Preprojective cluster variables of acyclic cluster algebras

Bin Zhu

Department of Mathematical Sciences,
Tsinghua University, 100084 Beijing, P. R. China

Dedicated to Professor Yingbo Zhang on the occasion of her 60th birthday

Abstract. It is proved that any cluster-tilted algebra defined in the cluster category $\mathcal{C}(H)$ has the same representation type as the initial hereditary algebra $H$. For any valued quiver $(\Gamma, \Omega)$, an injection from the subset $\mathcal{PI}(\Omega)$ of the cluster category $\mathcal{C}(\Omega)$ consisting of indecomposable preprojective objects, preinjective objects and the first shifts of indecomposable projective modules to the set of cluster variables of the corresponding cluster algebra $A_\Omega$ is given. The images are called preprojective cluster variables. It is proved that all preprojective cluster variables other than $u_i$ have denominators $u_i \dim M$ in their irreducible fractions of integral polynomials, where $M$ is the corresponding preprojective module or preinjective module. In case the valued quiver $(\Gamma, \Omega)$ is of finite type, the denominator theorem holds with respect to any cluster. Namely, let $x = (x_1, \cdots, x_n)$ be a cluster of the cluster algebra $A_\Omega$, and $V$ the cluster tilting object in $\mathcal{C}(\Omega)$ corresponding to $x$, whose endomorphism algebra is denoted by $\Lambda$. Then the denominator of any cluster variable $y$ other than $x_i$ is $y \dim M$, where $M$ is the indecomposable $\Lambda$-module corresponding to $y$. This result is a generalization of the corresponding result in [CCS2] to non-simply-laced case.

Key words. Cluster-tilted algebra, representation type, preprojective cluster variable, denominator of cluster variable.

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1. Introduction

Clusters and cluster algebras are defined and studied by Fomin and Zelevinsky [FZ1-2] [BFZ] in order to provide an algebraic framework for

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*Supported in part by the NSF of China (Grants 10471071).
†E-mail: bzhu@math.tsinghua.edu.cn
total positivity and canonical bases in semisimple algebraic groups. Recently there are many works to link representation theory of quivers with cluster algebras, see amongst others [MRZ], [BMRRRT], [BMR1-2], [BMRT], [CC], [CCS1-2], [K], [CK1-2], [Z1-3]. In [BMRRRT], also [CCS1], the authors have defined cluster categories and related cluster tilting theory with clusters and seeds. Now cluster categories have become a successful model to understand (acyclic) cluster algebras. For example, Caldero and Keller [CK1] [CK2] realize non simply-laced cluster algebras by using cluster categories of quivers via Hall algebra approach. From this realization, they proved that, see also [BMRT], Caldero-Chapoton’s formula [CC] gives a one-to-one correspondence from the set of exceptional objects in the cluster category to the set of cluster variables of the corresponding cluster algebra, this bijection maps the shift of indecomposable projective modules to the initial cluster variables of the initial seed, and sends cluster tilting objects to clusters. Cluster-tilted algebras are by definition, the endomorphism algebras of cluster tilting objects in the cluster categories. It provides a class of finite dimensional algebras which is close to the class of (quasi-)tilted algebras, but they are different. Their module categories have close connections with cluster categories [BMR1] [Z2] [ABS].

The main aims of the paper is the following: The first one is to determine the representation type of cluster-tilted algebras. It is proved that any two cluster-tilted algebras defined in the cluster category $C(H)$ have same representation type, in particular, they all have the same representation type as $H$. This result is a consequence of Krause [Kr] since any two cluster-tilted algebras defined in the same cluster category share a same factor category. The second aim is to study the denominators of cluster variables by applying the BGP-reflection functors defined in [Z1] and their corresponding isomorphisms of cluster algebras in [Z3]. Analogously with preprojective or preinjective modules of algebras, we introduce the notion of preprojective cluster variables. It is proved that the exponents of the denominators of all preprojective cluster variables which correspond preprojective modules or preinjective modules $M$ are $\dim M$. If the cluster algebras is of finite type, this denominator theorem for cluster variables holds with respect to any fixed cluster.

This paper is organized as follows: In Section 2, some basic notions which will be needed later on are recalled. In Section 3, it is proved that any two cluster-tilted algebras which are defined in the same cluster category share a same factor category, hence they have same representation type. In Section 4, the notation of preprojective cluster variables is introduced for
any acyclic cluster algebra. The preprojective cluster variables of a cluster
algebra correspond bijectively to the indecomposable preprojective objects
of the corresponding cluster category, i.e. the indecomposable preprojective
module, the indecomposable preinjective objects, or the first shift of inde-
composable projective modules in this cluster category. It is proved that
the denominator theorem for all preprojective cluster variables holds with
respect to an acyclic seed. For any cluster algebra of finite type, there is
a bijection between the set of indecomposable objects in the corresponding
cluster category and the set of cluster variables of this cluster algebras, see
Section 4 or [Z1, Z3], this bijection induces a bijection between the set of
cluster tilting objects to the set of clusters. Fix a cluster \( \mathbf{x} = (x_1, x_2, \cdots, x_n) \),
denote by \( V \) the (basic) cluster tilting object corresponding to \( \mathbf{x} \) and by \( B \)
the corresponding cluster-tilted algebra. It is proved that any cluster vari-
ables \( v \) other than \( x_i \) has denominator \( \mathbf{dim}^M \) with respect to the cluster
\( \mathbf{x} \), where \( M \) is the indecomposable \( B \)-module corresponding to \( v \) (compare
[FZ2] [CCS1-2]).

2. Basics on cluster categories and cluster algebras.

Let \( \mathcal{H} \) be a hereditary abelian category defined over a field \( K \) with finite
dimensional Homomorphism and extension spaces, and with tilting objects.
Assume that its Grothendieck group is isomorphic to \( \mathbb{Z}^n \) [HRS]. The endo-
morphism algebra of a tilting object in \( \mathcal{H} \) is called a quasi-tilted algebra.
Denote by \( \mathcal{D} = D^b(\mathcal{H}) \) the bounded derived category of \( \mathcal{H} \) with shift functor
\([1]\). \( \mathcal{D} \) has Auslander-Reiten triangles (AR-triangles for short), \( \tau \) is the AR-
translation. For any category \( \mathcal{T} \), we will denote by \( \text{ind}\mathcal{T} \) the subcategory
of isomorphism classes of indecomposable objects in \( \mathcal{T} \); depending on the
context we shall also use the same notation to denote the set of isomorphism
classes of indecomposable objects in \( \mathcal{T} \).

The orbit category \( \mathcal{D}/\tau^{-1}[1] \) is called the cluster category of type \( \mathcal{H} \),
which is denoted by \( \mathcal{C}(\mathcal{H}) \) ([BMRRT]), see also [CCS1]. It is a triangu-
lated category with shift functor \([1]\) [Ke] and has Auslander-Reiten triangles,
the AR-translation \( \tau \) is induced from AR-translaion of \( \mathcal{D} \) [BMRRT].
When \( \mathcal{H} \) is the module category of a hereditary algebra \( H \), or more gen-
eral, of a (quasi-)tilted algebra \( A \), or \( \mathcal{H} \) is the category of representations
of a valued quiver \( (\Gamma, \Omega) \), the corresponding cluster category is denoted by
\( \mathcal{C}(H) \), \( \mathcal{C}(A) \) or \( \mathcal{C}(\Omega) \) respectively. We use \( H \) to denote the tensor algebra
of the species \( \mathcal{M} \) of the valued quiver \( (\Gamma, \Omega) \), hence \( \text{mod}H \) is equivalent to
the category of representations of the species \( \mathcal{M} \) of \( (\Gamma, \Omega) \). We denote by
\( E_1, \cdots, E_n \) complete list of simple objects in \( \mathcal{H} \); by \( P_1, \cdots, P_n \) (or \( I_1, \cdots, I_n \))
the complete list of indecomposable projective (injective respectively) representations in $\mathcal{H}$. Denote $\text{Hom}_{\mathcal{C}(\mathcal{H})}(X, Y)$ simply by $\text{Hom}(X, Y)$, and define $\text{Ext}^1(X, Y) = \text{Hom}(X, Y[1])$.

An object $X$ in $\mathcal{C}(\mathcal{H})$ is called exceptional if $\text{Ext}^1(X, X) = 0$. The set of isomorphism classes of indecomposable exceptional objects in $\mathcal{C}(\mathcal{H})$ is denote by $\mathcal{E}(\mathcal{H})$. If $\mathcal{H}$ is the category of representations of a valued quiver $(\Gamma, \Omega)$, this set is denoted by $\mathcal{E}(\Omega)$ sometimes. An object $V$ in $\mathcal{C}(\mathcal{H})$ is called a cluster tilting object if it is exceptional and has $n$ non-isomorphic indecomposable direct summands. An exceptional object $M$ in $\mathcal{C}(\mathcal{H})$ with $n - 1$ non-isomorphic direct summands is called almost complete. For a cluster tilting object $V$ in $\mathcal{C}(\mathcal{H})$, the endomorphism ring $\text{End}V$ is called the cluster-tilted algebra of $V$ [BMRRT] [BMR1].

For a valued graph $\Gamma$, we denote by $\Phi = \Phi^+ \cup \Phi^-$ the set of roots of the corresponding Kac-Moody Lie algebra. Let $\Phi_{\geq -1} = \Phi^+ \cup \{-\alpha_i \mid i = 1, \cdots, n\}$ denote the set of almost positive roots, i.e. the positive roots together with the negatives of the simple roots. Let $s_i$ be the Coxeter generators of the Weyl group of $\Phi$ corresponding to $i \in \Gamma$. We recall from [FZ2] that the "truncated reflections" $\sigma_i$ of $\Phi_{\geq -1}$ are defined as follows:

$$\sigma_i(\alpha) = \begin{cases} \alpha & \alpha = -\alpha_j, \ j \neq i \\ s_i(\alpha) & \text{otherwise.} \end{cases}$$

When $\Gamma$ is a Dynkin graph, these truncated reflections were shown in [FZ2] to be one of the main ingredients of constructions (see also [MRZ]). For Dynkin graph $\Gamma$, the set of vertices can be divided into two completely disconnected subsets as $\Gamma = \Gamma^+ \sqcup \Gamma^-$, and one can define:

$$\sigma_\pm = \prod_{i \in \Gamma^\pm} \sigma_i.$$  

Then there is a so-called "compatibility degree" ($||$) defined on pairs of almost positive roots of $\Gamma$. It is uniquely defined by the following two properties:

$$(-\alpha_i||\beta) = \max(n_i(\beta), 0),$$

$$\langle \sigma_\pm \alpha||\alpha_\pm \beta \rangle = (\alpha||\beta),$$

for any $\alpha, \beta \in \Phi_{\geq -1}$, any $i \in \Gamma$, where $\beta = \sum_i n_i(\beta)\alpha_i$ [FZ2].
Now we define a map for any valued quiver \((\Gamma, \Omega)\) from \(\text{ind} C(\Omega)\) to \(\Phi_{\geq -1}\) as follows: for any \(X \in \text{ind}(\text{mod} H \setminus H[1])\),

\[
\gamma_{\Omega}(X) = \begin{cases} 
\dim X & \text{if } X \in \text{ind} H; \\
-\dim E_i & \text{if } X = P_i[1],
\end{cases}
\]

where \(\dim X\) denotes the dimension vector of the representation \(X\). In general, this map \(\gamma_{\Omega} : \text{ind} C(\Omega) \to \Phi_{\geq -1}\) is surjective, but not injective. Denote by \(\Phi_{sr_{-1}}\) the set of real Schur roots, i.e. \(\dim X\), where \(X\) is exceptional \(H\)-modules, together with negatives of simple roots. Hence \(\gamma_{\Omega}\) induces a bijection from \(E(\Omega)\) to \(\Phi_{sr_{-1}}\), which is still denoted by \(\gamma_{\Omega}\).

We recall some basic notation on cluster algebras which can be found in the papers by Fomin and Zelevinsky [FZ1-2]. The cluster algebras we deal with in this paper are defined on a trivial semigroup of coefficients.

Let \(F = \mathbb{Q}(u_1, u_2, \ldots, u_n)\) be the field of rational functions in indeterminates \(u_1, u_2, \ldots, u_n\). Set \(\underline{u} = (u_1, u_2, \ldots, u_n)\). Let \(B = (b_{ij})\) be an \(n \times n\) skew-symmetrizable integer matrix. A pair \((\underline{x}, B)\), where \(\underline{x} = (x_1, x_2, \ldots, x_n)\) is a transcendence base of \(F\) and where \(B\) is an \(n \times n\) skew-symmetrizable integer matrix, is called a seed. Fix a seed \((\underline{x}, B)\), \(z\) in the base \(\underline{x}\). Let \(z'\) in \(F\) be such that

\[
z z' = \prod_{b_{zz} > 0} x^{b_{zz}} + \prod_{b_{zz} < 0} x^{-b_{zz}}.
\]

Now, set \(\underline{z} := \underline{x} - \{z\} \cup \{z'\}\) and \(B' = (b'_{xy})\) such that

\[
b'_{xy} = \begin{cases} 
-b_{xy} & \text{if } x = z \text{ or } y = z, \\
b_{xy} + 1/2(|b_{xz}|b_{zy} + |b_{xz}|b_{zy}|) & \text{otherwise}.
\end{cases}
\]

The pair \((\underline{z}', B')\) is called the mutation of the seed \((\underline{x}, B)\) in direction \(z\), it is also a seed. The "mutation equivalence \(\approx\)" is an equivalence relation on the set of all seeds generated by the mutation.

The cluster algebra \(A_B\) associated to the skew-symmetrizable matrix \(B\) is by definition the subalgebra of \(F\) generated by all \(x_i\) in \(\underline{x}\) such that \((\underline{x}, B') \approx (\underline{u}, B)\). Such \(\underline{x} = (x_1, x_2, \ldots, x_n)\) is called a cluster of the cluster algebra \(A_B\) or simply of \(B\), and any \(x_i\) is called a cluster variable. The set of all cluster variables is denoted by \(\chi_B\). If the set \(\chi_B\) is finite, then the cluster algebra \(A_B\) is said to be of finite type. Cluster algebras of finite type can be characterized by Dynkin diagrams [FZ2].

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3. Cluster-tilted algebras.

Since any cluster tilting object $V$ in the cluster category $\mathcal{C}(H)$ is induced by a tilting object in a hereditary abelian category $\mathcal{H}'$, derived equivalent to $\mathcal{H}$, we may assume that, without loss the generality (compare [BM RRT] [Z2]), $V$ is a tilting object in $\mathcal{H}$, and then it is a cluster tilting object in $\mathcal{C}(H)$. We have the quasi-tilted algebras $A = \operatorname{End}_H V$ and the cluster-tilted algebra $\Lambda = \operatorname{End} V$.

The Hom functor $G = \operatorname{Hom}(V, -)$ induces a dense and full functor from the cluster category $\mathcal{C}(H)$ to $\Lambda - \operatorname{mod}$. It induces an equivalence $\mathcal{C}(H)/\operatorname{add}(\tau V)$ to $\Lambda - \operatorname{mod}$. This is proved in [BMR1] for the case where $H$ is module category over a hereditary algebra, and generalized to any hereditary category in [Z2]. Following [CB], [Kr], a ring $A$ is called generically wild if there is a generic $A$–module $M$ such that $\operatorname{End}_A(M)$ is not a PI-ring.

**Lemma 3.1 [BMR1][Z2].** Let $V$ be a cluster tilting object in $\mathcal{C}(H)$ and $\Lambda = \operatorname{End} V$ the cluster-tilted algebra. Then $G: \mathcal{C}(H)/\operatorname{add}(\tau V) \rightarrow \Lambda - \operatorname{mod}$ is an equivalence.

From this result, one can compare any two cluster-tilted algebras as following:

**Proposition 3.2.** Let $V$ and $V'$ be cluster tilting objects in $\mathcal{C}(H)$, $\Lambda = \operatorname{End} V$ and $\Lambda' = \operatorname{End} V'$ the corresponding cluster-tilted algebras. Then $\Lambda - \operatorname{mod}/\operatorname{add}(\operatorname{Hom}(V, \tau V')) \approx \Lambda' - \operatorname{mod}/\operatorname{add}(\operatorname{Hom}(V', \tau V))$, and the equivalence induces an isomorphism of the AR-quivers between $\Lambda - \operatorname{mod}/\operatorname{add}(\operatorname{Hom}(V, \tau V'))$ and $\Lambda' - \operatorname{mod}/\operatorname{add}(\operatorname{Hom}(V', \tau V))$.

Furthermore, $\Lambda$ is generically wild if and only if $\Lambda'$ is generically wild.

**Proof.** Denote by $G = \operatorname{Hom}(V, -)$. The induced functor $\bar{G}: \mathcal{C}(H)/\operatorname{add}(\tau V) \rightarrow \Lambda - \operatorname{mod}$ is an equivalence by Lemma 3.1. We consider the composition of functor $\bar{G}$ with the quotient functor $Q: \Lambda - \operatorname{mod} \rightarrow \Lambda - \operatorname{mod}/\operatorname{add}(\operatorname{Hom}(V, \tau V'))$, which is denoted by $G_1$. The functor $G_1$ is full and dense since $G$ and $Q$ are. Under the equivalence $\bar{G}$, $\tau V'$ corresponds to $\operatorname{Hom}(V, \tau V')$. For any morphism $f: X \rightarrow Y$ in the category $\mathcal{C}(H)/\operatorname{add}(\tau V')$, $\bar{G}(f): G(X) \rightarrow G(Y)$ factors through $\operatorname{add}(\operatorname{Hom}(V, \tau V'))$ if and only if $f$ factors through $\operatorname{add}(\tau V')$. Then $G_1$ induces an equivalence, denoted by $\bar{G}_1$, from the category $\mathcal{C}(H)/\operatorname{add}(\tau V')$ to the category $\Lambda - \operatorname{mod}/\operatorname{add}(\operatorname{Hom}(V, \tau V'))$. It can be proved in a similar way as Proposition 3.2. in [BMR1] that $\bar{G}$ preserves Auslander-Reiten sequences and
then $G_1$, induces an isomorphism of the AR-quivers between $\frac{C(H)}{\text{add}(\tau(V \oplus V')))}$ and $\frac{\Lambda\text{-mod}}{\text{add}(\text{Hom}(V, V'))}$. Similarly, we have a functor $G'_1$ from $\frac{C(H)}{\text{add}(\tau(V'))}$ to $\frac{\Lambda'\text{-mod}}{\text{add}(\text{Hom}(V', V)))}$. It induces an equivalence from the category $\frac{C(H)}{\text{add}(\tau(V \oplus V')))}$ to the category $\frac{\Lambda'\text{-mod}}{\text{add}(\text{Hom}(V', V)))}$ and this equivalence induces an isomorphism of the AR-quivers between $\frac{C(H)}{\text{add}(\tau(V \oplus V')))}$ and $\frac{\Lambda'\text{-mod}}{\text{add}(\text{Hom}(V', V)))}$.

Since $\Lambda$ and $\Lambda'$ are artin algebras, it follows from Corollary 3.4 in [Kr] that $\Lambda$ is generically wild if and only if $\Lambda'$ is generically wild. The proof is finished.

Corollary 3.3. Let $H$ be a finite dimensional hereditary algebra over an algebraically closed field, $V$ a cluster tilting object in the cluster category $C(H)$ and $\Lambda = \text{End}V$ the corresponding cluster-tilted algebra. Then

(1). $\Lambda$ is of representation finite type if and only if $H$ is of representation finite type; in this case, the numbers of the isomorphism classes of indecomposable modules over $H$ and over $\Lambda$ respectively are the same.

(2). $\Lambda$ is tame (or wild) if and only if $H$ is tame (or wild).

Proof. Part (1) is proved in Corollary 2.4 in [BMR1]. We prove part (2). Note that $H$ is a cluster-tilted algebra $\text{End}H$, then by Proposition 3.2., $\Lambda$ is generically wild if and only if $H$ is generically wild. From Theorem 4.1 in [CB], for a finite dimensional algebra over an algebraically closed field, it is wild if and only if it is generically wild. It implies that $\Lambda$ is wild if and only if $H$ is wild; and $\Lambda$ is tame if and only if $H$ is tame. The proof is finished.

Now we assume that $\bar{V}$ is an almost complete cluster tilting object in $\mathcal{C}(H)$ and $V_k$, $V^*_k$ are the two complements of $\bar{V}$, i.e. $V = \bar{V} \oplus V_k$ and $V' = \bar{V} \oplus V^*_k$ form a tilting mutation at $k$ [BMR2]. Denote the simple top of projective $\Lambda$–module $\text{Hom}(V, V_k)$ by $L_k$ and the simple top of projective $\Lambda'$–module $\text{Hom}(V', V^*_k)$ by $L'_k$. As a consequence of Proposition 3.2., we have the following (compare Theorem 4.2. in [BMR1]):

Corollary 3.4. Let $V$ and $V'$ be tilting objects in $\mathcal{C}(H)$. Suppose $V'$ is obtained from $V$ by a mutation at $k$, and $\Lambda$, $\Lambda'$ are the corresponding cluster-tilted algebras. Then $\frac{\Lambda\text{-mod}}{\text{add}L_k} \approx \frac{\Lambda'\text{-mod}}{\text{add}L'_k}$.

Proof. From Proposition 3.2, we have that $\frac{\Lambda\text{-mod}}{\text{add}(\text{Hom}(V, V^*_k))} \approx \frac{\Lambda'\text{-mod}}{\text{add}(\text{Hom}(V', V^*_k))}$. It is easy to see $\text{Hom}(V, V^*_k) \cong L_k$ and $\text{Hom}(V', V^*_k) \cong L'_k$ (compare
Lemma 4.1 in [BMR1]), which implies the equivalence what we want. The proof is finished.

In the rest of this section, we assume the hereditary category \( \mathcal{H} \) is the category of finite dimensional left module of a finite dimensional algebra \( H \) over a filed \( K \). Assume that \( V \) is a tilting \( H \)-module. Then \( \text{Hom}_H(V, -) \) induces a derived equivalence from \( D^b(H) \) to \( D^b(A) \), where \( A \) is the tilted algebra \( \text{End}_H V \). It induces a triangle equivalence from the cluster category \( \mathcal{C}(H) \) to the cluster category \( \mathcal{C}(A) = D^b(A)/\tau^{-1}[1] \). This equivalence is denoted by \( R_V [Z_1, Z_2] \).

**Proposition 3.5.** Let \( H, V, A \) be as above and \( \Lambda = \text{End} V \) the cluster-tilted algebra. Then we have the commutative diagram:

\[
\begin{array}{ccc}
\mathcal{C}(H) & \xrightarrow{R_V} & \mathcal{C}(A) \\
\downarrow & & \downarrow \\
H-\text{mod} & \cong & \Lambda-\text{mod} \\
\frac{\text{add}(\tau V)}{} & & \frac{\text{add}(\text{Hom}(V, \tau H))}{}
\end{array}
\]

where the left vertical map is \( Q_H \circ \text{Hom}(H, -) \), the right vertical map is \( Q_\Lambda \circ \text{Hom}(A, -) \), and \( Q_H : H-\text{mod} \to \frac{H-\text{mod}}{\text{add}(\tau V)}, Q_\Lambda : \Lambda-\text{mod} \to \frac{\Lambda-\text{mod}}{\text{add}(\text{Hom}(V, \tau H))} \) are the quotient maps.

**Proof.** \( R_V(V) = A \) is a cluster tilting object in \( \mathcal{C}(A) \), and \( \text{Hom}(V, -) = \text{Hom}(A, -) \circ R_V \). Then the commutativity of the diagram above follows from Proposition 3.2.

For any tilting \( H \)-module \( V \), from Lemma 3.1., we have that \( \text{ind}\mathcal{C}(H) = \text{ind}\Lambda \lor \{ V_i[1] \mid i = 1, \cdots n \} \). We define a map from \( \text{ind}\mathcal{C}(H) \) to \( \mathbb{Z}^n \) by generalizing the map \( \gamma_H \).

**Definition 3.6.** Let \( \gamma_V : \text{ind}\mathcal{C}(H) \to \mathbb{Z}^n \) be defined as follows: for any \( X \in \text{ind}\Lambda \lor \{ V_i[1] \mid i = 1, \cdots n \} \),

\[
\gamma_V(X) = \begin{cases} 
\dim X & \text{if } X \in \text{ind}\Lambda; \\
-\dim E_i & \text{if } X = V_i[1], 
\end{cases}
\]

\( \gamma_V \) is called the dimension vector map of the cluster category \( \mathcal{C}(H) \) associated to the tilting object \( V \). If \( X \in \text{ind}\Lambda \), then we say that \( \gamma_V(X) > 0 \). Denote by \( \Phi_V = \{ \gamma_V(X) \mid X \in \text{ind}\mathcal{C}(H) \} \) and \( \Phi_V^\geq -1 = \{ \gamma_V(X) \mid X \in \mathcal{E}(H) \} \).

In general, this map \( \gamma_V : \text{ind}\mathcal{C}(H) \to \Phi_V \) is not injective. When \( V = H \), \( \gamma_V = \gamma_H \) which maps from \( \text{ind}\mathcal{C}(H) \) to \( \Phi_{\geq -1} \).
Lemma 3.7. Suppose $V = \bigoplus_{i=1}^{n} V_i$ is a cluster tilting object, $M \in \text{ind}C(H)$ and $\tau M \rightarrow \oplus_j N_j \rightarrow M \rightarrow \tau M[1]$ is the AR-triangle ending at $M$, where $N_j$ is indecomposable in $C(H)$. Then

1. if $M \cong V_i$ for some $i$, we have $\gamma_V(M) = \dim E_i + \sum_{\gamma_V(N_j) > 0} \gamma_V(N_j)$;
2. if $M \cong V_i[1]$ for some $i$, we have $\gamma_V(\tau M) = \dim E_i + \sum_{\gamma_V(N_j) > 0} \gamma_V(N_j)$;
3. if $M \not\cong V_i$ and $M \not\cong V_i[1]$ for any $i$, we have $\gamma_V(M) + \gamma_V(\tau M) = \sum_{\gamma_V(N_j) > 0} \gamma_V(N_j)$.

Proof. By [BMR1], the sink map and the source map for any indecomposable $\Lambda$-module $M$ in $\Lambda$-mod are induced from the corresponding maps in the cluster category $C(H)$. Then the AR-triangle $\tau M \rightarrow \oplus N_j \rightarrow M \rightarrow \tau M[1]$ in $C(H)$ induces

1. the AR-sequence: $0 \rightarrow \text{Hom}(V, \tau M) \rightarrow \oplus_{\text{Hom}(V,N_j) \neq 0} \text{Hom}(V, N_j) \rightarrow \text{Hom}(V, M) \rightarrow 0$ provided $\text{Hom}(V, M) \neq 0$ and is not any projective $\Lambda$-module; or
2. the source map: $\text{Hom}(V, \tau M) \rightarrow \oplus_{\text{Hom}(V,N_i) \neq 0} \text{Hom}(V, N_i)$ provided $\text{Hom}(V, \tau M)$ is an injective $\Lambda$-module or
3. a sink map: $\oplus_{\text{Hom}(V,N_j) \neq 0} \text{Hom}(V, N_j) \rightarrow \text{Hom}(V, M)$ provided $\text{Hom}(V, M)$ is a projective $\Lambda$-module.

Then by the definition of $\gamma_V$, we have that (3). $\gamma_V(M) + \gamma_V(\tau M) = \sum_{\gamma_V(N_j) > 0} \gamma_V(N_j)$ if $M \not\cong V_i$ and $M \not\cong V_i[1]$ for any $i$; (2). $\gamma_V(\tau M) = \dim E_i + \sum_{\gamma_V(N_j) > 0} \gamma_V(N_j)$ if $M \cong V_i[1]$ for some $i$; and (3). $\gamma_V(M) = \dim E_i + \sum_{\gamma_V(N_j) > 0} \gamma_V(N_j)$ if $M \cong V_i$ for some $i$. The proof is finished.

Definition 3.8. Let $V$ be a cluster tilting object in $C(\Omega)$. We Define $\sigma_V : \Phi_{\geq -1}^\omega \rightarrow \Phi_{\geq -1}^\omega$ as the map $\gamma_\Omega(X) \mapsto \gamma_V(X)$ for any $X \in E(H)$.

Since the set of indecomposable exceptional objects in $C(H)$ consists of indecomposable exceptional $H$-modules and the shifts by 1 of indecomposable projective $H$-modules, they are determined by their dimensional vectors. Then the map $\sigma_V$ is well-defined. When $V$ is a BGP-tilting or APR-tilting module (at a sink or a source $k$), $\sigma_V = \sigma_k$.

Now we give an interpretation of Proposition 3.5. in terms of dimensional vector maps.

Proposition 3.9. Let $V$ be a tilting representation of $(\Gamma, \Omega)$, whose endomorphism algebra is denoted by $A$. Then we have the commutative
Proof. Since $R_V$ is a triangle equivalence from $\mathcal{C}(\Omega)$ to the cluster category $\mathcal{C}(A)$ of $A$, it maps exceptional objects to exceptional objects. Hence $R_V$ induces bijection from $\mathcal{E}(\Omega)$ to $\mathcal{E}(A)$. A = $R_V(V)$ is a cluster tilting object and $\Lambda = \text{End}V \cong \text{End}A$. By definition of dimensional vectors map, $\gamma_A = \gamma_V$. Then for any indecomposable exceptional object $X \in \mathcal{C}(\Omega)$, $\gamma_V R_V(X) = \gamma_V(X) = \sigma_V(\gamma_\Omega(X))$. the proof is finished.

4. Preprojective cluster variables

In this section, we assume that $\mathcal{H}$ is the category of finite dimensional representations over a field $K$ of a species $\mathcal{M}$ of the valued quiver $(\Gamma, \Omega)$. We always assume that $\Gamma$ contains no vertex loops and the quiver $(\Gamma, \Omega)$ contains no oriented cycles. Let $H$ denote the tensor algebra of the species $\mathcal{M}$. It is a hereditary finite dimensional $K-$algebra.

For any vertex $k \in \Gamma$, $s_k \Omega$ denotes the new orientation of $\Gamma$ by reversing the direction of arrows along all edges containing $k$ in $(\Gamma, \Omega)$. A vertex $k \in \Gamma$ is called a sink (or a source) with respect to $\Omega$ if there are no arrows starting (or ending) at vertex $k$.

We recall that $P_i$ (or $I_i$) is the projective (injective resp.) indecomposable $H-$module corresponding to the vertex $i \in \Gamma$, and $E_i$ is the corresponding simple module. If $k$ is a sink (or a source), then $P_k = E_k$ (resp. $I_k = E_k$) is a simple projective (resp. injective) $H-$module.

Let $V = \bigoplus_{i\in \Gamma-\{k\}} P_i \oplus \tau^{-1} P_k$. If $k$ is a sink, then $V$ is a tilting $H-$module which is called a BGP- or an APR-tilting module, $\text{Hom}_H(V, -)$ is the tilting functor induced by $V$, which is called a BGP-reflection functor and is denoted by $S^+_k$. The following theorem was proved in [Z1] [Z2] (in a more general case).

Theorem 4.1.[Z1, Z2] For any sink (or a source) $k$ of a valued quiver $(\Gamma, \Omega)$, the BGP-reflection functor $S^+_k$ (resp. $S^-_k$) induces a triangle equivalence $R(S^+_k)$ (resp.,$R(S^-_k)$) from $\mathcal{C}(\Omega)$ to $\mathcal{C}(s_k \Omega)$ and $\gamma_{s_k \Omega}(R(S^+_k)(X)) = \sigma_k(\gamma_\Omega(X))$. Moreover $R(S^+_k)$ induces a bijection from $\mathcal{E}(\Omega)$ to $\mathcal{E}(s_k \Omega)$. 


Definition 4.2. The triangle equivalence \( R(S_k^+) \) from \( \mathcal{C}(\Omega) \) to \( \mathcal{C}(s_k\Omega) \) induced from the reflection functor \( S_k^+ \) is called the BGP-reflection functor in \( \mathcal{C}(\Omega) \) at the sink \( k \), which is denoted simply by \( R_k^+ \). Dually for a source \( k \), we have the reflection functor \( R_k^- \) from \( \mathcal{C}(\Omega) \) to \( \mathcal{C}(s_k\Omega) \).

Let \( k_1, \ldots, k_n \) be an admissible sequence of sinks of \((\Gamma, \Omega)\). For simplicity, we assume that \( 1, 2, \ldots, n \) is such an admissible sequence of sinks of \((\Gamma, \Omega)\). Set \( C^+ = R_{n}^+ \cdots R_2^+ R_1^+ \), the composition of \( R_i^+ \). \( C^+ \) is a self-equivalence of \( \mathcal{C}(\Omega) \), it is called the Coxeter functor in the cluster category \( \mathcal{C}(\Omega) \) in \([Z3]\). For simplicity, we denote \( C^+ \) by \( C \). The inverse \( C^- \) of \( C \), which is also called the Coxeter functor in \( \mathcal{C}(\Omega) \) is \( C^- = R_n^- R_{n-1}^- \cdots R_1^- \). For any indecomposable object \( X \) in \( \mathcal{C}(\Omega) \), \( C(X) \simeq \tau X \), where \( \tau \) is the Auslander-Reiten translation in \( \mathcal{C}(\Omega) \). Let \( \mathcal{P}(\Omega) \) denote the set of isomorphism classes of indecomposable preprojective (preinjective resp.) \( H \)-modules, \( \mathcal{R} \) the set of isomorphism classes of indecomposable regular \( H \)-modules. We denote the union \( \mathcal{P} \vee \{ P_i[1] \mid i = 1, \ldots, n \} \vee \mathcal{I} \) by \( \mathcal{PI}(\Omega) \). Note that \( \mathcal{PI}(\Omega) = \{ C^m P_k[1] \mid m \in \mathbb{Z}, 1 \leq k \leq n \} \).

The objects in \( \mathcal{PI}(\Omega) \) are called preprojective objects in \( \mathcal{C}(\Omega) \).

If \( \Gamma \) is a Dynkin diagram, then \( \text{ind}\mathcal{C}(\Omega) = \mathcal{PI}(\Omega) \), otherwise \( \text{ind}\mathcal{C}(\Omega) = \mathcal{PI}(\Omega) \vee \mathcal{R} \).

Definition 4.3. Let \( A = (a_{ij}) \) be a generalized Cartan matrix corresponding to the diagram \( \Gamma \). For any index \( i \), we define an automorphism \( T_i \) of \( \mathcal{F} = \mathbb{Q}(u_1, \ldots, u_n) \) by defining the images of the indeterminates \( u_1, \ldots, u_n \) as follows:

\[
T_i(u_j) = \begin{cases} 
  u_j & \text{if } j \neq i, \\
  \prod_{a_{ik} < 0} u_k^{-a_{ik}+1} u_i & \text{if } j = i.
\end{cases}
\]

It is easy to check that all \( T_i \) are involutions of \( \mathcal{F} \), i.e. \( T_i^2 = \text{id}_\mathcal{F} \).

Let \( B \) be a skew-symmetrizable matrix corresponding to the valued quiver \((\Gamma, \Omega)\). Then the Cartan counterpart of \( B \) is \( A \) \([FZ2]\).

From the definition, all \( T_i \) are independent of orientations of the valued quiver \((\Gamma, \Omega)\) corresponding to \( B \), and depends only on the valued graph \( \Gamma \). Let \( k \) be a sink in \((\Gamma, \Omega)\). By Theorem 4.7 in \([Z3]\), \( T_k \) sends cluster variables and clusters in \( \chi_\Omega \) to those in \( \chi_{s_k\Omega} \) respectively and \( T_k \) induces an
isomorphism from the cluster algebra $A_\Omega$ to $A_{s_k\Omega}$ ($T_k$ induces a so-called strongly isomorphism from $A_\Omega$ to $A_{s_k\Omega}$).

Under the assumption that the sequence $1, 2, \cdots, n$ is an admissible sequence of sinks of $(\Gamma, \Omega)$, we define an automorphism $T_\Omega$ of $\mathcal{F}$ as $T_\Omega = T_n \cdots T_2 T_1$. $T_\Omega$ induces an automorphism of cluster algebra $A_\Omega$. $T_\Omega$ and its inverse $T_{\Omega^{-1}}$ are called the Coxeter automorphisms of the cluster algebra $A_\Omega$. $T_\Omega$ is simply denoted by $T$.

Set:

$$\chi_\Omega = \{ T^m(u_k) \mid m \in \mathbb{Z}, 1 \leq k \leq n \}.$$ 

When $\Gamma$ is of finite type, we have that $\chi'_\Omega = \chi_\Omega$ is the set of cluster variables of $(\Gamma, \Omega)$ and $C^m(P_i[1]) \mapsto T^m(u_i)$ is a bijection from $C(\Omega)$ to $\chi_\Omega$ which sends $P_i[1]$ to $u_i$ and sends tilting objects to clusters by Corollary 3.4. and Theorem 4.7. in [Z3]. For any valued quiver, it is proved in Theorem 4.13 of [Z3] that all elements $T^m(u_i)$ in $\chi'_\Omega$ are cluster variables of cluster algebra $A_\Omega$. The main aim in this section is to prove that the denominators of these clusters $T^m(u_i)$ as the reduced fractions of integral polynomials are $u_i^{\gamma_i(C^m(P_i[1]))}$. As an immediately consequence, the map $\phi_\Omega : C^k(P) \mapsto T^k(u_i)$, for any $k \in \mathbb{Z}$, $i = 1, \cdots, n$, is a bijection from $\mathcal{P}\mathcal{I}(\Omega)$ to $\chi'_\Omega$. Clusters and cluster variables in $\chi'_\Omega$ will be called preprojective clusters and preprojective cluster variables respectively.

Before we state the theorem, we recall a useful notation from [BMRT]. From [FZ1], any cluster variable $x$ ($\neq u_j \forall j$) is a Laurent polynomial, i.e. it can be written as a reduced fraction $f/m$ where $f$ is an integer polynomial in $u_1, \cdots, u_n$ and $m = u^a = u_1^{a_1} u_2^{a_2} \cdots u_n^{a_n}$, where $a = (a_1, \cdots, a_n)$ is a non-negative vector, is a monomial in $u_1, \cdots, u_n$. Following a technique used in [BMRT], we call a polynomial $f$ "positive" provided $f(e_i) > 0$ for any $e_i = (1, \cdots, 1, 0, 1, \cdots, 1)$ for $i = 1, \cdots, n$. It is clear that if $x = f/m$ with a "positive" polynomial $f$ and a monomial $m$, then $x$ is in reduced form [BMRT]. In this case or $x = u_i = \frac{1}{u_i}$ for some $i$, $x$ is called of the good reduced form.

**Theorem 4.4.** Any element $T^m(u_k)$ in $\chi'_\Omega$ is a cluster variable of the cluster algebra $A_\Omega$ with denominator $u_i^{\gamma_i(C^m(P_k[1]))}$. Furthermore, the assignment $\phi_\Omega : C^m(P_k[1]) \mapsto T^m(u_k), \forall m \in \mathbb{Z}$, $k \in \Gamma$, is a bijection from $\mathcal{P}\mathcal{I}(\Omega)$ to $\chi'_\Omega$ such that $P_k[1]$ corresponds to $u_k$.

**Proof.** Let $B$ be the skew-symmetrizable matrix corresponding to the valued quiver $(\Gamma, \Omega)$. If $k$ is a sink or a source, we denote by $s_k B$ the
skew-symmetrizable matrix corresponding to the valued quiver \((\Gamma, s_k \Omega)\). As above we assume that \(1, 2, \ldots, n\) is an admissible sequence of sinks of \((\Gamma, \Omega)\). By definition, \(T_1\) sends the seed \(((u_1, \ldots, u_n), B)\) (which is denoted by \((u, B)\) for simplicity) of cluster algebra \(A_{\Omega}\) to the seed \((T_1(u), B) = ((T_1(u_1), u_2, \ldots, u_n), B)\) of the cluster algebra \(A_{s_k \Omega}\), which can be viewed as one obtained by seed mutation in direction 1 from the seed \((u, s_1 B)\). Similarly, \((T_2T_1(u), B)\) is a seed of \(A_{s_2 s_1 \Omega}\). By induction, \((T(u), B)\) is a seed of \(A_{\Omega}\) since \(s_n \cdots s_1 \Omega = \Omega\). In this way, we prove that for any non-negative integer \(m\), \((T^m(u), B)\) is a seed of \(A_{\Omega}\). For \(m < 0\), we use \(T^{-1} = T_1 \cdots T_n\) to replace \(T\), the same argument implies \((T^{-m}(u), B)\) is a seed of \(A_{\Omega}\) for any positive integer \(m\). This proves the first statement in the theorem.

In the following, we will prove that all \(T^m(u)\) can be written as a good reduced form \(u^{\gamma_\Omega(C^1(P_{\Omega}(\Gamma)))} / f\) with \(f\) a ”positive” polynomial in \(u_1, \ldots, u_n\). Firstly we note that \(u_i = \frac{1}{u_i} = \frac{1}{u_i^{\gamma_\Omega(C^1(P_{\Omega}(\Gamma)))}}\) is a good reduced form; and if \(\Gamma\) is of finite type, then \(\phi_{\Omega}\) is a bijection which sends tilting objects to clusters, and sends \(P_i[1]\) to \(u_i\) (compare Theorem 4.7 and Remark 4.9 in \([Z3]\)).

Secondly, we note that if \(P_i[1]\) is a direct summand of a tilting object, i.e. \(V = \tilde{V} \oplus P_i[1]\), then \(\text{Hom}(P_i, \tilde{V}) = 0\), this means the \(i\)-th component of vector \(\gamma_\Omega(\tilde{V})\) is zero. Now we start at the \(n\)-tuple \(((u_1, \ldots, u_n-1, T(u_n)), s_n B)\) is a seed obtained from the initial seed \((u, B)\) by seed mutation in the direction \(n\), where \(n\) is a source in \((\Gamma, \Omega)\). We note that \(T(u_n) = T_n(u_n)\). Then \(T(u_n) u_n = \prod_{k < n} u_k^{-\gamma_\Omega(C^1(P_{\Omega}(\Gamma)))} u_{n-k+1}\) which is a good reduced form \(u^{\gamma_\Omega(C^1(P_{\Omega}(\Gamma)))} / f\) with that \(f\) is a ”positive” polynomial in \(u_1, \ldots, u_n\) since \(C^1(P_{\Omega}(\Gamma))\) is a simple injective module. Similarly, \(((u_1, \ldots, u_{n-2}, T(u_{n-1}), T(u_n)), s_{n-1} s_n B)\) is a seed obtained from the the seed \(((u_1, \ldots, u_n-1, T(u_n)), s_n B)\) by seed mutation in the direction \(n-1\), where \(n-1\) is a source in \((\Gamma, s_n \Omega)\). The reason is the following: Firstly, \(T_0 T_{n-1}(u_{n-1}) = T(u_{n-1})\) from the definition of \(T_i\); Secondly, from the definition of \(T_{n-1}\) we have \(T_{n-1}(u_{n-1}) u_{n-1} = \prod_{k \neq n-1} u_k^{-\gamma_\Omega(C^1(P_{\Omega}(\Gamma)))} + 1\) and then \(T(u_{n-1}) = T_n T_{n-1}(u_{n-1}) = \prod_{k < n-1} u_k^{-\gamma_\Omega(C^1(P_{\Omega}(\Gamma)))} u_{n-1}\).

Since \(C^1(P_{\Omega}(\Gamma))\) is an injective \(H\)-module and the AR-triangle starting at \(C^1(P_{\Omega}(\Gamma))\) is

\[
C^1(P_{\Omega}(\Gamma)) \rightarrow C^1(P_{\Omega}(\Gamma))^{-a_{n-1,n}} \oplus \bigoplus_{i \neq n-1, n} P_i[1]^{-a_{n-1,i}} \rightarrow P_{n-1}[1] \rightarrow C^1(P_{\Omega}(\Gamma))[1],
\]

by Lemma 3.8, we have that \(\gamma_\Omega(C^1(P_{\Omega}(\Gamma)))) = \dim E_{n-1} (-a_{n-1,n}) \gamma_\Omega(C^1(P_{\Omega}(\Gamma)))\).

Then \(T(u_{n-1}) = \frac{f_2}{u^{\gamma_\Omega(C^1(P_{\Omega}(\Gamma)))}}\) where \(f_2 = \prod_{k < n-1} u_k^{-\gamma_\Omega(C^1(P_{\Omega}(\Gamma)))} u_{n-1}\) is a ”positive” polynomial since \(f_1\) is ”positive". By induc
tion on $k$, we get $T(u_k) = \frac{f_k}{u^{\gamma(C^k(P_k[1]))}}$ with a "positive" polynomial $f_k$ for all $k$. So we have that $(T(u), B)$ is a seed with that all $T(u_i)$ can be written as a good reduced form with denominator $u^{\gamma(C^i(P_i[1]))}$.

By applying $h$, then it is a seed. The reason is the following: From the definition of $T$, $T \cdot \cdots \cdot T$, where $k \geq 1$, is a seed with that all cluster variables $T^k(u_j)$, $\forall j \leq i$ and $T^{k+1}(u_j) \forall j > i$ are of good reduced forms with denominators $u^{\gamma(C^k(P_k[1]))}$ or $u^{\gamma(C^{k+1}(P_{k+1}[1]))}$ respectively. We will show that the cluster variable $T^{k+1}(u_i)$ can be written as a good reduced form with denominator $u^{\gamma(C^{k+1}(P_{k+1}[1]))}$.

As before, the $n+1$-tuple $((T^k(u_1), \cdots, T^k(u_{n-1}), T^{k+1}(u_i), T^{k+1}(u_{i+1}), \cdots, T^{k+1}(u_n)), s_i, s_{i+1} \cdots s_n B)$ is a pair obtained from the seed $((T^k(u_1), \cdots, T^k(u_i), T^{k+1}(u_{i+1}), \cdots, T^{k+1}(u_n)), s_{i+1} \cdots s_n B)$ by seed mutation in the direction $i$, hence it is a seed. The reason is the following: From the definition of $T_i$, we have $T_i(u_i)u_i = \prod_{j \neq i} u_k^{a_{ik}} + 1$ which can be written as

$$T_iT_{i-1} \cdots T_1(u_i)u_i = \prod_{j > i} T_iT_{i-1} \cdots T_1(u_j)^{-a_{ij}} \prod_{j < i} u_j^{-a_{ij}} + 1.$$  

By applying $T_n \cdots T_{i+1}$ to the two sides of the equality, we get

$$T(u_i)u_i = \prod_{j > i} T(u_j)^{-a_{ij}} \prod_{j < i} u_j^{-a_{ij}} + 1.$$  

and then

$$T^{m+1}(u_i)T^{m}(u_i) = \prod_{j > i} T^{m+1}(u_j)^{-a_{ij}} \prod_{j < i} T^m(u_j)^{-a_{ij}} + 1.$$  

We have the following AR-sequence in $H$–mod:

$$0 \to C^{m+1}(P_{i}[1]) \to \oplus_{j > i} C^{m+1}(P_{j}[1])^{-a_{ij}} \oplus \oplus_{j < i} C^m(P_j[1])^{-a_{ij}} \to C^m(P_{i}[1]) \to 0.$$  

Then $\gamma_\Omega(C^{m+1}(P_{i}[1])) = \sum_{j > i} (-a_{ij}) \gamma_\Omega(C^{m+1}(P_{j}[1])) + \sum_{j < i} (-a_{ij}) \gamma_\Omega(C^{m}(P_j[1])) - \gamma_\Omega(C^m(P_{i}[1]))$. By assumption that $T^{m+1}(u_j) = \frac{f_j}{u^{\gamma(C^{k+1}(P_{k+1}[1]))}} \forall j > i$ and $T^{m}(u_j) = \frac{f_j}{u^{\gamma(C^k(P_k[1]))}} \forall j \leq i$, all forms are good reduced forms with "positive" polynomial $f_j$ respectively. Then we have that

$$T^{m+1}(u_i) = \prod_{j > i} (f_j/u^{\gamma(C^{k+1}(P_{k+1}[1]))})^{-a_{ij}} \cdot \prod_{j \leq i} (f_j/u^{\gamma(C^{k}(P_{k}[1]))})^{-a_{ij}+1}$$  

$$= \prod_{j > i} (f_j)^{-a_{ij}} \prod_{j \leq i} (u^{-a_{ij}})^{\gamma_\Omega(C^{k+1}(P_{k+1}[1]))} \prod_{j > i} u^{(-a_{ij})^{\gamma_\Omega(C^{k+1}(P_{k+1}[1]))}}$$  

$$\times \prod_{j \leq i} u^{(-a_{ij})^{\gamma_\Omega(C^{k}(P_{k}[1]))}} \prod_{j > i} u^{(-a_{ij})^{\gamma_\Omega(C^{k+1}(P_{k+1}[1]))}}.$$  

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The first factor is a polynomial by the "Laurent Phenomenon" [ZF1], which is easily to see to be "positive" since all $f_i$ are "positive"; and the second factor is a monomial

$$u \sum_{j > i} (-a_{ij}) \gamma(a(C^{m+1}(P_1[1])) + \sum_{j < i} (-a_{ij}) \gamma(a(C^m(P_1[1]))) - \gamma(a(C^m(P_1[1])))$$

which is $u^{\gamma(a(C^{m+1}(P_1[1])))}$. This proves that $T^{m+1}(u_i)$ is of a good reduced form with denominator $u^{\gamma(a(C^{m+1}(P_1[1])))}$.

If $\Gamma$ is a Dynkin diagram, there are two additional cases which may occur:

1. $T^{m+1}(u_i) = u_j$ for some $j$. In this case, $T^{m+1}(u_i)$ is a good reduced form $\frac{1}{u^{\gamma(a(C^{m+1}(P_1[1])))}}$, since $\phi_\Omega$ is a bijection sending $P_1[1]$ to $u_j$.

2. Some $u_j$ appears in the $n$-tuple $(T^k(u_1), \ldots, T^k(u_i), T^{k+1}(u_{i+1}), \ldots, T^{k+1}(u_n))$ as components. For simplicity, we assume that $T^k(u_{j_1}) = u_{j_1}$ for $j_1 \in J_1 \subset \{1, \ldots, i\}$ and $T^{k+1}(u_{j_2}) = u_{j_2}$ for $j_2 \in J_1 \subset \{i + 1, \ldots, n\}$. The AR-triangle ending at $C^m(P_1[1])$ is

$$C^{m+1}(P_1[1]) \rightarrow \oplus_{j > i} C^{m+1}(P_1[1])^{-a_{ij}} \oplus \oplus_{j < i} C^m(P_1[1])^{-a_{ij}} \rightarrow C^m(P_1[1]) \rightarrow C^{m+1}(P_1[1])[1].$$

This triangle induces the AR-sequence in $H -$mod (compare Lemma 3.7):

$$0 \rightarrow C^{m+1}(P_1[1]) \rightarrow \oplus_{j > i, j \notin J_2} C^{m+1}(P_1[1])^{-a_{ij}} \oplus \oplus_{j < i, j \notin J_1} C^m(P_1[1])^{-a_{ij}} \rightarrow C^m(P_1[1]) \rightarrow 0.$$

Then

$$\gamma(a(C^{m+1}(P_1[1]))) = \sum_{j > i, j \notin J_2} (-a_{ij}) \gamma(a(C^{m+1}(P_1[1]))) + \sum_{j < i, j \notin J_1} (-a_{ij}) \gamma(a(C^m(P_1[1]))) - \gamma(a(C^m(P_1[1]))).$$

As before we have the formula for $T^{m+1}(u_i)$:

$$T^{m+1}(u_i) = \prod_{j \geq i, j \notin J_1} (f_i / u^{\gamma(a(C^k(P_1[1])))} - a_{ij}) \prod_{j < i, j \notin J_1} u^{-a_{ij}} \prod_{j > i, j \notin J_2} u^{-a_{ij} - 1}$$

$$\prod_{j \geq i, j \notin J_2} (f_i / u^{\gamma(a(C^k(P_1[1])))} - a_{ij} - 1) \prod_{j < i, j \notin J_1} u^{-a_{ij}} \prod_{j > i, j \notin J_2} u^{-a_{ij}}$$

$$\times u^{-\gamma(a(C^k(P_1[1])))} \prod_{j > i, j \notin J_2} u^{-\gamma(a(C^{k+1}(P_1[1])))} \prod_{j < i, j \notin J_1} u^{-\gamma(a(C^k(P_1[1]))) \prod_{j > i, j \notin J_2} u^{-\gamma(a(C^{k+1}(P_1[1])))}}.$$
[ZF1], which is "positive" since all $f_i$ are "positive" and the monomials $\prod_{j \in J_1 \cup J_2} u_{t_j}^{a_{ij}}$ and $\prod_{j < i, j \not\in J_1} u_{t_j}^{(-a_{ij})} u_{t_i}^{m(C^{k+1}(P_j[1]))}$ are co-prime. The last statement follows from that $V = \bigoplus_{i < j} (C^m(P_j[1]) \bigoplus \bigoplus_{j > i} C^{m+1}(P_j[1]))$ is a cluster tilting object in $C(\Omega)$, for $j \in J_1 \cup J_2$. $P_j[1]$ is a direct summand of $V$, then the $j$-th components of $\gamma_1(C^{m+1}(P_j[1]))$ and of $\gamma_0(C^m(P_j[1]))$ are zero. This proves that $T^{m+1}(u_i)$ is a good reduced form with denominators $u^{\gamma_0(C^{m+1}(P_j[1]))}$.

By induction, we have all $T^m(u_i)$ are of good reduced forms with denominators $u^{\gamma_0(C^m(P_i[1]))}$ for any $m \geq 0, i = 1, \cdots n$. Dually, we have the same statement for $T^{-m}(u_i)$, i.e. they are of good reduced forms with denominators $u^{\gamma_0(C^{-m}(P_i[1]))}$, where $m > 0$. For any non-isomorphic objects $X, Y$ in $\mathcal{P}(\Omega)$, $\dim X \neq \dim Y$, then $\phi_\Omega(X) \neq \phi_\Omega(Y)$. This says that $\phi_\Omega$ is bijection. The proof is finished.

If $(\Gamma, \Omega)$ is a quiver (with trivial valuations), Caldero and Keller prove that the Caldero-Chapoton map $X_\gamma$ [CC] gives a bijection from $\mathcal{E}(H)$ to $\chi_B$ in Theorem 4.7 [CK2]. In this case, our map $\phi_\Omega$ coincides with the map $X_\gamma$ since $X_\gamma$ sends tilting objects to clusters, and then sends $C^m(P_i[1])$ to $T^m(u_i)$. Therefore $\phi_\Omega$ sends cluster tilting objects in $\mathcal{P}(\Omega)$ to clusters in $\chi_\Omega$.

In the following, we apply this theorem to the Dynkin diagrams $\Gamma$ to obtain a generalization of Fomin and Zelevinsky’s denominator theorem [FZ2]. First of all we have a direct consequence which is needed in the proof of the further generalization (see the next proposition).

**Corollary 4.6.** Let $\Gamma$ be a Dynkin diagram and $M$ an indecomposable $H$-module with $\dim M = \alpha$. Then $\phi_\Omega(M) = \frac{f_\alpha(u_1, \cdots, u_n)}{u_m^{\gamma_0(C^m(P_i[1]))}}$, where $f_\alpha(u_1, \cdots, u_n)$ is an integral polynomial and is not divided by $u_i$.

**Remark 4.7.** From Corollary 4.6, combining with Lemma 8.2. in [Ker], one can prove that different cluster monomials have different denominators with respect to a given acyclic clusters (compare [D]).

Now we strengthen the denominator statement in Corollary 4.6 to any seed. Namely we generalize the denominator Theorem in [CCS2] to non simply-laced Dynkin diagram by using the interpretation in [Z1] of the compatibility degree $(\| \|)$.

**Proposition 4.8.** Let $(\Gamma, \Omega)$ be a valued Dynkin quiver and $\underline{x} = (x_1, \cdots, x_n)$ a cluster of the cluster algebra $A_\Omega$. Let $V = \bigoplus_{i=1}^n V_i$ be the cluster tilting object corresponding to $\underline{x}$ under the map $\phi_\Omega$. Then for any indecomposable object $M$ in $\mathcal{C}(\Omega)$, $\phi_\Omega(M) = \frac{f_\alpha(x_1, \cdots, x_n)}{x_i^{\gamma_0(V_i[1])}}$, where $f_\alpha(x_1, \cdots, x_n)$ is an integral polynomial and is not divided by $x_i$.

**Proof.** For simplicity, $\gamma_0(M)$ is denoted by $\alpha$, $\gamma_0(V_i[1])$ by $\beta_i$. Let $-\alpha_i$ be the negative simple roots. Since the bijection $\phi_\Omega$ sends cluster tilting objects to clusters by Theorem 4.7, in [Z3], there is a cluster tilting object $V$ which corresponds to the cluster $\underline{x}$ under $\phi_\Omega$. The cluster variable $\phi_\Omega(M)$ of cluster algebra $A_\Omega$ can be written as $\frac{f_\alpha(x_1, \cdots, x_n)}{\prod_{i=1}^n x_i^{\gamma_0(V_i[1])}}$ by the Laurent phenomenon [FZ2], where $[\alpha, \beta_i, \underline{x}] \in \mathbb{Z}$.
From Lemmas 6.2, 6.3. in [CCS1] (the proofs of these lemmas there were given for simply-laced Dynkin case, but these proofs also work for non simply-laced Dynkin case without any changes), we have that $[\alpha, \beta_i, x] = [\alpha, \beta_i, y]$ provided $\beta_i \in \{x_j | j = 1, \cdots, n\} \bigcap \{y_j | j = 1, \cdots, n\}$ for clusters $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$. Then it is denoted by $[\alpha, \beta_i]$ and $[\sigma \pm \alpha, \sigma \pm \beta] = [\alpha, \beta]$. From Corollary 4.6, we know that $[\alpha, -\alpha_i] = n_i(\alpha)$ where $n_i(\alpha)$ is the $i$-th coefficient of expression of root $\alpha$ in simple roots. This proves $[\alpha, -\alpha_i] = (-\alpha_i|\alpha)$. By the uniqueness of $([|])$ in Section 3 in [FZ2], $[\alpha, \beta] = (\beta|\alpha)$ for $\beta \neq \alpha$. Then by Theorem 3.6 in [Z1], we have that $[\alpha, \beta_i] = (\beta_i|\alpha) = \dim_{\text{End}(V_i[1])} \text{Ext}^1_{C}(V_i[1], M)$ for $M \neq V_i[1]$. The latter equals $\dim_{\text{End}(V_i[1])} \text{Hom}_{C}(V_i[1], M[1]) = \dim_{\text{End}(V_i[1])} \text{Hom}_{C}(V_i, M)$, it is the $i$-th component of dimension vector $\gamma_V(M)$. The proof is finished.

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