INVERSE WAVE SCATTERING IN THE TIME DOMAIN FOR POINT SCATTERERS

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Abstract. Let \( \Delta_{\alpha,Y} \) be the bounded from above self-adjoint realization in \( L^2(\mathbb{R}^3) \) of the Laplacian with \( n \) point scatterers placed at \( Y = \{y_1, \ldots, y_n\} \subset \mathbb{R}^3 \), the parameters \( (\alpha_1, \ldots, \alpha_n) \equiv \alpha \in \mathbb{R}^n \) being related to the scattering properties of the obstacles. Let \( u_{\alpha,Y}^{\alpha} \) and \( u_{\alpha}^{\emptyset} \) denote the solutions of the wave equations corresponding to \( \Delta_{\alpha,Y} \) and to the free Laplacian \( \Delta \) respectively, with a source term given by a pulse \( f_\epsilon \) supported in \( \epsilon \)-neighborhoods of the points in \( X_N = \{x_1, \ldots, x_N\} \), \( X_N \cap Y = \emptyset \). We show that, for any fixed \( \lambda > \sup \sigma(\Delta_{\alpha,Y}) \geq 0 \), there exists \( N_0 \geq 1 \) such that the locations of the points in \( Y \) can be determined by the knowledge of a finite-dimensional scattering data operator \( F^\alpha_{\lambda} : \mathbb{R}^N \rightarrow \mathbb{R}^N, N \geq N_0 \). Such an operator is defined in terms of the limit as \( \epsilon \downarrow 0 \) of the Laplace transform of \( u_{\alpha,Y}^{\alpha}(t, x_k) - u_{\alpha}^{\emptyset}(t, x_k) \), \( k = 1, \ldots, N \). We exploit the factorized form of the resolvent difference \((-\Delta_{\alpha,Y} + \lambda)^{-1} - (-\Delta + \lambda)^{-1}\) and a variation on the finite-dimensional factorization in the MUSIC algorithm. Multiple scattering effects are not neglected; our model can be interpreted as the time-domain version of a frequency-domain scattering from an array of Foldy’s point-like obstacles.

1. Introduction.

Given the finite set \( Y = \{y_1, \ldots, y_n\} \subset \mathbb{R}^3 \) and \( 0 < r \ll 1 \), let

\[
\partial_{tt}u = \Delta_{Y^r}u
\]

be the wave equation describing the propagation of acoustic waves in the inhomogeneous medium made of a homogeneous one containing the array of \( n \) small spherical obstacles

\[
Y^r = B^1_r \cup \cdots \cup B^n_r, \quad B^i_r = \{x \in \mathbb{R}^3 : |x - y_i| < r\}.
\]

More precisely, \( \Delta_{Y^r} \) is the self-adjoint realization in \( L^2(\mathbb{R}^3) \) of the Laplacian with boundary conditions

\[
(1.1) \quad \gamma_i^0 u + \alpha_i(r) [\gamma_i^1] u = 0, \quad i = 1, \ldots, n,
\]

at the boundaries \( S^i_r = \{x \in \mathbb{R}^3 : |x - y_i| = r\} \). Here \( \gamma_i^0 \) and \( [\gamma_i^1] \), denote the Dirichlet trace at \( S^i_r \) and the jump across \( S^i_r \) of the Neumann trace \( \gamma_i^1 \) respectively; \( \alpha_1(r), \ldots, \alpha_n(r) \) are \( r \)-dependent parameters to be specified later.

In our previous work [17], we considered inverse wave scattering in the time domain for a wide class of self-adjoint Laplacians, including those with hard, soft and semi-transparent bounded obstacles with Lipschitz boundaries. By applying to \( \Delta_{Y^r} \) the results there provided (which build on our previous works [18], [19], [15], [16]), one gets the following: denoting by \( u_{f^r}^{\alpha} \) and \( u_{f}^{\emptyset} \) the solutions of the wave equations corresponding to \( \Delta_{Y^r} \) and to the free Laplacian \( \Delta \) respectively, with a source term \( f \) concentrated at time \( t = 0 \) (a pulse) one has
that for any fixed $\lambda \geq \lambda_0 > 0$ and any fixed open set $B \subset \subset \mathbb{R}^n \setminus Y_r$, the obstacle $Y^r$ can be reconstructed by the knowledge of the data operator $F^r_{\lambda} : L^2(B) \to L^2(B)$,

$$F^r_{\lambda}f := \int_0^\infty e^{-\sqrt{\lambda} t} 1_B (u^r_f(t, \cdot) - u^\varnothing_f(t, \cdot)) 1_B \, dt, \quad \text{supp}(f) \subset B.$$ 

Since the choice of the set $B$ where both the source and the detector are placed is arbitrary (beside the constraint $B \cap Y = \emptyset$), one is lead to choose $B$ having the same kind of shape as $Y^r$, i.e., $B = X^\epsilon$, where $X^\epsilon$ denotes the $\epsilon$-neighborhood of a set $X = \{x_1, \ldots, x_N\}$ such that $X \cap Y = \emptyset$. Thus, given $X$ and $Y$, once the parameters $\alpha_i(r)$ and $\lambda$ have been fixed, the data operator $F^r_{\lambda}X^\epsilon$ in (1.2) depends on $r$ and $\epsilon$ alone and a natural question arises: what happens whenever $r \searrow 0$ and $\epsilon \searrow 0$? In more detail:

1) is there a well defined limit self-adjoint operator $\Delta_{\alpha, Y}$ describing the propagation of acoustic waves in an otherwise homogeneous medium containing an array $Y$ of point scatterers?

2) is such an array $Y$ determined by a finite-dimensional scattering data operator $F^r_{\lambda}X$ corresponding to $X$ and to the wave dynamics generated by $\Delta_{\alpha, Y}$?

The answer to the first question has been known from a long time: by [7, Theorem 2] (see also [2, Lemma 2.2] and [23, Theorems 3.4 and 3.7] for the case of a single sphere), setting

$$\alpha_i(r) = r + 4\pi \alpha_i r^2 + o(r^2), \quad \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n,$$

in (1.1), $\Delta_{Y^r}$ converges as $r \searrow 0$ (in strong resolvent sense) to a well defined self-adjoint, bounded from above, operator $\Delta_{\alpha, Y}$. Such an operator $\Delta_{\alpha, Y}$ was firstly rigorously defined in the seminal paper [4] as a self-adjoint extension of the Laplacian restricted to smooth functions with compact support disjoint from $Y$. Since then it attracted an increasing attention and has been used in a wide range of applications: we refer to the huge list of references in [1], the main text devoted to this operator and its ramifications. In next Section 2 we will recall the definition of $\Delta_{\alpha, Y}$ and describe its main properties.

Although the origin of $\Delta_{\alpha, Y}$ has its root in Quantum Mechanics, as well as most of its applications (however see [21] for its connections with electrodynamics of point particles), in recent years it has been used to provide a rigorous mathematical framework for Foldy’s scattering of time-harmonic acoustic and elastic waves, see [10] and [11]. While Foldy’s approach considers wave scattering in the frequency domain (see [8, 20, Section 8.3]), our aim here is to work in the time domain.

Let us point out that the scaling (1.3) entering into the boundary conditions (1.1) is the only one leading to a not trivial limit dynamic, i.e., a dynamic different from propagation of free waves in the whole space. This is reminiscent of the case in which one approximates points scatterers with a scaled potential, where the limit dynamics is not trivial if and only if the unscaled potential has a zero-energy resonance (see [11, Section I.I.2]). There is an analogous phenomenon whenever one approximate a point scatterer with an obstacle modeled by a shrinking sphere: the boundary conditions (1.1) together with (1.3) provide a zero-energy resonance (see [23, Theorems 3.7]); different boundary conditions lead, in the limit $r \searrow 0$, to the free Laplacian. For example, whenever one consider Dirichlet boundary conditions on an array of shrinking spheres, one gets an expansion (w.r.t. the radius $r \ll 1$) of the scattered waves which contains no zero-order term (see [24]; notice that the coefficients
$C_j$ appearing there in the expansion (1.7) are proportional to the radii of the shrinking spheres.

Taking into account the limit $r \to 0$, one has then at disposal the well defined data operator $F^X_\lambda : L^2(X^\epsilon) \to L^2(X^\epsilon)$,

$$F^X_\lambda f := \lim_{r \to 0} F^Y_{\lambda,X^\epsilon} f = \int_0^\infty e^{-\sqrt{\lambda} t} 1_{X^\epsilon}(u_j^{\alpha,Y}(t,\cdot) - u_j^{\emptyset}(t,\cdot)) 1_{X^\epsilon} \ dt , \quad \text{supp}(f) \subset X^\epsilon .$$

where $u_j^{\alpha,Y}$ denotes the solutions of the wave equation corresponding to $\Delta_{\alpha,Y}$ with source term given by the pulse $f$ (see Remark 3.1).

As we will recall in Subsection 3.3 below, a relevant point is the fact that the limit wave equation generated by $\Delta_{\alpha,Y}$, i.e.,

$$\partial_{tt} u = \Delta_{\alpha,Y} u$$

can be recasted into the distributional form

$$\partial_{tt} u = \Delta u + \sum_{i=1}^n q_i(t) \delta_{y_i} .$$

Here $\delta_{y_i}$ denotes Dirac’s delta distribution at $y_i$ and the $q_i(t)$’s evolve according to a first-order retarded differential equation (see [14], [22] and (3.16) below); the terms containing the retardation provide the contributions due to multiple scattering.

Let us point out that, for a time-harmonic solution $u(t,x) = e^{-it\kappa} u_\kappa(x)$, the equation (1.4) recasts into the generalized eigenfunctions problem

$$(\Delta_{\alpha,Y} + \kappa^2) u_\kappa = 0 ,$$

whose solutions are known to be of the kind (see, e.g., [1] eq. (1.5.1), Section II.1.5)

$$u_\kappa(x) = u_\kappa^{in}(x) + \frac{1}{4\pi} \sum_{1 \leq j, \ell \leq n} \Lambda_{\kappa - \kappa^2 + i\epsilon}^{j\ell} u_\kappa^{in}(y_{\ell}) e^{i\kappa|x-y_{\ell}|}|x-y_{\ell}| ,$$

where $u_\kappa^{in}$ solves the free Helmholtz equation, while $\Lambda_{\kappa - \kappa^2 + i\epsilon}^{j\ell} := \lim_{\epsilon \to 0} \Lambda_{\kappa - \kappa^2 + i\epsilon}^{j\ell}$ and $\Lambda_{\kappa}$ is the inverse of the matrix (2.2) defined in the next Section. Since

$$\Lambda_{\kappa - \kappa^2 + i\epsilon} = \text{Diag}(\alpha_j - i\kappa/4\pi) A_\kappa ,$$

where the matrix $A_\kappa$ is defined as in [5] page 7, Section 2.3.1, formula (1.6) describes the scattering by an array $Y$ of Foldy’s point scatterers having scatterings coefficients

$$g_j = \frac{1}{\alpha_j - i\kappa/4\pi} ,$$

(see [5] eqs. (2.9) and (2.19)).

The aim of the present paper is to give a positive answer to the second question. We show that the limit data operator $F^X_\lambda := \lim_{\kappa \to 0} F^X_\lambda$ is a well defined map on $\mathbb{R}^N$ to itself (see Lemma 3.3 for the complete result). Moreover, denoting by $P^X_\lambda$ the orthogonal projector onto $\ker(F^X_\lambda)$ and by $\phi^X_\lambda(z) \in \mathbb{R}^N$ the vector with components

$$(\phi^X_\lambda(z))_k = \frac{e^{-\sqrt{\lambda}|x_k-z|}}{|x_k-z|} , \quad z \in \mathbb{R}^3 \setminus X , \quad x_k \in X ,$$

we have

$$(\phi^X_\lambda(z))_k = \frac{e^{-\sqrt{\lambda}|x_k-z|}}{|x_k-z|} , \quad z \in \mathbb{R}^3 \setminus X , \quad x_k \in X ,$$
we show in Theorem 4.2 (to which we refer for the precise statement) that the set \( Y \) is determined according to the relation
\[
Y = \{ \text{peak points of the function } \mathbb{R}^3 \setminus X \ni z \mapsto |P^X_\lambda \phi^X(z)|^{-1} \}.
\]
Our results can be read as a time-domain analogue of the inverse scattering by point-like scatterers in the Foldy regime studied in \([3\, \text{Section 2.3.1}]\); they provide the counterpart, in the case of point scatterers, of our previous results (see \([17]\)) on time-domain inverse scattering for extended obstacles.

The main ingredients in our proofs are the factorized form of the resolvent difference \((−\Delta_{α,Y} + λ)^{-1} − (−\Delta + λ)^{-1}\) and a variation on the factorization method approach to the MUSIC (MUltiple-SIgnal-Classification) algorithm provided by Kirsch in \([12\, \text{Section 2}]\) (see also \([13\, \text{Section 4.1}]\)); however here, contrarily to the frequency-domain case treated by Kirsch, the multiple scattering effects are not neglected.

2. LAPLACIANS WITH POINT SCATTERERS.

Given \( Y = \{y_1, \ldots, y_n\} \subset \mathbb{R}^3 \), to any \( α ≡ (α_1, \ldots, α_n) \in \mathbb{R}^n \) there corresponds the self-adjoint realization in \( L^2(\mathbb{R}^3) \) of the Laplacian with \( n \) point scatterers \( y_1, \ldots, y_n \) defined by
\[
\text{dom}(\Delta_{α,Y}) = \left\{ u \in L^2(\mathbb{R}^3) : u(x) = u_0(x) + \frac{1}{4\pi} \sum_{j=1}^n \frac{ξ_j}{|x - y_j|}, \ u_0 \in \dot{H}^2(\mathbb{R}^3) , \right\},
\]
\[
ξ ≡ (ξ_1, \ldots, ξ_n) \in \mathbb{C}^n , \ \lim_{x \to y_j} \left( u(x) − \frac{1}{4\pi} \frac{ξ_j}{|x - y_j|} \right) = α_jξ_j \}
\]
\[
Δ_{α,Y} : \text{dom}(Δ_{α,Y}) \subset L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3) , \quad Δ_{α,Y}u := Δu_0.
\]

We refer to \([1\, \text{Chapter II.1}]\) for more details and proofs. Here the homogeneous Sobolev space of order two \( Η^2(\mathbb{R}^3) \) is defined by
\[
\dot{Η}^2(\mathbb{R}^3) := \{ u \in Η_0(\mathbb{R}^3) : |∇u| \in L^2(\mathbb{R}^3), \ Δu \in L^2(\mathbb{R}^3) \}
\]
and its relation with the usual Sobolev space of order two \( Η^2(\mathbb{R}^3) \) is given by \( Η^2(\mathbb{R}^3) = \dot{Η}^2(\mathbb{R}^3) \cap L^2(\mathbb{R}^3) \).

The operator \( Δ_{α,Y} \) belongs to the set of self-adjoint extensions of the symmetric one \( S_Y \) given by the restriction of the free Laplacian to functions vanishing at the points in \( Y \), i.e., \( S_Y := Δ|C_0^\infty(\mathbb{R}^3 \setminus Y) ; \) the vector \( α \in \mathbb{R}^n \) plays the role of the extension parameter.

The extensions of the kind \( Δ_{α,Y} \) suffice for the description of the relevant physical models: by \([14\, \text{Theorem 4}]\), the wave equation \( ∂_t u = Au \) corresponding to a self-adjoint extension \( S_Y ⊂ A ⊂ S_Y^* \) has a finite speed of propagation if and only if \( A = Δ_{α,Y} \) for some \( α \in \mathbb{R}^n \). Moreover, finite speed of propagation holds if and only if the boundary conditions at \( Y \) specifying the self-adjointness domain are of local type, i.e., they do not couple scatterers placed at different points: the scatterers are independent of each other.

The vector \( α \in \mathbb{R}^n \), beside specifying the boundary conditions at \( Y \), is related to the scattering length \( a \) of the scatterers through the relation \( a = −(4\pi)^{-1} \sum_{i=1}^n α_i^{-1} \) (see \([1\, \text{Section II.1.5}]\)).

The resolvent of \( Δ_{α,Y} \) is given by
\[
(−Δ_{α,Y} + ζ)^{-1} = (−Δ + ζ)^{-1} + K_ζ , \quad ζ \notin σ(Δ_{α,Y}) ,
\]
where \( K_ζ : \text{dom}(Δ_{α,Y}) \to L^2(\mathbb{R}^3) \).
By \([\text{Section 2}]\), the quadratic form
\[
(\Delta + \zeta)^{-1} : L^2(\mathbb{R}^3) \to H^2(\mathbb{R}^3), \quad \zeta \in \mathbb{C} \setminus (-\infty, 0],
\]
is the resolvent of the free Laplacian with kernel function
\[
(\Delta + \zeta)^{-1}(x, y) = \frac{1}{4\pi} \frac{e^{-\sqrt{\zeta}|x-y|}}{|x-y|}, \quad \Re(\sqrt{\zeta}) > 0,
\]
and the finite-rank operator \(K_\zeta : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)\) has kernel function
\[
K_\zeta(x, y) = \frac{1}{(4\pi)^2} \sum_{1 \leq i, j \leq n} \Lambda_{\zeta}^{ij} \frac{e^{-\sqrt{\zeta}|x-y_i|}}{|x-y_i|} \frac{e^{-\sqrt{\zeta}|y-y_j|}}{|y-y_j|}.
\]
Here \(\Lambda_\zeta \equiv (\Lambda_{\zeta}^{ij})\) is the inverse of the \(n \times n\) matrix \(M_\zeta \equiv (M_{\zeta}^{ij})\) given by
\[
(2.2) \quad M_{\zeta}^{ij} = \left(\alpha_i + \frac{\sqrt{\zeta}}{4\pi}\right) \delta_{ij} - \frac{e^{-\sqrt{\zeta}|y_i-y_j|}}{4\pi |y_i-y_j|} (1 - \delta_{ij}),
\]
\(\delta_{ij}\) denoting Kronecker’s delta. Regarding the spectral profile, since the resolvent of \(\Delta_{a,Y}\) is a \(n\)-rank perturbation of the free resolvent, the essential spectrum of the free Laplacian is preserved and the discrete spectrum contains at most \(n\) distinct eigenvalues; in more detail
\[
\sigma_{ac}(\Delta_{a,Y}) = \sigma_{ess}(\Delta_{a,Y}) = (-\infty, 0], \quad \sigma_{disc}(\Delta_{a,Y}) = \{\lambda > 0 : \det M_\lambda = 0\}.
\]
For later purposes, we need to investigate the positiveness of the matrix in (2.2):

**Lemma 2.1.** Let the matrix \(M_\lambda \equiv (M_{\zeta}^{ij})\) be defined by (2.2) with \(\zeta = \lambda \in (0, +\infty)\). Then,
\[
\lambda > \lambda_{a,Y} := \sup \sigma(\Delta_{a,Y}).
\]

**Proof.** Let \(\lambda > 0\) and
\[
v_\lambda(\xi) := \frac{1}{4\pi} \sum_{j=1}^{n} \xi_j \frac{e^{-\sqrt{\lambda}|x-y_j|}}{|x-y_j|}, \quad \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n.
\]
By \([25]\) Section 2, the quadratic form \(Q_{a,Y}\) of \(-\Delta_{a,Y}\) has the \(\lambda\)-independent representation
\[
\text{dom}(Q_{a,Y}) = \{u \in L^2(\mathbb{R}^3) : u = u_\lambda + v_\lambda(\xi), \ u_\lambda \in H^1(\mathbb{R}^3), \xi \in \mathbb{C}^n\};
\]
\[
Q_{a,Y}(u) = \|\nabla u_\lambda\|_{L^2}^2 + \lambda \|u_\lambda\|_{L^2}^2 - \lambda \|u\|_{L^2}^2 + \langle \xi, M_\lambda \xi \rangle.
\]
If \(\lambda > \lambda_{a,Y}\) then, for any \(\xi \neq 0,
\[
\langle \xi, M_\lambda \xi \rangle = Q_{a,Y}(v_\lambda(\xi)) + \lambda \|v_\lambda(\xi)\|_{L^2}^2 > 0.
\]
Conversely, let \(\lambda > 0\) be such that \(M_\lambda\) is positive-definite and, given any \(u \in \text{dom}(Q_{a,Y}) \setminus \{0\}\), let use the decomposition \(u = u_\lambda + v_\lambda(\xi)\). Then
\[
Q_{a,Y}(u) + \lambda \|u\|_{L^2}^2 = \|\nabla u_\lambda\|_{L^2}^2 + \lambda \|u_\lambda\|_{L^2}^2 + \langle \xi, M_\lambda \xi \rangle
\]
\[
\geq \begin{cases} \langle \xi, M_\lambda \xi \rangle, & \xi \neq 0 \\ \|\nabla u_\lambda\|_{L^2}^2 + \lambda \|u_\lambda\|_{L^2}^2, & \xi = 0 \\ 0 & \end{cases}
\]
and so \(\lambda > \lambda_{a,Y}\). \(\square\)
Obviously, whenever \( Y \) is the singleton \( Y = \{ y \} \) one has

\[
\lambda_{\alpha,y} = \begin{cases} 
0 & \alpha \geq 0 \\
(4\pi\alpha)^2 & \alpha < 0.
\end{cases}
\]

The next result provides a simple rough estimate on \( \lambda_{\alpha,Y} \) whenever \( n > 1 \).

**Lemma 2.2.** Set

\[
\alpha_o := \min_{1 \leq i \leq n} \alpha_i, \quad d := \min_{i \neq j} |y_i - y_j|.
\]

Then

\[
0 \leq \lambda_{\alpha,Y} \leq \begin{cases} 
0, & 4\pi\alpha_o d \geq n - 1, \\
\lambda_o, & 4\pi\alpha_o d < n - 1,
\end{cases}
\]

where \( \lambda_o > 0 \) solves

\[
4\pi\alpha_o d + \sqrt{\lambda_o} d = (n - 1) e^{-\sqrt{\lambda_o} d}.
\]

**Proof.** The thesis is consequence of (2.3) and the inequality

\[
\langle \xi, M_{\lambda} \xi \rangle = \sum_{j=1}^{n} \left( \alpha_j + \frac{\sqrt{\lambda}}{4\pi} \right) |\xi_j|^2 - \sum_{j<k} \frac{e^{-\sqrt{\lambda}|y_j - y_k|}}{4\pi |y_j - y_k|} 2 \Re(\bar{\xi}_j \xi_k)
\]

\[
\geq \left( \alpha_o + \frac{\sqrt{\lambda}}{4\pi} \right) |\xi|^2 - \frac{e^{-\sqrt{\lambda}d}}{4\pi d} \sum_{j<k} 2 |\xi_j| |\xi_k|
\]

\[
\geq \left( \alpha_o + \frac{\sqrt{\lambda}}{4\pi} - (n - 1) \frac{e^{-\sqrt{\lambda}d}}{4\pi d} \right) |\xi|^2.
\]

\[\square\]

3. Wave scattering and the data operator.

3.1. **Abstract wave equations.** Let \( A : \text{dom}(A) \subset L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3) \) be self-adjoint and bounded from above; we consider the Cauchy problem for the corresponding wave equation

\[
\begin{cases} 
\partial_t^2 u(t) = Au(t) \\
u(0) = u_0 \in L^2(\mathbb{R}^3) \\
\partial_t u(0) = v_0 \in L^2(\mathbb{R}^3).
\end{cases}
\]

(3.1)

We say that \( u \in C(\mathbb{R}_+; L^2(\mathbb{R}^3)) \) is a mild solution of (3.1) whenever, for any \( t \geq 0 \), there holds

\[
\int_0^t (t-s)u(s) \, ds \in \text{dom}(A) \quad \text{and} \quad u(t) = u_0 + tv_0 + A \int_0^t (t-s)u(s) \, ds.
\]

By [3, Proposition 3.14.4, Corollary 3.14.8 and Example 3.14.16], the unique mild solution of (3.1) is given by

\[
u(t) = \cos_A(t) u_0 + \sin_A(t) v_0.
\]

(3.2)
where the \( \mathcal{B}(L^2(\mathbb{R}^3)) \)-valued functions \( t \mapsto \cos_A(t) \) and \( t \mapsto \sin_A(t) \) are defined through the \( \mathcal{B}(L^2(\mathbb{R}^3)) \)-valued (inverse) Laplace transform by the relations

\[
(3.3) \quad \sqrt{\lambda}(-A + \lambda)^{-1} = \int_0^\infty e^{-\sqrt{\lambda}t} \cos_A(t) \, dt, \quad \lambda > \lambda_A,
\]

\[
(3.4) \quad (-A + \lambda)^{-1} = \int_0^\infty e^{-\sqrt{\lambda}t} \sin_A(t) \, dt, \quad \lambda > \lambda_A,
\]

with

\[
\lambda_A := \sup \sigma(A).
\]

Notice that (see [3, relation (3.93)])

\[
(3.5) \quad \sin_A(t) = \int_0^t \cos_A(s) \, ds.
\]

If \( \lambda_A = 0 \), then, by functional calculus,

\[
\cos_A(t) = \cos(t(-A)^{1/2}), \quad \sin_A(t) = (-A)^{-1/2} \sin(t(-A)^{1/2}).
\]

Given \( \chi \in L^1(0, +\infty) \) and given \( g \in L^2(\mathbb{R}^3) \), let \( u^A_{\chi g} \) be the solution of the wave equation with the source \( \chi g \), i.e.,

\[
(3.6) \quad \begin{cases}
\partial_{tt} u^A_{\chi g}(t) = A u^A_{\chi g}(t) + \chi(t)g \\
u^A_{\chi g}(0) = 0 \\
\partial_t u^A_{\chi g}(0) = 0.
\end{cases}
\]

By [3] Proposition 3.1.16 (see also [6, Section II.4]),

\[
(3.7) \quad u^A_{\chi g}(t) = \int_0^t \sin_A(t-s) \chi(s) g \, ds.
\]

Let \( \chi_\tau \in L^1(0, +\infty) \) be an approximation of Dirac’s delta distribution at \( t = 0 \), i.e.,

\[
(3.8) \quad \chi_\tau(t) \geq 0, \quad \int_0^{+\infty} \chi_\tau(s) \, ds = 1, \quad \lim_{\tau \searrow 0} \int_0^{+\infty} s \chi_\tau(s) \, ds = 0.
\]

Two common choices are \( \chi_\tau(t) = \frac{1}{\tau} 1_{[0,\tau]}(t) \) and \( \chi_\tau(t) = \frac{1}{\tau} e^{-t/\tau} \).

Let \( u^A_g(t) \) be the solution of the homogenous Cauchy problem

\[
(3.9) \quad \begin{cases}
\partial_{tt} u^A_g(t) = A u^A_g(t) \\
u^A_g(0) = 0 \\
\partial_t u^A_g(0) = g.
\end{cases}
\]
By (3.2), (3.7), (3.5) and hypotheses (3.8), one gets
\[
\lim_{\tau \to 0} \left\| u_g^A(t) - u_{\chi r g}^A(t) \right\|_{L^2} = \lim_{\tau \to 0} \left\| \sin_A(t) g - \int_0^t \sin_A(t-s) \chi_\tau(s) g \, ds \right\|_{L^2} \\
\leq \lim_{\tau \to 0} \left( \int_0^t \left( \int_{t-s}^t \cos_A(r) \, dr \right) \chi_\tau(s) g \right) \, ds + \int_0^{+\infty} \left\| \sin_A(t) \chi_\tau(s) g \right\|_{L^2} \, ds \\
\leq \lim_{\tau \to 0} \left( \sup_{0 \leq r \leq t} \left\| \cos_A(r) \right\|_{L^2} \left\| g \right\|_{L^2} \int_0^t s \chi_\tau(s) \, ds + \left\| \sin_A(t) \right\|_{L^2} \left\| g \right\|_{L^2} \int_0^{+\infty} \chi_\tau(s) \, ds \right) \\
\leq c \lim_{\tau \to 0} \int_0^{+\infty} s \chi_\tau(s) \, ds = 0.
\]
Hence, the \( u_g^A(t) \) solving (3.9) can be interpreted as the solution of the inhomogeneous Cauchy problem
\[
\begin{cases}
\partial_t u_g^A(t) = A u_g^A(t) + \delta_0(t) g \\
u_g^A(0) = 0 \\
\partial_t u_g^A(0) = 0.
\end{cases}
\]

**Remark 3.1.** By (3.4), if \( A_n \) converges to \( A \) in strong resolvent sense as \( n \to +\infty \), then
\[
\lim_{n \to +\infty} \int_0^\infty e^{-\sqrt{\lambda} t} A_n^A \, dt = \int_0^\infty e^{-\sqrt{\lambda} t} A^A \, dt.
\]

### 3.2. The data operator with point scatterers.

Let \( u_{\chi r g}^\alpha \) and \( u_{\chi r g}^\varphi \) denote the solutions of the Cauchy problems
\[
\begin{align*}
\begin{cases}
\partial_t u_{\chi r g}^{\alpha,Y}(t, x) = \Delta_{\alpha,Y} u_{\chi r g}^{\alpha,Y}(t, x) + \chi_\tau(t) g(x) \\
u_{\chi r g}^{\alpha,Y}(0, x) = 0 \\
\partial_t u_{\chi r g}^{\alpha,Y}(0, x) = 0,
\end{cases}
& \quad \begin{cases}
\partial_t u_{\chi r g}^{\varphi,Y}(t, x) = \Delta u_{\chi r g}^{\varphi,Y}(t, x) + \chi_\tau(t) g(x) \\
u_{\chi r g}^{\varphi,Y}(0, x) = 0 \\
\partial_t u_{\chi r g}^{\varphi,Y}(0, x) = 0
\end{cases}
\end{align*}
\]

With respect to the previous subsection, here we use the notations
\[
u_{\alpha,...} \equiv u_{\alpha,...}^{\Delta}, \quad \nu_{\varphi,...} \equiv u_{\varphi,...}^{\Delta}.
\]

In typical scattering experiments one measures the scattered wave
\[
u_{\chi r g}^{\alpha,Y}(t, x) - u_{\chi r g}^{\varphi}(t, x)
\]
produced by a sharp pulse \( \chi_r g, \tau \ll 1 \). By the previous discussion leading to (3.10) (equivalently to (3.9)), in the ideal experiment in which the pulse is concentrated at \( t = 0 \), (3.11) is replaced by
\[
u_{\chi r g}^{\alpha,Y}(t, x) - u_{\chi r g}^{\varphi}(t, x),
\]
where \( u_{\chi r g}^{\alpha,Y} \) and \( u_{\chi r g}^{\varphi} \) solve
\[
\begin{align*}
\begin{cases}
\partial_t u_{\chi r g}^{\alpha,Y}(t, x) = \Delta_{\alpha,Y} u_{\chi r g}^{\alpha,Y}(t, x) \\
u_{\chi r g}^{\alpha,Y}(0, x) = 0 \\
\partial_t u_{\chi r g}^{\alpha,Y}(0, x) = g(x)
\end{cases}
& \quad \begin{cases}
\partial_t u_{\chi r g}^{\varphi,Y}(t, x) = \Delta u_{\chi r g}^{\varphi,Y}(t, x) \\
u_{\chi r g}^{\varphi,Y}(0, x) = 0 \\
\partial_t u_{\chi r g}^{\varphi,Y}(0, x) = g(x)
\end{cases}
\end{align*}
\]
Considering then an array of points $X_N = \{x_1, \ldots, x_N\} \subset \mathbb{R}^3 \setminus Y$, we now assume $g = f_\epsilon$ supported in an $\epsilon$-neighborhood of $X_N$, where

$$
(3.13) \quad f_\epsilon(x) := \sum_{k=1}^{N} f_k \varphi_\epsilon(x - x_k), \quad f \equiv (f_1, \ldots, f_N),
$$

$$
(3.14) \quad \varphi_\epsilon(x) = \frac{1}{\epsilon^3} \varphi \left( \frac{x}{\epsilon} \right), \quad \varphi \in \mathcal{C}_\text{comp}^\infty(\mathbb{R}^3), \quad \int_{\mathbb{R}^3} \varphi(x) \, dx = 1.
$$

By (3.13) and (3.14), $f_\epsilon$ converges, in distributional sense, to $\sum_{k=1}^{N} f_k \delta_{x_k}$ as $\epsilon \searrow 0$. Let us introduce the operator

$$
F_{\lambda}^{N,\epsilon} : \mathbb{R}^N \to \mathbb{R}^N, \quad (F_{\lambda}^{N,\epsilon} f)_k := \int_{0}^{\infty} e^{-\sqrt{\lambda t}} (u_{f_\epsilon}^{\alpha,Y}(t, x_k) - u_{f_\epsilon}^{\varnothing}(t, x_k)) \, dt, \quad \lambda > \lambda_{\alpha,Y}.
$$

$F_{\lambda}^{N,\epsilon}$ corresponds to the measurements at time $t$ and at points $x_1, \ldots, x_N$ of the scattered waves produced by pulses supported at $t = 0$ and in tiny (whenever $\epsilon \ll 1$) neighborhoods of the same points $x_1, \ldots, x_N$ (detectors and emitters are at the same places).

**Remark 3.2.** Since $f_\epsilon \in \mathcal{C}_\text{comp}^\infty(\mathbb{R}^3)$ belongs to the form domain of $\Delta_{\alpha,Y}$ (that is to dom$(Q_{\alpha,Y})$ as defined in the proof of Lemma 2.1), the solution $u_{f_\epsilon}^{\alpha,Y}(t, \cdot)$ entering in the definition of $F_{\lambda}^{N,\epsilon}$ is a strong one (see e.g. [9, Chapter 2, section 7]), i.e.,

$$
\begin{align*}
  &u_{f_\epsilon}^{\alpha,Y} \in \mathcal{C}(\mathbb{R}_+, \text{dom}(\Delta_{\alpha,Y})) \cap \mathcal{C}^1(\mathbb{R}_+, \text{dom}(Q_{\alpha,Y})) \cap \mathcal{C}^2(\mathbb{R}_+, L^2(\mathbb{R}^3)).
\end{align*}
$$

Since dom$(\Delta_{\alpha,Y}) \subset \mathcal{C}(\mathbb{R}^3 \setminus Y)$, the evaluation at the point $x_k$ in $(F_{\lambda}^{N,\epsilon} f)_k$ is a legitimate operation.

Next we show that $F_{\lambda}^{N,\epsilon}$ admits a well defined limit as $\epsilon \searrow 0$, so that one is allowed to consider the case in which the $N$ emitters and the $N$ detectors are both placed at the points $x_1, \ldots, x_N$.

**Lemma 3.3.** Let $u_{f_\epsilon}^{\alpha,Y}$ and $u_{f_\epsilon}^{\varnothing}$ be the solutions of the Cauchy problems in (3.12), where $g$ equals $f_\epsilon$ defined in (3.13). Then, the limits

$$
\lim_{\epsilon \searrow 0} (F_{\lambda}^{N,\epsilon} f)_k = \lim_{\epsilon \searrow 0} \int_{0}^{\infty} e^{-\sqrt{\lambda t}} (u_{f_\epsilon}^{\alpha,Y}(t, x_k) - u_{f_\epsilon}^{\varnothing}(t, x_k)) \, dt, \quad k = 1, \ldots, N,
$$

exist.
Proof. Since \( u_{f,\epsilon}^{\alpha,Y} \) and \( u_{f,\epsilon}^{\varphi} \) solve (3.12) with \( g = f_\epsilon \), by (3.2), (3.4) and the resolvent formula (2.1), one obtains

\[
\lim_{\epsilon \searrow 0} (F_{\lambda}^{N,\epsilon} f)_k = \lim_{\epsilon \searrow 0} \int_{0}^{\infty} e^{-\sqrt{\lambda} t} (u_{f,\epsilon}^{\alpha,Y}(t, x_k) - u_{f,\epsilon}^{\varphi}(t, x_k)) dt
\]

\[
= \lim_{\epsilon \searrow 0} \int_{0}^{\infty} e^{-\sqrt{\lambda} t} (\text{Sin}\Delta_{\alpha,Y}(t)f_\epsilon)(x_k) - ((-\Delta)^{-1/2} \sin(t(-\Delta)^{1/2}) f_\epsilon)(x_k) dt
\]

\[
= \lim_{\epsilon \searrow 0} \left( (-\Delta_{\alpha,Y} + \lambda)^{-1} f_\epsilon - (-\Delta + \lambda)^{-1} f_\epsilon \right)(x_k)
\]

\[
= \frac{1}{(4\pi)^2} \sum_{i,j=1}^{n} \Lambda_{ij}^{\alpha,Y} e^{-\sqrt{\lambda} |x_k - y_i|} \left( \lim_{\epsilon \searrow 0} \int_{\mathbb{R}^3} e^{-\sqrt{\lambda} |x - y_j|} f_\epsilon(x) dx \right)
\]

By (3.13) and (3.14),

\[
\lim_{\epsilon \searrow 0} (F_{\lambda}^{N,\epsilon} f)_k = \lim_{\epsilon \searrow 0} \int_{0}^{\infty} e^{-\sqrt{\lambda} t} (u_{f,\epsilon}^{\alpha,Y}(t, x_k) - u_{f,\epsilon}^{\varphi}(t, x_k)) dt
\]

\[
= \frac{1}{(4\pi)^2} \sum_{i,j=1}^{n} \Lambda_{ij}^{\alpha,Y} e^{-\sqrt{\lambda} |x_k - y_i|} \sum_{\ell=1}^{N} e^{-\sqrt{\lambda} |x_{\ell} - y_j|} f_\epsilon \cdot \quad \square
\]

3.3. A convenient representation of the scattered waves. In order to implement numerical tests, it is useful to have at disposal an explicit formula for the difference of the solutions of the two Cauchy problems (3.12), providing a convenient representation of the scattered waves entering in the definition of \( F_{\lambda}^{N,\epsilon} \). By [14] Theorem 3] (see also [22] Theorem 3.1], and, for the case of a single point scatterer, the antecedent result in [21, Theorem 3.2]) \( u_{f,\epsilon}^{\alpha,Y}(t) \) can be written in terms of \( u_{f,\epsilon}^{\varphi}(t) \) and of the solution of a system of inhomogeneous retarded first-order differential equations.

More precisely, if \( q_\epsilon(t) \equiv (q_\epsilon^1(t), \ldots, q_\epsilon^n(t)) \), \( t \geq 0 \), denotes the unique solution of the Cauchy problem (here the dot in \( \dot{q}_\epsilon^j(t) \) denotes the time-derivative and \( H \) is Heaviside’s function)

\[
\dot{q}_\epsilon^j(t) + \alpha_j q_\epsilon^j(t) = u_{f,\epsilon}^{\varphi}(t, y_j) + \sum_{i \neq j} \frac{H(t - |y_i - y_j|)}{4\pi |y_i - y_j|} q_\epsilon^i(t - |y_i - y_j|) ,
\]

\[
q_\epsilon^j(0) = 0, \quad j = 1, \ldots, n,
\]

then

\[
u_{f,\epsilon}^{\alpha,Y}(t, x) = u_{f,\epsilon}^{\varphi}(t, x) + \sum_{j=1}^{n} \frac{H(t - |x - y_j|)}{4\pi |x - y_j|} q_\epsilon^j(t - |x - y_j|) ,
\]

i.e., \( u_{f,\epsilon}^{\alpha,Y}(t) \) coincides with the solution \( u_{\epsilon}(t) \), of the inhomogeneous (distributional) Cauchy problem

\[
\begin{cases}
\partial_t u_{\epsilon}(t) = \Delta u_{\epsilon}(t) + \sum_{j=1}^{n} q_\epsilon^j(t) \delta_{y_j} + \delta_0(t)f_\epsilon \\
u_{\epsilon}(0) = 0 \\
\partial_\epsilon u_{\epsilon}(0) = 0 \, ,
\end{cases}
\]
where the \( q_i^j(t) \)'s solve (3.16). Notice that the retarded terms in (3.16) take into account the multiple scattering effects.

In conclusion,

\[
u_{f_k}^y(t, x_k) - u_{f_k}^y(t, x_k) = \sum_{j=1}^{n} \frac{H(t - |x_k - y_j|)}{4\pi |x_k - y_j|} q_i^j(t - |x_k - y_j|),
\]

where the \( q_i^j(t) \)'s solve (3.16).

4. **Inverse Wave Scattering in the Time Domain.**

Let us fix \( \lambda > \lambda_{o,Y} \), a compact set \( K \supset Y \) and a denumerable set

\[D = \{x_k, \ k \in \mathbb{N}\} \subset K \setminus Y;\]

\( D \) represents the points where the emitters/detectors can be placed. We introduce the following hypothesis regarding \( D \):

\[\text{(4.1)} \quad \text{the closure of } D \text{ contains a non empty open set.}\]

For any integer \( N > 0 \), we define the map

\[\phi_N^\lambda : K \setminus D \to \mathbb{R}^N, \quad \phi_N^\lambda(z) \equiv \begin{bmatrix} e^{-\sqrt{|x_1-z|}}/|x_1-z| \\ e^{-\sqrt{|x_2-z|}}/|x_2-z| \\ \vdots \\ e^{-\sqrt{|x_N-z|}}/|x_N-z| \end{bmatrix},\]

and the linear operator

\[\Phi_N^\lambda : \mathbb{R}^n \to \mathbb{R}^N, \quad \Phi_N^\lambda \equiv \begin{bmatrix} e^{-\sqrt{|x_1-y_1|}}/|x_1-y_1| & \cdots & e^{-\sqrt{|x_1-y_n|}}/|x_1-y_n| \\ \vdots & \ddots & \vdots \\ e^{-\sqrt{|x_N-y_1|}}/|x_N-y_1| & \cdots & e^{-\sqrt{|x_N-y_n|}}/|x_N-y_n| \end{bmatrix}.\]

**Lemma 4.1.** Let \( \phi_N^\lambda \) and \( \Phi_N^\lambda \) be the maps defined in (4.2) and (4.3). Then, there exists \( N_0 \geq 1 \) such that, for any \( N \geq N_0, \)

\[z \in Y \quad \iff \quad \phi_N^\lambda(z) \in \text{ran}(\Phi_N^\lambda).\]

**Proof.** (\( \Rightarrow \)) If \( z = y_k \in Y \), then \( \phi_N^\lambda(z) = \Phi_N^\lambda \xi \), where \( \xi \equiv (\xi_1, \ldots, \xi_n) \), \( \xi_j = \delta_{kj} \).

(\( \Leftarrow \)) Here we mimic the arguments provided in the proof of [13, Theorem 4.1]. Suppose that the implication is false, i.e.,

\[\forall N \geq 1, \exists z_N \in K \setminus Y \text{ such that } \phi_N^\lambda(z_N) \not\in \text{ran}(\Phi_N^\lambda).\]

Since \( \Phi_N^\lambda = [\phi_N^\lambda(y_1), \ldots, \phi_N^\lambda(y_n)] \),

\[\phi_N^\lambda(z_N) \in \text{ran}(\Phi_N^\lambda) \quad \iff \quad \phi_N^\lambda(z_N) \in \text{span}\{\phi_N^\lambda(y_1), \ldots, \phi_N^\lambda(y_n)\}.\]

an so there would exist sequences

\[\{N_i\}_{i=1}^\infty \subset \mathbb{N}, \ N_i \nearrow +\infty, \ \{\xi_i\}_{i=1}^\infty \subset \mathbb{R}^n, \ \{\eta_i\}_{i=1}^\infty \subset \mathbb{R}, \ |\xi_i|^2 + |\eta_i|^2 = 1, \ \{z_i\}_{i=1}^\infty \subset K \setminus Y,\]

...
such that
\[ \forall \ell \geq 1 , \quad \sum_{j=1}^{n} (\xi_\ell)_j \phi^{N_\ell}_\lambda (y_j) = \eta_\ell \phi^{N_\ell}_\lambda (z_\ell). \]

Therefore the analytic functions
\[ v_\ell : \mathbb{R}^3 \setminus (Y \cup \{z_\ell\}) \to \mathbb{R}, \quad v_\ell(x) := \eta_\ell \frac{e^{-\sqrt{\lambda}|x-z_\ell|}}{|x-z_\ell|} - \sum_{j=1}^{n} (\xi_\ell)_j \frac{e^{-\sqrt{\lambda}|x-y_j|}}{|x-y_j|} \]

would vanish on the set \( D \) and so, by our hypothesis \((4.1)\), they would be identically zero. Hence
\[ \forall \ell \geq 1 , \forall x \in \mathbb{R}^3 \setminus (Y \cup \{z_\ell\}), \quad \eta_\ell \frac{e^{-\sqrt{\lambda}|x-z_\ell|}}{|x-z_\ell|} = \sum_{j=1}^{n} (\xi_\ell)_j \frac{e^{-\sqrt{\lambda}|x-y_j|}}{|x-y_j|}. \]

Since the sequences \( \{\xi_\ell\}_{\ell=1}^{\infty}, \{\eta_\ell\}_{\ell=1}^{\infty} \) and \( \{z_\ell\}_{\ell=1}^{\infty} \) are bounded, one has, eventually considering subsequences, \( \xi_\ell \to \xi \in \mathbb{R}^n, \eta_\ell \to \eta \in \mathbb{R}, z_\ell \to z \in K \) as \( \ell \to +\infty \) and so one would get
\[ \forall x \in \mathbb{R}^3 \setminus (Y \cup \{z\}), \quad \eta \frac{e^{-\sqrt{\lambda}|x-z|}}{|x-z|} = \sum_{j=1}^{n} \xi_j \frac{e^{-\sqrt{\lambda}|x-y_j|}}{|x-y_j|}. \]

Let us now show that this is impossible, by considering separately the cases \( z \in K \setminus Y \) and \( z = y_i \in Y \).

If \( z \in K \setminus Y \) then, by considering the limit \( x \to y_k \) in \((4.5)\), one would get \( \xi_k = 0 \) for any \( k \) and hence \( \eta = 0 \); this is impossible, since \( |\xi|^2 + |\eta|^2 = 1 \).

If \( z = y_i \in Y \) then, by considering the limit \( x \to y_i \) in \((4.5)\), one would get \( \xi_i = \eta \) and therefore
\[ \forall x \in \mathbb{R}^3 \setminus (Y \cup \{z\}), \quad \sum_{j \neq i} \xi_j \frac{e^{-\sqrt{\lambda}|x-y_j|}}{|x-y_j|} = 0. \]

This would give \( \xi_j = 0 \) for any \( j \neq i \) and hence \( |\eta| = \frac{1}{\sqrt{2}} \). Now, let us re-write \((4.4)\) as
\[ \forall k \geq 1 , \quad \sum_{j \neq i} (\xi_\ell)_j \frac{e^{-\sqrt{\lambda}|x_k-y_j|}}{|x_k-y_j|} = \eta_\ell - (\xi_\ell)_k \frac{e^{-\sqrt{\lambda}|x_k-y_i|}}{|x_k-y_i|} \]
\[ \quad + \frac{\eta_\ell}{\varepsilon_\ell} \left( \frac{e^{-\sqrt{\lambda}|x_k-z_\ell|}}{|x_k-z_\ell|} - \frac{e^{-\sqrt{\lambda}|x_k-y_i|}}{|x_k-y_i|} \right), \]

where
\[ \varepsilon_\ell := \left( |\eta_\ell - (\xi_\ell)_i|^2 + \sum_{j \neq i} |(\xi_\ell)_j|^2 + |z_\ell - y_i|^2 \right)^{1/2}. \]

By
\[ \frac{e^{-\sqrt{\lambda}|x_k-z_\ell|}}{|x_k-z_\ell|} - \frac{e^{-\sqrt{\lambda}|x_k-y_i|}}{|x_k-y_i|} \]
\[ = \left( \sqrt{\lambda} + \frac{1}{|x_k-y_i|} \right) \frac{e^{-\sqrt{\lambda}|x_k-y_i|}}{|x_k-y_i|} \frac{x_k - y_i}{|x_k-y_i|} \cdot (z_\ell - y_i) + o(|y_i - z_\ell|), \]
and by (eventually considering subsequences)

\[
\left(\xi_j\right)_{\varepsilon_t} \rightarrow \tilde{\xi}_j, \quad j \neq i, \quad \left(\xi_i\right)_{\varepsilon_t} - \eta e \rightarrow \tilde{\xi}_i, \quad \frac{z_k - y_k}{\varepsilon_t} \rightarrow \tilde{z}_k, \quad |\tilde{\xi}_i|^2 + |\tilde{z}_k|^2 = 1,
\]

as \( \ell \not\to +\infty \), one would get, considering the limit \( \ell \not\to +\infty \) in the relation (4.7),

\[
\forall k \geq 1, \quad \sum_{j=1}^{n} \tilde{\xi}_j e^{-\sqrt{\lambda}|x_k-y_j|} \left| \begin{array}{c} x_k-y_j \end{array} \right| = \eta \left( \sqrt{\lambda} + \frac{1}{|x_k-y_j|} \right) e^{-\sqrt{\lambda}|x_k-y_i|} \left| \begin{array}{c} x_k-y_i \end{array} \right| \frac{x_k-y_i}{|x_k-y_i|} \cdot \tilde{z}_k.
\]

Again taking into account hypothesis (4.1) on the set \( D \), one would obtain

\[
(4.8) \quad \forall x \in \mathbb{R}^3 \setminus Y, \sum_{j=1}^{n} \tilde{\xi}_j e^{-\sqrt{\lambda}|x-y_j|} \left| \begin{array}{c} x-y_j \end{array} \right| = \eta \left( \sqrt{\lambda} + \frac{1}{|x-y_i|} \right) e^{-\sqrt{\lambda}|x-y_i|} \left| \begin{array}{c} y \end{array} \right| \cdot \tilde{z}_k.
\]

Considering the limits \( x \to y_j, \quad j \neq i \) in (4.8), one would get \( \tilde{\xi}_j = 0 \) for any \( j \neq i \) and so (4.8) would reduce to

\[
(4.9) \quad \forall x \in \mathbb{R}^3 \setminus Y, \quad \tilde{\xi}_i = \eta \left( \sqrt{\lambda} + \frac{1}{|x-y_i|} \right) \frac{x-y_i}{|x-y_i|} \cdot \tilde{z}_k.
\]

Considering the limit \( x \to y_i \) in (4.9), one would get \( \tilde{z}_k = 0 \) and hence \( \tilde{\xi}_i = 0 \). This is impossible, since \( |\tilde{\xi}_i|^2 + |\tilde{z}_k|^2 = 1 \).

Theorem 4.2. Let \( D = \{ x_k, \quad k \in N \} \subset K \setminus Y \) satisfy hypothesis (4.1) and let \( \lambda > \lambda_{\alpha,Y} \). Then there exists \( N_0 \geq 1 \) such that for any \( N \geq N_0 \) the data operator corresponding to \( X_N := \{ x_k \in D, \quad k \leq N \} \) defined by

\[
F_N^\lambda : \mathbb{R}^N \to \mathbb{R}^N, \quad (F_N^\lambda f)_k := \lim_{\varepsilon_t \not\to 0} \int_0^{\infty} e^{-\sqrt{\lambda}t} \left( u_{f_k}(t, x_k) - u_{f_k}(t, x_k) \right) dt
\]

determines \( Y \) according to

\[
z \in Y \quad \iff \quad \phi_N^\lambda(z) \in \ker(F_N^\lambda)^\perp.
\]

Equivalently, denoting by \( P_N^\lambda \) the orthogonal projector onto \( \ker(F_N^\lambda) \), one has

\[
Y = \{ \text{peak points of the function } z \mapsto |P_N^\lambda \phi_N^\lambda(z)|^{-1} \}.
\]

Proof. By (3.15), one has

\[
(4.10) \quad F_N^\lambda = (4\pi)^{-2} \Phi_N^\lambda \Lambda_\lambda^\frac{1}{2}(\Phi_N^\lambda)^{\ast}.
\]

Since \( M_\lambda \) is positive-definite by Lemma (2.1), \( \Lambda_\lambda = M_\lambda^{-1} \) is positive-definite as well and so it has a nonsingular square root. Hence

\[
F_N^\lambda = (4\pi)^{-2} \Phi_N^\lambda \Lambda_\lambda^\frac{1}{2}(\Phi_N^\lambda)^{\ast} \Lambda_\lambda^{-\frac{1}{2}}
\]

and

\[
\ker(F_N^\lambda)^\perp = \operatorname{ran}((F_N^\lambda)^\ast) = \operatorname{ran}(\Phi_N^\lambda \Lambda_\lambda^{\ast} (\Phi_N^\lambda)^{\ast}) = \operatorname{ran}(\Phi_N^\lambda \Lambda_\lambda^{\ast}) = \operatorname{ran}(\Phi_N^\lambda).
\]

The proof is then concluded by Lemma (4.1).
Remark 4.3. The arguments of proof of Lemma 4.1, being not constructive, do not provide any lower bound on the number $N_0$ of emitters/detectors needed to locate the $n$ point in the array $Y$. Since the $N \times N$ data matrix $F^N_\lambda$ is symmetric, by elementary dimensional considerations, in order to determine the unknown $3n$ coordinates, one at least would need $N_0$ such that $(N_0^2 + N_0)/2 \geq 3n$. This suggests the rough estimate

$$N_0 > \left\lfloor \frac{5\sqrt{n} - 1}{2} \right\rfloor,$$

where $\lfloor \cdot \rfloor$ denotes the integer part. At the best of our knowledge, a precise estimate of $N_0$ is a relevant open question which may require different techniques from the ones implemented here.

Remark 4.4. Suppose that the locations $y_1, \ldots, y_n$ of the scatterers have been determined. Let us now work with $N = n$ sources/detectors. Thus the $n \times n$ matrix $\Phi^n_\lambda$ is known.

Since the map $(0, +\infty) \ni \lambda \mapsto \det(\Phi^n_\lambda)$ is analytic, it has isolated zeroes; let us choose $\lambda > \lambda_{\alpha,Y}$ such that $\det(\Phi^n_\lambda) \neq 0$. Then, by (3.15), equivalently, by (4.10),

$$\Lambda_\lambda = M^{-1}_\lambda = (4\pi)^2 (\Phi^n_\lambda)^{-1} F^n_\lambda (\Phi^n_\lambda)^{-*} - F^n_\lambda (\Phi^n_\lambda)^{-1}.$$

and

$$\det(F^n_\lambda) = (4\pi)^{-2n} (\det(\Phi^n_\lambda))^2 \det(\Lambda_\lambda) \neq 0.$$

Therefore, using (2.2), one can recover the coefficients $\alpha_1, \ldots, \alpha_n$ from the knowledge of $F^n_\lambda$ by the relations

$$\alpha_i = \frac{1}{(4\pi)^2} \left( (\Phi^n_\lambda)^* (F^n_\lambda)^{-1} \Phi^n_\lambda \right)_{ii} - \frac{\sqrt{\lambda}}{4\pi}.$$

Acknowledgements. The authors thank Mourad Sini, Guanghui Hu and Ibtissem Ben Aïtcha for fruitful discussions. During the preparation of this work, A.M. was a member of the CNRS working group at WPI in Wien; he thanks the Pauli Institute for the hospitality and the support. We also gratefully acknowledge one anonymous referee for the stimulating remarks.

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