Kawasaki-type Dynamics: Diffusion in the kinetic Gaussian model

Han Zhu\(^2\) and Jian-Yang Zhu\(^1,3\)\(^*\)

\(^1\)CCAST (World Laboratory), Box 8730, Beijing 100080, China
\(^2\)Department of Physics, Nanjing University, Nanjing, 210093, China
\(^3\)Department of Physics, Beijing Normal University, Beijing 100875, China

Abstract

In this article, we retain the basic idea and at the same time generalize Kawasaki’s dynamics, spin-pair exchange mechanism, to spin-pair redistribution mechanism, and present a normalized redistribution probability. This serves to unite various order-parameter-conserved processes in microscopic, place them under the control of a universal mechanism and provide the basis for further treatment. As an example of the applications, we treated the kinetic Gaussian model and obtained exact diffusion equation. We observed critical slowing down near the critical point and found that, the critical dynamic exponent \(z = 1/v = 2\) is independent of space dimensionality and the assumed mechanism, whether Glauber-type or Kawasaki-type.

PACS number(s): 64.60.Ht, 05.70.Ln, 75.10.Hk, 68.35.Fx

\(^*\)Author to whom correspondence should be addressed. Address correspondence to Department of Physics, Beijing Normal University, Beijing 100875, China. Email address: zhujy@bnu.edu.cn
Irreversible dynamic systems exhibit complicated but interesting nonequilibrium phenomena near the critical point. In spite of their complexity, the interesting dynamic critical behaviors have been attracting a lot of researchers for many decades. Within the vast body of literature, two pioneering works completed by Glauber [1] and Kawasaki [2] have been regarded as a milestone. Great progress has been achieved with the application of Glauber’s single-spin flip mechanism and Kawasaki’s spin-pair exchange mechanism. These two schemes catch the inherent essential process bound in.

Kawasaki’s dynamics deals with a system consisting of an array of \( N \) coupled spins. The coupling between spins is represented by a set of transition probabilities of spin exchange. Initially, the focus was on Ising model. The spins may exchange with their nearest neighbors, and in this way, the system evolves while the total spin remains conserved. In later studies the idea of exchange has been proved very successful as it catches the essential nature. It has important applications in Ising model [3,4] and Ising-like models such as lattice gas model [5–7], Blume-Emery-Griffiths (BEG) model [8,9] and others [10,11]. These applications have contributed a lot to the understanding of the thermodynamics, and have erected itself as the basic mechanism in the class of order-parameter-conserved processes.

However, its initial embodiments was closely tied up with the simpleness of Ising model, in which the spins can only take two values, \( \pm 1 \), and there exists only nearest-neighbor interactions. It turned to be limited when applied to systems other than Ising or Ising-like models. The same situation exists in Glauber’s dynamics. Recently Zhu and Yang successfully generalized Glauber’s single-spin flip mechanism to single-spin transition mechanism and gave a normalized version of the transition probability [12]. The applications in several models have yielded encouraging results [12,13]. In this article, along the same line, we retain the basic idea and at the same time generalize Kawasaki’s dynamics, spin-pair exchange mechanism, to spin-pair redistribution mechanism, and also present a normalized redistribution probability.

The limit is to be removed and the spins may take various discrete values (discrete-spin model) or continuous values (continuous-spin model). Because in Ising model spins can only
have two opposite values, simply direct exchange may be enough to describe the way the system evolves. In other different models, such as Gaussian model, Potts model, or XY model, this simple picture may not be as capable. On the other hand, as is mentioned above, the conservation of the order-parameter has been regarded as the most important feature of this class of processes (and a necessary result of the exchange mechanism). Based on these considerations, in the generalized mechanism, spin pair redistribution mechanism, two neighboring spins, $\sigma_k$, $\sigma_{k+1}$, no longer merely exchange with each other. Instead, they may take any values, $\hat{\sigma}_k$, $\hat{\sigma}_{k+1}$, as long as their sum remains conserved, or, their sum is redistributed; and we call this, spin-pair redistribution.

The probability distribution function $P(\sigma_1, \ldots, \sigma_N; t)$, or simply $P(\sigma; t)$, denotes the probability of the $N$-spin system being in the state $(\sigma_1, \ldots, \sigma_N)$, or simply $\{\sigma\}$, at time $t$. The $j$th and $l$th spins are two neighboring spins, and let $W_{jl}(\sigma_j \sigma_l \rightarrow \hat{\sigma}_j \hat{\sigma}_l)$ be the probability per unit time that they are redistributed while the others remain unchanged. Then, on the supposition of neighboring spin-pair redistribution, we may write the time derivative of the function $P(\sigma; t)$ as

$$\frac{d}{dt}P(\sigma; t) = \sum_{j \neq l} \sum \{ -W_{jl}(\sigma_j \sigma_l \rightarrow \hat{\sigma}_j \hat{\sigma}_l) P(\sigma; t) + W_{jl}(\hat{\sigma}_j \hat{\sigma}_l \rightarrow \sigma_j \sigma_l) P(\{\sigma_i \neq j,l\}, \hat{\sigma}_j, \hat{\sigma}_l; t) \} \quad (1)$$

This is a probability equation, in which the first term in the right-hand side of (1) denotes the decrease of the probability distribution function $P(\sigma; t)$ per unit time due to the redistribution of the spin pair from initially $\sigma_j \sigma_l$ to various values $\hat{\sigma}_j \hat{\sigma}_l$, and the second term denotes the increase of $P(\sigma; t)$ due to the redistribution of the spin pair from initially various values $\hat{\sigma}_j \hat{\sigma}_l$ to finally $\sigma_j \sigma_l$. (Clearly, in Ising and Ising-like systems, it does become Kawasaki’s picture.) We shall refer to (1) as the master equation since its solution would contain the most complete description of the system available.

The key step for solving the master equation is the determination of the redistribution (or exchange, flip, transition, etc.) probability. Usually it can not be uniquely determined by the detailed balance condition, and thus, in Kawasaki’s as well as Glauber’s pioneering works,
some arbitrariness remained. We hope to make our choice of the spin-pair redistribution probability able to contain Kawasaki’s in the specific Ising model, and applicable to various other spin systems, while at the same time appropriate, clearer, and more definite. Now we consider it in both mathematical and physical aspects. In mathematics, generally speaking, the probability must be positive and able to be normalized; in physics, we often require that a system in thermodynamic equilibrium satisfy the detailed balance condition; we also require the probability be ergodic as long as the total spin keeps conserved through the redistribution. Based on these considerations, for arbitrary neighboring \( j \)th and \( l \)th spins, we can choose the redistribution probability \( W_{jl}(\sigma_j \sigma_l \rightarrow \hat{\sigma}_j \hat{\sigma}_l) \) to satisfy the following conditions:

(a) Ergodicity, positivity, and conservation of spin:

\[
W_{jl}(\sigma_j \sigma_l \rightarrow \hat{\sigma}_j \hat{\sigma}_l) > 0, \quad \forall \hat{\sigma}_j + \hat{\sigma}_l = \sigma_j + \sigma_l \\
W_{jl}(\sigma_j \sigma_l \rightarrow \hat{\sigma}_j \hat{\sigma}_l) = 0, \quad \forall \hat{\sigma}_j + \hat{\sigma}_l \neq \sigma_j + \sigma_l
\]

this may suggest a \( \delta \) function in the expression;

(b) Normalization:

\[
\sum_{\hat{\sigma}_j, \hat{\sigma}_l} W_{jl}(\sigma_j \sigma_l \rightarrow \hat{\sigma}_j \hat{\sigma}_l) = 1;
\]

(c) Detailed balance:

\[
\frac{W_{jl}(\sigma_j \sigma_l \rightarrow \hat{\sigma}_j \hat{\sigma}_l)}{W_{jl}(\hat{\sigma}_j \hat{\sigma}_l \rightarrow \sigma_j \sigma_l)} = \frac{P_{eq}(\sigma_1, \ldots, \hat{\sigma}_j, \hat{\sigma}_l, \ldots, \sigma_N)}{P_{eq}(\sigma_1, \ldots, \sigma_j, \sigma_l, \ldots, \sigma_N)},
\]

in which

\[
P_{eq} = \frac{1}{Z} \exp \left[ -\beta \mathcal{H}(\{\sigma\}) \right], \quad Z = \sum_{\{\sigma\}} \exp \left[ -\beta \mathcal{H}(\{\sigma\}) \right],
\]

where \( P_{eq} \) is the equilibrium Boltzmann distribution function, \( Z \) is the partition function, and \( \mathcal{H}(\{\sigma\}) \) is the system Hamiltonian.

Although the spin pair redistribution probabilities still are not determined uniquely by the above restriction, there is less room left. The consideration that we have is similar to
that in the generalization of Glauber’s dynamics \[12\], that the redistribution of a neighboring pair depends merely on the momentary values of the spins surrounding them, or, the spins of the system, and the influence of the heat bath. Based on this, we can similarly assume that the redistribution probability \( W_{jl}(\sigma_j \sigma_l \rightarrow \hat{\sigma}_j \hat{\sigma}_l) \) only depends on the heat Boltzmann factor of the system.

\[
W_{jl}(\sigma_j \sigma_l \rightarrow \hat{\sigma}_j \hat{\sigma}_l) \propto \delta_{\sigma_j,\sigma_l;\hat{\sigma}_j,\hat{\sigma}_l} \exp \left[-\beta H_{jl}(\hat{\sigma}_j, \hat{\sigma}_l, \{\sigma_m\}_{m \neq j,l})\right]
\]

or

\[
W_{jl}(\sigma_j \sigma_l \rightarrow \hat{\sigma}_j \hat{\sigma}_l) = \frac{1}{Q_{jl}} \delta_{\sigma_j,\sigma_l;\hat{\sigma}_j,\hat{\sigma}_l} \exp \left[-\beta H_{jl}(\hat{\sigma}_j, \hat{\sigma}_l, \{\sigma_m\}_{m \neq j,l})\right].
\] (2)

Using the normalization condition (b), we can write the normalization factor \( Q_{jl} \) as

\[
Q_{jl} = \sum_{\hat{\sigma}_j,\hat{\sigma}_l} \delta_{\sigma_j,\sigma_l;\hat{\sigma}_j,\hat{\sigma}_l} \exp \left[-\beta H_{jl}(\hat{\sigma}_j, \hat{\sigma}_l, \{\sigma_m\}_{m \neq j,l})\right].
\]

In fact, the Hamiltonian coming from the interaction of spins unrelated to \( \sigma_j, \sigma_l \) will be cancelled, and thus in the actual calculation, one needs only write the effective Hamiltonian. Obviously, \( Q_{jl} \) is related to the temperature, the surrounding spins and the conserved sum of \( \sigma_j, \sigma_l \).

Compared with Kawasaki’s expression, \( (2) \) is a normalized version. In his expression, there is an \( \alpha \) appearing in the exchange probability, which was assumed to be a constant. In Ising model, redistribution is in fact exchange and actually our expression is only a definite selection for constant \( \alpha \) by extra restriction and physical considerations.

Usually, we are interested in local magnetization. It is defined as

\[
q_k(t) = \langle \sigma_k(t) \rangle = \sum_{\{\sigma\}} \sigma_k P(\{\sigma\}; t).
\] (3)

According to the definition (3) and the master equation (1), and using the normalization condition (b), the time-evolving equation of \( q_k(t) \) can be derived as (see Appendix A)

\[
\frac{d}{dt}q_k(t) = -2dq_k(t) + \sum_{\{\sigma\}} \sum_w \left[ \sum_{\hat{\sigma}_k,\hat{\sigma}_{k+w}} \hat{\sigma}_k W_{k,k+w}(\sigma_k \sigma_{k+w} \rightarrow \hat{\sigma}_k \hat{\sigma}_{k+w}) \right] P(\{\sigma\}; t),
\] (4)
where \( d \) is the dimensionality of the system, and \( \sum_w \) means summation over the nearest neighbors (clearly it is related to the dimensionality, too).

Kawasaki’s exchange mechanism was initially designed for the study of diffusion constant, and he himself obtained an approximate result for Ising model by first deriving an expression of spin flux [2]. Now we will treat the kinetic Gaussian model as an application of our redistribution mechanism, while our method is a direct one.

The Gaussian model, proposed by T. H. Berlin and M. Kac, at first in order to make an Ising model more tractable, is a continuous-spin model. It has the same Hamiltonian form as the Ising model (three dimensional),

\[
-\beta H = K \sum_{i,j,k=1}^{N} \sum_w \sigma_{ijk} \left( \sigma_{i+w,jk} + \sigma_{ij,w,k} + \sigma_{ij,k+w} \right),
\]

where \( \sum_w \) means summation over nearest neighbors. Comparing it with the Ising model, there are two extensions: First, the spins \( \sigma_{ijk} \) can take any real value between \((-\infty, +\infty)\). Second, to prevent the spins from tending to infinity, the probability of finding a given spin between \( \sigma_{ijk} \) and \( \sigma_{ijk} + d\sigma_{ijk} \) is assumed to be the Gaussian-type distribution

\[
f(\sigma_{ijk}) d\sigma_{ijk} = \sqrt{\frac{b}{2\pi}} \exp \left( -\frac{b^2}{2} \sigma^2_{ijk} \right) d\sigma_{ijk},
\]

where \( b \) is a distribution constant independent of temperature. Although it is an extension of the Ising model, the Gaussian model is quite different. In the equilibrium case, on translational invariant lattices the Gaussian model was exactly solvable, and later as a starting point to study the unsolvable models it was also investigated with mean field theory and the momentum-space renormalization-group method.

In the 3D kinetic Gaussian model, the system Hamiltonian and the spin distribution probability are [3] and [3], respectively. In this case there are six combined terms in the 3D type time-evolving equation of the local magnetization, Eqs.[4], and the details of these complex calculations are in Appendix [3]. Here we give only the results,

\[
\sum_{\hat{\sigma}_{ijk}, \hat{\sigma}_{i+1,j,k}} \hat{\sigma}_{ijk} W_{i,j,k,i+1,j,k} \left( \sigma_{ijk} \sigma_{i+1,j,k} \rightarrow \hat{\sigma}_{ijk} \hat{\sigma}_{i+1,j,k} \right)
\]
Substituting them into the time-evolving equation of the local magnetization (4), we get

\[
\sum_{i<j<k} \hat{\sigma}_{ijk} W_{i,j,k;i-1,j,k} (\sigma_{ijk} \sigma_{i-1,j,k} \rightarrow \hat{\sigma}_{ijk} \hat{\sigma}_{i-1,j,k})
\]

\[
= \frac{1}{2(b+K)} [K (\sigma_{i,j,k+1} + \sigma_{i,j,k-1} + \sigma_{i,j,k+1} + \sigma_{i,j,k-1} + \sigma_{i,j,k+1} + \sigma_{i,j,k-1} + \sigma_{i,j,k+1} + \sigma_{i,j,k-1}) + b (\sigma_{ijk} + \sigma_{i-1,j,k})],
\]

\[
= \frac{1}{2(b+K)} [K (\sigma_{i,j,k+1} + \sigma_{i,j,k-1} + \sigma_{i,j,k+1} + \sigma_{i,j,k-1} + \sigma_{i,j,k+1} + \sigma_{i,j,k-1} + \sigma_{i,j,k+1} + \sigma_{i,j,k-1}) + b (\sigma_{ijk} + \sigma_{i-1,j,k})],
\]

Substituting them into the time-evolving equation of the local magnetization (4), we get
\[
\frac{d}{dt} q_{ijk}(t) = \frac{1}{2} \left( \frac{b}{b + K} \right)^2 b \left[ \left(q_{i,j,k+1} - q_{ijk}\right) - \left(q_{ijk} - q_{i,j,k-1}\right) \right] \\
+ \left[\left(q_{i+1,j,k} - q_{ijk}\right) - \left(q_{ijk} - q_{i-1,j,k}\right)\right] + \left[\left(q_{i,j+1,k} - q_{ijk}\right) - \left(q_{ijk} - q_{i,j-1,k}\right)\right] \\
+ \frac{K}{2} \left(2 \left(q_{i-1,j,k} - q_{i-1,j+1,k} - q_{i-1,j-1,k}\right) + \left(2q_{i-1,j,k} - q_{ijk} - q_{i-2,j,k}\right) \right) \\
+ 2 \left(2q_{i+1,j,k} - q_{i+1,j+1,k} - q_{i+1,j-1,k}\right) + \left(2q_{i+1,j,k} - q_{ijk} - q_{i+2,j,k}\right) \\
+ 2 \left(2q_{i,j-1,k} - q_{i,j-1,k+1} - q_{i,j-1,k-1}\right) + \left(2q_{i,j-1,k} - q_{ijk} - q_{i,j-2,k}\right) \\
+ 2 \left(2q_{i,j+1,k} - q_{i,j+1,k+1} - q_{i,j+1,k-1}\right) + \left(2q_{i,j+1,k} - q_{ijk} - q_{i,j+2,k}\right) \\
+ 2 \left(2q_{i,j,k-1} - q_{i,j,k-1} - q_{i+1,j,k-1}\right) + \left(2q_{i,j,k-1} - q_{ijk} - q_{i,j,k-2}\right) \\
+ 2 \left(2q_{i,j,k+1} - q_{i,j,k+1} - q_{i,j,k+1}\right) + \left(2q_{i,j,k+1} - q_{ijk} - q_{i,j,k+2}\right). \tag{7}
\]

With lattice constant \(a\) we can transform the above equation to be

\[
\frac{d}{dt} q(t) = \frac{a^2}{2(b + K)} b \left( \nabla_x^2 + \nabla_y^2 + \nabla_z^2 \right) q(t) \\
- \frac{a^2}{2(b + K)} K \left[ 2 \nabla_y^2 + \nabla_x^2 \right] + \left(2 \nabla_z^2 + \nabla_y^2 \right) \\
+ \left(2 \nabla_z^2 + \nabla_y^2 \right) + \left(2 \nabla_x^2 + \nabla_y^2 \right) \right] q(t) \\
= \frac{3a^2}{b + K} \left( \frac{b}{6} - K \right) \nabla^2 q(t). \tag{8}
\]

It is of the form of a diffusion equation

\[
\frac{dq(t)}{dt} = D \nabla^2 q(t),
\]

where

\[
D = \frac{3a^2}{b + K} \left( \frac{b}{6} - K \right) a^2. \tag{9}
\]

With the same treatment, we can easily obtain the diffusion equation of one and two dimensional kinetic Gaussian model. Here we give only the results,

\[
\frac{d}{dt} q_{\alpha}(t) = \frac{d a^2}{b + K} \left( \frac{b}{2d} - K \right) \nabla^2 q,
\]

where \(d\) is the system dimensionality. As we know that the critical point for \(d\)-dimensional Gaussian model is \(K_c = J/k_B T_c = b/2d\), thus, the diffusion equation we obtained reveals
that the diffusion will get much slower when it is near the critical point. The linear equations we obtained for a single spin can be directly solved, but the solution of the diffusion equations may already give us satisfying information. For example, in 1D case, it is

$$q(x, t) = \frac{1}{2\sqrt{D\pi t}} \int_{-\infty}^{\infty} q(\xi, 0) e^{-\frac{(x-\xi)^2}{4Dt}} d\xi.$$  

In a specific example of the diffusion of a Gaussian type packet, $q(\xi, 0) = e^{-\xi^2}$, one will obtain

$$q(x, t) = \sqrt{\frac{1}{1 + t/\tau}} \exp \left[ -\frac{x^2}{1 + t/\tau} \right],$$  \hspace{1cm} (11)

where

$$\tau = \frac{1}{4D} = \left[ \frac{4a^2}{b + K} \left( \frac{b}{2} - K \right) \right]^{-1}$$  \hspace{1cm} (12)

is the relaxation time. When $K \to K_c = b/2$, $D \to 0$ and $\tau \to \infty$, and this is a typical critical slowing down phenomenon. With the correlation length critical exponent $\nu = 1/2$ and the following dynamical scaling hypotheses

$$\xi \sim |T - T_c|^{-\nu}, \tau \sim \xi^z,$$

one can obtain the dynamic critical exponent $z = 2$. The same result can be obtained for 2D and 3D model. In earlier study [12] the same result $z = 1/\nu = 2$ for any dimensionality was obtained with Glauber-type mechanism. Thus we find that, in Kinetic Gaussian model, the critical dynamic exponent is independent of space dimensionality and the dynamic mechanism.

Before conclusion, we may note that actually, although a systematic formulation has been lacked, efforts towards wider application of Kawasaki’s dynamics have never been stopped. For example, in a Monte Carlo simulation of the three-dimensional ferromagnetic Heisenberg model [14], Zhang has suggested that two neighboring spins, $\sigma_i$ and $\sigma_j$, may rotate with their conserved sum being the axis. He found that the new scheme enabled the system to evolve to thermodynamic equilibrium faster, and commented that it might be more favorable in
reality. This is just an successful exploration of the spin-pair redistribution mechanism, and there are many other such examples in non-equilibrium statistics, though assuming different forms.

To summarize, in this article, we presented a systematic formulation of the Kawasaki-type dynamics: spin-pair redistribution. As a natural generalization of Kawasaki’s exchange mechanism, the new dynamics gives the system more freedom while keeping the order-parameter conserved. We have analyzed the redistribution in physics, given the master equation and a normalized redistribution probability determined by the heat Boltzmann factor. The presentation of this probability, being the key of the whole formulation, makes the mechanism mathematically well-organized and physically meaningful. As mentioned above, in many works focused on specific lattice systems there are already ideas of ”redistribution”, and these efforts often turned to be rather fruitful. However without a universal theoretical foundation, these works still remained tentative to some degree. This article compactly and systematically provides this foundation, upon which the generalized Kawasaki dynamics becomes universal and can be directly applied to microscopic systems. Without any extra requirements, it has its advantage compared with some earlier approaches covering the same ground, such as numerical Ginzberg Landau approaches.

The formulation is compact in mathematics, while on the other hand it is also quite open. People are able to introduce other elements into it. For example, for decades there have been attempts of the application of a combined mechanism, Glauber type and Kawasaki type both with a probability [15]. We can easily give the mathematical form of the combined mechanism, single-spin transition plus spin-pair redistribution. In fact for Gaussian model we have already obtained exact and physically clear results, which will be reported in later paper as further application. For another example we may note the recent interest in the small-world network [16] effect on transportation. One can also modify the redistribution mechanism and directly study these effects in non-equilibrium statistical dynamics.

The formulation of the mechanism is the chief purpose of this article. As an example of its applications we obtained exact diffusion equations in kinetic Gaussian model, and a
temperature-dependent diffusion coefficient which becomes zero at the critical point. We observed critical slowing down phenomenon in diffusion process, and found that at least in this specific case the critical dynamic exponent is independent of space dimensionality and the dynamic mechanism.

Up to this point we have successfully generalized Glauber’s single-spin flipping mechanism to single-spin transition mechanism, and Kawasaki’s spin-pair exchange mechanism to spin-pair redistribution mechanism. These two generalizations are of similar mathematical form and become counterparts of each other in nonconserved dynamics and conserved dynamics respectively.

The diffusion in Gaussian model studied in this article is only an example of the problems previously we were unable to treat. Based on the redistribution mechanism (or transition), one can also easily write the master equation and redistribution probability for Potts model, XY model, Heisenberg model and many other types (in principle arbitrary), and obtain the evolution of local magnetization (and correlation function, etc.) in these systems. Though exact treatment may be difficult, other methods can be later applied based on this foundation. At the same time this mechanism can be almost directly applied to conserved processes other than those in spin-lattice models. For example, it can be used to study the relaxation of granular material under shaking (mass conserved), or the activities of particles in space (particle number conserved). A feasible way may be like this: first we divide the system into small units and define the parameter (and its range) and the system Hamiltonian. The redistribution is not necessarily limited to neighboring units (we can modify the master equation accordingly) and the probability may assume a form other than the heat Boltzmann factor (but it has to be normalized). Then we write the master equation and the evolving equations of the parameters we are interested in. This serves to unite the various conserved processes in microscopic, place them under the control of a universal mechanism, and provide the basis for further treatment, either exact, approximation, or Monte Carlo. The same is for the single-spin transition mechanism. We think it is very important for the general progress in non-equilibrium statistical dynamics.
ACKNOWLEDGMENTS

This work was supported by the National Natural Science Foundation of China under Grant No. 10075025.

APPENDIX A: EQUATION DERIVATION

\[
\frac{dq_k(t)}{dt} = \frac{d}{dt} \sum_{\{\sigma\}} \sigma_k P (\{\sigma\} ; t) \\
= \sum_{\{\sigma\}} \sum_{j,l} \left[ -\sigma_k W_{jl} \left( \sigma_j \sigma_l \rightarrow \tilde{\sigma}_j \tilde{\sigma}_l \right) P \left( \{\sigma\} ; t \right) \right. \\
+ \sigma_k W_{jl} \left( \tilde{\sigma}_j \tilde{\sigma}_l \rightarrow \sigma_j \sigma_l \right) P \left( \{\sigma_{i\neq j,l}\} ; \tilde{\sigma}_j, \tilde{\sigma}_l ; t \right) \right]
\]

Taking summation over all \(\{\sigma\}\), one will find the terms of those pairs unrelated to \(\sigma_k\) cancel with each other. So one will only have

\[
\frac{dq_k(t)}{dt} = \sum_{\{\sigma\}} \sum_{w} \left[ \sum_{\tilde{\sigma}_k, \tilde{\sigma}_{k+w}} \sigma_k W_{k,k+w} \left( \tilde{\sigma}_k \tilde{\sigma}_{k+w} \rightarrow \sigma_k \sigma_{k+w} \right) P \left( \sigma_1, \ldots, \tilde{\sigma}_k, \tilde{\sigma}_{k+w}, \ldots, \sigma_N ; t \right) \right]
\]

\[
= -2dq_k(t) + \sum_{\{\sigma\}} \sum_{w} \left[ \sum_{\tilde{\sigma}_k, \tilde{\sigma}_{k+w}} \sigma_k W_{k,k+w} \left( \tilde{\sigma}_k \tilde{\sigma}_{k+w} \rightarrow \sigma_k \sigma_{k+w} \right) P \left( \sigma_1, \ldots, \tilde{\sigma}_k, \tilde{\sigma}_{k+w}, \ldots, \sigma_N ; t \right) \right]
\]

\[
= -2dq_k(t) + \sum_{w} \sum_{\sigma_1, \ldots, \sigma_k, \sigma_{k+w}, \tilde{\sigma}_k, \tilde{\sigma}_{k+w}, \ldots, \sigma_N} \sigma_k W_{k,k+w} \left( \tilde{\sigma}_k \tilde{\sigma}_{k+w} \rightarrow \sigma_k \sigma_{k+w} \right) P \left( \sigma_1, \ldots, \tilde{\sigma}_k, \tilde{\sigma}_{k+w}, \ldots, \sigma_N ; t \right)
\]

\[
= 2dq_k(t) + \sum_{\{\sigma\}} \sum_{w} \left[ \sum_{\tilde{\sigma}_k, \tilde{\sigma}_{k+w}} \tilde{\sigma}_k W_{k,k+w} \left( \sigma_k \sigma_{k+w} \rightarrow \tilde{\sigma}_k \tilde{\sigma}_{k+w} \right) P \left( \{\sigma\} ; t \right) \right].
\]

APPENDIX B: CALCULATIONAL DETAILS

We give the details for one of them, and the remaining terms follow the same way. The spin pair redistribution probability can be expressed as
The spins take continuous value, the summation for spin value turns into the integration.

Thus this combined term in Eqs. (4) becomes

$$W_{i,j,k;i+1,j,k} (\sigma_{ijk} \sigma_{i+1,j,k} \rightarrow \hat{\sigma}_{ijk} \hat{\sigma}_{i+1,j,k})$$

$$= \frac{1}{Q_{i,j,k;i+1,j,k}} \delta_{\hat{\sigma}_{ijk} \hat{\sigma}_{i+1,j,k}, \sigma_{ijk} + \sigma_{i+1,j,k}}$$

$$\times \exp \left\{ K \left[ \delta_{\hat{\sigma}_{ijk}} (\sigma_{i-1,j,k} + \sigma_{i,j+1,k} + \sigma_{i,j-1,k} + \sigma_{i,j,k-1} + \sigma_{i,j,k+1}) + \hat{\sigma}_{ijk} \hat{\sigma}_{i+1,j,k} 
+ \hat{\sigma}_{i+1,j,k} (\sigma_{i,j+1,k} + \sigma_{i,j,k+1} + \sigma_{i,j,k+1} + \sigma_{i+1,j,k-1}) \right] \right\},$$

and there are similar expressions for $W_{i,j,k;i-1,j,k}$, $W_{i,j,k;i,j+1,k}$, and $W_{i,j,k;i,j+1,k}$.

Because the spins take continuous value, the summation for spin value turns into the integration

$$\sum_{\sigma} \rightarrow \int_{-\infty}^{\infty} f(\sigma) \, d\sigma;$$

and then the normalization factor

$$Q_{i,j,k;i+1,j,k}$$

$$= \frac{b}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\hat{\sigma}_{ijk} d\hat{\sigma}_{i+1,j,k} \delta_{\hat{\sigma}_{ijk} + \hat{\sigma}_{i+1,j,k}, \sigma_{ijk} + \sigma_{i+1,j,k}}$$

$$\times \exp \left\{ K \left[ \delta_{\hat{\sigma}_{ijk}} (\sigma_{i,j+1,k} + \sigma_{i-1,j,k} + \sigma_{i,j-1,k} + \sigma_{i,j,k-1} + \sigma_{i,j,k+1}) + \hat{\sigma}_{ijk} \hat{\sigma}_{i+1,j,k} 
+ \hat{\sigma}_{i+1,j,k} (\sigma_{i,j+1,k} + \sigma_{i,j,k+1} + \sigma_{i,j,k+1} + \sigma_{i+1,j,k-1}) \right] \right\} - \frac{b}{2} \sigma_{ijk}^2 - \frac{b}{2} \sigma_{i+1,j,k}^2 \right\}$$

$$= \frac{b}{2\pi} \sqrt{\frac{\pi}{b + K}} \exp \left\{ K (\sigma_{ijk} + \sigma_{i+1,j,k}) 
\times (\sigma_{i+1,j+1,k} + \sigma_{i+2,j+1,k} + \sigma_{i+1,j-1,k} + \sigma_{i+1,j,k+1} + \sigma_{i+1,j,k-1}) \right\}$$

Thus this combined term in Eqs. (4) becomes

$$\sum_{\delta_{\hat{\sigma}_{ijk} \hat{\sigma}_{i+1,j,k}}} \hat{\sigma}_{ijk} W_{i,j,k;i+1,j,k} (\sigma_{ijk} \sigma_{i+1,j,k} \rightarrow \hat{\sigma}_{ijk} \hat{\sigma}_{i+1,j,k})$$

$$= \frac{b}{2\pi} \frac{1}{Q_{i,j,k;i+1,j,k}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\hat{\sigma}_{ijk} d\hat{\sigma}_{i+1,j,k} \delta_{\hat{\sigma}_{ijk} + \hat{\sigma}_{i+1,j,k}, \sigma_{ijk} + \sigma_{i+1,j,k}}$$

$$\times \hat{\sigma}_{ijk} \exp \left\{ K \left[ \delta_{\hat{\sigma}_{ijk}} (\sigma_{i,j+1,k} + \sigma_{i-1,j,k} + \sigma_{i,j-1,k} + \sigma_{i,j,k-1} + \sigma_{i,j,k+1}) + \hat{\sigma}_{ijk} \hat{\sigma}_{i+1,j,k} 
+ \hat{\sigma}_{i+1,j,k} (\sigma_{i,j+1,k} + \sigma_{i,j,k+1} + \sigma_{i,j,k+1} + \sigma_{i+1,j,k-1}) \right] \right\} - \frac{b}{2} \hat{\sigma}_{ijk}^2 - \frac{b}{2} \hat{\sigma}_{i+1,j,k}^2 \right\}$$

13
\[
= \frac{1}{2(b+K)} \left[ K \left( \sigma_{i,j+1,k} + \sigma_{i-1,j,k} + \sigma_{i,j-1,k} + \sigma_{i,j,k+1} + \sigma_{i,j,k-1} + \sigma_{i+1,j,k} + \sigma_{i+1,j,k-1} + \sigma_{i+1,j,k+1} + \sigma_{i+1,j,k} - \sigma_{i+1,j-1,k} - \sigma_{i+1,j,k+1} - \sigma_{i+1,j,k} - \sigma_{i+1,j,k} \right) + b \left( \sigma_{ijk} + \sigma_{i+1,j,k} \right) \right].
\]
REFERENCES

[1] R. J. Glauber, J. Math. Phys. 4, 294 (1963).

[2] K. Kawasaki, Phys. Rev. 145, 224 (1966).

[3] K. E. Bassler and Z. Racz, Phys. Rev. Lett. 73, 1320 (1994)

[4] P. Fratzl and O. Penrose, Acta Materialia, 43, 2921 (1995); 44, 3227 (1996)

[5] Y. He and R. B. Pandey, Phys. Rev. Lett. 71, 565 (1993).

[6] A. Szolnoki, G. Szabo and O. G. Mouritsen, Phys. Rev. E 55, 2255 (1997).

[7] S. Weinketz, Phys. Rev. E 58, 159 (1998).

[8] M. Porta, C. Frontera, E. Vives and T. Castan, Phys. Rev. B 56, 5261 (1997).

[9] J. F. F. Mendes., S. Cornell, M. Droz and E. J. S. Lage, J. Phys. A 25, 73 (1992).

[10] R. A. Denny and M. V. Sangaranarayanan, Chem. Phys. Lett. 239, 131 (1995).

[11] P. Fratzl and O. Penrose, Phys. Rev. B 55 R6101 (1997).

[12] Jian-Yang Zhu and Z. R. Yang, Phys. Rev. E 59, 1551 (1999).

[13] Jian-Yang Zhu and Z. R. Yang, Phys. Rev. E 61, 210 (2000), Phys. Rev. E 61, 6219 (2000).

[14] Zhengping Zhang, Phys. Rev. E 51, 4155 (1995).

[15] There has been continuing study in this direction. Some of the recent works on Ising model are: Ma Yuqiang and Liu Jiwen, Phys. Lett. A 238, 159 (1998); Yu-Qiang Ma, Ji-Wen Liu and W. Figueiredo, Phys. Rev. E 57, 3625 (1998); Attila Szolnoki, Phys. Rev. E 62, 7466 (2000); B. C. S. Grandi and W. Figueiredo, Phys. Rev. E 53, 5484 (1996), 54, 4722 (1996), 56, 5240 (1997).

[16] It refers to the introduction of a certain amount of random long-range connections into
an initially regular network. There are a lot of interesting and unexpected characteristics in such structures. D. J. Watts and S. H. Strogatz, Nature 393, 440 (1998).