On Understanding Generics: Towards a Simple and Accurate Domain-Theoretic Model of Generic Nominally-Typed OOP

Moez A. AbdelGawad
moez@cs.rice.edu

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College of Mathematics and Econometrics, Hunan University
Changsha 410082, Hunan, P.R. China
Informatics Research Institute, SRTA-City
New Borg ElArab, Alexandria, Egypt
(Began in 2010, while graduate student at Rice University)

The Gamma function is a metaphorical extension of the factorial function. One property, its recursion, becomes its most important feature, and serves as the basis for extending it. It’s a bit like calling an automobile a ‘horseless carriage’, preserving its essence of carrying and removing the unnecessary horsefulness, or like calling a railroad ferrovia in Italian or Eisenbahn in German, focusing on the fact that it’s still a road, but one made instead of iron.

~Adapted from a quote by Columbia Physicist Prof. Emanuel Derman

One important value behind developing and presenting NOOP [2, 3, 4, 12, 6, 5, 7, 9, 8], is to provide a more precise foundation on which further OOP research can be built. NOOP provides a capacity for better mathematical reasoning about mainstream OO languages.

The development of NOOP was mainly motivated by the lack of a precise model of OOP despite the prominence and domination of OOP among mainstream programmers, and the realization that extant models of OOP are deeply flawed, inadequate, or not enough for explaining the typing features of mainstream programming languages, and are thus hindering further development of these languages.\footnote{The development of the process of adding “closures” to the Java programming language is a prime example of how development of mainstream OOP languages is hindered. Also, research done on Java wildcards has been unable to present a proof that beyond doubt convinces us of the type safety of Java with wildcards. Also, Smith and Cartwright, in [18], present flaws in} A feature of mainstream OOP where such inadequacy and hindrance of development are clear is ‘generics’.

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Given that genericity is a useful feature for most OO developers, generics (i.e., generic classes) are supported in most mainstream OO languages. Generics offer OO developers with more expressive type systems.

1 Generics: A Summary of Developers Perspective

The dictionary definition of the word generic is: ‘Applicable to an entire class or group (not tied to particulars, or unconstrained)’. For example, if we purchase some generic dish soap, soap that has no brand name on it, we know that we are buying dish soap and expect it to help us clean our dishes, but we do not know what exact brand (if any) will be inside the bottle itself. We can treat it as dish soap, even though we don’t really have any idea of its exact contents.

Similarly, in OOP, generic classes, usually called generics, provide the programmers to abstract their classes over some types, and thus to define them “generically”, independent of the particular instantiations of them that class users later actually use. Generics move the decision as to what actual types to be used for some of the types used inside the class to the usage-sites of a class (i.e., are decided by the users of the class) rather than be declared and fixed at declaration-sites (i.e., decided by class developers).

Generics also offer OO programmers more flexibility, given that different type parameters of a generic class can be used at different usage sites, even in the same program. Without generics, such a capability could only be simulated by a cooperation between class developers and class users, depending on OO subtyping (which offers so-called ‘subtyping polymorphism’) and using the so-called ‘generic idiom’. Using the generic idiom is not a type-safe alternative to generics, given that using it involves requiring class users to insert downcasts by hand. Because they circumvent the type system, programs with downcasts can be type unsafe (See [14]).

Even further, at least as of version 1.6.0_16, the standard Java compiler, javac, has generics-related bugs. For example, the following code correctly compiles with no errors on the Eclipse Java compiler. The same code, however, generates a ‘Compile exception: java.lang.NullPointerException’ error message when compiled using javac.

```
class A<T> {}
class B<T> extends A<C<T>> {} // note that C<T> is used inside the supertype of B<T>
class C<T> extends B<T> {} // note that C<T> extends B<T>
```

(See Oracle Java compiler bug report [1] for an updated status of this bug in javac).

We believe all such problems are due to the lack of a precise conceptual mathematical model of mainstream OOP, particularly one that takes nominality in consideration. Even though the problems we mention are Java-specific problems, we believe similar problems (probably some of which are undiscovered) do exist in other mainstream OOP languages, given that the type systems of these languages are much similar to that of Java.
2 Modeling Generics

Building on how we modeled nominal OOP in NOOP, and given that generics is a typing feature of nominal OOP that crucially depends on nominality (as we will see shortly), it is natural to expect that modeling generics mathematically builds on top of the typing concepts we developed in NOOP, particularly on top of NOOP signatures.

In the following, we will see that this expectation is largely true. Compared to the modeling of nominal OOP in NOOP, the modeling of generics (or, more accurately, the modeling of ‘generic nominal OOP’) is about allowing type variables to be used inside signatures to stand for (abstract over) other signature names, making the plain names of signatures “gain some structure” by allowing them to be “applied” to other signature names, and, finally, to allow any valid combination of signature names to appear in any place inside a signature where a “plain” (i.e., NOOP) signature name was allowed before (e.g., field signatures, and method signatures). Combinations of signature names are formed, syntactically, by “applying” a generic signature name to other combinations of signature names.

Making signatures be generic (abstracting them over other signature names, using type variables) is sometimes called “generification”. Using generic signatures, we define ground object signatures, which, similar to NOOP object signatures, are paired with object records to construct generic objects (i.e., objects that carry instantiations of generic signatures as their nominal typing information).

Similar to how we constructed NOOP, we then build a model of generic nominal OOP, which we call GNOOP. GNOOP includes a domain of generic objects which we call GO. Domain GO uses ground signatures in the construction of its objects. Otherwise, GO is very similar to domain O of NOOP.

In the next two sections, we formally present generic signatures, and (a rough sketch for) the formal construction of GNOOP. Construction of GNOOP goes along the same lines of constructing NOOP.

As we will see in the next two sections, all features of generics, and their modeling, are centered around the main crucial idea that names of signatures are not plain but have some structure, indirectly implying, thus, that GNOOP objects could roughly be viewed as “NOOP objects with structured signature names” (rather than plain names used in NOOP).

3 Generic Signatures

Before giving the formal definitions for generic signatures, we first present an informal view of generic signatures to help motivate the later formal definitions.

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2 Generic signatures that take no parameters (“zeroary” signatures) are treated as being non-generic signatures. When used, they can be applied only to the empty sequence of signature names.
3.1 Informal View

A signature constructor is a parametrized (“generified”) $\mathcal{NOOP}$ signature. A signature constructor carries almost the same information a $\mathcal{NOOP}$ signature does (i.e., a name, supersignatures, member signatures, ... etc). Like a $\mathcal{NOOP}$ signature, a signature constructor is a syntactic typing entity.

Each signature constructor has a signature constructor name that is a plain label. Signature constructor names have a very similar purpose in $\mathcal{GNOOP}$ to that of signature names in $\mathcal{NOOP}$ signatures. Given its dependency on the use of names (of signature constructors), generics is also a feature of nominal OO programming that pivotally depends on nominality.

The main crucial difference between a signature constructor and a $\mathcal{NOOP}$ signature is that, inside a signature constructor, some generic signature names inside the signature constructor might be “missing” and have instead type variables as place holders (i.e., names are abstracted over by the type variables)\(^3\), and, further, structured generic signature names have to be used inside signature constructors (rather than plain unstructured signature names).

Following the name of a signature constructor, as an extra component in of a signature, all type variables that might be used inside the signature constructor are declared (in a sequence of type variables) ahead of the other components of the signature constructor (no nesting of type variable declarations occurs except at the top level, akin to a logical ‘prenex form’). The sequence of type variables is ordered (hence, a sequence) but allows no repetitions (i.e., all type variables of a signature constructor are distinct/have distinct names). Signature constructors are thus binding constructs. As such, two signature constructors are considered equivalent if they are equal modulo the consistent renaming of type variables throughout the two constructors (“alpha-renaming”).

No type variable can be used/referenced inside a signature constructor without it being declared in the type variables component of the constructor. Like a function expression in $\lambda$-calculus, the signature constructor, via its declared type variables, is said to abstract over the ground signature names that type variables, at usage-sites, can be instantiated to. Due to the fact that type variables cannot be “applied” (to other type names), a more accurate, and simpler, mathematical view of signature constructors is that they are signature schemes, rather than functions. We adopt this simpler view. We will see, in Section 3.5.3, how this view (together with type variables being distinct, and being declared in prenex form) helps us define a very simple notion of name substitution on signatures. All usage and properties of generic signatures in generic OOP depend on this simple notion of substitution.

Zeroary signature constructors, i.e., ones which take no type parameters, play the role of providing the base case for almost all inductive definitions or proofs that involve generic signatures. Given it takes no type parameters (i.e., has an empty sequence of type variables), and thus has no missing information,

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\(^3\)For the sake of familiarity, we keep using the name ‘type variables’ for these variables that abstract over signature names, although, strictly speaking, they should be called ‘ground signature name variables’ (See below).
a zeroary signature constructor, when its (empty) type variables component is dropped, corresponds directly to a non-generic \( \mathcal{NOOP} \) signature. Vice versa, by adding an empty sequence of type variables to a \( \mathcal{NOOP} \) signature it directly corresponds to a zeroary signature constructor. (In summary, thus, each zeroary \( \mathcal{GNOOP} \) signature constructor corresponds to a \( \mathcal{NOOP} \) signature, and each \( \mathcal{NOOP} \) signature corresponds to a zeroary \( \mathcal{GNOOP} \) signature constructor. There is a one-to-one correspondence between the two entities). Thus, in \( \mathcal{GNOOP} \), as in [14], ordinary non-generified (i.e., non-generic) signature names are identified with zero-ary signature constructor names. This is done merely as a technical convenience.

A signature constructor environment is a finite map (a table) that maps signature constructor names to signature constructors. Like all functions/mappings, by applying a signature constructor environment to a signature constructor name, the corresponding signature constructor (the one with the input name) is obtained.

To construct a generic signature name, for use either inside the supersignatures component, or the member signatures component of a signature constructor, a signature constructor name\(^4\) is paired with a (possibly empty) sequence of ‘type variables, or (nested) generic signatures names’. This sequence is called the sequence of type arguments that are “passed” (as parameters) to the signature constructor name to construct a generic signature name.

Relative to a certain signature constructor environment, a generic signature name is well-formed if all signature constructor names inside it are paired with sequences of type arguments that are the same length as the type variables components of the signature constructors in the environment corresponding to the signature constructor names\(^5\).

Generic signature names can be represented as labeled abstract syntax trees. The signature constructor name used to construct the generic signature name is the label for the root node of the tree. The subtrees of this node are trees that each represent an element of the type arguments sequence. Given that type variables cannot be applied, nodes representing type variables will always be leaf nodes in the abstract syntax tree representation of a generic signature name.

Inside a signature constructor, ‘a type variable or a generic signature name’ (usually just called a “type”, or, more accurately, a “type name”) can appear as the signature name of a field (in a field signature), and as the signature name of a method parameter or the signature name of a method return value (in a method signature). A type name can also appear as a supersignature name. To prevent the possibility of having circular subsigning hierarchies, a plain (also called “naked”) type variable is not allowed to appear in the supersignatures

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\(^4\)Whose corresponding signature constructor is then said to be ‘instantiated’, i.e., to be made an instance of.

\(^5\)That is, for a well-formed generic signature name, the number of type arguments must be equal to the declared number of type parameters. The declared number is based on the signature constructor information inside a signature constructor environment.
component of a signature constructor.\(^6\)

In the context of a certain signature constructor environment, well-formed generic signature names that have no type variables (are “variable-free”) are of special status, and they deserve some special attention: They are constructed only out of signature constructor names, nested inside each other via the type arguments component of a generic signature name. A generic signature name that contains no type variables (i.e., has no missing information) is called a *ground signature name*. Inductively, a ground signature name is, thus, constructed from a signature constructor name of a signature constructor that takes no type parameters (a zeroary signature constructor) paired to the empty sequence (of type names, as its type arguments), or is constructed from a signature constructor name of a (generic, non-zeroary) signature constructor paired to a sequence all of whose members are themselves ground signature names (as the type arguments of the signature constructor name).

Formally, we will see below that ground signature names are finite syntactic entities, and are members of an inductively defined set. As a subset of generic signature names, ground signature names also can be represented by labeled trees. Due to the absence of type variables, all leaves of a tree representation of a ground signature name are labeled by names of zeroary signature constructors. In fact, given that we noted above that type variables cannot occur elsewhere in the tree representing a generic signature name except as leaves, all nodes of the tree representing a ground signature name are labeled by signature constructors names only.

Relative to a specific signature constructor environment, well-formed generic signature names that are not ground signature names are called *non-ground signature names*. They are generic signature names that do contain type variables. They can only appear inside signature constructors (in member signatures, or as supersignatures).

Similar to NOOP object signatures, a *generic object signature* is a pair of a ground signature name and a signature constructor environment. Non-ground signature names are cannot be used to form object signatures.

A signature constructor whose (1) name is substituted by a ground signature name, that (2) has its type variables component removed, and (3) whose type variables are replaced (substituted) by ground signature names consistently throughout the signature constructor (i.e., throughout its supersignatures, field signatures, and method signatures components) is called a *ground signature*. Recalling that signature constructors are schemes, ground signatures are *instances* of signature constructors. Ground signatures have no type variables. Every zeroary signature can be trivially made into a ground signature.

\(^6\)In \[10\], it is discussed how MixGen in fact uses naked type variables in the supersignatures component of a signature to define signatures of “mixins” (as OO *components*) on top of first-class OO generics. In the context of mixins, there are in fact easily enforced rules that prevent circular nominal subtyping hierarchies. \[10\] presents an algorithm that MixGen uses to detect, and prevent, such circularity. Since we aim to model generics without mixins (i.e., to model “regular generics”) in this report, dealing with such a use of naked type variables is unnecessary.
The instantiation of a signature constructor, with ground signature names as type arguments, defines a ground signature, whose name is the ground signature name constructed from the signature constructor name paired with the sequence of type arguments.

Constructing ground signatures via instantiation of signature constructors is defined using a simple notion of *name substitution*. Name substitution substitutes (replaces) type variables inside a signature constructor with ground signature names.

We can easily see, thus, that in the context of a particular signature constructor environment, a ground signature name $ggsn$ whose first component, say $nm$, is the name of a signature constructor $sc$ defines (and is also the name of) a ground signature, say $gs$. $gs$ can be obtained from $sc$ and $ggsn$ by substituting the type variables $V$ inside $sc$ by the type arguments of $ggsn$ (the second component of $ggsn$, say $TN$). When we present formal definitions below, we denote this substitution/instantiation operation that defines the ground signature $gs$ corresponding to $ggsn$ by $gs = \{V \mapsto TN\}sc$, meaning that, if $sc$ has the same signature constructor name as $ggsn$, and $TN$ are the type arguments of $ggsn$, then $gs$ is defined by substituting the type variables $V$ inside $sc$ by the type arguments $TN$ of $ggsn$.

Name substitution plays a very important role in our mathematical (i.e., domain-theoretic) modeling of nominal OOP generics.

Given the use of ground signature names to define and name ground signatures, we should note that non-ground signature names do not name any actual signature names. Instantiation, and name substitution are only defined for ground signature names. Although, technically speaking, *generic signatures* could be defined (as a general notion that embodies signature constructors as well as ground signatures), they are not needed much in practice. This is why the instantiations of signature constructors that could define these generic signatures are disallowed.

Because non-ground signature names are names “with some missing information”, in instantiation and name substitution it has to always be guaranteed that whenever used, non-ground signature names (which may appear inside signature constructors) are used to construct ground signature names first, which are then the names used to refer to any ground signatures, and that when constructing ground signature names (to define ground signatures) non-ground signature names are never passed as type arguments “as is”.

In light of the informal definitions above, we can easily see that generic OOP depends on names (i.e., nominality) even more than non-generic OOP.

To summarize the above, for generics we have ten name-dependent defini-

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See Section 3.6 for one such important (theoretical/meta-linguistic) need, where, for checking for well-formedness of a signature constructor in a signature constructor environment, we need to make sure all instantiations of signature constructor satisfy a specific condition related to the corresponding instantiations of another signature constructor. This condition involves checking an infinite number of instantiations, which is generally not possible, unless we can abstract over all these instantiations (using type variables, which is indeed possible in this case).
tions. These are:

1. Signature constructors (which have names), signature constructor names, and signature constructor environments (mappings from names to constructors),

2. Type variables (which, incidentally, are also mere names, but ones not related to nominality of OOP),

3. Generic signature names (which are signature constructor names paired with sequences of type names),

4. Type names (which are type variables or generic signature names. Note the circular dependency on generic signature names),

5. Name substitution (which instantiates a generic signature name to a ground signature name, and a signature constructor to a ground signature),

6. Ground signature names, and ground signatures (which result from instantiation/name substitution), and

7. Generic object signatures (which are ground signature names paired with signature constructor environments).

Dependency on names (and nominality) allows named entities to be referenced and used before they are fully-defined (i.e., they allow circularity/mutual-recursiveness). It should be noted that generic signature entities have even a heavier dose of circularity than non-generic ones.8

Below we formally present all generic OOP definitions we informally discussed above.

### 3.2 Signature Constructors

Formally, corresponding to signature equations for $\mathcal{N}\mathcal{O}\mathcal{O}\mathcal{P}$, for $G\mathcal{N}\mathcal{O}\mathcal{O}\mathcal{P}$ signature constructors we have

$$
\mathcal{SC} = N \times X^* \times G N^* \times (L \times G N X)^* \times (L \times G N X^* \times G N X)^* 
$$

(3.1)

where $\mathcal{SC}$ is the set of signature constructors, $X$ is a non-empty set of type variables as plain names, and where sets $N$, $T$, $L$ and the set constructors $\times$ and $^*$ have the same meaning as in $\mathcal{N}\mathcal{O}\mathcal{O}\mathcal{P}$ signature equations.

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8Note, for example, that some of the definitions above mutually-recursively depend on each other. We have signature constructors (which have signature constructor names) using generic signature names, which, in turn, recursively use signature constructor names (a counterpart to this sort circularity exists for non-generic signatures, via the use of signature names). Also, generic signature names use type names, which, in turn, are type variables or, recursively, generic signature names (Due to the absence of type variables, this sort of circularity has no counterpart in non-generic signatures).

9Or, more accurately, type variable names, or most accurately, ground signature name variable names (given that “type” variables actually get instantiated only to ground signature names).
\(X^*\) is the set of (finite) sequences of type variables. As a component inside a signature constructor, a member of \(X^*\) is the sequence of type variables whose members can be used inside this signature constructor. Ordering of elements in an element (a sequence) of \(X^*\) does matter (type arguments are matched with type variables based on the order of each in their respective sequences). Repetitions is not allowed allowed in elements of \(X^*\). Similar to all sequences, elements of a sequence \(V \in X^*\) can be referenced by their indices, e.g., \(V_i\), where \(V_i \in X\). \(\#\) is a function whose value is the length of a sequence (e.g., the expression \(\#(V)\) gives the size of \(V\)). Given that repetition is not allowed, a sequence can also be viewed as a function from its members to their indices (natural numbers). Thus, for example, for \(Y \in X\), \(\overline{V}(Y)\) gives the index of type variable \(Y\) in the sequence \(V\) of type variables\(^{10}\). As a function, \(\overline{V}\) is undefined for indices equal to or larger than its size, nor for variable names that do not exist in its range.

For generic signature names, we formally have

\[
GN = N \times GN^*
\]  

(3.2)

where, for ‘a generic signature name or a type variable’ (which, inaccurately, is sometimes also called a type, or a type name), we have

\[
GNX = GN + X
\]  

(3.3)

(Note the mutual dependency between \(GN\) and \(GNX\), and that only members of \(N\), not \(X\), can be paired with members of \(GNX^*\)).

### 3.2.1 BNF Rules for Generic Signatures

Similar to BNF rules for \(NOOP\) signatures, the BNF rules (which allow us to name components of generic signatures) corresponding to the definitions above are:

| sc::= (nm, [X], [gnm], [gfs], [gms]) | signature constructors |
| gfs::= (a, gnmx) | generic field signatures |
| gms::= (b, [gnmx] \rightarrow gnmx) | generic method signatures |
| gnm::= (nm, [gnmx]) | generic signature names |
| gnmx::= (0, X)|(1, gnm) | type variables or generic signature names |

For members of \(gnmx\) when they are in printed form, the pairing with 0 and 1 is usually unnecessary, given that type variables and generic signature names are usually syntactically distinguishable.\(^{11}\) The pairing is, thus, usually elided when a member of \(gnmx\) is spelled out in this thesis.

\(^{10}\)Thus, if we have a sequence of signature names, say \(\overline{TN}\), assuming \(V\) is a sequence of type variables of the same length as \(\overline{TN}\), the expression \(\overline{TN}_{\overline{V}(Y)}\) gives the signature name in \(\overline{TN}\) corresponding to a type variable \(Y\) in \(\overline{V}\). We use this notation below to define name substitution.

\(^{11}\)Note that even if sets \(X\) and \(N\) where the same set (e.g., plain strings), if a non-algebraic notation (e.g., \(\langle x, y \rangle\) for pairing) is used for expressing generic signature names (as is the case in most mainstream generic OO languages), type variables (members of \(X\)) are syntactically
3.3 Ground Signature Names

A proper subset of generic signature names (i.e., a subset of set \( \text{GN} \)) that is of special interest (see Section 3.5 for more details) is one whose members are generic signature names that have no occurrences of type variables inside them. We call (members of) this subset \textit{ground signature names}. Ground signature names could, more simply, be viewed as “structured signature names”, as opposed to the \textit{plain} signature names used in constructing \textit{NOOP}.

Thus, formally, for the set \( \text{GGN} \) of ground signature names, we have

\[
\text{GGN} = \text{N} \times \text{GGN}^*\tag{3.4}
\]

(where the empty sequence of ground signature names provides the base case for the definition).

Each member of \( \text{GGN} \) is a member of \( \text{GN} \), but not necessarily vice versa (Members of the difference set, \( \text{GN} \setminus \text{GGN} \), are the non-ground signature names, which necessarily contain type variable occurrences).

The BNF rule corresponding to the definition of ground signature names is:

\[
\text{gnm} ::= (\text{nm}, [\text{gnm}])\text{ ground signature names}
\]

3.4 Signature Constructor Environments and Generic Object Signatures

Similar to \textit{NOOP} signature environments, signature constructor environments (SCEs, members of a set \( \text{SCE} \), for short) are finite functions from \( \text{N} \) (signature constructor names) to \( \text{SC} \) (signature constructors), i.e., SCEs are particular subsets of \( \text{N} \times \text{SC} \).

Within the context of a particular signature constructor environment \( \text{sce} \), a generic signature name (including ground signature names) \( \text{gnm} = (\text{nm}, [\text{gnmx}]) \) is well-formed if and only if (1) \( \text{sce}(\text{nm}) \) is defined (\( \text{nm} \) is in the domain of \( \text{sce} \)), (2) \( \#(\text{tvars}(\text{sce}(\text{nm}))) = \#(\text{gnmx}) \), and (3) all nested generic signature names in \( \text{gnmx} \) are also well-formed in \( \text{sce} \), where \( \text{tvars} \) is a projection function that extracts the type variables component of a signature constructor.

\[\text{distinguishable, using a context-free grammar, from generic signature names (members of G}\text{N}). \text{That is because type variables are not paired with a sequence of type arguments, since they cannot be “applied” and thus do not take type arguments (signature constructors are, thus, not “higher-order”), and also because, strictly speaking, signature constructors must be applied to a sequence of type arguments (even zeroary signature constructors have to be applied to the empty sequence of type arguments). \text{Raw types (see Section 5.1.1) and/or making the empty sequence of type arguments optional affects the easy distinction between the two entities, but the distinction is still possible using context-dependent information (e.g., when a signature constructor is used in a signature constructor environment, which provides the names of all signature constructors that could be referenced inside a signature constructor, and by, possibly, giving either N or X a higher priority [Java, for example, gives higher priority to set X])}.\]

\[\text{Incidentally, we might have used \( \rightarrow \) (the finite records constructor) to function also as a set constructor (making it ignore any ordering and repetitions of members of its input sets [the defining sets] and output sets [the defined sets]). If so, then we would have \( \text{SCE} = \text{N} \rightarrow \text{SC} \). We prefer, in this thesis, however, to limit the use of \( \rightarrow \) to the construction of semantic domains only.}\]
Similar to $\mathcal{NOOP}$ object signatures, generic object signatures are pairs of a ground signature name $ggnm$ and a well-formed (See Section 3.6) signature constructor environment $sce$ (where $ggnm$, by definition, is well-formed in $sce$). Generic object signatures are, thus, well-formed members of $GGN \times SCE$. Note that ground signature names, not the more general generic signature names, are used to define generic object signatures.

Similar to a $\mathcal{NOOP}$ signature, the informal “meaning” of a generic object signature is that its first component, the ground signature name, should be “interpreted” in the context of its second component, the signature constructor environment (In particular, signature constructor names referenced by the ground signature name are meant to refer to names of signature constructors in the signature constructor environment).

### 3.5 Ground Signatures

Ground signatures provide a connection (and a “middle ground”) between $\mathcal{NOOP}$ signatures and $\mathcal{GNOOP}$ signatures. Ground signatures are signature constructors that (1) use ground signature names as their names\(^{13}\), (2) have no type variables component, and (3) have type variables occurrences inside supersignatures, and member signatures substituted with ground signature names. Ground signatures could, equivalently, be viewed as $\mathcal{NOOP}$ signatures with structured signature names.

The equivalence of the two views of ground signatures is why they are considered a “middle ground” between (non-generic) $\mathcal{NOOP}$ signatures and (generic) $\mathcal{GNOOP}$ signatures.

Formally, we have $GGS$, the set of ground signatures, defined as

$$GGS = GGN \times GGN^* \times (L \times GGN)^* \times (L \times GGN^* \times GGN)^* \quad (3.5)$$

which is the same as the $\mathcal{NOOP}$ signature equation but uses ground signature names (members of $GGN$) in place of plain signature names (members of $N$), implying names of ground signatures have an inherent, desirable structure, and are not plain names as those of $\mathcal{NOOP}$ signatures.

The fact that ground signatures provide a middle ground (expressed mathematically in the equations for $S$, $SC$ and $GGS$) is behind the statement that “generics saves developers typing [as in writing] time” (which has the implication of decreasing the chances for coding errors, as well), and that “generics does not let developers do something fundamentally different than what they were able to do without generics (but with significantly more code written)\(^{14,15}\).”

\(^{13}\)That is, their names are members of $GGN$, rather than members of $N$.

\(^{14}\)However, as a refactoring technique, generics has the implication of decreasing the size of the code that OO developers would have to maintain if they were to “encode” generics into a non-generic OO language, e.g., using the so-called “generics idiom”.

\(^{15}\)Given that sets $N$ and $GGN$ have the same cardinality, $\aleph_0$, a one-to-one correspondence (mapping) between the two sets can be used to define a one-to-one correspondence between ground signatures and (non-generic) signatures (of $\mathcal{NOOP}$), and, accordingly, to define a mapping from ground object signatures (see below) to (non-generic) object signatures. An
The BNF rule corresponding to the definition above is:

| Symbol | Definition                        | Category     |
|--------|----------------------------------|--------------|
| ggs    | \((ggnm, [ggnm], [ggfs], [ggms])\) | ground signatures |
| ggsf   | \((a, ggnm)\)                    | ground field signatures |
| ggms   | \((b, [ggnm] \rightarrow ggnm)\) | ground method signatures |

### 3.5.1 Are Ground Signature Environments and Ground Object Signatures Needed?

Given the definition of ground signatures, as the result of instantiating the type variables inside a signature constructor, it may be tempting to extend this connection between generic and non-generic signatures to consider defining what could be called ‘ground signature environments’, which could, it may be thought, also provide a “middle ground” between signature constructor environments and \(NOOP\) signature environments, and could also be, it may be further thought, “\(NOOP\) signature environments with structured signature names”.

While defining more ground signature entities is possible mathematically, we should note that a hypothetical ground signature environment corresponding to a particular signature constructor environment could be an infinite function (from ground signature names to ground signatures), unlike a \(NOOP\) signature environment, which must be a finite function.

An example that demonstrates that a signature constructor environment can get translated to an infinite ground signature environment is the case where inside the signature constructor environment there is a signature constructor that has inside it a generic signature name in which the main signature constructor name is used to construct type arguments that are then passed again to the main signature constructor name (thus making rewriting/name substitution expansive. See [13]). Trying to instantiate this signature constructor to produce a ground signature, and, recursively, have ground signature names inside it produce ground signatures that should be in the co-domain of the ground signature environment will cause an infinite number of different instantiations, and thus the domain and co-domain (or, range) of the ground signature environment will also be infinite sets.\(^{16}\)

In any one generic nominal OOP program, up to any (finite) moment in time, however, only a finite number of instantiations of a signature constructor example of functions that map ground signature names to non-generic signatures names, and vice versa, are the so-called “name mangling” functions, such as the one used in the NextGen implementation of Java Generics (See [11]).

\(^{16}\)A sample valid Java code that demonstrates a case where rewriting, if done, would be infinite, is the following code:

```java
class C<T> {
    C<C<T>> m(){ return new C<C<T>>(); } // note the expansive return type
}
```

This example is similar to the recursive generics example in [11] (called recursive polymorphism in that paper), but this example, unlike the one in [11], demonstrates the use of an expansive type in the return type of a method, rather than only inside its body (if used only inside the body of a method, a recursively polymorphic type may not necessarily affect the typing (i.e., signature) of the method).
could take place, and thus the whole set (of ground signatures) resulting from all possible instantiations is not needed (or is “lazily” needed). This situation is much like an inductively defined data type (of, say, lists, or natural numbers), which has a finite set of data constructors that are then used to construct values of that data type. In fact, we could explicitly formalize ground signature names, relative to a fixed finite subset of \( \mathbb{N} \), as being elements of such a data type. Given its irrelevance to the goal of modeling generics, however, except for this brief note we do not present any further discussion of such a possibility in this thesis.

‘Ground object signatures’ could also be defined, by pairing ground signature names and (the undefined) ground signature environments. Given the impracticality of the latter, the former become impractical too, so we do not define them here.

### 3.5.2 The Import of Ground Signatures and Their Names

The reason behind defining ground signature names is that they are the names paired with signature constructor environments (SCEs) to construct generic object signatures. Generic object signatures, in turn, are paired with object records to construct the domain of (generic) objects, \( GO \), in \( G\text{V}O\text{OP} \) (See Section 4).

By using a more structured name for object signatures (i.e., ground signature names), when compared to non-generic OOP, generics could be viewed as basically offering a “structuring” of the namespace of signature names, and them providing access to a (finitely-constructed) infinite set of these signatures. Signature constructors, via generic signature names, then, enable abstracting over subsets of these ground signatures (subsets whose members share common signature constructor names). Signature constructors could, thus, be considered as functions from ground signature names to ground signatures, or, given their syntactic nature, as signature schemes rather than functions as we pointed out earlier. A signature constructor is an “abbreviation” for (abstracts over) an infinite set of structurally-similar ground signatures. By instantiating a signature constructor, we obtain an instance of the set of ground signatures that the signature constructor abbreviates (is a scheme for).

Thus, by having structured signature names, generics adds expressiveness to an OOP language by allowing multiple (ground) signatures with the same signature constructor name to be usable, in a type safe manner, in a single OO program.

\[\text{17In our case, i.e., for the “data type” of ground signature names, names of zeroary signature constructors provide the base elements of the data type, while the names of non-zeroary signature constructors will be used (with the more basic elements) to inductively construct the compound elements of the data type.}\]
3.5.3 Constructing Ground Signatures: Using Name Substitution to Instantiate Signature Constructors

This section presents the formal definition of the name substitution function. As explained above, name substitution uses ground signature names to map generic signature names to ground signature names, by replacing type variable occurrences inside the generic signature names with ground signature names, and, in turn, mapping a signature constructor to a ground signatures (i.e., to an instance of the set of ground signatures that the signature constructor abstracts over).

Given two special properties of how type variables are used in generics, generic name substitution has a simple definition (unlike the definition of substitution for $\beta$-reduction in $\lambda$-calculus, for example). These two special properties are:

1. Type variables are declared all at once, at the “outer level”, with nesting of type variables not being allowed except at that level (type variables later in a sequence of type variables are in the scope of type variables declared earlier in the sequence).

2. Type variables are required to have distinct names, and thus variable name clashes (and the use of shadowing, or overriding, or so) between type variables names are not possible (This property, together with the first property, allows substitution to be defined in a straightforward manner without concern for “capturing” or shadowing).

Formally, we use the following notation to denote the operation of substituting type variables $V$ in a signature constructor $sc$ with type arguments $TN$

$$\{V \mapsto TN\}sc$$

We first define name substitution on generic signature names, then extend this definition to define name substitution for signature constructors. For type variables $V \in X^*$, and $TN \in \text{GGN}$, we have

$$\{V \mapsto TN\}Y = \begin{cases} TN\!\!_{\overline{V}}(y) & Y \in V \\ Y & Y \notin V \end{cases}$$

(In fact, the second case, $Y \notin V$, should not occur. The substitution in this case should be undefined 18). For a generic signature name $gmx = (nm, [gmx])$, we have

$$\{V \mapsto TN\}gmx = (nm, [V \mapsto TN\!\!_{\overline{V}}gmx])$$

This definition extends naturally to signature constructors as follows:

18And an error should be reported, meaning, for example, that the name substitution is not the result of a proper signature constructor instantiation, or that the signature constructor is not well-formed.
For a signature constructor $sc = (nm, \{V\}, ssn, gfss, gmss)$, we have

$$\{V \mapsto TN\} sc = (nm, \{V \mapsto TN\} ssn, \{V \mapsto TN\} gfss, \{V \mapsto TN\} gmss)$$

(note that we require the sequence $V$ of the substituted variables to be the same as the type variables of $sc$, and also note that the substitution can be performed without need for a signature environment)

where for a generic member (field or method) signature $ms = (c, gcs)$, $\{V \mapsto TN\} ms = (c, \{V \mapsto TN\} gcs)$, and where for a name substitution on sequences of elements $[e]$, of size $n$, we have

$$\{V \mapsto TN\} [e] = [(\{V \mapsto TN\} e_0, \cdots), (\{V \mapsto TN\} e_i, \cdots), (\{V \mapsto TN\} e_{n-1})]$$

For instantiating a signature constructor to define a ground signature, we can thus, finally, now define

$$GenToGnd(sc, TN) = drop\_tvars(\{tvars(sc) \mapsto TN\} sc)$$

where $tvars$ returns the third (type variables) component of a signature constructor, and $drop\_tvars$ is a function that takes a sextuplet and returns a quintuplet without the third component of the input (drops it), and, by extension to generic object signatures, for a generic object signature $gos = (ggnm, sce)$ we define a ground signature $ggs$ where

$$GenToGnd(gos) = GenToGnd(sce(fst(ggnm)), snd(ggnm))$$

where $fst$ and $snd$ return the first and second component of a pair, respectively (and thus for the ground signature name $ggnm$ they return the signature constructor name used to construct $ggnm$, and the type arguments to that constructor name, respectively).

Claim 1. It is relatively clear that for a generic object signature $gos$, $GenToGnd(gos)$ is a ground signature with ground signature name $fst(gos)$.

For a generic object signature $gos$, one may also try defining a ground signature environment that, if paired with the ground signature name of $gos$, would define a ground object signature. This definition would have to produce infinite ground signature environments for some generic object signatures, i.e., ones containing signature constructors with ‘recursive generics’ in their signature constructor component.

It should be noted that the $GenToGnd$ function loses some information, and is thus not a reversible function. More concretely, this means that while a generic object signature can be mapped to a unique ground signature, it is not possible to reverse the work of $GenToGnd$ by mapping the resulting ground signature back to a unique generic object signature, and thus it cannot be mapped back to the same generic object signature that produced it. Intuitively, this is impossible because there are multiple ways to “genericify” a non-generic ground signature (i.e., many ways to make a non-generic signature be generic). Because of such an information loss, $GenToGnd$ is called an erasure function.
3.6 Well-formed Generic Signatures

Similar to what we have for NOOP, only well-formed generic signatures are used in constructing GNOOP. Well-formed generic signatures satisfy seven constraints.

(1-4) The first four well-formedness constraints for NOOP are almost the same for generic signature constructors and signature constructor environments, with the only difference being that names (members of set N) now refer to signature constructors not to signatures.

(5) We also require, for a signature constructor to be well-formed, that all generic signature names inside it to be well-formed (See Section 3.4), and for type variables inside the signature constructor to be members of its type variables component.

(6) The generic counterpart of the (No cycles in supersignatures) condition requires that when the supersignatures hierarchy of a signature constructor is recursively followed, via signature constructor names inside the signature constructors, and via the bindings of these in the signature environment, each signature constructor in the co-domain of a well-formed signature constructor environment has to have no cycles in its list of supersignatures (i.e., it has no ‘infinite ascending chains’ of “super” signature constructors). The names of the supersignature constructor names are obtained from the first component (the “head”) of the generic supersignature names.

(7) The generic counterpart of the (Signatures inherit signatures of members in supersignatures) condition is the most interesting well-formedness condition. Like for NOOP, members in well-formed signature constructors that are inherited from (have the same name in) supersignature constructors are required to have the same signature they have in the supersignature constructors when the supersignature constructors are instantiated with the (generic) signature names declared in the subsignature. This condition can be checked by making a name substitution (See Section 3.5.3) on the supersignature constructor using the generic signature name parameters used in the supersignatures component of the subsignature constructor. The signatures of members of the generic signature resulting from the substitution will be checked for being exact matches with signature of the corresponding members in the subsignature constructor. By using a generic signature name (which abstracts over all possible instantiation of the signature construction) in the substitution, it is made sure that all instantiations of the subsignature constructor satisfy the “member signatures matching” condition of the corresponding instantiation of the supersignature constructor. If signatures of all members of a signature constructor match the signatures of members the constructor shares with all its supersignature constructors, the signature constructor is well-formed.

3.7 Equality of Generic Signatures and Extension of SCEs

Equality of well-formed generic signature entities is defined not much differently from the definition of equality of NOOP signatures, except that the exact names
of type variables in signature constructors are irrelevant, and thus equality is defined modulo “alpha-renaming” of type variables. Otherwise, equality is defined component-wise using equality of components of generic signatures as syntactic elements (just like for NOOP signatures).

Given that well-formed signature constructor environments are “minimal”, by well-formedness conditions, extensions of signature constructor environments are defined exactly in a similar manner to their definition for NOOP signature environments. A signature constructor environment sce_1 extends a signature constructor environment sce_2 if, modulo alpha-renaming of type variables, all members of sce_1 (name/signature constructor pairs) are members of sce_1 (and thus sce_1 is a superset of sce_2).

3.8 Inheritance and Subsigning of Generic Signatures

In a signature constructor environment sce, a signature constructor sc has a supersignatures component, ss, that specifies for each instantiation of sc, which defines a ground signature gss, what ground signatures are the ground supersignatures of that instantiation. Ground supersignatures are decided by the supersignatures component of gss. When the substitution is, transitively, done throughout the supersignatures hierarchy (which is of finite size, by well-formedness rules), all ground supersignatures of gss can be obtained. In agreement with nominal subtyping, a pair of ground signatures (gss_1, gss_2) is in the subsigning relation if and only if the ground signature name of gss_2 is a member of the set of names of the ground supersignatures of gss_1.

The supersignatures component of a signature constructor is thus considered to define a pointwise subsigning relation, between ground signature instantiations of the signature constructor and instantiations of its explicitly declared supersignature constructors.

Thus, for two well-formed ground object signatures gos_1 = (ggnm_1, sce_1) and gos_2 = (ggnm_2, sce_2), we have

\[ gos_2 \subseteq gos_1 \Rightarrow sce_2 \sqsubseteq sce_1 \land (ggnm_2 = ggnm_1 \lor ggnm_1 \in gss(sce_2, ggnm_2)) \]

(3.6)

where gss is a projection function that computes the set of all ground supersignatures names of a given ground signature name (ggnm_2) in a given signature constructor environment (sce_2) by, transitively, going throughout the supersignature hierarchy. gss uses name substitution to obtain these ground supersignatures names.

4 GNOOP: A Domain Theoretic Model of Generic OOP

Construction of GNOOP mostly resembles the construction of NOOP. The main difference between generic objects of GNOOP and non-generic objects of NOOP is the use of generic object signatures in constructing generic objects.
in place of the (non-generic) object signatures used in constructing $\mathcal{N}OOP$
objects.

Given its great similarity to $\mathcal{N}OOP$ construction, we only skim through the
construction of $G\mathcal{N}OOP$ (in the next two subsections). We then discuss the
properties of $G\mathcal{N}OOP$ in Section 4.3.

4.1 $G\mathcal{N}OOP$ Domain Equation

The domain equation used to construct $G\mathcal{N}OOP$ is the same as the $\mathcal{N}OOP$
domain equation, with a domain $GO$, of generic objects, replacing domain $O$,
and using the (flat) domain of generic object signatures, $S_{GO}$, in place of domain
$SO$ of $\mathcal{N}OOP$ object signatures. The domain $L$ and domain constructors $\times$,
$\Rightarrow$, $\rightarrow$, $\ast$ retain their same meanings.

$$GO = S_{GO} \times (L \rightarrow GO) \times (L \rightarrow (GO \rightarrow GO))$$  \hfill (4.1)

Similar to the construction of $\mathcal{C}OOP$ [2, 3, 5], a model that represents the ‘core’
of $G\mathcal{N}OOP$ could be constructed, using the domain equation

$$GO = L \rightarrow (GO \rightarrow GO) + B$$  \hfill (4.2)

but the model constructed using this domain equation would be precisely the
same as $\mathcal{C}OOP$, because $\mathcal{C}OOP$ uses no signatures, and so does Equation (4.2).
Thus, like it does for $\mathcal{N}OOP$, $\mathcal{C}OOP$ functions as a simple, structural core of
$G\mathcal{N}OOP$ as well.

4.2 $G\mathcal{N}OOP$ Construction

The construction method of $G\mathcal{N}OOP$ is the same method used for constructing
$\mathcal{N}OOP$, where the construction proceeds in iterations, guided by the structure
of the right-hand-side of domain equation (4.1) (which has precisely the same
structure as that of the right-hand-side of the domain equation of $\mathcal{N}OOP$).

Similar to filtering of pre-$\mathcal{N}OOP$ to define $\mathcal{N}OOP$, $G\mathcal{N}OOP$ filtering is re-
ponsible for ensuring generic object records are matched with concrete generic
object signatures where member signatures in the generic object signatures
match those in the object records (note that generic object signatures define
ground signatures, whose member signatures are used to decide the matching).

4.3 Properties of $G\mathcal{N}OOP$

Similar to our investigation of some main properties of $\mathcal{N}OOP$, and using
terminology used there, we also interpret generic object signatures of $G\mathcal{N}OOP$
as denoting nominal object types, where we will have

$$G_{S}[gos] = \{ go \in GO \exists goss \in S_{GO}, gr \in G.R. (goss \subseteq gos) \land (go \sqsubseteq (goss, gr)) \}$$  \hfill (4.3)
where $GR$ is the generic object records counterpart of domain $R$ of $NOOP$ object records.

Similar to what we did for $NOOP$, it is easy, using essentially the proof we developed for $NOOP$ (only slightly appropriately adjusted for $GNOOP$), to prove that generic nominal types denoted by generic object signatures are weak ideals in domain $GO$.

Similar to what we did for $NOOP$, based on the definition of the interpretation of a $GNOOP$ generic object signatures and the definition of the interpretation of a $NOOP$ non-generic signature being very similar, we also use essentially the same proof we developed for $NOOP$ to prove that in $GNOOP$ the inheritance and generic nominal OO subtyping are completely reconcile (i.e., that adding generics to a non-generic nominal OO language preserves the reconciliation).

Formally, thus, similar to the ‘inheritance is subtyping’ theorem for $NOOP$, for $GNOOP$ we also have that

\[ gos_1 \sqsubseteq gos_2 \iff GS(gos_1) \subseteq GS(gos_2) \] (4.4)

That is, one fully instantiated class type is a subtype of another fully instantiated type only if its fully instantiated signature subsigns the former (the $\Leftarrow$ direction in 4.4). If subsigning holds, then the instantiated supertype signature name must appear in the chain of instantiated supertypes for the subtype. Then signature matching implies that the weak ideal of the subtype is contained in the weak ideal of the supertype (the $\Rightarrow$ direction in 4.4).

## 5 Modeling More Generic OOP Features

Although the modeling and analysis of more features of generic OOP is mostly considered future work and beyond the scope of this thesis, we here present a necessarily-incomplete basis for such future work and we present some main ideas we intend to build on and develop in the near future.

First, given that Java implements a somewhat-limited notion of generics on a virtual machine that has no notion of generics (the JVM \cite{JavaGenerics}), using a technique and concept called *erasure*\textsuperscript{19}, we would like to make a full assessment of this technique, by building a model for erased generics. Building on concepts we presented in earlier sections of this report, we present in Section 5.1 the basis on which we can build such a model of erased generics (which is called $EGNOOP$). We will see that erased generics (generics implemented via erasure) provides an approximation\textsuperscript{20} for full generics, since it basically ‘saves programmers from using the “generic idiom” but does not give them the full power of generics’, including support for generic type-dependent runtime operations (such as creating objects of type variable types [i.e., type specified using naked type variables]).

Next, in Section 5.2, we will discuss how to incorporate bounding type variables, so that rather than allowing a signature constructor to be instantiated

\textsuperscript{19}Generics, as implemented in Java is sometimes, thus, called “second-class generics”.

\textsuperscript{20}Not in a domain-theoretic sense!
using any ground signature name (which has been the case so far), some subtyping constraints about possible type argument instantiations is made on type variables, and these constraints are guaranteed to be satisfied for passed-in type arguments.

‘Raw types’ is not only an artifact of erasure, but also arise in Java from the need for supporting “migration compatibility”, where erased and non-erased code need to co-exist. We discuss raw types in Section 5.1.1.

Wildcards, as a main motivation behind developing NOOOP and GNOOP, try to increase the expressiveness of generics. Wildcards lessen the mismatch between subtyping polymorphism and generics, by allowing unknown types (with, or without known upper and lower bounds) to be passed as type arguments to signature constructors. Using generics terminology of Section 3, wildcards allow the naming and the referencing of more ground signatures. Wildcards are discussed in Section 5.3.

In Section 6, we briefly discuss how polymorphic methods could be modeled. Even though not a must, polymorphic methods are supported in virtually all OO languages that support generics. Generics allows abstracting full objects (i.e., all their members) over type arguments, while polymorphic methods allow abstracting individual methods of objects over type arguments.

While the discussions in the sections on modeling erasure and polymorphic methods are more technical, the discussions are less technical in the rest of the sections on modeling other generic OOP features.

We start the discussion of more generic OOP features by discussing erasure.

5.1 Erasure and Erased Generic OOP (The “Generics Idiom”)

Erasure as a technique using in some mainstream OO languages, most notably Java, for supporting generics when first-class generics is not possible to support. Erased generics thus only approximates full generics (when information is lost due to erasure, an operation the depends on the erased information is disallowed, or a warning is produced). Erasure produces code that a non-generic OO developer “would have written (using the ‘generic idiom’)” if they had no generics. This is the essential intuition behind erased generics in Java. The definitions accurately capture this intuition.

The generics idiom, and erased generics accordingly, crucially depend on the existence of a type (a signature constructor) at the top of the subtyping hierarchy (typically Object). The generics idiom, and erased generics, also depend on the observation that in the full-generics semantics all instantiations of a signature constructor have the same members, only with different member signatures.

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21This is why Java Generics does not support generics using primitive types, because Java primitive types of Java are outside the object subtyping hierarchy, and are not subtype of Object.

22The Java compiler tries to compensate for the difference in members signatures by using type cast operations that are never guaranteed to fail.
Using our modeling of generics presented in earlier sections, in an OO language where all object types have a common supertype (and without type variables having bounds—see §5.2), erasure is equivalent to instantiating a signature constructor with one type then using the signature constructor name to refer to this particular instantiation (rather than the ground signature name).

The erasure of a ground signature name $ggnm = (nm, [ggnm])$

$$GndToNonGen(ggnm) = nm$$

(That is, the erasure of the ground signature name $ggnm$ is simply the name of the signature constructor used to construct $ggnm$\(^{23}\)).

If we abbreviate the $GndToNonGen$ function to $\lceil \cdot \rceil$, then the erasure of a ground signature $ggs = (ggnm, \lceil ggnm \rceil, \lceil ggsf \rceil, \lceil ggms \rceil)$\(^{24}\) is

$$\lceil ggs \rceil = (\lceil ggnm \rceil, \lceil ggnm \rceil, \lceil ggsf \rceil, \lceil ggms \rceil)$$

For a signature constructor environment $sce = \{\ldots, (nm, sc), \ldots\}$, its erasure is

$$\lceil sce \rceil = \{(nm, ngs) | (nm, sc) \in sce \land ngs = \lceil GenToGnd(sc, Object) \rceil\}$$

(check Section 3.5.3 for the definition of the $GenToGnd$ function).

That is, the definition of erasure for signature constructor environments converts a signature constructor in the environment to a ground signature by instantiating it with a sequence of type arguments whose elements all having the same value: namely, signature constructor name $Object$ (which we assume is the type at the top of the subtyping hierarchy).

Equivalently, without making a detour via ground signatures, we could state that $ngs$ is the result of taking out the type variables component of $sc$, replacing each plain (i.e., naked) type variable by $Object$, and dropping the type arguments of all generic signature names (i.e., we only keep the outer signature constructor name). Given that plain type variables are not allowed in the supersignatures component of $sc$, this shows that the supersignatures component of the result signature will use the same signature constructors of the supersignatures component of $sc$ (This property plays an important role in preserving the inheritance and subtyping properties of the resulting erased-generic entities.)

For a generic object signature $gos = (ggnm, sce)$, its erasure, $egos$ is

$$egos = (\lceil ggnm \rceil, \lceil sce \rceil)$$

**Theorem 1.** If a $GNOOP$ generic object signature $gos$ is well-formed, its erasure, $egos$, is a well-formed non-generic $NNOOP$ object signature.

\(^{23}\)Erasure of $ggnm$ is a reminiscent reminder of the mathematical ceiling function, $\lceil \cdot \rceil$, which computes the smallest integer greater than or equal to a floating point number.

\(^{24}\)Erasure of a ground signature is like first converting the floating number 3.2 to the floating number 3.0 (corresponding to instantiating with $Object$), then referring to that new floating number using (the symbol for) the integer 3 (by dropping type arguments of generic signature names).
Proof. By induction on the structure of $gos$, and by noticing that $\lceil \cdot \rceil$ produces $\text{NOOP}$ signature name for given $\text{GNOOP}$ ground signature name, produces $\text{NOOP}$ signatures for $\text{GNOOP}$ signature constructors, and produces $\text{NOOP}$ signature environments for $\text{GNOOP}$ signature constructor environments.

A domain of erased-generics objects constructed using these erased-generics object signatures, based on $\text{NOOP}$ domain equations (or $\text{GNOOP}$ domain equations, for this matter), will produce a domain, $\text{EGO}$, isomorphic to the domain $\Omega$ of $\text{NOOP}$ of non-generic objects. It is this domain that is used to interpret generics in Java. The erasure of generic information (i.e., its unavailability at runtime) limits the operations a Java developer can perform on generic types (at compile time) to ones that will be type sound even without such information (dynamic dispatch is type safe, while `new` and `cast` operations cannot be performed on plain type variables).

5.1.1 Java Raw Types

`Raw types` refers to using non-structured plain signature constructor names as generic signature names, simultaneously while using (structured) generic signature names. Raw types are an artifact of Java erasure that are only motivated by attempting to maintain “migration compatibility”. Migration compatibility allows Java developers to simultaneously use generic and non-generic code, while motivating them to use generics. The Java type system gives unmodified non-generic code a generic interpretation. Raw types are best understood as generic signature names that are “missing type parametrization information”\(^\text{25}\). We relegate further analysis of raw types to future work.

5.2 Bounded Type Variables

For more expressive and more precise typing, type variables in generic OOP are usually provided with type bounds. An upper type bound on a type variable tells that all instantiations of the type variable will be guaranteed to be subtypes of the provided bound. A lower type bound of a type variable tells that the provided bound will be guaranteed to be a subtype of all instantiations of the type variable. Bounds on type variables of generic classes make use of the naturally-supported subtyping relation in OOP. In this section, and thus in

\(^\text{25}\)Raw types are much like “unbounded wildcard types, or erased types, with automatically-inserted type casts that are not guaranteed to succeed” (See Section 5.3 for a brief discussion of Java wildcards). Just like a Java program with downcasts, or “stupid casts” is not guaranteed to be free of type errors (see [14]), also a Java program with raw types (which are used as generic types) is not guaranteed (or cannot be decidably guaranteed) to be free of type errors. A Java program with raw types that are always used as non-generic types, as is the case in pre-generics Java (i.e., Java 4.0-), can be decidably guaranteed to be free of type errors. Generification helps the type system decide type safety. Generification can somewhat be viewed as providing the type checker with the “non-decidable” portion of the analysis it does while type checking. Otherwise generification would be an automatable process (Generification is believed to be an undecidable problem).
this thesis, we only discuss upper type bounds of generic type variables, and relegate a discussion of lower type bounds to future work.

An upper type bound of a type variable allows a class designer to assume that certain members do exist in objects of any valid instantiation of the type variable, and, because of nominality associating a behavioral contract with type names, to also assume that all instantiations of the type variable stick to the behavioral contract associated with the name of the type bound.\footnote{For example, a class that provides (i.e., whose instances provide) a generic sorting “service”, needs to be sure that whatever elements (objects) it orders do actually support ordering. The type of these elements will be provided to the “sorter” generically, as a type variable, but will be required, via an upperbound, to provide “proof” that its elements (of the type) do support ordering, e.g., by having a \texttt{compare()} or \texttt{leq()} (less-than-or-equal) method.}

Formally, adding upperbounds (upper type bounds) to type variables, amounts to a small addition to signature constructors, where the type variables component is not a sequence of type variables (names from set $X$), but a sequence of pairs of type variables and bounds, where bounds are members of the set $\mathbb{GN}$.\footnote{Not the set $\mathbb{GNX}$. We do not allow type variables to bound type variables, but only to appear as type arguments in the bounds. Even though the consequences of lifting this restriction could be interesting to investigate, we do not do so here.}

We thus define

$$UBX = X \times \mathbb{GN}$$

and we could now redefine signature constructors

$$SC = N \times UBX^* \times \mathbb{GN}^* \times (L \times \mathbb{GNX})^* \times (L \times \mathbb{GNX}^* \times \mathbb{GNX})^* \quad (5.1)$$

Other than this small change, all other definitions we had in Section 3 remain exactly the same. Type variable bounds, however, add few more definitions and concepts.

### 5.2.1 Valid Signature Constructor Instantiations and Valid Ground Signature Names

The main change done to our modeling of generics by having upper bounds is that it introduces, on top of the notion of well-formed signature constructor instantiations (which we have seen in Section 3), the notion of valid signature constructor instantiations. Rather than using generic object signatures that are only well-formed, valid generic object signatures (which use valid signature constructor instantiations) are now used to construct the domain of generic objects $GO$ of $\mathbb{GN\text{\text{\text{OOP}}}}$ to model objects of generic OOP where type variables have bounds. The addition of type variable bounds thus affects values in the domain.

A valid signature constructor instantiation is a well-formed generic object signature whose type arguments do satisfy the upper bound constraints on the corresponding type variables they instantiate. A type argument must be a sub-sign of the upper bound of the corresponding type variable. This condition cannot be considered a well-formedness condition. As we have seen in Section 3.6, and Section 3.8, subsigning is defined for well-formed signature constructor instantiations, by \textit{implicitly} assuming they are instantiated with valid
type arguments (all instantiations were valid at the time, and thus there was no need for an explicit validity check). Checking validity of an instantiation depends on the subsigning relation being defined. If validity were to be considered part of well-formedness then a circularity would ensue between the definition of subsigning and well-formedness. Luckily, this circularity can be resolved by introducing the notion of validity, which is defined after the subsigning relation is defined, which, in turn, is defined after well-formedness is defined. Subsigning (between two instantiations) is thus defined assuming that the instantiations are valid. A decision made regarding subsigning does not later change if the either of the instantiations is discovered to be invalid.28 29

Within the context of a signature constructor environment, thus, a valid ground signature name is a well-formed ground signature name with the additional condition that when the arguments are substituted (via name substitution) in the upper bounds of the signature constructors the required subsigning constraints will hold in the subsigning relation.

Formally, in the context of a signature constructor environment sce, \( \text{ggn} = (\text{nm}, \\{\text{ggn}\}) \) if \( \text{ggn} \) are valid ground signature names30, and if and only if when we define

\[
\text{ub} = \text{tvars}(\text{sce}(\text{nm}))
\]

we have

\[
\text{ggn} \subseteq \{\text{ub} \mapsto \text{ggn}\} \text{ub}(\text{ub})
\]

where \( \text{tvars} \) is a function that projects a signature constructor to its type variables component (now a member of \( \text{UBX} \)), \( \text{ub} \) returns the second component (the upper bound) of a pair (subsigning between pairs of ground signature constructor names is extended to sequences of pairs in the obvious way [i.e., by conjunction]). Note the use of \( \text{ggn} \) on both sides of the \( \subseteq \) relation, where it is used as the substitutant on the r.h.s of \( \subseteq \).

To construct a domain corresponding to these new instantiations, the only change in the construction is to redefine domain of signatures used to construct \( \text{GNOOP} \) to be the flat domain of valid generic object signatures (which use valid ground signature names).

Given that we allow type variables to appear as arguments in their bounds, substituting type arguments in their bounds then checking if those same arguments subsign these instantiated bounds, allows for a form of generics called

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28 This is the key observation behind not needing to redefine subsigning for valid instantiations, and then redefine valid instantiations and so on ad infinitum.

29 This split between well-formedness (as a more syntactic condition) and validity (as a more semantic condition), however, does introduce the annoying, but inevitable, notion of “well-formed yet invalid” ground signature names. An example from Java is the ground signature name \( \text{Enum<Object>} \), which is a well-formed instantiation of the generic \( \text{Enum} \) signature, but is invalid because \( \text{Object} \) is not a subsign of the bound on \( \text{Enum} \)’s type variable (in the case of this particular instantiation, the bound on the type variable of class \( \text{Enum} \), given its declaration in the Java class library, would, a little bit confusingly, be \( \text{Enum<Object>} \) itself. In Java, \( \text{Object} \) is not a subsignature of any other signature but itself [its supersignatures component is empty]).

30 Note that instantiations of zeroary signature constructors (only one such “instantiations” is possible) are all valid.
*F-bounded generics* (corresponding to the notion of F-bounded polymorphism, of structural OOP). F-bounded generics, arguably, provides more flexibility to generic nominal OOP by allowing better typing of so-called “binary methods”. Even though we have many interesting ideas related to them, we relegate further analysis of F-bounded generics, of so-called self-referential F-bounded generics (like in class `Enum` of Java), and of the generics-based solution to the problem of binary methods to future work.

### 5.2.2 Type Variable Bounds and Erasure

Upper type bounds on type variables offer a chance to do a little better regarding the information lost when interpreting generics using erasure (See Section 5.1). Due to the possible use, inside a class, of the members of the bound, having a better erasure precludes the use of “guaranteed-to-not-fail” type casts to make such accesses (method calls, or field accesses).

In this case, because it is more precise (more informative) than type `Object`, the erasure of the upper bound of a type variable is itself used, rather than type `Object`, as the erasure of the type variable when the type variable occurs naked inside a signature constructor.

### 5.3 Java Wildcards and Variance Annotations

‘Variance annotations’ is a feature of OOP generics that helps resolve the mismatch between subtyping polymorphism and generics, and to combine the power of the two type-abstraction features.

Because of supporting mutation (which we do not model in NOOP nor GNNOOP), a type like `LinkedList<Integer>` and `LinkedList<Number>` are unrelated by subtyping, even though type `Integer` is a subtype of type `Number`. Variance annotations is a feature that tries to resolve this mismatch. There are two styles for specifying variance annotations: declaration-site variance annotations, and usage-site variance annotations. Declaration-site variance annotations attach “annotations” to type variables of generic classes that limit how type variables can be used in the class (eg, as type of arguments to methods, or as return types of methods). Declaration-site variance annotations are used in languages such as Scala and C#. Usage-site variance annotations attach annotations to type arguments of generic class instantiations that limit how type variables used in the signature of methods of instantiations of a generic class are viewed outside (See [15] for a more detailed discussion of variance annotations). Java wildcards are the Java incarnation of usage-site variance annotations (See [19]).

Given the dependence of variance annotations on the nominal subtyping relation of OO languages, any accurate analysis of OO variance annotations should depend on having a precise model of nominal subtyping. Thus, having GNNOOP should help better reasoning about wildcards.

In this section we do not dig into modeling wildcards, but having GNNOOP at hand enables us to consider more precisely what could be involved if a full
model of wildcards is developed.

Our most critical observation is that with wildcards (or variance annotations, more generally), infinite chains of supertypes could now occur in the subtyping relation of signature constructor instantiations. Given the declaration of class Enum in Java, and with a typical class declaration that has the heading class C extends Enum<C>, consider, for example, the chain of Java types C, which has Enum<C> as a supertype (given the declaration of class C), which, by subtyping rules for wildcard types, has supertype Enum<? extends C>, which in turn has Enum<? extends Enum<C>> as a supertype, which has Enum<? extends Enum<? extends C>> as a supertype, which has Enum<? extends Enum<? extends Enum<C>>> as a supertype, ... and so on.

It is our opinion that having infinite supertype chains makes the semantics of full wildcards problematic, pending a detailed development of an elaborate model of wildcards. We do not see, however, a simpler way to handle full wildcards.

6 Polymorphic Methods

Polymorphic methods are usually associated with generics, but, despite some similarity, polymorphic methods are not an artifact of generics (generic classes) per se. Type variables of a generic class are shared by all methods. Polymorphic methods provide method-specific type variables. The declared type variables of a polymorphic method can be used in the signature and the code of that polymorphic method only. Polymorphic methods give more power and expressiveness to OO type systems.

Just like signature constructor type variables, method type variables allow a method signature to be abstracted over a set of signature names, allowing the same method to be used with different signature name instantiations.

In this section we give few core formal definitions for formally modeling polymorphic methods.

For signature constructors, we add an extra method type variables component Y* to sixth component of signature constructors (i.e., their method signatures component), and we use a separate set, GNY, for types that can be used in method signatures.

\[ SC = N \times X^* \times G^* \times (L \times GN^*)^* \times (L \times Y^* \times GNY^* \times GNY)^* \]

\[ GN = N \times GN^* \]
\[ GNX = GN + X \]

where for method signatures, we have

\[ GNM = N \times GNY^* \]
\[ GNY = GNM + X + Y \]
(as expected, method type variables cannot be used except in the types of the method that declares them, not in other parts of the signature constructor, whether that is other methods, fields, or the supersignatures of the signature constructor). Note that because of the X variant of GNY the type variables of a signature constructor can be used inside a method signature.

We leave the rest of research work needed to model polymorphic methods to future work. We expect it not to be too difficult since polymorphic methods merely involve an additional level of schematic abstraction and application.

7 Concluding Remarks

\textit{N}\textit{OOP} enables proper mathematical reasoning about mainstream OO languages. Developing \textit{GNOOP} based on \textit{N}\textit{OOP} has made ‘understanding generics’ an “application” demonstrating the utility of \textit{N}\textit{OOP} (\textit{i.e.}, of modeling nominal typing in particular). Reasoning about generics is not possible using earlier models of OOP. Because of them being structurally-typed and structurally-subtyped, earlier models of OOP, in fact, do not allow us to evaluate any interesting properties of nominal OO languages because the typing systems are fundamentally incompatible.

Even without using \textit{GNOOP} to reason about many generic OOP features, one immediate value of developing it is it dispelling a misconception that by supporting generics the type system of Java and similar languages is ‘a complex hybrid of nominal and structural type systems’ (See \cite[17, p.254]{17}). Looking at \textit{GNOOP}, it is clear that all OO typing features of Java, generic ones and ones unrelated to generics, are purely nominal. We believe the misconception that Java has become ‘a hybrid’ may have been due to confusing the ‘structure’ that generics does add to signature/type names (which is indeed important in \textit{GNOOP}) and the ‘structure’ of objects and classes (\textit{i.e.}, the set of object members and their signatures) which is the relevant structure that structural type systems refer to and are interested in.

8 Approach Extension

The author is also currently considering extending the above approach to generics and simplifying it. The extended approach is based on what is tentatively called ‘nominal intervals’ and ‘single-nested generics’. A nominal interval is a type variable name with both upper and lower bounds. In this approach, basically every type (including wildcard types) written inside a class is “captured” (not certain yet if that’s the right term) into a synthetic type variable (a nominal interval) of the class. For single-nested generics, because of the capturing in synthetic type variables, a type argument (like any type inside the class now) can always be a type variable (a nominal interval), not a generic type. In this simplified approach, class names/type names/signature constructor names will appear \textit{only} in bounds of type variables (interval bounds).
For this approach, the equations for generic signatures will be along lines of the following (\(Y\) is now the set of synthetic and original type variable names):

\[
GN = N \times Y^*
\]

\[
GNY = GN + Y
\]

(Note single-nesting of generic types. No mutual dependency between \(GN\) and \(GNY\) as that between \(GN\) and \(GNX\) above)

\[
YB = Y \times GN \times GN
\]

(Note nominal intervals. All type variables will have lower and upper bounds, in addition to a name. Can be extended to \(YB = Y \times GNY \times GNY\)?)

\[
SC = N \times YB^* \times GN^* \times (L \times Y)^* \times (L \times Y^* \times Y)^*
\]

(Note: all types of fields and methods are now replaced by references to type variables/nominal intervals).

We expect this approach to allow better modeling of Java wildcards and Java erasure.

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