Local strong solution for the viscous compressible and heat-conductive fluids with vacuum in 2D space

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Abstract

This paper considers the Cauchy problem of equations for the viscous compressible and heat-conductive fluids in the two-dimensional (2D) space. We establish the local existence theory of unique strong solution under some initial layer compatibility conditions. The initial data can be arbitrarily large, the initial density is allowed to vanish in any set and the far field state is assumed to be vacuum.

1 Introduction

The motion of a compressible viscous, heat-conducting, isotropic Newtonian fluid in the two-dimensional (2D) space is governed by the following system of equations (cf. [9, 23])

$$
\begin{cases}
\rho_t + \text{div} (\rho u) = 0, \\
(\rho u)_t + \text{div} (\rho u \otimes u) + \nabla P = \mu \triangle u + (\mu + \lambda) \nabla \text{div} u, \\
c_v [(\rho \theta)_t + \text{div} (\rho u \theta)] + P \text{div} u = \kappa \triangle \theta + \frac{\mu}{2} |\nabla u + (\nabla u)^{\text{tr}}|^2 + \lambda (\text{div} u)^2,
\end{cases}
$$

(1.1)

where $x \in \mathbb{R}^2, t > 0$, the unknown functions $\rho(x,t), u(x,t)$ and $\theta(x,t)$ denote the density, velocity and absolutely temperature, separately. The viscosity coefficients $\mu$ and $\lambda$ satisfy the physical requirements $\mu > 0$ and $\mu + \lambda \geq 0$; $\kappa > 0$ is the heat-conduction coefficient, and $c_v > 0$ denotes the heat capacity of the gas at constant volume.

In this paper, we focus on the polytropic fluids so that the pressure

$$P = R \rho \theta, \quad (R > 0).$$

(1.2)

We aim to develop a local existence result of strong solutions to the Cauchy problem of Eqs. (1.1) with the far field behavior

$$(u, \theta)(x, \cdot) \to 0, \quad \text{as } |x| \to \infty$$

(1.3)

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and the initial functions
\[(\rho, u, \theta)(x, t = 0) = (\rho_0, u_0, \theta_0)(x), \quad x \in \mathbb{R}^2. \quad (1.4)\]

As one of the most important systems in continuum mechanics, there is a vast literature studying the existence of solutions for Eqs. (1.1) with various boundary value conditions. When the density leaves vacuum, the global solutions and the large time behavior in 1D case has been well-studied extensively by many mathematicians, see [12,15,26,27] and the references therein. For high dimensions, the local existence of classical solutions was obtained by Nash [14] and Serrin [28], respectively; Matsumura-Nishida [24] showed the global existence of solutions in case of the initial data has a small disturbance around a non-vacuum equilibrium. If the initial density need not be positive and may vanish in open set, the global large weak solutions was first addressed by Li-ons [23] for isentropic flow when the adiabatic exponent \(\gamma\) is large. Later, the restriction on \(\gamma\) was relaxed by Feireisl-Novotny-Petzeltová [10]. If the temperature function is involved, we refer to [7] by Bresch-Desjardins and [9] by Feireisl for the global existence of weak solutions under different technique assumptions.

When it comes to the strong/classical solutions in the presence of vacuum, we refer to the papers [3–5] by Kim-Cho-Choe. In particular, for bounded or unbounded 3D domains, they obtained the local existence and the uniqueness of Eqs. (1.1) by imposing some initial compatibility conditions to remedy the degeneracy in time evolution in momentum equations (or energy equation), and derive the estimate for \(u_t\) (or \(\theta_t\)) in terms of \(\|\nabla u_t\|_{L^2}\) (or \(\|\nabla \theta_t\|_{L^2}\)) and Sobolev embedding inequalities. Huang-Li-Xin [16] proved that if the initial energy is small, the classical solution of the 3D Cauchy problem for the barotropic compressible flow exists globally in time.

However, the method developed in [3–5] fails to deal with the existence of strong/classical solutions in unbounded 2D domains. The reason is that the dimension two is critical and thus, it is hard to bound the \(L^p\)-norm of \(u\) (or \(\theta\)) just in terms of the \(L^2\)-norm of the gradient of it. Recently, Li-Liang [18] proved the existence of unique strong and classical solution to the 2D Cauchy problem for the barotropic compressible flow. See also the paper [20] for the incompressible case. In [18,20] we assume that the initial density has a quite decay when the spatial variables tends to infinity, and thereby, we estimate the momentum \(\rho u\) instead of velocity \(u\) itself. The key tool in the proof is the \(\mathcal{H}\)ardy type and Poincaré type inequalities.

For the fully compressible Navier-Stokes Eqs. (1.1), as far as we know, the existence of strong solution for 2D Cauchy problem is not available up to the publication. In this paper, we want to establish the local existence and uniqueness of strong solution to Eqs. (1.1) with the structural conditions (1.2)-(1.4). The theorem below states our main result

**Theorem 1.1** Define
\[\bar{x} = (\varepsilon + |x|^2)^{1/2} \ln^{1+\eta_0}(\varepsilon + |x|^2)\]  
for some small \(\eta_0 > 0\). Suppose the initial functions in (1.4) satisfy
\[0 \leq \bar{x}^p \rho_0 \in L^1(\mathbb{R}^2) \cap W^{1,q}(\mathbb{R}^2), \quad \nabla \sqrt{\bar{x}^p \rho} \in L^2(\mathbb{R}^2),\]
\[\sqrt{\rho_0} u_0 \in L^2(\mathbb{R}^2), \quad \bar{x}^{\frac{p}{2}} \nabla u_0 \in L^2(\mathbb{R}^2), \quad \nabla^2 u_0 \in L^2(\mathbb{R}^2),\]
\[\sqrt{\rho_0} \theta_0 \in L^2(\mathbb{R}^2), \quad \nabla \theta_0 \in H^1(\mathbb{R}^2)\]  
(1.6)
for numbers $a \in (1, 2)$, $b \in (0, \frac{4}{3})$ and $q \in (2, \infty)$. Suppose in addition that the initial compatibility conditions

\[
- \mu \Delta u_0 - (\mu + \lambda) \nabla \text{div} u_0 = \sqrt{\rho_0} g_1 \tag{1.7}
\]

and

\[
- \kappa \Delta \theta_0 - \frac{\mu}{2} |\nabla u_0 + (\nabla u_0)^{tr}|^2 - \lambda (\text{div} u_0)^2 = \sqrt{\rho_0} g_2 \tag{1.8}
\]

hold true for some $g_1, g_2 \in L^2(\mathbb{R}^2)$.

Then there is a small $T_*>0$, such that the Cauchy problem (1.1)-(1.4) has a unique strong solution $(\rho, u, \theta)$ over $[0, T_*) \times \mathbb{R}^2$, with properties

\[
\begin{align*}
\bar{\rho} \rho &\in L^\infty (0, T_*; L^1(\mathbb{R}^2) \cap W^{1,q}(\mathbb{R}^2)), \quad \nabla \sqrt{\bar{\rho}} \rho \in L^\infty (0, T_*; L^2(\mathbb{R}^2)), \\
\rho &\in C ([0, T_*]; L^1(\mathbb{R}^2) \cap H^1(\mathbb{R}^2) \cap W^{1,q}(\mathbb{R}^2)), \\
\sqrt{\rho} u, \sqrt{\bar{\rho}} u &\in L^\infty (0, T_*; L^2(\mathbb{R}^2)), \\
\bar{\rho} \nabla u &\in L^\infty (0, T_*; L^2(\mathbb{R}^2)), \\
\rho \nabla u |\nabla u| &\in L^\infty (0, T_*; L^2(\mathbb{R}^2)), \\
\nabla^2 u &\in L^2 (0, T_*; L^2(\mathbb{R}^2)) \cap L^2 (0, T_*; L^q(\mathbb{R}^2)), \\
\nabla \theta, \nabla \phi &\in L^2 (0, T_*; L^2(\mathbb{R}^2)),
\end{align*}
\]

where $\dot{f} = f_t + u \cdot \nabla f$ is the material derivative of $f$. Moreover,

\[
\inf_{t \in [0, T_*]} \left\| \rho(\cdot, t) \right\|_{L^1(B_{R^*})} \geq \frac{1}{4} \left\| \rho_0 \right\|_{L^1(\mathbb{R}^2)},
\]

with $B_{R^*} \triangleq \{ x \in \mathbb{R}^2 : |x| < R^* \}$ and $R^*$ being a large number.

Some remarks are in order:

**Remark 1.1** In contrast with the isentropic flows (See [18]), the initial compatibility conditions (1.7)-(1.8) can not be removed even we are seeking for strong solutions to Eqs.(1.1).

**Remark 1.2** Although we are focused on the perfect gas, the proof applies to more general state of equations

\[
P = \rho \frac{\partial e}{\partial \rho} + \theta \frac{\partial P}{\partial \theta}, \quad e = e(\rho, \theta),
\]

after some additional assumptions and slight modifications.

We are motivated by the papers [4-5] by Kim-Cho-Choe, as well as our previous works [18][20][21]. Our approach is to construct the approximate solutions to Eqs.(1.1) with positive density in any bounded balls $B_R$, and then by the domain expansion technique. As already mentioned, the arguments in [4-5] fail for unbounded 2D domains when the far field density is vacuum. In the light of [18][20], we assume a weighted initial density so that we can bound the weighted $L^p$-norm of velocity $u$ in terms of $|\nabla u|_{L^2}$ and $|\sqrt{\rho} u|_{L^2}$. However, the involvement of energy equation, especially the quadratic nonlinear term of which, makes the problem much complicated and thus, the
proof in [13][20] can not be generalized in a parallel. For example, if we first multiply Eq. (1.1) by \( \theta \) and obtain
\[
\frac{d}{dt} \int \rho \theta^2 + \int |\nabla \theta|^2 \leq C \int \theta |\nabla u|^2 + \text{other terms},
\]
we should impose additionally a weight on \( \nabla u \) (or else, on \( \theta \)) to deal with \( \int \theta |\nabla u|^2 \).
Next, if multiplied by \( \hat{\theta} \) in Eq. (1.1), it gives
\[
\|\nabla \theta\|_{L^2}^2 + \int_0^t \|\sqrt{\rho \hat{\theta}}\|_{L^2}^2
\leq C \|\sqrt{\rho \hat{u}}\|_{L^2}^2 + \int_0^t \|\nabla \theta\|_{L^2}^2 + \int_0^t \ldots \|\nabla \theta\|_{L^2}^2 + \ldots \quad \text{(see (3.14))}
\]
But when we multiply Eqs. (1.1) by \( \dot{u} \), it turns out that
\[
\|\sqrt{\rho \dot{u}}\|_{L^2}^2 + \int_0^t \|\nabla \dot{u}\|_{L^2}^2
\leq C \|\rho\|_{L^\infty} \int_0^t \|\sqrt{\rho \dot{\theta}}\|_{L^2}^2 + \int_0^t \ldots \|\nabla \theta\|_{L^2}^2 + \|\sqrt{\rho \dot{u}}\|_{L^2}^2 + \ldots \quad \text{(see (3.22))},
\]
that is, \( \|\sqrt{\rho \dot{u}}\|_{L^2} \) depends on \( \|\nabla \theta\|_{L^2} \) and vice versa. Fortunately, the strong coupling of energy and velocity fields is at a different level. With such observation, we follow the Hoff’s work in [13] and start with the basic energy estimates on both the (weighted) velocity and the temperature, as well as their material derivatives. Finally, combining such ideas with those due to [4][5][18][21], we achieve the desired \( a \ priori \) bounds on approximate solutions and the weighted density and the gradient of velocity, where all these bounds are independent of either the size of \( B_R \) or the lower bound of density.

The rest sections 2-4 are as follows: In Section 2, some useful lemmas are displayed, the Section 3 is devoted to obtaining the needed \( a \ priori \) estimates; and in the final Section 4, we complete the proof of Theorem 1.1.

2 Preliminaries

We begin with the local existence results of Eqs. (1.1) in bounded balls \( B_R \triangleq \{ x \in \mathbb{R}^2 : |x| < R \} \) with strictly positive initial density. The proof is similar to that in the paper [5] by Cho-Kim.

**Lemma 2.1** Assume that
\[
\rho_0 \in H^2(B_R), \quad \inf_{x \in B_R} \rho_0(x) > 0, \quad u_0, \theta_0 \in H^1_0 \cap H^2(B_R).
\]
Then the Eqs. (1.1) with the boundary conditions
\[
\begin{cases}
(\rho, u, \theta)(x, t = 0) = (\rho_0, u_0, \theta_0)(x), & x \in B_R, \\
u(x, t) = 0, & x \in \partial B_R, t > 0, \\
\theta(x, t) = 0, & x \in \partial B_R, t > 0
\end{cases}
\]
has a unique strong solution over \( B_R \times [0, T_R] \) for some small \( T_R > 0 \), satisfying
\[
\begin{cases}
\rho \in C \left( [0, T_R]; H^2 \right), \quad \rho(x, t) > 0 \text{ on } \overline{B_R} \times [0, T_R], \\
u, \theta \in C \left( [0, T_R]; H^1_0 \cap H^2 \right) \cap L^2 \left( 0, T_R; H^1 \right), \\
u_t, \theta_t \in L^\infty \left( 0, T_R; L^2 \right) \cap L^2 \left( 0, T_R; H^1_0 \right).
\end{cases}
\]


The following Gagliardo-Nirenberg inequalities will be used frequently throughout this paper.

**Lemma 2.2** [11, 17] Let $\Omega \subseteq \mathbb{R}^N$ be a bounded or unbounded domain with piecewise smooth boundaries. It holds that for any $v \in W^{1,q}(\Omega) \cap L^r(\Omega)$

$$\|v\|_{L^p(\Omega)} \leq C_1\|v\|_{L^r(\Omega)} + C_2\|\nabla v\|_{L^q(\Omega)}^{1-\gamma}\|v\|^{\gamma}_{L^r(\Omega)},$$  \hspace{1cm} (2.4)

where the constants $C_i$ ($i = 1, 2$) depend only on $p, q, r, \gamma$; and the exponents $0 \leq \gamma \leq 1$, $1 \leq q, r \leq \infty$ satisfy $\frac{1}{p} = \gamma\left(\frac{1}{q} - \frac{1}{N}\right) + (1 - \gamma)\frac{1}{r}$ and

$$\begin{cases} \min\{r, \frac{Nq}{N-q}\} \leq p \leq \max\{r, \frac{Nq}{N-q}\}, & \text{if } q < N; \\ r \leq p < \infty, & \text{if } q = N; \\ r \leq p \leq \infty, & \text{if } q > N. \end{cases}$$

Moreover, $C_1 = 0$ in case of $|\partial\Omega| = 0$ or $\int_{\Omega} v dx = 0$.

As an application of (2.4) for either $\Omega = \mathbb{R}^2$ or $\Omega = B_R$, we have

$$\|v\|_{L^p(\Omega)} \leq C\|v\|_{L^2(\Omega)}^2\|\nabla v\|_{H^1(\Omega)}^{p-2}, \quad \forall \ p \in [2, \infty),$$  \hspace{1cm} (2.5)

and

$$\|v\|_{L^\infty(\Omega)} \leq C\|v\|_{L^2(\Omega)} + C\|v\|_{L^2(\Omega)}^{\frac{q-2}{2}}\|\nabla v\|_{L^q(\Omega)}^{\frac{q}{2}}, \quad \forall \ q \in (2, \infty),$$  \hspace{1cm} (2.6)

provided the quantities of right-hand side are finite.

The next lemma provides an estimate on weighted $L^p$-norm for elements of the Hilbert space $\tilde{D}^{1,2}(\Omega) \triangleq \{v \in H^1_0(\Omega) : \nabla v \in L^2(\Omega)\}$ with either $\Omega = \mathbb{R}^2$ or $\Omega = B_R$ ($R \geq 1$).

**Lemma 2.3** For given numbers $m \in [2, \infty)$ and $\theta \in (2, \infty)$, there exists a constant $C$ such that for any $v \in \tilde{D}^{1,2}(\Omega)$

$$(\int_{\Omega} \frac{|v|^m}{(e + |x|^2)\ln^\theta(e + |x|^2)} dx)^{1/m} \leq C \left(\|v\|_{L^2(B_1)} + \|\nabla v\|_{L^2(\Omega)}\right).$$  \hspace{1cm} (2.7)

**Proof.** In [22] Theorem B.1, Lions originally proved (2.7) if $\theta \in (1 + \frac{m}{2}, \infty)$. Here we present an alternative approach and show that (2.7) is in fact valid for all $\theta \in (2, \infty)$.

Express the left-hand side of (2.7) as

$$I_1 + I_2 \triangleq \int_{\mathbb{R}^2} \frac{|v_{B_1}|^m}{(e + |x|^2)\ln^\theta(e + |x|^2)} + \int_{\mathbb{R}^2} \frac{|v - v_{B_1}|^m}{(e + |x|^2)\ln^\theta(e + |x|^2)},$$

where $u_{B_1} = \frac{1}{|B_1|}\int_{B_1} v(x) dx$ and $|B_1|$ symbols the measure of disc $B_1$ with radius $r$.

First, it satisfies for $\theta > 1$

$$I_1 \leq |v_{B_1}|^m \int_{\mathbb{R}^2} \frac{1}{(e + |x|^2)\ln^\theta(e + |x|^2)} \leq C\|v\|_{L^1(B_1)}^m.$$  \hspace{1cm} (2.8)

Next, putting $T_k = B_{2^k} \setminus B_{2^k-1}$, we decompose

$$I_2 = \left(\int_{B_1} + \sum_{k=1}^{\infty} \int_{T_k}\right) \frac{|v - v_{B_1}|^m}{(e + |x|^2)\ln^\theta(e + |x|^2)},$$  \hspace{1cm} (2.9)
By the definition of $BMO_m$ space ([25, Chapter IV]), one has for all $m \in [1, \infty)$
\[
\int_{B_1} \frac{|v - v_{B_1}|^m}{(e + |x|^2) \ln^\theta (e + |x|^2)} \leq C \int_{B_1} |v - v_{B_1}|^m \leq C\|v\|_{BMO_m(\mathbb{R}^2)}^m. \tag{2.10}
\]

Let us deal with the remainder terms of (2.9). Since $(e + |x|^2) \ln^\theta (e + |x|^2) \geq 4^{(k-1)} \ln^\theta (e + 4^{k-1})$ for all $x \in T_k$. Then,
\[
\int_{T_k} \frac{|v - v_{B_1}|^m}{(e + |x|^2) \ln^\theta (e + |x|^2)} \leq \frac{1}{4^{(k-1)} \ln^\theta (e + 4^{k-1})} \int_{B_{2k}} |v - v_{B_1}|^m. \tag{2.11}
\]

On the other hand,
\[
\int_{B_{2k}} |v - v_{B_1}|^m \leq \int_{B_{2k}} |v - v_{B_{2k}}|^m + \int_{B_{2k}} |v_{B_{2k}} - v_{B_1}|^m
\]
\[
\leq |B_{2k}| \left( \|v\|_{BMO_m(\mathbb{R}^2)}^m + \sum_{j=1}^k |v_{B_{2k}} - v_{B_{2k-1}}|^m \right) \tag{2.12}
\]
\[
\leq |B_{2k}|(1 + 4k)\|v\|_{BMO_m(\mathbb{R}^2)}^m,
\]
where the last inequality comes from
\[
|v_{B_{2k}} - v_{B_{2k-1}}|^m = \left| \frac{1}{|B_{2k-1}|} \int_{B_{2k-1}} (v_{B_{2k}} - v) \right|^m
\]
\[
\leq \frac{1}{|B_{2k-1}|} \int_{B_{2k-1}} |v_{B_{2k}} - v_{B_{2k-1}}|^m \quad \text{(Hölder inequality)}
\]
\[
= \frac{4}{|B_{2k}|} \int_{B_{2k}} |v_{B_{2k}} - v_{B_{2k-1}}|^m \leq 4\|v\|_{BMO_m(\mathbb{R}^2)}^m.
\]

The combination of (2.12) and (2.11) guarantees
\[
\int_{T_k} \frac{|v - v_{B_1}|^m}{(e + |x|^2) \ln^\theta (e + |x|^2)} \leq c_k\|v\|_{BMO_m(\mathbb{R}^2)}^m, \tag{2.13}
\]
where $c_k = \frac{|B_{2k}|(1 + 4k)}{4^{k-1} \ln (e + 4^{k-1})}$ satisfies
\[
c_k \leq \begin{cases} 
20\pi, & k = 1, \\
\frac{64\pi}{(k-1)^{\theta-1}}, & k \geq 2.
\end{cases}
\]

In conclusion, we have
\[
\sum_{k=1}^\infty \int_{T_k} \frac{|v - v_{B_1}|^m}{(e + |x|^2) \ln^\theta (e + |x|^2)} \leq \|v\|_{BMO_m(\mathbb{R}^2)}^m \sum_{k=1}^\infty c_k \leq C\|v\|_{BMO_m(\mathbb{R}^2)}^m,
\]
where we have used $\sum_{k=1}^\infty c_k < \infty$ because of $\theta > 2$. This and inequalities (2.8)-(2.10) ensure that
\[
\int_{\mathbb{R}^2} \frac{|v|^m}{(e + |x|^2) \ln^\theta (e + |x|^2)} dx \leq C\|v\|_{BMO_m(\mathbb{R}^2)}^m + C\|v\|_{L^1(B_1)}^m. \tag{2.13}
\]

On the other hand, Utilizing John-Nirenberg Lemma ([25 Page 246]) and Poincaré inequality give rise to
\[
\|v\|_{BMO_m(\mathbb{R}^2)} \leq C\|v\|_{BMO(\mathbb{R}^2)} \leq C\|v\|_{L^2(\mathbb{R}^2)}^m,
\]
which together with (2.13) give birth to the desired (2.7).
Remark 2.1 For the particular case $m = 2$, we may choose $\theta = 2$ in (2.7). Indeed,

$$2 \int_{\mathbb{R}^2} \frac{(1 - \phi^4)v^2}{|x|^2 \ln |x|^2} = 2 \int_{\mathbb{R}^2} \frac{(1 - \phi^4)(x \cdot \nabla)v \cdot v}{|x|^2 \ln |x|^2} - \int_{\mathbb{R}^2} \frac{x \cdot \nabla \phi^4v^2}{|x|^2 \ln |x|^2}$$

$$\leq \int_{\mathbb{R}^2} \frac{(1 - \phi^4)v^2}{|x|^2 \ln |x|^2} + C\|\nabla v\|_{L^2(\mathbb{R}^2)}^2 + C \int_{\{2 \leq |x| \leq 4\}} v^2,$$

where the cut-off function $\phi^R$ is defined in (3.5). However, whether $\theta = 2$ holds true or not for $m > 2$ is not clear.

An important usage of Lemma 2.3 is the inequalities (2.15) and (2.16) below, which plays a critical role in our analysis.

Lemma 2.4 Let $\eta_0, \bar{x}$ be as in (1.3), and $\Omega = \mathbb{R}^2$ or $\Omega = B_R (R \geq 1)$. Suppose a nonnegative function $\rho \in L^1(\Omega) \cap L^\infty(\Omega)$ such that

$$\int_{B_{N_1}} \rho dx \geq M > 0, \quad N_1 \in [1, R).$$

(2.14)

Then it satisfies that

$$\|v\bar{x}^{-1}\|_{L^2(\Omega)} \leq C \left( \|\sqrt{\rho}v\|_{L^2(\Omega)} + (1 + \|\rho\|_{L^\infty(\Omega)}) \|\nabla v\|_{L^2(\Omega)} \right)$$

(2.15)

and that, for any $\eta \in (0, 1]$

$$\|v\bar{x}^{-\eta}\|_{L^2(\Omega)} \leq C \left( \|\sqrt{\rho}v\|_{L^2(\Omega)} + (1 + \|\rho\|_{L^\infty(\Omega)}) \|\nabla v\|_{L^2(\Omega)} \right),$$

(2.16)

where the constant $C$ depends only on $M, N_1, \eta_0, \eta$.

Proof. As in [21] Lemma 2.4, we set $\bar{v}^2 = \frac{1}{|B_{N_1}|} \|v\|_{L^2(B_{N_1})}^2$. Utilizing (2.14), Poincaré and Hölder inequalities, we have

$$\frac{M}{|B_{N_1}|} \|v\|_{L^2(B_{N_1})}^2 \leq \int_{B_{N_1}} \rho \bar{v}^2 - v^2 + \int_{B_{N_1}} \rho \bar{v}^2$$

$$\leq C\|\rho\|_{L^\infty} \|v\|\|\nabla v\|_{L^1(B_{N_1})} + \int_{B_{N_1}} \rho \bar{v}^2$$

$$\leq \frac{M}{2|B_{N_1}|} \|v\|_{L^2(B_{N_1})}^2 + C\|\rho\|_{L^\infty}^2 \|\nabla v\|_{L^2(B_{N_1})}^2 + \int_{B_{N_1}} \rho \bar{v}^2,$$

i.e,

$$\|v\|_{L^2(\Omega)}^2 \leq C \left( \|\sqrt{\rho}v\|_{L^2(\Omega)}^2 + \|\rho\|_{L^\infty(\Omega)}^2 \|\nabla v\|_{L^2(\Omega)}^2 \right).$$

(2.17)

Insert it back into (2.7) yields the required (2.15) and (2.16). □

The following regularity theory for elliptic equations are also useful.

Lemma 2.5 If $v, w \in H^1(B_R) \cap W^{1,p}(B_R)$ satisfy respectively

$$\begin{cases}
- \mu \Delta v - (\mu + \lambda) \nabla \text{div } v = F, & \text{in } B_R, \\
v = 0, & \text{on } \partial B_R
\end{cases}$$

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and
\[
\begin{aligned}
-\kappa \Delta w &= G, \quad &\text{in } B_R, \\
w &= 0, \quad &\text{on } \partial B_R.
\end{aligned}
\]

Then for given functions \( F, G \in L^p(B_R) \) with \( p \in (1, +\infty) \), the inequalities
\[
\|\nabla^{k+2}v\|_{L^p(B_R)} \leq C\|F\|_{L^p(B_R)} \quad \text{and} \quad \|\nabla^{k+2}w\|_{L^p(B_R)} \leq C\|G\|_{L^p(B_R)} \tag{2.18}
\]
are fulfilled, where integers \( k \geq 0 \), and the \( C \) is independent of \( R \).

The final lemma of this section is responsible for the Caffarelli-Kohn-Nirenberg inequality in \( \mathbb{R}^2 \).

**Lemma 2.6** [2, 6] Given \( v \in C_0^\infty(\mathbb{R}^2) \), the inequality holds true for \( \tilde{b} \in (0, 2) \)
\[
\int_{\mathbb{R}^2} \frac{v^2}{|x|^{2-\tilde{b}}} \leq 4\tilde{b}^{-2} \int_{\mathbb{R}^2} |x|^\tilde{b} |\nabla v|^2, \tag{2.19}
\]
where the constant \( 4\tilde{b}^{-2} \) is optimal.

### 3 A priori estimates

Let \( p \in [1, \infty] \) and \( k = 1, 2 \). The simplified notations for Lebesgue and Sobolev spaces are used for convenience in this section.

\[
\int f = \int_{B_R} f dx, \quad L^p = L^p(B_R), \quad W^{k,p} = W^{k,p}(B_R), \quad H^k = W^{k,2}(B_R).
\]

In addition to (2.1), assume there is some large \( R_0 \in (1, \frac{R}{2}) \) such that
\[
\|\rho_0\|_{L^1(B_{R_0})} \geq \frac{1}{2}. \tag{3.1}
\]

By Lemma 2.1, the initial-boundary-value(IBM) problem (1.1), (2.1)-(2.2) has a unique strong solution \((\rho, u, \theta)\) over \( B_R \times [0,T_R] \) for some \( T_R > 0 \). The goal of this section is to prove Proposition 3.1 below, the *a priori* estimates for \((\rho, u, \theta)\) which are independent of \( R \) and the lower bound of the initial density.

Define
\[
\psi(t) \overset{def}{=} 1 + \sup_{0 \leq s \leq t} \left( \|\bar{x}^a \rho\|_{L^1 \cap W^{1,q}} + \|\nabla \sqrt{\bar{x}^a \rho}\|_{L^2} + \|\bar{x}^b \nabla u\|_{L^2} \right) + \int_0^t \|\nabla u\|_{L^q}^2 \tag{3.2}
\]
and
\[
\mathcal{E}_0 \overset{def}{=} \|\bar{x}^a \rho_0\|_{L^1 \cap W^{1,q}} + \|\nabla \sqrt{\bar{x}^a \rho_0}\|_{L^2} + \|\sqrt{\rho_0} u_0\|_{L^2} + \|\sqrt{\rho_0} \theta_0\|_{L^2} + \|\bar{x}^b \nabla u_0\|_{L^2} + \|\nabla^2 u_0\|_{L^2} + \|\nabla \theta_0\|_{H^1} + \|g_1\|_{L^2} + \|g_2\|_{L^2},
\]
where the constants \( a, b, q \), and the functions \( g_1, g_2 \) are assumed in Theorem 1.1.
**Proposition 3.1** Let \((\rho, u, \theta)\) be solutions to the IBV problem \((1.1), (2.1)-(2.2)\). There is a small time \(T_\ast > 0\) depending only on \(\mu, \lambda, \kappa, c_v, a, b, \eta_0, q, R_0\) and \(E_0\) such that

\[
\psi(T_\ast) + \sup_{0 \leq t \leq T_\ast} \left( \|\sqrt{\rho}\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 + \|\nabla \rho\|_{H^1}^2 + \|\sqrt{\rho} u\|_{L^2}^2 + \|\sqrt{\rho} \theta\|_{L^2}^2 \right) \\
+ \int_0^{T_\ast} \left( \|\nabla \dot{u}\|_{L^2}^2 + \|\nabla \dot{\theta}\|_{L^2}^2 + \|\nabla^2 \theta\|_{L^2}^2 + \|\nabla u\|_{L^\infty}^2 + \|\theta\|_{L^\infty}^2 \right) dt \\
\leq C,
\]

where, and in what follows, the \(C\) symbolizes a generic constant which may depend on \(\mu, \lambda, \kappa, c_v, a, b, \eta_0, q, R_0\) and \(T_\ast\); additionally, we use \(C_\alpha\) to emphasize that \(C\) relies especially upon \(\alpha\).

The validity of (3.3) follows directly from Lemmas 3.2-3.8 below, whose proof are divided into two steps.

**Step 1. The bound of \(\psi(t)\) in a short interval.**

First of all, the momentum Eqs. (1.1)\(_2\) and the energy Eq. (1.1)\(_3\) provide for all \(t \geq 0\)

\[
\int \left( \frac{1}{2} \rho |u|^2 + c_v \rho \theta \right) (x, t) = \int \left( \frac{1}{2} \rho |u|^2 + c_v \rho \theta \right) (x, 0) \leq C \varepsilon_0.
\]

Refer to [29, page 5], we introduce the function \(\phi^R \in C_0^\infty(\mathbb{R}^2)\) which satisfies

\[
\begin{cases} 
0 \leq \phi^R \leq 1, & \phi^R = 1 \text{ in } B_{\frac{3}{4}}, \quad \phi^R = 0 \text{ in } \mathbb{R}^2 \setminus B_R, \\
|\nabla^k \phi^R| \leq CR^{-k}, & k = 1, 2.
\end{cases}
\]

By (3.4), one calculates from Eq. (1.1)\(_1\) that

\[
\frac{d}{dt} \int \rho \phi^R = \int \rho u \cdot \nabla \phi^R \geq -C \|\rho\|_{L^1}^2 \|\sqrt{\rho} u\|_{L^2} \geq -C.
\]

Using (3.1) and selecting \(T_1 = \min\{1, (4C)^{-1}\}\), it satisfies

\[
\inf_{0 \leq t \leq T_1} \int_{B_{R_0}} \rho \geq \inf_{0 \leq t \leq T_1} \int_{B_{2R_0}} \rho \phi^{2R_0} \geq \int_{B_{2R_0}} \rho_0 \phi^{2R_0} - CT_1 \geq \int_{B_{R_0}} \rho_0 - CT_1 \geq \frac{1}{4},
\]

Inequalities (3.2), (3.4), (3.6) and Lemma 2.4 imply for any \(t \in [0, T_1]\)

\[
\|u \bar{x}^{-1}\|_{L^2} + \|u \bar{x}^{-\eta}\|_{L^\frac{2}{\eta}}^\frac{2}{\eta} \leq C_\eta \psi^2(t)
\]

and

\[
\|	heta \bar{x}^{-1}\|_{L^2} + \|	heta \bar{x}^{-\eta}\|_{L^\frac{2}{\eta}}^\frac{2}{\eta} \leq C_\eta \psi(t) \left( \|\sqrt{\rho} \theta\|_{L^2} + \|\nabla \theta\|_{L^2} \right),
\]

where \(\eta \in (0, 1]\).

**Lemma 3.2** Let \((\rho, u, \theta)\) be solutions of \((1.1), (2.1)-(2.2)\). Then for \(t \in [0, T_1]\)

\[
\|\sqrt{\rho} \theta\|_{L^2}^2 + \int_0^t \|\nabla \theta\|_{L^2}^2 \\
\leq C \int_0^t (1 + \|\nabla u\|_{L^\infty}) \|\sqrt{\rho} \theta\|_{L^2}^2 + C \int_0^t \left( \|\sqrt{\rho} u\|_{L^2}^5 + \psi^{12} \right) + C.
\]
By virtue of (2.18), (3.2), (3.8), it follows from Eqs. (1.1)

\[
\frac{c_v}{2} \frac{d}{dt} \int \rho \theta^2 + \kappa \int |\nabla \theta|^2 \leq C \int (\rho \theta^2 |\nabla u| + \theta |\nabla u|^2).
\]

(3.10)

Proof. Multiplying Eq. (1.1) by \( \theta \) yields

\[
\frac{c_v}{2} \frac{d}{dt} \int \rho \theta^2 + \kappa \int |\nabla \theta|^2 \leq C \int (\rho \theta^2 |\nabla u| + \theta |\nabla u|^2).
\]

By virtue of (2.13), (3.2), (3.8), it follows from Eqs. (1.1) that

\[
\| \nabla^2 u \|_{L^2} \leq C (\| \rho \dot{u} \|_{L^2} + \| \rho \nabla \theta \|_{L^2} + \| \theta \nabla \rho \|_{L^2})
\]

\[
\leq C \psi^2 \| \sqrt{\rho} \dot{u} \|_{L^2} + C \psi \| \nabla \theta \|_{L^2} + C \| \theta \|_{L^2} \| \dot{u} \|_{L^2} \| \sum_{i=1}^{q} \left| x^i \right| \| \theta \|_{L^2} \| \dot{u} \|_{L^2}^{2} \quad \text{(3.11)}
\]

which, along with (2.5) and (3.2), shows for \( r \in [2, \infty) \)

\[
\| \nabla u \|_{L^r} \leq C \| \nabla u \|_{L^2} \| \nabla u \|_{H^1}^{\frac{r-2}{2}}
\]

\[
\leq C \psi + C \psi \frac{\| \sqrt{\rho} \dot{u} \|_{L^2}^{r-2}}{2^{(r-1)}} (\| \sqrt{\rho} \dot{u} \|_{L^2} + \| \nabla \theta \|_{L^2})^{r-2} \quad \text{(3.12)}
\]

This combines with (3.2), (3.8) and Hölder inequality conclude

\[
C \int \theta |\nabla u|^2 \leq C \| \theta \|_{L^\frac{q}{2}} \| \nabla u \|_{L^q} \| \nabla u \|_{L^2}^{\frac{2}{q}}
\]

\[
\leq C \| \theta \|_{L^\frac{q}{2}} \| \nabla u \|_{L^q} \| \nabla u \|_{L^2}^{\frac{2}{q}}
\]

\[
\leq C \psi^2 \| \nabla \theta \|_{L^2} + C \| \nabla \theta \|_{L^2} \| \nabla u \|_{L^\frac{4}{3}}
\]

\[
\leq \varepsilon (\| \sqrt{\rho} \theta \|_{L^2}^2 + \| \nabla \theta \|_{L^2}^2) + C \| \sqrt{\rho} \dot{u} \|_{L^2} + C \psi^{12} \quad \text{(3.13)}
\]

Substituting (3.13) into (3.10) and choosing \( \varepsilon \) small yield the required (3.9).

Lemma 3.3 Let \((\rho, u, \theta)\) be solutions of (1.1), (2.1) - (2.2). Then for \( t \in [0, T_1] \)

\[
\begin{align*}
\| \sqrt{\rho} \theta \|_{L^2}^2 + \| \nabla \theta \|_{L^2}^2 + \frac{t}{\kappa} \int_0^t \left( \| \nabla \theta \|_{L^2}^2 + \| \sqrt{\rho} \dot{u} \|_{L^2}^2 \right) \leq C \psi^{12} (t) + C \| \sqrt{\rho} \dot{u} \|_{L^2}^b + C \int_0^t \left( \| \nabla \dot{u} \|_{L^2}^b + \| \sqrt{\rho} \dot{u} \|_{L^2}^3 \right) \\
+C \int_0^t \left( \psi^{14} + \| \nabla u \|_{L^\infty}^2 \right) (1 + \| \sqrt{\rho} \theta \|_{L^2} + \| \nabla u \|_{L^2})^2.
\end{align*}
\]

(3.14)

Proof. Multiplied by \( \dot{\theta} \), it gives from Eq. (1.1) that

\[
\kappa \frac{d}{dt} \int |\nabla \theta|^2 + c_v \int \rho |\dot{\theta}|^2
\]

\[
= \kappa \int \left( \frac{1}{2} |\nabla \theta|^2 \text{div} u - \partial_t \dot{\theta} \partial_i u^k \partial_k \theta \right) - \int P \text{div} u \dot{\theta}
\]

\[
+ \int \left( \frac{\mu}{2} |\nabla u + (\nabla u)^T|^2 + \lambda (\text{div} u)^2 \right) \dot{\theta},
\]

(3.15)

where \( \dot{\theta} = \dot{\theta}_t + u \cdot \nabla \theta \) is the material derivative of \( \theta \).
To avoid the $\|\nabla \theta\|_{L^2}$, we follow as in [8] and handle the last term

$$
\int \left( \frac{\mu}{2} |\nabla u + (\nabla u)^{tr}|^2 + \lambda (\text{div} u)^2 \right) \dot{\theta} + \frac{d}{dt} \int \left( \frac{\mu}{2} |\nabla u + (\nabla u)^{tr}|^2 + \lambda (\text{div} u)^2 \right) \theta 
- \int (\mu (\nabla u + (\nabla u)^{tr}) : (\nabla u + (\nabla u)^{tr})_t + 2\lambda \text{div} u \text{div} u) \theta 
+ \int \left( \frac{\mu}{2} |\nabla u + (\nabla u)^{tr}|^2 + \lambda (\text{div} u)^2 \right) u \cdot \nabla \theta.
$$

Notice that

$$
- \mu \int (\nabla u + (\nabla u)^{tr}) : (\nabla u + (\nabla u)^{tr})_t \theta 
= -2\mu \int \nabla u : \nabla u \theta - 2\mu \int \nabla u : (\nabla u)_t \theta = J_1 + J_2,
$$

and in particular,

$$
J_1 = -2\mu \int \nabla u : \nabla u \theta 
= -2\mu \int \left( \nabla u : \nabla \dot{u} - \partial_i u^j \partial_j u^k \partial_k u^i \right) \theta - \mu \int |\nabla u|^2 (\theta \text{div} u + u \cdot \nabla \theta),
$$

$$
J_2 = -2\mu \int \nabla u : (\nabla u)_t \theta 
= -2\mu \int \left( \nabla u : \nabla \dot{u} - \partial_i u^j \partial_j u^k \partial_k u^i \right) \theta - \mu \int \partial_i u^j \partial_j u^i (\theta \text{div} u + u \cdot \nabla \theta),
$$

Then,

$$
\left| -\mu \int (\nabla u + (\nabla u)^{tr}) : (\nabla u + (\nabla u)^{tr})_t \theta \right| 
\leq C \int \left( ||\nabla u|||\nabla \dot{u}| + |\nabla u|^3 \theta + |\nabla u|^2 |\nabla \theta| \right).
$$

A similar argument runs

$$
\left| -2\lambda \int (\text{div} u)(\text{div} u)_t \theta \right| \leq C \int \left( ||\nabla u|||\nabla \dot{u}| + |\nabla u|^3 \theta + |\nabla u|^2 |\nabla \theta| \right).
$$

Integration of (3.15) brings to

$$
\int |\nabla \theta|^2 + \int_0^t \int \rho |\dot{\theta}|^2 
\leq C + C \int \|\nabla u\|^2 \theta + C \int_0^t \|\nabla u\|_{L^\infty} \int |\nabla \theta|^2
+ C \int_0^t \int \left( \rho \dot{\theta} |\nabla u| \dot{\theta} + |\nabla u||\nabla \dot{u}| \theta + |\nabla u|^3 \theta + |\nabla u|^2 |\nabla \theta| u \right). \tag{3.16}
$$
In consideration of the last three inequalities, the (3.16) satisfies

\[ C \int \rho \theta |\nabla u| |\dot{\theta}| \leq \frac{1}{4} \| \sqrt{\rho} \theta \|_{L^2}^2 + C \| \nabla u \|_{L^\infty}^2 \| \sqrt{\rho} \theta \|_{L^2}^2. \]

For \( b \in (0, \frac{3}{2}) \), it follows from (3.7), (3.8) and (3.12) that

\[
C \int |\nabla \tilde{u}| |\nabla u| \theta \\
\leq C \| \nabla \tilde{u} \|_{L^2} \| \theta \tilde{x}^{-\frac{b}{2}} \|_{L^{\frac{16}{b}}} \| \tilde{x}^{\frac{b}{2}} \nabla \tilde{u} \|_{L^{\frac{16}{b}}} \\
\leq C \psi \| \nabla \tilde{u} \|_{L^2} (\| \sqrt{\rho} \theta \|_{L^2} + \| \nabla \theta \|_{L^2}) \| \tilde{x}^{\frac{b}{2}} \nabla u \|_{L^2} \| \nabla u \|_{L^\infty} \\\n\leq C \| \nabla \tilde{u} \|_{L^2} (\| \sqrt{\rho} \theta \|_{L^2} + \| \nabla \theta \|_{L^2}) \\
\left( \psi^2 + \psi^2 \| \sqrt{\rho} \tilde{u} \|_{L^2}^{\frac{b}{2}} + \psi^{\frac{4b+6}{8}} (\| \sqrt{\rho} \theta \|_{L^2} + \| \nabla \theta \|_{L^2})^{\frac{b}{8}} \right) \\
\leq C \psi^8 (1 + \| \sqrt{\rho} \theta \|_{L^2}^2 + \| \nabla \theta \|_{L^2}^2)^2 + C \left( \| \sqrt{\rho} \tilde{u} \|_{L^2}^{\frac{b}{2}} + \| \nabla \tilde{u} \|_{L^2}^{\frac{b}{2}} \right)
\]

and that

\[
C \int (|u| |\nabla u|^2 |\nabla \theta| + \theta |\nabla u|^3) \\
\leq C \left( \| \nabla \theta \|_{L^2} \| u \tilde{x}^{-\frac{b}{2}} \|_{L^{\frac{16}{b}}} + \| \nabla u \|_{L^2} \| \theta \tilde{x}^{-\frac{b}{2}} \|_{L^{\frac{16}{b}}} \right) \| \tilde{x}^{\frac{b}{2}} \nabla u \|_{L^2}^2 \| \nabla u \|_{L^{\frac{16}{b}}} \\
\leq C (\| \sqrt{\rho} \theta \|_{L^2} + \| \nabla \theta \|_{L^2}) \\
\left( \psi^4 + \psi^2 \| \sqrt{\rho} \tilde{u} \|_{L^2}^{\frac{4b+6}{8}} + \psi^{\frac{4b+6}{8}} (\| \sqrt{\rho} \theta \|_{L^2} + \| \nabla \theta \|_{L^2})^{\frac{4b+6}{8}} \right) \\
\leq C \psi^{14} (1 + \| \sqrt{\rho} \theta \|_{L^2}^2 + \| \nabla \theta \|_{L^2}^2)^2 + C \| \sqrt{\rho} \tilde{u} \|_{L^2}^2.
\]

In consideration of the last three inequalities, the (3.16) satisfies

\[
\int |\nabla \theta|^2 + \int_0^t \| \sqrt{\rho} \theta \|_{L^2}^2 \\
\leq C + C \int \theta |\nabla u|^2 + C \int_0^t \left( \| \nabla \tilde{u} \|_{L^2}^{\frac{b}{2}} + \| \sqrt{\rho} \tilde{u} \|_{L^2}^{\frac{b}{2}} \right) \\
\leq C \int_0^t \left( \psi^{14} + \| \nabla u \|_{L^\infty}^2 \right) (1 + \| \sqrt{\rho} \theta \|_{L^2}^2 + \| \nabla \theta \|_{L^2}^2)^2.
\]

Again using (3.13) for sufficiently small \( \varepsilon \), we receive the desired (3.14) from (3.9) and (3.17).

Lemma 3.4 Assume

\[ T_2 \leq T_1 \quad \text{and} \quad CT_2^{\frac{b-2}{(b-1)\psi}} \psi^{13} (T_2) \leq \frac{1}{2}.
\]

Then the solutions \((p, u, \theta)\) of (1.1), (2.1) - (2.2) satisfy

\[
\sup_t \left( \| \sqrt{\rho} \theta \|_{L^2}^2 + \| \nabla \theta \|_{L^2}^2 + \| \sqrt{\rho} \tilde{u} \|_{L^2}^2 \right) \\
\leq \int_0^t \left( \| \sqrt{\rho} \theta \|_{L^2}^2 + \| \nabla \tilde{u} \|_{L^2}^2 \right) \leq C \psi^{13} (t), \quad \forall \ t \in [0, T_2].
\]
Proof. By defining
\[ \tilde{g}_1 = \rho_0^{-\frac{1}{2}} (\mu \triangle u_0 + (\mu + \lambda) \nabla \text{div} u_0), \]
we compute
\[
\int \rho |\dot{u}|^2(x,0) \leq \limsup_{t \to 0^+} \int \rho^{-1} |\mu \triangle u + (\mu + \lambda) \nabla \text{div} u - R(\rho \theta)|^2 \\
\leq \| \tilde{g}_1 \|^2_{L^2} + C \left\| \nabla (\rho_0 \theta_0) \sqrt{\rho_0} \right\|^2_{L^2} \leq C,
\] (3.20)
in which the last inequality is valid because of
\[
\left\| \frac{\nabla (\rho_0 \theta_0)}{\sqrt{\rho_0}} \right\|^2_{L^2} \leq \left\| \rho_0 \nabla \theta_0 \right\|^2_{L^2} + \left\| \theta_0 \nabla \rho_0 \right\|^2_{L^2} \\
\leq C \| \nabla \theta_0 \|_{L^2} + C \| \theta_0 \nabla \rho_0 \|_{L^2} \\
\leq C + C \| \theta_0 \bar{x}^{-\frac{3}{2}} \|_{L^\infty} \| \rho_0 \bar{x}^{-\frac{3}{2}} \|_{H^1} \leq C.
\]

Operating \( \partial_t + \text{div}(u \cdot) \) to the \( j \)-th component of Eqs. (1.1)_2, we get
\[
\partial_t (\rho \dot{u}^j) + \text{div}(\rho u \dot{u}^j) - \mu \partial_t \partial_i \dot{u}^j - (\mu + \lambda) \partial_j \text{div} \dot{u} \\
= \mu \dot{u}_i \left( -\partial_t u^k \partial_k \dot{u}^j + \text{div} u \partial_i \dot{u}^j \right) \\
+ (\mu + \lambda) \partial_j \left( -\partial_t u^k \partial_k u^i + \text{div} u \dot{u}^i \right) - (\mu + \lambda) \partial_k \left( \partial_j u^k \text{div} \dot{u} \right) \\
- \partial_j \left( P_t + \text{div}(Pu) \right) + \partial_k (\partial_j u^k P).
\]
If we multiply it by \( \ddot{u}^i \), add them up, integrate by parts, utilize (3.8), (3.12) and
\[ P_t + \text{div}(Pu) = R \rho \dot{\theta}, \]
we infer
\[
\frac{d}{dt} \int \rho |\dot{u}|^2 + \int |\nabla \dot{u}|^2 \\
\leq C \left( \| \nabla u \|_{L^4}^4 + \| \rho \dot{\theta} \|_{L^2}^2 + \| \rho \theta \|_{L^4}^4 \right) \\
\leq C \left( \| \nabla u \|_{L^4}^4 + \| \rho \dot{\theta} \|_{L^2}^2 + \| \rho \bar{x}^{-\frac{3}{2}} \|_{L^\infty} \| \bar{x}^{-\frac{3}{2}} \theta \|_{L^4}^4 \right) \\
\leq C \psi^3 \| \sqrt{\rho} \dot{u} \|_{L^2}^2 + C \psi \| \sqrt{\rho} \dot{\theta} \|_{L^2}^2 + \psi^8 \left( 1 + \| \sqrt{\rho} \dot{\theta} \|_{L^2}^2 + \| \nabla \theta \|_{L^2}^2 \right)^2,
\]
which satisfies after integration in time
\[
\| \sqrt{\rho} \dot{u} \|_{L^2}^2 + \int_0^t \| \nabla \dot{u} \|_{L^2}^2 \\
\leq C + C \psi(t) \int_0^t \| \sqrt{\rho} \dot{\theta} \|_{L^2}^2 \\
+ C \int_0^t \psi^8 \left( 1 + \| \sqrt{\rho} \dot{\theta} \|_{L^2}^2 + \| \nabla \theta \|_{L^2}^2 \right) \left( 1 + \| \sqrt{\rho} \dot{u} \|_{L^2}^2 + \| \nabla \dot{u} \|_{L^2}^2 \right)^2.
\] (3.22)
By (3.14) one has
\[
\psi(t) \int_0^t \| \sqrt{\rho} \dot{\theta} \|_{L^2}^2 \\
\leq C \psi^{13}(t) + C \psi(t) \| \sqrt{\rho} \dot{u} \|_{L^2}^b + C \psi(t) \int_0^t \left( \| \nabla \dot{u} \|_{L^2}^2 + \| \sqrt{\rho} \dot{u} \|_{L^2}^2 \right) \\
+ C \psi(t) \int_0^t \psi^{14} \left( \| \nabla u \|_{L^\infty}^2 \right) \left( 1 + \| \sqrt{\rho} \dot{\theta} \|_{L^2}^2 + \| \nabla \theta \|_{L^2}^2 \right)^2.
\]
For some sufficiently large constant $K_1$, a sum of $K_1 \times (3.14) + (3.22)$ concludes that for $t \in [0, T_1]$

\[
\begin{align*}
\|\sqrt{\rho \theta}\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 + \|\nabla \dot{u}\|_{L^2}^2 + \int_0^t \left(\|\sqrt{\rho \theta}\|_{L^2}^2 + \|\nabla \dot{u}\|_{L^2}^2\right)
\leq C \psi^{13}(t) + C \psi \|\sqrt{\rho \theta}\|_{L^2}^2 + C \psi \int_0^t \|\nabla \dot{u}\|_{L^2}^2 \tag{3.23}
+ \psi(t) \int_0^t \left(\psi^{14} + \|\nabla u\|_{L^\infty}^2\right) (1 + \|\sqrt{\rho \theta}\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 + \|\sqrt{\rho \theta}\|_{L^2}^2)^2.
\end{align*}
\]

Noting

\[
\psi \|\sqrt{\rho \dot{u}}\|_{L^2}^2 + C \psi \int_0^t \|\nabla \dot{u}\|_{L^2}^2 \leq \frac{1}{2} \left(\|\sqrt{\rho \dot{u}}\|_{L^2}^2 + \int_0^t \|\nabla \dot{u}\|_{L^2}^2\right) + C \psi^4,
\]

we express (3.23) as the form

\[
\begin{align*}
f_1(t) + \int_0^t \left(\|\sqrt{\rho \theta}\|_{L^2}^2 + \|\nabla \dot{u}\|_{L^2}^2\right)
\leq C \psi^{13}(t) + C \psi(t) \int_0^t \left(\psi^{14} + \|\nabla u\|_{L^\infty}^2\right) f_1^2, \tag{3.24}
\end{align*}
\]

where

\[
f(t) \triangleq \|\sqrt{\rho \theta}\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 + \|\sqrt{\rho \theta}\|_{L^2}^2.
\]

Observe from (2.6) and (3.2) that

\[
C \psi(t) \int_0^t \left(\psi^{14} + \|\nabla u\|_{L^\infty}^2\right) \leq t \cdot C \psi^{15}(t) + C \psi^{\frac{2q-3}{q-1}} \int_0^t \|\nabla^2 u\|_{L^q}^{\frac{q}{q-1}}
\leq t \cdot C \psi^{15} + t^{\frac{q-2}{2(q-1)}} \cdot C \psi^{\frac{2q-6}{2(q-1)}} \leq t^{\frac{q-2}{2(q-1)}} \cdot \tilde{C} \psi^{15}(t),
\]

solving directly the integral inequality (3.24) yields

\[
\begin{align*}
\sup_{0 \leq s \leq t} f_1(s) + \int_0^t \left(\|\sqrt{\rho \theta}\|_{L^2}^2 + \|\nabla \dot{u}\|_{L^2}^2\right)
\leq \left(C \psi^{-13}(t) + C \psi(t) \int_0^t \left(\psi^{14} + \|\nabla u\|_{L^\infty}^2\right)\right)^{-1}
\leq \left(C \psi^{-13}(t) + \tilde{C} t^{\frac{q-2}{2(q-1)}} \psi^{15}\right)^{-1}
\leq C \psi^{13}(t),
\end{align*}
\]

as long as $0 \leq t \leq T_2$ with

\[
T_2 \leq T_1 \quad \text{and} \quad C^{-1} \tilde{C} t^{\frac{2q-2}{2(q-1)}} \psi^{28}(t) \leq \frac{1}{2}.
\]

In the next lemma, we will derive an estimates on $\|\sqrt{\frac{2}{3}} \nabla u\|_{L^2}$. 


Lemma 3.5 For all $t \in [0, T_2]$, the solutions $(\rho, u, \theta)$ of (2.11), (2.12), (2.22) satisfy

$$
\sup_t \left( \|x^\frac{b}{2} \nabla u\|_{L^2}^2 + \|x^\frac{b}{2} \rho \theta\|_{L^2}^2 \right) + \int_0^t \|x^\frac{b}{2} \sqrt{\rho} \dot{u}\|_{L^2}^2 
\leq C \left( 1 + t^\frac{b}{2} \psi^5(t) \right) \exp \left\{ t^\frac{b}{2} \psi^2(t) \right\}. 
$$

(3.26)

Proof. Multiplying Eqs. (1.11) by $\dot{x}^b \dot{u}$ and calculating it carefully, we receive

$$
\frac{1}{2} \frac{d}{dt} \int \dot{x}^b (\mu |\nabla u|^2 + (\mu + \lambda) (\text{div} u)^2) + \int \rho \dot{x}^b |\dot{u}|^2 
= \frac{d}{dt} \int P(\dot{x}^b \text{div} u + u \cdot \nabla \dot{x}^b) 
- \int (P_t + \text{div} (Pu)) (\dot{x}^b \text{div} u + u \cdot \nabla \dot{x}^b) + \int (u \cdot \nabla P)(u \cdot \nabla \dot{x}^b) 
+ \int P \partial_j u^k \partial_k u^j \dot{x}^b + \int P(u \cdot \nabla)(u \cdot \nabla \dot{x}^b) 
- \mu \int \dot{x}^b (\partial_i u^i \partial_j u^j - \frac{1}{2} (\partial_i u^j)^2 \text{div} u) + \frac{\mu}{2} \int (\partial_i u^j)^2 u \cdot \nabla \dot{x}^b 
- (\mu + \lambda) \int \dot{x}^b (\text{div} u \partial_i u^i \partial_k u^j - \frac{1}{2} (\text{div} u)^2) + \frac{\mu + \lambda}{2} \int (\text{div} u)^2 u \cdot \nabla \dot{x}^b 
= \frac{d}{dt} \int P(\dot{x}^b \text{div} u + u \cdot \nabla \dot{x}^b) + \sum_{i=1}^8 I_i. 
$$

(3.27)

We need to estimate the terms $I_i (i = 1 \sim 8)$. A simple calculation shows

$$
|\nabla \dot{x}^b| \leq C |\dot{x}^{b-1} \nabla \dot{x}| \leq C \dot{x}^{b-\frac{3}{2}} \quad \text{for} \quad 0 < b < \frac{a}{2} < 1. 
$$

(3.28)

By virtue of (3.7), (3.8), (3.12), (3.21) and (3.28), we estimate the first two inequalities as

$$
I_1 \leq C \int \rho \dot{\theta} (\dot{x}^b |\nabla u| + \dot{x}^{b+\frac{a}{2}} |u|) 
\leq C \left[ \rho \dot{x}^a \right]_{L^2} \left( \|\nabla u\|_{L^4} + \|\dot{x}^{b-\frac{a}{2}} u\|_{L^4} \right) 
\leq C \psi \frac{1}{2} \sqrt{\rho} |\nabla \dot{\theta}|_{L^2} \left( \psi^2 + \psi \frac{1}{2} \sqrt{\rho} \dot{u} \right)_{L^2} + \psi \frac{1}{2} \left( \|\sqrt{\rho} \dot{\theta}\|_{L^2} + \|\nabla \theta\|_{L^2} \right)_{L^2} 
\leq \frac{1}{2} \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + C \|\sqrt{\rho} \dot{\theta}\|_{L^2}^2 + C \|\dot{\nabla} \theta\|_{L^2}^2 + C \psi^{10} 
$$

and

$$
I_2 \leq C \int |u|^2 (\theta |\nabla \rho| + \rho |\nabla \theta|) 
\leq C \left[ u \dot{x}^a (0-\frac{a}{2}) \right]_{L^2} \left( \|u \dot{x}^\frac{1}{2} (0-\frac{a}{2})\|_{L^4} + \|\dot{u} \dot{x}^\frac{1}{2} (0-\frac{a}{2})\|_{L^4} + \|\nabla \theta\|_{L^2} + \|u \dot{x}^\frac{1}{2} (0-\frac{a}{2})\|_{L^b}^2 \right) 
\leq C \left( \|\sqrt{\rho} \dot{\theta}\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 \right) + C \psi^{12}. 
$$

Next,

$$
I_3 + I_4 \leq \left( \|\nabla u\|_{L^\infty} + \|u \dot{x}^{-\frac{a}{2}}\|_{L^\infty} \right) \int \dot{x}^b \left( |\nabla u|^2 + \rho^2 \theta^2 \right). 
$$

Next,
And the final four terms satisfy
\[
\sum_{i=5}^{8} I_i \leq C \int (\bar{x}^b|\nabla u|^3 + \bar{x}^{b-\frac{8}{3} \nabla u|^2 u) \\
\leq C \left( \|\nabla u\|_{L^\infty} + \|u \bar{x}^{-\frac{8}{3}}\|_{L^\infty} \right) \int \bar{x}^b|\nabla u|^2.
\]

Therefrom, we deduce from (3.27) that
\[
\|\bar{x}^{\frac{2}{3}}\nabla u\|^2_{L^2} + \int_0^t \|\bar{x}^{\frac{2}{3}} \sqrt{\rho} u\|^2_{L^2} \\
\leq C + C \int \rho \theta (\bar{x}^b|\nabla u| + |u|) \\
+ C \int_0^t \left( \|\nabla u\|_{L^\infty} + \|u \bar{x}^{-\frac{8}{3}}\|_{L^\infty} \right) \int \bar{x}^b (|\nabla u|^2 + \rho^2 \theta^2) \\
+ C \int_0^t \left( \psi^{12} + \|\sqrt{\rho} u\|^2_{L^2} + \|\nabla \theta\|^2_{L^2} + \|\sqrt{\theta}\|_{L^2}^\frac{4}{3} \right).
\] (3.29)

By (3.4), Cauchy and Hölder inequality, it satisfies
\[
C \int \rho \theta (\bar{x}^b|\nabla u| + |u|) \\
\leq \frac{1}{2} \int |\nabla u|^2 \bar{x}^b + C \int \rho^2 \theta^2 \bar{x}^b + C \|\sqrt{\rho} u\|_{L^2} \|\sqrt{\theta}\|_{L^2} \\
\leq \frac{1}{2} \int |\nabla u|^2 \bar{x}^b + C \int \rho^2 \theta^2 \bar{x}^b + C \|\sqrt{\theta}\|_{L^2},
\]
and whence, (3.29) satisfies
\[
\|\bar{x}^{\frac{2}{3}}\nabla u\|^2_{L^2} + \int_0^t \|\bar{x}^{\frac{2}{3}} \sqrt{\rho} u\|^2_{L^2} \\
\leq C + C \|\sqrt{\rho} \theta\|_{L^2} + C \|\bar{x}^{\frac{2}{3}} \rho \theta\|^2_{L^2} \\
+ C \int_0^t \left( \|\nabla u\|_{L^\infty} + \|u \bar{x}^{-\frac{8}{3}}\|_{L^\infty} \right) \left( \|\bar{x}^{\frac{2}{3}} \nabla u\|^2_{L^2} + \|\bar{x}^{\frac{2}{3}} \rho \theta\|^2_{L^2} \right) \\
+ C \int_0^t \left( \psi^{12} + \|\sqrt{\rho} \theta\|^2_{L^2} + \|\nabla \theta\|^2_{L^2} + \|\sqrt{\theta}\|_{L^2}^\frac{4}{3} \right).
\] (3.30)

To deal with the term \(\|\bar{x}^{\frac{2}{3}} \rho \theta\|^2_{L^2}\), we multiply Eq. (1.1) by \(\rho \theta \bar{x}^b\) and get
\[
\frac{c_v}{2} \frac{d}{dt} \int \rho^2 \theta^2 \bar{x}^b + \left( \frac{c_v}{2} + R \right) \int \rho^2 \theta^2 \bar{x}^b \text{div} u \\
= \frac{c_v}{2} \int \rho^2 \theta^2 u \cdot \nabla \bar{x}^b + \int \left( \frac{\mu}{2} |\nabla u + (\nabla u)^{tr}|^2 + \lambda (\text{div} u)^2 + \kappa \Delta \theta \right) \rho \theta \bar{x}^b.
\]

Since
\[
\kappa \int \Delta \theta \rho \theta \bar{x}^b
\]
\[
= -\kappa \int \nabla \theta \left( \rho \bar{x}^b \nabla \theta + \theta \bar{x}^b \nabla \rho + \rho \theta \nabla \bar{x}^b \right)
\]
\[
\leq C \|\rho \bar{x}^b\|_{L^\infty} \|\nabla \theta\|_{L^2}^2
\]
\[
+ C \|\nabla \theta\|_{L^2} \left( \|\bar{x}^b - \theta\|_{L^2(b \cdot 2^{1/2} - 1/2)} \|\bar{x}^a \nabla \rho\|_{L^2(b \cdot 2^{1/2} - 1/2)} + \|\bar{x}^a \rho\|_{L^2} \|\bar{x}^b - \theta\|_{L^2(b \cdot 2^{1/2} - 1/2)} \right)
\]
\[
\leq C \psi^2 \left( \|\sqrt{\rho \theta}\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 \right),
\]
then
\[
\|\bar{x}^b \rho \theta\|_{L^2}^2 \leq C + C \int_0^t \left( \|\nabla u\|_{L^\infty} + \|u \bar{x}^b - \theta\|_{L^\infty} \right) \left( \|\bar{x}^b \nabla u\|_{L^2}^2 + \|\bar{x}^b \rho \theta\|_{L^2}^2 \right)
\]
\[
+ C \int_0^t \psi^2 \left( \|\sqrt{\rho \theta}\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 \right) \tag{3.31}
\]

The combination of (3.30) with (3.31) leads to
\[
\|\bar{x}^b \nabla u\|_{L^2}^2 + \|\bar{x}^b \rho \theta\|_{L^2}^2 + \int_0^t \|\bar{x}^b \sqrt{\rho \theta}\|_{L^2}^2
\]
\[
\leq C + C \|\sqrt{\rho \theta}\|_{L^2}^2
\]
\[
+ C \int_0^t \left( \|\nabla u\|_{L^\infty} + \|u \bar{x}^b - \theta\|_{L^\infty} \right) \left( \|\bar{x}^b \nabla u\|_{L^2}^2 + \|\bar{x}^b \rho \theta\|_{L^2}^2 \right)
\]
\[
+ C \int_0^t \left( \psi^{12} + \psi^2 \|\sqrt{\rho \theta}\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 \right) \tag{3.32}
\]

Making use of (3.9) and
\[
C \int_0^t \|\sqrt{\rho \theta}\|_{L^2}^2 \leq \frac{1}{2} \int_0^t \|\bar{x}^b \sqrt{\rho \theta}\|_{L^2}^2 + C,
\]
we conclude from (3.32) that
\[
f_2(t) + \int_0^t \|\bar{x}^b \sqrt{\rho \theta}\|_{L^2}^2 \leq C \int_0^t \left( \|\nabla u\|_{L^\infty} + \|u \bar{x}^b - \theta\|_{L^\infty} + \psi^2 \right) f_2
\]
\[
+ C \int_0^t \left( \psi^{12} + \psi^2 \|\nabla \theta\|_{L^2}^2 + \|\sqrt{\rho \theta}\|_{L^2}^2 \right) + C, \tag{3.33}
\]
where
\[
f_2(t) \triangleq \|\bar{x}^b \nabla u\|_{L^2}^2 + \|\bar{x}^b \rho \theta\|_{L^2}^2 + \|\sqrt{\rho \theta}\|_{L^2}^2.
\]

Observe from (2.21), (3.7) and (3.8) that for any \( t \in [0, T_2] \)
\[
C \int_0^t \left( \psi^{12} + \psi^2 \|\nabla \theta\|_{L^2}^2 + \|\sqrt{\rho \theta}\|_{L^2}^2 \right) \tag{3.34}
\]
\[
\leq C t^{15} \psi^{15}(t) + C t^{4} \left( \|\sqrt{\rho \theta}\|_{L^2}^2 \right)^{\frac{4}{3}}
\]
\[
\leq C t^{15} \psi^{15}(t) + C t^{4} \psi^{4}(t) \leq C t^{4} \psi^{15}(t)
\]
and
\[
C \int_0^t \left( \| \nabla u \|_{L^\infty} + \| u \dot{x}^{-\frac{a}{2}} \|_{L^\infty} + \psi^2 \right)
\leq C \int_0^t \left( \| \nabla u \|_{L^2} + \| \nabla^2 u \|_{L^2} + \| u \dot{x}^{-\frac{a}{2}} \|_{L^4} + \| \nabla (u \dot{x}^{-\frac{a}{2}}) \|_{L^4} + \psi^2 \right)
\leq C \int_0^t (\| \nabla^2 u \|_{L^2} + \psi^2) \leq C t^\frac{1}{2} \psi^2(t),
\]
we get (3.26) by means of the Gronwall’s inequality and (3.33)-(3.35).

**Lemma 3.6** For \( q \in (2, \infty) \), it holds for \( t \in [0, T_2] \)
\[
\int_0^t \| \nabla u \|_{L^q}^2 \leq C t^\frac{q}{2} \psi^{38}(t).
\]

**Proof.** From (3.2) and (3.4) we obtain
\[
\| \nabla (\rho \dot{u}) \|_{L^2} \leq C \left( \| \rho \|_{L^\infty} \| \nabla \dot{u} \|_{L^2} + \| \dot{u} \dot{x}^{-a} \|_{L^2} \frac{\| \rho \dot{x}^a \|_{W^{1,q}}}{L^{\frac{1}{2}}} \right)
\leq C \psi^2 (\| \nabla \dot{u} \|_{L^2} + \| \sqrt{\rho} \dot{u} \|_{L^2}),
\]
and thus for \( q \in (2, \infty) \)
\[
\| \rho \dot{u} \|_{L^q} \leq C \| \rho \dot{u} \|_{L^2}^{\frac{q-2}{2}} \| \nabla (\rho \dot{u}) \|_{L^2}^{\frac{q}{2}}
\leq C \psi^3 \left( \| \sqrt{\rho} \dot{u} \|_{L^2} + \| \sqrt{\rho} \dot{u} \|_{L^2}^{\frac{q}{2}} \| \nabla \dot{u} \|_{L^2}^{\frac{q-2}{2}} \right).
\]

Utilizing (2.18) and (3.12), it gives from Eq. (1.11) that
\[
\| \nabla^2 \theta \|_{L^2}
\leq C \left( \| \dot{x} \|_{L^2} + \| \rho \theta \|_{L^2} + \| \nabla u \|_{L^4}^{\frac{1}{2}} \right)
\leq C \left( \psi^\frac{1}{2} \| \sqrt{\rho} \dot{u} \|_{L^2} + \| \dot{x} \|_{L^2}^{\frac{1}{2}} \| \dot{u} \|_{L^4} + \| \nabla u \|_{L^4}^{\frac{1}{2}} \right)
\leq C \left( \psi^\frac{1}{2} \| \sqrt{\rho} \dot{u} \|_{L^2} + \psi^2 + \| \sqrt{\rho} \dot{u} \|_{L^2} + \psi^4 \left( \| \sqrt{\rho} \dot{u} \|_{L^2} + \| \nabla \theta \|_{L^2} \right) \right).
\]

By this we have
\[
\| \nabla (\rho \theta) \|_{L^2} \leq C \| \dot{x} \|_{L^2} \| \theta \|_{W^{1,q}} \left( \| \dot{x} \|_{L^\infty} + \| \nabla \theta \|_{L^\infty} \right)
\leq C \psi \left( \| \dot{x} \|_{L^2} + \| \nabla \theta \|_{L^2} \right)
\leq C \psi^2 \left( \| \sqrt{\rho} \dot{u} \|_{L^2} + \| \theta \|_{L^2} + \| \nabla \theta \|_{L^2} \| \nabla^2 \theta \|_{L^2} \right)^{\frac{q-2}{q}}
\leq C \left( \psi^4 + \psi^6 \left( \| \sqrt{\rho} \dot{u} \|_{L^2} + \| \nabla \theta \|_{L^2} \right) + \psi^2 \| \sqrt{\rho} \dot{u} \|_{L^2} \right)^{\frac{q-2}{q}}
\]
\[
+ C \psi^{\frac{q-2}{2}} \left( \| \nabla \theta \|_{L^2} \| \sqrt{\rho} \dot{u} \|_{L^2} \right)^{\frac{q-2}{q}}.
\]
Therefore, it follows from (2.18), (3.37), (3.39) and Eqs. (1.1) that

$$\begin{align*}
C \int_{t_0}^{t} \| \nabla^2 u \|^2_{L^2} ds \\
\leq C \int_{t_0}^{t} \left( \| \rho \dot{u} \|^2_{L^2} + \| \nabla (\rho \theta) \|^2_{L^2} \right) \\
\leq C \int_{t_0}^{t} \left( \psi^6 \| \sqrt{\rho} \dot{u} \|^2_{L^2} \| \nabla \theta \|^2_{L^2} \| \sqrt{\rho} \theta \|^2_{L^2} + \psi^8 \| \nabla \theta \|^2_{L^2} \right) \\
+ C \int_{t_0}^{t} \psi^7 \| \sqrt{\rho} \dot{u} \|^2_{L^2} + \psi^8 \| \sqrt{\rho} \dot{\theta} \|^2_{L^2} \right). 
\end{align*}$$

From (3.18) we infer

$$\begin{align*}
C \int_{t_0}^{t} \left( \psi^6 \| \sqrt{\rho} \dot{u} \|^2_{L^2} \| \nabla \dot{u} \|^2_{L^2} (q - 2)q \right)
&\leq C t^2_{\psi^6} \psi^6 (t) \sup_{t} \left( \| \sqrt{\rho} \dot{u} \|^2_{L^2} \| \nabla \theta \|^2_{L^2} \right) \\
&\leq C \psi^2 \| \sqrt{\rho} \dot{u} \|^2_{L^2} \| \nabla \theta \|^2_{L^2} \right) \psi^2 \| \nabla \theta \|^2_{L^2} \right) \right) \\
&\leq C t^2_{\psi^6} \psi^6 (t)
\end{align*}$$

and

$$\begin{align*}
C \int_{t_0}^{t} \left( \psi^7 \| \sqrt{\rho} \dot{u} \|^2_{L^2} + \psi^8 \| \sqrt{\rho} \dot{\theta} \|^2_{L^2} \right) \\
\leq C t^2_{\psi^6} \psi^6 (t) \sup_{t} \left( \| \sqrt{\rho} \dot{u} \|^2_{L^2} \| \nabla \theta \|^2_{L^2} \right) \\
&\leq C t^2_{\psi^6} \psi^6 (t)
\end{align*}$$

The required (3.36) thus follows from the last three inequalities.

**Lemma 3.7** It satisfies for \( q \in (2, \infty) \)

$$\sup_{t} \left( \| \bar{x}^a \rho \|_{L^1} + \| \nabla \sqrt{\bar{x}^a} \rho \|_{L^2} \right) \leq C \exp \left\{ t^2 \psi^2 \right\}. \quad (3.40)$$

**Proof.** One derives from Eq. (1.1) that

$$\begin{align*}
(\rho \bar{x}^a)_t + \text{div}(u \rho \bar{x}^a) - au \rho \bar{x}^a \cdot \nabla \ln \bar{x} = 0.
\end{align*} \quad (3.41)$$

By \(3.4\), it has

$$\begin{align*}
\frac{d}{dt} \int \rho \bar{x}^a \leq C \left( \int \rho u^2 \right)^{\frac{1}{2}} \left( \int \rho \bar{x}^{2a-2} \ln^2 (e + |x|^2) \right)^{\frac{1}{2}} \\
\leq C \| \bar{x}^{a-2} \ln (e + |x|^2) \|_{L^\infty} \left( \int \rho \bar{x}^a \right)^{\frac{1}{2}} \\
\leq C \left( \int \rho \bar{x}^a \right)^{\frac{1}{2}},
\end{align*}$$

where the last inequality owes to \( a < 2 \). Thus,

$$\| \bar{x}^a \rho \|_{L^1} \leq C t. \quad (3.42)$$
Next, if we multiply (3.11) by \((\sqrt{x^a\rho})^{-1}\), differentiate it in \(x\), and then multiply the resulting expression by \(\nabla \sqrt{x^a\rho}\), we infer

\[
\frac{d}{dt} \|\nabla \sqrt{x^a\rho}\|_{L^2} \\
\leq C \left( \|\nabla u\|_{L^\infty} + \|u\nabla \ln \bar{x}\|_{L^\infty} \right) \|\nabla \sqrt{x^a\rho}\|_{L^2} + C \|\sqrt{\rho \bar{x}^a}\|_{L^{20}} \|\nabla^2 u\|_{L^q} \tag{3.43}
\]
\[+ C \|\rho \bar{x}^a\|_{L^4}^2 \left( \|\nabla u\nabla \ln \bar{x}\|_{L^\infty} + \|u\nabla^2 \ln \bar{x}\|_{L^\infty} \right).
\]

Making use of (3.35) and (3.42), we have

\[
(\|\nabla u\|_{L^\infty} + \|u\nabla \ln \bar{x}\|_{L^\infty} + \|u\nabla^2 \ln \bar{x}\|_{L^\infty})
\leq C \left( \|\nabla u\|_{L^\infty} + \|u\bar{x}^{-\frac{1}{2}}\|_{L^\infty} \right)
\leq C \left( \psi^2 + \|\nabla^2 u\|_{L^q} \right),
\]

by this, (3.43) is simplified as

\[
\frac{d}{dt} \left( 1 + \|\nabla \sqrt{x^a\rho}\|_{L^2} \right) \leq C \left( \psi^2 + \|\nabla^2 u\|_{L^q} \right) \left( 1 + \|\nabla \sqrt{x^a\rho}\|_{L^2} \right). \tag{3.44}
\]

Similarly, operating \(\nabla\) to (3.41), and multiplying it by \(q|\nabla (\rho \bar{x}^a)|^{-2} \nabla (\rho \bar{x}^a)\), to discover

\[
\frac{d}{dt} \left( 1 + \|\nabla (\rho \bar{x}^a)\|_{L^q} \right) \leq C \left( \psi^2 + \|\nabla^2 u\|_{L^q} \right) \left( 1 + \|\nabla (\rho \bar{x}^a)\|_{L^q} \right). \tag{3.45}
\]

By (3.35), exploiting the Gronwall’s inequality to (3.44) and (3.45) gives

\[
\sup_{0 \leq t \leq T_1} \left( \|\nabla \sqrt{x^a\rho}\|_{L^2} + \|\nabla (\rho \bar{x}^a)\|_{L^q} \right)
\leq C \exp \left\{ \int_0^t \left( \psi^2 + \|\nabla^2 u\|_{L^q} \right) \right\} \leq C \exp \left\{ \sqrt{t}\psi^2 \right\}.
\]

This combines with (3.42) complete the proof of Lemma 3.7.

By (3.2), the definition of \(\psi\), we deduce from Lemmas 3.5-3.7 that for any \(t \in [0, T_2]\)

\[
\psi(t) \leq C \left( 1 + t^\frac{2}{\alpha} \psi^{15}(t) \right) \exp \left\{ t^\frac{2}{\alpha} \psi^2(t) \right\} + C t^\frac{2}{\alpha} \psi^{38}(t) + C \exp \left\{ t^\frac{2}{\alpha} \psi^2 \right\}
\leq C \left( 1 + t^\alpha \psi^{38}(t) \right) \exp \left\{ t^\alpha \psi^2(t) \right\},
\]

where \(\alpha = \min\{\frac{2}{\alpha}, \frac{1}{\alpha}\}\). If choosing

\[
M = 2Ce \quad \text{and} \quad T_* = \min\{T_2, M^{-\frac{38}{\alpha}}\},
\]

we get

\[
\sup_{0 \leq t \leq T_*} \psi(t) \leq M. \tag{3.46}
\]

Once (3.46) is obtained, it yields from (3.11), (3.18) and (3.25) that

\[
\sup_{0 \leq t \leq T_*} \left( \|\sqrt{\rho \theta}\|_{L^2} + \|\nabla \theta\|_{L^2} + \|\nabla^2 u\|_{L^2} + \|\sqrt{\rho \bar{u}}\|_{L^2} \right)
\leq \int_0^{T_*} \left( \|\sqrt{\rho \theta}\|_{L^2}^2 + \|\nabla u\|_{L^\infty}^2 + \|\nabla \bar{u}\|_{L^2}^2 \right) \tag{3.47}
\]
\[\leq C_M.
\]

**Step 2. Higher order estimates for \(\theta\)**
Lemma 3.8  The solution $(\rho, u, \theta)$ of $(1.1)$, $(2.1)$ - $(2.2)$ satisfy

$$
sup_{0 \leq t \leq T} \left( \|\sqrt{\rho} \dot{\theta}\|_{L^2} + \|\nabla \theta\|_{H^1} \right)
+ \int_0^T \left( \|\nabla \dot{\theta}\|^2_{L^2} + \|\nabla^2 \theta\|^2_{L^2} + \|\nabla \theta\|_{L^\infty}^2 \right) \, ds \leq C. \tag{3.48}
$$

Proof. Define

$$
\tilde{g}_2 = \rho_0^{-\frac{1}{2}} \left( \kappa \Delta \theta_0 + \frac{\mu}{2} |\nabla u_0 + (\nabla u_0)^{tr}|^2 + \lambda (\text{div} u_0)^2 \right)
$$

and compute

$$
\int \rho|\dot{\theta}|^2(x, 0) \leq \limsup_{t \to 0^+} \int \rho^{-1} |\mu \Delta u + (\lambda + \mu) \text{div} u - R \text{div}(\rho \theta)|^2
\leq \|\tilde{g}_2\|^2_{L^2} + C \|\rho \theta_0|\nabla u_0\|_{L^2}
\leq C. \tag{3.49}
$$

Next, operating $\partial_t + \text{div}(u \cdot)$ to Eq. $(1.1)_3$ receives

$$
c_v \left( \partial_t (\rho \dot{\theta}) + \text{div}(u \rho \dot{\theta}) \right) - \kappa \Delta \dot{\theta}
= \kappa \text{div} u \Delta \theta - \kappa \partial_t (\partial_i u \cdot \nabla \theta) - \kappa \partial_i u \cdot \nabla \partial_i \theta
- R \rho \dot{\theta} \text{div} u - R \rho \theta \left( \text{div} \hat{\dot{u}} - \partial_k u^l \partial_l u^k \right)
+ \left( \frac{\mu}{2} |\nabla u + (\nabla u)^{tr}|^2 + \lambda (\text{div} u)^2 \right) \text{div} u
+ \mu \left( \partial_i u^l + \partial_j u^l \right) \left( \partial_i \hat{\dot{u}}^j + \partial_j \hat{\dot{u}}^i - \partial_i u \cdot \nabla \hat{\dot{u}}^j - \partial_j u \cdot \nabla \hat{\dot{u}}^i \right)
+ 2\lambda \text{div} u \left( \text{div} \hat{\dot{u}} - \partial_k u^l \partial_l u^k \right).
$$

After multiplied by $\dot{\theta}$, it gives

$$
\frac{d}{dt} \int \rho |\dot{\theta}|^2 + \int |\nabla \dot{\theta}|^2
\leq C \int |\dot{\theta}| |\nabla u| |\nabla^2 \theta| + C \int |\nabla u|^2 |\nabla \theta|^2
+ C \int |\dot{\theta}| \left( \rho|\dot{\theta}| |\nabla u| + \rho \theta |\nabla u|^2 + \rho \theta |\nabla \hat{\dot{u}}| + |\nabla u| |\nabla \hat{\dot{u}}| + |\nabla \hat{\dot{u}}|^3 \right). \tag{3.50}
$$

It follows from $(3.8)$, $(3.38)$, $(3.46)$ and $(3.47)$ that

$$
C \int |\dot{\theta}| |\nabla u| |\nabla^2 \theta|
\leq C \|\nabla^2 \theta\|_{L^2} \|\bar{x}^{-\frac{\theta}{2}} \dot{\theta}\|_{L^\frac{16}{8}} \|\bar{x}^{-\frac{\theta}{2}} \nabla u\|_{L^\frac{16}{8}}
\leq C \left( \|\sqrt{\rho} \dot{\theta}\|_{L^2} + 1 \right) \left( \|\sqrt{\rho} \theta\|_{L^z} + \|\nabla \theta\|_{L^2} \right) \|\bar{x}^{-\frac{\theta}{4}} \nabla u\|_{L^2} \|\nabla u\|_{L^\infty} \|\bar{x}^{-\frac{\theta}{4}} \nabla u\|_{L^\infty}
\leq C \|\sqrt{\rho} \dot{\theta}\|_{L^2}^2 + \frac{1}{6} \|\nabla \theta\|_{L^2}^2 + C,
$$
\[ C \int \left( \| \nabla u \|^2 \| \nabla \theta \|^2 + \rho |\dot{\theta}|^2 \| \nabla u \| + \rho \theta |\dot{\theta}| \| \nabla u \|^2 \right) \]
\[ \leq C \left( 1 + \| \nabla u \|^2_{L^\infty} \right) \left( \| \nabla \theta \|^2_{L^2} + \| \sqrt{\rho} \dot{\theta} \|^2_{L^2} + \| \sqrt{\rho} \theta \|^2_{L^2} \right) \]
\[ \leq C \left( 1 + \| \nabla u \|^2_{L^\infty} \right) \left( 1 + \| \sqrt{\rho} \dot{\theta} \|^2_{L^2} \right), \]

and
\[ C \int \rho \theta |\dot{\nabla} u| \leq C \| \rho \dot{x}^a \|_{L^\infty} \| \dot{x}^{-\frac{a}{2}} \theta \|_{L^4} \| \dot{x}^{-\frac{a}{2}} \theta \|_{L^4} \| \nabla \dot{u} \|_{L^2} \]
\[ \leq C \left( \| \sqrt{\rho} \dot{\theta} \|_{L^2} + \| \nabla \dot{\theta} \|_{L^2} \right) \| \nabla \dot{u} \|_{L^2} \]
\[ \leq C \| \sqrt{\rho} \dot{\theta} \|^2_{L^2} + C \| \nabla \dot{u} \|^2_{L^2} + \frac{1}{6} \| \nabla \dot{\theta} \|^2_{L^2}. \]

Again by (3.5), (3.46), (3.47), and interpolation theorem,
\[ C \int |\dot{\theta}| \| \nabla u \| \left( |\nabla \dot{u}| + |\nabla u|^2 \right) \]
\[ \leq C \left( \| \nabla \dot{u} \|^2_{L^2} + \| \nabla u \|^2_{L^2} \right) \| \dot{x}^{-\frac{a}{2}} \theta \|_{L^4} \| \dot{x}^{-\frac{a}{2}} \nabla u \|_{L^\infty} \]
\[ \leq C \left( \| \nabla \dot{u} \|^2_{L^2} + 1 \right) \left( \| \sqrt{\rho} \dot{\theta} \|_{L^2} + \| \nabla \dot{\theta} \|_{L^2} \right) \]
\[ \leq C \left( 1 + \| \sqrt{\rho} \dot{\theta} \|^2_{L^2} + \| \nabla \dot{u} \|^2_{L^2} \right) + \frac{1}{6} \| \nabla \dot{\theta} \|^2_{L^2}. \]

Substituting the last three inequalities back into (3.50) receives
\[ \frac{d}{dt} \| \sqrt{\rho} \dot{\theta} \|^2_{L^2} + \| \nabla \dot{\theta} \|^2_{L^2} \leq C \left( 1 + \| \nabla u \|^2_{L^\infty} \right) \left( 1 + \| \sqrt{\rho} \dot{\theta} \|^2_{L^2} \right) + C \| \nabla \dot{u} \|^2_{L^2}, \]
which yields by means of (3.49), (3.47) and Gronwall’s inequality
\[ \sup_{0 \leq t \leq T^*} \| \sqrt{\rho} \dot{\theta} \|_{L^2} + \int_0^{T^*} \| \nabla \dot{\theta} \|^2_{L^2} ds \leq C. \] (3.51)

As a result of (3.51), it satisfies from (3.38), (3.46) and (3.47) that
\[ \sup_{0 \leq t \leq T^*} \| \nabla \theta \|_{H^1} \leq C \] (3.52)
and
\[ \| \nabla (\rho \dot{\theta}) \|_{L^2} \leq C \left( \| \rho \|_{L^\infty} \| \nabla \dot{\theta} \|_{L^2} + \| \dot{x}^{-a} \|_{L^{2\frac{a}{q-2} = 2}} \| \rho \dot{x}^a \|_{W^{1,q}} \right) \]
\[ \leq C \| \nabla \dot{\theta} \|_{L^2} + C. \]
Furthermore, for \( q \in (2, \infty) \)
\[ \| \rho \dot{\theta} \|_{L^q} \leq C \| \rho \dot{\theta} \|_{L^2} + C \| \rho \dot{\theta} \|^\frac{2}{2} \| \nabla (\rho \dot{\theta}) \|_{L^2}^\frac{q-2}{2^2} \leq C \left( 1 + \| \nabla \dot{\theta} \|^\frac{q-2}{2} \right) \]
and
\[ \| \rho \theta \nabla u \|_{L^q} + \| \nabla u \|^2_{L^2} \leq C \left( \| \rho \theta \|^2_{L^2} + \| \nabla u \|^2_{L^2} \right) \leq C. \]
Hence, the last two inequalities and Eq.(1.1) guarantee that for $q \in (2, \infty)$
\[
\|\nabla^2 \theta\|_{L^q} \leq C \left( \|\rho \dot{\theta}\|_{L^q} + \|\rho \theta |\nabla u|\|_{L^q} + \|\nabla u\|_{L^2}^2 \right) 
\leq C \left( 1 + \|\nabla \theta\|_{L^2}^{\frac{q-2}{2}} \right),
\]
This, along with (3.51) and (3.52), shows
\[
\int_0^{T^*} (\|\nabla \theta\|_{L^\infty}^2 + \|\nabla^2 \theta\|_{L^q}^2) \, ds \leq C + C \int_0^{T^*} \|\nabla \theta\|_{L^2}^2 \leq C,
\]
which together with (3.51)-(3.52) imply (3.48). The proof is completed.

4 Proof of Theorem 1.1.

Now we are ready to prove Theorem 1.1, the existence and uniqueness of strong solution to the Cauchy problem (1.1)-(1.4).

Existence of strong solutions

Let $(\rho_0, u_0, \theta_0)$ be the functions defined in (1.4) which satisfy the hypotheses (1.6)-(1.8) imposed in Theorem 1.1. Without loss of generality, we may choose $\|\rho_0\|_{L^1(\mathbb{R}^2)} = 1$, and therefrom, for some large $R_0 \in [1, \mathbb{R}^2)$
\[
\int_{B_{R_0}} \rho_0(x) \, dx \geq \frac{3}{4}.
\]
Construct approximate functions $0 \leq \tilde{\rho}_R^0 \in C_0^\infty(\mathbb{R}^2)$ such that as $R \to \infty$
\[
\begin{cases}
\tilde{x}^\alpha \tilde{\rho}_R^0 \to \tilde{x}^\alpha \rho_0 & \text{in } L^1(\mathbb{R}^2) \cap W^{1,q}(\mathbb{R}^2), \\
\nabla \sqrt{\tilde{x}^\alpha \tilde{\rho}_R^0} \to \nabla \sqrt{\tilde{x}^\alpha \rho_0} & \text{in } L^2(\mathbb{R}^2), \\
\|\rho_0^R\|_{L^1(B_{R_0})} \geq \frac{1}{2}.
\end{cases}
\]
Consider the solution $u_0^R$ to the elliptic system
\[
\begin{cases}
-\mu \Delta u_0^R - (\mu + \lambda) \nabla \text{div} u_0^R + \rho_0^R u_0^R = \sqrt{\rho_0^R} h_1^R, & \text{in } B_R, \\
u_0^R = 0, & \text{on } \partial B_R,
\end{cases}
\]
where $\rho_0^R = \rho_0^R + R^{-1} e^{-|x|^2} > 0$ and $h_1^R = (\sqrt{\rho_0} u_0 + g_1) \ast j_{R^{-1}}$ with $j_{R^{-1}}$ being the standard mollifier of width $R^{-1}$. We claim
\[
\lim_{R \to \infty} \left( \|\nabla (u_0^R - u_0)\|_{H^1} + \left\| \sqrt{\rho_0^R} u_0^R - \sqrt{\rho_0} u_0 \right\|_{L^2} \right) = 0.
\]
In fact, extending $u_0^R$ to $\mathbb{R}^2$ by zero and multiplying (4.2) by $u_0^R$ yields
\[
\int_{\mathbb{R}^2} |\nabla u_0^R|^2 + \int_{\mathbb{R}^2} \rho_0^R |u_0^R|^2 \leq C \int_{\mathbb{R}^2} |h_1^R|^2 \leq C.
\]
With this inequality, (2.18), and (4.1)-(4.2), we furthermore deduce
\[
\|\nabla^2 u_0^R\|_{L^2} \leq C.
\]
Then, there is a limit function \( u_\infty \) such that, up to some subsequence, as \( R_j \to \infty \)

\[
\sqrt{\rho_0^R} \, u_\infty^R \rightarrow \sqrt{\rho_0^0} \, u_\infty \quad \text{in} \ L^2(\mathbb{R}^2), \quad \nabla u_0^R \rightarrow \nabla u_\infty \quad \text{in} \ H^1(\mathbb{R}^2). \tag{4.5}
\]

So, it is easy to check from (4.1), (4.2) and (4.5) that the \( u_\infty \) solves in \( D^{-1}(\mathbb{R}^2) \)

\[-\mu \Delta u_\infty - (\mu + \lambda) \nabla \text{div} u_\infty + \rho_0 u_\infty = \rho_0 u_0 + \sqrt{\rho_0 g_1}, \quad \text{in} \ \mathbb{R}^2.
\]

This together with (1.7) ensure

\[
\int_{\mathbb{R}^2} \nabla (u_\infty - u_0) \nabla \varphi \, dx + \int_{\mathbb{R}^2} \rho_0 (u_\infty - u_0) \varphi \, dx = 0, \quad \forall \ \varphi \in C_0^\infty (\mathbb{R}^2),
\]

and hence

\[
u_\infty = u_0. \tag{4.6}
\]

Multiply (4.2) by \( u_0^R \), and utilize (1.7), (4.1), (4.5), to receive

\[
\limsup_{R_j \to \infty} \left( \mu \int_{\mathbb{R}^2} |\nabla u_0^R|^2 + (\mu + \lambda) \int_{\mathbb{R}^2} |\text{div} u_0^R|^2 + \int_{\mathbb{R}^2} \rho_0 u_0^R |u_0^R|^2 \right) \leq \mu \int_{\mathbb{R}^2} |\nabla u_0|^2 + (\mu + \lambda) \int_{\mathbb{R}^2} |\text{div} u_0|^2 + \int_{\mathbb{R}^2} \rho_0 |u_0|^2,
\]

This inequality together with (4.5), (4.6), and the weakly lower semi-continuity of norms, provide that

\[
\lim_{R_j \to \infty} \int_{\mathbb{R}^2} |\nabla u_0^R|^2 = \int_{\mathbb{R}^2} |\nabla u_0|^2, \quad \lim_{R_j \to \infty} \int_{\mathbb{R}^2} \rho_0^R |u_0^R|^2 = \int_{\mathbb{R}^2} \rho_0 |u_0|^2. \tag{4.7}
\]

Due to the arbitrariness of subsequence, (4.5)–(4.7), we obtain

\[
\lim_{R \to \infty} \left( \|\nabla (u_0^R - u_0)\|_{L^2} + \left\| \sqrt{\rho_0^R} u_0^R - \sqrt{\rho_0^0} u_0 \right\|_{L^2} \right) = 0.
\]

By a similar argument, one has

\[
\lim_{R \to \infty} \left( \|\nabla^2 (u_0^R - u_0)\|_{L^2} \right) = 0.
\]

The (4.3) thus follows. Consequently, it gives from (1.6), (8.19) and (4.2) that

\[
\|\tilde{g}_1\|_{L^2} \leq C. \tag{4.8}
\]

Next, we consider the solution \( \theta_0^R \) of the following

\[
\begin{cases}
-\kappa \Delta \theta_0^R + \rho_0^R \theta_0^R \\
= \frac{\mu}{2} |\nabla u_0^R + (\nabla u_0^R)^T|^2 + \lambda (\text{div} u_0^R)^2 + \sqrt{\rho_0^R h_2^R}, & \text{in} \ B_R, \\
\theta_0^R = 0, & \text{on} \ \partial B_R,
\end{cases} \tag{4.9}
\]

where \( h_2^R = (\sqrt{\rho_0^0} + g_2) \ast j_{R^{-1}} \). We will prove

\[
\lim_{R \to \infty} \left( \|\nabla \theta_0^R - \nabla \theta_0\|_{H^1} + \left\| \sqrt{\rho_0^R} \theta_0^R - \sqrt{\rho_0^0} \theta_0 \right\|_{L^2} \right) = 0. \tag{4.10}
\]
To see this, multiplying (4.9) by $\theta_0^R$ gives

$$
\|
\nabla \theta_0^R \|_{L^2}^2 + \left\| \sqrt{\rho_0^R \theta_0^R} \right\|_{L^2}^2
\leq C \left\| \nabla u_0^R \theta_0^R \right\|_{L^1} + C \left\| \sqrt{\rho_0^R \theta_0^R} \right\|_{L^2} \left\| h_{R}^2 \right\|_{L^2(B_R)}
\leq C \| \theta_0^R x \frac{x}{2} \|_{L^\infty} \left\| \nabla u_0^R \right\|_{L^\infty} \left\| \nabla u_0^R \right\|_{L^\infty} + C \left\| \sqrt{\rho_0^R \theta_0^R} \right\|_{L^2(B_R)} \left\| h_{R}^2 \right\|_{L^2}
\leq \frac{1}{2} \left( \|
\nabla \theta_0^R \|^2_{L^2} + \left\| \sqrt{\rho_0^R \theta_0^R} \right\|^2_{L^2} \right) + C,
$$

where we have used (4.3) and (4.8). Thus,

$$
\|
\nabla \theta_0^R \|_{L^2} + \left\| \sqrt{\rho_0^R \theta_0^R} \right\|_{L^2} \leq C.
$$

This inequality, together with (4.3), (2.15), (4.9), deduces

$$
\|
\nabla^2 \theta_0^R \|_{L^2} \leq C.
$$

Therefore, there is a limit function $\theta_\infty$ such that

$$
\sqrt{\rho_0^R \theta_0^R_i} \to \sqrt{\rho_0 \theta_\infty} \quad \text{in } L^2(\mathbb{R}^2), \quad \nabla \theta_0^R_i \to \nabla \theta_\infty \quad \text{in } H^1(\mathbb{R}^2).
$$

(4.11)

It follows from (4.1), (4.3), (4.9), (4.11) and (1.8) that $\theta_\infty = \theta_0$. And a similar method runs that

$$
\lim_{R_j \to \infty} \int_{\mathbb{R}^2} \left| \nabla \theta_0^R_i \right|^2 = \int_{\mathbb{R}^2} \left| \nabla \theta_0 \right|^2,
$$

$$
\lim_{R_j \to \infty} \int_{\mathbb{R}^2} \rho_0^R |\theta_0^R_i|^2 = \int_{\mathbb{R}^2} \rho_0 |\theta_0|^2,
$$

which again with (4.11) ensure

$$
\lim_{R \to \infty} \left( \|
\nabla \theta_0^R - \nabla \theta_0 \|_{L^2(\mathbb{R}^2)} + \left\| \sqrt{\rho_0^R \theta_0^R} - \sqrt{\rho_0 \theta_0} \right\|_{L^2(\mathbb{R}^2)} \right) = 0
$$

and

$$
\|
\nabla^2 (\theta_0^R - \theta_0) \|_{L^2(\mathbb{R}^2)} \to 0,
$$

as $R \to \infty$. These last two inequalities give birth to the (4.10), and furthermore,

$$
\|
\tilde{g}_2 \|_{L^2} \leq C.
$$

(4.12)

In view of Lemma 2.1 the problem (1.11), (2.1)-(2.2) with the initial data replaced by $(\rho_0^R, u_0^R, \theta_0^R)$ has a unique solution $(\rho^R, u^R, \theta^R)$ over $B_R \times [0, T_R]$ for some $T_R > 0$. Moreover, all the estimates in Proposition 3.1 hold true for $(\rho^R, u^R, \theta^R)$ over $[0, T_*]$ for some $T_* > 0$ independent of $R$.

Extend $(\rho^R, u^R, \theta^R)$ to $\mathbb{R}^2$ by zero and denote

$$
\tilde{\rho}^R = (\phi^R)^2 \rho^R, \quad \tilde{u}^R = (\phi^R)^2 u^R, \quad \tilde{\theta}^R = (\phi^R)^2 \theta^R,
$$
where $\phi^R$ is taken from (3.3). We first deduce from (3.3) and (3.5) that
\[
\sup_{0 \leq t \leq T_*} \left( \| \sqrt{p^R u^R} \|_{L^2(\mathbb{R}^2)} + \| \sqrt{p^R \theta^R} \|_{L^2(\mathbb{R}^2)} + \| p^R \tilde{x}^a \|_{L^1 \cap L^\infty(\mathbb{R}^2)} \right) 
\leq C \sup_{0 \leq t \leq T_*} \left( \| \sqrt{p^R u^R} \|_{L^2(\mathbb{R}^2)} + \| \sqrt{p^R \theta^R} \|_{L^2(\mathbb{R}^2)} + \| p^R \tilde{x}^a \|_{L^1 \cap L^\infty(\mathbb{R}^2)} \right) \leq C, 
\]
and whence,
\[
\sup_{0 \leq t \leq T_*} \left( \| \nabla (\sqrt{p^R \tilde{x}^a}) \|_{L^2(\mathbb{R}^2)} + \| \nabla \sqrt{p^R \tilde{x}^a} \|_{L^2(\mathbb{R}^2)} \right) 
\leq C \sup_{0 \leq t \leq T_*} \left( \| \sqrt{p^R \tilde{x}^a} \|_{W^{1,q}(\mathbb{R}^2)} + \| \sqrt{p^R \tilde{x}^a} \|_{H^1(\mathbb{R}^2)} \right) \leq C. 
\]

By Hölder inequalities, it gives from (2.17), (3.3), (2.19) and (3.3) that
\[
\| \tilde{x}^\frac{1}{2} u^R \nabla \phi^R \|_{L^2(B_R)} 
\leq C_{N_0} \| u^R \|_{L^2(B_{N_0})} + CR^{-1} \| |x| \tilde{x}^\frac{1}{2} u^R \|_{L^2(B_R \setminus B_{N_0})} 
\leq C + C \| |x| \tilde{x}^\frac{1}{2} u^R \|_{L^2(B_R)} 
\leq C + C \tilde{b} \| \tilde{x}^\frac{1}{2} \nabla u^R \|_{L^2(B_R)} \leq C, \quad \text{for} \quad \tilde{b} \in (b, \frac{a}{2}).
\]

This and (3.3) imply
\[
\sup_{0 \leq t \leq T_*} \left( \| \tilde{x}^\frac{1}{2} \nabla u^R \|_{L^2(\mathbb{R}^2)} + \| \nabla^2 u^R \|_{L^2(\mathbb{R}^2)} \right) 
\leq C \sup_{0 \leq t \leq T_*} \left( \| \tilde{x}^\frac{1}{2} \nabla u^R \|_{L^2(\mathbb{R}^2)} + \| \nabla u^R \|_{H^1(\mathbb{R}^2)} + \| \tilde{x}^\frac{1}{2} u^R \nabla \phi^R \|_{L^2(B_R)} \right) 
\leq C, 
\]
\[
\sup_{0 \leq t \leq T_*} \| \nabla \tilde{\theta}^R \|_{H^1(\mathbb{R}^2)} \leq C \left( \| \nabla \theta^R \|_{H^1(\mathbb{R}^2)} + \| \nabla \phi^R \theta^R \|_{L^2(\mathbb{R}^2)} \right) \leq C. 
\]

Additionally,
\[
\int_{0}^{T_*} \left( \| \nabla^2 u^R \|_{L^q(\mathbb{R}^2)}^2 + \| \nabla^2 \theta^R \|_{L^q(\mathbb{R}^2)}^2 \right) 
\leq C \int_{0}^{T_*} \left( \| \nabla^2 u^R \|_{L^q(\mathbb{R}^2)}^2 + \| \nabla^2 \theta^R \|_{L^q(\mathbb{R}^2)}^2 \right) 
+ C \sup_{0 \leq t \leq T_*} \left( \| \nabla u^R \|_{H^1(\mathbb{R}^2)}^2 + \| \nabla \theta^R \|_{H^1(\mathbb{R}^2)}^2 + R^{-4} \| u^R \| + \theta^R \|_{L^q(\mathbb{R}^2)}^2 \right) 
\leq C. 
\]

Next, it gives from (3.35) and (3.40) that
\[
\int_{0}^{T_*} \| \tilde{x}^{\frac{1}{2}} \partial_t \tilde{\rho}^R \|_{L^2(\mathbb{R}^2)}^2 
\leq C \int_{0}^{T_*} \left( \| \tilde{x}^{\frac{1}{2}} u^R \nabla \tilde{\rho}^R \|_{L^2(\mathbb{R}^2)}^2 + \| \tilde{x}^{\frac{1}{2}} \rho^R \operatorname{div} u^R \|_{L^2(\mathbb{R}^2)}^2 \right) 
\leq C \int_{0}^{T_*} \left( \| u^R \tilde{x}^{-\frac{q}{2}} \|_{L^\infty(\mathbb{R}^2)}^2 + \| \operatorname{div} u^R \|_{L^\infty(\mathbb{R}^2)}^2 \right) \| \tilde{x}^{\frac{1}{2}} \rho^R \|_{W^{1,q}(\mathbb{R}^2)}^2 
\leq C. 
\]
Let $(\tilde{u}^R) = \partial_t \tilde{u}^R + \tilde{u}^R \cdot \nabla \tilde{u}^R$. It yields from (3.3), (3.7) and (3.3) that
\[
\sup_{0 \leq t \leq T_*} \| \sqrt{\rho^R(\tilde{u}^R)^*} \|_{L^2(\mathbb{R}^2)} \\
\leq C \sup_{0 \leq t \leq T_*} \left( \| \sqrt{\rho^R(u^R)} \|_{L^2(\mathbb{R}^2)} + \| \sqrt{\rho^R u^R} \|_{L^2(\mathbb{R}^2)} + \| \sqrt{\rho^R u^R} \|_{L^2(\mathbb{R}^2)} \right) \\
\leq C + C \| \rho^R(x^0) \|_{L^2(\mathbb{R}^2)} \sup_{0 \leq t \leq T_*} \left( \| \tilde{x}^R u^R \|_{L^4(\mathbb{R}^2)} \| \nabla u^R \|_{L^4(\mathbb{R}^2)} \right) \\
\leq C.
\] (4.19)

In terms of (2.19), (3.3), Poincaré inequality, we obtain
\[
\int_0^{T_*} \| \nabla (\tilde{u}^R)^* \|_{L^2(\mathbb{R}^2)} \\
\leq C \int_0^{T_*} \left( \| \nabla (u^R)^* \|_{L^2(\mathbb{R}^2)} + R^{-2} \| (u^R)^* \|_{L^2(\mathbb{R}^2)} \right) + C \sup_{0 \leq t \leq T_*} \left( \| \nabla u^R \|_{L^4(\mathbb{R}^2)} \| \nabla u^R \|_{L^4(\mathbb{R}^2)} + \| u^R \|_{L^2(\mathbb{R}^2)} \| \nabla u^R \|_{L^2(\mathbb{R}^2)} \right) \\
\leq C,
\] (4.20)

where in the last inequality we used
\[
R^{-1} \| u^R \|_{L^4(\mathbb{R}^2)} + \| u^R \|_{L^2(\mathbb{R}^2)} \leq C \| u^R \|_{W^{1,4}(\mathbb{R}^2)} \leq C.
\]

Thanks to (2.19), (3.8) and (3.3), a similar argument concludes that
\[
\sup_{0 \leq t \leq T_*} \| \sqrt{\rho^R(\tilde{\theta}^R)} \|_{L^2(\mathbb{R}^2)} + \int_0^{T_*} \| \nabla (\tilde{\theta}^R)^* \|_{L^2(\mathbb{R}^2)} \leq C. \tag{4.21}
\]

These inequalities (4.13)-(4.21) ensure that the sequence $(\tilde{\rho}^R, \tilde{u}^R, \tilde{\theta}^R)$ converges, up to some subsequences, to some $(\rho, u, \theta)$ in weak sense as $R \to \infty$,
\[
\begin{cases}
\frac{\tilde{x}^R - \tilde{x}^R}{R} \to \bar{x}^R & \text{weakly} \text{ in } L^\infty(0, T_*; W^{1,q}(\mathbb{R}^2)); \\
\nabla \frac{\sqrt{\tilde{\rho}^R \tilde{x}^R}}{R} \nabla \rho \to \nabla \rho & \text{weakly} \text{ in } L^\infty(0, T_*; L^2(\mathbb{R}^2)); \\
\nabla \frac{\tilde{u}^R}{R} \nabla u \to \nabla u & \text{weakly} \text{ in } L^\infty(0, T_*; L^2(\mathbb{R}^2)); \\
\nabla \frac{\tilde{\theta}^R}{R} \nabla \theta \to \nabla \theta & \text{weakly} \text{ in } L^\infty(0, T_*; L^2(\mathbb{R}^2)); \\
\n\frac{\tilde{x}^R}{R} \nabla \tilde{u} \to \nabla \tilde{u} & \text{weakly} \text{ in } L^\infty(0, T_*; L^2(\mathbb{R}^2)); \\
\n\frac{\tilde{u}^R}{R} \nabla \tilde{\theta} \to \nabla \tilde{\theta} & \text{weakly} \text{ in } L^\infty(0, T_*; L^2(\mathbb{R}^2)); \\
\n\frac{\tilde{u}^R - \tilde{\theta}^R}{R} \nabla \phi \to \nabla \tilde{\phi} & \text{weakly} \text{ in } L^\infty(0, T_*; L^2(\mathbb{R}^2)); \\
\n\frac{\tilde{\theta}^R}{R} \nabla \tilde{\phi} \to \nabla \tilde{\phi} & \text{weakly} \text{ in } L^\infty(0, T_*; L^2(\mathbb{R}^2)); \\
\end{cases}
\] (4.22)

and for any $N > 0$
\[
\frac{\tilde{x}^R}{R} \frac{\tilde{\theta}^R}{R} \to \bar{x}^R \bar{\phi} \text{ in } C \left( \overline{B_N \times [0, T_*]} \right).
\]

Therefore, for any $\phi \in C^\infty_0(\mathbb{R}^2 \times [0, T_*])$, we may choose $\phi(\phi^R)^7$ as a test function for the IBV problem (1.1), (2.1)-(2.2) with the initial data replaced by $(\rho_0^R, u_0^R, \theta_0^R)$. Via a standard limit procedure, we conclude that $(\rho, u, \theta)$ solves the original problem (1.1)-(1.4), and moreover, satisfies the property (1.9). So far, the existence part of strong solutions to the (1.1)-(1.4) is established.

Uniqueness
It remains to show the solution in Theorem 1.1 is unique in this regularity class. Firstly, for some large \( R^* \) we have

\[
\bar{x}^a \rho \in L^\infty(0, T_*; L^1(\mathbb{R}^2)) \quad \text{and} \quad \inf_{t \in [0, T_*]} \int_{B_{R^*}} \rho(x, t) \, dx \geq \frac{1}{4}.
\]

Next to show \((\rho, u, \theta)\) and \((\bar{\rho}, \bar{u}, \bar{\theta})\), the two solutions described in Theorem 1.1 must be identical. For this we define

\[\Phi = \rho - \bar{\rho}, \quad \Psi = u - \bar{u}, \quad \Theta = \theta - \bar{\theta}.\]

Subtracting Eq. (1.1) satisfied by \((\rho, u, \theta)\) and \((\bar{\rho}, \bar{u}, \bar{\theta})\) leads to

\[
\Phi_t + \bar{u} \cdot \nabla \Phi + \Phi \nabla \bar{u} + \rho \nabla \Psi + \Psi \cdot \nabla \rho = 0. \tag{4.23}
\]

Choosing \( \beta \in (1, a) \) such that \( 2 < \frac{2q}{q-(q-2)(a-\beta)} < q \), we multiply (4.23) by \( 2\Phi \bar{x}^{2\beta} \) and deduce

\[
\frac{d}{dt} \| \Phi \bar{x}^{2\beta} \|^2_{L^2} 
\leq C \left( \| \nabla \bar{u} \|_{L^\infty} + \| \bar{u} \bar{x}^{-\frac{3}{2}} \|_{L^\infty} \right) \| \Phi \bar{x}^{2\beta} \|_{L^2}^2 + C \| \rho \bar{x}^{\beta} \|_{L^\infty} \| \nabla \Psi \|_{L^2} \| \Phi \bar{x}^{\beta} \|_{L^2}
+ C \| \Phi \bar{x}^{\beta} \|_{L^2} \| \Psi \bar{x}^{-(a-\beta)} \|_{L^\infty} \| \bar{x}^a \nabla \rho \|_{L^\infty} \frac{2q}{q-(q-2)(a-\beta)} \| \bar{x}^a \nabla \rho \|_{L^\infty} \frac{2q}{q-(q-2)(a-\beta)} \right)
\leq C \left( 1 + \| \nabla^2 \bar{u} \|_{L^2} \right) \| \Phi \bar{x}^{2\beta} \|_{L^2}^2 + C \| \Phi \bar{x}^{\beta} \|_{L^2} \left( \| \nabla \Psi \|_{L^2} + \| \sqrt{\Theta} \Psi \|_{L^2} \right)
\leq C \left( 1 + \| \nabla^2 \bar{u} \|_{L^2} \right) \left( \| \Phi \bar{x}^{2\beta} \|_{L^2}^2 + \| \sqrt{\Theta} \Psi \|_{L^2}^2 \right)
\leq C \| \nabla \Psi \|_{L^2}^2 + C \| \sqrt{\Theta} \Psi \|_{L^2}^2.
\]

where the second inequality follows from (1.9), (3.7), and (3.5).

Second, it gives from the momentum equations that

\[
\rho \Psi_t + \rho u \cdot \nabla \Psi - \mu \Delta \Psi - \nabla \left( (\mu + \lambda) \nabla \Psi \right)
= -\rho \Psi \cdot \nabla \bar{u} - \Phi (\bar{u}_t + \bar{u} \cdot \nabla \bar{u}) - \nabla (\rho \Theta + \Phi \bar{\theta}).
\tag{4.25}
\]

Observe from (1.9), (3.7) and (3.8) that

\[
- \int_{\mathbb{R}^2} \nabla \left( \rho \Theta + \Phi \bar{\theta} \right) \Psi = \int_{\mathbb{R}^2} \left( \rho \Theta + \Phi \bar{\theta} \right) \nabla \Psi
\leq C \| \nabla \Psi \|_{L^2} \left( \| \rho \Theta \|_{L^2} + \| \bar{\theta} \bar{x}^{-\beta} \|_{L^\infty} \| \bar{x}^\beta \Phi \|_{L^2} \right)
\leq C \| \nabla \Psi \|_{L^2} \left( \| \sqrt{\Theta} \Psi \|_{L^2}^2 + \| \bar{x}^\beta \Phi \|_{L^2}^2 \right)
\]

and

\[
C \| \Phi \Psi \| \left( |\bar{u}_t| + |\bar{u}| |\nabla \bar{u}| \right) \|_{L^1}
\leq C \| \Phi \bar{x}^{\beta} \|_{L^2} \| \Psi \bar{x}^{-\frac{3}{2}} \|_{L^4} (|\bar{u}|) \bar{x}^{-\frac{3}{2}} \|_{L^4}
\leq C \| \Phi \bar{x}^{\beta} \|_{L^2} \left( \| \sqrt{\Psi} \|_{L^2} + \| \nabla \Psi \|_{L^2} \right) \left( 1 + \| \nabla \bar{u} \|_{L^2} \right).
\tag{4.26}
\]

With the last two inequalities, integrating (4.25) after multiplied by \( \Psi \) yields

\[
\frac{d}{dt} \| \sqrt{\Theta} \Psi \|_{L^2}^2 + \| \nabla \Psi \|_{L^2}^2
\leq C \| \nabla \bar{u} \|_{L^\infty} \| \sqrt{\rho} \Psi \|_{L^2}^2 + C \| \Phi \Psi \| \left( |\bar{u}_t| + |\bar{u}| |\nabla \bar{u}| \right) \|_{L^1}
- \int_{\mathbb{R}^2} \nabla \left( \rho \Theta + \Phi \bar{\theta} \right) \Psi
\leq C \left( 1 + \| \nabla^2 \bar{u} \|_{L^2} + \| \nabla \bar{u} \|_{L^2}^2 \right) \left( \| \sqrt{\Theta} \Psi \|_{L^2}^2 + \| \bar{x}^\beta \Phi \|_{L^2}^2 + \| \sqrt{\Theta} \|_{L^2}^2 \right).
\tag{4.27}
\]
Taking the above three inequalities into account, we arrive at
\[ 3 \rho \leq -\rho \nabla \bar{\theta} - \Phi (\bar{\theta}_t + \bar{\theta} \cdot \nabla \bar{\theta}) - R_p \partial \nabla \Psi - R (\rho \Theta + \Phi \partial \div \partial \bar{u}) + \mu \nabla \Psi : (\nabla \bar{u} + (\nabla \bar{u})^T + \bar{u}) + \mu \nabla \bar{u} : (\nabla \Psi)^T + \lambda \div \Psi \div (\partial + \bar{u}). \]

After multiplied by \( \Theta \), it takes the form
\[
d \frac{d}{dt} \| \sqrt{\rho} \Theta \|_{L^2}^2 + \| \nabla \Theta \|_{L^2}^2 \\
\leq C \| \nabla \bar{\theta} \|_{L^\infty} + \| \sqrt{\rho} \Theta \|_{L^2} + \| \nabla \Psi \|_{L^2} + C \| \Phi \| (\Theta) \| \Theta \|_{L^1} + C \| \rho \| \Psi \| \Theta \|_{L^1} \\
+ C \| (\rho \Theta + |\nabla \Theta|) \| \Theta \| \nabla \bar{u} \|_{L^1} + C \| \nabla \Psi \| (\| \nabla \bar{u} \| + |\nabla \bar{u}|) \| \Theta \|_{L^1}. \tag{4.28} \]

Similar argument as (4.29) runs that
\[
C \| \Phi \| (\Theta) \| \Theta \|_{L^1} + C \| (\rho \Theta + |\nabla \Theta|) \| \Theta \| \nabla \bar{u} \|_{L^1} \\
\leq C \| \Phi \bar{x}^\beta \|_{L^2} \| \bar{x}^{-\frac{1}{2}} \|_{L^4} \left( \| (\Theta) \| \bar{x}^{-\frac{1}{2}} \|_{L^4} + \| \nabla \bar{u} \|_{L^\infty} \| \bar{x}^{-\frac{1}{2}} \|_{L^4} \right) \\
+ C \| \nabla \bar{u} \|_{L^\infty} \| \sqrt{\rho} \Theta \|_{L^2}^2 \leq C \| \Phi \bar{x}^\beta \|_{L^2} \left( \| \sqrt{\rho} \Theta \|_{L^2} + \| \nabla \Theta \|_{L^2} \right) \left( 1 + \| \nabla \bar{\theta} \|_{L^2} + \| \nabla \bar{u} \|\right) \| \nabla \Theta \|_{L^2} \tag{4.29} \]

By (1.9) and (3.8), it satisfies
\[
C \| \rho \| \| \nabla \Psi \| \Theta \|_{L^1} \leq C \| \| \nabla \Psi \|_{L^2} |\rho \bar{x}^\alpha \|_{L^\infty} \| \bar{x}^{-\frac{1}{2}} \|_{L^\infty} \| \bar{x}^{-\frac{1}{2}} \|_{L^\infty} \| \sqrt{\rho} \Theta \|_{L^2} \\
\leq C \| \| \nabla \Psi \|_{L^2}^2 + C \| \sqrt{\rho} \Theta \|_{L^2}^2 \]

and
\[
C \| \| \nabla \Psi \| (|\nabla \bar{u} + \nabla \bar{u}|) \| \Theta \|_{L^1} \leq C \| \| \nabla \Psi \|_{L^2} |\bar{x}^{-\frac{1}{2}} \|_{L^\infty} \| \bar{x}^{-\frac{1}{2}} \|_{L^\infty} \| \bar{x}^{-\frac{1}{2}} \|_{L^\infty} \| \bar{x}^{-\frac{1}{2}} \|_{L^\infty} \| \bar{x}^{-\frac{1}{2}} \|_{L^\infty} \| \nabla \bar{u} \|_{L^2} \| \nabla \bar{u} \|_{L^2} \| \nabla \bar{u} \|_{L^2} \| \nabla \bar{u} \|_{L^2} \leq C \| \| \nabla \Psi \|_{L^2}^2 + C \| \sqrt{\rho} \Theta \|_{L^2}^2 + \frac{1}{4} \| \nabla \Theta \|_{L^2}^2 \]

Taking the above three inequalities into account, we arrive at
\[
d \frac{d}{dt} \| \sqrt{\rho} \Theta \|_{L^2}^2 + \| \nabla \Theta \|_{L^2}^2 \\
\leq C \left( 1 + \| \nabla \bar{\theta} \|_{L^2}^2 + \| \nabla \bar{u} \|_{L^2}^2 + \| \nabla \bar{u} \|_{L^2}^2 \right) G + C \| \| \nabla \Psi \|_{L^2}^2 \tag{4.30} \]

where
\[
G \triangleq \| \sqrt{\rho} \Theta \|_{L^2}^2 + \| \sqrt{\rho} \Theta \|_{L^2}^2 + \| \Phi \bar{x}^\beta \|_{L^2}^2.
\]

In conclusion, (4.27) + \( K_2 \times \) (4.27) + (4.30) provides that
\[
G'(t) \leq C \left( 1 + \| \nabla \bar{u} \|_{L^2}^2 + \| \nabla \bar{u} \|_{L^2}^2 + \| \nabla \bar{u} \|_{L^2}^2 + \| \nabla \bar{u} \|_{L^2}^2 \right) G, \tag{4.31}
\]
so long as $K_2$ is chosen sufficiently large. Owing to (1.9) and $G(0) = 0$, we integrate (4.31) and conclude $G(t) = 0$ for any $t \in [0, T^*_\ast]$, and therefrom,

$$
\rho = \bar{\rho}, \; u = \bar{u}, \; \theta = \bar{\theta}, \; a.e. \; (x, t) \in \mathbb{R}^2 \times (0, T^*_\ast).
$$

This is the end of the proof of Theorem 1.1.

References

[1] S. Agmon, A. Douglis, L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, I, Comm. Pure Appl. Math., 12 (1959), 623-727; II, Comm. Pure Appl. Math. 17 (1964), 35-92.

[2] L. Caffarelli, R. Kohn, L. Nirenberg, First order interpolation inequalities with weights, Composition Math., 53 (1984), 259-275.

[3] Y. Cho, H. Choe, H. Kim, Unique solvability of the initial boundary value problems for compressible viscous fluids, J. Math. Pures Appl., 83 (9) (2004), 243-275.

[4] Y. Cho, H. Kim, On classical solutions of the compressible Navier-Stokes equations with nonnegative initial densities, Manuscript Math., 120 (2006), 91-129.

[5] Y. Cho, H. Kim, Existence results for viscous polytropic fluids with vacuum, J. Differ. Eqs., 228 (2006), 377-411.

[6] F. Catrina, Z. Wang, On the Caffarelli-Kohn-Nirenberg inequalities: Sharp constants, existence (and nonexistence), and symmetry of extremal functions, Comm. Pure Appl. Math., 54 (2001), 229-258.

[7] D. Bresch, B. Desjardins, On the existence of global weak solutions to the Navier-Stokes equations for viscous compressible and heat conducting fluids, J. Math. Pures Appl., 87 (2007), 57-90.

[8] D. Fang, T. Zhang, R. Zi, A blow-up criterion for two dimensional compressible viscous heat-conductive flows, Nonlinear Analysis, T.M.A., 75(6) (2012), 3130-3141.

[9] E. Feireisl, Dynamics of viscous compressible fluids, Oxford University Press, 2004.

[10] E. Feireisl, A. Novotny, H. Petzeltová, On the existence of globally defined weak solutions to the Navier-Stokes equations, J. Math. Fluid Mech., 3(4) (2001), 358-392.

[11] E. Gagliardo, Ulteriori proprietà di alcune classi di funzioni in più variabili, Ricerche di Mat. Napoli., 8 (1959), 24-51.

[12] D. Hoff, Global existence for 1D, compressible, isentropic Navier-Stokes equations with large initial data, Trans. Amer. Math. Soc., 303 (1978) 169-181.

[13] D. Hoff, Global existence of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data, J. Differ. Eqs., 120 (1995), 215-254.
[14] J. Nash, *Le problème de Cauchy pour les équations différentielles d’un fluide général*, Bull. Soc. Math. France., 90 (1962) 487-497.

[15] V. Kazhikhov, V. Shelukhin.: *Unique global solution with respect to time of initial-boundary-value problems for one-dimensional equations of a viscous gas*, J. Appl. Math. Mech., 41 (1977), 273-282.

[16] X. Huang, J. Li, Z. Xin, *Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible Navier-Stokes equations*, Comm. Pure Appl. Math., 65 (2012), 549-585.

[17] O. Ladyzenskaja, V. Solonnikov, N. Ural’ceva, *Linear and quasilinear equations of parabolic type*, American Mathematical Society, Providence, RI (1968).

[18] J. Li, Z. Liang, *Local well-posedness of strong and classical solutions to Cauchy problem of the two-dimensional barotropic compressible Navier-Stokes equations with vacuum*, J. Math. Pures Appl., 102 (2014), 640-671.

[19] J. Li, Z. Xin, *Global well-posedness and large time asymptotic behavior of classical solutions to the compressible Navier-Stokes equations with vacuum*, [http://arxiv.org/abs/1310.1673](http://arxiv.org/abs/1310.1673).

[20] Z. Liang, *Local strong solution and blow-up criterion for the 2D nonhomogeneous incompressible fluids*, J. Differ. Eqns., 258 (2015), 2633-2654.

[21] Z. Liang, X. Shi, *Classical solution to the Cauchy problem for the 2D viscous polytropic fluids with vacuum and zero heat-conduction*, Comm. Math. Sci., 13 (2015), 327-345.

[22] P. Lions, *Mathematical Topics in Fluid Mechanics*, Vol.1, Incompressible Models, Oxford University Press, NewYork, 1996.

[23] P. Lions, *Mathematical Topics in Fluid Mechanics*, Volume 2, Compressible Models, Oxford Science Publication, Oxford, 1998.

[24] A. Matsumura, T. Nishida, *Initial value problems for the equations of motion of compressible viscous and heat-conductive fluids*, Proc. Japan Acad. Ser. A Math. Sci., 55 (1979), 337-342.

[25] C. Miao, *Harmonic analysis and its application in partial differential equations*, (in Chinese), Science Press, Beijing(2004).

[26] D. Serre, *Solutions faibles globales des équations de Navier-Stokes pour un fluide compressible*, C. R. Acad. Sci. Paris Ser. I Math., 303 (1986) 639-642.

[27] D. Serre, *Sur l’équation monodimensionnelle d’un fluide visqueux, compressible et conducteur de chaleur*, C. R. Acad. Sci. Paris Ser. I Math., 303 (1986) 703-706.

[28] J. Serrin, *On the uniqueness of compressible fluid motion*, Arch. Rational. Mech. Anal., 3 (1959), 271-288.

[29] Z. Wu, J. Yin, C. Wang, *Introduction to the elliptic and parabolic type equations*, (in Chinese) Beijing: Science Press, (2003).