GRAVITY-MODES IN ZZ CETI STARS
II. EFFECTS OF TURBULENT DISSIPATION
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ABSTRACT

We investigate dynamical interactions between turbulent convection and g-mode pulsations in ZZ Ceti variables (DAVs). Since our understanding of turbulence is rudimentary, we are compelled to settle for order of magnitude results. A key feature of these interactions is that convective response times are much shorter than pulsation periods. Thus the dynamical interactions enforce near uniform horizontal velocity inside the convection zone. They also give rise to a narrow shear layer in the region of convective overshoot at the top of the radiative interior. Turbulent damping inside the convection zone is negligible for all modes, but that in the region of convective overshoot may be significant for a few long period modes near the red edge of the instability strip. These conclusions are in accord with those reached earlier by Brickhill. Our major new result concerns nonlinear damping arising from the Kelvin-Helmholtz instability of the aforementioned shear layer. Amplitudes of overstable modes saturate where dissipation due to this instability balances excitation by convective driving. This mechanism of amplitude saturation is most effective for long period modes, and it may play an important role in defining the red edge of the instability strip.

1. INTRODUCTION

The current paper focuses on dynamical interactions between turbulent convection and g-mode pulsations in ZZ Ceti variables. It is the second of a series, and the last in which we rely on order of magnitude reasoning and the quasadiabatic approximation. The reader is referred to Goldreich & Wu (1998) (hereafter Paper I) for descriptions of the properties of g-modes and scaling relations appropriate to turbulent convection. Only results of immediate relevance to the current investigation are quoted here. Symbols taken from Paper I are defined in Table 1.

ZZ Ceti variables possess surface convection zones. Thus the interaction of their g-modes with turbulent convection requires attention. An important feature of convection in these stars is that its response time is much shorter than the periods of the observed modes. Thus to a good approximation, the convection adjusts to the instantaneous pulsational state. This behavior plays an essential role in the overstability mechanism (Brickhill 1990, Gautschy 1996, Paper I) which, following Brickhill, we refer to as convective driving.

The nonlinear advection term in the fluid momentum equation describes interactions between g-modes and turbulence. We model these interactions as giving rise to a turbulent viscosity which acts to reduce the magnitude of a mode’s shear tensor. We devote §2 to evaluating the effects of turbulent viscosity on g-modes, both inside the convection zone and in the region of convective overshoot at the top of the radiative interior. We apply these results in §3 to estimate the contributions of turbulent viscosity to mode damping. In §4, we analyze the saturation of the amplitude of an overstable mode due to instability of the narrow shear layer it forces at the top of the radiative interior. We conclude with a brief discussion in §5.

2. EFFECTS OF TURBULENT VISCOSITY

Since motion in the outer layers of a DA white dwarf is all that concerns us, it proves convenient to adopt a plane-parallel approximation. Perturbation theory is done with Lagrangian variables which are assumed to carry a time dependence $e^{-i\omega t}$ and a horizontal spatial dependence $e^{ik_\parallel x}$. Gravity modes involve mostly horizontal sloshing of fluid. Our principal concern is with the vertical derivative of the horizontal component of the displacement, $d z / dz$, because it is the largest component of the gradient of the displacement.

In §2.1, we calculate the magnitude of $d z / dz$ inside the convection zone. To begin, we estimate the magnitude that $d z / dz$ would have in the absence of turbulent viscosity. In the dual limit of an isentropic convection zone and a vanishing entropy perturbation, $|\partial_j z / \partial z|$ would be equal to $\gamma_j / \gamma R$ which is smaller than $\gamma_j / \gamma R$. However, a real convection zone is not an isentrope; its specific entropy increases with depth. Moreover, a g-mode produces a specific entropy perturbation. It turns out that the latter departure from isentropy is more important than the former in enhancing the magnitude of $d z / dz$. Without turbulent viscosity, the entropy perturbation could induce a $\partial_j z / \partial z$ as large as $\gamma / \gamma R$. Inclusion of turbulent viscosity reduces $\partial_j z / \partial z$ by a multiplicative factor of $1 / t_{cv}$.

In §2.2, we examine the velocity shear in the transition region between the base of the convection zone and the top of the radiative interior. The turbulent viscosity declines sharply with depth in this region. We approximate this decline by a discontinuous drop to zero viscosity. The application of appropriate boundary conditions then shows that the horizontal velocity is discontinuous at $z_\delta$. We evaluate the magnitude of this velocity jump and the associated jump in the gradient of the pressure perturbation.

2.1. Velocity Shear in the Convection Zone

The linearized equations of mass and momentum conservation, the latter in component form, read (cf. Paper I)

$$\nabla \cdot \mathbf{U} = -i k_h \frac{d z}{dz};
$$

(1)
Table 1
DEFINITIONS

| Symbol | Meaning |
|--------|---------|
| $R$    | stellar radius |
| $g$    | surface gravity |
| $r$    | radial distance from center of star |
| $z$    | depth below photosphere |
| $z_b$  | depth at bottom of convection zone |
| $z_!$  | depth at top of mode’s cavity, $z_!^2 = (gk_h^2)$ |
| $n$    | radial order of mode |
| $\lambda$ | angular degree of mode |
| $k_h$  | horizontal wave vector, $k_h^2 = (\lambda + 1)^2 = R^2$ |
| $\rho$ | mass density |
| $p$    | pressure |
| $s$    | specific entropy in units of $k_B m_p$ |
| $F$    | energy flux |
| $L$    | luminosity, $L = 4 \pi R^2 F$ |
| $c_s$  | adiabatic sound speed, $c_s^2 = (\partial \ln \rho/\partial s)_\rho$ |
| $\theta$ | denotes Lagrangian perturbation |
| $h$    | horizontal component of displacement vector |
| $z$    | vertical component of displacement vector |
| $v_{cv}$ | convective velocity, $v_{cv} (F) = \theta^3$ |
| $t_{cv}$ | response time for convection, $t_{cv} = z/v_{cv}$ |
| $A; B; C$ | turbulent kinematic viscosity, $z_{cv}$ |
| $A_{th}$ | thermal constant at depth $z$, $t_{cv} = \theta_{th} (v_{cv} c_{ev})^2$ in the convection zone |
| $b_{th}$ | unconventional thermal time constant at $z_b$, $b_{th} = 5$ at $z_b$. |
| $c$    | time constant of low pass filter for convection zone, $c = (B + C) b$ |
\[ \frac{d^2 h}{dz^2} = ik_h \frac{p}{p} - g z - \frac{f_h}{p} \]  
(2)

\[ \frac{1}{z^2} z = \frac{p}{d} - g z + \frac{p}{p} + ik_h h - \frac{f_z}{p} \]  
(3)

where \( f_z \) is the force per unit volume due to turbulent convection.

For the moment, we neglect the force due to turbulence. Then differentiating equation (2) and substituting for \( d z = dz \) using equation (1), we obtain

\[ \frac{d}{dz} = ik_h \frac{p}{d} - g z + \frac{p}{p} + ik_h h : \]  
(4)

With the aid of equation (3) and the equation of state,

\[ \frac{1}{z^2} = \frac{p}{c_s^2} + s : \]  
(5)

we recast equation (4) in a more revealing form

\[ \frac{d}{dz} = ik_h \frac{p}{d} - \frac{igk_h s}{z^2} - \frac{p dz}{g} + \frac{p}{p} - s : \]  
(6)

We can think of the \( p = p \) and \( s \) terms as providing adiabatic and nonadiabatic forcing of \( d h = dz \). Both terms vanish for adiabatic perturbations in an isentropic convection zone. In their absence, the perturbations are irrotational so \( d h = dz = ik_h z \). This value of shear would lead to negligible turbulent damping. To estimate \( d h = dz \) when the flow is rotational, we must relate \( p = p \) and \( s \) to \( h \). Here we appeal to Paper I which establishes that for \( z < z_0 < z_1 \),

\[ \frac{p}{p} \approx ik_h h ; \]  
(7)

and

\[ s = \frac{A(B+C)}{1-i} c \frac{p}{p} : \]  
(8)

The ratio of adiabatic to nonadiabatic forcing, as estimated at \( z = z_0 \), is given by

\[ \frac{p}{g} \frac{d s}{dz} = \frac{1}{p} h \frac{p}{b} \frac{1}{s} \frac{1}{h} \frac{p}{g} \frac{d s}{dz} \frac{[1+(i)^2]^{b+2}}{A(B+C)} \frac{1}{h} \frac{p}{b} \frac{A(B+C)}{c_s^2} \frac{v_c}{c_s} + \frac{[1+(i)^2]}{b} \]  
(9)

where we make use of the relations \( gds = dz \), \( (v_c = c_s)^2 \), and \( t_c = \frac{b}{b} (v_c = c_s)^2 \). This establishes nonadiabatic forcing as the principal driver of velocity shear in the convection zone. In an inviscid convection zone, \( \frac{gln}{z = dz} \) would be of order \( z_1^{-1} \).

Acting on a shear of this magnitude, turbulent damping would overwhelm convective driving.\(^3\)

Next, we assess the effects of turbulence on the shear. Given our present understanding, about the best we can do is to model the turbulent Reynolds stress by analogy with the stress due to molecular viscosity in a Newtonian fluid. The viscous force per unit volume is given by

\[ f_z = \frac{\theta}{\theta x} \frac{ij}{x} : \]  
(10)

where the stress tensor, \( \theta_{ij} \), is related to the g-mode’s displacement field by two scalar coefficients, the shear viscosity, \( \nu_c \), and the bulk viscosity, \( \nu_b \), according to

\[ \nu_{ij} = -i! \left[ \frac{\theta}{\theta x} + \frac{\theta}{\theta x} - \frac{2}{3} \frac{\theta}{\theta x} + \frac{\theta}{\theta x} i j : \right] \]  
(11)

When the effects of turbulent viscosity are neglected, \( d h = dz \) is by far the largest component of the gradient of the displacement. Since this term contributes to the shear but not to the divergence, we need not consider terms involving the bulk viscosity. Also, it is easy to show that \( f_z = g \) is much smaller than either \( p = p \) or \( ik_h h \). Therefore, we ignore the effects of turbulent viscosity in the vertical component of the momentum equation.

Turbulent viscosity has a profound effect on the horizontal component of the momentum equation. The dominant term in \( f_h \) takes the form\(^4\)

\[ f_h = -i! \frac{d}{dz} \frac{d h}{dz} - i! \nu_c \frac{d h}{dz} : \]  
(12)

Adding the approximate expression for \( f_h \) to the right hand side of equation (12) yields

\[ ! \frac{d h}{dz} = ik_h \text{p} \frac{p}{p} - g z + \frac{i! z d h}{t_c} : \]  
(13)

Differentiating this equation with respect to \( z \), and repeating the steps that led to equation (6), we obtain

\[ \frac{d h}{dz} = \frac{1}{1-i} \frac{t_c}{c_s} \frac{h}{h} \frac{g k h}{z^2} + \frac{p}{g} \frac{d s}{dz} \frac{p}{p} - s : \]  
(14)

Since \( ! t_c = 1 \), turbulent viscosity significantly reduces \( d h = dz \).

For completeness, we provide an estimate for \( d z = dz \). Combining equations (11), (12), (13), and (14) yields

\[ \frac{d z}{dz} = -ik_h h ; \]  
(15)

which is independent of the magnitude of the turbulent viscosity. The small value of \( d z = dz \) ensures that its contribution to mode damping is negligible, as are the contributions from \( ik_h h \) and \( k \).

\section{Shear Layer at the Base of the Convection Zone}

Here we show that there is a shear layer at the top of the radiative interior across which the horizontal displacement jumps by an amount similar to the depth integrated change it would undergo in an inviscid convection zone.

\(^3\)This is a linearized version of Kelvin’s circulation theorem.

\(^4\)Overtake modes have \( z > z_0 \). These are the only ones that concern us in this paper.

\(^5\)A similar conclusion is arrived at by Brickhill (1990).

\(^6\)We neglect the depth variation of \( h \) with respect to that of \( z \).
We assume that the turbulent viscosity drops discontinuously to zero across the boundary at $z = z_b$. Then the continuity of the tangential stress demands that

$$\frac{d h}{dz} = 0; \quad (16)$$

Next we multiply equation (13) by $p/p$ and integrate the resulting expression from the top of the stellar atmosphere to the bottom of the convection zone. This procedure yields

$$\frac{1}{2} \int_0^{z_b} dz \ h \ i k_b \ h \ \frac{p}{g} \ \frac{p}{p} - g \ z = 0; \quad (17)$$

We transform equation (17) in two steps. Integrating by parts, we obtain

$$\frac{1}{2} \int_0^{z_b} dz \ h \ i k_b \ h \ \frac{p}{g} \ \frac{p}{p} - g \ z = 0; \quad (18)$$

Then using equations (1), (3), and (5), and taking into account the continuity of $p$ and $z$ across $z_b$, we arrive at

$$\frac{1}{2} \int_0^{z_b} dz \ h \ i k_b \ h \ \frac{p}{g} \ \frac{p}{p} - g \ z = 0; \quad (19)$$

Since $t_{cv}$ is nearly constant within the convection zone, we establish the discontinuity of $h$ across $z_b$ to be

$$h = h_{cv} \ \frac{p}{p} - s; \quad (20)$$

It follows directly from equation (3) that

$$p \ \frac{d}{dz} \ h = -i k_b \ h_{cv} \ \frac{p}{p} - s; \quad (21)$$

Equation (21) merits a few comments. It may be simplified by taking advantage of the near constancy of $p/p$ and $s$ inside the convection zone. Both are due to rapid mixing by turbulent convection which occurs on the time-scale $t_{cv}$. Since the term involving the entropy perturbation dominates that due to the unperturbed entropy gradient, and $\frac{1}{2} s \ h \ \frac{p}{p} = p/p$ the fractional change in $h$ across $z_b$ is of order $z_b = z_1$. Thus the relative size of the jump is greatest for long period modes in cool stars. Finally, as advertised, comparison with equation (3) reveals that the discontinuity in $h$ is closely related to the total variation that $h$ would experience within the convection zone in the absence of turbulent viscosity.

3. TURBULENT DAMPING

Damping due to turbulent viscosity is evaluated inside the convection zone in §3.1 and in the region of convective overshoot in §3.2.7. The former is shown to be negligible for all modes. However, the latter may stabilize low frequency modes which would otherwise be overstable.

7We denote by $h$ quantities evaluated on the convective and radiative sides of the boundary.
8We drop the negligible $t_{cv}$ and $-f_{vis}$ terms in equation (3).
9Here we treat the region of convective overshoot as having a finite extent unlike what we did in §2.2.
10It might seem paradoxical, but provided $t_{cv} \ b$, turbulent damping is inversely proportional to the magnitude of the turbulent viscosity. This follows because the dissipation rate is proportional to $(z^2 - t_{cv} \ b) (d \ h/dz)^2$ and $g \ b = dz/\ b \ t_{cv}$ for $t_{cv} \ b$.1

3.1. Inside the Convection Zone

Damping due to turbulent viscosity within the convection zone may be estimated as

$$vis = \frac{1}{2} R^2 \ \frac{z_{cv} \ h}{\ b} \ \frac{d h}{dz} \ \frac{2}{\ b}; \quad (22)$$

Evaluating this integral with only the (dominant) entropy perturbation term from equation (3) retained on the right hand side of equation (14), we obtain

$$vis_{cv} = -0.3 \ \frac{1}{1 + \left(\frac{\ b}{\ b}\right)^2} \ \frac{z_{cv} \ L}{\ b} \ \frac{p}{p} \ \frac{2}{\ b}; \quad (23)$$

where $(p = p_b) = 1/(n + 1)$ is the normalized eigenfunction evaluated at $z_b$. In deriving equation (23), we integrate over an isentropic convection zone with adiabatic index $5/3$ and substitute for $A, B, C, v_{cv}$, and $t_{cv}$ according to Table 1. Radiative damping as evaluated in Paper I yields

$$rad = -0.3 \ \frac{1}{1 + \left(\frac{\ b}{\ b}\right)^2} \ \frac{z_{cv} \ L}{\ b} \ \frac{p}{p} \ \frac{2}{\ b}; \quad (24)$$

Since $t_{cv} = c_0$ 0.1 for DA white dwarfs crossing the ZZ Ceti instability strip, turbulent damping inside the convection zone is much smaller than radiative damping for all overstable g-modes.

It follows from equations (14) and (23) that turbulent damping of a given mode is maximized for $t_{cv} = 1$; it is negligible inside the convection zone because $t_{cv} = 1$ there.4 However, $t_{cv}$ rises with depth in the region of convective overshoot. This suggests that damping in this region may exceed that inside the convection zone.

3.2. In the Region Of Convective Overshoot

The intensity of turbulence in this region diminishes with depth. We estimate its contribution to mode damping by solving a model problem. We replace the bottom of the convection zone by a horizontal plate, and the region of convective overshoot by an underlying viscous fluid of uniform density whose viscosity decays exponentially with depth,

$$(z) = \rho e^{-(z_{cv} - z)/\ b}; \quad (25)$$

The distance below the convection zone over which turbulent mixing maintains the entropy gradient small and negative is of order $\ln(c_{\ e} \ v_{cv})$, or several times as large as $z_{cv}$.

The shear layer is driven by the plate which oscillates at frequency $\ b$; with displacement $\ h_0$ given by (20). The Navier-Stokes equation yields

$$\frac{d}{dz} \ \frac{d h}{dz} - i \ h_0 = 0; \quad (26)$$
which is to be solved subject to the boundary conditions $h = - h$ at $z = z_b$, and $h \neq 0$ as $z \to 1$. Next we define
\begin{equation}
\frac{1 - 2}{b} = 1; \quad (27)
\end{equation}
change independent variable to
\begin{equation}
y = 2 e^{(c_{z_a} z)^2} - 2i = 4; \quad (28)
\end{equation}
and dependent variable to
\begin{equation}
u = e^{-(c_{z_a} z)^2} h; \quad (29)
\end{equation}
With these changes, equation (36) is recast in a more familiar form
\begin{equation}
\frac{d^2 u}{dy^2} + \frac{1}{y} \frac{du}{dy} + \frac{1 - 1}{y^2} u = 0; \quad (30)
\end{equation}
which is Bessel’s equation for $n = 1$. The appropriate solution, which vanishes as $y \to 1$, is $H_1^{(2)} = J_1 - i Y_1$, where $J_1$, $Y_1$, and $H_1^{(2)}$ are the Bessel functions of the first kind, the second kind and the third kind, respectively (cf. Abramowitz & Stegun [1970]). The solution for $h$ may be written as
\begin{equation}
h = D y H_1^{(2)}(y); \quad (31)
\end{equation}
where the coefficient $D$ is evaluated by applying the boundary condition $h = - h$ at $z = z_b$. Since $y_b > 2$, we adopt the approximation $y_b H_1^{(2)}(y_b) = 2i = 1$, which implies
\begin{equation}
D \frac{i b}{2}; \quad (32)
\end{equation}
The viscous stress the plate exerts on the fluid is given by
\begin{equation}
S = i! b \frac{d}{dz} \frac{h}{2}; \quad (33)
\end{equation}
Making use of the identity
\begin{equation}
\frac{d}{dy}(y H_1^{(2)}) = y H_0^{(2)}; \quad (34)
\end{equation}
we find
\begin{equation}
S = - \frac{1}{4} b h^3 y H_0^{(2)}(y_b) 2! + i = 4; \quad (35)
\end{equation}
where $i$ is Euler’s constant.

The power per unit area supplied to the fluid, $P = A$, is equal to the product of $S$ with the plate’s velocity, $i! = h$. To obtain the time averaged rate of energy dissipation, we multiply $P = A$ by $4 R^2$ and average over both time and solid angle to arrive at
\begin{equation}
\frac{dE}{dt} = \frac{R^2}{4} b^3 j_1^3 j h^2 \frac{3}{4}; \quad (36)
\end{equation}
Note that $i dE = dt$ is independent of $b$ and proportional to $a$. This is not surprising. Close to the plate the fluid moves almost rigidly, so the rate of dissipation per unit volume is small, just as it is in the interior of the convection zone. This rate first increases with depth, reaching a peak where the viscous diffusion time is of order $t^{-1}$, and subsequently declines. Near the peak, $t^{-2}$, which implies a dissipation rate per unit volume $b^3 j_1^3 j h^2$. Taking the peak to have width $b$, and including a factor $R^2$ for the integration over area, we recover equation (36) up to a factor of order unity.

To relate $dE/dt$ to the damping rate due to convective overshoot, we make use of equations (31) and (32), and retain only the (dominant) entropy perturbation in equation (36). Adopting values and relations for $A$, $B$, $C$, $t_c$ and $v_{cv}$ from Table 1, we arrive at
\begin{equation}
\frac{dE}{dt} = \frac{1}{1 + (t/4)^2} \frac{L^2}{p^2} \frac{h}{b}; \quad (37)
\end{equation}

Figure 3 displays $dE/dt$ for $dE/dt = 10^{-4}$ for g-modes in stars of $T_{\text{eff}} = 12,400 \, \text{K}$ and $T_{\text{eff}} = 12,000 \, \text{K}$.[11] We choose $a = b = 1$ in order to maximize turbulent damping in the region of convective overshoot.[12] Even with this extreme choice for $a = b$, $dE/dt$ is smaller than $dE/dt$ for all the overstable g-modes.

4. AMPLITUDE LIMITATION DUE TO SHEAR INSTABILITY

Kelvin-Helmholtz instability of the shear layer at the top of the radiative interior is a source of nonlinear mode damping. As such, it acts to limit the amplitudes which overstable modes can achieve. We derive limiting amplitudes by balancing dissipation rates due to shear instability against excitation rates due to convective driving. In this section we deal with physical, as opposed to normalized, perturbations.

Even in carefully controlled laboratory settings, unstable shear layers are complex structures. Those of interest here are further complicated by externally driven turbulence associated with penetrative convection.[13] In particular, turbulent mixing reduces the Brunt-Väisälä frequency while turbulent viscosity lowers the effective Reynolds number. The former reduces stability while the latter enhances it. It is customary to express the nonlinear stress that maintains the shear in terms of the velocity jump across the layer, $v$, and a dimensionless drag coefficient, $C_D$, as (see, e.g., Landau & Lifshitz [1974], Tritton [1977])
\begin{equation}
S = \frac{1}{2} C_D (v)^2; \quad (38)
\end{equation}
$C_D$ depends only weakly on $v$ in turbulent shear layers. Terrestrial experiments indicate that $C_D$ varies logistically with the ratio of wall roughness to boundary layer width, with values as small as $10^{-3}$ being characteristic of flows over smooth walls. We might speculate that penetrative convection makes the upper boundary of the shear layer behave like a rough wall. Since we have no physical basis for assigning a reliable value to $C_D$, we treat it as a free parameter subject to the constraint $10^{-3} \leq C_D \leq 10^{-1}$.

The power per unit area input to the shear layer is $P = A = S v$. To obtain the time average rate of energy dissipation in the shear layer, we multiply $P = A$ by $4 R^2$, and carry out appropriate averages over both time and solid angle.[14] Next we set

[11] The DA white dwarf models used in this paper are provided by P. Bradley. For model details, see Bradley [1998].

[12] Helioseismology constrains the entire depth of the region of convective overshoot in the sun to be less than 0.05 pressure scale-heights (Basu [1997]).

[13] We distinguish turbulence associated with convective overshoot from that due to instability of a g-mode's velocity shear.

[14] The velocity jump associated with a g-mode shear layer has an angular dependence and varies harmonically with time.
FIG. 1.— Numerical solution to the toy model in §3.1 for \( \eta = 0 \). From top to bottom the panels display scaled dimensionless versions of the viscosity, the magnitude of the horizontal displacement, \( \eta \) the magnitude of the principal component of shear, \( \eta \) the local Reynolds number, \( \Omega \) and the viscous rate of energy dissipation per unit mass, \( \frac{dE}{dt} \), as functions of distance below \( z_b \) measured in units of \( \lambda \). Each of the curves plotted in the bottom four panels is a function of the single variable \( \eta \) and the parameter \( \eta \). Moreover, the dependence is weak (logarithmic) for \( \eta \). Note that the shear, the Reynolds number, and the rate of energy dissipation all peak near the depth where \( \eta \) = 1.

FIG. 2.— Ratio of the rate of linear turbulent damping in the overshoot region to the rate of radiative damping for two white dwarf models with \( T_{\text{eff}} = 12,000 \text{K} \) (solid triangles) and 12,000 K (open squares), plotted against mode period for \( \eta \) = 1 modes. Here, \( \eta \) is taken to equal \( z_b \) to maximize the effect of overshoot damping.

\( v = \eta \) where \( \eta \) is evaluated using equation (7), (8).
and (20) as in §3.2. This series of steps leads to

$$\frac{dE_{\text{vis}}}{dt} = \frac{0 \, dG_0}{[(1 \, e)^2 + 1]^{3/2}} \frac{z_b}{k_b \zeta^2} \frac{p}{p_b} \frac{3}{p} : \quad (39)$$

The saturation amplitude is obtained by balancing the damping due to shear instability by the net rate of energy gain due to convective driving plus radiative damping (see Paper I),

$$\frac{dE_{\text{net}}}{dt} = 0 \, dG_0 \frac{[(1 \, e)^2 - 1]}{[(1 \, e)^2 + 1]} \frac{L}{p} \frac{2}{p_b} : \quad (40)$$

This yields

$$\frac{p}{p_b} = \frac{0 \, d \left[ (1 \, e)^2 + 1 \right]^{1/2}}{C_D} \frac{1}{(1 \, e)^2 - 1} \frac{k_b \zeta^2}{z_b} : \quad (41)$$

Amplitudes of overstable modes limited by shear instability exhibit the following patterns. In a given star, \(( p=p_b)\) declines sharply with increasing period. For a fixed mode, the value of \(( p=p_b)\) rises as the host star cools. The fractional flux perturbation at the base of the convection zone is about \(2(c^2 - 1)\). The flux perturbation at the photosphere is smaller by the viscosity reduction factor \(1/c\). Realistic estimates of mode amplitudes must await consideration of additional amplitude limiting processes and the calculation of nonadiabatic growth rates.

Is it safe to assume, as we have been doing, that the shear layer is turbulent? Necessary conditions are that the unperturbed shear layer has both local shear large compared to \(c\) and local Reynolds number large compared to unity. These conditions can be expressed by the dimensionless relations \(\delta \gamma = dz \gamma\) and \(Re = (1 \, h^2) = (\delta \gamma = dz \gamma) 1\). Based on the discussion in §3.2, we know that the left hand side of each inequality attains a peak value \(h^2 = \gamma \zeta = R\) near to where \(2(c^2 - 1)\) (see Fig. 1). Thus the assumption of turbulence for low \(\gamma\) modes requires \(( p=p_b)\) \((z_b = c1) = R\). The right hand side of this inequality is of order \(z_b = R\) \(10^4\) for lowest frequency modes detected in stars near the red edge of the instability strip.

### 5. SUMMARY

Brickhill (1990) discussed the manner in which turbulent convection affects \(g\)-modes in ZZ Ceti stars. He recognized that turbulent viscosity forces the horizontal velocity to be nearly independent of depth inside the convection zone. Moreover, he deduced that turbulent damping inside the convection zone is negligible whereas that in the region of convective overshoot might stabilize long period modes near the red edge of the instability strip. Our investigation supports Brickhill’s conclusions, although it suggests that turbulent damping by penetrative convection is, at best, of minor significance.

We extend the investigation of turbulent dissipation to include that due to instability in the shear layer at the top of the radiative interior. Turbulence generated by Kelvin-Helmholtz instability depends upon the velocity shear, so the rate of energy dissipation depends nonlinearly on the mode energy; \(dE/dt = E^3/2\). Thus shear instability provides an amplitude saturation mechanism for overstable modes.

The applicability of results from this paper to ZZ Cetis must be viewed with caution. The quasidiabatic approximation, while useful, is of limited validity. Fully nonadiabatic results will be presented in Wu & Goldreich (1998).

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