Absolute simultaneity and invariant lengths: Special Relativity without light signals or synchronised clocks

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Abstract

It is demonstrated that the measured spatial separation of two objects, at rest in some inertial frame, is invariant under space-time transformations. This result holds in both Galilean and Special Relativity. A corollary is that there are no ‘length contraction’ or associated ‘relativity of simultaneity’ effects in the latter theory. A thought experiment employing four unsynchronised clocks and a single measuring rod reveals that the physical basis of the time dilatation effect is a relative velocity transformation law, not ‘length contraction’. Time dilatation, which is universal and translation invariant for all synchronised clocks at rest in any inertial frame, is the unique space-time phenomenon discriminating Special from Galilean Relativity.

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In the conventional presentation of Special Relativity Theory (SRT), following the approach of Einstein’s seminal paper on the subject [1], light signals and clock synchronisation play a crucial role in the development of predictions of space-time geometrical effects on the basis of the Lorentz transformation (LT). However, as early as 1910 [2], it was realised that Einstein’s methodology is not the only, and not the most economical, way (in terms both of the number of postulates and their simplicity) to formulate the foundations of SRT.

In a previous paper by the present author [3], using a similar approach to Ref. [2], it was shown that only two weak, and, from a naive point-of-view, ‘evident’ postulates, unrelated to classical electrodynamics or any other dynamical theory, suffice to derive the LT. This is true for events lying along the common $x, x'$ axis of two frames S and S' in which the latter frame moves with constant velocity along the $x$-axis in S. The symbol $t$ is used to denote the recorded time (epoch) of a clock a rest in S, and $t'$ to denote the epoch of a clock a rest in S'. An event in S is specified by $(x,t)$, one in S' by $(x',t')$, it being understood that that $y = y' = z = z' = 0$. The two postulates are that, the LT must be a single-valued function of its arguments, and a restricted form of the special relativity principle that may be called the ‘Measurement Reciprocity Postulate’ (MRP):

Reciprocal space-time measurements of similar measuring rods and clocks in two different inertial frames by observers at rest in these frames give identical results.
In order to derive the LT for events not lying along the common $x, x'$ axis, the further postulate of spatial isotropy is required. This is, to the present writer’s best knowledge, the minimum number of postulates, to date, from which the LT has been derived \[3\]^a

The aim of the present letter is to present a simple thought experiment involving four identical unsynchronised clocks A, B, A' and B' and a single measuring rod, R, that displays all essential predictions of SRT concerning measurements of length and time intervals in the inertial frames S and S'. The results obtained below from the analysis of this thought experiment are the same as those presented in previous papers [7, 8, 9, 10] by the present author. Some are quite different from those deduced, following Einstein, from the conventional interpretation of the LT in SRT. Although the experimentally-confirmed Time Dilatation (TD) effect is predicted in the same way as in conventional SRT, neither the Relativity of Simultaneity (RS) nor the Length Contraction (LC) effects are predicted to occur. Indeed, the distance between two physical objects at rest in S' as measured in either S or S' is found to be a Lorentz-invariant quantity. It is interesting to note that, at the time of this writing, there is no experimental evidence for the existence of either the RS or LC effects [7]. Two Earth-satellite experiments have recently been proposed by the present author to test for the existence of the RS effect [11].

In order to specify a length interval, two distinct physical objects or a single extended one are required. For the present study it is found more appropriate to consider two distinct objects, which may be chosen to be identical clocks at two different spatial locations. This choice will be found to be of importance for discussion of the existence (or not) of the RS effect. Since the analysis will involve length intervals in the two inertial frames S and S', it follows that the minimum number of distinct objects needed in the thought experiment is four, — two A,B at rest in S and two A',B' at rest in S'. The fixed separation of A and B (A'and B') specifies a length interval in S (S'). In order to discuss TD and RS it will be convenient to use similar unsynchronised clocks for the four objects A, B, A' and B', although for the analysis of length intervals as measured in S or S' (relevant to the existence, or not, of LC) it is of no importance that the objects are clocks. Any others — located at the same well-defined spatial locations— can be employed.

As a first step, the relative separation of just two objects A and A' as viewed in different inertial frames is considered. A' is placed at the origin of coordinates in the inertial frame S' so that $x'_{A'} = 0$, and is initially at the origin of S: $x_{A'}(t = 0) = 0$. The object A, permanently at rest in S, is placed near the $x$-axis at:

$$x = x_A = x_A - x_{A'}(t = 0) \equiv x_{AA'}(t = 0) \equiv D$$

The object A' undergoes a time-dependent proper acceleration, $\gamma(t') a_0$ (where $a_0$ is constant, $\gamma(t') \equiv 1/\sqrt{1 - (v_{A'}(t')/c)^2}$ and $v_{A'} \equiv dx_{A'}/dt$) along the positive $x$-axis during the time interval $t_{acc}$ in S, until it arrives at a position with the same $x$-coordinate as A. The corresponding proper time interval for A' is $t'_{acc}$. The world line of A', as viewed

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\(^a\)Einstein stated explicitly only two postulates in Ref. [1], The Special Relativity Principle and the constancy of the speed of light, but at least three other postulates: linearity of the equations, spatial isotropy, and the Reciprocity Principle (see below) were also tacitly assumed.

\(^b\)The present paper supersedes the previous version [4]. The analysis of Ref. [4] although correctly concluding that the relativity of simultaneity and length contraction effects are spurious, was invalidated by an important conceptual error. This was conflating of kinematical configurations in a primary experiment and its reciprocal with those in the frames S and S' in the primary experiment. See Refs. [5, 6] for the definition of ‘primary’ and ‘reciprocal’ experiments.
from S, is given by the equation, derived in the Appendix:

\[ x_{AA'}(t) \equiv x_A - x_{A'}(t) = D - \frac{c^2}{a_0} \left[ \sqrt{1 + \left( \frac{a_0 t}{c} \right)^2} - 1 \right] \tag{1} \]

where \( c \) is the speed of light in free space. Since \( x_{AA'}(t_{acc}) = 0 \), it follows from (1) that:

\[ t_{acc} = \frac{c}{a_0} \sqrt{\left( 1 + \frac{a_0 D}{c^2} \right)^2 - 1} \tag{2} \]

As shown in Figs. 1a and 1b, the equation

\[ x'_{AA'}(t') \equiv x'_A(t') - x'_{A'} = x_{AA'}(t) = D - x_{A'}(t) \tag{3} \]

holds when (see the Appendix) \( t \) and \( t' \) are related according to the equations:

\[ t'(t) = \frac{c}{a_0} \ln \left[ \frac{a_0 t}{c} + \sqrt{1 + \left( \frac{a_0 t}{c} \right)^2} \right] = \frac{c}{a_0} \text{arcsinh} \left( \frac{a_0 t}{c} \right), \quad t(t') = \frac{c}{a_0} \sinh \left( \frac{a_0 t'}{c} \right) \tag{4} \]

Then (2) and the first equation in (4) gives:

\[ t'_{acc} = \frac{c}{a_0} \text{arccosh} \left( 1 + \frac{a_0 D}{c^2} \right) \tag{5} \]

The world line of A as viewed from the proper frame of A', derived by combining Eqns(1),(3) and the second equation in (4) is:

\[ x'_A(t') = x_{AA'}(t(t')) = D - \frac{c^2}{a_0} \left[ \sqrt{1 + \left( \frac{a_0 t(t')}{c} \right)^2} - 1 \right] \]

\[ = D - \frac{c^2}{a_0} \left[ \cosh \left( \frac{a_0 t'}{c} \right) - 1 \right] \tag{6} \]

Since \( x'_A(t'_{acc}) = x'_{AA'}(t'_{acc}) = 0 \) it follows from (6) that

\[ t'_{acc} = \frac{c}{a_0} \ln \left[ 1 + \frac{a_0 D}{c^2} + \sqrt{\left( 1 + \frac{a_0 D}{c^2} \right)^2} - 1 \right] \]

\[ = \frac{c}{a_0} \text{arccosh} \left( 1 + \frac{a_0 D}{c^2} \right) \tag{7} \]

consistent with Eqn(5) above, thus verifying the correctness of Eqns(3) and (6). The world line of A’ as viewed from S, given by Eqn(1) is shown in Fig. 1a, and the world line of A as viewed from the proper frame of A’, given by Eqn(6), is shown in Fig. 1b, for \( a_0 = c = 1 \) and \( t_{acc} = \sqrt{3} \)

\[ ^* \text{Note that the world line of A’ corresponds to the ‘hyperbolic motion’ that has formerly been associated with a constant proper acceleration. Why this is incorrect is explained in the Appendix.} \]
Differentiating (6) gives:

\[ v_A'(t') \equiv \frac{dx_A'}{dt'} = -c \sinh \left( \frac{a_0 t'}{c} \right) = -a_0 t = -v_A(t) \gamma(t) \]  

(8)

In deriving (8), the second equation in (4) and the relation (see the Appendix)

\[ v_A'(t) \equiv \frac{dx_A'}{dt} = \frac{a_0 t}{\sqrt{1 + \left( \frac{a_0 t}{c} \right)^2}} = \frac{a_0 t}{\gamma(t)} \]  

(9)

has been used. The slopes of the world lines in Fig. 1a and Fig. 1b at the corresponding values of \( t \) and \( t' \) given by Eqns(4) are therefore related by the Lorentz factor \( \gamma(t) \). Thus the magnitude of the relative velocity of \( A \) and \( A' \) in the proper frame of \( A' \) is larger than that in the inertial frame \( S \), in which \( A \) is at rest, by the factor \( \gamma(t) \). An identical relation holds between the relative velocity of the two objects in \( S \) and that in the co-moving inertial frame of \( A' \). If therefore the acceleration is halted at the corresponding epochs \( t = t_{acc} \), \( t' = t'_{acc} \) the proper frame of \( A' \) at this instant becomes an inertial proper frame at all later times, so that subsequent time intervals \( \Delta t \) and \( \Delta t' \) satisfy the TD relation: \( \Delta t = \gamma(t_{acc}) \Delta t' \). Indeed just this relation is the basis of the derivation in the Appendix of the formulae (1), (4) and (9) above.

Since at any time after the acceleration program is halted,

\[ \frac{\Delta x_A'}{\Delta t'} = -\gamma(t_{acc}) \frac{\Delta x_{A'}}{\Delta t} \]  

(10)

where

\[ \Delta x_A' = x_A'(t' + \Delta t') - x_A'(t'), \quad \Delta x_{A'} = x_{A'}(t + \Delta t) - x_{A'}(t) \]

the TD relation requires that

\[ \Delta x_A' = -\Delta x_{A'} \]  

(11)

demonstrating the Lorentz invariance of corresponding length intervals in two inertial frames. Since both \( x_{A'} \) and \( D \) are constant, the relation (11) is already implicit in Eqn(3) above. This behaviour is illustrated in Fig. 1c and 1d where it is supposed that an acceleration procedure with a large value of \( a_0 \) is applied to \( A' \) so that it obtains the final velocity \( v_A'(t_{acc}) \) of Fig. 1a in a negligibly short interval of time. The proper frame of \( A' \) is subsequently the inertial frame \( S' \) and the world lines of \( A' \) and \( A \) are straight lines with gradients equal to those of the world lines in Fig. 1a and 1b at the times \( t_{acc} \) and \( t'_{acc} \) respectively. It is already clear from inspection of these plots and Eqns(10) and (11) that the physical basis of TD is not ‘length contraction’, as in the conventional interpretation, but rather the different relative velocities of \( A \) and \( A' \) in the frames \( S \) and \( S' \) according to Eqn(8).

The thought experiment involving the four clocks \( A, A', B \) and \( B' \) and the measuring rod \( R \) will now be described. At the start of the experiment, as shown in Fig. 2a, the clocks \( A' \) and \( B' \) are placed in \( S \) above the \( x \)-axis, a distance, \( L \), apart. The measuring rod, \( R \), of length, \( L \), is used to set the positions of the clocks. The clocks \( A \) and \( B \) are, similarly, set the same distance apart, using \( R \), below the \( x \)-axis. Again, using \( R \), the initial separation of \( B' \) and \( A \) is set to \( 2L \) and the origin of the \( x \) and \( x' \) axes is chosen midway between \( B' \) and \( A \). Initially, then, the coordinates of the clocks are: \( x_{A'} = x_{A'} = -2L \), \( x_{B'} = x_{B'} = -L \), \( x_A = x_A = L \) and \( x_B = x_B = 2L \). The clocks \( A' \) and \( B' \) are initially
The world lines of the objects A and A’ as viewed from the proper frame of A [a) and c) or that of A’ [b) and d)]. In a) and b) A’ is accelerated for a time interval $t_{acc}$ in the inertial proper frame, $S$, of A. Units and dimensions are chosen so that $a_0 = c = D = 1$ and $t_{acc} = \sqrt{3}$. In c) and d) a large value of $a$ is assumed so that A’ reaches the same velocity as in a) at $t = t_{acc} = \sqrt{3}$ in a negligibly short time. The slope of the world line of A’ in c) is then the same as the slope of the world line of A’ in a) at $t = t_{acc}$. Similarly the slope of the world line of A in d) is equal to that of the world line of A in b) at $t' = t'_{acc} = 1.317...$ The slopes of the world lines of A’ [A] in c) [d)] are the same as those of the objects A’, B’ [A, B] in Fig. 3c [Fig. 3b] during the phase of uniform motion. As given by Eqn(8), the slopes of the world lines of A in b) and d) at corresponding epochs are $-\gamma(t)$ times the slopes of the world lines in a) and c) respectively.
Figure 2: a) Initial configuration of the clocks A’ and B’ at rest in the frame S. The measuring rod, R, of length L is used to set the initial separation of the clocks. On pressing the button Bu, the signal generator, SG, sends simultaneous pulse-signals P(A’) and P(B’) along the identical cables C(A’) and C(B’) to A’ and B’, respectively. On arriving at the clocks, these signals initiate identical acceleration programmes at epochs t = 0 for the clocks as described in Eqns(1) and (9). b) After the time interval $t = t_{acc} = \sqrt{3}$ (in units where $c = a_0 = 1$), the clocks move with the uniform velocity $v_F = \sqrt{3}/2$ along the x-axis, separated by the distance L. The video camera, VC, records the passage of the clocks (see text for discussion).
Figure 3: Space-time trajectories of the clocks as observed in: a), S and b), S’. S’ is the instantaneous co-moving frame of A’ and B’. Units and dimensions with \(a_0 = c = L = 1\) are used. At the epochs \(t = T_{\text{simul}} = 4.15\), \(t' = T'_{\text{simul}} = 2.47\) clocks AA’ and BB’ are observed to be, simultaneously, in spatial coincidence in the \(x\)-direction, in both S and S’.
Figure 4: Spatial configurations of the clocks as observed in either $S$ or $S'$ at different epochs. 

a) $t = T_{\text{simul}} - L/v_F$, $t' = T'_{\text{simul}} - L/\left(\gamma_F v_F\right)$  
b) $t = T_{\text{simul}}$, $t' = T'_{\text{simul}}$  
c) $t = T_{\text{simul}} + L/v_F$, $t' = T'_{\text{simul}} + L/\left(\gamma_F v_F\right)$.

The spatial separation of $A$ and $B$ has been previously set to $L$, (see a)) using the measuring rod $R$. 

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at rest but are equipped with identical mechanisms that simultaneously accelerate them in the positive $x$-direction during the time interval $t_{acc}$ in the frame $S$. After this they move with the same uniform velocity $v_F$ (see Fig. 2b). It is therefore assumed that the same time-dependent acceleration program as discussed above, is applied simultaneously to both clocks in their common proper frame. The acceleration procedure is started simultaneously in $S$ by signals from the generator SG which is connected via the identical cables $C(A')$ and $C(B')$ and movable contacts to $A'$ and $B'$. When the button $Bu$ is pressed, simultaneous signal pulses $P(A')$ and $P(B')$ propagate along the cables (Fig. 2a) and are detected by the accelerating mechanism in $A'$ and $B'$. At time $t = 0$, according to a clock in $S$, both $A'$ and $B'$ start to accelerate in the direction of $A$ and $B$. The velocities and positions of the clocks are given by the formulae (9) and (1) above, respectively, as:

$$0 < t \leq t_{acc}$$

$$v_{A'}(t) = v_{B'}(t) = \frac{a_0 t}{\sqrt{1 + \left(\frac{a_0 t}{c}\right)^2}} \equiv v(t) \quad (12)$$

$$x_{A'}(t) + 2L = x_{B'}(t) = L = \frac{c^2}{a_0} \left[ \sqrt{1 + \left(\frac{a_0 t}{c}\right)^2} - 1 \right] \quad (13)$$

where $t_{acc} < t$

$$v_{A'}(t) = v_{B'}(t) = v(t_{acc}) \equiv v_F \quad (14)$$

$$x_{A'}(t) = x_{B'}(t) - L = x_{A'}(t_{acc}) + v_F(t - t_{acc}) \quad (15)$$

The $x\ v\ t$ plot of the motion of $A'$ and $B'$ in $S$ is shown in Fig. 3a. At time $t = T_{simul}$, $A,A'$ and $B,B'$ are, simultaneously, spatially contiguous while at $t = T_{simul} - L/v_F$ $A,B'$ are contiguous and at $t = T_{simul} + L/v_F$ $B,A'$ are contiguous (see Fig. 3a and Fig. 4).

The sequence of events as observed from the proper frame of of $A'$ and $B'$ is calculated with the aid of Eqns(6) and (7) above. These equations predict that the $x'$ versus $t'$ plots for $A,B,A'$ and $B'$ may be derived from those of Fig. 3a by reflection of the latter in the $x$ axis and by a suitable scaling of the slopes of the curves with different factors for the phases of accelerated and accelerated motion, the former scale factor, $-\gamma(t)$, being time-dependent and the latter, $-\gamma(t_{acc})$, constant. The space-time trajectories in Fig. 3b are then given by the formulae:

$$0 < t' \leq t'_{acc}$$

$$v'_{A}(t') = v'_{B}(t') = -c \sinh \frac{a_0 t'}{c} \quad (16)$$

$$L - x'_{A}(t') = 2L - x'_{B}(t') = \frac{c^2}{a_0} \left[ \cosh \frac{a_0 t'}{c} - 1 \right] \quad (17)$$

$$t'_{acc} < t'$$

$$v'_{A}(t') = v'_{B}(t') = v'_{A}(t'_{acc}) = -\gamma_F v_F \quad (19)$$

$$x'_{A}(t') = x'_{B}(t') - L = x'_{A}(t'_{acc}) - \gamma_F v_F (t' - t'_{acc}) \quad (20)$$
The curves shown in Fig. 3 recall a phenomenon that the reader may have noticed when sitting in a train that has halted at a station. On the adjacent track are the carriages of another train. At a certain moment, the person in the train, looking out of the window, thinks that his train has started, as the carriages of the adjacent train are seen to accelerate to the right. The last carriage passes and the person sees, with surprise, that his train is still stationary. Or, it may happen that instead the scenery outside is seen to be also moving to the right, at almost the same speed as the last carriage of the adjacent train, and the traveller is reassured that her journey is proceeding. In the first case the adjacent train accelerates to the right. In the second case the observer’s train accelerates, in an identical manner, to the left. A physicist equipped with an accelerometer, could, of course, easily distinguish the two cases. Inspection of Fig. 1a and Fig. 1b shows that, in Special Relativity, in contrast the Galilean relativity, if it is known that one of the two trains, but not which one, has the acceleration program considered above, observation of the world line of one train from the other reveals which of them is accelerated, and which is at rest. If the shape of the world line of the observed train is described by Eqn(1), as in Fig. 1a, it is being accelerated, whereas, if it is described by Eqn(6) as in Fig. 1b, it is the observer’s train which is undergoing acceleration. Since the functional dependence

\[
\frac{c^2}{a_0} \left[ \sqrt{1 + \left(\frac{a_0 t}{c}\right)^2} - 1 \right] = \frac{1}{2} a_0 t^2 - \frac{(a_0 t)^4}{8c^2} + \ldots \quad (21)
\]

and

\[
\frac{c^2}{a_0} \left[ \cosh \frac{a_0 t}{c} - 1 \right] = \frac{c^2}{2a_0} \left[ \left( \exp \left[ \frac{a_0 t}{c} \right] + \exp \left[ -\frac{a_0 t}{c} \right] \right) - 1 \right] = \frac{1}{2} a_0 t^2 + \frac{(a_0 t)^4}{24c^2} + \ldots \quad (22)
\]

so that the functional dependence differs only by terms of \(O((a_0 t)^4/c^2)\) and higher, which vanish in the Galilean limit where \(c \to \infty\). In Fig. 2, the video camera VC records the motion of the objects A’ and B’ as they accelerate to the right and then move with uniform speed \(v_F\). The motion of a particular point on the train follows the time dependence of Eqn(1) and (9) during the acceleration phase. If the VC were to be accelerated to the left in an identical manner, the motion of the same point on the train is instead described by Eqns(6) and (8), after reversing the direction of the \(x'\)-axis. In Galilean relativity, where only the \(a_0 t^2/2\) terms in (21) and (22) are retained the VC record is identical in the two cases.

The times recorded by the moving clocks A’ and B’, as viewed from the inertial frames S or S during the phase of uniform motion, will now be considered. It is assumed that they are identical, unsynchronised, digital clocks that display a periodically updated sequence of numbers which are identified with the epoch \(t_i\), for the clock \(i\). The times recorded by the clocks, at the events labelled in Figs. 3 and 4 by the proper times \(T_{simul} - L/v_F\), \(T_{simul} + L/v_F\) in S and \(T_{simul}' - L/\gamma_F v_F\), \(T_{simul}' + L/\gamma_F v_F\) in S’ are shown in Fig. 5, as observed in S, and in Fig. 6 as observed in S’. The epochs are: \(t_A(B'), t_A(A'), t_B(B'), t_B(A'), t_B'(A), t_B'(B)\) and \(t_B'(A)(B)\). Here \(t_i(j)\) is the epoch
Figure 5: Observations of the clocks from the frame $S$ at different epochs: a) $t = T_{\text{simul}} - L/v_F$, b) $t = T_{\text{simul}}$, c) $t = T_{\text{simul}} + L/v_F$. 
Figure 6: Observations of the clocks from the frame $S'$ at different epochs: 
a) $t' = T_{\text{simul}}' - L/\left(\gamma_F v_F\right)$, 
b) $t' = T_{\text{simul}}'$, 
c) $t' = T_{\text{simul}}' + L/\left(\gamma_F v_F\right)$. 
recorded by the clock \( i \) when it is in spatial coincidence, in the \( x \)-direction, with the clock \( j \). These eight observed epochs constitute the raw data of the thought experiment that will now be analysed using SRT.

Consider first the observation of the clock \( B' \) between the \( AB' \) and \( BB' \) spatial coincidences. In the frame \( S \) (see Fig. 5a and 5b):

\[
\Delta t(B', S) \equiv t_A(A') - t_A(B') = \frac{L}{v_F}
\]

while in \( S' \) (see Fig. 6a and 6b)

\[
\Delta t(B', S') \equiv t_{B'}(B) - t_{B'}(A) = \frac{L}{\gamma_F v_F}
\]

combining (23) and (24) gives

\[
\Delta t(B', S) = \gamma_F \Delta t(B', S')
\]

which is just the TD relation for the moving clock \( B' \). Similarly for \( A' \):

\[
\Delta t(A', S) \equiv t_B(A') - t_B(B') = \frac{L}{v_F}
\]

and

\[
\Delta t(A', S') \equiv t'_{A'}(B) - t'_{A'}(A) = \frac{L}{\gamma_F v_F}
\]

which give the TD relation for the moving clock \( A' \):

\[
\Delta t(A', S) = \gamma_F \Delta t(A', S')
\]

The TD effect for each of the moving clocks \( A' \) and \( B' \) involves two time intervals, one, \( \Delta t(A', S') \) or \( \Delta t(B', S') \), recorded by the clock itself and the other, \( \Delta t(A', S) \) or \( \Delta t(B', S) \), recorded by a clock at rest in \( S \). For \( \Delta t(A', S) \) (see Eqn(26)) this clock is \( B \), whereas for \( \Delta t(B', S) \), (see Eqn(23)) this clock is \( A \). The above analysis makes very clear the physical origin of the TD effect in the larger relative velocity of \( A \) or \( B \) relative to \( A' \) or \( B' \) in the frame \( S' \) than in the frame \( S \), the spatial separation between \( A' \) and \( B' \) and between \( A \) and \( B \) being equal at all times in both \( S \) and \( S' \).

It has been assumed above that observers exist in both \( S \) and \( S' \) in order to note the clock readings at the epochs of the \( AB' \), \( BB' \), \( AA' \) and \( BA' \) coincidence events as shown in Figs. 5 and 6. All times intervals appearing in the TD relations (25) and (28) may thus be identified with observations of clocks at rest by two different observers —one in frame \( S \), another in the frame \( S' \). However, in all actual experimental realisations of the TD effect performed to date —for example observation of the decay lifetime of an unstable elementary particle, \( P \), in motion— all observations are performed in the laboratory frame \( S \). As in the above example, \( \Delta t(P, S) \) is still actually, or effectively\(^d\), recorded by a clock at rest in \( S \), whereas \( \Delta t(P, S') \) is inferred from the measured values of \( \Delta t(P, S) \)

\(^d\)In the case of the muon decay experiment described in Ref. [13] the time interval is really measured by a clock in the laboratory frame. In more typical experiments [12] involving unstable particles with shorter lifetimes, it is deduced from the observed path length of the particle between production and decay and the velocity of the particle derived from kinematical measurements.
and $\gamma_P$. The TD relation is then verified by comparing the calculated mean proper decay lifetime $\bar{\tau}$ of a statistical ensemble, $P_i$, $i = 1, 2, 3...N$ of $N$ unstable particles:

$$\bar{\tau} = \frac{1}{N} \sum_i \frac{\Delta t(P_i, S)}{\gamma_{P_i}}$$

with the previously measured mean lifetime for decay at rest for the same type of particle [12, 13].

It is shown in Figs. 5 and 6 that the ‘Reciprocity Principle’ (RP) [14] that has hitherto been assumed to be valid in both Galilean and Special Relativity, does not hold in SRT. The RP states that: “If the velocity of an object $O'$ relative to an object $O$ in the rest frame of $O$ is $\vec{v}$ then the velocity of $O$ relative to $O'$ in the rest frame of $O'$ is $-\vec{v}$.” Eqn (8) above shows that this is not the case in SRT. As discussed at length in Refs. [5, 6], although the RP does not apply to observations of the same space-time experiment in different frames, as, for example in Figs. 5 and 6 above, it does describe correctly the effect of a kinematical (velocity) LT between two inertial frames. For example the RP can be derived from the parallel velocity transformation relation (A.17) of the Appendix. In this case what are related are not events in the same experiment as viewed in different inertial frames but rather the initial kinematical configurations of an experiment and its reciprocal as mentioned in the MRP above. If $S'$ moves with velocity $v$ along the positive $x$-axis, in $S$, then setting $u = 0$ in (A.17), for the origin of $S$, gives $u' = -v$. In the primary experiment therefore the ‘travelling frame’ $S'$ moves with speed $v$ along the positive $x$-axis in the ‘base frame’ $S$, whereas in the reciprocal experiment the travelling frame $S$ moves with speed $v$ along the negative $x$-axis in the base frame $S'$. The RP thus relates base frame configurations in the primary and reciprocal experiments, not observed velocities in the frames $S$ and $S'$ of the primary experiment [5, 6]. As shown for example, by the different nature of the TD effects—the slower running clock of the primary experiment becomes the faster running one of the reciprocal experiment—an experiment and its reciprocal are physically independent [5, 6].

Inspection of the geometry of Fig. 3 shows that the fundamental reason for the simultaneity in both $S$ and $S'$ of the AA' and BB' coincidence events, and the absence of any ‘length contraction’ effect, is simply translational invariance (or the homogeneity of free space). The world line of $B'$ is obtained from that of $A'$ in Fig. 3a, by the coordinate transformation $x \rightarrow x + L$, or that of $B$ from that of $A$ in Fig. 3b, by the transformation $x' \rightarrow x' + L$ —the separation of $B'$ from $A'$ in Fig. 3a and of $B$ from $A$ in Fig. 3b is then independent of both their velocity and of time. The simultaneity of the events AA' and BB' in both $S$ and $S'$ is then a necessary consequence of the invariance of the separation of $B$ from $A$ or $B'$ from $A'$ in both frames and of the equality of the two separations. All these consequences necessarily follow from the identical shapes of the world lines of $A$ and $B$ and of $A'$ and $B'$ in both frames. This identity of form follows directly from the assumed identical acceleration programs of $A'$ and $B'$, independently of the nature of the program. The choice of ‘hyperbolic motion’ considered above was made only for calculational convenience.

The fallacious nature of the conventional predictions, of SRT, following Ref. [1], of the RS and LC effects is explained, from different points of view, in Refs. [7, 8, 9, 10].
Appendix

Suppose that an object \( O \) moves with speed, \( u(O) \) along the positive \( x \)-axis in the frame \( S \), and that the object \( O' \) is at rest in the frame \( S' \) moving with (instantaneous) velocity \( v(O') \) along the positive \( x \)-axis. The observed velocity of \( O \) along the positive \( x' \)-axis in \( S' \) is then

\[
u'(O) = \gamma_v[u(O) - v(O')] \quad \text{(A.1)}
\]

where \( \gamma_v \equiv \frac{1}{\sqrt{1 - \beta_v^2}} \), \( \beta_v \equiv v(O')/c \). If \( u = 0 \) at all times (i.e. for all values of \( v(O') \)) then

\[
u'(O) \equiv -v' = -\gamma_v v(O') = -c\gamma_v \beta_v \quad \text{(A.2)}
\]

so that \( O \) is observed to move with speed \( c\gamma_v \beta_v \) along the negative \( x' \)-axis in \( S' \). Differentiating (A.2) with respect to the epoch, \( t' \), recorded by the clock a rest in \( S' \), noting the differential TD relation:

\[
dt = \gamma_v dt' \quad \text{(A.3)}
\]

gives

\[
\frac{dv'}{dt'} = c\frac{d(\gamma_v \beta_v)}{dt} = c\gamma_v [1 + (\gamma_v \beta_v)^2] \frac{d\beta_v}{dt} \gamma_v = \gamma_v^4 \frac{dv(O')}{dt} \quad \text{(A.4)}
\]

For the case of a constant proper acceleration of the frame \( S' \): \( dv'/dt' = a = \text{constant} \), the equation given by transposing (A.4)

\[
dt = \frac{c}{a} \frac{d\beta_v}{(1 - \beta_v^2)^2} \quad \text{(A.5)}
\]

may be integrated, making use of the identity:

\[
\frac{1}{(1-x)^2} \equiv \frac{1}{4} \left[ \frac{2+x}{(1+x)^2} + \frac{2-x}{(1-x)^2} \right] \quad \text{(A.6)}
\]

to yield the relation between \( t \) and \( \beta_v \) for any object at rest in the accelerated frame \( S' \):

\[
t(\beta_v) = \frac{c}{4a} \left[ \frac{2\beta_v}{(1 - \beta_v^2)} + \ln \left( \frac{1 + \beta_v}{1 - \beta_v} \right) \right] \quad \text{(A.7)}
\]

The inverse of this equation \( \beta_v(t) \) is therefore a transcendental function, preventing the determination, by analytical integration, of the equation of motion of \( O' \).

However, a simple analytical solution for the motion of \( O' \) may be obtained in the case that its proper acceleration is time-dependent:

\[
\frac{dv'}{dt'} = a(t') = a_0 \gamma_v(t') \quad \text{(A.8)}
\]

where \( a_0 = a(0) = \text{constant} \). In this case (A.4) gives:

\[
\frac{dv'}{dt'} = a_0 \gamma_v = \gamma_v^4 \frac{dv(O')}{dt} \quad \text{(A.9)}
\]

so that

\[
a_0 = \gamma_v^2 \frac{dv(O')}{dt} \quad \text{(A.10)}
\]
(A.10) may be integrated by noting the relation:

\[ d \left( \frac{\beta_v}{\sqrt{1 - \beta_v^2}} \right) = \frac{d\beta_v}{(1 - \beta_v^2)^{3/2}} \]  

(A.11)

to yield, for an object initially at rest in the frame S:

\[ \frac{\beta_v}{\sqrt{1 - \beta_v^2}} = \frac{a_0 t}{c} \]

(A.12)
or, transposing:

\[ \beta_v = \frac{1}{c} \frac{dx}{dt} = \frac{a_0 t}{\sqrt{c^2 + a_0^2 t^2}} \]

(A.13)

(A.13) is equivalent to Eqn(9) of the text. Integrating (A.13), assuming the same initial conditions as in (A.12) and (A.13) gives:

\[ x = \frac{c^2}{a_0} \left[ \sqrt{1 + \left(\frac{a_0 t}{c}\right)^2} - 1 \right] \]

(A.14)
as used in Eqn(1) of the text.

The relation between the epochs \( t \) and \( t' \) is given by integrating the TD relation:

\[ t'(t) = \int_0^t \frac{dt}{\gamma_v} = \int_0^t \frac{dt}{\sqrt{1 + \left(\frac{a_0 t}{c}\right)^2}} = \frac{c}{a_0} \ln \left[ at + \sqrt{1 + \left(\frac{a_0 t}{c}\right)^2} \right] = \frac{c}{a_0} \text{arcsinh} \frac{a_0 t}{c} \]

(A.15)

This is equivalent to Eqn(4) of the text. The relation \( \gamma_v = \sqrt{1 + \left(\frac{a_0 t}{c}\right)^2} \) used in the third member of (A.15) is obtained by transposing (A.13) and using the definition of \( \gamma_v \). The explicit \( t' \) dependence of the acceleration program is then, using (A.15):

\[ a(t') = a_0 \gamma_v(t') = a_0 \cosh \frac{a_0 t'}{c} \]

(A.16)

The formulae (A.13) and (A.14) describing ‘hyperbolic motion’ were first given by Born [15] and Sommerfeld [16]. A formula equivalent to (A.10) was given by Rindler [17], while Eqns(A.10),(A.13),(A.14) and (A.15) were all derived by Marder [18]. All of these authors associated these equations, not with the proper-time dependent proper acceleration of Eqn(A.16), but with a constant proper acceleration. The reason for these different predictions is that the analysis in the literature is based not on the relative-velocity transformation formula (A.2) but on the conventional parallel velocity addition formula of special relativity:

\[ u' = \frac{u - v}{1 - \frac{uv}{c^2}} \]

(A.17)
where $u \equiv u(O)$, $u' \equiv u'(O)$ and $v \equiv v(O')$. Differentiating (A.17) with respect to $t'$ gives, with use of (A.3)

$$
\frac{du'}{dt'} = -\gamma_v \left[ \frac{1}{1 - \frac{uv}{c^2}} - \frac{u}{c^2 \left(1 - \frac{uv}{c^2}\right)^2} \right] \frac{dv}{dt} + \gamma_v \left[ \frac{1}{1 - \frac{uv}{c^2}} + \frac{v}{c^2 \left(1 - \frac{uv}{c^2}\right)^2} \right] \frac{du}{dt}
$$

(A.18)

Since the condition $u = 0$ used to derive (A.2) from (A.1) holds at all times, $du/dt = 0$. Setting then $u = 0$ and $du/dt = 0$ in (A.18) gives

$$
\frac{du'}{dt'} = -\gamma_v \frac{dv}{dt}
$$

(A.19)

which is inconsistent with (A.4) and does not agree with (A.10) when $du'/dt' = a_0 = \text{constant}$.

In order to derive (A.10) starting from (A.17), the procedure used in Refs. [17, 18] was firstly to hold $v$ constant in differentiating (A.17) to give

$$
\frac{du'}{dt'} = \gamma_v \left[ \frac{1}{1 - \frac{uv}{c^2}} + \frac{v}{c^2 \left(1 - \frac{uv}{c^2}\right)^2} \right] \frac{du}{dt} \quad (v = \text{constant})
$$

(A.20)

and secondly to make the substitution $u = v$ in this equation to give

$$
\frac{du'}{dt'} = \gamma_v \frac{dv}{dt} = \gamma_v^3 \frac{dv}{dt}
$$

(A.21)

in agreement, up to a sign, with (A.10) when $du'/dt' = a_0 = \text{constant}$. There are two mathematical errors in the above derivation of (A.21):

(i) Since this is a discussion where the frame $S'$ is accelerated, $v$ is time-dependent, not constant. The postulate that $v$ is constant used to derive (A.20) from (A.18) therefore contradicts an initial assumption of the problem.

(ii) The condition $u = v$ implies that the object, the velocity of which is described by the left side of (A.21), is at rest in $S'$ and moves with speed $v$ in the direction of the positive $x$-axis in $S$, whereas what must be described by this transformation equation (as is the case for (A.4)) is the velocity of an object at rest in $S$ (so $u = 0$) as observed in the frame $S'$. In fact, since $u \equiv u(O)$ represents the velocity of the object $O$ whereas $v \equiv v(O')$ represents the velocity of the object $O'$, and the objects $O$ and $O'$ are distinct, the condition $u(O) = v(O')$ implies that $O$ and $O'$ (and hence $S$ and $S'$) are at rest relative to each other. The condition $u = v$ then also contradicts the assumed initial condition of the problem that $S'$ undergoes accelerated motion.

Apart from these purely mathematical errors, the use of the kinematical velocity transformation formula (A.17) does not correctly describe the velocity of $O$, as observed in the travelling frame $S$, in the space-time experiment in which $O'$ has velocity $v$ in the base frame $S$. This is correctly given, not by (A.17), but by (A.2). The formula (A.17) instead transforms, between two inertial frames, the base frame kinematical configurations of two physically independent space-time experiments, one primary and the other reciprocal. For further discussion of this essential conceptual point, see Refs. [5, 6].
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