Renormalization of the Yang-Mills theory in the ambiguity-free gauge.

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Abstract

The renormalization procedure for the Yang-Mills theory in the gauge free of the Gribov ambiguity is constructed. It is shown that all the ultraviolet infinities may be removed by renormalization of the parameters entering the classical Lagrangian and the local redefinition of the fields.

Keywords: Gribov ambiguity, BRST-symmetry, Renormalization

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1 Introduction

A problem of unambiguous quantization of nonabelian gauge theories remains unsolved. Although in the framework of perturbation theory a consistent quantization procedure was formulated by L.Faddeev and V.Popov [1] and B.DeWitt [2], for large fields as was pointed out by V.Gribov [3] the Coulomb gauge condition $\partial_i A_i = 0$ normally used in the process of quantization does not choose a unique representative in the class of gauge equivalent configurations. This result was later generalized by I.Singer [4] to arbitrary covariant gauge. At present it is not clear if this problem leads to serious physical consequences and different proposals how to overcome this difficulty were formulated (see for example [5]). However in our opinion a satisfactory solution has not been found.

The Gribov ambiguity usually arises when a gauge condition includes a differential operator, which leads to the existence of nontrivial solutions of the equation $L(A_\Omega, \phi^\Omega) = 0$, considered as the equation for the gauge function $\Omega$ at the surface $L(A, \phi) = 0$. So the most direct way to avoid this difficulty would be to consider so called algebraic gauges $\tilde{L}(A, \phi) = 0$, where the operator $\tilde{L}$ does not involve a differentiation. Example of such gauges is given by the condition $nA = 0$, where $n$ is some constant vector. Such gauges are known to be ghost free, but they violate explicitly the Lorentz invariance, resulting in the serious complications in analysis of the model. Moreover they lead to some additional problems which will not be discussed here.

The BRST quantization [6] avoids the problem of the gauge-fixing ambiguity. However in this case the gauge invariance is broken even at the classical level and one can prove the independence of observables on the gauge chosen only in the framework of perturbation theory.

On the other hand studies of QCD require a reliable gauge invariant method, valid beyond perturbation theory. It is highly desirable that this method preserves the manifest Lorentz invariance.

According to the common lore the quantization of the gauge invariant Yang-Mills theory always leads to the above mentioned problems. Algebraic gauges violate Lorentz invariance, whereas differential gauges are plagued by the Gribov ambiguity.

Recently a modified formulation of the Yang-Mills theory was proposed, which admits Lorentz invariant algebraic gauge conditions [7]. Contrary to the standard formulation in these gauges the ghost field Lagrangian is manifestly gauge invariant. A consistent quantization procedure was developed on the basis of this formulation [8]. However these gauges suffered from a new problem. Although the degree of divergency of arbitrary diagrams was limited, the number of primitively divergent diagrams was infinite and the standard perturbative renormalization of the model failed. Formally one could pass from the gauge proposed in [8] to the standard differential gauges like $\partial_\mu A_\mu = 0$, in which the theory is manifestly renormalizable, but this transition is legitimate only in the framework of perturbation theory. Of course for perturbative renormalization it is sufficient, but if one is planning to consider big fields as well, it is very useful to have a formulation which makes sense both in the framework of perturbation theory and beyond it. Moreover many modern attempts to study nonperturbative behaviour of QCD use some resummation of perturbation series. It requires a formulation of the model which may be used beyond perturbation theory and is perturbatively renormalizable. Such a formulation, which allows to perform a consistent quantization irrespectively of the validity of the perturbation theory was proposed in the papers [7], [8]. However renormalizability of the theory in this approach was not obvious. In the present paper we show that the proposed theory
is indeed renormalizable and therefore may serve as a starting point for non-perturbative approximations. In the framework of perturbation theory it produces the results coinciding with the standard formalism. Examples of the perturbative calculations confirming this statement were presented earlier in the paper [9].

Having this in mind in the present paper we studied the renormalization procedure for the ambiguity free formulation of the Yang-Mills theory. We show that all the ultraviolet divergencies may be removed by introducing a finite number of counterterms, supplemented by suitable fields redefinitions which preserve gauge invariance.

The paper is organized as follows. In the Section 2 we introduce the model and discuss its symmetries and the problem of unitarity. In the Section 3 we discuss the problem of renormalization of the Yang-Mills theory in the ambiguity free gauge. In the Section 4 the renormalization of the model is described and the renormalized Lagrangian is presented. In the Section 5 the problem of gauge independence of the renormalized theory is considered and equivalence of the present model to the standard Yang-Mills theory is proven. In the appendix B the symmetry preserving renormalization procedure which includes the field redefinition is explicitly constructed. Finally possible applications are briefly discussed.

2 The effective action and its symmetries

To save the place we consider the model invariant with respect to SU(2) gauge group. Generalization to SU(N) groups makes no problems.

To illustrate the main idea we consider firstly the classical Lagrangian of Yang-Mills theory supplemented by gauge-invariant couplings of the scalar fields \( \varphi, \chi, b, e \) [7, 8]

\[
\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + (D_\mu \varphi)^* D_\mu \varphi - (D_\mu \chi)^* D_\mu \chi - [(D_\mu b)^*(D_\mu e) + (D_\mu e)^*(D_\mu b)]
\] (1)

Here \( F^a_{\mu\nu} \) is the standard curvature tensor for the Yang-Mills field. The scalar fields \( \varphi, \chi, b, e \) form the complex SU(2) doublets parametrized as follows:

\[
\Phi = \left( i\Phi_1 + \Phi_2, \frac{\Phi_0 - i\Phi_3}{\sqrt{2}} \right)
\] (2)

where \( \Phi \) denotes any of doublets. The fields \( \varphi \) and \( \chi \) are commuting, and their components \( \varphi^\alpha, \chi^\alpha \) are hermitean. The fields \( e \) and \( b \) are anticommuting. By definition the components \( e^\alpha \) are hermitean, and the components \( b^\alpha \) are antihermitean. In the eq.(1) \( D_\mu \) denotes the usual covariant derivative.

As explained in [7], if the asymptotic states do not contain the excitations corresponding to the scalar fields \( \varphi, \chi, b, e \) one may perform in the path-integral

\[
S = \int \exp \left( i \int \mathcal{L} dx \right) \delta(\partial_i A_i) d\mu
\] (3)

the integration over the scalar fields. Then one gets the factor \( (|D^2|)^{-2} \) from the integration over the commuting fields \( \varphi \) and \( \chi \) and the factor \( (|D^2|^2)^2 \) from the integration over the anticommuting fields \( b, e \), so finally one ends up with the path-integral of the usual Yang-Mills theory in the Coulomb gauge.

Now we consider a different Lagrangian, which may be obtained from (1) by the following shift of the commuting scalar fields

\[
\varphi \rightarrow \varphi - g^{-1} \hat{m}; \quad \chi \rightarrow \chi + g^{-1} \hat{m}
\] (4)
The constant field $\hat{m}$ has a form
\[
\hat{m} = (0, m) .
\] (5)

Then we obtain the classical Lagrangian
\[
L = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + (D_\mu \varphi)^*(D_\mu \varphi) - (D_\mu \chi)^*(D_\mu \chi)
- g^{-1}[(D_\mu \varphi)^* + (D_\mu \chi)^*](D_\mu \hat{m}) - g^{-1}(D_\mu \varphi)^*[D_\mu \varphi + D_\mu \chi]
- [(D_\mu b)^*(D_\mu e) + (D_\mu e)^*(D_\mu b)]
\] (6)

Note that due to the negative sign of the $\chi$ field kinetic term, this field possesses negative energy. This is crucial in order to ensure the cancellation of the quadratic terms in $m$ in eq. (6) and therefore to provide a zero mass for the Yang-Mills fields. The factor $g^{-1}$ in the shift in eq. (4) is chosen in such a way that the scalar-gauge fields bilinears do not depend on $g$, and $g$ only enters into interaction vertices in eq. (6).

One may think that the massive parameter $m$ enters into expressions for observable gauge-invariant expectation values. We shall prove however that the observables do not depend on this parameter. A similar situation occurs in the Higgs model, if one works in the unitary gauge. To renormalize the theory at any given order of perturbation expansion one has to introduce a number of parameters (massless and massive). However the observable gauge-invariant quantities do not depend on these parameters, provided the renormalization preserves gauge invariance. The easiest way to see that is to pass to some manifestly renormalizable gauge (for example the Lorentz gauge $\partial_\mu A_\mu = 0$).

It is worth to notice that although the Lagrangian (6) may be obtained from the gauge invariant Lagrangian describing the interaction of the Yang-Mills field with the scalars $(\varphi^\pm, b, e)$ by the shift (4), the theory described by this Lagrangian may be absolutely different w.r.t. the unshifted one. Shift of the fields by a constant in general results in a new theory, inequivalent to the original one. A well-known example is given by the Higgs model. To prove the equivalence of our theory to the ordinary Yang-Mills theory one has to prove the decoupling of all the fields $(\varphi^\pm, b, e)$, temporal and longitudinal Yang-Mills quanta from the three dimensionally transversal Yang-Mills quanta. For unrenormalized theory it has been done in the papers [7, 8]. In this paper we shall prove it for the renormalized theory in the ambiguity free gauge.

The Lagrangian (6) is invariant with respect to the following gauge transformations inherited from the gauge symmetry of $\mathcal{L}$ in eq. (1)
\[
\delta A_\mu^a = \partial_\mu \eta^a + ge^{abc} A_\mu^b \eta^c
\]
\[
\delta \varphi_+^a = -\frac{g}{2} \varphi_+^a \eta^a
\]
\[
\delta \varphi_0^a = -\frac{g}{2} \varphi_0^a \eta^a
\]
\[
\delta \varphi_+^a = \frac{g}{2} e^{abc} \varphi_+^b \eta^c + \frac{g}{2} \varphi_0^a \eta^a
\]
\[
\delta \varphi_0^a = m \eta^a + \frac{g}{2} e^{abc} \varphi_0^b \eta^c + \frac{g}{2} \varphi_+^a \eta^a
\]
\[
\delta b^a = \frac{g}{2} e^{abc} b^b \eta^c + \frac{g}{2} b^0 \eta^a
\]
\[
\delta e^a = \frac{g}{2} e^{abc} e^b \eta^c + \frac{g}{2} e^0 \eta^a
\]
\[
\delta \eta^0 = -\frac{g}{2} b^a \eta^a
\]
\[
\delta e^0 = -\frac{g}{2} e^a \eta^a
\] (7)
Here the obvious notations
\[ \varphi^\alpha_\pm = \frac{\varphi^\alpha \pm \chi^\alpha}{\sqrt{2}} \]
are introduced.

This Lagrangian is also invariant with respect to the supersymmetry transformations
\[
\begin{align*}
\delta \varphi^a_- &= -b^a \\
\delta \varphi^0_- &= -b^0 \\
\delta e^a &= \varphi^a_+ \\
\delta e^0 &= \varphi^0_+ \\
\delta b &= 0 \\
\delta \varphi^a_+ &= 0
\end{align*}
\]  
(8)

This invariance plays a crucial role in the proof of the equivalence of the model described by the Lagrangian (6) to the standard Yang-Mills theory. It was shown in the papers [7, 8] that it provides the unitarity of the scattering matrix in the subspace which includes only three dimensionally transversal components of the Yang-Mills field.

The field \( \varphi^a_- \) is shifted under the gauge transformation by an arbitrary function \( m \eta^a \).

It allows to impose Lorentz invariant algebraic gauge condition \( \varphi^a_- = 0 \).

However imposing the Lorentz invariant gauge condition \( \varphi^a_- = 0 \) does not solve the problem of ambiguity completely. As it follows from the eq. (7) the field \( \varphi^a_- \) satisfying the condition \( \varphi^a_- = 0 \) is transformed by the gauge transformation to \( \varphi^a_- = (m + \frac{g}{2} \varphi^0_-) \eta^a \). For some \( x \) the factor \( (m + \frac{g}{2} \varphi^0_-(x)) \) may vanish, leading to nonuniqueness of the gauge fixing.

To avoid the problem of ambiguity completely we redefine the fields entering the Lagrangian (6) as follows

\[
\begin{align*}
\varphi^0_- &= 2m \left( \frac{gh}{2m} \right) - 1; \quad \varphi^a_- = \tilde{M} \tilde{\varphi}^a_- \\
\varphi^a_+ &= \tilde{M}^{-1} \tilde{\varphi}^a_+; \quad \varphi^0_+ = \tilde{M}^{-1} \tilde{\varphi}^0_+ \\
e &= \tilde{M}^{-1} \tilde{e}; \quad b = \tilde{M} \tilde{b}
\end{align*}
\]  
(9)

where

\[
\tilde{M} = 1 + \frac{g}{2m} \varphi^0_- = \exp \left\{ \frac{gh}{2m} \right\}
\]  
(10)

The new Lagrangian has the form

\[
\begin{align*}
\tilde{L} &= -\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \partial_\mu h \partial_\nu \tilde{\varphi}^0_+ - \frac{g}{2m} \partial_\mu h \partial_\mu h \tilde{\varphi}^0_+ \\
&+ m \tilde{\varphi}^a_+ \partial_\mu A_\mu^a - [(D \tilde{b})^a + \frac{g}{2m} \tilde{b}^a \partial_\mu h)(D \tilde{e} - \frac{g}{2m} \tilde{e} \partial_\mu h) + h.c.] \\
&+ \frac{mg}{2} A_\mu^a \tilde{\varphi}^0_+ + g \partial_\mu h A_\mu^a \tilde{\varphi}^a_+ \ldots
\end{align*}
\]  
(11)

Here \( \ldots \) denote the terms \( \sim \tilde{\varphi}^a_+ \), which are obviously polynomial.

The Lagrangian (11) by construction is invariant with respect to the gauge transformations generated by the transformations (7) after the change (9):

\[
\begin{align*}
\delta A_\mu^a &= \partial_\mu \eta^a + g\epsilon^{abc} A_\mu^b \eta^c \\
\delta \tilde{\varphi}^0_+ &= -\frac{g}{2} \tilde{\varphi}^a_+ \eta^a - \frac{g^2}{4m} \tilde{\varphi}^0_+ \tilde{\varphi}^a_- \eta^a
\end{align*}
\]
\[ \delta \hat{\varphi}_a^+ = \frac{g}{2} \epsilon^{abc} \hat{\varphi}_+^b \eta^c + \frac{g}{2} \hat{\varphi}_+^0 \eta^a - \frac{g^2}{4m} \hat{\varphi}_+^a \eta^b \]

\[ \delta \hat{\varphi}_-^a = m \eta^a + \frac{g}{2} \epsilon^{abc} \hat{\varphi}_-^b \eta^c + \frac{g^2}{4m} \hat{\varphi}_-^a \eta^b \]

\[ \delta \tilde{\varphi}_a^+ = \frac{g}{2} \epsilon^{abc} \tilde{\varphi}_+^b \eta^c + \frac{g}{2} \tilde{\varphi}_+^0 \eta^a + \frac{g^2}{4m} \tilde{\varphi}_+^a \eta^b \]

\[ \delta \tilde{\varphi}_-^a = \frac{g}{2} \epsilon^{abc} \tilde{\varphi}_-^b \eta^c + \frac{g}{2} \tilde{\varphi}_-^0 \eta^a + \frac{g^2}{4m} \tilde{\varphi}_-^a \eta^b \]

\[ \delta \tilde{\varphi}_a^0 = \frac{g}{2} \epsilon_{abc} \tilde{\varphi}_a^b \eta^c + \frac{g}{2} \tilde{\varphi}_a^0 \eta^a + \frac{g^2}{4m} \tilde{\varphi}_a^0 \eta^b \]

\[ \delta \tilde{\varphi}_-^0 = \frac{g}{2} \epsilon_{abc} \tilde{\varphi}_-^b \eta^c + \frac{g}{2} \tilde{\varphi}_-^0 \eta^a + \frac{g^2}{4m} \tilde{\varphi}_-^0 \eta^b \]

\[ \delta \eta^a = \frac{g}{2} \epsilon^a \eta^a - \frac{g^2}{4m} \tilde{\varphi}_-^a \eta^b \]

At the surface \( \tilde{\varphi}_-^a = 0 \) the gauge variation of the field \( \tilde{\varphi}_a^a \) is equal to \( m \eta^a \) and therefore the condition \( \tilde{\varphi}_-^a = 0 \) chooses the unique representative in the class of the gauge equivalent configurations.

Obviously the Lagrangian (11) is also invariant with respect to the supersymmetry transformations generated by the transformations (8) after the change (9):

\[ \delta h = -\tilde{\varphi}_-^0 \]

\[ \delta \tilde{\varphi}_a^+ = -\tilde{\varphi}_a^+ + \frac{g}{2m} \tilde{\varphi}_-^a \eta^b \]

\[ \delta \tilde{\varphi}_a^0 = -\frac{g}{2m} \tilde{\varphi}_a^0 \eta^b \]

\[ \delta \tilde{\varphi}_+^a = \tilde{\varphi}_a^a + \frac{g}{2m} \tilde{\varphi}_-^a \eta^b \]

\[ \delta \tilde{\varphi}_+^0 = \tilde{\varphi}_a^0 + \frac{g}{2m} \tilde{\varphi}_-^a \eta^b \]

\[ \delta \tilde{\varphi}_-^a = \tilde{\varphi}_a^a + \frac{g}{2m} \tilde{\varphi}_-^a \eta^b \]

\[ \delta \tilde{\varphi}_-^0 = \tilde{\varphi}_a^0 + \frac{g}{2m} \tilde{\varphi}_-^a \eta^b \]

\[ \delta \tilde{\varphi}_a^+ = -\frac{g}{2m} \tilde{\varphi}_a^+ \eta^b \]

\[ \delta \tilde{\varphi}_a^0 = -\frac{g}{2m} \tilde{\varphi}_a^0 \eta^b \]

\[ \delta \tilde{\varphi}_+^a = \tilde{\varphi}_a^a + \frac{g}{2m} \tilde{\varphi}_-^a \eta^b \]

\[ \delta \tilde{\varphi}_+^0 = \tilde{\varphi}_a^0 + \frac{g}{2m} \tilde{\varphi}_-^a \eta^b \]

\[ \delta \tilde{\varphi}_-^a = \tilde{\varphi}_a^a + \frac{g}{2m} \tilde{\varphi}_-^a \eta^b \]

\[ \delta \tilde{\varphi}_-^0 = \tilde{\varphi}_a^0 + \frac{g}{2m} \tilde{\varphi}_-^a \eta^b \]

\[ \delta \eta^a = \frac{g}{2} \epsilon^a \eta^a - \frac{g^2}{4m} \tilde{\varphi}_-^a \eta^b \]

Note however that imposing the gauge condition \( \hat{\varphi}_a^a = 0 \) we break the invariance of the effective action with respect to the supersymmetry transformation (13). To overcome this difficulty we consider in more details the effective action.

A canonical quantization in the gauge \( \tilde{\varphi}_-^a = 0 \) requires introduction of ultralocal ghosts. So the gauge fixing is introduced by adding to the action the term

\[ s \int d^4 x \tilde{c} \delta \varphi_\pm = \int d^4 x (\lambda^a \tilde{\varphi}_\pm^a - \hat{c}^a M^{ab} c^b) \] (14)

where

\[ M^{ab} = \delta^{ab} m + \frac{g}{2} \epsilon^{abc} \tilde{\varphi}_+^c + \frac{g^2}{4m} \tilde{\varphi}_-^a \tilde{\varphi}_-^b \] (15)

Here \( s \) is the BRST operator. In order to derive its action from eq. (12) some care is needed with the fermionic fields. One replaces first \( \eta^a \rightarrow \epsilon c^a \), where \( \epsilon \) is a constant
anticommuting parameter, and then drops $\epsilon$ once it has been moved to the left. Moreover one sets
\[
(s\epsilon)^{a} = -\frac{g}{2} e^{abc}c^{b}c^{c} \quad (sc)^{a} = \lambda^{a} \quad (s\lambda)^{a} = 0.
\tag{16}
\]

The gauge fixed action is obviously invariant with respect to this BRST transformation. It leads to some relations satisfied by the one particle irreducible diagrams, which will be discussed later.

However, as it was mentioned above, due to the presence of the term $\lambda^{a}\tilde{\varphi}^{a}$ this action is not invariant with respect to the supersymmetry transformation (13), which provides the physical unitarity of the model. As the transition from one gauge to the other one may be achieved by a gauge transformation, and in the gauge $\partial_{\mu}A_{\mu} = 0$ the effective action is invariant with respect to the supertransformation (13), in the gauge $\varphi^{a}_{-} = 0$ it also must be invariant with respect to some supertransformation.

To obtain this transformation we note that the Lagrangian eq.(11) which is invariant with respect to the BRST transformation corresponding to the gauge transformations eq. (12) and with respect to the supersymmetry transformations eq.(13) is also invariant with respect to the simultaneous change of the fields combining these two transformations:
\[
\delta A_{\mu}^{a} = \partial_{\mu}c^{a} + ge^{abc}A_{\mu}^{b}c^{c} \\
\delta \tilde{\varphi}^{0}_{+} = -\frac{g}{2} \varphi_{+}^{a}c^{a} - \frac{g^{2}}{4m} \varphi_{+}^{a}c^{a} - \frac{g}{2m} \varphi_{+}^{a}\tilde{b}^{0} \\
\delta h = -\frac{g}{2} \tilde{\varphi}^{a} - \tilde{b}^{0} \\
\delta \varphi_{+}^{a} = \frac{g}{2} e^{abc} \varphi_{+}^{b}c^{c} + \frac{g}{2} \varphi_{+}^{a}c^{a} - \frac{g^{2}}{4m} \varphi_{+}^{a}c^{a} - \frac{g}{2m} \varphi_{+}^{a}\tilde{b}^{0} \\
\delta \varphi_{-}^{a} = mc^{a} + \frac{g}{2} e^{abc} \varphi_{-}^{b}c^{c} + \frac{g^{2}}{4m} \varphi_{-}^{a}c^{a} - \frac{g}{2m} \varphi_{-}^{a}\tilde{b}^{0} \\
\delta \tilde{b}^{0} = -\frac{g}{2} e^{abc} \tilde{b}^{c}c^{a} - \frac{g}{2} \tilde{b}^{0}c^{a} + \frac{g^{2}}{4m} \tilde{b}^{0}c^{a} - \frac{g}{2m} \tilde{b}^{0}\tilde{b}^{0} \\
\delta \tilde{c}^{a} = -\frac{g}{2} e^{abc} \tilde{c}^{b}c^{c} - \frac{g}{2} \tilde{c}^{0}c^{a} + \frac{g^{2}}{4m} \tilde{c}^{0}c^{a} - \frac{g}{2m} \tilde{c}^{0}\tilde{c}^{0} \\
\delta \tilde{\varphi}^{a} = \frac{g}{2} e^{abc} \tilde{\varphi}^{b}c^{c} + \frac{g^{2}}{4m} \tilde{\varphi}^{a}c^{a} + \tilde{\varphi}_{+}^{a} + \frac{g}{2m} \tilde{\varphi}_{+}^{a}\tilde{\varphi}_{+}^{a} \\
\delta \tilde{\varphi}_{+}^{a} = +\frac{g}{2} \tilde{\varphi}_{+}^{a}c^{a} + \frac{g^{2}}{4m} \tilde{\varphi}_{+}^{a}c^{a} + \tilde{\varphi}_{+}^{a} + \frac{g}{2m} \tilde{\varphi}_{+}^{a}\tilde{\varphi}_{+}^{a}
\tag{17}
\]

It allows to use instead of the canonical gauge fixing eq.(14) the following gauge fixing
\[
s^{1} \int d^{4}x e^{a} \varphi_{-}^{a} = \int d^{4}x (\lambda^{a} \varphi_{-}^{a} - \tilde{c}^{a}(M^{ab}c^{b} - \tilde{b}^{0}))
\tag{18}
\]

where $s^{1}$ is the nilpotent operator defined by the eqs.(16) and (17).

The scattering matrix may be alternatively presented either in the canonical form
\[
S = \int \exp\{i \int [\tilde{L} + \lambda^{a} \tilde{\varphi}_{-}^{a} - \tilde{c}^{a}(M^{ab}c^{b} - \tilde{b}^{0})]d^{4}x\}d\mu
\tag{19}
\]

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or in the form

\begin{equation}
S = \int \exp\{i \int [\mathcal{L} + \lambda^a \varphi_+^a - \mathcal{E}^a M^{ab} c^b + \bar{c}^a \bar{b}^a] d^4x\} d\mu \tag{20}
\end{equation}

where \( L \) is the Lagrangian \((11)\).

The integral \((20)\) may be transformed to the form \((19)\) by the change of variables

\begin{equation}
c^a \to c^a + (M^{-1} \bar{b})^a \tag{21}
\end{equation}

Note that in the gauge \( \varphi_+^a = 0 \) the operator \( M^{ab} \) is equal to \( \delta^{ab} \).

As the variables \( \bar{c}, c \) are ultralocal, this transformation does not change the value of the integral. Hence for studying the structure of the counterterms necessary for perturbative renormalization of these integrals one may use the symmetries of both integrals \((19)\) and \((20)\).

Performing the integration over \( \bar{c}, c \) in the eq.\((20)\) we obtain in the exponent the effective action which is invariant with respect to the transformation obtained from the transformations eq.\((17)\) by substituting \( c^a = (M^{-1} \bar{b})^a \):

\begin{align*}
\delta A^a_{\mu} &= \frac{1}{m} (D_{\mu} \bar{b})^a \\
\delta \varphi_+^a &= 0 \\
\delta h &= -\bar{b}^0 \\
\delta \varphi_+^0 &= -\frac{g}{2m} (\varphi_+^a \bar{b}^a + \varphi_+^0 \bar{b}^0) \\
\delta c^a &= \frac{g}{2m} (\bar{c}^a \bar{b}^0 - \bar{c}^0 \bar{b}^a - \epsilon^{abc} \bar{c}^b \bar{b}^c) + \varphi_+^a \\
\delta \bar{b}^0 &= 0 \\
\delta \bar{c}^a &= \frac{g}{2m} (\bar{c}^a \bar{b}^0 + \bar{c}^0 \bar{b}^a) + \bar{c}^0 \\
\delta \bar{b}^a &= \frac{g}{2m} \epsilon^{abc} \bar{b}^b \bar{b}^c 
\end{align*}

(22)

One sees that these transformations do not change the field \( \varphi^a_+ \), hence the effective action in the gauge \( \varphi^a_- = 0 \) is invariant under these transformations.

The effective action in the gauge \( \varphi^a_- = 0 \) is given by the eq.\((11)\) with all terms \( \sim \varphi^a_- \) omitted. This action obviously is not invariant with respect to the global SU(2) rotations of all variables, but it is still invariant with respect to the SU(2) transformations which do not change the fields \( \varphi^0_+, \varphi^a_+, h \). This symmetry will be very helpful for analysis of possible counterterms.

The quadratic part of the effective action is

\begin{equation}
A_0 = \int d^4x \left[ \frac{1}{4} (\partial_{\mu} A^a_{\nu} - \partial_{\nu} A^a_{\mu})^2 + \partial_{\mu} h \partial_{\mu} \varphi^0_+ + m \varphi^a_+ \partial_{\mu} A^a_{\mu} - \bar{b}^0 \partial^2 c^0 - \bar{b}^a \partial^2 c^a \right] \tag{23}
\end{equation}

The nonvanishing propagators are:

\begin{align*}
\Delta (A^a_{\mu} A^b_{\nu}) &= -\frac{i \delta^{ab}}{p^2} T_{\mu \nu}, \quad \Delta (A^a_{\mu} \varphi^b_+) = -\delta^{ab} \frac{p_{\mu}}{mp^2} \\
\Delta (\varphi^0_+ h) &= \frac{i}{p^2}, \quad \Delta (\bar{b}^0 e^0) = \frac{i}{p^2}, \quad \Delta (\bar{b}^a e^b) = \frac{i \delta^{ab}}{p^2} \tag{24}
\end{align*}

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One sees that the complete asymptotic space of our model includes a number of unphysical excitations corresponding to the fields $\tilde{\phi}^{\pm}, \tilde{b}, \tilde{e}, h$ and longitudinal and temporal components of the Yang-Mills field. Note however that the Faddeev-Popov ghosts $\bar{c}^{a}, c^{a}$ are ultralocal and do not enter the asymptotic states. The unitarity of the model in the space including only three dimensionally transversal components of the Yang-Mills field is provided by the symmetry (22). The corresponding symmetry for asymptotic fields generates the nilpotent conserved charge $Q^{0}$, which acts on the asymptotic fields as follows

$$Q^{0} A^{a}_{\mu} = \frac{1}{m} \partial_{\mu} \tilde{b}^{a}$$

$$Q^{0} \tilde{e}^{a} = \tilde{\phi}^{a}_{-}$$

(25)

One sees that the field $\tilde{e}^{a}$ may be identified with the antighost field of the Yang-Mills theory. It is mapped into $\tilde{\phi}^{a}_{+}$ which plays the role of the Nakanishi-Lautrup field implementing the Lorentz gauge condition. The field $\tilde{b}^{a}$ plays the role of the Yang-Mills ghost. Therefore the transformations (22) asymptotically coincide with the usual BRST transformations which provide the decoupling of the fields $\tilde{\phi}^{a}_{+}, \tilde{e}^{a}, \tilde{b}^{a}$ and unphysical components of the Yang-Mills field. Moreover

$$Q^{0} \tilde{e}^{0} = \tilde{\phi}^{0}_{+}$$

$$Q^{0} \tilde{\phi}^{0}_{+} = 0$$

$$Q^{0} h = -\tilde{b}^{0}$$

$$Q^{0} \tilde{b}^{0} = 0$$

(26)

which guarantees the decoupling of $\tilde{e}^{0}, \tilde{b}^{0}, \tilde{\phi}^{0}_{+}, h$ from the physical subspace. The ultralocal fields $\lambda^{a}, \tilde{e}^{a}, c^{a}$ do not contribute to the physical asymptotic states.

Therefore formally the model described above has the same spectrum of observables as the usual Yang-Mills model, and one can show that the correlators of the gauge invariant operators also coincide. However our discussion so far was formal as we did not take into account the necessity of renormalization and dealt with ultraviolet divergent integrals.

The problem of renormalization in the gauge $\tilde{\phi}^{a}_{-} = 0$ appears to be quite nontrivial. Renormalization of our model requires not only changing the values of the parameters in the classical Lagrangian, but also a redefinition of the fields. So our model gives an example of a ”General Gauge Theory” considered in details by B.L.Voronov and I.V.Tyutin [10].

3 Renormalization

The divergency index of an arbitrary diagram is equal to

$$n = 4 - 2L_{\phi_{+}} - 2L_{\phi_{+}} - L_{\phi} - L_{\phi} - L_{\phi} - L_{\phi}$$

(27)

where $L_{\phi}$ is the number of the external lines of the field $\Phi$. For the diagrams with two or more external lines $n \leq 2$. Any diagram with more than four external lines is convergent, and the theory is manifestly renormalizable.

It is worth to notice that the transition to the variables (9), performed to get rid off the factor $m + \frac{g}{2} \phi^{0}_{-}$, which may lead to ambiguity, is needed also for a manifest renormalizability of the theory. If we quantize the theory in the original variables, the expression for the divergency index would be given by the eq. (27) without the last term. In that
case the divergency index does not depend on the number of external lines of the field \( \varphi^0 \) (\( h \)-lines in the new variables). As a result there are divergent diagrams with arbitrary number of external \( \varphi^0 \) lines. The transformation (9) cures simultaneously two diseases: it eliminates a possible source of ambiguity and makes the model manifestly renormalizable.

However the action

\[
A_{\text{ef}} = \int d^4x \tilde{L},
\]

where \( \tilde{L} \) is the Lagrangian in eq. (11), is not the most general classical action which is invariant under the transformation (22) and the global \( SU(2) \) transformations of the fields \( A_{\mu}, \varphi^a, \tilde{e}, \tilde{b} \). The following combination also respects the invariance under the transformation (11) as well as the residual \( SU(2) \) invariance:

\[
\mathcal{G} = \int d^4x \left[ (\tilde{\varphi}^0)^2 + (\varphi^a)^2 + \frac{g}{m} \tilde{\varphi}^0 \tilde{e}^0 \tilde{b}^0 + \tilde{e}^a \tilde{b}^a + \frac{g}{m} \tilde{\varphi}^a (\tilde{e}^a \tilde{b}^0 - \tilde{e}^0 \tilde{b}^a - \varepsilon^{abc} \tilde{e}^b \tilde{b}^c) \right.
\]

\[
- \frac{g^2}{2m^2} \left( - (2\varepsilon^{0b} \tilde{e}^0 \tilde{b}^b + \varepsilon^{abc} \varepsilon^{0b} \tilde{e}^c \tilde{b}^a - \varepsilon^{abc} \tilde{e}^b \tilde{b}^0 \tilde{e}^a) \right). \]  

Moreover it does not violate power-counting renormalizability and therefore it can be introduced in the action (28). \( \mathcal{G} \) can be made gauge-invariant by adding terms proportional to \( \varphi^a \). This can be easily done by performing in eq. (29) the substitutions

\[
\begin{align*}
\tilde{\varphi}^0_+ & \to \varphi^0_+ + \frac{g}{2m} \varphi^a \tilde{\varphi}^a_+ , & \varphi^0_+ & \to \varphi^0_+ - \frac{g}{2m} \varphi^a_+ \tilde{\varphi}^a , & \frac{g}{2m} \varphi^a_+ \tilde{\varphi}^a , \\
\tilde{e}^0 & \to \tilde{e}^0 + \frac{g}{2m} \varphi^a \tilde{e}^a , & \tilde{e}^a & \to \tilde{e}^a - \frac{g}{2m} \varphi^a \tilde{e}^a + \frac{g}{2m} \varepsilon^{abc} \tilde{e}^b \tilde{e}^c , \\
\tilde{b}^0 & \to \tilde{b}^0 + \frac{g}{2m} \varphi^a \tilde{b}^a , & \tilde{b}^a & \to \tilde{b}^a - \frac{g}{2m} \varphi^a \tilde{b}^a + \frac{g}{2m} \varepsilon^{abc} \tilde{b}^b \tilde{b}^c .
\end{align*}
\]

The expressions in the R.H.S. of eqs. (30) are invariant w.r.t. the gauge transformation eq. (12), as can be checked by direct computation. Gauge-invariance fixes uniquely the dependence on \( \varphi^a_- \).

The new action is

\[
A'_{\text{ef}} = A_{\text{ef}} + \frac{m^2}{2} \alpha \mathcal{G} .
\]

The prefactor \( m^2 \) in front of \( \mathcal{G} \) has been inserted for dimensional reasons and \( \alpha \) plays the role of the gauge-fixing parameter.

By keeping the full dependence on \( \varphi^a_- \) one explicitly finds

\[
A'_{\text{ef}} = \int d^4x \left\{ - \frac{1}{4} F_{\mu \nu} F_{\mu \nu} + \partial_{\mu} h \partial_{\mu} \varphi^0 + m \partial A^a \varphi^a_+ + \partial_{\mu} \varepsilon^{abc} \partial_{\mu} \varphi^a_+ - \frac{g}{2m} \partial_{\mu} h \partial_{\mu} \varphi^0_+ \\
+ g \partial_{\mu} h A^a_{\mu} \varphi^a_+ + \frac{gm}{2} A^2_{\mu \nu} \varphi^a_+ + \frac{g}{2m} \partial_{\mu} h (\partial_{\mu} \varphi^a_+ - \partial_{\mu} \varphi^a_-) \\
+ \frac{g}{2} A^a_{\mu} (\partial_{\mu} \varphi^0_+ - \varphi^a_+ \partial_{\mu} \varphi^0_- + \frac{g}{2m} \varepsilon^{abc} \partial_{\mu} \varphi^0_- - \partial_{\mu} \varphi^0_- \varphi^a_+ ) \\
+ \frac{g^2}{4} A^2_{\mu} \varphi^0_+ \varphi^0_+ - \frac{g^2}{4m^2} \partial_{\mu} h \partial_{\mu} h \varphi^a_+ \varphi^a_+ - \frac{g^2}{2m} \partial_{\mu} h \varphi^0_+ A^a_{\mu} \varphi^a_+ \\
+ \frac{g^2}{2m} \partial_{\mu} h \varepsilon^{abc} \varphi^a_+ \varphi^b_+ A^c_{\mu} [((D_{\mu} \tilde{b})^* + \frac{g}{2m} \tilde{b}^* \partial_{\mu} h)((D_{\mu} \tilde{e}) - \frac{g}{2m} \tilde{e} \partial_{\mu} h) + h.c] \right\} .
\]
which is manifestly polynomial.

The quadratic part of $A'_{ef}$ in the gauge $\bar{\varphi}^-_a = 0$ reads

$$A'_{ef,0} = \int d^4x \left[ -\frac{1}{4} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 + \frac{\alpha}{2} m^2 (\bar{\varphi}^0_+)^2 - \bar{\varphi}^+_0 \Box h + m\bar{\varphi}^+_0 \partial_\mu A^a_\mu + \frac{\alpha}{2} m^2 (\bar{\varphi}^+_0)^2 - \bar{\varphi}^+_0 \partial^2 \bar{e}^0 + \bar{\varphi}^+_a \partial^2 \bar{e}^a \right].$$

(33)

The nonvanishing propagators are

$$\Delta(A^a_\mu A^b_\nu) = -i\delta^{ab}/p^2 T_{\mu\nu} + i\delta^{ab}\alpha/p^2 L_{\mu\nu}, \quad \Delta(A^a_\mu \bar{\varphi}^+_b) = -\delta^{ab} p_\mu /mp^2, \quad \Delta(\bar{\varphi}^0_+ h) = i/p^2, \quad \Delta(h h) = -i\alpha m^2/p^4, \quad \Delta(\bar{b}^0 \bar{c}^0) = i/p^2, \quad \Delta(\bar{b}^a \bar{c}^b) = i\delta^{ab}/p^2.$$ (34)

The gauge field is quantized in the $\alpha$-gauge, while a dipole arises for $h$. However, we notice that the dependence on $\bar{\varphi}^0_0 + \frac{1}{\alpha} \Box h$ in eq. (33) can be recast as

$$\int d^4x \left[ \frac{\alpha}{2} \left( \bar{\varphi}^0_+ - \frac{1}{\alpha} \Box h \right)^2 + \frac{1}{2\alpha} \partial_\mu h \Box \partial_\mu h \right].$$ (35)

The combination $\bar{\varphi}^0_0 - \frac{1}{\alpha} \Box h$ is ultralocal. Moreover the decoupling of $\partial_\mu h$ from the physical states is guaranteed by the symmetry (26) since at the asymptotic level

$$Q h = -\bar{b}^c, \quad Q \bar{b}^0 = 0.$$ (36)

So the nonrenormalized action (31) preserves the unitarity in the subspace including only three dimensionally transversal components of the Yang-Mills field.

### 4 The structure of counterterms

In the previous Section we showed that the action (31) describes a renormalizable theory which formally is unitary in the physical subspace. The crucial role in the proof of the unitarity was played by the symmetry of the theory with respect to the transformations (22).

Now we are going to prove that renormalization does not violate this property. The counterterms needed to remove all ultraviolet divergences only change the values of the parameters entering the action (31) (modulo field redefinitions) and the transformations (22), but preserve the symmetry of the renormalized theory. Moreover we shall demonstrate that the renormalized action in the gauge $\bar{\varphi}^-_a = 0$ may be obtained by imposing the gauge condition on the invariant classical action (assuming of course that some gauge invariant intermediate regularization is introduced). Using this fact one can easily prove gauge independence of the observables. In particular in perturbation theory, when the Gribov ambiguity is absent, the correlation functions of gauge invariant operators constructed
from the Yang-Mills field calculated in the ambiguity free gauge and in the Lorentz gauge \( \partial_\mu A_\mu = 0 \) coincide.

In order to study the counterterms in the gauge \( \tilde{\phi}_a^\sigma = 0 \) let us introduce a new action \( \Gamma^{(0)} \), including apart from the classical action \( A'_{ef} \) in eq.(31) also the variation of the fields \( \Phi \), coupled to the external sources \( \Phi^* \), which are usually called "antifields" \( [11] \), i.e. we set

\[
\Gamma^{(0)} = A'_{ef} + \sum_\Phi \int d^4x \Phi^* \delta \Phi .
\]  

(37)

where \( \delta \) is defined in eq.(22). Then the invariance under the symmetry \( [22] \) is translated into the following functional identity for the 1-PI generating functional \( \Gamma \)

\[
S(\Gamma) = \int d^4x \sum_\Phi \frac{\delta \Gamma}{\delta \Phi^*(x)} \frac{\delta \Gamma}{\delta \Phi(x)} = 0 .
\]  

(38)

Then the invariance under the symmetry (22) is translated into the following functional identity for the 1-PI generating functional \( \Gamma \)

\[
S(\Gamma) = \int d^4x \sum_\Phi \frac{\delta \Gamma}{\delta \Phi^*(x)} \frac{\delta \Gamma}{\delta \Phi(x)} = 0 .
\]  

(38)

\( \Gamma \) is developed in the number of loops, i.e. \( \Gamma = \sum_{j=0}^{\infty} \hbar^j \Gamma^{(j)} \).

Assuming that some invariant regularization is introduced, eq.(38) holds if the effective action \( \hat{\Gamma} \) (tree-level plus counterterms) fulfills

\[
S(\hat{\Gamma}) = 0 .
\]  

(39)

Moreover we will also require invariance under the residual global \( SU(2) \) symmetry.

The most general solution of the eq.(39) compatible with the dimensional bounds and the residual \( SU(2) \) invariance may be written as follows. One should make the shift of the parameters \( g, m, \alpha \), which enter the classical action \( A'_{ef} \) and redefine the fields preserving the ultraviolet counting (eq.(27)):

\[
\begin{align*}
g' &= Z_g g , \quad m' = Z_m m , \quad \alpha' = \frac{Z_\alpha}{Z_m^2} \alpha , \\
\tilde{\epsilon}' &= Z_1 \tilde{\epsilon} , \quad \tilde{b}' = Z_m \tilde{b} , \quad A'_\mu = Z_2 A_\mu , \quad h' = Z_m Z_3 h , \\
\tilde{\phi}'_+ &= Z_4 \tilde{\phi}'_+ + Z_5 \partial \tilde{\phi}'_+ + Z_6 \frac{1}{m} \partial_\mu h A^\mu + Z_7 (\tilde{\epsilon} \tilde{b} - \tilde{\epsilon}' \tilde{b}') - \varepsilon^{abc} \tilde{\epsilon} \tilde{b}'_0 , \\
\tilde{\phi}'_0 &= Z_8 \tilde{\phi}'_0 + Z_9 \frac{1}{m^2} \square h + Z_{10} \frac{1}{m^2} \partial_\mu h \partial^\mu h + Z_{11} A^2 + Z_{12} (\tilde{\epsilon} \tilde{b} - \tilde{\epsilon}' \tilde{b}') .
\end{align*}
\]  

(41)

Since a multiplicative redefinition of \( \tilde{b} \) can always be compensated in \( A'_{ef} \) by a redefinition of \( \tilde{\epsilon} \), we do not rescale \( \tilde{b} \) apart from a factor \( Z_m \), which is introduced for convenience in such a way that \( Z_m \) multiplies in \( \hat{\Gamma} \) the global \( SU(2) \)- and \( \delta \)-invariant combination containing the kinetic term for \( \tilde{\epsilon} \), \( \tilde{b} \). Note that to satisfy eq.(39) the redefinition of the fields must be supplemented by the corresponding redefinition of the antifields.

For that purpose one may notice that the functional identity (38) can be formulated by means of the following bracket [12]

\[
(X, Y) = \int d^4x \sum_\Phi (-1)^{\epsilon(\Phi)\epsilon(X)} \left( \frac{\delta X}{\delta \Phi^*} \frac{\delta Y}{\delta \Phi} - (-1)^{\epsilon(X)+1} \frac{\delta X}{\delta \Phi} \frac{\delta Y}{\delta \Phi^*} \right)
\]  

(42)

where \( \epsilon \) denotes the statistics (1 for fermions, 0 for bosons). I.e. one has

\[
S(\Gamma) = \frac{1}{2} (\Gamma, \Gamma) = 0 .
\]  

(43)
Under eq. (42) the fields and the antifields are paired via the fundamental brackets

$$(\Phi_i, \Phi^*_j) = (-1)^{c(\Phi_j)} \delta_{ij}.$$  \hspace{1cm} (44)

Notice that our conventions on the antifields differs from the one of [12] by the redefinition $\Phi^* \rightarrow (-1)^{c(\Phi)} \Phi^*$, whence the sign factor in the r.h.s. of the above equation.

A redefinition of the fields and the antifields preserving eq. (44) is called a canonical transformation (w.r.t. the bracket (42)). It automatically preserves the bracket between any two functionals $X, Y$ and therefore also the functional identity (39).

A convenient way to complete the field redefinition (41) to a finite canonical transformation is to derive the latter via a suitable generating functional [12]. In the present case this is given by

$$G = \int d^4x \sum \Phi' (-1)^{c(\Phi')} \Phi^* \Phi' (\Phi)$$

and the field and antifield transformations are obtained by solving the equations

$$\Phi^* = (-1)^{c(\Phi)} \frac{\delta G}{\delta \Phi}, \quad \Phi' = (-1)^{c(\Phi')} \frac{\delta G}{\delta \Phi^*}.$$  \hspace{1cm} (45)

By construction $G$ generates the field redefinition (41), while the explicit expressions for the antifield redefinitions are presented in Appendix A.

Consequently the functional

$$\hat{\Gamma}[g', m', \alpha'; \Phi', \Phi^*] = \Gamma^{(0)}[Z_g g, Z_m m, Z_\alpha Z_m^2 ; \Phi(\Phi'), \Phi^*(\Phi', \Phi^*)]$$  \hspace{1cm} (46)

is a solution to eq. (39). One can verify it by explicit calculations.

It remains to be shown that all the divergences can be recursively removed by a suitable choice of the parameters $Z_g, Z_m, Z_\alpha$ and by changing the field renormalization constants $Z_j, j = 1, \ldots, 12$. This is done in Appendix B.

As announced, the renormalized effective action $\hat{\Gamma}$ is finally obtained by a shift in the parameters of the classical action (modulo field redefinitions).

5 Independence of observables on the gauge and comparison with the standard Yang-Mills theory

In the previous sections we proved that all the ultraviolet divergencies in the gauge invariant theory determined by the action (6) plus the additional term (29), quantized in the gauge $\tilde{\varphi}^a = 0$, may be removed by the renormalization of the parameters entering the classical action and the local redefinition of the fields. It was shown that the resulting (infrared regularized) scattering matrix is unitary in the subspace including only three dimensionally transversal components of the Yang-Mills field.

Now we want to show that the scattering matrix, obtained in this way, as well as other gauge invariant correlators depending only on the Yang-Mills field in the framework of perturbation theory may be transformed to the normally used differential gauges. In particular they can be calculated in the Lorentz gauge $\partial_{\mu} A_{\mu} = 0$. The values of these quantities also coincide with the corresponding values in the standard Yang-Mills theory.

The renormalized theory in the gauge $\tilde{\varphi}^a = 0$ is described by the effective action (46) with the proper chosen constants $Z_g, Z_m, Z_\alpha$ and $Z_j$. This renormalized action was obtained from the gauge invariant classical action by imposing the condition $\tilde{\varphi}^a = 0$ and redefining the fields and the parameters. As all the field redefinitions were local, in the path integral formulation they lead to the appearance of local jacobians, which
in regularizations like dimensional one are trivial. Having in mind this kind of invariant
regularization we can perform the inverse redefinition of the fields, resulting in the original
classical action depending on the renormalized parameters. Note that the redefinition of
the fields used above did not include the field $\tilde{\phi}^a$. Therefore the scattering matrix may be presented as the path integral

$$S = \int \exp\{i \int [L_{g.i.} + \lambda^a \phi^a] \, dx \} \, \det(M_{ab}) \, d\mu$$

(47)

where the local Jacobian $\det(M_{ab})$ replaces the usual Faddeev-Popov determinant and integration goes over all fields present in the Lagrangian. The boundary conditions for the three dimensionally transversal components of the Yang-Mills field are determined by the corresponding asymptotic fields and for all other fields we may use the vacuum (radiation) conditions, as the scattering matrix is unitary in the space including only three dimensionally transversal components of the Yang-Mills field. The gauge invariant Lagrangian $L_{g.i.}$ depends on the renormalized parameters

$$g_R = Z_g g; \quad m_R = Z_m m; \quad \alpha_R = \frac{\alpha}{Z_\alpha^2} (48)$$

As usual the jacobian $\det(M_{ab})$ may be presented as follows

$$(\det(M_{ab})^{-1})_{\phi^a=0} = \int \delta((\tilde{\phi})^a) \, d\Omega$$

(49)

Multiplying the integral (47) by "1"

$$1 = \Delta_L \int \delta(\partial_{\mu} A^a_{\mu}) \, d\Omega$$

(50)

and changing the variables

$$\Phi^\Omega = \Phi'$$

(51)

we obtain the expression for the scattering matrix in the Lorentz gauge:

$$S = \int \exp\{i \int [L_{g.i.}(x) + \lambda^a(x) \partial_{\mu} A^a_{\mu}(x) + \partial_{\mu} \tilde{c}^a] \partial_{\mu} c^a] \, dx \} \, d\mu$$

(52)

where $\tilde{c}, c$ are the usual Faddeev-Popov ghosts.

These reasonings show that the scattering matrix as well as the gauge invariant correlators depending only on the Yang-Mills field computed in the ambiguity free gauge $\tilde{\phi}^a = 0$ coincide with the corresponding objects in the Lorentz gauge. Strictly speaking this transition is legitimate only in the framework of the perturbation theory as beyond the perturbation theory the Faddeev-Popov determinant may have zeroes.

The last thing we wish to show is the equality of the scattering matrix and the gauge invariant correlators calculated on the basis of the action (6) plus the additional terms (29) in the Lorentz gauge $\partial_{\mu} A_{\mu} = 0$ to the corresponding objects in the standard Yang-Mills theory.

To do that we again make the inverse transformation of the fields, writing the eqs. (6) and (29) in terms of the original variables. The eq. (6) has the same form as before. The only difference is the change of the charge by the renormalized charge.
The expression for \( G \) may be written in terms of the original fields as follows

\[
G = \int d^4 x \left\{ \frac{g}{2m} \left( (\varphi_+^* + \hat{m}^*) \varphi_+ + \varphi_+^* (\varphi_- + \hat{m}) \right) + \frac{g}{2m} (b^* e + e^* b) \right\}^2 \\
+ \frac{g}{2m} \left( -i (\varphi_-^* + \hat{m}^*) \tau^a \varphi_+ + i \varphi_+^* \tau^a (\varphi_- + \hat{m}) + i \frac{g}{2m} (b^* \tau^a e - e^* \tau^a b) \right) \right\} \right\}
\]

(53)

Introducing the local field \( \mu^A \), \( A = 0, 1, 2, 3 \) one can represent the exponent \( \exp \left\{ i m^2 \alpha \frac{G}{2} \right\} \) as follows:

\[
\exp \left\{ i m^2 \alpha \frac{G}{2} \right\} = \int \exp \left\{ - i m^2 \alpha \int \left[ - \frac{g}{2m} \left( (\varphi_+^* + \hat{m}^*) \varphi_+ + \varphi_+^* (\varphi_- + \hat{m}) + (b^* e + e^* b) \right) \mu^0 + \frac{(\mu^0)^2}{2} \right. \\
- \frac{g}{2m} \left( -i (\varphi_-^* + \hat{m}^*) \tau^a \varphi_+ + i \varphi_+^* \tau^a (\varphi_- + \hat{m}) + i (b^* \tau^a e - e^* \tau^a b) \right) \mu^a + \frac{(\mu^a)^2}{2} \right] dx \right\} d\mu
\]

(54)

Changing the variables as in the paper [7]

\[
\varphi(x) = \varphi'(x) + g^{-1} \int D^{-2}(x, y) (D^2 \hat{m}(y)) dy \\
\chi(x) = \chi'(x) - g^{-1} \int D^{-2}(x, y) (D^2 \hat{m}(y)) dy
\]

(55)

and integrating over auxiliary fields \( \varphi, \chi, b, e \) we obtain the determinants which cancel each other and finally we are left with the expression which coincides with the standard Yang-Mills theory.

Note that in the equation (55) we are allowed to perform the integration by parts as the corresponding expressions are multiplied by the functions decreasing at infinity. We also may integrate explicitly over the auxiliary fields with the vacuum boundary conditions, as above we proved the unitarity of the scattering matrix in the subspace including only three dimensionally transversal components of the Yang-Mills field.

6 Discussion

In this paper we showed that the Yang-Mills theory allows a renormalizable formulation free of the Gribov ambiguity. It provides strong arguments in favour of the point of view according to which this ambiguity is an artefact of the quantization procedure and cannot produce some physical effects. From the technical point of view the model considered in this paper gives an interesting example of a nontrivial renormalizable theory whose renormalization requires nonmultiplicative field redefinition.

Of course a rigorous comparison of different formulations is possible only in the framework of the perturbation theory where the ambiguity is absent and it is not excluded that beyond the perturbation theory our formulation and the standard one describe different theories. However such a possibility seems to be rather unlikely. It is worth to mention that the studies of gluodynamics beyond the perturbation theory carried out now both by semi analytic methods, mainly based on the Schwinger-Dyson equations [13, 14, 15], and by computer simulations [16, 17], give controversial results concerning the infrared behaviour of the propagators. It would be interesting to carry out similar investigations in the present formulation.
The action of \( S_z \) transformation (45):

We collect here the explicit form of the antifield redefinitions induced by the canonical A Antifield Transformations RAS program “Nonlinear dynamics.”

\[
\Gamma(\Phi) \rightarrow \Gamma(\Phi) - \Gamma(\Phi^*) + \text{div} \Phi + \text{const}.
\]

This technical Appendix is devoted to the proof that all the divergences can be recursively removed up to order \( n \) in the loop expansion while preserving the global \( SU(2) \) invariance and eq. (38). Then at order \( n \) eq. (38) gives

\[
\mathcal{S}_0(\Gamma^{(n)}) = \int d^4x \sum_{\Phi} \left( \frac{\delta \Gamma^{(0)}}{\delta \Phi^*(x)} \frac{\delta}{\delta \Phi(x)} + \frac{\delta \Gamma^{(0)}}{\delta \Phi(x)} \frac{\delta}{\delta \Phi^*(x)} \right) \Gamma^{(n)} =
\]

\[
- \sum_{j=1}^{n-1} \int d^4x \sum_{\Phi} \frac{\delta \Gamma^{(n-j)}}{\delta \Phi^*(x)} \frac{\delta \Gamma^{(j)}}{\delta \Phi(x)} \tag{57}
\]

The second line of the above equation is finite since it contains only lower order terms which have already been subtracted. Hence one gets the following equation for the divergent part \( \Gamma_{\text{div}}^{(n)} \) at order \( n \)

\[
\mathcal{S}_0(\Gamma_{\text{div}}^{(n)}) = 0. \tag{58}
\]

The action of \( \mathcal{S}_0 \) on the fields is the same as that of \( \delta \):

\[
\mathcal{S}_0 \Phi = \frac{\delta \Gamma^{(0)}}{\delta \Phi^*} = \delta \Phi. \tag{59}
\]
Moreover $\mathcal{S}_0$ acts on the antifield $\Phi^*$ by mapping it into the classical e.o.m. of $\Phi$, namely

$$\mathcal{S}_0 \Phi^* = \frac{\delta \Gamma^{(0)}}{\delta \Phi}.$$  \hfill (60)

One can prove in the usual fashion [18] that $\mathcal{S}_0$ is nilpotent. This follows from the nilpotency of $\delta$ and the validity of the functional identity eq.(58) for $\Gamma^{(0)}$.

We are now going to prove that the divergences in $\Gamma^{(n)}_{\text{div}}$ can be reabsorbed by a shift of the factors $Z_g, Z_m, Z_\alpha$ and $Z_j, j = 1, \ldots, 12$ which appear in $\hat{\Gamma}$. For that purpose we notice that the most general solution to eq.(58) can be written as

$$\Gamma^{(n)}_{\text{div}} = A + \mathcal{S}_0 B$$  \hfill (61)

where $A$ cannot be presented in the form $\mathcal{S}_0 C$, with $C$ a local functional. Note that as $\Gamma^{(n)}$ is a Lorentz invariant functional, the functionals $A$ and $B$ also possess this invariance.

A convenient strategy for deriving the most general solution eq.(61) can be described as follows. We see that the Yang-Mills field-strength square

$$A = -\frac{a^{(n)}}{4} \int d^4x G_{\mu\nu} G^{\mu\nu}_a,$$  \hfill (62)

where the divergent coefficient $a^{(n)}$ is unconstrained by the symmetries, is a solution of the $A$-type. The r.h.s. of eq.(62) is obviously also invariant under the global $SU(2)$ symmetry.

We must now address the question of whether other type-$A$ solutions exist. One way to solve this problem is to compute the cohomology $H_F(\mathcal{S}_0)$ [20] of the nilpotent operator $\mathcal{S}_0$ in the space $F$ of Lorentz- and global $SU(2)$-invariant local functionals with dimension bounded by the power-counting. $H_F(\mathcal{S}_0)$ is defined as the quotient of the latter functional space w.r.t. to the equivalence relation

$$X \sim Y \Leftrightarrow X - Y = \mathcal{S}_0 (C)$$  \hfill (63)

for some Lorentz- and global $SU(2)$-invariant local functional $C$.

Clearly if we are able to prove that $H_F(\mathcal{S}_0)$ reduces to the equivalence class of the Yang-Mills field-strength squared (62), we have also established that the only type-$A$-solution to eq.(61) is given by eq.(62).

In order to evaluate $H_F(\mathcal{S}_0)$ we first perform the following change of variables

$$\begin{align*}
(\tilde{\varphi}^a_+)' &= \frac{g}{2m}(\tilde{e}^a \tilde{\theta}^0 - \epsilon^{abc} \tilde{e}^b \tilde{e}^c) + \tilde{\varphi}^a_+ \\
(\tilde{\varphi}^0_+)' &= \frac{g}{2m}(\tilde{e}^a \tilde{\theta}^0 + \epsilon^{abc} \tilde{e}^b \tilde{e}^c) + \tilde{\varphi}^0_+,
\end{align*}$$

which leads to the following $\mathcal{S}_0$-transforms

$$\begin{align*}
\mathcal{S}_0 h &= -\tilde{\theta}^0, \quad \mathcal{S}_0 (\tilde{\theta}^0) = 0 \\
\mathcal{S}_0 \tilde{e}_a &= (\tilde{\varphi}^a_+)', \quad \mathcal{S}_0 (\tilde{\varphi}^a_+)' = 0 \\
\mathcal{S}_0 \tilde{e}_0 &= (\tilde{\varphi}^0_+)', \quad \mathcal{S}_0 (\tilde{\varphi}^0_+)' = 0.
\end{align*}$$

The reason for carrying out such a field redefinition stems from the properties of the so-called doublet variables. A pair of variables $u, v$ such that $\mathcal{S}_0 u = v, \mathcal{S}_0 v = 0$ is called a
The Yang-Mills field-strength squared in $\hat{\Gamma}$ becomes $S_{\delta \Phi}$ the $\delta$-variation of a local Lorentz-invariant functional.

In the subspace where the doublets (and their antifields) are dropped, the action of $S_0$ on $A_{\alpha \mu}$, $\tilde{b}_a$ is the same as the standard gauge BRST transformation for the gauge group $SU(2)$, once one identifies the $SU(2)$ BRST ghosts with $\frac{1}{2} \tilde{b}_a$. As a consequence, the dependence on the antifields $A^{\alpha a}$, $\tilde{b}^{\alpha *}$ is also confined in the $S_0 B$-functional. So we conclude that the cohomology of $S_0$ is isomorphic to the one of the $SU(2)$ Yang-Mills theory, as one should expect. This result holds irrespectively of the dimensions of the local operators involved.

In the space of operators of dimension $\leq 4$ the only element of this cohomology is the Yang-Mills field strength squared in eq.(62). Therefore we see that there are no further $S_0B$.

They fall into two classes: those which do not involve the antifields and those which depend on the antifields. By imposing the dimensional bounds dictated by power-counting and the residual global $SU(2)$ invariance three invariants of the first class arise. The first one is

$$J_t = \int d^4 x S_0 \tilde{e}_0 = \int d^4 x \left( \frac{g}{2m} (\tilde{e}^a \tilde{b}^a + \tilde{e}^0 \tilde{b}^0) + \tilde{\phi}_0^+ \right).$$

It controls the tadpole for $\tilde{\phi}_0^+$ and never arises in dimensional regularization. The other two can be conveniently described in terms of the following operator insertions

$$J_m = \delta Z_m(n) \frac{\partial}{\partial Z_m} \tilde{\Gamma}, \quad J_\alpha = \delta Z_\alpha(n) \frac{\partial}{\partial Z_\alpha} \tilde{\Gamma},$$

where $\delta Z_m(n), \delta Z_\alpha(n)$ are divergent coefficients of order $n$.

The invariants of the $S_0B$-type involving the antifields are of the form (no sum over $\Phi$)

$$z_j^{(n)} \int d^4 x S_0(\Phi^* F_j(\Phi)) = z_j^{(n)} \int d^4 x \left( F_j(\Phi) \frac{\delta \Gamma^{(0)}}{\delta \Phi} - \Phi^* \frac{\delta F_j}{\delta \Phi} S_0 \Phi \right)$$

where again the divergent coefficient $z^{(n)}$ is of order $h^n$.

The possible $F_j$’s, $j = 1, \ldots, 12$ in eq.(63) are constrained by the rigid symmetries of the theory, quantum numbers and power-counting and have the same structure as the corresponding terms in eq.(41).

$Z_m, Z_\alpha, Z_\beta$ and $Z_j$, $j = 1, \ldots, 12$ are formal power series in $h$ of the general form $Z = 1 + \sum_{j=1}^{\infty} Z^{(j)}$. Their coefficients have been fixed up to order $n - 1$ due to the recursion assumption.

We must now prove that their $n$-th order coefficients can be chosen in such a way to remove the divergences in eq.(63).

For that purpose it is convenient to redefine $A^{a \mu}_\mu \rightarrow Z_g g A^{a \mu}_\mu$. Then the coefficient of the Yang-Mills field-strength squared in $\hat{\Gamma}$ becomes $-\frac{1}{4Z_g^2}$ and the term $A$ in eq.(62)
is reabsorbed by choosing $Z_g^{(n)} = \frac{a^{(n)}}{2}$. The terms in eq. (67) are recovered by choosing $Z_m^{(n)} = -\delta Z_m^{(n)}$ and $Z_a^{(n)} = -\delta Z_a^{(n)}$.

We now move to the terms in eq. (68). They can be reabsorbed by setting $Z_j^{(n)} = -z_j^{(n)}$, since by using eqs. (41) and (56) in eq. (46) one gets at the first non-vanishing order in $Z_j^{(n)}$

$$Z_j^{(n)} \int d^4 x \left( F_j(\Phi) \frac{\delta \Gamma^{(0)}}{\delta \Phi} \Phi - \Phi^* \frac{\delta F_j}{\delta \Phi} S_0 \Phi \right) + O(\hbar^{n+1}) =$$

$$- z_j^{(n)} \int d^4 x \left( F_j(\Phi) \frac{\delta \Gamma^{(0)}}{\delta \Phi} \Phi - \Phi^* \frac{\delta F_j}{\delta \Phi} S_0 \Phi \right) + O(\hbar^{n+1}) \quad (69)$$

Once the $n$-th order divergences have been removed, the procedure can be recursively applied. In fact $\hat{\Gamma}$ obeys eq. (39) and thus the functional identity will be fulfilled at the order $n + 1$. Since the divergences have been subtracted up to order $n$, eq. (57) holds at order $n + 1$ and the argument can be repeated.

So indeed the ultraviolet divergencies generated by the interaction may be removed by changing the parameters entering the classical action, expressed in terms of the redefined fields. Hence the renormalized theory is also unitary in the subspace including only three dimensionally transversal components of the Yang-Mills field.

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