Spectral and time-domain data representations in measurements

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Abstract. Classic stationary models and spectral representations of data are commonly used in practice, but they are not sufficient for solving modern problems of data processing in measurements. There are some kinds of non-stationary models, which are based on generalized spectral representations. Besides, some classes of time-domain representations are studied for non-stationary random functions. In the paper spectral and time-domain representations are compared in some aspects, including field of application, and efficiency for data processing in measurements. Regarding generalized spectral models, harmonizable processes and random processes with stationary increments are considered. The latter group seems to be especially useful for measurement problems. It is much wider, than the set of stationary processes, but these processes also have spectral representations. The extended model enables one to develop spectral methods of data processing, and also permits to investigate new data characteristics. For instance, Allan variance, which is widely used nowadays, is the estimate of structure function within this model.

As regards time-domain models, general canonical representations of the processes are considered, which may possess any spectral type and multiplicity. However, one-dimensional processes, which are used in practice, have multiplicity one.

1. Introduction

Stationary models of data are generally used in measurement practice. However, they are not sufficient for solving modern problems of data processing in measurements. Stationary processes have two kinds of integral expansion: spectral and time-domain representations [1]. They may be generalized in order to extend data models and devise data processing methods.

Some kinds of generalized spectral representations are developed [1, 2] for non-stationary random processes. Thus, it is reasonable to consider applications of these generalized representations for solving the measurement problems.

On the other hand, some general time-domain representations are devised for random functions [3]. These representations may also form a basis for development of data processing methods. But they are very general, so the practically important cases should be selected.

In this paper the spectral and time-domain representations are compared in several aspects, including the field of application, and the efficiency for data processing in measurements. It is reasonable to outline the main types of generalized representations, which are useful for measurement problems and may also provide basis for practical methods of data processing. While analyzing models, the formal theory of statistical model extension [4] may be also used.
2. Spectral representations of stationary and related processes

2.1. Stationary and harmonizable processes

Analysis of random signals and errors is commonly based on the model of second-order stationary random process. It is defined by the property that mean and correlation function do not change when shifted in time, so \( R(s, t) = R(s-t) \). The basic representation of this process is the spectral one [1]:

\[
X(t) = \int \exp(\imath t \lambda) \, dZ(\lambda),
\]

(1)

where \( Z(\lambda) \) is random process with non-correlated increments, \( \mathbb{E} |dZ(\lambda)|^2 = f(\lambda) \, d\lambda \).

The main characteristic of stationary process is spectral density \( f(\lambda) \), and the correlation function is quite defined by means of it:

\[
R(t) = \int \exp(\imath t \lambda) \, f(\lambda) \, d\lambda.
\]

(2)

So the spectral model of stationary process may be symbolically presented in the form:

\[
\Omega(X|S_t) = \{X(R) : R(t) = \Phi[f(\lambda)]\},
\]

(3)

where left side denote a set of stationary processes, and \( \Phi \) is a symbol of Fourier transform. The main methods of stationary data processing are generally based on spectral representations [5, 6].

However, in practice stationary condition is often too restrictive, and the use of common spectral methods in non-stationary case leads to significant systematic errors [2, 6]. So the proper extensions of the spectral representations (1) and (2) are called for.

Essential extension is a set of harmonizable processes, which are defined by (1), but with possible correlation of increments for \( Z(\lambda) \) [2, 6]. So the correlation function is defined by generalized spectral representation with bivariate spectral density \( f(\lambda, \mu) \):

\[
R(s, t) = \int \exp[i(t \lambda - s \mu)] \, f(\lambda, \mu) \, d\lambda \, d\mu.
\]

(4)

Similarly, spectral expansion of harmonizable process may be symbolically presented in the form:

\[
\Omega(X|H) = \{X(R) : R(s, t) = \Phi_2[f(\lambda, \mu)]\},
\]

(5)

where left side denote a set of harmonizable processes, \( \Phi_2 \) is a symbol of bivariate Fourier transform.

Unfortunately, the general representations (4) are much more tedious, than classical relation (2). Thus, general relation (4) may be used for analysis of harmonizable processes, which are similar to stationary processes, or are obtained by transformations of the stationary ones. Its effective application for data processing seems to be limited, so this model is not widely used in measurements.

2.2. Processes with stationary random increments

Sometimes non-stationary behavior of random process is presented in such a way, that the increments form a stationary process; for instance, it is valid for Wiener process. Thus, the model of process with stationary increments (SIP) seems to be promising [1, 2]. Such process is defined by the properties:

- mathematical expectation of the increments is a linear function: \( \mathbb{E}[X(s)-X(t)] = a \, (s-t) \);
- the structure function (that is, variance of the increments) is invariant under shift:

\[
D_\delta(\tau) = \mathbb{E}[X(t+\tau)-X(t)]^2.
\]

(6)

This process has a spectral representation of the form:

\[
X(t) = \hat{f} \exp(\imath t \lambda - \imath / \lambda) \, dZ(\lambda) + X_0,
\]

(7)

where \( Z(\lambda) \) is a random process with non-correlated increments.

Similarly, the structure function has spectral representation of the form:

\[
D_\delta(\tau) = 2 \int \frac{\hat{f} \, (1 - \cos \tau \lambda) / \lambda^2} {d\lambda}.
\]

(8)
where $dF(\lambda) = E \left| dZ(\lambda) \right|^2$. The function $F(\lambda)$ is bounded only in the case of a differentiable process. In a general case, function $F(\lambda)$ submits to weaker conditions [1], that is, for any $a>0$:

$$\int_{-\infty}^{\infty} dF(\lambda)/\lambda^2 < \infty, \quad \int_{-\infty}^{\infty} dF(\lambda)/\lambda^2 < \infty.$$  

(9)

Random processes with stationary $n$-th order increments are defined similarly. The structure function of the form (6) is determined by means of increments of the $n$-th order, and it also has the spectral representation, which is the generalization of (8).

So the spectral model of SIP-process may be symbolically presented in the form:

$$\Omega(X | SIP) = \{X(D): D(\tau) = \Phi_s [dF(\lambda)]\},$$

(10)

where left side denote set of SIP, and $\Phi_s$ is a symbol of transformation on the right side of (8). SIP space is a considerable extension of stationary set, since it includes both stationary and some non-stationary processes, such as the Wiener processes, white and flicker noises. On the other hand, the main methods of data processing may be extended to SIP with the use of spectral representations. Hence this generalization seems to be promising for measurement problems. Along with extension of spectral methods, there are also new aspects of data analysis, including definition and study of new characteristics. For instance, the Allan variance, which is widely used nowadays [7], was introduced empirically. Within SIP model, it gains a statistical meaning, as it turns out to be an estimate of the structure function, and it has a generalized spectral representation [8]. This property explains, why Allan variance proved to be more effective for non-stationary processes, than for stationary case. It is also useful for practical application of Allan variance.

3. Time-domain representations of data

Along with spectral representations of the form (1), the non-deterministic stationary process has also the time-domain representation [1, 2]:

$$x(t) = \int_{-\infty}^{t} g(t-u) \, dz(u),$$

(11)

where $z(t)$ - random process with non-correlated increments and spectral measure $dF(u) = E |dz(u)|^2 = du$. In the theory of random processes some classes of time-domain representations are studied [3], which are generalizations of (11). The second-order random process is considered as a curve in Hilbert space of random values: $H = \{ x: E|x|^2 < \infty \},$ with scalar product $(x, y) = E(x \cdot y)$. The general canonical time-domain representation is the direct sum of $M$ $(1 \leq M \leq \infty)$ stochastic integrals of innovation processes:

$$x(t) = \sum_{n=1}^{M} \int_{-\infty}^{t} g_n(t,u) \, dz_n(u),$$

(12)

where $z_n(t)$ – mutually orthogonal random process with non-correlated increments. The corresponding spectral measures are ordered: $dF_1 > dF_2 > \ldots , dF_M = E |dz_n(u)|^2,$ and $M$ is spectral multiplicity of the random process $x(t)$. The subspaces of Hilbert space $H$, generated by processes, form the direct sum:

$$H(x, t) = \sum_{n=1}^{M} H_n(z_n),$$

(13)

So, time-domain model of process may be symbolically presented in the form:

$$\Omega(X_{st}) = \{x: x(t) = \mathcal{S}_M \{ g_n(t,u), z_n(u), n = 1 \ldots M\},$$

(14)

where left side denote set of non-deterministic processes with multiplicity $M$, and $\mathcal{S}_M$ is just a symbol of integral transformation on right side of (12).

The representation (12) may be useful for development of data processing methods, since the main operations may be reduced to projections onto subspaces of Hilbert space $H$. Unfortunately, canonical representations are very general and tedious, so in practice extra information should be used.
In general, spectral multiplicity \( M \) may be arbitrary, from 1 to \( \infty \) [3]; however, the case of \( M=1 \) is the most important in practice:

\[
x(t) = \int_{-\infty}^{\infty} g(t,u) \, du.
\]

Moreover, all the known examples of the processes with multiplicity \( M>1 \) are very complicated and artificial [3, 9], so they are not of practical interest. It is also confirmed by the simple condition [10] of multiplicity increase while summing processes with \( M=1 \). Moreover, it was shown [11], that multiplicity of process would be \( M=1 \) under simple regularity conditions on (15). Therefore, in practice one may suppose that all the processes have multiplicity \( M=1 \), and accordingly use the representations (15).

4. Conclusion

1 Two kinds of general representations for random processes are analyzed, which are generalized spectral expansions and time-domain representations. These two groups are direct extensions of two classic representations for stationary processes. Each of group has its own advantages in some cases, and they are generally promising for solving the measurement problems.

2 The set of processes with stationary increments (SIP) seems to be especially useful for measurement problems. SIP set is much wider, than the set of stationary processes, but SIP also allow spectral representations. The extended model enables one to develop similar spectral methods of data processing, and also permits to investigate new characteristics. For instance, Allan variance, which is widely used nowadays, is the estimate of structure function for SIP, and it also has the generalized spectral representation.

3 The most general are time-domain canonical representations of non-deterministic processes, which may possess any spectral type and multiplicity \( M \) from 1 to \( \infty \). However, the processes, which are used in practice, have multiplicity 1; so these representations are also useful for measurement problems.

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Acknowledgments

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