On the Fay identity for KdV tau functions and the identity for the Wronskian of squared solutions of Sturm-Liouville equation

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Abstract. We show that the well known identity for the Wronskian of squared solutions of a Sturm-Liouville equation follows from the Fay identity. We also study some odd-order \((2^n - 1)\)-order, \(n = 2, 3, \ldots\) identities which are specific for tau functions, related to the KdV hierarchy.

I. Introduction.

We began this research as a study of the expression of the Wronskian of squared solutions of Sturm-Liouville equation by KdV tau functions and Fay identity. Now, when the desired result is obtained (cf. Theorem 1.1), we realize that this is a story of the common origin of the following three relations for the functions: \(x, \sin(x), \theta_{11}(x)\) respectively \((x, z_1, z_2 \in \mathbb{C})\):

\[
(z_2 - z_1) \left[ (x + z_1 + z_2)(x - z_1)(x - z_2) - (x - z_1 - z_2)(x + z_1)(x + z_2) \right] =
\]

\[
(z_1 + z_2) \left[ (x + z_1 - z_2)(x - z_1)(x + z_2) - (x - z_1 + z_2)(x + z_1)(x - z_2) \right],
\]

\[
\sin(z_2 - z_1) \left[ \sin(x + z_1 + z_2) \sin(x - z_1) \sin(x - z_2) - \
\sin(x - z_1 - z_2) \sin(x + z_1) \sin(x + z_2) \right] =
\]

\[
\sin(z_1 + z_2) \left[ \sin(x + z_1 - z_2) \sin(x - z_1) \sin(x + z_2) - \
\sin(x - z_1 + z_2) \sin(x + z_1) \sin(x - z_2) \right],
\]
\[
\theta_{11}(z_2 - z_1)\left[\theta_{11}(x + z_1 + z_2)\theta_{11}(x - z_1)\theta_{11}(x - z_2) - \right.
\theta_{11}(x - z_1 - z_2)\theta_{11}(x + z_1)\theta_{11}(x + z_2)\left.\right] = \\
\theta_{11}(z_1 + z_2)\left[\theta_{11}(x + z_1 - z_2)\theta_{11}(x - z_1)\theta_{11}(x + z_2) - \right.
\theta_{11}(x - z_1 + z_2)\theta_{11}(x + z_1)\theta_{11}(x - z_2)\left.\right]
\]

(we use the notations for theta functions from Ref. 1.

In the present paper we will prove an identity for general KdV tau functions (it will be a third-order identity for tau). The mentioned three types of functions are roughly speaking three types of KdV tau functions for stationary (t-independent) solutions of the KdV equation: \(u_t = 6uu_x + u_{xxx}\). So, the polynomial relation follows from this cubic identity for KdV taus, but there are some specific problems to translate the identity for the general KdV taus to the cases of trigonometric functions and elliptic theta functions. We will postpone the solution of these problems to some next publication.

This way, the status of the three relations is quite different: the first one is easy to prove directly (and it also follows from the cubic identity for KdV taus); the second one is not difficult to prove directly, using the well known trigonometric identities; and the third one is still conjectural (we could not derive it from the Riemann relations, we could only check it numerically, using the system Mathematica 3.0 at RIMS, Kyoto University).

Let \(\tau(t), t \equiv (t_1, t_2, t_3, \ldots) \in \mathbb{C}^\infty\), \(t_1 \equiv x\) is an arbitrary tau function, related to the Kadomtsev-Petviashvili (KP) hierarchy Ref. 2. Let us denote \((z \in \mathbb{C})\):

\[\tau(t + [z]) = \tau(t_1 + z, t_2 + z^2/2, t_3 + z^3/3, \ldots).\]

The following identity \((z_0, z_1, z_2, z_3 \in \mathbb{C})\):

\[
(z_0 - z_1)(z_2 - z_3)\tau(t + [z_0] + [z_1])\tau(t + [z_2] + [z_3]) + \\
(z_0 - z_2)(z_3 - z_1)\tau(t + [z_0] + [z_2])\tau(t + [z_3] + [z_1]) + \\
(z_0 - z_3)(z_1 - z_2)\tau(t + [z_0] + [z_3])\tau(t + [z_1] + [z_2]) = 0
\]

is called \textit{Fay identity} Ref. 3 for the KP tau function \(\tau\). It was firstly obtained Ref. 4 for theta functions related to Jacobians. In genus \(g = 1\) case its form is:

\[
\theta_{11}(z_0 - z_1)\theta_{11}(z_2 - z_3)\theta_{11}(t + z_0 + z_1)\theta_{11}(t + z_2 + z_3) + \\
\theta_{11}(z_0 - z_2)\theta_{11}(z_3 - z_1)\theta_{11}(t + z_0 + z_2)\theta_{11}(t + z_3 + z_1) + \\
\theta_{11}(z_0 - z_3)\theta_{11}(z_1 - z_2)\theta_{11}(t + z_0 + z_3)\theta_{11}(t + z_1 + z_2) = 0
\]

Afterwards it was used Ref. 1 in geometric treatment of soliton equations. Let it was generalized for tau functions Ref. 3.

Fay identity is fulfill also for tau functions related to \(n\) - th (\(n = 2, 3, 4, \ldots\)) Gel'fand-Dickey reduction of KP hierarchy. In the present paper we will consider...
only the \( n = 2 \) reduction, i.e. the KdV hierarchy. Such tau functions we will call KdV tau functions. They can be characterized by the conditions \( \partial_{x_k} \tau(t) = 0, \quad k = 1, 2, 3, \ldots \) which imply for every \( z \in \mathbb{C} \):

\[
\tau(t - |z|) = \tau(t + [z]).
\]

There are two main goals in the present article. The first aim is to show that the famous identity for the Wronskian \( W(f, g) := fg' - f'g \), \( \equiv \partial_x \equiv \frac{\partial}{\partial x} \) of squared solutions of Sturm-Liouville equation Ref. 5 follows from the Fay identity for KdV tau functions. The second aim is to obtain some specific relations for the KdV tau functions.

We came to these results when studying the problem to find a dictionary between the tau functions and some formulas related to squared solutions of Sturm-Liouville equation (especially the mentioned identity for the Wronskian of squared solutions-an important ingredient of this area Ref. 5, Ref. 6. Such dictionary will be useful in examining some features of Miura transformations. It is well known that squared solutions span the kernels of the Frechet derivatives \( M_{\pm}(v) = 2v \pm \partial_x \) of Miura transformations \( u_{\pm} = M_{\pm}v := v^2 \pm vx, \quad (vx \equiv \partial_x v) \), where \( v \) is a solution of the mKdV equation and \( u_{\pm} \) are solutions of the KdV equation. It is also well known Ref. 7 the interpretation of Miura transformations as projections from flag to corresponding subspace (in Sato Grassmannian), which is intimately connected to tau functions. Some parts of the dictionary were known: e.g. a formula, which expresses the squared solutions by means of \( \tau(t) \) and vertex operators Ref. 2 (in this paper the so-called Λ-operators Ref. 5 are also mentioned). There were no relations to the tau functions of the identity for the Wronskian of squared solutions, but it was well known an expression of the Wronskian of two solutions by means of \( \tau(t) \) Ref. 2, Ref. 8. Because of the fact that in the proof of the latest formula, the Fay identity was used, we expected that the same identity will be useful in the “paraphrase” of the Wronskian of squared solutions.

We need such dictionary, because we observed some similarities between Matsuo and Cherednik transformations (Ref. 9):

\[
\text{Knizhnik-Zamolodchikov equation} \rightarrow \text{quantum Calogero-Sutherland system}
\]

on the one hand and Miura transformation on the other hand. Our oppinion is that such similarities will be easier explained on the language of tau functions, flag and Grassmann manifolds, etc.. So, the “paraphrase” of the relations for the Wronskian of squared solutions of Sturm-Liouville equation is only the first step in this direction. We also expect that the presented connections between squared solutions and tau functions will be useful in another areas of the subject (cf. Ref.10, Ref. 11).

In order to explain the main results of the present article, let us remind some notations Ref. 2, Ref. 8. Let \( \psi(x, z) \) and \( \psi^*(x, z) \) are two linearly independent solutions (cf. Section II) of the Sturm-Liouville equation :

\[
(\partial_x^2 + u(x)) \psi(x, z) = z^2 \psi(x, z).
\]
Then the following relations:
\[ W(\psi(x, z_1)\psi^*(x, z_1), \psi(x, z_2)\psi^*(x, z_2)) = \]
\[-(z_1^2 - z_2^2)^{-1}\partial_x \left[ W(\psi(x, z_1), \psi(x, z_2))W(\psi^*(x, z_1), \psi^*(x, z_2)) \right] = \]
\[(1.4) \]
\[-(z_1^2 - z_2^2)^{-1}\partial_x \left[ W(\psi(x, z_1), \psi^*(x, z_2))W(\psi^*(x, z_1), \psi(x, z_2)) \right], \]

\((z_1, z_2 \in \mathbb{C})\) we will call Faddeev-Tahtajan identity. This relation has a long history. It was used in the theory of inverse spectral problems for the Sturm-Liouville operators. Afterwards Faddeev-Tahtajan identity played an important role in the first years of Soliton Theory. In Ref. 5 the origin of the identity is interpreted in terms of classical \(\tau\)-matrixes. Here we will explain the origin of Faddeev-Tahtajan identity using the language of tau functions.

The first main result in this paper is given in the following

**Theorem 1.1.** The Faddeev-Tahtajan identity (1.4) follows from the Fay identity (1.1) for KdV tau functions.

The second main result in the present article is given in the following

**Theorem 1.2.** Let \(\tau(t), t \in \mathbb{C}^\infty\) is an arbitrary KdV tau function. Then:

(i) for every \(z_1, z_2 \in \mathbb{C}\):
\[ (z_2 - z_1) \left[ \tau(t + z_1 + [z_2])\tau(t - [z_1])\tau(t - [z_2]) - \tau(t - [z_1])\tau(t + [z_1])\tau(t + [z_2]) \right] = \]
\[ (z_2 + z_1) \left[ \tau(t + [z_1] - [z_2])\tau(t - [z_1])\tau(t + [z_2]) - \tau(t - [z_1] + [z_2])\tau(t + [z_1])\tau(t - [z_2]) \right] ; \]

(ii) for every \(z \in \mathbb{C}\):
\[ \tau(t + 2[z])\tau^2(t - [z]) - \tau(t - 2[z])\tau^2(t + [z]) = \]
\[ 2 \sum_{k=0}^{\infty} z^{2k+1} \left[ \tau(t - [z]) W_{2k+1}(\tau(t), \tau(t + [z])) + \tau(t + [z]) W_{2k+1}(\tau(t), \tau(t - [z])) \right], \]
where we denote: \(W_{2k+1}(f, g) := f(\partial_{t_{2k+1}} g) - (\partial_{t_{2k+1}} f)g, k = 0, 1, 2, \ldots \).

**Remark 1.3** Let us mention that the identities from Theorem 1.2 are cubic in \(\tau\) relations (in contrary to the Fay identity, which is quadratic in \(\tau\) relation) and they are specific only for the KdV tau functions.

**Remark 1.4** The proof of Theorem 1.2 is based only on the following three facts:

(i) The Fay identity (1.1) (which is common for all tau functions),
(ii) The relation (1.2) (which is specific only for KdV tau functions),
(iii) The obvious identity for Wronskians:
\[ W(f_1f_2, g_1g_2) = f_1g_1W(f_2, g_2) + f_2g_2W(f_1, g_1) = \]
\[ f_1g_2W(f_2, g_1) + f_2g_1W(f_1, g_2). \]

The paper consists of four sections. In Section II we give some preliminary results. The proofs of the Theorem 1.1 and Theorem 1.2 are given in Section III. In Section IV we give some examples and comments of the main statements. A preliminary (and from different viewpoint) version of some of the results is presented in Ref. 12 and Ref. 13.
II. Preliminary results.
Firstly, let us mention some obvious relations for Wronskians.

Lemma 2.1.

(i) \[ W(e^{z_1 x} f, e^{z_2 x} g) = e^{(z_1 + z_2) x} \left[ W(f, g) - (z_1 - z_2) fg \right], \]

(ii) \[ W\left( \frac{f_1}{g}, \frac{f_2}{g} \right) = \frac{W(f_1, f_2)}{g^2}, \]

(iii) \[ \partial_x \left( \frac{f_1 f_2}{g^2} \right) = - \frac{f_1 W(f_2, g) + f_2 W(f_1, g)}{g^3}. \]

Instead of Fay identity (1.1) we will use the differential Fay identity Ref. 8 (\(z_1, z_2 \in \mathbb{C}\)):

\[
W(\tau(t + [z_1]), \tau(t + [z_2])) = (z_2^{-1} - z_1^{-1}) \left[ \tau(t + [z_1]) \tau(t + [z_2]) - \tau(t) \tau(t + [z_1] + [z_2]) \right].
\]

(2.1)

Shifting the argument \( t \) respectively to \((t - [z_1] - [z_2])\), \((t - [z_2])\) and \((t - [z_1])\) we could obtain expressions respectively for the following Wronskians:

\[ W(\tau(t - [z_1]), \tau(t - [z_2])), \quad W(\tau(t + [z_1] - [z_2]), \tau(t)), \quad W(\tau(t - [z_1] + [z_2]), \tau(t)). \]

But, shifting \( t \) we cannot obtain an expression e.g. for the Wronskian:

\[ W(\tau(t + [z_1]), \tau(t - [z_2])). \]

This is possible for KdV tau functions. Using (2.1) and (1.2), it is easy to see that:

\[
W(\tau(t + [z_1]), \tau(t - [z_2])) = -(z_2^{-1} + z_1^{-1}) \left[ \tau(t + [z_1]) \tau(t - [z_2]) - \tau(t) \tau(t + [z_1] - [z_2]) \right].
\]

This way we obtain the following expressions for the Wronskians of KdV tau functions.

Lemma 2.2.
Let \( \tau(t) \) be an arbitrary KdV tau function. Then we have:

\[
\begin{align*}
(i) \quad W(\tau(t + [z_1]), \tau(t + [z_2])) &= (z_2^{-1} - z_1^{-1})[\tau(t + [z_1])\tau(t + [z_2]) - \\
&- \tau(t)\tau(t + [z_1] + [z_2])], \\
W(\tau(t - [z_1]), \tau(t - [z_2])) &= -(z_2^{-1} - z_1^{-1})[\tau(t - [z_1])\tau(t - [z_2]) - \\
&- \tau(t)\tau(t - [z_1] - [z_2])], \\
W(\tau(t - [z_1]), \tau(t + [z_2])) &= (z_2^{-1} + z_1^{-1})[\tau(t - [z_1])\tau(t + [z_2]) - \\
&- \tau(t)\tau(t - [z_1] + [z_2])], \\
W(\tau(t + [z_1]), \tau(t - [z_2])) &= -(z_2^{-1} + z_1^{-1})[\tau(t + [z_1])\tau(t - [z_2]) - \\
&- \tau(t)\tau(t + [z_1] - [z_2])]; \\
(ii) \quad W(\tau(t + [z_1] - [z_2]), \tau(t)) &= (z_2^{-1} - z_1^{-1})[\tau(t + [z_1] - [z_2])\tau(t) - \\
&- \tau(t + [z_1])\tau(t - [z_2])] \\
W(\tau(t - [z_1] + [z_2]), \tau(t)) &= -(z_2^{-1} - z_1^{-1})[\tau(t - [z_1] + [z_2])\tau(t) - \\
&- \tau(t - [z_1])\tau(t + [z_2])], \\
W(\tau(t - [z_1] - [z_2]), \tau(t)) &= (z_2^{-1} + z_1^{-1})[\tau(t - [z_1] - [z_2])\tau(t) - \\
&- \tau(t - [z_1])\tau(t - [z_2])], \\
W(\tau(t + [z_1] + [z_2]), \tau(t)) &= -(z_2^{-1} + z_1^{-1})[\tau(t + [z_1] + [z_2])\tau(t) - \\
&- \tau(t + [z_1])\tau(t + [z_2])].
\end{align*}
\]

Let us define the wave functions \( \psi(t,z) \), \( \psi^*(t,z) \) Ref. 8 by expressions:

\[
\begin{align*}
\psi(t,z) &= \exp\left(\sum_{k=1}^{\infty} t_k z^k\right) \frac{\tau(t - [z^{-1}])}{\tau(t)}, \\
\psi^*(t,z) &= \exp\left(-\sum_{k=1}^{\infty} t_k z^k\right) \frac{\tau(t + [z^{-1}])}{\tau(t)}.
\end{align*}
\]

For an arbitrary KdV tau function \( \tau(t) \), denoting \( u(t) := 2\partial_t^2 \ln \tau(t) \), it is well known Ref. 8 that the wave functions \( \psi(t,z) \) and \( \psi^*(t,z) \) satisfy the Sturm-Liouville equation (1.3) \( (t_1 = x \text{ and } t_3, t_5, \ldots \text{ are parameters}) \). Using the relations of Lemma 2.2 we can explain the Wronskians of the wave functions \( \psi(t,z) \) and \( \psi^*(t,z) \) in terms of the tau-function \( \tau(t) \).

**Lemma 2.3.**

Let \( \tau(t) \) be an arbitrary KdV tau function and \( \psi(t,z) \), \( \psi^*(t,z) \) are the corresponding wave functions. Then we have \( (z_1, z_2 \in \mathbb{C}) \):
Then we have:

(i) \( W(\psi(t, z_1), \psi(t, z_2)) = (z_1 - z_2) \exp\left( \sum_{k=0}^{\infty} t_{2k+1}(z_1^{2k+1} + z_2^{2k+1}) \right) \frac{\tau(t - [z_1^{-1}] - [z_2^{-1}])}{\tau(t)} \),

(ii) \( W(\psi^*(t, z_1), \psi^*(t, z_2)) = -(z_1 - z_2) \exp\left( -\sum_{k=0}^{\infty} t_{2k+1}(z_1^{2k+1} + z_2^{2k+1}) \right) \frac{\tau(t + [z_1^{-1}] + [z_2^{-1}])}{\tau(t)} \),

(iii) \( W(\psi(t, z_1), \psi^*(t, z_2)) = (z_1 + z_2) \exp\left( \sum_{k=0}^{\infty} t_{2k+1}(z_1^{2k+1} - z_2^{2k+1}) \right) \frac{\tau(t - [z_1^{-1}] + [z_2^{-1}])}{\tau(t)} \),

(iv) \( W(\psi^*(t, z_1), \psi(t, z_2)) = -(z_1 + z_2) \exp\left( -\sum_{k=0}^{\infty} t_{2k+1}(z_1^{2k+1} - z_2^{2k+1}) \right) \frac{\tau(t + [z_1^{-1}] - [z_2^{-1}])}{\tau(t)} \).

**Proof:** Let us denote the functions:

\[ \varphi(t, z) := e^{xz} \frac{\tau(t - [z^{-1}])}{\tau(t)}, \quad \varphi^*(t, z) := e^{xz} \frac{\tau(t + [z^{-1}])}{\tau(t)}. \]

Then we have:

\[ \psi(t, z) = \exp\left( \sum_{k=1}^{\infty} t_{2k+1}z^{2k+1} \right) \varphi(t, z), \quad \psi^*(t, z) = \exp\left( -\sum_{k=1}^{\infty} t_{2k+1}z^{2k+1} \right) \varphi^*(t, z), \]

and consequently we have:

\[ W(\psi(t, z_1), \psi(t, z_2)) = \exp\left( \sum_{k=1}^{\infty} t_{2k+1}(z_1^{2k+1} + z_2^{2k+1}) \right) W(\varphi(t, z_1), \varphi(t, z_2)), \quad \text{etc.} \]

Using the relations of Lemma 2.1 and Lemma 2.2 we obtain:

\[ W(\varphi(t, z_1), \varphi(t, z_2)) = \]

\[ W\left( e^{z_1x} \frac{\tau(t - [z_1^{-1}])}{\tau(t)}, e^{z_2x} \frac{\tau(t - [z_2^{-1}])}{\tau(t)} \right) = \]

\[ e^{(z_1 + z_2)x} \left[ W\left( \frac{\tau(t - [z_1^{-1}])}{\tau(t)}, \frac{\tau(t - [z_2^{-1}])}{\tau(t)} \right) - (z_1 - z_2) \frac{\tau(t - [z_1^{-1}])\tau(t - [z_2^{-1}])}{\tau^2(t)} \right] = \]

etc.
\[
\begin{align*}
&\quad e^{(z_1 + z_2)t} \left[ \frac{W \left( \tau(t + [z_1^{-1}] \), \tau(t + [z_2^{-1}]) \right)}{\tau^2(t)} - (z_1 - z_2) \frac{\tau(t - [z_1^{-1}]) \tau(t - [z_2^{-1}])}{\tau^2(t)} \right] = \\
&= e^{(z_1 + z_2)t} \left[ (z_1 - z_2) \left( \tau(t - [z_1^{-1}]) \tau(t - [z_2^{-1}]) - \tau(t) \tau(t - [z_1^{-1}] - [z_2^{-1}]) \right) - \\
&\quad (z_1 - z_2) \tau(t - [z_1^{-1}]) \tau(t - [z_2^{-1}]) \right] = \\
&\quad (z_1 - z_2) e^{\tau(z_1 + z_2)} \frac{\tau(t - [z_1^{-1}] - [z_2^{-1}])}{\tau(t)}.
\end{align*}
\]

From here follows (i), because we have \( t_1 \equiv x \):
\[
e^{\tau(z_1 + z_2)} \exp \left( \sum_{k=1}^{\infty} t_{2k+1} (z_1^{2k+1} + z_2^{2k+1}) \right) = \exp \left( \sum_{k=0}^{\infty} t_{2k+1} (z_1^{2k+1} + z_2^{2k+1}) \right).
\]

It is easy to prove (ii), (iii) and (iv) in the same way. \( \square \)

III. Proof of the main results.

Proof of Theorem 1.2.

First we will prove the identity (i). Using the identities of Lemma 2.2 and (1.5), let us expand the following Wronskian:
\[
W \left( \tau(t + [z_1]) \tau(t - [z_1]), \tau(t + [z_2]) \tau(t - [z_2]) \right)
\]
in two different ways. From the first line of (1.5) we obtain:
\[
(z_2^{-1} - z_1^{-1}) \tau(t) \left[ \tau(t - [z_1] - [z_2]) \tau(t + [z_1]) \tau(t + [z_2]) - \\
\tau(t + [z_1] + [z_2]) \tau(t - [z_1]) \tau(t - [z_2]) \right],
\]
and from the second line of (1.5) we obtain:
\[
(z_2^{-1} + z_1^{-1}) \tau(t) \left[ \tau(t + [z_1] - [z_2]) \tau(t - [z_1]) \tau(t + [z_2]) - \\
\tau(t - [z_1] + [z_2]) \tau(t + [z_1]) \tau(t - [z_2]) \right].
\]

But \((z_2^{-1} - z_1^{-1}) = (z_1 - z_2)/z_1 z_2\) and \((z_2^{-1} + z_1^{-1}) = (z_1 + z_2)/z_1 z_2\), so we have:
\[
\frac{\tau(t)}{z_1 z_2} \left[ \text{l.h.s. of (i)} \right] = \frac{\tau(t)}{z_1 z_2} \left[ \text{r.h.s. of (i)} \right].
\]

The proof of the first identity of Theorem 1.2 is completed.

Now we will obtain the second identity (ii) of Theorem 1.2 letting \( z_2 \) to tend to \( z_1 \) in the first identity (i) (we will denote \( z_1 = z_2 = z \)). The l.h.s. of (ii) is clear.
In order to obtain the r.h.s. of (ii) we mention that:
\[
\partial_{z_2} \left( \tau(t + [z_1] - [z_2]) \right) \bigg|_{z_2 = z_1 = z} = \\
\partial_{z_2} \left[ \tau \left( (x + z_1) - z_2, (t_3 + \frac{z_1^3}{3}) - \frac{z_2^3}{3}, (t_5 + \frac{z_1^5}{5}) - \frac{z_2^5}{5}, \ldots \right) \right] \bigg|_{z_2 = z_1 = z} = \\
- \sum_{k=0}^{\infty} \varepsilon^{2k} \partial_{z_2} \tau(t).
\]
The same way we obtain:

\[ \partial_{z_2} (\tau(t - [z_1] + [z_2])) |_{z_2 = z_1} = \sum_{k=0}^{\infty} z^{2k} \partial_{t_{2k+1}} \tau(t), \]

\[ \partial_{z_2} (\tau(t + [z_2])) |_{z_2 = z} = \sum_{k=0}^{\infty} z^{2k} \partial_{t_{2k+1}} \tau(t + [z]), \]

\[ \partial_{z_2} (\tau(t - [z_2])) |_{z_2 = z} = -\sum_{k=0}^{\infty} z^{2k} \partial_{t_{2k+1}} \tau(t - [z]). \]

So, from the r.h.s. of (i) we obtain:

\[ \tau(t - [z]) \sum_{k=0}^{\infty} 2z^{2k+1} \left[ \tau(t) \partial_{t_{2k+1}} \tau(t + [z]) - \tau(t + [z]) \partial_{t_{2k+1}} \tau(t) \right] + \]

\[ \tau(t + [z]) \sum_{k=0}^{\infty} 2z^{2k+1} \left[ \tau(t) \partial_{t_{2k+1}} \tau(t - [z]) - \tau(t - [z]) \partial_{t_{2k+1}} \tau(t) \right], \]

which gives the r.h.s. of (ii).

**Proof of Theorem 1.1.**

On the one hand, using the expressions of the wave functions \( \psi(t, z) \) and \( \psi^*(t, z) \) in terms of tau function \( \tau(t) \) (in our case \( \tau \) is an arbitrary KdV tau function) we obtain from the first line of (1.4):

\[ W \equiv W[\psi(t, z_1)\psi^*(t, z_1), \psi(t, z_2)\psi^*(t, z_2)] = \]

\[ W \left( \frac{\tau(t + [z_1^{-1}])\tau(t - [z_1^{-1}])}{\tau^2(t)}, \frac{\tau(t + [z_2^{-1}])\tau(t - [z_2^{-1}])}{\tau^2(t)} \right) = \]

\[ \frac{1}{\tau^4(t)} W \left( \tau(t + [z_1^{-1}])\tau(t - [z_1^{-1}]), \tau(t + [z_2^{-1}])\tau(t - [z_2^{-1}]) \right). \]

From the proof of the identity (i) of the Theorem 1.2 we know that this equals either to:

\[ \frac{z_2 - z_1}{\tau^3(t)} \left[ \tau(t - [z_1^{-1}] - [z_2^{-1}])\tau(t + [z_1^{-1}])\tau(t + [z_2^{-1}]) - \tau(t + [z_1^{-1}] + [z_2^{-1}])\tau(t - [z_1^{-1}])\tau(t - [z_2^{-1}]) \right], \]

or to:

\[ \frac{z_2 + z_1}{\tau^3(t)} \left[ \tau(t + [z_1^{-1}] - [z_2^{-1}])\tau(t - [z_1^{-1}])\tau(t + [z_2^{-1}]) - \tau(t - [z_1^{-1}] + [z_2^{-1}])\tau(t + [z_1^{-1}])\tau(t - [z_2^{-1}]) \right]. \]
On the other hand, using the relations from Lemma 2.1, Lemma 2.2 and Lemma 2.3 we obtain from the second line of (1.4): 

\[ W_1 \equiv -(z_1^2 - z_2^2)^{-1} \partial_z \left[ W[\psi(t, z_1), \psi(t, z_2)] W[\psi^*(t, z_1), \psi^*(t, z_2)] \right] = (z_1 - z_2)^2 (z_1^2 - z_2^2)^{-1} \partial_z \left[ \frac{\tau(t - [z_1^{-1}] - [z_2^{-1}])\tau(t + [z_1^{-1}] + [z_2^{-1}])}{\tau^2(t)} \right] = \]

\[ -\frac{z_1 - z_2}{z_1 + z_2} \tau^{-3}(t) \left[ \frac{\tau(t - [z_1^{-1}] - [z_2^{-1}])W(\tau(t + [z_1^{-1}] + [z_2^{-1}]), \tau(t)) + \tau(t + [z_1^{-1}] + [z_2^{-1}])W(\tau(t - [z_1^{-1}] - [z_2^{-1}]), \tau(t))}{\tau(t - [z_1^{-1}] - [z_2^{-1}])}\right]. \]

and for the third line of (1.4): 

\[ W_2 \equiv -(z_1^2 - z_2^2)^{-1} \partial_z \left[ W[\psi(t, z_1), \psi^*(t, z_2)] W[\psi^*(t, z_1), \psi(t, z_2)] \right] = (z_1 + z_2)^2 (z_1^2 - z_2^2)^{-1} \partial_z \left[ \frac{\tau(t - [z_1^{-1}] + [z_2^{-1}])\tau(t + [z_1^{-1}] - [z_2^{-1}])}{\tau^2(t)} \right] = \]

\[ -\frac{z_1 + z_2}{z_1 - z_2} \tau^{-3}(t) \left[ \frac{\tau(t - [z_1^{-1}] + [z_2^{-1}])W(\tau(t + [z_1^{-1}] - [z_2^{-1}]), \tau(t)) + \tau(t + [z_1^{-1}] - [z_2^{-1}])W(\tau(t - [z_1^{-1}] + [z_2^{-1}]), \tau(t))}{\tau(t - [z_1^{-1}] + [z_2^{-1}])}\right]. \]

This way we obtain that \( W \) equals either to \( W_1 \) or to \( W_2 \), i.e. the Faddeev-Tahtajan identity is fulfilled.
IV. Conclusion remarks and examples.

Firstly we illustrate the identities from Theorem 1.2 by examples with polynomial KdV tau functions. The author thanks F.A. Grünbaum for suggestions to include these examples in the body of the paper.

Example 4.1 The first nontrivial polynomial KdV tau function is \( \tau_1(t) := t_1 \).
In this case the examination of the identities (i) and (ii) of Theorem 1. is easy to do directly and the result is: the both sides of (i) are equal to 2\( z_1^3 - 3z_2 \), the both sides of (ii) are equal to 4\( z_2^3 \).

Example 4.2 The next polynomial KdV tau function is of degree 3: \( \tau_3(t) := t_1^3 - 3t_3 \) and as is clear from the results, the examination of the identities (i) and (ii) of Theorem 1. in this case is difficult to do directly. We used the system Maple V Release 4 at RIMS, Kyoto University. So, the both sides of (i) are equal to:

\[
6(z_1z_2^3 - z_1^3z_2)t_1^6 + 36(z_1^3z_2^2 - z_1z_2^5)t_1^4 + 126(z_1z_2^3 - z_1^3z_2)t_1^3t_3 + \\
54(z_1^5z_2^2 - z_1z_2^5)t_1^4 + 54(z_1^7z_2 - z_1z_2^5)t_1t_3 + 54(z_1z_2^3 - z_1^3z_2)t_3^2,
\]

and the both sides of (ii) are equal to:

\[
12z_1^3t_1^6 - 144z_1^5t_1^4 + 252z_1^3z_2^3t_1^3 + 108z_1^7t_1^2 - 216z_1^5t_1t_3 + 108z_1^3t_3^2.
\]

There were some problems with fixing the correct form of the KdV tau function \( \tau_3(t) \) - polynomial of the form \( t_1^3 - at_3 \). The function \( t_1^3 - at_3 \) satisfies the Fay identity (1.1) iff \( a = 3 \).

Remark 4.3 Applying the identities (1.5) to the Wronskian:

\[
W(\tau(t+[z_1])\tau(t-[z_1])\tau(t+[z_2])\tau(t-[z_2]), \tau(t+[z_3])\tau(t-[z_3])\tau(t-[z_2])\tau(t-[z_4])),
\]

(\( z_1, z_2, z_3, z_4 \in \mathbb{C} \)) we can obtain 8 different (equivalent) expressions where we have Wronskians of two tau functions only (i.e. without any Wronskian of products of tau functions). The expressions are separated in two groups and applying Lemma 2.2 we could see that the resulting identities among the expressions in each group are easily obtained using the result of Theorem 1.2 (i). The equality of the given below expressions (from the two groups) is a non-trivial seventh-order (specific for KdV tau functions only) identity:

\[
(z_4^{-1} - z_3^{-1})\tau(t+[z_1])\tau(t-[z_1])\tau(t+[z_2])\tau(t-[z_2]) + \\
[\tau(t+[z_3])\tau(t+[z_4])\tau(t-[z_3])\tau(t-[z_4]) - \tau(t-[z_1])\tau(t-[z_2])\tau(t+[z_3])\tau(t-[z_4])] + \\
(z_2^{-1} - z_1^{-1})\tau(t+[z_3])\tau(t-[z_3])\tau(t+[z_4])\tau(t-[z_4]) + \\
[\tau(t+[z_1])\tau(t+[z_2])\tau(t-[z_1])\tau(t-[z_2]) - \tau(t-[z_3])\tau(t-[z_2])\tau(t+[z_1])\tau(t+[z_2])] = \\
(z_2^{-1} - z_1^{-1})\tau(t+[z_1])\tau(t-[z_1])\tau(t+[z_4])\tau(t-[z_4]) + \\
[\tau(t+[z_2])\tau(t+[z_3])\tau(t-[z_2])\tau(t-[z_3]) - \tau(t-[z_2])\tau(t-[z_3])\tau(t+[z_2])\tau(t+[z_3])] + \\
(z_4^{-1} - z_3^{-1})\tau(t+[z_2])\tau(t-[z_2])\tau(t+[z_3])\tau(t-[z_3]) + \\
[\tau(t+[z_1])\tau(t+[z_4])\tau(t-[z_1])\tau(t-[z_4]) - \tau(t-[z_1])\tau(t-[z_4])\tau(t+[z_1])\tau(t+[z_4])].
\]

It is clear that this way we can obtain generalized identities of order \( 2^n - 1 \) for any \( n = 4, 5, \ldots \). The identities from Theorem 1.2 and Remark 4.3 correspond to the cases \( n = 2 \) and \( n = 3 \) respectively.
Example 4.4 For the first polynomial tau function $\tau_1(t) = t_1$ the both sides of the identity from Remark 4.3 are equal to:

$$2(-z_1^2 + z_2^2 + z_3^2 + z_4^2)t_1^4 + 4(z_1^2 z_2^2 - z_1^2 z_4^2) t_1^2 + 2(-z_1^2 z_2^2 z_3^2 + z_1^2 z_3^2 z_4^2 - z_1^2 z_3^2 z_4^2 + z_2^2 z_3^2 z_4^2).$$

For the next polynomial KdV tau function $\tau_3(t) = t_1^3 - 3t_3$ the both sides of this identity have too many terms (more than 250).

Remark 4.5 As we mentioned in the Introduction, there are some problems to translate the identity (i) from Theorem 1.2 to the cases when KdV tau function is expressed by trigonometric functions or elliptic theta functions. The problems come roughly speaking from the fact that in the original Fay identity (i.e. for theta functions related to Jacobians) is used the “Prime Form” (e.g. in the $g = 1$ case: $\theta_1(z_0 - z_1)(\theta_1'(0))^{-1}$), but in the Fay identity (1.1) for KP tau functions is used the difference $(z_0 - z_1)$ instead. Our next task is to fix these problems and to find “geometric” explanation of the identities from the present paper. It will be done in some next article.

Remark 4.6 The “elliptic version” of the identity (ii) from Theorem 1.2 is the following relation:

$$\theta_{11}'(0) \left[ \theta_{11}(x + 2z)\theta_{11}^2(x - z) - \theta_{11}(x - 2z)\theta_{11}^2(x + z) \right] =$$

$$\theta_{11}(2z) \left[ \theta_{11}(x - z)W(\theta_{11}(x), \theta_{11}(x + z)) + \theta_{11}(x + z)W(\theta_{11}(x), \theta_{11}(x - z)) \right]$$

It is easily obtained from the elliptic version of the identity (i) from Theorem 1.2 (cf. the Introduction) letting $z_1 \to z_2$ and denoting $z_1 = z_2 \equiv z$.

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