DESIGN OF GREEN BONDS BY DOUBLE-BARRIER OPTIONS

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ABSTRACT. Green finance is an innovative model that can promote sustainable economic development. The green bonds also develop gradually as a part of green finance. The green bonds are designed to fund the projects of positive environmental impact. If the green bonds are superior to other debt securities, they will attract more investors’ participation in green energy projects. Thus, the design of green bonds is crucial to the development of green bonds market. This article assumes that the floating rate of green bonds is linked to carbon price, and carbon price is described by a jump diffusion process. The carbon price fluctuation can lead to interest rate fluctuation of green bonds. We set two boundary values of carbon price, and the coupon rate is revalued when the carbon price reaches the boundary. The higher the carbon price is, the higher the coupon rate is to be paid by issuers. Thus, the boundary can impel issuers to boost energy savings and emission-reduction, and the higher interest rate will also attract more investors to invest in green bonds. The lower the carbon price is, the lower the interest rate is to be paid by issuers. Accordingly, the boundary may encourage issuers to boost emission reduction. This design can monitor and incentivize issuers to make more contribution to green finance. Furthermore, the design is characterized by the double-barrier option, such that the interest rate of green bonds can be obtained by double-barrier option pricing. Subsequently, the central difference method and the composite trapezoidal formula are employed to obtain the numerical solution. Finally, we conduct the sensitivity analysis of the model.

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1. **Introduction.** Currently, energy sources, environment and sustainable development are worldwide problems. The long term usage of fossil fuels, such as oil, coal and natural gas, causes environment pollution and greenhouse effect. It also leads to global warming, species extinction, sea pollution and the resource exhaustion. Global warming will cause dramatic climate changes leading to frequent floods, heat waves, and droughts. The study carried out in (Burke et al. 2018) [6] found that the increase in temperature has a direct consequence to the suicide rate. The Paris Agreement signed by 196 nations is the first universal climate agreement (United Nations Climate Change Conference 2015) [37]. One of the highlights was to stabilize the global temperature increase at 1.5 degrees Celsius above the pre-industrial level. Scientists believe that this temperature is the minimum threshold of safety for the planet to reach net zero greenhouse gas emissions at the end of this century. Although the goal of limiting the increase of the temperature is by no more than two degrees Celsius above the pre-industrial level, most countries should try to stay under 1.5 degrees Celsius (Northrop 2016) [27]. However, the cost of achieving the goal is enormous. A solution to this problem is to tap private sectors to invest in the green projects so as to reduce the carbon emission. This is in addition to the government investment. Green bonds are the debt securities issued by institutions, such as European Investment Bank (EIB), the World Bank (WB), government entities, multilateral institutions, or corporations to raise capital. However, the risk in the investment of green bonds is high. This constitutes a barrier for traditional investors. Public institutions, such as EIB and WB, can provide investors with credit enhancement, and hence guiding the private sectors to contribute to the low carbon economy. There are innovative instruments, such as carbon-linked bonds, and investment projects linked with carbon trading. These bonds are initially conceived and subsequently issued by energy companies to hedge their risk. The structure of this bond provides the financing agents for raising low cost funding. It encourages developing countries to adopt a low emission growth path such that investors will get higher return on the coupons with the increase of carbon credit price (Bloch 2011) [5].

According to Hartmann et al. (2005) [17], “a green brand is defined by a specific set of brand attributes and benefits related to the reduced environmental impact of the brand and its perception as being environmentally sound”. Green brands are attractive to those consumers who are consciously seeking environmental friendly products. In 2014, a study based on data from 60 countries found that 55 % of online consumers were willing to pay more for products and services provided by companies committed to environment (Nielsen 2014) [26]. Limiting the global warming to no more than two degrees Celsius above the pre-industrial level makes financial sense to risk-neutral investors and even more to risk-averse investors (Dietz et al. 2016) [10]. The green bonds promote green growth, and the green growth is a motivating factor in Green Bond investments. In 2007, the European Investment Bank issued the first green bonds. Multinational financial organizations, such as the World Bank, the second issuer of green bonds, were the first to develop and support the spread of green bond issuances before private entities, such as Credit Agricole, Toyota, and Unilever, entered the market (Tripathy 2017) [35]. The green bonds are an example of an innovative fixed income investment product. It can pave the way for the development of next phase products for the mobilization of capital to finance the greatest challenges faced by our generation (Reichelt 2010) [30]. Issuing intertemporal green bonds can provide the funding for climate mitigation and
adaptation policies (Flaherty et al. 2016) [12]. The future growth of green bonds will depend on the capability of issuers, source of the projects, issue scale, stakeholder engagement, and financial support for green bonds (Chugan et al. 2017) [8]. The World Bank has issued more than USD 10.2 billion equivalent in green bonds through more than 135 transactions in 18 currencies since 2008 (The World Bank, 2017). Furthermore, climate change is not only an environmental challenge, but also a fundamental threat to economic development (Jim Yong Kim, President of the World Bank 2015). It is estimated that green bonds can supply up to 84 % of funding to the private, third-party capital required to finance the transition to a low carbon economy (As You Sow, 2014) [1]. Due to overissuing of green bond, it is urgently required for the investors to search for solutions (Inderst et al. 2012) [18]. The appearance of corporate green bonds has dramatically altered the market. The first corporate green bond was issued by Vasakronan, a Swedish real estate company, in November 2013. Since then, utilities, such as GDF Suez (now Engie) and Electricite de France (EDF), have issue green bonds. For example, the transportation company, Toyota, have made a record-breaking issuance of green bonds (Toyota 2014) [34]. The issuance of these green bonds has provided the financial support to the development of renewability and low-carbon production in such a way that they become core business models of the companies. In February 2016, Apple issued its first green bond, in the amount of 1.5 billion, the largest issuance up to date by a U.S. corporation. Green bonds have been issued in twenty-three countries and in twenty-three currencies. The United States, Canada, and Western Europe have well-established markets. The largest driver of market growth is China, which constitutes more than one-quarter of the global market. Due to skyrocketing domestic demand for low-carbon energy infrastructure, the green bond market in China is expected to grow continuously (Park 2018) [28].

The World Bank online report on climate bonds suggests that most coupons for green bonds are fixed. Green bonds are long-term investments, which can be adopted as pension funds. The most important reasons for investing in Green Bonds are to “improve brand/image” and “serve long-term company interests” (Park 2018) [28]. The other reason is to avoid being forced to deal with climate related problems in the future. By investing money in climate related projects, problems that otherwise could be costly in the future are dealt with today. Implementing a “green” branding strategy become a long-term plan for the company to attract environmentally aware consumers. This will contribute to the company’s long-term financial performance (Nielsen 2014) [26]. The green bonds will not attract investors if coupon rate, credit risk, and some other factors are not superior to other bonds. The growing green consciousness may promote the green bonds market, but it is not the main factor. The diversity of carbon bonds can attract more investors to participate in green energy projects, which can accelerate the development of green bond market. So, the design of the green bonds is important for the sustainability of the development of green finance. The interest rate is an important component of the green bonds, it will influence the future earnings of issuers and allure for investors. Black and Scholes (1973) [3] presented an explicit equilibrium model for valuing options. They indicated that a similar analysis could potentially be applied to all corporate securities. Also, both Merton (1973) [25] and Ross noted the broad applicability of option pricing arguments. In addition, Black and Cox (1976) [4] made some general statements on this valuation process for corporate
securities and investigated the effects of safety covenants, subordination arrangements, and restrictions on the interest of financing and dividend payments. Jiang (2008) [21] deduced the pricing formula of zero coupon bond. In order to attract investors to invest in green bonds, the design of the green bonds should be more profitable to the investor. The London Accord Project Group, which proposed the index-linked carbon bonds during the World Bank Government Borrowers’ Forum in May 2009, explained that the interest paid regularly may be linked to carbon price, government carbon emission targets, in-country fossil fuel prices, and tariff feed-in prices (Bloch 2011) [5]. Thus, the coupon rate mentioned in this study is considered to be linked to carbon price, which is described by a jump diffusion process. The carbon price fluctuates, which leads to the change in the interest rate of green bonds. Accordingly, we set two boundary values of carbon price, and the coupon rate is revalued when the carbon price reaches the boundary. The higher the carbon price is, the higher the coupon rate is to be paid by issuers. Thus, the boundary can impel issuers to boost energy savings and emission-reduction, and the higher interest rate will also attract more investors to invest in green bonds. Similarly, the lower the carbon price is, the lower the interest rate is to be paid by issuers. Accordingly, the boundary may encourage issuers to boost emission-reduction. This design can monitor and incentivize issuers to make more contributions to green finance. Furthermore, the design is characterized by the double-barrier option, such that the interest rate of the green bond can be obtained by double-barrier option pricing.

Barrier options are one of the most applicable types of exotic derivatives in the financial market, and are regarded as path-dependent options. The barrier option was first proposed by Merton (1973) [24] who derived the option pricing formula by continuously monitoring the lower knockout boundary. Then, the formulas for various types of path-dependent options were derived (Goldman et al. 1979) [16]. Barrier options have also been investigated by many researchers over the two past decades. For instance, the standard trinomial tree method was used to price path-dependent options (Kamrad and Ritchken 1991) [22]. An analytical solution for single-barrier option based on the Z-transform was proposed (Fusai 2006) [14]. Moreover, a numerical solution for discrete barrier options based on the combination of quadrature method and interpolation procedure was studied (Fusai 2007) [13]. Explicit formulas were derived for pricing double (single) barrier and touch options with time-dependent rebates assuming that the asset price follows a double-exponential jump diffusion process (Sepp 2004) [31]. A Laplace transform-based analytical solution was presented for pricing double-barrier options under a flexible hyper-exponential jump diffusion model (Cai et al. 2009) [7]. Expansion method was analyzed for the valuation of double-barrier options under a stochastic volatility model (Jeon 2017) [19]. What is more, the barrier option has been used for various applications. For example, price risk transfer mechanisms, which was between agricultural commodity futures market and option market, were compared by barrier option (Liu et al. 2005) [23]. The valuation model under the guarantee by the Government was proposed based on the barrier option (Gao et al. 2007) [15]. The barrier option was used to avoid foreign exchange risks (Jiang 2009) [21]. A barrier option was devised to hedge the risks of market price for the initiation of wind power generation (Xiao et al. 2016) [36]. Double-barrier options have also become popular to be used as derivatives in financial markets. A double-barrier option, which has a lower barrier and an upper barrier, is a combination of two
The present paper will design the green bonds based on the specific features of barrier option. The remainder of this paper is organized as follows. In Section 2, the mathematical model of the green bond interest rate is derived. In Section 3, the central difference and composite trapezoidal formula are proposed to numerically solve the models. Section 4 presents a detailed analysis of several numerical experimental results. Section 5 provides the summary.

2. Pricing model for green bond interest rates. Green bonds are different from other bonds, in that it is an important financing instrument for green finance. The coupon paid regularly for green bonds are reportedly linked to carbon price, government carbon emission targets, in-country fossil fuel prices, and tariff feed-in prices. From the green bonds issued by the World Bank, we observe that they have issued equity index-linked bonds, ethical index-linked bonds, and CMS-linked bonds (Figure 2.1). Black and Cox (1976) [4] value the corporate securities. The firm’s securities are affected by the value of the firm, and a partial differential equation (PDE) is established to price securities. The boundary condition is the minimization between the value of the firm and the final payment of the securities. Pelizzari and Cristian (2010) [29] propose a pricing model for inflation linked to bonds. The coupon is linked to the inflation, and a PDE is developed to price coupon, in which the boundary condition is the function of inflation. So, this article designs a kind of green bonds whose interest rate is variably linked to carbon price. The interest rate is \((S_T - K)^+\% \) at maturity, where \(K\) is a constant related to carbon price and \(S_T\) is the carbon price at maturity. This design ensures that the interest rate is non-negative at \(T\), and the investors can obtain the interest. The percent sign is omitted for ease of calculation, and the green bonds are the bonds with the face value of 1. The face value is 1, so the value of interest rate is equal to the value of coupon. We set two boundary values of carbon price, and the interest rate is

| Maturity Date | Issue Date | Volume | Category | Lead Manager | Maturity Date | Issue Date | Volume | Category | Lead Manager |
|---------------|------------|--------|----------|--------------|---------------|------------|--------|----------|--------------|
| 2021          | 27-Oct-14  | USD 22.753 Million | Equity Index-Linked | BNP | 2025          | 14-Mar-15  | USD 50 Million | Equity Index-Linked | BNP |
| 2022          | 12-Jan-15  | USD 91.046 Million | Equity Index-Linked | BNP Panbas | 2025 | 22-Feb-15  | USD 59.687 Million | Equity Index-Linked | BNP Panbas |
| 2022          | 20-Mar-17  | USD 13.75 Million | Equity Index-Linked | BNP Panbas | 2025 | 30-Jun-15  | USD 1.333 Million | Equity Index-Linked | BNP Panbas |
| 2022          | 10-Apr-15  | USD 10 Million | Equity Index-Linked | BNP Panbas | 2025 | 30-Oct-15  | USD 2.46 Million | Equity Index-Linked | BNP Panbas |
| 2022          | 10-Apr-15  | USD 40 Million | Equity Index-Linked | BNP Panbas | 2025 | 31-Dec-15  | USD 1.436 Million | Equity Index-Linked | BNP |
| 2022          | 30-Apr-15  | USD 10 Million | Equity Index-Linked | BNP Panbas | 2026 | 7-Oct-14   | USD 10 Million | Equity Index-Linked | BNP Panbas |
| 2023          | 30-Apr-15  | USD 103.275 Million | Equity Index-Linked | BNP Panbas | 2026 | 30-Sep-16  | USD 1.030 Million | Equity Index-Linked | BNP Panbas |
| 2024          | 7-Aug-14   | EUR 50 Million | Equity Index-Linked | BNP Panbas | 2028 | 8-Aug-16   | EUR 50 Million | Equity Index-Linked | Natixis |
| 2024          | 8-Jan-16   | USD 16.387 Million | Equity Index-Linked | BNP | 2034 | 28-Nov-14  | USD 20 Million | CMS Linked | CIB / MS |

Figure 1. Index-linked green bonds issued by World Bank
revalued when the carbon price reaches the boundary. The higher the carbon price, the higher the interest rate is to be paid by issuers. Thus, the boundary can impel issuers to boost energy savings and emission-reduction. The lower the carbon price, the lower the interest rate is to be paid by issuers. Accordingly, the boundary may encourage issuers to boost emission-reduction. The interest rate is related to future uncertainty and is characterized by contingent claims. Subsequently, we calculate the interest rate by using the option pricing model. We assume that the interest rate is the double knockout call option price, which underlying the carbon price. The mathematical models are provided in subsequent sections. In brief, the jump diffusion model was proposed by Press and was extended by Merton (1976) [24]. Carbon price is characterized by continuous changes and jumps at certain times. Moreover, for consistency between the barrier and the option payoff, we require

\[ dS_t = (r - \lambda k)dt + \sigma dW_t + U dq_t, \]

or

\[ d\ln S_t = (r - \lambda k - \frac{1}{2}\sigma^2)dt + \sigma dW_t + ln(1 + U) dq_t, \]

where \( k = E(U) \) and \( r \) denotes riskless rate. We consider a double knockout call option maturing at time \( T \) with the strike price \( K \), the lower barrier \( S_{min} \), and the upper barrier \( S_{max} \). We require \( S_{min} < S_0 < S_{max} \); otherwise, the option is worthless at time \( t_0 \). Moreover, for consistency between the barrier and the option payoff, we require \( S_{min} < K < S_{max} \). Parameter \( V(S, t) \) denotes the option price at time \( t \) given that \( S(t) = S \). For instance, we set \( V(S, t) = V \).

We consider a portfolio \( \Pi_t = V_t - \Delta_t S_t \), where \( \Delta_t \) is the share of the underlying asset. This portfolio is not free of risk during the jump, and the risk is unsystematic. Thus, we consider the expected return rate of the portfolio to be riskless, i.e., \( E(d\Pi_t) = r\Pi_t dt \), which implies that the price of the underlying asset has the following two possibilities:

1. When \( S_t \) does not jump, we set the event as \( \omega_1 \):

\[ d\Pi_t(\omega_1) = dV_t - \Delta_t dS_t = \left( \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \left( \frac{\partial V}{\partial S} - \Delta_t dS_t \right). \]

2. When \( S_t \) jumps, we set the event as \( \omega_2 \):

\[ d\Pi_t(\omega_2) = V(S_{t+}, t) - V(S_t, t) - \Delta_t(S_{t+} - S_t) = V((1 + U)S_t, t) - V(S_t, t) - \Delta_t US_t. \]
Thus,

\[ r \Pi dt = E(d \Pi_t) = (1 - \lambda dt)d \Pi(\omega_1) + \lambda dt d \Pi(\omega_2). \]  

(4)

Let \( \Delta_t = \frac{\partial V}{\partial S} \). Then, substitute Eqs. (2.1), (2.2) and (2.3) into Eq. (2.4). By taking the expected value of both sides of the resulting equation and omitting the term \((dt)^2\), it follows from Ito’s formula that:

\[
\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - \lambda k)S \frac{\partial V}{\partial S} - (r + \lambda)V + \lambda E_u(V(1+U)S, t) = 0. \]  

(5)

Barrier option pricing is a formulation of the B-S equation with a particular terminal-boundary value. The domain for Eq. (2.5) is \( D = (S,t) \mid S_{\min} \leq S \leq S_{\max}, 0 \leq t \leq T \).

When an underlying asset reaches either one of these barriers, the double-barrier knockout option is extinguished. Thus, we redefine the option price of the lower and upper barriers as 0 and \( S_{\max} - Ke^{-rt} \), respectively. The option pricing formula can be obtained as follows:

\[
\begin{aligned}
&\left\{ \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - \lambda k)S \frac{\partial V}{\partial S} - (r + \lambda)V + \lambda E_u(V(1+U)S, t) = 0, \\
&V(S,T) = \max\{ (S - K), 0 \}, \\
&V(S_{max}, t) = S_{max} - Ke^{-rt}, \\
&V(S_{min}, t) = 0. 
\end{aligned}
\]  

(6)

We can obtain the present value of the regular interest payments \( V_i(i = 1, 2, \cdots, n) \) by solving Eq. (2.6). The issue price of carbon bond is set to be \( PV \) and the principal at maturity is equal to the face value, so the issue price can be obtained by \( PV = \sum_{i=1}^{n} V_i e^{(-iT)} \). We apply the central difference and composite trapezoidal formula to discretize and solve Eq. (2.6) in Section 3.

3. Discretization. We let \( X = ln(S/K), \eta = ln(1+U) \sim N(\theta, \delta^2), \tau = T - t, V(Ke^X, T - t) = \nu(X, \tau) \), such that Eq. (2.6) can be transformed into Eq. (3.1) as follows:

\[
\begin{aligned}
&\left\{ \frac{\partial \nu}{\partial \tau} + \frac{1}{2}\sigma^2 X \frac{\partial^2 \nu}{\partial X^2} + (r - \lambda k - \frac{1}{2}\sigma^2) \frac{\partial \nu}{\partial X} - (r + \lambda)\nu + \lambda \int_{-\infty}^{\infty} \nu(X + \eta, \tau)p(\eta)d\eta = 0, \\
&V(X, 0) = \max\{ (Ke^X - K), 0 \}, \\
&V(X_{max}, t) = Ke^{X_{max}} - Ke^{-rt}, \\
&V(X_{min}, t) = 0. 
\end{aligned}
\]  

(7)

For the integral term of Eq. (3.1), we let \( X + \eta = Z \). Then, it follows that:

\[
\int_{-\infty}^{\infty} \nu(X + \eta, \tau)p(\eta)d\eta \\
= \int_{X_{min}}^{X_{max}} \nu(Z, \tau)p(Z - X)dZ + \int_{-\infty}^{X_{min}} \nu(Z, \tau)p(Z - X)dZ + \int_{X_{max}}^{\infty} \nu(Z, \tau)p(Z - X)dZ \\
= \nu_{\min}(X, \tau) + \nu_{\max}(X, \tau). 
\]  

(8)
where

\[ v_{loc}(X, \tau) = \int_{X_{\text{min}}}^{X_{\text{max}}} \nu(Z, \tau)p(Z - X)dZ, \]  

\[ \nu_c(X, \tau) = \int_{-\infty}^{X_{\text{min}}} \nu(Z, \tau)p(Z - X)dZ + \int_{X_{\text{max}}}^{\infty} \nu(Z, \tau)p(Z - X)dZ. \]

Eq. (3.3) can be discretized using the uniform mesh \( X_{\text{min}} = X_0 < X_1 < \cdots < X_{m-1} < X_m = X_{\text{max}} \), where \( m \) is a positive integer. Then, by using the composite trapezoidal formula, the resulting discrete form is obtained as follows:

\[ v_{loc}(X, \tau) = \frac{h_z}{2}(\nu(X, \tau)p(X_1 - X) + 2 \sum_{j=2}^{m-1} \nu(X_j, \tau)p(X_j - X) + \nu(X_n - X)). \]

Eq. (3.5) can be expressed in a matrix form as given below:

\[ \nu_{loc}(X, \tau) = \frac{h_z}{2} : \begin{bmatrix} p(0) & 2p(X_2 - X_1) & 2p(X_3 - X_1) & \cdots & 2p(X_{m-1} - X_1) & p(X_m - X_1) \\ p(X_1 - X_2) & 2p(0) & 2p(X_3 - X_2) & \cdots & 2p(X_{m-1} - X_2) & p(X_m - X_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ p(X_1 - X_{m-1}) & 2p(X_2 - X_{m-1}) & 2p(X_3 - X_{m-1}) & \cdots & 2p(X_m - X_{m-1}) & p(0) \end{bmatrix} \begin{bmatrix} \nu(X_1, \tau) \\ \nu(X_2, \tau) \\ \vdots \\ \nu(X_{m-1}, \tau) \\ \nu(X_m, \tau) \end{bmatrix}. \]

We know that \( \nu \to 0 \) when \( X \to -\infty \) (see the definition of \( X \)). Parameter \( \nu(Z, \tau) \) is set as a constant. We set \( Ke^{X_{\text{max}}} - Ke^{-\tau t} = \nu_u \), such that direct calculation yields the following expression:

\[ v_c(X, \tau) \approx \int_{X_{\text{max}}}^{+\infty} \nu(Z, \tau)p(Z - X)dZ = \nu_u \int_{X_{\text{max}}}^{+\infty} p(Z - X)dZ = \nu_u \delta(1 - \Phi(X_{\text{max}} - X - k)), \]

where \( \Phi \) is the standard normal cumulative distribution function. Let \( \epsilon(X, \tau) = \nu_u \delta(1 - \Phi(X_{\text{max}} - X - k)) \). Then it follows from Eq. (3.2) that:

\[ \int_{-\infty}^{-\infty} \nu(Z, \tau)p(Z - X)dZ \]

\[ = \frac{h_z}{2} \begin{bmatrix} p(0) & 2p(X_2 - X_1) & 2p(X_3 - X_1) & \cdots & 2p(X_{m-1} - X_1) & p(X_m - X_1) \\ p(X_1 - X_2) & 2p(0) & 2p(X_3 - X_2) & \cdots & 2p(X_{m-1} - X_2) & p(X_m - X_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ p(X_1 - X_{m-1}) & 2p(X_2 - X_{m-1}) & 2p(X_3 - X_{m-1}) & \cdots & 2p(X_m - X_{m-1}) & p(0) \end{bmatrix} \begin{bmatrix} \nu(X_1, \tau) \\ \nu(X_2, \tau) \\ \vdots \\ \nu(X_{m-1}, \tau) \\ \nu(X_m, \tau) \end{bmatrix} + \begin{bmatrix} \epsilon(X_1, \tau) \\ \epsilon(X_2, \tau) \\ \vdots \\ \epsilon(X_{m-1}, \tau) \\ \epsilon(X_m, \tau) \end{bmatrix}. \]

For the positive integers \( m \) and \( n \), we partition \((X_{\text{min}}, X_{\text{max}}) \times (0, T)\) into \( m \times n \) rectangles with nodes \((X_i, \tau_j)\) for \( i = 0, 1, \cdots, m \) and \( j = 0, 1, \cdots, n \) such that \( X_{\text{min}} = X_0 < X_1 < \cdots < X_{m-1} < X_m = X_{\text{max}} \) and \( 0 = \tau_0 < \tau_1 < \cdots < \tau_{n-1} < \tau_n = T \). Then we set \( \Delta X = X_{i+1} - X_i \) for \( i = 0, 1, \cdots, m - 1 \) and \( \Delta \tau = \tau_{j+1} - \tau_j \) for \( j = 0, 1, \cdots, n - 1 \). On this mesh, we can obtain the discrete form for the non-integral term in Eq. (3.1) as follows:
\[
\frac{\nu^{n+1} - \nu^n}{\Delta \tau} = \frac{1}{2} \sigma^2 \nu^{n+1} - 2 \nu^{n+1} + \nu^n}{\Delta X^2} + (r - \lambda k - \frac{1}{2} \sigma^2) \nu^{n+1} - \nu^n}{2 \Delta X} - (r + \lambda) \nu^{n+1} + \lambda \int_{-\infty}^{+\infty} \nu(X + \eta) p(\eta) d\eta
\]

That is,
\[
\left( -\frac{\sigma^2 \Delta \tau}{2 \Delta X} + \frac{\Delta \tau}{2 \Delta X} (r - \lambda k - \frac{\sigma^2}{2} \right) \nu^{n+1} + (1 + \frac{\Delta \tau \sigma^2}{2 \Delta X}) \nu^{n+1} + (r + \lambda) \Delta \tau \nu^{n+1} + (-\frac{\sigma^2 \Delta \tau}{2 \Delta X}) \nu^{n+1} \right) = \lambda \Delta \tau \int_{-\infty}^{+\infty} \nu(X + \eta) p(\eta) d\eta + \nu^n.
\]

Let:
\[
\alpha_i^n = -\frac{\sigma^2 \Delta \tau}{2 \Delta X} + \frac{\Delta \tau}{2 \Delta X} (r - \lambda k - \frac{\sigma^2}{2} \right), \beta_i^n = 1 + \frac{\Delta \tau \sigma^2}{2 \Delta X}, \gamma_i^n = -\frac{\sigma^2 \Delta \tau}{2 \Delta X} - \frac{\Delta \tau}{2 \Delta X} (r - \lambda k - \frac{\sigma^2}{2} \right).
\]

\[
A = \begin{bmatrix}
\beta_1^n & \gamma_1^n & 0 & 0 & \cdots & 0 & 0 & 0 \\
\alpha_1^n & \beta_2^n & \gamma_2^n & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \alpha_{n-1}^n & \beta_{m-1}^n & \gamma_{m-1}^n \\
0 & 0 & 0 & 0 & \cdots & \alpha_m^n & \beta_m^m & \gamma_m^m
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
p(0) & 2p(X_2 - X_1) & 2p(X_3 - X_1) & \cdots & 2p(X_{m-1} - X_1) & p(X_{m-1} - X_1) \\
p(X_1 - X_2) & 2p(0) & 2p(X_3 - X_2) & \cdots & 2p(X_{m-1} - X_2) & p(X_{m-1} - X_2) \\
\vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \vdots \\
p(X_1 - X_m) & 2p(X_2 - X_m) & 2p(X_3 - X_m) & \cdots & 2p(X_{m-1} - X_m) & p(0)
\end{bmatrix},
\]

\[
\tilde{\nu}^{n+1} = \begin{pmatrix} \nu_1^{n+1} \\ \nu_2^{n+1} \\ \vdots \\ \nu_m^{n+1} \end{pmatrix}, \tilde{\nu}^n = \begin{pmatrix} \nu_1^n \\ \nu_2^n \\ \vdots \\ \nu_m^n \end{pmatrix}, \tilde{\epsilon}^{n+1} = \begin{pmatrix} \epsilon_1^{n+1} \\ \epsilon_2^{n+1} \\ \vdots \\ \epsilon_m^{n+1} \end{pmatrix}.
\]

Then (3.1) can be written as:
\[
(A - \frac{1}{2} \lambda \Delta \tau \Delta X \cdot \frac{1}{2} B) \cdot \tilde{\nu}^{n+1} = \tilde{\nu}^n + \frac{1}{2} \lambda \Delta \tau \cdot \tilde{\epsilon}^{n+1}.
\]

In the subsequent section, numerical examples are implemented, where the values of the parameters of the models are calculated using real data available in the European carbon emission trading market.

4. Numerical experiments. Compared with other carbon emission trading markets, the European carbon emission trading market has a long history of development. Therefore, the trading system is relatively perfect. Asset prices are vulnerable to the impact of discrete random events in the capital market, and hence exhibiting...
jump phenomenon. The occurrence of this price fluctuation in the carbon prices, is due to the global financial crisis and the European debt crisis in recent years. Jump diffusion process is commonly used to study the jump in the carbon market. For Example, jump diffusion process was introduced to analyze spot sequence of carbon emissions with random walk (Daskalakis et al. 2009) [9]. Examples of such incidents that have taken place in around the time of the detected events: the 2008 global financial crisis; the September 2008 stock market crash; the 2008 and the 2012 crude oil price shocks; the 2010 earthquake in China; the 2010 Mexican oil spill; the 2011 Libyan war; the 2011 debt default risk of the US and Europe; the 2015 oil price decline, etc (Behmiri and Manera, 2015) [2]. The existence of jumps in the emission price index has been investigated in (Dutta 2018) [11]. The negative coefficient of the jump mean implies that the jump behavior driven by abnormal information has a negative impact on return, whereas the positive coefficient of the jump variation specifies that volatility driven by abnormal information has a positive effect on volatility of EUA return. From the 2012 to 2018 carbon price chart, we can see that after a brief rise in prices for several months, the price begins to decline, showing a downward trend. The brief rise can be seen as a jump phenomenon. EUA price is very susceptible to the influence of discrete stochastic events outside, causing jumps which can be positive or negative. In 2005, the EU has introduced the emission trading system (ETS) in order to reduce the emissions of greenhouse gas (GHG). Since then, the EU-ETS has emerged as the largest international emission allowance market representing 84 percent of the global carbon market value (Tian et al., 2016) [33]. We collect the data of the carbon price from 2012 to 2018 (Figure 2). Then, we let $S_i$ be the observations of $S(t)$ at time $t = t_i$, and use $t_i = i \Delta t$ ($i = 0, 1, 2, ..., n$) to discretize Eq. (2.1).

$$S_i \approx S_{i-1} + (r - \lambda k)S_{i-1} \Delta t + \sigma S_{i-1} dW_t + S_{i-1} Udq_t. \quad (13)$$

Modify Eq. (4.1):

$$\frac{S_i - S_{i-1}}{S_{i-1}} - Udq_t = (r - \lambda k)\Delta t + \sigma dW_t. \quad (14)$$

The likelihood function is obtained using Eq. (4.2). Then, the data of carbon price are inputed into the likelihood function, and its maximum value is solved using R software programming to obtain the values of the parameters (Table 1).

| Parameters | Values | Parameters | Values |
|------------|--------|------------|--------|
| $r$        | 0.019  | $\delta$  | 0.5    |
| $\sigma$   | 0.106  | $\mu$     | -0.06  |
| $\lambda$  | 0.04   |            |        |

We calculate the value of interest rate by using the values of the parameters shown in Table 1. Figure 3 shows the interest rate of the green bonds. The values of the interest rate are different in the two contrasting charts illustrated in Figure 3. The value of the interest rate increases when the carbon price is set to the upper barrier. Thus, the issuers of green bonds pay a higher interest rate when carbon price increases. This situation may compel issuers to exert more effort in developing green energy projects. The value of the interest rate declines when the
carbon price is set to the lower barrier. This finding implies that issuers pay a lower interest rate when carbon price decreases. Furthermore, this situation impels the issuers to make more contributions to green finance. As shown in Figure 4, the value of the interest rate changes when the barriers are adjusted, but the value of the non-barrier elements is not affected. Comparing Figure 3 and Figure 4, we can see that the interest rate increases when the upper barrier increases. Thus, the design of the barrier for interest rate can promote energy conservation and emission reduction, leading to sustainable development. Figure 5 shows that interest rate increases when risk-free interest rate \( r \) increases. The increase in risk-free interest rate may attract more investors to invest in risk-free assets. Thus, the issuers of green bonds have to increase the interest rate to draw in investors. interest rate increases when the parameter \( \sigma \) increases (where \( \sigma \) denotes the volatility of carbon
price). The higher the volatility of carbon price is, the higher the uncertainty of green bonds is. Figure 6 shows the effects of parameter $\lambda$ on the green bond interest rate. Parameter $\lambda$ reflects the intensity when a jump occurs. The carbon price also jumps when the market of carbon price is influenced by factors, such as economic and political factors. A high frequency of jumps will result in the loss of confidence among investors in the green bond market.

5. Conclusion. The growth of green bonds affects the green financial sustainable development, the design of green bonds is directly related to the issuance of green bonds. The interest rate is an important component of the green bonds, and the high or low interest rate will influence the future earnings of issuers and allure for investors. The aim of this paper is to design a new type of green bonds, where the main concern is on the floating rate of the bonds. In this study, we assume that the floating rate of green bonds is linked to carbon price. In addition, we introduce two boundaries (the upper and lower barriers) for carbon price. The upper barrier leads to the increase in interest rates and may compel issuers to focus on their carbon emission. The lower barrier will encourage issuers to make more contributions to green finance.
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Figure 4. The green bond interest rates when the barrier changes in non barriers and barriers.
Figure 5. The effects of parameters $r$ and $\sigma$ on the green bond interest rate.

Figure 6. The effects of parameters $\lambda$ on the green bond interest rate.
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