Manifolds with Density, applications and Gradient Schrödinger Operators

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Abstract

First, we briefly discuss the relationship of manifolds with density, Mean Curvature Flow, Ricci Flow and Optimal Transportation Theory.

The aim of this paper is twofold. On the one hand, the study of gradient Schrödinger operators on manifolds. We classify the space of solutions under global conditions. As an application, we extend the Naber-Yau Liouville Theorem. We relate those results to weighted $H_\phi$-stable hypersurfaces immersed in a manifold with density $(N, g, \phi)$ and we apply them to the classification of self-similar solutions to the mean curvature flow. In particular, we give conditions for the nonexistence of self-similar solutions (shrinker or expander) to the mean curvature flow under global hypothesis, extending the recent results of T. Colding and W. Minicozzi. The Naber-Yau Liouville Theorem will be used for the classification of gradient Ricci solitons. We extend the Hamilton-Ivey-Perelman classification of complete 3-dimensional shrinking gradient Ricci solitons of bounded curvature to higher dimensions, we relax the hypothesis to a $L^2$-type bound on its scalar curvature and an inequality between the scalar curvature and the Ricci curvature.

On the other hand, the topological and geometric classification of complete weighted $H_\phi$-stable surfaces immersed in a three-manifold with density $(N, g, \phi)$ whose Perelman scalar curvature, in short, P-scalar curvature, is nonnegative. Also, we classify weighted stable hypersurfaces in a manifold with density and a lower bound on its Bakry-Émery-Ricci tensor. Moreover, we study the classification of three-manifolds with density $(N, g, \phi)$ under the existence of a certain compact weighted area-minimizing surface and a lower bound of its P-scalar curvature. Here, the P-scalar curvature is defined as $R_\phi^\infty = R - 2\Delta_\phi \phi - |\nabla_\phi \phi|^2$, being $R$ the scalar curvature of $(N, g)$.

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1 Introduction

In a Riemannian manifold $(\mathcal{N}, g)$ there is a natural associated measure, that is, the Riemannian volume measure $dv_g \equiv dv$. More generally, we can consider Riemannian measure spaces, that is, triples $(\mathcal{N}, g, m)$, where $m$ is a smooth measure on $\mathcal{N}$. Equivalently by the Radon-Nikodým Theorem we can consider triples $(\mathcal{N}, g, \phi)$, where $\phi \in C^\infty(\mathcal{N})$ is a smooth function so that $dm = e^\phi dv$. The triple $(\mathcal{N}, g, \phi)$ is called a manifold with density $\phi$.

One of the first examples of a manifold with density appeared in the realm of probability and statistics, Euclidean space with the Gaussian density $e^{-\pi|x|^2}$ (see [8] for a detailed exposition in the context of isoperimetric problems). In 1985, D. Bakry and M. Émery [1] studied
manifolds with density in the context of diffusion equations. They introduced the so-called Bakry-Émery-Ricci tensor in the study of diffusion processes given by

\[ \text{Ric}_\phi^\infty = \text{Ric} - \nabla^2 \phi, \]  

(1.1)

where Ric is the Ricci tensor associated to \((\mathcal{N}, g)\) and \(\nabla^2\) is the Hessian with respect to the ambient metric \(g\). However, manifolds with density appear in many other fields of mathematics.

1.1 Hypersurfaces in manifolds with density

M. Gromov [25] considered manifolds with density as mm-spaces and introduced the generalized mean curvature of a hypersurface \(\Sigma \subset (\mathcal{N}, g, \phi)\) or weighted mean curvature as a natural generalization of the mean curvature, obtained by the first variation of the weighted area

\[ H_{\phi} = H + g(N, \nabla \phi), \]  

(1.2)

where \(H\) denotes the usual mean curvature of \(\Sigma\) and \(N\) is the unit normal vector field along \(\Sigma\).

**Remark 1.1.** In this paper, the mean curvature will be just the trace, and not the mean of the trace. Moreover, all the manifolds and submanifolds will be considered smooth, connected and orientable. Moreover, unless stated otherwise, they will be complete without boundary.

Let \((\mathcal{N}, g, \phi)\) be a Riemannian manifold with density \(\phi \in C^\infty(\mathcal{N}), \phi > 0\). For any Borel set \(\Omega \subset \mathcal{N}\) with boundary \(\Sigma := \partial \Omega\) and inward unit normal \(N\) along \(\Sigma\), the weighted volume of \(\Omega\) and the weighted area of \(\Omega\) are given by

\[ V_{\phi}(\Omega) := \int_\Omega e^\phi \, dv, \quad A_{\phi}(\Sigma) := \int_\Sigma e^\phi \, dv_{\Sigma}, \]

where \(dv\) and \(dv_{\Sigma}\) are the volume and area element with respect to \(g\) respectively. Note that the usual volume and area with respect to \(g\) are given when \(\phi \equiv 0\).

Let \(\Psi_t, \ t \in (-\varepsilon, \varepsilon)\), be a smooth family of diffeomorphisms of \(\mathcal{N}\) so that \(\Psi_0\) is the identity. Assume that

\[ \frac{d}{dt} \bigg|_{t=0} \Psi_{t|\Sigma} = X + u N, \]

where \(X\) is a tangential vector field along \(\Sigma\) and \(u\) is a smooth function with compact support on \(\Sigma\). An immersed hypersurface \(\Sigma \subset (\mathcal{N}, g, \phi)\) is weighted minimal, in short \(\phi\)-minimal, if it is a critical point of the weighted area functional, i.e., if

\[ \frac{d}{dt} \bigg|_{t=0} A_{\phi}(\Psi_t(\Sigma)) = 0, \]
for all compactly supported variations. From Bayle’s variational formulae [2] we can see that
\[
\frac{d}{dt} \bigg|_{t=0} A_{\phi}(\Psi_t(\Sigma)) = \int_{\Sigma} H_\phi u e^\phi \, dv_\Sigma,
\]
that is, $\Sigma$ is $\phi$–minimal if and only if the weighted mean curvature vanishes.

More generally, an immersed hypersurface $\Sigma$ has constant weighted mean curvature $H_\phi = H_0$ if and only if
\[
\frac{d}{dt} \bigg|_{t=0} (A_{\phi}(\Psi_t(\Sigma)) - H_0 V_\phi(\Psi_t(\Sigma))) = 0
\]
for all compactly supported variations $\{\Psi_t\}$.

**Definition 1.1.** We say that an open domain $\Omega \subset (\mathcal{N}, g, \phi)$ with boundary $\Sigma = \partial \Omega$ is **weighted stationary** if there exists a constant $H_0$ so that
\[
\frac{d}{dt} \bigg|_{t=0} (A_{\phi}(\Psi_t(\Sigma)) - H_0 V_\phi(\Psi_t(\Sigma))) = 0
\]
for all compactly supported variations $\{\Psi_t\}$.

As a first sight, the set of all weighted stationary domains is too big. So, we can focus on those which (locally) minimize the functional
\[
F_\phi(\Psi_t(\Sigma)) := A_{\phi}(\Psi_t(\Sigma)) - H_\phi V_\phi(\Psi_t(\Sigma))
\]
up to second order, that is, weighted stationary domains with nonnegative second variation for all compactly supported variations.

**Definition 1.2.** We say that a weighted stationary domain $\Omega \subset (\mathcal{N}, g, \phi)$ with boundary $\Sigma = \partial \Omega$ is **stable** if
\[
\frac{d^2}{dt^2} \bigg|_{t=0} (A_{\phi}(\Psi_t(\Sigma)) - H_\phi V_\phi(\Psi_t(\Sigma))) \geq 0,
\]
for all compactly supported variations $\{\Psi_t\}$. In this case, we say that $\Sigma$ is **weighted $H_\phi$–stable**, in short, $H_\phi$–**stable**.

Here, we are adopting the strong notion of stability and not the weak version that deals with isoperimetric problems, i.e., a weighted stationary domain $\Omega$ with boundary $\Sigma := \partial \Omega$ is weighted weakly stable if
\[
\frac{d^2}{dt^2} \bigg|_{t=0} (A_{\phi}(\Psi_t(\Sigma)) - H_\phi V_\phi(\Psi_t(\Sigma))) \geq 0,
\]
for all compactly supported variations $\{\Psi_t\}$ that fix the weighted volume. In this case, we say that $\Sigma$ is **weighted isoperimetric** (see [8] for details). From Bayle’s variational formula [2], one can prove:
Lemma 1.1. Consider a weighted stationary open set $\Omega \subset (N, g, \phi)$. Let $N$ be the inward unit normal along $\Sigma = \partial \Omega$, and $H_\phi$ the constant weighted mean curvature of $\Sigma$ with respect to $N$. Consider a compactly supported variation of $\Omega$, $\Psi_t$, with normal part $uN$ on $\Sigma$. Then, we have

$$\int_{\Sigma} (\text{Ric}_\phi^\infty(N,N) + |A|^2) u^2 e^\phi \, dv_\Sigma \leq \int_{\Sigma} |\nabla u|^2 e^\phi \, dv_\Sigma,$$

where $|A|^2$ is the squared norm of the second fundamental form and $\text{Ric}_\phi^\infty$ is the Bakry-Émery-Ricci curvature.

Equivalently, we have:

$$(A_\phi - H_\phi \, V_\phi)''(0) = - \int_{\Sigma} u \, L_\phi u \, e^\phi \, dv_\Sigma,$$

where

$$L_\phi u := \Delta u + g(\nabla \phi, \nabla u) + (|A|^2 + \text{Ric}_\phi^\infty(N,N))u.$$

Here $\Delta$ and $\nabla$ are the Laplacian and Gradient operators with respect to the induced metric on $\Sigma$, $A$ is the second fundamental form of $\Sigma$, $\text{Ric}$ is the Ricci curvature of the ambient manifold and $\bar{\nabla}^2$ is the Hessian operator with respect to the ambient metric.

1.2 Mean Curvature Flow

It is interesting to recall here that self-similar solutions to the mean curvature flow in $\mathbb{R}^{n+1}$ can be seen as weighted minimal hypersurfaces in the Euclidean space endowed with the corresponding density (c.f. G. Huisken [18] or T. Colding and W. Minicozzi [13, 14]). We will explain this in more detail.

Let $X : (0, T) \times \Sigma \to \mathbb{R}^{n+1}$ be a one parameter family of smooth hypersurfaces moving by its mean curvature, that is, $X$ satisfies

$$\frac{dX}{dt} = -HN$$

where $N$ is the unit normal along $\Sigma_t = X(t, \Sigma)$ and $H$ is its mean curvature. Self-similar solutions to the mean curvature flow are a special class of solutions, they correspond to solutions that a later time slice is scaled (up or down depending if it is expander or shrinker) copy of an early slice. In terms of the mean curvature, $\Sigma$ is said to be a self-similar solution if it satisfies the following equation

$$H = -\frac{c}{2}g(x, N),$$

where $c = \pm 1$, $x$ is the position vector in $\mathbb{R}^{n+1}$ and $g(\cdot, \cdot)$ is the standard Euclidean metric. Here, if $c = -1$ then $\Sigma$ is said a self-shrinker and if $c = +1$ then $\Sigma$ is called self-expander.

It is straightforward to check that self-shrinker (resp. self-expander) are weighted minimal hypersurfaces in $(\mathbb{R}^{n+1}, g_0, \phi_{-1})$ with density $\phi_{-1} := -\frac{|x|^2}{4}$ (resp. $\phi_{+1} := \frac{|x|^2}{4}$).
1.3 Ricci Flow

In G. Perelman’s work [38] on the Poincaré conjecture, he was able to formulate the Ricci flow as a gradient flow (we follow Topping’s Book [43]). Let us consider a (closed) manifold with density \((N, g, \phi)\) and introduce the following Fischer information functional

\[
F(g, \phi) := \int_N \left( R + |\nabla \phi|^2 \right) dm,
\]

where \(dm = e^\phi dv\) and \(R\) is the scalar curvature of \((N, g)\). Equivalently, integrating by parts,

\[
F(g, \phi) := \int_N R^\infty dm,
\]

here, \(R^\infty\) is the Perelman Scalar Curvature, in short P-scalar curvature, given by

\[
R^\infty = R - 2\Delta_g \phi - |\nabla \phi|^2, \tag{1.7}
\]

where \(\Delta_g\) and \(\nabla\) are the Laplacian and Gradient operators with respect to the ambient metric \(g\) respectively.

**Remark 1.2.** We focus on the compact case, i.e., \(N\) is closed. Nevertheless, the discussion can be extended to complete manifolds under appropriate conditions. Moreover, we will consider here the smooth case.

Let us consider the variation of \(F\) which preserves the distorted volume \(\phi dv\). So, the evolution of \(F\) is given by

\[
\frac{d}{dt} F(g, \phi) = -\int_N g \left( \text{Ric} - \nabla^2 \phi, \frac{\partial g}{\partial t} \right) e^\phi dv = -\int_N g \left( \text{Ric}^\infty, \frac{\partial g}{\partial t} \right) e^\phi dv.
\]

Therefore, if we have a solution to the coupled system

\[
\begin{cases}
\frac{\partial g}{\partial t} = -2\text{Ric}^\infty, \\
\frac{\partial \phi}{\partial t} = (R - \Delta_g \phi) \phi,
\end{cases} \tag{1.8}
\]

then \(F\) evolves as

\[
\frac{d}{dt} F(g, \phi) = 2\int_N |\text{Ric}^\infty|^2 e^\phi dv \geq 0.
\]

Summarizing, if we define \(\mu := e^\phi dv\), which is constant in time, we could view \(g\) as a gradient flow for the functional \(g \rightarrow F(g, \phi)\), where \(\phi\) is determined by \(\mu\) and \(dv\).
Perelman’s trick was to show that a solution of the coupled system (1.8) is somehow equivalent to a solution to the decoupled system

\[
\begin{align*}
\frac{\partial g}{\partial t} &= -2\text{Ric}, \\
\frac{\partial \phi}{\partial t} &= \left(R_{\phi}^\infty + \Delta_g \phi\right) \phi,
\end{align*}
\] (1.9)

One way to see this is that solutions to the coupled system (1.8) may be generated by pulling back solutions of the decoupled system (1.9) by an appropriate time-dependent diffeomorphism. We can also reverse the argument.

The aim of Perelman for introducing \(\mathcal{F}\) (and the more general \(\mathcal{W}\)-entropy) was the classification of (gradient) Ricci solitons, that is, self-similar solutions to the Ricci flow. They can be analytically described as manifolds with density \((\mathcal{N}, g, \phi)\) such that \(\text{Ric}_{\phi}^\infty = \lambda g\), with \(\lambda \in \mathbb{R}\). They are called shrinking, steady or expanding depending if \(\lambda > 0\), \(\lambda = 0\) or \(\lambda < 0\) respectively. Steady Ricci solitons appear as critical points of the \(\mathcal{F}\) functional (shrinking Ricci solitons appear as critical points of the \(\mathcal{W}\)-entropy functional). See [43] for a detailed exposition.

**Remark 1.3.** Perelman proved that \(R_{\phi}^\infty\) is not the trace of \(\text{Ric}_{\phi}^\infty\) but they are related by a Bianchi type identity. Moreover, Perelman’s entropy functional and manifolds with density have applications to String Theory (see [6]).

### 1.4 Optimal Transportation Theory

Recently, in Optimal Transport Theory (we follow Part II in Villani’s Book [44]), geometric problems on manifolds with density \((\mathcal{N}, g, \phi)\) has been related to correspondent problems in the Wasserstein space of probability measures equipped with the quadratic Wasserstein metric, which corresponds to a relaxed version of Monge’s optimal transportation problem. We will explain the relationship between them from a formal point of view, we will not be rigorous (see [44] and references therein). Moreover, we will assume all the objects involved are smooth and verify the appropriate convergence conditions to avoid technical problems and clarify the exposition.

Let \((\mathcal{N}, g)\) be a Riemannian manifold. Denote by \(dv\) and \(d\) the Riemannian measure and distance with respect to \(g\) respectively. The Wasserstein space \(\mathcal{P}_2(\mathcal{N})\) is the space of probability measures which have a finite moment of order 2, i.e.,

\[
\mathcal{P}_2(\mathcal{N}) = \left\{ \mu \in \mathcal{P}(\mathcal{N}) : \int_{\mathcal{N}} d(p_0, p)^2 d\mu < +\infty \right\},
\]

where \(p_0 \in \mathcal{N}\) is arbitrary and \(\mathcal{P}(\mathcal{N})\) is the space of probability measures. The Wasserstein space will be equipped with the Wasserstein distance

\[
\mathcal{W}_2(\mu, \nu) = \left( \inf \left\{ \int_{\mathcal{N}} d(p, q)^2 d\pi : \pi \in \Pi(\mu, \nu) \right\} \right)^{\frac{1}{2}},
\]

7
where the infimum is taken over all joint probability measures $\pi$ on $\mathcal{N} \times \mathcal{N}$ with marginals $\mu$ and $\nu$. Such joint measures are called transference plans; those achieving the infimum are called optimal transference plans.

**Remark 1.4.** We should think on $d^2$ as the cost function on the optimal transportation problem.

F. Otto [35] noticed that computations of Riemannian nature can shed light on Optimal Transportation Theory. So, the problem was to establish rules for formally perform differential calculus on $\mathcal{P}_2(\mathcal{N})$. We focus here on a certain class of functional that appear naturally on the theory, as for example, the Boltzman entropy.

Let us consider $\phi : \mathcal{N} \to \mathbb{R}$ a smooth function to distort the reference volume measure, i.e., $dm = e^\phi dv$, that is, we can consider a manifold with density $(\mathcal{N}, g, \phi)$. Consider a smooth function $U : \mathbb{R}^+ \to \mathbb{R}$ which will relate the values of the density of the probability measure and the value of the functional, i.e.,

$$U_m(\mu) = \int_\mathcal{N} U(\rho(p)) \, dm, \quad \mu = \rho \, dm.$$

**Remark 1.5.** We should think on $U_m$ as the internal energy of a fluid, i.e., it is like the energy contained on a fluid of density $\rho$. The function $U$ should be thought as a property of the fluid and it might mean some microscopic interaction. It is natural to assume $U(0) = 0$.

In analogy with thermodynamics, we can introduce the pressure as:

$$p(\rho) := \rho U'(\rho) - U(\rho).$$

**Remark 1.6.** The physical meaning of the pressure says that if a fluid is enclosed in a domain $\Omega$, then the pressure felt by the boundary $\partial \Omega$ at a point is normal and proportional to $p$ at that point.

So, one can consider the total pressure

$$\int_\mathcal{N} p(\rho) \, dm$$

and again, compute the variation of this functional with respect to small variations of the measure, which leads to the iterated pressure:

$$p_2(\rho) := \rho p'(\rho) - p(\rho).$$

One can see that the pressure and iterated pressure appear when we differentiate the energy functional $U_m$ to first and second order respectively. F. Otto [35] gave an explicit expression for the gradient and Hessian of the functional $U_m$. For a given measure $\mu$, the gradient of $U_m$ at $\mu$ is a tangent vector at $\mu$ in the Wasserstein space, and it is given by

$$\text{grad}_\mu U_m = - (\Delta p(\rho) + g(\nabla \phi, \nabla p(\rho))) \, dm,$$
where, remember, \( dm = e^\phi \, dv \). The Hessian of \( \mathcal{U}_m \) at \( \mu \) is a quadratic form on the tangent space \( T_\mu \mathcal{P}_2(\mathcal{N}) \), but its expression is rather complicated in general. We continue this discussion only for the Boltzman entropy functional, which is given by

\[
\mathcal{H}_m(\mu) := \int_\mathcal{N} \rho \ln \rho \, dm, \quad \mu = \rho \, dm.
\]

For the Boltzman entropy, its gradient is given by

\[
\text{grad}_\mu \mathcal{H}_m = - (\Delta \rho + g(\nabla \phi, \nabla \rho)) \, dm
\]

so, we can see that critical points of the Boltzman entropy correspond to positive solution to

\[
\Delta \rho + g(\nabla \phi, \nabla \rho) = 0,
\]

or, equivalently, they correspond to positive Jacobi functions for the gradient Schrödinger operator

\[
L_\phi \rho := \Delta \rho + g(\nabla \phi, \nabla \rho).
\]

## 2 Organization of the paper and Main results

We will describe here the organization of the paper and we will outline the main results. Moreover, we should remark that the required differentiability is much lower of that we consider here, we prefer to avoid technical details assuming smoothness for clarifying the exposition and ideas involved in this paper.

In Section 3 we develop the theory of gradient Schrödinger operators defined on a manifold \( \Sigma \) acting on piecewise smooth functions of compact support \( u \in C^\infty_0(\Sigma) \). That is, given \((\Sigma, g)\) a Riemannian manifold, complete or compact with boundary possibly empty, \( \phi \in C^\infty(\Sigma) \) and \( q \in C^\infty(\Sigma) \) called potential, we study differential operators on the form

\[
L_\phi u := \Delta u + g(\nabla \phi, \nabla u) + qu, \ u \in C^\infty_0(\Sigma). \tag{2.1}
\]

When \( \nabla \phi \equiv 0 \), that is, we are dealing with an usual Schrödinger operator \( L := \Delta + q \), it is known that \( L \) is self-adjoint with respect to the \( L^2 \)-inner product, but this is not longer true for a gradient Schrödinger operator. Nevertheless, we will see that \( L_\phi \) is self-adjoint with respect to a weighted \( L^2 \)-inner product, the weight depends on \( \phi \). We will show that gradient Schrödinger operators mimic most of the properties of the usual Schrödinger operators when we consider this weighted \( L^2 \)-inner product. When \( \Sigma \) is compact and there exists a non identically zero solution to \( L_\phi u = 0 \), then \( u \) vanishes nowhere and the linear space of such functions is one dimensional. We would like to extend this property to the noncompact case imposing a global condition on the triple \((\Sigma, g, \phi)\).
**Definition 3.2.** Let \((\Sigma, g)\) be a complete manifold and \(\phi \in C^\infty (\Sigma)\). We say that 
\((\Sigma, g, \phi)\) is of **finite type** if there exists a sequence of cut-off functions \(\{\psi_i\}_i \subset C^\infty_0 (\Sigma)\) such that

- \(0 \leq \psi_i \leq 1\) in \(\Sigma\).
- The compact sets \(\Omega_i := \psi_i^{-1}(1)\) form an increasing exhaustion of \(\Sigma\).
- The sequence of weighted energies \(\{\int_\Sigma |\nabla \psi_i|^2 e^\phi d\nu\}_i\) is bounded.

Moreover, given a complete manifold \((\Sigma, g)\) and smooth functions \(\phi, q \in C^\infty (\Sigma)\), we say that the gradient Schrödinger operator \(L_{\phi,q}\) with potential \(q\) given by (2.1) is of **finite type** if \((\Sigma, g, \phi)\) is of finite type.

In the case \(\nabla \phi = 0\), Definition 3.2 says that \((\Sigma, g)\) has finite capacity (see [24]). The following result, which extends [28, Theorem 2.3], tells us that finite type gradient Schrödinger operators behaves as in the compact case.

**Theorem 3.2.** Let \((\Sigma, g)\) be a complete manifold and take \(\phi, q_1, q_2 \in C^\infty (\Sigma)\). Assume that

1. The gradient Schrödinger operator \(L_{\phi,q_1} := \Delta + g(\nabla f, \nabla \cdot) + q_1\) is of finite type and there is a positive subsolution \(u\), i.e., \(u\) satisfies
\[L_{\phi,q_1} u \leq 0.\]

2. There exists a bounded function \(v \in C^\infty (\Sigma)\) so that
\[L_{\phi,q_2} v := \Delta v + g(\nabla f, \nabla v) + q_2 v \geq 0.\]

3. \(q_1 - q_2 \geq 0\) on \(\Sigma\).

Then, \(v/u\) is constant.

Moreover, we above technique allows us to extend the Naber-Yau Liouville Theorem [33]:

**Theorem 3.3.** Let \((\Sigma, g)\) be a complete manifold and take \(\phi, q \in C^\infty (\Sigma)\). Assume the gradient Schrödinger operator \(L_{\phi,q} := \Delta + g(\nabla \phi, \nabla \cdot) + q\) is of finite type.

1. Assume \(q \geq 0\). If there is a nonnegative (nonidentically zero) subsolution \(u\), i.e., \(u\) satisfies \(L_{\phi,q} u \leq 0\), then \(u\) is constant and either \(u \equiv 0\) or \(q \equiv 0\) on \(\Sigma\). In particular, any solution bounded above or below to \(L_{f,q} u = 0\) must be constant.

2. Assume \(q \leq 0\) and \(L_{\phi,q}\) is stable. If there is a bounded supersolution \(u\), i.e., \(u\) satisfies \(L_{\phi,q} u \geq 0\), then \(u\) is constant and either \(u \equiv 0\) or \(q \equiv 0\) on \(\Sigma\).
3. Assume \( q \leq 0 \) and there is a nonnegative supersolution \( u \), i.e., \( u \) satisfies \( L_{\phi,q} u \geq 0 \), such that the sequence \( \left\{ \int_{\Sigma} |\nabla \psi_i|^2 u^2 e^{\phi} \, dv \right\} \) is uniformly bounded. Then \( u \) is constant and either \( u \equiv 0 \) or \( q \equiv 0 \) on \( \Sigma \). Moreover, if \( q \equiv 0 \), we only need to assume that \( u \) is bounded below.

In Section 4, we will apply Theorem 3.2 to recover and extend a recent result of Colding-Minicozzi [13, 14] on the nonexistence of certain self-shrinkers. To understand properly the main results in this section, we need the following definition:

**Definition 4.2.** Let \( \Sigma \subset \mathbb{R}^{n+1} \) be a self-shrinker (resp. expander). We say that \( \Sigma \) is of finite type if \( (\Sigma, g, \phi_{-1}) \) (resp. \( (\Sigma, g, \phi_{+1}) \)) is of finite type. Here, \( g \) denotes the induced metric on \( \Sigma \).

Here, \( \phi_c = \frac{|x|^2}{4} \), where \( x \) is the position vector in \( \mathbb{R}^{n+1} \), and \( c = -1 \) if \( \Sigma \) is a self-shrinker and \( c = +1 \) if \( \Sigma \) is a self-expander. T. Colding and W. Minicozzi [13, 14] proved that there are no \( L \)-stable self-shrinker \( \Sigma \) with polynomial area growth. We say that a self-shrinker (resp. expander) \( \Sigma \) is \( L \)-**stable** if it is stable as a weighted minimal hypersurface in \( (\mathbb{R}^{n+1}, g_0, \phi_{-1}) \) (resp. \( (\mathbb{R}^{n+1}, g_0, \phi_{+1}) \)). Polynomial area growth implies that

\[
\int_{B(i+1)\setminus B(i)} e^{\phi_{-1}} \, dv \to 0,
\]

where \( B(r) \) denotes the geodesic ball of radius \( r \) centered at a fixed point. In particular, \( \Sigma \) is of finite type. So, we can prove:

**Theorem 4.1.** There are no complete \( L \)-stable self-shrinkers in \( \mathbb{R}^{n+1} \) of finite type.

As far as we know, there are no general classification results for \( L \)-stable self-expanders. We can, at least, give some condition for the nonexistence:

**Proposition 4.1.** There are no complete \( L \)-stable self-expanders in \( \mathbb{R}^{n+1} \) of finite type and \( |A|^2 \geq 1/2 \).

In fact, there is another notion of stability more related to graphs. Let \( \Sigma \subset \mathbb{R}^{n+1} \) be a self-shrinker (resp. expander). Let \( a \in \mathbb{R}^{n+1} \) be a constant vector and consider \( u_a := g(N, a) \), then \( u_a \) satisfies (see [13, 14]):

\[
L_0 u_a := \Delta u_a + g(\nabla \phi_c, \nabla u_a) + |A|^2 u_a = 0.
\]

If \( \Sigma \) is a complete multi-graph, say with respect to the \( e_{n+1} \) direction, then \( \nu := g(N, e_{n+1}) \) is positive and satisfies the above equation, therefore Lemma 3.1 yields that \( L_0 \) is stable in the sense of Definition 3.1. Therefore, a self-similar solution to the mean curvature flow is \( L_0 \)-stable when \( L_0 \) defined above is stable in the sense of Definition 3.1. Thus, this leads us to:
Theorem 4.2. The only complete self-similar solutions to the mean curvature flow that are $L^0$–stable and of finite type are hyperplanes.

In particular, The only self-similar solutions (shrinker or expander) to the mean curvature flow that are complete multi-graphs of finite type are hyperplanes. Moreover, the only self-shrinkers that are entire graphs are the hyperplanes.

Recently, L. Wang [45] proved that the only self-shrinkers that are entire graphs are hyperplanes. This result follows from us since a proper self-shrinker has polynomial volume growth [11, Theorem 1.3], and hence, it is of finite type.

In Section 5, we use Theorem 3.3 for classifying gradient Ricci solitons. Noncompact gradient Ricci solitons are fundamental in the proof of the Poincaré Conjecture. From the works of R. Hamilton [26], T. Ivey [27] and G. Perelman [38, 39] we have: The only three dimensional shrinking gradient Ricci solitons with bounded curvature are the finite quotients of $\mathbb{R}^3$, $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{S}^3$.

P. Petersen and W. Wylie [36] showed that the only complete shrinking gradient Ricci soliton such that $\int_{\Sigma} |\text{Ric}|^2 e^\phi dv_\Sigma < \infty$ and vanishing Weyl Tensor, i.e. $W \equiv 0$, are finite quotients of $\mathbb{R}^n$, $\mathbb{S}^{n-1} \times \mathbb{R}$ or $\mathbb{S}^n$ (see [36, 37] and references therein for a detailed exposition).

We extend the above result imposing only a $L^2$–type bound on the scalar curvature $R$. Moreover, they assume that the Weyl tensor vanishes identically, we change that condition by an inequality between the scalar curvature and the Ricci curvature.

Theorem 5.1. Let $(\Sigma, g, \phi, \lambda)$ be a complete shrinking soliton of dimension $n \geq 3$ so that $\lambda R \leq |\text{Ric}|^2$. Assume that there exists a sequence of cut-off functions $\{\psi_i\}_i \subset C^0_c(\Sigma)$ such that $0 \leq \psi_i \leq 1$ in $\Sigma$, the compact sets $\Omega_i := \psi_i^{-1}(1)$ form an increasing exhaustion of $\Sigma$, and the sequence of weighted energies $\{\int_{\Sigma} |\nabla \psi_i|^2 R^2 e^\phi dv_\Sigma\}_i$ is bounded.

Then, $(\Sigma, g)$ has constant scalar curvature and $\lambda R = |\text{Ric}|^2$.

So we recover the Hamilton-Ivey-Perelman classification assuming only a $L^2$–type bound on the scalar curvature and the inequality between the scalar curvature and the Ricci tensor. We also give the following classification result when the scalar curvature does not change sign:

Theorem 5.2.

- A complete gradient expander soliton of finite type and nonnegative scalar curvature is Ricci flat.
- A complete gradient steady soliton of finite type whose scalar curvature does not change sign is Ricci flat. Moreover, if $\phi$ is not constant then it is a product of a Ricci flat manifold with $\mathbb{R}$. Also, $\Sigma$ is diffeomorphic to $\mathbb{R}^n$. 

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A complete gradient shrinking soliton of nonpositive scalar curvature is Ricci flat.

In Section 6 we introduce the key tool for studying $H_\phi$–stable surfaces. We will associate a self-adjoint Schrödinger operator to any weighted $H_\phi$–stable surface which is stable in the sense of Schrödinger operators (see Definition 6.1), i.e., its first eigenvalue is nonnegative. Specifically, we prove

Lemma 6.1. Let $\Sigma \subset (\mathcal{N}^3, g, \phi)$ be a complete weighted $H_\phi$–stable surface. Then,

$$\int_{\Sigma} (V - aK) f^2 dv_{\Sigma} \leq \int_{\Sigma} |\nabla f|^2 dv_{\Sigma}$$

for any $f \in C_0^\infty(\Sigma)$ compactly supported piecewise smooth function, where

$$V := \frac{1}{3} \left( \frac{1}{2} R^\infty_\phi + \frac{1}{2} H^2_\phi + \frac{1}{2} |A|^2 + \frac{1}{8} |\nabla \phi|^2 \right) \text{ and } a := \frac{1}{3}.$$ 

In other words, the Schrödinger operator

$$L := \Delta - aK + V$$

is stable in the sense of Definition 6.1.

Section 7 is devoted to the classification of complete weighted $H_\phi$–stable surfaces in a manifold with density $\phi \in C^\infty(\Sigma)$ so that its P-scalar curvature is nonnegative. First, we extend the Rosenberg’s diameter estimate for weighted $H_\phi$–stable surfaces given a positive lower bound for $R^\infty_\phi + H^2_\phi$ (see [40] for the Riemannian case),

Theorem 7.1. Let $\Sigma \subset (\mathcal{N}, g, \phi)$ be a weighted $H_\phi$–stable surface with boundary $\partial \Sigma$. If $R^\infty_\phi + H^2_\phi \geq c > 0$ on $\Sigma$, then

$$d_{\Sigma}(p, \partial \Sigma) \leq \frac{2\pi}{\sqrt{3c}} \text{ for all } p \in \Sigma,$$

where $d_{\Sigma}$ denotes the intrinsic distance in $\Sigma$. Moreover, if $\Sigma$ is complete without boundary, then it must be topologically a sphere.

Theorem 7.1 says that the only complete noncompact weighted $H_\phi$–stable we shall consider are the minimal ones. So, the next step is to classify the complete noncompact weighted minimal stable surfaces in the spirit of the works of D. Fischer-Colbrie and R. Schoen [23] and R. Schoen and S.T. Yau [41]. The theorem we provide here extends a previous result of P. T. Ho [17]. In [17], the author needs the additional hypothesis that $|\nabla \phi|$ is bounded on $\Sigma$. Here, we drop that condition and characterize the stable cylinder as totally geodesic, flat and with constant density along $\Sigma$. We show:
Theorem 7.2. Let $\Sigma \subset (\mathcal{N}, g, \phi)$ be a complete (noncompact) weighted stable minimal surface where $R^\infty_\phi \geq 0$. Then, $\Sigma$ is conformally equivalent either to the complex plane $\mathbb{C}$ or to the cylinder $\mathbb{S}^2 \times \mathbb{R}$. In the latter case, $\Sigma$ is totally geodesic, flat and $\phi$ is constant and $R^\infty_\phi \equiv 0$ along $\Sigma$.

We finish Section 7 classifying $H_\phi$–stable hypersurfaces in a manifold with density satisfying a lower bound on the Bakry-Émery-Ricci tensor. Specifically:

Theorem 7.3. Let $\Sigma \subset (\mathcal{N}, g, \phi)$ be a complete $H_\phi$–stable hypersurface of finite type and assume that $\text{Ric}^\infty_\phi \geq k$, $k \geq 0$. Then, $\Sigma$ is totally geodesic and $\text{Ric}^\infty_\phi (N, N) \equiv 0$ along $\Sigma$.

In particular, Theorem 7.3 extends results obtained by X. Cheng, T. Mejia and D. Zhou [10].

In Section 8 we give local splitting results for three-manifolds with density $(\mathcal{N}, g, \phi)$ verifying some lower bound on its P-scalar curvature $R^\infty_\phi$. The idea is to extend the splitting theorems developed by Cai-Galloway [5], Bray-Brendle-Neves [4] and I. Nunes [34].

Theorem 8.1. Let $(\mathcal{N}^3, g, \phi)$ be a complete manifold with density containing a compact, embedded weighted area-minimizing surface $\Sigma$.

1. Suppose that $R^\infty_\phi \geq \lambda e^\phi$, for some positive constant $\lambda$, and $A_\phi(\Sigma) = \frac{8\pi}{\lambda}$. Then, $\Sigma$ has genus zero and it has a neighborhood in $\mathcal{N}$ which is isometric to the product $\mathbb{S}^2 \times (-\varepsilon, \varepsilon)$ with the product metric $g_{+1} + dt^2$ (up to scaling the metric $g$); here $g_{+1}$ is the metric of constant Guassian curvature $+1$. Moreover, $\phi$ is constant in $U$.

2. Suppose that $R^\infty_\phi \geq 0$ and $\Sigma$ has genus one. Then, $\Sigma$ has a neighborhood in $\mathcal{N}$ which is flat and isometric to the product $\mathbb{T}^2 \times (-\varepsilon, \varepsilon)$ with the product metric $g_0 + dt^2$; here $g_0$ is the metric of constant Guassian curvature $0$. Moreover, $\phi$ is constant in $U$.

3. Suppose that $R^\infty_\phi \geq -\lambda e^\phi$, for some positive constant $\lambda$, $\Sigma$ has genus $\gamma \geq 2$ and $A_\phi(\Sigma) = \frac{4\pi(\gamma-1)}{\lambda}$. Then, $\Sigma$ has a neighborhood in $\mathcal{N}$ which is isometric to the product $\Sigma \times (-\varepsilon, \varepsilon)$ with the product metric $g_{-1} + dt^2$ (up to scaling the metric $g$); here $g_{-1}$ is the metric of constant Guassian curvature $-1$. Moreover, $\phi$ is constant in $U$.

Finally, in Section 9 we establish the classification results for manifolds with density $\phi$ satisfying a lower bound on its P-scalar curvature and the existence of certain compact weighted area-minimizing surface in its homotopy class. First, we focus in the case the P-scalar curvature has a positive lower bound. We prove:
Theorem 9.1. Let \((N^3, g, \phi)\) be a compact manifold with density so that \(R_*^\phi\) is positive and \(\pi_2(N) \neq 0\). Define
\[
A(N, g, \phi) := \inf \left\{ A_\phi(f(S^2)) : f \in \mathcal{F} \right\},
\]
where \(\mathcal{F}\) is the set of all smooth maps \(f : S^2 \to N\) which represent nontrivial elements in \(\pi_2(N)\).
Then, we have
\[
A(N, g, \phi) \min \left\{ R_*^\phi(x) e^{\phi(x)} : x \in N \right\} \leq 8\pi.
\]
Moreover, if equality holds, the universal cover of \((N, g)\) is isometric (up to scaling) to the standard cylinder \(S^2 \times \mathbb{R}\) and \(\phi\) is constant in \(N\).

Second, when the P-scalar curvature is nonnegative:

Theorem 9.2. Let \((N^3, g, \phi)\) be a complete manifold with density so that \(R_*^\phi\) is nonnegative. If \((N, g, \phi)\) contains a weighted area minimizing compact surface in its homotopy class of genus greater than or equal to 1, then the product manifold \(T^2 \times \mathbb{R}\), where \(T^2\) is a torus equiped with the standard flat metric, is an isometric covering of \((N, g)\), and \(\phi\) is constant in \(N\). In particular, \((N, g)\) is flat.

And finally, when the P-scalar curvature has a negative lower bound:

Theorem 9.3. Let \((N^3, g, \phi)\) be a complete manifold with density so that \(R_*^\phi \geq -\lambda e^\phi\) for some positive constant \(\lambda\). Moreover, suppose that \(\Sigma \subset (N, g, \phi)\) is a two-sided compact embedded Riemannian surface of genus \(\gamma \geq 2\) which minimizes area in its homotopy class. Then,
\[
A_\phi(\Sigma) \geq \frac{4\pi(\gamma - 1)}{\lambda}.
\]
Moreover, if equality holds, the product manifold \(\Sigma_\gamma \times \mathbb{R}\), where \(\Sigma_\gamma\) is a compact surface of genus \(\gamma\) equiped with a metric of constant Gaussian curvature \(-1\), is an isometric covering (up to scaling) of \((N, g)\), and \(\phi\) is constant in \(N\).

Remark 2.1. Most of the above results can be generalized to the case that \(\Sigma\) has finite index, that is, there is only a finite dimensional space of normal variations which strictly decrease the weighted area (similar to results of Fischer-Colbrie [22]). Moreover, we could relax the hypothesis on the nonnegativity of the P-scalar curvature by either an integrability conditon or a decay condition as in [3] or [20].
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3 Gradient Schrödinger operators of finite type

We will follow the references [9, 15, 16]. Consider $(\Sigma, g)$ a compact Riemannian manifold with boundary $\partial \Sigma$ (possibly empty). Set $q \in C^\infty(\Sigma)$ and $X \in \mathfrak{X}(\Sigma)$ a smooth vector field along $\Sigma$. Consider the differential linear operator, called generalized Schrödinger operator, given by

$$L : C^\infty_0(\Sigma) \to C^\infty(\Sigma)$$

$$u \to Lu := \Delta u + g(X, \nabla u) + qu,$$

where $\Delta$ and $\nabla$ are the Laplacian and Gradient with respect to the Riemannian metric $g$ (respectively) and $C^\infty_0(\Sigma)$ stands for the linear space of compactly supported piecewise smooth functions on $\Sigma$. We also denote by $|\cdot|$ the norm with respect to the Riemannian metric $g$. Moreover,

- In the case that $X \equiv \nabla \phi$ for some smooth function $\phi$, i.e., $L_\phi u := \Delta u + g(\nabla \phi, \nabla u) + qu$, we call $L_\phi$ by gradient Schrödinger operator.

- In the case that $X \equiv 0$, i.e., $Lu := \Delta u + qu$, we call $L$ simply by Schrödinger operator.

Note that the generalized Schrödinger operator $L_\phi$, given by (2.1), is a gradient Schrödinger operator for $X := \nabla \phi$. In general, a gradient Schrödinger operator is not self-adjoint with respect to the $L^2$–inner product because of the first order term, but it is self-adjoint with respect to a weighted inner product.

**Proposition 3.1.** Let $(\Sigma, g)$ be a compact Riemannian manifold with boundary $\partial \Sigma$ (possibly empty). Set $q, \phi \in C^\infty(\Sigma)$ and consider

$$L_\phi u := \Delta u + g(\nabla \phi, \nabla u) + qu, \ u \in C^\infty_0(\Sigma).$$

Then, $L_\phi$ is self-adjoint acting on $u \in C^\infty_0(\Sigma)$ with respect to the weighted inner product given by

$$g(u, v)_\phi := \int_\Sigma u v e^\phi \, dv_\Sigma.$$
Proof. We have to prove that
\[ g(u, L_\phi v)_{\phi} = g(L_\phi u, v)_{\phi}, \quad \text{for all } u, v \in C_0^\infty(\Sigma). \]

Take \( u, v \in C_0^\infty(\Sigma) \), then
\[ g(u, L_\phi v)_{\phi} = \int_\Sigma u L_\phi v e^\phi \, dv_\Sigma = \int_\Sigma \left( u e^\phi \Delta v + u e^\phi g(\nabla \phi, \nabla v) + quv e^\phi \right) \, dv_\Sigma = \int_\Sigma \left( v \text{div}(e^\phi \nabla v) + quv e^\phi \right) \, dv_\Sigma = g(L_\phi u, v)_{\phi}, \]

where we have used the Divergence Theorem on the second line.

Let \( L_\phi \) be a gradient Schrödinger operator. We say that \( \lambda \in \mathbb{R} \) is an eigenvalue of \( L_\phi \) if there is a not identically zero function \( u \in C_0^\infty(\Sigma) \) so that
\[ L_\phi u = -\lambda u. \]

In the case of gradient Schrödinger operators, standard spectral theory (see [15, Pages 335–336]) gives:

**Theorem 3.1.** Let \((\Sigma, g)\) be a compact Riemannian manifold with boundary \( \partial \Sigma \) (possibly empty). Set \( q, \phi \in C^\infty(\Sigma) \) and consider
\[ L_\phi u := \Delta u + g(\nabla \phi, \nabla u) + qu, \quad u \in C_0^\infty(\Sigma). \]

Then,
\begin{itemize}
  \item \( L_\phi \) has real eigenvalues \( \lambda_1 < \lambda_2 \leq \ldots \) with \( \lambda_k \to +\infty \) as \( k \to +\infty \).
  \item There is an orthonormal basis \( \{u_k\} \subset C_0^\infty(\Sigma) \) for the weighted \( L^2 \) metric given in Proposition 3.1 so that \( L_\phi u_k = -\lambda_k u_k \).
  \item The lowest eigenvalue \( \lambda_1 \) is characterize by
    \[ \lambda_1(\Sigma, f) = \inf \left\{ \frac{\int_\Sigma \left( u \Delta u - qu^2 \right) e^\phi \, dv_\Sigma}{\int_\Sigma u^2 e^\phi \, dv_\Sigma} : \ u \in C_0^\infty(\Sigma) \setminus \{0\} \right\}. \]
  \item Any eigenfunction for \( \lambda_1 \) does not change sign and, consequently, if \( u \in C_0^\infty(\Sigma) \) is another solution to \( L_\phi u = -\lambda_1 u \), then \( u \) is a constant multiple of \( u_1 \).
\end{itemize}
In the case that \((\Sigma, g)\) is complete noncompact, \(\phi \in C^\infty(\Sigma)\), there may not be a lowest eigenvalue for \(L_\phi\), however, we can still define the bottom of the spectrum (we still call it \(\lambda_1\)) by
\[
\lambda_1(\Sigma, \phi) = \inf \left\{ \frac{\int_\Sigma (|\nabla u|^2 - qu^2) e^\phi \, dv_\Sigma}{\int_\Sigma u^2 e^\phi \, dv_\Sigma} : u \in C^\infty_0(\Sigma) \setminus \{0\} \right\},
\]
where now the infimum is taken over piecewise smooth functions of compact support and we must allow \(\lambda_1(\Sigma, \phi) = -\infty\). Another way to calculate \(\lambda_1(\Sigma, \phi)\) is
\[
\lambda_1(\Sigma, \phi) = \inf \{ \lambda_1(\Omega, \phi) : \Omega \subset \Sigma \}
\]
where the infimum is taken over all relatively compact domains \(\Omega \subset \Sigma\) with (at least) \(C^1\) boundary.

**Definition 3.1.** Let \((\Sigma, g)\) be a complete manifold, \(\phi \in C^\infty(\Sigma)\) and \(L_\phi\) a gradient Schrödinger operator acting on \(u \in C^\infty_0(\Sigma)\). We say that \(L_\phi\) is **stable** if \(\lambda_1(\Sigma, \phi) \geq 0\).

We continue characterizing stable gradient Schrödinger operators by a variation of an argument of Fischer-Colbrie and Schoen. T. Colding and W. Minicozzi [13, 14] proved a particular case of the following result in the context of stable self-shrinkers. We will prove here the general version.

**Lemma 3.1.** Let \((\Sigma, g)\) be a complete manifold and \(\phi \in C^\infty(\Sigma)\). Then, the following statements are equivalents:

1. \(L_\phi\) is stable.
2. There exists a positive function \(u\) on \(\Sigma\) so that \(L_\phi u = -\lambda_1(\Sigma, \phi)u\), \(\lambda_1(\Sigma, \phi) \geq 0\).

**Proof.** (1) implies (2): We argue as in [13, Lemma 9.25]. Since \(L_\phi\) is stable, Definition 3.1 yields that \(\lambda_1(\Sigma, \phi) \geq 0\).

Fix a point \(p \in \Sigma\) and let \(B(r_k)\) be the geodesic ball in \(\Sigma\) centered at \(p\) of radius \(r_k\). We can consider an increasing sequence \(\{r_k\}\) so that \(\partial B(r_k)\) is at least \(C^1\) and \(r_k \to +\infty\) as \(k \to +\infty\).

It is clear that \(\lambda_1(B(r_k), \phi) > \lambda_1(\Sigma, \phi) \geq 0\) and
\[
\lim_{k \to +\infty} \lambda_1(B(r_k), \phi) = \lambda_1(\Sigma, \phi) \geq 0.
\]

From Theorem 3.1, there exists a positive Dirichlet function \(u_k\) on \(B(r_k)\) so that
\[
L_\phi u_k = -\lambda_1(B(r_k), \phi)u_k,
\]
so that (after multiplying by a constant) \(u_k(p) = 1\). This holds for all \(k\).
Then, the sequence \( \{u_k\} \) is uniformly bounded on compact sets of \( \Sigma \) by the Harnack Inequality [16, Theorem 8.20]. Also, \( \{u_k\} \) has all its derivatives uniformly bounded on compact subsets of \( \Sigma \) by Schauder estimates [16, Theorem 6.2]. Therefore, Arzela-Ascoli’s Theorem and a diagonal argument give us that a subsequence (that we still denote by \( \{u_k\} \)) converges on compact subsets of \( \Sigma \) to a function \( u \in C^\infty(\Sigma) \) which satisfies

\[
\begin{cases}
L_\phi u = -\lambda_1(\Sigma,\phi)u & \text{in } \Sigma \\
u \geq 0 & \text{in } \Sigma \\
u(p) = 1 & \text{at } p \in \Sigma,
\end{cases}
\]

again, by the Harnack Inequality, we obtain that \( u > 0 \) in \( \Sigma \). This proves (1) implies (2).

(2) implies (1): We argue here as in [14, Proposition 3.2]. Let \( u \) be the positive smooth function given by item (2). Then, it satisfies \( L_\phi u \leq 0 \) since \( \lambda_1(\Sigma,\phi) \geq 0 \). Set \( w = \ln u \), then

\[
\Delta w = \frac{\Delta u}{u} - |\nabla w|^2 \leq -g(\nabla \phi, \nabla w) - q - |\nabla w|^2,
\]

which implies

\[
\text{div} \left( e^\phi \nabla w \right) \leq -\left( q + |w|^2 \right) e^\phi.
\]

Given \( \psi \in C^\infty_0(\Sigma) \), applying Stokes’ Theorem to \( \text{div} \left( \psi^2 e^\phi \nabla w \right) \) gives

\[
0 = \int_\Sigma \left( 2\psi e^\phi g(\nabla \psi, \nabla w) + \psi^2 \text{div} \left( e^\phi \nabla w \right) \right) \, dv_{\Sigma}
\]

\[
\leq \int_\Sigma \left( 2\psi g(\nabla \psi, \nabla w) - \psi^2 (q + |\nabla w|^2) \right) e^\phi \, dv_{\Sigma}
\]

\[
\leq \int_\Sigma \left( |\nabla \psi|^2 - \psi^2 q \right) e^\phi \, dv_{\Sigma}
\]

where we have used \( 2\psi g(\nabla \psi, \nabla w) \leq |\nabla \psi|^2 + \psi^2 |\nabla w|^2 \). Therefore, \( \lambda_1(\Sigma,\phi) \geq 0 \).

We can see from Theorem 3.1 that, if \( \Sigma \) is compact and there exists a non identically zero solution of \( L_\phi u = 0, \phi \in C^\infty(\Sigma) \), then \( u \) vanishes nowhere and the linear space of such functions is one dimensional. We would like to extend this property to the noncompact case imposing a global condition of the triple \((\Sigma, g, \phi)\).

**Definition 3.2.** Let \((\Sigma, g)\) be a complete manifold and \( \phi \in C^\infty(\Sigma) \). We say that \((\Sigma, g, \phi)\) is of finite type if there exists a sequence of cut-off functions \( \{\psi_i\}_i \subset C^\infty_0(\Sigma) \) such that

- \( 0 \leq \psi_i \leq 1 \) in \( \Sigma \).
- The compact sets \( \Omega_i := \psi_i^{-1}(1) \) form an increasing exhaustion of \( \Sigma \).
• The sequence of weighted energies \( \left\{ \int_{\Sigma} |\nabla \psi_i|^2 e^\phi \, dv_{\Sigma} \right\}_i \) is bounded.

Moreover, given a complete manifold \((\Sigma, g)\) and smooth functions \(\phi, q \in C^\infty(\Sigma)\), we say that the gradient Schrödinger operator \(L_\phi\) with potential \(q\), given by

\[
L_\phi u := \Delta u + g(\nabla \phi, \nabla u) + qu, \ u \in C_0^\infty(\Sigma),
\]

is of **finite type** if \((\Sigma, g, \phi)\) is of finite type.

We should point out that the above definition does not depend on the potential \(q \in C^\infty(\Sigma)\).

**Remark 3.1.** In the case \(\nabla \phi = 0\), i.e., when \(L\) is simply a Schrödinger operator, Definition 3.2 says that \((\Sigma, g)\) has finite capacity (see [24]). Moreover, we should remark that the required differentiability is much lower of that we consider here.

The following result, which extends [28, Theorem 2.3], tells us that finite type gradient Schrödinger operators behaves as in the compact case.

**Theorem 3.2.** Let \((\Sigma, g)\) be a complete manifold and take \(\phi, q_1, q_2 \in C^\infty(\Sigma)\). Assume that:

1. The gradient Schrödinger operator \(L_{\phi, q_1} := \Delta + g(\nabla \phi, \nabla \cdot) + q_1\) is of finite type and there is a positive subsolution \(u\), i.e., \(u\) satisfies

\[
L_{\phi, q_1} u \leq 0.
\]

2. There exists a bounded function \(v \in C^\infty(\Sigma)\) so that

\[
L_{\phi, q_2} v := \Delta v + g(\nabla \phi, \nabla v) + q_2 v \geq 0.
\]

3. \(q_1 - q_2 \geq 0\) on \(\Sigma\).

Then, \(v/u\) is constant.

**Proof.** Let \(u, v \in C^\infty(\Sigma)\) as in the statement of the theorem. A straightforward computation shows that

\[
\frac{v}{u} \text{div} \left( u^2 e^\phi \nabla (v/u) \right) = v \text{div}(e^\phi \nabla v) - \frac{v^2}{u} \text{div}(e^\phi \nabla u). \tag{3.1}
\]

Now, since \(u\) and \(v\) satisfies the differential inequalities in the statement of the theorem, multiplying by \(e^\phi\), we get

\[
\text{div}(e^\phi \nabla u) \leq -q_1 u e^\phi \quad \text{and} \quad \text{div}(e^\phi \nabla v) \geq -q_2 v e^\phi,
\]

with this information and (3.1), we obtain

\[
\frac{v}{u} \text{div} \left( u^2 e^\phi \nabla (v/u) \right) \geq (q_1 - q_2) v^2 e^\phi \geq 0. \tag{3.2}
\]
Consider the sequence \( \{ \psi_i \} \subset C_0^\infty(\Sigma) \) given by Definition 3.2. Then,

\[
0 = \int_{\Sigma} \text{div} \left( \left( \frac{\psi_i^2 u}{u^2} e^\phi \nabla(v/u) \right) \right) dv_{\Sigma} \\
= \int_{\Sigma} \psi_i^2 \frac{u}{u} \text{div} \left( \frac{u^2 e^\phi \nabla(v/u)}{u} \right) dv_{\Sigma} + \int_{\Sigma} g \left( \nabla \left( \frac{\psi_i^2 u}{u} \right), \frac{u^2 e^\phi \nabla(v/u)}{u} \right) dv_{\Sigma} \\
\geq 2 \int_{\Sigma} \psi_i u e^\phi g \left( \nabla \psi_i, \nabla(v/u) \right) dv_{\Sigma} + \int_{\Sigma} \psi_i^2 u^2 e^\phi |\nabla(v/u)|^2 dv_{\Sigma},
\]

that is, using Hölder Inequality,

\[
\int_{\Sigma} \psi_i^2 u^2 e^\phi |\nabla(v/u)|^2 dv_{\Sigma} \leq -2 \int_{\Sigma} \psi_i u e^\phi g \left( \nabla \psi_i, \nabla(v/u) \right) dv_{\Sigma} \\
= -2 \int_{\text{supp}(\psi_i) \setminus \Omega_i} \psi_i u e^\phi g \left( \nabla \psi_i, \nabla(v/u) \right) dv_{\Sigma} \\
\leq 2 \left( \int_{\text{supp}(\psi_i) \setminus \Omega_i} \psi_i^2 u^2 e^\phi |\nabla(v/u)|^2 dv_{\Sigma} \right)^{\frac{1}{2}} \left( \int_{\text{supp}(\psi_i) \setminus \Omega_i} v^2 e^\phi |\nabla \psi_i|^2 dv_{\Sigma} \right)^{\frac{1}{2}} \\
\leq 2 \left( \int_{\Sigma} \psi_i^2 u^2 e^\phi |\nabla(v/u)|^2 dv_{\Sigma} \right)^{\frac{1}{2}} \left( \int_{\Omega_i} v^2 e^\phi |\nabla \psi_i|^2 dv_{\Sigma} \right)^{\frac{1}{2}}.
\]

Now, since \( L_{\phi,q} \) is of finite type and \( v \) is bounded, there exists a constant \( C \) so that

\[
\int_{\Sigma} v^2 e^\phi |\nabla \psi_i|^2 dv_{\Sigma} \leq C \text{ for all } i,
\]

which implies from the previous inequality that

\[
\int_{\Sigma} \psi_i^2 u^2 e^\phi |\nabla(v/u)|^2 dv_{\Sigma} \leq 4C \text{ for all } i,
\]

therefore, we obtain

\[
\int_{\Sigma} u^2 e^\phi |\nabla(v/u)|^2 dv_{\Sigma} \leq 4C \tag{3.3}
\]

and

\[
\lim_{i \to +\infty} \int_{\text{supp}(\psi_i) \setminus \Omega_i} \psi_i^2 u^2 e^\phi |\nabla(v/u)|^2 dv_{\Sigma} = 0. \tag{3.4}
\]

Thus, (3.3) and (3.4) imply that

\[
\int_{\Omega_i} u^2 e^\phi |\nabla(v/u)|^2 dv_{\Sigma} \leq 2\sqrt{C} \left( \int_{\text{supp}(\psi_i) \setminus \Omega_i} \psi_i^2 u^2 e^\phi |\nabla(v/u)|^2 dv_{\Sigma} \right)^{\frac{1}{2}} \to 0
\]

as \( i \) goes to \( +\infty \), thus \( v/u \) is constant on \( \Sigma \).
As a consequence of the above technique, we can extend the Naber-Yau Liouville Theorem (see [36]).

**Theorem 3.3.** Let \((\Sigma, g)\) be a complete manifold and take \(\phi, q \in C^\infty(\Sigma)\). Assume the gradient Schrödinger operator \(L_{\phi,q} := \Delta + g(\nabla \phi, \nabla \cdot) + q\) is of finite type.

1. Assume \(q \geq 0\). If there is a nonnegative (nonidentically zero) subsolution \(u\), i.e., \(u\) satisfies \(L_{\phi,q}u \leq 0\), then \(u\) is constant and either \(u \equiv 0\) or \(q \equiv 0\) on \(\Sigma\). In particular, any solution bounded above or below to \(L_{f,q}u = 0\) must be constant.

2. Assume \(q \leq 0\) and \(L_{\phi,q}\) is stable. If there is a bounded supersolution \(u\), i.e., \(u\) satisfies \(L_{\phi,q}u \geq 0\), then \(u\) is constant and either \(u \equiv 0\) or \(q \equiv 0\) on \(\Sigma\).

3. Assume \(q \leq 0\) and there is a nonnegative supersolution \(u\), i.e., \(u\) satisfies \(L_{\phi,q}u \geq 0\), such that the sequence \(\left\{ \int_{\Sigma} |\nabla \psi_i|^2 u^2 e^\phi dv_{\Sigma} \right\}_i\) is uniformly bounded. Then \(u\) is constant and either \(u \equiv 0\) or \(q \equiv 0\) on \(\Sigma\). Moreover, if \(q \equiv 0\), we only need to assume that \(u\) is bounded below.

**Proof.** Let us prove each case:

1. Assume \(q \geq 0\). A straightforward computation, as above, shows that

\[
0 = \int_{\Sigma} \text{div} \left( \psi_i^2 e^{\phi-u} \nabla u \right) dv_{\Sigma} \\
= \int_{\Sigma} \psi_i^2 e^{-u} \text{div} \left( e^\phi \nabla u \right) dv_{\Sigma} + \int_{\Sigma} g \left( \nabla (\psi_i^2 e^{-u}), e^\phi \nabla u \right) dv_{\Sigma} \\
\leq 2 \int_{\Sigma} \psi_i e^{\phi-u} g \left( \nabla \psi_i, \nabla u \right) dv_{\Sigma} - \int_{\Sigma} \psi_i^2 e^{\phi-u} |\nabla u|^2 dv_{\Sigma} - \int_{\Sigma} \psi_i^2 e^{\phi-u} q uv dv_{\Sigma},
\]

therefore, using Hölder Inequality and \(u \geq 0\), we obtain

\[
\int_{\Sigma} \psi_i^2 e^{\phi-u} |\nabla u|^2 dv_{\Sigma} \leq \int_{\Sigma} \psi_i^2 e^{\phi-u} |\nabla u|^2 dv_{\Sigma} + \int_{\Sigma} \psi_i^2 e^{\phi-u} uq dv_{\Sigma} \\
\leq 2 \left( \int_{\text{supp}(\psi_i) \setminus \Omega_i} \psi_i^2 e^{\phi-u} |\nabla u|^2 dv_{\Sigma} \right)^{1/2} \left( \int_{\text{supp}(\psi_i) \setminus \Omega_i} e^{\phi-u} |\nabla \psi_i|^2 dv_{\Sigma} \right)^{1/2} \\
\leq 2 \left( \int_{\text{supp}(\psi_i) \setminus \Omega_i} \psi_i^2 e^{\phi-u} |\nabla u|^2 dv_{\Sigma} \right)^{1/2} \left( \int_{\text{supp}(\psi_i) \setminus \Omega_i} e^{\phi} |\nabla \psi_i|^2 dv_{\Sigma} \right)^{1/2}.
\]

Thus, arguing as Theorem 3.2, we obtain that \(u\) must be constant and either \(u \equiv 0\) or \(q \equiv 0\).
2. Assume \( q \leq 0 \) and \( u \) is bounded. Then, Theorem 3.2 implies that \( u > 0 \). Therefore, calculating as above:

\[
0 = \int_{\Sigma} \operatorname{div} \left( \psi_i^2 u e^\phi \nabla u \right) d\nu_{\Sigma} \\
= \int_{\Sigma} \psi_i^2 u \operatorname{div} \left( e^\phi \nabla u \right) d\nu_{\Sigma} + \int_{\Sigma} e^\phi g \left( \nabla (\psi_i^2 u), \nabla u \right) d\nu_{\Sigma} \\
\geq 2 \int_{\Sigma} \psi_i u e^\phi g \left( \nabla \psi_i, \nabla u \right) d\nu_{\Sigma} + \int_{\Sigma} \psi_i^2 e^\phi |\nabla u|^2 d\nu_{\Sigma} - \int_{\Sigma} \psi_i^2 e^\phi u^2 q d\nu_{\Sigma},
\]

that is,

\[
\int_{\Sigma} \psi_i^2 e^\phi |\nabla u|^2 d\nu_{\Sigma} - \int_{\Sigma} \psi_i^2 e^\phi u^2 q d\nu_{\Sigma} \leq -2 \int_{\Sigma} \psi_i u e^\phi g \left( \nabla \psi_i, \nabla u \right) d\nu_{\Sigma},
\]

and we can finish as in the previous case.

3. We argue as in item 2. That is, we have

\[
\int_{\Sigma} \psi_i^2 e^\phi |\nabla u|^2 d\nu_{\Sigma} - \int_{\Sigma} \psi_i^2 e^\phi u^2 q d\nu_{\Sigma} \leq -2 \int_{\Sigma} \psi_i u e^\phi g \left( \nabla \psi_i, \nabla u \right) d\nu_{\Sigma},
\]

which implies

\[
\int_{\Sigma} \psi_i^2 e^\phi |\nabla u|^2 d\nu_{\Sigma} \leq 2 \left( \int_{\supp(\psi_i) \setminus \Omega_i} \psi_i^2 e^\phi |\nabla u|^2 d\nu_{\Sigma} \right)^{1/2} \left( \int_{\supp(\psi_i) \setminus \Omega_i} u^2 e^\phi |\nabla \psi_i|^2 d\nu_{\Sigma} \right)^{1/2},
\]

therefore, we can finish as in Theorem 3.2.

Remark 3.2. Theorem 3.2 and Theorem 3.3 do not follow from [28] by doing a conformal change of metric.

4 Applications to Mean Curvature Flow

We will apply Theorem 3.2 to recover and extend a recent result of Colding-Minicozzi [13, 14] on the nonexistence of certain self-shrinkers. Set \( c = \pm 1 \), as we pointed out, a self-similar solution \( \Sigma \) to the mean curvature flow satisfies \( H = -\frac{2}{c} \langle x, N \rangle \), when \( c = -1 \), \( \Sigma \) is a self-shrinker and when \( c = +1 \), \( \Sigma \) is a self-expander. A self-shrinker (resp. expander) \( \Sigma \subset \mathbb{R}^{n+1} \) can be considered as a weighted minimal hypersurface in the manifold with density \((\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle, \phi_{-1})\) (resp. \((\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle, \phi_{+1})\)), where \( \langle \cdot, \cdot \rangle \) denotes the standard Euclidean metric and \( \phi_{-1} := -\frac{|x|^2}{4} \) (resp. \( \phi_{+1} := \frac{|x|^2}{4} \)). So,
**Definition 4.1.** We say that a self-shrinker (resp. expander) \( \Sigma \) is \( L \)-stable if it is stable as a weighted minimal hypersurface in \((\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle, \phi^{-1})\) (resp. \((\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle, \phi^{+1})\)).

We also need the following definition:

**Definition 4.2.** Let \( \Sigma \subset \mathbb{R}^{n+1} \) be a self-shrinker (resp. expander). We say that \( \Sigma \) is of finite type if \((\Sigma, g, \phi^{-1})\) (resp. \((\Sigma, g, \phi^{+1})\)) is of finite type. Here, \( g \) denotes the induced metric on \( \Sigma \).

So, we can prove:

**Theorem 4.1.** There are no complete \( L \)-stable self-shrinkers in \( \mathbb{R}^{n+1} \) of finite type.

**Proof.** Let \( \Sigma \subset (\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle, \phi^{-1}) \) be a self-shrinker solution, i.e., it satisfies \( H = \frac{1}{2} \langle x, N \rangle \). Moreover, we are assuming \( \Sigma \) is \( L \)-stable, that is, it is stable as a weighted minimal hypersurface, therefore the gradient Schrödinger operator (see Lemma 1.1) given by

\[
L := \Delta + g(\nabla \phi^{-1}, \nabla \cdot) + (|A|^2 + \frac{1}{2}),
\]

is stable in the sense of Definition 3.1. From Lemma 3.1, there exists a positive subsolution, that is, there exists \( u > 0 \) so that

\[
Lu \leq 0.
\]

Moreover, \( q := |A|^2 + \frac{1}{2} \geq \frac{1}{2} > 0 \). Thus, Theorem 3.3 says that \( u \) is constant and \( q \equiv 0 \), which is a contradiction.

T. Colding and W. Minicozzi [13, 14] proved that there are no \( L \)-stable self-shrinker \( \Sigma \) with polynomial area growth. Polynomial area growth implies that

\[
\int_{B(i+1) \setminus B(i)} \phi^{-1} \, dv \to 0,
\]

where \( B(r) \) denotes the geodesic ball of radius \( r \) centered at a fixed point. In particular, \( \Sigma \) is of finite type.

As far as we know, there are no general classification results for \( L \)-stable self-expanders. We can, at least, give some condition for the nonexistence:

**Proposition 4.1.** There are no complete \( L \)-stable self-expanders in \( \mathbb{R}^{n+1} \) of finite type and \( |A|^2 \geq 1/2 \).

In fact, there is another notion of stability more related to graphs. Let \( \Sigma \subset \mathbb{R}^{n+1} \) be a self-shrinker (expander). Let \( a \in \mathbb{R}^{n+1} \) be a constant vector and consider \( u_a := \langle N, a \rangle \), then \( u_a \) satisfies (see [13, 14]):

\[
L_0 u_a := \Delta u_a + g(\nabla \phi, \nabla u_a) + |A|^2 u_a = 0.
\]

If \( \Sigma \) is a complete multi-graph, say with respect to the \( e_{n+1} \) direction, then \( \nu := \langle N, e_{n+1} \rangle \) is positive and satisfies (4.1), therefore Lemma 3.1 yields that \( L_0 \) is stable in the sense of Definition 3.1. This motivates the following:
**Definition 4.3.** We say that a self-similar solution to the mean curvature flow $\Sigma$ is $L_0$–stable if the gradient Schrödinger operator

$$L_0 := \Delta + g(\nabla \phi, \nabla \cdot) + |A|^2,$$

is stable in the sense of Definition 3.1.

Therefore, the above discussion leads us to:

**Theorem 4.2.** The only complete self-similar solutions to the mean curvature flow that are $L_0$–stable and of finite type are hyperplanes.

In particular, the only self-similar solutions (shrinker or expander) to the mean curvature flow that are complete multi-graphs of finite type are hyperplanes. Moreover, the only self-shrinkers that are entire graphs are the hyperplanes.

**Remark 4.1.** It would be interesting to know if a self-shrinker that is a complete multi-graph must be an entire graph.

## 5 Applications to gradient Ricci solitons

As we said at the Introduction, gradient Ricci solitons (steady and shrinking) appear as critical points of the functional $\mathcal{F}$ (or $\mathcal{W}$–entropy) introduced by G. Perelman [38]. Gradient Ricci solitons are self-similar solutions of the Ricci flow and they arise as limits of dilatations of singularities of the Ricci flow. Analytically, we can define a Ricci soliton as a manifold with density $(N, g, \phi)$ so that $\text{Ric}^\phi = \lambda g$, $\lambda \in \mathbb{R}$, therefore we identify Ricci soliton with $(\Sigma, g, \phi, \lambda)$.

So, we say that the gradient Ricci soliton $(\Sigma, g, \phi, \lambda)$ is shrinking, steady or expanding if $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$ respectively.

The condition $\text{Ric}^\phi = \lambda g$ says

$$\text{Ric} - \nabla^2 \phi = \lambda g,$$

therefore, gradient Ricci solitons can be thought as a natural generalization of Einstein metrics.

One important consequence of F. Morgan result [32] is that gradient shrinking solitons are of finite type, which is the key condition to apply Theorem 3.3.

**Lemma 5.1.** Let $(\Sigma, g, \phi)$ be a complete manifold with density so that $\text{Ric}^\phi \geq \lambda$, $\lambda > 0$. Then

$$\int_N e^\phi \, dv < +\infty.$$  

In particular, $(\Sigma, g, \phi)$ is of finite type.

Now, we are ready to establish the main theorem in this section. Here, we extend a recent result of P. Petersen and W. Wylie [36] relaxing the hypothesis to a $L^2$–type bound of the scalar curvature $R$. Moreover, they assume that the Weyl tensor vanishes identically, we change that condition by an inequality between the scalar curvature and the Ricci curvature.

25
Theorem 5.1. Let $(\Sigma, g, \phi, \lambda)$ be a complete shrinking soliton of dimension $n \geq 3$ so that $\lambda R \leq |\text{Ric}|^2$. Assume that there exists a sequence of cut-off functions $\{\psi_i\}_i \subset C^\infty_0(\Sigma)$ such that $0 \leq \psi_i \leq 1$ in $\Sigma$, the compact sets $\Omega_i := \psi_i^{-1}(1)$ form an increasing exhaustion of $\Sigma$, and the sequence of weighted energies $\{\int_\Sigma |\nabla \psi_i|^2 R^2 e^\phi \, dv_\Sigma\}_i$ is bounded.

Then, $(\Sigma, g)$ has constant scalar curvature and $\lambda R = |\text{Ric}|^2$.

Proof. The proof follows from the following equation developed in [36]:

$$\Delta R + g(\nabla \phi, \nabla R) - 2\lambda R = -2|\text{Ric}|^2. \tag{5.1}$$

First, since the scalar curvature satisfies the boundedness condition, $R > 0$ or $(\Sigma, g)$ is flat (see [37]). So, we can continue assuming $R > 0$. Second, since $\lambda R - |\text{Ric}|^2 \geq 0$, (5.1) implies that

$$\Delta R + g(\nabla \phi, \nabla R) = 2\lambda R - 2|\text{Ric}|^2 \leq 0.$$

Thus, Theorem 3.3 implies that $R$ is constant, and so, $\lambda R = |\text{Ric}|^2$.

We also have the following classification result:

Theorem 5.2. • A complete gradient expander soliton of finite type and nonnegative scalar curvature is Ricci flat.

• A complete gradient steady soliton of finite type whose scalar curvature does not change sign is Ricci flat. Moreover, if $\phi$ is not constant then it is a product of a Ricci flat manifold with $\mathbb{R}$. Also, $\Sigma$ is diffeomorphic to $\mathbb{R}^n$.

• A complete gradient shrinking soliton of nonpositive scalar curvature is Ricci flat. Moreover, $(\Sigma, g)$ is isometric to $\mathbb{R}^n$ endowed with the standard Euclidean metric.

Proof. The proof follows from (5.1).

• Applying Theorem 3.3 to $u = R$ we obtain that $R$ vanishes identically on $\Sigma$, which implies that $(\Sigma, g)$ is Ricci flat from (5.1).

• As above, $R$ satisfies (5.1). So, taking either $u = -R$ if $R \leq 0$ or $u = R$ if $R \geq 0$, Theorem 3.3 implies that $R \equiv 0$. Therefore, $(\Sigma, g)$ is Ricci flat. The last sentence follows from [37, Proposition 4]. Moreover, $\Sigma$ is diffeomorphic to $\mathbb{R}^n$ by [6, Proposition 5.7].

• Taking $u = -R$, Theorem 3.3 implies that $R \equiv 0$. Therefore, $(\Sigma, g)$ is Ricci flat. Thus, [33, Lemma 7.1] implies that $(\Sigma, g)$ is isometric to $\mathbb{R}^n$ endowed with the standard Euclidean metric.
6 Weighted stable surfaces and Schrödinger operators

As we can see from Lemma 1.1, a weighted $H_{\phi}$–stable surface has a gradient Schrödinger operator, $L_{\phi}$ given by (2.1), associated to it. Nevertheless, we can associate a self-adjoint Schrödinger operator to a weighted $H_{\phi}$–stable surface which is stable in the sense of Definition 6.1. In this section we will work with surfaces.

When a Schrödinger operator $L := \Delta + q$ is self-adjoint, we can associate a bilinear quadratic form to $L$ given by

$$Q : C^\infty_0(\Sigma) \times C^\infty_0(\Sigma) \rightarrow \mathbb{R}$$

$$(u,v) \rightarrow \int_{\Sigma}(g(\nabla u, \nabla v) - q uv)dv_{\Sigma},$$

and so

$$(-Lu,v)_{L^2} = Q(u,v) = (-Lv,u)_{L^2},$$

therefore $L$ is self-adjoint with respect to the $L^2$–metric.

**Definition 6.1.** We say that the Schrödinger operator $L$ is **stable** if $Q(u,u) \geq 0$ for all $u \in C^\infty_0(\Sigma)$.

Now, we will see how to associate a self-adjoint Schrödinger operator to a weighted $H_{\phi}$–stable surface which is stable in the sense of Definition 6.1.

**Lemma 6.1.** Let $\Sigma \subset (\mathbb{N}^3, g, \phi)$ be a complete weighted $H_{\phi}$–stable surface. Then,

$$\int_{\Sigma}(V - aK)f^2 dv_{\Sigma} \leq \int_{\Sigma} |\nabla f|^2 dv_{\Sigma}$$

for any $f \in C^\infty_0(\Sigma)$ compactly supported piecewise smooth function, where

$$V := \frac{1}{3}\left(\frac{1}{2}R_{\phi}^\infty + \frac{1}{2}H_{\phi}^2 + \frac{1}{2}|A|^2 + \frac{1}{8}|\nabla \phi|^2\right) \quad \text{and} \quad a := \frac{1}{3}. \quad (6.2)$$

In other works, the Schrödinger operator

$$L := \Delta - aK + V$$

is stable in the sense of Definition 6.1.

**Proof.** A tedious but straightforward computation shows that

$$\text{Ric}^\infty_{\phi}(N, N) + |A|^2 = \frac{1}{2}R_{\phi}^\infty - K + \text{div}(X) + F,$$\quad (6.4)
where $K$ is the Gaussian curvature of $\Sigma$, $\text{div}$ is the divergence operator on $\Sigma$, and

\[
X = \nabla \phi,
\quad
F = \frac{1}{2} \left( H_{\phi}^2 + |A|^2 + |
\nabla \phi|^2 \right).
\]

We can also easily check that

\[
\frac{1}{2} R_{\phi}^\infty - K + \text{div}(X) + F = 3V - K + \text{div}(X) + \frac{3}{8} |
\nabla \phi|^2.
\]

Let $f \in C_0^\infty(\Sigma)$ and set $u := \phi^{-1/2} f \in C_0^\infty(\Sigma)$. Plugging $u = f \phi^{-1/2}$ in (1.3) and using the above equalities, we have

\[
\int_{\Sigma} \left( 3V - K + \text{div}(X) + \frac{3}{8} |
\nabla \phi|^2 \right) f^2 dv_{\Sigma} \leq \int_{\Sigma} \left( |\nabla f|^2 - f g(\nabla f, \nabla \phi) + \frac{f^2}{4} |\nabla \phi|^2 \right) dv_{\Sigma}.
\]

Now, the Divergence Theorem yields

\[
\int_{\Sigma} f^2 \text{div}(X) dv_{\Sigma} = -2 \int_{\Sigma} f g(\nabla f, \nabla \phi) dv_{\Sigma},
\]

and using the inequality

\[
f g(\nabla f, \nabla \phi) \leq 2|\nabla f|^2 + \frac{1}{8} |\nabla \phi|^2 f^2
\]

we obtain

\[
\int_{\Sigma} \left( 3V - K + \frac{3}{8} |\nabla \phi|^2 \right) f^2 dv_{\Sigma} \leq \int_{\Sigma} \left( 3|\nabla f|^2 + \frac{3}{8} |\nabla \phi|^2 f^2 \right) dv_{\Sigma},
\]

or equivalently,

\[
\int_{\Sigma} (V - aK) f^2 dv_{\Sigma} \leq \int_{\Sigma} |\nabla f|^2 dv_{\Sigma},
\]

as desired. \qed

Recently, the theory of stable Schrödinger operators has been extensively developed since the pioneer work of T. Colding and W. Minicozzi [12]. Next, we recall a Colding-Minicozzi type inequality (see [7, 21, 29]) we can derive from Lemma 6.1 for a particular choice of radial cut-off function.

As above, we denote by $\Sigma$ a connected Riemannian surface, with Riemannian metric $g$, and possibly with boundary $\partial \Sigma$. Let $p_0 \in \Sigma$ be a point of the surface and $D(p_0, s)$, for $s > 0$, denote the geodesic disk centered at $p_0$ of radius $s$. We assume that $\overline{D(p_0, s)} \cap \partial \Sigma = \emptyset$. Moreover, let $r$ be the radial distance of a point $p$ in $D(p_0, s)$ to $p_0$. We write $D(s) = D(p_0, s)$ if no confusion occurs.
We also denote

\[ l(s) = \text{Length}(\partial D(s)), \]

\[ A(s) = \text{Area}(D(s)), \]

\[ K(s) = \int_{D(s)} K dv, \]

\[ \chi(s) = \text{Euler characteristic of } D(s), \]

where length and area are measured with respect to the metric \( g \).

**Lemma 6.2** (Colding-Minicozzi stability inequality). Let \( \Sigma \) be a Riemannian surface possibly with boundary \( \partial \Sigma \). Let us fix a point \( p_0 \in \Sigma \) and a positive number \( 0 < s \) such that \( D(s) \cap \partial \Sigma = \emptyset \). Assume the Schrödinger operator

\[ L := \Delta - aK + V \]

is stable in the sense of Definition 6.1, where \( V \in C^\infty(\Sigma) \) and \( a \) is a positive constant, acting on \( f \in C_0^\infty(\Sigma) \).

Fix \( b \geq 1 \) and let \( f : D(s) \to \mathbb{R} \) be the nonnegative radial function given by

\[
    f(p) := \begin{cases} 
        \left(1 - \frac{r(p)}{s}\right)^b & \text{for } p \in D(s) \\
        0 & \text{if } p \in \Sigma \setminus D(s) 
    \end{cases} \tag{6.5}
\]

Then, the following holds:

\[
    \int_{D(s)} V f(p)^2 dv \leq 2a \pi G(s) + \frac{b(b(1 - 4a) + 2a)}{2b^2 - 2s^2} A(s/2), \tag{6.6}
\]

where

\[ G(s) := -\int_0^s (f(r)^2)' \chi(r) dr \leq 1. \]

**Proof.** Since \( L \) is stable, plugging \( f \) given by (6.5) in Lemma 6.1, we get

\[
    \int_{D(s)} V f^2 dv \leq \int_{D(s)} \left( |\nabla f|^2 + aK f^2 \right) dv. 
\]

So, from Co-Area Formula and Fubbini’s Theorem, we obtain

\[
    \int_{D(s)} V f^2 dv \leq \int_{D(s)} \left( |\nabla f|^2 + aK f^2 \right) dv \\
    = \int_0^s \left( f'(r)^2 \int_{\partial D(r)} 1 dl + a f(r)^2 \int_{\partial D(r)} K(r) dl \right) dr \\
    = \int_0^s \left( f'(r)^2 l(r) + a f(r)^2 K'(r) \right) dr.
\]
Now, from \([42]\) and since \((f(r)^2)' = 2f(r)f'(r) \leq 0\), we have
\[-(f(r)^2)'K(r) \leq (f(r)^2)'(l'(r) - 2\pi\chi(r)).\]

Integrating by parts and taking into account that \(\int_0^s (f(r)^2)' = -1\), we obtain
\[
\int_{D(s)} Vf^2dv \leq -2\pi a \int_0^s (f(r)^2)'\chi(r)dr + \int_0^s (a(f(r)^2)'l'(r)) dr
\]
\[
= 2\pi G(s) + \int_0^s ((1 - 2a)f'(r)^2 - 2af(r)f''(r)) l(r)dr
\]
\[
= 2\pi G(s) + \frac{b(b(1 - 4a) + 2a)}{s^2} \int_0^s \left(1 - \frac{r}{s}\right)^{2b - 2} l(r)dr
\]
\[
\leq 2\pi G(s) + \frac{b(b(1 - 4a) + 2a)}{s^2} A(s/2)
\]

Note that the bound on \(G(s)\) follows since the Euler characteristic of \(D(s)\) is less than or equal to 1.

Before we finish this section, we will see an useful Lemma:

**Lemma 6.3.** Under the conditions of Lemma 6.2, if \(\Sigma\) is complete and there exists \(s_0\) such that \(\chi(s) \leq -M\), \(M \geq 0\), for all \(s \geq s_0\), then
\[G(s) \leq -(M + 1)f(s_0)^2 + 1.\]

**Proof.** Assume there exists \(s_0\) so that for all \(s \geq s_0\), we have \(\chi(s) \leq -M\). Therefore, following Lemma 6.2, we have
\[
G(s) = -\int_0^s (f(r)^2)'\chi(r) = -\int_0^{s_0} (f(r)^2)'\chi(r) - \int_{s_0}^s (f(r)^2)'\chi(r)
\]
\[
\leq -\int_0^{s_0} (f(r)^2)' + M \int_{s_0}^s (f(r)^2)' = -(f(s_0)^2 - f(0)^2) + M (f(s)^2 - f(s_0)^2)
\]
\[
= -(M + 1)f(s_0)^2 + 1,
\]
since \(-(f(r)^2) \geq 0\) and \(\chi(r) \leq 1\) for all \(r\).  

**7 Manifolds with nonnegative Perelman Scalar Curvature**

In this Section, we will study complete weighted \(H_\phi\)-stable surfaces in three-manifolds with density \((\mathcal{N}, g, \phi)\) and nonnegative \(P\)-scalar curvature \(R_\phi^\infty \geq 0\).

The first thing we shall observe is when we can have a complete noncompact weighted \(H_\phi\)-stable surface under conditions on the \(P\)-scalar curvature. The following result is the extension of the diameter estimate for stable \(H\)-surfaces given by H. Rosenberg [40]:

30
Theorem 7.1. Let $\Sigma \subset (N, g, \phi)$ be a weighted $H_\phi$-stable surface with boundary $\partial \Sigma$. If $R_\phi^\infty + H_\phi^2 \geq c > 0$ on $\Sigma$, then

$$d_\Sigma (p, \partial \Sigma) \leq \frac{2\pi}{\sqrt{3c}}$$

for all $p \in \Sigma$,

where $d_\Sigma$ denotes the intrinsic distance in $\Sigma$. Moreover, if $\Sigma$ is complete without boundary, then it must be topologically a sphere.

Proof. The condition $R_\phi^\infty + H_\phi^2 \geq c > 0$ implies that $V \geq c > 0$ for some positive constant $c$. Therefore, [29, Theorem 2.8] with $a = 1/3$ implies the result. We will sketch the proof for the reader convinience.

Since $L$ is stable, there exists a positive function $u$ such that $Lu = 0$ [22]. Consider the conformal metric $\tilde{g} := u^6 g$, where $g$ is the metric on $\Sigma$. Denote by $\tilde{K}$ and $K$ the Gaussian curvature of $\tilde{g}$ and $g$ respectively. So, since $Lu = 0$, we get

$$\tilde{K} = 3u^{-6} \left( V + \frac{\nabla u^2}{u^2} \right) \geq 3u^{-6} \left( c + \frac{|\nabla u|^2}{u^2} \right). \quad (7.1)$$

Take $p \in \Sigma$ and let $\gamma$ be a $\tilde{g}$-geodesic ray emanating from $p$. Denote by $\tilde{l}$ and $l$ the length of $\gamma$ with respect to $\tilde{g}$ and $g$ respectively. Since $\gamma$ is a $\tilde{g}$-minimizing geodesic, the Second Variation Formula of the arc-length gives that

$$\int_0^{\tilde{l}} \left( \left( \frac{d\phi}{d\tilde{s}} \right)^2 - \tilde{K} \phi^2 \right) d\tilde{s} \geq 0, \quad (7.2)$$

for any smooth function $\phi : [0, \tilde{l}] \to \mathbb{R}$ such that $\phi(0) = \phi(\tilde{l})$. From (7.1), (7.2), $|\nabla u| \geq (u \circ \gamma)'(s) = u'(s)$ and changing variables $d\tilde{s} = u^3 ds$, we get

$$\int_0^l u(s)^{-3} \left( \phi'(s)^2 - 3 \left( c + \frac{u'(s)^2}{u(s)^2} \right) \phi(s)^2 \right) ds \geq 0. \quad (7.3)$$

Take $\phi = u^{3/2} \psi$, where $\psi : [0, l] \to \mathbb{R}$ is a smooth function such that $\psi(0) = \psi(l) = 0$. Then, the above (7.3) yields

$$\int_0^l (-4\psi''(s)\psi(s) - 3c\psi(s)^2) ds \geq 0, \quad (7.4)$$

for all $\psi \in C_0^\infty ([0, l])$. Taking $\psi(s) = \sin \left( \frac{\pi}{l} s \right)$ in (7.4), we get

$$\int_0^l \left( \frac{\pi^2}{l^2} - 3c \right) \sin^2 \left( \frac{\pi}{l} s \right) ds \geq 0,$$
which implies
\[ l \leq \pi \sqrt{\frac{4}{3c}}, \] 
which gives the desired estimate.

Now, if \( \Sigma \) is complete, then (7.5) and the Hopf-Rinow Theorem imply that \( \Sigma \) must be compact. Moreover, applying the operator \( L \) to the test function 1, we have
\[ \frac{1}{3} \int_{\Sigma} K \, dv_{\Sigma} \geq c \text{Area}(\Sigma), \]
which implies, by the Gauss-Bonnet Theorem, that \( \chi(\Sigma) > 0 \).

The above result says that, in a manifold with density \((\mathcal{N}, g, \phi)\) and \( R^\infty_\phi \geq 0 \), the only complete noncompact weighted \( H_\phi \)-stable surface we shall consider are the \( \phi \)-minimal ones. So, the next step is to extend the well-know result of D. Fischer-Colbrie and R. Schoen [23] on the topology and conformal type of complete noncompact stable minimal surfaces. P. T. Ho [17] extended Fischer-Colbrie-Schoen result under the additional hypothesis that \( |\nabla \phi| \) is bounded on \( \Sigma \).

Here, we drop that condition and characterize the weighted stable cylinder as totally geodesic, flat and the density must vanish along \( \Sigma \):

**Theorem 7.2.** Let \( \Sigma \subset (\mathcal{N}^3, g, \phi) \) be a complete (noncompact) weighted stable minimal surface where \( R^\infty_\phi \geq 0 \). Then, \( \Sigma \) is conformally equivalent either to the complex plane \( \mathbb{C} \) or to the cylinder \( \mathbb{S}^2 \times \mathbb{R} \). In the latter case, \( \Sigma \) is totally geodesic, flat and \( \phi \) is constant and \( R^\infty_\phi \equiv 0 \) along \( \Sigma \).

**Proof.** From Lemma 6.1, \( L := \Delta - aK + V \) is stable in the sense of operators with \( 1/4 < a = 1/3 \) and \( V \geq 0 \). From [7] or [29], we can see that \( \Sigma \) is conformally equivalent either to the plane or to the cylinder.

In the latter case, either [19, Lemma 2.1] or [3, Theorem 1.3] give us that \( K \equiv 0 \) and \( V \equiv 0 \), which implies that \( |A|^2 \equiv 0 \) and \( \phi \) is constant and \( R^\infty_\phi \equiv 0 \) along \( \Sigma \).

We will sketch the proof for the reader convenience. Let us consider the radial function
\[
 f(r) := \begin{cases} 
 (1 - r/s)^5 & r \leq s \\
 0 & r > s 
 \end{cases},
\]
where \( r \) denotes the radial distance from a point \( p_0 \in \Sigma \). Then, from Lemma 6.2, we have
\[
 \int_{D(s)} (1 - r/s)^{10} V \, dv_{\Sigma} \leq \frac{2}{3} \pi G(s) - \frac{1}{s^2} A(s/2). \tag{7.6}
\]
First, since $\Sigma$ is complete, from (7.6) and Lemma 6.3, we get that $\chi(\Sigma)$ equals either to 0 or 1, that is, $\Sigma$ is topologically either the plane or the cylinder.

Second, again from (7.6) and $V \geq 0$, we obtain

$$\frac{A(s)}{s^2} \leq C,$$

for some constant $C$, which yields that $\Sigma$ has quadratic area growth. Thus, since $\Sigma$ has finite topology and quadratic area growth, each end of $\Sigma$ is parabolic (see [12]). Therefore $\Sigma$ is conformally equivalent either to the plane or to the cylinder.

Now, suppose that $\Sigma$ is conformally equivalent to the cylinder. We will show that $\Sigma$ is flat, totally geodesic and $\phi$ is constant and $R^\infty_\phi \equiv 0$ along $\Sigma$.

• **Claim A:** $V$ vanishes identically on $\Sigma$. That is, $\Sigma$ is totally geodesic and $\phi$ is constant and $R^\infty_\phi \equiv 0$ along $\Sigma$.

Suppose there exists a point $p_0 \in \Sigma$ so that $V(p_0) > 0$. From now on, we fix the point $p_0$. Then, there exists $\epsilon > 0$ so that $V(q) \geq \delta$ for all $q \in D(\epsilon) = D(p_0, \epsilon)$. Since $\Sigma$ is topologically a cylinder, there exists $s_0 > 0$ so that for all $s > s_0$ we have $\chi(s) \leq 0$ (see [7, Lemma 1.4]).

Now, from the above considerations and (7.6), there exists $\beta > 0$ so that

$$0 < \beta \leq 2\pi G(s).$$

But, from Lemma 6.3, we can see that

$$G(s) = -f(s_0)^2 + 1 = -(1 - s_0/s)^2 + 1,$$

therefore,

$$G(s) \leq 1 - (1 - s_0/s)^2 \to 0, \text{ as } s \to +\infty,$$

which is a contradiction. Thus, $V$ vanishes identically along $\Sigma$.

• **Claim B:** $\Sigma$ is flat.

Since $\Sigma$ is a cylinder and (7.6), we have

$$\lim_{s \to +\infty} \frac{A(s)}{s^2} = 0,$$

therefore, $K \geq 0$ on $\Sigma$ (see [3, Proposition 5.1]). Thus, Cohn-Vossen Theorem yields that $K$ vanishes identically on $\Sigma$. 

$\Box$
7.1 Stable hypersurfaces

We finish this section classifying stable hypersurfaces in manifolds with density that verify a lower bound on its Bakry-Émery-Ricci tensor. Specifically:

**Theorem 7.3.** Let $\Sigma \subset (\mathcal{N}, g, \phi)$ be a complete $H_{\phi}$--stable hypersurface of finite type and assume that $\text{Ric}_{\phi}^\infty \geq k$, $k \geq 0$. Then, $\Sigma$ is totally geodesic and $\text{Ric}_{\phi}^\infty (N, N) \equiv 0$ along $\Sigma$.

**Proof.** Since $\Sigma \subset (\mathcal{N}, g, \phi)$ is $H_{\phi}$--stable, the gradient Schrödinger operator, given by (2.1)

$$L_{\phi} u := \Delta u + g(\nabla \phi, \nabla u) + q_{\phi} u,$$

where $q_{\phi} := \text{Ric}_{\phi}^\infty (N, N) + |A|^2$, is stable in the sense of Definition 6.1. Therefore, from Lemma 3.1, there exists a positive subsolution, i.e., there exists $u > 0$ such that $L_{\phi} u \leq 0$. Moreover, $q_{\phi} \geq 0$ since $\text{Ric}_{\phi}^\infty \geq 0$.

Thus, Theorem 3.3 implies that $q_{\phi} \equiv 0$, that is, $\Sigma$ is totally geodesic and $\text{Ric}_{\phi}^\infty (N, N) \equiv 0$ along $\Sigma$. \qed

8 Local splitting

In this section we study the topology and geometry of a manifold with density $(\mathcal{N}, g, \phi)$ verifying some lower bound on its P-scalar curvature $\text{R}^\infty_{\phi}$. The idea is to extend the splitting theorems developed by Cai-Galloway [5], Bray-Brendle-Neves [4] and I. Nunes [34]. Here, we will take the unified point of view considered by Micallef-Moraru [31].

First, we will generalize an area estimate for compact stable minimal surfaces in three-dimensional manifolds with a lower bound on its scalar curvature to the context of weighted stable minimal surfaces:

**Proposition 8.1.** Let $\Sigma \subset (\mathcal{N}, g, \phi)$ be a compact weighted stable minimal surface.

1. If $\text{R}^\infty_{\phi} \geq \lambda e^\phi$, for some positive constant $\lambda$, then $A_{\phi}(\Sigma) \leq \frac{8\pi}{\lambda}$. Moreover, if $A_{\phi}(\Sigma) = \frac{8\pi}{\lambda}$, then $\Sigma$ is totally geodesic, the normal Bakry-Émery-Ricci curvature of $(\mathcal{N}, g, \phi)$ vanishes and $\phi$ is constant along $\Sigma$. Moreover, $\text{R}^\infty_{\phi} = 2K = \lambda e^\phi$ along $\Sigma$.

2. If $\text{R}^\infty_{\phi} \geq 0$ and $\Sigma$ has genus one, then $\Sigma$ is totally geodesic, the normal Bakry-Émery-Ricci curvature of $(\mathcal{N}, g, \phi)$ vanishes and $\phi$ is constant along $\Sigma$. Moreover, $\text{R}^\infty_{\phi} = 2K = 0$ along $\Sigma$.

3. If $\text{R}^\infty_{\phi} \geq -\lambda e^\phi$, for some positive constant $\lambda$, and $\gamma = \text{genus}(\Sigma) \geq 2$, then $A_{\phi}(\Sigma) \geq \frac{4\pi(\gamma - 1)}{\lambda}$. Moreover, if $A_{\phi}(\Sigma) = \frac{4\pi(\gamma - 1)}{\lambda}$, then $\Sigma$ is totally geodesic, the normal Bakry-Émery-Ricci curvature of $(\mathcal{N}, g, \phi)$ vanishes and $\phi$ is constant along $\Sigma$. Moreover, $\text{R}^\infty_{\phi} = 2K = -\lambda e^\phi$ along $\Sigma$. 

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Proof. Since $\Sigma$ is compact and weighted stable, from Lemma 6.1 taking $f \equiv 1$ in (6.2), we get
\[
\int_{\Sigma} \left( \frac{1}{2} R_\phi^\infty + \frac{1}{2} |A|^2 + \frac{1}{8} |\nabla \phi|^2 \right) dv_{\Sigma} \leq \int_{\Sigma} K dv_{\Sigma} \tag{8.1}
\]
which yields, from Gauss-Bonnet formula, the following
\[
\frac{1}{2} \int_{\Sigma} R_\phi^\infty dv_{\Sigma} \leq 4\pi (1 - \gamma), \tag{8.2}
\]
where $\gamma := \text{genus}(\Sigma)$.

1. If $R_\phi^\infty \geq \lambda e^\phi$, for some constant $\lambda > 0$.

From (8.2), we have
\[
\frac{\lambda}{2} A_\phi(\Sigma) = \frac{\lambda}{2} \int_{\Sigma} e^\phi dv_{\Sigma} \leq \frac{1}{2} \int_{\Sigma} R_\phi^\infty dv_{\Sigma} \leq 4\pi,
\]
note that $\Sigma$ must be a topological sphere. Therefore, from this last equation, the inequality follows.

Moreover, in the case equality holds, (8.1) implies that $\Sigma$ is totally geodesic and $\phi$ is constant along $\Sigma$. This implies that the Schrödinger operator associated to $\Sigma$ given by Lemma 6.1 reads as
\[
L := \Delta + \frac{1}{6} \left( R_\phi^\infty - 2K \right).
\]

Therefore, the equality implies that
\[
\int_{\Sigma} (R_\phi^\infty - 2K) dv_{\Sigma} = 0,
\]
which yields that the first eigenvalue of $L$ is zero, i.e., $\lambda_1(-L) = 0$, and so, the constant functions are in the kernel of $L$, therefore
\[
R_\phi^\infty = 2K \text{ along } \Sigma.
\]

Thus, from (6.4) and the fact that $|A|^2 \equiv 0$ and $\phi$ is constant along $\Sigma$, we obtain
\[
\text{Ric}_\phi^\infty(N,N) = \text{Ric}_\phi^\infty(N,N) + |A|^2 = \frac{1}{2} R_\phi^\infty - K = 0 \text{ along } \Sigma,
\]
which finishes the proof of item 1.
2. If $R^\infty_\phi \geq 0$ and $\Sigma$ has genus one.

As above, taking $f \equiv 1$ in (8.1), we get

$$\int_\Sigma \left( \frac{1}{2} R^\infty_\phi + \frac{1}{2} |A|^2 + \frac{1}{8} |\nabla \phi|^2 \right) dv_\Sigma \leq \int_\Sigma K dv_\Sigma = 0,$$

which implies that $\Sigma$ is totally geodesic, $\phi$ is constant along $\Sigma$ and $R^\infty_\phi = 0$ along $\Sigma$. So, arguing as above, we get $K \equiv 0$ and $\text{Ric}^\infty_\phi(N,N) = 0$ along $\Sigma$ as well.

3. If $R^\infty_\phi \geq -\lambda e^\phi$, for some constant $\lambda > 0$.

This case is completely analogous to item 1.

Next, we shall prove the existence of a one parameter family of constant weighted mean curvature surfaces in a neighborhood of a totally geodesic compact surface verifying the conditions on Proposition 8.1.

**Proposition 8.2.** Let $\Sigma \subset (\mathcal{N}^3, g, \phi)$ be a compact immersed surface with unit normal vector field $N$. Assume that

- $\Sigma$ is totally geodesic and $\phi$–minimal,
- the normal Barky-Émery-Ricci curvature of $(\mathcal{N}, g, \phi)$ vanishes and $\phi$ is constant along $\Sigma$.

Then, there exists $\varepsilon > 0$ and a smooth function $w : \Sigma \times (-\varepsilon, \varepsilon) \to \mathbb{R}$ such that, for all $t \in (-\varepsilon, \varepsilon)$, the surfaces

$$\Sigma_t := \{ \exp_p(w(p,t)N(p)) : p \in \Sigma \}$$

have constant weighted mean curvature $H_\phi(t)$. Moreover, we have

$$w(p,0) = 0, \quad \frac{\partial}{\partial t} \bigg|_{t=0} w(p,t) = 1 \quad \text{and} \quad \int_\Sigma (w(\cdot, t) - t) dv_\Sigma = 0,$$

for all $p \in \Sigma$ and $t \in (-\varepsilon, \varepsilon)$.

**Proof.** We follow [34]. Fix $\alpha \in (0,1)$ and consider the Banach spaces

$$\mathcal{X}(n) := \left\{ u \in C^{n,\alpha}(\Sigma) : \int_\Sigma u dv_\Sigma = 0 \right\} \quad \text{for each} \ n \in \mathbb{N}. $$

For each $u \in C^{2,\alpha}(\Sigma)$ we define

$$\Sigma(u) := \{ \exp_p(u(p)N(p)) : p \in \Sigma \}.$$
Choose $\varepsilon_1 > 0$ and $\delta > 0$ so that $\Sigma(u + t)$ is a compact surface of class $C^{2,\alpha}$ for all $(t, u) \in (-\varepsilon_1, \varepsilon_1) \times B(0, \delta)$, here $B(0, \delta) := \{ u \in C^{2,\alpha}(\Sigma) : |u|_{C^{2,\alpha}} < \delta \}$. Denote by $H_\phi(u + t)$ the weighted mean curvature of $\Sigma(u + t)$.

Consider the map $\Phi : (-\varepsilon_1, \varepsilon_1) \times (B(0, \delta) \cap X(2)) \to X(0)$ given by

$$\Phi(t, u) := H_\phi(u + t) - \frac{1}{A_\phi(\Sigma)} \int_{\Sigma} H_\phi(u + t)e_\phi \, dv_{\Sigma}.$$ 

First, note that $\Phi(0, 0) = 0$ since $\Sigma(0) = \Sigma$. Second, we compute $D\Phi(0, 0) \cdot v$ for any $v \in X(2)$. We have

$$D\Phi(0, 0) \cdot v = \frac{d}{ds} \bigg|_{s=0} \Phi(0, sv) = \frac{d}{ds} \bigg|_{s=0} H_\phi(sv) - \frac{1}{A_\phi(\Sigma)} \int_{\Sigma} \frac{d}{ds} \bigg|_{s=0} H_\phi(sv)e_\phi \, dv_{\Sigma}$$

$$= L_\phi v - \frac{1}{A_\phi(\Sigma)} \int_{\Sigma} (L_\phi v)e_\phi \, dv_{\Sigma} = \Delta v - \frac{e_\phi}{A_\phi(\Sigma)} \int_{\Sigma} \Delta v \, dv_{\Sigma}$$

$$= \Delta v$$

since $\phi$ is constant along $\Sigma$ and

$$\frac{d}{ds} \bigg|_{s=0} H_\phi(sv) = L_\phi v = \Delta v$$

from (2.1) and the hypothesis.

So, since $\Delta : X(2) \to X(0)$ is a linear isomorphism, by the Implicit Function Theorem, there exists $0 < \varepsilon < \varepsilon_1$ and $u(t) := u(t, \cdot) \in B(0, \delta)$ for $t \in (-\varepsilon, \varepsilon)$ such that

$$u(0) = 0 \text{ and } \Phi(t, u(t)) = 0 \text{ for all } t \in (-\varepsilon, \varepsilon).$$

Finally, defining

$$w(t, p) = u(t, p) + t, \quad (t, p) \in (-\varepsilon, \varepsilon) \times \Sigma,$$

we obtain the result.

Now, we are ready to prove the local splitting result:

**Theorem 8.1.** Let $(N^3, g, \phi)$ be a complete manifold with density containing a compact, embedded weighted area-minimizing surface $\Sigma$.

1. Suppose that $R_\infty \geq \lambda e_\phi$, for some positive constant $\lambda$, and $A_\phi(\Sigma) = \frac{8\pi}{\lambda}$. Then $\Sigma$ has genus zero and it has a neighborhood in $N$ which is isometric to the product $S^2 \times (-\varepsilon, \varepsilon)$ with the product metric $g_{+1} + dt^2$ (up to scaling the metric $g$); here $g_{+1}$ is the metric of constant Gaussian curvature $+1$. Moreover, $\phi$ is constant in $U$. 

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2. Suppose that $R_{\infty} \geq 0$ and $\Sigma$ has genus one. Then $\Sigma$ has a neighborhood in $\mathcal{N}$ which is flat and isometric to the product $\mathbb{T}^2 \times (-\varepsilon, \varepsilon)$ with the product metric $g_0 + dt^2$; here $g_0$ is the metric of constant Guassian curvature 0. Moreover, $\phi$ is constant in $\mathcal{U}$.

3. Suppose that $R_{\infty} \geq -\lambda e^\phi$, for some positive constant $\lambda$, $\Sigma$ has genus $\gamma \geq 2$ and $A_\phi(\Sigma) = \frac{4\pi(\gamma-1)}{\lambda}$. Then $\Sigma$ has a neighborhood in $\mathcal{N}$ which is isometric to the product $\Sigma \times (-\varepsilon, \varepsilon)$ with the product metric $g_{-1} + dt^2$ (up to scaling the metric $g$); here $g_{-1}$ is the metric of constant Guassian curvature $-1$. Moreover, $\phi$ is constant in $\mathcal{U}$.

Proof. Let $\Sigma \subset (\mathcal{N}, g, \phi)$ be a compact embedded weighted area-minimizing surface under any of the conditions above. From Proposition 8.1 and Proposition 8.2, there exists $\varepsilon > 0$ and a smooth function $w : \Sigma \times (-\varepsilon, \varepsilon) \to \mathbb{R}$ such that, for all $t \in (-\varepsilon, \varepsilon)$, the surfaces

$$\Sigma_t := \{ \exp_p(w(p,t)N(p)) : p \in \Sigma \}$$

have constant weighted mean curvature $H_\phi(t)$. Moreover, we have

$$w(p, 0) = 0, \quad \frac{\partial}{\partial t} \bigg|_{t=0} w(p, t) = 1 \quad \text{and} \quad \int_\Sigma (w(\cdot, t) - t) d\nu_\Sigma = 0,$$

for all $p \in \Sigma$ and $t \in (-\varepsilon, \varepsilon)$.

On the one hand, from (2.1) we have:

$$L_\phi u = \Delta u + (Ric^\infty_\phi(N, N) + |A|^2)u + g(\nabla \phi, \nabla u)$$

$$= \Delta u + \left(\frac{1}{2} R^\infty_\phi - K + \text{div}X + F\right)u + g(\nabla \phi, \nabla u)$$

$$\geq \Delta u + \left(\frac{1}{2} R^\infty_\phi - K + \text{div}X + F\right)u - \frac{1}{2} |\nabla \phi|^2 u - \frac{1}{2} |\nabla u|^2$$

$$= \Delta u - \frac{|\nabla u|^2}{2u} + \left(\frac{1}{2} R^\infty_\phi - K + \text{div}X + F\right) - \frac{1}{2} |\nabla \phi|^2 u$$

$$= u \Delta \ln u + \frac{|\nabla u|^2}{2u} + \left(\frac{1}{2} R^\infty_\phi - K + \text{div}X + \frac{H^2_\phi}{2} + \frac{|A|^2}{2}\right)u,$$

that is,

$$L_\phi u \geq u \Delta \ln u + \frac{|\nabla u|^2}{2u} + \left(\frac{1}{2} R^\infty_\phi - K + \text{div}X + \frac{H^2_\phi}{2} + \frac{|A|^2}{2}\right)u. \quad (8.3)$$

Let

$$f_t(x) = \exp_p(w(p,t)N(p)), \quad p \in \Sigma, \quad t \in (-\varepsilon, \varepsilon),$$

thus $f_t(\Sigma) = \Sigma_t$ for each $t \in (-\varepsilon, \varepsilon)$ being $\Sigma_0 = \Sigma$. Define the lapse function $\rho_t : \Sigma \to \mathbb{R}$ by

$$\rho_t(p) = g \left( N_t(p), \frac{\partial}{\partial t} f_t(p) \right),$$

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where $N_t$ is a unit normal vector along $\Sigma_t$. Set $\xi: \Sigma \times (-\varepsilon, \varepsilon) \to N$ given by

$$
\xi(p, t) = f_t(p), \quad (p, t) \in \Sigma \times (-\varepsilon, \varepsilon).
$$

By the First Variation Formula for the Generalized Mean Curvature (see [2] and [8]), we have

$$
H'_{\phi}(t) = -L_{\phi} \rho_t.
$$

On the other hand, from (8.3), we obtain

$$
H'_{\phi}(t) \frac{1}{\rho_t} \leq -\Delta_t \ln \rho_t - \text{div} X_t + K_t - \frac{1}{2} R^\infty_{\phi}.
$$

(8.4)

1. If $R^\infty_{\phi} \geq \lambda e^\phi$, for some positive constant $\lambda$, and $A_{\phi}(\Sigma) = \frac{8\pi}{\lambda}$.

Since $R^\infty_{\phi} \geq \lambda e^\phi$, integrating (8.4) over $\Sigma_t$, we obtain for any $t \in (0, \varepsilon)$

$$
H'_{\phi}(t) \int_{\Sigma_t} \frac{1}{\rho_t} dv_{\Sigma_t} = \int_{\Sigma_t} H'_{\phi}(t) \frac{1}{\rho_t} dv_{\Sigma_t} \leq \int_{\Sigma_t} K_t dv_{\Sigma_t} - \frac{\lambda}{2} \int_{\Sigma_t} e^\phi dv_{\Sigma_t} = 4\pi - \frac{\lambda}{2} A_{\phi}(t),
$$

that is,

$$
H'_{\phi}(t) \int_{\Sigma_t} \frac{1}{\rho_t} dv_{\Sigma_t} \leq \frac{\lambda}{2} (A_{\phi}(0) - A_{\phi}(t)),
$$

where we have used that $H_{\phi}(t)$ is constant on each $\Sigma_t$. Since $\rho_0 \equiv 1$ on $\Sigma$, by continuity, there exists $\varepsilon > 0$ so that $1/2 < \rho_t < 2$. This yields that

$$
H'_{\phi}(t) \int_{\Sigma_t} \frac{1}{\rho_t} dv_{\Sigma_t} \leq \frac{\lambda}{2} (A_{\phi}(0) - A_{\phi}(t)) \leq 0
$$

since $\Sigma$ is a minimizer for the weighted area functional. Therefore, $H'_{\phi}(t) \leq 0$ for all $t \in (0, \varepsilon)$. So, by the First Variation Formula for the weighted area, we have

$$
A'_{\phi}(t) = \int_{\Sigma_t} H_{\phi}(t) \rho_t e^\phi dv_{\Sigma_t} \leq 0 \quad \text{for all } t \in (0, \varepsilon),
$$

and using that $\Sigma$ is a minimizer for the weighted area, we obtain $A_{\phi}(t) = A_{\phi}(0)$ for all $t \in (0, \varepsilon)$. Analogously, we can prove that $A_{\phi}(t) = A_{\phi}(0)$ for all $t \in (-\varepsilon, 0)$.

Thus, Proposition 8.1 implies that $\Sigma_t$ is weighted minimal, totally geodesic, $\phi \equiv \phi_t$ is constant along $\Sigma_t$ and $R^\infty_{\phi} = 2K_t = \lambda e^\phi_t$ (constant) along $\Sigma$ for each $t \in (-\varepsilon, \varepsilon)$. Now, since $\Sigma_t$ is weighted minimal and totally geodesic for each $t \in (-\varepsilon, \varepsilon)$, we get

$$
g(\nabla \ln \phi, N_t) = 0,
$$

which yields that $\phi = \phi_0$ is constant in $U := \xi(\Sigma \times (-\varepsilon, \varepsilon)) \subset N$. Moreover, from (8.4) and $\Sigma_t$ being totally geodesic, we obtain that $\rho_t$ is constant along each $\Sigma_t$, $t \in (-\varepsilon, \varepsilon)$, i.e.,
$\rho_t$ is a function of $t$ only. This yields that the vector field $N_t$ is parallel on $U$ which implies that the integral curves of $N_t$ are geodesics and so

$$\xi(t,p) = \exp_p(tN(p)) \text{ for all } p \in \Sigma,$$

furthermore, the above map is an isometry from $\Sigma \times (-\varepsilon, \varepsilon)$ to $U$, where we consider the product metric $g_\Sigma + dt^2$ in $\Sigma \times (-\varepsilon, \varepsilon)$. So, scaling $g$, we can assume that $K_t \equiv 1$ for all $\Sigma_t$, which finishes item 1.

2. If $R_\infty \geq 0$ and $\Sigma$ has genus one.

The proof is completely analogous to the one on item 1.

3. If $R_\infty \geq -\lambda e^\phi$, for some positive constant $\lambda$, $\Sigma$ has genus $\gamma \geq 2$ and $A_\phi(\Sigma) = \frac{4\pi(\gamma)}{\lambda}$.

As we did in item 1, integrating (8.4) over $\Sigma_t$, we obtain for any $t \in (0, \varepsilon)$

$$\int_{\Sigma_t} H_\phi'(t) \frac{1}{\rho_t} d\Sigma_t \leq \int_{\Sigma_t} K_t d\Sigma_t + \frac{\lambda}{2} \int_{\Sigma_t} e^\phi d\Sigma_t = 2\pi(1 - \gamma) + \frac{\lambda}{2} A_\phi(t),$$

that is,

$$H_\phi'(t) \int_{\Sigma_t} \frac{1}{\rho_t} d\Sigma_t \leq \frac{\lambda}{2} (A_\phi(t) - A_\phi(0)) = \frac{\lambda}{2} \int_0^t A_\phi'(s) ds,$$

so, using the first variation formula for the weighted area and that $H_\phi(t)$ is constant on each $\Sigma_t$ we get

$$H_\phi'(t) \int_{\Sigma_t} \frac{1}{\rho_t} d\Sigma_t \leq \int_0^t H_\phi(s) \left( \int_{\Sigma_s} \rho_s d\Sigma_s \right) ds. \quad (8.5)$$

Assume there exists $t_0 \in (0, \varepsilon)$ so that $H_\phi(t_0) > 0$. Define

$$I := \{ t \in [0, t_0] : H_\phi(t) \geq H_\phi(t_0) \}.$$

We will show that $\inf I = 0$. Assume by contradiction that $\inf I = \bar{t} > 0$. First, rewrite (8.5) as

$$H_\phi'(t) \int_{\Sigma_t} \frac{1}{\rho_t} d\Sigma_t \leq -\frac{1}{\psi(t)} \int_0^t H_\phi(s) \mu(s) ds,$$

where

$$\psi(t) := \int_{\Sigma_t} \frac{1}{\rho_t} d\Sigma_t \text{ and } \mu(t) = \int_{\Sigma_t} \rho_t e^\phi d\Sigma_t.$$
Since \( \rho_0 \equiv 1 \) and \( \phi \equiv \phi_0 \) on \( \Sigma \), by continuity, there exists \( \varepsilon > 0 \) so that \( 1/2 < \rho_t < 2 \) and \( e^{\phi_0}/2 < e^\phi < 2e^{\phi_0} \). This yields that
\[
\frac{A_\phi(t)}{2} < \mu(t) < 2A_\phi(t) \quad \text{for all } t \in (-\varepsilon, \varepsilon)
\]
and
\[
\frac{A_\phi(t)}{4e^{\phi_0}} < \psi(t) < 4e^{\phi_0}A_\phi(t) \quad \text{for all } t \in (-\varepsilon, \varepsilon)
\]
Shrinking \( \varepsilon \) if necessary, we can assume \( A_\phi(0)/2 < A_\phi(t) < 2A_\phi(0) \) for all \( t \in (-\varepsilon, \varepsilon) \).

Therefore, we obtain
\[
\frac{1}{\psi(t)} < \frac{8e^{\phi_0}}{A_\phi(0)} \quad \text{and } \mu(t) < 4A_\phi(0) \quad \text{for all } t \in (-\varepsilon, \varepsilon)
\]
Second, by the Mean Value Theorem, there exists \( t' \in (0, \bar{t}) \) so that
\[
H_\phi(\bar{t}) = H_\phi(t')\bar{t}.
\]
Therefore, from (8.6) and (8.7), we get
\[
H_\phi(t') \leq \frac{t'}{\psi(t)} \int_0^{\bar{t}} H_\phi(s) \mu(s) \, ds \leq 32e^{\phi_0}H_\phi(t')t'\bar{t} \leq 32e^{\phi_0}\varepsilon^2H_\phi(t'),
\]
so, if \( \varepsilon < \frac{1}{\sqrt{32e^{\phi_0}}} \), we get the desired contradiction.

Since \( \inf I = 0 \), it follows that \( H_\phi(0) > H_\phi(\bar{t}) > 0 \), which contradicts that \( \Sigma \) is weighted minimal. Therefore, \( H_\phi(t) \leq 0 \) for all \( t \in (0, \varepsilon) \). Now, we can do the same for \( t \in (-\varepsilon, 0) \). And we can finish as in item 1.

9 Topology of three-manifolds with density

This section is mostly based on Section 8 and the following observation:

**Proposition 9.1.** Let \( \Sigma \subset (\mathcal{N}^3, g, \phi) \) be a compact surface in a manifold with density. Then, \( \Sigma \) is weighted area-minimizing if and only if \( \Sigma \) is area-minimizing in \( (\mathcal{N}, \tilde{g} := e^\phi g) \).

First, we extend the rigidity result of Bray-Brendle-Neves [4]:
**Theorem 9.1.** Let \((N^3, g, \phi)\) be a compact manifold with density so that \(R^\infty_\phi\) is positive and \(\pi_2(N) \neq 0\). Define

\[A(N, g, \phi) := \inf \left\{ A_\phi(f(S^2)) : f \in F \right\},\]

where \(F\) is the set of all smooth maps \(f : S^2 \to N\) which represent nontrivial elements in \(\pi_2(N)\).

Then, we have

\[A(N, g, \phi) \min \left\{ R^\infty_\phi(x) e^{\phi(x)} : x \in N \right\} \leq 8\pi.\] (9.1)

Moreover, if equality holds, the universal cover of \((N, g)\) is isometric (up to scaling) to the standard cylinder \(S^2 \times \mathbb{R}\) and \(\phi\) is constant in \(N\).

**Proof.** Since \(R^\infty_\phi\) is positive and \(N\) is compact, we get

\[R^\infty_\phi \geq \lambda e^\phi, \quad \text{where} \quad \lambda := \min \left\{ \frac{R^\infty_\phi(x)}{e^{\phi(x)}} : x \in N \right\}.\]

So, from Theorem 8.1, we get the upper bound.

Now, assume the equality holds in (9.1). From Proposition 9.1 and results of Meeks-Yau [30], there exists a smooth immersion \(f \in F\) so that

\[A_\phi(f(S^2)) = A(N, g, \phi).\]

Denote \(\Sigma = f(S^2)\). Since we are assuming equality, Theorem 8.1 asserts that locally, in a neighborhood of \(\Sigma\), \(N\) splits as a product manifold and \(\phi\) is constant. We can also see that each leaf of the product structure is weighted area minimizer, so we can continue the process and we construct a foliation \(\{\Sigma_t\}_{t \in \mathbb{R}}\),

\[\Sigma_t := \left\{ \exp_{f(p)}(tN(p)) : p \in \Sigma \right\},\]

of embedded spheres which are totally geodesic, has the same constant Gaussian curvature and \(\phi\) and \(R^\infty_\phi\) are constant.

Now, it is not hard to see that the map

\[\Phi : S^2 \times \mathbb{R} \to N,\]

given by

\[\Phi(p, t) = \exp_{f(p)}(tN(p))\]

is a local isometry (see [4, Proposition 11]). From here, it follows that \(\Phi\) is a covering map and therefore, the universal cover of \((N, g)\) is isometric to \(S^2 \times \mathbb{R}\) equipped with the standard metric. This finishes the proof of Theorem (9.1) \(\square\)
Second, the rigidity result of Cai-Galloway [5]. The proof is completely similar to Theorem 9.1 so, we omit the proof.

**Theorem 9.2.** Let \((\mathcal{N}, g, \phi)\) be a complete manifold with density so that \(R_\phi^\infty\) is nonnegative. If \((\mathcal{N}, g, \phi)\) contains a weighted area minimizing compact surface in its homotopy class of genus greater than or equal to 1, then the product manifold \(\mathbb{T}^2 \times \mathbb{R}\), where \(\mathbb{T}^2\) is a torus equipped with the standard flat metric, is an isometric covering of \((\mathcal{N}, g)\), and \(\phi\) is constant in \(\mathcal{N}\). In particular, \((\mathcal{N}, g)\) is flat.

And finally, the rigidity result of Nunes [34]:

**Theorem 9.3.** Let \((\mathcal{N}, g, \phi)\) be a complete manifold with density so that \(R_\phi^\infty \geq -\lambda e^\phi\) for some positive constant \(\lambda\). Moreover, suppose that \(\Sigma \subset (\mathcal{N}, g, \phi)\) is a two-sided compact embedded Riemannian surface of genus \(\gamma \geq 2\) which minimizes area in its homotopy class. Then,

\[
A_\phi(\Sigma) \geq \frac{4\pi(\gamma - 1)}{\lambda}.
\]

Moreover, if equality holds, the universal cover of \((\mathcal{N}, g)\) is isometric (up to scaling) to the product manifold \(\Sigma_\gamma \times \mathbb{R}\), where \(\Sigma_\gamma\) is a compact surface of genus \(\gamma\) equipped with a metric of constant Gaussian curvature \(-1\), and \(\phi\) is constant in \(\mathcal{N}\).

### 10 Further comments: Optimal Transportation

As we said at the Introduction, given a manifold with density \((\mathcal{N}, g, \phi)\), \(\phi \in C^\infty(\mathcal{N})\), one can consider \(\phi\) as a function to distorted the reference volume measure \(dv\), i.e., \(dm = e^\phi dv\). So, we can consider the Boltzman entropy functional

\[
\mathcal{H}_m(\mu) := \int_{\mathcal{N}} \rho \ln \rho \, dm, \quad \mu = \rho = \rho \, dm \in \mathcal{P}_2(\mathcal{N}),
\]

where \(\mathcal{P}_2(\mathcal{N})\) is the Wasserstein space.

Following Otto’s calculus, we saw that critical points of the Boltzman entropy, i.e., \(\mu \in \mathcal{P}_2(\mathcal{N})\) so that

\[
\text{grad}_\mu \mathcal{H}_m = 0,
\]

correspond to positive solutions to the gradient Schrödinger operator

\[
L_\phi \rho := \Delta \rho + g(\nabla \phi, \nabla \rho).
\]

Now, since optimal transportation plans (for the Boltzman entropy functional) must be critical points, we call them **critical transportation plans**, we can locate the space of optimal transportation plans under global conditions on the initial data \((\mathcal{N}, g, \phi)\) using Theorem 3.2, that is:
Let \((N, g, \phi)\) be a complete manifolds with density of finite type. Then, critical transportation plans \(\mu\) for the Boltzmann entropy functional are \(\mu = \alpha dm\), \(\alpha \in \mathbb{R}^+\). Moreover, if we assume
\[
\int_N d\mu = \int_N dm,
\]
then, the only optimal transportation plan is \(\mu = \phi dv\).

Actually, this is the situation for the Gaussian measure space \((\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle, e^{-\pi|x|^2})\). Note that, if \(dm := e\phi dv \in \mathcal{P}_2(N)\), then \((N, g, \phi)\) is of finite type.

**Remark 10.1.** It would be interesting to investigate deeper this relationship, even for other functionals and not only for the Boltzmann entropy.

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