On integral weight spectra of the MDS codes  

cosets of weight 1, 2, and 3  

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Abstract. The weight of a coset of a code is the smallest Hamming weight of any vector in the coset. For a linear code of length $n$, we call integral weight spectrum the overall numbers of weight $w$ vectors, $0 \leq w \leq n$, in all the cosets of a fixed weight. For maximum distance separable (MDS) codes, we obtained new convenient formulas of integral weight spectra of cosets of weight 1 and 2. Also, we give the spectra for the weight 3 cosets of MDS codes with minimum distance 5 and covering radius 3.  

Keywords: cosets weight distribution, MDS codes.  

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1 Introduction  

Let $\mathbb{F}_q$ be the Galois field with $q$ elements, $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$. Let $\mathbb{F}_q^n$ be the space of $n$-dimensional vectors over $\mathbb{F}_q$. We denote by $[n, k, d]_q R$ an $\mathbb{F}_q$-linear code of length $n$, dimension $k$, minimum distance $d$, and covering radius $R$. If $d = n - k + 1$, it is a maximum distance separable (MDS) code. For an introduction to coding theory see [2,11,16,19].

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A *coset* of a code is a translation of the code. A coset $\mathcal{V}$ of an $[n, k, d]_q^R$ code $\mathcal{C}$ can be represented as

$$\mathcal{V} = \{ \mathbf{x} \in \mathbb{F}_q^n | \mathbf{x} = \mathbf{c} + \mathbf{v}, \mathbf{c} \in \mathcal{C} \} \subset \mathbb{F}_q^n \quad (1.1)$$

where $\mathbf{v} \in \mathcal{V}$ is a vector fixed for the given representation; see [2][11][16][17][19] and the references therein.

The weight distribution of code cosets is an important combinatorial property of a code. In particular, the distribution serves to estimate decoding results. There are many papers connected with distinct aspects of the weight distribution of cosets for codes over distinct fields and rings, see e.g. [1][7][9][10][12][15][20][21], [8] Sect. 6.3], [11] Sect. 7], [16] Sections 5.5, 6.6, 6.9], [17] Sect. 10] and the references therein.

For a linear code of length $n$, we call *integral weight spectrum* the overall numbers of weight $w$ vectors, $0 \leq w \leq n$, in all the cosets of a fixed weight.

In this work, for MDS codes, using and developing the results of [3], we obtain new convenient formulas of integral weight spectra of cosets of weight 1 and 2. The obtained formulas for weight 1 and 2 cosets, seem to be simple and expressive.

This paper is organized as follows. Section 2 contains preliminaries. In Section 3 we consider the integral weight spectrum of the weight 1 cosets of MDS codes with minimum distance $d \geq 3$. In Section 4 we obtain the integral weight spectrum of the weight 2 cosets of MDS codes with minimum distance $d \geq 5$. In Section 5 we give the spectra for the weight 3 cosets of MDS codes with minimum distance 5 and covering radius 3.

## 2 Preliminaries

### 2.1 Cosets of a linear code

We give a few known definitions and properties connected with cosets of linear codes, see e.g. [2][11][16][17][10] and the references therein.

We consider a coset $\mathcal{V}$ of an $[n, k, d]_q^R$ code $\mathcal{C}$ in the form (1.1). We have $\#\mathcal{V} = \#\mathcal{C} = q^k$. One can take as $\mathbf{v}$ any vector of $\mathcal{V}$. So, there are $\#\mathcal{V} = q^k$ formally distinct representations of the form (1.1); all they give the same coset $\mathcal{V}$. If $\mathbf{v} \in \mathcal{C}$, we have $\mathcal{V} = \mathcal{C}$. The distinct cosets of $\mathcal{C}$ partition $\mathbb{F}_q^n$ into $q^{n-k}$ sets of size $q^k$.

We remind that the *Hamming weight* of the vector $\mathbf{x} \in \mathbb{F}_q^n$ is the number of nonzero entries in $\mathbf{x}$.

**Notation 2.1.** For an $[n, k, d]_q^R$ code $\mathcal{C}$ and its coset $\mathcal{V}$ of the form (1.1), the following notation is used:

$$t = \left\lfloor \frac{d - 1}{2} \right\rfloor$$
the number of correctable errors;
$A_w(C)$ the number of weight $w$ codewords of the code $C$;
$A_w(V)$ the number of weight $w$ vectors in the coset $V$;
the weight of a coset the smallest Hamming weight of any vector in the coset;
$V^W$ a coset of weight $W$; $A_w(V^W) = 0$ if $w < W$;
a coset leader a vector in the coset of the smallest Hamming weight;
$\mathcal{A}_w^W(V^W)$ the overall number of weight $w$ vectors in all cosets of weight $W$;
$\mathcal{A}_w^\leq W(V)$ the overall number of weight $w$ vectors in all cosets of weight $\leq W$.

In cosets of weight $> t$, a vector of the minimal weight is not necessarily unique. Cosets of weight $\leq t$ have a unique leader.

The code $C$ is the coset of weight zero. The leader of $C$ is the zero vector of $\mathbb{F}_q^n$.

**Definition 2.2.** Let $C$ be an $[n, k, d]_q R$ code and let $V^W$ be its coset of weight $W$. Let $\mathcal{A}_w^W(V^W)$ be the overall number of weight $w$ vectors in all cosets of weight $W$. For a fixed $W$, we call the set $\{A_w^W(V^W)|w = 0, 1, \ldots, n\}$ integral weight spectrum of the code cosets of weight $W$.

Distinct representations of the integral weight spectra $\mathcal{A}_w^W(V^W)$ and values of $\mathcal{A}_w^\leq W(V)$ are considered in the literature, see e.g. [2, Th. 14.2.2], [5,6], [15, Lem. 2.14], [16, Th. 6.22]. For instance, in [5, Eqs. (11)-(13)], for an MDS code correcting $t$-fold errors, the value $D_u$ gives $\mathcal{A}_w^\leq t(V)$.

### 2.2 Some useful relations

For $w \geq d$, the weight distribution $A_w(C)$ of an $[n, k, d = n - k + 1]_q$ MDS code $C$ has the following form, see e.g. [11, Th. 7.4.1], [16, Th. 11.3.6]:

$$A_w(C) = \binom{n}{w} \sum_{j=0}^{w-d} (-1)^j \binom{w}{j} (q^{w-d+1-j} - 1). \quad (2.1)$$

In $\mathbb{F}_q^n$, the volume of a sphere of radius $t$ is

$$V_n(t) = \sum_{i=0}^{t} (q - 1)^i \binom{n}{i}. \quad (2.2)$$

The following combinatorial identities are well known, see e.g. [18, Sect. 1, Eqs. (I),(IV), Problem 9(a)]:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}. \quad (2.3)$$
\[
\binom{n}{m} \binom{m}{p} = \binom{n}{p} \binom{n-p}{m-p} = \binom{n}{m-p} \binom{n-m+p}{p}.
\]  
(2.4)

\[
\sum_{k=0}^{m} (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m}.
\]  
(2.5)

In Eqs. (11)–(13), for an \([n, k, d \geq 2t + 1]_q\) MDS code correcting \(t\)-fold errors, the following relations for \(\mathcal{A}_u^\Sigma(V \leq t)\) denoted by \(D_u\) are given:

\[
\mathcal{A}_u^\Sigma(V \leq t) = D_u = \binom{n}{u} \sum_{j=0}^{u-d+t} (-1)^j N_j, \quad d - t \leq u \leq n,
\]  
(2.6)

where

\[
N_j = \binom{u}{j} \left[ q^{u-d+1-j} V(t) - \sum_{i=0}^{t} \binom{u-j}{i} (q-1)^i \right] \quad \text{if} \quad 0 \leq j \leq u - d,
\]  
(2.7)

\[
N_j = \binom{u}{j} \left[ \sum_{w=d-u+j}^{t} \binom{n-u+j}{w} \sum_{i=0}^{w-d+u-j} (-1)^i \binom{w}{i} (q^{w-d+u-j-i+1} - 1) \right. \\
\left. \times \sum_{s=w}^{t} \binom{u-j}{s-w} (q-1)^{s-w} \right] \quad \text{if} \quad u - d + 1 \leq j \leq u - d + t.
\]  
(2.8)

3 The integral weight spectrum of the weight 1 cosets of MDS codes with minimum distance \(d \geq 3\)

In Sections 3–5, we represent the values \(\mathcal{A}_w^\Sigma(V^{(W)})\) in distinct forms that can be convenient in distinct utilizations, e.g. for estimates of the decoder error probability, see [5,6] and the references therein.

We use (with some transformations) the results of Eqs. (11)–(13) where, for an MDS code correcting \(t\)-fold errors, the value \(D_u\) gives the overall number \(\mathcal{A}_u^\Sigma(V \leq t)\) of weight \(u\) vectors in all cosets of weight \(\leq t\). We cite Eqs. (11)–(13)] by formulas (2.6)–(2.8), respectively.

In the rest of the paper we put that a sum \(\sum_{i=0}^{A} \ldots\) is equal to zero if \(A < 0\).

**Lemma 3.1.** Eqs. (11)–(13)] Let \(d-1 \leq w \leq n\). For an \([n, k, d = n - k + 1]_q\) MDS code \(C\) of minimum distance \(d \geq 3\), the overall number \(\mathcal{A}_w^\Sigma(V \leq 1)\) of weight \(w\) vectors in all cosets of weight \(\leq 1\) is as follows:

\[
\mathcal{A}_w^\Sigma(V \leq 1) = \binom{n}{w} \left[ \sum_{j=0}^{w-d} (-1)^j \binom{w}{j} \left[ q^{w-d+1-j}(1 + n(q-1)) - 1 - (w-j)(q-1) \right] \right] 
\]  
(3.1)
\[ -(-1)^{w-d} \binom{w}{d-1} (n-d+1)(q-1). \]

**Proof.** In the relations for \( D_u \) of [5] cited by (2.6)–(2.8), we put \( t = 1 \) and then use (2.2). In (2.8), we have \( j = u - d + 1 \) whence \( w = 1 \) in all terms. Finally, we change \( u \) by \( w \) to save the notations of this paper. \( \square \)

**Lemma 3.2.** The following holds:

\[
\sum_{j=0}^{m} (-1)^j \binom{w}{j} \binom{w-j}{v} = (-1)^m \binom{w}{v} \binom{w-v-1}{m}.
\] (3.2)

**Proof.** In (2.4), we put \( n = w, \ p = j, \ m - p = v \), and obtain

\[
\sum_{j=0}^{m} (-1)^j \binom{w}{j} \binom{w-j}{v} = \binom{w}{v} \sum_{j=0}^{m} (-1)^j \binom{w-v}{j}.
\]

Now we use (2.5). \( \square \)

**Lemma 3.3.** Let \( d - 1 \leq w \leq n \). The following holds:

\[
\sum_{j=0}^{w+1-d} (-1)^j \binom{w}{j} q^{w+1-d-j} = \sum_{j=0}^{w-d} (-1)^j \binom{w}{j} (q^{w+1-d-j} - (-1)^{w-d} \binom{w-1}{d-2}).
\]

**Proof.** We write the left sum of the assertion as

\[
\sum_{j=0}^{w-d} (-1)^j \binom{w}{j} (q^{w+1-d-j} - 1) - (-1)^{w-d} \binom{w}{d-1}.
\]

By (2.5),

\[
\sum_{j=0}^{w-d} (-1)^j \binom{w}{j} = (-1)^{w-d} \binom{w-1}{d-1}.
\]

Finally, we apply (2.3). \( \square \)

For an \([n, k, d]_q\) code \( C \), we denote

\[
\Omega_w^{(j)}(C) = (-1)^{w-d} \binom{n-j}{w-j} \binom{w-j-1}{d-j-2}.
\] (3.3)

Also, we denote

\[
\Phi_w^{(j)} = (-1)^{w-5} \left[ \binom{q+1}{w} \binom{w-1}{3} - \binom{q+1-j}{w-j} \binom{w-1-j}{3-j} \right].
\] (3.4)
Theorem 3.4. (integral weight spectrum 1)

Let \(d - 1 \leq w \leq n\). Let \(\mathcal{C}\) be an \([n, k, d = n - k + 1]_q\) MDS code of minimum distance \(d \geq 3\).

(i) For the code \(\mathcal{C}\), the overall number \(A_w^\Sigma(\mathcal{V}^{(1)})\) of weight \(w\) vectors in all weight 1 cosets is as follows:

\[
A_w^\Sigma(\mathcal{V}^{(1)}) = \binom{n}{w}(q-1) \left[ n \sum_{j=0}^{w+1-d} (-1)^j \binom{w}{j} q^{w+1-d-j} + (-1)^{w-d} \binom{w-2}{d-3} \right] \tag{3.5}
\]

\[
= n(q-1) \left[ \binom{n}{w} \sum_{j=0}^{w+1-d} (-1)^j \binom{w}{j} q^{w+1-d-j} + \Omega_{w}^{(1)}(\mathcal{C}) \right] \tag{3.6}
\]

\[
= n(q-1) \left[ \binom{n}{w} \sum_{j=0}^{w-d} (-1)^j \binom{w}{j} (q^{w+1-d-j} - 1) - \Omega_{w}^{(0)}(\mathcal{C}) + \Omega_{w}^{(1)}(\mathcal{C}) \right] \tag{3.7}
\]

\[
= n(q-1) \left[ A_w(\mathcal{C}) - \Omega_{w}^{(0)}(\mathcal{C}) + \Omega_{w}^{(1)}(\mathcal{C}) \right] \tag{3.8}
\]

\[
= n(q-1) \left[ A_w(\mathcal{C}) - (-1)^{w-d} \left( \binom{n}{w} \binom{w-1}{d-2} - \binom{n-1}{w-1} \binom{w-2}{d-3} \right) \right]. \tag{3.9}
\]

(ii) Let the code \(\mathcal{C}\) be a \([q+1, k, d = q + 2 - k]_q\) MDS code of length \(n = q + 1\) and minimum distance \(d \geq 3\). For \(\mathcal{C}\), the overall number \(A_w^\Sigma(\mathcal{V}^{(1)})\) of weight \(w\) vectors in all weight 1 cosets is as follows:

\[
A_w^\Sigma(\mathcal{V}^{(1)}) = \binom{q+1}{w}(q-1) \left[ q^{w+2-d} - \sum_{i=0}^{w-d} (-1)^i \left( \binom{w}{i+1} - \binom{w}{i} \right) q^{w+1-d-i} \right] \tag{3.10}
\]

\[
= (-1)^{w-d} \left( \binom{w}{d-1} - \binom{w}{d-3} \right), \quad d - 1 \leq w \leq q + 1.
\]

(iii) Let the code \(\mathcal{C}\) be a \([q+1, q - 3, 5]_q\) MDS code of length \(n = q + 1\) and minimum distance \(d = 5\). For \(\mathcal{C}\), the overall number \(A_w^\Sigma(\mathcal{V}^{(1)})\) of weight \(w\) vectors in all weight 1 cosets is as follows:

\[
A_w^\Sigma(\mathcal{V}^{(1)}) = (q^2 - 1) \left[ A_w(\mathcal{C}) - \Phi_{w}^{(1)} \right], \quad 4 \leq w \leq q + 1. \tag{3.11}
\]

Proof. (i) By the definition of \(A_w^\Sigma(\mathcal{V}^{\leq 1})\), see Notation 2.1, for the code \(\mathcal{C}\) of Lemma 3.1, we have

\[
A_w^\Sigma(\mathcal{V}^{(1)}) = A_w^\Sigma(\mathcal{V}^{\leq 1}) - A_w(\mathcal{C}). \tag{3.12}
\]

We subtract (2.1) from (3.11) that gives

\[
A_w^\Sigma(\mathcal{V}^{(1)}) = \binom{n}{w}(q-1) \left[ -(-1)^{w-d} \binom{w}{d-1} (n-d+1) \right] \tag{3.13}
\]
\begin{align*}
&+ \sum_{j=0}^{w-d} (-1)^j \binom{w}{j} \left( q^{w-d+1-j} n - w + j \right) \\
&= \binom{n}{w} (q - 1) \left[ n \sum_{j=0}^{w-d+1} (-1)^j \binom{w}{j} q^{w-d+1-j} - \sum_{j=0}^{w-d+1} (-1)^j \binom{w}{j} (w - j) \right].
\end{align*}

Here some simple transformations are missed out. Now, for the 2-nd sum \( \sum_{j=0}^{w-d+1} \ldots \), we use Lemma 3.2 and obtain (3.5).

To form (3.6) from (3.5), we change \( w^{(n)} \) by \( n^{(n-1)} \), see (2.4). To obtain (3.7) from (3.6), we apply Lemma 3.3. For (3.8), we use (2.1). Finally, (3.9) is (3.8) in detail.

(ii) We substitute \( n = q + 1 \) to (3.5) that implies (3.10) after simple transformations.

(iii) We substitute \( n = q + 1 \) and \( d = 5 \) to (3.9) that gives (3.11).

For \( A_{w}^{(V \leq 1)} \), we give a formula alternative to (3.1).

\textbf{Corollary 3.5.} Let \( V_n(1) \) be as in (2.2). Let \( C \) be an \( [n, k, d = n - k + 1]_q \) MDS code of minimum distance \( d \geq 3 \). Then for \( C \), the overall number \( A_{w}^{(V \leq 1)} \) of weight \( w \) vectors in all cosets of weight \( \leq 1 \) is as follows:

\begin{equation}
A_{w}^{(V \leq 1)} = A_w(C) \cdot V_n(1) - (-1)^w n(q - 1) \sum_{j=0}^{1} (-1)^j \binom{n-j}{w-j} \binom{w-j-1}{d-j-2}.
\end{equation}

\textit{Proof.} We use (3.12) and (3.9). \hfill \Box

\section{The integral weight spectrum of the weight 2 cosets of MDS codes with minimum distance \( d \geq 5 \)}

As well as in Lemma 3.1, we use the results of [5] with some transformations.

\textbf{Lemma 4.1.} [5, Eqs. (11)–(13)] Let \( d - 2 \leq w \leq n \). Let \( V_n(t) \) be as in (2.2). For an \( [n, k, d = n - k + 1]_q \) MDS code \( C \) of minimum distance \( d \geq 5 \), the overall number \( A_{w}^{(V \leq 2)} \) of weight \( w \) vectors in all cosets of weight \( \leq 2 \) is as follows:

\begin{equation}
A_{w}^{(V \leq 2)} = \binom{n}{w} \sum_{j=0}^{w-d} (-1)^j \binom{w}{j} \left[ q^{w-d+1-j} \cdot V_n(2) - V_{w-j}(2) \right].
\end{equation}

\begin{equation}
-(-1)^w \frac{n - d + 1}{2} \left( \binom{w}{d-1} [2 + (q - 1)(n + d - 2)] - \binom{w}{d} (n - d + 2) \right). \hfill \Box
\end{equation}

\textit{Proof.} In the relations for \( D_u \) of [5] cited by (2.6)–(2.8), we put \( t = 2 \) that gives, in (2.8), \( j = u - d + 1 \) and \( j = u - d + 2 \), whence \( w = 1, 2 \) and \( w = 2 \), respectively. Then we do simple transformations. Finally, we change \( u \) by \( w \) to save the notations of this paper. \hfill \Box
For an $[n, k, d]_q$ code $C$, we denote
\[
\Delta_w(C) = (-1)^{w-d} \binom{n}{w} \binom{w}{d-2} \binom{n-d+2}{2} (q-1); \quad (4.2)
\]
\[
\Delta^*_w(C) = \frac{\Delta_w(C)}{\binom{n}{2}(q-1)^2}.
\]

Lemma 4.2. The following holds:
\[
\Delta^*_w(C) = (-1)^{w-d} \binom{n-d+2}{n-w} \binom{n-2}{d-2} \frac{1}{q-1}. \quad (4.3)
\]

Proof. By (2.4), we have
\[
\binom{n}{w} \binom{w}{d-2} = \binom{n}{d-2} \binom{n-d+2}{w-d-2} = \binom{n}{d-2} \binom{n-d+2}{n-w},
\]
\[
\binom{n}{d} \binom{n-d+2}{2} = \binom{n}{d} \binom{d}{d-2} = \binom{n}{d} \binom{d}{2} = \binom{n}{d} \binom{n-2}{d-2}.
\]

\[
\begin{align*}
\text{Theorem 4.3. (integral weight spectrum 2)} \\
\text{Let } d - 2 \leq w \leq n. \text{ Let } C \text{ be an } [n, k, d = n - k + 1]_q \text{ MDS code of minimum distance } d \geq 5. \text{ Let } \Omega_w^{(j)}(C) \text{ and } \Phi_w^{(j)} \text{ be as in (3.3) and (3.4).}
\end{align*}
\]

\(\text{(i) For the code } C \text{, the overall number } A_w^\Sigma(V^{(2)}) \text{ of weight } w \text{ vectors in all weight 2 cosets is as follows:}
\]
\[
A_w^\Sigma(V^{(2)}) = \binom{n}{w}(q-1)^2 \left[ \binom{n}{2} \sum_{j=0}^{w-1-d} (-1)^j \binom{w}{j} q^{w+1-d-j} + (-1)^{w-d} \binom{w}{d} \binom{w-3}{d-4} \right] + \Delta_w(C). \quad (4.4)
\]
\[
= \binom{n}{2} (q-1)^2 \left[ \binom{n}{w} \sum_{j=0}^{w+1-d} (-1)^j \binom{w}{j} q^{w+1-d-j} + \Omega_w^{(2)}(C) \right] + \Delta_w(C). \quad (4.5)
\]
\[
= \binom{n}{2} (q-1)^2 \left[ \binom{n}{w} \sum_{j=0}^{w-d} (-1)^j \binom{w}{j} (q^{w+1-d-j} - 1) - \Omega_w^{(0)}(C) + \Omega_w^{(2)}(C) \right] + \Delta_w(C) \quad (4.6)
\]
\[
= \binom{n}{2} (q-1)^2 \left[ A_w(C) - \Omega_w^{(0)}(C) + \Omega_w^{(2)}(C) \right] + \binom{n}{2} (q-1)^2 \Delta_w^*(C) \quad (4.7)
\]
We subtract (3.1) from (4.1) that gives

\[ (\text{4.8}) \]

\[ \begin{align*}
\text{Proof.} \quad \text{(i)} \quad & \text{Applying Lemma 3.2 to the 2-nd sum due to (2.4) and (2.3), we have } \\
& \text{distance is as follows } \\
& \text{(ii) Let the code } C \text{ be a } [q + 1, q - 3, 5]_q \text{ MDS code of length } n = q + 1 \text{ and minimum distance } d = 5. \text{ For } C, \text{ the overall number } A_w(\mathcal{V}^1) \text{ of weight } w \text{ vectors in all weight 1 cosets is as follows } \\
& A_w(\mathcal{V}^2) = \left( \frac{q + 1}{2} \right) (q - 1)^2 \left[ A_w(C) - \Phi_w^{(2)} + (-1)^{w-3} \frac{1}{3} \left( \frac{q - 2}{w - 3} \right) \left( \frac{q - 2}{2} \right) \right], \quad (4.9) \\
& 3 \leq w \leq q + 1. \\
\end{align*} \]

By the definition of \( A_w(\mathcal{V}^2) \), see Notation 2.1 for the code \( C \) of Lemma 4.1, we have

\[ A_w^\Sigma(\mathcal{V}^2) = A_w^\Sigma(\mathcal{V}^2) - A_w^\Sigma(\mathcal{V}^1). \quad (4.10) \]

We subtract (3.1) from (4.1) that gives

\[ \begin{align*}
\text{(4.8)} & \text{ that gives } \\
A_w^\Sigma(\mathcal{V}^2) & = \left( \frac{n}{w} \right) (q - 1)^2 \sum_{j=0}^{w-d} (-1)^j \left( \begin{array}{c} w \\ j \end{array} \right) q^{w+1-d-j} \left( \begin{array}{c} n \\ 2 \end{array} \right) (q - 1)^2 - \left( \begin{array}{c} w - j \\ 2 \end{array} \right) (q - 1)^2 \\
& + (-1)^{w+1-d} \left( \begin{array}{c} w \\ d - 1 \end{array} \right) \frac{1}{2} (n - d + 1)(q - 1)^2(n + d - 2) + \Delta_w(C) \\
& = \left( \frac{n}{w} \right) (q - 1)^2 \left[ \left( \frac{n}{2} \right) \sum_{j=0}^{w-d} (-1)^j \left( \begin{array}{c} w \\ j \end{array} \right) q^{w+1-d-j} - \sum_{j=0}^{w-d} (-1)^j \left( \begin{array}{c} w \\ j \end{array} \right) \left( \begin{array}{c} w - j \\ 2 \end{array} \right) \\
& - (-1)^{w-d} \left( \begin{array}{c} w \\ d - 1 \end{array} \right) \left( \frac{1}{2} (n - d + 1)(n + d - 2) + \left( \begin{array}{c} n \\ 2 \end{array} \right) - \left( \begin{array}{c} n \\ 2 \end{array} \right) \right) \right] + \Delta_w(C). \\
\end{align*} \]

\[ \text{Applying Lemma 3.2 to the 2-nd sum } \sum_{j=0}^{w-d} \ldots, \text{ after simple transformations we obtain } \]

\[ A_w^\Sigma(\mathcal{V}^2) = \left( \frac{n}{w} \right) (q - 1)^2 \left[ \left( \frac{n}{2} \right) \sum_{j=0}^{w+1-d} (-1)^j \left( \begin{array}{c} w \\ j \end{array} \right) q^{w+1-d-j} - (-1)^{w-d} \left( \begin{array}{c} w \\ 2 \end{array} \right) \left( \begin{array}{c} w - 3 \\ w - d \end{array} \right) \\
& + (-1)^{w-d} \left( \begin{array}{c} w \\ d - 1 \end{array} \right) \left( \frac{d - 1}{2} \right) \right] + \Delta_w(C). \]

\[ \text{Due to (2.4) and (2.3), we have } \]

\[ \left( \begin{array}{c} w \\ d - 1 \end{array} \right) \left( \frac{d - 1}{2} \right) = \left( \begin{array}{c} w \\ 2 \end{array} \right) \left( \begin{array}{c} w - 2 \\ d - 3 \end{array} \right) = \left( \begin{array}{c} w \\ 2 \end{array} \right) \left[ \left( \begin{array}{c} w - 3 \\ d - 4 \end{array} \right) + \left( \begin{array}{c} w - 3 \\ d - 3 \end{array} \right) \right]. \]
Also, \( \binom{w-3}{w-d} = \binom{w-3}{d-3} \). Now we can obtain (4.4). Moreover, by (2.4), we have

\[
\binom{n}{w} \binom{w}{2} = \binom{n}{2} \binom{n-2}{w-2}
\]

that gives (4.5).

To obtain (4.6) from (4.5), we apply Lemma 3.3. For (4.7), we use (2.1). Finally, (4.8) is (4.7) in detail.

(ii) We substitute \( n = q + 1 \) and \( d = 5 \) to (4.8) that gives (4.9).

5 The integral weight spectrum of the weight 3 cosets of MDS codes with minimum distance \( d = 5 \) and covering radius \( R = 3 \)

**Theorem 5.1. (integral weight spectrum 3)**

Let \( d - 2 \leq w \leq n \). Let \( C \) be an \([n, n-4, 5]_{q^3}\) MDS code of minimum distance \( d = 5 \) and covering radius \( R = 3 \). Let \( V_n(t), \Phi_w^{(j)}, A_w^{V(1)}, \) and \( \Delta_w(C) \) be as in (2.2), (3.4), (4.1), and (4.2), respectively. Let \( A_w^{V(1)}(V(1)) \) and \( A_w^{V(2)}(V(2)) \) be as in Theorems 3.4 and 4.3, respectively.

(i) For the code \( C \), the overall number \( A_w^{V(3)}(V^3) \) of weight \( w \) vectors in all cosets of weight 3 is as follows:

\[
A_w^{V(3)}(V^3) = \binom{n}{w}(q-1)^w - A_w^{V(2)}(V(2)) \tag{5.1}
\]

\[
= \binom{n}{w}(q-1)^w - \left[ A_w(C) + A_w^{V(1)} + A_w^{V(2)} \right] \tag{5.2}
\]

\[
= \binom{n}{w}(q-1)^w - \left[ \binom{n}{w} \sum_{j=0}^{w-5} (-1)^j \binom{w}{j} [q^{w-1-j} \cdot V_n(2) - V_{w-j}(2)] \right] \tag{5.3}
\]

\[
= (-1)^{w-5} \binom{n-4}{2} (q-1) \binom{w}{4} \left[ 2 + (q-1)(n+3) - \binom{w}{3}(n-3) \right].
\]

(ii) Let the code \( C \) be a \([q+1, q-3, 5]_{q^3}\) MDS code of length \( n = q + 1 \), minimum distance \( d = 5 \), and covering radius \( R = 3 \). For \( C \), the overall number \( A_w^{V(3)}(V^3) \) of weight \( w \) vectors
in all weight 3 cosets is as follows

\[ A_w^\Sigma(V^{(3)}) = \binom{q+1}{w}(q-1)^w - \left( \binom{q+1}{w} \sum_{j=0}^{w-5} (-1)^j \binom{w}{j} [q^{w-4-j} \cdot V_{q+1}(2) - V_{w-j}(2)] - (-1)^{w-5} \frac{(q-3)(q-1)}{2} \left( \binom{w}{4} (q^2 + 3q - 2) - \binom{w}{3} (q - 2) \right) \right) \]

\[ = \binom{q+1}{w}(q-1)^w - \left[ V_{q+1}(2)A_w(C) - (q^2 - 1)\Phi_w^{(1)} - \frac{q+1}{2} (q - 1)^2 \Phi_w^{(2)} - \Delta_w(C) \right]. \]

(5.4)

\[ = \binom{q+1}{w}(q-1)^w - \left[ V_{q+1}(2)A_w(C) - (q^2 - 1)\Phi_w^{(1)} - 2(q + 1) \Phi_w^{(2)} - \Delta_w(C) \right]. \]

(5.5)

**Proof.** (i) Due to covering radius 3, in \( C \) there are not cosets of weight \( > 3 \); therefore for \( C \) we have (5.1) where \( \binom{n}{w}(q-1)^w \) is the total number of weight \( w \) vectors in \( \mathbb{F}_q^n \).

The relation (5.2) follows from (5.1), (3.12), and (4.10).

To form (5.3), we substitute (4.1) to (5.1) with \( d = 5 \).

(ii) We substitute \( n = q + 1 \) to (5.3) and obtain (5.4).

To obtain (5.5) from (5.2), we use (3.11), (4.9), (4.2), and (4.3) with \( n = q + 1, d = 5 \).

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