**Direction reversal of non-Hermitian skin effect via coherent coupling**

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(Dated: March 16, 2022)

Absolute negative mobility (ANM) in nonequilibrium systems depicts the possibility of particles propagating toward the opposite direction of an external force. We uncover in this work a phenomenon analogous to ANM regarding eigenstate localization and particle transport in non-Hermitian systems under the influence of the non-Hermitian skin effect (NHSE). A coherent coupling between two non-Hermitian chains individually possessing the same preferred direction of NHSE is shown to cause a direction reversal of NHSE for all eigenmodes. This concept is further investigated in terms of time evolution dynamics using a non-Hermitian quantum walk platform within reach of current experiments. Our findings are explained both qualitatively and quantitatively. The possible direction reversal of NHSE can potentially lead to interesting applications.

**Introduction.**- Non-Hermitian Hamiltonians provide an effective description of open quantum systems or wave systems with gain and loss [1–4]. One main feature of non-Hermitian lattice systems with nonreciprocity is the seminal non-Hermitian skin effect (NHSE) under open boundary conditions [5–8]. NHSE causes directional accumulation of eigenmodes at the system’s boundaries and has a rather deep connection with the point-gap topology of the complex spectrum of non-Hermitian systems [5–17]. NHSE has spurred considerable interest in condensed matter physics research because it challenged our conventional thinking of bulk-edge correspondence and has motivated the so-called non-Bloch band theory [6–8]. Much attention has also been paid to the interplay between the NHSE and other important physical effects, such as topological localization [14, 15], external electromagnetic fields [18–20], disorders and defects [21, 29].

Non-reciprocal hopping on a one-dimensional (1D) lattice defines a preferred direction analogous to a physical direction of an external force. The preferred boundary for bulk state localization as NHSE is thus intuitive, so does the preferred direction favoring particle transport [10, 15, 17, 80–84]. For example, if the strength of intercell hopping to the left is always larger than that to the right, then NHSE is expected to localize all states at the left boundary. On the other hand, we must take note of one remarkable dynamical phenomenon, namely, absolute negative mobility (ANM), where particles propagate toward the opposite direction of an external force. Seemingly contradictory to Newton’s second law, ANM has already been widely investigated and experimentally realized in various systems far from equilibrium [35–45]. Recognizing non-Hermitian systems as nonequilibrium systems, it is necessary to address the possibility of population accumulation or particle transport in a direction against the preferred direction indicated by the non-reciprocal hopping. This issue is not only of theoretical interest, but may offer versatile control knobs to manipulate NHSE for various applications, such as light funneling [46] and directional signal amplification [50, 51, 47].

In this work, we unveil a general scheme to induce 1D NHSE in a direction precisely opposite to the favored direction of non-reciprocal hopping. As shown below, this exotic phenomenon can be obtained at both the eigenstate level and the dynamics level. There can be multiple interpretations of why a direction reversal of NHSE occurs. Among them, a simple physical picture adopted below is based on the interference between multiple hopping pathways. Specifically, if two non-Hermitian lattices with the same preferred non-reciprocal direction are coupled, then multiple hopping pathways become available. The resulting interference between the multiple hopping pathways can counter-intuitively and drastically alter the effective strengths of hopping towards two directions, and hence one must reexamine the true physically favored direction of NHSE.

The direction reversal of NHSE by coherent coupling is in principle observable in a variety of quantum and classical platforms realizing NHSE [32–46]. In particular, reversed NHSE at the eigenstate level is already within the reach of classical platforms, such as circuits. However, how reversed NHSE is manifested at the dynamics level is less straightforward. We hence propose a non-unitary quantum walk setting directly addressing non-Hermitian dynamics [32–46], with the essential addition being an interchain hopping for the quantum walker along two chains. As elaborated below, even though the preferred direction of NHSE is no longer obvious in the quantum walk dynamics, the multiple propagation pathways induced by the interchain hopping can still lead to a direction reversal of particle transport.

**Direction reversal of eigenstate population accumulation.**- Our starting point is a minimal model depicting two coupled non-Hermitian chains [53] with different on-site potentials, as shown in Fig. 1. The real-space Hamiltonian reads

\[
\hat{H} = \sum_{x=1}^{L} \sum_{s=a,b} \left[ t_x e^{\alpha_x t_x} \hat{s}_x + t_x e^{-\alpha_x t_x} \hat{s}_{x-1} + \alpha_x (\hat{a}_x \hat{b}_{x+1} + \hat{b}_x \hat{a}_{x+1}) + \mu_x \hat{a}_x \hat{a}_{x+1} + \mu_x \hat{b}_x \hat{b}_{x+1} \right]
\]

with \(t_x\) and \(\alpha_x > 0\) determining the asymmetric hopping amplitudes on the two chains labeled by \(s = a, b\). Referring to Fig. 1 the preferred direction of the non-reciprocal hopping here is apparently to the left for both chains. The on-site potential is set to be \(\mu_a = -\mu_b = \mu\), with all other choices being...
The two chains are completely decoupled if $t_\perp = 0$, each displaying NHSE localization at the left edge, with an inverse localization length $\kappa_{a,b} = \alpha_{a,b}[6,8,14]$. An example depicting such a decoupling limit is illustrated in Fig. 2(a).

Upon switching on the interchain coupling ($t_\parallel \neq 0$), both the complex spectrum and eigenstate localization of the coupled system start to differ from that of the uncoupled case, no matter how small $t_\parallel$ is [16,54,55]. To allow for many hopping pathways from one site to its neighboring site, such as $a_x \rightarrow a_{x+1}$ and $a_x \rightarrow b_x \rightarrow b_{x+1} \rightarrow a_{x+1}$, to interfere significantly, here we investigate a strong coupling regime with sufficiently large $t_\parallel$. It is then found that all eigenmodes can now localize at the opposite edge as compared with that in the uncoupled case. This is clearly seen in Fig. 2(a)-(c) as the interchain coupling strength $t_\parallel$ increases from 0 to 6 and to 15.

To characterize the above-observed direction reversal of NHSE, we consider averages of the standard and directional inverse participation ratios (IPR and dIPR), defined as

$$
\bar{I} = \frac{1}{2L} \sum_{m} \sum_{x=1}^{L} \left( |\psi_{x,m}^a|^4 + |\psi_{x,m}^b|^4 \right),
$$

$$
\bar{I}_d = \frac{1}{2L} \sum_{m} \sum_{x=1}^{L} \left( x - (L + 1)/2 \right) \left( |\psi_{x,m}^a|^4 + |\psi_{x,m}^b|^4 \right) / (L - 1)/2,
$$

with $\psi_{x,m}^\alpha$ the wave amplitude of the $m$-th normalized right eigenmode at site $x$ of sublattice $\alpha$. Representative results are presented in Fig. 2(d). It is seen that the IPR (and the absolute value of dIPR) gets larger either for weaker or stronger $t_\perp$, indicating a stronger boundary accumulation of the eigenmodes, but with opposite accumulating directions, as evidenced by the signs of the dIPR. A reversal of the NHSE direction starts to occur when $\bar{I}_d = 0$, which is at $t_\perp \approx 6.3$ in the shown example. In the neighborhood of the transition point $\bar{I}_d = 0$ [see Fig. 2], the eigenmodes can possibly localize at both boundaries in a balanced manner as the bipolar NHSE [56,57]. More importantly, away from the transition point, all eigenmodes are now localized at the opposite boundary as compared with the uncoupled case. Meanwhile, the difference between the average distribution on the two lattices, defined as

$$
\bar{\delta}_\rho = \frac{1}{2L} \sum_{m} \left| \sum_{x=1}^{L} \left( |\psi_{x,m}^a|^2 - |\psi_{x,m}^b|^2 \right) \right|,
$$

is seen to decrease with increasing $t_\perp$, indicating a stronger hybridization between the two chains at larger $t_\perp$.

**Physics of reversed NHSE.** To confirm that the main physics behind reversed NHSE is the interference between multiple hopping pathways, we consider a straightforward first-order perturbation theory by treating $\hat{H}_\perp = t_\parallel \sum_x (\hat{a}_x^\dagger \hat{b}_x + \hat{b}_x^\dagger \hat{a}_x)$ as the unperturbed Hamiltonian. The unperturbed eigenmodes at site $x$ are simply given by local hybridized adiabatic modes of the coupled system, i.e.,

$$
|\mu_{x,\pm}\rangle = \hat{u}_{x,\pm}^\dagger |0\rangle = (a_{x,\pm}^\dagger \pm b_{x,\pm}^\dagger) |0\rangle / \sqrt{2}, \quad E_{x,\pm} = \pm t_\perp,
$$

with $|0\rangle$ the vacuum and $E_{x,\pm}$ the corresponding unperturbed eigenenergies due to the coherent coupling. By taking all the rest terms as a perturbation, we rewrite the non-reciprocal hopping Hamiltonian in the local adiabatic representation,
yielding
\[ \hat{H}_s = \sum_x (t_a e^{ix} + t_b e^{ix}) \hat{u}_{s,x}^\dagger \hat{u}_{s,x+1} + (t_a e^{-ix} + t_b e^{-ix}) \hat{u}_{s,x+1}^\dagger \hat{u}_{s,x} + (\mu_a \pm \mu_b) \hat{u}_{s,x}^\dagger \hat{u}_{s,x}. \] (6)

Interestingly, other than the local on-site energy being expectedly different, the "+" and "−" hybridized lattice sites have the same non-reciprocal hopping strengths, \((t_a e^{ix} + t_b e^{ix})\) to the left and \((t_a e^{-ix} + t_b e^{-ix})\) to the right. That is, the effective hopping amplitudes are seen to be a sum of two individual hopping amplitudes \(t_a e^{ix}\) and \(t_b e^{ix}\) or \(t_a e^{-ix}\) and \(t_b e^{-ix}\), thus clearly indicating an interference mechanism. Most importantly, if \(t_a\) and \(t_b\) are of different signs, then there is a destructive interference between the two favored amplitudes. This can then lead to
\[ |t_a e^{-ix} + t_b e^{-ix}| > |t_a e^{ix} + t_b e^{ix}|, \] (7)
meaning that NHSE here should accumulate/localize population to the right for \(all\) the eigenmodes, opposite to the NHSE direction on the uncoupled chains. Inequality (7) also suggests that reversed NHSE occurs within the following parameter regime
\[ \frac{t_a}{-t_b} \in (e^{ix} - e^{-ix}, e^{-ix} + e^{ix}), \] (8)
as shown by the solid lines in Fig. 2c). The transition lines obtained this way match well with the numerical results based on the sign of the average directional IPR \(I_d\). A momentum space perturbation theory together with the so-called generalized Brillouin zone [6] [7] yields the same prediction in theory, as detailed in Supplementary Materials. Note also that the role of the on-site potential difference \(\mu\) is not seen here due to our first-order treatment or the strong coupling assumption. The actual threshold value \(t_a\) to enter the reversed NHSE regime gradually increases with \(\mu\). Reversed NHSE may also be obtained under \(t_a t_b > 0\), if we introduce multiple hopping pathways in other manners, such as allowing for off-diagonal couplings between the two chains. These details can be found in Supplementary Materials.

**Reversed particle transport in non-Hermitian quantum walk.** So far, the reversed NHSE is investigated on the eigenstate level via population accumulation against the preferred direction of non-reciprocal hopping. To make a closer analogy to ANM, it is necessary to explore how this leads to particle transport along a reversed direction. To motivate experimental interest, we use an available and fruitful platform, namely, a discrete-time non-unitary quantum walk model realizing the NHSE through single-photon dynamics in a 1D chain [32]. We now propose a quantum walker along two chains, plus a local interchain exchange depending on the spin state.

Specifically, we consider the following two Floquet operators governing the quantum walk,
\[ U_0 = R(\theta_1) S R(\theta_2 + \theta_3) M R(\theta_2 + \theta_3) S_1 R(\theta_1), \] (9)
\[ U = R(\theta_1) S_2 R(\theta_3) M R(\theta_3) S_3 R(\theta_2) S_1 R(\theta_1). \] (10)

Here \(R(\theta)\) rotates the spin by \(\theta\) about the \(y\) axis, with \(R(\theta) = \sum_{i=-N}^{N} \sum_{s=a,b} \langle s, x| e^{-i\lambda s} \sigma^s_{\theta} |s, x\rangle \), \(s = a, b\) denoting the two chains, \(x\) the site index, and \(\lambda_a = 1\) and \(\lambda_b = -1\). The shift operators \(S_1\) and \(S_2\) are standard quantum walk operations, as they shift the walker to the left and right along either chain, if and only if the spin is up and down, respectively. Non-unitarity/non-Hermiticity is introduced through the operator \(M\), with
\[ M = \sum_{x=-N}^{N} \sum_{s=a,b} \langle s, x| \sigma^s_{\theta} \otimes (|\downarrow\rangle\langle\downarrow| + e^{-\gamma s}|\uparrow\rangle\langle\uparrow|) \),
\[ \text{describing the (quasi-)particle loss only for the spin-up component.} \]
\[ U_0 \text{ thus defined above yields exactly two copies of the quantum walk model realizing the NHSE in Refs. [32] [34], but with opposite spin rotation angles through the parameter } \lambda. \]
\[ \text{The } M \text{ operator alone seems to suggest that the spin-down channel is favored. This effect further interplays with the spin rotation operator } R(\theta) \text{ and the shift operators } S_{1,2} \text{ to yield non-reciprocal particle transport, with the preferred direction no longer obvious. Despite the difference in } \lambda \text{ between the two quantum walk copies, their preferred direction of transport is found to be always the same.} \]

We now couple the two chains accommodating \(U_0\), thus defining our quantum walk model \(U\). \(U\) is obtained by inserting \(S_3\) and \(S_4\) into \(U_0\). \(S_3\) and \(S_4\) are almost the same operations as \(S_1\) and \(S_2\) except that the walker is instructed to hop onto the other chain (of the same lattice index) when the spin state is up and down, respectively. Detailed definitions of these operations are shown in Supplementary Materials. \(S_3\) and \(S_4\) are expected to hybridize the two chains and

![FIG. 3. (a,b) Spatial distributions of the quantum walks for an initial state prepared in the middle of the system, governed by \(U_0\) and \(U\), the two quantum walks without and with the interchain hopping, respectively. (c,d) Average position of the final for the two quantum walks at \(T = 40\), versus the two angles \(\theta_2\) and \(\theta_3\). Yellow and blue (bright and dark) regimes indicate non-reciprocal pumping toward the directions of \(x = N\) and \(x = -N\) respectively. The red points in (c,d) represents the two cases of (a,c), with \(\theta_2 = \theta_3 = 0.4\pi\). Other parameters are \(\alpha_a = \alpha_b = 3, \theta_1 = 0.2\pi\) and \(N = 10\).](image-url)
induce interference between multiple hopping pathways. We aim to show that even though the two individual chains have the same preferred walk direction, their coupling can reverse the direction of transport, thus demonstrating direction reversal of NHSE via time evolution dynamics.

We consider an initial state prepared in the middle of the system, \( \Psi_{\text{ini}} = \frac{1}{\sqrt{2}} (|a, 0 \rangle \otimes |\uparrow\rangle + |b, 0 \rangle \otimes |\uparrow\rangle) \). In Fig. 3(a) and (b), we show the spatial distribution \( \rho(x) \) of the normalized final state \( \Psi_{\text{fin}} = U_0^T \Psi_{\text{ini}} \) and \( \rho(x) \) of \( \Psi_{\text{fin}} = U^T \Psi_{\text{ini}} \) for the quantum walk governed by \( U_0 \) and \( U \), respectively. Here \( T \) represents the number of steps of the quantum walk, and the normalized spatial distribution is defined as \( \rho(x) = \sum_{x, \sigma} \psi_{x, \sigma}^2 \), with \( \psi_{x, \sigma} \) the wave amplitude of the normalized final state,

\[
\tilde{\psi}_{x, \sigma} = \frac{\psi_{x, \sigma}}{\sqrt{\sum_{x, \sigma} \psi_{x, \sigma}^2}}.
\]

obtained from \( \Psi_{\text{fin}} = \sum_{x, \sigma} \psi_{x, \sigma} |x, x \rangle \otimes |\sigma\rangle \). It is clearly seen from Fig. 3(c) and (d) that introducing the interchain hopping reverses the propagation direction of the walker. In Fig. 3(c) and (d), we further examine the average position of the final state, defined as \( \tilde{x} = \sum_{x, \sigma} x |\tilde{\psi}_{x, \sigma}^2| \). Note that this average is over both chains. Without the interchain hopping operators \( S_{1,4} \), the quantum walk governed by \( U_0 \) exhibits the NHSE, of which the direction of non-reciprocal population accumulation is determined by the two rotation angles \( \theta_1 \) and \( \theta_2 + \theta_1 \) in Eq. (9). As seen in Fig. 3(c) and (d), the interchain hopping can reverse the direction of particle transport. That is, when the color in Fig. 3(c) mismatches that in Fig. 3(d), reversed particle transport is observed. Without the decoupled case, this switching can occur without changing the system parameters \( \theta_2, \theta_1 \) in the classical limit of non-hermitian systems. To further digest the direction reversal of NHSE, one may also investigate the winding behavior of the quasi-energy \( \epsilon_k \), obtained from \( U_k \Psi_k = e^{-i\alpha_k} \Psi_k \), with \( U_k \) being the Fourier transform of \( U \), \( \Psi_k \) the eigenvectors of \( U_k \), and \( k \) the Bloch momentum reflecting the translational invariance of the quantum walk model. The winding of the quasi-energy spectrum as \( k \) increases from 0 to \( 2\pi \) is shown in Fig. 4(a) to (f). The direction of the winding is seen to change when the system parameters \( \theta_2, \theta_1 \) move across the phase boundary identified in Fig. 4(a). This is because a jump of the spectral winding number between \( \pm 1 \) and 0, as we go from case (b) to case (f). In particular, as shown in Fig. 4(c) and (e), along the phase boundary, the quasi-spectrum in the \( k \)-space does not enclose any area, corresponding to a trivial spectral winding and the absence of NHSE. These results further verify that the above observed reversal of particle transport direction is due to reversed NHSE.

To conclude, we note that particle transport with a reversed direction, as illustrated in Fig. 3(a) and (b), can be observed within very few quantum walk steps. The required lattice size can also be small since there is no need to distinguish between bulk sites and edge sites. In Supplemental Materials, we even add an example where reversed NHSE in our quantum walk system can be observed using only two unit cells.

**Summary.**- We have shown that a coherent coupling between two 1D non-Hermitian chains can lead to direction reversal of NHSE for all the eigenmodes. We have demonstrated this concept using both the spatial profile of stationary solutions, as well as time evolution dynamics on a quantum walk platform within reach of current experiments. In our first model, the common and individual direction of NHSE is obvious, as observed from the non-reciprocal hopping on the two individual chains, yet a coupling between them yields a population accumulation along the reversed direction. This intriguing phenomenon is interpreted in terms of the interference between multiple hopping pathways and explained quantitatively via an adiabatic/hybridized representation. In our second working model aiming at an experimental proposal to observe reversed NHSE on the dynamics level, two individual chains hosting a quantum walker have the same preferred direction of particle transport, yet an interchain hopping can again reverse the direction of particle transport. In both models, we have witnessed how a physical phenomenon analogous to ANM may emerge in contexts or experimental platforms of non-Hermitian physics. Our findings should also offer useful schemes to manipulate the NHSE by tuning the coherent coupling between individual subsystems. It should be stimulating to extend our findings to higher dimensions, where NHSE becomes a rather universal property of non-Hermitian systems, with the boundary localization behavior of bulk eigenmodes...
strongly dependent on the system’s geometry [58].

Acknowledgements.— L. L. would like to thank Zhihuang Luo and Zhenhua Yu for helpful discussion. L. L. acknowledges funding support by the National Natural Science Foundation of China (12104519) and the Guangdong Basic and Applied Basic Research Foundation (2020A1515110773). J. G. acknowledges support from Singapore National Research Foundation (Grant No. NRF-NRFI2017-04).
Supplementary Materials for “Direction reversal of non-Hermitian skin effect via coherent coupling”

PERTURBATIVE GBZ SOLUTION FOR TWO COUPLED HATANO-NELSON CHAINS

To quantitatively unveil the reversed NHSE, we solve the corresponding generalized Brillouin zone (GBZ) of two coupled Hatano-Nelson chains through a perturbative calculation. The Hamiltonian in momentum space reads

\[ h(k) = \begin{pmatrix} 2t_a \cos(k - i\alpha) + \mu & t_\perp \\ t_\perp & 2t_b \cos(k - i\beta) - \mu \end{pmatrix}. \]  

(S1)

For large \( t_\perp \), we may take \( h_0 = t_\perp \sigma_z \) as the unperturbed Hamiltonian, and the rest terms \( h'(k) = h(k) - h_0 \) as perturbations. The first-order perturbation of the energies are given by

\[ \delta E_\pm(k) = \langle \psi_\pm | h'(k) | \psi_\pm \rangle \]

with \( \psi_\pm \) the two eigenvectors of \( h_0 \). Here we have \( \delta E_+ = \delta E_- := \delta E \), and the two branches of eigenmodes are distinguished by their unperturbed eigenenergies \( E_0 = \pm t_\perp \) plus the perturbation, i.e.

\[ E_\pm(k) \approx \delta E(k) \pm t_\perp. \]

To obtain the GBZ of our model, we introduce a complex deformation of the crystal momentum, \( k \to k + i\kappa \). According to the non-Bloch band theory [5, 7], the GBZ is given by certain values of \( \kappa \) (possibly \( k \)-dependent) so that each pair of eigenmodes with different \( k \) have the same eigenenergies. Note that in our model, the \( k \)-dependency of eigenenergies appear only in the first-order perturbation \( \delta E(k) \), which is the same for \( E_\pm(k) \). It means that the two bands are described by the same GBZ, determined solely by \( \delta E(k) \) with the complex deformation of \( k \). With some straightforward calculation, we obtain

\[ \delta E(k + i\kappa) = f_r \cos k + i f_i \sin k, \]

\[ f_r = \left( t_a e^{-\kappa} e^{\alpha} + t_b e^{-\kappa} e^{-\alpha} \right) \left( \frac{2}{t_a} \right), \]

\[ f_i = \left( t_a e^{-\kappa} e^{\alpha} - t_b e^{-\kappa} e^{-\alpha} \right) \left( \frac{2}{t_a} \right). \]

(S3)

We can see that \( \delta E(k + i\kappa) \) has its real part proportional to \( \cos k \), i.e. taking the same value at \( \pm k \), and its imaginary part proportional to \( \sin k \), i.e. taking the same value at \( \pi/2 \pm k \). Therefore, we may have \( \delta E(k + i\kappa) \) taking the same value at different \( k \) only when \( f_r, f_i = 0 \) [59], which leads to

\[ e^{i\kappa} = \left( \frac{t_a e^{\alpha} + t_b e^{-\alpha}}{t_a e^{-\alpha} + t_b e^{\alpha}} \right)^2, \]

\[ \kappa = \ln \sqrt{\frac{t_a e^{\alpha} + t_b e^{-\alpha}}{t_a e^{-\alpha} + t_b e^{\alpha}}}. \]

(S4)

For \( \alpha_{a,b} > 0 \), the direction reversal of NHSE occurs when \( \kappa < 0 \), which leads to the same result as that in the main text.

EXTENSIONS OF THE TWO COUPLED HATANO-NELSON CHAINS

Energy offset between the two chains

The model of two coupled Hatano-Nelson chains considered in the main text contains an energy offset between the two chains, labeled as \( \mu_a = -\mu_b = \mu \), which does not appear in the first-order perturbation correction. Physically, \( \mu \) tends to separate the two chains, making it more difficult for their eigensolutions to hybridize with each other. Thus the required \( t_\perp \) for entering the regime of reversed NHSE shall increase with \( \mu \). As seen in Fig. [S1] the critical value of \( t_\perp \) for the system to enter the regime of reversed NHSE, denoted as \( t_{\perp,c} \), exhibits a roughly linear dependence on \( \mu \) for \( \mu \gtrsim 1 \).

Off-diagonal interchain couplings

In the coupled Hatano-Nelson chains discussed in the main text, the direction reversal of NHSE occurs only when \( t_a \) and \( t_b \) take opposite signs, which is not a necessary condition in more general scenarios. To see this, here we consider an example with
FIG. S1. (a) The average directional inverse participation ratio (dIPR) over all eigenmodes of two coupled Hatano-Nelson chains, for several different values of $\mu$. See Eq. (3) in the main text for the definition of dIPR. The transition between normal and reversed NHSE can be identified with $\langle I_d \rangle = 0$, where the eigenmodes have a balanced distribution toward the two ends of the system. (b) The critical value of the interchain couplings, $t_{\perp,c}$, given by $\langle I_d(t_{\perp,c}) \rangle = 0$, i.e. the crossing between the solid and dash lines in (a). Other parameters are $t_a = 0.75$, $t_b = -1$, $\alpha_a = 0.5$, $\alpha_b = 0.2$.

FIG. S2. Phase diagrams with color indicating the directional IPR $\langle I_{d,\pm} \rangle$ at $t_{\perp} = 30$ with additional off-diagonal interchain couplings (a) $t_2 = 0.2$, (b) $t_2 = 1$, and (c) $t_2 = 2$. Black lines are obtained from the perturbation results of Eqs. (S11) and (S12). Other parameters are $\alpha_a = 0.5$, $\alpha_b = 0.2$, and $\mu_a = -\mu_b = 0.5$.

Extra off-diagonal couplings added to the model, described by

$$\hat{H}_2 = \sum_x \frac{t_2}{2} \hat{b}_x \hat{a}_{x+1} + \frac{t_2}{2} \hat{b}_x \hat{a}_{x-1} + \text{h.c.}$$

with h.c. denoting the Hermitian conjugate. The Hamiltonian of the system in momentum space now reads

$$h(k) = \begin{pmatrix} 2t_a \cos(k - i\alpha_a) + \mu & t_\perp + t_2 \cos k \\ t_\perp + t_2 \cos k & 2t_b \cos(k - i\alpha_b) - \mu \end{pmatrix}.$$  

(S6)

Similar to the previous case, here we take the term of $t_{\perp}$ as the unperturbed Hamiltonian. The first-order perturbation is now given by

$$\delta E^+ = t_a \cos(k - i\alpha_a) + t_b \cos(k - i\alpha_b) \pm t_2 \cos k.$$  

(S7)
With the complex deformation $k \rightarrow k + i\kappa$, we have

$$\delta E^\pm = f^\pm_r \cos k + i f^\pm_{i} \sin k,$$

(S8)

where

$$f^\pm_r = \left( t_a e^{\alpha x} + e^{\kappa} e^{-\alpha x} + \frac{t_b}{2} e^{i \delta} \pm t_2 \frac{e^{\kappa} + e^{-\kappa}}{2} \right),$$

(S9)

and

$$f^\pm_{i} = \left( t_a e^{\alpha x} + e^{\kappa} e^{-\alpha x} - \frac{t_b}{2} e^{i \delta} \pm t_2 \frac{e^{\kappa} - e^{-\kappa}}{2} \right).$$

Note that here the two bands have different dependencies on $k$, therefore we shall also rewrite $\kappa$ as $\kappa_\pm$ for the two bands. Requiring $f^+_r f^-_{i} = 0$, we obtained

$$\exp(i \kappa_\pm) = \left( \frac{t_a e^{\alpha x} + t_b e^{\alpha x} \pm t_2}{t_a e^{-\alpha x} + t_b e^{-\alpha x} \pm t_2} \right)^2,$$

and

$$\kappa_\pm = \ln \sqrt{\frac{t_a e^{\alpha x} + t_b e^{\alpha x} \pm t_2}{t_a e^{-\alpha x} + t_b e^{-\alpha x} \pm t_2}}.$$  

(S10)

The phase boundaries between normal and reversed NHSE for each band are given by $\kappa_+ = 0$, which leads to

$$t_a = -t_b(e^{\alpha x} - e^{-\alpha x}) \quad \text{and} \quad t_a = -t_b(e^{\alpha x} + e^{-\alpha x} - 2t_2),$$

and $\kappa_- = 0$, which leads to

$$t_a = -t_b(e^{\alpha x} - e^{-\alpha x}) \quad \text{and} \quad t_a = -t_b(e^{\alpha x} + e^{-\alpha x} + 2t_2).$$

(S11)

(S12)

The results for several different values of $t_2$ are shown in Fig. S2, together with the average directional IPR of the two bands, denoted as $I_{d,k}$ respectively. From this result, we can see that there are two interesting consequences due to the presence of $t_2$.

(i) Since now $\kappa_+ \neq \kappa_-$, it is possible to have the reversed non-reciprocal accumulation in only one band, i.e. for exactly half of the eigenmodes. In this way, the normal and reversed non-reciprocal accumulations are separated by a large energy gap as the two bands have eigenenergies around $\pm t_2$ respectively.

(ii) It is now also possible to have the reversed NHSE even when $t_a$ and $t_b$ take the same sign, which is impossible for the case with only $t_\pm$ (i.e. $t_2 = 0$). This is because $t_2$ takes different signs in determining $\kappa_\pm$ for the two bands, and it is possible to have one of $\kappa_\pm$ changing sign by tuning $t_2$. For example, in Fig. S2(c) with $t_2 = 2$, the reversed NHSE (denoted by blue color) is seen when $t_{ab} > 0$ for “−” band, and when $t_{ab} < 0$ for “+” band.

**DETAILS OF THE TWO-CHAIN QUANTUM WALK**

The Floquet operators governing the quantum walk in the main text,

$$U_0 = R(\theta_1) S_2 R(\theta_2 + \theta_3) M R(\theta_1) S_1 R(\theta_1),$$

(S13)

$$U = R(\theta_1) S_2 R(\theta_2) S_4 R(\theta_3) M R(\theta_3) S_3 R(\theta_2) S_1 R(\theta_1),$$

(S14)

are explicitly given by

$$R(\theta) = \sum_{x=-N}^{N} \sum_{s=a,b} |s, x\rangle \langle s, x| \otimes e^{-i \lambda_s \theta_2 / 2}, \quad M = \sum_{x=-N}^{N} \sum_{s=a,b} |s, x\rangle \langle s, x| \otimes \left( |\downarrow\rangle \langle \downarrow| + e^{\alpha x} |\uparrow\rangle \langle \uparrow| \right),$$

$$S_1 = \sum_{x=-N}^{N} \sum_{s=a,b} \left( |s, x\rangle \langle s, x| \otimes |\downarrow\rangle \langle \downarrow| + |s, x + 1\rangle \langle s, x + 1| \right), \quad S_2 = \sum_{x=-N}^{N} \sum_{s=a,b} \left( |s, x - 1\rangle \langle s, x - 1| \otimes |\downarrow\rangle \langle \downarrow| + |s, x\rangle \langle s, x| \otimes |\uparrow\rangle \langle \uparrow| \right),$$

$$S_3 = \sum_{x=-N}^{N} \sum_{s,a,b} \left( |s, x\rangle \langle s, x| \otimes |\downarrow\rangle \langle \downarrow| + |s, x\rangle \langle s, x| \right), \quad S_4 = \sum_{x=-N}^{N} \sum_{s,a,b} \left( |s, x\rangle \langle s, x| \otimes |\downarrow\rangle \langle \downarrow| + |s, x\rangle \langle s, x| \otimes |\uparrow\rangle \langle \uparrow| \right),$$

with $s = a, b$ denoting the two chains, $x$ the site index, and $\lambda_a = 1$ and $\lambda_b = -1$. 
FIG. S3. The normalized spatial distribution $\rho(x)$ or $\tilde{\rho}(x)$, $\tilde{\rho}_0(x)$ for a quantum walk with two unit cells, i.e. $x = 1$ or 2. The initial state is prepared as $\Psi_{\text{ini}} = |a, 1\rangle \otimes |\uparrow\rangle$. From left to right, the results are obtained with the system being (i) Hermitian and decoupled; (ii) Hermitian and coupled; (iii) non-Hermitian and decoupled, (iv) non-Hermitian and coupled. The parameters are $\alpha_a = \alpha_b = 0$ and $\alpha_a = \alpha_b = 3$ for Hermitian and non-Hermitian cases respectively.

**QUANTUM WALK IN A SYSTEM WITH TWO UNIT CELLS**

Since the NHSE describes a unidirectional transport of the (quasi-)particles or wave packets, direction reversal of NHSE in principle can manifest itself in very small systems, as long as a direction can be defined. Specifically, we consider the quantum walk discussed in the main text in a system with the position $x$ only taking 1 or 2, i.e. 4 qubits for the two chains. To distinguish the left- and right- propagations, the system must be under the “OBC”, where the shift operators $S_1$ and $S_2$ are also non-unitary. That is, $S_1$ will eliminate the (quasi-)particle at $|s, 2\rangle \otimes |\uparrow\rangle$, since $x = 2 + 1$ is not included in the system. For the same reason, $S_2$ will eliminate the amplitude at $|s, 1\rangle \otimes |\downarrow\rangle$. Therefore in such a small system, whether $S_1$ or $S_2$ acts first in the Floquet operator may also affect the results. To this end, we also consider another two sets of Floquet operators,

\[
\begin{align*}
\tilde{U} &= R(\theta_1)S_1R(\theta_2)S_2R(\theta_3)S_3R(\theta_2)S_2R(\theta_1), \\
\tilde{U}_0 &= R(\theta_1)S_1R(\theta_2 + \theta_3)MR(\theta_2 + \theta_3)S_2R(\theta_1),
\end{align*}
\]

where the roles of $S_1$ and $S_2$ are exchanged when compared with $U$ and $U_0$.

By comparing the dynamical evolutions governed by these four Floquet operators ($U$, $U_0$, $\tilde{U}$, and $\tilde{U}_0$) with or without the non-Hermitian loss, the direction reversal of NHSE induced by the interchain couplings can be justified even in this two-unit-cell system, as demonstrated in Fig. S3. Below we summarize the results. Starting from the left of Fig. S3

(i) in the first column, we choose $\alpha_{a,b} = 0$, and consider the quantum walks without the interchain couplings, governed by $U_0$ and $\tilde{U}_0$ respectively. There is no non-reciprocal pumping since there is no gain or loss in the system.

(ii) in the second column, we choose $\alpha_{a,b} = 0$, and consider the quantum walks with the interchain couplings, governed by $U$ and $\tilde{U}$ respectively. Even without any gain or loss in the system, we still observe some non-reciprocal accumulations. However, we can see that $U$ and $\tilde{U}$ (with $S_{1,2}$ exchanging their roles) induce different accumulating directions, showing that this is not the NHSE, but a consequence of the boundary effect induced by $S_1$ and $S_2$.

(iii) in the third column, we choose $\alpha_{a,b} = 3$, and consider the same quantum walks as that of the first column. We see a clear non-reciprocal accumulation from $x = 1$ to $x = 2$, for both Floquet operators. Therefore we conclude that this non-reciprocal accumulation reflects the NHSE, rather than the boundary effect induced by $S_1$ and $S_2$.

(iv) in the last column, we choose $\alpha_{a,b} = 3$, and consider the same quantum walks as that of the second column. A clear non-reciprocal accumulation is seen from $x = 2$ to $x = 1$, also for both Floquet operators. Compared with (iii), we can see that the direction of population accumulation is reversed when introducing $S_3$ and $S_4$, representing the direction reversal of NHSE induced by interchain couplings.
Hermitian non-Hermitian

FIG. S4. The normalized spatial distribution \( \rho(x) \) [or \( \rho_0(x) \), \( \bar{\rho}(x) \), \( \bar{\rho}_0(x) \)] for a quantum walk with two unit cells, i.e. \( x = 1 \) or \( 2 \). The initial state is prepared as \( \Psi_{\text{ini}} = |a, 2 \rangle \otimes |\uparrow \rangle \), which is different from that in Fig. S3. From left to right, the results are obtained with (i) Hermitian and decoupled; (ii) Hermitian and coupled; (iii) non-Hermitian and decoupled, (iv) non-Hermitian and coupled. Hermitian and non-Hermitian cases correspond to \( \alpha_a = \alpha_b = 0 \) and \( \alpha_a = \alpha_b = 3 \) respectively.

In Fig. S4, keeping parameters the same with those in Fig. S3, we present the results with a different initial state \( \Psi_{\text{ini}} = |a, 2 \rangle \otimes |\uparrow \rangle \) and we observe the same phenomena as shown in Fig. S3.

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