Delayed feedback control in quantum transport

Clive Emary

Institut für Theoretische Physik, Hardenbergstr. 36,
TU Berlin, D-10623 Berlin, Germany
(Dated: May 2, 2014)

Abstract

Feedback control in quantum transport has been predicted to give rise to several interesting effects, amongst them quantum state stabilisation and the realisation of a mesoscopic Maxwell’s daemon. These results were derived under the assumption that control operations on the system be affected instantaneously after the measurement of electronic jumps through it. In this contribution I describe how to include a delay between detection and control operation in the master equation theory of feedback-controlled quantum transport. I investigate the consequences of delay for the state-stabilisation and Maxwell’s-daemon schemes. Furthermore, I describe how delay can be used as a tool to probe coherent oscillations of electrons within a transport system and how this formalism can be used to model finite detector bandwidth.
The feedback control of quantum transport has recently been predicted to give rise to several interesting effects such as the freezing of current fluctuations \cite{3}, stabilisation of quantum states \cite{13, 18} and the realisation of a mesoscopic Maxwell’s daemon \cite{20}. In all these schemes, the feedback was of Wiseman-Milburn type, in which the control operations are performed instantaneously and directly after quantum jumps of the system \cite{23, 24}. The aim of this contribution is to analyse the effects of delay in the feedback control of quantum transport. In particular, we are interested in when the control operations follow the jump processes not directly, but rather after some time delay, be it originating from the finite-response time of feedback hardware, or introduced deliberately into the control loop.

Away from a transport setting, delay in quantum feedback has been considered by a number of authors, e.g. Amini et al. \cite{1}, Combes and Wiseman \cite{5}, Combes et al. \cite{6}, Nishio et al. \cite{17}. In particular, Wiseman \cite{23} considered, at a formal level, delay in Wiseman-Milburn control and has derived a delayed-feedback quantum master equation (QME) in the limit of small delay time. Here I rederive this delayed QME using an alternative approach that makes it clear that this equation can actually be valid for arbitrary delay, provided one assumes that if a jump occurs within the delay time of a previous control operation, then the interrupted control operation is skipped. This delayed control scheme results in a non-Markovian master equation which, by construction, has well-behaved solutions. To enable the calculation of transport properties, I generalise this result to make connection with full-counting statistics (FCS) of electron transfer \cite{14} by inclusion of counting fields (see Poeltl et al. \cite{18} for overview of this procedure without delay).

With this formalism in place, I investigate the consequences of feedback delay for the state-stabilisation of Poeltl et al. \cite{18} and Maxwell’s daemon of Schaller et al. \cite{20}. As an example of how delay in control need not necessarily be a negative thing \cite{21, 22}, I describe how a deliberately-swept delay can be used as a tool to probe coherent oscillations of electrons within a transport system. Finally, I show briefly how this delay formalism can be used to model a finite-bandwidth electron-counting detector.

I. DELAYED FEEDBACK IN THE QUANTUM MASTER EQUATION

Let us consider a transport system described by the QME, $\dot{\rho} = \mathcal{W} \rho$, with $\rho = \rho(t)$ the reduced density matrix of the system (e.g. quantum dot) at time $t$ and superoperator $\mathcal{W}$ the Liouvillian of the system \cite{2}. Let the Liouvillian be decomposed as $\mathcal{W} = \mathcal{W}_0 + \mathcal{J}$, where $\mathcal{J} = \sum_\alpha \mathcal{J}_\alpha$ describes quantum jump processes — in particular those in which electrons enter and leave the system —, and $\mathcal{W}_0$ describes the evolution without jumps. In terms of the Hilbert space jump operators $L_\alpha$, we have $\mathcal{W}_0 \rho = -i[H, \rho] - \frac{1}{2} \{\sum_\alpha L_\alpha^\dagger L_\alpha, \rho\}$ and $\mathcal{J}_\alpha \rho = L_\alpha \rho L_\alpha^\dagger$, such that $\mathcal{W}$ is of Lindblad form.

An intuitive picture of feedback control can be obtained by considering the solution of the QME in terms of quantum trajectories \cite{4}:

$$\rho(t) = \sum_{n=0}^{\infty} \int_0^t dt_n \cdots \int_0^{t_2} dt_1 \Omega_0(t_n - t_{n-1}) \mathcal{J} \Omega_0(t_2 - t_1) \mathcal{J} \Omega_0(t_1) \rho(0),$$

with $\Omega_0(t) = e^{\mathcal{W}_0 t}$. A sketch of a trajectory for transport through a quantum dot is shown in Fig.4.

Feedback control can be added to this scheme by considering that after each jump $\mathcal{J}_\alpha$ we operate on the system with control operator $C_\alpha$, assumed instantaneous. In standard
Wiseman-Milburn control, each jump is followed immediately with the control operation. The trajectories including control operations can then be resummed, such that the density matrix evolves under the action of the modified control Liouvillian $\mathcal{W}_C = \mathcal{W}_0 + \sum \alpha \mathcal{C}_\alpha \mathcal{J}_\alpha$.

We now want to consider a delay between jump $\mathcal{J}_\alpha$ and corresponding control operation $\mathcal{C}_\alpha$. It would seem a simple matter to simply insert the control operations into each trajectory a time $\tau$ after each jump. However, in doing so, one encounters a problem: what happens when a second jump occurs within the delay time of the first? If we assume that the control operation is applied regardless of whether this occurs or not, the trajectories become complicated and cannot be resummed as a master equation. To avoid this, we assume that in the case where a jump occurs in the delay time of a previous jump, the control operation of the first jump is simply skipped. In other words, if the time between two subsequent jumps is shorter than the delay time, no control operation for the first jump is performed. This I shall call the control-skipping assumption. This assumption is plausible: it is not hard to imagine that the feedback electronics be reset by the arrival of a second jump as they prepare to enact the control operation of the previous. Certainly, in the case where the delay is introduced deliberately, this feedback scheme could be arranged.

With this assumption, addition of delayed feedback control is effected by the replacement of the no-jump propagators following jumps $\mathcal{J}_\alpha$ as

$$
\Omega_0(t)\mathcal{J}_\alpha \rightarrow \begin{cases} 
\Omega_0(t)\mathcal{J}_\alpha & t < \tau_\alpha \\
\Omega_0(t-\tau_\alpha)\mathcal{C}_\alpha \Omega_0(\tau_\alpha)\mathcal{J}_\alpha & t \geq \tau_\alpha 
\end{cases},
$$

throughout all trajectories. In the first instance, time between the jumps is too short for the delayed control to be implemented, in the second line, there is sufficient time and the control operation is implemented. For the sake of generality, we have included here a subscript on the delay time such that each operation may have its own associated delay.

With this replacement, the controlled trajectories can still be resummed such that the
density matrix in Laplace space reads

$$\rho(z) = \int_0^\infty dt e^{-zt} \rho(t) = \frac{1}{z - W_0 - \sum \alpha D_\alpha(z) J_\alpha} \rho_0,$$

with delayed-control superoperator

$$D_\alpha(z) = 1 + (C_\alpha - 1) e^{(W_0 - z) \tau_\alpha}.$$  

Translating back into time-domain, we obtain the nonMarkovian QME

$$\dot{\rho}(t) = \int_0^t dt' \left[ W_0 \delta(t - t') + \sum \alpha D_\alpha(t - t') J_\alpha \right] \rho(t'),$$

with $D_\alpha(t) = \delta(t) + (C_\alpha - 1) e^{W_0 \tau_\alpha} \delta(t - \tau_\alpha)$. Evaluating the delta-functions we obtain the delayed QME:

$$\dot{\rho}(t) = W \rho(t) + \sum \alpha (C_\alpha - 1) e^{W_0 \tau_\alpha} J_\alpha \theta(t - \tau_\alpha) \rho(t - \tau_\alpha),$$

in which the time evolution of the density matrix $\rho(t)$ depends not only on the state of the system at time $t$ but also at previous times $\{t - \tau_\alpha\}$. The $\theta$-functions that accompany the delayed terms mean that, in accordance with the construction of this master equation from its specific solution, the history of the system up to time $t = 0$ is not needed. Furthermore, knowledge of the explicit solution allows us to conclude that no positivity or normalisation issues arise with this particular nonMarkovian QME. Eq. (6) is a slight generalisation of the form given in Wiseman [23]. There, this expression was derived in the framework of the stochastic Schrödinger equation as being valid only to first order in the delay time(s). The foregoing shows that Eq. (6) is actually valid for arbitrary delay, provided the additional control-skipping assumption is made.

To facilitate the calculation of the FCS of transport processes, we introduce the counting fields $\{\chi_\alpha\}$ associated with tunneling of electrons into/out of leads $\{\alpha\}$. With counting fields, and assuming that transport into each lead is unidirectional (infinite bias limit), the nonMarkovian Laplace-space delayed-control Liouvillian reads

$$W_{DC}(\chi, z) = W_0 + \sum \alpha D_\alpha(\chi, z) J_\alpha e^{i\chi_\alpha},$$

with delayed control operation

$$D_\alpha(\chi, z) = 1 + [C_\alpha(\chi) - 1] e^{(W_0 - z) \tau},$$

where $\chi$ without subscript refers to the complete set of counting fields. Note that in general the control operations $C_\alpha$, and hence $D_\alpha$, can transfer electrons and thus depend on the counting fields. Calculating the transport properties of nonMarkovian QMEs is well understood, see e.g. Emary and Aguado [7], Flindt et al. [10]. Providing that we start counting at $t = 0$, no inhomogeneous term is required in the QME.
FIG. 2. The length of the stationary Bloch-vector of the DQD transport qubit with feedback control as a function of delay time $\tau$. Results for two strategies of choosing control parameters are shown here: oblivious (black solid line) and optimised (blue dashed line). The value of $|\langle \sigma \rangle|$ without feedback control is shown (green dotted line). The parameters were $\epsilon = 0$ and $T_c = 0.24 \Gamma_R$.

II. NONEQUILIBRIUM STATE STABILISATION

As first application, let us re-analyse the feedback stabilization protocol of Poeltl et al. [18], this time with delay. The system consists of a double quantum dot (DQD) described by the three states: ‘empty’ $|0\rangle$ and left- and right- occupied states, $|L\rangle$ and $|R\rangle$. The Hamiltonian of the DQD reads $H_{\text{DQD}} = \frac{1}{2} \epsilon \sigma_z + T_C \sigma_x$ with pseudospin operators $\sigma_z = |L\rangle\langle L| - |R\rangle\langle R|$, $\sigma_x = |L\rangle\langle R| + |R\rangle\langle L|$. The two transport processes are tunneling into the DQD from the left lead, $L_L = \sqrt{\Gamma_L} |L\rangle\langle 0|$, and out to the right, $L_R = \sqrt{\Gamma_R} |0\rangle\langle R|$. We consider only the $\epsilon = 0$ case in the following.

In Poeltl et al. [18], the control operation was chosen as a coherent qubit rotation in $x$-$z$ plane conditioned on the tunnel of an electron into the DQD:

$$C_L = \exp \{ \theta_C [\sin \theta \Sigma_x + \cos \theta \Sigma_z] \}; \quad C_R = 0,$$

with rotations induced by Pauli matrices $\Sigma_\alpha \rho = -i [\sigma_\alpha, \rho]$. By choosing control parameters $\theta_C$ and $\theta$ correctly, this control operation was used to rotate the state of the incoming electron into an eigenstate of the effective Hamiltonian $\tilde{H} = H_{\text{DQD}} - \frac{i}{2} \sum_\alpha L_\alpha^\dagger L_\alpha$, a state protected from further evolution until the next jump. In the limit $\Gamma_L \rightarrow \infty$ where state $|0\rangle$ can be eliminated (the “transport qubit” limit), the system effectively spends all its time in this state, which is thus stabilised.

Introducing delay through the application of Eq. (7) and Eq. (5) we obtain the delayed-control kernel

$$\mathcal{W}_{DC}(\chi, z) = \mathcal{W}_0 + \mathcal{J}_R e^{i \chi_R} + \mathcal{D}_L(z) \mathcal{J}_L e^{i \chi_L},$$

with a delayed control operation that does not depend on counting fields

$$\mathcal{D}_L(z) = 1 + (C_L - 1) e^{(\mathcal{W}_0 - z) \tau}.$$
The central question is to what degree can the stationary state of the system (obtained from $W_{DC}(\chi = 0, z = 0)\rho_{stat} = 0$) be purified by feedback in the presence of delay. To this end we consider three strategies for choosing the control parameters:

- We simply use the values from the stabilisation protocol without delay, a strategy dubbed ‘oblivious control’ in Combes and Wiseman \[5\];
- We choose the angles such that the purity is maximised for a given delay time;
- Without delay, state stabilisation is correlated with Poissonian statistics of the transport process. We can therefore attempt to use this same criterion to set the control parameters in the presence of delay.

Results for the first two strategies are shown in Fig. 2, where we plot as a function of delay time the length of the Bloch vector of the stationary state, a measure of the states purity ($|\sigma| = 1$ for a pure state). Delay clearly serves to reduce the purity of the end state. With oblivious control at small delay times, the length of the Bloch vector drops linearly:

$$|\langle \sigma \rangle| \sim \begin{cases} 
1 - (\Gamma_R - 8T_c^2/\Gamma_R - \kappa)\tau & 4T_C < \Gamma_R \\
1 - \Gamma_R\tau/2 & 4T_C > \Gamma_R
\end{cases}$$

where the two cases arise from a bifurcation in the nature of the stabilisable states without delay. In the presence of delay, feedback-control only offers a purity improvement over the non-controlled steady-state for delays $\Gamma_R\tau \lesssim 0.2$. The reduction of the purity is strongest around $\tau\Gamma_R \sim 1$. For large delays $\tau\Gamma_R \gg 1$, the stationary state with control reverts to that without since, in this limit, the probability that successive jumps occur within the delay time is high and the control operation is only very rarely enacted.

Explicitly choosing the control parameters to maximise the purity shows a marked increase in the purity over the oblivious approach. As Fig. 2 shows, the the purity of the controlled state with strategy is significantly lies above that of its uncontrolled counterpart for most values of the delay.

Fig. 3 investigates the strategy of choosing the control angles based on the the FCS. Here, we just look for a (shotnoise) Fano factor $F$ equal to unity (the Poissonian value) to adjudge this. For each value of $\tau$ there exists multiple choices of control parameters that give $F = 1$. For small delay times $\Gamma_r \ll 1$, two of these solutions have high purity and are continuous with the stabilisable states of the $\tau = 0$ system. For small delays, relying on the FCS to locate useful control parameters remains a valid strategy. However, above a certain delay time, $\tau \gtrsim 0.1$ in Fig. 3a, these high-purity solutions disappear and selecting for $F = 1$ drives the system into one of two highly mixed states. This behaviour is explained in Fig. 3b. Away from the small $\tau$ limit then, the zeroes of $F - 1$ are unrelated with any kind of purity optimisation and this criterion should be avoided.

### III. Maxwell’s Daemon

Schaller et al. \[20\] described how a single-electron transistor (SET) with feedback can act as a Maxwell’s Daemon and transfer charge against a voltage gradient whilst no net work is performed on the system. Several feedback schemes were discussed in this work, but here I shall just discuss ‘Scheme IIa’ (details below) since this is of the appropriate type for our delay treatment.
FIG. 3. State stabilisation with control parameters based on the FCS. (a): Length of the Bloch vector as a function of delay time. Here the control angle $\theta$ was fixed at at a value $\theta = \pi/2$ (the non-delayed choice) and angle $\theta_C$ was chosen such that the Fano factor of transferred charge was exactly unity. For each value of the delay, there are several choices of control angles which give $F = 1$ and hence multiple solutions are plotted. The topmost two solutions correspond to the two stabilised pure states for $\tau \to 0$. Only for $\Gamma_R \tau \ll 0.1$ do there exist control parameters for which $F - 1$ is zero and the stationary state has a high purity. (b): This behaviour can be explained by considering the length of Bloch vector and Fano factor as functions of control angle $\theta_C$ (with $\theta = \pi/2$). Results are plotted for zero and finite ($\tau = 0.2\Gamma_R$) delays. At zero delay, the function $F - 1$ crosses the zero axis in four places; two of these solutions correspond to maximum purity states; the remaining two, low purity states. For larger delays, $\tau \gtrsim 0.1\Gamma_R$, the leftmost maximum of $F - 1$ drops below zero and the two high-purity unit-Fano-factor solutions disappear. Parameters were as Fig. 2.

The SET model consists of a quantum dot with just two states: ‘empty’, $|0\rangle$, and ‘full’, $|1\rangle$, connected to two leads at finite bias and temperature. A quantum point contact is used to monitor the charge state of the dot and without feedback, the system Liouvillian reads

$$W(\chi_L, \chi_R) = W_0 + J_I(\chi_L, \chi_R) + J_O(\chi_L, \chi_R)$$

where counting fields $\chi_{L,R}$ keep track of electron movements through the left and right barriers of the dot. Since the QPC can only tell us the occupation of the dot (but not e.g. from which lead the electron has tunneled), the two jump super-operators on which the control scheme is based are $J_I$ and $J_O$, which describe inward and outward jumps respectively. In the basis $\{|0\rangle, \{1\rangle\}$ we have

$$W_0 = \sum_\alpha \Gamma_\alpha F^0_\alpha; \quad J_I(\chi_L, \chi_R) = \sum_\alpha \Gamma_\alpha F^-_\alpha e^{-i\chi_\alpha}; \quad J_O(\chi_L, \chi_R) = \sum_\alpha \Gamma_\alpha F^+_\alpha e^{i\chi_\alpha};$$

$$F^0_\alpha = \begin{pmatrix} -f_\alpha & 0 \\ 0 & (1 - f_\alpha) \end{pmatrix}; \quad F^+_\alpha = \begin{pmatrix} 0 & (1 - f_\alpha) \\ 0 & f_\alpha \end{pmatrix}; \quad F^-_\alpha = \begin{pmatrix} f_\alpha & 0 \\ 0 & 0 \end{pmatrix},$$

where $\Gamma_\alpha$ is the tunnel rate associated with lead $\alpha = L, R$, and where $f_\alpha$ is the corresponding Fermi function, $f_\alpha = [e^{\beta(\epsilon - \mu_\alpha)} + 1]^{-1}$ with $\epsilon$ the dot level energy, $\mu_\alpha$ the chemical potential of lead $\alpha$ and $\beta = 1/kT$ the inverse thermal energy.
FIG. 4.  (a) Equilibrium current through the Maxwell daemon SET with delayed feedback as a function of delay time $\tau$. Results for several values of the Fermi function $f$ are shown. For $f = 1/2$, the decay of the current with $\tau$ is purely exponential.  (b) Power generated by the Maxwell daemon SET as a function of bias $eV$ applied symmetrically about the dot level: $\mu_L - \epsilon = \epsilon - \mu_R = eV/2$. Results for several values of the delay $\tau$ are shown (solid lines) and the maximum power point indicated (red dashed line). In both cases, the parameters were $\delta^I_R = \delta^O_R = \delta \rightarrow \infty$, $\delta^I_L = \delta^O_L = 0$ and $\Gamma_L = \Gamma_R = \Gamma$.

The control scheme IIa of Schaller et al. [20] is, on detection of a tunnel event either into or out of the system, to change the barrier heights and return them instantaneously. The corresponding control operations read

$$C_{I/O}(\chi_L, \chi_R) = \exp \left( \sum \alpha \delta^{I/O}_\alpha \left( F^0_\alpha + F^-_\alpha e^{-i\chi_\alpha} + F^+_\alpha e^{i\chi_\alpha} \right) \right),$$

where parameters $\delta^{I/O}_\alpha$ describe the ‘strength’ of the feedback transition involving lead $\alpha$ given an in/out jump. For simplicity, let’s use the “maximum feedback” case and set $\delta^I_R = \delta^O_R = \delta \rightarrow \infty$ and $\delta^I_L = \delta^O_L = 0$. We will also consider a symmetric SET, $\Gamma_L = \Gamma_R = \Gamma$. With delay, then, the controlled Liouvillian reads

$$\mathcal{W}_{DC}(\chi_L, \chi_R, z) = W_0 + \sum_{\alpha = I/O} D_\alpha(\chi_L, \chi_R, z) \mathcal{J}_\alpha(\chi_L, \chi_R)$$

with

$$D_{I/O}(\chi_L, \chi_R, z) = 1 + (C_{I/O}(\chi_L, \chi_R) - 1) e^{(W_0 - z)\tau}. \tag{17}$$

These equations are slightly different to Eq. (7) and Eq. (8) since here we have finite bias and bidirectional tunneling. The inclusion of delay is directly analogous, however.

Even with delayed feedback, the current through the SET can be obtained analytically. Let us first consider equilibrium conditions such that $f_L = f_R = f$. In this case, the current through the SET is

$$\langle I \rangle = \Gamma f (1 - f) \frac{f e^{2\Gamma \tau} - 2 f (1 - f) e^{2f \Gamma \tau} + (1 - f) e^{4f \Gamma \tau}}{e^{2(1 + f) \Gamma \tau} - f (1 - f) (e^{2f \Gamma \tau} + e^{4f \Gamma \tau})}. \tag{18}$$

8
This result is plotted in Fig. 4a. With the dot level placed on resonance with the chemical potential of the leads, we have \( f = 1/2 \) and the current assumes the simple form

\[
\langle I \rangle = \frac{\Gamma}{4} e^{-\Gamma \tau},
\]

such that it is clear that the current is exponentially suppressed by the delay. For \( \Gamma \tau \ll 1 \), however, the induced current remains close to the \( \tau = 0 \) value. The behaviour for \( f \neq 1/2 \) is a little more complicated but the basic trend is the same.

Away from equilibrium, the operation of the daemon may be assessed by considering the power generated by the device, \( P \equiv -\langle I \rangle V \). This power is shown for a symmetric bias configuration in Fig. 4b for the case of maximum feedback. Irrespective of the value of \( \tau \), this function shows a single maximum as a function of bias. Without delay, the maximum power generated by the device is obtained numerically as \( P \approx 0.084 \Gamma kT \). With increasing delay, the maximum power decreases approximately exponentially. The bias at which this maximum is reached also moves towards zero. For a delay time \( \Gamma \tau = 1 \) (which represents a large delay), the maximum power is \( P \approx 0.014 \Gamma kT \).

### IV. PROBING COHERENT OSCILLATIONS WITH DELAYED FEEDBACK

Whilst the previous two examples illustrate the negative effects of delay on previously established feedback schemes, delay may also be used constructively. In this section we investigate coherent oscillations of a DQD with delayed feedback as our probe. The model without control is the same as in Sec. II. The feedback scheme we use is to detect jumps through the left barrier and conditionally apply the control operation \( e^{K_R(\chi_R)} \rho = A \left( |0\rangle\langle R| \rho |R\rangle\langle 0| e^{i\chi_R} - \frac{1}{2} |R\rangle\langle R| \rho - \frac{1}{2} \rho |R\rangle\langle R| \right) \), which constitutes an instantaneous lowering and restoration of the right barrier. Parameter \( A \) is a measure of the feedback strength.

Fig. 5 shows the stationary current through the DQD as a function of delay time. When the ratio of right to left tunnel rates is small enough, the current shows a pronounced series of peaks that occur when the delay time is equal to odd-integer multiples of \( \pi/\Omega/ \). These peaks arise when the control operation is enacted just as the electron has completed a half-integer number of coherent oscillations starting from the left dot-state. The current consists of two components: that which occurs without control, and an extra component induced by the feedback operation. As \( \Gamma_R \) is decreased, the non-feedback component is reduced and the oscillations become more pronounced. Furthermore, since the only source of dephasing in this model is the coupling to the right lead, decreasing \( \Gamma_R \) also decreases this dephasing and this accounts for the decreased damping that accompanies the increased visibility of the oscillations.

It is clear then that this delayed-feedback scheme allows us to image the coherent oscillations taking place within the DQD. More generally, such schemes provide a way to investigate oscillatory behaviour in other transport systems. In this sense, delayed-feedback plus current measurement can provide an additional method for studying transport dynamics, complementary to the finite-frequency current correlations \( \[8, 13\] \) or pulsed operation \( \[12\] \).
FIG. 5. Stationary current through feedback-controlled DQD as a function of delay time, $\tau$. The delayed control scheme is such that a time $\tau$ after an electron tunnel into the system from the left, the right barrier is dropped and returned instantaneously. A series of peaks in the current is observed when the delay time is equal to odd-integer multiples of $\pi/\Omega$, the time taken for the electron to be coherently transferred across the DQD. Parameters: $\epsilon = 0$, $T_C = 3\Gamma_L$ and dimensionless feedback strength $A = 4$

V. MODELLING A FINITE-BANDWIDTH DETECTOR

The above delay formalism can be adapted to model the effects of a finite-bandwidth detector. In a generic (unidirectional) Liouvillian without control each jump operator $J_\alpha$ for which electrons are being counted is immediately followed by a counting-field factor $e^{i\chi_\alpha}$. This can be interpreted as the detector reacting instantaneously to the occurrence of a system jump. More realistic is that the detector takes a finite time $\tau$ to react such that jumps go undetected if several occur within this detector reaction time. This situation is very similar to the delayed-feedback situation described above but instead of having a control operator act at a time $\tau$ after the jump, we have a detection event described by a counting-field factor. The Liouvillian for this situation may thus be written

$$W_{BW}(\chi, z) = W_0 + \sum_\alpha D_\alpha (\chi_\alpha, z) J_\alpha,$$

(20)

with delayed counting factor

$$D_\alpha (\chi_\alpha, z) = 1 + (e^{i\chi_\alpha} - 1)e^{(W_0-z)\tau_\alpha}.$$  

(21)

The equations are the same as Eq. (7) and Eq. (8), but here the counting-field factors occurs not in Eq. (20) but rather in place on the control operation in Eq. (21).

As example, let us consider the SET of section III (without control) in the infinite bias limit $f_L \rightarrow 1$, $f_R \rightarrow 0$. For symmetric rates, $\Gamma_L = \Gamma_R = \Gamma$, the current detected by counting electrons with a detector reaction time of $\tau$ flowing through the SET reads

$$\langle I \rangle_{\text{detected}} = \frac{1}{2} \Gamma_R e^{-\Gamma_R\tau}.$$  

(22)
This shows an exponential suppression due to the detector lag over the actual current flowing, which is $\langle I \rangle = \Gamma_R/2$. With a reliable detector, we should have $\Gamma_R \tau \ll 1$ and the detected current reads $\langle I \rangle_{\text{detected}} \approx \frac{1}{2} \Gamma_R (1 - \Gamma_R \tau)$.

These results can be compared with the “detector-state model” [9, 11, 16]. Using an additional detector degree of freedom with detector switching rate $\Gamma_D$, the detected current of the symmetric SET was found to be

$$\langle I \rangle_{\text{detected}} = \frac{1}{2} \Gamma_R \frac{k}{1 + k}; \quad k = \frac{\Gamma_D}{2 \Gamma_R}$$  \hspace{1cm} (23)

For a fast detector, $\Gamma_D \gg \Gamma_R$, we may approximate $\langle I \rangle_{\text{detected}} \approx \frac{1}{2} \Gamma_R \left(1 - \frac{2 \Gamma_R}{\Gamma_D}\right)$. Thus, identifying the parameters of these two detector models as $\tau = 2/\Gamma_D$, the descriptions of the behaviour in the experimentally important regime are consistent. For larger values of $\tau$, the two models differ: the delayed-counting model predicts an exponential decay of the current whereas Eq. (23) predicts an algebraic one.

VI. CONCLUSIONS

The delay formalism described here contains two effects: the obvious one that the control operations follow a time $\tau$ after a jump, but also that control operations are rejected when the time between jumps is shorter than the delay time. This second means that in the limit $\tau \Gamma \to \infty$ (with $\Gamma$ the typical rate of the jump processes) the feedback control is completely frozen out. Whilst it is interesting to try to relax this second control-skipping assumption, without it, a master equation description (even a nonMarkovian one) is not possible. Such a control scheme could however be readily simulated. Extension to piecewise-constant control schemes, such as Scheme I of Schaller et al. [20] and Schaller [19] should also be possible.

Delay has been seen here to have a negative impact on the stabilisation and Maxwell daemon schemes, reducing the purity of the stabilised state and the power production, respectively. For small delay times, however, good results are still obtainable. Interestingly, the quantum stabilisation scheme appears to be impacted more severely than the effectively-classical Maxwell’s daemon. The purification effect based on FCS detection disappears completely for $\Gamma \tau \gtrsim 0.1$, whereas the Maxwell Daemon is capable of producing some power even with a very poor detector $\Gamma \tau > 1$. The power drops off exponentially with increasing $\Gamma \tau$, though.

In a more positive sense, we have shown how a deliberately-delayed control scheme introduces a extra time-scale into the system, which can be used as a probe of the transport dynamics.

Whilst we have concentrated on feedback control of quantum transport, these results should also be applicable to other systems, e.g. quantum optics.

Acknowledgements

I am grateful to T. Brandes, Gerold Kießlich, Christina Pöltl, and Gernot Schaller for useful discussions. This work was funded by the Deutsche Forschungsgesellschaft through
Amini, H., Somaraju, A., Dotsenko, I., Sayrin, C., Mirrahimi, M. and Rouchon, P. 2012 Feedback stabilization of discrete-time quantum systems subject to non-demolition measurements with imperfections and delays [arXiv:1201.1387].

Brandes, T. 2005 Coherent and collective quantum optical effects in mesoscopic systems Physics Reports 408, 315 - 474.

Brandes, T. 2010 Feedback Control of Quantum Transport Phys. Rev. Lett. 105, 060602.

Carmichael, H. 1993 An Open Systems Approach to Quantum Optics Springer.

Combes, J. & Wiseman, H. M. Quantum feedback for rapid state preparation in the presence of control imperfections 2011 J. Phys. B: At. Mol. Opt. Phys. 44, 154008.

Combes, J., Wiseman, H. M. & Scott, A. J. 2010 Replacing quantum feedback with open-loop control and quantum filtering Phys. Rev. A 81, 020301.

Emary, C. & Aguado, R. 2011 Quantum versus classical counting in non-Markovian master equations Phys. Rev. B 84, 085425.

Emary, C., Marcos, D., Aguado, R. & Brandes, T. 2007 Frequency-dependent counting statistics in interacting nanoscale conductors Phys. Rev. B 76, 161404R.

Flindt, C., Braggio, A. & Novotný, T. 2007 Non-Markovian dynamics in the theory of full counting statistics. In AIP Conference Proceedings (ed. Tacano, M., Yamamoto, Y. & Nakao, M.), AIP 922, 531-534.

Flindt, C., Novotný, T., Braggio, A., Sassetti, M. & Jauho, A.-P. 2008 Counting Statistics of Non-Markovian Quantum Stochastic Processes Phys. Rev. Lett. 100, 150601.

Gustavsson, S., Leturcq, R., Ihn, T., Ensslin, K., Reinwald, M. & Wegscheider, W. 2007 Measurements of higher-order noise correlations in a quantum dot with a finite bandwidth detector Phys. Rev. B 75, 075314.

Hayashi, T., Fujisawa, T., Cheong, H. D., Jeong, Y. H. & Hirayama, Y. 2003 Coherent Manipulation of Electronic States in a Double Quantum Dot Phys. Rev. Lett. 91, 226804.

Kießlich, G., Schaller, G., Emary, C. & Brandes, T. 2011 Charge Qubit Purification by an Electronic Feedback Loop Phys. Rev. Lett. 107, 050501.

Levitov, L. S., Lee, H. & Lesovik, G. B. 1996 Electron counting statistics and coherent states of electric current J. Math. Phys. 37, 4845-4866.

Marcos, D.; Emary, C.; Brandes, T. & Aguado, R. 2010 Finite-frequency counting statistics of electron transport: Markovian theory New J. Phys. 12, 123009.

Naaman, O. & Aumentado, J. 2006 Poisson Transition Rates from Time-Domain Measurements with a Finite Bandwidth Phys. Rev. Lett. 96, 100201.

Nishio, K., Kashima, K. & Imura, J. Effects of time delay in feedback control of linear quantum systems 2009 Phys. Rev. A 79, 062105.

Pöltl, C., Emary, C. & Brandes, T. 2011 Feedback stabilization of pure states in quantum transport Phys. Rev. B, 84, 085302.

Schaller, G. 2012 Fighting decoherence by feedback-controlled dissipation Phys. Rev. A 85, 062118.

Schaller, G., Emary, C., Kießlich, G., & Brandes, T. 2011 Probing the power of an electronic Maxwell’s demon: Single-electron transistor monitored by a quantum point contact Phys. Rev. B 84, 085418.
[21] Schöll, E. and Schuster, H. G. (Editors) 2008 *Handbook of Chaos Control* (Wiley-VCH, Weinheim).

[22] Schöll, E. 2009 Pattern formation and time-delayed feedback control at the nano-scale in *Nonlinear Dynamics of Nanosystems*, edited by G. Radons, B. Rumpf, and H. G. Schuster (Wiley-VCH, Weinheim).

[23] Wiseman, H. M. 1994 Quantum theory of continuous feedback Phys. Rev. A 49, 2133-2150.

[24] Wiseman, H. M. & Milburn, G. J. 2009 *Quantum Measurement and Control* (Cambridge University Press).