**Energy Dissipation Bounds in Autonomous Thermodynamic Systems**

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(Dated: March 19, 2019)

How much free energy is irreversibly lost when a system is moved through thermodynamic space? For systems with deterministic control, lower bounds on energy dissipation are established. Recent literature has also bounded the cost of moving a single degree of freedom with specified accuracy. Here we use stochastic thermodynamics to understand the combined costs of controlling a thermodynamic system, taking into account both the cost of nonconservative work done on the system and the entropic cost of precisely exerting control. We find an intriguing unification of previous results for an autonomously controlled near-equilibrium steady state system driven at a finite rate around a loop in thermodynamic space. In particular we demonstrate a lower bound for dissipation that is almost the sum of two previously proposed bounds. Our result suggests that even for an infinitely long control protocol, it is impossible to reach the adiabatic limit wherein thermodynamic change is energetically reversible.

Changing the state of a thermodynamic system requires the dissipation of free energy. Equilibrium systems obey detailed balance, moving forward and backward equally, preventing net changes. Moving in a directed manner requires consuming free energy to pay two distinct costs. Firstly, work must be done on a system to change its state, necessarily in excess of the resulting change in free energy [1–5]. In addition, free energy must be spent to break time-reversal symmetry, ensuring that change happens in a directed manner, moving forward more often than backwards [6–9]. The main aim of this work is to quantify the cost of dissipation for an autonomous method of control, wherein both these sources of dissipation are relevant.

The average work required to move a system through thermodynamic space under a deterministic protocol of control parameter \(\lambda(t)\) must in general be greater than the resulting change in free energy, dissipating energy given by \(\langle E_{\text{diss}} \rangle = \langle \Delta S \rangle = \langle W \rangle - \Delta F\), where here \(S\) is the combined entropy of the system and environment [10]. The last decades have seen substantial progress on understanding the statistics of \(\Delta S\) for deterministic protocols [1–3, 5]. Of particular relevance here, Ref. [3] found that the dissipation rate of a near-equilibrium system is:

\[
\langle \dot{S} \rangle \approx \frac{d\lambda^\alpha}{dt} \dot{g}_{\alpha\beta} \frac{d\lambda^\beta}{dt}
\]

where \(\dot{g}\) is the Kirkwood friction tensor [11]. The metric \(\dot{g}_{\alpha\beta}\) imposes a Riemannian structure on thermodynamic space in which optimal protocols are geodesics [12], giving a lower bound on the energy dissipation:

\[
S \geq \frac{\dot{L}^2(\lambda_i, \lambda_f)}{\Delta t}
\]

where \(\dot{L}\) is the length of the geodesic between \(\lambda_i\) and \(\lambda_f\) in the metric space and \(\Delta t\) is the total time of the protocol. In the quasistatic limit \(\Delta t \to \infty\), the excess energy cost of transforming the system becomes negligible. However, in this framework the breaking of time reversal symmetry arises from the deterministic trajectory \(\lambda(t)\), whose cost is neglected from this energetic book-keeping.

In parallel, a large body of recent work has studied the cost of breaking time reversal symmetry [6–9] and ensuring systems follow a precise trajectory [13–22]. In particular, Ref. [16, 17] suggest that infinitely precise control requires an infinite amount of energy. A self-contained thermodynamic machine must not only pay the cost of performing nonconservative work out of equilibrium, but it must also pay the cost of directing protocols in time with some degree of precision. These costs cannot be minimized at the same time; stochastic trajectories will deviate from the deterministic protocol that minimizes the nonconservative work. What happens when both of these are included in the total dissipation cost of thermodynamic control?

In a recent letter [23] we aimed to answer this question by studying a toy model where the control parameter was allowed to make discrete jumps. To break time reversal symmetry, there is a cost associated with biasing the system in the forward direction. After each jump, there is also some amount of dissipated energy associated with a sudden non-equilibrium change. We found that the average entropy production rate was given by:

\[
\langle \dot{S} \rangle = \frac{v^\mu v^\nu}{D_{\mu\nu}} + D_{\mu\nu} \dot{g}^\lambda_{\mu\nu}
\]

where \(v\) and \(D\) are the net drift and diffusion of the control parameter \(\lambda\). This led to a lower bound on the dissipation cost:

\[
S \geq 2\mathcal{L}(\lambda_i, \lambda_f)
\]

Here \(\mathcal{L}\) is the length of a geodesic in the thermodynamic space parametrized by \(\lambda\) with the metric \(\dot{g}^\lambda_{\mu\nu}\), the Fisher information metric:

\[
\dot{g}^\lambda_{\mu\nu}(\lambda) = \langle \delta\phi\mu \delta\phi\nu \rangle_{\text{eq}, \lambda}
\]
where $\delta \phi_\mu = \phi_\mu - (\phi_\mu)$ is the deviation of $\phi_\mu$, the force conjugate to $\lambda^\mu$, away from its equilibrium value.

Interestingly, this result defies the quasistatic limit, remaining finite even in the limit of an infinitely slow protocol ($u^\mu \to 0$). This finite energy cost is due to the tradeoff between the price of precise control ($u^\mu v^\nu / D^\mu v^\nu$) and the price of a diffusive control parameter ($D^\mu v^\nu \Delta^\mu \nu$). However, that setup took an unusual and poorly motivated continuum limit, and calculated the two forms of dissipation in very different ways. In addition, the formalism developed for that manuscript did not allow us to consider protocols which move at finite rate.

In this letter we use stochastic thermodynamics [24] to derive the entropy production rate for a stochastically controlled analog of the system studied in Ref. [3]. Using these tools we can calculate both the dissipation associated with nonconservative work and the dissipation associated with breaking time symmetry together. The aim is to explore how these different bounds interact in a system where both can be computed using the same framework. In particular, we consider a system of particles interacting through an energy of the form:

$$U(y) = \lambda^\alpha(\theta) \phi_\alpha(x) \quad y = (\theta, x)$$

where $x$ denotes the microstate of the system and $\theta$ parametrizes the path of the control functions around a cycle. The control functions $\lambda(\theta)$ are all $2\pi$ periodic in $\theta$, allowing the system to reach a steady state where the system cycles are fully self-contained. We suppose that $\theta$ moves with constant net drift velocity $v$ and diffusion constant $D$.

The result we will find is that in the steady-state limit, the entropy production rate of the system when it’s at the point $\theta$ is given by:

$$\dot{S}(\theta) \approx \left( \frac{d\lambda^\alpha}{dt} \right)_\theta \dot{\theta}^\alpha - \frac{v^2}{D} + Dg^\theta$$

where $(d\lambda^\alpha / dt)_\theta$ denote the time-derivatives of the expectation value of $\lambda$ under the Ito and reverse-Ito conventions of stochastic calculus when the system is in the known state $\theta$ (see equation (17)). We will see that the term containing these derivatives is the natural generalization of (1) to systems with stochastic control. The other two terms are identical to (3), only here they are one-dimensional with $g^\theta$ being the metric on $\theta$ inherited from its embedding in $\lambda$-space.

In considering a stochastic analog of Ref. [3], this work is similar to Ref. [4]. However, there they consider the stochasticity of the control parameter as external to the system, and thus do not count the precision of the control against the total energetic cost. They find a lower bound which can be made arbitrarily small by decreasing this noise.

We will conclude that (1) the dissipation rates found by Ref. [3] and that found by Ref. [23] are both applicable in the full consideration: (2) energetically optimal control protocols must be necessarily stochastic; (3) despite a modification to the rate found by Ref. [3] in the stochastic analog, the bound found in Ref. [23] is still a hard lower bound, and thus for a system with fully autonomous thermodynamic control, even quasistatic changes are not energetically reversible.

**Derivation** For the sake of simplicity, we will assume that $x$ and $\phi$ are one-dimensional (see Supplemental Material for the multidimensional generalization). We describe both $\theta$ and $x$ using overdamped Langevin equations [24]:

$$\dot{\theta} = v + \sqrt{2D\eta^\theta}$$

$$\dot{x} = D^x F(y) + \sqrt{2D^x \eta^x}$$

The terms $\eta^x$ and $\eta^\theta$ represent thermal noise. The overdamped force on $x$ due to the interaction energy $U$ is given by $F(y) = -\partial_x U(y)$. Note that by control parameter we mean precisely that the term $F_\theta(y) = -\partial_\theta U(y)$ is absent from the stochastic equation for $\dot{\theta}$; the dynamics of $\theta$ don’t receive feedback from the state of the system. Thus $\theta$ is not a degree of freedom in a larger thermodynamic system, but rather it can be thought of as a variable mediating a coupling between the system and a thermodynamic bath. A specific example of this is given in [23]. An analogous macroscopic system would be a series of switches each of which connects some source of energy to a thermodynamic system. The work required to flip each switch is independent of the state of the thermodynamic system being acted on. Other examples include calcium binding modulating the active force in muscle contraction [25] and ligand binding modulating the current of ligand-gated ion channels [26].

Denote $p(y, t)$ to be the probability of finding the system in the state $y$ at time $t$. Its dynamics obey the Fokker-Planck equation:

$$\partial_t p(y, t) = -\partial_y j^y(y, t) - \partial_x j^x(y, t)$$

$$j^\theta \equiv [v - D\partial_\theta]p(y, t)$$

$$j^x \equiv D^x F(y) - \partial_x p(y, t)$$

where $j^\theta$ and $j^x$ are the probability currents induced by the Langevin equations for $\theta$ and $x$ respectively.

Using stochastic thermodynamics [24], it can be shown that the total average entropy production rate, which includes changes in the internal entropy of the system and changes in the entropy of the environment due to dissipated heat, is given by:

$$\langle \dot{S}(t) \rangle = \int dy \left[ \frac{j^\theta(y, t) j^\theta(y, t)}{D\rho(y, t)} + \frac{j^x(y, t) j^x(y, t)}{D^x \rho(y, t)} \right]$$

By integrating by parts and assuming a steady-state, we find:

$$\langle \dot{S} \rangle = \frac{v^2}{D} - \int dy (\partial_x j^x) U(y)$$
where we have used the fact that \( p(y,t) \) goes to zero at the boundaries of \( x \). By rewriting \( U \) in terms of the free energy and the equilibrium Boltzmann distribution and using the Fokker-Plank equation, we find (see Supplemental Material):

\[
\langle \dot{S} \rangle_\theta = \frac{v^2}{D} - \langle (v \partial_\theta + D \partial_\theta^2) \log p_{eq}(x|\theta) \rangle_\theta \tag{12}
\]

where \( \langle \cdots \rangle_\theta \) indicates an average over all \( x \) with weight \( p(x|\theta) \). Here \( \langle \dot{S} \rangle_\theta \) represents the average entropy production rate when the value of the control parameter is \( \theta \). In the steady state, all values of \( \theta \) are equally likely, meaning that the overall average entropy production rate is just \( \langle \dot{S} \rangle = \frac{1}{2\pi} \int d\theta \langle \dot{S} \rangle_\theta \)

By expanding \( \log p_{eq} \) as \( F - U \), it can be shown that:

\[
\partial_\theta \log p_{eq}(x|\theta) = \left[ (\langle \phi \rangle_{eq,\theta} - \phi(x)) \right] \frac{d\lambda}{d\theta} \tag{13}
\]

\[
\partial_\theta^2 \log p_{eq}(x|\theta) = -g^\theta + \left( (\langle \phi \rangle_{eq,\theta} - \phi(x)) \right) \frac{d^2\lambda}{d\theta^2}
\]

where \( \langle \cdot \cdot \cdot \rangle_{eq,\theta} \) is an average over all \( x \) with weight \( p_{eq}(x|\theta) \) and where \( g^\theta \) is the Fisher information metric with respect to the \( \theta \) basis. In this system it takes the form: \( g^\theta = \frac{\partial^2}{\partial \lambda^2} \log p_{eq} \mid_{\theta} = \left( \frac{d\lambda}{d\theta} \right)^2 \langle \delta \phi^2 \rangle_{eq,\theta} \).

Using these relations we obtain:

\[
\langle \dot{S} \rangle_\theta = \frac{v^2}{D} + D g^\theta + \langle \delta \phi \rangle_\theta \left[ \frac{d\lambda}{d\theta} + \frac{D^2 \lambda}{d\theta^2} \right]_\theta \tag{14}
\]

where \( \langle \delta \phi \rangle_\theta = \langle \phi \rangle_\theta - \langle \phi \rangle_{eq,\theta} \) is the difference between the average conjugate force \( \phi \) when the system is at \( \theta \) and its value for a system at equilibrium at \( \theta \).

So far this is an exact result. To be able to put it into a more useful form, we will approximate \( \langle \delta \phi \rangle_\theta \) by assuming that \( D \) and \( v \) are small.

**Linear Response for Stochastic Control** To evaluate the term \( \langle \delta \phi \rangle_\theta \), we use a linear response approximation. For a fixed control parameter path \( \theta(\tau) \), the average linear response over all possible microstate paths is well understood [3]. Here we extend this result to an average over all possible microstate and control parameter paths.

If an ensemble of systems all undergo the same deterministc protocol \( \theta(\tau) \), then at time \( t \), the average linear response of \( \phi \) over this ensemble is given by [27]:

\[
\langle \delta \phi(t) \rangle_{\theta(\tau)} = \int_{-\infty}^{0} dt' C^\theta(t'|t') \frac{d\lambda}{d\theta} [\lambda(t + t')] \tag{15}
\]

where \( C^\theta(t') = \langle \delta \phi(0)\delta \phi(t') \rangle_{eq,\theta} \) is the equilibrium autocorrelation function for the conjugate force \( \phi \) and where we have written \( \lambda(t) \equiv \lambda(\theta(t)) \) for shorthand.

Assuming the protocol speed is much slower than the timescale of system relaxation, by integrating by parts we find [3]:

\[
\langle \delta \phi(t) \rangle_{\theta(\tau)} \simeq \frac{d\lambda}{dt} g^\lambda \rightarrow g^\lambda(\theta) = \int_{-\infty}^{0} dt' C^\theta(t') \tag{16}
\]

We now have to extend this result to an ensemble of stochastic protocols which are all located at the same point \( \theta \) at time \( t \). This process is a bit more challenging (see Supplemental Material). However the form of the result is manifestly the same: we only need to replace \( d\lambda/dt \) with the ensemble average of \( d\lambda/dt \), where the ensemble is over all control parameter trajectories \( \theta(\tau) \) such that \( \theta(t) = \theta \). This quantity is discontinuous at time \( t = 0 \). Its left and right-sided limits are:

\[
\left. \left( \frac{d\lambda}{dt} \right)_\theta \right|_{t'=t} = \lim_{t' \to t^-} \left( \frac{d\lambda}{dt} \right)(t + t') \equiv v \frac{d\lambda}{d\theta} \pm D \frac{d^2\lambda}{d\theta^2} \tag{17}
\]

which correspond to ensemble velocities under the reverse-Ito and Ito conventions [28]. Here we have dropped terms quadratic in \( v \) and \( D \) (See Supplemental Material for details). Since the domain of equation (15) is \( t' < 0 \), it the left-sided limit that is relevant in our calculation, and thus we obtain:

\[
\langle \delta \phi \rangle_\theta \simeq \left( \frac{d\lambda}{dt} \right)_\theta \rightarrow g^\lambda(\theta) \equiv \left( v \frac{d\lambda}{d\theta} - D \frac{d^2\lambda}{d\theta^2} \right) \tag{18}
\]

To obtain this result we have required that both the control parameter velocity and the diffusion rate are small with respect to the system relaxation timescale at equilibrium, \( \tau \). Explicitly, we demand:

\[
v \tau / L < 1 \quad \quad D \tau / L^2 < 1 \tag{19}
\]

where \( L \) is related to the length scale associated with the control function \( \lambda(\theta) \). In this limit, making use of equation (17), equation (14) becomes

\[
\langle \dot{S} \rangle_\theta = \left( \frac{d\lambda}{dt} \right)_\theta \rightarrow g^\lambda(\theta) \left( \frac{d\lambda}{dt} \right)_\theta + \frac{v^2}{D} + D g^\theta(\theta) \tag{20}
\]

which becomes (7) in the multidimensional case.

**Discussion** The first term in (20) represents the frictional dissipation arising from pushing the system out of equilibrium. This term is a generalization of the dissipation rate found in the deterministically controlled coupled system studied in Ref. [3]. To obtain that result, the authors made the assumption that the system moves much slower than the relaxation timescale of the system. Concretely, this is the assumption that \( v \tau / L \ll 1 \). In moving from (14) to (20), we make essentially the same assumption, but we also require that \( D \tau / L^2 \ll 1 \). The system of Ref. [3] can be seen as the limiting case where \( D \to 0 \) and the \( v^2/D \) term is ignored.

The other two terms in (20) are the same dissipation terms found in our earlier work [29]. The first of these, \( v^2/D \), is the energy required to break time symmetry in the control parameter, i.e., the energy required for “constancy” in the control clock [16, 29]. The final term \( D g^\theta \) can be thought of the energetic cost of straying from the optimal protocol (a geodesic [12]).
In equation (20) is an implicit energetic tradeoff. A control protocol that is very precise ($D \ll v$) pays a high energetic cost for strongly breaking time-reversal symmetry. However, a control protocol that is only weakly time-symmetry breaking ($D \gg v$), pays an energetic cost for undergoing suboptimal trajectories and performing redundant thermodynamic transitions. Ref. [3] investigated the energetically optimal control path $\lambda(t)$. This work shows that there is also the question of the energetically optimal “diffusive tuning” between $v$ and $D$ which minimizes this tradeoff. In particular, the optimal control protocol is not deterministic ($D \neq 0$).

As an example, consider a two-dimensional harmonic oscillator where the center is moved in a circle of radius $A$:

$$U(\theta, x) = \frac{1}{2}k(x - A\lambda)^2 \quad \lambda = (\cos \theta, \sin \theta)$$  \hspace{1cm} (21)

Here $\lambda(\theta) = (\cos \theta, \sin \theta)$ is the control function and $\phi(x) = -kAx$ is the conjugate force. The Fisher information metric is given by $g^\lambda = kA^2\mathbf{1}$ where $\mathbf{1}$ is the two-dimensional identity matrix and in the basis we have $g^\theta = \frac{d\lambda^x}{d\theta} g_{xx} \frac{d\lambda^x}{d\theta} + \frac{d\lambda^y}{d\theta} g_{yy} \frac{d\lambda^y}{d\theta} = kA^2$. Plugging these into (7), we find the average dissipation rate:

$$\langle S \rangle = \frac{v^2}{D}(1 + g^\theta D\tau) + Dg^\theta(1 - D\tau)$$  \hspace{1cm} (22)

If we require that the protocol takes an average of $\Delta t$ time per cycle, this fixes the net drift velocity $v = 2\pi/\Delta t$. To find the minimum dissipation, we then optimize $S$ with respect to $D$. This yields:

$$D^{(opt)} = \frac{v}{\sqrt{g^\theta}} \left(1 + \epsilon\right)$$  \hspace{1cm} (23)

where $|\epsilon| \ll 1$ is a small order correction due to the small $D\tau$ term. Plugging this in yields a total dissipation per cycle of:

$$\langle S \rangle \geq 2\mathcal{L} + \frac{\mathcal{L}^2}{\Delta t} \left(1 - \frac{(1 + \epsilon)^2}{g^\theta}\right)$$  \hspace{1cm} (24)

where $\mathcal{L} = 2\pi \sqrt{g^\theta}$ and $\bar{\mathcal{L}} = 2\pi \sqrt{g^\theta}\tau$ are the thermodynamic lengths of paths under the metrics $g^\lambda$ and $\hat{g}^\lambda$. In particular, we note that the dissipation remains bounded by $2\mathcal{L}$ in the limit of an infinitely long protocol $\Delta t \to \infty$.

In units where $\beta \neq 1$, the Fisher information metric is $g^\theta = kx_0^2\beta$. Thus the correction to the Sivak and Crooks bound can be neglected whenever the energy scale of the control is greater than the average thermal fluctuation.

The non-vanishing bound $2\mathcal{L}$ scales with $\sqrt{g^\lambda}$, the size of an average fluctuation in the system. Thus it is subextensive and disappears in the macroscopic limit. Its contribution is also dwarfed by thermodynamic friction when the control protocol is fast. Therefore it is expected that this bound should only become relevant in slow microscopic systems.

We also note that this analysis only applies to autonomous thermodynamic machines: those whose control is independent of environmental signals. In cases where the source of time-symmetry breaking occurs externally, e.g. a bacteria’s signaling network reacting to a time-varying external ligand concentration, this bound does not necessarily apply. This is because the origin of time-symmetry breaking in such networks is environmental and thus no energy must be expended to drive the system in a particular direction. Such reactive systems would be more appropriately characterized by Ref. [4], which likewise addressed the question of optimality and energetic bounds in stochastically controlled systems, only without taking into account the cost of breaking time symmetry. Those authors found a minimum bound on the energy of control which cannot be made arbitrarily small in the presence of a noisy control protocol. However, their bound is proportional to the magnitude of the control noise, and thus can be made arbitrarily small in the limit of noiseless protocols.

This work elucidates new constraints in the design of optimal thermodynamic machines. In previous studies, it has been found that systems must dissipate more energy to increase the accuracy of their output [17, 30]. The optimal diffusive tuning found here indicates that below a certain level of accuracy, increasing precision actually decreases the dissipation cost. In addition, the lower dissipation bound indicates that for very slow microscopic thermodynamic transformations, the dissipation cost no longer scales inversely with time. The diminishing energetic returns from increasing the length of control protocols perhaps sets a characteristic timescale for optimal microscopic machines without time constraints.

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Supplemental Material

Here we derive the main results in greater detail for the multidimensional case. The system’s energy is given by:

\[ U(y) = \lambda^\alpha(\theta) \phi_\alpha(x) \]  

(25)

Here \( \lambda, \phi \) are indexed by \( \alpha, \beta, \gamma \) and \( x \) is indexed by \( i, j, k \). We use natural units and unit temperature so that \( \beta = T = 1 \) and \( \mu = D = \gamma^{-1} \), where \( \mu \) is the mobility, \( D \) is the diffusion, and \( \gamma \) is the resistance. When \( \mu, D, \gamma \) and things like probability currents \( j \) appear without indices, they refer to \( \theta \). When they appear with roman indices \( (i, j, k) \), they refer to \( x \). The Langevin equations are:

\[ \dot{\theta} = v + \sqrt{2D} \eta \quad \dot{x}^j = D^{ij} F_j + \sqrt{2D^{ij}} \eta^j \]

\[ \langle \eta(t)\eta(t') \rangle = \delta(t - t') \quad \langle \eta^j(t)\eta^j(t') \rangle = \delta_{ij}\delta(t - t') \]  

(26)

where the \( \eta \) terms are white noise functions. The probability currents are given by:

\[ j = [v - D\partial_\theta] p = p D [\gamma v - \partial_\theta \log p] \]

\[ j^i = D^{ij} [F_j - \partial_j] p = p D^{ij} [F_j - \partial_j \log p] \]  

(27)

The total entropy production rate is the sum of the production rate for the two individual variables:

\[ \langle \dot{S} \rangle = \langle \dot{S}^x + \dot{S}^\theta \rangle = \int dy \left[ \frac{\gamma^2 \eta^j \partial_j}{p} + \frac{j^j \gamma^j}{p} \right] \]  

(28)

Starting with the \( \theta \) contribution:

\[ \langle \dot{S}^\theta \rangle = \int dy \frac{j^j \gamma_j}{p} = \int dy (v\gamma - \partial_\theta \log p) j \]

\[ = v\gamma \int dy j - \int dy j \partial_\theta \log p \]  

(29)

\[ = v^2 \gamma - v \int \int dy p \partial_\theta \log p - \int \int dy j \partial_\theta \log p \]

Since \( p \partial_\theta \log p = \partial_\theta p \) and \( \theta \) is periodic, the middle term disappears leaving:

\[ \langle \dot{S}^\theta \rangle = \frac{v^2}{D} - \int \int dy j \partial_\theta \log p \]  

(30)

Next we consider the \( x \) contribution:

\[ \langle \dot{S}^x \rangle = \int \int dy \frac{j^j \gamma_j}{p} = \int \int dy j^j (F_j - \partial_i \log p) \]

\[ = -\int \int \int dy j^i \partial_i U - \int \int dy j^i \partial_i \log p \]  

(31)

Giving us:

\[ \langle \dot{S} \rangle = \frac{v^2}{D} - \int \int \int dy [(j \partial_\theta + j^i \partial_i) \log p] \]  

(32)

We can rewrite \( U = F - \log p_{eq} \) where \( F(\theta) \) is the free energy for the system for a fixed \( \theta \) and \( p_{eq}(x|\theta) = e^{F(\theta) - U(x, \theta)} \) is the equilibrium Boltzmann distribution for fixed \( \theta \). Then \( \partial_i U = -\partial_\theta \log p_{eq} \) since \( F \) is a function of \( \theta \) only:

\[ \langle \dot{S} \rangle = \frac{v^2}{D} - \int \int \int dy [(j \partial_\theta + j^i \partial_i) \log p - (\partial_i j^i) \log p_{eq}] \]  

(33)

Now we may integrate by parts on each of the terms in the integral:

\[ \langle \dot{S} \rangle = \frac{v^2}{D} + \int \int \int dy [(\partial_\theta j + j \partial_i j^i) \log p - (\partial_i j^i) \log p_{eq}] \]  

(34)

where the boundary terms disappear because \( p(y, t) \) goes to zero at the boundaries of \( x \).

In the steady-state the Fokker-Plank equation yields \( \partial_\theta j + j \partial_i j^i = -\partial_t p = 0 \), allowing us to remove the first term in the integral and swap \( \partial_\theta j^i = -\partial_\theta j^i \) in the second:

\[ \langle \dot{S} \rangle = \frac{v^2}{D} + \int \int dy (\partial_\theta j^i) \log p_{eq} \]  

(35)
By expanding $\partial_\theta j^\theta$ and log $p_{eq}$ we get:

$$\langle \dot{S} \rangle = \frac{v^2}{D} + \int dy (v \partial_\theta p - D \partial_\theta ^2 p)(F - U)$$

(36)

Now we integrate by parts to move the $\partial_\theta$, $\partial_\theta ^2$ to the other term. Since $\theta$ is periodic, we can always neglect the boundary terms:

$$\langle \dot{S} \rangle = \frac{v^2}{D} - \int dy p \left[ v \partial_\theta + D \partial_\theta ^2 \right] [F - U]$$

(37)

We now need to compute the derivatives of $F$ and $U$ with respect to $\theta$:

$$\partial_\theta U(\theta, x) = \frac{\partial U}{\partial \lambda^\alpha} \frac{\partial \lambda^\alpha}{\partial \theta} = \phi_\alpha(x) \frac{\partial \lambda^\alpha}{\partial \theta}(\theta)$$

(38)

$$\partial_\theta ^2 U(\theta, x) = \phi_\alpha(x) \frac{\partial ^2 \lambda^\alpha}{\partial \theta ^2}(\theta)$$

(39)

$$\partial_\theta F(\theta) = \frac{\partial F}{\partial \lambda^\alpha} \frac{\partial \lambda^\alpha}{\partial \theta} = \langle \phi_\alpha \rangle_{eq, \theta} \frac{\partial \lambda^\alpha}{\partial \theta}(\theta)$$

(40)

$$\partial_\theta ^2 F(\theta) = \langle \phi_\alpha \rangle_{eq, \theta} \frac{\partial ^2 \lambda^\alpha}{\partial \theta ^2} - g_{\alpha \beta} \frac{\partial \lambda^\alpha}{\partial \theta} \frac{\partial \lambda^\beta}{\partial \theta}$$

(41)

Plugging these in gives:

$$\dot{S}(y) = \frac{v^2}{D} + \left[ v \frac{\partial \lambda^\alpha}{\partial \theta} + D \frac{\partial ^2 \lambda^\alpha}{\partial \theta ^2} \right] \delta \phi_\alpha + Dg^\theta \frac{\partial \lambda^\alpha}{\partial \theta} \frac{\partial \lambda^\beta}{\partial \theta}$$

(42)

Where $\delta \phi_\alpha = \phi_\alpha(x) - \langle \phi_\alpha \rangle_{eq, \theta}$. If we identify $g^\theta \equiv \frac{\partial \lambda^\alpha}{\partial \theta} \frac{\partial \lambda^\beta}{\partial \theta}$ as the Fisher information metric on $\theta$ inherited from $\lambda$, we get:

$$\dot{S}(y) = \frac{v^2}{D} + \left[ v \frac{\partial \lambda^\alpha}{\partial \theta} + D \frac{\partial ^2 \lambda^\alpha}{\partial \theta ^2} \right] \delta \phi_\alpha + Dg^\theta$$

(43)

To find the average dissipation rate when the system is at the point $\theta$, we average over all $x$ with weight $p(x|\theta)$. This immediately yields the multidimensional analog of (14):

$$\langle \dot{S} \rangle_\theta = \frac{v^2}{D} + Dg^\theta + \left[ v \frac{\partial \lambda^\alpha}{\partial \theta} + D \frac{\partial ^2 \lambda^\alpha}{\partial \theta ^2} \right] \langle \delta \phi_\alpha \rangle_\theta$$

Again, this is an exact expression. To go further we use a linear response approximation.

### Linear Response Approximation

Our starting point is the following expression [27] which gives the average linear response of $\phi$ to a specific control trajectory $\theta(t)$:

$$\langle \delta \phi_\alpha (t) \rangle_\theta \approx \int _{-\infty} ^0 \text{d}t' \left[ \frac{dC^\theta_{\alpha \beta}}{dt'} \right] \lambda(t)$$

(45)

with:

$$C^\theta_{\alpha \beta} (t) = \langle \delta \phi_\alpha (0) \delta \phi_\beta (t) \rangle_{eq, \theta}$$

(46)

This is the linear response function to a single path. To get the expression $\langle \delta \phi_\alpha \rangle_\theta$ we have to average this expression over all possible control paths $\theta(t)$ at time $t$. The important point is that only $\lambda(t + t')$ is trajectory dependent, the other parts of the expression only depend on the value of the trajectory at the moment $t$. Therefore we may write:

$$\langle \delta \phi_\alpha \rangle_\theta = \int _{-\infty} ^0 \text{d}t' \left[ \frac{dC^\theta_{\alpha \beta}}{dt'} \right] \lambda^\beta (t)$$

(47)

where here $\langle \cdots | \theta, t \rangle$ represents an average over all possible control paths $\theta(t)$ such that $\theta(t) = \theta$ weighted by their probability. By integrating by parts we are left with:

$$\langle \delta \phi_\alpha \rangle_\theta = \int _{-\infty} ^0 \text{d}t' C^\theta_{\alpha \beta} (t') \frac{d}{dt'} \lambda^\beta (t)$$

(48)

We can write the expectation value of $\lambda^\beta$ as:

$$\langle \lambda^\beta (t + t') | \theta, t \rangle = \int _{-\infty} ^\infty \text{d}t'' p(\theta + \theta', t + t' | \theta, t)$$

(49)

Note that despite the fact that $\theta$ is periodic, the integration bounds here are not. This is because $\theta' = 2\pi$ in this context refers to the control parameter making a full cycle in time $t'$ which is not the same as it not moving ($\theta' =$ 0).

Since the stochasticity of $\theta$ is driven by Gaussian noise, it’s trivial to write down $p$:

$$p(\theta + \theta', t + t' | \theta, t) = \frac{1}{\sqrt{4\pi D |t'|}} e^{-\frac{(\theta' - \theta_t)^2}{4 D |t'|}}$$

(50)

We also Taylor expand $\lambda^\beta (\theta + \theta')$ about $\theta$:

$$\lambda^\beta (\theta + \theta') = \lambda^\beta (\theta) + \theta' \partial_\theta \lambda^\beta (\theta) + \frac{1}{2!}(\theta')^2 \partial_\theta ^2 \lambda^\beta (\theta) + \cdots$$

(51)

Putting these together gives:

$$\langle \lambda^\beta (t + t') | \theta, t \rangle = \sum _{k=0} \frac{\partial_\theta ^k \lambda^\beta (\theta)}{k! \sqrt{4\pi D |t'|}} \int \text{d}t'' e^{-\frac{\theta'^2 - \theta t'}{2 D |t'|}}$$

(52)

Solving this integral for each value of $k$ will yield terms of the form:

$$(D |t'|)^m (v t')^n \partial_\theta ^{2m+n} \lambda^\beta (\theta)$$

(53)

Only keeping terms of order $m + n = 1$ yields:

$$\langle \lambda^\beta (t + t') | \theta, t \rangle = \frac{\lambda^\beta (t) v t'}{D |t'|} + \mathcal{O}(2)$$

(54)

We can see that to leading order, the time-derivative of this expression will be time-independent. Thus we can
We expect roughly that:

\[ \langle \frac{d\lambda^\alpha}{dt} \rangle_{\theta} = \lim_{\nu \to 0} \frac{d}{dt} \langle \lambda^\beta (t + t') | \theta, t \rangle \approx \nu \partial_{\theta} \lambda^\beta \pm D \partial^2_{\theta} \lambda^\beta \]  

(55)

As one may expect, this is exactly the formula for the net drift of \( \lambda^\alpha \) at \( t' = 0 \) given by the Ito formula under the Ito (+) and reverse-Ito (−) conventions [28].

Plugging this result into (48) and using the definition of thermodynamic friction:

\[ \dot{g}_{\alpha \beta}^\lambda = \tau_{\alpha \beta} \circ \dot{g}_{\alpha \beta}^\lambda = \int_{-\infty}^{0} dt' C^\theta_{\alpha \beta} (t') \]  

(56)

yields:

\[ \langle \delta \phi_{\alpha} \rangle_{\theta} = \dot{g}_{\alpha \beta}^\lambda \left( \frac{d\lambda^\beta}{dt} \right)_{\theta} \]  

(57)

which in turn gives an average entropy production rate:

\[ \langle \dot{S} \rangle_{\theta} = \left( \frac{d\lambda^\alpha}{dt} \right)_{\theta} \dot{g}_{\alpha \beta}^\lambda (\theta) \left( \frac{d\lambda^\beta}{dt} \right)_{\theta} - \frac{v^2}{D} + D g^\theta (\theta) \]  

(58)

What about the higher order terms in (54)? We can neglect them under the assumption that the speed of control is small compared to the excitation timescale \( \tau \) of the system at equilibrium. However, an explicit mathematical statement of this requirement is challenging because of the unknown form of \( \lambda (\theta) \). Since \( C \) is a decay function, we expect roughly that:

\[ \int_{-\infty}^{0} dt' (t')^k C^\theta_{\alpha \beta} (t') \sim g_{\alpha \beta}^\lambda \tau^{k+1} \]  

(59)

Thus in keeping terms of order \( m + n > 1 \) we would generate additional contributions to (57) of the form:

\[ (D\tau)^m (v\tau)^n \left[ g_{\alpha \beta}^\lambda \partial^2_{\theta} \lambda^\beta (\theta) \right] \]

(60)

Comparing these to the leading order terms \( (D\tau)g_{\alpha \beta}^\lambda \partial^2_{\theta} \lambda^\beta \) and \( (v\tau)g_{\alpha \beta}^\lambda \partial_t \lambda^\beta \), we can see that for a reasonably behaved function \( \lambda (\theta) \) with a characteristic length scale \( L \) we can summarize our assumption via the requirements:

\[ v\tau / L \ll 1 \quad D\tau / L^2 \ll 1 \]  

(61)

Essentially this is the requirement that \( \frac{d\lambda^\alpha}{dt} (t) \) (under the reverse-Ito convention) remains relatively constant over the system relaxation timescale \( \tau \). This is the natural stochastic generalization of the constraint imposed by [3].

However, because we have not explicitly given a definition for \( L \), this requirement is admittedly a little vague. One major complication in doing so arises from the fact that because we only care about the total dissipation bounds, we only want to keep the terms in (58) that contribute to the leading order behavior of the integral of (58). Thus, while it may be the case that for a specific point \( \theta_0 \), the second order term dominates: \( (v\tau)^2 \partial^2_{\theta} \lambda (\theta_0) \gg (v\tau) \partial_t \lambda (\theta_0) \), we still want to drop the higher order term because its contribution to the total integral is subleading. The actual formal constraints dictating when this approximation is appropriate is further complicated by the unconstrained behavior of \( \dot{g} (\theta) \) and \( \lambda (\theta) \). However, it should be clear that as we approach equilibrium behavior \( (D, v \to 0) \), the kept terms dominate over the dropped terms. We feel that the constraint given in (61) satisfactorily captures this idea.