Anomalous elasticity of disordered networks

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Continuum elasticity is a powerful tool applicable in a broad range of physical systems and phenomena. Yet, understanding how and on what scales material disorder may lead to the breakdown of continuum elasticity is not fully understood. We show, based on recent theoretical developments and extensive numerical computations, that disordered elastic networks near a critical rigidity transition, such as strain-stiffened fibrous biopolymer networks that are abundant in living systems, reveal an anomalous long-range elastic response below a correlation length. This emergent anomalous elasticity, which is non-affine in nature, is shown to feature a qualitatively different multipole expansion structure compared to ordinary continuum elasticity, and a slower spatial decay of perturbations. The potential degree of universality of these results, their implications (e.g. for cell-cell communication through biological extracellular matrices) and open questions are briefly discussed.

INTRODUCTION

Continuum elasticity is a powerful tool that describes a huge range of phenomena in diverse physical systems, over sufficiently large lengthscales \([1]\). On sufficiently small lengthscales, where the effect of material structure and disorder cannot be coarse-grained, continuum elasticity theory is expected to break down. The crossover between these two regimes is characterized by a correlation length \(\xi\). Indeed, recent work has identified \(\xi\) for glassy systems, particle packings and elastic networks \([2−6]\), and demonstrated the validity of continuum elasticity theory for lengthscales \(r\) that satisfy \(r \gg \xi\). In particular, the displacement response to a point force (monopole) perturbation decays as \(1/r\) in three dimensions (3D) for \(r \gg \xi\), while the displacement response to a force dipole — which is proportional to the spatial gradient of the point force response \([1]\) — decays as \(1/r^2\). Yet, it remains unclear whether a generic elastic response exists for \(r \ll \xi\) and if so, what form it takes.

For systems undergoing a glass transition upon cooling a melt, \(\xi\) has been shown to be of the order of 10 atomic distances and to vary mildly with glass formation history (e.g. the cooling rate) \([6]\). Consequently, it is difficult to imagine that a generic elastic response emerges on such a narrow range of scale and one expects that disorder and fluctuations dominate the elastic response for \(r < \xi\) in such systems. On the other hand, systems that undergo a critical rigidity transition — such as fibrous biopolymer networks that are abundant in living systems (e.g. collagen, fibrin and basement membrane) and that are known to undergo a dramatic stiffening transition when deformed to large enough strains — feature a macroscopically large \(\xi\) close to the rigidity transition \([3]\). In such cases, the regime \(r \ll \xi\) spans many orders of magnitude and might possibly accommodate a generic elastic response. In this brief report, we consider such disordered networks near their rigidity transition and study their elastic response for \(r \ll \xi\).

We show, based on recent theoretical developments and extensive numerical simulations, that the elastic response of disordered networks follows an anomalous power-law for \(r \ll \xi\), when \(\xi\) is sufficiently large, and that the decay is slower compared to the continuum elastic response for \(r \gg \xi\). Furthermore, we show that the anomalous elasticity for \(r \ll \xi\) features a qualitatively different multipole expansion structure compared to continuum elasticity. These results may have significant implications for long-range mechanical interactions between distant cells through biological extracellular matrices, inspire the design of heterogeneous structures with unusual properties and pose new basic questions, which are briefly discussed.

RESULTS

We consider isotropic disordered networks whose nodes are connected by relaxed Hookean springs and whose proximity to a rigidity (jamming-unjamming) transition is controlled by the connectivity \(z\) (average springs per node), which is close to (yet larger than) the critical Maxwell threshold \(z_c = 2d\) (where \(d\) is the spatial dimension). Such disordered elastic networks feature a shear modulus that scales linearly with \(z - z_c\) \([7]\) and

\[
\xi \sim \frac{1}{\sqrt{z - z_c}},
\]

which diverges as \(z \rightarrow z_c\) \([2, 4]\). Our goal is to understand the elastic response of such networks for \(r \ll \xi\).

The basic quantity we focus on is the displacement response \(u(r)\) to a force dipole applied at the origin. The reason we consider the dipole response (and not, for example, the monopole response) is three-fold. First, our results are relevant for fibrous biopolymer networks, which constitute extracellular matrices to which cells adhere in physiological contexts. Adherent cells apply to

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their surrounding extracellular matrices contractile forces that are predominantly dipolar [8]. Second, contact formation between solid particles during dense suspension flows, where the overdamped response is analogous to elastic response, generates a force dipole [9]. Finally, in many cases low-energy excitations in disordered systems are of dipolar nature, e.g. the universal nonphononic excitations in glasses [10].

Consider then the correlation function \( C(r) \sim \langle u(r) \cdot u(r) \rangle \), where \( \langle \cdot \rangle \) stands for an angular average, rendering \( C(r) \) a function of the distance \( r \) alone, and note that \( u(r) \) is normalized such that \( C(r) \) is dimensionless. Next, one can consider the integral \( \int_0^\xi C(r) \, \mathrm{d}^d r \) in \( d \) spatial dimensions and ask about its scaling with \( \xi \). If \( C(r) \sim \exp(-r/\xi) \) for \( r < \xi \), then one obtains \( \int_0^\xi C(r) \, \mathrm{d}^d r \sim \int_0^\xi C(r) \, r^{d-1} \, \mathrm{d}r \sim \xi^d \), i.e. the naive scaling with the correlation length \( \xi \). However, recent work [4, 9, 11, 12] suggested that in fact

\[
\int_0^\xi C(r) \, \mathrm{d}^d r \sim \xi^2 ,
\]

in any dimension \( d \).

The latter indicates the existence of long-range correlations, i.e. that in fact \( C(r) \sim r^{-2\beta} \exp(-r/\xi) \) for \( r < \xi \). Consistency with Eq. (2) requires that \( \beta = (d - 2)/2 \) [9]. Consequently, in 3D (\( d = 3 \)) we have for the dipolar displacement response

\[
u(r) \sim \begin{cases} \frac{1}{\sqrt{d}} & \text{for } r \ll \xi \\ \frac{1}{r^2} & \text{for } r \gg \xi \end{cases},
\]

where for \( r \gg \xi \) we just have the continuum elastic dipole response. Note that the dipolar response is predicted in Eq. (3) to decay significantly slower for \( r \ll \xi \) than for \( r \gg \xi \). The same argument leading to Eq. (3) has been spelled out in the context of flowing dense suspensions [9] (see the “Discussion and Conclusions” section therein, where the overdamped — not the elastic — response has been considered). A similar anomalous elastic response has been recently suggested using a Ginzburg-Landau description of the so-called overlap free-energy in a mean-field approach to glasses [12].

To test the prediction in Eq. (3), we set out to construct a correlation function that is slightly different than \( C(r) \). Our motivation for doing so is that we are not only interested in validating the form of \( u(r) \) for \( r \ll \xi \), but also in gaining insight into the structure of elasticity theory on these scales and the possible differences compared to continuum elasticity. The latter features a multipole expansion in which the \( n^{th} \) multipole order involves a spatial gradient \( \partial \) of the \((n-1)^{th}\) one. Consequently, we construct a correlation function \( c(r) \) (see Supplemental Materials for details) that scales as \( \partial u(r) \right|^2 \) for \( r \gg \xi \), and explore whether the corresponding structure persists also for \( r \ll \xi \). Moreover, \( \partial u(r) \right|^2 \) is proportional to the elastic energy density of the dipolar response (to quadratic order), and hence is a directly relevant physical quantity.

The response for \( r \ll \xi \) is expected to be strongly affected by disorder, and hence to be non-affine in na-

![FIG. 1.](image-url)
Consequently, we expect the angular average of spatial derivatives of the displacement vector $\mathbf{u}(\mathbf{r})$ to feature vectorial cancellations. As a result, one can hypothesize that $c(r)$ for $r \ll \xi$ inherits its scaling from $|\mathbf{u}(\mathbf{r})|^2$, not from $|\partial \mathbf{u}(\mathbf{r})|^2$. If true, then $c(r)$ takes the form

$$c(r) \sim \begin{cases} |\mathbf{u}(\mathbf{r})|^2 \sim \frac{1}{r^7} & \text{for } r \ll \xi \\ |\partial \mathbf{u}(\mathbf{r})|^2 \sim \frac{1}{r^5} & \text{for } r \gg \xi \end{cases},$$

where $c(r) \sim |\partial \mathbf{u}(\mathbf{r})|^2 \sim r^{-6}$ for $r \gg \xi$ follows (by the construction of $c(r)$) from $\mathbf{u}(\mathbf{r}) \sim r^{-2}$ of Eq. (3). Consequently, Eq. (4) suggests that the elastic response for $r \ll \xi$ is anomalous not just in its power-law decay, as predicted in Eq. (3), but also as it might not follow the multipole expansion structure of continuum elasticity.

Quantitatively testing the prediction in Eq. (4) requires very large networks that span a broad range of length scales and feature a large correlation length $\xi$, which is controlled by $z - z_c$. To that aim, we generated disordered elastic networks of 16 million nodes each for various $z - z_c$ values (see Supplemental Materials), and calculated their dipole response and subsequently $c(r)$ following its network-level definition (see Supplemental Materials). A single dipole response is shown in Fig. 1a. We first aim at verifying the continuum elastic response in Eq. (4) for sufficiently large scales, i.e. for $r \gg \xi$. In Fig. 1b, we plot $c(r)$ for several small values of $z - z_c$ (as indicated in the legend). We find that indeed for sufficiently large $r$, all curves follow $c(r) \sim r^{-6}$ (and, as expected, more so the larger $z - z_c$ is, corresponding to smaller $\xi$).

We then set out to test the prediction in Eq. (4) for $r \ll \xi$, which was our major goal. To that aim, we first note that assuming the spatial dependence of $c(r)$ in this regime (cf. Eq. (4)), one can also predict its $z - z_c$ dependence. The result reads $c(r) \sim (z - z_c)^3/r$ for $r \ll \xi$ (see Supplemental Materials). The latter implies that $r^6 c(r)/\sqrt{z - z_c} \sim (r^5 z - z_c)^5 \sim (r/\xi)^5$ for $r \ll \xi$. This prediction is tested in Fig. 1c and is shown to be in excellent quantitative agreement with the direct numerical calculations.

**DISCUSSION**

The results presented above, based on recent theoretical developments and extensive numerical simulations, provide strong evidence for the existence of anomalous elasticity in disordered networks close to their rigidity transition, below a correlation length $\xi$ that is expected to be macroscopically large. The elastic response for $r \ll \xi$ is anomalous, i.e. different from ordinary continuum elasticity that is valid for $r \gg \xi$, in at least two major respects; first, the spatial decay of perturbations is slower (i.e. characterized by a smaller inverse power-law exponent) compared to continuum elasticity. Second, the elastic response for $r \ll \xi$ is highly non-affine and hence does not appear to follow the ordinary multipole expansion structure of continuum elasticity, where a higher order multipole response is obtained by spatial gradients of a lower order one.

The anomalous elasticity discussed here may have significant implications for various systems such as flowing dense suspensions (where the overdamped response is analogous to the elastic response) and strain-stiffened fibrous biopolymer networks that are abundant in living systems, e.g. as extracellular matrices to which cells adhere and apply contractile forces [8]. In the latter context, our results may imply that for $r \ll \xi$ distant cells can mechanically communicate over significantly longer distances (compared to continuum elasticity) using their active contractility, a capability that might be important in various physiological processes (e.g. tissue development). Our predictions can be tested in experiments using, for example, collagen or fibrin networks that are posed just above their strain-stiffening transition. In addition, our findings may guide the design of heterogeneous materials and structures with unusual elastic properties.

Our results also pose new questions and open the way for additional research directions. First, we focused here on the response to force dipoles, which is well motivated from the physical and biological perspectives, as discussed above. Yet, force monopoles (point forces) are of interest as well, both because they are of practical relevance and because of the unusual multipole structure of the anomalous elastic response we discussed. In particular, in view of our findings, one can speculate that the monopole response is identical to the dipole response scaling-wise, which should be tested in future work. Second, our analysis considered the amplitude squared of the response, i.e. $|\mathbf{u}(\mathbf{r})|^2$. It would be interesting to understand whether the same scaling remains valid for the average of the vectorial response itself, i.e. $\mathbf{u}(\mathbf{r})$, which is obviously subjected to vectorial cancellations due to the non-affine nature of the response, see for example the results of [14]. Finally, the degree of universality of our results should be also tested in future work, e.g. in different disordered networks.

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Supplemental Materials for:
“Anomalous elasticity of disordered networks”

1. The correlation function \( c(r) \) and its scaling properties

We construct the correlation function \( c(r) \) as follows:

1. We define the dimensionless unit force dipole \( \mathbf{d}^{(ij)} \) applied on the pair of nodes \( i, j \) as

\[
\mathbf{d}^{(ij)}_k = \frac{\partial r_{ij}}{\partial x_k} = (\delta_{jk} - \delta_{ik}) \mathbf{n}_{ij}, \quad (S1)
\]

where the \( k \)th node coordinates are denoted by \( x_k \), \( r_{ij} \equiv |x_{ij}| \) is the (scalar) pairwise distance between nodes \( i \) and \( j \), \( x_{ij} \equiv x_j - x_i \) is the vector difference, and \( \mathbf{n}_{ij} \equiv x_{ij}/r_{ij} \) is the unit vector pointing from node \( i \) to node \( j \).

We note that the contraction of the dipole \( \mathbf{d} \) — as defined in Eq. (S1) above — with a field corresponds to taking the difference of the field across a bond, i.e. to a discrete (network-level) gradient in the bond direction.

2. The response of the network to a unit dimensionless force dipole \( \mathbf{d}^{(ij)} \) acting on the \((ij)\) spring is calculated through

\[
\mathbf{u}^{(ij)} = \mathbf{M}^{-1} \cdot \mathbf{d}^{(ij)}, \quad (S2)
\]

where \( \mathbf{M} \equiv \frac{\partial^2 U}{\partial x_i \partial x_j} \) is the Hessian matrix. An example of such a dipolar response is presented in Fig. 1a in the manuscript. We then define the normalized response as \( \tilde{\mathbf{u}}^{(ij)} \equiv \mathbf{u}^{(ij)}/\sqrt{\mathbf{u}^{(ij)}} \cdot \mathbf{u}^{(ij)} \) and subsequently use it.

3. We are next interested in the extension/compression of each spring connecting nodes \( m, n \), associated with the normalized response \( \tilde{\mathbf{u}}^{(ij)} \) to a dimensionless unit force dipole \( \mathbf{d}^{(ij)} \) applied on the pair of nodes \( i, j \). This quantity is given by

\[
A_{(ij), (nm)}^{(nm)} = \tilde{\mathbf{u}}^{(ij)} \cdot \mathbf{d}^{(nm)} = \frac{\mathbf{d}^{(ij)} \cdot \mathbf{M}^{-1} \cdot \mathbf{d}^{(nm)}}{\sqrt{\mathbf{d}^{(ij)} \cdot \mathbf{M}^{-2} \cdot \mathbf{d}^{(ij)}}} \quad (S3)
\]

Recall that a contraction of a vectorial field with \( \mathbf{d}^{(nm)} \) corresponds to the difference of the vectorial field across the nodes \( m, n \), projected on the direction of the \((mn)\) bond.

4. We are then interested in the average dimensionless energy density associated with the bond extension/compression of Eq. (S3) at a distance \( r \) from the position where the dimensionless unit force dipole \( \mathbf{d}^{(ij)} \) is applied. This quantity, denoted by \( c_{ij}(r) \), is given (to quadratic order) by

\[
c_{ij}(r) = \langle A_{(ij), (nm)}^{2} \rangle_{r_{ij, nm}=r}, \quad (S4)
\]

which is an angular average of \( A_{(ij), (nm)}^{2} \) over all \((mn)\) bonds at a distance \( r \) from the \((ij)\) bond.

5. Finally, the correlation function \( c(r) \) reported in Fig. 1b-c in the manuscript is given by an average of the individual functions \( c_{ij}(r) \) over a large set of random edges \( i, j \).

In Fig. 1c in the main text, we presented a scaling collapse of the products \( r^6 c(r) \) measured in 3D relaxed Hookean spring networks. This is achieved by rescaling the abscissa by \( 1/\sqrt{z-z_c} \) (since the correlation length \( \xi \sim 1/\sqrt{z-z_c} \) and the ordinate by \( \sqrt{z-z_c} \). The latter rescaling is motivated as follows.

First, we write the sum-of-squares of the dipole displacement response function \( \mathbf{u} \) as (omitting the \((ij)\) superscript for the ease of notation)

\[
\mathbf{u} \cdot \mathbf{u} = \mathbf{d} \cdot \mathbf{M}^{-2} \cdot \mathbf{d} = \sum_{\ell} \frac{(\psi_{\ell} \cdot \mathbf{d})^2}{\omega_{\ell}^4}, \quad (S5)
\]

where \( \psi_{\ell} \) are the eigenfunctions of the Hessian \( \mathbf{M} \), \( \omega_{\ell}^2 \) are the eigenvalues associated with the eigenfunctions \( \psi_{\ell} \) and \( \mathbf{d} \) is a dimensionless unit force dipole. It is known that in relaxed Hookean spring networks with \( z \to z_c \), one has \( \psi_{\ell} \cdot \mathbf{d} \sim \omega_{\ell} \). The sum above can thus be approximated by an integral over the vibrational density of states as [2]

\[
\sum_{\ell} \frac{(\psi_{\ell} \cdot \mathbf{d})^2}{\omega_{\ell}^4} \sim \int_{\omega_*} \frac{\omega^2 D(\omega)}{\omega^4} \sim \frac{1}{\omega_*} \sim \frac{1}{z-z_c}, \quad (S6)
\]

where \( \omega_* \) is a characteristic frequency in the unjamming of relaxed Hookean spring networks, whose scaling \( \omega_* \sim z-z_c \) is well established [11].

Combining the results in Eqs. (S5)-(S6), i.e. \( \mathbf{u} \cdot \mathbf{u} \sim (z-z_c)^{-1} \), with the spatial scaling of \( \mathbf{u}(r) \) discussed in the manuscript, \( \mathbf{u}(r) \sim r^{-(d-2)/2} \), we obtain for the normalized response \( \tilde{\mathbf{u}}(r) \) the following prediction

\[
|\tilde{\mathbf{u}}(r)| \sim \sqrt{z-z_c} |\mathbf{u}(r)| \sim \frac{\sqrt{z-z_c}}{r^{(d-2)/2}}. \quad (S7)
\]

To proceed, we consider the energy \( e \) associated with a unit dimensionless dipole response \( \tilde{\mathbf{u}} \) in a relaxed spring network, summed over all springs apart from the perturbed spring. It was shown that [16]

\[
e \sim \int_{a_0}^\xi (\mathbf{u} \cdot \mathbf{d})^2 r^{d-1} dr \sim z-z_c, \quad (S8)
\]

where \( a_0 \) is the bond length. If we then invoke the hypothesis that \( \mathbf{u} \cdot \mathbf{d} \sim 1/r^{(d-2)/2} \), spelled out in the manuscript, and account for coordination-dependence of \( \mathbf{u} \cdot \mathbf{d} \) by setting \( (\mathbf{u} \cdot \mathbf{d})^2 \equiv f(z)/r^{d-2} \), we obtain

\[
e \sim f(z) \int_{1}^\xi r^{-(d-2)} r^{d-1} dr \sim f(z) \xi^2 \sim z-z_c. \quad (S9)
\]
The last relation, together with \( \xi \sim 1/\sqrt{z-z_c} \), implies that \( f(z) \sim (z-z_c)^2 \). Consequently, \( A_{ij, (nm)} \) defined in Eq. (S3), which scales as \( u \cdot d \), satisfies

\[
A_{ij, (nm)} \sim \frac{\sqrt{z-z_c} \sqrt{f(z)}}{r(d-2)/2} \sim \frac{(z-z_c)^3/2}{r(d-2)/2}.
\]  

(S10)

Finally, since \( c(r) \) scales as \( (A_{ij, (nm)})^2 \), we obtain

\[
c(r) \sim (z-z_c)^3 r^{-(d-2)},
\]

for \( r < \xi \), which was used in the manuscript in 3D (\( d=3 \)).

II. Computer disordered networks

We created disordered networks of 16 million nodes each, composed of relaxed Hookean springs connecting (unit) point masses, with both positional and topological (i.e. degree of connectivity) disorder. This is achieved by adopting the interaction networks of simple, three-dimensional (3D) soft-spheres glasses (see Ref. [17] for a description of the soft-spheres model), where we place a Hookean spring between every pair of interacting particles in the original glass.

This procedure results in a disordered spring network of initial coordination \( z \approx 16 \), which is much larger than the Maxwell threshold \( z_c = 6 \) in 3D. We then systematically remove bonds (springs) by considering in each iteration the bond \( i,j \) whose combined connectivity \( z_i+z_j \) is largest. Since there are many bonds that share the same combined connectivity \( z_i+z_j \), we consider a secondary bond-removal criterion: amongst all bonds \( i,j \) whose \( z_i+z_j \) is maximal, we select to remove a bond whose difference \( |z_i-z_j| \) is smallest. These two criteria ensure that the connectivity fluctuations of the resulting disordered spring network are small. This procedure is iteratively applied until a target connectivity \( z \) is reached. We present the bond dilution algorithm in great detail next.

S-I. Network dilution algorithm

We assume having in hand a soft-sphere-packing-derived, highly coordinated initial network of \( N \) nodes, and a set of edges \( \mathcal{E} \). An edge \( e^{ij} \in \mathcal{E} \) connects between a pair of neighboring nodes \( i,j \), with ranks \( z^i \) and \( z^j \) respectively. For every edge \( e^{ij} \) we define the sum of ranks \( s^{ij} = z^i + z^j \) and the absolute difference \( d^{ij} = |z^i - z^j| \).

The algorithm first pre-processes the initial network as follows: each edge in the network is represented by a link which is stored in a data-structure — illustrated in Fig. S1 — that is tailored for efficient access to edges with respect to the sum and absolute difference of their ranks during the dilution process. The data-structure consists of a two dimensional array-of-linked-lists \( \mathcal{A} \) of dimensions \( s_{\text{max}} \times d_{\text{max}} \), where \( s_{\text{max}} \) is the maximal sum of ranks, and \( d_{\text{max}} \) is the maximal absolute difference of ranks, amongst all of the network edges. Each list is a concatenation of links that represent edges: each link holds a pointer to the next link in a list, and the two indices \( i,j \) that define the edge \( e^{ij} \) represented by that link, as illustrated in Fig. S1. Links that share same sum of ranks \( s \) and same absolute difference \( d \) are concatenated into a single list which is anchored at \( \mathcal{A}(s,d) \). The pre-processing is described by the following pseudo-code:

**Algorithm 1 Pre-processing : \( O(N) \)**

Assign an \( s_{\text{max}} \times d_{\text{max}} \) array-of-lists \( \mathcal{A} \)

for each edge \( e^{ij} \in \mathcal{E} \) do

\[
s^{ij} \leftarrow z^i + z^j
\]

\[
d^{ij} \leftarrow |z^i - z^j|
\]

Add chain link \( L(i,j) \) at \( \mathcal{A}(s^{ij}, d^{ij}) \)

end for

We can now proceed with deriving a network with some desired average connectivity \( \bar{z}_f \) from the initial network using the production stage of the procedure. The main idea is to proceed as follows: we will start from the first non-empty list \( \mathcal{A}(s,d) \) with the highest index \( s \) and lowest index \( d \); while a list is found and is non-empty, we will remove the first link from that list, and check for its validity in terms of its attributes \( s^{ij} \) and \( d^{ij} \) (the two latter could have changed by previous edge removals). If valid, we remove the represented edge from the network, or, otherwise, we will re-insert the link into the appropriate list. In this way edges are consecutively removed from...
the network until reaching the target mean connectivity \( \bar{z}_f \), following the min-max scheme discussed above.

The number of edges which need to be removed from the initial network in order to reach \( \bar{z}_f < \bar{z} \) is 
\[ k \equiv \lfloor N(\bar{z} - \bar{z}_f) \rfloor, \]
where \( \bar{z} \) is the mean connectivity of the initial network. The production stage is described by the following pseudo-code:

**Algorithm 2 Production : \( \mathcal{O}(N) \)**

\[
\begin{align*}
s & \leftarrow s_{\text{max}} \\
d & \leftarrow 0 \\
counter & \leftarrow 0 \\
\text{while} \ counter < k & \text{ do} \\
\quad \text{while} \ A(s, d) \text{ is empty} & \text{ do} \\
\quad \quad \text{if} \ d < d_{\text{max}} & \text{ then} \\
\quad \quad \quad \text{Increase} \ d \\
\quad \quad \text{else} & \\
\quad \quad \quad \text{Decrease} \ s \\
\quad \quad \quad d & \leftarrow 0 \\
\quad \text{end if} \\
\quad \text{end while} \\
\quad \text{Read} \ L(i, j) \text{ at} \ A(s, d) \\
\quad \text{if} \ (z^i + z^j = s) \text{ AND } (|z^i - z^j| = d) & \text{ then} \\
\quad \quad \text{Remove} \ L(i, j) \text{ from} \ A(s, d) \\
\quad \quad \text{Update} \ z^i \text{ and} \ z^j, \text{ and remove} \ e^{ij} \text{ from the network} \\
\quad \quad \text{Increase} \ counter \\
\quad \text{else} & \\
\quad \text{Insert} \ L(i, j) \text{ to} \ A(z^i + z^j, |z^i - z^j|) \\
\quad \text{end if} \\
\text{end while}
\end{align*}
\]

To estimate the complexity of this algorithm, we consider the worst case scenario in which each link visits all of the \( s_{\text{max}} \times d_{\text{max}} \) lists before removal, then the running time would be \( \propto s_{\text{max}} \times d_{\text{max}} \times N \). However, since both parameters \( s_{\text{max}} \) and \( d_{\text{max}} \) are independent of \( N \), the complexity remains \( \mathcal{O}(N) \).

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