GEOMETRIC CHARACTERIZATIONS OF
ASYMPTOTIC FLATNESS AND LINEAR MOMENTUM
IN GENERAL RELATIVITY

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Abstract. In 1996, Huisken-Yau proved that every three-dimensional Riemannian manifold can be uniquely foliated near infinity by stable closed surfaces of constant mean curvature (CMC) if it is asymptotically equal to the (spatial) Schwarzschild solution. Later, their decay assumptions were weakened by Metzger, Huang, Eichmair-Metzger, and the author. In this work, we prove the reverse implication, i.e. any three-dimensional Riemannian manifold is asymptotically flat if it possesses a CMC-cover satisfying certain geometric curvature estimates, a uniqueness property, and a weak foliation property. With the author’s previous result that every asymptotically flat manifold possesses a CMC-foliation, we conclude that asymptotic flatness is characterized by existence of such a CMC-cover. In particular, asymptotic flatness is a purely geometric property. Additionally, we use this characterization to give a geometric (i.e. coordinate-free) definition of a (CMC-)linear momentum and prove its compatibility with the linear momentum defined by Arnowitt-Deser-Misner.

Introduction

Surfaces of constant mean curvature (CMC) were for the first time used in mathematical general relativity by Christodoulou-Yau who studied quasi-local mass of asymptotically flat manifolds [CY88]. In 1996, Huisken-Yau proved the existence of a unique foliation by stable CMC-surfaces [HY96]. They considered Riemannian manifolds \((M, g, x)\) which are asymptotically equal to the (spatial) Schwarzschild solution, i.e. they assumed existence of a coordinate system \(x : M \setminus L \to \mathbb{R}^3 \setminus B_1(0)\) mapping the manifold (outside of some compact set \(L\)) to the Euclidean space (outside the closed unit ball) such that the push forward \(x_* g\) of the metric \(g\) is asymptotically equal to the (spatial) Schwarzschild metric. More precisely, they assumed that the \(k\)-th derivatives of the difference \(\tilde{g}_{ij} - \tilde{g}_{ij}\) of the metric \(\tilde{g}\) and the Schwarzschild metric \(\tilde{g} := (1 + m/2|x|)^{4/3} \tilde{g}\) decays in these coordinates like \(|x|^{-2-k}\) for every \(k \leq 4\), where the mass \(m\) was assumed to be positive and \(\tilde{g}\) denotes the Euclidean metric. This is abbreviated by \(\tilde{g} - \tilde{g} = O_4(|x|^{-2})\). Later, these decay assumptions were weakened by Metzger, Huang, Eichmair-Metzger, and the author [Met07, Hua10, EM12, Ner14]: It is sufficient to assume asymptotic flatness to ensure existence of a CMC-foliation (and its uniqueness in a well-defined class of surfaces). Here, being asymptotically flat means \(\tilde{g} - \tilde{g} = O_4(|x|^{-2-\epsilon})\) with \(\tilde{g} = O_3(|x|^{-3-\epsilon})\), where \(\tilde{g}\) is the scalar curvature of \(\tilde{g}\). Further properties of this foliation were studied by Huisken-Yau, Corvino-Wu, Eichmair-Metzger, the author, and others [HY96, CW08, EM12, Ner13, Ner14].
Inspired by an idea to use the CMC-foliation to define a unique coordinate system \( \mathfrak{g} : \tilde{M} \setminus K \to (\sigma_0; \infty) \times \mathbb{S}^2 \) which Huisken explained to the author, we prove that asymptotic flatness does not only imply the existence and uniqueness of a CMC-cover \( \{\sigma \Sigma\}_{\sigma \geq \sigma_0} \), but is characterized by it (Corollary 3.4). This means that any three-dimensional Riemannian manifold possessing a CMC-cover satisfying suitable curvature estimates is asymptotically flat if the leaves of the CMC-cover are locally unique and its Gauß curvature satisfies an integral assumption ensuring that the cover is a foliation (Theorem 3.1). This includes a topological result as we do a priori not assume that the CMC-cover is a CMC-foliation, i.e. that the leaves are pairwise disjoint and that they depend smoothly on their ‘mean curvature radius’ \( \sigma \). We furthermore give a corresponding characterization of \( W^{1,p} \)-asymptotic flatness (Theorem 3.5) for \( p \in (2; \infty) \), i.e. for asymptotic flatness in a Sobolev sense as defined by Bartnik [Bar86]. Note that in the latter setting, we do not impose pointwise assumptions on the Ricci curvature as we allow the Sobolev exponent \( p \) to be less than the dimension \( n = 3 \).

To best knowledge of the author, this is the first geometric characterization of asymptotic flatness without assuming that \((\tilde{M}, \mathfrak{g})\) corresponds to a stationary solution of the Einstein-equations a priori satisfying topological assumptions – compare with [Rei10, Rei13].

Additionally, we give a local version of this characterization (Theorem 4.1), i.e. if there is a CMC-cover \( \mathcal{M} = \{\sigma \Sigma\}_{\sigma \in (\sigma_0; \sigma_1)} \) of some part of a three-dimensional manifold for sufficiently large minimal radius \( \sigma_0 \), where \( \sigma \Sigma \) has mean curvature \( \sigma \mathcal{H} \equiv -\frac{2}{\sigma} \) which satisfies suitable curvature estimates, locally unique and satisfies a weak foliation property, then \( \mathcal{M} \) is a smooth CMC-foliation which is in a well-defined sense (asymptotically) rotational symmetric.

Furthermore, the characterization of asymptotic flatness can be used to define other quantities without using coordinates. Exemplary, we explain this for the linear momentum (Section 5): We define the CMC-linear momentum as a function \( \bar{P}_t \) on the initial data set \((\tilde{M}, \mathfrak{g}, \tilde{k}, \tilde{J}, \tilde{\rho})\) possessing a CMC-foliation. We prove that this function is well-defined outside of a compact set and that it characterizes the ADM-linear momentum calculated with respect to any asymptotically flat coordinate system. This means that the ADM-linear momentum can be interpreted as a coordinate expression of a geometric quantity: the CMC-linear momentum.

As a technical step in the proof which seems interesting for itself, we prove in Appendix A that every metric \( g \) on the two-dimensional sphere possesses a ‘good’ parametrization if it has a \( L^2 \)-almost constant Gauß curvature. This means if the Calabi-energy is sufficiently small, i.e. \( \|K - 1\|_{L^2(S^2, g)} \leq \varepsilon^2 \ll 1 \), and the area is bounded away from zero and \( 8\pi \), i.e. \( \mu(S^2) \in (\delta, 8\pi - \delta) \) for some \( \delta > 0 \), then there exists a conformal parametrization \( \varphi : S^2 \to S^2 \) satisfying \( \|u\|_{H^2(S^2, \Omega)} \leq C \|K - 1\|_{L^2(S^2, g)} \), where \( \varepsilon \) and \( C \) only depend on \( \delta \). Here, \( \mu \) is the measure on \( S^2 \) with respect to \( g \) and \( u \) is the corresponding conformal factor, i.e. \( \varphi^* g = \exp(2u) \Omega \) for the standard metric \( \Omega \) of the Euclidean unit sphere. This generalizes the well-known corresponding result for Gauß curvature pointwise bounded away from zero and infinity, see for example [CK93, Chap. 2].

1 up to a rotation
2 We explicitly use this idea in Theorem 4.1, the local version of the main theorem.
Acknowledgment. The author wishes to express gratitude to Gerhard Huisken for suggesting the topic of CMC-surfaces in asymptotically flat manifolds to the author and for many inspiring discussions. Further thanks are owed to Simon Brendle for suggesting the scaling argument in Appendix A. Finally, thanks goes to Carla Cederbaum for exchanging interesting ideas, and thoughts about CMC-foliations and to Mattias Dahl for bringing up the question whether it is necessary to a priori assume the existence of a smooth foliation rather than that of a cover.

Structure of the paper

In Section 1, we explain basic notations and definitions – these were also used in [Ner14]. We give main regularity arguments in Section 2, where we use Appendix A. The main result and its local version is stated and proven in Section 3 and Section 4, respectively. In Section 5, we give a coordinate-free definition of linear momentum and compare it with the linear momentum defined by Arnowitt-Deser-Misner (ADM). Finally, we prove existence of a ‘good’ parametrization for surfaces having $L^2$-almost constant Gauß curvature in Appendix A (see above).

1. Assumptions and notation

In order to study foliations (near infinity) of three-dimensional Riemannian manifolds by two-dimensional spheres, we will have to deal with different manifolds (of different or the same dimension) and different metrics on these manifolds, simultaneously. To distinguish between them, all three-dimensional quantities like the surrounding manifold $(\mathcal{M}, \mathcal{g})$, its Ricci and scalar curvature $\text{Ric}$ and $\mathcal{S}$, and all other derived quantities carry a bar, while all two-dimensional quantities like the CMC leaf $(\Sigma, g)$, the trace free part $\mathcal{k}^\circ$ of its second fundamental form $\mathcal{k}$, its Ricci, scalar, and mean curvature $\text{Ric}$, $\mathcal{S}$, and $\mathcal{H} := \text{tr} \mathcal{k}$, its outer unit normal $\nu$, and all other derived quantities do not. Furthermore, we stress that the sign convention used for the second fundamental form results in a negative mean curvature of the Euclidean coordinate sphere.

If different two-dimensional manifolds or metrics are involved, then the lower left index will always denote the mean curvature index $\sigma$ of the current leaf $\Sigma$, i.e. the leaf with mean curvature $\sigma \mathcal{H} \equiv -\mathcal{H}/\sigma$. We abuse notation and suppress this index, whenever it is clear from the context which metric we refer to. Furthermore, quantities carry the upper left index $e$ and $\Omega$ if they are calculated with respect to the Euclidean metric $e\mathcal{g}$ and the standard metric $\Omega$ of the Euclidean sphere $S^2_0(0)$, correspondingly.

Here, we interpret the second fundamental form and the normal vector of a hypersurface as quantities on the hypersurfaces (and thus as two-dimensional). For example, if $\Sigma$ is a hypersurface in $\mathcal{M}$, then $\nu$ denotes its normal (and not $\nu$). The same is true for the ‘lapse function’ and the ‘shift vector’ of hypersurfaces arising as a leaf of a given deformation or foliation.

Furthermore, we use upper case latin indices $I$, $J$, $K$, and $L$ for the two-dimensional range $\{2,3\}$ and lower case latin indices $i$, $j$, $k$, $l$, and $m$ for the three-dimensional range $\{1,2,3\}$. The Einstein summation convention is used accordingly.

Now, we give the main definition used within this work. Let us begin by recalling the Hawking mass [Haw03].
**Definition 1.1 (Hawking mass)**

Let \((\overline{M}, \overline{g})\) be a three-dimensional Riemannian manifold. For any closed hypersurface \(\Sigma \hookrightarrow (\overline{M}, \overline{g})\) the Hawking-mass is defined by

\[
m_H(\Sigma) := \sqrt{\frac{\lvert \Sigma \rvert}{16\pi}} \left(1 - \frac{1}{16\pi} \int H^2 \, d\mu \right),
\]

where \(H\) and \(\mu\) denote the mean curvature and measure induced on \(\Sigma\), respectively.

As there are different definitions of ‘asymptotically flat’, we now define the decay assumptions used in this paper.

**Definition 1.2 (\(C_{2+\varepsilon}^2\)-asymptotically flat Riemannian manifolds)**

Let \(\varepsilon \in (0; 1/2]\) be a constant and let \((\overline{M}, \overline{g})\) be a smooth Riemannian manifold. The tuple \((\overline{M}, \overline{g}, \overline{\tau})\) is called \(C_{2+\varepsilon}^2\)-asymptotically flat Riemannian manifold if \(\overline{\tau} : \overline{M} \setminus \mathcal{L} \rightarrow \mathbb{R}^3 \setminus B_1(0)\) is a smooth chart of \(\overline{M}\) outside a compact set \(\mathcal{L} \subseteq \overline{M}\) such that

\[
\left| \overline{g}_{ij} - \overline{\mathbb{E}}_{ij} \right| + \left| \overline{\tau} \right| \left| \Gamma_{ij}^k \right| + \left| \overline{\tau} \right|^2 \left| \overline{\mathcal{R}}_{ij} \right| + \left| \overline{\tau} \right|^3 \left| \overline{\mathcal{S}} \right| \leq \frac{c}{\left| \overline{\tau} \right|^{1+\varepsilon}} \quad \forall i, j, k \in \{1, 2, 3\}
\]

holds for some constant \(c \geq 0\), where \(\overline{g}\) denotes the Euclidean metric.

Arnowitt-Deser-Misner defined the \((\text{ADM-})\) mass of a \(C_{2+\varepsilon}^2\)-asymptotically Riemannian manifold \((\overline{M}, \overline{g}, \overline{\tau})\) by

\[
\overline{m}_{\text{ADM}} := \lim_{R \to \infty} \frac{1}{16\pi} \sum_{j=1}^3 \int_{S^2_R(0)} \left( \frac{\partial g_{ij}}{\partial \overline{\tau}} - \frac{\partial \overline{\tau}^i}{\partial \overline{\tau}} \frac{\partial g_{ij}}{\partial \overline{\tau}^j} \right) R \nu^j \, dR \mu,
\]

where \(R \nu\) and \(R \mu\) denote the outer unit normal and the area measure of \(S^2_R(0) \hookrightarrow (\overline{M}, \overline{g})\) \([\text{ADM61}]\).

In the literature, the ADM-mass is characterized using the curvature of \(\overline{g}\):

\[
\overline{m} := \lim_{R \to \infty} \frac{-R}{8\pi} \int_{S^2_R(0)} \overline{\mathcal{R}}(R \nu, R \nu) - \frac{3}{2} \, dR \mu,
\]

see the articles of Ashtekar-Hansen, Chruściel, and Schoen \([\text{AH78}, \text{Sch88}, \text{Chr86}]\). Miao-Tam recently gave a proof of this characterization \(\overline{m}_{\text{ADM}} = \overline{m}\) in the setting used within this paper, i.e. for any \(C_{2+\varepsilon}^2\)-asymptotically flat manifold \([\text{MT14}]\). We recall that this mass is also characterized by

\[
\overline{m} = \lim_{R \to \infty} m_H(S^2_R(0)).
\]

This can be seen by a direct calculation using the Gauß equation, the Gauß-Codazzi equation, and the decay assumptions on metric and curvatures. Here, \(m_H(S^2_R(0))\) denotes the Hawking-mass of \(S^2_R(0)\) which is for any closed hypersurface \(\Sigma \hookrightarrow \overline{M}\) defined by

\[
m_H(\Sigma) := \sqrt{\frac{\lvert \Sigma \rvert}{16\pi}} \left(1 - \frac{1}{16\pi} \int H^2 \, d\mu \right),
\]

where \(H\) and \(\mu\) denote the mean curvature and measure induced on \(\Sigma\), respectively \([\text{Haw03}]\).

\[3\text{The author thank Carla Cederbaum for bringing his attention to Miao-Tam’s article [MT14].}\]
We specify the definitions of Lebesgue and Sobolev norms on compact Riemannian manifolds which we will use throughout this article.

**Definition 1.3 (Lebesgue and Sobolev norms)**

If \((\Sigma, g)\) is a compact Riemannian manifold without boundary, then the *Lebesgue norms* are defined by

\[
\|T\|_{L^p(\Sigma)} := \left( \int_{\Sigma} |T|^p \, d\mu \right)^{\frac{1}{p}} \quad \forall \, p \in [1; \infty), \quad \|T\|_{L^\infty(\Sigma)} := \text{ess sup}_\Sigma |T|_g,
\]

where \(T\) is any measurable function (or tensor field) on \(\Sigma\). Correspondingly, \(L^p(\Sigma)\) is defined to be the set of all measurable functions (or tensor fields) on \(\Sigma\) for which the \(L^p\)-norm is finite. If \(r := (\Sigma/\omega_n)^{1/n}\) denotes the *area radius* of \(\Sigma\), where \(n\) is the dimension of \(\Sigma\) and \(\omega_n\) denotes the Euclidean surface area of the \(n\)-dimensional unit sphere, then the *Sobolev norms* are defined by

\[
\|T\|_{W^{k, p}(\Sigma)} := \|T\|_{L^p(\Sigma)} + r \|\nabla T\|_{W^{k-1, p}(\Sigma)}, \quad \|T\|_{W^{0, p}(\Sigma)} := \|T\|_{L^p(\Sigma)},
\]

where \(k \in \mathbb{N}_{\geq 0}, \ p \in [1; \infty]\) and \(T\) is any measurable function (or tensor field) on \(\Sigma\) for which the \(k\)-th (weak) derivative exists. Correspondingly, \(W^{k, p}(\Sigma)\) is the set of all such functions (or tensor fields) for which the \(W^{k, p}(\Sigma)\)-norm is finite. Furthermore, \(H^k(\Sigma)\) denotes \(W^{k, 2}(\Sigma)\) for any \(k \geq 1\) and \(H(\Sigma) := H^1(\Sigma)\).

Now, we can define the class of surfaces which we will use in the following.

**Definition 1.4 (Regular spheres)**

Let \((\overline{M}, \overline{g})\) be a three-dimensional Riemannian manifold. A *hypersurface* \(\Sigma \hookrightarrow (\overline{M}, \overline{g})\) is called *regular sphere with area radius* \(r := \sqrt{|\Sigma|/4\pi}\) and constants \(\kappa \in (1; 2), \ M > 0, \ p \in (2; \infty)\) and \(c_1, c_2 \geq 0\), in symbols \(\Sigma \in \mathcal{R}_{\kappa, r}(M, c_1, c_2)\), if \(\Sigma\) is a topological sphere satisfying \(|m_\Sigma(\Sigma)| \in [M^{-1}; M]\) and if there is a constant \(\mathcal{H} = \mathcal{H}(\Sigma)\) such that

\[
\|H - \mathcal{H}\|_{W^{1, \infty}(\Sigma)} \leq c_1 |\mathcal{H}|^\kappa, \quad \|\text{Ric}_\Sigma\|_{L^p(\Sigma)} + |\mathcal{H}|^{\frac{1}{2}} \|\mathcal{S}\|_{L^p(\Sigma)} \leq c_2 |\mathcal{H}|^{\kappa + 1 - \frac{2}{p}},
\]

where \(2/p = 0\) if \(p = \infty\). In this case \(\sigma := -2/\kappa\) is called *approximated mean curvature radius of* \(\Sigma\). If the mean curvature is constant with \(H \equiv \mathcal{H}\), then \(\sigma\) is called *mean curvature radius of* \(\Sigma\). Furthermore, \(\mathcal{R}_{\kappa, r}(M, c_1, c_2)\) denotes the radius independent family, i.e. \(\mathcal{R}_{\kappa, r}(M, c_1, c_2) := \bigcup_r \mathcal{R}_{\kappa, r}(M, c_1, c_2)\).

**Remark 1.5 (Lorentzian version).** The main motivation of asymptotically flat manifolds is spacelike hypersurfaces in a four-dimensional Lorentzian manifold solving the Einstein equations. However, we used in Definition 1.4 (and in the rest of this work) only the Riemannian data of such a time-slice. Here, we give alternative assumptions on the surfaces using data of the surrounding four-dimensional Lorentzian manifold:

If \((\overline{M}, \overline{g})\) is a spacelike hypersurface within a four-dimensional Lorentzian manifold \((M, \tilde{g})\) solving the Einstein equations \(8\pi \tilde{T} = \text{Ric} - \frac{1}{2} \tilde{S} \tilde{g}\) for the energy-momentum tensor \(\tilde{T}\), then the assumption on \(|\text{Ric}|_{\tilde{g}}\) in Definition 1.4 holds for a closed hypersurface \(\Sigma \hookrightarrow \overline{M}\) and \(p = \infty\) if

\[
\sup_{\Sigma} \left| \tilde{T} \right|_{\tilde{g}} \leq \frac{c_2}{4} |\mathcal{H}|^{\kappa + 1}, \quad \sup_{\Sigma} \left| \mathcal{K} \right|_{\tilde{g}} \leq \frac{c_2}{4} |\mathcal{H}|^{\frac{n+1}{2}}, \quad \sup_{\Sigma} \left| \mathcal{E}_{\tilde{g}} \mathcal{K} \right|_{\tilde{g}} \leq \frac{c_2}{4} |\mathcal{H}|^{\kappa + 1},
\]

with \(n\) being the dimension of \(\Sigma\).
where \( K \) is as in Definition 1.4, \( \tilde{K} \) denotes the second fundamental form of \((M, \tilde{g}) \rightarrow (M, \tilde{g})\), and \( \nabla_{\hat{\nu}} K \) denotes the Lie-derivative of \( K \) in direction of a unit normal field \( \hat{\nu} \) on \( \tilde{M} \).\(^4\) We can equivalently rewrite this assumption using a given foliation of space-time \( \tilde{M} \) by spacelike hypersurfaces \( \tilde{M} \) or for \( p \in (2; \infty) \).

For notation convenience, we use the following abbreviated form for the contraction of two tensor fields.

**Definition 1.6** (Tensor contraction)

Let \((\Sigma, g)\) be a Riemannian manifold. The *traced tensor product* of a \((0, k)\) tensor field \( S \) and a \((0, l)\) tensor field \( T \) on \((\Sigma, g)\) with \( k, l > 0 \) is defined by

\[
(S \circ T)_{I_1 \ldots I_{k-1} J_1 \ldots J_{l-1}} := S_{I_1 \ldots I_{k-1} K} T_{L J_1 \ldots J_{l-1}} g^{KL}.
\]

This definition is independent of the chosen coordinates. Furthermore, \( S \circ T \circ U \) is well-defined if \( T \) is a \((0, k)\) tensor field with \( k \geq 2 \), i.e. \( (S \circ T) \circ U = S \circ (T \circ U) \) for such a \( T \).

Finally, we infinitesimally characterize foliations in the following by their lapse functions and their shift vectors.

**Definition 1.7** (Lapse functions, shift vectors)

Let \( \theta > 0 \) and \( \sigma_0 \in \mathbb{R} \) be constants, \( I \supseteq (\sigma_0 - \theta \sigma; \sigma_0 + \theta \sigma) \) be an interval, and \((\tilde{M}, \tilde{g})\) be a Riemannian manifold. A smooth map \( \Phi : I \times \Sigma \rightarrow \tilde{M} \) is called *deformation* of the closed hypersurface \( \Sigma = \sigma_0 \Sigma = \Phi(\sigma_0, \Sigma) \subseteq \tilde{M} \) if \( \sigma \Phi(\cdot, \cdot) := \Phi(\sigma, \cdot) \) is a diffeomorphism onto its image \( \Sigma = \sigma \Phi(\Sigma) \) and \( \sigma_0 \Phi \equiv \operatorname{id}_\Sigma \). The decomposition of \( \partial_\sigma \Phi \) into its normal and tangential parts can be written as

\[
\frac{\partial \Phi}{\partial \sigma} = \sigma u + \sigma \beta,
\]

where \( \sigma u \) is the outer unit normal to \( \sigma \Sigma \). The function \( \sigma u : \Sigma \rightarrow \mathbb{R} \) is called *lapse function* and the vector field \( \sigma \beta \in \mathfrak{X}(\sigma \Sigma) \) is called *shift* of \( \Phi \). If \( \Phi \) is a diffeomorphism, then it is called a *foliation*.

### 2. Regularity of the hypersurfaces

In this section, we prove the regularity results for the hypersurfaces used within this work. The author proved that regular spheres \( S^2(\kappa, \rho) := \{ \sigma \Sigma = \Phi(\sigma, \Sigma) \subseteq \tilde{M} \} \) are asymptotically pointwise umbilic (as \( r \rightarrow \infty \)) \[\text{Ner14}\]. Using DeLellis-Müller's result [DLM05], this implies existence of conformal parametrizations \( r \phi : S^2 \rightarrow \Sigma \) such that the corresponding conformal factor \( r \nu \in W^{2,p}(S^2) \) is asymptotically constant, i.e. \( r \nu \rightarrow 1 \) in \( W^{2,p}(S^2) \) for \( r \rightarrow \infty \), where \( p \in [2; \infty) \) is arbitrary, \( r \phi^*, g = r^2 e^{2r \nu} \Omega \), and where \( \Omega \) denotes the standard metric of the Euclidean unit sphere. In the setting of a surrounding manifold \((\tilde{M}, \tilde{g}, \tilde{\tau})\) asymptotically equal to the (spatial) Schwarzschild solution, a similar result was previously proven by Metzger [Met07]. In Subsection 2.1 we prove the same result for regular spheres in an arbitrary three-dimensional Riemannian manifolds. To do so, we replace the

\[\text{\footnote{More exactly, } \nabla_{\hat{\nu}} K \text{ is the Lie-derivative of } \tilde{K}, \text{ where } \tilde{K}'(\exp_{\tilde{p}}(r \tilde{\nu} \tilde{p})) \text{ is the second fundamental form of graph } \tilde{\tau} := \{ \tilde{p} \in \tilde{M}_1 \rightarrow (\tilde{M}, \tilde{g}) \} \text{ in } \exp_{\tilde{p}}(r \tilde{\nu} \tilde{p}) \text{ which is well-defined in a } \hat{M}-\text{neighborhood of } \tilde{M}. \text{ Here, } \exp_{\tilde{p}} \text{ denotes the exponential map of } \tilde{M} \text{ in a point } \tilde{p} \in \tilde{M}. \text{ In particular, this Lie-derivative is well-defined}.} \]
crucial tool in the above argument, DeLellis-Müller’s result, by the arguments in Appendix A. There, we prove that metric $g$ on the Euclidean sphere converges (after conformal reparametrization) to the standard metric of the Euclidean sphere if the Gauß curvatures $\kappa$ converges in $L^2(S^2, \Omega)$ to 1. Note that the same result is well-known if the Gauß curvatures are pointwise bounded away from zero and infinity, see for example [CK93]. In Subsection 2.2 we then cite results and arguments from [Ner14] proving that the Eigenvalues of the stability operator are (asymptotically) controlled.

2.1. Conformal parametrization and umbilicness. We start by proving that any regular sphere $\Sigma$ has a ‘good’ conformal parametrization, i.e. the corresponding conformal factor is almost constant. To do so, we make the following three steps:

Lemma 2.2 prove that the sphere is in a $L^2$-sense almost umbilic and that the area of $\Sigma$ is (approximately) $4\pi\sigma^2$;

Lemma 2.3 prove that the sphere is in a $L^4$-sense almost umbilic and therefore the scalar curvature is in a $L^2$-sense asymptotically constant;

Proposition 2.4 prove that the surfaces is in a $W^{1,p}$-sense almost umbilic and that a ‘good’ parametrization exists – this uses Theorem 2.1 from Appendix A and Corollary 2.1 [Ner14, Prop 2.1].

Let us first look at the last step and rewrite the cited proposition [Ner14, Prop 2.1] in the notation used within this work.

Corollary 2.1 ($L^\infty$-estimates on the second fundamental form)

Let $(\Sigma, g) \in \mathcal{R}_{p,r}^p(M, c_1, c_2)$ be a hypersurface of a three-dimensional Riemannian manifold $(\overline{M}, \overline{g})$, where $\kappa \in (1 ; 2]$, $p \in (2 ; \infty]$, $c_1, c_2 \geq 0$, and $M > 0$ are constants. Assume there is a finite Sobolev constant $c_S \geq 0$, i.e.

$$\|f\|_{L^2(\Sigma)} \leq \frac{c_S}{r} \|f\|_{W^{1,1}(\Sigma)} \quad \forall f \in C^1(\Sigma).$$

There are constants $\sigma_0 = \sigma_0(\kappa, p, c_1, c_2, M, c_S)$ and $C = C(\kappa, p, c_1, c_2, M, c_S)$ such that

$$\|\kappa\|_{L^\infty(\Sigma)} + \sigma^{-1} \|\kappa\|_{H(\Sigma)} \leq \frac{C}{\sigma^\kappa}$$

if $\sigma \geq \sigma_0$, $|\kappa - r| \leq \frac{3 - \kappa}{5\sigma^\kappa}$, and $\|\kappa\|_{L^2(\Sigma)} \leq 1/c$.

Now, let us begin by a simple $L^2$-estimate for the second fundamental form and prove that all preliminaries of Corollary 2.1 except the Sobolev inequality (1) are satisfied if $\sigma$ is sufficiently large, too.

Lemma 2.2 ($L^2$-estimates for the second fundamental form)

Let $(\Sigma, g) \in \mathcal{R}_{p,r}^p(M, c_1, c_2)$ be a hypersurface of a three-dimensional Riemannian manifold $(\overline{M}, \overline{g})$, where $\kappa \in (1 ; 2]$, $p \in (2 ; \infty]$, $c_1, c_2 \geq 0$, and $M > 0$ are constants. There are constants $\sigma_0 = \sigma_0(\kappa, p, c_1, c_2, M)$ and $C = C(\kappa, p, c_1, c_2, M)$ such that

$$|\kappa - r| \leq C \sigma^{2-\kappa}, \quad \|\kappa\|_{L^2(\Sigma)} \leq \frac{C}{\sigma^{2-\kappa}}$$

if $\sigma \geq \sigma_0$.

Proof. With the assumption on the Hawking mass, we see that

$$\left| \int_\Sigma g^{1/2} \, \mathrm{d} \mu - 16\pi \right| \leq \frac{C}{\sigma^{\kappa-1}}, \quad \|\Sigma\| - 4\pi\sigma^2 \leq C \sigma^{3-\kappa}.$$
Thus, we conclude the claim by the Gauß equation and the Gauß-Bonnet theorem.

Now, we strengthen this to $L^4$-estimates for the second fundamental form. Here, we use the Simons identity also used in this context for example in [Met07] Prop. 3.3 and [Ner14] Prop. 2.1 where a corresponding result was proven under the assumption of asymptotic flatness of the surrounding manifold (in order to use DeLellis-Müller’s result [DLM05]).

**Lemma 2.3 (L^4-estimates for the second fundamental form)**

Let $(\Sigma, g) \in \mathcal{R}^{p, r}_{\kappa, \nu}(M, c_1, c_2)$ be a hypersurface of a three-dimensional Riemannian manifold $(\mathcal{M}, g)$, where $\kappa \in (1 ; 2]$, $p \in (2 ; \infty]$, $c_1, c_2 \geq 0$, and $M > 0$ are constants. There are constants $\sigma_0 = \sigma_0(\kappa, p, c_1, c_2, M)$ and $C = C(\kappa, p, c_1, c_2, M)$ such that

$$\|k\|^2_{L^2(\Sigma)} + \sigma^2 \|k_1\|^4_{L^4(\Sigma)} + \sigma^2 \|\nabla k\|^2_{L^2(\Sigma)} \leq \frac{C}{\sigma^{\kappa-1}}$$

if $\sigma > \sigma_0$.

**Proof.** Equivalent to [Met07] Prop. 3.3 and [Ner14] Prop. 2.1, we integrate $\text{tr}(\Delta k \circ k)$ and then integrate it by parts using the Simons-identity

$$\Delta k = \text{Hess} \mathcal{H} - \nabla \text{Ric}_\nu + \text{div}_2 \mathcal{F}_{\nu} + \frac{\mathcal{H}^2}{2} \circ k + \mathcal{H} \circ k - \|k\|^2_g k$$

and see that for every $\delta > 0$ there exists a constant $C \geq 0$ such that

$$\|\nabla k\|^2_{L^2(\Sigma)} \geq -\frac{C}{\sigma^\kappa} \|\nabla k\|^2_{L^2(\Sigma)} - \int \frac{\mathcal{H}^2}{2} |k|^2_g d\mu - \|\mathcal{H}\| \|\nabla k\|_{L^4(\Sigma)} \|k\|_{L^2(\Sigma)}$$

$$- \|k_1\|^3_{L^4(\Sigma)} \|\mathcal{H} - \mathcal{H}\|_{L^4(\Sigma)} + (1 - \delta) \|k_1\|^4_{L^4(\Sigma)} - \frac{C}{\sigma^\kappa} \|k\|^4_{L^2(\Sigma)}.$$ 

This means that for every $\delta > 0$ there exists a constant $C \geq 0$ such that

$$\|\nabla k\|^2_{L^2(\Sigma)} \geq -\frac{2 + \delta}{\sigma^2} \|k\|^2_{L^2(\Sigma)} - \frac{C}{\sigma^{2\kappa}} + (1 - \delta) \|k_1\|^4_{L^4(\Sigma)}.$$ 

On the other hand, we know

$$\|\nabla k\|^2_{L^2(\Sigma)} = -\int \text{tr}(\nabla \text{div} k \circ k) + \frac{S}{2} |k|^2_g d\mu = \int |\text{div} k|^2_g d\mu - \int \frac{S}{2} |k|^2_g d\mu.$$ 

Using the Codazzi equation $\text{div}(\mathcal{H} g - k) = \overline{\text{Ric}}(\nu, \cdot)$ and the assumptions on $\mathcal{H}$ and $\overline{\text{Ric}}(\nu, \cdot)$, this implies

$$\|\nabla k\|^2_{L^2(\Sigma)} \leq \frac{C}{\sigma^{2\kappa}} - \int \frac{S}{2} |k|^2_g d\mu.$$ 

Again using the Gauß equation, we conclude using the assumptions on $\overline{\text{Ric}}$

$$\|\nabla k\|^2_{L^2(\Sigma)} \leq \frac{C}{\sigma^{2\kappa}} - \frac{1}{\sigma^2} \|k\|^2_{L^2(\Sigma)} + \frac{3}{4} \|k_1\|^4_{L^4(\Sigma)}.$$ 

The claim of the lemma follows by combining this with Lemma 2.2 and [1].
Proposition 2.4 (Regularity of the spheres)
Let \( q < \infty \) be a constant and \((\Sigma, g) \in \mathcal{R}_p^q(M, c_1, c_2)\) be a hypersurface of a three-dimensional Riemannian manifold \((\overline{M}, \overline{g})\), where \( \kappa \in (1; 2], p \in (2; \infty), c_1, c_2 \geq 0, \) and \( M > 0 \) are constants. There are constants \( \sigma_0 = \sigma_0(p, c_1, c_2, M) \) and \( C = C(\kappa, p, c_1, c_2, M, q) \) and a conformal parametrization \( \varphi : S^2 \to \Sigma \) with corresponding conformal factor \( v \in H^2(S^2) \), i.e. \( \varphi^*g = \exp(2v)\sigma^2 \Omega \), such that

\[
\|v\|_{W^{2,p}(S^2, \sigma^2 \Omega)} \leq \frac{C}{\sigma^{\kappa+1} \frac{q}{p}}, \quad \|k\|_{W^{1,p}(\Sigma)} \leq \frac{C}{\sigma^\kappa \frac{q}{p}}
\]

if \( \sigma > \sigma_0 \), where \( \Omega \) denotes the standard metric of the Euclidean unit sphere.

Proof. By the Lemmata 2.2 and 2.3, we can use Theorem A.1 (after rescaling by the factor \( \sigma^{-1} \)) to conclude that the Sobolev inequality holds. Thus, we can use Corollary 2.1 and Lemma 2.2 to conclude (2). In particular, the Gauß equation implies \( \|3 - 2/\sigma^2\|_{L^p(\Sigma)} \leq C/\sigma^{\kappa+1} \) due to the assumptions on \( \mathbb{R}^n \) on \( \Sigma \). Thus, we can again use Theorem A.1 to conclude that a conformal parametrization exists whose conformal factor \( v \) satisfies the first inequality in (5). Now, we can use the regularity of the Laplace operator, see [CK93, Cor. 2.3.1.2] or the combination of [GM05 Thm 7.1] and [AF03 Thm 3.9], on the Simon’s identity [3] to conclude (5).

2.2. The stability operator. Now, we recall results from [Ner14], which we can use in this setting. As first step, we note that the eigenvalues of the stability operator of \( \Sigma \) are of order \( \sigma^{-2} \) except for three eigenvalues of order \( \sigma^{-3} \). As we will see in Proposition 2.6, the corresponding partition of \( H^2(\Sigma) \) (respectively \( L^2(\Sigma) \)) is (asymptotically) given as follows.

Definition 2.5 (Translational and deformational part of a function)
Let \((\Sigma, g) \in \mathcal{R}_p^q(M, c_1, c_2)\) be a hypersurface of a three-dimensional Riemannian manifold \((\overline{M}, \overline{g})\), where \( \kappa \in (1; 2], p \in (2; \infty), c_1, c_2 \geq 0, \) and \( M > 0 \) are constants. The translational part \( f^t \) of a function \( f \in L^2(\Sigma) \) is the \( L^2(\Sigma) \)-orthogonal projection of \( f \) on the linear span of eigenfunctions of the (negative) Laplace with eigenvalue \( \lambda \) satisfying \( |\lambda - 2/\sigma^2| \leq 1/\sigma^2 \), i.e.

\[
f^t := \sum_{|\lambda_i - 2/\sigma^2| \leq 1/\sigma^2} f_i \int_\Sigma f_i \, d\mu \quad \forall f \in L^2(\Sigma),
\]

where \( \{f_i\}_{i \in \mathbb{N}} \) is any complete orthogonal system of \( L^2(\Sigma) \) consisting of eigenfunctions of the (negative) Laplace operator with corresponding eigenvalue \( \lambda_i \), i.e. 

\(-\Delta f_i = \lambda_i f_i \) with \( \lambda_i \leq \lambda_{i+1} \). The deformational part \( f^d \) of such a function \( f \in L^2(\Sigma) \) is defined by \( f^d := f - f^t \).

The author explained in [Ner14 Prop. 4.5] reasons for calling this terms translational and deformational part. Note that \( L^2(\Sigma)^t := \{f^t : f \in L^2(\Sigma)\} \) is three-dimensional due to Proposition 2.4. Now, we can cite the announced stability proposition which is one of the central tools for the proof of the main theorem.

Proposition 2.6 (Stability (equivalent to [Ner14 Prop 2.7]))
Let \( q \in (2; \infty) \) be a constant and \((\Sigma, g) \in \mathcal{R}_p^q(M, c_1, c_2)\) be a hypersurface with constant mean curvature of a three-dimensional Riemannian manifold \((\overline{M}, \overline{g})\), where \( \kappa \in (1; 2], p \in (2; \infty), M > 0, c_1, \) and \( c_2 \) are constants. There are two constants...
\( \sigma_0 = \sigma_0(\kappa, q, c_1, c_2, M) \) and \( C = C(\kappa, q, c_1, c_2, M, p) \) such that
\[
\left| \int_{\Sigma} Lg^i h^i \mu - \frac{6m_H}{\sigma^3} \int_{\Sigma} g^i h^i \mu \right| \leq \frac{C}{\sigma^{2+\epsilon}} \left\| g^i \right\|_{L^2(\Sigma)} \left\| h^i \right\|_{L^2(\Sigma)} \quad \forall g, h \in L^2(\Sigma),
\]
if \( \sigma > \sigma_0 \). If \( f \in H^2(\Sigma) \) is a eigenfunction of \(-L\) with corresponding eigenvalue \( \gamma \) and \( \sigma > \sigma_0 \), then
\[
\left| \gamma \right| \geq \frac{3}{2\sigma^2} \quad \text{or} \quad \left\| g^i \right\|_{H^2(\Sigma)} \leq \frac{C}{\sigma^{2+\epsilon}} \left\| f \right\|_{H^2(\Sigma)}, \quad \left| \gamma - \frac{6m}{\sigma^3} \right| \leq \frac{C}{\sigma^{2+\epsilon}}.
\]
Furthermore, the corresponding \( W^{2,p} \)-inequalities
\[
\left\| g^i \right\|_{W^{2,p}(\Sigma)} \leq \left( \frac{\sigma^3}{6m_H} + C \sigma^{3-\epsilon} \right) \left\| Lg \right\|_{L^p(\Sigma)}, \quad \left\| g^i \right\|_{W^{2,p}(\Sigma)} \leq C \sigma^2 \left\| Lg \right\|_{L^p(\Sigma)}
\]
hold for every function \( g \in W^{2,p}(\Sigma) \) if \( \sigma > \sigma_0 \).

Proof. Exactly as the proof of [Ner14 Prop. 2.7] (respectively [Ner14 Lemma 2.5]), when we replace [Ner14 Prop. 2.4] by Proposition 2.4 ///

3. The main theorem

In this section, we prove the main theorem. We state the theorem first and explain afterwards the definitions used.

**Theorem 3.1** (Sufficient assumptions for asymptotic flatness)

Let \((\overline{M}, \overline{g})\) be a three-dimensional Riemannian manifold without boundary and let \( \overline{m} \neq 0, \epsilon \in (0;\frac{1}{2}], c > 0 \), and \( \sigma_0 > 0 \) be constants. Assume the existence of a family \( \mathcal{M} := \{\sigma \Sigma\}_{\sigma > \sigma_0} \subseteq \mathcal{R}_c(\overline{m}, 0, c) \) of regular spheres with total mass \( \overline{m} \), where \( \sigma \Sigma \) has mean curvature radius \( \sigma \), i.e., \( \sigma \Sigma \) has constant mean curvature \( \sigma H = -\frac{1}{\sigma} \), and that each ending covers \( \overline{M} \) outside a compact set, i.e.
\[
\overline{M} \setminus \bigcup_{\sigma > \sigma_1} \sigma \Sigma \quad \text{is relatively compact} \quad \forall \sigma_1 > \sigma_0.
\]

If \( \mathcal{M} \) is locally unique (locally complete), satisfies the foliation condition, and
\[
\left\| D_{\sigma^2} \mathcal{F} \right\|_{L^p(\sigma \Sigma)} \leq \frac{c_1}{\sigma^{2+\epsilon}} \quad \forall \sigma > \sigma_0
\]
holds for some unit normal field \( \sigma \nu \) of \( \Sigma \Sigma \Rightarrow (\overline{M}, \overline{g}) \) and some \( p > 1 \), then there exists a coordinate system \( \mathcal{F} : \overline{M} \setminus \mathcal{L} \to \mathbb{R}^3 \setminus \overline{B_1(0)} \) outside a compact set \( \mathcal{L} \subseteq \overline{M} \) such that \((\overline{M}, \overline{g}, \mathcal{F})\) is \( C^{2+\epsilon}_2 \)-asymptotically flat.

Now, we explain the terminology used above. Let us start by defining the total mass of a CMC-family.

**Definition 3.2** (Total mass of a family of regular spheres)

Let \( \mathcal{M} := \{\sigma \Sigma\}_{\sigma > \sigma_0} \subseteq \mathcal{R}_c(\overline{m}, c_1, c_2) \) be a family of regular spheres such that \( \sigma \Sigma \) has (approximated) mean curvature radius \( \sigma \), where \( \sigma_0 > 0, \kappa \in (1; 2], p \in (2; \infty], c_1, c_2 \geq 0 \), and \( M > 0 \) are constants. If the limit
\[
\overline{m} := \overline{m}(\mathcal{M}) := \lim_{\sigma \to \infty} m_H(\sigma \Sigma)
\]
exists, then it is called total mass of \( \mathcal{M} \).
The reason for calling this total mass is that \( \bar{m} \) is the total (ADM-)mass of \( \overline{M} \) in the coordinates constructed in Theorem 3.1 (and therefore in any \( C^2_{2 + \varepsilon} \)-asymptotically flat coordinates, see [Bar86]).

**Definition 3.3** (Locally uniqueness (completeness) and the foliation condition)

Let \( \mathcal{M} \subseteq \mathcal{R}^n_p(M, 0, c) \) be a family of regular spheres with constant mean curvature, where \( k \in (1; 2] \), \( p \in (2; \infty) \), \( c \geq 0 \), and \( M > 0 \) are constants. The family \( \mathcal{M} \) is called locally unique (or locally complete) if there exists a constant \( \delta = \delta(\Sigma) > 0 \) for every leaf \( \Sigma \in \mathcal{M} \) such that the implication

\[
\|f\|_{H^2(\Sigma)} \leq \delta, \quad \mathcal{H}(\text{graph } f) \equiv \mathcal{H}', \quad |\mathcal{H} - \mathcal{H}'| < \delta \quad \Rightarrow \quad \text{graph } f \in \mathcal{M}
\]

holds for every function \( f \in H^2(\Sigma) \), where \( \mathcal{H} \) and \( \mathcal{H}(\text{graph } f) \) denotes the mean curvature of \( \Sigma \) and graph \( f := \{ \Pi_p (f(p) \nu) : p \in \Sigma \} \), respectively. Furthermore, such a family satisfies the foliation condition if there exists a constant \( \eta \in (0; 1) \) satisfying

\[
\left| \int_{\Sigma} \mathcal{A} f_1 \, d\mu \right| \leq (1 - \eta) \left( \frac{\pi^3}{3} |\mu| \right)^{\frac{1}{2}} \quad \forall \, i \in \{1, 2, 3\}, \Sigma \in \mathcal{M},
\]

where \( \mathcal{A} \) denotes the Gauß curvature of \( \Sigma \) and \( \{ f_i \}_{i=0}^{\infty} \) is any complete orthonormal system of \( L^2(\Sigma) \) of eigenfunctions of the (negative) Laplace operator of \( \Sigma \) to increasing eigenvalue, i.e. \( \gamma \Delta f_i = -\gamma \lambda_i f_i \) with \( \gamma \lambda_i \leq \gamma \lambda_{i+1} \).

The reason for calling the first property locally completeness (or uniqueness) is quite obvious. The second property is called foliation property due to the fact that this property implies that \( \mathcal{M} \) is in fact not only a cover but a foliation (see Lemma 3.6), i.e. the spheres \( \sigma \Sigma \) are pairwise disjoint. In particular, we can replace the assumption ‘satisfies the foliation condition’ with the assumption that the spheres are pairwise disjoint. However, on first sight the latter seems to be the stronger assumption (a posterior they are equivalent).

In the uniqueness definition, we can assume that the implication is only true for functions \( f \in H^2(\Sigma) \) with well-defined graph. Additionally, we want to stress that we do not assume that any leaf of a CMC-family is a graph of some other leaf of the family.

Furthermore, it is important to see that the combination of [Ner14, Thm 3.1] and Theorem 3.1 implies the following characterization of a (slightly) stronger version of asymptotic flatness.

**Corollary 3.4** (Characterization of (strong) asymptotic flatness)

Let \( (\overline{M}, \overline{g}) \) be a three-dimensional Riemannian manifold without boundary and let \( \bar{m} \neq 0 \) and \( \varepsilon \in (0; 1/2) \) be constants. There is a coordinate system \( \overline{\pi} \) of \( \overline{M} \) outside a compact set such that \( (\overline{M}, \overline{g}, \overline{\pi}) \) is \( C^2_{2 + \varepsilon} \)-asymptotically flat with total mass \( \bar{m} \) and

\[
\exists \bar{\varepsilon} \geq 0 : \quad \left| \frac{\partial \overline{\pi}}{\partial \overline{\pi}} \right| \leq \frac{\varepsilon}{|\overline{\pi}|^{1/2 + \varepsilon}}
\]

if and only if a constant \( c \) and a family \( \mathcal{M} := \{ \sigma \Sigma \}_{\sigma > \sigma_0} \subseteq \mathcal{R}^\infty_{2 + \varepsilon}(\bar{c} \bar{m}, 0, c) \) exist which satisfy the assumptions of Theorem 3.1 and

\[
\left\| D_{\nu} \Sigma \right\|_{L^\infty(\sigma \Sigma)} + \| \nabla \Sigma \|_{L^\infty(\sigma \Sigma)} \leq \frac{c}{\sigma^{2 + \varepsilon}} \quad \forall \, \sigma > \sigma_0,
\]
where \( \sigma \nu \) and \( \sigma \mathcal{g} \) denotes a unit normal field and the metric of \( \sigma \Sigma \to (\overline{M}, \overline{\mathcal{g}}) \), respectively.

For readers familiar with Bartnik’s article about total mass and harmonic coordinates of asymptotically flat manifolds \([\text{Bar86}], \) we give an alternative version of Theorem 3.1 in the notation of weighted Sobolev spaces. For this theorem (and its proof), it is necessary to know Bartnik’s article, in particular the definition of weighted Sobolev spaces and their notation, \([\text{Bar86}], \) Def. 1.1, the corresponding Sobolev inequalities and regularity of the Laplace operator, \([\text{Bar86}, \text{Sect. 1}], \) and the existence and regularity of harmonic coordinates, \([\text{Bar86}, \text{Sect. 3}]. \) We prove this theorem by slightly altering the proof of Theorem 3.1 and one of Bartnik’s results \([\text{Bar86}, \text{Prop. 3.3}]. \)

**Theorem 3.5** (Characterization of \( W^{3,p}_{1/2} \)-asymptotic flatness)

Let \((\overline{M}, \overline{\mathcal{g}})\) be a three-dimensional Riemannian manifold without boundary and with integrable scalar curvature, i.e., \( \mathcal{S} \in L^1(\overline{M}) \), and let \( \mathcal{m} \neq 0 \), \( p \in (2; \infty) \), \( \eta \geq 1/2 \), and \( k \in \mathbb{N}_{\geq 0} \) be constants. There is a coordinate system \( \pi : \overline{M} \setminus \overline{\mathcal{L}} \to \mathbb{R}^{3} \setminus B_1(0) \) of \( \overline{M} \) outside of a compact set \( \overline{\mathcal{L}} \) such that \( \pi \circ \mathcal{g} = \mathcal{g} \in W^{3+k, p}(\mathbb{R}^{3} \setminus B_1(0)) \) and \((\overline{M}, \overline{\mathcal{g}}, \pi)\) has total mass \( \mathcal{m} \) and only if there are constants \( c \) and \( \sigma_0 \) and a locally unique family \( \mathcal{M} = \{ \sigma \Sigma \}_{\sigma > \sigma_0} \subseteq \mathcal{K}_2^p(c \mathcal{m}, 0, c) \) of CMC-surfaces, where \( \sigma \Sigma \) has mean curvature \( \mathcal{H} \equiv -\frac{2}{\sigma} \), which has total mass \( \mathcal{m} \) and satisfies the foliation property, the cover property \([\text{0}]), and

\[
\left\| \mathcal{Ric} \right\|_{L^p(\sigma \Sigma)} \leq \mathcal{C}(\sigma) \sigma^{-\frac{3}{2} + \frac{3}{p}}, \quad \sigma^{-3} \sum_{l=0}^{k+1} \int_{\sigma \Sigma} \left( \sigma^{n+l} \left| \nabla^l \mathcal{Ric} \right|_{\overline{\mathcal{g}}} \right)^p d\mu \leq \mathcal{C}'(\sigma)
\]

for all \( \sigma > \sigma_0 \) and some functions \( \mathcal{C}, \mathcal{C}' : (\sigma_0; \infty) \to [0; \infty) \) with \( \tau(\sigma) \to 0 \) for \( \sigma \to \infty \) and \( \tau' \in L^1((\sigma_0; \infty)) \), where \( \nabla^l \mathcal{Ric} \) denotes the \( l \)-th (three-dimensional) covariant derivative of the (three-dimensional) Ricci curvature \( \mathcal{Ric} \).

**Lemma 3.6** (The foliation property)

Let \((\overline{M}, \overline{\mathcal{g}})\) be a three-dimensional Riemannian manifold without boundary and let \( \mathcal{M} > 0, \varepsilon \in (0; \frac{1}{2}], p \in (2; \infty) \), and \( c > 0 \) be constants. There is a constant \( \sigma'_0 = \sigma'_0(M, \varepsilon, c) \) with the following property:

If \( \sigma_1 > \sigma_0 > \sigma'_0 \) are constants and \( \mathcal{M} := \{ \sigma \Sigma \}_{\sigma \in (\sigma_0; \sigma_1)} \subseteq \mathcal{K}_2^p(c \mathcal{m}, 0, c) \) is a family of regular CMC-spheres, where \( \sigma \Sigma \) has mean curvature radius \( \sigma \), which is locally unique (locally complete) and satisfies the foliation condition, then \( \mathcal{M} \) is a smooth foliation of its image, i.e., the elements of \( \mathcal{M} \) are pairwise disjoint and there is a \( C^1 \)-map \( \mathcal{F} : (\sigma_0; \sigma_1) \times S^2 \to \overline{M} \) such that \( \mathcal{F}(\sigma, S^2) = \sigma \Sigma \). In particular, \( \mathcal{F} \) is a \( C^1 \)-diffeomorphism onto its image.

It would be sufficient to assume that the family contains \( \{ \sigma \Sigma \}_{\sigma \in I} \), where \( I \) is a dense subset of \((\sigma_0; \sigma_1)\). However, this is a technical assumption and does not need any additional step in the proof as the uniqueness condition (a posteriori) implies that \( I \supseteq (\sigma'_0; \sigma_1) \) for some \( \sigma'_0 \geq \sigma_0 \).

**Proof of Lemma 3.6** We can assume that \( \sigma_0 \) is so large that we can use the Propositions 2.4 and 2.6 for each spheres \( \sigma \Sigma \in \mathcal{M} \). Fix a sphere \( \sigma \Sigma = \Sigma \) and suppress the corresponding index \( \sigma \). We know that the stability operator

\[
L : W^{2,q}(\Sigma) \to L^q(\Sigma) : \mathcal{f} \mapsto \Delta \mathcal{f} + \left( \left| k \right|^2 + \mathcal{Ric}(\nu, \nu) \right) \mathcal{f}
\]
is the Fréchet derivative of the mean curvature map
\[ H : W^2,q(\Sigma) \to L^q(\Sigma) : f \mapsto \mathcal{H}(\text{graph } f) \]
at \( f = 0 \) (for every \( q > 2 \)), where \( \mathcal{H}(\text{graph } f) \) denotes the mean curvature of the graph of \( f \) which we interpret as function on \( \Sigma \). By Proposition 2.6, the stability function theorem implies that \( H \) is bijective from a \( W^2,q(\Sigma) \)-neighborhood of \( 0 \in W^2,q(\Sigma) \) to a \( L^q(\Sigma) \)-neighborhood of \( \mathcal{H} \in L^q(\Sigma) \). In particular, there is a \( \eta > 0 \) and a curve \( \gamma : (\sigma - \eta; \sigma + \eta) \to W^2,q(\Sigma) \) such that \( H(\gamma(\mathcal{H} + \eta')) \equiv \mathcal{H} + \eta' \) for any \( |\eta'| < \eta \). By the uniqueness condition, this means that graph \( \gamma(\eta') = \eta + \eta' \Sigma \) for any \( |\eta'| < \eta \). In particular, every leaf \( \sigma \Sigma \) (with sufficiently large \( \sigma \)) is a graph of every other leaf \( \sigma' \Sigma \) with small enough \( |\sigma' - \sigma| \). Furthermore, this implies that the existence of a constant \( \sigma_0' \geq \sigma_0 \) and a \( C^1 \)-map \( \Phi : (\sigma_0'; \sigma_1) \times S^2 \to \overline{M} \) such that \( \Phi(\sigma, S^2) = \sigma \Sigma \) for any \( \sigma \in (\sigma_0'; \sigma_1) \). To prove that \( \Phi \) is a diffeomorphism onto its image, it is sufficient to show that
\[ \sigma u := \bar{\mathcal{F}} \left( \frac{\partial \Phi}{\partial \sigma}, \nu \right) > 0 \quad \forall \sigma > \sigma_0, \]
where \( \nu \) denotes the outer unit normal field of \( \sigma \Sigma \). Per Definition of \( \Phi \) and \( \sigma \Sigma \), we know
\[ Lu = \frac{\partial \sigma \mathcal{F}}{\partial \sigma} \equiv \frac{2}{\sigma^2}. \]
Thus, Proposition 2.4 implies
\[ |L(u - 1) - \mathcal{Ric}(\nu, \nu)| \leq \frac{C}{\sigma^{3+\varepsilon}}. \]
Hence, we know by Proposition 2.6
\[ \|u^d - 1\|_{H^2(\Sigma)} \leq C \sigma^{\frac{1}{2} - \varepsilon}, \quad \|u^1\|_{H^2(\Sigma)} \leq C \sigma^{\frac{1}{2} - \varepsilon}. \]
Therefore, \( \Phi \) is a diffeomorphism if \( \|u^1\|_{L^\infty(\Sigma)} \leq 1 - \eta \) with \( \eta > 0 \) (independent of \( \sigma \)). But again by Proposition 2.6 (for sufficiently large \( \sigma_0 \)), this is implied by
\[ \sum_{i=1}^{3} \int \mathcal{Ric}(\nu, \nu) f_i \, d\mu f_i \leq (1 - \eta) \frac{6|m_\mathcal{H}(\Sigma)|}{\sigma^3} \]
for some \( \eta > 0 \) (independent of \( \sigma \)). By comparing with the Euclidean sphere using Proposition 2.4 we see that this is the case if
\[ \int \mathcal{Ric}(\nu, \nu) f_i \, d\mu \leq (1 - \eta) \sqrt{\frac{16\pi}{3}} \frac{|m_\mathcal{H}(\Sigma)|}{\sigma^2} \quad \forall i \in \{1, 2, 3\} \]
for some \( \eta > 0 \) (independent of \( \sigma \)). Combining the decay of \( \mathcal{F} \) and \( \mathcal{K} \) with the fact that \( f_i \) is mean value free, this is true due to the Gauß equation and the assumed foliation property if \( \sigma \) is sufficiently large. Thus, \( \Phi \) is a diffeomorphism onto its image if \( \sigma_0 \) is sufficiently large and this proves the claim. //

As we will later construct the asymptotically flat coordinates by the first three eigenfunctions of the Laplace operator, we have to calculate their \( \sigma \)-derivatives'.

**Lemma 3.7** (\( \sigma \)-derivatives of \( f_i \))

Let \( (\overline{M}, \mathcal{F}) \) be a three-dimensional Riemannian manifold, \( \sigma_0 > 0, \varepsilon \in (0; 1/2], \eta > 0, \) and \( p \in (2; \infty) \) be constants, and let \( \mathcal{M} := \{\sigma \Sigma\}_{\sigma > \sigma_0} \) be a family of regular spheres
satisfying the assumptions of Theorem 3.1 or Theorem 3.5. Assume furthermore that

\[ \Phi: (\sigma_0 - \eta; \sigma_0 + \eta) \times \sigma_0 \Sigma \to \overline{M}: (\sigma, p) \mapsto \Phi(\sigma, p) \]

is the \( C^1 \)-map with \( \Phi|_{(\sigma_0, \cdot)} \) = id|_{\sigma_0 \Sigma} and

\[ \frac{\partial \Phi}{\partial \sigma} = \sigma \Phi^*(\sigma u, \nu + \sigma \nabla u) \quad \forall \sigma \in (\sigma_0 - \eta; \sigma_0 + \eta), \]

where \( \sigma \Phi := \Phi(\sigma, \cdot) \). There are \( L^2(\sigma \Sigma) \)-orthogonal functions \( \sigma f_i \in H^1(\sigma \Sigma) \) with

\[ \sigma f_i \in \text{lin}\left\{ f \in L^2(\sigma \Sigma) : \Delta f = -\lambda f, \; \lambda - \frac{2}{\sigma^2} \leq 1 \right\}, \quad \| \sigma f_i \|_{L^\infty(\sigma \Sigma)} = 1. \]

such that

\[ \left\| \frac{\partial(\sigma f \circ \sigma \Phi)}{\partial \sigma} \right\|_{W^{2,p}(\sigma \Sigma)} \leq \frac{C}{\sigma^{\frac{5}{3} - \frac{3}{p}}}, \]

where \( C = C(\varepsilon, c, m) \) does neither depend on \( \sigma \) nor on \( \eta \).

**Proof.** We suppress the index \( \sigma \). Using Proposition 2.4 to compare \((\Sigma, g)\) with the Euclidean sphere of radius \( \sigma \), we see

\[ \| \text{Hess} f^t - \frac{\Delta f^t}{2} g \|_{L^2(\Sigma)} \leq \frac{C}{\sigma^{\frac{5}{3} + \varepsilon}} \| f^t \|_{L^2(\Sigma)}. \]

With \( \Delta \nabla f^t = \frac{1}{2} \nabla \nabla f^t + \nabla \Delta f^t \), we strengthen this to

\[ \| \text{Hess} f^t - \frac{\Delta f^t}{2} g \|_{W^{1,p}(\Sigma)} \leq \frac{C}{\sigma^{\frac{5}{3} + \varepsilon}} \| f^t \|_{L^p(\Sigma)}. \]

We see

\[ \frac{\partial g}{\partial \sigma} = -2u \mathbf{k} + \sigma \text{Hess} u^i = \left( \frac{2u}{\sigma} + \sigma \Delta u^i \right) g - 2u \mathbf{k} + \sigma \text{Hess} u^i, \]

where the first term denotes the derivative of \( g \) along \( \Phi \). Combining this with the estimates \([5]\) on \( \mathbf{k}, [5] \) on \( u \), and \([10]\) on \( \text{Hess} u^i \), this implies

\[ \left\| \frac{\partial g}{\partial \sigma} + \frac{\sigma}{2} g \right\|_{W^{1,p}(\Sigma)} \leq \frac{C}{\sigma^{\frac{5}{3} + \varepsilon}} \| g \|_{W^{2,p}(\sigma \Sigma)}. \]

In particular, we get

\[ \left\| \left( \frac{\partial(\sigma \Delta \sigma g)}{\partial \sigma} + \frac{\sigma}{\sigma \sigma \Delta \sigma g} \right) \circ \Phi \right\|_{L^p(\sigma \Sigma)} \leq \frac{C}{\sigma^{\frac{5}{3} + \varepsilon}} \| g \|_{W^{2,p}(\sigma \Sigma)} \]

for every function \( g \in W^{2,p}(\Sigma) \) and its \( \Phi \)-constant expansion \( \sigma g := g \circ \sigma \Phi^{-1} \). This proves the claim as the \( \Phi \)-constant expansion \( \sigma g_i \) of the eigenfunction \( f_i \) of the Laplace operator on \( \Sigma \) can therefore be altered to become a eigenfunction of the Laplace operator on \( \sigma \Sigma \) and this alternation is controlled correspondingly to the above inequality. //

**Proof of Theorem 3.1.** We see \( \mathcal{M} \subseteq R_+^{2+\varepsilon}(c \overline{m}, 0, C) \) for every \( p \in (2; \infty) \) due to Lemma 2.2 and fix such \( p \in (2; \infty) \). By Lemma 3.6, we can without loss of generality assume that \( \mathcal{M} \) is a foliation of the entire space \( \overline{M} \) (replacing \( \overline{M} \) and \( \sigma_0 \) by \( \bigcup_{\sigma} \Sigma \) and \( \sigma_1 \), respectively). In particular, we can define a \( C^1 \)-map \( \sigma : \overline{M} \to (\sigma_0; \infty) \) such that \( \nu \in \sigma(\overline{M}) \) for every \( \nu \in \overline{M} \) and we can equally define a vector field \( \nu \) by the characterization \( \sigma \nu := \nu|_{\sigma \Sigma} \) is the outer unit normal of \( \sigma \Sigma \). By the inequalities \([8]\)
on the lapse function $u, \nu$ is at least continuously differentiable. Hence, the metric $\sigma g$ of $\sigma \Sigma$ depends smoothly on $\sigma$, i.e. the function

$$
\sigma g(X, Y) := \sigma g(X - \tilde{g}(X, \sigma \nu) \sigma \nu, Y - \tilde{g}(Y, \sigma \nu) \sigma \nu)
$$

is at least continuously differentiable in $\mathcal{M}$ for any smooth vector fields $X, Y \in T(\mathcal{M})$. For the same reasons, we furthermore see that $\sigma \mapsto \sigma g(X, Y) \in H(\Sigma)$ depends continuously on $\sigma$. In particular, we can choose differentiable functions $f_i : \mathcal{M} \to [-1; 1]$ such that $\sigma f_i := f_i|_{\Sigma}$ is as in Lemma 3.7. Now, we define

$$
\tilde{\xi} : \mathcal{M} \to \mathbb{R}^3 : \sigma \mapsto \left( \sqrt{\frac{3}{4\pi \sigma^2}} \int_{\Sigma} \sigma^{-1} \int_{\Sigma} \sigma f_i \sigma u \, d\mu(\sigma) \right)_{i=1}^3,
$$

and prove that the latter is an asymptotically flat coordinate system, where $u$ again denotes the lapse function. In the following, we identify $\tilde{\xi}$ and $\tilde{\xi} \circ \sigma^{-1}$.

**On each CMC-leaf:** First, let us proof that $\sigma \xi := \xi|_{\Sigma}$ are coordinates with respect to which the induced metric $\sigma g$ is asymptotically to the pullback of the corresponding Euclidean metric $\sigma \xi \tilde{g}$. We note that the estimates for the conformal parametrization in Proposition 2.4 imply

$$
\left| \int_{\Sigma} \frac{|\xi|}{\sigma^2} \, d\mu - 1 \right| = \sum_{i=1}^3 \frac{\|f_i\|_{L^2(\Sigma)}}{|\Sigma|} - 1 \leq C \sigma^{\frac{1}{2} + \epsilon},
$$

where $|\xi| := |\xi - \tilde{\xi}|$. Furthermore, Lemma 2.4 implies

$$
\left| \frac{\sigma \Delta |\xi|^2}{\sigma^2} + 6 \frac{|\xi|^2 - 1}{\sigma^4} \right| \leq \sum_{i=1}^3 \left( -\lambda_i f_i^2 + \frac{1}{\sigma^2} \left( 1 - f_i^2 \right) \right) + 6 \frac{|\xi|^2 - 1}{\sigma^4}.
$$

This means that $1 - |\xi|^2/\sigma^2$ is (asymptotically) an eigenfunction of the (negative) Laplace operator with eigenvalue $\theta/\sigma^2$. Again using Proposition 2.4, we see that there are five $L^2(\Sigma)$-orthonormal eigenfunctions $f_1, f_2, f_3, f_4, f_5$ of the Laplace operator such that the corresponding eigenvalues $\lambda_i$ satisfy $|\lambda_i - \theta/\sigma^2| \leq 1/\sigma^2$ and these satisfy $|\lambda_i - \theta/\sigma^2| \leq C/\sigma \frac{1}{2 + \epsilon}$ for some constant $C$. Again comparing with the corresponding Eigenfunctions of the Euclidean sphere, we see that

$$
\left| \int f_i \, d\mu \right| \leq \sum_{k=1}^5 \left| \int f_k \, d\mu \right| + \frac{C}{\sigma^{\frac{1}{2} + \epsilon}} \|f\|_{L^2(\Sigma)} \quad \forall \, i \in \{4, 5, 6, 7, 8\}, \, f \in L^2(\Sigma),
$$

where $g_1 := \sqrt{5} f_1 f_2, \, g_2 := \sqrt{5} f_2 f_3, \, g_3 := f_3^2 - \frac{1}{2} (f_1^2 + f_2^2), \, g_4 := \sqrt{5} f_1 f_3$ and $g_5 := \frac{1}{2} (f_1^2 - f_2^2)$. By calculating $\int |\xi|^2 \, g_i \, d\mu$, this implicates

$$
\left\| \frac{|\xi|^2}{\sigma^2} \right\|_{L^2(\Sigma)} \leq C \sigma^{-\frac{1}{2} + \epsilon}.
$$

Using the regularity of the Laplace operator this means

$$
\left\| \frac{|\xi|^2}{\sigma^2} - 1 \right\|_{W^{2, p}(\Sigma)} \leq \frac{C}{\sigma^{\frac{1}{2} + \epsilon - \frac{2}{p}}}.
$$
With
\[
\Delta \nabla \left( \frac{|\sigma|^2}{\sigma^2} \right) = \frac{1}{2} \Delta \nabla \left( \frac{|\sigma|^2}{\sigma^2} \right) + \nabla \Delta \left( \frac{|\sigma|^2}{\sigma^2} \right)
\]
\[
= \sum_{i=1}^{3} \left( \frac{3}{2} \nabla f_i^2 + \nabla \left( -2 \lambda_i f_i^2 + 2 |\nabla f_i|_g^2 \right) \right)
\]
\[
= \sum_{i=1}^{3} \left( \left( \frac{3}{2} - 3 \lambda_i \right) \left( \nabla f_i^2 \right) + 4 \nabla f_i (\nabla f_i, \cdot) \right),
\]
the inequalities for $S$, $W$, $\lambda_i$, and the above one for $|\sigma|^2$, we get
\[
\left| \Delta \nabla \left( \frac{|\sigma|^2}{\sigma^2} \right) \right| \leq \sum_{i=1}^{3} \left( \frac{3}{2} \nabla f_i^2 + 2 |\nabla f_i|_g^2 \right) + 4 \nabla f_i (\nabla f_i, \cdot) \right| + \frac{C}{\sigma^2 + \varepsilon}.
\]
Thus, we can strengthen the above inequality to
\[
\left| \frac{|\sigma|^2}{\sigma^2} - 1 \right| \leq \frac{C}{\sigma^2 + \varepsilon}.
\]
Hence, there is a function $f \in W^{3, p}(S^2_{\sigma}(\sigma\varepsilon))$ such that $\Sigma':=\Sigma(\sigma\varepsilon) = \text{graph}(\sigma f)$ and
\[
\|\sigma \|_{W^{3, p}(S^2_{\sigma}(\sigma\varepsilon))} \leq \frac{C}{\sigma^{2+\varepsilon}}
\]
where $F$ is the function of $f$, $\Omega = \sigma^2 \Omega$ is the standard metric on $S^2_{\sigma}(\sigma\varepsilon)$, and $\sigma\varepsilon = \varepsilon(p)$ for any (and therefore every) $p \in \Sigma$. In particular, $\sigma \Sigma := \Sigma(\sigma\varepsilon)$ are coordinates of $\Sigma$. Again, using the estimates for the conformal parametrization in Proposition 2.4, we furthermore know
\[
\left| g(\nabla_{\sigma \Sigma_i} \nabla_{\sigma \Sigma_j} - \sigma \Omega(\cdot \nabla \bar{y}_i, \cdot \nabla \bar{y}_j) \right| \leq \frac{C}{\sigma^{2+\varepsilon}},
\]
where $\bar{y} : R^3 \rightarrow R^3$ are the standard coordinates. This implies
\[
\left| \sigma f \right|_{W^{1, p}(\Sigma)} \leq \frac{C}{\sigma^{2+\varepsilon}}
\]
and we get
\[
\left| \sigma f \right|_{W^{1, p}(\Sigma)} \leq \frac{C}{\sigma^{2+\varepsilon}}
\]
by the same arguments. Doing a similar calculation as above for $\nabla f_i$ instead of $\nabla (|\sigma|^2 / \sigma^2)$, we strengthen this to
\[
\left| \nabla f_i \right|_{W^{1, p}(\Sigma)} \leq \frac{C}{\sigma^{2+\varepsilon}}.
\]
Thus, $\sigma \Sigma$ is a coordinate system of $\Sigma$ such that $\sigma \Sigma - \sigma \Sigma' \Sigma$ decays suitable fast.

On each leaf ($\sigma$-derivative): Now, we prove that the metrics on single leaves depend $C^1$ on $\sigma$ and that the corresponding derivative decays suitable fast. Using Lemma 3.7 on the deformation $\Phi$ satisfying $[\Phi]$ for each $\sigma \in (\sigma_0, \infty)$, we see
\[
\left| \frac{\partial f}{\partial \sigma} \right|_{W^{2, p}(S^2_{\sigma}(\sigma\varepsilon))} \leq \frac{C}{\sigma^{2+\varepsilon}}.
\]
where \( f \) is the function on \( S^2_{\sigma}(\sigma \vec{z}) \) with graph \( \sigma f = \sigma \Sigma \). Here, we used the map

\[
S^2_{\sigma}(\sigma \vec{z}) \rightarrow S^2_{\sigma'}(\sigma' \vec{z}) : p \mapsto \frac{\sigma'}{\sigma}(p - \sigma \vec{z}) + \sigma' \vec{z}
\]

to identify \( S^2_{\sigma}(\sigma \vec{z}) \) and \( S^2_{\sigma'}(\sigma' \vec{z}) \). Correspondingly, we get

\[
\left\| \frac{\partial g_{ij}}{\partial \sigma} - \frac{2}{\sigma} g_{ij} \right\|_{W^{1,p}(\Sigma)} \leq \frac{C}{\sigma^{\frac{1}{2}+\varepsilon}}
\]

where we used the graph function \( \sigma F \) of \( f \) and the above map to choose one coordinate system for every \( \sigma' \Sigma \) with sufficiently small \( |\sigma - \sigma'| \).

Radial direction: Again using Lemma 3.7 on the deformation \( \Phi \) satisfying (9) for each \( \sigma_1 \in (\sigma_0; \infty) \), we see

\[
\left| \left( \pi^* \mathcal{g} \right)(\nu, \nu) - 1 \right| \leq \frac{C}{\sigma^{\frac{1}{2}+\varepsilon}}.
\]

By the same argument, we see that the corresponding result holds for the \( \Sigma \)-tangential derivative, i.e.

\[
\left| D_X \left( \left( \pi^* \mathcal{g} \right)(\nu, \nu) \right) \right| \leq \frac{C}{\sigma^{\frac{1}{2}+\varepsilon}} \quad \forall \ X \in \mathfrak{X}(\sigma \Sigma),
\]

and that the \( \pi^* \mathcal{g} \)-tangential part of \( \pi^* \nu \) decays with \( C/\sigma^{\frac{1}{2}+\varepsilon} \) (and correspondingly for the first derivative). In particular \( \pi \) is a coordinate system of \( \mathbb{M} \).

Radial direction (\( \sigma \)-derivative): Thus, left to prove is

\[
\left| D_{\nu} \left( \left( \pi^* \mathcal{g} \right)(\nu, \nu) \right) \right| \leq \frac{C}{\sigma^{\frac{1}{2}+\varepsilon}}
\]

and that the corresponding result holds for the \( \pi^* \mathcal{g} \)-tangential part of \( \nu \). It is sufficient to prove

\[
\left\| \frac{\partial u}{\partial \sigma} \right\|_{L^\infty(\Sigma)} \leq \frac{C}{\sigma^{\frac{1}{2}+\varepsilon}},
\]

where we again used the map \( \Phi \) satisfying (9) for each \( \sigma_1 \in (\sigma_0; \infty) \) to define this derivative. Additional using the conformal parametrization of \( \sigma_0 \Sigma \) (for one \( \sigma_0 \)), we interpret \( \sigma \mathcal{g}, \sigma S \), etc. as quantities on \( S^2 \). As explained above, we know

\[
\left\| \frac{\partial g}{\partial \sigma} - \frac{\mathcal{H}}{2} \mathcal{g} \right\|_{W^{1,p}(\sigma \Sigma)} \leq \frac{C}{\sigma^{\frac{1}{2}+\varepsilon}} \quad \left\| \frac{\partial \mathcal{g}}{\partial \sigma} - \frac{\mathcal{H}}{2} \mathcal{g} \right\|_{L^\infty(\sigma \Sigma)} \leq \frac{C}{\sigma^{\frac{1}{2}+\varepsilon}}.
\]
It is well-known that
\[ \frac{\partial \tilde{k}}{\partial \sigma} = u \left( \overline{\text{Ric}} - \frac{s}{2} g - 2 k \otimes k + \mathcal{H} k \right) + \text{Hess} u + \sigma g (\nabla u', \nabla k). \]
Therefore, our inequalities on $\overline{\text{Ric}}$, $\tilde{k}$, and $s$ imply
\[ \left\| \frac{\partial \tilde{k}}{\partial \sigma} \right\|_{L^p(\Sigma)} = \frac{C}{\sigma^{\frac{3}{2}} - \frac{\varepsilon - p}{2}}. \]
Thus, Lemma 3.7 and
\[ Lu = \Delta u + \left( \frac{2}{\sigma^2} + |k|_g^2 + \overline{\text{Ric}}(\nu, \nu) \right) u = \frac{2}{\sigma^2} \]
imply
\[ \left\| L \left( \frac{\partial u}{\partial \sigma} \right) + \frac{\partial (\overline{\text{Ric}}(\nu, \nu))}{\partial \sigma} \right\|_{L^p(\Sigma)} \leq \frac{C}{\sigma^{\frac{3}{2}} + \varepsilon - \frac{p}{2}}. \]
Hence, we get by the regularity of the weak Laplace operator
\[ \left\| \frac{\partial u}{\partial \sigma} \right\|_{W^{1, q}(\Sigma)} \leq \frac{C}{\sigma^{\frac{3}{2}} + \varepsilon - \frac{p}{2}}, \]
if
\[ \left\| \frac{\partial (\overline{\text{Ric}}(\nu, \nu))}{\partial \sigma} \sigma u \right\|_{W^{-1, q}(\Sigma)} \leq \frac{C}{\sigma^{\frac{3}{2}} + \varepsilon - \frac{p}{2}} \]
for $\frac{1}{\mu} + \frac{1}{q} = 1$. The latter again is true if
\[ (12) \left\| \frac{\partial \sigma}{\partial \sigma} \left( \int_{\sigma} S \bar{f} d\sigma \right) + \frac{4}{\sigma^3} \int_{\sigma} S \bar{f} d\sigma \right\|_{W^{-1, q}(\Sigma)} \leq \frac{C}{\sigma^{\frac{3}{2}} + \varepsilon - \frac{p}{2}}, \]
where we used the Gauß equation, the estimates on $u$, the above control for the $\sigma$-derivative of $k$, and the assumed control for the $\sigma$-derivative of $\overline{\text{S}}$.
For the proof of (12), let $\Phi : (\sigma_1 - \eta; \sigma_1 + \eta) \times \sigma_0 \Sigma \to \overline{M}$ satisfy $\frac{\partial \Phi}{\partial \sigma} = u \nu$, where we fixed $\sigma_1 > \sigma_0$. Choosing conformal coordinates $x : \sigma_1 \Sigma \to S^2$, $\Phi$ gives raise coordinates $y = (\sigma, y^2, y^3) : M \to (\sigma_0; \infty) \times S^2$. Let $f \in W^{1, q}(\sigma_0 \Sigma)$ be arbitrary and denote by $\bar{f}$ its push-forward along $\Phi$ on $\text{im} \Phi \subseteq \overline{M}$. By integration by parts, we get
\[ \int_{\sigma_0} \sigma S \bar{f} d\sigma = \int_{\sigma_0} -\sigma \Gamma_{IJ}^K \sigma \text{div}(\sigma g^{IJ} \bar{f} e_K) + \sigma \Gamma_{IJ}^K \sigma \text{div}(\sigma g^{IJ} \bar{f} e_J) d\sigma \mu \]
\[ + \int_{\sigma_0} \sigma g^{IJ} \left( \sigma \Gamma_{KL} L \sigma \Gamma_{IJ}^L - \sigma \Gamma_{IJ} \sigma \Gamma_{KL} L \right) \bar{f} d\sigma \mu. \]
Using the above inequalities for the derivative of $g - \Omega$, i.e. the ones for $\tilde{k}$ and $u$, we conclude
\[ \left\| \frac{\partial}{\partial \sigma} \left( \int_{\sigma} \left( S - \frac{2}{\sigma^2} \right) \bar{f} d\mu \right) \right\| \leq \frac{C}{\sigma^{\frac{3}{2}} + \varepsilon - \frac{p}{2}} \int_{\Sigma} S \bar{f} d\sigma \mu. \]
As $\frac{\partial}{\partial \sigma} (d\mu - d^2 \tilde{k}) = -2 \mathcal{H} u \mu + \mathcal{H} d^2 \mu$, this implies (12) due to the estimates on $u$.
As explained above, this proves the claim. ///
Now, let us explain how to prove Theorem 3.5 by altering the above proof.

Proof of Theorem 3.5. If there exists a coordinate system \((\overline{M}, \overline{g}, \pi)\) such that \(\pi_* \overline{g} - \pi \overline{g} \in W^3,0_{ij}(\mathbb{R} \setminus B_1(0))\), then [Ner14] Thm 3.1, Remark 1.2 implies the existence of such a CMC-foliation.

First, we note that we can use Proposition 2.6 in this context on each CMC-surface \(\Sigma\) with sufficiently large mean curvature radius, where we have to replace \(C/\sigma^{3+\varepsilon}\) and \(C/\sigma^{3+\frac{\varepsilon}{2}}\) by \(C\sigma(\sigma)/\sigma^{3+\varepsilon}\) and \(C\sigma(\sigma)/\sigma^{3+\frac{\varepsilon}{2}}\), respectively, see [Ner14] Remark 1.2. Thus, we can repeat the arguments of the parts ‘On each CMC-leaf’, ‘On each leaf (\(\sigma\)-derivative)’, and ‘Radial direction’ of proof of Theorem 3.1, where we have to replace each \(\sigma^{-\varepsilon}\) by \(\sigma(\sigma)\). Thus, \(\pi\) as in \((11)\) is a coordinate system of \(\overline{M}\) outside of some compact set which we assume without loss of generality to be empty.

Now, we look at the last part, the ‘Radial direction (\(\sigma\)-derivative)’: As explained there, we know

\[
\left\| \frac{\partial \overline{g}_{jk}}{\partial x^i} \right\|_{L^2(p, (\Sigma))} \leq C \frac{\sigma(\sigma)}{\sigma^{2+\frac{\varepsilon}{2}}} + C \frac{\partial u^i}{\partial \sigma} \right\|_{W^{2, p}(\Sigma)} \quad \forall i, j, k \in \{1, 2, 3\}, \sigma > \sigma_0
\]

and

\[
\left\| \frac{\partial u^i}{\partial \sigma} \right\|_{W^{2, p}(\Sigma)} \leq C \frac{\sigma(\sigma)}{\sigma^{2+\frac{\varepsilon}{2}}} + C \sigma^2 \left\| \nabla \text{Ric} \right\|_{L^p(\Sigma)} \quad \forall \sigma > \sigma_0.
\]

Thus, we get

\[
\left\| \frac{\partial \overline{g}_{jk}}{\partial x^i} \right\|_{L^2(p, (\Sigma))} \leq C \frac{\sigma(\sigma)}{\sigma^{2+\frac{\varepsilon}{2}}} + C \sigma^2 \left\| \nabla \text{Ric} \right\|_{L^p(\Sigma)} \quad \forall i, j, k \in \{1, 2, 3\}, \sigma > \sigma_0.
\]

The assumptions on \(\nabla \text{Ric}\) therefore imply \(\overline{g}_{ij} \in W^{2, p}_{\frac{3}{2}+\varepsilon}(\overline{M})\) for every \(\varepsilon > 0\). Thus, Bartnik’s Sobolev-inequality [Bar86, Thm 1.2, (iv)] implies \(\pi_* \overline{g} - \overline{g} \in W^{2, p}_{\frac{3}{2}+\varepsilon}(E_R)\) for \(p^* = \frac{3p}{3-p} > 3\). Now, Bartnik’s existence and regularity result for harmonic coordinates, [Bar86, Prop. 3.3], and the assumptions on \(\text{Ric}\) imply existence of a harmonic coordinate system \(\overline{\eta} : M \setminus L' \to \mathbb{R}^3 \setminus B_R(0) =: E_R\), i.e. \(\overline{\Delta} \overline{\eta} = 0\), with \(\overline{\eta}, \overline{\eta} - \overline{g} \in W^{2, p}_{\frac{3}{2}+\varepsilon}(E_R)\), where \(L'\) is a compact subset of \(M\) and \(R \in (0, \infty)\). From now on, we only refer to this coordinate system. We know \(\Delta \overline{\eta}_{ij} = -2 \overline{\text{Ric}}_{ij} + 2 Q_{ij}(\overline{g}, \overline{\Gamma})\), where \(Q_{ij}\) is a polynomial in \(\overline{g}_{kl}, \overline{g}^{kl}\), and \(\overline{\Gamma}_{klm}\) and quadratic in \(\overline{\Gamma}_{klm}\), see [Bar86, Eq. (3.6)]. In particular, we get \(\Delta \overline{\eta}_{ij} \in W^{1, p}_{\frac{3}{2}+\varepsilon}(E_R)\) due to the Sobolev-inequality [Bar86, Thm 1.2, (iv)]. Therefore, the regularity of the Laplace operator implies \(\overline{\nabla}^2 \overline{\eta} - \overline{\eta} \in W^{3, p}_{3+\varepsilon}(E_R)\), see [Bar86, Prop 2.2]. Iterating this argument, we get \(\overline{\nabla} \overline{\eta} - \overline{\eta} \in W^{3, p}_{3+\varepsilon}(E_R)\).

4. A LOCAL VERSION OF THE MAIN THEOREM

Here, we state a local version of the main theorem which direct uses Huisken’s idea explain in the introduction. Note that we can equally localize Corollary 3.3 and Theorem 3.5.

Theorem 4.1 (Sufficient assumptions for asymptotic flatness (local version))

Let \((\overline{M}, \overline{g})\) be a three-dimensional Riemannian manifold without boundary and let \(M > 0, \varepsilon \in (0, \frac{1}{2})\), and \(c > 0\) be constants. There exists a constant \(\sigma_0' = \sigma_0'(\overline{M}, \varepsilon, c)\) with the following property:
If $\infty > \sigma_1 > \sigma_0 > \sigma'_1$ are constants and $M := \{\sigma \Sigma\}_{\sigma \in (\sigma_0 ; \sigma_1)} \subset \mathbb{R}^3_{\infty + \varepsilon}(M, 0, c)$ is a family of regular spheres which is locally unique (locally complete) and satisfies the foliation condition, then $M$ is a smooth foliation, i.e. there exists a $C^1$-map $\Phi : (\sigma_0 ; \sigma_1) \times \mathbb{S}^2 \to \mathbb{M}$ such that $\Phi$ is a corresponding $C^1$-map. Using conformal maps, we can assume that the balancing condition (20) is satisfied and

$$\|\Phi^* \mathcal{F} - \sigma^2 \Omega\|_{W^{2,p}(S^2)} \leq \frac{C}{\sigma^{2+\varepsilon-\frac{2}{p}}}, \quad \|\mathcal{F}(\partial_\sigma \Phi, \partial_\sigma \Phi) - 1\|_{W^{2,p}(S^2)} \leq \frac{C}{\sigma^{2-\frac{2}{p}}},$$

where $\Omega$ is the standard metric of the Euclidean unit sphere.

If furthermore (7) is satisfied, then there exists a parametrization $\vec{\mathcal{F}} : \bigcup_\sigma \sigma \Sigma \to \Omega \subset \mathbb{R}^3$ on the union of the CMC-surfaces to a subset $\Omega \subset \mathbb{R}^3$ diffeomorph to the annulus $B_{\sigma_1}(0) \setminus B_{\sigma_0}(0)$ such that $|\vec{\mathcal{F}} - \sigma| \leq C \sigma^{1-\varepsilon}$ for any $\sigma \in (\sigma_0 ; \sigma_1)$ and

$$|\vec{\mathcal{F}}|_{\Sigma}^{ij} - \vec{g}_{ij} + |\vec{\mathcal{F}}|_{\Sigma}^{k} |\mathrm{Ric}_{ij}^{k} | + |\vec{\mathcal{F}}|_{\Sigma}^{3} |3| \leq \frac{\varepsilon}{|\vec{\mathcal{F}}|_{\Sigma}^{i+j+k}} \quad \forall i, j, k \in \{1, 2, 3\}.$$

**Proof.** If we assume (7), then the same proof as for Theorem 3.1 implies existence of a coordinate system satisfying (13). Thus, we only have to prove the first part.

By Lemma 3.6, we can assume that $M$ is a smooth foliation of its union and denote by $\Phi : (\sigma_0 ; \sigma_1) \times \mathbb{S}^2 \to \mathbb{M}$ a corresponding $C^1$-map. Using diffeomorphisms of $\sigma \Sigma$, we can assume that $\Phi$ is orthogonal, i.e. $\partial_\sigma \Phi$ is orthogonal on $\sigma \Sigma$ for each $\sigma \in (\sigma_0 ; \sigma_1)$. Using Proposition 2.4, we can choose conformal parametrizations $\sigma \varphi : \mathbb{S}^2 \to \sigma \Sigma$ such that $|\sigma \varphi^* \mathcal{F} - \sigma^2 \Omega|_{W^{2,p}(S^2)} \leq \frac{C}{\sigma^{2+\varepsilon-\frac{2}{p}}}$ for each $p \in [2 ; \infty)$. Using conformal maps, we can assume that the balancing condition (20) is satisfied for each of these conformal parametrizations, see the proof of Theorem 3.1. In particular, these conformal parametrizations are unique up to a rotation (one for each $\sigma$). If we fix $\sigma_1 < \infty$ and $\sigma_1 \varphi$, then we can use rotations on $\sigma \varphi$ with $\sigma \neq \sigma_1$ such that

$$\left\|\frac{\partial(\sigma \varphi^* \mathcal{F})}{\partial \sigma}\right\|_{W^{1,p}(S^2)} \leq \frac{C}{\sigma^{2+\varepsilon-\frac{2}{p}}} \quad \forall p \in [2 ; \infty),$$

due to the estimates of the second fundamental form $k$ in (5) and of the Lapse function $u$ in (8). In particular, we can choose the above rotations such that $\sigma \varphi$ depends smoothly on $\sigma$ and

$$\left\|\frac{\partial(\sigma \varphi^* \mathcal{F})}{\partial \sigma}(X)\right\|_{W^{1,p}(\Sigma)} \leq \frac{C}{\sigma^{1+\varepsilon-\frac{2}{p}}} \|X\|_{L^\infty(\Sigma)} \quad \forall p \in [2 ; \infty), \ X \in H(\Sigma),$$

i.e. $\partial_\sigma \sigma \varphi$ is ‘almost’-orthogonal to $\sigma \Sigma$. The inequalities on the Lapse function in (8) now imply (13) for $\Phi : (\sigma_0 ; \infty) \times \mathbb{S}^2 \to \mathbb{M} : (\sigma, p) \mapsto \sigma \varphi(p)$ instead of $\Phi$. ///

5. **Characterizing other quantities: the linear momentum**

The results of Section 4 allow us to redefine other quantities without the use of coordinates. Here, we explain this by taking the example of the ADM-linear momentum $\vec{P} \in \mathbb{R}^3$ [ADM61] and $C^2_{\infty + \varepsilon}$-asymptotically flat manifolds. These results can also be used for other quantities and in the setting of $W^{1,p}_{x^2}$-asymptotically flat manifolds, see [Bar86] and Theorem 3.5.
For this purpose, let us briefly recall the definition of ADM-linear momentum.

**Definition 5.1 (ADM-linear momentum)**

Let \((\hat{M}, \hat{g}, \hat{J}, \hat{x}, \hat{\nabla})\) be a \(C^2_{\hat{g}}\)-asymptotically flat initial data set, i.e. \((\hat{M}, \hat{g})\) is a \(C^2_{\hat{g}}\)-asymptotically flat Riemannian manifold, the energy density \(\hat{\rho}\) satisfies the constraint equation \(\hat{\rho} = \frac{1}{2}(\hat{\kappa} - |\hat{\kappa}|^2)\), the exterior curvature \(\hat{\kappa}\) decays sufficiently fast \(|\hat{\kappa}| \leq \varepsilon|\hat{\kappa}^2|\), and the momentum density \(\hat{J}\) satisfying the constraint equation \(\hat{J} = \text{div}(\hat{H}\hat{g} - \hat{k})\) is integrable, i.e. \(\hat{J} \in L^1(\hat{M})\). The ADM-linear moment of \((\hat{M}, \hat{g}, \hat{J}, \hat{x}, \hat{\nabla})\) is defined by

\[
\hat{P}_i := \frac{1}{8\pi} \lim_{R \to \infty} \int_{S^3(0)} \hat{H} \nu_i - \hat{\kappa}(\nu, e_i) \, d\mu,
\]

where \(\hat{H} := \text{tr}\hat{\kappa}\) denotes the mean curvature of \((\hat{M}, \hat{g})\) [ADM61].

It is well-known that the linear momentum of a \(C^2_{\hat{g}}\)-asymptotically flat initial data set is always well-defined. This can be seen using the Gauß divergence theorem and the assumption on \(\hat{J}\). In this representation, the linear momentum is interpreted as tangential vector (of \(\hat{M}\)) at infinity. We see that this definition depends on the coordinate system, but it is well-known that it transforms correctly under coordinate changes [Chr88].

Now, let us reinterpret the ADM-linear momentum as a function on \(\hat{M}\) (well-defined near infinity): define the ADM-linear momentum function to be \(\hat{\mathcal{G}}(\hat{P}, \nu)\), where the vector field \(\nu\) is characterized by its restricted to the CMC-leaves being the outer unit normal vector field of the corresponding leaf, i.e. \(\nu|_{\sigma \Sigma} = \nu\) for each mean curvature radius \(\sigma\) and the corresponding CMC-leaf \(\sigma\Sigma\). Here, we identified the tangent vector near infinity \(\hat{P} \in \mathbb{R}^3\) with the constant vector field \(\pi^*\hat{P}\).

Note that the restriction of this function to a CMC-leaf \(\sigma\Sigma\) is (asymptotically as \(\sigma \to \infty\)) an eigenfunction of the (negative) Laplace operator on \(\sigma\Sigma\) with eigenvalue \(\hat{J}/\sigma^2\), i.e. \(\hat{\mathcal{G}}(\hat{P}, \nu) - \hat{\mathcal{G}}(\hat{P}, \nu)^4 \to 0\) on \(\sigma\Sigma\) for \(\sigma \to 0\). This means that this function (asymptotically) lays within a three-dimensional function space which is geometrically characterized.

As motivation for the above interpretation, we recall that CMC-foliations of \(C^2_{\hat{g}}\)-asymptotically flat initial data sets (asymptotically) evolve in time (under the Einstein equations) by a shift with lapse function asymptotically equal to the quotient of the ADM-linear momentum function and the Hawking mass of the leaf [Ner13]. Let us briefly explain this: assume that we have a temporal foliation \(\{(\hat{M}, \hat{g})\}_{t \in I}\) by \(C^2_{\hat{g}}\)-asymptotically flat initial data sets of a space-time \((\hat{M}, \hat{g})\) which satisfies the Einstein equations with respect to an asymptotically vanishing energy-momentum tensor and let \(\sigma\Sigma\) denote the corresponding CMC-foliations of one of these time-slices \((\hat{M}, \hat{g})\). For every parametrization \(\sigma\xi : I \times S^2 \to \hat{M}\) of the time evolution of one of these CMC-leaves \(\sigma\Sigma\), i.e. \(\sigma\xi(t, S^2) = \sigma\Sigma\), we can split its derivative \(\partial_t(\sigma\xi)\) in a part \(\hat{\mathcal{V}}\) orthogonal to \(\hat{M}\), which is a priori given by the temporal foliation, a part \(\hat{\mathcal{X}}\) tangential to the CMC-leaf \(\sigma\Sigma\), which depends on the specific parametrization we have chosen, and a part \(\hat{\mathcal{N}} = \hat{u} \hat{\nu}\) tangential to the time-slice \(\hat{M}\) but orthogonal to the CMC-leaf \(\sigma\Sigma\), which characterizes the evolution of the surface \(\sigma\Sigma\). The result cited above means that the latter part is (asymptotically) characterized by the ADM-linear momentum function \(\hat{\mathcal{G}}(\hat{P}, \nu)\), more precisely \(\hat{u}\) is (asymptotically) equal to \(\hat{\mathcal{G}}(\hat{P}, \nu)/\hat{m}_{\Sigma}\) (on \(\sigma\Sigma\)).
As the part \( J_t \) is given by the geometry of \((\overline{M}, \overline{\mathcal{J}})\), this means that the ADM-linear momentum function is (asymptotically) characterized by a geometric quantity – by a function on \( \sigma \Sigma \). Now, we define a (new) geometric linear momentum function using this quantity.\(^5\)

**Definition 5.2 (CMC-linear momentum)**

Let \((\overline{M}, \overline{\mathcal{J}}, \overline{\mathcal{K}}, \overline{\mathcal{P}}, \overline{\mathcal{J}})\) be a three-dimensional initial data set, \( \mathcal{M} := \{\sigma \Sigma\}_{\sigma > \sigma_{\text{eq}}} \) be a family of regular hypersurfaces satisfying the assumptions of Theorem 3.1 or 3.5. If \( \overline{\mathcal{K}} \) and \( \overline{\mathcal{J}} \) are continuous, then the **CMC-linear momentum** \( \overline{P}^i \in \mathbb{C}(\overline{M}) \) is defined by

\[
\overline{P}^i \big|_{\sigma \Sigma} := \sigma \overline{P}^i = \frac{\sigma^2}{6} \left( \sigma \text{div}(\overline{K}(\sigma \nu, \cdot)) - \sigma \overline{J}(\sigma \nu) + \sigma \text{tr}\overline{K} \right) \quad \forall \sigma > \sigma_{t_0},
\]

where \( \sigma_1 := \sup \{\sigma' > \sigma : \{\sigma \Sigma\}_{\sigma > \sigma_{t_0}} \text{ is a foliation of its union} \} \) and \( \sigma \text{div}, \sigma \nu, \) and \( \sigma \text{tr} \) denote the two-dimensional divergence, the outer unit normal, and the two-dimensional trace with respect to \( \sigma \Sigma \mapsto (\overline{M}, \overline{\mathcal{J}}) \), respectively.\(^6\)

**Theorem 5.3 (Characterization of the ADM-linear momentum)**

Let \((\overline{M}, \overline{\mathcal{J}}, \overline{\mathcal{K}}, \overline{\mathcal{P}}, \overline{\mathcal{J}})\) be a three-dimensional initial data set, \( \mathcal{M} := \{\sigma \Sigma\}_{\sigma > \sigma_{\text{eq}}} \) be a family of regular hypersurfaces satisfying the assumptions of Theorem 3.1 or 3.5. If the exterior curvature \( \overline{\mathcal{K}} \) and the momentum density \( \overline{\mathcal{J}} \) are continuous, then the CMC-linear momentum \( \overline{P}^i \in \mathbb{C}(\overline{M}) \) is well-defined outside a compact set \( \overline{\mathcal{K}} \subset \overline{M} \). If \((\overline{M}, \overline{\mathcal{J}}, \overline{\mathcal{K}}, \overline{\mathcal{P}}, \overline{\mathcal{J}})\) is a \( C^2_{2+\varepsilon} \)-asymptotically flat initial data set, then \( \overline{P}^i \) characterizes the ADM-linear momentum \( \overline{P} = (\overline{P}^1, \overline{P}^2, \overline{P}^3) \in \mathbb{R}^3 \), i.e.

\[
\lim_{\sigma \to \infty} \frac{3}{4\pi \sigma^4} \int_{\sigma \Sigma} \sigma \overline{P}^i \pi_i \, d\mu = \overline{P}^i \quad \forall i \in \{1, 2, 3\}.
\]

More precisely,

\[
\frac{1}{\sigma^p} \left\| \sigma \overline{P}^i - \overline{P}^i \sigma \nu_i \right\|_{W^{1, p}(\sigma \Sigma)} \leq \frac{C}{\sigma^\varepsilon} \quad \forall \sigma \in [1; \infty)
\]

where the constant \( C = C(\overline{M}, \varepsilon, \pi, p) \) depends on \( p \in [1; \infty) \) and \( \sigma \nu_i \) denotes the components of the outer unit normal of \( \sigma \Sigma \mapsto (\overline{M}, \overline{\mathcal{J}}) \) with respect to \( \pi \).

**Proof.** Using the results of Section 2, this is true due to an integration by parts – see [Ner13] for a detailed proof of this inequality.

As explained above, the CMC-spheres (asymptotically) evolve in time (under the Einstein equations) by \( \frac{\sigma^3}{m_{\Sigma}(\sigma \Sigma)} \sigma \nu \) and this is true in a pointwise sense. The corresponding error term between \( J_t \) and \( \sigma \overline{P}^i \) is of order \( \sigma^{-\varepsilon} \) and this is true for \( \| \overline{\mathcal{K}} \| \leq \gamma \sigma^{\frac{3}{2}+\varepsilon} \) and \( \overline{\mathcal{J}} \in L^4(\overline{M}) \). For the ADM-linear momentum this is only true with the additional assumption \( \| \overline{\mathcal{J}} \| \leq C/\sigma^\varepsilon \) [Ner13]. Thus, this definition of linear momentum seems better adapted to the evolution of the CMC-surfaces in time – the reason for this is the additional correction term \( \overline{J}(\sigma \nu) \) in the definition of the CMC-linear momentum.

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\(^5\) Note that we do not use the exact function \( J_t \), because we want the linear momentum to be completely characterized by the data of one initial data set – as it is true for the ADM-linear momentum – and not by data of the temporal foliation.

\(^6\) Note that the addend \( J(\sigma \nu) \) in the definition of the CMC-linear momentum is – compared to the ADM-linear momentum – a correction term, adapting the CMC-linear momentum to the CMC-foliation, see below.
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APPENDIX A. SPHERES WITH VANISHING CALABI ENERGY

The aim of this section is to prove a $H^2$-regularity of the Gauß curvature of the sphere, i.e. there exists a conformal parametrization with conformal factor $H^2$-close to 1 if the Gauß curvature of a metric on the Euclidean sphere is in $L^2$-close to 1. We note that the same result is well-known if the Gauß curvature is pointwise bounded away from zero and infinity, see for example [CK93, Chap 2]. However, the author is not aware of a corresponding result in $L^p$-spaces. Furthermore, we should note that we can not hope that every conformal factor on the sphere is close to a constant if its Gauß curvature is close to a constant. The reason for this lays in the action of the Möbius group – for more information, we refer to [CK93, Rem. 7], [Str02], and the citations therein. Note that besides the explained main result (Theorem A.1), two intermediate results (Proposition A.3 and A.4) are interesting for themselves.

The scaling argument used in the proof of Proposition A.3 was suggested to the author by Simon Brendle [Bre]. Furthermore, the first part of the proof of Proposition A.4 is analog to Struwe’s proof of [Str02, Thm 3.2] (Theorem A.2).

First, let us state the main result.

**Theorem A.1** ($W^{2,p}(S^2)$-regularity of the Gauß curvature)

*For each $p \in (1; \infty)$ and $\delta \in (0; 4\pi)$, there exist constants $C = C(p, \delta)$ and $\varepsilon = \varepsilon(\delta)$ with the following property: If a metric $g$ of the Euclidean unit sphere $S^2$ satisfies

$$\mu(S^2) = \left( \int_{S^2} d\mu \right) \in (\delta; 8\pi - \delta), \quad \int_{S^2} |\mathcal{K} - 1|^2 d\mu \leq \varepsilon,$$

where $\mathcal{K}$ and $\mu$ are the Gauß curvature and the measure on the sphere $S^2$ with respect to $g$, respectively, then there exists a conformal parametrization $\varphi : S^2 \rightarrow S^2$ with

$$\|v\|_{W^{2,p}(S^2, \Omega)} \leq C \|\mathcal{K} - 1\|_{L^p(S^2, g)}.$$

Here, $\Omega$ denotes the standard metric of the Euclidean unit sphere $S^2$ and $u \in H^2(S^2)$ is the corresponding conformal factor, i.e. $\varphi^* g = \exp(2u)\Omega$.

As main tools for the proof of Theorem A.1 we use Brezis-Merle’s famous inequality [BM91, Thm 1] (see Theorem A.6) and Chen-Li’s classification theorem [CL91, Thm 1]: Every solution $v$ of

$$-\Delta v = \exp(2v) \quad \text{in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} \exp(2v) \, dx < \infty$$

is given by $v(x) = \ln\left((2\lambda/|x-x_0|^2)\right)$ for some constant $\lambda > 0$ and some point $x_0 \in \mathbb{R}^2$. As an intermediate result, we prove a qualitative version of this characterization, Proposition A.4. If the Gauß curvature $\mathcal{K}_n := -\exp(-2v_n)\Delta v_n$ of a sequence of conformal factors converges in $L^2_{\text{loc}}(\mathbb{R}^2)$ to 1, the corresponding volumes $\int \exp(2v_n) \, dx$ are uniformly bounded, and they satisfy a non-concentration assumption, then $v_n$ converges in $H^2_{\text{loc}}(\mathbb{R}^2)$ to $v(x) := \ln\left((2\lambda/|x-x_0|^2)\right)$.

One of the main ideas of the proof of Theorem A.1 is to prove that any sequence of conformal factors is bounded in $H^2(S^2)$ or the mass (the area) of a subsequence is concentrated at some points if the corresponding Gauß curvatures converge in $L^2(S^2)$ to a constant. Here, concentration at a point means that the corresponding measures $\mu_n$ of the subsequence converge to a measure with non-trivial point measure at this point. Struwe proved the corresponding theorem in the context of the Calabi flow assuming only uniform boundedness of the Calabi energy of the

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sequence, however under our assumptions, we get a stronger control of the Dirac measures in the case of mass concentration [Str02 Thm 3.2]. Before we cite Struwe’s theorem, we recall that for a given closed Riemannian manifold \((\Sigma, g)\) with constant Gauß curvature \(K \equiv K_0\) and a conformal equivalent metric \(g' = \exp(2v')g\), the \textit{Calabi energy} of \(g'\) (respectively \(v'\)) with respect to \(g\) is defined by

\[
\text{Cal}_g(v') := \text{Cal}_g(g') := \int_{S^2} |K' - K|^2 \, d\mu = \|K' - K\|_{L^2(S^2, g)}^2,
\]

where \(K'\) denotes the Gauß curvature of \(\Sigma\) with respect to \(g'\).

**Theorem A.2** (Concentration compactness [Str02 Thm 3.2])

Let \((\Sigma, g)\) be a closed Riemannian manifold with constant Gauß curvature \(K\) and let \(u_n\) be a sequence of conformal factors with uniformly bounded Calabi energy and unit volume \(\mu_n(\Sigma) = \mu(\Sigma) = 1\), where \(\mu_n = \exp(2v_n)\mu\) and \(\mu\) is the measure induced on \(\Sigma\) by \(g\). Then either the sequence \(u_n\) is bounded in \(H^2(S^2, \Omega)\) or there exist points \(x_1, \ldots, x_L \in \Sigma\) and a subsequence \(k_n \to \infty\) such that

\[
\varrho_R(x_1) := \liminf_{n \to \infty} \int_{B_R(x_1)} \, d\mu_{k_n} \geq 2\pi \quad \forall R > 0, \ l \in \{1, \ldots, L\}.
\]

Moreover, there holds

\[
2\pi L \leq \limsup_{n \to \infty} \left( \int_{\Sigma} K_n^2 \, d\mu_n \right)^{\frac{1}{2}} < \infty
\]

and either \(u_{k_n} \to -\infty\) as \(n \to \infty\) locally uniformly on \(\Sigma \setminus \{x_1, \ldots, x_L\}\) or \((u_{k_n})\) is locally bounded in \(H^2(\Sigma \setminus \{x_1, \ldots, x_L\}, g)\).

Now, we strengthen this result in the case of the sphere and vanishing Calabi energy by proving that the amount \(L\) of critical points \(\{x_i\}_{i=1}^L\) satisfies in this setting

\[
4\pi L \leq \left( \int_{\Sigma} K^2 \, d\mu \right)^{\frac{1}{2}} = \lim_{n \to \infty} \left( \int_{\Sigma} K_n^2 \, d\mu_n \right)^{\frac{1}{2}}.
\]

Let us therefor recall that any Riemannian surfaces is locally conformal equivalent to the plane, i.e. we can look at metrics on the Euclidean space \(\mathbb{R}^2\) instead of \(S^2\) and get a new conformal factor \(v\) satisfying \(-\Delta v_n = K_n \exp(2v_n)\).

**Proposition A.3** (Concentration point for vanishing Calabi energy)

Let \(v_n\) be a family of smooth conformal factors on \(\mathbb{R}^2\). Assume that the corresponding volumes are bounded, i.e.

\[
\limsup_{n \to \infty} \int \exp(2v_n) \, dx =: c_v < \infty,
\]

and that their Calabi energy (compared to the standard unit sphere) is locally converging to zero, i.e.

\[
\lim_{n \to \infty} \int_K |K_n - 1|^2 \, d\mu_n = 0 \quad \forall K \subseteq \mathbb{R}^2 \text{ compact},
\]

where \(\mu_n = \exp(2v_n)\) \(dx\) and \(K_n = -\exp(-2v_n) \Delta v_n\) denotes the measure and the Gauß curvature with respect to \(\exp(2v_n)g\). Then each point \(x \in \mathbb{R}^2\) with \(\varrho(x) > 0\) satisfies \(\varrho(x) \geq 4\pi\), where

\[
\varrho(x) := \inf_{R > 0} \liminf_{n \to \infty} \int_{B_R(x)} \exp(2v_n) \, dx.
\]
In particular, there are not more than \( c_n/\pi \) many points \( x \in \mathbb{R}^2 \) with \( \rho(x) > 0 \).

We see that this implies (15). We will prove Proposition A.3 by a blowup argument in a neighborhood of any point \( x \in \mathbb{R}^3 \) with \( \rho(x) > 0 \), i.e. we rescale \( w_n \) around \( x \) by factors \( R_n \to \infty \) to functions \( w_n \) such that there is a fixed radius \( r > 0 \) with \( \int_{B_r(x)} \exp(2w_n) \, dx = \varepsilon(x)/2 \) and then prove that this already implies that there is a fixed radius \( R \) with \( \int_{B_r(x)} \exp(2w_n) \, dx \approx 4\pi \) (for sufficiently large \( n \)) – note that the radius is fixed in the scaled image, i.e. this implies \( \rho(x) \approx 4\pi \).

Let us start by the last argument, i.e. we assume that we already scaled the metric. This result should be understood as a qualitative analog of Chen-Li’s classification theorem [CL91, Thm 1].

**Proposition A.4** (Concentration point for vanishing Calabi energy – rescaled)
Let \( w_n \in C^2(\mathbb{R}^2) \) be a sequence of functions \( \mathbb{R}^2 \) with

\[
\limsup_{n \to \infty} \int_{\mathbb{R}^2} \exp(2w_n) \, dx =: c_\mu < \infty,
\]

\[
\lim_{n \to \infty} \int_K |\mathcal{K}_n - 1|^2 \exp(2w_n) \, dx = 0 \quad \forall \, K \subseteq \mathbb{R}^2 \text{ compact},
\]

where \( \mathcal{K}_n := -\exp(-2w_n)\Delta w_n \). If there are constants \( \varepsilon_0 \in (0; 2\pi) \), \( r > 0 \), and a sequence of positive numbers \( S_n > 0 \) converging to infinity, i.e. \( S_n \to \infty \) for \( n \to \infty \), such that

\[
\int_{B_r(x)} \exp(2w_n) \, dx \leq \int_{B_r(0)} \exp(2w_n) \, dx = \varepsilon_0 \quad \forall \, |x| \leq S_n,
\]

then

\[
\lim_{n \to \infty} w_n = w := \ln \left( \frac{2\lambda}{\lambda^2 + |x|^2} \right) \quad \text{in } W^{1,p}_{\text{loc}}(\mathbb{R}^2) \quad \forall \, p \in [1; \infty),
\]

where \( \lambda := \sqrt{\frac{4\pi - \varepsilon_0}{\varepsilon_0}} \cdot r \).

We see that this in particular implies the following corollary.

**Corollary A.5**
Let \( w_n \in C^2(\mathbb{R}^2) \) be a sequence of functions satisfying (16) and (17). If there exist constants \( r > 0 \) and \( \varepsilon_0 \in (0; \pi) \) with

\[
\int_{B_r(x)} \exp(2w_n) \, dx \leq \int_{B_r(0)} \exp(2w_n) \, dx \in [\varepsilon_0; 2\pi - \varepsilon_0] \quad \forall \, |x| \leq S_n
\]

for some sequence of constants \( S_n \) converging to infinity, then

\[
\liminf_{n \to \infty} \int_{\mathbb{R}^2} \exp(2w_n) \, dx \geq 4\pi.
\]

**Proof of Corollary A.5**  Assume \( w_n' := w_n'_{k_n} \) is the area minimizing sequence, i.e.

\[
\liminf_n \int_{\mathbb{R}^2} \exp(2w_n) \, dx = \lim_n \int_{\mathbb{R}^2} \exp(2w_n') \, dx.
\]

There is a subsequence \( w_n'' := w_n''_{k_n} \) such that \( \int_{B_r(0)} \exp(2w_n') \, dx \) converges to some number \( \varepsilon_0' \in [\varepsilon_0; 2\pi - \varepsilon_0] \).

Thus, Proposition A.4 implies \( w_n'' \to w'' = \ln(2\lambda/(\lambda^2 + |x|^2)) \) in \( W^{1,p}_{\text{loc}}(\mathbb{R}^2) \), where \( \lambda = \sqrt{\frac{4\pi - \varepsilon_0}{\varepsilon_0}} \cdot r \). Thus, for any \( \varepsilon > 0 \) there exist \( N > 0 \) and \( R > 0 \) such that

\[
\int_{B_R} \exp(2w_n''') \, dx \geq \int_{B_R} \exp(2w_n'') \, dx - \varepsilon \geq \int_{\mathbb{R}^2} \exp(2w_n') \, dx - 2\varepsilon = 4\pi - 2\varepsilon
\]

for any \( n \geq N \), i.e. \( \liminf_n \int_{\mathbb{R}^2} \exp(2w_n) \, dx = \lim_n \int_{\mathbb{R}^2} \exp(2w_n) \, dx \geq 4\pi. \)
The second part of the proof of Proposition A.4 uses Chen-Li’s classification theorem [CL91 Thm 1] for metrics of constant Gauss curvature in $\mathbb{R}^2$, while the first part of the proof is analogous to Struwe’s proof of [Str02 Thm 3.2]. We repeat it nonetheless for the readers’ convenience. As Struwe, we need the following result of

\[ \text{Theorem A.6 (BM91 Thm 1)} \]

Let $u$ be a distribution solution to the equation

\[-\Delta u = f \quad \text{in } B_1(0), \quad u \equiv 0 \quad \text{in } \partial B_1(0) = S^1_1(0),\]

where $f \in L^1(B_1(0))$. For every $p < 4\pi \|f\|^{-1}_{L^1(B_1(0))}$, there exists a constant $C$ with

\[ \int_{B_1(0)} \exp(p |u|) \, dx \leq C \left( \frac{4\pi - p \|f\|_{L^1(B_1(0))}}{4\pi} \right)^{-1}. \]

\[ \text{Proof of Proposition A.4} \]

We start by proving that the sequence $w_n$ is bounded in $H^2_0(R^2)$. As mentioned above, the proof of this part is analogous to Struwe’s proof of [Str02 Thm 3.2]. Let $N \gg 1$ be a constant, define $R := R_N := \inf_{n \geq N} S_n$, and note $R_N \to \infty$ for $N \to \infty$. Let $n \geq N$ be arbitrary and suppress the corresponding index $n$. Furthermore, let $x \in B_R(0)$ be arbitrary and choose functions $w^0, w^h \in C^2(B_r(x))$ with $w = w^0 + w^h$ such that

\[ \left\{ \begin{array}{ll}
-\Delta w^h = 0, & -\Delta w^0 = -\Delta w \quad \text{in } B_r(x) \\
\quad w^h = u, & w^0 = 0 \quad \text{in } \partial B_r(x),
\end{array} \right. \]

i.e. $w^h$ is the harmonic part of $w$ and $w^0$ is the rest having boundary value 0. We see that

\[ \int_{B_r(x)} |\Delta w^0| \, dx \leq \int_{B_r(x)} \exp(2w) \, dx + \frac{2\pi - \varepsilon_0}{2} \leq \frac{2\pi + \varepsilon_0}{2} < 2\pi \]

if $N$ is so large that $\int_{B_{R+\varepsilon}(0)} \exp(2w) |K-1| \, dx \leq \pi - \varepsilon_0/2$ holds for every $n \geq N$. In particular, we can choose $N$ independently of $x \in B_R(0)$. Fix a $q \in (1; \frac{4\pi}{2\pi + c_\mu})$ with $q \leq 2$. Brezis-Merle’s result, Theorem A.6, implies

\[ \int_{B_r(x)} \exp(2q |w^0|) \, dx \leq C, \]

where the constant $C$ depends on $q, r, \varepsilon_0$. In the following, we do not distinguish between constants $C$ depending on $q, r, \varepsilon_0, c_\mu$, and $R$. The above implies

\[ (18) \quad \int_{B_r(x)} |w^0| \, dx \leq \ln \left( \int_{B_r(x)} \exp |w^0| \, dx \right) \leq C \]

and we already know

\[ 2 \int_{B_r(x)} w \, dx \leq \ln \left( \int_{B_r(x)} \exp(2w) \, dx \right) \leq \ln c_{\mu} \leq C. \]

Thus, the mean value property of harmonic functions implies for \( y \in B_{r/2}(x) \)

\[ w^h(y) = \int_{B_{r/2}(y)} w^h \, dx \leq C, \]

\[ \text{Here, we state the same version of BM01 Thm 1} \] as Struwe [Str02 Thm 3.1] which is a slight modification of the original theorem.
i.e. \( w^h \leq C \) in \( B_{1/2}(x) \). We conclude
\[
\int_B |\Delta w^0|^q \, dx \leq \int_B \exp(qw) |\mathcal{X}|^q \exp(qu) \, dx \\
\leq \left( \int_B \exp(2w) |\mathcal{X}|^2 \, dx \right)^{\frac{q}{2}} \left( \int_B \exp \left( \frac{2q^2}{2-q} w \right) \, dx \right)^{\frac{2-q}{q}} \leq C,
\]
where \( B := B_{1/2}(x) \) and where we used \( 2q^2/(2-q) \leq q \) as \( q \leq 2 \). Hence, (18) implies \( \|w^0\|_{W^{2,q}(B)} \leq C \). In particular, the Sobolev inequalities imply \( w^h \leq w^h + |w^0| \leq C \) in \( B = B_{1/2}(x) \) and we therefore get
\[
\int_B |\Delta w|^2 \, dx = \int_B |\mathcal{X}|^2 \exp(4w) \, dx \leq C.
\]
As \( x \) was arbitrary in \( B_R(0) \), we conclude with the above estimates and the regularity of the Laplace operator
\[
\max_{B_R(0)} w \leq C + \min_{B_R(0)} w \leq C + \int_{B_R(0)} w \, dx \leq C + \frac{1}{2} \ln \left( \frac{e\mu}{4\pi R^2} \right) \leq C.
\]
Assuming without loss of generality that \( R > r \), we conclude
\[
4\pi \exp \left( 2 \max_{B_R(0)} w \right) R^2 \geq \int_{B_R(0)} \exp(2w) \, dx \geq \int_{B_r(0)} \exp(2w) \, dx \geq \varepsilon_0
\]
implying
\[
-C \leq \max_{B_R(0)} w \leq C + \min_{B_R(0)} w \leq C,
\]
i.e. \( |w| \leq C \) in \( B_R(0) \) – note that the constant \( C = C(R) \) depends on \( R \). Thus, we get the desired uniformly bound, as we proved
\[
(19) \quad \|w_n\|_{H^2(B_R(0))} \leq C(R), \quad \lim_{n \to \infty} \|\Delta w_n - \exp(2w_n)\|_{L^2(B_R(0))} = 0.
\]

By the compactness of the Sobolev embeddings, it is now sufficient that any in \( W^{1,p}_{\text{loc}}(\mathbb{R}^2) \) (for every \( p \in [1;\infty) \)) converging subsequence of \( w_n \) converges to \( \ln \left( \frac{2\lambda}{\lambda^2 + r^2} \right) \) for the fixed constant \( \lambda := \sqrt{\frac{4\pi - \varepsilon_0}{\varepsilon_0}} r \). Thus, we can assume that \( w_n \) converges in \( W^{1,p}_{\text{loc}}(\mathbb{R}^2) \) to some function \( w \in W^{1,p}_{\text{loc}}(\mathbb{R}^2) \), i.e.
\[
\lim_{n \to \infty} ||w_n - w||_{W^{1,p}(B_R(0))} = 0 \quad \forall p \in [1;\infty), \; R > 0.
\]
In particular, we know \( \exp(2w_n) \to \exp(2w) \) locally uniformly and that \( w \) is locally bounded. By the second inequality in (19), this implies \( \Delta w_n \to -\exp(2w) \) in \( L^2_{\text{loc}}(\mathbb{R}^2) \). Hence, we know that \( \Delta w = -\exp(2w) \) in the \( L^2 \)-weak sense in \( B_R(0) \) (for every \( R > 0 \)). The regularity of the Laplace operator and the locally boundedness of \( w_n \) (see above) therefore implies \( w \in C^\infty(\mathbb{R}^2) \) and \( \Delta w = -\exp(2w) \) pointwise everywhere. As we furthermore know
\[
\int_{B_R(0)} \exp(2w) \, dx \leq \lim_{n \to \infty} \int_{B_R(0)} \exp(2w_n) \, dx \leq \limsup_{n \to \infty} \int_{\mathbb{R}^2} \exp(2w_n) \, dx = C_{\mu},
\]
we can use the Chen-Li’s classification theorem [CL91, Thm 1] to conclude that there exist a point \( x_0 \) and a factor \( \kappa > 0 \) such that
\[
w(x) = \ln \left( \frac{2\kappa}{\kappa^2 + |x - x_0|^2} \right).
\]
We see that $x_0$ is uniquely determined by the fact that for any $R > 0$
\[
\int_{B_R(x)} \exp(2w) \, dx \leq \int_{B_R(x_0)} \exp(2w) \, dx \quad \forall x \in \mathbb{R}^2.
\]
We therefore conclude $x_0 = 0$ by
\[
\varepsilon_0 \geq \lim_{n \to \infty} \int_{B_r(x)} \exp(2w_n) \, dx = \int_{B_r(x)} \exp(2w) \, dx \quad \forall x \in \mathbb{R}^2
\]
and
\[
\varepsilon_0 = \lim_{n \to \infty} \int_{B_r(0)} \exp(2w_n) \, dx = \int_{B_r(0)} \exp(2w) \, dx,
\]
due to $\exp(2w_n) \to \exp(2w)$ in $L^1_{\text{loc}}(\mathbb{R}^2)$. In particular, we get
\[
\varepsilon_0 = \int_{B_r(0)} \exp(2w) \, dx = \frac{4\pi r^2}{\kappa^2 + r^2}
\]
implicating $\kappa = \lambda$.

Now we give the rescaling argument with which we prove Proposition A.3 using Corollary A.5.

**Proof of Proposition A.3** Let $x_0 \in \mathbb{R}^2$ be a point such that $\varepsilon_1 := g(x_0) > 0$ — without loss of generality $x_0 = 0$. Now, we want to rescale $v_n$ around some center point $y_n$ near 0 by a factor $\varrho_n \to \infty$ (for $n \to \infty$) and use Corollary A.5 on the rescaled functions $w_n(x) := v(n \varrho_n x + y_n) + \ln \varrho_n$. However, we cannot choose 0 as this center point (for all $n$) as it could be that the mass $\exp(2v_n) \, dx$ is only large around some point $y_n$ close to but not (necessarily) equal to 0 and this point $y_n$ could be ‘scaled away’ if we scaled around 0. Thus, we have to choose the center more carefully.

Now, we want to use the Brendle’s scaling argument [Bre]. As explained, we want to find a center point for the scaling done later and this center point should lay near 0. Thus, we have to fix a neighborhood of 0 such that the only ‘center’ $y_n$ (see below) of the mass (measure) $\exp(2v_n)$ $\exp(2v_n) \, dx$ within this neighborhood approximates 0. If $L \in \mathbb{N}$ is an integer, $r > 0$ is a constant, and $y_1, \ldots, y_L \in \mathbb{R}^2$ are points with $B_r(y_i) \cap \{y_1, \ldots, y_L\} = \{y_i\}$ and $g(y_i) \geq \min\{\pi, \varepsilon_1/2\} =: \varepsilon_0$ for every $i$, then
\[
c_n \geq \limsup_{n \to \infty} \int_{\mathbb{R}^2} \exp(2v_n) \, dx \geq \limsup_{n \to \infty} \sum_{i=1}^L \int_{B_r(y_i)} \exp(2v_n) \, dx \geq L \varepsilon_0.
\]
In particular, we know $L \leq c_n/\varepsilon_0 < \infty$. Thus, there is a radius $R > 0$ such that every $x \in B_{2R}(0)$ with $x \neq 0$ satisfies $g(x) < \varepsilon_0$. Now, we define the center points $y_n \in B_R(0) \subseteq \mathbb{R}^2$ as one of the points in which $\varrho_n$ is minimal, where
\[
\varrho_n(x) := \inf \left\{ r \in (0; 2R) \Bigg| \int_{B_r(x)} \exp(2v_n) \, dx \geq \varepsilon_0 \right\} \quad \forall x \in \mathbb{R}^2, |x| \leq R.
\]
Note that the minimum of $\varrho_n$ within $\{y : |y| \leq R\} =: B_R$ exists as $\varrho_n$ is continuous, i.e., we can choose such a (not necessarily uniquely defined) $y_n$ for every $n \gg 1$. As $(y_n)$ is a bounded sequence and every cluster point $y$ of it satisfies $g(y) \geq \varepsilon_0$ and (per definition of $R$) therefore $y = 0$ or $2R \leq |y| = \lim_n |y_n| \leq R$, we know $y_n \to 0$ for $n \to \infty$. Furthermore, we know $\varrho_n := \varrho_n(y_n) \leq \varrho_n(0) \to 0$ for $n \to \infty$. 

We rescale around \( y_n \) with the scaling factor \( \rho_n^{-1} \), i.e. we define
\[
\tilde{w}_n(x) := v_n(\rho_n x + y_n) + \ln \rho_n \quad \forall x \in \mathbb{R}^2
\]
and have to check the preliminaries of Corollary \([A.5]\) in order to use it. First, we see that
\[
\int_{\mathbb{R}^2} \exp(2 \tilde{w}_n) \, dx = \int_{\mathbb{R}^2} \tilde{\rho}^2 \exp(2 v_n(\rho_n x + y_n)) \, dx = \int_{\mathbb{R}^2} \exp(2 v_n) \, dx
\]
implying \( \limsup_n \int \exp(2 w_n) \, dx \leq c_\mu < \infty \). Furthermore, we get
\[
\Delta w_n = \tilde{\rho}^2(\Delta v_n)(\rho_n x + y_n) = -\mathcal{K}_n(\rho_n x + y_n) \exp(2 w_n) =: -\mathcal{K}_n'(x) \exp(2 w_n)
\]
and
\[
\int_{B_{\rho_n^{-1}(0)}} |\mathcal{K}_n' - 1|^2 \exp(2 w_n) \, dx = \int_{B_{\rho_n^{-1}(0)}} |\mathcal{K}_n - 1|^2 \exp(2 v_n) \, dx \xrightarrow{n \to \infty} 0.
\]
As \( \rho_n \to 0 \) for \( n \to \infty \), this implies that \( \int_K |\mathcal{K}_n' - 1|^2 \exp(2 w_n) \, dx \to 0 \) for any compact set \( K \subset \mathbb{R}^2 \). For the last preliminary, we note that every \( x \in B_{\rho_n \pi n}(0) \) and \( x_n := \rho_n x + y_n \) satisfies
\[
\int_{B_{\rho_n}(x_n)} \exp(2 w_n) \, dx = \int_{B_{\rho_n}(y_n)} \exp(2 v_n) \, dx \leq \int_{B_{\rho_n}(x_n)} \exp(2 v_n) \, dx = \varepsilon_0
\]
and
\[
\int_{B_{\rho_n}(y_n)} \exp(2 v_n) \, dx = \int_{B_{\rho_n}(y_n)} \exp(2 v_n) \, dx = \varepsilon_0.
\]
Thus, we can use Corollary \([A.5]\) for \( S_n = \theta^{-1/2} \) and \( r = 1 \) to conclude that \( w_n \to w \) in \( W^{1,3}_{\text{loc}}(\mathbb{R}^2) \) with \( \int_{\mathbb{R}^2} \exp(2 w) \, dx = 4\pi \). This means for every \( \varepsilon > 0 \) and \( \varepsilon' > 0 \) there is a radius \( R < \infty \) with
\[
4\pi \leq \int_{B_R(0)} \exp(2 w) \, dx + \frac{\varepsilon}{2} \leq \int_{B_R(0)} \exp(2 w_n) \, dx + \varepsilon
\]
\[
= \int_{B_{\rho_n}(y_n)} \exp(2 v_n) \, dx + \varepsilon \leq \int_{B_{\rho_n}(y_n)} \exp(2 v_n) \, dx + \varepsilon
\]
for every \( n \geq N \), where \( N \) is so large that \( \int_{B_{\rho_n}(y_n)} |\exp(2 w) - \exp(2 w_n)| \, dx \leq \varepsilon/2 \) and \( \rho_n R + |y_n| \leq \varepsilon' \). As \( \varepsilon > 0 \) was arbitrary, this implies
\[
\liminf_{n \to \infty} \int_{B_{\rho_n}(y_n)} \exp(2 v_n) \, dx \geq 4\pi \quad \forall \varepsilon' > 0,
\]
i.e. \( g(x_0) = g(0) \geq 4\pi \).

Now, we can prove the main theorem of the current section.

**Proof of Theorem** \([A.7]\) Using conformal maps, we can assume
\[
\mu(S^2 \cap \{ x_i \geq 0 \}) = \mu(S^2 \cap \{ x_i \leq 0 \}) \in \left( \frac{\delta}{2}, 4\pi - \frac{\delta}{2} \right) \quad \forall i \in \{1, 2, 3\},
\]
see for example \([DLM05\) Lemma 3.4\] for a proof of an analog claim. Let \( u \) denote a corresponding conformal factor, i.e. \( g = \exp(2 u) \Omega \), where \( \Omega \) denotes the standard metric of the Euclidean unit sphere. We first prove the implication
\[
\forall \varepsilon > 0 \quad \exists \varepsilon' = \varepsilon'(\varepsilon, \delta) > 0 : \quad \| \mathcal{K} - 1 \|_{L^1(S^2, g)} \leq \varepsilon' \quad \implies \quad \| u \|_{H^p(S^2, \Omega)} \leq \varepsilon
\]
by contradiction, i.e. we assume the existence of a constant \( \varepsilon > 0 \) and a sequence
with respect to \( \Omega \), and that

\[
\int_{\mathbb{R}^2} \exp(2v_n^p) \, dx \leq 8\pi - 2\delta, \quad \lim_{n \to \infty} \int_{\mathbb{R}^2} |\mathcal{K}_n - 1|^2 \, d\mu_n = 0
\]

and by the diffeomorphism invariance of \( \varrho \), we see \( \varrho'(0) = \varrho(p) > 0 \) and therefore Proposition A.3 implies \( \varrho(p) = \varrho'(0) \geq 4\pi \), where

\[
\varrho'(0) := \inf_{r > 0} \lim_{n \to \infty} \mu_n(B_r^p(0)) \geq 4\pi
\]

and where \( B_r(x) = \{ y \in \mathbb{R}^2 : |x| \leq r \} \) and \( \mu_n := \varrho^*\mu_n \) denote the Euclidean ball of radius \( r \) in \( \mathbb{R}^2 \) and the corresponding measure on \( \mathbb{R}^2 \), respectively. However, there is a direction \( i \in \{1, 2, 3\} \) and a fixed sign \( \pm \) such that \( B_{v_n^i}(0) \subseteq \{ \pm x_i \geq 0 \} \), i.e. we get the contradiction

\[
4\pi \leq \limsup_{n \to \infty} \mu_k(\{ U \} \leq \limsup_{n \to \infty} \mu_k(\mathbb{R}^3) \leq 4\pi - \delta.
\]

Thus, there is no concentration point, i.e. (23) holds.

Now, we prove a quantitative version of (23), i.e.

\[
\forall \varepsilon' > 0 \quad \exists r > 0 : \quad \forall p \in \mathbb{S}^2 : \quad \limsup_{n \to \infty} \mu_n(B_r^p(p)) \leq \varepsilon'.
\]

If such a radius did not exist, then there would exist a constant \( \varepsilon > 0 \) and a sequence \( y_n \in \mathbb{S}^2 \) such that \( \mu_k(B_{v_n}(y_n)) \geq \varepsilon' \) for some subsequence of \( \mu_k \). By the compactness \( \mathbb{S}^2 \), we can assume that \( y_n \) converges to some \( y \in \mathbb{S}^2 \) for which therefore

\[
\mu_k(B_{v_n}(y)) \geq \mu_k(B_{\pi}(y_n)) \geq \varepsilon' \quad \forall n > N
\]

holds if \( N \) is so large that \( r > |y_n - y| + 1/n \) for every \( n \geq N \) and where \( r > 0 \) is arbitrary. This implicates \( \varrho(y) \geq \varepsilon' \) contradicting (23). Hence, there exists such a radius for every \( \varepsilon' > 0 \), i.e. (24) holds. Therefore, we can without loss of generality assume that \( N := (1, 0, 0) \in \mathbb{S}^2 \) satisfies

\[
\mu_n(B_r^p(p)) \leq \mu_n(B_r^p(N)) \leq \pi \quad \forall n \in \mathbb{N}, \, p \in \mathbb{S}^2
\]

for some fixed radius \( r > 0 \).

Let \( P \in \{N, S\} \) be one of the poles, \( N := (1, 0, 0) \) and \( S := (-1, 0, 0) \), and again denote by \( \varrho^P \) and \( v_n^P \) the corresponding stereographic projection and conformal factor, respectively. By the existence of the above uniform radius \( r > 0 \), there
exists a \( s > 0 \) such that \( \int_{B_2(x)} \exp(2v_n^P) \, dx \leq \pi \) holds for every \( x \in \mathbb{R}^2 \). With \( \int_{\mathbb{R}^2} \exp(2v_n^P) \, dx \leq 8\pi - 2\delta < \infty \), we deduce the existence of a sequence \( y_n^P \) with

\[
\int_{B_2(x)} \exp(2v_n^P) \, dx \leq \int_{B_2(y_n^P)} \exp(2v_n^P) \, dx \leq \pi \quad \forall x \in \mathbb{R}^2
\]

and by (25) \( y_n^N = 0 \). Let us know prove that \( B_2^2(y_n^P) \) contains some positive mass bounded away from 0, i.e.

\[
\exists \varepsilon' > 0 : \quad \forall n \in \mathbb{N} : \quad \mu_n(B_2^2(y_n^P)) \geq \varepsilon'.
\]

Again, we use a contradiction argument and assume \( \liminf_n \int_{B_1(y_n^P)} \exp(2v_n^P) \, dx = 0 \). This implies

\[
\frac{\delta}{2} \leq \int_{B_{1}(0)} \exp(2v_n^P) \, dx \leq C \int_{B_1(y_n^P)} \exp(2v_n^P) \, dx \xrightarrow{n \to \infty} 0
\]

for some subsequence \( k_n \), where \( C \) depends only on \( s \). Here, we used that we can cover \( B_1(0) \) with finite many balls \( B_s(p) \) and that each of these balls contains less mass than \( B_s(y_n^P) \). As this is again a contradiction, we know

\[
\liminf_{n \to \infty} \int_{B_1(y_n^P)} \exp(2v_n^P) \, dx > 0.
\]

Now, we prove that \( u_n \) is bounded in \( H^2(S^2, \Omega) \). Let therefore \( u_{k_n} \to \infty \) be a \( H^2 \)-norm maximizing subsequence, i.e.

\[
\sup_{n \in \mathbb{N}} \|u_n\|_{H^2(S^2, \Omega)} = \sup_{n \in \mathbb{N}} \|u_{k_n}\|_{H^2(S^2, \Omega)}.
\]

With the same argument as in the proof of Corollary A.5, we can use Proposition A.4 to conclude that a subsequence \( v_{k_n}^P := v_{k_n}^P(x+y_n^P) \) of \( v_{k_n}^P(x+y_n^P) \) converges in \( W_{\text{loc}}^{1,2}(\mathbb{R}^2) \) to \( v^P \), where \( v^P \) is as in Proposition A.4. We see that this implies that \( \|v_{k_n}^P\|_{H^2(B_R(0))} \) is bounded for every \( R > 0 \), see the proof of Proposition A.4. This implies \( \|u_{k_n}\|_{H^2(S^2, \Omega)} \leq C \) for some constant \( C \geq 0 \) if \( \sup_n \|u_{k_n}^P\| < \infty \). But if there existed a subsequence \( y_{k_n}^P \), then

\[
\int_{B_1(0)} \exp(2v_{k_n}^P) \, dx = \int_{R^2\setminus B_2(0)} \exp(2v_{k_n}^S) \, dx \geq \int_{B_2(y_{k_n}^P)} \exp(2v_{k_n}^S) \, dx \geq \varepsilon'
\]

would again contradict \( \varrho(N) = 0 \), where we again used (26). Thus, all in all we know

\[
\sup_{n \in \mathbb{N}} \|u_n\|_{H^2(S^2, \Omega)} = \sup_{n \in \mathbb{N}} \|u_{k_n}\|_{H^2(S^2, \Omega)} =: c < \infty.
\]

By the compactness of the Sobolev embeddings, we can therefore assume that \( u_n \) converges in \( W^{1,3}(\mathbb{S}^2, \Omega) \) to a function \( u \in W^{1,3}(\mathbb{S}^2, \Omega) \) and only have to prove \( u \equiv 0 \). However, we know that

\[
-\Delta u_n = \mathcal{K}_n - \exp(-2u_n) \xrightarrow{n \to \infty} 1 - \exp(-2u),
\]

where we used that \( \frac{1}{C} \mu \leq \mu_n \leq C \mu \) due to the boundedness of \( u_n \). We get

\[
-\Delta u = 1 - \exp(-2u)
\]

weakly in \( \mathbb{S}^2 \) by combining the two above convergences. The convergence of \( u_n \) furthermore implies \( \|u\|_{W^{1,3}(\mathbb{S}^2)} \leq C \). Thus, we conclude \( u \in C^\infty(\mathbb{S}^2) \) and \( -\Delta u = 1 - \exp(2u) \) pointwise everywhere in \( \mathbb{S}^2 \) due to the regularity of the Laplace operator. Chen-Li’s
...converges to 0. Thus, the assumption on $K_\text{BM91}$ Ha"ım Brezis and Frank Merle. Uniform estimates and blow–up behavior for solutions $\text{AH78}$ Abhay Ashtekar and Richard O Hansen. A unified treatment of null and spatial infin-
...$p$-neighborhood of $\Omega$. Thus, we conclude $\|u\|_{W^2,p(S^2,\Omega)} \leq C\|K-1\|_{L^p(S^2,\Omega)}$ for some constant $C$ depending only on $p \in (1; \infty)$. Choosing sufficiently small $\varepsilon' \in (0; \eta')$, we conclude $\|u\|_{W^2,p(S^2,\Omega)} \leq C\|K-1\|_{L^p(S^2,\Omega)} \leq C\|\varepsilon - 1\|_{L^p(S^2,\Omega)}$. //

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