Laplacians on smooth distributions

Yu. A. Kordyukov

Abstract. Let $M$ be a compact smooth manifold equipped with a positive smooth density $\mu$ and let $H$ be a smooth distribution endowed with a fibrewise inner product $g$. We define the Laplacian $\Delta_H$ associated with $(H, \mu, g)$ and prove that it gives rise to an unbounded self-adjoint operator in $L^2(M, \mu)$. Then, assuming that $H$ generates a singular foliation $\mathcal{F}$, we prove that, for any function $\varphi$ in the Schwartz space $\mathcal{S}(\mathbb{R})$, the operator $\varphi(\Delta_H)$ is a smoothing operator in the scale of longitudinal Sobolev spaces associated with $\mathcal{F}$. The proofs are based on pseudodifferential calculus on singular foliations, which was developed by Androulidakis and Skandalis, and on subelliptic estimates for $\Delta_H$.

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§ 1. Introduction

The main purpose of the paper is to define and study certain natural geometric differential operators associated with an arbitrary smooth distribution on a compact manifold. Let $M$ be a connected compact smooth manifold of dimension $n$ equipped with a positive smooth density $\mu$. Let $H$ be a smooth rank $p$ distribution on $M$ (that is, $H$ is a smooth subbundle of the tangent bundle $TM$ of $M$) and let $g$ be a smooth fibrewise inner product on $H$. We define the horizontal differential $d_H f$ of a function $f \in C^\infty(M)$ to be the restriction of its differential $d f$ to $H \subset TM$. Thus, $d_H f$ is a section of the dual bundle $H^*$ of $H$, $d_H f \in C^\infty(M, H^*)$, and we get a first order differential operator $d^*_H: C^\infty(M) \to C^\infty(M, H^*)$. The Riemannian metric $g$ and the positive smooth density $\mu$ induce inner products in $C^\infty(M)$ and $C^\infty(M, H^*)$; this allows us to consider the adjoint $d^*_H: C^\infty(M, H^*) \to C^\infty(M)$ of $d_H$. Finally, the Laplacian $\Delta_H$ associated with $(H, g, \mu)$ is the second order differential operator on $C^\infty(M)$ given by

$$\Delta_H = d^*_H d_H.$$
If $X_k, k = 1, \ldots, p$, is a local orthonormal frame in $H$ defined on an open subset $\Omega \subset M$, then it is easy to check that the restriction of $\Delta_H$ to $\Omega$ is given by

$$\Delta_H|_{\Omega} = \sum_{k=1}^{p} X_k^* X_k.$$

The next theorem allows us to talk about spectral properties of the operator $\Delta_H$.

**Theorem 1.** The Laplacian $\Delta_H$, considered as an unbounded operator in the Hilbert space $L^2(M, \mu)$ with domain $C^\infty(M)$, is essentially self-adjoint.

Theorem 1 can be proved using a well-known result of Chernoff’s [6] based on the theory of first order linear symmetric hyperbolic systems. This proof is given in §2. We also present another proof of Theorem 1, which is more complicated, but we hope that the techniques used in this proof will be helpful in the study of more refined spectral properties of the operator $\Delta_H$.

If the distribution $H$ is completely integrable, then, by the Frobenius theorem, it gives rise to a smooth foliation $\mathcal{F}$ on $M$. In this case, the operator $\Delta_H$ is a formally self-adjoint longitudinally elliptic operator with respect to $\mathcal{F}$. Spectral properties of this operator, in particular its self-adjointness, have been studied in several papers (see, for instance, [7], [16], [35] and the references there). Here an important role is played by the longitudinal pseudodifferential calculus for foliations developed by Connes in [7]. On the other hand, if $H$ is completely nonintegrable (or bracket-generating), then using Hörmander’s sum of the squares theorem [13], we can show that the operator $\Delta_H$ is hypoelliptic, which easily implies that it is self-adjoint. The proof of Theorem 1 in the general case combines two approaches mentioned above. We assume that the distribution $H$ defines a singular foliation $\mathcal{F}$ in the sense of Stefan and Sussmann (see [31] and [32]). Then the operator $\Delta_H$ can be considered as a longitudinally hypoelliptic operator with respect to $\mathcal{F}$. In [3], Androulidakis and Skandalis developed a pseudodifferential calculus on singular foliations. Following Kohn’s proof of Hörmander’s sum of the squares theorem, we derive subelliptic estimates and prove longitudinal hypoellipticity for the operator $\Delta_H$ in the scale of longitudinal Sobolev spaces on $M$ associated with the singular foliation $\mathcal{F}$. Using these results, we can easily complete the proof of Theorem 1.

Theorem 1 allows us to consider more refined spectral properties of the Laplacian $\Delta_H$. First of all, by the spectral theorem we can consider functions of $\Delta_H$ such as the heat operator $e^{-t\Delta_H}$, the wave operator $e^{it\sqrt{\Delta_H}}$ and so on. Using the longitudinal hypoellipticity result mentioned above, we immediately get the following theorem.

**Theorem 2.** Suppose that the distribution $H$ defines a singular foliation $\mathcal{F}$. For any function $\varphi$ in the Schwartz space $\mathcal{S}(\mathbb{R})$, the operator $\varphi(\Delta_H)$ extends to a bounded operator from $H^s(\mathcal{F})$ to $H^t(\mathcal{F})$ for any $s, t \in \mathbb{R}$.

We can also use the spectral properties of the operator $\Delta_H$ to define invariants of smooth distributions. For instance, consider the class of distributions $H$ such that the spectrum of the associated Laplacian $\Delta_H$ has a gap near zero. It is easy to see that this property of $\Delta_H$ is independent of the choice of $g$ and $\mu$. For smooth foliations, it is apparently related with property (T) for its holonomy groupoid (see, for instance, the discussion in [18], Remark 10). To study more
refined invariants of distributions, it would be nice to have some natural way of choosing $\mu$ and $g$ that would give rise to the intrinsic Laplacian associated with $H$. The problem of determining the Laplacian and the intrinsic Laplacian was discussed extensively recently in sub-Riemannian geometry (see, for instance, [23], [1], [9], [11] and the references there). In the general case such an intrinsic choice is not always possible. For instance, in the case when $H$ is integrable, $g$ and $\mu$ appear to be completely independent: $g$ is responsible for the longitudinal structure and $\mu$ for the transverse one.

In [25], [27] and [30] (see also the references there) the authors studied global hypoellipticity of Hörmander’s sum of the squares operators. In the case when $H$ has transverse symmetries given by a Riemannian foliation orthogonal to $H$, the associated Laplacian (sometimes called the horizontal Laplacian) was studied in [4], [5], [17], [19], [20], [28] and the references in these papers. In particular, its self-adjointness was established in [17]. In [10] (see also [8]) the authors introduced the characteristic Laplacian associated with an arbitrary smooth distribution $H$ and a Riemannian metric on $M$. This operator coincides with the operator $\Delta_H$ in degree 0 if $g$ is the restriction of the Riemannian metric to $H$ and $\mu$ is the Riemannian volume form. The problem of constructing natural geometric operators on differential forms associated with an arbitrary smooth distribution is a very interesting open problem (see, for instance, [24], [26], [29] and the references there for some related results in sub-Riemannian geometry).

The paper is organized as follows. In §2 we state theorems on subelliptic estimates and longitudinal hypoellipticity for the Laplacian $\Delta_H$ and show how these results enable us to prove the main results of the paper. In §3 we give the necessary information about singular foliations and pseudodifferential calculus on singular foliations. Section 4 contains the proofs of the theorems on subelliptic estimates and longitudinal hypoellipticity stated in §2.

§2. Longitudinal hypoellipticity and the proofs of the main results

As above, let $M$ be a connected compact smooth manifold of dimension $n$ equipped with a positive smooth density $\mu$. Let $H$ be a smooth rank $p$ distribution on $M$ and $g$ a smooth fibrewise inner product on $H$. Consider the $C^\infty(M)$-module $C^\infty(M, TM)$ of smooth vector fields on $M$. It is a Lie algebra with respect to the Lie bracket. Denote by $C^\infty(M, H)$ the submodule of $C^\infty(M, TM)$ which consists of smooth vector fields, tangent to $H$ at each point. Let $\mathcal{F}$ be the minimal submodule of $C^\infty(M, TM)$ which contains $C^\infty(M, H)$ and is stable under Lie brackets. We assume that $\mathcal{F}$ is locally finitely generated. Then it is a singular foliation in the sense of Stefan and Sussmann. We will use classes $\Psi^m(\mathcal{F})$ of longitudinal pseudodifferential operators and the corresponding scale $H^s(\mathcal{F})$ of the longitudinal Sobolev space we associated with $\mathcal{F}$ (we refer the reader to §3 for the necessary information about singular foliations and pseudodifferential calculus on singular foliations).

First, we state subelliptic estimates for the operator $\Delta_H$.

**Theorem 3.** There exists $\epsilon > 0$ such that for any $s \in \mathbb{R}$

$$\|u\|_{s+\epsilon}^2 \leq C_s(\|\Delta_H u\|_s^2 + \|u\|_s^2), \quad u \in C^\infty(M),$$

where $C_s > 0$ is some constant and $\|\cdot\|_s$ denotes the norm in $H^s(\mathcal{F})$. 

As a consequence, we get the following longitudinal hypoellipticity result.

**Theorem 4.** If $u \in H^{-\infty}(\mathcal{F}) := \bigcup_{t \in \mathbb{R}} H^t(\mathcal{F})$ such that $\Delta_H u \in H^s(\mathcal{F})$ for some $s \in \mathbb{R}$, then $u \in H^{s+\epsilon}(\mathcal{F})$ where $\epsilon$ is the positive constant given by Theorem 3.

The proofs of Theorems 3 and 4 will be given in §4. Here we show how to prove Theorems 1 and 2 on the basis of these theorems.

**Proof of Theorem 1.** By the basic criterion of essential self-adjointness, it is sufficient to show that $\ker(\Delta_H^* + i) = \{0\}$, where $\Delta_H^*$ is the adjoint of $\Delta_H$ regarded as an unbounded linear operator in $L^2(M, \mu)$ with domain $C^\infty(M)$. Moreover, it is sufficient to show that $\ker(\Delta_H^* + i)$ is contained in the domain $\Dom \Delta_H$ of the closure of $\Delta_H$ in $L^2(M, \mu)$. Let $u \in \ker(\Delta_H^* + i)$. So we have $u \in L^2(M, \mu)$ and $(\Delta_H^* + i)u = 0$.

Since $\Delta_H$ is symmetric on $C^\infty(M)$, we obtain $(\Delta_H^* + i)u = 0$, where $\Delta_H u$ is understood in the distributional sense. Taking the fact that $u \in L^2(M, \mu) \subset H^{-\infty}(\mathcal{F})$ into account, and using Theorem 4, we see that $u$ is in $H^{\infty}(\mathcal{F}) := \bigcap_{t \in \mathbb{R}} H^t(\mathcal{F})$. This immediately completes the proof because it is easy to see that $H^2(\mathcal{F})$ is contained in $\Dom \Delta_H$ (see Theorem 9 and Proposition 2 below).

**Remark 1.** We note that [3] looks at Hilbert modules over the $C^*$-algebra $C^*(M, \mathcal{F})$ of the singular foliation $\mathcal{F}$. Unlike [3], we will not work with Hilbert modules and $C^*$-algebras, but with the concrete representation of the $C^*$-algebra $C^*(M, \mathcal{F})$ on $L^2(M)$. This enables us to use some results from the theory of linear operators in Hilbert spaces (first of all, the basic criterion for essential self-adjointness). It would be very interesting to prove a $C^*$-module version of Theorem 1, stating that $\Delta_H$ gives rise to a regular (unbounded) self-adjoint multiplier of $C^*(M, \mathcal{F})$.

For longitudinally elliptic operators on $\mathcal{F}$, this was proved in [3], extending a similar result for regular foliations by Vassout [35] (see also [16]). We also note that the proof of Theorem 3 can easily be extended to the Hilbert $C^*$-module setting.

**Proof of Theorem 2.** Let $\varphi \in \mathcal{S}(\mathbb{R})$. Then by Theorem 3

$$
\|\varphi(\Delta_H)u\|_2^2 \leq C\left(\|\Delta_H \varphi(\Delta_H)u\|^2 + \|\varphi(\Delta_H)u\|^2\right) \leq C_1(\varphi)\|u\|^2, \quad u \in C^\infty(M).
$$

Therefore, the operator $\varphi(\Delta_H)$ defines an operator from $L^2(M, \mu)$ to $H^s(\mathcal{F})$. Repeating this argument, we find that for any $\varphi \in \mathcal{S}(\mathbb{R})$ the operator $\varphi(\Delta_H)$ defines an operator from $L^2(M, \mu)$ to $H^s(\mathcal{F})$ and, by duality, from $H^{-s}(\mathcal{F})$ to $L^2(M, \mu)$ for any $s \geq 0$. It remains to show that for any $\varphi \in \mathcal{S}(\mathbb{R})$ the operator $\varphi(\Delta_H)$ defines an operator from $H^{-t}(\mathcal{F})$ to $H^s(\mathcal{F})$ for any $s, t \geq 0$.

The operator $\Delta_H + 1$ is invertible in $L^2(M, \mu)$, and by Theorem 3 the inverse $(\Delta_H + 1)^{-1}$ acts from $L^2(M, \mu)$ to $H^s(\mathcal{F})$:

$$
\|(\Delta_H + 1)^{-1}u\|_{\epsilon} \leq C\|u\|, \quad u \in C^\infty(M).
$$

Using Theorem 3 repeatedly, we obtain that, for any natural number $N$, the operator $(\Delta_H + 1)^{-N}$ acts from $L^2(M, \mu)$ to $H^{N\epsilon}(\mathcal{F})$ and, by duality, from $H^{-N\epsilon}(\mathcal{F})$ to $L^2(M, \mu)$:

$$
\|(\Delta_H + 1)^{-N}u\| \leq C\|u\|_{-N\epsilon}, \quad u \in C^\infty(M).
$$

Finally, for any $\varphi \in \mathcal{S}(\mathbb{R})$, $s > 0$ and natural number $N$, we get

$$
\|\varphi(\Delta_H)u\|_s = \|\varphi(\Delta_H)(\Delta_H + 1)^{N}(\Delta_H + 1)^{-N}u\|_s 
\leq C\|(\Delta_H + 1)^{-N}u\| \leq C\|u\|_{-N\epsilon}, \quad u \in C^\infty(M).
$$
Thus, the operator $\varphi(\Delta_H)$ defines an operator from $H^{-N_\epsilon}(\mathcal{F})$ to $H^s(\mathcal{F})$.

Theorem 2 is proved.

To conclude this section, we give the proof of Theorem 1 mentioned in the introduction, which is based on the theory of first order linear symmetric hyperbolic systems. Here we do not assume that the distribution $H$ defines a singular foliation.

\textbf{Proof of Theorem 1.} On the Hilbert space $H = L^2(M, \mu) \oplus L^2(M, H^*, \mu)$ consider the operator $A$, with domain $D(A) = C^\infty(M) \oplus C^\infty(M, H^*)$, given by the matrix

$$A = \begin{pmatrix} 0 & d_H^* \\ d_H & 0 \end{pmatrix}.$$ 

Observe that $A$ is symmetric. Applying Theorem 2.2 from [6] to the skew-symmetric operator $L = iA$, we see that every power of $A$ is essentially self-adjoint. Since

$$A^2 = \begin{pmatrix} d_H^*d_H & 0 \\ 0 & d_Hd_H^* \end{pmatrix},$$

the operator $d_H^*d_H$ is essentially self-adjoint on $C^\infty(M)$.

Theorem 1 is proved.

\section*{§ 3. Preliminaries}

In this section we give the information we need about singular foliations and describe the basic facts of pseudodifferential calculus on singular foliations, mostly given in [2] and [3] but adapted to a concrete representation in the $L^2$ space on the ambient manifold $M$.

\textbf{3.1. Foliations and bi-submersions.} Let $M$ be a smooth manifold. Consider the $C^\infty_c(M)$-module $C^\infty_c(M, TM)$ of smooth, compactly supported vector fields on $M$. As in [2], by a singular foliation $\mathcal{F}$ on $M$ we will mean a locally finitely generated $C^\infty_c(M)$-submodule of $C^\infty_c(M, TM)$, stable under Lie brackets. Here a submodule $\mathcal{E}$ of $C^\infty_c(M, TM)$ is said to be locally finitely generated if for any $p \in M$ there exists an open neighbourhood $U$ of $p$ in $M$ and vector fields $X_1, \ldots, X_k \in C^\infty_c(U, TU)$ such that, for any $f \in C^\infty_c(U)$ and $X \in C^\infty_c(M, TM)$, we have $fX|_U = \sum_{j=1}^k f_jX_j \in C^\infty_c(U, TU)$ for some $f_1, \ldots, f_k \in C^\infty_c(M)$.

Let $\mathcal{F}$ be a foliation on $M$ and $x \in M$. The tangent space of the leaf at $x$ is the image $F_x$ of $\mathcal{F}$ in $T_xM$ under the evaluation map $C^\infty_c(M, TM) \to T_xM$, $x \mapsto X(x)$. Put $I_x = \{f \in C^\infty_c(M) : f(x) = 0\}$. The fibre of $\mathcal{F}$ at $x$ is the quotient $\mathcal{F}_x = \mathcal{F}/I_x\mathcal{F}$. The evaluation map induces a short exact sequence of vector spaces

$$0 \to g_x \to \mathcal{F}_x \to F_x \to 0,$$

where $g_x$ is a Lie algebra. It can be shown that if, for $x \in M$, the images of $X_1, \ldots, X_n \in \mathcal{F}$ in $\mathcal{F}_x$ form a basis of $\mathcal{F}_x$, then there exists a neighbourhood $U$ of $x$ in $M$ such that the restriction of $\mathcal{F}$ to $U$ is generated by the restrictions of $X_1, \ldots, X_n$ to $U$.

For any smooth map $p : N \to M$ of a smooth manifold $N$ to $M$, we let $p^{-1}(\mathcal{F})$ denote the set of all vector fields on $N$ of the form $fY \in C^\infty_c(N, TN)$, where
f ∈ C^∞(N) and Y is a vector field on N, which is p-related to some X ∈ ℱ: dp_x(Y(x)) = X(p(x)) for any x ∈ N. It can be shown that p^{-1}(ℱ) is a submodule of C^∞(N,TN).

**Definition 1.** A bi-submersion of (M, ℱ) is a smooth manifold U endowed with two smooth maps s, t: U → M which are submersions and satisfy:

(a) s^{-1}(ℱ) = t^{-1}(ℱ);
(b) s^{-1}(ℱ) = C^∞(U, ker ds) + C^∞(U, ker dt).

**Definition 2.** A locally closed submanifold V ⊂ U is said to be an identity bisection of a bi-submersion (U,t,s) if the restriction s|_V: V → M (t|_V: V → M) of s (t, respectively) to V is a diffeomorphism onto an open subset s(V) (t(V), respectively) of M, and, moreover, s|_V = t|_V.

An important class of bi-submersions was constructed in [2], Proposition 2.10, (a). Let x ∈ M. Let X_1, ..., X_n ∈ ℱ be vector fields whose images in ℱ_x form a basis of ℱ_x. For y = (y_1, ..., y_n) ∈ R^n, put φ_y = exp(∑ y_i X_i) ∈ exp ℱ. Put W_0 = R^n × M, s_0(y,x) = x and t_0(y,x) = φ_y(x). It can be shown that there exists a neighbourhood W of (0,x) in W_0 such that (W, t, s) is a bi-submersion where s = s_0|_W and t = t_0|_W. Such a bi-submersion is called an identity bi-submersion.

A simple way to produce more bi-submersions, starting from the given one, is described in [2], Lemma 2.3. If (U,t,s) is a bi-submersion and p: W → U is a submersion, then (W,t ∘ p, s ∘ p) is a bi-submersion.

A morphism of bi-submersions (U_i,t_i,s_i), i = 1, 2, is a smooth map f: U_1 → U_2 such that, for any u ∈ U_1, s_1(u) = s_2(f(u)) and t_1(u) = t_2(f(u)). Any submersion p: W → U is a morphism of the bi-submersions (W,t ∘ p, s ∘ p) and (U,t,s).

As shown in [2], Proposition 2.10, (a), the identity bi-submersion W at x ∈ M provides a local model for any bi-submersion admitting a nonempty identity bisection containing x. More precisely, let (V,t_V,s_V) be a bi-submersion and W ⊂ V be an identity bisection. Then, for any v ∈ W with s_V(v) = x ∈ M, there exist an open neighbourhood V' of v in V and a submersion g: V' → W which is a morphism of bi-submersions, such that g(v) = (0,x).

A stronger statement is proved in [3], Lemma 2.5. Let (U_j,t_j,s_j), j = 1, 2, be bi-submersions, V_j ⊂ U_j identity bisections and let u_j ∈ V_j be such that s_1(u_1) = s_2(u_2). Then there exist an open neighbourhood U'_1 of u_1 in U_1 and a morphism of bi-submersions g: U'_1 → U_2 such that g(u_1) = u_2 and g(V_1 ∩ U'_1) ⊂ U_2.

Given any pair of bi-submersions (U_j,t_j,s_j), j = 1, 2, we define their composition (U_1,t_1,s_1) ∘ (U_2,t_2,s_2) = (U_1 ∘ U_2,t, s) as follows. The manifold U_1 ∘ U_2 is the fibre product

U_1 ∘ U_2 = U_1 ×_M U_2 = \{(u_1,u_2) ∈ U_1 × U_2: s_1(u_1) = t_2(u_2)\},

and s(u_1,u_2) = s_2(u_2) and t(u_1,u_2) = t_1(u_1). Given a bi-submersion (U,t,s), we define its inverse by (U,t,s)^{-1} = (U,s,t). We can show that U_1 ∘ U_2 and U^{-1} are bi-submersions.

Let ℳ_0 denote the set of bi-submersions generated by identity bi-submersions, that is, the minimal set of bi-submersions that contains all the identity bi-submersions and is closed under operations of composition and taking the inverse. ℳ_0 is called the path holonomy atlas.
Sometimes it is useful to extend the class of bi-submersions under consideration. We say that a bi-submersion \((W, t_W, s_W)\) is adapted to \(\mathcal{U}_0\) at \(w \in W\) if there exists an open subset \(W' \subset W\) containing \(w\), a bi-submersion \((U, t, s) \in \mathcal{U}_0\) and a morphism of bi-submersions \(W' \to U\). A bi-submersion \((W, t_W, s_W)\) is adapted to \(\mathcal{U}_0\) if for all \(w \in W\), \((W, t_W, s_W)\) is adapted to \(\mathcal{U}_0\) at \(w\).

3.2. Regularizing operators. From now on, we will assume that \(M\) is compact. In this section, we recall the definition of regularizing (or leafwise smoothing) operators on \(M\). Our constructions will be adapted to a certain Hilbert-space structure in \(L^2(M)\). Actually, we will describe a \(\ast\)-representation in \(L^2(M)\) of a certain involutive operator algebra associated with \(\mathcal{F}\), which was introduced in [2]. First, we fix a positive smooth density \(\mu\) on \(M\). For a vector space \(E\) and \(p \in (0, 1]\), we denote the space of \(p\)-densities on \(E\) by \(\Omega^pE\).

Suppose that \((U, t, s)\) is a bi-submersion. Denote the half-density bundle associated with the bundle \(\ker ds \otimes kernel dt\) on \(U\) by \(\Omega^{1/2}U\):

\[
\Omega^{1/2}U = \Omega^{1/2} \ker ds \otimes \Omega^{1/2} \ker dt.
\]

As was shown in [3], §3.2.1, for any quasi-invariant measure \(\mu\) on \(M\) there exists a measurable almost everywhere invertible section \(\rho^U\) of \(\Omega^{-1/2} \ker ds \otimes \Omega^{1/2} \ker dt\) on \(U\) such that for every \(f \in C_c(U, \Omega^{1/2}U)\)

\[
\int_M \left( \int_{s^{-1}(x)} (\rho^U)^{-1} \cdot f(u) \right) d\mu(x) = \int_M \left( \int_{t^{-1}(x)} \rho^U \cdot f(u) \right) d\mu(x).
\]

Here \((\rho^U)^{-1} \cdot f\) is a measurable section of \(\Omega^1 \ker ds\) on \(U\), which can be integrated along the fibres of \(s\), giving rise to a function on \(M\), and \(\rho^U \cdot f\) is a measurable section of \(\Omega^1 \ker dt\) on \(U\), which can be integrated along the fibres of \(t\).

If \(\mu\) is given by a smooth positive density on \(M\), \(\rho^U\) can be constructed in the following way, which also shows that it is smooth. First, for \(u \in U\), we have a short exact sequence

\[
0 \to \ker ds_u \to T_u U \xrightarrow{ds_u} TM_{s(u)} \to 0,
\]

which gives rise to an isomorphism

\[
\Omega^{1/2}T_u U \cong \Omega^{1/2} \ker ds_u \otimes \Omega^{1/2}TM_{s(u)}.
\]

Similarly, we get an isomorphism

\[
\Omega^{1/2}T_t U \cong \Omega^{1/2} \ker dt_u \otimes \Omega^{1/2}TM_{t(u)}.
\]

The smooth positive density \(\mu\) on \(M\) defines isomorphisms \(\Omega^{1/2}TM_{s(u)} \cong \mathbb{C}\) and \(\Omega^{1/2}TM_{t(u)} \cong \mathbb{C}\). Combining these isomorphisms we obtain a smooth invertible section \(\rho^U\) of the bundle \(\Omega^{-1/2} \ker ds \otimes \Omega^{1/2} \ker dt\).

Definition 3. Given a bi-submersion \((U, t_U, s_U)\), the regularizing operator \(R_U(k)\): \(L^2(M) \to L^2(M)\) associated with the longitudinal kernel \(k \in C_c^\infty(U, \Omega^{1/2}U)\) is defined as follows: for \(\xi \in L^2(M)\),

\[
R_U(k)\xi(x) = \int_{t^{-1}(x)} (\rho^U \cdot k)(u)\xi(s(u)), \quad x \in M.
\]
First, observe that two longitudinal kernels associated with the different bi-submersions can define the same operator in $L^2(M)$. Let $\varphi: M \to N$ be a submersion, and let $E$ be a vector bundle on $N$. Integration along the fibres gives rise to a linear map $\varphi_!: C_c(M, \Omega^1(\ker d\varphi) \otimes \varphi^*E) \to C_c(N, E)$ defined by

$$\varphi_!(f)(x) = \int_{\varphi^{-1}(x)} f, \quad x \in N.$$  

As was shown in [2], if $\varphi: U \to V$ is a morphism of bi-submersions which is a submersion, then for every $k \in C^\infty_c(U, \Omega^{1/2}U)$, we have $R_U(k) = R_V(\varphi(k))$. More generally, let $k_1 \in C^\infty_c(U_1, \Omega^{1/2}U_1)$ and $k_2 \in C^\infty_c(U_2, \Omega^{1/2}U_2)$. Assume that there exist a submersion $p: W \to U_1 \sqcup U_2$ that is a morphism of bi-submersions and a $k \in C^\infty_c(W, \Omega^{1/2}W)$ such that $p_!(k) = (k_1, k_2)$. Moreover, suppose that there exists a morphism $q: W \to V$ of bi-submersions, which is itself a submersion, such that $q(k) = 0$. Then $R_{U_1}(k_1) = R_{U_2}(k_2)$.

To describe the composition of regularizing operators, we first recall (see [2], §4.2) that for any bi-submersions $(U_j, t_j, s_j)$, $j = 1, 2$, there exists a canonical isomorphism

$$\Omega^{1/2}(U_1 \circ U_2)_{(u_1, u_2)} \cong \Omega^{1/2}(U_1)_{u_1} \otimes \Omega^{1/2}(U_2)_{u_2}.$$  

**Proposition 1** (see [2], §5.1). 1) Suppose that $k_1 \in C^\infty_c(U_1, \Omega^{1/2}U_1)$ and $k_2 \in C^\infty_c(U_2, \Omega^{1/2}U_2)$. Then

$$R_{U_1}(k_1) \circ R_{U_2}(k_2) = R_{U_1 \circ U_2}(k_1 \otimes k_2),$$

where $k_1 \otimes k_2 \in C^\infty_c(U_1, \Omega^{1/2}U_1) \otimes C^\infty_c(U_2, \Omega^{1/2}U_2) \cong C^\infty_c(U_1 \times U_2, \Omega^{1/2}(U_1 \circ U_2))$.

2) Suppose that $k \in C^\infty_c(U, \Omega^{1/2}U)$. Then

$$R_U(k)^* = R_{U^{-1}}(k^*),$$

where $k^* = \bar{k}$ via the canonical isomorphism $\Omega^{1/2}U^{-1} \cong \Omega^{1/2}U$.

### 3.3. Pseudodifferential operators.

In this section, we introduce the classes of pseudodifferential operators on $M$ associated with the singular foliation $\mathcal{F}$ and describe their properties, following [3]. We will keep the notation used in §3.2.

Let $(U, t, s)$ be a bi-submersion and $V \subset U$ the identity bisection. (Note that $V$ may be empty.) Let $p: N \to V$ be the normal bundle of the inclusion $V \hookrightarrow U$, that is, $N_v = T_vU/T_vV$, $v \in V$. Choose a tubular neighbourhood $(U_1, \phi)$ of $V$ in $U$. Thus, $U_1$ is a neighbourhood of $V$ in $U$ and $\phi: U_1 \to N$ is a local diffeomorphism such that $\phi(v) = (v, 0)$ for $v \in V$, and $d\phi|_V: T_vU \to T_{(v, 0)}N$ induces the identity isomorphism $N_v = T_vU/T_vV \cong N_v$. Let $p_N: N^* \to V$ be the conormal bundle. Set $N^*U_1 = \{(u, \eta) \in U_1 \times N^*: p(\phi(u)) = p_N(\eta)\}$.

The space $\mathcal{B}^m(U, V, \Omega^{1/2})$ of pseudodifferential kernels of order $m$ consists of all $k \in C^\infty_c(U, \Omega^{1/2}U) = C^\infty(U, \Omega^1(TU) \otimes \Omega^{-1/2}U)'$ of the form

$$\langle k, f \rangle = \int_U k_0(u)f(u) + (2\pi)^{-d} \int_{N^*U_1} e^{-i\langle \phi(u), \eta \rangle} \chi(u) \cdot a(p \circ \phi(u), \eta)f(u),$$

$$f \in C^\infty_c(U, \Omega^1(TU) \otimes \Omega^{-1/2}U),$$
where $k_0 \in C^\infty(U, \Omega^{1/2}U)$, $d = \text{rank } N = \dim U - \dim M$, $\chi \in C^\infty_c(U)$ is such that supp $\chi \subset U_1$ and $\chi \equiv 1$ in a neighbourhood of $V$, $a \in S^m_{cl}(V, N^*; \Omega^1 N^* \otimes \Omega^{1/2}U|_V)$.

Here, for any $v \in V$ the section $a(v, \cdot)$ is a smooth density on $N^*_v$ with values in the vector space $\Omega^{1/2}U_v$. In a local coordinate system on an open set $V_1 \subset \mathbb{R}^n \cong V_1 \subset V$ and a trivialization of the vector bundle $N^*$ over it, this smooth density can be written as $a(v, \eta)|d\eta|$, $v \in V_1$, $\eta \in \mathbb{R}^d$, where $a \in S^m(V \times \mathbb{R}^d; \Omega^{1/2}U)$ is a classical symbol of order $m$.

Note that in [3] the authors assume that the order $m$ is integer, but it is easy to see that all the results of [3] can easily be extended to the case of an arbitrary real $m$.

Remark 2. In [3], elements of $\mathcal{P}_c^m(U, V, \Omega^{1/2})$ are called generalized sections of the bundle $\Omega^{1/2}U$ with compact support and pseudodifferential singularities along $V$ of order $\leq m$. In fact, they are just conormal distributions on $U$ for the submanifold $V \subset U$ (see, for instance, [14]).

With any pseudodifferential kernel $k \in \mathcal{P}_c^m(U, V, \Omega^{1/2})$, we associate an operator $R_U(k) : C^\infty_c(s(U)) \to C^\infty_c(t(U))$ as follows: for $f \in C^\infty_c(s(U))$ we put

$$R_U(k)f(x) = R_U(k_0)f(x) + (2\pi)^{-d} \int_{N^*U^*_1} e^{-i\langle \phi(u), \xi \rangle} \chi(u) \rho_u^U : [a(p \circ \phi(u), \xi)]f(s_U(u)).$$

Here $N^*U^*_1 = \{(u, \eta) \in N^*U_1 : t(u) = x\}$.

Note that if $V$ is empty, then $R_U(k)$ is a regularizing operator. Using an appropriate cut-off function, the operator $R_U(k)$ can be uniquely extended to an operator $R(k)$ on $C^\infty(M)$.

Observe that the bundle $\Omega^1 N^* \otimes \Omega^{1/2}U|_V$ is canonically trivial. Indeed, since $V \subset U$ is an identity bisection of the bi-submersion $(U, t, s)$, by definition the restriction $s|_V : V \to M$ of $s$ to $V$ is a diffeomorphism onto an open subset $s(V)$. It follows that $d(s|_V)_v = ds_v|_{T_vV} : T_vV \cong T_{s(v)}M$. On the other hand, we have a short exact sequence $0 \to \ker ds_v \to T_vV \to T_{s(v)}M \to 0$, which implies that $\ker ds_v \cong T_vU/T_{s(v)}M$ hence implies that $T_vU/T_{s(v)}V = N_v$. Similarly, we get an isomorphism $\ker dt_v \cong N_v$. Therefore,

$$\Omega^1 N^* \otimes \Omega^{1/2}U|_V \cong \Omega^1 N^* \otimes \Omega^{1/2}(\ker ds)|_V \otimes \Omega^{1/2}(\ker dt)|_V \cong V \times \mathbb{C}.$$ 

Thus, we can consider the (full) symbol $a$ of the operator $R_U(k)$ as an element of $S^m_{cl}(V, N^*)$. The principal symbol $\tilde{\sigma}_m(R_U(k))$ of $R_U(k)$ is defined as the homogeneous component of degree $m$ of $a$:

$$\tilde{\sigma}_m(R_U(k))(v, \xi) = a_m(v, \xi), \quad v \in V, \quad \xi \in N^*_v \setminus \{0\}. \quad (3.1)$$

So $\tilde{\sigma}_m(R_U(k))$ is a smooth, degree $m$ homogeneous function on $N^*_v \setminus \{0\}$.

Definition 4. The class $\Psi^m(\mathcal{F})$ consists of operators $P$ in $C^\infty(M)$ of the form $P = \sum_{i=1}^d P_i$, where each operator $P_i$, $i = 1, \ldots, d$, has the form $P_i = R(k_i)$ and $k_i \in \mathcal{P}_c^m(U_i, V_i, \Omega^{1/2})$ for some bi-submersion $(U_i, t_i, s_i)$ and identity bisection $V_i \subset U_i$. 

In order to define the principal symbol of an operator from $\Psi^m(\mathcal{F})$, we first introduce the cotangent bundle of $\mathcal{F}$ as $\mathcal{F}^* = \bigcup_{x \in M} \mathcal{F}_x^*$, where, for any $x \in M$, $\mathcal{F}_x^*$ is the dual space of $\mathcal{F}_x$, the fibre of $\mathcal{F}$ at $x$. Note that $\mathcal{F}^*$ is not a vector bundle in the usual sense. It can be shown that $\mathcal{F}^*$ is a locally compact topological space.

Let $(U, t, s)$ be a bi-submersion and $V \subset U$ the identity bisection. Recall that $N \cong (\ker ds)|_V \cong (\ker dt)|_V$. Therefore, by Definition 1, for $v \in V$, $ds_v : N_v \to \mathcal{F}_x$, $x = s(v)$, is an epimorphism. So the dual map $ds^*_v$ embeds $\mathcal{F}^*_x$ in $N^*_v$. The longitudinal principal symbol of the operator $R_U(k)$ associated with the kernel $k \in P^m_c(U, V, \Omega^{1/2})$ is the homogeneous function $\sigma_m(R_U(k))$ of degree $m$ on $\mathcal{F}^* \setminus 0$, which is equal to zero on $\mathcal{F}^*_x \setminus \{0\}$ for $x \notin s(V)$ and for $x \in s(V)$ is defined on $\mathcal{F}^*_x \setminus \{0\}$ by

$$\sigma_m(R_U(k))(x, \xi) = \tilde{\sigma}_m(R_U(k))(v, ds^*_v(\xi)), \quad \xi \in \mathcal{F}^*_x \setminus \{0\}, \quad (3.2)$$

where $v = s^{-1}(x)$ and $\tilde{\sigma}_m(R_U(k)) \in C^\infty(N^* \setminus 0)$ is the (local) principal symbol of $R_U(k)$ defined by (3.1).

Extending the principal symbol map to $\Psi^m(\mathcal{F})$ by linearity we get the longitudinal principal symbol map $\sigma_m : \Psi^m(\mathcal{F}) \to C(\mathcal{F}^* \setminus 0)$. It can be shown that this map is well-defined.

**Theorem 5** (see Theorem 3.15 in [3]). *Given operators $P_i \in \Psi^{m_i}(\mathcal{F})$, $i = 1, 2$, their composition $P = P_1 \circ P_2$ lies in $\Psi^{m_1+m_2}(\mathcal{F})$ and $\sigma_{m_1+m_2}(P) = \sigma_{m_1}(P_1) \sigma_{m_2}(P_2)$.*

**Theorem 6.** *Given operators $P_i \in \Psi^{m_i}(\mathcal{F})$, $i = 1, 2$, the commutator $[P_1, P_2]$ lies in $\Psi^{m_1+m_2-1}(\mathcal{F})$.*

*Proof.* The proof of this theorem follows the arguments in [3], Theorem 3.15, using a slight modification of the proof of Proposition 1.10 in [3].

An operator $P \in \Psi^m(\mathcal{F})$ is said to be longitudinally elliptic if its longitudinal principal symbol $\sigma_m(P)$ is invertible.

**Theorem 7** (see Theorem 4.2 in [3]). *Given a longitudinally elliptic operator $P \in \Psi^m(\mathcal{F})$, there exists an operator $Q \in \Psi^{-m}(\mathcal{F})$ such that $1 - P \circ Q$ and $1 - Q \circ P$ are in $\Psi^{-\infty}(\mathcal{F})$.*

**Theorem 8** (see Theorem 5.3 in [3]). *Each operator $P \in \Psi^0(\mathcal{F})$ defines a bounded operator in $L^2(M)$.*

### 3.4. Examples

1. Suppose that $\mathcal{F}$ is a smooth foliation on a compact manifold $M$. Then we can define a bi-submersion $(U, t, s)$ as follows: $U = G$ is the holonomy groupoid of $\mathcal{F}$ (assume that it is Hausdorff) and $t, s : G \to M$ are the usual target and source maps of $G$. (We refer the reader to [21] and [22] for the basic concepts from the noncommutative geometry of foliations.) An identity bisection $V$ of this bi-submersion is given by the unit set $G^{(0)} \subset G$ of the groupoid $G$. The bundle $\Omega^{1/2}U$ is the leafwise half-density bundle associated with a natural 2$p$-dimensional foliation $\mathcal{G}$ on $G$, and the space $C^\infty_c(U, \Omega^{1/2}U)$ is a basic element for constructing operator algebras associated with $\mathcal{G}$. Finally, the space $\mathcal{P}^m_c(U, V, \Omega^{1/2})$ coincides with the space of kernels of $G$-pseudodifferential operators introduced in [7].
2. As above, suppose that $\mathcal{F}$ is a smooth foliation on a compact manifold $M$. Let $\phi: D \cong IP \times IQ$ and $\phi': D' \cong IP \times IQ$ be two compatible foliated charts on $M$ (here $I = (0, 1)$) and $W(\phi, \phi') \subset G \cong IP \times IP \times IQ$, the corresponding coordinate chart on the holonomy groupoid $G$ constructed in [7] (see also [21] and [22]). Then we have a bi-submersion $(U, t, s)$, where $U = W(\phi, \phi')$ and $t: W(\phi, \phi') \to D$ and $s: W(\phi, \phi') \to D'$ are the restrictions of the target and source maps of the holonomy groupoid $G$ to $W(\phi, \phi')$. In local coordinates, they are given by

$$t(x, x', y) = (x, y), \quad s(x, x', y) = (x', y), \quad (x, x', y) \in IP \times IP \times IQ.$$ 

In the charts $\phi$ and $\phi'$ the positive smooth density $\mu$ can be written as $\mu = \mu(x, y)|dx| |dy|$ and $\mu = \mu'(x', y')|dx'| |dy'|$, respectively. There are natural sections of the bundles $\Omega^{1/2} \ker ds$ and $\Omega^{1/2} \ker dt$, which can be written as $|dx|^{1/2}$ and $|dx'|^{1/2}$, respectively. Then $\rho^U \in C^\infty(U, \Omega^{-1/2} \ker ds \otimes \Omega^{1/2} \ker dt)$ is given by

$$\rho^U_{(x, x', y)} = \left(\frac{\mu'(x', y)}{\mu(x, y)}\right)^{1/2} |dx|^{-1/2} |dx'|^{1/2}, \quad (x, x', y) \in IP \times IP \times IQ.$$ 

Any kernel $k \in C^\infty_c(U, \Omega^{1/2}U)$ has the form $k = K(x, x', y)|dx|^{1/2} |dx'|^{1/2}$ with $K \in C^\infty_c(IP \times IP \times IQ)$, and the operator $R_U(k): C^\infty(D') \to C^\infty(D)$ is given by

$$R_U(k)f(x, y) = \int K(x, x', y) \left(\frac{\mu'(x', y)}{\mu(x, y)}\right)^{1/2} f(x', y) \, dx'.$$

In the case when $\phi = \phi'$, a nonempty identity bisection $V \subset W(\phi, \phi) \cong IP \times IP \times IQ$ is given by

$$V = \{(x, x', y) \in IP \times IP \times IQ : x = x'\} \cong IP \times IQ \cong D.$$ 

Then we have $N \cong IP \times IQ \times \mathbb{R}^p$ and can take a diffeomorphism $\phi: U_1 \subset U \to N$ in the form

$$\phi: (x, x', y) \in IP \times IP \times IQ \mapsto (x, y, x' - x) \in IP \times IQ \times \mathbb{R}^p.$$ 

Finally, a symbol $a \in S^m_{cl}(V, N^*)$ can be written as $a = a(x, y, \xi)$, where $(x, y, \xi) \in IP \times IQ \times (\mathbb{R}^p)^*$ and, for $f \in C^\infty_c(D') \cong C^\infty_c(IP \times IQ)$, the corresponding operator $P: C^\infty_c(D') \to C^\infty_c(D)$ is given by

$$Pf(x, y) = (2\pi)^{-p} \int_{IP} \int_{\mathbb{R}^p} e^{i(x - x', \xi)} \chi(x, x', y) a(x, y, \xi) f(x', y) \left(\frac{\mu'(x', y)}{\mu(x, y)}\right)^{1/2} |dx'| |d\xi|.$$ 

3. Suppose that $\mathcal{F}$ is a singular foliation on a compact manifold $M$. We will show that any vector field $X \in \mathcal{F}$, considered as a first order differential operator on $M$, belongs to $\Psi^1(\mathcal{F})$, and its principal symbol $\sigma_1(X) \in C(\mathcal{F}^* \setminus 0)$ is given by

$$\sigma_1(X)(\xi) = i(\xi, X), \quad \xi \in \mathcal{F}^*.$$ 

First, we consider an arbitrary bi-submersion $(U, t, s)$ and a nonempty identity bisection $V \subset U$ and assume that the support of $X \in \mathcal{F}$ lies in $s(V)$. 

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Since \( s: U \to M \) is a submersion, there exists a vector field \( \tilde{X} \in C^\infty(U, TU) \) such that \( ds_u(\tilde{X}(u)) = X(s(u)) \). Without loss of generality we can assume that \( \tilde{X} \in C^\infty(U, \ker dt) \). Indeed, \( \tilde{X} \) is defined up to \( C^\infty(U, \ker ds) \) and, by definition, \( \tilde{X} \in s^{-1}(\mathcal{F}) = C^\infty(U, \ker ds) + C^\infty(U, \ker dt) \). Then the restriction of \( \tilde{X} \) to \( V \) belongs to \( C^\infty(V, (\ker dt)|_V) \cong C^\infty(V, N) \), giving rise to a vector field \( \tilde{X} \in C^\infty(V, N) \).

Set
\[
a_X(v, \xi) = i(\xi, \tilde{X}(v)), \quad v \in V, \quad \xi \in N_v^*.
\]

The corresponding pseudodifferential operator
\[
P: C^\infty_c(s(U)) \to C^\infty_c(t(U))
\]
is given by
\[
Pf(x) = (2\pi)^{-d} \int_{Ux} \int_{N^*_p\phi(u)} e^{-i(\phi(u), \xi)} \chi(u) i(\xi, \tilde{X} \circ \phi(u)) \times f(s(u)) \rho^U \cdot 1, \quad x \in t(U),
\]
for \( f \in C^\infty_c(s(U)) \). Here \( 1 \) is a smooth section of the bundle \( \Omega^1 N^* \otimes \Omega^{1/2}U \) corresponding to its canonical trivialization. Thus, \( \rho^U \cdot 1 \) is a smooth section of \( \Omega^1 N^* \otimes \Omega^1 \ker dt \), which can be integrated over \( N^* U_1 \).

Since \( \tilde{X} \in C^\infty(U, \ker dt) \), it is tangent to \( U^x \), and we can use the formula
\[
\int_{Ux} (\tilde{X} F) \omega = - \int_{Ux} \text{div}\omega(\tilde{X}) F \omega,
\]
which holds for any function \( F \in C^\infty_c(U) \) and for any smooth positive section \( \omega \in C^\infty(U, \Omega^1 \ker dt) \). By this formula
\[
Pf(x) = (2\pi)^{-d} \int_{Ux} \int_{N^*_p\phi(u)} e^{-i(\phi(u), \xi)} (\tilde{X} + w)(\chi(u) f(s(u))) \rho^U \cdot 1
\]
for some \( w \in C^\infty(U) \). Now, using the inverse Fourier transform formula and observing that \( \phi(u) = 0 \iff u \in V \iff s_U(u) = x \), we obtain
\[
Pf(x) = (\tilde{X} + w)[f(s(u))]_{u=s^{-1}(x)} = Xf(x) + w(s^{-1}(x)) f(x).
\]
Thus, we conclude that \( X = P - (s^{-1})^* w \) has a kernel in \( \mathcal{P}^1(U, V, \Omega^{1/2}) \).

Using (3.2), we compute its longitudinal principal symbol:
\[
\sigma_1(X)(x, \xi) = a_X(v, ds_v^*(\xi)) = i(\xi, ds_v(\tilde{X}(v))) = i(\xi, X(x)), \quad \xi \in \mathcal{F}_x^*, \quad s(v) = x.
\]

Now take a finite family \( (U_\alpha, t_\alpha, s_\alpha) \), \( \alpha = 1, \ldots, d \), of bi-submersions equipped with identity bisections \( V_\alpha \subset U_\alpha \) such that \( M = \bigcup_{\alpha=1}^d s(V_\alpha) \), a partition of unity \( \phi_\alpha \in C^\infty(M) \), \( \alpha = 1, \ldots, d \), subordinate to the cover \( \{s(V_\alpha)\} \), \( \text{supp} \phi_\alpha \subset s(V_\alpha) \), and a family of smooth functions \( \psi_\alpha \in C^\infty(M) \), \( \alpha = 1, \ldots, d \), such that \( \text{supp} \psi_\alpha \subset s(V_\alpha) \), \( \phi_\alpha \psi_\alpha = \phi_\alpha \). Then we write \( X = \sum_{\alpha=1}^d \phi_\alpha X \psi_\alpha \) to see that \( X \) belongs to \( \Psi^1(\mathcal{F}) \).
3.5. Longitudinal Sobolev spaces. First, we observe that for any $s \in \mathbb{R}$ there exists a longitudinally elliptic operator $\Lambda_s$ of order $s$. To construct such an operator, first, as above, we take a finite family $(U_\alpha, t_\alpha, s_\alpha)$, $\alpha = 1, \ldots, d$, of bi-submersions equipped with identity bisections $V_\alpha \subset U_\alpha$ such that $M = \bigcup_{\alpha=1}^d s(V_\alpha)$, a partition of unity $\phi_\alpha \in C^\infty(M)$ subordinate to the cover of $M$, $\text{supp}\,\phi_\alpha \subset s(V_\alpha)$, and $\psi_\alpha \in C^\infty(M)$ such that $\text{supp}\,\psi_\alpha \subset s(V_\alpha)$, $\phi_\alpha \psi_\alpha = \phi_\alpha$. Then, for each $\alpha$, we consider an operator $P_\alpha$ defined by a pseudodifferential kernel $k_\alpha \in \mathcal{P}^s_c(U_\alpha, V_\alpha, \Omega^{1/2})$ with symbol $a(x, \xi) = (1 + |\xi|^s)$. Finally, we put $\Lambda_s = \sum_{\alpha=1}^d \phi_\alpha P_\alpha \psi_\alpha$.

We fix this operator $\Lambda_s$ for any $s$. Without loss of generality, we can assume that $\Lambda_s$ is formally self-adjoint and

$$\Lambda_s \circ \Lambda_{-s} = I + R_s \quad \text{and} \quad \Lambda_{-s} \circ \Lambda_s = I + R_s', \quad R_s, R_s' \in \Psi^{-\infty}(\mathcal{F}).$$

**Definition 5.** For $s \geq 0$ the Sobolev space $H^s(\mathcal{F})$ is defined as the domain of $\Lambda_s$ in $L^2(M)$:

$$H^s(\mathcal{F}) = \{ u \in L^2(M) : \Lambda_s u \in L^2(M) \}.$$

The norm in $H^s(\mathcal{F})$ is defined by the formula

$$\|u\|_s^2 = \|\Lambda_s u\|^2 + \|u\|^2, \quad u \in H^s(\mathcal{F}).$$

For $s < 0$ the space $H^s(\mathcal{F})$ is defined as the dual space of $H^{-s}(\mathcal{F})$.

Using Theorems 5 and 8 we immediately obtain the following result.

**Theorem 9.** For any $s \in \mathbb{R}$ an operator $A \in \Psi^m(\mathcal{F})$ determines a bounded operator $A : H^s(\mathcal{F}) \to H^{s-m}(\mathcal{F})$.

**Proposition 2.** For $s \in \mathbb{Z}$, the space $C^\infty(M)$ is dense in $H^s(\mathcal{F})$.

**Proof.** This can easily be proved using the standard Friedrichs’ mollifiers on $M$ (see [33], Ch. II, §7, for instance).

**Theorem 10.** Any formally self-adjoint longitudinally elliptic operator $P \in \Psi^m(\mathcal{F})$, $m > 0$, defines an unbounded self-adjoint operator in the Hilbert space $L^2(M, \mu)$ with domain $H^m(\mathcal{F})$.

We note that the results in this subsection can be obtained as consequences of the general results in [3], §6, applied to the natural representation of the $C^*$-algebra $C^*(M, \mathcal{F})$ of the singular foliation $\mathcal{F}$ on $L^2(M, \mu)$. In particular, the Sobolev space $H^k(\mathcal{F})$ is obtained as the image of $L^2(M, \mu)$ by the action of the Sobolev module $H^k \subset C^*(M, \mathcal{F})$ under this representation.

§ 4. The proofs of Theorems 3 and 4

The proof of Theorem 3 will closely follow Kohn’s proof of the subellipticity of the Hörmander’s operators [15] (see also [34] and [12]). We will keep the notation used in §2. The starting point is the following fact.

**Lemma 1.** For any $X \in C^\infty(M, H)$ there exists $C > 0$ such that

$$\|X u\|^2 \leq C((\Delta_H u, u) + \|u\|^2), \quad u \in C^\infty(M). \quad (4.1)$$
Proof. Let Ω be an open subset of M such that there exists a local orthonormal frame $X_1, \ldots, X_p \in C^\infty(\Omega, H|_{\Omega})$. Then, for any $u \in C^\infty_c(\Omega)$, we have

$$(\Delta_H u, u) = \|d_H u\|^2 = \sum_{j=1}^{p} \int_{\Omega} |X_j u(x)|^2 \, d\mu(x).$$

We can write $X(x) = \sum_{j=1}^{p} a_j(x) X_j(x)$, $x \in \Omega$ for some $a_j \in C^\infty(\Omega)$, $j = 1, \ldots, p$. Therefore, for any $u \in C^\infty_c(\Omega)$, we get

$$\|X u\|^2 = \int_{\Omega} |X u(x)|^2 \, d\mu(x) \leq C \sum_{j=1}^{p} \int_{\Omega} |X_j u(x)|^2 \, d\mu(x) = C(\Delta_H u, u).$$

To prove the estimate (4.1) in the general case we take a finite open cover $M = \bigcup_{\alpha=1}^{d} \Omega_\alpha$ of $M$ such that, for any $\alpha = 1, \ldots, d$, there exists a local orthonormal frame $X_1^{(\alpha)}, \ldots, X_p^{(\alpha)} \in C^\infty(\Omega_\alpha, H|_{\Omega_\alpha})$ and a partition of unity subordinate to this cover, and use the fact that for $\varphi \in C^\infty(M)$ the commutators $[X, \varphi]$ and $[d_H, \varphi]$ are zero order differential operators and are therefore bounded in $L^2$.

Lemma 1 is proved.

We start the proof of Theorem 3 by looking at the case $s = 0$.

**Proposition 3.** There exist $\epsilon > 0$ and $C > 0$ such that

$$\|u\|_\epsilon^2 \leq C(\|\Delta_H u\|^2 + \|u\|^2), \quad u \in C^\infty(M).$$

**Proof.** Let $\mathcal{P}$ be the set of all operators $P \in \Psi^0(\mathcal{F})$ such that there exist constants $\epsilon > 0$ and $C > 0$ such that

$$\|Pu\|_\epsilon^2 \leq C(\|\Delta_H u\|^2 + \|u\|^2), \quad u \in C^\infty(M). \quad (4.2)$$

We claim that $\mathcal{P}$ satisfies the following properties:

- (P0) $\bigcup_{m<0} \Psi^m(\mathcal{F})$ is in $\mathcal{P}$;
- (P1) $\mathcal{P}$ is a two-sided ideal in $\Psi^0(\mathcal{F})$;
- (P2) $\mathcal{P}$ is stable under taking adjoints;
- (P3) $X \Lambda_{-1} \in \mathcal{P}$ for $X \in C^\infty(M, H)$;
- (P4) if $P \in \mathcal{P}$ then $[X, P] \in \mathcal{P}$ for $X \in C^\infty(M, H)$.

**Proof of (P2).** First, observe that

$$\|\Lambda_\epsilon P^* u\|^2 = \langle PA_\epsilon^2 P^* u, u \rangle = \|\Lambda_\epsilon P u\|^2 + \langle (PA_\epsilon^2 P^* - P^* \Lambda_\epsilon^2 P) u, u \rangle.$$

It remains to note that $PA_\epsilon^2 P^* - P^* \Lambda_\epsilon^2 P \in \Psi^0(\mathcal{F})$ if $\epsilon < 1/2$.

**Proof of (P1).** First, observe that $\mathcal{P}$ is a left ideal by Theorem 9. Then, by (P2) it is a right ideal as well.

**Proof of (P3).** Using (4.1), we have

$$\|\Lambda_{-1} X u\|_1^2 \leq C \|X u\|_1^2 \leq C_1(\|\Delta_H u\|^2 + \|u\|^2),$$

which means that $\Lambda_{-1} X \in \mathcal{P}$. Therefore, $(\Lambda_{-1} X)^* = X^* \Lambda_{-1} \in \mathcal{P}$ by (P2).
Since $X^* = -X + c$ for some $c \in C^\infty(M)$, using (P0) and (P1) we get

$$X^* \Lambda_{-1} = -X \Lambda_{-1} + c \Lambda_{-1} \in \mathcal{P}.$$

**Proof of (P4).** Take $P \in \Psi^0(\mathcal{F})$ such that $P$ and $P^*$ satisfy (4.2) for some $\epsilon > 0$. For $\delta > 0$ we can write

$$\|[X, P]u\|_\delta^2 = \|[X, P]u, \Lambda_\delta^2[X, P]u\| + \|[X, P]u\|^2$$

$$= (XPu, T_{2\delta}u) - (PXu, T_{2\delta}u) + \|[X, P]u\|^2,$$ (4.3)

where $T_{2\delta} = \Lambda_\delta^2[X, P] \in \Psi^{2\delta}(\mathcal{F})$. The second term on the right-hand side of (4.3) satisfies

$$|(PXu, T_{2\delta}u)| = \|(Xu, P^*T_{2\delta}u)\| \leq \frac{1}{2}(\|Xu\|^2 + \|P^*T_{2\delta}u\|^2)$$

$$\leq \frac{1}{2}\|Xu\|^2 + \|T_{2\delta}P^*u\|^2 + \|[P^*, T_{2\delta}]u\|^2.$$

Assuming that $\delta < \min(1/2, \epsilon/2)$ we obtain

$$\|T_{2\delta}P^*u\|^2 \leq C\|P^*u\|^2_{2\delta} \leq C\|P^*u\|^2_\epsilon \leq C_1(\|\Delta_H u\|^2 + \|u\|^2)$$

and

$$\|[P^*, T_{2\delta}]u\|^2 \leq C\|u\|^2,$$

which proves the estimate

$$|(PXu, T_{2\delta}u)| \leq C_1(\|\Delta_H u\|^2 + \|u\|^2).$$ (4.4)

Similarly, the first term on the right-hand side of (4.3) satisfies

$$|(XPu, T_{2\delta}u)| = |(Pu, X^*T_{2\delta}u)| \leq |(Pu, XT_{2\delta}u)| + |(Pu, cT_{2\delta}u)|$$

$$\leq |(Pu, T_{2\delta} Xu)| + |(Pu, [X, T_{2\delta}]u)| + |(Pu, cT_{2\delta}u)|.$$

Now we proceed as follows, using $\delta < \epsilon/2$ and (4.1):

$$|(Pu, T_{2\delta}Xu)| = |(T_{2\delta}^* Pu, Xu)| \leq \|Pu\|_{2\delta} \|Xu\| \leq C_1(\|\Delta_H u\|^2 + \|u\|^2),$$

$$|(Pu, cT_{2\delta}u)| = |(T_{2\delta}^* c^* Pu, u)| \leq C\|Pu\|_{2\delta} \|u\| \leq C_1(\|\Delta_H u\|^2 + \|u\|^2)$$

and, finally,

$$|(Pu, [X, T_{2\delta}]u)| = |([X, T_{2\delta}]^* Pu, u)| \leq C\|Pu\|_{2\delta} \|u\| \leq C_1(\|\Delta_H u\|^2 + \|u\|^2).$$

We obtain

$$|(XPu, T_{2\delta}u)| \leq C_2(\|\Delta_H u\|^2 + \|u\|^2).$$ (4.5)

Plugging (4.4) and (4.5) into (4.3) we complete the proof of (P4).

Now we will complete the proof of Proposition 3. First, we claim that for any $X_1, \ldots, X_p \in C^\infty(M, H)$ the operator $[X_1, [X_2, \ldots, [X_{p-1}, X_p] \ldots]] \Lambda_{-1}$ belongs to $\mathcal{P}$. We proceed by induction. We write

$$[X_1, [X_2, \ldots, [X_{p-1}, X_p] \ldots]] = [X_1, Y], \quad Y = [X_2, \ldots, [X_{p-1}, X_p] \ldots]$$
and assume that, by the induction hypothesis, $Y\Lambda_{-1} \in \mathcal{P}$. Then we know from (P4) that $[X_1, Y\Lambda_{-1}] \in \mathcal{P}$. On the other hand we can write

$$[X_1, Y\Lambda_{-1}] = [X_1, Y]\Lambda_{-1} + Y[X_1, \Lambda_{-1}].$$

Since $\Lambda_{-1}\Lambda_1 = I + R_1$ with $R_1 \in \Psi^{-\infty}(\mathcal{F})$, we obtain

$$Y[X_1, \Lambda_{-1}] = Y\Lambda_{-1}\Lambda_1[X_1, \Lambda_{-1}] - YR_1[X_1, \Lambda_{-1}].$$

By (P0) and (P1), this immediately implies that $Y[X_1, \Lambda_{-1}] \in \mathcal{P}$ since $Y\Lambda_{-1} \in \mathcal{P}$, $\Lambda_1[X_1, \Lambda_{-1}] \in \Psi^0(\mathcal{F})$ and $YR_1[X_1, \Lambda_{-1}] \in \Psi^{-\infty}(\mathcal{F})$. Thus we conclude that $[X_1, Y]\Lambda_{-1}$ belongs to $\mathcal{P}$, which completes the proof.

By assumption, the $C^\infty(M)$-module $\mathcal{F}$ is generated by a finite set of vector fields $Y_1, \ldots, Y_N$ on $M$. Consider the operator $\Delta = \sum_{k=1}^N Y_k^* Y_k$, a Laplacian associated with $\mathcal{F}$. It is a formally self-adjoint, longitudinally elliptic, second order differential operator. Let $Q \in \Psi^{-2}(\mathcal{F})$ be its parametrix, that is, $Q\Delta = I - K_1$, $\Delta Q = I - K_2$, $K_1, K_2 \in \Psi^{-\infty}(\mathcal{F})$. Then we have

$$I = \sum_{j=1}^N QY_j^* Y_j + K_1.$$ 

Since $QY_j^* \in \Psi^{-1}(\mathcal{F})$, it follows from (P3) that $QY_j^* Y_j \in \mathcal{P}$. By (P0), $K_1 \in \mathcal{P}$. So $I \in \mathcal{P}$, which completes the proof of Proposition 3.

Now we extend the subelliptic estimates of Proposition 3 to arbitrary $s$, completing the proof of Theorem 3.

**Proof of Theorem 3.** By Proposition 3

$$\|u\|_{s+\epsilon}^2 \leq c(\|\Lambda_s u\|^2_\epsilon + \|u\|^2_s) \leq C(\|\Delta_H \Lambda_s u\|^2_\epsilon + \|u\|^2_s).$$

It remains to show that

$$\|\Delta_H \Lambda_s u\|^2 \leq C_s'(\|\Delta_H u\|^2_\epsilon + \|u\|^2_s). \quad (4.6)$$

**Lemma 2.** The operator $[\Delta_H, \Lambda_s]$ can be represented in the form

$$[\Delta_H, \Lambda_s] = \sum_{k=1}^N T_k^s X_k + T_0^s,$$

where $X_k \in C^\infty(M, H)$, $k = 1, \ldots, N$, and $T_k^s \in \Psi^s(\mathcal{F})$, $k = 0, \ldots, N$.

**Proof.** Let $M = \bigcup_{\alpha=1}^d \Omega_\alpha$ be a finite open cover of $M$ such that, for any $\alpha = 1, \ldots, d$, there exists a local orthonormal frame $X_1^{(\alpha)}, \ldots, X_p^{(\alpha)} \in C^\infty(\Omega_\alpha, H|_{\Omega_\alpha})$. As mentioned above, the restriction of $\Delta_H$ to $\Omega_\alpha$ can be written as

$$\Delta_H|_{\Omega_\alpha} = \sum_{j=1}^p (X_j^{(\alpha)})^* X_j^{(\alpha)}.$$
Let \( \phi_\alpha \in C^\infty(M) \) be a partition of unity subordinate to the cover, \( \text{supp} \phi_\alpha \subset U_\alpha \), and let \( \psi_\alpha \in C^\infty(M) \) be such that \( \text{supp} \psi_\alpha \subset U_\alpha \), \( \phi_\alpha \psi_\alpha = \phi_\alpha \). Then

\[
\Delta_H = \sum_{\alpha=1}^{d} \phi_\alpha(\Delta_H|_{\Omega_\alpha})\psi_\alpha = \sum_{\alpha=1}^{d} \sum_{j=1}^{p} \phi_\alpha(X_j^{(\alpha)})^* X_j^{(\alpha)} \psi_\alpha
\]

\[
= \sum_{\alpha=1}^{d} \sum_{j=1}^{p} \phi_\alpha(X_j^{(\alpha)})^* \psi_\alpha X_j^{(\alpha)} + \sum_{\alpha=1}^{d} \sum_{j=1}^{p} \phi_\alpha(X_j^{(\alpha)})^*[X_j^{(\alpha)}, \psi_\alpha].
\]

We can write

\[
\phi_\alpha(X_j^{(\alpha)})^* \psi_\alpha X_j^{(\alpha)} \Lambda_s = \phi_\alpha(X_j^{(\alpha)})^* \Lambda_s \psi_\alpha X_j^{(\alpha)} + \phi_\alpha(X_j^{(\alpha)})^*[\psi_\alpha X_j^{(\alpha)}, \Lambda_s]
\]

\[
= \Lambda_s \phi_\alpha(X_j^{(\alpha)})^* \psi_\alpha X_j^{(\alpha)} + [\phi_\alpha(X_j^{(\alpha)})^*, \Lambda_s] \psi_\alpha X_j^{(\alpha)} + [\psi_\alpha X_j^{(\alpha)}, \Lambda_s] \phi_\alpha(X_j^{(\alpha)})^* + [\phi_\alpha(X_j^{(\alpha)})^*, [\psi_\alpha X_j^{(\alpha)}, \Lambda_s]].
\]

Since \( (X_j^{(\alpha)})^* = -X_j^{(\alpha)} + c_j^{(\alpha)} \) for some \( c_j^{(\alpha)} \in C^\infty(M) \), we obtain

\[
\Delta_H \Lambda_s = \Lambda_s \Delta_H + \sum_{\alpha=1}^{d} \sum_{j=1}^{p} T_{1,j}^{s,(\alpha)} \psi_\alpha X_j^{(\alpha)} + \sum_{\alpha=1}^{d} \sum_{j=1}^{p} T_{2,j}^{s,(\alpha)} \phi_\alpha X_j^{(\alpha)} + T_0^s,
\]

where the operators

\[
T_{1,j}^{s,(\alpha)} = [\phi_\alpha(X_j^{(\alpha)})^*, \Lambda_s], \quad T_{2,j}^{s,(\alpha)} = -[\psi_\alpha X_j^{(\alpha)}, \Lambda_s]
\]

and

\[
T_0^s = \sum_{\alpha=1}^{d} \sum_{j=1}^{p} \left( [\psi_\alpha X_j^{(\alpha)}, \Lambda_s] \phi_\alpha c_j^{(\alpha)} + [\phi_\alpha(X_j^{(\alpha)})^*, [\psi_\alpha X_j^{(\alpha)}, \Lambda_s]] + [\psi_\alpha X_j^{(\alpha)}][X_j^{(\alpha)}, \psi_\alpha], \Lambda_s] \right)
\]

belong to \( \Psi^s(\mathcal{F}) \). Setting

\[
\{X_k, k = 1, \ldots, N\} = \{\psi_\alpha X_j^{(\alpha)}, \phi_\alpha X_j^{(\alpha)}, \alpha = 1, \ldots, d, j = 1, \ldots, p\},
\]

where \( N = 2dp \), completes the proof of the lemma.

It follows from Lemma 2 that there exists \( C > 0 \) such that

\[
\|\Delta_H \Lambda_s u\|^2 \leq C \left( \|\Delta_H u\|^2_s + \sum_{k=1}^{N} \|X_k u\|^2_s + \|u\|^2_s \right), \quad u \in C^\infty(M), \quad (4.7)
\]

For any \( k \) we have

\[
\|X_k u\|^2_s = \|\Lambda_s X_k u\|^2 + \|X_k u\|^2 \leq \|X_k \Lambda_s u\|^2 + \|[\Lambda_s, X_k] u\|^2 + \|X_k u\|^2
\]

\[
\leq \|X_k \Lambda_s u\|^2 + (\Delta_H u, u) + C\|u\|^2_s. \quad (4.8)
\]
Next, by (4.1), it follows that
\[ \|X_k \Lambda_su\|^2 \leq C(\|\Delta_H \Lambda_su, \Lambda_su\| + \|u\|^2_s) \]
\[ = C((\Delta_Hu, u)_s + ([\Delta_H, \Lambda_s]u, \Lambda_su) + \|u\|^2_s) \]
\[ = C\left( (\Delta_Hu, u)_s + \left( \sum_{k=1}^{N} T_k^s X_k + T_0^s \right) u, \Lambda_su \right) + \|u\|^2_s \]
\[ \leq C_1 \left( \|\Delta_Hu\|^2_s + \sum_{k=1}^{N} \|X_k u\||u||_s + \|u\|^2_s \right) \]
\[ \leq \varepsilon \sum_{k=1}^{N} \|X_k u\|^2_s + C_2(\varepsilon)(\|\Delta_Hu\|^2_s + \|u\|^2_s) \]  \hspace{1cm} (4.9)
for any \( \varepsilon > 0 \) and some \( C_2(\varepsilon) > 0 \). From (4.8) and (4.9) we immediately obtain
\[ \sum_{k=1}^{N} \|X_k u\|^2_s \leq C(\|\Delta_Hu\|^2_s + \|u\|^2_s). \]  \hspace{1cm} (4.10)
Plugging (4.10) into (4.7) we obtain (4.6).

**Proof of Theorem 4.** Following the standard construction of Friedrichs’ mollifiers (see, for instance, [33], Ch. II, § 7, or [34], Ch. II, § 4), we can construct a bounded family \( \{ J_\varepsilon \} \), \( 0 < \varepsilon \leq 1 \), of operators in \( \Psi^{-\infty}(\mathcal{F}) \) such that \( J_\varepsilon u \to u \) in \( L^2(M) \) as \( \varepsilon \to 0 \) for any \( u \in L^2(M) \) and, for any \( A \in \Psi^m(\mathcal{F}) \), the commutators \([A, J_\varepsilon] \in \Psi^{-\infty}(\mathcal{F})\), \( 0 < \varepsilon \leq 1 \), form a bounded family of operators in \( \Psi^{m-1}(\mathcal{F}) \). More precisely, we first construct such a family locally. Let \( (U, t, s) \) be a bi-submersion and \( V \subset U \) the identity bisection. In the notation of § 3.3 take a function \( \rho \in C^\infty(N^*) \) supported in a tubular neighbourhood \( \phi(U_1) \) in \( N^* \) such that \( \rho|_V \equiv 1 \). It can be shown that the operator family \( J_\varepsilon \), \( 0 < \varepsilon \leq 1 \), where the operator \( J_\varepsilon \) is defined by
\[ J_\varepsilon = R_U(k_\varepsilon) \]
with \( k_\varepsilon, 0 = 0 \) and \( a_\varepsilon(v, \eta) = \rho(v, \varepsilon \eta), v \in V, \eta \in N_v \), satisfies the desired conditions. The globally defined operator family \( J_\varepsilon \in \Psi^{-\infty}(\mathcal{F}), 0 < \varepsilon \leq 1 \), is obtained from families of this kind, constructed locally, by the usual gluing procedure (see, for instance, Example 3 of § 3.4).

As an easy consequence, we see that, for any \( s \in \mathbb{R} \), \( J_\varepsilon u \to u \) in \( H^s(\mathcal{F}) \) as \( \varepsilon \to 0 \) for any \( u \in H^s(\mathcal{F}) \) and, for any \( A \in \Psi^m(\mathcal{F}) \) and \( B \in \Psi^{m'}(\mathcal{F}) \), the operators \([B, [A, J_\varepsilon]] \in \Psi^{-\infty}(\mathcal{F}), 0 < \varepsilon \leq 1 \), form a bounded family of operators in \( \Psi^{m+m'-2}(\mathcal{F}) \). Then the proof of Theorem 4 follows easily, if we proceed, for instance, as in the proof of Lemma 5.3 in [34], Ch. II.

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Yuri A. Kordyukov
Institute of Mathematics with Computing Centre
of the Ufa Scientific Centre
of the Russian Academy of Sciences, Ufa
E-mail: yurikor@matem.anrb.ru

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