ON FAMILIES OF RIEMANN SURFACES WITH AUTOMORPHISMS

MILAGROS IZQUIERDO, SEBASTIÁN REYES-CAROCCA, AND ANITA M. ROJAS

Abstract. In this article we determine the maximal possible order of the automorphism group of the form $ag + b$, where $a$ and $b$ are integers, of a complex three and four-dimensional family of compact Riemann surfaces of genus $g$, appearing for all genus. In addition, we construct and describe explicit complex three and four-dimensional families possessing these maximal number of automorphisms.

1. Introduction

Let $\mathcal{M}_g$ denote the moduli space of compact Riemann surfaces of genus $g \geq 2$. It is classically known that $\mathcal{M}_g$ is endowed with an orbifold structure of dimension $3g - 3$ and that if $g \geq 3$ then its locus of orbifold-singular points is formed by those Riemann surfaces with non-trivial automorphisms.

The classification of groups of automorphisms of compact Riemann surfaces is a classical subject of study which has attracted broad interest ever since Schwarz and Hurwitz proved that the automorphism group of a compact Riemann surface of genus $g \geq 2$ is finite and its order is at most $84g - 84$.

Compact Riemann surfaces of genus $g$ with a group of automorphisms of order of the form $ag + b$ where $a, b$ are integers can be found in the literature plentiful supply. The most classical example concerning that is the class of the Riemann surfaces which possess exactly $84g - 84$ automorphisms. Nowadays, it is known that these Riemann surfaces correspond to regular covers of the projective line ramified over three values, marked with 2, 3 and 7. Another remarkable example is the cyclic case, which was firstly considered by Wiman. Indeed, he showed in [43] that the largest cyclic group of automorphisms of a compact Riemann surface of genus $g \geq 2$ has order at most $4g + 2$. Furthermore, the Riemann surface given by $y^2 = x^{2g+1} - 1$ shows that, for each value of $g$, this upper bound is attained; see also [18]. Moreover, in the early nineties, Kulkarni in [24] proved that, for $g$ sufficiently large, the aforementioned curve is the unique Riemann surface of genus $g$ with an automorphism of order $4g + 2$.

Riemann surfaces with $4g$ automorphisms have been classified in [9] by Bujalance, Costa and the first author; the Jacobian varieties of these surfaces were studied by the second author in [32]. Riemann surfaces with $8(g+3)$ automorphisms were considered independently by Accola in [1] and by Maclachlan in [27]. Under the assumption that $g - 1$ is a prime number, the case $ag - a$ has been classified by Belolipetsky and Jones in [5] for $a \geq 7$ and by the first and second authors in [23] and [33] for $a = 3, 4, 5$ and 6 (see [22] for the remaining cases). Recently, the case $3g - 3$ in which $g - 1$ is assumed to be the square of a prime number was classified in [11] by Carocca and the second author.

2010 Mathematics Subject Classification. 30F10, 14H37, 14H30, 14H40.

Key words and phrases. Riemann surfaces, Fuchsian groups, Group actions, Jacobian varieties.

The first and second authors were partially supported by Redes Grant 2017-170071. The second author was partially supported by Fondecyt Grants 11180024, 1190991. The third author was partially supported by Fondecyt Grant 1180073.
It is worth mentioning that the order of the automorphism group of a compact Riemann surface of genus \( g \) need not to be of the form \( ag + b \). For instance, in [16] it was proved the existence of a complex one-dimensional family of Riemann surfaces of genus \((g - 1)^2\) with \(4q^2\) automorphisms, for \( q \geq 3 \) prime.

This paper is aimed to address the problem of determining the maximal possible order of the automorphism group of the form \( ag + b \), where \( a \) and \( b \) are integers, of a family of compact Riemann surfaces of genus \( g \), appearing for all genus. To review known facts and then to state the results of this paper, inspired by Accola’s notation introduced in [1], we shall bring in the following definition.

**Definition.** For each \( d \geq 0 \) and \( A \subseteq \mathbb{N} - \{1\} \) we define

\[
N_d(g, A)
\]

to be the unique integer of the form \( ag + b \) where \( a, b \in \mathbb{Z} \), if exists, which satisfies:

1. for each \( g \in A \) there is a complex \( d \)-dimensional family of compact Riemann surfaces of genus \( g \) with a group of automorphisms of order \( N_d(g, A) \), and
2. there is no a complex \( d \)-dimensional family of compact Riemann surfaces of genus \( g \) with strictly more than \( N_d(g, A) \) automorphisms, for each \( g \in A \).

If \( A = \mathbb{N} - \{1\} \) then we simply write \( N_d(g) \) instead of \( N_d(g, A) \).

In the sixties, Accola [1] and Maclachlan [27] considered the zero-dimensional case; namely, they dealt with the problem of determining the largest order of the automorphism group of compact Riemann surfaces appearing for all genus. Independently, they proved that

\[
N_0(g) = 8g + 8
\]

by considering the Riemann surface given by the algebraic curve

\[
y^2 = x^{2g+2} - 1.
\]

Later, the uniqueness problem was addressed by Kulkarni in [24]. Concretely, he succeeded in proving that for \( g \not\equiv 3 \mod 4 \) sufficiently large, the aforementioned curve is the unique compact Riemann surface of genus \( g \) with \( 8g + 8 \) automorphisms.

Costa and the first author dealt with the one-dimensional case in [15]. For each \( g \geq 2 \), they found an equisymmetric complex one-dimensional family of hyperelliptic compact Riemann surfaces of genus \( g \) with a group of automorphisms isomorphic to \( D_{g+1} \times C_2 \) and then they proved that

\[
N_1(g) = 4g + 4.
\]

The uniqueness problem was also studied by noticing that for \( g \equiv 3 \mod 4 \), there exists another one-dimensional family with the same number of automorphisms. Besides, the two-dimensional case was addressed by the second author in [33], where a classification of compact Riemann surfaces of genus \( g \) endowed with a maximal non-large group of automorphisms was provided, under the assumption that \( g - 1 \) is prime. By means of this classification, it was noticed that

\[
N_2(g) = 4g - 4
\]

due to the existence, for all \( g \geq 2 \), of a complex two-dimensional family of compact Riemann surfaces of genus \( g \) with dihedral action. In addition, it was proved that if \( g - 1 \) is a prime number then the aforementioned family is the unique complex two-dimensional family with this number of automorphisms.

This article is devoted to extend the previous results by dealing with the complex three and four-dimensional cases. Concretely, we first prove that the equality

\[
N_3(g) = 2g - 2
\]
holds. We then observe that this case as well as the zero, one and two-dimensional cases are very much in contrast with the four-dimensional situation. Indeed, we prove that 

\[ N_4(g) \text{ does not exist.} \]

In proving the non-existence of \( N_g(4) \), we obtain the following facts, which are interesting in their own right: if \( A_1 \) and \( A_2 \) consist of those values of \( g \geq 2 \) that are odd and even respectively, then 

\[ N_4(g, A_1) = g - 1 \quad \text{and} \quad N_4(g, A_2) = g. \]

The strategy to prove the results is to find upper bounds for the number of automorphisms and then to construct in a very explicit manner complex three and four-dimensional families attaining these bounds. After that, we study in detail these families; concretely:

1. we address the uniqueness problem by providing conditions under which they turn into unique with this number of automorphisms,
2. we describe the families themselves as subset of the moduli space of Riemann surfaces in terms of the number of equisymmetric strata they consists of, and
3. we provide an isogeny decomposition of the corresponding families of Jacobian varieties in the moduli space of principally polarized abelian varieties.

It is worth emphasizing that, by definition, if 

\[ N_d(g, A) = ag + b \]

then there is no complex \( d \)-dimensional families of Riemann surfaces of genus \( g \) with more than \( ag + b \) automorphisms for all values \( g \in A \). We shall see throughout the article that the phrase for all values \( g \in A \) is vacuous for \( d = 3 \), but for \( d = 4 \) is not. Indeed, for the sake of completeness, we exhibit infinitely many odd and even values of \( g \) for which there is a complex four-dimensional family of Riemann surfaces of genus \( g \) with strictly more than \( g - 1 \) automorphisms and strictly more than \( g \) automorphisms, respectively.

This paper is organized as follows. In Section \( \S 2 \) we shall review the basic preliminaries; namely: Fuchsian groups, group actions on Riemann surfaces, the stratification of the moduli space and the decomposition of Jacobian varieties with group action. The three-dimensional case will be considered in Sections \( \S 3 \) and \( \S 4 \). The four-dimensional case will be considered in Sections \( \S 5, \S 6 \) and \( \S 7 \).

2. Preliminaries

2.1. Fuchsian groups. Let \( \Delta \) be a Fuchsian group; namely, a discrete group of automorphisms of \[ \mathbb{H} = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \}. \]

If the orbit space \( \mathbb{H}_\Delta \) given by the action of \( \Delta \) on \( \mathbb{H} \) is compact, then the algebraic structure of \( \Delta \) is determined by its so-called signature; namely, the tuple 

\[ \sigma(\Delta) = (h; m_1, \ldots, m_l), \]

where \( h \) denotes the genus of the quotient surface \( \mathbb{H}_\Delta \) and \( m_1, \ldots, m_l \) the branch indices in the associated universal projection \( \mathbb{H} \rightarrow \mathbb{H}_\Delta \). If \( l = 0 \) then it is said that \( \Delta \) is a surface Fuchsian group.

Let \( \Delta \) be a Fuchsian group of signature (2.1). Then

1. \( \Delta \) has a canonical presentation with generators \( \alpha_1, \ldots, \alpha_h, \beta_1, \ldots, \beta_h, x_1, \ldots, x_t \) and relations 
\[ x_1^{m_1} \cdots x_t^{m_t} = \Pi_{i=1}^h [\alpha_i, \beta_i] \Pi_{j=1}^l x_j = 1, \]

where \([u,v]\) stands for the commutator \( uvu^{-1}v^{-1} \).

2. the elements of \( \Delta \) of finite order are conjugate to powers of \( x_1, \ldots, x_t \), and

3. the Teichmüller space of \( \Delta \) is a complex analytic manifold homeomorphic to the complex ball of dimension \( 3h - 3 + l \).
Let $\Delta_2$ be a group of automorphisms of $\mathbb{H}$. If $\Delta$ is a subgroup of $\Delta_2$ of finite index then $\Delta_2$ is also Fuchsian. Moreover, if the signature of $\Delta_2$ is $(h_2; n_1, \ldots, n_s)$ then
\[ 2h_2 - 2 + \sum_{i=1}^{s}(1 - \frac{1}{n_i}) = [\Delta_2 : \Delta](2h_2 - 2 + \sum_{i=1}^{s}(1 - \frac{1}{n_i})). \]
The equality above is called the Riemann-Hurwitz formula. We refer to [19] and [42] for more details.

2.2. Group actions on Riemann surfaces. Let $S$ be a compact Riemann surface and let $\text{Aut}(S)$ denote its automorphism group. A finite group $G$ is said to act on $S$ if there is a group monomorphism $G \to \text{Aut}(S)$. The space of orbits $S_G$ of the action of $G$ on $S$ is endowed with a Riemann surface structure in such a way that the canonical projection $\pi_G : S \to S_G$ is holomorphic.

By uniformization theorem, there is a surface Fuchsian group $\Gamma$ such that $S$ and $\mathbb{H}_\Gamma$ are isomorphic. Moreover, Riemann’s existence theorem ensures that $G$ acts on $S \cong \mathbb{H}_\Gamma$ if and only if there is a Fuchsian group $\Delta$ containing $\Gamma$ together with a group epimorphism
\[ \theta : \Delta \to G \] such that $\ker(\theta) = \Gamma$.

Note that $S_G \cong \mathbb{H}_\Delta$. It is said that $G$ acts on $S$ with signature $\sigma(\Delta)$ and that this action is represented by the surface-kernel epimorphism $\theta$. For the sake of simplicity, we usually identify $\theta$ with the tuple of its images or generating vector: (see, for example, [7] and [37])
\[ \theta = (\theta(\alpha_1), \ldots, \theta(\alpha_h), \theta(\beta_1), \ldots, \theta(\beta_h), \theta(x_1), \ldots, \theta(x_i)). \]

Let $G$ be a subgroup of $G'$. The action of $G$ on $S$ is said to extend to an action of $G'$ on $S$ if:
(1) there is a Fuchsian group $\Delta'$ containing $\Delta$,
(2) the Teichmüller spaces of $\Delta$ and $\Delta'$ have the same dimension, and
(3) there exists a surface-kernel epimorphism
\[ \Theta : \Delta' \to G' \] in such a way that $\Theta|_\Delta = \theta$.

An action is called maximal if it cannot be extended in the previous sense. A complete list of pairs of signatures of Fuchsian groups $\Delta$ and $\Delta'$ for which it may be possible to have an extension as before was determined by Singerman in [41].

2.3. Equivalence of actions. Let $S$ be a compact Riemann surface and let $\text{Hom}^+(S)$ denote the group of orientation-preserving self-homeomorphisms of $S$. Two actions $\psi_1 : G \to \text{Aut}(S)$ are topologically equivalent if there exist $\omega \in \text{Aut}(G)$ and $h \in \text{Hom}^+(S)$ such that
\[ \psi_2(g) = h\psi_1(\omega(g))h^{-1} \] (2.3)
for all $g \in G$. Note that topologically equivalent actions necessarily have the same signature. Each homeomorphism $f$ satisfying (2.3) induces an outer automorphism $f^*$ of $\Delta$, where $\mathbb{H}_\Delta \cong S_G$. We denote the subgroup of $\text{Aut}(\Delta)$ consisting of those $f^*$ by $\mathfrak{B}$. It is known that the following two statements are equivalent (see [7], [19] and [28]).

(1) $\theta_1, \theta_2 : \Delta \to G$ define topologically equivalent actions.
(2) There are $\omega \in \text{Aut}(G)$ and $f^* \in \mathfrak{B}$ such that $\theta_2 = \omega \circ \theta_1 \circ f^*$

It is worth remarking for later use that, with the notations (2.2), if the genus $h$ of $S_G$ is zero, then $\mathfrak{B}$ is generated by the braid transformations $\Phi_i$ defined by:
\[ \Phi_i : x_i \mapsto x_{i+1}, \quad x_{i+1} \mapsto x_{i+1}^{-1}x_ix_{i+1} \quad \text{and} \quad x_j \mapsto x_j \text{ when } j \neq i, i + 1 \] (2.4)
for each $i \in \{1, \ldots, l - 1\}$. Meanwhile, if $h = 1$, then, in addition to (2.4), $\mathfrak{B}$ contains
\[ A_{1,n} : \alpha_1 \mapsto \alpha_1, \quad \beta_1 \mapsto \beta_1 \alpha_1^n, \quad x_j \mapsto x_j, \quad A_{2,n} : \alpha_1 \mapsto \alpha_1 \beta_1^n, \quad \beta_1 \mapsto \beta_1, \quad x_j \mapsto x_j \]
where $n \in \mathbb{Z}$, and the transformations
\[ C_{1,i} : \alpha_1 \mapsto x_i \alpha_1, \quad \beta_1 \mapsto \beta_1, \quad x_i \mapsto y_1 \alpha_1^{-1}, \quad x_j \mapsto x_j \text{ for each } j \neq i \]
Thereby, the classical Poincaré’s Reducibility theorem implies that there exists an abelian subvariety of a Riemann surface $S$, henceforth denoted by $\text{Prym}(S \to S_H)$ and called the Prym variety associated to $\pi$, such that $JS \sim JS_H \times \text{Prym}(S \to S_H)$, where $\sim$ stands for isogeny. See [6] for more details.
2.6. Decomposition of Jacobians with group action. It is classically known that if $G$ acts on a compact Riemann surface $S$ then this action induces a $\mathbb{Q}$-algebra homomorphism
\[ \rho : \mathbb{Q}[G] \to \text{End}_\mathbb{Q}(JS) = \text{End}(JS) \otimes \mathbb{Z} \mathbb{Q}, \]
from the rational group algebra of $G$ to the rational endomorphism algebra of $JS$.

For each $\alpha \in \mathbb{Q}[G]$ we define the abelian subvariety
\[ A_\alpha := \text{Im}(\alpha) = \rho(n\alpha)(JS) \subseteq JS \]
where $n$ is a suitable positive integer chosen in such a way that $n\alpha \in \mathbb{Z}[G]$.

Let $W_1, \ldots, W_r$ be the rational irreducible representations of $G$ and for each $W_i$ denote by $V_i$ a complex irreducible representation of $G$ associated to it. Following [25], the decomposition of 1 as the sum $e_1 + \cdots + e_r$ in $\mathbb{Q}[G]$, where $e_i$ is central idempotent associated to $W_i$, yields a $G$-equivariant isogeny
\[ JS \sim A_{e_1} \times \cdots \times A_{e_r}. \]  
Moreover, for each $l$ there are idempotents $f_{i1}, \ldots, f_{in_l}$ such that $e_l = f_{i1} + \cdots + f_{in_l}$ where $n_l = d_{V_l}/s_{V_l}$ is the quotient of the degree $d_{V_l}$ of $V_l$ and its Schur index $s_{V_l}$. These idempotents provide $n_l$ pairwise isogenous subvarieties of $JS$; let $B_i$ be one of them, for each $l$. Thus, the following isogeny is obtained
\[ JS \sim B_1^{e_1} \times \cdots \times B_r^{e_r} \]  
and is called the group algebra decomposition of $JS$ with respect to $G$. See [13] and also [35].

If $W_1$ denotes the trivial representation then $n_1 = 1$ and $B_1 \sim JS_G$.

If $H$ is a subgroup of $G$ then we will denote by $d^H_{V_i}$ the dimension of the vector subspace $V^H_i$ of $V_i$ consisting of those elements which are fixed under $H$. By Frobenius reciprocity theorem,
\[ d^H_{V_i} = (\text{Ind}^G_H, V_i)_G, \]
where $\text{Ind}^G_H$ stands for the representation of $G$ induced by the trivial one of $H$ and the brackets for the usual inner product of characters of $G$.

Following [13], the group algebra decomposition (2.5) induces the following isogenies.

(1) Let $H$ be a subgroup of $G$. The Jacobian variety $JS_H$ of the quotient $S_H$ decomposes as
\[ JS_H \sim B_1^{n_{11}} \times \cdots \times B_r^{n_{r1}} \]  
where $n_{il} = d_{V_i}^H/s_{V_i}$.

(2) Let $H_1 \leq H_2$ be subgroups of $G$. The Prym variety associated to $S_{H_1} \to S_{H_2}$ decomposes as
\[ \text{Prym}(S_{H_1} \to S_{H_2}) \sim B_1^{n_{11}:n_{12}} \times \cdots \times B_r^{n_{r1}:n_{r2}} \]  
where $n_{il} = n_{il}^{H_1} - n_{il}^{H_2}$.

The previous induced isogenies have been useful to provide decomposition of Jacobian varieties $JS$ whose factors are isogenous to Jacobians of quotients of $S$ and Pryms of intermediate coverings; see, for example, [10], [11] and [34].

Assume that $(\gamma; m_1, \ldots, m_l)$ is the signature of the action of $G$ on $S$ and that this action is represented by the surface-kernel epimorphism $\theta : \Delta \to G$, with $\Delta$ canonically presented as in (2.2). The third author proved in [37, Theorem 5.12] that the dimension of $B_i$ in (2.5) for $i \geq 2$ is given by
\[ \dim B_i = k_{V_i} [d_{V_i}(\gamma - 1) + \frac{1}{2} \sum_{k=1}^l (d_{V_i} - d_{V_i}^{(\theta(x_k))})] \]  
where $k_{V_i}$ is the degree of the extension $\mathbb{Q} \leq L_{V_i}$ with $L_{V_i}$ denoting a minimal field of definition for $V_i$. Note that the dimension of $B_1$ equals $\gamma$.

The decomposition of Jacobian varieties with group actions goes back to old works of Wirtinger, Schottky and Jung; see [40] and [44]. For decompositions of Jacobians with respect to special groups, we refer to the articles [2], [12], [17], [20], [21], [26], [29], [30] and [36].
Notation. We denote the cyclic group of order \( n \) by \( C_n \) and the dihedral group of order \( 2n \) by \( D_n \).

### 3. The three-dimensional case

**Theorem 1.** \( N_3(g) = 2g - 2 \).

The proof of the theorem will follow directly from Lemmata 3.1 and 3.2 stated and proved below.

**Lemma 3.1.** Let \( g \geq 2 \) be an integer. There is no complex three-dimensional family of compact Riemann surfaces of genus \( g \) with strictly more than \( 2(g - 1) \) automorphisms.

**Proof.** Assume the existence of a complex three-dimensional family of compact Riemann surfaces \( S \) of genus \( g \) with a group of automorphisms \( G \) of order strictly greater than \( 2(g - 1) \). If the signature of the action of \( G \) on \( S \) is \((h; m_1, \ldots, m_l)\) then, by the Riemann-Hurwitz formula, we have that

\[
2(g - 1) > 2(g - 1)[2h - 2 + \sum_{j=1}^{l} \left(1 - \frac{1}{m_j}\right)],
\]

or, equivalently,

\[
\sum_{j=1}^{l} \frac{1}{m_j} > 2h + l - 3.
\]

As the dimension \( 3h - 3 + l \) of the family is assumed to be 3, we notice that

\[
\sum_{j=1}^{l} \frac{1}{m_j} > 1 + \frac{l}{3} \text{ where } l \in \{0, 3, 6\}. \tag{3.1}
\]

If \( l = 0 \) then (3.1) turns into \( 0 > 1 \). Besides, if \( l = 3 \) or \( l = 6 \) then (3.1) turns into

\[
\sum_{j=1}^{3} \frac{1}{m_j} > 2 \text{ and } \sum_{j=1}^{6} \frac{1}{m_j} > 3
\]

respectively. In both cases this contradicts the fact that each \( m_j \) is at least 2. \( \square \)

**Lemma 3.2.** Let \( g \geq 2 \) be an integer. There is a complex three-dimensional family of compact Riemann surfaces \( S \) of genus \( g \) with a group of automorphisms \( G \) isomorphic to the dihedral group of order \( 2(g - 1) \) such that the signature of the action of \( G \) on \( S \) is \((0; 2, \ldots, 2)\).

**Proof.** Let \( \Delta \) be a Fuchsian group of signature \((0; 2, \ldots, 2)\) with canonical presentation

\[
\Delta = \langle x_1, \ldots, x_6 : x_1^2 = \cdots = x_6^2 = x_1 \cdots x_6 = 1 \rangle,
\]

and consider the dihedral group

\[
D_{g-1} = \langle r, s : r^{g-1} = s^2 = (sr)^2 = 1 \rangle
\]

of order \( 2(g - 1) \). Note that if \( g \geq 3 \) then the homomorphism

\[
\Delta \to D_{g-1} \text{ given by } x_1, \ldots, x_4 \mapsto s \text{ and } x_5, x_6 \mapsto sr \tag{3.2}
\]

is a surface-kernel epimorphism of signature \((0; 2, \ldots, 2)\). If \( g = 2 \) then the group is \( C_2 = \langle s \rangle \) and the surface-kernel epimorphism can be chosen to be \( x_j \mapsto s \) for each \( 1 \leq j \leq 6 \).

In addition, for each \( g \geq 2 \), the equality

\[
2(g - 1) = 2(g - 1)[0 - 2 + 6(1 - \frac{1}{g})]
\]

shows that the Riemann-Hurwitz formula is satisfied for a \( 2(g - 1) \)-fold regular covering map from a compact Riemann surface of genus \( g \) onto the projective line with six branch values marked with 2.

Thus, the existence of the desired family follows from Riemann’s existence theorem. \( \square \)

**Notation.** From now on, we shall denote the family of Lemma 3.2 by \( \mathcal{F}_g \).

**Remark 1.** Note that Lemma 3.1 does not ask any condition on \( g \). If follows that, as anticipated in the introduction, the phrase for all values of \( g \) in the definition of \( N_3(g) \) can be deleted.
4. The family $\mathcal{F}_g$

Proposition 1. Let $g \geq 3$ be an integer. If $g - 1$ is a prime number then $\mathcal{F}_g$ is the unique complex three-dimensional family of compact Riemann surfaces of genus $g$ with $2(g - 1)$ automorphisms.

Proof. Set $q = g - 1$. Let $\mathcal{F}$ be a complex three-dimensional family of compact Riemann surfaces of genus $g$ with a group of automorphisms $G$ of order $2q$. By considering the Riemann-Hurwitz formula and by arguing similarly as done in the proof of Lemma 3.1, one sees that the unique solution of

$$1 = 2h - 2 + \sum_{j=1}^{l} (1 - \frac{1}{m_j})$$

is $h = 0$, $l = 6$ and $m_j = 2$ for each $1 \leq j \leq 6$. Thus, the signature of the action of $G$ on each $S \in \mathcal{F}$ is necessarily equal to $(0; 2, 6, 2)$. If we now assume $q$ to be prime then $G$ is isomorphic to either the dihedral group or the cyclic group. We claim that the latter case is impossible. In fact, otherwise there would exist a surface-kernel epimorphism

$$\theta : \Delta \to C_{2q} \cong \begin{cases} C_4 & \text{if } q = 2 \\ C_2 \times C_q & \text{if } q \geq 3 \end{cases}$$

where $\Delta$ is a Fuchsian group of signature $(0; 2, 6, 2)$. This, in turn, would imply that the cyclic group of order $2q \geq 4$ can be generated by six involutions; a contradiction. It follows that $G$ is isomorphic to the dihedral group and therefore $\mathcal{F}$ agrees with the family $\mathcal{F}_g$ as desired. \qed

Proposition 2. Let $g \geq 4$ be an integer. If $g - 1$ is a prime number then $\mathcal{F}_g$ is equisymmetric.

Proof. Set $q = g - 1$ and assume $q$ to be prime. Let

$$\theta : \Delta \to D_q = \langle r, s : r^q = s^2 = (sr)^2 = 1 \rangle$$

be a surface-kernel epimorphism representing an action of $G$ on $S \in \mathcal{F}_g$. What we need to prove is that $\theta$ is equivalent to the surface-kernel epimorphism (3.2) of Lemma 3.2. To accomplish this task we shall introduce some notation. We write

$$sr^{n_j} = \theta(x_j) \text{ where } j = 1, \ldots, 6 \text{ and } n_j \in \{0, \ldots, q-1\},$$

and if $n_j \neq 0$ then we shall denote by $m_j$ its inverse in the field of $q$ elements. Also, we denote by $\phi_{\alpha, \beta}$ the automorphism of $D_q$ given by

$$(r, s) \mapsto (r^\alpha, sr^\beta) \text{ for } 1 \leq \alpha \leq q - 1 \text{ and } 0 \leq \beta \leq q - 1.$$ 

Claim 1. Let $j \in \{1, \ldots, 5\}$ fixed. If

$$n_j = 1 \text{ and } n_k = 0 \text{ for all } k < j$$

then, up to equivalence, we can assume that $n_{j+1} = 0$ or $n_{j+1} = 1$.

Assume $n_{j+1} \neq 0$. Then the transformation $\phi_{m_{j+1}, 0} \circ \Phi_j$ induces the correspondence

$$(s, i^{-1}, s, sr, sr^{n_{j+1}}) \mapsto (s, i^{-1}, s, sr, sr^{f(n_{j+1})}) \text{ where } f(u) = 2 - \frac{1}{u}.$$ 

The claim follows by noting that the rule $u \mapsto f(u)$ fixes 1 and has an orbit of length is $q - 1$ given by

$$f^{(n)}(2) = 1 + \frac{1}{n+1} \text{ with } 0 \leq n \leq q - 2.$$ 

Claim 2. Up to equivalence, we can assume $n_1 = n_2 = 0$.

Note that if $n_1 = n_2$ then it is enough to consider the automorphism $\phi_{1,-n_1}$ to obtain the claim. Thus, we shall assume that $n_1 \neq n_2$. If

$$\alpha := (n_2 - n_1)^{-1} \text{ and } \beta := n_1(n_1 - n_2)^{-1}$$

(where the inverses are taken in the field of $q$ elements) then the automorphism $\phi_{\alpha, \beta}$ ensures that, up to equivalence, $n_1 = 0$ and $n_2 = 1$. Now:
(a) if \( n_3 = 0 \) then \( \Phi_2 \) shows that we can assume \( n_1 = n_2 = 0 \), and
(b) if \( n_3 \neq 0 \) then, by Claim 1, we can assume \( n_3 = 1 \). We now apply \( \Phi_2 \circ \Phi_1 \circ \phi_{-1,1} \) to obtain that, up to equivalence, \( n_1 = n_2 = 0 \).

The proof of the claim is done.

We proceed by studying two cases separately, according to \( n_3 = 0 \) or \( n_3 \neq 0 \).

**Type 1.** Assume that \( n_3 = 0 \).

(a) If \( n_4 = 0 \) then necessarily \( n_5 \) and \( n_6 \) are equal and different from zero. We consider \( \phi_{m_5,0} \) to obtain that \( \theta \) is equivalent to (3.2).

(b) If \( n_4 \neq 0 \) then, we consider \( \phi_{m_4,0} \) to assume \( n_4 = 1 \). Now, by Claim 1, we can ensure that \( n_5 = 0 \) or \( n_6 = 1 \); thus, \( \theta \) is equivalent to either

\[
\theta_1 = (s, s, s, sr, s, sr^{-1}) \quad \text{or} \quad \theta_2 = (s, s, s, sr, s, s).
\]

Note that \( \theta_1 \) and \( \theta_2 \) are equivalent under \( \Phi_3 \) and that, in turn, \( \theta_1 \) is equivalent to (3.2) under the action of \( \phi_{-1,0} \circ \Phi_3 \).

**Type 2.** Assume that \( n_3 \neq 0 \). As before, by considering the automorphism \( \phi_{m_3,0} \), we can assume \( n_3 = 1 \). It follows, by Claim 1, that \( n_4 = 0 \) or \( n_4 = 1 \). The first case can be disregarded, since \( \phi_{-1,0} \circ \Phi_3 \) provides an equivalence with (4.1). Now, if \( n_4 = 1 \) then \( \theta \) is equivalent to

\[
\theta_u = (s, s, sr, sr^u, sr^u) \quad \text{for some} \quad u \in \{0, \ldots, q - 1\}.
\]

(1) if \( u \neq \pm 1 \) then define \( \alpha_u \) and \( \beta_u \) by

\[
\alpha_u(1 - u) \equiv 1 \mod q \quad \text{and} \quad \beta_u(1 + u) \equiv 1 \mod q.
\]

The transformation \( \phi_{\alpha_u,0} \circ \Phi^3_4 \circ \Phi^3_5 \circ \Phi^3_3 \circ \Phi^3_4 \) shows that \( \theta_u \) is equivalent to (3.2).

(2) if \( u = 1 \) or \( u = -1 \) then we consider the transformations

\[
\Phi_4 \circ \Phi_3 \circ \Phi_2 \circ \Phi_1 \circ \Phi_5 \circ \Phi_4 \circ \Phi_3 \circ \Phi_2 \circ \phi_{-1,1} \quad \text{and} \quad \Phi_4 \circ \Phi^2_5 \circ \Phi_3 \circ \Phi^3_4
\]

respectively (where \( 2\alpha = 1 \)) to see that \( \theta_u \) is equivalent to (3.2).

The proof of the proposition is done. \( \square \)

We shall denote the equisymmetric stratum corresponding to the action (3.2) defined by

\[
\Delta \to D_{g-1} \quad x_1, \ldots, x_4 \mapsto s \quad \text{and} \quad x_5, x_6 \mapsto sr
\]

by \( F_{g,1} \). Note that with this terminology the previous proposition can be rephrased as

\[
g - 1 \text{ odd prime} \implies F_{g,1} = F_g.
\]

In order to state the following proposition we need some notation. For each integer \( n \geq 2 \) we write

\[
\Omega(n) = \{d \in \mathbb{Z} : d \text{ divides } n \text{ and } 1 \leq d < n\}
\]

and for each \( n \geq 2 \) even we write

\[
\Omega(n) = \{d \in \mathbb{Z} : d \text{ divides } n \text{ and } 1 \leq d < \frac{n}{2}\},
\]

where \( \varphi \) stands for the Euler function.

**Proposition 3.** Let \( g \geq 4 \) be an integer.

(a) If \( g - 1 \) is odd and \( S \in F_g \) then \( JS \) decomposes, up to isogeny, as

\[
JS \sim A \times \Pi_{d \in \Omega(g-1)} B_d^2,
\]

where \( A \) is an abelian surface and \( B_d \) is an abelian variety of dimension \( \frac{1}{2} \varphi(d) \varphi(\frac{d-1}{2}) \). Moreover

\[
JS_{\langle r \rangle} \sim A \quad \text{and} \quad JS_{\langle s \rangle} \sim \Pi_{d \in \Omega(g-1)} B_d
\]
and therefore

\[ JS \sim JS_{(r)} \times JS_{(s)}^2. \]

(b) If \( g - 1 \) is even and \( S \in \mathcal{F}_{g,1} \) then \( JS \) decomposes, up to isogeny, as

\[ JS \sim E \times A \times \Pi_{d \in \Omega(g-1)} B_d^2, \]

where \( A \) is an abelian surface, \( B_d \) is an abelian variety of dimension \( \frac{1}{2} \varphi \left( \frac{d}{2} \right) \) and \( E \) is an elliptic curve. Moreover,

\[ JS_{(r)} \sim A, \quad JS_{(s)} \sim \Pi_{d \in \Omega(g-1)} B_d \quad \text{and} \quad JS_{(sr)} \sim E \times \Pi_{d \in \Omega(g-1)} B_d \]

and therefore

\[ JS \sim JS_{(r)} \times JS_{(s)} \times JS_{(sr)}. \]

**Proof.** We write \( n := g - 1 \).

We assume that \( n \) is odd. It is well-known that the complex irreducible representations of \( D_n = \langle r, s : r^n = s^2 = (sr)^2 = 1 \rangle \) are, up to equivalence (see, for example, [39, p. 36]):

1. two of degree 1: the trivial representation denoted by \( \chi_1 \) and \( \chi_2 : r \mapsto 1, \ s \mapsto -1. \)
2. \( \frac{n-1}{2} \) of degree 2, given by

\[ \psi_j : r \mapsto \text{diag}(\omega^j, \bar{\omega}^j) \quad \text{and} \quad s \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \]

where \( \omega \) is a primitive \( n \)th-root of unity and \( j = 1, \ldots, \frac{n-1}{2} \).

For \( d \in \Omega(n) \), we denote by \( K_d \) the character field of \( \psi_d \) (an extension of \( \mathbb{Q} \) of degree \( \frac{1}{2} \varphi \left( \frac{d}{2} \right) \)) and define

\[ W_d := \oplus_{\sigma \in G_d} \psi_d^\sigma, \quad (4.2) \]

where \( G_d \) stands for the Galois group associated to the extension of \( \mathbb{Q} \leq K_d \). Following for example [21, Section 2], up to equivalence, the rational irreducible representations of \( D_n \) are

\[ \chi_1, \chi_2 \text{ and } W_d \text{ with } d \in \Omega(n). \]

We recall that the Schur index of each representation of a dihedral group equals 1. Thus, as explained in \S 2.6, if \( S \in \mathcal{F}_g \) then the group algebra decomposition of \( JS \) with respect to \( G \) is

\[ JS \sim B_2 \times \Pi_{d \in \Omega(n)} B_d^2, \quad (4.3) \]

where the factor \( B_1 \) is disregarded since the genus of \( S_G \) is zero.

Note that as \( n \) is assumed to be odd, all the involutions of \( D_n \) are pairwise conjugate and therefore the dimension of the corresponding fixed subspaces agree. This simple fact implies that the dimension of each factor in (4.3) does not depend on the equisymmetric stratum to which \( S \) belongs. Then, in order to apply the formula (2.8) we only need to compute the dimension of the fixed subspaces of \( \chi_2 \) and \( \psi_d \) under the action of \( \langle s \rangle \). In the former case we have that

\[ \chi_2^{(s)} = 0 \quad \text{and therefore} \quad \dim B_2 = -1 + \frac{1}{2} (6(1-0)) = 2, \]

while in the latter case, for each \( d \in \Omega(n) \), we have

\[ \psi_d^{(s)} = 1 \quad \text{and therefore} \quad \dim B_d = \frac{1}{2} \varphi \left( \frac{d}{2} \right) (-2 + \frac{1}{2} (6(2-1)) = \frac{1}{2} \varphi \left( \frac{d}{2} \right). \]

Finally, we apply the induced isogeny (2.6) with \( H = \langle r \rangle \) and \( H = \langle s \rangle \) to obtain that

\[ JS_{(r)} \sim B_2 \quad \text{and} \quad JS_{(s)} \sim \Pi_{d \in \Omega(n)} B_d \]

respectively. The proof of the statement (a) follows after setting \( A = B_2 \).
We now assume that \( n \) is even and proceed analogously. The complex irreducible representations of \( D_n \) are, up to equivalence:

1. four of degree 1: the trivial one \( \chi_1 \), and
   \[ \chi_2 : r \mapsto 1, \ s \mapsto -1, \ \chi_3 : r \mapsto -1, \ s \mapsto 1 \text{ and } \chi_4 : r \mapsto -1, \ s \mapsto -1. \]
2. \( \frac{n}{2} - 1 \) of degree 2, given by \( \psi_j \) with \( j = 1, \ldots, \frac{n}{2} - 1 \).

Up to equivalence, the rational irreducible representations of \( D_n \) are
\[ \chi_1, \chi_2, \chi_3, \chi_4 \text{ and } W_d \text{ with } d \in \tilde{\Omega}(n), \]
where \( W_d \) is as in (4.2). If \( S \in \mathcal{F}_{g,1} \) then the group algebra decomposition of \( JS \) with respect to \( G \) is
\[ JS \sim B_2 \times B_3 \times B_4 \times \Pi_{d \in \tilde{\Omega}(n)} B_d^2, \]
where, as before, \( B_1 \) is not considered. Note that
\[ \chi_2^{(s)} = \chi_2^{(sr)} = 0, \ \chi_3^{(s)} = 1, \ \chi_3^{(sr)} = 0 \text{ and } \chi_4^{(s)} = 0, \ \chi_4^{(sr)} = 1 \]
and for each \( d \in \tilde{\Omega}(n) \)
\[ \psi_d^{(s)} = \psi_d^{(sr)} = 1. \]

Then, we apply (2.8) to conclude that
\[ \dim B_2 = 2, \ \dim B_3 = 0, \ \dim B_4 = 1 \text{ and } \dim B_d = \frac{1}{2} \varphi(\frac{d}{2}). \]

Finally, we consider the induced isogeny (2.6) with \( H = \langle r \rangle, \ H = \langle s \rangle \) and \( H = \langle sr \rangle \) to obtain that
\[ JS_{(r)} \sim B_2, \ JS_{(s)} \sim \Pi_{d \in \tilde{\Omega}(n)} B_d \text{ and } JS_{(sr)} \sim B_4 \times \Pi_{d \in \tilde{\Omega}(n)} B_d \]
respectively. The proof of the statement (b) follows after setting \( E = B_4 \) and \( A = B_2 \).

\[ \square \]

**Remark 2.** We end this section by pointing out some remarks concerning the family \( \mathcal{F}_g \).

1. Note that if \( g - 1 \) is an odd prime (or, more generally, odd) then \( D_{g-1} \) does not contain central subgroups of order two. Thus, generically, each \( S \in \mathcal{F}_g \) is non-hyperelliptic.
2. Note that \( \langle r \rangle \cong C_{g-1} \) acts freely on each \( S \in \mathcal{F}_g \) and \( S_{(r)} \) has genus two. Conversely, if \( C \rightarrow X \) is a unbranched covering map, where the genus of \( X \) is two, then its degree is \( g - 1 \) and, following [41], the automorphism group of \( C \) has order \( 2(g - 1) \) and acts on \( C \) with signature \((0; 2, 6, \ldots, 2)\). These facts show that, if \( g - 1 \) is prime, then \( \mathcal{F}_g \) corresponds to the family of compact Riemann surfaces of genus \( g \) that are cyclic unbranched covers of Riemann surfaces of genus two.
3. The family \( \mathcal{F}_3 \) consists of two equisymmetric strata: one of them represented by (3.2) and the other represented by \((s, r, r, sr, sr)\). See [7, Table 5, 3.h].
4. If \( g - 1 \) is not prime then Proposition 2 is not longer true. For instance, if \( g - 1 \) is even, then
\[ \langle r, \overline{r}, s, r, s, sr, sr \rangle \]
defines an action which is, clearly, non-equivalent to the one defined by (3.2). This shows that, in this case, the family \( \mathcal{F}_g \) consists of at least two equisymmetric strata.
5. We observe that, for each \( g \geq 2 \), the family \( \mathcal{F}_g \) (or, more precisely, the complement of its interior) contains the complex two-dimensional family with the maximal possible number of automorphisms. Indeed, following [33], this family is given by the surface-kernel epimorphism
\[ \Theta : \hat{\Delta} \rightarrow D_{2(g-1)} = \langle R, S : R^{2(g-1)} = S^2 = (SR)^2 = 1 \rangle \]
of MILAGROS IZQUIERDO, SEBASTIÁN REYES-CAROCCA, AND ANITA M. ROJAS

defined by
\[(y_1, y_2, y_3, y_4, y_5) \mapsto (S, S, SR^g, SR, R^{g-1}),\]
where \(\hat{\Delta}\) is a Fuchsian group of signature \((0; 2, \ldots, 2)\) canonically presented. Now, if
\[x'_1 := y_1, x'_2 := y_2, x'_3 := y_3y_1y_5, x'_4 := y_3y_2y_5, x'_5 := y_4y_1y_4\]
and \(x'_6 := (x'_1 \cdots x'_5)^{-1}\)
then \((x'_1, \ldots, x'_6)\) is a Fuchsian group of signature \((0; 2, \ldots, 2, 2)\) and the restriction of \(\Theta\) to it
\[\Theta|_{(x'_1, \ldots, x'_6)} : \langle x'_1, \ldots, x'_6 \rangle \to \langle R^2, S \rangle \cong D_{g-1}\]
is given by
\[(x'_1, x'_2, x'_3, x'_4, x'_5, x'_6) \mapsto (S, S, S, S, SR^2, SR^2).\]
Note that this surface-kernel epimorphism agrees with \((3.2)\) by setting \(r = R^2\) and \(s = S\).

(6) The group algebra decomposition of Jacobians of Riemann surfaces which belong to the same
family but lying in different equisymmetric strata may differ radically. For instance, if \(g \equiv 3 \mod 4\) and \(S\) belongs to the stratum defined by \((4.4)\), then the group algebra decomposition
of \(JS\) has three factors of dimension one, instead of only one as in the stratum \((3.2)\).

(7) Note that in Proposition 3(a) the Jacobian varieties \(JS_{(s)}\) and \(JS_{(sr)}\) are isomorphic. Thus,
independently of the parity of \(g\), if \(S \in F_g\), then \(JS\) is isogenous to \(JS_{(r)} \times JS_{(s)} \times JS_{(sr)}\).

(8) In [31] the second author proved that the maximal possible order of a nilpotent group of automor-
phisms of a complex three-dimensional family of compact Riemann surfaces of genus \(g\) is
\(2(g - 1)\). Note that if \(g - 1\) is a power of 2 then the dihedral group \(D_{g-1}\) is nilpotent, showing
that family \(F_g\) attains the aforementioned upper bound for infinitely many values of \(g\).

5. THE FOUR-DIMENSIONAL CASE

Lemma 5.1. Let \(g \geq 4\) be an even integer. If \(g - 1\) is a prime number then there is no complex four-
dimensional families of compact Riemann surfaces of genus \(g\) with strictly more than \(g\) automorphisms.

Proof. Assume the existence of a complex four-dimensional family of compact Riemann surfaces of
genus \(g\) with a group of automorphisms \(G\) of order strictly greater than \(g\). If the signature of the action
is \((h; m_1, \ldots, m_l)\), then the Riemann-Hurwitz formula ensures that
\[2(g - 1) > g(2h - 2 + l - \Sigma_{j=1}^l \frac{1}{m_j})\]
and, after straightforward computations, one can see that necessarily \(h = 0\) and \(l = 7\). Thus,
\[\Sigma_{j=1}^7 \frac{1}{m_j} > 3 + \frac{2}{g}\]  \((5.1)\)
If we denote the number of periods \(m_j\) that are different from 2 by \(v\), then \((5.1)\) implies that
\[\frac{12}{g} < 3 - v\] and therefore \(v \in \{0, 1, 2\}\).

If \(v = 0\) then the signature of the action is \((0; 2, \ldots, 2)\) and the order of \(G\) is \(\frac{1}{3}(g - 1)\). However, as
\(g - 1\) is assumed to be prime, we obtain that \(g = 4\), and this contradicts the assumption that the order
of \(G\) is strictly greater than the genus.

If \(v = 1\) then the signature of the action is \((0; 2, \ldots, 2, a)\) for some \(a \geq 3\) which satisfies, by \((5.1)\), the
inequality \(2a < g\). Note that the order of \(G\) is \(\frac{2a}{a-1}(g - 1)\), but, as \(g - 1\) is assumed to be prime, we see
that necessarily \(2a = g\): a contradiction.

Finally, if \(v = 2\) then the signature of the action is \((0; 2, \ldots, 2, a, b)\) for some \(a, b \geq 3\) that, by \((5.1)\),
satisfy \(\frac{1}{a} + \frac{1}{b} > \frac{1}{2}\). It follows that the signature of the action is either
\[(0; 2, \ldots, 2, 3, 3), (0; 2, \ldots, 2, 3, 4) \text{ or } (0; 2, \ldots, 2, 3, 5)\]
and, consequently, the order of $G$ is either
\[ \frac{12}{7}(g - 1), \frac{24}{7}(g - 1) \text{ or } \frac{60}{7}(g - 1). \]

Note that, as before, the assumption that $g - 1$ is prime, implies that $g$ equals 12, 24 or 60 respectively. The contradiction is obtained after noticing that, in every case, the order of $G$ agrees with the genus. \( \square \)

**Lemma 5.2.** For each even integer $g \geq 4$, there is a complex four-dimensional family of compact Riemann surfaces $S$ of genus $g$ with a group of automorphisms $G$ isomorphic to the dihedral group of order $g$ such that the signature of the action of $G$ on $S$ is $(0; 2, \ldots, 2, \frac{g}{2})$.

**Proof.** Let $\Delta$ be a Fuchsian group of signature $(0; 2, \ldots, 2, \frac{g}{2})$ with canonical presentation
\[ \Delta = \langle x_1, \ldots, x_7 : x_7^2 = \cdots = x_6^2 = x_5^2 = x_4 \cdots x_7 = 1 \rangle, \]
and consider the dihedral group
\[ D_{\frac{g}{2}} = \langle r, s : r^\frac{g}{2} = s^2 = (sr)^2 = 1 \rangle \]
of order $g$. Note that the homomorphism
\[ \Delta \rightarrow D_{\frac{g}{2}} \] given by $x_1, \ldots, x_5 \mapsto s, x_6 \mapsto sr^{-1}$ and $x_7 \mapsto r$
is a surface-kernel epimorphism of signature $(0; 2, \ldots, 2, \frac{g}{2})$. In addition, the equality
\[ 2(g - 1) = g(0 - 2 + 6(1 - \frac{1}{2}) + (1 - \frac{2}{7})) \]
shows that the Riemann-Hurwitz formula is satisfied for a $g$-fold regular covering map from a Riemann surface of genus $g$ onto the projective line with six branch values marked with 2 and with one branch value marked with $\frac{g}{2}$. Thus, the existence of the desired family follows from Riemann’s existence theorem. \( \square \)

**Notation.** From now on, we shall denote the family of Lemma 5.2 by $\mathcal{Y}'$.

**Lemma 5.3.** Let $g \geq 3$ be an odd integer. If $g - 1$ is a power of two then there is no complex four-dimensional families of Riemann surfaces of genus $g$ with strictly more than $g - 1$ automorphisms.

**Proof.** Assume the existence of a complex four-dimensional family of compact Riemann surfaces of genus $g$ with a group of automorphisms $G$ of order strictly greater than $g - 1$, and denote the signature of the action by $(h; m_1, \ldots, m_l)$. Then the Riemann-Hurwitz formula ensures that
\[ 4 > 2h + l - \sum_{j=1}^{l} \frac{1}{m_j}; \]
showing that $h = 0$ and $l = 7$, and consequently that
\[ 3 < \sum_{j=1}^{7} \frac{1}{m_j} \leq \frac{7}{2}. \]
By proceeding analogously as done in the proof of Lemma 5.1 one sees that the signature is either
\[(0; 2, \ldots, 2, 2), (0; 2, \ldots, 2, 2, a), (0; 2, \ldots, 2, 3, 3), (0; 2, \ldots, 2, 3, 4) \text{ or } (0; 2, \ldots, 2, 3, 5) \]
for some $a \geq 3$. It follows that the order of $G$ is either
\[ \frac{4}{3}(g - 1), \frac{2a}{2a-1}(g - 1), \frac{12}{11}(g - 1), \frac{24}{23}(g - 1) \text{ or } \frac{60}{59}(g - 1). \]
The contradiction is obtained after noticing that if $g - 1$ is a power of 2, then the aforementioned fractions are not integers. \( \square \)

**Lemma 5.4.** Let $g \geq 3$ be an odd integer. There are:

(a) a complex four-dimensional family of compact Riemann surfaces $S$ of genus $g$ with a group of automorphisms $G$ isomorphic to the cyclic group of order $g - 1$ such that the signature of the action of $G$ on $S$ is $(1; 2, \ldots, 2, 2)$, and
(b) a complex four-dimensional family of compact Riemann surfaces $S$ of genus $g$ with a group of automorphisms $G$ isomorphic to the dihedral group of order $g-1$ such that the signature of the action of $G$ on $S$ is $(1; 2,\ldots, 2)$.

Proof. Let $\Delta$ be a Fuchsian group of signature $(1; 2,\ldots, 4, 2)$ with canonical presentation

$$\Delta = \langle \alpha_1, \beta_1, x_1, x_2, x_3, x_4 : x_1^2 = x_2^2 = x_3^2 = x_4^2 = \alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}x_1x_2x_3x_4 = 1 \rangle,$$

and consider the cyclic and dihedral groups

$$C_{g-1} = \langle t : t^{g-1} = 1 \rangle \quad \text{and} \quad D_{2g-2} = \langle r, s : r^{2g-2} = s^2 = (sr)^2 = 1 \rangle$$

of order $g-1$. Note that the homomorphisms

$$\Delta \to C_{g-1} \quad \text{given by} \quad \alpha_1 \mapsto t, \beta_1 \mapsto 1, \ x_1, x_2, x_3, x_4 \mapsto t^{\frac{g-1}{2}}$$

and

$$\Delta \to D_{2g-2} \quad \text{given by} \quad \alpha_1, \beta_1 \mapsto 1, \ x_1, x_2 \mapsto s, \ x_3, x_4 \mapsto sr$$

are surface-kernel epimorphisms of signature $(1; 2,\ldots, 4, 2)$. In addition, the equality

$$2(g-1) = (g-1)[2 - 2 + 4(1 - \frac{1}{2})]$$

shows that the Riemann-Hurwitz formula is satisfied for a $(g-1)$-fold regular covering map from a Riemann surface of genus $g$ onto a Riemann surface of genus 1 with four branch values marked with 2.

Thus, the existence of the desired families follows from Riemann’s existence theorem. □

**Notation.** From now on, we shall denote the families of Lemma 5.2 by $\mathcal{U}^1_g$ and $\mathcal{U}^2_g$ respectively.

**Theorem 2.** $N_4(g)$ does not exist.

**Proof.** We shall proceed by contradiction. Let us assume that $N_4(g)$ exists and that

$$N_4(g) = ag + b \quad \text{for suitable (and fixed)} \quad a, b \in \mathbb{Z}.$$

We claim that $a = 1$. Indeed:

1. Clearly $a$ cannot be zero (consider Lemma 5.2 with $g = b + 1$).
2. If $a$ were negative (and therefore $b$ must be positive) then for each

$$g > \frac{-b}{a} + 1$$

the number $N_4(g)$ would be negative; a contradiction.
3. If $a$ were strictly greater than 1 then for

$$g > \begin{cases} 
2 & \text{if } b \geq 0 \\
\frac{2-b}{a-1} & \text{if } b < 0 
\end{cases}$$

the number $N_4(g)$ would exceed $g$; this fact contradicts Lemmata 5.1 and 5.3.

Furthermore, by Lemma 5.3, we see that necessarily $b \leq -1$.

If follows that for every $g \geq 2$, there is a complex four-dimensional family of compact Riemann surfaces $S$ of genus $g$ with a group of automorphisms $G$ of order $g + b$. If the signature of the action of $G$ on $S$ is $(b; m_1, \ldots, m_l)$ then each period $m_j$ must equal $g + b$, since otherwise

$$g \equiv -b \mod m_j \quad \text{for some} \quad m_j,$$

contradicting the fact that the family exists for all $g \geq 2$. In particular, we obtain that $G$ is necessarily isomorphic to the cyclic group and the Riemann-Hurwitz formula implies that

$$2(g-1) = (g+b)[2h - 2 + l(1 - \frac{1}{g+b})].$$
Hence
\[ b = \begin{cases} 
1 - \frac{g}{5} & \text{if } h = 0 \\
\frac{1}{3}(1 - g) & \text{if } h = 1 \\
\frac{1}{2}(g + 1) & \text{if } h = 2,
\end{cases} \]
showing that the existence of the family fails to be true for all genus.

Once the non-existence of \( N_4(g) \) has been proved, it makes sense to state the following theorem.

**Theorem 3.** Let
\[ A_1 = \{ g \in \mathbb{N} : g \geq 3 \text{ is odd} \} \quad \text{and} \quad A_2 = \{ g \in \mathbb{N} : g \geq 4 \text{ is even} \}. \]
Then
\[ N_4(g, A_1) = g - 1 \quad \text{and} \quad N_4(g, A_2) = g. \]

**Proof.** The proof follows directly from Lemmata 5.1, 5.2, 5.3 and 5.4.

**Remark 3.** As anticipated in the introduction of the article, the phrase *for all values of* \( g \in A_j \) *in the definition of* \( N_4(g, A_j) \) *is not vacuous. Indeed, it is not a difficult task to verify the following facts.

1. For each \( g \geq 7 \) such that \( g \equiv 3 \pmod{4} \) there exists a complex four-dimensional family of compact Riemann surfaces of genus \( g \) with a group of automorphisms isomorphic to the dihedral group of order \( g+1 \) such that the signature of the action is \((0; 2, \ldots, 2, \frac{4}{3}g)\).

2. For each \( g \geq 4 \) such that \( g \equiv 4 \pmod{6} \) there exists a complex four-dimensional family of compact Riemann surfaces of genus \( g \) with a group of automorphisms isomorphic to the dihedral group of order \( \frac{4}{3}(g-1) \) such that the signature of the action is \((0; 2, \ldots, 7, 2)\).

6. **The family** \( \mathcal{V}_g \)

**Proposition 4.** Let \( g \geq 4 \) be an even integer. *If* \( q \) *is a prime number then* \( \mathcal{V}_g \) *is the unique complex four-dimensional family of compact Riemann surfaces of genus* \( g \) *with* \( g \) *automorphisms.*

**Proof.** Let \( \mathcal{V} \) be a complex four-dimensional family of compact Riemann surfaces \( S \) of genus \( g \) with a group of automorphisms \( G \) of order \( g \). If the signature of the action of \( G \) on \( S \) is \((h; m_1, \ldots, m_l)\) then the Riemann-Hurwitz formula says that
\[ \Sigma_{j=1}^{l} \frac{1}{m_j} = (2h + l - 4) + \frac{2}{g}. \tag{6.1} \]
As argued in the proof of Lemma 5.1, the fact that the family \( \mathcal{V} \) is assumed to be of dimension four implies that necessarily \( h = 0 \) and \( l = 7 \). Thus, (6.1) turns into
\[ \Sigma_{j=1}^{7} \frac{1}{m_j} = 3 + \frac{2}{g}. \tag{6.2} \]
We denote the number of periods \( m_j \) that are different from \( 2 \) by \( v \). Clearly, \( v = 0 \) if and only if \( g = 4 \). We now assume \( q = \frac{2}{3} \) to be prime and notice that this fact implies that if some \( m_j \) is different from \( 2 \) then \( m_j \geq q \). We claim that \( v = 1 \) provided that \( g \geq 6 \). Indeed, if \( v \geq 2 \) then (6.2) implies that
\[ 3 + \frac{1}{3} \leq \frac{v}{q} + \frac{7-v}{2} \iff \frac{v-1}{2} \leq \frac{v-1}{q}, \]
and then \( g = 4 \). Thus, the only possible signature of the action of \( G \) on \( S \) is \((0; 2, \ldots, 7, 2, q)\).

Note that if \( S \) does not belong to the family \( \mathcal{V}_g \) then, as \( q \) is prime, the group \( G \) must be isomorphic to the cyclic group of order \( 2q \). However, this situation is impossible due to the fact that it is not possible to construct a surjective homomorphism from a Fuchsian group of signature \((0; 2, \ldots, 7, 2, q)\) onto the cyclic group of order \( 2q \). Thereby, \( \mathcal{V} \) and \( \mathcal{V}_g \) agree as desired.

**Proposition 5.** Let \( g \geq 6 \) be an even integer such that \( \frac{q}{2} \) is prime. Then family \( \mathcal{V}_g \) consists of at most \( \frac{2g+2}{q} \) equisymmetric strata.
Proof. Set \( q \geq 6 \) such that \( q = \frac{4}{2} \) is prime. Let
\[
\theta : \Delta \to D_q = \langle r, s : r^q = s^2 = (sr)^2 = 1 \rangle
\]
be a surface-kernel epimorphism representing an action of \( G \) on \( S \in \mathcal{V}_g \), with \( \Delta \) canonically presented as the proof of Lemma 5.2.

Similarly as done in the proof of Proposition 2 we shall introduce some notation. We write \( m \in \{1, \ldots, q-1\} \) and \( n_j \in \{0, \ldots, q-1\} \) for \( j = 1, \ldots, 6 \) such that
\[
sr^{n_j} = \theta(x_j) \quad \text{for} \quad j = 1, \ldots, 6 \quad \text{and} \quad r^m = \theta(x_7).
\]
If \( n_j \neq 0 \) then we shall denote by \( m_j \) its inverse in the field of \( q \) elements. Also, the automorphism of \( D_q \) given by
\[
(r, s) \mapsto (r^\alpha, sr^\beta) \quad \text{for} \quad 1 \leq \alpha \leq q-1 \quad \text{and} \quad 0 \leq \beta \leq q-1
\]
is denoted by \( \phi_{\alpha, \beta} \).

For the sake of clearness, we shall restate here two basic claims which were stated and proved in the proof of Proposition 2.

**Claim 1.** If \( n_j = 1 \) and \( n_k = 0 \) for \( k < j \) then we can assume \( n_{j+1} = 0 \) or \( n_{j+1} = 1 \).

**Claim 2.** Up to equivalence, we can assume \( n_1 = n_2 = 0 \).

We shall proceed by studying separately the cases \( n_3 = 0 \) and \( n_3 \neq 0 \).

**Type 1.** Suppose \( n_3 = 0 \).

Assume \( n_4 = 0 \). If \( n_5 \neq 0 \) then we consider the automorphism \( \phi_{m_5,0} \) to notice that, up to equivalence, \( n_5 = 1 \). Thus, \( \theta \) is equivalent to either
\[
(s, s, s, s, sr^{u}, r^{-u}) \quad \text{or} \quad (s, s, s, s, sr^{v}, r^{1-v})
\]
where \( u \neq 0 \) and \( v \neq 1 \), according to \( n_5 = 0 \) or \( n_5 = 1 \). Note that in the first case, as \( u \neq 0 \), the epimorphism is equivalent to the one in which \( u = 1 \); namely, equivalent to
\[
(s, s, s, s, sr, r^{-1}) \quad \quad (6.3)
\]
Meanwhile, in the latter case, by Claim 2, the epimorphism is equivalent to
\[
(s, s, s, s, sr, r) \quad \text{and, in turn, equivalent to} \quad (s, s, s, s, sr^{-1}, r). \quad \quad (6.4)
\]

Now, the transformation \( \phi_{-1,0} \circ \Phi_5 \) provides an equivalence between (6.3) and (6.4).

Assume \( n_4 \neq 0 \). Then we consider the automorphism \( \phi_{m_4,0} \) to notice that, up to equivalence, \( n_4 = 1 \) and, consequently, by Claim 2, we have that \( n_5 = 0 \) or \( n_5 = 1 \). Thereby, \( \theta \) is equivalent to either
\[
(s, s, s, sr, sr^{u}, r^{-1-u}) \quad \text{or} \quad \theta_v = (s, s, s, sr, sr^{v}, r^{-v})
\]
where \( u \neq -1 \) and \( v \neq 0 \). The first case is equivalent to (6.4); indeed, we can consider the transformation \( \phi_{-1,0} \circ \Phi_4 \) to see that \( \theta \) is equivalent to
\[
(s, s, s, s, sr^{-u}, r^{1+u})
\]
and therefore, by Claim 2, we can assume \( u = 0 \). For the second case, consider \( \Phi_6 \circ \Phi_6 \) to notice that \( \theta_v \) and \( \theta_{-v} \) are equivalent. It follows that there are at most \( \frac{q-1}{2} \) pairwise non-equivalent actions given by
\[
(s, s, s, sr, sr^{v}, r^{-v}) \quad \text{for some} \quad v \in \{1, \ldots, \frac{q-1}{2}\}.
\]

**Type 2.** Suppose \( n_3 \neq 0 \). As before, consider the automorphism \( \phi_{m_3,0} \) to assume \( n_4 = 1 \). Moreover, again by Claim 2, we see that, up to equivalence, \( n_4 = 0 \) or \( n_4 = 1 \). However, we only need to consider the case \( n_4 = 1 \) due to the fact that, if \( n_4 = 0 \) then the transformation \( \phi_{-1,0} \circ \Phi_3 \) provides an equivalence between \( \theta \) and either (6.4) or some (6.5). Thus, we assume that \( n_4 = 1 \). If \( n_5 = 0 \) then the
transformation \( \Phi_4 \circ \Phi_5 \circ \phi_{-1,0} \) shows that the epimorphism is equivalent to either (6.4) or some (6.5). Then, we can assume \( n_5 \neq 0 \) and therefore the epimorphism is equivalent to one of the form
\[
\theta_{u,v} = (s, s, sr, sr, s, sr^{u-v}, r^v)
\]
where \( u, v \in \{1, \ldots, q-1\} \). Note that the powers of \( \Phi_5 \) provide the equivalences
\[
\theta_{u,v} \cong \theta_{u-\lambda, v} \quad \text{where} \quad \lambda \in \{1, \ldots, q-1\}.
\]
We now choose \( \lambda = \frac{u-v}{v} \) to conclude that \( \theta \) is equivalent to
\[
\theta_{v,v} = (s, s, sr, sr, s, s, sr^{v}) \quad \text{for some} \quad v \in \{1, \ldots, q-1\}.
\]
Now, consider \( \phi_{-1,0} \circ \Phi_3 \circ \Phi_4 \circ \Phi_5 \) to conclude that \( \theta_{v,v} \) is equivalent to (6.5).

All the above says that \( \theta \) is equivalent to either (6.3) or some (6.5). Hence \( \mathcal{V}_g \) consists of at most
\[
\frac{q-1}{2} + 1 = \frac{q+2}{4}
\]
equisymmetric strata, as claimed. \( \square \)

**Proposition 6.** Let \( g \geq 6 \) be an even integer such that \( \frac{g}{2} \) is odd. If \( S \in \mathcal{V}_g \) then the Jacobian variety \( JS \) decomposes, up to isogeny, as
\[
JS \sim A \times \Pi_{d \in \Omega(\frac{g}{2})} B_d^2,
\]
where \( A \) is an abelian surface and \( B_d \) is an abelian variety of dimension \( \varphi(\frac{g}{2}) \). Moreover
\[
A \sim JS_{\langle r \rangle} \quad \text{and} \quad \Pi_{d \in \Omega(\frac{g}{2})} B_d \sim JS_{\langle s \rangle},
\]
and therefore
\[
JS \sim JS_{\langle r \rangle} \times JS_{\langle s \rangle}^2.
\]

**Proof.** Let \( S \in \mathcal{V}_g \) with \( g \geq 6 \) and \( n = \frac{q}{2} \) odd. As noticed in the proof of Proposition 3 and keeping the same notations as in there, the non-trivial rational irreducible representations of \( D_n \) are
\[
\chi_2 \text{ and } W_d \text{ with } d \in \Omega(n)
\]
and therefore the group algebra decomposition of each \( JS \) with respect to \( G \) is given by
\[
JS \sim B_2 \times \Pi_{d \in \Omega(n)} B_d^2.
\]

The fact that all the involutions of \( D_n \) are conjugate allows us to ensure that the dimension of the factors \( B_2 \) and \( B_d \) in (6.6) does not depend on the equisymmetric stratum to which \( S \) belongs. So, we assume the action of \( G \) on \( S \) to be represented by the surface-kernel epimorphism \((s, s, s, s, rs, r)\).

Now, we consider the equation (2.8) to see that:
\[
\chi_2^{(s)} = \chi_2^{(rs)} = 0, \quad \chi_2^{(r)} = 1 \quad \text{and therefore} \quad \dim B_2 = -1 + \frac{1}{2}[6(1-0) + (1-1)] = 2,
\]
and for each \( d \in \Omega(n) \)
\[
\psi_d^{(s)} = \psi_d^{(rs)} = 1, \quad \psi_d^{(r)} = 0 \quad \text{and therefore} \quad \dim B_d = \frac{1}{2}\varphi(\frac{g}{2})[-2 + \frac{1}{2}(6(2-1) + (2-0))] = \varphi(\frac{g}{2}).
\]
In addition, we consider the induced isogeny (2.6) with \( H = \langle r \rangle \) and \( H = \langle s \rangle \) to see that
\[
B_2 \sim JS_{\langle r \rangle} \quad \text{and} \quad \Pi_{d \in \Omega(n)} B_d \sim JS_{\langle s \rangle}
\]
respectively, and therefore the proof follows after setting \( A = B_2 \). \( \square \)

**Remark 4.** We end this section by remarking two facts concerning the family \( \mathcal{V}_g \).

(1) The behavior for \( g = 4 \) is completely different. Indeed, Costa and the first author noticed in [14] (see also [4]) that the family \( \mathcal{V}_4 \) consists of two equisymmetric strata, represented by
\[
\theta_1 = (r, r, r, r, s, sr) \quad \text{and} \quad \theta_2 = (r, r, r, s, s, sr).
\]
By proceeding analogously as done in the proof of Proposition 6, one sees that if \( S \in \mathcal{V}_4 \) then:
(a) if $S$ belongs to the equisymmetric stratum defined by $\theta_1$ then

$$JS \sim A_1 \times A_2,$$

where $A_1 \sim JS_{(s)}$ and $A_2 \sim JS_{(sr)}$ are abelian surfaces, and

(b) if $S$ belongs to the equisymmetric stratum defined by $\theta_2$ then

$$JS \sim E_1 \times E_2 \times A,$$

where $E_1 \sim JS_{(r)}$ and $E_2 \sim JS_{(s)}$ are elliptic curves and $A \sim JS_{(sr)}$ is an abelian surface.

(2) As the reader could expect, if $n = \frac{g}{2}$ is even then Propositions 5 and 6 are not longer true. For instance, the equisymmetric stratum defined by the surface-kernel epimorphism

$$(r^2, r^2, s, s, s, rs, r)$$

(6.7)

is not equivalent to any of the actions determined in Proposition 5. Furthermore, if $S$ belongs to the equisymmetric stratum defined by (6.7) then, by proceeding analogously as in the proof of Proposition 6, one sees that

(a) if $n \equiv 0 \mod 4$ then $JS$ contains two elliptic curves, and

(b) if $n \equiv 2 \mod 4$ then $JS$ contains two elliptic curves and an abelian surface.

7. The families $\mathbb{U}_g^1$ and $\mathbb{U}_g^2$

**Proposition 7.** Let $g \geq 11$ be an odd integer such that $g − 1$ is twice a prime number. Then $\mathbb{U}_g^1$ and $\mathbb{U}_g^2$ are the unique complex four-dimensional families with $g − 1$ automorphisms.

**Proof.** Let $g \geq 11$ be an odd integer and write $g − 1 = 2q$ where $q \geq 5$ is a prime number. As the cyclic and dihedral group are the unique groups of order $2q$, we only need to verify that $(1; 2, 4, 2)$ is the only possible signature for the action of a group $G$ of order $2q$ on a complex-four dimensional family of compact Riemann surfaces of genus $1 + 2q$.

A short computation shows that if signature of the action is not $(1; 2, 4, 2)$ then it is $(h, l; m_1, \ldots, m_l)$ where either $(h, l) = (2, 1)$ or $(h, l) = (0, 7)$. It is straightforward to see that the former case is impossible. So, we assume the signature of action to be $(0; m_1, \ldots, m_7)$ where, by the Riemann-Hurwitz formula,

$$\sum_{j=1}^{7} \frac{1}{m_j} = 3.$$

As argued in the proof of Lemma 5.1 and 5.3, one sees that the number $v$ of periods $m_j$ that are different from $2$ are either two of three.

(1) If $v = 2$ then the signature of the action is $(0; 2, 4, 2, a, b)$ where $a, b \geq 3$ satisfy

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{2} \quad \text{and therefore} \quad a = b = 4 \quad \text{or} \quad a = 3, b = 6.$$

(2) If $v = 3$ then the signature of the action is $(0; 2, 4, 2, a, b, c)$ where $a, b, c \geq 3$ satisfies

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1 \quad \text{and therefore} \quad a = b = c = 3.$$

It follows that the order of the group is divisible by $3, 4$ or $6$. Thus, $q = 2$ or $3$ and therefore the genus equals $g = 5$ or $g = 7$; a contradiction.

**Remark 5.** The exceptional signatures appearing in the proof of the proposition above are realized for the unconsidered cases $g = 5$ and $7$. Indeed:

(1) For $g = 5$, in addition to the families $\mathbb{U}_5^1$ and $\mathbb{U}_5^2$, there is another complex four-dimensional family with action of $C_4$ with signature $(0; 2, 4, 2, 4, 4)$. See [4, Lemma 8] for more details.
(2) For $g = 7$, in addition to the families $\mathcal{U}^1_7$ and $\mathcal{U}^2_7$, there are three complex four-dimensional families with action of a group of order six. Concretely, two families with action of $C_6$ and signatures $(0; 2, . . . , 2, 3, 3, 3)$ and $(0; 2, . . . , 2, 3, 6)$ and a family with action of $D_3$ and signature $(0; 2, . . . , 2, 3, 3, 3)$

**Proposition 8.** Let $g \geq 3$ be an odd integer. The family $\mathcal{U}^1_g$ is equisymmetric.

**Proof.** For $g = 3$ we refer to [7, Table 5, 3.b]. Assume $g \geq 5$. Let $\Delta$ be a Fuchsian group of signature $(1; 2, . . . , 2)$ canonically presented as in the proof of Lemma 5.4 and let

$$\theta : \Delta \to G = \langle t : t^{g-1} = 1 \rangle$$

be a surface-kernel epimorphism representing an action of $G$ on $S \in \mathcal{U}^1_g$. We have to prove that $\theta$ is equivalent to the surface-kernel epimorphism (5.2) in the proof of Lemma 5.4. Clearly

$$\theta(x_j) = t^{\frac{n_j}{2}} \quad \text{for} \quad j = 1, 2, 3, 4.$$

In addition, if we write

$$\theta(\alpha_1) = t^u \quad \text{and} \quad \theta(\beta_1) = t^v$$

then as $\theta$ is surjective, without loss of generality, we can assume $u$ to be coprime to $g - 1$. Now, we choose $w \in \{1, \ldots, g - 2\}$ such that $uw \equiv 1 \mod g - 1$ and consider the automorphism of $G$ given by $t \mapsto t^w$ to see that $\theta$ is equivalent to

$$\alpha_1 \mapsto t, \quad \beta_1 \mapsto t^l, \quad x_1, x_2, x_3, x_4 \mapsto t^{\frac{w^l-1}{2}}$$

for some $l = uv \in \{1, \ldots, g - 1\}$.

Now, we consider the transformation $A_{l,-l}$ (see §2.3) to see that $\theta$ is equivalent to (5.2), as desired. \qed

In order to state the following proposition, we introduce the following notation. For each even integer $n \geq 2$ we write

$$\Lambda(n) = \{1 \leq d < \frac{n}{2} : d \text{ divides } n \text{ and } \frac{dn}{2} \neq 0 \mod n\}.$$

**Proposition 9.** Let $g \geq 3$ be an odd integer. If $S \in \mathcal{U}^1_g$ then the Jacobian variety $JS$ decomposes, up to isogeny, as follows.

1. If $\frac{g-1}{2}$ is even

$$JS \sim E \times \prod_{d \in \Lambda(g-1)} B_d$$

where $E$ is an elliptic curve isogenous to $JS_G$ and $B_d$ is an abelian variety of dimension $2\varphi(\frac{g-1}{2})$.

2. If $\frac{g-1}{2}$ is odd

$$JS \sim E \times A \times \prod_{d \in \Lambda(g-1)} B_d$$

where $A$ is an abelian surface and $E$ and $B_d$ are as before.

**Proof.** Set $n = g - 1$ and let $\omega$ be a primitive $n$-th root of unity. For each $0 \leq j \leq n - 1$, we denote by $\chi_j$ the complex irreducible representation of $G = \langle t : t^n = 1 \rangle = C_n$ defined as

$$\chi_j : t \mapsto \omega^j.$$

After a routine computation, one sees that the collection

$$\chi_d \quad \text{where} \quad 1 \leq d \leq \frac{n}{2} \quad \text{and} \quad d \text{ divides } n$$

yields a maximal collection of non-trivial rational irreducible representations of $G$, up to equivalence.

Let $B_d$ denote the factor associated to $\chi_d$ in the group algebra decomposition of $JS$ with respect to $G$. Clearly, $B_0$ is an elliptic curve isogenous to $JS_G$. In addition, we observe that

$$\dim B_d = \varphi(\frac{g}{2}) \cdot 4(1 - \chi_d^{(\frac{g}{2})})$$
and therefore $B_d = 0$ if and only if

$$\chi_j(t^{\frac{1}{m}}) = \omega^{c^{d}} = 1$$

or, equivalently $\frac{a_d}{2} \equiv 0 \mod n$.

Hence, the group algebra decomposition of $JS$ with respecto to $G$ is

$$JS \sim JS_G \times B_{\frac{n}{2}} \times \prod_{d \in \Lambda(n)} B_d$$

where, for each $d \in \Lambda(n)$, the dimension of $B_d$ is $2\nu(\frac{d}{2})$. Finally, as $\chi_{\frac{n}{2}}(t) = -1$ we see that

$$\dim B_{\frac{n}{2}} = \frac{1}{2} \cdot 4(1 - \chi_{\frac{n}{2}}(t)) = \begin{cases} 2 & \text{if } \frac{n}{2} \text{ is odd} \\ 0 & \text{if } \frac{n}{2} \text{ is even} \end{cases}$$

and therefore the proof follows after setting $E = B_0$ and $A = B_{\frac{n}{2}}$ when $\frac{n}{2}$ is odd. $\Box$

**Proposition 10.** Let $g \geq 5$ be an odd integer such that $\frac{g - 1}{2}$ is a prime number. Then the family $\mathcal{U}_g^2$ consists of at most two equisymmetric strata.

**Proof.** Set $q = \frac{g - 1}{2}$ and assume $q$ to be prime. Let $\Delta$ be a Fuchsian group of signature $(1; 2, \ldots, 2)$ canonically presented as in Lemma 5.4 and let

$$\theta : \Delta \to G = D_q = \langle r, s : r^q = s^2 = (sr)^2 = 1 \rangle$$

be a surface-kernel epimorphism representing an action of $G$ on $S \in \mathcal{U}_g^2$. We write

$$a = \theta(\alpha_1), \quad b = \theta(\beta_1) \quad \text{and} \quad sr^{n_i} = \theta(x_i)$$

where $n_i \in \{0, \ldots, q - 1\}$ for $i = 1, 2, 3, 4$ and, as before, we identify $\theta$ with the 6-tuple

$$(a, b; sr^{n_1}, sr^{n_2}, sr^{n_3}, sr^{n_4}).$$

**Claim.** Up to equivalence, we can assume $(a, b) = (1, 1)$ or $(a, b) = (1, r)$.

Note that there are four cases to consider; namely $(a, b)$ equals to either $(r^u, r^v)$, $(sr^u, r^v)$, $(r^u, sr^v)$ or $(sr^u, sr^v)$ for some $u, v \in \{0, \ldots, q - 1\}$.

First of all, note the third and fourth cases can be disregarded, since $A_{1,1} \circ A_{2,1}$ and $A_{1,1}$ respectively (see §2.3), transform them into the second case.

Assume that $a = r^u$ and $b = r^v$.

1. If $u = 0$ then, up to an automorphism, we can assume $(a, b) = (1, 1)$ or $(1, r)$.
2. If $u \neq 0$ and $\hat{u}$ is its inverse in the field of $q$ elements, then the transformation

$$\phi_{-u,1} \circ A_{2,1} \circ A_{1,-1} \circ A_{1,-u\hat{u}}$$

allows us to assume that, up to equivalence, $(a, b) = (1, r)$.

Assume that $a = sr^u$ and $b = r^v$.

1. If $v = 0$ then, up to an automorphism, we can assume $(a, b) = (s, 1)$.
2. If $v \neq 0$ and $\hat{v}$ is its inverse in the field of $q$ elements, then

$$\phi_{v,0} \circ A_{2,-v\hat{u}}$$

shows that, up to equivalence, we can assume $(a, b) = (s, r)$.

The proof of the claim follows after noticing that the cases $(s, r)$ and $(1, s)$ are equivalent to the first case under the action of the transformations $C_{1,4}$ and $C_{2,4}$ respectively.

If $(a, b) = (1, 1)$ then we can assume $n_1 = 0$ and $n_2 = 1$. Thus, $\theta$ is equivalent to

$$(1, 1; s, sr, sr^{n_3}, sr^{n_3 - 1}).$$
Now, we apply transformation $\Phi_3$ to see that $\theta$ is equivalent to
\[
(1, 1; s, sr, sr, s) \quad \text{and therefore equivalent to} \quad (1, 1; s, s, sr, sr).
\tag{7.1}
\]
If $(a, b) = (1, r)$ then we can assume $n_1 = 0$ and therefore $\theta$ is equivalent to
\[
(1, r; s, sr^{n_2}, sr^{n_3}, sr^{n_4})
\]
(1) If $n_2 = 0$ then $n_3 = n_4$. If follows that $\theta$ is equivalent to $(1, r; s, s, s, s)$ or to
\[
\theta_j := (1, r^j; s, sr, s, sr) \quad \text{for some } j \in \{1, \ldots, q - 1\}.
\]
The latter case is equivalent to $(7.1)$, since $C_{2,3} \circ C_{2,2}$ identifies $\theta_j$ with $\theta_{j-1}$.
(2) If $n_2 \neq 0$ then $\theta$ is equivalent to
\[
(1, r^j; s, sr^{n_3}, sr^{n_3 - 1}) \quad \text{for some } j \in \{1, \ldots, q - 1\}.
\]
In addition, the transformation $\Phi_3$ shows that $\theta$ is equivalent to
\[
(1, r^j; s, sr, s, s) \quad \text{and therefore equivalent to } \theta_j.
\]
All the above guarantees that there are at most two equisymmetric strata, represented by
\[
\theta_1 = (1, 1; s, s, sr, sr) \quad \text{or } \quad \theta_2 = (1, r; s, s, s, s).
\]
\]

**Proposition 11.** Let $g \geq 7$ be an odd integer such that $\frac{g-1}{2}$ is odd. If $S \in \mathcal{U}_g^2$ then the Jacobian variety $JS$ decomposes, up to isogeny, as
\[
JS \sim E \times A \times \prod_{d \in \Omega\left(\frac{g-1}{2}\right)} B_d^2,
\]
where $A$ is an abelian surface, $B_d$ is an abelian variety of dimension $\varphi\left(\frac{g-1}{2}\right)$ and $E$ is an elliptic curve isogenous to $JS_G$. Furthermore
\[
\text{Prym}(S(r) \rightarrow S_G) \sim A \quad \text{and} \quad \text{Prym}(S(s) \rightarrow S_G)\prod_{d \in \Omega\left(\frac{g-1}{2}\right)} B_d
\]
and therefore
\[
JS \sim JS_G \times \text{Prym}(S(r) \rightarrow S_G) \times \text{Prym}(S(s) \rightarrow S_G)^2.
\]

**Proof.** Set $n = \frac{g-1}{2}$ and assume that $n$ is odd. Similarly as noticed in the proof of Proposition 3(a) dimension of the factors arising in the group algebra decomposition of $JS$ does not depend on the equisymmetric stratum to which $S$ belongs. So, we assume the action to be represented by $(1, r; s, s, s, s)$. Now, keeping the same notation as before, the rational irreducible representations of $G$ are $\chi_1, \chi_2$ and $\psi_d$ with $d \in \Omega(n)$ and therefore
\[
JS \sim B_1 \times B_2 \times \prod_{d \in \Omega(n)} B_d^2,
\]
where $B_1 \sim JS_G$ is an elliptic curve. The fact that $\chi_2^{(s)} = 0$ and $\psi_d^{(s)} = 1$ for $d \in \Omega(n)$ imply that
\[
\dim B_2 = \frac{1}{2}(4(1 - 0)) = 2 \quad \text{and} \quad \dim B_d = \frac{1}{2}\varphi\left(\frac{g-1}{2}\right)\frac{1}{2}(4(2 - 1)) = \varphi\left(\frac{g}{2}\right).
\]
Now, we consider the induced isogeny (2.7) with $H_1 = \langle r \rangle$ and $H_2 = G$ to see that
\[
B_2 \sim \text{Prym}(S(r) \rightarrow S_G).
\]
Similarly, consider the induced isogeny (2.7) with $H_1 = \langle s \rangle$ and $H_2 = G$ to see that
\[
\prod_{d \in \Omega(n)} B_d \sim \text{Prym}(S(s) \rightarrow S_G),
\]
and the result follows by setting $E = B_1$ and $A = B_2$. 
\]
Remark 6. The isogeny decomposition of the Jacobian varieties $JS$ for $S \in \mathcal{H}^g_2$ differs from the stated in Proposition 11 for the case $g \geq 7$. Furthermore, the decomposition depends on the equisymmetric stratum to which $S$ belongs (due to the fact that the involutions of $D_2 \cong C_2^2$ are, of course, non-conjugate). Indeed

(1) If $S$ belongs to the equisymmetric stratum represented by $\theta_1$ then

$$JS \sim E_1 \times E_2 \times E_3 \times A,$$

where $E_1 \sim JS_G$, $E_2 \sim JS_{(s)}$, and $E_3 \sim JS_{(sr)}$ are elliptic curves and $A \sim JS_{(r)}$ is an abelian surface.

(2) If $S$ belongs to the equisymmetric stratum represented by $\theta_2$ then

$$JS \sim E \times A_1 \times A_2,$$

where $E \sim JS_G$ is an elliptic curve and $A_1 \sim JS_{(r)}$, $A_2 \sim JS_{(sr)}$ are abelian surfaces.

Acknowledgments. The authors are grateful to [38], and its developers, for generously provide a useful software which allows to perform helpful experiments, towards general results.

References

[1] R. Accola, *On the number of automorphisms of a closed Riemann surface*, Trans. Am. Math. Soc., 131 (1968), 398–408.
[2] P. Barraza and A. M. Rojas, *The group algebra decomposition of Fermat curves of prime degree*, Arch. Math. (Basel) 104 (2015), no. 2, 145–155.
[3] G. Bartolini, *On the Branch Loci of Moduli Spaces of Riemann Surfaces*. Linköping Studies in Science and Technology Dissertations, 1440, Linköping 2012.
[4] G. Bartolini, A. F. Costa, and M. Izquierdo, *On the orbifold structure of the moduli space of Riemann surfaces of genera four and five*. Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM 108 (2014), no. 2, 769–793.
[5] M. V. Belolipetsky and G. A. Jones, *Automorphism groups of Riemann surfaces of genus $p+1$, where $p$ is prime*. Glasg. Math. J. 47 (2005), no. 2, 379–393.
[6] Ch. Birkenhake and H. Lange, *Complex Abelian Varieties*, 2nd edition, Grundl. Math. Wiss. 302, Springer, 2004.
[7] S. A. Broughton, *Classifying finite groups actions on surfaces of low genus*, J. Pure Appl. Algebra 69 (1990), no. 3, 233–270.
[8] S. A. Broughton, *The equisymmetric stratification of the moduli space and the Krull dimension of mapping class groups*, Topology Appl. 37 (1990), no. 2, 101–113.
[9] E. Bujalance, A. F. Costa and M. Izquierdo, *On Riemann surfaces of genus $g$ with $4g$ automorphisms*, Topology and its Applications 218 (2017) 1–18.
[10] A. Carocca, H. Lange and R. E. Rodríguez, *Jacobians with complex multiplication*. Trans. Amer. Math. Soc. 363 (2011), no. 12, 6159–6175.
[11] A. Carocca and S. Reyes-Carocca, *Riemann surfaces of genus $1 + q^2$ with $3q^2$ automorphisms*, Preprint, arXiv: 1911.04310v1.
[12] A. Carocca, S. Recillas and R. E. Rodríguez, *Dihedral groups acting on Jacobians*, Contemp. Math. 311 (2011), 41–77.
[13] A. Carocca and R. E. Rodríguez, *Jacobians with group actions and rational idempotents*. J. Algebra 306 (2006), no. 2, 322–343.
[14] A. F. Costa and M. Izquierdo, *Equisymmetric strata of the singular locus of the moduli space of Riemann surfaces of genus 4*, Geometry of Riemann surfaces, 120–138, London Math. Soc. Lecture Note Ser., 368, Cambridge Univ. Press, Cambridge, 2010.
[15] A. F. Costa and M. Izquierdo, *One-dimensional families of Riemann surfaces of genus $g$ with $4g+4$ automorphisms*, RACSAM (2018) 112, 623–631.
[16] A. F. Costa, M. Izquierdo, and D. Ying, *On cyclic $p$-gonal Riemann surfaces with several $p$-gonal morphisms*. Geom. Dedicata 147 (2010), 139–147.
[17] R. Donagi and E. Markman, *Spectral covers, algebraically completely integrable, Hamiltonian systems, and moduli of bundles*, in: Integrable Systems and Quantum Groups, Montecatini Terme, 1993, in: Lecture Notes in Math., vol. 1620, Springer, Berlin, 1996, pp. 1–119.
[18] J. Harvey, *Cyclic groups of automorphisms of a compact Riemann surface*. Quarterly J. Math. 17, (1966), 86–97.
[19] J. Harvey, *On branch loci in Teichmüller space*, Trans. Amer. Math. Soc. 153 (1971), 387–399.
[20] R. A. Hidalgo, L. Jiménez, S. Quispe and S. Reyes-Carocca, Quasiplatonic curves with symmetry group $\mathbb{Z}_2^2 \rtimes \mathbb{Z}_m$ are definable over $\mathbb{Q}$, Bull. London Math. Soc. 49 (2017) 165–183.

[21] M. Izquierdo, L. Jiménez, A. Rojas, Decomposition of Jacobian varieties of curves with dihedral actions via equisymmetric stratification, Rev. Mat. Iberoam. 35, No. 4 (2019), 1259–1279.

[22] M. Izquierdo, G. A. Jones and S. Reyes-Carocca, Groups of automorphisms of Riemann surfaces, maps and hypermaps of genus $p + 1$ where $p$ is prime. Preprint, arXiv:2003.05017

[23] M. Izquierdo and S. Reyes-Carocca, A note on large automorphism groups of compact Riemann surfaces. J. Algebra 547 (2020), 1–21.

[24] R. S. Kulkarni, A note on Wiman and Accola-Maclachlan surfaces. Ann. Acad. Sci. Fenn., Ser. A 1 Math. 16 (1) (1991) 83–94.

[25] H. Lange and S. Recillas, Abelian varieties with group actions. J. Reine Angew. Mathematik, 575 (2004) 135–155.

[26] H. Lange and S. Recillas, Prym varieties of pairs of coverings. Adv. Geom. 4 (2004) 373–387.

[27] C. Maclachlan, A bound for the number of automorphisms of a compact Riemann surface, J. London Math. Soc. 44 (1969), 265–272.

[28] A. McBeath, The classification of non-euclidean crystallographic groups, Canad. J. Math. 19 (1966), 1192–1205.

[29] J. Paulhus and A. M. Rojas, Completely decomposable Jacobian varieties in new genera, Experimental Mathematics 26 (2017), no. 4, 430–445.

[30] S. Recillas and R. E. Rodríguez, Jacobians and representations of $S_3$, Aportaciones Mat. Investig. 13, Soc. Mat. Mexicana, México, 1998.

[31] S. Reyes-Carocca, Nilpotent automorphism group of families of Riemann surfaces, Preprint, arXiv:2004.06506

[32] S. Reyes-Carocca, On the one-dimensional family of Riemann surfaces of genus $g$ with $4q$ automorphisms, J. of Pure and Appl. Algebra 223, no. 5 (2019), 2123–2144.

[33] S. Reyes-Carocca, On compact Riemann surfaces of genus $g$ with $4g - 4$ automorphisms, To appear in: Israel Journal of Mathematics (2020), arXiv:1812.00705

[34] S. Reyes-Carocca and R. E. Rodríguez, A generalisation of Kani-Rosen decomposition theorem for Jacobian varieties, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 19 (2019), no. 2, 705–722.

[35] S. Reyes-Carocca and R. E. Rodríguez, On Jacobians with group action and coverings, Math. Z. (2020) 294, 209–227.

[36] J. Ries, The Prym variety for a cyclic unramified cover of a hyperelliptic curve, J. Reine Angew. Math. 340 (1983) 59–69.

[37] A. M. Rojas, Group actions on Jacobian varieties, Rev. Mat. Iber. 23 (2007), 397–420.

[38] SageMath, the Sage Mathematics Software System (Version 9.0), The Sage Developers, 2019, www.sagemath.org.

[39] J. P. Serre, Linear Representations of Finite Groups. Graduate Texts in Maths 42.

[40] F. Schottky and H. Jung, Neue Sätze über Symmetralfunctionen und die Abel’schen Funktionen der Riemann’schen Theorie, S.B. Akad. Wiss. (Berlin) Phys. Math. Kl. 1 (1909), 282–297.

[41] D. Singerman, Finitely maximal Fuchsian groups, J. London Math. Soc. (2) 6, (1972), 29–38.

[42] D. Singerman, Subgroups of Fuchsian groups and finite permutation groups, Bull. London Math. Soc. 2, (1970), 319–323.

[43] A. Wiman, Über die hyperelliptischen Curven und diejenigen von Geschlechte $p$ - Welche eindeutige Transformationen in sich zulassen. Bihang till K. Svenska Vet.-Akad. Handlingar, Stockholm 21 (1895-6) 1–28.

[44] W. Wirtinger, Untersuchungen über Theta Funktionen, Teubner, Berlin, 1895.