Mean-square boundedness of stochastic networked control systems with bounded control inputs

Debasish Chatterjee, Saurabh Amin, Peter Hokayem, John Lygeros, and S. Shankar Sastry

Abstract—We consider the problem of controlling marginally stable linear systems using bounded control inputs for networked control settings in which the communication channel between the remote controller and the system is unreliable. We assume that the states are perfectly observed, but the control inputs are transmitted over a noisy communication channel. Under mild hypotheses on the noise introduced by the control communication channel and large enough control authority, we construct a control policy that renders the state of the closed-loop system mean-square bounded.

Index Terms—linear systems, stochastic stability, communication constraints

I. INTRODUCTION

Communication channels have become ubiquitous in control applications such as remotely operated robotic systems [1]. In such applications, measurement and control signals are exchanged via a lossy and noisy communication channel, which makes the system a networked control system (NCS). The research in NCS has branched into many different directions that deal with the effects of delays, limited information exchanged, and information losses on the stability of the plant, see, e.g., [2] and the references therein.

Control under information loss in the communication channel has been extensively studied within the Linear Quadratic Gaussian (LQG) framework [3], [4]. Typically, the communication channel(s) are modeled by an independent and identically distributed (i.i.d) Bernoulli process, which assign probabilities to the successful transmission of packets. Perhaps the most well known result in this setting is: When the transmission of sensor and control data packets happens over a network with TCP-like protocols, the closed-loop system under LQG controller can be mean-square stabilized provided that the probabilities of successful transmission are above a certain threshold. Since the TCP-like protocols enable the receiver to obtain an acknowledgment of whether or not the packets were successfully transmitted, the separation principle holds and the optimal LQG controller is linear in the estimated state. Thus, this result is a proper generalization of the classical LQG control problem to the networked control setting.

Within the LQG setting, control inputs are not assumed bounded and therefore linear state feedback is a permissible and optimal strategy. However, guaranteeing hard bounds on the control inputs is of paramount importance in applications. Consequently, many researchers have pursued the problem of optimal control and stabilization for linear systems with bounded control inputs [5], [6], [7], [8]. This problem has also received a renewed interest in recent years [9], [10], [11], [12], [13]. In the deterministic setting, it was argued in [15] that ensuring a mean-square bound for every initial condition is not possible for linear systems with bounded control elements if the system matrix $A$ is unstable. In [9] we established the existence of a policy with sufficiently large control authority that ensures mean-square boundedness of the states of the system under the assumption that $A$ is Lyapunov stable. Although Lyapunov stability of $A$ is a stronger requirement than the spectral radius of $A$ being at most 1, to the best of our knowledge, this is the current state of the art.

In this article we generalize the results of [9] to incorporate noisy control channels. We consider mean-square boundedness of stochastic linear systems under the following specification:

1) the communication channel between the controller and the system actuators is noisy whereas the communication channel between sensors and controller is noiseless, and
2) hard constraints on the control inputs must be satisfied.

We are thus concerned with a networked setting as proposed in [16], [17], [4] when generalized to incorporate bounded control inputs [9]. The control input $u^{(i)}$ for the $i$-th plant is communicated to the corresponding plant actuator via a lossy communication channel, which is characterized by the noise $\nu^{(i)}$ affecting the control input multiplicatively. We assume that the states are perfectly observed and are transmitted to the controller without any loss.

The remainder of this article is organized as follows. In Section II we formalize the main problem with all the underlying assumptions. In Section III we state the main
results and defer the proofs until Section IV. We conclude in Section V with a discussion of future work.

**Notation**

For any random vector \( \nu \) let \( \mu_\nu := E[\nu] \) denote its mean and \( \sigma_\nu := \text{var}(\nu) := E[[\nu - \mu_\nu]^2] \) denote its second moment. For a matrix \( M \) we let \( ||M|| \) denote the induced Euclidean norm of \( M \). We shall employ the standard notation \( \text{diam}(S) := \sup_{x,y \in S} ||x - y|| \) to denote the diameter of a subset \( S \) of Euclidean space. For \( n \in \mathbb{N} \), by \( 1_n \) we denote a vector of length \( n \) with all entries equal to 1. For \( r > 0 \) and \( n \in \mathbb{N} \), define the saturation function \( \text{sat}_r: \mathbb{R}^n \to \mathbb{R}^n \), as

\[
\text{sat}_r(z) = \begin{cases} 
  \frac{z}{rz/\|z\|} & \text{if } \|z\| < r \\
  z & \text{otherwise} 
\end{cases}
(1)
\]

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  \frac{z}{rz/\|z\|} & \text{if } \|z\| < r \\
  z & \text{otherwise} 
\end{cases}
(1)
\]

**II. Problem Setup**

Consider the following discrete-time stochastic linear system subjected to packet drops in the control communication channel

\[
x_{t+1} = Ax_t + B\tilde{u}_t + w_t, 
\]

\[
\tilde{u}_t := \nu_t \uplus u_t, 
\]

\[
t \in \mathbb{N}_0,
\]

where \( x_t \in \mathbb{R}^d \) is the state, \( u_t \in \mathbb{R}^m \) is the control input, \( A \in \mathbb{R}^{d \times d} \) is the dynamics matrix, \( B \in \mathbb{R}^{d \times m} \) is the input matrix, \( (w_t)_{t \in \mathbb{N}_0} \) is \( \mathbb{R}^d \)-valued random process noise, and \( (\nu_t)_{t \in \mathbb{N}_0} \) is an \( \mathbb{R}^m \)-valued random process modelling the uncertainty in the control communication channel, and \( \uplus \) denotes the Schur or Hadamard product of matrices.\(^1\) The initial condition \( x_0 = \bar{x} \) is given and the state \( x_t \) is perfectly observed by the controller.

The controller determines the control input \( u_t \) based on the history of \( k \) states \( X_{t,k} := (x_{t-k+1}, \ldots, x_{t-1}, x_t) \). (For \( t = 0, \ldots, k-2 \), \( X_{t,k} := (x_0, \ldots, x_{t+1}) \)), \( k \) times.

The controller synthesizes a deterministic control policy \( \pi = (\pi_t)_{t \in \mathbb{N}_0} \) which maps the states vector \( X \) into a control set \( \mathcal{U} \). To wit,

\[
u_t = \pi_t(X_{t,k}), 
\]

\[
t \in \mathbb{N}_0,
\]

where the maps \( \pi_t: \mathbb{R}^{kd} \to \mathcal{U} \subseteq \mathbb{R}^m \), \( t \in \mathbb{N}_0 \), are Borel measurable. Such a control policy \( \pi \) is known as a \textit{k-history dependent policy}. The control set \( \mathcal{U} \) is assumed to be nonempty, compact, and containing the origin. Any control policy \( \pi = (\pi_t)_{t \in \mathbb{N}_0} \) which guarantees that the control input sequence \( (u_t)_{t \in \mathbb{N}_0} \) satisfies

\[
u_t \in \mathcal{U}, 
\]

\[
t \in \mathbb{N}_0,
\]

is called an \textit{admissible k-history dependent policy}. In many practical situations involving saturating actuators and hard bounds on control inputs, \( \mathcal{U} \) is chosen to be a ball, i.e.,

\[
\mathcal{U} := \{ z \in \mathbb{R}^m \mid \|z\| \leq U_{\text{max}} \},
\]

where \( U_{\text{max}} > 0 \) is called the control authority available to the controller.

Our control objective is to synthesize an admissible \( k \)-history dependent policy which ensures that the second moment of the closed-loop system, for any initial condition \( \bar{x} \in \mathbb{R}^n \),

\[
x_{t+1} = Ax_t + B\nu_t \uplus \pi_t(X_{t,k}) + w_t, 
\]

\[
t \in \mathbb{N}_0,
\]

remains bounded for all \( t \in \mathbb{N}_0 \). We shall focus on the following problem:

\textbf{Problem 1:} Find, if possible, a control authority \( U_{\text{max}} \) and an admissible policy \( \pi = (\pi_t)_{t \in \mathbb{N}_0} \), with control authority \( U_{\text{max}} \), such that the following condition holds: for every initial condition \( \bar{x} \in \mathbb{R}^d \) there exists a constant \( \zeta > 0 \) such that the closed-loop system (5) satisfies

\[
E_x[\|x_t\|^2] \leq \zeta, \quad \forall t \in \mathbb{N}_0.
\]

In practice, a performance index that accounts for the average sum of cost-per-stage functions (involving the state and control inputs) of the system is often required to be minimized; however, in this article we are only concerned with the stability property defined in Problem 1.

We shall make the following standing hypotheses:

\textbf{Assumption 2:}

(i) The matrix \( A \) is Lyapunov stable, i.e., all the eigenvalues of \( A \) lie in the closed unit circle, and all eigenvalues \( \lambda \) satisfying \( |\lambda| = 1 \) have equal algebraic and geometric multiplicities.

(ii) The pair \((A, B)\) is stabilizable.

(iii) The process noise \((w_t)_{t \in \mathbb{N}_0}\) is an independent sequence, and has bounded fourth moment, i.e., \( C_4 := \sup_{t \in \mathbb{N}_0} E[\|w_t\|^4] < \infty \).

(iv) The control input sequence \((u_t)_{t \in \mathbb{N}_0}\) satisfies (3) and (4).

(v) The control channel noise \((\nu_t)_{t \in \mathbb{N}_0}\) is i.i.d. \( \diamond \)

It follows from Assumption 2-(iii) that there exists \( C_1 > 0 \) such that \( E[\|w_t\|] \leq C_1 \) for all \( t \in \mathbb{N}_0 \). (For instance, Jensen’s inequality shows that \( C_1 \leq \sqrt{C_4} \).) Note also that Assumption 2-(iii) does not require that the process noise vectors \((w_t)_{t \in \mathbb{N}_0}\) be identically distributed. The assumption of mutual independence of \((w_t)_{t \in \mathbb{N}_0}\) can also be relaxed, but we shall not pursue this line of generalization here.

Without any loss of generality, we also assume that \( A \) is in real Jordan canonical form (cf. [15]). Indeed, given a linear system described by system matrices \((A, B)\), there exists a coordinate transformation in the state-space that brings the pair \((A, B)\) to the pair \((A, B)\), where \( A \) is in real Jordan form [19, p. 150]. In particular, choosing a suitable ordering of the Jordan blocks, we can ensure that the pair \((A, B)\) has the form \((A_1 \ 0 \ 0 \ 0, A_2 \ 0 \ 0)\), where \( A_1 \in \mathbb{R}^{d_1 \times d_1} \) is Schur stable, and \( A_2 \in \mathbb{R}^{d_2 \times d_2} \) has its eigenvalues on the unit circle. By Assumption 2-(i), \( A_2 \) is therefore block-diagonal with elements on the diagonal being either \( \pm 1 \) or \( 2 \times 2 \) rotation matrices. As a consequence, \( A_2 \) is orthogonal. Moreover, since \((A, B)\) is stabilizable by Assumption 2-(ii), the pair \((A_2, B_2)\) must be reachable in a number of steps
κ ≤ d2 that depends on the dimension of A2 and the structure
of (A2, B2), i.e., \( \text{rank}(R_κ(A2, B2)) = d2 \), where

\[
R_κ(A2, B2) := [A2^{−1} B2 \ldots A2 B2] .
\]

The smallest such κ is called the controllability index of
(A2, B2) and is fixed throughout the rest of this article.
Summing up, we can start by considering that the state
equation (2) has the form

\[
\begin{bmatrix}
x(t+1) \\
x(t+1)
\end{bmatrix} = \begin{bmatrix} A1 & x(t) \\
A2 & x(t)
\end{bmatrix} + \begin{bmatrix} B \\
B
\end{bmatrix} u_t + \begin{bmatrix} w(t) \\
0
\end{bmatrix},
\]

where A1 is Schur stable, A2 is orthogonal, and the sub-
system (A2, B2) is reachable in κ steps. Since the matrix
\( R_κ(A2, B2) \) has rank full rank, its Moore-Penrose pseudoin-
verse exists and is given by

\[
R_κ(A2, B2)^+ := (R_κ(A2, B2)^T R_κ(A2, B2))^{-1}.
\]

III. MAIN RESULTS

We are ready to state the main result pertaining to the
existence of a policy of bounded authority that renders the
state of the system (2) mean-square bounded. Let us define
the normalized measure of dispersion or the noise-to-signal
ratio of the channel \( \Psi := \sqrt{\sigma_\nu}, \max_{i=1,...,m} |(\mu_\nu)|^{-1} \). We impose the following additional requirements:

Assumption 3: In addition to Assumption 2 we stipulate that:

(vi) The control channel noise has bounded range, i.e., \( \nu_0 \in \mathbb{T} \), where \( \mathbb{T} \) is a bounded subset of \( \mathbb{R}^m \), and that \( \mu_\nu \)
has nonzero entries.

(vii) The following two technical conditions hold:

\[
\text{(vi)} \quad \kappa \Psi \sqrt{\sigma_\nu} \| R_κ(A2, B2)^+ \| \| R_κ(A2, B2) \| < 1.
\]

\[
\text{(vii)} \quad \kappa \Psi \sqrt{\sigma_\nu} \| R_κ(A2, B2)^+ \| \| R_κ(A2, B2) \| < 1 - \kappa \Psi \| R_κ(A2, B2)^+ \| \| R_κ(A2, B2) \| .
\]

Proposition 4: Consider the system (2), and suppose that
Assumption 3 holds. Then there exists a \( \kappa \)-history dependent
policy \( \pi(t) \in \mathcal{N}_0 \) with control authority at most \( U_{\max} \), such that for every initial condition \( \bar{x} \) there exists a constant
\( \zeta = \zeta(\bar{x}, \kappa, \mu_\nu, \Psi, C_1) > 0 \) with

\[
\mathbb{E}_x(\|x_t\|^2) ≤ \zeta \quad \text{for all } t ∈ \mathbb{N}_0
\]
in closed-loop.

Remark 5: Proposition 4 assumes minimal structure from the set in which the control channel noise takes its values. In particular, we do not assume that the control channel noise takes values in a finite set—in fact, T may be uncountable.

While the standard choice of modelling uncertainty in the control channels has focussed on a multiplicative Bernoulli
\( \{0, 1\} \) random variable multiplying the entire control vector, there are cases in which the uncertainty model considered in (2) (i.e., different random variables multiplying the com-
ponents of the controller,) makes sense. For instance, the standard processes of control quantization or “binning” can
be viewed as introducing uncertainty to the controller—
components of the controller being multiplied by bounded
but not necessarily identically distributed random variables;
the set T has the natural interpretation of the “largest bin.”

In view of this, Assumption 3-(vii.a) is a technical condition
stipulated as a trade-off for the absence of any further
structure in the set T.

In Section IV we prove Proposition 4 by a constructive
method. It turns out that our policy (see (12) below) is
derived from the \( \kappa \)-subsampled system \( (x_κ(t))_{t∈\mathbb{N}_0} \), and is \( \kappa \)-
history dependent. To wit, for each \( n ∈ \mathbb{N}_0 \), at time \( \kappa n \), based on the state \( x_{\kappa n} \), the policy synthesizes a \( \kappa \)-long sequence
of control values for time steps \( \kappa n, \kappa n + 1, \ldots, \kappa (n+1) − 1 \).

Let us assume that the same uncertainty enters all the
control channels, i.e., \( \bar{u}_t = \nu_t u_t \), where \( \nu_t \in \mathbb{R} \). The
structure of our control policy permits us to transmit
the control data packets in a single burst each \( \kappa \) steps.
This however, necessitates the presence of a buffer at the actuator end of the plant to store the \( \kappa \) control
values \( \{v_{\kappa n} u_{\kappa n}, v_{\kappa n} u_{\kappa n+1}, \ldots, v_{\kappa n} u_{\kappa (n+1)−1}\} \) transferred in a burst at time \( \kappa n \), such that at each time \( t ∈ \{\kappa n, \ldots, \kappa (n+1) − 1\} \), the control \( v_{\kappa n} u_t \) can be applied.

Assumption 6: In addition to Assumption 2, we require that:

\[
\text{(vi)} \quad \text{Control signals are sent to the actuator every } \kappa \text{ steps, and for each } t ∈ \mathbb{N}_0, \text{the control channel noise is of the form } ν_κ t 1_{κ κn}, \text{with } ν_κ t ∈ \{0, 1\} \text{ and } P(ν_κ t = 1) = \nu_p ∈ [0, 1] \text{ for each } n ∈ \mathbb{N}_0.
\]

\[
\text{(vii)} \quad U_{\max} > \sqrt{C_1} \| R_κ(A2, B2)^+ \| \| R_κ(A2, B2) \| / p.
\]

Proposition 7: Consider the system (2), and suppose that
Assumption 6 holds. Then there exists a \( \kappa \)-history dependent
policy \( \pi(t) \in \mathcal{N}_0 \) with control authority at most \( U_{\max} \), such that for every initial condition \( \bar{x} \) there exists a constant
\( \zeta’ = \zeta’(\bar{x}, \kappa, p, C_1) > 0 \) with

\[
\mathbb{E}_x(\|x_t\|^2) ≤ \zeta’ \quad \text{for all } t ∈ \mathbb{N}_0
\]
in closed-loop.

Remark 8: We noted in Remark 5 that Proposition 4
assumes minimal structure from the bounded set \( T \). In
contrast, Proposition 7 assumes a rather specific structure
of the set \( T \)—that it consists of two elements (note that
\( ν_{\kappa n} ∈ \{0, 1\} \) for each \( n ∈ \mathbb{N}_0 \) in Assumption 6-(vi')).
The i.i.d Bernoulli assumption on \( ν_{\kappa t} \in \mathcal{N}_0 \) leads to a simpler
description of the control authority \( U_{\max} \) in Assumption 6-
(vii') compared to Assumption 3-(vii,b), and the analog of
Assumption 3-(vii,a) is not required here.

Example 9: Consider the scalar system \( x_{t+1} = x_t + \hat{x}_t + w_t \), \( t ≥ 0 \), with initial condition \( x_0 = \bar{x} \), \( \bar{u}_t = ν_0 u_t \).

Suppose that \( (\nu_t)_{t∈\mathbb{N}_0} \) is i.i.d Bernoulli \( \{0, 1\} \) with
\( P(ν_0 = 1) = p > 0 \), and let \( (w_t)_{t∈\mathbb{N}_0} \) be i.i.d, and satisfy

\[
\sup_{t′ ∈ \mathcal{N}_0} \mathbb{E}[|w_{t′}|^2] = C_4’ < ∞.
\]

This implies, in particular, that

\[
\sup_{t′ ∈ \mathcal{N}_0} \mathbb{E}[|w_{t′}|] ≤ C_1’ ≤ \sqrt{C_4’}.
\]

Suppose that \( U_{\max} > C_1’/p \), where \( u_t ∈ [-U_{\max}, U_{\max}] \) for all \( t \). With this much data it is
easy to verify the conditions of Assumption 6. We conclude
by Proposition 7 that there exists a policy with control
authority at most \( U_{\max} \) such that the system is mean-square

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bounded. In fact, we see that for every nonzero probability $p$ of transmission of the control signal, there exists a control authority $U_{\text{max}} > 0$ and a policy with control authority at most $U_{\text{max}}$, under which the state of the system is mean-square bounded.

Remark 10: Notice that Proposition 7 does not contradict the main results of [4], where it was proved (see [4, Lemma 5.4]) that there exists a threshold probability of i.i.d. Bernoulli packet drops such that a stabilizing linear feedback for unstable linear systems can be found provided the drop probability is less than that threshold. Indeed, in Assumption 2 we have specifically ruled out unstable $A$.

IV. PROOFS

For our proofs of Propositions 4 and 7 we shall employ the following immediate adaptation of [20, Theorem 1] on $L_2$ bounds of nonnegative random variables:

Proposition 11 ([20]): Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $(\mathcal{F}_t)_{t \in \mathbb{N}_0}$ be a filtration on $(\Omega, \mathcal{F}, P)$. Suppose that $(\xi_t)_{t \in \mathbb{N}_0}$ is a family of nonnegative random variables adapted to $(\mathcal{F}_t)_{t \in \mathbb{N}_0}$, such that there exist constants $\alpha, \gamma, \beta > 0$ such that $\mathbb{E}[\xi_t^\alpha] \leq c$.

Then there exists $c = c(a, J, M) > 0$ such that

$$\mathbb{E}^{\mathcal{F}_t}[\xi_{t+1} - \xi_t] \leq \alpha \quad \text{on the set} \{\xi_t > J\},$$

$$\mathbb{E}^{\mathcal{F}_t}[\xi_{t+1} - \xi_t^\gamma] \leq \gamma \xi_{t+1} \leq \gamma \xi_t \quad \text{on the set} \{\xi_t > J\}.$$  \hspace{1cm} (7)

(8)

Then there exists $c = c(a, J, M) > 0$ such that

$$\mathbb{E}[\xi_t^\alpha] \leq c.$$  \hspace{1cm} (9)

In what follows we let $I_2$ denote the $d_2 \times d_2$ identity matrix.

Lemma 12: Given the system (2), suppose that Assumption 2 holds, and consider the decomposition (6). Let $(\xi_t)_{t \in \mathbb{N}_0}$ be the $\sigma$-algebra generated by $(x_t)_{t \in \mathbb{N}_0}$. Then there exists $a, J > 0$ such that

$$\mathbb{E}^{\mathcal{F}_t}[\|x_{\kappa(t+1)}\|^2 - \|x_{\kappa(t)}\|^2] \leq \alpha \quad \text{on the set} \{\|x_{\kappa(t)}\|^2 > J\}$$

for all $t \in \mathbb{N}_0$.

Proof: To simplify notation we write compactly

$$\nu_{\kappa(t)} := \begin{bmatrix} \nu_{\kappa(t)} \\ \nu_{\kappa(t)+1} \\ \vdots \\ \nu_{\kappa(t+1)-1} \end{bmatrix} \quad \text{and} \quad u_{\kappa(t)} := \begin{bmatrix} u_{\kappa(t)} \\ u_{\kappa(t)+1} \\ \vdots \\ u_{\kappa(t+1)-1} \end{bmatrix}. \hspace{1cm} (10)$$

It follows from the system dynamics that

$$x_{\kappa(t+1)} = A_{\kappa(t)}^2 x_{\kappa(t)} + \mathcal{R}_\kappa(A_2, B_2) u_{\kappa(t)} + \mathcal{R}_\kappa(A_2, I_2) u_{\kappa(t),\kappa(t+1)-1}, \quad \text{for} \quad t \in \mathbb{N}_0,$$

where $u_{\kappa(t),\kappa(t+1)-1} := \begin{bmatrix} (w_{\kappa(t)})^T \\ \cdots \\ (w_{\kappa(t+1)-1})^T \end{bmatrix}^T$.

Therefore,

$$\mathbb{E}^{\mathcal{F}_t}[\|x_{\kappa(t+1)}\|^2 - \|x_{\kappa(t)}\|^2] \leq \mathbb{E}^{\mathcal{F}_t}\left[\left\|A_{\kappa(t)}^2 x_{\kappa(t)} + \mathcal{R}_\kappa(A_2, B_2) u_{\kappa(t)} + \mathcal{R}_\kappa(A_2, I_2) u_{\kappa(t),\kappa(t+1)-1}\right\|^2 \right].$$

In view of Assumption 3-(vii.a) we see that there exists $0 < \alpha := U_{\text{max}} \left(1 - \sqrt{\kappa} \mathbb{E}^{\mathcal{F}_t}[\|x_{\kappa(t)}\|^2] + \mathcal{R}_\kappa(A_2, I_2)\right)$. We now define

$$r := \alpha + \mathbb{E}^{\mathcal{F}_t}[\|x_{\kappa(t)}\|^2] U_{\text{max}} \sqrt{\mathbb{E}^{\mathcal{F}_t}[\|x_{\kappa(t)}\|^2] + \mathcal{R}_\kappa(A_2, I_2^t)\mathcal{C}_1].$$

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By Assumption 3-(vi), every entry of $\hat{\nu}$ is nonzero; we let $\hat{\nu}^{-1}$ be the vector of reciprocals of each entry of $\hat{\nu}$ (i.e., $(\hat{\nu}^{-1})_i = (\hat{\nu}_i)^{-1}$ for each $i$). We define our control policy

$$u_{\kappa t} := u_{\kappa t}(x^{(2)\kappa t}) := -\mathcal{R}_\kappa(A_2, B_2)^T \text{sat}_r(A_2^2 x^{(2)\kappa t}) \odot \hat{\nu}^{-1},$$

where sat$_r$ is the function defined in (1). Clearly, $u_{\kappa t}$ is $\mathcal{F}_{Nt}$-measurable. Substituting into (10) we see that

$$\mathbb{E} \left[ \sum_{t=0}^{N-1} \left\| x^{(2)\kappa t} - x^{(2)k t} \right\|^4 \right] \leq E \left[ \left\| x^{(2)\kappa t} - x^{(2)k t} \right\|^4 \right]$$

where the last inequality follows from the definition of $r$ above.

Thus, it only remains to see that the control policy defined in (12) satisfies the bound $\|u_{\kappa t}\| \leq U_{\max}$ for each $t$. But in view of the definition of $r$ in (11) and our policy (12), we see that $\|u_{\kappa t}\| \leq U_{\max}$, and the assertion follows.

**Lemma 13:** Given the system (2), suppose that Assumption 2 holds, and consider the decomposition (6). Then there exists $M > 0$ such that

$$\mathbb{E} \left[ \left\| x^{(2)\kappa t+1} - x^{(2)k t} \right\|^4 \right] \leq M$$

for all $t \in \mathbb{N}$.

**Proof:** We retain the notation $w^{(2)\kappa t} = \kappa_t^{(1)} - 1$ from the proof of Lemma 12. Fix $t \in \mathbb{N}$. Observe that since $A_2$ is orthogonal, $\|x^{(2)\kappa t}\| = \|A_2^2 x^{(2)\kappa t}\|$, and therefore,

$$\mathbb{E} \left[ \left\| x^{(2)\kappa t} - x^{(2)k t} \right\|^4 \right] \leq \mathbb{E} \left[ \left\| A_2^2 x^{(2)\kappa t} + \mathcal{R}_\kappa(A_2, B_2)^T\mathcal{N}_\kappa \odot \hat{\nu}^{-1} \right\|^4 \right]$$

where $\mathcal{R}_\kappa(A_2, B_2)^T\mathcal{N}_\kappa \odot \hat{\nu}^{-1}$ is mean-square bounded so long as the control is bounded, which by Assumption 3-(vi) clearly holds (i.e., there exists $\zeta(1) > 0$ such that $\mathbb{E}_x \left[ \left\| x^{(1)\kappa t} \right\|^2 \right] \leq \zeta(1)$ for all $t \in \mathbb{N}_0$). It thus suffices to concentrate on $(x^{(2)\kappa t})_t \in \mathbb{N}_0$. We let $\xi_t := \left\| x^{(2)\kappa t} \right\|$ for each $t \in \mathbb{N}_0$. We see that:

- the condition (7) of Proposition 11 holds with $J = \kappa$, where $\kappa$ is as defined in (11), by Lemma 12, and
- the condition (8) of Proposition 11 holds by Lemma 13.

Defining $J := \max \{ \left\| x^{(2)\kappa t} \right\| \}$, we see that by Proposition 11 there exists a $\tilde{\zeta}^{(2)} = \tilde{\zeta}^{(2)}(a, J, M, J)$ such that $\mathbb{E}_x \left[ \left\| x^{(2)\kappa t} \right\|^2 \right] \leq \tilde{\zeta}^{(2)}$ for all $t \in \mathbb{N}_0$. Then the $\kappa$-subsampled process $(x^{(2)\kappa t})_t \in \mathbb{N}_0$ is mean-square bounded, and $x^{(2)}$ is generated by a linear dynamical system, we conclude that there exists $\zeta(2) > 0$ such that $\mathbb{E}_x \left[ \left\| x^{(2)\kappa t} \right\|^2 \right] \leq \zeta(2)$ for all $t \in \mathbb{N}_0$. The assertion of Proposition 4 follows with $\zeta := \zeta(1) + \zeta(2)$ and noticing that $a$ and $J$ depend on $\tilde{\kappa}$, $\mu$, $\psi$ and $C_1$.

**Proof of Proposition 7** Let us consider the $\kappa$-subsampled system

$$x^{(2)\kappa t+1} = A_2^2 x^{(2)\kappa t} + \mathcal{R}_\kappa(A_2, B_2)^T\mathcal{N}_\kappa \odot \hat{\nu}^{-1} x^{(2)\kappa t} + \mathcal{R}_\kappa(A_2, B_2) w^{(2)\kappa t} x^{(2)\kappa t} \odot \hat{\nu}^{-1},$$

where $u^{(2)\kappa t} = \kappa_t^{(1)} - 1$ is independent of $(x^{(2)\kappa t})_t \in \mathbb{N}_0$ in view of Assumption 2-(iii), applying Jensen’s inequality to the right-hand side above yields

$$\mathbb{E} \left[ \left\| x^{(2)\kappa t} \right\|^4 \right] \leq \mathbb{E} \left[ \left\| x^{(2)\kappa t} \right\|^4 \right]$$

The assertion follows at once with $M$ equal to the right-hand side of the last inequality.

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2 This controller resembles in part the Ackermann’s formula in standard linear control theory [21, p. 477] employed in unconstrained deadbeat controllers.
for some \( r > 0 \) to be defined shortly. Let us verify the conditions of Proposition 11 for the process \(\|x^{(2)}_{n t}\|_{t \in \mathbb{N}_0}\) under the control policy proposed above. We see immediately that

\[
E^\mathbb{F}_{x t} \left[ \left| \left| x^{(2)}_{n t + 1} \right| - \left| x^{(2)}_{n t} \right| \right| \right] \\
\leq E^\mathbb{F}_{x t} \left[ \left| A^2_x x^{(2)}_{n t} + \mathcal{R}_x(A_2, B_2) u^{(2)}_{n t} \kappa^{(1)}(t+1) - \right| \left| x^{(2)}_{n t} \right| \right] \\
\leq \left| A^2_x \right| + E \left[ \left| \mathcal{R}_x(A_2, I_2) u^{(2)}_{n t} \kappa^{(1)}(t+1) - \right| \left| x^{(2)}_{n t} \right| \right] \\
= p \left( A^2_x \left| x^{(2)}_{n t} \right| + \mathcal{R}_x(A_2, B_2) u^{(2)}_{n t} \kappa^{(1)}(t+1) - \left| x^{(2)}_{n t} \right| \right) \\
+ (1-p) \left( A^2_u \left| x^{(2)}_{n t} \right| - \left| x^{(2)}_{n t} \right| \right) \\
+ E \left[ \left| \mathcal{R}_x(A_2, I_2) u^{(2)}_{n t} \kappa^{(1)}(t+1) - \right| \left| x^{(2)}_{n t} \right| \right] \\
= p \left( \sqrt{\kappa} \left| x^{(2)}_{n t} \right| + \mathcal{R}_x(A_2, B_2) u^{(2)}_{n t} \kappa^{(1)}(t+1) - \left| x^{(2)}_{n t} \right| \right) \\
+ \sqrt{\kappa} \left| \mathcal{R}_x(A_2, I_2) \kappa^{(2)}(t+1) - \right| \left| x^{(2)}_{n t} \right| \right] \\
= -pr + \sqrt{\kappa} \left| \mathcal{R}_x(A_2, I_2) \right| C_1,
\]

where we have employed orthogonality of \( A_2 \) to arrive at the second equality above. By Assumption 6-(vi') we see that there exists \( \alpha > 0 \) such that \( \left| \mathcal{R}_x(A_2, B_2) \right| (a + \sqrt{\kappa} C_1 \left| \mathcal{R}_x(A_2, I_2) \right| /p) \leq U_{max} \). Letting \( r := \alpha + \sqrt{\kappa} C_1 \left| \mathcal{R}_x(A_2, I_2) \right| /p \), we see that the condition (7) is verified with \( J = r \). The condition (8) follows readily from Lemma 13, since the elements of the control input are uniformly bounded. Letting \( J := \max \{ r, \left| x^{(2)}_{0} \right| \} \), we see that by Proposition 11 there exists a constant \( \zeta' > 0 \) such that \( E_{x} \left[ \left| x^{(2)}_{t} \right| ^2 \right] \leq \zeta' \) for all \( t \in \mathbb{N}_0 \). By the same argument involving the Schur stable part \( x^{(1)} \) as in the proof of Proposition 4, we see that there exists a constant \( \zeta' > 0 \) such that \( E_{x} \left[ \left| x^{(1)}_{t} \right| ^2 \right] \leq \zeta' \) for all \( t \in \mathbb{N}_0 \). In view of the fact that \( \alpha \) and \( J \) depend on \( \bar{x}, \kappa, p \), and \( C_1 \), this concludes the proof.

V. CONCLUSIONS

We considered a networked systems setup in which the control inputs are transmitted via a noisy communication channel. Under mild assumptions on the statistics of the channel noise and control authority, we constructed a causal \( \kappa \)-steps control strategy that renders the state of the closed-loop networked system mean-square bounded.

Future work will focus on extending the current results to encompass the case of imperfect and incomplete state measurements.

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