Testing the new BPS method in some models of nonabelian magnetic monopole

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Abstract. The proposed method in [\textit{Phys. Lett.} B768 351-358 (2017)], which can obtain BPS equations of some models of vortices, is used here to see whether it is still usable for some models of magnetic monopole. Other than the standard Yang-Mills-Higgs, here we report that the method is able to give us the BPS equations from two different magnetic monopole models.

1. Introduction

Monopole is a non-contractible defect whose second homotopy group over 2-sphere is nontrivial \cite{1-4}. Magnetic monopole has received much interest since Dirac \cite{5} proposed a certain vector potential in an abelian gauge theory. Generalized to a nonabelian one, \textquoteright{}t Hooft \cite{6} and Polyakov \cite{7,8} had shown that magnetic monopole arises from such non-abelian model. In their seminal paper, Prasad and Sommerfield \cite{9} showed that the corresponding second-order field equations can have analytic solutions in certain limit. Later Bogomolny \cite{10}, based on classical energy stability, showed that the field equations can be reduced to the first-order ones. Further, he proved that such solutions are stable since their energy is globally minimum. The corresponding first-order equations and exact solutions are called the BPS (Bogomolny-Prasad-Sommerfield) equations and solutions, respectively.

For certain models, some techniques to obtain BPS equations had been presented, such as \cite{11,12} for recent proposals. Though the method proposed in \cite{12} uses one of the results (i.e. BPS energy functional $Q$) from the On-Shell method proposed in \cite{11} as its basic motivation, the former is noticeably easier to use. Here we apply such method to magnetic monopoles. Although both vortices and monopoles depend on scalar and gauge fields configuration, the former is Abelian while the latter in not. We show that the method can be applied to finding the BPS equations for certain noncanonical models of magnetic monopoles.

The discussion in this proceeding paper are as follows. We review the magnetic monopole from a nonabelian gauge theory in section 2, then we describe the method from \cite{12} in section 3. Then we show the models whose BPS equations are able to be found using the method in the next three sections, i.e., standard Yang-Mills-Higgs, generalized Yang-Mills-Higgs, and Nakamura-Shiraishi DBI model \cite{13} in section 4, 5, and 6, respectively. Lastly since it is a work in progress, some remarks and future projections are discussed in section 7.
2. Magnetic monopole from nonabelian gauge field

We begin with action in a flat (1 + 3)-dimensional spacetime whose metric is $\eta_{\mu\nu}$ with signature $(+, -, -, -)$ with the Lagrangian density having a standard form

$$\mathcal{L} = \frac{1}{2} D_\mu \phi^a D^\mu \phi^a - V(|\phi|) - \frac{1}{4} F^{a\mu\nu} F_{a\mu\nu},$$

(1)

with the components of the adjoint covariant derivative and the field strength tensor are defined respectively as $(a = 1, 2, 3)$

$$D_\mu \phi^a = \partial_\mu \phi^a + e \epsilon^{abc} A^b_\mu \phi^c,$$

(2)

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + e \epsilon^{abc} A^b_\mu A^c_\nu.$$  

(3)

The scalar and Yang-Mills gauge field is written as $\phi = \phi^a T^a$ and $A = A_\mu dx^\mu = A^a_\mu T^a dx^\mu$, respectively, with $T^a \equiv \sigma^a/2$ denoting anti-Hermitian generators of $su(2)$ algebra, whose commutation relation is $[T^a, T^b] = i \epsilon^{abc} T^c$ and its trace is $\text{tr}(T^a T^b) = \frac{1}{2} \delta^{ab}$. Here we employ the 't Hooft-Polyakov hedgehog ansatz

$$\phi^a = \eta f(r) \frac{x^a}{r}, \quad A_0^a = 0,$$

(4)

$$A_m^a = \frac{1 - a(r)}{e} \epsilon_{amn} x^n / r^2,$$

(5)

where $x^a = (x, y, z)$ denotes Cartesian coordinate. In spherical coordinates $(r, \theta, \varphi)$, their components are

$$\phi = \eta f(\sin \theta (\cos \varphi T^1 + \sin \varphi T^2) + \cos \theta T^3),$$

(6)

$$A_r = 0, \quad A_\theta = \frac{1 - a}{e} (\sin \varphi T^1 - \cos \varphi T^2),$$

(7)

$$A_\varphi = \frac{1 - a}{e} \sin \theta (\cos \varphi T^1 + \sin \varphi T^2) - \sin \theta T^3).$$

(8)

Writing them this way make it easier to obtain the components of the adjoint covariant derivative of the scalar field $D_\mu \phi = \partial_\mu \phi - ie [A_\mu, \phi]$ and the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ie [A_\mu, A_\nu]$, which are

$$D_r \phi = \eta f'(r) (\sin \theta (\cos \varphi T^1 + \sin \varphi T^2) + \cos \theta T^3),$$

(9)

$$D_\theta \phi = \eta a f (\cos \theta (\cos \varphi T^1 + \sin \varphi T^2) - \sin \theta T^3),$$

(10)

$$D_\varphi \phi = \eta a f (\sin \theta (\cos \varphi T^1 + \cos \varphi T^2),$$

(11)

$$F_{r\varphi} = \frac{a}{e} (\sin \varphi T^1 + \cos \varphi T^2),$$

(12)

$$F_{r\varphi} = \frac{a}{e} (\sin \theta (\cos \varphi T^1 + \sin \varphi T^2) + \sin \theta T^3),$$

(13)

$$F_{\theta\varphi} = \frac{a^2 - 1}{e} \sin \theta (\cos \varphi T^1 + \sin \varphi T^2) + \cos \theta T^3),$$

(14)
with prime denoting derivative of its variable, e.g., $f'(r) \equiv df/dr$, so we obtain

$$\frac{1}{2} D_\mu \phi^a D^\mu \phi^a = \text{tr} \quad D_\mu \phi D^\mu \phi = \eta^{rr} (\eta f'(r))^2 + (\eta^\theta \eta^\varphi \sin^2 \theta \eta \alpha f)^2 \frac{1}{4} \text{tr} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= - \frac{(\eta f'(r))^2}{2} - \left( \frac{\eta \alpha f}{r} \right)^2,$$

$$\frac{1}{4} F_{\mu \nu}^a F^a_{\mu \nu} = \text{tr} \frac{1}{2} F_{\mu \nu} F^{\mu \nu} = \eta^{rr} (\eta^\theta \eta^\varphi \sin^2 \theta \eta \alpha f)^2 \left( \frac{a'(r)}{e r} \right)^2 + \frac{1}{2} \left( \frac{1 - \alpha^2}{e r^2} \right)^2.$$  \hspace{1cm} (15)

Hence the Lagrangian density becomes

$$\mathcal{L} = \text{tr} \quad \left( D_\mu \phi D^\mu \phi - \frac{1}{2} F_{\mu \nu} F^{\mu \nu} \right) - V(|\phi|)$$

$$= - \frac{\eta^2 f'(r)^2}{2} - \left( \frac{\eta \alpha f}{r} \right)^2 - V(f) - \left( \frac{a'(r)}{e r} \right)^2 - \frac{1}{2} \left( \frac{a^2 - 1}{e r^2} \right)^2.$$  \hspace{1cm} (17)

Using Bogomolny trick, i.e., when $V = 0$ we can arrange its static energy

$$E_{\text{static}} = -4 \pi \int r^2 dr \left[ -\frac{\eta^2 f'(r)^2}{2} - \left( \frac{\eta \alpha f}{r} \right)^2 - \left( \frac{a'(r)}{e r} \right)^2 - \frac{1}{2} \left( \frac{a^2 - 1}{e r^2} \right)^2 \right]$$

$$= 4 \pi \int r^2 dr \left[ \frac{1}{2} \left( \eta f'(r) \mp \frac{a^2 - 1}{e r^2} \right)^2 + \left( \frac{a'(r) \mp \eta \alpha f}{e r^2} \right)^2 \pm \frac{2 \eta a'(r) \alpha f}{e r^2} \pm \frac{\eta f'(r)(a^2 - 1)}{e r^2} \right]$$

$$\geq - \frac{4 \pi \eta}{e} \int r^2 dr \left[ \frac{(a^2 - 1) f'}{r^2} \right] = \pm \frac{4 \pi \eta}{e} (a^2 - 1) \left[ f \right]_{r \to 0}^{r \to \infty} = \pm \frac{4 \pi \eta}{e}.$$  \hspace{1cm} (18)

This lowest static energy $E_{\text{static}} = \frac{4 \pi \eta}{e}$ thus comes from the solution of the BPS equations

$$f'(r) = -\frac{a^2 - 1}{\eta e r^2},$$  \hspace{1cm} (19)

$$a'(r) = -\eta \alpha f.$$  \hspace{1cm} (20)

In the following discussion, we simplify our notation by setting $e = 1 = \eta$.

3. The method

Here we follow the prescription on obtaining BPS equations in [12]. Any effective Lagrangian density, whose fields\(^1\) are $\phi_i = (\phi_1, ..., \phi_N)$, can be added with some terms that do not change its Euler-Lagrange equations. These terms are called the boundary term(s) $\mathcal{L}_{bd}$ since their value is apparent only at the boundary of the space, while the Euler-Lagrange equations describe the system at all positions. Any static configuration has its Hamiltonian density equal to the

\(^1\) They need not be scalars.
negative of its Lagrangian density, so its total energy is 
\[ E_{\text{static}} = \int d^d x L, \]
with \( d \) its spatial dimension. Then one can write the static energy as

\[ E_{\text{static}} = \int d^d x \sum_i \Phi_i(\phi_1, ..., \phi_N, \partial \phi_i; r) + E_{\text{bd}}, \]  

(21)

with \( \Phi_i \) a set of positive-semidefinite functions that contains only one of its respective first derivative field and \( E_{\text{BPS}} \) is the lowest static energy available defined as \( E_{\text{bd}} = \int d^d x L_{\text{bd}} \). The BPS limit is when the static energy is equal to its lowest bound \( E_{\text{static}} = E_{\text{bd}} \), i.e., when \( \Phi_i(\phi_1, ..., \phi_N, \partial \phi_i; r) = 0 \). Then \( \Phi_i(\phi_1, ..., \phi_N, \partial \phi_i; r) = 0 \) are considered as the BPS equations and the boundary term in the Lagrangian density is then equal to BPS Lagrangian \( L_{\text{BPS}} \). Solutions from the BPS equations are solutions of the Euler-Lagrange equations at the BPS limit, thus we can write the Lagrangian density as

\[ L = -\left( \sum_i \Phi_i(\phi_1, ..., \phi_N, \partial \phi_i; r) \right) + L_{\text{bd}}, \]  

(22)

where at BPS limit, \( \Phi_i(\phi_1, ..., \phi_N, \partial \phi_i; r) = 0 \) and \( L_{\text{bd}} = L_{\text{BPS}} \). Hence at the BPS limit, we can obtain the BPS equations by employing \( L - L_{\text{BPS}} = 0 \).

A result from the On-Shell method [11] is that, at least for vortices [12], the lowest possible static energy at the BPS limit can be defined as

\[ E_{\text{bd}} = Q(r \to \infty) - Q(r \to 0) = \int_{r \to 0}^{r \to \infty} dQ, \]  

(23)

where \( Q \) is called as BPS energy functional. This, as suspected in [12], depends only on the fields since the ansatz employed in the effective Lagrangian density has no explicit coordinate expression, such as the radial coordinate \( r \). In this paper, the employed ansatz give us (9)-(14) that depend on \( r \) implicitly. Hence for general fields, \( Q \) is dependent only on the fields

\[ Q = \prod_{i=1}^{N} Q_i(\phi_i). \]  

(24)

which makes

\[ E_{\text{bd}} = \int \sum_i \partial Q \frac{d \phi_i}{d r} d r, \]  

(25)

and we can obtain \( L_{\text{BPS}} \).

Now after using a suitable \( Q \) to obtain \( L_{\text{BPS}} \) and then employing \( L - L_{\text{BPS}} = 0 \), we proceed to find all \( \Phi_i \). Notice that each \( \Phi_i \) must be positive-semidefinite, hence each must has \( \partial_r \phi_i \) as its roots, which if written as a set, the expression must be

\[ \partial_r \phi_i = \left\{ F_i^{(1)}, F_i^{(2)}, ..., F_i^{(m)} \right\}, \]  

(26)

with \( F_i^{(k)} = F_i^{(k)}(\phi_1, ..., \phi_N, r) \) \( (k = 1, ..., m) \) and \( m = \) even number. If \( \Phi_i = 0 \) is a quadratic equation, hence \( m = 2 \) and each BPS solution \( \phi_i \) will have a constraint \( F_i^{(1)} = F_i^{(2)} \).

Generally, one can also find the BPS equations as follows. By seeing \( L - L_{\text{BPS}} = 0 \) as a polynomial equation of one of the fields, say \( \partial_r \phi_1 \), one can find its roots to be

\[ \partial_r \phi_1 = \left\{ G_1^{(1)}, G_1^{(2)}, ..., G_1^{(m)} \right\}, \]  

(27)
with \( G^{(k)}_1 = G^{(k)}_1(\phi_1, ..., \phi_N, \partial_r \phi_1, \partial_r \phi_2, ..., \partial_r \phi_N, r) \) \((k = 1, ..., m \text{ and } m = \text{even number})\) with the underlined one is removed. If \( m = 2 \), then \( G^{(1)}_1 - G^{(2)}_1 = 0 \) is a polynomial function of another field \( \partial_r \phi_i \) with \( i > 1 \). If the total of all \( \partial_r \phi_i \) is \( m = 2 \), then by looping this steps until \( i = N \) we arrive at the last equation \( G^{(1)}_N - G^{(2)}_N = 0 \). Since the model is valid for all \( r \), we write \( G^{(1)}_N - G^{(2)}_N = 0 \) as terms with powers of \( r \) and we treat each term to be equal to zero. From each term, we can find each \( Q_i(\partial_i) \) and the BPS equations \( \partial \overline{\partial} \phi_i \) can then be found.

This method is shown to be adequate to obtain BPS equations for various models of vortices using BPS energy function \( Q = 2\pi F(f)A(a) \) as shown in [12] and here we investigate whether the method is also adequate for some models of 't Hooft-Polyakov magnetic monopoles or not.

4. Standard Yang-Mills-Higgs model
Looking at the standard Lagrangian density (17) and its ansatz, we guess that (24) is true so we define the BPS energy function for magnetic monopole to be

\[
Q = 4\pi F(f)A(a) \tag{28}
\]

Since \( E_{\text{BPS}} = -\int d^3x \mathcal{L}_{\text{BPS}} = -\int r^2 \sin \theta \, d\theta \, d\phi \, L_{\text{BPS}} = -4\pi \int r^2 dr \mathcal{L}_{\text{BPS}} \) we have

\[
\mathcal{L}_{\text{BPS}} = -\frac{F'(f)A}{r^2} f' - \frac{FA'(a)}{r^2} a'. \tag{29}
\]

From this point on, we will write \( A'(a) \) as \( A' \) and \( F'(f) \) as \( F' \) by keeping in mind that \( A' = dA/da \) and \( F' = dF/df \). We input this into \( \mathcal{L} - \mathcal{L}_{\text{BPS}} = 0 \) with \( \mathcal{L} \) from (17) so we have

\[
\frac{1}{2} f'^2 + \frac{a^2 f^2}{r^2} + V + \frac{1}{r^2} a^2 + \frac{(a^2 - 1)^2}{2r^4} - \frac{F' A}{r^2} f' - \frac{F A'}{r^2} a' = 0. \tag{30}
\]

Now we choose explicitly the expression of both \( \Phi_a \) and \( \Phi_f \). This can be done by choosing for \( \Phi_a \) and \( \Phi_f \), respectively,

\[
\frac{1}{r^2} a'^2 - \frac{F A'}{r^2} a' + \frac{a^2 f^2}{r^2} = 0, \tag{31}
\]

\[
\frac{1}{2} f'^2 - \frac{F' A}{r^2} f' + V + \frac{(a^2 - 1)^2}{2r^4} = 0. \tag{32}
\]

Each leads, respectively, to

\[
a'_\pm = \frac{F A'}{2} \pm \sqrt{\left(\frac{F A'}{2}\right)^2 - a^2 f^2}, \tag{33}
\]

\[
f'_\pm = \frac{F' A}{r^2} \pm \sqrt{\left(\frac{F' A}{r^2}\right)^2 - 2 \left(V + \frac{(a^2 - 1)^2}{2r^4}\right)}. \tag{34}
\]

Since we need \( \Phi_a \) to be positive-semidefinite then \( a'_+ = a'_- \) so

\[
FA' = \pm 2af \tag{35}
\]

which if we choose \( F = f \) then \( A = \pm (a^2 + kA) \). We also need \( f'_+ = f'_- \) so

\[
2V = \left(\frac{(a^2 + kA)^2}{r^2}\right)^2 - \frac{(a^2 - 1)^2}{r^4} = 0. \tag{36}
\]
The last equality is the condition for BPS equations to be found and it is true if \( k_A = -1 \). (If \( k_A \neq -1 \), then \( V = V(a) \), which is not true.) Hence the BPS equations are

\[
f' = \pm \frac{(a^2 - 1)}{r^2},
\]
\[
a' = \pm af.
\]  

We can also find with another approach which in [12] is said to be more general than what we have described in section 3. We can treat \( L - L_{\text{BPS}} = 0 \) as a quadratic equation of first derivative of one of the fields, where here we choose \( a' \)

\[
\frac{1}{r^2} a'^2 - \frac{FA'}{r^2} a' + \left[ \frac{1}{2} f'^2 + \frac{a^2 f^2}{r^2} + V + \frac{(a^2 - 1)^2}{2r^4} - \frac{F'A'}{r^2} f' \right] = 0,
\]

hence the roots of \( a' \) are

\[
a'_\pm = \frac{FA'}{2} \pm \sqrt{\frac{(FA')^2}{4} - r^2 \left[ \frac{1}{2} f'^2 + \frac{a^2 f^2}{r^2} + V + \frac{(a^2 - 1)^2}{2r^4} - \frac{F'A'}{r^2} f' \right]}.
\]

Since we want \( a'_+ = a'_- \), the square root must be zero, or after rearranging

\[
\frac{r^2}{2} f'^2 - F'A'f' - \frac{(FA')^2}{4} + a^2 f^2 + V + \frac{(a^2 - 1)^2}{2r^2} = 0.
\]

The roots are

\[
f'_\pm = \frac{F'A}{r^2} \pm \sqrt{\frac{(F'A)^2}{r^4} - \left( \frac{(FA')^2}{2r^2} + \frac{2a^2 f^2}{r^2} + \frac{2V}{r^2} + \frac{(a^2 - 1)^2}{r^4} \right)},
\]

which again from \( f'_+ = f'_- \) we need

\[
\frac{(F'A)^2}{r^4} - \left( \frac{(FA')^2}{2r^2} + \frac{2a^2 f^2}{r^2} + \frac{2V}{r^2} + \frac{(a^2 - 1)^2}{r^4} \right) = 0.
\]

This constraint can be solved by finding \( F \) and \( A \) through choosing each term with powers of \( r \) to be equal to zero, i.e.,

\[
\frac{(F'A)^2}{r^4} - \frac{(a^2 - 1)^2}{r^4} = 0,
\]
\[
- \frac{(FA')^2}{2r^2} + \frac{2a^2 f^2}{r^2} + \frac{2V}{r^2} = 0.
\]

If we choose \( F = f \) then from the former equation we have \( A = \pm (a^2 + k_A) \) with \( k_A \) a constant, and the latter equation give us

\[
2V = \frac{(\pm af)^2}{2} - 2a^2 f^2 = 0,
\]

which is the condition for BPS equations to be found. This leads to the same BPS equations as (37) and (38).
5. Generalized Yang-Mills-Higgs model

Now we consider the following Lagrangian density

\[ \mathcal{L} = \text{tr} \left( w(|\phi|) D_\mu \phi D^\mu \phi - \frac{1}{2} G(|\phi|) F_{\mu \nu} F^{\mu \nu} \right) - V(|\phi|) \]

\[ = - w(f) \left( \frac{f'(r)^2}{2} + \left( \frac{af}{r} \right)^2 \right) - V(f) - G(f) \left( \frac{a'(r)}{r} \right)^2 - \frac{1}{2} \left( \frac{a^2 - 1}{r^2} \right)^2, \]

(47)

that resembles the generalized Maxwell-Higgs model for vortices. Again by using the same \( \mathcal{L}_{\text{BPS}} \) and choosing \( \Phi_a \) and \( \Phi_f \), respectively, as

\[ -w \left( \frac{af}{r} \right)^2 + \frac{FA'}{r^2} a' - G \left( \frac{a'(r)}{r} \right)^2 = 0, \]

(48)

\[ -w \frac{f'^2}{2} + \frac{F' A}{w r^2} f' - V(f) - \frac{G}{2} \left( \frac{a^2 - 1}{r^2} \right)^2 = 0. \]

(49)

The former equation leads to

\[ a'_\pm = \frac{FA'}{2G} \pm \sqrt{\left( \frac{FA'}{2G} \right)^2 - \frac{w}{G} (af)^2}, \]

(50)

hence

\[ FA' = \pm 2af \sqrt{wG}, \]

(51)

which gives us \( F = f \sqrt{wG} \) and \( A = \pm (a^2 + k_A) \). The latter equation leads to

\[ f'_\pm = \frac{F' A}{wr^2} \pm \sqrt{\left( \frac{F' A}{wr^2} \right)^2 - \frac{2V}{w} - \frac{G}{w} \left( \frac{a^2 - 1}{r^2} \right)^2}, \]

(52)

which leads us to a constraint

\[ \left( \frac{\partial (\sqrt{wG}f)}{\partial f} \right)^2 = \frac{wr^4}{(a^2 + k_A)^2} \left[ 2V + G \left( \frac{a^2 - 1}{r^2} \right)^2 \right]. \]

(53)

For the case of \( V = 0 \) and \( k_A = -1 \), we have \( \left( \frac{\partial (\sqrt{wG}f)}{\partial f} \right)^2 = wG = const \). Hence the BPS equations are

\[ a' = \pm \sqrt{\frac{w}{G}} af, \]

(54)

\[ f' = \pm \sqrt{\frac{G}{w}} \frac{(a^2 - 1)}{r^2}. \]

(55)

Now we try using the second method as before. Arranging the equation to be

\[ \frac{G}{r^2} a'' - \frac{FA'(a)}{r^2} a' + \left[ \frac{w}{2} f'^2 - \frac{F'(f) A}{r^2} f' + \frac{w a^2 f^2}{r^2} + V + \frac{G}{2} \frac{(a^2 - 1)^2}{r^4} \right] = 0, \]

(56)

we obtain

\[ a'_\pm = \frac{FA'(a)}{2G} \pm \sqrt{\frac{F^2 A'(a)^2}{4G^2} - \frac{r^2}{G} \left[ \frac{w}{2} f'^2 - \frac{F'(f) A}{r^2} f' + \frac{w a^2 f^2}{r^2} + V + \frac{G}{2} \frac{(a^2 - 1)^2}{r^4} \right]}, \]

(57)
Now to have \(a'_+ = a'_-\), it must satisfy
\[
\frac{w}{2} f'^2 - \frac{F'(f) A}{r^2} f' + \left[ -\frac{F^2 A'(a)^2}{4Gr^2} + \frac{w a^2 f^2}{r^2} + V + \frac{G (a^2 - 1)^2}{2r^4} \right] = 0,
\]
whose roots are
\[
f'_\pm = \frac{F'(f) A}{w r^2} \pm \sqrt{\frac{F'(f)^2 A^2}{w^2 r^4} + \frac{2}{w} \left[ \frac{F^2 A'(a)^2}{4Gr^2} - \frac{w a^2 f^2}{r^2} - V - \frac{G (a^2 - 1)^2}{2r^4} \right]},
\]
hence we have
\[
\frac{F'(f)^2 A^2}{w^2 r^4} + \frac{F^2 A'(a)^2}{2wGr^2} - \frac{2a^2 f^2}{r^2} - \frac{2V}{w} - \frac{G (a^2 - 1)^2}{r^4} = 0
\]
to solve. Now we choose
\[
\frac{F^2 A'(a)^2}{2wGr^2} - \frac{2a^2 f^2}{r^2} = 0
\]
so that \(FA'(a) = \pm 2a\sqrt{wG}f\) and we choose \(F = \sqrt{wG}f\) so \(A = \pm (a^2 + k_A)\). The rest, i.e.,
\[
\frac{F'(f)^2 A^2}{w^2 r^4} - \frac{2V}{w} - \frac{G (a^2 - 1)^2}{2r^4} = 0
\]
leads to
\[
V = \left[ \frac{\partial(\sqrt{wG}f)}{\partial f} \right]^2 \frac{(a^2 + k_A)^2}{2wr^4} - \frac{G(a^2 - 1)^2}{2r^4}.
\]
Since \(V\) is a function of \(f\), then \(V = 0\), hence we have a constraint
\[
\frac{\partial(\sqrt{wG}f)}{\partial f} = \pm \sqrt{\frac{wG(a^2 - 1)^2}{(a^2 + k_A)^2}}.
\]
Since the left hand side is a function only depending on \(f\) thus it require \(k_A = -1\) which then makes \(wG = \text{const.}\), which, without loss of generality, can be normalized to unity. Thus,
\[
w = G^{-1}.
\]

The resulting BPS equations are the same as (54) and (55).

6. Nakamura-Shiraishi DBI model

We begin with a Lagrangian density suggested in [13] that is guaranteed to have the standard BPS equations. Here we show that our method is still usable for this form of Lagrangian density. The Lagrangian density has the following form
\[
\mathcal{L} = -\beta^2 \text{tr} \left( \sqrt{1 + \frac{2}{\beta^2} D_\mu \phi D^\mu \phi} \sqrt{1 + \frac{1}{\beta^2} F_{\mu\nu} F^{\mu\nu} - 1} \right) - V(|\phi|)
\]
\[
= -2\beta^2 \left( 1 + \frac{2}{\beta^2} \left( \frac{(\eta f'(r))^2}{4} - \frac{1}{2} \left( \eta a f \right)^2 \right) \right) \left( 1 + \frac{2}{\beta^2} \left( \frac{1}{e r^2} \left( \frac{a'(r)}{e r^2} \right)^2 + \frac{1}{4} \left( \frac{1 - a^2}{e r^2} \right)^2 \right) - 1 \right) - V(f),
\]
\[
= -2\beta^2 \left( \frac{1}{e r^2} \left( \frac{a'(r)}{e r^2} \right)^2 + \frac{1}{4} \left( \frac{1 - a^2}{e r^2} \right)^2 \right) - V(f).
\]
Setting \( e = 1 = \eta \) and subtracting with

\[
\mathcal{L}_{\text{BPS}} = -\frac{F'(f)A}{r^2} f' - \frac{F A'(a)}{r^2} a' \quad (67)
\]

we have

\[
\mathcal{L}_{\text{BPS}} - \mathcal{L} = 2\beta^2 \left( \sqrt{1 + \frac{1}{\beta^2} \left( \frac{(f')^2}{4} - \frac{1}{2} \left( \frac{af}{r} \right)^2 \right)} \sqrt{1 + \frac{1}{\beta^2} \left( \frac{a'}{r} \right)^2 + \frac{1}{4} \left( \frac{1 - a^2}{r^2} \right)^2} - 1 \right)
\]

\[
+ V(f) - \frac{F' A}{r^2} f' - \frac{F A'}{r^2} a' = 0. \quad (68)
\]

Now naming \( f' \equiv x, \; a' \equiv y, \; Qf \equiv F' A, \; Qa \equiv F A' \) so that we have

\[
a' = \pm \frac{\sqrt{D_0} + Qa Qf r^2 x - Qar^4 (V - 2\beta^2)}{r^2 (4a^2 f^2 - Qa^2 + 2r^2 (2\beta^2 + x^2))} \quad (69)
\]

with

\[
D_0 = r^2 \left( 2a^2 f^2 + r^2 (2\beta^2 + x^2) \right)
\]

\[
\times \left\{ -4a^6 f^2 + a^4 (8f^2 + Qa^2 - 2r^2 (2\beta^2 + x^2)) \right. \\
-2a^2 (f^2 (4\beta^2 r^4 + 2) + Qa^2 - 2r^2 (2\beta^2 + x^2)) \\
+2\beta^2 r^4 (Qa^2 + 4Qf x - 2r^2 (2V + x^2)) + Qa^2 \\
\left. +2r^2 (-2\beta^2 + (Qf^2 - 1) x^2 - 2Qfr^2 V x + r^4 V^2) \right\} \quad (70)
\]

that must be zero. This implies that, since we want to find \( F \) and \( A \), we need the terms inside the curly bracket to be zero. Then it produces

\[
f'_1 = \pm \frac{\sqrt{D_1} - 2Qf r^4 (V - 2\beta^2)}{2r^2 (a^4 - 2a^2 - Qf^2 + 2\beta^2 r^4 + 1)} \quad (71)
\]

with

\[
D_1 = r^2 \left( (a^2 - 1)^2 + 2\beta^2 r^4 \right)
\]

\[
\times \left\{ -4a^6 f^2 + a^4 (8f^2 + Qa^2 - 4\beta^2 r^4) + a^2 (4f^2 (Qf^2 - 2\beta^2 r^4 - 1) - 2Qa^2 + 8\beta^2 r^2) \right. \\
+Qa^2 (-Qf^2 + 2\beta^2 r^4 + 1) + 4\beta^2 r^2 (Qf^2 - 2r^2 V - 1) + 2r^6 V^2 \right\} \quad (72)
\]

that must be zero. Again, since we want to find \( F \) and \( A \), we need the terms inside the curly bracket to be zero. Since \( D_1 = 0 \) then the term inside curly bracket must be zero. Then expressing it as powers of \( r \) we have

\[
- \left( (a^2 - 1)^2 - Qf^2 \right) \left( 4a^2 f^2 - Qa^2 \right) + 2\beta^2 r^4 (Qa^2 - 4a^2 f^2) \\
+4\beta^2 r^2 \left( Qf^2 - (a^2 - 1)^2 \right) + 2r^6 V (V - 2\beta^2) = 0. \quad (73)
\]

Since the BPS energy function is valid at all \( r \), each term must be zero, which imply \( V = 0, \; Qa = \pm 2af, \; Qf = \pm (a^2 - 1) \). Hence we obtain the standard BPS energy function \( Q = 4\pi F(f)A(a) \) with \( F = f \) and \( A = \pm (a^2 - 1) \), and its standard BPS equations

\[
f' = \frac{Qf}{r^2} = \pm \frac{(a^2 - 1)}{r^2}, \quad (74)
\]

\[
a' = \frac{Qa}{2} = \pm af. \quad (75)
\]
7. Discussions
The method that had been proposed in [12] in obtaining BPS equations of some models of vortices is still able to obtain BPS equations of some nonabelian models of magnetic monopole. We are, at the moment, investigating the BPS equations in several other models using this method. Some of them (not shown here) refuse to give the BPS equations using our method, and thus we need to modify our formalism in order to enable it. For our next steps, we intend to find some other possible models that can be of interest, e.g., $V$ dependent on $|\phi|$ from some other models since all above models have $V = \text{constant}$.

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References
[1] Kibble T W B 1976 J. Phys. A 9 1387–1398
[2] Shellard E P S and Vilenkin A 1994 Cosmic strings and other topological defects (Cambridge: Cambridge University Press) p 64
[3] Rajaraman R 1982 Solitons and Instantons: an Introduction to Solitons and Instantons (Amsterdam: North-Holland) p 62
[4] Weinberg E J 2012 Classical solutions in quantum field theory: Solitons and Instantons in High Energy Physics (Cambridge: Cambridge University Press) p 89
[5] Dirac P A M 1931 Proc. Roy. Soc. Lond. A 133 60-72
[6] 't Hooft G 1974 Nucl. Phys. B 79 276–284
[7] Polyakov A M 1974 JETP Lett. 20 194–195 [Pisma Zh. Eksp. Teor. Fiz.20,430(1974)]
[8] Polyakov A M 1975 Zh. Eksp. Teor. Fiz. 68 1975
[9] Prasad M K and Sommerfield C M 1975 Phys. Rev. Lett. 35 760–762
[10] Bogomolny E B 1976 Sov. J. Nucl. Phys. 24 449 [Yad. Fiz.24,861(1976)]
[11] Atmaja A N and Ramadhan H S 2014 Phys. Rev. D 90 105009 (Preprint 1406.6180)
[12] Atmaja A N 2017 Phys. Lett. B 768 351–358 (Preprint 1511.01620)
[13] Nakamura A and Shiraishi K 1991 Hadronic J. 14 369–375 (Preprint 1309.0282)