On the Mattila–Sjölin distance theorem for product sets

Doowon Koh1 | Thang Pham2 | Chun-Yen Shen3

1Department of Mathematics, Chungbuk National University, Chungcheongbuk-do, South Korea
2University of Science, Vietnam National University, Hanoi, Vietnam
3Department of Mathematics, National Taiwan University and NCTS, Taipei City, Taiwan

Correspondence
Chun-Yen Shen, Department of Mathematics, National Taiwan University, Roosevelt Rd, Daan District, Taipei City 10617, Taiwan.
Email: cyshen@math.ntu.edu.tw

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1 INTRODUCTION

Let $E$ be a compact set in $\mathbb{R}^d$, we denote its distance set by $\Delta(E)$, namely,

$$\Delta(E) := \{ |x - y| : x, y \in E \}.$$ 

The classical Falconer distance conjecture says that if the Hausdorff dimension of $E$, denoted by $\dim_H(E)$, is greater than $\frac{d}{2}$, then the Lebesgue measure of the distance set $\Delta(E)$ is positive.
In 1986, Falconer [5] proved that if \( \dim_H(E) > \frac{d+1}{2} \), then \( L^1(\Delta(E)) > 0 \). For \( d = 2 \), the best current result is due to Guth, Iosevich, Ou and Wang [7]. More precisely, they showed that the condition \( \dim_H(E) > \frac{5}{4} \) would be enough. This result has been recently extended to all even dimensions \( d \geq 4 \) by Du, Iosevich, Ou, Wang and Zhang [2]. In odd dimensions, the current thresholds are \( \frac{3}{2} + \frac{3}{10} \) for \( d = 3 \) by Du, Guth, Ou, Wang, Wilson and Zhang [1], and \( \frac{d}{2} + \frac{d}{4d-2} \) for \( d \geq 5 \) by Du and Zhang [3].

In another direction, Mattila and Sjölin [11] proved a stronger result, namely, if \( \dim_H(E) > \frac{d+1}{2} \), then the distance set not only has positive Lebesgue measure, but it also contains an interval. This was done by showing that the distance measure has a continuous density. To the best of our knowledge, it is not known in the literature if the threshold \( \frac{d+1}{2} \) for general sets \( E \) is the best possible in order for \( \Delta(E) \) to have non-empty interior. But for the product of Cantor sets in \( \mathbb{R}^2 \), it was shown by Simon and Taylor [13] that one can do better. For instance, if \( E = C_{1/3} \times C_{1/3} \), where \( C_{1/3} \) is the classical middle thirds Cantor set, then their results show that \( \Delta(E) \) contains an interval (see Theorem 2.16 and Corollary 2.13 in [13]). In addition, Iosevich, Taylor and Mourougoglou [8] recovered Mattila and Sjölin’s result using a relatively simple Fourier analytic argument and extended it to more general metrics. In this paper, we make another progress in this direction. More precisely, let \( E = A \times A \cdots \times A \subset \mathbb{R}^d \) for a compact set \( A \subset \mathbb{R} \). It follows from Mattila and Sjölin’s theorem that if \( \dim_H(A) > \frac{d+1}{2d} \), then \( \Delta(A^d) \) has non-empty interior. We use \( \text{Int}(\Delta(A^d)) \) to denote the set of the interior points of the distance set \( \Delta(A^d) \). In the following theorems, we prove that the bound \( \frac{d+1}{2d} \) can be improved whenever \( d \geq 5 \).

First, for \( d \geq 10 \), we give an improvement of the threshold \( \frac{d+1}{2d} \). More precisely, we obtain the following.

**Theorem 1.1.** Let \( A \subset \mathbb{R} \) be compact and \( d \geq 10 \). Suppose that \( \dim_H(A) > \frac{d-2}{2(d-4)} - \frac{40}{57(d-4)} = \frac{d+1}{2d} - \frac{23d-228}{114d(d-4)} \), then \( \text{Int}(\Delta(A^d)) \neq \emptyset \).

An improvement of the bound \( \frac{d+1}{2d} \) for \( d \geq 5 \) is given as follows.

**Theorem 1.2.** Let \( A \subset \mathbb{R} \) be compact and \( d \geq 5 \). Suppose that \( \dim_H(A) > \frac{3d-2}{6d-8} = \frac{d+1}{2d} - \frac{d-4}{2d(3d-4)} \), then \( \text{Int}(\Delta(A^d)) \neq \emptyset \).

It follows from a direct computation that when dimension \( d \geq 27 \), the condition in Theorem 1.1 is better than that in Theorem 1.2. Hence, we obtain the following theorem by combining this fact with the result \( \frac{d+1}{2d} \) from Mattila and Sjölin’s theorem.

**Theorem 1.3.** Let \( A \) be a compact set in \( \mathbb{R} \). Then we have \( \text{Int}(\Delta(A^d)) \neq \emptyset \) provided that

\[
\dim_H(A) > \begin{cases} 
\frac{d+1}{2d} & \text{if } 2 \leq d \leq 4, \\
\frac{d+1}{2d} - \frac{d-4}{2d(3d-4)} & \text{if } 5 \leq d \leq 26, \\
\frac{d+1}{2d} - \frac{23d-228}{114d(d-4)} & \text{if } 27 \leq d.
\end{cases}
\]
2 | PROOFS OF THEOREM 1.1 AND THEOREM 1.2

Before we start proving our results, let us give an overview about the strategy to improve the dimensional thresholds. For $d \geq 5$, set $n = d - 3$. We first study the distance sets generated by the parabolic equation $P : \mathbb{R}^n \to \mathbb{R}$ defined by $P(x) = x_1^2 + \cdots + x_{n-1}^2 + x_n$, where $x = (x_1, \ldots, x_n)$. Then we show that the distance set $\Delta(A^d)$ can be understood via $P(\mathcal{E} - \mathcal{F})$ for some suitable chosen sets $\mathcal{E}$ and $\mathcal{F}$ in $\mathbb{R}^n$. More precisely, we define

$$\mathcal{E} = A^{d-4} \times SQ(\Delta(A^2)) \subset \mathbb{R}^{d-3}, \quad \mathcal{F} = A^{d-4} \times -SQ(\Delta(A^2)) \subset \mathbb{R}^{d-3},$$

where $SQ(X) = \{x^2 : x \in X\}$ for $X \subset \mathbb{R}$.

We observe that $SQ(\Delta(A^d)) = P(\mathcal{E} - \mathcal{F})$. Thus, if $P(\mathcal{E} - \mathcal{F})$ contains an interval, so does $\Delta(A^d)$. Notice that $\mathcal{E}$ and $\mathcal{F}$ are sets in $\mathbb{R}^{d-3}$. If $\dim_H(\mathcal{E})$ and $\dim_H(\mathcal{F})$ are greater than $\frac{d-3+1}{2} = \frac{d-2}{2}$, then we can use Lemma 2.2 below to conclude the proof. Therefore, the problem now is reduced to the question of showing that $\dim_H(\mathcal{E}), \dim_H(\mathcal{F}) > \frac{d-2}{2}$. These dimensional conditions will be satisfied by applying Lemma 2.4, and results of Shmerkin [12] and Liu [9] on the dimension of distance sets.

We are now ready to give more details. We first recall an important and well-known result, see Theorem 1.2.1 in [14] or the related paper [8].

**Lemma 2.1.** Let $S$ be a smooth convex hypersurface in $\mathbb{R}^n$, $n \geq 2$, with everywhere non-vanishing Gaussian curvature, and $\mu$ be a $C^\infty_0$ measure on $S$. Then

$$|\hat{\mu}(\xi)| \lesssim |\xi|^{-\frac{n+1}{2}}.$$

Given two compact sets $\mathcal{E}, \mathcal{F} \subset \mathbb{R}^n$, by scaling and translation we may assume $\mathcal{E}, \mathcal{F} \subset [0,1]^n$. Given the parabolic equation $P : \mathbb{R}^n \to \mathbb{R}$ defined by

$$P(x) = x_1^2 + \cdots + x_{n-1}^2 + x_n,$$

and since we are concerned with the parabolic distances set $P(\mathcal{E} - \mathcal{F})$, we can always assume the parabolic surface that we will work on throughout the paper is contained in the fixed bounded ball $B(0,R)$. Moreover, it is straightforward to check that the parabolic surface has everywhere non-vanishing Gaussian curvature. We now can prove the following lemma.

**Lemma 2.2.** Let $\mathcal{E}$ and $\mathcal{F}$ be compact sets in $\mathbb{R}^n$, $n \geq 2$, with $\dim_H(\mathcal{E}) > \frac{n+1}{2}$ and $\dim_H(\mathcal{F}) > \frac{n+1}{2}$. Let $P : \mathbb{R}^n \to \mathbb{R}$ defined by $P(x) = x_1^2 + \cdots + x_{n-1}^2 + x_n$. Then the set $P(\mathcal{E} - \mathcal{F}) := \{P(x-y) : x \in \mathcal{E}, y \in \mathcal{F}\}$ has non-empty interior.

The proof of Lemma 2.2 is roughly inspired by the work of Iosevich, Mourgoglou and Taylor [8]. The main new difference is the way we construct the $P$-distance measure $\nu_{\mathcal{E},\mathcal{F}} \in \mathcal{M}(P(\mathcal{E} - \mathcal{F}))$ which will satisfy our desired properties.

**Proof.** We first note that if $\dim_H(\mathcal{E}) > \frac{n+1}{2}$ and $\dim_H(\mathcal{F}) > \frac{n+1}{2}$, then as a consequence of [4, Theorem 1.8] for two sets, the set $P(\mathcal{E} - \mathcal{F})$ is of positive Lebesgue measure. We now show that it also has non-empty interior.
Let $\mu_\mathcal{E}$ and $\mu_\mathcal{F}$ be probability measures in $\mathcal{M}(\mathcal{E})$ and $\mathcal{M}(\mathcal{F})$, respectively, such that for any ball $B(x, r)$, we have

$$\mu_\mathcal{E}(B(x, r)) \lesssim r^{\frac{n+1}{2}}, \quad \mu_\mathcal{F}(B(x, r)) \lesssim r^{\frac{n+1}{2}},$$

and $I_{\frac{n+1}{2}}(\mu_\mathcal{E}) < \infty$, $I_{\frac{n+1}{2}}(\mu_\mathcal{F}) < \infty$, where we recall that $I_s(\mu) := \int \int |x - y|^{-s} d\mu(x) d\mu(y)$.

We now construct a measure in $\mathcal{M}(P(\mathcal{E} - \mathcal{F}))$. Let $J : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a positive function defined as follows: for $x = (x_1, \ldots, x_{n-1}, x_n), \ y = (y_1, \ldots, y_{n-1}, y_n) \in \mathbb{R}^n$,

$$J(x, y) = \sqrt{1 + 4(x_1 - y_1)^2 + \cdots + 4(x_{n-1} - y_{n-1})^2}.$$

Consider a measure $J(x, y) d\mu_\mathcal{E}(x) d\mu_\mathcal{F}(y)$ supported on $\mathcal{E} \times \mathcal{F}$ and define $\bar{P}(x, y) = P(x - y)$. Next we define the $P$-distance measure $\nu_{\mathcal{E}, \mathcal{F}} \in \mathcal{M}(P(\mathcal{E} - \mathcal{F}))$ by

$$\nu_{\mathcal{E}, \mathcal{F}}(B) = \int_{\bar{P}^{-1}(B)} J(x, y) d\mu_\mathcal{E}(x) d\mu_\mathcal{F}(y),$$

for Borel sets $B \subset \mathbb{R}$.

In other words, $\nu_{\mathcal{E}, \mathcal{F}}$ is the pushforward measure defined as the image of $J(\mathcal{E} \times \mathcal{F})$ under the map $\bar{P}(x, y) = P(x - y)$. Thus, for any continuous function $\varphi$ on $\mathbb{R}$, we have

$$\int \varphi d\nu_{\mathcal{E}, \mathcal{F}} = \int \int \varphi(P(x - y)) J(x, y) d\mu_\mathcal{E}(x) d\mu_\mathcal{F}(y).$$

It is not hard to show that the support of $\nu_{\mathcal{E}, \mathcal{F}}$ is contained in $P(\mathcal{E} - \mathcal{F})$, which is an important necessary condition for $P(\mathcal{E} - \mathcal{F})$ to have non-empty interior. Indeed, since $\nu_{\mathcal{E}, \mathcal{F}} = (\bar{P})_* (J(x, y) \mu_\mathcal{E}(x) \mu_\mathcal{F}(y))$, we have $\text{spt}(\nu_{\mathcal{E}, \mathcal{F}}) \subset \bar{P} (\text{spt}(J(\mu_\mathcal{E} \times \mu_\mathcal{F}))) = \bar{P} (\text{spt}(\mu_\mathcal{E} \times \mu_\mathcal{F})) \subset P(\mathcal{E} - \mathcal{F})$.

Moreover, we have a simple fact that if both $\mu'_\mathcal{E} \to \mu_\mathcal{E}$ and $\mu'_\mathcal{F} \to \mu_\mathcal{F}$ weakly, then $\nu'_{\mathcal{E}, \mathcal{F}} \to \nu_{\mathcal{E}, \mathcal{F}}$ weakly. For compactly supported smooth functions $f_1, f_2$ on $\mathbb{R}^n$, we can also define the pushforward measure $\nu_{f_1, f_2}$ by

$$\int g d\nu_{f_1, f_2} = \int \int g(P(x - y)) J(x, y) f_1(x) f_2(y) dx dy,$$

for any continuous function $g$ on $\mathbb{R}$. However, since $f_1$ and $f_2$ are smooth and compactly supported, the measure $\nu_{f_1, f_2}$ is actually a function defined by

$$\nu_{f_1, f_2}(t) = \int (\sigma_t * f_2)f_1,$$

where $\sigma_t, t \in \mathbb{R}$, denotes the surface measure on the surface $P_t := \{y \in \mathbb{R}^n : P(y) = t\}$. 

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1 For a Borel set $B$ in $\mathbb{R}^n$, we denote by $\mathcal{M}(B)$ the collection of Borel measures $\mu$ with a compact support $\text{spt}(\mu) \subset B$ and $0 < \mu(B) < \infty$. 
To see this, we first notice that for $y = (y', y_n) \in P_1$, we have

$$d\sigma_t(y) = \sqrt{1 + |\nabla \phi_t(y')|^2} \, dy' = \sqrt{1 + 4|y'|^2} \, dy' ,$$

where $\phi_t(y') = t - |y'|^2$ for $y' \in \mathbb{R}^{n-1}$. Next, we observe that for any continuous function $g$ with compact support in $\mathbb{R}$,

$$\int g(t) \int (\sigma_t \ast f_2)(x)f_1(x) \, dx \, dt = \int g(t) \int \int f_2(x - y)d\sigma_t(y)f_1(x) \, dx \, dt = \int g(t) \int \int f_2(x' - y', x_n - \phi_t(y'))\sqrt{1 + 4|y'|^2} \, dy' f_1(x) \, dx \, dt ,$$

which is by Fubini’s theorem and a change of variables, $y_n = \phi_t(y')$,

$$= \int \int g(P(y))f_2(x - y)\sqrt{1 + 4|y'|^2} \, dy f_1(x) \, dx ,$$

which is by a change of variables, replacing $y$ by $x - y$,

$$= \int \int g(P(x - y))f_2(y)\sqrt{1 + 4|(x' - y')|^2} \, dy f_1(x) \, dx ,$$

which is by Fubini’s theorem and definition of $J(x, y)$,

$$= \int \int g(P(x - y))J(x, y)f_1(x)f_2(y) \, dx \, dy = \int g \, dv_{f_1, f_2} .$$

Now let $\Psi$ be a smooth and compactly supported function in $\mathbb{R}^n$ with $\int \Psi = 1$, and denote $\Psi_\varepsilon(x) = \varepsilon^{-n}\Psi(\frac{x}{\varepsilon})$ and $\mu_\varepsilon = \Psi_\varepsilon \ast \mu$. Therefore, we have $\hat{\mu}_\varepsilon(\xi) = \hat{\Psi}(\frac{\xi}{\varepsilon})\hat{\mu}(\xi) \to \hat{\mu}(\xi)$ for all $\xi \in \mathbb{R}^n$. Now

$$\nu_{\mu_\varepsilon, \mu_\varepsilon}(t) = \int (\sigma_t \ast \mu_\varepsilon)\mu_\varepsilon = \int \hat{\sigma}_t \hat{\mu}_\varepsilon \hat{\mu}_\varepsilon ,$$

which is

$$\int \hat{\sigma}_t(\xi)|\hat{\Psi}(\frac{\xi}{\varepsilon})|^2|\hat{\mu}_\varepsilon(\xi)|\mu_\varepsilon(\xi) d\xi .$$

Recall that $I_{n+1}(\mu_\varepsilon) \sim \int |\xi|^{\frac{1-n}{2}} |\hat{\mu}_\varepsilon(\xi)|^2 d\xi < \infty$ and $I_{n+1}(\mu_\varepsilon) \sim \int |\xi|^{\frac{1-n}{2}} |\mu_\varepsilon(\xi)|^2 d\xi < \infty$. These facts together with $|\hat{\sigma}_t(\xi)| \leq |\xi|^{-\frac{n-1}{2}}$ and Hölder’s inequality and Lebesgue’s dominated convergence theorem imply that when $\varepsilon \to 0$, we have

$$\int \hat{\sigma}_t \hat{\mu}_\varepsilon \mu_\varepsilon \to \int \hat{\sigma}_t \mu_\varepsilon \mu_\varepsilon .$$
Since we also have \( \nu_{\mu_{\xi}, \mu_{\eta}}(t) \) converges weakly to \( \nu_{\xi, \eta} \), we conclude that \( \nu_{\xi, \eta} \) is a function and

\[
\nu_{\xi, \eta}(t) = \int \hat{\mu}_{\eta}(\xi) \hat{\sigma}_{\xi}(\xi) d\xi. \tag{2.1}
\]

We now prove that the \( P \)-distance measure \( \nu_{\xi, \eta} \) is continuous. In other words, we show that

\[
\nu_{\xi, \eta}(t + h) - \nu_{\xi, \eta}(t) = \int \hat{\mu}_{\eta}(\xi) \hat{\sigma}_{\xi}(\xi)(\hat{\sigma}_{\xi+h}(\xi) - \hat{\sigma}_{\xi}(\xi)) d\xi,
\]

which goes to 0 as \( h \to 0 \). As a result the \( P \)-distance set must contain an interval which is what we want. Indeed, the continuity of \( \nu_{\xi, \eta} \) will be derived by Lebesgue’s dominated convergence theorem with the following conditions:

1. \( \lim_{h \to 0} (\hat{\sigma}_{\xi+h}(\xi) - \hat{\sigma}_{\xi}(\xi)) = 0. \)
2. \( |\hat{\sigma}_{\xi}(\xi)| \leq c(t)|\xi|^{n-\frac{1}{2}}. \)
3. \( \int |\hat{\mu}_{\eta}(\xi)| |\hat{\sigma}_{\xi+h}(\xi) - \hat{\sigma}_{\xi}(\xi)| d\xi < \infty. \)

For condition (1), \( \hat{\sigma}_{\xi+h}(\xi) - \hat{\sigma}_{\xi}(\xi) = \int e^{-2\pi ix \cdot \xi} d(\sigma_{\xi+h}(x) - \sigma_{\xi}(x)) \) which is equal to

\[
\int (e^{-2\pi ih\xi} - 1)e^{-2\pi ix \cdot \xi} d\sigma_{\xi}(x).
\]

Therefore, by Lebesgue’s dominated convergence theorem, we see that

\[
\lim_{h \to 0} (\hat{\sigma}_{\xi+h}(\xi) - \hat{\sigma}_{\xi}(\xi)) = 0.
\]

Condition (2) follows from Lemma 2.1. To check condition (3), we apply the Cauchy–Schwarz inequality to obtain

\[
\left| \int \hat{\mu}_{\eta}(\xi) |\hat{\sigma}_{\xi+h}(\xi) - \hat{\sigma}_{\xi}(\xi)| d\xi \right| \leq \int |\hat{\mu}_{\eta}(\xi)||\hat{\sigma}_{\xi+h}(\xi) - \hat{\sigma}_{\xi}(\xi)| d\xi
\]

\[
\leq \int |\hat{\mu}_{\eta}(\xi)||\hat{\sigma}_{\xi}(\xi)| |\xi|^{n-\frac{1}{2}} d\xi
\]

\[
\leq I_{n+1}(\mu_{\eta})^{\frac{1}{2}} \cdot I_{n+1}(\mu_{\eta})^{\frac{1}{2}} < \infty. \]

\[ \square \]

Remark 2.3. We would like to thank Allan Greenleaf and Alex Iosevich for pointing out to us after posting this paper to Arxiv a couple of months that the result of our Lemma 2.2 can also be derived from Theorem 1.5 in their paper with Krystal Taylor [6]. Indeed, we can define \( \Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) by

\[
\Phi(x, y) = P(x - y) = \sum_{j=1}^{n-1} (x_j - y_j)^2 + (x_n - y_n) = |x' - y'| + (x_n - y_n).
\]
Given \( t_0 \in \mathbb{R} \), let \( Z_{t_0} = \{(x, y) : \Phi(x, y) = t_0\} \). Then we can solve \( y_n \) in terms of \( x', y' \) and \( x_n \) that is, \( y_n = -t_0 + x_n + |x' - y'| \). Furthermore,

\[
\nabla \Phi(x', x_n, y', y_n) = (2(x' - y'), 1, -2(x' - y'), -1).
\]

Hence, the canonical relation \( C_{t_0} \) given by

\[
\{(x', x_n, 2(x' - y')\theta, \theta, y', y_n(\cdot), 2(x' - y')\theta, \theta) : x \in \mathbb{R}^n, y' \in \mathbb{R}^{n-1}, \theta \in \mathbb{R} \setminus \{0\}\}
\]

can be checked to be a canonical graph. Thus, its associated Radon transform adds \( \frac{n-1}{2} \) derivatives on \( L^2 \) Sobolev spaces which in turn through their Theorem 1.5 gives that \( P(\mathcal{E} - F) \) contains an interval as long as \( \dim_H(\mathcal{E}) + \dim_H(F) > n + 1 \). We refer the interested reader to the paper [6] for more details.

We now proceed to prove our results. First, we also need the following simple lemma.

**Lemma 2.4.** Let \( X \) be a set in \( \mathbb{R} \). We have

\[
\dim_H(X) = \dim_H(SQ(X)) = \dim_H(-SQ(X)),
\]

where we recall that \( SQ(X) = \{x^2 : x \in X\} \), and \( -SQ(X) = \{-x^2 : x \in X\} \).

**Proof.** Since it is obvious that \( \dim_H(SQ(X)) = \dim_H(-SQ(X)) \), we only need to show that \( \dim_H(X) = \dim_H(SQ(X)) \). We use a well-known fact that if \( f \) is a bi-Lipschitz map from \( \mathbb{R}^n \to \mathbb{R}^n \), then \( f \) preserves the Hausdorff dimension. Now given a set \( X \subset \mathbb{R} \), and without loss of generality we may assume \( X \subset (0, \infty) \). Let \( X_k = X \cap (k, k+1] \) for \( k = 0, 1, ..., \). It is clear that except for \( k = 0 \), the function \( f(x) = x^2 \) is bi-Lipschitz on the set \( X_k \). Therefore, we may assume our set \( X \subset (0,1] \), otherwise we are done. However we can consider \( X_k = X \cap (\frac{1}{k},1] \) so that for each \( k \) we have \( \dim_H(f(X_k)) = \dim_H(X_k) \). Finally \( \dim_H(X) = \dim_H(\cup X_k) = \sup_k \dim_H(X_k) = \dim_H(f(X_k)) = \dim_H(\cup f(X_k)) = \dim_H(f(X)) \) which gives the result. \( \square \)

We now recall two lemmas below. The results of these two lemmas show that the Hausdorff dimension of the distance set \( \Delta(\Omega) \) has a nontrivial lower bound if we only assume \( \dim_H(\Omega) > 1 \), where \( \Omega \subset \mathbb{R}^2 \). As we mentioned at the beginning of Section 2, these results play an important role in our proofs of Theorem 1.1 and Theorem 1.2.

The first lemma is due to Shmerkin, and the second is due to Liu.

**Lemma 2.5** [12]. For \( \Omega \subset \mathbb{R}^2 \) with \( \dim_H(\Omega) > 1 \), then we have

\[
\dim_H(\Delta(\Omega)) \geq \frac{40}{57}.
\]

**Lemma 2.6** [9]. For \( \Omega \subset \mathbb{R}^2 \) with \( \dim_H(\Omega) > 1 \), then we have

\[
\dim_H(\Delta(\Omega)) \geq \min \left\{ \frac{4}{3} \dim_H(\Omega) - \frac{2}{3}, 1 \right\}.
\]
To compare these two lemmas, we remark that when $1 < \dim_H(\Omega)$ and very close to 1, the lower bound in Lemma 2.5 is better. More precisely, one has

$$\dim_H(\Delta(\Omega)) \geq \begin{cases} 1 & \text{if } \frac{5}{4} \leq \dim_H(\Omega) \\ \frac{4}{3} \dim_H(\Omega) - \frac{2}{3} & \text{if } \frac{39}{38} \leq \dim_H(\Omega) \leq \frac{5}{4} \\ \frac{40}{57} & \text{if } 1 < \dim_H(\Omega) \leq \frac{39}{38}. \end{cases}$$

**Remark 2.7.** Let $\Omega = A \times A$ for some $A \subset \mathbb{R}$. Since $\dim_H(\Omega) = \dim_H(A^2) \geq 2 \dim_H(A)$, we can invoke both Lemma 2.5 and Lemma 2.6 whenever $\dim_H(A) > 1/2$. Here we used a known fact that $\dim_H(A \times B) \geq \dim_H(A) + \dim_H(B)$ which, for example, can be found in [10, Theorem 8.10].

We are ready to prove Theorems 1.1 and 1.2.

**Proof of Theorem 1.1.** Let $A \subset \mathbb{R}$ be a compact set. For $d \geq 5$, set $\mathcal{E} = A^{d-4} \times SQ(\Delta(A^2)) \subset \mathbb{R}^{d-3}$ and $\mathcal{F} = A^{d-4} \times -SQ(\Delta(A^2)) \subset \mathbb{R}^{d-3}$. Let $P$ be the parabolic equation in $(d-3)$ variables defined by

$$P(x) = x_1^2 + \cdots + x_{d-4}^2 + x_{d-3}^2.$$

We now observe that

$$P(\mathcal{E} - \mathcal{F}) = SQ(\Delta(A^d)).$$

Indeed, given a point $x \in \mathcal{E}$ and $y \in \mathcal{F}$, we can write $x = (x_1, \ldots, x_{d-4}, z_{d-3})$ and $y = (y_1, \ldots, y_{d-4}, -z_{d-3}')$, where $z_{d-3} = (x_{d-3} - y_{d-3})^2 + (x_{d-2} - y_{d-2})^2$, $z_{d-3}' = (x_{d-1} - y_{d-1})^2 + (x_{d} - y_{d})^2 \in SQ(\Delta(A^2))$. Therefore

$$P(x - y) = P(x_1 - y_1, \ldots, x_{d-4} - y_{d-4}, z_{d-3} + z_{d-3}') \in P(\mathcal{E} - \mathcal{F})$$

which is equivalent to

$$(x_1 - y_1)^2 + \cdots + (x_{d-4} - y_{d-4})^2 + (x_{d-3} - y_{d-3})^2 + (x_{d-2} - y_{d-2})^2$$

$$+ (x_{d-1} - y_{d-1})^2 + (x_{d} - y_{d})^2 \in SQ(\Delta(A^d)).$$

Thus, if $P(\mathcal{E} - \mathcal{F})$ contains an interval, so does $\Delta(A^d)$. Since $\mathcal{E}, \mathcal{F} \subset \mathbb{R}^{d-3}$, it follows from Lemma 2.2 (with $n = d - 3$) that if $\dim_H(\mathcal{E}) > \frac{d-3+1}{2} = \frac{d-2}{2}$ and $\dim_H(\mathcal{F}) > \frac{d-3+1}{2} = \frac{d-2}{2}$, then $P(\mathcal{E} - \mathcal{F})$ contains an interval. Hence, our proofs are reduced to proving the conditions $\dim_H(\mathcal{E}), \dim_H(\mathcal{F}) > \frac{d-2}{2}$ under our hypotheses that $d \geq 10$ and $s := \dim_H(A) > \frac{d-2}{2(d-4)} - \frac{40}{57(d-4)}$.

As mentioned in Remark 2.7, we need $\dim_H(A) > \frac{1}{2}$ in order to use Lemma 2.5 and Lemma 2.6. It is easy to check that $\dim_H(A) = s > \frac{1}{2}$. Recall that $\mathcal{E} = A^{d-4} \times SQ(\Delta(A^2))$ and $\mathcal{F} = A^{d-4} \times -$.
−SQ(Δ(A²)). Now, using Lemmas 2.4 and 2.5 with Ω = A × A, we have
\[ \dim_H(\mathcal{E}), \dim_H(\mathcal{F}) \geq s(d - 4) + \dim_H(\Delta(A^2)) \geq s(d - 4) + \frac{40}{57}. \]
Hence, the condition \( s(d - 4) + \frac{40}{57} > \frac{d-2}{2} \) is satisfied if \( s > \frac{d-2}{2(d-4)} - \frac{40}{57(d-4)} \). The theorem then follows.

We proceed to prove Theorem 1.2 whose proof is almost identical with that of Theorem 1.1 except that we use Lemma 2.6 instead of Lemma 2.5.

**Proof of Theorem 1.2.** We adopt the same notations as in the proof of Theorem 1.1. The same ideas with Lemma 2.6 imply that for \( s := \dim_H(A) > 1/2 \),
\[ \dim_H(\mathcal{E}), \dim_H(\mathcal{F}) \geq s(d - 4) + \dim_H(\Delta(A^2)) \geq s(d - 4) + \min \left\{ \frac{8s}{3} - \frac{2}{3}, 1 \right\}. \]
As before, we need to find certain conditions on \( s \) and \( d \) such that
\[ s(d - 4) + \min \left\{ \frac{8s}{3} - \frac{2}{3}, 1 \right\} > \frac{d-2}{2}. \tag{2.2} \]

**(Case 1)** Assume that \( s \geq 5/8 \). Then \( \min\left\{ \frac{8s}{3} - \frac{2}{3}, 1 \right\} = 1 \), and so the inequality (2.2) holds if \( d \geq 5 \) and \( s > 1/2 \). This implies that if \( s \geq 5/8 \) and \( d \geq 5 \), then we obtain the desired result that \( \text{Int}(\Delta(A^d)) \neq \emptyset \).

**(Case 2)** Assume that \( 1/2 < s < 5/8 \). Then \( \min\left\{ \frac{8s}{3} - \frac{2}{3}, 1 \right\} = \frac{8s}{3} - \frac{2}{3} \), and thus the inequality (2.2) is the same as \( s > \frac{3d-2}{6d-8} \). Since \( \frac{1}{2} < \frac{3d-2}{6d-8} \), this implies that if \( \frac{3d-2}{6d-8} < s < \frac{5}{8} \), then we get the desirable conclusion that \( \text{Int}(\Delta(A^d)) \neq \emptyset \). Notice that we also need the condition on \( d \) such that \( \frac{3d-2}{6d-8} < \frac{5}{8} \), namely, \( d \geq 5 \).

By Case 1 and Case 2, we deduce that if \( d \geq 5 \) and \( s > \frac{3d-2}{6d-8} \), then \( \text{Int}(\Delta(A^d)) \neq \emptyset \). This completes the proof.

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