Exact Quantum Solutions of Extraordinary N-body Problems

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The wave functions of Boson and Fermion gases are known even when the particles have harmonic interactions. Here we generalise these results by solving exactly the N-body Schrödinger equation for potentials $V$ that can be any function of the sum of the squares of the distances of the particles from one another in 3 dimensions. For the harmonic case that function is linear in $r^2$. Explicit N-body solutions are given when $U(r) = -2M\hbar^{-2}V(r) = \zeta r^{-1} - \zeta_2 r^{-2}$. Here $M$ is the sum of the masses and $r^2 = \frac{1}{2}M^{-2}\sum\sum m_im_j(x_i - x_j)^2$. For general $U(r)$ the solution is given in terms of the one or two body problem with potential $U(r)$ in 3 dimensions.

The degeneracies of the levels are derived for distinguishable particles, for Bosons of spin zero and for spin $1/2$ Fermions. The latter involve significant combinatorial analysis which may have application to the shell model of atomic nuclei.

For large $N$ the Fermionic ground state gives the binding energy of a degenerate white dwarf star treated as a giant atom with an N-body wave function.

The N-body forces involved in these extraordinary N-body problems are not the usual sums of two body interactions, but nor are forces between quarks or molecules.

Bose-Einstein condensation of particles in 3 dimensions interacting via these strange potentials can be treated by this method.

1. Introduction

Exact Boson or Fermion solutions of the quantum N-body problem in which every particle interacts with every other in three dimensions are very rare. They are almost as rare in classical mechanics although Newton solved one in Principia (1687) (see Cajori 1934 \& Chandrasekhar 1995) and there are also some very special solutions such as Laplace’s in which the three unequal masses describe ellipses about their centre of mass while at each time they make an \textit{equilateral} triangle. However Newton’s solution was for all initial conditions when the force on body $I$ due to body $J$ was of the form $F_{i,j} = \frac{k}{m_im_j}(x_i - x_j)$. Newton reduced this problem to that of $N$ harmonic oscillators relative to the centre of mass. The quantum solution is similar to the $N$ oscillator solution for solid state
physics. The potential energy of Newton’s system is

\[ V = \frac{1}{2} \sum_{I < J} k m_I m_J (x_I - x_J)^2 = \frac{1}{2} k M \sum_I (x_I - \bar{x})^2 \equiv \frac{1}{2} k M^2 r^2. \]

Here we show that this solution may be generalised to systems in which the total potential energy \( V \) is any function of \( r \). We have already explored these systems and their generalisations in Classical Mechanics (Lynden-Bell & Lynden-Bell 1999). Except for Newton’s harmonic case all these systems give many-body forces in which the force between any two bodies is approximately linear for separations much less than the mean but with a coefficient that depends on that current mean. Despite the strange global nature of these force laws they may be the only non-trivial quantum many-body problems that have been solved exactly in 3 dimensions. Only in Newton’s harmonic case do the forces reduce to simple pair-wise interactions. It could be argued that such global forces are unnatural, however, in some respects the resulting behaviour mimics that found in Nature. Ruth Lynden-Bell (1995, 1996) showed that these systems can be used to give a simple model of a phase transition that can be calculated even when \( N \) is small. Also as we now show, such forces can be used to mimic some aspects of gravitation.

For a homogeneous sphere that may pulsate in radius, \( a \), the gravitational potential energy is \( V = -\frac{2}{5} GM^2/a \) and the gravitational force on unit mass not outside \( a \) is \( -\frac{4}{3} \pi G \rho x \). The mean square radius of the sphere is \( r^2 = 3a^2/5 \) so \( V = -(3/5)^{3/2} GM^2/r \). Now forget real gravity but adopt this form of \( V(r) \) for the potential of one of our extraordinary N-body problems. The force per unit mass on any particle at \( x \) is given by

\[ -M^{-1} \partial V/\partial x = -\left( \frac{3}{5} \right)^{3/2} GMx/r^3 = -GMx/a^3 = -\frac{4}{3} \pi G \rho x \]

where \( x \) stands for any one of the \( x_I - \bar{x} \).

Thus for homogeneous spheres this choice of \( V(r) \) in our extraordinary N-body problem exactly mimics the effect of true gravity both for global radial pulsations of the system and for the forces on masses within it. However if the system departs from homogeneity this mimicry is no longer exact. For inhomogeneous spherical systems the true gravitational potential energy \( V(r) \) can always be written \(-kGM^2/r\) with a \( k \) that depends on the radial profile. By taking that to be the \( V(r) \) in our extraordinary N-body problem its Virial theorem will perfectly mimic that of the gravitational problem but apart from the homogeneous case the forces on the individual particles of which the system is composed will not be the same in the mimic. Outside gravitational theory the concept of an effective potential is widely used in physics and chemistry, e.g., in the shell model of the nucleus, in quark-quark interactions at low energy, and in modelling intermolecular forces. Now we have shown that motion in these special potentials can be exactly calculated, they will no doubt be used as approximations in those applications, as well as many others. Since the N-body wave-function is known exactly, so is the correlation energy, but that may not be a useful general guide to correlation because in our systems the net force on each particle is directed radially toward the centre-of-mass whatever the configuration of the other particles may be. Furthermore in many real systems the interaction between any two...
particles is strongest when the particles are closest together while it is weakest for the systems discussed here. In spite of this it is possible to make systems that are strongly repulsive when all particles try to come close together and ones that behave like gravitating systems in the sense that the overall radius obeys the Virial Theorem for a self-gravitating system. Even without any repulsion the exclusion principle provides support for systems of Fermions so with a $V \propto -GM^2/r$ appropriate for gravity, we find configurations of White Dwarf type.

The N-body problems discussed here arose by direct generalisation of Newton’s work and so skipped the developments of the intervening centuries. We may nevertheless see how they fit into those developments. Liouville (1855) showed that if a system of $D$ degrees of freedom had $D$ integrals of the motion whose mutual Poisson Brackets vanished, then the remaining $D$ integrals of the motion could be found as quadratures. He also discovered a large class of such separable systems while Stäckel (1890) proved his necessary and sufficient conditions for separability. Whittaker (1904) gives a good description of those works while he, Eddington (1915) and Eisenhart (1934) helped to determine and classify such systems; De Zeeuw (1985) gives a good historical introduction in his thesis paper. Lynden-Bell (1962) and Hall (1985) developed different ideas for finding classes of systems with integrals or configuration invariants. Carter (1968) extended such results to the motion of charged particles in magnetic fields in General Relativity. Marshall and Wojciechowski (1988) determined those potentials in $D$ dimensions for which the motion of a classical particle separates in suitable coordinates and the hyper-spherical potential of the systems discussed here can be viewed as a highly degenerate member of their general $D$-dimensional ellipsoidally separable potentials. Evans (1990, 1991) has explored systems that are superintegrable, having more than $D$ integrals for $D$ degrees of freedom. They separate in several different coordinate systems and the integrals are the separation constants. This is the case for our hyperspherical systems (see Appendix).

In all these works separability was achieved by changing the coordinates only. The idea that separability might be achievable only via canonical transformations involving the momenta as well as the coordinates was not exploited. Thus Kovalevski’s top (1888) provided an unexpected new system in which the separation was not of the standard type. Linear soluble systems were known which were not of the simple separable type and Routh’s (1877) Adam’s prize essay on the Stability of Motion gives a very thorough discussion. Simple examples of both linear (Freeman 1966) and non-linear problems (Vandervoort 1979, Contopoulos & Vandervoort 1992) that need momentum dependent transformations arose in stellar dynamics but it was only recently, e.g., in the work of Sklyanin (1995) that more general ways of looking for such systems were found.

Meanwhile a whole body of work based around Lax (1968) operator pairs and the Inverse Scattering method showed that there were many previously unsuspected exact solutions in both classical and quantum mechanics. This field of endeavour is too large to be reviewed here so the reader is referred to the review volumes Solitons (Bullough & Caudrey eds., 1980), Dynamical Systems VI Integrable Systems (Arnold ed., 1995) and Soliton Theory: a survey of results (Fordy ed., 1990). The connections between soluble models of N-body problems and field theory are discussed in the book edited by Bazhanov & Burden (1995). In that volume quantum and classical integrable lattice models in one dimension

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are considered by Bullough & Timonen (1995) while two dimensional models in statistical mechanics are discussed by Baxter (1995).

Prominent among many exactly soluble N-body models in one dimension are the Toda lattice (Toda 1967, 1980) also discussed by Henon (1974) and the Calogero model and its generalisations. [Calogero (1971), Sutherland (1971), Bullough & Caudrey (1980), Olshanetsky & Perelomov (1995).] In two dimensions Baxter’s book (1982) and article (1995) contain much of interest and certain solvable models in 3 dimensions have been proposed by Baxter (1986) and Bazhanov & Baxter (1992, 1993). Probably the most prominent soluble field theory in 2 dimensions is that of Davey-Stewartson (1974), see Anker & Freeman (1978), the quantum version of which is considered in Pang et al. (1990).

Sklyanin (1995) holds out the hope that all these soluble systems may eventually be seen as special cases of the method of Separation of Variables and produces some supporting evidence.

Although the above models of interacting systems of many Fermions or Bosons can be solved exactly in one or two dimensions, the calculation that follows may be the only exactly solved non-trivial three dimensional N-body system yet known. Furthermore suitable choices of the function $V(r)$ will allow a study of the way the form of interaction (albeit one of our strange global type) affects Bose-Einstein condensation. It should also prove possible by these methods to study the effect of rotation on the condensation. However this paper is solely concerned with the solutions of Schrödinger’s equation with the correct symmetry in the wave function, so the statistical mechanics and Bose-Einstein condensation displayed by these models is not further discussed here. With the somewhat more realistic δ function interaction it has been studied previously for one dimensional chains and their continuum limits, see, e.g., Bogoliubov et al. (1994) and Bullough & Timonen (1998). It is of course the case that all soluble models are exceptional and a good example of the intricacies of non-soluble models was furnished by Henon (1969).

### 2. N-body Solutions of Schrödinger’s Equation

Let $m_I$ be the mass of the $I^{th}$ particle and $x_I$ its position vector. Writing $M = \Sigma m_I$ for the total mass, the centre of mass is given by

$$\mathbf{x} = \Sigma \mu_I x_I,$$

where $\mu_I = m_I/M$, which implies

$$\Sigma \mu_I = 1.$$  \hfill (2.1)

We define ‘mass weighted’ coordinates relative to the centre of mass $\mathbf{r}_I = \mu_I^{1/2} (x_I - \mathbf{x})$ and an associated $3N$ dimensional vector, $\mathbf{r}$, in the space of all the $\mathbf{r}_I$ by

$$\mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 \ldots \mathbf{r}_N).$$

The length of $\mathbf{r}$ is the mass weighted r.m.s. radius of the system since

$$r^2 = \Sigma \mu_I (x_I - \mathbf{x})^2.$$

\hfill (2.2)
This expression may be rewritten in terms of the mutual separations of the particles since
\[
\mathbf{r}^2 = \sum_I \mu_I (\mathbf{x}_I - \mathbf{x}) \cdot \mathbf{x}_I = \sum_I \sum_J \mu_I \mu_J (\mathbf{x}_I - \mathbf{x}_J) \cdot \mathbf{x}_J =
\]
\[
= \sum_J \sum_I \mu_J \mu_I (\mathbf{x}_J - \mathbf{x}_I) \cdot \mathbf{x}_J ,
\]
and by adding the last two expressions and halving the result
\[
\mathbf{r}^2 = \frac{1}{2} \sum_I \sum_J \mu_I \mu_J (\mathbf{x}_I - \mathbf{x}_J)^2 = \sum_{I<J} \mu_I \mu_J (\mathbf{x}_I - \mathbf{x}_J)^2 .
\]
In practice the \( \mathbf{r} \) vector is constrained by the fact that the centre of mass is at the origin so
\[
\sum_I \mu_I^{1/2} \mathbf{r}_I = \sum_I (\mathbf{x}_I - \mathbf{x}) = 0 .
\]

We define three mutually orthogonal unit vectors \( \hat{\mathbf{X}}, \hat{\mathbf{Y}} \) and \( \hat{\mathbf{Z}} \) in our 3\( N \) space by
\[
\hat{\mathbf{X}} = \left( \mu_1^{1/2}, 0, 0, \mu_2^{1/2}, 0, 0, \ldots, \mu_N^{1/2}, 0, 0 \right)
\]
\[
\hat{\mathbf{Y}} = \left( 0, \mu_1^{1/2}, 0, 0, \mu_2^{1/2}, 0, \ldots, 0, \mu_N^{1/2}, 0 \right) \tag{2.6}
\]
\[
\hat{\mathbf{Z}} = \left( 0, 0, \mu_1^{1/2}, 0, 0, \mu_2^{1/2}, \ldots, 0, 0, \mu_N^{1/2} \right).
\]
Then the constraints (2.5) can be rewritten
\[
\hat{\mathbf{X}} \cdot \mathbf{r} = \hat{\mathbf{Y}} \cdot \mathbf{r} = \hat{\mathbf{Z}} \cdot \mathbf{r} = 0 \tag{2.7}
\]
which show that \( \mathbf{r} \) is confined to three hyperplanes through the origin. Defining \( \hat{r} = \mathbf{r}/r \) then \( \hat{r} \) lies on the unit 3\( N \) sphere \( |\hat{r}|^2 = 1 \) but the \( \hat{\mathbf{X}} \) constraint confines it to the intersection of that sphere with the hyperplane \( \hat{\mathbf{X}} \cdot \mathbf{r} = 0 \), which is a sphere in 3\( N \)-1 space; similarly the \( \hat{\mathbf{Y}} \) constraint leaves it on the intersection of that 3\( N \)-1 sphere with the hyperplane \( \hat{\mathbf{Y}} \cdot \mathbf{r} = 0 \), which is a 3\( N \)-2 sphere and the third constraint leaves it on the 3\( N \)-3 sphere orthogonal to \( \hat{\mathbf{X}}, \hat{\mathbf{Y}} \) and \( \hat{\mathbf{Z}} \).

We are concerned with the \( N \)-body problems whose potential energies, \( V \), are functions of the magnitude \( r \) only, so Schrödinger’s equation takes the form
\[
-\frac{1}{2} \hbar^2 \sum_I m_I^{-1} \frac{\partial^2 \psi}{\partial x_I \cdot \partial x_I} + V \psi = E \psi .
\]
The key to solving this problem lies in the right choice of coordinates. In what follows upper case indices run over particle labels while lower case indices run over coordinate-vector components.

(a) Separation of \( \mathbf{r} \)

Let \( R_{ij} \) be an orthogonal unit 3\( N \times 3N \) rotation matrix which rotates the basis vectors of our 3\( N \) space so that \( \hat{\mathbf{X}}, \hat{\mathbf{Y}} \) and \( \hat{\mathbf{Z}} \) are the last three of the new orthogonal basis vectors. Thus with two alternative notations and assuming the
summation convention over lower case indices only,
\[ q_i = R_{ij} r_j = \sum_j R_{ij} \cdot r_j \]  \hspace{1cm} (2.9)

with
\[ q_{3N-2} = X \quad q_{3N-1} = Y \quad q_{3N} = Z \]  \hspace{1cm} (2.10)

Note
\[ R_{ij} R_{kj} = \delta_{ik} \quad \text{and} \quad \sum_J R_{iJ} \cdot R_{kJ} = \delta_{ik} \]  \hspace{1cm} (2.11)

Let \( a \) run from 1 to \( 3N-3 \) (rather than from 1 to \( 3N \)). Then the \( q_a \) together with the coordinates \( \mathbf{x} \) form a complete set of independent orthogonal coordinates for our system.

We shall need the partial derivatives from (2.1) and below (2.2)
\[ \frac{\partial \mathbf{x}}{\partial x_I} = \mu_I \delta, \]  \hspace{1cm} (2.12)
\[ \frac{\partial \mathbf{r}_I}{\partial x_I} = \sqrt{\mu_I} (\delta_{I\ell} - \mu_I) \delta, \]  \hspace{1cm} (2.13)

where \( \delta \) is the unit \( 3 \times 3 \) matrix. The centre of mass motion will separate so our wave functions may be taken in the form \( \psi = \tilde{\psi}(x) \tilde{\psi}(q_a \ldots) \) so
\[ \frac{\partial \psi}{\partial x_I} = \frac{\partial \tilde{\psi}}{\partial x_I} \mu_I \psi + \psi \frac{\partial \tilde{\psi}}{\partial x_I}. \]  \hspace{1cm} (2.14)

For Schrödinger’s equation we shall need
\[ \sum I \mu_I^{-1} \frac{\partial^2 \psi}{\partial x_I \cdot \partial x_I} = \frac{\partial^2 \tilde{\psi}}{\partial \mathbf{x} \cdot \partial \mathbf{x}} \tilde{\psi} + \tilde{\psi} \Sigma \mu_I^{-1} \frac{\partial^2 \tilde{\psi}}{\partial x_I \cdot x_I}, \]  \hspace{1cm} (2.15)

where the cross derivative term has vanished because \( \tilde{\psi} \) only involves differences of the coordinates \( x_I \) so \( \Sigma \partial \tilde{\psi} / \partial x_I = 0 \), i.e., \( \tilde{\psi} \) is independent of where the system-as-a-whole is.

To evaluate the second term we need from (2.9) and (2.13)
\[ \frac{\partial \tilde{\psi}}{\partial x_I} = \sum K \frac{\partial \tilde{\psi}}{\partial q_j} \frac{\partial q_j}{\partial x_I} \frac{\partial R_K}{\partial x_I} - \frac{\partial \tilde{\psi}}{\partial q_j} \left( R_{jI} \sqrt{\mu_I} - \sum K \sqrt{\mu_K} \mu_I R_{jK} \right). \]  \hspace{1cm} (2.16)

We check that indeed \( \Sigma \partial \tilde{\psi} / \partial x_I = 0 \) by summing this over \( I \) and noting that the two sums cancel because \( \Sigma \mu_I = 1 \). We now proceed to the last term in Schrödinger’s equation (2.15)
\[ \sum I \frac{1}{\mu_I} \frac{\partial^2 \tilde{\psi}}{\partial x_I \cdot \partial x_I} = \frac{\partial^2 \tilde{\psi}}{\partial q_j \partial q_j} \sum I R_{jI} \cdot R_{\ell I} - \sum K \sum I \frac{\partial^2 \tilde{\psi}}{\partial q_j \cdot \partial x_I} \sqrt{\mu_K} R_{jK}. \]

But the last term involves
\[ \frac{\partial}{\partial q_j} \left( \sum I \frac{\partial \tilde{\psi}}{\partial x_I} \right) \]

which is zero and
\[ \sum I R_{jI} \cdot R_{\ell I} = \delta_{j\ell} \]
because $\mathbf{R}$ is an orthogonal matrix. Hence we have the desired expression

$$
\sum_{I} \mu_{I}^{-1} \frac{\partial^{2} \psi}{\partial x_{I} \cdot \partial x_{I}} = \frac{\partial^{2} \tilde{\psi}}{\partial \mathbf{q} \cdot \partial \mathbf{q}} + \frac{\partial^{2} \tilde{\psi}}{\partial q_{a} \partial q_{a}} .
$$

$\alpha$ has replaced $j$ in the final term because $\tilde{\psi}$ is only dependent on the first 3N-3 of the $q_{j}$, and we remember that $q_{a}q_{a} = r^{2}$ since $X, Y$ and $Z$ are all zero.

On division by $\psi$ Schrödinger’s equation now takes the form

$$
-\frac{1}{2} \hbar^{2} M^{-1} \left( \frac{1}{\psi} \frac{\partial^{2} \tilde{\psi}}{\partial \mathbf{x} \cdot \partial \mathbf{x}} + \frac{1}{\psi} \frac{\partial^{2} \tilde{\psi}}{\partial \mathbf{q} \cdot \partial \mathbf{q}} \right) + V(r) = E_{T} ,
$$

where $\mathbf{q}$ stands for the $3(N-1)$ vector $q_{a}$. The equation clearly separates with the final three terms dependent on the $q_{a}$ only and the first dependent on $\mathbf{x}$ only, so it must be constant. Without loss of generality we can take the total momentum to be $\hbar \mathbf{K}$. Then $\tilde{\psi} = \exp(i \mathbf{K} \cdot \mathbf{x})$ and writing $E = E_{T} - \frac{1}{2} \hbar^{2} K^{2}/M$ we find

$$
- \frac{\hbar^{2}}{2M} \frac{\partial^{2} \tilde{\psi}}{\partial \mathbf{q} \cdot \partial \mathbf{q}} + V(r) \tilde{\psi} = E \tilde{\psi}
$$

where $\mathbf{q} \cdot \mathbf{q} = r^{2}$.

(b) Separation of Angular Coordinates

Equation (2.19) clearly separates again in hyperspherical polar coordinates but it is simplest to write them symbolically by putting $\mathbf{q} = r \hat{\mathbf{r}}$ and regarding $r$ as independent of the angular coordinates $\hat{\mathbf{r}}$. We need the partial differentials

$$
\frac{\partial r}{\partial \mathbf{q}} = \hat{\mathbf{r}}
$$

and by writing $\mathbf{r} = \mathbf{q}/r$ and using (2.20)

$$
\frac{\partial}{\partial \mathbf{q}} \cdot \hat{\mathbf{r}} = r^{-1}(3N - 4) .
$$

We write $\tilde{\psi} = \psi_{r}(r) \hat{\psi}(\hat{\mathbf{r}})$ and notice that $\hat{\psi}$ is constant on radial lines so that $\hat{\mathbf{r}} \cdot \partial \hat{\psi}/\partial \hat{\mathbf{r}} = 0$. Then

$$
\frac{\partial \tilde{\psi}}{\partial \mathbf{q}} = \frac{\partial \psi_{r}}{\partial r} \hat{\mathbf{r}} \hat{\psi} + r^{-1} \psi_{r} \frac{\partial \hat{\psi}}{\partial \hat{\mathbf{r}}}
$$

and

$$
\frac{\partial^{2} \tilde{\psi}}{\partial \mathbf{q} \cdot \partial \mathbf{q}} = \frac{\partial^{2} \psi_{r}}{\partial r^{2}} \hat{\mathbf{r}} \hat{\psi} + (3N - 4) r^{-1} \frac{\partial \psi_{r}}{\partial r} \hat{\mathbf{r}} \hat{\psi} + r^{-2} \psi_{r} \frac{\partial^{2} \hat{\psi}}{\partial \hat{\mathbf{r}} \cdot \partial \hat{\mathbf{r}}}
$$

dividing by $\psi$ Schrödinger’s equation now takes the form

$$
\psi_{r}^{-1} \left[ r^{2} \frac{\partial^{2} \psi_{r}}{\partial r^{2}} + (3N - 4) r \frac{\partial \psi_{r}}{\partial r} \right] - \alpha^{2} r^{2} + U(r) r^{2} = \psi_{r}^{-1} A(\psi) .
$$

$\alpha$ is given by $\alpha^{2} = -2 M E \hbar^{-2}$ and $U(r) = -2 M V(r) \hbar^{-2}$. The angular operator $A$ (the hyper-angular-momentum operator) is given by

$$
A(\hat{\psi}) = -\partial^{2} \hat{\psi}/\partial \hat{\mathbf{r}} \cdot \partial \hat{\mathbf{r}} .
$$

The angular operator $A$ also appears in the generalised $\nabla^{2}$ in 3N-3 dimensions.
viz $\partial^2/\partial \mathbf{q} \cdot \partial \mathbf{q}$, so we shall first study the hyper-spherically symmetric solutions of $\partial^2 \chi/\partial \mathbf{q} \cdot \partial \mathbf{q} = 0$. Evidently

$$\frac{d^2 \chi}{dr^2} + \frac{(3N - 4)}{r} \frac{d \chi}{dr} = 0$$

so $d\chi/dr = Cr^{-(3N-4)}$ therefore $\chi = Br^{-(3N-5)}$ where we have omitted a constant of integration which is irrelevant to our purpose. We see this is correct by considering the first non-trivial case in which terms other than $\nabla$ are involved which is $N = 2$. Then the space of the $q_0$ is 3 dimensional and the elementary solution has $\chi = Br^{-1}$. Now our generalised $\nabla^2$ knows no particular origin so if $\chi$ is a solution for $r \neq 0$ then so is $\bar{\chi} = B|\mathbf{r} - \mathbf{r}_0|^{-(3N-5)}$ for $\mathbf{r} \neq \mathbf{r}_0$. We expand such solutions both for $|\mathbf{r}| < |\mathbf{r}_0|$ and for $|\mathbf{r}| > |\mathbf{r}_0|$ in powers and the coefficients of these powers are $Y_L(\bar{\mathbf{r}})$ hyperspherical harmonics, just as they are in 3 dimensions. In particular the $L^{th}$ harmonics have a power in $r$ of either $r^L$ or $r^{L-(3N-5)}$. Looking for solutions of the form $r^L f(\bar{\mathbf{r}})$ to our generalised $\nabla^2 = 0$ we see that in $D = 3N - 3$ dimensions

$$r^2 \frac{d^2}{dr^2} \left( r^L f \right) + (D - 1)r \frac{d}{dr} \left( r^L f \right) = r^L A(f) ,$$

i.e.,

$$L(L + D - 2) f = A(f) \quad (2.27)$$

and hence the eigenvalues of $A(f)$ are $L(L + 3N - 5)$. Notice that for a two body problem this reduces to the $L(L + 1)$ in 3 dimensions that we know so well. We shall return later to look for the degeneracies of these different eigenstates but for solving Schrödinger’s equation the eigenvalues are sufficient. For the detailed separation of the $3N - 4$ angular coordinates each in turn see the Appendix.

(c) The Radial Equation

Schrödinger’s equation (2.25) now reads

$$d^2 \psi_r/dr^2 + (3N - 4)r^{-1}d\psi_r/dr - [\alpha^2 - U(r) + L(L + 3N - 5)r^{-2}]\psi_r = 0 . \quad (2.28)$$

Now in the corresponding classical N-body problem we showed that the solution for the radial pulsations of the whole N-body system could be found in terms of the radial pulsation of the corresponding two body problem with the same $U(r)$. With $N = 2$ we have the usual Schrödinger equation for a spherical potential with angular momentum $\ell$

$$d^2 \psi_2/dr^2 + 2r^{-1}d\psi_2/dr - (\alpha^2 - U(r) + \ell(\ell + 1)r^{-2})\psi_2 = 0 .$$

We shall suppose that this problem has been solved and the corresponding eigenvalues and eigenfunctions are known. Now put $\psi_2 = r^\beta \chi_2$; then $\chi_2$ obeys

$$\chi''_2 + (2\beta + 2)r^{-1}\chi'_2 - \left[\alpha^2 - U(r) + \ell(\ell + 1) - \beta(\beta + 1)\right]\chi_2 = 0 . \quad (2.29)$$

Writing $\ell(\ell + 1) - \beta(\beta + 1) = (\ell - \beta)(\ell + \beta + 1)$ one notices that putting

$$\beta = \frac{3}{2}(N - 2) \quad (2.30)$$

and $\ell = L + \beta$, transforms equation (2.29) into precisely (2.28). We deduce the surprising theorem below:
**Theorem:** If the energy levels of the usual Schrödinger equation in 3 dimensions are $E(P,\ell)$, where $P = 1, 2, 3, \ldots$ is the radial quantum number and $\ell$ is the angular momentum quantum number, then the energy levels of the $N$-body problem with the ‘same’ ‘potential’ $U(r)$ are $E \left( P, L + \frac{3}{2}(N-2) \right)$. Furthermore the radial parts of their wave functions are related by $\psi_{r}(P, L, r) = r^{-\frac{3}{2}(N-2)}\psi_{2} \left( P, L + \frac{3}{2}(N-2), r \right)$.

Notice that the ground states are not the same because $L + \frac{3}{2}(N-2)$ cannot be zero for $N > 2$. Thus the ground state of the $N$-body system corresponds to an $\ell$ of $\frac{3}{2}(N-2)$ which can be a high angular momentum state of the two body problem. We see at once that the $N$-body problem will have proper energy levels even if strongly attractive $r^{-2}$ potentials are added to $U$ just because of this effective increase in the $\ell$ of the ground state.

The theorem above is very powerful in that it enables us to use everything that is known about the solutions to the normal Schrödinger equation in spherical potentials and transform it into knowledge of our $N$-body problems. As is well known, not only the generalised Kepler potential $U(r) = \zeta r^{-1} - \zeta_{2} r^{-2}$, but also similarly generalised square well potentials, and the $U \propto \delta(r - r_{0})$, $\delta$-function potentials as well as oscillator potentials $U(r) = -\frac{1}{2} \kappa^{2} r^{2} - \zeta r^{-2}$ all have pretty solutions for the eigenvalues and eigenfunctions. All this takes over directly. However, when $\zeta_{2}$ is so negative that the two body problem has no ground state there is perhaps room for doubt as to whether we can use that potential’s higher angular momentum states for our $N$-body problem. To allay any such doubts and provide a pretty illustration of the truth of our general theorem, we now solve the $N$-body equation for the generalised Kepler potential.

**(d) Generalised Kepler Problem**

We have already shown that the wave function takes the form

$$\psi = \exp(i\mathbf{k} \cdot \mathbf{r}) Y_{L}(\hat{\mathbf{r}}) \psi_{r}(r)$$

where $Y_{L}$ may be given a further $3N-5$ indices corresponding to various components of $\mathbf{L}$. We must now determine $\psi_{r}$.

In solving equation (2.28) we follow the standard method of solution beautifully laid out in the book by Pauling & Wilson (1935). Setting $a\mathbf{r} = \tilde{\mathbf{r}}$ and $\zeta/\alpha = \tilde{\zeta}$ and keeping only the terms that dominate at large $\tilde{\mathbf{r}}$ we find

$$d^{2} \psi_{r}/d\tilde{r}^{2} - \psi_{r} \sim 0 .$$

So the asymptotic solutions behave as $\exp \pm \tilde{r}$. Only the minus sign is acceptable so we write $\psi_{r} = \eta(\tilde{r}) \exp(-\tilde{r})$ and derive the equation for $\eta$ valid for all $\tilde{r}$,

$$\eta'' + (3N-4)\tilde{r}^{-1}\eta' - 2\eta + \left\{ \tilde{\zeta} - (3N-4) \right\} \tilde{r}^{-1} - [\zeta_{2} + L(L + 3N-5)] \tilde{r}^{-2} \eta = 0 .$$

We look for power series solutions of the form

$$\eta = \sum_{p=0}^{\infty} a_{p}\tilde{r}^{s+p}$$

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with $a_0 \neq 0$ and find the recurrence relation

$$\left\{(p + s)(p + s + 3N - 5) - [\xi_2 + L(L + 3N - 5)]\right\}a_p =$$

$$= \left[2(p + s) + 3(N - 2) - \tilde{\xi}\right]a_{p-1}.$$  \hspace{1cm} (2.32)

The indicial equation has $a_{-1} = p = 0$ and yields a quadratic equation for $s$

$$s^2 + (3N - 5)s - [\xi_2 + L(L + 3N - 5)] = 0.$$  \hspace{1cm} (2.33)

In the pure Kepler case with $\xi_2 = 0$ this yields $s = L$ or $-(L + 3N - 5)$ of which only the positive $s = L$ solution obeys the boundary condition at the origin. For general $\xi_2$ the solutions are (cf (2.30))

$$s = -\frac{1}{2}(3N - 5) \pm \sqrt{\left[L + \frac{1}{2}(3N - 5)\right]^2 + \xi_2} = -\frac{1}{2} - \beta \pm \sqrt{\left(\frac{1}{2} + L + \beta\right)^2 + \xi_2},$$  \hspace{1cm} (2.34)

of which only that with the $+$ sign obeys the boundary condition at $r = 0$. When $\xi_2 < 0$ a more detailed discussion is given later. If the series for $\eta$ does not terminate the asymptotic form of the recurrence relation gives $a_p \approx 2a_{p-1}/p$ so $\eta \propto e^{2r}$ and $\psi_r$ is divergent at $\infty$; so the series must terminate at $a_{p-1}$ say and in (2.32)

$$\tilde{\xi} = 2(P + s) + 3(N - 2) = 2(P + s + \beta),$$  \hspace{1cm} (2.35)

with $s$ given by taking the upper sign in (2.34) (i.e., $s = L$ when $\xi_2 = 0$).

Remembering that $\tilde{\xi} = \xi/\alpha$ and that $\alpha^2 = -2M\hbar^{-2}$ expression (2.35) can be recast as the energy spectrum

$$E = -\frac{\hbar^2}{8M} \frac{\xi^2}{(P + s + \beta)^2} = -\frac{\hbar^2}{2M} \frac{\xi^2}{2P - 1 + \sqrt{(2L + 3N - 5)^2 + 4\xi_2}}.$$  \hspace{1cm} (2.36)

In accordance with our theorem the energy levels with general $N$ are given by putting $N = 2$ and then replacing $L$ by $L + \beta = L + \frac{1}{2}(N - 2)$. Of course if $\xi_2 = 0$ we have $s = L$ and the theorem is then obvious from the first form. Notice that the theorem really applies to $\alpha^2 = -2ME\hbar^{-2}$ thus we can only apply it to $E$ itself if we consider a two body problem with the same mass $M$ as the N-body problem; $\tilde{\xi}^2$ is also taken as unchanged since we require both problems to have the same $U(r)$. However this in no way restricts us to N-body problems with $M$ and $\xi$ independent of $N$; it merely means that we change correspondingly the masses $M$ and coefficients $\xi$ in the two body problems with which we compare N-body problems of different $N$.

Some may wish to see the precise Schrödinger hydrogen atom spectrum with the correct reduced mass emerging when $N = 2$; to get this we must evaluate $\xi$ in terms of $Z\hbar^2$. Our potential energy is $V = -\frac{1}{2}\hbar^2 M^{-1}\xi/r$ but this $r$ is not the separation of the nucleus from the electron, $R$, but the mass weighted r.m.s. separation of them from the centre of mass. Hence $r = \left(m m_n / M^2\right)^{1/2} R$ where $m$ and $m_n$ are the mass of the electron and the nucleus respectively. Setting $V = -\frac{Z\hbar^2}{R}$ we deduce that $\xi = 2(m m_n)^{1/2} \hbar^{-2} Z\hbar^2$. Inserting this $\xi$ into (2.36) along with $N = 2, \xi_2 = 0, n = P + L, M = m + m_n$ and putting the reduced mass
the energy levels of hydrogenic atoms are given by

\[ E = -\frac{m_e (Ze^2)^2}{2\hbar^2 n^2} \]

just as they should be.

We now return to the question of how negative ζ_2 can be. Since \( L \) can be zero the energy of the ground state ceases to be real if \( ζ_2 < -\left(\frac{3N-5}{2}\right)^2 \) which gives

\[ ζ_2 < -\frac{1}{4} \]

for the familiar case \( N = 2 \). Such strongly attractive forces cause the particles to propagate into the nucleus and the ground state ceases to exist. It may be seen that the limiting case has a wave function \( ψ ∝ r^{-\frac{1}{2}} \) near the origin which is easily still square integrable \( \int ψ ψ^* r^2 dr < ∞ \). This is also true for the limiting case \( ζ_2 = -\left(\frac{3N-5}{2}\right)^2 \) for then \( ψ ∝ r^{-\frac{1}{2}(3N-5)} \) and \( \int ψ ψ^* r^{3N-4} dr < ∞ \). The limits are surpassed for the attractions of magnetic monopoles on the magnetic moments of protons and for charged monopoles attracting spinning electrons (Lynden-Bell & Nouri-Zonoz 1998).

### 3. Level Degeneracies

In equations (2.26) and (2.27) and the attendant discussion we showed that the solutions of our Schrödinger equation consisted of a hyperspherical harmonic in \( 3N - 3 \) dimensions times a radial function. Furthermore the hyperspherical harmonics of degree \( L \) in \( D \) dimensions are the coefficients of \( r^L \) in the polynomial solutions of Laplace’s equation in \( D \) dimensions. Thus the degeneracy of the states of given \( L \) and given radial quantum number \( P \) will be equal to the number of independent polynomial solutions of Laplace’s equation of degree \( L \) in \( D = 3N - 3 \) dimensions (i.e., harmonic polynomials). To determine this number we first ask how many independent polynomials of degree \( L \) exist in \( D \) dimensions without the harmonic requirement. Each can be considered as a term of the form \( Π i x_i^{l_i} \) where \( i \) runs from 1 to \( D \) and \( Σ l_i = L \). That number of polynomials is equal to the number of ways of dividing \( L \) objects into \( D \) groups where a group is allowed to contain no objects. If we take \( L \) units and \( D - 1 \) dividing bars then the number of ways of ordering them is \( (L + D - 1)! \) and if we disregard the ordering of the \( D - 1 \) bars among themselves and the \( L \) units among themselves the number of ways of sorting them into groups is

\[ G(L, D) = \frac{(L + D - 1)!}{(D - 1)!L!} \]

so this is the number of independent polynomials of degree \( L \). Let \( f_L(x_a) \) be such a homogeneous polynomial of degree \( L \) in the \( x_a \). In general

\[ \nabla^2 f_L = \sum_a \frac{∂}{∂x_a} \cdot \frac{∂}{∂x_a} f_L = f_{L-2} \]

where \( f_{L-2} \) is such a polynomial of degree \( L - 2 \) which will have \( G(L - 2, D) \) independent coefficients. The condition that \( f_L \) be harmonic \( (\nabla^2 f_L = 0) \) thus imposes \( G(L - 2, D) \) constraints on the \( G(L, D) \) free coefficients in \( f_L \). Thus the
number of independent harmonic polynomials of degree \( L \) in \( D \) dimensions is
\[
g(L, D) = G(L, D) - G(L - 2, D) =
\]
\[
= \frac{(L + D - 3)!}{(L-1)!L!} \left[ (L + D - 1)(L + D - 2) - L(L - 1) \right] =
\]
\[
= \frac{(L + D - 3)!}{(D - 2)!L!} (2L + D - 2) .
\]
Notice that for the familiar case \( D = 3 \) this gives the correct answer, \( 2L + 1 \), for the degeneracy of the states of given \( L \).

When \( \zeta_2 = 0 \) we have the extra degeneracy between the \( s, p, d, f \) levels of hydrogen. Then a state of principal quantum number \( n \) can be obtained by combining a wave function of given \( L \) with a radial wave function with radial quantum number \( P = n - L \geq 1 \). Thus, to find the degeneracy of states with a given \( n \), we need to know the number of harmonic polynomials of degree less than or equal to \( n - 1 \), because the \( n - L - 1 \) extra quanta are taken up by different radial wave functions. To find this number we merely add a dummy group ‘o’ to our sorting of \( L \) objects into groups and ignore the number of units, \( n_o \), that falls into that group. Thus the required answer is
\[
g_H(n, D) = g(n - 1, D + 1) = \frac{(n + D - 3)!}{(D - 1)!(n - 1)!} (2n + D - 3) .
\]
For \( D = 3 \) this reduces to \( n^2 \), which is the familiar degeneracy of the \( n \)th hydrogen level before spin is considered. Thus for our problems the degeneracy of levels of hyper-angular-momentum \( L \) for a system of \( N \) particles is \( g(L, 3N - 3) \) with \( g \) given by (3.2), while for \( \zeta_2 = 0 \) the degeneracy of the \( n \)th level is
\[
g_H(n, 3N - 3) = g(n - 1, 3N - 2).
\]
The above degeneracies are worked out assuming that none of the particles are identical. In practice we are more interested in problems with \( N \) identical Bosons or \( N \) identical Fermions and they require wave functions with even or odd permutational symmetries so we now study that question.

4. Symmetry under permutation of particles

(a) Bosons

For Bosons we need wave-functions that are symmetrical for the interchange of any two particle labels \( \mathbf{x}_I, \mathbf{x}_J \). Both \( \mathbf{x} \) and \( r \) have the required symmetry when the particles are of equal mass. Just as the magnitude of the angular momentum treats \( x, y \) and \( z \) symmetrically in 3 dimensions, so the magnitude of the hyperangular momentum \( L \) is symmetrical for particle interchange. However the components of the term \( L_{ij} \) and the vector \( \hat{r} \) are not symmetrical under the interchange of particle labels. In §2 we found the solutions for our \( N \)-particle wave functions in the form
\[
\psi = \exp(i \mathbf{k} \cdot \mathbf{x}) Y_L(\hat{r}) \psi(r) ,
\]
\[ (4.1) \]
where \( \psi_\tau \) depends only on the scalar magnitude \( L \). The only term that is not automatically symmetrical for particle interchange is \( Y_L(\hat{r}) \) but even that will be automatically symmetrical when \( L = 0 \), because \( Y_0 \) is constant. Thus the ground state and all \( s \)-states are automatically symmetrical and are possible states for a system of identical Bosons.

The states we have been discussing are not the one-particle states commonly considered as components of \( N \)-particle product states (or, for Fermions, Slater determinants); rather our states are themselves \( N \)-particle states. To get a symmetrical \( N \)-particle state from one lacking that symmetry we merely add all the wave functions obtained by permuting the labels on the particles. But whereas each of our wave-functions thereby generates one boson \( N \)-particle state, such a state in general comes from a number of different unsymmetrical wave-functions so we can no longer count the degeneracies by the arguments of \( \S \) 3. However the arguments of \( \S \) 3 connect the number of \( Y_L \) functions with the number of polynomials that are homogeneous and both of degree \( L \) and harmonic. If we can count interchange symmetric polynomials independent of \( \mathbf{x} \) which are homogeneous of degree \( L \) and solutions of Laplace’s equation, we have the degeneracy of the quantum states of hyper-angular-momentum \( L \).

Let \( F_L(\mathbf{x}) = F_L(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N) \) be a homogeneous polynomial of degree \( L \) in \( \mathbf{x} \) which is symmetric under the interchange of any \( \mathbf{x}_I \) with \( \mathbf{x}_J \). Then \( F_L(\lambda \mathbf{x}_1, \lambda \mathbf{x}_2, \ldots, \lambda \mathbf{x}_N) = \lambda^L F_L(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N) = \lambda^L F_L(\mathbf{x}_1, \mathbf{x}_N, \ldots, \mathbf{x}_2) \), etc. Consider \( F_L(\mathbf{x} - \mathbf{x}) = F_L(\mathbf{x}_1 - \mathbf{x}, \ldots, \mathbf{x}_N - \mathbf{x}) \). It is also a homogeneous polynomial of degree \( L \) in \( \mathbf{x} \) and is also symmetric, but it has the property that it is invariant under the transformation \( \mathbf{x}_I \rightarrow \mathbf{x}_I + \Delta \) for all \( I \) (because then \( \mathbf{x} \rightarrow \mathbf{x} + \Delta \)). Thus such functions do not depend on the position of the centre of mass. However it can happen that \( F_L(\mathbf{x} - \mathbf{x}) \) is identically zero even when \( F_L(\mathbf{x}) \) is not. For this to happen \( F_L(\mathbf{x}) \) must be of the form \( F_L(\mathbf{x}) = \mathbf{x} \cdot \mathbf{F}_{L-1}(\mathbf{x}) \) where \( \mathbf{F}_{L-1}(\mathbf{x}) \) is a vector each of whose components is a polynomial of degree \( L - 1 \) in \( \mathbf{x} \) which is symmetric under interchange of \( \mathbf{x}_I \) and \( \mathbf{x}_J \). Now let \( \overline{G}(L) \) be the number of independent symmetric polynomials which are homogeneous of degree \( L \) in \( 3N \) dimensions. Then the number of such polynomials giving rise to non-zero \( F_L(\mathbf{x} - \mathbf{x}) \) will be \( \overline{G}(L) \) less the number of free coefficients in the \( -\mathbf{x} \cdot \mathbf{F}_{L-1} \) term which we might expect to be \( 3G(L - 1) \). However that is not quite right because a polynomial with a factor \( \mathbf{x} \cdot \mathbf{y} \) will occur as a possibility in both the \( x \) and \( y \) components of \( \mathbf{F}_{L-1} \) and in subtracting \( 3\overline{G}(L - 1) \) we will have subtracted its number of free coefficients not once but twice. The same double counting will have occurred for polynomials with factors \( \mathbf{y} \cdot \mathbf{z} \) or \( \mathbf{z} \cdot \mathbf{x} \) so we should be subtracting not \( 3\overline{G}(L - 1) \) but rather \( 3\overline{G}(L - 1) - 3\overline{G}(L - 2) \). However even that is not quite correct because there may be polynomials with a factor \( \mathbf{x} \cdot \mathbf{y} \cdot \mathbf{z} \). They will have been subtracted off three times in \( 3\overline{G}(L - 1) \) but added back in three times in \( 3\overline{G}(L - 2) \) so they will still be there and they should not be since they clearly vanish when \( \mathbf{x} - \mathbf{x} \) is written for \( \mathbf{x} \). Thus finally the number of independent symmetric polynomials which are homogeneous of degree \( L \) and independent of \( \mathbf{x} \) is

\[
\overline{G}_1(L) = \overline{G}(L) - 3\overline{G}(L - 1) + 3\overline{G}(L - 2) - \overline{G}(L - 3) .
\]

However, we still have to impose Laplace’s equation

\[
\sum_I \nabla^2_I F_L \equiv \nabla^2 F_L = 0 .
\]
Now in general $\nabla^2 F_L(x - x)$ will be a polynomial of degree $L - 2$ in $x$. However, since $F_L(x - x)$ is invariant to the transformation $x \rightarrow x + \nabla$, $\nabla^2 F_L(x - x)$ will also have that property. Thus in general we may write

$$\nabla^2 F_L(x - x) = F_{L-2}(x - x)$$

where $F_{L-2}$ is also symmetrical for particle label interchange since $\nabla^2$ does not destroy that property. Thus the condition $F_{L-2}(x - x) \equiv 0$ will put $G_{1}(L-2)$ constraints on the $G_{1}(L)$ free coefficients of the homogeneous $L$th degree polynomial $F_L(x - x)$. There will be just $G_{1}(L) - G_{1}(L-2)$ free coefficients left in $F_L(x - x)$ after imposing the harmonic condition so this is the degeneracy of the $Y_L$ that corresponds to the $(2L+1)$ with $L$ even for the 2 boson problem. Since $G_1$ is known in terms of $G$, we have reduced our problem to that of determining the number of exchange-symmetric homogeneous polynomials of degree $L$ in $3N$ dimensions. This is the crux of our problem and it took us considerable thought to solve it. Exchange symmetry involves exchanging vectors $x_I$ with $x_J$, so we do not need symmetry in all $3N$ dimensions but only between them taken in triples. We shall begin our considerations with the simpler case of $N$ bosons on a line with each having but one coordinate $x_I$. We then wish to know how many independent exchange-symmetric polynomials there are which are homogeneous of degree $L$ in $N$ dimensions.

Let $\Phi = S\Pi_{I} x_{I}^{\ell_{I}}$ be a symmetrical polynomial of degree $L$ with $N$ factors $x_{I}^{\ell_{I}}$. $S$ is the symmetrising operator which is the sum over all permutations of the particle labels $I$. Different symmetric polynomials are characterised by different sets of integers $(\ell_1, \ell_2, \ldots, \ell_N)$ or partitions of the integer $L$ into $N$ parts, some of which may be zero. We construct the generating function

$$B_1(u, x) = \sum_{n=1}^{\infty} \sum_{\ell=0}^{\infty} p(n, \ell) u^{n} x^{\ell}$$

where $p(N, L)$ is the number of partitions of $L$ into $N$ integers that may be zero, and for convenience we have defined $p(0, \ell) = 0$ and $p(n, 0) = 1$. We now show how the theory of partitions allows us to construct the function $B_1(u, x)$.

(b) Partitions of an integer $L$

We learn from Abramowitz & Stegun (1964) that ‘The number of decompositions of an integer $L$ into integer summands without regard to order is called $p(L)$’.

For example, five may be written

$$5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 = 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1,$$  \hspace{1cm} (4.3)$$

so we deduce that $p(5) = 7$. It is easiest to work with the generating function for the $p(L)$ which we call $A(x)$. For this there is a standard result see, e.g., Hardy & Ramanujan (1918),

$$A(x) = 1 + \sum_{L=1}^{\infty} p(L) x^{L} = \prod_{\ell=1}^{\infty} (1 - x^{\ell})^{-1}. \hspace{1cm} (4.4)$$

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For what follows it is essential to understand how this standard result comes about. To do so we rewrite the product by expansions in powers of $x$

$$A(x) = (1 + x^1 + x^2 + \ldots) (1 + (x^1)^2 + (x^2)^2 + \ldots) (1 + (x^3)^1 + (x^3)^2 \ldots)$$

\begin{align*}
&\quad (generates \text{ units}) \quad (generates \text{ twos}) \quad (generates \text{ threes}) \\
&\quad (1 + (x^4)^1 + \ldots) \quad (1 + (x^5)^1 + \ldots) \quad (\ldots) \\
&\quad (generates \text{ fours}) \quad (generates \text{ fives})
\end{align*}

(4.5)

To see how the coefficient of $x^5$ in this expression is $p(5) = 7$ we first notice that we must take the 1 from all brackets after the fifth, since otherwise we would get too high a power of $x$. In the 5th bracket we can take the $x^5$ term but then we must take the 1 from all earlier brackets. Alternatively we take the 1 in the 5th bracket. In the latter case we turn to the fourth bracket. Here we may take the $x^4$ term but that can only be coupled to the $x$ term in the first bracket in which case we get the split of 5 into 4 + 1. Turning now to the third bracket and taking the $x^3$ term we can take it with either the $(x^2)^1$ bracket of the second term to yield 3 + 2 or with the $(x^1)^2$ term of the first bracket to yield 3 + 1 + 1. Similarly from the second bracket we could take the $(x^2)^2$ term with the $x$ from the first bracket to give 2 + 2 + 1 or the $(x^2)^1$ term with the $x^3$ from the first to give 2 + 1 + 1 + 1. Finally we could take the $x^5$ from the first bracket to give 1 + 1 + 1 + 1 + 1. Thus the first bracket yields the number of ones in the sum, the second the number of twos, the third the number of threes, etc., and in this way the coefficient of $x^L$ yields $p(L)$.

However, we need the restricted partition of $L$ into $N$ or fewer non-zero integers $p(N, L)$. These are sums of partitions $p_1(N, L)$ into exactly $N$ non-zero integers. Looking at (4.3) we see, for example, that $p_1(2, 5) = 2 = p_1(3, 5)$. If we place a factor $u$ along with each factor $x^\ell$ in (4.5), then the power of $u$ in each term will tell us how many parts there are in the partition generated by a particular term. Thus in place of $A(x)$ we consider

$$A(u, x) = (1 + ux + (ux)^2 + (ux)^3 + \ldots) (1 + (ux^2)^1 + (ux^2)^2 + \ldots)$$

\begin{align*}
&\quad \quad (generates \text{ units}) \quad \quad (generates \text{ twos}) \\
&\quad (1 + (ux^3)^1 + (ux^3)^2 + \ldots) \quad \ldots \\
&\quad \quad (generates \text{ threes})
\end{align*}

(4.6)

Then the terms in $u^N x^L$ will have exactly $N$ integers in the corresponding partition of $L$ so (cf. 4.4)

$$A(u, x) = 1 + \sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} p_1(n, \ell) u^n x^\ell = \prod_{\ell=1}^{\infty} (1 - ux^\ell)^{-1}.$$ 

However, we want the number of partitions with $N$ or fewer integers, i.e., $p(N, L) = \sum_{n=1}^{N} p_1(n, L)$. These sums will be automatically generated if we multiply $A$ by $(1 + u + u^2 + u^3 \ldots)$ before taking the coefficient of $u^N x^L$ so

$$B_1(u, x) = \sum_{n=1}^{\infty} \sum_{\ell=0}^{\infty} p(n, \ell) u^n x^\ell = (1 + u + u^2 + \ldots) A(u, x) = \prod_{\ell=0}^{\infty} (1 - ux^\ell)^{-1}. \quad (4.7)$$
Readers will recognise the analogy of this expression with the grand partition function for a gas of non-interacting Bosons. So \( p(N, L) \) can be found from the product as the coefficient of \( u^N x^L \). Had we been interested in Bosons on a line then \( p(N, L) \) would have given us the desired function \( \mathcal{G}(L) \) but our problem is three dimensional. Instead of partitioning \( L \) into integers \( \ell_I \) we need it partitioned into integer triples \((\ell_{Ix}, \ell_{Iy}, \ell_{Iz})\) and the general term in our polynomial will be

\[
x_{\ell_{1x}}^{I1} y_{\ell_{1y}}^{I1} z_{\ell_{1z}}^{I1} \ldots x_{\ell_{x}}^{IN} y_{\ell_{y}}^{IN} z_{\ell_{z}}^{IN}.
\]

When we permute we do so by exchanging \((x_1, y_1, z_1)\) as a triple with say \((x_2, y_2, z_2)\). The degree of our polynomial is

\[
L = \sum_{i=1}^{N} (\ell_{Ix} + \ell_{Iy} + \ell_{Iz}).
\]

A triple \((2, 1, 0)\) corresponding to \( \ell_{1x} = 2, \ell_{1y} = 1, \ell_{1z} = 0 \) will not be permuted into \((1, 0, 2)\) by exchanging particle labels so such triples must be regarded as distinct. The number of partitions of 3 into triples, such that the order within a triple matters but the order of the different triples does not, \( p_3(L) \) with \( L = 3 \), is already quite a handful. Writing \( = \) for an equal total we have

\[
\begin{align*}
(3, 0, 0) &= (0, 3, 0) = (0, 0, 3) = (2, 1, 0) = (0, 2, 1) = (1, 0, 2) = (1, 2, 0) = (0, 1, 2) = \\
&= (2, 0, 1) = (1, 1, 0) = (2, 0, 1) + (1, 0, 0) = 8 \text{ more like that} \\
&= (1, 1, 0) + (0, 0, 1) = 8 \text{ more like that} \\
&= (1, 0, 0) + (1, 0, 0) + (1, 0, 0) = 9 \text{ more like that}
\end{align*}
\]

Hence there are 38 partitions of 3 into triples! Now let \( p_3(L) \) be the number of partitions of \( L \) into triples, every triple being counted as different but the different orderings of the same triples being regarded as the same. By analogy with (4.5) we consider the expression

\(\text{Phil. Trans. R. Soc. Lond. A (1996)}\)
\[ A(x, y, z) \equiv \]
\[ \equiv (1 + x + x^2 \ldots) \times (1 + (x^2)^1 + (x^2)^2 + \ldots) \times (1 + (x^3)^1 + (x^3)^2 + \ldots) \times \ldots \]
\[ \times (1 + y + y^2 \ldots) \times (1 + (y^2)^1 + (y^2)^2 + \ldots) \times (1 + (y^3)^1 + (y^3)^2 + \ldots) \times \ldots \]
\[ \times (1 + z + z^2 \ldots) \times (1 + (z^2)^1 + (z^2)^2 + \ldots) \times (1 + (z^3)^1 + (z^3)^2 + \ldots) \times \ldots \]

This column generates\(x \times (1 + (xz)^1 + (xz)^2 + \ldots) \times (1 + (yz)^1 + (yz)^2 + \ldots)\)

\[
\begin{array}{l}
(1) \text{ or } (0) \text{ or } (0) \\
(0) \text{ or } (1) \text{ or } (0) \\
(0) \text{ or } (0) \text{ or } (1)
\end{array}
\]
and multiple combinations thereof
generates\(x \times (1 + (xy)^1 + (xy)^2 + \ldots) \times (1 + (z^2x)^1 + (z^2x)^2 + \ldots) \times (1 + (x^2z)^1 + (x^2z)^2 + \ldots) \times \ldots\)

By analogy with the arguments beneath (4.5) one sees how the terms of the third degree generate all the partitions of 3 into triples that we have just enumerated. So the coefficient of \(t^\ell\) in \(A(t, t, t)\) will be \(p_3(L)\), the number of partitions of \(L\) into triples. Furthermore \(A(x, y, z)\) may be compactly written in the form

\[ A(x, y, z) = \prod_{p=0}^{\infty} \prod_{q=0}^{\infty} \prod_{r=0}^{\infty} (1 - x^py^qz^r)^{-1} \tag{4.8} \]

For our \(N\) boson problem we are interested not in such partitions into any triples but in partitions constrained to have \(N\) or fewer triples as summands. To get exactly \(N\) summands we merely insert a \(u\) in each term as was done in (4.6). While to get the sum of all terms with \(N\) or less summands we have to multiply by \((1 + u + u^2 \ldots)\) as in (4.7). Thus putting \(x = y = z = t\) the Boson generating function in 3 dimensions is, writing \(\ell = p + q + r\),

\[ B(u, t) = \prod_{p=0}^{\infty} \prod_{q=0}^{\infty} \prod_{r=0}^{\infty} (1 - ut^{p+q+r})^{-1} = \prod_{\ell=0}^{\infty} \left(1 - ut^{\ell}\right)^{-\frac{1}{2}(\ell+1)(\ell+2)} \tag{4.9} \]

where \(\frac{1}{2}(\ell+1)(\ell+2)\) is the number of terms in the triple product with \(p+q+r = \ell\).

The required function \(G(L)\) is the coefficient of \(u^Nt^L\) in \(B(u, t)\). Again if one only wants \(G(L)\) for \(L < L_{\text{max}}\) then the infinite product can be replaced by a finite product up to \(L_{\text{max}}\) without altering the required coefficients.

To get \(G_1(L)\) one merely takes the coefficient of \(u^Nt^L\) in \((1 - t)^3B(u, t)\) while \(G_1(L) - G_1(L - 2)\) is the coefficient of \(u^Nt^L\) in \((1 - t^2)(1 - t)^3B(u, t)\) = \((1 + t)(1 - t)^4B(u, t)\).

In summary the degeneracy \(g_B(L)\) of the \(N\) Boson state with hyperangular momentum \(L\) is the coefficient of \(u^Nt^L\) in the expression

\[ (1 + t)(1 - t)^4 \prod_{n=0}^{\infty} \left(1 - ut^{\ell}\right)^{-\frac{1}{2}(\ell+1)(\ell+2)} \tag{4.10} \]

For two particles \(N = 2\) we find that the coefficient of \(u^2\) is \((1 - 3t^2)(1 - t^2)^{-2}\).
The coefficient of \( t^L \) in this expression is \( 2L + 1 \) when \( L \) is even and zero when \( L \) is odd just as it should be. This is just as in the \( C^{12} - C^{12} \) homonuclear diatomic molecule of two Bosons with every odd rotational state missing as seen in Carbon star spectra.

(c) Fermions

The argument of § 3 relates the number of hyperspherical harmonics of degree \( L \) to the number of harmonic polynomials of that degree. Our experience in § 4a leads us to study first the number of antisymmetric polynomials of degree \( L \) in \( N \) dimensions.

If \( x_1^{\ell_1} x_2^{\ell_2} \ldots x_N^{\ell_N} \) is a term in such a polynomial then \( \sum \ell_I = L \). Furthermore the antisymmetric polynomial involving that term is the Slater determinant

\[
\begin{vmatrix}
  x_1^{\ell_1} & x_2^{\ell_1} & \ldots & x_N^{\ell_1} \\
  x_1^{\ell_2} & x_2^{\ell_2} & \ldots & x_N^{\ell_2} \\
  \vdots & \vdots & \ddots & \vdots \\
  x_1^{\ell_N} & x_2^{\ell_N} & \ldots & x_N^{\ell_N}
\end{vmatrix}
\]

Clearly if \( \ell_I = \ell_J \) for \( I \neq J \) then this determinant vanishes. Furthermore the determinant only changes sign (at most) if the \( \ell_I \) are permuted. So for any term that survives we may, without loss of generality, take \( \ell_1 < \ell_2 < \ell_3 \ldots < \ell_N \). Thus among the \( \ell_I \) only \( \ell_1 \) can be zero, \( \ell_2 \) must be at least 1, \( \ell_3 \) at least 2 and so on with \( \ell_N \) at least \( N - 1 \). For a non-zero result \( L = \sum \ell_I \geq \frac{1}{2} (N - 1)N \).

By analogy with our study of partitions \( p(L) \) for the Boson case we now study partitions of \( L \) into distinct parts. Let \( q(L) \) be the number of decompositions of \( L \) into distinct integer summands without regard to order. Thus \( 5 = 4 + 1 = 3 + 2 \) so that \( q(5) = 3 \). The generating functions for the \( q(L) \) is, setting \( q(0) = 1 \),

\[
\sum_0^\infty q(L)x^L = \prod_{\ell=1}^\infty (1 + x^\ell). \tag{4.11}
\]

However not all decompositions of \( L \) into distinct parts lead to antisymmetric polynomials in \( N \) dimensions. We need a decomposition into either \( 0 + (N - 1) \) unequal integers or into \( N \) unequal integers, i.e., we need the coefficient of \( u^N x^L \) in

\[
\prod_{\ell=0}^\infty \left(1 + ux^\ell\right) = \sum_{L=0}^\infty \sum_{N=0}^\infty q(L, N) u^N x^L, \tag{4.12}
\]

which is the expression analogous to (4.7) of the Boson case. Again it is the analogue of the grand partition function for Fermions.

The generalisation corresponding to three dimensions follows the argument for (4.8) and yields the generating function

\[
E(x, y, z) = \prod_{p=0}^\infty \prod_{q=0}^\infty \prod_{r=0}^\infty (1 + x^p y^q z^r), \tag{4.13}
\]

which leads analogously to (4.9) to the Fermion generating function

\[
F(u, t) = \prod_{p=0}^\infty \prod_{q=0}^\infty \prod_{r=0}^\infty \left(1 + ut^{p+q+r}\right) = \prod_{\ell=0}^\infty \left(1 + ut^{\ell}\right)^{\frac{1}{2}(\ell+1)(\ell+2)}. \tag{4.14}
\]
The coefficient of $u^N t^L$ in this expression gives the number of homogeneous polynomials of degree $L$ antisymmetric for interchanges of triples in $3N$ dimensions. For Fermions of spin $\frac{1}{2}$ we do not need complete antisymmetry but only antisymmetry between particles of the same spin state. To allow for the $\alpha$ and $\beta$ spin states being alternatives we generalise $E(x, y, z)$ to

$$
\prod_{p=0}^{\infty} \prod_{q=0}^{\infty} \prod_{r=0}^{\infty} (1 + x^p y^q z^r \alpha) (1 + x^p y^q z^r \beta)
$$

and in place of $F(u, t)$ we then find

$$
F^2 = \prod_{\ell=0}^{\infty} \left( 1 + ut^\ell \right)^{(\ell+1)(\ell+2)}.
$$

The coefficient of $u^N t^L$ in $F^2$ is the number of Fermionic polynomials for spin $\frac{1}{2}$ particles with the correct antisymmetry. The arguments relating the degeneracy of the $N$-Fermion spin $\frac{1}{2}$ wave function for the state of hyper-angular momentum $L$ to this expression is the same as for the Boson case. Thus $g_F(L)$ is coefficient of $u^N t^L$ in the expression

$$
(1 + t)(1 - t)^4 \prod_{\ell=0}^{\infty} \left( 1 + ut^\ell \right)^{(\ell+1)(\ell+2)}.
$$

To check this formula we take the coefficient of $u^2$, and find for the 2 Fermion problem the generating function

$$
\left( 1 + 9t + 3t^2 + 3t^3 \right) \left( 1 - t^2 \right)^{-2} = \sum_{\ell\text{ even}} (2\ell + 1)t^\ell + 3 \sum_{\ell\text{ odd}} (2\ell + 1)t^\ell,
$$

in which the coefficients of $t^\ell$ will be recognised as the degeneracies of the rotational states of the hydrogen molecule with 2 protons of spin $\frac{1}{2}$. The even $\ell$ values correspond to para-hydrogen and the odd $\ell$ values to ortho-hydrogen.

We now turn to the degeneracies in hydrogen-like potentials in which different $L$ states can be degenerate. Here we need the sum of the $g_F(L)$ for all $L \leq n - 1$ as in equation (3.3). If we multiply our generating function for $g_F(L)$ by $1 + t + t^2 + t^3 + \ldots = 1/(1 - t)$ and then pick the coefficient of $u^N t^{n-1}$ we will get the required sum so the hydrogenic Fermi degeneracy is that coefficient in

$$
(1 + t)(1 - t)^3 \prod_{\ell=0}^{\infty} \left( 1 + ut^\ell \right)^{(\ell+1)(\ell+2)}.
$$

To find the ground state we need the first energy level $n$ for which the coefficient of $u^N t^{n-1}$ is non-zero. To get $u^N$ and no more with the lowest power of $t$, we need to use the $ut^\ell$ terms rather than the 1 in all the low $\ell$ brackets since others come with higher powers of $t$. Thus if the highest $\ell$ needed is $\ell_m$ we require

$$
\sum_{\ell=0}^{\ell_m} (\ell + 1)(\ell + 2) = N.
$$

The sum is $\frac{1}{2}(\ell_m + 1)(\ell_m + 2)(\ell_m + 3)$ but $N$ may not be of precisely this form for
integer \( \ell_m \), in which case we take the lowest \( \ell_m \) that gives
\[
\frac{1}{4}(\ell_m + 1)(\ell_m + 2)(\ell_m + 3) \geq N
\]
so that there will be at least one term in \( u^N \). We are interested in the least power of \( t \) associated with this term. This will be
\[
n - 1 = \sum_{\ell=1}^{\ell_m} \ell(\ell + 1)(\ell + 2) = \frac{1}{4}\ell_m(\ell_m + 1)(\ell_m + 2)(\ell_m + 3) \tag{4.21}
\]
whenever \( N \) is of the special closed shell form given by the equality. Then \( \ell_m \simeq (3N)^{1/3} - 2 - (3N)^{-1/3} \) and \( n = \frac{3N}{4}\ell_m + 1 \rightarrow (3N)^{4/3}/4 \) as \( N \rightarrow \infty \). For large \( N \) we then find that the ground state energy behaves as
\[
E = -\frac{\hbar^2}{8M} \frac{\zeta^2}{\left[ \frac{(3N)^{1/3}}{4} \right]^2} \tag{4.22}
\]
To compare this energy with that of a white dwarf star we must first choose an appropriate value of \( \zeta \) so that the potential corresponds to gravity. We showed in an earlier paper (Lynden-Bell & Lynden-Bell 1999) that at high temperatures our systems have a Gaussian density profile at equilibrium. For such a profile the potential energy may be expressed in terms of the total mass \( Nm_H \) and the rms radius at equilibrium \( r \) and is
\[
V = \left( \frac{3}{4\pi} \right)^{1/2} G(Nm_H)^2/r \tag{4.23}
\]
We therefore choose
\[
\frac{\hbar^2}{2M} \zeta = \left( \frac{3}{4\pi} \right)^{1/2} G(Nm_H)^2 \tag{4.24}
\]
With this choice our ground state energy level is
\[
E = -\frac{2G^2m_H^4}{3\sqrt[3]{\pi} \hbar^2} N^{7/3} \tag{4.25}
\]
where we have written \( M = Nm_e \) for the mass of the degenerate electrons whose wave function we have been evaluating. This expression is of precisely the form found by the standard equation of state of a cold degenerate non-relativistic gas under its own gravity. We have, therefore, established that White Dwarfs may be regarded as ‘superatoms’ – systems with \( N \)-body wave functions which are solutions of Schrödinger’s equation in a central potential.

5. Conclusions

By treating the \( N \)-body problem as a separable system in 3\( N \) dimensions we have shown that it can be solved in appropriate potentials. Marshall & Wojciechowski (1988) have given the general form of potentials that allow separability in many dimensions and we have concentrated on the hyper-spherical one. Of the many others possible, a sub-class are symmetrical for exchange of the particles.

Whereas we have shown how to classify the wave functions by hyper-angular momentum, those interested in rotating systems will need to develop methods of classification involving the 3 dimensional angular momentum; here the methods of Dragt (1965) and the work by Louck & Galbraith (1972) may prove useful. It is
hoped that study of that problem will throw light on Bose-Einstein condensation of small clusters in rotating systems.

The fact that we are only able to treat non-relativistically degenerate white dwarfs suggests that an appropriate generalisation for relativistically moving particles should be sought for systems that do not radiate gravitational waves.

In our earlier paper on the classical mechanics of these systems we showed that the fundamental breathing oscillation in $r$ separates for the far-more-general potentials $V = V_0(r) + r^{-2}V_2(r)$ where $V_2$ is an arbitrary function of all the coordinates which is independent of scale, $\lambda$, when all the $x_I \rightarrow \lambda x_I$. While this is still the case in quantum mechanics the energy eigenvalues depend on the other motions so this separation does not by itself yield eigenvalues. However, the possibility of solid-like and liquid-like states where $V_0 = \frac{1}{2}kr^2$ and $r^{-2}V_2 = \sum_{T<J} \sum k' |r_I - r_J|^{-2}$ suggests that such systems are worthy of further study.

The statistical mechanics of the systems with $V = -\frac{GM^2}{r}$, $r < r_e$, gives negative specific heats just as those studied earlier as examples of phase transitions (Lynden-Bell & Lynden-Bell 1977). However, within one system there is no gravothermal catastrophe (Antonov 1962, Lynden-Bell & Wood 1968).

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Phil. Trans. R. Soc. Lond. A (1996)
Separation of $\nabla^2$ and commuting hyper-angular-momenta in $D$ dimensions

We write $q_1 = r \cos \theta_1$, $q_2 = r \sin \theta_1 \cos \theta_2$, $q_3 = r \sin \theta_1 \sin \theta_2 \sin \phi$, ..., $q_{D-1} = r \sin \theta_1 \sin \theta_2 \sin \theta_3 \sin \theta_{D-2} \cos \phi$ and $q_D = q_{D-1} \tan \phi$. 

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The metric is given by
\[ ds^2 = dq_1^2 + dq_2^2 + \ldots dq_D^2 = dr^2 + r^2 \left( d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \sin^2 \theta_1^2 d\theta_3^2 + \ldots + \prod_{i=1}^{D-2} \sin^2 \theta_i^2 d\phi^2 \right) = dr^2 + r^2 \left( h_1^2 d\theta_1^2 + h_2^2 d\theta_2^2 + \ldots + h_{D-1}^2 d\phi^2 \right). \]

\[ \nabla^2 = r^{-(D-1)} \frac{\partial}{\partial r} \left( r^{D-1} \frac{\partial}{\partial r} \right) + r^{-2} \left\{ \prod_{i=1}^{D-1} h_i \right\}^{-1} \sum_{j=1}^{D-1} \frac{\partial}{\partial \theta_j} \left[ \left( \prod_{k=1}^{D-1} h_k \right) h_j^{-2} \frac{\partial}{\partial \theta_j} \right]\right\}} \right\} = r^{-(D-1)} \frac{\partial}{\partial r} \left( r^{D-1} \frac{\partial}{\partial r} \right) - r^{-2} A. \]

Since the \( h_j \) depend only on \( \theta_i \) with \( i < j \) we can rewrite the angular operator \( A \) as follows
\[ A = \sum_{j=1}^{D-2} h_j^{-2} \left( \sin \theta_j^{-D+1+j} \right) \frac{\partial}{\partial \theta_j} \left[ \left( \sin \theta_j^{D-1-j} \right) \frac{\partial}{\partial \theta_j} \right] + h_{D-1}^{-2} \frac{\partial^2}{\partial \phi^2}. \]

Using (2.27), Schrödinger’s equation (2.25) separates giving in \( D \) dimensions
\[ L(L + D - 2) = \hat{\psi}^{-1}A(\hat{\psi}) \]
\[ \hat{\psi} \text{ separates into } \psi_\theta(\theta) \psi_\phi(\phi) \text{ and the separated equation for } \psi_\phi \text{ gives } \psi_\phi \propto \exp(\im \phi). \] If we instead multiply (A3) by \( h_{D-2}^2 \) then the last two terms are the only ones dependent on \( \theta_{D-2} \) so the system again separates and the \( \theta_{D-2} \) equation is Legendre’s equation
\[ \frac{1}{\sin \theta_{D-2}} \frac{\partial}{\partial \theta_{D-2}} \left( \sin \theta_{D-2} \frac{\partial \psi_{D-2}}{\partial \theta_{D-2}} \right) = -\ell_1 (\ell_1 + 1) \psi_{D-2} \]
with \( \ell_1 > |m| \).

Similarly multiplying (A2) by \( h_{D-3}^2 \) the \( \theta_{D-3} \) equation is
\[ \frac{1}{\sin^2 \theta_{D-3}} \frac{\partial}{\partial \theta_{D-3}} \left( \sin^2 \theta_{D-3} \frac{\partial \psi_{D-3}}{\partial \theta_{D-3}} \right) - \frac{\ell_1 (\ell_1 + 1) \psi_{D-3}}{\sin^2 \theta_{D-3}} = -\ell_2 (\ell_2 + 2) \psi_{D-3} \]
where the eigenvalue on the right comes from (A3) applied in 4 dimensions and \( \ell_2 \geq \ell_1 \).

The general equation is
\[ \frac{1}{\sin^{D-1-j} \theta_j} \frac{\partial}{\partial \theta_j} \left[ \sin^{D-1-j} \theta_j \frac{\partial \psi_{D-1-j}}{\partial \theta_j} \right] = -\ell_j (\ell_j + j) \psi_{D-1-j} \]
where the eigenvalues on the right come from (A3) applied in \( j + 2 \) dimensions.

The operators on the left of (A4), (A5) and (A6) all have simultaneous eigenvalues of the form \( -\ell_j (\ell_j + j) \) and commute with the energy and \( \partial / \partial \phi \). Thus we have \( D = 3N - 3 \) commuting operators whose eigenvalues are constants of the motion in \( D \) dimensions – we may of course add \( K \) and get \( 3N \) constants of the motion for our \( N \) particles. Thus our system is the quantum version of a system that is integrable by Liouville’s theorem.

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The above integrals of motion can all be made up of the hyper-angular-momenta components \( q_ap_b - q_bp_a \). Indeed the pattern of these operators is as follows: in 2 dimensions there is one angular momentum operator which is conserved, i.e., commutes with the Hamiltonian. In 3 dimensions there are two extra angular operators that are conserved but only the total angular momentum commutes with the 2\( D \) angular momentum chosen to start with. Likewise in 4 dimensions there are \( \frac{1}{4} \times 3 = 6 \) conserved angular momentum operators – 3 new ones – and again it is the total angular momentum that commutes with both the 3\( D \) total and the 2\( D \). In \( D \) dimensions there are \( \frac{1}{2}D(D-1) \) conserved angular momentum operators of which \( \frac{1}{2}D(D-1) - \frac{1}{2}(D-1)(D-2) = D-1 \) are new ones that did not occur in the \( D-1 \) dimensional case. Of these the grand total sum of \( \frac{1}{2}D(D-1) \) squares is conserved and commutes with all former totals. Thus in \( D \) dimensions there are \( D-1 \) independent mutually commuting angular momenta and these, together with the energy, give us the \( D \) commuting operators expected from Liouville’s theorem. We note that we could have chosen any pole for our spherical polar coordinates; each choice gives us a different complete set of operators.

Of course the Keplerian case \( V = -kM^2r^{-1} \) also separates in hyper-parabolic and hyper-spheroidal coordinates. For such systems we have a conserved Hamilton eccentricity vector (often called the Runge-Lenz vector in atomic physics) c.f. Lynden-Bell & Lynden-Bell (1999), equation (2.25K)

\[
    e_a = k^{-1}M^{-3} \sum_b (q_a p_b - q_b p_a) p_b - q_a / r .
\]