A THEOREM OF BOMBIERI-VINOGRADOV TYPE
WITH FEW EXCEPTIONAL MODULI

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Abstract. Let \( 1 \leq Q \leq x^{9/40} \) and let \( \mathcal{S} \) be a set of pairwise relatively prime integers in \( [Q,2Q) \). The prime number theorem for arithmetic progressions in the form

\[
\max_{y \leq x} \max_{(a,q)=1} \left| \sum_{n \equiv a \pmod{q}, n \leq y} \Lambda(n) - \frac{x}{\phi(q)} \right| < \frac{x}{\phi(q)(\log x)^A}
\]

holds for all \( q \) in \( \mathcal{S} \) with \( O((\log x)^{34+A}) \) exceptions.

1. Introduction

Let \( \Lambda(n) \) denote the von Mangoldt function. The prime number theorem in the form

\[
\sum_{n \leq x, n \equiv a \pmod{q}} \Lambda(n) = \frac{x}{\phi(q)} (1 + O_A((\log x)^{-A}))
\]

for every \( A > 0 \), holds for \( q \leq (\log x)^A, (a,q) = 1 \). The best-known average result for a set of moduli \( q \) is the Bombieri-Vinogradov theorem. Let

\[
E(x; q, a) = \sum_{n \leq x, n \equiv a \pmod{q}} \Lambda(n) - \frac{x}{\phi(q)},
\]

\[
E(x, q) = \max_{(a,q)=1} |E(x; q, a)|, E^*(x, q) = \max_{y \leq x} |E(y, q)|
\]

It is easy to deduce from the presentation of the Bombieri-Vinogradov theorem in [2] that

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for all integers $q$ in $[Q, 2Q)$ with at most $O(Q(\log x)^{-A})$ exceptions, provided that $Q \leq x^{1/2(\log x)^{-2A-6}}$.

It is of interest to restrict the size of this exceptional set further. Following Cui and Xue [1] we find that provided only prime moduli $q$ are considered, the exceptional set has cardinality $O(\mathcal{L}^{C+A})$ for some absolute constant $C$ when $Q \leq x^{1/5}$.

Glyn Harman has pointed out to me that one can obtain the result of [1] directly from Vaughan [9, Theorem 1] with $C = 3$.

In the present paper, the constant 1/5 is increased to 9/40 by adding a ‘Halasz-Montgomery-Huxley’ bound to the tools employed in [1]; see Lemma 3 below. We also relax the primality condition a little.

**Theorem.** Let $Q \leq x^{9/40}$. Let $S$ be a set of pairwise relatively prime integers in $[Q, 2Q)$. The number of $q$ in $S$ for which

$$E^*(x, q) > \frac{x}{\phi(q)(\log x)^A}$$

is $O((\log x)^{34+A})$.

As a simple example, we may take $S$ to be the set of prime powers in $[Q, 2Q)$. The constant 34 could be reduced with further effort. Constants implied by ‘$O$’, ‘$\ll$’ are absolute constants throughout the paper. We write $|E|$ for the cardinality of a finite set $E$ and

$$\mathcal{L} = \log x.$$ 

We suppose, as we may, that $x$ is large.

**2. A proposition which implies the theorem**

We write

$$\sum_{\chi \pmod{q}}' \ , \ \sum_{\chi \pmod{q}}^*$$

for a sum respectively over non-principal characters and primitive characters $\pmod{q}$. For $y \leq x$, let

$$\psi(y, \chi) = \sum_{n \leq y} \Lambda(n) \chi(n) \ , \ \psi_q(n) = \sum_{n \leq y \atop (n,q) = 1} \Lambda(n).$$

We note the identity, for $(a, q) = 1,$
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(2.1) \[ \sum_{n \leq y, n \equiv a \pmod{q}} \Lambda(n) - \frac{1}{\phi(q)} \psi_q(y) = \frac{1}{\phi(q)} \sum_{\chi \pmod{q}}' \bar{\chi}(a) \psi(y, \chi). \]

For brevity, we write \( \delta = 1/20. \)

**Proposition.** Let \( Q \leq x^{9/40}. \) Then

\[ S(Q) := \sum_{q < 2Q} \sum_{\chi \pmod{q}}' \max_{y \leq x} |\psi(y, \chi)| \ll xL^{34-\delta}. \]

The Proposition implies the Theorem. For if \( q \in [Q, 2Q), \) an argument on page 163 of [2] yields, for \( y \leq x, \)

(2.2) \[ \frac{1}{\phi(q)} \left( \sum_{n \leq y} \Lambda(n) - \psi_q(y) \right) \ll \frac{L^2 \log L}{Q}, \]

and

(2.3) \[ \frac{1}{\phi(q)} (\psi(y, \chi_1) - \psi(y, \chi)) \ll \frac{L^2 \log L}{Q} \]

where \( \chi \) is induced by the primitive character \( \chi_1. \) Let

\[ E^\dagger(x, q) = \max_{y \leq x} \max_{(a, q) = 1} \left| \sum_{n \leq y, n \equiv a \pmod{q}} \Lambda(n) - \frac{1}{\phi(q)} \sum_{n \leq y} \Lambda(n) \right|. \]

We combine all contributions to \( E^\dagger(x, q) \) made by an individual primitive character. We see from (2.1)–(2.3) that

\[ \sum_{q \in S} E^\dagger(x, Q) \ll \sum_{q \leq Q} L^2 \log L \]

\[ + \sum_{3 \leq q_1 \leq Q} \sum_{\chi_1 \pmod{q_1}}' \max_{y \leq x} |\psi(y, \chi_1)| \sum_{\frac{Q}{q_1} \leq k < \frac{2Q}{q_1}} 1 \frac{1}{\phi(kq_1)} \]

\[ \ll QL^3 + \frac{\log L}{Q} \sum_{3 \leq q_1 \leq Q} \sum_{\chi_1 \pmod{q_1}}' \max_{y \leq x} |\psi(y, \chi_1)| \sum_{\frac{Q}{q_1} \leq k < \frac{2Q}{q_1}} 1. \]
The inner sum is 0 or 1 by our hypothesis on $S$, and we obtain

$$\sum_{q \in S} E^\dagger(x, Q) \ll Q L^3 + \frac{\log L}{Q} S(Q) \ll \frac{L^{34}}{Q}.$$

The set $A$ of $q \in [Q, 2Q)$ for which

$$E^\dagger(x, q) > \frac{x}{2\phi(q)} L^{-A}$$

thus has cardinality

$$|A| \ll L^{34 + A}.$$

For $q \in S_Q - A, y \leq x, (a, q) = 1$ we have

$$\left| \sum_{n \leq y, n \equiv a \pmod{q}} \Lambda(n) - \frac{y}{\phi(q)} \right|$$

$$\leq \frac{x L^{-A}}{2\phi(q)} + \left| \sum_{n \leq y} \Lambda(n) - \frac{y}{\phi(q)} \right|$$

$$\leq \frac{x}{\phi(q)} L^{-A}$$

by the prime number theorem. This completes the proof of the theorem.

We now explain the initial stage of the proof of the proposition. For $\chi (\mod{q})$ a primitive character, $Q \leq q < 2Q$, choose $y(\chi)$ to maximize

$$\left| \sum_{n \leq y, n \equiv a(\chi) \pmod{q}} \Lambda(n) \chi(n) \right| \quad (y \leq x)$$

and $a(\chi)$ so that $|a(\chi)| = 1$, $a(\chi) \sum_{n \leq y(\chi)} \Lambda(n) \chi(n) = \sum_{n \leq y(\chi)} \Lambda(n) \chi(n)$.

Thus

$$S(Q) = \sum_{q < 2Q} \sum_{\chi (\mod{q})}^* a(\chi) \sum_{n \leq y(\chi)} \Lambda(n) \chi(n).$$
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From the discussion in Heath-Brown [3], $S(Q)$ is a linear combination, with bounded coefficients, of $O(L^8)$ sums of the form

$$S := \sum_{q<2Q} \sum_{\chi \pmod{q}} a(\chi) \sum_{\substack{n_1 \ldots n_8 \leq y(\chi) \atop n_i \in I_i}} (\log n_1) \mu(n_5) \ldots \mu(n_8) \chi(n_1) \ldots \chi(n_8)$$

in which $I_i = (N_i, 2N_i]$, $\prod_i N_i \leq x$ and $2N_i \leq x^{1/4}$ if $i > 4$. Some of the intervals $I_i$ may contain only the integer 1, and we replace these by $[1, 2)$ without affecting the upper bound $\prod_i N_i \leq x$. Now we need only bound $S$ by $O(xL^{26-\delta}/Q)$.

It is convenient to get rid of the factor $\log n_1$ in $S$. We have

$$S = \sum_{q,\chi} \frac{1}{v} \sum_{\substack{n_i \in I'_i \atop n_1 \ldots n_8 \leq y(\chi)}} \mu(n_5) \ldots \mu(n_8) \chi(n_1) \ldots \chi(n_8) dv,$$

where $I_1 = (\max(v, N_1), 2N_1]$ and $I'_i = I_i$ for $i > 1$.

Next we use Perron’s formula [8, Lemma 3.12]. Let

$$F_j(s, \chi) = F_j(s, \chi, v) = \sum_{n \in I'_j} a_j(n) \chi(n)n^{-s}$$

where $a_j(n) = 1$ ($j \leq 4$), $a_j(n) = \mu(n)$ ($j > 4$). Then

$$\sum_{\substack{n_i \in I'_i \atop n_1 \ldots n_8 \leq y(\chi)}} \mu(n_5) \ldots \mu(n_8) \chi(n_1) \ldots \chi(n_8)$$

$$= \frac{1}{2\pi i} \int_{1-L^{-1}-ix}^{1+L^{-1}+ix} F_1(s, \chi) \ldots F_8(s, \chi) \frac{y(\chi)^s}{s} ds + O(L^2).$$

We shift the path of integration to $\text{Re}(s) = 1/2$. We have

$$\left| \prod_{j=1}^{8} F_j(\sigma \pm ix, \chi) \right| \leq \prod_{j=1}^{8} N_j^{1-\sigma} \leq x^{1-\sigma},$$
so that the integral on the horizontal segments is $O(1)$. Thus

$$S = \frac{1}{2\pi i} \sum_{q,\chi} \int_1^{N_1} \int_{-x}^{x} \prod_{j=1}^{8} F_j \left( \frac{1}{2} + it, \chi \right) \frac{y(\chi)^{\frac{1}{2} + it}}{\frac{1}{2} + it} \ dt \ dv + O(Q^2 \mathcal{L}^2)$$

\[
\ll x^{1/2} \mathcal{L} \sum_{q,\chi,T} T^{-1} \int_{-T}^{T} \prod_{j=1}^{8} \left| F_j \left( \frac{1}{2} + it, \chi \right) \right| \ dt + O(Q^2 \mathcal{L}^2)
\]

where $T$ takes the values $2^k$, $1 \leq k \leq \mathcal{L}/\log 2$. Here $v$ is now fixed in $[1, N_1]$. Since $Q^4 < x$, we need only show that

\[
\sum_{q<2Q} \sum_{\chi \equiv \mod q} \int_{-T}^{T} \prod_{j=1}^{8} \left| F_j \left( \frac{1}{2} + t, \chi \right) \right| \ dt \ll T^{39/40} x^{1/2} \mathcal{L}^{25-\delta} \ (1 \leq T \leq x).
\]

This is done by grouping $F_1 \ldots F_8$ into two or three subproducts. It is time to state the lemmas we need on Dirichlet polynomials. For the rest of this section, let

$$S(s, \chi) = \sum_{n=N}^{N'} a_n \chi(n) n^{-s}$$

where $1 \leq N \leq x$, $N \leq N' \leq cN$ with an absolute constant $c$, and let

$$G = \sum_{n=N}^{N'} |a_n|^2.$$

**Lemma 1.** Let $1 \leq T, Q \leq x$. For a primitive character $\chi \ (\mod q)$ let $J_{\chi}$ be a set of numbers in $[-T,T]$ such that $|t - t'| \geq 1$ for distinct $t, t' \in J_{\chi}$. Then

\[
\sum_{q<2Q} \sum_{\chi \equiv \mod q} \sum_{t \in J_{\chi}} \left| S(it, \chi) \right|^2 \ll \mathcal{L}(Q^2 T + N) G.
\]

**Proof.** This follows at once from Theorem 7.3 of Montgomery [7]. \qed

**Lemma 2.** Let $a_n = 1$ $(N \leq n \leq N')$ in (2.4). Let $J_{\chi}$ be as in Lemma 1. Then

\[
\sum_{q<2Q} \sum_{\chi \equiv \mod q} \sum_{t \in J_{\chi}} \left| S \left( \frac{1}{2} + it, \chi \right) \right|^4 \ll Q^2 T \mathcal{L}^{10}.
\]
Proof. Following the argument of Liu and Liu [6], proof of Proposition 5.3, we find that

\[ M_1 := \sum_{q < 2Q} \sum_{\chi \pmod{q}}^{*} \int_{-T}^{T} \left| S \left( \frac{1}{2} + it, \chi \right) \right|^4 dt \ll Q^2 T \mathcal{L}^9 \]

and, for the derivative \( S' \),

\[ M_2 := \sum_{q < 2Q} \sum_{\chi \pmod{q}}^{*} \int_{-T}^{T} \left| S' \left( \frac{1}{2} + it, \chi \right) \right|^2 dt \ll Q^2 T \mathcal{L}^{13}. \]

We now appeal to Lemma 1.4 of [4] with \( S^2 \), \( 2SS' \) in place of \( S' \) This gives for the left-hand side of (2.7) the bound

\[
\sum_{q < 2Q} \sum_{\chi \pmod{q}}^{*} \left\{ \int_{-T}^{T} \left| S \left( \frac{1}{2} + it, \chi \right) \right|^4 dt \right. \\
+ \left( \int_{-T}^{T} \left| S \left( \frac{1}{2} + it, \chi \right) \right|^4 dt \right)^{1/2} \left( \int_{-T}^{T} \left| 2S \left( \frac{1}{2} + it, \chi \right) S' \left( \frac{1}{2} + it, \chi \right) \right|^2 dt \right) \}
\]

By the Cauchy-Schwarz inequality, the product contributes at most

\[
\sum_{q < 2Q} \sum_{\chi \pmod{q}}^{*} \left( \int_{-T}^{T} \left| S \left( \frac{1}{2} + it, \chi \right) \right|^4 dt \right)^{3/4} \left( \int_{-T}^{T} \left| S' \left( \frac{1}{2} + it, \chi \right) \right|^4 dt \right)^{1/4},
\]

which by Hölder’s inequality is at most \( M_1^{3/4} M_2^{1/4} \). The proof is now completed using (2.8), (2.9).

Lemma 3. Let \( \mathcal{B} \) be the set of \((q, \chi, t)\) with \( q \leq Q, \chi \pmod{q}, t \in J_\chi \) and

\[ |S(it, \chi)| \geq V > 0 \]

in Lemma 1. Then

\[ |\mathcal{B}| \ll G N V^{-2} \mathcal{L}^6 + G^3 N Q^2 T V^{-6} \mathcal{L}^{18}. \]

Proof. This is a very slight variant of Iwaniec and Kowalski [5, Theorem 9.18].
If \( S(it, \chi) \) is the product of \( b \) of the above functions \( F_j \left( \frac{1}{2} + it, \chi \right) \), it is clear that

\[
(2.11) \quad G \leq N^{-1} \sum_{n \leq cN} \tau_b^2(n) \ll L^{b^2-1}.
\]

The last step is a standard application of Perron’s formula to

\[
\sum_{n=1}^{\infty} \frac{\tau_b^2(n)}{n^s},
\]

which we can write as \( F(s) \zeta(s)^{b^2} \) with \( F \) analytic and bounded in \( \text{Re}(s) \geq 2/3 \). It follows that, with \( J_\chi \) as in Lemma 1,

\[
(2.12) \quad \sum_{q<2Q} \sum_{\chi (mod \ q)} \sum_{t \in J_\chi} \left| F_j \left( \frac{1}{2} + it, \chi \right) \right|^2 \ll x^{9/20} T L^{10}
\]

for \( N_j \ll x^{3/40} \) (using Lemma 1 and (2.11)) and for \( N_j > x^{1/4} \) (using Lemma 2). This explains the role of the ‘difficult interval’ \((9/40, 1/4)\) in Lemma 4 below.

3. Proof of the Proposition

**Lemma 4.** Let \( u_1 \geq \cdots \geq u_8 \geq 0 \), \( u_1 + \cdots + u_8 \leq 1 \). Then either

(a) there is a partition \( \{i\}, A_1, A_2 \) of \( \{1, \ldots, 8\} \) with \( \max(|A_1|, |A_2|) \leq 5 \),

\[
u_i \notin (9/40, 1/4) \quad \text{or} \quad \max \left( \sum_{j \in A_1} u_j, \sum_{j \in A_2} u_j \right) \leq 9/20,
\]

or

(b) there is a partition \( A_1, A_2 \) of \( \{1, \ldots, 8\} \) with \( \max(|A_1|, |A_2|) \leq 6 \),

\[
\sum_{j \in A_i} u_j \leq 11/20 \quad (i = 1, 2).
\]

**Proof.** If \( u_1 + \cdots + u_5 \leq 11/20 \) we have (b) with \( A_1 = \{1, \ldots, 5\} \) since \( u_6 + u_7 + u_8 \leq \frac{3}{8} \). Assume \( u_1 + \cdots + u_5 > 11/20 \). Let \( k \) be the least integer such that

\[
u_i + \cdots + u_k \geq 9/20.
\]

One of the following cases must occur.

(i) \( u_1 \notin (9/40, 1/4) \), \( u_2 + u_4 + u_6 + u_8 > 9/20 \).
(ii) \(u_1 \not\in (9/40, 1/4)\), \(u_2 + u_4 + u_6 + u_8 \leq 9/20\).

(iii) \(u_1 \in (9/40, 1/4)\), \(u_1 + \cdots + u_k \leq 11/20\).

(iv) \(u_1 \in (9/40, 1/4)\), \(u_1 + \cdots + u_k > 11/20\).

In Case (i) we have
\[
9/20 < u_2 + u_4 + u_6 + u_8 \leq 1/2
\]
and (b) holds with \(A_1 = \{2, 4, 6, 8\}\).

In Case (ii) we have \(u_3 + u_5 + u_7 \leq u_2 + u_4 + u_6\), and (a) holds with \(i = 1, A_1 = \{3, 5, 7\}\).

In Case (iii), (b) holds with \(A_1 = \{1, \ldots, k\}\).

In Case (iv), we have \(k \geq 3\). Now (a) holds with \(i = 2, A_1 = \{1, 3, \ldots, k\}\).

For \(u_2 \leq u_1 + u_2^2 < 9/40\) and \(u_1 + u_3 + \cdots + u_k \leq u_1 + u_2 + \cdots + u_{k-1} < 9/20\).

\[\square\]

Proof of the Proposition. In place of (2.4), it clearly suffices to show that, with \(J_\chi\) as in Lemma 1,
\[
E := \sum_{q < 2Q} \sum^*_{\chi (mod q)} \sum_{t \in J_\chi} \prod_{j=1}^8 \left| F_j \left( \frac{1}{2} + it, \chi \right) \right|
\]
\[
\ll T^{39/40} x^{1/2} L^{25-\delta}.
\]

We reorder \(N_1, \ldots, N_8\) so that \(N_1 \geq \cdots \geq N_8\) and write \(N_j = x^{u_j}\) with \(u_1 \geq \cdots \geq u_8 \geq 0, u_1 + \cdots + u_8 \leq 1\). Suppose we are in Case (b) of Lemma 4. Let us write
\[
\prod_{j \in A_1} N_j = M, \prod_{j \in A_2} N_j = N, S_\ell(it, \chi) = \prod_{j \in A_\ell} F_j \left( \frac{1}{2} + it, \chi \right), b = |A_2|.
\]

We bound \(E\) using Cauchy’s inequality and (2.10), (2.11) for \(S = S_1, S_2:\)
\[
E \ll \left( (Q^2 T + M) L^{(8-b)/2} \right)^{1/2} \left( (Q^2 T + N) L^{b/2} \right)^{1/2}
\]
\[
\ll (Q^2 T + x^{1/2} + (Q^2 T)^{1/2} (\max(M, N))^{1/2}) L^{20}.
\]
Now
\[ Q^2 T \ll T^{19/20} x^{1/2} \]
\[ (Q^2 T)^{1/2} (\max(M, N))^{1/2} \ll T^{1/2} x^{9/40 + 11/40} \ll T^{1/2} x^{1/2}. \]

This is acceptable in \((3.1)\).

Suppose now we are in Case (a) of Lemma 4. The argument mimics one due to Iwaniec \[4\]. We retain the notation \((3.2)\), and write
\[ L = x^{\mu_1}. \]

The contribution to \( E \) from those \( t \) with
\[ \min \left( \left| F_i \left( \frac{1}{2} + it, \chi \right) \right|, |S_1(it, \chi)|, |S_2(it, \chi)| \right) \leq x^{-1} \]

is at most
\[ Q^2 T x x^{-1} \ll T^{39/40} x^{1/2} L^{25 - \delta}. \]

By a simple splitting-up argument, there is a subset \( B \) of the set of triples \((q, \chi, t)\), \( q < 2Q \), \( \chi \) (mod \( q \)), \( t \in J_\chi \) in \((3.1)\) such that, for \((q, \chi, t) \in B\), we have
\[ U \leq \left| F_i \left( \frac{1}{2} + it, \chi \right) \right| < 2U, \]
\[ V \leq |S_1(it, \chi)| < 2V \]
\[ W \leq |S_2(it, \chi)| < 2W \]

for positive numbers \( U, V, W \) with
\[ x^{-1} \leq U \ll x^{\mu_1/2}, x^{-1} \leq V \ll M^{1/2}, x^{-1} \leq W \ll N^{1/2}, \]

while
\[ (3.3) \quad E \ll UVW |B| L^3 \]
\[ \ll UVWP L^3. \]
Here

\[ P = \min \left( \frac{(M + x^{9/20} T) L^{(7-b)^2}}{V^2}, \frac{(N + x^{9/20} T) L^{b^2}}{W^2}, \frac{x^{9/20} T L^{10}}{U^4}, \right. \\
\left. \frac{M}{V^2} L^{(7-b)^2+5} + \frac{M x^{9/20} T}{V^6} L^{3(7-b)^2+15}, \frac{N}{W^2} L^{b^2+5} + \frac{x^{9/20} T L^{3b^2+15}}{W^6}, \right. \\
\left. \frac{L^2}{U^4} L^9 + \frac{L^2 x^{9/20} T}{U^{12}} L^{27} \right), \]

and we have used (2.6), (2.7), (2.10), (2.11) in the second step in (3.3). It remains to show that

\[ (3.4) \quad UVWP \ll T^{39/40} x^{1/2} L^{22-\delta}. \]

We consider four cases.

**Case 1.** \( P \leq \frac{2M}{V^2} L^{(7-b)^2+5}, P \leq \frac{2N}{W^2} L^{b^2+5}. \)

In this case

\[ UVWP \leq 2UVW \min \left( V^{-2} M L^{(7-b)^2+5}, W^{-2} N L^{b^2+5} \right) \]

\[ \ll U(MN)^{1/2} L^{\frac{1}{2}((7-b)^2+b^2+10)} \ll x^{1/2} L^{20}. \]

**Case 2.** \( P > \frac{2M}{V^2} L^{(7-b)^2+5}, P > \frac{2N}{W^2} L^{b^2+5}. \) In this case,

\[ P \leq 2A_1 + 2B_1, \]

where

\[ A_1 = \min \left( x^{9/20} T V^{-2} L^{(7-b)^2+5}, x^{9/20} T W^{-2} L^{b^2+5}, x^{9/20} T M V^{-6} L^{3(7-b)^2+15}, x^{9/20} T N W^{-6} L^{(3b^2+15}, x^{9/20} T U^{-4} L^{10}, L^2 U^{-4} L^{9} \right) \]

and

\[ B_1 = \min \left( x^{9/20} T V^{-2} L^{(7-b)^2+5}, x^{9/20} T W^{-2} L^{b^2+5}, x^{9/20} T M V^{-6} L^{3(7-b)^2+15}, x^{9/20} T N W^{-6} L^{(3b^2+15}, x^{9/20} T U^{-4} L^{10}, x^{9/20} T L^2 U^{-12} L^{27}. \right) \]
We have, for a constant $K_1$,
\[
A_1 \leq \mathcal{L}^{K_1}(x^{9/20}TV^{-2})^{5/16}(x^{9/20}TW^{-2})^{5/16}(x^{9/20}TMV^{-6})^{1/16}.
\]
\[
(x^{9/20}TW^{-6})^{1/16} \min(x^{9/20}TU^{-4}, L^2U^{-4})^{1/4}
\]
\[
\ll \mathcal{L}^{K_1}(UVW)^{-1}TX^{9/20}(MN)^{1/16}\min(1, x^{-9/80}T^{-1/4}L^{1/2}).
\]

We bound the last minimum by $(x^{-9/80}T^{-1/4}L^{1/2})^{1/8}$, obtaining
\[
A_1 \ll \mathcal{L}^{K_1}(UVW)^{-1}T^{31/32}x^{319/640},
\]
which is acceptable in (3.4). Now
\[
B_1 \leq \min((x^{9/20}TV^{-2}L^{(7-b)^2+5})^{5/16}(x^{9/20}TW^{-2}L^{b^2+5})^{5/16}.
\]
\[
(x^{9/20}TMV^{-6}L^{3(7-b)^2+15})^{1/16}(x^{9/20}TNW^{-6}L^{3b^2+15})^{1/16}(x^{9/20}U^{-4}L^{10})^{1/4},
\]
\[
(x^{9/20}TV^{-2}L^{(7-b)^2+5})^{7/16}(x^{9/20}TW^{-2}L^{b^2+5})^{7/16}(x^{9/20}TMV^{-6}L^{3(7-b)^2+15})^{1/48},
\]
\[
(x^{9/20}TNW^{-6}L^{3b^2+15})^{1/48}(x^{9/20}TL^2U^{-12}L^{27})^{1/12}
\]
\[
\ll (UVW)^{-1}x^{9/20}T^{3/4}(MN)^{1/16}\min(\mathcal{L}^{K_2}, T^{1/4}L^{1/6}(MN)^{-1/24}\mathcal{L}^{K_3})
\]
where
\[
K_2 = ((7-b)^2 + b^2) \left( \frac{5}{16} + \frac{3}{16} \right) + \frac{50}{16} + \frac{30}{16} + \frac{10}{4} \leq 22,
\]
\[
K_3 = ((7-b)^2 + b^2) \left( \frac{7}{16} + \frac{3}{48} \right) + \frac{70}{16} + \frac{30}{48} + \frac{27}{12} \leq 22 - \frac{1}{4}.
\]

We bound the last minimum by $\mathcal{L}^{22-3/40}(T^{1/4}L^{1/6}(MN)^{-1/24})^{3/10}$, obtaining
\[
B_1 \ll (UVW)^{-1}T^{39/40}x^{1/2}\mathcal{L}^{22-\delta},
\]
which is acceptable in (3.4).

Case 3. $P > 2V^{-2}MC^{(7-b)^2+5}$, $P \leq 2W^{-2}NL^{b^2+5}$. In this case, for a constant $K_4$,
\[
P \leq \mathcal{L}^{K_4}(A_2 + B_2)
\]
where
\[
A_2 = \min(x^{9/20}TV^{-2}, NW^{-2}, x^{9/20}TMV^{-6}, x^{9/20}TU^{-4}, L^2U^{-4}),
\]
\[
B_2 = \min(x^{9/20}TV^{-2}, NW^{-2}, x^{9/20}TMV^{-6}, x^{9/20}TU^{-4}, x^{9/20}TL^2U^{-12}).
\]
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Now
\[ A_2 \leq \left( x^{9/20}TV^{-2} \right)^{1/8} (NW^{-2})^{1/2} \left( x^{9/20}TMV^{-6} \right)^{1/8} \]
\[ \min(x^{9/20}TU^{-4}, L^2U^{-4})^{1/4} \]
\[ = (UVW)^{-1} (x^{9/20}TN)^{1/2} M^{1/8} \min(1, x^{-9/80}T^{-1/4}L^{1/2}). \]

We bound the last minimum by \((x^{-9/80}T^{-1/4}L^{1/2})^{1/4}\), obtaining
\[ A_2 \ll (UVW)^{-1} T^{7/16}x^{157/320} \]
(using \(N \leq x^{9/20}\)), which is acceptable. Further,
\[ B_2 \ll \min(\left( x^{9/20}TV^{-2} \right)^{1/8} (NW^{-2})^{1/2} \left( x^{9/20}TMV^{-6} \right)^{1/8} (x^{9/20}TU^{-4})^{1/4}, \]
\[ (x^{9/20}TV^{-2})^{3/8} (NW^{-2})^{1/2} \left( x^{9/20}TMV^{-6} \right)^{1/24} (x^{9/20}L^2U^{-12})^{1/12} \]
\[ = (UVW)^{-1} (x^{9/20}TN)^{1/2} M^{1/8} \min(1, L^{1/6}M^{-1/12}). \]

We bound the last minimum by \((L^{1/6}M^{-1/12})^{1/2}\). Similarly to the bound for \(A_2\),
\[ B_2 \ll (UVW)^{-1} T^{1/2}x^{119/240}, \]
which is acceptable.

Case 4. \(P \leq 2V^{-2}ML^{(7-b)^2+5}, P > 2W^{-2}NL^{b^2+5}\). We proceed as in Case 3, interchanging the roles of \(S_1\) and \(S_2\).

This establishes (3.4) and completes the proof of the Proposition. \(\square\)

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