Inflation and quintessence: theoretical approach of cosmological reconstruction

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Abstract. In the first part of this paper, we outline the construction of an inflationary cosmology in the framework where inflation is described by a universally evolving scalar field \(\phi\) with potential \(V(\phi)\). By considering a generic situation that inflaton attains a nearly constant velocity, during inflation, \(m_p^{-1}|d\phi/dN| \equiv \alpha + \beta \exp(\beta N)\) (where \(N \equiv \ln a\) is the e-folding time), we reconstruct a scalar potential and find the conditions that have to be satisfied by the (reconstructed) potential to be consistent with the WMAP inflationary data. The consistency of our model with the WMAP result (such as \(n_s = 0.951 \pm 0.015, r < 0.3\)) would require \(0.16 < \alpha < 0.26\) and \(\beta < 0\). The running of the spectral index, \(\tilde{\alpha} \equiv d n_s/d \ln k\), is found to be small for a wide range of \(\alpha\).

In the second part of this paper, we introduce a novel approach of constructing dark energy within the context of the standard scalar–tensor theory. The assumption that a scalar field might roll with a nearly constant velocity, during inflation, can also be applied to quintessence or dark energy models. For the minimally coupled quintessence, \(\alpha_Q \equiv d A(Q)/d(\kappa Q) = 0\) (where \(A(Q)\) is the standard matter–quintessence coupling), the dark energy equation of state in the range \(-1 \leq w_{DE} < -0.82\) can be obtained for \(0 \leq \alpha < 0.63\). For \(\alpha < 0.1\), the model allows for only modest evolution of dark energy density with redshift. We also show, under certain conditions, that the \(\alpha_Q > 0\) solution decreases the dark energy equation of state \(w_Q\) with decreasing redshift as compared to the \(\alpha_Q = 0\) solution. This effect can be opposite in the \(\alpha_Q < 0\) case. The effect of the matter–quintessence coupling can be significant only if \(|\alpha_Q| \gtrsim 0.1\), while a small coupling \(|\alpha_Q| < 0.1\) will have almost no effect on cosmological parameters, including \(\Omega_Q, w_Q\) and \(H(z)\). The best fit value of \(\alpha_Q\) in our model is found to be \(\alpha_Q \simeq 0.06\), but it may contain significant numerical errors, namely \(\alpha_Q = 0.06 \pm 0.35\), which
clearly implies the consistency of our model with general relativity (for which $\alpha_Q = 0$) at 1σ level.

**Keywords:** dark energy theory, inflation, physics of the early universe

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### 1. Introduction and overview

It is true and remarkable that our understanding of the physical universe has deepened profoundly in the last few decades through thought, experiment and observation. Along with significant advancements in observational cosmology [1]–[3], Einstein’s general relativity has been established as a successful classical theory of gravitational interactions, from scales of millimetres through to kiloparsecs (1 pc = 3.27 light years). It has also been learnt that, at very short distance scales, large quantum fluctuations make gravity very strongly interacting, implying that general relativity cannot be used to probe spacetime (geometry) for distances close to Planck’s length, $l_P \sim 10^{-33}$ cm. In addition to this difficulty, three striking facts about nature’s clues suggest that we are missing a few important parts of the picture, notably the extreme weakness of gravity relative to the other forces, the huge size and flatness of the observable universe and the late-time cosmic acceleration.

Much is not understood: what is the nature of the mysterious smooth dark energy and the clumped non-baryonic dark matter, which respectively form 73% and 22% of the mass–energy in the universe? That means, we do not see and really understand yet about 95% of the total matter density of the universe. To understand the need for dark energy, or a mysterious force propelling the universe, and dark matter, one has to look at the different constituents of the universe, their properties and observational evidences (for reviews, see, e.g., [4]–[7]). The current standard model of cosmology somehow combines the original hot
big bang model and the early universe inflation, by virtue of the existence of a fundamental scalar field, called *inflaton*. The standard model of cosmology is, however, not completely satisfactory and it appears to have some gaps. If the universe is currently accelerating (on largest scales), which recent observations seem to indicate, then we need in the fabric of the cosmos a self-repulsive dark energy component, or a cosmological constant term, which had almost no role in the early universe, or need to modify Einstein’s theory of gravity on the largest scales in order to explain this acceleration.

When in 1917 Einstein proposed the field equations for general relativity:

\[ R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}, \]  

he had the choice of adding an extra term proportional to the metric \( g_{\mu\nu} \) either on the left-hand or right-hand side of equation (1.1). This extra term, the so-called cosmological constant \( \Lambda \), is not fixed by the structure of the theory. One also finds no good reason to set it to zero either, unless the underlying theory is purely supersymmetric. Adding the term \( \Lambda g_{\mu\nu} \) on the left-hand side of his famous equation, Einstein used it to tune the constant \( \Lambda \) in such a way that he would get a non-expanding solution. Einstein later dismissed the cosmological constant as his ‘greatest blunder’, when Hubble found a clear indication for an (ever) expanding universe. Today this constant is mainly written on the right-hand side of the Einstein equations but still with a positive sign, which therefore acts as an extra repulsive force (or dark energy) in cosmological (time-dependent) backgrounds.

Before presenting further thoughts on the nature of this puzzling form of energy, it is logical to recapitulate the independent pieces of evidence for its existence. The key measurements, leading to the result of the DE density fraction being \( \Omega_{\text{DE}} \approx 0.7 \) were made, rather unexpectedly, in 1998 by two independent groups (Supernova Cosmology Project and High-z Supernova Search Team) [1]. These observations revealed, for the first time, that the universe is not only expanding now but its expansion has been speeding up for the last 5–6 billion years, i.e. since the point the redshift \( z \) dropped below 0.85.

Evidence for the existence of dark energy also comes from observations of the cosmic microwave background (CMB) for which the most recent ones have been obtained by NASA’s Wilkinson Microwave Anisotropy Probe (WMAP) [3]. As first observed in 1992 by the COBE satellite [8] and afterwards by several other ground-and balloon-based experiments, the nearly perfect black body spectrum of the CMB has small temperature fluctuations of the order of \( \delta T/T \sim 18 \mu K/2.725 K \sim 10^{-5} \). The angular size of these fluctuations encodes the density and velocity fluctuations at the surface of last scattering, with redshift \( z \approx 1100 \). This corresponds to the cosmological epoch when the presently observed CMB photons first decoupled from matter. By plotting the square of the amplitude of CMB temperature fluctuations against their wavelengths (or multipoles in an equivalent Fourier power spectrum), there can be allocated several peaks at different angular sizes. The position of the first peak is often viewed as an indicator for the spatial curvature of the universe, which reveals that the present universe is nearly flat and homogeneous on large cosmological scales (>100 Mpc), meaning that \( \Omega_{\text{tot}} \approx 1 \) with high accuracy. However, when assuming a flat universe only containing pressureless dust (including DM) and assuming the current Hubble parameter to be \( h = 0.72 \pm 0.08 \) with \( H_0 = 100h \text{ km s}^{-1} \text{ Mpc}^{-1} \) (in agreement with observations of the Hubble Space Telescope Key project [9]), it is found that \( t_0 = 9 \pm 1 \text{ Gyr} \). This result, simply following from Einstein’s general relativity, implies that a flat universe without the cosmological constant...
term may suffer from a serious age problem. Introducing DE in the form of a constant \( \Lambda \),
with \( \Omega_{\Lambda,0} \approx 0.73 \), somehow resolves the problem, giving \( t_0 \approx 13.8 \) Gyr with \( h = 0.72 \).

When accepting the existence of DE, naturally the question arises of what it really is. Since the late 1960s when it was realized [10] that the zero-point vacuum fluctuations in quantum field theories are Lorentz-invariant, it has been attempted to associate this (quantum) vacuum energy with the present value of \( \Lambda \) but without much success. Even when placing a cutoff at some reasonable energy scale, this quantum vacuum energy is still several orders of magnitude larger than the mysterious dark energy today, \( \rho_\Lambda \sim 5 \times 10^{-47} \) GeV\(^4 \) or \( \rho_\Lambda \sim 10^{-123} \) in Planck units (for reviews, see, e.g., [11, 12]). Apparently, \( \rho_\Lambda^{1/4} \) is fifteen orders of magnitude smaller than the electroweak scale, \( m_{EW} \sim 10^{12} \) eV. No theoretical model, not even the most sophisticated such as supersymmetry or string theory, is able to explain the presence of a small positive \( \Lambda \).

Another hurdle in understanding the nature of dark energy is that only a very small window in the magnitude of the cosmological constant allows the universe to develop as it obviously has. It is still a mystery why \( \Omega_\Lambda \) has the value it has today. It could have been of the order of several magnitudes larger or smaller than the matter density today, instead of \( \Omega_\Lambda \approx 3 \Omega_m \). This is known as the cosmological coincidence problem.

At present the most common view is that dark energy is presumably constant and has a constant equation of state, \( w_{DE} = -1 \). But there remains the possibility that the cosmological constant (or the gravitational vacuum energy) is fundamentally variable. In a more realistic picture, at least from field theoretic viewpoints, dark energy should be dynamical in nature [13]. This is the case, for instance, with all time-dependent solutions arising out of evolving scalar fields, with an accelerated expansion coming from modified gravity models, holographic dark energy and the like.

Interestingly enough, the recent observations (WMAP + SDSS [3]) only demand that \(-1.04 < w_{DE} < -0.82\). In view of this wide range for the present value of the dark energy equation of state (EoS), it is certainly worth constructing an explicit model cosmology, where dark energy arises because of a dynamically evolving scalar field, and see what other consequences would arise from such a modification of Einstein’s general relativity.

2. Constructing inflationary cosmology

A complete model of the universe should perhaps feature a period of inflation in a distant past, leading to a generation of density (or scalar) perturbations via quantum fluctuations. This expectation has now received considerable observational support from measurements of anisotropies in the CMB as detected by WMAP and other experiments.

In the simplest class of inflationary models, inflation is described by a single scalar (or an inflaton) field \( \phi \), with some potential \( V(\phi) \). The corresponding action is

\[
S = \int d^4x \sqrt{-g} \left( \frac{R}{2\kappa^2} - \frac{1}{2}(\partial \phi)^2 - V(\phi) \right),
\]

(2.1)

where \( \kappa \equiv m_P^{-1} = (8\pi G_N)^{1/2} \) is the inverse Planck mass, with \( G_N \) being Newton’s constant, and \( \sqrt{-g} = \det g_{\mu\nu} \) is the determinant of the metric tensor.

Constructing concrete models of inflation and matching them to the CMB and large scale structure (LSS) experiments has become one of the major pursuits in cosmology. Most earlier studies regarding the form of an inflationary potential relied on a prior choice
of the potential \( V(\phi) \), or on slow-roll approximations in the calculation of power spectra and their relation to the mass of the field \( \phi \) during inflation (see [14] for a review). The latter approach can at best produce the tail of an inflationary potential, but not its full shape [15]. Indeed, recent studies show that the type or variety of scalar potential allowed by the array of WMAP inflationary data is still large [16]. Although, in order to understand the dynamics of inflation, the idea of utilizing one or the other form of the scalar field potential (motivated by physics beyond the standard model or even by theories of higher-dimensional gravity, such as string theory) is not bad at all, there might exist a more elegant way of confronting the WMAP inflationary data with a theoretical model.

In this paper we present a different and robust approach to tackle this problem: we do not make a specific choice for \( V(\phi) \), rather we make a simple ansatz for the scalar field \( \phi \) and then construct an inflationary potential, using the symmetry of Einstein’s field equations. Our approach would be novel in the sense that it provides a unique shape (and slope) to the scalar (or inflaton) potential. The model also makes falsifiable predictions. The basic ideas and some of the results were presented in a recent paper [17].

For simplicity, we consider a spatially flat Friedmann–Robertson–Walker spacetime. The evolution of the field \( \phi \) is then described by the equation (see, e.g., [18])

\[
\dot{\phi} = m_P (-2\dot{H})^{1/2} = -2m_P^2 \frac{d}{d\phi} H(\phi) \tag{2.2}
\]

and the evolution of the scalar potential \( V(\phi) \) is governed by

\[
\frac{V(\phi)}{m_P^2} = 3H^2(\phi) + \frac{d}{d\phi} H(\phi) = 3H^2(\phi) - 2m_P^2 \left( \frac{d}{d\phi} H(\phi) \right)^2, \tag{2.3}
\]

where \( H(\phi(t)) \equiv \dot{a}/a \) is the Hubble parameter, \( a(t) \) is the FRW scale factor and the dot denotes a derivative with respect to the cosmic time \( t \).

Let us first briefly discuss how the model that we are going to construct could satisfy inflationary constraints from the WMAP and other experiments. First, note that the term

\[
2m_P^2 \left( \frac{d}{d\phi} H(\phi) \right)^2
\]

is usually non-negligible (as compared to \( 3H^2(\phi) \)) at the onset of inflation. This would be the case, for instance, if the mass of the inflaton field, \( m_\phi \equiv (d^2V(\phi)/d\phi^2)^{1/2} \), is large enough initially, \( m_\phi \sim m_P \). Once the field \( \phi \) rolls satisfying \( |\dot{\phi}| \ll 3H|\phi| \), or equivalently, \( \Delta \phi \propto \ln[a(t)] \), the scalar potential is well approximated by an exponential term:

\[
V(\phi) \sim 3m_P^2 H^2(\phi) \sim \frac{H_0^2}{2} (6 - \phi^2) e^{2\kappa \phi'} e^{\kappa^2 \phi^2} \sim \frac{H_0^2}{2} (6 - \alpha^2) e^{2\alpha} e^{\alpha \phi}, \tag{2.4}
\]

where \( \kappa \phi' \equiv \alpha \) is the slope of the potential, during a slow-roll regime. The condition \( \kappa \dot{\phi} < \sqrt{6}H \) holds in general, so \( V(\phi) > 0 \). Inflation occurs as long as the condition

\[
\frac{\dot{a}}{a} = H^2(\phi) - 2m_P^2 \left( \frac{dH(\phi)}{d\phi} \right)^2 > 0
\]

holds, meaning that \( V(\phi) > \dot{\phi}^2 \). But, after a sufficient number of e-folds of expansion, inflation has to end. This is possible when the quantity \( (m_P/H(\phi))(dH(\phi)/d\phi) \) becomes comparable to (or even larger than) unity. Recent results from WMAP [3] indicate
that the spectral index of the scalar perturbations is consistent with an almost flat one, \( n_s = 0.951^{+0.016}_{-0.015} \). To a good approximation, \( 1 - n_s \approx \alpha^2 \), implying that \( \alpha < 0.25 \). This simple picture has obvious and intuitive appeal, which can be realized through an explicit construction.

To illustrate the construction, we make the following ansatz:

\[
\frac{\phi}{m_p} \equiv \text{const} - \alpha \ln \left( \frac{a}{a_i} \right) - \left( \frac{a}{a_i} \right)^\beta,
\]

(2.5)

where \( a_i \) is the initial value of the scale factor before inflation, and \( \alpha \) and \( \beta \) are free parameters for now. We take \( \beta < 0 \), so that after a few e-folds, since \( a \gg a_i \), the inflaton \( \phi \) naturally satisfies \( \phi'/m_p = (1/m_p)(d\phi/d\ln a) \simeq -\alpha \). One may think that the above choice for \( \phi \) is ad hoc and/or no more motivated than a particular choice of \( V(\phi) \), but it is not exactly! Indeed (2.5) is the property of an inflaton field in many well-motivated inflationary models that satisfy slow-roll conditions, after a few e-folds of inflation. It can also be compared to a generic solution for a dilaton (or modulus field), i.e. \( \phi(t) \sim \phi_0 + \alpha_0 \ln t + \alpha_1/t^\gamma \) (where \( \gamma > 0 \), in four-dimensional superstring models (see, e.g., [18,19]). Additionally, the ansatz (2.5) allows us to construct an explicit inflationary model, providing an appropriate shape (and slope) to the scalar field potential.

The evolution of \( \phi \) as given in equation (2.5) is provided by the Hubble parameter

\[
H(\phi) = H_0 \exp \left[ -\frac{\alpha^2}{2} N(\phi) - \alpha e^{-\beta N(\phi)} - \frac{\beta}{4} e^{-2\beta N(\phi)} \right],
\]

(2.6)

where \( N(\phi) \equiv \ln(a/a_i), a \equiv a(\phi(t)) \) and \( H_0 \) is an integration constant. We can easily evaluate the following two inflationary variables:

\[
\epsilon_H(\phi) = 2m_p^2 \left( \frac{1}{H} \frac{dH(\phi)}{d\phi} \right)^2 = \frac{1}{2} \left( \alpha + \beta e^{\beta N(\phi)} \right)^2,
\]

(2.7)

\[
\eta_H(\phi) = 2m_p^2 \frac{1}{H(\phi)} \left( \frac{d^2H(\phi)}{d\phi^2} \right) = \epsilon_H - \frac{\beta^2}{\beta + \alpha e^{-\beta N(\phi)}},
\]

(2.8)

(which are first order in slow-roll approximations). The magnitude of these quantities must be much smaller than unity, during inflation, in order to get a sufficient number of e-folds of expansion, like \( N_e \equiv \ln(a_f/a_i) \gtrsim 50 \). More precisely, we require \( |\epsilon_H| \ll 1, |\eta_H| < 1 \), except near to the exit from inflation where \( \epsilon_H \gtrsim 1 \). One may actually demand that \( 0 \leq \epsilon_H \leq 3 \), so that the scalar field potential

\[
V(\phi) = m_p^2 H^2(\phi) (3 - \epsilon_H)
\]

(2.9)

is non-negative. A typical shape of this potential is depicted in figure 1. The magnitude of \( H_0 \) (cf. equation (2.6)) can be fixed using the amplitude of density perturbations observed at the COBE experiments, using the normalization [20]

\[
(dV/d\phi)^{-1/2}/(\sqrt{15\pi m_p^3}) \simeq 1.92 \times 10^{-5}.
\]

Typically, with \( \alpha \sim 0.2 \) and \( N_e \equiv \ln(a_f/a_i) \sim 55 \), we find (assuming that \( \beta < 0 \))

\[
H_0 \sim 7.42 \times 10^{-5} m_p.
\]

(2.10)

This is a perfectly reasonable value, which also characterizes the average energy scale of inflation in most inflationary models.
As long as the parameter \( \epsilon_H(\phi) \) is slowly varying, the scalar curvature perturbation can be shown to be [21]

\[
P_R^{1/2}(k) = 2^{\nu - 3/2} \frac{\Gamma(\nu)}{\Gamma(3/2)} (1 - \epsilon_H)^{\nu - 1/2} \left( \frac{H^2}{2\pi|\dot{\phi}|} \right)^{1/2} a^{\nu - 1/2} \left( H^2 \right)^{-\nu} aH = k,
\]

(2.11)

where \( \nu = 3/2 + 1/(p - 1) \) and \( a \propto t^p \). The scalar spectral index \( n_s \) for \( P_R \) is defined by

\[
n_s(k) \equiv 1 + \frac{d \ln P_R}{d \ln k}. \tag{2.12}
\]

The fluctuation power spectrum is, in general, a function of wavenumber \( k \) and is evaluated when a given comoving mode crosses outside the (cosmological) horizon during inflation: \( k = aH = a_eH(\phi)e^{-\Delta N} \) is, by definition, a scale matching condition and \( a_e \) is the value of the scale factor at the end of inflation. Instead of specifying the fluctuation amplitude directly as a function of \( k \), it is convenient to specify it as a function of the number of e-folds \( N_e \) of expansion between the epoch when the horizon scale modes left the horizon and the end of inflation. To leading order in slow-roll parameters, \( n_s \) is given by [14]

\[
n_s - 1 = 2\eta_H - 4\epsilon_H = -\frac{\alpha^3 + 3\alpha^2\mu + 3\alpha\mu^2 + 2\beta\mu + \mu^3}{\alpha + \mu}, \tag{2.13}
\]

where \( \mu \equiv \beta e^{\beta N_e} \). In the conventional case that \( \beta = 0 \), which corresponds to a scenario where inflation is driven by a simple exponential potential, \( V(\phi) \propto e^{\alpha(\phi/m_P)} \), we obtain a well-known result that \( 1 - n_s \simeq \alpha^2 \). Here we shall assume that \( N_e \geq 47 \) and \( \beta < 0 \).

Let us also define the slope or running of the spectral index \( n_s \), which is given by

\[
\tilde{\alpha} \equiv \frac{dn_s}{d \ln k} = \frac{dn_s}{dN} \frac{dN}{d\phi} \frac{d\phi}{d \ln k}
\]

(2.14)

(a tilde is introduced here to avoid confusion with the exponent parameter \( \alpha \) introduced in equation (2.5)), where \( \phi \) and \( k \) are related by

\[
\frac{d\phi}{d \ln k} = -m_P \frac{\sqrt{2\epsilon_H(\phi)}}{(1 - \epsilon_H(\phi))}, \tag{2.15}
\]

while \( N \) and \( \phi \) are related by

\[
m_P \frac{dN}{d\phi} = -\frac{1}{\sqrt{2\epsilon_H(\phi)}}. \tag{2.16}
\]

These relations hold independent of our ansatz (2.5).
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Figure 2. The tensor-to-scalar ratio $r \simeq 16\epsilon_H$ versus the scalar spectral index $n_s$ with $\alpha = 0.2, 0.18$ and $0.16$ (top to bottom) and $\beta = (-0.5, 0)$. The solid (dotted) lines are for $N_e = 60$ ($N_e = 47$).

The WMAP bound on the tensor-to-scalar ratio, $r \simeq 16\epsilon_H < 0.3$ (95% confidence level), implies $\epsilon_H < 0.0187$. This bound is satisfied for

$$\alpha \lesssim 0.1936 \quad \text{and} \quad n_s \gtrsim 0.9624. \quad (2.17)$$

The spectral index obtained in this way is within the range indicated by three-year WMAP results [3]:

$$n_s = 0.958_{-0.019}^{+0.015}. \quad (2.18)$$

Of course, one may directly use the above bound for $n_s$ and find the corresponding bound on $r$. Again, by demanding that $N_e \gtrsim 50$ and $\beta \lesssim -0.2$, we find

$$\alpha = 0.2213_{-0.0360}^{+0.0360}, \quad r = 0.3918_{-0.1200}^{+0.1200}. \quad (2.19)$$

The smaller is the value of $\alpha$, the smaller will be the tensor-to-scalar ratio (see figure 2), allowing only a small running of the spectral index. For instance, if $|\alpha| \lesssim 0.1$ and $\beta \lesssim -0.2$, then we find

$$n_s \gtrsim 0.98, \quad r < 0.08. \quad (2.20)$$

The WMAP data requires a spectral index that is significantly less than the Harrison–Zel’dovich–Peebles scale-invariant spectrum ($n_s = 1, r = 0$). Thus, given that $\beta < 0$, consistency of our model (with WMAP result) seems to require $0.16 < \alpha < 0.26$.

It is also significant to note that, for $\beta < 0$, there exists a small window in the parameter space where

$$n_s \simeq 0.95, \quad r \sim \mathcal{O}(10^{-3} - 10^{-6}),$$

in which case, however, the slope parameters $\alpha$ and $\beta$ must be finely tuned. In figure 3 we show the contour plots with $N_e = 50$ and 60, representing such a case. In fact, in the case $|\alpha| < 0.05$, the gravity waves (or tensor modes) are almost nonexistent. On the right plot in figure 4 we show the running of spectral index $\tilde{\alpha}$, which is always very small in the parameter range $0.1 < \alpha < 1$ and $\beta < 0$.  

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Figure 3. Contour plots for $n_s = 0.95$ with $N_e = 50$ (left plot) and $N_e = 60$ (right plot).

Figure 4. The scalar spectral index $n_s$ (left plot) and its running $\tilde{\alpha}$ with respect to $\beta$ and $\alpha = 0.16, 0.20$ and 0.24 (top to bottom). The solid (dotted) lines are for $N_e = 60$ ($N_e = 47$). The running of $n_s$ could be large only if $|\alpha| \lesssim |\beta|$; for example, $\tilde{\alpha} \simeq -0.004$ for $\alpha \simeq 0.01$ and $\beta \simeq -0.05$.

In a model with more than one scalar field, the dependence of inflationary variables like $n_s$ and $r$ on the slope parameters $\alpha$ and $\beta$ could be more complicated than the simplest explanation provided above. Nonetheless, our approach has great significance as it generically leads to a spectrum of primordial scalar fluctuations that is slightly red-tilted ($n_s \lesssim 1$) and hence compatible with WMAP inflationary data.

3. Constructing quintessence cosmology

It is reasonable to assume that a late-time acceleration of the universe is driven by the same mechanism usually exploited to give early universe inflation, where the potential energy of a scalar field dominates its kinetic term. To this end, let us assume that the current expansion of the universe can be described by the action

$$ S = S_{\mathrm{grav}} + S_m = \int \mathrm{d}^4x \sqrt{-g} \left( \frac{R}{2\kappa^2} - \frac{1}{2} (\nabla Q)^2 - V(Q) \right) + S_m, \quad (3.1) $$
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where \( Q \) is a fundamental scalar (or dark energy) field, \( V(Q) \) is its potential, \( S_{\text{grav}} \) is the gravitational part of the action, \( S_m \) is the matter action describing the dynamics of ordinary fields (matter and radiation) and \( \nabla \) represents a four-dimensional covariant derivative. The matter part of the action (3.1) can be written as

\[
S_m = \int d^4x \mathcal{L}(\psi_m, A^2(Q)g_{\mu\nu}) \equiv \int d^4x \sqrt{-g} A^4(Q) \sum_i \rho_i, \tag{3.2}
\]

where \( \psi_m \) represents collectively the matter degrees of freedom and radiation. In the above definition of the matter Lagrangian, the implicit assumption is that matter couples to \( \tilde{g}_{\mu\nu} \equiv A(Q)^2 g_{\mu\nu} \), rather than the Einstein metric \( g_{\mu\nu} \) alone. This assumption then results in a non-minimal coupling between the scalar field \( Q \) and matter components \( (\rho_i) \). The matter–scalar coupling \( A(Q) \) may be understood as a natural modification of Einstein’s GR which can be motivated by, for instance, scalar–tensor theory. For further discussions on theoretical motivations of this coupling, see, for example, [23]–[26].

The coupling \( A(Q) \) actually generates a new term, namely

\[
-\frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_m^{(i)}}{\delta Q} \equiv \frac{dA(Q)}{dQ} T^\mu_{\mu(i)}, \tag{3.3}
\]

in the scalar wave equation for \( Q \). This expression also implies that radiation does not couple to the scalar field \( Q \) since its trace of the energy–momentum tensor equals zero. As we will show, the coupling \( dA(Q)/dQ \) introduces several qualitatively new cosmological features.

As is well known, the cosmological constant case (or, more generally, Einstein gravity with a cosmological term) arises as a special limit of the present model, for which

\[
\frac{\partial Q}{\partial t} = 0 \tag{3.4}
\]

and hence \( V(Q) = \text{const} \equiv \Lambda \) and \( A(Q) = \text{const} \). The model then reduces to the \( \Lambda \)CDM cosmology, given that dark matter is characterized by non-relativistic particles alone, \( w_m = 0 \). The cosmological term \( \Lambda \), which is governed by the equation

\[
m_p^2 G^\mu_{\mu} = 2\Lambda \delta^\mu_{\mu} + T^\mu_{\mu}, \tag{3.5}
\]

can clearly act as a source of gravitational repulsion or putative dark energy.

All the discussions so far have been made without making any particular choice of metric. Thus the nature of \( V(Q) \) acting as a repulsive force is rather general. For a more detailed treatment, it is necessary to evaluate the equations generated by variation of the total action \( S = S_{\text{grav}} + S_m \). Therefore a particular choice of a metric has to be made. We make a rather standard choice of a spatially flat FRW metric:

\[
ds^2 = -dt^2 + a(t)^2 dx^2, \tag{3.6}
\]

where \( a(t) \) is the scale factor of an FRW universe. This choice of the line element is well motivated by the observational fact that the universe is spatially flat on largest scales, which is consistent with the concept of inflation, discussed in the previous section. Of course, this choice of metric may lead to systematic errors in the calculation, as the universe actually is not homogeneous at smaller (or galactic) scales, as pointed out, for example, in [22], which is ignored in this simplified assumption.
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In the minimal coupling case, \( A(Q) \equiv 1 \), it is easy to see that
\[
\rho_i \propto [a(t)]^{-3(1+w_i)},
\]  
where \( w_i \equiv p_i/\rho_i \). In the non-minimal coupling case the modified scale factor \( \hat{a} \) is given by \( \hat{a} = a(t)A(Q) \). As a consequence, different equation of state parameters (cf. equation (3.3)) would cause different energy densities to evolve differently with changing scale factor:
\[
\rho_i \propto (a(t)A(Q))^{-3(1+w_i)},
\]  
(3.8)
This implies that \( \rho_m \propto (a(t)A(Q))^{-3} \) and \( \rho_r \propto (a(t)A(Q))^{-4} \), respectively, for ordinary matter and radiation. It also shows that radiation never directly couples to the scalar field, even with \( A(Q) \) being an arbitrary function of \( Q \). As explained in [17], the coupling \( A(Q) \) can be relevant, especially in a background where \( \rho_m \) is much larger than \( \rho_{\text{crit}} \) (where \( \rho_{\text{crit}} \equiv 3H_0^2/8\pi G \)), e.g. a galactic environment.

3.1. Basic equations

Taking a variation of the action (3.1) with respect to \( g_{\mu\nu} \) and then evaluating the \( tt \) and \( xx \) components of Einstein’s equation leads to the following two equations (cf. equation (A.1)):
\[
-\frac{3}{\kappa^2} H^2 + \frac{1}{2} \dot{Q}^2 + V(Q) + A^4(Q) \sum_i \rho_i = 0,
\]  
(3.9)
\[
\frac{1}{\kappa^2} \left( 2\dot{H} + 3H^2 \right) + \frac{1}{2} \dot{Q}^2 - V(Q) + A^4(Q) \sum_i (w_i \rho_i) = 0.
\]  
(3.10)
A variation with respect to the scalar field \( Q \), while considering an explicit matter–scalar coupling, yields the following equation of motion for \( Q \) (cf. equation (A.2)):
\[
\ddot{Q} + 3H \dot{Q} + \frac{dV(Q)}{dQ} - A^3 \frac{dA(Q)}{dQ} \sum_i (1 - 3w_i) \rho_i = 0,
\]  
(3.11)
the so-called Klein–Gordon equation for \( Q \). It shows that the scalar field \( Q \) couples to the trace of the energy–momentum tensor \( g^{\mu\nu}T_{\mu\nu} \) satisfying
\[
-\nabla^2 Q = \ddot{Q} + 3H \dot{Q} = -\frac{dV(Q)}{dQ} - A^3 \frac{dA(Q)}{dQ} T_{\mu}^\mu.
\]  
(3.12)
There is dissension about the sign of the coupling term between the scalar field and matter in the above equation in the way that it might be \( +A^3(dA(Q)/dQ)T_{\mu}^\mu \) instead of \( -A^3(dA(Q)/dQ)T_{\mu}^\mu \). In this paper, the negative sign, as written in equation (3.11), will be used.

The above set of equations can be supplemented by a fourth equation, arising from the equation of motion for a perfect barotropic fluid
\[
(aA) \frac{d\rho_i}{d(aA)} = -\rho_i 3 (1 + w_i).
\]  
(3.13)
This finally leads to (cf. equation (A.7), see also [27])

\[
\dot{\rho}_i + 3H (1 + w_i) \rho_i = \frac{\dot{Q} dA(Q)}{A dQ} (1 - 3w_i) \rho_i.
\] (3.14)

Out of the four equations (3.9)–(3.11) and (3.14), only three are independent, meaning the conservation equation of the perfect fluid (3.14) can be derived without the assumption (3.13) but only by combining (3.9)–(3.11). General covariance requires the conservation of the total energy density, \(\rho_{\text{tot}} = \rho_Q + \sum \rho_i\), which is obviously the case in our model (see also the appendix in [28]).

Next we make the following substitutions:

\[
\epsilon \equiv \frac{\dot{H}}{H^2} = \frac{H'}{H}, \quad \Omega_i \equiv \frac{\kappa^2 A_i^2 \dot{\rho}_i}{3H^2}, \quad \Omega_Q \equiv \frac{\kappa^2 \left(\frac{1}{2} \dot{Q}^2 + V(Q)\right)}{3H^2},
\] (3.15)

and

\[
w_Q \equiv \frac{(1/2) \dot{Q}^2 - V(Q)}{(1/2) \dot{Q}^2 + V(Q)} = \frac{p_Q}{\rho_Q}.
\] (3.16)

These substitutions and further simplifications lead to the set of four equations:

\[
\sum_i \Omega_i + \Omega_Q = 1,
\] (3.17)

\[
2\epsilon + 3 (1 + w_Q) \Omega_Q + 3 \sum_i (1 + w_i) \Omega_i = 0,
\] (3.18)

\[
\Omega_Q' + 2\epsilon \Omega_Q + 3\Omega_Q (1 + w_Q) + Q' \alpha_Q \sum_i (\eta_i \Omega_i) = 0,
\] (3.19)

\[
\sum_i \Omega_i' + 2\epsilon \sum_i \Omega_i + 3 \sum_i (\Omega_i (1 + w_i)) - Q' \alpha_Q \sum_i (\eta_i \Omega_i) = 0,
\] (3.20)

where, as above, the primes denote a derivative with respect to e-folding time \(N \equiv \ln[a(t)] + \text{const}\), \(Q' \equiv \dot{Q}/H\), \(\eta_i \equiv 1 - 3w_i\) and \(\alpha_Q \equiv d\ln[A(Q)]/(d\kappa Q)\). In the above we have used the relation

\[
\frac{\partial}{\partial N} = \frac{1}{H} \frac{\partial}{\partial t}.
\] (3.21)

Equations (3.17)–(3.20) represent the most general case of an evolving universe based on the general action (3.1). Equation (3.17) is simply the Friedmann constraint for the assumed flat universe. Changing the sign of the coupling \(\alpha_Q\) to the trace of the energy–momentum tensor \(T_{\mu}^{\mu}\) in equation (3.12) would cause a change of sign from \(+Q' \alpha_Q \sum_i (\eta_i \Omega_i)\) to \(-Q' \alpha_Q \sum_i (\eta_i \Omega_i)\) in equation (3.19). Adding equations (3.19) and (3.20), we find

\[
\Omega_Q' + 2\epsilon \Omega_Q + 3\Omega_Q (1 + w_Q) + \sum_i \Omega_i' + 2\epsilon \sum_i \Omega_i + 3 \sum_i (\Omega_i (1 + w_i)) = 0,
\] (3.22)

which can be interpreted as a global energy conservation equation. Thus, for not violating this principle of energy conservation (3.22), a sign change in (3.19) automatically implies a change in (3.20) as well.
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When having a particular solution of equations (3.17)–(3.20), it is of great interest to study how the corresponding potential looks like and how it affects the cosmic evolution of our universe. From the last expression in equation (3.15), we find

\[ V(Q) \equiv H^2 \left( \frac{3\Omega_Q}{\kappa^2} - \frac{1}{2} (Q')^2 \right), \tag{3.23} \]

which will be used later. Of course, in the case of a minimal coupling \((A(Q) \equiv 1)\), \(\alpha_Q\) vanishes, reducing the number of degrees of freedom in the system of equations (3.17)–(3.20) by one, which then makes the system easier to handle. Anyhow, in both cases \((\alpha_Q = 0 \text{ and } \alpha_Q \neq 0)\) it is not possible to find an analytical solution of this system without making some additional assumptions as there are more degrees of freedom than independent equations. In fact, the number of degrees of freedom depends on the number of matter components included in the analysis.

As the first check for compatibility of the model, it is useful to consider some simplified solution of equations (3.14)–(3.20) by expressing all matter fields as one component, \(w_i \equiv w_m\). By applying equation (3.21) to equation (3.14), and after a simple integration, we get

\[ \rho_m = \rho_m^{(0)} e^{-3\ln a} \exp \left[ \int (Q'\alpha_Q (3w_m - 1) + 3w_m) \, d\ln a \right], \tag{3.24} \]

with \(\rho_m^{(0)}\) being an arbitrary constant. The coupling \(\alpha_Q\) may be constrained by observations perhaps only in the combination \(Q'\alpha_Q\). One can study the effect of this coupling on both CMB temperature anisotropies and evolution of linear matter perturbations, as in [29]. In the minimal coupling case, one has

\[ \rho_m \propto a^{-3(1+w_m)}. \tag{3.25} \]

This is exactly the behaviour one would expect from general relativity. Equation (3.25) yields \(\rho_m \propto a^{-3}\) in a universe containing only ordinary matter (or dust), while for radiation \(\rho_r \propto a^{-4}\). Transposing equation (3.18) leads to a general expression for the equation of state of the DE component which can generally be written as

\[ w_Q = -2\epsilon + 3 \sum_i (1 + w_i) \Omega_i + 3\Omega_Q, \tag{3.26} \]

where all possible forms of matter are included, e.g. pressureless dust \((w_m = 0)\), radiation \((w_r = 1/3)\), stiff matter \((w_{sm} = 1)\), domain walls \((w_{dw} = -2/3)\), etc.

For further analysis, it is useful to introduce the so-called effective equation of state parameter \(w_{\text{eff}}\), which is defined by

\[ w_{\text{eff}} \equiv \frac{p_{\text{tot}}}{\rho_{\text{tot}}}, \quad p_{\text{tot}} \equiv p_Q + \tilde{p}_i, \quad \rho_{\text{tot}} \equiv \rho_Q + \tilde{\rho}_i, \tag{3.27} \]

whereas \(p_Q\) and \(\rho_Q\) are as defined in equation (3.16), and \(\tilde{p}_i \equiv A^4(Q)p_i\), \(\tilde{\rho}_i \equiv A^4(Q)\rho_i\). The meaning of \(w_{\text{eff}}\) is somehow that of a mean equation of state of all matter–energy components, including the dark energy component \(\rho_Q\). From equation (3.18), together with equations (3.15) and (3.16), we find that the total pressure is given by

\[ p_{\text{tot}} = \frac{3H^2}{\kappa^2} \left( -\frac{2\epsilon}{3} - \Omega_Q - \sum_i \Omega_i \right). \tag{3.28} \]
Combining this expression for $p_{\text{tot}}$ with the expression of total energy density $\rho_{\text{tot}}$, as defined by (3.27), and again using the substitutions (3.15) and (3.16), we get

$$w_{\text{eff}} = -1 - \frac{2\epsilon}{3}.$$  \hspace{1cm} (3.29)

This expression is valid even in the most general non-minimal coupling case. Similarly, we can define the deceleration parameter $q$ as

$$q \equiv -\frac{\ddot{a}}{aH^2} = -1 - \epsilon,$$  \hspace{1cm} (3.30)

showing that the name ‘deceleration parameter’ makes sense in such a way that $q > 0$ for a decelerated expansion ($\ddot{a} < 0$), while $q < 0$ for an accelerating expansion ($\ddot{a} > 0$).

As explained in the introduction, one reason for considering a universe containing a non-zero DE component, either in the form of a cosmological constant $\Lambda$ or a dynamically evolving scalar field $Q$, is the recently observed accelerated expansion of the universe by the Supernova Cosmology Project and the High-redshift Supernova Search team [1]. Thus we find it useful to study the solution of equations (3.17)–(3.20), which yields an accelerated expansion in a general context of non-trivial matter–scalar coupling.

Indeed, independent of any assumption or specific composition of the universe, simply the condition $w_{\text{eff}} < -1/3$ at some stage of cosmic evolution yields an accelerated solution. It should be noted at this stage that no such general connection can be established between the DE equation of state parameter $w_{\text{Q}}$ having a specific value (even like $w_{\text{Q}} = -1$) and the universe being in an accelerating phase.

In the notations used in this paper, both the non-baryonic (cold) dark matter and ordinary matter (pressureless dust) are combined in one matter constituent $\Omega_m$. As $w_{\text{DM}} \approx 0$ is a rather good approximation for the equation of state of cold dark matter (since it is non-relativistic) this combination seems to be reasonable. This assumption as regards the composition of the universe today implies that its only constituents are cold dark matter, ordinary matter, radiation and DE. Putting this composition ($\Omega_m \approx 0.27$, $\Omega_Q \approx 0.73$ and $\Omega_r \approx 10^{-4}$) of today’s universe into the very general expression of the DE equation of state parameter $w_{\text{Q}}$ (cf. equation (3.26)) and using again $w_m = 0$ and $w_Q \approx -1$, the value $\epsilon = -0.4$ is obtained. This implies that, for $w_{\text{Q}}$ at least being close to $-1$, the universe is in an accelerating phase today, which is what is observed.

The general considerations so far seem to be consistent with observations. As observations seem to indicate a value for $w_{\text{Q}}$ close to $-1$, the possibility of a dark energy component simply being a cosmological constant cannot be ruled out. But it is also important to realize that the effects of a slowly rolling scalar field would be almost indistinguishable from that of a pure cosmological constant if $\kappa Q' \equiv m_F^{-1}(\dot{Q}/H) \lesssim 0.1$ at present. Evidence for $w_{\text{Q}} \sim -1$ could actually imply that the field $Q$ is rolling only with a tiny velocity at present. This point should be more clear from the discussion below.

All the examinations so far have been in a rather general way without imposing any additional assumptions. Certainly, that is not really satisfying, as one might be interested in an analytic solution of the system of equations (3.17)–(3.20). As mentioned above this is not possible without further input because of the number of degrees of freedom exceeding the number of independent equations. In the next two subsections two different analytic solutions will be presented, making some simple additional assumptions. According to the present constitution of the universe being $\Omega_m \approx 0.27$ and $\Omega_Q \approx 0.73$, it is reasonable
to neglect the radiation component at least for redshift \( z \lesssim \mathcal{O}(10) \). Therefore the model universe assumed in the next two sections is thought to only consist of cold dark matter and ordinary matter combined in one component with a common equation of state \( w_m = 0 \) and a DE component represented by the scalar field \( Q \) with a variable EoS \( w_Q \).

The system of equations (3.17)–(3.20) can then be expressed in the form

\[
\Omega_m + \Omega_Q = 1, \tag{3.31}
\]
\[
2\epsilon + 3(1 + w_Q)\Omega_Q + 3\Omega_m = 0, \tag{3.32}
\]
\[
\Omega_Q' + 2\epsilon\Omega_Q + 3\Omega_Q (1 + w_Q) + Q'\alpha Q\Omega_m = 0, \tag{3.33}
\]
\[
\Omega_m' + 2\epsilon\Omega_m + 3\Omega_m - Q'\alpha Q\Omega_m = 0. \tag{3.34}
\]

The number of free parameters in this system is five (\( \Omega_m, \Omega_Q, w_Q, \epsilon \) and \( \alpha_Q \)), meaning two additional assumptions have to be made to find an analytic solution. To proceed further, we make the following assumption:

\[
Q = Q_0 + m_P \alpha \ln[a(t)] \equiv Q_0 + m_P \alpha (N + \text{const}), \tag{3.35}
\]

where \( \alpha \) is a constant which needs to be fixed by observations. This relation actually represents a generic situation that the field \( Q \) is rolling with a constant velocity, \( Q' = \text{const}^3 \). In the minimal coupling case this is enough, while in the non-minimal case (\( \alpha_Q \neq 0 \)) one more assumption is required, which will be discussed below.

Simply transposing (3.16) and utilizing the relation between \( \partial / \partial t \) and \( \partial / \partial N \), as given by (3.21), yields the following useful relation:

\[
w_Q = \frac{\kappa^2 Q^2 - 3\Omega_Q}{3\Omega_Q} \equiv \frac{\alpha^2 - 3\Omega_Q}{3\Omega_Q}. \tag{3.36}
\]

Supplementing equations (3.31)–(3.34) with this equation is an elegant way of imposing an additional constraint into the model.

### 3.2. Uncoupled quintessence

In the \( A(Q) = 1 \) case, the system of equations (3.31)–(3.34), supplemented by equation (3.35), can now be solved analytically. The explicit solution is given by

\[
\Omega_Q = 1 - \frac{\lambda}{c_1 \lambda \exp[\lambda N] + 3^{\lambda}}, \tag{3.37}
\]
\[
w_Q = \frac{-c_1 \lambda^2}{3\alpha^2 \exp[-\lambda N] + 3\lambda c_1}, \tag{3.38}
\]
\[
\epsilon = \frac{-c_1 \alpha^2 \lambda - 9 \exp[-\lambda N]}{2c_1 \lambda + 6 \exp[-\lambda N]}, \tag{3.39}
\]

\(^3\text{Note that we are demanding } Q' \equiv dQ/d\ln a = \dot{Q}/H \equiv \text{const, not } \dot{Q} = \text{const. For } \alpha < 0.6, \text{ our approach appears to give consistent results when applied to observational data; see also the review [30] for extensive discussions on various methods of reconstructing dark energy potentials. See [31] for a very different approach of dark energy reconstruction.}\)
where $N \equiv N(Q) = \ln[a(Q(t))]$ and we have made the substitution
\begin{equation}
\lambda \equiv 3 - \alpha^2.
\end{equation}

Using equations (3.29) and (3.30) we also evaluate
\begin{equation}
q = \frac{3 \exp[-\lambda N] - c_1(6 - 5\alpha^2 + \alpha^4)}{6 \exp[-\lambda N] + 2c_1\lambda},
\end{equation}
\begin{equation}
w_{\text{eff}} = \frac{-c_1\lambda^2}{9 \exp[-\lambda N] + 3c_1\lambda}.
\end{equation}

This general solution contains three free parameters ($N$, $\alpha$ and $c_1$). To keep the solution as general as possible it is useful to just fix one free parameter in terms of the other two. The integration constant $c_1$ can be fixed in terms of the field velocity $\alpha$ by using the observational input $\Omega_{m0} = 0.27$ at present. The e-folding time $N$ in relation to the cosmic time $t$ is only defined up to an arbitrary constant, so it needs to be normalized in some way. For simplicity, this will be done by taking $N = 0$ at present. Thus, the condition $\Omega_m[N = 0, \alpha, c_1] \equiv \Omega_{m0}$ yields
\begin{equation}
c_1 = \frac{3 - \alpha^2 - 3\Omega_{m0}}{(3 - \alpha^2)\Omega_{m0}},
\end{equation}
which now makes it possible to express equations (3.37)–(3.39) just in terms of the two free parameters $N$ and $\alpha$. For further analysis it is useful to parametrize the solution in terms of redshift $z$. By utilizing the dependence of the redshift on the scale factor $a(t)$, it is easy to obtain the relation between $N$ and $z$:
\begin{equation}
z + 1 = \frac{\lambda_{\text{obs}}}{\lambda_{\text{em}}} = \frac{a_0}{a(t)}, \quad \exp[N] = \frac{a(t)}{a_0} = \frac{1}{1 + z}.
\end{equation}

Here $t$ is the time when light was emitted, that is, observed now. Thus choosing $a_0$ to be the present scale factor automatically implies the normalization $N = 0$ at $z = 0$. From equations (3.37)–(3.39), one can clearly see that the solution is symmetric in $\alpha$ as only even powers of $\alpha$ occur. Thus, without loss of generality, in the further analysis only positive $\alpha$ will be considered. For the value of $c_1$ satisfying (3.43), and $\Omega_{m0} \simeq 0.27$, it is easy to see that $c_1 = 0$ for $\alpha^2 = 2.19$, implying that $\alpha_{\text{crit}} = 1.48$. This value of $\alpha$ has some significance, when looking at the evolution of $w_Q$ for different values of $\alpha$.

From the left plot in figure 5 it is easily seen that $\alpha = 1.48$ yields $w_Q \equiv 0$ whereas $w_Q < 0$ for $\alpha < 1.48$. Here $\alpha = 1.48$ separates solutions with the energy described by means of $Q$ being attractive or repulsive. Thus, if $\Omega_m = 0.27$ at $z = 0$, then a solution with self-repulsive DE requires $0 < \alpha < 1.48$, whereas $\alpha = 0$ equals the cosmological constant case with $w_Q \equiv -1$, which can also be seen in figure 5. WMAP data combined with the Supernova Legacy Survey (SNLS) data yields a significant constraint on the equation of state of the dark energy, $-1.04 < w_Q < -0.82$ (with 95% CL), see also [32, 33]. However, here we consider only the region $-1 < w_Q < -0.82$ so that $\dot{Q}^2 > 0$, that is, without going to a phantom regime. This would require $\alpha$ to be in the interval $0 < \alpha < 0.63$ which can also be inferred from figure 5. What can also be seen in figure 5 is that for all $\alpha$ in the range $0 < \alpha < 1.48$, $w_Q \to 0$ for $z \to \infty$, thus implying the DE component is indistinguishable from pressureless dust for high redshifts and only becomes the observed
Figure 5. $w_Q$ with respect to redshift $z$ (left plot) for $\alpha = 0, 0.5, 1.0, 1.48$ and $1.8$ and (right plot) $\alpha = 0, 0.15, 0.3, 0.45$ and $0.6$ (bottom to top).

Figure 6. The slow-roll parameter $\epsilon \equiv \dot{H}/H^2$ and deceleration parameter $q$ with respect to $z$, and $\alpha = 0, 0.7, 1.0, 1.48$ and $1.6$ (from top to bottom, left plot) or (bottom to top, right plot).

self-repulsive form of energy in recent time. For a further understanding of this solution it is useful to look at the deceleration parameter $q$.

In figure 6, it can be easily seen that, for all $0 \leq \alpha < 1.48$, $q$ gets negative somewhere between redshifts $z = 0$ and $1$, which implies that in this model accelerated expansion is a rather late-time phenomenon with the universe getting into an accelerated phase the earliest for $\alpha \equiv 0$, corresponding to the cosmological constant case. In the case of $\alpha = 1.48$, $q$ exactly equals $0.5$, corresponding to a decelerated expansion at constant deceleration. Finally, for $\alpha > \alpha_{\text{crit}} = 1.48$, $q$ is greater than $0.5$ and increases with decreasing redshift, yielding a decelerated expansion. This fits the evolution of the dark energy EoS $w_Q$, as seen in figure 5 (for $\alpha > \alpha_{\text{crit}}, w_Q > 0$).

From figures 7 and 8 we can see that for the solution which leads to a late-time acceleration ($w_{\text{eff}} < -1/3$) the universe is clearly dominated at high redshift by $\Omega_m$ with a transition to $\Omega_Q$ dominance in recent time leading to $\Omega_m = 0.27$ and $\Omega_Q = 0.73$ at $z = 0$. (That certainly does not come as a surprise, since that was the assumption made when fixing $c_1$). It is perhaps more interesting to note that for $\alpha = 1.48$ the ratio $\Omega_Q/\Omega_m$ remains constant for all $z$, whereas, for $\alpha < 1.48$, the early universe would be dominated by $\Omega_m$ with a shift to dark energy dominance in the recent epoch. The observed acceleration and DE dominance correspond best to values of $\alpha$ closer to zero.

Uncertainties in the current value of $\Omega_m$ affect $\alpha_{\text{crit}}$, to some extent, and hence the predicted value of $w_Q$ at some fixed redshift. That is, for a value of $\Omega_m$ different from $0.27$ at present, the critical value of $\alpha$, i.e. $\alpha_{\text{crit}} = \sqrt{2.19}$, can also be different. However, the general behaviour of the solution would be similar.
Figure 7. The effective EoS $w_{\text{eff}}$ with respect to redshift $z$: (left plot) $\alpha = 0, 0.7, 1.0, 1.48$ and $1.6$ (bottom to top) and (right plot) $\alpha = 0, 0.4, 0.6, 0.8$ and $1.0$ (bottom to top).

Figure 8. $\Omega_m$ and $\Omega_Q$ with respect to $\alpha$ and $z$. These quantities may not change with $z$ only if $\alpha = \alpha_{\text{crit}} = 1.48$, in which case obviously there will not be a cosmic acceleration.

Figure 9. $\Omega_Q$ with respect to $w_Q$ and $q$, for a varying $z = \{5, 0\}$ and $\alpha = \{1, 0\}$ (left plot) and $\alpha = 0, 0.15, 0.3, 0.45$ and $0.6$ (right plot, bottom to top). $w_Q$ decreases to $-1$ at a low redshift.

The left plot in figure 9 is a three-dimensional illustration of the above discussed fact, that a transition to the accelerated phase ($q < 0$) occurs for $w_Q$ tending to $-1$ and $\Omega_Q$ tending to $+1$. The right plot in figure 9 is a two-dimensional projection of the latter and thus just gives another illustration of the already discussed relation between $w_Q$ and $\Omega_Q$ for the accelerating case, where only accelerating solutions with $\alpha < 0.6$, which actually lead to $w_Q < -0.83$ at $z = 0$, are examined.

The discussion so far has been based on the idea of a dark energy as described by the scalar field $Q$ with some potential $V(Q)$. In obtaining the analytical solution (3.37)–
(3.39), no particular choice was made for the potential. The only assumption made was that the field might be rolling with a constant velocity $\alpha$, with respect to the e-folding time $N = \ln a$. Thus it would be worth looking at the shape of the potential as determined by this particular solution, following the idea of reconstruction underlying the focus of this paper. In obtaining the analytic expression of $V(Q)$, it is useful to consider the set of substitutions made in (3.15) and (3.16). By utilizing the additional constraint (3.36), it is easy to see that

$$Y \equiv \frac{\kappa^2 V(Q)}{H^2(Q)} = 3\Omega_Q - \frac{\alpha^2}{2}. \quad (3.45)$$

$Y$ is actually a dimensionless variable, which takes the value $Y = 3$ in a pure de Sitter space. The variation of $Y$ shown in figure 10 seems quite natural and can be understood in the following way. In order to get an accelerated expansion of the universe, with $w_Q$ close to $-1$ at a low redshift, $Y/3$ should exceed $\Omega_m$ in the recent past.

In order to find the potential, it is necessary first to evaluate the Hubble parameter $H$, which can be easily done by solving the equation

$$\epsilon[N]H[N] = H'[N]. \quad (3.46)$$

The analytic expression of $H$ is given by

$$H = c_2 \exp \left[ \frac{-N\alpha^2}{2} \right] \sqrt{3 \exp[-\lambda N]} + c_1 \lambda. \quad (3.47)$$

The numerical constant $c_2$ can be fixed by the assumption that $H[N = 0] = H_0$. Hence

$$c_2 = \frac{H_0}{\sqrt{3 + c_1 \lambda}}. \quad (3.48)$$

Finally, the quintessence potential takes the form

$$\kappa^2 V(Q(N)) = \frac{1}{2} c_2^2 \exp \left[ -\alpha^2 N \right] \left( 3\alpha^2 \exp[-\lambda N] + c_1 \left( 18 - 9\alpha^2 + \alpha^4 \right) \right). \quad (3.49)$$

In figure 11 it is clearly seen that $V(Q)$ increases exponentially with increasing redshift $z$, whereas this increment is steeper for larger values of $\alpha$. As it should be, in the $\alpha = 0$ case, the potential takes a constant value. In fact, the assumption of $Q$ rolling with
Figure 11. (Left plot) $\kappa^2 V / H_0^2$ with respect to $z$ and $\alpha$. (Right plot) $\kappa^2 V / H_0^2$ with respect to $z$ for $\alpha = 0.5, 1.0, 1.5$ and $2.0$ (from top to bottom, along the $y$ axis).

Figure 12. (Left plot) $\kappa^2 V(Q) / H_0^2$ with respect to $\kappa Q$ and $\alpha$, for $c_3 = 0$. (Right plot) $\kappa^2 V(Q) / H_0^2$ with respect to $\kappa Q$ and $c_3$, for $\alpha = 0.6$.

A constant velocity ($Q' \equiv \alpha$) yields that the potential $V(Q(z))$ must take a shape to cause this behaviour for $Q$. An exponential shape for the potential is no surprise. The quintessence potential constructed in this way takes the following form:

$$\kappa^2 V = \frac{c_3^2}{2} e^{-\alpha (c_3 + \kappa Q)} \left( 3\alpha^2 \exp \left( \frac{(\alpha^2 - 3) (c_3 + \kappa Q)}{\alpha} \right) + c_1 \left( 18 - 9\alpha^2 + \alpha^4 \right) \right).$$

(3.50)

The integration constant $c_3$ can be set to zero, without loss of generality, while $c_1$ and $c_2$ can be fixed in terms of $\alpha$ (and $H_0$), using equations (3.43) and (3.48). The potential can be brought into a form where it only depends on $Q$ and $\alpha$:

$$V(Q) = m_P^2 \exp \left( -\frac{\alpha Q}{m_P} \right) \left( V_0 \exp \left[ \frac{(\alpha^2 - 3) Q}{\alpha m_P} \right] + V_1 \right).$$

(3.51)

This potential is clearly double exponential in form and it would find interesting applications even for the early universe. As discussed in [34], one may be required to have $\alpha < 0.8$ in order to satisfy the bound on $\Omega_Q$ during big bang nucleosynthesis, namely $\Omega_Q(1$ MeV) < 0.1. It is also interesting to note that such a potential can easily arise from some fundamental theories of gravity in higher dimensions (see, e.g., [35]).
3.3. Coupled quintessence

As already mentioned above, when solving the system of equations (3.31)–(3.34) with the additional constraint (3.36) in the general case ($\alpha Q \neq 0$), one more constraint is needed to get an analytic solution. It is most canonical to assume $\kappa \alpha Q \equiv \text{const} \equiv \chi$, which represents the case of so-called exponential coupling between the scalar field $Q$ and matter, as $A(Q) \propto e^{\chi/2}$. This additional assumption then leads to a general analytic solution

$$\Omega_Q = 1 - \frac{\zeta}{3 + c_4 (3 + \alpha \chi - \alpha^2)} e^{\zeta N},$$  
(3.52)

$$w_Q = \frac{c_4 \zeta \exp[\zeta N] (\alpha^2 - 3) + 3\chi \alpha}{3c_4 \zeta \exp[\zeta N] + 3\alpha^2 - 3\chi \alpha},$$  
(3.53)

$$\epsilon = \frac{-\alpha^2}{2} + \frac{-3\zeta}{2c_4 \zeta \exp[\zeta N] + 6},$$  
(3.54)

where

$$\zeta \equiv 3 + \chi \alpha - \alpha^2.$$  
(3.55)

Further, the analytic expressions for $q$ and $w_{\text{eff}}$ are given by

$$q = -1 + \frac{\alpha^2}{2} + \frac{3\zeta}{2c_4 \zeta \exp[\zeta N] + 6};$$  
(3.56)

$$w_{\text{eff}} = -1 + \frac{\alpha^2}{3} + \frac{2\zeta}{2c_4 \zeta \exp[\zeta N] + 6}.$$  
(3.57)

By solving the differential equation (3.46), the Hubble parameter is found to be

$$H(Q) = c_5 \exp \left[ -\frac{N (3 + \alpha \chi)}{2} \right] \sqrt{c_4 \zeta \exp[\zeta N] + 3},$$  
(3.58)

where $N \equiv N(Q)$. The integration constant $c_5$ can be fixed by the assumption that $H[N = 0] \equiv H_0$. This yields

$$c_5 = \frac{H_0}{\sqrt{3 + c_4 \zeta}}.$$  
(3.59)

One normalizes $N$ such that $N = 0$ corresponds to $a \equiv a_0 = 1$. Further, insisting that $\Omega_m(N = 0, \alpha, c_4, \chi) \equiv \Omega_m^0$ at $z = 0$ fixes the integration $c_4$ in terms of $\alpha$ and $\chi$:

$$c_4 = \frac{3(1 - \Omega_m^0) + \alpha \chi - \alpha^2}{\Omega_m^0 (3 + \alpha \chi - \alpha^2)}.$$  
(3.60)

Compared to the minimal coupling case ($\chi = 0$), now the symmetry in the solution between positive and negative $\alpha$ is lost. However, a simultaneous change in sign of the parameters $\alpha$ and $\chi$ keeps $c_4$ unchanged. Thus in further analysis only the properties of a solution with positive $\alpha$ but either sign of $\chi$ will be examined. In the discussion that follows the case $\chi < 0$ will characterize solutions with $\alpha$ and $\chi$ having the opposite sign, while the case $\chi > 0$ will characterize solutions with $\alpha$ and $\chi$ having the same sign.
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Figure 13. The dark energy equation of state $w_Q$ with respect to $z$ and $\alpha$ for $\chi = -0.4$ (left plot) and $\chi = +0.4$ (right plot). The $\chi > 0$ solution yields a more negative $w_Q$ at a given redshift.

Figure 14. $\Omega_Q$ with respect to $z$ and $\alpha$ for $\chi = -0.6$ (left plot) and $\chi = +0.6$ (right plot).

As the parameters $q$, $w_{\text{eff}}$ and $\epsilon$ are all intimately connected by equations (3.29) and (3.30), only the $\chi$ dependence of dark energy EoS $w_Q$ will be examined as an exemplary.

As can be seen from figure 13, the $\chi > 0$ solution decreases $w_Q$, whereas the $\chi < 0$ solution increases $w_Q$ (for fixed $\alpha$ and $z$). That is, a negative $\chi$ causes the universe to get into an accelerating phase later than for positive $\chi$. In general, the decrease in $w_Q$ would be steeper for $\chi < 0$ than for $\chi \geq 0$. It is also important to realize that the coupling $\chi$ does not affect the value of $w_Q$ at $z \simeq 0$ but only at higher redshifts.

In figure 14 we show the variation of dark energy density with the field velocity $\alpha$ and the redshift $z$. It is found that, for fixed $\alpha (< \alpha_{\text{crit}})$, $\Omega_Q$ can be smaller (larger) at higher redshifts for $\chi > 0$ ($\chi < 0$). This behaviour would be somewhat opposite in an decelerating universe with $\alpha > \alpha_{\text{crit}}$. This behaviour is expected by the $\chi$ dependence of $q$, since an increase in matter density also increases $q$ and vice versa. For a better understanding of this situation, it is useful to study the behaviour of the potential $V(Q)$.

It is also worth examining the values of dark energy EoS $w_Q$ with a varying $\chi$. In the case $\chi < 0$, an increasing negative $\chi$ decreases $w_Q$, whereas an increasing positive $\chi$ will increase $w_Q$ with respect to the value it has in the minimal coupling case, $\chi = 0$; one may compare figure 15 with figure 5.

In analogy to the previous section $\kappa^2 V(Q)/(H^2(Q))$ can be obtained by using equation (3.45). In the $\chi \neq 0$ case, the effective potential consists of $V(Q)$ and an additional term depending on the matter–quintessence coupling $\alpha_Q$. The functional form of $V_{\text{eff}}(Q)$ can be obtained by integrating the right-hand side of equation (3.12) with
Figure 15. The dark energy EoS $w_Q$ with respect to $z$. Left plot: $\chi = +0.1$ and $\alpha = 0, 0.2, 0.4, 1.0$ and $1.4$ (bottom to top). Right plot: $\alpha = 0.4$ and $\chi = -0.2, -0.1, 0, 0.1$ and $0.2$ (top to bottom).

Figure 16. The effective potential $V_{\text{eff}}(Q)$ with respect to redshift $z$ and the slope parameter $\alpha$, in units $H_0 = 1 = \kappa$, for $\chi = -0.5$ (left plot) and $\chi = +0.5$ (right plot). We have taken $c_6 = 0$.

respect to $Q$. The result is given by

$$\kappa^2 V_{\text{eff}}(Q) = 0.5c_5^2 \exp\left[-\frac{(3 + \alpha\chi) (c_6 + \kappa Q)}{\alpha}\right]$$

$$\times \left(c_4\zeta \exp\left[\frac{\zeta (c_6 + \kappa Q)}{\alpha}\right]\right) \left(6 - \alpha^2\right) - 3 \left(6 + 4\alpha\chi - 3\alpha^2\right),$$

(3.61)

where $c_6$ is an integration constant. One can fix $c_4$ and $c_5$ using equations (3.60) and (3.59), and also equation (3.44). We exhibit the shape of this potential in figure 16.

As can be easily seen in figure 16, for a small $\alpha$ ($< \alpha_{\text{crit}}$), $V_{\text{eff}}(Q)$ increases with an increasing $z$, which allows the field to ‘roll down’ with a constant velocity $\alpha$, with the slope being zero for $\alpha = 0$, as in the $\chi = 0$ case. For large $\alpha$ (like $\alpha \gtrsim \sqrt{2}$), instead, $V_{\text{eff}}(Q)$ decreases with increasing $z$. This should not come as a surprise; this behaviour has its origin in the value of $\alpha_{\text{crit}}$ which is lowered for $\chi < 0$. For a given $\alpha$, the slope of the potential is shallower for $\chi > 0$ than for $\chi < 0$, with vanishing difference at lower redshifts.

We conclude this section with the following two remarks. Firstly, in our model, it is possible that the current acceleration of the universe is only transient. This can easily happen, for $\alpha_Q < 0$, when $\sum_i \int \alpha_Q (1 - 3w_i) \tilde{\rho}_i \text{d}Q$ (where $\tilde{\rho}_i \propto a^{-3 - 3w_i}$) becomes comparable to (or exceeds) $\kappa^2 V(Q)$, making the effective potential almost vanishing (or negative).
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Secondly, in the case both the ordinary and dark matter have the same coupling with the quintessence field \( Q \), current observational constraints (from Cassini experiments and the like) only demand that \( \alpha_Q^2 < 10^{-4} \), while this bound is significantly relaxed if dark matter can have much stronger coupling with \( Q \). It should be the astrophysical observations that decide whether \( \alpha_Q < 0 \) or \( \alpha_Q > 0 \). The answer to this question can have interesting cosmological effects which we aim to study in future work.

4. Confronting models with data

In this section we confront our models with recent cosmological datasets (Supernova Legacy Survey (SNLS) and SNIa Gold06 datasets) following the methods discussed, for example, in [36, 37].

In the minimal coupling case, since \( \dot{\rho}_Q + 3H(1 + w_Q)\rho_Q = 0 \) (i.e. \( \rho_Q \) and \( \rho_m \) are separately conserved), we get

\[
\rho_Q = \rho_{Q0} \exp \left[ 3 \int_0^z \frac{(1 + w(z_1))}{1 + z_1} \, dz_1 \right].
\]

Without any prior on \( w(z_1) \) or \( \rho_Q \), it can be shown that [38]

\[
H(z) = H_0 \left( \Omega_{m0}(1 + z)^3 + \Omega_{Q0} \exp \left[ 3 \int_0^{\ln(1+z)} (1 + w(z_1)) \, d \ln(1 + z_1) \right] \right)^{1/2},
\]

and

\[
w_Q(z) = \frac{(2/3)(1 + z)(d \ln H/dz) - 1}{1 - (H_0^2/H^2)\Omega_{m0}(1 + z)^3}.
\]

In our model we have assumed that \( m_P \dot{Q}/H \equiv \alpha \). In this particular case, with \( w_m = 0 \), the Hubble parameter \( H(z) \) as a function of the redshift \( z \) is given by (cf. equation (3.47))

\[
H(z) = H_0 \sqrt{\Omega_{m0}(1 + z)^3 + (1 - \Omega_{m0})(1 + z)^{\alpha^2}},
\]

where \( \Omega_{m0} \equiv 3/(3 + \tilde{c}_1) \). Using this expression of \( H(z) \), we show in figure 17 the best fit form of \( w(z) \) for the SNLS data with a prior \( \Omega_{m0} = 0.24 \). The dark energy equation of state \( w_Q(z) \) is given by

\[
w_Q(z) = \frac{(1 - \Omega_{m0})(\alpha^2 - 3) - \alpha^2 \Omega_{m0}(1 + z)^{3-\alpha^2}}{3(1 - \Omega_{m0}) + \alpha^2 \Omega_{m0}(1 + z)^{3-\alpha^2}}.
\]

Clearly, knowledge of \( \Omega_{m0} \) and \( \alpha \) would suffice to determine \( w_Q(z) \). In the \( \alpha = 0 \) case, \( w_Q(z) = w_\Lambda = -1 \). In tables 1 and 2 we present the best fit values of \( \alpha \) and \( w_Q \) for different choices of \( \Omega_{m0} \).

The Gold SNIa datasets could actually fit better with coupled quintessence (or interacting dark energy) models (cf. figure 18).

In the non-minimal coupling case, \( \rho_Q \) is not separately conserved, since \( \dot{\rho}_Q + 3H(1 + w_Q)\rho_Q = \alpha_Q H Q' \rho_m \); of course, the total energy is always conserved: \( \dot{\rho}_\text{tot} + 3H(\rho_\text{tot} + p_\text{tot}) = 0 \), where \( \rho_\text{tot} = \rho_m + \rho_Q \). Using the relations \( \partial/\partial t = H(\partial/\partial \ln a) \) and \( \ln a = -\ln(1 + z) \),
Figure 17. The best fit form of $w(z)$ for the SNLS datasets for a prior of $\Omega_{m0} = 0.24$ along with the $1\sigma$ errors (shaded region). The (black) solid line corresponds to the ansatz $w(z_1) = w_0 + w_1 z_1/(1 + z_1)$ (cf. equation (4.3)). The three other lines correspond to $\alpha = 0.4, 0.2109$ and 0 (top to bottom) and $w_Q(z)$ given by equation (4.5). With $\Omega_{m0} = 0.24$, $\alpha = 0.2109$ minimizes the $\chi^2_{\text{min}} (= 104.18)$. The SNLS data may favour a lower value of $\Omega_{m0}$ (as compared to the Gold SNIa dataset). Further, with a canonical quintessence, so that $w_Q(z = 0) \gtrsim -1$, we may require $\Omega_{m0} < 0.2592$.

Table 1. The best fit values of $w_Q(z)$ and $\alpha$ for the SNLS datasets for a given $\Omega_{m0}$.

| $\Omega_{m0}$ | $|\alpha|$ | $w_Q(z = 0)$ | $\chi^2_{\text{min}}$ |
|--------------|-----------|--------------|------------------|
| 0.22         | 0.2948    | -0.9628      | 104.23           |
| 0.23         | 0.2573    | -0.9713      | 104.21           |
| 0.24         | 0.2109    | -0.9805      | 104.18           |
| 0.25         | 0.1476    | -0.9903      | 104.16           |
| 0.259173     | 0.014     | -0.9999      | 104.14           |

Table 2. The best fit values of $w_Q(z)$ and $\alpha$ for the Gold SNIa dataset for a given $\Omega_{m0}$.

| $\Omega_{m0}$ | $|\alpha|$ | $w_Q(z = 0)$ | $\chi^2_{\text{min}}$ |
|--------------|-----------|--------------|------------------|
| 0.23         | 0.4001    | -0.9307      | 178.64           |
| 0.25         | 0.3376    | -0.9493      | 178.21           |
| 0.27         | 0.2544    | -0.9704      | 177.76           |
| 0.29         | 0.0933    | -0.9959      | 177.31           |
| 0.2929       | 0.0100    | -0.9999      | 177.25           |

we get

$$\rho_Q = \exp \left[ 3 \int_0^{z} \frac{(1 + w(z_1))}{1 + z_1} \, dz_1 \right] \left( \rho_{Q0} + \int_0^{z} \frac{Q'Q\rho_m}{1 + z_1} \exp \left[ -3 \int_0^{z} \frac{(1 + w(z_1))}{1 + z_1} \, dz_1 \right] \, dz_1 \right).$$

(4.6)
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Figure 18. The best fit form of $w(z)$ for the Gold SNIa dataset for a prior of $\Omega_{m0} = 0.27$ along with the 1σ errors (shaded region) with $H(z)$ given by equation (4.7) (left plot) and $H_{\text{obs}}(z)$ given by (4.10) (right plot); $\chi^2$ is minimized for $\alpha = 0.4735$ and $\alpha_{Q0} = 0.0633$. The (black) solid line corresponds to the best fit line with $\chi^2_{\text{min}} (\approx 177)$ and the three other lines represent $w_Q(z)$ (cf. equation (4.8)) with $\alpha = 0.6, 0.4$ and $0.2$ (top to bottom) and $\alpha_Q \equiv \chi = 0.4$.

In particular, with $m_P Q' \equiv \alpha$ and $\alpha_Q = \frac{d \ln A(Q)}{d(\kappa Q)} \equiv \chi$, the Hubble parameter $H(z)$ is found to be

$$H(z) = H_0 \sqrt{\Omega_{m0}(1 + z)^{3+\alpha\chi} + (1 - \Omega_{m0})(1 + z)^{\alpha^2}},$$

where $\Omega_{m0} \equiv 3/(3 + \tilde{c}_4 \zeta)$ and $\zeta \equiv 3 - \alpha^2 + \alpha \chi$. The dark energy equation of state is

$$w_Q(z) = \frac{\alpha \chi \Omega_{m0} + (1 - \Omega_{m0})(\alpha^2 - 3)(1 + z)^{-\zeta}}{3(1 - \Omega_{m0})(1 + z)^{-\zeta} + \alpha(\alpha - \chi)\Omega_{m0}}.$$  \hfill (4.8)

Next we briefly discuss an interesting possibility (leaving the details and further generalization to a forthcoming paper). In the non-minimal coupling case, the Hubble expansion parameter that one measures (in a physical Jordan frame) could actually be different than the one given by (4.7) by a conformal factor. Given that

$$H_{\text{obs}}(z) = \exp[\chi(Q/m_P)] \propto \exp[\chi \alpha \ln a] = a^{\alpha \chi} = (1 + z)^{-\alpha \chi}, \hfill (4.9)$$

we find

$$H_{\text{obs}}(z) = H_0 \sqrt{\Omega_{m0}(1 + z)^{3} + (1 - \Omega_{m0})(1 + z)^{\alpha^2 - \alpha \chi}}.$$ \hfill (4.10)

Using this expression of $H(z)$, we have presented in table 3 the best fit values of $\alpha$ and $\alpha_{Q0}$ which minimize the $\chi^2$ for the Gold SNIa, SNIa + CMB-shift (WMAP) + SDSS datasets for a given $\Omega_{m0} \equiv 0.27$.

The mean value of $w_{Q0}$ obtained above is within the range indicated by WMAP3+SDSS observations: $w_{DE} = -0.941^{+0.087}_{-0.101}$ [3]. The best fit value of $\alpha_Q$ is found to be $\alpha_Q \simeq 0.06$, but in our model it may contain significant numerical errors, namely $\alpha_Q = 0.06 \pm 0.35$, which thereby implies the consistency of our model with general relativity.
Figure 19. As in figure 18 (right plot) but with $\alpha_Q = 0$.

Table 3. The best fit values of $\alpha$ and $\alpha Q_0$, with 1$\sigma$ errors for $w_{Q_0} \equiv w_Q(z = 0)$.

|             | $\alpha$ | $\alpha Q_0$ | $w_{Q_0}$ (equation (4.8)) | $w_{Q_0}$ (equation (4.3)) |
|-------------|----------|--------------|----------------------------|----------------------------|
| SNIa        | 0.4735   | 0.0633       | $-0.90^{+0.35}_{-0.33}$    | $-0.94^{+0.10}_{-0.10}$    |
| SNIa + WMAP + SDSS | 0.5142   | 0.0583       | $-0.88^{+0.26}_{-0.31}$    | $-0.92^{+0.07}_{-0.08}$    |

(for which $\alpha Q = 0$) at the 1$\sigma$ level. To illustrate this result we show in figure 19 the best fit plot with $\alpha Q = 0$.

The post-Newtonian parameter $\tilde{\gamma}$ is related to $\alpha Q_0$ ($\equiv \chi$) through the relation [39]

$$\alpha Q_0^2 = \frac{1 - \tilde{\gamma}}{1 + \tilde{\gamma}}.$$ 

With the best fit value $\alpha Q_0 \simeq 0.06$, this yields $|\tilde{\gamma} - 1| \simeq 7.1 \times 10^{-3}$, which is not far from a constraint coming from the solar system experiments, i.e. $|\tilde{\gamma} - 1| < 2 \times 10^{-3}$. Moreover, in the non-minimal coupling case, with $A(Q) = e^{\chi(Q)/m_P}$ ($\chi \neq 0$) and $Q(t) \equiv \alpha \ln a + \text{const}$, there arise constraints on the time variation of Newton’s constant. With a scalar field $Q$ conformally coupled to the matter, the effective Newton’s constant (measured, e.g., in a Cavendish type experiment) can be given by

$$\frac{G_{\text{eff}}}{G} = A(Q)^2 (1 + \alpha Q^2_0) = (1 + z)^{-\alpha \chi}(1 + \chi^2).$$

(4.11)

The time derivative of Newton’s constant generally depends on the coupling $A(Q)$ and its derivative, $\alpha Q$. In our model, with $\alpha \chi > 0$, the case of decreasing $w_Q(z)$ (at a lower redshift) corresponds to an increasing Newton’s constant that boosts cosmic acceleration.

For the SNIa best fit value $(\alpha, \alpha Q) = (0.4735, 0.0633)$, the variation of $G_{\text{eff}}$ in the redshift range $z = \{0, 20\}$ is less than 10% (cf. figure 20) and $|dG_{\text{eff}}/dt|/G_{\text{eff}} = 0.029hH_0 \simeq 2.1 \times 10^{-12}$ yr$^{-1}$. We should mention that the current solar system constraint on $G_{\text{eff}}/G_{\text{eff}}$ could be more stringent than this, namely $(dG_{\text{eff}}/dt)/G_{\text{eff}} < 10^{-13}$ yr$^{-1}$ (see, e.g., [40] which derives constraints on $\dot{G}/G$ and $\ddot{G}/G$ for a model where the $Q$ field is explicitly coupled to the Einstein–Hilbert term); it is because the relevant background when studying the solar system is not cosmological but the solution of (3.12) corresponding to the galactic environment, where $\dot{Q}/H \approx 0$ and $\rho_{\text{gal}} \gg \rho_{\text{crit}} \equiv 3H_0^2/8\pi G$. In order to
properly address the question of time derivative (or variation) of Newton’s constant, one has to consider in detail the dynamical system where $\alpha_Q$ is time-varying. This is left for future studies.

5. Conclusion

In this paper we have outlined construction of an effective cosmological model for each of inflation and dark energy (or quintessence), within the framework of the standard scalar–tensor theory. The general assumption has been that the evolution of our universe can be described by Einstein’s gravity coupled to a fundamental scalar field plus matter, described by the general action (3.1). The gravitational part of the action, which is important for constructing a model of inflation, contains a scalar field Lagrangian. The matter part of the action contains all possible matter constituents in the form of a perfect fluid plus a coupling term $A(Q)$ which characterizes a universal coupling between a fundamental scalar field $Q$ and ordinary (plus dark) matter.

In section 2, we have presented an explicit model for inflation, by constructing an inflationary potential that, with proper choice of slope parameters, satisfies the main observational constraints from WMAP data, including the spectral index of scalar perturbations and tensor-to-scalar ratio.

In section 3, we have first derived a set of autonomous equations, by utilizing a fundamental variational principle, that in a compact form describes the evolution of different cosmological parameters, namely $\Omega_Q$, $w_Q$, $\Omega_i$, $w_i$, $\epsilon$ and $\alpha_Q$, as a system of four differential equations, of which only three are linearly independent (cf. (3.17)–(3.20)). By further general considerations, we have shown how the parameters $q$ and $w_{\text{eff}}$ can be determined from a solution of the above system. As discussed in the body of the text, the system of equations (3.17)–(3.20) could be analytically solved only by making a reduction in the number of free parameters or by imposing additional constraints. In this work, one of our aims was to keep the model as general as possible, but for being able to find analytic solutions the number of parameters was restricted to four, neglecting the radiation component, and making a reasonable additional assumption that $Q \equiv \alpha \ln a + \text{const}$ at the present epoch.

First, by examining the case with minimal coupling, $A(Q) = 1$, a class of exact (analytical) solutions has been found (cf. equations (3.37)–(3.42)), which find interesting applications for present-day cosmology. The general solution found in the minimal
coupling case has the behaviour that it is independent of the sign of $\alpha$ (i.e. the sign of $Q$). Thus the direction of a ‘rolling’ scalar field $Q$ does not seem to have any significant effect (which also directly followed when looking at the scalar field Lagrangian (cf. equation (3.1))), except in the shape of the potential. It is found that the critical value $\alpha_{\text{crit}} = 1.48$ separates the parameter spaces of $\alpha$ such that $\alpha < \alpha_{\text{crit}}$ allows a late-time acceleration while $\alpha > \alpha_{\text{crit}}$ does not. Thus the characteristic of the scalar field $Q$ acting as an additional self-repulsive or self-attractive form of energy is merely determined by the magnitude of the velocity of the field, $d(\kappa Q)/d\ln a \equiv \alpha$. In several interesting cases we have found a closed form expression for the (reconstructed) quintessence potential $V(Q)$.

As the combination of WAMP and type Ia supernova observations show a significant constraint on the present-day DE equation of state, $w_Q = -0.941^{+0.087}_{-0.101}$; for the mean value $\omega_Q \sim -0.941$, we require $|\alpha| \sim 0.4207 \sqrt{\Omega_Q} \sim 0.36$, while the WMAP+SSS bound $1 \leq w_Q < -0.82$ may be satisfied for $|\alpha| < 0.62$. Of course, $\alpha = 0$ simply represents the cosmological constant case ($w_Q = -1$). Claiming the same range of $-1 \leq w_Q < -0.82$ for $w_Q$ at redshift $z \geq 0$ imposes a more restrictive constraint on the slope of the potential $\alpha$ being smaller than 0.6. When looking at the evolution of different cosmological parameters ($\Omega_Q$, $\Omega_m$, $w_Q$, $\epsilon$, $w_{\text{eff}}$, $q$), we find that, for smaller values of $\alpha$, the model shows a late-time accelerated expansion (for $z < 1$), while a matter dominance at early times. These features are in agreement with recent WMAP and supernova observations.

To see how a non-minimal coupling, $\alpha_Q \neq 0$, might affect the cosmic expansion, we studied the simplest case of an exponential coupling $A(Q) \propto e^{Q/\sqrt{m^2 v^2}}$, which implies $\alpha_Q \equiv \chi$. In this case the solution is found to have a dependence on the sign of the slope parameter $\alpha$ and the coupling $\alpha_Q$. A replacement of $\alpha$ by $-\alpha$ is found to be equivalent to the replacement of $\alpha_Q$ by $-\alpha_Q$. Moreover, a positive coupling is found to decreases the dark energy equation of state $w_Q$, with respect to its value in the $\alpha_Q = 0$ case, while this effect is opposite for $\alpha_Q < 0$. Thus, for a fixed $\alpha$, the $\alpha_Q > 0$ solution could make the energy represented by $Q$ more repulsive, as compared to the $\alpha_Q = 0$ case. The coupling dependence of other parameters just resemble this fact ($\alpha_Q > 0$ in our convention just means $\alpha$ and $\alpha_Q$ having the same sign). For $|\alpha Q| \lesssim 0.1$, and at low redshifts, the present-day values of the cosmological parameters showed almost no $\alpha_Q$ dependence. That is, an observable effect on the evolution of cosmological parameters, such as $w_{\text{eff}}$ and $\Omega_Q$, can be expected to be seen only for a strong matter–scalar coupling, like $|\alpha Q| \gg 0.1$. The type Ia supernova data may favour a small value for matter–quintessence coupling, like $\alpha Q \sim 0.06$.

We have also shown how in principle a non-minimal matter–scalar coupling can alter the evolution of the cosmological parameters. In general the coupling $\alpha_Q$ always appears in combination with the matter density $\rho_m$ (cf. equation (3.34)). As the mass of the scalar field $Q$ can be determined by $(d^2 V_{\text{eff}}/dQ^2)^{1/2}$ evaluated at a local minimum and the scalar–matter coupling in $V_{\text{eff}}(Q)$ can involve a $\rho_m$-dependent term, the mass of a scalar field depends, in principle, on the ambient matter distribution. Thus in a more sophisticated model, not treating matter as an isotropic perfect fluid, the mass of the scalar field can vary locally due to a possibly strong local variation of $\rho_m$ on small scales.

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Appendix

Corresponding to the action (3.1), the equations of motion that describe gravity, the scalar field $Q$ and the background fields (matter and radiation) are given by

$$\frac{1}{2\kappa^2} \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) - \frac{1}{2} \left( \nabla_\mu Q \nabla_\nu Q \right) + \frac{1}{4} \left( \nabla Q \right)^2 g_{\mu\nu} + \frac{1}{2} V(Q) g_{\mu\nu} - \frac{1}{2} A^4(Q) T_{\mu\nu} = 0,$$

(A.1)

$$\nabla_\mu (g_{\mu\nu} \nabla_\nu Q) - \frac{dV(Q)}{dQ} + A^3 \frac{dA(Q)}{dQ} \sum_i \left( 1 - 3w_i \right) \rho_i = 0.$$

(A.2)

These equations may be supplemented with the equation of motion of a barotropic perfect fluid, which is given by

$$\frac{d(A^4 \rho_i)}{d(aA)} = \left( A^4 \dot{\rho}_i \frac{1}{(aA)} + A^3 \frac{\partial A}{\partial Q} (1 - 3w_i) \rho_i \frac{1}{(aA)} \right).$$

(A.3)

Combining the $(tt)$ and $(xx)$ components of equation (A.1), we get

$$-2\dot{H} = \kappa^2 \left( \frac{1}{2} \dot{Q}^2 + V(Q) + \frac{1}{2} \dot{Q}^2 - V(Q) + A^4 \sum_i (\rho_i + w_i \rho_i) \right).$$

(A.4)

Dividing this equation by $H^2$ and then using the substitution in (3.16) yields

$$-2\frac{\dot{H}}{H^2} = \kappa^2 \rho_Q \frac{H^2}{H^2} + w_Q \kappa^2 \frac{\rho_Q}{H^2} + \sum_i \left( \frac{\kappa^2 A^4 \rho_i}{H^2} (1 + w_i) \right).$$

(A.5)

Multiplying equation (3.11) with $\dot{Q}$ and using the identities

$$\dot{\rho}_Q = \dot{Q} \ddot{Q} + \dot{V}, \quad \rho_Q (1 + w_Q) = \dot{Q}^2,$$

(A.6)

which follow from equation (3.16), we get

$$\dot{\rho}_Q + 3H \rho_Q (1 + w_Q) = \dot{Q} A^3 \frac{dA(Q)}{dQ} \sum_i (1 - 3w_i) \rho_i.$$

(A.7)

Multiplying (A.7) by $\kappa^2/(3H^2)$ and then using equations (3.21), (3.15) and (3.16) leads to equation (3.19). Further, multiplying equation (3.14) by $\kappa^2/(3H^2)$ and then using equation (3.21) leads to

$$\frac{\kappa^2 \rho_i'}{3H^2} + \frac{\kappa^2 \rho_i}{3H^2} (1 + w_i) = \frac{\kappa^2 Q'}{3H^2} \frac{dA(Q)/dQ}{A(Q)} (1 - 3w_i) \rho_i.$$

(A.8)

Combining this equation with the identity

$$\Omega_i' \equiv \frac{\kappa^2 A^4}{3H^2} \rho_i' - 2\epsilon \Omega_i,$$

(A.9)

and then using the substitutions in (3.15) and (3.16) finally gives equation (3.20).
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