Congruence testing for odd subgroups of the modular group

Thomas Hamilton and David Loeffler

Abstract

We give a computationally effective criterion for determining whether a finite-index subgroup of \( \text{SL}_2(\mathbb{Z}) \) is a congruence subgroup, extending earlier work of Hsu for subgroups of \( \text{PSL}_2(\mathbb{Z}) \).

Recall that a finite-index subgroup of \( \text{SL}_2(\mathbb{Z}) \) is said to be a congruence subgroup if it is defined by congruence conditions on the entries of its elements; formally, a subgroup is congruence if it contains the subgroup \( \Gamma(N) \) of matrices congruent to the identity modulo \( N \), and the least such \( N \) is its level.

We are interested in the following question.

**Question.** Is there an efficient procedure that will determine whether a finite-index subgroup of \( \text{SL}_2(\mathbb{Z}) \) is congruence?

One such algorithm follows from the following theorem, proved in [3], which is an extension of a classical theorem of Wolfahrt.

**Theorem 1 (Kiming–Schütt–Verrill).** Let \( \Gamma \leq \text{SL}_2(\mathbb{Z}) \) be a finite-index subgroup, and let \( d \) be the lowest common multiple of the widths of the cusps of \( \Gamma \). If \( \Gamma \) is congruence, then its level is either \( d \) or \( 2d \).

(The case of level \( 2d \) can only occur if \( \Gamma \) is odd, that is does not contain \(-1\).)

In principle, one can now determine whether \( \Gamma \) is congruence by calculating explicitly a list of generators for \( \Gamma(N) \), where \( N = d \) or \( 2d \) as appropriate, and testing whether each of these is contained in \( \Gamma \). This approach is used in [3] in order to give explicit examples of non-congruence lifts to \( \text{SL}_2(\mathbb{Z}) \) of congruence subgroups of \( \text{PSL}_2(\mathbb{Z}) \). However, the number of generators of \( \Gamma(N) \) grows rather quickly with \( N \), so this algorithm rapidly becomes impractical for large values of \( N \).

We present the following alternative approach to the above problem. As has been noted by Hsu [2] and others, a convenient data structure for representing a subgroup of \( \text{SL}_2(\mathbb{Z}) \) of index \( m \) is by the homomorphism \( \text{SL}_2(\mathbb{Z}) \to S_m \) given by left multiplication on the cosets \( \text{SL}_2(\mathbb{Z})/\Gamma \). This, in turn, can be represented by two permutations giving the action of the generators \( L = ( \begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix} ) \) and \( R = ( \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} ) \) of \( \text{SL}_2(\mathbb{Z}) \) on the cosets \( \text{SL}_2(\mathbb{Z})/\Gamma \).

The computer algebra package Sage contains a library of routines for working with subgroups defined in this way, implemented by Vincent Delecroix and the second author based on an earlier implementation by Chris Kurth.

**Theorem 2.** Let \( N = d \) if \(-1 \in \Gamma \) and \( N = 2d \) otherwise. Then there exists an explicit list of relations \( \mathcal{L}_N \) in \( L \) and \( R \) (of length \( \leq 7 \)), such that \( \Gamma \) is congruence if and only if the permutation representation of \( \text{SL}_2(\mathbb{Z}) \) corresponding to \( \Gamma \) satisfies the relations in \( \mathcal{L}_N \).

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This theorem has been proved for subgroups containing $-1$ by Hsu [2]; our proof follows Hsu’s closely, except that we use the Kiming–Schütt–Verrill theorem (Theorem 1) in place of the classical theorem of Wolfahrt.

**Proposition 3.** Let $N ≥ 1$. There is an explicit finite set $\mathcal{L}_N$ of words in $L$ and $R$ whose image in $\text{SL}_2(\mathbb{Z})$ normally generates $\Gamma(N)$ (that is, $\Gamma(N)$ is the smallest normal subgroup of $\text{SL}_2(\mathbb{Z})$ containing the elements in $\mathcal{L}_N$).

**Proof.** See [2, Theorem 2.4]. The starting-point of the proof is the well-known fact that $\text{SL}_2(\mathbb{Z})$ has the presentation

$$\langle L, R \mid (LR^{-1}L)^2(R^{-1}L)^{-3}, (LR^{-1}L)^4 \rangle$$

where $L$ and $R$ correspond to the matrices given above. Thus if $\mathcal{L}$ is any set of words in $L$ and $R$, the group

$$\langle L, R \mid (LR^{-1}L)^2(R^{-1}L)^{-3}, (LR^{-1}L)^4, \mathcal{L} \rangle$$

(1)

is the largest quotient of $\text{SL}_2(\mathbb{Z})$ in which the elements in the image of $\mathcal{L}$ map to the identity, which is the quotient of $\text{SL}_2(\mathbb{Z})$ by the subgroup normally generated by the image of $\mathcal{L}$. In particular, the images of the elements of $\mathcal{L}$ normally generate $\Gamma(N)$ if and only if (1) is a presentation of the finite group $\text{SL}_2(\mathbb{Z}/N\mathbb{Z})$.

Explicit presentations of the groups $\text{SL}_2(\mathbb{Z}/N\mathbb{Z})$ for all $N$ in terms of the generators $L$ and $R$ are given in [2, Lemmas 3.3–3.5] (based on earlier work of Behr and Mennicke [1]), so it suffices to take $\mathcal{L}_N$ to be the set of relations appearing in these presentations.

**Proof of Theorem 2.** Let $N$ be as defined in the statement of the theorem. We know that $\Gamma$ is congruence if and only if it contains $\Gamma(N)$. Let $\Gamma'$ be the normal core of $\Gamma$, that is, the intersection of the conjugates of $\Gamma$ in $\text{SL}_2(\mathbb{Z})$; then, since the elements of $\mathcal{L}_N$ normally generate $\Gamma(N)$, it follows that $\Gamma$ is congruence if and only if $\mathcal{L}_N ⊂ \Gamma'$.

However, $\Gamma'$ is precisely the kernel of the map $\phi : \text{SL}_2(\mathbb{Z}) → S_m$ giving the permutation representation of $\Gamma$. So $\Gamma$ is congruence if and only if $\phi$ is trivial on the elements of $\mathcal{L}_N$.

(One could clearly adapt this argument to work with other explicit descriptions of $\Gamma$ as long as one has an algorithm for computing whether a given element of $\text{SL}_2(\mathbb{Z})$ lies in the normal core of $\Gamma$.)

We now reproduce, for the reader’s convenience, an explicit list of relations $\mathcal{L}_N$ as in Theorem 2, based on those given by Hsu.

- If $N$ is odd, one may take $\mathcal{L}_N$ to contain the single relation

$$\left(R^2L^{-1/2}\right)^3 = 1,$$

where $\frac{1}{2}$ is the multiplicative inverse of $2$ mod $N$. This follows from the fact that for $N$ odd,

$$\langle L, R \mid L^N = 1, (LR^{-1}L)^2 = (R^{-1}L)^3, (LR^{-1}L)^4 = 1, (R^2L^{-1/2})^3 = 1 \rangle$$

is a presentation of $\text{SL}_2(\mathbb{Z}/N\mathbb{Z})$, by [2, Lemma 3.3]. The relations $(LR^{-1}L)^2 = (R^{-1}L)^3$ and $(LR^{-1}L)^4 = 1$ are redundant; they are automatically satisfied by the permutation representation of $\text{SL}_2(\mathbb{Z})$ corresponding to $\Gamma$, since they are satisfied in $\text{SL}_2(\mathbb{Z})$ itself. The relation $L^N = 1$ is also automatically satisfied, since by definition $N$ is divisible by the widths of all of the cusps of $\Gamma$. (This case can, of course, only occur if $-1 ∈ \Gamma$ and is thus identical to the first case of Hsu’s Theorem 3.1.)
• If $N$ is a power of 2, let $S = L^{20}R^{1/5}L^{-4}R^{-1}$, where $\frac{1}{5}$ is the multiplicative inverse of 5 mod $N$. Then one may take $L_N$ to consist of the three relations

\[(LR^{-1}L)^{-1}S(LR^{-1}L) = S^{-1},
S^{-1}RS = R^{25},
(SR^5LR^{-1}L)^3 = (LR^{-1}L)^2.\]

As in the previous case, this follows from the fact that

\[(L, R | L^N = 1, (LR^{-1}L)^2 = (R^{-1}L)^3, (LR^{-1}L)^4 = 1, L_N)\]

is a presentation of $SL_2(\mathbb{Z}/N\mathbb{Z})$, by [2, Lemma 3.4], and the first three relations are automatically satisfied in the permutation relation corresponding to $\Gamma$.

(Note that if we assume that $-1 \in \Gamma$, we may replace the last relation with $(SR^5LR^{-1}L)^3 = 1$, which is the relation appearing in Hsu’s Theorem 3.1; but for odd subgroups we must use the slightly more complicated relation above.)

• If $N = em$ where $e$ is a power of 2, $m$ is odd and $e, m > 1$, then let $e, d$ be the unique integers mod $N$ such that $c = 0 \mod e, c = 1 \mod m, d = 1 \mod e, d = 0 \mod m$. Write $a = L^e, b = R^e, l = L^d, r = R^d$ and $s = l^{20}r^{1/5}l^{-4}r^{-1}$, where $\frac{1}{5}$ is interpreted mod $e$.

Then we may take $L_N$ to consist of the seven elements

\[[a, r] = 1, \]
\[(ab^{-1}a)^4 = 1, \]
\[(ab^{-1}a)^2 = (b^{-1}a)^3, \]
\[(ab^{-1}a)^2 = (b^2a^{-1/2})^3, \]
\[(lr^{-1}l)^{-1}s(lr^{-1}l) = s^{-1}, \]
\[s^{-1}rs = r^{25}, \]
\[(lr^{-1}l)^2 = (sr^5lr^{-1}l)^3. \]

As in the previous two cases, this follows from the presentation of the group $SL_2(\mathbb{Z}/N\mathbb{Z}) \cong SL_2(\mathbb{Z}/e\mathbb{Z}) \times SL_2(\mathbb{Z}/m\mathbb{Z})$ given in [2, Lemma 3.5].

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Thomas Hamilton
Premier Pensions Management
Corinthian House
17 Lansdowne Road
Croydon CR0 2BX
United Kingdom
d.a.loeffler@warwick.ac.uk

David Loeffler
Mathematics Institute
University of Warwick
Coventry CV4 7AL
United Kingdom

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