A unified and explicit approach for accurately characterizing the hardening-softening behavior of metals with a tension-compression strength asymmetry

H F Xi\textsuperscript{1,2,3}, L Zhan\textsuperscript{4}, S Y Wang\textsuperscript{1}, Z H Xu\textsuperscript{4} and H Xiao\textsuperscript{1,3}

\textsuperscript{1}School of Mechanics & Construction Engineering and MOE Lab for Disaster Forecast & Control in Engineering, Jinan University, 510632 Guangzhou, China
\textsuperscript{2}State Key Laboratory for Strength and Vibration of Mechanical Structures, Xi’an Jiaotong University, 710049 Xi’an, China

Email: xihuifeng@jnu.edu.cn (H F Xi); hxiao@jnu.edu.cn (H Xiao)

Abstract. A unified and explicit approach is proposed for accurately characterizing a complex hardening-softening behavior of metals with a tension-compression strength asymmetry. The proposed approach provides an explicit determining of the yield strength as plastic work function and contains the following three procedures: (i) a mode invariant is introduced to express the yield strength as a sum of two uncoupled parts for the tension and the compression case; (ii) a new expression for the plastic work is derived explicitly in terms of the uniaxial stress-strain function and, then, the yield strength function can be determined jointly from the just-mentioned two functions with the axial strain as parametric variable; and, finally, (iii) the axial stress-strain function is presented for representing the hardening-softening features. With these procedures, test data over the entire hardening-softening range can be fitted independently for the tension and the compression cases by directly treating the uniaxial stress-strain function. As such, tedious trial-and-error procedures are bypassed in identifying numerous unknown parameters. Numerical examples show that accurate simulations may be achieved with the proposed approach.

1. Introduction

Determination of the yield strength as function of the plastic work is the central issue for the purpose of characterizing complex hardening and softening effects of metals over the whole strain range up to failure. Because of undue complexities, a full treatment is still not available. That may be the case even for the tension behavior, let alone the more complex case with asymmetric effects in tension and compression.

Usual approaches are restricted to the limited case only for hardening effects. Either certain particular forms for the yield strength function are introduced or certain ad hoc forms of the evolution equation are presumed for specifying the yield strength. With these usual approaches, a number of unknown parameters should be introduced and need to be identified. Reference may be made to \cite{1-3} for details in these respects. Toward identifying numerous unknown parameters, tedious trial-and-error numerical procedures have to be repeatedly carried out in treating a coupled system of nonlinear elastoplastic rate equations, since the plastic work in the yield strength function is strongly coupled with each elastoplastic process.
To bypass the above complexity, a new and explicit approach will be proposed in this article. With this approach, the yield strength function may be explicitly determined over the entire range for hardening and softening effects and, in particular, the tension-compression asymmetry may be treated in a decoupled manner by independently treating the tension case and the compression case. This new approach is composed of the following three procedures: (i) the mode invariant is introduced to express the yield strength as a sum of two uncoupled parts for the tension and the compression case; (ii) a new expression for the plastic work is derived explicitly in terms of the uniaxial stress-strain function and, then, the yield strength function can be determined jointly from the just-mentioned two functions with the axial strain as parametric variable; and, finally, (iii) the axial stress-strain function is presented for representing the hardening-softening features. With these procedures, test data over the entire hardening-softening range can be fitted independently for the tension and the compression case by directly treating the uniaxial stress-strain function presented. As such, tedious trial-and-error procedures are bypassed in identifying numerous unknown parameters.

2. Elastoplastic J2-flow equations with tension-compression asymmetry

Finite strain effects should be treated for hardening and softening effects over the entire range up to failure. To this end, the self-consistent Eulerian rate formulation of finite elastoplasticity is used. Refer to [4-6] for details.

The starting point is the following separation of the stretching $D$:

$$D = D^e + D^p, \quad (1)$$

where $D^e$ and $D^p$ are the elastic and the plastic part of $D$, respectively. The elastic part $D^e$ is prescribed as follows:

$$D^e = \frac{1}{2G} \frac{\epsilon^\text{log}}{E} (\text{tr} \bar{\tau}^\text{log}) I, \quad (2)$$

In the above, $G$, $\nu$ and $E$ are the shear modulus, the Poisson ratio, and Young's modulus; $I$ is the identity tensor; and $\epsilon^\text{log}$ is the logarithmic rate of the Kirchhoff stress $\tau$. Let $J$ and $\sigma$ be the volumetric ratio and the Cauchy stress. Then, $\tau = J \sigma$.

On the other side, the plastic part is specified by the following flow rule:

$$D^p = \rho \frac{\bar{\tau}}{h} \frac{\partial f}{\partial \tau}, \quad (3)$$

Here, $\rho$ is the plastic index, which gives 1 and 0 for the loading case and the unloading case, respectively; $f$ is the von Mises yield function, viz.

$$f = \frac{1}{2} \text{tr} \bar{\tau}^2 - \frac{1}{3} q^2, \quad (4)$$

with the deviatoric part $\bar{\tau}$ of the Kirchhoff stress $\tau$ and the yield strength $q$. The latter relies on both the plastic work $\kappa$ and the mode invariant of the stress. The latter is designated by $\gamma$ and introduced for the purpose of incorporating the tension-compression asymmetry effects. Hence,

$$q = 0.5(1 + \gamma) q'(\kappa) + 0.5(1 - \gamma) q'(\kappa), \quad (5)$$

The plastic work $\kappa$ is prescribed by the evolution equation below:

$$\dot{\kappa} = \bar{\tau} : D^p, \quad (6)$$

and the mode invariant $\gamma$ is given by
\[
\gamma = \sqrt{6} J_3 J_2^{-1.5},
\]
where \(J_2\) and \(J_3\) are two basic invariants of the deviatoric stress and given as follows:
\[
J_2 = \text{tr}\bar{\tau}^2, \quad J_3 = \text{tr}\bar{\tau}^3.
\]

The mode invariant \(\gamma\) ranges from -1 to 1 for all stress modes. In particular, it gives -1 and 1 for the two cases of uniaxial tension and compression, respectively. Namely,
\[
\gamma = \begin{cases} 
+1 & \text{for uniaxial tension} \\
-1 & \text{for uniaxial compression}
\end{cases}
\]

As such, it may be evident that equation (5) supplies the yield strength functions \(q(t)\) and \(q(c)\) for uniaxial tension and compression, respectively, thus decoupling these two cases. Moreover,
\[
\hat{f} = \tau : \dot{\tau}, \quad \hat{h} = \frac{q''}{q} q' \left( 2q - \left( q' + q'' \right) \tau : \frac{\phi}{\varepsilon_T} \right),
\]

3. Determination of the yield strength via parametric variable

An explicit expression for the plastic work is derived at the first step. Let \(\tau\) and \(h\) be the axial Kirchhoff stress and the axial logarithmic strain, respectively. Then, the uniaxial stress-strain curve is prescribed by the stress-strain function \(\tau = \phi(h)\). In the uniaxial case, equations (1)-(9) may be reduced and then the following expression may be derived:
\[
\kappa = \int_{h_0}^{h} \phi(h)dh - \frac{1}{2E} \left( \phi(h)^2 - q_0^2 \right) \equiv \psi(h),
\]

Hence, two functions of the axial logarithmic strain \(h\) are obtained as follows:
\[
\begin{cases} 
q = \phi(h) \\
\kappa = \psi(h)
\end{cases}
\]

Thus, with the axial logarithmic strain \(h\) as the parameter variable, the above two supply two parametric equations and, accordingly, they jointly determine the yield strength \(q = q(\kappa)\) as function of the plastic work \(\kappa\). With such a function for the yield strength, the established elastoplastic equations can automatically simulate any given test data for the uniaxial stress-strain curve, whenever the uniaxial stress-strain function \(\tau = \phi(h)\) is chosen to fit such data. From equation (8) it may be deduced that test data for the two cases of tension and compression may be independently treated in a decoupled manner, and two yield strength functions, \(q(t)(\kappa)\) and \(q(c)(\kappa)\), are then obtained from equation (11), separately. This will be done below.

4. Stress-strain function for tension and compression

The stress-strain function \(\tau = \phi(h)\) that can represent the hardening-softening features for both tension and compression is presented as follows:
\[
\tau = \frac{1}{2} \tau_0 \left( 1 + \alpha (h - h_0) \right) \left( 1 - \tanh \beta (h - h_0) \right).
\]

The meanings of the parameters introduced are explained with reference to the tension and the compression case. Of them, the three parameters \(h_0, q_0\), and \(E\) are related to the initial yielding, namely, \(E\) is the Young modulus, \(q_0\) is the initial yield strength, and \(h_0\) is the initial yield strain. On the other side,
the three parameters $\alpha$, $\beta$ and $h_b$ characterize hardening effects for the compression case and both hardening-softening effects for the tension case. It should be noted that each such parameter may take different values for the tension and the compression case.

As mentioned above, values of these parameters may be separately chosen for the tension case and the compression case. With the new treatment here, the stress-strain data for these two cases can be fitted in a decoupled manner. This will be illustrated by the following numerical examples.

5. Numerical Examples

| Table 1. Parameter values for tension and compression. |
|-----------------------------------------------|
| **Parameters** | **Mode** | $\alpha$ | $\beta$ | $h_b$ |
|-----------------|---------|---------|---------|--------|
| Tension         |         | 150     | 115     | 0.014  |
| Compression     |         | 0.04    | 2.0     | 1.185  |

| Table 2. Material parameters for tension and compression |
|-----------------------------------------------|
| **Parameters** | **Mode** | $\tau_0$ | $h_0$ | $E$ |
|-----------------|---------|---------|-------|-----|
| Tension         |         | 4.8     | 0.0054| 888.89 |
| Compression     |         | 6       | 0.037 | 162.16 |

**Figure 1.** Simulation results of tension data (solid dots) for foamed metal

**Figure 2.** Simulation results of compression data (solid dots) for foamed metal

**Figure 3.** Plastic work of tension of foamed metal

**Figure 4.** Plastic work of compression of foamed metal
The experimental data for foamed metal displaying tension and compression asymmetry are taken into consideration. The parameter values in equation (12) are listed in tables 1 and 2 for both tension and compression. Simulation results of stress-strain curves and hardening work for tension-compression are shown in figures 1-4.

6. Summary
With the new approach proposed, accurate simulation is achieved for both tension and compression. In particular, that is the case for the tension case with both hardening and softening effects up to failure. The parameter values are obtained by directly fitting the stress-strain function equation (11) to the stress-strain data for tension and compression, separately. In doing so, cumbersome numerical procedures in handling elastoplastic rate equations are not involved.

In a broad sense, the stress-strain function $\tau = \varphi(h)$ in equations (8)-(9) may be given by a suitable interpolating function precisely fitting stress-strain data given. As a result, any given data over the whole hardening-softening range up to failure may be simulated without involving any adjustable parameters. Results will be reported elsewhere.

Acknowledgments
This work was carried out under support (Grant No. SV2018-KF-32) from the State Key Lab for Strength and Vibration of Mechanical Structures of Xi’an Jiaotong University, China.

References
[1] Hartley C S and Srinivasan R 1983 Constitutive equations for large plastic deformation of metals J. Eng. Mater. Techn. 105 162-7
[2] Sung J H, Kim J H and Wagoner R H 2010 A plastic constitutive equation incorporating strain, strain-rate, and temperature Int. J. Plasticity 26 1746-71
[3] Chaboche J L 2008 A review of some plasticity and viscoplasticity constitutive theories Int. J. Plasticity 24 1642-93
[4] Xiao H, Bruhns O T and Meyers A 2006 Elastoplasticity beyond small deformations Acta Mech. 182 31-111
[5] Xiao H, Bruhns O T and Meyers A 1998 Strain rates and material spins J. Elasticity 52 1-41
[6] Xiao H, Bruhns O T and Meyers A 2000 The choice of objective rates in finite elastoplasticity: general results on the uniqueness of the logarithmic rate Proc. R. Soc. London A. 456 1865-82