Baryon Wave Functions in Covariant Relativistic Quark Models

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Abstract

We derive covariant baryon wave functions for arbitrary Lorentz boosts. Modeling baryons as quark-diquark systems, we reduce their manifestly covariant Bethe-Salpeter equation to a covariant 3-dimensional form by projecting on the relative quark-diquark energy. Guided by a phenomenological multigluon exchange representation of a covariant confining kernel, we derive for practical applications explicit solutions for harmonic confinement and for the MIT Bag Model. We briefly comment on the interplay of boosts and center-of-mass corrections in relativistic quark models.

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One of the major goals of modern electron and hadron accelerators is the investigation of the internal structure of hadrons, in particular of baryons: detailed information is extracted from scattering experiments at large momentum transfers of typically 1 GeV/c and beyond. The corresponding form factors map out the various internal (generalized) charge distributions and provide stringent information on the underlying quark and gluon degrees of freedom. Presently various experiments are ongoing with electron or photon beams at MAMI, ELSA, MIT, JLAB und DESY (1) and with proton beams at COSY and CELSIUS (2) and other labs.

Practical calculations of form factors suffer in general from the pertinent problem of center-of-mass (CM) corrections for the many-body problem and from drastic effects from Lorentz contraction at increasing momentum transfers. While for the CM corrections various recipes have been developed and applied in practical calculations (3-6), less progress has been achieved in the formulation of covariant baryon wave functions suitable for practical calculations (7-24). A possible alternative, the evaluation of formfactors on the light cone, where Lorentz boosts are completely kinematical, has so far entered only selectively in practical applications at low scattering energies, beyond that such an approach suffers from other decreases, such as the loss of strict rotational invariance (25). As in general the construction of boosted, Lorentz contracted wavefunctions is nearly as complicated as the solution of the full problem, in most practical applications ad hoc and purely kinematical prescriptions for the rescaling of the coordinate along the direction of the momentum transfer are applied (examples are given ref. (26-27)). Thus, specific questions, as the dependence of Lorentz corrections on the confining kernel in quark models, are not addressed. In addition, to hopefully minimize the influence of Lorentz contractions formfactors are in general evaluated in the Breit frame, though experimentally they are measured in the lab system.

In this note we formulate an economical model for covariant baryon wave functions, which leads to results suitable for practical applications. As it our main goal to end up with analytical formulae, we model the baryon - in the following we use the word proton, though our approach is fairly general - as a quark-diquark system and restrict ourselves, without any loss of generality, to spin-isospin scalar diquarks (28).

Our starting point is the manifestly covariant 4-dimensional Bethe-Salpeter equation (29)

\[ \Gamma = K \Gamma \chi \quad \text{and} \quad \Psi = \Gamma \chi \]

(1)

with the vertex function and the Bethe-Salpeter amplitude \( \Gamma \) and \( \Psi \), respectively, and the interaction kernel \( K \). In the two-body Greens function for the quark with mass \( m \) and the diquark with mass \( m^* \) we fix the relative energy dependence from the covariant projection
on the diquark (30)

\[ G(P, q) = \frac{\hat{\sigma} + m}{q^2 - m^2 - i\epsilon} \sum_{\pm} i\pi \delta_+ \left( (P - q)^2 - m^2 \right) \]  

(2)

which results up to \( \left( \frac{q^2}{2M} \right) \) in the single particle Dirac equation for the quark for systems with arbitrary overall 4-momenta \( Q = (E(P) = \sqrt{P^2 + M^2}, 0, 0, P) \)

\[
 \left( \frac{M}{E(P)} \left( \epsilon + \frac{P}{M} q_z - \frac{q^2}{2M} \right) - (\alpha q + \beta m) \right) \varphi(Q, q) = \frac{1}{E(P)} \int K(Q, q, k) \varphi(Q, k) dk
\]  

(3)

Without any details we add a brief comment on the CM corrections in our model: evidently there is a direct coupling between the internal and external momenta \( q \) and \( P \), or equivalently, between boosts and the CM motion. In the rest system the leading center-of-mass corrections are absorbed for \( \epsilon = m + \epsilon_b \), where \( \epsilon_b \) is the binding energy of the quark in

\[
 \left( \frac{q^2}{2\mu} + \epsilon_b - V_n(r) \right) \varphi(r) = 0
\]  

(4)

with the reduced mass \( 1/\mu = 1/m + 1/(m + m^*) \) for an arbitrary quark potential \( V_n(r) \)
(a detailed discussion of CM corrections are presented in a separate paper).

The decisive step for a practical model is the formulation of a covariant interaction kernel in eq. (4). As the dynamics of the quark - quark interactions, particularly the microscopic nature of the confinement, lacks an understanding on the fundamental level of QCD, all current models in practical calculations rely on phenomenological formulations of the interaction kernel. Being unable to do better, we proceed here along similar lines: we assume that the interaction kernel can be presented as a superposition of appropriately weighted gluon exchange contributions; quantitative parameters can be extracted in comparison with studies to baryon spectroscopy, decay rates or form factors (31). Thus we start from the general kernel

\[
 K(P, q, k) = \sum_n \frac{k_n(P)}{((q - k)^2 - m^2 + i\epsilon)^{n+1}}
\]  

(5)

for arbitrary powers of \( n \) (which reflect different parametrizations of the confining kernel) (eq. (5) contains the linear confinement in the Cornell potential (32)). Upon projecting out the relative energy dependence this yields the covariant, 3-dimensional kernel

\[
 K_n(P, q, k) \propto \lim_{\mu \to 0} \left( \frac{d}{d\mu^2} \right)^n \frac{1}{\lambda^2(P)q_z^2 + q_\perp^2 + \mu^2 - i\epsilon}
\]  

(6)

with the ”quenching parameter”

\[
 \lambda(P) = M/\sqrt{M^2 + P^2} = M/E(P)
\]  

(7)
where we introduced the mass scale $\mu$ (to regularize the Fourier transform to coordinate space). Already simple power counting signals, that a kernel with the power $n$ leads to confinement with $\sim \mu^{2n}$. Upon performing the corresponding Fourier transform to coordinate space and performing the limit $\mu \to 0$ we find

$$K_n(P, q) \to (1 + \beta)/2 V_n(\sqrt{(z/\lambda(P))^2 + \rho^2})$$

where we introduced for convenience the particular Dirac structure of the kernel to facilitate the evaluation of the resulting Dirac equation. Eliminating the small component in eq. (3) with the kernel from eq. (6) and upon dropping CM corrections and $\epsilon^2$ terms for compactness, we end up with the Schroedinger type equation for the large component of the Dirac equation

$$
(2m\epsilon_b - \lambda^2(q_z - Pz)^2/\lambda^2 - q_{\perp}^2 - V_n(\sqrt{(z/\lambda(P))^2 + \rho^2}/R) \ u(z, \rho) = 0
$$

(with the typical length scale $R$; in the rest system the equation above reduces to the standard spherical Schrödinger type equation for a particle with mass $m$). The final equation defines with its connection to the small component by a simple differentiation the full relativistic covariant quark - diquark wave function for arbitrary Lorentz systems. In the equation above we see the shortcoming from the phenomenological nature of the interaction kernel: we absorb the explicit $\epsilon$ and $P$ dependence of the kernel in the definition of the energy scale $V_n$ for the confining force; including an explicit $P$ dependence in $V_n$ would require a detailed knowledge of its microscopic origin.

Approximate or numerical solutions for eq. (9) can be obtained for different confining scenarios (a more detailed investigation, such as also of the popular linear (heavy quark) confinement (33), is presented elsewhere). Here we enter only briefly into two scenarios, which allow a rigorous analytic solution for arbitrary systems: i. e. harmonic confinement and bag models in the limit $n \to \infty$ in eq(6).

- Harmonic confinement:

With the harmonic kernel defined as (34)

$$K(P, q, k) = -12/\pi \lim_{(\mu \to 0)} \left((d/d\mu^2)^2(\mu^2/2 + (d/d\mu^2)\mu^3/3)\right) - \left((1/\lambda(P))^2(d/dq_z)^2 + (d/dq_{\perp}^2)\right) \delta(q_z - k_z)\delta(\mathbf{q}_{\perp} - \mathbf{k}_{\perp}),$$

the solutions for arbitrary excitations of the baryon are easily obtained in momentum space. After a redefinition of the longitudinal momentum and upon separating the longitudinal and the perpendicular component, the general solution is given by a
product of confluent hypergeometric functions (35). Here we focus only on the nucleon as the quark-diquark ground state and obtain explicitly

$$u(q_z, q_{\perp}) = N e^{-\frac{a^2}{2}}(\lambda^2(q_z - \frac{M}{\pi P})^2 + q_{\perp}^2)^{1/2}$$

(11)

with the oscillator parameter $$a^2 = \frac{2}{\sqrt{V_c}}$$, with $$V_c \propto 1/R^4$$ being the confinement strength and with the ground state energy

$$\epsilon_b(P) \approx (1 + \frac{P^2}{M^2}) \cdot \frac{\sqrt{V_c}}{m}$$

(12)

As expected the standard solution for the spherical harmonic oscillator is recovered in the rest system, i.e. for $$P=0$$ and $$\lambda(0)=1$$. As the characteristic result we find a quenching of the effective P-dependent with size parameter

$$a^2(P)^2 = (\lambda(P)a)^2 = \frac{M^2}{P^2 + M^2} a^2$$

(13)

which leads to Lorentz quenching in coordinate space along the z-axis and thus to a significant increase of the longitudinal high momentum components with increasing P (Fig. 1(a,b));

- Bag Model:

As mentioned above we generate the Bag from the transition $$n \to \infty$$ in the power of gluon-exchange kernel. As we are unable to present an analytical solution for arbitrary n (a closed solution for the z-component exists only in the limit of vanishing binding $$\epsilon_b \to 0$$ (35)) we first perform the limit $$n \to \infty$$ and then solve the equation

$$\left(\epsilon_b + \frac{P}{M} q_z + m - (\alpha q + \beta m)\right) u(z, \rho) = 0$$

(14)

with the standard MIT boundary condition for the large and small components at $$z = \lambda(P) R$$ for the bag radius R. For the large component the ground state solution can be represented as

$$u(z, \rho) = N \cos(\frac{k_z}{\lambda} z) J_0(k_{\perp} \rho)$$

(15)

where the $$\lambda$$ dependence of the z-component again reflects the quenching of the bag (The extension to excited baryon states again is straightforward). The quenching for the bag along the boost momentum is also reflected in the boundary condition $$z = \lambda R$$ for $$\rho = 0$$, which for the deformed bag can be solved only numerically (36). Characteristic results for 3 different boost momenta are presented for the large and small component of the bag ground state solution in Fig. 2.
Comparing our findings with current more phenomenological recipes we find that a general and simple extension of the parametrization of the spherical wave functions and momentum distributions in the rest system to a boosted system, by rescaling the size parameter of the system, but keeping otherwise the spherical character of the solutions, is certainly very unsatisfactory and breaks down completely for boost momenta of typically $P/M \geq 1$. Only for very small boost momenta $P$ simple approximations, such as

$$u(z, \rho, a) \approx u \left( r, a / \left( \sqrt{3} \lambda(P) \right) \right) \quad \text{and}$$

$$u(z, \rho, R) = \exp \left( - \frac{r}{\sqrt{3} \lambda(P) R} \right)^2 u(r, R)$$

simulate very qualitatively Lorentz quenching of slowly moving systems.

With increasing boost momenta the breaking of the spherical symmetry for the quenched baryon leads for the ground (and all excited) state to the admixture of additional angular momenta, which drastically enhance the momentum spectrum of the ground state with increasing $q$. A characteristic result is shown in Fig. 3 for the d-wave admixture for different boost momenta.

Summarizing our main findings in this note, we have formulated covariant wave functions and their transformation properties in an analytical quark-diquark model for the baryon and we find characteristic modifications from the baryon rest system to moving Lorentz-systems for different confining kernels.

Our findings suggest possible extensions and basic shortcomings of the model. We feel that an extension of the model to mesons as $q\bar{q}$ systems, towards a more realistic quark-diquark description of baryons or to genuine 3-quark systems (together with a systematic inclusion of CM corrections) imposes only technical problems and is certainly feasible. Here we only mention that the quenching factor from eq.(7) is recovered in leading order for all current projections of the BS equation: as an example the Blankenbecler-Sugar reduction (30) for quarks with equal masses yields immediately

$$G_{BBS} (P, q) \sim \delta(q_0 - P/M q_z)$$

A more serious problem for confining kernels with a finite power in the interquark distance is the precise formulation of Lorentz quenching for the kernels itself (in Bag models the dependence is absorbed in the boundary condition). Here the unsurmountable problem is our current lack in understanding the confining mechanism; it is not clear, how the full $P$ dependence enters into the kernel (for an example compare ref.(37); however, different
szenarios may lead to quantitatively very different results for large P; for an example compare ref. (37)). We feel that a more realistic extension of present phenomenological quark models undoubtedly requires a much deeper analytical understanding of confinement. Here significant progress in various directions has been achieved recently, to mention only the modelling of confinement of QCD in the Coulomb gauge (38) or the extension of the concept of instantons to merons as solutions of the classical QCD equations (39).

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Figure Captions

Fig. 1: Z-dependence of the large component of the harmonic oscillator ground state in coordinate (a) and momentum space (b) for different boost momenta $P = 0$ (thin line), $M$ (middle line) and $2M$ (thick line).

Fig. 2: (a) Quenching and boundary conditions for the bag along the boost momenta $P=0$, $M$, $2M$. The functions $f(z)$ and $g(z)$ denote the large and small components of the ground state wave function (for a bag radius $R = 1$ fm).

Fig. 3: D-state admixture to the harmonic oscillator ground state. Compared are the s-wave distribution for $P=0$ with the d-state component for $P=M$ and $2M$ (middle and thick line, respectively).
