A Bayesian ILC Method for CMB B-mode Posterior Estimation and Reconstruction of Primordial Gravity Wave Signal

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Abstract

The cosmic microwave background (CMB) radiation B-mode polarization signal contains the unique signature of primordial metric perturbations produced during the inflation. The separation of the weak CMB B-mode signal from strong foreground contamination in observed maps is a complex task, and proposed new-generation low-noise satellite missions compete with the weak signal level of this gravitational background. In this paper, for the first time, we employ a foreground model-independent internal linear combination (ILC) method to reconstruct the CMB B-mode signal using simulated observations over large angular scales of the sky of six frequency bands of the future-generation CMB mission Probe of Inflation and Cosmic Origins (PICO). We estimate the joint CMB B-mode posterior density following the interleaving Gibbs steps of the B-mode angular power spectrum and cleaned map samples using the ILC method. We extend and improve the earlier reported Bayesian ILC method to analyze weak CMB B-mode reconstruction by introducing noise-bias corrections at two stages during the ILC weight estimation. By performing 200 Monte Carlo simulations of the Bayesian ILC method, we find that our method can reconstruct the CMB signals and the joint posterior density accurately over large angular scales of the sky. We estimate the Blackwell–Rao statistics of the marginal density of the CMB B-mode angular power spectrum and use them to estimate the joint density of the tensor-to-scalar ratio $r$ and a lensing power spectrum amplitude $A_{\text{tens}}$. Using 200 Monte Carlo simulations of the delensing approach, we find that our method can achieve an unbiased detection of the primordial gravitational-wave signal $r$ with more than $8\sigma$ significance for levels of $r \geq 0.01$ for input frequency maps with realistic lensing and foregrounds with constant spectral indices.

Unified Astronomy Thesaurus concepts: Cosmic microwave background radiation (322); Extragalactic astronomy (506); Cosmic inflation (319); Principal component analysis (1944)

1. Introduction

The discovery of the cosmic microwave background (CMB) in the second half of the last century led to an era of precision cosmology that resulted in the demise of many cosmological models and the survival of few. The current observations have put very stringent constraints (Planck Collaboration et al. 2020a, 2020b) on the various inflationary models (Martin et al. 2014), within the very successful inflationary paradigm for the origin of the primordial perturbations. The final Planck 2018 release (Planck Collaboration et al. 2020c) ruled out the perfect scale invariance for the spectral index of scalar perturbations at $8.4\sigma$. It negates the “running” and “running of the running” of the spectral index with 95% CL, consistent with the simplest slow-roll dynamics for the inflation. It also constrains the spatial curvature $\Omega_k$ to $-0.011 \pm 0.013$ at 95% CL. The BICEP2/Keck Array (BICEP2 Collaboration et al. 2016), together with Planck 2018, strongly disfavors monomial models with the inflationary potential $V(\phi) \propto \phi^p$, $p > 1$, natural inflation, and low-scale SUSY models. The observations have also established that the primordial perturbations are adiabatic to a very high degree (Planck Collaboration et al. 2020b), and the primordial power spectrum does not deviate from a pure power law (Planck Collaboration et al. 2020b). Further, the Planck likelihood, together with the B-mode polarization likelihood of the BICEP2/Keck Array, sets a stringent 95% CL upper limit of $r_{0.002} < 0.044$ (Tristram et al. 2021) corresponding to the energy scale of inflation of $V^{1/4} < 1.6 \times 10^{16}$ GeV with 95% CL bound.

To further constrain the cosmological models and probe the energy scale of the inflation and the existence of primordial gravitational waves, a new generation of CMB space missions, such as the Probe of Inflation and Cosmic Origins (PICO; Hanany et al. 2019), CORE (Delabrouille et al. 2018), LiteBIRD (Matsumura et al. 2014), and PIXIE (Kogut et al. 2011), have been proposed to detect the primordial CMB B mode on large angular scales at a level of $r < 10^{-3}$. The CMB B-mode signal given the current bound will have typical rms fluctuations $<0.1 \mu$K, which is 10 times weaker than the CMB E-mode signal, with the strong polarized Galactic foregrounds and instrumental systematic making their detection and reconstruction extremely difficult. To further add to the difficulties, gravitational lensing introduces spurious cosmic variance from the lens-induced B modes or the lensing B modes. The lensing B modes are due to the conversion of the CMB E mode to B mode due to weak gravitational lenses along the line of sight (Seljak & Hirata 2004). This biases not only the amplitude $r$ but also the cosmic variance of the primordial CMB B-mode power spectrum.

Over recent years, various studies dealing with foreground minimization in the context of the CMB B-mode sky have been undertaken (Baccigalupi et al. 2004; Betoule et al. 2009; Dunkley et al. 2009; Bonaldi & Ricciardi 2011; Katayama &Komatsu 2011; Armitage-Caplan et al. 2012; Errard et al. 2016; Remazeilles et al. 2016, 2018b; Hervías-Caimapo et al. 2017). Recently, new methods were proposed to investigate the joint posterior density of the CMB signal and corresponding theoretical angular power spectrum on large scales for CMB temperature (Sudevan & Saha 2018, 2020) and E-mode (Purkayastha et al. 2020, 2021) polarization.

In this paper, we extend (Sudevan & Saha 2020; Purkayastha et al. 2020) and develop a new nonparametric method by also
taking care of detector noise to reconstruct a clean CMB B-mode signal and corresponding angular power spectrum. We perform a joint analysis of the CMB B-mode signal and its angular power spectrum posterior density without considering any foreground model. Unlike the polarized CMB, we do not have an accurate enough model for the polarized galactic foreground, and the exact number of independent polarized foregrounds is not known (Remazeilles et al. 2016). Our nonparametric method avoids effects (Armitage-Caplan et al. 2012) due to inaccurate polarized Galactic foreground models. Using our method, we obtain the best-fit estimates of both the CMB B-mode map and its theoretical angular power spectrum, given the data, along with their confidence interval regions. We apply our Bayesian internal linear combination (ILC) method to recover the weak CMB B-mode signal from the simulated foreground and six noise-contaminated PICO frequency channels. We look into the ability of our Bayesian ILC method following the Gibbs procedure to reconstruct the primordial CMB B-mode signal and its theoretical angular power spectrum. We also perform correction for lensing bias in the CMB B-mode power spectrum without removing the lensing cosmic variance contribution to B modes. Throughout this paper, we call this correction of lensing bias from the power spectrum “delensing,” which is different from the delensing of maps (Remazeilles et al. 2018a). We use the samples of the theoretical CMB B-mode power spectrum generated at each Gibbs step of our method and simultaneously fit the amplitude of the primordial B-mode power spectrum parameter and the amplitude of lensing B-mode power spectrum in a Bayesian framework (Dickinson et al. 2009; Remazeilles et al. 2018a). The method removes the lensing bias on the posterior distribution of r and enables us to detect r with more than 8σ significance for a CMB B-mode satellite mission like PICO for r ≥ 0.01.

We organize our paper as follows. In Section 2, we illustrate the basic formalism of this work by describing our algorithm used to get the clean sky, angular power spectrum along with the Gibbs samples. In Section 3, we describe the procedure to get the foreground and noise-contaminated B-mode maps at six PICO frequencies. In Section 3.4, we discuss the method adopted to get the samples of the reconstructed CMB B-mode map and theoretical angular power spectrum along with the delensing procedure. In Section 4, we first present and discuss the results obtained for the cleaned map and angular power spectrum, and then we discuss and present the results for the delensing technique using the Blackwell–Rao approximation (Chu et al. 2005) to obtain the unbiased posterior distribution of r. Finally, in Section 5, we discuss and conclude.

2. Formalism

This section discusses the formalism used to estimate the joint posterior density of the CMB signal and its theoretical angular power spectrum given the observed data. We adopt and improve the formalism used in this work for component separation as in Sudevan & Saha (2020) and Purkayastha et al. (2020) so that it applies to the weak CMB B-mode signal. The method not only gives us the best-fit CMB B-mode map and its best-fit power spectrum, it also provides Markov Chain Monte Carlo Gibbs samples for a theoretical power spectrum $C_{\ell}^B$, which we utilize to delens the angular power spectrum and estimate the tensor-to-scalar ratio $r$.

2.1. Data Model

The observations of the linear polarization of the CMB are described using the Stokes parameters $Q$ and $U$, which are coordinate-dependent quantities. If polarization is represented by a complex number $P = Q + iU$, then under a rotation of the coordinate axes by an angle $\alpha$, $P \rightarrow Pe^{\pm 2i\alpha}$, implying that it is a spin-2 field (Hu & White 1997; Kamionkowski & Kovetz 2016). This spin-2 field, $P$, can be represented on the surface of a sphere using spin-2 basis functions $2Y_{\ell m}(k)$,

$$Q(k) \pm iU(k) = \sum_{\ell m} \pm 2a_{\ell m} Y_{\ell m}(k),$$

where $k$ is the pixel index and $\pm 2a_{\ell m}$ represents the $\pm 2$ spherical harmonics. We can define a spin-0 CMB B-mode map, which is coordinate-independent, as follows:

$$B(k) = \sum_{\ell m} a_{\ell m}^B Y_{\ell m}(k),$$

where

$$a_{\ell m}^B = \frac{1}{2}(\pm 2a_{\ell m} - \mp 2a_{\ell m}),$$

and $Y_{\ell m}(k)$ represents the spin-0 spherical harmonics.

In this paper, we work in the spin-0 spherical harmonics basis by first transforming the full-sky $Q$ and $U$ maps for each frequency channel to the corresponding B-mode maps. This avoids the additional spin-2 spherical harmonic transformations necessary to obtain the B-mode angular power spectrum at each Gibbs iteration and reduces the total disk storage requirement by half. Since we use full-sky $Q$, $U$ maps for converting to B-mode maps, there is no leakage between the E- and B-mode signals.

Assuming we have observations of the CMB B-mode signal $S$ at $n$ different frequencies in thermodynamic temperature units, we can write for an observed $i$th frequency map $X_i$,

$$X_i = S + F_i + N_i,$$

where $F_i$ is the net foreground contribution from all of the foreground components at the $i$th frequency channel, and $N_i$ is the corresponding detector noise. Each of the above boldfaced quantities is a column vector of size $N_{\text{pix}}$ representing a HEALPix map where $N_{\text{pix}} = 12N_{\text{side}}^2$, $N_{\text{side}}$ being the pixel resolution parameter, having a common beam and pixel resolution. Let $D$ denote the observed data set, i.e., $D = \{X_1, X_2, \ldots, X_n\}$.

2.2. CMB Posterior Estimation

Given the observed data, $D$, $P(S, C_{\ell}^B | D)$ represents the joint density of the CMB B-mode map, $S$, and the theoretical CMB B-mode angular power spectrum, $C_{\ell}^B$. As it is difficult to obtain the $P(S, C_{\ell}^B | D)$ analytically, we evaluate it by drawing samples from it. If we can sample from the conditional distributions $P(S | C_{\ell}^B, D)$ and $P(C_{\ell}^B | S, D)$ then utilizing Gibbs sampling approach, as discussed in Gelman & Rubin (1992), which says that samples $(S', C_{\ell}^{B'})$ can be drawn from the joint distribution $P(S, C_{\ell}^{B'})$ by iterating the following symbolic sampling

\footnote{Hierarchical Equal Area Isolatitude Pixellization of the sphere; e.g., see Górski et al. (2005).}
equations:
\[ S^{i+1} \leftarrow P(S | C^B_i, D), \]  
\[ C^B_{i+1} \leftarrow P(C^B_i | S^{i+1}). \]  
The symbol “\[ \rightarrow \]” implies that a sample of corresponding variables is drawn from the distribution on the right-hand side. Once the initial burn-in period is over, the samples will converge to being drawn from the required joint distribution.

2.2.1. Sampling the CMB Signal

We use a foreground model-independent method to draw samples of \( S \) given the CMB B-mode theory \( C^B_i \) and \( D \). We modify the global ILC method described in Sudevan & Saha (2018) and Purkayastha et al. (2020) to improve separating the weak CMB B-mode signal given the detector noise model. Let us assume that the mean corresponding to each frequency map \( \bar{X}_i \), as discussed in Section 2.1, has already been subtracted. The cleaned CMB B-mode map \( S \) can be obtained by a linear combination of \( n \) input maps \( \bar{X}_i \) with the weight factor \( w_i \), i.e.,
\[ S = \sum_{i=1}^{n} w_i \bar{X}_i = wX^T, \]  
where \( w = (w_1, w_2, \ldots, w_n) \) is a \( 1 \times n \) weight row vector and \( X = (\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_n) \) is an \( N_{\text{pix}} \times n \) matrix with each entry as a pixel map at a different frequency. Since the spectral distribution of CMB photons is a blackbody to an excellent approximation, the CMB anisotropy signal \( S \) (in thermodynamic temperature units) is independent of the frequency channel. In order to avoid multiplicative bias in the amplitudes of the CMB anisotropies, the sum of the weights is a constraint to unity, i.e., \( \sum_{i=1}^{n} w_i = 1 \). As discussed in Sudevan & Saha (2018), instead of minimizing the clean map variance \( S^T S \), we minimize
\[ \sigma^2 = S^T C^i S, \]  
where \( C \) represents the CMB B-mode theoretical covariance matrix, and \(^i\) denotes the Moore–Penrose generalized inverse (Penrose 1955). Using Equation (4) in Equation (5), we write
\[ \sigma^2 = W A W^T, \]  
where \( A \) is an \( n \times n \) matrix with its element \( A_{ij} \) given by
\[ A_{ij} = X_i^T C^i X_j. \]  
Equation (7) is simpler to compute:
\[ A_{ij} = \sum_{\ell=2}^{\ell_{\text{max}}} (2\ell + 1) \frac{\sigma_{ij}^{\ell}}{C_i^{\ell}}, \]  
where \( \ell_{\text{max}} \) denotes the maximum multipole used in the analysis, \( \sigma_{ij}^{\ell} \) denotes the angular cross-power spectrum between the \( \bar{X}_i \) and \( \bar{X}_j \) channel maps, and \( C_i^{\ell} \) represents the beam- and pixel-smoothed CMB B-mode theoretical power spectrum, i.e.,
\[ C_i^{\ell} = C_i^{BB} B_i^{\ell} P_i^{2}, \]  
where \( C_i^{BB} \) has no smoothing effect, and \( B_i^{\ell} \) and \( P_i \) are, respectively, the polarization beam and polarization pixel window functions. The ILC method for component separation performs well only in a low-noise environment, and since the CMB B-mode signal is even weaker than the CMB E mode by an order of magnitude, to minimize the residual noise bias in the output CMB–B-mode signal we initially subtract the noise autopower from the input frequency cross-power spectrum,
\[ A_{ij} = \sum_{\ell=2}^{\ell_{\text{max}}} (2\ell + 1) \frac{1}{C_i^{\ell}} (\sigma_{ij}^{\ell} - \delta_{ij} \sigma_{N,i}^{\ell}), \]  
where \( \sigma_{N,i}^{\ell} \) is the noise autopower corresponding to the detector at the \( i \)th frequency. We use the matrix \( A \), the components for which are given by Equation (11), in Equation (8) to obtain the row vector \( w \). We use the weights obtained to sample the foreground-minimized CMB B-mode signal \( S \) by linearly combining the input channel maps \( \bar{X}_i \) at every Gibbs step following Equation (4).

2.2.2. Sampling \( C_i^{BB} \)

The signal sample \( S \) can be represented mathematically in terms of spherical harmonics,
\[ S(\theta, \phi) = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} s_{lm} Y_{lm}(\theta, \phi), \]  
and then the realization-specific power spectrum is given by
\[ \hat{\delta}^{BB}_{\ell} = \frac{1}{2\ell + 1} \sum_{m=-\ell}^{\ell} |s_{\ell m}|^2. \]  
In order to minimize the noise bias in the sampled theory \( C_i^{BB} \), we further subtract the weighted noise power from \( \hat{\delta}^{BB}_{\ell} \) to obtain
\[ \hat{\delta}^{B}_{\ell} = \hat{\delta}^{BB}_{\ell} - \sum_{i=1}^{n} w_i^2 \sigma_{N,i}^{\ell}. \]  
Other than using Equations (11) and (14), one could also Gibbs sample the noise power spectrum, given the data. Gibbs sampling of the noise power spectrum can be performed provided a model for the noise power spectrum is assumed a priori. An alternative method of implementation would be to use cross-spectra for the noise-bias removal. But this would increase the noise variance of the input maps by a factor of 2, since the overall data are now divided into two sets with independent detector noise properties. We do not follow these approaches in the current paper. We will explore them as a possible route to noise-bias correction in a future paper.
The power spectrum only depends on the signal $S$ through $\hat{\sigma}_B^2$, not its phases; therefore, to draw samples of $C_B^l$ given $S$, we sample from $P(C_B^l | \hat{\sigma}_B^2)$. The conditional density $P(C_B^l | \hat{\sigma}_B^2)$ can be written (Sudevan & Saha 2020) as

$$P(C_B^l | \hat{\sigma}_B^2) = \left( \frac{1}{C_B^l} \right)^{(2l+1)/2} \exp \left[ -\frac{\hat{\sigma}_B^2 (2l + 1)}{2C_B^l} \right].$$  \hspace{1cm} (15)

In order to draw samples of $C_B^l$ using Equation (15), we first need to draw $x = \hat{\sigma}_B^2 (2l + 1)/C_B^l$, which is a $\chi^2$ distributed random variable having $2l + 1$ degrees of freedom. For this, we draw $2l + 1$ independent normal variables and then sum their squares. For a given $S$, we have estimates of $\hat{\sigma}_B^2$, and we can then obtain $C_B^l$ using $C_B^l = \hat{\sigma}_B^2 (2l + 1)/x$.

3. Frequency Maps

In this work, we simulate the foreground and noise-contaminated CMB B modes at the six least noisy frequency bands of the proposed satellite mission PICO in the frequency range 90–268 GHz. This will help reconstruct an unbiased estimator of $r$; at the same time, it will also help minimize the noise error on the reconstructed value of $r$. We list the frequency channel maps along with their instrumental specifications in Table 1.

### Table 1

| Frequency (GHz) | Beam FWHM (arcmin) | $Q$ and $U$ Noise rms ($\mu K_{\text{CMB}}$ arcmin) |
|----------------|-------------------|---------------------------------|
| 90            | 9.5               | 2.09                            |
| 108           | 7.9               | 1.70                            |
| 129           | 7.4               | 1.53                            |
| 155           | 6.2               | 1.28                            |
| 186           | 4.3               | 3.54                            |
| 268           | 3.2               | 2.63                            |

We generate foreground maps with constant spectral indices at all six PICO frequencies used in this work, corresponding to synchrotron and thermal dust components, which are two significant contributors to CMB polarized foreground. To generate them, we follow a procedure similar to that of Remazeilles et al. (2018a). We generate $Q$ and $U$ maps at each frequency and use them to obtain the corresponding B-mode maps for both foregrounds.

To generate the polarized Galactic synchrotron Stokes maps, we extrapolate the Wilkinson Microwave Anisotropy Probe (WMAP) 23 GHz (Page et al. 2007; Bennett et al. 2013) Stokes maps $Q_{23}$ and $U_{23}$ to the six PICO frequencies through a power-law frequency dependence:

$$Q_\nu^{\text{sync}}(p) = Q_{23}(p) \left( \frac{\nu}{23 \text{ GHz}} \right)^{\beta_q},$$  \hspace{1cm} (16)

$$U_\nu^{\text{sync}}(p) = U_{23}(p) \left( \frac{\nu}{23 \text{ GHz}} \right)^{\beta_u}.$$  \hspace{1cm} (17)

We use a constant spectral index $\beta_q = -3$, which is close to the typical mean values measured at CMB frequencies (Davies et al. 1996; Kogut et al. 2007; Miville-Deschênes et al. 2008; Dickinson et al. 2009; Bennett et al. 2013; Planck Collaboration et al. 2016b), and $p$ is the pixel index.

To simulate the Galactic polarized thermal dust Stokes maps, we extrapolate the generalized needlet ILC (GNILC) Planck 353 GHz thermal dust optical depth map (Planck Collaboration et al. 2016d) to the relevant PICO frequencies,

$$Q_\nu^{\text{dust}}(p) = f_d g_d(p) I_{\nu}^{\text{GNILC}}(p) \cos(2\gamma_p(p)),$$  \hspace{1cm} (18)

$$U_\nu^{\text{dust}}(p) = f_d g_d(p) I_{\nu}^{\text{GNILC}}(p) \sin(2\gamma_p(p)),$$  \hspace{1cm} (19)

where $f_d$ is the pixel-independent intrinsic dust polarization fraction that depends on the level of depolarization along the line of sight. Following Delabrouille et al. (2013) and Remazeilles et al. (2018a), we take it to be 0.15, where $g_d$ is the pixel-dependent geometric depolarization factor. To compute the polarization angle $\gamma_p$ (Delabrouille et al. 2013) at each pixel, we use the WMAP 23 GHz map after smoothing with a Gaussian beam of 3°,

$$\gamma_p(p) = \frac{1}{2} \tan^{-1} \left( \frac{-U_{23}(p)}{Q_{23}(p)} \right).$$  \hspace{1cm} (20)

We compute the depolarization factor $g_d$ using WMAP 23 GHz and the residual monopole-subtracted 408 MHz Haslam synchrotron template ($I_{408}$, 3) extrapolated to 23 GHz assuming a constant spectral index of $-3.0$. To compute it, we smooth the extrapolated map to a Gaussian beam of 3° at $N_{\text{side}} = 512$ and use

$$g_d(p) = \frac{\sqrt{Q_{23}^2(p) + U_{23}^2(p)}}{f_d I_{408}(p)(23.0/0.408)^{3\theta}}.$$  \hspace{1cm} (21)

where, for the spectral index used in above equation, the synchrotron polarization fraction $f_d = 0.75$. The $I_{\nu}^{\text{GNILC}}$ is the GNILC dust intensity map free from the cosmic infrared background at the frequency $\nu$ and is given by the modified blackbody spectrum,

$$I_{\nu}^{\text{GNILC}}(p) = \tau_{353}^{\text{GNILC}}(p) \left( \frac{\nu}{353 \text{ GHz}} \right)^{3.8} B_{\nu}(T_d),$$  \hspace{1cm} (22)

where $\tau_{353}^{\text{GNILC}}$ is the Planck GNILC dust optical depth at 353 GHz, the dust emissivity is $\beta_q = 1.6$, $T_d = 19.4$ K is the dust temperature, and $B_{\nu}(T_d)$ is the Planck function at the
thermal dust temperature \( T_d \) given by
\[
B_\nu(T_d) = \frac{2h^3}{c^2} \frac{1}{\exp \left( \frac{h}{kT_d} \right) - 1}.
\]

We use the above-obtained synchrotron and thermal dust Stokes \( Q \) and \( U \) maps to get B-mode synchrotron and thermal dust foreground maps at each of the PICO frequencies used in this paper at \( N_{\text{side}} = 16 \). We smooth the obtained maps by a polarized Gaussian beam of FWHM 9°.

### 3.3. Detector Noise Simulations

We simulate Gaussian, isotropic, and pixel-uncorrelated random realizations of detector \( Q \) and \( U \) noise maps for the six PICO (Young et al. 2018) frequency bands used in this work. We present detector specifications for each of the bands in Table 1. We further assume that the \( Q \) and \( U \) noise maps are pixel-uncorrelated, i.e.,
\[
\langle Q_i(p) U_i(p') \rangle = 0.
\]

We further assume that the pixel noise variances \( \sigma^2_{Q_i} \) for \( Q \) and \( \sigma^2_{U_i} \) for \( U \) maps at a frequency \( \nu_i \) are identical and given by
\[
\sigma^2_{Q_i} = \sigma^2_{U_i} = (\zeta \Delta T_d^2)/(\Delta \Omega),
\]
where \( \Delta T_d \) is the noise rms in arcminutes for the \( Q_i \) map, \( \zeta \) is the conversion factor from arcminutes to radians, and \( \Delta \Omega \) is the solid angle subtended by a single pixel at \( N_{\text{side}} = 16 \). We bring both the \( Q \) and \( U \) noise maps to the same beam resolution at \( N_{\text{side}} = 16 \) by multiplying the ratio of a polarized Gaussian beam of FWHM 9° and the polarized beam as given in Table 1 for the corresponding frequency channel. We finally convert the noise Stokes maps obtained to the full-sky B-mode noise map at each of the frequencies. In Figure 1, we show the PICO detector model noise power corresponding to the six channels, along with the lensed CMB B-mode theoretical power for \( r = 0.01 \) and 0.05. We can see that the noise power for all of the used frequency is well below the B-mode signal power spectrum at all multipoles used in this work. Finally, we obtain the simulated PICO input noisy foreground-contaminated CMB B-mode maps by combining all three components for each of the six frequencies using Equation (1).

### 3.4. Methodology

We implement our model-independent method on the simulated foreground and noise-contaminated B-mode maps obtained above after removing the monopole and dipole components from each of them. We smooth the theoretical \( C^B_\ell \) obtained from CAMB using \( \tau = 0.055 \), \( A^{\text{tens}} = 1 \), and the Planck 2015 best-fit values by a Gaussian beam of 9° and a polarization pixel window function corresponding to \( N_{\text{side}} = 16 \) as in Equation (10). Following the discussion above, we obtain 10,000 sky and power spectrum samples for each simulation. We perform this for 10 independent chains. We discard the initial 50 samples for the burn-in period in each chain. In total, we obtain 99,500 samples of \( C^B_\ell \) and \( S \). We perform this analysis on cases with 0.01 and 0.05 tensor-to-scalar ratios.

Using the samples obtained after applying our method, we forecast the proposed CMB space mission PICO’s ability to constrain \( r \) in the presence of realistic lensing and foregrounds with constant spectral indices. We simulate 200 different noise- and foreground-contaminated Gaussian random CMB B-mode realizations as described above in this section and apply the Gibbs ILC method to obtain 99,500 samples of \( C^B_\ell \) and \( S \) for each of them. We use a set of samples \( \{ C^B_{\ell i} \} \) to compute the posterior distribution of the tensor-to-scalar ratio, \( r \), and the amplitude of lensing, \( A^{\text{tens}} \), using the Blackwell–Rao estimator (Chu et al. 2005) for each of the 200 cases. We use sampled \( C^B_{\ell i} \) to obtain the best-fit value of the power spectrum for all 200 simulations and use them to study the bias in the recovered power spectrum. We also obtain a mean map and study the
reconstruction error in the recovered CMB B-mode maps using our method.

4. Results

This section presents the results obtained after applying our method to the simulated foreground- and noise-contaminated CMB B-mode map at six frequency channels of the proposed future CMB mission PICO with fiducial tensor-to-scalar ratios 0.05 and 0.01. We present our method’s ability to reconstruct the CMB B-mode map, CMB B-mode angular power spectrum, and power spectrum delensing in the following.

4.1. Cleaned Maps

In this subsection, we present the ability of our method to reconstruct the CMB B-mode maps. In Figures 2 and 3, we show the pixel standard deviation maps obtained using 200 CMB signal reconstruction following our method for \( r = 0.05 \) and 0.01, respectively. The second-to-last map at the bottom right labeled MEAN shows the mean of all 200 standard deviation maps obtained from the 200 simulations. The last map at the bottom right labeled STDEV shows a standard deviation map obtained using the 200 standard deviation maps from the 200 simulations. In all of the maps, the standard deviation’s maximum value is well within \( 10^{-3} \mu K \), which indicates accurate CMB signal reconstruction. The unit is in \( \mu K \) thermodynamic.

In Figure 4, we show the mean of 200 difference maps for both \( r \) values. We find a mean map using 99,500 samples of the map from a given simulation and subtract the input map to obtain the difference map corresponding to the simulation. From the mean difference maps, we find a small variation of order \( \leq 10^{-6} \mu K \) in pixel reconstruction error from one simulation to another for both cases of tensor-to-scalar ratios.

4.2. Angular Power Spectrum

In this subsection, we present our method’s ability to reconstruct the CMB B-mode angular power spectrum. We present normalized densities of the Gibbs samples of the CMB theoretical angular power spectrum given the observed data from multipoles 2–31, along with the input angular power spectrum (vertical black dashed line) and best-fit angular power spectrum (vertical red dashed line) for a randomly chosen simulation seed 1 with \( r = 0.05 \) in Figure 5. In the figure, the positions of most of the histogram peaks agree well with the input sky angular power spectrum. The deviation of the input angular power spectrum from the peaks in some of the histograms is due to presence of residual detector noise and foregrounds in the sampled \( C_\ell \). The plots in Figure 5 confirm the expected behavior of the \( C_\ell \) histograms at both low and high multipoles. For the tensor-to-scalar ratio of 0.05, we present in the top panel of Figure 6 the mean over 200 simulations of the input and best-fit angular power spectra along with the corresponding standard deviations to quantify...
the reconstruction error in the CMB B-mode angular power spectrum. We also plot in the bottom panel of Figure 6 the mean over 200 simulations of the difference between the best-fit and input angular power spectra along with the corresponding standard deviations to further quantify the reconstruction error in the recovered CMB B-mode angular power spectrum. From the top panel in Figure 6, we find that the mean input and best-fit power spectra agree very well for \( r = 0.05 \).

From the bottom panel of Figure 6, we find that the mean over the simulations of the absolute power reconstruction error at each multipole is \(<7 \times 10^{-6} \, \mu K^2\) for \( r = 0.01 \). We plot in Figure 9 the fractional bias \( \Delta C_{\ell}^{fb} \) in the recovered angular power spectrum calculated using 200 best-fit and input angular power spectra, defined as

\[
\Delta C_{\ell}^{fb} = \frac{\langle C_{\ell}^{\text{best-fit}} \rangle - \langle C_{\ell}^{\text{input}} \rangle}{\langle C_{\ell}^{\text{input}} \rangle}.
\]

From Figure 9, we find 2%–3% more bias in the reconstructed power spectrum for simulated CMB B-mode maps with a tensor-to-scalar ratio of 0.01 than 0.05. The fractional cosmic variance is the ratio of the cosmic variance and the theory power spectrum for each multipole. The fractional cosmic variance ranges from 17% to as large as 63% for \( \ell = 32 \) to 2, respectively. With respect to the fractional cosmic variance, we find that our method does not have significant bias, even when \( r = 0.01 \). This shows that our method performs very well in reconstructing the CMB B-mode angular power spectrum for both cases.

### 4.3. Reconstructing r

Using a set of Gibbs samples \( C_{\ell}^{B} \) and the Blackwell–Rao estimator (Chu et al. 2005) in a self-consistent Bayesian framework, we compute the joint posterior distribution \( P(r, A^{\text{ens}}) \) of the tensor-to-scalar ratio \( r \) and the amplitude of lensing \( A^{\text{ens}} \). To estimate the cosmological parameter \( r \) and
of the tensor-to-scalar ratio $P(\tau)$ (Dickinson et al. 2009; Remazeilles et al. 2018a). For a fiducial tensor-to-scalar ratio of 0.05, we plot in Figure 10 the normalized joint 2D Blackwell–Rao posterior density estimates $P(r, A^{\text{lens}})$ along with the normalized posterior distribution $P(\tau)$. We get $P(\tau)$ by slicing the joint 2D Blackwell–Rao posterior density $P(r, A^{\text{lens}})$, for a maximum likelihood of $A^{\text{lens}}$, using a set of Gibbs samples $\{C^{B,i}_\ell\}$ for each of the 200 different simulations. Since the true value of 0.05 is within 1σ of the normalized posterior, we conclude that our method performs well in reconstructing the angular power spectrum for $r = 0.05$ and hence delensing the angular power spectrum. Similarly, we plot the normalized joint 2D Blackwell–Rao posterior density $P(r, A^{\text{lens}})$ in the left panel and the posterior distribution $P(\tau)$ in the right panel of Figure 11 for a fiducial tensor-to-scalar ratio of 0.01. We find that the true value of 0.01 is within 1σ of the normalized posterior $P(\tau)$, establishing that our method also performs well in reconstructing the angular power spectrum for $r = 0.01$ and delensing the angular power spectrum.

To show the convergence of the posterior $P(\tau|D)$, we show the product of the 200 posteriors in Figures 12 and 13 as a shaded gray band for fiducial tensor-to-scalar ratios of 0.05 and 0.01, respectively. Since the fiducial value of $r$ in both cases is within the corresponding gray band’s width, we conclude that the chains converge, and our method is correct. We also show the posterior for some randomly chosen simulations, which we normalize arbitrarily in each of the plots to fit on the axes.

5. Discussions and Conclusion

We develop a new foreground model-independent approach to measure the CMB $B$-mode signal and angular power spectrum using simulated observations for the proposed future-generation PICO satellite mission in this work. Our new nonparametric method is useful, since spatial and spectral variations of the polarized foreground component may not be accurately known. In this paper, we extend and improve the earlier reported Bayesian ILC method to reconstruct weak CMB $B$-mode signals by introducing noise-bias corrections at two stages during the ILC weight estimation. We extensively test our new method’s ability to reconstruct the CMB $B$-mode sky signal and angular power spectrum and obtain the joint distribution of the tensor-to-scalar ratio and lensing amplitude for two different values of $r$. Proposed future-generation CMB $B$-mode missions, like PICO, can break the power spectrum degeneracy between the primordial and lensing $B$ modes by detecting the reionization bump. Utilizing this advantage, we further perform the delensing of the recovered CMB $B$-mode angular power spectrum. We use Gibbs samples of the CMB $B$-mode theoretical angular power spectrum obtained from our method to separate the primordial and lensing $B$-mode contribution to the recovered distribution of the tensor-to-scalar ratio in a Bayesian manner and have quantified the performance of our method at two different tensor-to-scalar ratios.

We summarize the findings of our method as follows.

1. From the mean standard deviation maps for $r = 0.05$ and 0.01, we find that the reconstruction bias along the galactic plane is $\lesssim 10^{-3}$ $\mu$K. From the standard deviation over the simulations of the standard deviation, we find a small variation of order $\lesssim 10^{-6}$ $\mu$K in the pixel reconstruction error from one simulation to another for both the
Figure 5. Normalized densities of the sampled CMB theoretical angular power spectrum obtained by Gibbs sampling for 2–31 multipoles for simulation seed 1 with $r = 0.05$. The horizontal axis for each subplot represents $\ell (\ell + 1) C_\ell / 2 \pi$ in units of $10^{-3} \mu K^2$. The histogram gives us the best estimates of the theoretical $C_\ell$ (vertical black dashed line) given the data. The vertical red dashed line is the value corresponding to input sky $C_\ell$. The positions of most of the histogram peaks agree well with the input sky $C_\ell$. The deviation of the input $C_\ell$ from the peaks in some of the histograms is due to presence of the residual detector noise and foreground along with the CMB. The plots confirm the expected behavior of the $C_\ell$ histograms at both low and high $\ell$.

Figure 6. For the tensor-to-scalar ratio $r = 0.05$, we plot the mean input and best-fit angular power spectra and corresponding standard deviation for 200 simulations in the top panel. We show the mean difference angular power spectrum and associated errors in the bottom panel. To get the difference angular power spectrum for a given simulation, we subtract the corresponding input power spectrum from the best-fit angular power spectrum. We find that the mean input and best-fit power spectra agree very well from the top panel. In the bottom panel, the mean over the simulations of the absolute power reconstruction error at each multipole is $< 2 \times 10^{-5} \mu K^2$. The small absolute reconstruction error at each multipole shows that our method performs very well in reconstructing the angular power spectrum.
Figure 7. Normalized densities of the sampled CMB theoretical angular power spectrum obtained by Gibbs sampling for 2–31 multipoles for simulation seed 1 with \( r = 0.01 \). The horizontal axis for each panel represents \( \ell (\ell + 1) C_\ell / 2\pi \) in units of \( 10^{-4} \mu K^2 \). The vertical black dashed line is the best-fit value, and the vertical red dashed line is the value corresponding to input \( C_\ell \). The positions of most of the histogram peaks agree well with the input sky \( C_\ell \). The deviation of the input \( C_\ell \) from the peaks in some of the histograms is due to the presence of the residual detector noise and foreground along with the CMB. The figure confirms the expected behavior of the \( C_\ell \) histograms at both low and high \( \ell \).

Figure 8. In the top panel, we show the mean input and best-fit angular power spectra and corresponding error bars. We show the mean difference angular power spectrum in the bottom panel and corresponding error bars. To get the difference angular power spectrum for a given simulation, we subtract the corresponding input power spectrum from the best-fit angular power spectrum. We find that the mean best-fit power spectrum from the top panel is slightly more than the mean input power spectrum at multipoles <7. In the bottom panel, the mean over the simulations of the absolute power reconstruction error at each multipole is \( <7 \times 10^{-6} \mu K^2 \). The small absolute reconstruction error at each multipole shows that our method performs very well in reconstructing the angular power spectrum, and there is no significant bias in angular power spectrum reconstruction for \( r = 0.01 \).
0.05 and 0.01 tensor-to-scalar ratios. We also find that the mean over the simulations of the absolute pixel reconstruction error is very small \((\approx 10^{-5} \mu K)\) for both cases. In light of the above, we conclude that our method accurately reconstructs the simulated primordial CMB B-mode sky for both the \(r = 0.05\) and 0.01 cases.

2. We find that the mean input and best-fit power spectra agree very well for the \(r = 0.05\) case, whereas for \(r = 0.01\), there is slightly more mean power in the reconstructed power spectrum at multipoles \(<7\). Using a fractional bias to quantify this positive bias, we find it to be only 2%–3% more for the case with \(r = 0.01\) than \(r = 0.05\), which indicates that our method does not have significant bias, even for \(r = 0.01\), and performs very well in reconstructing the CMB B-mode angular power spectrum for both cases.

3. In this work, we estimate the joint posterior for the CMB B-mode signal and its theoretical angular power spectrum over large angular scales. We also obtain the appropriate confidence intervals for the theoretical angular power spectrum necessary for cosmological parameter estimation. This is the first demonstration of the reconstruction of the CMB B-mode signal using an ILC approach following a Bayesian framework.

4. On fitting both \(r\) and \(A_{\text{ens}}\), we find the fiducial tensor-to-scalar values to be within \(1\sigma\) of the recovered distribution...
converges for a maximum likelihood of \( A_{\text{lens}} \). The vertical black dashed line represents the fiducial tensor-to-scalar ratio of 0.01. We arbitrarily normalize the posteriors to the product of the posteriors from all 200 simulations with the tensor-to-scalar ratio of 0.05. We expect that the gray band covers the fiducial value of a tensor-to-scalar ratio of 0.01, showing that the probability that the true values of \( r \) were 0.05 and 0.01, respectively, at low resolutions without using any \( A_{\text{lens}} \) prior.

5. Our method does not explicitly require a foreground model. Thus, any error due to an incorrect foreground B-mode model does not bias our results. Further, for an alternative model of foregrounds, for example, using synchrotron and thermal dust with varying spectral indices is expected to lead to further complexities in our foreground removal method. For foregrounds having varying spectral indices, we expect additional errors in determining \( r \) with our current method. In a future paper, we will investigate any other bias or larger uncertainty due to such complex foreground models.

6. Our method is computationally fast, efficient, and accurate in delensing and detecting significant unbiased detection for levels of \( r \gtrsim 10^{-2} \).

We use the publicly available HEALPix (Górski et al. 2005) package to perform spherical harmonic decomposition and for visualization purposes. We acknowledge the use of the Legacy Archive for Microwave Background Data Analysis (LAMBDA) and Planck Legacy Archive (PLA). LAMBDA is a part of the High Energy Astrophysics Science Archive Center (HEASARC), and the PLA contains all public products originating from the Planck mission, an ESA science mission with instruments and contributions directly funded by ESA Member States, NASA, and Canada. This research has made use of NASA’s Astrophysics Data System.

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