Non-ergodic subdiffusion from Brownian motion in an inhomogeneous medium

P. Massignan,1 C. Manzo,1 J. A. Torreno-Pina,1 M. F. García-Parajo,1,2 M. Lewenstein,1,2 and G. J. Lapeyre, Jr.1
1ICFO–Institut de Ciències Fotòniques, Mediterranean Technology Park, 08860 Castelldefels, Spain
2ICREA-Institució Catalana de Recerca i Estudis Avançats, Lluís Companys 23, 08010 Barcelona, Spain
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Non-ergodicity observed in single-particle tracking experiments is usually modeled by transient trapping rather than spatial disorder. We introduce models of a particle diffusing in a medium consisting of regions with random sizes and random diffusivities. The particle is never trapped, but rather performs continuous Brownian motion with the local diffusion constant. Under simple assumptions on the distribution of the sizes and diffusivities, we find that the mean squared displacement displays subdiffusion due to non-ergodicity for both annealed and quenched disorder. The model is formulated as a walk continuous in both time and space, similar to the Lévy walk.

Disordered systems exhibiting subdiffusion have been studied intensively for decades [1–5]. In these systems the ensemble averaged mean squared displacement (EMSD) grows for large times as

$$\langle x^2(t) \rangle \sim t^\beta \quad \text{with } 0 < \beta < 1,$$

(1)

whereas normal diffusion has $\beta = 1$. A broad class of systems show weak ergodicity breaking, that is, the MSD and the time averaged mean squared displacement (TMSD) differ. The prototypical framework for describing non-ergodic subdiffusion is the heavy-tailed continuous-time random walk (CTRW) [6, 7], in which a particle takes steps at random time intervals that are independently distributed with density

$$\psi(\tau) \sim \tau^{-\alpha - 1} \quad 0 < \alpha < 1.$$  

(2)

$\psi(\tau)$ has infinite mean, which leads to a subdiffusive MSD $\beta = \alpha$. Furthermore, the mean squared displacement (MSD) is non-ergodic because the particle experiences trapping times on the order of the observation time $T$ no matter how large $T$ is. The CTRW was introduced to describe charge carriers in amorphous solids [7], and has found wide application since. Recently, there has been a surge of work on the CTRW [8–11], triggered by single particle tracking experiments in biological systems [12–16] that display signatures of non-ergodicity.

A different approach to subdiffusion is to assume a deterministic diffusivity (i.e. diffusion coefficient) that is inhomogeneous in time [17, 18], or space [19–22]. But in fact, the anomalous diffusion in these works is also non-ergodic. Formulating models of inhomogeneous diffusivity is timely and important, given that recently measured spatial maps in the cell membrane often show patches of strongly varying diffusivity [23–28]. The presence of randomness in these experimental maps inspired us to consider disordered media. Thus, in this manuscript, we introduce a class of models of ordinary diffusion with a diffusivity that varies randomly but is constant on patches of random sizes. We call these models random patch models or just patch models. These models show subdiffusion with non-ergodic MSD, due to the diffusivity effectively changing at random times with a heavy-tailed distribution like that in (2) [29]. Note that ergodicity breaking is usually ascribed to energetic disorder that immobilizes the particle, e.g. via transient chemical binding [7, 30, 31]. But, in the patch models discussed here the particle constantly undergoes Brownian motion. The anomaly is introduced not by transient immobilization, but rather by a disordered medium. This is a crucial distinction because, although non-ergodicity and heterogeneity are often observed in the same system, the toolbox for describing them is rather spare [5]. Patch models address the pressing need to enlarge this toolbox.

After introducing the models, we explain the origin of the subdiffusivity (1), and the dependence of the exponent $\beta$ on the model parameters. Then we calculate $\beta$ for a patch model using Fourier-Laplace techniques. Next, we discuss the conditions under which the linear behavior observed in the time-ensemble averaged MSD (TEMSD) of the CTRW [8, 9] may occur in other models, and its appearance in patch models. Next we present our numerical results. Finally, we address future work.

The disorder in these models is introduced via independent and identically distributed pairs of random variables \{$(D_j, \tau_j)$\} or \{$(D_j, r_j)$\}. Here, $D_j$ is a diffusivity, $\tau_j$ is a transit time, and $r_j$ is a length scale (radius). For clarity, we concentrate on the one-dimensional case.

**Annealed transit time model (ATTM)** — In this model, the particle begins at $x = t = 0$ and diffuses for a time $\tau_1$ with diffusivity $D_1$. Then, a new pair $(D_2, \tau_2)$ is sampled, and from time $\tau_1$ to $\tau_1 + \tau_2$ the particle diffuses with diffusivity $D_2$. Diffusion then continues for the third pair and so on. We assume that the pairs \{$(D_j, \tau_j)$\} are distributed with a PDF $P_{D,\tau}(D, \tau) = P_D(D) P_{\tau}(\tau | D)$, such that as $D \to 0$,

$$P_D(D) \sim D^{\sigma - 1} \quad \text{with } \sigma > 0,$$

(3)

and that $P_D(D)$ decays rapidly for large $D$. Furthermore, we require that the PDF for $\tau$ given that we have sampled
Using (3), we find the PDF for the transit time
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 annealed patch models, a new pair (D, r) is sampled. After hitting the boundary,
the motion continues at the center of the new patch. We take P_{D,r}(D, r) = \phi(D) P_r(r|D),
where P_r(r|D) has mean E[r|D] = D^{(1-\gamma)/2}. Since \langle x^2(t) \rangle \propto Dt, this choice of the exponent ensures
that typical values of r_j are the same as those of \sqrt{D_j}\tau_j. As we will see, the average
behavior of the ARM and the ATTM is the same. In the annealed patch models, a new pair (D_j, r_j) or (D_j, \tau_j) is sampled
every time the particle hits a border. An example of annealed disorder is a protein subject to receptor-
ligand interactions or conformational changes that modulate the coupling with its environment [32, 33].
The result is a diffusivity that is not associated with a position on the membrane, but rather fluctuates in time.

Quenched radius model (QRM) — In this model, we have pairs (D_j, r_j) with the same PDF P_{D,r}(D, r) as in the ARM.
The difference is that the patches are fixed in space for the duration of each trajectory. Thus, if the particle
crosses a border from patch j with (D_j, r_j) to patch j + 1, and later crosses back to patch j, it will find again
the same (D_j, r_j). In fact, it may visit the same patch many times. An example of a system with quenched dis-
order is diffusion on liquid ordered/disordered phases of a lipid membrane [34]. Depending on dimension and de-
tails of the model, the difference between quenched and annealed disorder may drastically affect the dynamics.
We found that this is indeed the case for the QTM compared to the ATTM and ARM.

Anomalous exponents — As we will see, all patch models exhibit a regime of normal diffusion (0), and two anomalous regimes: (I) and (II). The corresponding exponents are summarized in Table I, and will be derived below. Their origin however may be understood in simple terms by considering the ATTM with the simplest PDF satisfying (4), \phi(D) \delta(\tau - D^{-\gamma}), that is \tau = D^{-\gamma}. Using (3), we find the PDF for the transit time
\[\psi(\tau) \, d\tau = P_D(D(\tau)) \frac{dD}{d\tau} \, d\tau \sim \tau^{-\gamma} \, d\tau,\]
which has a heavy tail for \sigma < \gamma. The density (5) will play the role of the waiting-time density (2) with
\alpha = \sigma/\gamma. In fact, if we observe the ATTM with a stroboscope that illuminates the particle only at the final
position on each patch, we see exactly a CTRW with waiting times \tau_j = D_j^{-\gamma} and step lengths with variance
\tau_j D_j = D_j^{(1-\gamma)/2}. Equivalently, we can generate \tau = r^2/D from a random radius \tau = D^{(1-\gamma)/2}
with PDF
\[P_\tau(r) \sim r^{-\frac{\gamma}{\gamma-1}},\]
which has a diverging variance when \sigma + 1 < \gamma. Similar arguments for the ARM and QRM, as well as for the asymptotic forms of other distributions for \phi(D) and P_r(r|D), result in the same boundaries between regimes as in the ATTM. These observations explain the regimes in Table I showing that regime (I) corresponds to divergent E[\tau] and finite E[r^2], while in regime (II), both E[\tau] and E[r^2] are divergent. In this way, regime (II) is similar to the Lévy walk [4].

Fourier-Laplace transform solution — Here we compute \langle x^2(t) \rangle in (1) for the ATTM using techniques for
analyzing CTRWs in which the waiting time and the step length are not independent [4, 35]. We again assume that
the PDF for \tau is concentrated on a point, i.e. \tau = D^{-\gamma}. To describe partially completed motion on a patch, we
write the probability density for a displacement x at time \tau on a patch with transit time \tau’ such that \tau \leq \tau’ [36]:
\[\psi(x, \tau’, \tau) = \phi(x|\tau’, \tau) \psi(\tau’).\]
We write the PDF for a displacement x at the end of a step, that is at time \tau, on a patch with transit time \tau, as
\[\phi(x|\tau) \equiv \phi(x|\tau, \tau).\]
Likewise, \psi(x, \tau) \equiv \psi(x, \tau, \tau). For the PDF of the displacement on a patch x at time \tau, in the case that the only information that we have on the transit time \tau’ is \tau < \tau’, we write
\[\Psi(x, \tau) = \int_\tau^\infty \psi(x, \tau’, \tau) \, d\tau’.\]
\[\Psi(x, \tau)\]
describes the displacement of the particle on the final, uncompleted, patch. Note that if \phi(x|\tau’, \tau)
is independent of \tau’, we have \Psi(x, \tau) = \phi(x|\tau’, \tau) \Psi(\tau),
where the survival probability \Psi(\tau) = \int_\tau^\infty \psi(x|\tau’ \, d\tau’ is the probability that a step is not completed by time \tau.
An example is the Lévy walk [4, 35], in which the walker undergoes rectilinear motion on each step; that is,
\[\psi(x, \tau’, \tau) = \delta(|x| - ct) \psi(\tau’),\]
where the speed c is independent of \tau’. In our case, however, D is not independent of \tau’ and this simplification cannot be made.

We denote by P(x,t) the PDF for the particle to be at x at time t, with the initial condition P(x,t=0) = \delta(x), and by \eta(x,t) the PDF of the particle’s position at time t just after having completed a step. Then \eta(x,t) = \delta(x) \delta(t) + \int_{-\infty}^t dx’ \int_0^t dt’ \eta(x’, t’|x - x’, t - t’).
Thus, accounting for the continuous motion does not affect the EMSD, which remains the same as in the CTRW. The inverse Laplace transform of (12) gives us

\[ \langle x^2(t) \rangle \sim t^\alpha \quad \text{for } q > \alpha. \]

Now we consider the case \( q < \alpha \). \( \psi''(k,s) |_{k=0} \) no longer satisfies the hypothesis of the Tauberian theorem. But its integral does, which leads to \( \psi''(k,s) |_{k=0} \sim c - bs^{\alpha-q} \).

Thus, the first term in (12) is \( \langle x^2(s) \rangle \sim (c - bs^{\alpha-q})/s^{\alpha+1} \), or for small \( s \), \( \langle x^2(s) \rangle \sim s^{-\alpha-1} \). A similar calculation again shows that the second term has the same exponent. The inverse Laplace transform gives

\[ \langle x^2(t) \rangle \sim t^\alpha \quad \text{for } q < \alpha. \]
(i) $x(t)$ has uncorrelated increments and

$\langle x^2(t)\rangle \sim t^\beta$ with $\beta \neq 1$,

then $x(t)$ has non-stationary increments and a non-ergodic MSD that satisfies

$$\langle x^2(t)\rangle_T \sim T^{\beta-1}t.$$  \hfill (18)

Brownian motion satisfies (i), but not (ii). Both fractional Brownian motion [41] with $\beta < 1$ and the random walk on a fractal [1] satisfy (ii), but not (i). The CTRW satisfies both (i) [12] and (ii). The CTRW on a fractal satisfies (ii), but not (i). It also shows non-ergodicity, but $\langle x^2(t)\rangle_T \sim T^{\beta-1}t$ [10]. The CTRW has already been shown to follow (18) \cite{8, 9}. Furthermore, the statistics of the time average $\langle x^2(t)\rangle_T$ for the CTRW, which does not converge to a constant random variable, have been studied in Ref. \cite{8}. We do not present a proof that patch models satisfy (i), but, in fact, our numerical results show they follow (18).

Simulations—The results of extensive computer simulations of all models are shown in Figs. 1 and 2. We used the gamma distribution for $P_D(D)$ in (3), and (normal and stretched) exponential, log-normal, and single-point distributions for $P_{D,r}(D, \tau)$ and $P_{D,r}(D, r)$. The exponent $\beta$ was determined for the EMSD by a linear fit of $\log(\langle x^2(t)\rangle)$ vs $\log(t)$. To analyze the TEMSD, we first determined the diffusivities by a linear fit of the TEMSD vs. the lag at given $T$. We then did a linear fit to a log-log plot of the resulting diffusivities vs. $T$ to get $\beta - 1$ in (18). The exponents $\beta$ obtained from the EMSD and TEMSD are in excellent agreement with Table I. The QRM in regime (II) clearly shows subdiffusion. But at present we have no explanation for $\beta$ in this regime.

To understand why, in the ATTM, we position the particle at the center of a new patch upon hitting a border, recall that a 1d Brownian path crosses a point infinitely many times before leaving any neighborhood [40]. Now, assume annealed disorder and that the particle enters a new patch at its boundary, as in the QRM. Because a new patch is sampled each time the border is crossed, the particle samples an infinite number of patches during the crossing. In this case, our simulations of the EMSD did not converge with decreasing step length. But the EMSD does converge for the QRM, which visits the same two patches an infinite number of times on crossing a border.

Outlook and Applications—Many questions remain to be addressed. For instance, what is the behavior at the boundaries of the parameter regimes, that is for $\gamma = \sigma$ and $\gamma = \sigma + 1$, as well as in regime (II) for the QRM? Regarding dimensions $d > 1$: The ATTM and ARM are the same for all $d$, and the EMSD for the quenched CTRW for $d > 1$ has the same exponent $\beta$ as the (annealed) CTRW, with logarithmic corrections for $d = 2$ \cite{2, 43}. But before analyzing the QRM in $d > 1$, a geometry of patches consistent with $P_{D,r}(D, \tau)$ must be found.

Patch models provide an alternative for describing non-ergodic diffusion in biological systems, one that is due to inhomogeneous diffusivity rather than transient trapping. But there are many similarities in the long-time behavior of the CTRW and the patch models. Thus, the main open problem is finding methods to distinguish them, and regime (I) from (II). Two promising leads in this direction are studying the survival time density and comparing the MSD exponents predicted by our models with those extracted from spatial maps of diffusivity.
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