( \omega \) is symplectomorphic to the negative part of the symplectization of a contact structure \( \zeta \) on the boundary sphere \( S^{2n-1} \). We will call \((S^{2n-1},\zeta)\) the link of the singularity \( p \). Equivalently, a symplectic structure \( \omega \) on \( X \) with a conical singularity at \( p \) can be viewed as a symplectic cobordism structure on \( \dot{X} = X \setminus \text{Int } B \), where \( B \) is a ball centered at \( p \), between a contact structure \( \zeta \) on the sphere \( \partial B \) as its negative end and \((\partial X,\xi)\) as the positive one.

Of course, if the contact structure \( \zeta \) is standard then the form extends to a nonsingular symplectic form on the whole \( X \). Note that if \( \zeta \) is overtwisted then according to Theorem 1.2.1 it is uniquely determined up to isotopy by the homotopy class of a tamed by \( \omega \) almost complex structure on a punctured neighborhood of \( p \).

The following result is a special case of Theorem 1.1.

Theorem 1.4. Let \( X \) be a compact \( 2n \)-dimensional manifold, with non-empty boundary \( \partial X \), \( \xi \) a contact structure on \( \partial X \), and \( \eta \) a non-degenerate 2-form on \( X \setminus p, p \in X \), which is equal do \( d\lambda \) near \( \partial X \) so that \( \xi = \{\lambda|_{\partial X} = 0\} \) and the Liouville vector field dual to \( \lambda \) is outward transverse to \( \partial X \). Let \( a \in H^2(X, \partial X) \) be a relative cohomology class. If \( n = 2 \) assume, in addition, that \( \xi \) is overtwisted. Then there exists a symplectic structure \( \omega \) on \( X \) with a conical singularity at \( p \) with an overtwisted link \((S^{2n-1},\zeta)\) and positive contact boundary \((\partial X,\xi)\), and such that the cohomology class
$[\omega] \in H^2(X \setminus p, \partial X) = H^2(X, \partial X)$ coincides with $a$, and the relative to $\partial X$ homotopy class of $\omega|_{X \setminus p}$ as a non-degenerate form coincides with $\eta$.

**Remark 1.5.**

1. In contrast with Theorem 1.4, construction of non-singular symplectic structures on $X$ is severely constrained. For instance, according to Gromov’s theorem in dimension 4 and Eliashberg-Floer-Mcduff’s theorem in higher dimensions, see [11, 13], any symplectic manifold $(X, \omega)$ bounded by the standard contact sphere and satisfying the condition $[\omega]|_{\pi_2(X)} = 0$ has to be diffeomorphic to a ball.

2. It is interesting to compare the flexibility phenomenon for symplectic structures with conical singularities with overtwisted links, exhibited in Theorem 1.4, with a similar flexibility phenomenon for Lagrangian manifolds with conical singularities with loose Legendrian links, which was uncovered in [7].

Theorem 1.4 implies the following

**Corollary 1.6.** Let $X$ be a closed manifold of dimension $2n > 4$ which admits an almost complex structure on $X \setminus p$, $p \in X$. Let $a \in H^2(X)$ be any cohomology class. Then given any closed symplectic $2n$-dimensional manifold $(Z, \omega)$ the connected sum $X \# Z$ admits a symplectic form with a conical singularity in the cohomology class $a + C[\omega]$ for a sufficiently large constant $C > 0$.

**Proof.** Choose a ball $B \subset X$ and apply Theorem 1.4 to get a symplectic structure with conical singularity $\omega_X$ on $X \setminus B$, so that $[\omega_X] = a$ and $\partial(X \setminus B, \omega_X) = \partial_+(X \setminus B, \omega_X) \cong (S^{2n-1}, \xi_{std})$. Then $(X \setminus B, \omega_X)$ can be implanted into any symplectic manifold $Z$ provided that it admits a symplectic embedding of a sufficiently large symplectic ball. \hfill $\square$

The assumption that in the 3-dimensional case the positive boundary is overtwisted is essential. In particular, see [17, 12].

**Proposition 1.7.** If there exists a 4-dimensional symplectic cobordism $(W, \omega, \xi_-, \xi_+)$ with an exact symplectic form $\omega$ and symplectically fillable contact structure $\xi_-$, then there is no exact symplectic cobordism structure $(W, \tilde{\omega}, \tilde{\xi}_-, \xi_+)$ with $\omega$ and $\tilde{\omega}$ in the same homotopy class of almost symplectic forms, so that $\tilde{\xi}_-$ is overtwisted. In particular, for any fillable contact manifold $(M, \xi)$ there is no symplectic concordance $(M \times [-1, 1], \lambda)$ in either direction between $(M, \xi)$ and $(M, \tilde{\xi})$ with an overtwisted $\tilde{\xi}$.

We do not know whether the condition $\partial_+ W \neq \emptyset$ in Theorem 1.1 is essential when $n > 2$. We note, however, that Theorem 1.1 implies that every overtwisted contact manifold $(M, \xi)$ of dimension $> 3$ admits a symplectic cap, i.e. there exists a
symplectic cobordism \((W, \omega)\) with \(\partial_+ W = \emptyset\) and \(\partial_-(W, \omega) = (M, \xi)\). Indeed, the groups of complex bordisms is trivial in odd dimensions, see [16]. Hence, according to Theorem 1.1 there is a symplectic cobordism between \((M, \xi)\) on the negative end and the standard contact sphere on the positive one, which then can be capped by any closed symplectic manifold, as in the proof of Corollary 1.6. A similar result for overtwisted contact 3-manifolds is proven in [9], and for general contact 3-manifolds in [6] and [8].

First constructions of symplectic cobordisms between contact manifolds were based on the Weinstein handlebody construction, see [5, 19, 3] and Theorem 2.2 below. Examples of Liouville domains with disconnected contact boundary, and hence with non-trivial topology above middle dimension were constructed in [15] in dimension 4, and then extended to higher dimensions in [10] and [14]. Related problems of (different flavors of) symplectic fillability and topology of symplectic fillings were extensively studied, especially in the contact 3-dimensional case. We refer the reader to Chris Wendl’s blog https://symplecticfieldtheorist.wordpress.com/author/lmpshd/ for a survey and a discussion of the related results and problems.

2 Direct and inverse Weinstein surgeries

In what follows we will also consider symplectic and Liouville cobordisms between manifolds \(\partial_{\pm} W\) with boundary, which we will usually view as sutured manifolds with corner along the suture. More precisely, we assume that the boundary \(\partial W\) is presented as a union of two manifolds \(\partial W_-\) and \(\partial_+ W\) with common boundary \(\partial^2 W = \partial_+ W \cap \partial_- W\), along which it has a corner. See Figure 2.1

**Fig. 2.1:** A Liouville cobordism \(W\) with corners. Note that \(\partial_{\pm} W\) may be disconnected. In the figure, \(\partial_- W\) has one closed component, and one component with boundary.

Let us recall some basic definitions and statements from the Weinstein surgery theory,
Let \((\mathbb{R}^{2n}, \sum_1^n dp_j \wedge dq_j)\) be the standard symplectic space. A Weinstein handle \(W_k\) of index \(k \leq n\) is the domain
\[
W_k = \left\{ \sum_1^k q_i^2 \leq 1, \sum_{k+1}^n q_i^2 + \sum_1^np_i^2 \leq 1 \right\} \subset \mathbb{R}^{2n}
\]
endowed with the Liouville form
\[
\lambda_k = \sum_1^k (2p_i dq_i + q_i dp_i) + \sum_{k+1}^n p_i dq_i.
\]
We have \(d\lambda_k = \sum_1^n dp_i \wedge dq_i\). The isotropic \(k\)-disc
\[
D := \{p_{k+1} = \cdots = p_n = 0, q_1 = \cdots = q_n = 0\} \subset W
\]
is called the core disc of the handle and the \((2n - k)\) coisotropic disc
\[
C := \{p_1 = \cdots = p_k = 0\} \subset W
\]
the co-core of the handle. We will denote \(\Lambda_- := \partial D\), \(\Lambda_+ := \partial C\). We also denote
\[
H^k_- := \Lambda_- \times C = \{ \sum_1^k q_i^2 = 1 \} \cap W_k, \quad H^k_+ := \Lambda_+ \times D = \{ \sum_{k+1}^n q_i^2 + \sum_1^np_i^2 = 1 \} \cap W_k,
\]
so that \(\partial W_k = H^k_+ \cup H^k_-\). The Liouville vector field
\[
Z_k = \sum_1^k \left( p_i \frac{\partial}{\partial p_i} - q_i \frac{\partial}{\partial q_i} \right) + \sum_{k+1}^n p_i \frac{\partial}{\partial p_i}
\]
is inward transverse to \(H^k_-\) and outward transverse to \(H^k_+\). We denote by \(\xi_{k,\pm}\) the contact structure on \(H^k_{\pm}\) defined by the contact form \(\lambda_k|_{H^k_{\pm}}\).

Let \((Y, \xi)\) be a contact manifold. Given a contact embedding \(h_- : (H^k_-, \xi_{k,-}) \to (Y, \xi)\), consider a contact form \(\alpha\) for \(\xi\) such that \(h^*_\alpha = \lambda_k|_{H^k_-}\). Consider a trivial symplectic cobordism \(([1 - \varepsilon, 1] \times Y, d(s\alpha))\) and attach to it the handle \(W_k\) with the embedding \(h_- : H^k_- \to Y = Y \times 1:\)
\[
W := ([1 - \varepsilon, 1] \times Y)_{h_-} \cup W_k.
\]
We can arrange that the Liouville vector field $Z_k$ on $W$ smoothly extends the Liouville vector field $s \frac{\partial}{\partial s}$ on $([1-\varepsilon, 1] \times Y)$. After smoothing the corners this gives us a Liouville cobordism with the contact manifold $(Y, \xi)$ on the negative end and a manifold $(\widehat{Y}, \widehat{\xi})$ on the positive one. We say that $(\widehat{Y}, \widehat{\xi})$ is obtained from $(Y, \xi)$ by a *direct* Weinstein surgery of index $k$ along $h_-(H^k_w)$.

Similarly, suppose we are given a contact embedding $h_+: (H^k_+, \xi_{k,+}) \to (Y, \xi)$ and choose a contact form $\alpha$ for $\xi$ such that $h^*_+ \alpha = \lambda_k|_{H^k_+}$. Consider a trivial symplectic cobordism $([1, 1+\varepsilon] \times Y, d(s\alpha))$ and attach to it the handle $W_k$ with the embedding $h_-: H^k_\epsilon \to Y = Y \times 1$:

$$W := ([1, 1+\varepsilon] \times Y) \cup W_k.$$ 

We can arrange that the Liouville vector field $Z_k$ on $W$ smoothly extends the Liouville vector field $s \frac{\partial}{\partial s}$ on $([1, 1+\varepsilon] \times Y)$. After smoothing the corners this gives us a Liouville cobordism with the contact manifold $(Y, \xi)$ on the positive end and a manifold $(\widehat{Y}, \widehat{\xi})$ on the negative one. We say that $(\widehat{Y}, \widehat{\xi})$ is obtained from $(Y, \xi)$ by an *inverse* Weinstein surgery of index $2n-k$ along $h_+(H^k_\epsilon)$ (or along the coisotropic sphere $h_+(\Lambda_+)$).

Given a Liouville cobordism $(W, \lambda)$ the above construction allows us to attach Weinstein handles to its positive or negative boundaries. In other words, if we are given a contact embedding $h_-: (H^k_\epsilon, \xi_{k,-}) \to (\partial_+ W, \xi_+)$, then we can attach an index $k$ Weinstein handle to get a new Liouville cobordism $(\widetilde{W}, \widetilde{\lambda}) \supset (W, \lambda)$ such that $(\partial_+ \widetilde{W} = \partial_+ \widetilde{W})$ and $(\widetilde{X} := \widetilde{W} \setminus \text{Int} W, \widetilde{\lambda}|_X)$ is an elementary Weinstein cobordism between $\partial_+ W$ and $\partial_+ \widetilde{W}$ with a single handle of index $k$. Similarly, given a contact embedding $h_+: (H^k_\epsilon, \xi_{k,+}) \to (\partial_- W, \xi_-)$, then we can attach an index $(2n-k)$ Weinstein handle to get a new Liouville cobordism $(\widetilde{W}, \widetilde{\lambda}) \supset (W, \lambda)$ such that $(\partial_+ W = \partial_+ \widetilde{W})$ and $(\widetilde{X} := \widetilde{W} \setminus \text{Int} W, \widetilde{\lambda}|_X)$ is an elementary Weinstein cobordism between $\partial_- \widetilde{W}$ and $\partial_- W$ with a single handle of index $k$.

On the other hand, if we are given a Liouville embedding of the whole handle, $G: (W_k, H^k_\epsilon; \lambda_k) \to (W, \partial_- W; \lambda)$ then one can *subtract* the handle, thus obtaining a new Liouville cobordism $(\widetilde{W}, \widetilde{\lambda}) \subset (W, \lambda)$ such that $(\partial_+ W = \partial_+ \widetilde{W})$ and $(\widetilde{X} := W \setminus \text{Int} \widetilde{W}, \widetilde{\lambda}|_X)$ is an elementary Weinstein cobordism between $\partial_- W$ and $\partial_- \widetilde{W}$ with a single handle of index $k$. In fact, this operation is just determined by the isotropic embedding $G|_D$, where $D \subset H^k_\epsilon$ is the core disc. Similarly, if we are given a Liouville embedding of the whole handle $G: (W_k, H^k_\epsilon; \lambda_k) \to (W, \partial_+ W; \lambda)$ then one can *subtract* the handle, thus obtaining a new Liouville cobordism $(\widetilde{W}, \widetilde{\lambda}) \subset (W, \lambda)$ such that $(\partial_- W = \partial_- \widetilde{W})$ and $(\widetilde{X} := W \setminus \text{Int} \widetilde{W}, \widetilde{\lambda}|_X)$ is an elementary Weinstein
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A cobordism between $\partial_+ \tilde{W}$ and $\partial_+ W$ with a single handle of index $k$. This operation is determined by the coisotropic embedding $G|_C$, where $C \subset H_k$ is the co-core disc. The following lemma follows from the classification of overtwisted contact structures in [1], see above Theorem 1.2.1.

**Lemma 2.1.** Let $(Y, \xi)$ be an overtwisted contact manifold. Then any formal contact embedding $h_\pm : (H^k_\pm, \xi_{k, \pm}) \to (Y, \xi)$ is isotopic to a genuine contact embedding. We can furthermore ensure that the complement of the image $h_\pm (H^k_\pm)$ is overtwisted.

Given a cobordism $(W, \partial^- W, \partial_+ W)$ we say that the relative Morse type of $(W, \partial^- W)$ is $\leq k$ if $W$ admits a Morse function $\phi : W \to \mathbb{R}$ with all critical points of index $\leq k$ and whose gradient with respect to some metric is inward transverse to $\partial^- W$ and outward transverse to $\partial_+ W$ (we do not require the function $\phi$ to be constant on $\partial_{\pm} W$, in order to incorporate the case of $\partial^2 W \neq \emptyset$). Given a manifold $W$ with boundary and a submanifold $(\Sigma, \partial \Sigma) \subset (W, \partial W)$ of positive codimension we say that the relative Morse type of $(W, \partial^- W)$ is $\leq k$ if the relative Morse type of $(\tilde{W} := W \setminus \text{Int } N, \partial \tilde{W} := \partial N \setminus \text{Int } (N \cap \partial W))$ is $\leq k$, where we denoted by $N$ a tubular neighborhood of $\Sigma$.

The following theorem is a corollary of Lemma 2.1 and the results from [5] (see also [3]) which were proved using Weinstein handlebody constructions:

**Theorem 2.2.** Any $2n$-dimensional almost complex cobordism $(W, J)$ such that the relative Morse type $(W, \partial W)$ is $\leq n$ admits a Liouville structure $(W, \lambda)$ such that the symplectic form $d\lambda$ is in the formal homotopy class determined by $J$, and $(\partial_+ W, \{\lambda|_{\partial_+ W} = 0\})$ and $(\partial^- W, \{\lambda|_{\partial^- W} = 0\})$ are its positive and negative overtwisted contact boundaries, which are uniquely determined up to isotopy by the homotopy class of the almost complex structure $J$.

The next proposition concerning an inverse Weinstein surgery on loose Legendrian knots is proven in [2] (see that paper, or [18] for a definition of loose Legendrians). For the convenience of the reader we provide a modified proof here.

**Proposition 2.3.** The inverse surgery on a loose Legendrian knot produces an overtwisted contact manifold.

Proposition 2.3 is an immediate corollary of its following special case.

**Lemma 2.4.** Let $\hat{\Lambda}$ be a loose Legendrian knot in $(H^n_+, \xi_{n, +})$ formally isotopic to $\Lambda_+$. Then the inverse surgery along $\hat{\Lambda}$ produces an overtwisted contact manifold.
Proof. Let $\Lambda \subset (Y, \xi)$ be a Legendrian sphere. Note that for an arbitrary neighborhood of the 0-section $U \subset T^*\Lambda_+$ the inclusion $\Lambda_+ \hookrightarrow H^+_n$ extends to a contact embedding $(U \times [-1, 1], pdq + dz) \rightarrow (H^+_n, \xi_{n, +})$. Present the sphere $\Lambda$ as a union $A \cup B := S^1 \times D^{n-2} \cup D^2 \times S^{n-3}$. Then $T^*\Lambda = T^*A \cup T^*B$ and $T^*A = T^*S^1 \times T^*D^{n-2}$. Hence, we can choose the above neighborhood $U$ which contains the product of arbitrary large neighborhoods $U_1$ and $U_2$ of the 0-sections in $T^*S^1$ and $T^*D^{n-2}$. We denote Liouville forms in $T^*S^1$ and $T^*D^{n-2}$ by $udt$ and $\sum_{1}^{n-2} p_i dq_i$, respectively. Let $\Gamma_{st}$ be a Legendrian stabilization, see [18, 5, 3], of the 0-section $\Gamma$ in the 3-dimensional contact manifold $(V_1 := U_1 \times [-1, 1], dz + udt)$. By subtracting the 4-dimensional handle $W_2$ along $\Gamma$ we get a symplectic cobordism $X$ with the negative contact boundary $V_1^-$ which is overtwisted, because a parallel copy of the original Legendrian knot $\Gamma$ is homological to 0 in $V_1^-$ but has a non-negative Thurston-Bennequin invariant. According to [18] there exists a Legendrian sphere $\Lambda_{st}$ in a neighborhood of $\Lambda$ which is isotopic to $\Lambda$ in $(Y, \xi)$ and which contains $\Gamma_{st} \times \{p = 0\} \subset V_1 \times T^*D^{n-2}$. Then the Liouville manifold which we get by subtracting the handle $W_n$ along $\Lambda_{st}$ contains the product $X \times U_2$, and hence its negative contact boundary contains $V_1^- \times U_2$. But the neighborhood of the 0-section $U_2 \subset T^*D^{n-2}$ can be chosen arbitrarily large, and hence the resulted contact manifold is overtwisted by Theorem 1.2.2.

Corollary 2.5 (see [2]). For any contact manifold $(Y, \xi)$ of dimension $> 3$ there exists a Weinstein cobordism structure on $W = Y \times [0, 1]$ between the contact structure $\xi$ on $\partial_+ W := Y \times 1$ and an overtwisted contact structure $\xi_{ot}$ on $\partial_- W := Y \times 0$.

Proof. On the trivial Weinstein cobordism $W = Y \times [0, 1]$ deform the Weinstein structure to create two critical points $a, b$ of index $n - 1$ and $n$, respectively. Let $Z$ be the intermediate level set for this cobordism. Let $\Gamma \subset Y \times 1$ be the unstable Legendrian sphere for the contact structure $\xi$. Consider a stabilization $\hat{\Gamma}$ of $\Gamma$, which is formally isotopic to $\Gamma$, [18]. By subtracting a handle $W_n$ from $(Y = Y \times 1, \xi)$ along $\hat{\Gamma}$ we construct a Liouville cobordism between $Z$ and $Y$ with an overtwisted contact structure $\xi_{ot}$ on $Z$. By Lemma 2.1 we can contactly embed the coisotropic unstable sphere of the critical point $p$ into $(Y, \xi_{ot})$. Hence one can further subtract the handle $W_{n-1}$ corresponding to $p$ from $Z$ to get a Weinstein structure on the cobordism $Y \times [0, 1]$ with the contact structure $\xi$ on the positive and an overtwisted contact structure on the negative one.
3 Proof of Theorem 1.1

We begin with two lemmas

**Lemma 3.1.** Let $\Sigma$ be a codimension 2 connected oriented submanifold of a $2n$-dimensional almost complex manifold $(W, J)$. Then $J$ is homotopic to $\tilde{J}$ for which $\Sigma$ is $\tilde{J}$-holomorphic (i.e. $T\Sigma \subset TW$ is $\tilde{J}$-invariant) in a complement of a $(2n-2)$-dimensional ball $D \subset \Sigma$. If $W$ is a manifold with boundary and $(\Sigma, \partial \Sigma) \subset (W, \partial W)$ a submanifold with non-empty boundary $\partial \Sigma$ then $J$ is homotopic to $\tilde{J}$ for which $W$ is $\tilde{J}$-holomorphic everywhere.

**Proof.** We construct $\tilde{J}$ inductively over skeleta of $\Sigma$. Suppose that $\sigma \subset \Sigma$ is a $l$-dimensional, $l < 2n-2$, simplex of $\Sigma$ and that we already deformed $J$ to make $\Sigma$ $\tilde{J}$-holomorphic near $\partial \sigma$. In order to to make $\Sigma \tilde{J}$-holomorphic on $O \sigma$ we choose a trivialization $e_1, \ldots, e_{2n}$ of $TW|O \sigma$ such that vector fields $e_1, \ldots, e_{2n-2}$ are tangent to $\Sigma$. Furthermore, we can assume that $\tilde{J}e_{2n-1} = e_{2n}$ on $O \sigma$. To make $\Sigma \tilde{J}$-holomorphic we need to arrange that $\tilde{J}e_{2n-1} = e_{2n}$ on $O \sigma$. The obstruction to this lies in $\pi_l(S^{2n-2}) = 0$ for $l < 2n-2$. In the case when $\partial \Sigma \neq \emptyset$ it is sufficient to make $\Sigma \tilde{J}$-holomorphic in a neighborhood of its $(2n-3)$-skeleton. \qed

The next lemma is the main step in the construction.

**Lemma 3.2.** Let $(W, J)$ be an almost complex $2n$-dimensional cobordism, with $\partial_+ W$ and $\partial_- W$ both non-empty. Let $(\Sigma, \partial \Sigma) \subset (W, \partial_+ W)$ be a codimension 2 submanifold, and $\lambda$ be a Liouville form on $\Sigma$ with an overtwisted conical singularity at $p \subset \text{Int} \Sigma$ and with a positive contact boundary $(\partial \Sigma, \{\lambda_{\Sigma} = 0\})$. Suppose that

(i) $(\Sigma, \partial \Sigma)$ is homologous to 0 in $H_{2n-2}(W, \partial_+ W)$;

(ii) $\Sigma$ has a trivial tubular neighborhood $N = \Sigma \times D^2$;

(iii) $d\lambda|_{\Sigma \setminus p}$ is in the homotopy class determined by $J$;

(iv) $(W \setminus \Sigma, \partial_- W)$ has its Morse type $\leq n$.

Then there exists a Liouville cobordism structure $\Lambda$ on $W$ with overtwisted positive and negative contact ends $(\partial_+ W, \xi_\pm)$, and such that $\Lambda|_{\partial \Sigma} = \lambda|_{\partial \Sigma}$ and the symplectic form $d\Lambda$ is in the homotopy class determined by $J$. 

\[ \]
Proof. Near \( p \in \Sigma \) we write \( \lambda = s\beta \), where \( \beta \) is an overtwisted contact form on \( S^{2n-3} \) and \( p = \{ s = 0 \} \). Let \((\sqrt{u}, t)\) be polar coordinates on the \( D^2 \) factor of the tubular neighborhood \( N = \Sigma \times D^2 = \{ u \leq \frac{1}{2} \} \), and define a 1-form \( \mu = (u - 1)dt + \lambda \) on \((\Sigma \setminus p) \times (D^2 \setminus 0)\). Notice that \( d\mu = du \wedge dt + d\lambda \) is symplectic for \( s > 0 \), and extends as a symplectic form over \( \{ u = 0 \} \setminus \{ s = 0 \} \) since \( du \wedge dt \) is defined at \( u = 0 \). The homological condition \((ii)\) ensures that the form \( dt \) can be extended from \( N \setminus \Sigma \) to all of \( W \setminus \Sigma \) as a closed 1-form \( \theta \), supported away from \( \partial \bar{W} \).

Fig. 3.1: The structure of the cobordism \( W \).

Let \((D = D^2(2), \partial D) \subset (W, \partial_+ W)\) be a 2-disc with boundary in \( \partial_+ W \) which intersects \( \Sigma \) transversely at \( p \) and which is homotopic rel. \( \partial D \) to a disc in \( \partial_+ W \). Matching the coordinates on \( N \), we let \((\sqrt{u}, t)\) be polar coordinates on \( D = \{ u \leq 2 \} \).

We can also ensure that \( \theta|_D = dt \). Extend \((D, \partial D) \hookrightarrow (W, \partial_+ W)\) to an inclusion \( j : U := (D \times D^{2n-2}, \partial D \times D^{2n-2}) \hookrightarrow (W, \partial_+ W) \), so that

\[
U \cap N = \{ s \leq 1, u \leq \frac{1}{2} \} \subset U = \{ s \leq 1, u \leq 2 \}.
\]

By deforming the almost complex structure \( J \) we can assume that \( j \) is \( J \)-holomorphic with respect to the product complex structure on \( D \times D^{2n-2} \subset \mathbb{C} \times \mathbb{C}^{n-1} \).

Denote \( H_+ := \partial D \times D^{2n-2} = \{ u = 2 \} \) and \( H_- := D \times \partial D^{2n-2} = \{ s = 1 \} \), so that \( \partial U = H_- \cup H_+ \). We extend the form \( \mu \) to \( U \setminus \{ u = 0 \} \) by the formula \( \mu = (u - 1)dt + s\beta \).
Denote $\hat{H}_- = H_- \setminus \text{Int } N$. Then $H_+$ and $\hat{H}_-$ are symplectically convex boundaries for the form $\mu$, which is Liouville away from $\{s = 0\}$.

We define a contact form $\alpha$ on $\partial_+ W \setminus (\text{Int } H_+ \cup \Sigma)$ as follows:

- On $N \cap \partial_+ W \setminus \Sigma$ let $\alpha = \mu = \lambda + (u - 1)dt$,
- Near $\partial H_+$, let $\alpha = s\beta + dt$,
- Elsewhere on $\partial_+ W \setminus (\text{Int } H_+ \cup \Sigma)$ we let $\alpha$ be any contact form in the homotopy class determined by $J$, which is overtwisted in the complement of the previous two domains.

Notice that $\alpha$ agrees with $\mu$ everywhere they overlap, therefore we extend $\mu$ to the manifold $Y := (\partial N \cup U) \cup (\partial_+ W \setminus (N \cup H_+))$ by $\mu|_Y = \alpha$. Denote $X := W \setminus \text{Int } (N \cup U)$, then $\partial X = Y \cup \hat{H}_-$. $\partial_+ W$ and $\partial^2 X = \partial \hat{H}_- = \{u = \frac{1}{2}; s = 1\}$. The Liouville vector field of $\mu$ points into $X$ on $\hat{H}_-$, and on $Y$ points out of $X$. Our next step is to extend $\mu$ to the structure of a Liouville cobordism with corners on $X$.

The relative Morse type of $(X, \partial X)$ is no more than $n$. Indeed, compared to the decomposition of $(W \setminus N, \partial_+ W)$ one needs just two additional handles of index 1 and 2. The contact manifold $(\hat{H}_-, \hat{\alpha} := \mu|_{\hat{H}_-})$ is overtwisted, if the form $\beta$ is chosen small enough according to a result from [2], see Theorem 1.2 in this paper and Theorem 10.2 in [1]. We chose $\mu|_Y = \alpha$ previously so that $(Y, \mu|_Y)$ is overtwisted, finally we let $\xi_-$ be the overtwisted contact structure on $\partial_- W$ in the homotopy class of $J|_{\partial_- W}$.

Since all boundary components are overtwisted, there is, in view of Theorem 2.2, a sequence of inverse Weinstein surgeries on $(Y, \mu|_Y)$ which produces a Liouville cobordism $(\tilde{X}, \tilde{\Lambda})$ with $(Y, \mu|_Y)$ on its positive side and an overtwisted $(\tilde{Y}, \tilde{\alpha})$ on the negative side, so that $\tilde{X}$ is diffeomorphic to $X$ and there exists a contactomorphism $h : (\hat{H}_-, \ker \hat{\alpha}) \amalg (\partial_- W, \xi_-) \to (\tilde{Y}, \ker \tilde{\alpha})$.

Moreover, we can arrange that the contactomorphism $h$ extends to a diffeomorphism $\tilde{h} : X \to \tilde{X}$ which is equal to the identity on $Y$ and that $h_* J$ is homotopic to an almost complex structure which is compatible with $d\tilde{\Lambda}$.

On $\hat{H}_-$ we have $h^* \tilde{\alpha} = \phi \hat{\alpha}$, where the positive function $\phi$ satisfies $\phi|_{\partial \hat{H}_-} = 1$ and $\phi|_{\partial \hat{H}_- \setminus \partial \hat{H}_-} > 1$. Take $\sigma \in (0, \min \phi)$, and denote $G_\sigma := \{s \leq \sigma, u \leq \sigma + 1\} \subset U$. Consider the symplectization $(\mathbb{R}_+ \times \hat{H}_-, d(\tau \hat{\alpha}))$ of $(\hat{H}_-, \hat{\alpha})$, and notice that the subset $(0, 1] \times \hat{H}_-$ can be identified with the cone $C_- := \{1 - \frac{s}{2} \leq u \leq 1 + s\} \subset U = \{s \leq 1, u \leq 2\}$, with the Liouville form $\tau \hat{\alpha}$ identified with $\mu$. Denote $\tilde{C}_- := \{\sigma \leq \tau \leq$
Denote $C_+ := \{u \geq 1 + s\} \subset U$. At this point, we have defined a Liouville form $\Lambda$ everywhere on $W \setminus (\Sigma \cup G_\sigma \cup C_+)$, by gluing together $\mu$ and $\Lambda$. On $G_\sigma$ we have $\Lambda = (u - 1)dt + s\beta$, which fails to be Liouville at $s = 0$. Near $\Sigma$ we have $\Lambda = (u - 1)dt + \lambda$, so $\Lambda$ is singular at $u = 0$. To fill the gap $C_+$ we will need the following

**Lemma 3.3.** There exists a contact form $\gamma$ on $H_\sigma^+ := \{s \leq \sigma, u = 1 + \sigma\}$ which for $s$ near $\sigma$ coincides with $s\beta + \sigma dt$ and such that $\gamma + dt$ is a contact form of the same sign as $\gamma$.

**Proof.** Note that for any contact form $\tilde{\gamma}$ the form $K\gamma + dt$ is contact if the constant $K$ is large enough. Take $a \in (0, \frac{\sigma^2}{1+\sigma})$ and let $\tilde{\gamma}$ be any contact form on $H_\sigma^+$ which is equal to $s\beta + \sigma dt$ for $s \geq a$. Let $K > 0$ be a constant such that $K\tilde{\gamma} + dt$ is contact over $H_\sigma^+ \cap \{s \leq a\}$. We claim that there exists a function $f : [a, \sigma] \to [1, K]$ such that $f = 1$ near $\sigma$ and to $K$ near $a$ and such that

$$\frac{d}{ds} \left( \frac{f(s)s}{\sigma f(s) + 1} \right) > 0,$$

(1)
or equivalently
\[ \sigma f^2(s) + f(s) + sf'(s) > 0. \]  
(2)

This inequality implies that the form
\[ \gamma := \begin{cases} f(s)(s\beta + \sigma dt), & s \in [a, \sigma], \\ K\gamma, & s < a \end{cases} \]
has the required properties. To construct \( f \) which satisfies (1) consider first the function
\[ f(s) := \frac{K(\sigma - a)}{(K - 1)s + \sigma - Ka}. \]
Then \( f(\sigma) = 1 \) and \( f(a) = K \). This function also satisfies the inequality (2). Indeed, first observe that
\[ (K - 1)s + \sigma - Ka = K(s - a) + (\sigma - s) > 0. \]
Then we have
\[ \frac{(K(s - a) + (\sigma - s))^2}{K(\sigma - a)}(\sigma f^2 + f + sf') = K\sigma^2 - aK\sigma + \sigma - Ka \]
\[ = K(\sigma + 1) \left( \frac{\sigma^2 + \frac{\sigma}{K}}{\sigma + 1} - a \right) > K(\sigma + 1) \left( \frac{\sigma^2 + \frac{\sigma}{K}}{\sigma + 1} - \frac{\sigma^2}{\sigma + 1} \right) > 0. \]

We also note that \( f > 1 \) and \( f'(s) < 0 \) if \( K > 1 \). Hence, the inequality (2) remains valid if one decreases \( |f'(s)| \) without changing \( f \). This allows us to to make the function \( f \) constant near the end points of the interval \([a, \sigma]\) without violating the inequality (2). \( \square \)

Using Lemma 3.3, we extend \( \Lambda \) to be equal to \( \tau\gamma \) on \( C_+ \), where we are identifying \( C_+ \) with the piece of symplectization \([1, \frac{1}{\sigma}] \times H^*_+ \), \( \tau\gamma \). Note that the form \( \Lambda|_{\{u=1+s\} \subset \partial C_+} \) coincides with \( \mu = (u - 1)dt + s\beta \). In particular, on \( \partial \partial H_+ \subset \partial_+ W \) the form \( \alpha \) coincides with the restriction of \( \Lambda \), and therefore we can extend \( \alpha \) to \( H_+ \) as equal to \( \Lambda|_{H_+} \).

Recall that on \( N \cap \partial_+ W \) we have \( \alpha = \Lambda|_{\partial_+ W} = (u - 1)dt + \lambda|_{\partial \Sigma} \), and \( \alpha \) is contact everywhere outside of \( \{u = 0\} \) where \( \alpha \) is undefined. We attach to \( W \) a cylindrical collar \( \partial_+ W \times [1, C] \) along \( \partial_+ W \) by identifying \( \partial_+ W \subset W \) with \( \partial_+ W \times \{1\} \subset \partial_+ W \times [1, C] \). On the set \( \{u \leq 1\} \times [1, C] \subset N \cap \partial_+ W \times [1, C] \) we extend \( \Lambda \) as \( \Lambda = \).
(u - 1)dt + r\lambda|_{\partial\Sigma} where r \in [1, C]. On (\partial_+ W \setminus \{u \leq 1\}) \times [1, C] we define \(\overline{\Lambda} = r\alpha\). The two definitions agree on \(A := (\partial_+ W \setminus \{u = 1\}) \times [1, C]\). Hence, by a small adjustment of \(\overline{\Lambda}\) on \(O_p A\) we can arrange that \(\overline{\Lambda}\) is a smooth Liouville form on the attached collar. We extend the closed 1-form \(\theta\) to the collar as its pull-back by the projection \(\partial_+ W \times [1, C] \to \partial_+ W\).

We now let \(\Lambda = \overline{\Lambda} + \theta\). \(\Lambda\) is contact on \(\partial G_\sigma = \{s = \sigma, u \leq 1 + \sigma\} \cup H^*_\sigma\), since it is equal to \(\beta + udt\) or \(\gamma + dt\), respectively. On \(\{u \leq 1\} \times \{C\} \subset \partial_+ W \times \{C\}\) we have \(\Lambda = C\alpha + \theta\), which is contact for sufficiently large \(C\). Also \(d\Lambda = d\overline{\Lambda}\), so \(\Lambda\) is symplectic everywhere on \(W \setminus G_\sigma\), that is, \(\Lambda\) is a Liouville structure on \(W \setminus G_\sigma\) with convex boundary \(\partial_+ W\) and concave boundary \(\partial_- W \amalg \partial G_\sigma\).

Finally, we choose an arc \(\ell\) and subtract from \(W\) the Weinstein handle \(W_1\) with the core disc \(D = \ell\), thus getting a Weinstein cobordism whose new negative boundary is a connected sum \(\partial_+ W_1 \# \partial G_\sigma\) along \(\ell\). The contact structure \(\ker(\Lambda|_{\partial_+ W_1 \# \partial G_\sigma})\) is isotopic to \(\xi_-\), since both are overtwisted contact structures in the same almost contact class, see [1] and the above Theorem 1.2. The contact structure \(\xi_+ := \{\Lambda|_{\partial_+ W} = 0\}\) is overtwisted by construction.

Proof of Theorem 1.1. First note that if \(n > 2\) then for any contact structure \(\xi\) on \(\partial_+ W\) there is a Liouville concordance between \(\xi\) on the positive end and an overtwisted contact structure \(\xi_\ot\) on the negative one, see Corollary 2.5 above. Hence, it is sufficient to prove Theorem 1.1 for the case when both contact structures \(\xi_\pm\) are overtwisted. We prove the theorem by induction over the dimension \(2n\). For \(n = 1\) the statement is trivially true. Suppose it is already proven in the \((2n - 2)\)-dimensional case.

It is sufficient to prove the result for an elementary cobordism \(W\) corresponding to a Morse function with just one critical point of index \(k = 1, \ldots, 2n - 1\). If \(k \leq n\) then the result follows from the Weinstein surgery theory, see [3, 5, 10] and Section 2 above.

Let \((W, J)\) be an elementary almost complex cobordism attaching a handle of index \(k > n\). Let \(\xi_\pm\) be overtwisted contact structures on \(\partial_\pm W\) compatible with \(J\). Let \((\Delta, \partial\Delta) \subset (W, \partial_+ W)\) be the co-core disc of dimension \(l = 2n - k < n\). The normal bundle \(\nu\) to \(\Delta\) in \(TW\) is trivial, so we can consider a splitting of its tubular neighborhood \(U = \Delta \times D^{k-1} \times D^1\). Denote \(\Sigma := \Delta \times \partial D^{k-1} \times 0 \subset U\). Using Lemma 3.1 we deform \(J\) to make \(\Sigma J\)-holomorphic in a complement of a ball \(B \subset \Sigma\). Using [1],
see Theorem 1.2.1 above, we can realize stable almost complex structures $J|_{\partial B}$ and $J|_{\partial \Sigma}$ as overtwisted contact structures $\zeta_{\pm}$, so that $(\partial B, \zeta_-)$ and $(\partial \Sigma, \zeta_+)$ are, negative and positive (or $J$-concave and $J$-convex) boundaries of the almost complex cobordism $(\Sigma \setminus \text{Int} B, J)$. Then the inductive hypothesis yields a Liouville structure $\lambda$ on $\Sigma \setminus \text{Int} B$ in the given almost complex class $J$ with contact ends $\zeta_{\pm}$. Moreover, the overtwistedness of $\xi_+$ allows us to realize $(\partial \Sigma, \zeta_+)$ as a contact submanifold of $(\partial \Sigma, \xi_+)$. Finally, we observe that $(W \setminus \Sigma, \partial_- W)$ has Morse type equal to $k + 1 \leq n$. Therefore, we can apply Lemma 3.2 to extend the Liouville form $\lambda$ to the required Liouville form $\Lambda$ on $W$. 

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3 Proof of Theorem 1.1

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