The spectral measure of certain elements of the complex group ring of a wreath product

Warren Dicks*  
Departament de Matemàtiques  
Universitat Autònoma de Barcelona  
08193 Bellaterra (Barcelona)  
Spain

Thomas Schick†  
FB Mathematik  
Universität Münster  
Einsteinstr. 62  
48159 Münster  
Germany

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Abstract

We use elementary methods to compute the \( L^2 \)-dimension of the eigenspaces of the Markov operator on the lamplighter group and of generalizations of this operator on other groups. In particular, we give a transparent explanation of the spectral measure of the Markov operator on the lamplighter group found by Grigorchuk-Zuk [4]. The latter result was used by Grigorchuk-Linnell-Schick-Zuk [3] to produce a counterexample to a strong version of the Atiyah conjecture about the range of \( L^2 \)-Betti numbers.

We use our results to construct manifolds with certain \( L^2 \)-Betti numbers (given as convergent infinite sums of rational numbers) which are not obviously rational, but we have been unable to determine whether any of them are irrational.

1 Notation and statement of main result

In this section we introduce notation that will be fixed throughout and will be used in the statement of the main result.

Let \( U \) denote a discrete group with torsion.

Let \( e \) be a nontrivial projection (so \( e = e^* = e^2 \), \( e \neq 0,1 \)) in \( \mathbb{C}[U] \). For example, \( U \) could be finite and nontrivial, and \( e \) could be the ‘average’ of the elements of \( U \),

\[
\text{avg}(U) := \frac{1}{|U|} \sum_{u \in U} u.
\]
This will be the example we shall make the most use of.

Let \( W = W(U, e) \) denote the inverse of the coefficient of 1 in the expression of \( e \) as a \( \mathbb{C} \)-linear combination of elements of \( U \). By results of Kaplansky and Zaleskii, \( W \) is a rational number greater than 1. For example, if \( U \) is finite and nontrivial, and \( e = \text{avg}(U) \), then \( W = |U| \).

For integers \( m, n, \) with \( 1 \leq m \leq n - 1 \), let \( \lambda_{m,n} := 2 \cos \left( \frac{m}{n} \pi \right) \).

For any integer \( n \geq 2 \), let \( M_n := \{ \lambda_{m,n} \mid 1 \leq m \leq n - 1, m \text{ coprime to } n \} \).

We write

\[
U \wr \mathbb{Z} := (\oplus_{i \in \mathbb{Z}} U) \rtimes C_{\infty},
\]

where \( C_{\infty} \) denotes an infinite cyclic group with generator \( t = t_U \) which acts on \( \oplus_{i \in \mathbb{Z}} U \) by the shift, i.e. \( t^{-1}((g_n)_{n \in \mathbb{Z}}) = (g_{n-1})_{n \in \mathbb{Z}} \). For each \( u \in U \), let \( a_u \) denote \( \ldots, 1, u, 1, \ldots \) \( \in \oplus_{i \in \mathbb{Z}} U \) where \( u \) occurs with index 0. Throughout, we identify \( u \) with \( a_u \). Thus \( U \) is a subgroup of \( U \wr \mathbb{Z} \). Notice that \( U \wr \mathbb{Z} \) is generated by \( t \) and \( U \).

Set

\[
T = T(U, e) := (et + t^{-1}e) \in \mathbb{C}[U \wr \mathbb{Z}].
\]

If \( U \) is finite and nontrivial, and \( e = \text{avg}(U) \), then \( T \) is two times the Markov operator of \( U \wr \mathbb{Z} \) with respect to the symmetric set of generators \( \{ut, (ut)^{-1} \mid u \in U \} \).

Let \( \mathcal{N}(U \wr \mathbb{Z}) \) denote the (von Neumann) algebra of bounded linear operators on the Hilbert space \( l^2(U \wr \mathbb{Z}) \) which commute with right multiplication by elements of \( U \wr \mathbb{Z} \). We identify each element \( x \) of \( \mathbb{C}[U \wr \mathbb{Z}] \) with an element of \( l^2(U \wr \mathbb{Z}) \) in the natural way, and also with the element of \( \mathcal{N}(U \wr \mathbb{Z}) \) given by left multiplication by \( x \). Thus \( \mathbb{C}[U \wr \mathbb{Z}] \) is viewed as a subset of \( l^2(U \wr \mathbb{Z}) \) and as a subalgebra of \( \mathcal{N}(U \wr \mathbb{Z}) \). For \( a \in \mathcal{N}(U \wr \mathbb{Z}) \) the (regularized) trace of \( a \) is defined as

\[
\text{tr}_{U \wr \mathbb{Z}}(a) := \langle a(1), 1 \rangle_{l^2(U \wr \mathbb{Z})}.
\]

Similar notation applies for any group.

Note that, if \( a \in \mathcal{N}(U \wr \mathbb{Z}) \) leaves invariant \( l^2(G) \) for a subgroup \( G \), then we can consider \( a \) to be an element of \( \mathcal{N}(G) \), and here \( \text{tr}_G(a) \) and \( \text{tr}_{U \wr \mathbb{Z}}(a) \) coincide.

Note also that, if \( a \) lies in \( \mathbb{C}[U \wr \mathbb{Z}] \), then \( \text{tr}_{U \wr \mathbb{Z}}(a) \) is the coefficient of 1 in the expression of \( a \) as a \( \mathbb{C} \)-linear combination of elements of \( U \wr \mathbb{Z} \).

The element (left multiplication by) \( T \) of \( \mathcal{N}(U \wr \mathbb{Z}) \) is self-adjoint. For each \( \mu \in \mathbb{R} \), let \( \text{pr}_\mu : l^2(U \wr \mathbb{Z}) \to l^2(U \wr \mathbb{Z}) \) denote the orthogonal projection onto \( \ker(T - \mu) \), so \( \text{pr}_\mu \in \mathcal{N}(U \wr \mathbb{Z}) \). The number

\[
\dim_{U \wr \mathbb{Z}} \ker(T - \mu) := \langle \text{pr}_\mu(1), 1 \rangle_{l^2(U \wr \mathbb{Z})} = \text{tr}_{U \wr \mathbb{Z}}(\text{pr}_\mu)
\]

is called the \( L^2 \)-multiplicity of \( \mu \) as an eigenvalue of \( T \).

Our main result is the following.

\textbf{1.1. Theorem.} With all the above notation, for any \( \mu \in \mathbb{R} \),

\[
\dim_{U \wr \mathbb{Z}} \ker(T - \mu) = \begin{cases} \frac{(W-1)^2}{W^2 - 1} & \text{if } n \geq 2 \text{ and } \mu \in M_n, \\ 0 & \text{if } \mu \notin \bigcup_{n \geq 2} M_n. \end{cases}
\]

Moreover, \( l^2(U \wr \mathbb{Z}) \) is the Hilbert sum of the eigenspaces of \( T \), i.e. the spectral measure of \( T \) off its eigenspaces is zero.
In [4, Corollary 3], Grigorchuk-Zuk proved the case of this result in which $U$ is (cyclic) of order two and $e = \text{avg}(U)$, so $W = 2$. This was used in [3] to give a counterexample to a strong version of the Atiyah conjecture about the range of $L^2$-Betti numbers. The argument in [4] is based on automata and actions on binary trees, while our proof is based on calculating traces of projections in the group ring $\mathbb{C}[U \wr \mathbb{Z}]$.

2 Preliminary matrix calculations

In this section, we introduce more notation which will be used throughout, and verify some identities which will be used in the proof.

For positive integers $i, j$, let

$$\alpha_{i,j} := \delta_{|i-j|,1} = \begin{cases} 1 & \text{if } i - j = \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

For each integer $n \geq 2$, let $A_n$ denote the $n - 1 \times n - 1$ matrix

$$A_n = (\alpha_{i,j})_{1 \leq i, j \leq n - 1} = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots \\ 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & 0 & \cdots \\ \cdots & 0 & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & 0 & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$  

Recall that $\lambda_{m,n}$ denotes $2\cos\left(\frac{m\pi}{n}\right)$.

2.1. Lemma. For each $n \geq 2$, the family of eigenvalues of $A_n$, with multiplicities, is $\{\lambda_{m,n} | 1 \leq m \leq n - 1\}$.

Proof. For a complex number $\mu$ different from $0, 1, -1$, one checks immediately by induction on $n$, and determinant expansion of the first row, that

$$\det(A_n + (\mu + \mu^{-1})I_{n-1}) = \frac{\mu^n - \mu^{-n}}{\mu - \mu^{-1}}.$$ 

Now, for $1 \leq m \leq n - 1$, taking $\mu = -e^{\frac{2\pi}{n}i}$ shows that $\lambda_{m,n}$ is an eigenvalue of $A_n$. Since we have $n - 1$ distinct eigenvalues for $A_n$, they all have multiplicity one. \qed

For $n \geq 2$, $A_n$ is a real symmetric matrix, so there exists a real orthogonal matrix $B_n = (\beta_{i,j}^{(n)})_{1 \leq i, j \leq n - 1}$ such that $B_nA_nB_n^*$ is a diagonal matrix $D_n$; here the diagonal entries are $\lambda_{m,n}, 1 \leq m \leq n - 1$, and we may assume the entries occur in this order, so $D_n = (\delta_{i,j}\lambda_{j,n})_{1 \leq i, j \leq n - 1}$. Since $B_nB_n^* = I_{n-1}$ and $B_nA_n = D_nB_n$, we have the identities

$$\sum_{j=1}^{n-1} \beta_{i,j}^{(n)} \beta_{j,k}^{(n)} = \delta_{i,k}, \quad 1 \leq i, k \leq n - 1, \quad (2.2)$$

$$\sum_{j=1}^{n-1} \beta_{i,j}^{(n)} \alpha_{j,k} = \lambda_{i,n} \beta_{i,k}^{(n)}, \quad 1 \leq i, k \leq n - 1. \quad (2.3)$$
3 Proof of the main result

We shall frequently use the following, which is well known and easy to prove.

3.1 Lemma. Let $G$ and $H$ be discrete groups, and let $p \in \mathcal{N}(G)$ and $q \in \mathcal{N}(H)$. Embed $G$ and $H$ in the canonical way into $G \times H$, so $p$ and $q$ become elements of $\mathcal{N}(G \times H)$. Then

$$\text{tr}_{G \times H}(pq) = \text{tr}_G(p) \cdot \text{tr}_H(q).$$

We need even more notation.

For each $i \in \mathbb{Z}$, we define, in $\mathbb{C}[U \wr \mathbb{Z}]$, $e_i := t^{-i}e^t$ and $f_i := 1 - e_i$. It is easy to see that all the $e_i$, $f_j$ are projections which commute with each other; moreover,

$$\text{tr}_{U \wr \mathbb{Z}}(e_i) = \text{tr}_{U \wr \mathbb{Z}}(e) = \frac{1}{W} \quad \text{and} \quad \text{tr}_{U \wr \mathbb{Z}}(f_i) = 1 - \frac{1}{W}. \quad (3.2)$$

For $n \geq 2$, let $q_n := f_1 e_2 e_3 \cdots e_{n-2} e_{n-1} f_n$. It is clear that $q_n$ is a projection. Moreover, the factors lie in $\mathbb{C}[t^{-i}U^n]$, $1 \leq i \leq n$, so, by Lemma 3.1,

$$\text{tr}_{U \wr \mathbb{Z}}(q_n) = \text{tr}_{U \wr \mathbb{Z}}(f_1) \text{tr}_{U \wr \mathbb{Z}}(e_2) \cdots \text{tr}_{U \wr \mathbb{Z}}(e_{n-1}) \text{tr}_{U \wr \mathbb{Z}}(f_n).$$

By (3.2),

$$\text{tr}_{U \wr \mathbb{Z}}(q_n) = (1 - \frac{1}{W})^2 (\frac{1}{W})^{n-2} = \frac{(W-1)^2}{W^n}. \quad (3.3)$$

3.4 Lemma. If $1 \leq m < n$ and $1 \leq m' < n'$ then

$$q_{n'}t^{-m'}q_n^m = \delta_{n,n'} \delta_{m,m'}q_n.$$

Proof. Note that $t^m q_n t^{-m} = f_{1-m} e_{2-m} \cdots e_{n-m-1} f_{n-m}$, and this is a projection. Thus

$$(t^m q_n t^{-m} \mid n \geq 2, 1 \leq m < n) = \{f_{-i} e_{-i+1} \cdots e_{j-1} f_j \mid -i \leq 0, 1 \leq j \}.$$ 

This is a family of pairwise orthogonal projections, since, if $-i, -i' \leq 0, 1 \leq j, j'$, then either $(i, j) = (i', j')$, or the product of $f_{-i} e_{-i+1} \cdots e_{j-1} f_j$ and $f_{-i'} e_{-i'+1} \cdots e_{j'-1} f_{j'}$ is zero since it contains a factor $e_{\alpha} f_n = 0$ for at least one $\alpha \in \{-i, -i', j, j'\}$. Since $t$ is invertible, the result follows.

Notice that, for $1 \leq m < n$,

$$T(t^m q_n) = e t^m q_n + t^{-1} e t^m q_n$$

$$= t^{m+1} e_{m+1} q_n + t^{m-1} e_m q_n$$

$$= t^{m+1} (1 - \delta_{m,n-1}) q_n + t^{m-1} (1 - \delta_{m,1}) q_n.$$ 

Hence

$$T(t^m q_n) = \sum_{i=1}^{n-1} \alpha_{m,i} t^i q_n. \quad (3.5)$$
For $1 \leq m \leq n - 1$, define $r_{m,n} := \sum_{i=1}^{n-1} \beta_{m,i} t_i q_n$ and $p_{m,n} := r_{m,n} r_{m,n}^*$. Observe that, if we identify the $i$th standard basis vector with $t_i q_n$, $1 \leq i \leq n - 1$, then $r_{m,n}$ is an eigenvector of $A_n$ with eigenvalue $\lambda_{m,n}$. Moreover, we have just checked that $T$ acts like $A_n$ on the span of the $t^m q_n$. This partially explains why the $r_{m,n}$ give rise to pairwise orthogonal projections with image contained in the eigenspace of $T$ for the eigenvalue $\lambda_{m,n}$, which is essentially the statement of the following lemma.

3.6. Lemma. $(p_{m,n} \mid n \geq 2, 1 \leq m \leq n - 1)$ is a family of pairwise orthogonal projections in $\mathbb{C}[U \wr \mathbb{Z}]$ which is complete, that is, $\sum_{n \geq 2} \sum_{m=1}^{n-1} \text{tr}_{U \wr \mathbb{Z}}(p_{m,n}) = 1$. Moreover, if $1 \leq m \leq n - 1$, then $T(p_{m,n}) = \lambda_{m,n} p_{m,n}$.

Proof. Let $1 \leq m \leq n - 1$ and $1 \leq m' \leq n' - 1$.

Here

$$r_{m,n}^* = q_n^* \sum_{i=1}^{n-1} (t^i)^* \beta_{m,i}^{(n)*} = q_n^* \sum_{i=1}^{n-1} t^{-i} \beta_{m,i}^{(n)}.$$ 

Thus

$$r_{m',n}^* r_{m,n} = q_{n'} \sum_{j=1}^{n'-1} t^{-j} \beta_{m',j}^{(n')} \sum_{i=1}^{n-1} \beta_{m,i}^{(n)} t^i q_n$$

$$= \delta_{n,n'} q_n \sum_{i=1}^{n-1} \beta_{m,i}^{(n)} \beta_{m,i}^{(n')} \text{ by Lemma 3.4}$$

$$= \delta_{n,n'} q_n \delta_{m,m'} \text{ by (2.2)}.$$ 

It follows that the $p_{m,n}$ are pairwise orthogonal. Moreover,

$$\text{tr}_{U \wr \mathbb{Z}}(p_{m,n}) = \text{tr}_{U \wr \mathbb{Z}}(r_{m,n} r_{m,n}^*) = \text{tr}_{U \wr \mathbb{Z}}(r_{m,n}^* r_{m,n})$$

$$= \text{tr}_{U \wr \mathbb{Z}}(q_n) = \frac{(W - 1)^2}{W^n} \text{ by (3.3)}.$$ 

Now,

$$\sum_{n \geq 2} \sum_{m=1}^{n-1} \text{tr}_{U \wr \mathbb{Z}}(p_{m,n}) = \sum_{n \geq 2} \sum_{m=1}^{n-1} \frac{(W - 1)^2}{W^n} = \sum_{n \geq 2} (n - 1) \frac{(W - 1)^2}{W^n}$$

$$= \sum_{n \geq 1} \frac{n(W - 1)^2}{W^{n+1}} = (1 - \frac{1}{W})^2 \sum_{n \geq 1} \frac{1}{W}^{n-1} = 1,$$

since, for $|x| < 1$, $\sum_{n \geq 1} nx^{n-1} = (\sum_{n \geq 0} x^n)' = (\frac{1}{1-x})' = \frac{1}{(1-x)^2}$. 
Also,
\[
T(r_{m,n}) = T(\sum_{j=1}^{n-1} \beta_{m,j}^{(n)} t^j q_n) = \sum_{j=1}^{n-1} \beta_{m,j}^{(n)} T(t^j q_n)
\]
\[
= \sum_{j=1}^{n-1} \beta_{m,j}^{(n)} \sum_{k=1}^{n-1} \alpha_{j,k} t^k q_n \text{ by (3.5)}
\]
\[
= \sum_{k=1}^{n-1} \left( \sum_{j=1}^{n-1} \beta_{m,j}^{(n)} \alpha_{j,k} t^k q_n \right)
\]
\[
= \sum_{k=1}^{n-1} \lambda_{m,n} \beta_{m,k}^{(n)} t^k q_n \text{ by (2.3)}
\]
\[
= \lambda_{m,n} r_{m,n}.
\]

Thus \( T(r_{m,n}) = \lambda_{m,n} r_{m,n} \), and, on right multiplying by \( r_{m,n}^* \), we see \( T(p_{m,n}) = \lambda_{m,n} p_{m,n} \).

We have now ‘diagonalized’ \( T \) in the sense that we have decomposed \( l^2(U \wr \mathbb{Z}) \) into the Hilbert sum of subspaces of the form \( p_{m,n}(l^2(U \wr \mathbb{Z})) \) on which \( T \) acts as multiplication by the scalar \( \lambda_{m,n} \).

Hence, for each \( \mu \in \mathbb{R} \), \( \ker(T - \mu) \) is the Hilbert sum of those \( p_{m,n}(l^2(U \wr \mathbb{Z})) \) such that \( \lambda_{m,n} = \mu \). Thus either \( \ker(T - \mu) = 0 \) or \( \mu = \lambda_{m_0,n_0} \) for some \( m_0, n_0 \) with \( 1 \leq m_0 \leq n_0 - 1 \).

We now consider the latter case. Here, for all \( (m,n) \), \( \lambda_{m,n} = \mu \) if and only if \( \frac{m}{n} = \frac{m_0}{n_0} \). We may assume that \( m_0 \) and \( n_0 \) are coprime, so \( \mu \in M_{n_0} \). Also, \( \lambda_{m,n} = \mu \) if and only if \( (m,n) = (im_0,in_0) \) for some \( i \geq 1 \). Thus \( \ker(T - \mu) \) is the Hilbert sum of the \( p_{im_0,in_0}(l^2(U \wr \mathbb{Z})) \) with \( i \geq 1 \); hence
\[
\dim_{U \wr \mathbb{Z}}(\ker(T - \lambda_{m_0,n_0})) = \sum_{i \geq 1} \dim_{U \wr \mathbb{Z}}(p_{im_0,in_0}(l^2(U \wr \mathbb{Z})))
\]
\[
= \sum_{i \geq 1} \text{tr}_{U \wr \mathbb{Z}}(p_{im_0,in_0}) = \sum_{i \geq 1} \frac{(W - 1)^2}{W^{im_0}} = \frac{(W - 1)^2}{W^{n_0} - 1}.
\]

Theorem 1.1 now follows.

3.7. Remarks. The hypothesis in Theorem 1.1 that \( U \) has torsion could be weakened to the assumption that \( \mathbb{C}[U] \) has a nontrivial projection; however, if \( U \) is torsion-free, it is conjectured, and known in many cases, that \( \mathbb{C}[U] \) does not contain any nontrivial projections.

It easy to show that the hypothesis in Theorem 1.1 that \( e \) is a nontrivial projection in \( \mathbb{C}[U] \) can be weakened to the assumption that \( e \) is a nontrivial projection in \( \mathcal{N}(U) \); here, the hypothesis that \( U \) has torsion should be weakened to the assumption that \( U \) is nontrivial.

4 Direct products of wreath products

We now produce even more unusual examples by taking direct products of the groups studied so far.
4.1. Theorem. Let $U$ and $V$ be groups with torsion, and $G = (U \times (V \times \mathbb{Z})).$
Let $e$ be a nontrivial projection in $C[U]$ and $f$ a nontrivial projection in $C[V].$
Let $X = (\text{tr}_U(e))^{-1}$ and $Y = (\text{tr}_V(f))^{-1}$, so $X > 1$, $Y > 1$. Let $T = T(U, e) \in C[U \times \mathbb{Z}] \subset C[G]$, and $S = T(V, f) \in C[V \times \mathbb{Z}] \subset C[G].$ Then

$$\dim_G(\ker(T - S)) = (X - 1)^2(Y - 1)^2 \sum_{m \geq 1} \text{gcd}(m, n) X^{mY^n} - (X - 1)(Y - 1).$$

(4.2)

Proof. By Lemma 3.6, there is a complete family $(p_{m, n} | n \geq 2, 1 \leq m < n)$ of pairwise orthogonal projections in $C[U \times \mathbb{Z}]$, such that, if $1 \leq m < n$, then $T(p_{m, n}) = \lambda, n = \rho_{m, n}$, and, by (3.3), $\text{tr}_{U(Z)}p_{m, n}) = \frac{(X - 1)^2}{X^n}.$

Similarly, there is a complete family $(q_{m, n} | n \geq 2, 1 \leq m < n)$ of pairwise orthogonal projections in $C[V \times \mathbb{Z}]$ such that, if $1 \leq m < n$, then $S(q_{m, n}) = \lambda, n, q_{m, n},$ and $\text{tr}_{V(X)}q_{m, n}) = \frac{(Y - 1)^2}{Y^n}.$

By Lemma 3.1, there is a complete family

$$(p_{m, n}q_{m', n'} | n, n' \geq 2, 1 \leq m < n, 1 \leq m' < n')$$

of pairwise orthogonal projections in $C[G]$, such that, if $1 \leq m < n$ and $1 \leq m' < n'$ then

$$T(p_{m, n}q_{m', n'}) = \lambda, m, n, q_{m', n'}$$

and

$$S(p_{m, n}q_{m', n'}) = \lambda, m', n, q_{m, n}.$$

and

$$\text{tr}_{G}(p_{m, n}q_{m', n'}) = \frac{(X - 1)^2(Y - 1)^2}{X^nY^{n'}}.$$

Thus $l^2(G)$ is the Hilbert sum of the subspaces of the form $p_{m, n}q_{m', n'}(l^2(G))$ where $T - S$ acts as multiplication by the scalar $\lambda, m - \lambda, m', n'.$

Hence $\ker(T - S)$ is the Hilbert sum of the $p_{m, n}q_{m', n'}(l^2(G))$ such that $\lambda, m, n = \lambda, m', n'.$

Therefore,

$$\dim_G(\ker(T - S)) = \sum_{n \geq 1} \sum_{n' \geq 1} b(n, n') \frac{(X - 1)^2(Y - 1)^2}{X^nY^{n'}}$$

where $b(n, n')$ is the number of pairs $(m, m')$ such that $1 \leq m < n, 1 \leq m' < n'$, and $\frac{m}{m'} = \frac{n}{n'}$. But such pairs correspond bijectively to the fractions of the form $\frac{\text{gcd}(n, n')}{\text{gcd}(n, n')}, 1 \leq n_0 < \text{gcd}(n, n')$. Thus $b(n, n') = \text{gcd}(n, n') - 1$. Hence

$$\dim_G(\ker(T - S)) = \sum_{n \geq 1} \sum_{n' \geq 1} \frac{(\text{gcd}(n, n') - 1)(X - 1)^2(Y - 1)^2}{X^nY^{n'}}$$

$$= \sum_{n \geq 1} \sum_{n' \geq 1} \frac{\text{gcd}(n, n')(X - 1)^2(Y - 1)^2}{X^nY^{n'}} - \sum_{n \geq 1} \sum_{n' \geq 1} \frac{(X - 1)^2(Y - 1)^2}{X^nY^{n'}}.$$

Since $\sum_{n \geq 1} \frac{1}{X^n} = X^{-1} \frac{1}{1 - X^{-1}} = \frac{1}{X - 1}$, the result follows.
4.3. Remarks. Recall that, for any positive integer \( n \), \( \phi(n) \) denotes the number of primitive \( n \)th roots of unity, so \( |M_n| = \phi(n) \).

For \( X > 1 \), \( Y > 1 \), the double infinite sum occurring in (4.2) has an expression as a single infinite sum,

\[
\sum_{m \geq 1} \sum_{n \geq 1} \frac{\gcd(m, n)}{X^m Y^n} = \sum_{k \geq 1} \frac{\phi(k)}{(X^k - 1)(Y^k - 1)},
\]

since

\[
\sum_{k \geq 1} \frac{\phi(k)}{(X^k - 1)(Y^k - 1)} = \sum_{k \geq 1} \frac{\phi(k)}{X^k - 1} \sum_{i \geq 1} X^{-ik} \sum_{j \geq 1} Y^{-jk} = \sum_{m \geq 1} \sum_{n \geq 1} \frac{a(m, n)}{X^m Y^n}
\]

where

\[
a(m, n) = \sum_{\{k \geq 1 : k|m, k|n\}} \phi(k) = \sum_{k | \gcd(m, n)} \phi(k) = \gcd(m, n).
\]

It follows that

\[
dim_{C}(\ker(T - S)) = (X - 1)^2(Y - 1)^2 \sum_{k \geq 2} \frac{\phi(k)}{(X^k - 1)(Y^k - 1)}. \quad \Box
\]

5 \( L^2 \)-Betti numbers

We previously observed that, by results of Kaplansky and Zaleskii, the traces of projections in complex, or rational, group algebras are rational numbers in the interval \([0, 1]\). In order to maximize the scope of Theorem 4.1 for producing examples of \( L^2 \)-Betti numbers, we need the following result which shows that the traces of projections in rational group algebras are precisely the rational numbers in the interval \([0, 1]\). We write \( C_n \) for a cyclic group of order \( n \), written multiplicatively, with generator \( t = t_n \).

5.1. Lemma. Let \( q \) be a rational number in the interval \([0, 1]\). Then there is an expression \( q = \frac{m}{n} \) where the denominator has the form \( n = 2^r s \) with \( s \) odd and \( 2^r \geq s - 1 \), and, for any such expression, \( \mathbb{Q}[C_n] \) contains some projection \( e \) with trace \( q \), and \( ne \in \mathbb{Z}[C_n] \).

Proof. By multiplying the numerator and denominator of \( q \) by a sufficiently high power of 2, we see that \( q \) has an expression of the desired type. Now consider any expression \( q = \frac{m}{n} \) where \( n = 2^r s \) with \( s \) odd and \( 2^r \geq s - 1 \).

We first show, by induction on \( r \), that, if \( 0 \leq c \leq 2^r \), then \( \mathbb{Q}[C_{2^r}] = \mathbb{Q}[t \mid t^{2^r} = 1] \) has an ideal whose dimension over \( \mathbb{Q} \) is \( c \). Since the orthogonal complement is then an ideal of dimension \( 2^r - c \) over the rationals, it amounts to the same if we consider only \( c \leq 2^{r-1} \). For \( r = 0 \), we can take the zero ideal; thus, we may assume that \( r \geq 1 \) and the result holds for smaller \( r \). Now \( \mathbb{Q}[C_{2^r}] \) has a projection \( e = \frac{1 + t^{2^r-1}}{2} \); this is \( \text{avg}(U) \) for the subgroup \( U \) of order 2 in \( C_{2^r} \). As rings

\[
e \mathbb{Q}[C_{2^r}] \simeq \mathbb{Q}[C_{2^r}]/(1 - e) \simeq \mathbb{Q}[C_{2^{r-1}}].
\]

By the induction hypothesis, the latter has an ideal of dimension \( e \) over \( \mathbb{Q} \), and viewed in \( e \mathbb{Q}[C_{2^r}] \) this is an ideal of \( \mathbb{Q}[C_{2^r}] \). This completes the proof.
by induction. Hence, if \( 0 \leq c \leq 2^r \), then \( \mathbb{Q}[C_{2^r}] \) has a projection \( e(c) \) with \( \text{tr}_{C_{2^r}}(e(c)) = \frac{c}{2^r} \).

Let \( f = \text{avg}(C_s) \in \mathbb{Q}[C_s] \), so \( \text{tr}_{C_s}(f) = \frac{1}{s} \), and \( \text{tr}_{C_s}(1 - f) = \frac{s - 1}{s} \).

By identifying

\[
\mathbb{Q}[C_n] = \mathbb{Q}[C^n_n \times C^n_n] = \mathbb{Q}[C_{2^r} \times C_s],
\]

we see that, for \( 0 \leq c \leq 2^r \), we have projections \( e(c) f \) and \( e(c)(1 - f) \) in \( \mathbb{Q}[C_n] \), with traces \( \frac{c}{2^r} = \frac{c}{n} \) and \( \frac{c(s - 1)}{2^r} = \frac{c(s - 1)}{n} \), respectively, by Lemma 3.1.

We claim there exist integers \( a, b \) with \( 0 \leq a, b \leq 2^r \) such that \( a + (s - 1)b = m \). We know that \( 0 \leq m \leq n = 2^r s \). If \( m \geq 2^r(s - 1) \), then \( m \in [2^r(s - 1), 2^r s] \), and we can take \( b = 2^r \) and \( a = m - (s - 1)b = m - 2^r(s - 1) \in [0, 2^r] \). If \( m < 2^r(s - 1) \), then, by the division algorithm, \( m = (s - 1)b + a \) with \( 0 \leq b < 2^r \), and \( 0 \leq a \leq s - 2 < 2^r \). This proves the claim.

Now let \( e = e(a)f + e(b)(1 - f) \), a sum of orthogonal projections. Thus, \( e \) is a projection and

\[
\text{tr}_{C_n}(e) = \text{tr}_{C_n}(e(a)f) + \text{tr}_{C_n}(e(b)(1 - f)) = \frac{a}{n} + \frac{b(s - 1)}{n} = \frac{a + b(s - 1)}{n} = \frac{m}{n},
\]

as desired.

It remains to show that \( e \) lies in \( \frac{1}{n} \mathbb{Z}[C_n] \), but it is well known that this holds for all the idempotents of \( \mathbb{Q}[C_n] \). Alternatively, it is straightforward to check that all the projections involved in the foregoing proof have the right denominators.

We now obtain the following special case of Theorem 4.1.

5.2. Corollary. Let \( p \) and \( q \) be rational numbers with \( 0 < p, q < 1 \). There exist positive integers \( n \) and \( m \), and projections

\[
e = e^* = e^2 \in \mathbb{Q}[C_m], \quad f = f^* = f^2 \in \mathbb{Q}[C_n]
\]

with \( \text{tr}_V(e) = p \), \( \text{tr}_V(f) = q \). Let

\[
G(p, q) := (C_m \wr \mathbb{Z}) \times (C_n \wr \mathbb{Z}),
\]

\[
T := T(U, e) \in \mathbb{C}[U \wr \mathbb{Z}] \subset \mathbb{C}[G], \quad \text{and} \quad S := T(V, f) \in \mathbb{C}[V \wr \mathbb{Z}] \subset \mathbb{C}[G].
\]

Let \( Z = Z(p, q) := mn(T - S) \), and let

\[
\kappa = \kappa(p, q) := (p^{-1} - 1)^2(q^{-1} - 1)^2 \sum_{k \geq 2} \frac{\phi(k)}{(p^{-k} - 1)(q^{-k} - 1)}
\]

\[
= (p^{-1} - 1)^2(q^{-1} - 1)^2 \left( \sum_{i \geq 1} \sum_{j \geq 1} \gcd(i, j)p^i q^j \right) - (p^{-1} - 1)(q^{-1} - 1).
\]

Then \( Z \in \mathbb{Z}[G] \) and \( \text{dim}_{\mathbb{C}}(\ker Z) = \kappa \).

5.3. Remarks. Let \( 0 < p, q < 1 \) be rational numbers. Let \( G = G(p, q) \), \( Z = Z(p, q) \) and \( \kappa = \kappa(p, q) \) as in Corollary 5.2.
By the Higman Embedding Theorem, any recursively presented group can be embedded in a finitely presented group, so $G$ can be embedded in a finitely presented group $H$. (Here it is easy to find an explicit suitable finitely presented group; see, for example, [2] or [3, Lemma 3]. This explicit supergroup has the additional nice property of being metabelian, that is, 2-step solvable. Moreover, one can precisely describe its finite subgroups.)

By Corollary 5.2, $Z \in \mathbb{Z}[G] \subseteq \mathbb{Z}[H]$ and $\dim_H(\ker Z) = \dim_G(\ker Z) = \kappa$.

It is then well known how to construct a finite CW-complex or a closed manifold $M$ with $\pi_1(M) \simeq H$ and with third $L^2$-Betti number $\kappa$; see, for example, [3].

Thus $\kappa(p, q)$ is an $L^2$-Betti number of a closed manifold. It is conceivable that this is a counterexample to Atiyah’s conjecture [1] that $L^2$-Betti numbers of closed manifolds are rational, but we have not been able to decide whether $\kappa(p, q)$ is rational or not.

5.4. Example. Consider $\kappa(\frac{1}{2}, \frac{1}{2}) = \sum_{k \geq 2} \frac{\phi(k)}{(2k-1)^2} = 0.1659457149\ldots$. If we sum the first 400 terms, then elementary methods show that the remaining tail is less than $10^{-201}$. This allows us to calculate the first 199 terms of the continued fraction expansion of $\kappa(\frac{1}{2}, \frac{1}{2})$. One consequence we find is that if $\kappa(\frac{1}{2}, \frac{1}{2})$ is rational then both the numerator and the denominator exceed $10^{100}$. It seems reasonable to assert that $\kappa(\frac{1}{2}, \frac{1}{2})$ is not obviously rational.

6 Power series

Throughout this section, let $\mathbb{C}((x, y))$ denote the field of (formal) Laurent series in two variables (with complex coefficients).

The expression

$$\Phi(x, y) := \sum_{m \geq 1} \sum_{n \geq 1} \gcd(m, n)x^my^n$$

arising from (4.2) can be viewed as an element of $\mathbb{C}((x, y))$. By Remarks 5.3, if there exist rational numbers $p, q$ in the interval $(0, 1)$ such that (the limit of) $\Phi(p, q)$ is irrational, then there exists a counterexample to the Atiyah conjecture; so it is of interest to know whether $\Phi(p, q)$ is always rational for such rational numbers $p, q$. One (traditionally successful) way to show that such an expression is rational would be to show that $\Phi(x, y)$ itself is rational, that is, lies in the subfield $\mathbb{Q}(x, y)$ of rational Laurent series over the rationals. In this section, we will eliminate this possibility by showing that $\Phi(x, y)$ is transcendental over $\mathbb{C}(x, y)$. In fact, we will show the stronger result that the specialization $\Phi(x, x)$ is transcendental over $\mathbb{C}(x)$.

The following result is well known, but we have not found a reference. The proof is left to the reader.

6.1. Lemma. Suppose that $f \in \mathbb{C}((x))$ is algebraic over $\mathbb{C}(x)$ of degree $d$. Then the subfield $\mathbb{C}(x, f)$ is closed under the usual derivation operation, $F \mapsto F' = \frac{dF}{dx}$, on $\mathbb{C}((x))$. Moreover, $\mathbb{C}(x, f)$ is a $d$-dimensional vector space over $\mathbb{C}(x)$, so the $d+1$ higher-order derivatives $f^{(i)} := (\frac{d}{dx})^i(f), 0 \leq i \leq d$, are $\mathbb{C}(x)$-linearly dependent. Hence $f$ satisfies some non-trivial order $d$ differential equation over $\mathbb{C}(x)$.\[\square\]
We can now apply this lemma to get a transcendentality criterion.

6.2. Proposition. Suppose that $a: \mathbb{N} \to \mathbb{C}$, $n \mapsto a(n)$, has the property that, for each $N \in \mathbb{N}$, there exist infinitely many $m \in \mathbb{N}$ such that, whenever $j \in \mathbb{Z}$ satisfies $1 \leq |j| \leq N$, 
\[
    |a(m)| > N |a(m + j)|.
\]
Then the power series $\sum_{n \geq 0} a(n)x^n \in \mathbb{C}((x))$ does not satisfy any non-trivial differential equation over $\mathbb{C}(x)$, so is transcendental over $\mathbb{C}(x)$.

Proof. Let $f := \sum_{n \geq 0} a(n)x^n \in \mathbb{C}((x))$, and suppose that $f$ satisfies a non-trivial differential equation over $\mathbb{C}(x)$,
\[
    \sum_{i=0}^{d} q_i f^{(i)} = 0 \quad (6.3)
\]
where $q_i \in \mathbb{C}(x)$, not all zero. By multiplying through by a common denominator, we may assume that all the $q_i$ lie in $\mathbb{C}[x]$. (Notice it is natural not to have a “constant term” on the right-hand side of (6.3) since it could be eliminated by iterated derivation of the equation.)

Viewing (6.3) as a collection of equations, one for each power $x^n$, we see that there exists some $N \in \mathbb{N}$, and polynomials $p_k(t) \in \mathbb{C}[t]$ such that
\[
    \sum_{k=0}^{N} p_k(n)a(n + k) = 0 \quad \text{for all } n \in \mathbb{N}. \quad (6.4)
\]

Choose $k_0$, with $0 \leq k_0 \leq N$, and $n_0 \in \mathbb{N}$ such that $|p_{k_0}(n)| \geq |p_k(n)|$ for all $n \geq n_0$, and all $k$ with $0 \leq k \leq N$. In other words, $p_{k_0}$ eventually dominates all the $p_k$, $0 \leq k \leq N$.

It follows from the hypothesis on the $a(n)$ that there exists $m \in \mathbb{N}$ such that $m \geq n_0 + k_0$, and $|a(m)| > N |a(m + j)|$ for all $j \in \mathbb{Z}$ with $1 \leq |j| \leq N$. Now take $n = m - k_0$. Then $n \geq n_0$, and
\[
    |a(n + k_0)| > \sum_{k=0}^{k_0-1} |a(n + k)| + \sum_{k=k_0+1}^{N} |a(n + k)|.
\]

Thus
\[
    |p_{k_0}(n)a(n + k_0)| > \sum_{k=0}^{k_0-1} |p_k(n)a(n + k)| + \sum_{k=k_0+1}^{N} |p_k(n)a(n + k)| \geq \left| \sum_{k=0}^{N} p_k(n)a(n + k) - p_{k_0}(n)a(n + k_0) \right| = |0 - p_{k_0}(n)a(n + k_0)| \quad \text{by } (6.4).
\]

This contradiction shows that $f$ does not satisfy any non-trivial differential equation over $\mathbb{C}(x)$, so, by Lemma 6.1, $f$ is not algebraic over $\mathbb{C}(x)$. \qed

We now record some important results from number theory that we shall require.
6.5. Lemma. For each positive integer \( i \), let \( p_i \) denote the \( i \)-th prime number. There exists an integer \( Q_0 \) such that, for all \( Q \geq Q_0 \), the following hold.

1. \( Q! \leq (\frac{Q}{2})^Q \).
2. \( \frac{3}{4} \leq \frac{p_Q}{Q \log Q} \leq \frac{5}{4} \).
3. \( \prod_{i=1}^{Q} (1 - \frac{1}{p_i}) \geq \frac{1}{Q} \).

Proof. In the following, \( f(Q) = o(g(Q)) \) means \( \lim_{Q \to \infty} f(Q)/g(Q) = 0 \), and \( f(Q) \sim g(Q) \) means \( \lim_{Q \to \infty} f(Q)/g(Q) = 1 \).

1. By Stirling’s formula, \( Q! \sim \sqrt{2\pi Q}Q^{\gamma}e^{-Q} \), and the latter is \( o((\frac{Q}{2})^Q) \), since \( e > 2 \). One can argue directly that \( \sum_{i=1}^{Q} \log i \leq \int_{1}^{Q+1} \log x \, dx \), so

\[
\log Q! \leq (Q+1) \log(Q+1) - Q,
\]

so

\[
Q! \leq (Q+1)^{Q+1}e^{-Q} = Q^Q (1 + \frac{1}{Q})^{Q} (Q+1)^{-Q} = o((\frac{Q}{2})^Q),
\]

since \( e > 2 \).

2. By the Prime Number Theorem, \( p_Q \sim Q \log Q \); see [5, Theorem 8, pages 10, 367].

3. By Mertens’ Theorem, \( \prod_{i=1}^{Q} (1 - \frac{1}{p_i}) \sim e^{-\gamma} \log Q \), where \( \gamma \) is Euler’s constant; see [5, Theorem 429, page 351]. By the Prime Number Theorem, \( \log p_Q \sim \log Q \), so

\[
\prod_{i=1}^{Q} (1 - \frac{1}{p_i}) \sim \frac{e^{-\gamma}}{\log Q}.
\]

Since \( \frac{1}{Q} = o(\frac{1}{\log Q}) \), we see that

\[
\frac{1}{Q} = o(\prod_{i=1}^{Q} (1 - \frac{1}{p_i}) = \frac{1}{Q}.
\]

The result now follows.

6.6. Theorem. \( \Phi(x, x) = \sum_{m \geq 1} \sum_{n \geq 1} \gcd(m, n)x^{m+n} \) and \( \sum_{n \geq 1} \sum_{d | n} \frac{\phi(d)}{d} n x^n \) are transcendental over \( \mathbb{C}(x) \).

Proof. For each positive integer \( n \), let \( a(n) := n \sum_{d | n} \frac{\phi(d)}{d} \). Thus

\[
a(n) = n \sum_{d | n} \frac{\phi(d)}{d} = n \sum_{d | n} \frac{\phi(d)}{d} = \sum_{d | n} \phi(d) = \sum_{d | n} \phi(d) = \sum_{i=1}^{n} \phi(d) = \sum_{i=1}^{n} \phi(d).
\]

Now

\[
\Phi(x, x) = \sum_{i \geq 1} \sum_{j \geq 1} \gcd(i, j)x^{i+j} = \sum_{n \geq 1} \sum_{i=1}^{n-1} \gcd(i, n-i)x^n,
\]

so

\[
\sum_{n \geq 1} a(n)x^n - \Phi(x, x) = \sum_{n \geq 1} nx^n = x(\sum_{n \geq 0} x^n)' = \frac{x}{(1 - x)^2}.
\]
Thus $\sum_{n \geq 1} a(n)x^n$ and $\Phi(x, x)$ differ by an element of $\mathbb{Q}(x)$, so it suffices to show that $\sum_{n \geq 1} a(n)x^n$ is transcendental over $\mathbb{C}(x)$.

By Proposition 6.2, it suffices to show that, for each $N \in \mathbb{N}$, there exist infinitely many $m \in \mathbb{N}$ such that, whenever $j \in \mathbb{Z}$ satisfies $1 \leq |j| \leq N$,

$$|a(m)| > N |a(m+j)|.$$ 

We may suppose that $N$ is fixed.

Remember the $p_i$ is the $i$th prime number. For each $Q \in \mathbb{N}$, let

$$m_Q := \prod_{i=1}^{Q} p_i \prod_{i=1}^{N} p_i^N. $$

We may now suppose that $j$ is fixed with $1 \leq |j| \leq N$, and it suffices to show that

$$\lim_{Q \to \infty} \frac{a(m_Q+j)}{a(m_Q)} = 0.$$ 

We use the notation of Lemma 6.5, concerning $Q_0$. Let

$$C_1 = \prod_{i=1}^{Q_0} p_i \prod_{i=1}^{N} p_i^N.$$ 

Now suppose that $Q$ is an integer with $Q \geq \max\{Q_0, N\}$, let $m = m_Q$ and let $m' = \prod_{i=1}^{Q} p_i$.

We wish to bound $a(m) = m \sum_{d|m} \phi(d) d$ from below. Recall that, for any positive integer $n$, $\phi(n) = \prod (1 - \frac{1}{p_i})$, where the product is over the distinct prime divisors $p$ of $n$. Thus $a(m) \geq m \sum_{d|m} \phi(m) m = m \prod_{i} \phi(m) m = m \cdot m \cdot m$, where $d(m)$ denotes the number of divisors $d$ of $m$. Also, $\phi(m) = \prod_{i} \frac{Q}{p_i}$, which, by Lemma 6.5(3), is at least $\frac{Q}{Q}$. Thus $a(m) \geq m \cdot m \cdot m$. Notice that $d(m) \geq d(m')$, since $m'$ divides $m$. From the definition of $m'$, we see that $d(m') = 2^Q$. Thus

$$a(m) \geq m 2^Q \frac{1}{Q}.$$ 

We next wish to bound $a(m+j)$ from above. Let $\Omega(m+j)$ be the number, counting multiplicity, of prime factors of $m$, and let

$$m+j = p_{i_1} p_{i_2} \cdots p_{i_{\Omega(m+j)}} $$

be the factorization of $m+j$ into prime factors. Then $d(m+j) \leq 2^{\Omega(m+j)}$, and

$$a(m+j) \leq (m+j) \sum_{d|(m+j)} \phi(d) d \leq (m+j) \sum_{d|(m+j)} 1 = (m+j) d(m+j)$$

$$\leq (m+j) 2^{\Omega(m+j)} \leq (m+N) 2^{\Omega(m+j)} \leq 2m 2^{\Omega(m+j)}.$$ 

Consider $1 \leq l \leq \Omega(m+j)$. If $i_l \leq Q$, then $p_{i_l}$ divides $m$ so $p_{i_l}$ divides $j$. But $1 \leq |j| \leq N$, so $p_{i_l} \leq N$, so $i_l \leq N$. Hence $p_{i_l}^N$ divides $m$, but
Spectral measure and wreath products

\[ p_i^N \geq 2^N > N \geq |j|, \text{ so } p_i^N \text{ cannot divide } j, \text{ so cannot divide } m + j. \] Thus, the number of \( i \) which are less than \( Q \) is at most \( N^N \). Let \( z = z(Q, j) \) denote the number of \( l \) such that \( i_l \geq Q \), so \( \Omega(m + j) \leq z + N^N \), and

\[ a(m + j) \leq 2m2^{\Omega(m + j)} \leq 2m2^{z+N^N}. \]

Thus

\[ \frac{a(m + j)}{a(m)} \leq \frac{2m2^{z+N^N}}{m2^Q} = Q2^{z-Q^N}. \]

Hence it remains to show that \( \lim_{Q \to \infty} Q2^{z-Q} = 0 \), or equivalently,

\[ \lim_{Q \to \infty} Q - z - \log_2 Q = \infty. \]

Since \( z \) is the number, counting multiplicity, of prime factors \( p_i \) of \( m + j \) with \( p_i \geq p_Q \),

\[ p_Q^z \leq m + j \leq m + N \leq 2m. \]

We can write

\[ m = \prod_{i=1}^{Q} p_i \prod_{i=1}^{N} p_i^N \leq \prod_{i=1}^{Q} p_i \prod_{i=2}^{Q} (\frac{5}{4} i \log i) \prod_{i=1}^{N} p_i^N = C_1 \prod_{i=2}^{Q} (\frac{5}{4} i \log i) \]

by Lemma 6.5(1). Thus

\[ (\frac{3}{4} Q \log Q)^z \leq p_Q^z \leq 2m \leq 2C_1 (\frac{5}{4} Q \log Q)^Q. \]

Hence

\[ (\frac{3}{4} Q \log Q)^{z-Q} \leq 2C_1 (\frac{4}{3} Q)^{Q(\frac{1}{2} Q(\frac{5}{4} Q = 2C_1 (\frac{5}{6} Q, \]

so \( (z - Q)(\log \frac{1}{4} + \log Q \log \log Q) \leq \log 2C_1 - Q \log(\frac{6}{5} Q, \) and

\[ -(Q - z) \leq \frac{\log 2C_1 - Q \log(\frac{6}{5} Q)}{\log \frac{1}{4} + \log Q + \log \log Q} \sim -\log(\frac{6}{5} Q). \]

It follows that

\[ \lim_{Q \to \infty} Q - z - \log_2 Q \geq \lim_{Q \to \infty} \log(\frac{6}{5} Q) - \log Q = \infty, \]

as desired. \( \square \)

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