Abstract. In this paper we prove the existence of purely log terminal blow-up for Kawamata log terminal singularity and obtain the criterion for a singularity to be weakly exceptional in terms of the exceptional divisor of plt blow-up.

Introduction

The main aim of this note is to prove two results of the paper [5] for non $\mathbb{Q}$-factorial case. The first one is the inductive blow-up existence theorem (theorem 1.5) and the second one is a criterion of weakly exceptionality (theorem 2.1). These blow-ups allow us to apply Shokurov’s inductive method to the study of singularities and in general case extremal contractions. Using this method we can reduce the questions on structure, complementness and exceptionality of singularity to a single exceptional divisor of purely log terminal blow-up. For any $\mathbb{Q}$-factorial singularity a plt blow-up is the unique one that allows to extend the complement of exceptional divisor to a global complement (remark 1.3). For non $\mathbb{Q}$-factorial klt singularity such blow-ups differ from plt blow-ups by a small flopping contraction (corollary 1.13). In studying any $\mathbb{Q}$-gorenstein singularities it is practically impossible to select $\mathbb{Q}$-factorial singularity class from the others. That is why we have to apply the theorems and constructions which are true in the general case. This paper also proves some results on the inductive method of any lc singularity studies.

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1. Purely log terminal blow-ups and their properties

All varieties are algebraic and are assumed to be defined over $\mathbb{C}$, complex number field. The results can be easily modified to the category of analytic spaces. We use the terminology and notations of Log Minimal
Model Program and the main properties of complements given in [4], [5]. A strictly lc singularity is called lc singularity, but is not klt singularity.

**Definition 1.1.** Let $X$ be a normal lc variety and let $f: Y \to X$ be a blow-up such that the exceptional locus of $f$ contains only one irreducible divisor $E$ ($\text{Exc}(f) = E$). Then $f: (Y, E) \to X$ is called a purely log terminal (plt) blow-up, if $K_Y + E$ is plt and $-E$ is $f$-ample.

**Remark 1.2.** In the definition 1.1 it is demanded that divisor $E$ must be $\mathbb{Q}$-Cartier. Hence $Y$ is a $\mathbb{Q}$-gorenstein variety.

**Remark 1.3.**
1. If $X$ is klt then $-(K_Y + E)$ is $f$-ample. Indeed, we have $K_Y + E = f^*K_X + (a(E, 0) + 1)E$ and $a(E, 0) + 1 > 0$.
2. If $X$ is strictly lc then $a(E, 0) = -1$.
3. [5, 2.2] If $X$ is $\mathbb{Q}$-factorial then $Y$ is also $\mathbb{Q}$-factorial and $\rho(Y/X) = 1$. Hence, in definition 1.1 for $\mathbb{Q}$-factorial singularity it is not necessary to demand that divisor $-E$ is $f$-ample, because ampleness takes place always. Note that every exceptional locus component has codimension 1 for any birational contraction to $\mathbb{Q}$-factorial variety.
4. By inversion of adjunction $K_E + \text{Diff}_E(0)$ is klt. If $K_E + \text{Diff}_E(0)$ is $n$-complementary then $K_Y + E$ is $n$-complementary and $K_X$ is too [3, 2.8].
5. [5, 2.2] Let $f_i: (Y_i, E_i) \to (X \ni P)$ be two plt blow-ups. If $E_1$ and $E_2$ define the same discrete valuation of function field $K(X)$, then $f_1$ and $f_2$ are isomorphic.

**Problem 1.4.** Describe the class of all weak log Del Pezzo surfaces, generically $\mathbb{P}^1$ and elliptic fibrations which can be exceptional divisors of some plt blow-ups of a terminal, canonical, $\varepsilon$-lt or lc singularities.

The existence of plt blow-up for klt singularity follows from the next theorem.

**Theorem 1.5.** Let $X$ be a klt variety and let $D \neq 0$ be a boundary on $X$ such that $(X, D)$ is lc, but is not plt. Suppose LMMP is true or $\dim X \leq 3$. Then there exists an inductive blow-up $f: Y \to X$ such that:

1. The exceptional locus of $f$ contains only one irreducible divisor $E$ ($\text{Exc}(f) = E$);
2. $K_Y + E + D_Y = f^*(K_X + D)$ is lc;
3. $K_Y + E + (1 - \varepsilon)D_Y$ is plt and anti-ample over $X$ for any $\varepsilon > 0$;
4. If $X$ is $\mathbb{Q}$-factorial then $Y$ is also $\mathbb{Q}$-factorial and $\rho(Y/X) = 1$. 

Proof. Let us consider the proof of this theorem for $\mathbb{Q}$-factorial singularities \cite{Kollar17, 2.9}. Let $g : \hat{Z} \to X$ be a minimal log terminal modification of $(X, D)$ and $\hat{E} = \sum \hat{E}_i$ be a reducible exceptional divisor \cite{Kollar17, 17.10}, \cite{Kollar17, 9.1}. By definition of such modification $\hat{Z}$ is $\mathbb{Q}$-factorial and $K_{\hat{Z}} + \hat{E} + D_{\hat{Z}} = g^*(K_X + D)$ is dlt. Since $X$ has only klt singularities then $K_{\hat{Z}} + \hat{E} = g^*K_X + \sum (a(\hat{E}_i, 0) + 1)\hat{E}_i$ cannot be $g$-nef by numerical properties of contractions \cite{Kollar17, 1.1}. Run $K_{\hat{Z}} + \hat{E}$-MMP over $X$. Hence at the last step we get a divisorial extremal contraction $g' : \hat{Z} \to X$ (see diagram (1)) and $K_{\hat{Z}} + \hat{E} + D_{\hat{Z}} = g'^*(K_X + D)$ is lc, where $\hat{E}$ is an irreducible divisor. Since $K_{\hat{Z}} + \hat{E}$ is plt then $(\hat{Z}, \hat{E} + (1 - \varepsilon)D_{\hat{Z}})$ is plt for any $\varepsilon > 0$.

If $X$ is $\mathbb{Q}$-factorial then $\text{Exc}(g') = \hat{E}$, $\rho(\hat{Z}/X) = 1$ and $-\hat{E}$ is $g'$-ample by remark \cite{Kollar17, 3}. Therefore $g'$ is an inductive blow-up.

\begin{equation}
(\hat{Z}, \hat{E} = \sum \hat{E}_i) \xrightarrow{\phi} (\hat{Z}, \hat{E}) \xrightarrow{g'} (Y', E') \xrightarrow{\psi} (Y, E)
\end{equation}

Assume that $\text{Exc}(g') = \hat{E} \cup \Delta$ where $\Delta \neq \emptyset$ and $\text{codim}_X \Delta \geq 2$. Obviously $K_{\hat{Z}} + D_{\hat{Z}} \equiv -\hat{E}$ over $X$. Thus $K_{\hat{Z}} + D_{\hat{Z}}$ is not $g'$-nef and it is not negative for curves lying on $\hat{E}$. Apply $K_{\hat{Z}} + D_{\hat{Z}}$-MMP. At the last step we get a divisorial contraction $f' : Y' \to X$ and $\text{Exc}(f') = E'$ is an irreducible divisor. Note also that the birational map $\varphi$ is a composition of log flips and $K_{Y'} + E' + (1 - \varepsilon)D_{Y'}$ is plt for any $\varepsilon > 0$. If $-E'$ is $f'$-ample then $f'$ is an inductive blow-up.

Let $-E'$ is not $f'$-ample. Since $-E'$ is $f'$-nef and $K_{Y'} + E' = f'^*K_X + (a(E', 0) + 1)E'$ where $a(E', 0) + 1 > 0$ then $-(K_{Y'} + E')$ is $f'$-nef. By Base Point Free Theorem \cite{Kollar17, 3.1.2} applied to klt divisor $K_{Y'} + (1 - \delta)E'$ $(0 < \delta \ll 1)$ the linear system $|-n(K_{Y'} + E')|$ is free over $X$ for $n \gg 0$. It gives small birational contraction $\psi : (Y', E') \to (Y, E)$. Let $C$ be an exceptional curve. Since $(K_{Y'} + E') \cdot C = 0$ then $E' \cdot C = 0$ and $K_{Y'} \cdot C = 0$. Therefore morphism $\psi$ contracts the curve $C$ if and only if $E' \cdot C = 0$. Clearly, the given blow-up $f : (Y, E) \to X$ is a required one. \hfill $\square$

**Definition 1.6.** Let $(X \ni P)$ be a lc singularity. It is said to be weakly exceptional if there exists only one plt blow-up (up to isomorphism). A lc pair $(X, D)$ is said to be exceptional, where $D$ is boundary, if there exists at most one divisor $E$ with discrepancy $a(E, D) = -1$. A lc
singularity \((X \ni P)\) is said to be \textit{exceptional} if \((X, D)\) is exceptional for any boundary \(D\) whenever \(K_X + D\) is lc.

The LMMP is also used in the next corollary from theorem 1.5.

**Corollary 1.7.** Let \(f : (Y, E) \to (X \ni P)\) be a plt blow-up of klt singularity and let \(\dim f(E) \geq 1\). Then there exists another plt blow-up of \((X \ni P)\). Therefore the singularity is not weakly exceptional.

\[\text{Proof.}\] Take two hyperplane sections \(H_1\) and \(H_2\) passing through the point \(P\) and not containing \(f(E)\). Let \(c > 0\) be a log canonical threshold of pair \((X, H_1 + H_2)\). Then \(K_X + c(H_1 + H_2)\) is not plt. The set \(f(E)\) is different from \(LCS(X,c(H_1 + H_2))\). Apply theorem 1.5 for \((X,c(H_1 + H_2))\). This completes the proof. \(\square\)

**Proposition 1.8.** Let \(f : (Y, E) \to (X \ni P)\) be a plt blow-up of strictly lc singularity and let \(\dim f(E) \geq 1\). Then \((X \ni P)\) is not exceptional singularity.

\[\text{Proof.}\] As in proof of corollary 1.7 there exists divisor \(D\) such that \((X, D)\) is lc, but is not plt and set \(f(E)\) is different from a minimal center \(LCS(X,D)\). Thus \((X \ni P)\) is not exceptional by definition. \(\square\)

The LMMP is used in order to prove the necessary condition in the following theorem.

**Theorem 1.9.** Let \((X \ni P)\) be a strictly lc singularity. Then

1. If there exists a plt blow-up then it is the unique (up to isomorphism).
2. The singularity is exceptional if and only if there exists a plt blow-up \(f : (Y, E) \to (X \ni P)\) such that \(f(E) = P\).

\[\text{Proof.}\] The first statement follows from the properties (2) and (5) of the remark 1.3. Let’s prove the second part of theorem.

\textit{Necessity.} Assume that the singularity \((X \ni P)\) is exceptional. We will construct a plt blow-up (cf. proof of theorem 1.3). Let \(g' : \tilde{Z} \to X\) be a minimal log terminal modification of \(X\) and \(\tilde{E} = \sum \tilde{E}_i\) be a reducible exceptional divisor. By definition of such modification \(\tilde{Z}\) is \(\mathbb{Q}\)-factorial and \(K_{\tilde{Z}} + \tilde{E} = g'^\ast K_X\) is dlt. Since the singularity is exceptional then \(\tilde{E}\) is irreducible divisor and \(K_{\tilde{Z}} + \tilde{E}\) is plt. Let \(Exc(g') = \tilde{E} \cup \Delta\), where \(\Delta \neq \emptyset\) and \(\text{codim}_X \Delta \geq 2\). Apply \(K_{\tilde{Z}}\)-MMP over \(X\). Hence at the last step we get a divisorial extremal contraction \(f' : Y' \to X\) and \(Exc(f') = E'\) is an irreducible divisor. Divisor \(K_{Y'} + E'\) is also plt. If \(-E'\) is \(f'\)-ample then \(f'\) is a required plt blow-up by proposition 1.8. Let \(-E'\) is not \(f'\)-ample. Since \(-E'\) is \(f'\)-nef and \(K_{Y'} \equiv -E'\)
over $X$, then $K_Y$ is $f'$-nef. By Base Point Free Theorem [3, 3.1.2] the linear system $|nK_Y|$ is free over $X$ for $n \gg 0$. It gives small birational morphism $h : (Y', E') \to (Y, E)$. The given blow-up $f : (Y, E) \to X$ is plt because $K_Y$ is $f$-ample. By proposition [1.8] $f(E) = P$.

**Sufficiency.** Conversely assume that there exists a required blow-up. Note that $E$ is an unique exceptional divisor with discrepancy $a(E, 0) = -1$. Let $(X, D)$ is any lc pair. Then $D = 0$ because $f(E) = P$.

**Corollary 1.10.** Let $(X \ni P)$ be a strictly lc exceptional singularity. Then the minimal index of complement is equal to gorenstein index of $(X \ni P)$.

**Remark 1.11.** A minimal index of complementary is bounded for three dimensional lc singularities [9, 7.1]. A hypothesis is that this index is not more then 66. For strictly lc exceptional singularities it was proved in papers [2] and [1]. For non-exceptional non-isolated strictly lc singularities the gorenstein index is not bounded [1, 5.1].

**Corollary 1.12.** [7, 2.4] Exceptional singularity is weakly exceptional.

**Proof.** The existence of plt blow-up follows from theorems 1.5 and 1.9. By [7, 2.4] such blow-up is unique.

We have the next corollary by proofs of theorems 1.5 and 1.9.

**Corollary 1.13.** Notation as in definition 1.1. Assume that we don't require $-E$ to be ample over $X$. Then such blow-up differs from a plt blow-up by a small flopping contraction.

2. **Criterion of weakly exceptionality**

To prove (3) $\Rightarrow$ (1) in the next theorem we use LMMP.

**Theorem 2.1.** Let $(X \ni P)$ be a klt blow-up and let $f : (Y, E) \to X$ be a plt blow-up of $P$. Then the following conditions are equivalent:

1. $(X \ni P)$ is not weakly exceptional;
2. There is an effective $\mathbb{Q}$-divisor $D \geq \text{Diff}_E(0)$ such that $-(K_E + D)$ is ample and $(E, D)$ is not klt;
3. There is an effective $\mathbb{Q}$-divisor $D \geq \text{Diff}_E(0)$ such that $-(K_E + D)$ is ample and $(E, D)$ is not lc.

**Proof.** The statements (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) follow from [5, 4.3]. Let’s prove (3) $\Rightarrow$ (1). By corollary 1.7 we can suppose $f(E) = P$. It was proved in [5, theorem 4.3] the existence of effective $\mathbb{Q}$-Cartier
divisor $B = \sum b_i B_i$ such that $K_Y + E + B$ is lc, but is not plt. Also $K_Y + E + B$ is anti-ample over $X$. We can take very ample divisor $H$ containing the minimal center of $LCS(Y, E + B)$. There is a small rational number $\varepsilon > 0$ such that $-(K_Y + E + B + \varepsilon H)$ is $f$-ample. Replacing $B$ by $c(B + \varepsilon H)$ we can assume without loss of generality that $b_i < 1$ for all $i (c < 1$ because $H$ contains a minimal center). Denote $L = f(B)$. Since $-(K_Y + E + B)$ is $f$-ample and all $b_i < 1$ then lc threshold $c'$ of pair $(X, L)$ is greater than $1$. If pair $(X, c'L)$ is plt then there is an effective $\mathbb{Q}$-Cartier divisor $L'\mathcal{O}$ that $(X, c'L + L')$ is lc, but is not plt. By theorem 1.5 we have an inductive blow-up $f': (Y', E') \to X$ of $(X, c'L + L')$. Moreover $K_Y + E + c'L_Y + L'_Y$ is not lc $(c' > 1)$. Thus $f$ and $f'$ are not isomorphic plt blow-ups.

**Example 2.2.** [5, 4.7], [6, 6.4] Two dimensional klt singularity is weakly exceptional if and only if it has type $D_n$, $E_6$, $E_7$ or $E_8$. Among them the singularities of type $D_n$ are not exceptional. Two dimensional strictly lc singularity is weakly exceptional (it is exceptional by theorem 1.9) if and only if it is simple elliptic or it has type $\tilde{D}_4$ (see the minimal resolution graph in fig. 1), $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$ (see the minimal resolution graph in fig. 2), where $(n_1, n_2, n_3) = (3, 3, 3), (2, 4, 4)$ or $(2, 3, 6)$ respectively.

[Fig. 1]

**Remark 2.3.** A three dimensional terminal singularity is not weakly exceptional [5, 4.8].

**Example 2.4.** Let $(X \ni P)$ be an $n$-dimensional ($n \geq 3$) canonical hypersurface singularity given by equation $(x_1^n + \cdots + x_n^n + x_{n+1}^{n+1} = 0) \subset (\mathbb{C}^{n+1}, 0)$. The weighted blow-up of $\mathbb{C}^{n+1}$ with weights $(n+1, \ldots, n+1, n)$ induces a plt blow-up of $P$. The obtained log Fano variety $(E, \text{Diff}_E(0))$ is

$$(x_1^n + \cdots + x_n^n + x_{n+1} \subset \mathbb{P}(1, \ldots, 1, n), \frac{n}{n+1}\{x_{n+1} = 0\}) = (\mathbb{P}^{n-1}, n+1\mathbb{Q}_n),$$

where $Q_n$ is smooth hypersurface of degree $n$ in $\mathbb{P}^{n-1}$. By theorem 2.1 the singularity $(X \ni P)$ is weakly exceptional. The divisor $\{x_{n+1} = 0\}$ is 1-complement being not plt. Therefore the singularity is not exceptional.
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