Abstract. We describe general methods for enumerating subsemigroups of finite semigroups and techniques to improve the algorithmic efficiency of the calculations. As a particular application we use our algorithms to enumerate all transformation semigroups up to degree 4. The values for degree 3 and 4 were not known before. Initial classification of these semigroups up to conjugacy and isomorphism, by size and rank, provides a solid base for further investigations of transformation semigroups.

1. Introduction

When studying certain finite structures it can be helpful to generate small examples using a computer program. By investigating these sample objects we can formulate new hypotheses. More diverse sample sets make the hypotheses stronger, or easier to falsify by finding counterexamples. Therefore, to maximize the usefulness of these small examples we naturally aim both to enumerate all objects of a certain parameter value, and to increase the value of this parameter.

Previous efforts for enumerating semigroups were focused on the abstract case, enumerating by size, by the order of the semigroups, proceeding to semigroups of size $n + 1$ after all semigroups of size $n$ are listed. The main idea for enumerating by size was finding all valid multiplication tables of the given size [2, 3, 7, 8, 10, 11, 12, 13]. Here we enumerate transformation semigroups, semigroups represented as functions of a finite set. The size of the set is the degree of the transformation and we proceed by enumerating transformation semigroups of degree $n$, then of degree $n + 1$, so we enumerate by degree. For degree $n$ this is the task of enumerating all valid subtables inside the multiplication table of the full transformation semigroup $T_n$. An analogue of Cayley’s Theorem states that every finite semigroup is isomorphic to a transformation semigroup; therefore enumerating by size and by degree both list the same set (the set of finite semigroups) eventually (with repetitions in the transformation case). However, the order in which the semigroups appear in the enumeration is radically different. There are 52,989,400,714,478 abstract semigroups of order 9 [2], so one could barely imagine the number of semigroups of order 27, where $T_3$ would first appear. On the other hand, our results show that there are only 25 different transformation semigroups on 3 points of order 9. Metaphorically speaking, enumeration by size and by degree go in completely different directions and proceed with different speed.

The complexity of the multiplication of two transformations of degree $n \in \mathbb{N}$ is linear in $n$. However, multiplication in a semigroup defined by a multiplication table has constant complexity. Hence, we choose multiplication tables as the main way of representing semigroups. This decision has two consequences. First, our algorithms fall into the class of semigroup algorithms that fully enumerate the elements. This
of course restricts us to relatively small semigroups. Second, multiplication table algorithms are completely representation agnostic, so they are widely applicable across different types of finite semigroups. For space efficiency, we store subsets as bitlists, encoding their characteristic functions.

The article is organised as follows. In Section 2 we describe multiplication table methods to calculate subsemigroups generated by a subset. In Section 3 we present generic search algorithms to enumerate subsemigroups of finite semigroups. In Section 4 we discuss techniques for improving the efficiency of the algorithms in Section 3, based on more specific algebraic results. Finally, in Section 5 we apply the developed methods for enumerating transformation semigroups acting on up to 4 points.

1.1. Notation. Let $S$ be a finite semigroup such that $|S| = n \in \mathbb{N}$. We fix an order on the semigroup elements $s_1, \ldots, s_n$, so we can refer to the elements by their indices. Then the multiplication table, or Cayley table of $S$ is a $n \times n$ matrix $M_S$ with entries from $\{1, \ldots, n\}$, such that $M_{i,j} = k$ if $s_is_j = s_k$. The subarray of $M_S$ spanned by $A \subseteq \{1, \ldots, n\}$ is denoted by $M_A$. The $i$th row is $M_i\Box$ and the $j$th column vector is $M_j\Box$. We denote the set of the entries of a vector $v$ by $\text{Set}(v) = \{x \mid x \text{ is an entry of } v\}$. Similarly for multiplication tables or subarrays, $\text{Set}(M_A) = \{x \mid x \text{ is an entry of } M_A\}$.

The set of all subsemigroups of $S$ is denoted by $\text{Sub}(S) = \{T \mid T \leq S\}$. We consider the empty set a semigroup, therefore $\emptyset \in \text{Sub}(S)$. $\text{Max}(S)$ is the set of maximal proper subsemigroups of $S$. For $A \subseteq S$, $\langle A \rangle$ is the least subsemigroup of $S$ containing $A$, the semigroup generated by $A$.

If $I$ is an ideal of $S$ then the Rees factor semigroup $S/I$ has elements $(S \setminus I) \cup \{0\}$ with multiplication the same as in $S$ if the product stays in $S \setminus I$ and 0 otherwise.

A transformation is a function $f : X \to X$ from a set to itself. The set is denoted by positive integers $\{1, \ldots, n\}$ and a transformation $t$ is denoted by simply listing the images of the points: $[t(1), t(2), \ldots, t(n)]$. A transformation semigroup $(X, S)$ of degree $n$ is a collection $S$ of transformations of an $n$-element set closed under function composition. The semigroup of all transformations of $n$ points is the full transformation semigroup $T_n$. The group consisting of all degree $n$ permutations is the symmetric group $S_n$.

2. Multiplication Table Algorithms

2.1. Closure Algorithm. A basic question in computational semigroup theory is: what subsemigroup does a subset $A \subseteq S$ generate? There is of course a simple algorithm for calculating $\langle A \rangle$: we keep multiplying elements of $A$, also with any new elements until all the possible products yield an element already in the collection. We can do this calculation directly on the table just by looking up products. Mirroring the notion of the subsemigroup generated by $A$ we define the closure of $A$ as the least subarray of $M_S$ containing $M_A$. To calculate the closure, first we need to determine the missing elements $m(B) = \text{Set}(M_B) \setminus B$, the products of elements of $B$ that are not in $B$. Then, recursively

$$\text{Cl}_1(A) = A \cup m(A)$$
$$\text{Cl}_{i+1}(A) = \text{Cl}_1(\text{Cl}_i(A)) \quad i \geq 1$$

and $\text{Cl}(A) = \text{Cl}_j(A)$, where $j$ is minimal for $\text{Cl}_j(A) = \text{Cl}_{j+1}(A)$, or equivalently $m(\text{Cl}_j(A)) = \emptyset$. 
The recursive definition describes an algorithm for calculating the closure, but not an efficient one. We can avoid the full calculation of the missing elements in the recursive steps. When extending the subarray $M_A$ by a single element $i$, if $m(A)$ is already calculated, then all new missing elements in $m(A \cup \{i\})$ can only come from the $i$th row or the $i$th column. So, for calculating the closure we can extend the subarray one-by-one using the elements of $m(A)$ and any new missing elements encountered during the recursion. This way each table entry is checked only once.

For example, let’s consider the multiplication table of $S_3$ induced by the ordering $(), (2,3), (1,2), (1,2,3), (1,3,2), (1,3)$ — the lexicographic order of the transformation notation.

|   | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 1 | 3 | 5 | 2 | 4 | 6 |
| 3 | 1 | 4 | 6 | 3 | 5 | 2 |
| 4 | 1 | 5 | 2 | 4 | 6 | 3 |
| 5 | 1 | 6 | 3 | 5 | 2 | 4 |
| 6 | 1 | 4 | 2 | 6 | 3 | 5 |

The subarray $M_{\{4\}}$ contains only one entry 5, which is different from 4, so the subarray is not closed. With the above notation $m(\{4\}) = \{5\}$, $m(\{4,5\}) = \{1\}$ and $m(\{1\}) = m(\{1,4,5\}) = \emptyset$.

### 2.2. Isomorphism and Anti-Isomorphism of Multiplication Tables

Though we have the luxury of having the full structure of the semigroup represented in the multiplication table, it is still not a trivial matter to decide isomorphism. Two semigroups $S$ and $T$ are isomorphic or anti-isomorphic if $M_S$ can be transformed into $M_T$ by rearranging its columns and rows without or with transposing the table. In other words, $S$ and $T$ are isomorphic if and only if they belong to the same orbit under the action of $\mathbb{Z}_2 \times S_n$. It is usually easier to check that $S$ and $T$ belong to the same orbit by calculating a canonical representative of the orbit of $S$ and the orbit of $T$, and then comparing these values. This is the approach in the SmallestImageSet function in GAP [5], which underlies the SmallestMultiplicationTable function in the SEMIGROPS package [9]. This method does not work well for larger semigroups, so we have to apply a bit more clever backtrack search.

By computing some global properties of the multiplication tables that are invariant under these rearrangements in some cases we can easily decide non-isomorphism. These properties can be any statistics that do not take into account any information of actual positions of elements, like frequency values. A frequency distribution takes a multiset and enumerates its distinct elements paired with the number of occurrences of the elements. For instance, the frequency distribution of a row vector $(2, 1, 2, 5, 5, 2)$ is $((1, 1), (2, 3), (5, 2))$. However, we cannot retain the element information as it depends on the sorting of the semigroup elements. We keep only the sorted frequency values $(1, 2, 3)$. Sorting is crucial here to decide whether two distributions are the same or not. If they are equal then we say that the multisets have the same type. For example, the vector $(2, 4, 2, 4)$ has the same type as $(1, 3, 3, 1)$ and $(2, 2, 4, 4)$, but $(2, 4, 4, 4)$ has different type. We can save storage space if frequency values are repeated many times by taking the frequency distribution of frequency values. This time it is appropriate to store the whole frequency distribution as it is derived from data containing no information on the ordering of the elements.
We can define the properties on the element and on the table level. The properties of an element are:

1. **Frequency**: the number of occurrences of the element in the table.
2. **Diagonal frequency**: the number of occurrences of the element in the diagonal of the table.
3. **Index-period**: the smallest values \( m \geq 1, r \geq 1 \) such that \( a^{m+r} = a^m \).
   The semigroup analogue of the order of a group element.
4. **Row type**: for \( i \in S \), the frequency values of elements in the row \( M_i,\square \).
5. **Column type**: for \( i \in S \), the frequency values of elements in the column \( \square, i \).
6. **Profile**: Combining together all previous properties.

Aggregating the properties of elements and other statistics of the table not containing information on actual positions determine the properties of a table, and thus a semigroup. Some of them are sensitive enough to tell apart groups, though group multiplication tables are Latin squares.

1. **Frequency distribution of frequency values of elements**.
2. **Column and row types**: The frequency distribution of column types and row types of elements.
3. **Diagonal frequencies**: This can even tell some groups apart: \( \mathbb{Z}_4 \mapsto \langle 2, 2 \rangle \), \( \mathbb{Z}_2 \times \mathbb{Z}_2 \mapsto \langle 4 \rangle \), but it assigns \( \langle 2, 6 \rangle \) to both \( D_8 \) and \( Q_8 \).
4. **Element Profiles**: In the group case this invariant reduces to the order of elements, but it can distinguish between \( D_8 \) and \( Q_8 \). However, this invariant fails to detect the difference between some direct and semidirect products. For instance, \( \mathbb{Z}_8 \times \mathbb{Z}_2 \) and \( \mathbb{Z}_8 \rtimes \mathbb{Z}_2 \) both have 1 element of order 1, 3 of order 2, 4 of order 4, and 8 of order 8.

For deciding non-isomorphism we can check these properties; if any one of these is different then the semigroups are not isomorphic. If all invariants check out, then we can use backtrack to find out whether one of the multiplication tables can be rearranged to get the other one. An element of \( \mathbb{Z}_2 \times S_n \) can witness the isomorphism or anti-isomorphism. Fortunately we do not have to search through the whole symmetric group, we only need to consider some special permutations that respect the profile of an element.

3. **Basic Search Algorithms for Subsemigroup Enumeration**

**Problem 3.1.** For a semigroup \( S \), find all of its subsemigroups:

\[
\text{Sub}(S) = \{ X \subseteq S \mid \langle X \rangle = X \},
\]

Thinking in terms of the multiplication table \( M_S \), we are looking for all subarrays \( M_X \) that are also multiplication tables, i.e. they do not contain elements not in \( X \), \( \text{Set}(M_X) \subseteq X \).

The obvious brute-force algorithm for constructing \( \text{Sub}(S) \) is the enumeration of the powerset \( 2^S \) and checking each subset whether it is closed or not. This works only for small cases as \( 2^n \) grows fast. It is also inefficient in the sense that in general only a fraction of the subsets are closed under multiplication (but not always, e.g. left zero semigroups).
Algorithm 1: Finding subsemigroups by minimal extensions. Depending on how \texttt{exts}, the storage for extensions, behaves under the \texttt{Store/Retrieve} operations we get different search strategies. Stack gives depth-first, while queue data structure gives breadth-first search.

\begin{algorithm}
\caption{Finding subsemigroups by minimal extensions.}
\begin{algorithmic}[1]
\State \textbf{input} : $S$, the ambient semigroup
\hspace{1em} $T \subseteq S$, a subsemigroup to be extended
\hspace{1em} $X \subseteq S$, a set of elements to extend with
\State \textbf{output}: all $T' \subseteq S$ such that $T' = \langle T \cup Y \rangle$ for some $Y \subseteq X$
\State $\text{SubSemigroupsByMinimalExtensions} (T, S, X)$:
\State \hspace{1em} $\text{subs} \leftarrow \{ T \}$
\State \hspace{1em} $\text{exts} \leftarrow \emptyset$
\For {$s \in (S \setminus T) \cap X$}
\State \hspace{1em} \text{Store} ($\text{exts}$, $T \cup \{ s \}$)
\EndFor
\While {$|\text{exts}| > 0$}
\State $T' \leftarrow \langle \text{Retrieve}(\text{exts}) \rangle$
\If {$T' \notin \text{subs}$}
\State $\text{subs} \leftarrow \text{subs} \cup \{ T' \}$
\For {$s \in (S \setminus T') \cap X$}
\State \hspace{1em} \text{Store} ($\text{exts}$, $T' \cup \{ s \}$)
\EndFor
\EndIf
\EndWhile
\State \textbf{return} $\text{subs}$
\end{algorithmic}
\end{algorithm}

3.1. Enumerating by Minimal Generating Sets. The rank of a semigroup is the least size of a generating set. The rank of a subsemigroup can be bigger than the rank of the semigroup itself. For example, the full transformation semigroup has rank 3 \cite{[4]}, but its minimal ideal, which is a left zero semigroup has rank $n$. Assuming that we know the maximum rank for subsemigroups of $S$, we can check all subsets of $S$ with cardinality up to that value to see what subsemigroups they generate. The same subsemigroup may be generated by many generating sets but the maximality guarantees that we get $\text{Sub}(S)$. On each level $k$ we check $\binom{|S|}{k}$ many generator sets. Therefore the method is only feasible if the maximum value of the ranks of the subsemigroups is known to be small. The maximal rank of a subsemigroup of $T_n$ is not known.

What can we do if we do not know the maximum rank value? We can keep going until no new semigroup is generated. First we check all subsemigroups generated by one element. Then all those generated by two elements. Then we subtract the previous set from the latter one to get the set of rank-2 subsemigroups. Then continue up to $n$ where the set of $n$-generated semigroups is empty. Unfortunately this last step is wasted, unless $\text{rank}(S) = S$, e.g. left zero semigroups, in which case, this is just the brute-force search.

3.2. Enumerating by Minimal Extensions. A minimal extension of a subsemigroup $T \subseteq S$ is a subsemigroup $\langle T \cup \{ u \} \rangle$, where $u \in S \setminus T$. We simply add a new element to $T$ and calculate the closure. If we recursively do minimal extensions for all $u \in S \setminus T$, then we enumerate all subsemigroups of $S$ containing $T$.

This algorithm is a graph search. The nodes are the subsemigroups. There is a directed edge labelled $u$ from $T$ to $T'$ if $T' = \langle T \cup \{ u \} \rangle$. In general, there are many
incoming edges to a subsemigroup. The efficiency of the algorithm comes from the fact that the search tree is cut when the search encounters a subsemigroup already known, simply by making no further extensions. See Algorithm 1 for details. A full subsemigroup enumeration can be done by starting the algorithm with parameters $T = \emptyset$, the ambient semigroup is simply $S$, and $X = S$. This is simply extending the empty set by all elements of $S$ recursively.

Optionally, we can also keep track of the generating sets of the subsemigroups. When using the breadth-first search strategy, the generating set is minimal, so Algorithm 1 can easily be modified to enumerate minimal generating sets. Little consideration shows that this is a more efficient version of the minimal generating sets algorithm (Section 3.1), but it does not escape checking generating sets one bigger than the maximal rank.

4. Advanced Algorithmic Techniques

Since we are dealing with well-studied algebraic structures, we have many mathematical results to exploit, improving the efficiency of any basic search algorithm.

4.1. Parallel Enumeration in Ideal Quotients. In general, an ideal $I$ divides a subsemigroup $T$ into two parts: a subsemigroup contained in the ideal, $L = T \cap I$, and a subset outside the ideal, $U = T \cap (S \setminus I)$. We call these the lower torso and upper torso, respectively (Fig. 1).

The upper torso is only a subset in general, but it can be made into a subsemigroup by adjoining a zero element. This is the well-known Rees quotient. Subsemigroup enumeration can be done in parallel in $I$ and $S/I$ and then we can combine the results.

Lemma 4.1. Let $I$ be an ideal of $S$, then

$$\text{Sub}(S) = \{ \langle (U \setminus \{0\}) \cup T \rangle \mid U \in \text{Sub}(S/I), \ T \in \text{Sub}(I) \}.$$
Figure 2. Calculating lower torsos for subsemigroup $T$ by minimal extensions. First extending with $x_1$ then by $x_2$, elements from $I$. The idea is that $|T| << |\langle T \cup \{x_1\} \rangle| << |\langle T \cup \{x_1, x_2\} \rangle|$. The jumps in size are due to the upper torso acting on the elements of the ideal.

**Proof.** Let $T \in \text{Sub}(S)$ and put $L = T \cap I$ and $U = T \cap (S \setminus I) \cup \{0\}$. Then $T = L \cup (U \setminus \{0\}) = \langle L \cup (U \setminus \{0\}) \rangle$. This establishes the forward set containment. The other is trivial. \qed

The method suggested by Lemma 4.1 requires calculating $|\text{Sub}(S/I)| \cdot |\text{Sub}(I)|$ set unions and subsequent closures.

### 4.2. Lower Torso Enumeration

For an ideal $I$ of semigroup $S$, suppose we enumerated $\text{Sub}(S/I)$. Removing all the zero elements from these we get all upper torsos. Next question is finding all the matching lower torsos. In Section 4.1 we enumerated $\text{Sub}(I)$ and checked what the combinations generated. We can do better. The idea is that the upper torso acts on the elements of the ideal, so if we do a minimal extension search (Section 3.2) the extensions will be ‘large jumps’ (Fig. 2). We can use Algorithm 1 starting from $T$ and extending only by the elements from the ideal. In practice, for the full transformation semigroups, this is a very useful trick.

### 4.3. Maximal Subsemigroups

Assuming that we have the maximal subsemigroups calculated, we can parallelize enumerating subsemigroups by enumerating subsemigroups of its maximal subsemigroups and merge the results.

**Fact 4.2.** $\text{Sub}(S) = (\bigcup_{T \in \text{Max}(S)} \text{Sub}(T)) \cup \{S\}$

To compute $\text{Max}(S)$, a list of properties that every maximal subsemigroup of a finite semigroup must have was described in [6]. Algorithms, based on these properties, are being implemented in [9].

However, the sets of subsemigroups of the maximal subsemigroups do overlap in general, therefore the same subsemigroup gets enumerated many times and merging is a non-trivial step. Also, recursively iterating the maximal subsemigroups is a variant of the depth-first search algorithm.
4.4. Exploiting Symmetries. If we know all the symmetries of \( S \), the semigroup’s automorphism group, then we can accelerate any subsemigroup enumeration algorithm. Whenever a subsemigroup is found, we can generate its conjugate subsemigroups. Conjugation is defined in a way analogous to the group theoretical notion: \( t^g = g^{-1}tg \) and for sets of semigroup elements the conjugation is done pointwise.

**Fact 4.3.** \( T \in \text{Sub}(S) \) and \( g \in \text{Aut}(S) \) then \( T^g \in \text{Sub}(S) \).

There is an algorithm for computing the automorphism group of a finite semigroup [1].

If \( G \) is a group of automorphisms of \( S \), then we denote the set of conjugacy class representatives of the subsemigroups of \( S \) by \( \text{Sub}_{G}(S) \).

4.5. Equivalent Generators. We define an equivalence relation on \( S \) by

\[ s \equiv t \iff \langle s \rangle = \langle t \rangle, \]

so semigroup elements are equivalent if they generate the same subsemigroup.

For distinct elements this can only happen to nontrivial elements of cyclic groups of prime order. However, a semigroup can contain many copies of those. Beyond the obvious copies with fixed points, we have examples that also move the point not in the cycle. For instance, in the singular transformation semigroup of degree 4, \([2, 3, 1, 1]\) is equivalent to \([3, 1, 2, 2]\), both generating \( \{[2, 3, 1, 1], [3, 1, 2, 2], [1, 2, 3, 3] \} \).

In a search algorithm, if \( s \equiv t \), then after extending by \( s \) we can simply omit extending by \( t \).

5. Enumerating Transformation Semigroups of Degree 2, 3 and 4

In order to enumerate all transformation semigroups on \( n \) points we construct all subsemigroups of the full transformation semigroup \( T_n \). We use its ideal structure to make the enumeration more efficient by making the calculation parallel.

The **rank** of a transformation \( t \) is \( |\text{im}(t)| \). The ideal of \( T_n \) containing all elements of rank at most \( i \) is denoted by \( K_{n,i} \). The ideal structure of \( T_n \) is a linear order of nested ideals:

\[ \emptyset \subset K_{n,1} \subset K_{n,2} \subset \ldots \subset K_{n,n-1} \subset T_n. \]

The ideal \( K_{n,n-1} \) is also called the **singular transformation semigroup** of degree \( n \), all transformations but the permutations.

For \( T_n \), the automorphism group is \( S_n \), so we are primarily interested in calculating \( \text{Sub}_{S_n}(T_n) \).

We can make a few observations on the multiplication table of \( T_n \).

**Fact 5.1.** Let \( M \) be the multiplication table of \( T_n \). If \( t \) is in the row \( M_i, \square \), then it appears there \( n^k \) times, for some \( 0 \leq k \leq n \).

**Proof.** The \( i \)th row of the multiplication table is the right ideal \( s_iT_n \). The element \( t \) appears in \( M_i, \square \) as many times as the number of solutions to the equation \( s_iu = t \). Let \( r \) be the rank of \( s_i \). Since \( s_i \) and \( t \) are fixed, any transformation \( u \) such that \( s_iu = t \) must satisfy \( (x)s_iu = (x)t \) for all \( x \in \{1, \ldots, n\} \). If there are \( r \) distinct elements \( (x)s_i \), then there are \( n-r \) points where \( u \) can be defined arbitrarily, which yields \( n^{n-r} \) possible solutions \( u \).

Let \#s denote the number of occurrences of \( s \) in \( M_{T_n} \).

**Fact 5.2.** For all \( s \in T_n \), \#s is a multiple of \( n \).
Proof. Every element appears once in a row indexed by a permutation. So we always get $n!$ and some nontrivial powers of $n$ by Fact 5.1.

5.1. $\mathcal{T}_2$, The Pen and Paper Case. $\mathcal{T}_2$ has only four elements and consequently the search space size is only $2^4 = 16$. It is an easy exercise to find all of its subsemigroups. We order the elements lexicographically, $1=[1,1]$, $2=[1,2]$, $3=[2,1]$, $4=[2,2]$. Here are the closed subarrays:

\[
\begin{align*}
1144 & 124 \quad 12 \quad 14 \\
124 & 11 \quad 4 \quad 12 \\
14 & 4 \quad 4 \quad 1 \\
124 & 14 \quad 4 \quad 12 \\
1144 & 124 \quad 12 \quad 14 \\
124 & 11 \quad 4 \quad 12 \\
14 & 4 \quad 4 \quad 1 \\
124 & 14 \quad 4 \quad 12 \\
\end{align*}
\]

Using these we can draw the subsemigroup lattice (Fig. 3). It also shows that it is possible that isomorphic subsemigroups are not conjugate.

An obvious classification of $\text{Sub}(\mathcal{T}_2)$ can be done according to the sizes of the subsemigroups (Fig. 3). It turns out that another way of partitioning the elements will also be important for higher degrees. One big chunk of the subsemigroup lattice of $\mathcal{T}_n$ formed by the subsemigroups of the singular part, $\text{Sub}(K_{n,n-1})$, and this has an order-isomorphic copy when we adjoin the identity of $\mathcal{T}_n$ to each subsemigroup. The remaining part is the set of subsemigroups that contain nontrivial permutations. Since we have no problems with fully calculating and displaying $\text{Sub}(\mathcal{T}_2)$, this division has no significance, but can be visualized easily (Fig. 4).
Figure 4. The subsemigroup lattice of $\mathcal{T}_2$, alternative classification. The singular part is indicated by the lowest light gray blob. The other light gray group is an identical copy of the singular part, just the identity adjoined to each subsemigroup. The dark grey part consists of the subsemigroups containing nontrivial permutations. The size of the dark group gets smaller relative to the singular part for higher degrees.

5.2. $\mathcal{T}_3$, The Brute-force Doable. The search space size for $\mathcal{T}_3$ is $2^{3^3} = 134217728$, approximately 134.2 million, thus exhaustive enumeration of subsets is still possible, though it takes many hours on a desktop computer. In contrast, using the minimal extension method (Section 3.2) together with the equivalent generators trick (Section 4.5) the calculation is seemingly instantaneous. For generating the 283 conjugacy classes only 5362 subsets need to be checked. For all the 1299 subsemigroups 25041 checks are required. This demonstrated efficiency of the graph search algorithm makes our approach feasible.

| Order | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 |
|-------|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| #conj | 1 | 3 | 10 | 19 | 28 | 38 | 42 | 38 | 30 | 25 | 14 | 12 | 7 | 3 | 1 | 3 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| #isom | 1 | 1 | 5 | 15 | 24 | 37 | 42 | 38 | 30 | 25 | 14 | 12 | 7 | 3 | 1 | 3 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

Table 1. The frequency distribution of conjugacy and isomorphism classes of $\text{Sub}(\mathcal{T}_3)$. 
Classifying according to the sizes of the subsets is summarized in Table 1. It is easy to see why there are no transformation semigroups on 3 points of size 25 and 26. The biggest maximal subsemigroup is of order 24, so there is nothing in between that and $T_3$. On the other hand, we have no such explanation for orders 18, 19 and 20. The same argument may apply for the maximal subsemigroups, but that would still require some more fundamental explanation.

| rank | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|------|---|---|---|---|---|---|---|
| #conjugacy classes | 1 | 7 | 46 | 101 | 85 | 36 | 7 |

Table 2. Rank value distribution of conjugacy classes of $\text{Sub}(T_3)$.
The number of generators in a minimal size generating set and the number of conjugacy classes with that many minimal generators.

We can also apply the minimal generating sets method (Section 3.1), yielding Table 2.

A semigroup $S$ is nilpotent if $S^k = \{0\}$ for some $k \in \mathbb{N}$. It is $k$-nilpotent if $k$ is the minimal such number. In other words, being $k$-nilpotent means that any product with at least $k$ elements yields the zero element, but some product of $k-1$ elements does not. The empty semigroup is not nilpotent. To decide $k$-nilpotency algorithmically, in general we do not need to calculate the power $S^k$. We can take a random $k$-tuple of semigroup elements, evaluate it as a product and assume that this value is the zero element. If we find any other $k$-tuple evaluating to a different value, then the semigroup is not $k$-nilpotent. The worst case is when $S$ is indeed $k$-nilpotent, we end up checking all $k$-tuples.

It turns out that that there are only 4 different nilpotent transformation semigroups on 3 points. The trivial monoid is 1-nilpotent and it can be realized by three different ways: by the identity transformation, by a constant map, and by a conjugate of $[1, 1, 3]$. The only 2-nilpotent conjugacy class has the representative $\{[1, 1, 1], [1, 1, 2]\}$.

5.3. $T_4$, The Parallel Possible. Since $|T_4| = 4^4 = 256$, the search space is already enormous, $2^{256}$ is a 78-digit number. Therefore, the practical calculation of $\text{Sub}(T_4)$ is done by climbing up the ideal hierarchy and do calculations in parallel whenever possible. We can jump over the first ideal.

(1) Calculate $\text{Sub}_{S_4}(K_{4,3}/K_{4,2})$ by the minimal extension algorithm (Section 3.2). There are 10 002 390, slightly more than 10 million conjugacy classes.

(2) In parallel, enumerate all lower torsos for all the upper torsos derived from $\text{Sub}_{S_4}(K_{4,3}/K_{4,2})$ with the limited enumeration method (Section 4.2). This gives $\text{Sub}_{S_4}(K_{4,3})$, with 65 997 018 conjugacy classes. The calculation is truly parallel since the upper torsos always differ, so there is no need for merging the elements. By extending the empty upper torso we get $\text{Sub}_{S_4}(K_{4,2})$.

(3) To get the isomorphic copy of the singular part, we simply adjoin the identity to all subsemigroups in $\text{Sub}_{S_4}(K_{4,3})$. Call this set $C$. Purely administrative step.

(4) Calculate $\text{Sub}_{S_4}(S_4)$ with minimal extensions. These are all closed upper torsos. This is a lot easier subgroup enumeration problem.
In parallel, find all lower torsos for all nontrivial subgroups in \( \text{Sub}_{S_4}(S_4) \). Let \( P \) be the set of subsemigroups of \( T_4 \) with nontrivial permutations, including the subgroups as well. This part corresponds to the dark blob on Figure 4. Though the search space is the set of subsets (or in this case the subsemigroups) of \( K_{4,3} \), the search is surprisingly quick. This is due to the fact that a subgroup acts on the singular part, making each minimal extension into a huge step. In other words, we take each (conjugacy class representative) subgroup \( 1 \neq G \leq S_4 \) and look for subsemigroups of \( K_{3,3} \) closed under the products with \( G \). Even a single nontrivial permutation makes the closure relatively big. For instance, there are only 71,146 lower torsos in \( K_{4,3} \) for \( \mathbb{Z}_2 \).

\( \text{Sub}(T_4) = \text{Sub}(K_{4,3}) \cup C \cup P. \) The set of subsemigroups of the ideal part, its copy with the identity adjoined to each subsemigroup, and the subsemigroups with permutations.

The size distribution of \( \text{Sub}(T_4) \) shows an interesting pattern (Fig. 5). For subgroups of a group only the divisors of its order have nonzero frequency values. If we considered the size distribution of all subsets, the maximal binomial coefficient would define the peak value. For \( \text{Sub}(T_4) \) the situation is more involved. The
numbers are big and they make the impression of continuous change with several
peaks. To explain the shape of the distribution a systematic study of the size classes
is needed.

There are only 22 nonempty nilpotent subsemigroups of \( T_4 \) up to isomorphism,
4 of them are 1-nilpotent, 7 are 2-nilpotent and 11 are 3-nilpotent. The biggest
3-nilpotent subsemigroup has 6 elements:

\[
\{ [1,1,1,1], [1,1,1,2], [1,1,1,3], [1,1,2,1], [1,1,2,2], [1,1,2,3] \}.
\]

It is an interesting question whether the pattern seen here is a general recipe for
constructing maximal 3-nilpotent transformation semigroups.

6. Summary and Conclusion

We enumerated and classified all transformation semigroups up to degree 4. It
turns out while enumerating abstract semigroups yields mostly 3-nilpotent semi-
groups, enumerating transformation semigroups gives mostly non-nilpotent ones.
Both methods enumerates the same set, the set of all finite semigroups, but they go
in different directions and proceed with different speed – metaphorically speaking.

The methods developed here, with more concentrated effort and computational
power, may be able to enumerate \( \text{Sub}(T_5) \) or the subsemigroups of some of its
ideals/Rees quotients. The closure algorithms in Section 2 need full complexity
analysis before attacking the next degree.

However, a better usage of the results would be to investigate the possibility
of a more constructive theory of all transformation semigroups. For instance, by
studying how many different ways \( \text{Sub}(T_n) \) is embedded into \( \text{Sub}(T_{n+1}) \), we can
probably estimate \( |\text{Sub}(T_{n+1})| \), or even construct some recursive formula.

In any case, the raw data and the initial classification gives us plenty to investi-
gate and think about.

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