ON THE STABILITY OF REAL SCALAR BOSON STARS

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We discuss spherically symmetric static solutions of the Einstein-Klein-Gordon equations for a real scalar field with a mass and a quartic self-interaction term. As for the massless case the solutions have a naked singularity at the origin. However, linear stability analysis shows that these solutions as well as the massless ones are dynamically unstable.

1 Introduction

The recent developments in particle physics and cosmology suggest that scalar fields may have played an important role in the evolution of the early universe, for instance in primordial phase transitions, and that they may make up part of the dark matter. These facts motivated the study of gravitational equilibrium configurations of scalar fields, in particular for massive complex fields, which form so-called boson stars (for a review on this subject see Refs. 1 and 2). Solutions to the Einstein-Klein-Gordon equations for massless real scalar fields were first studied by Buchdahl and more recently by Wyman. All known static solutions for the real scalar field to the Einstein-Klein-Gordon equations are either topologically non-trivial or have a logarithmic singularity. The latter is an example of a so-called naked singularity.

Here we present the static, spherically symmetric solutions for a real scalar field obtained by including a mass and a quartic self-interaction term. We also discuss the dynamical stability of the equilibrium solutions using linear perturbation theory. Using the variational principle, from which the pulsation equation is derived, we find for all equilibrium solutions a negative upper bound for the lowest eigenvalue. We, therefore, conclude that all equilibrium configurations, including the massless ones, are dynamically unstable.

2 Basic equations and their properties

We consider the action of a real scalar field \( \phi \) with mass and a quartic self-interaction term (\( \lambda > 0 \)) minimally coupled to gravity with the following spherically symmetric metric

\[
d s^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) .
\]

The static equilibrium configurations are determined by a coupled system of equations, which can be derived from the action, by varying with respect to the various
fields. For all solutions we require as boundary conditions that the metric is asymptotically flat and that \( \phi \) vanishes at infinity. It turns out that at the origin \( \phi \) is singular and behaves for all cases as: \( \phi \sim -\ln r \). For \( r \to 0 \): \( e^{\lambda} \sim r^a \) (\( a \) is real and \( a > 1 \)) and \( e^{\lambda} \to 0 \), whereas \( e^{\nu} \sim r^b \) (\( b \) is real and \( b > -1 \)). For \( e^{\nu} \) there are three different possible behaviors: \( e^{\nu} \equiv 1 \), \( e^{\nu} \to \infty \) or \( e^{\nu} \to 0 \). See Ref. 5 for a plot of the results of the numerical integration for \( \phi \) and the metric functions. The corresponding ADM mass of the boson star has been also computed numerically.

3 Dynamical stability

We turn now to the problem of dynamical stability and consider small time dependent radial perturbations, which still preserve spherical symmetry. The equations governing the linear perturbations are obtained by expanding all functions to first order \( (\lambda(r,t) = \lambda_0(r) + \delta\lambda(r,t); \nu(r,t) = \nu_0 + \delta\nu(r,t); \phi(r,t) = \phi_0(r) + \delta\phi(r,t)) \) and by linearizing the Einstein-Klein-Gordon equations. We make also use of the \( G_{01} \) component of the Einstein equation, which can be integrated once in time. This way the metric functions \( \delta\lambda \) and \( \delta\nu \) can be eliminated from the linearized scalar wave equation. Furthermore, we suppose a time dependence of the form \( e^{i\sigma t} \) for \( \delta\phi(r,t) \). Thus \( \delta\phi(r,t) = e^{i\sigma t}\Psi(r)/r \), where we denote the radial part of \( \delta\phi \) by \( \Psi(r)/r \). Performing a change of variable defined by \( \frac{d\rho}{dr} = e^{(\lambda_0 - \nu_0)/2} \) with \( \rho(r = 0) = 0 \), the pulsation equation transforms to the following Schrödinger-type equation

\[
-\frac{d^2\Psi}{d\rho^2} + V(r(\rho))\Psi = \sigma^2\Psi,
\]

where

\[
V(r) = e^{\nu_0 - \lambda_0}((\nu'_0 - \lambda'_0)/2r - 4\pi Gr\phi_0^2(2/r + \nu'_0 - \lambda'_0) + 16\pi Gr\phi_0 e^{\lambda_0}(m^2\phi_0 + \tilde{\lambda}\phi_0^3) + e^{\lambda_0}(m^2 + 3\tilde{\lambda}\phi_0^3))
\]  

\((r = r(\rho))\). Dynamical instability occurs whenever the lowest eigenvalue \( \sigma_0^2 \) is negative. Indeed, the perturbation \( \delta\phi \approx e^{\sigma_0 t} \) will then grow exponentially. Since the asymptotic behavior of the solution \( \phi_0(r) \) does not depend on the mass nor on the quartic self-interaction term, the behavior of \( V(\rho) \) will not depend on \( m \) and \( \lambda \) and thus will be the same for all types of solutions. One finds \( V(\rho) \approx -\frac{1}{4\rho^2} \) for \( \rho \to 0 \). It is useful to write the potential \( V(\rho) \) as: \( V(\rho) = -\frac{1}{4\rho^2} + \tilde{V}(\rho) \). This way we can write the Hamilton operator as \( H = H_0 + \tilde{V}(\rho) \), where \( H_0 = -\frac{d^2}{d\rho^2} - \frac{1}{4\rho^2} \). \( H_0 \) is not selfadjoint on its natural domain of definition \( \mathcal{D}(H_0) = \mathcal{D}(\frac{d^2}{d\rho^2}) \cap \mathcal{D}(\frac{1}{4\rho^2}) \), since \( \mathcal{D}(H_0) \neq \mathcal{D}(H_0^*) \). This problem can be solved by extending in an appropriate way the domain of definition of \( H_0 \), such that for the corresponding operator \( H_{0,a} \) (\( a \) is real) \( \mathcal{D}(H_{0,a}) \subset \mathcal{D}(H_0^*) \) and \( \mathcal{D}(H_{0,a}) = \mathcal{D}(H_{0,a}) = \mathcal{D}(H_{0,a}) = \mathcal{D}(H_{0,a}) \). We define

\[
\mathcal{D}(H_{0,a}) = \mathcal{D}(H_0) + \Psi_a
\]

with

\[
\Psi_a = \sqrt{\rho}(H_0^{(1)}[\exp(i\frac{\pi}{4})\rho] + \exp(ia)H_0^{(1)}[\exp(i\frac{\pi}{4})\rho])
\]

and \( H_0^{(1)} \) is the Hankel function. The spectrum of \( H_{0,a} \) depends on the value of \( a \), which parameterizes the extension. For our problem \( \Psi_a \) must be real, since it
describes a perturbation of the equilibrium solution for a real scalar field. Ψ_a is real only for a = 0. On this domain of definition D(H_{0,a=0}) H_0 (means H_{0,a=0} from now on) has a discrete eigenvalue E = 0, whose eigenfunction Ψ_0=Ψ_{a=0} is in \mathcal{L}^2(d\rho,(0,\infty)). From the scaling property of H_0, it follows that this eigenvalue is infinitely degenerated.

Using Ψ_0(\rho) as a trial function we get upper bounds for the eigenvalue \sigma_0^2:

\sigma_0^2 \leq \frac{\langle \Psi_{0,\beta} | H_0 + \tilde{V} | \Psi_{0,\beta} \rangle}{\langle \Psi_{0,\beta} | \Psi_{0,\beta} \rangle} = \frac{\langle \Psi_{0,\beta} | \tilde{V} | \Psi_{0,\beta} \rangle}{\langle \Psi_{0,\beta} | \Psi_{0,\beta} \rangle} \quad \text{(5)}

For all equilibrium solutions we easily find a value for \beta such that \langle \Psi_{0,\beta} | \tilde{V} | \Psi_{0,\beta} \rangle is negative. Indeed, for large \beta, \Psi_0(\beta \rho) reaches its maximum at smaller values of \rho than for \beta = 1. \tilde{V}(\rho) is negative at the origin and behaves as \sim const/\rho for \rho \to 0. Thus by appropriately choosing \beta, \Psi_0(\beta \rho) has most of its support where \tilde{V}(\rho) is negative, which leads then to a negative value for \langle \Psi_{0,\beta} | \tilde{V} | \Psi_{0,\beta} \rangle. Therefore, we conclude that all the static solutions are dynamically unstable.

This result is in agreement with the cosmic censorship conjecture, which excludes spacetimes with naked singularities.

4 Conclusion

From the above considerations it follows that only with complex scalar fields one may hope to form boson stars, and it is thus natural to study their formation and properties within the standard cosmological model. To that purpose an analysis of the coupled Einstein-Klein-Gordon equations using the Friedmann-Lemaître metric has been carried out for a complex field in Ref. 9. Moreover, the time evolution of the perturbations of the complex scalar field and the metric has been analyzed in Ref. 10. It turns out, that during the oscillatory phase after inflation the perturbations of the complex scalar field at best oscillate. Therefore, the formation of boson stars in a universe driven by the same scalar field is not possible.

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