WEIGHTED COMPOSITION OPERATORS ON FOCK
SPACES AND THEIR DYNAMICS

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Abstract. Bounded weighted composition operators, as well as compact weighted composition operators, on Fock spaces have been characterised. This characterisation is refined to the extent that the question of whether weighted composition operators on the Fock space can be supercyclic is answered in the negative.

1. Introduction

The study of composition operators \( C_\varphi f = f \circ \varphi \) acting on spaces of analytic functions has been an active area of research since the pioneering work of Nordgren [29] and Kamowitz [25, 26]. Although this class of operators is usually tractable, there exists a rich literature which reveals many interesting properties in a variety of settings, including in the Hardy, Bergman and Dirichlet spaces [28], [15], [31], [18].

A natural generalisation of the class of composition operators are the weighted composition operators \( W_{\psi,\varphi} f = \psi \cdot (f \circ \varphi) \), which have recently attracted much research interest [14], [1], [20], [21], [8], [23], [33]. In particular, the bounded and compact weighted composition operators, acting on Fock spaces of entire functions, were characterised by Ueki [34, 35], Le [27], and Hai and Khoi [22]. Analogous results for unweighted composition operators acting on the classical Fock space were previously identified by Carswell et al. [13].

Hitherto, characterisations of the boundedness and compactness of \( W_{\psi,\varphi} \) in the Fock space setting have required that the symbol \( \psi \) and the multiplier \( \varphi \) satisfy certain uniform conditions. In this article we study \( W_{\psi,\varphi} \) from the perspective of the order and type of the multiplier \( \psi \). This approach allows us to obtain more explicit boundedness and compactness conditions.

The second theme of this article is to investigate the linear dynamical properties of weighted composition operators acting on Fock spaces. To this end we take a short detour to identify a class of paranormal weighted composition operators, before we employ our refined characterisation of

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boundedness and compactness to show that in this setting $W_{\psi,\omega}$ cannot be supercyclic. This reveals contrasting dynamical behaviour of weighted composition operators acting on different spaces of analytic functions.

2. The Fock spaces

In this section we recall the pertinent notation and results, taken from Zhu [37], which are needed in the sequel.

For $\alpha$ positive and $1 \leq p < \infty$, the Fock space $\mathcal{F}_\alpha^p$ comprises of entire functions $f$ for which the norm

$$\|f\|_{p,\alpha} = \left( \frac{p\alpha}{2\pi} \int_{\mathbb{C}} |f(z)|^p e^{-\frac{\alpha}{2}|z|^2} \, dm(z) \right)^{1/p}$$

is finite. Here $dm$ is area measure in the complex plane $\mathbb{C}$. For $\alpha$ positive and $p = \infty$, the Fock space $\mathcal{F}_\alpha^\infty$ is composed of entire functions $f$ for which the norm

$$\|f\|_{\infty,\alpha} = \sup \left\{ |f(z)| e^{-\frac{\alpha}{2}|z|^2} : z \in \mathbb{C} \right\} < \infty.$$ 

Henceforth, the dependence of the norm on $\alpha$ and on $p$ is suppressed in the notation. An entire function $f$ belongs to the Fock space $\mathcal{F}_\alpha^p$ when the function $f(z)e^{-\alpha|z|^2/2}$ is in the space $L^p(\mathbb{C})$ of Lebesgue measurable functions on $\mathbb{C}$.

For $1 \leq p \leq \infty$, the Fock space $\mathcal{F}_\alpha^p$ is a Banach space with dual $(\mathcal{F}_\alpha^p)^* = \mathcal{F}_\alpha^{\frac{p}{p-1}}$, where $p$ and $q$ are the usual conjugate exponents with $1/p + 1/q = 1$. The function $g \in \mathcal{F}_\alpha^{\frac{p}{p-1}}$ acts on $f \in \mathcal{F}_\alpha^p$ according to

$$\langle f, g \rangle = \frac{\alpha}{\pi} \int_{\mathbb{C}} f(z) \overline{g(z)} e^{-\alpha|z|^2} \, dm(z).$$

For $z \in \mathbb{C}$, we define the function $k_z(w) = e^{\alpha \overline{w} z}$, where $w \in \mathbb{C}$. It has norm $\|k_z\| = e^{\alpha|z|^2/2}$ and it belongs to $\mathcal{F}_\alpha^p$ for $1 \leq p \leq \infty$. Furthermore, the point evaluation of $f \in \mathcal{F}_\alpha^p$ at $z$ is given by

$$f(z) = \frac{\alpha}{\pi} \int_{\mathbb{C}} f(w) \overline{k_z(w)} e^{-\alpha|w|^2} \, dm(w).$$

In the case $p = 2$, the Fock space $\mathcal{F}_\alpha^2$ is a reproducing kernel Hilbert space with reproducing kernel at $z$ given by $k_z$. The normalised point evaluations act weakly on functions $f$ in the Fock space $\mathcal{F}_\alpha^p$ if $p < \infty$, that is

$$|\langle f, k_z/\|k_z\| \rangle| = |f(z)| e^{-\alpha|z|^2/2} \to 0, \quad \text{as} \quad |z| \to \infty.$$ 

The entire function $f(z) = e^{\beta z^2}$ belongs to the Fock space $\mathcal{F}_\alpha^p$, $1 \leq p < \infty$, if and only if $|\beta| < \alpha/2$, as can be seen by computing the norm (2.1) in polar coordinates and first integrating radially. The function $f(z) = e^{\beta z^2}$ belongs to $\mathcal{F}_\alpha^\infty$ if and only if $|\beta| \leq \alpha/2$. The fact that $f(z) = e^{\alpha z^2/2}$ belongs to $\mathcal{F}_\alpha^\infty$ but not to $\mathcal{F}_\alpha^p$ for $p < \infty$ will play an important role in our later observations.
We note that some articles only consider the Fock spaces corresponding to the case \( \alpha = 1 \). The extension to general \( \alpha \) is straightforward and when we recall results in this generality we do so without additional comment.

3. Weighted composition operators on Fock spaces

Carswell et al. [13] characterised the bounded (unweighted) composition operators acting on the Fock space \( \mathcal{F}_\alpha^2 \) of functions holomorphic on \( \mathbb{C}^n \). For our present purposes, we state their result for the base case \( n = 1 \). The composition operator \( C_\varphi \) with symbol \( \varphi \) is defined as

\[
C_\varphi f = f \circ \varphi,
\]

for \( f \in \mathcal{F}_\alpha^2 \). The interested reader can find comprehensive introductions to the study of composition operators in the monographs by Shapiro [32], and Cowen and MacCluer [16].

The pertinent result from [13] is as follows.

**Theorem A** (Carswell, MacCluer and Schuster [13]). The composition operator \( C_\varphi \) is bounded on the Fock space \( \mathcal{F}_\alpha^2 \) only when \( \varphi(z) = a + \lambda z \) with \( |\lambda| \leq 1 \).

Conversely, suppose that \( \varphi(z) = a + \lambda z \). If \( |\lambda| = 1 \) then \( C_\varphi \) is bounded on \( \mathcal{F}_\alpha^2 \) if and only if \( a = 0 \). If \( |\lambda| < 1 \) then \( C_\varphi \) acts compactly on \( \mathcal{F}_\alpha^2 \).

Following the initial work of Ueki [34], Le [27] generalised the results of Carswell et al. [13] to weighted composition operators acting on the Fock space \( \mathcal{F}_\alpha^2 \). The extension to \( \mathcal{F}_\alpha^p \), for \( 1 \leq p \leq \infty \), was carried out by Hai and Khoi [22]. The weighted composition operator \( W_{\psi,\varphi} \), with symbol \( \varphi \) and multiplier \( \psi \), is defined as

\[
W_{\psi,\varphi} f = \psi \cdot (f \circ \varphi)
\]

for \( f \in \mathcal{F}_\alpha^p \). We assume throughout that the multiplier \( \psi \) is not identically zero.

A necessary condition for the boundedness of \( W_{\psi,\varphi} \) acting on \( \mathcal{F}_\alpha^p \) is deduced from the following elementary observation. If \( W_{\psi,\varphi} \) is bounded on \( \mathcal{F}_\alpha^p \) then, by applying \( W_{\psi,\varphi} \) to the constant function 1, one finds that \( \psi \) itself must belong to \( \mathcal{F}_\alpha^p \).

The action of the adjoint \( W_{\psi,\varphi}^* \) on the point evaluations \( k_z \) is straightforward and is given by

\[
W_{\psi,\varphi}^* k_z = \psi(z) k_{\varphi(z)}
\]

for \( z \in \mathbb{C} \). As a consequence

\[
\frac{\| W_{\psi,\varphi}^* k_z \|^2}{\| k_z \|^2} = \frac{|\psi(z)|^2 \| k_{\varphi(z)} \|^2}{\| k_z \|^2} = |\psi(z)|^2 e^{\alpha(|\varphi(z)|^2 - |z|^2)}
\]

is uniformly bounded in \( z \).

For \( z \in \mathbb{C} \) and the entire functions \( \psi \) and \( \varphi \), we set

\[
M_z(\psi, \varphi) := |\psi(z)|^2 e^{\alpha(|\varphi(z)|^2 - |z|^2)}
\]
and
\[ M(\psi, \varphi) := \sup \{ M_z(\psi, \varphi) : z \in \mathbb{C} \}. \]
The main result from [27], extended in [22], characterising the boundedness of \( W_{\psi, \varphi} \) is as follows.

**Theorem B** (Le [27], Han and Khoi [22]). Let \( 1 \leq p \leq \infty \). The weighted composition operator \( W_{\psi, \varphi} \) is bounded on the Fock space \( F_p^\alpha \) if and only if the following hold
\[ (i) \, \psi \in F_p^\alpha, \]
and
\[ (ii) \, M(\psi, \varphi) < \infty. \]

Note that condition (ii) does not depend on \( p \). In the course of proving Theorem B, it is shown for entire functions \( \psi \) and \( \varphi \), with \( \psi \) not identically zero, that if \( M(\psi, \varphi) < \infty \) then \( \varphi \) is of the form \( \varphi(z) = a + \lambda z \), where \(|\lambda| \leq 1\) and \( a \in \mathbb{C} \). Moreover, if \(|\lambda| = 1\) then the weighted composition operator \( W_{\psi, \varphi} \) is bounded on the Fock space \( F_p^\alpha \) if and only if the multiplier \( \psi \) is given by
\[ (3.3) \quad \psi(z) = \psi(0)e^{-\alpha|\lambda|z} = \psi(0)k_{-\alpha\lambda}(z). \]

In the case \( p = 2 \) and \(|\lambda| = 1\), it was also shown in [27] that \( W_{\psi, \varphi} \) is a constant multiple of a unitary operator.

The second main result of Le [27] was the characterisation of the compact weighted composition operators acting on the Fock space \( F_2^\alpha \). The compact weighted composition operators on \( F_\infty^\alpha \) had already been described by Ueki [35]. These characterisations were extended to the Fock spaces \( F_p^\alpha \) for \( 1 \leq p < \infty \) by Han and Khoi [22]. When \( \lambda = 0 \) and \( \varphi(z) = a \), a constant, the operator \( W_{\psi, \varphi} \) reduces to the rank one operator \( W_{\psi, f} = f(a) \psi \), in which case the condition for boundedness and compactness is simply that \( \psi \in F_p^\alpha \). The various results on compactness are combined into the following theorem.

**Theorem C** (Le [27], Ueki [35], Han and Khoi [22]). The weighted composition operator \( W_{\psi, \varphi} \) acts compactly on the Fock space \( F_p^\alpha \), \( 1 \leq p \leq \infty \), if and only if \( \varphi(z) = a + \lambda z \), where \( \lambda = 0 \) and \( \psi \in F_p^\alpha \), or \( 0 < |\lambda| < 1 \) and
\[ (3.4) \quad M_z(\psi, \varphi) \longrightarrow 0, \text{ as } |z| \rightarrow \infty. \]

We remark that the statement of Theorem C does not require the condition that \( \psi \in F_\infty^\alpha \) in the case \( \lambda \neq 0 \), since this follows immediately from the assumptions that \(|\lambda| < 1\) and \((3.4)\).

4. The multiplier \( \psi \) in the case \(|\lambda| < 1\)

We investigate here some of the implications of compactness condition \((3.4)\). It is crucial to observe that in contrast to the case of unweighted composition operators, the condition \(|\lambda| < 1\) on the symbol \( \varphi(z) = a + \lambda z \) is
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in itself not sufficient to guarantee compactness of the weighted composition operator $W_{\psi,\varphi}$. This is perhaps best illustrated by a simple example.

Example 4.1. Given $\lambda$ with $0 < \lambda < 1$, set $\beta = 1 - \lambda^2$, so that $0 < \beta < 1$. We also set

$$\psi(z) = e^{\frac{\alpha}{2}\beta z^2} \quad \text{and} \quad \varphi(z) = \lambda z$$

for $z \in \mathbb{C}$. Since $\beta < 1$, the function $\psi$ is in the Fock space $\mathcal{F}_\alpha^p$ for $1 \leq p \leq \infty$. We compute

$$M(\psi, \varphi) = \sup \left\{ |\psi(z)|^2 e^{\alpha(|\varphi(z)|^2 - |z|^2)} : z \in \mathbb{C} \right\}$$

$$= \sup \left\{ |\psi(z)|^2 e^{\alpha(\lambda^2 - 1)|z|^2} : z \in \mathbb{C} \right\}$$

$$= \sup \left\{ M_{\psi}(r)^2 e^{-\alpha \beta r^2} : r > 0 \right\}$$

where $M_{\psi}(r) := \max \{ |\psi(z)| : |z| = r \}$ denotes the maximum modulus function of $\psi$. Since $M_{\psi}(r)^2 = e^{\alpha \beta r^2}$, we see that $M_{\psi}(r)^2 e^{-\alpha \beta r^2} = 1$ for each $r$. In particular, $M(\psi, \varphi) < \infty$ and so, by Theorem 3, the weighted composition operator $W_{\psi,\varphi}$ is bounded on the Fock space. However, (3.4) does not hold and so $W_{\psi,\varphi}$ does not act compactly on $\mathcal{F}_\alpha^p$. \hfill \Box

The parameter $\beta$ from Example 4.1 will appear frequently in the sequel, so for $|\lambda| < 1$ we set

$$\beta := 1 - |\lambda|^2$$

which in particular gives that $0 < \beta \leq 1$.

In this section we will show that the order and type of the multiplier $\psi$ can shed some light on whether or not the operator $W_{\psi,\varphi}$ is compact. We recall, the order $\rho$ of an entire function $f$ is defined as

$$\rho := \lim \sup_{r \to \infty} \frac{\log \log M_f(r)}{\log r}$$

and if $f$ is of finite order $\rho$, its type $\sigma$ is defined to be

$$\sigma := \lim \sup_{r \to \infty} \frac{\log M_f(r)}{r^\rho}.$$ 

For example, every polynomial has order zero, while the function $f(z) = e^{az^2}$ has order 2 and type $|a|$.

Bénétou et al. [3] have previously investigated the order and type of functions in the Fock space. They showed that each function $f \in \mathcal{F}_\alpha^2$ has order at most 2, and that if $f$ has order 2 then its type is at most $\alpha/2$. Conversely, if $f$ has order less than 2, or if it has order 2 and type less than $\alpha/2$ then $f$ is in the Fock space $\mathcal{F}_\alpha^2$.

However, membership of the Fock space cannot be characterised in terms of order and type alone. In [3] examples of functions in the liminal case are provided, that is functions of order 2 and type $\alpha/2$ that belong to the Fock space $\mathcal{F}_\alpha^2$, and examples of such functions that do not. It is also shown that
if $f \in \mathcal{F}_\alpha^2$ has order two and type $\alpha/2$ then $f$ must have infinitely many zeros.

Similarly, condition (3.4) can almost be characterised in terms of the growth of the function $\psi$. Our first main result demonstrates that there is a balance between the growth of the symbol $\varphi$ and that of the multiplier $\psi$ in that the more contractive is $\varphi$, the faster $\psi$ is permitted to grow.

**Theorem 4.2.** Let $1 \leq p \leq \infty$, and $\varphi(z) = a + \lambda z$, with $|\lambda| < 1$. If $\psi$ has order strictly less than 2, or if $\psi$ has order 2 and type strictly less than $\frac{\alpha}{2} \beta$, then $W_{\psi,\varphi}$ is a compact weighted composition operator on $\mathcal{F}_\alpha^p$.

If $\psi$ has order greater than 2, or if $\psi$ has order 2 and type strictly greater than $\frac{\alpha}{2} \beta$, then $W_{\psi,\varphi}$ is not bounded on $\mathcal{F}_\alpha^p$.

**Proof.** Suppose that $\psi$ is of order strictly less than 2, that is

$$\rho = \rho(\psi) = \limsup_{r \to \infty} \frac{\log \log M_\psi(r)}{\log r} = 2 - 2\varepsilon,$$

where $\varepsilon$ is positive. Then there exists $r_0$ such that

$$\frac{\log \log M_\psi(r)}{\log r} \leq 2 - \varepsilon$$

for $r \geq r_0$ and thus $|\psi(z)| \leq \exp\left(\frac{r^{2-\varepsilon}}{r}\right)$ for $|z| = r \geq r_0$. Next notice that

$$|\varphi(z)|^2 - |z|^2 = |a + \lambda z|^2 - |z|^2 \leq -\beta |z|^2 + |a|^2 + c_1 |z|,$$

where $c_1 = 2|a||\lambda|$. Thus, for $|z| = r \geq r_0$,

$$|\psi(z)|^2 e^{\alpha(|\varphi(z)|^2 - |z|^2)} \leq c_2 \exp\left(r^2 [-\alpha \beta + 2r^{-\varepsilon} + \alpha c_1/r]\right)$$

where $c_2 = e^{\alpha|a|^2}$. This shows that (3.4) holds and so it follows from Theorem (C) that $W_{\psi,\varphi}$ is compact.

Now suppose that $\psi$ has order 2 and type $\sigma$ with $\sigma < \frac{\alpha}{2} \beta$, say $\sigma = \frac{\alpha}{2} \beta - \varepsilon$ where $\varepsilon$ is positive. Since $\psi$ has order 2 and type $\sigma$, there exists $r_0$ such that

$$\frac{\log M_\psi(r)}{r^2} \leq \frac{\alpha}{2} \beta - \frac{\varepsilon}{2}, \quad r \geq r_0.$$

Thus $M_\psi(r) \leq e^{r^2(\frac{\alpha}{2} \beta - \frac{\varepsilon}{2})}$, for $r \geq r_0$, so that $|\psi(z)|^2 \leq e^{r^2(\alpha\beta - \varepsilon)}$ for $|z| = r \geq r_0$. Again, for $|z| = r \geq r_0$ we have that

$$|\psi(z)|^2 e^{\alpha(|\varphi(z)|^2 - |z|^2)} \leq c_2 \exp\left(r^2 [-\alpha \beta + \alpha \beta - \varepsilon] + \alpha c_1/r]\right).$$

So we again see that (3.4) holds and $W_{\psi,\varphi}$ is compact.

Next, suppose that $\psi$ has order $\rho = \rho(\psi)$ strictly greater than 2, say $\rho = 2 + 2\varepsilon$ where $\varepsilon$ is positive. Then there exists a sequence $r_n \to \infty$ such that

$$\frac{\log \log M_\psi(r_n)}{\log r_n} \geq 2 + \varepsilon.$$
that is $M_\psi(r_n) \geq e^{r_n^2+\varepsilon}$. Thus, there exists a sequence $\{z_n\}$, with $|z_n| = r_n \to \infty$, for which

$$|\psi(z_n)|e^{-\frac{\alpha}{2}|z_n|^2} \geq e^{|z_n|^2+\varepsilon-\frac{\alpha}{2}|z_n|^2}.$$  

Since the expression on the right becomes unbounded as $n \to \infty$, $\psi$ fails to belong to $\mathcal{F}_\alpha^\infty$, and since $\psi$ fails to satisfy \((2.2)\), $\psi$ does not belong to $\mathcal{F}_\alpha^p$ for $1 \leq p < \infty$ either. It follows by Theorem B that the weighted composition operator $W_\psi,\varphi$ is not bounded on $\mathcal{F}_\alpha^p$.

Finally, suppose that $\psi$ has order 2 and type $\sigma$ where $\sigma > \frac{\alpha}{2}\beta$, say $\sigma = \frac{\alpha}{2}\beta + \varepsilon$ where $\varepsilon$ is positive. There then exists a sequence $r_n \to \infty$ such that

$$\frac{\log M_\psi(r_n)}{r_n^2} \geq \frac{\alpha}{2} + \frac{\varepsilon}{2},$$

that is $M_\psi(r_n) \geq e^{r_n^2(\frac{\alpha}{2} + \frac{\varepsilon}{2})}$. Thus, there exists a sequence $\{z_n\}$, with $|z_n| = r_n \to \infty$, such that $|\psi(z_n)|^2 \geq e^{r_n^2(\alpha\beta + \varepsilon)}$. Then

$$|\psi(z_n)|^2 e^{\alpha(\varphi(z_n)^2 - |z_n|^2)} \geq c_2 \exp \left((\alpha\beta + \varepsilon)r_n^2 - \alpha\beta r_n^2 - \alpha c_1 r_n \right)$$

which is unbounded as $n \to \infty$. Again, $M(\psi, \varphi)$ is not finite and it follows that the weighted composition operator $W_{\psi,\varphi}$ is not bounded on $\mathcal{F}_\alpha^p$. \(\square\)

When $\varphi(z) = a + \lambda z$ for $|\lambda| < 1$ and $\psi$ has order 2 and type $\frac{\alpha}{2}\beta$, it is not possible to determine based on this information alone whether or not the operator $W_{\psi,\varphi}$ is compact, or even bounded. As illustrated in Example \ref{example1.1}, the choice of $\psi$ and $\varphi$ may give rise to an operator $W_{\psi,\varphi}$ that is bounded on the Fock space but not compact.

The subtlety of this question is further revealed by considering $\psi(z) = z e^{\frac{\alpha}{2}\beta z^2}$ and $\varphi(z) = \lambda z$, which gives that $M(\psi, \varphi) = \infty$ and so $W_{\psi,\varphi}$ is not even bounded on $\mathcal{F}_\alpha^p$. On the other hand, if we take $\psi(z) = (e^{\frac{\alpha}{2}\beta z^2} - 1)/z$ then \((3.4)\) holds and $W_{\psi,\varphi}$ acts compactly on the Fock space.

It is, however, possible to give a full description of the zero-free multipliers $\psi$ associated with the symbol $\varphi(z) = a + \lambda z$, for $|\lambda| < 1$. For such multipliers, the operator $W_{\psi,\varphi}$ is compact if and only if $\psi$ has order less than two, or has order two and type strictly less than $\frac{\alpha}{2}\beta$, so that a converse to the sufficient compactness condition in Theorem \ref{theorem1.2} holds for zero-free multipliers. This is the main case to be considered when we later investigate the linear dynamical properties of weighted composition operators acting on the Fock space.

**Theorem 4.3.** Let $1 \leq p \leq \infty$, $\varphi(z) = a + \lambda z$ with $|\lambda| < 1$, and suppose that $\psi$ is non-vanishing.

(a) The operator $W_{\psi,\varphi}$ is compact on $\mathcal{F}_\alpha^p$ if and only if $\psi$ has the form

$$\psi(z) = e^{a_0 + a_1 z + a_2 z^2}$$

and $|a_2| < \frac{\alpha}{2}\beta$.

(b) Let $t = a_1 + a\alpha\lambda$. The operator $W_{\psi,\varphi}$ is bounded but not compact on $\mathcal{F}_\alpha^p$ if and only if $\psi$ has the form \((4.2)\) with $|a_2| = \frac{\alpha}{2}\beta$, and either
\( (i) \) \( t = 0, \)
\[ \text{or} \]
\( (ii) \) \( t \neq 0, \) and \( a_2 = -\frac{\alpha}{2} \beta \frac{t^2}{|t|^2}. \)

**Proof.** Before we begin we record some facts that are required in the proof.

If \( W_{\psi,\varphi} \) is bounded then it follows from Theorem 4.2 that \( \psi \) has order at most 2. If \( \psi \) is non-vanishing, it follows from the Hadamard Factorisation Theorem that
\[
\psi(z) = e^{g(z)}
\]
where \( g(z) = a_0 + a_1 z + a_2 z^2 \) is a polynomial of degree at most 2 (the order of \( \psi \)). Consequently, the type of \( \psi \) is \( |a_2| \). Theorem 4.2 gives that the type of \( \psi \) is at most \( \frac{a}{2} \beta \) and thus
\[
|a_2| \leq \frac{a}{2} \beta.
\]

We also note, for \( \psi(z) = e^{a_0 + a_1 z + a_2 z^2} \) with \( |a_2| \leq \frac{a}{2} \beta \), that we have
\[
M_z(\psi, \varphi) = C \exp \left( 2 \Re(tz) + 2 \Re(a_2 z^2) - \alpha \beta |z|^2 \right),
\]
where \( C = \exp \left( 2 \Re(a_0) + \alpha |a|^2 \right) \). If \( a_2 \neq 0 \), we write \( a_2 = |a_2| e^{-2i\theta_2} \) and we replace \( z \) by \( e^{i\theta_2} w \) in (4.4) to give
\[
M_w(\psi, \varphi) = C \exp \left( 2 \Re(te^{i\theta_2} w) + 2 |a_2| \Re(w^2) - \alpha \beta |w|^2 \right).
\]

**(a)** Suppose that \( W_{\psi,\varphi} \) is compact. Since compact operators are bounded, it is immediate that \( \psi \) is of the form (4.2) with \( |a_2| \leq \frac{a}{2} \beta \).

If \( |a_2| = \frac{a}{2} \beta \), observe for \( w = u \) real that (4.5) gives
\[
M_u(\psi, \varphi) = C \exp \left( 2u \Re(te^{i\theta_2}) \right).
\]

If \( \Re(te^{i\theta_2}) = 0 \) then \( M_u(\psi, \varphi) \equiv C \). If \( \Re(te^{i\theta_2}) > 0 \) then \( M_u(\psi, \varphi) \to \infty \) as \( u \to \infty \), and if \( \Re(te^{i\theta_2}) < 0 \) then \( M_u(\psi, \varphi) \to \infty \) as \( u \to -\infty \). So compactness condition (3.1) is not satisfied in these cases, and it thus follows that \( |a_2| \leq \frac{a}{2} \beta \).

Conversely, suppose that \( \psi(z) = e^{a_0 + a_1 z + a_2 z^2} \) with \( |a_2| < \frac{a}{2} \beta \). It follows from (4.4) for \( |z| = r \) that
\[
M_z(\psi, \varphi) \leq C \exp \left( 2 |t| r - (\alpha \beta - 2 |a_2|) r^2 \right)
\]
which gives that \( M_z(\psi, \varphi) \to 0 \) as \( r \to \infty \). Theorem C thus gives that \( W_{\psi,\varphi} \) is compact.

**(b)** We assume \( W_{\psi,\varphi} \) is bounded and non-compact with \( \psi \) non-vanishing, so it follows that \( \psi \) has the form (4.2) with \( |a_2| \leq \frac{a}{2} \beta \). In fact in this case it holds that \( |a_2| = \frac{a}{2} \beta \), since part (a) of this theorem gives that \( W_{\psi,\varphi} \) is compact when \( |a_2| < \frac{a}{2} \beta \).
It follows from (4.5) that the boundedness and non-compactness of \( W_{\psi,\varphi} \) is equivalent to the condition that

\[
M_w(\psi, \varphi) = C \exp \left( 2 \Re(te^{i\theta_2}w) + \alpha \beta \left( \Re(w^2) - |w|^2 \right) \right)
\]

is uniformly bounded in \( w \), but does not vanish as \( |w| \to \infty \).

We again let \( a_2 = \frac{\alpha}{2} \beta e^{-2i\theta_2} \). When \( w = u \) is real in (4.7), we obtain (4.6) and, as before in part (a), this leads to the necessary condition that \( \Re(te^{i\theta_2}) = 0 \) for boundedness of \( W_{\psi,\varphi} \). Hence, a necessary condition for \( W_{\psi,\varphi} \) to be bounded and non-compact is that \( |a_2| = \frac{\alpha}{2} \beta \) and \( \Re(te^{i\theta_2}) = 0 \).

Conversely we assume that \( \psi \) is of the form (4.2), with \( |a_2| = \frac{\alpha}{2} \beta \), and that either (i) or (ii) holds. Part (a) of the theorem gives that \( W_{\psi,\varphi} \) cannot be compact, so we only need to check the boundedness of \( W_{\psi,\varphi} \).

We first check case (i), so we assume that \( t = 0 \). Then (4.7) becomes, for \( w \in \mathbb{C} \),

\[
M_w(\psi, \varphi) = C \exp \left[ \alpha \beta \left( \Re(w^2) - |w|^2 \right) \right] \leq C
\]

and it thus follows from Theorem B that \( W_{\psi,\varphi} \) is bounded in this case.

Next we check case (ii). That \( \frac{\alpha}{2} \beta e^{-2i\theta_2} = a_2 = -\frac{\alpha}{2} \beta \frac{t^2}{|t|^2} \) implies that \( (te^{i\theta_2})^2 = -|t|^2 \). Thus, \( te^{i\theta_2} \) is purely imaginary, say \( te^{i\theta_2} = iy \), for some \( y \in \mathbb{R} \). Then (4.7) becomes

\[
M_w(\psi, \varphi) = C \exp \left[ -2y \Im(w) + \alpha \beta \left( \Re(w^2) - |w|^2 \right) \right]
= C \exp \left[ -2 \left( yv + \alpha \beta v^2 \right) \right]
\leq C \exp \left[ y^2/(2\alpha \beta) \right] < \infty.
\]

This shows that \( M_w(\psi, \varphi) \) is uniformly bounded in \( w \), and hence \( W_{\psi,\varphi} \) is also bounded in this case. \( \square \)

5. Paranormal weighted composition operators

The normal weighted composition operators acting on the Hilbert space \( \mathcal{F}_\alpha^2 \) were characterised in [27, Theorem 3.3]. In this section we consider a generalisation of normality for a particular class of weighted composition operators acting on the Banach spaces \( \mathcal{F}_\alpha^p \).

For a Banach space \( X \), the bounded linear operator \( U : X \to X \) is said to be paranormal if

\[
\|Ux\|^2 \leq \|U^2x\| \cdot \|x\|
\]

for all \( x \in X \). Paranormality is a generalisation to the Banach space setting of the Hilbert space property of normality.

The following theorem uses the translation operator \( T_b : \mathcal{F}_\alpha^p \to \mathcal{F}_\alpha^p \), for \( 1 \leq p \leq \infty \), which is defined for any \( b \in \mathbb{C} \) as

\[
(T_b f)(z) = e^{-\alpha |b|^2} f(z + b),
\]
where \( z \in \mathbb{C} \) and \( f \in \mathcal{F}_\alpha^p \). We note that \( T_b \) is an isometry on the Fock spaces \( \mathcal{F}_\alpha^p \) (cf. [32, Proposition 2.38]). For \(|\lambda| = 1\), we compose \( T_b \) with a rotation by \( \lambda \) to define the operator \( T_{\lambda,b} : \mathcal{F}_\alpha^p \to \mathcal{F}_\alpha^p \) by

\[
(T_{\lambda,b}f)(z) = e^{-\alpha|\lambda|z - \frac{b}{2}|z|^2}f(\lambda z + b)
\]

and it can easily be seen that \( T_{\lambda,b} \) is also an isometry.

**Theorem 5.1.** For \( 1 \leq p \leq \infty \), let \( W_{\psi,\varphi} \) be a bounded weighted composition operator on the Fock space \( \mathcal{F}_\alpha^p \), with \( \varphi(z) = a + \lambda z \) and \(|\lambda| = 1\). Then \( W_{\psi,\varphi} \) is paranormal.

**Proof.** We know from [33] that in the case \(|\lambda| = 1\) the function \( \psi \) is of the form \( \psi(z) = \psi(0)e^{-\alpha|\lambda|z} \). So it follows for any \( f \in \mathcal{F}_\alpha^p \) that

\[
(W_{\psi,\varphi}^2) f(z) = \psi(0)^2 e^{-\alpha|\lambda|a|z|^2} e^{-\alpha|\lambda|(1+\lambda)z}f(\lambda^2 z + a(1 + \lambda))
\]

Letting \( b = a(1 + \lambda) \) and using the fact that \((1 + \lambda)/(1 + \lambda) = \lambda\), notice that

\[
e^{-\alpha|\lambda|(1+\lambda)z}f(\lambda^2 z + a(1 + \lambda)) = e^{-\alpha|\lambda|z}f(\lambda^2 z + b)
\]

\[
e^{-\alpha|\lambda|z}e^{\alpha|\lambda|^2z + \frac{b}{2}|\lambda|^2} (T_{\lambda^2,b}f)(z)
\]

Next, substituting \( a \) back into the exponential term and using the fact that the operator \( T_{\lambda^2,b} \) is an isometry we obtain that

\[
\| W_{\psi,\varphi}^2 f \| \| f \| = |\psi(0)|^2 e^{\frac{b}{2}|\lambda|^2(1+\lambda)^2} \| T_{\lambda^2,b} f \| \| f \|
\]

\[
= |\psi(0)|^2 e^{\alpha|a|^2} \| f \|^2.
\]

A calculation similar to above gives that

\[
\| W_{\psi,\varphi} f \|^2 = |\psi(0)|^2 e^{\alpha|a|^2} \| T_{\lambda,a} f \|^2 = |\psi(0)|^2 e^{\alpha|a|^2} \| f \|^2
\]

and thus it follows that the weighted composition operator \( W_{\psi,\varphi} \) is paranormal.

At this juncture it is convenient to state the following corollary to Theorem 5.1, which follows from results by Bourdon [9] that paranormal operators cannot be supercyclic (cf. [19, p. 159]). (The definition of supercyclicity can be found in Section 6.)

**Corollary 5.2.** For \( 1 \leq p \leq \infty \), let \( W_{\psi,\varphi} \) be a bounded weighted composition operator on the Fock space \( \mathcal{F}_\alpha^p \), with \( \varphi(z) = a + \lambda z \) and \(|\lambda| = 1\). Then \( W_{\psi,\varphi} \) cannot be supercyclic.

6. The Fock Spaces do not support supercyclic weighted composition operators

In this section we investigate the linear dynamical properties of weighted composition operators acting on the Fock spaces \( \mathcal{F}_\alpha^p \). The linear dynamics of composition operators have previously been studied by Bernal González...
and Montes-Rodríguez [4], Gallardo-Gutiérrez and Montes-Rodríguez [17], Bernardes et al. [5] and Bés [6]. The weighted composition operators case has been investigated by Yousefi and Rezaei [36], Kamali et al. [24], Rezaei [30] and Bés [7].

The central notion of linear dynamics is hypercyclicity, and we recall that a bounded linear operator $T$ acting on a separable Banach space $X$ is hypercyclic if there exists a vector $x \in X$ with a dense $T$-orbit in $X$. The operator $T$ is said to be supercyclic if there exists a vector $x \in X$, called a supercyclic vector for $T$, such that its projective orbit

$$\{ \zeta T^n x : \zeta \in \mathbb{C}, n \in \mathbb{N} \}$$

is dense in $X$. It is well known that the class of hypercyclic operators is strictly contained in the class of supercyclic operators. Comprehensive introductions to linear dynamics can be found in [2] and [19].

We recall that Bourdon and Shapiro [11, 12] proved a composition operator, induced by an automorphism $\varphi$ of the disc, is hypercyclic on the Hardy space if and only if $\varphi$ has no fixed points in the disc (cf. [19, Theorem 4.48]). In contrast, the main result of this section shows that the more general weighted composition operators are not even supercyclic in the Fock space setting.

**Theorem 6.1.** Let $W_{\psi,\varphi}$ be a bounded weighted composition operator on the Fock space $F^p_\alpha$, for $1 \leq p \leq \infty$. Then $W_{\psi,\varphi}$ cannot be supercyclic.

For a bounded weighted composition operator $W_{\psi,\varphi}$, with $\varphi(z) = a + \lambda z$ for $|\lambda| \leq 1$, verification of Theorem 6.1 comes down to checking a number of cases depending on the value of $\lambda$ and the form of the multiplier $\psi$. This is unavoidable since, as we shall see, the orbits of $W_{\psi,\varphi}$ have this dependence.

An initial observation is that $W_{\psi,\varphi}$ can only be supercyclic if the multiplier $\psi$ is zero-free. To see this assume that $\psi$ has a zero at $z$. It then follows for any $f \in F^p_\alpha$ that for $n \geq 1$

$$W^n_{\psi,\varphi} f(z) = \psi(z)W^{n-1}_{\psi,\varphi}(f \circ \varphi)(z) = 0$$

and hence the orbit of any $f \in F^p_\alpha$ is contained in $\{f\} \cup \text{ker}(k_z)$. Such a set cannot be projectively dense in $F^p_\alpha$ and hence $W_{\psi,\varphi}$ cannot be supercyclic. We may therefore assume that the multiplier $\psi$ has the form (4.2).

The following lemma and propositions form the core of the proof of Theorem 6.1. We write $z_0 = a/(1 - \lambda)$ for the fixed point of $\varphi$ and we denote the $n$-fold composition of $\varphi$ with itself as

$$\varphi_n = \underbrace{\varphi \circ \varphi \circ \cdots \circ \varphi}_{n\text{-times}}.$$

**Lemma 6.2.** Suppose that $W_{\psi,\varphi}$ is bounded on $F^p_\alpha$, $1 \leq p \leq \infty$, that $\varphi(z) = a + \lambda z$ where $|\lambda| < 1$, and that $\psi$ is zero-free and of the form (4.2). Then there exist convergent sequences of complex numbers $\{c_{0,n}\}$ and $\{c_{1,n}\}$, with
limits $c_0$ and $c_1$ respectively, such that, for $f$ in $\mathcal{F}_p^\alpha$ and $n \geq 1$,

\[
(W^n_{\psi,\varphi} f)(z) = \psi(z_0)^n \exp \left( c_{0,n} + c_{1,n} z + a_2 \frac{1 - \lambda^{2n}}{1 - \lambda^2} z^2 \right) f(\varphi_n(z)).
\]

Moreover,

\[
c_1 = \frac{1}{1 - \lambda} \left( a_1 + \lambda \frac{2a_2}{1 - \lambda^2} \right).
\]

**Proof.** We outline the main steps of the proof which is effectively a computation. While the quadratic term in the exponential of (6.1) that primarily concerns us, achieving the constant factor $\psi(z_0)^n$ is also important.

By Theorem 4.3, the multiplier $\psi$ has the form (4.2), and we write $q(z) = a_0 + a_1 z + a_2 z^2$. Then, for $f \in \mathcal{F}_p^\alpha$,

\[
(W^n_{\psi,\varphi} f)(z) = \left( \prod_{k=0}^{n-1} \psi(\varphi_k(z)) \right) f(\varphi_n(z))
\]

\[
= \left( \prod_{k=0}^{n-1} e^{q(\varphi_k(z))} \right) f(\varphi_n(z))
\]

\[
= \psi(z_0)^n \left( \prod_{k=0}^{n-1} e^{q(\varphi_k(z)) - q(z_0)} \right) f(\varphi_n(z)).
\]

For the last step we used the fact that $\psi(z_0) = e^{q(z_0)}$. Also, for $k \geq 1$,

\[
\varphi_k(z) = a \left( \frac{1 - \lambda^k}{1 - \lambda} \right) + \lambda^k z = z_0(1 - \lambda^k) + \lambda^k z.
\]

Next we note that

\[
\prod_{k=0}^{n-1} e^{q(\varphi_k(z)) - q(z_0)} = \prod_{k=0}^{n-1} e^{a_1(\varphi_k(z) - z_0) + a_2(\varphi_k^2(z) - z_0^2)}
\]

\[
= \exp \left[ \frac{a_1}{k=0}^n \sum_{k=0}^{n-1} (\varphi_k(z) - z_0) + a_2 \sum_{k=0}^{n-1} (\varphi_k^2(z) - z_0^2) \right]
\]

(6.4)

Now it follows from (6.3) that

\[
\varphi_k(z) - z_0 = \lambda^k (z - z_0)
\]

and

\[
\varphi_k^2(z) - z_0^2 = [\varphi_k(z) - z_0][\varphi_k(z) + z_0] = \lambda^{2k}(z - z_0)^2 + 2\lambda^k z_0(z - z_0).
\]

Thus,

\[
\sum_{k=0}^{n-1} (\varphi_k^2(z) - z_0^2) = \frac{1 - \lambda^n}{1 - \lambda}(z - z_0)
\]

(6.5)
and
\[
\sum_{k=0}^{n-1} \left( \varphi_k^2(z) - z_0^2 \right) = \frac{1 - \lambda^{2n}}{1 - \lambda^2} (z - z_0)^2 + \frac{2 - \lambda^n}{1 - \lambda} z_0 (z - z_0).
\]

Formula (6.1) then follows from (6.4), (6.5) and (6.6), with explicit expressions for the coefficients $c_{0,n}$ and $c_{1,n}$. The only quadratic term comes from (6.6). Computing the term $c_{1,n}$, taking the limit as $n \to \infty$ (so that $\lambda^n \to 0$), and finally replacing $z_0$ by $a/(1 - \lambda)$ leads to (6.2).

Next we determine the orbit $W^n(z, \psi, \varphi f)$ in the case that $\lambda$ is real. Interestingly, there are two further cases depending on whether $p$ is finite or $p = \infty$, reflecting the fact that the function $e^{\alpha z^2/2}$ belongs to $\mathcal{F}_\alpha^p$, but not to $\mathcal{F}_\alpha^p$ for $p < \infty$.

**Proposition 6.3.** Suppose that $W(z, \psi, \varphi f)$ is bounded, that $\psi(z) = a + \lambda z$ where $-1 < \lambda < 1$ and that $\psi$ has the form (1.2) with $|a_2| = \frac{\pi}{2}$. Suppose that $f \in \mathcal{F}_\alpha^p$, $1 \leq p < \infty$, and that $f(z_0) \neq 0$ where $z_0$ is the fixed point of $\psi$. Then
\[
|\psi(z)|^{-n} \|W^n(z, \psi, \varphi f)\| \to \infty, \text{ as } n \to \infty.
\]

**Proof.** Since $f(z_0) \neq 0$, we may choose $r_0 > 0$ so that $|f(z)| \geq \frac{1}{2}|f(z_0)| = c$ for $z \in D(z_0, r_0)$, the disc of radius $r_0$ centred at $z_0$. Note that $c$ is positive.

Set $c_{2,n} = a_2(1 - \lambda^{2n})/(1 - \lambda^2)$ and $a_2 = \frac{\pi}{2} \beta e^{-2i\theta_2}$, with $0 \leq \theta_2 < \pi$, so that
\[
c_{2,n} = \frac{\pi}{2} (1 - \lambda^{2n}) e^{-2i\theta_2}.
\]

Here it is crucial that $\beta = 1 - |\lambda|^2 = 1 - \lambda^2$, a consequence of the assumption that $\lambda$ is real.

We suppose, as we may by Lemma 6.2, that $|c_{0,n}| \leq 2c_0$ for each $n$. Then, by (6.1),
\[
|\psi(z)|^{-n} \|W^n(z, \psi, \varphi f)\| = \|\exp (c_{0,n} + c_{1,n}z + c_{2,n}z^2) f (\varphi_n(z))\|
\geq e^{-c_0} \|\exp (c_{1,n}z + c_{2,n}z^2) f (\varphi_n(z))\|.
\]

Next, for $1 \leq p < \infty$,
\[
\|\exp (c_{1,n}z + c_{2,n}z^2) f (\varphi_n(z))\|^p
= \frac{pc_0}{2\pi} \int_C \exp \left[ p\Re(c_{1,n}z) + p\Re(c_{2,n}z^2) \right] |f (\varphi_n(z))|^p e^{-\frac{ap}{2}|z|^2} \, dm(z)
\geq \frac{c^ppc_0}{2\pi} \int_{D_{r_0}} \exp \left[ p\Re(c_{1,n}z) + p\Re(c_{2,n}z^2) \right] e^{-\frac{ap}{2}|z|^2} \, dm(z)
\]
for sufficiently large $n$. Here $D_{r_0} = D(0, \frac{1}{2}r_0|\lambda|^{-n})$ denotes the disc of radius $\frac{1}{2}r_0|\lambda|^{-n}$ centred at the origin. If $z \in D_{r_0}$ then by (6.3)
\[
|\varphi_n(z) - \frac{a}{1 - \lambda}| = |\lambda^n z - \frac{a\lambda^n}{1 - \lambda}| = |\lambda|^n |z - z_0|
\leq |\lambda|^n |z| + |\lambda|^n |z_0| \leq \frac{1}{2} r_0 + |\lambda|^n |z_0| < r_0
\]

and
\[
\sum_{k=0}^{n-1} \left( \varphi_k^2(z) - z_0^2 \right) = \frac{1 - \lambda^{2n}}{1 - \lambda^2} (z - z_0)^2 + \frac{2 - \lambda^n}{1 - \lambda} z_0 (z - z_0).
\]
for sufficiently large \(n\). Thus, \(\varphi_n(z) \in D(z_0, r_0)\) for \(z \in D_n\) and so \(|f(\varphi_n(z))| > c\), thereby justifying the inequality \((6.9)\).

Notice that
\[
\exp \left[ p\Re(c_{1,n}z) + p\Re(c_{2,n}z^2) \right] + \exp \left[ p\Re(c_{1,n}(-z)) + p\Re(c_{2,n}(-z)^2) \right] \\
= (\exp \left[ p\Re(c_{1,n}z) \right] + \exp [-p\Re(c_{1,n}z)]) \exp \left[ p\Re(c_{2,n}z^2) \right] \\
\geq 2 \exp \left[ p\Re(c_{2,n}z^2) \right].
\]
(6.10)

We set
\[
S := \left\{ re^{it} : \theta_2 - \frac{\pi}{8} < t < \theta_2 + \frac{\pi}{8} \right\}
\]
and by summing the integrals over \(D_n \cap S\) and \(D_n \cap (-S)\) we get that
\[
\int_{D_n} \exp \left[ p\Re(c_{1,n}z) + p\Re(c_{2,n}z^2) \right] e^{-\frac{\alpha p}{2}|z|^2} \, dm(z) \\
\geq 2 \int_{D_n \cap S} \exp \left[ p\Re(c_{2,n}z^2) - \frac{\alpha p}{2}|z|^2 \right] \, dm(z) \\
\geq 2 \int_{D_n \cap S} \exp \left[ \frac{\alpha p}{2}(1 - \lambda^{2n})\Re(e^{-2i\theta z}z^2) - \frac{\alpha p}{2}|z|^2 \right] \, dm(z).
\]
(6.11)

Set
\[
g_n(z) = \exp \left[ \frac{\alpha p}{2}(1 - \lambda^{2n})\Re(e^{-2i\theta z}z^2) - \frac{\alpha p}{2}|z|^2 \right] \mathbb{1}_{D_n \cap S}
\]
and
\[
g(z) = \exp \left[ \frac{\alpha p}{2}\Re(e^{-2i\theta z}z^2) - \frac{\alpha p}{2}|z|^2 \right] \mathbb{1}_S.
\]
(6.12)

Since \(\Re(e^{-2i\theta z}z^2) > 0\) on the sector \(S\) and \(1 - \lambda^{2n}\) increases to 1, the functions \(g_n\) increase monotonically to \(g\) on \(S\). Hence, by monotone convergence,
\[
\int_{D_n \cap S} \exp \left[ \frac{\alpha p}{2}(1 - \lambda^{2n})\Re(e^{-2i\theta z}z^2) - \frac{\alpha p}{2}|z|^2 \right] \, dm(z) \\
\rightarrow \int_S \exp \left[ \frac{\alpha p}{2}\Re(e^{-2i\theta z}z^2) - \frac{\alpha p}{2}|z|^2 \right] \, dm(z) = \infty
\]
as \(n \to \infty\). The computation showing that this final integral is unbounded is essentially the same as showing that the function \(f(z) = e^{\alpha z^2/2}\) is not in \(\mathcal{F}_\alpha^p\) for \(1 \leq p < \infty\). This last estimate, together with the estimates \((6.8)\), \((6.9)\) and \((6.11)\), establish \((6.7)\) for \(1 \leq p < \infty\). \(\square\)

**Proposition 6.4.** Suppose that \(W_{\psi, \varphi}\) is bounded, that \(\varphi(z) = a + \lambda z\) with \(-1 < \lambda < 1\) and that \(\psi\) has the form \((6.2)\) where \(|a_2| = \frac{\alpha}{2}\beta\). Then the function
\[
F(z) = \exp \left( c_1 z + a_2 \beta^{-1}z^2 \right)
\]
belongs to \(\mathcal{F}_\alpha^\infty\), where \(c_1\) is as in the statement of Lemma \((6.2)\).

**Proof.** We again write \(a_2 = \frac{\alpha}{2}\beta e^{-2i\theta_2}\), with \(0 \leq \theta_2 < \pi\), and since \(\lambda\) is real we have that \(2a_2/(1 - \lambda^2) = \alpha e^{-2i\theta_2}\).
We again let \( t = a_1 + \alpha a \lambda \). By Theorem 4.3 (b), the assumption that \( W_{\psi, \varphi} \) is bounded gives that either \( t = 0 \), or \( t \neq 0 \) and \( a_2 = -\frac{\alpha}{2} \beta t^2 / |t|^2 \).

We first consider the case \( t = 0 \), so that \( a_1 = -\alpha a \lambda \). It follows from (6.2) that

\[
c_1 = \frac{-\alpha a \lambda + \alpha a \lambda e^{-2i\theta_2}}{1 - \lambda} = e^{-i\theta_2} \frac{\alpha \lambda}{1 - \lambda} 2i \Im \left( ae^{-i\theta_2} \right).
\]

By definition of the norm on \( F^\infty_{\alpha} \)

\[
\| F \| = \sup_{z \in \mathbb{C}} \left\{ \left| \exp \left( c_1 z + a_2 \beta^{-1} z^2 \right) \right| e^{-\frac{\alpha}{2} |z|^2} \right\}
\]

\[
= \sup_{z \in \mathbb{C}} \left\{ \exp \left[ \Re(c_1 z) + \Re\left( \frac{\alpha}{2} e^{-2i\theta_2} z^2 \right) - \frac{\alpha}{2} |z|^2 \right] \right\}
\]

where we substituted \( w = e^{-i\theta_2} z \). By (6.15)

\[
\Re(c_1 e^{i\theta_2} w) = \frac{2 \alpha \lambda}{1 - \lambda} \Im(ae^{-i\theta_2}) \Im(w) = A \Im(w),
\]

where \( A \) is real. Writing \( w = u + iv \),

\[
A \Im(w) + \frac{\alpha}{2} \left( \Re(w^2) - |w|^2 \right) = Av - \alpha v^2
\]

which is uniformly bounded on \( \mathbb{C} \), thereby showing that \( \| F \| < \infty \), as required.

Next we consider the case \( t \neq 0 \), with \( a_2 = -\frac{\alpha}{2} \beta t^2 / |t|^2 \). The expression (6.2) for \( c_1 \) becomes, in terms of \( t \),

\[
c_1 = \frac{1}{1 - \lambda} \left( a_1 - \alpha a \lambda \frac{t^2}{|t|^2} \right).
\]

Then

\[
\| F \| = \sup_{z \in \mathbb{C}} \left\{ \exp \left[ \Re(c_1 z) - \Re\left( \frac{\alpha}{2} \frac{t^2}{|t|^2} z^2 \right) - \frac{\alpha}{2} |z|^2 \right] \right\}
\]

where we substituted \( w = tz / |t| \). Starting from (6.17) and replacing \( a_1 \) by \( t - \alpha a \lambda \), we find that

\[
c_1 \frac{|t|}{t} = \frac{1}{1 - \lambda} \left( t - \alpha a \lambda - \alpha a \lambda \frac{t^2}{|t|^2} \right) \frac{|t|}{t} \frac{1}{|t| (1 - \lambda)} \left[ |t|^2 - 2 \alpha a \Re(at) \right] = B,
\]

where \( B \) is real. Since, again, \( B \Re(w) - \frac{\alpha}{2} \left[ \Re(w^2) + |w|^2 \right] \) is uniformly bounded on \( \mathbb{C} \), (6.15) shows that \( \| F \| \) is finite in this case. \( \square \)
The existence of eigenvalues plays a significant role in the theory of linear dynamics. A widely known result states if an operator $T$ acting on a separable Banach space is hypercyclic, or if $T$ is compact and supercyclic, then the point spectrum of the adjoint $T^*$ is empty (cf. [2, p. 29 and Propositions 1.17 and 1.26]).

Another useful necessary condition for supercyclicity is the following corollary of the Angle Criterion, which can be found in [2, Corollary 9.3]. Let $T$ be an operator acting on a separable Banach space $X$. Assume for all $x$ in some non-empty open subset $U \subset X$ that one can find a non-zero linear functional $x^* \in X^*$ such that

$$
\lim_{n \to \infty} \frac{\| (T^*)^n x^* \|}{\| T^n x \|} = 0.
$$

Then $T$ is not supercyclic.

We are now ready to prove that the Fock spaces do not support supercyclic weighted composition operators. The fact that no such operator is hypercyclic is straightforward, as can be seen in the proof of Theorem 6.1.

**Proof of Theorem 6.1.** Suppose that $W_{\psi,\varphi}$ is a bounded weighted composition operator on the Fock space $F^p_\alpha$. Then $\varphi(z) = a + \lambda z$ and $|\lambda| \leq 1$. We also assume, as we may, the necessary condition that $\psi$ is non-vanishing.

For the case $|\lambda| = 1$, we know from Corollary 5.2 that $W_{\psi,\varphi}$ cannot be supercyclic (and therefore not hypercyclic).

We next assume that $|\lambda| < 1$. Then the function $\varphi$ has a fixed point at $z_0 = a/(1 - \lambda)$ and we know from (3.1) that $\psi(z_0)$ is an eigenvalue of the adjoint $W^*_{\psi,\varphi}$ with corresponding eigenvector $k_{z_0}$. (At this point we can conclude that $W_{\psi,\varphi}$ cannot be hypercyclic.)

So, in the case that $W_{\psi,\varphi}$ is compact it follows from the above mentioned spectral condition that $W_{\psi,\varphi}$ cannot be supercyclic.

It thus follows from Theorem 4.3 that we are left to investigate the case when $|\lambda| < 1$, for $\psi$ of the form (4.2) with $|a_2| = a_2/2\beta$ (that is, $W_{\psi,\varphi}$ is a bounded and non-compact weighted composition operator). Interestingly, the iterates of $W_{\psi,\varphi}$ have different asymptotic behaviour depending on whether $\lambda$ is real and $p$ is finite, $\lambda$ is real and $p = \infty$, or whether $\lambda$ is complex, and we now check these three subcases.

In the case $\lambda$ is real and $p$ is finite, we employ the above mentioned corollary of the Angle Criterion. We choose as our linear functional the point evaluation $k_{z_0}$, where $z_0$ denotes the fixed point of $\varphi$. Iterating (3.1) gives that

$$
\| (W_{\psi,\varphi}^*)^n k_{z_0} \| = \| \psi(z_0)^n k_{z_0} \| = |\psi(z_0)|^n e^{a|z_0|^2/2}.
$$

It follows from (6.3) that $\varphi(z) \to z_0$ as $k \to \infty$ for each $z$, so we set

$$
U = \{ f \in F^p_\alpha : f(z_0) \neq 0 \}.
$$

Note that $U = \delta_{z_0}^{-1}(\mathbb{C} \setminus \{0\})$, where $\delta_{z_0}$ is the point evaluation at $z_0$. The point evaluation is a continuous linear functional and $\mathbb{C} \setminus \{0\}$ is an open set, so $U$ is an open subset of $\mathcal{F}_0$. Observe that it follows from (6.20) and Proposition 6.3 for any $f \in U$, that

$$
\frac{\| (W^n_{\psi,\phi})^m k_{z_0} \|}{\| W^n_{\psi,\phi} f \|} = \frac{\| \psi(z_0) \|^{e^{\alpha(z_0)^2}/2}}{\| W^n_{\psi,\phi} f \|} \to 0.
$$

Thus (6.19) holds and it follows from the corollary of the Angle Criterion that $W_{\psi,\phi}$ cannot be supercyclic in this case.

Next we consider the case when $\lambda$ is real and $p = \infty$. Suppose, to the contrary, that $f_0 \in \mathcal{F}_0^{\infty}$ is a supercyclic vector for $W_{\psi,\phi}$.

It is then also a supercyclic vector for $\tilde{W}_{\psi,\phi} = \psi(z_0)^{-1} W_{\psi,\phi}$. We note that a necessary condition on $f_0$ is that $f_0(z_0) \neq 0$. If not, $(\tilde{W}^n_{\psi,\phi} f_0)(z_0) = 0$ for $n \geq 1$ so that the projective orbit of $f_0$ under $\tilde{W}_{\psi,\phi}$ is contained in $\text{span}\{f_0\} \cup \ker(k_{z_0})$.

By Lemma 6.2 the iterates of $f_0$ under $\tilde{W}_{\psi,\phi}$ are given by

$$(\tilde{W}^n_{\psi,\phi} f_0)(z) = \exp \left( c_0, n + c_1, n z + a_2 \beta^{-1}(1 - \lambda^{2n}) z^2 \right) f_0(\phi_n(z)).$$

Moreover, $c_{0,n} \to c_0$ and $c_{1,n} \to c$ as $n \to \infty$, where $c_1$ is given by (6.14).

Since $\phi_n(z) \to z_0$ uniformly on compact subsets of $\mathbb{C}$, it is natural to define an operator $\tilde{W}$ by

$$(\tilde{W} f)(z) = \exp \left( c_0 + c_1 z + a_2 \beta^{-1} z^2 \right) f(z_0), \quad f \in \mathcal{F}_0^{\infty}.$$ 

By Lemma 6.4 $\tilde{W}$ is bounded on $\mathcal{F}_0^{\infty}$ and further note that $\tilde{W}^n_{\psi,\phi} f \to \tilde{W} f$ in the topology of local uniform convergence on $\mathbb{C}$ for each $f \in \mathcal{F}_0$.

Since the norm in $\mathcal{F}_0^{\infty}$ is given by a supremum, we may choose $z = z_0$ in this supremum and conclude that

$$\| \tilde{W}^n_{\psi,\phi} f_0 \| \geq \| \exp \left( c_0, n + c_1, n z_0 + a_2 \beta^{-1}(1 - \lambda^{2n}) z_0^2 \right) \| \cdot \| f(z_0) \| \exp(-\frac{c}{2} |z_0|^2)$$

which is in turn greater, for all $n$, than $c |f(z_0)|$ for a suitable positive constant $c$.

Suppose that $g \in \mathcal{F}_0^{\infty}$ lies in the projective orbit of $f_0$ under $\tilde{W}_{\psi,\phi}$, so there exist sequences $\{n_k\}_k \subset \mathbb{N}$ and $\{\zeta_{n_k}\}_k \subset \mathbb{C}$ such that

$$\zeta_{n_k} \tilde{W}^n_{\psi,\phi} f_0 \to g \text{ in } \mathcal{F}_0^{\infty}.$$ 

In particular,

$$\| g \| = \lim_{k \to \infty} \| \zeta_{n_k} \tilde{W}^n_{\psi,\phi} f_0 \| = \lim_{k \to \infty} |\zeta_{n_k}| \| \tilde{W}^n_{\psi,\phi} f_0 \|.$$ 

Since $\| \tilde{W}^n_{\psi,\phi} f_0 \| \geq c |f(z_0)|$, it follows for all sufficiently large $k$ that

$$|\zeta_{n_k}| \leq \frac{2\| g \|}{c |f(z_0)|} = C_0.$$
Convergence in $F_α^{∞}$ implies locally uniform convergence, thus for each positive $R$
\begin{equation}
(6.22) \quad \sup_{|z| \leq R} \left| \zeta_{n_k} (\tilde{W}^{_{\psi,\varphi}} f_0)(z) - g(z) \right| \to 0
\end{equation}
as $k \to \infty$. Consequently
\[
\sup_{|z| \leq R} \left| \zeta_{n_k} (\tilde{W} f_0)(z) - g(z) \right|
\leq \sup_{|z| \leq R} |\zeta_{n_k}| \left| (\tilde{W} f_0)(z) - (\tilde{W}^{_{\psi,\varphi}} f_0)(z) \right| + \sup_{|z| \leq R} \left| \zeta_{n_k} (\tilde{W}^{_{\psi,\varphi}} f_0)(z) - g(z) \right|
\leq C_0 \sup_{|z| \leq R} \left| (\tilde{W} f_0)(z) - (\tilde{W}^{_{\psi,\varphi}} f_0)(z) \right| + \sup_{|z| \leq R} \left| \zeta_{n_k} (\tilde{W}^{_{\psi,\varphi}} f_0)(z) - g(z) \right|.
\]
By the locally uniform convergence of $\tilde{W}^{_{\psi,\varphi}} f_0$ to $\tilde{W} f_0$ and by (6.22) we see that $\zeta_{n_k} \tilde{W} f_0(z) \to g(z)$ locally uniformly, so that $g \in \text{span}\{\tilde{W} f_0\}$, contradicting the assumption that $f_0$ is a supercyclic vector for $W^{_{\psi,\varphi}}$.

Finally, we consider the case where $|\lambda| < 1$ and $\lambda$ is not real. It follows from the proof of Lemma 6.24 that, for any $f \in F_α$, $W^{_{\psi,\varphi}} f = \psi(z_0)^2 \exp (P(z)) f(a(1 + \lambda) + \lambda^2 z)$ where explicitly

\[
P(z) = a_1(1 + \lambda)(z - z_0) + a_2 \left[ (1 + \lambda^2)(z - z_0)^2 + 2(1 + \lambda)z_0(z - z_0) \right].
\]
This implies $W^{_{\psi,\varphi}}$ is itself a weighted composition operator with symbol $\tilde{\varphi}(z) = a(1 + \lambda) + \lambda^2 z$ and multiplier $\tilde{\psi}(z) = \psi(z_0)^2 \exp [P(z)]$. The modulus of the coefficient $a_2$ of $z^2$ in the quadratic $P$ is

\[
|a_2| = |a_2| \left| 1 + \lambda^2 \right| = \frac{a}{\beta} \left( 1 - |\lambda|^2 \right) \left| 1 + \lambda^2 \right| < \frac{a}{\beta} \left( 1 - |\lambda|^2 \right) \left| 1 + \lambda^2 \right| = \frac{a}{\beta} \tilde{\beta}.
\]
The last inequality is a consequence of the assumption that $\lambda$ is not real. So it now follows from Theorem 3.1 that $W^{_{\psi,\varphi}}$ is compact.

The remarkable work of Bourdon and Feldman includes the following fact: if the projective orbit is somewhere dense then it is everywhere dense [10, Theorem 3.1]. In particular, this gives that any power of a supercyclic operator is itself supercyclic with the same supercyclic vectors. Since $W^{_{\psi,\varphi}}$ is a compact weighted composition operator on $F_α$ we know that it cannot be supercyclic. Hence $W^{_{\psi,\varphi}}$ itself cannot be supercyclic in this case. $\Box$

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