Abstract. We introduce a family of dense subalgebras of the Toeplitz algebra and give conditions under which our algebras are quasi-free. As a corollary, we show that the smooth Toeplitz algebra introduced by Cuntz is quasi-free.

1. Introduction

Quasi-free algebras were introduced by Schelter [37] under the name of “smooth algebras”. The main idea behind this notion is that, for a number of reasons, quasi-free algebras can be viewed as noncommutative analogs of smooth affine varieties (or, more exactly, as analogs of algebras of functions on smooth affine varieties in the category of all associative algebras). This point of view was further developed by Cuntz and Quillen [7], who coined the name “quasi-free” for this class of algebras, mostly because they behave like free algebras with respect to nilpotent extensions. Cuntz and Quillen gave several useful characterizations of quasi-free algebras, provided many examples, and proved a number of interesting properties of such algebras. Quasi-free algebras and their generalizations were also considered in the functional analytic context, both in the locally convex [6, 33] and in the bornological settings [23, 24, 26, 41, 42]. They play an important role in cyclic homology theory [6, 8, 9, 23, 24, 26, 36, 41, 42] and in some other aspects of noncommutative geometry [3, 15, 22].

In this paper, we study quasi-freeness for some dense subalgebras of the Toeplitz algebra, i.e., the universal $\mathcal{T}$-algebra generated by an isometry. Here we understand quasi-freeness in the setting of locally convex algebras, but we believe that the bornological approach is also possible. The starting point for our study is the fact (probably due to Meyer [28]) that the algebraic Toeplitz algebra, i.e., the $*$-subalgebra $\mathcal{T}_{\text{alg}}$ of $\mathcal{T}$ algebraically generated by the “universal” isometry $v \in \mathcal{T}$, is quasi-free (see Proposition 2.3 below). On the other hand, $\mathcal{T}$ itself is not quasi-free. This is a simple corollary of a very general result due to Aristov [1]. In retrospective, this is not surprising at all, because there are numerous results (going back to Helemskii’s global dimension theorem [10]) showing that, for Banach algebras, the property of being quasi-free is a rare phenomenon.

We will be interested in locally convex algebras continuously embedded in $\mathcal{T}$ and containing $\mathcal{T}_{\text{alg}}$. The best known example of such an algebra is the smooth Toeplitz algebra introduced by Cuntz [5]. Another natural example is the holomorphic Toeplitz algebra recently considered by Panarin [29]. Our main goal is to construct a family $\{\mathcal{T}_{P,Q}\}$ of locally convex subalgebras of $\mathcal{T}$ (where $P$ and $Q$ are Köthe sets satisfying some natural conditions, see Section 3 for details) and to give a sufficient condition for the inclusion $\mathcal{T}_{\text{alg}} \hookrightarrow \mathcal{T}_{P,Q}$ to be a homological epimorphism. This will imply the quasi-freeness of $\mathcal{T}_{P,Q}$.

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As a corollary, we show that the smooth Toeplitz algebra and the holomorphic Toeplitz algebra are quasi-free.

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2. Preliminaries

2.1. Locally convex algebras and modules. This subsection gives a brief account of homological algebra in categories of locally convex modules. Our main reference is [7]; some details can also be found in [13, 18, 19, 35, 36].

Throughout, all vector spaces and algebras are assumed to be over the field $\mathbb{C}$ of complex numbers. All algebras are assumed to be associative and unital. By a $\hat{\otimes}$-algebra we mean an algebra $A$ endowed with a complete locally convex topology in such a way that the multiplication $A \times A \to A$ is jointly continuous. Note that the multiplication uniquely extends to a continuous linear map $A \hat{\otimes} A \to A$, $a \otimes b \mapsto ab$, where the symbol $\hat{\otimes}$ stands for the completed projective tensor product (whence the name “$\hat{\otimes}$-algebra”). If the topology on $A$ can be determined by a family of submultiplicative seminorms (i.e., a family $\{\| \cdot \|_\lambda : \lambda \in A\}$ of seminorms such that $\|ab\|_\lambda \leq \|a\|_\lambda \|b\|_\lambda$ for all $a, b \in A$), then $A$ is said to be locally $m$-convex (or an Arens-Michael algebra). A Fréchet algebra is a $\hat{\otimes}$-algebra $A$ whose underlying locally convex space is metrizable. The category of all $\hat{\otimes}$-algebras and continuous algebra homomorphisms will be denoted by $\hat{\otimes}$-alg.

Let $A$ be a $\hat{\otimes}$-algebra. A left $A$-$\hat{\otimes}$-module is a left $A$-module $M$ endowed with a complete locally convex topology in such a way that the action $A \times M \to M$ is jointly continuous. We always assume that $1_A \cdot x = x$ for all $x \in M$, where $1_A$ is the identity of $A$. Left $A$-$\hat{\otimes}$-modules and their continuous $A$-module morphisms form a category denoted by $A$-mod. The categories $\text{mod}$-$A$ and $A$-$\text{mod}$-$A$ of right $A$-$\hat{\otimes}$-modules and of $A$-$\hat{\otimes}$-bimodules are defined similarly. Note that $A$-$\text{mod}$-$A \cong A^e$-$\text{mod}$ $\cong \text{mod}$-$A^e$, where $A^e = A \hat{\otimes} A^{\text{op}}$, and where $A^{\text{op}}$ stands for the algebra opposite to $A$. Given $M, N \in A$-mod (respectively, $\text{mod}$-$A$, $A$-$\text{mod}$-$A$), the space of morphisms from $M$ to $N$ will be denoted by $A\text{h}(M, N)$ (respectively, $h_A(M, N)$, $A h_A(M, N)$).

If $M$ is a right $A$-$\hat{\otimes}$-module and $N$ is a left $A$-$\hat{\otimes}$-module, then their $A$-module tensor product $M \hat{\otimes} A N$ is defined to be the completion of the quotient $(M \hat{\otimes} N)/L$, where $L \subset M \hat{\otimes} N$ is the closed linear span of all elements of the form $x \cdot a \otimes y - x \otimes a \cdot y$ ($x \in M$, $y \in N$, $a \in A$). As in pure algebra, the $A$-module tensor product can be characterized by the universal property that, for each complete locally convex space $E$, there is a natural bijection between the set of all jointly continuous $A$-balanced bilinear maps from $M \times N$ to $E$ and the set of all continuous linear maps from $M \hat{\otimes} A N$ to $E$.

A chain complex $C = (C_n, d_n)_{n \in \mathbb{Z}}$ in $A$-mod is admissible if it is split exact in the category of topological vector spaces, i.e., if it has a contracting homotopy consisting of continuous linear maps. Geometrically, this means that each $d_n$ is an open map of $C_n$ onto $\text{Ker} \, d_{n-1}$ (which implies, in particular, that $C$ is exact in the purely algebraic sense), and that $\text{Ker} \, d_n$ is a complemented subspace of $C_n$ for each $n$. The category $A$-mod together with the class of all short admissible sequences is an exact category in the sense of Quillen [23]. This implies that the derived categories $D(A$-mod), $D^+(A$-mod), and $D^b(A$-mod) are defined (see [24, 25] for details). The same is true of $\text{mod}$-$A$ and $A$-$\text{mod}$-$A$. 

...
A left $A\hat{\otimes}$-module $P$ is projective if the functor $A\text{h}(P, -)$ takes admissible sequences of $A\hat{\otimes}$-modules to exact sequences of vector spaces. A projective resolution of $M \in A\text{-mod}$ is a pair $(P, \varepsilon)$, where $P$ is a nonnegative chain complex consisting of projective $A\hat{\otimes}$-modules and $\varepsilon$ is a morphism from $P_0$ to $M$ such that the sequence $P \xrightarrow{\varepsilon} M \rightarrow 0$ is an admissible complex. The length of $P$ is the minimum integer $n$ such that $P_i = 0$ for all $i > n$, or $\infty$ if there is no such $n$. It is a standard fact that $A\text{-mod}$ has enough projectives, i.e., each left $A\hat{\otimes}$-module has a projective resolution. The same is true of $\text{mod}-A$ and $A\text{-mod}-A$.

The projective homological dimension of $M \in A\text{-mod}$ is the minimum integer $n = \text{dh}_A M \in \mathbb{Z}_+ \cup \{\infty\}$ (where $\mathbb{Z}_+$ is the set of all nonnegative integers) with the property that $M$ has a projective resolution of length $n$. The bidimension of $A$ is defined by $\text{db} A = \text{dh}_A e^A A$.

Given a $\hat{\otimes}$-algebra $A$ and an $A\hat{\otimes}$-bimodule $M$, we let $\text{Der}(A, M)$ denote the space of all continuous derivations of $A$ with values in $M$. We will need the following standard characterization of derivations. Let us equip $A \times M$ with a $\hat{\otimes}$-algebra structure by letting $(a, m)(b, n) = (ab, an + mb)$ ($a, b \in A$, $m, n \in M$). Let $p_i : A \times M \rightarrow A$ denote the projection given by $(a, m) \mapsto a$. We then have a natural isomorphism

$$\text{Der}(A, M) \cong \{ \varphi \in \text{Hom}_{\hat{\otimes}\text{-alg}}(A, A \times M) : p_1 \varphi = 1_A \}.$$ (2.1)

Explicitly, the above isomorphism takes each derivation $D : A \rightarrow M$ to the homomorphism $A \rightarrow A \times M$ given by $a \mapsto (a, D(a))$ (see, e.g., [33]).

If $A$ is a $\hat{\otimes}$-algebra, then the bimodule of noncommutative differential 1-forms over $A$ is an $A\hat{\otimes}$-bimodule $\Omega^1 A$ together with a derivation $d_A : A \rightarrow \Omega^1 A$ such that for each $A\hat{\otimes}$-bimodule $M$ and each derivation $D : A \rightarrow M$ there exists a unique $A\hat{\otimes}$-bimodule morphism $\Omega^1 A \rightarrow M$ making the following diagram commute:

$$\begin{array}{ccc}
\Omega^1 A & \xrightarrow{d_A} & M \\
\downarrow & & \downarrow \RL{D} \\
A & & \\
\end{array}$$

In other words, we have a natural isomorphism

$$A\text{h}_A(\Omega^1 A, M) \cong \text{Der}(A, M) \quad (M \in A\text{-mod}-A).$$

It is a standard fact (see, e.g., [4, 33]) that $\Omega^1 A$ exists and is isomorphic to the kernel of the multiplication map $\mu_A : A \otimes A \rightarrow A$. Under the above identification, the universal derivation $d_A : A \rightarrow \Omega^1 A$ acts by the rule $d_A(a) = 1 \otimes a - a \otimes 1$ ($a \in A$). Thus we have an exact sequence

$$0 \rightarrow \Omega^1 A \xrightarrow{j_A} A \hat{\otimes} A \xrightarrow{\mu_A - A} A \rightarrow 0$$ (2.2)

in $A\text{-mod}-A$, where $j_A$ is uniquely determined by $j_A(d_A(a)) = 1 \otimes a - a \otimes 1$ ($a \in A$). Note that (2.2) splits in $A\text{-mod}$ and in $\text{mod}-A$ ([33], cf. also [4]). In particular, (2.2) is admissible.

Let $B$ be a $\hat{\otimes}$-algebra. By an extension of $B$ we mean a continuous open homomorphism $\sigma : A \rightarrow B$, where $A$ is a $\hat{\otimes}$-algebra. It is convenient to interpret $(B, \sigma)$ as an exact sequence

$$0 \rightarrow I \xrightarrow{\sigma} A \xrightarrow{\sigma} B \rightarrow 0,$$ (2.3)
where \( I = \text{Ker} \sigma \) and \( i \) is the inclusion map. We say that (2.3) \emph{splits} (respectively, that (2.3) is \emph{admissible}) if there exists a \( \hat{\otimes} \)-algebra homomorphism (respectively, a continuous linear map) \( j: B \to A \) such that \( \sigma j = 1_B \). We say that (2.3) is a \emph{square-zero} extension if \( I^2 = 0 \).

Following [7] (see also [6, 36, 43, 44]), we say that a \( \hat{\otimes} \)-algebra \( A \) is \emph{quasi-free} if it satisfies any (hence all) of the following equivalent conditions:

1. Each admissible square-zero extension of \( A \) splits.
2. For each admissible square-zero extension \( 0 \to I \to B \to C \to 0 \) of \( \hat{\otimes} \)-algebras and for each \( \hat{\otimes} \)-algebra homomorphism \( A \to C \) there exists a \( \hat{\otimes} \)-algebra homomorphism \( A \to B \) making the following diagram commute:

\[
\begin{array}{ccc}
A & \to & B \\
\downarrow & & \downarrow \\
0 & \to & C
\end{array}
\]

3. \( \Omega^1 A \) is projective in \( A\text{-mod}-A \).
4. \( \text{db} A \leq 1 \).

The above list of equivalent conditions can be extended; see, e.g., [6, 7, 23, 36].

\textbf{Remark 2.1.} Of course, the notion of a quasi-free algebra makes sense in the purely algebraic case as well (i.e., in the case of algebras not equipped with a topology), and this is exactly the case where they were introduced for the first time [7, 37]. In this respect, let us note that each algebra \( A \) of at most countable dimension becomes a \( \hat{\otimes} \)-algebra under the strongest locally convex topology [4, A.2.8], and that \( A \) is quasi-free as a \( \hat{\otimes} \)-algebra if and only if \( A \) is quasi-free in the purely algebraic sense. This readily follows, for example, from [34, Corollary 8.5].

By a \( \hat{\otimes} \)-algebra \emph{epimorphism} we mean an epimorphism in the category of all \( \hat{\otimes} \)-algebras, i.e., a \( \hat{\otimes} \)-algebra homomorphism \( f: A \to B \) such that, whenever \( C \) is a \( \hat{\otimes} \)-algebra and \( g, h: B \to C \) are \( \hat{\otimes} \)-algebra homomorphisms satisfying \( gf = hf \), we have \( g = h \). Equivalently, \( f \) is an epimorphism if and only if the canonical map \( B \hat{\otimes}_A B \to B \) induced by the multiplication on \( B \) is an isomorphism in \( B\text{-mod}-B \) (the proof of this fact given in [38, XI.1] for the category of rings holds verbatim for \( \hat{\otimes} \)-algebras). Following [14], we say that \( f \) is a \emph{homological epimorphism} if the canonical morphism \( B \hat{\otimes}_A^L B \to B \) induced by the multiplication on \( B \) is an isomorphism in \( D^{-}(B\text{-mod}-B) \) (where \( \hat{\otimes}_A^L \) is the total left derived functor of \( \hat{\otimes}_A \)). Explicitly, this means that for some (or, equivalently, for each) projective resolution \( P \to A \) of \( A \) in \( A\text{-mod}-A \) the complex \( B \hat{\otimes}_A P \hat{\otimes}_A B \to B \) is admissible. Homological epimorphisms were introduced by J. L. Taylor [40] under the name of \emph{absolute localizations}. Since then, they were rediscovered several times under different names (see [6, 12, 14, 25, 28]), both in the purely algebraic and in the functional analytic contexts.

\textbf{Remark 2.2.} For future reference, observe that, if \( f: A \to B \) is a homological epimorphism, then we obviously have \( \text{db} B \leq \text{db} A \). In particular, if \( A \) is quasi-free, then so is \( B \).

\textbf{2.2. A survey of Toeplitz algebras.} Recall that the \emph{Toeplitz algebra} is the universal unital \( C^* \)-algebra \( \mathcal{T} \) generated by an isometry. This means that there is an isometry
Moreover, \( K \) is defined as the \( C^* \)-subalgebra of \( \mathcal{B}(l^2) \) generated by the right shift operator \( v \). The equivalence of the above definitions follows from Coburn’s theorem (see, e.g., \( \text{[27]} \) for details). Since \( v^*v = 1 \), we have

\[
T = \text{span}\{v^k(v^*)^\ell : k, \ell \in \mathbb{Z}_+\} \subset \mathcal{B}(l^2).
\]

The Toeplitz algebra can also be characterized in terms of Toeplitz operators as follows. Given a continuous function \( f \) on the circle \( \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} \), let \( T_f \) denote the corresponding Toeplitz operator on the Hardy space \( H^2 = H^2(\mathbb{T}) \). We then have

\[
T = \{T_f + K : f \in C(\mathbb{T}), \ K \in \mathcal{K}(H^2)\}.
\]

More exactly, the map

\[
C(\mathbb{T}) \oplus \mathcal{K}(H^2) \to T, \quad (f, K) \mapsto T_f + K,
\]

is a vector space isomorphism (see, e.g., \( \text{[11, 27]} \)).

The algebraic Toeplitz algebra \( \text{[3, 10, 24]} \) is the unital \( * \)-subalgebra \( T_{\text{alg}} \) of \( T \) generated (as a \( * \)-algebra) by \( v \). It can also be interpreted in terms of Toeplitz operators as follows. Let \( M_\infty \) denote the algebra of infinite complex matrices \( a = (a_{ij})_{i,j \in \mathbb{Z}_+} \) such that \( a_{ij} = 0 \) for all but finitely many \( i, j \in \mathbb{Z}_+ \). We identify \( M_\infty \) with a subalgebra of \( \mathcal{K}(H^2) \) by associating to each \( a \in M_\infty \) the operator on \( H^2 \) whose matrix w.r.t. the trigonometric basis \( \{z^k : k \in \mathbb{Z}_+\} \) is \( a \). Then

\[
T_{\text{alg}} = \{T_f + K : f \in \mathbb{C}[z, z^{-1}], \ K \in M_\infty\},
\]

where the algebra \( \mathbb{C}[z, z^{-1}] \) of Laurent polynomials is interpreted as the algebra of trigonometric polynomials on \( \mathbb{T} \).

Recall also \( \text{[10]} \) that \( T_{\text{alg}} \) is isomorphic to the unital algebra generated by two elements \( u, v \) with relation \( uv = 1 \). It is well known and easy to show that the elements \( v^i u^j \) \((i, j \in \mathbb{Z}_+)\) form a basis of \( T_{\text{alg}} \). Since the dimension of \( T_{\text{alg}} \) is countable, we may and will consider \( T_{\text{alg}} \) as a \( \mathbb{T}\)-algebra with respect to the strongest locally convex topology (cf. Remark \( \text{[2, 1]} \)).

As was mentioned in the Introduction, our main objects are locally convex algebras sitting in between \( T_{\text{alg}} \) and \( T \). The best known example of such an algebra is the smooth Toeplitz algebra \( T_{\text{smth}} \) introduced by Cuntz \( \text{[3]} \) (see also \( \text{[3, 10, 21, 24]} \)). To define \( T_{\text{smth}} \), consider the algebra

\[
\mathcal{K}_{\text{smth}} = \left\{a = (a_{ij})_{i,j \in \mathbb{Z}_+} : a_{ij} \in \mathbb{C}, \ |a|_n = \sum_{i,j} |a_{ij}|(1 + i + j)^n < \infty \ \forall n \in \mathbb{Z}_+\right\}
\]

of smooth compact operators introduced by Phillips \( \text{[34]} \). Recall that \( \mathcal{K}_{\text{smth}} \) is an algebra under the usual matrix multiplication and is a Fréchet algebra for the topology generated by the norms \( \|\cdot\|_n \) \((n \in \mathbb{Z}_+)\). Now \( T_{\text{smth}} \subset T \) is defined as follows:

\[
T_{\text{smth}} = \{T_f + K : f \in C^\infty(\mathbb{T}), \ K \in \mathcal{K}_{\text{smth}}\}.
\]

The restriction of \( \text{[2, 4]} \) to \( C^\infty(\mathbb{T}) \oplus \mathcal{K}_{\text{smth}} \) is a vector space isomorphism between \( C^\infty(\mathbb{T}) \oplus \mathcal{K}_{\text{smth}} \) and \( T_{\text{smth}} \). This enables us to topologize \( T_{\text{smth}} \) so that it becomes a Fréchet space. Moreover \( \text{[3]}, T_{\text{smth}} \) is a Fréchet-Arens-Michael algebra.
Another natural example is the holomorphic Toeplitz algebra $\mathcal{T}_{\text{hol}}$ recently introduced by Panarin [29]. The definition of $\mathcal{T}_{\text{hol}}$ is similar to that of $\mathcal{T}_{\text{smth}}$. Namely, we consider the algebra

$$\mathcal{H}_{\text{hol}} = \left\{ a = (a_{ij})_{i,j \in \mathbb{Z}^+} : a_{ij} \in \mathbb{C}, \|a\|_n = \sum_{i,j} |a_{ij}| n^{i+j} < \infty \forall n \in \mathbb{Z}^+ \right\}$$

of holomorphic compact operators, and define $\mathcal{T}_{\text{hol}}$ as follows:

$$\mathcal{T}_{\text{hol}} = \{ T_f + K : f \in \mathcal{O}(\mathbb{C}^\times), K \in \mathcal{H}_{\text{hol}} \}.$$  

Here $\mathcal{O}(\mathbb{C}^\times)$ denotes the algebra of holomorphic functions on $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. Similarly to $\mathcal{T}_{\text{smth}}$, $\mathcal{T}_{\text{hol}}$ becomes a Fréchet-Arens-Michael algebra if we identify the underlying vector space of $\mathcal{T}_{\text{hol}}$ with $\mathcal{O}(\mathbb{C}^\times) \oplus \mathcal{H}_{\text{hol}}$ via the isomorphism $(f, K) \mapsto T_f + K$.

The following result is probably due to Meyer (see [29], where a much more general result is proved). Since this is the main motivation for the present paper, we give a proof here for the reader’s convenience. This elementary proof is due to Aristov (private communication).

**Proposition 2.3.** $\mathcal{T}_{\text{alg}}$ is quasi-free.

**Proof.** By Remark 2.1, it suffices to show that $\mathcal{T}_{\text{alg}}$ is quasi-free in the purely algebraic sense. Let

$$0 \rightarrow I \rightarrow R \xrightarrow{p} \mathcal{T}_{\text{alg}} \rightarrow 0$$

be a square-zero extension of $\mathcal{T}_{\text{alg}}$. Choose $a, b \in R$ such that $p(a) = u$ and $p(b) = v$. Then $ab = 1 + c$ for some $c \in I$. Since $I^2 = 0$, we have $ab(1 - c) = 1 - c^2 = 1$. Letting $b' = b(1 - c)$, we see that $ab' = 1$. By the universal property of $\mathcal{T}_{\text{alg}}$, there is a unique homomorphism $j : \mathcal{T}_{\text{alg}} \rightarrow R$ such that $j(u) = a$ and $j(v) = b'$. Since $p(a) = u$ and $p(b') = p(b - bc) = p(b) = v$, we have $pj = 1_{\mathcal{T}_{\text{alg}}}$. Thus (2.3) splits, which completes the proof. \hfill \Box

3. A family of locally convex Toeplitz algebras

In this section we construct a family $\{ \mathcal{T}_{P,Q} \}$ of locally convex subalgebras of $\mathcal{T}$, where $P$ and $Q$ are Köthe sets satisfying some natural conditions. We will also show that $\mathcal{T}_{\text{alg}}$, $\mathcal{T}_{\text{smth}}$, and $\mathcal{T}_{\text{hol}}$ are special cases of our construction.

Let $I$ be any set, and let $P$ be a set of nonnegative real-valued functions on $I$. For $p \in P$ and $i \in I$, we write $p_i$ for $p(i)$. Recall that $P$ is a Köthe set on $I$ if the following axioms are satisfied (see, e.g., [32]):

$$\forall i \in I \quad \exists p \in P \quad p_i > 0; \quad \text{(P1)}$$

$$\forall p, q \in I \quad \exists r \in P \quad \max\{p_i, q_i\} \leq r_i \quad (i \in I). \quad \text{(P2)}$$

Given a Köthe set $P$, the Köthe space $\lambda(P) = \lambda(I, P)$ is defined as follows [loc. cit.]:

$$\lambda(P) = \left\{ x = (x_i) \in \mathbb{C}^I : \|x\|_P = \sum_i |x_i| p_i < \infty \quad \forall p \in P \right\}.$$  

This is a complete locally convex space with the topology determined by the family $\{ \| \cdot \|_p : p \in P \}$ of seminorms. Clearly, $\lambda(P)$ is a Fréchet space if and only if $P$ contains an at most countable cofinal subset.

For each $i \in I$ denote by $e_i$ the function on $I$ which is 1 at $i$ and 0 elsewhere. Obviously, $x = \sum_i x_i e_i$ for each $x \in \lambda(P)$.
Given Köthe sets $P \subset [0, +\infty)^I$ and $Q \subset [0, +\infty)^J$, let $P \times Q$ denote the Köthe set on $I \times J$ consisting of all functions of the form $(i, j) \mapsto p_i q_j \,(p \in P, \, q \in Q)$. By [31], there exists a topological isomorphism

$$\lambda(P) \widehat{\otimes} \lambda(Q) \cong \lambda(P \times Q), \quad e_i \otimes e_j \mapsto e_{(i,j)},$$

(3.1)

Moreover, if we identify $\lambda(P) \widehat{\otimes} \lambda(Q)$ with $\lambda(P \times Q)$ via (3.1), then for each $p \in P$ and $q \in Q$ the seminorm $\| \|_{(p,q)}$ on $\lambda(P \times Q)$ is identified with the projective tensor seminorm $\| \|_{p \otimes \pi} \| \cdot \|_q$ on $\lambda(P) \widehat{\otimes} \lambda(Q)$.

From now on, we concentrate on Köthe sets on $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. As a first step towards constructing the locally convex Toeplitz algebras $\mathcal{T}_{P,Q}$, we define auxiliary power series algebras, which will play the role of “building blocks” for $\mathcal{T}_{P,Q}$. More exactly, we would like to make $\lambda(P)$ into an algebra under convolution. This requires one more condition on the Köthe set $P$.

**Definition 3.1.** Let $P$ be a Köthe set on $\mathbb{Z}_+$. We say that $P$ is a weighted set if for each $p \in P$ we have $p_0 = 1$, and if for each $p \in P$ there exist $p' \in P$ and $C > 0$ such that

$$p_{i+j} \leq C p'_i p'_j \quad (i, j \in \mathbb{Z}_+).$$

If the above property holds with $p' = p$ and $C = 1$, then we say that $P$ is an $m$-weighted set.

**Proposition 3.2.** For each weighted set $P$ on $\mathbb{Z}_+$ there exists a unique multiplication on $\lambda(P)$ satisfying $e_{i+j} = e_i e_j \,(i, j \in \mathbb{Z}_+)$ and making $\lambda(P)$ into a $\widehat{\otimes}$-algebra. If, moreover, $P$ is an $m$-weighted set, then each seminorm $\| \cdot \|_p$ is submultiplicative, and so $\lambda(P)$ is an Arens-Michael algebra.

The proof is straightforward and is therefore omitted. Observe also that $\lambda(P)$ is unital and that $e_0$ is the identity of $\lambda(P)$.

For our purposes, it will be convenient to let $z = e_1$ and to denote the algebra $\lambda(P)$ with the above multiplication by $\lambda(z, P)$ (so that $z$ plays the role of a “formal variable”). Thus we have

$$\lambda(z, P) = \left\{ a = \sum_{i=0}^{\infty} a_i z^i : a_i \in \mathbb{C}, \quad \|a\|_P = \sum_{i} |a_i| p_i < \infty \quad \forall \, p \in P \right\}.$$

Obviously, we have a chain of algebra embeddings $\mathbb{C}[z] \subset \lambda(z, P) \subset \mathbb{C}[[z]]$, and both embeddings are dense.

**Example 3.3.** Let $P = \{p^{(1)} \, p^{(2)}, \ldots\}$, where $p^{(k)} = (1, \ldots, 1, 0, 0, \ldots)$ (with 1 repeated $k$ times). Clearly, $P$ is an $m$-weighted set, and we have $\lambda(z, P) = \mathbb{C}[[z]]$, both algebraically and topologically.

**Example 3.4.** Let $P = \{p^{(1)} \, p^{(2)}, \ldots\}$, where $p^{(k)}_n = (1+n)^k \,(n \in \mathbb{Z}_+)$. It is easily seen that $P$ is an $m$-weighted set, and that $\lambda(z, P)$ is topologically isomorphic (via the Fourier transform) to the subalgebra of $C^\infty(\mathbb{T})$ consisting of those smooth functions whose negative Fourier coefficients vanish.

**Example 3.5.** Let $R \in (0, +\infty]$, and let $P = \{p^{(r)} : 0 < r < R\}$, where $p^{(r)}_n = r^n$ for all $n \in \mathbb{Z}_+$. Obviously, $P$ is an $m$-weighted set, and $\lambda(z, P)$ is nothing but the algebra $\mathcal{O}(\mathbb{D}_R)$ of holomorphic functions on the open disk $\mathbb{D}_R = \{z \in \mathbb{C} : |z| < R\}$. 


Thus for each pair of monomials $i,j \in \mathbb{Z}_+$. Indeed, let $p'_0 = 1$, and assume that we have already constructed $p'_0, \ldots, p'_k$ satisfying (3.2) whenever $i,j \leq k$. Clearly, there exists $p'_{k+1} > 0$ such that $p_{i+k+1} \leq p'_0p'_{k+1}$ for all $i \leq k+1$. Hence (3.2) holds for all $i,j \leq k+1$, and the induction argument completes the proof. Thus $P$ is a weighted set. We clearly have $\lambda(z,P) = \mathbb{C}[z]$, and the topology on $\lambda(z,P)$ determined by $P$ is the strongest locally convex topology. It is well known that $\mathbb{C}[z]$ is not locally $m$-convex, and so $P$ is not an $m$-weighted set.

Now we are ready to define our family $\{T_{P,Q}\}$ of locally convex Toeplitz algebras. Let us identify the underlying vector space of $T_{\text{alg}}$ with $\mathbb{C}[v] \otimes \mathbb{C}[u]$ by sending each monomial $v^iu^j$ to $v^i \otimes u^j$ ($i,j \in \mathbb{Z}_+$). Given weighted sets $P$ and $Q$, we let $T_{P,Q} = \lambda(v,Q) \hat{\otimes} \lambda(u,P)$. Clearly, $T_{\text{alg}}$ is a dense vector subspace of $T_{P,Q}$. By using (3.1), we see that

$$T_{P,Q} = \left\{ a = \sum_{i,j \in \mathbb{Z}_+} c_{ij}v^iu^j : c_{ij} \in \mathbb{C}, \|a\|_{q,p} = \sum_{i,j \in \mathbb{Z}_+} |c_{ij}|q_i p_j < \infty \forall q \in Q, \forall p \in P \right\}.$$  

In order to make $T_{P,Q}$ into a $\hat{\otimes}$-algebra containing $T_{\text{alg}}$ as a subalgebra, we need to impose one more condition on the Köthe sets $P$ and $Q$.

**Definition 3.7.** Let $P$ be a weighted set. We say that $P$ is **monotone** if for each $p \in P$ and each $i \in \mathbb{Z}_+$ we have $p_i \leq p_{i+1}$.

**Proposition 3.8.** For each pair $P,Q$ of monotone weighted sets on $\mathbb{Z}_+$, there exists a unique jointly continuous multiplication on $T_{P,Q}$ that extends the multiplication on $T_{\text{alg}}$. If, moreover, $P$ and $Q$ are $m$-weighted sets, then $T_{P,Q}$ is an Arens-Michael algebra.

**Proof.** Given $p \in P$ and $q \in Q$, choose $p' \in P$, $q' \in Q$, and $C > 0$ such that for each $i,j \in \mathbb{Z}_+$ we have

$$p_{i+j} \leq Cp'_i p'_j, \quad q_{i+j} \leq Cq'_i q'_j.$$  

Without loss of generality, we may assume that $p \leq p'$ and $q \leq q'$. For each pair $v^i u^j$ and $v^k u^\ell$ of monomials in $T_{\text{alg}}$, we clearly have

$$(v^i u^j)(v^k u^\ell) = \begin{cases} v^{i+j-k+\ell} & \text{if } j \geq k; \\ v^{i+k-j+\ell} & \text{if } j \leq k. \end{cases}$$  

(3.3)

If $j \geq k$, then

$$\|(v^i u^j)(v^k u^\ell)\|_{q,p} = \|v^{i+j-k+\ell}\|_{q,p} = q_i p_{j-k+\ell} \leq q_i p_{j+\ell} \leq Cq'_i q'_k p'_j p'_\ell = C\|v^i u^j\|_{q',p'}\|v^k u^\ell\|_{q',p'}.$$  

On the other hand, if $j \leq k$, then

$$\|(v^i u^j)(v^k u^\ell)\|_{q,p} = \|v^{i+k-j+\ell}\|_{q,p} = q_i p_{k-j+\ell} \leq q_i p_{k+\ell} \leq Cq'_i q'_k p'_j p'_\ell = C\|v^i u^j\|_{q',p'}\|v^k u^\ell\|_{q',p'}.$$  

Thus for each pair of monomials $a,b \in T_{\text{alg}}$ we have

$$\|ab\|_{q,p} \leq C\|a\|_{q',p'}\|b\|_{q',p'}.$$  

(3.4)
Observe now that, if we take any $a \in T_{\text{alg}}$ and decompose it as a finite sum $a = \sum_i a_i$ of linearly independent monomials $a_i$, then we have

$$\|a\|_{q,p} = \sum_i \|a_i\|_{q,p}. \quad (3.5)$$

This implies that (3.4) actually holds for each $a, b \in T_{\text{alg}}$. Therefore the multiplication on $T_{\text{alg}}$ is jointly continuous for the topology inherited from $T_{P,Q}$, and so it uniquely extends by continuity to $T_{P,Q}$. Finally, if both $P$ and $Q$ are $m$-weighted sets, then we can repeat the above argument with $p' = p, q' = q$, and $C = 1$, which implies that each seminorm $\| \cdot \|_{q,p}$ is submultiplicative. This completes the proof. □

Let us now show that $T_{\text{smth}}$ and $T_{\text{hol}}$ are special cases of the above construction.

**Proposition 3.9.** Let $P = \{p^{(1)}, p^{(2)}, \ldots\}$, where $p^{(k)}_n = (1 + n)^k$ for all $k \in \mathbb{N}, n \in \mathbb{Z}_+$. (see Example 3.4). Then the identity map of $T_{\text{alg}}$ uniquely extends to a topological algebra isomorphism $T_{P,P} \cong T_{\text{smth}}$.

**Proof.** For each $i, j \in \mathbb{Z}_+$, let $e_{i,j} \in M_\infty$ denote the respective matrix unit (i.e., the matrix whose $(i,j)$th entry is 1 and the other entries are 0). The restriction of (2.4) to $\mathbb{C}[z, z^{-1}] \oplus M_\infty$ is a vector space isomorphism between $\mathbb{C}[z, z^{-1}] \oplus M_\infty$ and $T_{\text{alg}}$, which acts on the basis $\{z^k, e_{i,j} : k \in \mathbb{Z}, i, j \in \mathbb{Z}_+\}$ as follows (see [3]):

$$z^k \mapsto v^k \quad (k \geq 0), \quad z^{-k} \mapsto u^k \quad (k \geq 0), \quad e_{i,j} \mapsto v^i(1 - vu)u^j. \quad (3.6)$$

Identifying the underlying vector spaces of $\mathbb{C}[z, z^{-1}] \oplus M_\infty$ and $T_{\text{alg}}$ via the above isomorphism, we conclude from (1.0) that

$$v^iu^j = \begin{cases} z^{i-j} - (e_{i-j,0} + e_{i-j+1,1} + \cdots + e_{i-1,j-1}) & \text{if } i \geq j, \\ z^{i-j} - (e_{0,j-i} + e_{1,j-i+1} + \cdots + e_{i-1,j-1}) & \text{if } i < j. \end{cases} \quad (3.7)$$

By definition, the topology on $T_{\text{alg}}$ inherited from $T_{P,P}$ is given by the family $\{\| \cdot \|_k : k \in \mathbb{Z}_+\}$ of norms, where

$$\left\| \sum_{i,j \in \mathbb{Z}_+} c_{ij}v^iu^j \right\|_k = \sum_{i,j \in \mathbb{Z}_+} |c_{ij}|(1 + i)^k(1 + j)^k \quad (c_{ij} \in \mathbb{C}).$$

Since $T_{\text{smth}} = C^\infty(\mathbb{T}) \oplus \mathcal{K}_{\text{smth}}$ as locally convex spaces, we see that the topology on $T_{\text{alg}}$ inherited from $T_{\text{smth}}$ is given by the family $\{\| \cdot \|'_k : k \in \mathbb{Z}_+\}$ of norms, where

$$\left\| \sum_{i,j \in \mathbb{Z}_+} a_{ij}e_{i,j} + \sum_{p \in \mathbb{Z}} b_p z^p \right\|'_k = \sum_{i,j \in \mathbb{Z}_+} |a_{ij}|(1 + i + j)^k + \sum_{p \in \mathbb{Z}} |b_p|(1 + |p|)^k.$$

Thus, to complete the proof, it suffices to show that the families

$$\{\| \cdot \|_k : k \in \mathbb{Z}_+\} \quad \text{and} \quad \{\| \cdot \|'_k : k \in \mathbb{Z}_+\} \quad (3.8)$$

of norms are equivalent on $T_{\text{alg}}$.

If $i, j \in \mathbb{Z}_+$ and $i \geq j$, then we see from (3.7) that

$$\|v^iu^j\|'_k = (1 + i - j)^k + (1 + i - j)^k + (3 + i - j)^k + \cdots + (|1 + i - j|^k) \leq (j + 1)(1 + i + j)^k \leq (1 + i + j)^{k+1} \leq (1 + i)^{k+1}(1 + j)^{k+1} = \|v^iu^j\|_{k+1}. $$
A similar argument shows that \( \|v^iu^j\|_k \leq \|v^iu^j\|_{k+1} \) whenever \( i, j \in \mathbb{Z}_+ \) and \( i < j \). Now for each \( a = \sum_{i,j} c_{ij}v^iu^j \in \mathcal{T}_{alg} \) we have
\[
\|a\|_k' \leq \sum_{i,j} |c_{ij}| \|v^iu^j\|_k' \leq \sum_{i,j} |c_{ij}| \|v^iu^j\|_{k+1} = \|a\|_{k+1}.
\]
(3.9)

On the other hand, for each \( i, j \in \mathbb{Z}_+ \) we have
\[
\|e_{i,j}\|_k = \|v^iu^j - v^{i+1}u^{j+1}\|_k = (1+i)^k(1+j)^k + (2+i)^k(2+j)^k \\
\leq (1+i+j)^{2k} + 4^k(1+i+j)^{2k} = (4^k + 1)\|e_{i,j}\|_{2k}.
\]
(3.10)

Also, for each \( p \in \mathbb{Z}_+ \) we have
\[
\|z^p\|_k = \|v^p\|_k = (1+p)^k = \|z^p\|'_k,
\]
and, similarly, \( \|z^{-p}\|_k = \|z^{-p}\|'_k \). Together with (3.10), this implies that for each \( a = \sum_{i,j} a_{ij}e_{i,j} + \sum_p b_pz^p \in \mathcal{T}_{alg} \) we have
\[
\|a\|_k \leq (4^k + 1) \sum_{i,j} |a_{ij}|\|e_{i,j}\|_{2k} + \sum_p |b_p|\|z^p\|_{2k} \leq (4^k + 1)\|a\|_{2k}.
\]

Comparing this with (3.9), we conclude that the families (3.8) are indeed equivalent. This completes the proof. \( \square \)

**Proposition 3.10.** Let \( P = \{p^{(1)}, p^{(2)}, \ldots\} \), where \( p^{(k)}_n = k^n \) for all \( k \in \mathbb{N}, n \in \mathbb{Z}_+ \) (see Example 3.5). Then the identity map of \( \mathcal{T}_{alg} \) uniquely extends to a topological algebra isomorphism \( \mathcal{T}_{P,P} \cong \mathcal{T}_{hol} \).

We omit the proof, because it is similar to that of Proposition 3.9.

**Remark 3.11.** Let \( P \) denote the collection of all positive nondecreasing sequences \( p \in (0, +\infty)^{\mathbb{Z}_+} \) satisfying \( p_0 = 1 \). It easily follows from Example 3.6 that \( \mathcal{T}_{P,P} = \mathcal{T}_{alg} \), both algebraically and topologically.

**Remark 3.12.** Proposition 3.10 easily implies that \( \mathcal{T}_{hol} \) is the Arens-Michael envelope of \( \mathcal{T}_{alg} \) (see, e.g., [17, Chap. V] or [33] for general information on Arens-Michael envelopes). In other words, \( \mathcal{T}_{hol} \) is the universal Arens-Michael algebra with two distinguished elements \( u \) and \( v \) satisfying \( uv = 1 \) (cf. [29, Theorem 2.14]). For a similar characterization of \( \mathcal{T}_{smth} \), see [6, Satz 6.1].

### 4. Calculation of \( \Omega^1\mathcal{T}_{alg} \)

In this section, we calculate explicitly the bimodule \( \Omega^1\mathcal{T}_{alg} \) of noncommutative differential 1-forms over the algebraic Toeplitz algebra. This result will be applied in Section 6 to finding sufficient conditions for the quasi-freeness of \( \mathcal{T}_{P,Q} \).

**Lemma 4.1.** Let \( A \) be an algebra and \( p \in A \) an idempotent. Then for each left \( A \)-module \( M \) there exists a vector space isomorphism
\[
_\varphi(\mathcal{A}h)(Ap, M) \to pM, \quad \varphi \mapsto \varphi(p).
\]
(4.1)

**Proof.** For each \( \varphi \in _\varphi(\mathcal{A}h)(Ap, M) \) we have \( (1-p)\varphi(p) = \varphi(0) = 0 \), hence \( \varphi(p) \in pM \). Now observe that for each \( m \in pM \) we have an \( A \)-module morphism \( \varphi_m : Ap \to M \) given by \( \varphi_m(ap) = am \). Indeed, if \( ap = 0 \), then \( am = (a-ap)m = a(1-p)m = 0 \), so \( \varphi_m \) is well defined. Now it is easy to see that the map \( m \mapsto \varphi_m \) is the inverse of (4.1). \( \square \)
From now on, we let \( e' = vu \in \mathcal{T}_{\text{alg}} \) and \( e = 1 - vu \in \mathcal{T}_{\text{alg}} \). Clearly, both \( e' \) and \( e \) are idempotents.

**Theorem 4.2.** Let \( A = \mathcal{T}_{\text{alg}} \). There exists an \( A \)-bimodule isomorphism

\[
\Omega^1 A \cong (A \otimes A) \oplus (Ae \otimes A). \tag{4.2}
\]

Under the above identification, the universal derivation \( d_A: A \to (A \otimes A) \oplus (Ae \otimes A) \) acts as follows:

\[
d_A(u) = (1 \otimes 1, 0), \quad d_A(v) = (-v \otimes v, e \otimes 1). \tag{4.3}
\]

**Proof.** For each \( A \)-bimodule \( M \), every derivation \( d: A \to M \) is uniquely determined by the elements \( m = du \) and \( n = dv \). Conversely, given \((m, n) \in M \times M \), we conclude from (4.2) that a derivation \( d: A \to M \) taking \( u \) to \( m \) and \( v \) to \( n \) exists if and only if we have \((u, m)(v, n) = (1, 0)\) in \( A \times M \), i.e., if and only if

\[
un + mv = 0. \tag{4.4}
\]

Thus we have a vector space isomorphism

\[
\text{Der}(A, M) \cong \left\{ (m, n) \in M \times M : un + mv = 0 \right\}, \quad d \mapsto (du, dv). \tag{4.5}
\]

Now observe that for each \( m \in M \) the element \( n = -vmv \) satisfies (4.4). Also, for each \( \ell \in eM \) we have \( u\ell = vu \ell = u\ell \ell = 0 \), so \( n = -vmv + \ell \) also satisfies (4.4). Thus we have a linear map

\[
\varphi: M \oplus eM \to \left\{ (m, n) \in M \times M : un + mv = 0 \right\}, \quad (m, \ell) \mapsto (m, -vmv + \ell).
\]

On the other hand, if \((m, n) \in M \times M \) satisfies (4.4), then

\[
e'(n + vmv) = vun + vuvmv = v(un + mv) = 0,
\]

so \( n + vmv \in eM \), and we have a linear map

\[
\psi: \left\{ (m, n) \in M \times M : un + mv = 0 \right\} \to M \oplus eM, \quad (m, n) \mapsto (m, n + vmv).
\]

Clearly, \( \varphi \psi = 1 \) and \( \psi \varphi = 1 \), whence \( \varphi \) and \( \psi \) are isomorphisms. Thus we have

\[
\left\{ (m, n) \in M \times M : un + mv = 0 \right\} \cong M \oplus eM. \tag{4.6}
\]

Let now \( p = e \otimes 1 \in A^e \). Since \( p \) is an idempotent, Lemma 4.1 yields vector space isomorphisms

\[
M \oplus eM = M \oplus pM \cong _{A^e}h(A^e, M) \oplus _{A^e}h(A^e p, M)
\]

\[
\cong _{A^e}h(A^e \oplus A^e p, M) \cong h_A((A \otimes A) \oplus (Ae \otimes A), M). \tag{4.7}
\]

Since all the above isomorphisms are natural in \( M \), we conclude from (4.3), (4.6), and (4.7) that \( \Omega^1 A \cong (A \otimes A) \oplus (Ae \otimes A) \), as required.

To complete the proof, let \( M = (A \otimes A) \oplus (Ae \otimes A) \), and observe that the universal derivation \( d_A: A \to M \) corresponds to the identity map \( 1_M \) under the composition of isomorphisms (4.2), (4.6), and (4.7). Reading (4.7) from right to left, we see the pair \((m, \ell) \in M \oplus eM\) corresponding to \( 1_M \) is given by \( m = (1 \otimes 1, 0) \) and \( \ell = (0, e \otimes 1) \). Hence

\[
(d_Au, d_Av) = \varphi(m, \ell) = (m, -vmv + \ell) = ((1 \otimes 1, 0), (-v \otimes v, e \otimes 1)),
\]

as required. \( \square \)
5. Homological epimorphisms from quasi-free algebras

Our next goal is to give a convenient criterion for a \( \hat{\otimes} \)-algebra homomorphism \( \varphi: A \to B \) to be a homological epimorphism, assuming that \( A \) is quasi-free and that \( \varphi \) is an epimorphism. In the next section, we apply this result to the embedding of \( T_{\text{alg}} \) into \( T_{P,Q} \).

Let \( A \) and \( B \) be \( \hat{\otimes} \)-algebras, and let \( \varphi: A \to B \) be a continuous homomorphism. The universal property of \( \Omega^1 A \) yields an \( A-\hat{\otimes} \)-bimodule morphism \( \Omega^1 A \to \Omega^1 B \) uniquely determined by \( d_A a \mapsto d_B(\varphi(a)) \). Tensoring \( \Omega^1 A \) by \( B \) on both sides, we get a \( B-\hat{\otimes} \)-bimodule morphism

\[
\alpha(\varphi): B \hat{\otimes} \Omega^1 A \hat{\otimes} B \to \Omega^1 B, \quad 1 \otimes d_A a \otimes 1 \mapsto d_B(\varphi(a)). \tag{5.1}
\]

**Theorem 5.1.** Let \( \varphi: A \to B \) be a \( \hat{\otimes} \)-algebra epimorphism. Suppose that \( A \) is quasi-free. Then the following conditions are equivalent:

(i) \( \alpha(\varphi) \) is an isomorphism;
(ii) there exists a derivation \( D: B \to B \hat{\otimes} A \Omega^1 A \hat{\otimes} A B \) making the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{d_A} & \Omega^1 A \\
\varphi \downarrow & & \downarrow \beta \\
B & \xrightarrow{D} & B \hat{\otimes} A \Omega^1 A \hat{\otimes} A B
\end{array}
\]

commute, where \( \beta \) is given by \( \omega \mapsto 1 \otimes \omega \otimes 1 \);
(iii) \( \varphi \) is a homological epimorphism.

*If the above conditions are satisfied, then \( B \) is quasi-free.*

To prove Theorem 5.1, we need the following simple lemma.

**Lemma 5.2.** Let \( \varphi: A \to B \) be a \( \hat{\otimes} \)-algebra epimorphism, and let \( M \) be a \( B-\hat{\otimes} \)-bimodule. Suppose that \( d_1, d_2: B \to M \) are derivations such that \( d_1 \varphi = d_2 \varphi \). Then \( d_1 = d_2 \).

**Proof.** Define \( \hat{\otimes} \)-algebra homomorphisms \( \psi_1, \psi_2: B \to B \times M \) by \( \psi_i(b) = (b, d_i(b)) \) \((b \in B, i = 1, 2)\); see (2.1). We clearly have \( \psi_1 \varphi = \psi_2 \varphi \), whence \( \psi_1 = \psi_2 \) and \( d_1 = d_2 \). \( \square \)

**Proof of Theorem 5.1.** (i) \( \implies \) (ii). Define a derivation \( D: B \to B \hat{\otimes} A \Omega^1 A \hat{\otimes} A B \) by \( D = \alpha^{-1} d_B \), where \( \alpha = \alpha(\varphi) \). To show that (5.2) commutes, it suffices to prove that \( \alpha D \varphi = \alpha \beta d_A \), which is equivalent to \( d_B \varphi = \alpha \beta d_A \). However, the latter relation is precisely the definition of \( \alpha \) (see (5.1)).

(ii) \( \implies \) (i). The universal property of \( \Omega^1 B \) yields a \( B-\hat{\otimes} \)-bimodule morphism

\[
\tau: \Omega^1 B \to B \hat{\otimes} A \Omega^1 A \hat{\otimes} A B, \quad \tau d_B = D.
\]

We claim that \( \tau \) is the inverse of \( \alpha \). Indeed, for each \( a \in A \) we have

\[
(\tau \alpha)(1 \otimes d_A a \otimes 1) = \tau(d_B(\varphi(a))) = D(\varphi(a)) = 1 \otimes d_A a \otimes 1.
\]

Since \( B \hat{\otimes} A \Omega^1 A \hat{\otimes} A B \) is generated (as a \( B-\hat{\otimes} \)-bimodule) by elements of the form \( 1 \otimes d_A a \otimes 1 \) \((a \in A)\), we conclude that \( \tau \alpha = 1 \). To prove that \( \alpha \tau = 1 \), it suffices to show that \( \alpha \tau d_B = d_B \) (by the universal property of \( \Omega^1 B \)). Since both \( d_B \) and \( \alpha \tau d_B \) are derivations, Lemma 5.2 implies that \( \alpha \tau d_B = d_B \) whenever \( \alpha \tau d_B \varphi = d_B \varphi \). For each \( a \in A \), we have

\[
(\alpha \tau d_B \varphi)(a) = (\alpha D \varphi)(a) = (\alpha \beta d_A)(a) = (d_B \varphi)(a).
\]
In view of the above remarks, this proves that $\alpha \tau = 1$. Hence $\alpha$ is an isomorphism.

(i) $\iff$ (iii). Since $A$ is quasi-free, we see that (2.2) is a projective resolution of $A$ in $A$-mod-$A$. Tensoring (2.2) by $B$ on both sides, we conclude that $\varphi$ is a homological epimorphism if and only if the resulting sequence

$$
0 \rightarrow B \hat{\otimes} A \Omega^1 A \hat{\otimes} A B \rightarrow B \hat{\otimes} B \rightarrow B \rightarrow 0
$$

is admissible (where $k_B$ is induced by $j_A$ in the obvious way). Clearly, (5.3) fits into the commutative diagram

$$
\begin{array}{cccc}
0 & \rightarrow & B \hat{\otimes} A \Omega^1 A \hat{\otimes} A B & k_B \\
& & \downarrow \alpha & \\
0 & \rightarrow & B \hat{\otimes} B & \rightarrow B \rightarrow 0
\end{array}
$$

Since the bottom row of (5.4) is admissible, we conclude that (5.3) is admissible if and only if $\alpha$ is an isomorphism.

To complete the proof, recall that (iii) together with the assumption that $A$ is quasi-free implies that $B$ is quasi-free as well (see Remark 2.2). □

Remark 5.3. Observe that we have (i) $\implies$ (ii) for any $\hat{\otimes}$-algebra homomorphism, and (ii) $\implies$ (i) for any $\hat{\otimes}$-algebra epimorphism (see the proof of Theorem 5.1). In both cases, the assumption that $A$ is quasi-free is inessential.

6. The main result

Given two sequences $p, q \in [0, +\infty)^{\mathbb{Z}_+}$, we define their convolution $p \ast q \in [0, +\infty)^{\mathbb{Z}_+}$ by

$$(p \ast q)_k = \sum_{i+j=k} p_i q_j \quad (k \in \mathbb{Z}_+).$$

If $P$ and $Q$ are Köthe sets on $\mathbb{Z}_+$, we let

$$P \ast Q = \{p \ast q : p \in P, q \in Q\}.$$ 

Clearly, $P \ast Q$ is a Köthe set as well. Finally, we say that $P$ is dominated by $Q$ and write $P \prec Q$ if for each $p \in P$ there exist $q \in Q$ and $C > 0$ such that $p_i \leq Cq_i$ ($i \in \mathbb{Z}_+$).

**Theorem 6.1.** Let $P$ and $Q$ be monotone weighted sets on $\mathbb{Z}_+$ such that $P \ast P \prec P$ and $Q \ast Q \prec Q$. Then the embedding of $T_{\text{alg}}$ into $T_{P,Q}$ is a homological epimorphism. As a consequence, $T_{P,Q}$ is quasi-free.

**Proof.** Let $A = T_{\text{alg}}$ and $B = T_{P,Q}$. To prove the result, we will show that the embedding of $A$ into $B$ satisfies condition (ii) of Theorem 5.1. Using Theorem 1.2, we see that diagram (5.2) looks as follows:

$$
\begin{array}{ccc}
A & \xrightarrow{d_A} & (A \otimes A) \oplus (Ae \otimes A) \\
\downarrow & & \downarrow \\
B & \xrightarrow{D} & (B \hat{\otimes} B) \oplus (Be \hat{\otimes} B)
\end{array}
$$

Hence, to show that $D$ exists, it suffices to prove the continuity of $d = d_A$ for the topologies on $A$ and $(A \otimes A) \oplus (Ae \otimes A)$ inherited from $B$ and $(B \hat{\otimes} B) \oplus (Be \hat{\otimes} B)$, respectively.
Using (3.1), we can identify the underlying topological vector space of $B \otimes B$ with $\lambda(Q \times P \times Q \times P)$. Since $Be$ is a topological direct summand of $B$, we can interpret $Be \otimes B$ as a subspace of $B \otimes B$. Given $q \in Q$ and $p \in P$, define a seminorm $\| \cdot \|_{q,p}$ on $(B \otimes B) \oplus (Be \otimes B)$ by

$$
\| (c, d) \|_{q,p} = \| c \|_{q,p,q,p} + \| d \|_{q,p,q,p} \quad (c \in B \otimes B, \; d \in Be \otimes B).
$$

Clearly, $\{ \| \cdot \|_{q,p} : q \in Q, \; p \in P \}$ is a defining family of seminorms on $(B \otimes B) \oplus (Be \otimes B)$. Let $i, j \in Z_+$. Using (4.3), we see that

$$
\begin{align*}
\| (c, d) \|_{q,p} &= \| c \|_{q,p,q,p} + \| d \|_{q,p,q,p} \\
&= (i-1) \sum_{k=0}^{i-1} v^k \cdot dv \cdot v^{i-k-1} w^j = \sum_{k=0}^{i-1} (-v^{k+1} \otimes v^{i-k-1} u^j, v^k \otimes v^{i-k-1} u^j) \\
&= \sum_{k=0}^{i-1} (-v^{k+1} \otimes v^{i-k-1} u^j, v^k \otimes v^{i-k-1} u^j - v^{k+1} u \otimes v^{i-k-1} u^j).
\end{align*}
$$

Similarly,

$$
\begin{align*}
\| v^i d(u^j) \|_{q,p} &= \sum_{\ell=0}^{j-1} v^i u^\ell \cdot dv \cdot u^{j-\ell-1} = \sum_{\ell=0}^{j-1} (v^i u^\ell \otimes u^{j-\ell-1}, 0).
\end{align*}
$$

Take any $p \in P$, $q \in Q$ and find $C > 0$, $p' \in P$, and $q' \in Q$ such that $p * p \leq Cp'$, $q * q \leq Cq'$. Choose also $q'' \in Q$ and $C_1 > 0$ such that $q''_k q''_\ell \leq C_1 q''_k q''_\ell$ ($k, \ell \in Z_+$). Without loss of generality, we may also assume that $p \leq p'$ and $q \leq q'$. Using (5.1), we obtain

$$
\| d(v^i u^j) \|_{q,p} = \sum_{k=0}^{i-1} (q_{k+1} q_{i-k} p_\ell + q_k q_{i-k-1} p_\ell + q_{k+1} p_1 q_{i-k-1} p_\ell) \\
\leq p_j ((q * q)_i + (q * q)_{i-1} + p_1(q * q)_i) \\
\leq C p_j (q''_i q''_j + q''_\ell + p_1(q''_j) \leq C p_j (C_1 q''_i q''_j + q''_\ell + p_1(q''_j) = C_2 \| v^i u^j \|_{q'', p'},
$$

where $C_2 = C(C_1 q''_i + p_1 + 1)$. Similarly, (6.2) implies that

$$
\| v^i d(u^j) \|_{q,p} = \sum_{\ell=0}^{j-1} q_i p_\ell p_{j-\ell-1} = q_i (p * p)_{j-1} \leq C q_i p''_j = C \| v^i u^j \|_{q'', p'}.
$$

Combining (3.3) and (3.4), we see that

$$
\| d(v^i u^j) \|_{q,p} = \| d(v^i u^j) + v^i d(u^j) \|_{q,p} \leq 2C_2 \| v^i u^j \|_{q'', p'}.
$$

Taking into account (3.5), we conclude that

$$
\| d(a) \|_{q,p} \leq 2C_2 \| a \|_{q'', p'} \quad (a \in T_{alg}).
$$

Hence $d$ is continuous for the topologies determined by $P$ and $Q$. In view of the above remarks, this implies the existence of $D$ and completes the proof. \hfill \square

**Corollary 6.2.** The embedding of $T_{alg}$ into $T_{smth}$ is a homological epimorphism. As a consequence, $T_{smth}$ is quasi-free.
Proof. By Proposition 3.10, we have $\mathcal{T}_{\text{smth}} = \mathcal{T}_{P,P}$, where $P = \{p^{(k)}\}_{k \in \mathbb{N}}$ and $p^{(k)}_n = (1+n)^k$ for all $k \in \mathbb{N}$, $n \in \mathbb{Z}_+$. Given $k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we have

$$ (p^{(k)} \ast p^{(k)})_n = \sum_{i+j=n} (1+i)^k (1+j)^k \leq (1+n)(1+n)^k = p^{(2k+1)}_n. $$

Hence $P \ast P \preceq P$, and Theorem 6.1 implies the result. \qed

Corollary 6.3. The embedding of $\mathcal{T}_{\text{alg}}$ into $\mathcal{T}_{\text{hol}}$ is a homological epimorphism. As a consequence, $\mathcal{T}_{\text{hol}}$ is quasi-free.

Proof. By Proposition 3.11, we have $\mathcal{T}_{\text{hol}} = \mathcal{T}_{P,P}$, where $P = \{p^{(k)}\}_{k \in \mathbb{N}}$ and $p^{(k)}_n = n^k$ for all $k \in \mathbb{N}$, $n \in \mathbb{Z}_+$. Given $k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we have

$$ (p^{(k)} \ast p^{(k)})_n = \sum_{i+j=n} k^i n^j = (n+1)n^k \leq 2^n n^k = p^{(2k)}_n. $$

Hence $P \ast P \preceq P$, and Theorem 6.1 implies the result. \qed

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ALEXEI YU. PIRKOVSKII, FACULTY OF MATHEMATICS, HSE UNIVERSITY, 6 USACHEVA, 119048 MOSCOW, RUSSIA
Email address: aupirkovskii@hse.ru