On the Approximation and Simulation of Iterated Stochastic Integrals and the Corresponding Lévy Areas in Terms of a Multidimensional Brownian Motion

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Abstract

A new algorithm for the approximation and simulation of twofold iterated stochastic integrals together with the corresponding Lévy areas driven by a multidimensional Brownian motion is proposed. The algorithm is based on a truncated Fourier series approach. However, the approximation of the remainder terms differs from the approach considered by Wiktorsson (2001). As the main advantage, the presented algorithm makes use of a diagonal covariance matrix for the approximation of one part of the remainder term and has a higher accuracy due to an exact approximation of the other part of the remainder. This results in a significant reduction of the computational cost compared to, e.g., the algorithm introduced by Wiktorsson. Convergence in $L^p(\Omega)$-norm with $p \geq 2$ for the approximations calculated with the new algorithm as well as for approximations calculated by the basic truncated Fourier series algorithm is proved and the efficiency of the new algorithm is analyzed.

1 Introduction

Iterated stochastic integrals play an important role in the context of stochastic processes driven by Brownian motion. They appear, e.g., in stochastic Taylor expansions [9], the Wiener chaos expansion [16] and they are needed for approximation schemes of higher orders like the Milstein scheme [14, 15] or stochastic Runge-Kutta methods [20] for stochastic differential equations (SDEs) as well as for stochastic partial differential equations (SPDEs) [4, 8]. The approximation and simulation of iterated stochastic integrals driven by a multidimensional Brownian motion is still subject of ongoing research and there is a demand for efficient algorithms. Milstein (1995) [15] as well as Kloeden, Platen and Wright (1992) [10], see also [9], proposed an approximation algorithm for twofold iterated stochastic integrals in case of an $m$-dimensional driving Brownian motion for arbitrary $m \geq 1$ that is based on a truncated Fourier series expansion of the Brownian bridge process and they prove the exact $L^2$-error for that approach. However, the convergence rate of the truncated Fourier series approach is not good enough for a reasonable application with, e.g., the Milstein scheme compared to the Euler-Maruyama scheme [9, 15] if the computational cost are taken into account in the finite dimensional SDE setting. It has to be mentioned that Gaines and Lyons (1994) [3] proposed an algorithm for the special case of a 2-dimensional driving Brownian motion based on a generalization of Marsaglia’s rectangle-wedge-tail method that is, however, complicated to implement. Moreover, in the papers [13] and
the authors approximate the inverse of the distribution function by polynomials in order to simulate realizations of the twofold iterated stochastic integral in the case of a 2-dimensional driving Brownian motion.

For the general $m$-dimensional case with $m \geq 1$, Wiktorsson (2001) [22] improved the Fourier series approach by taking into account an approximation of the truncation term. Wiktorsson’s idea is a breakthrough on the way to increase the efficiency of such approximation algorithms and allows for an improvement of the strong order of convergence of, e.g., the Milstein scheme compared to the Euler Maruyama scheme in case of SDEs driven by a multidimensional Brownian motion. Recently, Pleis (2020) [18] proposed an improved algorithm that is based on the approach of Wiktorsson [22]. This algorithm is of the same order of convergence as that of Wiktorsson, however Pleis obtained some smaller constant for his error estimate and thus needs less computational effort compared to Wiktorsson’s algorithm. Moreover, convergence in $L^p$-norm for $p \geq 2$ is proved for this algorithm. For the infinite dimensional setting, the algorithm proposed in [10, 9, 15] as well as the algorithm in [22] have been generalized by Leonhard and Rößler (2019) [11] for the approximation of iterated stochastic integrals driven by a $Q$-Wiener process like they are needed, e.g., in the setting of SPDEs. We refer to [4, 5] and [8] for a detailed discussion of this topic for Milstein type schemes applied to SPDEs without commutative noise.

In the present paper, we propose an algorithm for the approximation of twofold iterated stochastic integrals driven by a multidimensional Brownian motion. This new algorithm is based on the Fourier series approach and makes use of the seminal idea due to Wiktorsson [22] to approximate the truncation term by some appropriate multivariate Gaussian random variable. However, in contrast to Wiktorsson’s algorithm, we split the truncation term into two parts. The first part can be exactly simulated using Gaussian random variables, see [10, 9, 15], while that of the second part is conditionally Gaussian and approximated by replacing the exact conditional covariance matrix by a deterministic diagonal matrix similar to the approach by Wiktorsson [22]. The idea to split the truncation term was first considered by Milstein [15] as well as by Kloeden, Platen and Wright [10]. Pleis [18] combined this idea together with Wiktorsson’s idea for the approximation of the truncation terms. In contrast to the approach by Pleis, we combine these two ideas in a different way in order to minimize conditional dependencies of the appearing random variables. As a result of this, we derive a different algorithm with better error estimates and thus less computational effort. Compared to Wiktorsson’s algorithm, the main advantage of the newly proposed algorithm are error estimates with some smaller constant and that the covariance matrix for the approximate simulation of the truncation error is a constant diagonal matrix. This has significant consequences if the iterated stochastic integrals need to be simulated on many time intervals of some time discretization for, e.g., a numerical scheme like the Milstein scheme. If Wiktorsson’s algorithm is applied, one has to recalculate the square root of the covariance matrix for the approximation of the truncation term each time step as it depends on the increments of Brownian motion. In contrast to this, the covariance matrix of our algorithm is simply the identity matrix multiplied by some constant. Thus, it is not necessary to recalculate the covariance matrix and to calculate the square root of this matrix for each time step. Furthermore, due to the smaller error constant less normally distributed random variables are needed by the proposed algorithm. Beside enormous savings of computational cost the newly proposed approach allows for a dynamical scaling of the dimension of the driving Brownian motion without a recalculation of already derived approximations due to the diagonal structure of the covariance matrix.
The paper is organized as follows: In Section 2, we briefly describe the Fourier series expansion of the Brownian bridge process, which leads to the approximation algorithm proposed by Milstein [15] and by Kloeden, Platen and Wright [10]. Given their approach, we analyze the conditional distribution of the truncation terms in Section 3, which turn out to have a Gaussian distribution. This feature is used in Section 4 to define the newly proposed approximation algorithm for twofold iterated stochastic integrals. Since the Hilbert space setting allows for sharper error estimates, we first prove error estimates in $L^2$-norm in Section 4.1 whereas $L^p$-norm estimates for $p > 2$ are proved in Section 4.2 for the well known truncated Fourier series approach by Milstein [15] and by Kloeden, Platen and Wright [10] as well as for the newly proposed algorithm. In contrast to the approximation problem, we consider the simulation of iterated stochastic integrals with the newly proposed algorithm in Section 5.1. Moreover, we compare the efficiency and the computational cost of our new algorithm with the cost of the algorithm proposed by Wiktorsson [22] and the algorithm due to Milstein [15] and Kloeden, Platen and Wright [10].

2 A series expansion for Lévy areas

In the following, let a complete probability space $(\Omega, \mathcal{F}, P)$ be given and let $(W_t)_{t \geq 0}$ denote an $m$-dimensional Brownian motion where $W_t = (W^1_t, \ldots, W^m_t)^\top$ and $(W^i_t)_{t \geq 0}$ for $i = 1, \ldots, m$ are independent scalar Brownian motions. We want to approximate the iterated stochastic Itô integrals

$$I_{(i,j)}(t, t + h) = \int_t^{t+h} \int_t^s \text{d}W^i_u \text{d}W^j_s$$

or, in case of Stratonovich calculus, the iterated stochastic Stratonovich integrals

$$J_{(i,j)}(t, t + h) = \int_t^{t+h} \int_t^s \text{d}W^i_u \circ \text{d}W^j_s$$

for $i, j \in \{1, \ldots, m\}$ and some $0 \leq t < t + h < \infty$. The idea is to consider a Fourier series expansion of the corresponding Brownian bridge process that can be used to replace the integrand of the twofold iterated stochastic integral. Then, a detailed analysis for a suitable truncation of this Fourier series as well as an approximation of the truncation term will be applied in order to develop the approximation algorithm that we propose in the following.

Let $\Delta W^i(t, t + h) = W_{t+h}^i - W^i_t$ denote the increment of the $i$th component of $(W_t)_{t \geq 0}$ for $i = 1, \ldots, m$. We point out that the joint distribution of $I_{(i,j)}(t, t + h)$, $\Delta W^i(t, t + h)$ and $W^j(t, t + h)$ for $i, j = 1, \ldots, m$ does not depend on $t$. Therefore, w.l.o.g. we choose $t = 0$ in the following and we simply write $I_{(i,j)}(h)$ and $\Delta W^i(h)$ for $I_{(i,j)}(t, t + h)$ and $\Delta W^i(t, t + h)$, respectively. The same applies to the iterated Stratonovich integral $J_{(i,j)}(h)$. Further, let $\Delta W(h) = (\Delta W^i(h))_{1 \leq i \leq m}$ denote the $m$-dimensional vector of the increments of the Brownian motion and let $I(h) = (I_{(i,j)}(h))_{1 \leq i,j \leq m}$ and $J(h) = (J_{(i,j)}(h))_{1 \leq i,j \leq m}$ denote the $m \times m$ matrices of the iterated stochastic Itô and Stratonovich integrals, respectively.

Twofold iterated stochastic integrals can be expressed by so-called Lévy stochastic area integrals and increments of the driving Brownian motions. Therefore, the approximation of these iterated stochastic integrals is synonymous with the approximation of the corresponding Lévy area. Let $A(h) = (A_{(i,j)}(h))_{1 \leq i,j \leq m}$ denote the $m \times m$ matrix of the Lévy areas $A_{(i,j)}(h)$ that are defined as

$$A_{(i,j)}(h) = \frac{1}{2}(I_{(i,j)}(h) - I_{(j,i)}(h))$$

(3)
for \(i, j = 1, \ldots, m\) (see, e.g., [3]). Then, it holds that
\[
\Delta W^i(h) \Delta W^j(h) = I_{(i,j)}(h) + I_{(j,i)}(h),
\]
\[
A_{(i,j)}(h) = -A_{(j,i)}(h),
\]
\[
A_{(i,i)}(h) = 0
\]
for \(i \neq j\) and
\[
I_{(i,j)}(h) = \frac{1}{2}(\Delta W^i(h) \Delta W^j(h) - h \mathbb{1}_{i=j}) + A_{(i,j)}(h),
\]
\[
J_{(i,j)}(h) = \frac{1}{2}(\Delta W^i(h) \Delta W^j(h)) + A_{(i,j)}(h)
\]
for any \(i, j \in \{1, \ldots, m\}\). Since \(J_{(i,j)}(h) = I_{(i,j)}(h)\) for \(i \neq j\) and \(J_{(i,i)}(h) = I_{(i,i)}(h) + \frac{h}{2}\), we can restrict our considerations to iterated stochastic Itô integrals \(I_{(i,j)}(h)\) in the following.

In order to obtain a Fourier series expansion of the Brownian motion \((W^i_t)_{t \geq 0}\), we start with the so-called Brownian bridge process \((\tilde{W}^i_t)_{t \in [0,h]}\) defined by
\[
\tilde{W}^i_t = W^i_t - \frac{t}{h} W^i_h, \quad t \in [0,h].
\]
Milstein [15] as well as Kloeden, Platen and Wright [10] considered the Fourier series expansion of the Brownian bridge process \((\tilde{W}^i_t)_{t \in [0,h]}\) which results in
\[
W^i_t = \frac{t}{h} W^i_h + \frac{a_{i,0}}{2} + \sum_{k=1}^{\infty} a_{i,k} \cos \left( \frac{2\pi k t}{h} \right) + b_{i,k} \sin \left( \frac{2\pi k t}{h} \right)
\]
for \(t \in [0, h]\) with random coefficients
\[
a_{i,k} = \frac{2}{h} \int_0^h \left( W^i_s - \frac{s}{h} W^i_h \right) \cos \left( \frac{2\pi k s}{h} \right) \, ds,
\]
\[
b_{i,k} = \frac{2}{h} \int_0^h \left( W^i_s - \frac{s}{h} W^i_h \right) \sin \left( \frac{2\pi k s}{h} \right) \, ds
\]
for \(k \in \mathbb{N}_0\) and for \(i \in \{1, \ldots, m\}\). The series on the right hand side of equation (10) converges in the \(L^2(\Omega)\)-norm [10, 9]. Since the sample paths of a Brownian motion are almost surely continuous, the right hand side of (10) converges also \(\text{P.-a.s.}\) in the \(L^2([0,h])\)-norm to \((W^i_t)_{t \in [0,h]}\). The coefficients \(a_{i,k}\) and \(b_{i,k}\) as well as the increments \(\Delta W^i(h)\) for \(i \in \{1, \ldots, m\}\) and \(k \in \mathbb{N}\) are all independent Gaussian random variables with \(a_{i,k} \sim N\left(0, \frac{h}{2\pi^2 k^2}\right)\), \(b_{i,k} \sim N\left(0, \frac{h}{2\pi^2 k^2}\right)\) and \(\Delta W^i(h) \sim N(0, h)\). Further, it holds \(a_{i,0} = -2 \sum_{k=1}^{\infty} a_{i,k}\) with \(a_{i,0} \sim N\left(0, \frac{h}{k}\right)\).

Next, the integrand \(W^i_t\) in (1) is replaced by its Fourier series (10). Integrating this expression results in (7) with representation
\[
A_{(i,j)}(h) = \frac{1}{2}(a_{i,0} \Delta W^j(h) - a_{j,0} \Delta W^i(h)) + \pi \sum_{k=1}^{\infty} k(a_{i,k} b_{j,k} - a_{j,k} b_{i,k})
\]
for \(i, j \in \{1, \ldots, m\}\), see also [10, 9, 15]. Series (11) is the starting point for the construction of several approximation algorithms. The main idea is to truncate the series in an appropriate way. In order to simplify notation, we replace the random variables in (11) by standard Gaussian random variables such that
\[
A_{(i,j)}(h) = \Delta W^i(h) \sqrt{\frac{h}{2\pi}} \sum_{k=1}^{\infty} \frac{1}{k} X_{j,k} - \Delta W^j(h) \sqrt{\frac{h}{2\pi}} \sum_{k=1}^{\infty} \frac{1}{k} X_{i,k} + \frac{h}{2\pi} \sum_{k=1}^{\infty} \frac{1}{k} (X_{i,k} Y_{j,k} - X_{j,k} Y_{i,k})
\]
where \( a_{i,k} = \frac{\sqrt{\pi}}{\sqrt{2\pi k}} X_{i,k} \) and \( b_{i,k} = \frac{\sqrt{\pi}}{\sqrt{2\pi k}} Y_{i,k} \) with independent and identically \( N(0,1) \) distributed random variables \( X_{i,k} \) and \( Y_{i,k} \). For some \( n \in \mathbb{N} \) define the truncated series

\[
A_{(i,j)}^{(n)}(h) = \Delta W^i(h) \frac{\sqrt{h}}{\sqrt{2\pi}} \sum_{k=1}^{n} \frac{1}{k} X_{j,k} - \Delta W^j(h) \frac{\sqrt{h}}{\sqrt{2\pi}} \sum_{k=1}^{n} \frac{1}{k} X_{i,k} + \frac{h}{2\pi} \sum_{k=1}^{n} \frac{1}{k} (X_{i,k} Y_{j,k} - X_{j,k} Y_{i,k})
\]

(12)

and we split the remainder term into two parts denoted as

\[
R_{(i,j)}^{1,(n)}(h) = \Delta W^i(h) \frac{\sqrt{h}}{\sqrt{2\pi}} \sum_{k=n+1}^{\infty} \frac{1}{k} X_{j,k} - \Delta W^j(h) \frac{\sqrt{h}}{\sqrt{2\pi}} \sum_{k=n+1}^{\infty} \frac{1}{k} X_{i,k},
\]

(13)

\[
R_{(i,j)}^{2,(n)}(h) = \frac{h}{2\pi} \sum_{k=n+1}^{\infty} \frac{1}{k} (X_{i,k} Y_{j,k} - X_{j,k} Y_{i,k})
\]

(14)

such that \( A_{(i,j)}(h) = A_{(i,j)}^{(n)}(h) + R_{(i,j)}^{1,(n)}(h) + R_{(i,j)}^{2,(n)}(h) \) for all \( i, j \in \{1, \ldots, m\} \).

An approximation for the Lévy area \( A_{(i,j)}(h) \) can be calculated by simply truncating the Fourier series after \( n \) summands, i.e., calculating \( A_{(i,j)}^{(n)}(h) \). However, as we will see in Section 3, the conditional distribution of \( R_{(i,j)}^{1,(n)}(h) \) given the increments \( \Delta W^i(h) \) and \( \Delta W^j(h) \) is Gaussian. So, this term can be included for the approximation as well resulting in some smaller error constant and this is exactly the algorithm that has been proposed by Milstein [15] and it is also considered in Kloeden, Platen and Wright [10] in a broader context, see also [9]. For \( n \in \mathbb{N} \), let

\[
I_{(i,j)}^{FS,(n)}(h) = \frac{1}{2}(\Delta W^i(h)\Delta W^j(h) - h \mathbb{1}_{i=j}) + A_{(i,j)}^{(n)}(h) + R_{(i,j)}^{1,(n)}(h)
\]

(15)

denote the truncated Fourier series approximation for \( I_{(i,j)}(h) \). Then, the sequence \( I_{(i,j)}^{FS,(n)}(h) \) converges in \( L^2(\Omega) \)-norm to \( I_{(i,j)}(h) \) as \( n \to \infty \) with

\[
\left( \mathbb{E} \left( \left| I_{(i,j)}(h) - I_{(i,j)}^{FS,(n)}(h) \right|^2 \right) \right)^{1/2} = \left( \frac{h^2}{12} - \frac{h^2}{2\pi^2} \sum_{k=1}^{n} \frac{1}{k^2} \right)^{1/2} \leq \frac{h}{\pi \sqrt{2n}}
\]

(16)

for \( i, j \in \{1, \ldots, m\} \) with \( i \neq j \), see [10, 9, 15]. The last estimate follows due to \( \sum_{k=n+1}^{\infty} \frac{1}{k^2} \leq \frac{1}{n} \).

As a result of this, for the discrete time approximation of the solutions of some SDE with an strong order 1 method like the Milstein scheme [9, 15] or a stochastic Runge-Kutta method [20] one has to choose \( n \approx h^{-1} \) in order to preserve the strong order of convergence when replacing \( I_{(i,j)}(h) \) by \( I_{(i,j)}^{FS,(n)}(h) \) in the approximation scheme.

### 3 Conditional distribution of the truncation terms

For the derivation of the approximation algorithm, it is sufficient to approximate \( A_{(i,j)}(h) \) for the case \( j > i \) due to properties (5) and (6) of the Lévy areas. To begin with, we analyze the joint conditional distribution of the truncation terms \( R_{(i,j)}^{1,(n)}(h) \) and \( R_{(i,j)}^{2,(n)}(h) \) for \( j > i \). Following the notation in [11, 22], we first introduce some matrix operations that will be used in the following. For \( B = (b_{i,j}) \in \mathbb{R}^{m \times n} \) and \( C \in \mathbb{R}^{p \times q} \), let \( B \otimes C = (b_{i,j} C) \) denote the Kronecker product of \( B \) and \( C \), which is in \( \mathbb{R}^{mp \times nq} \). Further let vec: \( \mathbb{R}^{m \times n} \to \mathbb{R}^{mn \times 1} \) denote
the vectorization operator such that vec\((B)\) ∈ \(\mathbb{R}^{mn\times 1}\) is the vector that one obtains by stacking the columns of matrix \(B\) one upon the other, i.e.,

\[
\text{vec}(B) = \begin{pmatrix} b_{1,1} \\ b_{1,2} \\ \vdots \\ b_{1,n} \\ b_{2,1} \\ b_{2,2} \\ \vdots \\ b_{2,n} \\ \vdots \\ b_{m,1} \\ b_{m,2} \\ \vdots \\ b_{m,n} \end{pmatrix}. 
\]

For a \(m \times n\) matrix, the inverse operator of vec is denoted as mat_{m,n} : \(\mathbb{R}^{mn\times 1} \rightarrow \mathbb{R}^{m\times n}\) such that mat_{m,n}(vec\((B)\)) = \(B\). For simplification of notation, we make use of the permutation matrix \(P_m \in \mathbb{R}^{m^2 \times m^2}\) defined as

\[
P_m = \sum_{i=1}^{m} e_i^T \otimes (I_m \otimes e_i),
\]

where \(e_i \in \mathbb{R}^{m}\) denotes the \(i\)-th unit vector and \(I_m\) is the \(m \times m\) identity matrix. Then, it holds \(P_m = P_m^T\) and \(P_m P_m^T = I_m\), see also [11, 12, 22]. Moreover, \(P_m(u \otimes v) = v \otimes u, P_m(B \otimes u) = u \otimes B\) and \(P_m(B \otimes C)P_m = C \otimes B\) for \(u, v \in \mathbb{R}^{m}\) and \(B, C \in \mathbb{R}^{m \times m}\), see [12, Theorem 3.1]. Due to relation (5), it is sufficient to approximate \(A_{(i,j)}(h)\) for \(i < j\) and we denote

\[
\hat{A}(h) = (A_{(1,2)}(h), \ldots, A_{(1,m)}(h), A_{(2,3)}(h), \ldots, A_{(2,m)}(h), \ldots, A_{(l+1,l)}(h), \ldots, A_{(l,m)}(h), \ldots, A_{(m-1,m)}(h))^T,
\]

which is a vector of length \(M = \frac{m(m-1)}{2}\). Next, we introduce the selection matrix \(H_m\) of size \(M \times m^2\) defined as

\[
H_m = \begin{pmatrix}
0_{m-1\times 1} & I_{m-1} & 0_{m-1\times m(m-1)} \\
0_{m-2\times m+2} & I_{m-2} & 0_{m-2\times m(m-2)} \\
\vdots & \vdots & \vdots \\
0_{m-l\times (l+1)m+l} & I_{m-l} & 0_{m-l\times m(m-l)} \\
\vdots & \vdots & \vdots \\
0_{1\times (m-2)m+m-1} & 1 & 0_{1\times m} 
\end{pmatrix}. 
\]

Then, it holds \(H_m H_m^T = I_M\) and \(H_m P_m H_m^T = 0_{M \times M}\), see [22]. If \(B\) is some \(m \times m\) matrix, then \(H_m\) applied to vec\((B)\) picks out exactly those \(M\) elements of vec\((B)\) that belong to the lower triangle matrix of \(B\), i.e., the elements \(B_{i,j}\) with \(i > j\), and gives them back in a vector of length \(M\) in the same order as they are given in the vector vec\((B)\). Therefore, it holds

\[
\hat{A}(h) = H_m \text{vec}\left(A(h)^T\right),
\]

which is a vector of length \(M\). Analogously, we can write (12) as well as the remainder terms (13) and (14) as

\[
\hat{A}^{(n)}(h) = H_m \text{vec}\left(A^{(n)}(h)^T\right)
= \frac{\sqrt{h}}{\sqrt{2\pi}} \sum_{k=1}^{n} \frac{1}{k} H_m(J_m^2 - P_m)(\Delta W(h) \otimes X_k) + \frac{h}{2\pi} \sum_{k=1}^{n} \frac{1}{k} H_m(P_m - I_m^2)(Y_k \otimes X_k)
\]

(18)
\[ \hat{R}^{1, (n)}(h) = H_m \text{vec} \left( R^{1, (n) T} \right) = \frac{\sqrt{h}}{2\pi} \sum_{k=n+1}^{\infty} \frac{1}{k} H_m (I_{m^2} - P_m) (\Delta W(h) \otimes X_k) \]  

(19)

\[ \hat{R}^{2, (n)}(h) = H_m \text{vec} \left( R^{2, (n) T} \right) = \frac{h}{2\pi} \sum_{k=n+1}^{\infty} \frac{1}{k} H_m (P_m - I_{m^2}) (Y_k \otimes X_k) \]  

(20)

with \( X_k = (X_{1,k}, \ldots, X_{m,k})^T \) and \( Y_k = (Y_{1,k}, \ldots, Y_{m,k})^T \). Here, \( X_k \sim N(0_m, I_m) \), \( Y_k \sim N(0_m, hI_m) \) for \( k \in \mathbb{N} \) and \( \Delta W(h) \sim N(0_m, hI_m) \) are all independent Gaussian random variables. Thus, given \( \Delta W(h) \), the random vector \( \hat{A}^{(n)}(h) \) is conditionally independent from \( \hat{R}^{1, (n)}(h) \) and \( \hat{R}^{2, (n)}(h) \). Further, it holds \( \hat{A}(h) = \hat{A}^{(n)}(h) + \hat{R}^{1, (n)}(h) + \hat{R}^{2, (n)}(h) \).

Firstly, we consider the truncation term \( \hat{R}^{1, (n)}(h) \). The series \( \sum_{k=n+1}^{\infty} \frac{1}{k} X_k \) converges \( \text{P-a.s.} \), see e.g. [2, Theorem 7.5]. Since the random variables \( X_{1,k}, \ldots, X_{m,k} \) for \( k \in \mathbb{N} \) are all i.i.d. Gaussian random variables, it follows that \( \sum_{k=n+1}^{\infty} \frac{1}{k} X_k \) is Gaussian with expectation \( 0_m \) and covariance matrix

\[ \Sigma^{1, (n)} = \left( \frac{\pi^2}{6} - \sum_{k=1}^{n} \frac{1}{k^2} \right) I_m , \]

see also [10, 9, 15]. Thus, we have

\[ \sum_{k=n+1}^{\infty} \frac{1}{k} X_k = (\Sigma^{1, (n)})^{1/2} \Psi^{1, (n)} \]

where \( \Psi^{1, (n)} \) is a \( N(0_m, I_m) \) distributed random vector given by

\[ \Psi^{1, (n)} = (\Sigma^{1, (n)})^{-1/2} \sum_{k=n+1}^{\infty} \frac{1}{k} X_k . \]  

(21)

Therefore, it follows that we can rewrite the truncation term \( \hat{R}^{1, (n)}(h) \) as

\[ \hat{A}^{1, (n)}(h) = \frac{\sqrt{h}}{\sqrt{2\pi}} \left( \frac{\pi^2}{6} - \sum_{k=1}^{n} \frac{1}{k^2} \right)^{1/2} H_m (I_{m^2} - P_m) (\Delta W(h) \otimes \Psi^{1, (n)}) . \]  

(22)

Next, we analyze the second truncation term \( \hat{R}^{2, (n)}(h) \). We proceed in a similar way as in [22], however differing in some crucial points. First of all, for each \( k \in \mathbb{N} \) and given the random vector \( X_k \) one observes that the summand \( (P_m - I_{m^2})(Y_k \otimes X_k) \) is conditionally Gaussian distributed with conditional mean

\[ \mathbb{E}((P_m - I_{m^2})(Y_k \otimes X_k) \mid X_k) = (P_m - I_{m^2})(\mathbb{E}(Y_k) \otimes X_k) = 0_{m^2} \]

and conditional covariance matrix \( \Sigma(X_k) \) that is given by

\[ \Sigma(X_k) = \mathbb{E}((P_m - I_{m^2})(Y_k \otimes X_k)(P_m - I_{m^2})(Y_k \otimes X_k)) \mid X_k) = (P_m - I_{m^2})(\mathbb{E}(Y_k Y_k^T \mid X_k) \otimes (X_k X_k^T))(P_m - I_{m^2})^T = (P_m - I_{m^2})(I_m \otimes (X_k X_k^T))(P_m - I_{m^2})^T . \]

Thus, given \( X^{(n)} = (X_k)_{k \geq n+1} \) the conditional distribution of \( \hat{R}^{2, (n)}(h) \) is Gaussian with conditional mean \( 0_M \) and conditional covariance matrix

\[ \Sigma^{2, (n)}(X^{(n)}) = \frac{h^2}{4\pi^2} \sum_{k=n+1}^{\infty} \frac{1}{k^2} H_m \Sigma(X_k) H_m^T . \]  

(23)
since the summands in (20) are independent. As a result of this, it holds

\[ \tilde{R}^{2,(n)}(h) = \frac{h}{2\pi} \left( \sum_{k=n+1}^{\infty} \frac{1}{k^2} H_m \Sigma(X_k)H_m^T \right)^{1/2} \psi^{2,(n)} \]

with some N(0_M, I_M) distributed random vector \( \psi^{2,(n)} \) that is given by

\[ \psi^{2,(n)} = \frac{2\pi}{h} \left( \sum_{k=n+1}^{\infty} \frac{1}{k^2} H_m \Sigma(X_k)H_m^T \right)^{-1/2} \tilde{R}^{2,(n)}(h). \]  

(24)

An explicit representation of the conditional covariance matrix \( H_m \Sigma(X_k)H_m^T \) is derived in Appendix A. Note that for each \( r, s \in \{1, \ldots, M\} \) the series \( \frac{h^2}{2\pi} \sum_{k=n+1}^{\infty} \frac{1}{k^2} (H_m \Sigma(X_k)H_m^T)_{r,s} \) in (23) converges absolutely in \( L^p(\Omega) \)-norm for \( p \geq 1 \) and thus it also converges P-a.s. [1, Prop. 8.7].

These considerations build the basis for the construction of our approximation algorithm for iterated stochastic integrals. Note that \( \psi^{1,(n)} \) depends only on the random variables \( X_k \) for \( k \geq n + 1 \), while \( \psi^{2,(n)} \) is independent from the random variables \( X_k \). Therefore, \( \psi^{1,(n)} \) and \( \psi^{2,(n)} \) are stochastically independent, which is essential for our algorithm. However, in contrast to truncation term \( \tilde{R}^{1,(n)}(h) \) we do not know the conditional distribution of \( \tilde{R}^{2,(n)}(h) \) completely as it still depends on the random variables \( X_k \) for \( k \geq n + 1 \).

4 Approximation algorithm for iterated stochastic integrals and error estimates

In order to approximate truncation term \( \tilde{R}^{2,(n)}(h) \), we define the \( M \times M \) matrix \( \Sigma^{2,\infty} \) as the mean of the conditional covariance matrix of \( \tilde{R}^{2,(n)}(h) \). Due to Lebesgue’s dominated convergence theorem (see, e.g., [6, Theorem 9.2]) together with Appendix A, we have

\[ \Sigma^{2,\infty} = E(\Sigma^{2,(n)}(X^{(n)})) \]

\[ = \frac{h^2}{4\pi^2} \sum_{k=n+1}^{\infty} \frac{1}{k^2} E(H_m \Sigma(X_1)H_m^T) \]

\[ = \frac{h^2}{4\pi^2} \sum_{k=n+1}^{\infty} \frac{1}{k^2} H_m (P_m - I_m)(I_m \otimes E(X_1X_1^T))(P_m - I_m)^T H_m^T \]

\[ = \frac{h^2}{4\pi^2} \sum_{k=n+1}^{\infty} \frac{1}{k^2} H_m (P_m - I_m)(P_m - I_m)^T H_m^T \]

\[ = \frac{h^2}{4\pi^2} \sum_{k=n+1}^{\infty} \frac{1}{k^2} 2I_M. \]  

(25)

The matrix \( \Sigma^{2,\infty} \) serves as an approximation of the conditional covariance matrix \( \Sigma^{2,(n)}(X^{(n)}). \) Therefore, we define

\[ A^{2,(n)}(h) = (\Sigma^{2,\infty})^{1/2} \psi^{2,(n)} = \frac{h}{\sqrt{2\pi}} \left( \frac{\pi^2}{6} - \sum_{k=1}^{n} \frac{1}{k^2} \right)^{1/2} \psi^{2,(n)} \]  

(26)

as an approximation of the truncation term \( \tilde{R}^{2,(n)}(h) \), where \( \psi^{2,(n)} \) is the N(0_M, I_M) distributed random vector in (24).
Now, we can define the approximation of the iterated stochastic integrals as follows: For the $m \times m$ matrix $I(h) = (I_{(i,j)}(h))_{1 \leq i,j \leq m}$ of all iterated stochastic integrals it holds

$$
\text{vec} \left( I(h)^T \right) = \frac{1}{2}(\Delta W(h) \otimes \Delta W(h) - h \text{vec}(I_m)) + \text{vec} \left( A(h)^T \right).
$$

Let

$$
I(h) = H_m \text{vec} \left( I(h)^T \right)
= \frac{1}{2}H_m(\Delta W(h) \otimes \Delta W(h) - h \text{vec}(I_m)) + \hat{A}(h).
$$

Now, we approximate the $M$-dimensional vector $\hat{I}(h)$ containing all iterated stochastic integrals $I_{(i,j)}(h)$ for $j > i$ by the newly proposed algorithm that is defined by the approximation

$$
\hat{I}^{(n)}(h) = \frac{1}{2}H_m(\Delta W(h) \otimes \Delta W(h) - h \text{vec}(I_m)) + \hat{A}^{(n)}(h) + \hat{A}^{1,(n)}(h) + \hat{A}^{2,(n)}(h)
$$

(27)

with $\hat{A}^{(n)}(h), \hat{A}^{1,(n)}(h)$ and $\hat{A}^{2,(n)}(h)$ defined in (18), (22) and (26), respectively.

Now, from (27) one can rebuild the full $m^2$-dimensional approximation vector by

$$
\text{vec} \left( I^{(n)}(h)^T \right) = \frac{1}{2}(\Delta W(h) \otimes \Delta W(h) - h \text{vec}(I_m))
+ (I_{m^2} - P_m)H_m^T(\hat{A}^{(n)}(h) + \hat{A}^{1,(n)}(h) + \hat{A}^{2,(n)}(h))
$$

and with $I^{(n)}(h) = (\text{mat}_{m,m}(\text{vec}(I^{(n)}(h)^T)))^T$ one obtains the $m \times m$ matrix containing all approximations $I^{(n)}_{(i,j)}$ for $1 \leq i,j \leq m$, which is an approximation of the matrix $I(h)$. The approximation $I^{(n)}(h)$ is unbiased, i.e., it holds:

**Proposition 4.1.** For any $n \in \mathbb{N}$ it holds for the approximations $I^{(n)}_{(i,j)}(h)$ of the iterated stochastic integral $I_{(i,j)}(h)$ that

$$
|E \left( I_{(i,j)}(h) - I^{(n)}_{(i,j)}(h) \right) | = 0
$$

(28)

for all $1 \leq i,j \leq m$ and $h > 0$.

**Proof.** Let $n \in \mathbb{N}$ and $h > 0$ arbitrarily fixed. For $i = j$ the assertion obviously holds because $I_{(i,i)}(h) = I^{(n)}_{(i,i)}(h)$. Therefore, it is sufficient to consider the case $i \neq j$. Since $\hat{A}(h) = \hat{A}^{(n)}(h) + \hat{R}^{1,(n)}(h) + \hat{R}^{2,(n)}(h)$, it holds

$$
E \left( \hat{I}(h) - I^{(n)}(h) \right)
= E \left( \hat{A}(h) - \hat{A}^{(n)}(h) - \hat{A}^{1,(n)}(h) - \hat{A}^{2,(n)}(h) \right)
= E \left( \hat{R}^{1,(n)}(h) + \hat{R}^{2,(n)}(h) - \hat{A}^{1,(n)}(h) - \hat{A}^{2,(n)}(h) \right)
= E \left( \frac{h}{2\pi} \left( \sum_{k=n+1}^{\infty} \frac{1}{k^2}H_m \Sigma(X_k)H_m^T \right)^{1/2} \Psi^{2,(n)} - \frac{h}{\sqrt{2\pi}} \left( \frac{\pi^2}{6} - \sum_{k=1}^{n} \frac{1}{k^2} \right)^{1/2} \Psi^{2,(n)} \right)
= 0
$$

because $\hat{R}^{1,(n)}(h) = \hat{A}^{1,(n)}(h)$ P-a.s. and since $\Psi^{2,(n)}$ is $N(0_M, I_M)$ distributed. \qed
4.1 Error estimates in $L^2$-norm

In order to analyze the error of the approximation $\hat{I}^{(n)}(h)$ given by the proposed algorithm in (27) subject to the parameters $h$ and $n \in \mathbb{N}$, we need some auxiliary results first.

Proposition 4.2. The sequence of conditional covariance matrices $(\Sigma^{2,(n)}(X^{(n)}))_{n \in \mathbb{N}}$ converges to the matrix $\Sigma^{2,\infty}$ in $L^2(\Omega, \| \cdot \|_F)$-norm as $n \to \infty$. Further, it holds:

(i) For any $n \in \mathbb{N}$ and $r \in \{1, \ldots, M\}$ it holds

$$\sum_{q=1}^{M} E\left(\left|\left(\Sigma^{2,(n)}(X^{(n)}) - \Sigma^{2,\infty}\right)_{r,q}\right|^2\right) = \frac{h^4 m^2 (m - 1)}{16\pi^4} \sum_{k=n+1}^{\infty} k^{-4}. \quad (29)$$

(ii) For any $n \in \mathbb{N}$ it holds

$$E\left(\left\|\Sigma^{2,(n)}(X^{(n)}) - \Sigma^{2,\infty}\right\|_F^2\right) = \frac{h^4 m^2 (m - 1)}{16\pi^4} \sum_{k=n+1}^{\infty} k^{-4}. \quad (30)$$

Proof. Due to (25), it holds $E(\Sigma^{2,(n)}(X^{(n)})) = \Sigma^{2,\infty}$. Let $r \in \{1, \ldots, M\}$ be arbitrarily fixed. Then, it follows with Lebesgue’s dominated convergence theorem (see, e.g., [6, Theorem 9.2]) together with Appendix A that

$$\sum_{q=1}^{M} E\left(\left|\left(\Sigma^{2,(n)}(X^{(n)}) - \Sigma^{2,\infty}\right)_{r,q}\right|^2\right) = \sum_{q=1}^{M} E\left(\left|\Sigma^{2,(n)}_{r,q}(X^{(n)}) - \Sigma^{2,\infty}_{r,q}\right|^2\right)$$

$$= \sum_{q=1}^{M} E\left(\left|\Sigma^{2,(n)}_{r,q}(X^{(n)}) - E\left(\Sigma^{2,(n)}_{r,q}(X^{(n)})\right)\right|^2\right)$$

$$= \sum_{q=1}^{M} \text{Var}\left(\Sigma^{2,(n)}_{r,q}(X^{(n)})\right)$$

$$= \frac{h^4}{16\pi^4} \sum_{q=1}^{M} \sum_{k=n+1}^{\infty} \frac{1}{k^4} \text{Var}\left(H_m \Sigma(X_k) H_m^T\right)_{r,q}$$

since $H_m \Sigma(X_k) H_m^T$ are i.i.d. for $k \geq n + 1$. Calculating the symmetric $M \times M$ matrix $H_m \Sigma(X_1) H_m^T$ explicitly, see Appendix A, it follows that each row as well as each column has exactly $2m - 4$ entries of type $X_{i,1} X_{j,1}$, each of them with different indices $i \neq j$, and one entry on the diagonal position of type $X_{i,1}^2 + X_{j,1}^2$ for some $i \neq j$. The remaining $M - 2m - 3$ entries are zero. Therefore, we get

$$\sum_{q=1}^{M} \text{Var}\left(H_m \Sigma(X_1) H_m^T\right)_{r,q} = \text{Var}\left(X_{1,1}^2 + X_{2,1}^2\right) + (2m - 4) \text{Var}\left(X_{1,1} X_{2,1}\right)$$

$$= E\left(\left(X_{1,1}^2 + X_{2,1}^2\right)^2\right) - \left(E\left(X_{1,1}^2 + X_{2,1}^2\right)\right)^2$$

$$+ (2m - 4)\left(E\left(X_{1,1} X_{2,1}\right)\right)^2$$

$$= 2m,$$

and (i) follows directly. Moreover, applying (i) we calculate

$$E\left(\left\|\Sigma^{2,(n)}(X^{(n)}) - \Sigma^{2,\infty}\right\|_F^2\right) = \sum_{r=1}^{M} \sum_{q=1}^{M} E\left(\left|\left(\Sigma^{2,(n)}(X^{(n)}) - \Sigma^{2,\infty}\right)_{r,q}\right|^2\right)$$

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which completes the proof of statement (ii). \( \square \)

The following auxiliary Lemma will be useful for the proof of error estimates for the proposed algorithm. A similar result under slightly different assumptions can be found in [18, 22].

**Lemma 4.3.** Let \( A, B \in \mathbb{R}^{q \times q} \) be symmetric commuting matrices, i.e., \( AB = BA \), such that \( A \) is positive semi-definite and \( B \) is positive definite. Further, let \( C \in \mathbb{R}^{r \times q} \) and let \( \lambda_{\text{min}}^B \) denote the smallest eigenvalue of \( B \). Then, it holds:

1. \( \|C(A^{1/2} - B^{1/2})\|_2 \leq \frac{1}{\sqrt{\lambda_{\text{min}}^B}}\|C(A - B)\|_2 \)
2. \( \|C(A^{1/2} - B^{1/2})\|_F \leq \frac{1}{\sqrt{\lambda_{\text{min}}^B}}\|C(A - B)\|_F \)
3. \( \|C(A^{1/2} - B^{1/2})\|_F \leq \frac{1}{\lambda_{\text{min}}^B}\|C(A - B)\|_F \)

**Proof.** By the assumptions of Lemma 4.3 it follows that \( A \) and \( B \) are normal and that \( A^{1/2} + B^{1/2} \) is symmetric and positive definite. Moreover, it follows that \( A^{1/2}B^{1/2} = B^{1/2}A^{1/2} \). Then, making use of the fact that the applied matrix norm is sub-multiplicative, we get

\[
\|C(A^{1/2} - B^{1/2})\|_2 = \|C(A^{1/2} - B^{1/2})(A^{1/2} + B^{1/2})(A^{1/2} + B^{1/2})^{-1}\|_2 \\
= \|C(A - B)(A^{1/2} + B^{1/2})^{-1}\|_2 \\
\leq \|C(A - B)\|_2 \|(A^{1/2} + B^{1/2})^{-1}\|_2. \tag{31}
\]

Since \( A \) and \( B \) are symmetric and commute, they are both simultaneously diagonalizable, i.e., there exist a unitary matrix \( U \) such that \( A = U D_A U^T \) and \( B = U D_B U^T \) where \( D_A = \text{diag}(\lambda_1^A, \ldots, \lambda_q^A) \) and \( D_B = \text{diag}(\lambda_1^B, \ldots, \lambda_q^B) \) with eigenvalues \( \lambda_k^A \) of \( A \) and \( \lambda_k^B \) of \( B \) for \( 1 \leq k \leq q \). Then, it follows

\[
\|(A^{1/2} + B^{1/2})^{-1}\|_2 = \|(U(D_A^{1/2} + D_B^{1/2})U^T)^{-1}\|_2 \\
= \|(U D_A^{1/2} + D_B^{1/2})^{-1}U^T\|_2 \\
= \left\| \text{diag}\left( \frac{1}{\sqrt{\lambda_1^A + \lambda_1^B}}, \ldots, \frac{1}{\sqrt{\lambda_q^A + \lambda_q^B}} \right) \right\|_2 \\
= \max_{1 \leq k \leq q} \frac{1}{\sqrt{\lambda_k^A + \lambda_k^B}} \leq \frac{1}{\sqrt{\lambda_{\text{min}}^B}}. \tag{32}
\]

Then, (i) follows from (31) and (32). Because \( \|H\|_2 \leq \|H\|_F \) for any \( H \in \mathbb{R}^{q \times q} \), (ii) follows from (i). Finally, it holds

\[
\|C(A^{1/2} - B^{1/2})\|_F = \|C(A^{1/2} - B^{1/2})(A^{1/2} + B^{1/2})(A^{1/2} + B^{1/2})^{-1}\|_F \\
= \|((A^{1/2} + B^{1/2})^{-1})^T(A - B)^T C^T\|_F \\
\leq \|(A^{1/2} + B^{1/2})^{-1}\|_2 \|C(A - B)\|_F, \tag{33}
\]

and (iii) follows from (33) and (32). \( \square \)
We consider the error for the approximation of the iterated stochastic integrals calculated by the proposed algorithm w.r.t. the $L^p(\Omega)$-norm for $p \geq 2$. However, we treat the case $p = 2$ separately because we can obtain a sharper upper bound for this case compared to the general case due to a more sophisticated proof.

**Theorem 4.4.** Let $n \in \mathbb{N}$ and let $I_{(i,j)}^{(n)}(h)$ be the approximation given by (27) of the iterated stochastic integral $I_{(i,j)}(h)$ for $1 \leq i, j \leq m$. Further, let $I(h) = (I_{(i,j)}(h))_{1 \leq i,j \leq m}$ and $I^{(n)}(h) = (I_{(i,j)}^{(n)}(h))_{1 \leq i,j \leq m}$ denote the corresponding $m \times m$ matrices. Then, it holds:

(i) $$\max_{1 \leq i,j \leq m} \mathbb{E}(|I_{(i,j)}(h) - I_{(i,j)}^{(n)}(h)|^2) \leq \frac{h^2 m^2 (m - 1)}{4 \pi^2} \sum_{k=n+1}^{\infty} \frac{k^{-4}}{k^{-2}}$$

(ii) $$\mathbb{E}(\|I(h) - I^{(n)}(h)\|_F^2) \leq \frac{h^2 m^2 (m - 1)}{4 \pi^2} \sum_{k=n+1}^{\infty} \frac{k^{-4}}{k^{-2}}$$

**Proof.** Let $n \in \mathbb{N}$. Because $I_{(i,j)}(h) = I_{(i,i)}^{(n)}(h)$ for all $i \in \{1, \ldots, m\}$ and with (5), it holds

$$\max_{1 \leq i,j \leq m} \mathbb{E}(|I_{(i,j)}(h) - I_{(i,j)}^{(n)}(h)|^2) = \max_{1 \leq r \leq M} \mathbb{E}(\langle \hat{A}^{(n)}(h) - \hat{A}^{(n)}(h) - \hat{A}^{(1)}(h) - \hat{A}^{(2)}(h) \rangle_r^2)
= \max_{1 \leq r \leq M} \mathbb{E}(\langle \hat{R}^{(n)}(h) - \hat{A}^{(2)}(h) \rangle_r^2)
= \max_{1 \leq r \leq M} \mathbb{E}(\langle (\Sigma^{(2)}(X^{(n)}))^{1/2} \Sigma^{(2)}(n) - (\Sigma^{(2,\infty)})^{1/2} \Sigma^{(2,\infty)} \rangle_r^2).
$$

Let $r \in \{1, \ldots, M\}$ be arbitrarily fixed. To simplify notation, let $C(r) \in \mathbb{R}^{1 \times M}$ with $C_{1,r}(r) = 1$ and $C_{l,r}(r) = 0$ for $l \neq r$. Further, denote $S(X^{(n)}) = \left(\Sigma^{(2)}(X^{(n)})\right)^{1/2} - \left(\Sigma^{(2,\infty)}\right)^{1/2}$. We define

$$\varphi_r(X^{(n)}, \Sigma^{(2)}(n)) = \sum_{s=1}^{M} \left(\langle (\Sigma^{(2)}(X^{(n)}))^{1/2} - (\Sigma^{(2,\infty)}\rangle_{r,s} \varphi^{(s)}(n)
$$

and we first prove $|\varphi_r(X^{(n)}, \Sigma^{(2)}(n))|^2 \in L^1(\Omega)$. Since $\Sigma^{2,\infty}$ and $\Sigma^{(2)}(X^{(n)})$ commute, we can apply Lemma 4.3 and get

$$\mathbb{E}(\varphi_r(X^{(n)}, \Sigma^{(2)}(n))^2) = \mathbb{E}(\|C(r) S(X^{(n)}) \Sigma^{(2)}(n)\|_F^2)
= \mathbb{E}(\|C(r) S(X^{(n)})\|_F^2) \mathbb{E}(\|\Sigma^{(2)}(n)\|_F^2)
= \lambda^{2,\infty}_{\min}^{-1} \sum_{s=1}^{M} \mathbb{E}(\langle (\Sigma^{(2)}(X^{(n)}))^{1/2} - (\Sigma^{(2,\infty)}\rangle_{r,s}^2)
$$

where $\lambda^{2,\infty}_{\min} = \frac{h^2}{2 \pi^2} \sum_{k=n+1}^{\infty} \frac{1}{k^{-2}}$ is the minimal eigenvalue of $\Sigma^{2,\infty}$. Now, by applying Proposition 4.2 (i) we get that $\mathbb{E}(\varphi_r(X^{(n)}, \Sigma^{(2)}(n))|^2)$ is uniformly bounded for all $r \in \{1, \ldots, M\}$ and all $n \in \mathbb{N}_0$. Thus, we can take the conditional expectation of $|\varphi_r(X^{(n)}, \Sigma^{(2)}(n))|^2$ w.r.t. $X^{(n)}$. Since $\Sigma^{(2,\infty)}$ is independent of $X^{(n)}$, it follows

$$\mathbb{E}(\|\langle (\Sigma^{(2)}(X^{(n)}))^{1/2} \Sigma^{(2)}(n) - (\Sigma^{(2,\infty)})^{1/2} \Sigma^{(2,\infty)}\rangle_r^2
= \mathbb{E}\left( \sum_{s,t=1}^{M} \left( S(X^{(n)}) \varphi^{(s)}(n)(S(X^{(n)}))_{r,t} \varphi^{(t)}(n) \right) \right)$$

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\[ E\left( \sum_{s=1}^{M} \left( S(X^{(n)}) \right)_{r,s}^2 \right) = E\left( \left\| C(r) \left( \sum_{r}^{2}(X^{(n)}) \right)^{1/2} \right\|_{L^2}^2 \right). \]

Together with Proposition 4.2 (i) and Lemma 4.3 we obtain the estimate
\[ E\left( \left\| C(r) \left( \sum_{r}^{2}(X^{(n)}) \right)^{1/2} \right\|_{L^2}^2 \right) \leq \left( \frac{h^2}{2\pi^2} \sum_{k=n+1}^{\infty} \frac{1}{k^2} \right)^{-1} E\left( \left\| C(r) \left( \sum_{r}^{2}(X^{(n)}) - \sum_{r}^{2} \right) \right\|_{L^2}^2 \right). \]

which proves (i). Due to \( I_{(i,j)}(h) = I_{(i,j)}^{(n)}(h) \) for \( i, j \leq m \), (ii) follows directly from (i).

\textbf{Corollary 4.5.} Let \( n \in \mathbb{N} \) and let \( I_{(i,j)}^{(n)}(h) \) be the approximation defined in (27) of the iterated stochastic integral \( I_{(i,j)}(h) \) for \( 1 \leq i, j \leq m \). Then, it holds:

(i) \[ \max_{1 \leq i, j \leq m} \left( E\left( \left| I_{(i,j)}(h) - I_{(i,j)}^{(n)}(h) \right|^2 \right) \right)^{1/2} \leq \frac{\sqrt{m} h}{\sqrt{12} \pi n} \] (36)

(ii) \[ \left( E\left( \left\| I(h) - I^{(n)}(h) \right\|_{L^2}^2 \right) \right)^{1/2} \leq \frac{\sqrt{m} h}{\sqrt{12} \pi n} \] (37)

\textbf{Proof.} The assertions follow from Theorem 4.4 and the estimate
\[ \frac{\sum_{k=n+1}^{\infty} \frac{1}{k^2}}{\sum_{k=n+1}^{\infty} \frac{1}{k^2}} \leq \frac{\int_{n+\frac{1}{3}}^{n+\frac{2}{3}} \frac{1}{x^2} dx}{\int_{n+\frac{1}{3}}^{n+\frac{2}{3}} \frac{1}{x^2} dx} = \frac{n + \frac{2}{3}}{3(n + \frac{3}{4})^3} \leq \frac{1}{3n^2} \]

for all \( n \in \mathbb{N} \), see also [22] for this estimate.

\[ \square \]

\subsection*{4.2 Error estimates in \( L^p \)-norm}

Next to the error estimates in \( L^2(\Omega) \)-norm, we also give some error estimate in \( L^p(\Omega) \)-norm for arbitrary \( p > 2 \) in the following. Therefore, we first prove \( L^p \) convergence for the truncated Fourier series approach proposed in [10, 9, 15], which is a new result on its own.

\textbf{Proposition 4.6.} Let \( n \in \mathbb{N} \) and \( p \geq 2 \). Then, for the truncated Fourier series approximation \( I_{(i,j)}^{FS,(n)}(h) \) in (15) it holds
\[ \max_{1 \leq i, j \leq m} \left( E\left( \left| I_{(i,j)}(h) - I_{(i,j)}^{FS,(n)}(h) \right|^p \right) \right)^{1/p} \leq \left( \frac{p-1}{\sqrt{2\pi}} \left( \Gamma\left( \frac{p}{2} + 1 \right) \right)^{1/p} \left( \frac{\pi^2}{6} - \sum_{k=1}^{n} \frac{1}{k^2} \right)^{1/2} \right)^{1/p} \] (38)

with equality for \( p = 2 \) if \( i \neq j \) and \( E\left( \left| I_{(i,i)}(h) - I_{(i,i)}^{FS,(n)}(h) \right|^p \right) \) for \( i \in \{1, \ldots, m\} \).
For the proof of Proposition 4.6, the following lemma on the $p$th moment of $\chi_r^2$ distributed random variables is needed first.

**Lemma 4.7.** Let $X$ and $Y$ be independent $\mathcal{N}(0,1)$ distributed random variables on $(\Omega, \mathcal{F}, P)$, let $p > -1$ and $c \in \mathbb{R}$. Then, for the $\chi^2$ distributed random variable $X^2 + Y^2$ with 2 degrees of freedom it holds

$$E(|X^2 + Y^2 - c|^p) = 2^p e^{-\frac{1}{2}} \left( \Gamma(p + 1) + \int_0^\infty |t|^p e^{t^2} dt \right).$$

**Proof.** Let $p > -1$ and $c \in \mathbb{R}$ be fixed. Using polar coordinates, we calculate due to the independence of $X$ and $Y$ that

$$E(|X^2 + Y^2 - c|^p) = \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty |x^2 + y^2 - c|^p e^{-\frac{1}{2}(x^2+y^2)} dx dy$$

$$= \frac{2}{\pi} \int_0^\infty \int_0^\infty |x^2 + y^2 - c|^p e^{-\frac{1}{2}(x^2+y^2)} dx dy$$

$$= \frac{2}{\pi} \int_0^\infty \int_0^\infty |r^2 - c|^p r e^{-\frac{1}{2}r^2} dr d\varphi$$

$$= \int_0^\infty |r^2 - c|^p r e^{-\frac{1}{2}r^2} dr.$$

Substituting $z = \frac{1}{2}(r^2 - c)$, we get

$$\int_0^\infty |r^2 - c|^p r e^{-\frac{1}{2}r^2} dr = 2^p e^{-\frac{1}{2}} \left( \int_{-\infty}^\infty |z|^p e^{-z^2} dz + \int_0^\infty z^p e^{-z^2} dz \right).$$

Substituting $t = -z$ in the first integral and expressing the second integral by the gamma function completes the proof. \qed

Note that one may choose $c = 0$ (or $c = E(X^2 + Y^2) = 2$) in order to get a formula for the $p$th absolute moment of a (centered) $\chi_r^2$ random variable.

**Proof of Proposition 4.6.** Let $n \in \mathbb{N}$ and $p \geq 2$ be arbitrarily fixed. Then, it holds for $i, j \in \{1, \ldots, m\}$ with $i \neq j$ that

$$I_{(i,j)}(h) = I_{(i,j)}^{FS,(n)}(h) = \int_0^h \int_0^s dW_i^a dW_j^b$$

$$- \left( \frac{1}{2} W_i^a W_j^b + \frac{a_i,0}{2} W_j^b - \frac{a_j,0}{2} W_i^a + \pi \sum_{k=1}^n k(a_{i,k} b_{j,k} - a_{j,k} b_{i,k}) \right)$$

$$= \int_0^h W_i^a - \left( \frac{s}{h} W_i^a + \frac{a_i,0}{2} + \sum_{k=1}^n a_{i,k} \cos \left( \frac{2\pi k s}{h} \right) + b_{i,k} \sin \left( \frac{2\pi k s}{h} \right) \right) dW_j^b.$$

With the Burkholder-Davis-Gundy inequality we get

$$E\left( |I_{(i,j)}(h) - I_{(i,j)}^{FS,(n)}(h)|^p \right)$$

$$\leq (p-1)^p E\left( \left( \int_0^h \left| W_i^a - \frac{s}{h} W_i^a - \left( \frac{a_i,0}{2} + \sum_{k=1}^n a_{i,k} \cos \left( \frac{2\pi k s}{h} \right) + b_{i,k} \sin \left( \frac{2\pi k s}{h} \right) \right) ds \right|^p \right)^{p/2}.$$

Taking into account that $\left\{ \frac{1}{\sqrt{n}} \sqrt{\frac{2}{n}} \sin \left( \frac{2\pi k s}{n} \right), \sqrt{\frac{2}{n}} \cos \left( \frac{2\pi k s}{n} \right) : k \in \mathbb{N}, s \in [0, h] \right\}$ is an orthonormal basis of $L^2([0, h])$, that $W^i = (W_i^a - \frac{s}{h} W_i^a)_{s \in [0, h]}$ has $\mathbb{P}$-a.s. continuous paths and thus
belongs P-a.s. to $L^2([0, h])$, we get with Parseval’s equality, Lebesgue’s dominated convergence theorem and triangle inequality for the right hand side

\[(p - 1)^p E \left( \left( \int_0^h \left| W^i - \frac{s}{h} W^i_h \right| \frac{a_{i,0}}{2} + \sum_{k=1}^n a_{i,k} \cos \left( \frac{2\pi k s}{h} \right) + b_{i,k} \sin \left( \frac{2\pi k s}{h} \right) \right)^2 \right)^{p/2} \]

\[= (p - 1)^p \left( \frac{h}{2} \right)^{p/2} \left( \int_0^h \left( \sum_{k=n+1}^\infty \left| a_{i,k} \right|^2 + \left| b_{i,k} \right|^2 \right)^{p/2} \right) \]

\[\leq (p - 1)^p \left( \frac{h}{2} \right)^{p/2} \left( \sum_{k=n+1}^\infty \left( E \left( \left| a_{i,k} \right|^2 + \left| b_{i,k} \right|^2 \right)^{p/2} \right) \right)^{2/p} \]

because \( E(\|\hat{W}^i\|_{L^2([0, h])}^p) < \infty \). Since \( a_{i,k}, b_{i,k} \sim N(0, \frac{h}{2\pi^2 k^2}) \) are i.i.d., it follows that \( X_{i,k} = \frac{\sqrt{\pi k} a_{i,k}}{\sqrt{h}} \sim N(0, 1) \) and \( Y_{i,k} = \frac{\sqrt{\pi k} b_{i,k}}{\sqrt{h}} \sim N(0, 1) \) are i.i.d. Then, due to Lemma 4.7 it holds

\[ (p - 1)^p \left( \frac{h}{2} \right)^{p/2} \left( \sum_{k=n+1}^\infty \left( E \left( \left| a_{i,k} \right|^2 + \left| b_{i,k} \right|^2 \right)^{p/2} \right) \right)^{2/p} \]

\[= (p - 1)^p \left( \frac{h}{2} \right)^{p/2} \left( \sum_{k=n+1}^\infty \frac{h}{2\pi^2 k^2} \left( E \left( \left( X_{i,k}^2 + Y_{i,k}^2 \right)^{p/2} \right) \right)^{2/p} \right)^{p/2} \]

\[= (p - 1)^p \left( \frac{h}{2\pi} \right)^p \left( \sum_{k=n+1}^\infty \frac{1}{k^2} \right)^{p/2} \frac{\Gamma(p/2 + 1)}{2^{p/2}}. \]

The case \( i = j \) follows directly due to \( A_{(i,i)}^{(n)}(h) = 0 \) and \( R_{(i,i)}^{(n)}(h) = 0 \) for all \( n \in \mathbb{N} \). \( \square \)

Now, we state the main result of this section on the approximation error of \( \hat{I}^{(n)}(h) \) in (27) for the iterated stochastic integral \( \hat{I}(h) \) in the \( L^p(\Omega) \)-norm.

**Theorem 4.8.** Let \( p > 2 \) and \( n \in \mathbb{N} \). Then, for the approximation \( I_{(i,j)}^{(n)}(h) \) defined in (27) of the iterated stochastic integral \( I_{(i,j)}(h) \) for \( 1 \leq i, j \leq m \) and for the corresponding \( m \times m \) matrices \( I(h) = (I_{(i,j)}(h))_{1 \leq i, j \leq m} \) and \( I^{(n)}(h) = (I^{(n)}_{(i,j)}(h))_{1 \leq i, j \leq m} \) it holds

(i)

\[ \max_{1 \leq i, j \leq m} \left( E \left( \left| I_{(i,j)}(h) - I_{(i,j)}^{(n)}(h) \right| \right) \right)^{1/p} \leq c_{m,p} \frac{\sqrt{p - 1} h}{\pi^{p/2}} \left( \sum_{k=n+1}^\infty \frac{k^{-4}}{k^{-2}} \right)^{1/2}, \]

(ii)

\[ \left( E \left( \left| I(h) - I^{(n)}(h) \right| \right) \right)^{1/p} \leq c_{m,p} \frac{\sqrt{p - 1}(m^2 - m) h}{\pi^{2p + 1}} \left( \sum_{k=n+1}^\infty \frac{k^{-4}}{k^{-2}} \right)^{1/2}, \]

where the constant \( c_{m,p} \) is given by

\[ c_{m,p} = \frac{\Gamma(p+1)}{\pi^{p/2}} \left( e^{-2/p} \left( \Gamma(p+1) + \frac{e}{p+1} \right)^{2/p} + \frac{2m - 4}{\pi^{2p}} \left( \Gamma(p+1/2) \right)^{4/p} \right)^{1/2}. \]

For the proof of Theorem 4.8 we need the following result that can be easily proved, see also [7, p. 5 (1.1)] and [18, Lemma V.20].
Lemma 4.9. Let $Z \sim N(0, \sigma^2)$ be a real-valued Gaussian random variable with some finite $\sigma > 0$. Then, for $p \in [1, \infty]$ it holds

$$E(|Z|^p) = \frac{(2\sigma)^p}{\sqrt{\pi}} \Gamma\left(\frac{p+1}{2}\right).$$

Proof. Let $p \geq 1$ and $\sigma > 0$ be arbitrarily fixed. Substituting $x = \frac{z^2}{2\sigma^2}$, it follows that

$$E(|Z|^p) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} |z| e^{-\frac{z^2}{2\sigma^2}} \, dz = \frac{2}{\sqrt{2\pi}\sigma} \int_{0}^{\infty} z^p e^{-\frac{z^2}{2\sigma^2}} \, dz = \frac{(2\sigma)^{p+1}}{\sqrt{2\pi\sigma}} \int_{0}^{\infty} x^{\frac{p-1}{2}} e^{-x} \, dx.$$

For $p = 1$ the assertion follows directly because $\Gamma(1) = 1$. In case of $p > 1$, the assertion follows due to $\Gamma(s) = \int_{0}^{\infty} t^{s-1} e^{-t} \, dt > 0$.

Now, we are prepared for the proof of the error estimate in $L^p(\Omega)$-norm stated in Theorem 4.8.

Proof of Theorem 4.8. Let $p > 2$ and $n \in \mathbb{N}$ arbitrarily fixed. Then, we get due to Proposition 4.6 that

$$\max_{1 \leq i,j \leq n} E(|I(i,j)(h) - I^{(n)}(i,j)(h)|^p) = \max_{1 \leq r \leq M} E\left(||(\hat{A}(h) - (\hat{A}^{(n)}(h) + \hat{A}^{1,(n)}(h) + \hat{A}^{2,(n)}(h))_r||_p\right) = \max_{1 \leq r \leq M} E\left(||(\hat{A}^{(n)}(h) - \hat{A}^{2,(n)}(h))_r||_p\right) = \max_{1 \leq r \leq M} E\left(||(\Sigma^{2,(n)}(X^{(n)})^{1/2} \Psi^{2,(n)} - (\Sigma^{2,\infty})^{1/2} \Psi^{2,(n)})_{r,s}\right).$$

Let $r \in \{1, \ldots, M\}$ be arbitrarily fixed. Given $X^{(n)}$, the real-valued random variable

$$\varphi_r(X^{(n)}, \Psi^{2,(n)}) = \left((\Sigma^{2,(n)}(X^{(n)})^{1/2} \Psi^{2,(n)} - (\Sigma^{2,\infty})^{1/2} \Psi^{2,(n)})_r\right) = \sum_{s=1}^{M} \left((\Sigma^{2,(n)}(X^{(n)})^{1/2} - (\Sigma^{2,\infty})^{1/2})_{r,s}\Psi^{2,(n)}_{s}\right)$$

is conditionally Gaussian distributed with conditional expectation 0 and conditional variance

$$\sigma_{\varphi_r}^2(X^{(n)}) = \sum_{s=1}^{M} \left((\Sigma^{2,(n)}(X^{(n)})^{1/2} - (\Sigma^{2,\infty})^{1/2})_{r,s}\right)^2.$$

First of all, we prove that $|\varphi_r(X^{(n)}, \Psi^{2,(n)})|^p \in L^1(\Omega)$. Let $C(r) \in \mathbb{R}^{1 \times M}$ with $C_{1,r}(r) = 1$ and $C_{1,l}(r) = 0$ for $l \neq r$. Then, making use of Lemma 4.3 and taking into account that the minimal eigenvalue of $\Sigma^{2,\infty}$ is $\lambda_{\Sigma^{2,\infty}}^2 = \frac{k^2}{\sum_{n=1}^{\infty} 1/k}$, we obtain

$$E(||\varphi_r(X^{(n)}, \Psi^{2,(n)})||_p^p) = E(||C(r) \left((\Sigma^{2,(n)}(X^{(n)})^{1/2} - (\Sigma^{2,\infty})^{1/2})_{r,s}\right)^p||_p^p) \leq E\left(||C(r) \left((\Sigma^{2,(n)}(X^{(n)})^{1/2} - (\Sigma^{2,\infty})^{1/2})_{r,s}\right)^p||_p^p\right) \leq (\lambda_{\Sigma^{2,\infty}}^2)^{-p/2} E\left(||C(r) \left((\Sigma^{2,(n)}(X^{(n)}) - (\Sigma^{2,\infty})_{r,s}\right)^p||_p^p\right).$$

with $E(||\Psi^{2,(n)}||_p^p) \leq M^{p/2-1} \sum_{s=1}^{M} E(||\Psi^{2,(n)}_{s}||_p^p) < \infty$ due to Lemma 4.9. Moreover, with the triangle inequality it holds

$$E\left(||\sum_{s=1}^{M} \left((\Sigma^{2,(n)}(X^{(n)}) - (\Sigma^{2,\infty})_{r,s}\right)^p||_p^p\right) = ||\sum_{s=1}^{M} \left((\Sigma^{2,(n)}(X^{(n)}) - (\Sigma^{2,\infty})_{r,s}\right)^p||_{L^p(\Omega)}^p).$$
\[
\leq \left( \sum_{s=1}^{M} \left\| (\Sigma^{2,(n)}(X^{(n)}) - \Sigma^{2,\infty})_{r,s} \right\|_{L^p(\Omega)} \right)^{p/2}
= \left( \sum_{s=1}^{M} \left\| (\Sigma^{2,(n)}(X^{(n)}) - \Sigma^{2,\infty})_{r,s} \right\|_{L^p(\Omega)}^{2} \right)^{p/2}.
\]

Making use of the structure of the matrix \(\Sigma^{2,(n)}(X^{(n)})\) detailed in Appendix A and applying Lemma 4.9, we get

\[
\left\| (\Sigma^{2,(n)}(X^{(n)}) - \Sigma^{2,\infty})_{r,s} \right\|_{L^p(\Omega)} \leq \frac{h^2}{4\pi^2} \sum_{k=n+1}^{\infty} \frac{1}{k^2} \left\| (H_m \Sigma(X_1) H_m^T - 2 I_M)_{r,s} \right\|_{L^p(\Omega)}
\leq \frac{h^2}{4\pi^2} \frac{\pi^2}{6} \max \left( \left\| X_{1,1}^2 + X_{2,1}^2 - 2 \right\|_{L^p(\Omega)}, \left\| X_{1,1} X_{2,1} \right\|_{L^p(\Omega)} \right)
\leq \frac{h^2}{24} \max \left( 2 \left( \frac{\sqrt{p}}{\sqrt{\pi}} \Gamma \left( \frac{p+1}{2} \right) \right)^{1/p} + 2, \left( \frac{\sqrt{p}}{\sqrt{\pi}} \Gamma \left( \frac{p+1}{2} \right) \right)^{2/p} \right).
\]

Thus, \(E(|\varphi_r(X^{(n)}, \Psi^{2,(n)})|^p)\) is uniformly bounded for all \(1 \leq r \leq M\) and therefore it holds \(|\varphi_r(X^{(n)}, \Psi^{2,(n)})|^p \in L^1(\Omega)\) for all \(1 \leq r \leq M\).

Since \(X^{(n)}\) and \(\Psi^{2,(n)}\) are stochastically independent and because \(|\varphi_r(X^{(n)}, \Psi^{2,(n)})|^p \in L^1(\Omega)\), it follows by the substitution property of the conditional expectation [1, Theorem 2.10] that

\[
E\left(|\varphi_r(X^{(n)}, \Psi^{2,(n)})|^p \mid X^{(n)} \right) = h(X^{(n)})
\]

with \(h(x) = E\left(|\varphi_r(x, \Psi^{2,(n)})|^p \right)\). Since \(\varphi_r(x, \Psi^{2,(n)})\) has a Gaussian distribution with expectation 0 and variance \(\sigma^2_{\varphi_r}(x)\) it follows with Lemma 4.9 that \(h(x) = \frac{(v_{\varphi_r}(x))^p}{\sqrt{\pi}} \Gamma \left( \frac{p+1}{2} \right) \). Thus, it holds

\[
E\left(|\left(\Sigma^{2,(n)}(X^{(n)})\right)^{1/2} \Psi^{2,(n)} - \left(\Sigma^{2,\infty}\right)^{1/2} \Psi^{2,(n)}\right|^p)
= E\left(E\left(\sum_{s=1}^{M} \left| \left(\Sigma^{2,(n)}(X^{(n)})\right)^{1/2} - \left(\Sigma^{2,\infty}\right)^{1/2}\right|_{r,s} \Psi^{2,(n)} \right|^p \mid X^{(n)}\right)
= \frac{2p^2}{\sqrt{\pi}} \Gamma \left( \frac{p+1}{2} \right) E\left(\sum_{s=1}^{M} \left| \left(\Sigma^{2,(n)}(X^{(n)})\right)^{1/2} - \left(\Sigma^{2,\infty}\right)^{1/2}\right|_{r,s}^2 \right)^{p/2}
= \frac{2p^2}{\sqrt{\pi}} \Gamma \left( \frac{p+1}{2} \right) E\left(\left|\left(\Sigma^{2,(n)}(X^{(n)})\right)^{1/2} - \left(\Sigma^{2,\infty}\right)^{1/2}\right|_F^p \right).
\]

Next, we make use of Lemma 4.3 taking into account that the minimal eigenvalue of \(\Sigma^{2,\infty}\) is \(\lambda^{2,\infty}_{\min} = \frac{h^2}{2\pi^2} \sum_{k=n+1}^{\infty} \frac{1}{k^2}\). Then, we obtain

\[
\frac{2p^2}{\sqrt{\pi}} \Gamma \left( \frac{p+1}{2} \right) E\left(\left|\left(\Sigma^{2,(n)}(X^{(n)})\right)^{1/2} - \left(\Sigma^{2,\infty}\right)^{1/2}\right|_F^p \right)
\leq \frac{2p^2}{\sqrt{\pi}} \Gamma \left( \frac{p+1}{2} \right) \left( \lambda^{2,\infty}_{\min} \right)^{-p/2} E\left(\left|\left(\Sigma^{2,(n)}(X^{(n)}) - \Sigma^{2,\infty}\right)\right|_F^p \right)
= \frac{2p^2}{\sqrt{\pi}} \Gamma \left( \frac{p+1}{2} \right) \left( \lambda^{2,\infty}_{\min} \right)^{-p/2} E\left(\sum_{s=1}^{M} \left| \left(\Sigma^{2,(n)}(X^{(n)}) - \Sigma^{2,\infty}\right)_{r,s} \right|^2 \right)^{p/2}
\]

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Now, Lemma 4.7 and Lemma 4.9 are applied. Further, due to $e^t \leq e^1$ for all $t \in [0, 1]$, it follows that

\[
\left(\|X^2_{1,1} + X^2_{2,1} - 2\|_{L_p(\Omega)}^2 + (2m - 4)\|X_{1,1}X_{2,1}\|_{L_p(\Omega)}^2\right)^{p/2} \leq \left(\frac{4}{e^{2/p}} \left(\Gamma(p + 1) + \int_0^1 t^{p - 1} e^t \, dt\right)^{2/p} + (2m - 4) \left(\frac{\sqrt{2}}{\sqrt{\pi}} \Gamma\left(\frac{p + 1}{2}\right)\right)^{4/p}\right)^{p/2}.
\]

Thus, (i) is proved. For (ii), observe that

\[
\left(\mathbb{E}\left(\|I(h) - I^{(n)}(h)\|_{L_p(\Omega)}^p\right)\right)^{1/p} \leq \left(\sum_{i,j=1}^m \left\|I_{(i,j)}(h) - I^{(n)}_{(i,j)}(h)\right\|_{L_p(\Omega)}^{p/2}\right)^{1/2} \leq \left(\sum_{i,j=1}^m \left\|I_{(i,j)}(h) - I^{(n)}_{(i,j)}(h)\right\|_{L_p(\Omega)}^{p/2}\right)^{1/2} \leq (m^2 - m)^{1/2} \max_{1 \leq i,j \leq m} \left\|I_{(i,j)}(h) - I^{(n)}_{(i,j)}(h)\right\|_{L_p(\Omega)}.
\]

because $I_{(i,i)}(h) - I^{(n)}_{(i,i)}(h) = 0$ P-a.s. for all $i = 1, \ldots, m$.

From Theorem 4.8 the following corollary directly follows by estimating the series on the right hand side in the same way as in the proof of Corollary 4.5.
Corollary 4.10. Let $p > 2$ and $n \in \mathbb{N}$. Then, for the approximation $I_{(i,j)}^{(n)}(h)$ defined in (27) of the iterated stochastic integral $I_{(i,j)}(h)$ for $1 \leq i, j \leq m$ it holds

(i) \[
\max_{1 \leq i,j \leq m} \left( \mathbb{E} \left( |I_{(i,j)}(h) - I_{(i,j)}^{(n)}(h)|^p \right) \right)^{1/p} \leq c_{m,p} \frac{\sqrt{p-1}h}{\sqrt{3\pi} \frac{2p+1}{2p} n}, \tag{42}
\]

(ii) \[
\left( \mathbb{E} \left( \|I(h) - I^{(n)}(h)\|_F^p \right) \right)^{1/p} \leq c_{m,p} \frac{\sqrt{(p-1) (m^2 - m)} h}{\sqrt{3\pi} \frac{2p+1}{2p} n}, \tag{43}
\]

where the constant $c_{m,p}$ is the same as the one given in Theorem 4.8.

5 Simulation of iterated stochastic integrals

For many applications like, e.g., stochastic models described by SDEs or SPDEs, one needs to simulate realizations of the approximate solutions. Here it has to be pointed out that, in general, the simulation problem is different from the approximation problem. For the approximation problem considered in Section 4 we make use of information about the realization of the driving Brownian motion like the values of the increments of the Brownian motion, of the Fourier coefficients and of $\Psi^{1,(n)}$ in (21) as well as of $\Psi^{2,(n)}$ in (24). This information has to be provided or needs to be known and is a priori fixed for the approximation problem. On the other hand, for the simulation problem we are free to generate or choose a realization of the driving Brownian motion to be considered, i.e., to generate the whole necessary information ourselves. For the simulation problem, one only has to take care to sample from the correct distribution when this information is generated.

Since the distribution of the iterated stochastic integrals is, in general, not explicitly known, it is necessary to apply an approximation algorithm. Depending on the considered error criterion, the realizations generated by the approximation algorithm need to be close enough to corresponding realizations that may come into existence based on the exact distribution. This feature is guaranteed if the algorithm under consideration also applies to the approximation problem, which is the case for algorithm (27) proposed in Section 4 as convergence in $L^p(\Omega)$-norm is proved. Therefore, we can now describe how to apply this algorithm for simulating approximations for iterated stochastic integrals with some prescribed precision.

5.1 Simulation algorithm

Assume that we want to simulate the twofold iterated stochastic integrals $I_{(i,j)}(h)$ together with the increments $\Delta W^i(h)$ for $1 \leq i, j \leq m$ and some $h > 0$ such that $L^p$-accuracy of at least $\varepsilon > 0$ is guaranteed, i.e., such that

\[
\max_{1 \leq i,j \leq m} \left( \mathbb{E} \left( |I_{(i,j)}(h) - I_{(i,j)}^{(n)}(h)|^p \right) \right)^{1/p} \leq \varepsilon . \tag{44}
\]

Therefore, we have to determine $n \in \mathbb{N}$ as small as possible under the condition that (44) is fulfilled. Let $\hat{c}_{m,p} = \frac{\sqrt{m}}{\sqrt{12 \pi}}$ for $p = 2$ due to (36) and $\hat{c}_{m,p} = c_{m,p} \frac{\sqrt{p-1}}{\sqrt{3\pi} \frac{2p+1}{2p}}$ if $p > 2$ due to (42). Then, it follows that

\[
n \geq \frac{\hat{c}_{m,p}}{\varepsilon} \frac{h}{\varepsilon} \tag{44}
\]
Note that for the simulation of $I_{(i,j)}(h)$ and $\Delta W^i(h)$ for $i, j \in \{1, \ldots, m\}$ we choose $n = \lceil \epsilon_m, \rho^\frac{n}{2} \rceil$.

The algorithm for the simulation of $I_{(i,j)}(h)$ and $\Delta W^i(h)$ follows: Let $h > 0$ and $n \in \mathbb{N}$ be given.

1. Simulate $V \sim \mathcal{N}(0_m, I_m)$ and let
   \[ \Delta W^i(h) = \sqrt{h} V_i \]
   for $i = 1, \ldots, m$.

2. For $k = 1, \ldots, n$ simulate $X_k, Y_k \sim \mathcal{N}(0_m, I_m)$ and calculate
   \[ A_{(i,j)}^{(n)}(h) = \frac{h}{2\pi} \sum_{k=1}^{n} \frac{1}{k} \left( X_{i,k} \left( Y_{j,k} - \sqrt{\frac{2}{h}} \Delta W^j(h) \right) - X_{j,k} \left( Y_{i,k} - \sqrt{\frac{2}{h}} \Delta W^i(h) \right) \right) \]
   for $1 \leq i < j \leq m$.

3. Simulate $\Psi_{1,(n)} \sim \mathcal{N}(0_m, I_m)$ and compute
   \[ A_{(i,j)}^{1,(n)}(h) = \frac{\sqrt{h}}{\sqrt{2\pi}} \left( \frac{\pi^2}{6} - \sum_{k=1}^{n} \frac{1}{k^2} \right)^{1/2} \left( \Delta W^i(h) \Psi_{1,(n)}^{1,(n)}(\psi_j^{1,(n)}) - \Delta W^j(h) \Psi_{1,(n)}^{1,(n)}(\psi_i^{1,(n)}) \right) \]
   for $1 \leq i < j \leq m$.

4. Let $M = \frac{m(m-1)}{2}$. Simulate $\Psi_{2,(n)} \sim \mathcal{N}(0_M, I_M)$ and compute
   \[ A_{(i,j)}^{2,(n)}(h) = \frac{h}{\sqrt{2\pi}} \left( \frac{\pi^2}{6} - \sum_{k=1}^{n} \frac{1}{k^2} \right)^{1/2} \Psi_{2,(n)}^{2,(n)} \]
   for $1 \leq i < j \leq m$ with $r = (i-1)n + j - \sum_{k=1}^{i} k$.

5. Compute the approximation $I_{(i,j)}^{(n)}(h)$ of $I_{(i,j)}(h)$ as
   \[ I_{(i,j)}^{(n)}(h) = \frac{1}{2} \Delta W^i(h) \Delta W^j(h) + A_{(i,j)}^{(n)}(h) + A_{(i,j)}^{1,(n)}(h) + A_{(i,j)}^{2,(n)}(h) \]
   and let
   \[ I_{(j,i)}^{(n)}(h) = \frac{1}{2} \Delta W^j(h) \Delta W^i(h) - A_{(i,j)}^{(n)}(h) - A_{(i,j)}^{1,(n)}(h) - A_{(i,j)}^{2,(n)}(h) \]
   for $1 \leq i < j \leq m$ and further let $I_{(i,i)}^{(n)}(h) = \frac{1}{2} ((\Delta W^i(h))^2 - h)$ for $i = 1, \ldots, m$.

Note that for the simulation of $I_{(i,j)}^{(n)}(h)$ and $\Delta W^i(h)$ for $i, j \in \{1, \ldots, m\}$ one has to simulate $2m(n+1) + \frac{m(m-1)}{2}$ realizations of independent and identically $\mathcal{N}(0,1)$ distributed random variables. Moreover, it has to be pointed out that in contrast to the algorithm proposed by Wiktorsson [22] the presented algorithm permits to simulate the iterated stochastic integrals $I_{(i,j)}^{(n)}(h)$ sequentially and even to increase the dimension $m$ during the simulation without recomputing the already simulated iterated stochastic integral. This feature is due to the covariance matrix $\Sigma_{2,\infty}$ which is a diagonal matrix. As a result of this, the approximations for the truncation terms $A_{(i,j)}^{2,(n)}(h)$ are stochastically independent for $i < j$. 
5.2 Computational cost

Now, we take a closer look at the computational cost that we measure as the number of independent realizations of standard Gaussian random variables needed for one realization of $\Delta W(h)$ together with the matrix $I^{(n)}(h)$ for some $h > 0$ in case of an $m$-dimensional Brownian motion. Given some prescribed error bound $\varepsilon$ for the $L^p(\Omega)$-error \((44)\), the improved algorithm proposed in Section 5.1, which is denoted as IA, has computational cost

$$\text{cost}_{IA}(\varepsilon) = 2m \left( \left\lfloor \frac{\sqrt{5}(m-1)m}{6} \frac{h}{\varepsilon} \right\rfloor + 1 \right) + \frac{m(m-1)}{2}$$

for the simulation of one realization of the increments $\Delta W^i(h)$ together with the iterated stochastic integrals $I^{(n)}_{(i,j)}(h)$ for all $i,j = 1, \ldots, m$. Especially, note that $\hat{c}_{m,2} = \frac{\sqrt{5}}{2\sqrt{5\pi}}$ if $p = 2$.

For a comparison, we consider the cost of the algorithm proposed by Wiktorsson [22] denoted as WIK, for which upper $L^2(\Omega)$-error bounds are known. Therefore, choosing $n$ as in [22, (4.9)] for this algorithm it holds that

$$\text{cost}_{WIK}(\varepsilon) = 2m \left( \left\lfloor \frac{\sqrt{5}(m-1)m}{24\pi} \frac{h}{\varepsilon} \right\rfloor + 1 \right) + \frac{m(m-1)}{2}$$

if the error bound in \((44)\) has to be fulfilled for $p = 2$. Finally, we consider algorithm \((15)\) denoted as FS that is proposed in \([10, 9, 15]\). For $p \geq 2$ it follows with Proposition 4.6 that, in order to fulfill the error bound \((44)\), one has to choose $n \geq \left\lfloor \frac{(p-1)^2}{2\pi^2} \left( \Gamma\left(\frac{p}{2} + 1\right) \right)^{2/p} \frac{h^2}{\varepsilon^2} \right\rfloor$. Choosing $n = \left\lfloor \frac{(p-1)^2}{2\pi^2} \left( \Gamma\left(\frac{p}{2} + 1\right) \right)^{2/p} \frac{h^2}{\varepsilon^2} \right\rfloor$ results in the computational cost

$$\text{cost}_{FS}(\varepsilon) = 2m \left( \left\lfloor \frac{(p-1)^2}{2\pi^2} \left( \Gamma\left(\frac{p}{2} + 1\right) \right)^{2/p} \frac{h^2}{\varepsilon^2} \right\rfloor + 1 \right).$$

Comparing the computational costs of the algorithms IA, WIK and FS as $\varepsilon \to 0$, it follows that the costs of IA and WIK are of order $O(h \varepsilon^{-1})$ while the cost of FS is of order $O(h^2 \varepsilon^{-2})$ for some fixed $h > 0$. Thus, algorithms IA and WIK possess a higher order of convergence than algorithms FS if their errors versus costs are compared. Having a closer look at algorithms IA and WIK, there is asymptotically a reduction by the factor $\sqrt{\frac{5}{2}m(m-1)}$ for the cost of algorithm IA compared to the cost of WIK if the error estimates in \([22, \text{Theorem 4.1}] \) are applied in the case of $p = 2$. It is worth mentioning that in case of the Frobenius norm as in \((37)\) the computational cost for IA is reduced by the factor $\sqrt{5}$ compared to algorithm WIK. This reduction of the computational cost for algorithm IA originates in the exact approximation of the Fourier coefficients $a_{i,0}$ represented by the truncation term $\hat{R}^{1,(n)}$ that is conditionally Gaussian distributed. Moreover, it has to be pointed out that algorithm IA does not need the calculation of the square root of a $M \times M$ covariance matrix as it is the case for algorithm WIK, see \([22, (4.7)] \) because the covariance matrix $\Sigma^{2,\infty}$ in \((25)\) for algorithm IA is a multiple of the identity matrix whereas the covariance matrix for algorithm WIK even depends on $\Delta W(h)$ and thus needs to be recalculated for each realization. As a result of this, there is a substantial improvement in applying algorithm IA compared to the algorithm WIK by Wiktorsson \([22]\), on the one hand by a reduction of the number of necessary realizations of Gaussian random variables and on the other hand by saving the calculation of the square root of the covariance matrix.

As an example, if one of the algorithms FS, WIK or IA is applied together with a numerical scheme for the computation of $L^2$-approximations $Y^h$ of solutions $X$ for some SDE with a root mean square error (RMSE) of order $O(h)$ w.r.t. step size $h$ like the Milstein scheme, i.e., such that

$$(\mathbb{E}\|X_T - Y^h_T\|^2)^{1/2} = O(h)$$
at some time point \( T > 0 \) as \( h \to 0 \), then one has to choose \( n \) such that the convergence rate \( \mathcal{O}(h) \) is preserved if the twofold iterated stochastic integrals in the numerical scheme are replaced by the approximated ones. This means that the RMSE for the approximated iterated stochastic integrals has to be of order \( \mathcal{O}(h^{3/2}) \), see [9, Cor. 10.6.5] or [15, Lem. 6.2]. Choosing \( \varepsilon = h^{3/2} \) results in 

\[
n = \left\lceil \frac{m}{\sqrt{12\pi h^{-1/2}}} \right\rceil
\]

and the total cost for the computation of one realization of \( Y^h(T) \) that is based on approximations on \( T/h \) time intervals amounts to

\[
\text{cost}_{IA}(h) = \mathcal{O}(h^{-3/2}).
\]

Analogously, the computational cost for algorithm WIK is

\[
\text{cost}_{WIK}(h) = \mathcal{O}(h^{-3/2}).
\]

In contrast to that, the computational cost for algorithm FS is

\[
\text{cost}_{FS}(h) = \mathcal{O}(h^{-2}),
\]

which is of the same order as if one would apply the strong order \( 1/2 \) Euler-Maruyama scheme with step size \( h^2 \) resulting in \( \text{cost}_{EM}(h^2) = \mathcal{O}(h^{-2}) \) in order to obtain the same RMSE of order \( \mathcal{O}(h) \). Thus, instead of applying, e.g., the Milstein scheme together with algorithm FS, one may use the Euler-Maruyama scheme with step size \( h^2 \), which is easier to implement and in general needs less computational effort. However, if the Milstein scheme is combined with the algorithms IA or WIK, then a higher order of convergence than that of the Euler-Maruyama scheme is attained, see also discussions in [20, 22]. Especially, if the RMSE versus computational costs are considered, then the resulting so-called effective order of convergence for the Milstein scheme together with algorithm IA or WIK is \( p_{\text{eff}} = 3/2 \), whereas the effective order of convergence for the Milstein scheme together with algorithm FS is \( p_{\text{eff}} = 1/2 \), which is the same as that for the Euler-Maruyama scheme. Thus, the algorithms IA and WIK allow to improve the order of convergence if they are combined with the Milstein scheme. Since algorithm IA needs significantly less computational effort than algorithm WIK in order to guarantee the same RMSE, the combination of the Milstein scheme with algorithm IA outperforms the combination of the Milstein scheme with algorithm WIK.

A Appendix: The covariance matrix

For the proofs of convergence, it is useful to have a closer look at the matrix \( H_m \Sigma(X_k) H_m^T \). Observe, that we can factorize this matrix by

\[
H_m \Sigma(X_k) H_m^T = H_m (P_m - I_m^2) (I_m \otimes (X_k X_k^T)) (P_m - I_m^2)^T H_m^T
\]

\[
= H_m (P_m - I_m^2) (I_m \otimes X_k) [H_m (P_m - I_m^2) (I_m \otimes X_k)]^T
\]

\[
= H_m (X_k \otimes I_m - I_m \otimes X_k) [H_m (X_k \otimes I_m - I_m \otimes X_k)]^T.
\]
Now, we can easily calculate the $m^2 \times m$ matrix

\[
X_k \otimes I_m - I_m \otimes X_k = \begin{pmatrix}
0 & -X_{2,k} & X_{1,k} \\
-X_{3,k} & X_{1,k} & & \\
-X_{4,k} & & X_{1,k} & \\
\vdots & & & \\
-X_{m,k} & & & X_{1,k} \\
X_{2,k} & & & 0 \\
& -X_{3,k} & X_{2,k} & \\
& -X_{4,k} & & X_{2,k} \\
& \vdots & & \vdots \\
& -X_{m,k} & & X_{2,k} \\
\vdots & & & \\
X_{l,k} & & & -X_{1,k} \\
& X_{l,k} & & -X_{2,k} \\
& & \ddots & \\
& & & 0 \\
& & & \vdots \\
& & & \vdots \\
& & & X_{m,k} \\
X_{m,k} & & & -X_{1,k} \\
& X_{m,k} & & -X_{2,k} \\
& & \ddots & -X_{3,k} \\
& & \vdots & \vdots \\
& & \vdots & \vdots \\
& & & X_{m,k} \\
& & & -X_{m-1,k} \\
& & & 0
\end{pmatrix}
\]

The selection matrix $H_m$ applied to $X_k \otimes I_m - I_m \otimes X_k$ picks out the $M$ rows that we obtain by deleting in the first block the first row, in the second block the first two rows, in the $l$-th block the first $l$ rows and so on, until the $m-1$-th block, were we only take the last row while the $m$-th block is deleted completely. Finally, we have to multiply $H_m(X_k \otimes I_m - I_m \otimes X_k)$ with itself transposed which results in the symmetric $M \times M$ block matrix

\[
H_m \Sigma(X_k) H_m^T = \begin{pmatrix}
B_{1,1}^k & B_{1,2}^k & \cdots & B_{1,m-1}^k \\
B_{2,1}^k & B_{2,2}^k & \cdots & B_{2,m-1}^k \\
\vdots & \vdots & \ddots & \vdots \\
B_{m-1,1}^k & B_{m-1,2}^k & \cdots & B_{m-1,m-1}^k
\end{pmatrix},
\]
For $l \in \{1, \ldots, m-1\}$ the $l$-th diagonal block can be calculated as the symmetric $(m-l) \times (m-l)$ matrix

$$B_{l,l}^k = \begin{pmatrix}
X_{l,k}^2 + X_{l+1,k}^2 & X_{l+1,k}X_{l+2,k} & X_{l+1,k}X_{l+3,k} & \cdots & \cdots & X_{l+1,k}X_{m,k} \\
X_{l+2,k}X_{l+1,k} & X_{l+2,k}^2 + X_{l+3,k}^2 & X_{l+2,k}X_{l+3,k} & \cdots & \cdots & X_{l+2,k}X_{m,k} \\
X_{l+3,k}X_{l+1,k} & X_{l+3,k}X_{l+2,k} & X_{l+3,k}^2 + X_{l+4,k}^2 & \cdots & \cdots & X_{l+3,k}X_{m,k} \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
X_{m,k}X_{l+1,k} & X_{m,k}X_{l+2,k} & X_{m,k}X_{l+3,k} & \cdots & \cdots & X_{m,k}^2 + X_{m-1,k}^2
\end{pmatrix}$$

For $1 \leq r < s \leq m-1$ the non-diagonal block at position $(r, s)$ is the $(m-r) \times (m-s)$ matrix that can be calculated as

$$B_{r,s}^k = \begin{pmatrix} 0_{(s-r-1)\times (m-s)} \\
-b_{r,s}^k \\
d_{r,s}^k
\end{pmatrix}$$

with the $1 \times (m-s)$ vector $b_{r,s}^k = (X_{r,k}X_{s+1,k}, X_{r,k}X_{s+2,k}, X_{r,k}X_{s+3,k}, \ldots, X_{r,k}X_{m,k})$ and the $(m-s) \times (m-s)$ diagonal matrix $d_{r,s}^k = \text{diag}(X_{r,k}X_{s,k}, \ldots, X_{r,k}X_{m,k})$. Further, it holds $B_{s,r}^k = (B_{r,s}^k)^T$. With (45) it follows that the matrix $H_m \Sigma(X_k)H_m^T$ is a symmetric $M \times M$ matrix such that $(H_m \Sigma(X_k)H_m^T)_{p,q} \in \{X_{i,k}^2 + X_{j,k}^2, \pm X_{i,k}X_{j,k}, 0\}$ for some $i \neq j$. Moreover, each row (column) has exactly $2m - 4$ matrix entries of type $\pm X_{i,k}X_{j,k}$ and one diagonal entry of type $X_{i,k}^2 + X_{j,k}^2$ for some $i, j \in \{1, \ldots, m\}$ with $i \neq j$, respectively, while the remaining $M - 2m + 3$ entries in each row (column) are equal to 0.

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