Experimental observation of non-Abelian topological acoustic semimetals and their phase transitions

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Topological phases of matter connect mathematical principles to real materials, and may shape future electronic and quantum technologies. So far, this discipline has mostly focused on single-gap topology described by topological invariants such as Chern numbers. Here, based on a tunable kagome model, we observe non-Abelian band topology and its transitions in acoustic semimetals, in which the multi-gap Hilbert space plays a key role. In non-Abelian semimetals, the topological charges of band nodes are converted through the braiding of nodes in adjacent gaps, and their behaviour cannot be captured by conventional topological band theory. Using kagome acoustic metamaterials and pump–probe measurements, we demonstrate the emergence of non-Abelian topological nodes, identify their dispersions and observe the induced multi-gap topological edge states. By controlling the geometry of the metamaterials, topological transitions are induced by the creation, annihilation, merging and splitting of band nodes. This reveals the underlying rules for the conversion and transfer of non-Abelian topological charges in multiple bandgaps. The resulting laws that govern the evolution of band nodes in non-Abelian multi-gap systems should inspire studies on multi-band topological semimetals and multi-gap topological out-of-equilibrium systems.
real-valued eigenstates of the Hamiltonian can be interpreted as a three-dimensional (3D) orthonormal frame that is defined up to a sign for each eigenstate\(^1\). The evolution of such a 3D orthonormal frame along a path in the \(k\)-space encircling a band node defines the topological frame charge of the node\(^4\). Such a frame charge can be directly calculated by decomposing the 3D rotation matrix along a path encircling the node, which thus reflects the geometry of SO(3) rotations\(^6\). According to homotopy group theory, the frame charge of a band node in the first gap around one in the second gap (III) (schematically indicated by the brown trajectory) changes its charge by a factor of \(-1\), resulting in a spectrum with nodes of the same charge in each gap (II). The Euler class \(\xi\) integrated over a Brillouin zone patch (the green box) then quantifies the stability of the pair of nodes (III and IV).

Non-Abelian topological nodes: (I) a Dirac node with a frame charge \(q = \pm i\) or \(\pm j\); (II) a quadratic node with Euler class \(\xi = \pm 1\); (III) a linear triply-degenerate point with frame charge \(q = \pm k\) or \(-1\); (IV) a quadratic triply-degenerate point with frame charge \(q = 1\).

**Kagome tight-binding model**

As a key insight to observe this new physics, we depart from a carefully designed model that elucidates this multi-gap physics by virtue

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**Fig. 1 | Conventional single-gap topology versus multi-gap non-Abelian topology.** a. Conventional (Abelian) topological invariants are defined on a single bandgap separating the conduction and valence bands (red and blue, respectively). Starting from a trivial insulator (I), a topological insulator (III) can be obtained from a band inversion (II) that changes the symmetry character (schematically denoted as \(\pm\)) of the valence bands when going from one high-symmetry momentum \(G\) to another, \(G\). Non-Abelian band topology is defined for multi-gap systems where the topological invariants depend on the braiding between the band nodes in adjacent gaps. Starting from a trivial gapped system, band nodes can be induced in both gaps (I). These nodes are oppositely charged (open/filled symbols) within a gap and behave as quaternions \(\pm i\) and \(\pm j\) with respect to the charges in the other gap (red circles versus blue triangles). As a result, braiding a band node in the first gap around one in the second gap (III) (schematically indicated by the brown trajectory) changes the braiding between the band nodes in adjacent gaps. Starting from a trivial gapped system, band nodes can be induced in both gaps (I). These nodes are oppositely charged (open/filled symbols) within a gap and behave as quaternions \(\pm i\) and \(\pm j\) with respect to the charges in the other gap (red circles versus blue triangles). As a result, braiding a band node in the first gap around one in the second gap (III) (schematically indicated by the brown trajectory) changes its charge by a factor of \(-1\), resulting in a spectrum with nodes of the same charge in each gap (II). The Euler class \(\xi\) integrated over a Brillouin zone patch (the green box) then quantifies the stability of the pair of nodes (III and IV). Non-Abelian topological nodes: (I) a Dirac node with a frame charge \(q = \pm i\) or \(\pm j\); (II) a quadratic node with Euler class \(\xi = \pm 1\); (III) a linear triply-degenerate point with frame charge \(q = \pm k\) or \(-1\); (IV) a quadratic triply-degenerate point with frame charge \(q = 1\).
of an intricate interplay with crystalline symmetries\cite{23-27}. We consider a kagome tight-binding model with tunable nearest-neighbour (NN) and next-nearest-neighbour (NNN) couplings, denoted as $t$ and $t'$, respectively (Fig. 2a). There are three sites in each unit cell, labelled A, B and C. Each of these sites has an $s$ orbital, leading to a three-band system described by real-valued Hamiltonian due to the $C_3T$ symmetry. The explicit Hamiltonian is given in the Methods. By tuning the NN and NNN couplings, the evolution of the band nodes in the first and second gaps can be triggered, as illustrated in Fig. 2b–h.

We start the tuning process with the familiar case where only the NN coupling is finite, that is, $(t, t') = (-1, 0)$. In this case, there are linear (Dirac) nodes at the K and K' points in the first bandgap (denoted henceforth as 'gap I') and a quadratic node at the G point in the second bandgap (denoted 'gap II'). The Dirac nodes in gap I have frame charges $q = \pm i$, whereas the quadratic node in gap II has frame charge $q = 1$ and a patch Euler class of $\xi = \pm 1$ (Fig. 2b). We find that the quadratic node is rather resilient and remains in all cases in Fig. 2 during the tuning process, which directly reflects the topological robustness of the topological Euler class (see Methods and Supplementary Note 1 for analysis).

As shown in Fig. 2b–h, by continuously decreasing the strength of the NN coupling and increasing the strength of the NNN coupling, rich non-Abelian topological phases can emerge. First, six pairs of Dirac nodes with frame charges $q = \pm j$ can be created in gap II out of the G point. These Dirac nodes can be created together because they have a total frame charge of $q = 1$, demonstrating a case where Dirac nodes can be created or annihilated together. Constrained by the crystalline symmetry (that is, the six-fold rotation symmetry), they are located on the G'M and G'K lines in the 2D Brillouin zone (Fig. 2c). Further tuning the NN and NNN couplings moves the Dirac nodes on the G'K line towards the K point. Merging three Dirac nodes in gap II (with frame charges $-j$, $-j$ and $j$, separately) together with the Dirac node in gap I (with frame charge $i$) at the K point leads to a linear triply-degenerate point (LTDP)\cite{9-12} with frame charge $q = k$ (Fig. 2d). This process can be regarded as the annihilation of a pair of Dirac nodes in gap II with opposite frame charges ($-j$ and $j$) taking place simultaneously with the merging of a node in gap II and another node in gap I, leading to a total frame charge of $q = -j \times i = k$ for the LTDP at the K point. Similarly, another LTDP appears at the K' point with the same frame charge.

Further tuning the system splits the LTDPs and leads to six Dirac nodes in gap I on the MK line with frame charges $q = \pm i$ (Fig. 2e). Meanwhile, notably, the Dirac nodes at the K and K' points are transferred from gap I to gap II. Going still further along the tuning process, the six Dirac nodes in gap I and the six Dirac nodes in gap II merge together at the M, M' and M'' points to form three LTDPs with $q = -1$ (Fig. 2f). By further tuning the couplings, the three LTDPs at the M, M' and M'' points split into 12 Dirac nodes (which can also be regarded as the transfer of non-Abelian charges between different gaps); the six Dirac nodes in gap I move to the G'M line, while the other six Dirac nodes in gap II move to the MK line (Fig. 2g) and eventually reach the K and K' points. Finally, the coalescence of the Dirac nodes in gap II leads to the emergence of quadratic nodes with frame charge $q = -1$ and the Euler class $\xi = \pm 1$ at the K and K' points (Fig. 2h).

At this stage, it is important to summarize the salient features in the above. First, the evolution process includes the creation, annihilation, merging and splitting of nodes in two gaps, while the non-Abelian charges are converted and transferred between these gaps. Such features, which cannot be understood via conventional topological band theory, faithfully reveal the non-Abelian nature of the multi-gap topology. Detailed theory and calculations that account for the above evolution of the non-Abelian topological charges are presented in the Methods, Extended Data Figs. 4 and 5, and Supplementary Note 1. We emphasize that the emergent triply-degenerate points play a central role in the transfer of the non-Abelian charges between different bandgaps. Second, the emergence of the LTDPs and the quadratic nodes at the K and K' points cannot be explained by symmetry analysis alone, but originates from the intrinsic non-Abelian band topology (Supplementary Note 1 provides more analysis). Third, we find that, in our model,
the non-Abelian topological charges are often connected with the dispersion of the nodes. As illustrated in Figs. 1c and 2, Dirac nodes in gap I (gap II) correspond to frame charges \( q = \pm i q = \pm j \). Quadratic nodes are characterized by frame charge \( q = -1 \) and Euler class \( \xi = \pm 1 \). However, the triply-degenerate points are distinct. For instance, they do not have a Euler class, but are described by the frame charges \( q = \pm k \) or \( -1 \). Besides, they can have both linear and quadratic dispersions with the same frame charge \( q = -1 \) (Supplementary Note 1).

**Tunable kagome acoustic metamaterials**

To study the non-Abelian topological phases in experiments, we designed a type of kagome acoustic metamaterial that can reproduce all the salient features in the tight-binding model. Recently, acoustic metamaterials have been demonstrated as a versatile platform for the study of topological phenomena\[^5\]-\[^8\]. In particular, they give access to spectral property measurements in a wide frequency range, with no limitation imposed by the Fermi–Dirac filling effect, which is advantageous for the study of multi-gap non-Abelian topology.

In our design, the A, B and C sites in the kagome model are realized by cylindrical acoustic resonators of the same size: the air region has height \( H = 24 \text{ mm} \) and diameter \( D = 19.2 \text{ mm} \), encapsulated by a shell made of photosensitive resin that confines the acoustic waves. The NN (NNN) couplings are realized by horizontal tubes with diameter \( d_{i} \) (\( d_{j} \)) that connect the NN (NNN) resonators. The lattice constant is \( a = 36\sqrt{3} \text{ mm} \), while the other geometry parameters are specified in the caption of Fig. 3. By tuning \( d_{i} \) and \( d_{j} \), the NN and NNN couplings can be effectively controlled (Fig. 3b; Supplementary Note 2 provides details).

Six acoustic metamaterials, as snapshots of the tuning process, were fabricated using 3D printing technology (Fig. 3c presents a photograph of a sample). Each sample has 200 unit cells, giving a rectangular geometry with dimensions of \( 1,050 \text{ mm} \times 710 \text{ mm} \) in the \( x-y \) plane. Their geometry parameters \( (d_{i}, d_{j}) \) and acoustic band structures are presented in Fig. 3d–l. The acoustic band

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**Fig. 3 | Kagome acoustic metamaterials and non-Abelian topological nodes.** a, Forming kagome lattices using connected cylindrical acoustic resonators of height \( H = 24 \text{ mm} \) and diameter \( D = 19.2 \text{ mm} \). The NN (NNN) couplings are realized by horizontal tubes of length \( l_{h} = 18\sqrt{3} \text{ mm} \) (\( l_{h} = 54 \text{ mm} \)) and diameter \( d_{i} (d_{j}) \). b, Diameters \( d_{i} \) and \( d_{j} \) are varied to tune the NN and NNN couplings. c, Photograph of the experimental set-up for measurements of the bulk and edge band structures. The upper and left boundaries (labelled by green lines) form armchair and zigzag edges, whereas the other two boundaries are open. The source and detector that are connected with the network analyser (not shown) are depicted for the set-up of measuring the bulk Bloch

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structures are measured using a pump–probe scheme, with an acoustic source placed into a resonator at the centre of the sample to excite the acoustic waves at various frequencies from 5 Hz to 3 kHz. Assisted by an Agilent network analyser, a microphone is used to detect the acoustic wavefunctions (specifically, the complex-valued acoustic pressure distributions) at all other resonators. Via the Fourier transformation of the detected acoustic wavefunctions at each excitation frequency, we obtain the dispersions of the acoustic Bloch waves (see Methods and Supplementary Note 3 for details).

Figure 3d–i shows that the measured acoustic band structures agree excellently with the calculated ones (details of the calculations are provided in the Methods), especially when the intrinsic dissipation of acoustic waves and the finite-sample-size effect are taken into account (these effects introduce, respectively, the broadening in the frequency dimension and the wavevector space in the measured acoustic dispersions). The calculated configurations of the band nodes and their non-Abelian topological charges for the six meta-materials are presented in the insets. The consistent experiments and calculations in Fig. 3 confirm the creation, moving, merging and splitting of the nodes in both gaps I and II. Therefore, the results also provide indirect evidence for the associated conversion and transfer of non-Abelian topological charges between different bandgaps which are constrained by the crystalline symmetry and governed by the basic rules for the evolution of the non-Abelian topological charges (Methods and Supplementary Note 1).

It is worth mentioning that the acoustic band structures in Fig. 3 and the tight-binding band structures in Fig. 2 are visibly different, yet they share exactly the same topological features. This property reflects the resilience of the observed topological phenomena, that is, the non-Abelian multi-gap topology and its transitions. We also remark that, from material science aspects, except the band structure in Fig. 3d, all other band structures, particularly the LTDPs in Fig. 3e,g and the multiple quadratic degeneracy in Fig. 3i (not only at the Γ point but also at the K and K’ points), are unconventional dispersions in kagome materials. These exotic band structures reveal that kagome materials embrace rich physical properties that are attractive for fundamental science (for example, strongly correlated quantum materials based on a kagome lattice) and potential applications (for example, Dirac materials with tunable bands for wave manipulations37).

Figure 4 presents the dispersions of the LTDPs and the quadratic nodes emerging in Fig. 3e,g,i, separately. Specifically, the dispersion

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**Fig. 4 | Dispersions of non-Abelian topological nodes.** a,d, Measured (a) and calculated (d) acoustic dispersions around the LTDP at the K point for the case in Fig. 3e. b,e, Measured (b) and calculated (e) acoustic dispersions around the LTDP at the M point for the case in Fig. 3g. c,f, Measured (c) and calculated (f) acoustic dispersions around the quadratic node at the K point for the case in Fig. 3i. In a–c, the dispersions are presented in iso-frequency contours (brighter colours indicate stronger signals in the experiments). White dashed lines represent calculated iso-frequency contours, while green lines are a guide to the eye to illustrate the dispersion from calculation. In d–f, the dispersions are presented by the surfaces, while the black lines give the iso-frequency contours. $k_1$ and $k_2$ are the wavevectors along the reciprocal primitive vectors.
of the LTDP at the K point with frame charge \( q = +k \) for the sample in Fig. 3c is shown in Fig. 4a (measurements) and Fig. 4d (calculation). The dispersion of the LTDP at the M point with frame charge \( q = -1 \) for the sample in Fig. 3g is shown in Fig. 4b (measurements) and Fig. 4c (calculation). The quadratic node at the K point in gap II with frame charge \( q = -1 \) and Euler class \( \xi = \pm 1 \) for the sample in Fig. 3l is presented in Fig. 4e (measurements) and 4f (calculation). These unique dispersions and the associated non-Abelian topological charges reveal unprecedented properties of kagome semimetals. We emphasize that the quadratic nodes at the K and K’ points cannot be explained by symmetry analysis alone, because the K and K’ points only have \( C_3 \) rotation symmetry. They are in fact due to the Euler class \( \xi = \pm 1 \) formed by the non-Abelian braiding process (Methods). Similarly, the LTDPs also cannot be explained by the symmetry analysis alone. They are unavoidable intermediate states for the braiding that leads to the transfer of non-Abelian topological charges from one gap to another. In the tight-binding model, these features may come from the simplicity of the model. However, in acoustic metamaterials, where the Bloch bands cannot be described quantitatively by the tight-binding model⁹, the emergence of the LTDPs at the K and M points (and so on) reflects the profound origin of these degeneracies as due to the (symmetry-constrained) braiding and the non-Abelian topology.

Multi-gap bulk-edge physics

From the bulk topological charge configuration and its evolution, one can expect observable evolution of the edge states through the braiding process. Inversely, the evolution of the edge states through the different steps of the phase diagram provides an indirect manifestation of the bulk braiding of the band nodes. Topological band nodes have direct consequences on the edge states in both gap I and gap II. In each gap, for a given momentum along the edge boundary, a Zak phase can be defined¹⁰. For both the zigzag and armchair boundaries, such a Zak phase is quantized and switches from 0 to \( \pi \) or vice versa when passing through an odd number of Dirac nodes. Because a quadratic node is equivalent to two Dirac nodes, it does not affect the Zak phase. On the other hand, an LTDP with frame charge \( q = \pm k \) \((q = -1)\) is equivalent to an odd (even) number of Dirac nodes in both gap I and gap II. For the edge boundaries studied in our experiments, Zak phase \( 0 \) \((\pi)\) leads to emergent (vanishing) edge states¹⁰. Therefore, the emergence and disappearance of edge states reflect the projections of the band nodes in the edge Brillouin zone. Using the advantages that acoustic systems are not limited by the Fermi–Dirac filling effect, we can measure the edge states in both gap I and gap II and study the multi-gap bulk-edge physics.

We note, in this regard, a recent work on a 1D gapped real-valued three-band Hamiltonian realized in electric circuits¹⁰, where the relation between Zak phases and non-Abelian charges was observed. Within this work, we consider genuine Euler class semimetallic materials, a topological phase that has no 1D equivalent, and the underlying braiding process (which also requires two spatial dimensions) that leads to such non-Abelian topological semimetal phases. Moreover, the discussed unavoidable bulk features, such as the stability of the quadratic nodes, the non-Abelian charge transfer effect and the triple degeneracies upon braiding of nodes, are universal and applicable outside the realm of metamaterials. We emphasize that, while relations between the Zak phases and edge states are well defined, a general bulk–boundary correspondence in terms of the non-Abelian frame charges is yet to be formally understood.

In the measurements of the edge states, the acoustic source is placed at the centre of the edge boundary (labelled by yellow stars in Fig. 3c). A microphone is used to detect the acoustic wavefunctions in the resonators around the edge boundary of concern (the zigzag
or the armchair edge boundary). Via Fourier transformations of the detected acoustic wavefunctions along the edge boundary for each excitation frequency, we can obtain the acoustic dispersions of the edge states in both gaps I and II (Supplementary Note 4).

We present the measured and calculated dispersions of the zigzag edge states along the y direction in Fig. 5 for the six samples studied in Fig. 3. The results show that the experiments and calculations agree reasonably well with each other for the first bandgap. The emergence and disappearance of the edge states are consistent with the projections of the topological nodes. For example, in Fig. 5a, because the Dirac nodes switch the Zak phase between 0 and π, they act as the terminations of the edge states. In Fig. 5b, the LTDPs with frame charge $q = \pm k$ play the same role as the Dirac nodes. Figure 5c–f is consistent with the picture that two Dirac nodes projected into the same wavevector in the edge Brillouin zone do not change the Zak phase nor the edge states in gap I. Similar conclusions are arrived at by observing the armchair edge states in gap I and comparing them with calculations (Supplementary Note 4).

Owing to the smallness of the projected gap II for most of the samples, the observation of the edge states in gap II is much more difficult. Nevertheless, Fig. 5a still convincingly evidences the emergence of the edge states in gap II, consistent with our theory and calculation. In addition, for armchair edges, the edge states in gap II apparently appear in more samples and agree well with the calculations (Supplementary Note 4) due to the much larger projected gap II. We note the slight deviations between the experimental data and calculations are mainly due to the finite size effect and the sample quality at the interfaces, which have more effect when the bandgap is small and the edge dispersions are flat.

The observed edge states in gap II for both the zigzag and armchair boundaries indicate that the edge states can also provide much information about the topological nodes in gap II. For example, in Fig. 5a, the quadratic node at the Γ point in gap II does not terminate the edge states, because a quadratic node cannot change the Zak phase. Therefore, the results in Fig. 5 and Supplementary Note 4 confirm that the projections of the topological nodes are reflected by the emergence and disappearance of the edge states in multiple bandgaps. These observations also indicate that the multi-gap bulk-edge physics provides important information on the configurations and (partly) the non-Abelian topological charges of the band nodes.

Conclusions and outlook

We theoretically predict simple models with new multi-gap observables due to an interplay with kagome symmetries. More importantly, using these results, we experimentally observe non-Abelian topological semimetals and their evolutions using acoustic Bloch bands in kagome acoustic metamaterials. By tuning the geometry of the metamaterials, we experimentally confirm the creation, annihilation, moving, merging and splitting of the topological band nodes in multiple bandgaps and the associated non-Abelian topological phase transitions (for example, the conversion and transfer of non-Abelian charges of nodes between different bandgaps). The underlying laws governing the evolution and transitions of non-Abelian topological charges of band nodes are crucial for the understanding of topological semimetals with multiple inter-connected bands, quench dynamics in inversion time-reversal symmetric systems and multi-gap topological phenomena in non-equilibrium bosonic systems (for example, phonons, photons and magnons).

Online content

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Patch Euler class. While complex gapped Hamiltonians are generally characterized by the Chern class in 2D, the Euler class similarly characterizes the topology of real gapped Hamiltonians. However, one specific feature of the Euler class is that it is only defined over \(C_2\) and \(C_4\) and not over \(C_3\) and \(C_6\) and other bands.

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patch (Extended Data Fig. 2). For example, if the patch Euler class for two nodes, say in the second gap, is computed to be 0, we can assign either two opposite charges within the patch, as in Extended Data Fig. 2a, or the same charges but with one opposite charge in the non-opposite gap, as in Extended Data Fig. 2b. If the patch Euler class is ±1, we can assign either the same charge to both nodes, as in Extended Data Fig. 2c, or opposite charges with one adjacent string crossing the patch, as in Extended Data Fig. 2d. The rule of the game is to first fix the charges and the Dirac string of one patch and then to reconstruct the rest of the entire topological configuration like a puzzle, using the outputs of the connected patch Euler class over every patch and the rules presented above.

Similar to the previous discussion of the patch Euler class, the sign of a frame charge is gauge-dependent, as it can be flipped by changing the gauge sign of the Bloch eigenvectors. Furthermore, the configuration (which node connects to which node) and trajectory (crossing or avoiding one patch) of the Dirac string can be changed, as these are a result of the local gauge freedom introduced above. We now proceed with the phase transitions discussed in the main text.

**Branding processes on the torus.** Our aim is to predictively study the successive semimetals phase transitions reached through the displacement of nodes. One crucial process here is the branding of nodes on the Brillouin zone torus, leading to gauge-invariant charge conversion, which either allows or forbids the physical process of node annihilation. In particular, branding results in a change of patch Euler class. We emphasize that branding is a physical process through which the nodes must be displaced, in contrast to a change between equivalent topological configurations associated to the gauge freedoms discussed above.

We illustrate two generic branding processes that can happen on the Brillouin zone torus and from which all the topological transitions can be deduced. First, we consider the simple branding of one of the second gap around one node of the first gap shown in Extended Data Fig. 3a. Panel A in Extended Data Fig. 3a shows an example with an initial patch Euler class of zero for a pair of nodes in the second gap. At this stage, the pair of nodes can be annihilated within the grey patch. Panels B, C and D in Extended Data Fig. 3a show then the effect of branding node C around node ▲. By crossing the adjacent Dirac string (blue dashed line), it must flip its charge to ●, and the final patch Euler class now gives \(|\mathcal{C}| = 1\), indicating that the two nodes are now stable (that is, they cannot be annihilated in the grand patch). We note that the patch Euler class has been changed through the branding from Extended Data Fig. 3a A to Extended Data Fig. 3a D.

Extended Data Fig. 3b shows similar processes, but after one of the nodes has travelled through one cyclic direction of the torus, creating a full Dirac string in the process. In panels A and B of Extended Data Fig. 3b there is no branding, and node ○ annihilates with node ● in the end. In panels C and D of Extended Data Fig. 3b, node ○ instead braids with the adjacent nodes, leading to a change of the final patch Euler class (compare Extended Data Fig. 3b panel D and Extended Data Fig. 3a panel A). In the following we refer to ‘branding’ whenever the nodes of one gap must cross the Dirac string of an adjacent gap, leading to a conversion of charges.

**Topological phase diagram of the kagome model.** Equipped with the non-Abelian nodal charges of simple nodes, the Dirac strings connecting them and the patch Euler class of multiple nodes, we are in a position to fully unveil the intricate topology hidden in the above-presented kagome model, the highlights of which are presented in the main text. We start from the simplest gapped phase and then introduce band nodes to study the evolution and branding of the nodes along a designed trajectory in parameter space. To this end, consider the gapped atomic insulating phase that breaks C3 symmetry, by setting \(\epsilon_b = \epsilon_c = 1\) (1, 0, 0) and \((l, t) = (0, 0)\). This system then has band energies \(E_1 = 1\), \(E_0 = 0\) and \(E_{-1} = 1\), and the corresponding Bloch states \(|\epsilon_b, k\rangle = |\epsilon_c, k\rangle = |\epsilon_a, k\rangle\). In the following, we refer to gap 1 as the (partial) energy gap between bands 1 and 2, and gap II as the (partial) energy gap between bands 2 and 3. The Berry phase factors for each band \(\psi_i = (\Gamma + K + \Gamma^-)\) where \(K\) is a reciprocal lattice vector, that is, \(\exp\left(i\mathcal{C}_\mathcal{N}\right)|\psi_i, k\rangle = (\Gamma + K + \Gamma^-)|\psi_i, k\rangle\), with the nth band projector \(P_n|\psi_i, k\rangle = |\psi_n, k\rangle\).\(\psi_i|\psi_j, k\rangle = \delta_{ij}\), and \(|\psi_i, k\rangle\), are then readily obtained by

\[
\psi_2|\psi_1, k\rangle = (\Gamma + K + \Gamma^-)|\psi_2, k\rangle = \epsilon_b k_x,
\]

\[
\psi_2|\psi_3, k\rangle = (\Gamma + K + \Gamma^-)|\psi_2, k\rangle = \epsilon_c k_y,
\]

\[
\psi_2|\psi_4, k\rangle = (\Gamma + K + \Gamma^-)|\psi_2, k\rangle = \epsilon_a k_z,
\]

where we have used the periodic gauge (that is, \(\epsilon_b + \epsilon_c + \epsilon_a = 0\)) for the Wilson loop integration. Berry phases on a non-contractible path are also referred to as Zak phases. Taking \(K = b\) and \(b\) being the writing vector of \(\mathcal{C}_\mathcal{N}\) for each band sum to \(0\mod 2\pi\) in each direction—the ‘flag bundle’ (where we consider each band separately) is non-orientable, as indicated by the non-zero Zak phases. Each \(\pi\) Zak phase indicates the presence of an odd number of Dirac strings crossing perpendicular to the path of integration. The corresponding conjugates \(\pi\) non-Dirac strings are non-orientable gaps, as shown in Extended Data Fig. 4a, where the Dirac string across gap I (DSI) is shown as a blue dashed line, and the Dirac string across gap II (DSII) as a red solid line. It is remarkable that, as long as \(\Gamma/\mathcal{C}_\mathcal{N}\) is preserved, the system cannot exhibit a fully trivial and orientable band structure, as will become clear in the following.

Remarkably, we conclude from the last three phases (Extended Data Fig. 4d–f) that the triply-degenerate points permit the transfer of nodes (and their non-Abelian charges) from one gap to the other.
In the next step, the six nodes of gap II on the Γ-M lines merge with the six nodes of gap I on the KM lines at the M points, leading to triply-degenerate points as shown in Extended Data Fig. 4g. To determine their frame charges, we consider the patch Euler class of a patch that contains one M point in Extended Data Fig. 4f and, in its vicinity, two nodes of gap II and two nodes of gap I. By first merging the nodes of gap I, one Dirac string (blue) must remain between the nodes of gap II (red), such that the latter cannot be annihilated, implying a patch Euler class \( \{\} \). We then conclude that the frame charge of the triply-degenerate points at \( M \) (Extended Data Fig. 4g) has frame charge \( \pi \) (see also below, where we compute the frame charge independently). Importantly, we note that the dispersion is again linear for two of the bands of the triply-degenerate points at \( M \), as for the triply-degenerate points at \( K \), despite their distinct frame charges.

In Extended Data Fig. 4h, each triply-degenerate point at the \( M \) points splits into two nodes in gap II and two nodes in gap I. The latter three successive phases (Extended Data Fig. 4f-h) thus illustrate the merging and splitting (that is, scattering) of stable nodes. We then readily predict that the nodes of gap II (red) will merge at \( K \) and \( K' \), giving rise to patch Euler classes \( \pi = 1 \) and \( \pi = -1 \), respectively. We thus predict the existence of quadratic nodes at the \( K \) points in the band structure of Extended Data Fig. 4h. This is further discussed in the main text.

We note that the tight-binding model allows us to further explore the phase diagram as discussed in Supplementary Note 1.4. In particular, we show the transfer of the quadratic node at \( \Gamma \) from gap II to gap I, mediated by the formation of a triply-degenerate point.

**Computation of the frame charge of the triply-degenerate points.** We compute the frame charge of the triply-degenerate points at \( K \) and \( M \), shown in Extended Data Fig. 5a,b, in a way that is complementary to the above evaluation based on the patch Euler class \( \{\} \). Defining a loop \( lX \) around one triply-degenerate point \( \chi = \{K, M\} \) that we parameterize with polar angle \( \theta_X \in [0, 2\pi] \) choosing \( X \) as the radial origin (Extended Data Fig. 5a,b), we obtain, numerically, the frame of eigenvectors over the loop \( R(\theta_X) = (u_1(\theta_X) \quad u_2(\theta_X) \quad u_3(\theta_X)) \). We then define the handedness of the frame at the initial point \( \theta_X = 0 \) through hand (0) = \( u_1(0) \times u_2(0) \times u_3(0) \) \( \in \{+, -\} \), and set \( u_0 \) to hand (0) \( u_1(0) \) to guarantee the right-handedness of the frame. We then iteratively define the overlap between the frames at two successive points of the loop \( \varphi(\theta_X) = \text{diag}\{u_1, u_2, u_3\} \times R(\theta_X) \times \text{diag}\{u_1, u_2, u_3\} \times R(\theta_X) \times \text{diag}\{u_1, u_2, u_3\} \) and express the overlap with the frame at the origin through \( \varphi(\theta_X) = \text{diag}\{u_1, u_2, u_3\} \times R(\theta_X) = \text{diag}\{u_1, u_2, u_3\} \times R(\theta_X) = \text{diag}\{u_1, u_2, u_3\} \times R(\theta_X) \), where we have fixed the handedness through multiplication with the gauge gauge \( s_i = s_i[\{\theta_X + \delta\}] \) for \( t = 1, 2, 3 \). We finally obtain the accumulated geometric phase of the frame over the loop through \( \varphi(\theta_X) = \text{diag}\{u_1, u_2, u_3\} \times R(\theta_X) = \text{diag}\{u_1, u_2, u_3\} \times R(\theta_X) = \text{diag}\{u_1, u_2, u_3\} \times R(\theta_X) \), which we have accumulated frame angles over \( k_1 \) and \( l_1 \) in Extended Data Fig. 5c. The total frame angle of the triply-degenerate point \( K \) is \( \pi \), indicating the non-Abelian frame charge of \( -1 \).

**Simulations.** All simulations of acoustic wave dynamics and calculations of acoustic energy bands were performed using the commercial finite-element solver COMSOL Multiphysics. The theoretical calculations were performed using the tight-binding model. The field of a single unit cell with Bloch boundary conditions in the 3D y-z plane. The edge band structures are calculated by solving the same equation in ribbon-like structures with the zigzag or armchair boundary in one direction and the periodic boundary condition in the other (see next section). In our set-up, the zigzag boundary along the \( x \) direction (Fig. 3c).

**Experiments.** The six experimental samples were all fabricated by commercial 3D printing technology based on photosensitive resin provided by a company in Shenzhen city. The fabrication precision was 0.1 mm. All samples were constructed using an FDM printer. The resonators were half-opened at the top, and the bottom surfaces of the resonators were half-open (but the top and bottom surfaces remained unchanged), so the acoustic waves were able to propagate into free space.

An acoustic source was placed into a resonator at the centre of the sample for excitation of the bulk modes. The acoustic wavefunction (technically, the acoustic pressure field) distributions were manually detected for each resonator through a portable probe microphone with diameter of \( \sim 7 \) mm, which perfectly fit the size of the top hole in each resonator. The amplitude and phase distributions of the acoustic wavefunctions (precisely, the acoustic pressure fields) were detected and then simultaneously recorded by an Agilent network analyser (Keysight E5061B). By Fourier transforming the detected acoustic pressure distributions at each excitation frequency, we obtained the acoustic bulk band structure for each sample (the results are presented in Fig. 3). To measure theedge band structures for the zigzag (armchair) edge boundary, we placed the acoustic source into a resonator in the middle of the zigzag (armchair) boundary.

**Data availability.** The data that support the findings of this study are available from the corresponding authors on reasonable request.

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**Author contributions**

A.B. and R.-J.S. performed the theory analysis underpinning the project. B.J. and J.-H.J. designed the metamaterials. B.J., Z.-K.L., X.Z., B.H., E.L. and J.-H.J. performed the experiments. A.B., R.-J.S. and J.-H.J. wrote the manuscript and the Supplementary Information, with input from all authors.

**Competing interests**

The authors declare no competing interests.

**Additional information**

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Extended Data Fig. 1 | Schematic illustration of the rules for the momentum-space braiding of non-Abelian topological nodes and their charge conversions with the Dirac strings (DSs). The left column shows the effect of a DS residing in the second gap on nodes in the first gap. The charge inversion processes of pulling a first-gap node through this DS or subsequently retracting the string over the other DS that connects the pair in the first gap is illustrated in the bottom panels, respectively. The right column depicts the recombination rules of Dirac strings as outlined in the main text. Every panel is merely a different representation of the same physics.
Extended Data Fig. 2 | Schematic illustration of the rules for the momentum-space braiding of non-Abelian topological nodes and their charge conversions with the Dirac strings (DSs). The left column shown the effect of a DS residing in the second gap on nodes in the first gap. The charge inversion processes of pulling a first-gap node through this DS or subsequently retracting the string over the other DS that connects the pair in the first gap is illustrated in the bottom panels, respectively. The right column depicts the recombination rules of Dirac strings as outlined in the main text. Every panel is merely a different representation of the same physics.
Extended Data Fig. 3 | Obstructions to nodes annihilation through the braiding of nodes on the (Brillouin zone) torus. In a the moving node does not cross the whole Brillouin zone, in b (AB) and b (CD) the moving node crosses the Brillouin zone along one of the cyclic direction of the torus leaving a non-contractible DS behind. The patch Euler class ($\xi(D)$) and the topological configuration after braiding are fully determined by the nodes charges and the Dirac strings of the initial configuration.
Extended Data Fig. 4 | Multi-gap topology in kagome models. a, Taking \((\epsilon_A, \epsilon_B, \epsilon_C) = (1, 0, -1)\) and \((t, t') = (0, 0)\) gives crossing Dirac strings (DS) in both gaps (blue for first gap, red for second gap). b and c, As the next step, turning off the onsite potentials and switching on the hopping terms induces band nodes. The nodes in the second gap (filled/empty red circles indicating ± topological charges) cross the DS in the first gap, forming a stable pair as the double node at \(\Gamma\) (brown circle) in (c) which has finite patch Euler class \(\xi = 1\). Meanwhile, the first gap features nodes at \(K\) points (triangles). c-i, Braiding process and transfer of band nodes from one gap to another through triple points. Band nodes and DS strings evolve such that the degeneracy at \(K\) in the first gap (blue triangles) is tuned into a double node configuration that has finite patch Euler class in the second gap (brown circles).
Extended Data Fig. 5 | Triply-degenerate points at K and M, and their frame charges. Band structure in the vicinity of the triply-degenerate points K (a) and M (b), with the encircling base loops $l_K$ and $l_M$. c The accumulated geometric frame angle computed over the base loops $l_K$ (full line) and $l_M$ (dashed line). The triply-degenerate point at K exhibits a total frame angle of $\pi$ indicating the non-Abelian frame charge $k$, while the triply-degenerate point at M exhibits a total frame angle of $2\pi$ indicating the non-Abelian frame charge of $-1$. 