A NOTE ON POLYDEGREE $(n,1)$ RATIONAL INNER FUNCTIONS, SLICE MATRICES, AND SINGULARITIES

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ABSTRACT. We analyze certain compositions of rational inner functions in the unit polydisk $D^d$ with polydegree $(n,1)$, $n \in \mathbb{N}^{d-1}$, and isolated singularities in $T^d$. Provided an irreducibility condition is met, such a composition is shown to be a rational inner function with singularities in precisely the same location as those of the initial function, and with quantitatively controlled properties. As an application, we answer a $d$-dimensional version of a question posed in [9] in the affirmative.

1. Introduction

Background. This note is concerned with certain bounded holomorphic functions on the unit polydisk in $\mathbb{C}^d$,

$$D^d = \{ z = (z_1, \ldots, z_d) \in \mathbb{C}^d : |z_j| < 1, \ j = 1, \ldots, d \},$$
called rational inner functions, and their singularities on the $d$-torus

$$T^d = \{ \zeta = (\zeta_1, \ldots, \zeta_d) \in \mathbb{C}^d : |\zeta_j| = 1, \ j = 1, \ldots, d \};$$

here and throughout, $d \in \mathbb{N}$. By Fatou’s theorem for polydisks (see e.g. [20, Chapter 3]), any bounded holomorphic function $\phi : D^d \to \mathbb{C}$ has non-tangential boundary values $\phi^*(\zeta) = \angle \lim_{D^d \ni z \to \zeta} \phi(z)$ at almost every point $\zeta \in T^d$. If these boundary values satisfy $|\phi^*(\zeta)| = 1$ for almost every $\zeta \in D^d$, we say that $\phi$ is an inner function.

Inner functions of the form $\phi = q/p$, where $q, p \in \mathbb{C}[z_1, \ldots, z_d]$ and $p$ has no zeros in $D^d$, are called rational inner functions (RIFs). In one variable, RIFs are precisely the finite Blaschke products in the unit disk $\mathbb{D}$. Blaschke products play a central role in function theory, see for instance [11] for an overview of the very rich theory of these functions. In two and more variables, RIFs are a concrete class of bounded holomorphic functions that is amenable to detailed study [20].
Chapter 5], and appears naturally in several setting, for instance in connections with interpolation problems [1].

A classical result of Rudin and Stout (see [20, Chapter 5]) states that any RIF in $D^d$ admits a representation of the form

$$\phi(z) = e^{ia \cdot z} \frac{\tilde{p}(z)}{p(z)},$$

where $a \in \mathbb{R}$, $m = (m_1, \ldots, m_d) \in \mathbb{N}^d$, and $\tilde{p}$ is the reflection of a polynomial $p$ with no zeros in $D^d$ known as a stable polynomial. The reflection polynomial is defined as

$$\tilde{p}(z) = z_1^{n_1} \cdots z_d^{n_d} \tilde{\bar{p}}\left(\frac{1}{z_1}, \ldots, \frac{1}{z_d}\right).$$

The vector $(n_1, \ldots, n_d)$ is referred to as the polydegree of $p$; each $n_j = \deg_{z_j}(p)$ is the degree of $p$ in the variable $z_j$. In this note, we shall strip out monomial factors and consider RIFs $\phi = e^{ia \tilde{p}/p}$; this simplifies formulas and is not material for the problem we study.

RIFs as well as more general bounded rational functions in two or more variables have been considered by a number of authors in recent years, often in connection with stable polynomials, representation formulas, and operator-theoretic problems. We cannot give a full overview here, but a sampler of related work might include papers of Anderson, Dritschel, and Rovnyak [3]; Ball, Sadosky, and Vinnikov [4]; Knese [13, 14, 15], and Kollár [17].

A series of recent papers with Bickel and Pascoe [7, 8, 9]; Bickel, Knese, and Pascoe [10]; and Tully-Doyle [21] deal with aspects of RIF theory that are particular to dimensions $d \geq 2$. Namely, unlike in one dimension, RIFs in two or more variables can have singularities on the $d$-torus, arising at points $\zeta \in \mathbb{T}^d$ where $p(\zeta) = 0$ and $\tilde{p}(\zeta) = 0$ vanish without having common factors that cancel out. A $d$-dimensional example (see [15, Section 5] and [9, Example 2.5]) is given by

$$\phi_d(z) = \frac{\tilde{p}(z)}{p(z)} = \frac{d \prod_{k=1}^d z_k - \sum_{j \in J} z_{j_1} \cdots z_{j_d-1}}{d - \sum_{k=1}^d z_k} \quad (d \geq 2)$$

which has a singularity at $(1, \ldots, 1) \in \mathbb{T}^d$. Here, $J = \{(j_1, \ldots, j_{d-1}) \in \mathbb{N}^d: 1 \leq j_1 < j_2 < \cdots < j_{d-1} \leq d\}$.

One would like to describe RIF singularities in detail, and there are different ways of doing this. The papers [7, 8, 9], as well as [10], investigate for which $p \geq 1$ the partial derivative of a RIF has $\frac{\partial \phi}{\partial z_d} \in L^p(\mathbb{T}^d)$. Roughly speaking, the smaller the maximal $p$ for which integrability holds, the stronger the singularity of $\phi$; for the example (1.2), the maximal integrability index is $p = \frac{1}{2}(d + 1)$; see [9] and [10] for comprehensive discussions. The paper [7] and the work of Bergqvist [5] also consider other notions of derivative integrability corresponding to norms of Dirichlet type.
Overview of results. The purpose of this short note is to present some straightforward observations regarding $d$-variable RIFs of polydegree $(n, 1)$, $n = (n_1, \ldots, n_{d-1}) \in \mathbb{N}^{d-1}$, and their singularities. This restricted class of functions is often singled as more amenable to analysis, see for instance [13, 9, 6]. If $\hat{\zeta} = (\zeta_1, \ldots, \zeta_{d-1}) \in \mathbb{T}^{d-1}$ is kept fixed and $\varphi = \tilde{p}/p$ is a RIF in $\mathbb{D}^d$, the resulting one-variable function $\phi_{\hat{\zeta}}(z_d)$ is either a Möbius transformation mapping the unit disk onto itself, or else is a unimodular constant. By encoding this fact in a $2 \times 2$ matrix-valued function of $\hat{\zeta}$, and expressing the determinant of this matrix in terms of $\hat{\zeta}$-polynomials extracted from $p$ and $\tilde{p}$, we are able to read off certain geometric characteristics of such $\varphi$.

This allows us to exhibit $d$-variable RIFs with prescribed singularity types, and hence derivative integrability properties, while keeping the $z_d$-degree of the resulting functions equal to 1. As a specific application, we are able to answer a stronger version of [9, Question 3] in the affirmative.

2. Preliminaries

Polydegree $(n, 1)$ RIFs and their singularities. Let $p$ be an irreducible stable polynomial in $\mathbb{D}^d$, the latter meaning that $\mathcal{Z}(p) = \{ z \in \mathbb{C}^d : p(z) = 0 \}$ does not intersect $\mathbb{D}^d$. We assume throughout that $p$ has polydegree $(n, 1)$ where $n = (n_1, \ldots, n_d) \in \mathbb{N}^{d-1}$ and that $p$ is atoral, which in this context means that $p$ and $\tilde{p}$ share no common factor, see [9, Section 1.2]. Then we can decompose $p$ as a sum

$$p(z) = p_1(z_1, \ldots, z_{d-1}) + z_d p_2(z_1, \ldots, z_{d-1})$$

where $p_1(\hat{z})$ and $p_2(\hat{z})$ are in $\mathbb{C}[z_1, \ldots, z_{d-1}]$, and similarly

$$\tilde{p}(z) = \tilde{p}_2(\hat{z}) + z_d \tilde{p}_1(\hat{z}), \quad \tilde{p}_1, \tilde{p}_2 \in \mathbb{C}[z_1, \ldots, z_{d-1}].$$

As we are interested in singular RIFs $\phi = \tilde{p}/p$, we assume there exists at least one $\zeta \in \mathbb{T}^d$ such that $p(\zeta) = 0$. A result of Pascoe [13, Corollary 1.7] shows that if we assume $p$ is irreducible, then any zero $p$ on $\mathbb{T}^d$ gives rise to a singularity of $\phi$. We restrict attention to the class of such $p$ for which we have the additional property that $\mathcal{Z}(p) \cap \mathbb{T}^d$ is finite; we call the corresponding $\phi = \tilde{p}/p$ finite-singularity RIFs.

Definition 1. Suppose $\phi = \tilde{p}/p$ is a finite-singularity RIF in $\mathbb{D}^d$ with a singularity at $(1, \ldots, 1) \in \mathbb{T}^d$. We say $p^* \geq 1$ is a local $z_d$-derivative integrability index of $\phi$ if

$$p^* = \sup_{p \geq 1} \left\{ p : \frac{\partial \phi}{\partial z_d} \in L^p_{\text{loc}}(\mathbb{T}^d) \right\},$$

where each $L^p_{\text{loc}}(\mathbb{T}^d)$ is a standard local Lebesgue space on the $d$-torus. The global $z_d$-derivative integrability index of $\phi$ is the maximum of all the local $z_d$-derivative integrability indices of the finite-singularity RIF $\phi$. 
Because of the argument principle, $\frac{\partial \phi}{\partial z_d}$ is integrable for any RIF so the assumption that $p \geq 1$ is justified; see [7, 5] for details. In a similar way, we can define $z_j$-derivative indices. To keep this note as elementary as possible, we focus on the $z_d$-derivative integrability index of a $(n, 1)$ RIF.

It is not a straight-forward task to determine local $z_j$-derivative indices of a $d$-variable RIF, or their global counterparts. Two-dimensional RIFs are much better understood than their general $d$-dimensional counterparts: for instance, the $z_1$ and $z_2$-derivative indices of a RIF coincide when $d = 2$, but this is false when $d \geq 3$, and their values are determined by a geometric characteristic of $p$ at its zeros. See [7] and [10] for comprehensive presentations of the two-variable theory.

As explained in [9], the $z_d$-derivative integrability of a polydegree $(n, 1)$ RIF $\phi$ is controlled by the rate at which the zero set of $\tilde{p}$ approaches $T^d$ from inside $D^d$. To make this statement precise, we return to the one-variable function $\hat{\phi}(\hat{\zeta})$ and note that the $L^p$ norm of the derivative of a Möbius transformation is proportional to the distance to $T$ of the point $\psi^0 \in D$ for which $\phi(\hat{\zeta}, \psi^0) = 0$; see [8, Lemma 4.2].

Solving $\tilde{p}(\hat{\zeta}, \psi^0) = 0$ yields $\psi^0(\hat{\zeta}) = -\tilde{p}_2(\hat{\zeta})/\tilde{p}_1(\hat{\zeta})$, where $\tilde{p}_1, \tilde{p}_2$ are the polynomials from (2.2).

Therefore, we set

$$\rho(\hat{\zeta}) = 1 - |\psi^0(\hat{\zeta})|^2 = \frac{|\tilde{p}_1(\hat{\zeta})|^2 - |\tilde{p}_2(\hat{\zeta})|^2}{|\tilde{p}_1(\hat{\zeta})|^2}.$$ 

Note that since $\phi$ was assumed to be a finite-singularity RIF, the polynomial $\tilde{p}_1$ has no zeros in $T^{d-1}$; otherwise $Z(\tilde{p}) \cap T^d$ would contain a vertical line [9, Section 3], which is impossible since zeros of $\tilde{p}$ on $T^d$ are also zeros of $p$. Hence the vanishing of $\rho$ near a singularity is determined by the vanishing of its numerator. As a consequence of this discussion and [9, Theorem 2.1], we obtain the following criterion.

**Theorem 1.** Suppose $\phi$ is a finite-singularity RIF with polydegree $(n, 1)$ and a singularity at $(1, \ldots, 1) \in T^d$. Then $\frac{\partial \phi}{\partial z_d} \in L^p_{loc}(T^d)$ at $(1, \ldots, 1)$ if and only if

$$\int_U \left[ |\tilde{p}_1(\hat{\zeta})|^2 - |\tilde{p}_2(\hat{\zeta})|^2 \right]^{1-p} dm(\hat{\zeta}) < \infty,$$

where $U \subset T^{d-1}$ is any sufficiently small open set in $T^{d-1}$ containing $(1, \ldots, 1)$.

**Polydegree $(n, 1)$ rational inner functions and $2 \times 2$ matrices.** Suppose $\phi = \tilde{p}/p$ is a finite-singularity RIF of polydegree $(n, 1)$, and consider, for $\hat{\zeta} \in T^{d-1}$ fixed, the one-variable function

$$\phi_{\hat{\zeta}}(z_d) = \phi(\hat{\zeta}, z_d).$$
Then, $\phi_\zeta(z_d)$ is a rational function in $\mathbb{D}$, which attains unimodular boundary values at every point $\zeta_d \in \mathbb{T}$ by a theorem of Knese [16, Theorem C]. Hence $\phi_\zeta$ is either a Möbius transformation of the unit disk, or else $\phi_\zeta(z_d)$ is constant, and equal to some element of $\mathbb{T}$. The former obtains generically, but the latter possibility certainly occurs on some exceptional sets, as can be checked by considering $\phi_d(1, \ldots, 1, \zeta_d)$, where $\phi_d$ is the function in (1.2).

Guided by this discussion, we make the following definition.

Definition 2. The slice matrix of $\phi$ is the function $M_\phi: \mathbb{T}^{d-1} \to M_{2,2}(\mathbb{C})$ given by

$$M_\phi(\hat{\zeta}) = \left( \begin{array}{cc} \hat{p}_1(\hat{\zeta}) & \hat{p}_2(\hat{\zeta}) \\ p_2(\hat{\zeta}) & p_1(\hat{\zeta}) \end{array} \right).$$

The slice determinant of $\phi$ is the function $P_\phi: \mathbb{T}^{d-1} \to \mathbb{C}$ given by

$$P_\phi(\hat{\zeta}) = \det M_\phi(\hat{\zeta}).$$

Formally, the numerator and the denominator of $\phi_\zeta(z_d)$ can be read off from $M_\phi(\hat{\zeta})(z_d, 1)^T$. The slice determinant allows us to detect singularities of $\phi$ as well as their finer properties.

Lemma 2. The function $\phi_\zeta$ is constant if and only if $P_\phi(\hat{\zeta}) = 0$, and this happens if and only if $(\hat{\zeta}, \eta)$ is a singularity of $\phi$ for some value of $\eta \in \mathbb{T}$. Moreover, $\frac{\partial \phi}{\partial z_d} \in L^p_{\text{loc}}(\mathbb{T}^d)$ at $(\hat{\zeta}, \eta)$ if and only if $\int_{B(\hat{\zeta})} |P_\phi(\hat{\zeta})|^{1-p}dm(\hat{\zeta}) < \infty$ for sufficiently small $\epsilon > 0$.

Proof. The first assertion is a direct consequence of the following well-known facts that, for $a, b, c, d$ complex, $m(z) = (az + b)/(cz + d)$ furnishes a non-trivial Möbius transformation of the Riemann sphere if and only if $ad - bc \neq 0$; if $ad - bc = 0$ then $m$ is constant. See [12] for a comprehensive treatment of Möbius transformations and their connections with matrix groups.

The second assertion is a consequence of the results in [6, Subsection 3.2], see in particular [6, Lemma 3.3].

The third assertion essentially amounts to a computation. Namely,

$$\det M_\phi(\hat{\zeta}) = \hat{p}_1(\hat{\zeta})p_1(\hat{\zeta}) - \hat{p}_2(\hat{\zeta})p_2(\hat{\zeta}).$$

Observing that $\zeta_j = 1/\zeta_j$, $j = 1, \ldots, d-1$, and examining the definition of reflection polynomials, the expression on the right-hand side can be rewritten (in standard multi-index notation) as

$$\hat{\zeta}^n \hat{p}_1(\hat{\zeta})p_1(\hat{\zeta}) - \hat{\zeta}^n \hat{p}_2(\hat{\zeta})p_2(\hat{\zeta}) = \hat{\zeta}^n \left( |\hat{p}_1(\hat{\zeta})|^2 - |\hat{p}_2(\hat{\zeta})|^2 \right).$$

The result now follows after taking moduli and appealing to Theorem [12].
3. Compositions and local properties of singularities

Given an \((n, 1)\) finite-singularity RIF, we define the following sequence of functions. See [21] for a fuller study of dynamical properties of mappings, especially skew-products, whose components are RIFs.

**Definition 3.** Let \(\phi = \tilde{p}/p\) be a finite-singularity RIF of polydegree \((n, 1)\). Then \(\phi^2 : \mathbb{D}^d \to \mathbb{C}\) is defined as

\[
\phi^2(z) = (\phi \circ \phi)(z), \quad (\hat{z}, z_d) \in \mathbb{D}^d.
\]

For any \(N \in \mathbb{N}\) with \(N \geq 3\), \(\phi^N\) is defined inductively as

\[\phi^N \hat{\zeta} = \phi^{N-1} \hat{\zeta} \circ \cdots \circ \phi \hat{\zeta}.
\]

The functions \(\phi^N\) are clearly rational and holomorphic in \(\mathbb{D}^d\). As can be seen from (2.1) and (2.2), the \(z_j\)-degree of \(\phi^N\) is at most \(N \cdot n_j\) for \(j = 1, \ldots, d-1\), and \(\deg_{z_d}(\phi^N) \leq 1\). One complication that may arise is that the numerator and the denominator of the composite function may initially share a common factor. We always assume any such factors present are cancelled, in which case we get a polydegree drop in \(\phi^N\).

**Lemma 3.** Suppose \(\phi^N = \tilde{p}_N/p_N\) is as in Definition 3 and does not experience a polydegree drop. Then \(\phi^N\) is a finite-singularity RIF with the same singularities as \(\phi\).

**Proof.** Since \(\phi\) maps \(\mathbb{T}^d\) onto \(\mathbb{T}\), Knese’s theorem implies that each \((\phi^N)^*\) is unimodular. Hence \(\phi^N\) is inner.

Next, for \(\hat{\zeta} \in \mathbb{T}^{d-1}\), computing the slice matrix of \(\phi^N\) amounts to taking the matrix power \(M^N_{\phi}(\hat{\zeta}) = M_{\phi}(\hat{\zeta}) \cdots M_{\phi}(\hat{\zeta})\), see [12]. The assumption that \(\phi^N\) has full polydegree implies there are no common factors in the matrices that would be cancelled in \(\phi^N\). Then, by multiplicativity of determinants, \(\det M^N_{\phi}(\hat{\zeta})\) vanishes if and only if \(\det M_{\phi}(\hat{\zeta})\) does. Thus, the \(\hat{\zeta}\)-coordinates of the singularities of \(\phi^N\) are the same as those of \(\phi\). Since \(\phi^N\) has degree 1 in \(z_d\), and since \(\phi\) has a singularity on the line \(\{\hat{\zeta}\} \times \mathbb{T}\), each such \(\hat{\zeta}\) determines a unique \(\eta \in \mathbb{T}\) such that \((\hat{\zeta}, \eta) \in \mathbb{T}^d\) is a singularity of \(\phi^N\).

The following example illustrates that common factors can be eliminated by rotating \(\phi\) by a suitable factor \(e^{ia}\), \(a \in \mathbb{R}\), or in other words by replacing \(\tilde{p}\) by \(e^{ia}\tilde{p}\). Doing this only affects \(P_{\phi}\) up to a unimodular factor.

**Example 4.** Consider \(\phi = -(2z_1 z_2 - z_1 - z_2)/(2 - z_1 - z_2)\). As is shown by induction in [21] Example 1, each \(\phi^N, N = 1, 2, 3, \ldots\), has bidegree \((1, 1)\).

Next, consider \(\phi = (2z_1 z_2 - z_1 - z_2)/(2 - z_1 - z_2)\). Then

\[
\psi = \phi^2 = \frac{4z_1^2 z_2 - z_1^2 - 3z_1 z_2 - z_1 + z_2}{4 - 3z_1 - z_2 - z_1 z_2 + z_1^2},
\]
is a RIF that often features as a second example of a singular RIF on the bidisk; see for instance [2, 4, 8]. In particular, while \( \frac{\partial^2 \phi}{\partial z^2} \in L^p(T^2) \) if and only if \( p < 3/2 \), it was shown by direct computation in [7, Example 2] that \( \frac{\partial^2 \psi}{\partial z^2} \in L^p(T^2) \) if and only if \( p < 5/4 \). We now give a conceptual explanation for this finding.

**Theorem 5.** Suppose \( \phi = \tilde{p}/p \) is a finite-singularity RIF of polydegree \((n, 1)\), with a singularity at \( \vec{1} = (1, \ldots, 1) \). Suppose the local \( z_d \)-derivative integrability index of \( \phi \) at \( \vec{1} \) is equal to \( p^* = 1 + q^* \), where \( q^* \geq 0 \).

If \( N \in \mathbb{N} \) and \( \phi^N \) has full polydegree, then the RIF \( \phi^N = \tilde{p}_N/p_N \) has local \( z_d \)-derivative integrability index equal to \( 1 + q^*/N \) near \( \vec{1} \).

**Proof.** By Lemma 3, \( \phi^N \) is a RIF with the same singularities as \( \phi \), and in particular \( \phi^N \) has a singularity at \( \vec{1} \). Since \( \tilde{p}_N \) and \( p_N \) have no common factors that can be cancelled, the slice matrix of \( \phi^N \) is equal to \( M^N_{\phi} \), the \( N \)-fold power of the slice matrix of \( \phi \). Hence the order of vanishing of the slice determinant of \( \phi^N \) is equal to \( N \) times the order of vanishing of the slice determinant of \( \phi \). In other words, \( \frac{\partial^d \phi}{\partial z_d^d} \in L^p(T^d) \) precisely when

\[
\int_U \left( |\tilde{p}_1^2(\hat{\zeta})|^2 - |\tilde{p}_2(\hat{\zeta})|^2 \right)^{N(1-p)} dm(\hat{\zeta})
\]

is finite for \( U \supset \vec{1} \) sufficiently small. By our assumption on \( \phi \), this holds if \( N(1-p) > -q^* \) and fails when \( N(1-p) < -q^* \), and the result follows. \( \square \)

When \( d = 2 \) and \( \phi \) has a singularity at \((e^{im_1}, e^{im_2})\), it can be shown that \( 1 - |\psi^0(e^{i\theta_1})|^2 \asymp (\theta_1 - \eta_1)^{2K} \) for some \( K \in \mathbb{N} \). The number \( 2K \) is called the \( z_2 \)-contact order of \( \phi \) at \((e^{im_1}, e^{im_2})\); see [7, 8] for definitions and proofs. The assumption that \( \phi \) has finitely many singularities becomes superfluous in two variables by Bézout’s theorem, and we obtain the following.

**Corollary 6.** Suppose \( \phi = \tilde{p}/p \) is a bidegree \((n_1, 1)\) RIF in \( \mathbb{D}^2 \) with \( s \) singularities having associated contact orders \( \{2K_1, \ldots, 2K_s\} \), and suppose \( \phi^N = \tilde{p}_N/p_N \) has full polydegree.

Then \( \phi^N \) has \( s \) singularities with contact orders \( \{2NK_1, \ldots, 2NK_s\} \).

### 4. Applications

**Finding extraneous zeros of two-variable RIF denominators.** In [19], Pascoe presents a way of constructing two-variable RIFs with at least one singularity where the local contact order can be prescribed to take any value \( 2n \), \( n \in \mathbb{N} \). (Strictly speaking, the construction is given in the setting of the bi-upper half-plane, but it can readily be transferred to the bidisk by means of conjugation by a suitable Möbius
map. See [8, Section 7].) In particular, any positive even integer is the contact order of some RIF in \( \mathbb{D}^2 \).

However, Pascoe’s construction may produce additional singularities in \( \phi \) and, to the author’s knowledge, does not appear to give any immediate information about their location or nature. In principle, this can be addressed by finding all zeros of the two-variable denominator \( p \), and then using the techniques in [8, 10] to determine the associated contact orders. By examining the matrix-valued function \( \zeta_1 \mapsto M_\phi(\zeta_1) \) we can detect any such extraneous singularities and determine their contact orders in a fairly simple way. First, we compute \( P_\phi(\zeta_1) = \det M_\phi(\zeta_1) \) and find the zeros \( \{\zeta_1^1, \ldots, \zeta_1^s\} \) of the one-variable polynomial \( p(\zeta_1^j, z_2) \) that are located on the unit circle. Plugging these values into \( p \), we find the point \( \zeta_2 \in \mathbb{T} \) at which the polynomial \( p(\zeta_1^j, z_2) \) vanishes as a function of \( z_2 \). Finally, the order of vanishing of \( P_\phi \) gives us the \( z_2 \)-contact order of \( \phi \) at each singularity. By [8, Section 4], this is equal to the \( z_1 \)-contact order of \( \phi \) as well, allowing us to read off the derivative integrability of \( \phi \) at each singularity.

**Example 7** (Example 7.4 of [8]). Consider the two-variable RIF

\[
\phi(z_1, z_2) = \frac{4z_1^3 z_2 + z_1^3 - z_2^2 + 3z_1 + 1}{4 + z_2 - z_1 z_2 + 3z_1^2 z_2 + z_1^3 z_2},
\]

which is obtained using Pascoe’s method. His construction guarantees that \( \phi \) has a singularity at \((-1, -1)\) with contact order equal to 4. The slice matrix associated with \( \phi \) is

\[
M_\phi(\zeta_1) = \begin{pmatrix}
4\zeta_1^3 & \zeta_1^3 - \zeta_1^2 + 3\zeta_1 + 1 \\
\zeta_1^3 + 3\zeta_1^2 - \zeta_1 + 1 & 4
\end{pmatrix}
\]

and has determinant

\[
P_\phi(\zeta_1) = \det M_\phi(\zeta_1) = -(\zeta_1 - 1)^2(\zeta_1 + 1)^4.
\]

We immediately discern that \( \phi \) has an additional singularity at \((1, -1)\), with contact order equal to 2, as was checked in an ad hoc way in [8].

**Further derivative integrability cutoffs of \( d \)-variable RIFs.** In [9], a glueing construction from [8, Section 7] was adapted to three variables and was used to exhibit a three-variable RIF with a single isolated singularity and worse derivative integrability properties than the three-variable instance of (1.2). The drawback of that example is that the RIF so constructed has tridegree \((2, 2, 2)\), which in turn causes the verification of its claimed derivative integrability cutoff to involve lengthy computations. Thus, [9, Question 3] asked whether there exist tridegree \((n_1, n_2, 1)\) RIFs manifesting the same phenomenon. The example below answers this in the affirmative, in all dimensions \( d = 3, 4, 5, \ldots \).

**Example 8**. For \( d \geq 2 \) fixed and \( N \in \mathbb{N} \), we consider the RIF in (1.2) and its associated compositions RIFs \( \phi_d^N = \bar{p}_{d,N}/p_{d,N} \), all of degree
1 in \( z_d \). Reading off the slice matrix of \( \phi_d \) from (1.2), we check that \( M_{\phi_d}(\mathbf{1})^N \) has non-zero entries. This means there are no common factors vanishing at \( \mathbf{1} \) to cancel in \( \phi_N^d \), and we can proceed as in Theorem 5.

In [9, Example 2.5], it was shown that near \( (1, \ldots, 1) \in T^{d-1} \),

\[
1 - |\psi^0(e^{i\theta_1}, \ldots, e^{i\theta_{d-1}})|^2 \asymp \sum_{k=1}^{d-1} \theta_k^2,
\]

and hence

\[
[|\tilde{p}_1(e^{i\theta_1}, \ldots, e^{i\theta_{d-1}})|^2 - |\tilde{p}_2(e^{i\theta_1}, \ldots, e^{i\theta_{d-1}})|^2]_N \asymp \left( \sum_{k=1}^{d-1} \theta_k^2 \right)^N.
\]

Then, \( \frac{\partial \phi_{d,N}}{\partial z_d} \in L^p(T^d) \) if and only if

\[
\int_{B_\epsilon(\mathbf{0})} \left( \sum_{k=1}^{d-1} \theta_k^2 \right)^{N(1-p)} dm
\]

converges for small \( \epsilon > 0 \). In \( d - 1 \)-dimensional polar coordinates, this corresponds to the convergence of

\[
\int_{0}^{1} r^{2N(1-p)} r^{d-2} dr = \int_{0}^{1} r^{2N+d-2-2Np} dr.
\]

This integral is finite if and only if \(-1 < 2N + d - 2 - 2Np\), and hence we obtain the \( z_d \)-derivative integrability cutoffs

\[
(4.1) \quad p^*(d, N) = 1 + \frac{d - 1}{2N}.
\]

This extends the work in [9, Example 2.5], where it was shown that \( p^*(d, 1) = 1 + (d-1)/2 = (d+1)/2 \). Next, since \( p^*(2, N) = 1 + 1/(2N) \), we observe that any contact order is realized by a RIF with a unique singularity at \( (1, 1) \in T^2 \).

Finally, the fact that \( p^*(3, 2) = 3/2 \) shows that \( \phi_{3,2} \), with denominator

\[
p_{3,2}(z) = 9 - 6z_1 - 6z_2 - 3z_3 + z_1^2 + z_2^2 + 3z_1z_2 + 2z_1z_3 + 2z_2z_3 - 3z_1z_2z_3,
\]

provides an example of a RIF providing a positive answer to [9 Question 3].

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