QUANTUM GROUPS AND DOUBLE QUIVER ALGEBRAS

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Abstract

For a finite dimensional semisimple Lie algebra $\mathfrak{g}$ and a root $q$ of unity in a field $k$, we associate to these data a double quiver $\overline{Q}$. It is shown that a restricted version of the quantized enveloping algebras $U_q(\mathfrak{g})$ is a quotient of the double quiver algebra $k\overline{Q}$.

Introduction

Let $U_q(\mathfrak{g})$ be the Drinfeld-Jimbo quantum group, which is a deformation of the universal enveloping algebra of a finite dimensional semisimple Lie algebra $\mathfrak{g}$. In the generic case, i.e. the parameter $q$ is not a root of unity, several models have been raised to realize it. For example, the Ringel-Hall algebra approach is one successful model among them, see [10, 5, 11]. The case where $q$ is a root of unity is of particular interest since it is related with the modular representation theory. It is remarkable that a finite dimensional Hopf algebra, so-called restricted version of $U_q(\mathfrak{g})$ arose naturally when Lusztig considered this class of quantum

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group at roots of unity. See [7, 8, 9]. Just as the generic case, it is interesting to find an algebra method to realize \( U_q(\mathfrak{g}) \), or its restricted version. In this sense, Cibils [1, 2] found that some quotient of a particular path algebra can realize the positive part of the restricted quantized enveloping algebra corresponding to \( \mathfrak{g} \). Gordon in [6] extended several results in [2]. We remark that the restricted form of \( U_q(\mathfrak{sl}_2) \) is realized in the paper [12] by applying the deformation of preprojective algebra introduced in [3].

Let \( k \) be a field, \( \ell = n \) if \( n \) is odd and \( \ell = n/2 \) if \( n \) is even, where \( n \geq 5 \). Let \( \mathfrak{g} \) be a finite dimensional semisimple simple laced Lie algebra and \( q \) an \( n \)-th primitive root of unity. By \( C \) we denote the associated Cartan matrix of \( \mathfrak{g} \).

For the Cartan matrix \( C = (a_{ij})_{t \times t} \), there is an associated quantum algebra \( u_q(C) \), which by definition, is an associative \( k \)-algebra with generators \( K_i, E_i, F_i \) for \( 1 \leq i \leq t \), subjecting to the relations

\[
\begin{align*}
K_i^n &= 1, \quad K_i K_j = K_j K_i; \\
K_i E_j K_i^{-1} &= q^{a_{ij}} E_j, \quad K_i F_j K_i^{-1} = q^{-a_{ij}} F_j; \\
E_i F_j - F_j E_i &= \delta_{ij} K_i - K_i^{-1}; \\
E_i^\ell &= 0, \quad F_i^\ell = 0; \\
\sum_{s=0}^{1-a_{ij}} (-1)^s \left[ \frac{1-a_{ij}}{s} \right] q^{E_i^{1-a_{ij}-s} E_j E_i^s} &= 0, \text{ if } i \neq j; \\
\sum_{s=0}^{1-a_{ij}} (-1)^s \left[ \frac{1-a_{ij}}{s} \right] q^{F_i^{1-a_{ij}-s} F_j F_i^s} &= 0, \text{ if } i \neq j,
\end{align*}
\]

where \([m]_x = \frac{x^m - x^{-m}}{x - x^{-1}}, [m]_x! = [m]_x \cdots [1]_x, [0]_x! = 1\), and \( \left[ \begin{array}{c} m \\ s \end{array} \right]_x = \frac{[m]_x!}{[s]_x! [m-s]_x!} \) for an indeterminate \( x \). The algebra \( u_q(C) \) is a Hopf algebra, where the comultiplication, counit, and antipode are given as follows

\[
\begin{align*}
\Delta(K_i) &= K_i \otimes K_i, \\
\Delta(E_i) &= E_i \otimes 1 + K_i \otimes E_i, \\
\Delta(F_i) &= F_i \otimes K_i^{-1} + 1 \otimes F_i, \\
\varepsilon(K_i) &= 1, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0, \\
S(K_i) &= K_i^{-1}, \quad S(E_i) = -K_i^{-1} E_i, \quad S(F_i) = -F_i K_i.
\end{align*}
\]

It would be noted that the restricted quantum group in the sense of Lusztig is some quotient of \( u_q(C) \) [8, 6].

As a continuation of [12], the aim of this paper is to construct the quantum group \( u_q(C) \) from a quiver up to Hopf algebra isomorphism for an arbitrary finite dimensional semisimple simply laced Lie algebra \( \mathfrak{g} \). Accordingly, we first define a double quiver \( \overline{Q} \) associated to \( C \).
and $q$, and we get the double quiver algebra $k\mathcal{Q}$. Then we describe an algebra $\Pi^C$ obtained from the algebra $k\mathcal{Q}$, which is actually a Hopf algebra. Finally, we construct a quotient $u_q^C$ of $\Pi^C$, which inherits the Hopf algebra structure. We will show that $u_q(C)$ is isomorphic to $u_q^C$ as Hopf algebras. It is mentioned that the results in [2] are extended. If $\mathfrak{g}$ is the three dimensional simple Lie algebra, $\Pi^C$ is just a deformation of preprojective algebra in [3] and $u_q^C$ is the algebra $u_q$ described in [12].

1 Preliminaries

In this section, we list some notations for the convenience of the statement.

Let $k$ be a fixed field. Let us fix an integer $n \geq 5$, an $n$-th primitive root $q$ of unity and a positive definite symmetric Cartan matrix $C = (a_{ij})_{t \times t}$. By $c_i$ we denote the $i$-th column of $C$, by $\mathbb{Z}_s$ the cyclic group $\mathbb{Z}^s$ for any positive integer $s$, and by $\mathcal{T}$ the set \{1, 2, $\cdots$, $t$\}.

For $\alpha = (a_1, \cdots, a_t), \beta = (b_1, \cdots, b_t) \in \mathbb{Z}_n^t$, we denote $\alpha \cdot \beta = a_1b_1 + \cdots + a_tb_t$. The following lemma will be used later on.

**Lemma 1.1** If $0 \neq \beta \in \mathbb{Z}_n^t$, then $\sum_{\alpha \in \mathbb{Z}_n^t} q^{\alpha \cdot \beta} = 0$.

**Proof.** Obviously, if $n \nmid j$, then $\sum_{i=0}^{n-1} q^{ij} = 0$ since $q$ is an $n$-th root of 1. In general, if $\beta = (b_1, b_2, \cdots, b_t) \neq 0$, then there exists $i$ such that $b_i \neq 0$. We have

$$\sum_{\alpha \in \mathbb{Z}_n^t} q^{\alpha \cdot \beta} = (\sum_{a_1 \in \mathbb{Z}_n} q^{a_1b_1}) \cdots (\sum_{a_t \in \mathbb{Z}_n} q^{a_tb_t}) = 0.$$ 

The lemma follows. $\square$

Given a positive integer $m$ and a variable $x$, we denote

$$(m)_x = 1 + x + x^2 + \cdots + x^{m-1},$$

and

$$(m)!_x = (m)_x(m-1)_x \cdots (1)_x.$$ 

We set $(0)!_x = 1$,\[ \binom{m}{u}_x = \frac{(m)!_x}{(m-u)_x(u)_x} \] if $u \leq m$ are positive integers. The notations $[m]_x, [0]!_x, [m]!_x$, and $\left[ \begin{array}{c} m \\ s \end{array} \right]_x$ are as stated before.
A quiver $Q$ is an oriented graph given by two sets $Q_0$ and $Q_1$ of vertices and arrows and two maps $s, t : Q_1 \to Q_0$ providing each arrow with its source and terminal vertex. A path $\gamma$ is a finite sequence of concatenated arrows $\gamma = a_n \cdots a_1$, which means that $t(a_i) = s(a_{i+1})$ for $i = 1, \cdots, n - 1$. We set $s(\gamma) = s(a_1)$ and $t(\gamma) = t(a_n)$. It is noted that a vertex $u$ coincides with its source and terminal vertex. The length of a path means the length of its arrow sequence; vertices mean zero-length paths.

The path algebra of $Q$ is a $k$-vector space $kQ$ with the basis $\{p \mid p$ is a path of $Q\}$, where the multiplication of two paths $p$ and $q$ is defined by

$$p \cdot q = \begin{cases} pq, & \text{if } s(p) = t(q), \\ 0, & \text{otherwise.} \end{cases}$$

Let $\overline{Q}$ be the double quiver of $Q$. Namely, $\overline{Q}$ is obtained by adding a reverse arrow $a^* : j \to i$ for each arrow $a : i \to j$ in $Q$. For each vertex $i$, let $e_i$ be the associated trivial path.

We always assume that $\ell = n$ if $n$ is odd and $\ell = n/2$ if $n$ is even.

## 2 Deformations of double quiver algebras

For each pair $(\mathcal{C}, \mathbb{Z}_n)$, we associate to it a quiver $Q = Q(\mathcal{C}, \mathbb{Z}_n)$ as follows. The set of vertices has a one-to-one correspondence with the group $\mathbb{Z}_n^t$. The set of arrows is

$$\{a(x, i) : x - c^t \to x \mid x \in \mathbb{Z}_n^t, i \in \mathcal{C}\}.$$ 

It is noticed that $e_u a(u, i) = a(u, i) e_{u - c^t} = a(u, i)$, $a(u, i)^* e_u = e_{u - c^t} a(u, i)^* = a(u, i)^*$ in the double quiver algebra $k\overline{Q}$.

The following lemma is well known.

**Lemma 2.1** ([2], Proposition 2.3) *The path algebra $k\overline{Q}$ is a Hopf algebra where the comultiplication, counit, and antipode are given by the following*

$$\begin{align*}
\Delta(e_x) &= \sum_{u+v=x} e_u \otimes e_v, \\
\Delta(a(x, i)) &= \sum_{u+v=x} q^{ui} e_u \otimes a(v, i) + \sum_{u+v=x} a(u, i) \otimes e_v, \\
\varepsilon(e_0) &= 1, \quad \varepsilon(e_x) = 0 \text{ for all } x \neq 0, \quad \varepsilon(a(x, i)) = 0, \\
S(a(x, i)) &= -q^{x_i - 2} a(-x + c^t, i), \quad S(e_x) = e_{-x}, \text{ for all } x, i.
\end{align*}$$
We consider the quotient algebra $\Pi^C = k\mathcal{Q}/I$, where $I$ is the ideal generated by the following relations

\begin{align}
  a(x,i)a(x,i)^* - a(x+c^i,i)^*a(x+c^i,i) - \frac{q^{x_i} - q^{-x_i}}{q - q^{-1}}e_x, \\
  a(x,j)^*a(x,i) - a(x-c^i,i)a(x-c^i,j)^* 
\end{align}

for all $x \in \mathbb{Z}_n^t, i, j \in \mathbb{L}$ with $i \neq j$.

The following fact is important to our aim.

**Lemma 2.2** The algebra $\Pi^C$ is a Hopf algebra with comultiplication, counit, and antipode given by

\begin{align*}
  \Delta(e_x) &= \sum_{u+v=x} e_u \otimes e_v, \\
  \Delta(a(x,i)) &= \sum_{u+v=x} q^{u_i}e_u \otimes a(v,i) + \sum_{u+v=x} a(u,i) \otimes e_v, \\
  \Delta(a(x,i)^*) &= \sum_{u+v=x} e_u \otimes a(v,i)^* + \sum_{u+v=x} q^{-v_i}a(u,i)^* \otimes e_v, \\
  \varepsilon(e_0) &= 1, \quad \varepsilon(e_x) = 0, \text{ for all } x \neq 0, \\
  \varepsilon(a(x,i)) &= 0, \quad \varepsilon(a(x,i)^*) = 0, \\
  S(a(x,i)) &= -q^{x_i-2}a(-x+c^i,i), \quad S(a(x,i)^*) = -q^{-x_i+2}a(-x+c^i,i)^*, \quad S(e_x) = e_{-x}, \text{ for all } x \in \mathbb{Z}_n^t, \ i \in \mathbb{L}. 
\end{align*}

**Proof.** Let $\mathcal{Q}^{op}$ be the reverse quiver of $\mathcal{Q}$. Then we have a path algebra $k\mathcal{Q}^{op}$. By duality and Lemma 2.1, one can prove that $k\mathcal{Q}^{op}$ is a Hopf algebra equipped with

\begin{align*}
  \Delta(a(x,i)^*) &= \sum_{u+v=x} e_u \otimes a(v,i)^* + \sum_{u+v=x} q^{-v_i}a(u,i)^* \otimes e_v, \\
  \varepsilon(e_0) &= 1, \quad \varepsilon(e_x) = 0, \text{ for all } x \neq 0; \\
  \varepsilon(a(x,i)^*) &= 0, \\
  S(a(x,i)^*) &= -q^{-x_i+2}a(-x+c^i,i)^*, \quad \text{for all } x, i. \\
  S(e_x) &= e_{-x}, \text{ for } x \in \mathbb{Z}_n^t. 
\end{align*}

To see that $\Pi^C$ is a Hopf algebra, it is sufficient to show that the given maps $\Delta$ and $\varepsilon$ keep
For the relations (2.1) and (2.2). Indeed, for the relation (2.1), we have

\[
\begin{align*}
\Delta(a(x, i))\Delta(a(x, i)^*) & - \Delta(a(x + c^i, i))\Delta(a(x + c^i, i)^*) \\
& = \sum_{u + v = x} q^u e_u \otimes a(v, i)a(v, i)^* + \sum_{u + v = x} q^{-v_i} a(u, i)a(u, i)^* \otimes e_v \\
& - \sum_{u + v = x + c^i} q^u e_u \otimes a(v, i)^*a(v, i) - \sum_{u + v = x + c^i} q^{-v_i} a(u, i)^*a(u, i) \otimes e_v \\
& = \sum_{u + v = x} q^u e_u \otimes [a(v, i)a(v, i)^* - a(v + c^i, i)^*a(v + c^i, i)] \\
& + \sum_{u + v = x} q^{-v_i} [a(u, i)a(u, i)^* - a(u + c^i, i)^*a(u + c^i, i)] \otimes e_v \\
& = \sum_{u + v = x} q^u e_u \otimes [a(v, i)a(v, i)^* - a(v + c^i, i)^*a(v + c^i, i)] \\
& + \sum_{u + v = x} q^{-v_i} [a(u, i)a(u, i)^* - a(u + c^i, i)^*a(u + c^i, i)] \otimes e_v \\
& = \left( \frac{q^{x_i} - q^{-x_i}}{q - q^{-1}} \right) \sum_{u + v = x} e_u \otimes e_v.
\end{align*}
\]

Hence \(\Delta\) keeps the relation (2.1). For the relation (2.2), it is easy to see that

\[
\begin{align*}
\Delta(a(x, j)^*)\Delta(a(x, i)) & - \Delta(a(x - c^i, i))\Delta(a(x - c^i, j)^*) \\
& = \sum_{u + v = x} q^u e_u \otimes [a(v, j)^*a(v, i) - a(v - c^i, i)a(v - c^i, j)^*] \\
& + \sum_{u + v = x} q^{-v_i} [a(u, j)^*a(u, i) - a(u - c^i, i)a(u - c^i, j)^*] \otimes e_v \\
& = 0
\end{align*}
\]

The arguments for the counit \(\varepsilon\) are similar. It remains to show that \(S\) is an antipode. For example,

\[
S(a(x, i)^*)S(a(x, i)) - S(a(x + c^i, i))S(a(x + c^i, i)^*) = a(-x + c^i, i)^*a(-x + c^i, i) - a(-x, i)a(-x, i)^* \\
= -\frac{q^{-x_i} - q^{x_i}}{q - q^{-1}} e_{-x} = \frac{q^{x_i} - q^{-x_i}}{q - q^{-1}} S(e_x).
\]

For the relation (2.2) the argument for \(S\) is similar (we note that \(a_{ij} = a_{ji}\)). Hence \(S\) is the antipode of \(\Pi^c\). The lemma follows. \(\square\)

We denote the paths \(a(x, i_1)a(x-c^{i_1}, i_2)\cdots a(x-c^{i_1}\cdots c^{i_s+1}, i_s)\) of \(\mathcal{Q}\) by \(a(x, i_1i_2\cdots i_s)\). For brevity, \(a(x, i)a(x-c^i, i)\cdots a(x-(s-1)c^i, i)\) is denoted by \(a(x, i^s)\). Similarly, we use the notations \(a(x, i_1i_2\cdots i_s)^*\) and \(a(x, i^s)^*\).
Lemma 2.3 We have the following formulae:

\[ \Delta(a(x,i^m)) = \sum_{u+v=x \atop u+1 \neq m} \binom{m}{s} q^{u_i} a(u,i^s) \otimes a(v,i^t); \]
\[ \Delta(a(x,i^m)^*) = \sum_{u+v=x \atop u+1 \neq m} \binom{m}{s} q^{-sv_i} a(u,i^s)^* \otimes a(v,i^t)^*. \]

In particular, we have

\[ \Delta(a(x,i^\ell)) = \sum_{u+v=x} q^{\ell u_i} e_u \otimes a(v,i^\ell) + \sum_{u+v=x} a(u,i^\ell) \otimes e_v; \]
\[ \Delta(a(x,i^\ell)^*) = \sum_{u+v=x} q^{-\ell v_i} a(u,i^\ell)^* \otimes e_v + \sum_{u+v=x} e_u \otimes a(v,i^\ell)^*. \]

Proof. It is easy to see that the formula

\[ \binom{m+1}{u} x = \binom{m}{u} x + x^{m-u+1} \binom{m}{u-1} x \]

holds. By this formula, the proof is completed by induction on \( m \). \( \square \)

We set \( \kappa = 1 - a_{ij} \) and for all \( x \in \mathbb{Z}_n^t, \ i, j \in \mathbb{T} \) with \( i \neq j \),

\[ \omega_{ij}(x) = \sum_{t=0}^{\kappa} (-1)^t \binom{\kappa}{t} q \ a(x,i^{\kappa-t}j; i^t); \]
\[ \omega_{ij}(x)^* = \sum_{t=0}^{\kappa} (-1)^t \binom{\kappa}{t} q \ a(x,i^{\kappa-t}j; i^t)^*. \]

We have

Lemma 2.4 For all \( x \in \mathbb{Z}_n^t, \ i, j \in \mathbb{T} \) with \( i \neq j \),

\[ \Delta(\omega_{ij}(x)) = \sum_{u+v=x} q^{\kappa v_i} e_u \otimes \omega_{ij}(v) + \sum_{u+v=x} \omega_{ij}(u) \otimes e_v \]
\[ \Delta(\omega_{ij}(x)^*) = \sum_{u+v=x} e_u \otimes \omega_{ij}(v)^* + \sum_{u+v=x} q^{-\kappa v_i} \omega_{ij}(u)^* \otimes e_v \]

Proof. The proof is considerably straightforward. We compute the formula \( \Delta(\omega_{ij}(x)) \) where \( a_{ij} = -1 \). In this case,

\[ \omega_{ij}(x) = a(x,i^2j) - (q+q^{-1})a(x,iji) + a(x,j^2i). \]
We have
\[
\Delta(a(x, i^2 j)) = \\
\sum_{u+v=x} q^{2u+uj} e_u \otimes a(v, i^2 j) + \sum_{u+v=x} q^{2u_i} a(u, j) \otimes a(v, i^2) \\
+ \sum_{u+v=x} (q + q^{-1}) q^{u_i+uj} a(u, i) \otimes a(v, ij) \\
+ \sum_{u+v=x} (1 + q^{-2}) q^{u_i} a(u, i) \otimes a(v, i) \\
+ \sum_{u+v=x} q^{u_i+j} a(u, i^2) \otimes a(v, j) + \sum_{u+v=x} a(u, i^2 j) \otimes e_v.
\]

Similarly,
\[
\Delta(a(x, ji^2)) = \\
\sum_{u+v=x} q^{2u+uj} e_u \otimes a(v, j i^2) + \sum_{u+v=x} q^{2u_i+2} a(u, j) \otimes a(v, i^2) \\
+ \sum_{u+v=x} (1 + q^{-2}) q^{u_i+uj} a(u, i) \otimes a(v, ji) \\
+ \sum_{u+v=x} (q + q^{-1}) q^{u_i} a(u, ji) \otimes a(v, i) \\
+ \sum_{u+v=x} q^{u_i} a(u, i^2) \otimes a(v, j) + \sum_{u+v=x} a(u, j i^2) \otimes e_v,
\]

and
\[
\Delta(a(x, ij i)) = \\
\sum_{u+v=x} q^{2u+uj} e_u \otimes a(v, ij i) + \sum_{u+v=x} q^{u_i+uj-1} a(u, i) \otimes a(v, ji) \\
+ \sum_{u+v=x} q^{2u_i+1} a(u, j) \otimes a(v, i^2) + \sum_{u+v=x} q^{u_i-1} a(u, ij) \otimes a(v, i) \\
+ \sum_{u+v=x} q^{u_i+uj} a(u, i) \otimes a(v, ij) + \sum_{u+v=x} q^{uj+1} a(u, i^2) \otimes a(v, j) \\
+ \sum_{u+v=x} q^{u_i} a(u, ji) \otimes a(v, i) + \sum_{u+v=x} a(u, ji i) \otimes e_v.
\]

Therefore,
\[
\Delta(\omega_{ij}(x)) = \sum_{u+v=x} q^{2u+uj} e_u \otimes \omega_{ij}(v) + \sum_{u+v=x} \omega_{ij}(u) \otimes e_v.
\]

The cases when \(a_{ij} = 0\) and \(\omega_{ij}(x)\) are similar. \(\square\)

Let \(J\) be the ideal of \(\Pi^C\) generated by
\[
a(x, i^2), a(x, i^i), \omega_{ij}(x), \omega_{ij}(x)^* \tag{2.3}
\]
for all \( x \in \mathbb{Z}_n^t \) and \( i, j \in \mathcal{T} \) with \( i \neq j \). By the definition of the antipode, Lemmas 2.3 and 2.4, it is easy to see that \( J \) is a Hopf ideal. We denote by \( u_q^C \) the quotient algebra \( \Pi^C/J \), which can be presented by generators and relations as follows. As an algebra, \( u_q^C \) is generated by \( \{e_x, a(x, i), a(x, i)^* \mid x \in \mathbb{Z}_n^t, i \in \mathcal{T} \} \) with the following relations: for all \( x \in \mathbb{Z}_n^t \) and \( i, j \in \mathcal{T} \) with \( i \neq j \),

\[
\begin{align*}
    e_x e_y &= \delta_{x y} e_x, e_x a(y, i) = a(y, i) e_x - c_i = \delta_{x, y} a(x, i), \\
    a(x, i)^* e_y &= e_{y - c_i} a(x, i)^* = \delta_{x, y} a(x, i)^*, \\
    a(x, i) a(x, i)^* - a(x + c^i, i)^* a(x + c^i, i) &= \frac{q^{x_i} - q^{-x_i}}{q - q^{-1}} e_x, \\
    a(x, j)^* a(x, i) &= a(x - c^j, i) a(x - c^j, j)^*, \\
    a(x, i^j) &= 0, a(x, i^j)^* = 0, \\
    \omega_{ij}(x) &= 0, \omega_{ij}(x)^* = 0.
\end{align*}
\]

We yield the result as follows.

**Theorem 2.5** \( u_q^C \) is a Hopf algebra, of which the comultiplication, counit, and antipode are defined by

\[
\begin{align*}
    \Delta(e_x) &= \sum_{u + v = x} e_u \otimes e_v, \\
    \Delta(a(x, i)) &= \sum_{u + v = x} q^{u_i} e_u \otimes a(v, i) + \sum_{u + v = x} a(u, i) \otimes e_v, \\
    \Delta(a(x, i)^*) &= \sum_{u + v = x} e_u \otimes a(v, i)^* + \sum_{u + v = x} q^{-v_i} a(u, i)^* \otimes e_v, \\
    \varepsilon(e_0) &= 1, \varepsilon(e_x) = 0, \text{ for all } x \neq 0, \\
    \varepsilon(a(x, i)) &= 0, \varepsilon(a(x, i)^*) = 0, S(a(x, i)) = -q^{x_i - 2} a(-x + c^i, i), \\
    S(a(x, i)^*) &= -q^{-x_i + 2} a(-x + c^i, i)^*, \text{ for all } x, i, \\
    S(e_x) &= e_{-x}, \text{ for all } x.
\end{align*}
\]

### 3 Isomorphisms between \( u_q(C) \) and \( u_q^C \)

In this section, we keep all notations as before.

For \( x = (x_1, \cdots, x_t) \), let \( K_x = K_1^{x_1} \cdots K_t^{x_t} \) and \( \epsilon_x = \frac{1}{m} \sum_{y \in \mathbb{Z}_n^t} q^{-x y} K_y \). By Lemma 1.1, it
is easy to see that for any \( x, y \in \mathbb{Z}_n^t \),

\[
K_y \epsilon_x = q^{x \cdot y} \frac{1}{n^t} \sum_{\beta \in \mathbb{Z}_n^t} q^{-x \cdot (\beta + y)} K_{\beta + y} = q^{x \cdot y} \epsilon_x,
\]

\[
\epsilon_x \epsilon_y = \frac{1}{n^t} \sum_{\gamma \in \mathbb{Z}_n^t} q^{-x \cdot \gamma} K_\gamma \epsilon_y = \frac{1}{n^t} \sum_{\gamma \in \mathbb{Z}_n^t} q^{-(x \cdot y) \cdot \gamma} \epsilon_y = \delta_{x,y} \epsilon_x,
\]

\[
\sum_{x \in \mathbb{Z}_n^t} \epsilon_x = \frac{1}{n^t} \sum_{x \in \mathbb{Z}_n^t} \sum_{\beta \in \mathbb{Z}_n^t} q^{-x \cdot \beta} K_\beta = 1 + \frac{1}{n^t} \sum_{\beta \in \mathbb{Z}_n^t} \left( \sum_{x \in \mathbb{Z}_n^t} q^{-x \cdot \beta} K_\beta \right) = 1,
\]

\[
E_i \epsilon_x = \epsilon_{x + c_i} E_i, \quad F_i \epsilon_x = \epsilon_{x - c_i} F_i,
\]

\[
\Delta(\epsilon_x) = \sum_{u + v = x} \epsilon_u \otimes \epsilon_v, \quad S(\epsilon_x) = \epsilon_{-x}.
\]

We have constructed the Hopf algebra \( u^C_q \). The relationship between \( u^C_q \) and \( u_q(C) \) is given as follows.

**Theorem 3.1** The map \( \tau : u^C_q \rightarrow u_q(C) \) defined by

\[
\tau(e_x) = \epsilon_x, \quad \tau(a(x, i)) = \epsilon_x E_i, \quad \tau(a(x, i)^*) = F_i \epsilon_x
\]

is a Hopf algebra isomorphism.

**Proof.** We prove the theorem in several steps.

**Step 1:** The map \( \tau \) is well-defined. We should first verify that \( \tau \) keeps the basic relations for path algebra. That is, \( e_x \epsilon_y = \delta_{xy} \epsilon_x, \epsilon_x a(y, i) = a(y, i) \epsilon_x = \epsilon_x a(y, i), \) and \( a(x, i)^* \epsilon_y = e_{y - c_i} a(x, i)^* = \delta_{xy} a(x, i)^* \). For the first one,

\[
\tau(e_x) \tau(e_y) = \epsilon_x \epsilon_y = \delta_{x,y} \epsilon_x = \tau(\delta_{xy} \epsilon_x) = \tau(e_x \epsilon_y).
\]

For the second one,

\[
\tau(e_x) \tau(a(y, i)) = \epsilon_x \epsilon_y E_i = \delta_{x,y} \epsilon_y E_i = \tau(\delta_{xy} a(y, i)).
\]

The rest are similar.

It remains to show that \( \tau \) keeps the relations (2.1)-(2.3). For example, for the relation
For the relation (2.2), it is similar. For the relation (2.3),

\[ \tau(a(x, i)) \cdots \tau(a(x - (t - 1)c^j), i) = (\epsilon_x E_i) \cdots (\epsilon_x E_{1-(t-1)c^j}) = \epsilon_x E_i \cdots = 0, \]

and

\[
\sum_{t=0}^{\kappa} (-1)^t \left[ \frac{\kappa}{t} \right] \tau(a(x, i)) \cdots \tau(a(x - (\kappa - t - 1)c^j), i) \\
\times \tau(a(x - (\kappa - t)c^j, j)) \\
\times \tau(a(x - (\kappa - t)c^j - c^j), i) \cdots \tau(a(x - (\kappa - 1)c^j - c^j, i)) \\
= \sum_{t=0}^{\kappa} (-1)^t \left[ \frac{\kappa}{t} \right] (\epsilon_x E_i) \cdots (\epsilon_x E_{1-(\kappa-1)c^j}) \\
\times (\epsilon_x E_{1-(\kappa-1)c^j}) \cdots (\epsilon_x E_{1-c^j} E_i) \\
= \epsilon_x \left( \sum_{t=0}^{\kappa} (-1)^t \left[ \frac{\kappa}{t} \right] E_i^{\kappa-t} E_j^t \right) = 0.
\]

The rest relations in (2.3) are similar.

**Step 2:** Define an algebra map \( \sigma : u_q(C) \to u_q^C \) by

\[
\sigma(K_c) = \sum_{x \in \mathbb{Z}_n^t} q^{c_x} e_x, \quad \sigma(E_i) = \sum_{x \in \mathbb{Z}_n^t} a(x, i), \quad \sigma(F_i) = \sum_{x \in \mathbb{Z}_n^t} a(x, i)^*.
\]

The aim is to show that \( \sigma \) is the inverse of \( \tau \). The map \( \sigma \) is also well defined. Indeed, for the relation (0.1),

\[
\sigma(K_i)^n = \left( \sum_{x \in \mathbb{Z}_n^t} q^{x_i} e_x \right)^n = \sum_{x \in \mathbb{Z}_n^t} q^{nx_i} e_x = \sum_{x \in \mathbb{Z}_n^t} e_x = 1,
\]
since \( q \) is an \( n \)-th root of unity. For the relation (0.2), we have

\[
\sigma(K_i)\sigma(E_j)\sigma(K_i^{-1}) = \left( \sum_{x \in \mathbb{Z}_n^t} q^{x_i} e_x \right) \left( \sum_{x \in \mathbb{Z}_n^t} a(x, j) \right) \left( \sum_{x \in \mathbb{Z}_n^t} q^{-x_i} e_x \right)
\]

\[
= \sum_{x \in \mathbb{Z}_n^t} q^{x_i} e_x a(x, j) q^{-x_i + a_{ij}} e_{x-c^i}
\]

\[
= q^{a_{ij}} \sum_{x \in \mathbb{Z}_n^t} a(x, j) = q^{a_{ij}} \sigma(E_j).
\]

The another relation in (0.2) is similar. For the relation (0.3),

\[
\sigma(E_i)\sigma(F_j) - \sigma(F_j)\sigma(E_i)
\]

\[
= \left( \sum_{x \in \mathbb{Z}_n^t} a(x, i) \right) \left( \sum_{x \in \mathbb{Z}_n^t} a(x, j)^* \right) - \left( \sum_{x \in \mathbb{Z}_n^t} a(x, j)^* \right) \left( \sum_{x \in \mathbb{Z}_n^t} a(x, i) \right)
\]

\[
= \left\{\begin{array}{ll}
\sum_{x \in \mathbb{Z}_n^t} [a(x, i) a(x, i)^* - a(x + c^i, i)^* a(x + c^i, i)], & \text{if } i = j; \\
- \sum_{x \in \mathbb{Z}_n^t} [a(x, j)^* a(x, i) - a(x - c^j, i) a(x - c^j, j)^*], & \text{if } i \neq j
\end{array}\right.
\]

\[
= \left\{\begin{array}{ll}
\sum_{x \in \mathbb{Z}_n^t} q^{x_i-q^{-x_i}} e_x, & \text{if } i = j; \\
0, & \text{if } i \neq j
\end{array}\right.
\]

\[
= \sigma\left( \delta_{ij} \frac{K_i - K_i^{-1}}{q-q^{-1}} \right).
\]

The relation (0.4) is due to the the first two relations of (2.3). As for the quantum Serre relations, if \( a_{ij} = 0 \), we have

\[
\sigma(E_i)\sigma(E_j) - \sigma(E_j)\sigma(E_i)
\]

\[
= \left( \sum_{x \in \mathbb{Z}_n^t} a(x, i) \right) \left( \sum_{x \in \mathbb{Z}_n^t} a(x, j) \right) - \left( \sum_{x \in \mathbb{Z}_n^t} a(x, j) \right) \left( \sum_{x \in \mathbb{Z}_n^t} a(x, i) \right)
\]

\[
= \sum_{x \in \mathbb{Z}_n^t} [a(x, ij) - a(x, ji)] = 0;
\]
if $a_{ij} = -1$, we have

$$\sigma(E_i)^2 \sigma(E_j) - (q + q^{-1}) \sigma(E_i) \sigma(E_j) \sigma(E_i) + \sigma(E_j) \sigma(E_i)^2$$

$$= \left( \sum_{x \in \mathbb{Z}_n^t} a(x, i) \right)^2 \left( \sum_{x \in \mathbb{Z}_n^t} a(x, j) \right) - (q + q^{-1}) \left( \sum_{x \in \mathbb{Z}_n^t} a(x, i) \right)$$

$$\times \left( \sum_{x \in \mathbb{Z}_n^t} a(x, j) \right) \left( \sum_{x \in \mathbb{Z}_n^t} a(x, i) \right) + \left( \sum_{x \in \mathbb{Z}_n^t} a(x, j) \right) \left( \sum_{x \in \mathbb{Z}_n^t} a(x, i) \right)^2$$

$$= \sum_{x \in \mathbb{Z}_n^t} \omega_{ij}(x) = 0.$$  

The arguments of the rest relations are similar.

By direct calculation, we have $\sigma \circ \tau = 1$ and $\tau \circ \sigma = 1$. Hence $\tau$ is an algebra isomorphism.

**Step 3:** $\tau$ is a Hopf algebra homomorphism. It is enough to verify that $\tau$ is also a coalgebra map since bialgebra homomorphisms are Hopf homomorphisms. It suffices to check it on the generators, but this is considerably direct. We check only one of them.

$$\Delta(\tau(a(i, x))) = \Delta(\varepsilon x E_i) = \Delta(\varepsilon) \Delta(E_i)$$

$$= \left( \sum_{u+v=x} \varepsilon_u \otimes \varepsilon_v \right) \left( E_i \otimes 1 + K_i \otimes E_i \right)$$

$$= \sum_{u+v=x} \tau(a(u, i)) \otimes \tau(e_v) + \sum_{u+v=x} K_i \varepsilon_u \otimes \tau(a(v, i))$$

$$= \sum_{u+v=x} \tau(a(u, i)) \otimes \tau(e_v) + \sum_{u+v=x} q^{uv} \tau(e_u) \otimes \tau(a(v, i))$$

$$= (\tau \otimes \tau) \Delta(a(x, i)).$$

The others are similar.

The proof is completed. \qed

For the Cartan matrix $C = (a_{ij})_{t \times t}$, there is an associated Hopf algebra $u_q^+$ generated by $K_i$, $E_i$ for $1 \leq i \leq t$, subjecting to the relations

$$K_i^{n_i} = 1, \ K_i K_j = K_j K_i,$$

$$K_i E_j K_i^{-1} = q^{a_{ij}} E_j,$$

$$\Delta(K_i) = K_i \otimes K_i,$$

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i,$$

$$\varepsilon(K_i) = 1, \ \varepsilon(E_i) = 0,$$

$$S(K_i) = K_i^{-1}, \ S(E_i) = -K_i^{-1} E_i.$$
Similarly, there is an associated Hopf algebra $u_q^{-}$ generated by $K_i$, $F_i$ for $1 \leq i \leq t$, subjecting to the relations

\begin{align*}
K_i^n &= 1, 
K_iK_j = K_jK_i; \\
K_iF_jK_i^{-1} &= q^{-a_{ij}}F_j; \\
\Delta(K_i) &= K_i \otimes K_i, \\
\Delta(F_i) &= F_i \otimes K_i^{-1} + 1 \otimes F_i, \\
\varepsilon(K_i) &= 1, 
\varepsilon(F_i) = 0, \\
S(K_i) &= K_i^{-1}, 
S(F_i) &= -F_iK_i.
\end{align*}

By [11], there is a skew Hopf pairing $u_q^+ \times u_q^- \rightarrow k$ and we have the Drinfeld double $\mathcal{D}(u_q)$ (see [11], Sect. 2). Let $I = \langle K_i \otimes 1 - 1 \otimes K_i \mid i \in \mathcal{I} \rangle$ be the ideal of $\mathcal{D}(u_q)$ generated by $K_i \otimes 1 - 1 \otimes K_i$, $i \in \mathcal{I}$. It is easy to see that $I$ is a Hopf ideal of $\mathcal{D}(u_q)$ and $\Pi^C \cong \mathcal{D}(u_q)/I$ as Hopf algebras by Theorem 3.1. Furthermore, if $t = \{1\}$, the relation (2.2) automatically vanishes and $\Pi^C$ is just a deformation of preprojective algebra associated to the quiver $Q$.

We have found an algebraic realization of the quantum group $u_q(C)$. This method is very intuitive. It is natural to expect that the presentation via a double quiver will help to study the representation theory, probably by consulting the theory of deformed preprojective algebras. This will be considered in the forthcoming papers.

### 4 Some Remarks

In the representation theory of finite dimensional algebras, finite dimensional basic algebras can always be constructed via quivers with admissible relations, according to Gabriel’s Theorem. Recall that a path relation is called admissible if the length of the paths involved are at least two. We remark that our relation (2.1) is not admissible. Actually, the quantum groups $u_q(C)$ are not basic and hence there is no hope to present them via quivers with admissible relations. However it will be of interest to consider the Ext-quiver of the quantum groups and compare with the double quivers obtained.

For $\mathfrak{sl}_2$ the quiver obtained by the authors is the same as the quiver described in arXiv: math.RT/0410017, which appeared after the submission of our paper. The authors thank the referee for pointing out the reference.

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