Interconversion of exceptional points between different orders in non-Hermitian systems

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Abstract
Singularities of non-Hermitian systems typified by exceptional points (EPs) are critical for understanding non-Hermitian topological phases and trigger many intriguing phenomena. However, it remains unexplored what happens when EPs meet one another. Here, in a typical four-level model with both touching and crossing intersections of EP hypersurfaces, we report the interconversion mechanisms between EPs of different orders. By examining both the eigenvalues and eigenvectors, we show analytically that all EPs of higher orders are formed at the touching intersections of two different types of EP hypersurfaces of lower orders. Contrarily, the crossing intersection of EP structures lowers the order of EPs. The mechanisms of the increase and decrease in defectiveness discovered here are expected to hold for EPs of any order in various non-Hermitian systems, providing a comprehensive understanding of EPs and inspiration toward advanced applications such as biosensing and information processing.

1. Introduction

The topological physics of matter has been investigated intensively over the past few decades in condensed matter physics. Within the Hermitian framework, various topological invariants have been used to characterize different types of nodal singularities (e.g. Dirac and Weyl points) of topological semimetals [1, 2–4], robust edge states of topological insulators [5–11] and Majorana zero modes of topological superconductors [12–14]. Recently, considerable efforts have been devoted to uncovering non-Hermitian topological phases [15–26] in a broad range of wave systems, such as photonic, phononic and mechanical devices [27–35]. Exotic features of non-Hermitian systems that have no counterparts in Hermitian systems challenge our current understanding of topological physics.

Different from Hermitian systems where degeneracies frequently originate from symmetry, the occurrence of non-Hermitian degeneracies such as exceptional points (EPs) is ubiquitous and their existence does not require special symmetries. At the EPs, the Hamiltonian matrix becomes defective and two or more eigenvalues and their corresponding eigenvectors coalesce [25, 36–39]. In the vicinity of EPs, self-intersecting Riemann surfaces can emerge resulting in a fractional topological invariant [38]. The presence of EPs not only enriches the topological classification in the existing non-Hermitian framework providing more profound understandings of non-Hermitian degeneracies [40–44], but also enables practical applications, such as lasing [45–47], enhanced sensing [30, 33, 48, 49], mode switching [28, 50–52] and photonic routing [53]. To date, the most discussed in the literature are EPs of order two/three (EP2/EP3). EPs of higher orders are difficult to achieve. An EP3 has been created by adding non-Hermiticity to a Hermitian system possessing triple degeneracy [54, 55]. It was also found in three-level systems subject to PT symmetry [29, 56]. More
Generally, an EP of higher order can be created at the intersection of EPs of lower orders [46, 54–58]. However, we are not aware of any a priori reason why intersecting EPs should result in a higher-order EP. Understanding what will happen when EPs meet one another will deepen our understanding of the interconnection between non-Hermitian degeneracies of different orders and types.

To facilitate our discussion, we consider a typical four-level non-Hermitian model with five degrees of freedom (DOFs) in the parameter space. The system supports EP4s in the form of curves in a five-dimensional (5D) parameter space (figure 1(a)). The appearance of EP4 curves is made possible by the presence of sublattice symmetry (SLS) in the system [59]. In addition, the system supports two types of EP2s. We shall call one type 'single EP2s', at which two eigenvectors with zero eigenvalues coalesce (the middle panel of figure 1(b)), and the other type 'paired EP2s', where the eigenvalues of two EP2s are opposite in sign (the right panel of figure 1(b)).

The presence of rich singularity structures allows us to study the interconnection between EPs of different orders and types. By analyzing both the eigenvalue and eigenvector, we find the interconversion mechanisms of EPs between different orders. For the increase in defectiveness (i.e. EP order), we show analytically that all EP4s in the system are created at the touching intersections of two EP2 hypersurfaces of different types at which two eigenvectors of the EP2s further coalesce. As for the decrease in defectiveness, we find that the crossing intersection of four EP4 curves gives rise to a degenerate EP2 pair, and the crossing intersection of two EP2 hypersurfaces of the same type produces nondefective degeneracies.

2. Non-Hermitian model with EP4 curves

We start with a four-level non-Hermitian system comprising nearest-neighbor coupled entities (labeled as 1–4), as shown in figure 1(a). Onsite gain and loss are employed to induce non-Hermiticity [58]. Its Hamiltonian can be written as

\[
H = \begin{pmatrix}
-M_1 & \kappa & 0 & 0 \\
\kappa & -M_0 & \beta\kappa & 0 \\
0 & \beta\kappa & M_0 & \kappa \\
0 & 0 & \kappa & M_1
\end{pmatrix},
\]

where \(M_{0,1} = m_{0,1} + i\gamma_{0,1}\) denote complex onsite energies, with \(\gamma_{0,1}\) being the gain/loss parameters. In practice, differential loss is usually used as a surrogate for gain/loss without changing the physics, noting that gain is difficult to implement. \(\kappa\) represents the coupling coefficient while \(\beta\) is a coupling detuning parameter. The system can be realized by resonant cavities [60, 61], waveguides [62–65] or superconducting qubits [66, 67]. For example, in coupled acoustic resonators, complex onsite energies, couplings and loss can be conveniently engineered by tuning the size of resonators, coupling tubes and absorbing materials (e.g. sponge), respectively [68]. Without loss of generality, we assume that \(\kappa = 1\) and is real. The system has five DOFs denoted by \(P = (m_0, \gamma_0, m_1, \gamma_1, \beta)\). The Hamiltonian \(H\) respects SLS defined by

\[
\Gamma H(P) \Gamma^{-1} = -H(P),
\]
where \( \Gamma = \sigma_x \otimes \sigma_y \) with \( \sigma_x, \sigma_y \) being the Pauli matrices. Hence the eigenvalues of \( H \) appear in \((-\lambda, \lambda)\) pairs, and \( \lambda = \pm \sqrt{g_1 \pm \sqrt{g_2}} \), where \( g_{1,2} \) are given as

\[
g_1 = \left[ M_0^2 + M_1^2 + \beta^2 + 2 \right] / 2 \text{and } g_2 = g_1^2 - (M_0^2 + \beta^2) M_1^2 + 2M_0M_1 - 1. \tag{3}
\]

SLS also guarantees three types of non-Hermitian singularities, namely, EP4s, single EP2s and paired EP2s, as displayed in figure 1(b). EPs of order \( \mu \) can be located by requiring the characteristic polynomial \( f_\mu(\lambda) = \det [H(P) - \lambda] \) and its successive derivatives \( f_\mu'(\lambda) := \partial f_\mu / \partial \lambda \) with \( j = 1, 2, \ldots, \mu - 1 \) to vanish simultaneously (section I in the supplemental material [69]). A general conclusion is that the number of constraints for an EP\( \mu \) to exist can be reduced from \( 2\mu - 2 \) to \( \mu - 1 \) for even (odd) \( \mu \) by the presence of SLS [69]. For our system, the number of constraints on the existence of EP4s is reduced to 4 [69]. Therefore, EP4 curves are expected to occur since we have five DOFs. The hypersurface of general single EP2s in this system is given by

\[
(M_0M_1 - 1)^2 + \beta^2 M_1^2 = 0, \tag{4}
\]

which is obtained by substituting \( g_1^2 = g_2 \) into equation (3), while the hypersurface of paired EP2s fulfills \( g_2 = 0 \) (section I in the supplemental material [69]). The constraints on EP4s can be derived as \( g_{1,2} = 0 \), which is mathematically equivalent to forming touching intersections of two hypersurfaces of single EP2s and paired EP2s (section II in the supplemental material [69]). Thus, all EP4s are associated with \( \lambda = 0 \) due to SLS and lie at the touching intersections of two hypersurfaces, given by

\[
g_2 = 0 \text{ and } g_1^2 = g_2. \tag{5}
\]

At the touching point it can be shown that the tangents of two hypersurfaces are identical in all directions, forcing the two coalescing eigenvectors on each hypersurface to become parallel to each other, therefore, coalesce into the only eigenvector of EP4. Thus, the tangential intersection can be considered as the mechanism of achieving a higher-order EP. The mathematical derivation as well as the physical explanation of the phenomenon is given in section II of the supplemental material [69]. This upward process from EP2s of different types to EP4 applies to all EP4s.

To see the downward process, we can consider two parameter subspaces, i.e. \((m_0, \gamma_0, \beta)\) and \((m_1, \gamma_1, \beta)\), as the projections of the 5D parameter space. The solution of equation (5) is illustrated in figures 2(a) and (b). There are totally eight EP4 curves in the 5D parameter space, accompanied by two crossing intersections located at \( P = (0, \pm 1, 0, \mp 1, 0) \), which are marked by two stars. Three representative cross sections are displayed in the insets of figures 2(a) and (b). At the stars with \( \beta = 0 \), the system is decoupled into two independent subsystems. Each subsystem supports an EP2 with zero eigenvalue. We call these two EP2s as degenerate EP2 pairs. Thus, a crossing intersection of higher-order EP curves yields lower-order EPs. The tangents at these crossing intersections cannot be uniquely identified, causing the only eigenvector of EP4s to be preserved as two noncoalescing eigenvectors (section III in the supplemental material [69]). This downward process from EP4s to EP2s applies to the entire parameter space.

To characterize the topological properties of EP4s and degenerate EP2 pairs, we study their winding numbers based on both eigenvalues and eigenvectors [68, 70–75]. We find that each EP4 (degenerate EP2 pair) carries a quantized winding number of eigenvalues 1 (2) and fractional winding number of eigenvectors 3/4 (1/2) (section III in the supplemental material [69]). Besides, we note that these EP4 curves preserve mirror symmetries: \( m_{0,1} \rightarrow -m_{0,1}, \gamma_{0,1} \rightarrow -\gamma_{0,1} \) and \( \beta \rightarrow -\beta \), which is also reflected in system eigenvectors, i.e. \( |\psi^R_1\rangle = (\phi_1, \phi_2)^T \) with \( \phi_1(\beta) = \phi_1(-\beta) \) and \( \phi_2(\beta) = -\phi_2(-\beta) \) (section VI in the supplemental material [69]). Figures 2(c) and (d) illustrate eigenvector behaviors on eight curves along two opposite directions with respect to \( \beta \), where different colors denote independent eigenvectors. Four EP4 curves meet at each ‘star’ point for positive (negative) \( \beta \), which manifests as a crossing in the neighborhood of \( \beta = 0^+ \) (\( \beta = 0^- \)). At an EP4, four eigenvectors coalesce into one. At a crossing intersection of two EP4 paths, each EP4 path contributes to one eigenvector, therefore two noncoalescing eigenvectors are generated indicating the occurrence of EP2 pairs. Specifically, one ‘star’ point has two eigenvectors \( |\psi^R_1\rangle, |\psi^R_2\rangle = (i, -i, \pm i, \pm 1)^T \), while the other one has two eigenvectors \( (|\psi^R_1\rangle)^* \) and \( (|\psi^R_2\rangle)^* \). This allows only one degenerate EP2 pair to exist at an intersection, where the fourfold spectral degeneracy is protected by SLS (section V in the supplemental material [69]). To conclude, it is the presence of crossing intersections that reduces the order of EPs from 4 to 2.

Not every parameter point that satisfies the above constraint equations of single EP2s or paired EP2s exhibits non-Hermitian degeneracies with coalescing eigenvectors. Nondefective degeneracy points (NDPs) occur when two hypersurfaces of single EP2s or paired EP2s cross each other with \( \beta = 0 \). At the crossing point two hypersurfaces have different tangents, leading to two independent eigenvectors, therefore, making
degenerate eigenvalues of $H$ with larger geometric multiplicities and producing single NDPs or paired NDPs (section VI in the supplemental material [69]). This downward process is opposite to the upward process discussed earlier where two hypersurfaces of different types touch with the same tangent making degenerate eigenvalues of $H$ with smaller geometric multiplicities.

To obtain the condition of single NDPs, we set $\beta = 0$ in the constraint of $g_2 = g_2$ and obtain $M_0 M_1 = 1$. For the paired NDPs, we set $\beta = 0$ in the constraint of and obtain $M_0 + M_1 = 0$. When, the system becomes Hermitian, single (paired) NDPs reduce to single (paired) generalized diabolic points (GDPs). In figure 3(a), we display the curves of single GDPs (orange) and paired GDPs (green) for $\gamma_0, \gamma_1 = 0$. Since these two curves do not intersect, there is no fourfold degeneracy in the Hermitian limit. However, it is easy to show that single NDPs and paired NDPs do meet when $M_0 = \pm i$ and $M_1 = \mp i$, which are exactly degenerate EP2 pairs marked by two stars (figure 3(b)).

We note that the formation of degenerate EP2 pairs is not due to the crossing of single NDPs and paired NDPs. At the crossing point each of two eigenvectors in a subsystem coalesces with the other eigenvector in the same subsystem forming an EP2, giving rise to a degenerate EP2 pair. This process holds separately for single NDPs and paired NDPs (section VI in the supplemental material [69]).

3. EP4s in reduced parameter subspaces with four DOFs

To further reveal hierarchical EP4s, we study EP4 structures in reduced parameter spaces with four DOFs by fixing one of five DOFs. Five fixed parameters uncover various allowable numbers of EP4s and degenerate EP2 pairs as displayed in table 1. According to Abel–Ruffini theorem, equation (5) can be reduced to polynomials of degree four with solutions in radicals (section VII in the supplemental material [69]) [76]. A rigorous derivation based on equation (5) proves that for $M_0$, $\lim_{\beta \to \pm \infty} |m_0| = 0$ and $\lim_{\beta \to \pm \infty} |\gamma_0| = \pm \infty$ always hold. For, $\lim_{\beta \to \pm \infty} |m_1| = \pm \infty$ and $\lim_{\beta \to \pm \infty} |\gamma_1| = 0$ or $\pm \infty$ always hold (section VIII in the supplemental material [69]). Assisted by known EP4 curves in figures 2(a) and (b), we have an infinity.
Figure 3. GDPs and NDPs. (a) The curves of single GDPs (orange) and paired GDPs (green) for $\gamma_0,1=0$. (b) The curves of single NDPs (red) and paired NDPs (blue) for $m_0,1=0$.

Table 1. Allowable numbers of EP4s (I) and degenerate EP2 pairs (II) in reduced parameter spaces with four DOFs. Different numbers of I and II are obtained by fixing one of five different parameters.

| Fixed parameters | $m_0$ | $\gamma_0$ | $m_1$ | $\gamma_1$ | $\beta$ |
|------------------|-------|------------|-------|------------|--------|
| Nos. of I        | 0,4,8,EP4 lines | 2,4,8 | 4,EP4 lines | 0,2,6 | 0,8    |
| Nos. of II       | 0,2   | 0,1        | 0,2   | 0,1        | 0,2    |

number of EP4s if and only if $m_0=0$ or $m_1=0$ in reduced parameter spaces with four DOFs. Besides, there is no EP4 in the system if and only if $|m_0| \gtrsim 0.47$ or $\gamma_1=0, \pm 1$ [69]. Except for above two cases, only a discrete and limited number of EP4s is allowed in reduced parameter spaces with four DOFs.

4. Various singularities in reduced parameter subspaces with three DOFs

As the parameter space is reduced to three DOFs, both non-Hermitian and Hermitian degeneracies emerge intuitively. Parameter spaces with three DOFs can be obtained by fixing two of five DOFs. An intuitive way is to select specific parameter points in $\mathcal{M}_{0,1}$ to study the remaining DOFs. By projecting all solutions of equation (5) to $\mathcal{M}_{0,1}$, we show the possible number of EP4s and degenerate EP2 pairs in fixed parameter spaces with two DOFs: (a) the points on $\mathcal{M}_0$ (figure 4(a)) allow the absence of EP4s, two EP4s, two EP4s and one degenerate EP2 pair, and four EP4s; (b) the points on $\mathcal{M}_1$ (figure 4(c)) allow the absence of EP4s, one degenerate EP2 pair, and two EP4s.

We further study detailed singularity structures in a reduced parameter space with three DOFs. All scenarios can be categorized as a combination of four cases: (a) the curves of single EP2s cross themselves to form single NDPs (or single GDPs); (b) the curves of paired EP2s cross themselves to form paired NDPs (or paired GDPs); (c) the curves of single EP2s and paired EP2s touch to form an EP4; (d) single NDPs and paired NDPs merge together to form degenerate EP2 pairs. In this sense, panels (A) (figure 4(b)) and (E) (figure 4(d)) can be understood as the simultaneous appearance of (a) and (b). Panel (B) (figure 4(b)) is read as the simultaneous appearance of (b) and (c). Panel (C) (figure 4(b)) corresponds to the simultaneous appearance of (c) and (d) [77]. Panels (D) (figure 4(b)) and (F) (figure 4(d)) show the single appearance of (c) and (d), respectively. Panels (G) (figure 4(d)) and (H) (figure 4(d)) reveal the single appearance of (d) and the simultaneous appearance of (a), (b) and (c), respectively. These abundant results reflect the reconfiguration of different degeneracies, spanning non-Hermitian and Hermitian degeneracies, as well as low-order and higher-order non-Hermitian degeneracies.

Notably, the crossing intersection of non-Hermitian degeneracies always raise the number of noncoalescing eigenvectors due to the presence of non-unique tangents (e.g. the formation of single NDPs, paired NDPs and degenerate EP2 pairs), while the touching intersection of non-Hermitian degeneracies of different types always causes the eigenvectors to coalesce because of sharing the same tangent (e.g. the formation of EP4s) (section II in the supplemental material [69]). This clarifies the interconversion
mechanism of various non-Hermitian and Hermitian degeneracies and provides guidance for the constructions of other complex degeneracies.

5. Discussion

Using a typical four-level non-Hermitian model with SLS, we have presented the interconversion mechanisms of EPs between different orders with five DOFs in the parameter space. Such a system can be implemented by modulating relative losses in four coupled optical or acoustic cavities. Under this minimal model with touching and crossing intersections simultaneously, we demonstrated that the touching intersection of lower-order EP structures of different types leads to higher-order EPs, whereas the crossing intersection of higher-order EP structures always gives rise to lower-order EPs. EPs of different orders can be identified with winding numbers of eigenvalues and eigenvectors. The self-crossing intersection of single (paired) EP2 structures always yields single (paired) NDPs or GDPs. These processes reveal that the touching (crossing) behavior of EPs allows the upward (downward) process of non-Hermitian degeneracies with
different orders. Our findings constitute a fundamental understanding of non-Hermitian topological physics with high DOFs, promising for implementing sensitivity-tunable sensing in one system and robust chiral mode conversions for information technologies [33].

**Data availability statement**

All data that support the findings of this study are included within the article (and any supplementary files).

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