Distributed Generalized Nash Equilibrium Seeking for Energy Sharing Games

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Abstract—With the proliferation of distributed generators and energy storage systems, traditional passive consumers in power systems have been gradually evolving into the so-called “prosumers”, i.e., proactive consumers, which can both produce and consume power. To encourage energy exchange among prosumers, energy sharing is increasingly adopted, which is usually formulated as a generalized Nash game (GNG). In this paper, a distributed approach is proposed to seek the Generalized Nash equilibrium (GNE) of the energy sharing game. To this end, we convert the GNG into an equivalent optimization problem. A Krasnosel’skiĭ-Mann iteration type algorithm is thereby devised to solve the problem and consequently find the GNE in a distributed manner. The convergence of the proposed algorithm is proved rigorously based on the nonexpansive operator theory. The performance of the algorithm is validated by experiments with three prosumers, and the scalability is tested by simulations using 123 prosumers.

Index Terms—Energy sharing, prosumer, distributed algorithm, generalized Nash game, generalized Nash Equilibrium.

I. INTRODUCTION

It has been recognized that our power system is undergoing a fundamental transition due to: 1) The proliferation of distributed generations (DGs), such as wind turbines, photovoltaics, energy storage systems and electric vehicles [1]–[5]; 2) The advancement of communications and control in consumer-level via smart appliances and energy management systems [6]–[8]. Together, these changes allow traditional passive consumers to convert into the so-called “prosumers”, i.e., proactive consumers, that can actively regulate their generation and consumption [9]–[13]. Conventionally, a hierarchical structure is utilized in the power system energy management, usually in a centralized way. However, the centralized manner may face great challenges raised by the ever-increasing number of prosumers and uncertainties of renewable generations. This essentially advocates a distributed paradigm [14]–[17] in energy management. Particularly, energy sharing turns to be a promising form of the market that encourages energy trading among prosumers. As prosumers typically belong to different owners, the inherent competition may lead to strategic behaviors in energy sharing. In this situation, each prosumer intends to optimize its own profit while maintaining the power balance over the whole system. This leads to a generalized Nash game (GNG) with global constraints. In this regard, it is desirable to investigate distributed approaches to seeking the generalized Nash equilibrium (GNE) of the GNG.

There are many papers investigating distributed methods of GNE seeking, which can be roughly divided into two categories in terms of methods used: gradient-based algorithms [18]–[23], and proximal-point algorithms [24]–[27]. In the first type, the pseudo-gradient of each player’s disutility function is utilized to seek the GNE, including continuous-time algorithms [18], [19] and discrete algorithms [20]–[23]. In [18], a distributed continuous-time projection-based algorithm is proposed to seek the GNE of aggregative games with linear coupled constraints. This is extended to the case where players have nonsmooth payoff functions in [19]. In [20], two distributed primal-dual algorithms are proposed for computing a GNE in noncooperative games with shared affine constraints. The operator splitting method is utilized to prove the convergence of the algorithms. It is further improved in [21] to consider communication time delays and partial-decision information. In [22], three stochastic gradient strategies are developed to seek GNE where agents are subject to randomness in the environment of unknown statistical distribution. In [23], the relations between Nash and Wardrop equilibria of the aggregative game are investigated and two algorithms are proposed to seek the equilibrium.

In the proximal-point algorithms, the Alternating Direction Method of Multipliers (ADMM) method is widely used. An inexact-ADMM algorithm is proposed in [24]. This method is improved in [27], where each player only has partial information of their opponents and the communication graph is not necessarily the same as the cost dependency graph of each player. In [25], two double-layer preconditioned proximal-point algorithms are proposed to seek GNE with both coupled equality and inequality constraints, respectively. For the GNG with a special structure, i.e., the coupling in the cost functions of the agents is linear, a distributed proximal-point algorithm is developed in [26] for GNG with maximally monotone pseudo-gradient.

The aforementioned works have made great progress in the context of distributed GNE seeking. In most of the works, if the objective function of one player is associated with decisions of all players, each player is required to communicate with all of the other players, i.e., full information is needed. Then, the communication network is very dense. To address this problem, a local estimation of the overall decision profile for each player is added in [21], [28]. However, this will increase one order in the algorithm, making it more complicated.

In this paper, we offer a different perspective for seeking the GNE in the GNG when full information is included in the objective functions. The energy sharing game is transformed into an equivalent optimization problem. Instead of dealing with the GNG directly, we alternatively solve an equivalent optimization problem of the original game problem. Both experiments and simulations are utilized to validate our method. The main contributions are as follows.

- A systematic way is proposed to transform the energy sharing game into an equivalent centralized optimization problem, which could be solved in a fully distributed
way with neighboring communication. Moreover, the existence and uniqueness of the GNE of the energy sharing game are proved.

- A distributed method is devised based on Krasnosel’skii-Mann iteration to solve the equivalent counterpart instead of solving the original game problem directly. By constructing a firmly nonexpansive operator, we prove that the proposed distributed algorithm converges to the GNE of the original energy sharing game.

The rest of this paper is organized as follows. In Section II, we briefly introduce some necessary preliminaries. Section III formulates the energy sharing model. In Section IV, the existence and uniqueness of GNE are analyzed. A distributed GNE seeking algorithm is proposed in Section V. The convergence of the algorithm is proved in Section VI. The effectiveness of the algorithm is verified in Section VII by experiments and simulation studies. Section VIII concludes the paper.

II. Preliminaries

In this paper, we use $\mathbb{R}^n$ to denote the $n$-dimensional Euclidean space. For a matrix $A$, $[A]_{ij}$ is the entry in the $i$-th row and $j$-th column of $A$. For vectors $x, y \in \mathbb{R}^n$, $x^Ty = \langle x, y \rangle$ denotes the inner product of $x$ and $y$. $\|x\|_2 = \sqrt{x^Tx}$ denotes the Euclidean norm of $x$. Denote the inner product under a positive definite matrix $Q$ by $\langle x, y \rangle_Q = (Qx, y)$. Similarly, the norm induced by $Q$ is $\|x\|_Q = \sqrt{(Qx, x)}$. The following relationship holds for a $Q$-induced norm.

$$\|a - c\|_Q^2 - \|b - c\|_Q^2 = 2\langle a - b, a - c \rangle_Q - \|a - b\|_Q^2 \tag{1}$$

which can be obtained by the equation $\|a + b\|_Q^2 = \|a\|_Q^2 + 2\langle a, b \rangle_Q + \|b\|_Q^2$.

The identity matrix with dimension $n$ is denoted by $I_n$. Use $\prod_{i=1}^m \Omega_i$ to denote the Cartesian product of the sets $\Omega_i$, $i = 1, \cdots, n$. Define the projection of $x$ onto a set $\Omega$ as $P_\Omega(x) = \arg \min_{y \in \Omega} \|x - y\|_2 \tag{2}$

Use $I_d$ to denote the identity operator, i.e., $I_d(x) = x$, $\forall x$. Define $N_\Omega(x) = \{y : \langle y - x, \Omega \rangle \leq 0, \forall y \in \Omega\}$. We have $P_\Omega(x) = (I_d + N_\Omega)^{-1}(x) \tag{29}$ Chapter 23.1).

For a single-valued operator $T : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$, a point $x \in \Omega$ is a fixed point of $T$ if $T(x) = x$. The set of fixed points of $T$ is denoted by $\text{Fix}(T)$. $T$ is nonexpansive if $\|T(x) - T(y)\| \leq \|x - y\|, \forall x, y \in \Omega$. For $\alpha \in (0, 1)$, $T$ is called $\alpha$-averaged if there exists a nonexpansive operator $R$ such that $T = (1 - \alpha)I_d + \alpha R$. Use $A(\alpha)$ to denote the class of $\alpha$-averaged operators. If $T \in A(\alpha)$, $T$ is called firmly nonexpansive. The graph of $T$ is $\text{gra} T = \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^n | u \in T(x)\}$. $T$ is monotone if $\forall (x, u), \forall (y, v) \in \text{gra} T$, $(x - y, u - v) \geq 0$. $T$ is maximally monotone if $\text{gra} T$ is not strictly contained in the graph of any other monotone operator.

III. Energy Sharing Game

In this section, the model of prosumers is introduced. Then, we formulate the energy sharing game.

A. Model of prosumers

In this paper, we consider the energy sharing problem among a set of prosumers, denoted by $N = \{1, 2, \cdots, n\}$. The communication edge is denoted by $E \subseteq N \times N$. For a prosumer $i$, the set of its neighbors is denoted by $N_i$. If $j \in N_i$, prosumers $i$ and $j$ can communicate directly. The Laplacian matrix of the communication graph is denoted by $L$ and we have $I^TD = 0$, where $I$ is a vector with all of the components as 1.

The scenario is that each prosumer has a power shortage or surplus, denoted by $D_i$. To balance its power, it can produce power itself, denoted by $p_i$, or buy power from (sell power to) the grid, denoted by $q_i$. In the sharing market, the demand function of each prosumer can be expressed by

$$q_i = a_i \mu_c + b_i \tag{3}$$

where $\mu_c$ is the market clearing price, $b_i$ is the willingness to buy. $a_i < 0$ implies the price elasticity. The market clears when the net quantity $\sum_i q_i = 0$ and the obtained sharing price if

$$\mu_c = -\frac{1}{N} \frac{b}{a} \tag{4}$$

where $\bar{a} = \frac{1}{N} a_i^2$ is the average value of $a_i$. In the rest of the paper, $b_{-i} = (b_1, b_2, \cdots, b_{i-1}, b_{i+1}, \cdots, b_n)^T$.

B. Energy sharing game

In the energy sharing game, each prosumer intends to minimize its cost while maintaining the global power balance. The optimization problem of each prosumer is

$$\begin{align*}
\min_{p_i, b_i} & f_i(p_i, b) = h_i(p_i) + (a_i \mu_c(b) + b_i) \mu_c(b) \tag{5a} \\
\text{s.t.} & p_i + a_i \mu_c(b) + b_i = D_i \tag{5b} \\
& \sum_{i \in N_i} (a_i \mu_c(b) + b_i) = 0 \tag{5c} \\
& \underline{p_i} \leq p_i \leq \overline{p_i} \tag{5d}
\end{align*}$$

The disutility function consists of two parts, where $h_i(p_i)$ is the cost of prosumer $i$ to produce $p_i$ and $(a_i \mu_c(b) + b_i) \mu_c(b)$ is the cost of buying power $q_i$.

The Lagrangian of (5) is

$$\begin{align*}
\mathcal{L}(p_i, b, \lambda_i, \eta_i) = f_i(p_i, b) + \lambda_i (p_i + a_i \mu_c(b) + b_i - D_i) + \sum_{i \in N_i} (a_i \mu_c(b) + b_i) \lambda_i = 0 \tag{6}
\end{align*}$$

where $\lambda_i, \eta_i$ are the Lagrangian multipliers.

Regarding the energy sharing game (5), we have the following assumptions.

**Assumption 1.** The functions $h_i(p_i)$ are convex and differentiable.

**Assumption 2.** For a given $b_{-i}$, the Slater’s condition of the problem (5) holds [30 Chapter 5.2.3], i.e., problem (5) is feasible.

**Assumption 3.** The generation satisfies $I^TP < I^TD < I^T\overline{p}$.

For Assumption 1, the power generation cost $h_i(p_i)$ is usually quadratic, which is convex and differentiable. For Assumption 2, it is a common assumption, otherwise the problem is infeasible. The Assumption 3 implies that the energy production can strictly satisfy the load demand. As

\footnote{Given a collection of $z_i$ for $i$ in a certain set $Z$, $z$ denotes the column vector $z := (z_i, i \in Z)$ with a proper dimension with $z_i$ as its components.}
long as the total regulation capacity of all prosumers is strictly larger than the total load demand, Assumption 3 holds. Assumption 3 is stricter than Assumption 2, which is used to determine the uniqueness of the GNE.

In summary, the energy sharing game among all prosumers is composed of the following elements:
- Player: all prosumers, denoted by \( N = \{1, 2, \ldots, n\} \);
- Strategy: generation power \( p_i \) and power interaction \( b_i \);
- Payoff: the disutility function \( f_i(p_i, b_i), \forall i \in N \).

The GNE \((p^*, b^*)\) of the game in (5) is defined as:
\[
(p^*_i, b^*_i) = \arg \min_{p_i, b_i} f_i(p_i, b_i) + g_i(b_i, b_{-i}), \quad \text{s.t.} \quad (5b), (5c), \quad \forall i \in N
\]

**Remark 1.** The energy sharing game model (5) basically comes from \([13]\), and is modified by additionally considering local capacity constraints (5d). Mathematically, the energy sharing game (5) is indeed a GNG by noting that the global constraint (5c) couples the strategies of players. As proved in \([13]\), if the constraint (5d) is not taken into account, the energy sharing game can be equivalently converted into a standard Nash game instead of a GNG without any coupling constraints. However, when (5d) is considered, the coupling constraints cannot be eliminated. This is the key difference between this paper and \([13]\) in the modeling.

**IV. EQUIVALENT PROBLEM AND UNIQUENESS OF GNE**

In this section, we analyze the property of the Nash equilibrium and transform the game into an equivalent optimization problem. Replacing \( \mu_c \) in (5) using (4), then the sharing game problem (5) can be rewritten as
\[
\min_{p_i, b_i} f_i(p_i, b_i) + g_i(b_i, b_{-i}) \quad \text{(7a)}
\]
\[
\text{s.t.} \quad p_i + \frac{1^T a - a_i}{1^T a} b_i - \frac{a_i}{1^T a} \sum_{j \neq i} b_j = D_i \quad \text{(7b)}
\]
\[
p_i \leq p_i \leq \bar{p}_i \quad \text{(7c)}
\]
where
\[
g_i(b_i, b_{-i}) = (a_i \mu_c(b_i) + b_i) \mu_c(b_i)
\]
\[
= -\left(\frac{1^T a - a_i}{1^T a} b_i - \frac{a_i}{1^T a} \sum_{j \neq i} b_j\right) b_i + \frac{1^T a}{1^T a} \sum_{j \neq i} b_j
\]
\[
= \frac{a_i - 1^T a b_i}{N^2 a^2} b_i^2 + \left(\frac{2 a_i - 1^T a}{N^2 a^2}\right) b_i \sum_{j \neq i} b_j
\]
\[
+ \frac{a_i}{N^2 a^2} \left(\sum_{j \neq i} b_j\right)^2
\]
\[
= \alpha_i b_i^2 + \beta_i b_i \sum_{j \neq i} b_j + \delta_i \left(\sum_{j \neq i} b_j\right)^2
\]
(8)
with
\[
\alpha_i = \frac{a_i - 1^T a b_i}{N^2 a^2}, \quad \beta_i = \frac{2 a_i - 1^T a}{N^2 a^2}, \quad \delta_i = \frac{a_i}{N^2 a^2}
\]
Because \( a_i < 0 \), we have \( \alpha_i > 0, \beta_i > 0, \delta_i < 0 \). Then, \( g_i(b_i, b_{-i}) \) is strictly convex with respect to \( b_i \) given any \( b_{-i} \). The objective function \( g_i(b_i, b_{-i}) \) is associated with all of other players. When solving (7) directly, it requires each player to be able to access all other players decisions, i.e., full information is needed. This is very difficult to implement in practice. In the rest of this work, we give a different perspective by solving an equivalent optimization problem of (7).

The pseudo-gradient of \( f_i(p_i, b_i), \forall i \in N \), denoted by \( F(p, b) \), is defined as
\[
F(p, b) = \text{col} (\nabla_{p_i} f_i(p_i, b_i), \cdots, \nabla_{p_n, b_n} f_N(p_n, b_n))
\]
where \( \text{col} (x_1, \cdots, x_n) \) is the column vector stacked with column vectors \( x_1, \cdots, x_n \).

Define the sets
\[
\Omega_i \ := \{p_i | \min_{j \neq i} p_j \leq p_i \leq \bar{p}_i \}, \quad \Omega := \prod_{i=1}^n \Omega_i
\]
and
\[
X^*_i := \left\{ (p_i, b_i) | p_i + \frac{1^T a - a_i}{1^T a} b_i - \frac{a_i}{1^T a} \sum_{j \neq i} b_j = D_i \right\}
\]
\[
X^e = \prod_{i=1}^n X_i^e, \quad X_i = \Omega_i \cap X_i^e, \quad X = \Omega \cap X^e
\]
(10)

For any given \( b_{-i} \), the feasible set of prosumer \( i, X_i \), is closed and convex. Then, we have the following result.

**Lemma 1.** If assumptions [1] and [2] hold, a point \((p, b)\) is an equilibrium if and only if it is a solution of the variational inequality \( VI(X, F(p, b)) \).

**Proof.** For any given \( b_{-i} \), the disutility function (5) is continuous, differentiable and convex with respect to \((p_i, b_i)\). Moreover, the feasible set \( X_i \) is closed and convex. Thus, we have this assertion by [31] Theorem 3.3] \(\square\).

For the existence and uniqueness of the GNE, we have the following results.

**Theorem 2.** If Assumption [1] and Assumption [2] hold, for the generalized Nash game (7), we have
1) the generalized Nash equilibrium exists;
2) if the Assumption [3] also holds, the generalized Nash equilibrium is unique.

The detailed proof of Theorem 2 is given in Appendix A. From the proof of Theorem 2 an equivalent optimization problem is obtained, which is
\[
\min_{p} \hat{h}(p) = \sum_{i \in N} \left( h_i(p_i) + \frac{p_i^2}{2(a_i - 1^T a)} - \frac{D_i}{a_i - 1^T a} p_i \right)
\]
\[
\text{s.t.} \quad \sum_{i \in N} p_i = \sum_{i \in N} D_i, \quad \mu_c
\]
\[
p_i \leq p_i \leq \bar{p}_i
\]
(12)
with \( \mu_c \) is the Lagrangian multiplier of constraint (14b). The “equivalent” here means that we can first solve (14a) to get \( p^*, \mu^*_c \), and then get \( b^* \) with \( b^*_i = D_i - p^*_i - a_i \mu^*_c \). Finally, the GNE is obtained.

**Remark 2.** The game (7) is difficult to solve by existing methods due to the full information in the objective function. Now, we can alternatively consider the equivalent counterpart (14), which could be solved in a distributed way as long as \( 1^T a \) is known. In this regard, the market coordinator (or a third-part platform) is required to broadcast the sum of \( a_i \) of all prosumers. That is needed merely when new prosumers join or existing ones quit, which can be known by the market.
coordinator or the third-part platform. After getting \( p_i^*, b_i^* \) can be obtained from \( b_i^* = D_i - p_i^* - a_i\mu_i^* \). Then, the energy sharing game problem \([7]\) can be solved.

V. DISTRIBUTED ALGORITHM FOR EQUILIBRIUM SEEKING

In this section, we first propose a distributed algorithm based on Krasnosel’skii-Mann iteration to solve the problem \([14]\), i.e., to solve the generalized game \([7]\). Then, we use the nonexpansive operator theory to prove the convergence of the proposed algorithm.

A. Algorithm design

Before giving the algorithm, we first define a matrix and a function. Define the matrix

\[
\Theta := \begin{bmatrix}
\Gamma & 0 & -I_n \\
0 & \alpha_z^{-1}I_n & -L \\
-I_n & -L & \alpha_{\mu}^{-1}I_n
\end{bmatrix}
\]

where the matrix \( \Gamma = \text{diag}(\gamma_i), \gamma_i, \alpha_z \) and \( \alpha_{\mu} \) are constant to make \( \Theta \) positive definite.

Define the function

\[
H_i(p_i) = \frac{\partial \Phi_i}{\partial p_i}(p_i) + \gamma_i p_i
\]

where \( \Phi_i(p_i) = \tilde{h}_i(p_i) = h_i(p_i) + \frac{p_i^2}{2(\alpha_z-1)\alpha_z} - \frac{D_i}{\alpha_z} p_i \). We have the follow result.

Lemma 3. The function \( H_i(p_i) \) is strictly monotone. Moreover, its inverse function \( H_i^{-1}(p_i) \) exists and is also strictly monotone.

The proof is straightforward as \( \tilde{h}_i(p_i) \) is strongly convex, which is omitted here.

We propose the following algorithm, which is denoted by SGNE (Seeking the Generalized Nash Equilibrium).

The algorithm has the Krasnosel’skii-Mann iteration type \([29]\), Chapter 5), which consists of two phases: prediction phase and update phase. In the prediction phase, each bus uses the local stored information of the last two steps to get predictive variables by a linear extrapolation. In the update phase, the predictive variables are utilized to proceed the next iteration. In the algorithm, only communications with neighbors are needed, which means that it is fully distributed.

B. Algorithm reformulation

From \([17d]\), we know

\[
H_i^{-1}(p_i) = \begin{cases}
\leq p_i, & p_i < p_i + 1 < \bar{p}_i \\
p_i + 1, & p_i + 1 < p_i + 1 < \bar{p}_i \\
\geq p_i + 1, & p_i + 1 < \bar{p}_i
\end{cases}
\]

Because \( H_i^{-1}(p_i) \) is monotone, we have

\[
H_i(p_i + 1) = \begin{cases}
\geq 0, & p_i + 1 = p_i \\
0, & p_i < p_i + 1 < \bar{p}_i \\
\leq 0, & p_i + 1 = \bar{p}_i
\end{cases}
\]

It is equivalent to

\[
p_i + 1 = \bar{p}_i, \quad p_i + 1 = \bar{p}_i
\]

3It should be noted that such \( \gamma_i, \sigma_z, \sigma_\mu \) always exist to make \( \Theta \) diagonally dominant, or positive definite.

Algorithm 1 SGNE

Prediction phase:

For prosumer \( i \), it computes

\[
\begin{align}
\tilde{p}_i, t & = p_i, t + \eta(p_i, t - p_{i, t-1}) \\
\tilde{z}_i, t & = z_i, t + \eta(z_i, t - z_{i, t-1}) \\
\tilde{\mu}_i, t & = \mu_i, t + \eta(\mu_i, t - \mu_{i, t-1})
\end{align}
\]

Update phase:

For prosumer \( i \), it computes

\[
\begin{align}
\tilde{p}_i, t+1 & = \tilde{p}_i, t + \sigma_{\gamma}(p_i, t + 1 - \tilde{p}_i, t + 1) + \tilde{\mu}_i, t \\
\tilde{z}_i, t+1 & = \tilde{z}_i, t + \sigma_z(\tilde{z}_i, t - \tilde{z}_i, t) + \tilde{\mu}_i, t
\end{align}
\]

Recalling \( P_{\Omega}(x) = (I_d + \Omega_{\Omega})^{-1}(x) \), we have

\[
\begin{align}
\gamma_i(\tilde{p}_i, t - \tilde{p}_i, t + 1) & = \tilde{\mu}_i, t + \tilde{\mu}_i, t + 1 \in \Omega_{\Omega}(p_i, t + 1) \\
& + \nabla h_i(p_i, t + 1) + \mu_i, t + 1
\end{align}
\]

From \([17d]\), we have

\[
\begin{align}
\sum_{j \in N_i} (\tilde{\mu}_i, t - \tilde{\mu}_i, t + 1) & = \sum_{j \in N_i} (\mu_i, t + 1 - \mu_i, t + 1) \\
+ \sum_{j \in N_i} (\mu_i, t + 1 - \mu_j, t + 1) & = \sum_{j \in N_i} (\mu_i, t + 1 - \mu_j, t + 1)
\end{align}
\]

From \([17d]\), we have

\[
\begin{align}
\gamma_i^{-1}(\tilde{z}_i, t + 1 - \tilde{z}_i, t) & = \tilde{\mu}_i, t + \tilde{\mu}_i, t + 1 \\
- \sum_{j \in N_i} (\tilde{z}_i, t - \tilde{z}_j, t) & = \tilde{\mu}_i, t + \tilde{\mu}_i, t + 1 \\
& = -\tilde{p}_i, t + 1 + D_i + \sum_{j \in N_i} (\tilde{z}_i, t + 1 - \tilde{z}_j, t + 1)
\end{align}
\]

The compact form of the algorithm is

\[
\begin{align}
\tilde{p}_i, t & = p_i, t + \eta(p_i, t - p_{i, t-1}) \\
\tilde{z}_i, t & = z_i, t + \eta(z_i, t - z_{i, t-1}) \\
\tilde{\mu}_i, t & = \mu_i, t + \eta(\mu_i, t - \mu_{i, t-1})
\end{align}
\]

Define the following operator

\[
U : \begin{bmatrix}
p \\
z \\
\mu
\end{bmatrix} \mapsto \begin{bmatrix}
\mu + \Omega_{\Omega}(p) + \nabla h(p) \\
L\mu \\
-p + D - Lz
\end{bmatrix}
\]

Then, the algorithm \([24]\) is rewritten as

\[
\dot{\omega}_i = \omega_i + \eta(\omega_i - \omega_{i-1})
\]
\[ \Theta(\tilde{\omega}_t - \omega_{t+1}) \in \mathcal{U}(\omega_{t+1}) \quad (26b) \]

where \( \Theta \) is defined in (15). For the operator \( \mathcal{U} \), it has the following properties

**Lemma 4.** Take step sizes \( \Gamma, \alpha_z, \) and \( \alpha_\mu \) such that \( \Theta \) is positive definite. We have the following properties:

1. Operator \( \mathcal{U} \) is maximally monotone;
2. \( \Theta^{-1}\mathcal{U} \) is maximally monotone under the \( \Theta \)-induced norm \( \| \cdot \|_\Xi \);
3. \((\text{Id} + \Theta^{-1}\mathcal{U})^{-1}\) exists and is firmly nonexpansive.

**Proof.** 1) The operator \( \mathcal{U} \) can be rewritten as

\[
\mathcal{U} = \begin{bmatrix} 0 & 0 & I \\ 0 & 0 & L \\ -I & -L & 0 \end{bmatrix} \begin{bmatrix} p \\ z \\ \mu \end{bmatrix} + \begin{bmatrix} N_\Theta(p) + \nabla \hat{h}(p) \\ 0 \\ D \end{bmatrix}
\]

Thus, \( \mathcal{U} \) is a skew-symmetric matrix, \( \mathcal{U} \) is maximally monotone [29, Example 20.35]. Moreover, \( N_\Theta(z) \) is maximally monotone by [29, Example 20.26]. In addition, \( 0 \) and \( \mathcal{D} \) are monotone and continuous, which are all maximally monotone [29, Corollary 20.28]. As \( \hat{h}(p) \) is convex, \( \nabla \hat{h}(p) \) is also maximally monotone [29, Theorem 20.25]. Thus, \( \mathcal{U}_2 \) is also maximally monotone. Then, by [29, Corollary 25.5], we have \( \mathcal{U} \) is maximally monotone.

2) As \( \Theta \) is symmetric positive definite and \( \mathcal{U} \) is maximally monotone, we can prove that \( \Theta^{-1}\mathcal{U} \) is maximally monotone under the \( \Theta \)-induced norm by the similar analysis in Lemma 5.6 of (20).

3) As \( \Phi^{-1}\mathcal{U} \) is maximally monotone, \((\text{Id} + \Phi^{-1}\mathcal{U})^{-1}\) exists and is firmly nonexpansive by [29, Proposition 23.8]. \( \square \)

By the third assertion of Lemma 4 (26) is equivalent to

\[ \tilde{\omega}_t = \omega_t + \eta(\omega_t - \omega_{t-1}) \quad (28a) \]

\[ \omega_{t+1} = (\text{Id} + \Theta^{-1}\mathcal{U})^{-1}\tilde{\omega} \quad (28b) \]

Up to now, we transform the algorithm SGNE to a fixed point problem with a nonexpansive operator \((\text{Id} + \Theta^{-1}\mathcal{U})^{-1}\), which provides fundamental support for the convergence proof.

### VI. NASH EQUILIBRIUM SEEKING AND CONVERGENCE

In this section, we address the optimality of the equilibrium point and the convergence of the algorithm SGNE, i.e., the discrete dynamic system (17).

#### A. Nash Equilibrium

First, we define the equilibrium of the algorithm SGNE.

**Definition 1.** A point \( w^* = (w_i^*, i \in \mathcal{N}) = (p_i^*, z_i^*, \mu_i^*) \) is an equilibrium point of (17) if \( \lim_{t \to +\infty} w_{t,i} = w_i^* \) holds for all \( i \).

Recalling the (A.5), we have the following result.

**Theorem 5.** Suppose assumptions \( 1, 2 \) and \( 3 \) hold. At the equilibrium of the algorithm SGNE, we have \( \mu_i^* = z_i^* = \mu_0, \forall i, j \) is the clearing price and \( (p_i^*, b_i^*) \) is the GNE of the game, where \( \mu_0 \) is constant.

**Proof.** From Definition 1 and (24), we have

\[ 0 \in N_\Theta(p^*) + \nabla \hat{h}(p^*) + \mu^* \quad (29a) \]

\[ 0 = L\mu^* \quad (29b) \]

\[ 0 = -p^* + D - Lz^* \quad (29c) \]

From (29b), we have

\[ \mu_i^* = \mu_j^* = \mu_0, \forall i, j \quad (30) \]

From (29c), we have

\[ 0 = -1^T p^* + 1^T D - 1^T Lz^* = -1^T p^* + 1^T D \quad (31) \]

The equations (29a), (30) and (31) are the KKT condition (A.7). This completes the proof. \( \square \)

#### B. Convergence

In this subsection, we analyze the convergence of the algorithm SGNE based on the compact form (28). First, we give the following result.

**Theorem 6.** Suppose Assumption 1 and Assumption 2 hold. Given a parameter \( \eta \) satisfying \( 0 < \eta < \frac{1}{\mathcal{N}} \) and the step sizes \( \Gamma, \alpha_z \) and \( \alpha_\mu \) such that \( \Theta \) is positive definite. Then with SGNE, \( \omega_t \) converges to a primal-dual optimal solution \( \omega^* \) of the problem (14). Then, \( (p^*, b^*) \) is the GNE of \( \mathcal{G} \).

**Proof.** Frist, we prove that \( \lim_{t \to +\infty} (\omega_{t+1} - \omega_t) = 0 \). Given any equilibrium point \( \omega^* \), use the equation (11), and we have

\[ \|\omega_t - \omega^*\|_\Theta^2 - \|\omega_{t+1} - \omega^*\|_\Theta^2 \]

\[ = 2(\omega_t - \omega_{t+1}, \omega_t - \omega^*)_\Theta - \|\omega_t - \omega_{t+1}\|_\Theta^2 \]

\[ = 2(\omega_t - \omega_{t+1}, \omega_t - \omega^*)_\Theta + \|\omega_t - \omega_{t+1}\|_\Theta^2 \]

\[ = \|\omega_t - \omega_{t+1}\|_\Theta^2 + 2(\omega_t - \omega_{t+1}, \omega_t - \omega^*)_\Theta - 2\eta(\omega_t - \omega_{t+1}, \omega_t - \omega^*)_\Theta \quad (32) \]

where the last equation is derived by integrating (28a). By (26b), we have

\[ 0 \in \mathcal{U}(\omega^*) \quad (33) \]

Since \( \mathcal{U} \) is maximally monotone, we have

\[ \langle (\Theta^{-1}\omega_t + \Theta^{-1} \omega_{t+1} - \omega^* \rangle_\Theta \geq 0 \quad (34) \]

Thus, we have

\[ \|\omega_{t+1} - \omega^*\|_\Theta^2 - \|\omega_t - \omega^*\|_\Theta^2 \]

\[ \leq 2\eta(\omega_t - \omega_{t+1}, \omega_t - \omega^*)_\Theta - \|\omega_t - \omega_{t+1}\|_\Theta^2 \]

Moreover,

\[ \eta(\|\omega_t - \omega^*\|_\Theta^2 - \|\omega_{t+1} - \omega^*\|_\Theta^2) \]

\[ = \eta(2(\omega_t - \omega_{t+1}, \omega_t - \omega^*)_\Theta - \|\omega_t - \omega_{t+1}\|_\Theta^2) \]

From (35) and (36), we have

\[ \|\omega_{t+1} - \omega^*\|_\Theta^2 \leq \|\omega_t - \omega^*\|_\Theta^2 - \eta(\|\omega_t - \omega_{t+1}\|_\Theta^2) \]

\[ \leq 2\eta(\omega_t - \omega_{t+1}, \omega_t - \omega^*)_\Theta - \|\omega_t - \omega_{t+1}\|_\Theta^2 \]

\[ - \eta(\|\omega_t - \omega_{t+1}\|_\Theta^2 - \|\omega_{t+1} - \omega^*\|_\Theta^2) \]

\[ = - \|\omega_t - \omega_{t+1}\|_\Theta^2 + 2\eta(\omega_t - \omega_{t+1}, \omega_t - \omega^*)_\Theta \]

\[ + \eta(\|\omega_t - \omega_{t+1}\|_\Theta^2 - \|\omega_{t+1} - \omega^*\|_\Theta^2) \]

\[ \leq (\eta - 1)(\|\omega_t - \omega_{t+1}\|_\Theta^2 + 2\|\omega_t - \omega_{t+1}\|_\Theta^2) \quad (37) \]

Define a sequence \( s_k = \|\omega_t - \omega^*\|_\Theta^2 - \eta(\|\omega_{t+1} - \omega^*\|_\Theta^2 + 2\|\omega_t - \omega_{t+1}\|_\Theta^2) \)

\[ s_{k+1} - s_k = \|\omega_{t+1} - \omega^*\|_\Theta^2 - \|\omega_t - \omega^*\|_\Theta^2 \]

\[ - \eta(\|\omega_t - \omega_{t+1}\|_\Theta^2 - \|\omega_{t+1} - \omega^*\|_\Theta^2) \]

\[ + 2\eta(\omega_t - \omega_{t+1}, \omega_t - \omega^*)_\Theta - 2\eta(\omega_t - \omega_{t+1}, \omega_t - \omega^*)_\Theta \]

\[ \leq (3\eta - 1)(\|\omega_{t+1} - \omega^*\|_\Theta^2) \quad (38) \]
From (38), we have 
\[(1 - 3\eta)^k \sum_{i=1}^k \|\omega_{i+1} - \omega_i\|_{\Theta}^2 \leq s_1 - s_{k+1} + \eta \|\omega_1 - \omega^*\|_{\Theta}^2\] 
As \(0 < \eta < \frac{1}{3}\), we have \(s_{k+1} \leq s_k \leq s_1\) and \(\|\omega_1 - \omega^*\|_{\Theta}^2 - \eta \|\omega_1 - \omega^*\|_{\Theta}^2 \leq s_k \leq s_1\). 
Then, \(\|\omega_1 - \omega^*\|_{\Theta}^2 \leq \eta^k(\|\omega_1 - \omega^*\|_{\Theta}^2 - \frac{\mu}{1-\alpha}) + \frac{\mu_1}{1-\alpha}\), i.e., \(\omega_1\) is bounded. 
Thus, \((1 - 3\eta)^k \sum_{i=1}^k \|\omega_{i+1} - \omega_i\|_{\Theta}^2 \leq s_1 + \eta^{k+1}(\|\omega_1 - \omega^*\|_{\Theta}^2 - \frac{\mu}{1-\alpha}) + \frac{\mu_1}{1-\alpha}\). 
As \(0 < \eta < \frac{1}{3}\), we have 
\[\sum_{i=1}^\infty \|\omega_{i+1} - \omega_i\|_{\Theta}^2 < \infty\] 
(39)

Thus, \(\lim_{t \to \infty} (\omega_{t+1} - \omega_t) = 0\).

Then, we prove that \(\|\omega_t - \omega^*\|_{\Theta}^2\) also converges by the similar analysis in [20] Theorem 6.3. We also write it here for completeness.

Denote \(\varphi_t = \max\{0, \|\omega_t - \omega^*\|_{\Theta}^2 - \|\omega_{t+1} - \omega^*\|_{\Theta}^2\}\) and \(\zeta_t = 2\eta \|\omega_t - \omega_{t-1}\|_{\Theta}^2\). We know that \(\varphi_t\) is lower bounded. 
Recall (37), and we have \(\varphi_{t+1} \leq \eta \varphi_t + \zeta_t\). Apply this relationship recursively, and we have 
\[\varphi_{t+1} \leq \eta^t \varphi_1 + \sum_{i=0}^{t-1} \eta^t \zeta_{t-i}\]
Adding both sides of (40) from \(t = 1 \to \infty\), we have 
\[\sum_{i=1}^\infty \varphi_i \leq \frac{\varphi_1}{1-\eta} + \frac{1}{1-\eta} \sum_{i=1}^\infty \zeta_i\]
(41)
From (39), we know \(\sum_{i=1}^\infty \zeta_i < \infty\). This implies that \(\sum_{i=1}^\infty \varphi_i\) is bounded, non-decreasing and converges.

Consider another sequence \(\{\|\omega_t - \omega^*\|_{\Theta}^2 - \sum_{t=1}^T \varphi_i\}\), which is lower bounded. Moreover, 
\[\|\omega_{t+1} - \omega^*\|_{\Theta}^2 - \sum_{t=1}^{k+1} \varphi_i = \|\omega_{t+1} - \omega^*\|_{\Theta}^2 - \varphi_{k+1} - \sum_{t=1}^{k} \varphi_i\]
\[\leq \|\omega_{t+1} - \omega^*\|_{\Theta}^2 - \|\omega_{t+1} - \omega^*\|_{\Theta}^2 + \|\omega_t - \omega^*\|_{\Theta}^2 - \sum_{t=1}^{k} \varphi_i\]
\[= \|\omega_t - \omega^*\|_{\Theta}^2 - \sum_{t=1}^{k} \varphi_i\]
This implies that \(\{\|\omega_t - \omega^*\|_{\Theta}^2 - \sum_{t=1}^{k+1} \varphi_i\}\) is a non-increasing sequence and also converges. 
\(\|\omega_t - \omega^*\|_{\Theta}^2\) is the sum of two convergent sequences, and also converges.

Since \(\omega_t\) is bounded, it has a convergent subsequence \(\omega_{n_t}\) converging to a point \(\hat{\omega}^*\). 
From \(\lim_{t \to \infty} (\omega_{t+1} - \omega_t) = 0\), we have 
\(\lim_{t \to \infty} (\omega_{n_t+1} - \omega_{n_t}) = 0\) and \(\lim_{t \to \infty} (\omega_{n_t} - \omega_{n_t+1}) = 0\).

Due to the continuity of the right-hand side of (28), we have 
\(\hat{\omega}^* = (\text{Id} + \Theta^{-1}L\gamma)^{-1}\omega^*\). Thus, \(\omega^*\) is an equilibrium point of the sequence \(\omega_t\). This also implies that \(\|\omega_t - \omega^*\|_{\Theta}^2\) converges. Because \(\|\omega_{n_t} - \omega^*\|_{\Theta}^2\) converges to zero, \(\|\omega_t - \omega^*\|_{\Theta}^2\) also converges to zero.

Based on Theorem 5, we have \((p^*, b^*)\) is the GNE of (7). This completes the proof.

Invoking Theorem 3 if Assumption 3 also holds, the algorithm will converge to the unique GNE of the generalized Nash game (7), equivalently the energy sharing game (5).

VII. EXPERIMENTS AND SIMULATION STUDIES
In this section, experiments and numerical simulations are introduced to verify the effectiveness of the proposed method. First, experiments with three prosumers are carried out to illustrate the basic properties of the algorithm. Then, a case with 123 prosumers is investigated to test the scalability, where

![Fig. 1. Experiment platform based on dSPACE RTI 1202 controller](image1)

![Fig. 2. Topology of the experiment system](image2)

the communication topology is identical to the topology of the IEEE 123-bus system [33].

A. EXPERIMENTAL RESULTS
The proposed method is verified on an experimental platform based on the dSPACE RTI 1202 controller, which is presented in Fig.1. It is composed of three inverters, one dSPACE RTI 1202 controller, two switchable loads, and one host computer. Each inverter represents a prosumer, which can both produce and consume power. The system topology is given in Fig.2. In the experiments, the breaker B0 is open, i.e., the system operates in an isolated mode. One load is connected at the bus of Prosumer1 and the other is at the bus of Prosumer2. Three prosumers are connected through impedances. The communication topology is Prosumer1 ↔ Prosumer2 ↔ Prosumer3 ↔ Prosumer1. The disutility function is 
\(h_i(p_i) = \frac{1}{2}c_i p_i^2 + d_i p_i\) with 
\(c_1 = 0.00075\), 
\(c_2 = 0.0006\), 
\(c_3 = 0.001\), 
\(d_i = 0\). The price elasticity of each prosumer is set as \(\alpha_i = -1000\). The load demand of each prosumer is 
\(D_1 = 730\) W, \(D_2 = 365\) W, \(D_3 = 0\).

The simulation scenario is: 1) at \(t = 10\) s, two loads are connected; 2) at \(t = 30\) s, load 2 is disconnected. Then, each DG regulates its generation to balance the power difference. The frequency dynamics are illustrated in Fig.3. When loads are connected, the frequency drops to about 49.2 Hz and recovers to the nominal value in four seconds. On the contrary, when load 2 is switched off, the frequency increases and recovers in 3 seconds.

The GNE of the first stage is 
\((p_1^* = 505.4\) W, \(b_1^* = 103.3\) W), \((p_2^* = 408.4\) W, \(b_2^* = 355.0\) W), \((p_3^* = 177.8\) W, \(b_3^* = 310.0\) W). Dynamics of seeking the GNE is illustrated in Fig.4 where the left part is the generation \(p_i\) of each prosumer and
the right part is the purchase willingness $b_i$. They vary slightly around the equilibrium. In this stage, Prosumer1 buys power from Prosumer2 and Prosumer3. The GNE of the second stage is $(p_1^1 = 440.6 W, b_i^1 = -41.7 W), (p_2^1 = 168.9 W, b_i^1 = -310.0 W), (p_3^1 = 123.9 W, b_i^1 = -444.2 W)$. Compared with the results obtained from the centralized method, the steady state generations are optimal to the problem (14). This shows that the GNE is obtained. Moreover, real power can always trace the reference value. This verifies that the proposed method can get the correct results in the experiment. Dynamics of the payoff functions are given in Fig. 4 which also converges in five seconds. In the first stage, Prosumer3 earns profit by selling power to Prosumer1 and Prosumer2. In the second stage, there is only Prosumer1 buying power, while others selling. The cost of Prosumer2 changes from positive to negative, which implies that it earns profit by selling power to Prosumer1.

\[ \eta = 1 + \frac{1}{T} \]

The effect of $\eta$ is also investigated, which is set as $0, 0.1, 0.2, 0.33$ respectively. The convergence of $p_i(t)$ to the equilibrium point is given in Fig. 5. With $\eta = 0$, it takes more than 20,000 iterations to make the relative error smaller than 1%. With $\eta = 0.33$, it takes about 14,000 iterations, which is reduced by 30%. It is also observed that a larger $\eta$ results in a higher convergence rate.

VIII. CONCLUSION

In this work, the energy sharing game among prosumers is formulated, whose objective function for each prosumer is associated with decisions of all the prosumers. To solve it in a fully distributed way, the game is transformed to an equivalent optimization problem. Then, a Kränsosel’skiĭ–Mann iteration type algorithm is devised to solve the problem and consequently find the GNE in a distributed manner with neighboring communication. The convergence of the algorithm is proved rigorously. Experimental results with three prosumers show that the GNE can be obtained at a fast speed. Simulations on 123 prosumers verify the scalability of the proposed method.

This paper offers a different perspective for seeking the GNE in the GNG. If the objective function of one prosumer is associated with too many other prosumers, it is possible to transform the GNG to an equivalent optimization problem to avoid dense communication. In our future work, it is interesting to investigate the conditions of the equivalent transformation for a more general GNG.

REFERENCES

[1] Y. M. Chiang, “Building a better battery,” Science, vol. 330, no. 6010, pp. 1485–1486, 2010.
[2] P. Mani, J. Lee, K. Kang, and Y. H. Joo, “Digital controller design via Lmis for direct-driven surface mounted pmg-based wind energy conversion system,” IEEE Transactions on Cybernetics, pp. 1–12, 2019.
[3] Z. Wang, Y. Chen, S. Wei, S. Huang, and Y. Xu, “Optimal expansion planning of isolated microgrid with renewable energy resources and controllable loads,” IET Renewable Power Generation, vol. 11, no. 7, pp. 931–940, 2017.
[4] M. Tran, D. Banister, J. D. K. Bishop, and M. D. McCulloch, “Realizing the electric-vehicle revolution,” Nature Climate Change, vol. 2, no. 5, pp. 328–333, 2012.
[5] Y. Zhang, L. Chu, Y. Ou, C. Guo, Y. Liu, and X. Tang, “A cyber-physical system-based velocity-profile prediction method and case study of application in plug-in hybrid electric vehicle,” IEEE Transactions on Cybernetics, pp. 1–12, 2019.
[6] D. M. Han and J. H. Lim, “Smart home energy management system using iee 802.15.4 and zigbee,” IEEE Transactions on Consumer Electronics, vol. 56, no. 3, pp. 1403–1410, 2010.
[7] S. Weng, D. Yue, C. Dou, J. Shi, and C. Huang, “Distributed event-triggered cooperative control for frequency and voltage stability and power sharing in isolated inverter-based microgrid.” IEEE transactions on cybernetics, no. 99, pp. 1–13, 2018.
[8] H. Zhang, D. Yue, C. Dou, K. Li, and X. Xie, “Event-triggered multiagent optimization for two-layered model of hybrid energy system with price bidding-based demand response,” IEEE Transactions on Cybernetics, pp. 1–12, 2019.
[9] A. Dimeas, S. Drenkard, N. Hatziargyriou, S. Karnouskos, K. Kok, J. Ringelstein, and A. Weidlich, “Smart houses in the smart grid: Developing an interactive network,” IEEE Electrification Magazine, vol. 2, no. 1, pp. 81–93, 2014.

[10] Y. Parag and B. K. Sovacool, “Electricity market design for the prosumer era,” Nature Energy, vol. 1, no. 4, pp. 1–6, 2016.

[11] T. Morstyn, N. Farrell, S. J. Darby, and M. D. McIcullough, “Using peer-to-peer energy-trading platforms to incentivize prosumers to form federated power plants,” Nature Energy, vol. 3, no. 2, pp. 94–101, 2018.

[12] N. Liu, X. Yu, C. Wang, C. Li, L. Ma, and J. Lei, “Energy-sharing model with price-based demand response for microgrids of peer-to-peer prosumers,” IEEE Trans. Power Syst., vol. 32, no. 5, pp. 3569–3583, 2017.

[13] Y. Chen, S. Mei, F. Zhou, S. H. Low, and F. Liu, “An energy sharing game in prosumers based on generalized demand bidding: Model and properties,” IEEE Transactions on Smart Grid, in press, 2019.

[14] Z. Wang, F. Liu, S. H. Low, C. Zhao, and S. Mei, “Distributed frequency control with operational constraints, part ii: Network power balance,” IEEE Trans. Smart Grid, vol. 10, no. 1, pp. 53–64, Jan 2019.

[15] F. Dorfler, J. W. Simpson-Porco, and F. Bullo, “Breaking the hierarchy: Distributed control and optimization in microgrids,” IEEE Trans. Control Network Syst., vol. 3, no. 3, pp. 241–253, 2016.

[16] Z. Wang, F. Liu, Y. Chen, S. H. Low, and S. Mei, “Unified distributed control of stand-alone dc microgrids,” IEEE Trans. Smart Grid, vol. 10, no. 1, pp. 1013–1024, Jan 2019.

[17] Z. Wang, F. Liu, S. H. Low, P. Yang, and S. Mei, “Distributed load-side control: Coping with variation of renewable generations,” Automatica, in press, 2019.

[18] S. Liang, P. Yi, and Y. Hong, “Distributed Nash equilibria seeking for aggregative games with coupled constraints,” Automatica, vol. 85, pp. 179–185, 2017.

[19] X. Zeng, J. Chen, S. Liang, and Y. Hong, “Generalized Nash equilibrium seeking strategy for distributed nonsmooth multi-cluster game,” Automatica, vol. 103, pp. 20–26, 2019.

[20] P. Yi and L. Pavel, “An operator splitting approach for distributed generalized Nash equilibria computation,” Automatica, vol. 102, pp. 111–121, 2019.

[21] P. Yi and L. Pavel, “Asynchronous distributed algorithms for seeking generalized Nash equilibria under full and partial-decision information,” IEEE transactions on cybernetics, in press, 2019.

[22] C. Yu, M. van der Schaar, and A. H. Sayed, “Distributed learning for stochastic generalized Nash equilibrium problems,” IEEE Transactions on Signal Processing, vol. 65, no. 15, pp. 3899–3908, Aug 2017.

[23] D. Paccagnan, B. Gentile, F. Parise, M. Kamgarpour, and J. Lygeros, “Nash and Wardrop equilibrium in aggregative games with coupling constraints,” IEEE Transactions on Automatic Control, vol. 64, no. 4, pp. 1373–1388, April 2019.

[24] F. Salesisadaghi and L. Pavel, “Distributed Nash equilibrium seeking via the alternating direction method of multipliers,” IFAC-PapersOnLine, vol. 50, no. 1, pp. 6166 – 6171, 2017.

[25] P. Yi and L. Pavel, “Distributed generalized Nash equilibria computation of monotone games via double-layer pre-conditioned proximal-point algorithms,” IEEE Transactions on Control of Network Systems, vol. 6, no. 1, pp. 299–311, 2018.

[26] G. Belgioioso and S. Grammatico, “A distributed proximal-point algorithm for Nash equilibrium seeking in generalized potential games with linearly coupled cost functions,” in 2019 19th European Control Conference (ECC), June 2019, pp. 1–6.

[27] F. Salesisadaghi, W. Shi, and L. Pavel, “Distributed Nash equilibrium seeking under partial-decision information via the alternating direction method of multipliers,” Automatica, vol. 103, pp. 27 – 35, 2019.

[28] L. Pavel, “Distributed self seeking under partial-decision information over networks via a doubly-augmented operator splitting approach,” IEEE Transactions on Automatic Control, vol. 65, no. 4, pp. 1584–1597, 2020.

[29] H. Bauschke and P. L. Combettes, Convex Analysis and Monotone Operator Theory in Hilbert Spaces. Springer, 2017.

[30] S. Boyd and L. Vandenberghe, Convex optimization. Cambridge university press, 2004.

[31] F. Facchinei and C. Kanzow, “Generalized nash equilibrium problems,” Annals of Operations Research, vol. 175, no. 1, pp. 177–211, 2010.

[32] W. H. Kersting, “Radial distribution test feeders,” IEEE Transactions on Power Systems, vol. 6, no. 3, pp. 975–985, 1991.

[33] A. P. Ruszczynski and A. Ruszczynski, Nonlinear optimization. Princeton university press, 2006, vol. 13.

APPENDIX A
PROOFS OF THEOREM 2

Proof. 1) From Lemma 1, we need to find $x^* := (p_i^*, b_i^*) \in X$ such that
\[
\langle F(x^*), x - x^* \rangle \geq 0, \forall x \in X \tag{A.1}
\]
Now, we check the KKT condition for $V(X, F(p, b))$ in (A.1). In fact, $x^*$ is a solution to $V(X, F(p, b))$ in (A.1) if and only if $x^*$ is the optimal solution to the following optimization problem:
\[
\min_{x \in \mathbb{R}^n} \langle F(x^*), x \rangle, \quad \text{s.t.} \ x \in X \tag{A.2}
\]
If $x^*$ solves (A.2), there exists $\lambda^* \in \mathbb{R}$ such that the following optimality conditions (KKT) are satisfied [33 Theorem 3.25]

\[
0 \in h_i(p_i^*) + \lambda_i^* + N_{\Omega_i}(p_i^*) \tag{A.3a}
\]
\[
0 = - (2a_i \mu_c + b_i^* + a_i \lambda_i^*) \frac{1}{1^T a} + \mu_c + \lambda_i^* \tag{A.3b}
\]
\[
0 = p_i^* - a_i \mu_c^* + b_i^* - D_i \tag{A.3c}
\]
\[
0 = 1^T a \mu_c^* + b_i^* + \sum_{j \neq i} b_j^* \tag{A.3d}
\]
From (A.3b), we know
\[
0 = - (2a_i \mu_c^* + b_i^* + a_i \lambda_i^*) \frac{1}{1^T a} + \mu_c + \lambda_i^*
\] \Rightarrow \lambda_i^* = \frac{1^T a - 2a_i}{a_i - 1^T a} \mu_c^* - \frac{1}{a_i - 1^T a} b_i^* \tag{A.4}
\]
From (A.3c), we know
\[
b_i^* = D_i - p_i^* - a_i \mu_c^* \tag{A.5}
\]
Combining (A.4) and (A.5), we have
\[
\lambda_i^* = \frac{1^T a - 2a_i}{a_i - 1^T a} \mu_c^* - \frac{1}{a_i - 1^T a} D_i
\]
\[
\mu_c^* = \frac{a_i - 1^T a}{a_i - 1^T a} \mu_c^* + \frac{1}{a_i - 1^T a} D_i \tag{A.6}
\]
Then, the KKT condition (A.3) is
\[
0 \in h_i(p_i^*) + \frac{p_i^*}{a_i - 1^T a} - \frac{D_i}{a_i - 1^T a} - \mu_c^* + N_{\Omega_i}(p_i^*) \tag{A.7a}
\]
\[
0 = \sum_{i \in N} D_i - \sum_{i \in N} p_i \tag{A.7b}
\]
It is also the KKT condition of the following problem
\[
\min_{p} h(p) = \sum_{i \in N} \left( h_i(p_i) + \frac{p_i^2}{2(a_i - 1^T a)} - \frac{D_i}{a_i - 1^T a} p_i \right) \tag{A.8a}
\]
s.t. \[
\sum_{i \in N} p_i = \sum_{i \in N} D_i, \quad \mu_c \tag{A.8b}
\]
where $\mu_c$ is the Lagrangian multiplier.
Define
\[
\hat{h}_i(p_i) = h_i(p_i) + \frac{p_i^2}{2(a_i - 1^T a)} - \frac{D_i}{a_i - 1^T a} p_i \tag{A.9}
\]
We know $\hat{h}_i(p_i)$ is strongly convex and the Slater’s condition holds by Assumption 2. Thus, $p_i^*$ and $\mu_c^*$ exist. By (A.5), $b_i^*$ also exists.

2) If Assumption 3 holds, there exists at least one prosumer $i$ with $p_i < p_i^* < \tilde{p}_i$. For this $i$, we have
\{0\} = N_{\Omega_i}(p^*_i). Then, by (A.7a), \(\mu^*_e\) is unique. The unique \(b^*_i\) can be obtained from (A.5). \(\square\)