SECOND MAIN THEOREMS FOR MEROMORPHIC MAPPINGS INTERSECTING MOVING HYPERPLANES WITH TRUNCATED COUNTING FUNCTIONS AND UNICITY PROBLEM

SI DUC QUANG

ABSTRACT. In this article, we show some new second main theorems for the mappings and moving hyperplanes of $\mathbb{P}^n(\mathbb{C})$ with truncated counting functions. Our results are improvements of recent previous second main theorems for moving hyperplanes with the truncated (to level $n$) counting functions. As their application, we prove a unicity theorem for meromorphic mappings sharing moving hyperplanes.

1. Introduction

The theory of the Nevanlinna’s second main theorem for meromorphic mappings of $\mathbb{C}^m$ into the complex projective space $\mathbb{P}^n(\mathbb{C})$ intersecting a finite set of fixed hyperplanes or moving hyperplanes in $\mathbb{P}^n(\mathbb{C})$ was started about 70 years ago and has grown into a huge theory. For the case of fixed hyperplanes, maybe, the second main theorem given by Cartan-Nochka is the best possible. Unfortunately, so far there has been a few second main theorems with truncated counting functions for moving hyperplanes. Moreover, almost of them are not sharp.

We state here some recent results on the second main theorems for moving hyperplanes with truncated counting functions.

Let $\{a_i\}_{i=1}^q$ be meromorphic mappings of $\mathbb{C}^m$ into the dual space $\mathbb{P}^n(\mathbb{C})^*$ in general position. For the case of nondegenerate meromorphic mappings, the second main theorem with truncated (to level $n$) counting functions states that.

**Theorem A** (see [4, Theorem 2.3] and [6, Theorem 3.1]). Let $f : \mathbb{C}^m \rightarrow \mathbb{P}^n(\mathbb{C})$ be a meromorphic mapping. Let $\{a_i\}_{i=1}^q$ be meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})^*$ in general position such that $f$ is linearly nondegenerate over $\mathcal{R}(\{a_i\}_{i=1}^q)$. Then

$$\left| \mathcal{T}_f(r) \right| \leq \sum_{i=1}^{q} N_{f,a_i}^n(r) + o(\mathcal{T}_f(r)) + O(\max_{1 \leq i \leq q} \mathcal{T}_{a_i}(r)).$$

We note that, Theorem A is still the best second main theorem with truncated counting functions for nondegenerate meromorphic mappings and moving hyperplanes available at present. In the case of degenerate meromorphic mappings, the second main theorem for moving hyperplanes with counting function truncated to level $n$ was first given by M. Ru-J. Wang [5] in 2004. After that in 2008, D. D. Thai-S. D. Quang [7] improved the result of M. Ru-J. Wang by proved the following second main theorem.
Theorem B (see [7 Corollary 1]). Let \( f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C}) \) be a meromorphic mapping. Let \( \{a_i\}_{i=1}^q \) (\( q \geq 2n+1 \)) be \( q \) meromorphic mappings of \( \mathbb{C}^m \) into \( \mathbb{P}^n(\mathbb{C})^* \) in general position such that \( (f, a_i) \neq 0 \) \( (1 \leq i \leq q) \). Then

\[
|| \frac{q}{2n+1} \cdot T_f(r) \leq \sum_{i=1}^q N^{[2]}_{(f,a_i)}(r) + O\left(\max_{1 \leq i \leq q} T_{a_i}(r)\right) + O\left(\log^+ T_f(r)\right).
\]

These results play very essential roles in almost all researches on truncated multiplicity problems of meromorphic mappings with moving hyperplanes. However, in our opinion, the above mentioned results of these authors are still weak.

Our main purpose of the present paper is to show a stronger second main theorem of meromorphic mappings from \( \mathbb{C}^m \) into \( \mathbb{P}^n(\mathbb{C}) \) for moving targets. Namely, we will prove the following.

**Theorem 1.1.** Let \( f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C}) \) be a meromorphic mapping. Let \( \{a_i\}_{i=1}^q \) (\( q \geq 2n-k+2 \)) be meromorphic mappings of \( \mathbb{C}^m \) into \( \mathbb{P}^n(\mathbb{C})^* \) in general position such that \( (f, a_i) \neq 0 \) \( (1 \leq i \leq q) \), where \( k+1 = \text{rank}_{\mathbb{R}(a)}(f) \). Then the following assertions hold:

(a) \( || \frac{q}{2n-k+2} T_f(r) \leq \sum_{i=1}^q N^{[k]}_{(f,a_i)}(r) + o(T_f(r)) + O(\max_{1 \leq i \leq q} T_{a_i}(r)), \)

(b) \( || \frac{q-n+2k-1}{n+k+1} T_f(r) \leq \sum_{i=1}^q N^{[k]}_{(f,a_i)}(r) + o(T_f(r)) + O(\max_{1 \leq i \leq q} T_{a_i}(r)). \)

We may see that Theorem 1.1(a) is a generalization of Theorem A and also is an improvement of Theorem B. Theorem 1.1(b) is really stronger than Theorem B.

**Remark.**

1) If \( k \geq \frac{n+1}{2} \) then Theorem 1.1(a) is stronger than Theorem 1.1(b). Otherwise, if \( k < \frac{n+1}{2} \) then Theorem 1.1(b) is stronger than Theorem 1.1(a).

2) If \( k = 0 \) then \( f \) is constant map, and hence \( T_f(r) = 0 \).

3) Setting \( t = \frac{2n-k+2}{3n+3} \) and \( \lambda = \frac{n+k+1}{3n+3} \), we have \( t + \lambda = 1 \). Thus, for all \( 1 \leq k \leq n \) we have

\[
\max\left\{ \frac{q}{2n-k+2}, \frac{q-n+2k-1}{n+k+1} \right\} \geq \frac{q}{2n-k+2} \cdot t + \frac{q-n+2k-1}{n+k+1} \cdot \lambda
\]

\[
= \frac{2q-n+2k-1}{3n+3} \geq \frac{2q-n+1}{3n+3}.
\]

4) If \( k \geq 1 \), we have the following estimates:

- \( \min_{1 \leq k \leq \frac{n+1}{2}} \left( \frac{q}{2n-k+2} \right) \geq \frac{q}{2n-\frac{n+1}{2}+2} = \frac{2q}{3(n+1)} \).

- \( \min_{1 \leq k \leq \frac{n+1}{2}} \left( \frac{q-n+2k-1}{n+k+1} \right) = \min_{1 \leq k \leq \frac{n+1}{2}} \left( \frac{q-3n-3}{n+k+1} + 2 \right) \).
Corollary 1.2. Let $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic mapping. Let $\{a_i\}_{i=1}^q$ $(q \geq 2n+1)$ be meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})^*$ in general position such that $(f, a_i) \neq 0$ $(1 \leq i \leq q)$.

(a) Then we have
$$
\| \frac{2q - n + 1}{3(n+1)} T_f(r) \| \leq \sum_{i=1}^{q} N^{[n]}_{(f, a_i)}(r) + o(T_f(r)) + O(\max_{1 \leq i \leq q} T_{a_i}(r)).
$$

(b) If $q \geq 3n + 3$ then
$$
\| \frac{2q}{3(n+1)} T_f(r) \| \leq \sum_{i=1}^{q} N^{[n]}_{(f, a_i)}(r) + o(T_f(r)) + O(\max_{1 \leq i \leq q} T_{a_i}(r)).
$$

(c) If $q < 3n + 3$ then
$$
\| \frac{q - n + 1}{n + 2} T_f(r) \| \leq \sum_{i=1}^{q} N^{[n]}_{(f, a_i)}(r) + o(T_f(r)) + O(\max_{1 \leq i \leq q} T_{a_i}(r)).
$$

As applications of these second main theorems, in the last section we will prove a unicity theorem for meromorphic mappings sharing moving hyperplanes regardless of multiplicities. To state our main result, we give the following definition.

Let $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic mapping. Let $k$ be a positive integer or maybe $+\infty$. Let $\{a_i\}_{i=1}^q$ be “slowly” (with respect to $f$) moving hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position such that
$$
\dim \{ z \in \mathbb{C}^m : (f, a_i)(z) \cdot (f, a_j)(z) = 0 \} \leq m - 2 \quad (1 \leq i < j \leq q).
$$

Consider the set $\mathcal{F}(f, \{a_i\}_{i=1}^q, k)$ of all meromorphic maps $g : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ satisfying the following two conditions:

(a) $\min \{ \nu_{(f, a_i)}(z), k \} = \min \{ \nu_{(g, a_i)}(z), k \}$ $(1 \leq i \leq q)$, for all $z \in \mathbb{C}^m$,

(b) $f(z) = g(z)$ for all $z \in \bigcup_{i=1}^{q} \text{Zero}(f, a_i)$.

We will prove the following

Theorem 1.3. Let $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic mapping. Let $\{a_i\}_{i=1}^q$ be slowly (with respect to $f$) moving hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position such that
$$
\dim \{ z \in \mathbb{C}^m : (f, a_i)(z) \cdot (f, a_j)(z) = 0 \} \leq m - 2 \quad (1 \leq i < j \leq q).
$$
Then the following assertions hold:

a) If \( q > \frac{9n^2 + 9m + 4}{4} \) then \( \# F(f, \{a_i\}_{i=1}^q, 1) \leq 2 \),

b) If \( q > 3n^2 + n + 2 \) then \( \# F(f, \{a_i\}_{i=1}^q, 1) = 1 \).

Acknowledgements. This work was done during a stay of the author at Vietnam Institute for Advanced Study in Mathematics. He would like to thank the institute for their support.

2. Basic notions and auxiliary results from Nevanlinna theory

(a) Counting function of divisor.

For \( z = (z_1, \ldots, z_m) \in \mathbb{C}^m \), we set \( \|z\| = \left( \sum_{j=1}^m |z_j|^2 \right)^{1/2} \) and define

\[
B(r) = \{ z \in \mathbb{C}^m; \|z\| < r \}, \quad S(r) = \{ z \in \mathbb{C}^m; \|z\| = r \},
\]

\[
d^e = \frac{\sqrt{-1}}{4\pi} (\partial - \partial), \quad \sigma = (dd^c \|z\|^2)^{m-1},
\]

\[
\eta = d^c \log \|z\|^2 \wedge (dd^c \log \|z\|)^{m-1}.
\]

Throughout this paper, we denote by \( \mathcal{M} \) the set of all meromorphic functions on \( \mathbb{C}^m \). A divisor \( E \) on \( \mathbb{C}^m \) is given by a formal sum \( E = \sum \mu_\nu X_\nu \), where \( \{X_\nu\} \) is a locally family of distinct irreducible analytic hypersurfaces in \( \mathbb{C}^m \) and \( \mu_\nu \in \mathbb{Z} \). We define the support of the divisor \( E \) by \( \text{Supp}(E) = \bigcup_{\nu \neq 0} X_\nu \). Sometimes, we identify the divisor \( E \) with a function \( E(z) \) from \( \mathbb{C}^m \) into \( \mathbb{Z} \) defined by \( E(z) := \sum_{X_\nu \ni z} \mu_\nu \).

Let \( k \) be a positive integer or \( +\infty \). We define the truncated divisor \( E^{[k]} \) by

\[
E^{[k]} := \sum_{\nu} \min\{\mu_\nu, k\} X_\nu,
\]

and the truncated counting function to level \( k \) of \( E \) by

\[
N^{[k]}(r, E) := \int_1^r \frac{n^{[k]}(t, E)}{t^{2m-1}} dt \quad (1 < r < +\infty),
\]

where

\[
n^{[k]}(t, E) := \begin{cases} 
\int_{\text{Supp}(E) \cap B(t)} E^{[k]} \sigma & \text{if } m \geq 2, \\
\sum_{|z| \leq t} E^{[k]}(z) & \text{if } m = 1.
\end{cases}
\]

We omit the character \( [k] \) if \( k = +\infty \).

For an analytic hypersurface \( E \) of \( \mathbb{C}^m \), we may consider it as a reduced divisor and denote by \( N(r, E) \) its counting function.

Let \( \varphi \) be a nonzero meromorphic function on \( \mathbb{C}^m \). We denote by \( \nu_\varphi^0 \) (resp. \( \nu_\varphi^\infty \)) the divisor of zeros (resp. divisor of poles) of \( \varphi \). The divisor of \( \varphi \) is defined by

\[
\nu_\varphi = \nu_\varphi^0 - \nu_\varphi^\infty.
\]
We have the following Jensen’s formula:
\[
N(r, \nu^0_\varphi) - N(r, \nu^\infty_\varphi) = \int_{S(r)} \log|\varphi|\eta - \int_{S(1)} \log|\varphi|\eta.
\]

For convenience, we will write \(N_\varphi(r)\) and \(N^{[k]}_\varphi(r)\) for \(N(r, \nu^0_\varphi)\) and \(N^{[k]}(r, \nu^0_\varphi)\), respectively.

(b) The first main theorem.

Let \(f\) be a meromorphic mapping of \(\mathbb{C}^m\) into \(\mathbb{P}^n(\mathbb{C})\). For arbitrary fixed homogeneous coordinates \((w_0 : \cdots : w_n)\) of \(\mathbb{P}^n(\mathbb{C})\), we take a reduced representation \(f = (f_0 : \cdots : f_n)\), which means that each \(f_i\) is holomorphic function on \(\mathbb{C}^m\) and \(f(z) = (f_0(z) : \cdots : f_n(z))\) outside the analytic set \(I(f) := \{z; f_0(z) = \cdots = f_n(z) = 0\}\) of codimension at least 2.

Denote by \(\Omega\) the Fubini Study form of \(\mathbb{P}^n(\mathbb{C})\). The characteristic function of \(f\) (with respect to \(\Omega\)) is defined by
\[
T_f(r) := \int_1^r \frac{dt}{t^{2m-1}} \int_{B(t)} f^* \Omega \wedge \sigma, \quad 1 < r < +\infty.
\]

By Jensen’s formula we have
\[
T_f(r) = \int_{S(r)} \log|f|\eta + O(1),
\]
where \(|f| = \max\{|f_0|, \ldots, |f_n|\}\).

Let \(a\) be a meromorphic mapping of \(\mathbb{C}^m\) into \(\mathbb{P}^n(\mathbb{C})^*\) with reduced representation \(a = (a_0 : \cdots : a_n)\). We define
\[
m_{f,a}(r) = \int_{S(r)} \log\frac{|f| \cdot |a|}{|(f,a)|} \eta - \int_{S(1)} \log\frac{|f| \cdot |a|}{|(f,a)|} \eta,
\]
where \(|a| = (|a_0|^2 + \cdots + |a_n|^2)^{1/2}\) and \((f,a) = \sum_{i=0}^n f_i \cdot a_i\).

Let \(f\) and \(a\) be as above. If \((f,a) \neq 0\), then the first main theorem for moving hyperplanes in value distribution theory states
\[
T_f(r) + T_a(r) = m_{f,a}(r) + N_{(f,a)}(r) + O(1) \quad (r > 1).
\]

For a meromorphic function \(\varphi\) on \(\mathbb{C}^m\), the proximity function \(m(r, \varphi)\) is defined by
\[
m(r, \varphi) = \int_{S(r)} \log^+ |\varphi|\eta,
\]
where \(\log^+ x = \max\{\log x, 0\}\) for \(x \geq 0\). The Nevanlinna’s characteristic function is defined by
\[
T(r, \varphi) = N(r, \nu^\infty_\varphi) + m(r, \varphi).
\]
We regard \(\varphi\) as a meromorphic mapping of \(\mathbb{C}^m\) into \(\mathbb{P}^1(\mathbb{C})^*\), there is a fact that
\[
T_{\varphi}(r) = T(r, \varphi) + O(1).
\]

(c) Lemma on logarithmic derivative.

As usual, by the notation \("|\ |\ ^P\) we mean the assertion \(P\) holds for all \(r \in [0, \infty)\) excluding a Borel subset \(E\) of the interval \([0, \infty)\) with \(\int_E dr < \infty\). Denote by \(\mathbb{Z}_+\) the set
of all nonnegative integers. The lemma on logarithmic derivative in Nevanlinna theory is stated as follows.

**Lemma 2.1** (see [8, Lemma 3.11]). Let \( f \) be a nonzero meromorphic function on \( \mathbb{C}^m \). Then
\[
\left| m\left(r, \frac{D^\alpha(f)}{f}\right) \right| = O(\log^+ T_f(r)) \quad (\alpha \in \mathbb{Z}^m_+).
\]

(d) Family of moving hyperplanes.

We assume that throughout this paper, the homogeneous coordinates of \( \mathbb{P}^n(\mathbb{C}) \) is chosen so that for each given meromorphic mapping \( a = (a_0 : \cdots : a_n) \) of \( \mathbb{C}^m \) into \( \mathbb{P}^n(\mathbb{C})^* \) then \( a_0 \neq 0 \). Set
\[
\tilde{a}_i = \frac{a_i}{a_0} \text{ and } \tilde{a} = (\tilde{a}_0 : \tilde{a}_1 : \cdots : \tilde{a}_n).
\]

Let \( f : \mathbb{C}^m \rightarrow \mathbb{P}^n(\mathbb{C}) \) be a meromorphic mapping with the reduced representation \( f = (f_0 : \cdots : f_n) \). We put \( (f, a) := \sum_{i=0}^n f_i a_i \) and \( (f, \tilde{a}) := \sum_{i=0}^n f_i \tilde{a}_i \).

Let \( \{a_i\}_{i=1}^q \) be \( q \) meromorphic mappings of \( \mathbb{C}^m \) into \( \mathbb{P}^n(\mathbb{C})^* \) with reduced representations \( a_i = (a_{i0} : \cdots : a_{in}) \) \((1 \leq i \leq q)\). We denote by \( \mathcal{R}(\{a_i\}) \) (for brevity we will write \( \mathcal{R} \) if there is no confusion) the smallest subfield of \( \mathcal{M} \) which contains \( \mathbb{C} \) and all \( a_i / a_k \) with \( a_{ik} \neq 0 \).

**Definition 2.2.** The family \( \{a_i\}_{i=1}^q \) is said to be in general position if \( \dim(\{a_{i_0}, \ldots, a_{i_n}\})_{\mathcal{M}} = n+1 \) for any \( 1 \leq i_0 \leq \cdots \leq i_n \leq q \), where \( \{a_{i_0}, \ldots, a_{i_n}\}_{\mathcal{M}} \) is the linear span of \( \{a_{i_0}, \ldots, a_{i_N}\} \) over the field \( \mathcal{M} \).

**Definition 2.3.** A subset \( \mathcal{L} \) of \( \mathcal{M} \) (or \( \mathcal{M}^{n+1} \)) is said to be minimal over the field \( \mathcal{R} \) if it is linearly dependent over \( \mathcal{R} \) and each proper subset of \( \mathcal{L} \) is linearly independent over \( \mathcal{R} \).

Repeating the argument in ([11, Proposition 4.5]), we have the following:

**Proposition 2.4** (see [11, Proposition 4.5]). Let \( \Phi_0, \ldots, \Phi_k \) be meromorphic functions on \( \mathbb{C}^m \) such that \( \{\Phi_0, \ldots, \Phi_k\} \) are linearly independent over \( \mathbb{C} \). Then there exists an admissible set \( \{a_i = (a_{i1}, \ldots, a_{in})\}_{i=0}^k \subset \mathbb{Z}^n_+ \) with \( |a_i| = \sum_{j=1}^n |a_{ij}| \leq k \) \((0 \leq i \leq k)\) such that the following are satisfied:

(i) \( \{D^{\alpha_0}\Phi_0, \ldots, D^{\alpha_k}\Phi_k\}_{i=0}^k \) is linearly independent over \( \mathcal{M} \), i.e., \( \det(D^{\alpha_i}\Phi_j) \neq 0 \).

(ii) \( \det(D^{\alpha_i}(h\Phi_j)) = h^{k+1}\det(D^{\alpha_i}\Phi_j) \) for any nonzero meromorphic function \( h \) on \( \mathbb{C}^m \).

3. **Proof of Theorem 1.1**

In order to prove Theorem 1.1 we need the following.

**Lemma 3.1.** Let \( f : \mathbb{C}^m \rightarrow \mathbb{P}^n(\mathbb{C}) \) be a meromorphic mapping. Let \( \{a_i\}_{i=1}^q \) \((q \geq n+1)\) be \( q \) meromorphic mappings of \( \mathbb{C}^m \) into \( \mathbb{P}^n(\mathbb{C})^* \) in general position. Assume that there exists a partition \( \{1, \ldots, q\} = I_1 \cup I_2 \cup \cdots \cup I_t \) satisfying:

(i) \( \{(f, \tilde{a}_i)\}_{i \in I_t} \) is minimal over \( \mathcal{R} \), and \( \{(f, \tilde{a}_i)\}_{i \in I_t} \) is linearly independent over \( \mathcal{R} \) \((2 \leq t \leq l)\),
(ii) For any $2 \leq t \leq l$, $i \in I_t$, there exist meromorphic functions $c_i \in \mathcal{R} \setminus \{0\}$ such that

$$\sum_{i \in I_t} c_i(f, \tilde{a}_i) \in \left(\bigcup_{j=1}^{t-1} \bigcup_{i \in I_j} (f, \tilde{a}_i)\right)_\mathcal{R}.$$ 

Then we have

$$T_f(r) \leq \sum_{i=1}^{q} N_{(f,a_i)}^{[k]} + o(T_f(r)) + O(\max_{1 \leq i \leq q} T_{a_i}(r)),$$

where $k + 1 = \text{rank}_\mathcal{R}(f)$.

**Proof.** Let $f = (f_0 : \cdots : f_n)$ be a reduced representation of $f$. By changing the homogeneous coordinate system of $\mathbb{P}^n(\mathbb{C})$ if necessary, we may assume that $f_0 \neq 0$. Without loss of generality, we may assume that $I_1 = \{1, \ldots, k_1\}$ and $I_t = \{k_{t-1} + 1, \ldots, k_t\}$ ($2 \leq t \leq l$, where $1 = k_0 < \cdots < k_l = q$).

Since $\{(f, \tilde{a}_i)\}_{i \in I_1}$ is minimal over $\mathcal{R}$, there exist $c_{1i} \in \mathcal{R} \setminus \{0\}$ such that

$$\sum_{i=1}^{k_1} c_{1i} \cdot (f, \tilde{a}_i) = 0.$$

Define $c_{1i} = 0$ for all $i > k_1$. Then

$$\sum_{i=1}^{k_t} c_{1i} \cdot (f, \tilde{a}_i) = 0.$$

Because $\{c_{1i}(f, \tilde{a}_i)\}_{i=k_0+1}^{k_1}$ is linearly independent over $\mathcal{R}$, Lemma 2.31 yields that there exists an admissible set $\{\alpha_{1(k_0+1)}, \ldots, \alpha_{1k_1}\} \subset \mathbb{Z}^n_+$ ($|\alpha_{1i}| \leq k_1 - k_0 - 1 \leq \text{rank}_\mathcal{R} f - 1 = k$) such that the matrix

$$A_1 = (D^{\alpha_{1i}}(c_{1j}(f, \tilde{a}_j)); k_0 + 1 \leq i, j \leq k_1)$$

has nonzero determinant.

Now consider $t \geq 2$. By constructing the set $I_t$, there exist meromorphic mappings $c_{ti} \neq 0$ ($k_{t-1} + 1 \leq i \leq k_t$) such that

$$\sum_{i=k_{t-1}+1}^{k_t} c_{ti} \cdot (f, \tilde{a}_i) \in \left(\bigcup_{j=1}^{t-1} \bigcup_{i \in I_j} (f, \tilde{a}_i)\right)_\mathcal{R}.$$ 

Therefore, there exist meromorphic mappings $c_{ti} \in \mathcal{R}$ ($1 \leq i \leq k_{t-1}$) such that

$$\sum_{i=1}^{k_t} c_{ti} \cdot (f, \tilde{a}_i) = 0.$$

Define $c_{ti} = 0$ for all $i > k_t$. Then

$$\sum_{i=1}^{k_t} c_{ti} \cdot (f, \tilde{a}_i) = 0.$$
Since \( \{c_l(f, \tilde{a}_i)\}_{i=k_l-1+1}^{k_l} \) is \( R \)-linearly independent, by again Lemma \[2, 4\] there exists an admissible set \( \{\alpha_{l(k_l-1+1)}, \ldots, \alpha_{l k_l}\} \subset \mathbb{Z}_+^m \) \( |\alpha_{l i}| \leq k_i - k_{i-1} - 1 \leq \text{rank}_R f - 1 = k \) such that the matrix

\[
A_t = (\mathcal{D}^\alpha_1(c_{1j}(f, \tilde{a}_j)); k_{t-1} + 1 \leq i, j \leq k_t)
\]

has nonzero determinant.

Consider the following \((k_t - 1) \times k_l\) matrix

\[
T = (\mathcal{D}^\alpha_{11}(c_{1j}(f, \tilde{a}_j)); k_0 + 1 \leq i \leq k_t, 1 \leq j \leq k_l)
\]

Denote by \( D_i \) the subsquare matrix obtained by deleting the \((i + 1)\)-th column of the minor matrix \( T \). Since the sum of each row of \( T \) is zero, we have

\[
det D_i = (-1)^{i-1} det D_1 = (-1)^{i-1} \prod_{j=1}^{l} det A_j.
\]

Since \( \{a_i\}_{i=1}^{q} \) is in general position, we have

\[
det(\tilde{a}_{i j}, 1 \leq i \leq n + 1, 0 \leq j \leq n) \neq 0.
\]

By solving the linear equation system \((f, \tilde{a}_i) = \tilde{a}_{i0} \cdot f_0 + \ldots + \tilde{a}_{in} \cdot f_n \) \((1 \leq i \leq n + 1)\), we obtain

\[
f_v = \sum_{i=1}^{n+1} A_v i f_i (\tilde{a}_i) \quad (A_{vi} \in \mathcal{R}) \text{ for each } 0 \leq v \leq n.
\]

Put \( \Psi(z) = \sum_{i=1}^{n+1} \sum_{v=0}^{n} |A_{vi}(z)| \) \((z \in \mathbb{C}^m)\). Then

\[
||f(z)|| \leq \Psi(z) \cdot \max_{1 \leq i \leq n+1} \left|\langle f, \tilde{a}_i \rangle (z)\right| \leq \Psi(z) \cdot \max_{1 \leq i \leq q} \left|\langle f, \tilde{a}_i \rangle (z)\right| \quad (z \in \mathbb{C}^m),
\]
and
\[
\int_{S(r)} \log^+ \Psi(z) \eta \leq \sum_{i=1}^{n+1} \sum_{v=0}^{n} \int_{S(r)} \log^+ |A_{vi}(z)| \eta + O(1)
\]
\[
\leq \sum_{i=1}^{n+1} \sum_{v=0}^{n} T(r, A_{vi}) + O(1)
\]
\[
= O(\max_{1 \leq i \leq q} T_{a_i}(r)) + O(1).
\]

Fix \( z_0 \in C^m \setminus \bigcup_{j=1}^{q} \left( \text{Supp} (\nu_{(f, \tilde{a}_j)}^0) \cup \text{Supp} (\nu_{(f, \tilde{a}_j)}^\infty) \right) \). Take \( i (1 \leq i \leq q) \) such that
\[
|(f, \tilde{a}_i)(z_0)| = \max_{1 \leq j \leq q} (|f, \tilde{a}_j|(z_0)|).
\]

Then
\[
\frac{|\det D_1(z_0)| \cdot ||f(z_0)||}{\prod_{j=1}^{q} |(f, \tilde{a}_j)(z_0)|} = \frac{|\det D_i(z_0)|}{\prod_{j=1, j \neq i}^{q} |(f, \tilde{a}_j)(z_0)|} \cdot \left( \frac{||f(z_0)||}{|(f, \tilde{a}_i)(z_0)||} \right)
\]
\[
\leq \Psi(z_0) \cdot \frac{|\det D_i(z_0)|}{\prod_{j=1, j \neq i}^{q} |(f, \tilde{a}_j)(z_0)||}
\]
This implies that
\[
\log \frac{|\det D_1(z_0)| \cdot ||f(z_0)||}{\prod_{j=1}^{q} |(f, \tilde{a}_j)(z_0)|} \leq \log^+ \left( \Psi(z_0) \cdot \left( \frac{|\det D_i(z_0)|}{\prod_{j=1, j \neq i}^{q} |(f, \tilde{a}_j)(z_0)|} \right) \right)
\]
\[
\leq \log^+ \left( \frac{|\det D_i(z_0)|}{\prod_{j=1, j \neq i}^{q} |(f, \tilde{a}_j)(z_0)|} \right) + \log^+ \Psi(z_0).
\]
Thus, for each \( z \in C^m \setminus \bigcup_{j=1}^{q} \left( \text{Supp} (\nu_{(f, \tilde{a}_j)}^0) \cup \text{Supp} (\nu_{(f, \tilde{a}_j)}^\infty) \right) \), we have
\[
\log \frac{|\det D_i(z)| \cdot ||f(z)||}{\prod_{i=1}^{q} |(f, \tilde{a}_i)(z)|} \leq \sum_{i=1}^{q} \log^+ \left( \frac{|\det D_i(z)|}{\prod_{j=1, j \neq i}^{q} |(f, \tilde{a}_j)(z)|} \right) + \log^+ \Psi(z)
\]
Hence
\[
(3.3) \quad \log ||f(z)|| + \frac{|\det D_1(z)|}{\prod_{i=1}^{q} |(f, \tilde{a}_i)(z)|} \leq \sum_{i=1}^{q} \log^+ \left( \frac{|\det D_i(z)|}{\prod_{j=1, j \neq i}^{q} |(f, \tilde{a}_j)(z)|} \right) + \log^+ \Psi(z).
\]
Note that
\[
\det D_i \prod_{j=1, j \neq i}^q (f, \tilde{a}_j) = \frac{\det D_i / f_0^{q-1}}{\prod_{j=1, j \neq i}^q (f, \tilde{a}_j) / f_0}
\]

\[
= \begin{bmatrix}
\mathcal{D}^{a_{i1}} \left( \frac{c_{11}(f, \tilde{a}_1)}{f_0} \right) & \cdots & \mathcal{D}^{a_{ik}} \left( \frac{c_{ik}(f, \tilde{a}_k)}{f_0} \right) \\
\frac{(f, \tilde{a}_1)}{f_0} & \ddots & \cdots \\
\mathcal{D}^{a_{ik}} \left( \frac{c_{11}(f, \tilde{a}_1)}{f_0} \right) & \cdots & \mathcal{D}^{a_{ik}} \left( \frac{c_{ik}(f, \tilde{a}_k)}{f_0} \right)
\end{bmatrix}
\]

(The determinant is counted after deleting the i-th column in the above matrix).

Each element of the above matrix has a form
\[
\mathcal{D}^a \left( \frac{c(f, \tilde{a}_j)}{f_0} \right) = \mathcal{D}^a \left( \frac{c(f, \tilde{a}_j)}{f_0} \right) \cdot c \ (c \in \mathbb{R}).
\]

By lemma on logarithmic derivative lemma, we have
\[
m \left( r, \frac{\mathcal{D}^{a_{i1}} \left( \frac{c(f, \tilde{a}_j)}{f_0} \right)}{f_0} \right) \leq m \left( r, \frac{\mathcal{D}^{a} \left( \frac{c(f, \tilde{a}_j)}{f_0} \right)}{f_0} \right) + m(r, c)
\]

\[
= O \left( \log^+ T \left( r, \frac{c(f, \tilde{a}_j)}{f_0} \right) \right) + O(\max_{1 \leq i \leq q} T(r, a_i))
\]

\[
= O(\log^+ T_f(r)) + O(\max_{1 \leq i \leq q} T(r, a_i)).
\]

This yields that
\[
m \left( r, \frac{\det D_i}{\prod_{j=1, j \neq i}^q (f, \tilde{a}_j)} \right) = O(\log^+ T_f(r)) + O(\max_{1 \leq j \leq q} T_{a_j}(r)) \quad (1 \leq i \leq q).
\]

Hence
\[
\sum_{i=1}^q m \left( r, \frac{\det D_i}{\prod_{j=1, j \neq i}^q (f, \tilde{a}_j)} \right) = O(\log^+ T_f(r)) + O(\max_{1 \leq j \leq q} T_{a_j}(r)).
\]
Integrating both sides of the inequality (3.3), we have

\[
\left\| \int_{S(r)} \log \left| \frac{\det D_0}{\prod_{i=1}^q |(f, \tilde{a}_i)|} \right| \eta + \int_{S(r)} \log \left( \frac{|\det D_0|}{\prod_{i=1}^q |(f, \tilde{a}_i)|} \right) \eta \right\| 
\leq \sum_{i=1}^q \int_{S(r)} \log^+ \left( \frac{|\det D_i|}{\prod_{j=1,j\neq i}^q |(f, \tilde{a}_j)|} \right) \eta + \int_{S(r)} \log^+ \Psi(z) \eta 
= \sum_{i=1}^q m \left( r, \frac{|\det D_i|}{\prod_{j=1,j\neq i}^q |(f, \tilde{a}_j)|} \right) + O(\max_{1 \leq i \leq q} T_{a_i}(r))
= O(\log^+ T_f(r)) + O(\max_{0 \leq i \leq q} T_{a_i}(r)).
\]

Hence

\[
\left\| T_f(r) + \int_{S(r)} \log \left( \frac{|\det D_1|}{\prod_{i=1}^q |(f, \tilde{a}_i)|} \right) \eta \right\| = O(\log^+ T_f(r)) + O(\max_{1 \leq i \leq q} T_{a_i}(r)), \text{ i.e.,}
\]

\[
\left\| T_f(r) \right\| = \int_{S(r)} \log \left( \frac{\prod_{i=1}^q |(f, \tilde{a}_i)|}{|\det D_1|} \right) \eta + O(\log^+ T_f(r)) + O(\max_{1 \leq i \leq q} T_{a_i}(r))
= \int_{S(r)} \log \prod_{i=1}^q |(f, \tilde{a}_i)| \eta - \int_{S(r)} \log |\det D_1| \eta + O(\log^+ T_f(r)) + O(\max_{1 \leq i \leq q} T_{a_i}(r))
\leq N_{\prod_{i=1}^q (f, \tilde{a}_i)}(r) - N(r, \nu_{\det D_1}) + O(\log^+ T_f(r)) + O(\max_{1 \leq i \leq q} T_{a_i}(r)).
\]

Claim 3.5. \( || \prod_{i=1}^q (f, \tilde{a}_i) || N(r, \nu_{\det D_1}) \leq \sum_{i=1}^q N_{(f, \tilde{a}_i)}^\ell(r) + O(\max_{1 \leq i \leq q} T_{a_i}(r)). \)

Indeed, fix \( z \in C^m \setminus I(f) \), where \( I(f) = \{ f_0 = \cdots = f_n = 0 \} \). We call \( i_0 \) the index satisfying

\[
\nu_{(f, \tilde{a}_{i_0})}^0(z) = \min_{1 \leq i \leq n+1} \nu_{(f, \tilde{a}_i)}^0(z).
\]

For each \( i \neq i_0, i \in I_s \), we have

\[
\nu_{D^{\alpha_{sk_s+1}}(c_s(f, \tilde{a}_i))}^0(z) \geq \min_{\alpha_{sk_s+1}, \beta \in \mathbb{Z}^n} \left\{ \nu_{D^{\alpha_{cs}}D^{\alpha_{sk_s+1}}(f, \tilde{a}_i)}(z) \right\}
\geq \min_{\alpha_{sk_s+1}, \beta \in \mathbb{Z}^n} \left\{ \max \{ 0, \nu_{(f, \tilde{a}_i)}^0(z) - |\alpha_{sk_s+1} - \beta| - (\beta + 1) \nu_{c_s}^\infty(z) \} \right\}
\geq \max \{ 0, \nu_{(f, \tilde{a}_i)}^0(z) - k \} - (k + 1) \nu_{c_s}^\infty(z)
\]

On the other hand, we also have

\[

\nu_{D^{\alpha_{sk_s+1}}(c_s(f, \tilde{a}_i))}^\infty(z) \leq (|\alpha_{sk_s+1}| + 1) \nu_{c_s}^\infty(z) \leq (k + 1) \nu_{c_s}^\infty(z) + \nu_{a_{i_0}}^0(z).
\]

Thus

\[
\nu_{D^{\alpha_{sk_s+1}}(c_s(f, \tilde{a}_i))}(z) \geq \max \{ 0, \nu_{(f, \tilde{a}_i)}^0(z) - k \} - (k + 1) (2 \nu_{c_s}^\infty(z) + \nu_{a_{i_0}}^0(z))
\]
Since each element of the matrix $D_{i_0}$ has a form $\mathcal{D}^{\alpha_{b_k s_i-1}^j} (c_{s_i} (f, \tilde{a}_i)) (i \neq i_0)$, one estimates

\[
\nu_{D_i} (z) = \nu_{D_{i_0}} (z) \geq \sum_{i \neq i_0} \left( \max \{0, \nu_{0}^0 (f, \tilde{a}_i) (z) - k \} - (k + 1) \left( 2 \nu_{\infty}^\infty (z) + \nu_{a,0}^0 (z) \right) \right).
\]

We see that there exists $v_0 \in \{0, \ldots, n\}$ with $f_{v_0} (z) \neq 0$. Then by (3.2), there exists $i_1 \in \{1, \ldots, n + 1\}$ such that $A_{v_0 i_1} (z) \cdot (f, \tilde{a}_i) (z) \neq 0$. Thus

\[
\nu_{0}^0 (f, \tilde{a}_{i_1}) (z) \leq \nu_{0}^0 (f, \tilde{a}_i) (z) \leq \nu_{\infty}^\infty (z) \leq \sum_{A_{v_0} \neq 0} \nu_{A_{v_0}}^\infty (z).
\]

Combining the inequalities (3.6) and (3.7), we have

\[
\nu_{0}^0 \prod_{i=1}^q (f, \tilde{a}_i) (z) - \nu_{\det D_i} (z)
\leq \sum_{i \neq i_0} \left( \min \{ \nu_{0}^0 (f, \tilde{a}_i) (z), k \} + (k + 1) \left( 2 \nu_{\infty}^\infty (z) + \nu_{a,0}^0 (z) \right) \right) + \sum_{A_{v_0} \neq 0} \nu_{A_{v_0}}^\infty (z)
\leq \sum_{i=1}^q \left( \min \{ \nu_{0}^0 (f, \tilde{a}_i) (z), k \} + (k + 1) \left( 2 \nu_{\infty}^\infty (z) + \nu_{a,0}^0 (z) \right) \right) + \sum_{A_{v_0} \neq 0} \nu_{A_{v_0}}^\infty (z),
\]

where the index $s$ of $c_{s_i}$ is taken so that $i \in I_s$. Integrating both sides of this inequality, we obtain

\[
|| N_{\prod_{i=1}^q (f, \tilde{a}_i)} (r) - N(r, \nu_{\det D_i}) ||
\leq \sum_{i=1}^q \left( \nu_{(f, \tilde{a}_i)}^r (r) + (k + 1) \left( 2 \nu_{f,0}^\infty (r) + \nu_{a,0}^0 (r) \right) \right) + \sum_{A_{v_0} \neq 0} \nu_{A_{v_0}}^\infty (r)
= \sum_{i=1}^q \nu_{(f, \tilde{a}_i)}^r (r) + O(\max_{1 \leq i \leq q} T_{a_i} (r)).
\]

The claim is proved.

From the inequalities (3.4) and the claim, we get

\[
|| T_f (r) \leq \sum_{i=1}^q \nu_{(f, \tilde{a}_i)}^r (r) + O(\log^+ T_f (r)) + O(\max_{1 \leq i \leq q} T_{a_i} (r)).
\]

The lemma is proved. \qed

**Proof of Theorem 1.1**

(a) We denote by $\mathcal{I}$ the set of all permutations of $q$–tuple $(1, \ldots, q)$. For each element $I = (i_1, \ldots, i_q) \in \mathcal{I}$, we set

\[
N_I = \{ r \in \mathbb{R}^+; \nu_{(f, \tilde{a}_i)}^r (r) \leq \cdots \leq \nu_{(f, \tilde{a}_i)}^r (r) \}.
\]

We now consider an element $I = (i_1, \ldots, i_q)$ of $\mathcal{I}$. We will construct subsets $I_t$ of the set $A_1 = \{ 1, \ldots, 2n - k + 2 \}$ as follows.

We choose a subset $I_t$ of $A$ which is the minimal subset of $A$ satisfying that $\{(f, \tilde{a}_i)\}_{j \in I_t}$ is minimal over $\mathcal{R}$. If $\mathcal{I}_t \geq n + 1$ then we stop the process.

Otherwise, set $A_2 = A_1 \setminus I_1$. We consider the following two cases:
Therefore, from the above two cases, we see that there exist nonzero meromorphic functions $c_i \in \mathcal{R} (i \in I_2)$, such that for any $2 \leq t \leq l$, $j \in I_t$, there exist meromorphic functions $c_j \in \mathcal{R} \setminus \{0\}$ such that

\[ \sum_{j \in I_t} c_j(f, 1) \in \left( \bigcup_{i \in I_t \cup I_2} (f, 1) \right) \]

Continuing this process, we get the subsets $I_1, \ldots, I_l$, which satisfy:

- $\sharp(\bigcup_{i=1}^{l} I_i) \leq k+1$ and $\sharp(I_1 \cup \cdots \cup I_3) \leq \min\{2n-k+2, n+k+1\}$.
Then the family of subsets $I_1, \ldots, I_t$ satisfies the assumptions of the Lemma 3.1. Therefore, we have

$$
|| T_f(r) \leq \sum_{j \in J} N_{(f,a_j)}^{[k]}(r) + o(T_f(r)) + O(\max_{1 \leq i \leq q} T_a(r)),
$$

where $J = I_1 \cup \cdots \cup I_t$. Then for all $r \in N_I$ (may be outside a finite Borel measure subset of $\mathbb{R}^+$) we have

$$
|| T_f(r) \leq \frac{\#J}{q - (2n - k + 2)} + \frac{\#J}{q - 2n + k + 2} \left( \sum_{j \in J} N_{(f,a_j)}^{[k]}(r) + \sum_{j = 2n - k + 3}^{q} N_{(f,a_j)}^{[k]}(r) \right)
+ o(T_f(r)) + O(\max_{1 \leq i \leq q} T_a(r)).
$$

(3.9)

Since $\#J \leq 2n - k + 2$, the above inequality implies that

$$
(3.10) \quad || T_f(r) \leq \frac{2n - k + 2}{q} \sum_{i=1}^{q} N_{(f,a_i)}^{[k]}(r) + o(T_f(r)) + O(\max_{1 \leq i \leq q} T_a(r)), \quad r \in N_I.
$$

We see that $\bigcup_{I \in \mathcal{I}} N_I = \mathbb{R}^+$ and the inequality (3.10) holds for every $r \in N_I, I \in \mathcal{I}$. This yields that

$$
T_f(r) \leq \frac{2n - k + 2}{q} \sum_{i=1}^{q} N_{(f,a_i)}^{[k]}(r) + o(T_f(r)) + O(\max_{1 \leq i \leq q} T_a(r))
$$

for all $r$ outside a finite Borel measure subset of $\mathbb{R}^+$. Thus

$$
|| \frac{q}{2n - k + 2} T_f(r) \leq \sum_{i=1}^{q} N_{(f,a_i)}^{[k]}(r) + o(T_f(r)) + O(\max_{1 \leq i \leq q} T_a(r)).
$$

The assertion (a) is proved.

(b) We repeat the same argument as in the proof of the assertion (a). If $n + k + 1 > 2n - k + 1$ then the assertion (b) is a consequence of the assertion (a). Then we now only consider the case where $n + k + 1 \leq 2n - k + 1$.

From (3.9) with a note that $\#J \leq n + k + 2$, we have

$$
|| T_f(r) \leq \frac{n + k + 1}{q - (2n - k + 2) + n + k + 1} \sum_{i=1}^{q} N_{(f,a_i)}^{[k]}(r) + o(T_f(r)) + O(\max_{1 \leq i \leq q} T_a(r))
= \frac{n + k + 1}{q - n + 2k - 1} \sum_{i=1}^{q} N_{(f,a_i)}^{[k]}(r) + o(T_f(r)) + O(\max_{1 \leq i \leq q} T_a(r)) \quad r \in N_I.
$$

Repeating again the argument in the proof of assertion (a), we see that the above inequality holds for all $r \in \mathbb{R}^+$ outside a finite Borel measure set. Then the assertion (b) is proved.

\[
\square
\]

4. Proof of Theorem 1.3

In order to prove Theorem 1.3 we need the following.
4.1. Let \( f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C}) \) be a meromorphic mapping with a reduced representation \( f = (f_0 : \ldots : f_n) \). Let \( \{a_i\}_{i=1}^q \) be “slowly” (with respect to \( f \)) moving hyperplanes of \( \mathbb{P}^n(\mathbb{C}) \) in general position such that
\[
\dim \{ z \in \mathbb{C}^m : (f, a_i)(z) = (f, a_j)(z) = 0 \} \leq m - 2 \quad (1 \leq i < j \leq q).
\]

For \( M + 1 \) elements \( f^0, \ldots, f^M \in \mathcal{F}(f, \{ a_j \}_{j=1}^q, 1) \), we put
\[
T(r) = \sum_{k=0}^M T(r, f^k).
\]

Assume that \( a_i \) has a reduced representation \( a_i = (a_{i0} : \cdots : a_{in}) \). By changing the homogeneous coordinate system of \( \mathbb{P}^n(\mathbb{C}) \), we may assume that \( a_{i0} \neq 0 \) (1 \( \leq i \leq q \)).

We set \( F_i^{jk} := \frac{(f^k, a_j)}{(f^k, a_i)} \) (1 \( \leq i, j \leq q \), 0 \( \leq k \leq M \)).

**Lemma 4.1.** Suppose that \( q \geq 2n + 1 \). Then
\[
\| T_g(r) = O(T_f(r)) \text{ for each } g \in \mathcal{F}(f, \{ a_i \}_{i=1}^q, 1).
\]

**Proof.** By Corollary I.2(a), we have
\[
\| \frac{2q - n + 1}{3(n + 1)} T_g(r) \leq \sum_{i=1}^q N_{(g, a_i)}^{[n]}(r) + o(T_g(r) + T_f(r))
\]
\[
\leq n \sum_{i=1}^q N_{(g, a_i)}^{[1]}(r) + o(T_g(r) + T_f(r))
\]
\[
= \sum_{i=1}^q n N_{(f, a_i)}^{[1]}(r) + o(T_g(r) + T_f(r))
\]
\[
\leq q n T_f(r) + o(T_g(r) + T_f(r)).
\]

Hence \( \| T_g(r) = O(T_f(r)) \). \( \square \)

**Definition 4.2** (see [2, p. 138]). Let \( F_0, \ldots, F_M \) be nonzero meromorphic functions on \( \mathbb{C}^m \), where \( M \geq 1 \). Take a set \( \alpha := (\alpha^0, \ldots, \alpha^{M-1}) \) whose components \( \alpha^k \) are composed of \( m \) nonnegative integers, and set \( |\alpha| = |\alpha^0| + \ldots + |\alpha^{M-1}| \). We define Cartan’s auxiliary function by
\[
\Phi^\alpha \equiv \Phi^\alpha(F_0, \ldots, F_M) := F_0 F_1 \cdots F_M \left| \begin{array}{cccc}
1 & 1 & \cdots & 1 \\
\mathcal{D}^{\alpha^0}(\frac{1}{F_0}) & \mathcal{D}^{\alpha^0}(\frac{1}{F_1}) & \cdots & \mathcal{D}^{\alpha^0}(\frac{1}{F_M}) \\
\vdots & \vdots & \ddots & \vdots \\
\mathcal{D}^{\alpha^{M-1}}(\frac{1}{F_0}) & \mathcal{D}^{\alpha^{M-1}}(\frac{1}{F_1}) & \cdots & \mathcal{D}^{\alpha^{M-1}}(\frac{1}{F_M})
\end{array} \right|
\]

**Lemma 4.3** (see [2 Proposition 3.4]). If \( \Phi^\alpha(F, G, H) = 0 \) and \( \Phi^\alpha(\frac{1}{F}, \frac{1}{G}, \frac{1}{H}) = 0 \) for all \( \alpha \) with \( |\alpha| \leq 1 \), then one of the following assertions holds:

(i) \( F = G, G = H \) or \( H = F \)

(ii) \( \frac{F}{G}, \frac{G}{H}, \text{ and } \frac{H}{F} \) are all constant.
**Lemma 4.4** (see [3, Lemma 4.7]). Suppose that there exists $\Phi^\alpha = \Phi^\alpha(F_{i_0}^{j_0}, \ldots, F_{i_0}^{j_0^M}) \neq 0$ with $1 \leq i_0, j_0 \leq q$, $|\alpha| \leq \frac{M(M - 1)}{2}$, $d \geq |\alpha|$. Assume that $\alpha$ is a minimal element such that $\Phi^\alpha(F_{i_0}^{j_0}, \ldots, F_{i_0}^{j_0^M}) \neq 0$. Then, for each $0 \leq k \leq M$, the following holds:

$$\|N^{[d-|\alpha|]}_{(f_k,a_{j_0})}(r) + M \sum_{j \neq j_0,a_{j_0}} N^{[1]}_{(f_k,a_j)}(r) \leq N_{\Phi^\alpha}(r) \leq T(r) - M \cdot N^{[1]}_{(f_k,a_{j_0})}(r) + o(T(r)).$$

And hence

$$\|N^{[d-|\alpha|]}_{(f_k,a_{j_0})}(r) + M \sum_{j \neq j_0,a_{j_0}} N^{[1]}_{(f_k,a_j)}(r) \leq T(r) + o(T(r)).$$

### 4.2. Proof of Theorem 4.3

a) Assume that $q > \frac{9n^2 + 9n + 2}{2}$. Suppose that there exist three distinct elements $f^0, f^1, f^2 \in \mathcal{F}(f, \{a_j\}_{j=1}^q, 1)$.

Suppose that there exist two indices $i, j \in \{1, \ldots, q\}$ and $\alpha = (\alpha_0, \alpha_1) \in (\mathbb{Z}^*_n)^2$ with $|\alpha| \leq 1$ such that $\Phi^\alpha(F_{i}^{j_0}, F_{j_1}^{i_1}, F_{j_2}^{i_2}) \neq 0$. By Lemma 4.4 we have

$$2 \sum_{t \neq i} N^{[1]}_{(f_{j_0}^{i_0}, a_t)}(r) \leq T(r) + o(T_f(r)).$$

Hence, by Corollary 4.2(b) we have

$$\| T(r) \geq \frac{2}{3} \sum_{k=1}^{3} \sum_{t \neq i} N^{[1]}_{(f_k^{i_1}, a_t)}(r) + o(T_f(r)) \geq \frac{2}{3n} \sum_{k=1}^{3} \sum_{t \neq i} N^{[n]}_{(f_k^{i_1}, a_t)}(r) + o(T_f(r))$$

$$\geq \frac{4(q-1)}{9n(n+1)}T(r) + o(T_f(r)).$$

Letting $r \to +\infty$, we get $1 \geq \frac{4(q-1)}{9n(n+1)}$, i.e., $q \leq \frac{9n^2 + 9n + 4}{4}$. This is a contradiction.

Then for two indices $i, j$ ($1 \leq i < j \leq q$), we have

$$\Phi^\alpha(F_{i}^{j_0}, F_{j_1}^{i_1}, F_{j_2}^{i_2}) \equiv 0 \text{ and } \Phi^\alpha(F_{i}^{j_0}, F_{j_1}^{i_1}, F_{j_2}^{i_2}) \equiv 0$$

for all $\alpha = (\alpha_0, \alpha_1)$ with $|\alpha| \leq 1$. By Lemma 4.3 there exists a constant $\lambda$ such that

$$F_{i}^{j_0} = \lambda F_{i}^{i_1}, F_{j_1}^{i_1} = \lambda F_{j_1}^{i_2}, \text{ or } F_{j_2}^{i_2} = \lambda F_{j_2}^{j_0}.$$ 

For instance, we assume that $F_{i}^{j_0} = \lambda F_{i}^{i_1}$. We will show that $\lambda = 1$.

Indeed, assume that $\lambda \neq 1$. Since $F_{i}^{j_0} = F_{j_1}^{i_1}$ on the set $\bigcup_{k \neq j} \{z : (f, a_k)(z) = 0\}$, we have that $F_{i}^{j_0} = F_{j_1}^{i_1} = 0$ on the set $\bigcup_{k \neq j} \{z : (f, a_k)(z) = 0\}$. Hence $\bigcup_{k \neq j} \{z : (f, a_k)(z) = 0\} \subset \{z : (f, a_i)(z) = 0\}$. It follows that $\{z : (f, a_k)(z) = 0\} = \emptyset$ ($k \neq i, j$). We obtain that

$$\| \frac{2(q-2)}{3(n+1)}T_f(r) \leq \sum_{k \neq i, k \neq j} N^{[n]}_{(f_k,a_{i_k})}(r) + o(T_f(r)) = o(T_f(r)).$$

This is a contradiction. Thus $\lambda = 1$ ($1 \leq i < j \leq q$).

Define

$$I_1 = \{i \in \{1, \ldots, q-1\} : F_{q}^{i_0} = F_{q}^{i_1}\}, $$

$$I_2 = \{i \in \{1, \ldots, q-1\} : F_{q}^{i_1} = F_{q}^{i_2}\}, $$
$I_3 = \{i \in \{1, \ldots, q - 1\} : F^i_q = F^0_q\}$.

Since $\sharp(I_1 \cup I_2 \cup I_3) = \sharp\{1, \ldots, q - 1\} = q - 1 \geq 3n - 2$, there exists $1 \leq k \leq 3$ such that $\sharp I_k \geq n$. Without loss of generality, we may assume that $\sharp I_1 \geq n$. This implies that $f^0 = f^1$. This is a contradiction.

Thus, we have $\sharp \mathcal{F}(f, \{a_i\}_{i=1}^q) \leq 2$.

b) Assume that $q > 3n^2 + n + 2$.

Take $g \in \mathcal{F}(f, \{a_i\}_{i=1}^q)$. Suppose that $f \neq g$. By changing indices if necessary, we may assume that

$$
\begin{align*}
(f, a_1) & \equiv (f, a_2) \equiv \cdots \equiv (f, a_k) \neq (f, a_{k+1}) \equiv \cdots \equiv (f, a_{k+q}) \\
(g, a_1) & \equiv (g, a_2) \equiv \cdots \equiv (g, a_k) \\
\neq (g, a_{k+1}) & \equiv \cdots \equiv (g, a_{k+q}),
\end{align*}
$$

where $k_s = q$.

For each $1 \leq i \leq q$, we set

$$
\sigma(i) = \begin{cases} 
  i + n & \text{if } i + n \leq q, \\
  i + n - q & \text{if } i + n > q
\end{cases}
$$

and

$$
P_i = (f, a_i)(g, a_{\sigma(i)}) - (g, a_i)(f, a_{\sigma(i)}).
$$

By supposition that $f \neq g$, the number of elements of each group is at most $n$. Hence $(f, a_i)$ and $(f, a_{\sigma(i)})$ belong to distinct groups. This means that $P_i \neq 0$ ($1 \leq i \leq q$).

Fix an index $i$ with $1 \leq i \leq q$. It is easy to see that

$$
\nu_{P_i}(z) \geq \min\{\nu_{(f, a_i)}, \nu_{(g, a_i)}\} + \min\{\nu_{(f, a_{\sigma(i)})}, \nu_{(g, a_{\sigma(i)})}\} + \sum_{\nu \neq i, \sigma(i)}^{q} \nu_{(f, a_{\nu})}^{[1]}(z)
$$

outside a finite union of analytic sets of dimension $\leq m - 2$. Since $\min\{a, b\} + n \geq \min\{a, n\} + \min\{b, n\}$ for all positive integers $a$ and $b$, the above inequality implies that

$$
N_{P_i}(r) \geq \sum_{\nu = i, \sigma(i)}^{q} \left( N_{(f, a_{\nu})}^{[n]}(r) + N_{(g, a_{\nu})}^{[n]}(r) - nN_{(f, a_{\nu})}^{[1]}(r) \right) + \sum_{\nu = 1}^{q} N_{(f, a_{\nu})}^{[1]}(r).
$$

On the other hand, by the Jensen formula, we have

$$
N_{P_i}(r) = \int_{S(r)} \log |P_i| \eta + O(1)
$$

$$
\leq \int_{S(r)} \log \left( |(f, a_i)|^2 + |(f, a_{\sigma(i)})|^2 \right) \frac{1}{2} \eta + \int_{S(r)} \log \left( |(g, a_i)|^2 + |(g, a_{\sigma(i)})|^2 \right) \frac{1}{2} \eta + O(1)
$$

$$
\leq T_f(r) + T_g(r) + o(T_f(r)).
$$
This implies that
\[ T_f(r) + T_g(r) \geq \sum_{v=i, \sigma(i)} \left( N^{[n]}_{(f,a_v)}(r) + N^{[n]}_{(g,a_v)}(r) - nN^{[1]}_{(f,a_v)}(r) \right) \]
\[ + \sum_{v=1}^{q} N^{[1]}_{(f,a_v)}(r) + o(T_f(r)). \]

Summing-up both sides of the above inequality over \( i = 1, \ldots, q \) and by Corollary 1.2(b), we have
\[ q(T_f(r) + T_g(r)) \geq 2 \sum_{v=i}^{q} \left( N^{[n]}_{(f,a_v)}(r) + N^{[n]}_{(g,a_v)}(r) \right) \]
\[ + (q - 2n - 2) \sum_{v=1}^{q} N^{[1]}_{(f,a_v)}(r) + o(T_f(r)) \]
\[ \geq (2 + \frac{q - 2n - 2}{2n}) \sum_{v=1}^{q} \left( N^{[n]}_{(f,a_v)}(r) + N^{[n]}_{(g,a_v)}(r) \right) + o(T_f(r)) \]
\[ \geq (2 + \frac{q - 2n + 2}{2n}) \frac{2q}{3(n + 1)} (T_f(r) + T_g(r)) + o(T_f(r)). \]

Letting \( r \to \infty \), we get \( q \geq (2 + \frac{2n-2}{2n}) \frac{2q}{3(n + 1)} \Leftrightarrow q \leq 3n^2 + n + 2 \). This is a contradiction.

Then \( f = g \). This implies that \( \sharp F(f, \{a_i\}_{i=1}^q, 1) = 1 \). The theorem is proved. \( \square \)

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