THB-spline approximations for turbine blade design with local B-spline approximations

Cesare Bracco\textsuperscript{a}, Carlotta Giannelli\textsuperscript{a}, David Großmann\textsuperscript{b}, Sofia Imperatore\textsuperscript{a}, Dominik Mokriš\textsuperscript{b}, Alessandra Sestini\textsuperscript{a}

\textsuperscript{a} Dipartimento di Matematica e Informatica “U. Dini”, Università degli Studi di Firenze, v.le Morgagni 67/A, 50134 Firenze, Italy
\textsuperscript{b} MTU Aero Engines AG, Munich, Germany

emails: \{cesare.bracco, carlotta.giannelli, sofia.imperatore, alessandra.sestini\}@unifi.it,
\{David.GROSSMANN, Dominik.Mokris\}@mtu.de

Abstract

We consider adaptive scattered data fitting schemes with truncated hierarchical B-splines (THB-splines) based on two-stage methods for the adaptive reconstruction of industrial models. The first stage of the scheme is devoted to the computation of local least squares B-spline approximations. A simple fairness functional is suitably exploited to handle data distributions with a varying density of points locally available. Hierarchical spline quasi-interpolation based on THB-splines is considered in the second stage of the method to construct the adaptive spline surface approximating the scattered data set. A selection of examples on geometric models representing components of aircraft turbine blades highlights the performance of the scheme. The tests include a scattered data set with voids and the adaptive reconstruction of a cylinder-like surface.

1 Introduction

Scattered data fitting is nowadays of fundamental importance in a variety of fields, ranging from geographic applications to medical imaging and geometric modeling, see for example \cite{8, 13, 14}. The topic can be addressed in different approximation spaces, either by using spline spaces or radial basis functions which are particularly attractive in high dimension \cite{19}. In particular, thanks to their low computational cost and also to their better control of conditioning, two stage schemes relying on local approximations have received a lot of attention, being formulated either as partition of unity interpolation or as two–stage approximation methods, see for example \cite{3, 5, 6, 7, 15} and references therein.

In this work we are interested in surface reconstruction of industrial models starting from scattered data obtained by optical scanning acquisitions. In order to reduce the noise, the available data are always preprocessed, since the interest is in highly accurate reconstructions. Anyway, since the available data can not
be considered exact, interpolation is not required and a less costly approximation scheme can be used to compute any local fitting. In particular, we are interested in using extensions of tensor-product B-splines, see e.g., [16], the standard choice for industrial computer-aided design applications.

As well known, an approximating spline can be obtained by using different approximation approaches and also operating in different kinds of spline spaces. The approximation schemes can be divided into two main classes. The first is the class of global schemes which simultaneously use all the given information and usually require the solution of a linear system whose size is equal to the cardinality of the considered spline space. The final approximation defined with the second approach can be collocated in the field of Quasi-Interpolation (QI) which can require the solution of small linear systems whose size does not depend on the cardinality of the entire data set. These local systems descend from the application of local approximation schemes, each one considering a small number of data whose associated parameter values belong to a small portion of the domain intersecting the support of a compactly supported basis function. The computational advantage of the QI approach is evident, since these local linear systems are independent of each other. The quality of the final spline approximation depends in this case not only on the considered spline space but also on the approximation power of the local approximation scheme and on the choice of the local data sets.

Since the data can be characterized by a highly varying distribution, by also including voids, a flexible and reliable approach which automatically (re-)constructs the geometric model, by suitably adapting the solution to the shape and configuration of the given point clouds, strongly improves the efficiency of the overall scheme. We then consider adaptive spline approximations based on the hierarchical extension of multivariate B-splines, usually defined in commercial software tools via the tensor-product approach, which does not provide the possibility of a full local mesh refinement. In particular, in our construction we exploit the capabilities of truncated hierarchical B-splines (THB-splines) [11], since they allow an easy extension to hierarchical spline spaces of a quasi–interpolation scheme formulated in a standard spline space [17, 18].

The first proposal based on adaptive THB-spline fitting of scattered data for the reconstruction of industrial models was introduced in [12], where a global adaptive smoothed least-squares scheme was developed. Successively, in order to increase its locality and reduce the computational cost, the same problem has been addressed in different papers by combining a two-stage approach with hierarchical spline approximations. The first contribution where this kind of schemes was used by some of the authors was presented in [2] to deal with gridded data of Hermite type. In [3] these kinds of approximants were extended for the first time to scattered data, by using in the first stage of the scheme local polynomial least squares approximations of variable degree. A preliminary application of this scheme to industrial data reconstruction was given in [4], where its theoretical analysis was also presented.

In this paper a variant of the approach considered in [3, 4] is presented, to further decrease the number of degrees of freedom necessary to reach a certain accuracy and also to reduce the artifacts in the resulting surface. Since the local polynomials used in [3, 4] need to be converted into B-spline form for being usable by the quasi-interpolation approach adopted in the second stage of scheme, we now work directly in local spline spaces. In this way, the algorithm for the first stage of the
scheme is simplified. In order to avoid rank-deficiency problems by simultaneously controlling the smoothness of local approximants, a smoothing term is added to the least squares objective function. To improve the stability of the proposed method for general data configuration, we also inserted an automatic control on the choice of the local data sets.

The structure of the paper is as follows. Section 2 presents the model problem, while the first stage of the new scheme, devoted to the computation of local smoothed least squares B-spline approximations, is described in Section 3. Section 4 introduces the construction of the adaptive THB-spline surface approximating the scattered data set. A selection of examples on geometric models representing components of aircraft turbine blades is presented in Section 5 and compared with the results obtained in [4]. The numerical experiments include a new scattered data set with voids and the adaptive reconstruction of a surface closed in one parametric direction.

2 The problem

The industrial models here considered are components of aircraft turbine blades which can be suitably represented in parametric form by using just one map, with the possibility of being periodic in one parametric direction. The problem can be mathematically described as follows. Let

$$F := \{f_i \in \mathbb{R}^3, i = 1, \ldots, n\}$$

be a scattered set of distinct points in the 3D space which can be reasonably associated by a one-to-one map to a set $X := \{x_i := (x_i, y_i) \in \Omega \subset \mathbb{R}^2, i = 1, \ldots, n\}$ of distinct parameter values belonging to a closed planar parametric domain $\Omega$. Since the choice of a suitable parametrization method for the definition of the set $X$, which can naturally influence the quality of the final approximation, is not our focus, in this paper we relate to classical parametrization methods based on a preliminary triangulation of the scattered data set $F$, see e.g., [9, 10]. Consequently, both $F$ and $X$ are considered input data for the approximation problem.

Focusing on two-stage spline approximation schemes, we can introduce the basic lines referring for simplicity to their formulation in a standard space $V$ of tensor product splines of bi-degree $d$. With this kind of methods, a quasi-interpolation operator $Q$ is defined so that $Q(F, X) = s$, with $s$ denoting a vector function, possibly periodic in one parametric direction, with components in the spline space $V$. Using a suitable spline basis $B := \{B_J\}_{J \in \Gamma}$ of $V$, such vector spline $s$ can be expressed as follows,

$$s(x) := \sum_{J \in \Gamma} \lambda_J(F_J, X_J)B_J(x), \quad x \in \Omega,$$

where each coefficient vector $\lambda_J(F_J, X_J), J \in \Gamma$, is computed in the first stage of the scheme by using a certain local subset $F_J \subset F$ of data and the corresponding set of parameter values $X_J \subset X$, so that $s(x_i) \approx f_i, i = 1, \ldots, n$.

When dealing with discrete data, measuring the accuracy of the spline approximation with the maximum of the errors $||s(x_i) - f_i||_2$ at each parameter site can appear reasonable at the first sight. However, the quality of the approximant is also
strictly related to the lack of unwanted artifacts, a feature of fundamental importance for industrial applications of any approximation scheme. In this context, it is then a common practice to require the error under a prescribed tolerance only at a certain percentage of sites in \(X\).

### 3 First stage: local B-spline approximations

For computing each vector coefficient \(\lambda_J, J \in \Gamma\), necessary in (1) to define the approximation \(s\), we consider a local data subset \(F_J \subset F\),

\[
F_J := \{f_i : i \in I_J\} \quad \text{with} \quad I_J := \{i : x_i \in X \cap \Omega_J\},
\]

associated to the set

\[
X_J := \{x_i : i \in I_J\}
\]

of parameter values in a local subdomain \(\Omega_J\) of \(\Omega\) which has non empty intersection with the support of the basis function \(B_J\), namely \(\Omega_J \cap \text{supp}(B_J) \neq \emptyset\). By denoting with \(B_J := \{B_I : I \in \Lambda_J \subset \Gamma\}\) the set of B-splines in \(B\) not vanishing in \(\Omega_J\) (which necessarily includes \(B_J\)), the value of \(\lambda_J\) is defined as the vector coefficient \(c^{(J)}_J\) associated with \(B_J\) in the local spline approximation

\[
s_J(x) := \sum_{I \in \Lambda_J} c^{(J)}_I B_I(x)
\]

minimizing the objective function

\[
\sum_{i \in I_J} \|s_J(x_i) - f_i\|^2 + \mu E(s_J), \tag{2}
\]

where \(\mu > 0\) is a smoothing coefficient and \(E(s_J)\) the thin-plate energy,

\[
E(s_J) := \int_{\Omega_J} \left( \left\| \frac{\partial^2 s_J}{\partial x^2} \right\|^2 + 2 \left\| \frac{\partial^2 s_J}{\partial x \partial y} \right\|^2 + \left\| \frac{\partial^2 s_J}{\partial y^2} \right\|^2 \right) \, dx dy.
\]

As well known, the assumption of a positive \(\mu\) ensures that this local approximation problem admits always a unique solution, regardless of the distribution of \(X_J\).

Since the scheme is locally applied, an automatic (data-dependent) selection of the parameter \(\mu\) could be considered. For example, the choice may take into account the cardinality \(|X_J|\) of the local sample or the area of \(\Omega_J\), which influence the value of the first and of the second addend in (2), respectively. In view of this influence however, we may observe that a constant value of \(\mu\) implies that the balancing between the fitting and the smoothing term in the objective function usually increases when \(|X_J|\) or the area of \(\Omega_J\) increases, being this true in the second case because second derivatives are involved in the smoothing term. Both these behaviors seem reasonable and are confirmed by the quality of the results obtained in our experiments, where a constant value for \(\mu\) is suitably chosen.

Differently from [3, 4], in order to better avoid overfitting, a lower bound \(n_{\text{min}}\) for the cardinality of \(X_J\) is now required. Obviously, the value of \(n_{\text{min}}\) has to be reasonably chosen with respect to the data distribution. To fulfill this condition, \(\Omega_J\) is initialized as \(\text{supp}(B_J)\) and enlarged until \(|X_J| \geq n_{\text{min}}\). The refinement strategy presented in Section 4 automatically guarantees that \(|X_J| \geq n_{\text{min}}\) and prevents an
excessive enlargement of the set $\Omega_J$, which would compromise the locality of the approximation. For this reason, it is not necessary to set a maximum value for controlling the enlargement of the local data set. Summarizing, the computation of each $\lambda_J$, $J \in \Gamma$, is done according to the following algorithm.

**Inputs**
- $F \subset \mathbb{R}^3$: scattered data set;
- $X \subset \Omega \subset \mathbb{R}^2$: set of parameter values corresponding to the data in $F$;
- $V$: tensor-product spline space defined on $\Omega$;
- $G$: tensor-product mesh associated with the space $V$;
- $J \in \Gamma$: index of the considered basis function $B_J \in \mathcal{B}$;
- $n_{\text{min}} \leq n$: minimum number of data required for the local approximation;

**Local smoothing spline approximant**
1. initialize $\Omega_J = \text{supp}(B_J)$;
2. initialize $I_J = \{i : x_i \in X \cap \Omega_J\}$, $F_J = \{f_i : i \in I_J\}$ and $X_J = \{x_i : i \in I_J\}$;
3. while $|F_J| < n_{\text{min}}$
   - (a) enlarge $\Omega_J$ with the first ring of cells in $G$ surrounding $\Omega_J$;
   - (b) update $I_J = \{i : x_i \in X \cap \Omega_J\}$, $F_J = \{f_i : i \in I_J\}$ and $X_J = \{x_i : i \in I_J\}$;
4. compute the local approximation $s_J$ for the data $F_J$ and $X_J$ by minimizing (2);
5. set $\lambda_J = c_J^{(J)}$.

**Output**
- vector coefficient $\lambda_J$.

Note that exploiting a regularized least square approximation and, as a consequence, being able to directly employ the local spline space has significantly simplified the algorithm originally proposed for the first stage in \[3, 4\], where a variable-degree local polynomial approximation was considered. In particular, the new scheme does not require the selection of a suitable degree for the computation of any coefficient $\lambda_J$ and eliminates the conversion of the computed approximant from the polynomial to the B-spline basis.

In the following section, after introducing the THB-spline basis, the operator $Q$ is easily extended to hierarchical spline spaces, by also introducing the automatic refinement algorithm here considered. Note that this extension rule ensures that the coefficient associated to a THB-spline basis function remains unchanged on a refined hierarchical mesh if this function remains active on the updated hierarchical configuration.

4 Second stage: THB-spline approximation

Let us consider a sequence $V^0 \subset \ldots \subset V^{M-1}$ of $M$ spaces of tensor-product splines of bi–degree $d := (d_1, d_2)$ defined on the closed domain $\Omega$ and a sequence of closed domains $\Omega^0 \supset \ldots \supset \Omega^M$, with $\Omega^0 := \Omega$ and $\Omega^M := \emptyset$. Any $\Omega^\ell$, for $\ell = 1, \ldots, M - 1$
is the union of cells of the tensor-product grid of level \( \ell - 1 \). The hierarchical mesh \( \mathcal{G}_H \) is the set of active cells at different levels.

We denote by \( B_{a}^{\ell} := \{ B_{J}^{\ell} \} \) the B-spline basis of \( V^\ell \). For any \( s \in V^\ell \), \( \ell = 0, \ldots, M - 2 \), let
\[
s = \sum_{J \in \Gamma_{a}^{\ell+1}} \sigma_{J}^{\ell+1} B_{J}^{\ell+1}
\]
be its representation in terms of B-splines of the refined space \( V^{\ell+1} \). We define the truncation of \( s \) with respect to level \( \ell + 1 \) and its (cumulative) truncation with respect to all finer levels as
\[
\text{trunc}^{\ell+1}(s) := \sum_{J \in \Gamma_{a}^{\ell+1}: \text{supp}(B_{J}^{\ell+1}) \not\subseteq \Omega^{\ell+1}} \sigma_{J}^{\ell+1} B_{J}^{\ell+1},
\]
and
\[
\text{Trunc}^{\ell+1}(s) := \text{trunc}^{M-1}(\text{trunc}^{M-2}(\ldots(\text{trunc}^{\ell+1}(s))\ldots)),
\]
respectively. For convenience, we also define \( \text{Trunc}^{M}(s) := s \) for \( s \in V^{M-1} \). The THB-spline basis \( T_{a}(\mathcal{G}_H) \) of the hierarchical space \( S_H \) was introduced in [11] and can be defined as
\[
T_{a}(\mathcal{G}_H) := \left\{ T_{J}^{\ell} := \text{Trunc}^{\ell+1}(B_{J}^{\ell}) : J \in A_{a}^{\ell}, \ell = 0, \ldots, M - 1 \right\}
\]
where
\[
A_{a}^{\ell} := \left\{ J \in \Gamma_{a}^{\ell} : \text{supp}(B_{J}^{\ell}) \subseteq \Omega^{\ell} \land \text{supp}(B_{J}^{\ell}) \not\subseteq \Omega^{\ell+1} \right\}
\]
is the set of active multi-indices, and \( \text{supp}(B_{J}^{\ell}) \) denotes the intersection of the support of \( B_{J}^{\ell} \) with \( \Omega^{0} \). The B-spline \( B_{J}^{\ell} \) is called the mother B-spline of the truncated basis function \( T_{J}^{\ell} \).

By following the general approach introduced in [18], we construct the vector THB-spline approximation of the scattered data set in terms of the hierarchical quasi-interpolant \( s = Q(F, X) \) as follows,
\[
s(x) := \sum_{\ell=0}^{M-1} \sum_{J \in A_{a}^{\ell}} \lambda_{J}(F_{J}, X_{J}) B_{J}^{\ell}(x), \quad (3)
\]
where each vector coefficient \( \lambda_{J} \) is the one of the mother function \( B_{J}^{\ell} \) and is obtained by computing the local regularized B-spline approximation \( s_{J}^{\ell} \) on the data set \( F_{J}^{\ell} \) associated to \( B_{J}^{\ell} \) as described in Section [3].

In order to define an adaptive approximation scheme, we need to specify how to iteratively construct the hierarchical mesh \( \mathcal{G}_H \) and the corresponding spline space \( S_H \).

At the first iteration we always start from a hierarchical mesh \( \mathcal{G}_H \) coinciding with a uniform mesh \( \mathcal{G}^{0} \), and therefore the spline space \( S_H \) is a tensor-product space \( V^{0} \). At each iteration the coefficients of the hierarchical QI defined by [3] are computed, and the approximation \( s \) is then evaluated at each data point \( x_{i} \in X \) to get the errors
\[
e_{i} := \|s(x_{i}) - f_{i}\|_{2}, \quad i = 1, \ldots, n.
\]
The hierarchical mesh is refined in the areas of the domain where these errors exceed the given tolerance \( \epsilon \) and a refinement criterion is satisfied. The refinement
criterion has been motivated by the observation that, when the parameter values corresponding to the local data set $F^\ell_I$ are concentrated in a small part of the support of $B^\ell_I$, the quality of the approximation may be affected. More precisely, we consider a splitting of the two sides of $\text{supp}(B^M_{M-1})$ in $n_1$ and $n_2$ uniform segments, respectively, and subdivide the support of $B^M_{M-1}$ in the resulting $n_1n_2$ subregions. We then dyadically refine (in the two parametric directions) the support of the B-splines $B^\ell_J$, $J \in A^d_\ell$, $\ell = 0, \ldots, M - 1$, which contains at least one point $x_i$ where $e_i > \epsilon$ and at least $\lceil n_{\text{loc}}/(n_1n_2) \rceil$ data points in any of the $n_1n_2$ subregions, being $n_{\text{loc}}$ a given nonnegative integer. To simplify the usage of the algorithm by default we set $n_1 = n_2 = 1$ but different values can be chosen if suitable, see e.g. the data set with voids considered in Example 3 of Section 5.

Concerning the parameter $n_{\text{loc}}$, we may observe that the requirement $n_{\text{loc}} \geq n_{\text{min}}$ guarantees that the points needed to compute the coefficients associated with the new functions in the first stage of the next iteration can be found not too far from the support of the functions themselves. Indeed in the algorithm introduced in the previous section, after a few enlargements, $\Omega_J$ will surely include the support of a refined function of the previous level intersecting $\text{supp}(B_J)$. As a consequence, analogously to $n_{\text{min}}$, a high value of $n_{\text{loc}}$ contributes to the reduction of oscillations deriving from overfitting, but this value should also be low enough to guarantee that the refinement strategy can generate a hierarchical spline space with enough degrees of freedom for satisfying the given tolerance $\epsilon$.

Once the hierarchical mesh is refined, the hierarchical space is updated accordingly. This loop is repeated until the tolerance is satisfied for a given percentage of the data points. If this condition is not satisfied within a maximum number of levels, the procedure stops anyway.

The proposed adaptive approximation method also extends to the case, not addressed in the previous works, of surfaces closed in one (or even two) parametric directions. Note that the local nature of the considered approximation approach makes the implementation especially easy, since coefficients associated with a THB present at successive steps of the adaptive refinement procedure (even if possibly further truncated) do not depend on such steps.

5 Examples

We present a selection of tests for the approximation of industrial data obtained by an optical scanning process of different aircraft engine parts. The parameter values are computed in all examples with standard parametrization methods based on a triangulation of the scattered data sets, see e.g. [9, 10]. The results highlight the effects of considering a minimum number of local data points (also) in the first stage of method, as well as the improvements obtained by introducing a regularized B-spline approximation for each local fitting with respect to the scattered data fitting scheme considered in [4]. By combining these two changes, the two-stage approximation algorithm is more stable and unwanted oscillations are further reduced.

Example 1 (Tensile). In this example, we consider THB-spline approximations to reconstruct a part of a tensile from the set of 9281 scattered data shown in Figure 1 (top). We compare the new local scheme based on local B-spline approximations with the algorithm based on local polynomial approximations of variable degree
presented in [4], where this test was originally considered.

As the first test, we ran both algorithms with the same setting considered in [4], namely, by starting with a $4 \times 4$ tensor-product mesh with $d = (2, 2)$, tolerance $\epsilon = 5 \cdot 10^{-5}$ m, and $n_{loc} = 20$. The algorithm with local polynomial approximations with the parameter choice considered in [4] ($\sigma = 10^8$) led to an approximation with 2506 degrees of freedom, 96.25% of points below the tolerance and a maximum error of $1.22332 \cdot 10^{-4}$ m. The new scheme based on local B-spline approximations with $n_{min} = 6$ and $\mu = 10^{-6}$ generated a THB-spline surface with 1855 degrees of freedom that satisfies the required tolerance in 98.88% of points with a maximum error of $8.06007 \cdot 10^{-5}$ m.

As the second test, we ran both algorithms by starting with a $16 \times 4$ tensor-product mesh with $d = (2, 2)$, tolerance $\epsilon = 5 \cdot 10^{-5}$ m, and $n_{loc} = 15$. The algorithm with local polynomial approximations led to an approximation with 5922 degrees of freedom, 98.18% of points below the tolerance and a maximum error of $1.44222 \cdot 10^{-4}$ m. The surface and the corresponding hierarchical mesh are shown in Figure 1 (center). This approximation clearly shows strong oscillations on the boundary of the reconstructed surface, due to a lack of available data points for the local fitting in correspondence of high refinement levels. The scheme based on local B-spline approximations, with $n_{min} = 7$ and $\mu = 10^{-6}$, produced a THB-spline surface with 2605 degrees of freedom that satisfies the required tolerance in all data points (100% of points below the tolerance) with a maximum error of $4.91520 \cdot 10^{-5}$ m. The surface, free of unwanted oscillations along the boundary, and the corresponding hierarchical mesh are shown in Figure 1 (bottom).

**Example 2 (Blade).** In this example, we test the second example considered in [4] on the set of 27191 scattered data representing a scanned part of a blade shown in Figure 2 (top). Again, to compare the new local scheme with the algorithm based on local polynomial approximations there considered, we ran both algorithms with the same setting of [4], namely, by starting with a $4 \times 4$ tensor-product mesh with $d = (3, 3)$, tolerance $\epsilon = 2 \cdot 10^{-5}$ m, and $n_{loc} = 60$. The algorithm with local polynomial approximations with the parameter choice considered in [4] ($\sigma = 10^8$) led to an approximation with 12721 degrees of freedom, 97.02% of points below the tolerance and a maximum error of $4.88418 \cdot 10^{-5}$ m. The new scheme based on local B-spline approximations with $n_{min} = 6$ and $\mu = 10^{-8}$ generated a THB-spline surface with 8533 degrees of freedom that satisfies the required tolerance in 100.00% of points with a maximum error of $1.99780 \cdot 10^{-5}$ m. The surface and the corresponding hierarchical mesh are shown in Figure 2 (bottom).

**Example 3 (Endwall).** In this example, we illustrate the behavior of the adaptive algorithm on data sets with voids by considering the reconstruction of an end-wall part from the scattered data set of 43869 points shown in Figure 3 (top). The figure shows that in this case the data set represents a model with three different holes, where no input data are available. The aim of this reconstruction is to avoid artifacts due to lack of points and obtain a sufficiently regular surface (eg. by avoiding self-intersections), that can be post-processed with standard geometric software tools to obtain a suitably trimmed model. Consequently, not only the number of points in the local data sets is important to reach this aim, but also their distribution. To properly address this issue, we consider a real density parameter with value between 0 and 1 which determines whether the distribution of the points in the local set is reliable or not for the fitting. The distribution of the local data
Fig. 1 Example 1: scattered data set corresponding to a critical part of a tensile (top), the reconstructed surfaces and the corresponding hierarchical meshes obtained with the algorithm presented in [4] (center) and the new local scheme (bottom).
points is computed as the number of mesh cells of level $\ell$ inside the support of $B^\ell$ or its enlargement, which contain at least one point, over the total number of mesh cells, either in the support of $B^\ell$ or its enlargement. If this ratio is below a density threshold, then more data points are required and the function support is enlarged for the computation of the local approximation in the first-stage of the method. The approximation is developed by starting from a $32 \times 32$ tensor-product mesh, with $d = (3, 3)$, $n_{loc} = 15$, $n_{min} = 11$, $\mu = 10^{-6}$ and $n_1 = n_2 = 2$. A choice of the density parameter equals to 0.3 permits to take care of the distribution of data points in the construction of the approximation. By considering a tolerance $\epsilon = 5 \cdot 10^{-5}$ m, the refinement generated a THB-spline approximation with 14817 degrees of freedom, $99.36\%$ of points below the threshold and a maximum error of $2.30094 \cdot 10^{-4}$ m. The surface and the corresponding hierarchical mesh are shown in Figure 3 (bottom).

Example 4 (Airfoil). This example illustrates the behavior of the new adaptive fitting algorithm with local B-spline approximations for surfaces closed in one parametric direction. We test the scheme to reconstruct a blade airfoil from the set of 19669 scattered data shown in Figure 4 (top). We ran the algorithm by starting with a $32 \times 4$ tensor-product mesh with $d = (3, 3)$, tolerance $\epsilon = 5 \cdot 10^{-5}$ m, and

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**Fig. 2** Example 2: scattered data set corresponding to a critical part of a blade (top), the reconstructed surfaces and the corresponding hierarchical meshes obtained with the new local scheme (bottom).
Fig. 3 Example 3: scattered data set corresponding to a critical part of an endwall (top), the reconstructed surfaces and the corresponding hierarchical meshes obtained with the new local scheme (bottom).

\( n_{\text{loc}} = 30 \). The local algorithm was used in this case with \( n_{\text{min}} = 12 \), \( \mu = 10^{-6} \), and produced an approximation with 3446 degrees of freedom, that satisfies the required tolerance in 97.71% of the data points with maximum error \( 9.48753 \cdot 10^{-5} \text{ m} \). The surface and the corresponding hierarchical mesh are shown in Figure 4 (bottom).

By trying to force additional refinement, some oscillations appear. In this case, they are consistent with the data distribution since there are clusters of high density points, due to scan noise. Consequently, they do not represent artifacts caused by regions with very low density of data and cannot be prevented by exploiting the bound for cardinality of the local data sample governed by \( n_{\text{loc}} \).

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Fig. 4 Example 4: scattered data set corresponding to a critical part of an airfoil (top), the reconstructed surfaces and the corresponding hierarchical meshes obtained with the new local scheme (bottom).

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