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Posterior linearisation smoothing with robust iterations

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Abstract

This paper considers the problem of robust iterative Bayesian smoothing in nonlinear state-space models with additive noise using Gaussian approximations. Iterative methods are known to improve smoothed estimates but are not guaranteed to converge, motivating the development of more robust versions of the algorithms. The aim of this article is to present Levenberg–Marquardt (LM) and line-search extensions of the classical iterated extended Kalman smoother (IEKS) as well as the iterated posterior linearisation smoother (IPLS). The IEKS has previously been shown to be equivalent to the Gauss–Newton (GN) method. We derive a similar GN interpretation for the IPLS. Furthermore, we show that an LM extension for both iterative methods can be achieved with a simple modification of the smoothing iterations, enabling algorithms with efficient implementations. Our numerical experiments show the importance of robust methods, in particular for the IEKS-based smoothers. The computationally expensive IPLS-based smoothers are naturally robust but can still benefit from further regularisation.

Index Terms

Bayesian state estimation, robust smoothing, posterior linearisation, Levenberg–Marquardt

I. INTRODUCTION

SMOOTHING is a form of state estimation, where past states of a stochastically evolving process are estimated from a noisy measurement sequence. It has wide-ranging applications in navigation, target tracking and communications [3], [11]. From a Bayesian perspective, the objective is to obtain the posterior probability density function (PDF) for a sequence of states, or the marginal PDF for a specific state, given all measurements [26].

General Gaussian Rauch-Tung-Striebel (RTS) smoothers form a family of methods which utilises the problem structure to efficiently calculate a Gaussian marginal posterior over the states. The name stems from the RTS smoother which, for linear/affine and Gaussian systems, computes the exact smoothing distributions [22], [26]. For non-linear systems, general Gaussian RTS smoothers make a linear approximation and subsequent closed-form RTS smoothing of the approximated systems [27].

The choice of linearisation method defines the members of this smoother family. Examples include first-order Taylor expansion for the Extended Kalman Smoother EKS [26] and statistical linear regression (SLR) [2] with sigma point methods for the Unscented RTS smoother [13], [25] and Cubature RTS smoother [1].

The point, around which linearisation is done, can greatly impact the resulting estimated trajectory. In general, we want to perform linearisation around points close to the posterior estimates, motivating iterative extensions of aforementioned smoothers [10]. A full smoothed trajectory is repeatedly computed, with linearisations done around the most recent estimates. The increased computational complexity can lead to significantly better performance. Examples of iterative methods are the Iterated EKS (IEKS) and the Iterated posterior linearisation smoother (IPLS) [3], [10].

To improve the robustness and convergence properties of iterative methods, damping or regularisation can be used. A common approach is to relate the smoothing procedure to an optimisation method, such as Gauss–Newton (GN) [4]–[6], [8], [21], [24]. Regularised versions can then be created by guaranteeing a non-increasing cost function, e.g., with the Levenberg–Marquardt (LM) method, an extension of the GN method [19]. Existing works have investigated robust alternatives, in particular for the related filtering problem. A more numerically stable update step for the IEKF, as well as the use of LM regularisation was proposed in [6]. The Damped IPLF (DIPLF) [21] also used a GN algorithm to improve the convergence of the Iterated posterior linearisation filter (IPLF). Similar LM extensions as explored in this paper were also developed in [7], [12], [16].

In this paper, we present LM regularised versions of both the IEKS and the IPLS, called LM–IEKS and LM–IPLS, as well as line-search versions, called LS–IEKS and LS–IPLS. The results concerning the IEKS were first presented in [28]. This paper is an extension of that work and contains the following contributions

1) In Section IV-A, we show that the IPLS can be interpreted as performing GN optimisation of a sequence of cost functions.
2) In Section IV-B, we show that LM regularisation can be achieved with a simple modification of a state-space model.
3) Section IV-C, we introduce a class of smoother methods based on a simple line-search algorithm.
4) In Section IV-D, we combine these results to introduce the algorithmic descriptions of the LM-regularised smoothers LM–IEKS and LM–IPLS as well as the line-search smoothers LS–IEKS and LS–IPLS. Finally in Section V, we demonstrate with numerical simulations the importance of more robust versions of iterative smoothing algorithms.

II. PROBLEM FORMULATION

In smoothing, we consider the problem of estimating a sequence of latent states in a Markov process \((x_1, x_2, \ldots, x_K)\), \(x_k \in \mathbb{R}^{d_x}\), \(k = 1, \ldots, K\), based on noisy measurements \((y_1, y_2, \ldots, y_K)\), \(y_k \in \mathbb{R}^{d_y}\), \(k = 1, \ldots, K\), where \(K\) is the final time step. A lower-case letter, e.g. \(x\), denotes a column vector and capitalised letters, e.g. \(P\), denote matrices. We introduce the shorthand notations for the sequences \(x_{1:K} := (x_1, x_2, \ldots, x_K)\), \(y_{1:K} := (y_1, y_2, \ldots, y_K)\).

We approximate the states as normally distributed, parameterised by their state mean and covariance, for every time step \(k = 1, \ldots, K\). We use the shorthand notations \(\hat{x}_{k|k}\), \(\hat{P}_{k|k}\) for the state mean and covariance estimates at time step \(k\) based on the measurements up to and including time step \(k\). For smoothing, the aim is to estimate the smoothed mean \(\hat{x}_{1|K}\) and covariance \(\hat{P}_{1|K}\), for all timesteps \(k\), that is, based on all measurements \(y_{1:K}\).

The state-space model is specified by its motion (also known as dynamic or process) and measurement models

\[
x_{k+1} = f_k(x_k) + q_k \\
y_k = h_k(x_k) + r_k
\]

where \(f_k : \mathbb{R}^{d_x} \rightarrow \mathbb{R}^{d_x}\), \(q_k \in \mathbb{R}^{d_x \times d_q}\), \(h_k : \mathbb{R}^{d_x} \rightarrow \mathbb{R}^{d_y}\), \(r_k \in \mathbb{R}^{d_y \times d_r}\) are assumed to be known and the process and measurement noise, \(q_k \in \mathbb{R}^{d_x}\) and \(r_k \in \mathbb{R}^{d_y}\), are assumed to be independent. The initial prior \(x_1 \sim \mathcal{N}(\bar{x}_{1|0}, \bar{P}_{1|0})\) is assumed to be known.

An important special case of space-space models are the linear (technically affine) and Gaussian models:

\[
x_{k+1} = F_k x_k + b_k + \omega_k + q_k \\
y_k = H_k x_k + c_k + \gamma_k + r_k
\]

where for all time steps, \(F_k \in \mathbb{R}^{d_x \times d_x}\), \(b_k \in \mathbb{R}^{d_x}\) and \(H_k \in \mathbb{R}^{d_y \times d_x}\), \(c_k \in \mathbb{R}^{d_y}\) constitute the linear mappings and the random variables \(q_k, \omega_k, r_k, \gamma_k\) are mutually independent. For such systems, the linear RTS smoother computes the exact marginal posterior PDF in closed-form [26].

General Gaussian RTS smoothers can handle non-linear/non-Gaussian systems [26], [27]. These smoothers can all be described as performing two steps to obtain a tractable smoothing algorithm. First, they approximate the original models in (1) as linear and Gaussian models of the form in (2). Second, they compute the posterior densities for this approximate model exactly using the RTS smoother. We refer to the linearisation as an enabling approximation since it enables us to find a closed form solution to the smoothing problem. The family of general Gaussian RTS smoothers includes the EKS, the unscented RTS smoother and the cubature RTS smoother, and the algorithms in this family only differ in how the linearisation parameters \(\Theta_{1:K} = (F_k, b_k, \Omega_k, H_k, c_k, \Gamma_k)\), \(k = 1, \ldots, K\), are chosen.

Iterative extensions of general gaussian RTS smoothers perform repeated steps of linearisation and RTS smoothing. The sequence of estimates produced by iterative smoothers are denoted \(\hat{x}_{1:K}^{(i)}\), where the superscript \((i)\) indicates that these are the smoothed estimates for iteration \(i\). A superscript without parentheses refers to a special iterate, labelled by the superscript, for instance a fix point \(\hat{x}_{1:K}^{+}\). The starting point is the initial estimates \(\hat{x}_{1:K}^{(0)}, \hat{P}_{1:K}^{(0)}\), which form the basis for selecting the initial linearisation parameters \(\Theta_{1:K}^{(0)}\). We iteratively find new estimates \(\hat{x}_{1:K}^{(i)}, \hat{P}_{1:K}^{(i)}\) with closed form RTS smoothing, and use them to select new linearisation parameters \(\Theta_{1:K}^{(i+1)}\). The iterative process is repeated until some convergence criteria are met. In this paper, we propose robust versions of such iterated smoothers.

III. BACKGROUND

A. Linearisations used in smoothing

All general Gaussian RTS smoothers use the linear RTS smoother. What sets them apart is that they use different methods of linearising the state-space model, which makes them apply the RTS smoother on slightly different state-space models.

The EKS is a well-known general Gaussian RTS smoother [26]. It makes the enabling linear approximation through an analytical linearisation (exemplified here by the motion model in (1))
where $J_f(\hat{x})$ is the Jacobian of $f$, evaluated at $\hat{x}$. Linearisation is done around some estimate of the mean of the state $\hat{x}_k$. For instance, in the ordinary EKS it is done around the updated means $\hat{x}_{k|k}$. Note that for the EKS and its extensions, linearisation is done around a point estimate of the state at time step $k$.

Another category of smoothers are those that use statistical linear regression (SLR) for the enabling approximation, such as the Unscented RTS and cubature RTS smoothers. With SLR, the enabling approximation is

$$F_k(\hat{x}_k, \hat{P}_k) = \Psi_{f_k}^T \hat{P}_k^{-1}$$  \hspace{1cm} (4a)

$$b_k(\hat{x}_k, \hat{P}_k) = \bar{x}_k - F_k \hat{x}_k$$ \hspace{1cm} (4b)

$$\Omega_k(\hat{x}_k, \hat{P}_k) = \Phi_{f_k} - A\hat{P}_k A^T,$$  \hspace{1cm} (4c)

where

$$\bar{x}_k = \int f_k(x_k)p(x_k)dx_k$$

$$\Psi_{f_k} = \int (x_k - \bar{x}_k)(f_k(x_k) - \bar{x}_k)^T p(x_k)dx_k$$

$$\Phi_{f_k} = \int (f_k(x_k) - \bar{x}_k)(f_k(x_k) - \bar{x}_k)^T p(x_k)dx_k.$$  \hspace{1cm} (5)

Since the linearisation is done with respect to the distribution $p(x_k) = \mathcal{N}(x_k; \bar{x}_k, \hat{P}_k)$, the SLR approximation depends, on both the mean $\bar{x}_k$ and the covariance $\hat{P}_k$. The moments in (5) are not tractable for a general function $f_k$ and some form of approximation is needed, such as Monte Carlo methods or, more commonly, sigma point methods [14], [26]. We refer to these basic SLR smoothers as versions of the prior linearisation smoother (PrLS), following the nomenclature introduced in [9].

For non-iterative methods such as the basic EKS or the PrLS, the linearisation (2) is done around the predicted and filtered estimates for the filtering and smoothing passes respectively. With non-linear motion or measurement models the risk is that the linearisation is a poor fit. For instance, the choice of linearisation point based on the predicted mean $\hat{x}_{k|k-1}$ is not informed by the measurements $y_{1:k}$. In effect, they select the parameters $\Theta_{1:k}$ with respect to the prior distribution of the states. Previous work has noted that the approximation is sensitive to the linearisation point when the model is non-linear and the measurement noise is low [18].

Ideally, the linearisation point would be chosen with respect to the posterior distribution of the states, thereby selecting the parameters $\Theta_{1:k}$ of the enabling approximation optimally, taking into account all the measurements $y_{1:k}$. However, we cannot directly select parameters with respect to the true posterior, since it is unknown.

Iterative smoothers take this insight into account to improve estimation performance. By iteratively refining the linearisation (2), the algorithms can improve the estimates by using successively better linearisations. Given an estimate of the posterior moments $\hat{x}_{1:k}, \hat{P}_{1:k}$, we obtain a new linear approximation $\Theta_{1:k}$, from which new estimates of the moments $\hat{x}_{1:k}, \hat{P}_{1:k}$ are obtained, using RTS smoothing [26]. The two-step process is iterated

$$\Theta_{1:k}^{(i)} = \text{Linearisation} \left( \hat{x}_{1:k}^{(i-1)}, \hat{P}_{1:k}^{(i-1)} \right)$$

$$\left( \hat{x}_{1:k}^{(i)}, \hat{P}_{1:k}^{(i)} \right) = \text{RTS smoother} \left( \Theta_{1:k}^{(i)}, y_{1:k} \right).$$  \hspace{1cm} (6)

The starting estimate of the moments are commonly, but not necessarily, the output of the corresponding non-iterative smoother. Hopefully, with every iteration, the estimates of the posterior grow closer to the true posterior until the linearisation point is approximately chosen with respect to the true posterior.

The EKS is a well-known iterative extension of the EKS. The linearisation step in (6) is done with a first-order Taylor expansion around the estimated means from the previous iteration.

The IPLS is a more recent iterative smoother, first introduced in [10]. In IPLS, $\Theta_{1:k}^{(i)}$ is selected by performing SLR as in (4a) to (4c) and (5) with $p(x_k) = \mathcal{N}(x_k; \tilde{x}_k^{(i)}, \tilde{P}_k^{(i)})$. It is the iterative version of the family of sigma point smoothers we call PrLS. It should be noted that the SLR described in (4a) to (4c) and (5) has a simple interpretation. The parameters in the linear approximation of the motion model are

$$(F_k, b_k) = \min_{F_k, b_k} \mathbb{E} \left[ \left\| f_k(x_k) - F_k^+ x_k - b_k^+ \right\|^2_2 | y_{1:k} \right]$$  \hspace{1cm} (7)

and the covariance matrix of the linearisation error for $f_k$ is

$$\Omega_k = \mathbb{E} \left[ \left\| f_k(x_k) - F_k x_k - b_k \right\|^2_2 | y_{1:k} \right].$$  \hspace{1cm} (8)

A similar equation holds for the measurement model.
B. The Gauss–Newton (GN) method

It is useful to view the iterated smoothers as optimisation algorithms, by identifying a cost function that they minimise. An important advantage with this is that it enables us to make use of other optimisation methods, to minimise the same cost function, which have better convergence properties. Formulated as a general optimisation problem, many techniques can be applied but the smoothers offer the advantage of exploiting the problem structure.

Gauss–Newton (GN) is a well-known optimisation method which is used to solve problems on the form

\[ x^* = \arg \min_x L_{GN}(x) \]

\[ = \arg \min_x \frac{1}{2} \| \rho(x) \|_2^2 = \arg \min_x \frac{1}{2} \rho(x) \top \rho(x), \tag{9} \]

where \( \rho(x) \) is a general function. Starting from an initial guess, the method iteratively finds the exact solution to the approximate objective

\[ \hat{L}^{(i)}_{GN}(x) = \frac{1}{2} \| \hat{\rho}^{(i)}(x) \|_2^2 = \frac{1}{2} \hat{\rho}^{(i)}(x) \top \hat{\rho}^{(i)}(x), \tag{10} \]

defined by the first order approximation of \( \rho(\cdot) \) around \( \hat{x}^{(i)} \)

\[ \rho(x) \approx \hat{\rho}^{(i)}(x) := \rho(\hat{x}^{(i)}) + J_\rho(\hat{x}^{(i)})(x - \hat{x}^{(i)}). \tag{11} \]

Linearisation is done around the current iterate \( \hat{x}^{(i)} \) and the next iterate is the solution to the approximate problem [19].

C. Levenberg–Marquardt regularisation

The GN method is not guaranteed to converge rather, like many iterative methods, it can diverge. A more robust extension of GN is the Levenberg–Marquardt (LM) method [15], [17], [19]. In the LM method, robustness is achieved by extending \( L_{GN} \) with a regularisation term

\[ L^{(i)}_{LM}(x) = L_{GN}(x) \]

\[ + \frac{1}{2} \lambda^{(i)}(x - \hat{x}^{(i)}) \top S^{(i)}^{-1}(x - \hat{x}^{(i)}), \tag{12} \]

where \( \lambda^{(i)} > 0 \) is an adaptable regularisation parameter and \( S^{(i)} \) is a sequence of positive definite regularisation matrices. The matrices \( S^{(i)} \) can be selected to scale the problem suitably [19], [23]. In this paper, we assume that the matrices are given while \( \lambda^{(i)} \) is adapted as part of the optimisation algorithm. The regularisation term encourages a new iterate to be close to the previous one, hopefully in a region where the approximation is acceptable. A new iterate is accepted only if it decreases the cost function. The level of regularisation is controlled by adapting \( \lambda^{(i)} \) from some initial value; reducing \( \lambda^{(i)} \) when an iterate is accepted and increasing it on rejection. We introduce \( \nu > 1 \) as a parameter to control the adaptation and increase or reduce \( \lambda^{(i)} \) with a factor \( \nu \) on accepting or rejecting an iterate respectively [17], [20].

D. The IEKS as a GN method

The IEKS can be interpreted as an optimisation method. Consider the problem of minimising the cost function

\[ L_{IEKS}(x_{1:K}) \]

\[ = \frac{1}{2} \left( (x_1 - \hat{x}_{1|0}) \top \hat{P}_{1|0}^{-1}(x_1 - \hat{x}_{1|0}) \right) \]

\[ + \sum_{k=1}^{K} (y_k - h_k(x_k)) \top R_k^{-1}(y_k - h_k(x_k)) \]

\[ + \sum_{k=1}^{K-1} (x_{k+1} - f_k(x_k)) \top Q_k^{-1}(x_{k+1} - f_k(x_k)) \). \tag{13} \]

GN optimisation of this function is equivalent to running the IEKS for the corresponding state-space model. This was first shown in [4], and we highlight this result via the following proposition:

**Proposition III.1.** The IEKS inference of the state-space model in (1) is a GN method for minimising the function \( L_{IEKS} \) in (13).

At each iteration the GN algorithm linearises \( L_{GN}(x_{1:K}) \) in (13) with a first-order Taylor expansion around the current estimate \( \hat{x}^{(i)}_{1:K} \) and computes the next estimate \( \hat{x}^{(i+1)}_{1:K} \) as the closed form solution of the linearised objective. The IEKS performs iterative smoothing of the state-space model in (1) by linearising it, also using first-order Taylor expansion around \( \hat{x}^{(i)}_{1:K} \) and analytically computing the next estimate \( \hat{x}^{(i+1)}_{1:K} \) with RTS smoothing.
Figure 1. Visualisation of the connection between GN optimisation and a general Gaussian smoother. Both the Gauss-Newton algorithm and the general Gaussian smoothers work by 1) linearising the problem and 2) solving the linearised problem analytically. The idea in the proofs of props. III.1 and IV.1 is to show that the linearised problems are identical under certain conditions. This implies that the problems must have the same closed form solution and the algorithms therefore yield the same output.

The proof of prop. III.1 can be separated into three parts: First, constructing $\rho(x_{1:K})$ for the cost function in (13) and linearising it around $\tilde{x}^{(i)}_{1:K}$ to obtain $\tilde{L}^{(i)}_{GN}(x_{1:K})$. Second, linearising the state-space model, which for the IEKS is done with a first-order Taylor expansion around the estimated means $\hat{x}^{(i)}_{1:K}$ from the previous iteration. Third, noting that $L^{(i)}_{GN}(x_{1:K})$ is the negative log-posterior of the linearised state-space model in (2) (up to a constant). Then, since the GN algorithm minimises $\tilde{L}^{(i)}_{GN}$ in closed form, whereas the IEKS minimises the negative log-posterior, it follows that $\hat{x}^{(i)}_{1:K}$ must be identical for both algorithms, making them equivalent. This structure is outlined in Fig. 1 and we will reuse it when proving a similar connection between GN and the IPLS.

IV. ROBUST ITERATED POSTERIOR LINEARISATION SMOOTHERS

A. Iterated Posterior Linearisation Smoother IPLS

Inspired by the IEKS, we seek to establish a similar connection between the IPLS and the GN method. To this end, we require a cost function that leads to a GN method, corresponding to the IPLS.

A key difference between the IEKS and the IPLS is the role the covariances play. In the IEKS, the analytical linearisation is done only with respect to the estimated means $\hat{x}^{(i)}_{1:K}$, whereas the SLR linearisation of the IPLS is based on both the estimated means $\bar{x}^{(i)}_{1:K}$ and covariances $\hat{P}^{(i)}_{1:K}$. The IEKS further assumes that the linearisation error covariances are zero, whereas the IPLS estimates them as $\hat{\Omega}^{(i)}_{k}, \hat{\Gamma}^{(i)}_{k}$ respectively. These matrices appear in the approximate state-space model of (2) and must therefore be included in a cost function. The estimated covariances $\tilde{P}^{(i)}_{1:K}$ are implicitly included in the approximate state-space model, since they are used in the SLR linearisation.

The inclusion of the covariance matrices complicates the construction of a matching GN objective, which should be on a quadratic form in the state sequence $x_{1:K}$ (or some function of it). The matrix defining the quadratic form will depend on the (estimated means of the) state sequence and the cost function will also, implicitly, depend on the covariance sequence.

We propose to use the cost function

$$L_{IPLS}^{(i)}(x_{1:K}) = \frac{1}{2} \left( \sum_{k=1}^{K-1} (x_{k+1} - \bar{x}_k(x_k))^\top (Q_k + \hat{\Omega}^{(i)}_k)^{-1} (x_{k+1} - \bar{x}_k(x_k)) + \sum_{k=1}^{K} (y_k - \tilde{y}_k(x_k))^\top (R_k + \hat{\Gamma}^{(i)}_k)^{-1} (y_k - \tilde{y}_k(x_k)) \right),$$

(14)

where $\bar{x}(\cdot), \tilde{y}(\cdot)$ are the SLR estimated expectations:

$$\bar{x}_k(x_k) = \int f_k(x)\mathcal{N}(x; x_k, \tilde{P}^{(i)}_k)dx,$$

(15a)

$$\tilde{y}_k(x_k) = \int h_k(x)\mathcal{N}(x; x_k, \tilde{P}^{(i)}_k)dx,$$

(15b)

and $\hat{\Omega}^{(i)}_k, \hat{\Gamma}^{(i)}_k$ are computed with (4c) with respect to $\bar{x}^{(i)}_{1:K}, \tilde{P}^{(i)}_{1:K}$ using $f_k$ and $h_k$ respectively.

It is clear that the cost function in (14) depends on the most recent estimate of the sequence of covariance matrices, both through the SLR expectations in (15a) and (15b) and through the estimated linearisation errors $\hat{\Omega}^{(i)}_k$ and $\hat{\Gamma}^{(i)}_k$.

Proposition IV.1. The output of one iteration of GN optimisation of the cost function $L_{IPLS}^{(i)}(x_{1:K})$ in (14), defined by the current GN estimate $\hat{x}^{(i)}_{1:K}$ and the covariance matrices $\tilde{P}^{(i)}_{1:K}$, is the same as the means estimated by RTS smoothing of (1), linearised with SLR around $\bar{x}^{(i)}_{1:K}, \hat{P}^{(i)}_{1:K}$, that is, one iteration of the IPLS.
Proof. The proof follows the steps outlined in Fig. 1. We construct $\rho^{(i)}(x_{1:K})$ such that $L_{1\text{PLS}}^{(i)}(x_{1:K}) = \frac{1}{2} \|\rho^{(i)}(x_{1:K})\|_2^2$ and then linearise $\rho^{(i)}(x_{1:K})$ to form the approximate objective $\tilde{L}_{1\text{PLS}}^{(i)}(x_{1:K})$. Secondly, the state-space model is linearised with SLR, according to the IPLS. Finally, we compare the approximate GN objective to the linearised state-space model and note that they correspond to the same minimisation problem.

Note that the covariance matrices $\Omega_k^{(i)}$ and $\Gamma_k^{(i)}$ in $L_{1\text{PLS}}^{(i)}(x_{1:K})$ depend on the current estimates $\hat{x}_{1:K}$ and not on the optimisation variable $x_{1:K}$. Therefore, $L_{1\text{PLS}}^{(i)}(x_{1:K})$ is on the form of (9) and can be optimised with the GN method.

We construct $\rho^{(i)}(x_{1:K})$ by collecting states and measurements in a vector and group the covariance matrices in a single block diagonal matrix:

$$z_2(x_{1:K}) = \begin{pmatrix} x_1 - \hat{x}_{1:0} \\ x_2 - \hat{x}_1(x_1) \\ \vdots \\ x_K - \hat{x}_{K-1}(x_{K-1}) \\ y_1 - \hat{y}_1(x_1) \\ y_2 - \hat{y}_2(x_2) \\ \vdots \\ y_K - \hat{y}_K(x_K) \end{pmatrix}, \quad (16a)$$

$$\Sigma_{z_2}^{-1} = \text{diag}(\tilde{P}_{1:10}, (Q_1 + \Omega_1^{(i)})^{-1}, \ldots, (Q_{K-1} + \Omega_{K-1}^{(i)})^{-1}, (R_1 + \Gamma_1^{(i)})^{-1}, \ldots, (R_K + \Gamma_K^{(i)})^{-1}). \quad (16b)$$

Defining

$$\rho^{(i)}(x_{1:K}) = \Sigma_{z_2}^{-T/2} z_2(x_{1:K}), \quad (17)$$

with $\Sigma_{z_2}^{-1} = \Sigma_{z_2}^{-T/2} \Sigma_{z_2}^{-1} = \Sigma_{z_2}^{-1}$, we confirm that

$$\frac{1}{2} \|\rho^{(i)}(x_{1:K})\|_2^2 = \frac{1}{2} z_2^T(x_{1:K}) \Sigma_{z_2}^{-1} z_2(x_{1:K}) = L_{1\text{PLS}}^{(i)}(x_{1:K}).$$

Next, we linearise $\rho^{(i)}$ by computing the first-order approximation around $\hat{x}_{1:K}$

$$\rho^{(i)}(x_{1:K}) \approx \tilde{\rho}^{(i)}(x_{1:K})$$

$$= \rho^{(i)}(\hat{x}_{1:K}) + J_p^{(i)}(\hat{x}_{1:K})(x_{1:K} - \hat{x}_{1:K})$$

$$= \Sigma_{z_2}^{-T/2} z_2(\hat{x}_{1:K}) + \Sigma_{z_2}^{-T/2} J_{z_2}(x_{1:K})(x_{1:K} - \hat{x}_{1:K})$$

$$= \Sigma_{z_2}^{-T/2} \hat{z}_2^{(i)}(x_{1:K}), \quad (18)$$

where

$$\hat{z}_2^{(i)}(x_{1:K}) = \begin{pmatrix} x_1 - \hat{x}_{1:0} \\ x_2 - \left(F_1(\hat{x}_{1:K}) x_1 + b_1(\hat{x}_{1:K})\right) \\ x_3 - \left(F_2(\hat{x}_{2:K}) x_2 + b_2(\hat{x}_{2:K})\right) \\ \vdots \\ x_K - \left(F_{K-1}(\hat{x}_{K-1:K}) x_{K-1} + b_{K-1}(\hat{x}_{K-1:K})\right) \\ y_1 - \left(H_1(\hat{x}_{1:K}) x_1 + c_1(\hat{x}_{1:K})\right) \\ \vdots \\ y_K - \left(H_K(\hat{x}_{K:K}) x_K + c_K(\hat{x}_{K:K})\right) \end{pmatrix}$$

and $F_k(x_k), H_k(x_k)$ are the SLR Jacobians of $f_k, h_k$ respectively, see the supplementary material for the derivation details.

The approximate GN objective becomes

$$\tilde{L}_{1\text{PLS}}^{(i)}(x_{1:K})$$

$$= \frac{1}{2} \hat{z}_2^{(i)}(x_{1:K})^T \Sigma_{z_2}^{-1} \hat{z}_2^{(i)}(x_{1:K}) \left(\hat{z}_2^{(i)}(x_{1:K})\right). \quad (19)$$
To perform one iteration of IPlS we instead first linearise the state-space model in (1) using SLR with respect to \( \mathcal{N}(x_k; \hat{x}^{(i)}_k, \hat{P}^{(i)}_k) \). The resulting approximate state-space model is on the form (2) with linearisation parameters \( \Theta^{(i)}_{1:K} \) selected using (4a) to (4c). The next iterate \( \hat{x}^{(i+1)}_1 \) is computed as the closed-form output of RTS smoothing.

Examining \( L^{(i)}_{\text{IPLS}}(x_{1:K}) \), we note that it is the negative log-posterior of the SLR linearised state-space model (up to a constant). The next GN iterate will be the closed-form solution to this minimisation problem. Since the two methods, GN and IPlS, in a single iteration computes the exact solution to the same optimisation problem, their output \( \hat{x}^{(i+1)}_{1:K} \) must be the same.

\[ \hat{x}^{(i)}_k = x_k + e_k, \quad e_k \sim \mathcal{N}(0, (\lambda^{(i)})^{-1} S^{(i)}_k) \] (20)

is a GM method with the LM-regularisation defined in (12), if \( S^{(i)} \) is a sequence of block-diagonal regularisation matrices: \( S^{(i)} = \text{diag}(S^{(i)}_1, \ldots, S^{(i)}_K), S^{(i)}_k \in \mathbb{R}^{d_x \times d_x}, \forall k = 1, \ldots, K \).

Proof. The proof follows the structure of the earlier proofs and we can use the results of props. III.1 and IV.1 to simplify it. First, we construct \( \rho^{(i)}_{\text{LM}}(x_{1:K}) \) such that \( L^{(i)}_{\text{LM}}(x_{1:K}) = 1 \frac{1}{2} \left\| \rho^{(i)}_{\text{LM}}(x_{1:K}) \right\|_2^2 \) and show that the linearisation of \( \rho^{(i)}_{\text{LM}}(x_{1:K}) \) results in an approximate objective which is \( \tilde{L}^{(i)}_{\text{GN}}(x_{1:K}) \) plus an extra term. Second, we introduce the measurement in (20) into the state-space model (1) and linearise. Third, we confirm that the LM optimisation and iterative smoothers solve the same minimisation problem at each iteration.

In step 1 we derive the approximate LM objective. From (12) we have that the LM objective \( L^{(i)}_{\text{LM}}(x_{1:K}) \) is the sum of the GN objective \( L^{(i)}_{\text{GN}}(x_{1:K}) \) and a regularisation term. By simply extending \( \rho(x_{1:K}) \) in (17), we can construct

\[ \rho^{(i)}_{\text{LM}}(x_{1:K}) = \begin{pmatrix} \rho(x_{1:K}) \\ \left( (\lambda^{(i)})^{-1} S^{(i)} \right)^{-T/2}(x_{1:K} - \hat{x}^{(i)}_{1:K}) \end{pmatrix} \] (21)

such that \( L^{(i)}_{\text{LM}}(x_{1:K}) = \frac{1}{2} \left\| \rho^{(i)}_{\text{LM}}(x_{1:K}) \right\|_2^2 \).

Since the regularisation term is linear in \( x_{1:K} \), the resulting linearisation is

\[ \tilde{L}^{(i)}_{\text{LM}}(x_{1:K}) \approx \begin{pmatrix} \tilde{\rho}^{(i)}(x_{1:K}) \\ \left( (\lambda^{(i)})^{-1} S^{(i)} \right)^{-T/2}(x_{1:K} - \hat{x}^{(i)}_{1:K}) \end{pmatrix} \] (22)

where we note that the approximate objective is indeed the approximate GN objective in (19) with added LM regularisation

\[ \tilde{L}^{(i)}_{\text{LM}}(x_{1:K}) = \frac{1}{2} \left\| \begin{pmatrix} \tilde{\rho}^{(i)}(x_{1:K}) \\ \left( (\lambda^{(i)})^{-1} S^{(i)} \right)^{-T/2}(x_{1:K} - \hat{x}^{(i)}_{1:K}) \end{pmatrix} \right\|_2^2 \]

\[ = \tilde{L}^{(i)}_{\text{GN}}(x_{1:K}) + \frac{1}{2} \lambda^{(i)} (x_{1:K} - \hat{x}^{(i)}_{1:K})^T S^{(i)} x_{1:K} - \hat{x}^{(i)}_{1:K} \] (23)

In step 2 we derive the negative log-posterior for the linearised state-space model with the measurement \( \hat{x}^{(i)}_k \) in (20). The additional measurement will, for each timestep, contribute with a term \( \frac{1}{2} \lambda^{(i)} (x_k - \hat{x}^{(i)}_k)^T S^{(i)} x_k - \hat{x}^{(i)}_k \) to the negative log-posterior. All the extra measurements can be combined into a single term \( \frac{1}{2} \lambda^{(i)} (x_{1:K} - \hat{x}^{(i)}_{1:K})^T S^{(i)} x_{1:K} - \hat{x}^{(i)}_{1:K} \), with the block-diagonal matrix \( S^{(i)} = \text{diag}(S^{(i)}_1, S^{(i)}_2, \ldots, S^{(i)}_K), S^{(i)}_k \in \mathbb{R}^{d_x \times d_x} \). Note that we restrict ourselves to \( S^{(i)} \) on this form, whereas other suitable options exist [19], [23]. The measurement model for \( \hat{x}^{(i)}_{1:K} \) is linear in \( x_{1:K} \) so the log-posterior of the approximated state-space model produced by IEKS or IPlS linearisation will simply be extended with this term.

In step 3 we compare the linearisations. We know from props. III.1 and IV.1 that the log-posterior without the measurement is \( \tilde{L}^{(i)}_{\text{LM}}(x_{1:K}) \) and after introducing the measurement the log-posterior (up to a constant) is therefore \( L^{(i)}_{\text{LM}}(x_{1:K}) \) in (23).
As in the earlier proofs, we conclude that the algorithms yield the same result since they minimise the same loss function in closed form. It follows that an iteration of IEKS or IPLS for this extended state-space model is equivalent to an iteration of LM optimisation of the corresponding cost function.

In other words, we achieve a more robust version of the smoother by imposing this additional modelling assumption.

C. Line-search

Another way to improve the GN method is to introduce a line-search procedure [19] to the algorithm. The line-search version of the IEKS was described in [28] and here we extend it to the IPLS. Line-search can be implemented by introducing a parameter $\alpha > 0$ to restrict the iterative update of the estimates. That is, given $\hat{x}_{1:K}^{(i+1)}$ and $\hat{x}_{1:K}^{(i)}$, we define $\Delta\hat{x}_{1:K}^{(i+1)} = \hat{x}_{1:K}^{(i+1)} - \hat{x}_{1:K}^{(i)}$, and we obtain the line-search update of the estimates:

$$\hat{x}_{1:K}^{(i+1)}(\alpha) = \hat{x}_{1:K}^{(i)} + \alpha \Delta\hat{x}_{1:K}^{(i+1)}$$

where

$$\alpha = \arg \min_{\alpha \in [0,1]} L^{(i)}_{\text{GN}}(\hat{x}_{1:K}^{(i)} + \alpha \Delta\hat{x}_{1:K}^{(i+1)})$$

We can also use an inexact version of line-search which only requires a sufficient decrease in the cost function by using Armijo or Wolfe conditions [19]. An algorithm for inexact line-search with Armijo conditions was presented in [28, Alg. 4] and the Wolfe conditions can be implemented analogously.

By selecting a suitable estimate on a line between the previous estimate and the new one proposed by the smoothing update, the methods become more robust since the size of the update step is allowed to decrease for iterations that risk diverging. Potentially, this could also lead to faster convergence, albeit with the extra computational demand incurred by finding the optimal $\alpha$.

D. Algorithms

We can now give a full description of the robust versions of the iterative smoothers.

1) LM-regularisation: The method is an iterative smoother which uses estimates from the previous iteration to linearise the motion and measurement models, giving an approximate affine state-space model as in (2). A new estimate is then proposed through RTS smoothing of the affine state-space model, with an extra measurement of the state, corresponding to LM regularisation, see prop. IV.2. The new estimate is accepted if it results in a lower value for an associated cost function, see (13) and (14).

A single iteration step for the linearised models is described in alg. 1. The full algorithm is simply the iteration of steps taken in alg. 1, using the accepted estimates in the previous step as the estimates used for linearisation. The complete procedure is described in alg. 2. For readability, we omit some model parameters in the algorithmic description, the origins of which should be clear from the context.

The differences between the variants of the LM smoother stems from the different method of linearisation: SLR defined in (4a) to (4c) for the LM–IPLS and Taylor expansion in (3a) to (3c) for the LM–IEKS. Apart from the obvious difference in the computed linearisation, the SLR also requires some extra steps in the algorithm, which we detail below.

The IPlS’s use of both the estimated means and covariances for the linearisation requires a sequence of cost functions (instead of a single cost function), see Section IV-A. The cost function is changed when the current estimated covariances are updated. In practice, the algorithm controls this by adding an inner loop in alg. 2.

The inner loop allows for an arbitrary number of LM-iteration steps, where the estimated means are updated while the covariances are kept fixed. After some provided termination condition is fulfilled, the covariances are updated, thereby moving on to a new cost function. We have found that the simplest setting, to exit the inner loop after a single iteration, works well in practice but more elaborate conditions are possible, such as requiring a sufficient decrease in $L^{(i)}_{\text{LM}}$ or a sufficiently large $\lambda^{(i)}$. For the LM–IEKS this inner loop has no effect since it uses the same cost function throughout the optimisation.

In alg. 2 the LM–IEKS and LM–IPLS differ only in the different methods of linearisation that are applied when estimating the affine approximations.

2) Line-search: The basis of the line-search algorithms LS–IEKS and LS–IPLS is to optimise the line that connects the previous and proposed estimates, see (24) and (25). However, when we use the IEKS/IPLS implementation of the GN method, there is no increment computed in the same sense as in the classical formulation of the GN method. Fortunately, given the previous iterate $\hat{x}_{1:K}^{(i)}$ and the proposed iterate $\hat{x}_{1:K}^{(i+1)}$, we can compute the corresponding increment via $\Delta\hat{x}_{1:K}^{(i+1)} = \hat{x}_{1:K}^{(i+1)} - \hat{x}_{1:K}^{(i)}$. The proposed iterate $\hat{x}_{1:K}^{(i+1)}$ is computed with a standard step of the GN optimisation, which is equivalent to running alg. 1 with $\lambda^{(i)} = 0$. A combined line-search algorithm is described in alg. 3.

Similar to the LM-regularised methods, there is some difference between the IEKS and IPLS based versions of the line-search algorithm. The LS–IEKS and LS–IPLS use their respective version of the smoother iteration in alg. 1, detailed in Section IV-D1. For the LS–IEKS, only the estimated means $\hat{x}_{1:K}^{(i)}$ are used in the linearisation and the estimated covariances are disregarded.
The cost function at every iteration. Unless otherwise stated we use with the lowest cost. Alternatively, we can use inexact line-search and only require a sufficient decrease of the cost function can always perform an approximative exact line-search with a grid-search, comparing a fixed number of candidates for the properties of the different smoothers.

For the same reason, the inner loop which enables repeated optimisation of the same cost function, that is with covariances of the IPLS methods is more computationally expensive than its IEKS counterpart. The large spread in the metrics for the IEKS methods comes from the fact that they diverge for a significant number of realisations.

\[ L_{i,j}^{L}(\hat{x}_{i,j}) < L_{i,j}^{L}(\hat{x}_{i,j}) \]

\[ \lambda \leftarrow \lambda/\nu \]

\[ \lambda \leftarrow \nu \lambda \]

\[ \text{end procedure} \]

V. SIMULATION RESULTS

We demonstrate the usefulness of a robust iterative approach for hard smoothing problems and make an analysis of the properties of the different smoothers.

In the experiments we use a scheme where we update the estimates after each accepted new iterate, that is updating the cost function at every iteration. Unless otherwise stated we use \( \lambda^{(i)} = 0.01, \nu = 10, S^{(i)} = \text{diag}(S_{1}^{(i)}, \ldots, S_{K}^{(i)}) \) with \( S_{k}^{(i)} = I, k = 1, \ldots, K \) as parameters for the LM-regularisation and perform line-search through a grid-search with 10 uniformly spaced candidates for \( \alpha \).

Implementations for all the experiments are available here.

A. Coordinated turn (CT) model with bearings only measurements

We extend the experiment from [28] to include the IPLS methods along with the IEKS method of the original paper. We also include an extended analysis of different metrics across iterations, averaged over 100 independent realisations. The experiment setup is the same as in the original experiment with a true trajectory simulated from a coordinated turn model and measurements of bearings only. The bearings measurements come from two sensors placed at \((-1.5, 0.5)^{T}\) and \((1, 1)^{T}\) respectively, with relatively high noise with variance \( \sigma^{2} = 1/2^{2} \) rad². A single realisation is shown in Fig. 2, along with examples of estimated trajectories from the different models.

For this particular realisation, all algorithms perform similarly, largely following the true trajectory. To see a discrepancy between the models we measure root mean square error (RMSE) and normalised estimation error squared (NEES) [3], averaged over 100 independent realisations. The results are displayed in Fig. 3. From these results it is clear that the IPLS methods consistently perform better and require fewer iterations to reach a good trajectory. It should again be noted that a single iteration of the IPLS methods is more computationally expensive than its IEKS counterpart. The large spread in the metrics for the IEKS methods comes from the fact that they diverge for a significant number of realisations.
Figure 2. Simulated coordinated turn with bearings only measurements. The bearings measurements come from two sensors placed at \((-1.5, 0.5)^T\) and \((1, 1)^T\) respectively.

Figure 3. Simulated coordinated turn model with bearings only measurements, see Fig. 2 for setup. The plot shows averaged RMSE and NEES across iterations averaged over 100 independent trials. Both the true trajectory and measurements are resampled at every trial. Error bars correspond to the standard error, i.e. the estimated standard deviation scaled by \(1/\sqrt{100}\).
relies on using a fine grid of points to approximate the cost function accurately along the line-segment. A cost function which is costly to compute, since the SLR expectations need to be reestimated for every new sequence of estimated means.

We show that LM-regularisation can be achieved with a simple LM–IEKS, LM–IPLS and line-search smoothers LS–IEKS, LS–IPLS. We build on existing work connecting the IEKS to GN optimisation and derive a similar interpretation for the IPLS. We examine the impact of regularisation, we analyse a special case of the coordinated turn experiment in Section VI.

The experiments further show that the IPLS based smoothers perform better than their IEKS counterparts, albeit with a higher computational complexity. This is especially true for the grid-search version of the LS–IPLS since its performance relies on using a fine grid of points to approximate the cost function accurately along the line-segment. A cost function which is costly to compute, since the SLR expectations need to be reestimated for every new sequence of estimated means.

VI. DISCUSSION AND CONCLUSION

In this paper we present more robust versions of the IEKS and ILS smoothers in the form of LM-regularised smoothers LM–IEKS, LM–IPLS and line-search smoothers LS–IEKS, LS–IPLS. We build on existing work connecting the IEKS to GN optimisation and derive a similar interpretation for the IPLS. We show that LM-regularisation can be achieved with a simple modification of the state-space model in the form of an added pseudo-measurement of the state.

We present simulation results that show the importance of the robustness achieved by our smoother methods, as well as providing further support for the conclusion of existing work that suggests the benefit of iterative smoothers in general.

The experiments further show that the ILS based smoothers perform better than their IEKS counterparts, albeit with a higher computational complexity. This is especially true for the grid-search version of the LS–IPLS since its performance relies on using a fine grid of points to approximate the cost function accurately along the line-segment. A cost function which is costly to compute, since the SLR expectations need to be reestimated for every new sequence of estimated means.
Algorithm 3 Iterative smoothing with line-search

**Input:** Initial moments \( \hat{x}^{(0)}_{1:K}, P^{(0)}_{1:K} \), priors \( \hat{x}_{1:0} \) and \( P_{1:0} \), measurements \( y_{1:K} \), smoother type \( t \in \{LM–IEKS, LM–IPLS\} \) and implicitly a cost function \( L^{(i)} \), motion and measurement models and parameters: \( f_{1:K}, Q_{1:K}, h_{1:K}, R_{1:K} \).

**Output:** The smoothed trajectory \( \hat{x}^{(i)}_{1:K}, \hat{P}^{(i)}_{1:K} \).

1: procedure LS–IEKS/LS–IPLS
2:     Set \( i \leftarrow 0 \)
3:     repeat
4:         if \( t == \) LS–IEKS then
5:             Set \( \Omega_{1:K}, \Gamma_{1:K} \) to 0, see (3c)
6:         else if \( t == \) LS–IPLS then
7:             Est. \( \Omega_{1:K}, \Gamma_{1:K} \) using \( \hat{x}^{(i)}_{1:K}, \hat{P}^{(i)}_{1:K} \) in (4c)
8:         end if
9:         // Init tracking cov. estimates in the inner loop.
10:        \( \hat{P}^{(i)}_{1:K} \leftarrow \hat{P}^{(i)}_{1:K} \)
11:     repeat
12:         if \( t == \) LM–IEKS then
13:             // Affine approx. using (3a) and (3b).
14:             Est. \( F_{1:K}, b_{1:K}, H_{1:K}, c_{1:K}, \Gamma_{1:K} \) using \( \hat{x}^{(i)}_{1:K} \)
15:         else if \( t == \) LM–IPLS then
16:             // Affine approx. using (4a) and (4b).
17:             Est. \( F_{1:K}, b_{1:K}, H_{1:K}, c_{1:K} \), using \( \hat{x}^{(i)}_{1:K}, \hat{P}^{(i)}_{1:K} \)
18:         end if
19:         \( \Theta^{(i)}_{1:K} \leftarrow F_{1:K}, b_{1:K}, \Omega_{1:K}, H_{1:K}, c_{1:K}, \Gamma_{1:K} \)
20:         // LM-iter with \( \lambda^{(i)} = 0 \) corresp. to a GN-step.
21:         \( \hat{x}^{(i)}_{1:K}, \hat{P}^{(i)}_{1:K} \leftarrow \text{LM-iter}(\hat{x}^{(i)}_{1:K}, \hat{P}^{(i)}_{1:K}, y_{1:K}, 0, \Theta^{(i)}_{1:K}) \)
22:         \( \Delta \hat{x}^{(i)}_{1:K}, \Delta \hat{P}^{(i)}_{1:K} \leftarrow \hat{x}^{(i)}_{1:K} - \hat{x}^{(i)}_{1:K}, \hat{P}^{(i)}_{1:K} - \hat{P}^{(i)}_{1:K} \)
23:         \( \alpha \leftarrow \arg \min L^{(i)}(\hat{x}^{(i)}_{1:K} + \alpha \Delta \hat{x}^{(i+1)}_{1:K}) \) \( \alpha \epsilon [0,1] \)
24:         \( \hat{x}^{(i+1)}_{1:K} \leftarrow \hat{x}^{(i)}_{1:K} + \alpha \Delta \hat{x}^{(i+1)}_{1:K} \)
25:         // Track cov. est. w/o updating the cost.
26:         \( \hat{P}^{(i)}_{0:K} \leftarrow \hat{P}^{(i)}_{0:K} + \alpha \Delta \hat{P}^{(i)}_{1:K} \)
27:         \( \hat{P}^{(i+1)}_{1:K} \leftarrow \hat{P}^{(i)}_{1:K} \)
28:         \( i \leftarrow i + 1 \)
29:     until Inner loop termination condition met
30:     \( \hat{P}^{(i)}_{1:K} \leftarrow \hat{P}^{(i)}_{1:K} \)
31:     until Converged
32:     return \( \hat{x}^{(i)}_{1:K}, \hat{P}^{(i)}_{1:K} \)
33: end procedure

The increased complexity is somewhat ameliorated by the empirical observation that the IPLS methods require fewer iterations to find an acceptable estimate of the trajectory.

The benefit of regularisation appears to be more important for the IEKS based smoothers. This is probably due to the increased complexity IPLS method. In a sense, the ordinary IPLS is in itself regularised since it will have less trust in an update in a region where the linearisation of motion or measurement is uncertain due to non-linearities.

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Figure 4. Visualisation of a single realisation of the CT experiment with varying bearings only measurements. The measurements at $k = 50, 100, \ldots, 500$ are low noise bearings measurements from a single sensor at $(1,1)^T$. Note that for this particular realisation, it is only the LM-regularised smoothers, LM–IEKS and LM–IPLS that estimate the general shape of the true trajectory.

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Figure 5. Simulated coordinated turn model with bearings only measurements. The plot shows averaged RMSE and NEES across iterations. Note that the first iterations are omitted to better display the relative performance in the end result. Error bars correspond to the standard error, that is, the estimated standard deviation scaled by $1/\sqrt{100}$.

APPENDIX

A. Derivation of the GN-IPLS

The proof of prop. IV.1 is similar in structure to prop. III.1 in that it requires finding the derivative of the function $\rho^{(i)}$ and constructing the approximation $\tilde{\rho}^{(i)}$. Here, we provide details of these two steps.

The Jacobians of the SLR expectations $\bar{x}(\cdot), \bar{y}(\cdot)$ are

$$J_{\bar{x}}(x_k) = J(\mathbb{E}[f(x_k)])$$

$$= J \left( \int f(x)\mathcal{N}(x; x_k, \hat{\rho}^{(i)})dx \right)$$

$$= \int f(x)J(\mathcal{N}(x; x_k, \hat{\rho}^{(i)}))dx$$

$$= \int f(x)(x - x_k)^T \mathcal{N}(x; x_k, \hat{\rho}^{(i)})dx \left( \hat{\rho}^{(i)} \right)^{-1}$$

Here, we use equation (S-9 in [21]) and note that

$$\int \bar{x}_k(x_k) (x - x_k)^T \mathcal{N}(x; x_k, \hat{\rho}^{(i)})dx$$

$$= \bar{x}_k(x_k) \left( \int x^T \mathcal{N}(x; x_k, \hat{\rho}^{(i)})dx - x_k^T \int \mathcal{N}(x; x_k, \hat{\rho}^{(i)})dx \right)$$

$$= \bar{x}_k(x_k) \left( x_k^T - x_k^T \right) = 0.$$
We substitute (27) into the derivative in (26) to obtain
\[
J_{\hat{F}}(x_k) = \int (f(x) - \hat{x}_k(x_k)) (x - x_k)^T N(x; x_k, \hat{\rho}^{(i)}) dx \left( \hat{\rho}^{(i)} \right)^{-1} = \Psi_{h_k}(x_k) \left( \hat{\rho}^{(i)} \right)^{-1} = F_k(x_k).
\]
(28)

Analogously,
\[
J_{\hat{H}}(x_k) = \Psi_{h_k}^T(x_k) \left( \hat{\rho}^{(i)} \right)^{-1} = H_k(x_k)
\]
(29)

Given the definition of \(\rho^{(i)}\) in (17), we compute the Jacobian \(J_{\rho^{(i)}}(x_{1:K}) : \mathbb{R}^{Kd_x} \to \mathbb{R}^{N \times Kd_x}\), in terms of the above Jacobians:
\[
J_{\rho^{(i)}}(x_{1:K}) = J_{\Sigma^{-T/2}z_2}(x_{1:K}) = \Sigma^{-T/2}_z J_2(x_{1:K}) = \Sigma^{-T/2}_z \begin{pmatrix} F(x_{1:K}) \\ H(x_{1:K}) \end{pmatrix}
\]
(30)

where,
\[
F(x_{1:K}) := \begin{pmatrix} I_d & 0 & \ldots & 0 \\ -F_1(x_1) & I_d & \ldots & 0 \\ 0 & -F_2(x_2) & I_d & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & \ldots & I_d & 0 \\ 0 & \ldots & 0 & 0 & I_d \end{pmatrix}
\]

and
\[
H(x_{1:K}) := \text{diag}(-H_1(x_1), -H_2(x_2), \ldots, -H_K(x_K)).
\]

Now, we can make the GN approximation, linearising around the current smoothed state sequence estimate \(\hat{x}_{1:K}^{(i)}\):
\[
\rho^{(i)}(x_{1:K}) \approx \hat{\rho}^{(i)}(x_{1:K}) = \rho^{(i)}(\hat{x}_{1:K}^{(i)}) + J_{\rho^{(i)}}(\hat{x}_{1:K}^{(i)})(x_{1:K} - \hat{x}_{1:K}^{(i)})
\]
\[
= \Sigma^{-T/2}_z \begin{pmatrix} F(\hat{x}_{1:K}^{(i)}) \\ H(\hat{x}_{1:K}^{(i)}) \end{pmatrix} (x_{1:K} - \hat{x}_{1:K}^{(i)})
\]
\[
+ \Sigma^{-T/2}_z \begin{pmatrix} \hat{x}_{1:K}^{(i)} - \hat{x}_{1:K}^{(i)} \\ \hat{x}_{1:K}^{(i)} - \hat{x}_{1:K}^{(i)} \\ \hat{x}_{1:K}^{(i)} - \hat{x}_{1:K}^{(i)} \\ \hat{x}_{1:K}^{(i)} - \hat{x}_{1:K}^{(i)} \\ \hat{x}_{1:K}^{(i)} - \hat{x}_{1:K}^{(i)} \\ \hat{x}_{1:K}^{(i)} - \hat{x}_{1:K}^{(i)} \\ \hat{x}_{1:K}^{(i)} - \hat{x}_{1:K}^{(i)} \\ \hat{x}_{1:K}^{(i)} - \hat{x}_{1:K}^{(i)} \\ \hat{x}_{1:K}^{(i)} - \hat{x}_{1:K}^{(i)} \end{pmatrix}
\]
\[
= \Sigma^{-T/2}_z \begin{pmatrix} x_{1:K} - \hat{x}_{1:K}^{(i)} \\ x_{2:K} - \hat{x}_{2:K}^{(i)} + b_1(x_1^{(i)}) \\ x_{3:K} - \hat{x}_{3:K}^{(i)} + b_2(x_2^{(i)}) \\ \vdots \\ x_{K:K} - \hat{x}_{K:K}^{(i)} + b_{K-1}(x_{K-1}^{(i)}) \\ y_1 - \hat{y}_1(x_1^{(i)}) + H_1(x_1^{(i)})(x_1 - \hat{x}_1^{(i)}) \\ y_2 - \hat{y}_2(x_2^{(i)}) + H_2(x_2^{(i)})(x_2 - \hat{x}_2^{(i)}) \\ \vdots \\ y_K - \hat{y}_K(x_K^{(i)}) + H_K(x_K^{(i)})(x_K - \hat{x}_K^{(i)}) \end{pmatrix}
\]
\[
= \Sigma^{-T/2}_z \begin{pmatrix} x_{1:K} - \hat{x}_{1:K} \\ x_{2:K} - \left( F_1(\hat{x}_1^{(i)}) x_{1:K} + b_1(\hat{x}_1^{(i)}) \right) \\ x_{3:K} - \left( F_2(\hat{x}_2^{(i)}) x_{2:K} + b_2(\hat{x}_2^{(i)}) \right) \\ \vdots \\ x_{K:K} - \left( F_{K-1}(\hat{x}_{K-1}^{(i)}) x_{K:K} + b_{K-1}(\hat{x}_{K-1}^{(i)}) \right) \\ y_1 - \left( H_1(x_1^{(i)}) x_{1:K} + c_1(x_1^{(i)}) \right) \\ y_2 - \left( H_2(x_2^{(i)}) x_{2:K} + c_2(x_2^{(i)}) \right) \\ \vdots \\ y_K - \left( H_K(x_K^{(i)}) x_{K:K} + c_K(x_K^{(i)}) \right) \end{pmatrix}
\]
\[
= \Sigma^{-T/2}_z \begin{pmatrix} \hat{z}_{1:K}^{(i)} \end{pmatrix}
\]
(31)

where, in the second to last equality, we use the relation in (4b) to express \(\hat{\rho}^{(i)}(x_{1:K})\) in terms of the offsets \(b_{1:K}(\cdot)\) and \(c_{1:K}(\cdot)\).