TUNNELING BEHAVIORS OF TWO MUTUAL FUNDS

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ABSTRACT. In practice, the mutual fund manager charges asset based management fee as the incentives. Meanwhile, we suppose that the investor could sustainedly obtain the fixed proportions of the fund values as the rewards. In this perspective, the objectives of the investor and the manager seem to be consistent. Unfortunately, it is a common situation that the fund managers have private relations, and they transfer the assets illegally. In this paper, we study the optimal tunneling behaviors of the two fund managers to maximize the overall performance criterions. It is the first time to use two prototypes whether the management fee rates are consistent with the investment returns to study the impacts of the two factors on the tunneling behaviors. We firstly study the problem without transaction cost between funds, and it is formalized as a two-dimensional stochastic optimal control problem, whose semi-analytical solution is derived by the dynamic programming methods. Furthermore, the transaction cost is considered, and we explore the penalty method and the finite difference method to establish the numerical solutions. The results show that the well performed and high rewarded fund manager obtains most of the total assets by tunneling, and only keep the other fund at the brink of maximal withdraws for the liquidity considerations. Moreover, the well performed and low rewarded fund manager obtains most of the total assets. Being inconsistent with the instinct, the high management fee rate could neither make the fund managers work efficiently, nor induce the beneficial tunneling behaviors.

1. Introduction. The mutual fund is one of the major investment channels and is favored by the investors for the flexible investment strategies and attractive investment returns. Being donated a lump-sum of money by the investors, the fund manager dynamically chooses the asset allocation strategies to achieve the objectives. As a reward, the fund manager could obtain some proportion of the fund asset
value as the management fee. It is an incentive of fund managers to take effort to increase the fund value. According to the literatures of Demski and Feltham (1978) [8], Stoughton (1993) [25], Gomez-Mejia and Wiseman (1997) [14], and Foster and Peyton Young (2010) [11], the moral hazard problem generated by the principle-agent problem is less severe in the management fee regime, comparing to the performance fee regime. In this paper, we suppose that the fund managers only obtain the asset based management fee as the rewards.

In practice, the mutual funds could charge flexible management fees according to their heterogeneous past performances and reputations. Gil-Bazo and Ruiz-Verdú (2009) [12] study the relationship between the management fee and the performance of the fund. The results show that the negative correlation between the management fee and the performance is robust as the strategic setting with different degrees of sensitivity to performance. Furthermore, the fund with better fund governance may bring fees more in line with performance. In this paper, we both study the scenarios that the management fees are in line and not in line with the performances of the fund managers.

In the mutual fund management, there are two ways to realize the rewards of the investor and the manager. In the open-ended way, the investor could input or withdraw endowments according to the actual value of the fund. The objective of the investor is to maximize the total withdraws minus the total endowments. In this way, the manager could obtain higher rewards by enlarging the fund size. In the close-ended way, the investor could only obtain the fixed proportion of the fund value sustainedly as the rewards. The objective of the investor is to maximize the total rewards. In this way, the manager could obtain higher rewards by increasing the fund value. In this perspective, the objectives of the investor and the manager seem to be more consistent under the close-ended way. In this paper, we suppose that the rewards of the investor are obtained in the close-ended way. Furthermore, there exists the maximal withdraw level to protect the interest of the investor. The fund will be compulsorily liquidated when the level is reached.

As the moral hazard problem generated by the principle-agent relationship between the investor and the fund manager is so common in practice, the illegal behaviors of the manager have been widely studied in the literatures. The studies of Eisenhards (1989) [9], Maiden (2003) [21], and Houge and Wellman (2005) [16] show that the behavior of the fund manager varies illegally due to the differed objectives between the manager and the investor. Reynolds et al. (2006) [23] point that, there are two different principal-agent relationships existing in most fund company scenarios. The fund manager is an agent to both the fund company and the fund investors. Thus, the fund manager’s behavior may align with the profit maximization of the company rather than the investors. Under the private relationships of the fund managers, the objectives of the investor and the manager will differ, and the moral hazard problem is then generated. Furthermore, the tunneling behaviors induce unfair reward distributions and loss accommodations between the two funds. In practice, it is a common situation that the fund managers have private relations, and they transfer the assets illegally. The empirical literatures including Brown et al. (1996) [5], Sirri and Tufano (1998) [24], and Fant and O’Neal (2000) [10] verify the positive relationships among the net asset transfer, the investment return and the reputation. Meanwhile, Goetzmann and Ibbotson (1994) [13] show that the investment return may reverse in longer time, and the asset will be transferred from the well performed fund to the badly performed fund.
In this paper, the problem is studied in the framework of the optimal control of the two cooperative companies. There has been a sustainable increasing attention in studying the optimal control problem with two related counterparts. Ruin probability expressions under a two-dimensional risk process framework are studied in Avram et al. (2007) [3] for simultaneous claim arrivals. Avanz (2009) [2], and Azcue and Muler [4] (2013) study the problem of optimally transferring capital between two portfolios with transaction costs. Jin, Yang and Yin (2017) [17] study the optimal dividend policy with transaction costs under the numerical approach. Albrecher et al. [1] (2015) study a two-dimensional optimal dividend problem of two insurance companies which collaborate by covering each other’s deficit when necessary. In this paper, we study both cases that the two mutual fund managers could transfer the assets frictionless or with transaction cost, and they choose the optimal tunneling behaviors to achieve the overall performance criterions, i.e., the maximization of the total rewards before the liquidation. Furthermore, Davis et al. (2007) [7] use empirical evidence to test the theoretical hypothesis that the fund management fees and control structures have great impacts on the illegal behaviors within fund organizations. As the rate of the management fee is an important factor effecting the behaviors of the managers, we study both the scenarios that the fees are in line and not in line with the performances of the funds.

Using the dynamic programming methods in Øksendal and Sulem (1984) [22], we solve the two-dimensional stochastic optimal control problem, and establish the semi-analytical solutions for the frictionless case. Inspired by the numerical methods in the literatures of Li and Wang (2009) [18], and Witte and Reisinger (2011) [26], the case with transaction cost is solved by penalty method and finite difference method. The results of Lions and Szmitman (1984) [20] guarantee the dynamics of two funds are uniquely determined by the stochastic differential equations. We also simulate the heterogenous tunneling behaviors of two funds under different scenarios. Furthermore, we evaluate the changes of the rewards of the investors and the managers, and the durations of the funds. The results could be important tools in detecting the tunneling behaviors of the fund managers and evaluating the impacts on the utilities of the investors.

The paper is organized as follows: In Section 2, we study the value function of the fund manager in one mutual fund case as the benchmark. In Section 3, we establish the two-dimensional stochastic optimal control model to formalize the optimal tunneling behaviors of the two mutual funds without transaction cost. The stochastic maximum principle and dynamic programming methods are used to derive the semi-analytical solutions of the problem. In Section 4, the problem is formalized in the framework with transaction cost. The numerical solution is established by the penalty method and the finite difference method. In Section 5, we simulate the heterogenous tunneling behaviors of two funds under different scenarios, and the impacts of the tunneling behaviors on the utilities of the investors are also evaluated by Monte Carlo Methods. The conclusions are in the last section.

2. Benchmark of one fund case. In this section, we study the value function of the mutual fund manager under the one-dimensional framework. We suppose that the dynamics of the investment return of the fund evolves according to the fund’s past performance. For simplicity, the dynamical allocations of the fund are not considered in the model. The mutual fund is locked up after it has been raised successfully. The fixed proportion of the fund asset is given to the investor...
sustainably as the rewards. In order to protect the interest of the investor, there exists a maximal withdraw level. If the fund value reaches the maximal withdraw level, there will be compulsory liquidation. Meanwhile, the rate of the management fee is an exogenous variable, and it is charged under the asset backed basis. In this perspective, the rewards of the fund manager and the investor are fully correlated, and the principle-agent problem seems minor.

In practice, the rate of the management fee is determined by the past performance of the fund and the strategic planning of the fund manager. The higher management fee rate represents the higher reward rate of the fund manager, as well as the higher level of the fund and the strategic planning of the fund manager. The higher management fee is an exogenous variable, and it is charged under the asset backed basis. In this paper, the rate of the management fee represents the higher reward rate of the fund manager, and we are curious on the impacts of the heterogenous risk of early compulsory liquidation. In this paper, the rate of the management fee is an exogenous variable, and we are curious on the impacts of the heterogenous management fees on the behaviors of the two correlated mutual fund managers.

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) be a filtered complete probability space satisfying the usual conditions. All the processes on \((\Omega, \mathcal{F}, \mathbb{P})\) below are adapted to the filtration \(\{\mathcal{F}_t\}_{t \geq 0}\). We suppose that \(X(t)\) is the value of a mutual fund at time \(t\) and satisfies
\[
dX(t) = (\mu - l - c)X(t)dt + \sigma X(t)dW(t), \quad X(0) = x_0,
\]
where \(\mu\) and \(\sigma\) are the expected return and the volatility of the mutual fund, respectively. They are estimated according to the past performance of the fund. \(l\) is the rewards rate of the investors, which is calculated as a fixed proportion of the fund value. \(c\) is the rate of the management fee of the fund manager and it is an exogenous variable. \(\{W(t), t \geq 0\}\) is a standard Brownian motion on \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\). \(x_0\) is the initial value of the mutual fund. Denote the compulsory liquidation time of the fund by \(\tau\), which is defined by \(\tau \triangleq \inf \left\{ t \geq 0 : X(t) < ax_0 \right\}\), where \(0 < a < 1\) is a constant representing the maximal withdraw level. The utility function of the fund manager is to maximize the present value of the total management fees before liquidation. In the single fund scenario, the utility function could be calculated directly as the non-existence of control variables.

\[
V^{x_0}(x_0) \triangleq E_{x_0} \left\{ \int_0^\tau e^{-\beta s}cX(s)ds \right\}, \quad (1)
\]
where \(\beta\) is the discount rate.

**Theorem 2.1.** At time \(t\), if the fund value is \(X(t) = x\), we define the present value of the total management fees of the fund manager as the utility of the manager, i.e.,

\[
V^{x_0}(x) \triangleq E_{x_t} \left\{ \int_t^\tau e^{-\beta(s-t)}cX(s)ds \right\},
\]
then it satisfies the following HJB equations:

\[
\frac{1}{2}\sigma^2 x^2 V_{xx} + (\mu - l - c)x V_x - \beta V + cx = 0, \quad x \geq ax_0, \quad (2)
\]
with boundary conditions \(V^{x_0}(ax_0) = 0\).

**Proof.** The proof is classical and we omit it here. \(\Box\)

**Theorem 2.2.** Under the assumption that \(\mu - l - c - \beta < 0\), \(V^{x_0}(x)\) is given by

\[
V^{x_0}(x) = \frac{cx}{\mu - l - c - \beta} \left( \exp \left\{ \ln \frac{x}{ax_0} - \frac{-(\mu - l - c + 0.5\sigma^2)}{\sigma^2} \right\} - 1 \right).
\]
Proof. Assume \( V^{x_0}(x) \) has the following form:

\[
V^{x_0}(x) = Ax \left( \frac{x}{a x_0} \right)^\alpha - 1,
\]

where \( A \) and \( \alpha \) are coefficients to be determined. Submitting this form into (2), letting the coefficients of \( x \) and \( x^{\alpha+1} \) be zero and noticing that \( V^{x_0}(x) > 0 \), we get

\[
\begin{aligned}
A &= \frac{c}{\mu - l - c - \beta}, \\
\alpha &= \frac{-(\mu - l - c + 0.5\sigma^2) - \sqrt{(\mu - l - c + 0.5\sigma^2)^2 - 2\sigma^2(\mu - l - c - \beta)}}{\sigma^2}.
\end{aligned}
\]

Thus, we finish the proof.

As a corollary of Theorem 2.2, we have the following:

**Corollary 1.** Under the assumption that \( \mu - l - c - \beta < 0 \), the rewards \( V^{x_0}(x_0) \) of the fund manager at time 0 are given by

\[
V^{x_0}(x_0) = \frac{c x_0}{\mu - l - c - \beta} \left( \exp \left\{ \ln \frac{1}{a} \frac{-(\mu - l - c + 0.5\sigma^2)}{\sigma^2} \right. \
- \left. \sqrt{(\mu - l - c + 0.5\sigma^2)^2 - 2\sigma^2(\mu - l - c - \beta)}}{\sigma^2} \right\} - 1 \right).
\]

Indeed, Corollary 1 states that the change of the fund value equals to the change of the maximal withdraw level \( a \) for the fixed initial value \( x_0 \) of the fund from the perspective of compulsory liquidation probability. Thus, we derive the analytical form of the utility function of the fund manager in the one-dimensional case. In this circumstance, the rewards of the investor are fully correlated with the rewards of the fund manager, and value function of the investor could be simply calculated.

3. **Two funds case without transaction cost.**

3.1. **The model.** As the rewards of the investor and the fund manager are both calculated on the asset backed basis, there seems no moral hazard problem generated by the principle-agent relations. However, taking the tunneling behaviors between the two mutual funds into consideration, severe moral hazard problem raises. In fund management practice, it is a common situation that the two mutual funds have private relations. The two funds may belong to the same company, and the rewards of the two funds could be concentrated and then be re-settled. The managers of the two funds may also have private benefit sharing plans. So, the objective of the two correlated funds becomes the optimization of the total rewards of the two funds. It is no longer consistent with the objective of the investors, i.e., the tunneling behaviors raise.

The following behaviors could be observed. If one well performed fund reaches the maximal withdraw level under some unexpected shocks, the other would like to transfer assets to avoid the compulsory liquidation. Especially, when the management fee of the fund is much higher due to the better performance, the tunneling behavior could happen more timely. Meanwhile, if one fund is badly performed, the fund manager may prefer to transfer the assets to the other fund to increase the total rewards. In both of the scenarios, the reward distributions and loss accommodations between the investors of the two funds are altered. Furthermore, taking the management fee rate issue into consideration, the results could be more tricky. If the management fee rate of one fund is higher, and the performance is worse,
whether the fund manager could transfer the asset into this fund to obtain higher rewards in the short term, or into the other fund to obtain sustainable rewards in the long term? Obviously, it is an important issue to study the tunneling behaviors of the two funds with heterogeneous investment returns and management fee rates, and evaluate the impacts of the behaviors on the utilities of the managers and the investors.

In this section, we study the optimal tunneling behaviors of the fund managers in the two-dimensional case. We assume the fund manager could transfer the assets to the other fund without any transaction fees and they cooperatively choose the transfer policies as control variables to maximize the total rewards. The tunneling problem is formalized into a two-dimensional stochastic optimal control problem, and the semi-analytical solutions could be derived by the dynamic programming methods.

The values of the two cooperatively managed funds are $X_1(t)$ and $X_2(t)$, respectively, and the dynamics of the two funds evolve according to the following stochastic differential equations (SDEs):

\[
\begin{align*}
\frac{dX_1(t)}{dt} &= (\mu_1 - l_1 - c_1)X_1(t)dt + \sigma_1X(t)dW_1(t), \\
X_1(0) &= x_{10}, \\
\frac{dX_2(t)}{dt} &= (\mu_2 - l_2 - c_2)X_2(t)dt + \sigma_2X(t)dW_2(t), \\
X_2(0) &= x_{20}.
\end{align*}
\]

(3)

For $i = 1, 2$, $\mu_i$ and $\sigma_i$ are the expected return and the volatility of the fund $i$. $l_i$ is the rewards rate to the fund investor. $c_i$ is the management fee rate of the fund manager, and it is an exogenous variable. $\{W_1(t) : t \geq 0\}$ and $\{W_2(t) : t \geq 0\}$ are independent standard Brownian motions on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$. $x_{i0}$ is the initial value of the fund $i$. Denote the compulsory liquidation time of the fund $i$ by $\tau_i$, which is defined by $\tau_i \triangleq \inf \left\{ t \geq 0 : X_{i}(t) < a_ix_{i0} \right\}$, where $0 < a_i < 1$ is a constant representing the maximal withdraw level of the fund $i$.

The fund will be compulsorily liquidated when its value reaches the maximal withdraw level. As observed in practice, if the well-performed fund reaches the maximal withdraw level due to some unexpected shocks, the cooperatively managed fund would like to transfer assets to it to avoid the liquidation. Thus, in the model, we suppose that when one fund reaches the maximal withdraw level, the other fund manager could choose whether to help it from liquidation. Moreover, the badly performed fund manager may prefer to transfer assets to the other fund to increase the overall profitability. This event could happen even when there are no compulsory liquidations. Thus, we also suppose that even if neither of the two funds reaches the maximal withdraws, the transfers of the assets are also allowed. Besides that, one fund is allowed to transfer its assets to the other and reaches the maximal withdraw level strategically. After one fund is liquidated, the other fund will still exist and generate its management fee. The two fund managers cooperatively control the asset transfer policies to maximize the total rewards.

**Remark 1.** In fact, two funds with asset transfer behaviors are always dependent in practice. So, it is more practical to assume that $W_1$ and $W_2$ are correlated. In Section 4, the dependent case will be considered. Besides that, the proportional transaction fee is also considered in the model. In this section, the two important issues are simply treated to establish the semi-analytical solutions.
Under the above hypothesis, the associated fund value processes \((X_1(t), X_2(t))\) with initial values \((x_{10}, x_{20})\) are

\[
\begin{align*}
\text{d}X_1(t) &= (\mu_1 - l_1 - c_1)X_1(t)\text{d}t + \sigma_1 X(t)\text{d}W_1(t) + \text{d}C_{21}(t) - \text{d}C_{12}(t), \\
\text{d}X_2(t) &= (\mu_2 - l_2 - c_2)X_2(t)\text{d}t + \sigma_2 X(t)\text{d}W_2(t) + \text{d}C_{12}(t) - \text{d}C_{21}(t),
\end{align*}
\]

where \(C_{21}(t)\) corresponds to the cumulative amount of assets transferred from Fund Two to Fund One up to time \(t\); \(C_{12}(t)\) corresponds to the cumulative amount of assets transferred from Fund One to Fund Two up to time \(t\), and \(t \leq \min\{\tau_1, \tau_2\}\). For simplicity, the transaction fees are not considered.

Besides that, one fund is also allowed to transfer all its excess assets to the other and reach the maximal withdraw level strategically. Then, it will be liquidated immediately and there is only one fund after that time. Because of this assumption, \(\min\{\tau_1, \tau_2\}\) is also a control variable.

Let \(G_+\) denote the set \(\{(x_1, x_2) : +\infty > x_1 \geq a_1 x_{10}, +\infty > x_2 \geq a_2 x_{20}\}\). We call a transfer strategy \(C(\cdot) = \{(C_{12}(t), C_{21}(t)) : t \geq 0\}\) admissible with initial value \((x_1, x_2) \in G_+\), denoted as \(C(\cdot) \in \pi_{(x_1, x_2)}\), if

(i) \(\{(C_{12}(t), C_{21}(t)) : t \geq 0\}\) is progressively measurable w.r.t. \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) and non-negative, and \((C_{12}(t), C_{21}(t))\) is non-decreasing function of \(t\) that is right continuous with left limits;

(ii) For any given initial value \((x_1, x_2) \in G_+\), the SDEs (4) have a unique solution.

The objective of the fund managers is to maximize the present value of the total management fees of the two funds before the compulsory liquidation. The two fund managers cooperatively choose the optimal asset transfer policies to achieve the objectives, i.e., to maximize the utility function of the two fund managers.

\[
V^{x_{10},x_{20}}(x_{10}, x_{20}) = \sup_{C(\cdot) \in \pi_{(x_{10}, x_{20})}} \left\{ V^{C,x_{10},x_{20}}(x_{10}, x_{20}) \right\},
\]

where

\[
V^{C,x_{10},x_{20}}(x_{10}, x_{20}) \triangleq \mathbb{E}_{x_{10},x_{20}} \left\{ \int_0^{\tau_1} e^{-\beta s} c_1 X_1(s)\text{d}s + \int_0^{\tau_2} e^{-\beta s} c_2 X_2(s)\text{d}s \right\},
\]

where \(c_1\) and \(c_2\) are the management fee rates of Fund One and Fund Two, and \(X_i(t)\) is the state process with strategy \(C(\cdot)\), respectively. \(x_{10}, x_{20}\) in the superscript of \(V^{C,x_{10},x_{20}}(x_{10}, x_{20})\) means its compulsory liquidation level is \(a_1 x_{10}\) and \(a_2 x_{20}\) in the brackets means the value of \(X_i(0)\) is \(x_{i0}\). Since the management fee is charged according to the asset backed basis, the fund managers have two ways to achieve the objectives. One way is to increase the assets of the fund with higher investment return or higher management fee rate, leading to increase the management fee. The other way is to postpone the liquidation time, and extend the duration of the well performed fund. The two cooperatively managed funds illegally transfer the assets to achieve their own objectives.

Define \(\tau^1 \triangleq \min\{\tau_1, \tau_2\}\) and \(\tau^2 \triangleq \max\{\tau_1, \tau_2\}\). Notice that after time \(\tau^1\), the fund manager can only obtain the management fee from the remaining fund until its liquidation. We have established the rewards of the fund under this case in Theorem 2.2. Define \(V^{x_{i0}}_i(X_i(t)), i = 1, 2\) as the value function in Theorem 2.2 with the initial value \(x_{i0}\) at time \(t\), that is,

\[
V^{x_{i0}}_i(X_i(t)) = \frac{c_i X_i(t)}{\mu_i - l_i - c_i - \beta} \left( \exp \left\{ \ln \frac{X_i(t)}{a_i x_{i0}} \frac{-(\mu_i - l_i - c_i + 0.5\sigma^2_i)}{\sigma^2_i} \right\} \right)
\]
So, we can derive the value function (5) for any initial fund values \((x_1, x_2) \in G_+\) at time \(t\) by
\[
V^{x_{20}}(x_{20}(t)) = \sup_{C(\cdot) \in \pi(x_{10}, x_{20})} \left\{ \int_0^{\tau_1} e^{-\beta s} c_1 X_1(s) ds + \int_0^{\tau_2} e^{-\beta s} c_2 X_2(s) ds \right\}
\]
and
\[
V^{x_{20}}_2(x_{20}(t)) = \frac{c_2 X_2(t)}{\mu_2 - l_2 - c_2 - \beta} \left( \exp \left\{ \ln \frac{X_2(t)}{a_2 x_{20}} \frac{-\left(\mu_2 - l_2 - c_2 + 0.5 \sigma_2^2\right)}{\sigma_2^2} \right\} - 1 \right).
\]

So, we can derive the value function (5) for any initial fund values \((x_1, x_2) \in G_+\) at time \(t\) by
\[
V^{x_{10}, x_{20}}(x_{10}, x_{20}) = \sup_{C(\cdot) \in \pi(x_{10}, x_{20})} \left\{ \int_0^{\tau_1} e^{-\beta s} c_1 X_1(s) ds + \int_0^{\tau_2} e^{-\beta s} c_2 X_2(s) ds \right\}
\]
\[
= \sup_{C(\cdot) \in \pi(x_{10}, x_{20})} \left\{ \int_0^{\tau_1} e^{-\beta s} [c_1 X_1(s) + c_2 X_2(s)] ds \right\}
\]
\[
+ \int_0^{\tau_1} e^{-\beta s} [c_1 X_1(s) + c_2 X_2(s)] ds \right\}
\]
\[
\sup_{C(\cdot) \in \pi(x_{10}, x_{20})} \left\{ \int_0^{\tau_1} e^{-\beta s} [c_1 X_1(s) + c_2 X_2(s)] ds \right\}
\]
\[
+ e^{-\beta \tau_1}[V^{x_{10}}_1(X_1(\tau_1), \tau_1) + V^{x_{20}}_2(X_2(\tau_1), \tau_1)]
\]
\[\vdots\]
\[
V^{x_{10}, x_{20}}(x_{10}(t), x_{20}(t)) = \sup_{C(\cdot) \in \pi(x_{10}(t), x_{20}(t))} \left\{ \int_0^{\tau_1} e^{-\beta s} [c_1 X_1(s) + c_2 X_2(s)] ds \right\}
\]
\[
+ e^{-\beta \tau_1}[V^{x_{10}}_1(X_1(\tau_1), \tau_1) + V^{x_{20}}_2(X_2(\tau_1), \tau_1)]
\]
\[\vdots\]

The following lemma is concluded without proof from the definition of the value function, and the assumption that the assets can be transferred between the two funds at any time before \(\tau^1\) without any transaction cost.

**Proposition 1.** If \(x_1 > a_1 x_{10}\) and \(x_2 > a_2 x_{20}\), the value function \(V^{x_{10}, x_{20}}(x_1, x_2, t)\) is increasing with respect to both \(x_1\) and \(x_2\), and
\[
V^{x_{10}, x_{20}}(x_1 + \Delta x, x_2 - \Delta x) = V^{x_{10}, x_{20}}(x_1, x_2),
\]
\[\forall \Delta x \in \mathbb{R} \text{ satisfying } x_1 + \Delta x \geq a_1 x_{10} \text{ and } x_2 - \Delta x \geq a_2 x_{20}.\]

3.2. **Hamilton-Jacobi-Bellman equation.** Since the value function of the two-dimensional optimal control problem satisfies the Bellman’s principle of optimality, we can derive the following results by solving the associated Hamilton-Jacobi-Bellman (HJB) equations. From Eq. (9), we know that \(V^{x_{10}, x_{20}}(x_1, x_2) = V^{x_{10}, x_{20}}_{1}(x_1, x_2) = 0\). The value function at point \((x_1, x_2)\) is given by the following lemma if neither of the two funds is compulsorily liquidated.
Lemma 3.1. If neither of the two funds is liquidated at \((x_1, x_2)\) immediately, then the value function \(V^{x_{10}, x_{20}}(x_1, x_2)\) satisfies the following conditions:

\[
\max \left\{ \mathcal{L}V^{x_{10}, x_{20}}(x_1, x_2), V^{x_{10}, x_{20}}_{x_1} - V^{x_{10}, x_{20}}_{x_2}, V^{x_{10}, x_{20}}_{x_2} - V^{x_{10}, x_{20}}_{x_1} \right\} = 0, \tag{10}
\]

where

\[
\mathcal{L}V^{x_1, x_2} = c_1 x_1 + c_2 x_2 - \beta V + (\mu_1 - l_1 - c_1)x_1V_{x_1},
\]

\[
+ (\mu_2 - l_2 - c_2)x_2V_{x_2} + \frac{1}{2}\sigma_1^2 x_1^2 V_{x_1x_1} + \frac{1}{2}\sigma_2^2 x_2^2 V_{x_2x_2}.
\]

Proof. Since Bellman's principle of optimality holds, we assume the strategy \(C(\cdot)\) between \((0, \Delta t)\) is \(dC_{12}(t) = c_{12} dt, dC_{21}(t) = c_{21} dt\), where \(0 \leq t \leq \Delta t\). \(c_{12}, c_{21}\) are constants, and no matter what the values of \(c_{12}, c_{21}\) are, the value function with this strategy is not better than the optimal value function. Thus, we can use variational methods to get the HJB equation. The details are referred in Yong and Zhou (1999)[27]. □

Next, we write \(C_{ij}(t)\) as

\[
C_{ij}(t) = \int_0^t dC_{ij}(s) + \sum_{\{\bar{x}_1(s^-) \neq \bar{x}_1(s), \bar{x}_2(s^-) \neq \bar{x}_2(s)\} \cap \{s \leq t\}} (C_{ij}(s) - C_{ij}(s^-))
\]

and consider the behaviors of the fund manager at point \((x_1, x_2)\). Basically, the fund manager has the following three options:

- **Option One:** Fund Two manager transfers \(x_2 - a_2x_2\) to Fund One, and Fund Two is compulsorily liquidated immediately. In this situation, only Fund One still runs, and the value function is \(V^{x_{10}, x_{20}}_{One}(x_1, x_2) = V^{x_{10}, x_{20}}_{One}(x_1 + x_2 - a_2x_2, a_2x_2) = V^{x_{10}}_{One}(x_1 + x_2 - a_2x_2)\).

- **Option Two:** Similarly, Fund One manager transfers \(x_1 - a_1x_1\) to Fund Two, and Fund One is compulsorily liquidated immediately. In this situation, only Fund Two still runs, and the value function \(V^{x_{10}, x_{20}}_{Two}(x_1, x_2) = V^{x_{10}, x_{20}}_{Two}(a_1x_1, x_2 + x_1 - a_1x_1) = V^{x_{20}}_{Two}(x_2 + x_1 - a_1x_1)\).

- **Option Three:** Whether one fund transfers some assets to the other or not, neither of the two funds will be compulsorily liquidated at this point. In this situation, the value function \(V^{x_{10}, x_{20}}(x_1, x_2)\) is given by Eq.(10).

Based on the above discussions, we have the following theorem:

**Theorem 3.2.** At any point \((x_1, x_2) \in \mathbf{G}_+\), the value function \(V^{x_{10}, x_{20}}(x_1, x_2)\) in (7) satisfies the following HJB equation:

\[
\max \left\{ \mathcal{L}V^{x_{10}, x_{20}}(x_1, x_2), V^{x_{10}, x_{20}}_{x_1} - V^{x_{10}, x_{20}}_{x_2}, V^{x_{10}, x_{20}}_{x_2} - V^{x_{10}, x_{20}}_{x_1} \right\} = 0,
\]

with boundary condition \(V^{x_{10}, x_{20}}(a_1x_1, a_2x_1) = 0\).

### 3.3. A heuristic algorithm for determining the solution of HJB equation

Since \(V^{x_{10}, x_{20}}_{x_1} = V^{x_{10}, x_{20}}_{x_2}\) and \(V^{x_{10}, x_{20}}(a_1x_1, a_2x_2) = 0\), to determine the value of \(V^{x_{10}, x_{20}}(x_1, x_2)\) at any point \((x_1, x_2) \in \mathbf{G}_+\), we only need to determine its value at line \(\{(x_1, x_2) : +\infty > x_1 \geq a_1x_1, x_2 = a_2x_2\}\). Assume that we have already determined the value of \(V^{x_{10}, x_{20}}(x_1, a_2x_2)\) from \((a_1x_1, a_2x_2)\) to \((x_1, a_2x_2)\), where
Then, if $\hat{x}_1 > a_1 x_{10}$. In order to determine the value of $V^{x_{10},x_{20}}(\hat{x}_1 + \varepsilon, a_2 x_{20})$, where $\varepsilon$ is very small, we need to compare the outcomes of the three options: Option One and Option Two (transferring the assets from one fund to the other, and making one of the funds compulsorily liquidated immediately) lead $V^{x_{10},x_{20}}(\hat{x}_1 + \varepsilon, a_2 x_{20})$ to be $V_1^{x_{10}}(\hat{x}_1 + \varepsilon, a_2 x_{20} - a_1 x_{10})$. Meanwhile, if the fund manager chooses Option Three, by Lemma 3.1 and Eq.(10), the value function $V^{x_{10},x_{20}}(\hat{x}_1 + \varepsilon, a_2 x_{20})$ is given by the following equation:

$$
\begin{aligned}
&\max \left\{ \mathcal{L}^{V^{x_{10},x_{20}}(x_1, x_2)}, V^{x_{10},x_{20}}(x_1, x_2) - V^{x_{10},x_{20}}(x_2, x_2) - V^{x_{10},x_{20}}(x_1, x_2) - V^{x_{10},x_{20}}(x_2, x_2) \right\} = 0, \\
&V^{x_{10},x_{20}}(x_1, a_2 x_{20})|_{x_1 = \bar{x}_2} = V^{x_{10},x_{20}}(\hat{x}_1, a_2 x_{20}),
\end{aligned}
$$

where $(x_1, x_2) \in [\hat{x}_1, \hat{x}_1 + \varepsilon] \times \{a_2 x_{20}\}$, and the boundary condition $V^{x_{10},x_{20}}(\hat{x}_1, a_2 x_{20})$ has already been established.8pt

The fund manager will compare the outcomes of the three options and choose the best one to execute. Starting from the point $(a_1 x_{10}, a_2 x_{20})$, when $\varepsilon$ is small enough, the three options mentioned above will make $V^{x_{10},x_{20}}(a_1 x_{10} + \varepsilon, a_2 x_{20})$ to be $\varepsilon \frac{dV^{x_{10},x_{20}}(a_1 x_{10})}{dx}$, $\varepsilon \frac{dV^{x_{10},x_{20}}(a_2 x_{20})}{dx}$ and $\varepsilon \frac{dV^{x_{10},x_{20}}(a_1 x_{10}, a_2 x_{20})}{dx}$, respectively. Here, $V^{x_{10},x_{20}}(x_1, x_2)$ is the solution of the following equation:

$$
\begin{aligned}
&\max \left\{ \mathcal{L}^{V^{x_{10},x_{20}}(x_1, x_2)}, V^{x_{10},x_{20}}(x_1, x_2) - V^{x_{10},x_{20}}(x_2, x_2) - V^{x_{10},x_{20}}(x_1, x_2) - V^{x_{10},x_{20}}(x_2, x_2) \right\} = 0,
&V^{x_{10},x_{20}}(a_1 x_{10}, a_2 x_{20}) = 0.
\end{aligned}
$$

Thus, we only need to compare the values of $\frac{dV^{x_{10},x_{20}}(a_1 x_{10})}{dx}$, $\frac{dV^{x_{10},x_{20}}(a_2 x_{20})}{dx}$ and $\frac{dV^{x_{10},x_{20}}(a_1 x_{10}, a_2 x_{20})}{dx}$.

**Remark 2.** It is difficult to estimate the value of $\frac{dV^{x_{10},x_{20}}(a_1 x_{10}, a_2 x_{20})}{dx}$ directly, we firstly use finite difference method to solve equation (12) and derive the value of $V^{x_{10},x_{20}}(a_1 x_{10}, a_2 x_{20})$ in the whole domain. Then, we use discretization approximation to approximate it as in (10), the details of the method are explained in Subsection 4.3.

### 3.3.1. Case 1:

\[dV^{x_{10},x_{20}}(a_1 x_{10}) = \max \left( \frac{dV^{x_{10},x_{20}}(a_1 x_{10})}{dx}, \frac{dV^{x_{10},x_{20}}(a_2 x_{20})}{dx}, \frac{dV^{x_{10},x_{20}}(a_1 x_{10}, a_2 x_{20})}{dx} \right)\].

In this case, the outcome of Option One outperforms the other two options. The fund manager will choose Option One in this situation. Since all of the three derivatives are continuous, there exists an $\varepsilon$ small enough that $V^{x_{10},x_{20}}(a_1 x_{10}, a_2 x_{20}) = V_1^{x_{10}}(x_1)$, $x_1 \in [a_1 x_{10}, a_1 x_{10} + \varepsilon]$.

We define $n_1^*$ by

$$
n_1^* \triangleq \sup \left\{ n : x_1 \in [a_1 x_{10}, n] \text{ and } V^{x_{10},x_{20}}(x_1, a_2 x_{20}) = V_1^{x_{10}}(x_1) \right\}.
$$

Then, if $x_1 < n_1^*$, Option One is the optimal choice for the fund manager, and if $x_1 > n_1^*$, Option Two or Option Three is the optimal choice. So, at the point $(n_1^*, a_2 x_{20})$, either the derivative of $V_1^{x_{10}}(n_1^*)$ equals to the derivative getting from Eq.(10) which starts from the point $(n_1^*, a_2 x_{20})$ or $V_1^{x_{10}}(n_1^*)$ equals to $V_2^{x_{20}}(n_1^* + a_2 x_{20} - a_1 x_{10})$.

Define $V^{x_{10},x_{20}}(x_1, x_2; n, x(n))$ as the solution of the following equation:

$$
\begin{aligned}
&\max \left\{ \mathcal{L}^{V^{x_{10},x_{20}}(x_1, x_2; n, x(n))}, V^{x_{10},x_{20}}(x_1, x_2; n, x(n)) - V^{x_{10},x_{20}}(x_2, x_2) - V^{x_{10},x_{20}}(x_1, x_2) - V^{x_{10},x_{20}}(x_2, x_2) \right\} = 0,
&V^{x_{10},x_{20}}(n, a_2 x_{20}; n, x(n)) = x(n),
\end{aligned}
$$
where \((n, x(n))\) in \(V^{x_{10},x_{20}}(x_1, x_2; n, x(n))\) are two parameters that record the initial condition of \(V^{x_{10},x_{20}}\), whose value at \((n, a_2x_{20})\) is \(x(n)\), and \(n \geq a_1x_{10}\) and \(x(n) \geq 0\).

The following theorem gives the method to determine the value of \(V^{x_{10},x_{20}}(x_1, x_2)\) on \(\{(x_1, x_2) : a_1x_{10} \leq x_1 \leq x^*_1, x_2 = a_2x_{20}\}\).

**Theorem 3.3.** For all \(n < n^*_1\), we have
\[
V^{x_{10},x_{20}}(n, a_2x_{20}) = V_1^{x_{10}}(n),
\]
and
\[
V_1^{x_{10}}(n) > V_2^{x_{20}}(n + a_2x_{20} - a_1x_{10})
\]
as well as
\[
\frac{dV_1^{x_{10}}(n)}{dx} > \frac{dV_1^{x_{10}}(n, a_2x_{20}; n, V_1^{x_{10}}(n))}{dx_1}.
\]

When \(n = n^*_1\), we have
\[
V_1^{x_{10}}(n^*_1) = V_2^{x_{20}}(n^*_1 + a_2x_{20} - a_1x_{10})
\]
or
\[
\frac{dV_1^{x_{10}}(n^*_1)}{dx} = \frac{dV_1^{x_{10}}(n^*_1, a_2x_{20}; n^*_1, V_1^{x_{10}}(n^*_1))}{dx_1}.
\]

**3.3.2. Case 2:** \(\frac{dV_2^{x_{20}}(a_2x_{20})}{dx} = \max\left\{ \frac{dV_1^{x_{10}}(a_1x_{10})}{dx}, \frac{dV_2^{x_{20}}(a_2x_{20})}{dx}, \frac{dV_1^{x_{10}}(a_1x_{10}, a_2x_{20})}{dx_1} \right\}\).

This case is similar to Case 1. Define \(n^*_1\) by
\[
n^*_1 = \sup \left\{ n : x_1 \in [a_1x_{10}, n], V^{x_{10},x_{20}}(x_1, a_2x_{20}) = V_2^{x_{20}}(x_1 + a_2x_{20} - a_1x_{10}) \right\}.
\]
The following theorem establishes the methods to derive the value of \(V^{x_{10},x_{20}}(x_1, x_2)\) on \(\{(x_1, x_2) : a_1x_{10} \leq x_1 \leq n^*_1, x_2 = a_2x_{20}\}\).

**Theorem 3.4.** For all \(n < n^*_1\), we have
\[
V^{x_{10},x_{20}}(n, a_2x_{20}) = V_2^{x_{20}}(n + a_2x_{20} - a_1x_{10}),
\]
and
\[
V_2^{x_{20}}(n + a_2x_{20} - a_1x_{10}) > V_1^{x_{10}}(n)
\]
as well as
\[
\frac{dV_2^{x_{20}}(n + a_2x_{20} - a_1x_{10})}{dx} > \frac{dV_1^{x_{10},x_{20}}(n, a_2x_{20}; n, V_2^{x_{20}}(n + a_2x_{20} - a_1x_{10}))}{dx_1}.
\]

When \(n = n^*_1\), we have
\[
V_2^{x_{20}}(n^*_1 + a_2x_{20} - a_1x_{10}) = V_1^{x_{10}}(n^*_1)
\]
or
\[
\frac{dV_2^{x_{20}}(n^*_1 + a_2x_{20} - a_1x_{10})}{dx} = \frac{dV_1^{x_{10},x_{20}}(n^*_1, a_2x_{20}; n^*_1, V_2^{x_{20}}(n^*_1 + a_2x_{20} - a_1x_{10}))}{dx_1}.
\]
3.3.3. Case 3: \[
\frac{dV^{x_{10},x_{20}}(x_1,x_2)}{dx_1} = \max \left\{ \frac{dV^{x_{10}}(a_1,x_{10})}{dx_1}, \frac{dV^{x_{20}}(a_2,x_{20})}{dx_1}, \frac{dV^{x_{10},x_{20}}(a_1,a_2,x_{10},x_{20})}{dx_1} \right\}.
\]
In this case, the outcome of Option Three outperforms the other two options. The fund manager will choose Option Three in this situation. Define \( n_1^* \) by
\[
n_1^* = \sup \left\{ n : x_1 \in [a_1,x_{10}, n], V^{x_{10},x_{20}}(x_1,a_2,x_{20}) = V^{x_{10},x_{20}}(x_1,a_2,x_{20};a_1,x_{10},0) \right\}.
\]
The following theorem establishes the methods to derive the value of \( V^{x_{10},x_{20}}(x_1,x_2) \) on \( \{(x_1,x_2) : a_1 x_{10} \leq x_1 \leq n_1^*, x_2 = a_2 x_{20}\} \).

**Theorem 3.5.** For all \( n < n_1^* \), we have
\[
V^{x_{10},x_{20}}(n,a_2,x_{20}) = V^{x_{10},x_{20}}(n,a_2,x_{20};a_1,x_{10},0) > \max \left\{ V^{x_{10}}(n), V^{x_{20}}(n + a_2 x_{20} - a_1 x_{10}) \right\}.
\]
When \( n = n_1^* \), we have
\[
V^{x_{10},x_{20}}(n_1^*,a_2,x_{20};a_1,x_{10},0) = \max \left\{ V^{x_{10}}(n_1^*), V^{x_{20}}(n_1^* + a_2 x_{20} - a_1 x_{10}) \right\}.
\]

3.3.4. Value after \( n_1^* \). We have determined the value of \( V^{x_{10},x_{20}}(x_1,x_2) \) on \( \{(x_1, x_2) : x_1 \geq x_1 \geq a_1 x_{10}, x_2 = a_2 x_{20}\} \) under the three cases, respectively. To determine the value of \( V^{x_{10},x_{20}}(x_1,x_2) \) on \( \{(x_1,x_2) : x_1 > n_1^*, x_2 = a_2 x_{20}\} \), we repeat the similar procedures except for using \( (n_1^*,a_2 x_{20}) \) instead of \( (a_1 x_{10},a_2 x_{20}) \) as the initial point. Similarly, there exists an \( \varepsilon \) small enough such that one of the following three cases holds:

1. \( V^{x_{10},x_{20}}(x_1,a_2,x_{20}) = V_1^{x_{10}}(x_1), x_1 \in [n_1^*,n_1^* + \varepsilon] \),
2. \( V^{x_{10},x_{20}}(x_1,a_2,x_{20}) = V_2^{x_{20}}(x_1 + a_2 x_{20} - a_1 x_{10}), x_1 \in [n_1^*,n_1^* + \varepsilon] \),
3. \( V^{x_{10},x_{20}}(x_1,a_2,x_{20}) = V_1^{x_{10}}(x_1,a_2,x_{20};n_1^*,a_2 x_{20}), x_1 \in [n_1^*,n_1^* + \varepsilon] \).

Define \( n_2^* \) by
\[
\begin{cases}
\sup \left\{ n : x_1 \in [n_1^*,n], V^{x_{10},x_{20}}(x_1,a_2,x_{20}) = V_1^{x_{10}}(x_1) \right\}, \text{if } (1), \\
\sup \left\{ n : x_1 \in [n_1^*,n] \right\} \text{ and } \\
n_2^* = \sup \left\{ n : x_1 \in [n_1^*,n], V^{x_{10},x_{20}}(x_1,a_2,x_{20}) = V_2^{x_{20}}(x_1 + a_2 x_{20} - a_1 x_{10}) \right\}, \text{if } (2), \\
\sup \left\{ n : x_1 \in [n_1^*,n] \right\} \text{ and } \\
V^{x_{10},x_{20}}(x_1,a_2,x_{20}) = V_1^{x_{10}}(x_1,a_2,x_{20};n_1^*,a_2 x_{20}), V^{x_{10},x_{20}}(n_1^*,a_2 x_{20}), x_1 \in [n_1^*,n_1^* + \varepsilon], \\
\end{cases}
\]
The following theorem establishes the methods to derive the value of \( V^{x_{10},x_{20}}(x_1,x_2) \) on \( \{(x_1,x_2) : n_1^* < x_1 \leq n_2^*, x_2 = a_2 x_{20}\} \).

**Theorem 3.6.** If (1) holds, for all \( n_1^* < n < n_2^* \), we have
\[
V^{x_{10},x_{20}}(n,a_2,x_{20}) = V_1^{x_{10}}(n),
\]
if (1),
\[
\sup \left\{ n : x_1 \in [n_1^*,n] \right\} \text{ and } \\
n_2^* = \sup \left\{ n : x_1 \in [n_1^*,n], V^{x_{10},x_{20}}(x_1,a_2,x_{20}) = V_2^{x_{20}}(x_1 + a_2 x_{20} - a_1 x_{10}) \right\}, \text{if } (2), \\
\sup \left\{ n : x_1 \in [n_1^*,n] \right\} \text{ and } \\
V^{x_{10},x_{20}}(x_1,a_2,x_{20}) = V_1^{x_{10}}(x_1,a_2,x_{20};n_1^*,a_2 x_{20}), V^{x_{10},x_{20}}(n_1^*,a_2 x_{20}), x_1 \in [n_1^*,n_1^* + \varepsilon], \\
\end{cases}
\]
If 2n
\[ \frac{dV_x^{T_0}}{dx} > \frac{dV_x^{T_0}(n, a_2 x_20; n, V_x^{T_0}(n))}{dx_1} \]
When \( n = n_2^* \), we have
\[ V_x^{T_0}(n_2^*) = V_x^{T_0}(n_2^* + a_2 x_20 - a_1 x_10) \]
or
\[ \frac{dV_x^{T_0}(n_2^*)}{dx} = \frac{dV_x^{T_0}(n_2^*, a_2 x_20; n_2^*, V_x^{T_0}(n_2^*))}{dx_1} \]
If 2 holds, for all \( n_1^* < n < n_2^* \), we have
\[ V_x^{T_0}(n, a_2 x_20) = V_x^{T_0}(n + a_2 x_20 - a_1 x_10), \]
\[ V_x^{T_0}(n + a_2 x_20 - a_1 x_10) > V_x^{T_0}(n) \]
and
\[ \frac{dV_x^{T_0}(n + a_2 x_20 - a_1 x_10)}{dx} > \frac{dV_x^{T_0}(n, a_2 x_20; n, V_x^{T_0}(n + a_2 x_20 - a_1 x_10))}{dx_1} \]
When \( n = n_2^* \), we have
\[ V_x^{T_0}(n_2^* + a_2 x_20 - a_1 x_10) = V_x^{T_0}(n_2^*) \]
or
\[ \frac{dV_x^{T_0}(n_2^* + a_2 x_20 - a_1 x_10)}{dx} = \frac{dV_x^{T_0}(n_2^*, a_2 x_20; n_2^*, V_x^{T_0}(n_2^* + a_2 x_20 - a_1 x_10))}{dx_1} \]
If 3 holds, for all \( n_1^* < n < n_2^* \), we have
\[ V_x^{T_0}(n, a_2 x_20) = \tilde{V}_x^{T_0}(n, a_2 x_20; n_1^*, V_x^{T_0}(n_1^*, a_2 x_20)) \]
\[ > \max \left\{ V_x^{T_0}(n), V_x^{T_0}(n + a_2 x_20 - a_1 x_10) \right\} \]
When \( n = n_2^* \), we have
\[ \tilde{V}_x^{T_0}(n_2^*, a_2 x_20; n_1^*, V_x^{T_0}(n_1^*, a_2 x_20)) \]
\[ = \max \left\{ V_x^{T_0}(n_2^*), V_x^{T_0}(n_2^* + a_2 x_20 - a_1 x_10) \right\} \]
According to the discussions above, the value of \( V_x^{T_0}(x_1, x_2) \) on \( \{ (x_1, x_2) : n_1^* < x_1 \leq n_2^*, x_2 = a_2 x_20 \} \) has been determined. Repeating the procedures, we can find the sequence \( \{ n_3^*, n_4^*, ..., n_k^*... \} \) to determine the value of \( V_x^{T_0}(x_1, x_2) \) on \( \{ (x_1, x_2) : a_1 x_10 \leq x_1 < +\infty, x_2 = a_2 x_20 \} \). Finally, the value of \( V_x^{T_0}(x_1, x_2) \) on \( G_+ \) is established. Thus, the solutions of the HJB equation (11) have been derived on \( G_+ \). More specifically, different boundary values on \( \{ (x_1, x_2) : a_1 x_10 \leq x_1 < +\infty, x_2 = a_2 x_20 \} \) are determined for different scenarios. Finally, the exact expression of \( V_x^{T_0}(x_1, x_2) \) has been derived.

The proof of the verification theorem for the case without transaction cost is a special result of the proof of the verification theorem with transaction cost. The verification theorem with transaction cost is derived in next section, so we omit the proof here.
4. Two funds case with transaction cost.

4.1. The model. In this section, we study the optimal tunneling behaviors of the two fund managers with the transaction cost, and the correlation between the two fund value processes. The tunneling problem is also formalized into a two-dimensional stochastic optimal control problem. We firstly derive the HJB equation by the dynamic programming methods. Secondly, the numerical procedures are implemented by the penalty method and the finite difference method, which is inspired by the studies of Witte and Reisinger (2011) [26], Li and Wang (2013) [19], and Jin, Yang and Yin (2017) [17]. Finally, the verification theorem is derived. All the assumptions are similar to the assumptions in Section 3 except that there is the proportional transaction cost when transferring assets. The coefficient is denoted as \( k_i \), namely if fund \( i \) plans to transfer \( x \) amount asset to the other fund, it has to pay \( k_i x \).

Thus, the associated fund value processes \((X_1(t), X_2(t))\) with initial values \((x_{10}, x_{20})\) become,

\[
\begin{align*}
\dot{X}_1(t) &= (\mu_1 - l_1 - c_1)X_1(t)dt + \sigma_1 X_1(t) dW_1(t) + dC_1(t) - k_1 dC_1(t), \\
\dot{X}_2(t) &= (\mu_2 - l_2 - c_2)X_2(t)dt + \sigma_2 X_2(t) dW_2(t) + C_2 dt - k_2 dC_2(t),
\end{align*}
\]

where \( \{W_1(t) : t \geq 0\} \) and \( \{W_2(t) : t \geq 0\} \) are dependent standard Brownian motions, and their correlation coefficient is \( \rho \).

The following Proposition 2 is concluded without the proof from the definition of the value function, and the assumption that the assets could be transferred between the two funds at any time before \( \tau^1 \) with proportional transaction costs.

**Proposition 2.** If \( x_1 > a_1 x_{10} \) and \( x_2 > a_2 x_{20} \), then the value function \( V^{x_{10}, x_{20}}(x_1, x_2) \) is increasing with respect to both \( x_1 \) and \( x_2 \), and

\[
\begin{align*}
V^{x_{10}, x_{20}}(x_1 + \Delta x, x_2 - k_2 \Delta x) &\leq V^{x_{10}, x_{20}}(x_1, x_2), \\
V^{x_{10}, x_{20}}(x_1 - k_1 \Delta x, x_2 + \Delta x) &\leq V^{x_{10}, x_{20}}(x_1, x_2),
\end{align*}
\]

\( \forall \Delta x \in \mathbb{R} \) satisfying \( x_1 + \Delta x \geq a_1 x_{10} \) \( x_1 - k_1 \Delta x \geq a_1 x_{10} \) and \( x_2 - k_2 \Delta x \geq a_2 x_{20} \), \( x_2 + \Delta x \geq a_2 x_{20} \).

4.2. Hamilton-Jacobi-Bellman equation. At point \((x_1, x_2)\), if neither of the two mutual funds is compulsorily liquidated, the value function is given by the following lemma.

**Lemma 4.1.** If neither of the two funds is liquidated at \((x_1, x_2)\) immediately, then the value function \( V^{x_{10}, x_{20}}(x_1, x_2) \) satisfies the following conditions:

\[
\max \left\{ \mathcal{L} V^{x_{10}, x_{20}}(x_1, x_2), V^{x_{10}, x_{20}} - k_2 V^{x_{10}, x_{20}} - k_1 V^{x_{10}, x_{20}} \right\} = 0,
\]

where

\[
\mathcal{L} V(x_1, x_2) = c_1 x_1 + c_2 x_2 - \beta V + (\mu_1 - l_1 - c_1) x_1 V_{x_1}
\]

\[
+ (\mu_2 - l_2 - c_2) x_2 V_{x_2} + \frac{1}{2} \sigma_1^2 x_1^2 V_{x_1 x_1} + \frac{1}{2} \sigma_2^2 x_2^2 V_{x_2 x_2} + \rho \sigma_1 \sigma_2 x_1 x_2 V_{x_1 x_2}.
\]

**Proof.** The proof is similar to the proof of Lemma 3.1 and we omit it here. \( \square \)

We have the following theorem:
Theorem 4.2. At any point \((x_1, x_2) \in G_+\), the value function \(V^{x_{10}, x_{20}}(x_1, x_2)\) in (7) satisfies the following HJB equation:

\[
\max \left\{ \mathcal{L} V^{x_{10}, x_{20}}(x_1, x_2), V^{x_{10}, x_{20}}(x_1, x_2) - k_2 V^{x_{10}, x_{20}}(x_{2}, x_1), V^{x_{10}, x_{20}}(x_1, x_2) - k_1 V^{x_{10}, x_{20}}(x_1, x_2), \right. \\
V^{x_{10}}(x_1 + \frac{x_2 - a_2 x_{20}}{k_2}, x_2) - V^{x_{10}, x_{20}}(x_1, x_2), \\
V^{x_{20}}(x_2 + \frac{x_1 - a_1 x_{10}}{k_1}, x_2) - V^{x_{10}, x_{20}}(x_1, x_2) \left. \right\} = 0
\]

with boundary conditions:

\[
V^{x_{10}, x_{20}}(x_1, 0, x_{20}) = V^{x_{10}}(x_1), \\
V^{x_{10}, x_{20}}(0, x_{10}, x_2) = V^{x_{20}}(x_2).
\]

We now describe some properties of the optimal value function.

Proposition 3. The optimal value function \(V^{x_{10}, x_{20}}(x_1, x_2)\) is continuous in \(G_+\).

Proof. Assume \(h_1 > 0\) and \(h_2 > 0\), consider the following strategy \(\hat{C}()\) and let \((\hat{X}_1(t), \hat{X}_2(t))\) be the controlled surplus process associated with \(\hat{C}()\): there is no asset transfer before \(\hat{t} \triangleq \inf\{t \geq 0 : \hat{X}_1(t) \geq x_1 + h_1 \text{ and } \hat{X}_2(t) \geq x_2 + h_2\}\). Then

\[
V^{x_{10}, x_{20}}(x_1, x_2) \geq E_{x_{10}, x_{20}}(1_{\{\hat{t} < \tau_1\}} e^{-\beta(\hat{t} \wedge \tau_1)}) V^{x_{10}, x_{20}}(x_1 + h_1, x_2 + h_2).
\]

Hence,

\[
0 \leq V^{x_{10}, x_{20}}(x_1 + h_1, x_2 + h_2) - V^{x_{10}, x_{20}}(x_1, x_2) \leq \left( \frac{1}{E_{x_{10}, x_{20}}(1_{\{\hat{t} < \tau_1\}} e^{-\beta(\hat{t} \wedge \tau_1)})} - 1 \right) V^{x_{10}, x_{20}}(x_1, x_2).
\]

When \(h_1, h_2 \to 0\), we have \(\hat{t} \wedge \tau_1 \to 0\) and \(1_{\{\hat{t} < \tau_1\}} \to 1\). Thus, \(V^{x_{10}, x_{20}}(x_1 + h_1, x_2 + h_2) - V^{x_{10}, x_{20}}(x_1, x_2) \to 0\). The cases \(h_1 > 0, h_2 < 0, h_1 < 0, h_2 < 0, h_1 < 0, h_2 > 0\) are similar, so \(V^{x_{10}, x_{20}}(x_1, x_2)\) is continuous in \(G_+\).

4.3. Numerical solution to the HJB equation.

4.3.1. Penalty method. In this part, we propose a penalty method to deal with the inequality constraints in the HJB equation. Applying the penalty method directly to (14) would result in the following penalty approximation:

\[
\mathcal{L} V^{x_{10}, x_{20}}(x_1, x_2) + K_1 \max \left( V^{x_{10}, x_{20}}(x_1, x_2) - k_2 V^{x_{10}, x_{20}}(x_{2}, x_1), 0 \right) \\
+ K_2 \max \left( V^{x_{10}, x_{20}}(x_{2}, x_1) - k_1 V^{x_{10}, x_{20}}(x_1, x_2), 0 \right) \\
+ K_3 \max \left( V^{x_{10}}(x_1 + \frac{x_2 - a_2 x_{20}}{k_2}, x_2) - V^{x_{10}, x_{20}}(x_1, x_2), 0 \right) \\
+ K_4 \max \left( V^{x_{20}}(x_2 + \frac{x_1 - a_1 x_{10}}{k_1}, x_2) - V^{x_{10}, x_{20}}(x_1, x_2), 0 \right) = 0,
\]

where \(K_i, i = 1, 2, 3, 4\) are penalty terms, which are positive numbers large enough.
4.3.2. PDE discretization. To approximate the solution of the above partial differential equation (PDE) (15), we use the finite difference method (FDM) for both $X_1$ and $X_2$ discretizations. Dividing both $X_1$ direction and $X_2$ direction by $N$ grids, we let $X_1^{max} = 10x_1$ and $X_2^{max} = 10x_2$ be the upper bound of $X_1$ and $X_2$. The lower bound is $a_1x_{10}$ and $a_2x_{20}$, and the $X_1$ step is $\Delta X_1 = \frac{X_1^{max} - a_1x_{10}}{N}$, and $X_2$ step is $\Delta X_2 = \frac{X_2^{max} - a_2x_{20}}{N}$. We denote $x_{1i} = a_1x_{10} + (i - 1)\Delta X_1$, $x_{2j} = a_2x_{20} + (j - 1)\Delta X_2$, $(i, j = 1, ..., N + 1)$ and define $V_j^i \approx V^{x_{1i}x_{2j}}(x_{1i}, x_{2j})$ to be the approximation of the solution of (15) at point $(x_{1i}, x_{2j})$.

The derivatives in $\mathcal{L}V^{x_{10}x_{20}}(x_{1i}, x_{2j})$ are approximated by central difference method:

$$
\frac{\partial V_j^i}{\partial x_1} = \frac{V_j^{i+1} - V_j^{i-1}}{2\Delta X_1}, \quad \frac{\partial V_j^i}{\partial x_2} = \frac{V_j^{i+1} - V_j^{-1}}{2\Delta X_2},
$$

$$
\frac{\partial^2 V_j^i}{\partial x_1^2} = \frac{V_j^{i+1} - 2V_j^i + V_j^{i-1}}{(\Delta X_1)^2}, \quad \frac{\partial^2 V_j^i}{\partial x_2^2} = \frac{V_j^{i+1} - 2V_j^i + V_j^{i-1}}{(\Delta X_2)^2},
$$

$$
\frac{\partial^2 V_j^i}{\partial x_1 \partial x_2} = \frac{V_j^{i+1} + V_j^{-1} - V_j^{i-1} - V_j^{i+1}}{4\Delta X_1 \Delta X_2}.
$$

So the equation (15) at point $V_j^i$ is approximated by

$$
c_{1}x_{1i} + c_{2}x_{2j} - \beta V_j^i + (\mu_1 - l_1 - c_1)x_{1i} \frac{V_j^{i+1} - V_j^{i-1}}{2\Delta X_1}$$

$$
+ (\mu_2 - l_2 - c_2)x_{2j} \frac{V_j^{i+1} - V_j^{i-1}}{2\Delta X_2} + \frac{1}{2} \sigma^2 x_{1i} \frac{V_j^{i+1} - 2V_j^i + V_j^{i-1}}{(\Delta X_1)^2}$$

$$
+ \frac{1}{2} \sigma^2 x_{2j} \frac{V_j^{i+1} - 2V_j^i + V_j^{i-1}}{(\Delta X_2)^2} + \rho \sigma_1 \sigma_2 x_{1i} x_{2j} \frac{V_j^{i+1} + V_j^{-1} - V_j^{i-1} - V_j^{i+1}}{4\Delta X_1 \Delta X_2}$$

$$
+ K_1^{i,j} \max(\frac{V_j^{i+1} - V_j^{i-1}}{2\Delta X_1} - k_2 \frac{V_j^{i+1} - V_j^{i}}{2\Delta X_2}, 0)
$$

$$
+ K_2^{i,j} \max(\frac{V_j^{i+1} - V_j^{i-1}}{2\Delta X_2} - k_1 \frac{V_j^{i+1} - V_j^{i}}{2\Delta X_1}, 0)
$$

$$
+ K_3^{i,j} \max(\bar{V}_{1}^{x_{10}}(x_{1i} + \frac{x_{2j} - a_2x_{20}}{k_2}) - V_j^i, 0)
$$

$$
+ K_4^{i,j} \max(\bar{V}_{2}^{x_{20}}(x_{2j} + \frac{x_{1i} - a_1x_{10}}{k_1}) - V_j^i, 0) = 0.
$$

We use penalty method to solve the HJB equation. The penalty term is defined as a large quantity related to $tol$, which is a sufficiently small tolerance level:

$$
K_1^{i,j} = \begin{cases} \frac{1}{tol} \quad \text{if } \frac{V_j^{i+1} - V_j^{i-1}}{2\Delta X_1} > k_1 \frac{V_j^{i+1} - V_j^{i}}{2\Delta X_1}, \\ 0 \quad \text{if } \frac{V_j^{i+1} - V_j^{i-1}}{2\Delta X_1} \leq k_1 \frac{V_j^{i+1} - V_j^{i}}{2\Delta X_1}, \end{cases}
$$

and

$$
K_2^{i,j} = \begin{cases} \frac{1}{tol} \quad \text{if } \frac{V_j^{i+1} - V_j^{i-1}}{2\Delta X_2} > k_2 \frac{V_j^{i+1} - V_j^{i}}{2\Delta X_2}, \\ 0 \quad \text{if } \frac{V_j^{i+1} - V_j^{i-1}}{2\Delta X_2} \leq k_2 \frac{V_j^{i+1} - V_j^{i}}{2\Delta X_2}, \end{cases}
$$

and

$$
K_3^{i,j} = \begin{cases} \frac{1}{tol} \quad \text{if } V_j^i < \bar{V}_{1}^{x_{10}}(x_{1i} + \frac{x_{2j} - a_2x_{20}}{k_2}), \\ 0 \quad \text{if } V_j^i \geq \bar{V}_{1}^{x_{10}}(x_{1i} + \frac{x_{2j} - a_2x_{20}}{k_2}), \end{cases}
$$
and
\[ K_{4}^{i,j} = \begin{cases} 
\frac{1}{10} & \text{if } V_{j}^{i} < V_{2}^{x_{20}}(x_{2j} + \frac{x_{11} - a_{1}x_{10}}{k_{1}}), \\
0 & \text{if } V_{j}^{i} \geq V_{2}^{x_{20}}(x_{2j} + \frac{x_{11} - a_{1}x_{10}}{k_{1}}). 
\end{cases} \]

4.3.3. Upper bound conditions. To ensure the approximated solution derived by FDM to (15) is unique, we need to give an approximated value of \( V \) on upper bound \((X_{1}^{\max}, x_{2j}), j = 1, \ldots, N + 1\) and \((x_{1i}, X_{2}^{\max}), i = 1, \ldots, N + 1\). After that, the problem is converted to the PDE with Dirichlet boundary condition.

To ensure an approximated value of \( V_{j}^{N+1} \), namely \( V \) on \((X_{1}^{\max}, x_{2j}), j = 1, \ldots, N + 1\), we use backward difference method instead of central difference method, then (17) becomes

\[
\begin{align*}
&c_{1}X_{1}^{\max} + c_{2}x_{2j} - \beta V_{j}^{N+1} + \left( \mu_{1} - l_{1} - c_{1} \right) X_{1}^{\max} \frac{V_{j}^{N+1} - V_{j}^{N-1}}{2\Delta X_{1}} \\
&+ \left( \mu_{2} - l_{2} - c_{2} \right) x_{2j} \frac{V_{j}^{N+1} - V_{j}^{N-1}}{2\Delta X_{2}} \\
&+ \frac{1}{2} \sigma_{1}^{2}(X_{1}^{\max})^{2} \frac{V_{j}^{N+1} - 2V_{j}^{N} + V_{j}^{N-1}}{(\Delta X_{1})^{2}} \\
&+ \frac{1}{2} \sigma_{2}^{2} x_{2j}^{2} \frac{V_{j}^{N+1} - 2V_{j}^{N} + V_{j}^{N-1}}{(\Delta X_{2})^{2}} \\
&+ \rho \sigma_{1} \sigma_{2} X_{1}^{\max} x_{2j} \frac{V_{j}^{N+1} + V_{j}^{N-1} - V_{j}^{N-1} - V_{j}^{N+1}}{4\Delta X_{1}\Delta X_{2}} \\
&+ K_{1}^{N+1,j} \max \left( \frac{V_{j}^{N+1} - V_{j}^{N-1}}{2\Delta X_{1}}, 0 \right) \\
&+ K_{2}^{N+1,j} \max \left( \frac{V_{j}^{N+1} - V_{j}^{N-1}}{2\Delta X_{2}}, 0 \right) \\
&+ K_{3}^{N+1,j} \max \left( V_{x_{10}}(X_{1}^{\max} + \frac{x_{2j} - a_{2}x_{20}}{k_{2}}), V_{x_{10}}(x_{1}(N+1), x_{21}) = V_{x_{10}}(x_{1}^{\max}, a_{2}x_{20}) = V_{x_{10}}(X_{1}^{\max}), \right. \\
&\left. + K_{4}^{N+1,j} \max \left( V_{x_{20}}(x_{2j} + \frac{X_{1}^{\max} - a_{1}x_{10}}{k_{1}}), V_{x_{20}}(X_{1}^{\max}, a_{1}) \right) - V_{j}^{N+1} \right) = 0.
\end{align*}
\]

We have \( V_{1}^{N+1} = V_{x_{10}}(x_{1}(N+1), x_{21}) = V_{x_{10}}(X_{1}^{\max}, a_{2}x_{20}) = V_{x_{10}}(X_{1}^{\max}), \) and we approximate

\[
V_{2}^{N+1} \approx \max \left\{ V_{1}^{x_{10}}(X_{1}^{\max} + \frac{\Delta X_{2}}{k_{2}}), V_{2}^{x_{20}}(\frac{X_{1}^{\max}}{k_{1}} + \Delta X_{2}) \right\}.
\]

This is reasonable. When \( \frac{x_{1}}{k_{1}} + \frac{x_{2}}{k_{2}} \) is large enough, the optimal strategy is transferring the total asset to the better performed fund, which is verified by Monte Carlo simulation. Then for \( j = 3, 4, \ldots, N + 1 \), we have \( N - 1 \) equations like (18). Similarly, to ensure an approximated value of \( V \) on upper bound \((x_{1i}, X_{2}^{\max}), i = 1, \ldots, N\), we also use backward difference method instead of central difference method, and get \( N - 2 \) equations.

4.3.4. Solution to Dirichlet problem. We get \( 2N - 3 \) equations on the upper boundary, and for \( i, j = 2, \ldots, N \) we have \((N - 1) \ast (N - 1)\) equations like (17). So, we get \( N^{2} - 2 \) equations finally. The value to be determined are \( V_{j}^{i}, i, j = 2, \ldots, N, V_{j}^{N+1}, j = 3, \ldots, N + 1 \) and \( V_{N+1}^{i}, i = 3, \ldots, N, \) exactly \( N^{2} - 2 \) unknown values of \( V \).
The Newton iteration method is implemented to solve this \(N^2 - 2\) equations and get the approximated value of \(V\).

To be specific, for \(i, j = 2, ..., N\), we have

\[
A^i_j V^i_j + B^i_j V^{i+1}_j + C^i_j V^{i-1}_j + D^i_j V^i_{j+1} + E^i_j V^i_{j-1} = f^i_j,
\]

where

\[
A^i_j = -\beta - \frac{\sigma^2_i x^2_{1i}}{\Delta X^2_1} - \frac{\sigma^2_i x^2_{2j}}{\Delta X^2_2},
\]

\[
B^i_j = \frac{(\mu_1 - l_1 - c_1)x_{1i}}{2\Delta X_1} + \frac{\sigma^2_i x^2_{1i}}{2\Delta X^2_1},
\]

\[
C^i_j = -\frac{(\mu_1 - l_1 - c_1)x_{1i}}{2\Delta X_1} + \frac{\sigma^2_i x^2_{2j}}{2\Delta X^2_2},
\]

\[
D^i_j = \frac{(\mu_2 - l_2 - c_2)x_{2j}}{2\Delta X_2} + \frac{\sigma^2_i x^2_{2j}}{2\Delta X^2_2},
\]

\[
E^i_j = -\frac{(\mu_2 - l_2 - c_2)x_{2j}}{2\Delta X_2} + \frac{\sigma^2_i x^2_{2j}}{2\Delta X^2_2},
\]

\[
f^i_j = f^i_j (V^{i+1}_j, V^{i-1}_j, V^i_{j+1}, V^i_{j-1})
\]

\[
= c_1 x_{1i} + c_2 x_{2j} + \rho \sigma_1 \sigma_2 x_{1i} x_{2j} - \frac{V^i_{j+1} + V^i_{j-1} - V^{i-1}_{j+1} - V^{i-1}_{j-1}}{4\Delta X_1 \Delta X_2}
\]

\[
\left( K^1_{i,j} \max(\frac{V^i_{j+1} - V^i_{j-1}}{2\Delta X_1} - k_2 \frac{V^i_{j+1} - V^i_{j-1}}{2\Delta X_2}, 0) + K^2_{i,j} \max(\frac{V^i_{j+1} - V^i_{j-1}}{2\Delta X_2} - k_1 \frac{V^i_{j+1} - V^i_{j-1}}{2\Delta X_1}, 0) + K^3_{i,j} \max(V^i_{j+1} - k_2 x_{2j} - a_2 x_{20}, 0) - V^i_j, 0) + K^4_{i,j} \max(V^i_{j+1} - k_1 x_{1i} - a_1 x_{10}, 0) - V^i_j, 0) \right).
\]

For \(i = N + 1, j = 3, ..., N + 1\), we have

\[
A^i_j V^i_j + C^i_j V^{i-1}_j + F^i_j V^{i-2}_j + E^i_j V^i_{j-1} + G^i_j V^i_{j-2} = f^i_j,
\]

where

\[
A^i_j = -\beta + \frac{\sigma^2_i (X^i_{1\max})^2}{2\Delta X^2_1} + \frac{\sigma^2_i x^2_{2j}}{2\Delta X^2_2} + \frac{(\mu_1 - l_1 - c_1)X^i_{1\max}}{2\Delta X_1} + \frac{(\mu_2 - l_2 - c_2)x_{2j}}{2\Delta X_2},
\]

\[
C^i_j = -\frac{\sigma^2_i (X^i_{1\max})^2}{\Delta X^2_1},
\]

\[
E^i_j = -\frac{\sigma^2_i x^2_{2j}}{\Delta X^2_2},
\]

\[
F^i_j = -\frac{(\mu_1 - l_1 - c_1)X^i_{1\max}}{2\Delta X_1} + \frac{\sigma^2_i (X^i_{1\max})^2}{2\Delta X^2_1}.
\]
\[ G_j^i = -\frac{(\mu_2 - l_2 - c_2)x_{2j}}{2\Delta X_2} + \frac{\sigma_2^2 x_{2j}^2}{2\Delta X_2}, \]
\[ f_j^i = f_j^i(V_j^i, V_{j-1}^i, V_{j-2}^i, V_{j-3}^i, V_{j-4}^i) = c_1 x_1^{\text{max}} + c_2 x_{2j} \]
\[ + \rho \sigma_1 \sigma_2 x_1^{\text{max}} x_{2j} \cdot \frac{V_{j+1}^i + V_{j-1}^i - V_j^i - V_{j-2}^i}{4\Delta X_1 \Delta X_2} \times \]
\[ \left( K_1^{N+1,j} \max \left( \frac{V_{j+1}^i - V_j^i}{2\Delta X_1} - k_2 \frac{V_j^i - V_{j-2}^i}{2\Delta X_2}, 0 \right) \right. \]
\[ + K_2^{N+1,j} \max \left( \frac{V_{j+1}^i - V_{j-2}^i}{2\Delta X_2} - k_1 \frac{V_{j-2}^i - V_j^i}{2\Delta X_1}, 0 \right) \]
\[ + K_3^{N+1,j} \max (V_j^{x_{10}}(X_1^{\text{max}} + \frac{x_{2j} - a_2 x_{20}}{k_2}) - V_j^{N+1}, 0) \]
\[ + K_4^{N+1,j} \max (V_j^{x_{20}}(x_{2j} + \frac{X_1^{\text{max}} - a_1 x_{10}}{k_1}) - V_j^{N+1}, 0) \). \]

Then for \( i = 3, ..., N, j = N + 1 \), we also have
\[ A_j^i V_j^i + C_j^i V_j^{i-1} + F_j^i V_j^{i-2} + E_j^i V_j^{i-3} + G_j^i V_j^{i-4} = f_j^i, \]
and the parameters could be derived with the same procedures in (19).

Denoting \( \mathbf{V}_j = (V_j^2, V_j^3, ..., V_j^N, V_j^{N+1})' \), and \( \mathbf{V} = (V_2, V_3, ..., V_{N+1})' \), \( \mathbf{V} \) is a vector of \( N^2 \) dimension. Denoting \( \mathbf{f} = (f_j^2, f_j^3, ..., f_j^N, f_j^{N+1})' \), and \( \mathbf{f} = (f_2, f_3, ..., f_{N+1})' \), it is also a vector of \( N^2 \) dimension. The equations above can be rewritten as:
\[ \mathbf{K} \mathbf{V} = \mathbf{f}, \] (20)

where
\[ \mathbf{K} = \begin{pmatrix}
K_2 & M_2 & 0 & 0 & \ldots \\
L_3 & K_3 & M_3 & 0 & \ldots \\
0 & L_4 & K_4 & \ddots & \ldots \\
\vdots & \vdots & \ddots & \ddots & \mathbf{M}_N \\
0 & 0 & 0 & L_{N+1} & \mathbf{K}_{N+1}
\end{pmatrix}. \]

is a \( N^2 \times N^2 \) matrix, and \( \mathbf{K}(i), \mathbf{L}(i), \mathbf{M}(i) \) are \( N \times N \) matrixes. For \( i = 2, ..., N, \)
\[ \mathbf{K}_i = \begin{pmatrix}
A_i^2 & B_i^2 & 0 & 0 & \ldots \\
C_i^3 & A_i^3 & B_i^3 & 0 & \ldots \\
0 & C_i^4 & A_i^4 & \ddots & \ldots \\
\vdots & \vdots & \ddots & \ddots & \mathbf{B}_i^N \\
0 & 0 & 0 & C_i^{N+1} & A_i^{N+1}
\end{pmatrix}, \]
\[ \mathbf{L}_i = \begin{pmatrix}
E_i^2 & 0 & 0 & 0 & \ldots \\
0 & E_i^3 & 0 & 0 & \ldots \\
0 & 0 & E_i^4 & \ddots & \ldots \\
\vdots & \vdots & \ddots & \ddots & 0 \\
0 & 0 & 0 & 0 & E_i^{N+1}
\end{pmatrix}. \]
To implement the upper condition $i = N + 1$, let

$$
\mathbf{K}_i = \begin{pmatrix}
0 & 0 & 0 & 0 & \cdots \\
C_i^3 & A_i^3 & 0 & 0 & \cdots \\
F_i^4 & C_i^4 & A_i^4 & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
0 & 0 & F_i^{N+1} & C_i^{N+1} & A_i^{N+1}
\end{pmatrix}.
$$

and add matrix $\mathbf{G}_{N+1}$ in front of $\mathbf{L}_{N+1}$ and $\mathbf{K}_{N+1}$. Thus, the last row of $\mathbf{K}$ becomes $(0, 0, \ldots, \mathbf{G}_{N+1}, \mathbf{L}_{N+1}, \mathbf{K}_{N+1})$, where

$$
\mathbf{G}_{N+1} = \begin{pmatrix}
0 & 0 & 0 & 0 & \cdots \\
0 & G_{N+1}^4 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
0 & 0 & 0 & 0 & G_{N+1}^{N+1}
\end{pmatrix}.
$$

Dealing with the upper condition $j = N + 1$ similarly, we finally get equation (20), where $\mathbf{V}$ is a $N^2$ vector with rank $N^2 - 2$. Since we know the values of $V_{2N+1}$ and $V_{2N+1}$, $\mathbf{K}$ is a $N^2 \times N^2$ matrix of rank $N^2 - 2$ with tow rows are zero. Then, the Newton iteration method is implemented to solve equation (20) and derive the value of $\mathbf{V}$.

4.4. The verification theorem.

**Definition 4.3.** A continuous function $\bar{u}(\mathbf{y}) : \mathbf{G}_+ \to \mathbf{R}$ is a viscosity supersolution(subsolution) of (14) at $(x_1, x_2) \in \mathbf{G}_+$ if any twice continuously differentiable function $\phi : \mathbf{G}_+ \to \mathbf{R}$ with $\phi(x_1, x_2) = \bar{u}(x_1, x_2)(\bar{u}(x_1, x_2))$ such that $\bar{u} - \phi(\bar{u} - \phi)$ reaches the minimum(maximum) at $(x_1, x_2)$ satisfies

$$
\max \left\{ L\phi(x_1, x_2), \phi_{x_1} - k_2 \phi_{x_2}, \phi_{x_2} - k_2 \phi_{x_1}, V_{2}^{x_{10}}(x_1 + \frac{x_2 - a_2 x_{10}}{k_1}) - \phi(x_1, x_2), \right. \\
V_{2}^{x_{10}}(x_2 + \frac{x_1 - a_1 x_{10}}{k_1}) - \phi(x_1, x_2) \left. \right\} \leq 0(\geq 0).
$$

Since $C_{12}, C_{21}$ are right continuous with left limits, we can write

$$
C_{ij}(t) = \int_0^t dC_{ij}^c(s) + \sum_{\mathbf{x}_i(s) \neq \mathbf{x}_i(s) \lor \mathbf{x}_2(s) \neq \mathbf{x}_2(s) | s \leq t_i} (C_{ij}(s) - C_{ij}(s-)).
$$

Then the following Lemma is a direct application of Itô’s formula for semi-martingales and the optional sampling theorem.

**Lemma 4.4.** Let $\mathbf{X}_1(t), \mathbf{X}_2(t)$ be the controlled state processes with control $C(\cdot)$ and initial values $x_1, x_2$. For any twice continuously differentiable function $\phi$ on
Let $\phi$ be a test function. Assume $(x_1, x_2)$ is the minimum point of $V - \phi$ satisfying $V^{x_{10},x_{20}}(x_1, x_2) = 0$. For any fixed $c_{12}, c_{21} > 0$, consider the following strategy $C(t) = \{dC_{12}(t) = c_{12}dt, dC_{21}(t) = c_{21}dt : \Delta t \geq t \geq 0\}$, and let $(\bar{X}_1(\cdot), \bar{X}_2(\cdot))$ be the controlled fund value process associated with $C(t)$. With initial fund value $(x_1, x_2)$, the transfer strategy is $C(t)$ until time $\Delta t \cap \tau^1$. Then

$$
\phi(x_1, x_2) = V^{x_{10},x_{20}}(x_1, x_2)
\geq E_{x_1, x_2} \left\{ \int_0^{\tau^1 \cap \Delta t} e^{-\beta s}[c_1\bar{X}_1(s) + c_2\bar{X}_2(s)]ds 
+ e^{-\beta s}[\phi(x_1(\cdot), x_2(\cdot))] \right\}
\geq E_{x_1, x_2} \left\{ \int_0^{\tau^1 \cap \Delta t} e^{-\beta s}[c_1\bar{X}_1(s) + c_2\bar{X}_2(s)]ds 
+ E_{x_1, x_2} \left[ e^{-\beta s}[\phi(x_1(\cdot), x_2(\cdot))] \right] \right\}.
$$

Using Lemma 4.4 to $e^{-\beta s}[\phi(x_1(\cdot), x_2(\cdot))]$ and letting $\Delta t \to 0$, we have

$$
0 \geq E_{x_1, x_2} \left[ e^{-\beta s}[\phi(x_1(\cdot), x_2(\cdot))] \right] = \{\phi(x_1, x_2) - k_2\phi x_2, \phi x_2 - k_1\phi x_1\} = \{c_1, c_2\}.\]

Let $c_{11} = c_{12} = 0$, we get \(L\phi(x_1, x_2) \leq 0\). Let $c_{21} \to +\infty(c_{21} \to +\infty)$, we get $\phi x_1 - k_2\phi x_2 \leq 0(\phi x_2 - k_1\phi x_1 \leq 0)$. Notice $\phi(x_1, x_2) = V^{x_{10},x_{20}}(x_1, x_2) \geq V^{x_{10}}(x_1 + \frac{x_2 - a_2 x_{20}}{k_2})$, and $\phi(x_1, x_2) = V^{x_{10},x_{20}}(x_1, x_2) \geq V^{x_{20}}(x_2 + \frac{x_1 - a_1 x_{10}}{k_1})$. We finally get

$$
\max \left\{ L\phi(x_1, x_2), \phi x_1 - k_2\phi x_2, \phi x_2 - k_1\phi x_1, V^{x_{10}}(x_1 + \frac{x_2 - a_2 x_{20}}{k_2}) - \phi(x_1, x_2), \right\} \leq 0,
$$

and finish the proof.}

**Proposition 5.** The optimal value function $V$ is a viscosity subsolution of (14).

**Proof.** Let $\varphi$ be a test function. Assume $(x_1, x_2)$ is the maximum point of $V - \varphi$ satisfying $V^{x_{10},x_{20}}(x_1, x_2) = \varphi(x_1, x_2)$ and $V^{x_{10},x_{20}} \leq \varphi$. Then for any $\varepsilon > 0$ and
\(\Delta t > 0\) with \(\Delta t\) small enough, by the definition of value function (8) and Bellman's principle of optimality, we can find a strategy \(C^\ast(t) = \{(C^{12}(t), C^{21}(t)) : \Delta t \geq t \geq 0\}\) such that
\[
V^{x_{10},x_{20}}(x_1, x_2) \leq \varepsilon(\tau^1 \cap \Delta t) + E_{x_1, x_2} \left\{ \int_{0}^{\tau^1 \cap \Delta t} e^{-\beta s} [c_1(s)X^c_1(s-1) + c_2(s)X^c_2(s-1)]ds \\
+ e^{-\beta \tau^1 \cap \Delta t} V^{x_{10},x_{20}}(X^c_1(\tau^1 \cap \Delta t), X^c_2(\tau^1 \cap \Delta t)) \right\}.
\]

Then,
\[
\varphi(x_1, x_2) = V^{x_{10},x_{20}}(x_1, x_2) \\
\leq \varepsilon(\tau^1 \cap \Delta t) + E_{x_1, x_2} \left\{ \int_{0}^{\tau^1 \cap \Delta t} e^{-\beta s} [c_1(s)X^c_1(s-1) + c_2(s)X^c_2(s-1)]ds \\
+ e^{-\beta \tau^1 \cap \Delta t} V^{x_{10},x_{20}}(X^c_1(\tau^1 \cap \Delta t), X^c_2(\tau^1 \cap \Delta t)) \right\}.
\]

Applying Lemma 4.4 to \(e^{-\beta \tau^1 \cap \Delta t} \varphi(X^c_1(\tau^1 \cap \Delta t), X^c_2(\tau^1 \cap \Delta t))\) in (21) and letting \(\Delta t \to 0\), we have
\[
\mathcal{L}\varphi(x_1, x_2) + (\phi_{x_1} - k_2\phi_{x_2})C^{21}(0) + (\phi_{x_2} - k_1\phi_{x_1})C^{12}(0) \geq 0.
\]

Since \(C^{21}(0) \geq 0\) and \(C^{12}(0) \geq 0\), we have
\[
\max \left\{ \mathcal{L}\varphi(x_1, x_2), \phi_{x_1} - k_2\phi_{x_2}, \phi_{x_2} - k_1\phi_{x_1} \right\} \geq 0,
\]
and finish the proof. \(\square\)

We summarize the results above into the following theorem.

**Theorem 4.5.** The optimal value function \(V\) is a continuous viscosity solution of (14).

For the opposite direction of the verification theorem, the following theorem adds some necessary conditions and completes the proof.

**Theorem 4.6.** If \(\bar{u}\) is a viscosity supersolution of (14) with bounded second derivatives, then for any strategy \(C(\cdot) \in \pi(x_1, x_2)\), the corresponding value function \(V^{C, x_{10}, x_{20}}(x_1, x_2) \leq \bar{u}\). Furthermore, consider a family of admissible strategies \(C^*(\cdot) \in \pi(x_1, x_2)\), if the corresponding function \(V^{C^*, x_{10}, x_{20}}(x_1, x_2)\) is a viscosity supersolution of (14) at \(\forall(x_1, x_2) \in \mathbf{G}_+\), and \(V^{C^*, x_{10}, x_{20}}(x_1, x_2)\) has bounded second derivatives, then \(V^{C^*, x_{10}, x_{20}}(x_1, x_2)\) is the optimal value function.

**Proof.** Let \(\bar{u}\) be a viscosity supersolution of (14) at \((x_1, x_2)\) with bounded second derivatives. Since \(\bar{u}\) is twice differentiable at \((x_1, x_2)\), we use Taylor approximation
to estimate the value of $\bar{u}(x_1 + \Delta x_1, x_2 + \Delta x_2)$, that $(x_1 + \Delta x_1, x_2 + \Delta x_2)$ is near $(x_1, x_2)$:

$$\bar{u}(x_1 + \Delta x_1, x_2 + \Delta x_2) = \bar{u}(x_1, x_2) + \bar{u}_x(x_1, x_2) \Delta x_1$$

$$+ \bar{u}_x(x_1, x_2) \Delta x_2 + \frac{1}{2} \bar{u}_{xx}(x_1, x_2)(\Delta x_1)^2$$

$$+ \frac{1}{2} \bar{u}_{xx}(x_1, x_2)(\Delta x_2)^2 + o((\Delta x_1)^2 + (\Delta x_2)^2).$$

When $(\Delta x_1)^2 + (\Delta x_2)^2$ is sufficiently small, define

$$\chi_n(x_1 + \Delta x_1, x_2 + \Delta x_2) = \bar{u}(x_1, x_2) + \bar{u}_x(x_1, x_2) \Delta x_1 + \bar{u}_x(x_1, x_2) \Delta x_2$$

$$+ \frac{1}{2}(\bar{u}_{xx}(x_1, x_2) - \frac{1}{n})(\Delta x_1)^2 + \frac{1}{2}(\bar{u}_{xx}(x_1, x_2) - \frac{1}{n})(\Delta x_2)^2.$$  

Hence $\chi_n(x_1, x_2) = \bar{u}(x_1, x_2)$ and $\bar{u} - \chi_n \geq 0$ in a neighborhood of $(x_1, x_2)$. As $\bar{u}$ is continuous, we can extend $\chi_n$ to $\mathcal{G}_+$ such that $\chi_n$ is twice continuously differentiable and $\bar{u} - \chi_n$ reaches its minimum at $(x_1, x_2)$. Therefore, $\chi_n$ is a test function of $\bar{u}$ in $(x_1, x_2)$. We have

$$\max \left\{ L\chi_n(x_1, x_2), \chi_{n_x} - k_2 \chi_{n_{xx}}, \chi_{n_{xx}} - k_1 \chi_{n_{x}}, V_1^{x_{10}}(x_1 + \frac{x_1 - a_2 x_{20}}{k_2}) - \chi_n(x_1, x_2) \right\} \leq 0.$$  

Let $n \to +\infty$, for $\forall (x_1, x_2) \in \mathcal{G}_+$, we get

$$\max \left\{ L\bar{u}(x_1, x_2), \bar{u}_x - k_2 \bar{u}_{xx}, \bar{u}_{xx} - k_1 \bar{u}_x, V_1^{x_{10}}(x_1 + \frac{x_1 - a_2 x_{20}}{k_2}) - \bar{u}(x_1, x_2), V_2^{x_{20}}(x_2 + \frac{x_1 - a_1 x_{10}}{k_1}) - \bar{u}(x_1, x_2) \right\} \leq 0.  \tag{22}$$

In particular, $L\bar{u}(x_1, x_2) \leq 0$. Then, for any strategy $C(\cdot) \in \pi(x_1, x_2)$ and its corresponding state process $(X_1^C(s), X_2^C(s))$, applying Lemma 4.4 to $e^{-\beta s} \bar{u}(X_1^C(s), X_2^C(s))$ and combing (22), we finally get

$$\bar{u}(x_1, x_2) \geq V_{C, x_{10}, x_{20}}(x_1, x_2),$$

and finish the proof.  

\[ \square \]

5. \textbf{Sensitivity analysis.}

5.1. \textbf{The simple case: No transaction cost.} In this subsection, we study the optimal tunneling behaviors of the fund managers, and the impacts of the behaviors on the utilities of the investors without the transaction cost under the simulation framework. First, we assign the parameters according to the empirical data and the literatures. As the performance and the management fee rate may both alter the behaviors of the fund managers, two prototypes of scenarios are considered in the simulation. In one scenario, the management fee rate is consistent with the performance of the fund, i.e., the better performed fund charges higher management fee rate. In the other scenario, the management fee rate is not consistent with the performance. The circumstance may exist due to the excellent marketing strategies or strong reversal expectations. Next, we calculate the optimal asset transfer policies according to the principles in Subsection 3.3. As the optimal transfer policies are irrelevant to the time, the optimal policies are established under the static basis.
The last, we use Monte Carlo Methods to simulate the tunneling behaviors and the total rewards of the fund managers. The statistical results are also used to evaluate the impacts on the utilities of the investors.

First, the parameters are assigned according to the empirical data and the literatures, as in Chevalier and Ellison (1997)[6], and Guercio and Tkac (2002)[15]. In order to protect the interest of the investor and control the risk, there usually exist the 80% maximal withdraw levels in the mutual fund industry. So, the maximal withdraw levels are \(a_1 = 0.8, a_2 = 0.8\). The risk-free interest rate is \(\beta = 0.03\). In the paper, the fund is locked-up after being raised successfully. The investor obtains the rewards by sustainably receiving the fixed proportions of the fund assets. As the reward rates of the investors should neither be too low, nor too high. \(l_1 = 0.04\) and \(l_2 = 0.04\) are assigned by averaging the common return of the mutual fund industry and the risk-free interest rate. Furthermore, the expected returns and the volatilities of the two funds are estimated according to their past performances. The magnitude and the range of two parameters are consistent with the empirical data and the literatures. For simplicity, the volatilities of the two funds are controlled as \(\sigma_1 = \sigma_2 = 0.3\). In order to study the impacts of the investment return and the management fee rate on the tunneling behaviors of the fund manager, the following two prototypes of scenarios are studied:

**Case 1.** The management fee rates are positively correlated with the investment returns. In this scenario, the expected returns and the management fee rates of Fund One and Fund Two are \(\mu_1 = 0.1, c_1 = 0.05; \mu_2 = 0.08, c_2 = 0.03\), respectively.

**Case 2.** The management fee rates are negatively correlated with the investment returns, and \(\mu_2 = 0.06, c_1 = 0.05; \mu_2 = 0.08, c_2 = 0.03\), respectively.

The initial values of the two funds are \(x_{10} = x_{20} = 10.7\). So, the values of the parameters are summarized in the following Table 1:

Next, the optimal asset transfer policies are established under the two scenarios in Case 1 and Case 2. The transfer policies are the functions of the assets levels of the two funds, but irrelevant to the time. So, the tunneling behaviors are established as the static two-dimensional functions. The principles to calculate the optimal transfer policies are given in Subsection 3.3. In Figure 1, we calculate the optimal asset transfer policies in Case 1 that the management fee rates are consistent with the investment returns. The results show that Fund Two transfers almost its total assets to Fund One and only maintains the minimal assets to avoid compulsory liquidation. Obviously, there exits a linear relationship between the asset level of Fund Two and the amount transferred from Fund Two to Fund One. Furthermore, when the asset level of Fund Two approaches the maximal withdraw level, Fund One immediately transfers the minimal assets required to it to avoid the liquidation. In this case, the well performed and high rewarded fund manager obtains most of the total assets by tunneling, and only keeps the other fund at the brink of liquidation. The other fund could be treated as a risk-independent asset, and is expected to provide liquidity when the well performed fund encounters the unexpected shocks. To make the results more clear, the two-dimensional figures are given in the appendix, that the assets of Fund One and Fund Two are controlled as \(x_1 = x_{10} * 80\%\), \(x_{10}, 100, 500\) and \(x_2 = x_{20} * 80\%\), \(x_{20}, 100, 500\), respectively.

In Figure 2, we calculate the optimal asset transfer policies in Case 2 that the management fee rates are not consistent with the investment returns. In Fund One, the expected return is lower, but the management fee rate is higher. As the
Table 1. Values of the parameters in the model

| Parameter | Value |
|-----------|-------|
| $a_1$     | 0.8   |
| $a_2$     | 0.8   |
| $\beta$   | 0.03  |
| $l_1$     | 0.04  |
| $l_2$     | 0.04  |
| $\sigma_1$| 0.3   |
| $\sigma_2$| 0.3   |
| $x_{10}$  | 10.7  |
| $x_{20}$  | 10.7  |

Case 1.
| $\mu_1$  | 0.1   |
| $c_1$     | 0.05  |
| $\mu_2$  | 0.08  |
| $c_2$     | 0.03  |

Case 2.
| $\mu_1$  | 0.06  |
| $c_1$     | 0.05  |
| $\mu_2$  | 0.08  |
| $c_2$     | 0.03  |

Figure 1. The amount of assets transferred from Fund Two to Fund One in Case 1 (without transaction cost).
Figure 2. The amount of assets transferred from Fund Two to Fund One in Case 2 (without transaction cost).

fund volatilities and the investor reward rates are both controlled in the model, the fund with lower return and higher management fee rate has lower growth potential. Furthermore, this will reduce the long term profitability of the fund and increase the probability of early liquidation. However, the higher management fee rate may generate higher short term rewards for the fund manager. The optimal tunneling results show that Fund One transfers almost its total assets to Fund Two and only maintains the minimal assets to avoid liquidation. Similarly, there exits a linear relationship between the asset level of Fund One and the amount transferred from Fund One to Fund Two. Furthermore, when Fund One approaches the maximal withdraw level, Fund Two helps it to avoid the liquidation. To make the results more clear, the two-dimensional figures are given in the appendix, that the assets of Fund One and Fund Two are controlled as \( x_1 = x_{10} \times 80\%, x_{10}, 100, 500 \) and \( x_2 = x_{20} \times 80\%, x_{20}, 100, 500 \), respectively.

Fortunately, in this case, the net expected return (expected return minus management fee rate) of the well performed fund is attractive, the growth of the fund asset could be explosive, and the duration of the fund could be much longer. So, this fund could generate enough rewards for the manager to offset the adverse impacts of the lower management fee rate. Similarly, the badly performed and high rewarded fund is kept slightly from liquidation to be used as a spare liquidity provider.

In the following part, we use the Monte Carlo methods to study the impacts of the tunneling behaviors on the utilities of the fund managers and the investors under the two scenarios. The two fund managers cooperatively control the asset transfer policies to maximize the utility function in Eq.(6). The respective utility is calculated as in the single fund scenario in Eq.(1), and the state process Eq.(4) under the above optimal transfer policies. As discussed above, under the private relations between the two fund managers, they may transfer the assets illegally from
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one to the other. This will improve the utilities of the fund managers and reduce the utilities of the investors. There are two keynotes to evaluate the impacts of the tunneling behaviors: the changes of the total rewards and the durations of the funds with and without tunneling behaviors.

First, the Monte Carlo methods are implemented to generate 10000 tracks of two independent Browning motion processes \( \{W_1(t), t \geq 0\} \) and \( \{W_2(t), t \geq 0\} \), which represent the exact outcomes of yearly returns of Fund One and Fund Two, respectively. Second, in the case without the tunneling behaviors, the asset levels of two funds are calculated according to Eq.(3) and the stochastically generated Brownian motion processes. At each step, the rewards of the fund managers and the investors are simultaneously distributed. We trace the evolutions of the asset levels of the two funds until the last one reaches the maximal withdraws. Third, in the case with the tunneling behaviors, the exact investment returns are calculated according Eq.(4) by using the above Brownian motion processes to maintain the comparability. Besides that, the asset transfer behaviors are calculated according to the optimal tunneling policies in Subsection 3.3. At each step, we re-calculate of the asset levels of the funds by combinations of the investment returns, the distributed rewards and the transferred assets. After one fund is compulsorily liquidated, the other fund evolves as the single fund case. Finally, we statistically calculate the average rewards of the fund mangers and the average durations of the funds of the 10000 stochastic tracks. As the rewards of the investors are proportional to the rewards of the fund managers, we omit the repeated statistics here. The results are summarized in Table 2 and Table 3.

**Table 2.** The impacts of the tunneling behaviors in Case 1.

| Keynote | Value  |
|---------|--------|
| The management fee of Fund One without tunneling: \( V_1^{x_{10}} \) | 15.55 |
| The management fee of Fund Two without tunneling: \( V_2^{x_{20}} \) | 9.65 |
| The total management fee without tunneling: \( V_1^{x_{10}} + V_2^{x_{20}} \) | 25.2 |
| The management fee of Fund One with tunneling: \( \tilde{V}_1^{x_{10}} \) | 31.02 |
| The management fee of Fund Two with tunneling: \( \tilde{V}_2^{x_{20}} \) | 0.46 |
| The total management fee with tunneling: \( V_1^{x_{10}, x_{20}} = \tilde{V}_1^{x_{10}} + \tilde{V}_2^{x_{20}} \) | 31.48 |
| The duration of the Fund One without tunneling(year): \( E_{x_{10}} \{ T_1 \} \) | 47.14 |
| The duration of the Fund Two without tunneling(year): \( E_{x_{20}} \{ T_2 \} \) | 49.28 |
| The duration of the Fund One with tunneling(year): \( E_{x_{10}} \{ \tau_1 \} \) | 49.22 |
| The duration of the Fund Two with tunneling(year): \( E_{x_{20}} \{ \tau_2 \} \) | 49.19 |
| The average exchange amount(Fund Two to Fund One): \( E_{x_{10}, x_{20}} \int_{T_1}^{T_2} \delta(s)ds \) | 0.012 |

The impacts of the tunneling behaviors are summarized in Table 2 under the case that the management fee rates are consistent with the investment returns. In this scenario, the net investment returns are equal. So, the expected durations of the two funds are almost the same in the scenario without tunneling behaviors. Furthermore, the differences of the rewards are only due to the differed management fee rates. However, in the scenario with tunneling behaviors, most of the total assets have been transferred into Fund One, which is well performed and high rewarded. In this circumstance, the assets of the well performed fund are explosive, and the rewards of the manager are huge. The rewards of the badly performed and low rewarded fund decline sharply at the same time. Being consistent with
the instinct, the two fund managers increase the total rewards slightly by using the tunneling behaviors. However, this harms the investors in the badly performed fund enormously. Though the duration of this fund extends, it is only kept around the brink of the liquidation for the purpose of being a risk-independent liquidity provider.

| Keynote                                                                 | Value   |
|------------------------------------------------------------------------|---------|
| The management fee of Fund One without tunneling: \( V_1^{x_1} \)       | 3.05    |
| The management fee of Fund Two without tunneling: \( V_2^{x_2} \)       | 9.39    |
| The total management fee without tunneling: \( V_1^{x_1} + V_2^{x_2} \)  | 12.44   |
| The management fee of Fund One with tunneling: \( \tilde{V}_1^{x_1} \)  | 0.71    |
| The management fee of Fund Two with tunneling: \( \tilde{V}_2^{x_2} \)  | 17.41   |
| The total management fee with tunneling: \( V_1^{x_1-\tilde{V}_1^{x_1}} + V_2^{x_2-\tilde{V}_2^{x_2}} \) | 18.12   |
| The duration of the Fund One without tunneling(year): \( E_{x_1}^{T_1} \) | 6.84    |
| The duration of the Fund Two without tunneling(year): \( E_{x_2}^{T_2} \) | 45.87   |
| The duration of the Fund One with tunneling(year): \( E_{x_1}^{\tau_1} \) | 45.13   |
| The duration of the Fund Two with tunneling(year): \( E_{x_2}^{\tau_2} \) | 45.17   |
| The average exchange amount(Fund Two to Fund One): \( E_{x_1,x_2} \int_{T_1}^{\tau_1} \delta(s)ds \) | -0.013  |

In Table 3, we study the impacts of the tunneling behaviors in the case that the management fee rates are not consistent with the investment returns. In this scenario, the duration of the fund with bad performance and high management fee rate is much shorter than the other fund. Meanwhile, the rewards of this kind of fund manager are much lower. Being a bit inconsistent with the instinct, in the scenario with tunneling behaviors, most of the total assets have been transferred into Fund Two, which is better performed and lower rewarded. In this circumstance, the assets of the well performed fund are explosive. The growth potential of the fund is so attractive that it could offset the adverse impacts of the lower management fee rate. In this scenario, the two fund managers increase the total rewards and extend both of the durations enormously by using the tunneling behaviors. Similarly, the behavior further reduces the rewards of the investors in the badly performed fund. The longer duration of the badly performed fund is just due to the liquidity issues.

It is notable that the overall improvement of the fund managers’ rewards by the tunneling behaviors is minor in Case 1. So, if the transaction fee issue is taken into consideration, then the fund manager may give up the illegal behaviors. On the contrary, the improvement of the rewards is so attractive in Case 2 that even the fund managers are hard to give up. Thus investors could pay attention to the fund with poor past performance as well as abnormal high management fee rate when making investment plans. The high management fee rate could neither make the fund managers work efficiently, nor induce the beneficial tunneling behaviors. The private relations between the fund managers and the illegal tunneling behaviors will make the rewards of the investors even worse in this circumstance.

5.2. Impacts of the Transaction cost. In this subsection, we study the impacts of the transaction cost on the optimal tunneling behaviors. In order to demonstrate the impacts clearly, we choose the relatively large transaction cost, as \( k_1 = k_2 = 2 \). The results in Figure 3 and Figure 4 show the amount of assets transferred from
Fund Two to Fund One under the scenarios that the management fees are in line and not in line with the performances, respectively. The results are consistent with the results in Figure 1 and Figure 2, except that the transfer amounts are reduced, and there even do not exist the transfers in some circumstances. Larger transaction cost reduces the amount of asset transfer, and moderates the tunneling behaviors. Furthermore, the tunneling behavior vanishes as the transaction cost is large enough. In this perspective, larger transaction cost is favorable to the investors.

5.3. Impacts of the risk dependent. In this subsection, we study the impacts of the risk dependent on the optimal tunneling behaviors. In order to demonstrate the impacts clearly, we study the extreme case, that the two Brownian motions in Eq.(13) are fully correlated, as $\rho = 1$. The results in Figure 5 and Figure 6 are consistent with the results in Figure 1 and Figure 2, except that the total asset is transferred to the better performed fund and the other fund is liquidated immediately. Technically, no matter what the combinations of $x_1$ and $x_2$ are, the optimal tunneling strategy is Option One or Option Two. In this extreme scenario, the two funds have the same risk source. The optimal control is to transfer the total assets to the better performed fund, as the diversified investment can not reduce the risk. In this circumstance, the role of the independent liquidity provider could not be provided by the badly performed fund.

6. Conclusion. In this paper, we study the optimal tunneling behaviors of the two mutual fund managers who have private relations. In order to study the impacts of investment return and the management fee rate on the tunneling behaviors, we study two prototypes of scenarios whether the management fee rates are consistent
Figure 4. The amount of assets transferred from Fund Two to Fund One in Case 2 (with transaction cost).

Figure 5. The amount of assets transferred from Fund Two to Fund One in Case 1 (with correlation).
with the investment returns. The results show that, the well performed and high rewarded fund manager obtains most of the total assets by tunneling, and only keeps the other fund at the brink of maximal withdraws for the liquidity considerations. Meanwhile, the overall improvement of the rewards of the fund managers is minor in the case that the investment return and management fee rate are positively correlated. On the contrary, the well performed and low rewarded fund manager obtains most of the total assets by tunneling. The growth potential of the fund is so attractive that it could offset the impacts of lower management fee rate. The overall improvement of the rewards of the fund managers is enormous in the case that the investment return and management fee rate are negatively correlated. Being inconsistent with the instinct, the high management fee rate could neither make the fund managers work efficiently, nor induce the beneficial tunneling behaviors.

**Appendix.** In this appendix, we give the two-dimensional figures to make the results in Figure 1 and Figure 2 more clear. The assets of Fund One and Fund Two are controlled as $x_1 = x_{10} \times 80\%, x_{10}, 100, 500$ and $x_2 = x_{20} \times 80\%, x_{20}, 100, 500$, respectively.

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The amount of assets transferred from Fund Two to Fund One in Case 1 (without transaction cost).

The amount of assets transferred from Fund Two to Fund One in Case 2 (without transaction cost).

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