W* Dynamics of Infinite Dissipative Quantum Systems

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Abstract

We formulate the dynamics of an infinitely extended open dissipative quantum system, \( \Sigma \), in the Schroedinger picture. The generic model on which this is based comprises a \( \mathcal{C}^\star \)-algebra, \( \mathcal{A} \), of observables, a folium, \( \mathcal{F} \), of states on this algebra and a one-parameter semigroup, \( \tau \), of linear transformations of \( \mathcal{F} \) that represents its dynamics and is given by a natural infinite volume limit of the corresponding semigroup for a finite system. On this basis, we establish that the dynamics of \( \Sigma \) is given by a one parameter group of completely positive linear transformations of the \( W^\star \)-algebra dual to \( \mathcal{F} \). This result serves to extend our earlier formulation [1] of infinitely extended conservative systems to open dissipative ones.

1. Introduction.

The dynamics of a finite dissipative quantum system has been formulated by Lindblad [2] and Gorini, Kossakowski and Sudarshan [3] as a one parameter semigroup of completely positive (CP) linear transformations of its observables. The aim of this article is to provide a corresponding formulation of infinitely extended dissipative quantum systems, which may provide a natural basis for a treatment of nonequilibrium statistical thermodynamics. We remark that, for the case where these reduce to conservative systems, such a treatment has been made [1] for the evolution of a folium, \( \mathcal{F}_{\text{con}} \) of states, which was shown to be governed by the action of a one parameter group of \( \star \)-automorphisms of the \( W^\star \)-algebra dual to \( \mathcal{F}_{\text{con}} \). The essential result of the present article is that, in the general dissipative case, the quantum evolution of a folium, \( \mathcal{F} \), of states is given instead by a one parameter semigroup of completely positive identity preserving linear transformations of the \( W^\star \)-algebra dual to \( \mathcal{F} \): in general these are quite different, due to their dissipative character, from the \( \star \)-automorphisms of the conservative case.

We base our treatment of \( \Sigma \) on a generic operator algebraic model of an infinitely extended open, dissipative quantum system, as represented by a triple \((\mathcal{A}, \mathcal{F}, \tau)\) where \( \mathcal{A} \) is a \( \mathcal{C}^\star \)-algebra of observables, \( \mathcal{F} \) is a folium** of its states and \( \tau \) is a one parameter semigroup.
semigroup of transformations of $\mathcal{F}$ as given by an infinite volume limit of the dynamical semigroup of a finite version of $\Sigma$. The system is thus an infinite volume counterpart of the finite model formulated in [2] and [3], and is designed to be applicable to statistical mechanics and quantum field theory.

We provide further specifications of the above model in subsequent Sections. Thus, in Section 2 we pass to a formulation of the algebra $\mathcal{A}$ in terms of a standard quasi-local structure, and in Section 3 we provide corresponding specifications of the folium $\mathcal{F}$ and the dynamical semigroup $\tau$. This leads to our main result, namely the Proposition of Section 3, which establishes and identifies the $W^*$-dynamics of the model.

2. The Algebraic Structure.

We assume that $\Sigma$ occupies a space $X$, which may be either $\mathbb{R}^d$ or $\mathbb{Z}^d$, with $d$ finite. We denote by $L$ the set of bounded open subregions of $X$ and to each $\Lambda \in L$ we assign a $W^*$-algebra, $\mathcal{A}(\Lambda)$, whose self-adjoint elements represent the bounded observables localised in that region. We assume that the algebras $\{\mathcal{A}(\Lambda)|\Lambda \in L\}$ are type I factors that satisfy the standard requirements of isotony and local commutativity. We define $\mathcal{A}_L$, the algebra of local observables of $\Sigma$, to be $\bigcup_{\Lambda \in L} \mathcal{A}(\Lambda)$; and we define $\mathcal{A}$, the norm completion of $\mathcal{A}_L$, to be the $C^*$-algebra of quasi-local bounded observables of the system. We assume that each of the local algebras $\mathcal{A}(\Lambda)$ is equipped with a one parameter semigroup $\{\gamma_t(\Lambda)|t \in \mathbb{R}_+\}$ of completely positive (CP) [2, 5] identity preserving transformations, which represent the dynamics of the finite version, $\Sigma(\Lambda)$, of $\Sigma$ confined to the region $\Lambda$.

3. The Folium $\mathcal{F}$ and the dynamical semigroup $\tau$.

It follows from these specifications [4] that the linear span, $[\mathcal{F}]$, of $\mathcal{F}$ is the predual of the bicommutant of a certain representation*, $\pi$, of $\mathcal{A}$, i.e. that $\mathcal{F}$ is the set of normal states on $\pi(\mathcal{A})''$. We assume that the dynamics of $\Sigma$, in the Schroedinger representation, is given by a one parameter semigroup, $\tau$, of affine transformations of $\mathcal{F}$. Hence, by duality, this semigroup induces a corresponding one, $\tau^*$, of affine transformations of $\pi(\mathcal{A})''$, as defined by the formula

$$\langle f; \tau_t^*B \rangle = \langle \gamma_t f; B \rangle \quad \forall f \in [\mathcal{F}], \quad B \in \pi(\mathcal{A})'', \quad t \in \mathbb{R}_+,$$

where $[\mathcal{F}]$ is the linear span of $\mathcal{F}$. We assume that $\tau^*$ is just that canonically induced by the local semigroup $\{\gamma_t(\Lambda)|t \in \mathbb{R}_+\}$ in the limit $\Lambda \uparrow X$, i.e.

$$\tau_t^* \pi(A) = s : \lim_{\Lambda \uparrow X} \pi(\gamma_t(\Lambda)A) \quad \forall A \in \mathcal{A}_L, \quad t \in \mathbb{R}_+. \quad (3.2)$$

Equivalently, defining $\tilde{\gamma}_t(\Lambda)$ to be the transformation of $\pi(\mathcal{A}(\Lambda))$ given by the formula

$$\tilde{\gamma}_t(\Lambda) \pi(A) := \pi(\gamma_t(\Lambda)A) \quad \forall A \in \mathcal{A}_L, \quad t \in \mathbb{R}_+, \quad (3.3)$$

as $B$ runs through $\mathcal{A}$.

* Specifically, $\pi$ is any element of the quasi-equivalence class of the direct sum of the GNS representations of the states comprising $\mathcal{F}$.
the condition (3.2) may be expressed in the form
\[ \tau^* \pi(A) = s : \lim_{\Lambda \uparrow X} \tilde{\gamma}_t(\Lambda) \pi(A) \forall A \in \mathcal{A}_L, \ t \in \mathbb{R}_+. \] \tag{3.4}

Suppose now that \( \mathcal{M} \) is a finite dimensional matrix algebra. Then since any element, \( C, \) of \( \pi(\mathcal{A}_L) \otimes \mathcal{M} \) may be expressed in the form \( \sum_r \pi(A_r) \otimes M_r, \) where the \( M_r \)'s form an operator valued basis in \( \mathcal{M} \) and the \( A_r \)'s are elements of \( \pi(\mathcal{A}_L), \) it follows that the condition (3.4) implies that
\[ [\tau^* \otimes I] C = s : \lim_{\Lambda \uparrow X} [\tilde{\gamma}_t(\Lambda) \otimes I] C \forall C \in \pi(\mathcal{A}_L) \otimes \mathcal{M}, \ t \in \mathbb{R}_+. \] \tag{3.5}

**Proposition.** Under the above specifications of the model., the action of the dynamical semigroup \( \tau^* \) on the algebra \( \pi(\mathcal{A})'' \) is completely positive and identity preserving.

**Lemma.** Given \((t, \Lambda) (\in \mathbb{R}_+ \times \mathbb{L})\) the transformation \( \tilde{\gamma}_t(\Lambda) \) of \( \pi(\mathcal{A})^{\Lambda} \) is CP and identity preserving.

**Proof of Lemma.** Since \( \mathcal{A}(\Lambda) \) is a primary \( W^*- \) algebra, it follows from Krauss’s formula [6] that the action of \( \gamma_t(\Lambda) \) on this algebra may be expressed in the form
\[ \gamma_t(\Lambda) C = \sum_{n \in \mathbb{N}} W_n^* CW_n \forall C \in \mathcal{A}(\Lambda) \] \tag{3.6},
where \( \{W_n\} \) is a sequence of elements of \( \mathcal{A}(\Lambda) \) such that
\[ \sum_{n \in \mathbb{N}} W_n^* W_n = I \] \tag{3.7}
and \( \sum_{n \in \mathbb{N}} \) is taken to be the strong limit in the case where the number of terms is infinite. Hence, by the normality of \( \pi \) and Equs. (3.3) and (3.6),
\[ \tilde{\gamma}_t(\Lambda) \pi(C) = \sum_{n \in \mathbb{N}} \tilde{W}_n^* \pi(C) \tilde{W}_n \] \tag{3.8}
where
\[ \tilde{W}_n := \pi(W_n) \] \tag{3.9}
and
\[ \sum_{n \in \mathbb{N}} \tilde{W}_n^* \tilde{W}_n = I. \] \tag{3.10}
Now let \( \mathcal{M} \) be a finite dimensional matrix algebra. Then any element \( \tilde{C} \) of \( \pi(\mathcal{A}(\Lambda)) \otimes \mathcal{M} \) may be expressed as a finite sum
\[ \tilde{C} = \sum_{r \in J} \pi(C_r) \otimes M_r \] \tag{3.11},
where \( J \) is a finite index set, the \( C_r \)'s are elements of \( \mathcal{A}(\Lambda) \) and the \( M_r \)'s form an operator basis for \( \mathcal{M} \). Hence, by Equs. (3.8)-(3.11),
\[ [\tilde{\gamma}_t(\Lambda) \otimes I](\tilde{C}^* \tilde{C}) = [\tilde{\gamma}_t(\Lambda \otimes I)] \sum_{r,s \in J} \pi(C_r^* C_s) \otimes M_r^* M_s = \]
\[
\sum_{r,s \in J, n \in \mathbb{N}} \tilde{W}^* n \pi(C^*_r C_s) \tilde{W} n \otimes M^*_r M_s = \sum_{n \in \mathbb{N}} D^*_n D_n
\]  
(3.12),

where
\[
D_n = \sum_{r \in J} \pi(C_r) \tilde{W} n \otimes M_r.
\]
(3.13)

Hence \( \tilde{\gamma}_t(\Lambda) \otimes I \) is positive and therefore \( \tilde{\gamma}_t(\Lambda) \) is CP. Further, by Eqs. (3.8) and (3.10), it is identity preserving.

**Proof of Proposition.** It follows immediately from the Lemma, together with the definition (3.3) of \( \tilde{\gamma}_t(\Lambda) \) and the complete positivity of \( \gamma_t(\Lambda) \), that the transformation \( \tilde{\gamma}_t(\Lambda) \) is both CP and identity preserving.

To prove that the same is true for \( \tau_t^* \), we first infer from Eqs. (3.3) and (3.4) that its identity preserving property follows from that of \( \gamma_t(\Lambda) \).

Next we note that, by the finite dimensionality of \( \mathcal{M} \), elements \( \tilde{B} \) of \( \pi(\mathcal{A}_L) \otimes \mathcal{M} \) take the form
\[
\tilde{B} = \sum_J \pi(B_J) \otimes M_J
\]
where \( J \) is a finite index set. Hence
\[
[\tau^*_t \otimes I] \tilde{B} = \sum_J \tau^*_t \pi(B_J) \otimes M_J
\]
and hence, by Equ. (3.5),
\[
[\tau^*_t \otimes I] \tilde{B} = s - \lim_{\Lambda \uparrow X} (\tilde{\gamma}_t(\Lambda) \otimes I) \tilde{B}
\]
Therefore
\[
[\tau^*_t \otimes I] (\tilde{B}^* \tilde{B}) = s - \lim_{\Lambda \uparrow X} \tilde{\gamma}_t(\Lambda) (\tilde{B}^* \tilde{B})
\]
Since it follows from the Lemma that \( \tilde{\gamma}_t(\Lambda) \) is CP, i.e. that \( \tilde{\gamma}_t(\Lambda) \otimes I \) is positive, it follows from the last equation that the same is true for \( \tau^*_t \otimes I \), for all values of the dimensionality of \( \mathcal{M} \). In other words, \( \tau^* \) is CP.

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