On the Analytical Tractability of Hexagonal Network Model with Random Traffic Distribution

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Abstract—Explicit derivation of interferences in hexagonal wireless networks has been widely considered intractable and requires extensive computations with system level simulations. In this paper, we fundamentally tackle this problem and explicitly evaluate interference factor $f$ for any mobile location $m$ in a hexagonal network composed of omni-directional or tri-sectorized sites. The explicit formula of $f$ is a very convergent series on $m$ and involves the use of hypergeometric and Hurwitz Riemann zeta functions. Besides, we establish simple identities that well approximate this convergent series and turn out quite useful compared to other approximations in literature. The derived expression of $f$ is used to show further that the SINR distribution is explicit as well and it is provided for any arbitrary mobile location distribution, reflecting the spatial traffic density in the network. Knowing explicitly about interferences and SINR distribution is very useful in capacity and coverage planning of wireless cellular networks and particularly for macro-cells' layer that forms almost a regular point pattern.

Index Terms—Wireless cellular networks, Interferences, Hexagonal model, SINR distribution, Performance analysis, Traffic distribution.

I. INTRODUCTION

INTERFERENCE in wireless networks is the prime concern of telecommunication actors which continuously attempt to minimize it in all the stages of the technology conception and deployment (from the standardization to the process of network design, planning and exploitation). Interferences are often attributed to co-channel interferences which are the result of the scarce spectrum reuse in different cells of the network. In the network design phase, planning engineers try to understand the behavior of the system in terms of radio capacity and relate it to interferences. Often, they need to have a quick answer on its estimation without having recourse to a simulator. This answer should come up with a closed form expression of interferences, required to be directly used in some tools essential for capacity and coverage planning, such as link budget tool [1].

In literature, co-channel interferences are often represented by the interference factor parameter, named also $f$-factor, which is defined as the normalized intercell interferences against the received useful power (received power from the serving cell). The $f$-factor parameter constitutes the chestnut of the performance analysis, in terms of SIR (Signal-to-Interference-Ratio) and SINR (Signal-to-Interference-plus-Noise-Ratio). Its evaluation in hexagonal wireless networks has been widely addressed but it is still a fresh topic interesting both academic and industrial communities [2]–[19]. Besides, interference factor evaluation is known as intractable and requires extensive numerical computations with simulations. Several works have been made to give approximations of interferences in regular hexagonal networks based on observed simulation results [2], [3], [7], [19]. To the best of our knowledge, no closed exact formula, mathematically and rigorously proved, was found.

By way of examples, Chan et al. provided in [4] an approximation of the inter-cell interference distribution assuming that the traffic follows a Poisson model and interference calculations were only based on path-loss. In [5], Haenggi and Ganti provided a good review of interferences in regular and random networks and in particular they gave lower and upper bounds of the cumulated interferences in triangular lattices assuming that the receiver is placed at the origin [5, pp. 19]. In the search of the best approximation that well fits the results of simulations, Karray provided in [7] a comparison between some approximations of $f$-factor in regular hexagonal networks.

To make the hexagonal network model tractable, authors, in [2], [8], transformed the hexagonal model to the fluid model: interfering sites are continuously and uniformly distributed in the plane. The interference approximation, given in [2], was shown to be close to that of hexagonal network. In [9], it was stated that the fluid model is a good approximation of the intractable hexagonal model. Almost all works about interference evaluation in hexagonal fluid model assume that sites in the network are omni-directional. A comparison was held in [11], between the hexagonal fluid model and the flower model: an improvement of the fluid model including not only the distance to sites but also the angle between the mobile location and the interfering sites. In [11], it was shown that interference approximation in fluid model is more accurate for omni-directional than for tri-sectorized sites.

Unlike in the hexagonal model, interference analysis is tractable when sites are organized according to a Homogeneous Poisson Point Process (HPPP), i.e., the number of sites in a given area follows a Poisson distribution with constant site density [5], [12]–[18]. An excellent survey of the mathematical theory of interferences in Poisson network of interferers can be found in [5], [12], [16], [17]. Even if HPPP model allows a tractable interference analysis, it can not always fit with the geometry of real wireless networks and mainly for macro-cells that form a more regular point pattern than the HPPP [18]. This is related to the fact that radio engineers start the design of the network based on a hexagonal model. The deviation of the real network structure from the hexagonal model is therefore
linked to the constraints imposed by engineering rules, terrain imperfections and government charters. And so, it turns out credible to use the hexagonal model (at least the perturbed hexagonal, see [18]) for network performance analysis and as the basis for system level simulations [3], [19].

In this work, we give a mathematical framework, including new explicit formulas, of the downlink interference factor in hexagonal wireless networks with omni-directional sites. We prove that the f-factor, as a function of the mobile location \( m = re^{i\theta} \) admits an absolutely convergent Fourier series on \( \theta \) and an entire analytic expansion on \( r \). The average of the f-factor over the angle \( \theta \) is rigorously derived without any approximation and involves the use of the well-known hypergeometric functions [20] and Hurwitz Riemann Zeta functions [21]. Different identities and accurate approximations of f-factor are also given and compared with other approximations in [2], [7]. The f-factor expression in omni-directional networks is used to derive an accurate approximation of f-factor in tri-sectorized networks.

The second contribution of this work is the explicit derivation of the network performances in terms of SINR distribution in hexagonal network with an arbitrary traffic distribution (spatial distribution of mobile locations) [22]. The finding of SINR distribution entails of course the inversion of the f-factor function. To show the utility of the explicit formula of the SINR distribution, we numerically investigate the SINR distribution for two scenarios of traffic: uniform and Log-normal traffic distributions.

We derive the explicit formulas of f-factor for omni-directional networks is used to derive an accurate approximation of f-factor in tri-sectorized networks.

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The proof is simple using the geometry of the network and it is given in Appendix A.

A mobile location in the plane is denoted by \( m \). Likewise, label \( m \) is used to denote the geographical location of the mobile. In the complex plane, \( m = re^{i\theta} \); where \( r = |m| \) is the absolute value of \( m \) representing its distance to the central site and \( \theta \) is the angle coordinate of the mobile relative to the horizontal axis.

For hexagonal omni-directional network, we assume that location \( m \) is connected to central cell \( S \) covering a disk of radius \( R < \delta \). The choice of \( R \) is problematic because the disk circumscribed the hexagon over-estimates interferences whereas the disk inscribed in the hexagon under-estimates interferences and generates coverage holes. For all the theoretical study, we take an arbitrary \( R \), such that \( R < \delta \), but for the numerical results we use \( R = \delta \sqrt{3} \sqrt{3} \), which is the radius of the disk having the same area of the hexagon representing the cell.

Note that when the mobile location varies inside the disk of radius \( R \), the hexagonal network model can not be stationary. A hexagonal network is stationary only when \( m \) varies uniformly inside the hexagon [18, Def. 1 pp.4].
B. Location model of tri-sectorized sites in hexagonal network

Unlike for omni-directional sites, tri-sectorized sites are located at the corner of the hexagon and composed of 3 sectors covering each one a cell. Sites are arranged to form always a hexagonal grid (see Fig. 1). This model is called regular tri-sectorized network. Since each site is divided into three sectors, we shall use $S_{l,k,j,c}$ to identify the sector $c$, $1 \leq c \leq 3$, of the site $j$ in the tier $k$ and located in region $\Omega_l$. The sector $c$ in the central site is simply denoted $S_c$. The azimuth of the antenna, in which the radiation is at its maximum, is taken relative to the geographical North. Let $\phi_{l,k,j} = \arg(m - S_{l,k,j})$ be the angle between site $S_{l,k,j}$ and mobile $m$, then the angle $\phi_{l,k,j,c}$ between the azimuth of sector $c$ in the site $S_{l,k,j}$ and the mobile $m$ is given by

$$\phi_{l,k,j,c} = \frac{\pi}{3} (2c - 3) + \phi_{l,k,j}$$  \hspace{1cm} (3)

We denote by $G(\cdot)$ the antenna mask of a site (relative to its azimuth) and we define the antenna mask of a site, denoted $G_s$, by the sum of all antenna masks of sectors belonging to the same site

$$G_s(\phi_{l,k,j}) = \sum_{c=1}^{3} G(\phi_{l,k,j,c})$$  \hspace{1cm} (4)

C. Propagation model

In this study, we use simplified Ukumura-Hata propagation model [24], i.e., the path-loss between the site $S_{l,k,j}$ and the mobile $m$ is expressed as

$$L_{l,k,j}(m) = a |m - S_{l,k,j}|^{2b}$$  \hspace{1cm} (5)

where $b > 1$ is the amplitude loss exponent, (i.e., $2b$ is the pathloss exponent) [16], $a$ is a constant dependent on the type of the environment (indoor, outdoor, rural, urban,...). We omit also the lower index of $L_{0,0,0}$ and we use simply the notation $L$.

In this paper, we will neglect both shadowing and fast fading in the propagation model. The first is omitted because we assume that the random variability of the network would be related only to traffic distribution; i.e., site location and received signal are deterministic, only mobile location is a random variable. Fast fading is neglected because of two reasons: 1) its impact appears only at low time scale; 2) it is often taken into account in the link level simulation with the link curve that gives the throughput as a function of SINR for different types of propagation environment and different user equipment speeds and categories.

D. Definition of interference factors

Definition 1: Let $m = re^{i\theta}$ be a location in the plane $\mathbb{C}$ and let $S_{l,k,j}$, as defined in Lemma 2.1, be the location of a given site in a regular hexagonal network. We define the individual inter-site interference factor of site $S_{l,k,j}$, for $k \geq 1$, relative to sector $c = 1$ of central site $S$, the quantity

$$f_{l,k,j}(m) = \frac{L(m)}{L_{l,k,j}(m)} G_s(\phi_{l,k,j})$$  \hspace{1cm} (6)

Note that for omni-directional site, there is only one sector and hence $G_s = G = 1$.

With the homogeneity hypothesis that all sectors transmit with the same power, $f_{l,k,j}(m)$ is the normalized received power from site $S_{l,k,j}$, against the useful received power at location $m$ served by the first sector of central site $S$.

Definition 2: The inter-site f-factor parameter of any location $m$, served by the first sector of central site $S$, is defined as the sum of all individual interference factors

$$f(m) = \sum_{0 \leq l,k \leq k-1, 0 \leq j \leq k-1} f_{l,k,j}(m)$$  \hspace{1cm} (7)

III. MATHEMATICAL ANALYSIS AND EXPLICIT DERIVATION OF F-FACTOR

In this section we use mathematical tools to derive the Fourier and Maclaurin series expansion of $f(m)$. We first analyze f-factor for omni-directional network and we point out its approximation in tri-sectorized one at the end of this section.

A. F-factor in hexagonal networks with omni-directional sites

1) Explicit formulas of f-factor:

Using the definition of the individual interference factor and substituting $L_{l,k,j}$ by its form in (5), we shall establish the following mathematical results.

Proposition 3.1: For a given location $m = re^{i\theta}$ connected to the central cell, the individual interference factor, defined in (6), is given for omni-directional sites by

$$f_{l,k,j}(m) = \sum_{n=-\infty}^{+\infty} \left( \frac{r}{D_{k,j}} \right)^{2b+|n|} e^{-\pi c n (\theta - \theta_{l,k,j} - \frac{3\pi}{2})} \cdot \frac{\Gamma(b + |n|)}{\Gamma(b)\Gamma(1 + |n|)}$$

$$\times \; 2F1 \left( b, b + |n|, 1 + |n|, \left( \frac{r}{D_{k,j}} \right)^{2} \right)$$  \hspace{1cm} (8)

where $2F1(\cdot, \cdot, \cdot)$ and $\Gamma(\cdot)$ are respectively the Hypergeometric and Euler Gamma functions [20, pp.561].

Proof: We first replace $L_{l,k,j}(m)$ in (6) by its expression in (5) and use (2), we write down

$$f_{l,k,j}(m) = \left( \frac{r}{D_{k,j}} \right)^{2b} \left( 1 - \frac{r}{D_{k,j}} e^{i(\theta - \theta_{l,k,j} - \frac{3\pi}{2})} \right)^{-2b}$$  \hspace{1cm} (9)

The form of $f_{l,k,j}(m)$ in (9) can be developed to obtain

$$f_{l,k,j}(m) = \left( \frac{r}{D_{k,j}} \right)^{2b} \left( 1 - \frac{r}{D_{k,j}} e^{i(\theta - \theta_{l,k,j} - \frac{3\pi}{2})} \right)^{-b} \times$$

$$\left( 1 - \frac{r}{D_{k,j}} e^{-i(\theta - \theta_{l,k,j} - \frac{3\pi}{2})} \right)^{-b}$$  \hspace{1cm} (10)
Since \( \frac{r}{D_{k,j}} < 1 \), each part of the product in (10) is expanded to an absolutely convergent series
\[
f_{l,k,j}(m) = \sum_{n=0}^{+\infty} \left( \frac{r}{D_{k,j}} \right)^{n+h} e^{in(\theta \theta_{k,j} - l\frac{\pi}{2})} \frac{\Gamma(b+n)}{\Gamma(b)(1+n)} \times \sum_{h=0}^{+\infty} \left( \frac{r}{D_{k,j}} \right)^{h+b} e^{-in(\theta \theta_{k,j} - l\frac{\pi}{2})} \frac{\Gamma(b+h)}{\Gamma(b)(1+h)}
\]
Arranging the product of the two series in order to get a Fourier series expansion of \( f_{l,k,j} \) on \( \theta \), we have then
\[
f_{l,k,j}(m) = \sum_{n=-\infty}^{+\infty} \left( \frac{r}{D_{k,j}} \right)^{2b+|n|} e^{in(\theta \theta_{k,j} - l\frac{\pi}{2})} \times \Gamma(b+|n|+h) \frac{\Gamma(b+|n|)}{\Gamma(b)(1+|n|)}
\]
Using the definition of the Hypergeometric function [20, pp.561], we complete the proof.

**Theorem 3.2:** Let \( f \), defined as in (7) by \( f(m) \), be the factor function of a location \( m = re^{i\theta} \) in a regular hexagonal network with infinite number of omni-directional sites. Let \( \delta \) be the inter-site distance and let \( b \) be as in (5). Take \( x = \frac{\theta}{\delta} \), then for \( b > 1 \) and \( |x| < 1 \), \( f \) admits an absolutely convergent Fourier series expansion on \( \theta \)
\[
f(m) = H_0(x) + 2 \sum_{n=1}^{+\infty} H_n(x) \cos(6n\theta)
\]
where
\[
H_0(x) = \frac{6x^{2b}}{\Gamma(b)^2} \sum_{h=0}^{+\infty} \frac{\Gamma(b+h)^2}{\Gamma(h+1)^2} \lambda(b+h)x^{2h}
\]
\[
H_n(x) = \frac{6\Gamma(b+6n)}{\Gamma(b)(1+6n)} \sum_{k=1}^{+\infty} \frac{\Gamma(b+6n)}{(1-x^2)^k}, \quad \text{as} \quad n \to +\infty
\]
and
\[
\lambda(b) = 3^{-b}\zeta(b) \left( \zeta(b, \frac{1}{3}) - \zeta(b, \frac{2}{3}) \right)
\]
with \( \zeta(.) \) and \( \zeta(.,.) \) are respectively the Riemann Zeta and Hurwitz Riemann Zeta functions [21, pp. 1036].

The explicit formulas of \( f \)-factor in (13) and (14) are rapidly convergent series and were unknown before this work. Also, it is important to note that the interference received at the origin is simply the quantity \( \frac{6\Gamma(b)}{\sqrt{\pi}}x^{2b} \). In [5, pp. 19], Haenggi and Ganti gave only lower and upper bounds of the interferences at the origin of \( C \).

To prove theorem 3.2, we first establish the following lemma, proved in Appendix B.

**Lemma 3.3:** Let \( z \) be any complex number such that \( \Re(z) > 1 \) then
\[
\lambda(z) = \sum_{k=1}^{+\infty} \sum_{j=0}^{k-1} \frac{1}{(k^2+j^2-jk)^2} = 3^{-z}\zeta(z) \left( \zeta(z, \frac{1}{3}) - \zeta(z, \frac{2}{3}) \right)
\]
where \( \zeta(.) \) and \( \zeta(.,.) \) are respectively the Riemann Zeta and Hurwitz Riemann Zeta functions.

**Proof of theorem 3.2:**
Using the definition of \( f(m) \) in (7) and the explicit form of \( f_{l,k,j}(m) \) in (8), we obtain
\[
f(m) = \sum_{0 \leq l \leq k \leq b-1} \sum_{n=-\infty}^{+\infty} \left( \frac{r}{D_{k,j}} \right)^{2b+|n|} e^{in(\theta \theta_{k,j} - l\frac{\pi}{2})} \times \frac{\Gamma(b+|n|)}{\Gamma(b)(1+|n|)^2} 2F_1 \left( b, b + |n|, 1 + |n|, \left( \frac{r}{D_{k,j}} \right)^2 \right)
\]
The sum over \( l \) is simple to evaluate because it vanishes for every integer \( n \) not multiple of 6. In addition, the infinite hexagonal network is symmetric with respect to the real axis, then the \( f \)-factor function \( f \) stays unchanged when we substitute the location \( m \) by its complex conjugate. It follows that \( f \) is an even function on \( \theta \) and can be written as
\[
f(m) = 6 \sum_{0 \leq l \leq k \leq b-1} \sum_{n=-\infty}^{+\infty} \left( \frac{r}{D_{k,j}} \right)^{2b+|n|} \cos(6n\theta) e^{-6n\theta_{k,j}} \times \frac{\Gamma(b+6|n|)}{\Gamma(b)(1+6|n|)^2} 2F_1 \left( b, b + 6|n|, 1 + 6|n|, \left( \frac{r}{D_{k,j}} \right)^2 \right)
\]
It follows that \( f \) can be written in a closed form
\[
f(m) = H_0(x) + 2 \sum_{n=1}^{+\infty} H_n(x) \cos(6n\theta)
\]
where
\[
H_n(x) = \frac{6\Gamma(b+6n)}{\Gamma(b)(1+6n)} \sum_{k=1}^{+\infty} \sum_{j=0}^{+\infty} \frac{\Gamma(b+6n)}{(1-x^2)^k} \lambda(b+6n)x^{2h} \times 2F_1 \left( b, b + 6n, 1 + 6n, \left( \frac{x}{D_{k,j}} \right)^2 \right)
\]
To prove that the Fourier series expansion of \( f \) is absolutely convergent, we shall explicitly evaluate \( H_n \) and thus show (14) and (15).

We first substitute \( D_{k,j} \) with \( \delta \sqrt{k^2+j^2-jk} \) and \( r \) with \( \delta \). Expanding then hypergeometric function \( 2F_1 \) in (21) and using lemma 3.3 yield
\[
H_0(x) = \frac{6x^{2b}}{\Gamma(b)^2} \sum_{h=0}^{+\infty} \frac{\Gamma(b+h)^2}{\Gamma(h+1)^2} \lambda(b+h)x^{2h}
\]
and
\[
H_n(x) = \frac{6x^{2b+6n}}{\Gamma(b)^2} \sum_{h=0}^{+\infty} \frac{\Gamma(b+h)\Gamma(b+6n+h)}{\Gamma(h+1)\Gamma(1+6n+h)} x^{2h} \times \sum_{k=1}^{+\infty} \sum_{j=0}^{+\infty} \cos(6n\theta_{k,j}) \left( \frac{\Gamma(k^2+j^2-jk)^{b+6n+h+3n}}{(k^2+j^2-jk)^{b+6n+h+3n}} \right)
\]
Now when \( 6n \) is sufficiently high, we have the following equivalences [20, pp.262]:
\[
\frac{\Gamma(b+6n+h)}{\Gamma(1+6n+h)} \sim \frac{\Gamma(b+6n)}{\Gamma(1+6n)} \sim (6n)^{b-1}
\]
and
\[ \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} \frac{\cos(6n\theta_{k,j})}{(k^2 + j^2 - jk)^{b+h+3n}} \sim 1 \]  
(25)

Replacing the previous relations (24) and (25) in the expression (23) of \( H_n(x) \) yields
\[ H_n(x) \sim 6\Gamma(b+6n) \frac{x^{2b+6n}}{\Gamma(b+1+6n)} \sum_{k=0}^{\infty} \frac{\Gamma(b+h)}{\Gamma(b+1)} x^{2h} \]  
(26)

The last sum in (26) converges uniformly in the unit disk to \( \frac{1}{(1-x^2)^2} \). We obtain then
\[ H_n(x) \sim 6\Gamma(b+6n) \frac{x^{2b+6n}}{\Gamma(b+1+6n)} (1-x^2)^{-2b} \]  
(27)

It follows that the Fourier series expansion of f-factor in (13) converges like the series
\[ H_0(x) + \frac{12x^{2b}}{\Gamma(b)(1-x^2)^b} \sum_{n=1}^{\infty} \frac{\Gamma(b+6n)}{\Gamma(b+1+6n)} x^{6n} \cos(6n\theta) \]  
(28)

which converges absolutely for \( |x| < 1 \).

**Corollary 3.4:** Let \( f \), as defined in (7), be the f-factor function of location \( m \) in a regular hexagonal network with infinite number of omni-directional sites. Let \( b \) and \( x \) be defined as in theorem 3.2, then we have for \( b > 1 \) and \( |x| < 1 \)
\[ f(m) \approx H_0(x) - \frac{12x^{2b}}{(1-x^2)^b} + \frac{2x^{2b}}{(1-x^2)^{2b}} x \sum_{i=0}^{\infty} \Re \left[ \left( -x e^{i(\theta+i\pi/3)} \right)^{-b} \right] \]  
(29)

where \( \Re[z] \) is the imaginary part of the complex number \( z \). \( H_0(x) \) is the average of f-factor over \( \theta \) and given by (14).

The proof of corollary 3.4 is simple. We have just to replace \( H_n(x) \) in the expression of \( f(m) \) by the approximation of (15), valid for \( n \geq 1 \), and then evaluate the sum in (28).

**Remark:**
- As can be observed from equation (15), \( H_n(x) \) decays very fast with \( n \) and becomes very small for \( n \geq 1 \). Thus, the coefficients \( H_n(x) \), for \( n \geq 1 \), contribute very little on the value of \( f(m) \). The f-factor \( f \) is then a very slowly varying function on \( \theta \). This observation was pointed out in [7] based on the curve of \( f \).
- The f-factor \( f(m) \) can be well approximated only by \( H_0(x) \) with an error equal to \( O(x^{2b+2b}) \); where \( O \) is the Knuth's notation. This error is at most in the order of \( \frac{2x^{2b}}{(1-x^2)^{2b}} \), because \( x \) can not exceed \( \frac{1}{\sqrt{3}} \) for a regular omni-directional hexagonal network.

**Corollary 3.5:** Let \( f \) be the f-factor defined in (7) and given explicitly by (13) and (29). Let \( H_0(x) \) be the average of \( f(m) \) over \( \theta \) and given explicitly by (14). Then, for \( b > 1 \) and \( |x| < 1 \), the following approximations hold.

\[ f(m) \approx H_0(x) \]  
(30)

\[ H_0(x) = 6x^{2b} (\lambda(b) + b(b+1)2^2x^2 + O(x^4)) \]  
(31)

\[ H_0(x) \approx 6x^{2b} \left( \zeta(2b-1)x^{2b} + \lambda(b) - 1 \right) \]  
(32)

\[ H_0(x) \approx 6x^{2b} \left( \frac{1}{(1-x^2)^{2b}} + \lambda(b) - 1 \right) \]  
(33)

**Proof:** Approximation (30) arises directly from the previous remark. Approximation (31) results from the Maclaurin series expansions of \( H_0 \) provided in (14) and we take only the development to order 2.

To prove (32), let \( \lambda \) be the function in (17), take the expansion of \( H_0 \) in (14) and use the approximation \( \lambda(b+h) \approx 1 \), for \( h \geq 1 \).

To prove (33), we start from (32) and use Euler transformation formula of the Hypergeometric function [20, pp.564]:
\[ 2F_1(b,b,1, x) = (1-x^2)^{1-2b} 2F_1(1-b, 1-b, 1, x^2) \]  
(34)

To complete the proof, take only the Maclaurin series development of order 2 in \( 2F_1(1-b, 1-b, 1, x^2) \) because, for \( x < 1 \) and \( b > 1 \), the other terms of higher order are neglected.

**2) Validation of the approximations:**

In Fig. 2, we plot the approximation (29) of the interference factor and the exact expression (7) with different amplitude exponent coefficients \( b = 1.25, b = 1.4 \) and \( b = 2 \); and for two cases of angle \( \theta \in \{0, \pi/6\} \). We notice that the explicit simple form (29) of \( f \) is almost similar to the exact expression of the f-factor, for all \( x < 1 \) and \( b > 1 \), but in practice we focus only on the part where the location \( m \) is served by the central cell, i.e., \( x = \frac{\sqrt{3}}{2} \).

It is worth noting that the f-factor undergoes small variations with \( \theta \) except at cell edge (\( x \approx \frac{\sqrt{3}}{2} \)) where its impact becomes significant. Moreover, the influence of \( \theta \) on the f-factor increases with \( b \).

Fig. 3 represents the exact expression of \( H_0 \) in (14) compared with their development of order 0, namely \( 6\lambda(b)x^{2b} \) and the \( 2^{nd} \) order given in (31) for \( b = 1.1, b = 1.3 \) and \( b = 2 \). For \( b \) close to 1, both the order 0 and the \( 2^{nd} \) order match well with the exact form of \( H_0 \). However, for high value of \( b \), the exact value of \( H_0 \) begins to distance from its first orders’ developments and mainly from order 0. This is explained by the fact that when \( b \) increases, the coefficient \( \frac{\Gamma(b+6n)}{\Gamma(b+1+6n)} \) of the series expansion of \( H_0 \) increases also. Hence, we need to go for higher order and notably at the cell edge area. To sum up, the \( 2^{nd} \) order development (31) is a good approximation of \( H_0 \) in the covered area of the cell for \( 1 < b \leq 1.5 \).

**3) Comparison with other approximations in literature:**

In addition to the explicit calculation (14) of \( H_0 \), this paper gives a simple approximation of \( H_0 \) in (33) that would match very well with \( H_0 \) for every \( b > 1 \). To show the effectiveness of this approximation, we numerically compare it with other approximations provided in [2] and [7]. The approximation of the f-factor in [2, Eq.15] is
\[ f_1(m) = \frac{2\pi x^{2b}}{\sqrt{3(b-1)}} (1-x)^{2-2b} \]  
(35)

Whereas, the provided one in [7, Eq.7] is
\[ f_2(m) = \frac{\zeta(2b-1)x^{2b}}{(1-x)^{2b}} \left( 1 + 4(1-x)^b + \frac{(x-1)^2}{2} \right) \]  
(36)
In Fig. 4, numerical results show that, for any value of \( x \leq \frac{1}{\sqrt{3}} \), the average f-factor \( H_0 \) is well captured and precisely estimated by the proposed simple approximation (33). The approximation \( f_2 \), in [7], is valid for low values of \( x \) but is limited at the cell edge (with error higher than 20%) for all \( b > 1 \). The approximation \( f_1 \), in [2], is good only when \( b \) approaches 1. For higher \( b \) (e.g., close to 2), \( f_1 \) moves away from \( H_0 \) and becomes inefficient to estimate the interferences in hexagonal network. To reduce the gap between \( f_1 \) and the exact value of \( H_0 \), Kelif et al. introduced in [2, Eq.16, pp.7] a corrective term obtained by simulation. Since the proposed approximation (33) is accurate and simple, we recommend its use in the analytical performance evaluations of wireless cellular networks such as LTE or LTE advanced. Approximation (33) can be used in capacity and coverage dimensioning tools as well.

### B. F-factor in hexagonal networks with tri-sectored sites

Sectorization effect on interferences was largely investigated but mostly with simulations [27]–[30]. In regular hexagonal networks with tri-sectored sites, interferences obviously depend on the antenna radiation patterns of the sectors in both serving and interfering sites. In general, both horizontal and vertical radiation patterns influence interferences, but here we give an approximation of f-factor considering only the horizontal pattern. The vertical one is neglected and we assume that its effect is taken in the constant term of the antenna gain.

As mentioned earlier, antenna pattern (in dB scale) is the sum of a constant gain and a mask depending on the angle between the mobile location and the antenna’s boresight. The antenna mask \( G_s \) of a site, as defined in (4), is a periodic function with period \( \frac{2\pi}{3} \) if all sectors forming the site have the same antenna pattern. In tri-sectored network, the relative interference \( F \), received in location \( m \), is the sum over all interferences received from each site \( S_{l,k,j} \) (for \( 0 \leq l \leq 5 \), \( k \geq 1 \) and \( 0 \leq j \leq k - 1 \)) and from all collocated sectors of the same site.

\[
F(m) = -1 + \frac{G_s(\theta)}{G(\theta - \frac{2\pi}{3})} + f_s(m) 
\]

where the quantity \(-1 + \frac{G_s(\theta)}{G(\theta - \frac{2\pi}{3})}\) corresponds to the intra-site interference factor generated by sectors belonging to the same site and \( f_s(m) \) is the inter-site interference factor as defined in sub-section II-D. A good approximation of \( F(m) \) is provided in the following proposition.

**Proposition 3.6:** The f-factor \( F(m) \), received at location \( m \) connected to the first sector of site \( S \) in a regular hexagonal network with infinite number of tri-sectored sites, has the following approximation:

\[
F(m) \approx -1 + \frac{G_s(\theta)}{G(\theta - \frac{2\pi}{3})} + \frac{\alpha_0 f(m)}{G(\theta - \frac{2\pi}{3})} - \frac{2\alpha_1 x^{2b}}{G(\theta - \frac{2\pi}{3})} \sum_{l=0}^{5} |e^{i\frac{2\pi}{3}} - xe^{i\theta}|^{2b+3} 
\]

where \( f \) is recalled the f-factor function for omni-directional network, given in theorem 3.2, and \( x = \frac{r}{\delta} \). The coefficients \( \alpha_0 \) and \( \alpha_1 \) are given by

\[
\alpha_p = \frac{3}{\pi} \int_{0}^{\frac{2\pi}{3}} G_s(\theta) \cos(3p\theta) d\theta, \quad p \in \{0,1\} 
\]

**Proof:** Given that \( G_s \) is a periodic function on \( \theta \) with period \( \frac{2\pi}{3} \), we assume that it has a convergent Fourier series expansion

\[
G_s(\theta) = \alpha_0 + 2 \sum_{p=1}^{+\infty} \alpha_p \cos(3p\theta) 
\]
where
\[ \alpha_p = \frac{3}{\pi} \int_{0}^{\frac{\pi}{3}} G_s(\theta) \cos(3p\theta) \, d\theta, \quad \forall p \in \mathbb{Z} \] (41)

Let again \( m \) be an arbitrary location in \( S \) and recall that \( e^{j(m,k,j)} = (m - S_{i,k,j})/[m - S_{i,k,j}] \). Using the expression of pathloss in (5), we have
\[ \frac{L(m)G_s(\phi_{i,k,j})}{L_{i,k,j}(m)} = r^{2b} \sum_{p \in \mathbb{Z}} \alpha_p \frac{(m - S_{i,k,j})^{3p}}{|m - S_{i,k,j}|^{2b+4p}} \] (42)

We neglect the coefficient \( \alpha_p \) for \( p \geq 2 \), we obtain
\[ \frac{L(m)G_s(\phi_{i,k,j})}{L_{i,k,j}(m)} \approx \alpha_0 \frac{L(m)}{L_{i,k,j}(m)} + 2\alpha_1 r^{2b} \Re \left[ \frac{(m - S_{i,k,j})^3}{|m - S_{i,k,j}|^{2b+4}} \right] \] (43)

We now sum all individual interference factors. For the second term having the coefficient \( \alpha_1 \), we limit the sum only to the first tier \( k = 1 \) because it behaves like an interference factor with path loss exponent equals \( 2b + 3 \). To prove the latter assumption, we develop \( r^{2b} \Re \left[ (m - S_{i,k,j})^3 \right] |m - S_{i,k,j}|^{-2b-3} \) as we did in proposition 3.1. It follows that
\[ G(\theta - \frac{\pi}{3}) f_s(m) \approx \sum_{0 \leq \ell \leq 5, k \geq 1, 0 \leq \delta \leq \ell - k} \frac{L(m)}{L_{i,k,j}(m)} \frac{1}{2\alpha_0 f_0(m)} + 2\alpha_1 r^{2b} \sum_{l=0}^{5} \Re \left[ \frac{(m - S_{i,1,0})^{3l}}{|m - S_{i,1,0}|^{2b+3l}} \right] \] (44)

Replacing \( S_{i,1,0} \) by \( \delta e^{j\frac{\pi}{3}} \) and \( m \) by \( r e^{j\theta} \), we complete the proof. It is important to note that the approach, used to give the closed formula (38) of f-factor in tri-sectorized network, is applicable to any sectorization level such as quadri or hexa-sectorization.

In order to assess the validity of f-factor approximation in tri-sectorized network, we use an antenna mask from a real pattern

The used antenna mask is drawn in Fig. 5. The coefficients of the site mask calculated in (39) for the considered antenna are \( \alpha_0 = 0.62 \) and \( \alpha_1 = -0.19 \).

In Fig. 6, we draw both the exact calculation \( (7) \) and the explicit form \( (38) \) of the f-factor in a hexagonal tri-sectorized network for different values of \( b \) and for the extreme values of \( \theta \): \( 0 \) and \( \frac{\pi}{3} \). For \( \theta = 0 \), \( r \leq \frac{2}{3} \) whereas for \( \theta = \frac{\pi}{3} \), \( r \) varies up to \( \frac{2\pi}{3} \). The simple explicit formula \( (38) \) approximates very well the f-factor for all locations in the cell except with small difference when \( m \) approaches \( \frac{2\pi}{3} e^{j\frac{\pi}{3}} \) and for high value of \( b \). In addition, for \( \theta = 0 \), the signal coming from the serving sector is almost equal to the signal form the collocated one and consequently f-factor is higher than 1 even for locations close to site center. Furthermore, we observe a high gap between f-factor in \( \theta = 0 \) and \( \theta = \frac{\pi}{3} \). Hence, the impact of \( \theta \) is no longer negligible and it plays a crucial role in the calculation of f-factor.

In Fig. 7, the inter-site interference factor \( G(\theta - \frac{\pi}{3}) f_s(m) \), in (44), is represented as a function of \( \theta \) and \( r \). In contrast to the small impact of \( \theta \) in the omni-directional network, we notice further that the inter-site f-factor in tri-sectorized network depends on \( \theta \) as much as on \( r \). In fact, the antenna mask defines the envelope curve of the interference factor calculated in omni-directional network. Furthermore, Fig. 7 shows that the inter-site interference is periodic with period \( \frac{2\pi}{3} \). This result is easy to be mathematically proved.

IV. APPLICATION TO PERFORMANCE ANALYSIS: SINR DISTRIBUTION

The performance analysis of wireless networks definitely entails the evaluation of the SINR distribution. Its knowledge allows to estimate the throughput which is related to the SINR by a continuous, differentiable, and strictly monotonically increasing function. This function is often called the link level curve of the system and of course should account for different
parameters such as device categories and system characteristics (available spectrum bandwidth, the use of MIMO, fast fading...).

In this section, we give the explicit formula of the Complementary Cumulative Distribution Function (CCDF) of SINR for a general location distribution. Considering an arbitrary traffic location distribution is of great interest for capacity dimensioning because it was shown that the traffic is practically very heterogeneous between cells of the network and even inside cells [22], [26].

A. Explicit form of SINR distribution

Before the explicit calculation of the SINR CCDF, we give the following definitions.

Definition 3: The SINR of location m connected to central cell S in a regular omni-directional hexagonal network is the ratio between the useful received power and the total received interferences including thermal noise power

\[ \text{SINR} = \left( \eta f(m) + y_0 x^{2b} \right)^{-1} \]  

where \( y_0 = \frac{2P_r}{\pi} \delta^{2b} \), \( P_r \) is recalled the transmitted power of the cell, including the antenna gain, \( a \) and \( b \) are the path loss parameters given in (5), \( \delta \) is recalled the inter-site distance, \( x = \frac{r}{\delta} \), \( \eta \) is the average load of the interfering cells and \( P_N \) is the thermal noise power.

Note that \( P \) and \( P_N \) should be calculated in the same spectrum bandwidth.

Definition 4: The CCDF of SINR for any scenario of traffic distribution is given by the following integral

\[ \Psi(y) = \mathbb{P}(\text{SINR} > y) = \int_{S} \mathbb{I}(\text{SINR} > y) dt(m) \]  

where \( S \) is the area of the central cell assumed to be a disk of radius \( R < \delta \), \( \mathbb{I}() \) is the indicator function that takes 1 if the condition "SINR > y" is verified and 0 else. \( t \) is the probability measure of the location variable \( m \) and reflects the spatial traffic distribution in cell \( S \).

We assume that \( t \) is a probability measure on \( S \) and consequently

\[ \int_{S} dt(m) = 1 \]  

The CCDF of SINR is widely called the coverage probability because it gives the percentage of locations in which the condition "SINR > y" is guaranteed.

When the thermal noise power is neglected with respect to interferences, \( y_0 \approx 0 \) and the SINR might be similar to SIR. Hence, SIR CCDF is a particular case of SINR CCDF.

In the calculation of the SINR distribution, we consider the expression of f-factor in (30) since we showed earlier that, for an omni-directional network, the angle \( \theta \) has a little impact. This latter would be further smoothed when looking for the distribution of any function of f-factor. Moreover, the calculation of SINR CCDF requires the inverse function of \( g : x \mapsto \eta f(x) + y_0 x^{2b} \) that realizes a continuous bijection form \([0, \rho]\) to \([0, g(\rho)]\) for every real number \( \rho \in [0, 1) \). The inversion of \( g \) can be explicitly given by the series reversion method or numerically by reversing axis \( x \) with \( y \) [23].

**Proposition 4.1:** Let \( f \) be the f-factor function defined in (30) by \( H_0(x) \). Let \( y_0 \) be as in (45) and \( b \) be defined as in (5). Recall \( \delta \) be the inter-site distance and \( R \) be the radius of the cell. Take \( g(x) = \eta f(x) + y_0 x^{2b} \). Then the SINR CCDF is explicitly given by

\[ \Psi(y) = T(\Lambda(y)), \forall y > 0 \]  

with

\[ \Lambda(y) = \min\left( \delta \times g^{-1}(\frac{1}{y}), R \right) \]  

The function \( T(.) \) is the marginal Cumulative Distribution Function of the location random variable \( m \) and is obtained form the probability measure \( t(m) = t(r, \theta) \) by \( dT(r) = \int_0^{2\pi} dt(r, \theta) \).

**Proof:** Taking the definition of SINR and \( \Psi \) in respectively (45) and (46) and with the assumption that the impact of \( \theta \) on the f-factor is neglected, we have

\[ \Psi(y) = \int_0^{2\pi} \int_0^R \mathbb{I}\left( \frac{1}{g(\frac{1}{y})} > y \right) dt(r, \theta) \]  

\[ = \int_0^{\min(\delta \times g^{-1}(\frac{1}{y}), R)} \int_0^{2\pi} dt(r, \theta) \]  

\[ = \int_0^{\min(\delta \times g^{-1}(\frac{1}{y}), R)} dT(r) \]  

**Remark:** Recall that when \( b \) is close to 1, the average f-factor \( H_0 \) can be approximated by its Maclaurin series expansion of order 0 (see Fig. 3). In this case \( g^{-1}(y) \approx \left( \frac{y}{\eta \lambda(b) + y_0} \right)^{\frac{1}{b}} \) and consequently \( \Psi \) is simply given by

\[ \Psi(y) \approx T\left( \min\left( \delta \times \left( 6\eta \lambda(b) + y_0 \right)^{\frac{1}{b}}, R \right) \right) , \forall y > 0 \]  

Furthermore, this approximation is still valid for every value of \( b \) and in any scenario of traffic distribution except for a hotspot at cell edge. The approximation is obviously also valid in a noise-limited environment, i.e., the thermal noise dominates interferences.

B. Numerical results for different traffic scenarios

In order to evaluate and validate the analytical expression of SINR distribution \( \Psi \) for different scenarios of traffic, we consider a uniform and non-uniform traffic distributions in the cell. Note that in practice, the estimation of the traffic weight (density) in each location can be based on processing probes or on the manipulation of some key performance indicators of the network as in [22]. Furthermore, authors in [26] showed with real data measurements that the spatial traffic density can be approximated by a Log-normal distribution. Therefore, numerical results of this work are provided for both uniform and Log-normal traffic distribution scenarios.

Besides, for each traffic scenario, we simulate interferences in a hexagonal omni-directional network using Monte-Carlo method. The comparison between theoretical and simulation
results are provided for 2 types of environment considering an LTE system with 20MHz of spectrum bandwidth:

1) Outdoor suburban environment with parameters: \( \delta = 1Km, a = 130dB, P_N = -93dBm \) and \( P = 60dBm \).

2) Deep indoor urban environment with parameters: \( \delta = 1Km, a = 166dB, P_N = -93dBm \) and \( P = 60dBm \).

For each scenario, we set the radius of the cell to \( R = \frac{\sqrt{\delta}}{2\pi} = 0.525\delta \) and we assume that interfering cells are fully loaded, i.e., \( \eta = 1 \). Furthermore, We investigate results for 3 different values of \( b = 1.25, 1.5 \) and 2.

1) SINR distribution for uniform traffic scenario:

When the traffic distribution is uniform in the disk (representing the cell) of radius \( R \), we have \( dt(m) = \frac{2rdrd\theta}{\pi R^2} \). The distribution \( \Psi \) of SINR in (48) becomes

\[
\Psi(y) = \frac{\Lambda(y)^2}{R^2} \quad (52)
\]

In Fig. 8, we present the curve of \( \Psi \) in (52) and the SINR CCDF obtained by Monte Carlo simulations for both outdoor suburban environment (Fig.8a) and deep indoor urban environment (Fig. 8b).

The results show that the expression of \( \Psi \) calculated in (52) fits with simulations results of the SINR CCDF quite well. Moreover, as expected the impact of \( \theta \) on \( \Psi \) is smoothed and thus neglected. In addition, it is important to note, from the comparison between Fig. 8a and Fig. 8b, that in the deep indoor urban environment, the impact of the value \( y_0 \) on the SINR is clear and tends to decrease it by 2 to 3dB depending on \( b \). This is explained by the high value of the deep indoor penetration margin that makes the path loss very high and thus the signal arrives very low at indoor locations. Conversely, for outdoor suburban environment, \( y_0 \) is low and SINR CCDF should be quite similar to SINR CCDF.

2) SINR distribution for Log-normal traffic scenario:

For a Log-normal traffic, we assume that variable \( r \) follows a Log-normal distribution \( \sim ln\mathcal{N}(\mu, \sigma) \) with mean \( \mu \) and standard deviation \( \sigma \). Variable \( \theta \) is always uniform \( \sim \mathcal{U}(0, 2\pi) \).

The probability measure of the traffic is written by

\[
dt(m) = \frac{e^{-(\ln(r)-\mu)^2/2\sigma^2}}{\sigma\sqrt{2\pi}}drd\theta
\]

and consequently the expression (48) of \( \Psi \) becomes

\[
\Psi(y) = \frac{Q\left(\frac{1}{2}(\ln(\Lambda(y)) - \mu)\right)}{Q\left(\frac{1}{2}(\ln(R) - \mu)\right)} \quad (53)
\]

where \( Q \) is the cumulative distribution function of the standard normal distribution and is given by the following integral.

\[
Q(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{u^2}{2}}du, \forall z \in \mathbb{R}
\]

The parameters \( \mu \) and \( \sigma \) of the Log-normal distribution allow to deal with different scenarios of traffic hotspots.

\(^3\) \( P_N = E_n + 10\log(W) + NF \); where \( E_n \) is the energy of the thermal noise equal to \(-174dBm/Hz\), \( W \) is the spectrum bandwidth equal to 20MHz and \( NF \) is the noise figure of the transmitter set here to 8dB. \( P = 60dBm \) corresponds to \( 43dBm \) from the transmitter power amplifier and \( 17dB \) for the antenna gain of the transmitter.

Besides, \( \mu \) and \( \sigma \) can be tuned to precisely approximate the distribution of SINR in each cell of a real network. Here, we show the SINR distribution for two cases of traffic hotspots. The first case is a Log-normal hotspot close to cell center with parameters \( (\mu, \sigma) = (-2, 0.5) \), whereas the second one is a Log-normal traffic hotspot located at cell edge and having parameters \( (\mu, \sigma) = (-0.75, 0.1) \). Both hotspots are depicted in Fig. 9.

In Fig. 10, we show the CCDF of the SINR for the previously cited environments (Fig. 10a for outdoor suburban and Fig. 10b for deep indoor urban environment) considering a hotspot close to the cell center (Fig. 9-a and c). Simulation Results confirm once again the precision brought by the analytical form of SINR CCDF in (48). We notice indeed that numerical and analytical curves of SINR CCDF are very close. This consolidates also the different observations from the study of uniform traffic distribution, notably regarding the negligible influence of \( \theta \) on the calculation of the SINR distribution and the importance of the penetration margin that deteriorates more or less the SINR in deep indoor environment.

In Fig. 11, we present the SINR CCDF for a Log-normal traffic hotspot located at cell edge (see Fig.9- b and d). Even
Fig. 9: Plots of the two traffic hotspots following Log-normal distribution: The first is close to cell center (Curves (a,c)) with \((\mu, \sigma) = (-2, 0.5)\) and the second is at cell edge (Curves (b,d)) with \((\mu, \sigma) = (-0.75, 0.1)\).

if \(\theta\) influences the f-factor value in cell edge (see Fig. 2), the curve of SINR CCDF shows to be still insensitive to \(\theta\) for traffic hotspot at cell border because the simulation result considering \(\theta\) is almost similar to the analytical expression (48) which does not account for \(\theta\) variation. All locations in a hotspot at cell edge suffer from degraded SINR. We note also that the SINR CCDF curve of a hotspot close to cell center is more scattered than that of a hotspot at cell edge. This is related to the value of \(\sigma\) which is higher for the first hotspot. The same previous conclusions regarding the impact of the penetration margin in indoor environment are still valid for hotspot at cell edge. We could say that the SINR CCDF for deep indoor urban is obtained form that of outdoor suburban environment by a translation of almost \(3\) dB to low SINR values.

V. CONCLUSIONS

This paper has analytically and fundamentally investigated interferences in hexagonal wireless networks. Several results have been established for the f-factor \(f\) and the SINR distribution. Given a location \(m = re^{i\theta}\), we have proved that the
Apart from the fact that any new result for such a network model is scientifically interesting, there are additional motivations for looking on it:

- Perturbed hexagonal network model approximates more than any other model the geometry of the real network [18]. Consequently, performances of perturbed hexagonal model would be closer to the performances of a given real network.
- In addition to the parameters of the traffic distribution, considered in this paper, the perturbed model introduces a new parameter, named the average perturbation. SINR or throughput distribution for a given real cell can be precisely determined with tuning the average perturbation and the first moments of the traffic distribution.

**APPENDIX A**

**PROOF OF LEMMA 2.1**

Let $\delta$ be the inter-site distance and $(x_{0,k,j}, y_{0,k,j})$ be the Cartesian coordinates of the site $S_{0,k,j}$ located in the tier $k$ and region $\Omega_0$, then $x_{0,k,j} = \delta(k - \frac{1}{2})$ and $y_{0,k,j} = \delta \sqrt{\frac{3}{2}}$, for $0 \leq j \leq k - 1$ and $k \geq 1$. Transforming the Cartesian coordinates into the Polar coordinates yields $S_{0,k,j} = D_{k,j} e^{i \theta_{k,j}}$; where $D_{k,j} = \delta \sqrt{k^2 + j^2 - jk}$ and $\theta_{k,j} = \text{atan}(\frac{\sqrt{3}}{k})$.

Since every region $\Omega_k$ is obtained from rotating the region $\Omega_0$ by an angle equal to $\frac{\pi}{2}$, every site $S_{l,k,j}$ in the tier $k$ and region $\Omega_k$ is obtained from $S_{0,k,j}$ by an isometric map of angle $\frac{\pi}{2}$. Therefore $S_{l,k,j} = S_{0,k,j} e^{il\pi}$, for $0 \leq l \leq 5$.

Replacing the expression of $S_{0,k,j}$, we complete the proof.

**APPENDIX B**

**PROOF OF LEMMA 3.3**

Let $z$ be any complex number such that $Re(z) > 1$, define

$$
\lambda(z) = \sum_{k=1}^{+\infty} \sum_{j=0}^{k-1} \frac{1}{(k^2 + j^2 - jk)^z}
$$

(54)

Observe that $\lambda(z) \Gamma(z)$ is the Mellin transform of the function $\vartheta$ defined for all complex number $t$, such that $Re(t) > 0$, by

$$
\vartheta(t) = \sum_{k=1}^{+\infty} \sum_{j=0}^{k-1} e^{-t(k^2 + j^2 - jk)} = \sum_{k=1}^{+\infty} \sum_{j=0}^{k-1} e^{-t(k^2 + j^2 - jk)}
$$

(55)

Now using the Borwein cubic theta function defined by the bilateral sum [31], we have

$$
\sum_{k,j \in \mathbb{Z}} e^{-t(k^2 + j^2 + jk)} = 1 + 2 \sum_{k=1}^{+\infty} \sum_{j=0}^{k-1} e^{-t(k^2 + j^2 + jk)} + 2 \sum_{k=1}^{+\infty} \sum_{j=0}^{k-1} e^{-t(k^2 + j^2 - jk)}
$$

(56)

Dividing the last term in the right hand side of (56) into two parts and manipulating the indexes give the following identity

$$
\sum_{k=1}^{+\infty} \sum_{j=0}^{k-1} e^{-t(k^2 + j^2 - jk)} = 2\vartheta(t)
$$

(57)
It follows, from the last identity and (56), that
\[ \sum_{k,j \in \mathbb{Z}} e^{-t(k^2+j^2+jk)} = 1 + 6 \vartheta(t) \]  
(58)

Using the Lambert Series form of Borwein cubic theta function [31], we obtain
\[ \vartheta(t) = \sum_{k=0}^{\infty} \left( \frac{1}{e^{t(3k+1)} - 1} - \frac{1}{e^{t(3k+2)} - 1} \right) \]  
(59)

Finally, applying the Mellin transform to both right and left hand sides of the last equation yields
\[ \lambda(z) \Gamma(z) = 3^{-z} \zeta(z) \left( \zeta(z, \frac{1}{3}) - \zeta(z, \frac{2}{3}) \right) \Gamma(z) \]  
(60)

which completes the proof of Lemma 3.3.

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