Superconformal Symmetry, Correlation Functions and the Operator Product Expansion

F.A. Dolan and H. Osborn

Department of Applied Mathematics and Theoretical Physics, Silver Street, Cambridge, CB3 9EW, England

Superconformal transformations are derived for the $\mathcal{N} = 2, 4$ supermultiplets corresponding to the simplest chiral primary operators. These are applied to two, three and four point correlation functions. When $\mathcal{N} = 4$, results are obtained for the three point function of various descendant operators, including the energy momentum tensor and $SU(4)$ current. For both $\mathcal{N} = 2$ or 4 superconformal identities are derived for the functions of the two conformal invariants appearing in the four point function for the chiral primary operator. These are solved in terms of a single arbitrary function of the two conformal invariants and one or three single variable functions. The results are applied to the operator product expansion using the exact formula for the contribution of an operator in the operator product expansion in four dimensions to a scalar four point function. Explicit expressions representing exactly the contribution of both long and possible short supermultiplets to the chiral primary four point function are obtained. These are applied to give the leading perturbative and large $N$ corrections to the scale dimensions of long supermultiplets.

PACS no: 11.25.Hf

Keywords: Superconformal symmetry, Chiral primary operators, Correlation functions, Operator product expansion.

† address for correspondence: Trinity College, Cambridge, CB2 1TQ, England
emails: fad20@damtp.cam.ac.uk and ho@damtp.cam.ac.uk
1. Introduction

Amongst four dimensional quantum field theories the $\mathcal{N} = 4$ supersymmetric theories for any non abelian gauge group enjoy a very special status. As was surmised from their advent they are finite and have the maximal possible supersymmetry if gravity is excluded. For a gauge group $SU(n)$ the theory is parameterised by the coupling $g$ together with the associated vacuum angle $\theta$ and the moduli corresponding to non zero expectation values of various scalar fields. In this case there are exact duality symmetries as was also conjectured long ago by Montonen and Olive [1]. Conversely at the special point where all expectation values are zero and there are no mass scales the theory has $\mathcal{N} = 4$ superconformal symmetry.

Despite its intrinsic interest a full analysis of the consequences of $\mathcal{N} = 4$ superconformal symmetry in four dimensions, which corresponds to the supergroup $PSU(2,2|4)$, has yet to be completed, primarily due to the lack of a straightforward superfield formalism. Primarily for $\mathcal{N} = 2$, superconformal identities for correlation functions of various operators [2,3,4,5,6] have been obtained using harmonic superspace methods and such results have also been extended to the $\mathcal{N} = 4$ case. It is particularly desirable to explore fully the implications for four point correlation functions since in this case the ordinary conformal group leaves these undetermined up to one or more arbitrary functions of two conformal invariants $u,v$.

Previously [7], see also [8], we discussed the four point function of four $\mathcal{N} = 1$ chiral superfields when eight linear differential equations involving four functions of $u,v$ were derived. Taking into account crossing symmetry constraints there was a unique solution up to an overall constant. However this four point function was unphysical in that for it to be non zero the scale dimensions of the fields had to be below their unitarity bound. In general superconformal symmetry relates correlation functions for the different fields in each supermultiplet which are expressed in terms of the correlation functions for the lowest weight (or dimension) operators in each supermultiplet. For $\mathcal{N} = 2,4$ superconformal symmetry there are potential additional constraints on correlation functions of the lowest weight operators themselves when some of these fields belong to short supermultiplets of the superconformal group. Any supermultiplet forming a representation of the superconformal group may be generated by the successive action of superconformal transformations on a lowest weight operator with the scale dimension increasing by $\frac{1}{2}$ between each level. Here operators belonging to level $n$ are obtained by $n$ superconformal transformations acting on the lowest weight operators. The operators present at each level are naturally classified in terms of representations of the $R$ symmetry and spin group. If $d$ is the number of lowest weight operators with scale dimension $\Delta$, corresponding to an irreducible representation of the $R$-symmetry group and spin group with dimension $d$, then in a generic long multiplet,
without any restriction on $\Delta$, there are $2^8 d$ or $2^{16} d$, for $\mathcal{N} = 2$ or $\mathcal{N} = 4$, operators in the supermultiplet and $n_{\text{max}} = 8$ or 16 respectively. For the simplest BPS-like short supermultiplets one of the representations, which would be present in a general multiplet, is missing at the first level, so that there are less than $8d$ or $16d$, for $\mathcal{N} = 2$ or $\mathcal{N} = 4$, operators with dimension $\Delta + \frac{1}{2}$. Under the action of superconformal transformations there are consequential restrictions on the representation content at higher levels leaving then a truncated multiplet with a reduced maximum level. Such short multiplets are present only if the lowest weight operator belongs to particular representations and has a corresponding value of $\Delta \ [9]$.

For $\mathcal{N} = 4$, the AdS/CFT correspondence may be applied most simply to obtain correlation functions for chiral primary operators which are scalar fields transforming under the $SU(4)$ $R$-symmetry according to the representations with Dynkin labels $[0, p, 0]$, $p = 2, 3, \ldots$, and having scale dimension $\Delta = p$. These are the lowest weight operators for short supermultiplets whose total dimension is proportional to $2^8$ and $n_{\text{max}} = 4$. The existence of constraints at the first level under superconformal transformations leads to integrability conditions for four point correlation functions of such chiral primary operators which become linear differential relations on the associated invariant functions of $u, v$.

In this paper we mostly consider $\mathcal{N} = 4$ superconformal symmetry for correlation functions involving the short supermultiplet for $p = 2$, when the representation $[0, 2, 0]$ has dimension 20, which is the simplest case of relevance. This example is of particular significance since this supermultiplet contains the energy momentum tensor as well as the $R$-symmetry current. Even in this case the short multiplets are quite large so in some cases we consider also examples with $\mathcal{N} = 2$ and even $\mathcal{N} = 1$ superconformal symmetry. Our discussion is based on using directly the superconformal transformation properties of the component fields in each supermultiplet without any need for considering superspace. Although in general superspace techniques provide considerable calculational simplifications this is perhaps less evident for $\mathcal{N} = 4$. The crucial integrability conditions arise just by considering the action of superconformal transformations on the lowest weight operators. However for illustration we give the full superconformal transformations for all fields in the $p = 2$ supermultiplet since this allows us to obtain some results for correlation functions involving the energy momentum tensor itself.

For the four point function of four chiral primary operators with $p = 2$ the integrability conditions necessary for superconformal symmetry give six linear differential relations on the initially six independent functions of $u, v$ corresponding to the number of irreducible representations appearing in the tensor product $[0, 2, 0] \otimes [0, 2, 0]$. The corresponding result for $\mathcal{N} = 2$ using harmonic superspace was given in [3] and this was extended to $\mathcal{N} = 4$ for the four point function of interest here in [4]. Without imposing any conditions following
from crossing symmetry the linear partial differential equations may be solved in terms of one arbitrary function $\mathcal{G}$ of $u,v$ as well as some single variable functions. Here we express these as functions of $z,x$ which are related to $u,v$ by $u = zx$, $v = (1 - z)(1 - x)$. The essential dynamics is then contained in $\mathcal{G}(u,v)$.

It is our aim to relate the general analysis of superconformal invariance for the four point function to the operator product expansion. Previous discussions, both for weak coupling using $O(g^2)$ perturbative results [11] and also for strong coupling for large $N$ using results from the AdS/CFT correspondence [12], have considered the contributions in the operator product expansion to the four point function of operators belonging to different $SU(4)$ representations independently. The various operators in the same supermultiplet must of course have related scale dimensions and hence for a long supermultiplet the same anomalous dimensions. However for operators of free dimension 4 and higher in $\mathcal{N} = 4$ supersymmetric gauge theories there are in general several with same spin and belonging to the same $SU(4)$ representation. In such situations there can then be a complicated mixing problem to solve before the independent anomalous dimensions can be disentangled. In this work we consider the contribution of complete supermultiplets, in both long and short representations, to the operator product expansion which thereby avoids having to consider operator mixing unless there are degenerate supermultiplets in the free theory. The contribution of a supermultiplet in the operator product expansion to the four point correlation function satisfies the necessary superconformal identities, although not the crossing symmetry properties, of the full correlation function. Using a basis for the six invariant functions of $u,v$, denoted by $A_R(u,v)$, which correspond to contributions of operators in the operator product expansion belonging to $SU(4)$ representations labelled by their dimensions $R = (1, 15, 20, 84, 105, 175)$, then constraints arise since they each may be expressed in terms of just the single function $\mathcal{G}(u,v)$.

For application to long supermultiplets the essential observation is that, if the lowest weight operator is a $SU(4)$ singlet, there is just one operator in the supermultiplet belonging to the representation of dimension 105. It is convenient to therefore take $A_{105}(u,v) = u^4 G(u,v)$ which then gives the remaining $A_R(u,v)$ in terms of $\mathcal{G}(u,v)$ multiplied by simple polynomials in $u,v$. For a single long supermultiplet we take $A_{105}(u,v) = u^{\Delta + 4 - \ell} G^{(\ell)}_{\Delta + 4}(u,v)$, which represents the contribution of a single operator of dimension $\Delta + 4$ and spin $\ell$ (so that the $SO(3,1)$ representation is $(j,j)$ where $\ell = 2j$), together with its derivatives, in the operator product expansion. Explicit simple formulae, in four dimensions, for $G^{(\ell)}_{\Delta}(u,v)$ were found by us [13] earlier which are naturally expressed in terms of the variables $z,x$ mentioned above. Using various recurrence relations, which may be derived from this explicit result, the other $A_R(u,v)$ may then be expressed as a sum of contributions involving $G^{(\ell)}_{\Delta'}(u,v)$, for suitable $\Delta', \ell'$, corresponding
to operators in the operator product expansion with dimension $\Delta'$ and spin $\ell'$. Depending on $R$, $\Delta'$ ranges from $\Delta$ to $\Delta + 8$ and $\ell'$ from $\ell - 4$ to $\ell + 4$. The associated operators correspond exactly with those expected in a long supermultiplet where the lowest weight operator is a singlet of dimension $\Delta$ and spin $\ell$. A crucial consistency check is that the coefficients of all $G^{(\ell)}_{\Delta'}(u,v)$ are positive, as required by unitarity.

We also consider the role of the single variable functions of $z,x$ which are present in the general solution of the superconformal identities as well as $G(u,v)$. There are two such functions for the case of interest here and they are related to the operator product expansion for various short multiplets which have no anomalous dimensions. This ties in nicely with the argument [10] that these functions receive no perturbative corrections beyond the free theory form.

An alternative approach to discussing the compatibility of the operator product expansion with superconformal symmetry, as expressed in the reduction of the operator product contributions for a representation $R$ to the four point function $A_R$ in terms of a single unknown conformal invariant $G(u,v)$ were given in [14]. Our results are in accord with [14] but go further in that they identify a special role for $A_{105}$ and make extensive use of the explicit form for $G^{(\ell)}_{\Delta}(u,v)$ obtained in [13]. In our approach, besides the contributions of the well known [3] short multiplets with lowest weight operators scalar fields belonging to the 20, 105 and 84 dimensional representations, there are also in general supermultiplets with lowest weight operators in the 20 and 15-representations which have protected scale dimensions $\Delta = \ell + 4$, for spin $\ell = 0, 2, \ldots$ and $\ell = 1, 3, \ldots$ respectively. The existence of a protected scalar operator with $\Delta = 4$ contributing to the operator product expansion for the four point function was shown in [12,11] and discussed further in [14,15,16].

With the aid of this formalism it is easy to rederive and extend results for both weak and strong coupling. Thus the Konishi supermultiplet and its higher spin partners, which in free field theory have scale dimension $\Delta = \ell + 2$, $\ell = 0, 2, \ldots$, develop an anomalous dimension to first order in $g^2$,

$$\eta_\ell = \frac{g^2 N}{2\pi^2} \sum_{j=1}^{\ell+2} \frac{1}{j} ,$$

which extends the results of Anselmi [17] for the first three cases. Our results allow this to be extended to cases when the free field dimension is $\Delta = \ell + 2t$, $t = 2, 3, \ldots$ when the anomalous dimensions are $O(g^2 N/N^2)$ for large $N$. Corresponding results are for large $g^2 N$ and large $N$ using results [18] from supergravity calculations in the AdS/CFT correspondence for $G(u,v)$. In this discussion we adopt the simplest assumption of neglecting any possible mixing between supermultiplets contributing to the operator product expansion having the same quantum numbers at zeroth order in $g^2$ or $1/N$. This requirement is of
course not valid for individual operators in $SU(N)\,\mathcal{N}=4$ supersymmetric gauge theories but, without as yet a detailed analysis, does not seem to be invalid for complete supermultiplets. If this does not hold, and further the assumption that the lowest dimension operator in a long supermultiplet with $\Delta$ unrestricted is a $SU(4)_R$ singlet needs modification, then analysis of the four point function of chiral primary operators belonging to the $p=2$ supermultiplet by itself is not sufficient to determine the full spectrum of operators contributing to the operator product expansion. Our results are fully consistent with $U(1)_Y$ bonus symmetry \cite{19,20}, where $U(1)_Y$ is an external automorphism of $PSU(2,2|4)$ which acts analogously to the $U(1)_R$ R-symmetry group in $\mathcal{N}=1$ superconformal symmetry and also to $U(1) \subset U(2)_R$ for $\mathcal{N}=2$.

The arrangement of this paper is then as follows. In the next section, following a simpler discussion for $\mathcal{N}=1,2$ superconformal symmetry we list the fields belonging to the $p=2$ short supermultiplet containing the energy momentum tensor and then describe their transformations under $\mathcal{N}=4$ superconformal symmetry verifying the required closure of the superconformal algebra. In section 3 we consider the two and three point functions for this supermultiplet showing how these may be determined by using superconformal transformations in terms of the two and three point functions for the lowest dimension chiral primary operators. By successive superconformal transformations the three point function of the energy momentum tensor is derived starting from the basic three point function for three chiral scalars belonging to the 20 dimensional representation. In section 4 the analysis is extended to the case when one of the operators belongs to a long supermultiplet. We recover from the superconformal transformation rule for a lowest weight scalar field belonging to a $SU(4)_R\,[q,p,q]$ representation the usual conditions for multiplet shortening by determining the values of the dimension $\Delta$ when superconformal transformations on the lowest weight operator have a non trivial cokernel at the first level. Singlet operators may contribute to the three point function for arbitrary scale dimension $\Delta$ but we show that for non singlet operators $\Delta$ is constrained. In section 5 we obtain the necessary integrability conditions on the four point function of chiral primary operators which follow from $\mathcal{N}=2$ and $\mathcal{N}=4$ superconformal symmetry. The resulting linear differential equations are solved in section 6 and used to express the different $A_R$, which correspond to contributions from representation $R$ to the four point function, in terms of a single invariant function $G(u,v)$. Sections 7 and 8 then describe the analysis in terms of the operator product expansion making clear how the contributions of long and various short supermultiplets may be separated. In accord with section 4 we find that there are contributions associated with operators belonging to the 20 and 15 dimensional representations where $\Delta = 4 + \ell$, for $\ell$ even or odd respectively. These results are applied in the $\mathcal{N}=4$ case for weak coupling and formulae for anomalous dimensions are obtained including (1.1). Where appropriate they are identical with previous results.
Several technical details are contained in four appendices. Appendix A lists our conventions for $SU(4)$ gamma matrices together with other relevant notation and also contains some of the algebraic details necessary for the derivation of the integrability conditions for the $N = 4$ four point function in section 5. Appendix B describes the short multiplet based on a $[0,p,0]$ chiral primary operator and also discusses the shortening conditions for multiplets whose lowest weight operator belongs to the $(j,0)$ spin representation, extending the results of section 4. Appendix C exhibits the result obtained in [13] for the contribution of an operator $\Delta$ and spin $\ell$ in the operator product expansion for the four point function and then derives recurrence relations, using properties of standard hypergeometric functions, for $\frac{1}{2}(1 \pm v)G^{(\ell)}(u,v)$ in terms of $G^{(\ell')}(u,v)$ for suitable $\Delta', \ell'$. These relations, which are analogous in the ordinary partial wave expansion to results such as $(2\ell + 1)zP\ell(z) = (\ell + 1)P_{\ell + 1}(z) - \ell P_{\ell - 1}(z)$ for Legendre polynomials, are required to obtain the operator product expansion results compatible with superconformal symmetry in sections 7 and 8. Finally in appendix D we use general relations for AdS/CFT integrals to show how the supergravity results in [18] may be reduced to a form in agreement with superconformal symmetry and also significantly simplified to a single $\mathcal{D}$ function defined by integrals on AdS$_5$.

2. Superconformal Transformations on Fields

The action of an infinitesimal conformal transformation on fields with a finite number of components is straightforward. A quasi-primary field $O(x)$, following [21], transforms as

$$\delta O = -v \cdot \partial O - \frac{1}{2} \hat{\omega}^{ab} s_{ab} O - \Delta \hat{\lambda} O,$$

where $\Delta$ is the scale dimension and $s_{ab} = -s_{ba}$ are the appropriate spin matrices, obeying the algebra of $SL(2,\mathbb{C})$ or $O(3,1)$, acting on $O$. For a general conformal transformation we have

$$v^a(x) = a^a + \omega^a x^b + \lambda x^a + b^a x^2 - 2b \cdot x x^a,$$

$$\dot{\lambda}(x) = \lambda - 2b \cdot x, \quad \dot{\omega}^{ab}(x) = \omega^{ab} + 4b^a x^b,$$

where $v^a$ is a conformal Killing vector and $\dot{\lambda}, \dot{\omega}^{ab}$ represent associated local infinitesimal scale transformations, rotations. In four dimensions for extension to supersymmetry it is more useful to rewrite in terms of spinor notation\footnote{Thus 4-vectors are identified with $2 \times 2$ matrices using the hermitian $\sigma$-matrices $\sigma_a, \tilde{\sigma}_a, \sigma_a \tilde{\sigma}_b = -\eta_{ab} 1$, $x^a \to x_a \hat{\alpha} = x^a (\sigma_a)_{a \hat{\alpha}}$, $\hat{x}^{\hat{\alpha} a} = x^a (\tilde{\sigma}_a)^{\hat{\alpha}} = \epsilon^{a \beta} \epsilon^{\hat{\beta} \hat{\alpha}} x_{\beta \hat{\beta}}$, with inverse $x^a = -\frac{1}{2} \text{tr}(\sigma^a \hat{x})$. We have $x \cdot y = x^a y_a = -\frac{1}{2} \text{tr}(\hat{x} y)$.}:

$$\hat{v}(x) = \hat{a} + \hat{\omega} \hat{x} - \hat{x} \omega + \lambda \hat{x} + \hat{x} b \hat{x},$$

$$\hat{\lambda}(x) = \lambda - 2b \cdot x, \quad \hat{\omega}^{\hat{a} b}(x) = \omega^{ab} + 4b^a x^b,$$

where $\hat{a}$ is a conformal Killing spinor and $\hat{\lambda}, \hat{\omega}^{\hat{a} b}$ represent associated local infinitesimal scale transformations, rotations. In four dimensions for extension to supersymmetry it is more useful to rewrite in terms of spinor notation\footnote{Thus 4-vectors are identified with $2 \times 2$ matrices using the hermitian $\sigma$-matrices $\sigma_a, \tilde{\sigma}_a, \sigma_a \tilde{\sigma}_b = -\eta_{ab} 1$, $x^a \to x_a \hat{\alpha} = x^a (\sigma_a)_{a \hat{\alpha}}$, $\hat{x}^{\hat{\alpha} a} = x^a (\tilde{\sigma}_a)^{\hat{\alpha}} = \epsilon^{a \beta} \epsilon^{\hat{\beta} \hat{\alpha}} x_{\beta \hat{\beta}}$, with inverse $x^a = -\frac{1}{2} \text{tr}(\sigma^a \hat{x})$. We have $x \cdot y = x^a y_a = -\frac{1}{2} \text{tr}(\hat{x} y)$.}.

6
where \( \omega^a_b \rightarrow \omega^\alpha_\beta = -\frac{1}{4} \omega^{ab}(\sigma_a \bar{\sigma}_b)_{\alpha\beta} \), \( \bar{\omega}^{\dot{\alpha}\dot{\beta}} = -\frac{1}{4} \omega^{ab}(\bar{\sigma}_a \sigma_b)^{\dot{\alpha}\dot{\beta}} \). With the definition in (2.2) we then have

\[
\hat{\omega}^\alpha_\beta (x) = \omega^\alpha_\beta + \frac{1}{2} (x \bar{b} - b \bar{x})_{\alpha\beta}, \quad \hat{\bar{\omega}}^{\dot{\alpha}\dot{\beta}} (x) = \bar{\omega}^{\dot{\alpha}\dot{\beta}} + \frac{1}{2} (\bar{x} b - \bar{b} x)^{\dot{\alpha}\dot{\beta}}. \tag{2.4}
\]

For the superconformal group \( SU(2,2|\mathcal{N}) \) in four dimensions the conformal group \( SU(2,2) \) is extended by the usual Grassman supertranslations \( \epsilon^i_\alpha \), \( \bar{\epsilon}^{\dot{i}\dot{\alpha}} \) and also their conformal extensions \( \eta^i_\alpha \), \( \bar{\eta}\dot{i}\dot{\alpha} \), \( i = 1, \ldots, \mathcal{N} \), as well as the appropriate \( R \)-symmetry group. The critical cases are of course \( \mathcal{N} = 2 \) and \( \mathcal{N} = 4 \), when the superconformal group reduces to \( PSU(2,2|4) \) by removal of an ideal, although for orientation we briefly consider \( \mathcal{N} = 1 \). The formulae for infinitesimal supertransformations are naturally written in terms of Killing spinors,

\[
\hat{\epsilon}^\alpha_\alpha (x) = \epsilon^\alpha_\alpha - i \eta^i_\alpha \bar{x}^{\dot{\dot{\alpha}}}, \quad \hat{\bar{\epsilon}}^{\dot{\alpha}\dot{\alpha}} (x) = \bar{\epsilon}^{\dot{\alpha}\dot{\alpha}} + i \bar{x}^{\dot{\dot{\alpha}}} \eta^i_\alpha. \tag{2.5}
\]

For \( \mathcal{N} = 1 \) we consider just the complex component fields, \( \varphi, \psi_\alpha, F \) belonging to a chiral scalar superfield for which the crucial transformation rules, dropping the index \( i \) for this case, are

\[
\delta \varphi = \epsilon^\alpha \psi_\alpha, \quad \delta \psi_\alpha = 2i \partial_{\alpha\dot{\alpha}} \varphi \hat{\epsilon}^{\dot{\alpha}} + 4q \varphi \eta_\alpha + \epsilon_{\alpha\beta} \bar{\epsilon}^{\beta} F, \quad \delta F = 2i \epsilon_{\dot{\beta}\dot{\alpha}} \partial_{\dot{\alpha\beta}} \psi_\beta \hat{\epsilon}^{\dot{\beta}} - 4(q - 1) \epsilon^{\alpha\beta} \psi_\beta \eta_\alpha, \tag{2.6}
\]

The algebra is readily seen to close on an ordinary conformal transformation together with a \( U(1)_R \) transformation,

\[
[\hat{\delta}_2, \hat{\delta}_1] \varphi = -v \cdot \partial \varphi - q \sigma \varphi, \quad [\hat{\delta}_2, \hat{\delta}_1] \psi_\alpha = -v \cdot \partial \psi_\alpha + \hat{\omega}^\alpha_\beta \psi_\beta - ((q - \frac{1}{2}) \sigma + \bar{\sigma}) \psi_\alpha, \quad [\hat{\delta}_2, \hat{\delta}_1] F = -v \cdot \partial F - ((q - 1) \sigma + 2\bar{\sigma}) F, \tag{2.7}
\]

for

\[
\bar{v} = 4i(\hat{\epsilon}^1_1 \hat{\epsilon}_2 - \hat{\epsilon}_2^0 \hat{\epsilon}_1^0), \quad \tag{2.8}
\]

and

\[
\sigma = 4(\hat{\epsilon}_1^1 \eta_1 - \hat{\epsilon}_1^0 \eta_2), \quad \bar{\sigma} = 4(\bar{\eta}_1^0 \hat{\epsilon}_1^2 - \bar{\eta}_2^1 \hat{\epsilon}_1^1), \quad \hat{\omega}^\alpha_\beta - \frac{1}{2} \delta^\alpha_\beta \sigma = 4(\eta_1^0 \hat{\epsilon}_1^2 - 1 \leftrightarrow 2). \tag{2.9}
\]

It is easy to see that (2.8) can be expanded in the form (2.3) showing that the right hand side of (2.7) is of the form of a conformal transformation as in (2.1) with \( \hat{\lambda} = \frac{1}{2}(\sigma + \bar{\sigma}) \), and with the coefficient of \( \sigma - \bar{\sigma} \) corresponding to the \( U(1)_R \) charge. In (2.8) \( q \) is then the
In detail the transformation formulae are
\[ \delta \varphi^i_j = \epsilon^i (\psi^j) + \epsilon^{i k} \epsilon^{j l} \chi (k \bar{\epsilon}_l), \]
\[ \delta \psi^i_\alpha = 2i \partial_{\alpha \dot{\alpha}} \varphi^i_j \dot{\epsilon}_j^\dot{\alpha} + 8 \varphi^{i j} \eta_{j \alpha} + \rho \varepsilon_{\alpha \beta} \dot{\epsilon}^i_j + \epsilon^{i j} J_{\alpha \dot{\alpha}}, \]
\[ \delta \chi_{i \dot{\alpha}} = -2i \epsilon^{i \alpha} \partial_{\dot{\alpha}} \dot{\epsilon}_j^j + 8 \varphi^{i j} \dot{\epsilon}_j^\dot{\alpha} \eta_{j \alpha} + \rho \varepsilon_{\alpha \beta} \dot{\epsilon}^i_j + \epsilon^{i j} \chi (k \bar{\epsilon}_l), \]
\[ \delta \varphi^i_j = 2i \epsilon^{j \alpha} \dot{\epsilon}^i_j \rho \varepsilon_{\alpha \beta} \dot{\epsilon}_j^j - 4 \epsilon^{i \alpha} \psi_\beta \eta_{j \alpha} + 2i \epsilon \epsilon_{\alpha \beta} \dot{\epsilon}_l^l \chi (k \bar{\epsilon}_l), \]
\[ \delta J_{\alpha \dot{\alpha}} = -i \epsilon^{i j} \partial_{\alpha \dot{\alpha}} \psi_{j \beta} + 2i \epsilon \epsilon_{\alpha \beta} \dot{\epsilon}_l^l \chi (k \bar{\epsilon}_l) + 6 \epsilon_{i j} \psi_\beta \eta_{j \alpha} - 6 \epsilon^{i j} \dot{\epsilon}_l^l \chi (k \bar{\epsilon}_l). \]

The coefficients are determined by closure of the algebra to give
\[ [\delta_2, \delta_1] \varphi^i_j = -v \partial \varphi^{i j} - (\sigma + \bar{\sigma}) \varphi^{i j} + \dot{\epsilon}_k \varphi^{i k} + \dot{\epsilon}^i_j \varphi^{i j}, \]
\[ [\delta_2, \delta_1] \psi^i_\alpha = -v \partial \psi^i_\alpha + \hat{\omega}_\alpha \dot{\epsilon}^i_j \psi^{j \beta} - (\sigma + \bar{\sigma}) \psi^i_\alpha + \dot{\epsilon}^i_j \psi^j_\alpha, \]
\[ [\delta_2, \delta_1] \chi_{i \dot{\alpha}} = -v \partial \chi_{i \dot{\alpha}} - \bar{\chi}_{i \dot{\alpha}} \hat{\omega}^{\beta \dot{\alpha}} \alpha - (\frac{3}{2} \sigma + \bar{\sigma}) \chi_{i \dot{\alpha}} = \bar{\chi}_{i \dot{\alpha}} \dot{\epsilon}_j^j, \]
\[ [\delta_2, \delta_1] \rho = -v \partial \rho - (\sigma + 2 \bar{\sigma}) \rho, \]
\[ [\delta_2, \delta_1] \bar{\sigma} = -v \partial \bar{\sigma} - (2 \sigma + \bar{\sigma}) \bar{\sigma}, \]
\[ [\delta_2, \delta_1] J_{\alpha \dot{\alpha}} = -v \partial J_{\alpha \dot{\alpha}} + \hat{\omega}^{\alpha \dot{\alpha}} \dot{\epsilon}_j^j \bar{J}_{\dot{\alpha}} - J_{\alpha \dot{\alpha}} \hat{\omega}^{\alpha \dot{\alpha}} \dot{\epsilon}_j^j - \frac{3}{2} (\sigma + \bar{\sigma}) J_{\alpha \dot{\alpha}}. \]

Here, in addition to (2.8) and (2.9) with implicit summation over the SU(2) indices, we have
\[ \hat{\omega}^{\alpha \dot{\alpha}} + \frac{1}{2} \delta^{\alpha \dot{\alpha}} \sigma = 4 (\hat{\epsilon}_1 \eta_{2 \alpha} - 1 \leftrightarrow 2), \]
\[ \dot{\epsilon}_j^j (\sigma - \bar{\sigma}) = 4 (\hat{\epsilon}_1 \eta_{2 j} + \eta_{1 \alpha} \dot{\epsilon}_2 \bar{\alpha} - 1 \leftrightarrow 2). \]
\( \hat{t}_j, \hat{t}_i = 0 \), represents an infinitesimal \( SU(2) \) transformation of the fields which is part of the \( U(2) \) \( R \)-symmetry group in this case. In (2.11) the result for \( \delta J_{\alpha\bar{\alpha}} \) is also compatible with the conservation equation \( \partial \tilde{\alpha}_\alpha J_{\alpha\bar{\alpha}} = 0 \), which is necessary to obtain the result for \([\delta_2, \delta_1]J_{\alpha\bar{\alpha}} \). For each field (2.12) is of the required form

\[
[\delta_2, \delta_1]O = -v \partial O - \Delta f(\sigma + \bar{\sigma}) O + r f\frac{1}{2}(\sigma - \bar{\sigma}) O - \frac{1}{2} \tilde{\omega}^{ab} s_{ab} O + \hat{t}_a R_a O ,
\]

where \( r \) is the \( U(1)_R \) charge, \( R_a \) are the appropriate generators of \( SU(2)_R \) with \( \hat{t}_a = \text{tr}(\hat{t}_a) \), for \( \tau_a \) the usual Pauli matrices.

For \( \mathcal{N} = 4 \) the fields in the corresponding self-conjugate short supermultiplet proliferate. For the appropriate \( SU(4)_R \) \( R \)-symmetry and spin representations they are listed, with the necessary constraints, in the table below.

| \( SU(4) \) rep | \( SU(4) \) dim | \( (j_1, j_2) \) | field | field constraints | field dim |
|----------------|----------------|----------------|-------|------------------|----------|
| \([0, 0, 0]\)    | 20                     | (0, 0)            | \( \varphi_{rs} \) | \( \varphi_{rs} = \varphi_{(rs)}, \varphi_{rr} = 0 \) | 20       |
| \([0, 1, 1]\)    | 20                     | \( \frac{1}{2}, 0 \) | \( \psi_{r\alpha} \) | \( \gamma_r \psi_\tau = 0 \) | 40       |
| \([1, 1, 0]\)    | 20                     | \( 0, \frac{1}{2} \) | \( \overline{\psi}^{r\bar{\alpha}} \) | \( \overline{\psi}_r \gamma_\tau = 0 \) | 40       |
| \([0, 1, 0]\)    | 6                      | (1, 0)            | \( f_{r\alpha\beta} \) | \( f_{r\alpha\beta} = f_{r(\alpha\beta)} \) | 18       |
| \([0, 1, 0]\)    | 6                      | \( 0, 1 \)         | \( \overline{f}_{r\alpha\beta} \) | \( \overline{f}_{r\alpha\beta} = \overline{f}_{r(\alpha\beta)} \) | 18       |
| \([0, 0, 0]\)    | 10                     | (0, 0)            | \( \rho_{ij} \) | \( \rho_{ij} = \rho_{(ij)} \) | 10       |
| \([2, 0, 0]\)    | 10                     | (0, 0)            | \( \overline{\psi}_{ij} \) | \( \overline{\psi}_{ij} = \overline{\psi}_{(ij)} \) | 10       |
| \([1, 0, 1]\)    | 15                     | \( \frac{1}{2}, \frac{1}{2} \) | \( J_{rsa\bar{\alpha}} \) | \( J_{rsa\bar{\alpha}} = J_{[rs]\alpha\bar{\alpha}}, \partial \tilde{\alpha}_\alpha J_{rsa\bar{\alpha}} = 0 \) | 45       |
| \([0, 0, 1]\)    | 4                      | \( \frac{1}{2}, 0 \) | \( \lambda_{io} \) | \( \partial_\alpha = 0 \) | 8        |
| \([1, 0, 0]\)    | 4                      | \( 0, \frac{1}{2} \) | \( \chi_{i\alpha} \) | \( \partial_\alpha = 0 \) | 8        |
| \([1, 0, 0]\)    | 4                      | \( \frac{1}{2}, \frac{1}{2} \) | \( \Psi^i_{\alpha\beta\bar{\alpha}} \) | \( \Psi_{\alpha\beta\bar{\alpha}} = \Psi_{(\alpha\beta)\bar{\alpha}}, \partial \tilde{\alpha}_\alpha \Psi_{\alpha\beta\bar{\alpha}} = 0 \) | 16       |
| \([0, 0, 1]\)    | 4                      | \( \frac{1}{2}, 1 \) | \( \overline{\psi}^{i\alpha\beta\bar{\alpha}} \) | \( \overline{\psi}^{i\alpha\beta\bar{\alpha}} = \overline{\psi}^{(i\alpha\beta)\bar{\alpha}}, \partial \tilde{\alpha}_\alpha \overline{\psi}^{i\alpha\beta\bar{\alpha}} = 0 \) | 16       |
| \([0, 0, 0]\)    | 1                      | (0, 0)            | \( \Phi \) | \( \partial_\alpha = 0 \) | 1        |
| \([0, 0, 0]\)    | 1                      | (0, 0)            | \( \overline{\Phi} \) | \( \partial_\alpha = 0 \) | 1        |
| \([0, 0, 0]\)    | 1                      | (1, 1)            | \( T_{\alpha\beta\bar{\alpha}\bar{\beta}} \) | \( T_{\alpha\beta\bar{\alpha}\bar{\beta}} = T_{(\alpha\beta)\bar{\alpha}\bar{\beta}}, \partial \tilde{\alpha}_\alpha T_{\alpha\beta\bar{\alpha}\bar{\beta}} = 0 \) | 5        |

Here \( i, j = 1, \ldots, 4 \) are \( SU(4) \) indices and \( r, s = 1, \ldots, 6 \) correspond to \( SO(6) \), other details of the notation are given in appendix A. The matrices \( \gamma_i^{ij} = -\gamma_i^{ji} \) and \( \gamma_{rij} = -\gamma_{rji} \) give the explicit identification of the 6-dimensional representation of \( SO(6) \) with the antisymmetric tensor products \( 4 \times 4 \) and \( 4 \times 4 \) for \( SU(4) \). From the last column it is easy to see that there are 128 bosonic and also 128 fermionic degrees of freedom. This is the simplest gauge invariant supermultiplet arising in \( \mathcal{N} = 4 \) supersymmetric gauge theories, besides the \( SU(4) \) current it of course contains the energy momentum tensor. For \( \mathcal{N} = 4 \) gauge theories \( \Phi \propto F^2 + iF\tilde{F} \), where \( F \) is the field strength and \( \tilde{F} \) is its dual.

In a similar fashion to (2.10) the supersymmetry transformations on this multiplet
may be represented diagrammatically by

\[ \Delta \\
\varphi_{rs} \leftarrow \psi_{ra} \rightarrow \overline{\psi}_{r\dot{a}} \\
\frac{5}{2} \quad f_{ra\dot{\beta}, \rho_{ij}} \leftarrow J_{rsa\dot{\alpha}} \rightarrow J_{r\dot{\alpha}\beta}, \overline{\rho}^{ij} \\
\frac{7}{2} \quad \lambda_{i\alpha} \leftarrow \psi_{i\alpha\dot{a}} \rightarrow \overline{\Psi}_{ia\dot{a}\dot{\beta}} \rightarrow \overline{\lambda}_{\dot{a}} \\
4 \quad \Phi \leftarrow T_{\alpha\beta\dot{a}\dot{\beta}} \rightarrow \overline{\Phi} \]

where we list the scale dimension \( \Delta \) and also the \( Y \)-charge corresponding to the \( U(1)_Y \) bonus symmetry \([19]\). Disregarding those which may be obtained by conjugation the relevant supersymmetry transformations are, suppressing explicit \( SU(4) \) indices when convenient by assuming for example \( \psi_{ra} \) is a column vector with \( \psi_{ra}^t \) its transpose, then

\[
\delta \varphi_{rs} = -\dot{\epsilon}\gamma_{(rs)}\psi_s + \overline{\psi}_{(rs)}\dot{\gamma}_s \dot{\epsilon}, \\
\delta \psi_{ra} = i\partial_{a\dot{\alpha}}\varphi_{rs} \hat{\gamma}_s \hat{\epsilon}^\alpha + 4 \varphi_{rs} \hat{\gamma}_s \eta_{a} - f_{ra\dot{\beta}} \hat{\epsilon}^\beta t - \frac{1}{6} f_{sa\dot{\beta}} \hat{\gamma}_r \gamma_s \hat{\epsilon}^\beta t + \rho \epsilon_{a\dot{\alpha}} \hat{\epsilon}^\beta t + J_{rsa\dot{\alpha}} \hat{\gamma}_s \hat{\epsilon}^\alpha + \frac{1}{6} J_{sta\dot{\alpha}} \hat{\gamma}_r \gamma_s \hat{\epsilon}^\alpha, \\
\delta f_{ra\dot{\beta}} = 2i \psi_{r(a}^{t} \overline{\partial}(\overline{\partial})_{\dot{\alpha}\dot{\beta}} \hat{\epsilon}^\alpha + 12 \psi_{r(\alpha} \eta_{\beta)} - \epsilon_{(a\dot{\alpha}} \hat{\epsilon}^{\gamma t} \gamma_r \lambda_{\beta)} + 2 \psi_{a\dot{\alpha}\dot{\beta}} \hat{\gamma}_r \hat{\epsilon}^\alpha, \\
\delta \rho_{ij} = -i \epsilon_{a\dot{\alpha}} \hat{\epsilon}^k \hat{\gamma}_{r(k} \partial_{\dot{\alpha}\dot{\beta}j)} \hat{\epsilon}^\alpha + 2 \epsilon_{a\dot{\alpha}} \hat{\epsilon}^k \epsilon_{r(k} \psi_{j)} + \epsilon_{i(\lambda} \hat{\epsilon}^\alpha, \\
\delta J_{rsa\dot{\alpha}} = 2i \epsilon_{[r\dot{\alpha}\dot{\beta}]a} \partial_{\dot{\alpha}\dot{\beta}} \hat{\epsilon}^\alpha - i \epsilon_{[r\dot{\alpha}\dot{\beta}]a} \partial_{\dot{\alpha}\dot{\beta}} - 6 \epsilon^{[r\dot{\alpha}\dot{\beta}]} \gamma_{[r\dot{\alpha}\dot{\beta}]} \hat{\epsilon}^\alpha + 2 i \partial_{a\dot{\alpha}} \overline{\psi}_{[r\dot{\alpha}\dot{\beta}]} \hat{\epsilon}^\beta - i \partial_{a\dot{\alpha}} \overline{\psi}_{[r\dot{\alpha}\dot{\beta}]} \hat{\epsilon}^\beta + 6 \overline{\psi}_{[r\dot{\alpha}\dot{\beta}]} \hat{\epsilon}^\alpha + \epsilon^\beta \gamma_{[r\dot{\alpha}\dot{\beta}]} \psi_{a\dot{\alpha}} + \epsilon_{a\dot{\alpha}} \hat{\epsilon}^\beta \epsilon_{[r\dot{\alpha}\dot{\beta}]} \hat{\epsilon}^\beta, \\
\delta \lambda_{a\dot{\alpha}} = i \epsilon_{\dot{\beta}a} \epsilon_{r\dot{\alpha}\dot{\beta}} \hat{\gamma}_r \hat{\epsilon}^\alpha + 2 \epsilon_{\dot{\beta}a} \epsilon_{r\dot{\alpha}\dot{\beta}} \hat{\gamma}_r \hat{\epsilon}^\alpha + 2 i \partial_{a\dot{\alpha}} \rho \hat{\epsilon}^\alpha + 12 \rho \eta_{a} + \Phi \epsilon_{a\dot{\alpha}} \hat{\epsilon}^\beta t, \\
\delta \Psi_{a\dot{\alpha}\dot{\beta}} = -\frac{1}{2} i f_{ra\dot{\beta}} \overline{\partial}_{\dot{\alpha}\dot{\beta}} \hat{\epsilon}^\alpha + \frac{1}{3} i f_{r(\alpha} \overline{\partial}_{\dot{\beta}]\dot{\alpha}} \gamma_r \hat{\epsilon}^\alpha t + \frac{4}{3} f_{ra\dot{\beta}} \gamma_r \hat{\epsilon}^\alpha t - \frac{1}{4} i \partial_{(a\dot{\alpha}} J_{r\beta)\dot{\beta}} \gamma_r \gamma_s \hat{\epsilon}^\gamma + \frac{1}{12} i \partial_{(a\dot{\alpha}} J_{r\beta)\dot{\beta}} \gamma_r \gamma_s \hat{\epsilon}^\gamma - \frac{4}{3} J_{rs(a\dot{\alpha}} \gamma_{r\beta)\dot{\beta}} \gamma_r \gamma_s \hat{\epsilon}^\gamma + T_{a\dot{\alpha}\dot{\beta}} \hat{\epsilon}^\beta, \\
\delta \Phi = 2i \epsilon_{a\dot{\alpha}} \hat{\epsilon}^t \hat{\epsilon}^\dot{\beta} \hat{\gamma}_{\dot{a}} \lambda_{a} - \Phi \epsilon_{a\dot{\alpha}} \hat{\epsilon}^t \lambda_{a}, \\
\delta T_{a\dot{\alpha}\dot{\beta}} = 2i \hat{\gamma} \partial_{(a} \hat{\epsilon}^\beta \overline{\psi}_{a\dot{\alpha}\dot{\beta})} - i \hat{\epsilon}^\gamma \partial_{(a\dot{\alpha}} \hat{\gamma}_{(a\dot{\alpha}} \hat{\epsilon}^\beta t) - 10 \hat{\eta}_{(\dot{a}} \hat{\epsilon}^\beta t), \\
- 2i \overline{\psi}_{(a\dot{\alpha}} \partial_{(a\dot{\alpha}} \hat{\gamma}_{(a\dot{\alpha}} \hat{\epsilon}^\beta t) + i \overline{\psi}_{(a\dot{\alpha}} \hat{\gamma}_{(a\dot{\alpha}} \hat{\gamma}_{(a\dot{\alpha}} \hat{\epsilon}^\gamma - 10 \overline{\psi}_{(a\dot{\alpha}} \hat{\gamma}_{(a\dot{\alpha}} \hat{\gamma}_{(a\dot{\alpha}} \hat{\epsilon}^\gamma). 
\]

(2.16)
These formulae are consistent with the conservation equations listed in the table and lead to

\[
[\delta_2, \delta_1] \varphi_{rs} = -v \partial \varphi_{rs} - 2\hat{\lambda} \varphi_{rs} + \hat{t}_{rt} \varphi_{ts} + \hat{t}_{st} \varphi_{rt},
\]

\[
[\delta_2, \delta_1] \psi_{ira} = -v \partial \psi_{ira} + \hat{\omega}_a^\beta \psi_{ir\beta} - \frac{5}{2} \hat{\lambda} \psi_{ira} + \hat{t}_i \psi_{jra} + \hat{t}_{rs} \psi_{isa},
\]

\[
[\delta_2, \delta_1] f_{ra\beta} = -v \partial f_{ra\beta} + \hat{\omega}_a^\gamma f_{r\gamma\beta} + \hat{\omega}_r^\gamma f_{\gamma a\beta} - 3\hat{\lambda} f_{ra\beta} + \hat{t}_{rs} f_{sa\beta},
\]

\[
[\delta_2, \delta_1] \rho_{ij} = -v \partial \rho_{ij} - 3\hat{\lambda} \rho_{ij} + \hat{t}_i \rho_{jk} + \hat{t}_j \rho_{ik},
\]

\[
[\delta_2, \delta_1] J_{r s a \bar{a}} = -v \partial J_{r s a \bar{a}} + \hat{\omega}_a^\beta J_{r s \bar{a} \bar{a}} - J_{r s a \bar{a}} \hat{\omega}_{\bar{a}}^\beta - 3\hat{\lambda} J_{a \bar{a}} + \hat{t}_{rt} J_{ts a \bar{a}} + \hat{t}_{st} J_{rta \bar{a}},
\]

\[
[\delta_2, \delta_1] \lambda_{i a} = -v \partial \lambda_{i a} + \hat{\omega}_a^\beta \lambda_{i \beta} - \frac{7}{2} \hat{\lambda} \lambda_{i a} + \hat{t}_i \lambda_{j a},
\]

\[
[\delta_2, \delta_1] \Psi_{a \bar{b} \bar{a}} = -v \partial \Psi_{a \bar{b} \bar{a}} + \hat{\omega}_a^\gamma \Psi_{\gamma \bar{b} \bar{a}} + \hat{\omega}_r^\gamma \Psi_{r \gamma \bar{a}} - \Psi_{a \bar{b} \bar{a}} \hat{\omega}_a^\gamma + 2\hat{\lambda} \Psi_{a \bar{b} \bar{a}} - \Psi_{a \bar{b} \bar{a}} \hat{t}_j \hat{t}_j,
\]

\[
[\delta_2, \delta_1] T_{a \bar{b} \bar{a} \bar{\beta}} = -v \partial T_{a \bar{b} \bar{a} \bar{\beta}} + \hat{\omega}_a^\gamma T_{\gamma \bar{b} \bar{a} \bar{\beta}} + \hat{\omega}_r^\gamma T_{r \gamma \bar{a} \bar{\beta}} - T_{a \bar{b} \bar{a} \bar{\beta}} \hat{\omega}_a^\gamma - T_{a \bar{b} \bar{a} \bar{\beta}} \hat{\omega}_{\bar{a}}^\gamma - \hat{\lambda} T_{a \bar{b} \bar{a} \bar{\beta}},
\]

\[
[\delta_2, \delta_1] \Phi = -v \partial \Phi - 4\hat{\lambda} \Phi .
\]

The definition of \( v^a \) is as in (2.8) while

\[
\hat{t}_{rs} = -2(\hat{\epsilon}_1 \gamma_{[r} \hat{\gamma}_{s]} \eta_2 - \overline{\eta}_2 \gamma_{[r} \hat{\gamma}_{s]} \hat{\epsilon}_1 - 1 \leftrightarrow 2), \quad \hat{t}_i \hat{j}_j = 4(\hat{\epsilon}_{1i} \eta_2 \hat{j}_j - \overline{\eta}_2 \hat{\epsilon}_{1j} - \text{trace}(ij) - 1 \leftrightarrow 2),
\]

generate local \( SU(4) \) transformations and

\[
\hat{\omega}_a^\beta = 4(\eta_1 \hat{i}_a \hat{e}_{2i}^\beta - \text{trace}(\alpha \beta) - 1 \leftrightarrow 2), \quad \hat{\omega}_{\bar{a}}^\beta = 4(\hat{\epsilon}_1 \hat{i} \eta_2 \overline{\eta}_1 \hat{e}_{2i}^\beta - \text{trace}(\alpha \beta) - 1 \leftrightarrow 2), \quad \hat{\lambda} = 2(\hat{\epsilon}_2 \eta_1 + \overline{\eta}_1 \hat{\epsilon}_2 - 1 \leftrightarrow 2),
\]

represent infinitesimal local rotations and scale transformations.

3. Two and Three Point Correlation Functions

The superconformal transformations obtained in (2.6), (2.11) and (2.16) allow correlation functions of the fields in the relevant supermultiplets to be related. This is shown most clearly in the case of two point functions which are a precursor to considering higher point functions later.

For the simplest \( \mathcal{N} = 1 \) case we start from the two point correlation function for the scalar field \( \varphi \) in the basic chiral supermultiplet

\[
\langle \varphi(x_1) \overline{\varphi}(x_2) \rangle = \frac{1}{r_{12}^2}, \quad (3.1)
\]
with a convenient choice of normalisation, and where
\[ r_{ij} = (x_i - x_j)^2. \] (3.2)

The basic superconformal identity here requires
\[ \langle \psi_\alpha(x_1) \delta \varphi(x_2) \rangle + \langle \delta \psi_\alpha(x_1) \varphi(x_2) \rangle = 0. \] (3.3)

Using (2.6) in (3.3) with (3.1) then gives
\[ \langle \psi_\alpha(x_1) \bar{\psi}_{\dot{\alpha}}(x_2) \rangle \hat{\bar{\epsilon}}^{\dot{\alpha}}(x_2) = -2i \partial_\alpha \frac{1}{r_{12}^q} \hat{\bar{\epsilon}}^{\dot{\alpha}}(x_1) - 4q \frac{1}{r_{12}^q} \eta_\alpha = 4iq \frac{X_{12\alpha\dot{\alpha}}}{r_{12}^{q+1}} \hat{\bar{\epsilon}}^{\dot{\alpha}}(x_2), \] (3.4)

where we use \( \hat{\bar{\epsilon}}^{\dot{\alpha}}(x_1) = \hat{\bar{\epsilon}}^{\dot{\alpha}}(x_2) + i \bar{x}_{12}^{\dot{\alpha}} \eta_\alpha \). Hence it is clear that
\[ \langle \psi_\alpha(x_1) \bar{\psi}_{\dot{\alpha}}(x_2) \rangle = 4iq \frac{X_{12\alpha\dot{\alpha}}}{r_{12}^{q+1}}, \] (3.5)

and in a similar fashion
\[ \langle F(x_1) \bar{F}(x_2) \rangle = 16q(q - 1) \frac{1}{r_{12}^{q+1}}. \] (3.6)

Clearly positivity requires \( q \geq 1 \).

For \( \mathcal{N} = 2 \) similar calculations give for the two point functions of the fields in short multiplet shown in (2.10) together with their conjugates,
\[ \langle \varphi^{ij}(x_1) \varphi_{kl}(x_2) \rangle = \delta^{(i} \delta^{(j} \tau^{k} \tau^{l} \frac{1}{r_{12}^2} ; \] \[ \langle \psi^{i}_{\alpha}(x_1) \psi_{j\dot{\alpha}}(x_2) \rangle = \langle \chi^{i}_{\dot{\alpha}}(x_1) \chi_{j\alpha}(x_2) \rangle = 8i \delta^{ij} \frac{X_{12\alpha\dot{\alpha}}}{r_{12}^3} ; \] \[ \langle \rho(x_1) \rho(x_2) \rangle = \langle \tau(x_1) \tau(x_2) \rangle = 32 \frac{1}{r_{12}^3} ; \] \[ \langle J_{\alpha\dot{\alpha}}(x_1) \bar{J}_{\beta\dot{\beta}}(x_2) \rangle = 48 \frac{X_{12\alpha\dot{\alpha}} X_{21\beta\dot{\beta}}}{r_{12}^4}. \] (3.7)

For \( \mathcal{N} = 4 \) the two point functions are also determined for the short multiplet given
in the table in terms of that for \( \varphi_{rs} \). Choosing a normalisation coefficient \( \hat{N} \) we have

\[
\langle \varphi_{rs}(x_1) \varphi_{uv}(x_2) \rangle = \frac{1}{2} \hat{N} \left( \frac{1}{2} (\delta_{ru} \delta_{sv} + \delta_{rv} \delta_{su}) - \frac{1}{6} \delta_{rs} \delta_{du} \right) \frac{1}{r_{12}^2},
\]

\[
\langle \psi_{r\alpha}(x_1) \bar{\psi}_{s\dot{\alpha}}(x_2) \rangle = 2 \hat{N} \dot{\gamma}_r \frac{1 + 4 \hat{\gamma}_r \gamma_s}{6} \frac{X_{12\alpha\dot{\alpha}}}{r_{12}^3},
\]

\[
\langle f_{r\alpha\beta}(x_1) \bar{f}_{s\dot{\alpha}\dot{\beta}}(x_2) \rangle = -24 \hat{N} \frac{X_{12(\alpha\dot{\alpha})} X_{12(\beta\dot{\beta})}}{r_{12}^4},
\]

\[
\langle \hat{\rho}_{ij}(x_1) \hat{p}^{kl}(x_2) \rangle = 4 \hat{N} \frac{1}{r_{12}^3},
\]

\[
\langle J_{rs\alpha\dot{\alpha}}(x_1) J_{uv\beta\dot{\beta}}(x_2) \rangle = -6 \hat{N} \left( \delta_{ru} \delta_{sv} - \delta_{rv} \delta_{su} \right) \frac{X_{12\alpha\dot{\alpha}} X_{21\beta\dot{\beta}}}{r_{12}^4}, \tag{3.8}
\]

\[
\langle \lambda_{\alpha}(x_1) \bar{\lambda}_{\dot{\alpha}}(x_2) \rangle = 48 \hat{N} \frac{1}{r_{12}^4},
\]

\[
\langle \Psi_{\alpha\beta\dot{\alpha}}(x_1) \bar{\Psi}_{\gamma\dot{\gamma}\dot{\beta}}(x_2) \rangle = 16 \hat{N} \frac{1}{r_{12}^5} \frac{X_{12(\alpha\dot{\alpha})} X_{12(\beta\dot{\beta})} X_{21(\gamma\dot{\gamma})}}{r_{12}^4},
\]

\[
\langle \Phi(x_1) \bar{\Phi}(x_2) \rangle = 384 \hat{N} \frac{1}{r_{12}^4},
\]

\[
\langle T_{\alpha\beta\dot{\alpha}\dot{\beta}}(x_1) T_{\gamma\delta\dot{\gamma}\dot{\delta}}(x_2) \rangle = 160 \hat{N} \frac{X_{12(\alpha\dot{\alpha})} X_{12(\beta\dot{\beta})} X_{21(\gamma\dot{\gamma})} X_{21(\delta\dot{\delta})}}{r_{12}^6}. \tag{3.9}
\]

The associated three point functions contain the essential information necessary to obtain the operator product expansion. It is therefore of interest to consider the three point functions for component fields in the short multiplet listed in the table above. Using \( \mathcal{N} = 4 \) superconformal symmetry we show how various three point functions involving descendant fields such as the \( SU(4) \) symmetry current and the energy momentum tensor may be uniquely obtained. With notation described in appendix A, so that \( \mathcal{C}_r \) is a basis for symmetric traceless tensors, and defining \( \varphi^I \equiv C^I_{rs} \varphi_{rs} \) the starting point is

\[
\langle \varphi^{I_1}(x_1) \varphi^{I_2}(x_2) \varphi^{I_3}(x_3) \rangle = \hat{N} \frac{C^{I_1 I_2 I_3}}{r_{12} r_{13} r_{23}}, \tag{3.9}
\]

where \( C^{IJK} = \text{tr}(C^I C^J C^K) \). In (3.9) we have taken into account that the normalisation is fixed once that of the two point function is given as a consequence of non-renormalisation theorems \([23]\). For a \( SU(N) \) gauge theory with \( \mathcal{N} = 4 \) superconformal symmetry then (3.8) and (3.9) are valid if \( \hat{N} = N^2 - 1 \).

The superconformal transformations given in (2.16) relate correlation functions of the component fields in which \( \sum_i \Delta_i \) differ by one. At the first step we may use the condition \( \delta \langle \psi_{r\alpha}(x_1) \varphi^{I_2}(x_2) \varphi^{I_3}(x_3) \rangle = 0 \) and keep just the terms involving \( \hat{\epsilon} \) from (2.16) which then gives

\[
\langle \psi_{r\alpha}(x_1) \bar{\psi}_{s\dot{\alpha}}(x_2) \varphi^{I_3}(x_3) \rangle C^{I_2}_{s t} \gamma_t \hat{\epsilon} \dot{\alpha}(x_2) = \langle \psi_{r\alpha}(x_1) \varphi^{I_2}(x_2) \bar{\psi}_{s\dot{\alpha}}(x_3) \rangle C^{I_3}_{s t} \gamma_t \hat{\epsilon} \dot{\alpha}(x_3)
\]

13
\[ + \langle J_{rs\alpha\dot{a}}(x_1) \varphi^{I_2}(x_2) \varphi^{I_3}(x_3) \rangle \tilde{\gamma}_s \tilde{\epsilon}^\alpha(x_1) + \frac{1}{4} \langle J_{st\alpha\dot{a}}(x_1) \varphi^{I_2}(x_2) \varphi^{I_3}(x_3) \rangle \tilde{\gamma}_r \tilde{\gamma}_s \tilde{\gamma}_t \tilde{\epsilon}^\alpha(x_1) \]
\[ = 2\hat{N} i \left( \frac{1}{r_{12}} x_{12\alpha\alpha} C^{I_2}_{rs} \tilde{\gamma}_s \tilde{\epsilon}^\alpha(x_2) + \frac{1}{r_{13}} x_{13\alpha\alpha} C^{I_2}_{rs} \tilde{\gamma}_s \tilde{\epsilon}^\alpha(x_3) \right), \tag{3.10} \]

with \(C^{IJ}_{rs}\) the symmetric traceless part of \((C^I C^J)_{rs}\). To solve this we introduce, for three points \(x_i, x_j, x_k\),

\[ \dot{X}_{ij|k} = \frac{x_{ij} \tilde{x}_{jk} x_{ki}}{r_{ij} r_{ik}} = \frac{1}{r_{ij}} x_{ij} - \frac{1}{r_{ik}} x_{ik}, \tag{3.11} \]

which transforms under conformal transformations as a vector at \(x_i\) and is antisymmetric in \(jk\). It is important to note the relations

\[ \dot{X}_{ij|k} \tilde{\alpha}(x_i) = \frac{1}{r_{ij}} \dot{x}_{ij} \tilde{\alpha}(x_j) - \frac{1}{r_{ik}} \dot{x}_{ik} \tilde{\alpha}(x_k), \tag{3.12} \]

With the aid of the definition (3.11) and (3.12) we then find

\[ \langle \psi_{ra}(x_1) \bar{\psi}_{sa}(x_2) \varphi^I(x_3) \rangle = 2\hat{N} i \frac{x_{12\alpha\alpha}}{r_{12} r_{13} r_{23}} (C^{I}_{rs} 1 + \frac{1}{6} \tilde{\gamma}_r \tilde{\gamma}_t C^{I}_{ts} + \frac{1}{6} C^{I}_{rt} \tilde{\gamma}_s), \tag{3.13} \]

In a similar fashion other three point functions may be determined iteratively. Thus from \(\delta \langle J_{rs\alpha\dot{a}}(x_1) \varphi^{I}(x_2) \bar{\psi}_{v\dot{\beta}}(x_3) \rangle = 0\) we find

\[ \dot{\epsilon}^\beta(x_1) \gamma_{[r} \tilde{\gamma}_{s]} \langle \bar{\psi}_{\beta a\dot{a}}(x_1) \varphi^{I}(x_2) \bar{\psi}_{v\dot{\beta}}(x_3) \rangle - \dot{\epsilon}^\beta(x_2) \gamma_{t} C^{I}_{tu} \langle J_{rs\alpha\dot{a}}(x_1) \psi_{u\beta}(x_2) \bar{\psi}_{v\dot{\beta}}(x_3) \rangle \]
\[ + \dot{\epsilon}^\beta(x_3) \gamma_{u} \langle J_{rs\alpha\dot{a}}(x_1) \varphi^{I}(x_2) J_{u\beta\dot{\beta}}(x_3) \rangle \]
\[ = 4\hat{N} \frac{1}{r_{12} r_{13} r_{23}} \left( 6 \dot{\epsilon}^\beta(x_3) x_{31\beta\dot{\alpha}} x_{31\alpha\dot{\beta}} \frac{1}{r_{13}} - 2 \dot{\epsilon}^\beta(x_1) \left( 2 X_{123\alpha\dot{\beta}} X_{13\alpha\dot{\beta}} - X_{123\alpha\dot{\alpha}} X_{13\dot{\alpha}\dot{\beta}} \right) \right) \]
\[ \times \gamma_{[r} (C^{I}_{s} u + \frac{1}{6} \tilde{\gamma}_{s]} \gamma_{t} C^{I}_{lu} + \frac{1}{6} C^{I}_{s} \tilde{\gamma}_{[r} \gamma_{t]u}) \]
\[ + 2\hat{N} \frac{1}{r_{12} r_{13} r_{23}} \left( 6 \dot{\epsilon}^\beta(x_3) x_{31\beta\dot{\alpha}} x_{31\alpha\dot{\beta}} \frac{1}{r_{13}} + X_{312\beta\dot{\beta}} X_{123\alpha\dot{\alpha}} + 2 \dot{\epsilon}^\beta(x_1) x_{13\beta\dot{\beta}} X_{123\alpha\dot{\alpha}} \frac{1}{r_{13}} \right) \]
\[ \times \gamma_{u} (C^{I}_{u[r} \delta_{s]} u + C^{I}_{v[r} \delta_{s]} u). \tag{3.14} \]

Using (3.12) again this may be decomposed to give

\[ \langle J_{rs\alpha\dot{a}}(x_1) \varphi^{I}(x_2) J_{u\beta\dot{\beta}}(x_3) \rangle = 4\hat{N} \frac{1}{r_{12} r_{13} r_{23}} \left( \frac{1}{r_{13}} x_{13\beta\dot{\beta}} x_{31\beta\dot{\alpha}} x_{31\alpha\dot{\beta}} - X_{123\alpha\dot{\beta}} X_{312\beta\dot{\beta}} \right) C^{I}_{[u[r} \delta_{s]} u), \]
\[ \langle \bar{\psi}_{\alpha\beta\dot{a}}(x_1) \varphi^{I}(x_2) \bar{\psi}_{v\dot{\beta}}(x_3) \rangle = \frac{4}{3} \hat{N} \frac{1}{r_{12} r_{13} r_{23}} X_{123\alpha\dot{\alpha}} x_{13\beta\dot{\beta}} \gamma_{u} C^{I}_{uv}, \tag{3.15} \]
and also

\[ \langle J_{rs\alpha\dot{\beta}}(x_1) \psi_{u\beta}(x_2) \bar{\psi}_{v\dot{\gamma}}(x_3) \rangle = -2\tilde{N} \left\{ \frac{2}{r_{12}} \frac{X_{1[23]\alpha\dot{\beta}}}{r_{r_{12}} r_{13} r_{23}^2} \left( \delta_{u[r} \delta_{s]} v + \frac{1}{6} \gamma_{u \gamma} [r \delta_{s]} v + \frac{1}{6} \delta_{u[r} \gamma_{s]} \gamma_{v]} + \frac{1}{30} \gamma_{u \gamma} [r \gamma_{s]} \gamma_{v]} \right) + \frac{x_{21} \beta_{\dot{\alpha}}}{r_{12}} \frac{x_{13} \dot{\alpha} \dot{\beta}}{r_{12}} \left( \delta_{u v} \gamma_{[r \gamma_{s]} + \frac{2}{3} \gamma_{u \gamma} [r \delta_{s]} v + \frac{2}{3} \delta_{u[r} \gamma_{s]} \gamma_{v]} + \frac{5}{36} \gamma_{u \gamma} [r \gamma_{s]} \gamma_{v]} \right) \right\} \]  

(3.16)

Similarly from \( \delta \langle \Psi_{\alpha\beta\dot{\alpha}}(x_1) \varphi_{I2}(x_2) \varphi_{I3}(x_3) \rangle = 0 \) we get

\[ \langle T_{\alpha\dot{\beta}\dot{\alpha}}(x_1) \varphi_{I2}(x_2) \varphi_{I3}(x_3) \rangle = -\frac{4}{3} \tilde{N} \frac{1}{r_{12}} \frac{1}{r_{13}} \frac{1}{r_{23}} X_{1[23]}(\alpha \dot{\alpha} X_{1[23]} \beta \dot{\beta} \delta_{I2 I3}. \]  

(3.17)

At the next stage, for \( \sum \Delta_i = 9 \), we obtain from \( \delta \langle J_{rs\alpha\dot{\beta}}(x_1) J_{uv\beta\dot{\beta}}(x_2) \bar{\psi}_{w\dot{\gamma}}(x_3) \rangle = 0 \), after some calculation,

\[ \langle J_{rs\alpha\dot{\beta}}(x_1) \Psi_{\beta\gamma\dot{\beta}}(x_2) \bar{\psi}_{w\dot{\gamma}}(x_3) \rangle = \frac{8}{3} \tilde{N} i \left\{ \left( \frac{1}{r_{12}} x_{12} \beta_{\dot{\beta}} x_{23} (\beta \dot{\beta} - X_{1[23]} \alpha \dot{\alpha} x_{2[31]} (\beta \dot{\beta}) \right) x_{23} \gamma_{\dot{\gamma}} \frac{2}{r_{12}} \frac{\delta_{I2 I3}}{r_{13} r_{23}} x_{13} \gamma_{\dot{\gamma}} x_{21} (\beta \dot{\beta}) \right\} \]

\[ \times \left( 1 \frac{1}{r_{12}} \frac{1}{r_{13}} \frac{1}{r_{23}} \left( \gamma_{[r \delta_{s]} w + \frac{1}{6} \gamma_{[r \gamma_{s]} \gamma_{w]} \right) \right\}, \]  

(3.18a)

\[ \langle J_{rs\alpha\dot{\beta}}(x_1) J_{uv\beta\dot{\beta}}(x_2) J_{\mu\nu\gamma\dot{\gamma}}(x_3) \rangle = 8\tilde{N} i \left\{ 5 \left( x_{12} \beta_{\dot{\beta}} x_{23} \gamma_{\dot{\gamma}} x_{31} \gamma_{\dot{\alpha}} - x_{13} \gamma_{\dot{\alpha}} x_{32} \gamma_{\dot{\beta}} x_{21} \beta_{\dot{\alpha}} \right) \frac{1}{r_{12} r_{13} r_{23}} \right. \]

\[ - 2 X_{1[23]} \alpha \dot{\alpha} x_{2[31]} (\beta \dot{\beta} x_{3[12]} \gamma_{\dot{\gamma}} \right\} \left( \frac{1}{r_{12}} \frac{1}{r_{13}} \frac{1}{r_{23}} \delta_{[r \delta_{s]} [u \delta_{v} w] \right) \]

\[ + 4\tilde{N} \left( x_{12} \beta_{\dot{\beta}} x_{23} \gamma_{\dot{\gamma}} x_{31} \gamma_{\dot{\alpha}} + x_{13} \gamma_{\dot{\alpha}} x_{32} \gamma_{\dot{\beta}} x_{21} \beta_{\dot{\alpha}} \right) \frac{1}{r_{12} r_{13} r_{23}} \bar{\varepsilon}_{r s u v e t w} \]  

(3.18b)

To achieve this form requires the use of the identity

\[ \left( x_{12} \beta_{\dot{\beta}} x_{23} \gamma_{\dot{\gamma}} x_{31} \gamma_{\dot{\alpha}} - x_{13} \gamma_{\dot{\alpha}} x_{32} \gamma_{\dot{\beta}} x_{21} \beta_{\dot{\alpha}} \right) \frac{1}{r_{12} r_{13} r_{23}} \]

\[ = \frac{1}{r_{12}} x_{12} \beta_{\dot{\beta}} x_{21} \beta_{\dot{\alpha}} x_{3[12]} \gamma_{\dot{\gamma}} + \frac{1}{r_{13}} x_{13} \gamma_{\dot{\alpha}} x_{31} \gamma_{\dot{\alpha}} x_{2[31]} (\beta \dot{\beta} + \frac{1}{r_{23}} x_{23} \gamma_{\dot{\gamma}} x_{32} \gamma_{\dot{\beta}} x_{1[23]} \alpha \dot{\alpha} \right) \]

\[ + X_{1[23]} \alpha \dot{\alpha} x_{2[31]} (\beta \dot{\beta} x_{3[12]} \gamma_{\dot{\gamma}} \right) \]

(3.19)

Using also \( \delta \langle T_{\alpha\dot{\beta}\dot{\alpha}}(x_1) \varphi_{rs}(x_2) \bar{\psi}_{u\dot{\gamma}}(x_3) \rangle = 0 \) we obtain

\[ \langle T_{\alpha\dot{\beta}\dot{\alpha}}(x_1) \psi_{r\gamma}(x_2) \bar{\psi}_{s\dot{\gamma}}(x_3) \rangle = -\frac{8}{3} \tilde{N} i \left( \frac{1}{r_{23}} X_{1[23]} (\alpha \dot{\alpha} x_{1[23]} \beta \dot{\beta} x_{23} \gamma_{\dot{\gamma}} - \frac{3}{r_{12}} X_{1[23]} (\alpha \dot{\alpha} x_{13} \beta \dot{\beta} x_{21} \gamma_{\dot{\gamma}} \right) \]

\[ \times \frac{1}{r_{12} r_{13} r_{23}} \left( \delta_{r s} 1 + \frac{1}{6} \gamma_{r \gamma_{s}} \right) \]  

(3.20)
For $\sum \Delta_i = 10$, using now $\delta \langle T_{\alpha\beta\dot{\alpha}\dot{\beta}}(x_1) J_{rs\gamma\dot{\gamma}}(x_2) \bar{\psi}_{uv\delta}(x_3) \rangle = 0$, we obtain from (3.18d) and (3.20)

$$\langle T_{\alpha\beta\dot{\alpha}\dot{\beta}}(x_1) J_{rs\gamma\dot{\gamma}}(x_2) J_{uv\delta\dot{\delta}}(x_3) \rangle$$

$$= -\frac{16}{3} \hat{N} \frac{1}{r_{12} r_{13} r_{23}} \left\{ \frac{2}{r_{13}^2} x_{12}(\alpha^\gamma x_{13}\beta^\delta x_{21}\gamma(\dot{\alpha} x_{31}\dot{\beta})) + X_{1[23]}(\alpha^\dot{\alpha} X_{1[23]}^\beta \beta^\dot{\beta}) \left( 3X_{2[3]}(\gamma^\gamma X_{3[12]} \delta + \frac{1}{r_{23}} x_{23}\gamma^\delta x_{32}\dot{\gamma} \right) \right. \right.$$

$$\left. - \frac{8}{r_{12} r_{13} r_{23}} X_{1[23]}(\alpha^\dot{\alpha} (x_{12}\beta^\dot{\gamma} x_{23}\gamma^\delta x_{31}\dot{\beta} - x_{13}\beta^\dot{\delta} x_{23}\dot{\gamma} x_{21}\gamma^\beta)) \right\} \delta_{uv\delta\dot{\delta}}.$$ 

(3.21)

Further from $\delta \langle \Psi_{\alpha\dot{\alpha}}(x_1) J_{uv\beta\dot{\beta}}(x_2) J_{tw\gamma\dot{\gamma}}(x_3) \rangle = 0$ we may determine

$$\langle \Psi_{\alpha\dot{\alpha}}(x_1) J_{uv\beta\dot{\beta}}(x_2) \bar{\Psi}_{\gamma\dot{\gamma}}(x_3) \rangle$$

$$= \frac{8}{9} \hat{N} \left\{ \frac{1}{r_{12} r_{13} r_{23}} \left( 4 X_{1[23]}(\alpha^\dot{\alpha} X_{2[3]}^\beta \beta^\dot{\beta} x_{3[12]}(\gamma^\gamma + X_{2[3]}(\beta^\gamma x_{13}(\alpha^\dot{\gamma} x_{31}\dot{\alpha}) \frac{1}{r_{13}} x_{21}\alpha^\dot{\alpha} - 4 x_{12}(\beta^\alpha x_{23}(\gamma^\gamma x_{31}\dot{\alpha}) \frac{1}{r_{13}} x_{21}\beta) \right) \right.$$

$$+ \left. \frac{2}{r_{12} r_{13} r_{23}} X_{1[23]}(\alpha^\dot{\alpha} X_{3[12]}(\gamma^\gamma x_{12}\delta) x_{23}\beta) \right\} \gamma[\dot{u}\dot{v}] \dot{v}. \tag{3.22}$$

Considering now $\sum \Delta_i = 11$ we analyse $\delta \langle T_{\alpha\beta\dot{\alpha}\dot{\beta}}(x_1) J_{rs\gamma\dot{\gamma}}(x_2) \bar{\Psi}_{\epsilon\dot{\epsilon}\eta}(x_3) \rangle = 0$ to obtain

$$\langle T_{\alpha\beta\dot{\alpha}\dot{\beta}}(x_1) \Psi_{\gamma\dot{\gamma}}(x_2) \bar{\Psi}_{\epsilon\dot{\epsilon}\eta}(x_3) \rangle$$

$$= -\frac{64}{3} \hat{N} \left\{ \frac{1}{r_{12} r_{13} r_{23}} x_{1[23]}(\alpha^\dot{\alpha} \left( \frac{1}{4} x_{1[23]}^\beta \beta x_{23}(\gamma^\gamma x_{31}\dot{\alpha}) \frac{1}{r_{23}} x_{23}(\gamma^\gamma x_{31}\dot{\alpha}) \right. \right.$$

$$+ X_{2[3]}(\gamma^\gamma x_{13}\beta) \left( x_{31}(\epsilon^\gamma) x_{23}(\gamma^\gamma x_{13}\beta) \frac{1}{r_{13}} \right. \left. + X_{3[12]}(\epsilon^\gamma x_{12}\beta) \frac{1}{r_{12}} X_{21}(\gamma^\beta x_{12}\beta) \frac{1}{r_{12}} \right. \right.$$

$$+ \left. \left( \frac{7}{9} x_{12}(\gamma^\gamma x_{23}(\gamma^\gamma x_{31}\dot{\beta}) - \frac{8}{3} x_{13}(\gamma^\gamma x_{31}\dot{\beta}) x_{21}(\gamma^\beta) \frac{1}{r_{12}} r_{13} r_{23} \right) x_{23}(\gamma^\gamma x_{12}(\gamma^\beta) \frac{1}{r_{12}} r_{13} r_{23} \right) \right.$$

$$+ \left. \frac{1}{r_{12}^2} x_{31}(\gamma^\gamma x_{13}(\gamma^\gamma x_{31}(\epsilon^\gamma) x_{21}(\gamma^\gamma x_{31}(\epsilon^\gamma) x_{21}(\gamma^\beta) \frac{1}{r_{12}} r_{13} r_{23} \right) x_{23}(\gamma^\gamma x_{12}(\gamma^\beta) x_{13}(\gamma^\beta) \frac{1}{r_{12}} r_{13} r_{23} \right) \right.$$

$$+ \left. \left( \frac{7}{9} x_{12}(\gamma^\gamma x_{23}(\gamma^\gamma x_{31}(\epsilon^\gamma) x_{21}(\gamma^\beta) x_{13}(\gamma^\beta) \frac{1}{r_{12}} r_{13} r_{23} \right) x_{23}(\gamma^\gamma x_{12}(\gamma^\beta) x_{13}(\gamma^\beta) \frac{1}{r_{12}} r_{13} r_{23} \right) \right. \right.$$

$$\right\} \right. \right.$$

(3.23)

where the symmetrisations act on pairs of indices at the same point. To achieve the form (3.23) we use the identity (3.19) as well as

$$\begin{aligned}
X_{1[23]}(\alpha^\dot{\alpha} X_{2[3]}(\gamma^\gamma x_{13}\beta) &\frac{1}{r_{13}} - X_{1[23]}(\alpha^\dot{\alpha} x_{12}\beta) x_{23}(\gamma^\gamma x_{12}\beta) \frac{1}{r_{12} r_{23}} \right.
+ X_{12}(\alpha^\gamma x_{21}(\gamma^\beta) x_{13}\beta) \frac{1}{r_{12} r_{13}} = 0,
\end{aligned} \tag{3.24}$$
together with various permutations.

Finally from \(\delta \langle T_{\alpha\beta\dot{\alpha}\dot{\beta}}(x_1) T_{\gamma\delta\dot{\gamma}\dot{\delta}}(x_2) \nabla_{\epsilon\epsilon\eta}(x_3) \rangle = 0\) we determine the energy momentum tensor three point function, with similar conventions on symmetrisations of indices,

\[
\langle T_{\alpha\beta\dot{\alpha}\dot{\beta}}(x_1) T_{\gamma\delta\dot{\gamma}\dot{\delta}}(x_2) T_{\epsilon\epsilon\eta}(x_3) \rangle = \frac{128}{3} \hat{N} \left\{ -3 X_{1[23]}(\alpha\dot{\alpha} X_{1[23]} X_{2[31]}(\gamma\dot{\gamma} X_{2[31]} \dot{\delta} X_{3[12]}(\epsilon\dot{\epsilon} X_{3[12]}) \eta \right.
\]

\[
\left. + X_{1[23]}(\alpha\dot{\alpha} X_{1[23]} X_{2[31]}(\gamma\dot{\gamma} X_{2[31]} \dot{\delta} X_{3[12]}(\epsilon\dot{\epsilon} X_{3[12]}) \eta \frac{1}{r_{23}} \eta \right)
\]

\[
\left. + X_{2[31]}(\gamma\dot{\gamma} X_{2[31]} \dot{\delta} X_{3[12]}(\epsilon\dot{\epsilon} X_{3[12]} \eta \frac{1}{r_{13}} \eta \right)
\]

\[
\left. + X_{3[12]}(\epsilon\dot{\epsilon} X_{3[12]} \eta \frac{1}{r_{12}} \eta \right) \right\}.
\]

This satisfies the necessary symmetry requirements. The same three point function has also been calculated directly using the AdS/CFT correspondence in [24].

As a check on the above results for two and three point functions we have verified in each case the appropriate Ward identities which restrict the operator product expansions involving the SU(4) current and the energy momentum tensor. From [21] we have

\[
x^a J_{rs a}(x) \mathcal{O}(0) \sim i \frac{1}{x^2} T_{rs} \mathcal{O}(0) + \ldots, \quad J_{rs a} = -\frac{1}{2} (\hat{\sigma}_a)^{\dot{\alpha}\dot{\beta}} J_{rs \alpha\dot{\alpha}},
\]

where \(T_{rs} = -T_{sr}\) are the generators of SU(4) acting on \(\mathcal{O}\) obeying

\[
[T_{rs}, T_{uv}] = -\delta_{ru} T_{sv} + \delta_{rv} T_{su} + \delta_{su} T_{rv} - \delta_{sv} T_{ru},
\]

(3.27)

(for a 6-vector \(v_p T_{rs} \to 2\delta_{p[r} \delta_{s]q}\) and for a spinor \(\psi_i T_{rs} \to -\frac{1}{2} (\tilde{\gamma}_{[r} \gamma_{s]} )^i j \) and also

\[
x^a x^b T_{ab}(x) \mathcal{O}(0) \sim -\frac{1}{x^2} \Delta \mathcal{O}(0) + \ldots, \quad T_{ab} = \frac{1}{4} (\hat{\sigma}_a)^{\dot{\alpha}\dot{\beta}} (\hat{\sigma}_b)^{\dot{\alpha}\dot{\beta}} T_{\alpha\beta\dot{\alpha}\dot{\beta}}.
\]

In (3.26) and (3.28) . . . denote terms which vanish on integration over \(d\Omega_\frac{3}{2}\). More generally for the operator product expansion of the energy momentum tensor itself we have

\[
x^a x^b T_{ab}(x) T_{cd}(0) \sim -\frac{1}{x^2} \left( A T_{cd}(0) + B \frac{1}{x^2} x^e (x(c T_d) x(e) - \frac{1}{4} \eta_{cd} x^f T_{ef}(0)) \right.
\]

\[
\left. + C \frac{1}{(x^2)^2} (x_c x_d - \frac{1}{4} \eta_{cd} x^2) x^e x^f T_{ef}(0) \right),
\]

(3.29)
For compatibility with (3.28) we must have
\[ A + \frac{1}{4} B + \frac{1}{12} C = 4. \] (3.30)
The coefficients \( A, B, C \), along with the coefficient \( C_T \) of the energy momentum tensor two point function, determine fully the conformal three point function of the energy momentum tensor. From (3.25) we have, along with \( C_T = 40 \bar{N} \),
\[ A = \frac{106}{45}, \quad B = \frac{268}{45}, \quad C = \frac{28}{15}. \] (3.31)
These satisfy (3.30) and also
\[ A - \frac{7}{4} B + \frac{121}{28} C = 0, \] (3.32)
which is a necessary condition for \( \mathcal{N} = 1 \) superconformal symmetry [25]. Furthermore in terms of the coefficients \( a, c \) of the energy momentum tensor trace on curved space we have, from the results in [21],
\[ \frac{a}{c} = \frac{1}{288} (124A - B + C), \] (3.33)
and (3.31) gives \( a = c \) as expected for \( \mathcal{N} = 4 \) superconformal symmetry.

4. Further Applications to Three Point Functions

Superconformal symmetry leads to further constraints on the correlation functions involving operators belonging to short multiplets. We here discuss the conditions for a three point function for two chiral primary operators \( \varphi_{rs} \), whose superconformal transformation properties were described in section two, and a third operator belonging to a long multiplet.

For simplicity we consider first a self-conjugate scalar operator \( \Phi^I = \bar{\Phi}^I \) where \( I \) is an index for the \( SU(4) \) representation space, the representation having Dynkin labels \([q, p, q]\). Under a superconformal transformation we assume
\[ \delta \Phi^I = \epsilon^\alpha_i \Psi^i_{\alpha} + \bar{\Psi}^{i\dot{\alpha}} \hat{\epsilon}_{i\dot{\alpha}}, \] (4.1)
where in general \( \Psi^i_{\alpha}, \bar{\Psi}^{i\dot{\alpha}} \) transform under reducible \( SU(4) \) representations. To ensure closure of the algebra we take
\[
\delta \epsilon \Psi^I_{\alpha} = i \partial_{\alpha \dot{\alpha}} \Phi^I \hat{\epsilon}^\dot{\alpha} + 2 \Delta \eta_\alpha \Phi^I - \gamma_{[r\bar{\gamma}s]} \eta_\alpha (T_{rs})^I J \Phi^J - \frac{1}{2 \Delta} (T_{rs})^I J i \partial_{\alpha \dot{\alpha}} \Phi^J \gamma_{[r\bar{\gamma}s]} \hat{\epsilon}^\dot{\alpha} + J^I_{\alpha \dot{\alpha}} \hat{\epsilon}^\dot{\alpha}, \tag{4.2a}
\]
\[
\delta \epsilon \bar{\Psi}^I_{\dot{\alpha}} = - \hat{\epsilon}^{\alpha i} i \partial_{\alpha \dot{\alpha}} \Phi^I + 2 \Delta \Phi^I \bar{\eta}_{\dot{\alpha}} + (T_{rs})^I J \Phi^J \bar{\eta}_{\dot{\alpha}} \gamma_{[r\bar{\gamma}s]} + \frac{1}{2 \Delta} \hat{\epsilon}^{\alpha \dot{\alpha}} \gamma_{[r\bar{\gamma}s]} (T_{rs})^I J i \partial_{\alpha \dot{\alpha}} \Phi^J + \hat{\epsilon}^{\alpha \dot{\alpha}} J^I_{\alpha \dot{\alpha}} \tag{4.2b} \]

\footnote{In terms of the coefficients in [21], with the normalisations here, \( C_T A = \frac{1}{4} (A - 3 C) \), \( C_T B = 5 A + \frac{1}{2} B - 6 C \), \( C_T C = - \frac{7}{2} (B - 2 C) \). For free scalars \( A = \frac{7}{2}, B = \frac{88}{9}, C = \frac{28}{9} \), for free spin-\( \frac{1}{2} \) fields \( A = \frac{3}{2}, B = 10, C = 0 \) and for free vectors \( A = 4, B = C = 0 \).}
where $\mathcal{J}^{I}_{j\alpha \dot{\alpha}}$ is a new vector field. From (4.1) and (4.2a, b) we obtain
\begin{equation}
[\delta_{2}, \delta_{1}] \Phi^{I} = -v \cdot \partial \Phi^{I} - \Delta \hat{\lambda} \Phi^{I} + \frac{1}{2} \hat{\epsilon}_{rs} (T_{rs})^{I}_{J} \Phi^{J},
\end{equation}
where $\hat{\lambda}$ and $\hat{\epsilon}_{rs}$ are given by (2.19) and (2.18) and $v$ by (2.8). The result (1.3) is just as required by closure of the $\mathcal{N} = 4$ superconformal algebra for $T_{rs}$ the appropriate generators of $SU(4)$ obeying (3.27). The terms in the second lines of (4.3) do not contribute to the right hand side of (1.3) but are necessary for $\delta_{\delta} \Psi^{I}_{\alpha}$ and $\delta_{\delta} \bar{\Psi}^{I}_{\dot{\alpha}}$ to transform covariantly. To see this we consider the conformal transformations, following (2.1),
\begin{equation}
\delta_{v} \Phi^{I} = -v \cdot \partial \Phi^{I} - \Delta \hat{\lambda} \Phi^{I}, \quad \delta_{v} \bar{\Psi}^{I}_{\dot{\alpha}} = -v \cdot \partial \bar{\Psi}^{I}_{\dot{\alpha}} - (\Delta + \frac{1}{2}) \hat{\lambda} \bar{\Psi}^{I}_{\dot{\alpha}} - \bar{\Psi}^{I}_{\beta} \hat{\omega}^{\beta}_{\dot{\alpha}},
\end{equation}
where $v, \lambda$ and $\hat{\omega}^{\beta}_{\dot{\alpha}}$ are as in (2.2) and (2.4). The additional terms involving $1/\Delta$ are necessary to achieve
\begin{equation}
[\delta_{\delta}, \delta_{v}] \bar{\Psi}^{I}_{\dot{\alpha}} = \delta_{v} \bar{\Psi}^{I}_{\dot{\alpha}},
\end{equation}
where\footnote{To verify (1.5) it is useful to note that $(\partial_{\alpha \dot{\alpha}} v) \cdot \partial = \hat{\omega}^{\beta}_{\alpha} \partial_{\alpha \beta} - \hat{\omega}^{\beta}_{\dot{\alpha}} \partial_{\beta \dot{\alpha}} + \partial \partial_{\alpha \dot{\alpha}}$.}
\begin{equation}
\hat{\epsilon}'^{\alpha} = -v \cdot \partial \hat{\epsilon}^{\alpha} + \frac{1}{2} \hat{\lambda} \hat{\lambda}^{\alpha} - \hat{\lambda} \hat{\omega}^{\beta}_{\alpha},
\end{equation}
which may easily be decomposed as in (2.5) into $\eta'$ and
\begin{equation}
\bar{\gamma}^{I}_{\dot{\alpha}} = -\frac{1}{2} \hat{\lambda} \bar{\gamma}^{I}_{\dot{\alpha}} - \bar{\gamma}^{I}_{\dot{\beta}} \hat{\omega}^{\beta}_{\dot{\alpha}} + i \hat{\epsilon}_{I}^{\alpha} b_{\alpha \dot{\alpha}}.
\end{equation}

It is perhaps appropriate to remark that from (4.2a) it is straightforward to recover the usual conditions for obtaining a short supermultiplet belonging to the C series \footnote{To verify (1.5) it is useful to note that $(\partial_{\alpha \dot{\alpha}} v) \cdot \partial = \hat{\omega}^{\beta}_{\alpha} \partial_{\alpha \beta} - \hat{\omega}^{\beta}_{\dot{\alpha}} \partial_{\beta \dot{\alpha}} + \partial \partial_{\alpha \dot{\alpha}}$.} with a scalar lowest dimension operator (the B series is discussed in appendix B). The relevant requirement is that, for a suitable $\Delta$,
\begin{equation}
2\Delta \eta \Phi^{I} - \gamma_{[r} \bar{\gamma}_{s]} \eta (T_{rs})^{I}_{J} \Phi^{J},
\end{equation}
should not, for arbitrary $\eta^{i}$, span the full representation space corresponding to the direct product $[1, 0, 0] \otimes [q, p, q] = [q+1, p, q] \oplus [q-1, p+1, q] \oplus [q, p-1, q+1] \oplus [q, p, q-1]$, so that there is a non zero cokernel. To illustrate this we may consider the $[0, p, 0]$ representation when we take
\begin{equation}
\Phi^{I} \rightarrow \varphi(u_{1}...u_{p}) = \varphi(u_{1}...u_{p}), \quad \varphi(u_{1}...u_{p-2} uu) = 0.
\end{equation}
Since $(T_{rs})^{I}_{J} \Phi^{J} \rightarrow p(\delta_{r}(u_{2} \varphi_{u_{2}...u_{p}}s) - \delta_{s}(u_{1} \varphi_{u_{2}...r_{p}})r)$ we may obtain for this case
\begin{equation}
\gamma_{[r} \bar{\gamma}_{s]} \eta (T_{rs})^{I}_{J} \Phi^{J} \rightarrow 2p \gamma(u_{1} \varphi_{u_{2}...u_{p}}s) \bar{\gamma}_{s} \eta + 2p \varphi(u_{1}...u_{p}) \eta.
\end{equation}
Choosing \( \Delta = p \) then projects out in (4.8) the \([1, p, 0]\) representation. In (4.2) the terms involving \( \hat{e} \) have an identical form to (4.8) and therefore we may restrict, for \( \Delta = p \) and with suitable conditions also on \( J_{\alpha \tilde{\alpha}} \), \( \Psi^I_\alpha \) to the \( SU(4) [0, p-1, 1] \) representation by letting

\[
\Psi^I_\alpha \rightarrow \Psi^{i_1 \ldots i_q}_{u_1 \ldots u_p \alpha}, \quad \Psi^{i_1 \ldots i_q}_{u_1 \ldots u_p \alpha} = \frac{1}{4p} \gamma u \tilde{\gamma} (u_1 \Psi^{i_2 \ldots i_q}_{u_2 \ldots u_p} u_\alpha), \quad \Rightarrow \Psi^{i_1 \ldots i_q}_{u_1 \ldots u_p \alpha} = -\gamma (u_1 \Psi^{i_2 \ldots i_q}_{u_2 \ldots u_p} u_\alpha), \quad \gamma u \Psi^{i_1 \ldots i_q}_{u_1 \ldots u_p \alpha} = 0. \tag{4.11}
\]

For the \([q, 0, q]\) representation then similarly

\[
\Phi^I \rightarrow \hat{\varphi}^{i_1 \ldots i_q}_{j_1 \ldots j_q} = \varphi^{i_1 \ldots i_q}_{j_1 \ldots j_q}, \quad \hat{\varphi}^{i_1 \ldots i_q}_{j_1 \ldots j_q} = 0. \tag{4.12}
\]

For this case \((T_{rs})^I_{j} \hat{\Phi}^J \rightarrow -\frac{1}{2} q (\gamma [r \tilde{\gamma} s])^{i_1 j_{1 \ldots j_q}} + \frac{1}{2} q \varphi^{i_1 \ldots i_q}_{j_1 \ldots j_q} (\gamma [r \tilde{\gamma} s])^{j_{1 \ldots j_q}} \gamma [r \tilde{\gamma} s])^{i_{1 \ldots i_q}}_{j_{1 \ldots j_q}} = -8\delta^{i_1 j_1} \delta^{i_q j_q} + 2\delta^{i_1 j_q} \delta^{i_q j_1} \), we have

\[
(\gamma [r \tilde{\gamma} s])^{i_1 j_{1 \ldots j_q}} (T_{rs})^I_{j} \hat{\Phi}^J \rightarrow 4q (\varphi^{i_1 \ldots i_q}_{j_1 \ldots j_q} \eta_{i_q}) - \delta^{i_1 j_1} \delta^{i_q j_q} \eta_{i_q}). \tag{4.13}
\]

It is then easy to see that in (4.8) choosing \( \Delta = 2q \) removes the components appropriate to the \([q+1, 0, q]\) representation. In this case \( \Psi^I_\alpha \) is restricted to belong to the \([q-1, 1, q]\) and \([q, 0, q-1]\) representations,

\[
\Psi^I_\alpha \rightarrow \Psi^{i_1 \ldots i_q}_{j_1 \ldots j_q \alpha}, \quad \Psi^{i_1 \ldots i_q}_{j_1 \ldots j_q \alpha} = 0. \tag{4.14}
\]

The general case follows by a combination of the above two examples giving \( \Delta = p + 2q \) for a short supermultiplet for lowest weight scalar field belonging to the self-conjugate \([q, p, q]\) representation.

To illustrate how further constraints, beyond those arising from conformal invariance arise, we consider then the three point function

\[
\langle \varphi_{rs}(x_1) \varphi_{uv}(x_2) \Phi^I(x_3) \rangle = \frac{1}{r_{12}^2} \left( \frac{r_{12}}{r_{13} r_{23}} \right)^{\frac{1}{2} \Delta} D^I_{rs,uv}, \tag{4.15}
\]

for \( D^I_{rs,uv} \) a \( SU(4) \) invariant tensor. The symmetry condition \( D^I_{rs,uv} = D^I_{uv,rs} \) here implies that \( \Phi^I \) is restricted to belong to the singlet, 20, 84 or 105 dimensional representations. The conditions for superconformal invariance follow in a similar fashion to the previous section. We first consider \( \delta \langle \psi_{r\alpha}(x_1) \varphi_{uv}(x_2) \Phi^I(x_3) \rangle = 0 \) using (2.16) and (4.11). For \( \Delta \neq 2 \) the analysis is simplified by taking account of the requirement that \( \langle J_{rs\alpha \tilde{\alpha}}(x_1) \varphi_{uv}(x_2) \Phi^I(x_3) \rangle = 0 \) in order to comply with the conservation equation \( \partial_{\alpha} J_{rs\alpha \tilde{\alpha}} = 0 \). Using this we then get

\[
\langle \psi_{r\alpha}(x_1) \bar{\psi}_{u\tilde{\alpha}}(x_2) \Phi^I(x_3) \rangle \tilde{\gamma} v = (4 - \Delta) \frac{i \chi_{12\alpha \tilde{\alpha}}}{r_{12}^3 - \frac{3}{2} \Delta r_{13}^2 + \frac{1}{2} \Delta r_{23}^2} D^I_{rs,uv} \tilde{\gamma}_s, \tag{4.16}
\]

\[
\langle \psi_{r\alpha}(x_1) \varphi_{uv}(x_2) \Phi^I_{\tilde{\alpha}}(x_3) \rangle = \Delta \frac{i \chi_{13\alpha \tilde{\alpha}}}{r_{12}^2 - \frac{3}{2} \Delta r_{13}^2 + \frac{1}{2} \Delta r_{23}^2} D^I_{rs,uv} \tilde{\gamma}_s.
\]
The critical conditions follow from \( \delta \langle \varphi_{rs}(x_1) \varphi_{uv}(x_2) \bar{\Psi}^I_{i\alpha}(x_3) \rangle = 0 \). Using (4.21) we may obtain

\[
\langle \varphi_{rs}(x_1) \varphi_{uv}(x_2) \delta \bar{\Psi}^I_{i\alpha}(x_3) \rangle = \hat{\epsilon}^\alpha(x_3) \langle \varphi_{rs}(x_1) \varphi_{uv}(x_2) \mathcal{J}^I_{\alpha\dot{\alpha}}(x_3) \rangle
\]

\[
- \frac{i}{r_{12}^{2-\frac{1}{2}\Delta}} \frac{i}{r_{13}^{1+\frac{1}{2}\Delta}} \left( \hat{\epsilon}^\alpha(x_1) x_{13\alpha\dot{\alpha}} \frac{1}{r_{13}} + \hat{\epsilon}^\alpha(x_2) x_{23\alpha\dot{\alpha}} \frac{1}{r_{23}} \right) (\Delta D^I_{rs,uv} + \frac{1}{2}(T_{tw})^I_J D^J_{rs,uv})
\]

(4.17)

Since \( D^I_{rs,uv} \) is a SU(4) invariant we have

\[
(T_{tw})^I_J D^J_{rs,uv} = -2\delta_{v[t} D^I_{uv]}_{w,s} - 2\delta_{s[t} D^I_{w,v]}_{r,u} + 2D^I_{rs,u} [\delta_{w} v] + 2D^I_{rs,v} [\delta_{u} w],
\]

(4.18)

so that, with the aid of (4.14), applying (4.17) in the superconformal invariance condition leads to

\[
\hat{\epsilon}^\alpha(x_3) \langle \varphi_{rs}(x_1) \varphi_{uv}(x_2) \mathcal{J}^I_{\alpha\dot{\alpha}}(x_3) \rangle
\]

\[
= \frac{i}{r_{12}^{2-\frac{1}{2}\Delta}} \frac{i}{r_{13}^{1+\frac{1}{2}\Delta}} \left( \hat{\epsilon}^\alpha(x_1) x_{13\alpha\dot{\alpha}} \frac{1}{r_{13}} ((\Delta - 2)X^I_{rs,uv} - 2Y^I_{rs,uv}) \right)
\]

\[+ \hat{\epsilon}^\alpha(x_2) x_{23\alpha\dot{\alpha}} \frac{1}{r_{23}} ((\Delta - 2)Y^I_{rs,uv} - 2X^I_{rs,uv}) \right),
\]

(4.19)

for

\[
X^I_{rs,uv} = \frac{1}{2} \gamma_{[r} \tilde{\gamma}_{t]} D^I_{ts,uv} + \frac{1}{2} \gamma_{[s} \tilde{\gamma}_{t]} D^I_{tr,uv}, \quad Y^I_{rs,uv} = \frac{1}{2} \gamma_{[u} \tilde{\gamma}_{v]} D^I_{rs,uv} + \frac{1}{2} \gamma_{[v} \tilde{\gamma}_{t]} D^I_{rs,ts}.
\]

(4.20)

For superconformal invariance the right hand side of (4.19) must involve \( \hat{\epsilon} \) just in the form \( \hat{\epsilon}^\alpha(x_3) X_{[12]}^I_{\alpha\dot{\alpha}} \) and by virtue of (3.12) this requires

\[
(\Delta - 4)X^I_{rs,uv} = -(\Delta - 4)Y^I_{rs,uv}.
\]

(4.21)

This is trivially satisfied if \( \Delta = 4 \) which includes the cases of short multiplets belonging to the 105 and 84 representations and also the for \( \Phi^I \) belonging to the 20 dimensional representation for which there is no shortening for this \( \Delta \). For \( \Phi^I \rightarrow \Phi \), a SU(4) singlet, the conditions arising from (4.19) are satisfied for any scale dimension \( \Delta \) since then \( X_{rs,uv} = -Y_{rs,uv} \) (for the singlet case the right side of (4.19) just involves \( \Delta X_{rs,uv} \) and \( \Delta Y_{rs,uv} \)).

We now extend this discussion to the case of a self-conjugate operator of spin \( \ell \), \( \Phi^I_{\alpha_1...\alpha_{\ell} \dot{\alpha}_1...\dot{\alpha}_{\ell}} = \Phi^I_{(\alpha_1...\alpha_{\ell}), (\dot{\alpha}_1...\dot{\alpha}_{\ell})} \). In this case (4.1) becomes

\[
\delta \Phi^I_{\alpha_1...\alpha_{\ell} \dot{\alpha}_1...\dot{\alpha}_{\ell}} = \hat{\epsilon}^\beta_{\dot{\epsilon}} \Psi^I_{\beta \alpha_1...\alpha_{\ell} \dot{\alpha}_1...\dot{\alpha}_{\ell} \dot{\alpha}_1...\dot{\alpha}_{\ell}} + \bar{\Psi}^I_{i\alpha_1...\alpha_{\ell} \dot{\alpha}_1...\dot{\alpha}_{\ell} \dot{\alpha}_1...\dot{\alpha}_{\ell}} \hat{\epsilon}^i_{\dot{\epsilon}}.
\]

(4.22)
Corresponding to (1.26) we now have

\[ \delta_\ell \hat{\Psi}^I_{\alpha_1...\alpha_\ell,\dot{\alpha}_1...\dot{\alpha}_\ell} = -\varepsilon^2 \Phi^I_{\alpha_1...\alpha_\ell,\dot{\alpha}_1...\dot{\alpha}_\ell} + 2(\Delta - \ell) \Phi^I_{\alpha_1...\alpha_\ell,\dot{\alpha}_1...\dot{\alpha}_\ell} \eta_{\beta} + 4\ell \Phi^I_{\alpha_1...\alpha_\ell,\beta}(\dot{\alpha}_1...\dot{\alpha}_{\ell-1}\eta_{\dot{\alpha}_\ell}) \\
+ \frac{\ell}{\Delta - 1} \varepsilon^2 \left( \partial_{(\alpha_1 \beta} \Phi^I_{\beta)(\alpha_2...\alpha_\ell),\dot{\alpha}_1...\dot{\alpha}_\ell} - \partial_{\beta}(\dot{\alpha}_1 \Phi^I_{\alpha_1...\alpha_\ell,\dot{\alpha}_2...\dot{\alpha}_\ell}) \right) \\
+ (T_{rs})^I_J \Phi^J_{\alpha_1...\alpha_\ell,\dot{\alpha}_1...\dot{\alpha}_\ell} \eta_{\beta} \gamma_{[r\tilde{s}]} \\
- \varepsilon^2 \hat{\gamma}_{[r\tilde{s}]} (T_{rs})^I_J \hat{\Phi}^J_{\dot{\alpha}_1...\dot{\alpha}_\ell} + \ell \hat{\omega}_{(\alpha_1 \beta} \Phi^I_{\beta)(\alpha_2...\alpha_\ell),\dot{\alpha}_1...\dot{\alpha}_\ell} + \ell \Phi^I_{\alpha_1...\alpha_\ell,\beta}(\dot{\alpha}_1...\dot{\alpha}_{\ell-1}\hat{\omega}_{\dot{\alpha}_\ell}) \right) \\
+ \hat{\beta}^J_{\beta\alpha_1...\alpha_\ell,\dot{\alpha}_1...\dot{\alpha}_\ell} \right), \tag{4.23} \]

while the equivalent version of (4.24) is given by its conjugate. The structure of (4.23) is determined as before by the requirement of closure of the algebra

\[ [\delta_2, \delta_1] \Phi^I_{\alpha_1...\alpha_\ell,\dot{\alpha}_1...\dot{\alpha}_\ell} = -(\nu \partial + \Delta \hat{\lambda}) \Phi^I_{\alpha_1...\alpha_\ell,\dot{\alpha}_1...\dot{\alpha}_\ell} + \frac{1}{2} \hat{\omega}_{rs} (T_{rs})^I_J \Phi^J_{\alpha_1...\alpha_\ell,\dot{\alpha}_1...\dot{\alpha}_\ell} \right), \tag{4.24} \]

and also that \( \delta_\ell \hat{\Psi}^I_{\alpha_1...\alpha_\ell,\dot{\alpha}_1...\dot{\alpha}_\ell} \) has the correct form under conformal transformations which requires

\[ [\delta_\ell, \delta_v] \hat{\Psi}^I_{\alpha_1...\alpha_\ell,\dot{\alpha}_1...\dot{\alpha}_\ell} = \delta_v \hat{\Psi}^I_{\alpha_1...\alpha_\ell,\dot{\alpha}_1...\dot{\alpha}_\ell}, \tag{4.25} \]

where, with notation as in (2.2) and (2.4),

\[ \delta_v \Phi^I_{\alpha_1...\alpha_\ell,\dot{\alpha}_1...\dot{\alpha}_\ell} = -(\nu \partial + \Delta \hat{\lambda}) \Phi^I_{\alpha_1...\alpha_\ell,\dot{\alpha}_1...\dot{\alpha}_\ell} + \ell \hat{\omega}_{(\alpha_1 \beta} \Phi^I_{\beta)(\alpha_2...\alpha_\ell),\dot{\alpha}_1...\dot{\alpha}_\ell} - \ell \Phi^I_{\alpha_1...\alpha_\ell,\beta}(\dot{\alpha}_1...\dot{\alpha}_{\ell-1}\hat{\omega}_{\dot{\alpha}_\ell}) \right) \right), \tag{4.26} \]

The last requirement determines the coefficients \( c, d, e \) in (4.24) through the relations

\[ (\Delta + \ell - 2)e + 2\ell d = 0, \quad e + (\Delta - 1)d + \ell c = 0, \quad 4d + 2(\Delta - \ell)c = 1. \tag{4.27} \]

To apply these results in the context relevant for this paper we extend (4.13) to

\[ \langle \varphi_{rs}(x_1) \varphi_{uv}(x_2) \Phi^I_{\alpha_1...\alpha_\ell,\dot{\alpha}_1...\dot{\alpha}_\ell} \rangle = \frac{1}{r_{12}^2} \left( \frac{r_{12}}{r_{13} r_{23}} \right)^{2(\Delta - \ell)} X_3[12](\alpha_1 \dot{\alpha}_1 \cdots X_3[12] \dot{\alpha}_\ell) D^I_{rs, uv}, \tag{4.28} \]
which is uniquely determined by conformal invariance up the $SU(4)$ invariant $D^I_{rs,uv}$ satisfying

$$D^I_{rs,uv} = (-1)^{\ell} D^I_{uv,rs}.$$  

(4.29)

With the aid of (4.23) and (4.27) we may then calculate

$$\langle \varphi_{rs}(x_1) \varphi_{uv}(x_2) \delta \tilde{\Psi}^I_{\alpha_1 \ldots \alpha_\ell, \dot{\alpha}_1 \ldots \dot{\alpha}_\ell}(x_3) \rangle = \hat{\epsilon}^\beta(x_3) \langle \varphi_{rs}(x_1) \varphi_{uv}(x_2) \mathcal{J}^I_{\beta \alpha_1 \ldots \alpha_\ell, \dot{\alpha}_1 \ldots \dot{\alpha}_\ell}(x_3) \rangle$$

$$- \frac{i}{r_{12}^2} \left( \frac{r_{12}}{r_{13}^2 r_{23}} \right)^{\frac{1}{2}} (\Delta - \ell) \left\{ \left( \hat{\epsilon}^\beta(x_1) x_{13} x_{13} \hat{\epsilon}^\beta(x_2) x_{23} x_{23} \hat{\epsilon}^\beta(x_3) \right) X_{3[12]}(\alpha_1 \alpha_1 \ldots X_{3[12]} \alpha_\ell \dot{\alpha}_\ell) \right.$$  

$$\times \left( (\Delta - \ell) D^I_{rs,uv} + \frac{1}{2} (T_{tu})^I_{,J} D^J_{rs,uv} \gamma_{[t \bar{\gamma} u]} \right)$$

$$+ 2\ell \left( \hat{\epsilon}^\beta(x_1) x_{13} x_{13} \hat{\epsilon}^\beta(x_2) x_{23} x_{23} \hat{\epsilon}^\beta(x_3) \right) X_{3[12]}(\alpha_1 \beta \ldots X_{3[12]} \alpha_\ell \dot{\alpha}_\ell) D^I_{rs,uv} \right\}. \tag{4.30}$$

Assuming

$$\langle \psi_{r,\beta}(x_1) \varphi_{uv}(x_2) \hat{\Psi}^I_{\alpha_1 \ldots \alpha_\ell, \dot{\alpha}_1 \ldots \dot{\alpha}_\ell}(x_3) \rangle$$

$$= - \frac{i}{r_{12}^2} \left( \frac{r_{12}}{r_{13}^2 r_{23}} \right)^{\frac{1}{2}} (\Delta - \ell) \left( x_{13} x_{13} x_{3[12]}(\alpha_1 \alpha_1 \ldots X_{3[12]} \alpha_\ell \dot{\alpha}_\ell) P^I_{rs,uv} \right.$$  

$$+ x_{13} x_{13} x_{3[12]}(\alpha_1 \beta \ldots X_{3[12]} \alpha_\ell \dot{\alpha}_\ell) Q^I_{rs,uv} \right) \right), \tag{4.31}$$

where

$$\gamma_r P^I_{rs,uv} = 0, \quad \gamma_r Q^I_{rs,uv} = 0 \tag{4.32}$$

then since conformal invariance requires $\langle \varphi_{rs}(x_1) \varphi_{uv}(x_2) \mathcal{J}^I_{\beta \alpha_1 \ldots \alpha_\ell, \dot{\alpha}_1 \ldots \dot{\alpha}_\ell}(x_3) \rangle$ to be expressed in terms of $X_{3[12]} \beta \beta X_{3[12]}(\alpha_1 \alpha_1 \ldots X_{3[12]} \alpha_\ell \dot{\alpha}_\ell)$ and $X_{3[12]} \beta \dot{\alpha} X_{3[12]}(\alpha_1 \beta \ldots X_{3[12]} \alpha_\ell \dot{\alpha}_\ell)$ we may decompose, using (3.12), $\delta \langle \varphi_{rs}(x_1) \varphi_{uv}(x_2) \hat{\Psi}^I_{\alpha_1 \ldots \alpha_\ell, \dot{\alpha}_1 \ldots \dot{\alpha}_\ell}(x_3) \rangle = 0$ into the following conditions

$$2(\Delta - \ell) D^I_{rs,uv} + (T_{tu})^I_{J} D^J_{rs,uv} \gamma_{[t \bar{\gamma} u]} = \gamma_r P^I_{s,uv} + (-1)^{\ell} \gamma_{(u} P^I_{v),rs}, \tag{4.33a}$$

$$4\ell D^I_{rs,uv} = \gamma_r Q^I_{s,uv} + (-1)^{\ell} \gamma_{(u} Q^I_{v),rs}. \tag{4.33b}$$

In general from (1.18) and (4.20) we may write,

$$(T_{tu})^I_{J} D^J_{rs,uv} = - 4(X^I_{rs,uv} + Y^I_{rs,uv})$$

$$D^I_{rs,uv} = - \gamma_r D^I_{s,t,uv} \bar{\gamma}_t + X^I_{rs,uv} = - \gamma_{(u} D^I_{rs,v)} \bar{\gamma}_t + Y^I_{rs,uv}. \tag{4.34}$$

Using this (4.33a) may be simplified to

$$2(\Delta - \ell) D^I_{rs,uv} = \gamma_r \bar{P}^I_{s,uv} + (-1)^{\ell} \gamma_{(u} \bar{P}^I_{v),rs}, \quad \bar{P}^I_{r,uv} = P^I_{r,uv} + 4D^I_{rs,uv} \bar{\gamma}_s. \tag{4.35}$$
For further constraints we consider also \( \delta \langle \psi_{r\beta}(x_1) \varphi_{uv}(x_2) \Phi^{I}_{\alpha_1...\alpha_\ell,\dot{\alpha}_1...\dot{\alpha}_\ell} \rangle = 0 \). Using (4.28) and (4.31) this may be decomposed into terms involving \( \hat{\epsilon}(x_2) \) which give

\[
\left\langle \psi_{r\beta}(x_1) \bar{\psi}_{u\dot{\beta}}(x_2) \Phi^{I}_{\alpha_1...\alpha_\ell,\dot{\alpha}_1...\dot{\alpha}_\ell} \gamma_v \rightangle = \frac{i}{r_{12}^2 r_{13}} \frac{1}{r_{12} r_{23}} \frac{1}{(\Delta - \ell)} \begin{cases} 
\frac{1}{r_{12}} x_{123}\beta X_{3[12]i \alpha_1} \cdots X_{3[12]\alpha_\ell} \tilde{P}^{I}_{r,uv} \\
- \frac{1}{r_{13} r_{23}} x_{13}\beta x_{32}(\alpha_1\beta X_{3[12]}(\alpha_2\dot{\alpha}_2 \cdots X_{3[12]}(\alpha_\ell) \dot{\alpha}_\ell) Q^{I}_{r,uv} \end{cases},
\]

and also if

\[
\left\langle J^I_{rs\beta\dot{\beta}}(x_1) \varphi_{uv}(x_2) \Phi^{I}_{\alpha_1...\alpha_\ell,\dot{\alpha}_1...\dot{\alpha}_\ell} \right\rangle = \frac{i}{r_{12}^2} \frac{1}{r_{13} r_{23}} \frac{1}{(\Delta - \ell)} \begin{cases} 
X_{1[23]}\beta X_{3[12]}(\alpha_1\dot{\alpha}_1 \cdots X_{3[12]}(\alpha_\ell) \dot{\alpha}_\ell) J^{I}_{rs,uv} \\
+ \frac{1}{r_{12}} x_{13}\beta x_{31}(\alpha_1\beta X_{3[12]}(\alpha_2\dot{\alpha}_2 \cdots X_{3[12]}(\alpha_\ell) \dot{\alpha}_\ell) K^{I}_{rs,uv} \end{cases},
\]

then from terms involving \( \tilde{\epsilon}(x_1) \)

\[
J^{I}_{rs,uv} \tilde{\gamma}_s + \frac{1}{6} J^{I}_{st,uv} \tilde{\gamma}_r \gamma_s \tilde{\gamma}_t = - (\Delta - \ell) D^{I}_{rs,uv} \tilde{\gamma}_s - P^{I}_{r,uv},
\]

\[
K^{I}_{rs,uv} \tilde{\gamma}_s + \frac{1}{6} K^{I}_{st,uv} \tilde{\gamma}_r \gamma_s \tilde{\gamma}_t = 2 \ell D^{I}_{rs,uv} \tilde{\gamma}_s + Q^{I}_{r,uv}.
\]

An additional condition is provided by the conservation of the current \( J^I_{rs\beta\dot{\beta}} \) and in (4.37) this requires

\[
2(\Delta - 2) J^{I}_{rs,uv} - (\Delta - \ell - 4) K^{I}_{rs,uv} = 0.
\]

If we combine (4.33d), (4.35), (4.38) and (4.39) we get

\[
(\Delta + \ell - 2)(\Delta - \ell - 4)(D^{I}_{rs,uv} + \gamma_r D^{I}_{s,uv}) \tilde{\gamma}_t + (-1)^\ell \gamma_s D^{I}_{r,uv}) \tilde{\gamma}_t = 0.
\]

For general \( \Delta \) this requires, from (4.20),

\[
X^{I}_{rs,uv} + Y^{I}_{rs,uv} = 0,
\]

which, by virtue of (4.34), is only possible for a singlet operator. For this case, disregarding an overall constant factor, we have

\[
D^{I}_{rs,uv} \rightarrow D_{rs,uv} = \delta_{r(u)} \delta_{v}s - \frac{1}{6} \delta_{r}s \delta_{uv},
\]

and then (4.33a, b) is easily solved for any \( \Delta \) and even \( \ell \) by taking

\[
P_{r,uv} = - (\Delta - \ell) D_{rs,uv} \tilde{\gamma}_s = - (\Delta - \ell) \left( \delta_{r(u)} + \frac{1}{6} \tilde{\gamma}_r \gamma(u) \tilde{\gamma}_v \right), \quad Q_{r,uv} = - 2 \ell D_{rs,uv} \tilde{\gamma}_s.
\]
This is compatible with (4.36) and (4.38) ensures that \( J^I_{rs,uv} = K^I_{rs,uv} = 0 \) so that the current three-point function (4.37) is zero as expected by \( SU(4) \) symmetry. Apart from (4.42) and (4.43) it is also easy to see that for \( J^I_{rs,uv} = K^I_{rs,uv} = 0 \) the only alternative solution, without any constraint on \( D^I_{rs,uv} \), is for \( \Delta = 4, \ell = 0 \) as was found earlier (it is easy to derive in this case \( \ell(X^I_{rs,uv} + Y^I_{rs,uv}) = (\Delta - 4)(X^I_{rs,uv} + Y^I_{rs,uv}) = 0) \).

To find explicit results for non-singlet operators we introduce

\[
C^I_{rs} = (-1)^\ell C^I_{sr}, \quad C^I_{rr} = 0, \tag{4.44}
\]

and then define

\[
D^I_{rs,uv} = \delta_{(r(u C^I_{s})v)} - \frac{1}{6} \delta_{rs} C^I_{(uv)} - \frac{1}{6} \delta_{uv} C^I_{(rs)}, \tag{4.45}
\]

satisfying (4.29). For \( \ell \) even this represents an operator in the 20-representation while \( \ell \) odd corresponds to the 15-representation. If we let

\[
G^I_{r,u} = C^I_{ru} + \frac{1}{6} \gamma_r \gamma_t C^I_{tu} + \frac{1}{6} C^I_{rt} \gamma_r \gamma_u + \frac{1}{36} \gamma_r \gamma_t C^I_{tw} \gamma_w \gamma_u, \tag{4.46}
\]

then

\[
\gamma_r G^I_{(s),(u \bar{\gamma} v)} + (-1)^\ell \gamma_r G^I_{(u \bar{\gamma} v)} = -2D^I_{rs,uv}. \tag{4.47}
\]

Hence (4.33) with (4.34) is solved by taking

\[
\bar{P}^I_{r,uv} = -(\Delta - \ell - 4)G^I_{r,(u \bar{\gamma} v)} \quad \text{and} \quad Q^I_{r,uv} = -2\ell G^I_{r,(u \bar{\gamma} v)}, \tag{4.48}
\]

in accord with (4.32) and (4.36). Now letting

\[
J^I_{rs,uv} = \delta_{[r(u C^I_{s})v]} + \frac{1}{6} \delta_{uv} C^I_{[rs]}, \tag{4.49}
\]

then

\[
J^I_{rs,uv} \gamma_s + \frac{1}{6} J^I_{st,uv} \gamma_r \gamma_s \gamma_t = D^I_{rs,uv} \gamma_s - G^I_{r,(u \bar{\gamma} v)}, \tag{4.50}
\]

and using (4.48) in (4.38) gives, with the required symmetries on \( rs \) and \( uv \),

\[
J^I_{rs,uv} = -(\Delta - \ell - 4)J^I_{rs,uv} \quad \text{and} \quad K^I_{rs,uv} = 2\ell J^I_{rs,uv}. \tag{4.51}
\]

Substituting this into (4.39) gives finally

\[
\Delta = 4 + \ell, \quad \Delta = 2 - \ell. \tag{4.52}
\]

The second solution is only relevant for \( \ell = 0 \) when \( \Phi^I \) may be identified with the chiral primary operator \( \varphi^I \). In general therefore non-singlet 15 or 20-representation operators can contribute to the three-point function only for special values of \( \Delta \), in agreement with [5].
5. Four Point Functions

The primary interest in this paper is to consider the constraints arising from superconformal symmetry on four point functions for superfields belonging to short representations. For four points there are two invariants under the conformal group which we here take as

\[ u = \frac{r_{12} r_{34}}{r_{13} r_{24}}, \quad v = \frac{r_{14} r_{23}}{r_{13} r_{24}}. \]  

(5.1)

In general the four point function depends on arbitrary functions of \( u \) and \( v \) which may be analysed using the operator product expansion in terms of operators with appropriate spins and dimensions. As will be shown superconformal invariance provides further conditions which lead to differential constraints.

For simplicity we consider the \( N = 1 \) case first although there are no new constraints in this case. For chiral fields we consider then

\[ \langle \varphi(x_1) \overline{\varphi}(x_2) \varphi(x_3) \overline{\varphi}(x_4) \rangle = \frac{1}{r_{12} r_{34}} a(u, v). \]  

(5.2)

Applying the superconformal condition \( \delta\langle \psi_{\alpha}(x_1) \overline{\varphi}(x_2) \varphi(x_3) \overline{\varphi}(x_4) \rangle = 0 \) using \( (2.6) \) and its conjugate then leads to

\[
\begin{align*}
\langle \psi_{\alpha}(x_1) \overline{\psi}_{\dot{\alpha}}(x_2) \varphi(x_3) \overline{\varphi}(x_4) \rangle &+ \langle \psi_{\alpha}(x_1) \overline{\psi}_{\dot{\alpha}}(x_4) \varphi(x_3) \overline{\varphi}(x_2) \rangle \hat{\epsilon}^\alpha(x_2) \\
&= 4i \frac{1}{r_{12} r_{34}} \left( (qa - u \partial_u a) \frac{1}{r_{12}} x_{12\alpha\dot{\alpha}} \hat{\epsilon}^\alpha(x_2) - v \partial_v a \frac{1}{r_{14}} x_{14\alpha\dot{\alpha}} \hat{\epsilon}^\alpha(x_4) \right) \\
&\quad + (u \partial_u + v \partial_v) a \frac{1}{r_{13}} x_{13\alpha\dot{\alpha}} \hat{\epsilon}^\alpha(x_3). \tag{5.3}
\end{align*}
\]

To solve this relation we use

\[ r_{24} x_{13\alpha\dot{\alpha}} \hat{\epsilon}^\alpha(x_3) = (x_{13\bar{x}_{34}x_{42}})_{\alpha\dot{\alpha}} \hat{\epsilon}^\alpha(x_2) + (x_{13\bar{x}_{32}x_{24}})_{\alpha\dot{\alpha}} \hat{\epsilon}^\alpha(x_4), \]  

(5.4)

and hence

\[
\begin{align*}
\langle \psi_{\alpha}(x_1) \overline{\psi}_{\dot{\alpha}}(x_2) \varphi(x_3) \overline{\varphi}(x_4) \rangle &\quad = 4i \frac{1}{r_{12} r_{34}} \left\{ \frac{1}{r_{12}} x_{12\alpha\dot{\alpha}} (qa - u \partial_u a) - \frac{1}{r_{13} r_{24}} x_{13\bar{x}_{34}x_{42}} x_{14\alpha\dot{\alpha}} (u \partial_u + v \partial_v) a \right\}, \\
\langle \psi_{\alpha}(x_1) \overline{\psi}_{\dot{\alpha}}(x_4) \varphi(x_3) \overline{\varphi}(x_2) \rangle &\quad = 4i \frac{1}{r_{12} r_{34}} \left\{ \frac{1}{r_{14}} x_{14\alpha\dot{\alpha}} v \partial_v a - \frac{1}{r_{13} r_{24}} x_{13\bar{x}_{32}x_{24}} x_{14\alpha\dot{\alpha}} (u \partial_u + v \partial_v) a \right\}. \tag{5.5}
\end{align*}
\]

Imposing symmetry under \( x_2 \leftrightarrow x_4 \) in \( (5.2) \) requires \( a(v, u) = (v/u)^q a(u, v) \) and this leads to the corresponding relation between the two expressions in \( (5.3) \). The lack of further
constraints in this case is easily understood in terms of $\mathcal{N} = 1$ superspace. The scalar fields $\varphi(x)$ and $\overline{\varphi}(x)$ are then the lowest components of chiral and anti-chiral superfields $\phi(x_+, \theta)$ and $\phi(x_-, \bar{\theta})$. The four point function $\langle \phi(x_{1+}, \theta_1) \overline{\phi}(x_{2-}, \bar{\theta}_2) \phi(x_{3+}, \theta_3) \overline{\phi}(x_{4-}, \bar{\theta}_4) \rangle$ depends on an unconstrained function of two variables since the invariants $u, v$ in (3.1) may be extended to superconformal invariants respecting the required chirality conditions. As shown in [25] for two points $(x_i, \theta_i, \bar{\theta}_i)$ and $(x_j, \theta_j, \bar{\theta}_j)$ we may define $x_{ij\alpha\dot{\alpha}}$ which is superconformally covariant and depends on $(x_{i+}, \theta_i)$ and $(x_{j-}, \bar{\theta}_j)$. The superconformal invariants are the $\tilde{\phi}(x_{12}, x_{34})$ and $\tilde{v}(x_{12}, x_{34})$.

For $\mathcal{N} = 2$ we consider the four point function

$$\langle \varphi^{i_1 j_1} (x_1) \overline{\varphi}(i_2 j_2)(x_2) \varphi^{i_3 j_3}(x_3) \overline{\varphi}(i_4 j_4)(x_4) \rangle$$

$$= \delta^{(i_1 i_2)} \frac{1}{r_{12} r_{34}} a(u, v) + \delta^{(i_1 i_4)} \frac{1}{r_{14} r_{23}} b(u, v)$$

$$+ \delta^{(i_2 i_4)} \frac{1}{r_{12} r_{23} r_{34} r_{14}} c(u, v), \quad (5.6)$$

and show how the present discussion leads to the conditions of Eden et al [3,10]. In a similar fashion to (2.11) we may obtain using (2.11)

$$\langle \psi^{i_1 \alpha} (x_1) \overline{\psi}_{i_2 j_2}(x_2) \varphi^{i_3 j_3}(x_3) \overline{\varphi}(i_4 j_4)(x_4) \rangle$$

$$= \frac{1}{r_{12} r_{34}} \left( V_{i_2 j_2 i_4 j_4} \frac{1}{r_{12}} x_{12} \alpha \hat{\beta} k \hat{\alpha}(x_2) + V_{i_2 j_2 i_4 j_4} \frac{1}{r_{14}} x_{14} \alpha \hat{\beta} k \hat{\alpha}(x_4) \right) \quad (5.7)$$

where the right hand side is determined from (5.6) giving

$$V_{i_2 j_2 i_4 j_4}^{i_1 k i_3 j_3} = \delta^{(i_1 i_2)} \frac{1}{r_{12}} x_{12} \alpha \hat{\beta} k \hat{\alpha}(x_2) + \frac{1}{r_{12}} x_{12} \alpha \hat{\beta} k \hat{\alpha}(x_4)$$

$$V_{i_2 j_2 i_4 j_4}^{i_1 k i_3 j_3} = - \delta^{(i_2 i_4)} \frac{1}{r_{12}} x_{12} \alpha \hat{\beta} k \hat{\alpha}(x_2) + \frac{1}{r_{12}} x_{12} \alpha \hat{\beta} k \hat{\alpha}(x_4)$$

$$W_{i_2 j_2 i_4 j_4}^{i_1 k i_3 j_3} = \delta^{(i_1 i_2)} \frac{1}{r_{12}} x_{12} \alpha \hat{\beta} k \hat{\alpha}(x_2) + \frac{1}{r_{12}} x_{12} \alpha \hat{\beta} k \hat{\alpha}(x_4)$$

$$W_{i_2 j_2 i_4 j_4}^{i_1 k i_3 j_3} = - \delta^{(i_2 i_4)} \frac{1}{r_{12}} x_{12} \alpha \hat{\beta} k \hat{\alpha}(x_2) + \frac{1}{r_{12}} x_{12} \alpha \hat{\beta} k \hat{\alpha}(x_4)$$

(5.8)
On the left hand side (5.7) we may write in general

\[
\langle \psi^i_\alpha(x_1) \bar{\psi}_i_\bar{\alpha}(x_2) \varphi^{i_3;j_3}(x_3) \bar{\varphi}_{i_4;j_4}(x_4) \rangle = 4i \left( \frac{X_{i_2} \bar{\alpha}}{r_{12}^2 r_{34}^2} R_{i_2 i_4;j_4}^{\alpha_3} + \frac{X_{13} \bar{\alpha}_4}{r_{12} r_{13} r_{14} r_{23} r_{24} r_{34}} S_{i_2 i_4;j_4}^{\alpha_3} \right),
\]

(5.9)

This contains just two independent terms as a consequence of the identity

\[
x_{13} \bar{\alpha}_3 x_{42} + x_{14} \bar{\alpha}_4 x_{32} = r_{34} x_{12}. 
\]

(5.10)

The SU(2) dependence may be decomposed as

\[
R_{i_2 i_4;j_4} = \delta^{i_1}_{i_2} \delta(i_3) \delta(j_3) R_1 + \delta^{i_1}_{i_2} \delta(i_3) \delta(j_3) R_2,
\]

\[
S_{i_2 i_4;j_4} = \delta^{i_1}_{i_2} \delta(i_3) \delta(j_3) S_1 + \delta^{i_1}_{i_2} \delta(i_3) \delta(j_3) S_2,
\]

(5.11)

where \( R_1, R_2, S_1, S_2 \) are functions of \( u, v \). A similar expansion to (5.9), letting \( x_2, i_2, j_2 \leftrightarrow x_4, i_4, j_4 \), may also be written for \( \langle \psi^i_\alpha(x_1) \bar{\psi}_i_\bar{\alpha}(x_4) \varphi^{i_3;j_3}(x_3) \bar{\varphi}_{i_4;j_4}(x_2) \rangle \), defining in this case \( R'_1, R'_2, S'_1, S'_2 \). In addition we may write

\[
\langle \psi^i_\alpha(x_1) \bar{\psi}_i_\bar{\alpha}(x_2) \varphi^{i_3;j_3}(x_3) \bar{\varphi}_{i_4;j_4}(x_4) \rangle = 4i \left( \frac{X_{13} \bar{\alpha}_3}{r_{13}^2 r_{24}^2} T_{i_2 i_4;j_4}^{i_1 \alpha} + \frac{X_{12} \bar{\alpha}_4}{r_{12} r_{13} r_{14} r_{23} r_{24} r_{34}} U_{i_2 i_4;j_4}^{i_1 \alpha} \right),
\]

(5.12)

\[
T_{i_2 i_4;j_4}^{i_1 \alpha} = \delta^{i_1}_{i_2} \varepsilon(j_2)(i_4) \varepsilon(j_4) i_3 T_1 + \delta^{i_1}_{i_2} \varepsilon(j_2)(i_4) \varepsilon(j_4) i_3 T_2,
\]

\[
U_{i_2 i_4;j_4}^{i_1 \alpha} = \delta^{i_1}_{i_2} \varepsilon(j_2)(i_4) \varepsilon(j_4) i_3 U_1 + \delta^{i_1}_{i_2} \varepsilon(j_2)(i_4) \varepsilon(j_4) i_3 U_2,
\]

and, with the definition (3.11) noting that \( X_{1[24]} = X_{1[23]} + X_{1[34]} \),

\[
\langle J_{\alpha \bar{\alpha}}(x_1) \bar{\varphi}_{i_2;j_2}(x_2) \varphi^{i_3;j_3}(x_3) \bar{\varphi}_{i_4;j_4}(x_4) \rangle = 4i \delta^{i_3}_{i_2} \varepsilon(j_2)(i_4) \varepsilon(j_4) \frac{1}{r_{13}^2 r_{24}^2} \left( \frac{1}{r_{12} r_{34}} X_{1[24]} \alpha \bar{\alpha} A + \frac{1}{r_{14} r_{23}} X_{1[34]} \alpha \bar{\alpha} A \right). 
\]

(5.13)

Using (5.9), with (5.11), (5.12) and (5.13) we may analyse the linear equations (5.7). The terms involving \( S_{i_2 i_4;j_4}^{i_1 \alpha_3} \), \( S_{i_2 i_4;j_4}^{i_1 \alpha_3} \), and \( U_{i_2 i_4;j_4}^{i_1 \alpha_3} \) have no equivalent on the right hand side. Using the relations (5.4) and also

\[
(x_{12} \bar{\alpha}_3 x_{24}) \alpha \bar{\alpha} = (x_{13} \bar{\alpha}_3 x_{24}) \alpha \bar{\alpha} = r_{34} x_{12} \alpha \bar{\alpha},
\]

(5.14)

this leads to the constraints

\[
S_{i_2 i_4;j_4}^{i_1 \alpha_3 j_3} \delta^{j_4}_{j_2} - S_{i_2 i_4;j_4}^{i_1 \alpha_3 j_3} \delta^{j_4}_{j_2} = \varepsilon(i_3 \varepsilon(j_3) k U_{i_2 i_4;j_4}^{i_1 \alpha_3 j_3}. 
\]

(5.15)
With this and \((3.12), (5.1)\) reduces to
\[
\begin{align*}
R_{i_1i_3j_3}^{k} & + \frac{u^2}{v} \varepsilon^{(i_1l \varepsilon j_3 k) U^{i_1}_{i_2j_2i_4j_4}} + u \varepsilon^{(i_1k} \delta^{(i_3}_{(i_2 \varepsilon j_2)(i_4 \delta_{j_4})} A = V^{i_1k}_{i_2j_2i_4j_4}, \\
R'_{i_1i_3j_3}^{k} & - \frac{v^2}{u} \varepsilon^{(i_1l \varepsilon j_3 k) U^{i_1}_{i_2j_2i_4j_4}} - v \varepsilon^{i_1k} \delta^{(i_3}_{(i_2 \varepsilon j_2)(i_4 \delta_{j_4})} A' = V'^{i_1k}_{i_2j_2i_4j_4}, \\
\varepsilon^{(i_1l \varepsilon j_3 k) T^{i_1}_{i_2j_2i_4j_4}} & + \frac{1}{uv} S^{i_1i_3j_3}_{i_2i_4j_4} \delta^{k}_{j_2} - \varepsilon^{i_1k} \delta^{(i_3}_{(i_2 \varepsilon j_2)(i_4 \delta_{j_4})} (\frac{1}{u} A - \frac{1}{v} A') = W^{i_1i_3j_3}_{i_2j_2i_4j_4}.
\end{align*}
\]
(5.16)

The conditions \((5.15)\) give
\[
S_1 + U_1 = S_2 - U_2 = S'_1 - U_2 = S'_2 + U_1 = 0,
\]
(5.17)
and from the terms in \((5.16)\) antisymmetric in \(i_1, k\) we have also
\[
\begin{align*}
R_2 - \frac{u^2}{v} (U_1 - U_2) &= uA, && R'_2 - \frac{v^2}{u} (U_1 - U_2) = vA', \\
T_1 - T_2 - \frac{1}{uv} S_2 &= \frac{1}{u} A - \frac{1}{v} A'.
\end{align*}
\]
(5.18)

With \((5.8)\) the remaining equations become
\[
\begin{align*}
R_1 - \frac{u^2}{v} U_1 &= 2a - u \partial_v a, && U_2 = \frac{u}{v} \partial_u b, && R_2 + \frac{u^2}{v} (U_1 + U_2) = \frac{u}{v} (c - u \partial_u c), \\
R'_1 + \frac{v^2}{u} U_2 &= 2b - v \partial_u b, && U_1 = -\frac{v}{u} \partial_v a, && R'_2 - \frac{v^2}{u} (U_1 + U_2) = \frac{v}{u} (c - v \partial_v c), \\
T_1 - \frac{1}{uv} S_1 &= -\frac{1}{u^2} (u \partial_u + v \partial_v) a, && T_2 = -\frac{1}{v^2} (u \partial_u + v \partial_v) b, \\
T_1 + T_2 + \frac{1}{uv} S_2 &= \frac{1}{uv} (u \partial_u + v \partial_v) c.
\end{align*}
\]
(5.19)

In conjunction with \((5.17)\), which implies \(S_1 + S_2 = U_2 - U_1\), and \((5.18)\), eliminating \(A\) and \(A'\), there are two constraints which may be expressed in the form
\[
\begin{align*}
\partial_u c &= \frac{v}{u} \partial_v a - \partial_v b + \frac{1}{v} (1 - u - v) \partial_u b, \\
\partial_v c &= \frac{u}{v} \partial_u b - \partial_u a + \frac{1}{u} (1 - u - v) \partial_v a.
\end{align*}
\]
(5.20)

These equations are invariant under \(u \leftrightarrow v, a \leftrightarrow b\), as is consistent with the crossing symmetry conditions for the four point function in \((5.6)\), \(a(u, v) = b(v, u), c(u, v) = c(v, u)\).

With these relations we may easily find
\[
\begin{align*}
A &= 2 \partial_v a + \frac{1}{v} (c - u \partial_u c), && A' &= 2 \partial_u b + \frac{1}{u} (c - v \partial_v c),
\end{align*}
\]
(5.21)
and the conservation equation $\partial^{\dot{\alpha} \alpha} J_{\alpha \dot{\alpha}} = 0$ requires

$$2uv^2 \partial_u A - (1 - u - v)v^2 \partial_v A = 2u^2 v \partial_v A' - (1 - u - v) u^2 \partial_u A'. \quad (5.22)$$

As also checked in [3] this is satisfied by virtue of (5.20). For $\mathcal{N} = 4$ superconformal symmetry we again consider the four point function for the simplest short multiplet whose superconformal transformation properties were described in section 2. In this case we may write, with notation defined in appendix A,

$$\langle \varphi^{I_1}(x_1) \varphi^{I_2}(x_2) \varphi^{I_3}(x_3) \varphi^{I_4}(x_4) \rangle = \frac{\delta^{I_1 I_2} \delta^{I_3 I_4}}{r_{12}^2 r_{34}^2} a_1 + \frac{\delta^{I_1 I_3} \delta^{I_2 I_4}}{r_{13}^2 r_{24}^2} a_2 + \frac{\delta^{I_1 I_4} \delta^{I_2 I_3}}{r_{14}^2 r_{23}^2} a_3$$

$$+ \frac{C^{I_1 I_2 I_3 I_4}}{r_{12} r_{14} r_{23} r_{34}} c_1 + \frac{C^{I_1 I_3 I_2 I_4}}{r_{13} r_{14} r_{23} r_{24}} c_2 + \frac{C^{I_1 I_2 I_4 I_3}}{r_{12} r_{13} r_{24} r_{34}} c_3, \quad (5.23)$$

involving six functions of $u, v$. This basis is convenient since for free fields $a_1 = a_2 = a_3$ and $c_1 = c_2 = c_3$ are both constants. The relevant superconformal identity follows from

$$\delta \langle \psi_{\alpha}(x_1) \varphi^{I_2}(x_2) \varphi^{I_3}(x_3) \varphi^{I_4}(x_4) \rangle = 0 \quad \text{which may be expanded using (2.16) as}

\begin{align*}
\langle \psi_{\alpha}(x_1) \bar{\psi}_{\dot{\alpha}}(x_2) \varphi^{I_3}(x_3) \varphi^{I_4}(x_4) \rangle C_{I_2}^{I_3} \bar{\gamma}_s \bar{\hat{\epsilon}}^{\dot{\alpha}}(x_2) &+ \langle \psi_{\alpha}(x_1) \bar{\psi}_{\dot{\alpha}}(x_3) \varphi^{I_2}(x_2) \varphi^{I_4}(x_4) \rangle C_{I_3}^{I_4} \bar{\gamma}_s \bar{\hat{\epsilon}}^{\dot{\alpha}}(x_3) \\
+ \langle \psi_{\alpha}(x_1) \bar{\psi}_{\dot{\alpha}}(x_4) \varphi^{I_3}(x_3) \varphi^{I_2}(x_2) \rangle C_{I_4}^{I_3} \bar{\gamma}_s \bar{\hat{\epsilon}}^{\dot{\alpha}}(x_4) &+ \frac{1}{6} \langle J_{\alpha \bar{\alpha}}(x_1) \varphi^{I_2}(x_2) \varphi^{I_3}(x_3) \varphi^{I_4}(x_4) \rangle \bar{\gamma}_s \bar{\hat{\epsilon}}^{\dot{\alpha}}(x_1)
\end{align*}

$$= 2i \left( \frac{X_{12 \alpha \dot{\alpha}}}{r_{12}^2 r_{34}^2} C_{I_2}^{I_1} \bar{\gamma}_s \bar{\hat{\epsilon}}^{\dot{\alpha}}(x_2) Q_{I_2}^{I_1} Q_{I_3}^{I_4} + \frac{X_{13 \alpha \dot{\alpha}}}{r_{13}^2 r_{24}^2} C_{I_3}^{I_1} \bar{\gamma}_s \bar{\hat{\epsilon}}^{\dot{\alpha}}(x_3) Q_{I_2}^{I_3} Q_{I_4}^{I_1} + \frac{X_{14 \alpha \dot{\alpha}}}{r_{14}^2 r_{23}^2} C_{I_4}^{I_1} \bar{\gamma}_s \bar{\hat{\epsilon}}^{\dot{\alpha}}(x_4) Q_{I_2}^{I_4} Q_{I_3}^{I_1} \right), \quad (5.24)$$

4 Using (5.21) for $A$ and (5.20) for $\partial u c$ in the resulting term involving $\partial u \partial u c$ the left hand side of (5.22) becomes $(1 - u - v)(uv_{c, uv} - uc, u - vc, v + c - 2u^2 b_{uu} - 2v^2 a_{vv}) + 2uv^2 a_{uv} + 2u^2 vb_{uu} + 2v^2 a_{v} + 2u^2 b_{u}$. This is symmetric under $u \leftrightarrow v$, $a \leftrightarrow b$ and so is equal to the right hand side.
where, from (5.23), we have

\[ Q_2^{II_2I_3I_4} = \delta^{II_2}\delta^{I_3I_4}(2a_1 - u\partial_u a_1) - \delta^{II_3}\delta^{I_2I_4} u^3\partial_u a_2 - \delta^{II_4}\delta^{I_2I_3} u^3 \frac{v^2}{v^2} \partial_u a_3 \]
\[ + C^{II_2I_3I_4} \frac{u}{v}(c_1 - u\partial_u c_1) - C^{II_3I_2I_4} \frac{u^3}{v^2} \partial_u c_2 + C^{II_2I_4I_3} v(u_3 - u\partial_u c_3), \]

\[ Q_4^{II_2I_3I_4} = -\delta^{II_2}\delta^{I_3I_4} v^3 \frac{u^2}{u^2} \partial_v a_1 - \delta^{II_3}\delta^{I_2I_4} v^3 \partial_v a_2 + \delta^{II_4}\delta^{I_2I_3} (2a_3 - v\partial_v a_3) \]
\[ + C^{II_2I_3I_4} \frac{v}{u}(c_1 - v\partial_v c_1) + C^{II_3I_2I_4} (v(c_2 - v\partial_v c_2) + C^{II_2I_4I_3} v^3 \frac{u^2}{u^2} \partial_v c_3), \quad (5.25) \]

\[ Q_3^{II_2I_3I_4} = \delta^{II_2}\delta^{I_3I_4} \frac{1}{u^2}(u\partial_u + v\partial_v)a_1 + \delta^{II_3}\delta^{I_2I_4} (2a_2 + (u\partial_u + v\partial_v)a_2) \]
\[ + \delta^{II_4}\delta^{I_2I_3} \frac{1}{v^2}(u\partial_u + v\partial_v)a_3 + C^{II_2I_3I_4} \frac{1}{u^2}(u\partial_u + v\partial_v)c_1 \]
\[ + C^{II_2I_4I_3} \frac{1}{v}(c_2 + (u\partial_u + v\partial_v)c_2) + C^{II_2I_4I_3} \frac{1}{u}(c_3 + (u\partial_u + v\partial_v)c_3). \]

On the left hand side of (5.24) we may write

\[ \langle \psi_{r\alpha}(x_1) \bar{\psi}_{s\dot{\alpha}}(x_4) \phi^J(x_j) \phi^K(x_k) \rangle = 2i \left( X_{1i\alpha\dot{\alpha}} \right) \frac{1}{r_1^3} \frac{1}{r_2^3} \frac{1}{r_3^2} \frac{1}{r_4^2} R_{i,r,s}^{JK} + \frac{1}{r_{12}^3 r_{14}^3 r_{23}^2 r_{24}^2 r_{34}^4} S_{i,r,s}^{JK} \right), \]

\[ \langle J_{rs\alpha\dot{\alpha}}(x_1) \phi^J(x_2) \phi^J(x_3) \phi^K(x_4) \rangle = 2i \frac{1}{r_1^3 r_{24}^2} \left( \frac{1}{r_{12}^3 r_{23}^4} X_{1i[23]j\alpha\dot{\alpha}} A_{i,r,s}^{JK} + \frac{1}{r_{14}^2 r_{23}^4} X_{1[34]i\alpha\dot{\alpha}} B_{i,r,s}^{JK} \right). \]

Using the invariants \( T_{r,s}^{(n)JK} \) defined in appendix A we have

\[ R_{i,r,s}^{JK} = \sum_{n=1}^{6} R_{i,s}^{(n)} T_{r,s}^{(n)JK}, \quad S_{i,r,s}^{JK} = \sum_{n=1}^{6} S_{i,s}^{(n)} T_{r,s}^{(n)JK}, \quad (5.27) \]

and \( A_{i,r,s}^{JK}, B_{i,r,s}^{JK} \) may also be expanded in terms of a basis formed by \( (C^I C^J C^K)_{[rs]}, \)

\( (C^J C^I C^K)_{[rs]} \) and \( (C^I C^K C^J)_{[rs]} \).

As before there are conditions necessary to cancel terms involving \( x_{1j} \bar{x}_{jk} x_{ki} \) from (5.26). Using (5.4) and (5.14) this requires

\[ - (S_{2,I_3I_4} C_{I_2})_{rs} + (S_{3,I_2I_4} C_{I_3})_{rs} + (S_{4,I_2I_3} C_{I_4})_{rs} \gamma_{i,s} = 0. \]

Using (5.27) and the relations in (A.13,16) this decomposes into 12 linear relations for 18
variables which may be written as

\[
\begin{align*}
S_2^{(1)} - \frac{1}{6} S_2^{(2)} &= - S_3^{(5)} - S_4^{(5)}, \\
S_3^{(1)} - \frac{1}{6} S_3^{(2)} &= - S_2^{(5)} + S_4^{(6)}, \\
S_4^{(1)} - \frac{1}{6} S_4^{(2)} &= - S_2^{(6)} + S_3^{(6)}, \\
S_2^{(2)} &= 2 \left( S_2^{(5)} + S_2^{(6)} + S_3^{(6)} + S_4^{(6)} \right), \\
S_3^{(2)} &= 2 \left( S_2^{(6)} + S_3^{(5)} + S_3^{(6)} - S_4^{(5)} \right), \\
S_4^{(2)} &= 2 \left( S_2^{(5)} - S_3^{(5)} + S_4^{(5)} + S_4^{(6)} \right), \\
S_2^{(3)} &= S_3^{(3)} = - S_4^{(3)} = 2 \left( S_2^{(5)} - S_2^{(6)} + S_3^{(5)} - S_3^{(6)} - S_4^{(5)} + S_4^{(6)} \right), \\
S_2^{(4)} &= - S_3^{(5)} + S_4^{(5)}, \\
S_3^{(4)} &= - S_2^{(5)} - S_4^{(6)}, \\
S_4^{(4)} &= - S_2^{(6)} - S_3^{(6)}.
\end{align*}
\] (5.29)

There are 9 further relations which are necessary to cancel terms of the form \( E_{rJK} \), as defined on (A.14), which cannot occur elsewhere in (5.24). This gives 9 relations

\[
\begin{align*}
R_2^{(6)} &= \frac{u^2}{v} S_3^{(6)}, & R_4^{(5)} &= - \frac{v^2}{u} S_3^{(5)}, & R_3^{(6)} &= \frac{1}{uv} S_2^{(6)}, \\
R_2^{(5)} &= \frac{u^2}{v} S_3^{(4)}, & R_4^{(6)} &= \frac{v^2}{u} S_3^{(4)}, & R_3^{(5)} &= \frac{1}{uv} S_2^{(4)}, \\
R_2^{(4)} &= \frac{u^2}{v} S_3^{(5)}, & R_4^{(4)} &= \frac{v^2}{u} S_3^{(6)}, & R_3^{(4)} &= \frac{1}{uv} S_2^{(6)}.
\end{align*}
\] (5.30)

The remaining equations from (5.24) which are symmetric in \( rs \) correspond to the terms on the right hand side. With (5.25), and using (5.29) and (5.30) for simplification we have

\[
\begin{align*}
2 \left( S_2^{(6)} - S_3^{(5)} + S_3^{(6)} + S_4^{(5)} \right) &= (u \partial_u + v \partial_v) c_1, \\
2 \left( - S_2^{(5)} + S_2^{(6)} + S_3^{(6)} - S_4^{(4)} \right) &= -u \partial_u c_2, \\
2 \left( S_2^{(5)} + S_3^{(5)} - S_4^{(5)} + S_4^{(6)} \right) &= v \partial_v c_3, \\
2 S_2^{(5)} &= -uv \partial_u a_2, & 2 S_2^{(6)} &= -\frac{u}{v} (u \partial_u + v \partial_v) a_3, & 2 S_3^{(5)} &= -\frac{v}{u} \partial_v a_1, \\
2 S_3^{(6)} &= \frac{u}{v} \partial_u a_3, & 2 S_4^{(5)} &= -\frac{v}{u} (u \partial_u + v \partial_v) a_1, & 2 S_4^{(6)} &= -uv \partial_u a_2.
\end{align*}
\] (5.31)
as well as

\begin{align*}
R_2^{(2)} - R_2^{(3)} &= \frac{2u}{v} (c_1 - u \partial_u c_1) + u \partial_v a_1, \\
R_2^{(2)} + R_2^{(3)} &= 2u (c_3 - u (\partial_u + \partial_v) c_3) - u \partial_v a_1, \\
R_4^{(2)} - R_4^{(3)} &= 2v (c_1 - v \partial_v c_1) + v \partial_u a_3, \\
R_4^{(2)} - R_4^{(3)} &= 2v (c_2 - v (\partial_u + \partial_v) c_2) - v \partial_u a_3, \\
R_3^{(2)} - R_3^{(3)} &= \frac{1}{v} (c_2 + (u \partial_u + v \partial_v) c_1) + \partial_v a_2, \\
R_3^{(2)} + R_3^{(3)} &= \frac{1}{u} (c_3 + (u \partial_u + v \partial_v) c_3 - \partial_v c_3) - \partial_v a_2, \\
R_4^{(1)} - \frac{1}{6} R_4^{(2)} &= 2a_1 - u \partial_u a_1 - \frac{1}{2} u \partial_v a_1, \\
R_4^{(1)} - \frac{1}{6} R_4^{(2)} &= 2a_3 - v \partial_v a_3 - \frac{1}{2} v \partial_u a_3, \\
R_3^{(1)} - \frac{1}{6} R_3^{(2)} &= 2a_2 + (u \partial_u + v \partial_v) a_2 - \frac{1}{2} u \partial_v a_2.
\end{align*}

(5.32)

Finally there are three constraints which arise from the requirement that $\langle J_{rs} \varphi^{I_2} \varphi^{I_3} \varphi^{I_4} \rangle$ has the conformally covariant form given by (5.26). Using (5.29) and (5.30) these give

\begin{align*}
\frac{v}{2u} (R_2^{(2)} - R_2^{(3)}) - \frac{u}{2v} (R_4^{(2)} + R_4^{(3)}) - 3u S_3^{(5)} - 3v S_3^{(6)} + 2 (S_2^{(6)} + S_3^{(5)} + S_3^{(6)} - S_4^{(5)}) &= 0, \\
\frac{1}{2} (R_3^{(2)} - R_3^{(3)}) - \frac{1}{2v^2} (R_4^{(2)} - R_4^{(3)}) - \frac{3}{uv} S_2^{(5)} + \frac{3}{u} S_3^{(6)} + 2 \left( \frac{1}{v} - \frac{1}{u} \right) (S_2^{(5)} + S_3^{(5)} + S_4^{(6)}) &= 0, \\
\frac{1}{2} (R_4^{(2)} + R_4^{(3)}) - \frac{u^2}{2} (R_3^{(2)} + R_3^{(3)}) - \frac{3u}{v} S_2^{(5)} + \frac{3u^2}{v} S_3^{(5)} + 2 \left( \frac{u^2}{v} + \frac{u}{v} - u \right) (S_2^{(5)} - S_3^{(5)} + S_4^{(5)} + S_4^{(6)}) &= 0.
\end{align*}

(5.33)

It is clear that (5.31) provides three constraints and using (5.32) in (5.33) gives three more which are essentially integrability conditions. With some simplification these may
be written as
\[
\begin{align*}
\partial_u c_1 &= \frac{v}{u} \partial_v a_1 + \frac{1}{v} ((1 - v) \partial_u a_3 - (u \partial_u + v \partial_v) a_3), \\
\partial_v c_1 &= \frac{u}{v} \partial_u a_3 + \frac{1}{u} ((1 - u) \partial_v a_1 - (u \partial_u + v \partial_v) a_1), \\
\partial_u c_2 &= -v(\partial_u + \partial_v) a_2 + \partial_v a_3 - \frac{1}{v} (1 - u) \partial_u a_3, \\
\partial_v c_2 &= u \partial_u a_2 - (1 - u) \partial_v a_2 - \frac{1}{v} (u \partial_u + v \partial_v) a_3, \\
\partial_u c_3 &= v \partial_v a_2 - (1 - v) \partial_u a_2 - \frac{1}{u} (u \partial_u + v \partial_v) a_1, \\
\partial_v c_3 &= \partial_u a_1 - \frac{1}{u} (1 - v) \partial_v a_1 - u(\partial_u + \partial_v) a_2. 
\end{align*}
\tag{5.34}
\]

These equations are symmetric under \( u \leftrightarrow v, a_1 \leftrightarrow a_3, c_2 \leftrightarrow c_3 \) and also for \( u \rightarrow u/v, v \rightarrow 1/v, a_2 \leftrightarrow a_3, c_1 \leftrightarrow c_3 \). This is consistent with the crossing symmetry relations expected in (5.23),
\[
\begin{align*}
a_1(u, v) &= a_3(v, u), \\
a_2(u, v) &= a_2(v, u), \\
c_1(u, v) &= c_1(v, u), \\
c_2(u, v) &= c_3(v, u), 
\end{align*}
\tag{5.35}
\]
and also
\[
\begin{align*}
a_1(u, v) &= a_1(u', v'), \\
a_2(u, v) &= a_3(u', v'), \\
c_1(u, v) &= c_3(u', v'), \\
c_2(u, v) &= c_2(u', v'), 
\end{align*}
\tag{5.36}
\]

The results obtained for \( N = 2 \) in (5.20) are obviously a subset of (5.34) obtained for \( c_1 \rightarrow c, a_1 \rightarrow a, a_3 \rightarrow b \).

6. Solution of Conformal Constraint Equations

In this section we show how the linear equations (5.20) and (5.34) can be simply solved. To this end we introduce new variables \( z, x \) defined by
\[
u = x z, \quad v = (1 - x)(1 - z). \tag{6.1}
\]

With these variables (5.20) can be rewritten in the form
\[
\begin{align*}
\frac{\partial}{\partial x} \left( c - \frac{1 - z}{z} a - \frac{z}{1 - z} b \right) &= 0, \\
\frac{\partial}{\partial z} \left( c - \frac{1 - x}{x} a + \frac{x}{1 - x} b \right) &= 0. 
\end{align*}
\tag{6.2}
\]

\(^5\) These may be inverted by defining \( \lambda^2 = (z - x)^2 = (1 - u - v)^2 - 4uv \) and then \( z = \frac{1}{2} (1 - v + u + \lambda), x = \frac{1}{2} (1 - v + u - \lambda) \). In the Euclidean regime \( \sqrt{u} + \sqrt{v} \geq 1 \) so that \( \lambda^2 < 0 \) and \( x = z^* \).
The solution is then trivial

\[ c - \frac{1 - z}{z} a - \frac{z}{1 - z} b = f(z), \quad c - \frac{1 - x}{x} a - \frac{x}{1 - x} b = f(x), \]  

(6.3)

where we impose symmetry under \( z \leftrightarrow x \) as is essential since \( a, b, c \) depend just on \( u, v \) which, from (6.1), are invariant under this interchange. Eliminating \( c \) or \( a \) in (6.3) gives

\[ \frac{1}{u} a - \frac{1}{v} b = \frac{f(z) - f(x)}{z - x}, \quad \frac{1}{v} c - \frac{1 - v - u}{v^2} b = \frac{\frac{z}{1 - z} f(z) - \frac{x}{1 - x} f(x)}{z - x}, \]  

(6.4)

For later use we define amplitudes corresponding to \( SU(2)_R \) quantum numbers \( R = 0, 1, 2 \) in the decomposition of \( \varphi^{i,j_1}(x_1) \varphi^{i,j_2}(x_2) \) in (5.2),

\[ A_0 = a + \frac{u^2}{3v^2} b + \frac{u}{2v} c, \quad A_1 = \frac{u^2}{v^2} b + \frac{u}{v} c, \quad A_2 = \frac{u^2}{v^2} b. \]  

(6.5)

Using (6.4) we then get

\[ A_0 = (\frac{1}{2}(1 + v) - \frac{1}{6} u) G - \frac{1}{2} u \frac{2 - z}{z} \tilde{f}(z) - \frac{2 - x}{x} \tilde{f}(x) \]  

(6.6)

\[ A_1 = (1 - v) G - u \frac{\tilde{f}(z) - \tilde{f}(x)}{z - x}, \]

\[ A_2 = u G, \]

where we define

\[ G(u, v) = \frac{u}{v^2} b(u, v), \quad \tilde{f}(z) = \frac{z}{z - 1} f(z). \]  

(6.7)

For the \( \mathcal{N} = 4 \) case the equations (5.34) can be similarly simplified using the variables \( z, x \), as given in (6.1), to three pairs of equations which have the form (6.2), involving single partial derivatives with respect to \( z \) and \( x \) and which are symmetric under \( z \leftrightarrow x \). The corresponding solutions to (6.3) then involve three initially independent single variable functions \( f_{1,2,3} \),

\[ c_1 - \frac{1 - z}{z} a_1 - \frac{z}{1 - z} a_3 = f_1(z), \quad c_1 - \frac{1 - x}{x} a_1 - \frac{x}{1 - x} a_3 = f_1(x), \]

\[ c_2 + (1 - z) a_2 + \frac{1}{1 - z} a_3 = f_2(z), \quad c_2 + (1 - x) a_2 + \frac{1}{1 - x} a_3 = f_2(x), \]

\[ c_3 + z a_2 + \frac{1}{z} a_1 = f_3(z), \quad c_3 + x a_2 + \frac{1}{x} a_1 = f_3(x). \]  

(6.8)

These may be solved to give relations between any pair of functions of \( u, v \). For instance eliminating \( c_{1,2,3} \) it is easy to see that

\[ \frac{1}{u} a_1 - \frac{1}{v} a_3 = \frac{f_1(z) - f_1(x)}{z - x}, \quad -\frac{1}{v} a_3 = \frac{f_2(z) - f_2(x)}{z - x}, \quad \frac{1}{u} a_1 = \frac{f_3(z) - f_3(x)}{z - x}. \]  

(6.9)
Either from (6.9) or directly from (6.8) we must have

\[ f_1(x) + f_2(x) + f_3(x) = k, \quad (6.10) \]

for \( k = \sum_i (a_i + c_i) \), a constant.

In order to discuss the operator product expansion for \( \varphi^{I_1}(x_1) \varphi^{I_2}(x_2) \) in terms of operators belonging to the different possible \( SU(4) \) representations, which are here labelled by their dimensions \( R = 1, 20, 84, 105, 15, 175 \), we must consider the corresponding decomposition of the four point function (5.23),

\[ \langle \varphi^{I_1}(x_1) \varphi^{I_2}(x_2) \varphi^{I_3}(x_3) \varphi^{I_4}(x_4) \rangle = \frac{1}{r_1 r_2 r_3 r_4} \sum_R A_R(u, v) P_{R}^{I_1 I_2 I_3 I_4}, \quad (6.11) \]

where \( P_{R}^{I_1 I_2 I_3 I_4} \) are projection operators which are given explicitly in appendix A. In terms of (5.23) we have

\[ A_1 = 20a_1 + u^2 a_2 + \frac{u^2}{v^2} a_3 + \frac{10}{3} \left( \frac{u}{v} c_1 + uc_3 \right) + \frac{u^2}{3v} c_2, \]
\[ A_{20} = u^2 a_2 + \frac{u^2}{v^2} a_3 + \frac{5}{3} \left( \frac{u}{v} c_1 + uc_3 \right) + \frac{u^2}{6v} c_2, \]
\[ A_{84} = u^2 a_2 + \frac{u^2}{v^2} a_3 - \frac{u^2}{2v} c_2, \]
\[ A_{105} = u^2 a_2 + \frac{u^2}{v^2} a_3 + \frac{u^2}{v} c_2, \]
\[ A_{15} = u^2 a_2 - \frac{u^2}{v^2} a_3 - 2 \left( \frac{u}{v} c_1 - uc_3 \right), \]
\[ A_{175} = u^2 a_2 - \frac{u^2}{v^2} a_3. \quad (6.12) \]

Assuming (5.36) we have

\[ A_R(u, v) = \begin{cases} 
A_R(u', v'), & R = 1, 20, 84, 105; \\
-A_R(u', v'), & R = 15, 175.
\end{cases} \quad (6.13) \]
Taking account of (6.8) we may express $A_R$ just in terms of $A_{105}$ in the form

$$A_1 = \frac{1}{3} (u^2 + 10(1-v)^2 - 8u(1+v) + 60v) \mathcal{G}$$

$$- \frac{10}{3} (1-v) \frac{\tilde{f}_2(z) - \tilde{f}_2(x)}{z-x} + \frac{8}{3} u \frac{\frac{2-z}{z} \tilde{f}_2(z) - \frac{2-x}{x} \tilde{f}_2(x)}{z-x}$$

$$+ 5u \frac{(\frac{2-z}{z})^2 \tilde{f}(z) - (\frac{2-x}{x})^2 \tilde{f}(x)}{z-x} - \frac{5}{3} u \frac{\tilde{f}(z) - \tilde{f}(x)}{z-x} - 20 \big( f_2(z) + f_2(x) \big) + 20k,$$

$$A_{20} = \frac{1}{6} (u^2 + 10(1-v)^2 - 5u(1+v)) \mathcal{G}$$

$$- \frac{5}{3} (1-v) \frac{\tilde{f}_2(z) - \tilde{f}_2(x)}{z-x} + \frac{5}{6} u \frac{\frac{2-z}{z} \tilde{f}_2(z) - \frac{2-x}{x} \tilde{f}_2(x)}{z-x}$$

$$+ \frac{5}{3} u \frac{\tilde{f}(z) - \tilde{f}(x)}{z-x},$$

$$A_{84} = \frac{1}{2} u (3(1+v) - u) \mathcal{G} - \frac{3}{2} u \frac{\frac{2-z}{z} \tilde{f}_2(z) - \frac{2-x}{x} \tilde{f}_2(x)}{z-x},$$

$$A_{105} = u^2 \mathcal{G},$$

$$A_{15} = - (1-v)(2(1+v) - u) \mathcal{G} + 2(1-v) \frac{\frac{2-z}{z} \tilde{f}_2(z) - \frac{2-x}{x} \tilde{f}_2(x)}{z-x}$$

$$- \frac{u}{3} \frac{(\frac{2-z}{z})^2 \tilde{f}_2(z) - (\frac{2-x}{x})^2 \tilde{f}_2(x)}{z-x} - 2u \frac{\frac{2-z}{z} \tilde{f}(z) - \frac{2-x}{x} \tilde{f}(x)}{z-x},$$

$$A_{175} = - u(1-v) \mathcal{G} + u \frac{\tilde{f}_2(z) - \tilde{f}_2(x)}{z-x}.$$

To achieve the form (6.14), which is convenient for application to the operator product expansion subsequently, we use the definitions,

$$\tilde{f}_2(z) = \frac{z^2}{1-z} f_2(z), \quad \tilde{f}(z) = z f_3(z) - \frac{z}{z-1} f_1(z).$$

(6.15)

The symmetry requirements (6.13) are satisfied if

$$\mathcal{G}(u, v) = \frac{1}{v^2} \mathcal{G}(u', v'), \quad \tilde{f}_2(z) = \tilde{f}_2(z'), \quad \tilde{f}(z) = -\tilde{f}(z'), \quad z' = \frac{z}{z-1}. \quad (6.16)$$

7. Operator Product Expansion, $N = 2$

The four point function for quasi-primary fields $\phi_1, \phi_2, \phi_3, \phi_4$ in a conformal field theory has an expansion in terms of the contributions of all fields occurring in the operator
product expansion of $\phi_1 \phi_2$ and $\phi_3 \phi_4$. For simplicity taking $\phi_i = \phi$, a scalar field of dimension $\Delta_\phi$, this has the form

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \frac{1}{(r_{12} r_{34})^{\Delta_\phi}} \sum_{\Delta, \ell} a_{\Delta, \ell} u^{\frac{1}{2}(\Delta-\ell)} G_{\Delta}^{(\ell)}(u, v) . \quad (7.1)$$

The functions $G_{\Delta}^{(\ell)}(u, v)$ are analytic in $u, 1 - v$ and represent the contributions arising from a quasi-primary operator, and all its derivatives, of dimension $\Delta$ which transforms under the rotation group $O(d)$ in $d$ Euclidean dimensions as a symmetric traceless rank $\ell$ tensor. For the identity operator $G_0^{(0)} = 1$. Corresponding to $x_1 \leftrightarrow x_2$ or $x_3 \leftrightarrow x_4$ we have

$$G_{\Delta}^{(\ell)}(u, v) = (-1)^\ell v^{-\frac{1}{2}(\Delta-\ell)} G_{\Delta}^{(\ell)}(u', v'), \quad u' = u/v, \ v' = 1/v . \quad (7.2)$$

Recently [13] we obtained compact explicit expressions, using the variables $z, x$ defined in (6.1), in four dimensions which are given here in appendix C.

With superconformal symmetry the fields appearing in the operator product expansion belong to supermultiplets under the superconformal group which link the contributions of differing $\ell$. We first discuss the $\mathcal{N} = 2$ case for the four point function shown in (5.6) and analyse the contributions to $A_R$, as defined in (6.5), for $R = 0, 1, 2$. For a long multiplet whose lowest dimension operator, with dimension $\Delta$ and spin $\ell$, has $R = 0$ the list of operators which may contribute to the four point function are (only operators in four dimensions with representation $(j_1, j_2)$ can appear if $j_1 = j_2 = \frac{1}{2} \ell$ and the $U(1)_R$ charge $r = 0$ so this is a subset of the full set of $2^8(\ell + 1)^2$ fields),

$$R = 0 \quad \Delta_\ell \quad (\Delta + 1)_{\ell\pm 1} \quad (\Delta + 2)_{\ell\pm 2}, (\Delta + 2)^2 \quad (\Delta + 3)_{\ell\pm 1} \quad (\Delta + 4)_\ell$$

$$R = 1 \quad (\Delta + 1)_{\ell\pm 1} \quad (\Delta + 2)^\ell \quad (\Delta + 3)_{\ell\pm 1}$$

$$R = 2 \quad (\Delta + 2)^\ell . \quad (7.3)$$

It is evident that for $R = 2$ only a single operator contributes from this supermultiplet. Consequently in this case the corresponding contribution, choosing here the overall coefficient to be one, is just

$$A_2(u, v) = u^{\frac{1}{2}(\Delta+2-\ell)} G_{\Delta+2}^{(\ell)}(u, v) . \quad (7.4)$$

According to (6.6) this gives trivially $G(u, v) = u^{\frac{1}{2}(\Delta-\ell)} G_{\Delta+2}^{(\ell)}(u, v)$. Setting $f(z) = 0$ this then determines $A_0$ and $A_1$ in (6.6). Using the results of appendix C allows $A_0, A_1$ to be expressed in terms of a sum of contributions $G_{\Delta}^{(\ell)}$, for suitable $\Delta, \ell$, with coefficients
depending on $\Delta, \ell$. The results are then

$$A_1(u, v) = -u^{\frac{1}{2}(\Delta - \ell)} \left( 2G_{\Delta+1}^{(\ell+1)}(u, v) + \frac{1}{2}uG_{\Delta+1}^{(\ell-1)}(u, v) \right) + \frac{(\Delta + \ell + 2)^2}{8(\Delta + \ell + 1)(\Delta + \ell + 3)} uG_{\Delta+3}^{(\ell+1)}(u, v) + \frac{(\Delta - \ell)^2}{32(\Delta - \ell - 1)(\Delta - \ell + 1)} u^2G_{\Delta+3}^{(\ell-1)}(u, v),$$

$$A_0(u, v) = u^{\frac{1}{2}(\Delta - \ell)} \left( G_{\Delta}^{(\ell)}(u, v) + \frac{1}{12}uG_{\Delta+2}^{(\ell)}(u, v) \right) + \frac{(\Delta + \ell + 2)^2}{4(\Delta + \ell + 1)(\Delta + \ell + 3)} G_{\Delta+2}^{(\ell+2)}(u, v) + \frac{(\Delta - \ell)^2}{64(\Delta - \ell - 1)(\Delta - \ell + 1)} u^2G_{\Delta+2}^{(\ell-2)}(u, v) + \frac{(\Delta + \ell + 2)^2(\Delta - \ell)^2}{256(\Delta + \ell + 1)(\Delta + \ell + 3)(\Delta - \ell - 1)(\Delta - \ell + 1)} u^2G_{\Delta+4}^{(\ell)}(u, v) \right).$$

These are exactly the contributions expected in an expansion as in (7.1) corresponding to the operators listed in (7.3) with the constraint that the operators contributing to $A_1$ have $(-1)^{\ell}$ differing in sign from those appearing in $A_0, A_2$. The conditions (the overall minus sign in $A_1$ arises as a consequence of (7.2) and the constraint on $\ell$ just noticed since the positivity requirement applies directly to $\langle \varphi^{ij_1j_2}(x_1) \varphi^{i_2j_3j_4}(x_2) \varphi^{i_3j_3j_4}(x_3) \varphi^{i_4j_4}(x_4) \rangle$ which is related to (5.6) by $x_3 \leftrightarrow x_4$ and hence $u \to u/v, v \to 1/v$) necessary for a unitary theory clearly require only $\Delta > \ell + 1$. For a generic $\mathcal{N} = 2$ supermultiplet the unitarity condition for a lowest weight operator with dimension $\Delta$, belonging to a $SU(2)_R$ representation $R$, $U(1)_R$ charge $r$ and with $j_1 = j_2 = \frac{1}{2} \ell$ is

$$\Delta \geq 2 + \ell + 2R + |r|, \quad (7.6)$$

and in application to the four point function of interest here $r = 0$ and $R = 0, 1, 2$.

In (7.5) $\ell = 0, 1$ are special cases but the results may be obtained from (7.3) by using, as shown in appendix C,

$$G_{\Delta}^{(-1)}(u, v) = 0, \quad \frac{1}{4}uG_{\Delta}^{(-2)}(u, v) = -G_{\Delta}^{(0)}(u, v).$$

Thus we may obtain for $\ell = 0$,

$$A_0(u, v) = u^{\frac{1}{2}\Delta} \left( G_{\Delta}^{(0)}(u, v) + \frac{\Delta^2 - 4}{48(\Delta^2 - 1)} uG_{\Delta+2}^{(0)}(u, v) + \frac{(\Delta + 2)^2}{4(\Delta + 1)(\Delta + 3)} G_{\Delta+2}^{(2)}(u, v) + \frac{\Delta^2(\Delta + 2)^2}{256(\Delta + 1)^2(\Delta + 3)(\Delta - 1)} u^2G_{\Delta+4}^{(0)}(u, v) \right). \quad (7.8)$$
Unitarity here requires $\Delta \geq 2$ in accord with (7.6) for $R = \ell = 0$.

There are also short multiplets in which the spectrum of $SU(2)_R$ representations is reduced since some superconformal transformations annihilate the operators with lowest dimension. For such supermultiplets $\Delta$ and $\ell$ are related. The contribution of such operators in the operator product expansion to the four point function involves considering the function $\tilde{f}(z)$ which arises in the general solution of the superconformal identities exhibited in (6.6). The terms arising from the function $\tilde{f}$ may also be written in terms of the operator expansion functions $G^{(\ell)}_{\Delta}(u,v)$ but only in particular cases where $\Delta$ is given in terms of $\ell$. From the results in [13], as quoted here in in appendix C, the essential expression, which matches the form of the contributions appearing in (6.6), is

$$G^{(\ell)}_{\ell+2}(u,v) = \frac{g_{\ell+1}(z) - g_{\ell+1}(x)}{z - x}, \quad g_{\ell}(z) = \left( -\frac{1}{2} z \right)^{\ell - 1} z F(\ell, \ell; 2\ell; z), \quad (7.9)$$

with $F$ a hypergeometric function. Using the formuale in appendix C or standard hypergeometric identities this satisfies the crucial relations

$$\frac{2 - z}{z} g_{\ell}(z) = -g_{\ell-1}(z) - \frac{\ell^2}{(2\ell - 1)(2\ell + 1)} g_{\ell+1}(z), \quad g_{\ell}(z) = (-1)^\ell g_{\ell}(z'), \quad (7.10)$$

with $z'$ as in (6.16).

We first in (6.6) set $G = 0$ and therefore

$$A_2(u,v) = 0, \quad (7.11)$$

so that there are no contributions from $R = 2$ operators. If we then choose in (6.6) $\tilde{f}(z) = g_{\ell+2}(z)$ we would have, by virtue of (7.9),

$$A_1(u,v) = -u G^{(\ell+1)}_{\ell+3}(u,v). \quad (7.12)$$

Using (7.10) in (6.6) further gives, in conjunction with (7.11) and (7.12),

$$A_0(u,v) = \frac{1}{2} u G^{(\ell)}_{\ell+2}(u,v) + \frac{(\ell + 2)^2}{2(2\ell + 3)(2\ell + 5)} u G^{(\ell+2)}_{\ell+4}(u,v). \quad (7.13)$$

The results (7.11), (7.12) and (7.13) thus represent the contribution of a restricted multiplet in which the lowest dimension field has spin $\ell$ and $R = 0, \Delta = 2 + \ell$, saturating the bound (7.6). In both (7.12) and (7.13) the relevant operators have twist, $\Delta - \ell$, two. For $\ell = -1$ these results reduce to

$$A_0(u,v) = \frac{1}{6} u G^{(1)}_3(u,v), \quad A_1(u,v) = -u G^{(0)}_2(u,v), \quad A_2(u,v) = 0, \quad (7.14)$$
in which the $R = 1$ operator has the lowest dimension. This corresponds to the short supermultiplet exhibited in (2.10). If $\ell = 0$ instead we get

$$A_0(u, v) = \frac{1}{2} u G_2^{(0)}(u, v) + \frac{2}{15} u G_4^{(2)}(u, v), \quad A_1(u, v) = -u G_3^{(1)}(u, v), \quad A_2(u, v) = 0,$$

which represents the contribution of the supercurrent supermultiplet containing the energy momentum tensor (the $U(1)_R$ current does not appear in the operator product expansion in this case).

If in (6.6) we take $\tilde{f} = 1$ then

$$A_0 = 1, \quad A_1 = A_2 = 0,$$

(7.16)
corresponding to the identity operator.

Besides the case represented by (7.11), (7.12) and (7.13) for any $\ell$ there are also other restricted supermultiplets, with dimensions not involving a factor $2^8$ as in a generic long multiplets. A further example may be found by considering the contributions to the four point function in the operator product expansion obtained by subtracting twice (7.12) and (7.13) from (7.5) for $\Delta = \ell + 2$. This gives, with (7.11), the results

$$A_2(u, v) = u^2 G_{\ell+4}^{(\ell)}(u, v), \quad A_1(u, v) = -\frac{1}{2} u^2 G_{\ell+3}^{(\ell-1)}(u, v) - \frac{1}{24} u^3 G_{\ell+5}^{(\ell-1)}(u, v) - \frac{(\ell + 2)^2}{2(2\ell + 3)(2\ell + 5)} u^2 G_{\ell+5}^{(\ell+1)}(u, v), \quad A_0(u, v) = \frac{1}{12} u^2 G_{\ell+4}^{(\ell)}(u, v) + \frac{1}{48} u^3 G_{\ell+4}^{(\ell-2)}(u, v) + \frac{(\ell + 2)^2}{48(2\ell + 3)(2\ell + 5)} u^3 G_{\ell+6}^{(\ell)}(u, v),$$

(7.17)

which would correspond to a supermultiplet in which the lowest dimension operator has spin $\ell - 1$ and $R = 1$, $\Delta = \ell + 3$ containing operators of twist 4, 6. For such multiplets the bound (7.6) is again saturated for the lowest weight operator.

Taking $\ell = 0$ in (7.17) gives

$$A_0(u, v) = \frac{1}{180} u^3 G_6^{(0)}(u, v), \quad A_1(u, v) = -\frac{2}{15} u^2 G_5^{(1)}(u, v), \quad A_2(u, v) = u^2 G_4^{(0)}(u, v),$$

(7.18)

where the lowest dimension operator has $R = 2$. For $\mathcal{N} = 2$ the superconformal group $SU(2,2|2)$ has short multiplets corresponding to lowest weight operators which are scalars with $\Delta = 2R$, [9], and can be represented as in (2.10), which is the special case when

---

6 This can also be realised by (7.12) and (7.13) if $\ell = -2$ since, by virtue of (7.7), $\frac{1}{2} u G_0^{(-2)}(u, v) = -2 G_0^{(0)}(u, v) = -2$. 

41
\[ R = 1, \text{ giving} \]

\[
\begin{align*}
\Delta & \quad 2R \\
2R + \frac{1}{2} & \quad (R - \frac{1}{2})(\frac{1}{2},0) \quad (R - \frac{1}{2})(0,\frac{1}{2}) \\
2R + 1 & \quad (R - 1)(0,0) \quad (R - 1)(\frac{1}{2},\frac{1}{2}) \quad (R - 1)(0,0) \\
2R + \frac{3}{2} & \quad (R - \frac{3}{2})(0,\frac{1}{2}) \quad (R - \frac{3}{2})(\frac{1}{2},0) \\
2R + 2 & \quad (R - 2)(0,0)
\end{align*}
\]

where the representations are denoted by \( R(j_1,j_2) \) with \((j_1,j_2)\) determining the spin, \( R \) the \( SU(2)_R \) representation and for \( r \) the \( U(1)_R \) charge. The total dimension is \( 16(2R - 1) \).

For \( R = 2 \) the operators listed in (7.19) with zero \( r \) are just those required to give the operator product contributions in (7.18).

For a general four point function, represented as in (6.8), the superconformal operator product expansion can thus be written simply as

\[
G(u,v) = \sum_{\Delta,\ell} a_{\Delta,\ell} (-1)^\ell u^{\frac{1}{2}(\Delta - \ell)} G_{\Delta+2}^{(\ell)}(u,v), \quad \tilde{f}(z) = A + \sum_{\ell \geq -1} a_\ell (-1)^\ell g_{\ell+2}(z). \quad (7.20)
\]

For generic \( \Delta \) unitarity requires \( a_{\Delta,\ell} > 0 \). From the above it is sufficient if \( \Delta = \ell + 2 \) to impose \( a_\ell + 2a_{\ell+2,\ell} \geq 0 \), for \( \ell = 0, 1, \ldots \), and also \( A > 0 \).

As an illustration we may consider the result for a free theory when in (5.6) we have a trivial solution of the superconformal constraints, \( a = b \) and \( c \) constants. In this case, from (6.3) and (6.7), we have

\[
G(u,v) = a \frac{u}{v^2}, \quad \tilde{f}(z) = a \left( 1 + \frac{z^2}{(1-z)^2} \right) - c \frac{z}{1-z}. \quad (7.21)
\]

Using our previous results [13] we have

\[
a_{\Delta,\ell} = a 2^{\ell-1} \frac{(\ell + t - 1)!((\ell + t)!((t - 1)!)^2}{(2\ell + 2t - 1)!(2t - 2)!} \frac{(\ell + 1)(\ell + 2t) \delta_{\Delta,\ell+2t}}{\delta_{\Delta,\ell+2t}}, \quad \ell = 0, 1, 2 \ldots \quad (7.22)
\]

Furthermore, by analysing the expansion of \( \tilde{f}(z) \) in powers of \( z \),

\[
A = a, \quad a_\ell + 2a_{\ell+2,\ell} = c 2^{\ell+1} \frac{((\ell + 1)!^2}{(2\ell + 2)!}, \quad \ell = -1, 0, \ldots. \quad (7.23)
\]
Clearly positivity conditions are satisfied for \( a, c > 0 \). From (7.23) it is evident that the short multiplets represented by (7.11), (7.12), (7.13) and also (7.17), including the special cases (7.14) and (7.18), are relevant in the operator product expansion for this free theory.

8. Operator Product Expansion, \( \mathcal{N} = 4 \)

For the analysis of the operator product expansion for the four point function for the simplest chiral primary operators as in (5.23), the implications of \( \mathcal{N} = 4 \) superconformal symmetry can be derived following a similar procedure as in the \( \mathcal{N} = 2 \) case. This depends essentially on the representation (3.14) for the contributions of operators belonging to different representations of the \( SU(4)_R \) symmetry.

We first consider the contributions of long supermultiplets, without any constraints, in which the lowest dimension operator is a \( SU(4)_R \) singlet of dimension \( \Delta \). According to (8.1) and also section 4 this is the only possibility for generic \( \Delta \). For the case when this operator also has spin zero the decomposition of all \( 2^{16} \) operators into representations of \( SU(4)_R \) and also \( SO(3,1) \), which are labelled \((j_1, j_2)\), was listed by Andrianopoli and Ferrara (26). Here we are interested in just those operators contributing in the operator product expansion to the scalar four point function, which have \( j_1 = j_2 = \frac{1}{2} \ell \), and we also consider only those operators with \( Y = 0 \), corresponding to those occurring in a superfield expansion with equal numbers of \( \theta \)'s and \( \bar{\theta} \)'s. With a similar notation as in (7.3) these operators are

\[
\begin{array}{ccccccc}
R & 1 & 15 & 20 & 84 & 175 & 105 \\
\Delta_0 & (\Delta + 1)_{1,1} & (\Delta + 1)_{1,1} & (\Delta + 2)_{0,2} & (\Delta + 2)_{0,2} & (\Delta + 3)_{1,3} & (\Delta + 3)_{1,3} \\
(\Delta + 2)_{0,2} & (\Delta + 2)_{0,2} & (\Delta + 2)_{0,2} & (\Delta + 2)_{0,2} & (\Delta + 3)_{1,3} & (\Delta + 3)_{1,3} & (\Delta + 3)_{1,3} \\
(\Delta + 3)_{1,3} & (\Delta + 3)_{1,3} & (\Delta + 3)_{1,3} & (\Delta + 3)_{1,3} & (\Delta + 3)_{1,3} & (\Delta + 3)_{1,3} & (\Delta + 3)_{1,3} \\
(\Delta + 4)_{0,2,4} & (\Delta + 4)_{0,2,4} & (\Delta + 4)_{0,2,4} & (\Delta + 4)_{0,2,4} & (\Delta + 4)_{0,2,4} & (\Delta + 4)_{0,2,4} & (\Delta + 4)_{0,2,4} \\
(\Delta + 5)_{1,3} & (\Delta + 5)_{1,3} & (\Delta + 5)_{1,3} & (\Delta + 5)_{1,3} & (\Delta + 5)_{1,3} & (\Delta + 5)_{1,3} & (\Delta + 5)_{1,3} \\
(\Delta + 6)_{0,2} & (\Delta + 6)_{0,2} & (\Delta + 6)_{0,2} & (\Delta + 6)_{0,2} & (\Delta + 6)_{0,2} & (\Delta + 6)_{0,2} & (\Delta + 6)_{0,2} \\
(\Delta + 7)_{1,1} & (\Delta + 7)_{1,1} & (\Delta + 7)_{1,1} & (\Delta + 7)_{1,1} & (\Delta + 7)_{1,1} & (\Delta + 7)_{1,1} & (\Delta + 7)_{1,1} \\
(\Delta + 8)_{0,0} & (\Delta + 8)_{0,0} & (\Delta + 8)_{0,0} & (\Delta + 8)_{0,0} & (\Delta + 8)_{0,0} & (\Delta + 8)_{0,0} & (\Delta + 8)_{0,0} \\
\end{array}
\]

For the case of the lowest dimension operator having spin \( \ell \) we may easily obtain, since there are no constraints, the corresponding table by tensoring with \((j, j)\), \( \ell = 2j \). Thus in (8.1), for any \( \Delta \), we have \( \Delta_0 \to \Delta_\ell, \Delta_1 \to \Delta_{\ell \pm 1}, \Delta_2 \to \Delta_{\ell \pm 2}, \Delta_3 \to \Delta_{\ell \pm 3}, \ell_\pm 1 \) and \( \Delta_4 \to \Delta_{\ell \pm 4}, \ell_\pm 2, \ell \). This representation is unitary if \( \Delta \geq \ell + 2 \).
According to \((6.14)\) this now gives \(G(u,v) = u^{\frac{1}{2}}(\Delta - \ell)G^{(\ell)}_{\Delta+4}(u,v)\). Discarding \(\tilde{f}(z)\) and \(\tilde{f}_2(z)\) we may now determine the remaining \(A_{R}\) from \((6.14)\), in a similar fashion to \((7.3)\), by using the results of appendix C. We give the results in order of increasing complexity. For \(R = 175\),

\[
A_{175}(u,v) = u^{\frac{1}{2}}(\Delta + 2 - \ell) \left( 2G^{(\ell+1)}_{\Delta+3}(u,v) + \frac{1}{2}uG^{(\ell-1)}_{\Delta+3}(u,v) \right)
\]

\[
+ \frac{(\Delta + \ell + 4)^2}{8(\Delta + \ell + 3)(\Delta + \ell + 5)} uG^{(\ell+1)}_{\Delta+5}(u,v)
\]

\[
+ \frac{(\Delta - \ell + 2)^2}{32(\Delta - \ell + 1)(\Delta - \ell + 3)} u^2G^{(\ell-1)}_{\Delta+5}(u,v).
\]

For \(R = 84\),

\[
A_{84}(u,v) = 3u^{\frac{1}{2}}(\Delta + 2 - \ell) \left( G^{(\ell)}_{\Delta+2}(u,v) + \frac{1}{12}uG^{(\ell)}_{\Delta+4}(u,v) \right)
\]

\[
+ \frac{(\Delta + \ell + 4)^2}{4(\Delta + \ell + 3)(\Delta + \ell + 5)} uG^{(\ell+2)}_{\Delta+4}(u,v)
\]

\[
+ \frac{(\Delta - \ell + 2)^2}{64(\Delta - \ell + 1)(\Delta - \ell + 3)} u^2G^{(\ell-2)}_{\Delta+4}(u,v)
\]

\[
+ \frac{(\Delta + \ell + 4)^2(\Delta - \ell + 2)^2}{256(\Delta + \ell + 3)(\Delta + \ell + 5)(\Delta - \ell + 1)(\Delta - \ell + 3)} u^2G^{(\ell)}_{\Delta+6}(u,v)
\].

For \(R = 20\),

\[
A_{20}(u,v) = \frac{5}{3}u^{\frac{1}{2}}(\Delta - \ell) \left( 4G^{(\ell+2)}_{\Delta+2}(u,v) + uG^{(\ell)}_{\Delta+2}(u,v) + \frac{1}{4}u^2G^{(\ell-2)}_{\Delta+2}(u,v) \right)
\]

\[
+ \frac{(\Delta + \ell + 4)^2}{4(\Delta + \ell + 3)(\Delta + \ell + 5)} uG^{(\ell+2)}_{\Delta+4}(u,v)
\]

\[
+ \frac{(\Delta - \ell + 2)^2}{64(\Delta - \ell + 1)(\Delta - \ell + 3)} u^3G^{(\ell-2)}_{\Delta+4}(u,v) + \frac{1}{10} u^2G^{(\ell)}_{\Delta+4}(u,v)
\]

\[
+ \frac{1}{8} \left( \frac{1}{(\Delta + \ell + 1)(\Delta + \ell + 5)} + \frac{1}{(\Delta - \ell - 1)(\Delta - \ell + 3)} \right) u^2G^{(\ell)}_{\Delta+4}(u,v)
\]

\[
+ \frac{(\Delta + \ell + 4)^2(\Delta + \ell + 6)^2}{64(\Delta + \ell + 3)(\Delta + \ell + 5)^2(\Delta + \ell + 7)} u^2G^{(\ell+2)}_{\Delta+6}(u,v)
\]

\[
+ \frac{(\Delta + \ell + 4)^2(\Delta - \ell + 2)^2}{256(\Delta + \ell + 3)(\Delta + \ell + 5)(\Delta - \ell + 1)(\Delta - \ell + 3)} u^3G^{(\ell)}_{\Delta+6}(u,v)
\]

\[
+ \frac{(\Delta - \ell + 2)^2(\Delta - \ell + 4)^2}{210(\Delta - \ell + 1)(\Delta - \ell + 3)^2(\Delta - \ell + 5)} u^4G^{(\ell-2)}_{\Delta+6}(u,v)
\).

\[
44
\]
For $R = 15,$

$$A_{15}(u, v) = u^{\frac{1}{2}}(\Delta - \ell) \left( 8G^{(\ell+1)}_{\Delta+1}(u, v) + 2uG^{(\ell-1)}_{\Delta+1}(u, v) ight)$$

$$+ \frac{2(\Delta + \ell + 4)^2}{(\Delta + \ell + 3)(\Delta + \ell + 5)} G^{(\ell+3)}_{\Delta+3}(u, v) + \frac{(\Delta + \ell + 3)^2 - 3}{(\Delta + \ell + 1)(\Delta + \ell + 5)} uG^{(\ell+1)}_{\Delta+3}(u, v)$$

$$+ \frac{(\Delta - \ell + 2)^2}{4(\Delta - \ell - 1)(\Delta - \ell + 3)} u^2G^{(\ell-1)}_{\Delta+3}(u, v)$$

$$+ \frac{(\Delta + \ell + 4)^2}{32(\Delta - \ell + 1)(\Delta - \ell + 3)} u^3G^{(\ell-3)}_{\Delta+3}(u, v)$$

$$+ \frac{8(\Delta + \ell + 3)(\Delta + \ell + 5)^2(\Delta + \ell + 7)}{(\Delta + \ell + 4)^2((\Delta - \ell + 1)^2 - 3)} u^2G^{(\ell+1)}_{\Delta+5}(u, v)$$

$$+ \frac{(\Delta - \ell + 2)^2((\Delta + \ell + 3)^2 - 3)}{16(\Delta - \ell - 1)(\Delta - \ell + 3)(\Delta + \ell + 3)(\Delta + \ell + 5)} u^3G^{(\ell-1)}_{\Delta+5}(u, v)$$

$$+ \frac{(\Delta - \ell + 3)^2(\Delta - \ell + 5)^2}{64(\Delta - \ell + 1)(\Delta - \ell + 3)(\Delta + \ell + 1)(\Delta + \ell + 5)} u^4G^{(\ell-3)}_{\Delta+5}(u, v)$$

$$+ \frac{(\Delta + \ell + 4)^2(\Delta + \ell + 6)^2}{2^9(\Delta - \ell - 1)(\Delta - \ell + 3)^2(\Delta - \ell + 5)} u^4G^{(\ell+1)}_{\Delta+7}(u, v)$$

$$+ \frac{(\Delta - \ell + 2)^2(\Delta - \ell + 4)^2(\Delta + \ell + 4)^2}{211(\Delta - \ell + 1)(\Delta - \ell + 3)^2(\Delta - \ell + 5)(\Delta + \ell + 3)(\Delta + \ell + 5)} u^4G^{(\ell-1)}_{\Delta+7}(u, v)$$

(8.6)

Finally the singlet contribution is

$$A_1(u, v)$$

$$= \frac{1}{3} u^{\frac{1}{2}}(\Delta - \ell) \left( 60G^{(\ell)}_{\Delta}(u, v) + \frac{10(\Delta + \ell + 2)(\Delta + \ell + 4)}{(\Delta + \ell + 1)(\Delta + \ell + 5)} G^{(\ell+2)}_{\Delta+2}(u, v) ight)$$

$$+ 4 uG^{(\ell)}_{\Delta+2}(u, v) + \frac{5(\Delta - \ell)(\Delta - \ell + 2)}{8(\Delta - \ell - 1)(\Delta - \ell + 3)} u^2G^{(\ell-2)}_{\Delta+2}(u, v)$$

$$+ \frac{15(\Delta + \ell + 4)^2(\Delta + \ell + 6)^2}{4(\Delta + \ell + 3)(\Delta + \ell + 5)^2(\Delta + \ell + 7)} G^{(\ell+4)}_{\Delta+4}(u, v)$$

$$+ \frac{(\Delta + \ell + 4)^2}{(\Delta + \ell + 3)(\Delta + \ell + 5)} uG^{(\ell+2)}_{\Delta+4}(u, v)$$

$$+ \frac{5(\Delta - \ell)(\Delta - \ell + 2)(\Delta + \ell + 2)(\Delta + \ell + 4)}{48(\Delta - \ell - 1)(\Delta - \ell + 3)(\Delta + \ell + 1)(\Delta + \ell + 5)} u^2G^{(\ell)}_{\Delta+4}(u, v) + \frac{1}{3} u^2G^{(\ell)}_{\Delta+4}(u, v)$$

$$+ \frac{(\Delta - \ell + 2)^2}{16(\Delta - \ell + 1)(\Delta - \ell + 3)} u^3G^{(\ell-2)}_{\Delta+4}(u, v)$$

45
\[ + \frac{15(\Delta - \ell + 2)^2(\Delta - \ell + 4)^2}{2^{10}(\Delta - \ell + 1)(\Delta - \ell + 3)^2(\Delta - \ell + 5)} u^4 G_{\Delta+4}^{(\ell-4)}(u, v) \]
\[ + \frac{5(\Delta - \ell)(\Delta - \ell + 2)(\Delta + \ell + 4)^2(\Delta + \ell + 6)^2}{128(\Delta - \ell - 1)(\Delta - \ell + 3)(\Delta + \ell + 5)(\Delta + \ell + 7)} u^2 G_{\Delta+6}^{(\ell+2)}(u, v) \]
\[ + \frac{(\Delta - \ell + 2)^2(\Delta + \ell + 4)^2}{64(\Delta - \ell + 1)(\Delta - \ell + 3)(\Delta + \ell + 5)} u^3 G_{\Delta+6}^{(\ell)}(u, v) \]
\[ + \frac{5(\Delta - \ell + 2)^2(\Delta - \ell + 4)^2(\Delta + \ell + 2)(\Delta + \ell + 4)}{2^{14}(\Delta - \ell + 1)(\Delta - \ell + 3)^2(\Delta - \ell + 5)(\Delta + \ell + 5)^2(\Delta + \ell + 7)} u^4 G_{\Delta+8}^{(\ell)}(u, v) \]}

It is evident that these results correspond exactly to what would be expected for a long multiplet whose lowest dimension operator is a singlet of spin \( \ell \). The symmetry conditions (6.13), using (7.2), require \( \ell \) to be even.

For later use a more succinct notation is useful so the \( A_R \) are assembled as a vector,
\[ \mathcal{A} = (A_1, A_{15}, A_{20}, A_{84}, A_{175}, A_{105}), \] (8.8)
and then the equations (8.2), (8.3), (8.4), (8.5), (8.6) and (8.7) are expressed as
\[ \mathcal{A}(u, v) = G_{\Delta, \ell}(u, v), \] (8.9)
defining implicitly the vector \( G_{\Delta, \ell} \).

If \( \ell = 0 \) then the above results are valid if we use (7.7) and also
\[ u^3 G_{\Delta}^{(-4)}(u, v) = -64 G_{\Delta}^{(2)}(u, v), \quad u^2 G_{\Delta}^{(-3)}(u, v) = -16 G_{\Delta}^{(1)}(u, v). \] (8.10)

The corresponding results are then
\[ A_{105}(u, v) = u^{\frac{1}{2}}(\Delta+4) G_{\Delta+4}^{(0)}(u, v), \]
\[ A_{175}(u, v) = u^{\frac{1}{2}}(\Delta+2) \left( 2 G_{\Delta+3}^{(1)}(u, v) + \frac{(\Delta + 4)^2}{8(\Delta + 3)(\Delta + 5)} u G_{\Delta+5}^{(1)}(u, v) \right), \]
\[ A_{84}(u, v) = u^{\frac{1}{2}}(\Delta+2) \left( 3 G_{\Delta+2}^{(0)}(u, v) + \frac{\Delta(\Delta + 4)}{16(\Delta + 1)(\Delta + 3)} u G_{\Delta+4}^{(0)}(u, v) \right) \]
\[ + \frac{3(\Delta + 4)^2}{4(\Delta + 3)(\Delta + 5)} G_{\Delta+4}^{(2)}(u, v) + \frac{3(\Delta + 2)^2(\Delta + 4)^2}{256(\Delta + 3)^2(\Delta + 5)(\Delta + 1)} u^2 G_{\Delta+6}^{(0)}(u, v), \]
\[ A_{20}(u, v) = u^{\frac{1}{2}}(\Delta) \left( \frac{20}{3} G_{\Delta+2}^{(2)}(u, v) + \frac{5(\Delta + 4)^2}{12(\Delta + 3)(\Delta + 5)} u G_{\Delta+4}^{(2)}(u, v) \right) \]
\[
A_1(u, v) = u^{1/2} \Delta \left( 8G_{\Delta+1}^{(1)}(u, v) + \frac{2(\Delta + 4)^2}{(\Delta + 3)(\Delta + 5)} G_{\Delta+3}^{(3)}(u, v) + \frac{(\Delta + 4)^2}{2(\Delta + 3)(\Delta + 5)} u G_{\Delta+3}^{(1)}(u, v) 
+ \frac{(\Delta + 4)^2(\Delta + 6)^2}{8(\Delta + 3)(\Delta + 5)} u G_{\Delta+5}^{(3)}(u, v) + \frac{\Delta^2(\Delta + 4)^2}{32(\Delta - 1)(\Delta + 3)(\Delta + 5)} u^2 G_{\Delta+1}^{(1)}(u, v) 
+ \frac{(\Delta + 2)^2(\Delta + 4)^2(\Delta + 6)^2}{2^9(\Delta + 1)(\Delta + 3)^2(\Delta + 5)^2(\Delta + 7)} u^3 G_{\Delta+7}^{(1)}(u, v) \right),
\]

\[
A_{15}(u, v) = u^{1/2} \Delta \left( \frac{\Delta(\Delta + 4)}{16(\Delta - 1)(\Delta + 5)} u^2 G_{\Delta+4}^{(0)}(u, v) + \frac{5(\Delta + 4)^2(\Delta + 6)^2}{192(\Delta + 3)(\Delta + 5)^2(\Delta + 7)} u^2 G_{\Delta+6}^{(2)}(u, v) \right),
\]

The different contributions correspond exactly with those expected according to (8.11) with \( \ell \) restricted to be even or odd for the 1, 20, 84, 105 or 15, 175-representations.

We now turn to an analysis of the of the contributions corresponding to the functions \( \tilde{f}(z) \), \( \tilde{f}_2(z) \) in (6.14). As in the \( \mathcal{N} = 2 \) case these are associated with operators in which \( \Delta \) is related to \( \ell \). First if we restrict to only the contribution of the term \( k \) in (6.14) we have, setting \( k = 1 \),

\[
A_1 = 20, \quad A_{105} = A_{175} = A_{84} = A_{20} = A_{15} = 0 \quad \Rightarrow \quad \mathcal{A}(u, v) = \mathcal{I},
\]

where \( \mathcal{I} = (20, 0, 0, 0, 0, 0) \). This is obviously the contribution corresponding to the identity operator.

If we consider now in (6.14) just the function \( \tilde{f} \) we must have

\[
A_{105} = A_{175} = A_{84} = 0.
\]
Assuming $\tilde{f}(z) = g_{\ell+1}(z)$, in the notation of (7.9), we have using (7.10),

\begin{align*}
A_{20}(u, v) &= \frac{5}{3} uG_{\ell+2}(u, v), \\
A_{15}(u, v) &= 2 uG_{\ell+1}(u, v) + \frac{2(\ell + 1)^2}{(2\ell + 1)(2\ell + 3)} uG_{\ell+3}(u, v), \\
A_1(u, v) &= 5 uG_{\ell-2}(u, v) + \frac{10\ell(\ell + 1)}{3(2\ell - 1)(2\ell + 3)} uG_{\ell+2}(u, v) \\
&\quad + \frac{5(\ell + 1)(\ell + 2)^2}{(2\ell + 1)(2\ell + 3)(2\ell + 5)} uG_{\ell+4}(u, v). \tag{8.14}
\end{align*}

The results given by (8.13) and (8.14), with the notation in (8.8), may now be written as

\[ \mathcal{A}(u, v) = \mathcal{B}_\ell(u, v), \] 

defining the vector $\mathcal{B}_\ell$. The operators which give (8.14) in the operator product expansion have twist two and to comply with (6.16) $\ell$ must be even.

If we consider $\ell = 0$ in (8.14) this gives

\begin{align*}
A_{20}(u, v) &= \frac{5}{3} uG^{(0)}_2(u, v), \quad A_{15}(u, v) = \frac{2}{3} uG_3^{(1)}(u, v), \\
A_1(u, v) &= \frac{4}{9} uG_4^{(2)}(u, v) - 20. \tag{8.16}
\end{align*}

The term $-20$ in $A_1$ corresponds to the identity operator so that from (8.15) and (8.12) this may be written as

\[ \mathcal{B}_0(u, v) = \hat{\mathcal{B}}_0(u, v) - \mathcal{I}. \tag{8.17} \]

The result given by $\hat{\mathcal{B}}_0$ then represents precisely the contribution in the operator product expansion of the basic short multiplet exhibited in (2.15) where the lowest dimension operator is belongs to a $SU(4)_R$ 20-representation with $\Delta = 2$. The expression for $A_1$ arises from the energy momentum tensor in the operator product expansion and for $A_{15}$ from the $SU(4)_R$ conserved current.

For $\tilde{f}_2$ the results are more involved. We first take $\tilde{f}_2(z) = g_{\ell+2}(z)$ in (6.14) with $\ell$ even to ensure (5.16) holds. By virtue of (7.10) this leads to $f_2$ being a linear combination of $g_\ell, g_{\ell+2}, g_{\ell+4}$. Using (7.10) and also

\begin{align*}
g_\ell(z) + g_\ell(x) &= -2 G^{(\ell)}_\ell(u, v) + \frac{1}{2} uG^{(\ell-2)}_\ell(u, v) + \frac{\ell^2}{2(2\ell - 1)(2\ell + 1)} uG^{(\ell+2)}_{\ell+2}(u, v), \tag{8.18}
\end{align*}

48
which is derived in appendix C, we may then obtain

\[ A_{105}(u, v) = 0, \quad A_{175}(u, v) = uG^{(\ell+1)}_{\ell+3}(u, v), \quad A_{20}(u, v) = \frac{10}{3} G^{(\ell+2)}_{\ell+2}(u, v), \]

\[ A_{84}(u, v) = \frac{3}{2} uG^{(\ell)}_{\ell+2}(u, v) + \frac{3(\ell + 2)^2}{2(2\ell + 3)(2\ell + 5)} uG^{(\ell+2)}_{\ell+4}(u, v), \]

\[ A_{15}(u, v) = 4 G^{(\ell+1)}_{\ell+1}(u, v) + \frac{4(\ell + 2)^2}{(2\ell + 3)(2\ell + 5)} G^{(\ell+3)}_{\ell+3}(u, v), \]

\[ A_{1}(u, v) = 10 G^{(\ell)}_{\ell}(u, v) + \frac{20(\ell + 1)(\ell + 2)}{3(2\ell + 1)(2\ell + 5)} G^{(\ell+2)}_{\ell+2}(u, v) \]

\[ + \frac{10(\ell + 2)^2(\ell + 3)^2}{(2\ell + 3)(2\ell + 5)^2(2\ell + 7)} G^{(\ell+4)}_{\ell+4}(u, v) \]

\[ - \frac{5}{2} uG^{(\ell-2)}_{\ell}(u, v) - \frac{3\ell(\ell + 2)}{2(2\ell - 1)(2\ell + 5)} uG^{(\ell)}_{\ell+2}(u, v) \]

\[ - \frac{3(\ell + 1)(\ell + 2)^2(\ell + 3)}{2(2\ell + 1)(2\ell + 3)(2\ell + 5)(2\ell + 7)} uG^{(\ell+2)}_{\ell+4}(u, v) \]

\[ - \frac{5(\ell + 2)^2(\ell + 3)^2(\ell + 4)^2}{2(2\ell + 3)(2\ell + 5)^2(2\ell + 7)^2(2\ell + 9)} uG^{(\ell+4)}_{\ell+6}(u, v). \]  

(8.19)

For this case these results define the vector \( C_\ell \) so that (8.19) becomes

\[ A(u, v) = C_\ell(u, v). \]  

(8.20)

The results (8.19) are not acceptable in isolation since the corresponding operators do not satisfy the necessary unitarity conditions. This precludes any operators with twist \( \Delta - \ell = 0 \). To eliminate such terms, and also to obtain positive coefficients, we consider the combination

\[ D_\ell = G_{\ell, \ell} - 2C_\ell - B_\ell - \frac{(\ell + 2)^2}{(2\ell + 3)(2\ell + 5)} B_{\ell+2}, \]  

(8.21)

which removes all twist 0 and twist 2 terms. The detailed results for \( D_\ell \) are then

\[ A_{105}(u, v) = u^2G^{(\ell)}_{\ell+4}(u, v), \]

\[ A_{175}(u, v) = \frac{1}{4} u^2G^{(\ell-1)}_{\ell+3}(u, v) + \frac{(\ell + 2)^2}{2(2\ell + 3)(2\ell + 5)} u^2G^{(\ell+1)}_{\ell+5}(u, v) + \frac{1}{16} u^3G^{(\ell-1)}_{\ell+5}(u, v), \]

\[ A_{84}(u, v) = \frac{1}{4} u^2G^{(\ell)}_{\ell+4}(u, v) + \frac{1}{16} u^3G^{(\ell-2)}_{\ell+4}(u, v) + \frac{(\ell + 2)^2}{16(2\ell + 3)(2\ell + 5)} u^3G^{(\ell)}_{\ell+6}(u, v), \]

\[ A_{20}(u, v) = \frac{5}{12} u^2G^{(\ell-2)}_{\ell+2}(u, v) + \frac{5}{144} u^3G^{(\ell-2)}_{\ell+4}(u, v) + \frac{7}{72} u^2G^{(\ell)}_{\ell+4}(u, v) \]

\[ + \frac{5}{24(2\ell + 1)(2\ell + 5)} u^2G^{(\ell)}_{\ell+4}(u, v) + \frac{5(\ell + 2)^2(\ell + 3)^2}{12(2\ell + 3)(2\ell + 5)^2(2\ell + 7)} u^2G^{(\ell+2)}_{\ell+6}(u, v) \]

\[ + \frac{5(\ell + 2)^2}{144(2\ell + 3)(2\ell + 5)} u^3G^{(\ell)}_{\ell+6}(u, v) + \frac{1}{432} u^4G^{(\ell-2)}_{\ell+6}(u, v), \]

which are then
A_{15}(u, v) = \frac{1}{6} u^2 G^{(\ell-1)}_{\ell+3}(u, v) + \frac{1}{24} u^3 G^{(\ell-3)}_{\ell+3}(u, v) + \frac{(\ell + 2)^2}{6(2\ell + 3)(2\ell + 5)} u^2 G^{(\ell+1)}_{\ell+5}(u, v) \\
+ \frac{(2\ell + 3)^2 - 3}{48(2\ell + 1)(2\ell + 5)} u^3 G^{(\ell-1)}_{\ell+5}(u, v) + \frac{1}{360} u^4 G^{(\ell-3)}_{\ell+5}(u, v) \\
+ \frac{(\ell + 2)^2(\ell + 3)^2}{24(2\ell + 3)(2\ell + 5)^2(2\ell + 7)} u^3 G^{(\ell+1)}_{\ell+7}(u, v) \\
+ \frac{(\ell + 2)^2}{360(2\ell + 3)(2\ell + 5)} u^4 G^{(\ell-1)}_{\ell+7}(u, v), \\
A_1(u, v) = \frac{1}{9} u^2 G^{(\ell)}_{\ell+4}(u, v) + \frac{1}{36} u^3 G^{(\ell-2)}_{\ell+4}(u, v) + \frac{1}{144} u^4 G^{(\ell-4)}_{\ell+4}(u, v) \\
+ \frac{(\ell + 2)^2}{36(2\ell + 3)(2\ell + 5)} u^3 G^{(\ell)}_{\ell+6}(u, v) + \frac{(\ell + 1)(\ell + 2)}{216(2\ell + 1)(2\ell + 5)} u^4 G^{(\ell-2)}_{\ell+6}(u, v) \\
+ \frac{(\ell + 2)^2(\ell + 3)^2}{144(2\ell + 3)(2\ell + 5)^2(2\ell + 7)} u^4 G^{(\ell)}_{\ell+8}(u, v). (8.22)

For \( \ell = 2, 4 \ldots \) these results correspond to the lowest dimension operator having \( \Delta = \ell + 2 \), belonging to the 20-representation, with maximum dimension \( \Delta = \ell + 8 \). \( \ell = 0 \) is again a special case. Using (7.7) and (8.10) we obtain

\[
A_{105}(u, v) = u^2 G_4^{(0)}(u, v), \quad A_{175}(u, v) = \frac{2}{15} u^2 G_5^{(1)}(u, v), \quad A_{84}(u, v) = \frac{1}{60} u^3 G_6^{(0)}(u, v), \quad A_{20}(u, v) = -\frac{5}{3} u^2 G_2^{(0)}(u, v) + \frac{1}{35} u^2 G_6^{(2)}(u, v), \\
A_{15}(u, v) = -\frac{2}{3} u^2 G_3^{(1)}(u, v) + \frac{1}{350} u^3 G_7^{(1)}(u, v), \quad A_1(u, v) = -\frac{4}{3} u^2 G_4^{(2)}(u, v) + \frac{1}{2100} u^4 G_8^{(0)}(u, v). (8.23)
\]

The negative terms present in the results for \( A_1, A_{15}, A_{20} \) in (8.23) are just those corresponding to the short supermultiplet built on the scalar with \( \Delta = 2 \) belonging to the 20-representation, as given in (8.10). Thus we may write

\[
\mathcal{D}_0 = -\hat{B}_0 + \hat{D}_0, \quad (8.24)
\]

where the terms in \( \hat{D}_0 \) then represent the contribution for a lowest dimension scalar operator with \( \Delta = 4 \) belonging to the 105-representation. This is a chiral primary operator and the contributions to \( \mathcal{D}_0 \) correspond exactly to the short supermultiplet operators with \( Y = 0 \) for this case, as listed in appendix B for \( p = 4 \).

When \( \Delta = \ell + 2 \) the contributions given by the operator product expansion in \( \mathcal{G}_{\Delta, \ell} \) can also be decomposed since we may write

\[
\mathcal{G}_{\ell+2, \ell} = 4 \mathcal{B}_{\ell+2} + \mathcal{E}_\ell, \quad (8.25)
\]
where $E_\ell$ corresponds to operators with twist $\geq 4$. The detailed results for $E_\ell$ are then given by

$$A_{105}(u,v) = u^3 G^{(\ell)}_{\ell+6}(u,v),$$

$$A_{175}(u,v) = 2u^2 G^{(\ell+1)}_{\ell+5}(u,v) + \frac{1}{2} u^3 G^{(\ell-1)}_{\ell+5}(u,v) + \frac{(\ell + 3)^2}{2(2\ell + 5)(2\ell + 5)} u^3 G^{(\ell+1)}_{\ell+7}(u,v) + \frac{1}{20} u^4 G^{(\ell-1)}_{\ell+7}(u,v),$$

$$A_{84}(u,v) = 3u^2 G^{(\ell)}_{\ell+4}(u,v) + \frac{1}{4} u^3 G^{(\ell)}_{\ell+6}(u,v) + \frac{3(\ell + 3)^2}{(2\ell + 5)(2\ell + 7)} u^3 G^{(\ell+2)}_{\ell+6}(u,v) + \frac{1}{20} u^4 G^{(\ell-2)}_{\ell+6}(u,v) + \frac{(\ell + 3)^2}{20(2\ell + 5)(2\ell + 7)} u^4 G^{(\ell)}_{\ell+8}(u,v),$$

$$A_{20}(u,v) = \frac{5}{3} u^2 G^{(\ell)}_{\ell+4}(u,v) + \frac{5}{12} u^3 G^{(\ell-2)}_{\ell+4}(u,v) + \frac{5(\ell + 3)^2}{3(2\ell + 5)(2\ell + 7)} u^3 G^{(\ell+2)}_{\ell+6}(u,v) + \frac{5(\ell + 3)^2}{24(2\ell + 3)(2\ell + 7)} u^3 G^{(\ell)}_{\ell+6}(u,v) + \frac{1}{36} u^4 G^{(\ell-2)}_{\ell+6}(u,v) + \frac{5(\ell + 3)^2}{12(2\ell + 5)(2\ell + 7)^2(2\ell + 9)} u^3 G^{(\ell+2)}_{\ell+8}(u,v) + \frac{(\ell + 3)^2}{36(2\ell + 5)(2\ell + 7)} u^4 G^{(\ell)}_{\ell+8}(u,v) + \frac{1}{560} u^5 G^{(\ell-2)}_{\ell+8}(u,v),$$

$$A_{15}(u,v) = 2u^2 G^{(\ell-1)}_{\ell+3}(u,v) + \frac{(2\ell + 5)^2 - 3}{(2\ell + 3)(2\ell + 7)} u^2 G^{(\ell+1)}_{\ell+5}(u,v) + \frac{(2\ell + 5)^2}{(2\ell + 3)(2\ell + 7)^2} u^2 G^{(\ell+3)}_{\ell+7}(u,v) + \frac{2(\ell + 3)^2}{2(2\ell + 5)(2\ell + 7)^2(2\ell + 9)} u^2 G^{(\ell+3)}_{\ell+7}(u,v) + \frac{3(\ell + 3)^2}{10(2\ell + 5)(2\ell + 7)} u^3 G^{(\ell+1)}_{\ell+7}(u,v) + \frac{3(\ell + 3)^2}{60(2\ell + 3)(2\ell + 7)} u^3 G^{(\ell+1)}_{\ell+7}(u,v) + \frac{3(\ell + 3)^2}{30(2\ell + 5)(2\ell + 7)^2(2\ell + 9)} u^4 G^{(\ell+1)}_{\ell+9}(u,v) + \frac{3(\ell + 2)^2}{1400(2\ell + 5)(2\ell + 7)} u^5 G^{(\ell-1)}_{\ell+9}(u,v),$$

$$A_1(u,v) = \frac{4}{3} u^2 G^{(\ell)}_{\ell+4}(u,v) + \frac{1}{3} u^3 G^{(\ell-2)}_{\ell+4}(u,v) + \frac{4(\ell + 3)^2}{3(2\ell + 5)(2\ell + 7)} u^2 G^{(\ell-2)}_{\ell+6}(u,v) + \frac{(2\ell + 5)^2 - 3}{6(2\ell + 3)(2\ell + 7)} u^3 G^{(\ell)}_{\ell+6}(u,v) + \frac{1}{45} u^4 G^{(\ell-2)}_{\ell+6}(u,v) + \frac{(\ell + 3)^2}{560(2\ell + 5)(2\ell + 7)^2(2\ell + 9)} u^5 G^{(\ell)}_{\ell+10}(u,v).$$

In this the lowest dimension operator occurs in the 15-representation with $\Delta = \ell + 3$.\[\text{(8.26)}\]
If we use (8.17), (8.21), (8.24), assuming
where, along with ˆ
Thus for \( \mathcal{E}_0 \) the lowest dimension relevant operator is a scalar belonging to the 84-representation with \( \Delta = 4 \), corresponding to a well known short multiplet.

As a result of the above considerations we expect in general expansions of the form

\[
\mathcal{G}(u, v) = \sum_{\Delta, \ell} a_{\Delta, \ell} u^{\frac{1}{2}(\Delta - \ell)} G_{\Delta + 4}^{(\ell)}(u, v),
\]

\[
\tilde{f}(z) = \sum_{\ell} b_{\ell} g_{\ell+1}(z), \quad \tilde{f}_2(z) = \sum_{\ell} c_{\ell} g_{\ell+2}(z), \quad \ell = 0, 2, 4 \ldots, \tag{8.28}
\]

where, along with \( k \), \( a_{\Delta, \ell}, b_{\ell}, c_{\ell} \) determine the operator product expansion coefficients for all \( A_R \) since we may write

\[
A(u, v) = k \mathcal{I} + \sum_{\ell} (b_{\ell} \mathcal{B}_\ell(u, v) + c_{\ell} \mathcal{C}_\ell(u, v)) + \sum_{\Delta, \ell} a_{\Delta, \ell} \mathcal{G}_{\Delta, \ell}(u, v). \tag{8.29}
\]

If we use (8.17), (8.21), (8.24), assuming \( c_{\ell} + 2a_{\ell, \ell} = 0 \) as is necessary for unitarity, this may be rewritten in the following form

\[
A(u, v) = C \mathcal{I} + b_0 \mathcal{B}_0(u, v) + d_0 \mathcal{D}_0(u, v) + e_0 \mathcal{E}_0(u, v)
+ \sum_{\ell \geq 2} (d_{\ell} \mathcal{D}_\ell(u, v) + e_{\ell} \mathcal{E}_\ell(u, v)) + \sum_{\Delta, \ell} \hat{a}_{\Delta, \ell} \mathcal{G}_{\Delta, \ell}(u, v), \tag{8.30}
\]

where

\[
C = k - b_0 + \frac{1}{2} c_0, \quad d_\ell = -\frac{1}{2} c_\ell, \quad e_\ell = -\frac{1}{4} b_{\ell+2} + \frac{1}{8} c_{\ell+2} + \frac{(\ell + 2)^2}{8(2\ell + 1)(2\ell + 3)} c_\ell, \tag{8.31}
\]

and \( \hat{a}_{\Delta, \ell} \) is identical with \( a_{\Delta, \ell} \) save that \( \hat{a}_{\ell, \ell} = 0 \) and \( \hat{a}_{\ell+2, \ell} = a_{\ell+2, \ell} - e_\ell \). In the first line of (8.30) the contribution of various short multiplets is explicit. The range of scale dimensions and spins of the sets of operators which contribute to the different terms in (8.30) are summarised in the following table,
As an illustration of these results we consider the free case when $a_1 = a_2 = a_3 = a$ and $c_1 = c_2 = c_3 = c$ are both constants. From (6.12) and (6.14) we then have

$$G(u,v) = a\left(1 + \frac{1}{v^2}\right) + c\frac{1}{v},$$

and from (6.8), (6.10) and (6.15),

$$k = 3(a + c),$$

$$\tilde{f}(z) = a(z^2 - z'^2) + c(z - z'), ~ \tilde{f}_2(z) = a(z^2 + z'^2) - c(z + z'), ~ z' = \frac{z}{z - 1}. \quad (8.33)$$

The expansion coefficients in (8.28) for this case may be determined in terms of our previous results [13],

$$a_{\Delta,\ell} = 2^{\ell\ell}(\ell + t)!((\ell + t + 1)!\ell)! a(\ell + 1)(\ell + 2t + 2) + c(-1)^t \delta_{\Delta,\ell + 2t}, \quad \ell = 0, 2, \ldots, t = 0, 1, \ldots, \quad (8.34)$$

and also

$$b_\ell = 2^{\ell+1}\frac{(\ell)!}{(2\ell)!}\left(-\ell(\ell + 1)a + c\right), ~ c_\ell = -2^{\ell+1}\frac{\ell!(\ell + 1)!}{(2\ell + 1)!}\left((\ell + 1)(\ell + 2)a + c\right), \quad \ell = 0, 2, \ldots. \quad (8.35)$$

Clearly the condition $2a_{\ell,\ell} + c_\ell = 0$ is satisfied and in (8.30) we have by using (8.31)

$$C = a, \quad b_0 = 2c, \quad d_\ell = 2^{\ell\ell}\frac{\ell!(\ell + 1)!}{(2\ell + 1)!}\left((\ell + 1)(\ell + 2)a + c\right),$$

$$e_\ell = 2^{\ell\ell}\frac{(\ell + 2)!}{(2\ell + 4)!}\left((\ell + 1)(\ell + 4)a - 3c\right), \quad (8.36)$$

and furthermore

$$\hat{a}_{\ell + 2,\ell} = 2^{\ell+1}\frac{(\ell + 2)!}{(2\ell + 4)!} c. \quad (8.37)$$
9. Results for Weak Coupling and Large $N$

We show here how the results just exhibited in general in section 7 for the operator product expansion may be used to recover and extend previous results [11] obtained in an analysis of perturbative corrections to the four point correlation function in (5.23). It is convenient first to rewrite the solution of the superconformal identities in the form

\begin{align*}
a_1 &= \frac{1}{4} \hat{N}^2 + \frac{1}{2} \hat{N} u F, \\
a_2 &= \frac{1}{4} \hat{N}^2 + \frac{1}{2} \hat{N} F, \\
a_3 &= \frac{1}{4} \hat{N}^2 + \frac{1}{2} \hat{N} v F, \\
c_1 &= \hat{N} \left( 1 + \frac{1}{2} (1 - v - u) F \right), \\
c_2 &= \hat{N} \left( 1 - \frac{1}{2} (1 + v - u) F \right), \\
c_3 &= \hat{N} \left( 1 - \frac{1}{2} (1 - v + u) F \right). \tag{9.1}
\end{align*}

Here we have used the normalisation convention of (3.8) and also, following [10], assumed that the contributions corresponding to the functions $f_1, f_2, f_3$ are given just by free field results. The non-trivial dynamical results are therefore contained in the function $F(u, v)$ which for (5.36) and (5.35) must satisfy $F(u, v) = F(v, u) = F(u', v')/v$. From (6.12) and (6.14) we have

\begin{align*}
G(u, v) &= \frac{1}{4} \hat{N}^2 \left( 1 + \frac{1}{v^2} \right) + \hat{N} \frac{1}{v} \left( 1 + \frac{1}{2} u F(u, v) \right). \tag{9.2}
\end{align*}

The contributions of short supermultiplets to the operator product expansion are therefore given just by the free results and we use the formulæ obtained in section 7 with $a = \frac{1}{4} \hat{N}^2$, $c = \hat{N}$. Denoting the operator product coefficients for operators in representation $R$ by $a^R_{\Delta, \ell}$ the identity operator and the three short multiplets are given, from (8.36), by

\begin{align*}
a_{0,0}^1 &= \frac{1}{4} \hat{N}^2, \\
a_{0,0}^{20} &= \frac{10}{3} \hat{N}, \\
a_{0,0}^{84} &= \frac{1}{2} \hat{N}^2 \left( 1 - \frac{3}{N} \right), \\
a_{0,0}^{105} &= \frac{1}{2} \hat{N}^2 \left( 1 + \frac{2}{N} \right). \tag{9.3}
\end{align*}

where we have absorbed the relevant coefficients from (8.16), (8.27) for $R = 20, 84$. There are also infinite sequences of unrenormalised operators associated with the coefficients $d_{\ell}, e_{\ell}$ (8.30) which correspond to

\begin{align*}
a_{\ell+4, \ell}^{20} &= \hat{N}^2 2\ell \frac{5(\ell + 2)! (\ell + 3)!}{3(2\ell + 5)!} \left( \frac{1}{4}(\ell + 3)(\ell + 4) + \frac{1}{N} \right), \\
a_{\ell+5, \ell+1}^{15} &= \hat{N}^2 2\ell^3 \frac{((\ell + 4))!^2}{(2\ell + 8)!} \left( \frac{1}{4}(\ell + 3)(\ell + 6) + \frac{3}{N} \right). \tag{9.4}
\end{align*}

The formulæ (9.3) and (9.4) are in agreement with the results in [11].

The $O(g^2)$ result for the four point function in (5.23) in a $SU(N)$ $N = 4$ superconformal theory, when $\hat{N} = N^2 - 1$ is the dimension of the adjoint representation, has been
calculated by various authors \[27,28,29\]. It is reducible to a standard conformal integral giving for
\[ F(u, v) = -\tilde{\lambda} \Phi^{(1)}(u, v), \quad \tilde{\lambda} = \frac{g^2 N}{4\pi^2}, \quad (9.5) \]
where, using the variables \( z, x \) defined in (6.1),
\[ \Phi^{(1)}(u, v) = \frac{1}{z - x} \left( \ln zx \ln \frac{1 - z}{1 - x} + 2\ln_2(z) - 2\ln_2(x) \right), \quad (9.6) \]
with \( \ln_2 \) the dilogarithm function. The \( \ln u = \ln zx \) term in (9.6) generates a shift in the dimensions of long supermultiplets, the dilogarithm terms generate a series which is analytic in \( u, 1 - v \). To analyse the consequences of the \( \ln u \) term we expand
\[ -\frac{u}{v} \frac{1}{z - x} \ln \frac{1 - z}{1 - x} = \sum_{t, \ell} \epsilon_{t,\ell} u^t G^{(\ell)}_{\ell+2t+4}(u, v). \quad (9.7) \]
By matching terms in a power series expansion we find
\[ \epsilon_{t,\ell} = 2^{\ell+3} \frac{((\ell + t + 1)! t!)^2}{(2\ell + 2t + 2)! (2t)!} \sum_{j=1}^{\ell+t+1} \frac{1}{j}, \quad \ell = 0, 2, \ldots \\
\epsilon_{t,\ell} = -2^{\ell+2} \frac{((\ell + t)! (t + 1)! t!)^2}{(2\ell + 2t + 1)! (2t)!} \sum_{j=1}^{t} \frac{1}{j}, \quad t = 0, 2, \ldots \]
(9.8)

For long multiplets the dimensions of the lowest weight operators in each supermultiplet may be written as
\[ \Delta_{t,\ell} = 2t + \ell + \eta_{t,\ell}, \quad t = 1, 2, \ldots, \ell = 0, 2, \ldots. \quad (9.9) \]
We first consider \( t = 1 \). Using (8.37), with \( c = \hat{N} \), then the perturbative results give
\[ \eta_{1,\ell} = \frac{\epsilon_{1,\ell}}{a_{\ell+2,2}} \tilde{\lambda} \hat{N} = 2 \sum_{j=1}^{\ell+2} \frac{1}{j} \tilde{\lambda}, \quad (9.10) \]
which is the result given in the introduction in (1.1). Note that \( \eta_{1,0} = 3\tilde{\lambda}, \eta_{1,2} = \frac{25}{6} \tilde{\lambda}, \eta_{1,4} = \frac{49}{10} \tilde{\lambda} \) which coincide with the results of Anselmi [17]. For \( \ell = 0 \) this corresponds to the Konishi supermultiplet. This and also \( \eta_{1,2} \) were obtained in [11] by analysis of the perturbative four point function, but here there is no need to separate carefully the contributions of the three free field theory \( \Delta = 4, \ell = 2 \) operators, which include the energy momentum tensor and a descendant of the Konishi scalar operator. For \( t > 1 \) we
just quote the large $N$ result based on the tacit assumption that there is just one long supermultiplet contributing to the operator product expansion in each case,

\[ \eta_{t,\ell} = \frac{16}{(\ell + 1)(\ell + 2t + 2)} \sum_{j=1}^{\ell+t+1} \frac{1}{j} \tilde{\lambda} N^2, \quad \ell = 0, 2\ldots, t = 3, 5, \ldots, \]

and

\[ \eta_{t,\ell} = -\frac{16(t+1)}{(\ell + 1)(\ell + t + 1)(\ell + 2t + 2)} \sum_{j=1}^{t} \frac{1}{j} \tilde{\lambda} N^2, \quad \ell = 0, 2\ldots, t = 2, 4, \ldots. \]

These results correspond to ‘double trace’ operators.

We now consider the analogous results for large $N$ and also large $\tilde{\lambda}$ which may be obtained from supergravity calculations on $AdS_5 \times S^5$ using the AdS/CFT correspondence. Arutunov and Frolov [18] calculated the leading large $N$ behaviour of the four point function for chiral primary operators in the $20$-representation which is expressed here in the form (5.23). As shown in appendix D their result may be simplified to the form given in (9.1) with

\[ \mathcal{F}(u, v) = -2\mathcal{D}_{2224}(u, v), \]

where $\mathcal{D}_{2224}(u, v)$ are functions of the conformal invariants $u, v$ which arise from Feynman type integrals on AdS space. For our purposes we make use of the analysis in [13] to write

\[ \mathcal{D}_{2224}(u, v) = \frac{1}{3} F(2, 1; 4; 1-v) + G(u, v) u \ln u + O(u), \]

where the discarded terms are analytic in $u, 1-v$. From [13] the function $G$ has the expansion

\[ G(u, v) = \frac{1}{5} \sum_{m,n=0} (3m(4)_m (3m+n(2)_m+n+u^{m+n+1}(1-v)^n, \quad (\alpha)_m = \frac{\Gamma(\alpha + m)}{\Gamma(\alpha)}. \]

The $\ln u$ terms in (9.13) are absorbed by a shift in the scale dimensions of the operators appearing in the operator product expansion. To determine these we first expand

\[ \frac{u^2}{v} G(u, v) = \sum_{t,\ell} \tilde{\epsilon}_{t,\ell} u^t G^{(\ell)}_{t+2t+4}(u, v). \]

\[ G(u, v) = \left( \frac{120 uv^2}{(z-x)^7} + \frac{12v}{(z-x)^5}(1 + u + v) \right) \ln \frac{1-z}{1-x} + \frac{60uv}{(z-x)^6}(1 + v - u) + \frac{2}{(z-x)^4} \left( (1 + v)(1 + v - u) + 16v \right). \]

\[ 56 \]
By using the expansion (9.14) the coefficients $\tilde{\epsilon}_{t,\ell}$ may be determined iteratively and can then be fitted to the formula
\[
\tilde{\epsilon}_{t,\ell} = 2^\ell - 2(t - 1) \frac{t!(t + 2)! (\ell + t)! (\ell + t + 1)!}{(2t - 1)! (2\ell + 2t + 1)!}, \quad \ell = 0, 2 \ldots
\]
\[
t = 2, 3, \ldots.
\] (9.16)

Assuming a single lowest weight operator for a long representation for any $\Delta, \ell$ the order $1/N^2$ corrections to the dimensions in (9.9) may be determined from $\eta_{t,\ell} = -2N^2 \tilde{\epsilon}_{t,\ell}/a_{\Delta,\ell}$ where $a_{\Delta,\ell}$ is given by (8.34) for $t = 2, 3 \ldots$ and $a \to \frac{1}{4}N^4$,
\[
\eta_{t,\ell} = -\frac{4(t - 1)t(t + 1)(t + 2)}{(\ell + 1)(\ell + 2t + 2)} \frac{1}{N^2}.
\] (9.17)

For $t = 2$ this coincides with the result of Hoffman, Mesref and Rühl [30] who considered the four point function involving, in our notation, the descendant fields $\Phi$ and $\bar{\Phi}$. For the simplest case, $t = 2, \ell = 0$, $\eta_{2,0} = -16/N^2$ was derived earlier in [31] and also from the results of [18] in [12].

We may also take account of the first term in (9.13) by expanding
\[
\frac{u}{3v} F(2, 1; 4; 1 - v) = \sum_{\ell=0} b_{\ell} u G_{\ell+6}^{(t)} (u, v) + \text{higher twist},
\] (9.18)
or equivalently
\[
\frac{1}{1 - z} \frac{1}{3} F(2, 1; 4; z) = \sum_{\ell=0} b_{\ell} (-\frac{1}{2}z)^{\ell} F(\ell + 3, \ell + 3; 2\ell + 6; z),
\] (9.19)
which gives
\[
b_{\ell} = 2^{\ell+1} \frac{((\ell + 2)!)^2}{(2\ell + 4)!}, \quad \ell = 0, 2, \ldots.
\] (9.20)

The contribution to $a_{\Delta,\ell}$ in (8.28) is then $-b_{\ell}N^2$ for $\Delta = \ell + 2$. This just cancels $\hat{a}_{\ell+2,\ell}$ in (8.37) which is a reflection of the fact that the supermultiplet containing the Konishi scalar and its partners disappear from the spectrum in the large $N$ limit [12].

10. Discussion

We have endeavoured to show in this paper the exact compatibility with superconformal identities for the four point function of chiral primary operators belonging to the simplest short representations of $\mathcal{N} = 2$ and $\mathcal{N} = 4$ superconformal symmetry with the representation content of the various possible supermultiplets of operators which may contribute in the operator product expansion. In both cases the superconformal identities are
solved in terms of a single function $G(u, v)$ of the two conformal invariants which may be expanded in terms of operators belonging to long supermultiplets whose lowest dimension operator is a singlet under the appropriate $R$ symmetry group with arbitrary $\Delta$. This is in agreement with the analysis of three point functions in [4, 5, 6]. In our case the analysis depended on the result that for such supermultiplets there was just one operator contributing to the operator product expansion belonging to the $R = 2$ representation for $\mathcal{N} = 2$ and the 105-dimensional representation for $\mathcal{N} = 4$. Although we have not shown in detail the impossibility of incorporating alternative long multiplets in the operator product expansion for the four point functions here it is clear, from the structure of (6.6) and (6.14), that for contributions for general $\Delta$ alone it is impossible to have operators with a finite range of dimensions and spins if any other representation is assumed to contain just one operator.

In addition the contribution of possible short supermultiplets involve the single variable functions which are present in the solution of the superconformal identities. The arguments of Eden et al [10] show that such contributions have no perturbative corrections and are essentially given by the results for free field theory. Thus for $\mathcal{N} = 2$ we have exhibited the contributions for lowest weight operators scalars with $\Delta = 2R$, for $R = 1, 2$, and for $\mathcal{N} = 4$ for chiral primary operators belonging to the $[q, p, q]$ representation, $\Delta = p + 2q$, for the relevant cases here of $q = 0, p = 1, 2$ and $q = 2, p = 0$. The terms appearing in the operator product expansion tie in with the expected representation content. For these short representations the crucial restrictions arise at the first level.

There are also contributions for arbitrary spin $\ell$ which arise for $\mathcal{N} = 2$ when the basic inequality (7.6) is saturated for $R = 0, 1, r = 0$. For $\mathcal{N} = 4$ the corresponding unitarity inequality for a lowest weight operator with spin $\ell$ belonging to the $[q, p, q]$-representation is [9],

$$\Delta \geq 2 + \ell + p + 2q$$

(10.1)

In the results in section 8 we have found contributions to the operator product expansion when this is an equality for $\ell$ even for the 20-dimensional representation, $p = 2, q = 0$, represented by $\mathcal{D}_{\ell+2}$, and for $\ell$ odd for the 15-dimensional representation, $p = 0, q = 1$, represented by $\mathcal{E}_{\ell+1}$. The necessity for a protected operator in the 20-representation with $\Delta = 4$ was first shown in [12]. The full supermultiplet structure does not seem to have been exhibited in these cases but our results seem to imply that they do not correspond to a full long supermultiplet with dimensions proportional to $2^{16}$. The constraint on the dimensions is perhaps an extension of the condition that the conservation condition on a current is only compatible with conformal invariance in $d$-dimensions if $\Delta = d - 1$. A related discussion $\mathcal{N} = 4$ supersymmetric theories is given in [33]. The simplest example in

---

8 For a recent discussion of short multiplets with $q \neq 0$ in $\mathcal{N} = 4$ theories see [34].
this context is the supercurrent multiplet in $\mathcal{N} = 2$ superconformal symmetry [34]. With a similar notation to (7.19), where the representations are denoted by $R_{(j_1,j_2)}$, this has the structure

\begin{equation}
\begin{array}{c}
\Delta \\
2 \\
\frac{5}{2} \\
3 \\
\frac{7}{2} \\
r
\end{array}
\begin{array}{c}
0_{(0,0)} \\
0_{\left(\frac{1}{2},0\right)} \\
0_{\left(\frac{1}{2},\frac{1}{2}\right)}, 1_{\left(\frac{1}{2},\frac{1}{2}\right)} \\
0_{\left(1,\frac{1}{2}\right)} \\
0_{(1,1)} \\
0 \\
-\frac{1}{2} \\
-1
\end{array}
\end{equation}

This contains the conserved currents for $U(2)_R$ symmetry and the conserved energy momentum tensor as well as the conserved spinorial supercurrents so that there are 24 bosonic and 24 fermionic degrees of freedom. There is however no shortening at the first level, so there is no BPS-like condition in this case, although shortening appears at level 2. The operators appearing in (10.2) for $r = 0$ match those necessary for the operator product expansion results for the four point function in (7.15). In general when (10.1) becomes an equality there are possible constraints on superfields [33] which may imply shortening at level 2.

A remaining issue is whether the conditions derived in section 5, and solved in section 6, exhaust all the implications of superconformal symmetry in these cases. This seems highly probable in that the conditions derived reflect directly the shortening conditions at the first level in the supermultiplets to which the chiral primary operators belong but although they are necessary there is no guarantee of sufficiency as yet. Of course in the future it would be very nice to investigate more general correlation functions than those considered in this paper.

**Acknowledgements**

One of us (FAD) would like to thank the EPSRC, the National University of Ireland and Trinity College, Cambridge for support and is also very grateful to David Grellscheid for help with Mathematica. HO would like to thank Massimo Bianchi, Anastasios Petkou and Johanna Erdmenger for useful conversations.
Appendix A. SU(4) formulae

The link between the 4-dimensional indices $i, j, \ldots$ and the 6-dimensional indices $r, s, \ldots$ is given by the antisymmetric gamma matrices,

$$
\gamma_{r}^{ij} = -\gamma_{r}^{ji}, \quad \bar{\gamma}_{rij} = \frac{1}{2} \varepsilon_{ijkl} \gamma_{r}^{kl}, \tag{A.1}
$$

where we impose the completeness/orthogonality relations

$$
\gamma_{r}^{ij} \bar{\gamma}_{sij} = 4 \delta_{rs}, \quad \gamma_{r}^{ij} \bar{\gamma}_{rkl} = 4 \delta_{[i} \delta_{j]} \Rightarrow \bar{\gamma}_{rij} \bar{\gamma}_{rkl} = 2 \varepsilon_{ijkl}. \tag{A.2}
$$

It is then easy to see that we have the usual $\gamma$-matrix algebra

$$
\gamma_{r} \bar{\gamma}_{s} + \gamma_{s} \bar{\gamma}_{r} = -2 \delta_{rs} 1, \quad \bar{\gamma}_{r} \gamma_{s} + \bar{\gamma}_{s} \gamma_{r} = -2 \delta_{rs} 1, \tag{A.3}
$$

with 1 the $4 \times 4$ unit matrix. These $\gamma$-matrices also satisfy

$$
\gamma_{[r} \bar{\gamma}_{s]} \gamma_{[u} \bar{\gamma}_{v]} = i \varepsilon_{rstuv} 1, \tag{A.4}
$$

involving the six dimensional antisymmetric symbol. This leads to the useful relations

$$
\gamma_{[r} \bar{\gamma}_{s]} \gamma_{[u} \bar{\gamma}_{v]} = -i \varepsilon_{rstuv} \gamma_{w}, \quad \gamma_{[r} \bar{\gamma}_{s]} \gamma_{[u} \bar{\gamma}_{v]} = -\frac{1}{2} i \varepsilon_{rstuv} \bar{\gamma}_{w}, \tag{A.5}
$$

For discussion of the four point function for the 20-dimensional representation formed by $\varphi_{rs}$ it is also convenient to define a basis by $C_{rs}^{I} = C_{sr}^{I}$, $C_{rr}^{I} = 0$ satisfying

$$
C_{rs}^{I} C_{rs}^{J} = \delta^{IJ}, \quad C_{rs}^{I} C_{uv}^{I} = \frac{1}{2} \left( \delta_{ru} \delta_{sv} + \delta_{rv} \delta_{su} \right) - \frac{1}{6} \delta_{rs} \delta_{uv}. \tag{A.6}
$$

For the correlation functions of $\varphi^{I} \equiv C_{rs}^{I} \varphi_{rs}$ the relevant invariant tensors are then

$$
\text{tr} \left( C^{I_{1}} C^{I_{2}} \ldots C^{I_{n}} \right) = C^{I_{1} I_{2} \ldots I_{n}} = C^{I_{n} I_{n-1} \ldots I_{1}} = C^{I_{n} \ldots I_{1} I_{n-1}}, \tag{A.7}
$$

and we also define

$$
C^{I_{1} I_{2} \ldots I_{n}} = C_{rs}^{I} C^{I_{1} I_{2} \ldots I_{n}} = (C^{I} \ldots C^{I})_{(rs)} - \frac{1}{6} \delta_{rs} C^{I_{1} I_{2} \ldots I_{n}}. \tag{A.8}
$$

Since $[0, 2, 0] \otimes [0, 2, 0] = [1, 2, 1] \oplus [0, 4, 0] \oplus [2, 0, 2] \oplus [0, 2, 0] \oplus [1, 0, 1] \oplus [0, 0, 0]$ ($20 \times 20 = 175 + 105 + 84 + 20 + 15 + 1$), the four point function $\langle \varphi^{I_{1}} \varphi^{I_{2}} \varphi^{I_{3}} \varphi^{I_{4}} \rangle$ may be decomposed
into six irreducible representations for which the associated projection operators\footnote{With a different normalisation these were given in\cite{11}.} are

\begin{align}
P_{1^{1}I_{2}I_{3}I_{4}}^{1} &= \frac{1}{20} \delta_{I_{1}I_{2}} \delta_{I_{3}I_{4}}, \\
P_{1^{1}I_{2}I_{3}I_{4}}^{15} &= -\frac{1}{4} (C_{I_{1}I_{2}I_{3}I_{4}} - C_{I_{2}I_{1}I_{3}I_{4}}), \\
P_{20}^{1}I_{2}I_{3}I_{4} &= \frac{3}{10} (C_{I_{1}I_{2}I_{3}I_{4}} + C_{I_{2}I_{1}I_{3}I_{4}}) - \frac{1}{10} \delta_{I_{1}I_{2}} \delta_{I_{3}I_{4}}, \\
P_{84}^{1}I_{2}I_{3}I_{4} &= \frac{1}{3} (\delta_{I_{1}I_{3}} \delta_{I_{2}I_{4}} + \delta_{I_{2}I_{3}} \delta_{I_{1}I_{4}}) + \frac{1}{30} \delta_{I_{1}I_{2}} \delta_{I_{3}I_{4}} \\
&\quad - \frac{2}{3} C_{I_{1}I_{3}I_{2}I_{4}} - \frac{1}{6} (C_{I_{1}I_{2}I_{3}I_{4}} + C_{I_{2}I_{1}I_{3}I_{4}}), \\
P_{105}^{1}I_{2}I_{3}I_{4} &= \frac{1}{6} (\delta_{I_{1}I_{3}} \delta_{I_{2}I_{4}} + \delta_{I_{2}I_{3}} \delta_{I_{1}I_{4}}) + \frac{1}{60} \delta_{I_{1}I_{2}} \delta_{I_{3}I_{4}} \\
&\quad + \frac{2}{3} C_{I_{1}I_{3}I_{2}I_{4}} - \frac{1}{15} (C_{I_{1}I_{2}I_{3}I_{4}} + C_{I_{2}I_{1}I_{3}I_{4}}), \\
P_{175}^{1}I_{2}I_{3}I_{4} &= \frac{1}{2} (\delta_{I_{1}I_{3}} \delta_{I_{2}I_{4}} - \delta_{I_{2}I_{3}} \delta_{I_{1}I_{4}}) + \frac{1}{4} (C_{I_{1}I_{2}I_{3}I_{4}} - C_{I_{2}I_{1}I_{3}I_{4}}),
\end{align}

which satisfy

\[ P_{R}^{I_{1}I_{2}I_{3}I_{4}} P_{R'}^{J_{1}J_{2}J_{3}J_{4}} = \delta_{RR'} P_{R}^{I_{1}I_{2}I_{3}I_{4}}, \quad \sum_{R} P_{R}^{I_{1}I_{2}I_{3}I_{4}} = \delta_{I_{1}I_{3}} \delta_{I_{2}I_{4}}, \quad P_{R}^{I_{1}J_{1}J_{2}J_{3}J_{4}} = R. \quad (A.10)\]
arise from the two 15-representations. For the analysis of the four point function we need

\[ T^{(1)IJ}_{rt} C^{K}_{rs} \tilde{\gamma}_{s} = \delta^{IJ} C^{K}_{rs} \tilde{\gamma}_{s} , \]

\[ T^{(2)IJ}_{rt} C^{K}_{rs} \tilde{\gamma}_{s} = - \frac{1}{6} \delta^{IJ} C^{K}_{rs} \tilde{\gamma}_{s} + \frac{1}{2} (C^{IJK}_{rs} + C^{JKI}_{rs}) \tilde{\gamma}_{s} + \frac{1}{2} (D^{IJK}_{r} + D^{JKI}_{r}) , \]

\[ \tilde{T}^{(3)IJ}_{rt} C^{K}_{rs} \tilde{\gamma}_{s} = \frac{1}{2} (C^{IJK}_{rs} - C^{JKI}_{rs}) \tilde{\gamma}_{s} + \frac{7}{12} (D^{IJK}_{r} - D^{JKI}_{r}) + \frac{1}{24} E^{KIJ}_{r} , \]

\[ \tilde{T}^{(4)IJ}_{rt} C^{K}_{rs} \tilde{\gamma}_{s} = - \frac{1}{2} (C^{IJK}_{rs} - C^{JKI}_{rs}) \tilde{\gamma}_{s} - \frac{7}{12} (D^{IJK}_{r} - D^{JKI}_{r}) + \frac{1}{24} E^{KIJ}_{r} , \]

\[ \tilde{T}^{(5)IJ}_{rt} C^{K}_{rs} \tilde{\gamma}_{s} = \delta^{IK} C^{I}_{rs} \tilde{\gamma}_{s} - \frac{1}{6} (C^{IJK}_{rs} - C^{JKI}_{rs}) \tilde{\gamma}_{s} - D^{IJK}_{r} + D^{JKI}_{r} + E^{IJK}_{r} , \]

\[ \tilde{T}^{(6)IJ}_{rt} C^{K}_{rs} \tilde{\gamma}_{s} = - \delta^{IK} C^{I}_{rs} \tilde{\gamma}_{s} - \frac{1}{6} (C^{IJK}_{rs} - C^{JKI}_{rs}) \tilde{\gamma}_{s} - D^{IJK}_{r} - D^{JKI}_{r} + E^{IJK}_{r} , \]

where

\[ D^{IJK}_{r} = (C^{I} C^{J} C^{K})_{[rs]} + \frac{1}{6} (C^{I} C^{J} C^{K})_{[ts]} \tilde{\gamma}_{r} \tilde{\gamma}_{t} \tilde{\gamma}_{s} , \]

\[ E^{IJK}_{r} = C^{I}_{r} [C^{J} C^{K}]_{uv} \tilde{\gamma}_{t} \tilde{\gamma}_{u} \tilde{\gamma}_{v} + \frac{1}{6} (C^{I} C^{J} C^{K} - C^{I} C^{K} C^{J})_{[ts]} \tilde{\gamma}_{r} \tilde{\gamma}_{t} \tilde{\gamma}_{s} , \]

which necessarily satisfy, from (A.12), \( \gamma_{r} D^{IJK}_{r} = \gamma_{r} E^{IJK}_{r} = 0 \). As a consequence of (A.13) it is more convenient to use a basis where

\[ T^{(3)IJ}_{rs} = \frac{1}{36} (34 \tilde{T}^{(3)IJ}_{rs} - \tilde{T}^{(4)IJ}_{rs}) , \quad T^{(4)IJ}_{rs} = \frac{1}{3} (2 \tilde{T}^{(3)IJ}_{rs} + \tilde{T}^{(4)IJ}_{rs}) , \]

which from (A.13) satisfy

\[ T^{(3)IJ}_{rt} C^{K}_{rs} \tilde{\gamma}_{s} = \frac{1}{2} (C^{IJK}_{rs} - C^{JKI}_{rs}) \tilde{\gamma}_{s} + \frac{1}{2} (D^{IJK}_{r} - D^{JKI}_{r}) , \]

\[ \tilde{T}^{(4)IJ}_{rt} C^{K}_{rs} \tilde{\gamma}_{s} = D^{IJK}_{r} - D^{JKI}_{r} + \frac{1}{2} E^{KIJ}_{r} . \]
Appendix B. Superconformal Short Multiplets

The component fields in four dimensional $\mathcal{N} = 4$ supersymmetry may be denoted by $[k, p, q]_{(j_1, j_2)}$ where $[k, p, q]$ are the Dynkin labels for the $SU(4)$ $R$-symmetry representation and $(j_1, j_2)$ label $(2j_1 + 1)(2j_2 + 1)$ dimensional representations of the Lorentz group. For a short multiplet of the superconformal group starting from a scalar $[0, p, 0]$ field with dimension $\Delta = p$ the field content is obtained by superconformal transformations $\epsilon$ ($\bar{\epsilon}$) and $\bar{\epsilon}$ ($\bar{\epsilon}$), following [33], as follows

\[
\begin{array}{c}
\Delta \\
p & [0, p, 0]_{(0, 0)} \\
p + \frac{1}{2} & [0, p - 1, 1]_{(\frac{1}{2}, 0)} & [1, p - 1, 0]_{(0, \frac{1}{2})} \\
p + 1 & [0, p - 2, 1]_{(\frac{1}{2}, 0)} & [1, p - 2, 1]_{(\frac{1}{2}, \frac{1}{2})} & [0, p - 1, 0]_{(0, 1)} & [2, p - 2, 0]_{(0, 0)} \\
p + \frac{3}{2} & [0, p - 2, 1]_{(\frac{1}{2}, 0)} & [1, p - 2, 1]_{(\frac{1}{2}, \frac{1}{2})} & [0, p - 2, 1]_{(\frac{1}{2}, 1)} & [2, p - 3, 1]_{(\frac{1}{2}, 0)} & [1, p - 2, 0]_{(0, \frac{1}{2})} \\
p + 2 & [0, p - 2, 0]_{(0, 0)} & [1, p - 3, 1]_{(\frac{1}{2}, \frac{1}{2})} & [0, p - 2, 0]_{(1, 1)} & [2, p - 3, 1]_{(1, 0)} & [0, p - 3, 1]_{(0, 1)} & [1, p - 3, 1]_{(\frac{1}{2}, 1)} & [0, p - 2, 0]_{(0, 0)} \\
p + \frac{5}{2} & [1, p - 3, 0]_{(0, \frac{1}{2})} & [0, p - 2, 0]_{(1, \frac{1}{2})} & [1, p - 3, 1]_{(\frac{1}{2}, \frac{1}{2})} & [1, p - 3, 1]_{(\frac{1}{2}, 1)} & [0, p - 3, 1]_{(\frac{1}{2}, 0)} \\
p + 3 & [0, p - 3, 0]_{(0, 1)} & [2, p - 4, 1]_{(\frac{1}{2}, 0)} & [1, p - 3, 0]_{(1, \frac{1}{2})} & [1, p - 4, 1]_{(\frac{1}{2}, \frac{1}{2})} & [0, p - 3, 0]_{(1, 0)} & [0, p - 4, 1]_{(\frac{1}{2}, 0)} \\
p + \frac{7}{2} & [1, p - 4, 0]_{(0, \frac{1}{2})} & [0, p - 4, 1]_{(\frac{1}{2}, 0)} \\
p + 4 & [0, p - 4, 0]_{(0, 0)} \\
\end{array}
\]

This representation has dimension $2^8 \times \frac{1}{12} p^2 (p^2 - 1)$ for $p = 2, 3, \ldots$. We may note that $\frac{1}{12} p^2 (p^2 - 1) = \dim [0, p - 2, 0]$ which may be understood since the whole supermultiplet
may be generated from the representation with $Y = 2$ using $\tilde{e}^{i\alpha}$ supersymmetry transformations, the dimensions of the representations in each column add to $\left(\frac{8}{8-4Y}\right)^\frac{1}{12} p^2 (p^2 - 1)$.

The lowest weight operator, belonging to the $[0, p, 0]$-representation, can be represented by symmetric, traceless tensor, $\varphi_{r_1...r_p}$. The closure of the superconformal algebra follows directly from the considerations of section 4, from (4.1), (4.9) and (4.11), and if

$$\delta \varphi_{r_1...r_p} = -\hat{\epsilon} \gamma_{(r_p} \psi_{r_1...r_{p-1})} + \bar{\psi}_{(r_1...r_{p-1})} \hat{\bar{\gamma}}_p \hat{\epsilon},$$

where $\psi_{r_1...r_{p-1} \alpha}$ and $\bar{\psi}^{i_{r_1...r_{p-1} \dot{\alpha}}}$ belong to the $[0, p - 1, 1]$ and $[1, p - 1, 0]$-representations respectively. From (4.2a, 4)

$$\delta \psi_{r_1...r_{p-1} \alpha} = i \partial_{\alpha \dot{a}} \varphi_{r_1...r_{p-1} s} \hat{\gamma}_s \hat{\epsilon}^{\dot{\alpha}} + 2p \varphi_{r_1...r_{p-1} s} \hat{\gamma}_s \eta_\alpha + \ldots,$$

$$\delta \bar{\psi}^{i_{r_1...r_{p-1} \dot{\alpha}}} = \hat{\epsilon}^{\dot{\alpha}} \gamma_s i \partial_{\dot{a} \alpha} \varphi_{r_1...r_{p-1} s} - 2p \varphi_{r_1...r_{p-1} s} \hat{\eta}^{\dot{\alpha}} \gamma_s + \ldots,$$

so that we may then obtain

$$[\delta_2, \delta_1] \varphi_{r_1...r_p} = -v \partial \varphi_{r_1...r_p} - p \hat{\lambda} \varphi_{r_1...r_p} + ip \hat{t}_{r_p \{ s} \varphi_{r_1...r_{p-1} \} s},$$

with the notation of (2.18) and (2.19).

Following the discussion in section 4 we may also for completeness consider supermultiplets with the lowest weight operators belonging to $(j, 0)$ or $(0, j)$ spin representations and redefine the usual shortening conditions on $\Delta$ for these cases. Thus for a field $\Phi^I_{\alpha_1...\alpha_{2j}} = \Phi^I_{(\alpha_1...\alpha_{2j})}$ transforming according to a $(j, 0)$ representation (4.11) is extended to

$$\delta \Phi^I_{\alpha_1...\alpha_{2j}} = \epsilon^\beta \Psi^I_{\beta \alpha_1...\alpha_{2j}} + \Lambda^I_{\alpha_1...\alpha_{2j} \beta} \hat{\epsilon}^{i \beta}.$$

The relevant superconformal variations of $\Psi^I_{\beta \alpha_1...\alpha_{2j}}$ and $\Lambda^I_{\alpha_1...\alpha_{2j} \beta}$ are then

$$\delta \epsilon \Psi^I_{\beta \alpha_1...\alpha_{2j}} = (2 - \nu) i \partial_{\beta \dot{a}} \Phi^I_{\alpha_1...\alpha_{2j}} \hat{\epsilon}^{\dot{\beta}} + 2(\Delta - 2j) \Phi^I_{\alpha_1...\alpha_{2j}, \eta_\beta} + 8j \Phi^I_{\alpha_1...\alpha_{2j-1} \eta_{\alpha_{2j}}}$$

$$- (T_{rs})^I_j \Phi^J_{\alpha_1...\alpha_{2j}} \gamma_{[\eta_\gamma] \eta_\beta} + a i \partial_{(\alpha_1 \beta} \Phi^I_{\beta | \alpha_{2j-1} \eta_{\alpha_{2j}}}$$

$$- \frac{1}{2 \Delta} (T_{rs})^I_j (\nu i \partial_{\beta \dot{a}} \Phi^J_{\alpha_1...\alpha_{2j}} - a i \partial_{(\alpha_1 \beta} \Phi^I_{\beta | \alpha_{2j-1} \eta_{\alpha_{2j}}}) \gamma_{[\eta_\gamma] \hat{\epsilon}^{\dot{\beta}} + \ldots (B.6a)$$

$$\delta \epsilon \Lambda^I_{\alpha_1...\alpha_{2j} \beta} = - \nu \epsilon^\beta i \partial_{\beta \dot{a}} \Phi^I_{\alpha_1...\alpha_{2j}} + 2 \Delta \Phi^I_{\alpha_1...\alpha_{2j}, \eta^\beta} + (T_{rs})^I_j \Phi^J_{\alpha_1...\alpha_{2j}} \eta^\beta$$

$$+ a \epsilon^\beta i \partial_{(\alpha_1 \beta} \Phi^I_{\beta | \alpha_{2j-1} \eta_{\alpha_{2j}}}$$

$$- \frac{1}{2 \Delta} \epsilon^\beta \gamma_{[\eta_\gamma]} (T_{rs})^I_j (\nu i \partial_{\beta \dot{a}} \Phi^J_{\alpha_1...\alpha_{2j}} - a i \partial_{(\alpha_1 \beta} \Phi^I_{\beta | \alpha_{2j-1} \eta_{\alpha_{2j}}}) + \ldots (B.6b)$$

with $T_{rs}$ the appropriate $SU(4)$ generators and the coefficients $\nu, a$ are given by

$$\nu = \frac{\Delta(\Delta + j - 1)}{(\Delta + j)(\Delta - j - 1)}; \quad a = \frac{2j}{(\Delta + j)(\Delta - j - 1)}.$$
The form of \([B.6a, b]\), with \([B.7]\), is determined by the requirement of closure of the algebra
\[
[\delta_2, \delta_1] \Phi^I_{\alpha_1 \ldots \alpha_{2j}} = -(v \cdot \partial + \Delta \hat{\lambda}) \Phi^I_{\alpha_1 \ldots \alpha_{2j}} + 2j \hat{\omega}(\alpha_1 \beta \Phi^I_{\beta \alpha_2 \ldots \alpha_{2j}}) + \frac{1}{2} \hat{t}_{rs}(T_{rs})^I_J \Phi^J_{\alpha_1 \ldots \alpha_{2j}} ,
\]
with notation as in \([2.18]\) and \([2.19]\), and also by requiring
\[
[\delta_{\varepsilon}, \delta_1] \Lambda^I_{\alpha_1 \ldots \alpha_{2j}, \hat{\beta}} = \delta_{\varepsilon} \Lambda^I_{\alpha_1 \ldots \alpha_{2j}, \hat{\beta}} , \quad [\delta_{\varepsilon}, \delta_1] \Psi^I_{\beta \alpha_1 \ldots \alpha_{2j}} = \delta_{\varepsilon} \Psi^I_{\beta \alpha_1 \ldots \alpha_{2j}} , \quad (B.8)
\]
for \(\varepsilon'\) defined in \([4.6]\) and, as in \([2.1]\) with \([2.2]\) and \([2.4]\),
\[
\delta_1 \Phi^I_{\alpha_1 \ldots \alpha_{2j}} = -(v \cdot \partial + \Delta \hat{\lambda}) \Phi^I_{\alpha_1 \ldots \alpha_{2j}} + 2j \hat{\omega}(\alpha_1 \beta \Phi^I_{\beta \alpha_2 \ldots \alpha_{2j}}) , \quad \delta_1 \Psi^I_{\beta \alpha_1 \ldots \alpha_{2j}} = -(v \cdot \partial + \Delta \hat{\lambda}) \Psi^I_{\beta \alpha_1 \ldots \alpha_{2j}} + 2j \hat{\omega}(\alpha_1 \gamma \Psi^I_{\gamma \beta \alpha_2 \ldots \alpha_{2j}}) + \hat{\omega}_\gamma \Psi^I_{\gamma \alpha_1 \ldots \alpha_{2j}} , \quad \delta_1 \Lambda^I_{\alpha_1 \ldots \alpha_{2j}, \hat{\beta}} = -(v \cdot \partial + \Delta \hat{\lambda}) \Lambda^I_{\alpha_1 \ldots \alpha_{2j}, \hat{\beta}} + 2j \hat{\omega}(\alpha_1 \gamma \Lambda^I_{\gamma \alpha_2 \ldots \alpha_{2j}, \hat{\beta}}) - \Lambda^I_{\alpha_1 \ldots \alpha_{2j} \gamma \hat{\beta}} . \quad (B.10)
\]
As a consequence of \([B.6b]\) it is easy to see that multiplet shortening occurs if, suppressing the now irrelevant spinorial indices,
\[
2 \Delta \Phi^I \bar{\eta} + (T_{rs})^I_J \Phi^J \bar{\eta}[r \bar{s}] , \quad (B.11)
\]
does not span the full representation space. As an illustration we consider the \([k, 0, q]-\)representation so that, similarly to \([4.12]\), \(\Phi^I \rightarrow \phi^{i_1 \ldots i_k}_{j_1 \ldots j_q}\) and
\[
(T_{rs})^I_J \Phi^J \bar{\eta}(\gamma[r \bar{s}])^i_J \rightarrow 4k \delta^{i_{1} \ldots i_{k}}_{j_{1} \ldots j_{q}} \phi^{i_{k}}_{j_{1} \ldots j_{q}} \bar{\eta} - 4q \phi^{i_{1} \ldots i_{k}}_{j_{1} \ldots j_{q} \gamma} \bar{\eta} + (k - q) \phi^{i_{1} \ldots i_{k}}_{j_{1} \ldots j_{q}} \bar{\eta} . \quad (B.12)
\]
It is then easy to see that in \([B.11]\) that if
\[
2 \Delta = 3q + k , \quad (B.13)
\]
then the \([k, 0, q+1]-\)representation does not appear on the right hand side of \([B.6b]\) allowing \(\Lambda^I_{\alpha_1 \ldots \alpha_{2j}, \hat{\beta}}\) to be restricted to the \([k, 1, q-1]\) and \([k-1, 0, q]-\)representations.

Appendix C. Recurrence Relations for Conformal Partial Waves

In four dimensions we recently \([13]\) derived an explicit formula for the contribution of a quasi-primary operator of dimension \(\Delta\) and belonging to a \((j, j)\) representation under the Lorentz group including all its derivative descendent to a four point function for scalar fields \(\langle \phi_1(x_1) \phi_2(x_2) \phi_3(x_3) \phi_4(x_4) \rangle\). Assuming for simplicity the dimensions of \(\phi_i\) satisfy

\[
65
\]
\[ \Delta_1 = \Delta_2, \Delta_3 = \Delta_4 \] and letting \( \ell = 2j \) the contribution of such a conformal block is given by \( u^{\frac{1}{2}(\Delta-\ell)}G^{(\ell)}(u,v) \) where
\[
G^{(\ell)}(u,v) = \frac{1}{z - x} \left( -\frac{1}{2}z \right)^{\ell} \left( \frac{1}{2} \right)^{\ell} \left( \frac{1}{2} \right)^{\ell} F\left( \frac{1}{2}(\Delta + \ell), \frac{1}{2}(\Delta + \ell); \Delta + \ell; z \right)
\times F\left( \frac{1}{2}(\Delta - \ell - 2), \frac{1}{2}(\Delta - \ell - 2); \Delta - \ell - 2; x \right) - z \leftrightarrow x \right),
\]
where the conformal invariants are as in (5.1) and \( z, x \) are defined in (6.1). Formally the definition (C.1) extends to \( \ell < 0 \) and it is easy to see that then
\[
\left( \frac{1}{4}u \right)^{\ell-1} G^{(-\ell)}(u,v) = -G^{(\ell-2)}(u,v).
\]
Using (C.2) it is straightforward to obtain (7.7) and (8.10).

The crucial results obtained here depend entirely on the following relation for hypergeometric functions of the form which appear in (C.1),
\[
(1 - \frac{1}{2}z) F\left( \frac{1}{2} \delta, \frac{1}{2} \delta; \delta; z \right) = F\left( \frac{1}{2} \delta - 1, \frac{1}{2} \delta - 1; \delta - 2; z \right) + \frac{\delta^2}{16(\delta - 1)(\delta + 1)} z^2 F\left( \frac{1}{2} \delta + 1, \frac{1}{2} \delta + 1; \delta + 2; z \right).
\]
This is easily verified by considering the power series expansion of both sides, or using combinations of standard hypergeometric identities.

For \( \Delta = \ell + 2 \) (C.2) gives directly (7.9). For \( \Delta = \ell \) we have, with the definition of \( g_{\ell} \) in (7.9),
\[
G^{(\ell)}(u,v) = \frac{1}{z - x} \left( -\frac{1}{2}z \right) \left( 1 - \frac{1}{2}x \right) g_{\ell}(z) - z \leftrightarrow x \right),
\]
\[
= -\frac{1}{2} \left( g_{\ell}(z) + g_{\ell}(x) \right) - \frac{1}{4} u \frac{2z - zg_{\ell}(z) - zg_{\ell}(x)}{x - z}.
\]
With the aid of (7.10), which follows from (C.3), we may obtain (8.18).

Other identities also follow from (C.3). To derive a relation for \(-\frac{1}{2}(1 - v)G^{(\ell)}(u,v)\), we first obtain
\[
\frac{(\Delta + \ell)^2}{8(\Delta + \ell - 1)(\Delta + \ell + 1)} u G^{(\ell+1)}(u,v)
= -\frac{zx}{z - x} \left( -\frac{1}{2} z \right)^{\ell} \left( 1 - \frac{1}{2} z \right) F\left( \frac{1}{2}(\Delta + \ell), \frac{1}{2}(\Delta + \ell); \Delta + \ell; z \right)
\times F\left( \frac{1}{2}(\Delta - \ell - 2), \frac{1}{2}(\Delta - \ell - 2); \Delta - \ell - 2; x \right) - z \leftrightarrow x \right)
- \frac{1}{2} u G^{(\ell-1)}(u,v),
\]
and hence, since \( 1 - v = z + x - zx \),

\[
- \frac{1}{2}(1 - v)G^{(\ell)}_{\Delta}(u, v) - \frac{(\Delta + \ell)^2}{16(\Delta + \ell - 1)(\Delta + \ell + 1)} uG^{(\ell + 1)}_{\Delta+1}(u, v) - \frac{1}{4}u G^{(\ell - 1)}_{\Delta-1}(u, v)
\]

\[
= - \frac{1}{z - x} \left( z^2 \left( - \frac{1}{2}z \right)^{\ell+1}(1 - \frac{1}{2}x) F\left( \frac{1}{2}(\Delta + \ell), \frac{1}{2}(\Delta + \ell); \Delta + \ell; z \right)
\times F\left( \frac{1}{2}(\Delta - \ell - 2), \frac{1}{2}(\Delta - \ell - 2); \Delta - \ell - 2; x \right) - z \leftrightarrow x \right).
\]  

Using (C.3) once again then gives

\[
- \frac{1}{2}(1 - v)G^{(\ell)}_{\Delta}(u, v) = G^{(\ell + 1)}_{\Delta-1}(u, v) + \frac{(\Delta + \ell)^2}{16(\Delta + \ell - 1)(\Delta + \ell + 1)} uG^{(\ell + 1)}_{\Delta+1}(u, v)
\]

\[
+ \frac{1}{4}u G^{(\ell - 1)}_{\Delta-1}(u, v) + \frac{(\Delta - \ell - 2)^2}{64(\Delta - \ell - 3)(\Delta - \ell - 1)} u^2 G^{(\ell - 1)}_{\Delta+1}(u, v).
\]

We also consider similarly \( \frac{1}{2}(1 + v)G^{(\ell)}_{\Delta}(u, v) \). We first obtain

\[
G^{(\ell + 2)}_{\Delta}(u, v) = \frac{1}{z - x} \left( z^2 \left( - \frac{1}{2}z \right)^{\ell+2}(1 - \frac{1}{2}x) F\left( \frac{1}{2}(\Delta + \ell), \frac{1}{2}(\Delta + \ell); \Delta + \ell; z \right)
\times F\left( \frac{1}{2}(\Delta - \ell - 2), \frac{1}{2}(\Delta - \ell - 2); \Delta - \ell - 2; x \right) - z \leftrightarrow x \right)
\]

\[
- \frac{(\Delta - \ell - 2)^2}{64(\Delta - \ell - 3)(\Delta - \ell - 1)} u^2 G^{(\ell)}_{\Delta+2}(u, v),
\]  

(C.8)

and hence, with \( \frac{1}{2}(1 + v) - (1 - \frac{1}{2}x)(1 - \frac{1}{2}z) = \frac{1}{4}u \),

\[
\frac{1}{2}(1 + v)G^{(\ell)}_{\Delta}(u, v)
\]

\[
- \frac{(\Delta + \ell)^2}{4(\Delta + \ell - 1)(\Delta + \ell + 1)} \left( G^{(\ell+2)}_{\Delta}(u, v) + \frac{(\Delta - \ell - 2)^2}{64(\Delta - \ell - 3)(\Delta - \ell - 1)} u^2 G^{(\ell)}_{\Delta+2}(u, v) \right)
\]

\[
= \frac{1}{4}u G^{(\ell)}_{\Delta}(u, v)
\]

\[
- \frac{1}{z - x} \left( z \left( - \frac{1}{2}z \right)^{\ell}(1 - \frac{1}{2}x) F\left( \frac{1}{2}(\Delta + \ell - 2), \frac{1}{2}(\Delta + \ell - 2); \Delta + \ell - 2; z \right)
\times F\left( \frac{1}{2}(\Delta - \ell - 2), \frac{1}{2}(\Delta - \ell - 2); \Delta - \ell - 2; x \right) - z \leftrightarrow x \right).
\]  

(C.9)

With the aid of (C.3) once more we may then derive

\[
\frac{1}{2}(1 + v)G^{(\ell)}_{\Delta}(u, v) = \frac{1}{4}u G^{(\ell)}_{\Delta}(u, v) + \frac{(\Delta + \ell)^2}{4(\Delta + \ell - 1)(\Delta + \ell + 1)} G^{(\ell+2)}_{\Delta}(u, v)
\]

\[
+ \frac{(\Delta + \ell)^2(\Delta - \ell - 2)^2}{256(\Delta + \ell - 1)(\Delta + \ell + 1)(\Delta - \ell - 3)(\Delta - \ell - 1)} u^2 G^{(\ell)}_{\Delta+2}(u, v)
\]

\[
+ G^{(\ell)}_{\Delta-2}(u, v) + \frac{(\Delta - \ell - 2)^2}{64(\Delta - \ell - 3)(\Delta - \ell - 1)} u^2 G^{(\ell-2)}_{\Delta}(u, v).
\]  

(C.10)
As a consistency check it is easy to verify that (C.7) and (C.10) are invariant under (C.2) and we may also check compatibility with (7.2).

The results (C.7) and (C.10) are also true for \( \ell = 0, 1 \) by using (7.7). Thus (C.10) implies

\[
\frac{1}{2}(1 + v)G^{(0)}(u, v) = \frac{\Delta^2}{4(\Delta - 1)(\Delta + 1)} G^{(2)}(u, v)
+ \frac{\Delta^2(\Delta - 2)^2}{256(\Delta - 1)^2(\Delta + 1)(\Delta - 3)} u^2 G^{(0)}_{\Delta+2}(u, v)
+ G^{(0)}_{\Delta-2}(u, v) + \frac{3(\Delta - 2)^2 - 4}{16(\Delta - 3)(\Delta - 1)} u G^{(0)}(u, v). \tag{C.11}
\]

For further application in the text we also use (C.10) and (C.7) twice to calculate

\[
\frac{1}{4}(1 + v)^2 G^{(\ell)}(u, v) - \frac{1}{4}(1 - v)^2 G^{(\ell)}(u, v)
\]

\[
\theta G^{(\ell)}_\Delta(u, v) = G^{(\ell)}_{\Delta-4}(u, v)
- \frac{(\Delta + \ell - 1)^2 - 5}{2(\Delta + \ell - 3)(\Delta + \ell + 1)} G^{(\ell+2)}_{\Delta-2}(u, v)
- \frac{(\Delta - \ell - 3)^2 - 5}{32(\Delta - \ell - 5)(\Delta + \ell - 1)} u^2 G^{(\ell-2)}_{\Delta-2}(u, v)
+ \frac{(\Delta + \ell)^2(\Delta + \ell + 2)^2}{16(\Delta + \ell - 1)(\Delta + \ell + 1)(\Delta + \ell + 3)} G^{(\ell+4)}_{\Delta}(u, v)
+ \frac{(\Delta - \ell)^2(\Delta - \ell - 2)^2}{2^{12}(\Delta - \ell - 3)(\Delta - \ell - 1)^2(\Delta - \ell + 1)} u^4 G^{(\ell-4)}_{\Delta}(u, v)
+ \frac{((\Delta + \ell - 1)^2 - 5)((\Delta - \ell - 3)^2 - 5)}{64(\Delta + \ell - 3)(\Delta + \ell + 1)(\Delta - \ell - 5)(\Delta + \ell - 1)} u^2 G^{(\ell)}_{\Delta}(u, v)
- \frac{(\Delta + \ell)^2(\Delta + \ell + 2)^2((\Delta - \ell - 3)^2 - 5)}{2^9(\Delta + \ell - 1)(\Delta + \ell + 1)^2(\Delta + \ell + 3)(\Delta - \ell - 5)(\Delta - \ell - 1)} u^2 G^{(\ell+2)}_{\Delta+2}(u, v)
- \frac{(\Delta - \ell)^2(\Delta - \ell - 2)^2((\Delta + \ell - 1)^2 - 5)}{2^{13}(\Delta - \ell - 3)(\Delta - \ell - 1)^2(\Delta - \ell + 1)(\Delta + \ell - 3)(\Delta + \ell + 1)} u^4 G^{(\ell-2)}_{\Delta+2}(u, v)
+ \frac{(\Delta + \ell)^2(\Delta + \ell + 2)^2(\Delta - \ell - 2)^2(\Delta - \ell - 2)^2}{2^{16}(\Delta + \ell - 1)(\Delta + \ell + 1)^2(\Delta + \ell + 3)(\Delta - \ell - 3)(\Delta - \ell - 1)^2(\Delta - \ell + 1)} u^4 G^{(\ell)}_{\Delta+4}(u, v). \tag{C.12}
\]

**Appendix D. Simplification of strong coupling result**

By virtue of the AdS/CFT correspondence Arutyunov and Frolov [18] calculated the four point function (5.23) in the leading large \( N \) limit for large \( g^2 N \). The result
is rather complicated and expressed in terms of conformal integrals over AdS with four bulk/boundary propagators. We here define the corresponding functions of \( u, v \) \( \overline{D}_{\Delta_1 \Delta_2 \Delta_3 \Delta_4} \) by,

\[
\frac{\prod_{i=1}^{4} \Gamma(\Delta_i)}{\Gamma(\Sigma - \frac{1}{2}d)} \frac{2}{\pi^{d/2}} \int_{0}^{\infty} dz \int d^{d}x \prod_{i=1}^{4} \left( \frac{z}{z^{2} + (x - x_i)^{2}} \right)^{\Delta_i} = \frac{r_{14}^{\Sigma - \Delta_1 - \Delta_4} r_{34}^{\Sigma - \Delta_3 - \Delta_4}}{r_{13}^{\Sigma - \Delta_4} r_{24}^{\Delta_2}} \overline{D}_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(u, v),
\]

where

\[
\Sigma = \frac{1}{2} \sum_{i=1}^{N} \Delta_i. \tag{D.2}
\]

In \( \overline{D}_{\Delta_1 \Delta_2 \Delta_3 \Delta_4} \) is independent of the dimension \( d \). The \( \overline{D} \)-functions satisfy the identities

\[
\overline{D}_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(u, v) = \overline{D}_{\Sigma - \Delta_1 \Sigma - \Delta_2 \Sigma - \Delta_3 \Sigma - \Delta_4}(u, v) \tag{D.3a}
\]

\[
v^{-\Delta_2} \overline{D}_{\Delta_1 \Delta_2 \Delta_4 \Delta_3}(u/v, 1/v) \tag{D.3b}
\]

\[
v^{\Delta_4 - \Sigma} \overline{D}_{\Delta_2 \Delta_1 \Delta_3 \Delta_4}(u/v, 1/v) \tag{D.3c}
\]

\[
v^{\Delta_2 + \Delta_4 - \Sigma} \overline{D}_{\Delta_1 \Delta_3 \Delta_2 \Delta_4}(u, v) \tag{D.3d}
\]

\[
v^{\Delta_3 + \Delta_4 - \Sigma} \overline{D}_{\Delta_4 \Delta_3 \Delta_2 \Delta_1}(u, v) \tag{D.3e}
\]

which reflect permutation symmetries of the basic integral in \( \overline{D}_{\Delta_1 \Delta_2 \Delta_3 \Delta_4} \). We also have the relations

\[
(\Delta_2 + \Delta_4 - \Sigma) \overline{D}_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(u, v) = \overline{D}_{\Delta_1 \Delta_2 + 1 \Delta_3 \Delta_4 + 1}(u, v) - \overline{D}_{\Delta_1 \Delta_2 + 1 \Delta_3 \Delta_4 + 1}(u, v),
\]

\[
(\Delta_1 + \Delta_4 - \Sigma) \overline{D}_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(u, v) = \overline{D}_{\Delta_1 \Delta_2 + 1 \Delta_3 \Delta_4 + 1}(u, v) - v \overline{D}_{\Delta_1 \Delta_2 + 1 \Delta_3 \Delta_4 + 1}(u, v),
\]

\[
(\Delta_3 + \Delta_4 - \Sigma) \overline{D}_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(u, v) = \overline{D}_{\Delta_1 \Delta_2 \Delta_3 + 1 \Delta_4 + 1}(u, v) - u \overline{D}_{\Delta_1 \Delta_2 \Delta_3 + 1 \Delta_4 + 1}(u, v),
\]

and

\[
\Delta_4 \overline{D}_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(u, v) = \overline{D}_{\Delta_1 \Delta_2 \Delta_3 + 1 \Delta_4 + 1}(u, v) + \overline{D}_{\Delta_1 \Delta_2 + 1 \Delta_3 \Delta_4 + 1}(u, v) + \overline{D}_{\Delta_1 \Delta_2 + 1 \Delta_3 \Delta_4 + 1}(u, v). \tag{D.5}
\]

There are also relations which arise since if \( \Delta_i = 0 \) the integral \( \overline{D}_{\Delta_1 \Delta_2 \Delta_3 \Delta_4} \) reduces to a three point function. Thus

\[
\Delta_2 \overline{D}_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}|_{\Delta_2 = 0} = \Gamma(\Sigma - \Delta_1) \Gamma(\Sigma - \Delta_3) \Gamma(\Sigma - \Delta_4), \tag{D.6}
\]

and by using \( \overline{D}_{\Delta_1 \Delta_2 \Delta_3 \Delta_4} \) we have

\[
(\Sigma - \Delta_4) \overline{D}_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}|_{\Sigma - \Delta_4 = 0} = \Gamma(\Delta_1) \Gamma(\Delta_2) \Gamma(\Delta_3). \tag{D.7}
\]
From (D.5) and the sum of eqs. (D.4) this may be rewritten as

\[ \begin{align*}
(D_{\Delta_1+1 \Delta_2 \Delta_3+1 \Delta_4} + uD_{\Delta_1+1 \Delta_2+1 \Delta_3 \Delta_4} + vD_{\Delta_1 \Delta_2+1 \Delta_3+1 \Delta_4})|_{\Delta_4 = \Delta_1+\Delta_2+\Delta_3} &= \Gamma(\Delta_1)\Gamma(\Delta_2)\Gamma(\Delta_3).
\end{align*} \] (D.8)

For compatibility with superconformal symmetry the results obtained in [18] by using the AdS/CFT correspondence must be expressible in the form (9.1) in terms of a single function \( F \). This was demonstrated by Eden et al [10] by direct calculation starting from

\[ D_{1111}(u,v) = \Phi(1)(u,v), \] (D.9)

where \( \Phi(1) \) is given here by (9.6). We show here that this requirement also follows from the above \( D \) identities and that the resulting expression for \( F \) may be simplified to a single \( D \) function.

For \( a_1 \) in (5.23), with the normalisation conventions as in (9.1), expression obtained in [18] may be written in terms of \( D \) functions in the form

\[ a_1 = \frac{1}{4} \hat{N}^2 + \frac{1}{2} \hat{N}u^2 \left( -D_{2211} + 3D_{2222} + (1 + v - u)D_{3322} \right). \] (D.10)

Using (D.37) this gives from (9.1),

\[ F = -D_{1122} + 3uD_{2222} + (1 + v - u)D_{2233}. \] (D.11)

Now using (D.4) to eliminate \( u, v \) from (D.11) and also for \( D_{2233} - D_{1324} \) we get

\[ F = -4D_{1122} + 5D_{1133} - D_{2123} - D_{1223} + D_{1324} + D_{3124} - D_{1144}. \] (D.12)

From (D.5) we may obtain \( D_{1324} + D_{3124} - D_{1144} = 3(D_{2123} + D_{1223} - D_{1133}) - 2D_{2224} \) so that (D.12) becomes

\[ F = -4D_{1122} + 2(D_{1133} + D_{2123} + D_{1223}) - 2D_{2224} = -2D_{2224}, \] (D.13)

using (D.3) again. Thus we obtain (9.12) where \( F \) is reduced to a single \( D \) function. The result of [18] for \( c_1 \) may similarly be expressed as

\[ c_1 = \hat{N}(uv(D_{2211} + 2D_{2223} - 3D_{2222}) - v(1 - v)D_{2222} \\
+ \frac{1}{2}(D_{1212} - D_{2112} - vD_{1221}) + u \leftrightarrow v)
= \hat{N}(2uv(D_{2233} + D_{3223}) + (v^2 + u^2 - v - u - 6uv)D_{2222} \\
+ (u + v)D_{1212} + (u - v)(D_{2112} - D_{1122})), \] (D.14)
using (D.3d,e,f). To simplify (D.14) we first consider the terms with $\Sigma = 5$,

\[
2uv(\overline{D}_{2233} + \overline{D}_{3223}) = -u\overline{D}_{3214} - v\overline{D}_{1234} - (u + v)\overline{D}_{2123} + u\overline{D}_{3113} + v\overline{D}_{1133},
\]

\[
-u\overline{D}_{3214} - v\overline{D}_{1234} = (u + v)\overline{D}_{2224} + u\overline{D}_{2314} + v\overline{D}_{1324} - 3u\overline{D}_{2213} - 3v\overline{D}_{1223},
\]

(D.15)

\[
= 1 - (1 - u - v)\overline{D}_{2224} - 3u\overline{D}_{2213} - 3v\overline{D}_{1223},
\]

where in the last line we have used (D.8) for $\Delta_1 = \Delta_3 = 1$, $\Delta_2 = 2$. The $\Sigma = 4$ terms may then be written using (D.4) as

\[
-(u + v)\overline{D}_{2123} + u\overline{D}_{3113} + v\overline{D}_{1133} - 3u\overline{D}_{2213} - 3v\overline{D}_{1223} + (v^2 + u^2 - v - u - 6uv)\overline{D}_{2222}
\]

\[
= - (u + v)(\overline{D}_{2213} + \overline{D}_{1223} + \overline{D}_{1313}) + (u - 2v)(\overline{D}_{1133} + \overline{D}_{1223} + \overline{D}_{2123})
\]

\[
+ (v - 2u)(\overline{D}_{3113} + \overline{D}_{2213} + \overline{D}_{2123})
\]

\[
- (u - 3v)\overline{D}_{1122} - (v - 3u)\overline{D}_{2112} + (u + v)\overline{D}_{1212}.
\]

(D.16)

Using (D.5) the $\Sigma = 3$ terms cancel leaving just the result in (9.1) with (9.12) again.

It is perhaps worth noting that if $F$ is expressible as a single $\overline{D}$ function then the symmetry properties (D.3c,d) determine

\[
F(u, v) = K \overline{D}_{1+s1+s1+s1+3s}(u, v),
\]

(D.17)

depending on a single parameter $s$, which includes both the weak and strong coupling results for $s = 0, 1$. 

71
References

[1] C. Montonen and D. Olive, *Magnetic Monopoles as Gauge Particles?*, Phys. Lett. B72 (1977) 117.

[2] P.S. Howe, E. Sokatchev and P.C. West, *3-Point Functions in $N = 4$ Yang-Mills*, Phys. Lett. B444 (1998) 341, [hep-th/9808162](https://arxiv.org/abs/hep-th/9808162).

[3] B. Eden, P.S. Howe, A. Pickering, E. Sokatchev and P.C. West, *Four-point functions in $N = 2$ superconformal field theories*, Nucl. Phys. B581 (2000) 523, [hep-th/0001138](https://arxiv.org/abs/hep-th/0001138).

[4] G. Arutyunov, B. Eden and E. Sokatchev, *On Non-renormalization and OPE in Superconformal Field Theories*, Nucl. Phys. B619 (2001) 359, [hep-th/0105254](https://arxiv.org/abs/hep-th/0105254).

[5] B. Eden and E. Sokatchev, *On the OPE of 1/2 BPS Short Operators in $N = 4$ SCFT$_4$*, Nucl. Phys. B618 (2001) 259, [hep-th/0106249](https://arxiv.org/abs/hep-th/0106249).

[6] P.J. Heslop and P.S. Howe, *OPEs and 3-point correlators of protected operators in $N = 4$ SYM*, Nucl. Phys. B626 (2002) 265, [hep-th/0107212](https://arxiv.org/abs/hep-th/0107212).

[7] F.A. Dolan and H. Osborn, *Implications of $N = 1$ Superconformal Symmetry for Chiral Fields*, Nucl. Phys. B593 (2001) 599, [hep-th/0006098](https://arxiv.org/abs/hep-th/0006098).

[8] A. Pickering and P. West, *Chiral Green’s Functions in Superconformal Field Theory*, Nucl. Phys. B569 (2000) 303, [hep-th/9904076](https://arxiv.org/abs/hep-th/9904076).

[9] S. Ferrara and A. Zaffaroni, *Superconformal Field Theories, Multiplet Shortening, and the AdS$_5$/SCFT$_4$ Correspondence*, Proceedings of the Conférence Moshé Flato 1999, vol. 1, ed. G. Dito and D. Sternheimer, Kluwer Academic Publishers (2000), [hep-th/9908163](https://arxiv.org/abs/hep-th/9908163).

[10] B. Eden, A.C. Petkou, C. Schubert and E. Sokatchev, *Partial non-renormalisation of the stress-tensor four-point function in $N = 4$ SYM and AdS/CFT*, Nucl. Phys. B607 (2001) 191, [hep-th/0009106](https://arxiv.org/abs/hep-th/0009106).

[11] G. Arutyunov, S. Frolov and A.C. Petkou, *Perturbative and instanton corrections to the OPE of CPOs in $N = 4$ SYM$_4$*, Nucl. Phys. B602 (2001) 238, [hep-th/0010137](https://arxiv.org/abs/hep-th/0010137); (E) Nucl. Phys. B609 (2001) 540.

[12] G. Arutyunov, S. Frolov and A.C. Petkou, *Operator Product Expansion of the Lowest Weight CPOs in $N = 4$ SYM$_4$ at Strong Coupling*, Nucl. Phys. B586 (2000) 547, [hep-th/0005182](https://arxiv.org/abs/hep-th/0005182); (E) Nucl. Phys. B609 (2001) 539.

[13] F.A. Dolan and H. Osborn, *Conformal four point functions and the operator product expansion*, Nucl. Phys. B599 (2001) 459, [hep-th/0011040](https://arxiv.org/abs/hep-th/0011040).
[14] G. Arutyunov, B. Eden, A.C. Petkou and E. Sokatchev, *Exceptional non-renormalization properties and OPE analysis of chiral four-point functions in $\mathcal{N} = 4$ SYM$_4$*, Nucl. Phys. B620 (2002) 380, [hep-th/0103230](http://arxiv.org/abs/hep-th/0103230).

[15] M. Bianchi, S. Kovacs, G. Rossi and Y.S. Stanev, *Properties of the Konishi multiplet in $\mathcal{N} = 4$ SYM theory*, JHEP 0105 (2001) 042, [hep-th/0104016](http://arxiv.org/abs/hep-th/0104016).

[16] S. Penati and A. Santambrogio, *Superspace approach to anomalous dimensions in $\mathcal{N} = 4$ SYM*, Nucl. Phys. B614 (2001) 367, [hep-th/0107071](http://arxiv.org/abs/hep-th/0107071).

[17] D. Anselmi, *The $\mathcal{N} = 4$ Quantum Conformal Algebra*, Nucl. Phys. B541 (1999) 369, [hep-th/9809192](http://arxiv.org/abs/hep-th/9809192).

[18] G. Arutyunov and S. Frolov, *Four-point Functions of Lowest Weight CPOs in $\mathcal{N} = 4$ SYM$_4$ in Supergravity Approximation*, Phys. Rev. D62 (2000) 064016, [hep-th/0002170](http://arxiv.org/abs/hep-th/0002170).

[19] K. Intriligator, *Bonus Symmetries of $\mathcal{N} = 4$ Super-Yang-Mills Correlation Functions via AdS Duality*, Nucl. Phys. B551 (1999) 575, [hep-th/9811047](http://arxiv.org/abs/hep-th/9811047).

[20] K. Intriligator and W. Skiba, *Bonus Symmetry and the Operator Product Expansion of $\mathcal{N} = 4$ Super-Yang-Mills*, Nucl. Phys. B559 (1999) 165, [hep-th/9905020](http://arxiv.org/abs/hep-th/9905020).

[21] J. Erdmenger and H. Osborn, *Conserved Currents and the Energy Momentum Tensor in Conformally Invariant Theories for General Dimensions*, Nucl. Phys. B483 (1997) 431, [hep-th/9605009](http://arxiv.org/abs/hep-th/9605009).

[22] B.P. Conlong and P.C. West, *Anomalous dimensions of fields in a supersymmetric quantum field theory at a renormalization group fixed point*, J. Phys. A 26 (1993) 3325.

[23] S. Lee, S. Minwalla, M. Rangamani and N. Seiberg, *Three-Point Functions of Chiral Operators in $D = 4$, $\mathcal{N} = 4$ SYM at Large $N$*, Adv. Theor. Math. Phys. 2 (1998) 697, [hep-th/9806074](http://arxiv.org/abs/hep-th/9806074).

[24] G. Arutyunov and S. Frolov, *Three-point function of the stress-tensor in the AdS/CFT correspondence*, Phys. Rev. D60 (1999) 026004, [hep-th/9901121](http://arxiv.org/abs/hep-th/9901121).

[25] H. Osborn, *$\mathcal{N} = 1$ Superconformal Symmetry in Four-Dimensional Quantum Field Theory*, Ann. Phys. (N.Y.) 272 (1999) 243, [hep-th/9808041](http://arxiv.org/abs/hep-th/9808041).

[26] L. Andrianopoli and S. Ferrara, *On short and long $SU(2,2/4)$ multiplets in the AdS/CFT correspondence*, Lett. Math. Phys. 48 (1999) 145, [hep-th/9812067](http://arxiv.org/abs/hep-th/9812067).

[27] F. Gonzalez-Rey, I. Park and K. Schalm, *A note on four-point functions of conformal operators in $N = 4$ Super-Yang Mills*, Phys. Lett. B448 (1999) 37, [hep-th/9811153](http://arxiv.org/abs/hep-th/9811153).
[28] B. Eden, P.S. Howe, C. Schubert, E. Sokatchev and P.C. West, *Four-point functions in $N = 4$ supersymmetric Yang-Mills theory at two loops*, Nucl. Phys. B557 (1999) 355, [hep-th/9811172]. *Simplifications of four-point functions in $N = 4$ supersymmetric Yang-Mills theory at two loops*, Phys. Lett. B466 (1999) 20, [hep-th/9906051].

[29] M. Bianchi, S. Kovacs, G. Rossi and Y.S. Stanev, *On the logarithmic behaviour in $\mathcal{N} = 4$ SYM theory*, JHEP 9908 (1999) 020, [hep-th/9906188].

[30] L. Hoffmann, L. Mesref and W. Rühl, *Conformal partial wave analysis of AdS amplitudes for dilaton-axion four-point functions*, Nucl. Phys. B608 (2001) 177, [hep-th/0012153].

[31] E. D’Hoker, S.D. Mathur, A. Matsusis and L. Rastelli, *The Operator Product Expansion of $N = 4$ SYM and the 4-point Functions of Supergravity*, Nucl. Phys. B589 (2000) 38, [hep-th/9911222].

[32] A.V. Ryzhov, *Quarter BPS Operators in $\mathcal{N} = 4$ SYM*, JHEP 0111 (2001) 046, [hep-th/0109064].

E. D’Hoker and A.V. Ryzhov, *Three Point Functions of Quarter BPS Operators in $\mathcal{N} = 4$ SYM*, JHEP 0202 (2002) 047, [hep-th/0109063].

[33] P.J. Heslop and P.S. Howe, *A note on composite operators in $N = 4$ SYM*, Phys. Lett. 516B (2001) 367, [hep-th/0106238].

[34] M.F. Sohnius, *The multiplet of currents for $N = 2$ extended supersymmetry*, Phys. Lett. 81B (1979) 8.

[35] M. Günyaydin and N. Marcus, *The spectrum of the $S^5$ compactification of the chiral $N = 2$, $D = 10$ supergravity and the unitary supermultiplets of $U(2, 2/4)$*, Class. and Quantum Gravity, 2 (1985) L11.