On $B^*c$ – adherent points and its properties

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Abstract. In this paper, we introduced study about a basic properties of adherent points by the class $B^*c$ – open. Also, we showed properties accumulation points by using $B^*c$ – open sets and we provided the relation among adherent points, $\beta$ – adherent points and $B^*c$ – adherent points. Also, accumulation points, $\beta$ – accumulation points and $B^*c$ – accumulation points.

Keywords: $B^*c$ – open set, $B^*c$ – closed set, $B^*co(X)$, $B^*cc(X)$, $B^*c$ – adherent points and $B^*c$ – accumulation points.

Mathematics subject classification: 54xx.

1. Introduction:

The main idea from studying these points on this set is to generalize the properties to show the relations with other sets and to use them in proof of many theorems. In 1963 Levine [1], introduced the concept of semi open sets.

In 1982 Mashhour , Abd Elmonsef and El-Deeb [2], introduced the concept of pre-open sets. In 1983 Abd El -Monsef M. and et. al. [3], introduced class of sets $\beta$ – open, $\beta$ – closed, also give some topological properties related to these sets.

In 1986, Andrijecivic D. [4], introduced the concept of semi pre-open sets (which equivalence of the definition $\beta$ – open sets). Theorems and notions of semi pre-open sets verified by a lot of researchers.

In 2018 , karim. F. Radhi , [5], introduced a survey about the concept $B^*c$ – open set and through it, introduced proof many of theorems as the set $B^*c$ – open set which it can lead to set $\beta$ – open set. where they are considered as input to study the set of class $B^*c$ – open, $B^*c$ – closed and we introduced the adherent points and the accumulation points as property of $B^*c$ – open set. In 1977 J.N. Sharma [6], introduced the adherent points and the accumulation points as input to study the concept to $\beta$ – adherent points, $B^*c$ – adherent points $\beta$ – accumulation points, $B^*c$ –accumulation points and we study the relation among them.

2. On basic properties by $B^*c$ – adherent points:
Definition (2.1):[3]

Let X be a topological space. Then a subset A of X is said to be:

i) a $\beta$–open set if $A \subseteq \text{cl} (\text{int}(\text{cl} A))$.

ii) a $\beta$–closed set if $A \supseteq \text{int} (\text{cl} (\text{int} A))$.

The family of all $\beta$–open (resp. $\beta$–closed) set subsets of a space X will be as always denoted by $\beta o(X)$ (resp. $\beta c(X)$).

Definition (2.2):[3]

Let X be a topological space and $A \subseteq X$. Then a $\beta$–open set A is called a $B^\ast c$–open set if for all $x \in A$ there exists $F_x$ closed set such that $x \in F_x \subseteq A$. A is a $B^\ast c$–closed set if $A^c$ is a $B^\ast c$–open set, the family of all $B^\ast c$–open (resp. $B^\ast c$–closed) set subset of a space X will be as always denoted by $B^\ast co(X)$ (resp. $B^\ast cc(X)$).

Definition (2.3):[6]

Let X be a topological space and let $A \subseteq X$. A point $y \in X$ is called an adherent point of A if $N \cap A \neq \emptyset$ for all open set N such that $y \in N$.

The set of all adherent point of A will be as always denoted by $\text{cl}(A)$.

Definition (2.4):

Let X be a topological space and let $A \subseteq X$. A point $y \in X$ is called $\beta$–adherent point of A if $N \cap A \neq \emptyset$ for all $\beta$–open set N such that $y \in N$. The set of all $\beta$–adherent point of A will be denoted by $\beta cl(A)$.

Definition (2.5):

Let X be a topological space and let $A \subseteq X$. A point $y \in X$ is called $B^\ast c$–adherent point of A if $N \cap A \neq \emptyset$ for all $B^\ast c$–open set N such that $y \in N$.

The set of all $B^\ast c$–adherent point of A will be denoted by $B^\ast c cl(A)$.

Example (2.6):

Let $X = \{a,b,c\}$, $\tau = \{\emptyset , X, \{a\}, \{b\}, \{a,b\}\}$. $\beta o(X) = \{\emptyset , X, \{a\}, \{b\}, \{a,b\}\}$. $B^\ast co(X) = \{\emptyset , X, \{a,c\}, \{b,c\}\}$. Let $A = \{a,c\}$. Note that

i) $a \in X$, $N \cap A \neq \emptyset$ for all $\beta$–open set N such that $a \in N$. Hence a is a $\beta$–adherent point of A.

Also, $a \in X$, $N \cap A \neq \emptyset$ for all $B^\ast c$–open set N such that $a \in N$. Hence a is $B^\ast c$–adherent point of A.

ii) $b \in X$, there exists $N = \{b\}$ $\beta$–open set such that $b \in N$ and $N \cap A = \emptyset$. Hence b is not $\beta$–adherent point of A.
Also, b ∈ X, N ∩ A ≠ ∅ for all B∗c – open set N such that b ∈ N. Hence b is B∗c – adherent point of A.

iii) c ∈ X, N ∩ A ≠ ∅ for all β – open set N such that c ∈ N. Hence c is β – adherent point of A.

Also, c ∈ X, N ∩ A ≠ ∅ for all B∗c – open set N such that c ∈ N. Hence c is B∗c – adherent point of A.

**Theorem (2.7):[3]**

Let X be a topological space. Then every open set is β – open set.

**Theorem (2.8):**

Let X be a topological space. Then every β – adherent point of A is adherent point of A.

**Proof:**

Let y be a β – adherent point of A, then N ∩ A ≠ ∅ for all β – open set N such that y ∈ N. Since every open set is β – open set (by theorem 2.7), then N ∩ A ≠ ∅ for all open set N such that y ∈ N. Therefore y is an adherent point of A.

The converse of above theorem is not true in general.

**Example (2.9):**

In example (2.6) let A = {b}. Note that c ∈ X, N ∩ A ≠ ∅ for all open set N such that c ∈ N. Hence c is adherent point of A. But c is not β – adherent point of A because there exists β – open set N = {a, c} such that c ∈ N and N ∩ A = ∅.

**Remark (2.10):[5]**

Let X be a topological space. Then every B∗c – open set is β – open set.

**Theorem (2.11):**

Let X be a topological space. Then every β – adherent point of A is B∗c – adherent point of A.

**Proof:**

Let y be a β – adherent point of A, then N ∩ A ≠ ∅ for all β – open set N such that y ∈ N. Since every B∗c – open set is β – open set (by remark 2.10), then N ∩ A ≠ ∅ for all B∗c – open set N such that y ∈ N. Therefore y is B∗c – adherent point of A.

The converse of above theorem is not true in general.

**Example (2.12):**

In example (2.6) let A = {b, c}. Note that a ∈ X, N ∩ A ≠ ∅ for all B∗c – open set N such that a ∈ N. Hence a is B∗c – adherent point of A. But a is not β – adherent point of A because there exists β – open set N = {a} such that a ∈ N and N ∩ A = ∅.
Remark (2.13): 

Let X be a topological space. Then the adherent point of A and $B^*c$ – adherent point of A are independent in general.

The following example showing that

Example (2.14):

In example (2.6)

i) let $A = \{a\}$. Note that $c \in X$, $N \cap A \neq \emptyset$ for all open set $N$ such that $c \in N$. Hence $c$ is adherent point of $A$. But $c$ is not $B^*c$ – adherent point of $A$ because their exists $B^*c$ – open set $N = \{b, c\}$ such that $c \in N$ and $N \cap A = \emptyset$.

ii) let $A = \{c\}$. Note that $a \in X$, $N \cap A \neq \emptyset$ for all $B^*c$ – open set $N$ such that $a \in N$. Hence $a$ is $B^*c$ – adherent point of $A$. But $a$ is not adherent point of $A$ because their exists open set $N = \{a\}$ such that $a \in N$ and $N \cap A = \emptyset$.

The following diagram shows the relation among types of points.

3. On basic properties by $B^*c$ – accumulation points:

Definition (3.1):[6],[7]

Let X be a topological space and let $A \subseteq X$. A point $y \in X$ is called an accumulation point of $A$ if $(N \cap A) - \{y\} \neq \emptyset$ for all open set $N$ such that $y \in N$.

Definition (3.2):

Let X be a topological space and let $A \subseteq X$. A point $y \in X$ is called an $\beta$ – accumulation point of $A$ if $(N \cap A) - \{y\} \neq \emptyset$ for all $\beta$ – open set $N$ such that $y \in N$.

Definition (3.3):

Let X be a topological space and let $A \subseteq X$. A point $y \in X$ is called an $B^*c$ – accumulation point of $A$ if $(N \cap A) - \{y\} \neq \emptyset$ for all $B^*c$ – open set $N$ such that $y \in N$. 
**Example (3.4):**

In example (2.6). Note that

1) \( a \in X \), there exists \( N = \{a\} \) open set such that \( a \in N \) and \( (N \cap A) - \{a\} = \emptyset \). Hence \( a \) is not accumulation point of \( A \).

2) \( a \in X \), their exists \( N = \{a\} \) \( \beta \) – open set such that \( a \in N \) and \( (N \cap A) - \{a\} = \emptyset \). Hence \( a \) is not \( \beta \) – accumulation point of \( A \).

3) \( a \in X \), \( (N \cap A) - \{a\} \neq \emptyset \) for all \( B^*c \) – open set \( N \) such that \( a \in N \). Hence \( a \) is \( B^*c \) – accumulation point of \( A \).

**Theorem (3.5):**

Let \( X \) be a topological space. Then every \( \beta \) – accumulation point of \( A \) is accumulation point of \( A \).

**Proof:**

Let \( y \) be a \( \beta \) – accumulation point of \( A \), then \( (N \cap A) - \{y\} \neq \emptyset \) for all \( \beta \) – open set \( N \) such that \( y \in N \). Since every open set is \( \beta \) – open set (by theorem 2.7), then
\[ (N \cap A) - \{y\} \neq \emptyset \] for all open set \( N \) such that \( y \in N \). Therefore \( y \) is accumulation point of \( A \).

The converse of above theorem is not true in general.

**Example (3.6):**

See example (3.4) (iii).

**Theorem (3.7):**

Let \( X \) be a topological space. Then every \( \beta \) – accumulation point of \( A \) is \( B^*c \) – accumulation point of \( A \).

**Proof:**
Let y be a $\beta$–accumulation point of $A$, then $(N \cap A) - \{y\} \neq \emptyset$ for all $\beta$–open set $N$ such that $y \in N$. Since every $B^*c$–open set is $\beta$–open set (by remark 2.10), then $(N \cap A) - \{y\} \neq \emptyset$ for all $B^*c$–open set $N$ such that $y \in N$. Therefore $y$ is $B^*c$–accumulation point of $A$.

The converse of above theorem is not true in general.

**Example (3.8):**

See example (3.4) (i).

**Remark (3.9):**

Let $X$ be a topological space. Then the accumulation point of $A$ and $B^*c$–accumulation point of $A$ are independent in general.

The following example showing that

**Example (3.10):**

In example (3.4). Note that

i) $c \in X (N \cap A) - \{c\} \neq \emptyset$ for all open set $N$ such that $c \in N$. Hence $c$ is accumulation point of $A$.

But $c$ is not $B^*c$–accumulation point of $A$ because their exists $N = \{b, c\}$ $B^*c$–open set such that $c \in N$ and $(N \cap A) - \{c\} = \emptyset$.

ii) $b \in X (N \cap A) - \{b\} \neq \emptyset$ for all $B^*c$–open set $N$ such that $b \in N$. Hence $b$ is $B^*c$–accumulation point of $A$.

But $b$ is not accumulation point of $A$ because their exists $N = \{b\}$ open set such that $b \in N$ and $(N \cap A) - \{b\} = \emptyset$.

The following diagram shows the relation among types of points.

4. The relation between $B^*c$– adherent points and $B^*c$–accumulation points:
Theorem (4.1):[6]

Let X be a topological space. Then every an accumulation point of A is an adherent point of A.

Theorem (4.2):

Let X be a topological space. Then every an $\beta$ – accumulation point of A is $\beta$ – adherent point of A.

Proof:

Let $y$ be a $\beta$ – accumulation point of A, then $(N \cap A) - \{y\} \neq \emptyset$ for all $\beta$ – open set N such that $y \in N$. Then $N \cap A \neq \emptyset$ for all $\beta$ – open set N such that $y \in N$. Therefore $y$ is $\beta$ – adherent point of A.

The converse of above theorem is not true in general.

Example (4.3):

In example (2.6). Note that $c \in X, N \cap A \neq \emptyset$ for all $\beta$ – open set N such that $c \in N$. Hence c is a $\beta$ – adherent point of A. But c is not $\beta$ – accumulation point of A because their exists $N = \{b, c\}$ $\beta$ – open set such that $c \in N$ and $(N \cap A) - \{c\} = \emptyset$.

Theorem (4.4):

Let X be a topological space. Then every an $B^*c$ – accumulation point of A is an $B^*c$ – adherent point of A.

Proof:

Let $y$ be a $B^*c$ – accumulation point of A, then $(N \cap A) - \{y\} \neq \emptyset$ for all $B^*c$ – open set N such that $y \in N$. Then $N \cap A \neq \emptyset$ for all $B^*c$ – open set N such that $y \in N$. Therefore y is $B^*c$ – adherent point of A.

The converse of above theorem is not true in general.

Example (4.5):

In example (2.6). Note that $c \in X, N \cap A \neq \emptyset$ for all $B^*c$ – open set N such that $c \in N$. Hence c is $B^*c$ – adherent point of A.

But c is not $B^*c$ – accumulation point of A because their exists $N = \{b, c\}$ $B^*c$ – open set such that $c \in N$ and $(N \cap A) - \{c\} = \emptyset$.

5. References:

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