Casimir Energy of a Ball
and Cylinder in the Zeta Function Technique

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Abstract

A simple method is proposed to construct the spectral zeta functions required
for calculating the electromagnetic vacuum energy with boundary conditions
given on a sphere or on an infinite cylinder. When calculating the Casimir
energy in this approach no exact divergencies appear and no renormalization
is needed. The starting point of the consideration is the representation of the
zeta functions in terms of contour integral, further the uniform asymptotic
expansion of the Bessel function is essentially used. After the analytic con-
tinuation, needed for calculating the Casimir energy, the zeta functions are
presented as infinite series containing the Riemann zeta function with rapidly
falling down terms. The spectral zeta functions are constructed exactly for
a material ball and infinite cylinder placed in an uniform endless medium
under the condition that the velocity of light does not change when crossing
the interface. As a special case, perfectly conducting spherical and cylindri-
cal shells are also considered in the same line. In this approach one succeeds,

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specifically, in justifying, in mathematically rigorous way, the appearance of
the contribution to the Casimir energy for cylinder which is proportional to
$\ln(2\pi)$.

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I. INTRODUCTION

A considerable achievement in theoretical investigations of the Casimir effect [1,2] was its calculation for massive fields (scalar and spinor) with boundary conditions on a sphere [3,4]. The various divergent contributions had been discussed in detail from the point of view of the general theory of adiabatic expansions (resp. heat kernel expansion). In a subsequent paper [5] it was clarified in which cases the calculation of the Casimir energy, after the proper renormalization, yields a meaningful (unique) result and in which not independently of the regularization used. For a massive field a well defined result can be obtained in any case using the normalization condition proposed there. Instead, for a massless field the heat kernel coefficient $a_2$ must vanish in order to allow for a meaningful calculation of the Casimir energy. For instance, this is the case for a material body characterized by a polarizability and a permittivity when the speeds of light inside and outside are the same or their difference is small. The vanishing of $a_2$ for the Dirichlet and Robin boundary conditions (and as a consequence for the conductor and bag boundary conditions) when taking inside and outside contributions together made Boyer’s [6] and all subsequent calculations possible and meaningful. When using a clever regularization (like the zetafunctional one [7]) it is even possible to avoid the appearance of divergencies other than that in the Minkowski space contribution at all.

Practically every problem on calculation of the Casimir energy (or force) has been considered multiply with employment of more and more effective and elaborated mathematical methods. For example, the first calculation of the Casimir energy of a perfectly conducting spherical shell carried out by T.H. Boyer in 1968 [6] has required computer calculations during 3 years [9]. Later this problem was considered in many papers [10–12]. By making use of the modern methods [13] it can be solved practically without numerical calculations (with a precision of a few percent). It requires only the application of the uniform asymptotic expansion for the Bessel functions.

In recent papers [14,15] the Casimir energy of a compact ball [16] and infinite cylinder has been calculated by making use of the mode–by–mode summation technique. In these problems two sums appear, over the roots of radial frequency equation at fixed value of angular momentum and then over angular momentum. The either of these sums is divergent. In papers [14,15] for each of these summation a separate regularization has been used. The first summation was carried out by applying the contour integration with subsequent subtraction of the contribution of an infinite homogeneous space. The second sum was evaluated by making use of the Riemann $\zeta$ function technique. However the procedure of analytic continuation, required by rigorous approach, has not been considered there.

The present paper pursues the aim to eliminate the minor points of preceding considerations, i.e., the Casimir energy for two configurations mentioned above will be calculated by the rigorous $\zeta$ function techniques, and the analytic continuation of the relevant spectral $\zeta$ functions will be carried out exactly. An essential advantage of this regularization procedure is that no manifestly divergent expressions arise in its framework, and it gives final finite result without any subtractions (renormalizations).

The layout of this paper is as follows. In Sec. II, the spectral zeta function is constructed for a compact ball placed in uniform endless medium when the light velocity is the same inside and outside the ball. As a special case the zeta function for perfectly conducting
spherical shell is also considered. In Sect. III the spectral functions for infinite cylinder are constructed under the same conditions. These results provide a firm footing for the previous calculations of the Casimir energy for given boundary conditions by making use of a "naive" zeta function method. In Sect. IV the obtained results are shortly discussed.

II. CASIMIR ENERGY OF A COMPACT BALL UNDER THE CONDITION \( \varepsilon \mu = 1 \)

In the \( \zeta \) function method [7,8] the Casimir energy \( E_C \) is defined in the following way. Let \( \omega_p \)'s be the eigenfrequencies of the quantum field system under the influence of the boundary conditions, and let \( \bar{\omega}_p \)'s be the same frequencies when the boundaries are removed. By making use of this spectrum one defines the \( \zeta \) function for the problem in hand
\[ \zeta(s) = \sum_{\{p\}} (\omega_p^{-s} - \bar{\omega}_p^{-s}) . \] (2.1)
Here the summation (or integration) should be done over all the quantum numbers \( \{p\} \) specifying the spectrum. The parameter \( s \) is considered at first to belong to region of the complex plane \( s \) where the sum (2.1) exists. Further the analytic continuation of (2.1) to the point \( s = -1 \) should be constructed. After that one puts
\[ E_C = \frac{1}{2} \zeta(s = -1) . \] (2.2)

Let us consider a solid ball of radius \( a \), consisting of a material which is characterized by permittivity \( \varepsilon_1 \) and permeability \( \mu_1 \). The ball is assumed to be placed in an infinite medium with permittivity \( \varepsilon_2 \) and permeability \( \mu_2 \). The eigenfrequencies of electromagnetic field for this configuration are determined by the frequency equation for the TE–modes \[ \Delta_{l}^{\text{TE}}(a\omega) \equiv \sqrt{\varepsilon_1\mu_2} \tilde{s}_l(k_1a) \tilde{e}_l(k_2a) - \sqrt{\varepsilon_2\mu_1} \tilde{s}_l(k_1a) \tilde{e}_l'(k_2a) = 0 , \] (2.3)
and the analogous equation for the TM–modes
\[ \Delta_{l}^{\text{TM}}(a\omega) \equiv \sqrt{\varepsilon_2\mu_1} \tilde{s}_l'(k_1a) \tilde{e}_l(k_2a) - \sqrt{\varepsilon_1\mu_2} \tilde{s}_l(k_1a) \tilde{e}_l'(k_2a) = 0 , \] (2.4)
where \( k_i = \sqrt{\varepsilon_i\mu_i} \omega, i = 1, 2 \) are the wave numbers inside and outside the ball, respectively. Here \( \tilde{s}_l(x) \) and \( \tilde{e}_l(x) \) are the Riccati–Bessel functions
\[ \tilde{s}_l(x) = \sqrt{\frac{\pi}{2x}} J_{l+1/2}(x) , \quad \tilde{e}_l(x) = \sqrt{\frac{\pi}{2x}} H_{l+1/2}^{(1)}(x) , \] (2.5)
and prime stands for the differentiation with respect to their arguments, \( k_1a \) or \( k_2a \). The orbital momentum \( l \) in Eqs. (2.3) and (2.4) assumes the values \( 1, 2, \ldots \).

As usual when one is dealing with an analytic continuation, it is convenient to represent the sum (2.1) in terms of the contour integral
\[ \zeta_C(s) = \frac{2l + 1}{2\pi i} \lim_{\mu \to 0} \oint_C dz (z^2 + \mu^2)^{-s/2} \frac{d}{dz} \ln \frac{\Delta_{l}^{\text{TE}}(az)\Delta_{l}^{\text{TM}}(az)}{\Delta_{l}^{\text{TE}}(\infty)\Delta_{l}^{\text{TM}}(\infty)} , \] (2.6)
where the contour $C$ surrounds, counterclockwise, the roots of the frequency equations in the right half-plane. For brevity we write in (2.6) simply $\Delta_l(\infty)$ instead of $\lim_{a \to \infty} \Delta_l(az)$.

Transition to the complex frequencies $z$ in Eq. (2.6) is accomplished by introducing the unphysical photon mass $\mu$

$$\omega \to (z^2 + \mu^2)^{s/2}_{\mu \to 0}.$$  \hfill (2.7)

Extension to the complex $z$–plane of the frequency equations $\Delta_{l}^{\text{TE}}(az)$ and $\Delta_{l}^{\text{TM}}(az)$ should be done in the following way. In the upper (lower) half–plane the Hankel functions of the first (second) kind $H^{(2)}_\nu(az)$ ($H^{(1)}_\nu(az)$) must be used [18]. Location of the roots of Eqs. (2.3) and (2.4) enables one to deform the contour $C$ into a segment of the imaginary axis ($-i\Lambda, i\Lambda$) and a semicircle of radius $\Lambda$ in the right half–plane. When $\Lambda$ tends to infinity the contribution along the semicircle into $\zeta_{\text{ball}}(s)$ vanishes because the argument of the logarithmic function in the integrand tends in this case to 1. As a result we obtain

$$\zeta_{\text{ball}}(s) = -\frac{(2l + 1)a^s}{2\pi i} \lim_{\mu \to 0} \int_{-i\infty}^{+i\infty} dz (z^2 + \mu^2)^{-s/2} \frac{d}{dz} \ln \frac{\Delta_{l}^{\text{TE}}(z)\Delta_{l}^{\text{TM}}(z)}{\Delta_{l}^{\text{TE}}(i\infty)\Delta_{l}^{\text{TM}}(i\infty)}. \hfill (2.8)$$

Now we impose the condition that the velocity of light inside and outside the ball is the same

$$\varepsilon_1\mu_1 = \varepsilon_2\mu_2 = c^{-2}. \hfill (2.9)$$

Under this assumption the argument of the logarithm in (2.8) can be simplified considerably [14] with the result

$$\zeta_{\text{ball}}(s) = \left(\frac{c}{a}\right)^{-s} \sum_{l=1}^{\infty} (2l + 1) \frac{\sin(\pi s/2)}{\pi} \int_{0}^{\infty} dy y^{-s} \frac{d}{dy} \ln[1 - \xi^2 \sigma_l^2(y)], \hfill (2.10)$$

where

$$\xi = \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}, \quad \sigma_l(y) = \frac{d}{dy}[s_l(y) e_l(y)]. \hfill (2.11)$$

Here $s_l(y)$ and $e_l(y)$ are the modified Riccati–Bessel functions

$$s_l(x) = \sqrt{\pi x} I_{\nu}(x), \quad e_l(x) = \sqrt{2x/\pi} K_{\nu}(x), \quad \nu = l + 1/2. \hfill (2.12)$$

More details concerning the contour integral representation of the spectral $\zeta$ functions can be found in [3,19,21].

Further the analytic continuation of Eq. (2.10) is accomplished by expressing the sum over $l$ in terms of the Riemann $\zeta$ function. This cannot be done in a closed form. Making use of the uniform asymptotic expansion (UAE) for the Bessel functions in increase powers of $1/\nu$ enables one to construct the analytic continuation looked for in the form of the series, the terms of which are expressed through the Riemann $\zeta$ function. The problem of the convergence of this series does not arise because its terms go down very fast.
We demonstrate this keeping only two terms in UAE for the product of the Bessel functions
\[ I_\nu(\nu z)K_\nu(\nu z) \approx \frac{t}{2\nu} \left[ 1 + \frac{t^2(1 - 6t^2 + t^4}{8\nu^2} + \ldots \right], \quad t = \frac{1}{\sqrt{1 + z^2}}. \quad (2.13) \]

After changing the integration variable \( y = \nu z \) in Eq. (2.10) we substitute (2.13) into this formula and expand the logarithm function up to the order \( \nu^{-3} \) keeping at the same time only the terms linear in \( \xi^2 \). The last assumption is not principal. It is introduced for simplicity and in order to have possibility of a direct comparison with the results of Ref. [14]. Thus we have

\[
\frac{d}{dz}\ln \left\{ 1 - \xi^2 \left[ \frac{d}{dz}(zI_\nu(\nu z)K_\nu(\nu z)) \right]^2 \right\} = (2.14)
\]

\[
= \frac{3\xi^2}{2\nu^2} z t^8 + \frac{\xi^2}{16\nu^4} z t^8 (-12 + 216t^2 - 600t^4 + 420t^6) + O(\nu^{-6}).
\]

Integration over \( z \) can be done by making use of the formula

\[
\int_0^\infty z^{-\alpha-1} e^{-\beta z} dz = \frac{1}{2} \frac{\Gamma\left(\frac{\alpha + \beta}{2}\right)\Gamma\left(-\frac{\alpha}{2}\right)}{\Gamma\left(\frac{\beta}{2}\right)}. \quad (2.15)
\]

Also the properties of the \( \Gamma \) function

\[
\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z}, \quad \Gamma(1+z) = z\Gamma(z)
\]

prove to be useful. After simple calculations we arrive at the result

\[
\zeta_{ball}(s) \approx \frac{\xi^2}{32} \left(\frac{c}{a}\right)^{-s} (2 + s)(4 + s) \left( \sum_{l=1}^{\infty} \nu^{-1-s} + p(s) \nu^{-3-s} \ldots \right),
\]

\[
\nu = l + 1/2,
\]

where

\[
p(s) = -\frac{1}{2} \left[ 1 - \frac{9}{4}(6 + s) + \frac{5}{8}(6 + s)(8 + s) - \frac{7}{192}(6 + s)(8 + s)(10 + s) \right]. \quad (2.18)
\]

The zeta function \( \zeta_{ball}(s) \) represented in the form (2.17) is defined for Re \( s > 0 \) due to the first sum over \( l \). This term corresponds to the order \( 1/\nu \) in the uniform asymptotic expansion (2.13). The second sum in (2.17), defined at Re \( s > -2 \), has been generated by the term \( \sim 1/\nu^3 \) in Eq. (2.13). It is clear that the terms of order \( 1/\nu^{2k+1} \) in (2.13) will give rise to the singularity of \( \zeta_{ball}(s) \) at the points \( s = -2k, \quad k = 0, 1, 2, \ldots \). Due to the multipliers in front of the square brackets in (2.17) the first three singularities are really the indefinitenesses like \( 0 \cdot \infty \).
The analytic continuation of Eqs. (2.17), (2.18) into the region $\text{Re } s \leq 0$ can be accomplished by expressing the sums over angular momentum $l$ through the Riemann $\zeta$ function according to the formula \[23\]

$$
\sum_{l=1}^{\infty} \nu^{-s} = (2^s - 1)\zeta(s) - 2^s, \quad \nu = l + 1/2.
$$

(2.19)

As a result one gets

$$
\zeta_{\text{ball}}(s) \simeq \frac{\xi^2}{32} \left(\frac{c}{a}\right)^s \frac{s(2+s)(4+s)}{(2^{1+s} - 1)\zeta(1+s) - 2^{1+s} - p(s)[(2^{3+s} - 1)\zeta(1+s) - 2^{3+s}] + \ldots}.
$$

(2.20)

The singularities in Eq. (2.17) are transformed in (2.20) into the poles of the Riemann $\zeta$ functions at the points $s = 2k$, $k = 0, 1, 2, \ldots$

$$
\zeta(1+s) \simeq \frac{1}{s} + \gamma + \ldots, \quad s \to 0,
$$

$$
\zeta(3+s) \simeq \frac{1}{s+2} + \gamma + \ldots, \quad s \to -2,
$$

$$
\zeta(5+s) \simeq \frac{1}{s+4} + \gamma + \ldots, \quad s \to -4,
$$

(2.21)

where $\gamma$ is the Euler constant. The first three poles are annihilated by the multipliers in front of the curly brackets in Eq. (2.20). The first surviving singularity (simple pole) appears only at the point $s = -6$. Thus the formula (2.20) affords the required analytic continuation of the function $\zeta_{\text{ball}}(s)$ into the region $\text{Re } s < 0$. In view of Eq. (2.2) we are interested in the point $s = -1$ where $\zeta_{\text{ball}}(s)$ given by (2.20) is regular

$$
\zeta_{\text{ball}}(-1) = \frac{3\xi^2 c}{32a} \left[1 + \frac{9}{128} \left(\frac{\pi^2}{2} - 4\right) + \ldots\right].
$$

(2.22)

It is exactly the first two terms in Eq. (3.10) of Ref. [14]. The procedure of analytic continuation presented above can be extended in a straightforward way to the arbitrary order of the uniform asymptotic expansion (2.13). Certainly in this case analytical calculations should be done by making use of Mathematica or Maple.

The problem under consideration with $\xi = 1$ is of a special interest because in this case it gives the Casimir energy of a perfectly conducting spherical shell. As it was noted above, this configuration has been considered by many authors. We present here the basic formulae which afford the analytical continuation of the corresponding spectral $\zeta$ function. We again content ourselves two terms in the UAE (2.13). In the next formula (2.14) it is impossible to put simply $\xi = 1$ in the next formula (2.14). One has to do the expansion here anew keeping all the terms $\sim 1/\nu^4$. This gives

$$
\frac{d}{dz} \ln \left\{1 - \left[\frac{d}{dz}(zI_\nu(\nu z)K_\nu(\nu z))\right]^2\right\} =
$$

(2.23)
\[
\sum_{l=1}^{\infty} \frac{1}{(2l+1)^2} = \frac{\pi^2}{8} - \frac{1}{12} + O(\nu^{-6})
\]

After integration and elementary simplifications we arrive at the following result for the spectral function in hand

\[
\zeta_{\text{shell}}(s) \simeq \frac{1}{32a^{-s}} \cdot 32a^{-s} (2 + s)(4 + s) \left[ \sum_{l=1}^{\infty} \nu^{-1-s} + q(s) \sum_{l=1}^{\infty} \nu^{-3-s} + \ldots \right], 
\]

where

\[
q(s) = \frac{1}{3840} (480 + 868s + 504s^2 + 71s^3). 
\]

Obviously formula (2.23) has the same singularities as Eq. (2.17), i.e., it is defined for \(\text{Re } s > 0\). The analytic continuation is accomplished by making use of Eq. (2.19)

\[
\zeta_{\text{shell}}(s) \simeq \frac{1}{32a^{-s}} \cdot 32a^{-s} (2 + s)(4 + s) \left\{ (2^{1+s} - 1)\zeta(1 + s) - 2^{1+s} + q(s)(2^{3+s} - 1)\zeta(3 + s) - 2^{3+s} \right\} + \ldots \}
\]

The nearest singularity in this formula is simple pole at \(s = -6\). As above it is originated in the term \(\sim 1/\nu^2\) in the UAE (2.13). At the point \(s = -1\) the spectral zeta function \(\zeta_{\text{shell}}(s)\) is regular and gives the following value for the Casimir energy of a perfectly conducting spherical shell

\[
E_{\text{shell}}(-1) = \frac{1}{2} \zeta_{\text{shell}}(-1) = \frac{3}{64a} \left[ 1 - \frac{3}{256} \left( \frac{\pi^2}{2} - 4 \right) + \ldots \right] = \frac{1}{a} 0.046361 \ldots . 
\]

Without considering the analytic continuation and do not carrying out the analysis of the singularities in the complex \(s\) plane this result has been obtained in [13].

Undoubtedly, the calculation of the Casimir energy of a nonmagnetic dielectric ball \((\varepsilon_1\mu_1 \neq \varepsilon_2\mu_2)\) by a rigorous \(\zeta\) function method is also of a special interest. However, in this case the very definition of the spectral zeta function should probably be changed in order to incorporate the contact terms which seem to be essential in this problem [24–26].

### III. Vacuum Energy of Electromagnetic Field with Boundary Conditions Given on an Infinite Cylinder

Calculation of the Casimir energy of an infinite cylinder [28,15] proves to be a more involved problem in comparison with that for sphere. In this section the spectral zeta function \(\zeta_{\text{cyl}}(s)\), for this configuration will be constructed its analytical continuation into the left half–plane of the complex variable \(s\) will be carried out, and relevant singularities will be analyzed.

Thus we are considering an infinite cylinder of radius \(a\) which is placed in an uniform unbounded medium. The permittivity and the permeability of the material making up the cylinder are \(\varepsilon_1\) and \(\mu_1\), respectively, and those for surrounding medium are \(\varepsilon_2\) and
\[ \mu_2. \] We assume again that the condition \( \text{[2.3]} \) is fulfilled. In this case the electromagnetic oscillations can again be divided into the transverse–electric (TE) modes and transverse–magnetic (TM) modes. In terms of the cylindrical coordinates \((r, \theta, z)\) the eigenfunctions of the given boundary value problem contain the multiplier

\[
\exp\left(\pm i\omega t + ik_z z + i\theta\right)
\]

(3.1)

and their dependence on \( r \) is described by the Bessel functions \( J_n \) for \( r < a \) and by the Hankel functions \( H_n^{(1)} \) or \( H_n^{(2)} \) for \( r > a \). The frequencies of TE– and TM–modes are determined, respectively, by the equations \[ \text{[17]} \]

\[
\Delta_n^{\text{TE}}(\lambda, a) \equiv \lambda [\mu_1 J_n'(\lambda a) H_n(\lambda a) - \mu_1 J_n(\lambda a) H_n'(\lambda a)] = 0,
\]

(3.2)

\[
\Delta_n^{\text{TM}}(\lambda, a) \equiv \lambda [\varepsilon_1 J_n'(\lambda a) H_n(\lambda a) - \varepsilon_1 J_n(\lambda a) H_n'(\lambda a)] = 0,
\]

(3.3)

where \( n = 0, \pm 1, \pm 2, \ldots \). Here \( \lambda \) is the eigenvalue of the corresponding transverse (membrane–like) boundary value problem

\[
\lambda^2 = \frac{\omega^2}{c^2} - k_z^2.
\]

(3.4)

In a complete analogy with the preceding Section we define the Casimir energy per unit length of the cylinder through the spectral zeta function

\[
E_{cyl} = \frac{1}{2} \zeta_{cyl}(-1).\]

(3.5)

Let \( \lambda_{nr} \) be the roots of the frequency equations \( \text{[3.2]} \) and \( \text{[3.3]} \), then the function \( \zeta_{cyl}(s) \) is introduced in the following way

\[
\zeta_{cyl}(s) = c^{-s} \int_{-\infty}^{+\infty} \frac{dk_z}{2\pi} \sum_{n,r} (\lambda_{nr}(a) + k_z^2)^{-s/2} - (\lambda_{nr}(\infty) + k_z^2)^{-s/2}.
\]

(3.6)

In terms of the contour integral it can be represented in the form

\[
\zeta_{cyl}(s) = c^{-s} \int_{-\infty}^{+\infty} \frac{dk_z}{2\pi} \sum_{n=-\infty}^{+\infty} \oint_C \frac{d\lambda}{(\lambda^2 + k_z^2)^{-s/2}} \ln \frac{\Delta_n^{\text{TE}}(\lambda a) \Delta_n^{\text{TM}}(\lambda a)}{\Delta_n^{\text{TE}}(\infty) \Delta_n^{\text{TM}}(\infty)}.
\]

(3.7)

Again we can take the contour \( C \) to consist of the imaginary axis \((+i\infty, -i\infty)\) closed by a semicircle of an infinitely large radius in the right half–plane. Continuation of the expressions \( \Delta_n^{\text{TE}}(\lambda a) \) and \( \Delta_n^{\text{TM}}(\lambda a) \) into the complex plane \( \lambda \) should be done in the same way as in the preceding Section, i.e., by using \( H_n^{(1)}(\lambda) \) for \( \text{Im} \lambda < 0 \) and \( H_n^{(2)}(\lambda) \) for \( \text{Im} \lambda > 0 \). On the semicircle the argument of the logarithm in Eq. \( \text{[3.7]} \) tends to 1. As a result this part of the contour \( C \) does not give any contribution into the zeta function \( \zeta_{cyl}(s) \). When integrating along the imaginary axis we choose the branch line of the function \( \phi(\lambda) = (\lambda^2 + k_z^2)^{-s/2} \) to run between \(-ik_z\) and \(+ik_z\), where \( k_z = +\sqrt{k_z^2} > 0 \) and use further that branch of this function which assumes real values when \(|y| < k_z\), where \( y = \text{Im} \lambda \). More precisely we have

\[
\phi(iy) = \begin{cases} 
  e^{-i\pi s/2}(y^2 - k_z^2)^{-s/2}, & y > k_z, \\
  (k_z^2 - y^2)^{-s/2}, & |y| < k_z, \\
  e^{i\pi s/2}(y^2 - k_z^2)^{-s/2}, & y < -k_z.
\end{cases}
\]

(3.8)
Employment of the Hankel functions $H_n^{(1)}(\lambda)$ and $H_n^{(2)}(\lambda)$ by extending the expressions $\Delta_n^{\text{TE}}(\lambda)$ and $\Delta_n^{\text{TM}}(\lambda)$ into the complex plane $\lambda$, as it was noted above, gives rise to the argument of the logarithm function depending only on $y^2$ on the imaginary axis. It means that the derivative of the logarithm is odd function of $y$. As a result the segment of the imaginary axis ($-ik_z, +ik_z$) gives zero, and Eq. (3.7) acquires the form

$$\zeta_{\text{cyl}}(s) = \frac{c^{-s}}{\pi^2} \sin \frac{\pi s}{2} \sum_{n=-\infty}^{+\infty} \int_{0}^{\infty} dk_z \int_{k_z}^{\infty} (y^2 - k_z^2)^{-s/2} dy \ln \frac{\Delta_{n\lambda}^{\text{TE}}(i\infty) \Delta_{n\lambda}^{\text{TM}}(i\infty)}{\Delta_{n\lambda}^{\text{TE}}(ay) \Delta_{n\lambda}^{\text{TM}}(ay)}. \quad (3.9)$$

Changing the order of integration of $k_z$ and $y$ and taking into account the value of the integral

$$\int_{0}^{y} dk_z (y^2 - k_z^2)^{-s/2} = \frac{\sqrt{\pi}}{2} y^{1-s} \frac{\Gamma \left( 1 - \frac{s}{2} \right)}{\Gamma \left( \frac{3-s}{2} \right)}, \quad \text{Re } s < 2, \quad (3.10)$$
we obtain after the substitution $ay \to y$

$$\zeta_{\text{cyl}}(s) = \frac{1}{2\sqrt{\pi} a \Gamma \left( \frac{s}{2} \right) \Gamma \left( \frac{3-s}{2} \right)} \left( \frac{c}{s} \right)^{-s} \sum_{n=-\infty}^{+\infty} \int_{0}^{\infty} dy y^{1-s} \frac{d}{dy} \ln[1 - \xi^2 \mu_n^2(y)], \quad (3.11)$$

where

$$\mu_n(y) = y(I_n(y)K_n(y))', \quad \xi = \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}. \quad (3.12)$$

We shall again content ourselves with the first two terms in the uniform asymptotic expansion (2.13) and take into account only the terms linear in $\xi^2$. In this approximation, upon changing the integration variable $y = nz, n = \pm 1, \pm 2, \ldots$, we have

$$\ln \left\{ 1 - \xi^2 \left[ z \frac{d}{dz} (I_n(nz)K_n(nz)) \right]^2 \right\} =$$

$$= -\xi^2 \frac{z^4 t^6}{4n^2} \left[ 1 + \frac{t^2}{4n^2} (3 - 30t^2 + 35t^4) + O(n^{-4}) \right]. \quad (3.13)$$

Now we substitute (3.13) into all the terms in (3.11) with $n \neq 0$. The term with $n = 0$ in this sum will be treated by subtracting and adding to the logarithmic function the quantity

$$-\xi^2 \frac{y^4}{4 (1 + y^2)^3}. \quad (3.14)$$

As a result the zeta function $\zeta_{\text{cyl}}(s)$ can be presented now as the sum of three terms

$$\zeta_{\text{cyl}}(s) = Z_1(s) + Z_2(s) + Z_3(s), \quad (3.15)$$

where
In view of the sum over 

In these equations $Z_1(s)$ has accumulated the term with $n = 0$ from Eq. (3.11) subtracted by (3.14); $Z_2(s)$ involves the contribution of the term of order $1/n^2$ in expansion (3.13) and the added expression (3.14); $Z_3(s)$ is generated by the terms of order $1/n^4$ in the expansion (3.13).

Taking into account that

the integration by parts in Eq. (3.16) can be done for $-3 < \text{Re } s < 1$ with the result

With allowance for (3.19) one infer easily that the function $Z_1(s)$ is an analytic function of the complex variable $s$ in the region $-3 < \text{Re } s < 1$. In the linear order of $\xi^2$ it reduces to

This function is also analytic in the region $-3 < \text{Re } s < 1$. Integration in Eq. (3.17) can be accomplished exactly by making use of the formula

This gives for $Z_2(s)$ in (3.17)

In view of the sum over $n$ in (3.23) the function $Z_2(s)$ is defined only for $s > 0$. 

\[ Z_1(s) = \frac{(c/a)^{-s}}{2\sqrt{\pi}a \Gamma \left(\frac{s}{2}\right) \Gamma \left(\frac{3-s}{2}\right)} \int_0^\infty dy y^{1-s} \frac{d}{dy} \left\{ \ln[1 - \xi^2 \mu_0^2(y)] + \frac{\xi^2}{4} y^4 t^6 \right\}, \]  

\[ Z_2(s) = -\xi^2 \left(\frac{c}{a}\right)^{-s} \frac{2}{8\sqrt{\pi}a \Gamma \left(\frac{s}{2}\right) \Gamma \left(\frac{3-s}{2}\right)} \int_0^\infty dz z^{1-s} \frac{d}{dz} (z^4 t^6), \]  

\[ Z_3(s) = -\xi^2 \frac{1}{32\sqrt{\pi}a \Gamma \left(\frac{s}{2}\right) \Gamma \left(\frac{3-s}{2}\right)} \int_0^\infty dz z^{1-s} \frac{d}{dz} (z^4 t^6 (3 - 30t^2 + 35t^4)). \]
For simplicity we apply in Eq. (3.18) the integration by parts which is correct for \(-3 < \text{Re} \, s < 2\) and leads to the result

\[
Z_3(s) = \xi^2 \left( \frac{c}{a} \right)^{-s} \frac{(1-s)(3-s)(7s^2 - 4s - 27)}{6144\sqrt{\pi} a} \frac{\Gamma\left( \frac{3+s}{2} \right)}{\Gamma\left( \frac{s}{2} \right)} \sum_{n=1}^{\infty} n^{-s-3}.
\] (3.24)

Again the sum over \(n\) in (3.24) gives the restriction \(\text{Re} \, s > -2\) for definition of the function \(Z_3(s)\).

Thus the spectral zeta function \(\zeta_{cyl}(s)\) in the linear approximation with respect to \(\xi^2\) and with allowance for the first two terms in the UAE (3.13) is given by

\[
\zeta_{cyl}(s) = Z_{1}^{\text{lin}}(s) + Z_2(s) + Z_3(s),
\] (3.25)

where the \(Z\)'s are presented in Eqs. (3.21), (3.23) and (3.24), respectively. Summing up all the restrictions on the complex variable \(s\) which have been imposed when deriving Eqs. (3.21), (3.23), and (3.24), we infer that \(\zeta_{cyl}(s)\) is defined in the strip \(0 < \text{Re} \, s < 1\). In order to continue these equations into the surroundings of the point \(s = -1\), it is sufficient to express the sum in Eq. (3.24) in terms of the Riemann \(\zeta\) function

\[
Z_2(s) = \xi^2 \left( \frac{c}{a} \right)^{-s} \frac{(1-s)(3-s)}{64\sqrt{\pi} a} \left[2\zeta(s+1) + 1\right] \frac{\Gamma\left( \frac{1+s}{2} \right)}{\Gamma\left( \frac{s}{2} \right)}.
\] (3.26)

It is left now to take the limit \(s \to -1\) in Eqs. (3.21), (3.23) and (3.24). A special care should be paid when calculating this limit in (3.24) in view of the poles of the function \(\Gamma((1+s)/2)\) at this point. Using the values

\[
\zeta(0) = -\frac{1}{2}, \quad \zeta'(0) = -\frac{1}{2}\ln 2\pi, \quad \Gamma'(1) = \gamma,
\] (3.27)

one derives

\[
\lim_{s \to -1} [2\zeta(1+s) + 1] \Gamma\left( \frac{1+s}{2} \right) = \lim_{s \to -1} [2\zeta(0) + 2\zeta'(0)(1+s) + O((1+s)^2) + 1] \left[ \frac{2}{1+s} + \gamma + O(1+s) \right] = -2\ln(2\pi).
\] (3.28)

With allowance for this we obtain from (3.26)

\[
Z_2(-1) = \frac{c\xi^2}{2\pi a^2} \frac{1}{4} \ln(2\pi).
\] (3.29)

The appearance of the finite term proportional to \(\ln(2\pi)\) is remarkable for the problem under consideration. It is derived here in a consistent way by making use of an analytic continuation of the relevant spectral zeta function. In Ref. [15] it was obtained in a more transparent though not rigorous way.

Gathering together Eqs. (3.21), (3.24) with \(s = -1\) and Eq. (3.24) we have
\[ \zeta_{\text{cyl}}(-1) = \frac{\xi^2}{2\pi a^2} \left\{ \int_0^\infty y dy \left[ \frac{y^4}{4(1+y^2)^3} - \mu_0^2(y) \right] + \frac{1}{48} \sum_{n=1}^{\infty} \frac{1}{n^2} + \frac{1}{4} \ln(2\pi) \right\} \]

\[ = \frac{\xi^2}{2\pi a^2} \left( -0.490878 + 0.034269 + 0.459469 \right) \]

\[ = \frac{\xi^2}{2\pi a^2} 0.002860. \quad (3.30) \]

This result is not the final answer in the problem in hand. The point is that in view of severe cancellations in (3.30) the contribution of the next term in the UAE (3.13) proves to be essential. Its account gives \[ \zeta_{\text{cyl}}(-1) = 0. \quad (3.31) \]

Thus the Casimir energy of a compact cylinder possessing the same speed of light inside and outside proves to be zero. The consideration presented in this Section can be extended to the next term of order \( \sim 1/n^6 \) in the UAE (3.13) in a straightforward way. Therefore we shall not present here these rather cumbersome expressions \[27\].

Now we address to the consideration of a special case when \( \xi = 1 \). It corresponds to a perfectly conducting cylindrical shell \[15\]. Instead of the expansion (3.13) we have

\[ \ln \left\{ 1 - \left[ \frac{z}{\sqrt{\pi a^2}} (I_n(nz)K_n(nz)) \right]^2 \right\} = -\frac{z^4t^6}{4n^2} \left[ 1 + \frac{t^2}{4n^2} \left( 3 - 30t^2 + 35t^4 + \frac{1}{2} z^4t^4 \right) + O(n^{-4}) \right]. \quad (3.32) \]

Proceeding in the same way as above we obtain for the spectral zeta function concerned

\[ \zeta_{\text{cyl}}^{\text{shell}}(s) = Z_1(s) + Z_2(s) + Z_3(s), \quad (3.33) \]

where \( Z_1(s) \) is given by Eq. (3.20) with \( \xi = 1 \), \( Z_2(s) \) is the same as in Eq. (3.26), and \( Z_3(s) \) now is

\[ Z_3(s) = \frac{(1-s)(3-s)(71s^2 - 52s - 235)}{61440\sqrt{\pi a^1-s}} \frac{\Gamma \left( \frac{3+s}{2} \right)}{\Gamma \left( \frac{s}{2} \right)} \sum_{n=1}^{\infty} n^{-3-s}. \quad (3.34) \]

At the point \( s = -1 \) it has the value

\[ \zeta_{\text{cyl}}^{\text{shell}}(-1) = \frac{1}{2\pi a^2} (-0.6517) + \frac{1}{2\pi a^2} \frac{7}{480} \sum_{n=1}^{\infty} \frac{1}{n^2} + \frac{1}{8\pi a^2} \ln(2\pi) \]

\[ = \frac{1}{2\pi a^2} (-0.6517 + 0.0240 + 0.4595) \]

\[ = -\frac{1}{a^2} 0.0268. \quad (3.35) \]

This exactly reproduce the contribution of the first two terms in calculations of the Casimir energy for cylindrical shell in Ref. \[15\]. With higher accuracy this energy is given by \[28\]
\[ E_{cyl}^{shell} = -\frac{1}{a^2} 0.01356. \] (3.36)

In a recent paper [29] the vacuum energy of a perfectly conducting cylindrical surface has been calculated to much higher accuracy by making use of another version of the zeta function technique. By integrating over \( dk_z \) directly in Eq. (3.3) the authors reduced this problem to investigation of the zeta function for circle, which has been considered earlier by introducing the partial wave zeta functions for interior and exterior region separately. In this respect our approach dealing only with one spectral zeta function for given boundary conditions proves to be more simple and straightforward.

IV. CONCLUSION

The method for constructing the spectral zeta functions developed here proceeds from the contour integral representation with a subsequent employment of the uniform asymptotic expansions for the Bessel functions. Upon an analytic continuation the zeta functions prove to be presented as (infinite) series over the Riemann \( \zeta \) functions with rapidly decreasing terms (see, for example, Eqs. (2.20), (2.26)).

We did not pursue here the goal of obtaining high accuracy when calculating the Casimir energy. In fact we seek to present the consideration in such a form that no manifest divergencies appear. An obvious advantage of the regularization method in hand does not need any renormalization.

By treating the boundary condition given on an infinite cylinder, we have clearly demonstrated the importance of a consistent analytic continuation of the relevant spectral zeta function, in contrast to identifying simply the sum of the type \( \sum_{n=1}^{\infty} n^{-s} \) with Riemann \( \zeta \) function, in order to involve correctly the contributions to the Casimir energy proportional to \( \ln(2\pi) \).

Consideration in this framework of the same configuration of vacuum electromagnetic field but with different velocities of light inside and outside the boundaries probably will demand the modification of the definition of the spectral zeta functions for incorporating in a proper way the contact terms important in this case [24–26].

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