Coordinated Nonorthogonal Pilot Design for Massive MIMO

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Abstract—Pilot contamination caused by the nonorthogonality of pilots is a main limiting factor in multi-cell massive multiple-input multiple-output (MIMO) systems because it can significantly impair channel estimation. Recent works have suggested coordinating pilot sequence design across multiple cells in order to reduce the channel estimation error due to pilot contamination. This paper further investigates this approach using a new optimization framework. Specifically, we reformulate the problem of choosing the pilots to minimize the weighted mean-squared error (MSE) in channel estimation as a matrix fractional program. This problem can be efficiently approximated as a sequence of convex optimizations via a matrix fractional programming approach. The proposed algorithm, named fractional programming pilot design (FP-Pilot), provides fast convergence to a stationary point of the weighted MSE objective. The paper further characterizes the matrix fractional programming approach to allow the derivation of the optimal pilot sequence. Finally, we derive a relation between rate maximization and MSE minimization, which provides insight into the appropriate setting of weights in weighted MSE minimization. Simulations demonstrate the advantage of the proposed pilot design over alternative approaches in enhancing channel estimation and data throughput.

Index Terms—Massive MIMO, nonorthogonal pilots, weighted minimum mean-squared error for channel estimation.

I. INTRODUCTION

Acquisition of channel state information is crucial in massive multiple-input multiple-output (MIMO) wireless networks. A main challenge in channel estimation is that due to the limited coherence time, pilot sequences assigned to multiple users across multiple cells cannot all be orthogonal. The nonorthogonality of the pilots (e.g., due to the reuse of pilots across cells) causes the channel estimate of a particular user terminal to be affected by the pilots of other users—a phenomenon known as pilot contamination [2], [3].

This paper pursues a strategy of designing the pilot sequences of the users across the cells as a function of their large-scale fading (assuming that they are relatively stationary) in order to minimize pilot contamination. Following the recent works of [4], [5], the idea is that the effect of pilot contamination depends on the large-scale fading between the users and the base stations (BSs). For example, if an interfering pilot signal is weak, then it can afford to have higher correlation with the pilots of the desired users. Thus, judicious pilot design for the different users across multiple cells can help control pilot contamination.

The works [4], [5] suggested mitigating pilot contamination by designing the sequences to optimize a suitable system metric. In particular, the authors considered minimizing the sum of mean-squared errors (MSEs) in channel estimation. This paper further considers a metric of minimizing the sum of weighted MSEs in channel estimation across a multi-cell massive MIMO network. It is shown that this problem is a sum-of-functions-of-matrix-ratio problem that can be efficiently solved by using a recently developed matrix fractional programming (FP) technique [6]. In contrast to [4], [5] that design the pilots of all users in a greedy symbols-by-symbols fashion, here we consider a joint optimization of all pilot symbols based on a matrix FP formulation. We also address a discrete symbol case in which the pilot symbols are restricted to a fixed constellation. Finally, we examine the resulting data rate when our nonorthogonal pilots are used for channel estimation, and provide a method of choosing the MSE weights that takes data rates into account. Numerical results show that our coordinated pilot sequence design strategy significantly outperforms both the conventional pilot-reuse approach and the greedy pilot design technique of [4], [5], e.g., our algorithm reduces the sum MSE by more than 9 dB as compared to the conventional pilot-reuse approach.

Pilot contamination has been studied extensively in the literature over the past decade. To quantify the performance, many works [7]–[14] consider maximization of throughput, while others suggest minimization of the MSE in channel estimation [4], [5], [15]–[18]. Our work starts with the MSE minimization objective, and then connects it to rate maximization. The blind or semi-blind pilot methods in [7], [15], [19] aim to bypass the pilot contamination problem by estimating the channels directly without the pilots. The primary idea of [20] is to optimize the set of downlink pilots by using a precoding matrix, while other works, e.g., [8]–[12], [16], seek a contamination-aware allocation of a given set of orthogonal pilots.

Following the recent works of [4], [5], [21], our system model allows arbitrary sequences (under the power and length constraints) to be used as pilots, while prior works mostly assume orthogonal pilots within the cell in order to eliminate the intra-cell interference in channel estimation. As illustrated
in Fig. 1, the orthogonal scheme eliminates interfering pilots from the home cell, but results in pilot contamination from neighboring cells (when the same set of orthogonal pilots are reused in each cell). In comparison, the nonorthogonal scheme provides more flexibility in pilot design, thereby avoiding high correlation with the desired pilot. Simulations show that the proposed fractional programming pilot design (FP-Pilot) method outperforms the orthogonal sequences significantly in channel estimation due to the effective control of pilot contamination. Furthermore, the orthogonal scheme requires that the pilot sequence must be sufficiently long otherwise the users of the same cell cannot have orthogonal pilots; the nonorthogonal design does not have this constraint.

Three main questions are considered throughout this work: (i) What are the optimal nonorthogonal pilots that minimize the sum of weighted MSEs in channel estimation? (ii) What is the data rate that can be achieved for this set of nonorthogonal pilots? (iii) How should the MSE weights be chosen in order to maximize the overall data rates? To optimize the pilot sequences, we treat the weighted minimum mean-squared error (MMSE) problem as a matrix fractional program. We then use the matrix quadratic transform proposed in [6] to approximate the objective by a sequence of convex problems. With low computational costs, FP-Pilot is guaranteed to attain a stationary point of the weighted MMSE objective. In contrast, the computational complexity of the greedy algorithm in [4], [3] grows with the number of BSs as well as the pilot sequence length. In addition, the performance is difficult to analyze. We provide an analytic expression for the achievable rate that takes into account the nonorthogonality of the pilots, which is easier to interpret than the instantaneous ergodic rate. The proposed rate expression is an extension of the existing asymptotic rate expression of the orthogonal scheme [3], [22].

This rate expression is amenable to network optimization (e.g., power control) because of its deterministic form due to channel hardening in the massive MIMO regime [23]. Finally, by using the Lagrangian dual transform of [24], we relate rate maximization to MSE minimization, and thereafter propose a rate-aware MSE weighting strategy.

The main contributions of this paper are summarized as follows:

- **Weighted Sum-MSE Metric**: We extend the problem formulation of [4], [5] to allow for different weights in the MSE terms. Our simulations show that the MSE weights play a key role in improving data rates for the cell-edge users.

- **Matrix FP Approach**: We propose a new approach for pilot design when minimizing the weighted sum MSEs of channel estimation. The main idea is to reformulate the original nonconvex problem as a sequence of convex problems using the matrix quadratic transform [6].

- **Achievable Data Rate**: Nonorthogonal pilots have been previously considered in [4], [5], [21], but a formal study of the resulting data rate has not been performed. An instantaneous ergodic rate expression can be readily obtained, but it is difficult to interpret because of the expectation (taken over the channel randomness) outside the Gaussian capacity function. This work provides a deterministic achievable rate expression, which, unlike the instantaneous ergodic rate expression, can be computed in closed form directly, thus facilitating network optimization at the data transmission stage. In particular, our rate expression encompasses the known orthogonal case [3], [22] as a special case.

- **Setting of MSE Weights**: Grounded on the above contributions, this work further explores the relation between MSE minimization and rate maximization. We use the Lagrangian dual transform of [24] as the main tool to approximate the weighted sum rates maximization as a weighted sum MSE minimization. This relation provides insight into the MSE weight setting, e.g., it explains why the normalized MMSE scheme outperforms the MMSE metric in maximizing data rate as empirically shown in [18].

**Notation**: We use bold lower-case (or upper-case) letters to denote vectors (or matrices), \( \| \cdot \| \) the Euclidean norm, \( \langle \cdot \rangle \) the entry-wise conjugate of vector or matrix, \( \langle \cdot \rangle^\dagger \) the transpose, \( \langle \cdot \rangle^* \) the conjugate transpose, \( \text{vec}(\cdot) \) the vectorization, \( \text{tr}(\cdot) \) the trace, and \( \langle \cdot \rangle_{2} \) the square root of a matrix. Let \( \mathbb{E} \) be the expectation, \( \text{Var} \) be the variance, \( \mathbb{R} \) the set of real numbers, \( \mathbb{R}_{+} \) the set of nonnegative numbers, \( \mathbb{R}_{++} \) the set of strictly positive numbers, \( \mathbb{C}^{m \times n} \) the set of \( m \times n \) complex matrices, \( \mathbb{H}^{n} \) the set of \( n \times n \) positive-definite Hermitian matrices, \( \mathbb{D}^{n} \) the diagonal matrix, \( j \) the imaginary unit, \( \mathbb{R} \) (or \( \mathbb{I} \)) the real (or imaginary) part of a complex number, \( I_{n} \) the \( n \times n \) identity matrix, and \( N(m, \sigma^{2}) \) (or \( CN(m, \sigma^{2}) \)) a (complex) Gaussian distribution with mean value \( m \) and variance \( \sigma^{2} \). Finally, we use underline to denote a collection of variables, e.g., \( \underline{X} = \{X_1, X_2, \ldots, X_n\} \). For two random variables \( X \) and \( Y \), \( X \perp Y \) means they are independent.

II. PROBLEM FORMULATION

Consider a total of \( L \) BSs each associated with \( K \) user terminals. We refer to the area occupied by each BS and its user terminals as a **cell**. The BSs estimate the uplink channels.
based on the uplink pilots transmitted from the user terminals. We seek a coordinated pilot design that minimizes the channel estimation error throughout the network.

We use \( i \) or \( j \) to denote the index of each cell and its BS, and \((i, k)\) the index of the \( k \)th user in cell \( i \). Assume that every BS has \( M \) antennas and every user terminal has a single antenna. Let \( h_{j,ik} \in \mathbb{C}^{M} \) be the channel from user \((i, k)\) to BS \( j \), and let \( \mathbf{H}_{ji} \in \mathbb{C}^{M \times K} \) be the channel matrix:

\[
\mathbf{H}_{ji} = \begin{bmatrix}
    h_{j,i1} & h_{j,i2} & \cdots & h_{j,iK}
\end{bmatrix}.
\] (1)

Following the previous works [4, 5], we model each channel \( \mathbf{H}_{ji} \) by using the Kronecker structure with a partially separable correlation, i.e.,

\[
\mathbf{H}_{ji} = \mathbf{Q}_{j}^{\frac{1}{2}} \mathbf{G}_{ji} \mathbf{P}_{ji}^{\frac{1}{2}},
\] (2)

where the receiver-side channel correlation \( \mathbf{Q}_{j} \in \mathbb{C}^{M \times M} \) is a deterministic positive semi-definite (PSD) matrix, the large-scale channel strength \( \mathbf{P}_{ji} \in \mathbb{C}^{K \times K} \) is a deterministic diagonal PSD matrix

\[
\mathbf{P}_{ji} = \text{diag}([\beta_{j,i1}, \beta_{j,i2}, \ldots, \beta_{j,iK}])
\] (3)

with \( 0 \leq \beta_{j,ik} \leq 1 \) between any pair of BS \( j \) and user \((i, k)\), and the small-scale fading \( \mathbf{G}_{ji} \in \mathbb{C}^{M \times K} \) is a random matrix with i.i.d. entries distributed as \( \mathcal{CN}(0, 1) \). This channel assumption is experimentally justified in [25]. A common channel model as considered in [3] is one with \( \mathbf{Q}_{j} = \mathbf{I}_{M} \) for each \( j = 1, 2, \ldots, L \), which is a special case of (2).

Assume that each pilot sequence consists of \( \tau \) symbols. Let \( s_{ik} \in \mathbb{C}^{\tau} \) be the pilot sequence transmitted from user \((i, k)\), and further denote

\[
\mathbf{S}_{i} = \begin{bmatrix}
    s_{i1} & s_{i2} & \cdots & s_{iK}
\end{bmatrix}.
\] (4)

Let \( s_{ik}[\ell] \in \mathbb{C} \) be the \( \ell \)th symbol of the pilot sequence \( s_{ik} \), i.e., \( s_{ik} = (s_{ik}[0], s_{ik}[1], \ldots, s_{ik}[\tau - 1]) \). Assuming that the full spectrum band is reused in every cell, the received pilot signal at BS \( i \) is

\[
\mathbf{V}_{i} = \mathbf{H}_{ii} \mathbf{S}_{i}^{\dagger} + \sum_{j=1, j \neq i}^{L} \mathbf{H}_{ji} \mathbf{S}_{j}^{\dagger} + \mathbf{Z}_{i},
\] (5)

where \( \mathbf{Z}_{i} \in \mathbb{C}^{M \times \tau} \) is additive noise with i.i.d. entries distributed as \( \mathcal{CN}(0, \sigma^2) \) for the fixed noise power level \( \sigma^2 \).

Upon receiving the uplink pilot signals from the users, each BS \( i \) estimates its \( \mathbf{H}_{ii} \) using the MMSE criterion from [4, 5]. Let \( \hat{\mathbf{H}}_{ii} \) be the MMSE estimate of \( \mathbf{H}_{ii} \); it is determined as

\[
\text{vec}(\hat{\mathbf{H}}_{ii}) = \left( \mathbf{P}_{ii} \mathbf{S}_{i}^{\dagger} \otimes \mathbf{I}_{M} \right) \left( \mathbf{D}_{i} \otimes \mathbf{I}_{M} \right)^{-1} \text{vec}(\mathbf{V}_{i}),
\] (6)

where

\[
\mathbf{D}_{i} = \sigma^2 \mathbf{I}_{\tau} + \sum_{j=1}^{L} \mathbf{S}_{j} \mathbf{P}_{ij} \mathbf{S}_{j}^{\dagger}.
\] (7)

The resulting MSE of user \((i, k)\) is

\[
\text{MSE}_{ik} = \mathbb{E}[\|\hat{\mathbf{h}}_{i,ik} - \mathbf{h}_{i,ik}\|^2],
\] (8)

where \( \mathbf{h}_{i,ik} \) is the \( k \)th column of \( \mathbf{H}_{ii} \). We aim to choose the pilot sequences to minimize the sum of weighted MSEs, i.e.,

\[
\text{minimize } \sum_{i=1}^{L} \sum_{k=1}^{K} w_{ik} \text{MSE}_{ik}
\] (9)

for a set of fixed nonnegative weights \( w_{ik} \geq 0 \). For instance, we may set \( w_{ik} = 1 \) to minimize the sum of MSEs as in [4], or \( w_{ik} = 1/\beta_{i,ik} \) to minimize the sum of normalized MSEs as in [18]. For now, we focus on optimizing \( \mathbf{S} \) for the fixed weights \( w_{ik} \). In Section IV-B, we discuss the choice of \( w_{ik} \) for maximizing the achievable rate.

Following the steps in [5], we can formalize problem (9) as

\[
\begin{align}
\text{maximize} & \quad \sum_{i=1}^{L} \sum_{k=1}^{K} \alpha_{i} \text{tr}(\mathbf{W}_{i} \mathbf{P}_{ii} \mathbf{S}_{i}^{\dagger} \mathbf{D}_{i}^{-1} \mathbf{S}_{i} \mathbf{P}_{ii}) \\
\text{subject to} & \quad ||\mathbf{s}_{ik}||^2 \leq \rho_{ik}, \quad \text{for all } (i, k),
\end{align}
\] (10a, 10b)

where \( \alpha_{i} = \text{tr}(\mathbf{Q}_{i}) \), \( \mathbf{W}_{i} = \text{diag}(w_{i1}, w_{i2}, \ldots, w_{iK}) \), and \( \rho_{ik} \) is the power constraint of user \((i, k)\).

Problem (10) is a difficult optimization problem, because the choice of pilot sequences \( \mathbf{S}_{i} \) appears in both the numerator and the denominator of a matrix fraction in (10a). The authors in [5] propose a greedy sum of ratio traces maximization (GSRMT) algorithm to optimize each row of \( \mathbf{S}_{i} \) sequentially. Here we suggest a matrix-FP approach to optimize the entire set of matrices \( \mathbf{S}_{i} \) jointly. We illustrate via simulations that this leads to more accurate channel estimation.

As studied in [4, 5], the above problem formulation can be extended to the case of reduced radio-frequency (RF) chains, i.e., when each BS \( i \) uses an RF chain combiner \( \mathbf{U}_{i} \in \mathbb{C}^{N \times M} \) (for \( N < M \)) to reduce the dimensionality of the received signal \( \mathbf{V}_{i} \). As already shown in [4, 5], the weight \( \alpha_{i} \) in problem (10) then becomes \( \text{tr}(\mathbf{Q}_{i} \mathbf{U}_{i}^{\dagger} (\mathbf{Q}_{i} \mathbf{U}_{i}^{\dagger})^{-1} \mathbf{U}_{i} \mathbf{Q}_{j}) \). Our work focuses on optimizing the pilot variable \( \mathbf{S} \) given the weights \( \alpha_{i} \) in (10), regardless of how each \( \alpha_{i} \) is determined.

III. PILOT DESIGN USING FRACTIONAL PROGRAMMING

A. Matrix FP

We begin by reviewing the matrix FP technique of [6]. We then show how this approach can be applied to pilot design. We start with the definition of a matrix ratio. For a pair of matrices \( \mathbf{A} \in \mathbb{C}^{m \times n} \) and \( \mathbf{B} \in \mathbb{C}^{m \times n} \), \( \mathbf{A}^{\ast} \mathbf{B}^{-1} \mathbf{A} \) is said to be the ratio between the numerator matrix \( \mathbf{A} \mathbf{A}^{\ast} \) and the denominator matrix \( \mathbf{B} \). The following theorem from [6] decouples the numerator and denominator of the matrix ratio.

\textbf{Theorem 1 (Matrix Quadratic Transform [6])}: Given a nonempty constraint set \( \mathcal{X} \) as well as a sequence of functions \( \mathbf{A}_{i}(x) \in \mathbb{C}^{m \times n} \) and functions \( \mathbf{B}_{i}(x) \in \mathbb{C}^{m \times n} \), nondecreasing functions \( f_{i} : \mathbb{R}^{n} \rightarrow \mathbb{R} \) in the sense that \( f_{i}(\mathbf{R}_{1}) \geq f_{i}(\mathbf{R}_{2}) \) if \( \mathbf{R}_{1} \succeq \mathbf{R}_{2} \), for \( i = 1, 2, \ldots, L \), the sum-of-functions-of-matrix-ratio problem

\[
\text{maximize } \sum_{i=1}^{L} f_{i}(\mathbf{A}_{i}(x) \mathbf{B}_{i}^{-1}(x) \mathbf{A}_{i}(x))
\] (11)
is equivalent to

\[
\max_{\mathbf{x} \in \mathbb{R}^n} \sum_{i=1}^{L} f_i \left( 2 \Re \{ \mathbf{A}_i^* (\mathbf{x}) \mathbf{Y}_i \} - Y_i^* \mathbf{B}_i (\mathbf{x}) \mathbf{Y}_i \right),
\]

(12)

where \( \mathbf{Y}_i \in \mathbb{C}^{m \times n} \) is an auxiliary variable introduced for each matrix ratio term.

**Proof:** By completing the square, each \( f_i (\mathbf{R}) \) can be monotonically increasing function of \( \mathbf{R} \); (ii) \( \mathbf{A}_i (\mathbf{x}) \) is a convex function. In this case, the new problem (12) is convex for fixed \( \mathbf{Y}_i \), and solving the new problem (12) when the following three conditions hold: (i) \( f_i (\mathbf{R}) \) is a convex function. In this case, the new problem (12) is convex in \( \mathbf{x} \) for fixed \( \mathbf{Y}_i \). It turns out that the weighted MMSE problem (10) satisfies the above three conditions. Furthermore, as shown below, the reformulated problem of (10) has an analytic solution.

**B. Iterative Optimization via Matrix FP**

In light of Theorem 1, we can reformulate problem (10) as follows.

**Proposition 1:** Problem (10) is equivalent to

\[
\max_{\mathbf{S}, \mathbf{Y}} \sum_{i=1}^{L} \alpha_i \text{tr} \left( \mathbf{W}_i \left( 2 \Re \{ \mathbf{P}_i^* \mathbf{S}_i^* \mathbf{Y}_i \} - \mathbf{Y}_i^* \mathbf{D}_i \mathbf{Y}_i \right) \right)
\]

subject to \( \| \mathbf{s}_{ik} \|^2 \leq \rho_{ik} \), for all \( (i, k) \),

\[
\mathbf{Y}_i \in \mathbb{C}^{r \times K}, \text{ for all } i,
\]

(13a)

(13b)

(13c)

where \( \mathbf{Y}_i \) is the auxiliary variable.

**Proof:** The new objective function (13a) is derived by treating \( \mathbf{S}_i \mathbf{P}_i^* \) and \( \mathbf{D}_i \) as \( \mathbf{A}_i \) and \( \mathbf{B}_i \) in Theorem 1 respectively, with \( f_i (\mathbf{R}) = \alpha_i \text{tr} (\mathbf{W}_i \mathbf{R}) \).

In the rest of the paper, we use \( \mathbf{y}_{ik} \) to denote the \( k \)th column of \( \mathbf{Y}_i \). To solve problem (13) in Proposition 1 we propose to optimize \( \mathbf{S} \) and \( \mathbf{Y} \) alternatively. When \( \mathbf{S} \) is held fixed, each \( \mathbf{Y}_i \) can be optimally determined by completing the square for \( \mathbf{Y}_i \) in (13a), i.e.,

\[
\mathbf{Y}_i = \mathbf{D}_i^{-1} \mathbf{S}_i \mathbf{P}_{ii}.
\]

Next, we optimize \( \mathbf{S} \) for fixed \( \mathbf{Y} \). The key step is to rewrite the objective function (13a) in a quadratic form with respect to each \( \mathbf{s}_{ik} \), as specified in the following proposition.

**Proposition 2:** The objective function (13a) is equivalent to

\[
f (\mathbf{S}, \mathbf{Y}) = \sum_{(i,k)} \xi_{ik} + c (\mathbf{Y}),
\]

(15)

where

\[
\xi_{ik} = 2 \Re \{ w_{ik} \beta_{i,ik} \mathbf{s}_{ik} \mathbf{y}_{ik} \} - \mathbf{s}_{ik}^* \left( \sum_{j=1}^{L} \beta_{j,ik} \mathbf{Y}_j \mathbf{Y}_j^* \right) \mathbf{s}_{ik},
\]

(16)

and

\[
c (\mathbf{Y}) = \sigma^2 \sum_{i=1}^{L} \text{tr} (\mathbf{Y}_i^* \mathbf{Y}_i).
\]

(17)

**Algorithm 1:** Fractional Programming Pilot Design (FP-Pilot)

1. Initialize all the variables to feasible values;
2. repeat
3. Update the auxiliary variable \( \mathbf{Y} \) by (14);
4. Update the pilot variable \( \mathbf{S} \) by (19) along with the Lagrangian multiplier \( \lambda_{ik} \) in (20), or by simultaneous scaling when \( \sigma^2 = 0 \);
5. until the variables \( (\mathbf{S}, \mathbf{Y}) \) converge;

The proof is deferred to Appendix A.

Combining Proposition 2 with the power constraint (13b), we arrive at a Lagrangian function for problem (13):

\[
\mathcal{L} (\mathbf{S}, \mathbf{Y}, \lambda) = \sum_{(i,k)} \left( \xi_{ik} - \lambda_{ik} (\| \mathbf{s}_{ik} \|^2 - \rho_{ik}) \right) + c (\mathbf{Y}),
\]

(18)

where each \( \lambda_{ik} \) is a Lagrangian multiplier for constraint (13b).

By completing the square in (18), the optimal \( \mathbf{s}_{ik} \) is given by

\[
\mathbf{s}_{ik} = \left( \sum_{j=1}^{L} \beta_{j,ik} \mathbf{Y}_j \mathbf{Y}_j^* + \lambda_{ik} I \right)^{-1} w_{ik} \beta_{i,ik} \mathbf{y}_{ik},
\]

(19)

where \( \lambda_{ik} \) is determined by complementary slackness, i.e.,

\[
\lambda_{ik} \begin{cases} 0, & \text{if } \| \mathbf{s}_{ik} \|^2 \leq \rho_{ik}; \\
\lambda_{ik} \rho_{ik} > 0, & \text{otherwise.}
\end{cases}
\]

(20)

For the second case of (20) wherein \( \| \mathbf{s}_{ik} \|^2 = \rho_{ik} \), the optimal \( \lambda_{ik} \) can be computed efficiently by a bisection search because \( \| \mathbf{s}_{ik} \|^2 \) in (19) is monotonically decreasing with \( \lambda_{ik} > 0 \). Computation of \( \mathbf{s}_{ik} \) can be simplified in the special case in which \( \mathbf{Z}_i \) is negligible (i.e., \( \sigma^2 = 0 \)). As shown in [5], in this case, the sum of weighted MSEs remains the same if each pilot \( \mathbf{s}_{ik} \) is multiplied by the same nonzero factor \( \alpha \). Thus, we can enforce the power constraint (13b) by scaling all the \( \mathbf{s}_{ik} \)'s (assuming that \( \lambda_{ik} = 0 \)) simultaneously with a sufficiently small positive factor, instead of going through the computation of \( \lambda_{ik} \).

When \( \mathbf{Y} \) is held fixed, maximizing the objective function (15) over \( \mathbf{S} \) is a convex problem, so \( (\mathbf{S}, \lambda) \) obtained from (19) and (20) are jointly optimal from a Lagrangian dual theoretic perspective.

The matrix-FP reformulation presented here mimics that of [6] until (14). Proposition 2 is needed in order to complete the square, due to the difference in how \( \mathbf{P}_i \) is specified in the following proposition.

**Proposition 3:** The sum of weighted MSEs is monotonically nonincreasing per iteration in FP-Pilot; the pilot variable \( \mathbf{S} \) converges to a stationary point of problem (10).

**Proof:** The iterative update in FP-Pilot can be interpreted as a sequence of minorization-maximization (MM) steps [26], [27]. Specifically, when \( \mathbf{Y} \) is held fixed, \( \mathbf{S} \) in (19) is the globally optimal solution that maximizes \( f (\mathbf{S}, \mathbf{Y}) \) in (15), namely the maximization phase; when \( \mathbf{S} \) is held fixed, \( f (\mathbf{S}, \mathbf{Y}) \) is less than or equal to the original objective in (13a), where
$$R_{ik} = E \left[ \log_2 \left( 1 + \frac{\|\hat{h}_{i,ik}\|^4 \rho_{ik}}{\sum_{(j,\ell) \neq (i,k)} \|\hat{h}_{j,ik}\| \|h_{i,ik}\|^2 + \sigma^2 \|\hat{h}_{i,ik}\|^2 + \|h_{i,ik}^\dagger (\hat{h}_{i,ik} - h_{i,ik})\|^2 \} \right) \right].$$ (25)

the equality holds iff $\mathbf{Y}$ meets (14), namely the minorization phase. The above proposition then directly follows by the property of the MM algorithm [26], [27]. More details are referred to [6].

We next compare FP-Pilot with the GSRTM algorithm proposed in [3]. The main idea behind GSRTM is to optimize one row of the matrix $S_i$ at a time while fixing all other rows. Because the rows of $S_i$ are not optimized jointly in GSRTM, this greedy method is prone to being trapped in a local optimum. Furthermore, it can be shown that FP-Pilot has a computational complexity scaling of $O(\tau^3 LKT)$ where $T$ is the number of iterations, while GSRTM has computational complexity scaling of $O(\tau^3 L^2 K + \tau^4 L)$, so that GSRTM is more sensitive to $\tau$ and $L$. Moreover, the convergence property of GSRTM is difficult to analyze, whereas FP-Pilot is guaranteed to converge to a stationary-point solution. In Section V we present numerical results that demonstrate the performance advantage of FP-Pilot over GSRTM.

C. Discrete Pilot Sequence Design

Arbitrary complex-valued pilot sequences may be difficult to implement in practice. In this section, we restrict the choice of each pilot symbol to a 4-QAM constellation $C = \{\varepsilon(1 + j), \varepsilon(1 - j), -\varepsilon(1 + j), -\varepsilon(1 - j)\}$ with a power control factor $\varepsilon = \sqrt{\rho/2\tau}$. Such sequences are referred to as discrete pilot sequences.

To design optimal discrete pilot sequences, we maximize the objective function $f(S, \mathbf{Y})$ in (13) for fixed $\mathbf{Y}$ (which has been updated by (14)) over the QAM constellation as follows:

$$s_{ik}' = \arg \min_{s \in C^\tau} \left\| \sum_{j=1}^L \beta_{j,ik} Y_j W_j Y_j^* \right\| (s' - s_{ik}),$$ (21)

where $s_{ik}$ has the same form as (19) but with $\lambda_{ik} = 0$ (since there is no power constraint on $s_{ik}$ in this case); $C^\tau$ refers to a Cartesian power of set $C$. The projection of $s_{ik}$ onto $C^\tau$ may be computationally complex in practice as the size of $C^\tau$ grows exponentially with the pilot length $\tau$. Thus, we propose a suboptimal solution of simply rounding each $s_{ik}[t]$ to $C$, i.e.,

$$s_{ik}'[t] = \varepsilon \cdot \text{sgn}(\Re\{s_{ik}[t]\}) + j \varepsilon \cdot \text{sgn}(\Im\{s_{ik}[t]\}) \quad (22)$$

where $\text{sgn}(\cdot)$ is the sign function. Observe that the heuristic in (22) amounts to $s_{ik}' = \arg \min_{s \in C^\tau} \|s' - s_{ik}\|$. As compared to (21), this heuristic is in essence assuming that

$$\sum_{j=1}^L \beta_{j,ik} Y_j W_j Y_j^* \approx \theta I_{\tau},$$ (23)

for some positive scalar $\theta > 0$. Although above approximation is not tight in general, but the resulting discrete pilot sequences are often quite effective in reducing the channel estimation error.

IV. ACHIEVABLE DATA RATE

This section consists of two parts. We first introduce two different achievable rate expressions both accounting for the nonorthogonal pilots; one has a simpler derivation while the other is easier to optimize. We then establish a connection between rate maximization and MSE minimization, which we use to guide the MSE weight setting in problem (10).

A. Achievable Rate Expression

We turn our attention to data transmission since this is what channel estimation is ultimately used for. The main result of this section is to characterize the achievable data rate in uplink transmission given a set of nonorthogonal pilots used for channel estimation.

Our analysis rests on two assumptions. First, the data signal $x_{ik}$ of each user $(i,k)$ has an i.i.d. Gaussian distribution $CN(0, 1)$. Second, each BS $i$ multicasts the received signal of user $(i,k)$ with the complex conjugate of its channel estimate $\hat{h}_{i,ik}$, namely maximum ratio receiver (MRC). We use $\hat{\rho}_{ik}$ to denote the transmit power level of the data signal of user $(i,k)$. For user $(i,k)$, the data signal received at the target BS $i$ after MRC processing is

$$\tilde{v}_{ik} = \hat{h}_{i,ik}^* \left( \sum_{(j,\ell) \neq (i,k)} \sqrt{\hat{\rho}_{j}\hat{\rho}_{\ell}} h_{i,ik} x_{j,\ell} + z_{ik} \right)$$

$$= \sqrt{\hat{\rho}_{ik}} \|\hat{h}_{i,ik}\|^2 x_{ik} + \hat{h}_{i,ik} \left( \sum_{(j,\ell) \neq (i,k)} \sqrt{\hat{\rho}_{j}\hat{\rho}_{\ell}} h_{i,ij} x_{\ell} \right)$$

$$+ \hat{h}_{i,ik}^* \left( z_{ik} + \sqrt{\hat{\rho}_{ik}} (h_{i,ik} - \hat{h}_{i,ik}) x_{ik} \right),$$ (24)

where $z_{ik} \sim CN(0, \sigma^2 I_M)$ is an i.i.d. additive Gaussian noise of the data transmission phase.

By treating interference as noise, we obtain an achievable data rate in (25) for user $(i,k)$ as displayed at the top of the page, where the expectation is taken over the random fadings $\{h_{i,j}\}$ for a large number of coherence intervals with independent small-scale fading. The pilots affect $R_{ik}$ through $h_{i,ik}$. Following [23], we refer to the above $R_{ik}$ as the instantaneous ergodic rate. As pointed out in [23], the instantaneous ergodic rate is hard to evaluate because of the expectation of the logarithm rate expression.

In the following, we take advantage of the channel hardening effect [23] in the massive MIMO regime to derive a deterministic achievable rate expression that does not involve expectation and is also more interpretable and more amenable to optimization. Note that we average the rate expression in (25) because the effective channel gain $\|\hat{h}_{i,ik}\|^2$ used in (24)
depends on each realization of \( h_{i,ik} \). The main step of our approach is to replace \( |h_{i,ik}|^2 \) with an artificial channel gain
\[
\tilde{h}_{i,ik} = \mathbb{E}[\hat{h}_{i,ik}^* h_{i,ik}],
\]
which reflects the average effective channel gain when the MMSE estimate \( \hat{h}_{i,ik} \) is used. Observe that \( \tilde{h}_{i,ik} \) is a metric that can be determined \textit{a priori} based on the channel distribution. It can be shown that the randomness caused by the small-scale fading can all be encompassed in \( \tilde{h}_{i,ik} \), so the resulting data rate is a deterministic function of the large-scale fading, as stated in the following theorem.

**Theorem 2:** For a set of pilots \( \{s_{ik}\} \), the data rate shown in (27) at the top of the page is achievable, where
\[
\mu_{ik} = \beta_{i,ik}^2 s_{ik}^* D^{-1}_i s_{ik}
\]
and
\[
F_i(\hat{\mu}) = \sum_{j=1}^L \left( S_j P_{ij} \cdot \text{diag} [\hat{\beta}_j, \hat{\beta}_j, \ldots, \hat{\beta}_j] \cdot S_j^* \right).
\]

We remark that when pilots are held fixed, the rate expression in (27) has the form
\[
\tilde{R}_{ik} = \log_2 \left( 1 + \sum_{(j,t)} a_{jt} \hat{\mu}_{jt} + 1 \right)
\]
for a set of positive coefficients \( a_{jt} \). Thus, common optimizations (e.g., power control and scheduling) that have been well studied in the literature for the scalar deterministic channel case can now be readily applied. In effect, (27) is an approximation of the instantaneously ergodic rate expression (25), which is difficult to use for subsequent analysis. Before proceeding to the proof of Theorem 2, we further consider an asymptotic case of \( \tilde{R}_{ik} \) when each BS has an infinite number of antennas.

**Corollary 1:** In the limit that the number of antennas at each BS, \( M \), goes to infinity, the achievable data rate \( \tilde{R}_{ik} \) in (27) tends to
\[
\tilde{R}_{ik,\infty} = \log_2 \left( 1 + \frac{\mu_{ik}^2 \tilde{\mu}_{ik}}{\beta_{i,ik}^2 s_{ik}^* D^{-1}_i \cdot F_i(\hat{\rho}) \cdot D_i^{-1} s_{ik} - \mu_{ik}^2 \hat{\mu}_{ik}} \right).
\]
We also point out that \( \tilde{R}_{ik} \leq \tilde{R}_{ik,\infty} \) in general.

The nonorthogonality of pilots comes into (31) through \( D_i^{-1} \cdot F_i(\hat{\rho}) \cdot D_i^{-1} \) which would be a diagonal matrix if the pilots are orthogonal. Furthermore, (31) reduces to well-known results if \( r \geq K \) and the same set of orthogonal pilots \( \{s_1^*, s_2^*, \ldots, s_K^*\} \) is reused in each cell. In this case, the above \( \tilde{R}_{ik,\infty} \) reduces to
\[
\tilde{R}_{ik,\infty} = \log_2 \left( 1 + \frac{\beta_{i,ik}^2 \tilde{\mu}_{ik}}{\sum_{j \neq i} \beta_{i,ik}^2 \hat{\mu}_{ij(k)} \hat{\mu}_{ij(k)}} \right),
\]
where \( \kappa(j, ik) \) outputs the index of the user in cell \( j \) assigned the same pilot sequence as user \( (i, k) \). The asymptotic rate in \( \Delta_{ik} \) is a well-known result in the massive MIMO literature [3], [22].

The rest of this section aims to establish the achievability of Theorem 2. First, inspired by the decoding method (under the orthogonal scheme) in [8], we rewrite the received signal after MRC processing as
\[
\tilde{v}_{ik} = \sqrt{\rho_{ik}} \tilde{h}_{i,ik} x_{ik} + \Delta_{ik},
\]
where
\[
\Delta_{ik} = \sum_{(j,t)} \sqrt{\rho_{jt}} \tilde{h}_{i,jt} s_{jt} + \tilde{h}_{i,ik} \tilde{z}_{ik} - \sqrt{\rho_{ik}} \tilde{h}_{i,ik} x_{ik}.
\]
In (33), we express the received signal \( \tilde{v}_{ik} \) as if \( x_{ik} \) passed through the known channel \( \tilde{h}_{i,ik} \). Note that the BS knows \( \tilde{h}_{i,ik} \), although it is not aware of every realization of \( h_{i,ik} \).

The following lemma is crucial in deriving the achievable data rate we proposed in Theorem 2.

**Lemma 1:** \( \mathbb{E}[\tilde{h}_{i,ik} x_{ik}] \Delta_{ik} = 0 \) with the expectation taken over \( (H, Z, x, \tilde{Z}) \).

**Proof:** It can be seen that
\[
\mathbb{E}[\tilde{h}_{i,ik} x_{ik}] \Delta_{ik} = \mathbb{E}[\tilde{h}_{i,ik} x_{ik}] \mathbb{E}[\sqrt{\rho_{ik}} \tilde{h}_{i,ik} x_{ik} - \sqrt{\rho_{ik}} \tilde{h}_{i,ik} x_{ik}] \
= \mathbb{E}[\tilde{h}_{i,ik} x_{ik}] \mathbb{E}[\sqrt{\rho_{ik}} \tilde{h}_{i,ik} x_{ik}] - \mathbb{E}[\sqrt{\rho_{ik}} \tilde{h}_{i,ik} x_{ik}]^2
\]
by (35), where (a) follows since \( x_{ik} \) and \( x_{ik} \) are uncorrelated. Finally, (b) follows by substituting (26) in the first expectation term.

From Lemma 1, \( \Delta_{ik} \) can be recognized as an uncorrelated noise added to the desired signal \( \sqrt{\rho_{ik}} h_{i,ik} x_{ik} \). The early work [28] shows that in terms of channel capacity, the worst-case uncorrelated additive noise under a variance constraint has a Gaussian distribution. Henceforth, with respect to (33), a lower bound on the achievable data rate of user \( (i, k) \) is
\[
\tilde{R}_{ik} = \log_2 \left( 1 + \frac{\mathbb{E}[\tilde{h}_{i,ik} x_{ik}^2]}{\mathbb{E}[\Delta_{ik}^2]} \right),
\]
where the expectation of \( |\tilde{h}_{i,ik} x_{ik}|^2 \) is taken over \( x_{ik} \) while the expectation of \( |\Delta_{ik}|^2 \) is taken over \( (H, Z, x, \tilde{Z}) \).

It remains to compute \( \mathbb{E}[|\tilde{h}_{i,ik} x_{ik}|^2] \) and \( \mathbb{E}[|\Delta_{ik}|^2] \), both of which depend on the MMSE channel estimation \( \hat{h}_{i,ik} \). Recall that \( \hat{h}_{i,ik} \) corresponds to the \( k \)th column of \( \text{vec}(H_{i,ik}) \) in (6).

The following lemma provides a simpler form for \( \hat{h}_{i,ik} \).

**Lemma 2:** The MMSE estimate of channel \( h_{i,ik} \) can be rewritten as
\[
\hat{h}_{i,ik} = \beta_{i,ik} V_i \tilde{D}_i^{-1} s_{ik},
\]
Proof: The right-hand side of (37) can be rewritten as
\[
\left( P_i s_i^* \otimes I_M \right) \left( D_i \otimes I_M \right)^{-1} \text{vec}(V_i) \\
\overset{(a)}{=} \left( P_i s_i^* \otimes I_M \right) \left( D_i^{-1} \otimes I_M \right) \text{vec}(V_i) \\
\overset{(b)}{=} \left( P_i s_i^* I_M \otimes I_M \right) \cdot \text{vec}(V_i) \\
\overset{(c)}{=} \text{vec} \left( V_i (P_i s_i D_i^{-1})^T \right),
\]
(38)
where (a) follows as \((A \otimes B)^{-1} = A^{-1} \otimes B^{-1}\), (b) follows since \((A \otimes B) \cdot (A' \otimes B') = (AA') \otimes (BB')\), and (c) is a result of \((A^T \otimes B) \cdot \text{vec}(C) = \text{vec}(ACB)\). The identity in (37) is thus established.

This new form of \(h_{i,ik}\) makes the computation of (36) much easier. After some algebra as in Appendix B and Appendix C, the achievable data rate \(R_{i,ik}\) in (27) can be obtained from (36).

To examine the achievable rate expressions in (25) and (27) more deeply, we further provide a mutual information perspective, the subject of the next subsection.

B. Setting of Weights via Rate Maximization

This section aims to find a set of MSE weights \(w_{i,ik}\) in (30) that account for data rates; we consider the asymptotic rate in (31) for ease of discussion. Toward this end, we first explore the relation between MSE minimization and rate maximization. The primary idea here is to rewrite the rate expression (31) in a weighted MSE form by using a transformation technique from the recent work [24] on the logarithmic fractional program:

**Theorem 3 (Lagrangian Dual Transform [24]):** Given a nonempty constraint set \(\mathcal{X} \subseteq \mathbb{C}^d\), \(N\) pairs of nonnegative function \(A_i : \mathbb{C}^d \mapsto \mathbb{R}_+\) and positive function \(B_i : \mathbb{C}^d \mapsto \mathbb{R}_{++}\), and a set of nonnegative weights \(\eta_i \geq 0\), for \(i = 1, 2, \ldots, N\), where \(d \in \mathbb{N}\), the logarithmic fractional program

\[
\begin{align*}
\text{maximize} & \quad \sum_{i=1}^{N} \eta_i \log \left( 1 + \frac{A_i(x)}{B_i(x)} \right) \\
\text{subject to} & \quad x \in \mathcal{X}
\end{align*}
\]
(39a)

is equivalent to

\[
\begin{align*}
\text{maximize} & \quad \sum_{i=1}^{N} \eta_i \left( \log(1 + \gamma_i) - \gamma_i + \frac{(1 + \gamma_i) A_i(x)}{A_i(x) + B_i(x)} \right) \\
\text{subject to} & \quad x \in \mathcal{X}, \quad \gamma_i \geq 0, \text{ for } i = 1, 2, \ldots, N
\end{align*}
\]
(40a)

where \(\gamma_i\) is an auxiliary variable introduced for each ratio \(A_i(x)/B_i(x)\).

The role of the Lagrangian dual transform is to “move” the fractional terms to outside of the logarithms, thereby reformulating the logarithmic fractional problem as an easier sum-of-ratios problem (in aid of the auxiliary variables \(\gamma_i\)). By virtue of the Lagrangian dual transform, the weighted sum rates maximization, with \(\eta_{i,ik} \geq 0\) being the rate weight of user \((i, k)\), is recast into a new form, i.e.,

\[
\text{maximize} \quad \sum_{(i,k)} \eta_{i,ik} \bar{R}_{i,ik} \iff \text{maximize} \quad \sum_{(i,k)} \eta_{i,ik} T_{i,ik}
\]
(41)

where

\[
T_{i,ik} = \log(1 + \gamma_{i,ik}) - \gamma_{i,ik} + \frac{(1 + \gamma_{i,ik}) \mu_{i,ik}^2 \delta_{i,ik}}{\beta_{i,ik}^2 s_{i,ik} D_i^{-1} \cdot F_i(\hat{\rho}) \cdot D_i^{-1} s_{i,ik}}.
\]
(42)

The auxiliary variable \(\gamma_{i,ik}\) can be interpreted as the signal-to-interference-and-noise ratio (SINR) of user \((i, k)\). It turns out that the optimal \(\gamma_{i,ik}\) in (41) coincides with the real SINR in (31).

To make it tractable, we further assume that the data signal is stronger than any individual interfering signal, i.e.,

\[
\beta_{i,ik} \hat{\rho}_{ik} \geq \beta_{i,j\ell} \hat{\rho}_{j\ell}, \quad \text{for all } (i, k) \text{ and } (j, \ell).
\]
(43)

This is a reasonable assumption for a massive MIMO system with proper power control. Now, assume the use of the channel inversion power control [29], the desired signals received at each particular BS would be of the same strength, i.e.,

\[
\beta_{i,ik} \hat{\rho}_{ik} = \varrho_i \text{, for some } \varrho_i \geq 0.
\]
(44)

We then obtain an upper bound on \(F_i(\hat{\rho})\) as

\[
F_i(\hat{\rho}) = \sum_{j=1}^{L} \left( S_j P_{ij}^2 \cdot \text{diag}[\hat{\rho}_{j1}, \hat{\rho}_{j2}, \ldots, \hat{\rho}_{jK}] \cdot S_j^* \right)
\]

\[
\leq \varrho_i \sum_{j=1}^{L} \left( S_j P_{ij} S_j^* \right) = \varrho_i D_i,
\]
(45)

which further leads to a lower bound on \(T_{i,ik}\):

\[
T_{i,ik} \geq \log(1 + \gamma_{i,ik}) - \gamma_{i,ik} + \frac{(1 + \gamma_{i,ik}) \mu_{i,ik}^2 \delta_{i,ik}}{\varrho_i^2 \beta_{i,ik}^2 s_{i,ik} D_i^{-1} s_{i,ik}} = \log(1 + \gamma_{i,ik}) - \gamma_{i,ik} + \frac{1 + \gamma_{i,ik}}{\beta_{i,ik}^2} \mu_{i,ik}.
\]
(46)

If the optimal auxiliary variables \(\gamma_{i,ik}\) are already determined, then the weighted sum-rate maximization problem in (41) can be converted to

\[
\text{maximize} \quad \sum_{(i,k)} \eta_{i,ik} \left( \frac{1 + \gamma_{i,ik}}{\beta_{i,ik}^2} \right) \mu_{i,ik}
\]
(47)

by using the lower bound (46) to approximate \(T_{i,ik}\). Contrasting (10) and (47) gives the following strategy for setting the appropriate MSE weights:

\[
w_{i,ik} = \eta_{i,ik} \left( \frac{1 + \gamma_{i,ik}}{\beta_{i,ik}^2} \right), \text{ for all } (i, k).
\]
(48)

Now, the auxiliary variable \(\gamma_{i,ik}\), which represents the optimal SINR of user \((i, k)\), is unknown \textit{a priori} in general. To resolve this issue, we suggest some heuristic methods, e.g., (i) set \(\gamma^*\) to some target SINR; (ii) update \(\gamma^*\) iteratively with (48).

We further argue that in many cases, the exact \(\gamma_{i,ik}^*\) is not required. As shown in [29], if the pilot contamination has been suppressed effectively, then the users from the same cell would achieve similar data rates under the channel inversion power control \(\delta_{i,ik} = \delta/\beta_{i,ik}\) for some positive constant \(\delta > 0\). We further argue that this similarity in achievable rate holds throughout the massive MIMO system, i.e., \(\gamma_{i,ik}^* \approx \gamma_{j\ell}^*\), for any
\[(i, j, k, \ell)\), so long as every cell has a similar setup (e.g., cell size, channel condition, and user distribution). In this case, the MSE weight in (48) is equivalent to

\[ w_{ik} = \frac{\eta_{ik}}{\beta_{i,ik}}. \]

Hence, the normalized MMSE scheme with \(w_{ik} = 1/\beta_{i,ik}\) as suggested in [18] is suitable for maximizing the sum rates under the channel inversion power control.

V. NUMERICAL RESULTS

We validate the performance of the proposed method in a 7-cell wrapped-around network. Each cell consists of a 100-antenna BS located at the center and 9 single-antenna user terminals uniformly distributed in a hexagonal area. The BS-to-BS distance is 1000 meters. Let \(\tau = 16\) and \(\rho_{ik} = 1\). Following [4, 5], we assume that the background noise is negligible and that \(\beta_{j,ik} = \varphi_{j,ik}/(d_{j,ik})^3\) where \(\varphi_{j,ik}\) is an i.i.d. log-normal random variable according to \(\mathcal{N}(0, \zeta^2)\) with \(\zeta = 8\) dB and \(d_{j,ik}\) is the distance between user \((i, k)\) and BS \(i\).

In addition to the GSRTM algorithm with a random dictionary (see [5]), we introduce two baseline methods as follows:

- **Orthogonal Method**: Fix a set of 16 orthogonal pilots; randomly choose 9 pilots out of them for each cell.
- **Random Method**: Generate the pilots randomly and independently according to a Gaussian distribution.

The orthogonal method is used to initialize FP-Pilot. For data signals, we adopt the channel inversion power control in [29], i.e., \(\tilde{\rho}_{ik} = \delta/\beta_{i,ik}\) and we set \(\delta = 1\) without loss of generality.

Fig. 2 compares the sum of MSEs for the various methods. According to the figure, the coordinated approach in FP-Pilot reduces the sum of MSEs sharply as compared to the conventional orthogonal method. Furthermore, around 75% of the sum-MSE reduction is obtained after just 10 iterations. It can also be seen that the discrete pilot strategy already improves upon the baseline methods and GSRTM, albeit not by as much as the infinite precision coordinated approach. Fig. 3 takes a closer look at the cumulative distribution of the MSE. Observe that the coordinated approach is far superior to all the
other techniques in that it yields the smallest MSE in all the percentiles.

We now consider minimizing the sum of weighted MSEs across the network. Because the absolute value of MSE is proportional to the channel magnitude, weighting MSEs equally would give preference to the users with strong channels. To provide some measure of fairness, a possible heuristic is to weight the MSEs by \( w_{ik} = 1/\beta_{i,ik} \). (It is possible to use \( \beta_{i,ik} \) then iteratively update \( \gamma_{ik} \) in setting the weights, but such a strategy does not make much difference in our case.)

Fig. 4 shows a scatter plot of MSE vs. channel strength for this weighted coordinated approach as compared to the sum-MSE version of FP-Pilot and the orthogonal method. Although the sum-MSE coordinated approach has considerable advantage over the orthogonal method in minimizing the overall MSEs as shown in the previous results, its performance in the weak-channel region (e.g., when \( \beta_{i,ik} < -75 \) dB) is close to or even slightly worse than that of the orthogonal method as shown in Fig. 4. The reason is that using the sum of MSEs as the objective does not take into account the difference in channel strengths among the users, while the weighted coordinated approach is able to improve the MSE for the cell-edge users (which are more vulnerable to pilot contamination) at a slight cost to the cell-center users. Indeed, the weighted coordinated approach is inferior to the orthogonal method when the channel strength is very strong (\( \beta_{i,ik} > -68 \) dB), but only a very small portion of users have such strong channels. Thus, there is an overall benefit for the weighted coordinated approach.

Finally, we compare the data rates achieved by the different pilot strategies. We use Monte Carlo method to evaluate the instantaneous ergodic rate for each of the algorithms. In particular, for the weighted FP-Pilot, we compare the instantaneous ergodic rate with the deterministic rate. Since the spectrum bandwidth is normalized in simulations, we also refer to data rate as spectral efficiency. Fig. 5 shows the cumulative distributions of data rates. As shown in the figure, the weighted FP-Pilot outperforms the other methods significantly, especially in the low-rate regime. For instance, at the 10th percentile point, the spectral efficiency of the weighted FP-Pilot (with \( w_{ik} = 1/\beta_{i,ik} \)) is at most 5 times higher than that of any other algorithm. Observe also that the achievable rate by Theorem 2 is close to the instantaneous ergodic rate. Note that for the weighted FP-Pilot, these two types of achievable rates both have the cumulative distribution curve as staircase. This is because under the channel inversion power control, the users in the same cell ought to have similar SINRs if the pilot contamination is sufficiently low, and then the cumulative distribution of data rates have 7 “jumps” across the 7 cells; this observation agrees with the conclusion of [29]. This staircase phenomenon however does not occur to the other methods because their channel estimations are not as accurate. Observe also that the orthogonal method achieves higher throughput than the random method and GSRTM, even though it has the worst performance in minimizing the sum of MSEs according to Fig. 4. Hence, in terms of the data rate objective, the sum of MSEs may not be a suitable metric for pilot design.

The paper proposes a matrix-FP approach for coordinating the uplink pilots across multiple cells in order to mitigate pilot contamination in a massive MIMO system. The proposed algorithm optimizes the pilots iteratively in closed form, guaranteeing a monotonic reduction of the sum of weighted MSEs of channel estimation throughout the network. The paper further examines the achievable data rates by using nonorthogonal pilots. The proposed achievable rate expression is more amenable to network optimization than the simultaneous ergodic rate because it does not include expectation. Furthermore, we show a relation between rate maximization and MSE minimization whereby the MSE weights can be properly chosen according to the data rate objective. Numerical results show that the proposed algorithm outperforms the conventional orthogonal pilot reuse method significantly particularly for the cell-edge users and also improves upon a recently proposed greedy method.

**Appendix A**

**Proof of Proposition 2**

Let \( \pi_i = 2 \cdot \text{tr} \left( W_i \Re \{ P_i S_i^* Y_i \} \right) \) and \( \nu_i = \text{tr} \left( W_i Y_i^* \left( \sum_{j=1}^{L} S_j P_j S_j^* Y_j \right) \right) \). We may then rewrite the objective function in (13a) as \( \sum_{i=1}^{L} (\pi_i - \nu_i) + c(Y) \). We can further express each term \( \pi_i \) as

\[
\pi_i = 2 \cdot \text{tr} \left( \text{diag} \left( w_{i1} \beta_{i1,ik}, w_{i2} \beta_{i2,ik}, \ldots, w_{iK} \beta_{iK,ik} \right) \cdot \Re \{ S_i^* Y_i \} \right)
\]

and rewrite each term \( \nu_i \) as

\[
\nu_i = \sum_{j=1}^{L} \text{tr} \left( W_i Y_i^* \left( S_j P_j S_j^* Y_j \right) \right)
\]

\[
= \sum_{j=1}^{L} \text{tr} \left( P_j S_j^* Y_i W_i Y_i^* S_j \right)
\]

\[
= \sum_{k=1}^{K} \sum_{j=1}^{L} \beta_{i,jk}^s s_{jk}^* Y_i W_i Y_i^* s_{jk}.
\]

Combining the above results yields

\[
\sum_{i=1}^{L} (\pi_i - \nu_i)
\]

\[
= \sum_{(i,k)} \left( 2 w_{ik} \beta_{i,ik} s_{ik}^* Y_i - \sum_{j=1}^{L} \beta_{i,jk}^s s_{jk}^* Y_i W_i Y_i^* s_{jk} \right)
\]

\[
= \sum_{(i,k)} \left( 2 w_{ik} \beta_{i,ik} s_{ik}^* Y_i - \sum_{j=1}^{L} \beta_{j,ik}^s Y_j W_j Y_j^* s_{ik} \right)
\]

\[
= \sum_{(i,k)} \left( 2 w_{ik} \beta_{i,ik} s_{ik}^* Y_i - s_{ik}^* \left( \sum_{j=1}^{L} \beta_{j,ik}^s Y_j W_j Y_j^* s_{ik} \right) \right).
\]

Therefore, the objective function (13a) is equivalent to \( \sum_{(i,k)} \xi_{ik} + c(Y) \).
APPENDIX B
COMPUTATION OF $\mathbb{E}[|\hat{h}_{i,ik}|^2]$ IN (36)

Note that the virtual channel $\hat{h}_{i,ik}$ is deterministic; it can be evaluated as

$$
\hat{h}_{i,ik} = \mathbb{E}[\hat{h}_{i,ik}^* h_{i,ik}]
= \mathbb{E}[\beta_{i,ik} s_{ik}^* D_i^{-1} V_i^* h_{i,ik}]
= \mathbb{E} \left[ \beta_{i,ik} s_{ik}^* D_i^{-1} \left( \sum_{j=1}^L \bar{S}_{j} H_{ij} + Z_i^* \right) h_{i,ik} \right]
= M \beta_{i,ik}^2 \bar{s}_{ik}^* D_i^{-1} s_{ik}
= M \mu_{ik},
$$

(53)

where the expectation is taken over $H$. We then commutate the numerator of the SINR term in (36) by taking expectation over $Z$:

$$
\mathbb{E}[|\hat{h}_{i,ik} x_{ik}|^2] = |\hat{h}_{i,ik}|^2 \cdot \mathbb{E}[|x_{ik}|^2]
= M^2 \mu_{ik}^2.
$$

(54)

The subsequent appendix considers the denominator of the SINR term.

APPENDIX C
COMPUTATION OF $\mathbb{E}[|\Delta_{ik}|^2]$ IN (36)

Recall that the interference-plus-noise strength $\mathbb{E}[|\Delta_{ik}|^2]$ in (36) is the expectation over the random variables $(H, Z, x, \hat{z})$. We expand $\mathbb{E}[|\Delta_{ik}|^2]$ as follows:

$$
\mathbb{E}[|\Delta_{ik}|^2] = \mathbb{E} \left[ \left( \sum_{(j,\ell)} \sqrt{\rho_{j\ell}} h_{i,ik}^* \hat{h}_{i,j\ell} x_{j\ell} + \hat{h}_{i,ik} \hat{h}_{i,j\ell} \hat{z}_{ik} - \sqrt{\bar{\rho}_{ik}} \bar{h}_{i,ik} x_{ik} \right)^2 \right]
= \sum_{(j,\ell)} \rho_{j\ell} \mathbb{E}[|\hat{h}_{i,ik} h_{i,j\ell}|^2] + \sigma^2 \mathbb{E}[||\hat{h}_{i,ik}||^2]
= \rho_{i,j\ell} \mathbb{E}[|\hat{h}_{i,ik} x_{ik}|^2],
$$

(55)

where (a) follows by taking expectation over $(x, \hat{z})$. Observe that the last term of (55) is already computed in Appendix B, so the rest of this appendix focuses on computing the other two terms.

The second term of (55) can be computed as

$$
\sigma^2 \mathbb{E}[||\hat{h}_{i,ik}||^2]
= \sigma^2 \mathbb{E}[\hat{h}_{i,ik}^* \hat{h}_{i,ik}]
= \sigma^2 \mathbb{E}[\beta_{i,ik}^2 s_{ik}^* D_i^{-1} V_i^* V_i D_i^{-1} s_{ik}]
= \sigma^2 M^2 \beta_{i,ik}^2 s_{ik}^* D_i^{-1} \left( \sum_{j=1}^L \bar{S}_{j} P_{ij} S_j^* \right) D_i^{-1} s_{ik}
= \sigma^2 M^2 \beta_{i,ik}^2 s_{ik}^* D_i^{-1} s_{ik}
= M \mu_{ik} \sigma^2.
$$

(56)

To determine the first term of (55), we start with each expected channel strength after the MRC processing, that is

$$
\mathbb{E}[|\hat{h}_{i,ik} h_{i,j\ell}|^2]
= \mathbb{E}[|\hat{h}_{i,ik} h_{i,j\ell}^* h_{i,j\ell}|^2]
= \mathbb{E} \left[ \beta_{i,ik}^2 s_{ik}^* D_i^{-1} V_i^* h_{i,j\ell} h_{i,j\ell}^* V_i D_i^{-1} s_{ik} \right]
= \beta_{i,ik}^2 s_{ik}^* D_i^{-1} \cdot \mathbb{E} \left[ V_i^* h_{i,j\ell} h_{i,j\ell}^* V_i \right] \cdot D_i^{-1} s_{ik}
= \beta_{i,ik}^2 s_{ik}^* D_i^{-1} \cdot \mathbb{E} \left[ \sum_{j=1}^L \bar{S}_{j} H_{ij}^* h_{i,j\ell} h_{i,j\ell}^* H_{ij} S_j^* \right]
+ M \beta_{i,j\ell} I_t \cdot D_i^{-1} s_{ik},
$$

(57)

where the expectation part in the middle can be further computed as

$$
\mathbb{E} \left[ \sum_{j=1}^L \bar{S}_{j} H_{ij}^* h_{i,j\ell} h_{i,j\ell}^* H_{ij} S_j^* + M \beta_{i,j\ell} I_t \right]
= \beta_{i,j\ell} \left( \sum_{j=1}^L \bar{S}_{j} \cdot \mathbb{E}[P_{ij} S_j^*] + I_t \right)
+ M^2 \mathbb{E} \left[ \bar{S}_{j} \cdot \mathbb{E}[\hat{z}_{ik}] \cdot \mathbb{E}[P_{ij} S_j^*] \right]
= M \beta_{i,j\ell} D_i + M^2 \mathbb{E} \left[ \bar{S}_{j} \cdot \mathbb{E}[\hat{z}_{ik}] \cdot \mathbb{E}[P_{ij} S_j^*] \right],
$$

(58)

The substitution of (57) and (58) into the first term of (55) yields

$$
\sum_{(j,\ell)} \rho_{j\ell} \mathbb{E}[|\hat{h}_{i,ik} h_{i,j\ell}|^2]
= \sum_{(j,\ell)} \beta_{i,ik}^2 \beta_{i,j\ell} \rho_{j\ell} s_{ik}^* D_i^{-1} s_{ik} + \sum_{j=1}^L \left( M^2 \beta_{i,ik}^2 s_{ik}^* D_i^{-1} \bar{S}_{j} \cdot \mathbb{E}[P_{ij} S_j^*] \right)
= \sum_{(j,\ell)} \beta_{i,ik} \beta_{i,j\ell} \rho_{j\ell} s_{ik}^* D_i^{-1} s_{ik} + \sum_{j=1}^L \left( M^2 \beta_{i,ik}^2 s_{ik}^* D_i^{-1} \bar{S}_{j} \cdot \mathbb{E}[P_{ij} S_j^*] \right)
= M \mu_{ik} \sum_{(j,\ell)} \beta_{i,ik} \rho_{j\ell} + M^2 \mu_{ik} \beta_{i,ik}^2 s_{ik}^* D_i^{-1} \cdot \mathbb{E}[P_{ij} S_j^*] \cdot D_i^{-1} s_{ik}.
$$

(59)

Finally, combining (55), (56), and (59), along with (54) from Appendix B establishes the achievability of the proposed data rate in (27).

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