1 Transition probabilities in the hull dynamics of the majority vote model (Eq. 3–5 in main text)

Let the hull be parameterized as \((x_1, y_1), \ldots, (x_l, y_l)\) by the left-turning walk described in the main text. Figure 1c of the main text shows that the hull in \(\text{MV}_{0.8}\) separates a predominantly white region on the left from a similarly dense black region on the right. This observation justifies a “solid-on-solid” approximation \([1]\) where we ignore

- any overhangs in the interface (i.e., parts of the left-turning walk that move towards smaller \(y\)-coordinates),
- any isolated islands of the minority color to the left and right of the hull.

In this approximation, the hull at time step \(t\) is completely characterized by

\[
h(t, y) = \min \{ x_k | y_k = y, k = 1, \ldots, l \} - \frac{1}{2} \tag{1}
\]

because every site \((x, y)\) with \(x < h(t, y)\) will be white and every site with \(x > h(t, y)\) black. We now have to distinguish three cases.

1.1 Case 1: \(h(t, y)\) is neither a strict local minimum nor maximum

In this case, one of the following four conditions must be met

- \(h(t, y - 1) = h(t, y)\),
- \(h(t, y) = h(t, y + 1)\),
- \(h(t, y - 1) < h(t, y) < h(t, y + 1)\), or
- \(h(t, y - 1) > h(t, y) > h(t, y + 1)\).

Let \(A\) be the white site in the \(y\)-th row with \(x\)-coordinate \(h(t, y) - \frac{1}{2}\) and \(B\) the black site at \(h(t, y) + \frac{1}{2}\) (see Fig. 1). In all cases listed above, both \(A\) and \(B\) have at least two neighbors of their own color, namely one in the \(y\)-th row and one in a neighboring row. Including their own vote, the local majority supports their current opinion. As a consequence, in the deterministic majority vote model \(\text{MV}_1\) neither \(A\) nor \(B\) will change color and thus \(h(t + 1, y) = h(t, y)\). In the stochastic model \(\text{MV}_{1-r}\) with \(r > 0\), the probability that \(A\) becomes black is

\[
\Pr[A \text{ black at } t + 1 | h(t, y) = x] = r \left[ g \left( x - \frac{1}{2} \right) + p_c \right], \tag{2}
\]

and the probability that \(B\) becomes white

\[
\Pr[B \text{ white at } t + 1 | h(t, y) = x] = r \left[ 1 - g \left( x + \frac{1}{2} \right) - p_c \right]. \tag{3}
\]
Figure 1: Examples where \( h(t, y) \) is neither a strict local minimum nor maximum. In each case, the white and black sites at the interface, \( A \) and \( B \), have at least two neighbors of their own color so that in MV\(_1\) there is no change of the hull position (i.e., \( h(t, y) = h(t+1, y) \)). In MV\(_{1-r}\), there is a \( O(r) \) probability that the hull moves one site to the left or right. All other probabilities are \( O(r^2) \).

Figure 2: Examples where \( h(t, y) \) is a strict local (a) minimum or (b) maximum. The minimal or maximal site of the protruding opinion is in a local minority, but all other sites in row \( y \) will have at least two neighbors of the same opinion. Thus, in MV\(_1\) only the front site will change opinion between time steps \( t \) and \( t + 1 \). In MV\(_{1-r}\) the probability of the hull shifting one step towards the (a) right, (b) left is \( 1 - O(r) \). The probability of no change or two steps to the (a) right, (b) left is \( O(r) \). All other transitions have probabilities \( O(r^2) \).

In the solid-on-solid approximation, the hull can shift exactly one step to the left only if \( A \) turns black, while all other sites keep their colors with a probability \( 1 - O(r) \). Because the probabilities are independent, we can multiply them and obtain

\[
\Pr[h(t+1, y) = x - 1 \mid h(t, y) = x] = r \left[ g \left( x - \frac{1}{2} \right) + p_c \right] + O(r^2).
\]

(4)

Similarly,

\[
\Pr[h(t+1, y) = x + 1 \mid h(t, y) = x] = r \left[ 1 - g \left( x + \frac{1}{2} \right) - p_c \right] + O(r^2).
\]

(5)

Because shifts of more than one step to either side have probabilities \( O(r^2) \) and the sum of the probabilities must equal one,

\[
\Pr[h(t+1, y) = x \mid h(t, y) = x] = 1 + r(g-1) + O(r^2).
\]

(6)

1.2 Case 2: \( h(t, y) \) is a strict local minimum

Here the leftmost black site \( A \) in row \( y \) is in a local minority (see Fig. 2a). It stays black with probability

\[
\Pr[A \text{ black at } t+1 \mid h(t, y) = x] = r \left[ g \left( x + \frac{1}{2} \right) + p_c \right].
\]

(7)
Its right neighbor $B$ is black, which is the local majority because at least its two neighbors in row $i$ are black. Hence, it becomes white with probability

$$\Pr[B \text{ white at } t + 1 \mid h(t, y) = x] = r \left[ 1 - g \left( x + \frac{3}{2} \right) - p_c \right].$$

(8)

In the solid-on-solid approximation, the hull can only stay at the same position if none of the sites in row $i$ changes its color. With the only exception of $A$, an individual opinion change has probability $1 - O(r)$ for all sites so that

$$\Pr[h(t, y) = x \mid h(t, y) = x] = r \left[ g \left( x + \frac{1}{2} \right) + p_c \right] + O(r^2).$$

(9)

The hull can only shift two sites to the right if $A$ and $B$ become white. The former has probability $1 - O(r)$, the latter is given by Eq. 8, and all other probabilities are $1 - O(r)$. Therefore,

$$\Pr[h(t + 1, y) = x + 2 \mid h(t, y) = x] = r \left[ 1 - g \left( x + \frac{3}{2} \right) - p_c \right] + O(r^2).$$

(10)

All other shifts further to the left and right are $O(r^2)$, so that the only remaining transition of one step to the right has probability

$$\Pr[h(t + 1, y) = x + 1 \mid h(t, y) = x] = 1 + r(g - 1) + O(r^2).$$

(11)

1.3 Case 3: $h(t, y)$ is a strict local maximum

In analogy to case 2, we find

$$\Pr[h(t + 1, y) = x \mid h(t, y) = x] = r \left[ 1 - g \left( x - \frac{1}{2} \right) - p_c \right] + O(r^2),$$

(12)

$$\Pr[h(t + 1, y) = x - 2 \mid h(t, y) = x] = r \left[ g \left( x - \frac{3}{2} \right) + p_c \right] + O(r^2),$$

(13)

$$\Pr[h(t + 1, y) = x - 1 \mid h(t, y) = x] = 1 + r(g - 1) + O(r^2).$$

(14)

1.4 Summary

We can summarize the results so far with the notation

$$K_y = \begin{cases} 
+1 & \text{if } h(t, y) \text{ is a strict minimum,} \\
-1 & \text{if } h(t, y) \text{ is a strict maximum,} \\
0 & \text{otherwise.} 
\end{cases}$$

(15)

Neglecting terms $O(r^2)$,

$$\Pr[h(t + 1, y) = x - 1 + K_y \mid h(t, y) = x] = r \left[ g \left( x - \frac{1}{2} \right) + p_c \right] + rgK_y,$$

(16)

$$\Pr[h(t + 1, y) = x + K_y \mid h(t, y) = x] = 1 + r(g - 1),$$

(17)

$$\Pr[h(t + 1, y) = x + 1 + K_y \mid h(t, y) = x] = r \left[ 1 - g \left( x + \frac{1}{2} \right) - p_c \right] - rgK_y.$$ 

(18)

Because there are no isolated clusters in the solid-on-solid approximation and the dynamics is symmetric under interchange of black and white, $p_c$ equals $\frac{1}{2}$. This assumption is consistent with our numerical results for the full model $p_c = 0.5000(4)$. Equations 16–18 with $p_c = \frac{1}{2}$ yield Eq. 3–5 in the main text.

2 The stochastic differential equation for the hull evolution (Eq. 6 in main text)

An alternative formulation of Eq. 16–18 is

$$h(t + 1, y) = h(t, y) + K_y + \zeta_y,$$

(19)
where

\[
\begin{align*}
\Pr(\zeta_y = -1) &= r \left[ \frac{1}{2} + g \left( h(t, y) - \frac{1}{2} + K_y \right) \right], \\
\Pr(\zeta_y = 0) &= 1 + r(g - 1), \\
\Pr(\zeta_y = 1) &= r \left[ \frac{1}{2} - g \left( h(t, y) + \frac{1}{2} + K_y \right) \right].
\end{align*}
\]

(20)

The expectation value of \(\zeta_y\) is

\[
\langle \zeta_y \rangle = -2gr(h + K_y),
\]

(23)

so that we can rephrase Eq. 19 as

\[
h(t + 1, y) - h(t, y) = K_y - 2gr(h + K_y) + \zeta_y - \langle \zeta_y \rangle.
\]

(24)

Our objective is to take the continuum limit of Eq. 24 in the following manner. With the notation

\[
\begin{align*}
\Delta_+ &= h(t, y + 1) - h(t, y), \\
\Delta_- &= h(t, y) - h(t, y - 1),
\end{align*}
\]

(25)

(26)

we can express \(K_y\) of Eq. 15 using the Heaviside step function

\[
\theta(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{otherwise} \end{cases}
\]

(27)

as

\[
K_y = [1 - \theta(-\Delta_+)] [1 - \theta(\Delta_-)] - [1 - \theta(\Delta_+)] [1 - \theta(-\Delta_-)].
\]

(28)

The discontinuous Heaviside function can be written as the limit \(\epsilon \to 0\) of the differentiable function [2]

\[
\theta_\epsilon(x) = \epsilon \ln \left( \frac{\exp \left( \frac{x + 1}{\epsilon} \right) + 1}{\exp \left( \frac{x}{\epsilon} \right) + 1} \right).
\]

(29)

Simultaneously with the limit of the Heaviside function, we take the continuum limit of the space and time variables,

\[
\begin{align*}
\tilde{t} &= \epsilon^k t, \\
\tilde{y} &= \epsilon^l y, \\
\tilde{h}(\tilde{t}, \tilde{y}) &= \epsilon^m h(t, y),
\end{align*}
\]

(30)

(31)

(32)

with \(k, l, m > 0\) and let \(g\) approach zero as

\[
g = \epsilon^n \tilde{g}
\]

(33)

with \(n > 0\). We will now determine the leading terms in the individual parts of Eq. 24, which will give us conditions for these exponents.

### 2.1 The discrete time derivative \(A\) in Eq. 24

Assuming that \(h\) is a smooth function, we can expand \(A\) as

\[
A = \epsilon^{-m} \left( \tilde{h}(\tilde{t} + \epsilon^k) + \tilde{h}(\tilde{t}) \right) = \epsilon^{k-m} \frac{\partial \tilde{h}}{\partial \tilde{t}} + O(\epsilon^{2k-m}).
\]

(34)
2.2 The variable $K_y$ encoding a strict minimum or maximum

For the derivatives of $\theta$, of Eq. 29, we find

$$\theta_i(0) = 1 - \epsilon \ln(2) + O \left( \epsilon e^{-1/\epsilon} \right),$$

$$\frac{d\theta_i(0)}{dx} = \frac{1}{2} + O \left( \epsilon^{-1/\epsilon} \right),$$

$$\frac{d^2\theta_i(0)}{dx^2} = \frac{1}{4\epsilon} + O \left( \epsilon^{-1/\epsilon} \right),$$

$$\frac{d^3\theta_i(0)}{dx^3} = O \left( \epsilon^{-2} e^{-1/\epsilon} \right).$$

Inserting these derivatives into Eq. 28 we obtain

$$K_{y,\epsilon} = \left[ \epsilon \ln(2) + \frac{1}{2}\Delta_+ - \frac{1}{8\epsilon}\Delta_+^2 + O \left( \epsilon^{-3}\Delta_+^4 \right) \right] \left[ \epsilon \ln(2) - \frac{1}{2}\Delta_- - \frac{1}{8\epsilon}\Delta_-^2 + O \left( \epsilon^{-3}\Delta_-^4 \right) \right]$$

$$= \epsilon \ln(2) \left( \Delta_+ - \Delta_- \right) + \frac{1}{8\epsilon}\Delta_+ \Delta_- \left( \Delta_+ - \Delta_- \right) + O \left( \epsilon^{-3}\Delta_+ \Delta_-^5 \right).$$

From the Taylor expansions of $\tilde{h}$ we obtain

$$\Delta_+ - \Delta_- = \epsilon^{2l-m} \frac{\partial^2 \tilde{h}}{\partial y^2} + O \left( \epsilon^{4l-m} \right),$$

$$\Delta_+ \Delta_- (\Delta_+ - \Delta_-) = \epsilon^{4l-3m} \left( \frac{\partial \tilde{h}}{\partial y} \right)^2 \frac{\partial^2 \tilde{h}}{\partial y^2} + O \left( \epsilon^{6l-3m} \right),$$

so that

$$K_{y,\epsilon} = \epsilon^{2l-m+1} \ln(2) \frac{\partial^2 \tilde{h}}{\partial y^2} + O \left( \epsilon^{4l-3m-1} \right),$$

where the expansion converges only if

$$l - m > 1.$$  

2.3 The gradient term $B$ in Eq. 24

From Eq. 32, 33 and 42, we obtain

$$B = 2\epsilon^n \tilde{g} \left[ \epsilon^{-m} \tilde{h}(\tilde{t}, \tilde{y}) + O \left( \epsilon^{2l-m+1} \right) \right] = 2\epsilon^n \tilde{g} \tilde{r} \tilde{h}(\tilde{t}, \tilde{y}) + O \left( \epsilon^{2l-m+n+1} \right).$$

2.4 The noise term $C$ in Eq. 24

The covariance of the noise is

$$\langle [\zeta_y(t) - \langle \zeta_y(t) \rangle] \times [\zeta_y'(t') - \langle \zeta_y'(t') \rangle] \rangle = [r(1-g) + O(r^2)] \delta_{t,t'} \delta_{y,y'}.$$  

In the continuum limit, the Kronecker deltas transform as

$$\delta_{t,t'} = \epsilon^k \delta(\tilde{t} - \tilde{t}'),$$

$$\delta_{y,y'} = \epsilon^l \delta(\tilde{y} - \tilde{y}').$$

Dropping terms of order $O(r^2)$,

$$\langle \mathcal{C}(\tilde{t}, \tilde{y}) \mathcal{C}(\tilde{t}', \tilde{y}') \rangle = \epsilon^{k+l} \delta(\tilde{t} - \tilde{t}') \delta(\tilde{y} - \tilde{y}') + O \left( \epsilon^{k+l+n} \right).$$
where $C(\tilde{t}, \tilde{y}) = \zeta(\tilde{t}) - \langle \zeta(\tilde{t}) \rangle$ is defined as in Eq. 24. If we introduce the rescaled noise
\[
\eta(\tilde{t}, \tilde{y}) = \frac{\epsilon^{-(k+1)/2}}{\sqrt{\tau}} C(\tilde{t}, \tilde{y}),
\] (49)
then the covariance
\[
\langle \eta(\tilde{t}, \tilde{y}) \eta(\tilde{t}', \tilde{y}') \rangle = \delta(\tilde{t} - \tilde{t}') \delta(\tilde{y} - \tilde{y}') + O(\epsilon^n)
\] (50)
is to highest order independent of $\epsilon$.

2.5 Summary

Including only the leading terms, Eq. 24 becomes
\[
\epsilon^{k-m} \frac{\partial h}{\partial t} = \epsilon^{2l-m+1} \ln(2) \frac{\partial^2 h}{\partial y^2} - 2\epsilon^{n-m} g \tau \eta(\tilde{t}, \tilde{y}) + \sqrt{\tau} \epsilon^{(k+1)/2} \eta(\tilde{t}, \tilde{y}).
\] (51)
The four different terms scale with the same power of $\epsilon$ if
\[
l = \frac{1}{2} (k - 1),
\] (52)
\[
m = \frac{1}{4} (k + 1),
\] (53)
\[
n = k.
\] (54)
The inequality of Eq. 43 can be satisfied by $k > 7$. Dividing Eq. 51 by $\epsilon^{k-m} = \epsilon^{2l-m+1} = \epsilon^{n-m} = \epsilon^{(k+1)/2}$ yields Eq. 6 in the main text.

3 The derivation of the hull width (Eq. 7 in main text)

We consider Eq. 6 in the main text
\[
\frac{\partial h}{\partial t} = D \frac{\partial^2 h}{\partial y^2} - Eg h + F \eta(t, y)
\] (55)
with periodic boundaries at $y = 0$ and $y = L$. The solution $G(t, y)$ of the deterministic equation
\[
\frac{\partial G}{\partial t} = D \frac{\partial^2 G}{\partial y^2} - Eg G
\] (56)
with initial condition $\lim_{t \to 0} G(t, y) = \delta(y - y_0)$ is
\[
G(t, y; y_0) = \frac{1}{L} \sum_{n=-\infty}^{\infty} \exp \left[ -(Dk_n^2 + Eg) t + ik_n (y - y_0) \right],
\] (57)
where $k_n = 2\pi n / L$. The hull position can be expressed in terms of $G$ as
\[
h(t, y) = F \int_0^t dt' \int_0^y dy' G(t - t', y; y_0) \eta(t', y_0).
\] (58)
Combining the last two expression, we can derive
\[
\langle h(t, y) h(t, y') \rangle = \frac{F^2}{2L} \sum_{n=-\infty}^{\infty} \frac{1 - \exp \left( -2 \left( Dk_n^2 + Eg \right) t \right)}{Dk_n^2 + Eg} \exp \left[ ik_n (y' - y) \right]
\] (59)
The hull width is

\[
    w^2(L) = \lim_{t \to \infty} \left( \frac{1}{L} \int_0^L dy h^2(t, y) - \left( \frac{1}{L} \int_0^L dy h(t, y) \right)^2 \right) 
\]

\[
    = \frac{F^2}{L} \sum_{n=1}^{\infty} \left( Dk_n^2 + Eg \right)^{-1} 
\]

\[
    = \frac{F^2}{2} \left( \coth \left( \frac{L}{2 \sqrt{DEg}} \right) - \frac{1}{EgL} \right). 
\]

In the limit of large system size,

\[
    \lim_{L \to \infty} w^2(L) = \frac{F^2}{4\sqrt{DEg}}, 
\]

which is Eq. 7 in the main text.

4 UR and UR are in the IP universality class

In the main text, we show collapse plots for the maximum cluster size \(s_{\text{max}}\) (Fig. 3a and 3b) and the cluster size distribution \(p_{cs}\) (Fig. 4a and 4b) for IP and MV. In these cases, the data points lie on a single curve when we plot \(s_{\text{max}}L^{d_f} \) versus \(L^{\nu/(\nu+1)}\) and \(p_{cs}s^{-\tau} \) versus \(sg^{1/(\nu+1)}\). The crucial observation is that the collapse occurs when inserting the IP critical exponents \(d_f = 91/48, \nu = 4/3, \tau = 187/91\) and \(\sigma = 36/91\) in these expressions, a telltale sign that MV is indeed in the IP universality class.

In Fig. 3 of this supplement we make the equivalent plots for UR and UR with the same exponents. The data again fall on a single curve in each case. Moreover, we have already seen in Fig. 2 of the main text that for both of these models \(w \propto g^{-\nu/(\nu+1)}\) and \(b \propto g^{-1/\nu}\) as in IP. Thus, all numerical evidence points to both UR and UR (unlike MV) belonging to the IP universality class.
References

[1] D. M. Kroll, Z. Phys. B 41, 345 (1981).

[2] D. D. Vvedensky, Phys. Rev. E 67, 025102(R) (2003).