ESTIMATES FOR THE SHIFTED CONVOLUTION SUM INVOLVING FOURIER COEFFICIENTS OF CUSP FORMS OF HALF-INTEGRAL WEIGHT

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Abstract. In this article, we obtain certain estimate for the shifted convolution sum involving the Fourier coefficients of half-integral weight cusp forms.

1. Introduction

The estimates for the shifted convolution sums involving the Fourier coefficients of automorphic forms have been investigated by several authors. Selberg [15] started the study of shifted convolution sums and obtained the analytic properties of the \( L \)-function associated with cusp forms. Goldfeld [4], by using the analytic and arithmetic property of Poincaré series obtained an estimate of the following shifted convolution sum:

\[
\sum_{n \geq 1} a_f(n)a_g(n + m)e^{-n/X},
\]

where \( f \) and \( g \) are cusp forms of weight \( k \) and \( l \), respectively for the full modular group, and \( a_f(n) \) (respectively \( a_g(n) \)) denotes the \( n \)-th Fourier coefficient of \( f \) (respectively \( g \)).

Hafner [5] extended the result of Goldfeld [4] for congruence subgroups by using spectral decomposition method. Similar sums have also been considered for other kinds of automorphic forms, see the list [2, 7, 10, 13, 14, 17, 18]. Recently, Luo [11] obtained an estimate for the following shifted convolution sum:

\[
S(f, g, b) := \sum_{n \geq 1} a_f(n + b)a_g(n)G(n),
\]

where \( f \) is a cusp form of weight \( k + \frac{1}{2} \) for the group \( \Gamma_0(4N) \), \( g \) is a cusp form of weight \( l \) or a Maass cusp form for the group \( \Gamma_0(1) \), and \( G \) is a smooth function with the support in \([\frac{X}{2}, \frac{3X}{2}]\) satisfying \( G^{(p)}(x) \ll (\frac{X}{P})^{-p} \) for all integer \( p \geq 0 \), where \( P \) is a real number with \( 1 \leq P \leq X \). The aim of this paper is to obtain an estimate for \( S(f, g, b) \) when \( f \) and \( g \) are both half-integral weight cusp forms by using a similar method as in [11]. Now, we state the main result of the paper.

Let \( f \) be a cusp form of weight \( k + \frac{1}{2} \) and level \( 4N \) with Fourier series expansion

\[
f(\tau) = \sum_{n \geq 1} a_f(n)n^{k/2-1/4}e(n\tau),
\]
and $g$ be a newform of weight $l + \frac{1}{2}$ and level $4N$ with Fourier series expansion

$$g(\tau) = \sum_{n \geq 1} a_g(n)n^{l+1/4}e(n\tau).$$

For a fixed positive integer $b$ and a smooth function $G(x)$ as in (1), we consider the following sum:

$$S(f, g, b) = \sum_{n \geq 1} a_f(n+b)a_g(n)G(n).$$

**Theorem 1.1.** Let $f, g$ and $G$ be as above, and $N$ be an odd and squarefree positive integer. Then, we have

$$S(f, g, b) \ll_{\epsilon,f,g,b,G} X^{\frac{3}{4}+\epsilon} P\frac{3}{2},$$

2. **Notation and Preliminaries**

Let $\mathcal{H}$ denote the complex upper-half plane. For a complex number $\tau$, we use the notation $e(\tau) := e^{2\pi i \tau}$. The full modular group $SL_2(\mathbb{Z})$ and the congruence subgroup $\Gamma_0(N)$ of level $N \in \mathbb{N}$ is defined as follows;

$$SL_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\},$$

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}.$$

The group $SL_2(\mathbb{Z})$ acts on the complex upper half-plane $\mathcal{H}$ via fractional linear transformation as follows;

$$\gamma \tau := \frac{a\tau + b}{c\tau + d}, \text{ where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \text{ and } \tau \in \mathcal{H}.$$

For $k \in \mathbb{Z}_+$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N)$, and a holomorphic function $f : \mathcal{H} \to \mathbb{C}$, define the weight $k + \frac{1}{2}$ slash operator as follows;

$$f|_{k+\frac{1}{2}} \gamma(\tau) := \left( \frac{c}{d} \right) \epsilon_d^{2k+1}(c\tau + d)^{-\left(k+\frac{1}{2}\right)} f(\gamma \tau),$$

where $\left( \frac{c}{d} \right)$ is the Kronecker symbol and $\epsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \pmod{4}, \\ i & \text{if } d \equiv 3 \pmod{4}. \end{cases}$

**Definition 2.1.** A modular form of weight weight $k + \frac{1}{2}$ for $\Gamma_0(4N)$ is a complex-valued holomorphic function $f : \mathcal{H} \to \mathbb{C}$ satisfying the following properties:

1. $f|_{k+\frac{1}{2}} \gamma(\tau) = f(\tau)$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N)$.

2. $f$ is holomorphic at the cusps of $\Gamma_0(4N)$ with the Fourier series expansion given by

$$f(\tau) = \sum_{n \geq 0} a_f(n)e(n\tau).$$

Further, a modular form $f$ of weight $k + \frac{1}{2}$ for $\Gamma_0(4N)$ is said to be a cusp form if it vanishes at every cusp of $\Gamma_0(4N)$. 
We denote by $M_{k+\frac{1}{2}}(\Gamma_0(4N))$ and $S_{k+\frac{1}{2}}(\Gamma_0(4N))$ the space of modular forms and the space of cusp forms of weight $k + \frac{1}{2}$ for $\Gamma_0(4N)$, respectively. The Kohnen plus space $S^+_{k+\frac{1}{2}}(\Gamma_0(4N))$ is the subspace of cusp forms in $S_{k+\frac{1}{2}}(\Gamma_0(4N))$ whose $n$-th Fourier coefficient vanishes whenever $(-1)^k n \equiv 2, 3 \pmod{4}$. Kohnen using certain operators developed the theory of newforms of half-integral weight parallel to the Atkin-Lehner theory of newforms in integral weight case. For more details on the theory of modular forms and newforms of half-integral weight, we refer to [8, 9, 12, 16]. We assume the Ramanujan-Petersson conjecture for the Fourier coefficients of half-integral weight newforms, i.e., for any $\epsilon > 0$, we have $a_f(n) \ll \epsilon n^{k/2-1/4+\epsilon}$.

3. Preparatory Lemmas

In this section, we state some lemmas and recall some of the properties of Poincaré series which will be used in the proof of Theorem [11]. First we state the Poisson-Voronoi summation formula for half-integral weight cusp forms.

For $f(\tau) = \sum_{n \geq 1} a_f(n) n^{k/2-1/4} e(n \tau) \in S_{k+\frac{1}{2}}(4N)$ and a smooth function $G(x)$ with compact support in $(0, \infty)$, the Poisson-Voronoi summation formula is given by the following lemma.

**Lemma 3.1.** For any positive integers $c$ and $a$ with $\gcd(a, c) = 1$, we have

$$
\sum_{n=1}^{\infty} a_f(n) e\left(\frac{an}{c}\right) G(n) = \frac{2\pi i^{k+\frac{1}{2}}}{c} \left(\frac{c}{d}\right) e^{2k+1} \sum_{d=1}^{\infty} a_f(n) e\left(\frac{dn}{c}\right) H(n),
$$

where $H(n) = \int_{0}^{\infty} G(x) J_{k+\frac{1}{2}} \left(\frac{4\pi\sqrt{nx}}{c}\right) dx$, $d$ is the multiplicative inverse of a modulo $c$, and $J_{\nu}(z)$ denotes a Bessel function of order $\nu$.

**Proof.** For a proof, we refer to [3, Section 5]. \hfill \square

For every positive integer $m$, we define the $m$-th Poincaré series $P_m(\tau)$ of weight $k + \frac{1}{2}$ on the congruence subgroup $\Gamma_0(4N)$ by

$$
P_m(\tau) := \sum_{\gamma \in \Gamma_{\infty,4N} \setminus \Gamma_0(4N)} e^{2\pi im\tau} \left|_{k+\frac{1}{2}} \gamma(\tau)\right.,
$$

where $\tau \in \mathcal{H}$ and $\Gamma_{\infty,4N} = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} | n \in \mathbb{Z} \right\} \cap \Gamma_0(4N)$.

It is well-known that $P_m(\tau) \in S_{k+\frac{1}{2}}(\Gamma_0(4N))$ for $m \geq 1$. The $m$-th Poincaré series $P_m(\tau)$ has a Fourier series expansion given by

$$
P_m(\tau) = \sum_{n=1}^{\infty} a_{P_m}(n) n^{k/2-1/4} e(n \tau),
$$

where $a_{P_m}(n)$ is defined to be

$$
m^{k/2-1/4} a_{P_m}(n) = \delta_{m,n} + 2\pi i^{k+1/2} \sum_{c \geq 1, 4N | c} \epsilon^{-1} S_{k}(m, n; c) J_{k-\frac{1}{2}} \left(\frac{4\pi\sqrt{mn}}{c}\right),
$$
where $J_\nu(z)$ denotes a Bessel function of order $\nu$, and $S_k(m, n; c)$ denotes the Kloosterman sum defined by

$$S_k(m, n; c) = \sum_{a \equiv (m, c) \mod{c}, \gcd(a,c)=1} e^{-2\pi i (2k+1)(\frac{c}{a})} e^{\left(\frac{md + na}{c}\right)},$$

here $d$ denotes the multiplicative inverse of $a$ modulo $c$. For more details on Poincaré series we refer to [3, Section 6].

The Weil-Salié bound for the Kloosterman sum is given by [11, pp. 241 3):

$$S_k(m, n; c) \ll \gcd(m, n, c)^{1/2} d(c)^{1/2},$$

where for a positive integer $n$, $d(n)$ denotes the number of positive divisors of $n$.

**Lemma 3.2.** For any integer $p \geq 0$ and fixed $m, b \in \mathbb{N}$, we have

$$\int_0^\infty G(x) J_{k-1/2} \left( \frac{4\pi \sqrt{m(x+b)}}{c} \right) J_{l-1/2} \left( \frac{4\pi \sqrt{nx}}{c} \right) dx \ll X (|P_c(Xn)|^{-\frac{1}{2}} + n^{-\frac{1}{2}})$$

$$\times \min \left( \left( \frac{\sqrt{X}}{c} \right)^{1 \over 2}, \left( \frac{\sqrt{X}}{c} \right)^{k-1 \over 2} \right) \min \left( \left( \frac{\sqrt{nx}}{c} \right)^{-1 \over 2}, \left( \frac{\sqrt{nx}}{c} \right)^{l-1 \over 2} \right).$$

**Proof.** For a proof, we refer to [11, Lemma 3].

**Lemma 3.3.** For sufficiently large $X$ and any arbitrarily small $\epsilon > 0$, we have

$$\sum_{c \leq X} d(c) \frac{1}{c} = \frac{1}{2} (\log X)^2 + 2\gamma \log X + O(1) \ll (\log X)^2 \ll X^\epsilon,$$

where $\gamma$ is the Euler constant.

**Proof.** Proof is an application of Abel’s partial summation formula and we refer to [11, pp.70].

**4. Proof of Theorem 1.1**

It is sufficient to obtain the estimate of $S(f, g, b)$ when $f$ is a Poincaré series, because the space of cusp forms $S_{k+\frac{1}{2}}(\Gamma_0(4N))$ is generated by the Poincaré series $\{P_m(\tau) \mid m \geq 1\}$. Thus, we obtain the estimate for $S(P_m, g, b) = \sum_{n \geq 1} a_{P_m}(n + b) a_g(n) G(n)$, where $a_{P_m}(n)$ is the $n$-th Fourier coefficient of $m$-th Poincaré series. We assume that $X$ is sufficiently large depending on $m$.

Substitute the expression of $a_{P_m}(n)$ from (9) to obtain

$$S(P_m, g, b) = \sum_{n \geq 1} a_{P_m}(n + b) a_g(n) G(n) \left( \frac{1}{2\pi i^{k+1/2}} \delta_{m,n+b} + \sum_{c \geq 1, 4N \mid c} c^{-1} S_k(m, n+b; c) J_{k-\frac{1}{2}} \left( \frac{4\pi \sqrt{m(n+b)}}{c} \right) \right).$$
which yields

\[ |S(P_m, g, b)| \ll \sum_{c \geq 1, 4N|c} c^{-1} \sum_{a \equiv (\text{mod} \ c)} \epsilon_{a}^{-(2k+1)} \left( \frac{c}{a} \right) e \left( \frac{md + ba}{c} \right) \times \sum_{n \geq 1} a_{g}(n)G(n) e \left( \frac{na}{c} \right) J_{k-\frac{1}{2}} \left( \frac{4\pi \sqrt{m(n+b)}}{c} \right) | . \]

Now, we apply Poisson-Voronoi summation formula to obtain

\[ |S(P_m, g, b)| \ll \sum_{c \geq 1, 4N|c} c^{-2} \sum_{n \geq 1} a_{g}(n)S_{k}(m - n, b; c) \times \int_{0}^{\infty} G(x)J_{k-\frac{1}{2}} \left( \frac{4\pi \sqrt{m(x+b)}}{c} \right) J_{l-\frac{1}{2}} \left( \frac{4\pi \sqrt{nx}}{c} \right) dx | . \]

(11)

Without loss of generality, we may assume \( n \ll X^{4} \), for a fixed large constant \( A > 0 \). We now break the sum in (11) into three parts as follows:

**Part I:** \( n \ll X^{4} \).

**Part II:** \( n \gg X^{4} \) and \( Pc < (nX)^{1/2}X^{-\epsilon} \).

**Part III:** \( n \gg X^{4} \) and \( Pc \geq (nX)^{1/2}X^{-\epsilon} \).

**Estimate for Part I:** In this case, the contribution for the sum (11) denoted by \( S_{1} \), is at most \( X^{\frac{3}{2}+\epsilon} \) which is obtained as follows:

\[
S_{1} = \sum_{c \geq 1, 4N|c} c^{-2} \sum_{n \ll X^{4}} a_{g}(n)S_{k}(m - n, b; c) \times \int_{0}^{\infty} G(x)J_{k-\frac{1}{2}} \left( \frac{4\pi \sqrt{m(x+b)}}{c} \right) J_{l-\frac{1}{2}} \left( \frac{4\pi \sqrt{nx}}{c} \right) dx .
\]

We apply the Weil-Salié bound for the Kloosterman sum \( S_{k}(m - n, b; c) \) and the Ramanujan-Petersson bound for the Fourier coefficients \( a_{g}(n) \), and then use Lemma 3.2 (with \( p = 0 \));

\[
|S_{1}| \ll \sum_{c \geq 1, 4N|c} c^{-2+1/2}d(c) \sum_{n \ll X^{4}} n^{4}X([Pc(Xn)^{-1/2}]^{p} + n^{-p/2}) \times \min((\sqrt{X}/c)^{-1/2}, (\sqrt{X}/c)^{k-1/2}) \times \min((\sqrt{nx}/c)^{-1/2}, (\sqrt{nx}/c)^{l-1/2}),
\]
Here we have used Lemma 3.3 and partial summation formula to obtain the estimate for first sum, and the last sum is an absolutely convergent series.

Estimate for Part II: In this case, the integral in Lemma 3.2 is of order $O(X^{-p\epsilon})$. Therefore, by choosing sufficiently large $p$, we see that the integral is of order $O(X^{-A})$. Hence the contribution for the sum in (11) is negligible.

Estimate for Part III: In this case, we again decompose the sum

$$\sum_{c \geq 1, 4Nc} c^{-2} \sum_{n > X^{4\epsilon}} \sum_{P \geq (nX)^{1/2}X^{-\epsilon}} a_g(n)S_k(m - n, b; c)$$

into sub-sums of type $M \leq n \leq 2M$ using the dyadic division method and break the sum over $c$ into the following two parts;

**Part (a):** $c > \sqrt{2MX}$.

**Part (b):** $\sqrt{MX^X} \leq c \leq \sqrt{2MX}$.

Estimate for Part (a): In this case, the contribution for the sum, denoted by $S_a$, is at most $X^{\frac{3}{2} + \epsilon}$ which is obtained as follows:

$$S_a = \log X \max_{X^{4\epsilon} \leq M \leq X^4} \sum_{c > \sqrt{MX^{2\epsilon}}} c^{-2} \sum_{M \leq n \leq 2M} |a_g(n)S_k(m - n, b; c)|$$

$$\times \left| \int_0^{\infty} G(x)J_{k - \frac{1}{2}} \left( \frac{4\pi \sqrt{m(x + b)}}{c} \right) J_{l - \frac{1}{2}} \left( \frac{4\pi \sqrt{nx}}{c} \right) dx \right|.$$
Now, apply the Weil-Salïé bound for the Kloosterman sum, Ramanujan-Petersson bound for the Fourier coefficients $a_q(n)$, and then use Lemma 3.2 to obtain

\[
S_a \ll \log X \max_{X^{4^e} \leq M \leq X^A} \sum_{c > \sqrt{2M}/4N^e} c^{-2+1/2} d(c) \sum_{M \leq n \leq 2M} n^{\epsilon} \left( [Pc(Xn)^{-1/2}]^p + n^{-p/2} \right)
\]

\[
\times \min \left( \left( \frac{\sqrt{X}}{c} \right)^{-\frac{1}{2}}, \left( \frac{\sqrt{X}}{c} \right)^{k-1/2} \right) \min \left( \left( \frac{\sqrt{MX}}{c} \right)^{-\frac{1}{2}}, \left( \frac{\sqrt{MX}}{c} \right)^{l-1/2} \right),
\]

\[
\ll X^{1+\epsilon} \max_{X^{4^e} \leq M \leq X^A} \left( M^{1+\epsilon} \sum_{c > \sqrt{2M}} c^{-3/2} d(c)(\sqrt{X}/c)^{k-1/2} \right) \times (\sqrt{MX}/c)^{l-1/2}.
\]

Finally (by taking $p = 0$), we obtain

\[
S_a \ll X^{1+\epsilon} \max_{X^{4^e} \leq M \leq X^A} \left( M^{1+\epsilon} \sum_{c > \sqrt{2M}} c^{-3/2} d(c)(\sqrt{X}/c)^{k-1/2} \right)
\]

\[
\ll X^{1+\epsilon} \max_{X^{4^e} \leq M \leq X^A} \left( M^{1+\epsilon} \int_{x=\sqrt{2M}}^{\infty} x^{-k+1+\epsilon} dx \right),
\]

\[
\ll X^{1+\epsilon} \max_{X^{4^e} \leq M \leq X^A} \left( M^{1+\epsilon} \frac{k-1/2}{2} (MX)^{-k/2+\epsilon} \right),
\]

\[
\ll X^{1+\epsilon} \max_{X^{4^e} \leq M \leq X^A} \left( M^{1+\epsilon} X^{-\frac{1}{2}} (X^{1+\epsilon}) \right) \ll X^{1+\epsilon}.
\]

**Estimate for Part (b):** $\sqrt{MX^{1-\epsilon}}/p \leq c \leq \sqrt{2MX}$. 
If $\sqrt{X} \geq \sqrt{MX^{1-\epsilon}}/p$, then we have $M \leq P^2 X^{2\epsilon}$. In this case, the contribution, denoted by $S_{b,1}$, is at most $X^{1+\epsilon} P^2 \ll X^{1+\epsilon}$. Which is obtained as follows:

\[
S_{b,1} = \log X \max_{X^{4^e} \leq M \leq P^2 X^{2\epsilon}} \sum_{\sqrt{MX^{1-\epsilon}}/4N^e \leq c \leq \sqrt{2MX}} c^{-2} \sum_{M \leq n \leq 2M} |a_q(n)S_k(m - n, b, c)|
\]

\[
\times \left| \int_0^\infty G(x) J_{k-\frac{1}{2}} \left( \frac{4\pi \sqrt{m(x + b)}}{c} \right) J_{l-\frac{1}{2}} \left( \frac{4\pi \sqrt{n x}}{c} \right) dx \right|.
\]
Now, apply the Weil-Salié bound for the Kloosterman sum, Ramanujan-Petersson bound for the Fourier coefficients $a_g(n)$, and then use Lemma 3.2 to obtain

$$S_{b,1} \ll \log X \max_{X^{4\varepsilon} \leq M \leq P^2 X^{2\varepsilon}} \sum_{\sqrt{M} X^{1/4} \leq \nu \leq \sqrt{2MX}} c^{-2+1/2} d(c) \sum_{M \leq n \leq 2M} n^{\epsilon} X([Pc(Xn)^{-1/2}]^p + n^{-p/2})$$

$$\times \min \left( \left( \frac{X}{c} \right)^{-\frac{1}{2}}, \left( \frac{X}{c} \right)^{k-1/2} \right) \min \left( \left( \frac{MX}{c} \right)^{-\frac{1}{2}}, \left( \frac{MX}{c} \right)^{l-1/2} \right),$$

$$\ll X^{1+\varepsilon} \max_{X^{4\varepsilon} \leq M \leq P^2 X^{2\varepsilon}} M^{1+\varepsilon} \left\{ \sum_{\sqrt{M} X^{1/4} \leq \nu \leq \sqrt{2MX}} c^{-3/2} d(c) \min \left( \left( \frac{X}{c} \right)^{-\frac{1}{2}}, \left( \frac{X}{c} \right)^{k-1/2} \right) \right. \left. \times \left( \frac{MX}{c} \right)^{-1/2} \right\}$$

$$\ll X^{1+\varepsilon} \max_{X^{4\varepsilon} \leq M \leq P^2 X^{2\varepsilon}} M^{1+\varepsilon} \left\{ \sum_{\sqrt{M} X^{1/4} \leq \nu \leq X} c^{-3/2} d(c) (\sqrt{X}/c)^{k-1/2} \times (\sqrt{MX}/c)^{-1/2} + \right. \left. \sum_{\sqrt{M} X^{1/4} \leq \nu \leq \sqrt{X}} c^{-3/2} d(c) (\sqrt{X}/c)^{-1/2} \times (\sqrt{MX}/c)^{-1/2} \right\}$$

$$\ll X^{1+\varepsilon} \max_{X^{4\varepsilon} \leq M \leq P^2 X^{2\varepsilon}} M^{1+\varepsilon} \left\{ (MX)^{-1/4} \sum_{\sqrt{X} \leq \nu \leq \sqrt{2MX}} c^{-1} d(c) \right. \right.$$  

$$X^{-1/2+1/4} M^{-1/4} \sum_{\sqrt{MX} X^{1/4} \leq \nu \leq \sqrt{X}} \left. \frac{d(c)}{c} \right\},$$

$$\ll X^{1+\varepsilon} \max_{X^{4\varepsilon} \leq M \leq P^2 X^{2\varepsilon}} \left( M^{1+\varepsilon} \times (MX)^{-1/4} \sum_{\sqrt{MX} X^{1/4} \leq \nu \leq \sqrt{2MX}} \frac{d(c)}{c} \right),$$

$$\ll X^{1+\varepsilon} \max_{X^{4\varepsilon} \leq M \leq P^2 X^{2\varepsilon}} \left( M^{1+\varepsilon} \times (MX)^{3/2} \right) \ll X^{1+\varepsilon} P^{3/2}.$$  

If $\sqrt{X} \leq \sqrt{MX} X^{-\varepsilon} P^{-1}$, then we have $M \geq P^2 X^{2\varepsilon}$. In this case, the contribution, denoted by $S_{b,2}$, is at most $X^{1+\varepsilon} P^{3/2}$ which is obtained as follows (by taking $p = 0$):

$$S_{b,2} = \log X \max_{P^2 X^{2\varepsilon} \leq M \leq X^A} \sum_{\sqrt{M} X^{1/4} \leq \nu \leq \sqrt{2MX}} c^{-2} \sum_{M \leq n \leq 2M} |a_g(n) S_k(m - n, b; c)|$$

$$\times \left| \int_0^\infty G(x) J_{k-\frac{1}{2}} \left( \frac{4\pi \sqrt{m(x + b)}}{c} \right) J_{l-\frac{1}{2}} \left( \frac{4\pi \sqrt{nx}}{c} \right) dx \right|.$$
Now, apply the Weil-Salié bound for the Kloosterman sum, Ramanujan-Petersson bound for the Fourier coefficients $a_q(n)$, and then use Lemma 3.2 to obtain

$$S_{b,2} = \log X \max_{p^2 X^{2\epsilon} \leq M \leq X^A} \sum_{\sqrt{M} \leq \frac{X^{\epsilon}}{4N|c|} \leq \sqrt{XM}} c^{-2} \sum_{M \leq n \leq 2M} |n^\epsilon c^{1/2}d(c)| \times X([Pc(Xn)^{-1/2}] + n^{-p/2})$$

$$\times \min((\sqrt{X}/c)^{-1/2}, (\sqrt{X}/c)^{k-1/2}) \times \min((\sqrt{nX}/c)^{-1/2}, (\sqrt{nX}/c)^{l-1/2}),$$

$$\ll X^{1+\epsilon} \max_{p^2 X^{2\epsilon} \leq M \leq X^A} M^{1+\epsilon} \sum_{\sqrt{M} \leq \frac{X^{\epsilon}}{4N|c|} \leq \sqrt{XM}} c^{-3/2}d(c)(\sqrt{X}/c)^{k-1/2} \times (\sqrt{MX}/c)^{-1/2}.$$ 

Since $k \geq 2$ and $\sqrt{MXX^{-\epsilon}} \leq c$, i.e., $\frac{\sqrt{X}}{c} \leq \frac{PX^\epsilon}{\sqrt{M}} \leq 1$, therefore

$$S_{b,2} \ll X^{1+\epsilon} \max_{p^2 X^{2\epsilon} \leq M \leq X^A} M^{1+\epsilon} \sum_{\sqrt{M} \leq \frac{X^{\epsilon}}{4N|c|} \leq \sqrt{XM}} c^{-3/2}d(c) \left(\frac{PX^\epsilon}{\sqrt{M}}\right)^{3/2} \times (\sqrt{MX}/c)^{-1/2},$$

$$\ll X^{1+\epsilon} P^{3/2} \max_{p^2 X^{2\epsilon} \leq M \leq X^A} M^{1+\epsilon} \sum_{\sqrt{M} \leq \frac{X^{\epsilon}}{4N|c|} \leq \sqrt{XM}} M^{-\frac{3}{4} - \frac{1}{2}\epsilon} c^{-1}d(c),$$

$$\ll X^{1+\epsilon} P^{3/2} \max_{p^2 X^{2\epsilon} \leq M \leq X^A} (M)^\epsilon \log(MX) \ll X^{1+\epsilon} P^{3/2} X^{A\epsilon} \ll X^{1+\epsilon} P^{3/2}.$$ 

Finally, all these estimates give us the required result.

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