Complex analysis

Weak solutions to complex Monge–Ampère equations on compact Kähler manifolds

Solutions faibles des équations de Monge–Ampère complexes sur des variétés de Kähler compactes

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A B S T R A C T
We show a general existence theorem of solutions to the complex Monge–Ampère type equation on compact Kähler manifolds.

R É S U M É
Nous montrons un théorème général d'existence et d'unicité de solution d'une équation de type Monge–Ampère complexe sur des variétés de Kähler compactes.

1. Introduction

Let \((X, \omega)\) be a compact Kähler manifold of dimension \(n\). Throughout this note, \(\theta\) denotes a smooth closed form of bidegree \((1, 1)\) which is nonnegative and big, i.e. such that \(\int_X \theta^n > 0\). Recall that a \(\theta\)-plurisubharmonic (\(\theta\)-psf for short) function is an upper semi-continuous function \(\varphi\) such that \(\theta + \ddbar{\varphi}\) is nonnegative in the sense of currents. The set of all \(\theta\)-psf functions \(\varphi\) on \(X\) will be denoted by \(PSH(X, \theta)\) and endowed with the weak topology, which coincides with the \(L^p(X)\)-topology. We shall consider the existence and uniqueness of the weak solution to the following complex Monge–Ampère equations

\[
(\theta + \ddbar{\varphi})^n = F(\varphi, \cdot) d\mu
\]

where \(\varphi\) is a \(\theta\)-psf function, \(F(t, x) \geq 0\) is a measurable function on \(\mathbb{R} \times X\) and \(\mu\) is a positive measure. It is well known that we cannot always make sense to the left-hand side of \((1)\) as a nonnegative measure. But according to [4] (see also [6,7,12]), we can define the non-pluripolar product \((\theta + \ddbar{u})^n\) as the limit of \(1_{(u > -j)}(\theta + \ddbar{(\max(u, -j)})^n\). It was shown in [7] that its trivial extension is a nonnegative closed current and

\[
\int_X (\theta + \ddbar{u})^n \leq \int_X \theta^n.
\]
Denote by $\mathcal{E}(X, \theta)$ the set of all $\theta$-psh with full non-pluripolar Monge–Ampère measure, i.e. the $\theta$-psh functions for which the last inequality becomes an equality.

For $F$ smooth and $\mu = dV$ is a smooth positive volume form, the equation has been studied extensively by various authors, see for example [1,2,7,15,13,14,16], etc., and references therein. Recently, Kołodziej treated the case $F$ bounded by a function independent of the first variable and $\mu = \omega^n$, where $\omega$ is a Kähler form on $X$. In this paper, we consider a more general case. Our main purpose is to prove the following theorem.

**Main Theorem.** Assume that $F: \mathbb{R} \times X \to [0, +\infty)$ is a measurable function such that:

1) for all $x \in X$ the function $t \mapsto F(t, x)$ is continuous and nondecreasing;
2) $F(t, \cdot) \in L^1(X, d\mu)$ for all $t \in \mathbb{R}$;
3) $\lim_{t \to +\infty} \int_X F(t, x) \, d\mu \leq \int_X \theta^n < \lim_{t \to -\infty} \int_X F(t, x) \, d\mu$.

Then there exists a unique (up to additive constant) $\theta$-psh function $\varphi \in \mathcal{E}(X, \theta)$ solution to

$$\left(\theta + dd^c\varphi\right)^n = F(\varphi, \cdot) \, d\mu.$$ 

2. **Proof**

**Lemma 2.1.** Let $\mu$ be a positive measure on $X$ vanishing on all pluripolar subsets of $X$ and $u_j \in \mathcal{E}(X, \theta)$ such that $u_j \geq u_0$ for some $u_0 \in \mathcal{E}(X, \theta) \cap L^1(d\mu)$.

If $u_j \to u$ in $L^1(X)$, then

$$\lim_{j \to +\infty} \int_X u_j \, d\mu = \int_X u \, d\mu.$$

**Proof.** Since $u_0 \in L^1(d\mu)$ and the measure $\mu$ puts no mass on pluripolar subsets of $X$, then

$$\int_{\alpha}^{+\infty} \int_{(u_j < t)} \, d\mu \, dt \leq \int_{\alpha}^{+\infty} \int_{(u_0 < t)} \, d\mu \, dt \to 0 \quad \text{as } \alpha \to +\infty.$$

Hence, by the Dunford–Petit theorem (see for example [10] p. 274), we have that the sequence $(u_j)$ is weakly relatively compact in $L^1(d\mu)$. Let $\tilde{u} \in L^1(d\mu)$ be a cluster point of $(u_j)$. After extracting a subsequence, we may assume that $(u_j)$ converges to $\tilde{u}$ weakly in $L^1(d\mu)$. On the other hand, we have $u_j \to u$ in $L^1(X)$. So, choosing a subsequence if necessary, we can assume that $u_j \to u$ point-wise on $X \setminus A$, where $A = \limsup_{j \to +\infty} u_j < u)$. But $A$ is negligible, hence, by [3] $A$ is pluripolar subset of $X$, thus $\mu(A) = 0$. It follows from Lebesgue’s dominated convergence theorem that $u_j \to u$ weakly in $L^1(d\mu)$. Therefore $\tilde{u} = u \mu$-a.e. Hence $u$ is the unique cluster point of $(u_j)$, which means that $(u_j)$ converges to $u$ weakly in $L^1(d\mu)$ and the proof is complete. □

The following corollary is the global version of Corollary 1.4 in [8].

**Corollary 2.2.** Let $\mu$ be a nonnegative measure that puts no mass on pluripolar sets of $X$. Then for any sequence $u_j \in \mathcal{E}(X, \theta)$ converging weakly, one can extract a subsequence that converges pointwise $d\mu$-almost everywhere.

**Proof of the Main Theorem.** The set of $\varphi \in \text{PSH}(X, \theta)$ normalized by $\sup_X \varphi = 0$ is compact (cf. [11,12]). Then there exists a positive constant $C_0 > 0$ such that

$$\int_X -u \theta^n \leq C_0, \quad \forall u \in \text{PSH}(X, \theta); \quad \sup_X u = 0.$$

Consider the set

$$\mathcal{H} := \left\{ \varphi \in \text{PSH}(X, \theta); \varphi \leq 0 \text{ and } \int_X -\varphi \theta^n \leq C_0 \right\}$$

It is obvious that $\mathcal{H}$ is a compact convex subset of $L^1(X)$. 
From the conditions of the main theorem, there exists a real number $c_0$ such that
\[
\int_X F(c_0, \cdot) \, d\mu = \int_X \theta^n.
\]
Fix a function $\varphi \in \mathcal{H}$. Then there exists a real number $c_\varphi \geq c_0$ such that
\[
\int_X F(\varphi + c_\varphi, \cdot) \, d\mu = \int_X \theta^n.
\]
Since $F(\varphi + c_\varphi, \cdot) \in L^1(X, d\mu)$ and $\mu$ vanishes on pluripolar sets, it follows by [7,5] that there exists a function $\tilde{\varphi} \in \mathcal{E}(X, \theta)$ such that $\sup_X \tilde{\varphi} = 0$ and
\[
(\theta + dd^c \tilde{\varphi})^n = F(\varphi + c_\varphi, \cdot) \, d\mu.
\]
The function $\tilde{\varphi}$ does not depend on the constant $c_\varphi$. Indeed, assume that there exist two constant $c_\varphi$ and $c_\varphi'$ such that
\[
\int_X F(\varphi + c_\varphi, \cdot) \, d\mu = \int_X F(\varphi + c_\varphi', \cdot) \, d\mu = \int_X \theta^n.
\]
If $c_\varphi \leq c_\varphi'$ then $F(\varphi + c_\varphi, \cdot) \, d\mu \leq F(\varphi + c_\varphi', \cdot) \, d\mu$. Therefore
\[
F(\varphi + c_\varphi, \cdot) \, d\mu = F(\varphi + c_\varphi', \cdot) \, d\mu.
\]
By the uniqueness result in [7] and [9], we get that $\tilde{\varphi}$ is unique and therefore independent of the constant $c_\varphi$.

From the definition of $\mathcal{H}$ we have $\tilde{\varphi} \in \mathcal{H}$. Consider the map $\Phi : \mathcal{H} \to \mathcal{H}$ defined by $\varphi \mapsto \tilde{\varphi}$. In fact, the range of $\Phi$ is equal to $\mathcal{H} \cap \mathcal{E}(X, \theta)$.

We claim that $\Phi$ is continuous. Indeed, let $\varphi_j \in \mathcal{H}$ be a converging sequence with limit $\varphi \in \mathcal{H}$ in $L^1(X)$-topology. Let $\psi$ be any cluster point of the sequence $\tilde{\varphi}_j := \tilde{\varphi}(\varphi_j)$. We may assume, up to extracting, that $\tilde{\varphi}_j$ converges towards $\psi$ in $L^1(X)$. Since the measure $\mu$ vanishes on pluripolar subsets, then by Corollary 2.2 above, we can extract a subsequence, which is still denoted by $\varphi_j$, so that $\varphi_j \to \varphi$ $\mu$-a.e. We claim that the sequence $c_{\varphi_j}$ is bounded. Indeed, by construction we have $c_{\varphi_j} \geq c_0$. Now if $c_{\varphi_j} \to +\infty$ then
\[
\int_X \theta^n = \liminf_{j \to +\infty} \int_X F(\varphi_j + c_{\varphi_j}, \cdot) \, d\mu > \int_X \theta^n,
\]
which is impossible.

So by passing to a subsequence, we may assume that $c_{\varphi_j} \to c$. Therefore $F(\varphi_j + c_{\varphi_j}, \cdot) \to F(\varphi + c, \cdot)$ in $L^1(d\mu)$. Since $\tilde{\varphi}_j \to \psi$ in $L^1(X)$, then $\psi = (\limsup_{j \to +\infty} \tilde{\varphi}_j)^*$ and therefore by Hartogs’ lemma $\sup_X \psi = 0$. Let denote $\psi := (\sup_{k \geq j} \tilde{\varphi}_k)^* = (\lim_{k \to +\infty} \max_{l \geq k} \tilde{\varphi}_k)^*$. Since the set $(\sup_{k \geq j} \tilde{\varphi}_k < (\sup_{k \geq j} \tilde{\varphi}_k)^*)$ is pluripolar, then by the continuity of the complex Monge–Ampère operator along monotonic sequences, we have:
\[
(\theta + dd^c \psi)^n = \lim_{j \to +\infty} \left( (\theta + dd^c \tilde{\varphi}_j)^n \right)
\]
\[
= \lim_{j \to +\infty} \lim_{l \to +\infty} \left( (\theta + dd^c \max_{l \geq k \geq j} \tilde{\varphi}_k)^n \right)
\]
\[
\geq \lim_{j \to +\infty} \lim_{l \to +\infty} \min \lim F(\varphi_k + c_{\varphi_k}, \cdot) \, d\mu
\]
\[
= \lim_{j \to +\infty} \min \lim F(\varphi_j + c_{\varphi_j}, \cdot) \, d\mu
\]
\[
= (\theta + dd^c \psi)^n.
\]
Thence $(\theta + dd^c \psi)^n = (\theta + dd^c \tilde{\varphi})^n$. By uniqueness (shown in [7]), we get $\tilde{\varphi} = \psi$ and therefore $\Phi$ is continuous. Now, Shauder’s fixed point theorem implies that there exists a function $u \in \mathcal{H}$ such that $\Phi(u) = u$. Since $\Phi(\mathcal{H}) \subset \mathcal{E}(X, \theta)$ we have $u \in \mathcal{E}(X, \theta)$ and
\[
(\theta + dd^c u)^n = F(u + c_u, \cdot) \, d\mu.
\]
The function $\varphi := u + c_u$ is the required solution.

Uniqueness follows in a classical way from the comparison principle [3] and its generalization [7]. Indeed assume that there exist tow solutions $\varphi_1$ and $\varphi_2$ in $\mathcal{E}(X, \theta)$ such that
\[
(\theta + dd^c \varphi_i)^n = F(\varphi_i, \cdot), \quad i = 1, 2.
\]
Then, since $F$ is non-decreasing with respect to the first variable, we have

$$F(\varphi_1, \ldots) d\mu \leq F(\varphi_2, \ldots) d\mu \quad \text{on } (\varphi_1 < \varphi_2).$$

On the other hand, by the comparison principle we have

$$\int_{(\varphi_1 < \varphi_2)} (\theta + dd^c \varphi_2)^n \leq \int_{(\varphi_1 < \varphi_2)} (\theta + dd^c \varphi_1)^n.$$  \hfill (3)

Combining (2), (3) and (4), we get

$$\int_{(\varphi_1 < \varphi_2)} F(\varphi_1, \ldots) d\mu \leq \int_{(\varphi_1 < \varphi_2)} F(\varphi_2, \ldots) d\mu \leq \int_{(\varphi_1 < \varphi_2)} F(\varphi_1, \ldots) d\mu.$$  \hfill (4)

Hence

$$F(\varphi_1, \ldots) = F(\varphi_2, \ldots) \mu\text{-almost everywhere} \quad \text{on } (\varphi_1 < \varphi_2).$$

In the same way, we get the equality on $\varphi_1 > \varphi_2$ and then on $X$. Hence

$$(\theta + dd^c \varphi_1)^n = (\theta + dd^c \varphi_2)^n.$$

Once more, the uniqueness result of [7] and [9] implies that $\varphi_1 - \varphi_2 = \text{Cst.}$ \hfill \(\Box\)

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