Towards a classification of the tridiagonal pairs

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Abstract

Let $\mathbb{K}$ denote a field and let $V$ denote a vector space over $\mathbb{K}$ with finite positive dimension. Let $\text{End}(V)$ denote the $\mathbb{K}$-algebra consisting of all $\mathbb{K}$-linear transformations from $V$ to $V$. We consider a pair $A, A^* \in \text{End}(V)$ that satisfy (i)–(iv) below:

(i) Each of $A, A^*$ is diagonalizable.
(ii) There exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of $A$ such that $A^* V_i \subseteq V_{i-1} + V_i + V_{i+1}$ for $0 \leq i \leq d$, where $V_{-1} = 0$ and $V_{d+1} = 0$.
(iii) There exists an ordering $\{V^*_i\}_{i=0}^\delta$ of the eigenspaces of $A^*$ such that $A^* V^*_i \subseteq V^*_{i-1} + V^*_i + V^*_{i+1}$ for $0 \leq i \leq \delta$, where $V^*_{\delta-1} = 0$ and $V^*_{\delta+1} = 0$.
(iv) There is no subspace $W$ of $V$ such that $AW \subseteq W$, $A^* W \subseteq W$, $W \neq 0$, $W \neq V$.

We call such a pair a tridiagonal pair on $V$. Let $E^*_0$ denote the element of $\text{End}(V)$ such that $(E^*_0 - I)V^*_i = 0$ and $E^*_0 V^*_i = 0$ for $1 \leq i \leq d$. Let $\mathcal{D}$ (resp. $\mathcal{D}^*$) denote the $\mathbb{K}$-subalgebra of $\text{End}(V)$ generated by $A$ (resp. $A^*$). In this paper we prove that the span of $E^*_0 \mathcal{D} \mathcal{D}^* E^*_0$ equals the span of $E^*_0 \mathcal{D} E^*_0 \mathcal{D} E^*_0$, and that the elements of $E^*_0 \mathcal{D} E^*_0$ mutually commute. We relate these results to some conjectures of Tatsuro Ito and the second author that are expected to play a role in the classification of tridiagonal pairs.

1 Tridiagonal pairs

Throughout the paper $\mathbb{K}$ denotes a field and $V$ denotes a vector space over $\mathbb{K}$ with finite positive dimension.

We begin by recalling the notion of a tridiagonal pair. We will use the following terms.

Let $\text{End}(V)$ denote the $\mathbb{K}$-algebra consisting of all $\mathbb{K}$-linear transformations from $V$ to $V$. For $A \in \text{End}(V)$ and for a subspace $W \subseteq V$, we call $W$ an eigenspace of $A$ whenever $W \neq 0$ and there exists $\theta \in \mathbb{K}$ such that $W = \{ v \in V \mid Av = \theta v \}$; in this case $\theta$ is the eigenvalue of $A$ associated with $W$. We say $A$ is diagonalizable whenever $V$ is spanned by the eigenspaces of $A$.

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Definition 1.1 [19] By a tridiagonal pair on $V$ we mean an ordered pair $A, A^* \in \text{End}(V)$ that satisfy (i)–(iv) below:

(i) Each of $A, A^*$ is diagonalizable.

(ii) There exists an ordering $\{V_i\}_{i=0}^{d}$ of the eigenspaces of $A$ such that

$$A^*V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d),$$

where $V_{-1} = 0$ and $V_{d+1} = 0$.

(iii) There exists an ordering $\{V^*_i\}_{i=0}^{\delta}$ of the eigenspaces of $A^*$ such that

$$AV^*_i \subseteq V^*_{i-1} + V^*_i + V^*_{i+1} \quad (0 \leq i \leq \delta),$$

where $V^*_{-1} = 0$ and $V^*_{\delta+1} = 0$.

(iv) There is no subspace $W$ of $V$ such that $AW \subseteq W$, $A^*W \subseteq W$, $W \neq 0$, $W \neq V$.

Note 1.2 It is a common notational convention to use $A^*$ to represent the conjugate-transpose of $A$. We are not using this convention. In a tridiagonal pair $A, A^*$ the linear transformations $A$ and $A^*$ are arbitrary subject to (i)–(iv) above.

Let $A, A^*$ denote a tridiagonal pair on $V$, as in Definition 1.1. By [19, Lemma 4.5] the integers $d$ and $\delta$ from (ii), (iii) are equal; we call this common value the diameter of the pair.

We refer the reader to [1–3, 5, 19–21, 27, 35–37, 45, 47, 67] for background on tridiagonal pairs. See [4, 6–11, 14, 15, 17, 18, 22–26, 29–33, 48–50, 52–54, 56, 65, 70] for related topics. The following special case has received a lot of attention. A tridiagonal pair $A, A^*$ is called a Leonard pair whenever the eigenspaces for $A$ (resp. $A^*$) all have dimension 1. See [12, 13, 16, 34, 38–44, 46, 55, 57–64, 66, 68, 69] for information about Leonard pairs.

2 Tridiagonal systems

When working with a tridiagonal pair, it is often convenient to consider a closely related object called a tridiagonal system. To define a tridiagonal system, we recall a few concepts from linear algebra. Let $A$ denote a diagonalizable element of $\text{End}(V)$. Let $\{V_i\}_{i=0}^{d}$ denote an ordering of the eigenspaces of $A$ and let $\{\theta_i\}_{i=0}^{d}$ denote the corresponding ordering of the eigenvalues of $A$. For $0 \leq i \leq d$ let $E_i : V \to V$ denote the linear transformation such that $(E_i - I)V_i = 0$ and $E_iV_j = 0$ for $j \neq i$ ($0 \leq j \leq d$). Here $I$ denotes the identity of $\text{End}(V)$. We call $E_i$ the primitive idempotent of $A$ corresponding to $V_i$ (or $\theta_i$). Observe that (i) $\sum_{i=0}^{d} E_i = I$; (ii) $E_i E_j = \delta_{i,j} E_i$ ($0 \leq i, j \leq d$); (iii) $V_i = E_i V$ ($0 \leq i \leq d$); (iv) $A = \sum_{i=0}^{d} \theta_i E_i$. Moreover

$$E_i = \prod_{\substack{0 \leq j \leq d \\theta_j \neq \theta_i}} \frac{A - \theta_j I}{\theta_i - \theta_j}.$$  \hspace{1cm} (3)

We note that each of $\{E_i\}_{i=0}^{d}$, $\{A^i\}_{i=0}^{d}$ is a basis for the $K$-subalgebra of $\text{End}(V)$ generated by $A$. 


Now let $A, A^*$ denote a tridiagonal pair on $V$. An ordering of the eigenspaces of $A$ (resp. $A^*$) is said to be standard whenever it satisfies (1) (resp. (2)). We comment on the uniqueness of the standard ordering. Let $\{V_i\}_{i=0}^d$ denote a standard ordering of the eigenspaces of $A$. Then the ordering $\{V_{d-i}\}_{i=0}^d$ is standard and no other ordering is standard. A similar result holds for the eigenspaces of $A^*$. An ordering of the primitive idempotents of $A$ (resp. $A^*$) is said to be standard whenever the corresponding ordering of the eigenspaces of $A$ (resp. $A^*$) is standard.

**Definition 2.1** By a tridiagonal system on $V$ we mean a sequence

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

that satisfies (i)–(iii) below.

(i) $A, A^*$ is a tridiagonal pair on $V$.

(ii) $\{E_i\}_{i=0}^d$ is a standard ordering of the primitive idempotents of $A$.

(iii) $\{E_i^*\}_{i=0}^d$ is a standard ordering of the primitive idempotents of $A^*$.

We will use the following notation.

**Notation 2.2** let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a tridiagonal system on $V$. We denote by $D$ (resp. $D^*$) the $K$-subalgebra of End($V$) generated by $A$ (resp. $A^*$). For $0 \leq i \leq d$ let $\theta_i$ (resp. $\theta_i^*$) denote the eigenvalue of $A$ (resp. $A^*$) associated with the eigenspace $E_i V$ (resp. $E_i^* V$). We observe $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) are mutually distinct and contained in $K$.

Referring to Notation 2.2, it has been conjectured that $E_0 V$ has dimension 1, provided that $K$ is algebraically closed [20]. A more recent and stronger conjecture is that $E_0 T E_0^*$ is commutative, where $T$ is the $K$-subalgebra of End($V$) generated by $D$ and $D^*$ [47]. There is more detailed version of this conjecture which reads as follows:

**Conjecture 2.3** [47, Conjecture 12.2] With reference to Notation 2.2 the following (i), (ii) hold.

(i) $E_0 T E_0^*$ is generated by $E_0^* D E_0^*$.

(ii) The elements of $E_0^* D E_0^*$ mutually commute.

The following special case of Conjecture 2.3 has been proven. Referring to Notation 2.2, there is a well known parameter $q$ associated with $A, A^*$ that is used to describe the eigenvalues [19,56]. In [28], Conjecture 2.3 is proven assuming $q$ is not a root of unity and $K$ is algebraically closed. In this paper we use a different approach to prove part (ii) of Conjecture 2.3 without any extra assumptions. We also obtain a result which shed some light on why part (i) of the conjecture should be true without any extra assumptions. We now state our main theorem.
**Theorem 2.4** With reference to Notation 2.2 the following (i), (ii) hold.

(i) The span of $E_0^*D^kD^*E_0^*$ is equal to the span of $E_0^*DE_0^*D^kE_0^*$.

(ii) The elements of $E_0^*DE_0^*$ mutually commute.

Our proof of Theorem 2.4 appears in Section 11. In Sections 3–10 we obtain some results that will be used in the proof. Our point of departure is the following observation.

**Lemma 2.5** With reference to Notation 2.2 the following (i), (ii) hold for $0 \leq i, j, k \leq d$.

(i) $E_i^*A^kE_j^* = 0$ if $k < |i - j|$.

(ii) $E_iA^kE_j = 0$ if $k < |i - j|$.

**Proof.** Routinely obtained using lines (1), (2) and Definition 2.1. □

### 3 The $D_4$ action

Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a tridigonal system on $V$. Then each of the following is a tridigonal system on $V$:

- $\Phi^* := (A^*; \{E_i^*\}_{i=0}^d; A; \{E_i\}_{i=0}^d)$,
- $\Phi \downarrow := (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$,
- $\Phi \downarrow \downarrow := (A; \{E_{d-i}\}_{i=0}^d; A^*; \{E_{d-i}^*\}_{i=0}^d)$,
- $\Phi \uparrow := (A^*; \{E_i^*\}_{i=0}^d; A; \{E_i\}_{i=0}^d)$,
- $\Phi \uparrow \uparrow := (A^*; \{E_{d-i}^*\}_{i=0}^d; A; \{E_i\}_{i=0}^d)$,
- $\Phi \uparrow \downarrow := (A^*; \{E_{d-i}^*\}_{i=0}^d; A; \{E_{d-i}\}_{i=0}^d)$.

Viewing $\star, \downarrow, \downarrow \downarrow$ as permutations on the set of all tridiagonal systems,

\[
\star^2 = \downarrow^2 = \downarrow\downarrow = 1, \quad (4)
\]

\[
\downarrow \star = \star \downarrow, \quad \downarrow \star \downarrow = \star \downarrow \downarrow = 1. \quad (5)
\]

The group generated by symbols $\star, \downarrow, \downarrow \downarrow$ subject to the relations (4), (5) is the dihedral group $D_4$. We recall that $D_4$ is the group of symmetries of a square, and has 8 elements. Apparently $\star, \downarrow, \downarrow \downarrow$ induce an action of $D_4$ on the set of all tridiagonal systems. Two tridiagonal systems will be called *relatives* whenever they are in the same orbit of this $D_4$ action. The relatives of $\Phi$ are as follows:

| name | relative |
|------|----------|
| $\Phi$ | $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ |
| $\Phi^\downarrow$ | $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ |
| $\Phi^\downarrow \downarrow$ | $(A; \{E_{d-i}\}_{i=0}^d; A^*; \{E_{d-i}^*\}_{i=0}^d)$ |
| $\Phi^\uparrow$ | $(A^*; \{E_i^*\}_{i=0}^d; A; \{E_i\}_{i=0}^d)$ |
| $\Phi^\uparrow \uparrow$ | $(A^*; \{E_{d-i}^*\}_{i=0}^d; A; \{E_i\}_{i=0}^d)$ |
| $\Phi^\uparrow \downarrow$ | $(A^*; \{E_{d-i}^*\}_{i=0}^d; A; \{E_{d-i}\}_{i=0}^d)$ |
| $\Phi^\uparrow \downarrow \downarrow$ | $(A^*; \{E_{d-i}^*\}_{i=0}^d; A; \{E_{d-i}\}_{i=0}^d)$ |
4 Some polynomials

Let $\lambda$ denote an indeterminate and let $\mathbb{K}[\lambda]$ denote the $\mathbb{K}$-algebra consisting of all polynomials in $\lambda$ that have coefficients in $\mathbb{K}$.

**Definition 4.1** With reference to Notation 2.2, for $0 \leq i \leq d$ we define the following polynomials in $\mathbb{K}[\lambda]$:

$$\tau_i = (\lambda - \theta_0)(\lambda - \theta_1) \cdots (\lambda - \theta_{i-1}),$$

$$\eta_i = (\lambda - \theta_d)(\lambda - \theta_{d-1}) \cdots (\lambda - \theta_{d-i+1}),$$

$$\tau_i^* = (\lambda - \theta_0^*)(\lambda - \theta_1^*) \cdots (\lambda - \theta_{i-1}^*),$$

$$\eta_i^* = (\lambda - \theta_d^*)(\lambda - \theta_{d-1}^*) \cdots (\lambda - \theta_{d-i+1}^*).$$

Note that each of $\tau_i, \eta_i, \tau_i^*, \eta_i^*$ is monic with degree $i$.

The following lemmas show the significance of these polynomials. We will focus on $\tau_i, \eta_i$; of course similar results hold for $\tau_i^*, \eta_i^*$.

**Lemma 4.2** With reference to Notation 2.2, each of $\{\tau_i(A)\}_{i=0}^d, \{\eta_i(A)\}_{i=0}^d$ form a basis for $\mathcal{D}$.

**Proof.** The sequence $\{A^i\}_{i=0}^d$ is a basis for $\mathcal{D}$ and each of $\tau_i, \eta_i$ has degree $i$ for $0 \leq i \leq d$. $\square$

**Lemma 4.3** With reference to Notation 2.2 for $0 \leq i \leq d$ we have

$$\tau_i(A) = \sum_{j=0}^d \tau_i(\theta_j)E_j,$$

$$\eta_i(A) = \sum_{j=0}^{d-i} \eta_i(\theta_j)E_j,$$

$$E_i = \sum_{j=0}^d \frac{\eta_{d-j}(\theta_i)\tau_j(A)}{\tau_i(\theta_i)\eta_{d-i}(\theta_i)},$$

$$E_i = \sum_{j=d-i}^d \frac{\tau_{d-j}(\theta_i)\eta_j(A)}{\tau_i(\theta_i)\eta_{d-i}(\theta_i)}.$$  

**Proof.** To get the equation on the left in (10), observe that

$$\tau_i(A) = \sum_{j=0}^d \tau_i(A)E_j = \sum_{j=0}^d \tau_i(\theta_j)E_j$$

and $\tau_i(\theta_j) = 0$ for $0 \leq j \leq i - 1$. Concerning the equation on the right in (10), first observe by (3) that

$$E_i = \frac{\tau_i(A)\eta_{d-i}(A)}{\tau_i(\theta_i)\eta_{d-i}(\theta_i)}.$$  

By [41, Lemma 5.4] or a routine induction on $d$ we find

$$\eta_{d-i} = \sum_{j=i}^d \eta_{d-j}(\theta_i)\tau_j^{-1} \tau_j.$$
Therefore

\[ \tau_i \eta_{d-i} = \sum_{j=i}^{d} \eta_{d-j}(\theta_i) \tau_j. \]  \hspace{1cm} (13)

Evaluating the right-hand side of (12) using (13) we obtain the equation on the right in (10). We have now obtained (10). Applying (10) to \( \Phi \) we obtain (11). \hspace{1cm} \square

**Lemma 4.4** With reference to Notation 2.2 the following (i)–(iii) hold for \( 0 \leq i \leq d \).

(i) \( \text{Span}\{A^h | 0 \leq h \leq i\} = \text{Span}\{\tau_h(A) | 0 \leq h \leq i\} \).

(ii) \( \text{Span}\{E_h | i \leq h \leq d\} = \text{Span}\{\eta_h(A) | i \leq h \leq d\} \).

(iii) \( \tau_i(A) \) is a basis for \( \text{Span}\{A^h | 0 \leq h \leq i\} \cap \text{Span}\{E_h | 0 \leq h \leq d\} \).

**Proof.** (i): Recall that \( \tau_h \) has degree \( h \) for \( 0 \leq h \leq d \).

(ii): Follows from Lemma 4.3

(iii): Immediate from (i), (ii) above. \hspace{1cm} \square

Applying Lemma 4.4 to \( \Phi \) we obtain the following result.

**Lemma 4.5** With reference to Notation 2.2 the following (i)–(iii) hold for \( 0 \leq i \leq d \).

(i) \( \text{Span}\{A^h | 0 \leq h \leq i\} = \text{Span}\{\eta_h(A) | 0 \leq h \leq i\} \).

(ii) \( \text{Span}\{E_h | i \leq h \leq d-i\} = \text{Span}\{\eta_h(A) | i \leq h \leq d\} \).

(iii) \( \eta_i(A) \) is a basis for \( \text{Span}\{A^h | 0 \leq h \leq i\} \cap \text{Span}\{E_h | 0 \leq h \leq d-i\} \).

5 Some bases for \( \mathcal{D} \) and \( \mathcal{D}^* \)

In this section we give some bases for \( \mathcal{D} \) and \( \mathcal{D}^* \) that will be useful later in the paper. We will state our results for \( \mathcal{D} \); of course similar results hold for \( \mathcal{D}^* \).

**Lemma 5.1** With reference to Notation 2.2 consider the following basis for \( \mathcal{D} \):

\[ E_0, E_1, \ldots, E_d. \] \hspace{1cm} (14)

For \( 0 \leq n \leq d \), if we replace any \((n+1)\)-subset of (14) by \( I, A, A^2, \ldots, A^n \) then the result is still a basis for \( \mathcal{D} \).

**Proof.** Let \( \Delta \) denote a \((n+1)\)-subset of \{0, 1, \ldots, d\} and let \( \overline{\Delta} \) denote the complement of \( \Delta \) in \{0, 1, \ldots, d\}. We show

\[ \{A^i\}_{i=0}^{n} \cup \{E_i\}_{i \in \overline{\Delta}} \] \hspace{1cm} (15)

is a basis for \( \mathcal{D} \). The number of elements in (15) is \( d + 1 \) and this equals the dimension of \( \mathcal{D} \). Therefore it suffices to show the elements (15) span \( \mathcal{D} \). Let \( S \) denote the subspace of
$\mathcal{D}$ spanned by (15). To show $\mathcal{D} = S$ we show $E_i \in S$ for $i \in \Delta$. For $0 \leq j \leq n$ we have $A^j = \sum_{i=0}^{d} \theta_i^j E_i$. In these equations we rearrange the terms to find

$$
\sum_{i \in \Delta} \theta_i^j E_i \in S \quad (0 \leq j \leq n).
$$

In the linear system (16) the coefficient matrix is Vandermonde and hence nonsingular. Therefore $E_i \in S$ for $i \in \Delta$. Now $S = \mathcal{D}$ and the result follows. □

6 The space $R$

Definition 6.1 With reference to Notation 2.2 we consider the tensor product $\mathcal{D} \otimes \mathcal{D}^* \otimes \mathcal{D}$ where $\otimes = \otimes_{\mathbb{K}}$. We define a $\mathbb{K}$-linear transformation

$$
\pi : \mathcal{D} \otimes \mathcal{D}^* \otimes \mathcal{D} \rightarrow \text{End}(V)
$$

$$
X \otimes Y \otimes Z \mapsto E_0^* X Y Z E_0^*
$$

We note that the image of $\pi$ is the span of $E_0^* \mathcal{D} \mathcal{D}^* \mathcal{D} E_0^*$.

Definition 6.2 With reference to Notation 2.2 let $R$ denote the sum of the following three subspaces of $\mathcal{D} \otimes \mathcal{D}^* \otimes \mathcal{D}$:

1. $\text{Span}\{A_i \otimes E_j^* | 0 \leq i < j \leq d\} \otimes \mathcal{D}$,
2. $\mathcal{D} \otimes \text{Span}\{E_j^* \otimes A_i | 0 \leq i < j \leq d\}$,
3. $\text{Span}\{E_i \otimes A^*^t \otimes E_j | 0 \leq i, j, t \leq d, t < |i - j|\}$.

Lemma 6.3 With reference to Definitions 6.1 and 6.2 the space $R$ is contained in the kernel of $\pi$.

Proof. Routinely obtained using Lemma 2.5. □

With reference to Notation 2.2 and Lemma 6.3 we desire to understand the kernel of $\pi$. To gain this understanding we systematically investigate $R$. We proceed as follows.

Lemma 6.4 With reference to Notation 2.2,

$$
\mathcal{D} \otimes \mathcal{D}^* \otimes \mathcal{D} = \sum_{t=0}^{d} \mathcal{D} \otimes \tau_t^* (A^*) \otimes \mathcal{D} \quad \text{(direct sum)}.
$$

Proof. Applying Lemma 4.2(i) to $\Phi^*$ we find that $\{\tau_t^* (A^*)\}_{t=0}^{d}$ is a basis for $\mathcal{D}^*$. The result follows. □
Definition 6.5 With reference to Notation 2.2 and Definition 6.2 for 0 \leq t \leq d let \( R_t \) denote the intersection of \( R \) with \( D \otimes \tau^*_t(A^*) \otimes D \).

With reference to Notation 2.2 and Definition 6.2 our next goal is to show \( R = \sum_{t=0}^{d} R_t \) (direct sum). The following lemma will be useful.

Lemma 6.6 With reference to Notation 2.2 the following (i)–(iii) hold.

(i) The space (17) is the direct sum over \( t = 0, 1, \ldots, d \) of the following subspaces:
\[
\text{Span}\{A^i | 0 \leq i < t\} \otimes \tau^*_t(A^*) \otimes D.
\] (20)

(ii) The space (18) is the direct sum over \( t = 0, 1, \ldots, d \) of the following subspaces:
\[
D \otimes \tau^*_t(A^*) \otimes \text{Span}\{A^i | 0 \leq i < t\}.
\] (21)

(iii) The space (19) is the direct sum over \( t = 0, 1, \ldots, d \) of the following subspaces:
\[
\text{Span}\{E_i \otimes \tau^*_t(A^*) \otimes E_j | 0 \leq i, j \leq d, t < |i - j|\}.
\] (22)

Proof. (i): Applying Lemma 4.4(ii) to \( \Phi^* \) we obtain
\[
\text{Span}\{A^i \otimes E^*_t | 0 \leq i < t \leq d\}
\]
\[
= \sum_{i=0}^{d} A^i \otimes \text{Span}\{E^*_i | i < t \leq d\}
\]
\[
= \sum_{i=0}^{d} A^i \otimes \text{Span}\{\tau^*_t(A^*) | i < t \leq d\}
\]
\[
= \sum_{t=0}^{d} \text{Span}\{A^i | 0 \leq i < t\} \otimes \tau^*_t(A^*).
\]

In the above lines we tensor each term on the right by \( D \) to find that the space (17) is the sum over \( t = 0, 1, \ldots, d \) of the spaces (20). The sum is direct by Lemma 6.4.

(ii): Similar to the proof of (i) above.

(iii): Applying Lemma 4.4(i) to \( \Phi^* \) we obtain
\[
\text{Span}\{E_i \otimes A^{*t} \otimes E_j | 0 \leq i, j, t \leq d, t < |i - j|\}
\]
\[
= \sum_{i=0}^{d} \sum_{j=0}^{d} E_i \otimes \text{Span}\{A^{*t} | 0 \leq t < |i - j|\} \otimes E_j
\]
\[
= \sum_{i=0}^{d} \sum_{j=0}^{d} E_i \otimes \text{Span}\{\tau^*_t(A^*) | 0 \leq t < |i - j|\} \otimes E_j
\]
\[
= \sum_{t=0}^{d} \text{Span}\{E_i \otimes \tau^*_t(A^*) \otimes E_j | 0 \leq i, j \leq d, t < |i - j|\}.
\]

In other words the space (19) is the sum over \( t = 0, 1, \ldots, d \) of the spaces (22). This sum is direct by Lemma 6.4. \( \square \)
Therefore \( E \).

Moreover the sets (23)–(25) equals the span of (20)–(22) and this equals 

Proof. For \( 0 \leq t \leq d \) let \( R'_t \) denote the sum of the spaces (20), (21), (22). Note that \( R'_t \) is contained in \( \mathcal{D} \otimes \tau_t^s(A^*) \otimes \mathcal{D} \), and that \( R = \sum_{t=0}^d R'_t \) by Lemma 6.6. By these comments and Lemma 6.4 we find \( R'_t = R_t \) for \( 0 \leq t \leq d \). The result follows. \( \square \)

7 A basis for \( R_t \) and \( \mathcal{D} \otimes \tau_t^s(A^*) \otimes \mathcal{D} \)

With reference to Notation 2.2 and Definition 6.5 for \( 0 \leq t \leq d \) we display a basis for \( R_t \) and extend this to a basis for \( \mathcal{D} \otimes \tau_t^s(A^*) \otimes \mathcal{D} \).

Theorem 7.1 With reference to Notation 2.2 and Definition 6.5 for \( 0 \leq t \leq d \) the following sets of vectors together form a basis for \( \mathcal{D} \otimes \tau_t^s(A^*) \otimes \mathcal{D} \):

\[
\{ A^i \otimes \tau_t^s(A^*) \otimes A^j | 0 \leq i \leq d, 0 \leq j < t \}, \quad (23)
\]

\[
\{ A^i \otimes \tau_t^s(A^*) \otimes A^j | 0 \leq i < t, t \leq j \leq d \}, \quad (24)
\]

\[
\{ E_i \otimes \tau_t^s(A^*) \otimes E_j | 0 \leq i, j \leq d, t < |i - j| \}, \quad (25)
\]

\[
\{ E_i \otimes \tau_t^s(A^*) \otimes E_i | 0 \leq i \leq d - t \}. \quad (26)
\]

Moreover the sets (23)–(25) together form a basis for \( R_t \).

Proof. The span of (23)–(25) equals the span of (20)–(22) and this equals \( R_t \) by Theorem 6.7(i). The dimension of \( \mathcal{D} \otimes \tau_t^s(A^*) \otimes \mathcal{D} \) is \( (d + 1)^2 \). The cardinality of the sets (23)–(26) is \( t(d + 1), t(d - t + 1), (d - t)(d + t - 1) \), and the sum of these numbers is \( (d + 1)^2 \). Therefore the number of vectors in (23)–(26) is equal to the dimension of \( \mathcal{D} \otimes \tau_t^s(A^*) \otimes \mathcal{D} \). Consequently to finish the proof it suffices to show that (23)–(26) together span \( \mathcal{D} \otimes \tau_t^s(A^*) \otimes \mathcal{D} \). Let \( S \) denote the span of (23)–(26). We first claim that for \( 0 \leq i \leq d - t \), both

\[
E_i \otimes \tau_t^s(A^*) \otimes \mathcal{D} \subseteq S, \quad \mathcal{D} \otimes \tau_t^s(A^*) \otimes E_i \subseteq S.
\]

To prove the claim, by induction on \( i \) we may assume

\[
E_j \otimes \tau_t^s(A^*) \otimes \mathcal{D} \subseteq S, \quad \mathcal{D} \otimes \tau_t^s(A^*) \otimes E_j \subseteq S \quad (0 \leq j < i). \quad (27)
\]

By Lemma 5.1 the following vectors together form a basis for \( \mathcal{D} \):

\[
E_0, E_1, \ldots, E_{i-1}; \quad E_i; \quad I, A^2, \ldots, A^{t-1}; \quad E_{i+t+1}, E_{i+t+2}, \ldots, E_d.
\]

Therefore \( E_i \otimes \tau_t^s(A^*) \otimes \mathcal{D} \) is the sum of the following spaces:

\[
E_i \otimes \tau_t^s(A^*) \otimes \text{Span}\{ E_0, E_1, \ldots, E_{i-1} \}, \quad (28)
\]

\[
E_i \otimes \tau_t^s(A^*) \otimes \text{Span}\{ E_i \}, \quad (29)
\]

\[
E_i \otimes \tau_t^s(A^*) \otimes \text{Span}\{ I, A^2, \ldots, A^{t-1} \}, \quad (30)
\]

\[
E_i \otimes \tau_t^s(A^*) \otimes \text{Span}\{ E_{i+t+1}, E_{i+t+2}, \ldots, E_d \}. \quad (31)
\]
The space (28) is contained in $S$ by (27), the space (29) is contained in $S$ by (26), the space (30) is contained in $S$ by (23), and the space (31) is contained in $S$ by (25). Therefore $E_i \otimes \tau^*_t(A^*) \otimes D$ is contained in $S$. Similarly one shows that $D \otimes \tau^*_t(A^*) \otimes E_i$ is contained in $S$ and the claim is proved. Next we claim that $E_i \otimes \tau^*_t(A^*) \otimes D$ is contained in $S$ for $d - t < i \leq d$. By Lemma 5.1 the following vectors together form a basis for $D$:

$$E_0, E_1, \ldots, E_{d-t}; \quad I, A, A^2, \ldots, A^{t-1}.$$  

Therefore $E_i \otimes \tau^*_t(A^*) \otimes D$ is the sum of the following spaces:

$$E_i \otimes \tau^*_t(A^*) \otimes \text{Span}\{E_0, E_1, \ldots, E_{d-t}\}, \quad (32)$$
$$E_i \otimes \tau^*_t(A^*) \otimes \text{Span}\{I, A, A^2, \ldots, A^{t-1}\}. \quad (33)$$

The space (32) is contained in $S$ by the first claim, and the space (33) is contained in $S$ by (23). Therefore $E_i \otimes \tau^*_t(A^*) \otimes D$ is contained in $S$ and the second claim is proved. By the two claims and since $\{E_i\}_{i=0}^d$ is a basis for $D$, we find $D \otimes \tau^*_t(A^*) \otimes D$ is contained in $S$. In other words (23)–(26) together span $D \otimes \tau^*_t(A^*) \otimes D$ as desired. The result follows. □

**Corollary 7.2** With reference to Notation 2.2 and Definition 6.5 the following (i)–(iv) hold.

(i) For $0 \leq t \leq d$ the dimension of $R_t$ is $d^2 + d + t$.

(ii) For $0 \leq t \leq d$ the codimension of $R_t$ in $D \otimes \tau^*_t(A^*) \otimes D$ is $d - t + 1$.

(iii) The dimension of $R$ is $d(d+1)(2d+3)/2$.

(iv) The codimension of $R$ in $D \otimes D^* \otimes D$ is $(d+1)(d+2)/2$.

**Proof.** (i), (ii): The dimension of $D \otimes \tau^*_t(A^*) \otimes D$ is $(d+1)^2$. By (26) the codimension of $R_t$ in $D \otimes \tau^*_t(A^*) \otimes D$ is $d - t + 1$. The result follows.

(iii): Sum the dimension in (i) over $t = 0, 1, \ldots, d$.

(iv): Sum the codimension in (ii) over $t = 0, 1, \ldots, d$. □

8 The map $\dagger$

**Definition 8.1** With reference to Notation 2.2 we define a $K$-linear transformation

$$\dagger: \quad D \otimes D^* \otimes D \quad \rightarrow \quad D \otimes D^* \otimes D$$

$$X \otimes Y \otimes Z \quad \mapsto \quad Z \otimes Y \otimes X$$

We call $\dagger$ the transpose map. We observe that $\dagger$ is an involution.
Proposition 8.2  With reference to Notation 2.2 and Definition 8.1 the following (i)–(iii) hold.

(i) \( R \) is invariant under \( \dagger \).

(ii) For \( 0 \leq t \leq d \) the space \( D \otimes \tau^*_t(A^*) \otimes D \) is invariant under \( \dagger \).

(iii) For \( 0 \leq t \leq d \) the space \( R_t \) is invariant under \( \dagger \).

Proof. (i): By Definition 6.2 the space \( R \) is the sum of (17)–(19). The map \( \dagger \) exchanges (17), (18) and leaves (19) invariant. The result follows.

(ii): Clear.

(iii): By Theorem 6.7(i) the space \( R_t \) is the sum of (20)–(22). The map \( \dagger \) exchanges (20), (21) and leaves (22) invariant. The result follows. \( \square \)

Theorem 8.3  With reference to Notation 2.2 and Definition 8.1 the following (i), (ii) hold.

(i) For \( 0 \leq t \leq d \) the image of \( D \otimes \tau^*_t(A^*) \otimes D \) under \( 1 - \dagger \) is contained in \( R_t \).

(ii) The image of \( D \otimes D^* \otimes D \) under \( 1 - \dagger \) is contained in \( R \).

Proof. (i): Let \( C \) denote the subspace of \( D \otimes \tau^*_t(A^*) \otimes D \) spanned by the elements (26). By Theorem 7.1 the space \( D \otimes \tau^*_t(A^*) \otimes D \) is the direct sum of \( R_t \) and \( C \). By Proposition 8.2(iii) the image of \( R_t \) under \( 1 - \dagger \) is contained in \( R_t \). By (26) the image of \( C \) under \( 1 - \dagger \) is zero. The result follows.

(ii): Combine Lemma 6.4, Theorem 6.7(ii), and (i) above. \( \square \)

9  A complement for \( R \) in \( D \otimes D^* \otimes D \)

With reference to Notation 2.2 and Definition 6.2 our goal in this section is to show that the elements \( \{ E_i \otimes \tau^*_t(A^*) \otimes E_j | 0 \leq i \leq j \leq d \} \) form a basis for a complement of \( R \) in \( D \otimes D^* \otimes D \). We begin with a slightly technical lemma.

Lemma 9.1  With reference to Notation 2.2 and Definition 6.2, for \( 0 \leq t \leq d \) and \( 0 \leq i < j \leq i + t \leq d \) the space

\[ R_t + \text{Span}\{ E_h \otimes \tau^*_t(A^*) \otimes E_h | 0 \leq h < i \} \]  \hspace{1cm} (34)

contains both

\[ f^t_{ij}(\theta_i)E_i \otimes \tau^*_t(A^*) \otimes E_i + f^t_{ij}(\theta_j)E_i \otimes \tau^*_t(A^*) \otimes E_j, \]  \hspace{1cm} (35)

\[ f^t_{ij}(\theta_i)E_i \otimes \tau^*_t(A^*) \otimes E_i + f^t_{ij}(\theta_j)E_j \otimes \tau^*_t(A^*) \otimes E_i, \]  \hspace{1cm} (36)

where \( f^t_{ij} = \prod_{h=i+1, h \neq j}^{i+t}(\lambda - \theta_h). \)

Proof. We fix \( t \) and show by induction on \( i = 0, 1, \ldots, d - t \) that each of (35), (36) is contained in (34) for \( i < j \leq i + t \). Concerning (35), in the equation \( f^t_{ij}(A) = \sum_{n=0}^d f^t_{ij}(\theta_n)E_n \) we tensor each term on the left by \( E_i \otimes \tau^*_t(A^*) \) to get

\[ E_i \otimes \tau^*_t(A^*) \otimes f^t_{ij}(A) = \sum_{n=0}^d f^t_{ij}(\theta_n)E_i \otimes \tau^*_t(A^*) \otimes E_n. \]  \hspace{1cm} (37)
We examine the terms in (37). The left side of (37) is in $R_t$ by (23) and since $f^t_{ij}$ has degree $t-1$. For $0 \leq n \leq d$ consider the $n$-summand on the right in (37). First assume $0 \leq n < i - t$. Then the $n$-summand is in $R_t$ by (25). Next assume $i - t \leq n < i$. By (36) and induction,

$$f^t_{ni}(\theta_n)E_n \otimes \tau_t^s(A^*) \otimes E_n + f^t_{ni}(\theta_i)E_i \otimes \tau_t^s(A^*) \otimes E_n \in R_t + \text{Span}\{E_h \otimes \tau_t^s(A^*) \otimes E_h \mid 0 \leq h < n\}.$$ 

By this and since $f^t_{ni}(\theta_i)$ is nonzero,

$$E_i \otimes \tau_t^s(A^*) \otimes E_n \in R_t + \text{Span}\{E_h \otimes \tau_t^s(A^*) \otimes E_h \mid 0 \leq h \leq n\}. \tag{38}$$

Therefore our $n$-summand is in (34). Next assume $i + 1 \leq n \leq i + t$, $n \neq j$. Then the $n$-summand is 0 since $f^t_{ij}(\theta_n) = 0$. Next assume $i + t < n \leq d$. Then the $n$-summand is in $R_t$ by (25). By these comments the expression (35) is contained in (34). By this and Theorem 8.3(i) the expression (36) is contained in (34).

\[\square\]

Proposition 9.2 With reference to Notation 2.2 and Definition 6.5, for $0 \leq t \leq d$ the vectors

$$\{E_i \otimes \tau_t^s(A^*) \otimes E_{i+t} \mid 0 \leq i \leq d - t\} \tag{39}$$

form a basis for a complement of $R_t$ in $D \otimes \tau_t^s(A^*) \otimes D$.

**Proof.** Consider the quotient vector space

$$V_t = D \otimes \tau_t^s(A^*) \otimes D/R_t.$$ 

We show the vectors

$$E_i \otimes \tau_t^s(A^*) \otimes E_{i+t} + R_t \quad (0 \leq i \leq d - t) \tag{40}$$

form a basis for $V_t$. By Theorem 7.1 the vectors

$$E_i \otimes \tau_t^s(A^*) \otimes E_i + R_t \quad (0 \leq i \leq d - t) \tag{41}$$

form a basis for $V_t$. Write the elements (40) in terms of the basis (41). By Lemma 9.1 the resulting coefficient matrix is upper triangular with all diagonal entries nonzero. Therefore (40) is a basis for $V_t$ and the result follows. \[\square\]

Theorem 9.3 With reference to Notation 2.2 and Definition 6.2 the vectors

$$\{E_i \otimes \tau_{j-i}^s(A^*) \otimes E_j \mid 0 \leq i \leq j \leq d\} \tag{42}$$

form a basis for a complement of $R$ in $D \otimes D^* \otimes D$.

**Proof.** The set (42) is the disjoint union over $t = 0, 1, \ldots, d$ of the sets (39). The result follows in view of Lemma 6.4, Theorem 6.7(ii), and Proposition 9.2. \[\square\]

12
10 The space $\mathcal{D} \otimes E_0^* \otimes \mathcal{D}$

With reference to Notation 2.2 and Definition 6.2, in this section we show that the elements $\{E_i \otimes E_j | 0 \leq i \leq j \leq d\}$ form a basis for a complement of $R$ in $\mathcal{D} \otimes \mathcal{D}^* \otimes \mathcal{D}$. We will use the following lemma.

Lemma 10.1 With reference to Notation 2.2 for $0 \leq t \leq d$ and $0 \leq i \leq d - t$ the element

$$E_i \otimes \tau_t^*(A^*) \otimes E_{i+t} - (\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_t^*) E_i \otimes E_0^* \otimes E_{i+t}$$

is contained in

$$R + \sum_{n=1}^{d} \mathcal{D} \otimes \tau_n^*(A^*) \otimes \mathcal{D}. \quad (43)$$

Proof. Applying the equation on the right in (10) to $\Phi^*$,

$$E_0^* = \sum_{n=0}^{d} \frac{\eta_{d-n}(\theta_0^*) \tau_n^*(A^*)}{\eta_d^*(\theta_0^*)}.$$

In this equation we tensor each term on the left by $E_i$ and on the right by $E_{i+t}$ to get

$$E_i E_0^* E_{i+t} = \sum_{n=0}^{d} \frac{\eta_{d-n}(\theta_0^*)}{\eta_d^*(\theta_0^*)} E_i \otimes \tau_n^*(A^*) \otimes E_{i+t}. \quad (44)$$

For $0 \leq n \leq d$ consider the $n$-summand on the right in (44). For $0 \leq n \leq t - 1$ the $n$-summand is in $R$ by (25). For $t + 1 \leq n \leq d$ the $n$-summand is in (43) by construction. The result follows from these comments and since

$$\eta_d^*(\theta_0^*) = (\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_t^*) \eta_{d-t}(\theta_0^*).$$

□

Theorem 10.2 With reference to Notation 2.2 and Definition 6.2 the vectors

$$\{E_i \otimes E_j \otimes E_j | 0 \leq i \leq j \leq d\} \quad (45)$$

form a basis for a complement of $R$ in $\mathcal{D} \otimes \mathcal{D}^* \otimes \mathcal{D}$.

Proof. The cardinality of the set (45) is $(d + 1)(d + 2)/2$, and by Corollary 7.2(iv) this is the codimension of $R$ in $\mathcal{D} \otimes \mathcal{D}^* \otimes \mathcal{D}$. Therefore, it suffices to show that $R$ and the elements (45) together span $\mathcal{D} \otimes \mathcal{D}^* \otimes \mathcal{D}$. Let $S$ denote the subspace of $\mathcal{D} \otimes \mathcal{D}^* \otimes \mathcal{D}$ spanned by $R$ and the elements (45). To show that $S = \mathcal{D} \otimes \mathcal{D}^* \otimes \mathcal{D}$ we show $\mathcal{D} \otimes \tau_t^*(A^*) \otimes \mathcal{D} \subseteq S$ for $0 \leq t \leq d$. We show this by induction on $t = d, d - 1, \ldots, 0$. Let $t$ be given. By Proposition 9.2

$$\mathcal{D} \otimes \tau_t^*(A^*) \otimes \mathcal{D} = R_t + \text{Span}\{E_i \otimes \tau_t^*(A^*) \otimes E_{i+t} | 0 \leq i \leq d - t\}.$$  

By construction $R_t \subseteq R \subseteq S$. For $0 \leq i \leq d - t$ we have $E_i \otimes \tau_t^*(A^*) \otimes E_{i+t} \in S$ by Lemma 10.1 and induction on $t$. By these comments $\mathcal{D} \otimes \tau_t^*(A^*) \otimes \mathcal{D} \subseteq S$ and the result follows.

□
11 The proof of Theorem 2.4

Using the results in earlier sections we can now easily prove Theorem 2.4.

Proof of Theorem 2.4. (i): By Definition 6.1 the image $\pi(D \otimes D^* \otimes D)$ is the span of $E_0^* D D^* D E_0^*$. Similarly the image $\pi(D \otimes E_0^* \otimes D)$ is the span of $E_0^* D E_0^* D E_0^*$. We show

$$\pi(D \otimes D^* \otimes D) = \pi(D \otimes E_0^* \otimes D).$$

Let $C$ denote the subspace of $D \otimes D^* \otimes D$ spanned by the elements (45). By Theorem 9.3 $D \otimes D^* \otimes D$ is the direct sum $C + R$. By the construction $C$ is contained in $D \otimes E_0^* \otimes D$. By Lemma 6.3 the space $R$ is contained in the kernel of $\pi$. Therefore

$$D \otimes D^* \otimes D = D \otimes E_0^* \otimes D + \text{Ker}(\pi).$$

Applying $\pi$ to this equation we get (46) and the result follows.

(ii): For $X, Y \in D$ we show $E_0^* X E_0^*, E_0^* Y E_0^*$ commute. By Theorem 8.3(ii),

$$X \otimes E_0^* Y - Y \otimes E_0^* X \in R.$$

In the above line we apply the map $\pi$ and use Lemma 6.3 to find

$$E_0^* X E_0^* Y E_0^* = E_0^* Y E_0^* X E_0^*.$$

By this and since $E_0^* E_0^* = E_0^*$ the elements $E_0^* X E_0^*, E_0^* Y E_0^*$ commute. □

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