TORSION OF A FINITE QUASIGROUP QUANDLE IS ANNIHILATED BY ITS ORDER

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Abstract. We prove that if $Q$ is a finite quasigroup quandle, then $|Q|$ annihilates the torsion of its rack homology.

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1. Introduction

1.1. History of the problem.

It is a classical result in reduced homology of finite groups that the order of a group annihilates its homology [Bro]. Namely, we consider the chain homotopy $(g_1, \ldots, g_n) \mapsto \sum_{y \in G} (y, g_1, \ldots, g_n)$ between $|G|\text{Id}$ and the zero map.

From the very beginning of the rack homology (between 1990 and 1995 [FRS-1, FRS-2, Fenn]) the analogous result was suspected. The first general results in this direction were obtained independently about 2001 by Litherland and Nelson [L-N], and P. Etingof and M. Graña [E-G]. We give here an outline of known results.\(^1\) In [L-N] it is proven that if $(Q; \ast)$ is a finite homogeneous rack (this includes quasigroup

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\(^1\)The necessary definitions about rack and quandle homology are given in the next subsection.
racks) then the torsion of $H_n^R(Q)$ is annihilated by $|Q|^n$. In [E-G] it is proven that if $(X; A)$ is a finite rack and $N = |G^0_Q|$ is the order of the group of inner automorphisms of $Q$, then the only primes which can appear in the torsion of $H_n(X)$ are those dividing $N$ (the case of connected Alexander quandles was proven before by T. Mochizuki [Moch]). The results in [L-N] and [E-G] are independent as the latter is for all finite racks and the former is for only homogeneous racks but gives concrete approximation for torsion. The result in [L-N] is generalized in [N-P-1] and in particular, it is proven there that the torsion part of the homology of the dihedral quandle $R_3$ is annihilated by 3. In [N-P-2] it is conjectured that for a finite quasigroup quandle, torsion of its homology is annihilated by the order of the quandle. The conjecture is proved by T. Nosaka for finite Alexander quasigroup quandles [Nos].

In this paper we prove the conjecture in full generality (Theorem 2.1).

1.2. Racks, quandles, and their homology.

In this section we review some definitions and preliminary facts. The algebraic structure $(X; *)$ with a universe $X$ and a binary operation $* : X \times X \to X$ is called a magma. If the binary operation satisfies the right self-distributive property, $(a * b) * c = (a * c) * (b * c)$ for any $a, b, c \in X$, then the magma is said to be a shelf. Let $b \in X$ and $*_b : X \to X$ be a map given by $*_b(a) = a * b$. If $*_b$ is invertible for any $b \in X$, then the shelf is called a rack. We use the notation $*_b = *^{-1}_b$ and $a*_b = *_b(a)$, thus if $a * b = c$ then $c*_b = a$. If the binary operation $*$ is idempotent, then the rack is said to be a quandle. The three axioms of a quandle are motivated by the three Reidemeister moves [Joy, Matv]. If we fix a magma $(X; *)$ and color arcs of the diagram by elements of $X$ (with the convention of Figure 1(i)) and if we want to preserve the cardinality of the set of the colorings by Reidemeister moves, we have to assume that the magma satisfies the quandle axioms.

Quandles can be used to classify classical knots [Joy, Matv]. If the quandle has the quasigroup property, i.e. for any $a, b \in X$, the equation $a * x = b$ has a unique solution, then it is called a quasigroup quandle [2].

If $(X; *)$ is a quasigroup quandle, then, at any crossing, coloring of two arcs determines the color of the third arc (see Figure 1(ii)).

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[2] In the theory of quasigroups, the following standard notation is used: if we start from a magma $(X; *)$ and if $a * b = c$ then $a = c / b$, and $b = a \backslash c$. In knot theory one uses $\tilde{*}$ for $/$ and $\circ$ for $\backslash$. See [Gal] for a review of quasigroups.
Example 1.1. If $G$ is an abelian group, we define a quandle called a Takasaki quandle (or kei) of $G$, denoted by $T(G)$, by taking $a \ast b = 2b - a$. If $G = \mathbb{Z}_n$, then we denote $T(\mathbb{Z}_n)$ by $R_n$(dihedral quandle).

Fenn, Rourke, and Sanderson [FRS-3] first defined the rack homology theory and a modification to quandle homology theory was introduced by Carter, Jelsovsky, Kamada, Langford, and Saito [CJKLS] to define knot invariants in a state-sum form (so-called cocycle knot invariants). We review the definition of rack and quandle homology after [CKS-2].

Definition 1.2. Let $C^R_n(X)$ be the free abelian group generated by n-tuples $(x_1, \ldots, x_n)$ of elements of a rack $X$, i.e. $C^R_n(X) = \mathbb{Z}X^n = (\mathbb{Z}X)^{\otimes n}$. We define a boundary homomorphism $\partial : C^R_n(X) \to C^R_{n-1}(X)$ by

$$\partial(x_1, \ldots, x_n) = \sum_{i=1}^{n} (-1)^i (d_i^{(e)} - d_i^{(s)})(x_1, \ldots, x_n)$$

where $d_i^{(e)}(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ and $d_i^{(s)}(x_1, \ldots, x_n) = (x_1 * x_i, \ldots, x_{i-1}*x_i, x_{i+1}, \ldots, x_n)$. Then $(C^R_n(X), \partial)$ is said to be a rack chain complex of $X$.

Definition 1.3. For a quandle $X$, we have the subset $C^D_n(X)$ of $C^R_n(X)$ generated by n-tuples $(x_1, \ldots, x_n)$ of elements of $X$ with $x_i = x_{i+1}$ for some $i = 1, \ldots, n - 1$. Then $(C^D_n(X), \partial)$ is the subchain complex of a rack chain complex $(C^R_n(X), \partial)$ and it is called a degenerated chain complex of $X$. Then we have the quotient chain complex $(C^Q_n(X), \partial) = (C^R_n(X)/C^D_n(X), \partial)$ and it is called the quandle chain complex.

Definition 1.4. For an abelian group $G$, we define the chain complex $C^W_*(X; G) = C^W_*(X) \otimes G$ with $\partial = \partial \otimes \text{id}$ for $W=R$, $D$, and $Q$. Then
the nth rack, degenerate, and quandle homology groups of a quandle \( X \) with coefficient group \( G \) are defined as

\[
H_n^W(X; G) = H_n(C^W_*(X; G)) \quad \text{for } W=R, D, \text{ and } Q.
\]

The free parts of rack, degenerate, and quandle homology groups of finite racks or quandles were computed in [E-G, L-N].

**Theorem 1.5.** [E-G, L-N] Let \( O \) be the set of orbits of a rack \( X \) with respect to the action of \( X \) on itself by right multiplication. Then

1. \( \text{rank} H^n_R(X) = |O|^n \) for a finite rack \( X \),
2. \( \text{rank} H^n_Q(X) = |O|^{n-1} \) for a finite quandle \( X \),
3. \( \text{rank} H^n_D(X) = |O|^n - |O|(|O| - 1)^{n-1} \) for a finite quandle \( X \).

Homology of quandles ((co)cycle invariants) are used successfully in higher dimensional knot theory [CKS-2, P-R].

2. **The proof of Theorem 2.1**

**Theorem 2.1.** Let \( Q \) be a finite quasigroup quandle. Then the torsion subgroup of \( H^n_R(Q) \) is annihilated by \( |Q| \).

**Proof.** We consider two chain maps, \( f^j_r \) and \( f^j_s \) defined as below. Let \( x = (x_1, \ldots, x_n) \in Q^n \). Then we define the (repeater) chain map \( f^j_r : C^n_R(Q) \to C^n_R(Q) \) by

\[
f^j_r(x) = |Q|(x_j, \ldots, x_j, x_{j+1}, \ldots, x_n) \quad \text{for } 1 \leq j \leq n,
\]

and define the (symmetrizer) chain map \( f^j_s : C^n_R(Q) \to C^n_R(Q) \) by

\[
f^j_s(x) = \sum_{y \in Q} (y, \ldots, y, x_{j+1}, \ldots, x_n) \quad \text{for } 0 \leq j \leq n.
\]

We first prove that \( f^j_r \) is chain homotopic to \( f^j_s \) for \( 1 \leq j \leq n \), by using the following chain homotopy:

\[
D^j_n(x) = \sum_{y \in Q} (x_j, \ldots, x_j, y, x_{j+1}, \ldots, x_n) \quad \text{for } 1 \leq j \leq n.
\]

If \( i \leq j \), then

\[
d^{(s)}_i D^j_n(x) = \sum_{y \in Q} (x_j, \ldots, x_j, y, x_{j+1}, \ldots, x_n)
\]

so that the formula above does not depend on the binary operation \( * \), in particular \( (d^{(s)}_i - d^{(eq)}_i)D^j_n = 0 \).

If \( i = j + 1 \), then since \( Q \) satisfies the quasigroup property,

\[
d^{(s)}_i D^j_n(x) = \sum_{y \in Q} (y, \ldots, y, x_{j+1}, \ldots, x_n) = f^j_s(x),
\]
Note that therefore, we have the equality

\[ d_i^{(s)} D^j_n(x) = \sum_{y \in Q} (x_j * x_{i-1}, \ldots, x_j * x_{i-1}, y, x_{j+1} * x_{i-1}, \ldots, x_{i-2} * x_{i-1}, x_i, \ldots, x_n). \]

On the other side, if \( i \leq j \), then

\[ D^j_{n-1} d_i^{(s)}(x) = \sum_{y \in Q} (x_j, \ldots, x_j, y, x_{j+1}, \ldots, x_n), \]

thus this formula does not depend on the binary operation *, and

\[ D^j_{n-1}(d_i^{(s)} - d_i^{(s_0)}) = 0. \]

If \( j + 1 \leq i \), then

\[ D^j_{n-1} d_i^{(s)}(x) = \sum_{y \in Q} (x_j * x_{i-1}, \ldots, x_j * x_{i-1}, y, x_{j+1} * x_{i-1}, \ldots, x_{i-2} * x_{i-1}, x_i, \ldots, x_n). \]

Note that \( d_{i+1}^{(s)} D^j_n = D^j_{n-1} d_i^{(s)} \) and \( d_{i+1}^{(s_0)} D^j_n = D^j_{n-1} d_i^{(s_0)} \) if \( j + 1 \leq i \leq n \).

Therefore, we have the equality

\[ \partial_{n+1} D^j_n(x) + D^j_{n-1} \partial_n(x) = (-1)^i (f^j_s(x) - f^j_r(x)). \]

We next will prove that \( f^j_{s-1} \) is chain homotopic to \( f^j_r \) for \( 1 \leq j \leq n \), and the chain homotopy is given by the formula:

\[ F^j_n(x) = \sum_{y \in Q} (x_j, \ldots, x_j, y, x_j, \ldots, x_n) \text{ for } 1 \leq j \leq n. \]

If \( i \leq j - 1 \), then

\[ d_i^{(s)} F^j_n(x) = \sum_{y \in Q} (x_j, \ldots, x_j, y, x_j, \ldots, x_n) \]

so this formula does not depend on the binary operation *, in particular \( (d_i^{(s)} - d_i^{(s_0)}) F^j_n = 0. \)

If \( i = j \), then since \( Q \) satisfies the quasigroup property,

\[ d_i^{(s)} F^j_n(x) = \sum_{y \in Q} (y, \ldots, y, x_j, \ldots, x_n) = f^j_{s-1}(x), \]

\[ d_i^{(s_0)} F^j_n(x) = |Q|(x_j, \ldots, x_j, x_{j+1}, \ldots, x_n) = f^j_r(x). \]

If \( i = j + 1 \), then \( (d_i^{(s)} - d_i^{(s_0)}) F^j_n = 0. \)

Last, if \( j + 2 \leq i \leq n + 1 \), then

\[ d_i^{(s)} F^j_n(x) = \sum_{y \in Q} (x_j * x_{i-1}, \ldots, x_j * x_{i-1}, y, x_j * x_{i-1}, \ldots, x_{i-2} * x_{i-1}, x_i, \ldots, x_n). \]
On the other hand, if $i \leq j$, then

$$F^j_{n-1}d_i^{(s)}(x) = \sum_{y \in Q} (x_j, \ldots, x_j, y, x_j, \ldots, x_n),$$

so that the formula above does not depend on the binary operation $\ast$, and $F^j_{n-1}(d^{(s)}_i - d^{(s)}_{i+1}) = 0$.

If $j + 1 \leq i$, then

$$F^j_{n-1}d_i^{(s)}(x) = \sum_{y \in Q} (x_j \ast x_i, \ldots, x_j \ast x_i, y, x_j \ast x_i, \ldots, x_{i-1} \ast x_i, x_{i+1}, \ldots, x_n).$$

Notice that $d_{i+1}^{(s)}F^j_n = F^j_{n-1}d_i^{(s)}$ and $d_{i+1}^{(s)}F^j_n = F^j_{n-1}d_{i+1}^{(s)}$ if $j + 1 \leq i \leq n$.

Hence we have the following equality

$$\partial_{n+1}F^j_n(x) + F^j_{n-1}\partial_n(x) = (-1)^j(f^j_r(x) - f^{j-1}_s(x)).$$

Then from the above, we obtain a sequence of chain homotopic chain maps,

$$|Q|Id = f^0_s = f_1^r = f_2^r = \ldots = f^n_r = f^n_s,$$

where $f^n_s = \sum_{y \in Q}(y, \ldots, y)$. Then, on homology level, we have the same induced chain maps $|Q|Id = (f^n_s)_*: H^n_R(Q) \to H^n_R(Q)$. Recall that free($H^n_R(Q)$) is $\mathbb{Z}$ and it is generated by $(y, \ldots, y)$ for $y \in Q$. Therefore $|Q|tor(H^n_R(Q)) = 0$.

**Corollary 2.2.** The reduced quandle homology\(^3\) of a finite quasigroup quandle is annihilated by its order, i.e., $|Q|H^n_R(Q) = 0$.

**Proof.** The homology of a quandle splits into degenerate and quandle parts, i.e. $H^n_R(Q) = H^n_D(Q) \oplus H^n_Q(Q)$ (see [L-N]), therefore, by Theorem 2.1, $|Q|$ annihilates the torsion of $H^n(D(Q)$ and $H^n_Q(Q)$. Furthermore, since $Q$ is a connected quandle, rank($H^n_Q(Q)$) = 0 for $n > 1$ and rank($H^1_Q(Q)$) = 1. Therefore the reduced quandle homology of $Q$ is a torsion group annihilated by $|Q|$.

**Remark 2.3.** We recall that if a quandle $Q$ has the quasigroup property, then it is a connected quandle. But the opposite does not hold. For example, the 6-elements quandle $QS(6)$ (see [CKS-1] and [CKS-2]) is a connected quandle but not a quasigroup quandle. This example also shows that Theorem 2.1 does not hold when we replace the condition

\(^3\)Reduced homology is obtained from augmented chain complex $\cdots \to C_1 \xrightarrow{\partial_1} \mathbb{Z}$ where $\partial_1(x) = 1$; reduced quandle homology of a finite quandle has trivial free part(compare Theorem [L-S] 2).
“quasigroup” with “connected” in the theorem, because $H_3^S(QS(6)) = \mathbb{Z}_{24}$, see [CKS-1].

3. Presimplicial homotopy

We can express our computation in the language of presimplicial homotopy between $f^n_s$ and $f^n_n$. First, we start from the definitions [Lod].

Definition 3.1. A presimplicial module $C$ is a collection of modules $C_n$, $n \geq 0$, together with maps, called face maps or face operators,

$$d_i : C_n \to C_{n-1}, i = 0, ..., n$$

such that

$$d_id_j = d_{j-1}d_i \text{ for } 0 \leq i < j \leq n.$$

Note that $(\mathbb{Z}X^n, d_i^{(s)})$ is a shifted presimplicial module (to get presimplicial module we could take $(\mathbb{Z}X^{n+1}, d_i^{(s)})$).

Lemma 3.2. Let $\partial = \sum_{i=0}^n (-1)^id_i$, then $\partial\partial = 0$. In other words $(C_*, \partial)$ is a chain complex.

A map of presimplicial modules $f : C \to C'$ is a collection of maps $f : C_n \to C'_n$ such that $f_{n-1} \circ d_i = d_i \circ f_n$. It implies that $f_{n-1} \circ d = d \circ f_n$ and so induces a map of complexes $f : C_* \to C'_*$. On homology the induced map is denoted $f_* : H_*(C_*) \to H_*(C'_*)$.

Definition 3.3. A presimplicial homotopy $h$ between two presimplicial maps $f$ and $g : C \to C'$ is a collection of maps $h_i : C_n \to C'_{n+1}$, $i = 0, ..., n$ such that

$$d_ih_j = h_{j-1}d_i \text{ for } i < j$$

$$d_ih_i = d_ih_{i-1} \text{ for } i \leq n \text{ (the case } i = j \text{ and } i = j + 1),$$

$$d_ih_j = h_jd_{i-1} \text{ for } i > j + 1,$$

$$d_0h_0 = f \text{ and } d_{n+1}h_n = g.$$

Lemma 3.4. If $h$ is a presimplicial homotopy from $f$ to $g$, then $h = \sum_{i=0}^n (-1)^ih_i$ is a chain homotopy from $f$ to $g$ and therefore on homology $f_* = g_*$. 

Proposition 3.5. Define the map $G^n_j = D^n_j + F^n_j$ for $1 \leq j \leq n$ ($D^n_j$ and $F^n_j$ as in the proof of Theorem [2.1]). Then the collection of maps $G^n_j : C^n_R(Q) \to C^n_{R+1}(Q)$, denoted by $G$, is a presimplicial homotopy between two presimplicial maps $|Q|\text{Id}$ and $f^n_s$ from $C^n_R(Q)$ to $C^n_R(Q)$. 

Then the map $G_n = \sum_{j=1}^{n} (-1)^j G_n^j$ is a chain homotopy from presimplicial maps $f_s^0 = |Q|Id$ to $f_s^n$, therefore $|Q|Id = (f_s^n)_*$ on homology.

### 4. Multi-term homology

Our main result can be extended from rack homology of quandles to multi-term homology of multi-quandles.

Recall after [Prz] that $\text{Bin}(X)$ denotes a monoid of binary operations on a set $X$ with a composition $\ast_1 \ast_2$ defined by $a \ast_1 b = (a \ast_1 b) \ast_2 b$ and the identity element $\ast_0$ given by $a \ast_0 b = a$. We say that a subset $S \subset \text{Bin}(X)$ is a distributive set if any pair $\ast_1, \ast_2 \in S$ satisfies the distributivity property: $(a \ast_1 b) \ast_2 c = (a \ast_2 c) \ast_1 (b \ast_2 c)$ for any $a, b, c \in X$. We call $(X; S)$ a multi-shelf (or a multi-right-distributive system (RDS)). If each $(X; \ast_i)$ is a quandle, then it is called a multi-quandle containing the identity operation.

We define a chain complex $(C_n, \partial_n)$ by putting $C_n = ZX^n$, $a_0, ..., a_k$ integers, and $\partial_n^{(a_0, a_1, ..., a_k)} = \sum_{i=0}^{k} a_i \partial_n^{(\ast_i)}$. Homology of this multi-quandle is called the multi-term rack homology and denoted by $H_n^{(a_0, a_1, ..., a_k)}$.

We generalize Theorem 2.1 as follows:

**Theorem 4.1.** Let $(X; S)$ be a multi-quandle where $X$ is a finite set and $S = (\ast_0, \ast_1, ..., \ast_k)$ satisfying the following conditions:

(i) $\sum_{i=0}^{k} a_k = 0$,

(ii) $\ast_0$ is the trivial operation and $a_0 \neq 0$, and

(iii) $(X; \ast_i)$ is a quasigroup quandle for $i \geq 1$.

Then $a_0 |X|$ annihilates the torsion of the multi-term homology $H_n^{(a_0, a_1, ..., a_k)}$.

**Proof.** We follow the proof of Theorem 2.1 by properly generalizing chain homotopies $D_n^i$ and $F_n^j$. Namely we put:

$$D_n^i(x_1, ..., x_n) = \sum_{i=0}^{k} a_i \sum_{y \in X} (x_j, ..., x_j, y, x_{j+1}, ..., x_n)$$

and

$$F_n^j(x_1, ..., x_n) = \sum_{i=0}^{k} a_i \sum_{y \in X} (x_j, ..., x_j, y, x_{j+1}, ..., x_n).$$
Combining homotopies $D^n_j$ and $F^n_j$ together, as in the proof of Theorem 2.1, we obtain the chain homotopy between $a_0|X|Id$ and $a_0 \sum_{y \in X} (y, \ldots, y)$.

\[ \square \]

5. Future research

Not much is known about the torsion of rack homology group in the case a quandle is not a quasigroup.

As we noted in Remark 2.3, the main theorem, Theorem 2.1, does not generalize directly to non-quasigroup quandles. More data is needed to make conjectures in the general case. However in [N-P-2], we make the specific conjecture that the number $k$ annihilates $\text{tor} H_n(R_{2k})$, unless $k = 2^t$, $t \geq 1$; the number $2k$ is the smallest number annihilating $\text{tor} H_n(R_{2k})$ for $k = 2^t$, $t \geq 1$.

For one-term rack homology, we found in [CPP] many shelves with non-trivial torsion in homology. However all of them are non-connected. It is still an open problem whether there is a connected shelf with non-trivial torsion in one-term distributive homology.

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