PRICING AMERICAN OPTIONS UNDER PROPORTIONAL TRANSACTION COSTS USING A PENALTY APPROACH AND A FINITE DIFFERENCE SCHEME

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Abstract. In this paper we propose a penalty method combined with a finite difference scheme for the Hamilton-Jacobi-Bellman (HJB) equation arising in pricing American options under proportional transaction costs. In this method, the HJB equation is approximated by a nonlinear partial differential equation with penalty terms. We prove that the viscosity solution to the penalty equation converges to that of the original HJB equation when the penalty parameter tends to positive infinity. We then present an upwind finite difference scheme for solving the penalty equation and show that the approximate solution from the scheme converges to the viscosity solution of the penalty equation. A numerical algorithm for solving the discretized nonlinear system is proposed and analyzed. Numerical results are presented to demonstrate the accuracy of the method.

1. Introduction. In a financial market, an option is a contract in which one party (the writer) sells to another party (the holder) the right, but not the obligation, to buy (call option) or sell (put option) a specified amount of an underlying asset such as a stock at a fixed price (the exercise or strike price) on or before a given date (expiry date). There are two major types of options: European options and American options. A European option gives the holder the right to buy (for a call option) or to sell (for a put option) the underlying asset at the strike price on the expiry date, while an American option, in contrast to a European option, allows the holder to exercise it at any time before or on the expiry date.

Options are tradable in a financial market. Thus, how to determine the price of an option is an important topic in financial engineering. In a complete market, Black and Scholes [2] used a no-arbitrage argument to price a European option on a stock under the conditions that the interest rate is constant, the underlying stock price is a geometric Brownian motion and there are no transaction costs on trading the underlying stock and the bond. The Black-Scholes model has been widely used for valuing simple European options. However, in the presence of transaction costs on trading in the bond or/and stock, the Black-Scholes option pricing methodology is no longer valid. In the open literature, there are four main approaches to the problem of option pricing with transaction costs (see, for example,
One of them is the utility based option pricing approach which is an optimal portfolio selection problem to maximize an investor’s expect utility of terminal wealth.

The utility based option pricing approach was first proposed in [15] to price European options under proportional transaction costs. It was further developed in [10] in which the authors showed that computing the reservation price of a European option involves solving two singular stochastic optimal control problems and the value functions of these problems are the unique viscosity solutions to a fully nonlinear HJB equation satisfying respectively different boundary conditions. Based on the study in [10], Davis and Zariphopoulou [11] showed that calculating the reservation purchase price of an American option under proportional transaction costs involves solving a combination of a singular stochastic control and an optimal stopping problems. Furthermore, they proved that the value functions of the singular stochastic control problems with an optimal stopping are the unique viscosity solutions of the corresponding fully nonlinear HJB equations. Thus, using utility based method to price a European option and an American option is equivalent to finding the solutions to the HJB equations in [10] and [11], respectively.

Since closed-form solutions of the HJB equations are not available, numerical methods are necessary for approximating the solutions. In the open literature, most authors (see, for example, [9, 10, 22, 32, 33, 5]) used Markov chain approximation schemes for solving the HJB equations. These schemes are explicit in time and thus computationally very expensive. Recently, we proposed a penalty approach to solve the HJB equations arising in the European option pricing model under proportional transaction costs (cf. [20]). In the approach, we first approximated the HJB equation by a nonlinear PDE with penalty terms which penalize the part of the solution violating the constraints and showed that the viscosity solution to the penalty equation converges to that of the HJB equation based on the power penalty methods proposed in [30, 31, 16]. We then proposed and analyzed an upwind finite difference method for solving the penalty equations arising from the penalty method [21]. In this paper, we extend the penalty method combined with the finite difference scheme proposed in [20, 21] for the HJB equation in the European option pricing model to that arising in pricing American options under proportional transaction costs. In this approach, we first approximate the HJB equation by a quasi-nonlinear parabolic PDE containing three penalty terms with a penalty constant $\rho$ and show that the viscosity solution to the penalty equation converges to that of the original HJB equation as $\rho$ approaches infinity. We will then propose a finite difference scheme for the resulting penalty equation and prove that the solution to the discretized equation system converges to that of the penalty equation.

The rest of this paper is organized as follows. In the next section, we give a brief account of the formulation of the American options valuation problem as HJB equations using the utility maximization theory. In Section 3, we propose a penalty method for the HJB equations and establish a convergence theory for the penalty method. In Section 4, we present a discretization scheme for the penalty problem and show that the solution from the discretization scheme converges to the viscosity solution of the penalized problem. An iterative algorithm and its convergence will be presented in Section 5. In Section 6 we present some numerical results to illustrate the theoretical findings.
2. The American option pricing model. In this section, we briefly discuss the utility based option pricing model for valuing American call and put options with proportional transaction costs. A more detailed discussion can be found in [11, 9].

2.1. American option pricing via utility maximization. Consider a time interval \([0, T]\) and a continuous-time economy with a risky stock and a risk-less bond. Assume that the price of the stock at time \(u\), denoted as \(S_u\), evolves according to the following geometric Brownian motion

\[
dS_u = \mu du + \sigma dZ_u, \tag{1}
\]

where \(\mu\) and \(\sigma\) are respectively constant drift rate and volatility, \(Z_u\), representing a single source of uncertainty in the market, is a standard Brownian motion on a filtered probability space denoted as \((\Omega, \mathcal{F}, (\mathcal{F}_u)_{0 \leq u \leq T}, P)\). We also assume that the price of the bond, \(B(u)\), at time \(u\) is determined by the following ordinary differential equation

\[
 dB(u) = rB(u)du,
\]

where \(r \geq 0\) is a constant interest rate.

We suppose that the investors in the economy must pay transaction costs when buying or selling the stock and the transaction costs are proportional to the amount transferred from the stock to the bond.

Let \(\beta_u\) denote the amount the investors hold in the bond and \(\alpha_u\) the number of shares of the stock held by the investors at time \(u \in [0, T]\), then the evolution equations for \(\beta_u\) and \(\alpha_u\) are

\[
 d\beta_u = r\beta_u du - (1 + \theta)S_u dL_u + (1 - \theta)S_u dM_u, \tag{2}
\]

\[
 d\alpha_u = dL_u - dM_u, \tag{3}
\]

where \(\theta \in [0, 1)\) represents the transaction costs in percentage of the traded amount in the stock, and \(L_u\) and \(M_u\) denote respectively the cumulative number of shares bought and sold up to \(u\). At time \(u\) the liquidated cash value of the stock is \(S_u(\alpha_u - \theta|\alpha_u|)\) and the investors’ wealth, denoted as \(W_u\), is given by

\[
 W_u(\alpha_u, \beta_u, S_u) = \beta_u + S_u(\alpha_u - \theta|\alpha_u|). \tag{4}
\]

We now describe the utility based option pricing approach. The idea of the utility based option pricing approach is to consider the optimal portfolio problem of an investor whose objective is to find an admissible trading strategy so that the utility of the terminal wealth is maximized. To use this approach to value reservation purchase price of American call and put options, we need to define the following three different utility maximization problems.

**Problem 1** (Utility maximization for an investor without an option). Consider an investor who trades in the underlying stock and the bond. Assume that the investor holds \(\beta\) dollars in the bond and \(\alpha\) shares of the stock whose price is \(S\) at time \(t \in [0, T]\). The objective of the investor is to maximize the expected utility of wealth at the terminal time \(T\) over the set of feasible strategies, i.e.,

\[
 V^0(t, \alpha, \beta, S) = \sup_{\mathcal{A}^0(t, \alpha, \beta, S)} E_t[U(W_T)] \quad (0 \leq t \leq T), \tag{5}
\]

where \(V^0(t, \alpha, \beta, S)\) denotes the investor’s time \(t\) maximum expected utility of terminal wealth (also known as value function), \(E_t\) denotes the expectation operator conditional on the time \(t\) information \((\alpha, \beta, S)\), \(U(\cdot)\) is a utility function and
\( \Lambda^0(t, \alpha, \beta, S) \) is the set of admissible strategies available to the investor, defined as the set of right-continuous, measurable, \( F \)-adapted, increasing processes, \( L_u \) and \( M_u \) \((t \leq u \leq T)\), such that the following conditions are satisfied:

1. The associated processes \((\alpha^{L_u,M_u}, \beta^{L_u,M_u}, S_u)\) satisfy (1), (2) and (3) in \([t, T]\) with the initial state \((t, \alpha, \beta, S)\).
2. \( W_u(\alpha^{L_u,M_u}, \beta^{L_u,M_u}, S_u) > 0, \forall u \in [t, T] \), where \( W_u \) is defined in (4).

The choice of the utility function \( U \) is non-unique. In this work, we use the following exponential utility function:

\[
U(W) = 1 - \exp(-\gamma W),
\]
where \( \gamma > 0 \) is a constant risk aversion parameter. Problem 1 is a utility maximization problem without any options. Let \( u^+ := \max\{0, u\} \). In what follows, we define two other optimization problems with buying an American call option and an American put option respectively.

**Problem 2** (Utility maximization for an investor buying a call option). Consider an investor who trades in the market for the stock and the bond, and in addition, purchases a cash-settled American call option written on the stock with strike price \( K \) and expiry date \( T \). If the investor exercises the option at a stopping time \( \tau \in [t, T] \), then he/she receives the amount of \((S_\tau - K)^+\) dollars and faces the utility maximization problem without an option. Thus, the investor’s time \( \tau \) expected utility of terminal wealth is given by

\[
V_{cex}(\tau, \alpha_\tau, \beta_\tau, S_\tau) = V^0(\tau, \alpha_\tau, \beta_\tau + (S_\tau - K)^+, S_\tau) = \sup_{\Lambda^0(\tau, \alpha_\tau, \beta_\tau + (S_\tau - K)^+, S_\tau)} E_{\tau}[U(W_{\tau})]. \tag{7}
\]

The investor’s objective is to choose an admissible trading strategy and an exercise time to maximize (7), i.e.,

\[
V_{cex}(\tau, \alpha, \beta, S) = \sup_{\Lambda^0(\tau, \alpha, \beta; S_t), \tau} E_{\tau}[V_{cex}(\tau, \alpha_\tau, \beta_\tau, S_\tau)], \tag{8}
\]
where \( \Lambda^0(\tau, \alpha, \beta; S_t) \) denotes the investor’s admissible strategies which are defined as the set of right-continuous, measurable, \( F \)-adapted, increasing processes, \( L_u \) and \( M_u \) \((t \leq u \leq \tau)\), such that the following conditions are satisfied:

1. The associated processes \((\alpha^{L_u,M_u}, \beta^{L_u,M_u}, S_u)\) satisfy (1) to (3) in \([t, \tau]\) with the initial state \((t, \alpha, \beta, S)\).
2. \( W_u(\alpha^{L_u,M_u}, \beta^{L_u,M_u}, S_u) + (S_u - K)^+ > 0, \forall u \in [t, \tau] \).

**Problem 3** (Utility maximization for an investor buying a put option). Assume that the investor trades in the market for the stock and the bond, and in addition, purchases a cash-settled American put option written on the stock with strike price \( K \) and expiry date \( T \). If the investor choose to exercise the option at time \( \tau \), he receives the payoff \((K - S_\tau)^+\) dollars and faces the utility maximization problem without an option. That is, the investor’s value function at time \( \tau \) is defined as

\[
V_{pex}(\tau, \alpha_\tau, \beta_\tau, S_\tau) = V^0(\tau, \alpha_\tau, \beta_\tau + (K - S_\tau)^+, S_\tau) = \sup_{\Lambda^0(\tau, \alpha_\tau, \beta_\tau + (K - S_\tau)^+, S_\tau)} E_{\tau}[U(W_{\tau})]. \tag{9}
\]
The investor’s problem is to choose an admissible trading strategy and an exercise time to maximize (9), i.e.,

\[ V^{ax}(t, \alpha, \beta, S) = \sup_{\Lambda^*(t,\alpha,\beta,S)} E_t[V_p^{ax}(\tau, \alpha_\tau, \beta_\tau, S_\tau)], \]  

where \( \tau \in [t,T] \) is a stopping time and \( \Lambda^*_p(t,\alpha,t,\beta,t,S_t) \) is the investor’s admissible strategies which are defined as the set of right-continuous, measurable, \( \mathcal{F} \)-adapted, increasing processes, \( L_u \) and \( M_u \) \((t \leq u \leq \tau)\), such that the following conditions are satisfied:

1. The associated processes \((\alpha^L_u,M_u,\beta^L_u,M_u,S_u)\) satisfy (1) to (3) in \([t,\tau]\) with the initial state \((t,\alpha,\beta,S)\).
2. \( W_u(\alpha^L_u,M_u,\beta^L_u,M_u,S_u) + (K - S_u)^+ > 0, \forall u \in [t,\tau]. \)

**Remark 1.** In this model, the investor’s wealth is required to be positive at any time \( u \in [0,T] \), i.e., there is no bankruptcy in our economy. Item 2 in each of Problems 2–3 represents the no-bankruptcy restriction. These conditions ensure that the investor’s wealth is positive at all trading times. Since an American option can be exercised at any time before or on the expiry date, the model allows the investor to pursue trading strategies for which the liquidated cash value of the stock and bond, \( W_u \), is negative, as long as the amounts from exercising the option are large enough to cover the negative liquidated value.

Using the above definitions, we now define the reservation purchase prices of American call and put options as follows.

**Definition 2.1** (reservation purchase price of an American call option). Consider an investor who starts trading at time \( t = 0 \) with holding \( \beta \) dollars in the bond and \( \alpha \) shares of the stock whose price is \( S \). Assume that the investor only can buy the option at the initial time \( t = 0 \), then the investor’s reservation purchase price of an American call option is defined as the amount, \( P_c \), such that

\[ V^0(0, \alpha, \beta, S) = V^{ax}_c(0, \alpha, \beta - P_c, S). \]

**Definition 2.2** (reservation purchase price of an American put option). Consider an investor who starts trading at time \( t = 0 \) with holding \( \beta \) dollars in the bond and \( \alpha \) shares of the stock whose price is \( S \). Assume that the investor only can buy the option at the initial time \( t = 0 \), then the investor’s reservation purchase price of an American put option is defined as the amount, \( P_p \), such that

\[ V^0(0, \alpha, \beta, S) = V^{ax}_p(0, \alpha, \beta - P_p, S). \]

Clearly, the definitions are based on the no-arbitrage principle, that is, the expected return for an investor does not depend on whether he/she buys options or not. From the above definitions we see that computing the reservation purchase prices of American call and put options involves three value functions defined in (5), (8) and (10) respectively. By the principle of dynamic programming, Davis and Zarphopolous [11] have derived HJB equations governing these value functions, as given in the next subsection.

### 2.2. The HJB equations

It has been shown in [11] that each of the value functions \( V^0, V^{ax}_c \) and \( V^{ax}_p \) satisfies a set of HJB equations. In what follows, we list these equations without proof. A detailed deduction of these HJB equations can be found in [11].
Let \( \mathcal{L}_k, k = 1, 2, 3 \), be the linear differential operators defined respectively by

\[
\mathcal{L}_1 = -\left( \frac{\partial}{\partial t} + r\beta \frac{\partial}{\partial \beta} + \mu S \frac{\partial}{\partial S} + \frac{1}{2} \sigma^2 S \frac{\partial^2}{\partial S^2} \right),
\]

(11)

\[
\mathcal{L}_2 = -\frac{\partial}{\partial \alpha} + (1 + \theta) S \frac{\partial}{\partial \beta},
\]

(12)

\[
\mathcal{L}_3 = -\frac{\partial}{\partial \alpha} - (1 - \theta) S \frac{\partial}{\partial \beta}.
\]

(13)

Then, the value function \( V^0 \) solves the following HJB equation

\[
\min \{ \mathcal{L}_1 V, \mathcal{L}_2 V, \mathcal{L}_3 V \} = 0, \quad (t, \alpha, \beta, S) \in [0, T) \times \Omega^0
\]

(14)

with the terminal condition

\[
V(T, \alpha, \beta, S) = U(\beta + S(\alpha - \theta|\alpha|)), \quad (\alpha, \beta, S) \in \Omega^0,
\]

(15)

where

\[
\Omega^0 = \{(\alpha, \beta, S) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+ : \beta + S \alpha - S\theta|\alpha| > 0\}.
\]

(16)

For \( i = c,p \), the value function \( V_{i}^{\alpha x} \) is a solution to the HJB equation

\[
\min \{ V - V_{i}^{\alpha x}, \mathcal{L}_1 V, \mathcal{L}_2 V, \mathcal{L}_3 V \} = 0, \quad (t, \alpha, \beta, S) \in [0, T) \times \Omega_{i}^{\alpha x}
\]

(17)

with the terminal condition

\[
V(T, \alpha, \beta, S) = V_i(T, \alpha, \beta, S), \quad (\alpha, \beta, S) \in \Omega_{i}^{\alpha x},
\]

(18)

where \( V^{\alpha x}_c \) and \( V^{\alpha x}_p \) are the value functions defined in (7) and (9), respectively, and

\[
V_c(T, \alpha, \beta, S) = U(\beta + S(\alpha - \theta|\alpha|) + (S_T - K)^+),
\]

(19)

\[
V_p(T, \alpha, \beta, S) = U(\beta + S(\alpha - \theta|\alpha|) + (K - S_T)^+),
\]

(20)

\[
\Omega_{c}^{\alpha x} = \{(\alpha, \beta, S) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+ : \beta + S \alpha - S\theta|\alpha| + (S_T - K)^+ > 0\},
\]

(21)

\[
\Omega_{p}^{\alpha x} = \{(\alpha, \beta, S) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+ : \beta + S \alpha - S\theta|\alpha| + (K - S_T)^+ > 0\}.
\]

(22)

Note that (14) and (17) are nonlinear and they do not have in general classical solutions. Using the notion of viscosity solution, Davis and Zarphopoulou [11] showed that the value function \( V^0 \) is the unique viscosity solution of (14) satisfying the terminal condition (15) and \( V_{i}^{\alpha x}, i = c,p \) are the unique viscosity solutions of (17) satisfying, respectively, the terminal conditions (18)–(20). We will discuss this briefly in the next subsection.

2.3. Unique solvability of the HJB equations. In this subsection we first introduce the definitions of constrained viscosity solutions and then we will present a brief account of the results on the unique solvability of the HJB problems (14)-(20).

The concept of viscosity solution was first introduced in [6] for handling weak solutions of nonlinear first order PDEs. For a general introduction to the viscosity solution theory, we refer to [7, 14]. The concept of constrained viscosity solution was introduced in [24, 18] to handle control problems with state constrains.

In order to introduce the notion of viscosity solution, let us consider a fully non-linear second order PDE of the following form:

\[
F(X, W(X), DW(X), D^2W(X)) = 0, \quad \text{for } X \in [0, T) \times \Omega
\]

(23)

with the terminal condition

\[
W(X) = g(X), \quad \text{for } X \in \{T\} \times \Omega,
\]

(24)
where $\Omega \subseteq \mathbb{R}^n$ is an open set, $W : [0, T] \times \Omega \rightarrow \mathbb{R}$ is an unknown function, and $DW$ and $D^2W$ denote respectively the gradient and Hessian of $W$ with respect to $(t, x_1, ..., x_n)$. Let $S^n$ denote the set of $n \times n$ symmetric matrices, then $F$ is a mapping

$$F : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^n \times S^n \rightarrow \mathbb{R}.$$  

For any $A, B \in S^n$, we say $A \leq B$ if $\xi^T A \xi \leq \xi^T B \xi$ for all $\xi \in \mathbb{R}^n$. We assume that $F$ is continuous in all its arguments and satisfies the following degenerate ellipticity condition:

$$F(X, Y, Z, A) \leq F(X, Y, Z, B) \quad \text{whenever} \quad B \leq A$$

for $X \in [0, T] \times \Omega, Y \in \mathbb{R}, Z \in \mathbb{R}^n$, and $A, B \in S^n$.

Using the above notation, we now state the definition of constrained viscosity solution.

**Definition 2.3** (Viscosity sub/sup-solutions and constrained viscosity solution).

1. An upper semi-continuous function (USC) $W : [0, T] \times \Omega \rightarrow \mathbb{R}$ is a viscosity sub-solution of (23) on $[0, T] \times \Omega$ if, for every $\psi \in C^{1,2}([0, T] \times \Omega)$ and every local maximum point $X_0 \in [0, T] \times \Omega$ of $W - \psi$, we have

$$F(X_0, W(X_0), D\psi(X_0), D^2\psi(X_0)) \leq 0.$$  

2. A lower semi-continuous function (LSC) $W : [0, T] \times \Omega \rightarrow \mathbb{R}$ is a viscosity super-solution of (23) on $[0, T] \times \Omega$ if for every $\psi \in C^{1,2}([0, T] \times \Omega)$ and every local minimum point, $X_0 \in [0, T] \times \Omega$ of $W - \psi$, we have

$$F(X_0, W(X_0), D\psi(X_0), D^2\psi(X_0)) \geq 0.$$  

3. A continuous function $W : [0, T] \times \Omega \rightarrow \mathbb{R}$ is a constrained viscosity solution of (23) if it is both a viscosity sub-solution of (23) on $[0, T] \times \Omega$ and a super-solution of (23) on $[0, T] \times \Omega$. Furthermore, $W$ is a constrained viscosity solution of (23) if it is a viscosity solution of (23) satisfying (24).

**Remark 2.** In the above definition, we can replace “local maximum (minimum) point” by “strict local maximum (minimum) point" , “global maximum (minimum) point” or “strict global maximum (minimum) point”. We can also assume that the extremum of $W - \psi$ has the value zero.

With the definitions of viscosity solutions, we can now state the following theorems:

**Theorem 2.4.** Assume that the value function $V^0$ is continuous on $[0, T] \times \bar{\Omega}^0$, where $\Omega^0$ is defined in (16). Then, $V^0$ is the unique constrained viscosity solution of (14) and (15).

**Proof.** See the proof of Theorem 3.2 in [20].

We now state the comparison result in the following lemma which will be used in the proof of the unique solvability of (17) and (18). For a proof of this lemma, we refer to [11].

**Lemma 2.5.** Let $i \in \{c, p\}$ and $u^i$ be a bounded upper semi-continuous viscosity subsolution of (17) on $[0, T] \times \bar{\Omega}^{ix}$, and $v^i$ be a lower semi-continuous function which is bounded from below, exhibits sublinear growth and is a viscosity supersolution of (17) in $[0, T] \times \bar{\Omega}^{ix}$ such that

$$u^i(T, \alpha, \beta, S) \leq v^i(T, \alpha, \beta, S), \quad \forall (\alpha, \beta, S) \in \bar{\Omega}^{ix}.$$  

Then $u^i \leq v^i$ on $[0, T] \times \bar{\Omega}^{ix}$. 
Theorem 2.6. Let \( i \in \{c, p\} \) and assume that the value function \( V_i^{\alpha x} \) is continuous on \([0, T] \times \bar{\Omega}_i^{\alpha x}\), where \( \Omega_i^{\alpha x} \) are defined in (21) and (22) respectively for \( i = c \) and \( p \). Then, \( V_i^{\alpha x} \) is the unique constrained viscosity solutions of (17) and (18).

Proof. A proof of the existence of a viscosity solution to (17)–(18) can be found in [11]. We now use Lemma 2.5 to show that the viscosity solution is unique.

For each \( i \in \{c, p\} \), suppose that there are two viscosity solutions, \( V_i^{\alpha x} \) and \( W_i^{\alpha x} \), to (17)–(18). Since \( V_i^{\alpha x} \) is a subsolution and \( W_i^{\alpha x} \) is supersolution to (17)–(18), we have \( V_i^{\alpha x} \leq W_i^{\alpha x} \) by Lemma 2.5. On the other hand, \( V_i^{\alpha x} \) is a supersolution and \( W_i^{\alpha x} \) is subsolution to (17)–(18). Thus, \( V_i^{\alpha x} \geq W_i^{\alpha x} \). Combining these two inequalities gives \( V_i^{\alpha x} = W_i^{\alpha x} \). \( \square \)

3. The penalty method and its analysis. Penalty methods have been successfully used for option pricing problems without transaction costs and HJB equations in both infinite and finite dimensions (cf., for example, [30, 31, 16, 17, 34, 35]). Recently, we propose a penalty method in [20] for the European option pricing problem under proportional transaction costs. The HJB problem (14)-(15) arising from Problem 2.1 is of the form considered in [20]. Using the penalty method, the HJB equation (14)-(15) is approximated by

\[
\mathcal{L}_1 V_\lambda + \lambda |\mathcal{L}_2 V_\lambda|^- + \lambda |\mathcal{L}_3 V_\lambda|^- = 0
\]

(25)

for \( (t, \alpha, \beta, S) \in [0, T) \times \Omega^0 \) with the terminal condition

\[
V_\lambda(T, \alpha, \beta, S) = U(\beta + S(\alpha - \theta|\alpha|)), \quad (\alpha, \beta, S) \in \Omega^0,
\]

(26)

where \( \lambda > 0 \) is a penalty parameter and \(|v|^- := \min\{v, 0\}\) for any function \( v \). It has been proved in [20] that the viscosity solution of the above penalized PDEs converges to that of the original HJB equation as the penalty parameter \( \lambda \to \infty \). Thus, in what follows, we will discuss a penalty method for the HJB equations (17)-(18) only.

Motivated by the penalty equation (25), we consider the following equation approximating (17)-(18):

\[
\mathcal{L}_1 V_\rho + \rho (|V_\rho - V_i^{\alpha x}|^- + |\mathcal{L}_2 V_\rho|^- + |\mathcal{L}_3 V_\rho|^-) = 0
\]

(27)

for \( (t, \alpha, \beta, S) \in [0, T) \times \Omega_i^{\alpha x} \) with the terminal condition

\[
V_\rho(T, \alpha, \beta, S) = V_i(T, \alpha, \beta, S), \quad (\alpha, \beta, S) \in \Omega_i^{\alpha x},
\]

(28)

where \( \rho > 1 \) is a penalty constant. The terminal conditions \( V_i(T, \alpha, \beta, S) \) for \( i = c \) and \( p \) are defined in (19)-(20). Clearly, (27) is a nonlinear equation containing a penalty term \( \rho (|V_\rho - V_i^{\alpha x}|^- + |\mathcal{L}_2 V_\rho|^- + |\mathcal{L}_3 V_\rho|^-) \) which penalizes any of the negative parts of \( V_\rho - V_i^{\alpha x} \), \( \mathcal{L}_2 V_\rho \) and \( \mathcal{L}_3 V_\rho \). Also note that \( \mathcal{L}_1 V_\rho \geq 0 \) for any \( \rho > 1 \).

We next prove that for each \( i \in \{c, p\} \) the viscosity solution to (27)-(28) converges to that of (17)-(18) as \( \rho \to \infty \). We start this discussion with the following lemma whose proof is the same as that in [7, 14].

Lemma 3.1. For \( i \in \{c, p\} \) and \( \rho > 1 \), let \( v_\rho^i \) be the constrained viscosity solution of (27)-(28) and let \( \bar{v}^i \) and \( \underline{v}^i \) be respectively the upper and lower weak limits of \( v_\rho^i \).
defined by
\[
\overline{v}^i(X) = \limsup_{\rho \to \infty} \sup_{Y \in [0, T] \times \bar{\Omega}^{ax}_i} \left\{ v^i_{\rho}(Y) \mid Y \to (X) \right\}
\]
and
\[
\underline{v}^i(X) = \liminf_{\rho \to \infty} \inf_{Y \in [0, T] \times \bar{\Omega}^{ax}_i} \left\{ v^i_{\rho}(Y) \mid Y \to (X) \right\}
\]
Then, we have the following results.

1.: The upper and the lower weak limits, \(\overline{v}^i\) and \(\underline{v}^i\), are respectively USC and LSC on \([0, T] \times \bar{\Omega}^{ax}_i\).

2.: Let \(v^i_{\rho}\), the solution to \((27)-(28)\), be locally bounded uniformly in \(\rho\). If \(X_0 \in [0, T] \times \bar{\Omega}^{ax}_i\) is a strict local maximum point of \(\overline{v}^i - \psi\) (respectively minimum point of \(v^i_{\rho} - \psi\)) for a \(\psi \in C^{1,2}([0, T] \times \bar{\Omega}^{ax}_i)\), then, there exist subsequences, \(X_\rho \to X_0\) and \(v^i_{\rho}(X_\rho) \to \overline{v}^i(X_0)\) (respectively \(v^i_{\rho}(X_\rho) \to \underline{v}^i(X_0)\)) as \(\rho \to \infty\), such that \(X_\rho\) is a local maximum (respectively minimum) point of \(v^i_{\rho} - \psi\) for each \(\rho\) for \(i \in \{c, p\}\).

3.: If \(\overline{v}^i = \overline{v} = v\) on a compact subset of \([0, T] \times \bar{\Omega}^{ax}_i\), then \(v\) is continuous and \(v^i_{\rho} \to v\) as \(\rho \to \infty\) on the compact set.

Using Lemma 3.1, we are ready to establish a convergence theory for the penalty approach. This is given in the following theorem.

**Theorem 3.2.** Let \(i \in \{c, p\}\) and \(V_i\) be the unique constrained viscosity solution of \((17)-(18)\). For each \(\rho > 1\), let \(v^i_{\rho}\) be the constrained viscosity solution of \((27)-(28)\) for \(i \in \{c, p\}\). Then \(v^i_{\rho} \to V_i\) as \(\rho \to \infty\).

**Proof.** We only consider the case that \(i = c\) as the proof for \(i = p\) is essentially the same as that for \(i = c\).

To prove this theorem, we first need to show that the solution \(v^c_{\rho}\) to \((27)-(28)\) is locally bounded uniformly in \(\rho\). Since the proof of this requires a result in the next section, it will be given in Appendix B.

Let \(\overline{v} := \overline{v}^c\) and \(\underline{v} := \underline{v}^c\) denote respectively the upper and lower weak limits of \(v^c_{\rho}\) defined by \((29)\) and \((30)\) respectively. In what follows, we show that \(\underline{v} = \overline{v}\).

First, from the definitions of upper and lower weak limits in \((29)\) and \((30)\) we see that
\[
\underline{v} \leq \overline{v}. \tag{31}
\]
Thus, it remains to show that \(\underline{v} \geq \overline{v}\). To achieve this, it suffices to prove that \(v\) is a supersolution of \((17)\) on \([0, T] \times \bar{\Omega}^{ax}_c\) and \(\overline{v}\) a subsolution of \((17)\) on \([0, T] \times \bar{\Omega}^{ax}_c\) with \(\overline{v}(T, \alpha, \beta, S) = \overline{v}(T, \alpha, \beta, S) = V_c(T, \alpha, \beta, S)\), for \((\alpha, \beta, S) \in \bar{\Omega}^{ax}_c\). This is given below.

\(\overline{v}\) is a subsolution of \((17)\)

To prove \(\overline{v}\) is a subsolution of \((17)\) on \([0, T] \times \bar{\Omega}^{ax}_c\), we need to show that, from the definition of subsolution, for all \(\psi \in C^{1,2}([0, T] \times \bar{\Omega}^{ax}_c)\) and a strict local maximum
point, denoted as \( X_0 = (t_0, \alpha_0, \beta_0, S_0) \in [0, T) \times \Omega^e_c \), of \( \overline{\psi}(X) - \psi(X) \), the following inequality holds:

\[
\min \{ \overline{\psi}(X_0) - V_c^{e x}(X_0), (\mathcal{L}_1 \psi)(X_0), (\mathcal{L}_2 \psi)(X_0), (\mathcal{L}_3 \psi)(X_0) \} \leq 0.
\]

We prove this by contradiction. Assume (32) is not satisfied, i.e.

\[
\min \{ \overline{\psi}(X_0) - V_c^{e x}(X_0), (\mathcal{L}_1 \psi)(X_0), (\mathcal{L}_2 \psi)(X_0), (\mathcal{L}_3 \psi)(X_0) \} > 0.
\]

We have

\[
\overline{\psi}(X_0) - V_c^{e x}(X_0) > 0, \quad \text{and} \quad (\mathcal{L}_i \psi)(X_0) > 0
\]

for \( i = 1, 2, 3 \). Note that \( \psi \in C^{1,2}(\mathbb{R}^2) \). By continuity, there exists a neighborhood of \( X_0, B(X_0) \subset [0, T) \times \Omega^e_c \), such that the following inequalities also hold

\[
(\mathcal{L}_i \psi)(X) > 0, \quad i = 1, 2, 3
\]

for all \( X \in B(X_0) \). From Lemma 3.1(2) we see that there exists a subsequence \( \{X_{\rho}\} \) satisfying \( X_{\rho} \to X_0 \) and \( v_{\rho}^c(X_{\rho}) \to \overline{\psi}(X_0) \) as \( \rho \to \infty \) such that \( X_{\rho} \) is a local maximum point of \( v_{\rho}^c - \psi \) for each \( \rho \). This implies that, when \( \rho \) is sufficiently large, we have

\[
X_\rho \in B(X_0)
\]

and

\[
v_{\rho}^c(X_\rho) - V_c^{e x}(X_\rho) \to \overline{\psi}(X_0) - V_c^{e x}(X_0).
\]

Since \( \overline{\psi}(X_0) - V_c^{e x}(X_0) > 0 \), we imply that

\[
v_{\rho}^c(X_\rho) - V_c^{e x}(X_\rho) > 0
\]

when \( \rho \) is sufficiently large. Combining (33) with (34) and (35) we have

\[
\min \{ v_{\rho}^c(X_{\rho}) - V_c^{e x}(X_{\rho}), (\mathcal{L}_1 \psi)(X_{\rho}), (\mathcal{L}_2 \psi)(X_{\rho}), (\mathcal{L}_3 \psi)(X_{\rho}) \} > 0
\]

when \( \rho \) is sufficiently large. This inequality implies

\[
(\mathcal{L}_1 \psi)(X_{\rho}) + \rho \left[ v_{\rho}^c(X_{\rho}) - V_c^{e x}(X_{\rho}) \right] \geq (\mathcal{L}_1 \psi)(X_{\rho}) > 0
\]

when \( \rho \) is sufficiently large, which is a contradiction, since we have assumed that \( v_{\rho}^c \) is a constrained viscosity solution of (27). Thus, we conclude that \( \overline{\psi} \) is a subsolution of (17) on \([0, T) \times \Omega^e_c\).

**\( \psi \) is a supersolution of (17)**

We now prove that \( \psi \) is a supersolution of (17) on \([0, T) \times \Omega^e_c\). Let \( \psi \in C^{1,2}(\mathbb{R}^2) \) and suppose \( X_0 = (t_0, \alpha_0, \beta_0, S_0) \in [0, T) \times \Omega^e_c \) is a strict local minimum point of \( \psi(X) - \psi(X) \). We need to show that

\[
\min \{ \psi(X_0) - V_c^{e x}(X_0), (\mathcal{L}_1 \psi)(X_0), (\mathcal{L}_2 \psi)(X_0), (\mathcal{L}_3 \psi)(X_0) \} \geq 0.
\]

We will prove (36) by contradiction. Suppose

\[
\min \{ \psi(X_0) - V_c^{e x}(X_0), (\mathcal{L}_1 \psi)(X_0), (\mathcal{L}_2 \psi)(X_0), (\mathcal{L}_3 \psi)(X_0) \} < 0.
\]

Then, we consider the following four combinations of sign patterns of the terms in the left-hand side of (36).

**Case 1:** \( (\mathcal{L}_1 \psi)(X_0) < 0 \) and others are arbitrary.

Since \( \psi \in C^{1,2}(\mathbb{R}^2) \) and \( (\mathcal{L}_1 \psi)(X_0) < 0 \), there exists a neighborhood of \( X_0, B(X_0) \subset [0, T) \times \Omega^e_c \), such that for all \( X \in B(X_0), (\mathcal{L}_1 \psi)(X) < 0 \). Furthermore, by Lemma 3.1(2), there exists a subsequence, \( X_{\rho} \to X_0 \) as \( \rho \to \infty \), such that \( X_{\rho} \) is
Case 3

That means that when \( \rho \) is sufficiently large, \( X_\rho \in B(X_0) \), and so

\[
(L_1 \psi)(X_\rho) < 0.
\]

From this inequality we have

\[
(L_1 \psi)(X_\rho) + \rho [V^{\text{ex}}(X_\rho) - V^{\text{ex}}(X_\rho)] > \rho [(L_2 \psi)(X_\rho)] - \rho [(L_3 \psi)(X_\rho)] \\
\text{where } \rho \text{ is sufficiently large, since } [z]^{-} \leq 0 \text{ for any } z. \text{ This is a contradiction, since we assumed that } V^{\text{ex}} \text{ is a constrained viscosity solution of (27).}
\]

Case 2: \((L_1 \psi)(X_0) = k \geq 0, V(X_0) - V^{\text{ex}}(X_0) = c < 0 \) and others are arbitrary.

The fact that \( \psi \in C^{1,2}([0,T] \times \Omega^{\text{ex}}_{\text{c}}) \) implies that for any \( \epsilon > 0 \), there exists a neighborhood of \( X_0, B_{\epsilon}(X_0) \subset [0,T] \times \Omega^{\text{ex}}_{\text{c}} \) such that, for all \( X \in B_{\epsilon}(X_0) \),

\[
k - \epsilon \leq (L_1 \psi)(X) \leq k + \epsilon.
\]

As before, we can find a subsequence, \( X_\rho \to X_0 \) and \( V^{\text{ex}}(X_\rho) \to \psi(X_0) \) as \( \rho \to \infty \), such that \( X_\rho \) is a local minimum point of \( V^{\text{ex}} - \psi \) for every \( \rho \).

1. there exists \( \rho_0 > 1 \) such that \( X_\rho \in B_{\epsilon}(X_0) \) when \( \rho > \rho_0 \);
2. there exists \( \rho_1 > 1 \) such that

\[
(V(X_0) - V^{\text{ex}}(X_0)) - \epsilon \leq V^{\text{ex}}(X_\rho) - V^{\text{ex}}(X_\rho) \leq (V(X_0) - V^{\text{ex}}(X_0)) + \epsilon
\]

when \( \rho > \rho_1 \).

Thus, we have

\[
c - \epsilon \leq V^{\text{ex}}(X_\rho) - V^{\text{ex}}(X_\rho) \leq c + \epsilon, \quad \text{and} \quad k - \epsilon \leq (L_1 \psi)(X_\rho) \leq k + \epsilon
\]

when \( \rho \geq \max\{\rho_0, \rho_1\} = \bar{\rho} \). Since \( \epsilon \) is arbitrary, we choose \( \epsilon = \min\{\frac{1}{2}|c|, \frac{3k+1}{2}\} > 0 \) and have from (37)

\[
\frac{3}{2} c \leq V^{\text{ex}}(X_\rho) - V^{\text{ex}}(X_\rho) \leq \frac{1}{2} c < 0,
\]

\[
\frac{3}{2} k - \frac{1}{2} \leq (L_1 \psi)(X_\rho) \leq \frac{3}{2} k + \frac{1}{2}
\]

when \( \rho \geq \bar{\rho} \). Therefore, when \( \rho > \max\{\bar{\rho}, \frac{3k+1}{|c|}\} \), the above inequalities imply

\[
(L_1 \psi)(X_\rho) + \rho [V^{\text{ex}}(X_\rho) - V^{\text{ex}}(X_\rho)] \leq \rho [(L_2 \psi)(X_\rho)] - \rho [(L_3 \psi)(X_\rho)] \\
< \frac{3}{2} k + \frac{1}{2} + \frac{3k+1}{|c|} \frac{1}{2} c = 0
\]

which contradicts the assumption that \( V^{\text{ex}} \) is a constrained viscosity solution of (27).

Case 3: \((L_1 \psi)(X_0) = k \geq 0, (L_2 \psi)(X_0) = l < 0 \) and others are arbitrary.

Since \( \psi \in C^{1,2}([0,T] \times \Omega^{\text{ex}}_{\text{c}}) \), for any \( \epsilon > 0 \), there exists a neighborhood of \( X_0, B_{\epsilon}(X_0) \subset [0,T] \times \Omega^{\text{ex}}_{\text{c}} \) such that, for all \( X \in B_{\epsilon}(X_0) \),

\[
l - \epsilon \leq (L_2 \psi)(X) \leq l + \epsilon \quad \text{and} \quad k - \epsilon \leq (L_1 \psi)(X) \leq k + \epsilon.
\]

From Lemma 3.1(2) we see there exists a subsequence, \( X_\rho \to X_0 \) as \( \rho \to \infty \), such that \( X_\rho \) is a local minimum point of \( V^{\text{ex}} - \psi \) for every \( \rho \). Therefore, there exists a constant \( \rho_0 > 1 \) such that \( X_\rho \in B_{\epsilon}(X_0) \) when \( \rho \geq \rho_0 \). This implies that the
inequalities in (38) are also satisfied by $X = X_\rho$ when $\rho \geq \rho_0$. Since $\varepsilon$ is arbitrary, choosing $\varepsilon = \min\{\frac{1}{2}|l|, \frac{k+1}{2}\} > 0$, we obtain from (38)

$$\frac{3}{2} l \leq (L_2\psi)(X_\rho) \leq \frac{1}{2} l < 0$$

$$\frac{1}{2} k - \frac{1}{2} \leq (L_1\psi)(X_\rho) \leq \frac{3}{2} k + \frac{1}{2}$$

when $\rho \geq \rho_0$. Therefore, when $\rho > \max\{\rho_0, \frac{3k+1}{2l}\}$,

$$(L_1\psi)(X_0) + \rho \left[ v_{\rho}^c(X_\rho) - V_{\text{ax}}^c(X_\rho) \right]^- + \rho \left[ (L_2\psi)(X_\rho) \right]^- + \rho \left[ (L_3\psi)(X_\rho) \right]^- < \frac{3}{2} k + \frac{1}{2} + \frac{3k+1}{2l} = 0,$$

which is a contradiction, since we assume that $v_{\rho}^c$ is a constrained viscosity solution of (27).

**Case 4**: $(L_1\psi)(X_0) = k \geq 0$, $(L_3\psi)(X_0) = h < 0$ and others are arbitrary. By symmetry, the proof of Case 3 also applies to this one.

Thus, (36) is proved and we conclude that $\upsilon$ is a supersolution of (17) on $[0, T] \times \Omega_{\text{ax}}^c$.

**$\upsilon$ and $\overline{\upsilon}$ satisfy the terminal condition**

$$\overline{\upsilon}(T, \alpha, \beta, S) = \upsilon(T, \alpha, \beta, S) = U(\beta + S(\alpha - \theta|\alpha|) + (S - K)^+).$$

Since $v_{\rho}^c$ is a constrained viscosity solution of (27) and (28), we have that for all $\rho > 1$,

$$v_{\rho}^c(T, \alpha, \beta, S) = U(\beta + S(\alpha - \theta|\alpha|) + (S - K)^+), \quad \text{for } (\alpha, \beta, S) \in \Omega_{\text{ax}}^c.$$

Combining the continuity of $v_{\rho}^c$ and the definitions of $\upsilon$ and $\overline{\upsilon}$, we have

$$\upsilon(T, \alpha, \beta, S) = U(\beta + S(\alpha - \theta|\alpha|) + (S - K)^+) = \overline{\upsilon}(T, \alpha, \beta, S)$$

for any $(\alpha, \beta, S) \in \Omega_{\text{ax}}^c$.

Using the comparison result in Lemma 2.5 and the above arguments, we obtain

$$\upsilon \geq \overline{\upsilon} \quad \text{on} \quad [0, T] \times \Omega_{\text{ax}}^c.$$ 

Combining this with (31) gives

$$\upsilon = \overline{\upsilon} =: V_c.$$

Hence, the theorem is proved.

4. **Discretization and convergence.** Several efficient discretization schemes can be found in the open literature for Black-Scholes operators similar to that in (11). Two major ones are upwind finite difference and fitted finite volume schemes (cf., for example, [27, 29]). In this section, we will develop numerical schemes for solving the HJB equations in the previous sections. Since the discretization of (25)–(26) has been discussed in [21], in what follows we only consider the discretization of (27)–(28). For brevity, we only show the case that $i = c$ in (27)–(28).

Before presenting the scheme, we first rewrite (27)–(28) as the following equivalent form:

$$L_1 V_\rho + \min_{m_0 \in [0, \rho]} m_0 (V_\rho - V_{\text{ax}}^c) + \min_{m_2 \in [0, \rho]} m_2 L_2 V_\rho + \min_{m_3 \in [0, \rho]} m_3 L_3 V_\rho = 0 \quad (39)$$
for \((t, \alpha, \beta, S) \in [0, T] \times \Omega_c^{\infty}\) with the terminal condition
\[
V_\rho(T, \alpha, \beta, S) = U(\beta + S(\alpha - \theta |\alpha|) + (S_T - K)^+), \quad (\alpha, \beta, S) \in \Omega_c^{\infty}. \quad (40)
\]

Note that the solution domain \(\Omega_c^{\infty}\) is an unbounded region. In practice, we are only interested in option prices defined in a finite region \(\Omega_R\) given below:
\[
\Omega_R := (-R_\alpha, R_\alpha) \times (-R_\beta, R_\beta) \times (0, R_S),
\]
where \(R_\alpha, R_\beta, R_S\) and \(R_S\) are positive constants. Now, we define a uniform mesh for \(\Omega_R\) with mesh nodes
\[
\alpha_i = -R_\alpha + (i - 1) \times h_1, \quad i = 1, 2, ..., E,
\]
\[
\beta_j = -R_\beta + (j - 1) \times h_2, \quad j = 1, 2, ..., M,
\]
\[
S_k = (k - 1) \times h_3, \quad k = 1, 2, ..., P,
\]
where \(E, M\) and \(P\) are positive integers, and
\[
h_1 = \frac{R_\alpha + R_\beta}{E - 1}, \quad h_2 = \frac{R_\beta + R_\beta}{M - 1}, \quad h_3 = \frac{R_S}{P - 1}.
\]

Let \(h = \max\{h_1, h_2, h_3\}\) and \(Z_h\) denote respectively the index sets of the internal and boundary mesh nodes defined respectively by
\[
Z_h = \{(i, j, k) : i = 2, 3, ..., E - 1, j = 2, 3, ..., M - 1, k = 2, 3, ..., P - 1\}
\]
and
\[
\partial Z_h = \{(i, j, k), (E, j, k), (i, 1, k), (i, M, k), (i, j, 1), (i, j, P) : i = 1, 2, ..., E, j = 1, 2, ..., M, k = 1, 2, ..., P\}.
\]
Clearly, each grid point \((i, j, k) \in Z_h = \partial Z_h \cup Z_h\) corresponds to a state \((\alpha_i, \beta_j, S_k)\).

We divide the time interval \([0, T]\) into \(N\) sub-intervals with time points \(t_n = n \times \Delta t\) for \(n = 0, 1, ..., N\), where \(\Delta t = T/N\). We let \(Z_{\Delta t} = \{0, 1, 2, ..., N\}\) denote the index set of the time mesh points.

Let \(V_{ijk}^n\) denote the approximation (to be determined) to the solution of (39)–(40) at the mesh node \((t_n, \alpha_i, \beta_j, S_k)\). Using the finite difference operators
\[
D_t V_{ijk}^n = \frac{V_{ijk}^{n+1} - V_{ijk}^n}{\Delta t},
\]
\[
D_i^+ V_{ijk}^n = \frac{V_{i+1jk}^n - V_{ijk}^n}{h}, \quad D_i^- V_{ijk}^n = \frac{V_{ijk}^n - V_{i-1jk}^n}{h},
\]
\[
D_j^+ V_{ijk}^n = \frac{V_{ij+1k}^n - V_{ijk}^n}{h}, \quad D_j^- V_{ijk}^n = \frac{V_{ijk}^n - V_{ij-1k}^n}{h},
\]
\[
D_k^+ V_{ijk}^n = \frac{V_{ij(k+1)}^n - V_{ijk}^n}{h}, \quad D_k^- V_{ijk}^n = \frac{V_{ijk}^n - V_{ij(k-1)}^n}{h},
\]
\[
D_{kk}^+ V_{ijk}^n = \frac{V_{ijk(k+1)}^n - 2V_{ijk(k)}^n + V_{ijk(k-1)}^n}{h^2},
\]
we propose a discretization scheme for (39) as follows:
\[
\mathcal{L}_1^\Delta (i, j, k)V_{ijk}^n + \min_{m_0 \in [0, \rho]} m_0 (V_{ijk}^n - V_{ijk}^{xx,n}) + \min_{m_2 \in [0, \rho]} m_2 \mathcal{L}_2^\Delta (i, j, k)V_{ijk}^n
\]
\[
+ \min_{m_3 \in [0, \rho]} m_3 \mathcal{L}_3^\Delta (i, j, k)V_{ijk}^n = 0
\]
\[
(41)
\]
for \( n = N - 1, N - 2, \ldots, 0 \) and \((i, j, k) \in Z_h\), where \( V_{ijk}^{ex,n} \) denotes an approximation to \( V_{ex}(t_n, \alpha_i, \beta_j, S_k) \), \( v^- = \min\{v, 0\} \) for any function \( v \), and \( \mathcal{L}_1^\Delta, \mathcal{L}_2^\Delta, \mathcal{L}_3^\Delta \) are defined as follows:

\[
\mathcal{L}_1^\Delta(i, j, k) := -(D_t + (r\beta_j)^+D_j^+ + (r\beta_j)^-D_j^- + \mu S_k D_k^+ + \frac{1}{2} \sigma^2 S_k^2 D_{kk}), \tag{42}
\]

\[
\mathcal{L}_2^\Delta(i, j, k) := -D_i^+ + (1 + \theta)S_k D_j^-, \tag{43}
\]

\[
\mathcal{L}_3^\Delta(i, j, k) := D_i^- - (1 - \theta)S_k D_j^+. \tag{44}
\]

Clearly, the above are difference operators approximating respectively the continuous ones in (11)–(13). The discretization of the first-order spatial derivatives used above is based on the well-known upwind finite differencing technique (cf., for example, [28]).

**Remark 3.** Note that from (7) we see that \( V_{ex}^0(t, \alpha, \beta, S) = V_0^0(t, \alpha, \beta + (S - K)^+, S) \). Let \( V_{h,\Delta}^{0,\lambda}(t, \alpha, \beta, S) \) be the numerical solution to (25)–(26) by the numerical scheme in [21] using a comparable mesh as defined above. We then define

\[
V_{ijk}^{ex,n} := V_{ijk}^{ex,\lambda,n} = V_{h,\Delta}^{0,\lambda}(t_n, \alpha_i, \beta_j + (S_k - K)^+, S_k) \tag{45}
\]

for all feasible \((n, i, j, k)\) and \( \lambda > 1 \).

Based on (28) and (40), the terminal and the boundary conditions for (41) are defined respectively as

\[
V_{ijk}^N = \begin{cases} 
U(\beta_j + S_k(\alpha_i - \theta|\alpha_i|) + (S_k - K)^+), & (i, j, k) \in Z_h, x_{i,j,k} \in \bar{\Omega}^{ax}_i, \\
0, & (i, j, k) \in Z_h, x_{i,j,k} \notin \bar{\Omega}^{ax}_i,
\end{cases} \tag{46}
\]

\[
V_{ijk}^n = \begin{cases} 
U(\beta_j + S_k(\alpha_i - \theta|\alpha_i|) + (S_k - K)^+), & (i, j, k) \in \partial Z_h, x_{i,j,k} \in \bar{\Omega}^{ax}_i, \\
0, & (i, j, k) \in \partial Z_h, x_{i,j,k} \notin \bar{\Omega}^{ax}_i.
\end{cases} \tag{47}
\]

for all feasible \((n, i, j, k)\).

Before presenting the convergence analysis of the scheme, we first state the following theorem:

**Theorem 4.1.** Let \( V_{ijk}^{ex,\lambda,n} \) be defined in (45) for any feasible \((n, i, j, k)\) and \( \lambda > 1 \). Then, \( V_{ijk}^{ex,\lambda,n} \) converges to \( V_{ex}(t_n, \alpha_i, \beta_j, S_k) \) as \( \Delta := (h, \Delta t) \to (0, 0) \) and \( \lambda \to \infty \).

**Proof.** The proof of this theorem is just a combination of the convergence results in [20] and [21] and thus it is omitted. \( \square \)

The following comparison principle for (27) is also needed for the proof of our convergence result.

**Lemma 4.2** (Comparison principle). Let \( u^i \) be a bounded upper semi-continuous viscosity subsolutions of (27) on \( [0, T] \times \bar{\Omega}^{ax}_i \), and \( v^i \) be a lower semi-continuous function which is bounded from below, exhibits sublinear growth and is a viscosity supersolution of (27) on \( [0, T] \times \bar{\Omega}^{ax}_i \) such that \( u^i(T, \alpha, \beta, S) \leq v^i(T, \alpha, \beta, S) \), \( \forall (\alpha, \beta, S) \in \bar{\Omega}^{ax}_i \), then \( u^i \leq v^i \) on \( [0, T] \times \bar{\Omega}^{ax}_i \).

**Proof.** The proof of Lemma 4.2 is identical to that of Theorem 4.3 in [11] and therefore it is omitted. \( \square \)

With the above theorem and lemma, we establish the following theorem:

**Theorem 4.3.** Let \( u^\Delta \) be the solution to (41)–(47), then \( u^\Delta \) converges uniformly to the viscosity solution of (27)–(28)/(39)–(40) as \( \Delta \to (0, 0) \).

The proof is rather lengthy and we will put it in Appendix A.
5. Decoupling algorithm and its convergence. In this section we consider the implementation of the scheme (41)-(47). Note that (41)-(47) is a coupled non-linear system. In computation, we use the following algorithm to decouple it.

Algorithm A:

1. Initialize $V_{ijk}^N$ for all $(i, j, k) \in Z_h$ using the terminal and boundary conditions (46)- (47) and let $n = N-1$.
2. Let $V_{ijk}^{n,0} = V_{ijk}^{n+1}$ for all $(i, j, k) \in Z_h$ and evaluate

$$m_0^0(n, i, j, k) = \arg \left( \min_{m_0 \in [0, \rho]} m_0(V_{ijk}^{n+1} - V_{ijk}^{ex,n,\lambda}) \right),$$
$$m_2^0(n, i, j, k) = \arg \left( \min_{m_2 \in [0, \rho]} m_2 \mathcal{L}_2 \Delta \ V_{ijk}^{n+1} \right),$$
$$m_3^0(n, i, j, k) = \arg \left( \min_{m_3 \in [0, \rho]} m_3 \mathcal{L}_3 \Delta \ V_{ijk}^{n+1} \right).$$

3. Given a tolerance $\varepsilon > 0$, set $l = 0$.
4. Solve the following system along with the boundary conditions for $(V_{ijk}^{n,l+1})_{(i,j,k) \in Z_h}$:

$$\mathcal{L}_1 \Delta V_{ijk}^{n,l+1} + m_0^l(n, i, j, k)(V_{ijk}^{n,l+1} - V_{ijk}^{ex,n,\lambda}) + m_2^l(n, i, j, k) \mathcal{L}_2 \Delta V_{ijk}^{n,l+1} + m_3^l(n, i, j, k) \mathcal{L}_3 \Delta V_{ijk}^{n,l+1} = 0,$$

where, when applied to $V_{ijk}^{n,l+1}$, the finite difference operator $D_t$ involved in $\mathcal{L}_1$ is understood as

$$D_t V_{ijk}^{n,l+1} = \frac{V_{ijk}^{n+1} - V_{ijk}^{n,l+1}}{\Delta t}.$$  

5. For all $(i, j, k) \in Z_h$, evaluate

$$m_0^{l+1}(n, i, j, k) = \arg \left( \min_{m_0 \in [0, \rho]} m_0(V_{ijk}^{n,l+1} - V_{ijk}^{ex,n,\lambda}) \right),$$
$$m_2^{l+1}(n, i, j, k) = \arg \left( \min_{m_2 \in [0, \rho]} m_2 \mathcal{L}_2 \Delta V_{ijk}^{n,l+1} \right),$$
$$m_3^{l+1}(n, i, j, k) = \arg \left( \min_{m_3 \in [0, \rho]} m_3 \mathcal{L}_3 \Delta V_{ijk}^{n,l+1} \right).$$

6. If $\max_{(i,j,k) \in Z_h} |V_{ijk}^{n,l+1}(i, j, k) - V_{ijk}^{n,l}(i, j, k)| \geq \varepsilon$, set $l = l + 1$ and go to Step 4. Otherwise, go to Step 7.
7. Set $V_{ijk}^n = V_{ijk}^{n,l+1}$ for $(i, j, k) \in Z_h$ and $m_0(n, i, j, k) = m_0^{l+1}(n, i, j, k)$, $m_2(n, i, j, k) = m_2^{l+1}(n, i, j, k)$ and $m_3(n, i, j, k) = m_3^{l+1}(n, i, j, k)$ for $(i, j, k) \in Z_h$. Let $n = n - 1$ and go to Step 2.

Before showing that the iterative scheme is convergent, we first prove that (48) is uniquely solvable, which is equivalent to that (41)-(47) is uniquely solvable with any given $m_0(n, i, j, k) \in [0, \rho], m_2(n, i, j, k) \in [0, \rho]$ and $m_3(n, i, j, k) \in [0, \rho]$ for any $(i, j, k) \in Z_h$.

**Lemma 5.1.** For any $\Delta > (0, 0)$ the system (48) for any given $m_0(n, i, j, k) \in [0, \rho], m_2(n, i, j, k) \in [0, \rho]$ and $m_3(n, i, j, k) \in [0, \rho]$ has a unique solution and the solution is bounded uniformly on $Z_h \times Z_\Delta$. 


Proof. The proof is similar to that of Theorem 4.1 in [21]. For notational simplicity, we omit the superscript \( l \) and \( l + 1 \) in (48) in this proof.

Introduce an index transformation \( q = q(i, j, k) \) for \( i = 2, ..., E - 1, j = 2, ..., M - 1 \) and \( k = 2, ..., P - 1 \) such that all the interior nodes of the mesh, i.e., those having indexes in \( Z_h \), are re-ordered consecutively in such a way that \( q(2, 2, 2) = 1, q(3, 2, 2) = 2, ..., q(E - 1, M - 1, P - 1) = (E - 2) \times (M - 2) \times (P - 2) =: Q \). Let

\[
\begin{align*}
\omega_n^0(q(i, j, k)) &= \frac{\Delta t}{h_1} m_2(n, i, j, k), \\
\omega_n^0(q(i, j, k)) &= \frac{\Delta t}{h_1} m_3(n, i, j, k), \\
\omega_n^0(q(i, j, k)) &= \frac{\Delta t}{h_2} [(r\beta_j)^+ + m_3(n, i, j, k)(1 - \theta)S_k], \\
\omega_n^0(q(i, j, k)) &= \frac{\Delta t}{h_2} [(r\beta_j)^- + m_2(n, i, j, k)(1 + \theta)S_k], \\
\omega_n^0(q(i, j, k)) &= \frac{\Delta t}{h_3} (\mu S_k) + \frac{1}{2} \sigma S_k^2 \frac{\Delta t}{h_3^2}, \\
\omega_n^0(q(i, j, k)) &= \frac{1}{2} \sigma S_k^2 \frac{\Delta t}{h_3^2}, \\
\omega_n^0(q(i, j, k)) &= 1 + \sum_{l=2}^7 \omega_n^0(q(i, j, k)) + m_0(n, i, j, k).
\end{align*}
\]

It is clear that

\[
w_n^0(q(i, j, k)) \geq 0 \quad l \in \{2, 3, ..., 7\},
\]

\[
w_n^0(q(i, j, k)) = 1 + \sum_{l=2}^7 \omega_n^0(q(i, j, k)) + m_0(n, i, j, k) \geq 1.
\]

Using the above notation, it is easy to verify that (48) can be written as the following form:

\[
V_{ij}^{n+1} = V_{ij}^n w_n^0(q(i, j, k)) - V_{i+1,j}^n w_n^0(q(i, j, k)) - V_{i,j+1}^n w_n^0(q(i, j, k)) - V_{i,j,k+1}^n w_n^0(q(i, j, k)) - V_{i,j,k-1}^n w_n^0(q(i, j, k)) - m_0(n, i, j, k)V_{i,j,k}^{\text{ex},m,n},
\]

where \( i = 2, ..., E - 1, j = 2, ..., M - 1, k = 2, ..., P - 1 \). Re-arranging the terms, we see that (55) can be written as the following matrix form:

\[
A^n V^n = b^n + c^{n+1} + v^n
\]

for \( n = N - 1, N - 2, ..., 0 \), where \( A^n = (a^n_{pq})_{p,q=1}^Q \) with \( Q := (E - 2) \times (M - 2) \times (P - 2) \) is a septa-diagonal matrix with the non-zero entries given by

\[
\begin{align*}
a_{qq}^n &= w_n^1(q), \quad a_{q,q+1}^n = -w_2^1(q), \\
a_{q,q}^n &+ (E - 2) = -w_3^1(q), \quad a_{q,q}^n + (E - 2) \times (M - 2) = -w_5^1(q), \\
a_{q,q}^n &+ (E - 2) \times (M - 2) = -w_5^1(q), \quad a_{q,q}^n - (E - 2) \times (M - 2) = -w_5^1(q)
\end{align*}
\]

for \( q = 1, 2, ..., Q \) and all admissible column indexes \( q \pm 1, q \pm (E - 2) \) and \( q \pm (E - 2) \times (M - 2) \). \( b^n \) and \( c^{n+1} \) are \( Q \times 1 \) column vectors representing, respectively, the contribution from the Dirichlet boundary conditions at the \( n \)th level, the left-hand side of (55) involving the approximate solution at \( n + 1 \), \( v^n \) is a \( Q \times 1 \) column vector representing the term \( m_0(i, j, k)V_{i,j,k}^{\text{ex},m,n} \) and \( V^n \) is the unknown vector. This is a
Q \times Q linear system in V^n. From (53), (54) and (57)-(59) it is easy to see that A^n is diagonally dominant and irreducible, and has positive diagonal and non-positive off-diagonal elements. Thus, A^n is an M-matrix (cf., for example, [26]) and is nonsingular. Therefore, we conclude that there exists a unique solution to (56) and so (48) is uniquely solvable.

We now prove that for any \Delta > (0,0), the solution is uniformly bounded. Our strategy is to prove that if the terminal conditions satisfy

\[
\max_{(i,j,k)\in Z_h} |V_{i,j,k}^N| < C < +\infty,
\]

then

\[
\max_{(i,j,k)\in Z_h} |V_{i,j,k}^n| < C, \quad n = 1, 2, \ldots, N - 1.
\]  

By (46)-(47) and (6), we have

\[
\max_{(i,j,k)\in Z_h} |V_{i,j,k}^N| < 1.
\]

Letting \(\max_{(i,j,k)\in Z_h} |V_{i,j,k}^n| < 1\) hold for an \(n \leq N\), we will show that

\[
\max_{(i,j,k)\in Z_h} |V_{i,j,k}^{n-1}| < 1.
\]

We prove it by contradiction. Suppose that \(\max_{(i,j,k)\in Z_h} |V_{i,j,k}^{n-1}| \geq 1\). Then, there exists an index triple \((i_0, j_0, k_0) \in Z_h\), such that

\[
|V_{i_0,j_0,k_0}^{n-1}| \geq 1 \quad \text{and} \quad |V_{i,j,k}^{n-1}| \leq |V_{i_0,j_0,k_0}^{n-1}|, \quad \forall (i,j,k) \in Z_h.
\]

From (53) and (55), we have

\[
|V_{i_0,j_0,k_0}^{n}| \geq \left| V_{i_0,j_0,k_0}^{n-1} \left( 1 + \sum_{l=2}^{7} u_{l}^{n-1}(q_0) \right) \right| + \left| V_{i_0,j_0,k_0}^{n-1} m_0(q_0) \right| - \left| V_{i_0,j_0,k_0}^{n-1} m_0(q_0) u_{2}^{n-1}(q_0) \right|
\]

where \(q_0 = q(i_0, j_0, k_0)\). From Theorem 4.1 in [21], we have \(|V_{i_0,j_0,k_0}^{ex,(n-1),\lambda}| < 1\). Thus, we have from the above

\[
|V_{i_0,j_0,k_0}^{n}| \geq |V_{i_0,j_0,k_0}^{n-1}| \geq 1.
\]

Clearly, this contradicts our assumption that \(\max_{(i,j,k)\in Z_h} |V_{i,j,k}^{n-1}| < 1\). Thus, we have \(\max_{(i,j,k)\in Z_h} |V_{i,j,k}^{n-1}| < 1\). By the mathematical induction principle, (60) holds and the theorem is proved.

We now prove that the iterative scheme (48)-(52) generates a sequence \(\{V_{i,j,k}^{n}\}_{i=0}^\infty\) that converges to the solution of (41)-(47).

**Theorem 5.2.** The iterative scheme (48)-(52) generates a sequence \(\{V_{i,j,k}^{n}\}_{i=0}^\infty\) that converges to the solution of (41)-(47).
Proof. We will follow the notation in Algorithm A. To prove this theorem, we first show that the sequence \( \{V^{n,l}\}_{l=0}^{\infty} \) generated by the iterative method is monotonically increasing, i.e., \( V^{n,l} \leq V^{n,l+1} \) for \( l \geq 1 \).

From (48) we have
\[
L_1^\Delta V_{ijk}^{n,l} + m_0^{-1}(n, i, j, k)(V_{ijk}^{n,l} - V_{ijk}^{ex,n,\lambda}) \\
+ m_1^{-1}(n, i, j, k)L_2^\Delta V_{ijk}^{n,l} + m_2^{-1}(n, i, j, k)L_3^\Delta V_{ijk}^{n,l} = 0,
\]
for \((i, j, k) \in Z_h \) and \( l = 1, 2, \ldots \). (Recall the convention for \( D_i \) defined in (49).)

This can be rewritten as
\[
L_1^\Delta V_{ijk}^{n,l} + m_0(n, i, j, k)(V_{ijk}^{n,l} - V_{ijk}^{ex,n,\lambda}) + m_1(n, i, j, k)L_2^\Delta V_{ijk}^{n,l} + m_2(n, i, j, k)L_3^\Delta V_{ijk}^{n,l} \\
= [m_0(n, i, j, k) - m_0^{-1}(n, i, j, k)](V_{ijk}^{n,l} - V_{ijk}^{ex,n,\lambda}) \\
+ [m_1(n, i, j, k) - m_2^{-1}(n, i, j, k)]L_2^\Delta V_{ijk}^{n,l} + [m_2(n, i, j, k) - m_3^{-1}(n, i, j, k)]L_3^\Delta V_{ijk}^{n,l} \leq 0, \quad \forall (i, j, k) \in Z_h,
\]
(61)
since, by (50)–(52),
\[
m_0(n, i, j, k) = \arg \left( \min_{m_0 \in [0, \rho]} m_0(V_{ijk}^{n,l} - V_{ijk}^{ex,n,\lambda}) \right), \\
m_1(n, i, j, k) = \arg \left( \min_{m_2 \in [0, \rho]} m_2 L_2^\Delta V_{ijk}^{n,l} \right), \\
m_2(n, i, j, k) = \arg \left( \min_{m_3 \in [0, \rho]} m_3 L_3^\Delta V_{ijk}^{n,l} \right).
\]

Note that both (48) and (61) have the same boundary conditions, i.e., \( V_{ijk}^{n,l} = V_{ijk}^{n,l+1} \) for any \((i, j, k) \in \partial Z_h \). Thus, using the notation in the proof of Lemma 5.1, we may write (48) and (61) as the following respective matrix forms:
\[
A^{n,l}V^{n,l+1} = b^n + c^{n+1} + v^n, \quad A^{n,l}V^{n,l} \leq b^n + c^{n+1} + v^n,
\]
where \( A^{n,l}, b^n, c^{n+1} \) and \( v^n \) are as defined in (56) with \( A^{n,l} \) an \( M \)-matrix. Therefore, from the above we have
\[
A^{n,l}(V^{n,l+1} - V^{n,l}) \geq 0.
\]
Since \( A^{n,l} \) is an \( M \)-matrix, we have
\[
V^{n,l+1} - V^{n,l} \geq 0.
\]
Therefore, the monotonicity of iteration process is proved.

From Lemma 5.1 we have that \( V^{n,l} \) is bounded for any \( l = 0, 1, 2, \ldots \). Combining the boundedness and monotonicity we see that \( V^{n,l} \) is convergent. Finally, from the construction of (48) and (52) it is obvious that \( V_{ijk}^{n,l}, m_0(n, i, j, k), m_2(n, i, j, k) \) and \( m_1(n, i, j, k) \) solve (41) when \( l \to \infty \). Thus, we have proved the theorem. \( \Box \)

6. Numerical results. In this section, we use the scheme (41)-(47) and that in [21] to compute the value functions \( V_{i}^{ex}(i = c, p) \) and \( V^0 \) respectively, and present the computed reservation purchase prices, \( P_c \) and \( P_p \), of American call and put options with different values of the risk aversion parameter \( \gamma \) in utility function (6).

We now illustrate the performance of the schemes using the following test example:
Test Example: Reservation purchase prices of American call and put options with the strike price $K = 2.6$, expiry date $T = 1$, the initial price of stock $S_0 = 2.6$ and various values of $\alpha_0, \beta_0$ and $\gamma$. Other parameters are: the interest rate $r = 0.05$, the drift parameter $\mu = 0.1$, the volatility $\sigma = 0.2$ and the proportional transaction cost parameter $\theta = 0.05$.

To solve this problem, we choose $R_{\alpha} = 2, R_{\beta} = 6, R_S = 5$ and $R_{\beta} = 10$. The discretization and penalty parameters are chosen to be $E = 41, M = 61, P = 26, N = 25$ and $\rho = 1000$. Comparable mesh and penalty parameters were used for computing $V^0$ using the scheme in [21]. The problem is solved by a MATLAB code in double precision under the Linux environment. We compute the reservation purchase prices of an American call option for $\gamma = 0.1$ and various values of initial holdings in the bond $\beta_0$ and share $\alpha_0$. The results are reported in Table 1 and Figure 1.

Please note that the price of an American option is higher than or equal to that of its European counterpart because there are more exercising opportunities for an American option than for a European option. It is known that in the economy without transaction costs, the price of an American call option is equivalent to that of its European counterpart. In order to compare the reservation purchase prices of an American call option and its European counterpart involving transaction costs, we also compute the prices of European options, denoted by $P_{b,c}$, using the scheme in [21]. The results for both American and European options are listed in Table 1.

| $\alpha_0$ | 0   | 1   | 2   | 3   |
|------------|-----|-----|-----|-----|
| $P_{b,c}$  | 0.3730 | 0.3463 | 0.3213 | 0.2981 |
| $P_c$      | 0.3730 | 0.3463 | 0.3213 | 0.2981 |

Table 1. Computed reservation purchase price $P_c$ of the American call option for $\gamma = 0.1$ and the reservation purchase price $P_{b,c}$ of its European counterpart.

From Table 1 and Figure 1, we see that reservation purchase price of this American call option with transaction costs is the same as that of its European counterpart, coinciding with the case of options without transaction costs. Both of purchase prices of an American and a European call options decrease as the initial holding of the stock $\alpha_0$ increases. This conforms to the theory of supply and demand that the more stocks an investor holds at $t = 0$, the less the investor wants to purchase the stocks and therefore the investor will reduce the call option price. From Figure 1 we also see that the reservation purchase prices are independent of $\beta_0$, i.e., the initial holding in the bond does not affect the reservation price of an American call option.

To investigate the influence of the risk aversion parameter $\gamma$ on the reservation prices, we assume that the investor’s initial holding of the share is zero, i.e., $\alpha_0 = 0$, and compute the reservation purchase prices of the American call option and its European counterpart for $\gamma = 0.01, 0.1, 0.3,$ and $0.5$. The results are listed and depicted in Table 2 and Figure 2 respectively. It is clear from Table 2 and Figure 2 that the reservation purchase prices of an American call option are identical to the corresponding purchase prices of a European call option.

From Table 2 and Figure 2 we observe that the purchase price is a decreasing function of $\gamma$. Noting that the investment risk is also a decreasing function of $\gamma$,
Figure 1. Computed reservation purchase price of the American call option for $\gamma = 0.1$ and the reservation purchase price of its European counterpart.

This result is very realistic and true as when an investor is willing to take more risk, he/she is willing to purchase a call option at a higher price, and vice versa. As in the case in Figure 1, the purchase prices are independent of the initial bond holding $\beta$.

Finally, let us discuss the influence of $\gamma$ on the reservation purchase price of an American put option and compare the price to that of its European counterpart. Assume that the investor’s initial holding in the stock is zero, i.e., $\alpha_0 = 0$. We compute the reservation purchase prices of an American put option and its European counterpart, denoted as $P_{b,p}$, for $\gamma = 0.01, 0.1, 0.3$ and 0.5. The results are showed in Table 3 and Figure 3. From the table and the figure, we have the following observations:

1. The reservation purchase price of an American put option is greater than that of its European counterpart. This coincide with what we commented earlier in this section.

| $\gamma$ | 0.01  | 0.1   | 0.3   | 0.5   |
|----------|-------|-------|-------|-------|
| $P_{b,c}$ | 0.3843 | 0.3730 | 0.3493 | 0.3277 |
| $P_{c}$   | 0.3843 | 0.3730 | 0.3493 | 0.3277 |

Table 2. Computed reservation purchase price of the American call option for $\alpha_0 = 0$ and the reservation purchase price of its European counterpart.
Figure 2. Computed reservation purchase price $P_c$ of the American call option for $\alpha_0 = 0$ and the reservation purchase price $P_{b,c}$ of its European counterpart.

2. The purchase prices are a decreasing function of $\gamma$ as we observed earlier from the results in Figure 2.

3. As in the previous cases, the reservation purchase prices of American put options are independent of the initial holding in the bond $\beta_0$.

| $\gamma$ | 0.01 | 0.1  | 0.3  | 0.5  |
|----------|------|------|------|------|
| $P_{b,p}$ | 0.1155 | 0.1145 | 0.1128 | 0.1116 |
| $P_p$    | 0.1370 | 0.1365 | 0.1360 | 0.1338 |

Table 3. Computed reservation purchase prices of the American put option for $\alpha_0 = 0$ and the reservation purchase prices of its European counterpart.

Before closing this section, we have the following comments.

Remark 4. It is worth pointing out that each of the HJB equations in the original problem is defined on an infinite domain and does not have any Dirichlet boundary conditions. However, we solve it numerically on a finite region such as $\Omega_R$ defined in Section 4 and define an artificial (homogeneous) Dirichlet boundary condition on each of the boundary segments as the exact boundary condition is unknown. The accuracy of the numerical solution depends on the choice of the boundary conditions. From the numerical experiments we have found that the computational errors are essentially located near the boundary of the finite domain $\Omega_R$. A theoretical justification for this artificial boundary conditions is given in [23]. Thus, in the
numerical results presented above, we only plot the computed values at the mesh points which are some distances away from the boundary segments.

Remark 5. We have observed from our computational results that the reservation purchase prices of both American call and put options under proportional transaction costs are independent of $\beta_0$.

7. Conclusion. In this paper we have studied the problem of American option pricing with proportional transaction costs. We propose a penalty approach combined with an upwind finite difference scheme to compute the reservation purchase prices of American call and put options. We have proved that both the penalty method and the discretization scheme are convergent. An iterative scheme for the discretized nonlinear system has also been proposed and analyzed. Numerical results show that the reservation purchase price of an American call option is equivalent to that of its European counterpart and the reservation purchase price of an American put option is greater than that of its European counterpart. This means that premature exercise of an American call option is not optimal.

Appendix A: Proof of Theorem 4.3. A standard method for proving the convergence of a finite difference scheme was developed by Barles and Souganidis [1] who showed that any monotone, stable and consistent scheme converges to the exact solution provided that there exists a comparison principle. In what follows, we will show that the scheme (41) is stable, monotone and consistent which, combined with Lemma 4.2, guarantee the convergence of the solution.
Stability. To prove the stability of the scheme (41), we need to show that for any 
\( \Delta = (\Delta, h) > (0, 0) \), the solution \( u^\Delta \) to (41) is uniformly bounded.

From the proof of Lemma 5.1, it is easy to see that \(|u^\Delta| \leq 1 \) for any \( \Delta t \) and \( h \).

Monotonicity. To prove the monotonicity of the scheme (41), we need to show that the scheme (41) satisfies

\[
L^\Delta_1(i, j, k)v^n_{ijk} + \min_{m_o \in [0, \rho]} (v^n_{ijk} - V^{ex, \lambda, n}_{ijk})m_2(i, j, k)L^\Delta_2(i, j, k)
\]

\[
v^n_{ijk} + m_3(i, j, k)L^\Delta_3(i, j, k)v^n_{ijk}
\]

\[
\leq L^\Delta_1(i, j, k)w^n_{ijk} + \min_{m_o \in [0, \rho]} (w^n_{ijk} - V^{ex, \lambda, n}_{ijk})m'_2(i, j, k)L^\Delta_2(i, j, k)w^n_{ijk}
\]

\[
+ m'_3(i, j, k)L^\Delta_3(i, j, k)w^n_{ijk},
\]

whenever \( v^n_{ijk} = w^n_{ijk} = V^n_{ijk} \) and

\[
v := (v^n_{ijk}, v^n_{(i+1)jk}, v^n_{(i-1)jk}, v^n_{(j+1)k}, v^n_{(j-1)k}, v^n_{ijk(k+1)}, v^n_{ijk(k-1)})^T
\]

\[
\geq w := (w^n_{ijk}, w^n_{(i+1)jk}, w^n_{(i-1)jk}, w^n_{(j+1)k}, w^n_{(j-1)k}, w^n_{ijk(k+1)}, w^n_{ijk(k-1)})^T,
\]

(62)

where

\[
m_2(n, i, j, k) = \arg \min_{m_2 \in [0, \rho]} m_2L^\Delta_2(i, j, k)v^n_{ijk},
\]

\[
m_3(n, i, j, k) = \arg \min_{m_3 \in [0, \rho]} m_3L^\Delta_3(i, j, k)v^n_{ijk},
\]

\[
m'_2(n, i, j, k) = \arg \min_{m'_2 \in [0, \rho]} m'_2L^\Delta_2(i, j, k)w^n_{ijk},
\]

\[
m'_3(n, i, j, k) = \arg \min_{m'_3 \in [0, \rho]} m'_3L^\Delta_3(i, j, k)w^n_{ijk},
\]

and \( L^\Delta_1, L^\Delta_2 \) and \( L^\Delta_3 \) are the operators defined in (42)–(44) respectively.

Noting \( \min_{m_o \in [0, \rho]} (v^n_{ijk} - V^{ex, \lambda, n}_{ijk}) = \min_{m_o \in [0, \rho]} (w^n_{ijk} - V^{ex, \lambda, n}_{ijk}) \), to prove (62), we show that when \( v \geq w \),

\[
L^\Delta_1(i, j, k)v^n_{ijk} \leq L^\Delta_1(i, j, k)w^n_{ijk},
\]

(63)

\[
m_2(n, i, j, k)L^\Delta_2(i, j, k)v^n_{ijk} \leq m'_2(n, i, j, k)L^\Delta_2(i, j, k)w^n_{ijk},
\]

(64)

\[
m_3(n, i, j, k)L^\Delta_3(i, j, k)v^n_{ijk} \leq m'_3(n, i, j, k)L^\Delta_3(i, j, k)w^n_{ijk}.
\]

(65)

From the definition of \( L^\Delta_1 \) it is easy to see that (63) holds when (62) is satisfied as \( L^\Delta_1 v^n_{ijk} \) consists of the elements of \( v \) with negative coefficients. We now prove (64). From the definition of \( L^\Delta_2 \) it is clear

\[
-\frac{v^n_{(i+1)jk} - V^n_{ijk}}{h_1} + (1 + \theta)S_k \frac{V^n_{ijk} - v^n_{(j-1)k}}{h_2}
\]

\[
\leq -\frac{w^n_{(i+1)jk} - V^n_{ijk}}{h_1} + (1 + \theta)S_k \frac{V^n_{ijk} - w^n_{(j-1)k}}{h_2},
\]

or

\[
L^\Delta_2 v^n_{ijk} \leq L^\Delta_2 w^n_{ijk}.
\]

(66)

We now look at the following two cases:

**Case 1:** \( L^\Delta_1 v^n_{ijk} \cdot L^\Delta_2 w^n_{ijk} \geq 0 \).

In this case, it is easy to see that either \( m_2(n, i, j, k) = m'_2(n, i, j, k) = 0 \) or \( m_2(n, i, j, k) = m'_2(n, i, j, k) = \rho \). Thus, from (66), we have (64).
Case 2: $\mathcal{L}_n^{\Delta} v_{ij}^n < 0 < \mathcal{L}_2^{\Delta} w_{ijk}^n$.

This implies $m_2(n, i, j, k) = \rho$ and $m_2'(n, i, j, k) = 0$. Thus (64) still holds since $\mathcal{L}_2^{\Delta} w_{ijk}^n < 0$.

In a similar way, we can prove (65) and thus (62) holds.

Consistency. From Theorem 4.1 we have
\[
V_{\Delta}^{ex,n,\lambda} \rightarrow V^{ex}(t_n, \alpha_i, \beta_j, S_k)
\]
when $\Delta \rightarrow (0, 0)$ and $\lambda \rightarrow \infty$. Combining (67) and the definitions of the discretization operators used in (42)-(44) it is easy to see that for any $\psi(X) \in C^{2,1}([0, T] \times \Omega_{ac}^{ex})$
\[
\mathcal{L}_1^{\Delta} \psi_{ij}^n + m_0(n, i, j, k)(\psi_{ij}^n - V_{\Delta}^{ex,n,\lambda}) + m_2(n, i, j, k)\mathcal{L}_2^{\Delta} \psi_{ijk}^n + m_3(n, i, j, k)\mathcal{L}_3^{\Delta} \psi_{ijk}^n \rightarrow (\mathcal{L}_1^{\Delta} \psi)(X) + \min_{m \in [0, \rho]} m_0(\psi(X) - V^{ex}(X)) + \min_{m \in [0, \rho]} m_2(\mathcal{L}_2^{\Delta} \psi)(X) + \min_{m \in [0, \rho]} m_3(\mathcal{L}_3^{\Delta} \psi)(X)
\]
as $\Delta \rightarrow (0, 0)$ and $(t_n, \alpha_i, \beta_j, S_k) \rightarrow X \in (0, T) \times \Omega_{ac}^{ex}$, where $\psi_{ijk}^n = \psi(t_n, \alpha_i, \beta_j, S_k)$ for all admissible $(n, i, j, k)$. Therefore, the scheme is consistent.

Finally, using Theorem 2.1 in [1] and Lemma 4.2, we have that $u^{\Delta}$ converges to the unique viscosity solution of (27)-(28)/(39)-(40) as $\Delta \rightarrow (0, 0)$, $\Omega_L \rightarrow \Omega$ and $\lambda \rightarrow \infty$. This completes the proof.

Appendix B: Proof of uniform boundedness of $v^\rho_{\Delta}$.

Proof. For each $\rho > 1$, let $u^{\Delta}_\rho$ be the solution to (41)-(47). From Theorem 4.3, we see that $u^{\Delta}_\rho \rightarrow v^\rho_{\Delta}$, where $v^\rho_{\Delta}$ is the viscosity solution to (27)-(28). Also, from the proof of Theorem 4.3, we find that $|u^{\Delta}_\rho| < 1$ for any $\rho > 1$ and $\Delta t, h > 0$. Thus, we have
\[
|v^\rho_{\Delta}| \leq 1.
\]

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