On Classical and Bayesian Asymptotics in State Space Stochastic Differential Equations

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Abstract

In this article we investigate consistency and asymptotic normality of the maximum likelihood and the posterior distribution of the parameters in the context of state space stochastic differential equations (SDE’s). We then extend our asymptotic theory to random effects models based on systems of state space SDE’s, covering both independent and identical and independent but non-identical collections of state space SDE’s. We also address asymptotic inference in the case of multidimensional linear random effects, and in situations where the data are available in discretized forms. It is important to note that asymptotic inference, either in the classical or in the Bayesian paradigm, has not been hitherto investigated in state space SDE’s.

Keywords: Asymptotic normality; Kullback-Leibler divergence; Posterior consistency; Random effects; State space stochastic differential equations; Stochastic stability.

1 Introduction

State-space models are well-known for their versatility in modeling complex dynamic systems in the context of discrete time, and have important applications in various disciplines like engineering, medicine, finance and statistics. As is also well-known, most time series models of interest can be expressed in the context of discrete time, and have important applications in various disciplines like engineering, medicine, finance and statistics. As is also well-known, most time series models of interest can be expressed in the context of discrete time, and have important applications in various disciplines like engineering, medicine, finance and statistics.

Discrete time state space models are characterized by a latent, unobserved stochastic process, \( X \) of the following type:

\[
\begin{align*}
X(t) &= b_X(X(t), t) dt + \sigma_X(X(t), t) dW_X(t), \\
Y(t) &= b_Y(Y(t), t) dt + dW_Y(t),
\end{align*}
\]

where \( W_Y \) and \( W_X \) are independent standard Wiener processes, \( b_Y, b_X \) are real-valued drift functions, and \( \sigma_X \) is the real-valued diffusion coefficient. The SDE’s are assumed to satisfy the usual regularity conditions that guarantee existence of strong solutions; see, for example, Arnol’d (1974), Øksendal.

\[\boxed{b_Y} \]

\[\boxed{b_X} \]

\[\boxed{\sigma_X} \]

\[\boxed{W_Y} \]

\[\boxed{W_X} \]

This approach could be interpreted as continuous time versions of the traditional discrete time hidden Markov model based approach. Continuous time models closely resembling the above-mentioned type exists in the literature, but rather than estimating relevant parameters, filtering theory has been considered. For instance, Stratonovich (1968), Jazwinski (1970), Maybeck (1979), Maybeck (1982), Särkkä (2006), Crisan and Rozovskii (2000) consider the filtering problem in state space SDE’s of the following type:

\[
\begin{align*}
\frac{dY(t)}{dt} &= b_Y(X(t), t) dt + dW_Y(t); \\
\frac{dX(t)}{dt} &= b_X(X(t), t) dt + \sigma_X(X(t), t) dW_X(t),
\end{align*}
\]

However, when the time is continuous, research on state space or hidden Markov models seem to be much scarce. Ideally, one should consider a pair of stochastic differential equations (SDE’s) whose solutions would be the continuous time processes \( Y = \{Y(t) : t \in [0, T]\} \) and \( X = \{X(t) : t \in [0, T]\} \).

In fact, the SDE with solution \( Y \) should depend upon \( X \). Since the solutions of SDE’s under general regularity conditions are Markov processes (see, for example, Mad (2011)), \( X \) would turn out to be a Markov process, and conditionally on \( X \), \( Y \) would also be a Markov process. Thus, such an approach could be interpreted as continuous time versions of the traditional discrete time hidden Markov model based approach.

\[
\boxed{Y(t)} \]

\[
\boxed{X(t)} \]

\[
\boxed{W_Y(t)} \]

\[
\boxed{W_X(t)} \]

\[
\boxed{b_Y} \]

\[
\boxed{b_X} \]

\[
\boxed{\sigma_X} \]

\[
\boxed{W_Y(t)} \]

\[
\boxed{W_X(t)} \]

\[
\boxed{b_Y} \]

\[
\boxed{b_X} \]

\[
\boxed{\sigma_X} \]

\[
\boxed{W_Y(t)} \]

\[
\boxed{W_X(t)} \]
The purpose of filtering theory is to compute the posterior distribution of the latent process conditional on the observed process. This can be obtained from the continuous-time optimal filtering equation, which is, in fact, the Kushner-Stratonovich equation (Kushner (1964), Bucy (1965)). Note that (see Särkkä (2012), for example) it is possible to obtain the latter as continuous-time limits of the Bayesian filtering equations. The so-called Zakai equation (Zakai (1969)) provides a simplified form by removing the non-linearity in the Kushner-Stratonovich equation. In the special case of (1.1) and (1.2), with \( b_Y(X(t),t) = L(t)X(t), b_X(X(t),t) = H(t)X(t) \) and \( \sigma_Y(X(t),t) = \sigma_Y(t) \), exact solution of the filtering problem, known as the Kalman-Bucy filter (Kalman and Bucy (1961)), can be obtained. In the non-linear cases various approximations are employed; see Crisan and Rozovski (2000), Särkkä (2007), Särkkä and Sarmavuori (2013), among others.

In pharmocokinetic/pharmacodynamic contexts, the following type of model is regarded as the state space model, assuming \( \{Y_1, \ldots, Y_n\} \) are observed at discrete times \( \{t_1, \ldots, t_n\} \):

\[
\begin{align*}
Y_j &= b_Y(X_{t_j}, \theta) + \sigma_Y(X_{t_j}, \theta)\varepsilon_j; \quad \varepsilon_j \overset{ind}{\sim} N(0, 1); \\
X(t) &= b_X(X(t), t, \theta)dt + \sigma_X(X(t), \theta)dW_X(t),
\end{align*}
\]

where \( b_Y \) and \( \sigma_Y \) are appropriate real-valued functions, and \( \theta \) denotes the set of relevant parameters. The standard choices of \( \sigma_Y \) are \( \sigma_Y(x, \theta) = \sigma \) (homoscedastic model) and \( \sigma_Y(x, \theta) = a + \sigma_Y(x, \theta) \) (heterogeneous model), and \( b_Y \) is usually chosen to be a linear function. Thus, even though the latent process \( X \) is described as the solution of the SDE (1.4), the model for the (discretely) observed data is postulated to be arising from independent normal distributions, conditional on the discretized version of the diffusion process \( X \). This simplifies inference proceedings to a large extent, particularly when the Markov transition model associated with (1.4) is available explicitly. Here we recall that under suitable regularity conditions, the solution of (1.4) is a continuous time Markov process (see, for example, Arnold (1974), Øksendal (2003), Mao (2011)). If the Markov transition model is not available in closed form, then various approximations are proposed in the literature to approximate the likelihood of \( \theta \), using which the MLE of \( \theta \) or the posterior distribution of \( \theta \) is obtained. Under special cases, for instance, when \( \sigma_Y(x, \theta) = \sigma \), \( b_Y(x, \theta) = b_0x \), \( \sigma_X(x, \theta) = \sigma_0 \), \( b_X(x_t, t, \theta) = a_0x_t + c_0(t) \), an explicit form of the likelihood (based on discretization) is available, and the resulting MLE has been shown to be consistent and asymptotically normal by Favetto and Samson (2010), but in more general, non-linear situations, theoretical results do not seem to be available. A comprehensive account of the methods of approximating the MLE and posterior distribution of \( \theta \), with discussion of related computational issues and theoretical results, have been provided in Donnet and Samson (2013).

Our interest in this article is primarily the investigation of asymptotic parametric inference, as \( T \to \infty \), from both classical and Bayesian perspectives, in the context of state space models where the models for the observed data as well as the latent process, are both described by SDE’s. Indeed, to our knowledge, asymptotic inference in such models has not been hitherto investigated. In our proceedings we assume a somewhat generalized version of the state space SDE’s described by (1.1) and (1.2) in that the drift function \( b_Y \) depends upon \( Y(t) \), in addition to \( X(t) \) and \( t \); moreover, we assume that there is a diffusion coefficient \( \sigma_Y(Y(t), X(t), t) \) associated with the Wiener process \( W_Y(t) \) that drives the observational SDE (1.1); a practical instance of such a state space model in the case of bacterial growth can be found in Muller et al. (2012). We further assume that there is a common set of parameters \( \theta \) associated with both the SDE’s, which are of interest. In particular, we assume that there exist appropriate real-valued, known, functions for \( \theta, \psi_Y(\theta) \) and \( \psi_X(\theta) \), such that the drift functions are \( \psi_Y(\theta)b_Y(Y(t), X(t), t) \) and \( \psi_X(\theta)b_X(X(t), t) \), respectively. In Section 2 we clarify that \( \psi_Y(\theta) \) and \( \psi_X(\theta) \) offers very general scope of parameterizations by mapping the perhaps high-dimensional (although finite-dimensional) quantity \( \theta \) to appropriate real-valued functional forms composed of the elements of \( \theta \). We also assume that the diffusion coefficients of the respective SDE’s are independent of \( \theta \). A key assumption in our approach to asymptotic investigation is that \( X \) is stochastically stable. In
a nutshell, in this article, by stochastic stability of \( X \) we mean that
\[
|x(t)| \leq \xi \lambda(t) \quad \text{for all } t \geq 0,
\] (1.5)
after all initial values \( x(0) \in \mathbb{R} \), where \( \lambda(t) \to 0 \) as \( t \to \infty \), and \( \xi \) is a non-negative, finite random variable depending upon \( x(0) \). For comprehensive details regarding various versions of stochastic stability of solutions of SDE’s, see [Mac (2011)].

It is to be noted that our model clearly corresponds to a dependent set-up, and establishment of asymptotic results are therefore not be achievable by the state-of-the-art methods that typically deal with at least independent situations. For Bayesian asymptotics we find the consistency results of [Shalizi (2009)] and the general result on posterior asymptotic normality of [Schervish (1995)] useful for our purpose, while for classical asymptotics we obtain a suitable asymptotic approximation to the target log-likelihood, which helped us establish strong consistency, as well as asymptotic normality of the MLE.

Once we establish classical and Bayesian asymptotic results associated with our state space SDE model, we then extend our model to random effects state space model (see [Delattre et al. (2013)], for instance, for SDE based random effects model), where we model each time series data available on \( n \) individuals using our state space model, assuming that the effects \( \psi_Y(\theta) \) and \( \psi_X(\theta) \) for individual \( i \) are parameterized by \( \theta \), which is the parameter of interest. From the classical point of view, this is not a random effects model technically since \( \theta \) is treated as a fixed quantity, but from the Bayesian viewpoint, a prior on \( \theta \) renders the effects random. Slightly abusing terminology for the sake of convenience, we continue to call the model random effects stochastic SDE, from both classical and Bayesian perspectives. Under such random effects SDE model we seek asymptotic classical and Bayesian inference on \( \theta \) as both number of individuals, \( n \), and the domain of observations \( [0,T_i] \); \( i = 1, \ldots, n \) increase indefinitely. For our purpose we assume \( T_i = T \) for each \( i \). Here we remark that [Donnet and Samson (2013)] discuss population SDE models with measurement errors; see also [Overgaard et al. (2005)]. [Donnet and Samson (2008), Yan et al. (2014), Leander et al. (2015)]; for the \( i \)-th individual such models are of the same form as (1.3) and (1.4), but specifics depending upon \( i \), and with \( \theta \) replaced with \( \phi_i \), where \( \{\phi_1, \ldots, \phi_n\} \) are independently and identically distributed with some distribution with parameter \( \theta \), say, which is one of the parameters of interest. This is a genuine random effects model unlike ours, but here only the latent process \( X \) is based upon SDE. Theoretical results do not exist for this set-up; see [Donnet and Samson (2013)]. On the other hand, even though our random effects state space SDE model is completely based upon SDEs, the simplified form of the effects, parameterized by a common \( \theta \), enables us to obtain desired asymptotic results for both classical and Bayesian paradigms. Indeed, in our case it is certainly possible to postulate a genuine random effects state space SDE model by replacing \( \psi_Y(\theta) \) and \( \psi_X(\theta) \) with iid random effects \( \phi_Y_i \) and \( \phi_X_i \), having distributions parameterized by quantities of inferential interest \( \theta_Y \) and \( \theta_X \), say, but in this set-up complications arise regarding handling the observed integrated likelihood and its associated bounds, which does not assist in our asymptotic investigations.

Discretization of our state space SDE models is essential for practical applications such as in fields of pharmacokinetics/pharmacodynamics, where continuous time data are usually unavailable. We show in the supplement that the same asymptotic results go through in discretized situations.

In our proceedings with each set-up, we first investigate Bayesian consistency, then consistency and asymptotic normality of the MLE, and finally asymptotic posterior normality. One reason behind this sequence is that the proofs of the results on posterior normality depend upon the proofs of the results of consistency and asymptotic normality of MLE, which, in turn, depend upon the proofs associated with Bayesian posterior consistency. Moreover, adhering to this sequence allows us to introduce the assumptions in a sequential manner, so that an overall logical order could be maintained throughout the paper.

The rest of our article is organized as follows. In Section 2 we introduce our state space SDE model, detail the necessary assumptions including stochastic stability of the solution of the latent SDE, state a lemma necessary for the proceedings whose proof is provided in supplement, and derive bounds on the true, data-generating model associated with the true parameter value \( b_0 \), and the modeled likelihood,
where $\theta$ is unknown. Next, in Sections 3 and 4 we prove posterior consistency of $\theta$: we proceed by stating the sufficient conditions and a theorem of Shalizi (2009) in Sections 3 and then by proving validity of the stated sufficient conditions in Section 4. Here, for our purpose, we needed to introduce extra assumptions and prove a lemma. We prove strong consistency and asymptotic normality of the MLE in Section 5; several extra assumptions are needed here, and the method of proving a key result on uniformly approximating the log-likelihood uses an idea introduced in Section 3 for validating one of the assumptions of Shalizi (2009). In Section 6 we establish asymptotic posterior normality of $\theta$ under some extra assumptions; again we needed an uniform approximation result, the method of which we borrow from the previous section. We introduce random effects state space SDE models in Section 7. With the introduction of further extra assumptions, we are able to prove posterior consistency, strong consistency and asymptotic normality of the MLE, and asymptotic posterior normality of the parameters in this set-up in the same section. Finally, in Section 8 we provide a brief summary of our work, discuss some key issues, and identify future research agenda. The extension of our theory for state space SDE models with multidimensional linear random effects and in the case of discretized data are discussed in the supplement.

## 2 State space SDE

First consider the following “true” state space SDE:

$$
\begin{align*}
    dY(t) &= \phi_{Y,0}b_Y(Y(t), X(t), t)dt + \sigma_Y(Y(t), X(t), t)dW_Y(t); \\
    dX(t) &= \phi_{X,0}b_X(X(t), t)dt + \sigma_X(X(t), t)dW_X(t),
\end{align*}
$$

for $t \in [0, b_T]$, where $b_T \to \infty$, as $T \to \infty$. The first SDE, namely, (2.1) is the true observational SDE and is associated with the observed data. The second SDE (2.2) is the true evolutionary, unobservable SDE. In the above two equations, we assume that $\phi_{Y,0}$ and $\phi_{X,0}$ are both explained by a “true” set of parameters $\theta_0$, through known but perhaps different functions of $\theta_0$. In other words, we assume that $\phi_{Y,0} = \psi_Y(\theta_0)$ and $\phi_{X,0} = \psi_X(\theta_0)$, where $\psi_Y$ and $\psi_X$ are known functions. Note that this is a general formulation, where we allow the possibility $\theta_0 = (\theta_{Y,0}, \theta_{X,0})$ and choice of $\psi_Y$ and $\psi_X$ such that $\psi_Y(\theta_0) = \theta_{Y,0}$ and $\psi_X(\theta_0) = \theta_{X,0}$, for scalars $\theta_{Y,0}$ and $\theta_{X,0}$. In this instance, the observational and evolutionary SDEs have their own sets of parameters. We also allow common subsets of the parameter vector $\theta_0$ to feature in the two SDEs. For instance, $\psi_Y(\theta_0) = \theta_{Y,0} + \theta_{X,0}$ and $\psi_X(\theta_0) = \theta_{X,0}$. Indeed, $\theta_0$ can be any finite-dimensional vector, appropriately mapped to the real line by $\psi_Y$ and $\psi_X$. We wish to learn about the set of parameters $\theta_0$, which would enable learning about $\phi_{Y,0}$ and $\phi_{X,0}$ simultaneously. For our purpose, we assume that $(\psi_Y(\theta), \psi_X(\theta))$ is identifiable in $\theta$, that is, $(\psi_Y(\theta_1), \psi_X(\theta_1)) = (\psi_Y(\theta_2), \psi_X(\theta_2))$ implies $\theta_1 = \theta_2$.

Our modeled state space SDE is analogously given, for $t \in [0, b_T]$ by:

$$
\begin{align*}
    dY(t) &= \phi_Yb_Y(Y(t), X(t), t)dt + \sigma_Y(Y(t), X(t), t)dW_Y(t); \\
    dX(t) &= \phi_Xb_X(X(t), t)dt + \sigma_X(X(t), t)dW_X(t),
\end{align*}
$$

where $\phi_Y = \psi_Y(\theta)$ and $\phi_X = \psi_X(\theta)$.

Throughout, we assume that the initial values associated with the SDEs (2.1), (2.2), (2.3) and (2.4), are non-random. It is worth mentioning in this context that for stochastic stability it is enough to assume non-randomness of the initial value; see Mao (2011), page 111, for a proof of this.

We wish to establish consistency and asymptotic normality of the maximum likelihood estimator (MLE) and the posterior distribution of $\theta$, as $T \to \infty$. For technical reasons we shall consider the likelihood for $t \in [a_T, b_T]$, where $a_T \to \infty$ and $(b_T - a_T) \to \infty$, as $T \to \infty$. In particular, we assume that $(b_T - a_T) \geq T$.
2.1 Assumptions regarding $b_Y$ and $\sigma_Y$

(H1) For every $T > 0$ and integer $\eta \geq 1$, given any $x$, there exists a positive constant $K_{Y,x,T,\eta}$ such that for all $t \in [0, b_T]$ and all $(y_1, y_2)$ with $\max\{|y_1, y_2| \leq \eta,$
$$\max\left\{|b_Y(y_1, x, t) - b_Y(y_2, x, t)|^2, |\sigma_Y(y_1, x, t) - \sigma_Y(y_2, x, t)|^2\right\} \leq K_{Y,x,T,\eta}|y_1 - y_2|^2.$$\)

(H2) For every $T > 0$, given any $x$, there exists a positive constant $K_{x,T}$ such that for all $(y, t) \in \mathbb{R} \times [0, T]$,
$$\max\left\{b_Y^2(y, x, t), \sigma_Y^2(y, x, t)\right\} \leq K_{x,T}(1 + y^2).$$

(H3) For every $T > 0$, there exist positive constants $K_{Y,1,T}, K_{Y,2,T}, \alpha_{Y,1}, \alpha_{Y,2}$ such that for all $(x, t) \in \mathbb{R} \times [0, b_T]$,
$$K_{Y,1,T}\left(1 - \alpha_{Y,1}x^2\right) \leq \frac{b_Y^2(y, x, t)}{\sigma_Y^2(y, x, t)} \leq K_{Y,2,T}\left(1 + \alpha_{Y,2}x^2\right),$$
where $K_{Y,1,T} \rightarrow K_Y$ and $K_{Y,2,T} \rightarrow K_Y$ as $T \rightarrow \infty$; $K_Y$ being a positive constant. We further assume that for $j = 1, 2$, $(b_T - a_T)|K_{Y,j,T} - K_Y| \rightarrow 0$, as $T \rightarrow \infty$.

In (H3) we have assumed that the bounds of $\frac{b_Y^2(y,x,t)}{\sigma_Y^2(y,x,t)}$ do not depend upon $y$, which is somewhat restrictive. Dependence of the bounds on $y$ can be insisted upon, but at the cost of the assumption of stochastic stability of $Y$ in addition to that of $X$. See Section 8 for details regarding the modified assumption. All our results remain intact under the modified assumption. It is also important to clarify that the lower bound in (H3), when utilized in our SDE context, becomes non-negative after possibly a few time steps, thanks to the stochastic stability assumption which ensures (1.5).

2.2 Assumptions regarding $b_X$ and $\sigma_X$

(H4) $b_X(0, t) = 0 = \sigma_X(0, t)$ for all $t \geq 0$.

(H5) For every $T > 0$, and integer $\eta \geq 1$, there exists a positive constant $K_{T,\eta}$ such that for all $t \in [0, b_T]$ and all $(x_1, x_2)$ with $\max\{|x_1, x_2| \leq \eta,$
$$\max\left\{|b_X(x_1, t) - b_X(x_2, t)|^2, |\sigma_X(x_1, t) - \sigma_X(x_2, t)|^2\right\} \leq K_{T,\eta}|x_1 - x_2|^2.$$\)

(H6) For every $T > 0$, there exists a positive constant $K_T$ such that for all $(x, t) \in \mathbb{R} \times [0, b_T]$,
$$\max\left\{b_X^2(x, t), \sigma_X^2(x, t)\right\} \leq K_T(1 + x^2).$$

(H7) For every $T > 0$, there exist positive constants $K_{X,1,T}, K_{X,2,T}, \alpha_{X,1}, \alpha_{X,2}$ such that for all $(x, t) \in \mathbb{R} \times [0, b_T]$,
$$K_{X,1,T}\left(1 - \alpha_{X,1}x^2\right) \leq \frac{b_X^2(x, t)}{\sigma_X^2(x, t)} \leq K_{X,2,T}\left(1 + \alpha_{X,2}x^2\right),$$
where $K_{X,1,T} \rightarrow K_X$ and $K_{X,2,T} \rightarrow K_X$, as $T \rightarrow \infty$; $K_X$ being a positive constant. We also assume that for $j = 1, 2$, $(b_T - a_T)|K_{X,j,T} - K_X| \rightarrow 0$, as $T \rightarrow \infty$.

2.3 Further assumptions ensuring almost sure stochastic stability of $X(t)$

Let $C$ denote the family of all continuous nondecreasing functions $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $f(0) = 0$ and $f(r) > 0$ when $r > 0$. 

5
Let $S_h = \{ x \in \mathbb{R} : |x| < h \}$ and $C(S_h \times [0, \infty); \mathbb{R}^+)$ denote the family of all continuous functions $V(x, t)$ from $S_h \times [0, \infty)$ to $\mathbb{R}^+$ with continuous first partial derivatives with respect to $x$ and $t$. Also, let $C(S_h \times [0, \infty); \mathbb{R}^+)$, where $0 < h \leq \infty$, denote the family of non-negative functions $V(x, t)$ defined on $S_h \times \mathbb{R}^+$ such that they are continuously twice differentiable in $x$ and once in $t$. Let

$$LV(x, t) = V_t(x, t) + V_x(x, t)b_X(x, t) + \frac{1}{2}\sigma_X^2(x, t)V_{xx}(x, t),$$

where $V_t = \frac{\partial V}{\partial t}$, $V_x = \frac{\partial V}{\partial x}$, and $V_{xx} = \frac{\partial^2 V}{\partial x^2}$.

With these definitions and notations, we now make the following assumption:

(H8) Let $p > 0$ and let there exist a function $V \in C(S_h \times [0, \infty); \mathbb{R}^+)$, a continuous non-decreasing function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\gamma(t) \to \infty$ as $t \to \infty$, and a continuous function $\bar{\eta} : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\int_0^\infty \bar{\eta}(t) < \infty$. Assume that for $x \neq 0$, $t \geq 0$,

$$\gamma(t)|x|^p \leq V(x, t)$$

and $LV(x, t) \leq \bar{\eta}(t)$.

Thanks to Theorem 6.2 of [Mag 2011] (page 145), assumption (H8) ensures that stochastic stability of $X$ of the form $|x(t)| \leq \xi \lambda(t)$ for all $t \geq 0$ holds almost surely, for all initial values $x(0) \in \mathbb{R}$ with

$$\lambda(t) = [\gamma(t)]^{-\frac{1}{p}},$$

where $\xi$ is a non-negative, finite random variable depending upon $x(0)$.

Let us define

$$v_{Y\mid X,T} = \int_{aT}^{bT} \frac{bT}{\sigma_Y^2(Y(s), X(s), s)} ds$$

(2.7)

$$u_{Y\mid X,T} = \int_{aT}^{bT} \frac{bT}{\sigma_Y^2(Y(s), X(s), s)} dY(s)$$

(2.8)

$$v_{X,T} = \int_{aT}^{bT} \frac{bT}{\sigma_X^2(X(s), s)} ds$$

(2.9)

$$u_{X,T} = \int_{aT}^{bT} \frac{bT}{\sigma_X^2(X(s), s)} dX(s).$$

(2.10)

Due to (H3) and (H7), the following hold:

$$K_{Y,\xi,T,1} \leq v_{Y\mid X,T} \leq K_{Y,\xi,T,2};$$

(2.11)

$$K_{X,\xi,T,1} \leq v_{X,T} \leq K_{X,\xi,T,2},$$

(2.12)

where

$$K_{Y,\xi,T,1} = K_{Y,1,T} \left( (b_T - a_T) - \alpha_{Y,1} \xi^2 \int_{aT}^{bT} \lambda^2(s) ds \right);$$

(2.13)

$$K_{Y,\xi,T,2} = K_{Y,2,T} \left( (b_T - a_T) + \alpha_{Y,2} \xi^2 \int_{aT}^{bT} \lambda^2(s) ds \right);$$

(2.14)

$$K_{X,\xi,T,1} = K_{X,1,T} \left( (b_T - a_T) - \alpha_{X,1} \xi^2 \int_{aT}^{bT} \lambda^2(s) ds \right);$$

(2.15)

$$K_{X,\xi,T,2} = K_{X,2,T} \left( (b_T - a_T) + \alpha_{X,2} \xi^2 \int_{aT}^{bT} \lambda^2(s) ds \right).$$

(2.16)

To proceed, we shall make use of the following relationships between $u_{Y\mid X,T}$, $v_{Y\mid X,T}$ and $u_{X,T}$,
$v_{X,T}$ under the true state space SDE model described by (2.1) and (2.2):

\[ u_{Y|X,T} = \phi_{Y,0}v_{Y|X,T} + \int_{\mathcal{W}} \frac{b_Y(Y(s), X(s), s)}{\sigma_Y(Y(s), X(s), s)} dW_Y(s); \]  
\[ u_{X,T} = \phi_{X,0}v_{X,T} + \int_{\mathcal{W}} \frac{b_X(X(s), s)}{\sigma_X(X(s), s)} dW_X(s). \]  

Let

\[ I_{Y,X,T} = \int_{\mathcal{W}} \frac{b_Y(Y(s), X(s), s)}{\sigma_Y(Y(s), X(s), s)} dW_Y(s); \]  
\[ I_{X,T} = \int_{\mathcal{W}} \frac{b_X(X(s), s)}{\sigma_X(X(s), s)} dW_X(s). \]  

Because of (2.11), (2.12), (2.17) and (2.18) the following hold:

\[ \phi_{Y,0}K_{Y,t,T,1} + I_{Y,X,T} \leq u_{Y|X,T} \leq \phi_{Y,0}K_{Y,2,T,2} + I_{Y,X,T}; \]  
\[ \phi_{X,0}K_{X,t,T,1} + I_{X,T} \leq u_{X,T} \leq \phi_{X,0}K_{X,2,T,2} + I_{X,T}. \]

### 2.4 True model

First note that $\exp \left( \phi_{Y,0}u_{Y|X,T} - \frac{\phi_{Y,0}^2}{2}v_{Y|X,T} \right)$ is the conditional density of $Y$ given $X$, with respect to $Q_{T,Y|X}$, the probability measure associated with (2.1) on $[a_T, b_T]$, assuming null drift. Also, $\exp \left( \phi_{X,0}u_{X,T} - \frac{\phi_{X,0}^2}{2}v_{X,T} \right)$ is the marginal density of $X$ with respect to $Q_{T,X}$, the probability measure associated with the latent state SDE (2.2) on $[a_T, b_T]$, but assuming null drift. These are standard results; see for example, Lipster and Shiryaev (2001), Øksendal (2003), Delattre et al. (2013).

It then follows that the marginal likelihood under the true model (2.1) and (2.2) is the marginal density of $\{Y(t) : t \in [a_T, b_T]\}$, given by

\[ p_T(\theta_0) = \int \exp \left( \phi_{Y,0}u_{Y|X,T} - \frac{\phi_{Y,0}^2}{2}v_{Y|X,T} \right) \times \exp \left( \phi_{X,0}u_{X,T} - \frac{\phi_{X,0}^2}{2}v_{X,T} \right) dQ_{T,X} \]
\[ = E_{T,X} \left[ \exp \left( \phi_{Y,0}u_{Y|X,T} - \frac{\phi_{Y,0}^2}{2}v_{Y|X,T} \right) \times \exp \left( \phi_{X,0}u_{X,T} - \frac{\phi_{X,0}^2}{2}v_{X,T} \right) \right], \]

where $E_{T,X}$ denotes expectation with respect to $Q_{T,X}$. The following lemma proved in supplement formalizes the dominating measure with respect to which $p_T(\theta_0)$ is the Radon-Nikodym derivative.

**Lemma 1** The likelihood given by (2.23) is the density of $\{Y(t) : t \in [a_T, b_T]\}$ with respect to $Q_{T,Y}$, where for any relevant measurable set $A$,

\[ Q_{T,Y}(A) = \int_{\mathcal{X}_T} dQ_{T,Y|X}(A) dQ_{T,X} = \int_A \int_{\mathcal{X}_T} dQ_{T,Y|X} dQ_{T,X}. \]

In the above, $\mathcal{X}_T$ stands for the sample space of $\{X(t) : t \in [a_T, b_T]\}$.

It is important to remark that our likelihood (2.23) is of a very general form and does not usually admit a closed form expression, but this is not at all a requirement for our asymptotic purpose. Closed form expressions may be necessary when it is of interest to directly maximize the likelihood with respect to the parameters, and in such cases, more stringent assumptions regarding the SDEs are necessary. See, for example, Frydman and Lakner (2003); see also Kailath and Zakai (1971). Also, observe that
our dominating measure $Q_{T,Y}$ is not the Wiener measure, unlike the aforementioned papers, albeit it reduces to the Wiener measure if $\sigma_Y \equiv 1$ and $\sigma_X \equiv 1$.

### 2.4.1 Asymptotic approximation of $p_T(\theta_0)$

Using (2.11) and (2.21) we obtain

$$B_{L,T}(\theta_0) \leq p_T(\theta_0) \leq B_{U,T}(\theta_0),$$

where

$$B_{L,T}(\theta_0) = E_{T,X} (Z_{L,T,\theta_0}(X));$$

$$B_{U,T}(\theta_0) = E_{T,X} (Z_{U,T,\theta_0}(X)),$$

where

$$Z_{L,T,\theta_0}(X) = \exp \left( \phi_{Y,0}^2 K_{Y,\xi,T,1} + \phi_{Y,0} I_{Y,X,T} - \frac{\phi_{Y,0}^2}{2} K_{Y,\xi,T,2} \right)$$

$$\times \exp \left( \phi_{X,0}^2 K_{X,\xi,T,1} + \phi_{X,0} I_{X,T} - \frac{\phi_{X,0}^2}{2} K_{X,\xi,T,2} \right)$$

and

$$Z_{U,T,\theta_0}(X) = \exp \left( \phi_{Y,0}^2 K_{Y,\xi,T,2} + \phi_{Y,0} I_{Y,X,T} - \frac{\phi_{Y,0}^2}{2} K_{Y,\xi,T,1} \right)$$

$$\times \exp \left( \phi_{X,0}^2 K_{X,\xi,T,2} + \phi_{X,0} I_{X,T} - \frac{\phi_{X,0}^2}{2} K_{X,\xi,T,1} \right).$$

The expressions (2.27) and (2.28) have the same asymptotic form. We first provide the intuitive idea and then rigorously prove our result on asymptotic approximation. Note that, by (H3) and (H7), (2.13), (2.14), (2.15), (2.16), the facts that $\frac{1}{b_T - a_T} \int_{a_T}^{b_T} \lambda^2(s) ds \to 0$ as $T \to \infty$, and $\xi$ is a finite random variable, that $K_{Y,\xi,T,1} \overset{a.s.}{\sim} (b_T - a_T) K_Y$, $K_{Y,\xi,T,2} \overset{a.s.}{\sim} (b_T - a_T) K_Y$, $K_{X,\xi,T,1} \overset{a.s.}{\sim} (b_T - a_T) K_X$ and $K_{X,\xi,T,2} \overset{a.s.}{\sim} (b_T - a_T) K_X$, where, for any two random sequences $\{A_T : T \geq 0\}$ and $\{B_T : T \geq 0\}$, $A_T \overset{a.s.}{\sim} B_T$ stands for $A_T / B_T \to 1$, almost surely, as $T \to \infty$. Also, as we show, the distributions of $(b_T - a_T)^{-\frac{1}{2}} I_{Y,X,T}$ and $(b_T - a_T)^{-\frac{1}{2}} I_{X,T}$ are asymptotically normal with zero means and variances $K_Y$ and $K_X$, respectively. Heuristically substituting these in (2.27) and (2.28) yields the form

$$\hat{p}_T(\theta_0) = \exp \left( \frac{(b_T - a_T) K_Y \phi_{Y,0}^2}{2} + \phi_{Y,0} \sqrt{K_Y} (W_Y(b_T) - W_Y(a_T)) + (b_T - a_T) K_X \phi_{X,0}^2 \right).$$

(2.31)

### 2.5 Modeled likelihood and its asymptotic approximation

Our modeled likelihood associated with the state space model described by (2.3) and (2.4) is given by:

$$L_T(\theta) = \int \exp \left( \phi_{Y} u_Y | X, T - \frac{\phi_{Y}^2}{2} v_Y | X, T \right) \times \exp \left( \phi_{X} u_X | X, T - \frac{\phi_{X}^2}{2} v_X | X, T \right) dQ_{T,X}.$$  

(2.32)

Using the same method of obtaining bounds of $p_T(\theta_0)$, we obtain the following bounds for $L_T(\theta)$:

$$\tilde{B}_{L,T}(\theta) \leq L_T(\theta) \leq \tilde{B}_{U,T}(\theta),$$

(2.33)
where

\[ \hat{B}_{L,T}(\theta) = E_{T,X} \left( \hat{Z}_{L,T,\theta}(X) \right); \quad (2.34) \]

\[ \hat{B}_{U,T}(\theta) = E_{T,X} \left( \hat{Z}_{U,T,\theta}(X) \right), \quad (2.35) \]

where

\[ \hat{Z}_{L,T,\theta}(X) = \exp \left( \phi_Y \phi_{Y,0} K_{Y,\xi,T,1} + \phi_Y I_{Y,X,T} - \frac{\phi_Y^2}{2} K_{Y,\xi,T,2} \right) \]
\[ \times \exp \left( \phi_X \phi_{X,0} K_{X,\xi,T,1} + \phi_X I_{X,T} - \frac{\phi_X^2}{2} K_{X,\xi,T,2} \right) \quad (2.36) \]

and

\[ \hat{Z}_{U,T,\theta}(X) = \exp \left( \phi_Y \phi_{Y,0} K_{Y,\xi,T,2} + \phi_Y I_{Y,X,T} - \frac{\phi_Y^2}{2} K_{Y,\xi,T,1} \right) \]
\[ \times \exp \left( \phi_X \phi_{X,0} K_{X,\xi,T,2} + \phi_X I_{X,T} - \frac{\phi_X^2}{2} K_{X,\xi,T,1} \right). \quad (2.37) \]

It follows as before that the modeled likelihood can be approximated as

\[ \hat{L}_T(\theta) = \exp \left( (b_T - a_T) K_Y \phi_Y \phi_{Y,0} + \phi_Y \sqrt{K_Y} (W_Y(b_T) - W_Y(a_T)) \right) \]
\[ - \frac{(b_T - a_T) K_Y \phi_Y^2}{2} + (b_T - a_T) K_X \phi_X \phi_{X,0} \right). \quad (2.38) \]

2.6 A briefing on the formal results on the asymptotic approximations

Formal proof of the results \( p_T(\theta_0) \overset{a.s.}{\sim} \hat{p}_T(\theta_0) \) and \( L_T(\theta) \overset{a.s.}{\sim} \hat{L}_T(\theta) \) requires the following two additional assumptions:

(H9) There exists an integer \( k_0 \geq 1 \) such that \[ \sum_{T=1}^{\infty} \delta_T^{-2k_0} (b_T - a_T)^{k_0-1} \int_{a_T}^{b_T} \lambda^2(s) ds < \infty, \]
where \( \delta_T \downarrow 0 \) as \( T \to \infty \) is a specific sequence decreasing fast enough so that it satisfies, because of continuity of the exponential function, the following: for any \( \epsilon > 0 \),

\[ \sum_{T=1}^{\infty} \left| \exp(I_{Y,X,T}) - \exp(\sqrt{K_Y} (W_Y(b_T) - W_Y(a_T))) \right| < \epsilon \]

Also assume that \( E|\xi|^{2k_0} < \infty \).

(H10) \[ \sup_{T>0} E \left( \frac{Z_{L,T,\theta_0}(X)}{p_T(\theta_0)} \right) < \infty, \sup_{T>0} E \left( \frac{Z_{U,T,\theta_0}(X)}{p_T(\theta_0)} \right) < \infty, \sup_{T>0, \theta \in \Theta} E \left( \frac{\hat{Z}_{L,T,\theta}(X)}{L_T(\theta)} \right) < \infty \]
\[ \sup_{T>0, \theta \in \Theta} E \left( \frac{\hat{Z}_{U,T,\theta}(X)}{L_T(\theta)} \right) < \infty. \]

The following lemma shows that under assumptions (H1) – (H9), \( \exp(I_{Y,X,T}) \) and \( \exp(I_{X,T}) \) are asymptotically independent of \( X \).

Lemma 2 Under assumptions (H1) – (H9),

\[ \left| \exp(I_{Y,X,T}) - \exp(\sqrt{K_Y} (W_Y(b_T) - W_Y(a_T))) \right| \overset{a.s.}{\longrightarrow} 0; \quad (2.40) \]

\[ \left| \exp(I_{X,T}) - \exp(\sqrt{K_Y} (W_X(b_T) - W_X(a_T))) \right| \overset{a.s.}{\longrightarrow} 0. \quad (2.41) \]
The following corollary of Lemma 2 shows asymptotic normality of the relevant quantities involved in the asymptotic approximations.

**Corollary 3** Since \((b_T - a_T)^{-\frac{1}{2}} \sqrt{K_Y} (W_Y(b_T) - W_Y(a_T))\) and \((b_T - a_T)^{-\frac{1}{2}} \sqrt{K_X} (W_X(b_T) - W_X(a_T))\) are normally distributed with mean zero and variances \(K_Y\) and \(K_X\), respectively, it follows that

\[
(b_T - a_T)^{-\frac{1}{2}} I_{Y,X,T} \xrightarrow{a.s.} N(0, K_Y); \quad \text{(2.42)}
\]

\[
(b_T - a_T)^{-\frac{1}{2}} I_{X,T} \xrightarrow{a.s.} N(0, K_X). \quad \text{(2.43)}
\]

Finally, our asymptotic approximation result is given by the following theorem, which requires assumptions (H1) – (H10).

**Theorem 4** Assume (H1) – (H10). Then

\[
p_T(\theta_0) \xrightarrow{a.s.} \hat{p}_T(\theta_0); \quad \text{(2.44)}
\]

\[
L_T(\theta) \xrightarrow{a.s.} \hat{L}_T(\theta), \text{ for all } \theta \in \Theta. \quad \text{(2.45)}
\]

The proofs of Lemma 2 and Theorem 4 are presented in the supplement.

### 3 Convergence of the posterior distribution of \(\theta\)

In order to prove convergence of our posterior distribution we verify the conditions of the theorem proved in Shalizi (2009) which take account of dependence set-ups and misspecifications. Before stating the assumptions and the theorems of Shalizi (2009), we bring in some required notations.

Let \(\mathcal{F}_T = \sigma(\{Y(s); s \in [a_T, b_T]\})\) denote the \(\sigma\)-algebra generated by \(\{Y(s); s \in [a_T, b_T]\}\). Let \(\Theta\) and \(\mathcal{T}\) denote the parameter space and the associated \(\sigma\)-algebra.

#### 3.1 Assumptions and theorem of Shalizi in the context of our state space SDE

(A1) Consider the following likelihood ratio:

\[
R_T(\theta) = \frac{L_T(\theta)}{p_T(\theta_0)}. \quad \text{(3.1)}
\]

Assume that \(R_T(\theta)\) is \(\mathcal{F}_T \times \mathcal{T}\)-measurable for all \(T > 0\).

(A2) For each \(\theta \in \Theta\), the generalized or relative asymptotic equipartition property holds, and so, almost surely,

\[
\lim_{T \to \infty} \frac{1}{b_T - a_T} \log R_T(\theta) = -h(\theta),
\]

where \(h(\theta)\) is given in (A3) below.

(A3) For every \(\theta \in \Theta\), the Kullback-Leibler divergence rate

\[
h(\theta) = \lim_{T \to \infty} \frac{1}{b_T - a_T} E\left( \log \frac{p_T(\theta_0)}{L_T(\theta)} \right), \quad \text{(3.2)}
\]

exists (possibly being infinite) and is \(\mathcal{T}\)-measurable.

(A4) Let \(I = \{\theta : h(\theta) = \infty\}\). The prior \(\pi\) satisfies \(\pi(I) < 1\).
Following the notation of Shalizi (2009), for $A \subseteq \Theta$, let

$$h(A) = \text{ess inf}_{\theta \in A} h(\theta); \quad (3.3)$$

$$J(\theta) = h(\theta) - h(\Theta); \quad (3.4)$$

$$J(A) = \text{ess inf}_{\theta \in A} J(\theta). \quad (3.5)$$

(A5) There exists a sequence of sets $G_T \to \Theta$ as $T \to \infty$ such that:

1. $\pi(G_T) \geq 1 - \alpha \exp(-\beta(b_T - a_T))$, for some $\alpha > 0$, $\beta > 2h(\Theta)$; \quad (3.6)

2. The convergence in (A3) is uniform in $\theta$ over $G_T \setminus I$.

3. $h(G_T) \to h(\Theta)$, as $T \to \infty$.

For each measurable $A \subseteq \Theta$, for every $\delta > 0$, there exists a random natural number $\tau(A, \delta)$ such that

$$(b_T - a_T)^{-1} \int_A R_T(\theta)\pi(\theta)d\theta \leq \delta + \limsup_{(b_T - a_T) \to \infty} (b_T - a_T)^{-1} \int_A R_T(\theta)\pi(\theta)d\theta,$$ \quad (3.7)

for all $(b_T - a_T) > \tau(A, \delta)$, provided $\limsup_{(b_T - a_T) \to \infty} (b_T - a_T)^{-1} \log \pi(\mathbb{1}_A R_T) < \infty$. Regarding this, the following assumption has been made by Shalizi:

(A6) The sets $G_T$ of (A5) can be chosen such that for every $\delta > 0$, the inequality $(b_T - a_T) > \tau(G_T, \delta)$ holds almost surely for all sufficiently large $T$.

(A7) The sets $G_T$ of (A5) and (A6) can be chosen such that for any set $A$ with $\pi(A) > 0$,

$$h(G_T \cap A) \to h(A),$$ \quad (3.8)

as $T \to \infty$.

Under the above assumptions, the following version of the theorem of Shalizi (2009) can be seen to hold.

**Theorem 5** (Shalizi (2009)) Consider assumptions (A1)–(A7) and any set $A \in \mathcal{T}$ with $\pi(A) > 0$. If $\beta > 2h(A)$, where $\beta$ is given in (3.6) under assumption (A5), then

$$\lim_{T \to \infty} \frac{1}{b_T - a_T} \log \pi(A|\mathcal{F}_T) = -J(A),$$ \quad (3.9)

where $\pi(\cdot|\mathcal{F}_T)$ is the posterior distribution given $\mathcal{F}_T = \sigma(\{Y_s; s \in [a_T, b_T]\})$.

### 4 Verification of the assumptions of Shalizi

Before proceeding further, we make the following assumptions regarding $\psi_Y$ and $\psi_X$:

(H11) (i) For every $\theta \in \Theta \cup \{\theta_0\}$, $\psi_Y(\theta)$ and $\psi_X(\theta)$ are finite and satisfy $(\psi_Y(\theta_1), \psi_X(\theta_1)) = (\psi_Y(\theta_2), \psi_X(\theta_2))$ implies $\theta_1 = \theta_2$.

(ii) $|\psi_Y|$ is coercive, that is, for every sequence $\{\theta_T : T > 0\}$ such that $\|\theta_T\| \to \infty$, $|\psi_Y(\theta_T)| \to \infty$.

(iii) For every sequence $\{\theta_T : T > 0\}$ such that $\|\theta_T\| \to \infty$, $|\psi_Y(\theta_T)|^2 (b_T - a_T)|K_{Y,j,T} - K_Y| \to 0$ and $|\psi_X(\theta_T)|^2 (b_T - a_T)|K_{X,j,T} - K_X| \to 0$, for $j = 1, 2$, and $C_1(b_T - a_T) \leq (\psi_Y(\theta_T) - \psi_Y(\theta_0))^\beta \leq C_2(b_T - a_T)$, for some constants $C_1, C_2 > 0$, as $T \to \infty$. 

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is clearly

We consider the likelihood ratio

We now obtain the limit of the quantity

\[ \psi_Y(\theta) \] is assumed to have finite expectation with respect to the prior \( \pi(\theta) \).

(v) \( \psi_X(\theta) \leq |\psi_X(\theta_0)| \) for all \( \theta \in \Theta \).

(vi) The first and second derivatives of \( \psi_X \) vanish at \( \theta = \theta_0 \).

(vii) \( \psi_Y \) and \( \psi_X \) are at least thrice continuously differentiable.

\subsection*{4.1 Verification of (A1)}

Recall that our likelihood \( L_T(\theta) \) is given by (2.32). In the same way as the proof of the second part of Proposition 2 of Delattre et al. (2013), it can be proved that the first factor of the integrand of (2.32) is a measurable function of \( \{Y(s); s \in [a_T, b_T]\} \), \( \{X(s); s \in [a_T, b_T]\} \), \( \theta \). Also, by the same result of Delattre et al. (2013) the second factor of the integrand is a measurable function of \( \{\{X(s); s \in [a_T, b_T]\}; \theta \}. Thus, the integrand is a measurable function of \( \{\{Y(s); s \in [a_T, b_T]\} \), \( \{X(s); s \in [a_T, b_T]\} \), \( \theta \). Since the associated measure spaces are \( \sigma \)-finite, \( L_T(\theta) \) is clearly \( \mathcal{F}_T \times \mathcal{T} \)-measurable for all \( T > 0 \).

\subsection*{4.2 Verification of (A2)}

We consider the likelihood ratio \( R_T(\theta) \) given by (3.7). Using Theorem 4 we obtain that

\[
\frac{1}{b_T - a_T} R_T(\theta) \overset{a.s.}{\to} -\frac{K_Y}{2} (\phi_Y - \phi_{Y,0})^2 + \sqrt{K_Y} (\phi_Y - \phi_{Y,0}) \left( \frac{W_Y(b_T) - W_Y(a_T)}{b_T - a_T} \right)
- \frac{K_X}{2} (\phi_X - \phi_{X,0})^2 + \frac{K_X}{2} \left( \phi^2_X - \phi^2_{X,0} \right).
\] (4.1)

Since \( \frac{W_Y(b_T) - W_Y(a_T)}{b_T - a_T} \overset{a.s.}{\to} 0 \), it follows that, almost surely,

\[
\frac{1}{b_T - a_T} \log R_T(\theta) \to -\frac{1}{2} \left[ K_Y (\phi_Y - \phi_{Y,0})^2 + K_X (\phi_X - \phi_{X,0})^2 + K_X (\phi^2_X - \phi^2_{X,0}) \right].
\] (4.2)

Let

\[
\psi(\theta) = \frac{1}{2} \left[ K_Y (\phi_Y - \phi_{Y,0})^2 + K_X (\phi_X - \phi_{X,0})^2 + K_X (\phi^2_X - \phi^2_{X,0}) \right]
= \frac{1}{2} \left[ K_Y (\psi_Y(\theta) - \psi_Y(\theta_0))^2 + K_X (\psi_X(\theta) - \psi_X(\theta_0))^2 + K_X (\psi^2_X(\theta) - \psi^2_X(\theta_0)) \right].
\] (4.3)

Note that due to (H11) (v), \( \psi(\theta) \geq 0 \), for all \( \theta \in \Theta \). Thus (A2) holds.

\subsection*{4.3 Verification of (A3)}

We now obtain the limit of the quantity

\[
\frac{1}{b_T - a_T} E_{\theta_0} \left( \log \frac{p_T(\theta_0)}{L_T(\theta)} \right) = -\frac{1}{b_T - a_T} E_{\theta_0} \left( \log R_T(\theta) \right),
\]

where \( E_{\theta_0} \) is the expectation with respect to the true likelihood \( p_T(\theta_0) \). Proceeding in the same way as in the case of \( R_T(\theta) \) and noting that \( E_{\theta_0} (W_Y(b_T) - W_Y(a_T)) = 0 \), it is easy to see that

\[
\frac{1}{b_T - a_T} E_{\theta_0} \left( \log \frac{p_T(\theta_0)}{L_T(\theta)} \right) \to \psi(\theta),
\] (4.4)

as \( T \to \infty \).
4.4 Verification of (A4)

To verify (A4) we reformulate the original parameter space \( \Theta \) as \( \Theta \setminus I \). Abusing notation, we continue to denote \( \Theta \setminus I \) as \( \Theta \). Hence, the prior \( \pi \) on \( \Theta \) clearly satisfies \( \pi(I) = 0 \).

4.5 Verification of (A5)

4.5.1 Verification of (A5) (i)

Now consider \( \mathcal{G}_T = \{ \theta \in \Theta : |\psi_Y(\theta)| \leq \exp(\beta(b_T - a_T)) \} \), where \( \beta \) is chosen such that \( \beta > 2h(\Theta) \). Coerciveness of \( ||\psi_Y|| \) implies compactness of \( \mathcal{G}_T \), for every \( T > 0 \).

The above theorem guarantees that (4.6) admits the following approximation:

\[
\pi(\mathcal{G}_T) > 1 - E(|\psi_Y(\theta)|) \exp(-\beta(b_T - a_T)) = 1 - \alpha \exp(-\beta(b_T - a_T)),
\]

where the first inequality is due to Markov’s inequality and \( \alpha = E(|\psi_Y(\theta)|) > 0 \). The expectation, which is with respect to the prior \( \pi \), exists by (H11) (iv).

4.5.2 Verification of (A5) (ii)

We now show that convergence of (4.2) is uniform in \( \theta \) over \( \mathcal{G}_T \setminus I \). First note that \( \mathcal{G}_T \setminus I = \mathcal{G}_T \), since we have already removed \( I \) from \( \Theta \). Now note that, because of compactness of \( \mathcal{G}_T \) and continuity of \( \frac{1}{b_T - a_T} \log R_T(\theta) + h(\theta) \) in \( \theta \), there exists \( \theta_T \in \mathcal{G}_T \) such that

\[
\sup_{\theta \in \mathcal{G}_T \setminus I} \left| \frac{1}{b_T - a_T} \log R_T(\theta) + h(\theta) \right| = \left| \frac{1}{b_T - a_T} \log R_T(\theta_T) + h(\theta_T) \right|. \tag{4.6}
\]

Note that \( \theta_T \) depends upon the data. However, under the additional condition (H11) (iii), it is clear from the proof of Theorem 4 (see Section S-3 of the supplement) that our asymptotic approximation of \( L_T(\theta_T) \) remains valid even in this case. Formally,

**Theorem 6** Assume (H1) (H10) and (H11) (iii). Consider any, perhaps, data-dependent sequence \( \{ \theta_T : T > 0 \} \), where either \( ||\theta_T|| \) remains finite almost surely or \( ||\theta_T|| \to \infty \), almost surely, as \( T \to \infty \). Then \( L_T(\theta_T) \overset{a.s.}{\to} L_T(\theta_T) \).

The above theorem guarantees that (4.6) admits the following approximation:

\[
\left| \frac{1}{b_T - a_T} \log R_T(\theta_T) + h(\theta_T) \right| \overset{a.s.}{\approx} \sqrt{\frac{R_Y}{\sqrt{b_T - a_T}}} \left| \frac{\psi_Y(\theta_T) - \psi_Y(\theta_0)}{\sqrt{b_T - a_T}} \right| \times \frac{W_Y(b_T) - W_Y(a_T)}{\sqrt{b_T - a_T}}. \tag{4.7}
\]

By Corollary 3 and (H11) (iii), the right hand side of (4.7) goes to zero almost surely, as \( T \to \infty \). Hence the convergence of (4.2) is uniform in \( \theta \) over \( \mathcal{G}_T \setminus I \).

4.5.3 Verification of (A5) (iii)

We now show that \( h(\mathcal{G}_T) \to h(\Theta) \), as \( T \to \infty \). Due to compactness of \( \mathcal{G}_T \) and continuity of \( h(\theta) \), it follows that there exists \( \theta_T \in \mathcal{G}_T \) such that \( h(\mathcal{G}_T) = h(\theta_T) \). Also, since \( \mathcal{G}_T \) is a non-decreasing sequence of sets, \( h(\theta_T) \) is non-increasing in \( T \). Since \( \mathcal{G}_T \to \Theta \), it follows that \( h(\mathcal{G}_T) \to h(\Theta) \), as \( T \to \infty \).
4.6 Verification of (A6)

Under (A1) – (A3), which we have already verified, it holds that (see equation (18) of Shalizi (2009)) for any fixed $\mathcal{G}$ of the sequence $\mathcal{G}_T$, for any $\epsilon > 0$ and for sufficiently large $T$,

$$\frac{1}{b_T - a_T} \log \int_{\mathcal{G}} R_T(\theta) \pi(\theta) d\theta \leq -h(\mathcal{G}) + \epsilon + \frac{1}{b_T - a_T} \log \pi(\mathcal{G}). \eqno{(4.8)}$$

It follows that $\tau(\mathcal{G}_T, \delta)$ is almost surely finite for all $T$ and $\delta$. We now argue that for sufficiently large $T$, $\tau(\mathcal{G}_T, \delta) > (b_T - a_T)$ only finitely often with probability one. By equation (41) of Shalizi (2009),

$$\sum_{T=1}^{\infty} P(\tau(\mathcal{G}_T, \delta) > (b_T - a_T)) \leq \sum_{T=1}^{\infty} \sum_{m=T+1}^{\infty} P\left( \frac{1}{b_m - a_m} \log \int_{\mathcal{G}_T} R_m(\theta) \pi(\theta) d\theta > \delta - h(\mathcal{G}_T) \right). \eqno{(4.9)}$$

Now, by compactness of $\mathcal{G}_T$, $h(\mathcal{G}_T) = h(\hat{\theta}_T)$, for $\hat{\theta}_T \in \mathcal{G}_T$, and by the mean value theorem for integrals,

$$\frac{1}{b_m - a_m} \log \int_{\mathcal{G}_T} R_m(\theta) \pi(\theta) d\theta = \frac{1}{b_m - a_m} \log R_m(\hat{\theta}_T) \pi(\mathcal{G}_T),$$

for $\hat{\theta}_T \in \mathcal{G}_T$ depending upon the data, so that

$$\frac{1}{b_m - a_m} \log \int_{\mathcal{G}_T} R_m(\theta) \pi(\theta) d\theta > \delta - h(\mathcal{G}_T)$$

implies, since $h(\hat{\theta}_T) \geq h(\hat{\theta}_T)$, that

$$\frac{1}{b_m - a_m} \log R_m(\hat{\theta}_T) + h(\hat{\theta}_T) > \delta - \frac{1}{b_m - a_m} \log \pi(\mathcal{G}_T) > \delta.$$ 

Thus, it follows from (4.9) and Chebychev’s inequality, that

$$\sum_{T=1}^{\infty} P(\tau(\mathcal{G}_T, \delta) > (b_T - a_T))$$

$$\leq \sum_{T=1}^{\infty} \sum_{m=T+1}^{\infty} P\left( \frac{1}{b_m - a_m} \log R_m(\hat{\theta}_T) + h(\hat{\theta}_T) > \delta \right)$$

$$\leq \sum_{T=1}^{\infty} \sum_{m=T+1}^{\infty} \delta^{-6} E \left( \frac{1}{b_m - a_m} \log R_m(\hat{\theta}_T) + h(\hat{\theta}_T) \right)^6. \eqno{(4.10)}$$

From equation (4.1) and (4.3) it is clear that

$$\frac{1}{b_m - a_m} \log R_m(\hat{\theta}_T) + h(\hat{\theta}_T) \overset{a.s.}{\sim} \sqrt{K_Y} \frac{\psi_Y(\hat{\theta}_T) - \psi_Y(\theta_0)}{\sqrt{b_m - a_m}} \times \frac{W_Y(b_m) - W_Y(a_m)}{\sqrt{b_m - a_m}} \eqno{(4.11)}$$

Now, let $Z_m = \frac{1}{b_m - a_m} \log R_m(\hat{\theta}_T) + h(\hat{\theta}_T)$ and $Z_m = \sqrt{K_Y} \frac{\psi_Y(\hat{\theta}_T) - \psi_Y(\theta_0)}{\sqrt{b_m - a_m}} \times \frac{W_Y(b_m) - W_Y(a_m)}{\sqrt{b_m - a_m}}.$

Then

$$\frac{Z_m^6 - \tilde{Z}_m^6}{E\left(\tilde{Z}_m^6\right)} = \frac{Z_m^6 - \tilde{Z}_m^6}{E\left(\tilde{Z}_m^6\right)} \times \frac{\tilde{Z}_m^6}{E\left(\tilde{Z}_m^6\right)} \overset{a.s.}{\to} 0 \text{ as } m \to \infty. \eqno{(4.12)}$$

because, due to (4.11) the first factor on the right hand side of (4.12) tends to zero almost surely, while by
(H11) (iii) the second factor is bounded above by a constant times standard normal distribution raised to the power 6. It can be easily verified using (H11) (iii) that \( \sup_{m \geq 1} E \left[ \frac{Z^n_m - \tilde{Z}^n_m}{E(Z^n_m)} \right]^2 < \infty \), so that \( \frac{Z^n_m - \tilde{Z}^n_m}{E(Z^n_m)} \) is uniformly integrable. Hence, it follows from (4.12) that
\[
\frac{E \left( Z^n_m - \tilde{Z}^n_m \right)}{E (\tilde{Z}^n_m)} \to 0, \text{ as } m \to \infty.
\]
In other words, as \( m \to \infty \),
\[
E \left( Z^n_m \right) \overset{a.s.}{\to} E \left( \tilde{Z}^n_m \right). \tag{4.13}
\]
Now note that for studying convergence of the double sum (4.10), it is enough to investigate convergence of
\[
S_{T_0} = \sum_{T=T_0}^{\infty} \sum_{m=T+1}^{\infty} E \left( \frac{1}{b_m - a_m} \log R_m(\hat{\theta}_T) + h(\hat{\theta}_T) \right)^6,
\]
for some sufficiently large \( T_0 \). By virtue of (4.13) it is then enough to study convergence of
\[
\tilde{S}_{T_0} = \sum_{T=T_0}^{\infty} \sum_{m=T+1}^{\infty} E \left( \tilde{Z}^n_m \right)
= \tilde{c} \sum_{T=T_0}^{\infty} \sum_{m=T+1}^{\infty} \frac{(\psi_Y(\hat{\theta}_T) - \psi_Y(\theta_0))^6}{(b_m - a_m)^3}, \tag{4.14}
\]
where \( \tilde{c} (> 0) \) is a constant. By (H11) (iii), for sufficiently large \( T \), \( (\psi_Y(\hat{\theta}_T) - \psi_Y(\theta_0))^6 \leq C_2 (b_T - a_T) \), for some \( C_2 > 0 \). Hence,
\[
S_{T_0} \leq C_Y \sum_{T=T_0}^{\infty} \sum_{m=T+1}^{\infty} \frac{b_T - a_T}{(b_m - a_m)^3}, \tag{4.15}
\]
where \( C_Y (> 0) \) is a constant. Now note that, since \( (b_T - a_T) \) is increasing in \( T \), \( (b_{T_0 + j} - a_{T_0 + j}) < (b_{T_0 + j + 1} - a_{T_0 + j + 1}) \) for \( j \geq 0 \), so that
\[
\sum_{T=T_0}^{\infty} \sum_{m=T+1}^{\infty} \frac{b_T - a_T}{(b_m - a_m)^3} = \frac{(b_{T_0} - a_{T_0})}{(b_{T_0+1} - a_{T_0+1})^3} + \frac{(b_{T_0} - a_{T_0})}{(b_{T_0+2} - a_{T_0+2})^3} + \frac{(b_{T_0} - a_{T_0})}{(b_{T_0+3} - a_{T_0+3})^3} + \cdots
\]
\[
\leq \sum_{k=1}^{\infty} \frac{k}{(T_0 + k)^2}
\leq \sum_{k=1}^{\infty} \frac{k}{(T_0 + k)^2}
\to \int_0^1 \frac{x}{(1+x)^2} \, dx = \frac{3}{8}, \text{ as } T_0 \to \infty. \tag{4.16}
\]
That is, \( \tilde{S}_{T_0} < \infty \) for sufficiently large \( T_0 \). In other words, (A6) holds.
4.7 Verification of (A7)

For any set $A \subseteq \Theta$ with $\pi(A) > 0$, it follows that $\mathcal{G}_T \cap A \rightarrow \Theta \cap A = A$. Since $h(\mathcal{G}_T \cap A)$ is non-increasing as $T$ increases, it follows that $h(\mathcal{G}_T \cap A) \rightarrow h(A)$, as $T \rightarrow \infty$.

To summarize, we have the following theorem.

**Theorem 7** Assume that the data was generated by the true model given by (2.1) and (2.2), but modeled by (2.3) and (2.4). Assume (H1)–(H10) and (H11) (i) – (v). Consider any set $A \in T$ with $\pi(A) > 0$. Let $\beta > 2h(A)$, where $\beta$ is given in Section 4.5.1. Then,

$$\lim_{T \rightarrow \infty} \frac{1}{b_T - a_T} \log \pi(A | \mathcal{F}_T) = -J(A). \quad (4.17)$$

5 Consistency and asymptotic normality of the maximum likelihood estimator

Now we make the following further assumption:

(H12) The parameter space $\Theta$ is compact.

Let

$$g_{Y,T}(\theta) = -\frac{K_Y}{2} (\psi_Y(\theta) - \psi_Y(\theta_0))^2 + \sqrt{K_Y} (\psi_Y(\theta) - \psi_Y(\theta_0)) \frac{W_Y(b_T) - W_Y(a_T)}{b_T - a_T}; \quad (5.1)$$

$$g_{X,T}(\theta) = -\frac{K_X}{2} (\psi_X(\theta) - \psi_X(\theta_0))^2 + \frac{K_X}{2} (\psi_X(\theta) - \psi_X(\theta_0))^2. \quad (5.2)$$

Then note that

$$\sup_{\theta \in \Theta} \left| \frac{1}{b_T - a_T} \log R_T(\theta) - g_{Y,T}(\theta) - g_{X,T}(\theta) \right| = \frac{1}{b_T - a_T} \log R_T(\theta^*_T) - g_{Y,T}(\theta^*_T) - g_{X,T}(\theta^*_T), \quad (5.3)$$

for some $\theta^*_T \in \Theta$ where $\theta^*_T$ is dependent on data. Proceeding in the same way as in Section 4.5.2 it is easily seen that (5.3) tends to zero almost surely with respect to both $Y$ and $X$, as $T \rightarrow \infty$. Hence, the maximum likelihood estimator (MLE) can be approximated by maximizing the function

$$\tilde{g}_T(\theta) = g_{Y,T}(\theta) + g_{X,T}(\theta)$$

with respect to $\theta$.

5.1 Strong consistency of the maximum likelihood estimator of $\theta$  

Observe that for $k = 1, \ldots, d$,

$$\frac{\partial \tilde{g}_T(\theta)}{\partial \theta_k} = -K_Y (\psi_Y(\theta) - \psi_Y(\theta_0)) \frac{\partial \psi_Y(\theta)}{\partial \theta_k} - K_X (\psi_X(\theta) - \psi_X(\theta_0)) \frac{\partial \psi_X(\theta)}{\partial \theta_k} + K_X \psi_X(\theta) \frac{\partial \psi_X(\theta)}{\partial \theta_k} + \sqrt{K_Y} \frac{\partial \psi_Y(\theta)}{\partial \theta_k} \frac{W_Y(b_T) - W_Y(a_T)}{b_T - a_T}. \quad (5.4)$$

Let

$$\tilde{g}_T(\theta) = \left( \frac{\partial \tilde{g}_T(\theta)}{\partial \theta_1}, \ldots, \frac{\partial \tilde{g}_T(\theta)}{\partial \theta_d} \right)^T.$$
Also, let \( \tilde{g}_k''(\theta) \) denote the matrix with \((j, k)\)-th element given by

\[
\frac{\partial^2 \tilde{g}_k''(\theta)}{\partial \theta_j \partial \theta_k} = -K_Y \left[ \frac{\partial \psi_Y(\theta)}{\partial \theta_k} \frac{\partial \psi_Y(\theta)}{\partial \theta_j} + \left( \psi_Y(\theta) - \psi_Y(\theta_0) \right) \frac{\partial^2 \psi_Y(\theta)}{\partial \theta_j \partial \theta_k} \right]
- K_X \left[ \frac{\partial \psi_X(\theta)}{\partial \theta_j} \frac{\partial \psi_X(\theta)}{\partial \theta_k} + \left( \psi_X(\theta) - \psi_X(\theta_0) \right) \frac{\partial^2 \psi_X(\theta)}{\partial \theta_j \partial \theta_k} \right]
+ K_X \left[ \frac{\partial \psi_X(\theta)}{\partial \theta_j} \frac{\partial \psi_X(\theta)}{\partial \theta_k} + \psi_X(\theta) \frac{\partial^2 \psi_X(\theta)}{\partial \theta_j \partial \theta_k} \right]
+ \sqrt{K_Y} \frac{\partial^2 \psi_Y(\theta) W_Y(b_T) - W_Y(a_T)}{b_T - a_T}.
\]

(5.5)

Letting \( \hat{\theta}_T \) denote the MLE, note that

\[
0 = \tilde{g}'_T(\hat{\theta}_T) = \tilde{g}'_T(\theta_0) + \tilde{g}'_T(\theta_T^*) (\hat{\theta}_T - \theta_0),
\]

(5.8)

where \( \theta_T^* \) lies between \( \theta_0 \) and \( \hat{\theta}_T \). From (5.7) it is clear that

\[
\left[ \frac{\partial^2 \tilde{g}_k''(\theta)}{\partial \theta_j \partial \theta_k} \right]_{\theta = \theta_0} \overset{a.s.}{\longrightarrow} -K_Y \left[ \frac{\partial \psi_Y(\theta)}{\partial \theta_k} \frac{\partial \psi_Y(\theta)}{\partial \theta_j} \right]_{\theta = \theta_0},
\]

as \( T \to \infty \). Let \( \mathcal{I}(\theta) \) denote the matrix with \((j, k)\)-th element given by

\[
\{ \mathcal{I}(\theta) \}_{jk} = K_Y \left[ \frac{\partial \psi_Y(\theta)}{\partial \theta_j} \frac{\partial \psi_Y(\theta)}{\partial \theta_k} \right].
\]

(5.9)

From (5.6) it is obvious that \( \{ \mathcal{I}(\theta_0) \}_{jk} \) is the covariance between the \( j \)-th and the \( k \)-th components of \( \sqrt{b_T - a_T} \tilde{g}_T'(\theta_0) \), and so \( \mathcal{I}(\theta_0) \) is non-negative definite. We make the following assumptions:

(H13) The true value \( \theta_0 \in \text{int}(\Theta) \), where by \( \text{int}(\Theta) \) we mean the interior of \( \Theta \).

(H14) The matrix \( \mathcal{I}(\theta) \) is positive definite for \( \theta \in \text{int}(\Theta) \).

Hence, from (5.8) we obtain, after pre-multiplying both sides of the relevant equation with \( \mathcal{I}^{-1}(\theta_T^*) \), the following:

\[
- \mathcal{I}^{-1}(\theta_T^*) \tilde{g}_T''(\theta_T^*) (\hat{\theta}_T - \theta_0) = \mathcal{I}^{-1}(\theta_T^*) \tilde{g}_T'(\theta_0).
\]

(5.10)

Since as \( T \to \infty \), \( \tilde{g}_T'(\theta_0) \overset{a.s.}{\longrightarrow} 0 \) and \( -\mathcal{I}^{-1}(\theta_T^*) \tilde{g}_T''(\theta_T^*) \overset{a.s.}{\longrightarrow} \mathcal{I}_d \), \( \mathcal{I}_d \) being the identity matrix of order \( d \), it holds that

\[
\hat{\theta}_T \overset{a.s.}{\longrightarrow} \theta_0.
\]

(5.11)
as \( T \to \infty \), showing that the MLE is strongly consistent. The result can be formalized as the following theorem.

**Theorem 8** Assume that the data was generated by the true model given by (2.1) and (2.2), but modeled by (2.3) and (2.4). Assume conditions (H1)–(H14). Then the MLE of \( \theta \) is strongly consistent in the sense that (5.11) holds.

5.2 **Asymptotic normality of the maximum likelihood estimator of \( \theta \)**

Since \( \hat{\theta}_T \xrightarrow{a.s.} \theta_0 \) and \( \theta^* \) lies between \( \theta_0 \) and \( \hat{\theta}_T \), it follows that \( \theta^* \xrightarrow{a.s.} \theta_0 \) as \( T \to \infty \). This, and the fact that \( (W_Y(b_T) - W_Y(a_T))/\sqrt{b_T - a_T} \sim N(0,1) \), guarantee that

\[
- \sqrt{b_T - a_T} I^{-1}(\theta^*_T) \tilde{g}'_T(\theta_0) \xrightarrow{\mathcal{D}} N\left(0, I^{-1}(\theta_0)\right),
\]

(5.12)

Thus, we can present the following theorem.

**Theorem 9** Assume that the data was generated by the true model given by (2.1) and (2.2), but modeled by (2.3) and (2.4). Assume conditions (H1)–(H14). Then the MLE of \( \theta \) is asymptotically normal in the sense that (5.13) holds.

6 **Asymptotic posterior normality**

Let \( \ell_T(\theta) = \log L_T(\theta) \) stand for the log-likelihood, and let

\[
\Sigma^{-1}_T = \begin{cases} 
-\ell''_T(\hat{\theta}_T) & \text{if the inverse and } \hat{\theta}_T \text{ exist} \\
\mathcal{I}_d & \text{if not,}
\end{cases}
\]

(6.1)

where for any \( z \),

\[
\ell''_T(z) = \left( \left| \frac{\partial^2}{\partial \theta_i \partial \theta_j} \ell_T(\theta) \right|_{\theta=z} \right).
\]

(6.2)

Thus, \( \Sigma^{-1}_T \) is the observed Fisher’s information matrix.

6.1 **Regularity conditions and a theorem of Schervish (1995)**

(1) The parameter space is \( \Theta \subseteq \mathbb{R}^d \) for some finite \( d \).

(2) \( \theta_0 \) is a point interior to \( \Theta \).

(3) The prior distribution of \( \theta \) has a density with respect to Lebesgue measure that is positive and continuous at \( \theta_0 \).

(4) There exists a neighborhood \( \mathcal{N}_0 \subseteq \Theta \) of \( \theta_0 \) on which \( \ell_T(\theta) = \log L_T(\theta) \) is twice continuously differentiable with respect to all co-ordinates of \( \theta \), a.s. \( [P_{\theta_0}] \).

(5) The largest eigenvalue of \( \Sigma_T \) goes to zero in probability.

(6) For \( \delta > 0 \), define \( \mathcal{N}_0(\delta) \) to be the open ball of radius \( \delta \) around \( \theta_0 \). Let \( \rho_T \) be the smallest eigenvalue of \( \Sigma_T \). If \( \mathcal{N}_0(\delta) \subseteq \Theta \), there exists \( K(\delta) > 0 \) such that

\[
\lim_{T \to \infty} P_{\theta_0} \left( \sup_{\theta \in \Theta \setminus \mathcal{N}_0(\delta)} \rho_T [\ell_T(\theta) - \ell_T(\theta_0)] < -K(\delta) \right) = 1.
\]

(6.3)
For each \( \varepsilon > 0 \), there exists \( \delta(\varepsilon) > 0 \) such that
\[
\lim_{T \to \infty} P_{\theta_0} \left( \sup_{\theta \in N(\delta(\varepsilon)), \|\gamma\|=1} \left| 1 + \gamma^T \Sigma_T^{-1/2} \ell_T'(\theta) \Sigma_T^{-1/2} \right| < \varepsilon \right) = 1. \tag{6.4}
\]

**Theorem 10 (Schervish (1995))** Assume the above seven regularity conditions. Then denoting \( \Psi_T = \Sigma_T^{-1/2} (\theta - \hat{\theta}_T) \), for each compact subset \( B \) of \( \mathbb{R}^d \) and each \( \varepsilon > 0 \), the following holds:
\[
\lim_{T \to \infty} P_{\theta_0} \left( \sup_{\Psi_T \in B} \left| \pi(\Psi_T|\mathcal{F}_T) - \varrho(\Psi_T) \right| > \varepsilon \right) = 0, \tag{6.5}
\]
where \( \varrho(\cdot) \) denotes the density of the standard normal distribution.

### 6.2 Verification of the seven regularity conditions for posterior normality

Also we assume that \( \Theta \) is compact (assumption (H11)) which enables us to uniformly approximate \( \frac{1}{b_T - a_T} \log R_T(\theta) \) by \( g_Y T(\theta) + g_X T(\theta) \) for \( \theta \in \Theta \); see Section 5. As a consequence, \( \frac{1}{b_T - a_T} \ell_T'(\theta) \) can be uniformly approximated by \( g_Y T(\theta) + g_X T(\theta) + \frac{1}{b_T - a_T} \log p_T(\theta_0) \), for \( \theta \in \Theta \). Let
\[
\frac{1}{b_T - a_T} \ell_T'(\theta) = g_Y T(\theta) + g_X T(\theta) + \frac{1}{b_T - a_T} \log p_T(\theta_0). \tag{6.6}
\]

Henceforth, we shall be working with \( \frac{1}{b_T - a_T} \ell_T'(\theta) \) whenever convenient. With this, the first four regularity conditions presented in Section 6.1 trivially hold.

To verify regularity condition (5), note that, since \( \hat{\theta}_T \xrightarrow{a.s.} \theta_0 \),
\[
\frac{1}{b_T - a_T} \ell_T'(\theta_T) \xrightarrow{a.s.} -\mathcal{I}(\theta_0). \tag{6.7}
\]

Hence, almost surely,
\[
\Sigma_T^{-1} \sim (b_T - a_T) \times \mathcal{I}(\theta_0),
\]
so that
\[
\Sigma_T \xrightarrow{a.s.} 0,
\]
as \( T \to \infty \). Thus, regularity condition (5) holds.

For verifying condition (6), observe that
\[
\rho_T [\ell_T(\theta) - \ell_T(\theta_0)] = \rho_T (b_T - a_T) \times \frac{1}{b_T - a_T} \log R_T(\theta), \tag{6.8}
\]
where \( \rho_T(b_T - a_T) \to c \), for some \( c > 0 \) and, due to \(4.2\),
\[
\rho_T [\ell_T(\theta) - \ell_T(\theta_0)] \xrightarrow{a.s.} -\frac{c}{2} \left[ K_Y (\psi_Y(\theta) - \psi_Y(\theta_0))^2 + K_X (\psi_X(\theta) - \psi_X(\theta_0))^2 + K_X (\psi_X^2(\theta_0) - \psi_X^2(\theta)) \right], \tag{6.9}
\]
for all \( \theta \in \Theta \setminus \mathcal{N}_0(\delta) \). Now note that

\[
\lim_{T \to \infty} P_{\theta_0} \left( \sup_{\theta \in \Theta \setminus \mathcal{N}_0(\delta)} \rho_T \left[ \ell_T(\theta) - \ell_T(\theta_0) \right] < -K(\delta) \right) \\
\geq \lim_{T \to \infty} P_{\theta_0} \left( (\rho_T(b_T - a_T)) \times \frac{1}{b_T - a_T} \log R_T(\theta) < -K(\delta) \right) \\
= 1,
\]

the last step following due to (6.9). Thus, regularity condition (6) is verified. For verifying condition (7), we note that \( \theta \in \mathcal{N}_0(\delta(\epsilon)) \) can be represented as \( \theta = \theta_0 + \delta_2 \frac{\theta_0}{\|\theta_0\|} \), where \( 0 < \delta_2 \leq \delta(\epsilon) \). Therefore, Taylor's series expansion around \( \theta_0 \) yields

\[
\frac{\tilde{\ell}_T''(\theta)}{b_T - a_T} = \frac{\tilde{\ell}_T''(\theta_0)}{b_T - a_T} + \delta_2 \frac{\tilde{\ell}_T''(\theta^*)\theta_0}{(b_T - a_T)\|\theta_0\|},
\]

where \( \theta^* \) lies between \( \theta_0 \) and \( \theta \). As \( T \to \infty \), \( \frac{\tilde{\ell}_T''(\theta_0)}{b_T - a_T} \) tends to \( -I(\theta_0) \), almost surely. Now notice that

\[
\frac{\|\tilde{\ell}_T''(\theta^*)\theta_0\|}{(b_T - a_T)\|\theta_0\|} \leq \frac{\|\tilde{\ell}_T''(\theta^*)\|}{b_T - a_T}.
\]

Because of (H11) (vii) and compactness of \( \Theta \) it follows that \( \frac{\|\tilde{\ell}_T''(\theta^*)\|}{b_T - a_T} \to 0 \) as \( T \to \infty \). Hence, it follows that \( \tilde{\ell}_T''(\theta) = O\left(- (b_T - a_T) \times I(\theta_0) + (b_T - a_T)\delta_2\right) \), almost surely. Since \( \sum_{T}^{\infty} \) is asymptotically almost surely equivalent to \( (b_T - a_T)^{-1/2} \sum_{T}^{\infty} \), condition (7) holds. We summarize our result in the form of the following theorem.

**Theorem 11** Assume that the data was generated by the true model given by (2.1) and (2.2), but modeled by (2.3) and (2.4). Assume (H1) – (H14). Then denoting \( \Psi_T = \sum_{T}^{\infty} \left( \theta - \hat{\theta}_T \right) \), for each compact subset \( B \) of \( \mathbb{R}^d \) and each \( \epsilon > 0 \), the following holds:

\[
\lim_{T \to \infty} P_{\theta_0} \left( \sup_{\Psi_T \in B} \left| \pi(\Psi_T|\mathcal{F}_T) - \phi(\Psi_T) \right| > \epsilon \right) = 0,
\]

where \( \phi(\cdot) \) denotes the density of the standard normal distribution.

7 Asymptotics in random effects models based on state space SDEs

We now consider the following “true” random effects models based on state space SDEs: for \( i = 1, \ldots, n \), and for \( t \in [0, b_T] \),

\[
dY_i(t) = \phi_{Y_{i,0}}b_Y(Y_i(t), X_i(t), t)dt + \sigma_Y(Y_i(t), X_i(t), t)dW_{Y,i}(t); \tag{7.1}

dX_i(t) = \phi_{X_{i,0}}b_X(X_i(t), t)dt + \sigma_X(X_i(t), t)dW_{X,i}(t). \tag{7.2}
\]

In the above, \( \phi_{Y_{i,0}} = \psi_Y(\theta_0) \) and \( \phi_{X_{i,0}} = \psi_X(\theta_0) \), where \( \psi_Y \) and \( \psi_X \) are known functions; \( \theta_0 \) is the true set of parameters.

Our modeled state space SDE is given, for \( t \in [0, b_T] \) by:

\[
dY_i(t) = \phi_{Y_{i,0}}b_Y(Y_i(t), X_i(t), t)dt + \sigma_Y(Y_i(t), X_i(t), t)dW_{Y,i}(t); \tag{7.3}

dX_i(t) = \phi_{X_{i,0}}b_X(X_i(t), t)dt + \sigma_X(X_i(t), t)dW_{X,i}(t). \tag{7.4}
\]
where \( \phi_{Y,i} = \psi_{Y_i}(\theta) \) and \( \phi_{X,i} = \psi_{X_i}(\theta) \). As before, we wish to learn about the set of parameters \( \theta \). Note that for simplicity of our asymptotic analysis we assumed the same time interval \([0, b_T]\) for \( i = 1, \ldots, n \).

We make the following extra assumptions.

(H15) For every \( \theta \in \Theta \cup \{\theta_0\} \), \( \psi_{Y_i}(\theta) \) and \( \psi_{X_i}(\theta) \) are finite for all \( i = 1, \ldots, n \). And

\[
\begin{align*}
\psi_{Y_i}(\theta) &\to \bar{\psi}_Y(\theta); \\
\psi_{X_i}(\theta) &\to \bar{\psi}_X(\theta),
\end{align*}
\]

as \( i \to \infty \), for all \( \theta \in \Theta \). Also,

\[
\begin{align*}
K_{Y,i} &\to \bar{K}_Y; \\
K_{X,i} &\to \bar{K}_X,
\end{align*}
\]

as \( i \to \infty \).

(H16) Assume that \( \bar{\psi}_Y \) and \( \bar{\psi}_X \) satisfy (H11) (i) – (v).

### 7.1 True likelihood

Here the true likelihood on \([a_T, b_T]\) is given by

\[
\bar{p}_{n,T}(\theta_0) = \prod_{i=1}^{n} p_{T,i}(\theta_0),
\]

(7.5)

where

\[
p_{T,i}(\theta_0) = \int \exp \left( \phi_{Y_i,0} u_{Y_i|X_i,T} - \frac{\phi_{Y,i,0}^2}{2} v_{Y_i|X_i,T} \right) \times \exp \left( \phi_{X_i,0} u_{X_i,T} - \frac{\phi_{X,i,0}^2}{2} v_{X_i,T} \right) \, dQ_{T,X_i}
\]

(7.6)

It follows as in Section 2.4.1 that \( \bar{p}_{n,T}(\theta_0) \xrightarrow{a.s.} \prod_{i=1}^{n} \hat{p}_{T,i}(\theta_0) \), where

\[
\hat{p}_{T,i}(\theta_0) = \exp \left( \frac{(b_T - a_T) K_{Y,i} \phi_{Y,i,0}^2}{2} + \phi_{Y,i,0} \sqrt{K_{Y,i}} (W_{Y_i}(b_T) - W_{Y_i}(a_T)) + (b_T - a_T) K_{X,i} \phi_{X,i,0}^2 \right).
\]

(7.7)

### 7.2 Modeled likelihood

The modeled likelihood in this set-up is given by

\[
L_{n,T}(\theta) = \prod_{i=1}^{n} L_{T,i}(\theta),
\]

where

\[
L_{T,i}(\theta) = \int \exp \left( \phi_{Y_i,u_{Y_i|X_i,T}} - \frac{\phi_{Y,i}^2}{2} v_{Y_i|X_i,T} \right) \times \exp \left( \phi_{X_i,u_{X_i,T}} - \frac{\phi_{X,i}^2}{2} v_{X_i,T} \right) \, dQ_{T,X_i}
\]

(7.8)

As in Section 2.5 here it holds that \( \hat{L}_{n,T}(\theta) \xrightarrow{a.s.} \prod_{i=1}^{n} \hat{L}_{T,i}(\theta) \), where

\[
\hat{L}_{T,i}(\theta) = \exp \left( (b_T - a_T) K_{Y,i} \phi_{Y,i,0}^2 + \phi_{Y,i} \sqrt{K_{Y,i}} (W_{Y_i}(b_T) - W_{Y_i}(a_T)) \\
- \frac{(b_T - a_T) K_{Y,i} \phi_{Y,i,0}^2}{2} + (b_T - a_T) K_{X,i} \phi_{X,i,0} \right).
\]

(7.9)
7.3 Bayesian consistency

We now proceed to verify the assumptions of [Shalizi (2009)]. First note that $L_{T,i} (\theta)$ is measurable with respect to $\mathcal{F}_{T,i} \times \mathcal{T}$, where $\mathcal{F}_{T,i} = \sigma \left( \{ Y_{i,s} ; s \in [a_T, b_T] \} \right)$, the smallest $\sigma$-algebra with respect to which $\{ Y_{i,s} ; s \in [a_T, b_T] \}$ is measurable. Let $\mathcal{F}_{n,T} = \sigma \left( \{ Y_{i,s} ; i = 1, \ldots, n ; s \in [a_T, b_T] \} \right)$. Then for each $i = 1, \ldots, n$, $L_{T,i} (\theta)$ is also $\mathcal{F}_{n,T} \times \mathcal{T}$-measurable. It follows that the likelihood $L_{n,T}(\theta) = \prod_{i=1}^{n} L_{T,i}(\theta)$ is measurable with respect to $\mathcal{F}_{n,T} \times \mathcal{T}$. Hence, (A1) holds.

Let $\bar{R}_{n,T}(\theta) = \frac{L_{n,T}(\theta)}{p_{n,T}(\theta_0)}$. Then
\[
\frac{1}{n(b_T - a_T)} \log \bar{R}_{n,T}(\theta) = \frac{1}{n(b_T - a_T)} \sum_{i=1}^{n} \log R_{T,i},
\]
where
\[
R_{T,i} = \frac{L_{T,i} (\theta)}{p_{T,i}(\theta_0)}.
\]
Since
\[
\frac{1}{b_T - a_T} \log R_{T,i} (\theta) \to -\frac{1}{2} \left[ K_{Y,i} (\psi_{Y,i}(\theta) - \psi_{Y,i}(\theta_0))^2 + K_{X,i} (\psi_{X,i}(\theta) - \psi_{X,i}(\theta_0))^2 + K_{X} (\psi_{X,i}(\theta_0) - \psi_{X,i}^2(\theta)) \right]
\]
for each $i$ as $T \to \infty$, it follows, using (H15), that
\[
\lim_{n \to \infty} \lim_{T \to \infty} \frac{1}{n(b_T - a_T)} \log \bar{R}_{n,T}(\theta) = -\bar{h} (\theta),
\]
almost surely, where
\[
\bar{h} (\theta) = \frac{1}{2} \left[ K_{Y} (\bar{\psi}_{Y}(\theta) - \bar{\psi}_Y (\theta_0))^2 + K_{X} (\bar{\psi}_{X}(\theta) - \bar{\psi}_X (\theta_0))^2 + K_{X} (\bar{\psi}_{X}^2(\theta_0) - \bar{\psi}_{X}^2(\theta)) \right].
\]
Thus, (A2) holds, and noting that $E (W_{Y,i}(b_T) - W_{Y,i}(a_T)) = 0$, it is easy to see that (A3) also holds.

We define, in our current context, the following:
\[
\bar{h} (A) = \text{ess inf}_{\theta \in A} \bar{h} (\theta);
\]
\[
\bar{J} (\theta) = \bar{h} (\theta) - \bar{h} (\Theta);
\]
\[
\bar{J} (A) = \text{ess inf}_{\theta \in A} \bar{J} (\theta).
\]

The way of verification of (A4) remains the same as in Section 4.4, with $I = \{ \theta : \bar{h} (\theta) = \infty \}$. To verify (A5) (i) we define $\bar{G}_{n,T} = \{ \theta : | \bar{\psi}_Y (\theta) | \leq \exp \left( \beta n (b_T - a_T) \right) \}$, where $\beta > 2 \bar{h} (\Theta)$. Coerciveness of $\bar{\psi}_Y$ ensures compactness of $\bar{G}_{n,T}$, and clearly, $\bar{G}_{n,T} \to \Theta$, as $n, T \to \infty$. Moreover,
\[
\pi \left( \bar{G}_{n,T} \right) > 1 - \bar{\alpha} \exp \left( -\beta n (b_T - a_T) \right),
\]
where $0 < \bar{\alpha} = E \left( | \bar{\psi}_Y (\theta) | \right) < \infty$. Verification of (A5) (ii) follows in the same way as in Section 4.5.2 assuming (H10) holds for every $i$, and (A5) (iii) holds in the same way as in Section 4.5.3 with $\bar{h}$ replaced with $\bar{\psi}_Y$ and $\bar{G}_{T}$ replaced with $\bar{G}_{n,T}$. Similarly as in Section 4.6 (A6) holds by additionally replacing $R_{T}$ and $R_{m}$ with $R_{n,T}$ and $R_{n,m}$, respectively. Now, here $Z_m = \frac{1}{n(b_m - a_m)} \log \bar{R}_{n,m}(\theta_T) + \bar{h}(\theta_T)$ and
\[
\tilde{Z}_m = \sqrt{\frac{\bar{K}_Y (\bar{\psi}_Y (\theta_T) - \bar{\psi}_Y (\theta_0))}{n \sqrt{b_m - a_m}}} \times \sum_{i=1}^{n} \frac{W_{Y,i}(b_m) - W_{Y,i}(a_m)}{\sqrt{b_m - a_m}}.
\]
Note that

\[
\frac{\tilde{Z}_m^6}{E(\tilde{Z}_m^6)} = \frac{\left( \sum_{i=1}^{n} \frac{W_Y(b_m) - W_Y(a_m)}{\sqrt{b_m - a_m}} \right)^6}{E\left( \left[ \sum_{i=1}^{n} \frac{W_Y(b_m) - W_Y(a_m)}{\sqrt{b_m - a_m}} \right]^6 \right)} = \frac{E\left( \left[ \sum_{i=1}^{n} \frac{W_Y(b_m) - W_Y(a_m)}{\sqrt{b_m - a_m}} \right]^6 \right)}{E\left( \sum_{i=1}^{n} \frac{W_Y(b_m) - W_Y(a_m)}{\sqrt{b_m - a_m}} \right)^6}, \tag{7.17}
\]

Hence, even in this case,

\[
\frac{Z_m^6 - \tilde{Z}_m^6}{E(\tilde{Z}_m^6)} = \frac{Z_m^6 - \tilde{Z}_m^6}{\tilde{Z}_m^6} \times \frac{\tilde{Z}_m^6}{E(\tilde{Z}_m^6)} \overset{a.s.}{\to} 0 \text{ as } m \to \infty, \tag{7.18}
\]

where the first factor on the right hand side of (7.18) tends to zero almost surely as in Section 4.6, while by the fact that \( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} W_Y(b_T) - W_Y(a_T) \sim N(0, 1) \), the second factor is bounded above by a constant times standard normal distribution raised to the power 6. The rest of the verification is the same as in Section 4.6. It is also easy to see that (A7) holds, as in Section 4.7.

We summarize our results in the form of the following theorem.

**Theorem 12** Let the true, data-generating model be given by (7.1) and (7.2), but let the data be modeled by (7.3) and (7.4). Assume that (H1)–(H10) hold (for each \( i = 1, \ldots, n \), whenever appropriate); also assume (H13)–(H16). Consider any set \( A \in \mathcal{T} \) with \( \pi(A) > 0 \), and let \( \beta > 2\hat{h}(A) \). Then,

\[
\lim_{n \to \infty, T \to \infty} \frac{1}{n(b_T - a_T)} \log \pi(A | \mathcal{F}_n, T) = - \bar{J}(A). \tag{7.19}
\]

### 7.4 Strong consistency and asymptotic normality of the maximum likelihood estimator of \( \theta \)

We now replace (H16) with

(H16') Assume that \( \tilde{\psi}_Y \) and \( \tilde{\psi}_X \) satisfy (H11) (i) – (vii).

Let

\[
\bar{g}_{Y,T}(\theta) = -\frac{K_Y}{2} (\tilde{\psi}_Y(\theta) - \tilde{\psi}_Y(\theta_0))^2 + \sqrt{K_Y} (\tilde{\psi}_Y(\theta) - \tilde{\psi}_Y(\theta_0)) \frac{1}{n} \sum_{i=1}^{n} \frac{W_Y(a_T) - W_Y(b_T)}{b_T - a_T}; \tag{7.20}
\]

\[
\bar{g}_{X,T}(\theta) = -\frac{K_X}{2} (\tilde{\psi}_X(\theta) - \tilde{\psi}_X(\theta_0))^2 + \frac{K_X}{2} (\tilde{\psi}_X(\theta) - \tilde{\psi}_X^2(\theta)) \tag{7.21}
\]

Then note that

\[
\sup_{\theta \in \Theta} \frac{1}{n(b_T - a_T)} \log \bar{R}_{n,T}(\theta) - \bar{g}_{Y,T}(\theta) - \bar{g}_{X,T}(\theta) = \frac{1}{n(b_T - a_T)} \log \bar{R}_{n,T}(\tilde{\theta}_{n,T}) - \bar{g}_{Y,T}(\tilde{\theta}_{n,T}) - \bar{g}_{X,T}(\tilde{\theta}_{n,T}) \tag{7.22}
\]

for some \( \tilde{\theta}_{n,T} \in \Theta \). As before despite the dependence of \( \tilde{\theta}_{n,T} \) on data it can be shown that (7.22) tends to zero as \( T \to \infty \). So, it is permissible to approximate the MLE by maximizing

\[
\bar{g}_{n,T}(\theta) = \tilde{g}_{Y,T}(\theta) + \tilde{g}_{X,T}(\theta)
\]

with respect to \( \theta \).

Let

\[
\tilde{g}_{n,T}^T(\theta) = \left( \frac{\partial \bar{g}_{n,T}(\theta)}{\partial \theta_1}, \ldots, \frac{\partial \bar{g}_{n,T}(\theta)}{\partial \theta_d} \right)^T;
\]

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and let $\tilde{g}_{n,T}''(\theta)$ be the matrix of second derivatives. The relevant elements at $\theta = \theta_0$ are given by

$$
\left[ \frac{\partial^2 \tilde{g}_{n,T}''(\theta)}{\partial \theta_j \partial \theta_k} \right]_{\theta = \theta_0} = \sqrt{K_Y} \left[ \frac{\partial \tilde{w}_Y(\theta)}{\partial \theta_j} \frac{\partial \tilde{w}_Y(\theta)}{\partial \theta_k} \right]_{\theta = \theta_0} + \sqrt{K_Y} \left[ \frac{\partial^2 \tilde{w}_Y(\theta)}{\partial \theta_j \partial \theta_k} \right]_{\theta = \theta_0} \frac{1}{n} \sum_{i=1}^{n} \frac{W_{Y_i}(b_T) - W_{Y_i}(a_T)}{b_T - a_T},
$$

(7.23)

$$
\left[ \frac{\partial \tilde{g}_{n,T}''(\theta)}{\partial \theta_k} \right]_{\theta = \theta_0} = \sqrt{K_Y} \left[ \frac{\partial \tilde{w}_Y(\theta)}{\partial \theta_k} \right]_{\theta = \theta_0} \frac{1}{n} \sum_{i=1}^{n} \frac{W_{Y_i}(b_T) - W_{Y_i}(a_T)}{b_T - a_T},
$$

(7.24)

In this case, the $(j, k)$-th element of the matrix $\mathcal{I}(\theta_0)$ is given by

$$
\mathcal{I}(\theta_0)_{jk} = \sqrt{K_Y} \left[ \frac{\partial \tilde{w}_Y(\theta)}{\partial \theta_j} \frac{\partial \tilde{w}_Y(\theta)}{\partial \theta_k} \right]_{\theta = \theta_0},
$$

(7.25)

and the MLE $\hat{\theta}_{n,T}$ satisfies

$$
\mathcal{I}^{-1}(\hat{\theta}_{n,T}) \tilde{g}_{n,T}''(\theta_0) (\hat{\theta}_{n,T} - \theta_0) = - \mathcal{I}^{-1}(\hat{\theta}_{n,T}) \tilde{g}_{n,T}''(\theta_0),
$$

(7.26)

where $\theta_{n,T}$ lies between $\theta_0$ and $\hat{\theta}_{n,T}$. It is easily seen as in Section 5.1 that

$$
\hat{\theta}_{n,T} \xrightarrow{a.s.} \theta_0,
$$

(7.27)

as $n \to \infty$, $T \to \infty$.

**Theorem 13** Let the true, data-generating model be given by (7.1) and (7.2), but let the data be modeled by (7.3) and (7.4). Assume that (H1)–(H10), (H12)–(H15) and (H16') hold (for each $i = 1, \ldots, n$, whenever appropriate). Then the MLE of $\theta$ is strongly consistent in the sense that (7.27) holds.

Moreover, following the same ideas presented in Section 5.2 and employing (H15'), it is easily seen that asymptotic normality also holds. Formally, we have the following theorem.

**Theorem 14** Let the true, data-generating model be given by (7.1) and (7.2), but let the data be modeled by (7.3) and (7.4). Assume that (H1)–(H10), (H12)–(H15) and (H16') hold (for each $i = 1, \ldots, n$, whenever appropriate). Then

$$
\sqrt{n(b_T - a_T)} (\hat{\theta}_{n,T} - \theta_0) \xrightarrow{d} N_d \left( 0, \mathcal{I}^{-1}(\theta_0) \right),
$$

(7.28)

as $n \to \infty$, $T \to \infty$.

**7.5 Asymptotic posterior normality**

From Section 7.4 (see (7.22)) it is evident that $\frac{1}{n(b_T - a_T)} \ell_{n,T}(\theta)$, where $\ell_{n,T}(\theta) = \log L_{n,T}(\theta)$, can be uniformly approximated by

$$
\frac{1}{n(b_T - a_T)} \tilde{\ell}_{n,T}(\theta) = \tilde{g}_{Y,T}(\theta) + \tilde{g}_{X,T}(\theta) + \frac{1}{n(b_T - a_T)} \log \tilde{p}_{n,T}(\theta),
$$

(7.29)

for $\theta \in \Theta$. With this approximate version, it is again easy to see that the first four regularity conditions presented in Section 6.1 trivially hold.

We now verify regularity condition (5). Since, as $n \to \infty$, $T \to \infty$, $\hat{\theta}_{n,T} \xrightarrow{a.s.} \theta_0$,

$$
\frac{1}{n(b_T - a_T)} \tilde{\ell}_{n,T}'(\hat{\theta}_{n,T}) \xrightarrow{a.s.} -\mathcal{I}(\theta_0),
$$

(7.30)
Thus, as before, almost surely,
\[
\Sigma_{n,T}^{-1} \sim n(b_T - a_T) \times \mathcal{I}(\theta_0),
\]
where
\[
\Sigma_{n,T}^{-1} = \begin{cases} 
- \bar{\ell}'_{n,T}(\hat{\theta}_{n,T}) & \text{if the inverse and } \hat{\theta}_{n,T} \text{ exist} \\
\mathcal{J}_d & \text{if not},
\end{cases}
\]
(7.31)
Hence,
\[
\Sigma_{n,T} \xrightarrow{a.s.} 0,
\]
as \(n \to \infty, T \to \infty\). Thus, regularity condition (5) holds.

For verifying condition (6), observe that
\[
\rho_n \left[ \ell_{n,T}(\theta) - \ell_{n,T}(\theta_0) \right] = \rho_n(b_T - a_T) \times \frac{1}{n(b_T - a_T) \log R_{n,T}(\theta)},
\]
(7.32)
where \(\rho_n\) is the smallest eigenvalue of \(\Sigma_{n,T}\), and, as in Section 5.2, \(\rho_n \to \bar{c}\), for some \(\bar{c} > 0\). Then, as in (6.9), it holds that
\[
\rho_n \left[ \ell_{n,T}(\theta) - \ell_{n,T}(\theta_0) \right] \xrightarrow{a.s.} - \frac{\bar{c}}{2} \left[ \bar{K}_Y \left( \bar{\psi}_Y(\theta) - \bar{\psi}_Y(\theta_0) \right)^2 + \bar{K}_X \left( \bar{\psi}_X(\theta) - \bar{\psi}_X(\theta_0) \right)^2 \right],
\]
for all \(\theta \in \Theta \setminus \mathcal{N}_0(\delta)\). Then, in the same way as in (6.10) it follows that
\[
\lim_{n \to \infty, T \to \infty} P_{\theta_0} \left( \sup_{\theta \in \Theta \setminus \mathcal{N}_0(\delta)} \rho_n \left[ \ell_{n,T}(\theta) - \ell_{n,T}(\theta_0) \right] < -K(\delta) \right) = 1.
\]
In other words, condition (6) holds.

Condition (7) can be verified essentially in the same way as in Section 5.2. As in Section 5.2 using continuity of the third derivatives of \(\bar{\psi}_Y\) and \(\bar{\psi}_X\), as assumed in (H15’), it can be shown that
\[
\bar{\ell}''_{n,T}(\theta) = O \left( -n(b_T - a_T) \times \mathcal{I}(\theta_0) + n(b_T - a_T)\bar{\delta}_2 \right),
\]
almost surely. It is also easy to see that \(\Sigma_{n,T}^{3/2}\) is asymptotically almost surely equivalent to \(n^{-2} (b_T - a_T)^{-1/2} \bar{\mathcal{I}}^{-1/2}(\theta_0)\). Thus, condition (7) holds.

We summarize our result in the form of the following theorem.

**Theorem 15** Let the true, data-generating model be given by (7.1) and (7.2), but let the data be modeled by (7.3) and (7.4). Assume that (H1) – (H10), (H12)–(H15) and (H16’) hold (for every \(i = 1, \ldots, n\), whenever appropriate). Then denoting \(\bar{\Psi}_{n,T} = \Sigma_{n,T}^{-1/2} \left( \theta - \hat{\theta}_{n,T} \right)\), for each compact subset \(B \) of \(\mathbb{R}^d\) and each \(\epsilon > 0\), the following holds:
\[
\lim_{n \to \infty, T \to \infty} P_{\theta_0} \left( \sup_{\bar{\Psi}_{n,T} \in B} \left| \pi(\bar{\Psi}_{n,T}|\mathcal{F}_{n,T}) - \theta(\bar{\Psi}_T) \right| > \epsilon \right) = 0.
\]
(7.34)

## 8 Summary and discussion

In this paper, we have investigated the asymptotic properties of the MLE and the posterior distribution of the set of parameters associated with state space SDE’s and random effects state space SDE’s. In particular, we have established posterior consistency based on Shalizi (2009) and asymptotic posterior normality based on Schervish (1995). In addition, we have also established strong consistency and asymptotic normality of the MLE associated with our state space SDE models. Acknowledging the importance of discretization in practical scenarios, we have shown (in Section S-5 of the supplement) that our results go through even with discretized data.

In the case of our random effects SDE models we only required independence of the state space models for different individuals. That is, our approach and the results remain intact if the initial values
for the processes associated with the individuals are different. This is in contrast with the asymptotic works of Maitra and Bhattacharya (2016a) and Maitra and Bhattacharya (2015) in the context of independent but non-identical random effects models for the individuals. Although not based on state space SDE’s, their approach required the simplifying assumption that the sequence of initial values is a convergent subsequence of some sequence in some compact space.

In fact, the relative simplicity of our current approach is due to the assumption of stochastic stability of the latent processes of our models, the key concept that we adopted in our approach to alleviate the difficulties of the asymptotic problem at hand. Specifically, we adopted the conditions of Theorem 6.2 provided in Mao (2011), as sufficient conditions of our results. Indeed, there is a large literature on stochastic stability of solutions of SDE’s, with very many existing examples (see, for example, Mao (2011) and the references therein), which indicate that the assumption of stochastic stability is not unrealistic.

In our work we have assumed stochastic stability of $X$ only. If, in addition, asymptotic stability of $Y$ is also assumed, then our results hold good by replacing (H3) in Section 2 with the following assumptions:

(H3(i)) $b_Y(0, 0, t) = 0 = \sigma_Y(0, 0, t)$ for all $t \geq 0$.

(H3(ii)) For every $T > 0$, there exist positive constants $K_{1,T}$, $K_{2,T}$, $\alpha_1$, $\alpha_2$, $\beta_1$ and $\beta_2$ such that for all $(x, t) \in \mathbb{R} \times [0, T]$,

$$K_{Y,1,T} \left(1 - \alpha_1 x^2 - \beta_1 y^2\right) \leq \frac{b_Y^2(y, x, t)}{\sigma_Y^2(y, x, t)} \leq K_{Y,2,T} \left(1 + \alpha_2 x^2 + \beta_2 y^2\right),$$

where $K_{Y,1,T} \to K_Y$ and $K_{Y,2,T} \to K_Y$ and as $T \to \infty$; $K_Y$ being a positive constant as mentioned in (H3).

In this case the bounds of $\frac{b_Y^2(y, x, t)}{\sigma_Y^2(y, x, t)}$ are somewhat more general than in (H3) in that they depend upon both $x$ and $y$, while in (H3) the bounds are independent of $y$.

To our knowledge, our work is the first time effort towards establishing asymptotic results in the context of state space SDE’s, and the results we obtained are based on relatively general assumptions which are satisfied by a large class of models. Since the notion of stochastic stability is valid for any dimension of the associated SDE, it follows that our results admit straightforward extension to high-dimensional state space SDE’s. Corresponding results in the multidimensional extension of the random effects is provided briefly in Section S-4 of the supplement.

As we mentioned in the introduction, our random effects state space SDE model can not be interpreted as a bona fide random effects model from the classical perspective, and that introduction of actual random effects would complicate our method of asymptotic investigation. Also, in this article we have assumed that the diffusion coefficient are free of parameters, which is not a very realistic assumption. We are working on these issues currently, and will communicate our findings subsequently.

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We only need to verify that for any measurable and integrable function $g_T : \mathcal{Y}_T \rightarrow \mathbb{R}$, $E_{T,X} \left[ E_{T,Y|X} \{g_T(Y)\} \right] = E_{T,Y} \{g(Y)\}$, where $\mathcal{Y}_T$ denotes the sample space of $\{Y(t) : t \in [a_T, b_T]\}$, $E_{T,X}$ denotes the marginal expectation with respect to the Girsanov formula based density dominated by $Q_{T,X}$, $E_{T,Y|X}$ is the conditional expectation with respect to the Girsanov formula based conditional density dominated by $Q_{T,Y|X}$, and $E_{T,Y}$ stands for the marginal expectation with respect the proposed density $p_T(\theta_0)$ and the proposed dominating measure $Q_{T,Y}$. All the quantities are associated with $[a_T, b_T]$. Note that $E_{T,X} \left[ E_{T,Y|X} \{|g_T(Y)|\} \right] < \infty$ if and only if $E_{T,Y} \{|g_T(Y)|\} < \infty$, which easily follows from Tonelli’s theorem related to interchange of orders of integration for non-negative integrands.

Now, due to Fubini’s theorem, for such integrable measurable function $g_T$,

$$E_{T,Y} \{g_T(Y)\} = \int_{\mathcal{Y}_T} g_T(y)p_T(\theta_0)dQ_{T,Y}$$

$$= \int_{\mathcal{Y}_T} g_T(y) \int_{\mathcal{X}_T} \exp \left( \phi_{Y,0}u_{Y|X,T} - \frac{\phi_{Y,0}^2}{2}v_{Y|X,T} \right) \times \exp \left( \phi_{X,0}u_{X,T} - \frac{\phi_{X,0}^2}{2}v_{X,T} \right) dQ_{T,X} dQ_{T,Y}$$

$$= \int_{\mathcal{X}_T} \left[ \int_{\mathcal{Y}_T} g_T(y) \exp \left( \phi_{Y,0}u_{Y|X,T} - \frac{\phi_{Y,0}^2}{2}v_{Y|X,T} \right) \right] dQ_{T,X} dQ_{T,Y|X}$$

$$= \int_{\mathcal{X}_T} \left\{ \int_{\mathcal{Y}_T} g_T(y) \exp \left( \phi_{Y,0}u_{Y|X,T} - \frac{\phi_{Y,0}^2}{2}v_{Y|X,T} \right) \right\} dQ_{T,Y|X}$$

$$\times \exp \left( \phi_{X,0}u_{X,T} - \frac{\phi_{X,0}^2}{2}v_{X,T} \right) dQ_{T,X}$$

$$= \int_{\mathcal{X}_T} E_{T,Y|X} \{g_T(Y)\} \times \exp \left( \phi_{X,0}u_{X,T} - \frac{\phi_{X,0}^2}{2}v_{X,T} \right) dQ_{T,X}$$

$$= E_{T,X} \left[ E_{T,Y|X} \{g_T(Y)\} \right] \text{ (S-1.1)}$$

since $\int_{\mathcal{X}_T} dQ_{T,X} = 1$ as $Q_{T,X}$ is a probability measure. In particular, letting $g_T(y) = \mathbb{I}_{\mathcal{Y}_T}(y)$, where for any set $A$, $\mathbb{I}_A$ denotes the indicator function of $A$, the right hand side of (S-1.1) becomes 1, showing that $p_T(\theta_0)$ is the correct density with respect to $Q_{T,Y}$.

S-2 Proof of Lemma 2 of MB

Since the proofs of (2.40) and (2.41) of MB are same, we provide the proof of (2.40) only.
Consider the sequence $\delta_T \downarrow 0$ introduced in (H9). Then due to continuity of the exponential function,

$$P \left( \left| \exp \left( I_{Y,X,T} \right) - \exp \left( \sqrt{K_Y} \left( W_Y(b_T) - W_Y(a_T) \right) \right) \right| > \epsilon \right)$$

$$\leq P \left( \left| I_{Y,X,T} - \sqrt{K_Y} \left( W_Y(b_T) - W_Y(a_T) \right) \right| > \delta_T \right)$$

$$+ P \left( \left| I_{Y,X,T} - \sqrt{K_Y} \left( W_Y(b_T) - W_Y(a_T) \right) \right| \leq \delta_T, \right.$$

$$\left| \exp \left( I_{Y,X,T} \right) - \exp \left( \sqrt{K_Y} \left( W_Y(b_T) - W_Y(a_T) \right) \right) \right| > \epsilon \right).$$  \hspace{1cm} (S-2.1)

The choice of $\delta_T \downarrow 0$ guarantees via (H9) that the terms (S-2.2) yield a convergent sum.

We now turn to $P \left( \left| I_{Y,X,T} - \sqrt{K_Y} \left( W_Y(b_T) - W_Y(a_T) \right) \right| > \delta_T \right)$. Note that, almost surely, it holds due to (H3) and (1.5) of MB, that

$$\left| \frac{b_Y(Y(s), X(s), s)}{\sigma_Y(Y(s), X(s), s)} - \sqrt{K_Y} \right| \leq \frac{b_Y(Y(s), X(s), s)}{\sigma_Y(Y(s), X(s), s)} \leq \sqrt{K_Y}$$

$$\leq \sqrt{K_{Y,2,T}} \sqrt{1 + \alpha_{Y,2,T} \xi^2 \lambda^2(s) + \sqrt{K_Y}}$$

$$\leq 2 \max \left\{ \sqrt{K_{Y,2,T}}, \sqrt{K_Y} \right\} \sqrt{1 + \alpha_{Y,2,T} \xi^2 \lambda^2(s)}. \hspace{1cm} (S-2.3)$$

Now, noting the fact that for $k_0 \geq 1$, $\lambda^{2k_0} Y(s) \leq \lambda^2(s)$ since $\lambda(t) \to 0$ as $t \to \infty$ and (H9), it holds due to (S-2.3), that

$$\delta_T^{-2k_0} (b_T - a_T)^{k_0-1} E \int_{a_T}^{b_T} \left| \frac{b_Y(Y(s), X(s), s)}{\sigma_Y(Y(s), X(s), s)} - \sqrt{K_Y} \right|^{2k_0} ds$$

$$\leq 2^{2k_0} \max \left\{ K_{Y,2,T}^{-k_0}, K_Y^{k_0} \right\} \delta_T^{-2k_0} (b_T - a_T)^{k_0-1} E \int_{a_T}^{b_T} \left( 1 + \alpha_{Y,2,T} \xi^2 \lambda^2(s) \right)^{k_0} ds \hspace{1cm} (S-2.4)$$

$$< \infty. \hspace{1cm} \text{(due to (H9))} \hspace{1cm} (S-2.5)$$

Due to (S-2.5) it follows that (see, Theorem 7.1 of Mao (2011), page 39)

$$\delta_T^{-2k_0} (b_T - a_T)^{k_0-1} E \int_{a_T}^{b_T} \left| \frac{b_Y(Y(s), X(s), s)}{\sigma_Y(Y(s), X(s), s)} - \sqrt{K_Y} \right|^{2k_0} dW_Y(s)$$

$$\leq (k_0(2k_0 - 1))^{k_0} \delta_T^{-2k_0} (b_T - a_T)^{k_0-1} E \int_{a_T}^{b_T} \left| \frac{b_Y(Y(s), X(s), s)}{\sigma_Y(Y(s), X(s), s)} - \sqrt{K_Y} \right|^{2k_0} ds. \hspace{1cm} (S-2.6)$$

Hence, using Chebychev’s inequality, it follows using (S-2.6) that for any $\epsilon > 0$,

$$P \left( \left| \int_{a_T}^{b_T} \left| \frac{b_Y(Y(s), X(s), s)}{\sigma_Y(Y(s), X(s), s)} - \sqrt{K_Y} \right| dW_Y(s) \right| > \delta_T \right)$$

$$< (k_0(2k_0 - 1))^{k_0} \delta_T^{-2k_0} (b_T - a_T)^{k_0-1} E \int_{a_T}^{b_T} \left| \frac{b_Y(Y(s), X(s), s)}{\sigma_Y(Y(s), X(s), s)} - \sqrt{K_Y} \right|^{2k_0} ds. \hspace{1cm} (S-2.7)$$

Using (S-2.4) and (H9), it follows that

$$\sum_{T=1}^{\infty} P \left( \left| \int_{a_T}^{b_T} \left| \frac{b_Y(Y(s), X(s), s)}{\sigma_Y(Y(s), X(s), s)} - \sqrt{K_Y} \right| dW_Y(s) \right| > \delta_T \right) < \infty. \hspace{1cm} (S-2.8)$$
Combining (2.39) of (H9) and (S-2.8) it follows that for all \( \varepsilon > 0 \),
\[
\sum_{T=1}^{\infty} P \left( \left| \exp (I_{Y,X,T}) - \exp \left( \sqrt{K_Y} (W_Y(b_T) - W_Y(a_T)) \right) \right| > \varepsilon \right) < \infty,
\]
proving that
\[
\exp (I_{Y,X,T}) - \exp \left( \sqrt{K_Y} (W_Y(b_T) - W_Y(a_T)) \right) \overset{a.s.}{\longrightarrow} 0.
\]

**S-3 Proof of Theorem 4 of MB**

Since the proofs of (2.44) and (2.45) of MB are similar, we prove only (2.44).

By Lemma 2
\[
\frac{\exp (I_{Y,X,T}) - \exp \left( \sqrt{K_Y} (W_Y(b_T) - W_Y(a_T)) \right)}{\exp \left( \sqrt{K_Y} (W_Y(b_T) - W_Y(a_T)) \right)} \overset{a.s.}{\longrightarrow} 1.
\]

Similarly,
\[
\frac{\exp (I_{X,T}) - \exp \left( \sqrt{K_X} (W_X(b_T) - W_X(a_T)) \right)}{\exp \left( \sqrt{K_X} (W_X(b_T) - W_X(a_T)) \right)} \overset{a.s.}{\longrightarrow} 1.
\]

By (H3) and (H7), \( ((b_T - a_T)|K_{Y,j,T} - K_Y| \rightarrow 0 \) and \( ((b_T - a_T)|K_{X,j,T} - K_X| \rightarrow 0 \), for \( j = 1, 2 \).

Hence, it also holds that \( ((b_T - a_T)|K_{Y,1,T} - K_{Y,2,T}| \rightarrow 0 \) and \( ((b_T - a_T)|K_{X,1,T} - K_{X,2,T}| \rightarrow 0 \).

Also, by (H9), \( \int_{a_T}^{b_T} \lambda^2(s)ds \rightarrow 0 \). Hence, it follows, as \( \xi \) is a finite random variable, that
\[
\exp \left( \phi^2_{Y,0} |K_{Y,\xi,T,2} - \frac{\phi^2_{Y,0} K_{Y,\xi,T,1}}{2} \right) \exp \left( (b_T - a_T) K_{Y,\xi,T} \phi^2_{Y,0} \right)
\]
\[
= \exp \left[ \frac{\phi^2_{Y,0}}{2} (b_T - a_T)(K_{Y,2,T} - K_Y) + \frac{\phi^2_{Y,0}}{2} (b_T - a_T)(K_{Y,2,T} - K_{Y,1,T}) \right]
\]
\[
+ \frac{\phi^2_{Y,0}}{2} (K_{Y,2,T} \alpha_{Y,2} - K_{Y,1,T} \alpha_{Y,1}) \xi^2 \int_{a_T}^{b_T} \lambda^2(s)ds \right] \overset{a.s.}{\longrightarrow} 1,
\]
and, similarly,
\[
\exp \left( \phi^2_{X,0} |K_{X,\xi,T,2} - \frac{\phi^2_{X,0} K_{X,\xi,T,1}}{2} \right) \exp \left( (b_T - a_T) K_{X,\xi,T} \phi^2_{X,0} \right) \overset{a.s.}{\longrightarrow} 1,
\]
as \( T \rightarrow \infty \).
Now, let
\[
\hat{Z}_{T,\theta_0}(W_X) = \exp \left( \frac{(b_T - a_T)K_Y \phi_{Y,0}^2}{2} + \phi_{Y,0} \sqrt{K_Y} (W_Y(b_T) - W_Y(a_T)) \right)
\]
\[\times \exp \left( \frac{(b_T - a_T)K_X \phi_{X,0}^2}{2} + \phi_{X,0} \sqrt{K_X} (W_X(b_T) - W_X(a_T)) \right) .
\]

From (S-3.1), (S-3.2), (S-3.3) and (S-3.4) it follows that \(Z_{U,T,\theta_0(X)} / \hat{Z}_{T,\theta_0}(W_X) \to 1\), almost surely with respect to \(W_X\) and \(X\), as \(T \to \infty\), given any fixed \(W_Y\) in the respective non-null set. That is, given any sequences \(\{Z_{U,T,\theta_0(X)} : T > 0\}\) and \(\{\hat{Z}_{T,\theta_0}(W_X) : T > 0\}\) associated with the complement of null sets, for any \(\epsilon > 0\), there exists \(T_0(\epsilon, W_Y, W_X) > 0\) such that for \(T \geq T_0(\epsilon, W_Y, W_X)\),
\[
(1 - \epsilon) \hat{Z}_{T,\theta_0}(W_X) \leq Z_{U,T,\theta_0(X)} \leq (1 + \epsilon) \hat{Z}_{T,\theta_0}(W_X), \quad (S-3.5)
\]
for almost all \(X\). Thus, letting \(g_{T,\theta_0}(W_X) = E[Z_{U,T,\theta_0(X)}|W_X]\), it follows that \(g_{T,\theta_0}(W_X) - \hat{Z}_{T,\theta_0}(W_X) \xrightarrow{a.s.} 0\). In fact,
\[
g_{T,\theta_0}(W_X) - \hat{Z}_{T,\theta_0}(W_X) \xrightarrow{a.s.} 0.
\]

By (H10),
\[
\sup_{T > 0} E \left( \frac{Z_{U,T,\theta_0(X)} - \hat{Z}_{T,\theta_0}(W_X)}{\hat{p}_T(\theta_0)} \right) \leq \sup_{T > 0} E \left( \frac{Z_{U,T,\theta_0(X)}}{\hat{p}_T(\theta_0)} \right) + 1 < \infty.
\]

Minor modification of Lemma B.119 of Schervish (1995) then guarantee that \(g_{T,\theta_0}(W_X) - \hat{Z}_{T,\theta_0}(W_X) / \hat{p}_T(\theta_0)\) is uniformly integrable. Hence,
\[
E \left( \frac{g_{T,\theta_0}(W_X) - \hat{Z}_{T,\theta_0}(W_X)}{\hat{p}_T(\theta_0)} \right) \to 0,
\]
so that
\[
E \left( \frac{g_{T,\theta_0}(W_X)}{\hat{p}_T(\theta_0)} \right) \to 1, \text{ as } T \to \infty.
\]
In other words, for almost all \(W_Y\),
\[
B_{U,T}(\theta_0) / \hat{p}_T(\theta_0) \to 1, \text{ as } T \to \infty. \quad (S-3.6)
\]

In the same way it follows that for almost all \(W_Y\),
\[
B_{L,T}(\theta_0) / \hat{p}_T(\theta_0) \to 1, \text{ as } T \to \infty. \quad (S-3.7)
\]
Combining (S-3.6) and (S-3.7) it follows that \(p_T(\theta_0) \sim \hat{p}_T(\theta_0)\) for almost all \(W_Y\).
S-4 Asymptotic theory for multidimensional linear random effects

We now consider the following true, multidimensional linear random effects models based on state space SDEs: for \( i = 1, \ldots, n \), and for \( t \in [0, b_T] \),

\[
dY_i(t) = \phi_{Y,0}^T b_Y(Y_i(t), X_i(t), t)dt + \sigma_Y(Y_i(t), X_i(t), t)dW_{Y,i}(t); \\
dX_i(t) = \phi_{X,0}^T b_X(X_i(t), t)dt + \sigma_X(X_i(t), t)dW_{X,i}(t).
\]

(S-4.1) (S-4.2)

In the above, \( \phi_{Y,0} = \phi_{Y,0}(\theta_0) = (\psi_{Y,1}(\theta_0), \ldots, \psi_{Y,R_Y}(\theta_0))^T \) and \( \phi_{X,0} = \phi_{X,0}(\theta_0) = (\psi_{X,1}(\theta_0), \ldots, \psi_{X,R_X}(\theta_0))^T \), where \{ \psi_{Y,j}; j = 1, \ldots, R_Y \} and \{ \psi_{X,j}; j = 1, \ldots, R_X \} are known functions, \( R_Y (> 1) \) and \( R_X (> 1) \) are dimensions of the multivariate functions \( \phi_{Y,0} \) and \( \phi_{X,0}; \theta_0 \) is the true set of parameters. Also, \( b_Y(y, x) = (b_{Y,1}(y, x), \ldots, b_{Y,R_Y}(y, x))^T \) and \( b_X(y, x) = (b_{X,1}(y, x), \ldots, b_{X,R_X}(y, x))^T \) are \( R_Y \) and \( R_X \) dimensional functions respectively.

Our modeled state space SDE is given, for \( t \in [0, b_T] \) by:

\[
dY_i(t) = \phi_{Y,0}^T b_Y(Y_i(t), X_i(t), t)dt + \sigma_Y(Y_i(t), X_i(t), t)dW_{Y,i}(t); \\
dX_i(t) = \phi_{X,0}^T b_X(X_i(t), t)dt + \sigma_X(X_i(t), t)dW_{X,i}(t),
\]

where \( \phi_{Y,i} = \phi_{Y,i}(\theta) = (\psi_{Y,1}(\theta), \ldots, \psi_{Y,R_Y}(\theta))^T \) and \( \phi_{X,i} = \phi_{X,i}(\theta) = (\psi_{X,1}(\theta), \ldots, \psi_{X,R_X}(\theta))^T \).

In this section we generalize our asymptotic theory in the case of the above multidimensional random effects models based on state space SDEs.

Let \( \tilde{b}_Y(y, x, \theta_0) = \phi_{Y,0}^T b_Y(y, x) \) and \( b_Y(y, x, \theta) = \phi_{Y,\theta}^T b_Y(y, x) \). Also let \( \tilde{b}_X(x, \theta_0) = \phi_{X,0}^T b_X(x) \) and \( b_X(x, \theta) = \phi_{X,\theta}^T b_X(x) \). We assume that given any \( \theta \in \Theta, \tilde{b}_{Y,i} \) and \( \tilde{b}_{X,i} \) satisfy conditions (H1) – (H7) of MB. However, we replace (H3) and (H7) with the following:

(H3') For every pair \( (j_1, j_2); j_1 = 1, \ldots, R_Y; j_2 = 1, \ldots, R_Y \), and for every \( T > 0 \), there exist positive constants \( K_{Y,1,T,j_1,j_2}, K_{Y,2,T,j_1,j_2}, \alpha_{Y,1,j_1,j_2}, \alpha_{Y,2,j_1,j_2} \) such that for all \( (x, t) \in \mathbb{R} \times [0, b_T] \),

\[
K_{Y,1,T,j_1,j_2} \left(1 - \alpha_{Y,1,j_1,j_2} x^2 \right) \leq \frac{b_{Y,j_1}(y, x, t)b_{Y,j_2}(y, x, t)}{\sigma_Y^2(y, x, t)} \leq K_{Y,2,T,j_1,j_2} \left(1 + \alpha_{Y,2,j_1,j_2} x^2 \right),
\]

where \( K_{Y,1,T,j_1,j_2} \rightarrow K_{Y,j_1,j_2} \) and \( K_{Y,2,T,j_1,j_2} \rightarrow K_{Y,j_1,j_2} \) as \( T \rightarrow \infty \); \( K_{Y,j_1,j_2} \) being positive constants. We further assume for \( k = 1, 2, (b_T - a_T)|K_{Y,k,T,j_1,j_2} - K_{Y,j_1,j_2}| \rightarrow 0 \), as \( T \rightarrow \infty \).

(H7') For every pair \( (j_1, j_2); j_1 = 1, \ldots, R_X; j_2 = 1, \ldots, R_X \), and for every \( T > 0 \), there exist positive constants \( K_{X,1,T,j_1,j_2}, K_{X,2,T,j_1,j_2}, \alpha_{X,1,j_1,j_2}, \alpha_{X,2,j_1,j_2} \) such that for all \( (x, t) \in \mathbb{R} \times [0, b_T] \),

\[
K_{X,1,T,j_1,j_2} \left(1 - \alpha_{X,1,j_1,j_2} x^2 \right) \leq \frac{b_{X,j_1}(x, t)b_{X,j_2}(x, t)}{\sigma_X^2(x, t)} \leq K_{X,2,T,j_1,j_2} \left(1 + \alpha_{X,2,j_1,j_2} x^2 \right),
\]

where \( K_{X,1,T,j_1,j_2} \rightarrow K_{X,j_1,j_2} \) and \( K_{X,2,T,j_1,j_2} \rightarrow K_{X,j_1,j_2} \) as \( T \rightarrow \infty \); \( K_{X,j_1,j_2} \) being positive constants. We also assume for \( k = 1, 2, (b_T - a_T)|K_{X,k,T,j_1,j_2} - K_{X,j_1,j_2}| \rightarrow 0 \), as \( T \rightarrow \infty \).

For each \( i = 1, \ldots, n \), we re-define \( u_{Y,i}|X_i,T \) and \( u_{X,i,T} \) as \( r_Y \) and \( r_X \) dimensional vectors, with elements given by:

\[
u_{Y,i}|X_i,T = \int_{a_T}^{b_T} \frac{b_{Y,j}(Y_i(s), X_i(s), s)}{\sigma_Y^2(Y_i(s), X_i(s), s)}dY_i(s); \quad j = 1, \ldots, R_Y; \\
u_{X,i,T} = \int_{a_T}^{b_T} \frac{b_{X,j}(X_i(s), s)}{\sigma_X^2(X_i(s), s)}dX_i(s); \quad j = 1, \ldots, R_X.
\]

(S-4.5) (S-4.6)

Also, for \( i = 1, \ldots, n \), let us define \( r_Y \times r_Y \) and \( r_X \times r_X \) matrices \( u_{Y,i}|X_i,T \) and \( u_{X,i,T} \) with \((j_1,j_2)\)-th
Here the true likelihood is given by

\[ p_{\theta}^{T} = \prod_{i=1}^{n} p_{\theta,T}(\theta_{0}) \]  

(S-4.9)

where

\[ p_{\theta,T}(\theta_{0}) = \int_{\Theta} \exp \left( \phi_{Y,i,0}^{T} Y_{i}| X_{i}, T - \frac{1}{2} \phi_{Y,i,0}^{T} \sigma_{Y}^{2}(Y_{i}| X_{i}, s) \phi_{Y,i,0} \right) \times \exp \left( \phi_{X,i,0}^{T} X_{i}, T - \frac{1}{2} \phi_{X,i,0}^{T} \sigma_{X}^{2}(X_{i}, s) \phi_{X,i,0} \right) dQ_{X} \times \]  

(S-4.10)

We assume that

\[ \frac{b_{Y,i,j}(Y_{i}| X_{i}, T)}{\sigma_{Y}^{2}(Y_{i}| X_{i}, s)} \]

\[ \frac{b_{X,i,j}(X_{i}, s)}{\sigma_{X}^{2}(X_{i}, s)} \]

We also replace (H16) of MB with the following.

\[ \text{(H16')} \quad \forall \theta \in \Theta \cup \{ \theta_{0} \}, \psi_{Y,i,j}(\theta) \text{ and } \psi_{X,i,j}(\theta) \text{ are finite for all } i = 1, \ldots, n; j = 1, \ldots, r_{Y}; j = 1, \ldots, r_{X}. \And

\[ \psi_{Y,i,j}(\theta) \rightarrow \psi_{Y}(\theta, j); j = 1, \ldots, r_{Y} \]

\[ \psi_{X,i,j}(\theta) \rightarrow \psi_{X}(\theta, j); j = 1, \ldots, r_{X} \]

as \( i \rightarrow \infty \), for all \( \theta \in \Theta \), where \( \psi_{Y}(\theta) = (\psi_{Y}(\theta, 1), \ldots, \psi_{Y}(\theta, r_{Y}))^{T} \) and \( \psi_{X}(\theta) = (\psi_{X}(\theta, 1), \ldots, \psi_{X}(\theta, r_{X}))^{T} \) are H-coercive functions with continuous third derivatives. Here, by coerciveness we mean \( \| \psi_{Y}(\theta) \| \) and \( \| \psi_{X}(\theta) \| \) tend to infinity as \( \| \theta \| \rightarrow \infty \). We additionally assume that \( \| \psi_{Y}(\theta) \| \) has finite expectation with respect to the prior \( \pi(\theta) \). Also, for \( k = 1, 2 \) letting \( K_{Y,k,i} \) and \( K_{X,k,i} \) denote matrices with \( (j_{1}, j_{2}) \)-th elements \( K_{Y,k,j_{1},j_{2}} \) and \( K_{X,k,j_{1},j_{2}} \) as in (H3') and (H7'), we assume

\[ K_{Y,k,i} \rightarrow \bar{K}_{Y} \]

\[ K_{X,k,i} \rightarrow \bar{K}_{X} \]

as \( i \rightarrow \infty \), where \( \bar{K}_{Y} \) and \( \bar{K}_{X} \) are positive definite matrices, and that

\[ (\psi_{X}(\theta_{0}) - \psi_{X}(\theta))^{T} \bar{K}_{X} (\psi_{X}(\theta_{0}) + \psi_{X}(\theta)) \geq 0 \]

for all \( \theta \in \Theta \). We assume that for every sequence \( \{ \theta_{T}, T > 0 \} \) such that \( \| \theta_{T} \| \rightarrow \infty \), as \( T \rightarrow \infty \),

(i) \( (b_{T} - a_{T}) (\phi_{Y,1}(\theta_{T}))^{T} (K_{Y,1,1} - K_{Y,1,i}) (\phi_{Y,1}(\theta_{T})) \rightarrow 0 \), for every \( i = 1, 2, \ldots, \) and for \( k = 1, 2 \);

(ii) \( (b_{T} - a_{T}) (\phi_{X,1}(\theta_{T}))^{T} (K_{X,1,1} - K_{X,1,i}) (\phi_{X,1}(\theta_{T})) \rightarrow 0 \), for every \( i = 1, 2, \ldots, \) and for \( k = 1, 2 \);

(iii) \( C_{1} (b_{T} - a_{T}) \leq \| \psi_{Y}(\theta_{T}) - \psi_{Y}(\theta_{0}) \|^{6} \leq C_{2} (b_{T} - a_{T}) \), for some \( C_{1}, C_{2} > 0 \).

**S-4.1 True and modeled likelihood in the multidimensional case**

Here the true likelihood is given by

\[ p_{\theta,T}(\theta_{0}) = \prod_{i=1}^{n} p_{T,i}(\theta_{0}), \]

(S-4.10)
The modeled likelihood is given by \( \hat{L}_{n,T}(\theta) = \prod_{i=1}^{n} \hat{L}_{T,i}(\theta) \), where

\[
\hat{L}_{T,i}(\theta) = \int \exp \left( \phi^{T}_{Y,i} u_{Y,i} - \frac{1}{2} \phi^{T}_{Y,i} v_{Y,i} \phi_{Y,i} - \frac{1}{2} \phi^{T}_{X,i} u_{X,i} \phi_{X,i} \right) dQ_{T,X}.
\]

(S-4.11)

The inequalities (2.11) and (2.12) of MB hold for \( i = 1, \ldots, n \), but now \( K_{Y,1,T} \) and \( K_{Y,2,T} \) in (2.13) and (2.14) of MB are matrices with \((j_1, j_2)\)-th elements \( K_{Y,1,T,j_1,j_2} \) and \( K_{Y,2,T,j_1,j_2} \), respectively, as described in (H7').

As before the likelihood \( \hat{L}_{n,T}(\theta) = \prod_{i=1}^{n} \hat{L}_{T,i}(\theta) \) is easily seen to be measurable, so that (A1) of MB holds.

It is easily seen, as before, that

\[
\lim_{n \to \infty} \lim_{T \to \infty} \frac{1}{nT} \log \hat{R}_{n,T}(\theta) = -\bar{h}(\theta),
\]

almost surely, where

\[
\bar{h}(\theta) = \frac{1}{2} \left[ (\bar{\psi}_Y(\theta) - \bar{\psi}_Y(0))^T K_Y (\bar{\psi}_Y(\theta) - \bar{\psi}_Y(0)) + (\bar{\psi}_X(\theta) - \bar{\psi}_X(0))^T K_X (\bar{\psi}_X(\theta) - \bar{\psi}_X(0)) \right]
\]

\[
+ (\bar{\psi}_X(\theta) - \bar{\psi}_X(0))^T K_X (\bar{\psi}_X(\theta) + \bar{\psi}_X(\theta)).
\]

(S-4.12)

Thus, (A2) of MB holds, and as before, (A3) of MB is also clearly seen to hold.

The way of verification of (A4) of MB remains the same as in Section 4.4 of MB, with \( I = \{ \theta : h(\theta) = \infty \} \). Condition (A5) can be seen to hold as before, and defining \( \Theta_{n,T} = \{ \theta : \| \bar{\psi}_Y(\theta) \| \leq \exp (\beta n(bT - aT)) \} \), where \( \beta > 2\bar{h}(\Theta) \) and \( \alpha = E [ \| \bar{\psi}_Y(\theta) \| ] \), (A5) is verified as before. That (A6) and (A7) of MB hold can be argued as it is verified in section 4.6 and 4.7 of MB. We thus have the following theorem.

**Theorem 16** Assume that the data was generated by the true model given by (S-4.1) and (S-4.2), but modeled by (S-4.3) and (S-4.4). Assume that (H1)–(H10), (H12)–(H15) of MB hold (for each \( i = 1, \ldots, n \)), whenever appropriate), with (H3) and (H7) replaced with (H3') and (H7'), respectively. Also assume (H13)–(H15) of MB, (H16') and (H17). Consider any set \( A \in \mathcal{T} \) with \( \pi(A) > 0 \); let \( \beta > 2\bar{h}(A) \). Then,

\[
\lim_{n \to \infty, T \to \infty} \frac{1}{nT} \log \pi(A|\bar{\mathcal{F}}_{n,T}) = -\bar{J}(A).
\]

(S-4.13)

**S-4.2 Strong consistency and asymptotic normality of the maximum likelihood estimator of \( \theta \)**

We now assume, in addition to (H16') that

(H16') \( \bar{\psi}_Y(\theta) \) and \( \bar{\psi}_X(\theta) \) are thrice continuously differentiable and that the first two derivatives of \( \bar{\psi}_X(\theta) \) vanish at \( \theta_0 \).

In this case, the \( MLE \) satisfies (7.26) of MB with the appropriate multivariate extension as detailed above where the \((j, k)\)-th element of \( \mathcal{I}(\theta) \) is given by

\[
\{ \mathcal{I}(\theta) \}_{jk} = \left( \frac{\partial \bar{\psi}_Y(\theta)}{\partial \theta_j} \right)^T K_Y \left( \frac{\partial \bar{\psi}_Y(\theta)}{\partial \theta_k} \right).
\]

(S-4.14)

Thus, as before, it can be shown that strong consistency of the \( MLE \) of the form (7.27) of MB holds. Formally, we have the following theorem.

**Theorem 17** Assume that the data was generated by the true model given by (S-4.1) and (S-4.2), but modeled by (S-4.3) and (S-4.4). Assume that (H1)–(H10), (H12)–(H15) of MB hold (for each \( i =
1, . . . , n, whenever appropriate), with (H3) and (H7) replaced with (H3′) and (H7′), respectively. Also assume (H16′), (H16″) and (H17). Then the MLE is strongly consistent, that is,

$$\hat{\theta}_{n,T} \xrightarrow{a.s.} \theta_0,$$

as \( n \to \infty, T \to \infty \).

Asymptotic normality of the form (7.28) of MB also holds, where the elements of the information matrix \( I(\theta_0) \) are given by (S-4.14). Here the formal theorem is given as follows.

**Theorem 18** Assume that the data was generated by the true model given by (S-4.1) and (S-4.2), but modeled by (S-4.3) and (S-4.4). Assume that (H1)–(H10), (H12) – (H15) of MB hold (for each \( i = 1, \ldots, n \), whenever appropriate), with (H3) and (H7) replaced with (H3′) and (H7′), respectively. Also assume (H16′), (H16″) and (H17). Then

$$\sqrt{n(b_T - a_T)} \left( \hat{\theta}_{n,T} - \theta_0 \right) \xrightarrow{d} N_d \left( 0, I^{-1}(\theta_0) \right),$$

as \( n \to \infty, T \to \infty \).

**S-4.3 Asymptotic posterior normality in the case of multidimensional random effects**

All the conditions (1)–(7) of MB can be verified exactly as in Section 7.5 of MB, only noting the appropriate multivariate extensions. Hence, the following theorem holds:

**Theorem 19** Assume that the data was generated by the true model given by (S-4.1) and (S-4.2), but modeled by (S-4.3) and (S-4.4). Assume that (H1)–(H10), (H12)–(H15) of MB hold (for each \( i = 1, \ldots, n \), whenever appropriate), with (H3) and (H7) replaced with (H3′) and (H7′), respectively. Also assume (H16′), (H16″) and (H17). Then denoting \( \Psi_{n,T} = \bar{\Sigma}_{n,T}^{1/2} \left( \theta - \hat{\theta}_{n,T} \right) \), for each compact subset \( B \) of \( \mathbb{R}^d \) and each \( \epsilon > 0 \), the following holds:

$$\lim_{n \to \infty, T \to \infty} P_{\theta_0} \left( \sup_{\Psi_{n,T} \in B} \left| \pi(\Psi_{n,T} | \bar{F}_{n,T}) - \vartheta(\Psi_T) \right| > \epsilon \right) = 0.$$

**S-5 Asymptotics in the case of discrete data**

In similar lines as Delattre et al. (2013) suppose that we observe data at times \( t_k^n = t_k = k \frac{b_T - a_T}{m} ; \quad k = 0, 1, 2, \ldots \). We then set

$$u_{Y}^{m,n} | X, T = \sum_{k=0}^{m-1} b_Y^2(Y(t_k), X(t_k), t_k) \frac{\sigma_Y^2(Y(t_k), X(t_k), t_k)}{(t_{k+1} - t_k)} (t_{k+1} - t_k)$$

(S-5.1)

$$u_{Y}^{m,n} | X, T = \sum_{k=0}^{m-1} b_Y(Y(t_k), X(t_k), t_k) \frac{\sigma_Y^2(Y(t_k), X(t_k), t_k)}{(Y(t_{k+1}) - Y(t_k))} (Y(t_{k+1}) - Y(t_k))$$

(S-5.2)

$$v_{X}^{m,n} | T = \sum_{k=0}^{m-1} b_X^2(X(t_k), t_k) \frac{\sigma_X^2(X(t_k), t_k)}{(t_{k+1} - t_k)} (t_{k+1} - t_k)$$

(S-5.3)

$$u_{X}^{m,n} | T = \sum_{k=0}^{m-1} b_X(X(t_k), t_k) \frac{\sigma_X^2(X(t_k), t_k)}{(X(t_{k+1}) - X(t_k))} (X(t_{k+1}) - X(t_k)).$$

(S-5.4)

For any given \( T \), the actual MLE or the posterior distribution can be obtained (perhaps numerically) after replacing (2.7) – (2.10) of MB with (S-5.1) – (S-5.4) in the likelihood.
For asymptotic inference we assume that $m = m(T)$, and that $\frac{m(T)}{\alpha_T} \to \infty$, as $T \to \infty$. Then note that, since $\frac{1}{\alpha_T} \log R_T(\theta)$ can be uniformly approximated by $\tilde{g}_T(\theta) = \tilde{g}_{Y,T}(\theta) + \tilde{g}_{X,T}(\theta)$ (as in MB) for $\theta \in G_T \setminus T$ in the case of Bayesian consistency and for $\theta \in \Theta$ for $\Theta$ compact, for asymptotics of MLE and asymptotic posterior normality, and since $\tilde{g}_T(\theta)$ involve the data only through $(W_Y(b_T) - W_Y(a_T))/\sqrt{b_T - a_T}$, asymptotically the discretized version agrees with the continuous version. This implies that, even with discretization, all our Bayesian and classical asymptotic results remain valid in all the SDE set-ups considered in MB.

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