Broken Chiral Symmetry on a Null Plane

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ABSTRACT: On a null-plane (light-front), all effects of spontaneous chiral symmetry breaking are contained in the three Hamiltonians (dynamical Poincaré generators), while the vacuum state is a chiral invariant. This property is used to give a general proof of Goldstone’s theorem on a null-plane. Focusing on null-plane QCD with $N$ degenerate flavors of light quarks, the chiral-symmetry breaking Hamiltonians are obtained, and the role of vacuum condensates is clarified. In particular, the null-plane Gell-Mann-Oakes-Renner formula is derived, and a general prescription is given for mapping all chiral-symmetry breaking QCD condensates to chiral-symmetry conserving null-plane QCD condensates. The utility of the null-plane description lies in the operator algebra that mixes the null-plane Hamiltonians and the chiral symmetry charges. It is demonstrated that in a certain non-trivial limit, the null-plane operator algebra reduces to the symmetry group $SU(2N)$ of the constituent quark model.

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1 Introduction

Spontaneous symmetry breaking is usually treated as a phenomenon that arises from properties of an asymmetric quantum mechanical vacuum state. In particular, the non-invariance of the vacuum state with respect to a symmetry is said to lead to spontaneous symmetry breakdown. While this picture is clearly valid and useful, it is not generally appreciated that in relativistic theories of quantum mechanics, it is strictly a matter of convention which arises from the (usually implicit) choice of quantization surface [1]. Indeed, the standard viewpoint—the instant form—arises from choosing to view dynamics in Minkowski space as the evolution of families of parallel spaces at various instants of time. An alternate view of dynamics is to consider the evolution of families of parallel spaces tangent to the light cone; i.e. null planes [1–6]. In this viewpoint—the front form—the momentum operator has a spectrum confined to the open positive half-line and therefore the vacuum of the interacting theory may be regarded as the structureless Fock-space vacuum, which is an invariant with respect to all internal symmetries, and spontaneous symmetry breaking must be attributed to properties of the dynamical Poincaré generators. Therefore in the front form, spontaneous chiral symmetry breaking is a property of operators rather than of a complicated vacuum state. Naturally one expects that physics is independent of the choice of quantization surface. However, for theories like QCD where the detailed dynamics are largely intractable, one may suppose that the two forms of dynamics lead to distinct insights into the behavior of the theory at strong coupling. Our goal in this paper is to argue that this is indeed the case.

The fundamental point which we wish to emphasize in this paper is that, in contrast to the instant form, where spontaneous symmetry breaking lives entirely in the non-trivial vacuum, in the front form, symmetry breaking is expressed entirely through the fact that the Hamiltonians, or dynamical Poincaré generators, do not commute with the internal symmetry charges. The resulting commutation relations among space-time generators and internal symmetry generators in QCD imply powerful constraints on the spectrum and spin of the hadronic world [7–10]. There have been many studies of spontaneous chiral symmetry breaking on null planes [7, 8, 11–41]. In many cases the emphasis has been on learning detailed information about the dynamical mechanism of chiral symmetry breaking in QCD and in models. Here our approach is much less ambitious; we assume that chiral symmetry is broken spontaneously by complicated and not-well understood dynamics, and we then determine the constraints that follow from this assumption. In particular, we are interested primarily in formulating the model-independent consequences of chiral symmetry breaking on null-planes. A fundamental assumption we make is that physics must be independent of the choice of quantization surface. Nowhere in this study do we find anything resembling a contradiction of this basic assumption. Indeed, this assumption of what one might call “form invariance” leads to various constraints which reveal a great deal about the nature and consistency of chiral symmetry breaking on null-planes. On general grounds, the null-plane chiral symmetry charges annihilate the vacuum. Therefore, in order that spontaneous chiral symmetry breaking take place, the chiral symmetry axial-vector current on the null-plane cannot be conserved [11, 12]. This property leads to a simple proof of Goldstone’s
theorem on a null-plane, which is completely decoupled from any assumptions about the formation of symmetry-breaking condensates. A second consistency condition is that the part of the QCD vacuum energy that is dependent on the quark masses should be invariant with respect to the choice of coordinates. This condition recovers the Gell-Mann-Oakes-Renner relation [42] in the null-plane description, and leads to a general prescription for relating all instant-form chiral-symmetry breaking condensates to the vacuum expectation values of chiral singlet null-plane QCD operators.

It is difficult to find a general solution of the null-plane operator algebra [7–10]. However, there is a non-trivial limit in which a solution can be found. One expects that, in general, the chiral symmetry breaking part of the null-plane energy has an energy scale comparable to $\Lambda_{QCD}$ and therefore is not parametrically small. However, assuming that this is small (which is the case parametrically for baryon operators at large-$N_c$), while the chiral symmetry breaking part of the spin Hamiltonians is of natural size, allows a non-trivial solution of the operator algebra which closes to the Lie brackets of $SU(2N)$, thus recovering the basic group theoretical structure of the constituent quark model. This result, originally found by Weinberg [10] working with current-algebra sum rules in special Lorentz frames, is shown in this context to be a general consequence of the null-plane QCD Lie algebraic constraints which are valid in any Lorentz frame.

The paper is organized as follows. In section 2, the null-plane coordinates and conventions are introduced, and the front-form Poincaré algebra is obtained. The null-plane Hamiltonians and the Lie brackets that they satisfy are identified, and the momentum eigenstates are constructed. In section 3 the null-plane internal symmetry charges are introduced, and the commutators that mix Poincaré and chiral generators are obtained. Using these commutators, a general proof of Goldstone’s theorem is given, and a polology analysis is given which elucidates the structure of the axial-vector current on the null-plane. The special case of QCD with $N$ flavors of light quarks is considered in section 4. The QCD Lagrangian is expressed in the null-plane coordinates, and the chiral symmetry breaking Hamiltonians and the constraints that they satisfy are derived. The issue of condensates in the null-plane formulation is addressed in detail; the Gell-Mann-Oakes-Renner formula is recovered in the front-form and a general method for relating instant-form condensates to front-form condensates is presented. Section 5 explores the consequences of the QCD null-plane operator algebra. In particular, a simple solution of the operator algebra is given which contains the spin-flavor symmetries of the constituent quark model. In section 6 we summarize our findings and conclude.

Nota bene: We have made use of the many general reviews of null-plane (or light-front) quantization [43–55], as well as reviews that focus primarily on chiral symmetry related issues [26, 27, 56]. In order to provide a self-contained description of the subject of chiral-symmetry breaking on a null-plane, there is a significant amount of review material in this paper.
A null plane is a surface tangent to the light cone. The null-plane Hamiltonians map the initial light-like surface onto some other surface and therefore describe the dynamical evolution of the system. The energy $P^-$ translates the system in the null-plane time coordinate $x^+$, whereas the spin Hamiltonians $F^r$ rotate the initial surface about the surface of the light cone.

2 Space-time symmetry in the front form

2.1 A null plane defined

In the front-form of relativistic Hamiltonian dynamics, one chooses the initial state of the system to be on a light-like plane, or null-plane, which is a hypersurface of points $x$ in Minkowski space such that $x \cdot n = \tau$ (see fig. 1). Here $n$ is a light-like vector which will be chosen below, and $\tau$ is a constant which plays the role of time. We will refer to a null-plane as $\Sigma^\tau_n$. The subgroup of the Poincaré group that maps $\Sigma^\tau_n$ to itself is called the stability group of the null-plane and determines the kinematics within the null-plane. The remaining three Poincaré generators map $\Sigma^\tau_n$ to a new surface, $\Sigma^\tau_{n'}$, and therefore describe the evolution of the system in time. The front-form is special in that it has seven kinematical generators, the largest stability group of all of the forms of dynamics [1]. It stands to reason that in complicated problems in relativistic quantum mechanics one would prefer a formulation which has the fewest number of Hamiltonians to determine.
2.2 Choice of coordinates

Consider the light-like vectors \( n^\mu \) and \( n^*\mu \) which satisfy \( n^2 = n^*2 = 0 \) and \( n \cdot n^* = 1 \). Here we will choose these vectors such that

\[
n^\mu = \frac{1}{\sqrt{2}} (1, 0, 0, -1) , \quad n^*\mu = \frac{1}{\sqrt{2}} (1, 0, 0, 1) .
\]

We will take the initial surface to be the null-plane \( \Sigma_0 \). A coordinate system adapted to null-planes is then given by

\[
x^+ \equiv x \cdot n = \frac{1}{\sqrt{2}} (x^0 + x^3) , \quad x^- \equiv x \cdot n^* = \frac{1}{\sqrt{2}} (x^0 - x^3)
\]

which we take as the time variable and “longitudinal” position, respectively \(^1\). The remaining coordinates, \( x_\perp = (x^1, x^2) \) provide the “transverse” position. Denoting the null-plane contravariant coordinate four-vector by \( \tilde{x}^\mu = (x^+, x^1, x^2, x^-) = (x^+, x_\perp, x^-) \), then one can write

\[
\tilde{x}^\mu = C^\mu_{\nu} x^\nu .
\]

The matrix \( C^\mu_{\nu} \), given explicitly in Appendix A, allows one to transform all Lorentz tensors from instant-form to front-form coordinates. In particular, the null-plane metric tensor is given by

\[
\tilde{g}_{\mu\nu} = (C^{-1})^\alpha_{\mu} g_{\alpha\beta} (C^{-1})^\beta_{\nu} .
\]

The energy, canonical to the null-plane time variable \( x^+ \) is \( p^- = p_+ \), and the momentum canonical to the longitudinal position variable \( x^- \) is \( p^+ = p_- \). Therefore, the on-mass-shell condition for a relativistic particle of mass \( m \) yields the null-plane dispersion relation:

\[
p^- = \frac{p_1^2 + m^2}{2p^+} .
\]

This dispersion relation reveals several interesting generic features of the null-plane formulation. Firstly, the dispersion relation resembles the non-relativistic dispersion relation of a particle of mass \( p^+ \) in a constant potential. Secondly, we see that the positivity and finiteness of the null-plane energy of a free massive particle requires \( p^+ > 0 \). Only massless particles with strictly vanishing momentum can have \( p^+ = 0 \). This implies that pair production is subtle, and the vacuum state is in some sense simple, with the exception of contributions that are strictly from \( p^+ = 0 \) modes [43–49, 51, 52, 54, 55].

2.3 The null-plane Poincaré generators

In this section we will review the Lie brackets of the Lorentz generators in the front form \(^2\). The Poincaré algebra in our convention is:

\[
\begin{align*}
\left[ P^\mu , P^\nu \right] &= 0 , \\
\left[ M_{\mu\nu} , P_\rho \right] &= i \left( g_{\nu\rho} P_\mu - g_{\mu\rho} P_\nu \right) , \\
\left[ M_{\mu\nu} , M_{\rho\sigma} \right] &= i \left( g_{\mu\sigma} M_{\nu\rho} + g_{\nu\rho} M_{\mu\sigma} - g_{\mu\rho} M_{\nu\sigma} - g_{\nu\sigma} M_{\mu\rho} \right) .
\end{align*}
\]

\(^1\)This is known as the Kogut-Soper convention [5]. Our metric and other notational conventions can be found in Appendix A and in Ref. [48].

\(^2\)Here we follow closely the development of Refs. [4–6]. See also Ref. [57].
where $M_{ij} = \epsilon_{ijk}J_k$ and $M_{0i} = K_i$ with $J_i$ and $K_i$ the generators of rotations and boosts, respectively. Using $C_\nu^\mu$, we can transform from the instant-form to the front-form giving $\tilde{P}_\mu = (P^+, P^1, P^2, P^-)$, $\tilde{M}_{+} = -\tilde{M}_{-} = F_r$, $\tilde{M}_{r} = E_r$, $\tilde{M}_{rs} = \epsilon_{rs}J_3$, and $\tilde{M}_{+} = -\tilde{M}_{-} = K_3$; where we have defined

$$
P^\pm = \frac{1}{\sqrt{2}} (P^0 \pm P^3) \quad , \quad P^- = \frac{1}{\sqrt{2}} (P^0 - P^3) ;$$

$$
E_r = \frac{1}{\sqrt{2}} (K_r + \epsilon_{rs}J_s) \quad , \quad F_r = \frac{1}{\sqrt{2}} (K_r - \epsilon_{rs}J_s) . \quad (2.7)
$$

Here $P_+ = P^-$ is the null-plane energy while $P_- = P^+$ is the longitudinal momentum. (Note that the indices $r, s, t, \ldots$ are transverse indices that range over 1, 2. See Appendix A.)

It is straightforward to show that $P^+, P_r, K_3, E_r$, and $J_3$ are kinematical generators that leave the null plane $x^+ = 0$ intact. These seven generators form the stability group of the null plane. It is useful to classify the subgroups of the Poincaré algebra by considering the transformation properties of the generators with respect to longitudinal boosts, which serve to rescale the generators. Writing

$$
[K_3, A] = -i\gamma A \quad (2.8)
$$

where $A$ is a generator, one finds $E_r$ and $P^+$ have $\gamma = 1$, $J_3$, $K_3$ and $P_r$ have $\gamma = 0$, and $P^-$ and $F_r$ have $\gamma = -1$. The Poincaré generators have subgroups $G_\gamma$ labeled by $\gamma$, and there exist two seven-parameter subgroups $S_\pm$ with a semi-direct product structure $S_\pm = G_0 \times G_\pm$. Therefore the stability group coincides with the subgroup $S_+$. The non-vanishing commutation relations among these generators are:

$$
[K_3, E_r] = -iE_r \quad , \quad [K_3, P^+] = -iP^+ ;

[J_3, E_r] = i\epsilon_{rs}E_s \quad , \quad [J_3, P_r] = i\epsilon_{rs}P_s ;

[E_r, P_3] = -i\delta_{rs}P^+ . \quad (2.9)
$$

By contrast, $P^-$ and $F_r$ are the Hamiltonians which consist of the subgroup $G_- 1$; they are the dynamical generators which move physical states away from the $x^+ = 0$ surface (see fig. 1). The non-vanishing commutators among the stability group generators and the Hamiltonians are:

$$
[K_3, P^-] = iP^- \quad , \quad [E_r, P^-] = -iP_r ;

[K_3, F_r] = iF_r \quad , \quad [J_3, F_r] = i\epsilon_{rs}F_s ;

[P_r, F_3] = i\delta_{rs}P^- \quad , \quad [P^+, F_r] = iP_r ;

[E_r, F_3] = -i(\delta_{rs}K_3 + \epsilon_{rs}J_3) . \quad (2.10)
$$

This algebraic structure is isomorphic to the Galilean group of two-dimensional quantum mechanics where one identifies $\{ P^-, E_r, P_r, J_3, P^+ \}$ with the Hamiltonian, Galilean boosts, momentum, angular momentum, and mass, respectively. This isomorphism is responsible for the similarities between the front form and nonrelativistic quantum mechanics that we noted in the dispersion relation, and was originally noted in the context of the infinite momentum frame of instant-form dynamics $[2, 3]$ which has a similar dispersion relation.
2.4 Null-plane momentum states and reduced Hamiltonians

As momentum is a kinematical observable, it is convenient to work with momentum eigenstates, such that
\[
    P_r | p^+ , p_\perp \rangle = p_r | p^+ , p_\perp \rangle ; \quad (2.11)
\]
\[
    P^+ | p^+ , p_\perp \rangle = p^+ | p^+ , p_\perp \rangle . \quad (2.12)
\]
The action of the boosts on momentum states follows directly from the commutation relations in eq. 2.9 and is given by
\[
    e^{-i \nu_r E_r} e^{-i \omega K_3} | p^+ , p_\perp \rangle = | e^{i \omega} p^+ , p_\perp + p^+ v_\perp \rangle . \quad (2.13)
\]
One can then define the unitary boost operator
\[
    U(p^+, p_r) = e^{-i \beta_r E_r} e^{-i \beta_3 K_3} , \quad (2.14)
\]
with \( \beta_r \equiv p_r / p^+ \) and \( \beta_3 \equiv \log(\sqrt{2} p^+ / M) \) which boosts the state at rest to one with arbitrary momentum:
\[
    U(p^+, p_r) | M / \sqrt{2} , 0 \rangle = | p^+ , p_\perp \rangle . \quad (2.15)
\]
The action of the boosts on the momentum states is then easily found to be
\[
    E_r | p^+ , p_\perp \rangle = i p^+ \frac{d}{dp_r} | p^+ , p_\perp \rangle ; \quad (2.16)
\]
\[
    K_3 | p^+ , p_\perp \rangle = i p^+ \frac{d}{dp^+} | p^+ , p_\perp \rangle . \quad (2.17)
\]
Unitarity of the boost operators fixes the normalization of the momentum states up to a constant. We assume the covariant normalization:
\[
    \langle p^{+'} , p_\perp' | p^+ , p_\perp \rangle = (2\pi)^3 2 p^+ \delta(p^{+'} - p^+) \delta^2(p_\perp' - p_\perp) , \quad (2.18)
\]
and the corresponding completeness relation
\[
    1 = \int \frac{dp^+ dp_\perp}{(2\pi)^3 2 p^+} | p^+ , p_\perp \rangle \langle p^+ , p_\perp | . \quad (2.19)
\]
We can now find angular momentum operators, \( J_r \) and \( J_3 \), that are valid in any frame by boosting from an arbitrary momentum state to a state at rest, acting with the angular momentum generators \( J_r = \epsilon_{rs} (F_s - E_s) / \sqrt{2} \) and \( J_3 \), and then boosting back to the arbitrary momentum state. That is,
\[
    J_i | p^+ , p_\perp \rangle = U(p^+, p_r) J_i U^{-1}(p^+, p_r) | p^+ , p_\perp \rangle . \quad (2.20)
\]
Using this procedure one finds angular momentum operators that are valid in any frame:
\[
    J_3 = J_3 + \epsilon_{rs} E_r P_s (1 / P^+) ; \quad (2.21)
\]
\[
    J_r = \epsilon_{rs} \left[ P^+ F_s - P^- E_s + \epsilon_{st} P_t J_3 + P_s K_3 \right] (1 / M) . \quad (2.22)
\]
Inverting eq. 2.22 one then finds the following expressions for the Hamiltonians:

\[
\begin{align*}
P^- &= (1/2 P^+) \left[ P_1^2 + P_2^2 + M^2 \right]; \\
F_1 &= (1/P^+) \left[ - P_1 K_3 + P^- E_1 - P_2 J_3 - M J_2 \right]; \\
F_2 &= (1/P^+) \left[ - P_2 K_3 + P^- E_2 + P_1 J_3 + M J_1 \right].
\end{align*}
\]

A striking feature of the null-plane formulation is that the fundamental dynamical objects are the *products* \( M^2 \) and \( M J_r \), rather than the generators themselves. Following Ref. [6], we will refer to these objects as reduced Hamiltonians. The reduced Hamiltonians, together with \( J_3 \), commute with all kinematical generators and satisfy the algebra of \( U(2) \). This is conveniently demonstrated by making use of the Pauli-Lubanski vector

\[
W^{\mu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma},
\]

which satisfies \( W^{\mu} P_\mu = 0 \) and the non-trivial commutation relations:

\[
\begin{align*}
[M^{\mu\nu}, W_\rho] &= i \left( g_{\nu\rho} W_\mu - g_{\mu\rho} W_\nu \right); \\
[W^{\mu}, W^{\nu}] &= -i \epsilon^{\mu\nu\rho\sigma} W_\rho P_\sigma.
\end{align*}
\]

One then finds general, compact expressions for the angular momentum operators:

\[
J_3 = W^+/P^+, \quad M J_r = W_r - P_r W^+/P^+.
\]

By considering the commutation relations among \( W_\mu, P^{\mu} \) and \( M^{\mu\nu} \) one confirms that

\[
\begin{align*}
[J_3, M J_r] &= i \epsilon_{rs} M J_s, \\
[M J_r, M J_s] &= i \epsilon_{rs} M^2 J_3, \\
[M^2, M J_r] &= 0.
\end{align*}
\]

Hence, the reduced Hamiltonians together with the stability group generator \( J_3 \) satisfy the algebra of \( U(2) \), and the problem of finding a Lorentz invariant description of a relativistic quantum mechanical system is thus equivalent to finding a representation of the three reduced Hamiltonians which satisfy this algebra. Since the essence of Lorentz invariance resides in these Lie brackets, and they require knowledge of the reduced Hamiltonians, in theories with complicated dynamics like QCD, the formulation of the theory at weak coupling — where QCD is defined as a continuum quantum field theory — will lack manifest Lorentz invariance, which is tied up with the detailed dynamics of the theory, and is as complicated to achieve as finding the spectrum of the theory.

We can write a general momentum eigenstate as:

\[
|p^+, p^\perp; \lambda, n\rangle = |p^+, p^\perp\rangle \otimes |\lambda, n\rangle.
\]

Here \( n \) are additional variables that may be needed to specify the state of a system at rest, and \( \lambda \) is helicity, the eigenvalue of \( J_3 \):

\[
J_3 |p^+, p^\perp; \lambda, n\rangle = \lambda |p^+, p^\perp; \lambda, n\rangle,
\]

\footnote{Since the mass operator, \( M = \sqrt{p_\mu p^\mu} \), commutes with the spin operators, this algebra can clearly be expressed in the canonical form: \([J_i, J_j] = i \epsilon_{ijk} J_k\) and \([M, J_i] = 0\).}
and therefore, using eq. 2.21, we have

$$J_3 | p^+, p_\perp; \lambda, n \rangle = \left( \lambda + i \epsilon_s p_r \frac{d}{dp_s} \right) | p^+, p_\perp; \lambda, n \rangle ,$$

(2.31)

which completes the catalog of the action of the stability group generators on the momentum states. It is useful to write

$$| p^+, p_\perp; \lambda, n \rangle = U(p^+, p_r) M/\sqrt{2} | 0 \rangle \equiv a_n^\dagger | p^+, p_\perp; \lambda \rangle ,$$

(2.32)

where $a_n^\dagger$ is an operator that creates the momentum state when acting on the null-plane vacuum, $| 0 \rangle$. What is special about the null-plane description is that the kinematical generators (with the exception of $J_3$) act on states in a manner independent of the inner variables $n$. And the reduced Hamiltonians act exclusively on the inner variables in a manner independent of the momentum. Therefore, one may view the Poincaré algebra by the direct sum of $K$ and $D$, where $K = \{ E_r, P_r, K_3, P^+ \}$ contains all stability group generators with the exception of $J_3$ which is grouped with the reduced Hamiltonians, $D = \{ J_3, M J_r, M^2 \}$ [58].

The structure of the Poincaré algebra in the front-form is well suited to the study of systems with complicated dynamics like QCD, as the dynamical generators are directly related to the most important observable quantities, namely the energy and the angular momentum of the system, while momenta and boosts are purely kinematical and therefore are easy to implement. The reduced Hamiltonians will have a fundamental role to play in the description of chiral symmetry breaking on null planes.

3 Chiral symmetry in the front form

3.1 Null plane charges and the chiral algebra

Consider a Lagrangian field theory that has an $SU(N)_R \otimes SU(N)_L$ chiral symmetry. Let us assume that this system has a null-plane Lagrangian formulation which allows one, by the standard Noether procedure, to obtain the currents $\tilde{J}_\mu(x)$ and $\tilde{J}_5(x)$, which are related to the symmetry currents via $\tilde{J}_\mu^{\alpha} = (\tilde{J}_\mu^{\alpha} - \tilde{J}_5^{\alpha})/2$ and $\tilde{J}_5^{\alpha} = (\tilde{J}_\mu^{\alpha} + \tilde{J}_5^{\alpha})/2$. We will further assume that the Lagrangian contains an operator that explicitly breaks the chiral symmetry in the pattern $SU(N)_R \otimes SU(N)_L \rightarrow SU(N)_F$ and is governed by the parameter $\epsilon$ such that as $\epsilon \rightarrow 0$, the symmetry is restored at the classical level. The general relation between currents and their associated charges is given by

$$Q(n \cdot x) = \int d^4 y \delta(n \cdot (x - y)) n \cdot J(y) ,$$

(3.1)

where the vector $n_\mu$ selects the initial quantization surface, which we take to be the null plane $\Sigma_n^\mu$. Therefore, the null-plane chiral symmetry charges are

$$\tilde{Q}_\alpha = \int d x^- d^2 x_\perp \tilde{J}_\alpha(x^-) ;$$

(3.2)

$$\tilde{Q}_5(x^+) = \int d x^- d^2 x_\perp \tilde{J}_5(x^-, x^+) ;$$

(3.3)

4By contrast, in the instant form of dynamics, the energy and the boosts are dynamical. As boosts are not among the observables, one refers only to the one Hamiltonian corresponding to energy.
where the axial charges have been given explicit null-plane time dependence as they are not conserved due to the explicit breaking operator in the Lagrangian. These charges satisfy the $SU(N)_{R} \otimes SU(N)_{L}$ chiral algebra,

$$[\tilde{Q}^{\alpha}, \tilde{Q}^{\beta}] = i f^{\alpha\beta\gamma} \tilde{Q}^{\gamma} ; \quad [\tilde{Q}^{\alpha}_{5}(x^{+}), \tilde{Q}^{\beta}] = i f^{\alpha\beta\gamma} \tilde{Q}^{\gamma}_{5}(x^{+}) ;$$

(3.4)

$$[\tilde{Q}^{\alpha}_{5}(x^{+}), \tilde{Q}^{\beta}_{5}(x^{+})] = i f^{\alpha\beta\gamma} \tilde{Q}^{\gamma}_{5} .$$

(3.5)

We further assert that both types of chiral charges annihilate the vacuum. That is,

$$\tilde{Q}^{\alpha} |0\rangle = \tilde{Q}^{\alpha}_{5} |0\rangle = 0 .$$

(3.6)

This is the statement that the front-form vacuum is invariant with respect to the full $SU(N)_{R} \otimes SU(N)_{L}$ symmetry. In particular, this implies that there can be no vacuum condensates that break $SU(N)_{R} \otimes SU(N)_{L}$ on a null-plane. This may seem to be an odd assumption, since the chiral charge is directly related to the axial-vector current through eq. 3.3, and in general one would expect that this current has a Goldstone boson pole contribution, in turn implying that the chiral charges acting on the vacuum state excite massless Goldstone bosons. Below we will confirm the assertion, eq. 3.6, by using standard current-algebra polology to show that indeed the Goldstone boson pole contribution to the null-plane axial-vector current is absent.

### 3.2 Symmetries of the reduced Hamiltonians

Mixed commutators among the Poincaré generators and internal symmetry generators can be expressed generally as [13]:

$$[Q_{\alpha}(n \cdot x), P^{\mu}] = -i n^{\mu} \int d^{4}y \delta(n \cdot (x - y)) \partial_{\nu} J^{\nu}_{\alpha}(y) ;$$

(3.7)

$$[Q_{\alpha}(n \cdot x), M^{\mu\nu}] = i \int d^{4}y \delta(n \cdot (x - y)) (n^{\mu} y^{\nu} - n^{\nu} y^{\mu}) \partial_{\kappa} J^{\kappa}_{\alpha}(y) .$$

(3.8)

From these expressions one then obtains the mixed commutator between the Pauli-Lubanski vector and the internal symmetry charges:

$$[Q_{\alpha}(n \cdot x), W_{\nu}] = \frac{i}{2} \varepsilon_{\nu\lambda\rho\sigma} \int d^{4}y \delta(n \cdot (x - y)) \left[ M^{\lambda\sigma} n^{\rho} - \left( n^{\lambda} y^{\rho} - n^{\rho} y^{\lambda} \right) P^{\sigma} \right] \partial_{\nu} J^{\sigma}_{\alpha}(y).$$

(3.9)

Using these expressions, one finds the commutation relations between null-plane chiral charges and the reduced Hamiltonians:

$$[\tilde{Q}^{\alpha}_{5}(x^{+}), M^{2}] = -2i P^{+} \int dx^{-} d^{2}x_{\perp} \partial_{\mu} \tilde{J}^{\mu}_{\alpha}(x^{-}, \vec{x}_{\perp}, x^{+}) ;$$

(3.10)

$$[\tilde{Q}^{\alpha}_{5}(x^{+}), M J_{r}] = i \varepsilon_{rs} P^{+} \int dx^{-} d^{2}x_{\perp} \Gamma_{s} \partial_{\mu} \tilde{J}^{\mu}_{\alpha}(x^{-}, \vec{x}_{\perp}, x^{+}) ,$$

(3.11)

where $\Gamma_{s} \equiv E_{s} - P^{+} x_{s}$. Here and in what follows, we are assuming that $SU(N)_{F}$ is unbroken and therefore $\partial_{\mu} \tilde{J}^{\mu}_{\alpha} = 0$ and the reduced Hamiltonians commute with the $SU(N)_{F}$ charges:

$$[\tilde{Q}_{\alpha}, M^{2}] = [\tilde{Q}_{\alpha}, M J_{r}] = 0 .$$

(3.12)
3.3 Goldstone’s theorem on a null plane

In the instant form, a symmetry has three possible fates in the quantum theory: the symmetry remains exact and the current is conserved, the symmetry is spontaneously broken and again the current is conserved, or the symmetry is anomalous and the current is not conserved. The front form realizes a fourth possibility: the symmetry is spontaneously broken and the associated current is not conserved. This fourth possibility is necessary in the front form because the vacuum is invariant with respect to all internal symmetries. In general, we can write

$$\partial_\mu \tilde{J}_\alpha^\mu(x^-, \vec{x}_\perp, x^+) = \epsilon \chi \tilde{P}_\alpha(x^-, \vec{x}_\perp, x^+) ,$$

where $\epsilon \chi$ is the parameter that gauges the amount of chiral symmetry breaking that is present in the Lagrangian. Using the short hand,

$$| h \rangle \equiv | p^+, \vec{p}_\perp; \lambda, h \rangle ,$$

for the momentum eigenstates, we take the matrix element of eq. 3.10 between momentum eigenstates, which gives

$$\langle h' | [ \tilde{Q}_\alpha^5(x^+), M^2 ] | h \rangle = -2i p^+ \epsilon \chi \int dx^- d^2 \vec{x}_\perp \langle h' | \tilde{P}_\alpha(x^-, \vec{x}_\perp, x^+) | h \rangle ;$$

If the right hand side of this equation vanishes for all $h$ and $h'$, then there can be no chiral symmetry breaking of any kind. Therefore, in order that the chiral symmetry be spontaneously broken, the chiral current cannot be conserved and we have the following constraint [22, 26] in the limit $\epsilon \chi \to 0$:

$$\int dx^- d^2 \vec{x}_\perp \langle h' | \tilde{P}_\alpha(x^-, \vec{x}_\perp, x^+) | h \rangle \to \frac{1}{\epsilon \chi} + \ldots ,$$

where the dots represent other possible terms that are non-singular in the limit $\epsilon \chi \to 0$. Now we will show that this condition implies the existence of $N^2 - 1$ Goldstone bosons $^5$. We will assume that $\tilde{P}_\alpha$ is an interpolating operator for Lorentz-scalar fields $\phi_i^\alpha$, and therefore we can write

$$\tilde{P}_\alpha(x) = \sum_i Z_i \phi_i^\alpha(x)$$

where the $Z_i$ are overlap factors. Using the reduction formula we relate the matrix elements of field operators between physical states to transition amplitudes. Of course here it is understood that there is no selection rule which would forbid these transitions. The S-matrix element for the transition $h(p) \to h'(p') + \phi_i^\alpha(q)$ can be defined by

$$\langle h'; \phi_i^\alpha(q) | S | h \rangle \equiv i(2\pi)^4 \delta^4(p - p' - q) \mathcal{M}_i^\alpha(p', \lambda', h'; p, \lambda, h)$$

$$= i \int d^4 x e^{-iq\cdot x} \left( -q^2 + M_i^2 \right) \langle h' | \phi_i^\alpha(x) | h \rangle$$

$^5$Note that if we took eq. 3.16 as a constraint on the operator $\tilde{P}_\alpha$ rather than on its matrix elements, then this constraint would be viewed as a constraint on the zero-modes of the operator [22, 26]. Here we work entirely with matrix elements.
where $\mathcal{M}_\alpha^i$ is the Feynman amplitude and in the second line we have used the reduction formula. It then follows that

$$
\langle h' | \phi^i_\alpha(x) | h \rangle = -e^{iq\cdot x} \frac{1}{q^2 - M^2_{\phi^i}} \mathcal{M}_\alpha^i(q).
$$

(3.19)

Using this formula together with eq. 3.17 in eq. 3.15 then gives

$$
\langle h' | [\tilde{Q}_\alpha^5(x^+), M^2] | h \rangle = 2i \not{p}^+ (2\pi)^3 \delta(q^+ + \vec{q}_\perp) e^{ix^+ + \vec{q}_\perp} \sum_i \frac{\epsilon_{\chi} Z_i}{q^2 - M^2_{\phi^i}} \mathcal{M}_\alpha^i(q)
$$

$$
= -2i \not{p}^+ (2\pi)^3 \delta(q^+ + \vec{q}_\perp) e^{ix^+ + \vec{q}_\perp} \sum_i \frac{\epsilon_{\chi} Z_i}{M^2_{\phi^i}} \mathcal{M}_\alpha^i(q^-),
$$

(3.20)

where in the second line we have used the momentum delta functions. In order that the right hand side not vanish in the symmetry limit, there must be at least one field $\phi^i_\alpha$ whose mass-squared vanishes proportionally to $\epsilon_{\chi}$ as $\epsilon_{\chi} \to 0$. We will denote this field as $\pi^\alpha \equiv \phi^i_\alpha$ with

$$
M^2_{\pi} = c_p \epsilon_{\chi},
$$

(3.21)

where $c_p$ is a constant of proportionality. There are therefore $N^2 - 1$ massless fields $\pi^\alpha$ in the symmetry limit, which we identify as the Goldstone bosons. It is noteworthy that this proof relies entirely on physical matrix elements; i.e. there is no need to assume the existence of a vacuum condensate that breaks the chiral symmetry. Of course, in instant-form QCD, we know that the proportionality constant in eq. 3.21 contains the quark condensate. This issue will be resolved below in section 4. While we have carried out this proof in the case of $SU(N)_R \otimes SU(N)_L$ broken to $SU(N)_V$, it is clearly easily generalized to other systems.

We can now write $\tilde{P}_\alpha = \bar{Z} \pi^\alpha + \ldots$ where the dots represent non-Goldstone boson fields, and

$$
\langle h' | \partial_\mu \tilde{J}_{\alpha\alpha}^\mu(x) | h \rangle = \langle h' | \bar{Z} M^2_{\pi} \pi^\alpha(x) | h \rangle,
$$

(3.22)

where $\bar{Z} \equiv Z/c_p$. Here, as in the usual current algebra manipulations, we have assumed that only the Goldstone bosons couple to the axial-vector current, and it is now a standard exercise to determine the overlap factor. First, define the Goldstone-boson decay constant, $F_\pi$, via

$$
\langle 0 | \tilde{J}_{\alpha\alpha}^\mu(x) | \pi_\beta \rangle \equiv -i p^\mu \delta_{\alpha\beta} e^{ip\cdot x},
$$

(3.23)

where $| \pi_\beta \rangle \equiv | p^+, \vec{p}_\perp; 0, \pi_\beta \rangle$. Taking the divergence of the current and raising eq. 3.22 to an operator relation yields

$$
\langle 0 | \bar{Z} M^2_{\pi} \pi^\alpha(x) | \pi_\beta \rangle = F_\pi M^2_{\pi} \delta_{\alpha\beta} e^{ip\cdot x},
$$

(3.24)

The normalization of the Goldstone-boson field,

$$
\langle 0 | \pi^\alpha(x) | \pi_\beta \rangle = \delta_{\alpha\beta} e^{ip\cdot x},
$$

(3.25)
then gives $\tilde{Z} = F_\pi$ and we recover the standard operator relation
\[ \partial_\mu \tilde{J}_{5\alpha}^\mu (x) = F_\pi M_\pi^2 \pi_\alpha(x). \tag{3.26} \]
We can now express the mixed Lie bracket, eq. 3.20, as
\[ \langle h' | [\tilde{Q}_5^\alpha (x^+), M^2] | h \rangle = -2i p^+ (2\pi)^3 \delta(q^+) \delta^2(q_\perp) e^{ix^+q^-} F_\pi M_\pi(q^-), \tag{3.27} \]
where here $M_\alpha(q^-)$ is the Feynman amplitude for the transition $h(p) \rightarrow h'(p') + \pi_\alpha(q)$. We see that while the chiral current is not conserved, its divergence is proportional to an S-matrix element. Noting that
\[ \langle h' | [\tilde{Q}_5^\alpha (x^+), M^2] | h \rangle = 2 p^+ q^- \langle h' | \tilde{Q}_5^\alpha (x^+) | h \rangle = -2p^+ \langle h' | i \frac{d}{dx^+} \tilde{Q}_5^\alpha (x^+) | h \rangle, \tag{3.28} \]
and from the definition of the chiral charge, eq. 3.3,
\[ \langle h' | \tilde{Q}_5^\alpha (x^+) | h \rangle = (2\pi)^3 \delta(q^+) \delta^2(q_\perp) e^{ix^+q^-} \langle h' | \tilde{J}_{5\alpha}^\mu (0) | h \rangle, \tag{3.29} \]
one finds, using eq. 3.27,
\[ M_\alpha(q^-) = \frac{i q^-}{F_\pi} \langle h' | \tilde{J}_{5\alpha}^\mu (0) | h \rangle, \tag{3.30} \]
or, in Lorentz-invariant form,
\[ M_\alpha(q) = \frac{i q_\mu}{F_\pi} \langle h' | \tilde{J}_{5\alpha}^\mu (0) | h \rangle, \tag{3.31} \]
which is the standard current-algebra result. In order to confirm some of these properties in a better-known fashion, and to address the assumption we have made that the chiral charges annihilate the vacuum, we will now consider current algebra polology on the null-plane.

### 3.4 Polology and the chiral invariant vacuum

Our starting point is the matrix element between hadronic states $h$ and $h'$ of the axial-vector current, which can be written in a general way as \[59, 60\]
\[ \langle h' | \tilde{J}_{5\alpha}^\mu (0) | h \rangle = \frac{i F_\pi q^\mu}{q^2 - M_\pi^2} M_\alpha + \langle h' | \tilde{J}_{5\alpha}^\mu (0) | h \rangle_N \tag{3.32} \]
where as before $q = p - p'$. Using translational invariance, we have
\[ \langle h' | \tilde{J}_{5\alpha}^\mu (x) | h \rangle = e^{iqx} \langle h' | \tilde{J}_{5\alpha}^\mu (0) | h \rangle. \tag{3.33} \]
It follows that
\[ \langle h' | \partial_\mu \tilde{J}_{5\alpha}^\mu (x) | h \rangle = i q_\mu \langle h' | \tilde{J}_{5\alpha}^\mu (x) | h \rangle = e^{iqx} \left[ -\frac{F_\pi q^2}{q^2 - M_\pi^2} M_\alpha + i q_\mu \langle h' | \tilde{J}_{5\alpha}^\mu (0) | h \rangle_N \right], \tag{3.34} \]
Figure 2. Above shows the standard instant-form pololoy; the matrix element of the chiral current has a Goldstone-boson pole piece, and a non-pole piece. These two contributions cancel in the symmetry limit ensuring a conserved chiral current. Below shows the standard front-form pololoy; the Goldstone-boson pole contribution is absent and therefore the current is not conserved but rather has a divergence which is proportional to the matrix element for the emission or absorption of a Goldstone boson.

and using

\[ \langle h' \mid \partial_\mu \tilde{J}_{5a}^\mu(x) \mid h \rangle = \langle h' \mid F_\pi M_\pi^2 \pi_\alpha(x) \mid h \rangle, \]  

(3.35)

and the reduction formula, eq. 3.19, reproduces eq. 3.31. Note that in null-plane coordinates eq. 3.32 gives

\[ \langle h' \mid \tilde{J}_{5a}^+ (0) \mid h \rangle = \frac{iF_\pi q^+}{2q^+ q^- - \bar{q}_\perp^2 - M_\pi^2} \mathcal{M}_\alpha + \langle h' \mid \tilde{J}_{5a}^+ (0) \mid h \rangle_N. \]  

(3.36)

We therefore have

\[ \lim_{q^+ \to 0} \langle h' \mid \tilde{J}_{5a}^+ (0) \mid h \rangle = \langle h' \mid \tilde{J}_{5a}^+ (0) \mid h \rangle_N. \]  

(3.37)

By comparing with eq. 3.29, it is clear that the null-plane chiral charges, by construction, do not excite the Goldstone boson states. The property, eq. 3.6, of vacuum annihilation which we assumed above, is therefore a general property of the null-plane chiral charges.

Again consider the space-integrated current divergence in the front-form, but now using eq. 3.34. One finds

\[ \int dx^+ d^2x_\perp \langle h' \mid \partial_\mu \tilde{J}_{5a}^\mu(x) \mid h \rangle = (2\pi)^3 \delta(q^+) \delta^2(\bar{q}_\perp) e^{ix^+ q^-} \langle h' \mid \partial_\mu \tilde{J}_{5a}^\mu (0) \mid h \rangle 
\]

\[ = (2\pi)^3 \delta(q^+) \delta^2(\bar{q}_\perp) e^{ix^+ q^-} \left[ \frac{-F_\pi (2q^+ q^- - \bar{q}_\perp^2)}{2q^+ q^- - \bar{q}_\perp^2 - M_\pi^2} \mathcal{M}_\alpha + i q_\alpha \langle h' \mid \tilde{J}_{5a}^+ (0) \mid h \rangle_N \right] 
\]

\[ = (2\pi)^3 \delta(q^+) \delta^2(\bar{q}_\perp) e^{ix^+ q^-} i q_\alpha \langle h' \mid \tilde{J}_{5a}^+ (0) \mid h \rangle_N 
\]

\[ = (2\pi)^3 \delta(q^+) \delta^2(\bar{q}_\perp) e^{ix^+ q^-} F_\pi \mathcal{M}_\alpha(q^-) \],  

(3.38)
where in the third line the momentum delta functions have been used, and in the last line we have used eq. 3.31 and eq. 3.37. Now using eq. 3.10, we see that we have recovered eq. 3.27. In this derivation we see explicitly that the Goldstone-boson pole does not contribute to the divergence of the axial-current. It is for this reason that the current cannot be conserved. For purposes of comparison, recall that in the instant form, one has

\[
\int d^3x \langle h' \mid \partial_\mu J^\mu_{5a}(x) \mid h \rangle = (2\pi)^3 \delta^3(\vec{q}) \langle h' \mid \partial_\mu, J^\mu_{5a}(0) \mid h \rangle ;
\]

\[
= (2\pi)^3 \delta^3(\vec{q}) \left[ -\frac{F_\pi \vec{q}_0^2}{q_0^2 - M^2_\pi} \mathcal{M}_\alpha + i q_\mu \langle h' \mid J^\mu_{5a}(0) \mid h \rangle_N \right] ;
\]

\[\xrightarrow{M_\pi \to 0} (2\pi)^3 \delta^3(\vec{q}) \left[ -F_\pi \mathcal{M}_\alpha + i q_\mu \langle h' \mid J^\mu_{5a}(0) \mid h \rangle_N \right] ;
\]

\[= 0 ,
\]

(3.39)

where in the last line, eq. 3.31 has once again been used. Here there is a cancellation between the pole and non-pole parts of the matrix element which ensure that the integrated current divergence vanishes in the chiral limit. This analysis, which is expressed pictorially in fig. 2, suggests that the front-form and instant-form axial-vector currents are related, at the operator level, through

\[
\tilde{J}^\mu_{5a} = J^\mu_{5a} - (J^\mu_{5a})_{GB \text{ pole}}
\]

(3.40)

where the second term on the right is the purely Goldstone-boson pole part of the axial-vector current. We will see that this peculiar realization of chiral symmetry does indeed emerge in QCD.

It is useful to define new objects which give a matrix-element representation of the internal-symmetry charges [7, 8]:

\[
\langle h' \mid \tilde{Q}^5_\alpha(x^+) \mid h \rangle = (2\pi)^3 2p^+ \delta(q^+) \delta^2(\vec{q}_\perp) [X_\alpha(\lambda)]_{h'h} \delta_{\lambda'\lambda} ;
\]

(3.41)

\[
\langle h' \mid \tilde{Q}_\alpha \mid h \rangle = (2\pi)^3 2p^+ \delta(q^+) \delta^2(\vec{q}_\perp) [T_\alpha]_h \delta_{hh'} \delta_{\lambda'\lambda} .
\]

(3.42)

These definitions are particularly useful as they allow the preservation of the Lie-algebraic structure of the operator algebra in the case where correlation functions are given purely by single-particle states. The matrix element for Goldstone boson emission and absorption is:

\[
\mathcal{M}_\alpha(p', \lambda', h'; p, \lambda, h) = \frac{i}{F_\pi} (M^2_h - M^2_{h'}) [X_\alpha(\lambda)]_{h'h} \delta_{\lambda'\lambda} .
\]

(3.43)

As one might expect, in the limit that chiral symmetry is restored through a second-order phase transition, the matrix \([X_\alpha(\lambda)]_{h'h}\) becomes a true symmetry generator [61]. In this limit, one also expects that the states \(h'\) and \(h\) become degenerate. In order that the matrix element of eq. 3.43 not vanish in this limit, \(F_\pi\) must approach zero in the symmetry limit in precisely the same way [61]. The role of \(F_\pi\) as an order parameter of chiral symmetry breaking is then apparent in eq. 3.27, as the mixed-Lie bracket vanishes as \(F_\pi \to 0\). Therefore, \(F_\pi\) is an order parameter of chiral symmetry breaking on the null-plane.
3.5 Broken chiral symmetry and spin

Using the results of the previous two sections one finds

\[
\langle h', \lambda' | \tilde{Q}_\alpha^5(x^+) , M^2 | h , \lambda \rangle = \delta_{\lambda',\lambda} (2\pi)^3 2 p^+ \delta(q^+) \delta^2(\tilde{q}_\perp) \left( M_h^2 - M_h^2 \right) [ X_\alpha(\lambda) ]_{h/k} \quad (3.44)
\]

and

\[
\langle h', \lambda' | \tilde{Q}_\alpha^5(x^+) , M J_\pm | h , \lambda \rangle = \delta_{\lambda',\lambda \pm 1} (2\pi)^3 2 p^+ \delta(q^+) \delta^2(\tilde{q}_\perp) \times \left[ M_h c_{ \pm}(h, \lambda) [ X_\alpha(\lambda \pm 1) ]_{h/k} - M_h c_{ \mp}(h', \lambda') [ X_\alpha(\lambda) ]_{h/k} \right]. \quad (3.45)
\]

where \( J_\pm \equiv J_1 \pm i J_2 \) and \( c_{ \pm}(h, \lambda) \equiv \sqrt{J_h (J_h + 1)} - \lambda(\lambda \pm 1) \). Eq. 3.45 has been obtained by a direct evaluation of the left-hand side using the usual angular momentum ladder relations and eq. 3.41. Written in this form, it is clear that in the presence of spontaneous symmetry breaking, the mixed Lie brackets between the reduced Hamiltonians and the chiral charge are directly related to Goldstone-boson transition amplitudes and are non-vanishing in the symmetry limit. The spin reduced Hamiltonians imply constraints on Goldstone-boson transitions that change the helicity by one unit.

An important consequence of eqs. 3.44 and 3.45 which will prove useful below is that chiral symmetry breaking remains relevant even when there are no mass splittings. If we take \( M_h = M_{h'} \), then chiral symmetry breaking arises solely through the transverse spin operator, \( J_r \), which is dynamical on the null-plane. That is,

\[
\langle h', \lambda' | \tilde{Q}_\alpha^5(x^+) , J_\pm | h , \lambda \rangle = \delta_{\lambda',\lambda \pm 1} (2\pi)^3 2 p^+ \delta(q^+) \delta^2(\tilde{q}_\perp) \times \left[ c_{ \pm}(h, \lambda) [ X_\alpha(\lambda \pm 1) ]_{h/k} - c_{ \mp}(h', \lambda') [ X_\alpha(\lambda) ]_{h/k} \right]. \quad (3.46)
\]

In this case, Goldstone’s theorem must be obtained from the relation

\[
\langle h' | M [ \tilde{Q}_\alpha^5(x^+) , J_r ] | h \rangle = -i \epsilon_{rs} p^+ \epsilon_\chi \int dx^- d^2 \mathbf{x}_\perp \langle h' | x_\alpha \tilde{P}_\alpha(x^-, \mathbf{x}_\perp , x^+) | h \rangle, \quad (3.47)
\]

and its corresponding constraint

\[
\int dx^- d^2 \mathbf{x}_\perp \langle h' | x_\alpha \tilde{P}_\alpha(x^-, \mathbf{x}_\perp , x^+) | h \rangle \to \frac{1}{\epsilon_\chi} + \ldots \quad (3.48)
\]

in the symmetry limit, \( \epsilon_\chi \to 0 \). Following the same steps as for the mass-squared reduced Hamiltonian, we have

\[
\langle h' | [ \tilde{Q}_\alpha^5(x^+) , J_r ] | h \rangle = -\epsilon_{rs} \frac{p^+}{M_h} (2\pi)^3 \delta(q^+) \delta^2(\tilde{q}_\perp) e^{ix^+q^+} \sum_i \frac{\epsilon_\chi Z_i}{M_{q_i}^2} \left( \frac{\partial}{\partial q^i} M_{q_i}(q) \right) \quad (3.49)
\]

which again leads, via the same logic presented above, to Goldstone’s theorem. Therefore, even if \( M^2 \) commutes with the chiral charges, the chiral symmetry breaking contained in the spin Hamiltonians implies the presence of massless states. Evaluating eq. 3.49 in the rest frame, where \( p^+ \to M_h/\sqrt{2} \) and \( J_r \to J_r \), and using eq. 3.31, gives

\[
\langle h' | [ \tilde{J}^+_s, J^-_r ] | h \rangle = i \frac{1}{\sqrt{2}} \epsilon_{rs} \langle h' | J^+_s | h \rangle, \quad (3.50)
\]

which is simply the statement that the axial current transform as a vector operator.
3.6 General operator algebra and the chiral basis

A physical system with an $SU(N)_R \otimes SU(N)_L$ chiral symmetry broken to the vector subgroup $SU(N)_F$ may be expressed as a dynamical Hamiltonian system which evolves with null-plane time, whose reduced Hamiltonians satisfy the $U(2)$ algebra of eq. 2.28, and in addition have non-vanishing Lie brackets with the non-conserved chiral charges. In operator form the reduced Hamiltonians satisfy:

\[ [\tilde{Q}_5^\beta(x^+), M^2] \neq 0 ; \quad [\tilde{Q}_5^\beta(x^+), M J_\pm] \neq 0 , \]  

(3.51)

which express the spontaneous breaking of the chiral symmetry. Eq. 3.51 has the general operator solution

\[ M^2 = M^2_1 + \sum_\mathcal{R} M^2_\mathcal{R} ; \]
\[ M J_\pm = (M J_\pm)_1 + \sum_\mathcal{R} (M J_\pm)_\mathcal{R} \]  

(3.52)

where $\mathbf{1}$ denotes the singlet $SU(N)_R \otimes SU(N)_L$ representation, $(\mathbf{1}, \mathbf{1})$, and $\mathcal{R} = (\mathcal{R}_R, \mathcal{R}_L)$ is a non-trivial representation. Note that all three symmetry-breaking reduced Hamiltonians must transform in the same way. This follows directly from eqs. 3.10 and 3.11.

It is useful to give a heuristic description of the consequences of this algebraic structure. Consider an interpolating field operator, $a_h^\dagger$ which creates a momentum state $h$ out of the vacuum; that is,

\[ a_h^\dagger |0\rangle = |h\rangle . \]  

(3.53)

Here and below for simplicity we will suppress the flavor indices. Because the null-plane chiral charges annihilate the vacuum, $\tilde{Q}_5|0\rangle = 0$, one has

\[ \tilde{Q}_5 |h\rangle = [\tilde{Q}_5, a_h^\dagger]|0\rangle . \]  

(3.54)

Now we will assume that the interpolating field operator $a_h^\dagger$ has definite chiral transformation properties with respect to the chiral charge in the sense that

\[ [\tilde{Q}_5, a_h^\dagger] = C' a_{h'}^\dagger + C'' a_{h''}^\dagger + \ldots , \]  

(3.55)

where $C', C'', \ldots$ are group-theoretic factors. This is simply the statement that the field operators $\{a_h, a_{h'}, a_{h''}, \ldots\}$ are in a non-trivial $SU(N)_R \otimes SU(N)_L$ representation. It then follows from eq. 3.54 that

\[ \tilde{Q}_5 |h\rangle = C' |h'\rangle + C'' |h''\rangle + \ldots , \]  

(3.56)

and therefore the states $\{h, h', h'', \ldots\}$ are also in an $SU(N)_R \otimes SU(N)_L$ representation.

Note that the instant-form interpolating operators also fill out $SU(N)_R \otimes SU(N)_L$ representations. However, the instant-form charges do not annihilate the vacuum, i.e. $Q_5|0\rangle \equiv |\omega\rangle$, it follows that $Q_5|h\rangle = [Q_5, a_h^\dagger]|0\rangle + |h; \omega\rangle$. Therefore $Q_5^2|h\rangle = C' |h'\rangle + C'' |h''\rangle + \ldots + |h; \omega\rangle$ and the utility of chiral symmetry as a classification symmetry is lost.
One then has, for instance,

\[ \langle h' | \tilde{Q}_5 | h \rangle = C' ; \tag{3.57} \]

\[ \langle h'' | \tilde{Q}_5 | h \rangle = C'' , \tag{3.58} \]

which are, via eq. 3.41, Goldstone-boson transition matrix elements. If \( \{h, h', h'', \ldots \} \) are in an irreducible representation, then the \( C \)'s are completely determined by the symmetry (i.e. are Clebsch-Gordon coefficients), while if the representation is reducible, then the \( C \)'s will depend on the mixing angles which mix the irreducible representations. Therefore through the study of Goldstone-boson transitions one learns about the chiral representations filled out by the physical states \(^7\). To learn more about the chiral representations, one considers the mixed Lie brackets, eqs. 3.44 and 3.45. Knowledge of the transformation properties of the chiral-symmetry breaking reduced Hamiltonians gives information about how the hadron masses and spins are related, and therefore in how the irreducible representations mix with each other when the symmetry is broken.

A natural null-plane basis can be written as

\[ | k^+ , \vec{k}_\perp ; \lambda , h , (R_R , R_L) \rangle \]. \tag{3.59} \]

While the mass eigenstates are eigenstates of helicity, they clearly are not eigenstates of \( SU(N)_R \otimes SU(N)_L \) when the symmetry is spontaneously broken. Nevertheless, the chiral basis is useful when the state \( h \) can only appear in a finite number of chiral representations, even though \( h \) may be in an infinite-dimensional reducible chiral representation, as is the case generally in QCD at large-\( N_c \) [9, 62]. In the chiral basis, the reduced Hamiltonian matrix \( M^2 \) is then of finite rank, even though there can be submatrices of infinite rank (and therefore the Fock expansion in the number of constituents is infinite). Ultimately, the utility of the chiral basis is determined by comparison with experiment [7–10, 63–68].

4 QCD in the front form

4.1 Basic instant-form conventions

In this section, we will review the relevant symmetry properties of the instant-form QCD Lagrangian for purposes of establishing conventions which will clarify the null-plane description. Consider the QCD Lagrangian with \( N \) flavors of light quarks and \( N_c \) colors:

\[ L_{\text{QCD}}(x) = \bar{\psi}(x) \left[ \frac{i}{2} \left( \vec{D}_\mu - \vec{D}_\mu \right) \gamma^\mu - \mathbb{M} \right] \psi(x) - \frac{1}{4} F_{\mu\nu}^a(x) F^{\mu\nu}_a(x) \] \tag{4.1} \]

where \( \mathbb{M} \) is the quark mass matrix, for now taken as a diagonal \( N \times N \) matrix, and the covariant derivatives are

\[ \vec{D}_\mu = \partial_\mu - ig t^a A_\mu^a (x) , \quad \vec{D}_\mu = \partial_\mu + ig t^a A_\mu^a (x) , \] \tag{4.2} \]

\(^7\)Here it should be stressed that the chiral multiplet structure of the states is useful only when the null-plane chiral charges mediate transitions between single-particle states [7, 8]. Multi-particle states obscure the algebraic consequences of null-plane chiral symmetry.
where \( g \) is the strong coupling constant, and indices \( a, b, \ldots \) are taken as adjoint indices of the \( SU(3) \)-color gauge group. The Lagrangian is invariant with respect to baryon number and singlet axial transformations

\[
\psi \to e^{-i\theta} \psi \quad , \quad \psi \to e^{-i\theta} \gamma_5 \psi ,
\]

with associated currents

\[
J^\mu = \bar{\psi} \gamma^\mu \psi \quad , \quad J^\mu_5 = \bar{\psi} \gamma^\mu \gamma_5 \psi ,
\]

and with divergences

\[
\partial_\mu J^\mu = 0 \quad , \quad \partial_\mu J^\mu_5 = 2i\bar{\psi} M \gamma_5 \psi - N \frac{g^2}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{tr} (F_{\mu\nu} F_{\rho\sigma}) ,
\]

where the singlet axial symmetry is of course anomalous. In addition, the Lagrangian is invariant with respect to the symmetry transformations

\[
\psi \to e^{-i\theta_a T_a} \psi \quad , \quad \psi \to e^{-i\theta_a \gamma_5 \psi} ,
\]

where the \( T_a \) are \( SU(N) \) generators (see appendix). By the standard Noether procedure one defines the associated currents,

\[
J^\mu_a = \bar{\psi} \gamma^\mu T_a \psi \quad , \quad J^\mu_5 a = \bar{\psi} \gamma^\mu \gamma_5 T_a \psi ,
\]

respectively, with divergences

\[
\partial_\mu J^\mu_a = -i\bar{\psi} [M, T_a] \psi \quad , \quad \partial_\mu J^\mu_5 a = i\bar{\psi} \{ M, T_a \} \gamma_5 \psi .
\]

Therefore, with \( N \) degenerate flavors the QCD Lagrangian is \( SU(N)_F \) invariant and in the chiral limit where \( M \) vanishes, there is an \( SU(N)_R \otimes SU(N)_L \) chiral symmetry generated by the currents \( J^\mu_a = (J^\mu_a - J^\mu_5 a) / 2 \) and \( J^\mu_R a = (J^\mu_a + J^\mu_5 a) / 2 \).

The energy-momentum tensor may be written as

\[
T^{\mu\nu} = -g^{\mu\nu} \mathcal{L}_{\text{QCD}} - F^{\mu\rho}_{a} F_{a\rho}^{\nu} + \frac{i}{2} \bar{\psi} \tilde{D}^\mu \gamma^\nu \psi .
\]

From the energy-momentum tensor we can form the Hamiltonian,

\[
P^0 = \int d^3 x T^{00} .
\]

Here we will assume that chiral symmetry is spontaneously broken through the formation of the condensate

\[
\mathcal{M} \langle \Omega | \bar{\psi} \psi | \Omega \rangle = \mathcal{M} \langle \Omega | \frac{\partial T^{00}}{\partial M} | \Omega \rangle = \mathcal{M} \frac{\partial \mathcal{E}_0}{\partial M} \neq 0 ,
\]

where we have used the Feynman-Hellmann theorem, \( | \Omega \rangle \) represents the (complicated) instant-form QCD vacuum state, and \( \mathcal{E}_0 \) is the QCD vacuum energy. It is straightforward to show that the condensate transforms as the \( (\mathbb{N}, \mathbb{N}) \oplus (\mathbb{N}, \bar{\mathbb{N}}) \) representation of \( SU(N)_R \otimes \mathbb{R}^2 \).
SU(N)_L. We can compute the vacuum energy in the low-energy effective field theory; i.e. chiral perturbation theory (χPT) [69, 70], as well. And therefore,

\[ \mathcal{M} \frac{\partial \mathcal{E}_0}{\partial \mathcal{M}} = \mathcal{M} \frac{\partial \mathcal{E}_0^{\chi PT}}{\partial \mathcal{M}} , \]  

(4.12)

where \( \mathcal{E}_0^{\chi PT} \) is the χPT vacuum energy. In the non-linear realization of the chiral group the Goldstone boson field may be written as \( U(x) = \exp (i \pi_n(x) T_a / F_\pi) \), and the leading quark mass contribution to the χPT Lagrangian is

\[ \mathcal{L}_{QCD}^{\chi PT} = v \text{tr} \left( U \mathcal{M} \dagger + U \mathcal{M} \right) + \ldots , \]  

(4.13)

where \( \mathcal{M} \) is the χPT vacuum energy. In the non-linear realization of the chiral group the Goldstone boson field may be written as \( U(x) = \exp (i \pi_n(x) T_a / F_\pi) \), and the leading quark mass contribution to the χPT Lagrangian is

\[ \mathcal{L}_{QCD}^{\chi PT} = v \text{tr} \left( U \mathcal{M} \dagger + U \mathcal{M} \right) + \ldots , \]  

(4.13)

where \( v = M_\pi^2 F_\pi^2 / \mathcal{M} \) and with \( M_\pi \) the Goldstone boson mass. One then obtains the Gell-Mann-Oakes-Renner formula [42].

\[ - \mathcal{M} \langle \Omega | \bar{\psi} \psi | \Omega \rangle = \frac{1}{2} N M_\pi^2 F_\pi^2 + \ldots . \]  

(4.14)

It will be a principle goal in what follows to determine what takes the place of this relation in null-plane QCD.

### 4.2 Null plane representation

The QCD Lagrangian in the null-plane coordinates is obtained by generalizing the results given in Appendices B and C to the interacting case. (Note that we work in light-cone gauge, \( A^+ = 0 \), throughout.) The QCD equations of constraint for the non-dynamical degrees of freedom are

\[ \psi_- = \frac{1}{2i \partial^+} \left( -i \gamma^r \tilde{D}^r + \mathcal{M} \right) \gamma^+ \psi_+ , \quad \psi_+^\dagger = \psi_+^\dagger \gamma^{-} \left( i \gamma^r \tilde{D}^r - \mathcal{M} \right) \frac{1}{2i \partial^+} \]  

(4.15)

for the redundant quark degrees of freedom, and

\[ \partial^+ A_a^- = \frac{1}{\partial^+} D^r_{ab} \partial^+ A_b^r - g \frac{1}{\partial^+} \bar{\psi}_+ \gamma^+ t^a \psi_+ , \]  

(4.16)

for the redundant gauge degrees of freedom. The null-plane QCD Lagrangian can then be expressed in terms of the dynamical degrees of freedom as

\[ \bar{\mathcal{L}}_{QCD} = i \bar{\psi}_+ \gamma^+ \partial^- \psi_+ - \frac{i}{2} \bar{\psi}_+ \gamma^+ \gamma^s D^r \frac{1}{\partial^+} D^s \psi_+ \]

\[ + \frac{i}{2} \bar{\psi}_+ \gamma^+ \mathcal{M}^2 \frac{1}{\partial^+} \psi_+ + \frac{i}{2} \bar{\psi}_+ \gamma^+ \mathcal{M} \left( \gamma^r g t^a A_a^r \right) \frac{1}{\partial^+} \psi_+ - \frac{i}{2} \bar{\psi}_+ \gamma^+ \mathcal{M} \left( \gamma^r g t^a A_a^r \psi_+ \right) \]

\[ - \frac{1}{4} F_{a}^{rs} F_{a}^{rs} + \left( \partial^+ A_a^r \right) \left( \partial^- A_a^r \right) - \frac{1}{2} \left( \frac{1}{\partial^+} D^r_{ab} \partial^+ A_b^r - g \frac{1}{\partial^+} \bar{\psi}_+ \gamma^+ t^a \psi_+ \right)^2 . \]  

(4.17)

The price to pay for working with the physical degrees of freedom in the null-plane coordinates is a loss of manifest Lorentz covariance, as well as the appearance of operators which that appear to be non-local in the longitudinal coordinate. As in the instant-form,

---

*We follow the notation and conventions of Ref. [55].*
one should view this Lagrangian as providing a perturbative definition of QCD at large momentum transfers, where the longitudinal zero modes play no role. Notice that in null-plane QCD there are two kinds of operators that depend on the quark-mass matrix. One is a kinetic term, quadratic in the quark masses, and the other is a spin-flip quark-gluon interaction that is linear in the quark masses.

Naturally we expect that null-plane QCD has the same symmetries as instant-form QCD. Consider the $U(1)_R \otimes U(1)_L$ transformations,

$$\psi_+ \rightarrow e^{-i\theta} \psi_+ \quad , \quad \psi_+ \rightarrow e^{-i\theta_5} \psi_+ .$$

While baryon number is unaltered in moving to the null-plane coordinates, this is clearly not the same chiral transformation that we had in the instant form, as that transformation acts on the non-dynamical degrees of freedom, $\psi_-$, in a distinct manner and is therefore complicated on the null-plane. That the chiral symmetry transformations are different in the two forms of dynamics is essential for what follows. We will return below to the relation between the chiral symmetries in the instant-form and the front form, as this will be important in understanding the role of condensates on the null-plane. The $U(1)_A$ current and its divergence are

$$\tilde{J}_5^\mu = J_5^\mu - i\bar{\psi} \gamma_5 \gamma_5 \frac{1}{\partial^+} \psi_+ ;$$

with divergences

$$\partial_\mu \tilde{J}_5^\mu = \frac{1}{2} \bar{\psi} \gamma_5 \gamma_5 \frac{1}{\partial^+} \psi_+ + \frac{1}{2} \bar{\psi} \gamma_5 \gamma_5 \{ \mathcal{M} , T_\alpha \} \frac{1}{\partial^+} \psi_+ .$$

For $N$ degenerate flavors, the quark mass matrix is proportional to the identity, the vector current is conserved, and the axial current and the divergence of the axial current are

$$\tilde{J}_{5a}^\mu = J_{5a}^\mu - i\bar{\psi} \gamma_5 \gamma_5 T_\alpha \frac{1}{\partial^+} \psi_+ ;$$

with divergences

$$\partial_\mu \tilde{J}_{5a}^\mu = \frac{1}{2} \bar{\psi} \gamma_5 \gamma_5 \frac{1}{\partial^+} \psi_+ + \frac{1}{2} \bar{\psi} \gamma_5 \gamma_5 \{ \mathcal{M} , T_\alpha \} \frac{1}{\partial^+} \psi_+ .$$

To minimize clutter, it will prove convenient to define the operator

$$\frac{1}{\partial^+} (\gamma^a g t^a A_5^\mu) \psi_+ \equiv (\gamma^a g t^a A_5^\mu) \frac{1}{\partial^+} \psi_+ - \frac{1}{\partial^+} (\gamma^a g t^a A_5^\mu) \psi_+ .$$

9To minimize clutter, it will prove convenient to define the operator
\[
\partial_\mu \tilde{J}_5^\mu = \bar{\psi}_+ \gamma^+ \gamma_5 T_\alpha M \frac{1}{i \partial^+} (\gamma^\mu g t^\alpha A^\mu_5) \psi_+ .
\]  
(4.28)

Here note in particular that the null-plane axial-vector current in null-plane QCD evidently takes the form, eq. 3.40, expected on general grounds.

### 4.3 Null-plane charges

The null-plane singlet axial charge is defined as
\[
\tilde{Q}^5 = \int dx^- d^2 x_\perp \tilde{J}^+_5 = \int dx^- d^2 x_\perp \bar{\psi}_+ \gamma^+ \gamma_5 \psi_+ ,
\]  
(4.29)

where we have used eq. 4.20. Using the momentum-space representation of \( \psi_+ \), given in eq. B.20, one finds
\[
\tilde{Q}^5 = \sum_{\lambda = \uparrow \downarrow} 2\lambda \int \frac{dk^+ d^2 k_\perp}{2k^+ (2\pi)^3} \left\{ b^\dagger_\lambda(k^+, k_\perp) b_\lambda(k^+, k_\perp) + d^\dagger_\lambda(k^+, k_\perp) d_\lambda(k^+, k_\perp) \right\} .
\]  
(4.30)

Comparison with eq. B.31 one sees that the singlet axial charge coincides (up to a factor of two) with the free-fermion helicity operator. This of course explains why the quark mass term in the free-fermion theory is a chiral invariant; on the null-plane, chiral symmetry breaking in the free-fermion theory implies breaking of rotational invariance in the transverse plane.

Similarly, the null-plane non-singlet vector and chiral charges are, respectively,
\[
\tilde{Q}_\alpha = \int dx^- d^2 x_\perp \tilde{J}^+_{\alpha} = \int dx^- d^2 x_\perp \bar{\psi}_+ \gamma^+ T_\alpha \psi_+ ;
\]  
(4.31)

\[
\tilde{Q}_5^\alpha = \int dx^- d^2 x_\perp \tilde{J}^5_{\alpha} = \int dx^- d^2 x_\perp \bar{\psi}_+ \gamma^+ \gamma_5 T_\alpha \psi_+ ,
\]  
(4.32)

and using the momentum-space representation of \( \psi_+ \), given in eq. B.20, one finds
\[
\tilde{Q}_\alpha = \sum_{\lambda = \uparrow \downarrow} \int \frac{dk^+ d^2 k_\perp}{2k^+ (2\pi)^3} \left\{ b^\dagger_\lambda(k^+, k_\perp) T_\alpha b_\lambda(k^+, k_\perp) - d^\dagger_\lambda(k^+, k_\perp) T_\alpha^T d_\lambda(k^+, k_\perp) \right\} ;
\]  
(4.33)

\[
\tilde{Q}_5^\alpha = \sum_{\lambda = \uparrow \downarrow} 2\lambda \int \frac{dk^+ d^2 k_\perp}{2k^+ (2\pi)^3} \left\{ b^\dagger_\lambda(k^+, k_\perp) T_\alpha b_\lambda(k^+, k_\perp) + d^\dagger_\lambda(k^+, k_\perp) T_\alpha^T d_\lambda(k^+, k_\perp) \right\} .
\]  
(4.34)

One readily checks that the null-plane chiral algebra, eqs. 3.4 and 3.5, is satisfied by these charges. As these charge are written as sums of number operators that count the number of quarks and anti-quarks, both chiral charges annihilate the vacuum, and we have
\[
\tilde{Q}_\alpha |0\rangle = \tilde{Q}_5^\alpha |0\rangle = 0 ,
\]  
(4.35)

as expected on the general grounds presented above. One then has
\[
[\tilde{Q}_\alpha , \psi_+] = -T_\alpha \psi_+ ; \quad [\tilde{Q}_5^\alpha , \psi_+] = -\gamma_5 T_\alpha \psi_+ .
\]  
(4.36)

Breaking down the fields into left- and right-handed components,
\[
\psi_{+R} = \frac{1}{2} (1 + \gamma_5) \psi_+ , \quad \psi_{+L} = \frac{1}{2} (1 - \gamma_5) \psi_+ .
\]  
(4.37)
and, using the results of Appendix B, one verifies the fermion transformation properties with respect to $SU(N)_R \otimes SU(N)_L$:

\[
\begin{align*}
\psi^+_R &= \psi^\dagger_+ \in (1, N), \\
\psi_+^\dagger_R &= \psi_+^\dagger \in (1, N), \\
\psi_+^+_L &= \psi_+^\dagger_+ \in (N, 1), \\
\psi_+^{\dagger}_L &= \psi_+^\dagger_+ \in (\bar{N}, 1),
\end{align*}
\] (4.38)

and the helicity eigen-equations of the quarks

\[
\begin{align*}
\Sigma_{12} \psi^{\dagger}_+ &= \frac{1}{2} \psi^{\dagger}_+ ; \\
\Sigma_{12} \psi^+_+ &= -\frac{1}{2} \psi^+_+ ,
\end{align*}
\] (4.40)

where the helicity operator, $\Sigma_{12}$, is defined in Appendix B.

4.4 Chiral symmetry breaking Hamiltonians

Using the results of the previous section, it is straightforward to find the transformation properties of the symmetry-breaking parts of the reduced Hamiltonians. Define the operators \(\tilde{D}_{5a}\):

\[
\tilde{D}_{5a} \equiv \partial_{\mu} \tilde{j}_{5a}^{\mu} = \bar{\psi} \gamma^+ \gamma_5 T_a \frac{1}{\partial_{\bar{M}}} \left( \gamma^r g t^a A^r_{\alpha} \right) \psi^+ ;
\] (4.42)

\[
\tilde{D}_5 \equiv \bar{\psi} \gamma^+ \gamma_5 \frac{1}{\partial_{\bar{M}}} \left( \gamma^r g t^a A^r_{\alpha} \right) \psi^+ ;
\] (4.43)

\[
\tilde{D}_a \equiv \bar{\psi} \gamma^+ T_a \frac{1}{\partial_{\bar{M}}} \left( \gamma^r g t^a A^r_{\alpha} \right) \psi^+ ;
\] (4.44)

\[
\tilde{D} \equiv \bar{\psi} \gamma^+ \frac{1}{\partial_{\bar{M}}} \left( \gamma^r g t^a A^r_{\alpha} \right) \psi^+ .
\] (4.45)

It is then a textbook exercise to find

\[
[\tilde{Q}^a_5, \tilde{D}^b_5] = \frac{1}{N} \delta^{a\beta} \tilde{D}^\beta + d^{a\beta\gamma} \tilde{D}^\gamma ;
\] (4.46)

\[
[\tilde{Q}^a_5, \tilde{D}] = 2 \tilde{D}^a ;
\] (4.47)

\[
[\tilde{Q}^a_5, \tilde{D}^b_5] = \frac{1}{N} \delta^{a\beta} \tilde{D}_5^\beta + d^{a\beta\gamma} \tilde{D}^\gamma_5 ;
\] (4.48)

\[
[\tilde{Q}^a_5, \tilde{D}_5] = 2 \tilde{D}^a .
\] (4.49)

It follows that the $2N^2$ operators ($\tilde{D}_{5a}, \tilde{D}_5, \tilde{D}_a, \tilde{D}$) fill out the $(\bar{N}, N) \oplus (N, \bar{N})$ representation of $SU(N)_R \otimes SU(N)_L$.

The null-plane Hamiltonian $P^-$ is:

\[
P^- = \int dx^- d^2 \mathbf{x} T^{--} ,
\] (4.50)

\footnote{From here forward we will use the definition:}

\[
\frac{1}{\partial_{\bar{M}}} \equiv M \frac{1}{\partial^r} .
\]
and therefore the chiral-symmetry breaking part of this Hamiltonian is given by:

$$P^{-}_{(N,N)} \equiv -\frac{i}{2} \int dx^- d^2x_\perp \tilde{D}.$$  \hfill (4.51)

One readily checks that this is consistent with eqs. 3.10 and 4.42.

One then finds the symmetry breaking parts of the reduced QCD Hamiltonians:

$$M^2_{(N,N)} = -iP^+ \int dx^- d^2x_\perp \bar{\psi}_+ \gamma^\perp \left( \frac{1}{\partial^+_M} \frac{1}{\partial^+_M} (\gamma^r g t^a A^r_a) \right) \psi_+ ; \hfill (4.52)$$

$$[M^J_r]_{(N,N)} = \frac{1}{2} \epsilon_{rs} P^+ \int dx^- d^2x_\perp \bar{\psi}_+ \gamma^\perp \left( \frac{1}{\partial^+_M} \frac{1}{\partial^+_M} (\gamma^r g t^a A^r_a) \right) \psi_+ , \hfill (4.53)$$

where, in addition, we have used eqs. 3.11 and 4.28 to obtain the reduced Hamiltonian for spin. All chiral symmetry breaking in null-plane QCD is contained in these two operators.

Using eqs. 3.10, 3.11 and 4.46 one finds

$$[ \tilde{Q}_5^\alpha, [ \tilde{Q}_5^\beta, M^2 ]] = -2iP^+ \int dx^- d^2x_\perp \bar{\psi}_+ \gamma^\perp \left( \frac{1}{N} \delta^{\alpha\beta} \tilde{D} + d^{\alpha\beta\gamma} \tilde{D} \gamma \right) ; \hfill (4.54)$$

$$[ \tilde{Q}_5^\beta, [ \tilde{Q}_5^\alpha, M^2 ]] = i\epsilon_{\alpha\beta} P^+ \int dx^- d^2x_\perp \bar{\psi}_+ \gamma^\perp \left( \frac{1}{N} \delta^{\alpha\beta} \tilde{D} + d^{\alpha\beta\gamma} \tilde{D} \gamma \right) . \hfill (4.55)$$

Acting on these equations with $\delta^{\alpha\beta}$ and $d^{\alpha\beta\gamma}$, and using the identities in Appendix D gives

$$-2iP^+ \int dx^- d^2x_\perp \tilde{D} = \frac{N}{N^2 - 1} [ \tilde{Q}_5^\alpha, [ \tilde{Q}_5^\beta, M^2 ]] ; \hfill (4.56)$$

$$-2iP^+ \int dx^- d^2x_\perp \tilde{D} \gamma = d_{\alpha\beta\gamma} \frac{N}{N^2 - 4} [ \tilde{Q}_5^\alpha, [ \tilde{Q}_5^\beta, M^2 ]] . \hfill (4.57)$$

Therefore, eq. 4.54 can be written as

$$[ \tilde{Q}_5^\beta, [ \tilde{Q}_5^\alpha, M^2 ]] = \frac{1}{N^2 - 1} \delta^{\alpha\beta} [ \tilde{Q}_5^\gamma, [ \tilde{Q}_5^\gamma, M^2 ]] + \frac{N}{N^2 - 4} d^{\alpha\beta\gamma} d^{\mu\nu\gamma} [ \tilde{Q}_5^\alpha, [ \tilde{Q}_5^\beta, M^2 ]], \hfill (4.58)$$

and eq. 4.55 takes the same form but with $M^2$ replaced by $M^J_{\pm}$. Defining the projection operator

$$P^{\alpha\beta;\mu\nu} \equiv \delta^{\alpha\mu} \delta^{\beta\nu} - \frac{1}{N^2 - 1} \delta^{\alpha\beta} \delta^{\mu\nu} - \frac{N}{N^2 - 4} \epsilon^{\alpha\beta\gamma} d^{\mu\nu\gamma} , \hfill (4.59)$$

we can express the constraints on the reduced Hamiltonians in compact notation as:

$$P^{\alpha\beta;\mu\nu} [ \tilde{Q}_5^\mu, [ \tilde{Q}_5^\nu, M^2 ]] = P^{\alpha\beta;\mu\nu} [ \tilde{Q}_5^\mu, [ \tilde{Q}_5^\nu, M^J_{\pm} ] ] = 0 . \hfill (4.60)$$

These are quite possibly the most important equations in null-plane QCD, as they are the mathematical expression of the specific way in which the internal symmetries and Poincaré symmetries intersect. These equations were obtained originally in Refs. [7–9] by considering the most general form of Goldstone-boson-hadron scattering amplitudes in specially-designed Lorentz frames, and using input from Regge-pole theory expectations of their high-energy behavior. Note that the projection operator, $P^{\alpha\beta;\mu\nu}$, has four adjoint
indices and is, as shown in Ref. [7] related to the interactions of Goldstone bosons (in the t-channel of Goldstone-boson-hadron scattering), which are in the adjoint of SU(N)F and whose scattering amplitudes therefore transform as the product of two adjoints. In the case of two flavors, where \(3 \otimes 3 = 1 \oplus 3 \oplus 5\), it projects out the 5-dimensional representation \((I = 2)\) and in the case of three flavors, where \(8 \otimes 8 = 1 \oplus 8 \oplus 10 \oplus 10 \oplus 27\), it projects out the 10, \(\bar{10}\), and 27-dimensional representations. As shown above, these are the representations that cannot be formed from a single quark bilinear; i.e. they are not contained in \((N, \bar{N}) \oplus (N, N)\), as is clear from direct inspection of eqs. 4.54 and 4.55.

4.5 Gell-Mann-Oakes-Renner relation recovered

We are now in a position to address the fate of instant-form QCD chiral-symmetry breaking condensates in null-plane QCD. Again using the Feynman-Hellmann theorem we find

\[
\mathcal{M} \langle 0 | \frac{\partial T^{-+}}{\partial \mathcal{M}} | 0 \rangle = \mathcal{M} \frac{\partial \hat{E}_0}{\partial \mathcal{M}} = \mathcal{M} \frac{\partial \hat{E}_0}{\partial \mathcal{M}} = \mathcal{M} \frac{\partial \hat{E}_0}{\partial \mathcal{M}},
\]

where \(|0\rangle\) represents the null-plane QCD vacuum state, and \(\hat{E}_0\) is the null-plane QCD vacuum energy. In this equation we have also expressed that physics is independent of the choice of coordinates. Therefore calculation of the leading quark-mass contribution to the vacuum energy must be independent of the quantization surface, and should be the same whether one works with the fundamental degrees of freedom, or with the Goldstone bosons in the infrared. One then has

\[
\mathcal{M} \frac{\partial \hat{E}_0}{\partial \mathcal{M}} = -\mathcal{M} \langle 0 | i \bar{\psi} \gamma^+ \frac{1}{\partial \mathcal{M}} \gamma^0 \psi_+ | 0 \rangle + \langle 0 | \frac{i}{2} \bar{\psi} \gamma^+ \frac{1}{\partial \mathcal{M}} (\gamma^r g t^a \gamma_0^r \gamma^a) \psi_+ | 0 \rangle.
\]

The second term must vanish as the chiral charges annihilate the vacuum and therefore there can be no chiral-symmetry breaking condensates. Operationally one sees this directly by taking the vacuum expectation value of eq. 4.46 which gives

\[
\langle 0 | \frac{i}{2} \bar{\psi} \gamma^+ \frac{1}{\partial \mathcal{M}} (\gamma^r g t^a \gamma_0^r \gamma^a) \psi_+ | 0 \rangle = 0.
\]

We are then left with the null-plane expression of the Gell-Mann-Oakes-Renner relation:

\[
\mathcal{M} \langle 0 | i \bar{\psi} \gamma^+ \frac{1}{\partial \mathcal{M}} \gamma^0 \psi_+ | 0 \rangle = \frac{1}{2} N M^2 \pi F_\pi^2 + \ldots.
\]

Hence, a chiral-symmetry breaking condensate in the instant-form formulation of QCD has been replaced by a chiral-symmetry conserving condensate in the null-plane formulation. Note that while the operator naively vanishes in the chiral limit, the matrix element is infrared singular and therefore it need not, and indeed cannot, vanish in the chiral limit. It would be very interesting to define the relevant operator non-perturbatively and calculate this condensate directly, perhaps using transverse lattice gauge theory methods [71–79]. Note that a \textit{a priori} knowledge of the singlet condensate in eq. 4.64 is not very different to \textit{a priori} knowledge of the symmetry-breaking quark condensate in eq. 4.14. In both cases, it is necessary to keep the quark masses finite and only at the very end take the chiral limit [80].

\[11\] This expression of the Gell-Mann-Oakes-Renner formula was found previously in Ref. [29] using the methods that will be described below.
4.6 Condensates on a null-plane

We will now derive the Gell-Mann-Oakes-Renner relation in a different way which will suggest a general prescription for expressing all instant-form condensates with null-plane condensates. While the left- and right-handed components of $\psi_+$ transform irreducibly with respect to the null-plane chiral charges, the transformation properties of $\psi$ are complicated by the presence of the non-dynamical component $\psi_-$. Indeed one finds

$$[\tilde{Q}_5^\alpha, \psi] = -\gamma_5 T^\alpha \psi - i \gamma_5 \gamma^+ T^\alpha \frac{1}{\partial_M^+} \psi,$$  \hspace{1cm} (4.65)$$

from which it follows that

$$\psi_R, \psi_L \in (1, N) \oplus (N, 1), \quad \psi_R^\dagger, \psi_L^\dagger \in (1, \bar{N}) \oplus (\bar{N}, 1). \hspace{1cm} (4.66)$$

Since the left- and right-handed components of the quark field transform reducibly with respect to the chiral group, generally products of bilinear operators of the form $\bar{\psi} \Gamma \psi$ will have complicated reducible chiral transformation properties. However, QCD operators built out of these bilinears will always have a component that transforms as a chiral singlet. We will now see, for the simplest example, that this is essential to the consistency of the null-plane formulation. Consider the transformation properties of the following set of bilinears:

$$D_5^\alpha \equiv \bar{\psi} \gamma_5 T^\alpha \psi, \quad D^5 \equiv \bar{\psi} \gamma_5 \psi; \hspace{1cm} (4.67)$$

$$D^\alpha \equiv \bar{\psi} T^\alpha \psi, \quad D \equiv \bar{\psi} \psi. \hspace{1cm} (4.68)$$

Is is again simple to check that these operators fill out the $(\bar{N}, N) \oplus (N, \bar{N})$ representation of $SU(N)_R \otimes SU(N)_L$ with respect to the instant-form chiral charges $Q_5^\alpha$. Now consider the transformation properties of these operators with respect to the null-plane chiral charges. One finds

$$[\tilde{Q}_5^\alpha, D_5^\beta] = -\frac{1}{N} \delta^{\alpha\beta} \left( D + i \bar{\psi}_+ \gamma^+ \frac{1}{\partial_M^+} \psi_+ \right) - d^{\alpha\beta\gamma} \left( D^\gamma + i \bar{\psi}_+ \gamma^+ T^\gamma \frac{1}{\partial_M^+} \psi_+ \right); \hspace{1cm} (4.69)$$

$$[\tilde{Q}_5^\alpha, D] = -2 D_5^\alpha; \hspace{1cm} (4.70)$$

$$[\tilde{Q}_5^\alpha, D^\beta] = -\frac{1}{N} \delta^{\alpha\beta} D_5 - d^{\alpha\beta\gamma} D_5^\gamma + f^{\alpha\beta\gamma} \bar{\psi}_+ \gamma^+ \gamma_5 T^\gamma \frac{1}{\partial_M^+} \psi_+; \hspace{1cm} (4.71)$$

$$[\tilde{Q}_5^\alpha, D_5] = -2 D^\alpha - 2i \bar{\psi}_+ \gamma^+ T^\alpha \frac{1}{\partial_M^+} \psi_+. \hspace{1cm} (4.72)$$
To close the algebra we must add, in addition, the commutation relations:

\[
\begin{align*}
[ \tilde{Q}_5^\alpha, \bar{\psi}^+ \gamma^+ \frac{1}{\partial_M} \psi^+ ] &= 0 ; \\
[ \tilde{Q}_5^\alpha, \bar{\psi}^+ \gamma^+ T^\beta \frac{1}{\partial_M} \psi^+ ] &= i f^{\alpha \beta \gamma} \bar{\psi}^+ \gamma^+ T^\gamma \frac{1}{\partial_M} \psi^+ ; \\
[ \tilde{Q}_5^\alpha, \bar{\psi}^+ \gamma^+ \gamma_5 T^\beta \frac{1}{\partial_M} \psi^+ ] &= i f^{\alpha \beta \gamma} \bar{\psi}^+ \gamma^+ T^\gamma \frac{1}{\partial_M} \psi^+ .
\end{align*}
\] (4.73, 4.74, 4.75)

Hence the full set of operators transform as the reducible \(4N^2\)-dimensional \((1, 1) \oplus (A, 1) \oplus (\bar{N}, N) \oplus (N, \bar{N})\) representation of \(SU(N_R) \otimes SU(N_L)\), where here \(A\) denotes the \(SU(N)\) adjoint representation. In particular one see that

\[
\bar{\psi} \psi \in (\bar{N}, N) \oplus (N, \bar{N}) \oplus (1, 1) \oplus \ldots ,
\] (4.76)

and therefore transforms reducibly. This is verified by direct calculation which gives

\[
M \bar{\psi} \psi = -i M \bar{\psi} \gamma^+ \frac{1}{\partial_M} \psi^+ + \frac{i}{2} \bar{\psi} \gamma^+ \frac{1}{\partial_M} (\gamma^r g t^a A_a^r \gamma^r \psi^+ .
\] (4.77)

Taking the vacuum expectation value of eq. 4.77 (or eq. 4.69) gives the general solution [29]

\[
\langle 0 | \bar{\psi} \psi | 0 \rangle = -(0 | i \bar{\psi} \gamma^+ \frac{1}{\partial_M} \psi^+ | 0 \rangle.
\] (4.78)

Therefore only the singlet part of \(\bar{\psi} \psi\) can acquire a vacuum expectation value on the null plane, as must be the case since \(SU(N_R) \otimes SU(N_L)\) is a symmetry of the null-plane vacuum state. This argument readily generalizes to any chiral symmetry breaking Lorentz scalar operator, \(O\), that one can build out of products of quark bilinears in instant-form QCD. One can write

\[
O = \sum_R O_R = \sum \tilde{R} O_{\tilde{R}} + O_1
\] (4.79)

where \(R\) is a non-trivial chiral representation with respect to the instant-form chiral charges, \(Q_{5a}\), and \(\tilde{R} (1)\) is a non-trivial (the singlet) representation with respect to the front-form chiral charges, \(\tilde{Q}_{5a}\). Unless protected by another symmetry, \(O\) has a non-vanishing vacuum expectation value, which can be expressed as

\[
\langle \Omega | O | \Omega \rangle = \langle \Omega | \sum_R O_R | \Omega \rangle = \langle 0 | O_1 | 0 \rangle \neq 0 .
\] (4.80)

Note that the final equality expresses an equivalence between a matrix element evaluated in the instant form and one in the front form. This equality ensures that physics is unmodified in moving between the two forms of dynamics. Therefore all instant-form chiral symmetry
breaking QCD condensates map to chiral symmetry conserving condensates in the front-
form. The presence of the singlet part of the operator can always be traced to the reducible
chiral transformation property of $\psi$ given in eq. 4.65. For the case at hand, with $\mathcal{O} = \bar{\psi}\psi$, we have

$$\langle \Omega | \bar{\psi}\psi | \Omega \rangle = \langle 0 | \bar{\psi}\psi | 0 \rangle ,$$

which together with eq. 4.78, provides the desired link between the instant-form and front-
form expressions of the Gell-Mann-Oakes-Renner relation.

The general relation, eq. 4.80 is important for the consistency of null-plane QCD, as it
demonstrates that, as expected, the QCD vacuum energy is unaltered in moving from the
instant-form to the front-form description, and these relations must, of course, exist in order
that the operator product expansion be independent of the choice of quantization surface.
We see that a symmetry-breaking condensate can form in the instant-form coordinates with
an asymmetric vacuum which is equal to a corresponding symmetry-preserving condensate
in the null-plane description with a symmetric vacuum. The condensate relation eq. 4.81
is one of an infinite number of relations which translates condensates which break chiral
symmetry in the instant form to null-plane condensates which transform as chiral singlets.

5 Consequences of the operator algebra

5.1 Summary of the null-plane QCD description

Before considering the consequences of the null-plane QCD operator algebra, we will sum-
mmarize the picture of chiral symmetry breaking that we have so far established. While the
null-plane QCD vacuum state is chirally invariant, chiral symmetry is spontaneously bro-
ken by the three reduced Hamiltonians that have contributions, $M^2_{(N,N)}$ and $(M \mathcal{J}_r)_{(N,N)}$, which transform as $(\bar{N}, N) \oplus (N, \bar{N})$ with respect to $SU(N)_R \otimes SU(N)_L$. The three reduced Hamiltonians satisfy the constraints, eq. 4.60. In addition to these signatures of chiral sym-
metry breaking, the three reduced Hamiltonians, together with the generator of rotations
on the transverse plane together generate the $U(2)$ dynamical sub-group of the null-plane
Poincaré algebra, eq. 2.28. And finally, the null-plane vector and chiral charges satisfy the
$SU(N)_R \otimes SU(N)_L$ algebra, eqs. 3.4 and 3.5. The entire set of Lie-brackets provide all of
the constraints that exist among the generators of the internal and space-time symmetries
in null-plane QCD. The consequences of chiral symmetry breaking for the spectrum and
spin of QCD are contained in the symmetry-breaking parts of the reduced Hamiltonians.

5.2 Recovery of spin-flavor symmetries

In searching for solutions of the algebraic system that mixes the chiral charges and the
reduced Hamiltonians, one may worry about the existence of no-go theorems that forbid
non-trivial algebras that mix space-time and internal symmetries. In the null-plane for-
mulation the no-go theorems are avoided because it is only the dynamical part, $\mathcal{D}$, of
the null-plane Poincaré algebra that mixes with the internal symmetry generators [58]. Unfor-
nately, a direct general solution of the null-plane QCD operator algebra in the general
case appears difficult. However, there is a limiting case in which the algebra yields an important non-trivial solution. Here we will treat the QCD operator algebra as an abstract operator algebra and consider the limit in which the chiral-symmetry breaking part of the reduced Hamiltonian $M^2$ can be treated as a perturbation. However, one should keep in mind that matrix elements of the operator relations between hadronic states must eventually be taken in order to extract observables. We first define

$$[\tilde{Q}_5^\alpha, M] = \epsilon^\alpha,$$

and neglect terms of $O(\epsilon)$. This implies that all chiral symmetry breaking occurs in the spin Hamiltonians. This limit is non-trivial, as we have shown above in section 3.5 that the spin Hamiltonians alone imply the presence of Goldstone bosons. In this limit, the QCD operator algebra reduces to

$$[J_i, J_j] = i\epsilon_{ijk} J_k,$$

which generates $SU(2)$ spin, and the $SU(N)_R \otimes SU(N)_L$ algebra,

$$[\tilde{Q}_5^\alpha, \tilde{Q}_5^\beta] = if_{\alpha\beta\gamma} \tilde{Q}_5^\gamma, [\tilde{Q}_5^\alpha, \tilde{Q}_5^\beta] = if_{\alpha\beta\gamma} \tilde{Q}_5^\gamma, [\tilde{Q}_5^\alpha, \tilde{Q}_5^\beta] = if_{\alpha\beta\gamma} \tilde{Q}_5^\gamma. \quad (5.3)$$

The remaining non-trivial mixed commutator is for the spin Hamiltonian:

$$P_{\alpha\beta;\mu\nu} [\tilde{Q}_5^\alpha, [\tilde{Q}_5^\beta, J_3]] = 0. \quad (5.4)$$

Now this simplified algebra can be put into a more familiar form. Consider an operator $G_{\alpha i}$ which transforms in the adjoint of $SU(N)$ and as a rotational vector in the sense that

$$[J_i, G_{\alpha j}] = i\epsilon_{ijk} G_{\alpha k}; \quad (5.5)$$

$$[\tilde{Q}_5^\alpha, G_{\beta i}] = if_{\alpha\beta\gamma} G_{\gamma i}. \quad (5.6)$$

In general, the commutator of $G^{\alpha i}$ with itself may be expressed as

$$[G_{\alpha i}, G_{\beta j}] = if_{\alpha\beta\gamma} A_{ij,\gamma} + i\epsilon_{ijk} B_{\alpha\beta,k}, \quad (5.7)$$

where $A_{ij,\gamma} = A_{ji,\gamma}$ and $B_{\alpha\beta,k} = B_{\beta\alpha,k}$. Now we identify $G^{\alpha 3} \equiv \tilde{Q}_5^\alpha$. From eq. 5.3 it then follows that $A_{33,\alpha} = \tilde{Q}_\alpha$. Rotational invariance then implies $A_{ij,\alpha} = \delta_{ij} \tilde{Q}_\alpha$. By considering Jacobi identities of $J_i$ and $\tilde{Q}_\alpha$ with the commutator in eq. 5.7 one finds, respectively,

$$[\tilde{Q}_\gamma, B_{\alpha\beta,i}] = if_{\gamma\beta\mu} B_{\alpha\mu,i} + if_{\gamma\alpha\mu} B_{\beta\mu,i}; \quad (5.8)$$

$$[J_i, B_{\alpha\beta,j}] = i\epsilon_{ijk} B_{\alpha\beta,k}^i, \quad (5.9)$$

which simply indicate that $B_{\alpha\beta,i}$ transforms as a rank-two $SU(N)$ tensor and a rotational vector.

To obtain $B_{\alpha\beta,i}$ we use eq. 5.4 to find:

$$P_{\alpha\beta;\mu\nu} [G_{\alpha 3}, G_{\beta 1} \pm iG_{\beta 2}] = 0, \quad (5.10)$$

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from which it follows that $B_{\alpha\beta,2}$ and $B_{\alpha\beta,1}$ have a piece proportional to $\delta_{\alpha\beta}$ and a piece proportional to $d_{\alpha\beta}$. Rotational invariance then determines that $B_{\alpha\beta,1}$ is a linear combination of $\delta_{\alpha\beta}\mathcal{J}_l$ and $d_{\alpha\beta}\gamma\mathcal{G}_{\gamma\ell}$. The coefficients of these terms are determined by considering the Jacobi identity of $G_{\alpha\ell}$ with the commutator in eq. 5.7, together with the relation among $SU(N)$ structure constants given in Appendix D. Finally, one obtains

$$[G_{\alpha\ell}, G_{\beta\gamma}] = i\delta_{ij} f_{\alpha\beta\gamma} \mathcal{Q}_{ij} + \frac{2}{N} i\delta_{\alpha\beta} \epsilon_{ijk} \mathcal{J}_k + i\epsilon_{ijk} d_{\alpha\beta\gamma} \mathcal{G}_{\gamma\ell} \mathcal{G}_{\gamma\ell} \mathcal{G}_{\gamma\ell},$$  \hspace{1cm} (5.11)

which together with

$$[\tilde{Q}_\alpha, G_{\beta\ell}] = i f_{\alpha\beta\gamma} G_{\gamma\ell} \mathcal{Q}_{\alpha\ell}, \quad [\mathcal{J}_l, G_{\alpha\ell}] = i \epsilon_{ijk} G_{\alpha\ell} \mathcal{J}_k; \quad (5.12)$$

$$[\tilde{Q}_\alpha, \mathcal{Q}_\beta] = i f_{\alpha\beta\gamma} \mathcal{Q}_{\gamma\ell} \mathcal{Q}_{\alpha\ell}, \quad [\mathcal{J}_l, \mathcal{J}_k] = i \epsilon_{ijk} \mathcal{J}_k$$  \hspace{1cm} (5.13)

close the algebra of the symmetry group $SU(2N)$. To find the consequences of this algebra for observable quantities like the mass-squared matrix and the matrix elements for Goldstone boson emission and absorption, one takes matrix elements of this algebra between hadron states $h'$ and $h$, and neglecting transitions from single-particle to multi-particle states in the completeness sums over intermediate states, one recovers the same algebra with the replacements $\tilde{Q}_\alpha \rightarrow [T_a][h', h]$ and $\tilde{Q}_{5\alpha} \rightarrow [X_\alpha(\lambda)] [h', h]$, and corresponding replacements for $G_{\beta\ell}$ and $\mathcal{J}_k$. This result, originally found by Weinberg [10], is here shown to be a general consequence of the null-plane QCD operator algebra, valid in any Lorentz frame.

It is important to emphasize that the $SU(2N)$ symmetry found here is only operative in the full interacting field theory. It is therefore unrelated to the $SU(2N)$ invariance of the QCD Lagrangian in the limit of no interaction. Indeed we have show above in section 3.5 that eq. 5.4, the main ingredient in the derivation of $SU(2N)$, in itself implies the existence of Goldstone bosons. In addition, in a special case, this symmetry does emerge in a well-defined limit of QCD. As $\langle h' | e^a | h \rangle \sim M_h - M_{h'}$, and baryons within a given large-$N_c$ multiplet have mass splittings that scale as $1/N_c$ [81], the large-$N_c$ QCD scaling rules suggest that for baryons $e^a \sim 1/N_c$. Of course, as the matrix element of chiral charges between baryon states scales as $N_c$, the $SU(2N)$ symmetry reduces to the contracted $SU(2N)$ [10, 82] for baryons in the large-$N_c$ limit, as one expects on general grounds [83–85].

It is instructive to consider a simple example. Consider the case $N = 3$. Using the chiral transformation properties of the quarks, eq. 4.39, one sees that a $\lambda = 3/2$ baryonic operator $\psi_{\lambda\lambda} \psi_{\lambda\lambda} \psi_{\lambda\lambda}$ transforms as $(1, 1)$, $(1, 8)$, or $(1, 10)$ with respect to $SU(3)_R \otimes SU(3)_L$. Therefore, if the baryon is a decuplet of $SU(3)_F$ with its $\lambda = 3/2$ part in the $(1, 10)$, then one easily checks that its $\lambda = 1/2$ part must transform as $(3, 6)$ or $(6, 3)$. However, the different helicity states are unrelated by chiral symmetry in itself. It is the mixed Lie bracket, eq. 5.4, the expression of broken chiral symmetry in the spin Hamiltonian, that relates the helicities. Indeed taking the $\lambda = 1/2$ decuplet to transform as $(3, 6)$ together with an octet spin-1/2 field and their negative-helicity partners in $(10, 1) \oplus (6, 3)$ together fill out the 56-dimensional representation of $SU(6)$ as is familiar from the quark model. The difference here is that this symmetry arises from QCD symmetries and their pattern of
breaking, and, in particular, has nothing to do with the non-relativistic limit. Hence we see that starting from the formal null-plane QCD operator algebra, the simple assumption that the part of the null-plane reduced Hamiltonian, $M^2$, that breaks chiral symmetry is small implies all of the usual consequences of the non-relativistic quark model, without the need of any further assumption like the existence of constituent quark degrees of freedom [10].

6 Conclusion

Usually one views the spontaneous breaking of a symmetry as the non-invariance of the vacuum state with respect to the symmetry. However, in relativistic theories of quantum mechanics, this picture is purely a matter of convention. We have seen that the front-form vacuum is a singlet with respect to all symmetries and yet spontaneous symmetry breaking can occur via non-conserved currents whose divergences are directly proportional to S-matrix elements for the emission and absorption of Goldstone bosons. One may view the null-plane description as a change of coordinates which moves dynamical information out of the vacuum state and into the interaction operators of the theory. The primary advantage of working with the null-plane description is that broken chiral symmetry constraints become manifest in the sense that there are non-trivial Lie brackets between the Poincaré generators and the broken symmetry generators. In the instant-form, the chiral constraints that appear naturally in the front-form are present, but require one to work in special Lorentz frames and to make assumptions about the asymptotic behavior of Goldstone-boson scattering amplitudes.

Here we will restate the main conclusions of this paper:

• In the front-form, spontaneous chiral symmetry breaking is contained entirely in the three null-plane reduced Hamiltonians, which encode the mass spectrum and spin content of a given theory. This must be the case as the null-plane chiral charges annihilate the vacuum state, and therefore chiral symmetry breaking cannot be attributed to the formation of chiral-symmetry breaking condensates. In null-plane QCD, all chiral symmetry breaking arises from the symmetry breaking parts of the reduced Hamiltonians, given explicitly in eqs. 4.52 and 4.53.

• Goldstone’s theorem on the null-plane follows directly from the Lie-brackets between the null-plane Hamiltonians and the chiral charges. A consistent null-plane description of spontaneous symmetry breaking requires that a small explicit symmetry-breaking operator be included and that this explicit symmetry breaking be taken to zero only at the level of matrix elements of operators. The divergence of the axial-vector current is proportional to the explicit symmetry breaking. Therefore, as the current cannot be conserved in the symmetry limit, the existence of massless states arises as a consequence of the need to cancel the explicit breaking parameter that appears in its divergence.

• The Gell-Mann-Oakes-Renner relation is recovered in null-plane QCD and a general
A prescription exists for translating all chiral-symmetry breaking condensates in instant-form QCD to chiral-singlet condensates in null-plane QCD. It is therefore simplistic to say that the vacuum is trivial in the front-form, since there are necessarily symmetry-preserving condensates which arise from modes with strictly zero longitudinal momentum. In particular, in contrast with claims in the literature [32–34, 39], we expect that the QCD vacuum energy is unaltered in moving from the instant-form to the front-form descriptions of QCD, as is essential for the consistency of null-plane QCD.

• A simple solution of the null-plane operator algebra recovers the spin-flavor symmetry of the constituent quark model. This result was obtained originally in Ref. [10], which obtained the algebra of charges and Hamiltonians by working with sum rules obtained in special Lorentz frames, and using input from Regge-pole theory expectations of the asymptotic behavior of scattering amplitudes involving Goldstone bosons. The results of the present work may be viewed as an attempt to clarify this original work by formulating it in a Lorentz frame-independent manner which follows directly from null-plane QCD.

In the null-plane formulation of QCD, the loss of manifest Lorentz invariance and locality are, operationally, a result of integrating out non-dynamical degrees of freedom. Physically, it is clear that the loss of Lorentz invariance is tied to the fact that the essence of Lorentz invariance lies in the Poincaré Lie brackets that must be satisfied by the spin generators, and, of course, on the null-plane spin is dynamical and therefore requires the solution of the theory to properly implement. By contrast, the non-locality of the theory would appear to be related to the fact that the null-plane chiral symmetry constraints on observables are properly formulated as sum rules which span many energy scales, and therefore do not exhibit the separation of scales that allows a useful description in terms of local Lagrangian effective field theory. Indeed, it appears that, in some sense, scattering amplitudes are the fundamental objects in the null-plane formulation. This is particularly clear from the Lie-brackets that mix the Poincaré and chiral symmetry generators, which are given by the S-matrix elements for Goldstone boson emission and absorption. From a theoretical standpoint, the most interesting consequences of the results obtained in this paper are apparent only in the large-$N_c$ limit, which will be treated separately.

I thank Ulf-G. Meißner for valuable comments on the manuscript, and T. Becher, G. Colangelo, H. Leutwyler, F. Niedermayer, and U. Wenger for useful discussions. I am particularly grateful to the Institute for Theoretical Physics at the University of Bern for providing a stimulating work environment during academic year 2010/2011. The Albert Einstein Center for Fundamental Physics is supported by the Innovations- und Kooperationsprojekt C-13 of the Schweizerische Universitätskonferenz SUK/CRUS. I gratefully acknowledge the hospitality of HISKP-theorie and the support of the Mercator programme of the Deutsche Forschungsgemeinschaft during academic year 2012/2013. This work was supported in part by NSF CAREER Grant PHY-0645570 and continuing grant PHY1206498.
A Null-plane conventions

We adopt the metric convention:

\[ g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]  

(A.1)

which takes the contravariant coordinate four-vector \( x^\mu = (x^0, x^1, x^2, x^3) = (t, x, y, z) \) to the covariant coordinate four-vector \( x_\mu = g_{\mu\nu} x^\nu = (x_0, -x_1, -x_2, -x_3) \). With \( x^+ \equiv x \cdot n \) and \( x^- \equiv x \cdot n^* \), we denote the null-plane contravariant coordinate four-vector by \( \tilde{x}^\mu = (x^+, x^1, x^2, x^-) \). Then we have

\[ \tilde{x}^\mu = C^\mu_\nu x^\nu , \]  

(A.2)

with

\[ C^\mu_\nu = \begin{pmatrix} 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 0 & -1/\sqrt{2} \end{pmatrix} . \]  

(A.3)

This matrix transforms all Lorentz tensors in the instant-form notation to the front-form notation. For instance, the null-plane metric tensor is given by

\[ \tilde{g}^{\mu\nu} = (C^{-1})^{\mu}_{\alpha} g_{\alpha\beta} (C^{-1})^{\beta}_{\nu} \]  

(A.4)

which gives

\[ \tilde{g}^{\mu\nu} = \tilde{g}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} . \]  

(A.5)

We can now form the scalar product

\[ x \cdot p = x^\mu p_\mu = x^+ p_+ + x^- p_- + x^1 p_1 + x^2 p_2 = x^+ p^- + x^- p^+ - \mathbf{x}_\perp \cdot \mathbf{p}_\perp . \]  

(A.6)

The indices \( i, j, k, \ldots \) are spatial indices that range over 1, 2, 3, and \( r, s, t, \ldots \) are transverse indices that range over 1, 2. We place all transverse coordinates, momenta and fields in boldface, and additionally label coordinates and momenta with the \( \perp \) symbol. The totally antisymmetric symbol is

\[ \epsilon_{+12} = 1 = \epsilon_{+12} = 1 . \]  

(A.7)

Note that \( \partial_+ = \partial^- \) is a time-like derivative \( \partial / \partial x^+ = \partial / \partial x^- \) as opposed to \( \partial_- = \partial^+ \), which is a space-like derivative \( \partial / \partial x^- = \partial / \partial x^+ \). Many more useful relations can be found in Ref. [48].
Consider the Lagrangian of a free fermion of mass $m$,
\[ \mathcal{L}(x) = \bar{\psi}(x) \left[ \frac{i}{2} \left( \partial_\mu - \partial^\mu \right) \gamma^\mu - m \right] \psi(x). \]  
(B.1)

The Dirac equations of motion for the fermion and anti-fermion fields are:
\[ \left( i \gamma^\mu \partial_\mu - m \right) \psi(x) = 0, \quad \bar{\psi}(x) \left( i \gamma^\mu \partial_\mu + m \right) = 0. \]  
(B.2)

In order to express the Lagrangian in null-plane coordinates such that the null-plane dispersion relation is recovered, the fermion field is decomposed into two components,
\[ \psi = \Pi^+ \psi + \Pi^- \psi \equiv \psi_+ + \psi_-, \]  
(B.3)

where the projection operator is defined as
\[ \Pi^\pm \equiv \frac{1}{2} \gamma^\pm, \quad \gamma^0 = \gamma^0, \quad \gamma^3 = \gamma^3 \]  
(B.4)

Application of the projection operator to the Dirac equation then gives
\[ 2i \partial^\pm \psi_\pm = \left( -i \gamma^r \partial^r + m \right) \gamma^+ \psi_+, \quad 2i \psi^\dagger_\pm \partial^\pm = \psi^\dagger_\pm \left( i \gamma^r \partial^r - m \right), \]  
(B.5)

which reveals that the $\psi_-$ field is non-dynamical. One can solve for $\psi_-$ by inverting the longitudinal coordinate derivative operator to give
\[ \psi_- = \frac{1}{2i} \left( -i \gamma^r \partial^r + m \right) \gamma^+ \psi_+, \quad \psi^\dagger_- = \psi^\dagger_+ \left( i \gamma^r \partial^r - m \right), \]  
(B.6)

where $(1/\partial^+) \partial^+ = \partial^+ (1/\partial^+) = 1$. An explicit representation of this operator can be taken as:
\[ \left( \frac{1}{\partial^+} \right) f(x^+, x^-, x_\perp) = \frac{1}{4} \int_{-\infty}^{+\infty} dy^- f(x^+, x^-, y^-) f(x^+, y^-, x_\perp), \]  
(B.7)

where $\epsilon(z) = -1, 0, 1$ for $x > 0, = 0, < 0$, respectively. Now, using eq. B.3 and the constraint equation, eq. B.6, gives the null-plane free-fermion Lagrangian,
\[ \tilde{\mathcal{L}}(x) = -\psi^\dagger_+(x) \frac{\Box^2 + m^2}{\sqrt{2i\partial^+}} \psi_+(x), \]  
(B.8)

where $\Box^2 \equiv 2\partial^+ \partial^- - \partial^r \partial^r$.

It is useful to list the Poincaré generators in the free fermion theory. We take
\[ T^{\mu\nu} = -g^{\mu\nu} \mathcal{L} + \frac{i}{2} \bar{\psi} \gamma^\nu \partial^\mu \psi \]  
(B.9)

as the free-fermion energy-momentum tensor. The free-fermion Poincaré generators are then obtained via
\[ \hat{P}^\mu = \int dx^- d^2 x_\perp T^{\mu+}; \]  
(B.10)
\[ \hat{M}^{\mu\nu} = \int dx^- d^2 x_\perp \left( x^\mu T^{\nu+} - x^\nu T^{\mu+} + \frac{i}{4} \bar{\psi} \{ \gamma^\nu, \sigma^{\mu\nu} \} \psi \right), \]  
(B.11)
where $\sigma^{\mu\nu} = i[\gamma^\mu, \gamma^\nu]/2$. The free-fermion stability group generators are [14]:

\[ P^r = i\sqrt{2} \int dx^\perp d^2 x_\perp \psi^\dagger_+(x) \partial^r \psi_+(x); \tag{B.12} \]

\[ P^+ = i\sqrt{2} \int dx^\perp d^2 x_\perp \psi^\dagger_+(x) \partial^+ \psi_+(x); \tag{B.13} \]

\[ E^r = i\sqrt{2} \int dx^\perp d^2 x_\perp \psi^\dagger_+(x) \left( x^r \partial^r - x^+ \partial^r \right) \psi_+(x); \tag{B.14} \]

\[ K^3 = i\sqrt{2} \int dx^\perp d^2 x_\perp \psi^\dagger_+(x) \left[ -x^+ \frac{1}{2\partial^+} \left( -\partial^r \partial^r + m^2 \right) - x^- \partial^+ + \frac{1}{2} \right] \psi_+(x); \tag{B.15} \]

\[ J^3 = i\sqrt{2} \int dx^\perp d^2 x_\perp \psi^\dagger_+(x) \epsilon^r \left( x^r \partial^r + \frac{1}{3} \gamma^r \gamma^3 \right) \psi_+(x), \tag{B.16} \]

and the Hamiltonians are:

\[ P^- = i\sqrt{2} \int dx^\perp d^2 x_\perp \psi^\dagger_+(x) \frac{1}{2\partial^+} \left( -\partial^r \partial^r + m^2 \right) \psi_+(x); \tag{B.17} \]

\[ F^r = i\sqrt{2} \int dx^\perp d^2 x_\perp \psi^\dagger_+(x) \left[ -x^r \frac{1}{2\partial^+} \left( -\partial^r \partial^r + m^2 \right) - x^- \partial^r \right. \]

\[ \left. - \frac{\gamma^r}{2\partial^+} \left( -\gamma^a \partial^a + im \right) \right] \psi_+(x). \tag{B.19} \]

It is clear that the null-plane dispersion relation, eq. 2.5, is correctly reproduced by eq. B.17.

The dynamical fermion field $\psi_+$ can be expressed in momentum space as

\[ \psi_+(x) = \sum_{\lambda = \uparrow, \downarrow} \int \frac{dk^+ d^2 k_\perp}{2k^+ (2\pi)^3} \left\{ b_\lambda(k^+, k_\perp) u_+(k, \lambda) e^{-ik^+ x} + d_\lambda^\dagger(k^+, k_\perp) v_+(k, \lambda) e^{ik^+ x} \right\}, \quad \tag{B.20} \]

where $b_\lambda(k^+, k_\perp)$ destroys a fermion and $d_\lambda^\dagger(k^+, k_\perp)$ creates an antifermion. This decomposition is meaningful only on the initial surface, $x^+ = 0$, where the fermions are free. The creation/destruction operators satisfy the anti-commutation relations

\[ \{ b_\lambda(k^+, k_\perp), b_\lambda^\dagger(k'^+, k'_\perp) \} = 2k^+ (2\pi)^3 \delta(k^+ - k'^+) \delta^2(k_\perp - k'_\perp) \delta_{\lambda\lambda'}; \tag{B.21} \]

\[ [ d_\lambda(k^+, k_\perp), d_\lambda^\dagger(k'^+, k'_\perp) ] = 2k^+ (2\pi)^3 \delta(k^+ - k'^+) \delta^2(k_\perp - k'_\perp) \delta_{\lambda\lambda'}. \tag{B.22} \]

which in turn imply that the fermion field $\psi_+$ satisfies

\[ \{ \psi_+(x), \psi^\dagger_+(y) \}_{x^+ = y^+} = \frac{1}{\sqrt{2}} \Pi^+ \delta(x^+ - y^+) \delta^2(x_\perp - y_\perp). \tag{B.23} \]

The solutions of the free Dirac equation in the chiral representation of the gamma matrices are [5]:

\[ u(k, \uparrow) = \frac{1}{2^{1/4} \sqrt{k^+}} \begin{pmatrix} \sqrt{2} k^+ \\ k_\perp \\ m \\ 0 \end{pmatrix}, \quad u(k, \downarrow) = \frac{1}{2^{1/4} \sqrt{k^+}} \begin{pmatrix} 0 \\ m \\ -k_\perp \\ \sqrt{2} k^+ \end{pmatrix}; \tag{B.24} \]

\[ v(k, \uparrow) = \frac{1}{2^{1/4} \sqrt{k^+}} \begin{pmatrix} 0 \\ -m \\ -k_\perp \\ \sqrt{2} k^+ \end{pmatrix}, \quad v(k, \downarrow) = \frac{1}{2^{1/4} \sqrt{k^+}} \begin{pmatrix} \sqrt{2} k^+ \\ k_\perp \\ -m \\ 0 \end{pmatrix}. \tag{B.25} \]
where $k_1 \equiv k_1 + ik_2$ and $\bar{k}_1 \equiv k_1 - ik_2$. Projecting out the dynamical spinors gives

$$u_+ (k, \uparrow) = \Pi^+ u(k, \uparrow) = 2^{1/4} \sqrt{k^+} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = v_+ (k, \downarrow) = \Pi^+ v(k, \downarrow); \quad (B.26)$$

$$u_+ (k, \downarrow) = \Pi^+ u(k, \downarrow) = 2^{1/4} \sqrt{k^+} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = v_+ (k, \uparrow) = \Pi^+ v(k, \uparrow), \quad (B.27)$$

which leads to the eigenvalue equations,

$$u_+^\dagger (k, \lambda) \gamma^5 u_+ (k, \lambda) = u_+^\dagger (k, \lambda) 2\Sigma_{12} u_+ (k, \lambda) = 2\lambda \sqrt{2} k^+; \quad (B.28)$$

$$v_+^\dagger (k, \lambda) \gamma^5 v_+ (k, \lambda) = v_+^\dagger (k, \lambda) 2\Sigma_{12} v_+ (k, \lambda) = -2\lambda \sqrt{2} k^+, \quad (B.29)$$

where $\Sigma_{12} \equiv \gamma_1 \gamma_2 / 2$. The relation between chirality and helicity in the null-plane formulation arises from these relations which arise from the fact that each of the fields has only a single non-vanishing component. Now it is a straightforward matter to express the Poincaré generators in the momentum-space representation. For instance, comparing eqs. 2.30, 2.31, and B.16 gives the free-fermion helicity operator,

$$\mathcal{J}^3 = i\sqrt{2} \int dx^- d^2 x_\perp \psi_+^\dagger (x) \Sigma^{12} \psi_+ (x), \quad (B.30)$$

which, using eqs. B.20 and B.29, is found to have the momentum-space representation

$$\mathcal{J}^3 = \sum_{\lambda=\uparrow \downarrow} \lambda \int \frac{dk^+ d^2 k_\perp}{2k^+(2\pi)^3} \left\{ b_\lambda^\dagger (k^+, k_\perp) b_\lambda (k^+, k_\perp) + d_\lambda^\dagger (k^+, k_\perp) d_\lambda (k^+, k_\perp) \right\}. \quad (B.31)$$

This operator explicitly counts the helicity of the fermions and the antifermions.

### C Free gauge fields decomposed

Consider the Lagrangian of a free gluon field,

$$\mathcal{L}(x) = -\frac{1}{4} F_{\mu\nu}^a(x) F_{\mu\nu}^a(x). \quad (C.1)$$

The equation of motion is

$$D_{\mu}^{ab} F_{b}^{\mu\nu} = 0, \quad (C.2)$$

where $D_{\mu}^{ab} = \delta^{ab} \partial_\mu + g f^{abc} A_\mu^c$, where here $f^{abc}$ are $SU(3)$ structure constants. The gauge potential can be expressed in null-plane coordinates as

$$A^\mu = (A^+, A, A^-) \quad (C.3)$$
where $A^+ = n \cdot A$, $A^- = n^* \cdot A$ and $A = (A^1, A^2)$. Working in light-cone gauge, $A^+ = 0$, one finds
\[
\partial^+ A^- = -\frac{1}{\partial^+} D_{ab}^{\mu} \partial^+ A_b^\mu. \tag{C.4}
\]
Therefore $A^-_a$ is non-dynamical and can be integrated out, giving
\[
\mathcal{L}(x) = -\frac{1}{4} F_{a}^{rs} F_{a}^{rs} + (\partial^+ A^r_a) (\partial^- A^r_a) - \frac{1}{2} \left( \frac{1}{\partial^+} D_{ab}^{\mu} \partial^+ A_b^\mu \right)^2. \tag{C.5}
\]
The light-cone gauge does not fix the gauge entirely and therefore to eliminate all redundancy one should assign a boundary condition to the transverse gauge field; e.g. $A^r_a(x^+, x_\perp, x^- = \infty) = 0$.

D  \textit{SU}(N) conventions

The fundamental representation $SU(N)$ generators $T_a$ with $\alpha = 1, \ldots, N^2 - 1$ satisfy:
\[
[T_a, T_\beta] = i f_{a\beta\gamma}; \tag{D.1}
\]
\[
\{ T_\alpha, T_\beta \} = \frac{1}{N} 1^\alpha + d_{a\beta\gamma} T_\gamma, \tag{D.2}
\]
where $1$ is the $N \times N$ unit matrix, and hence are normalized such that $\text{Tr}(T_a T_\beta) = \delta_{a\beta}/2$.
The structure constants satisfy the relations:
\[
f_{\alpha\mu\nu} f_{\beta\mu\nu} = N \delta_{a\beta}; \tag{D.3}
\]
\[
d_{\alpha\mu\nu} d_{\beta\mu\nu} = \frac{N^2 - 4}{N} \delta_{a\beta}. \tag{D.4}
\]
An additional useful relation is:
\[
f_{\alpha\beta\nu} f_{\gamma\mu\nu} = \frac{2}{N} (\delta_{\alpha\gamma} \delta_{\beta\mu} - \delta_{\alpha\mu} \delta_{\beta\gamma}) + d_{\alpha\gamma\nu} d_{\beta\mu\nu} - d_{\beta\gamma\nu} d_{\alpha\mu\nu}. \tag{D.5}
\]
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