Magnetohydrodynamic Turbulence: Generalized Energy Spectra

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Abstract A general framework that incorporates the Iroshnikov-Kraichnan (IK) and Goldreich-Sridhar (GS) phenomenologies of magnetohydrodynamic (MHD) turbulence is developed. This affords a clarification of the regimes of validity of IK and GS phenomenologies and hence help resolve some controversies in this problem.

Magnetohydrodynamic (MHD) flows that occur naturally (like astrophysical situations) and in modern technological systems (like fusion reactors) show turbulence. Early theoretical investigations of MHD turbulence considered the isotropic case. On the latter premise, Iroshnikov [1] and Kraichnan [2] (IK) made arguments à la Kolmogorov [3] and proposed that statistical properties of the small-scale components of the velocity and magnetic fields

*are controlled by the shear Alfvén wave dynamics;
*show, in the limit of large viscous and magnetic Reynolds numbers, some universality in the inertial range.

and gave for the total energy spectral density $E(k)$, the behavior $E(k) \sim k^{-3/2}$. Montgomery et al. [4] and [5], Goldreich and Sridhar [6] - [8] (GS) pointed out that the isotropy assumption in the IK theory is not a very sound one in the MHD case, thanks to the magnetic field of the large-scale eddies, and gave for the energy spectrum in the plane transverse to the magnetic field the behavior $E(k_{\perp}) \sim k_{\perp}^{-5/3}$ [8]. However, DNS of MHD turbulence in a strong applied magnetic field (Maron and Goldreich [9], Muller et al. [10] and [11]) showed that the transverse energy spectrum is close
to the IK theory $E(k_{\perp}) \sim k_{\perp}^{-3/2}$. On the other hand, the 3D DNS (Muller et al. [12], Haugen et al. [13]) of MHD turbulence and solar wind measurements (Leamon et al. [14], Goldstein et al. [15]) confirm the GS spectrum $E(k_{\perp}) \sim k_{\perp}^{-5/3}$. A resolution of this apparent conflict requires clarification of the regimes of validity of IK and GS phenomenologies which is the objective of this paper. A general framework that incorporates IK and GS phenomenologies is developed to accomplish this objective. The IK and GS hypotheses can be shown to follow from the formal analogy between the hydrodynamic and MHD spectral energy density expressions.

Let us write the spectral energy density $E(k)$ as

$$E(k) \sim \frac{V}{k^2 \tau} \quad (1)$$

$V$ being given the characteristic velocity of the spectral element $k$. The hydrodynamic eddy turn-over time $\tau$ given by

$$\tau \sim \frac{1}{kV} \quad (2)$$

then becomes

$$\tau \sim \frac{1}{k^{4/3}E^{2/3}} \quad (3)$$

(2) implies that the energy transfer in the hydrodynamic case is local in the spectral space which reflects the fact that a large-scale velocity field can be transformed away via Galilean invariance.

If we use the relation

$$\tau \sim \frac{E k}{\varepsilon} \quad (4)$$

$\varepsilon$ being the mean energy transfer rate, (3) leads to the Kolmogorov [16] spectrum

$$E(k) \sim \varepsilon^{4/3} k^{-5/3} \quad (5)$$

One may write for the MHD case, in analogy with (1) (Shivamoggi [17]),

$$E(k) \sim \frac{C_A}{k^{2/3} \tilde{\tau}} \quad (6)$$

$\tilde{\tau}$ being the MHD turn-over time, and $C_A$ being the velocity of Alfvén waves in the total magnetic field -

$$C_A = C_{A_0} + \tilde{C}_A \quad (7)$$
where,

\[ C_{A_0} \equiv \frac{B_0}{\sqrt{\rho}}, \quad \tilde{C}_A \equiv \frac{\tilde{B}}{\sqrt{\rho}}. \]  

(8)

\( B_0 \) is the applied magnetic field and \( \tilde{B} \) is the magnetic field of the large-scale eddies. Combining (6) with (3), we obtain

\[ \hat{\tau} \sim \tau \left(1 + \frac{\tau}{\tau_A}\right) \]  

(9)

where,

\[ \tau_A \sim \frac{1}{kC_{A_0}}, \quad \tau \sim \frac{1}{kV} \sim \frac{1}{k\tilde{C}_A}. \]  

(10)

(9) implies that the energy transfer in the MHD case is non-local in the spectral space which reflects the fact that a large-scale magnetic field cannot be transformed away via Galilean invariance [2]. We have from (9) further

\[ \hat{\tau} \sim \begin{cases} \frac{\tau_A^2}{\tau_A} & \tau \gg \tau_A \\
\tau & \tau \ll \tau_A \end{cases} \]  

(11)

The first case in (11) corresponds to the case with a very strong applied magnetic field \( (B_0 \gg \tilde{B}) \) and represents the IK hypothesis, while the second case corresponds to the case with a very weak applied magnetic field \( (B_0 \ll \tilde{B}) \) and represents the GS situation.

Using (7) - (10), (6) leads to the generalized energy spectrum -

\[ E(k) \sim \varepsilon \frac{1}{2} C_{A_0}^{\frac{1}{2}} k^{-\frac{3}{2}} \left[1 + \left(\frac{E_k}{C_{A_0}^2}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} \]  

(12a)

or alternatively,

\[ E(k) \sim \varepsilon \frac{3}{2} k^{-\frac{3}{2}} \left[1 + \left(\frac{C_A^2}{E_k}\right)^{\frac{1}{2}}\right]^{\frac{3}{2}} \]  

(12b)

For the IK regime which corresponds to \( \left(\frac{E_k}{C_{A_0}^2}\right) \ll 1 \), (which can be verified a posteriori), (12) reduces to

\[ E(k) \sim \varepsilon \frac{1}{2} C_{A_0}^{\frac{1}{2}} k^{-\frac{3}{2}} \]  

(13)
while for the GS regime which corresponds to $\left( \frac{E_k}{C_A^2} \right) \gg 1$, (12) reduces to

$$E(k) \sim \varepsilon^{\frac{2}{3}} k^{-\frac{2}{3}}$$

as well known.

On the other hand, (12)a,b indicate that the energy spectrum, in the general case, shows no power-law behavior!
References

[1] P.S. Iroshnikov: *Sov. Astron.* **7**, 566, (1964).

[2] R.H. Kraichnan: *Phys. Fluids* **8**, 1385, (1965).

[3] A.N. Kolmogorov: *Dokl. Akad. Nauk. SSSR* **30**, 4, (1941).

[4] D.C. Montgomery: *Phys. Scr.* **T2/1**, 83, (1982).

[5] J.V. Shebalin, W.H. Matthaeus, and D.C. Montgomery: *J. Plasma Phys.* **29**, 525, (1983).

[6] S. Sridhar and P. Goldreich: *Astrophys. J.* **432**, 612, (1994).

[7] P. Goldreich and S. Sridhar: *Astrophys. J.* **438**, 763, (1995).

[8] P. Goldreich and S. Sridhar: *Astrophys. J.* **485**, 680, (1997).

[9] J. Maron and P. Goldreich: *Astrophys. J.* **554**, 1175, (2001).

[10] W.C. Muller, D. Biskamp, and R. Grappin: *Phys. Rev. E.* **67**, 066302, (2003).

[11] W.C. Muller and R. Grappin: *Phys. Rev. Lett.* **95**, 114502, (2005).

[12] W.C. Muller and D. Biskamp: *Phys. Rev. Lett.* **84**, 475, (2000).

[13] N.E.L. Haugen, A. Brandenburg, and W. Dobler: *Phys. Rev. E* **70**, 016308, (2004).

[14] R.J. Leamon, C.W. Smith, N.F. Ness, W.H. Matthaeus, and H.K. Wong: *J. Geophys. Res.* **103**, 4775, (1998).

[15] M.L. Goldstein and D.A. Roberts: *Phys. Plasmas* **6**, 4154, (1999).

[16] A.N. Kolmogorov: *Dokl. Akad. Nauk. SSSR* **30**, 301, (1941).

[17] B.K. Shivamoggi: *Ann. Phys.* **253**, 239, (1997); Erratum in *Ann. Phys.* **312**, 270, (2004).