A timing formula for main-sequence star binary pulsars

Norbert Wex

Max-Planck-Institut für Radioastronomie, Auf dem Hügel 69, D-53121 Bonn, Germany

ABSTRACT
In binary radio pulsars with a main-sequence star companion, the spin-induced quadrupole moment of the companion gives rise to a precession of the binary orbit. As a first approximation one can model the secular evolution caused by this classical spin-orbit coupling by linear-in-time changes of the longitude of periastron and the projected semi-major axis of the pulsar orbit. This simple representation of the precession of the orbit neglects two important aspects of the orbital dynamics of a binary pulsar with an oblate companion. First, the quasiperiodic effects along the orbit, due to the anisotropic \(1/r^3\) nature of the quadrupole potential. Secondly, the long-term secular evolution of the binary orbit which leads to an evolution of the longitude of periastron and the projected semi-major axis which is non-linear in time.

In this paper a simple timing formula for binary radio pulsars with a main-sequence star companion is presented which models the short-term secular and most of the short-term periodic effects caused by the classical spin-orbit coupling. I also give extensions of the timing formula which account for long-term secular changes in the binary pulsar motion. It is shown that the short-term periodic effects are important for the timing observations of the binary pulsar PSR B1259–63. The long-term secular effects are likely to become important in the next few years of timing observations of the binary pulsar PSR J0045–7319. They could help to restrict or even determine the moments of inertia of the companion star and thus probe its internal structure.

Finally, I reinvestigate the spin-orbit precession of the binary pulsar PSR J0045–7319 since the analysis given in the literature is based on an incorrect expression for the precession of the longitude of periastron. A lower limit of 20° for the inclination of the B star with respect to the orbital plane is derived.

Key words: pulsar timing – binary pulsars – classical spin-orbit coupling – pulsars: individual: PSR J0045–7319, PSR B1259–63

1 INTRODUCTION
Timing observations of pulsars, i.e. the measurement of the time-of-arrival (TOA) of pulsar signals at a radio telescope, is one of the few high-precision experiments in astronomy and therefore has a wide-ranging field of interesting applications (Bell 1996). The first evidence for the existence of gravitational waves as predicted by Einstein’s theory of gravity (Taylor & Weisberg 1989) and the first discovery of extrasolar planets (Wolszczan & Frail 1992) are just the most striking examples for the achievements of high-precision pulsar-timing observations. Approximately 10% of the known pulsars are members of binary star system, i.e. in orbit around a white dwarf, neutron star or main-sequence star companion. Timing observations of these binary pulsars is a powerful tool to study various physical and astrophysical effects related to binary star motion and stellar evolution. To extract the maximum possible information from pulsar timing observations, one needs an appropriate model (timing formula) for transforming each measured topocentric TOA, \(t_{\text{obs}}\), to the corresponding time of emission, \(T\), measured in the reference frame of the pulsar. Various timing formulae, particularly for relativistic binary pulsars, have been developed to describe radio-pulsar timing observations.

With the discovery of PSR B1259–63 during a high frequency survey of the Galactic plane by Johnston et al. (1992) the first radio pulsar with a massive, non-degenerate companion was found. PSR B1259–63 is a 48-ms pulsar in a highly eccentric orbit with the main-sequence Be star SS 2883. The second known radio pulsar with a massive, non-degenerate companion is PSR J0045–7319, discovered in a systematic search of the Magellanic Clouds for radio pulsars (McConnell et al. 1991, Kaspi et al. 1994). Some of the parameters of these two main-sequence star binary pulsars are listed in Table 1. For both binary systems significant deviations from a Keplerian orbit have been detected which are
Table 1. Parameters of the main-sequence star radio binary pulsars (Johnston et al. 1994, Kaspi et al. 1994, Bell et al. 1995). $P_b$, $x$, $e$, $\omega$, $i$, $m_*$ and $R_*$ denote respectively the orbital period, the projected semi-major axis, the eccentricity of the orbit, the longitude of periastron, the inclination of the orbit with respect to the line-of-sight, the mass of the companion main-sequence star and its radius.

|               | B1259–63 | J0045–7319 |
|---------------|----------|------------|
| $P_b$ (days)  | 1237     | 51.17      |
| $x$ (sec)     | 1296     | 174.3      |
| $e$           | 0.870    | 0.808      |
| $\omega$ (deg) | 138.7    | 115.3      |
| $i$ (deg)     | 36 or 144 | 44 or 136  |
| $m_*$ ($M_\odot$) | $\sim$ 10 | $8.8 \pm 1.8$ |
| $R_*$ ($R_\odot$) | $\sim$ 6 | $6.4 \pm 0.7$ |

Most easily explained by classical spin-orbit coupling (Lai et al. 1995, Kaspi et al. 1996, Wex et al. 1997). Due to their high proper rotation the main-sequence star companions of PSRs B1259–63 and J0045–7319 show rotational deformation and thus give rise to a gravitational quadrupole field. As a result of this a coupling between the orbital angular momentum and the spin of the companion takes place and leads to a precession of the binary orbit.

In this paper I will shown that the present timing formulae represent only crude approximations to the orbital dynamics caused by the classical spin-orbit coupling and that there is a need for a new timing formula to model the timing observations. Before I focus on the construction of a new timing formula for main-sequence star binary pulsars a short presentation of the most important timing models is given.

In a simple spin-down law the pulsar proper time is related to the phase, $N$, of the pulsar by

$$N(T) = N_0 + \nu T + \frac{1}{2} \nu^2 T^2 + \frac{1}{6} \nu^3 T^3,$$

where $\nu$, $\dot{\nu}$, and $\ddot{\nu}$ are the rotation frequency of the pulsar, its first and second time derivative, respectively (spin parameters).

For a single pulsar the timing formula includes terms related to the position ($\alpha, \delta$), proper motion ($\mu_\alpha, \mu_\delta$) and parallax ($\pi$) of the pulsar. Moreover, it contains terms related to relativistic time dilation and light propagation effects in the solar system and also corrects for propagation effects in the interstellar medium (Backer 1989, Taylor 1989, Doroshenko & Kopeikin 1990):  

$$T = t_{\text{obs}} + \Delta_C - D/f_0^2 + \Delta_{R_\odot}(\alpha, \delta, \mu_\alpha, \mu_\delta, \pi) + \Delta_{E_\odot} + \Delta_{S_\odot}(\alpha, \delta).$$

(2)

$\Delta_C$ corrects for the offset between the observatory clock and the ‘Terrestrial Dynamical Time’ represented by the best terrestrial standard of time. $D/f_0^2$ corrects for the dispersive delay in the interstellar medium at the (barycentric) frequency $f_0$ where $D$ is proportional to the column density of free electrons between the pulsar and the observer. $\Delta_{R_\odot}$ describes the so called Roemer delay, a change in the time of flight of the radio signal caused by the motion of the observer in the solar system reference frame. $\Delta_{E_\odot}$ represents the transformation between ‘Terrestrial Dynamical Time’ and ‘Barycentric Dynamical Time’ (Fairhead & Brengnon 1990). Finally, $\Delta_{S_\odot}$ describes the Shapiro delay in the gravitational field of the Sun (Shapiro 1964).

For binary pulsars the timing formula (3) has to be extended by terms representing orbital motion and light propagation effects in the binary system (Blandford & Teukolsky 1976, Damour & Deruelle 1986, Damour & Taylor 1992):

$$T = t_{\text{obs}} + \Delta_C - D/f_0^2 + \Delta_{R_\odot} + \Delta_{E_\odot} + \Delta_{S_\odot} + \Delta_R + \Delta_E + \Delta_S,$$

where the major effect is the Roemer delay $\Delta_R$ which depends on the orbital motion of the pulsar and the orientation of the pulsar orbit with respect to the line of sight. If the binary motion is purely Keplerian then the Roemer delay depends on 5 (Keplerian) parameters:

- $P_b$, the orbital period of the binary system,
- $x \equiv a_p \sin i/c$, the projected semi-major axis,
- $e$, the eccentricity of the orbit,
- $\omega$, the longitude of periastron,
- $T_0$, the time of periastron passage.

$\omega$ is the semi-major axis of the pulsar orbit, $i$ is the inclination of the orbital plane with respect to the line of sight, where $i = 90^\circ$ implies edge on, and $c$ is the speed of light. The Roemer delay caused by the Keplerian motion of a binary system is given by

$$\Delta_R = x[(\cos U - e) \sin \omega + (1 - e^2)^{1/2} \sin U \cos \omega],$$

(4)

where $U$, the eccentric anomaly, is related to time, $T$, by the Kepler equation

$$U - e \sin U = 2\pi \frac{T - T_0}{P_b}.$$  

(5)

Soon after the discovery of PSR B1913+16 (Hulse & Taylor 1975) it was clear that a pure Keplerian timing model is not appropriate to analyse the timing observation of this 7.8-hour orbital-period binary pulsar. For this purpose Blandford & Teukolsky (1976) derived a timing model (BT model) which contains the largest short-period relativistic effect, the ‘Einstein delay’ $\Delta_E$, a combination of special-relativistic time dilation and gravitational redshift. They also included secular drifts of the main orbital parameters by following linear-in-time expressions:

$$P_b = P_{b0} + \dot{P}_b(T - T_0),$$

(6)

$$x = x_0 + \dot{x}(T - T_0),$$

(7)

$$e = e_0 + \dot{e}(T - T_0),$$

(8)

$$\omega = \omega_0 + \dot{\omega}(T - T_0).$$

(9)

Based on a remarkably simple analytic solution of the post-Newtonian two-body problem (Damour & Deruelle 1985) Damour & Deruelle (1986) derived an improved timing formula (DD model) for relativistic binary pulsars. The DD model differs from the BT model in two ways: it contains the Shapiro delay $\Delta_S$, which is of particular importance for $i$ close to $90^\circ$, and it allows for periodic effects in the orbital motion, e.g. in the BT model only the secular drift of the longitude of periastron is taken into account (equation (4)), whereas the DD model describes both the secular and quasi-periodic changes in $\omega$ according to

$$\omega = \omega_0 + k A_\omega(U),$$

with $\omega = 2\pi k/P_b$.
A timing formula for main-sequence star binary pulsars

In Section 5 I reinvestigate the orbital precession of PSR J0045–7319 since results presented so far in the literature were based on an incorrect formula for the precession of the longitude of periastron. In Section 6 the conclusions are given.

2 THE ORBITAL MOTION OF MAIN-SEQUENCE STAR BINARY PULSARS

The quadrupole of a main-sequence star companion is given by the difference between the moments of inertia about the spin-axis, \( I_2^* \), and an orthogonal axis, \( I_1^* \). It is proportional to the mass of the companion, \( m_\star \), the (polar) radius of the rotating star, \( R_\star \), squared, and to the spin squared (Cowling 1938, Schwarzschild 1958):

\[
I_1^* - I_2^* = 4k m_\star R_\star^2 \hat{\Omega}^2 \equiv m_\star q ,
\]

where \( k \) is the apsidal motion constant and \( \hat{\Omega} \equiv \Omega/(Gm_\star/R_\star^3)^{1/2} \) is the dimensionless proper rotation of the companion. \( \Omega \) is the angular velocity of the companion’s proper rotation. A main-sequence star of 10\( M_\odot \) has \( k \lesssim 0.03, R_\star \sim 6R_\odot \). If the star is rotating at 70\% of its break-up velocity, which appears to be a typical value for Be stars (Porter 1996), one finds for its quadrupole moment

\[
q = \frac{2}{5} k R_\star^2 \hat{\Omega}^2 \lesssim 0.35 R_\odot^2 = 7.6 \times 10^{-6} \text{AU}^2.
\]

The spin-induced quadrupole moment of the companion leads to an additional \( 1/r^3 \) potential term in the gravitational interaction between the two components which implies an apsidal motion and, when the spin of the companion is not aligned with the orbital angular momentum, to a precession of the orbital plane. The general expressions for the rates of apsidal motion and orbital precession can be found in Smarr & Blandford (1976) and Kopal (1978), (see Section 4 in this paper). The expressions in Smarr & Blandford (1976) and Kopal (1978) are derived by averaging the orbital dynamics over a full orbital period, in order to get the secular changes in the orbit. This procedure, by definition, neglects all short-term periodic effects. But, as will be shown in this paper, for main-sequence star binary pulsars with a long orbital period and a high eccentricity, like PSRs B1259–63 and J0045–7319, these short-term periodic effects are important.

For a study of the short-term periodic effects one needs the orbital motion in full detail. In the centre-of-mass system the (Newtonian) orbital dynamics of a binary pulsar with an oblate companion star is given by the Hamiltonian (Barker & O’Connell 1975)

\[
\mathcal{H} = \frac{p^2}{2\mu} - \frac{GM\mu}{r} \left( 1 + \frac{q}{2\pi} \left[ 1 - 3(\hat{n} \cdot \mathbf{s})^2 \right] \right) ,
\]

where the linear momentum \( p \) is related to the linear momenta of pulsar and companion by \( p = p_\mu = -p_\nu \mathbf{r} \) is a vector pointing from the companion to the pulsar, \( r \equiv |\mathbf{r}|, \hat{n} \equiv \mathbf{r}/r, M = m_\mu + m_\nu \) is the total mass of the system, \( \mu \equiv m_\mu m_\nu/M \) is the reduced mass and \( \mathbf{s} \) is the unit vector in direction of the spin of the companion.

Before studying the special case of motion in the equatorial plane. Although the two known main-sequence star binary radio pulsars do not

where

\[
A_e(U) = 2 \arctan \left[ \frac{1 + \epsilon}{1 - \epsilon} \right]^{1/2} \tan \frac{U}{2} ,
\]

and \( U \) is a function of \( T \) given by the solution of the (generalised) Kepler equation

\[
U - \epsilon \sin U = 2\pi \left( \frac{T - T_0}{P_\delta} - \frac{P_\delta}{2} \left( \frac{T - T_0}{P_\delta} \right)^2 \right) .
\]

\( P_\delta \) accounts for any secular change in the orbital period, like the damping caused by tidal dissipation or the emission of gravitational waves. The post-Keplerian Roemer delay is given by

\[
\Delta_R = x \left[ \cos U - c(1 + \delta_r) \sin \omega(U) \right] + \left[ 1 - \epsilon^2(1 + \delta_r)^2 \right]^{1/2} \sin U \cos \omega(U) ,
\]

where \( \omega(U) \) is given by equation (10). The post-Keplerian parameters \( \delta_r \) and \( \delta_r \) represent periodic post-Newtonian changes in the orbital motion, i.e. periodic changes of order \( (v/c)^2 \) where \( v \) is a typical orbital velocity of the binary star system.

Damour and Taylor (1992) defined an improved version of the BT model (BT+ model) where equation (9) is replaced by equation (10). The advantage of the BT+ model is that it contains the same number of parameters as the BT model but is able to account for quasi-periodic changes in \( \omega \).

To date, more than 50 radio pulsars are known which are members of a binary star system. The vast majority of these binary pulsars have a compact degenerate companion which is either a helium white dwarf or a neutron star. Timing observations for most of these binary pulsars can be fully explained by the timing models above. In a first approximation these models can be used for timing observations of the two main-sequence star binary pulsars, PSRs B1259–63 and J0045–7319, by fitting for the five Keplerian parameters, and for \( \omega \) and \( z \). However, this approximation models only the short-term secular changes of the binary orbit correctly.

In this paper an improved timing model for binary radio pulsars with main-sequence star companions is presented. The new timing formula accounts for the short-term secular precessional effects, for most of the short-term periodic orbital effects and for the long-term secular effects which are caused by classical spin-orbit coupling. In Section 2 a detailed investigation of the orbital dynamics of binary systems with classical spin-orbit coupling is given. First, a simple analytic solution is presented for the case that the orbital motion takes place in the equatorial plane of the massive companion. This solution will be helpful when developing the new timing formula. Then, the orbital dynamics of the general case, i.e. arbitrary orientation of the orbit with respect to the companion, is studied. In Section 3 it is shown how the orbital dynamics influences the timing observation of the main-sequence star binary pulsars PSRs B1259–63 and J0045–7319. As a consequence the new timing formula is developed. In Section 4 the long-term validity of the old and new timing formulae is studied. Extensions which are quadratic in time and take into account long-term precessional effects in the binary orbit are presented. It is shown that observations of such long-term secular effects have the potential to probe the internal structure of the companion.

Before studying the special case of motion in the equatorial plane. Although the two known main-sequence star binary radio pulsars do not
orbit their companion in the equatorial plane the comparably simple solution of this problem will be helpful in developing a new timing model in the next section.

2.1 The equatorial motion

For motion in the equatorial plane ($\dot{s} \cdot \hat{n} = 0$) the Hamiltonian (14) reduces to

$$\mathcal{H}_e = \frac{P_b^2}{2\mu} \frac{GM\mu}{r} \left( 1 + \frac{q}{2r^2} \right).$$

(17)

The invariance of this Hamiltonian under time translation and spatial rotations implies the conservation of the (reduced) energy, $E \equiv \mathcal{H}_e/\mu$, and the (reduced) total angular momentum, $\mathbf{J} \equiv \mathbf{r} \times \mathbf{p}/\mu$. If one introduces polar coordinates, $\mathbf{r} = r(\cos f, \sin f, 0)$, and makes use of the conserved quantities one finds the following equations of motion:

$$r^2 = 2E + \frac{2GM}{r} - \frac{J^2}{r^2} + \frac{GMq}{r^3},$$

(18)

$$\dot{f} = \frac{J}{r^2},$$

(19)

where $J \equiv |\mathbf{J}|$. These equations of motion can be solved to first order in $q$ by a quasi-Keplerian trigonometric parametrisation (cf. Damour & Deruelle 1985). For bound orbits, $E < 0$, one finds

$$2\pi \frac{t - t_0}{P_b} = U - e_\ell \sin U,$$

(20)

$$r = a(1 - e, \cos U),$$

(21)

$$f - f_0 = (1 + k)A_e f(U),$$

(22)

where $A_e f(U)$ is given by equation (14) and

$$P_b = \frac{2\pi GM}{(-2E)^{3/2}}.$$  

(23)

$$e_\ell = \left( 1 + \frac{2EJ^2}{G^2M^2} - \frac{2Eq}{J^2} \right)^{1/2},$$

(24)

$$a = -\frac{GM}{2E} \left( 1 + \frac{Eq}{J^2} \right),$$

(25)

$$e_\ell \equiv (1 + \delta e) e_\ell = \left( 1 - \frac{Eq}{J^2} \right) e_\ell,$$

(26)

$$k = \frac{3}{2} \left( \frac{GM}{J^2} \right)^2 q,$$

(27)

$$e_f \equiv (1 + \delta k)e_\ell = \left( 1 - \frac{2Eq}{J^2} \right) e_\ell.$$  

(28)

The motion of the pulsar is

$$\mathbf{r}_p = \frac{m_n}{M} r(\cos f, \sin f, 0).$$

(29)

Therefore, if the orbital angular momentum and the spin of the companion are (nearly) aligned then the orbital motion of the binary system can be described to first order in $q$ by a simple trigonometric parametrisation which is identical to the ‘quasi-Keplerian’ parametrisation given by Damour & Deruelle (1985) to solve the post-Newtonian two-body problem. Further, the timing observations of such a system can be fully explained using the DD timing model.

2.2 The general case

The full dynamics given by the Hamiltonian (14) is a well known problem in the theory of Earth satellite motion. Due to the anisotropic nature of the quadrupole potential one does not expect any simple analytic solution as in the previous section. Various methods have been developed to solve this problem (see e.g. Hagihara 1970, Roy 1978). To first order in $q$ the dynamics given by equation (14) can be solved by the method of the variation of the elements (osculating orbits). The following six elements are used to represent the osculating orbit:

- $a$, the semi-major axis of the (relative) orbit
- $e$, the eccentricity of the orbit
- $\theta$, the inclination of the orbital plane
- $\phi$, the longitude of the ascending node
- $\psi$, the longitude of periapsis
- $\lambda$, the mean anomaly

The angles are defined with respect to the equatorial plane of the oblate companion (see Fig. 1).

LaGrangian’s planetary equations for this problem can be written in the form

$$\frac{da}{dt} = \frac{2}{na} \frac{\partial R}{\partial \lambda},$$

(30)

$$\frac{dc}{dt} = \frac{1}{na^2}e \left( \frac{1 - e^2}{\sin^2 \phi} - \frac{\partial R}{\partial \phi} \right),$n

(31)

$$\frac{d\theta}{dt} = \frac{1}{na^2(1 - e^2)^2 \sin \phi} \left( \cos \phi \frac{\partial R}{\partial \phi} - \frac{\partial R}{\partial \phi} \right),$$

(32)

$$\frac{d\phi}{dt} = \frac{1}{na^2(1 - e^2)^2 \sin \phi} \left( \frac{\partial R}{\partial \phi} - \cos \phi \frac{\partial R}{\partial \phi} \right),$n

(33)

$$\frac{d\psi}{dt} = \frac{1}{na^2(1 - e^2)^2 \sin \phi} \left( \frac{1 - e^2}{e \sin \phi} \frac{\partial R}{\partial e} + a \frac{\partial R}{\partial \mu} \right).$$

(34)

$$\frac{dM}{dt} = n - \frac{2}{na^2} \left( \frac{1 - e^2}{2e} \frac{\partial R}{\partial e} + a \frac{\partial R}{\partial \mu} \right).$$

(35)

The relevant parts of the disturbing function $R \equiv R_+ + R_p$ are
A timing formula for main-sequence star binary pulsars

\[ \Delta a_s = \frac{2e_0 a_0}{1 - e_0^2} \{ C_{e_0, \theta_0, f} \} \]
\[ \Delta e_s = Q \{ C_{e_0, \theta_0, f} \} \]
\[ \Delta \phi_s = Q \{ \cos \theta \sin f \} \]
\[ \Delta \psi_s = Q \{ \frac{1}{2} \sin^2 \theta \} \]
\[ \Delta M_s = Q \{ 1 + Q \{ 1 - \frac{3}{2} \sin^2 \theta \} \{ 1 - e_0^2 \} \} \]

where the definition

\[ Q = \frac{3g_0}{2p_0} \]

was used. The zero subscript indicates evaluation at the initial condition. \( n_0 \) is related to the initial semi-major axis, \( a_0 \), by

\[ n_0^2 a_0^3 = GM \]

To simplify the representation of the first-order short-periodic perturbations I define the functions

\[ S_{e, \theta, f} \equiv \left( 1 - \frac{3}{2} \sin^2 \theta \right) \left( 1 - \frac{e^2}{4} \sin f + \frac{e}{2} \sin 2f + \frac{e^2}{12} \sin 3f \right) \]
\[ C_{e, \theta, f} \equiv \left( 1 - \frac{3}{2} \sin^2 \theta \right) \left( 1 + \frac{e^2}{4} \cos f + \frac{e}{2} \cos 2f + \frac{e^2}{12} \cos 3f \right) \]

and

\[ S_{e, \theta, f} \equiv a_0 \sin(2\psi - f) + \sum_{m=1}^{5} a_m \sin(2\psi + mf) \]
\[ C_{e, \theta, f} \equiv a_0 \cos(2\psi - f) + \sum_{m=1}^{5} a_m \cos(2\psi + mf) \]

and

\[ W_{e, f} \equiv f - M + e \sin f \]

To derive the first-order short-periodic perturbations, the disturbing function \( R \) in equations (30) to (33) is replaced by \( R_p \). Integration of the resulting equations leads to the following expressions for the six elements (Kozai 1959, Fitzpatrick 1970):

\[ \Delta e_p = Q \{ C_{e_0, \theta_0, f} \} \]
\[ + \sin^2 \theta_0 C_{e_0, \theta_0, f} \left( \frac{e_0^2}{16}, \frac{2 + 11/2}{16}, \frac{5}{4}, \frac{28 + 17/2}{48}, \frac{5}{8}, \frac{25}{16} \right) \]
\[ \Delta \phi_p = Q \{ \cos \theta \sin f \} \]
\[ \Delta \psi_p = Q \{ \frac{1}{2} \sin^2 \theta \} \]

The mean and true anomaly are connected by the equations

\[ \Delta \psi_s = \frac{1}{2} \sin^2 \theta \]
\[ \Delta \phi_s = \frac{1}{2} \sin^2 \theta \]

The distance between the main-sequence star and the pulsar is

\[ r = a(1 - e \cos U) = \frac{a(1 - e^2)}{1 + e \cos f} \]

The position vector of the pulsar at the time \( t \) with respect to the equatorial coordinate system is

\[ r = \left( \begin{array}{ccc} \cos \phi & - \sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{array} \right) \cos(\psi + f) \sin(\psi + f) \sin \theta \]

where the (variable) parameters \( \alpha, e, \theta, \phi, \psi, \mathcal{M} \) are given by

\[ \xi(t) = \xi_0 + [\Delta \xi_s(t) - \Delta \xi_s(0)] + [\Delta \xi_p(t) - \Delta \xi_p(0)] \]

(\( \xi \) stands for one of these six parameters). To calculate \( \Delta \xi_s \) one uses \( \mathcal{M} = \mathcal{M}_0 + n_0 t, e = e_0 \) in equations (53) and (54) to obtain an approximated value for \( f \) which then is used in evaluating equations (53) to (54).

As an example I substitute the orbital elements of the main-sequence star binary pulsar PSR B1259-63 (Table 1) into equations (12) to (15). For the initial value of \( \theta (\theta_0) \) I take 60°, which is a realistic value for this binary system (see Wex et al. 1997). The values for \( \phi_0 \) and \( \psi_0 \) can be estimated from \( \omega, i, \) and \( \lambda_s \):

\[ \cos \phi = \frac{\cos i - \cos \theta \cos \lambda_s}{\sin \theta \sin \lambda_s} \]
\[ \sin(\omega - \psi) = - \frac{\sin \lambda_s \sin \phi}{\sin i} \]
\[ \cos(\omega - \psi) = \frac{\sin \lambda_s \cos \lambda_s - \cos \theta \sin \lambda_s \cos \phi}{\sin i} \]
The results for two full orbits of PSR B1259–63 are given in Fig. 2 and Fig. 3. Fig. 2 presents the changes of the four ‘periodic’ parameters, \( a, e, \theta, M \), which take their initial value after every full period. Fig. 3 presents the changes of \( \phi \) and \( \psi \). I call \( \phi \) and \( \psi \) ‘secular’ parameters since they show both periodic and secular changes. It is obvious that for all parameters the major changes take place very close to the periastron passages, as expected from the \( 1/r^3 \) nature of the quadrupole potential.

3 TIMING MODELS FOR MAIN-SEQUENCE STAR BINARY PULSARS I. SHORT-TERM PERIODIC EFFECTS

Let \( \mathbf{K}_0 = (0, -\sin \lambda_\ast, \cos \lambda_\ast) \) be the unit vector which indicates the direction of sight (see Fig. 1). The Roemer delay measured by an observer on Earth is then given by

\[
\Delta R = \frac{1}{c} \mathbf{K}_0 \cdot \mathbf{r}_p ,
\]

where \( c \) is the speed of light and

\[
\mathbf{r}_p = \frac{m_\ast r}{M}
\]

is the position vector of the pulsar originating in the centre of mass of the binary system. Using equation (61) leads to

\[
\Delta R = \frac{m_\ast r}{M c} \left\{ -\sin \lambda_\ast \sin \phi \cos(\psi + f) \\
+ \cos \phi \sin(\psi + f) \cos \theta \right\}.
\]

As mentioned in the introduction, the simplest timing model for binary pulsars is the BT model where changes in the longitude of periastron, \( \omega \), and changes in the projected semi-major axis of the pulsar orbit, \( x \), are assumed to be linear in time. The discussion in the previous section clearly showed that the (osculating) parameters of a binary system with an oblate companion do not change linearly in time (cf. Fig. 2 and Fig. 3). Thus one does not expect that the application of the BT model leads to a perfect fit. Fig. 4 shows the difference between equations (61), (7), (9) and the actual Roemer delay expected for the binary pulsars PSRs B1259–63 and J0045–7319. (Using the DD model instead leads to similar results). The typical precision in the measurement of the arrival time of pulsar signals is of the order of 100 \( \mu s \) for PSR B1259–63 (Johnston et al. 1994) and a few ms for PSR J0045–7319 (Kaspi et al. 1994). For both pulsars the deviations given in Fig. 4 are larger than the error in the TOAs. For PSR B1259–63 it is more than a factor of ten. Therefore the BT model is only a very crude approximation for these two binary pulsars and there is the need for an improved timing model, for PSR B1259–63 in particular.

To extract reliable information from timing observations of main-sequence star binary pulsars one could construct a timing model that contains the full orbital dynamics given in the previous section. In the (unlikely) case that the orbital motion takes place in the equatorial plane of the Be star one can use the DD timing model as shown by the solution in section 2.1. For the general case one has to use the solution of Section 2.2 which leads to a rather complicated timing formula with a comparably high number of parameters to fit for where some of these parameters are only indirectly related with observable effects. Therefore, given the limited number of TOAs and the finite size of their measurement errors, one sees that in general a timing formula based on the equations of Section 2.2 is not a practical procedure. What one is looking for is a simple timing model which is a very good approximation to reality. In the ideal case the number of parameters should be the same as in the BT model.
The main deficit of the BT model is the use of equations (7) and (9) to describe the precession of the orbit. Even for a purely equatorial motion (see Section 2.1) equation (9) has to be replaced by (10) according to the DD timing model. If the orbit is tilted with respect to the equatorial plane the precession of the orbit leads to a change in the inclination of the orbital plane with respect to the line of sight. Fig. 5 gives this change of $i$ for PSR B1259–63 as a function of the time, $t$, and as a function of the true anomaly, $f$. The change of $i$ is neither linear in $t$ nor linear in $f$. But the assumption of linearity in $f$ is obviously much closer to reality than the assumption of linearity in $t$. The same is true for the change of $x$ which is a function of $i$.

Therefore I define a new timing model which I call BT++ model. Analogue to the construction of the BT+ model by Damour and Taylor (1992) the BT++ model is defined by replacing equations (1) and (3) in the BT model by

$$x = x_0 + \xi A_x(U), \quad \dot{x} \equiv 2\pi \xi / P_b,$$

and

$$\omega = \omega_0 + k A_\omega(U), \quad \dot{\omega} \equiv 2\pi k / P_b.$$  

A comparison of the expected Roemer delay and the Roemer delay as used in the BT timing model minus Roemer delay as expected for PSRs B1259–63 (upper figure) and J0045–7319 (lower figure). For B1259–63 I used the same parameters as in Fig. 2. For J0045–7319 I used the parameters of Table 1 ($i_0 = 44^\circ$, $\lambda_\ast = 15^\circ$, $\phi_0 = 30^\circ$, $\psi_0 = 156^\circ$, $\psi_0 = 106^\circ$, and $q = 2 \times 10^{-6}$ AU$^2$).

I conclude that one should certainly use the BT++ model instead of the BT model to fit the TOAs of PSR B1259–63. The BT++ model has the same number of parameters as the BT model, but is able to account for the fact that changes of the binary parameters happen mainly close to periastron. The BT++ model was already applied successfully to fit the TOAs of PSR B1259–63 (Wex et al. 1997). I expect a slight improvement of the residuals for PSR J0045–7319 when one uses the BT++ model.

As concluded in Section 2.1, the correct timing model for equatorial orbits is the DD model. The DD model takes into account all the periodic effects of the (equatorial) orbital motion. One can now try to construct an even better timing model for main-sequence star binary pulsars, say DD+, by replacing equation (7) in the DD model by equation (64). The result is a timing model which combines the advantages of the BT++ model in describing the precession of the orbital plane and the DD model in describing periodic orbital effects. The representation of the Roemer delay in the DD+ model contains one more parameter than in the BT++ model. The DD+ model has the same number of parameters as the DD model. From Fig. 7 one sees that the DD+ model represents a nearly perfect fit for most parts of the orbit and close to periastron it is an improvement by a factor of 2 with respect to the BT++ model. At present
the measurement precision for the TOAs for PSRs B1259–63 and J0045–7319 does not allow to fit for the (full) DD+ model.

Finally, the upper figure of Fig. 5 indicates that a step-function model ($\omega$ and $x$ change discontinuously at each periastron) will also give a good approximation to reality. In fact, Fig. 8 shows that a step-function model is clearly better than the BT model (lower figure of Fig. 4). On the other hand, comparison between the lower figure of Fig. 6 and Fig. 8 implies that a step-function model is clearly worse than the BT++ model, in particular for observations close to periastron.

4 TIMING MODELS FOR MAIN-SEQUENCE STAR BINARY PULSARS II. LONG-TERM SECULAR EFFECTS

So far only the short-term effects in the orbital motion have been dealt with. I have shown the advantage of the BT++ (and DD+) model in taking into account the short-term periodic effects of the orbital motion. In the BT, BT+, DD, BT++ and DD+ model the secular changes in $\omega$ and $x$ are assumed to be linear in time. This approximation will hold as long as there are only small changes in $\omega$ and $x$. In this
The change of the orientation of the main-sequence star is caused by the spin induced quadrupole of the focus only on the secular changes in the orbit of the binary system. Thus, although of order 1, the orientation of the main-sequence star changes by 000 if averaged over one full period (c.f. Section 2.2). For simplicity I will skip the bar on top of the averaged quantities for the rest of this section. Equations (69) and (70) can be derived directly from equations (64) and (65).

For the inclination of the orbit with respect to the line of sight, \( i \), and the longitude of periastron, \( \omega \), one finds the relations

\[
\cos i = \cos i_0 \cos \theta_J - \sin i_0 \sin \theta_J \cos \Phi, \\
\sin \omega = \frac{1}{\sin i} \left[ (\sin \theta_J \cos i_0 + \cos \theta_J \sin i_0 \cos \Phi) \sin \Psi + \sin i_0 \sin \Phi \cos \Psi \right], \\
\cos \omega = \frac{1}{\sin i} \left[ (\sin \theta_J \cos i_0 + \cos \theta_J \sin i_0 \cos \Phi) \cos \Psi - \sin i_0 \sin \Phi \sin \Psi \right]
\]

(see Fig. 9 for the definition of \( i_0, \theta_J, \Phi \) and \( \Psi \)). The angles \( i_0 \) and \( \theta_J \) are conserved quantities. The angles \( \Phi \) and \( \Psi \) evolve linearly in time, \( t \),

\[
\Phi = \Phi_0 + \Phi(t - t_0) \quad \text{and} \quad \Psi = \Psi_0 + \Psi(t - t_0).
\]

Equations (69) to (72) give the full (secular) evolution of the projected semi-major axis, \( x = a_0 \sin i/c \), and the longitude of periastron, \( \omega \). This evolution is clearly non-linear in time since changes in \( \Psi \) couple in a complicated way with changes in \( \Phi \) to produce the secular changes in \( i \) and \( \omega \).

A first approximation of this non-linear behaviour can be given by

\[
x(t) \simeq x_0 + \dot{x}_0 (t - t_0) + \frac{1}{2} \ddot{x}_0 (t - t_0)^2,
\]

and

\[
\omega(t) \simeq \omega_0 + \dot{\omega}_0 (t - t_0) + \frac{1}{2} \ddot{\omega}_0 (t - t_0)^2.
\]

The quantities \( \dot{x}_0, \ddot{x}_0, \dot{\omega}_0, \ddot{\omega}_0 \), if measured in timing observations, contain information about the orientation and the quadrupole moment of the companion star.

In general the orbital angular momentum is much larger than the spin of the companion star and thus \( \theta_J \ll 1 \). In case of the binary pulsars PSRs B1259–63 and J0045–7319, \( |S|/\mathcal{L} \sim 0.1 \) and \( \theta_J \lesssim 6^\circ \). If \( \theta_J \ll 1, \pi - i \) one finds after an (Laurent) expansion of equations (69) to (72) with respect to the small angle \( \theta_J \):

\[
i = i_0 + \theta_J \cos \Phi + O(\theta_J^2),
\]

\[
\omega = \Psi + \Phi - \theta_J \cot i \sin \Phi + O(\theta_J^2),
\]

and

\[
\dot{x}_0 = nQx_0 \cot i \sin \theta \cos \theta \sin \Phi_0 + O(\theta_J),
\]

\[
\ddot{x}_0 = -n^2 \mathcal{Q}^2 x_0 \cot \left( \frac{\sin^2 \theta \cos^2 \theta \sin \Phi_0}{\sin \theta_J} \right) + O(\theta_J^2),
\]

\[
\dot{\omega}_0 = nQ \left( 1 - \frac{3}{2} \sin^2 \theta + \cot i \sin \theta \cos \theta \sin \Phi_0 + O(\theta_J) \right).
\]

Figure 9. Definition of various angles in the total-angular-momentum reference frame. The invariable \((X-Y)\) plane is perpendicular to the total angular momentum \(J = L + S\). The line-of-sight \((\hat{K}_0)\) is in the \(Y-Z\) plane. \(J\) is a conserved quantity and, if averaged over one full period, \(|L|\) and \(|S|\) are conserved quantities. Thus \(i, \theta_J\), and \(\theta\) are conserved. The angles \(\Phi\) and \(\Psi\) change linearly with time.
\[ \dot{\omega}_0 = n^2 Q^2 \cot \alpha \left( \frac{\sin^2 \theta \cos^2 \theta \sin \Phi_0}{\sin \theta} \right) + O(\theta_i^2). \] (80)

While doing these expansions it is important to keep in mind that \( \dot{\Phi} \) and the leading term of \( \dot{\Psi} \) are of order \( \theta_i^{-1} \) but \( \Phi + \dot{\Psi} \) is only of order \( \theta_i^0 \). A fact which has been overlooked by Smarr and Blandford (1976) and Lai et al. (1995) leading to a wrong result for \( \dot{\omega}_0 \).

Since \( Q \) is a quantity which is not well known, it is not possible to extract the angles \( \theta \) and \( \Phi_0 \) just from the measurement of \( \dot{x}_0 \) and \( \omega_0 \). But, if the masses, and therefore \( i \), are known within a certain accuracy (which I assume for the following considerations) the measurement of \( \dot{x}_0 \) and \( \omega_0 \) restricts possible values of \( \theta \) and \( \Phi_0 \) to a small region in the \( \theta-\Phi_0 \) plane by dividing equation (77) by equation (79), which leads to

\[ \frac{\omega_0 x_0}{x_0} \sin \Phi_0 - \cos \Phi_0 = \frac{1 + 3 \cos 2\theta}{2 \cot i \sin 2\theta}. \] (81)

The sign of \( \dot{x}_0 \) or \( \dot{\omega}_0 \) will help to exclude further regions in the \( \theta-\Phi_0 \) plane. Since the right-hand sides of equations (77) and (79) have to be positive or negative depending on the sign of \( \dot{x}_0 \) and \( \omega_0 \) respectively (c.f. Fig. 11).

If one is able to measure either \( \dot{x} \) or \( \dot{\omega}_0 \), in addition to \( \dot{x} \) and \( \dot{\omega}_0 \), one can make use of the following restrictions on \( \Phi_0 \) and \( \omega_0 \):

\[ \frac{x_0^2 \tan i}{x_0 x_0} = -\sin \Phi_0 \tan \Phi_0 \theta_j \] (82)

or

\[ \frac{x_0^2 \tan i}{\omega_0 x_0} = \sin \Phi_0 \theta_j. \] (83)

If one is able to measure \( \dot{x}, \dot{\omega}, \ddot{x}, \) and \( \ddot{\omega} \) for a binary pulsar with a main-sequence star companion, then

\[ \frac{\dot{\omega}_0 x_0}{\dot{x}} = -\tan \Phi_0 \] (84)

will give \( \Phi_0 \). Equation (82) or (83) will then give \( \theta_j \) and equation (84) will give \( \dot{\theta}_j \). Therefore all angles in the geometry of the binary system are determined and one can further get \( Q \) from equation (77) or (79) and the spin of the companion using \( |L|, \theta_j \) and \( \theta \) (see Fig. 9). This allows to determine the moments of inertia \( I_1 \) and \( I_2 \) of the companion, which are related with its internal structure.

For PSR B1259–63 the changes of \( i \) and \( \omega \) are typically of the order of one second of arc per orbit (see Fig. 5). Since the orbital period is 3.4 years, the quadratic-in-time changes of these angles will be absolutely negligible for the next few decades.

For PSR J0045–7319 the situation is different. The changes in \( x \) and \( \omega \) are two orders of magnitude larger than for PSR B1259–63. Fig. 10 shows the result of fitting for simulated pulse arrival times for PSR J0045–7319. The first figure (10a) gives the pre-fit residuals for a BT model that leads to a good fit for the first few orbits. After four years the model is off by about 40 ms. Fitting for the whole time span of 1500 days using the BT model one finds the post-fit residuals given in the second figure (10b). Most of the deviations of Fig. 10a are absorbed in the spin parameters by changing them according to

\[ \Delta P = -2.1 \times 10^{-10} \, s, \quad \Delta \dot{P} = -3.3 \times 10^{-18}. \] (85)

The residuals are in the order of the present measurement precision. For future observations the BT model (and therefore the BT++ and DD+ models) will fail to explain the observations and one has to fit for higher derivatives in \( x \) and \( \omega \). The result of such a fit is presented in the last figure (10c).

Finally, it should be mentioned that fitting for a \( P_\text{b} \) instead of \( \dot{x} \) and \( \dot{\omega} \) improves the residuals only marginally and gives a \( P_\text{b} \) which is two orders of magnitudes smaller than the one observed in this system. Therefore the nonlinear drifts of \( x \) and \( \omega \) cannot explain the observed \( P_\text{b} \).
5 PSR J0045–7319 — EVIDENCE FOR A NEUTRON-STAR BIRTH KICK?

The orbital precession of the binary pulsar PSR J0045–7319 was seen as a direct evidence that the neutron star of this system received a kick of at least 100 km/s at the moment of birth (Kaspi et al. 1996). Since the theoretical analysis in this paper is based on the calculations of Smarr and Blandford (1976) and Lai et al. (1995) an incorrect formula for \( \dot{\omega} \), the precession of the longitude of periastron, was used (equation (1) in Kaspi et al. 1996). Therefore their limits on the angle between the spin axis of the B star and the orbital angular momentum, \( 25^\circ < \theta < 41^\circ \), are incorrect and one has to reinvestigate the question whether this binary star system provides an evidence for a neutron-star birth kick.

Using the values of Table 1 in Kaspi et al. (1996) one finds for PSR J0047–7319

\[
\frac{\dot{\omega}_0 x_0}{x_0} = 1.80 \pm 0.05 .
\]  

(86)

\( \dot{\omega} \) was corrected for the general relativistic contribution of 0.004°/yr. The uncertainty in the masses leads to

\[
i = 44^\circ \pm 5^\circ \quad \text{(or) } i = 136^\circ \pm 5^\circ .
\]  

(87)

Equation (83) restricts the values of \( \Phi_0 \) and \( \theta \) as shown in Fig. 11 and one finds as a lower limit on the inclination of the B star with respect to the orbital plane

\[
\theta > 20^\circ .
\]  

(88)

This limit is only slightly smaller than the one given in Kaspi et al. (1996) and so does not change their major conclusion, i.e. that the binary system PSR J0045–7319 provides direct evidence for a neutron-star birth kick of at least 100 km/s. However, their conclusion that \( \dot{\omega}_0 > 0 \) implies \( \theta < 55^\circ \) is incorrect. In principle one can have \( \theta \) up to \( 70^\circ \), although \( \theta \) being close to \( 70^\circ \) requires an unphysically high apsidal motion constant \( k \) for the B star. Numerical simulations show that \( \theta > 65^\circ \) is excluded by the fact that present timing observations are still in agreement with a simple \( \dot{x} - \dot{\omega} \) model. Similar arguments apply for the case \( \theta > 90^\circ \).

6 CONCLUSIONS

In this paper I have presented a timing formula for main-sequence star binary pulsars which takes into account most of the periodic variations along the orbit caused by the anisotropic nature of the \( 1/r^3 \) potential of the spin-induced quadrupole moment of the companion star. The new timing formula contains the same number of parameters as the Blandford-Teukolsky timing formula. I have shown by numerical simulations, that the new timing formula leads to much better results in case of the long-orbital period binary pulsar PSR B1259–63 then the Blandford-Teukolsky or the Damour-Deruelle timing formula. Only very close to periastron the new timing formula shows deviations are slightly greater than the typical measurement error in the time-of-arrival of the pulsar signals. But so far there are no timing observations of PSR B1259–63 close to periastron, due to the eclipse of the pulsar caused by the circumstellar material. For PSR J0045–7319 these periodic variations are of the order of the measurement precision.

I have given quadratic-in-time extensions of the timing formula which account for long-term secular changes in the orientation of the binary-pulsar orbit. In particular for the binary pulsar PSR J0045–7319 these extensions might be important in the next few years of timing observations depending on the, so far unknown, orientation of the B star spin and the total angular momentum of the binary system. I have concluded that the measurement of these long-term secular effects has the potential to probe the internal structure of the companion.

Finally I have reinvestigated the classical spin-orbit precession of the binary pulsar PSR J0045–7319 since the theoretical analysis of this binary system given in Lai et al. (1995) and Kaspi et al. (1996) is based on an incorrect expression for the precession of the longitude of periastron. I have found \( 20^\circ \) as a lower limit for the inclination of the B star with respect to the orbital plane which does not change the conclusions concerning the neutron-star birth kick in this system given in Kaspi et al. (1996).

ACKNOWLEDGMENTS

I thank Peter Müller for many stimulating discussions and Simon Johnston for carefully reading the manuscript.

REFERENCES

Backer D.C., 1989, in Ögelman H., van den Heuvel E.P.J., eds, Timing of Neutron Stars, Kluwer, Dordrecht, p. 3

Barker B.M., O’Connell R.F., 1975, Phys. Rev. D, 12, 329

Bell J.F., 1996, to appear in Satellite and Ground Based Studies
N. Wex

of Radio Pulsars, proceedings of the 31st Scientific Assembly of COSPAR, astro-ph/9610142

Bell J.F., Bessell M.S., Stappers B.W., Bailes M., Kaspi V.M., 1995, ApJ, 447, L117
Blandford R., Teukolsky S.A., 1976, ApJ, 205, 580
Cowling T.G., 1938, MNRAS, 98, 734
Damour T., Deruelle N., 1985, Ann. Inst. Henri Poincaré, 43, 107
Damour T., Deruelle N., 1986, Ann. Inst. Henri Poincaré, 44, 263
Damour T., Taylor J.H., 1992, Phys. Rev. D, 45, 1840
Doroshenko O., Kopeikin S., 1990, SvA, 34, 496
Fairhead L., Bletnagnon P., 1990, A&A, 229, 240
Fitzpatrick P.M., 1970, Principles of Celestial Mechanics, Academic Press, New York and London
Hagihara Y., 1970, Celestial Mechanics I, MIT Press
Hulse R.A., Taylor J.H., 1975, ApJ, 195, L51
Johnston S., Manchester R.N., Lyne A.G., Bailes M., Kaspi V., Qiao G., D’Amico N., 1992, ApJ, 387, L37
Johnston S., Manchester R.N., Lyne A.G., Nicastro L., Spyromilio J., 1994, MNRAS, 268, 430
Kaspi V.M., Bailes M., Manchester R.N., Stappers B.W., Bell J.F., 1996, Nature, 381, 584
Kaspi V.M., Johnston S., Bell J.F., Manchester R.N., Bailes M., Bessell M., Lyne A.G., D’Amico N., 1994, ApJ, 423, L43
Kopal Z., 1978, Dynamics of Close Binary Systems, D. Reidel Publishing Company, Dordrecht
Kozai Y., 1959, AJ, 64, 367
Lai D., Bildsten L., Kaspi V.M., 1995, ApJ, 452, 819
McConnell D., McCulloch P.M., Hamilton P.A., Ables J.G., Hall P.J., Jacka C.E., Hunt A.J., 1991, MNRAS, 249, 654
Porter J. M., 1996, MNRAS, 280, L31
Roy A.E., 1978, Orbital Motion, Adam Hilger Ltd, Bristol
Schwarzschild M., 1958, Structure and evolution of stars, Princeton Univ. Press, Princeton
Shapiro I.I., 1964, Phys. Rev. Lett., 13, 789
Smarr L.L., Blandford, R.D., 1976, ApJ, 207, 574
Taylor J.H., 1989, in Ögelman H., van den Heuvel E.P.J., eds, Timing of Neutron Stars, Kluwer, Dordrecht, p. 17
Taylor J.H., Weisberg J.M., 1989, ApJ, 345, 434
Wex N., Johnston S., Manchester R.N., Lyne A.G., Stappers B.W., Bailes M., 1997, submitted to MNRAS
Wolszczan A., Frail D.A., 1992, Nature, 355, 145