EIGENVALUES OF STURM-LIOUVILLE OPERATORS WITH DISTRIBUTIONAL POTENTIALS

JUN YAN, GUOLIANG SHI, AND JIA ZHAO

Abstract. We introduce a novel approach for dealing with eigenvalue problems of Sturm-Liouville operators generated by the differential expression

\[ Ly = \frac{1}{r} (-\left(p\left[y'+sy\right]\right)' + sp\left[y'+sy\right] + qy) \]

which is based on norm resolvent convergence of classical Sturm-Liouville operators. This enables us to describe the continuous dependence of the \(n\)-th eigenvalue on the space of self-adjoint boundary conditions and the coefficients of the differential equation after giving the inequalities among the eigenvalues. Moreover, oscillation properties of the eigenfunctions are also characterized. In particular, our main results can be applied to solve a class of Sturm-Liouville problems with transmission conditions.

Introduction

The prime motivation behind this paper is to discuss the properties of eigenvalues of self-adjoint Sturm-Liouville operators generated by the differential expression (0.1)

\[ Ly = \frac{1}{r} (-\left(p\left[y'+sy\right]\right)' + sp\left[y'+sy\right] + qy), \quad \text{on} \quad J = (a, b), \quad -\infty < a < b < \infty, \]

where the coefficients \(p, q, r, s\) are real-valued and (0.2) \(1/p, q, r, s \in L(J, \mathbb{R}), p > 0, r > 0\) a.e. on \(J\).

Note that when \(p(x) \equiv 1\), the definition and the self-adjoint domain of (0.1) have been characterized by A. M. Savchuk and A. A. Shkalikov in [1] and [2]. Moreover, in the special case \(s \equiv 0\) this differential expression reduces to the standard one, that is, one obtains,

\[ Ly = \frac{1}{r} (-\left(p y'\right)' + qy). \]

In the paper [3], J. Eckhardt, F. Gesztesy, R. Nichols and G. Teschl have given a description of all the self-adjoint operators generated by the expression (0.1). Following [3], we introduce the quasi-derivative \(y[1] = p[y' + sy]\), the self-adjoint boundary conditions are given as follows:

(0.4) \[ A \begin{pmatrix} y(a) \\ y[1](a) \end{pmatrix} + B \begin{pmatrix} y(b) \\ y[1](b) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \]

2010 Mathematics Subject Classification. Primary 34B24; Secondary 34L15, 34L05, 34L10, 34C10.

Key words and phrases. Sturm-Liouville problems, eigenvalue inequalities, distributional potentials, oscillation properties.

*This research was supported by the National Youth Scientific Foundation of China under Grant No. 11601372.
where the complex $2 \times 2$ matrices $A$ and $B$ satisfy:

$$\text{(0.5)}$$

the $2 \times 4$ matrix $(A|B)$ has full rank, and $AEA^* = BEB^*$, $E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Note that $A^*$ is the complex conjugate transpose of the complex matrix $A$. The boundary conditions (0.4) can be divided into three classes of boundary conditions as follows:

1. Separated self-adjoint boundary conditions:

$$\text{(0.6)}$$

$$S_{\alpha,\beta} : \left\{ \begin{array}{ll}
\cos \alpha y(a) - \sin \alpha y^{[1]}(a) = 0, & \alpha \in [0, \pi), \\
\cos \beta y(b) - \sin \beta y^{[1]}(b) = 0, & \beta \in (0, \pi].
\end{array} \right.$$  

2. All real coupled self-adjoint boundary conditions:

$$\text{(0.7)}$$

$$Y(b) = KY(a), \ K \in \text{SL}(2, \mathbb{R}).$$

3. All complex coupled self-adjoint boundary conditions:

$$\text{(0.8)}$$

$$Y(b) = e^{i\gamma} KY(a), \ -\pi < \gamma < 0 \text{ or } 0 < \gamma < \pi, \ K \in \text{SL}(2, \mathbb{R}),$$

where

$$K \in \text{SL}(2, \mathbb{R}) =: \left\{ \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} : k_{ij} \in \mathbb{R}, \ \det K = 1 \right\}, \ Y(\cdot) = \begin{pmatrix} y(\cdot) \\ y^{[1]}(\cdot) \end{pmatrix}.$$

Actually, (0.7) can be treated as a case of (0.8) when $\gamma = 0$.

In the last decades, Schrödinger operators with distributional potentials have attracted tremendous interest since they can be used as solvable models in many situations. We should mention that there were actually earlier papers dealing with Schrödinger operators involving strongly singular and oscillating potentials, such as, M.-L. Baeteman and K. Chadan [4], [5], M. Combescure [6], M. Combescure and J. Ginibre [7], D. B. Pearson [8], F. S. Rofe-Beketov and E. H. Hristov [9], [10], and a more recent contribution treating distributional potentials by J. Herczynski [11]. In addition, numerous results on the case of point interactions can be found in some standard monographs by S. Albeverio, F. Gesztesy, R. Høg-Hrohn, and H. Holden [12] and S. Albeverio and P. Kurasov [13]. It was not until 1999 that A. M. Savchuk and A. A. Shkalikov [1] started a new development for Schrödinger operators with distributional potential coefficients. And the operators with distribution potentials proposed by A.M. Savchuk and A.A. Shkalikov have been received enormous attention. We also emphasize that similar differential expressions have already been studied by C. Bennewitz and W. N. Everitt [14] (see also [15]).

In the paper [3] and [16], J. Eckhardt, F. Gesztesy, R. Nichols and G. Teschl have given a systematical development of Weyl–Titchmarsh theory and inverse spectral theory for singular differential operators on arbitrary intervals $(a, b) \subset \mathbb{R}$ associated with the differential expressions (0.1). Under the assumption (0.2) on the coefficients, the discreteness and boundedness from below of the spectrum has been proved in [3] for the self-adjoint differential operators associated with the differential expression (0.1). In this paper, we will continue to discuss the properties of eigenvalues of the self-adjoint differential operators with distributional potentials under the assumption (0.2).
Actually, in this paper, by a different method, we will also show that the spectrum of the self-adjoint differential operators associated with the differential expression (0.1) is discrete, and the eigenvalues can also be ordered to form a non-decreasing sequence

$$\lambda_0, \lambda_1, \lambda_2, \lambda_3, \ldots$$

approaching $+\infty$ so that the number of times an eigenvalue appears in the sequence is equal to its multiplicity. Here multiplicity refers to both the algebraic and geometric multiplicities, since in this paper we will show that the two multiplicities are equal for self-adjoint boundary conditions. Note that for the Sturm-Liouville problems with distributional potentials, the algebraic multiplicity of an eigenvalue we introduce is the order of its zero as a root of the characteristic function discussed in Lemma 1.10. The geometric multiplicity of an eigenvalue is naturally the number of the linearly independent eigenfunctions of this eigenvalue.

As we have known, for Sturm-Liouville operators with regular potentials, i.e., the operators generated by the differential expression (0.3), M.S.P. Eastham, Q. Kong, H. Wu, and A. Zettl (17, 18 and 19) have characterized the inequalities among the eigenvalues corresponding to different self-adjoint boundary conditions, the continuity region of the $n$-th eigenvalue as a function on the space of self-adjoint boundary conditions, the dependence of the $n$-th eigenvalue on the coefficients of the differential equation, and the oscillation properties of the eigenfunctions. In contrast, such theory for Sturm-Liouville operators with distributional potentials has not yet been developed, and it is precisely the purpose of this paper to have a discussion on the corresponding properties.

Enlightened by the space introduced by Q. Kong and A. Zettl in [19], in this paper we will introduce a more general “boundary value problem space” with a metric to study the Sturm-Liouville problems with distributional potentials. Let $\Omega = \{ \omega = (A, B, 1/p, q, r, s); (12) \text{ and } (1.5) \text{ hold} \}$. For the topology of $\Omega$ we use a metric defined as follows: For $\omega = (A, B, 1/p, q, r, s) \in \Omega$, $\omega_0 = (A_0, B_0, 1/p_0, q_0, r_0, s_0) \in \Omega$, define $d(\omega, \omega_0) = \|A - A_0\| + \|B - B_0\| + \int_a^b \left( \left\| \frac{1}{p} - \frac{1}{p_0} \right\| + |q - q_0| + |r - r_0| + |s - s_0| \right)$ where $\|\cdot\|$ denotes any matrix norm. Denote the space of all complex self-adjoint boundary conditions by $B_{CS}$ which is the similar to the space associated with the Sturm-Liouville problems with regular potentials introduced firstly in [24]. Under such a topology, we will have a research on the continuity region of the $n$-th eigenvalue as a function on $B_{CS}$, the differentiability and monotonicity of the $n$-th eigenvalue with respect to $\alpha, \beta$ in the separated boundary conditions are also given.

We will prove the continuous dependence and differentiability of the $n$-th eigenvalue with respect to the coefficients $1/p, q, r, s$ in the sense of Frechet derivative in the Banach space $L(J, \mathbb{R})$.

It is worth mentioning that, in order to analyze the eigenvalues of Sturm-Liouville operators with distributional potentials, we introduce an approach that relies heavily on the “norm resolvent convergence” and an asymptotic form of the fundamental solutions of the equation (1.2) for sufficiently negative $\lambda$, which is different from that for Sturm-Liouville operators with regular potentials. In this paper, we will find a sequence of Sturm-Liouville operators $L_m$ with regular potentials to approximate the Sturm-Liouville operator $L$ with a distributional potential in norm resolvent convergence (Lemma 2.11). Furthermore, in Lemma 2.15 we will show that the lowest eigenvalues of $L_m$ are uniformly semi-bounded from below, this will
guarantee the sequence of the $n$-th eigenvalues of Sturm-Liouville operators $L_m$ converges to the $n$-th eigenvalue of the operator $L$ (Lemma 2.11). In a word, our approach not only enables us to obtain a series of results, but also yields a relation between the eigenvalues of Sturm-Liouville operators with distributional potentials and the eigenvalues of Sturm-Liouville operators with regular potentials. Moreover, the main conclusions obtained in this paper can be applied to solve the eigenvalue problems of Sturm-Liouville operators with transmission conditions which have been an important research topic in mathematical physics [20, 21, 22, 23].

This paper is organized as follows. In Section 1, we recall some basic results, and prove a condition for norm resolvent convergence. In Section 2, some preliminary and important lemmas for the main results are stated and proved. In Section 3, we give a comment on the continuity region, the differentiability and monotonicity of the $n$-th eigenvalue with respect to $\alpha, \beta$ in the separated boundary conditions. Oscillation properties of the eigenfunctions of all the self-adjoint Sturm-Liouville problems are given in Section 5 after discussions on the inequalities among eigenvalues in Section 4. Section 6 is devoted to describe the continuity region of the $n$-th eigenvalue as a function on the space of self-adjoint boundary conditions. In Section 7, we also comment on the continuous dependence and differentiability of the $n$-th eigenvalue on the coefficients of the differential equation. Finally, in Section 8, we solve some eigenvalue problems of a class of Sturm-Liouville operators with transmission conditions.

1. Notation and prerequisites results

We introduce the quasi-derivative $y^{[1]} = p [y' + sy]$ and rewrite expression (0.1) in the form

$$L y = \frac{1}{r} \left(-y^{[1]} + sy^{[1]} + qy\right), \quad x \in (a, b).$$

Let $\phi_1$ and $\phi_2$ be the fundamental solutions of

$$-y^{[1]} + sy^{[1]} + qy = \lambda r y, \quad x \in (a, b),$$

determined by the initial conditions

$$\phi_1(a, \lambda) = \phi_2^{[1]}(a, \lambda) = 1, \quad \phi_2(a, \lambda) = \phi_1^{[1]}(a, \lambda) = 0, \quad \lambda \in \mathbb{C}.$$

Denote

$$\Phi(x, \lambda) = \left(\begin{array}{cc} \phi_1(x, \lambda) & \phi_2(x, \lambda) \\ \phi_1^{[1]}(x, \lambda) & \phi_2^{[1]}(x, \lambda) \end{array}\right),$$

then $\Phi(x, \lambda)$ is the fundamental matrix solution of

$$Y^{'}(x) = [P(x) - \lambda W(x)]Y(x), \quad Y(a) = I, \quad x \in (a, b),$$

where

$$P(x) = \left(\begin{array}{cc} -s(x) & \frac{1}{p(x)} \\ q(x) & s(x) \end{array}\right), \quad W(x) = \left(\begin{array}{cc} 0 & 0 \\ r(x) & 0 \end{array}\right).$$
For $K \in \text{SL}(2, \mathbb{R})$, and $\lambda \in \mathbb{C}$, we define

\[
D(\lambda) = k_{11}\phi_2^{[1]}(b, \lambda) - k_{21}\phi_2(b, \lambda) + k_{22}\phi_1(b, \lambda) - k_{12}\phi_1^{[1]}(b, \lambda),
\]

\[
A(\lambda) = k_{11}\phi_1^{[1]}(b, \lambda) - k_{21}\phi_1(b, \lambda),
\]

\[
B(\lambda) = k_{11}\phi_2^{[1]}(b, \lambda) + k_{12}\phi_1^{[1]}(b, \lambda) - k_{21}\phi_2(b, \lambda) - k_{22}\phi_1(b, \lambda),
\]

\[
D_1(\lambda) = k_{11}\phi_2^{[1]}(b, \lambda) - k_{21}\phi_2(b, \lambda),
\]

\[
D_2(\lambda) = k_{22}\phi_1(b, \lambda) - k_{12}\phi_1^{[1]}(b, \lambda),
\]

\[
C(\lambda) = k_{22}\phi_2(b, \lambda) - k_{12}\phi_2^{[1]}(b, \lambda).
\]

Note that

\[
K^{-1}\Phi(b, \lambda) = \left( \begin{array}{cc}
D_2(\lambda) & C(\lambda) \\
A(\lambda) & D_1(\lambda)
\end{array} \right).
\]

For $K \in \text{SL}(2, \mathbb{R})$, consider the separated boundary conditions:

\[
y(a) = 0, \quad k_{22}y(b) - k_{12}y^{[1]}(b) = 0,
\]

\[
y^{[1]}(a) = 0, \quad k_{22}y(b) - k_{12}y^{[1]}(b) = 0,
\]

and denote the $n$-th eigenvalue for (1.7) and (1.8) by $\mu_n = \mu_n(K)$ and $\nu_n = \nu_n(K)$ respectively, $n \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$. Denote the $n$-th eigenvalue for (0.7), (0.8) by $\lambda_n(K)$, $\lambda_n(\gamma, K)$ respectively, $n \in \mathbb{N}_0$.

Note that if $ps$ is a smooth function, then the operators generated by the differential expression (0.4) with the boundary conditions (0.4) are Sturm-Liouville operators with regular potentials.

We also introduce the conditions: (i) Dirichlet boundary conditions $S_{0,\pi}$: $y(a) = y(b) = 0$, (ii) periodic boundary conditions (when $K = I$) : $y(a) = y(b), y^{[1]}(a) = y^{[1]}(b)$, (iii) semi-periodic boundary conditions (when $K = -I$): $y(a) = -y(b), y^{[1]}(a) = -y^{[1]}(b)$.

Now we recall some notations introduced by T. Kato in [26]. Consider closed linear manifolds $M$ and $N$ of a Banach space $X$. We denote by $S_M$ the unit sphere of $M$ (the set of all $u \in M$ with $\|u\| = 1$). For any two closed linear manifolds $M$, $N$ of $X$, we set

\[
\delta(M, N) = \sup_{u \in S_M} \text{dist}(u, N), \quad \delta(M, N) = \max[\delta(M, N), \delta(N, M)].
\]

Consider the set $\mathcal{C}(X, Y)$ of all closed operators from a Banach space $X$ to a Banach space $Y$. If $T, S \in \mathcal{C}(X, Y)$, their graphs $G(T), G(S)$ are closed linear manifolds of the product space $X \times Y$. We set

\[
\delta(T, S) = \delta(G(T), G(S)), \quad \hat{\delta}(T, S) = \hat{\delta}(G(T), G(S)) = \max[\delta(T, S), \delta(S, T)].
\]

$\hat{\delta}(T, S)$ will be called the gap between $T$ and $S$. We recall that $T_n$ **converges to $T$ in the generalized sense** if $\hat{\delta}(T_n, T) \to 0$.

Now we recall the definition of the norm resolvent convergence. Let $T$ and $\{T_m\}_{m \in \mathbb{N}}$ be closed operators in Hilbert space $H$. We say that the sequence of the operators $T_m$ converges to $T$ in the sense of norm resolvent convergence, i.e. $T_m \xrightarrow{\text{r}} T$, if there is a number $\mu \in \mathbb{C}$ belonging to the resolvent sets $\rho(T)$ and $\rho(T_m)$ for all sufficiently large $m$ and the sequence of bounded operators $(T_m - \mu)^{-1}$ converges uniformly to the operator $(T - \mu)^{-1}$. 

Lemma 1.1. Let $T, T_m \in \mathcal{C}(X, Y)$, $n \in \mathbb{N}$. We denote by $\mathcal{B}(X, Y)$ the set of all bounded operators on $X$ to $Y$.

1. If $T^{-1}$ exists and belongs to $\mathcal{B}(Y, X)$, $T_m \to T$ in the generalized sense if and only if $T_m^{-1}$ exists and belongs to $\mathcal{B}(Y, X)$, for sufficiently large $m$ and $\|T_m^{-1} - T^{-1}\| \to 0$.

2. $T_m \to T$ in the generalized sense and if $A \in \mathcal{B}(X, Y)$, then $T_m + A \to T + A$ in the generalized sense.

Proof. See [26, Theorem 2.23].

Remark 1.2. If $T_m \xrightarrow{R} T$, from the above claim (1), it can be seen that $T_m - \mu \to T - \mu$ in the generalized sense, $\mu \in \mathbb{C}$ is a number belonging to the resolvent sets $\rho(T)$ and $\rho(T_m)$ for all sufficiently large $m$. From the above claim (2), it follows that $T_m \to T$ in the generalized sense.

Lemma 1.3. Let $T \in \mathcal{C}(X)$ and let $\Gamma$ be a compact subset of the resolvent set $\rho(T)$. Then there is a $\delta > 0$ such that $\Gamma \subset \rho(S)$ for any $S \in \mathcal{C}(X)$ with $\delta(S, T) < \delta$.

Proof. See [26, Theorem 3.1].

Corollary 1.4. For self-adjoint operators $T_m$ and $T$, $T_m \xrightarrow{R} T$.

1. If $\lambda(m) \in \sigma(T_m)$, and $\lambda(m) \to c$, as $m \to \infty$, then $c \in \sigma(T)$.

2. If $\lambda \in \sigma(T)$, then there must exist $\lambda(m) \in \sigma(T_m)$, such that $\lambda(m) \to \lambda$, as $m \to \infty$.

Proof. (1) If $c \in \rho(T) \cap \mathbb{R}$, since $\rho(T)$ is open, so there exists a $\gamma > 0$ such that $[c - \gamma, c + \gamma] \subset \rho(T)$. So from Lemma 1.3, there is a $\delta > 0$ such that $[c - \gamma, c + \gamma] \subset \rho(S)$ for any $S \in \mathcal{C}(X)$ with $\delta(S, T) < \delta$. If $T_m \xrightarrow{R} T$, then $T_m \to T$ in the generalized sense. So there exists $N > 0$ such that if $m > N$, $\delta(T_m, T) < \delta$. Then $[c - \gamma, c + \gamma] \subset \rho(T_m)$ if $m > N$. This contradicts to the fact that $\lambda(m) \to c$, as $m \to \infty$.

(2) See [28, Theorem VIII.24].

Lemma 1.5. For self-adjoint operators $T_m$ and $T$, $T_m \xrightarrow{R} T$, if $\lambda_0$ is an isolated eigenvalue of the operator $T$ with finite geometric multiplicity $\chi$, then there are finitely many eigenvalues of the operators $T_m$ in an arbitrary sufficiently small $\delta$-neighborhood of the point $\lambda_0$ if $m$ is large enough. Moreover, their total geometric multiplicity equals $\chi$.

Proof. See [26, 4.3.4 and 4.3.5] and [1, Lemma 5]. Note that for self-adjoint operators, from [27, Proposition 6.3], the geometric multiplicity of an eigenvalue is equal to the multiplicity described in [26] and [1].

Lemma 1.6. For self-adjoint operators $T_m$ and $T$, $T_m \xrightarrow{R} T$, as $m \to \infty$, $m \in \mathbb{N}$, the spectrum of $T_m$ are discrete and uniformly semi-bounded from below, denote the $n$-th eigenvalue of $T_m$ by $\lambda_n(m)$, $n \in \mathbb{N}_0$. Then we obtain the following conclusions:

1. The spectrum of $T$ is discrete and semi-bounded from below.

2. The sequence of the $n$-th eigenvalues $\lambda_n(m)$ of the operators $T_m$ converges to the $n$-th eigenvalue $\lambda_n(0)$ of the operator $T$, i.e., $\lambda_n(m) \to \lambda_n(0)$, as $m \to \infty$. (Note that the eigenvalues are ordered with geometric multiplicities.)
Proof. (1) For self-adjoint operators $T_m$, the spectrum of $T_m$ is discrete if and only if $T_m$ has compact resolvent, together with the fact $T_m \xrightarrow{R} T$, so $T$ has compact resolvent and the spectrum of $T$ is discrete. Denote $r$ the uniform bound of the spectrum of $T_m$. Assume the spectrum of $T$ is not semi-bounded from below, there must exist an eigenvalue $\lambda$ of $T$ such that $\lambda < r - 1$. Since $T_m \xrightarrow{R} T$, for the eigenvalue $\lambda$ of $T$, there must exist $\lambda(m) \in \sigma(T_m)$, such that

$$\lambda(m) \to \lambda < r - 1, \text{ as } m \to \infty.$$ 

Now we reach a contradiction to obtain our claim.

(2) Next, we will show that for $n \in \mathbb{N}_0$, $\lambda_n(m) \to \lambda_n(0)$, as $m \to \infty$. For simplicity, we assume $\lambda_0(0)$ is geometrically simple and $\lambda_1(0)$ is geometrically double, and only prove the claim for $\lambda_0(0)$ and $\lambda_1(0)$, the proofs for other cases follow from a similar process.

(i) For the simple eigenvalue $\lambda_0(0)$ of $T$, there exist $\Lambda_0(m) \in \sigma(T_m)$ such that

$$\Lambda_0(m) \to \lambda_0(0), \text{ as } m \to \infty.$$ 

It suffices to show that $\Lambda_0(m) = \lambda_0(m)$ for sufficiently large $m$. Assume the contrary, there exists a subsequence $\{\lambda_0(m_j)\}_{j=1}^\infty$ such that

$$\Lambda_0(m_j) > \lambda_0(m_j).$$

Since the spectrum of $T_m$ are uniformly semi-bounded from below, without loss of generality, assume $\lambda_0(m_j) \to c$, then $\lambda_0(0) \geq c \in \sigma(T)$. Since $\lambda_0(0)$ is geometrically simple, from Lemma 1.5 one deduces that $c < \lambda_0(0)$. This contrary implies

$$\lambda_0(m) \to \lambda_0(0), \text{ as } m \to \infty.$$ 

(ii) For the double eigenvalue $\lambda_1(0) = \lambda_2(0)$, there exist eigenvalues $\Lambda_1(m)$ and $\Lambda_2(m)$ of $T_m$ such that

$$\Lambda_1(m) \to \lambda_1(0), \text{ } \Lambda_2(m) \to \lambda_2(0), \text{ as } m \to \infty,$$

and

$$\Lambda_1(m) \leq \Lambda_2(m).$$

Then it suffices to show that for sufficiently large $m$, 

$$\Lambda_1(m) = \lambda_1(m), \text{ } \Lambda_2(m) = \lambda_2(m).$$

Assume there exists a subsequence $\{\lambda_1(m_j)\}_{j=1}^\infty$ such that $\Lambda_1(m_j) > \lambda_1(m_j)$. Since $\{\lambda_1(m_j)\}_{j=1}^\infty$ is bounded, without loss of generality, assume

$$\lambda_1(m_j) \to c, \text{ as } j \to \infty.$$ 

Hence it follows from Corollary 1.4 that $c \in \sigma(T)$ and $\lambda_0(0) \leq c \leq \lambda_1(0)$. Since $\lambda_0(0)$ is geometrically simple and $\lambda_1(0)$ is geometrically double, from Lemma 1.5 one deduces that

$$\lambda_0(0) < c < \lambda_1(0).$$

This contrary implies $\Lambda_1(m) = \lambda_1(m)$ for sufficiently large $m$.

Assume there exists a subsequence $\{\lambda_2(m_j)\}_{j=1}^\infty$ such that $\Lambda_2(m_j) > \lambda_2(m_j)$. Since $\{\lambda_2(m_j)\}_{j=1}^\infty$ is bounded, without loss of generality, assume

$$\lambda_2(m_j) \to c, \text{ as } j \to \infty.$$
Lemma 1.9. For any \( t \in \mathbb{R} \) and \( |t| < \varepsilon \), it is deduced from Lemma 1.5 that \( \lambda_1(0) \leq c \leq \lambda_2(0) \). Since \( \lambda_1(0) \) is geometrically double, from Lemma 1.5 one deduces a contradiction to imply \( \Lambda_2(m) = \lambda_2(m) \) when \( m \) is sufficiently large.

Proceeding as in the proof for \( \lambda_0(0) \) and \( \lambda_1(0) \), this theorem will be completed. \( \square \)

Lemma 1.7. For any \( x_0 \in [a, b] \), the initial problem consisting of equation (1.2) with the initial value

\[
y(x_0, \lambda) = c_1, \quad y^{[1]}(x_0, \lambda) = c_2,
\]

where \( c_1, c_2 \in \mathbb{C} \), has a unique solution \( y(x, \lambda) \). And each of the functions \( y(x, \lambda) \) and \( y^{[1]}(x, \lambda) \) is continuous on \( [a, b] \times \mathbb{C} \), in particular, the functions \( y(x, \lambda) \) and \( y^{[1]}(x, \lambda) \) are entire functions of \( \lambda \in \mathbb{C} \).

Proof. The proof is similar to the proof of Sturm-Liouville problems with regular potentials, see [25]. The last conclusion can also be found in [3]. \( \square \)

Lemma 1.8. Consider the initial value problem consisting of the equation (1.2) and the initial conditions

\[
y(c) = h, \quad y^{[1]}(c) = k, \quad c \in [a, b].
\]

Then, given \( c_j \in [a, b] \), \( h_j, k_j \in \mathbb{C} \), \( 1/p_j, q_j, r_j, s_j \in L(J, \mathbb{R}) \), \( j = 1, 2 \), and given \( \epsilon > 0 \), there exists a number \( \delta > 0 \) such that if

\[
\int_a^b \left[ \left| 1/p_1 - 1/p_2 \right| + |q_1 - q_2| + |r_1 - r_2| + |s_1 - s_2| \right] \, dx + |c_1 - c_2| + |h_1 - h_2| + |k_1 - k_2| < \delta,
\]

then

\[
|y(t, c_2, h_2, k_2, 1/p_2, q_2, r_2, s_2) - y(t, c_1, h_1, k_1, 1/p_1, q_1, r_1, s_1)| < \epsilon
\]

and

\[
|y^{[1]}(t, c_2, h_2, k_2, 1/p_2, q_2, r_2, s_2) - y^{[1]}(t, c_1, h_1, k_1, 1/p_1, q_1, r_1, s_1)| < \epsilon
\]

uniformly for all \( t \in [a, b] \).

Proof. This is a consequence of [25] Theorem 1.6.2. \( \square \)

Note that in this paper, we will denote the norm in \( L(J, \mathbb{R}) \) by \( \| \cdot \|_1 \).

Lemma 1.9. For \( m \in \mathbb{N} \), let \( L_m \) denote the operators generated by the expression (1.1) and the self-adjoint boundary conditions (1.4), with the coefficients \( p, q, s \) replaced by \( p_m, q_m, s_m \), respectively. The coefficients \( p_m, q_m, s_m \) are real-valued and

\[
1/p_m, \quad q_m, \quad s_m \in L(J, \mathbb{R}), \quad p_m > 0 \text{ a.e. on } J.
\]

For the operator \( L \) generated by the expression (1.1) and the self-adjoint boundary conditions (1.4), if

\[
\|1/p_m - 1/p\|_1 \to 0, \quad \|q_m - q\|_1 \to 0, \quad \|s_m - s\|_1 \to 0, \quad \text{as } m \to \infty,
\]

then \( L_m \xrightarrow{R} L \).
Proof. Let \( \lambda \in \mathbb{C} \setminus \mathbb{R} \), then \( \lambda \in \rho(L) \cap \rho(L_m) \). Denote by \( \phi_{1,m} \) and \( \phi_{2,m} \) the pair of solutions of the equation

\[-(p_m [y' + s_m y])' + s_m (p_m [y' + s_m y]) + q_m y = \lambda r y, \ x \in (a, b),\]
determined by the initial conditions

\[\phi_{1,m}(a) = \phi_{1,m}^{[1]}(a) = 1, \ \phi_{2,m}(a) = \phi_{1,m}^{[1]}(a) = 0.\]

According to Theorem 1.6.1 in [25], (let \( y^{[0]} = y \)), we have for \( j = 0, 1 \),

\[1.13 \quad \| \phi^{[j]}_{1,m}(t) - \phi^{[j]}_{1}(t) \| + \| \phi^{[j]}_{2,m}(t) - \phi^{[j]}_{2}(t) \| \leq C (\| 1/p_m - 1/p \|_1 + \| q_m - q \|_1 + \| s_m - s \|_1 ),\]

where \( \phi_1 \) and \( \phi_2 \) are defined at the beginning of this section and \( C \) depends only on the chosen number \( \lambda \) and the fixed functions \( 1/p, q, s, r \).

Since the Wronskian of the pair \( \phi_{1,m} \) and \( \phi_{2,m} \) equals 1 identically, a straightforward calculation shows that the function

\[z_m(x) = \int_a^x (\phi_{1,m}(x)\phi_{2,m}(\xi) - \phi_{2,m}(x)\phi_{1,m}(\xi)) r(\xi)f(\xi)d\xi\]
satisfies the resolvent equation

\[1.14 \quad \frac{1}{r}(- (p_m [y' + s_m y])' + s_m (p_m [y' + s_m y]) + q_m y) - \lambda y = f \in L^2(J, \mathbb{R}), \ x \in (a, b).\]

Also, the function

\[z(x) = \int_a^x (\phi_1(x)\phi_2(\xi) - \phi_2(x)\phi_1(\xi)) r(\xi)f(\xi)d\xi\]
satisfies the resolvent equation

\[1.15 \quad \frac{1}{r}(-(p [y' + s y])' + s (p [y' + s y]) + q y) - \lambda y = f \in L^2(J, \mathbb{R}), \ x \in (a, b).\]

As before, the solution is understood in the sense of Lemma 1.7. By the estimate 1.13, we have that

\[|z_m(x) - z(x)| \leq C (\| 1/p_m - 1/p \|_1 + \| q_m - q \|_1 + \| s_m - s \|_1 )\int_a^b |r(t)f(t)| dt \leq C_1 (\| 1/p_m - 1/p \|_1 + \| q_m - q \|_1 + \| s_m - s \|_1 )\int_a^b r(t) |f(t)|^2 dt,\]

where \( C_1 \) depends only on the chosen number \( \lambda \) and the fixed functions \( 1/p, q, s, r \).

A general solution of equation 1.14 and 1.15 has the representation

\[y_m(x) = z_m(x) + a_1(m)\phi_{1,m}(x) + a_2(m)\phi_{2,m}(x)\]

and

\[y(x) = z(x) + a_1\phi_1(x) + a_2\phi_2(x),\]

respectively.
Substituting $y_m(x)$ and $y(x)$ into the boundary conditions, denote
\[
A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},
\]
\[
U_j(y) = \begin{pmatrix} a_{1j} y(a) + a_{j2} y^{[1]}(a) + b_{j1} y(b) + b_{j2} y^{[1]}(b) \end{pmatrix}_j, \quad j = 1, 2,
\]
we get
\[
a_1(m) = \triangle^{-1} \begin{pmatrix} U_1(\phi_{2,m}) & U_1(z_m) \\ U_2(\phi_{2,m}) & U_2(z_m) \end{pmatrix}, \quad a_2(m) = \triangle^{-1} \begin{pmatrix} U_1(z_m) & U_1(\phi_{1,m}) \\ U_2(z_m) & U_2(\phi_{1,m}) \end{pmatrix},
\]
\[
a_1 = \triangle^{-1} \begin{pmatrix} U_1(\phi_2) & U_1(z) \\ U_2(\phi_2) & U_2(z) \end{pmatrix}, \quad a_2 = \triangle^{-1} \begin{pmatrix} U_1(z) & U_1(\phi_1) \\ U_2(z) & U_2(\phi_1) \end{pmatrix},
\]
where $\Delta_m$ and $\triangle$ are defined as follows:
\[
\Delta_m = \begin{pmatrix} U_1(\phi_{1,m}) & U_1(\phi_{2,m}) \\ U_2(\phi_{1,m}) & U_2(\phi_{2,m}) \end{pmatrix}, \quad \triangle = \begin{pmatrix} U_1(\phi_1) & U_1(\phi_2) \\ U_2(\phi_1) & U_2(\phi_2) \end{pmatrix}.
\]
Note that $\Delta_m(\lambda) \neq 0$ and $\triangle(\lambda) \neq 0$, otherwise, the chosen complex number $\lambda$ is an eigenvalue of the operators $L_m$ and $L$.

From the estimate (1.13), we have that
\[
|U_j(\phi_{1,m}) - U_j(\phi_1)| + |U_j(\phi_{2,m}) - U_j(\phi_2)| \\
\leq C \left( \|1/p_m - 1/p\|_1 + \|q_m - q\|_1 + \|s_m - s\|_1 \right), \quad j = 1, 2,
\]
and therefore
\[
|\Delta_m - \triangle| \leq C \left( \|1/p_m - 1/p\|_1 + \|q_m - q\|_1 + \|s_m - s\|_1 \right),
\]
where $C$ depends only on the chosen number $\lambda$ and the fixed functions $1/p$, $q$, $s$, $r$.

Consequently,
\[
|a_1(m) - a_1| + |a_2(m) - a_2| \\
\leq C \left( \|1/p_m - 1/p\|_1 + \|q_m - q\|_1 + \|s_m - s\|_1 \right) \int_a^b r(t) |f(t)|^2 \, dt,
\]
where $C$ depends only on the chosen number $\lambda$ and the fixed functions $1/p$, $q$, $s$, $r$.

The estimates obtained show that the solutions
\[
y_m = (L_m - \lambda)^{-1} f
\]
are subject to the inequality
\[
\| (L_m - \lambda)^{-1} f - (L - \lambda)^{-1} f \|_{L^2(J,R)} \\
= \| y_m - y \|_{L^2(J,R)} = \left( \int_a^b r(t) |y_m - y(t)|^2 \, dt \right)^{\frac{1}{2}} \\
\leq C \max_{t \in [a,b]} |y_m(t) - y(t)| \\
\leq C \left( \|1/p_m - 1/p\|_1 + \|q_m - q\|_1 + \|s_m - s\|_1 \right) \int_a^b r(t) |f(t)|^2 \, dt,
\]
where $C$ depends only on the chosen number $\lambda$ and the fixed functions $1/p$, $q$, $s$, $r$.

This estimate implies the norm resolvent convergence of the operators $L_m$. □
Lemma 1.10. A number $\lambda$ is an eigenvalue of Sturm-Liouville problem consisting of (1.2) and (0.4) if and only if
\[ \Delta(\lambda) = \det(A + B\Phi(b, \lambda)) = 0. \]

A number $\lambda$ is an eigenvalue of Sturm-Liouville problem consisting of (1.2) and (0.8) if and only if $D(\lambda) = 2\cos\gamma$. A number $\lambda$ is an eigenvalue of Sturm-Liouville problem consisting of (1.2) and (1.7) if and only if $D(\lambda) = 2$.

Proof. The proof is similar to the proof of the classical Sturm-Liouville problem, see Lemma 3.2.2, Lemma 3.2.6 in [25].

Lemma 1.11. For a fixed $\lambda \in \mathbb{R}$, the function $y(x, \lambda)$ is a real-valued solution to the initial problem consisting of equation (1.2) with the initial value $y(x_0, \lambda) = 0$, $y^{[1]}(x_0, \lambda) = c$, $c > 0$. Then there exists $\varepsilon > 0$ such that $y(x, \lambda) < 0$ for all $x \in (x_0 - \varepsilon, x_0)$ and $y(x, \lambda) > 0$ for all $x \in (x_0, x_0 + \varepsilon)$.

Proof. See [3, Lemma 11.2]. Since $y^{[1]}(x) = p(x)[y'(x) + s(x)y(x)]$, it follows that $y'(x) = p(x)^{-1}y^{[1]}(x) - s(x)y(x)$, from the knowledge of the differential equation, we have
\[ y(x) = e^{-S(x)} \int_{x_0}^{x} e^{S(t)} p(t)^{-1} y^{[1]}(t) dt, \quad S(x) = \int_{x_0}^{x} s(t) dt, \]
thus the claim is obvious.

Lemma 1.12. Let $y_1(x, \lambda_1)$ and $y_2(x, \lambda_2)$ be two real-valued solutions of the equation (1.2), and $\lambda_2 \geq \lambda_1$, if $x_1$ and $x_2$ are two adjacent zeros of $y_1(x)$, then $y_2(x)$ has at least one zero on $[x_1, x_2]$.

Proof. The Lagrange’s formula in [3] gives
\[ (\lambda_2 - \lambda_1) \int_{x_1}^{x_2} y_1 y_2 r dt = \int_{x_1}^{x_2} (y_1 Ly_2 - y_2 Ly_1) r dt \]
(1.16)\[ = y_1^{[1]}(x_2)y_2(x_2) - y_1^{[1]}(x_1)y_2(x_1). \]
Suppose $y_2(x)$ has no zeros on $[x_1, x_2]$, then without loss of generality, we assume that $y_2(x) > 0$ on $[x_1, x_2]$ and $y_1(x) > 0$ on $(x_1, x_2)$, thus Lemma 1.11 implies that $y_1^{[1]}(x_2) < 0$ and $y_1^{[1]}(x_1) > 0$. The last term of (1.16) is negative, but the first term of the equality is not negative, so the contradiction proves the lemma. In fact, if $\lambda_2 > \lambda_1$, through the similar process, we can obtain $y_2(x)$ has at least one zero on $(x_1, x_2)$.

2. Preliminary lemmas for the main results

In this section, we will give several lemmas that will be used in the proofs of our main results.

To study the Sturm-Liouville problem consisting of (1.2) and (0.4), we introduce a “boundary value problem space” with a metric. Let $\Omega = \{ \omega = (A, B, 1/p, q, r, s); (1.2) \text{ and } (0.7) \text{ hold} \}$. For the topology of $\Omega$ we use a metric $d$ defined as follows:

For $\omega = (A, B, 1/p, q, r, s) \in \Omega$, $\omega_0 = (A_0, B_0, \frac{1}{p_0}, q_0, r_0, s_0) \in \Omega$, define
\[ d(\omega, \omega_0) = \| A - A_0 \| + \| B - B_0 \| + \int_{a}^{b} \left( \frac{1}{p} - \frac{1}{p_0} \right) + |q - q_0| + |r - r_0| + |s - s_0|, \]
where $\| \cdot \|$ denotes any matrix norm.
Note that an element $\omega = (A, B, 1/p, q, r, s) \in \Omega$ can be used to represent a Sturm-Liouville problem consisting of (1.2) and (0.4). By an eigenvalue of $\omega \in \Omega$ we mean an eigenvalue of the Sturm-Liouville problem consisting of (1.2) and (0.4).

**Lemma 2.1.** (Continuity of the zeros of an analytic function). Let $A$ be an open set in the complex plane $\mathbb{C}$, $F$ a metric space, $f$ a continuous complex valued function on $A \times F$ such that for each $\alpha \in F$, the map $z \to f(z, \alpha)$ is an analytic function on $A$. Let $B$ be an open subset of $A$ whose closure $B$ in $\mathbb{C}$ is compact and contained in $A$, and let $a_0 \in F$ be such that no zero of $f(z, a_0)$ is on the boundary of $B$. Then there exists a neighborhood $W$ of $a_0$ in $F$ such that:

(a) For any $\alpha \in W$, $f(z, \alpha)$ has no zero on the boundary of $B$.

(b) For any $\alpha \in W$, the sum of the orders of the zeros of $f(z, \alpha)$ contained in $B$ is independent of $\alpha$.

**Proof.** See [24, 9.17.4].

**Lemma 2.2.** Let $\omega_0 = (A_0, B_0, \frac{1}{p_0}, q_0, r_0, s_0) \in \Omega$. Assume that $\lambda(\omega_0)$ is an eigenvalue of the problem (1.2), (0.4). Then, given any $\epsilon > 0$, there exists a $\delta > 0$ such that if $\omega = (A, B, 1/p, q, r, s) \in \Omega$ satisfies

$$d(\omega, \omega_0) < \delta,$$

then the Sturm-Liouville problem $\omega$ has an eigenvalue $\lambda(\omega)$ satisfying

$$|\lambda(\omega) - \lambda(\omega_0)| < \epsilon.$$

**Proof.** On the basis of Lemma 1.7, Lemma 1.8 and Lemma 1.10 the proof is similar to the proof of the classical Sturm-Liouville problem with regular potentials, see [19, Theorem 3.1].

Note that for the Sturm-Liouville problem with distributional potentials consisting of (1.2) and (0.4), the algebraic multiplicity of an eigenvalue is the order of the eigenvalue as a zero of the characteristic function $\Delta(\lambda)$ discussed in Lemma 1.10.

**Lemma 2.3.** Let $\omega_0 = (A_0, B_0, \frac{1}{p_0}, q_0, r_0, s_0) \in \Omega$. Assume that $r_1$ and $r_2$, $r_1 < r_2$, are any two real numbers such neither of them is an eigenvalue of $\omega_0$ and $n \geq 0$ is the number of eigenvalues of $\omega_0$ in the interval $(r_1, r_2)$. Then there exists a neighborhood $O$ of $\omega_0$ in $\Omega$ such that any $\omega \in O$ also has $n$ eigenvalues in the interval $(r_1, r_2)$ (Here the eigenvalues are counted with algebraic multiplicity.)

**Proof.** On the basis of Lemma 1.7, Lemma 1.8 and Lemma 1.10 this is a direct consequence of Lemma 2.1.

**Remark 2.4.** Furthermore, we can obtain that each algebraically simple eigenvalue is on a locally unique continuous eigenvalue branch, while each algebraically double eigenvalue is on two locally unique continuous eigenvalue branches, the number of the eigenvalue branches is counted with algebraic multiplicity.

**Lemma 2.5.** (i) Assume the eigenvalue $\lambda(\omega_0)$ is geometrically simple for some $\omega_0 \in \Omega$ and let $w = w(\cdot, \omega_0)$ denote a normalized eigenfunction of the eigenvalue $\lambda(\omega_0)$. Then there exist normalized eigenfunctions $w = w(\cdot, \omega)$ of $\lambda(\omega)$ such that

$$w(\cdot, \omega) \to w(\cdot, \omega_0), \quad w^{[1]}(\cdot, \omega) \to w^{[1]}(\cdot, \omega_0), \quad \text{as } \omega \to \omega_0 \text{ in } \Omega,$$
both uniformly on the interval \([a, b]\).

(ii) Assume that \(\lambda(\omega)\) is a geometrically double eigenvalue for all \(\omega\) in some neighborhood \(M\) of \(\omega_0\) in \(\Omega\). Let \(w = w(\cdot, \omega_0)\) be any normalized eigenfunction of the eigenvalue \(\lambda(\omega_0)\). Then there exist normalized eigenfunctions \(w = w(\cdot, \omega)\) of \(\lambda(\omega)\) such that

\[
w(\cdot, \omega) \to w(\cdot, \omega_0), \quad w^{[1]}(\cdot, \omega) \to w^{[1]}(\cdot, \omega_0), \quad \text{as} \quad \omega \to \omega_0 \text{ in } \Omega,
\]

both uniformly on the interval \([a, b]\).

**Proof.** On the basis of Lemma 2.2 and Lemma 1.8 the proof is similar to [19, Theorem 3.2]. \(\square\)

**Lemma 2.6.** The eigenvalues \(\{\lambda_n, n \in \mathbb{N}_0\}\) of the separated Sturm-Liouville problems consisting of (1.2) and (0.6) are all geometrically simple, real, and form a sequence accumulating to \(+\infty\):

\[-\infty < \lambda_0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots,\]

the number of times an eigenvalue appears in the sequence is equal to its geometric multiplicity; the eigenfunction \(\Psi_n(x)\) corresponding to the eigenvalue \(\lambda_n\) has exactly \(n\) zeros on the interval \((a, b)\).

**Proof.** The assertion of the eigenvalues and the eigenfunctions can be proved by the Prüfer transformation introduced in [30]. Denote \(y_1 = y, y_2 = y^{[1]},\) assume

\[
\begin{align*}
  y_1(t, \lambda) &= \rho(t, \lambda) \sin \theta(t, \lambda), \\
  y_2(t, \lambda) &= \rho(t, \lambda) \cos \theta(t, \lambda),
\end{align*}
\]

we can obtain

\[(2.1) \quad \theta'(t, \lambda) = \frac{1}{\rho(t)} \cos^2 \theta(t, \lambda) - s(t) \sin2\theta(t, \lambda) + (\lambda r(t) - q(t)) \sin^2 \theta(t, \lambda).\]

For the equation (2.1) with the initial condition

\[(2.2) \quad \theta(a, \lambda) = \alpha, \quad 0 \leq \alpha < \pi, \quad \lambda \in \mathbb{R},\]

using the method introduced in [25, Theorem 4.5.3], we can obtain the properties of the unique real valued solution \(\theta(t, \lambda)\) defined on \([a, b]\):

1. For fixed \(t \in (a, b]\), \(\theta(t, \lambda)\) is continuous and strictly increasing in \(\lambda\).
2. If \(\theta(c, \lambda) = k\pi\), for some \(c \in (a, b), \lambda \in \mathbb{R}\), and some \(k \in \mathbb{N}\), then \(\theta(t, \lambda) > k\pi\) for \(c < t \leq b\).
3. \(\theta(b, \lambda) \to \infty\), as \(\lambda \to \infty\).
4. For all \(t \in (a, b]\), \(\theta(t, \lambda) \to 0\), as \(\lambda \to -\infty\).

To prove part (1), assume \(\lambda_1 > \lambda_2\), \(\theta_1 = \theta(t, \lambda_1), \theta_2 = \theta(t, \lambda_2), V = \theta_1 - \theta_2,\) then \(V' = fV + h,\) where

\[
f = -2s(t) \frac{\sin 2\theta_1 - \sin 2\theta_2}{2\theta_1 - 2\theta_2} + \left[\lambda_2 r(t) - q(t) - \frac{1}{\rho(t)}\right] (\sin \theta_1 + \sin \theta_2) \frac{\sin \theta_1 - \sin \theta_2}{\theta_1 - \theta_2},
\]

\[
h = (\lambda_1 r(t) - \lambda_2 r(t)) \sin^2 \theta_1.
\]

Then the claim follows from the proof that is similar to the proof in [25, Theorem 4.5.1 and 4.5.2]. The proof of (2) is similar to that of part (1). To prove part
(2), we consider the following equation, let \( \theta = \theta(t, \lambda) \), \( V(t) = \theta(t, \lambda) - k\pi \), then \( V' = fV + h \), where

\[
f(t) = -2s(t)\frac{\sin 2\theta - \sin 2k\pi}{2\theta - 2k\pi} + \left[ \lambda r(t) - q(t) - \frac{1}{p(t)} \right] (\sin \theta + \sin k\pi)\frac{\sin \theta - \sin k\pi}{\theta - k\pi},
\]

\[
h(t) = \frac{1}{p(t)}.
\]

To prove part (3) let \( \tan \phi = \lambda \frac{1}{p} \tan \theta \) for \( \lambda > 0 \) and determine \( \phi \) uniquely by requiring \( |\theta - \phi| < \frac{\pi}{2} \). Then

\[
\phi' = \lambda \frac{1}{p} \cos^2 \phi + \lambda \frac{1}{r} \sin^2 \phi - \lambda^{-\frac{1}{2}} q \sin^2 \phi - s \sin 2\phi,
\]

\[
\phi(b, \lambda) - \phi(a, \lambda) = \lambda \frac{1}{p} \int_a^b \left( \frac{1}{p} \cos^2 \phi + r \sin^2 \phi \right) - \lambda^{-\frac{1}{2}} \int_a^b q \sin^2 \phi dt - \int_a^b s \sin 2\phi dt,
\]

\[
\phi(b, \lambda) \geq \phi(a, \lambda) + \lambda \frac{1}{p} \min_{a \leq t \leq b} \left\{ \frac{1}{p} r \right\} - \lambda^{-\frac{1}{2}} \int_a^b |q| - \int_a^b |s|,
\]

hence \( \phi(b, \lambda) \to \infty \), as \( \lambda \to \infty \). Therefore \( \theta(b, \lambda) \to \infty \), as \( \lambda \to \infty \).

Finally, for the proof of part (4), let \( \theta_{-\infty}(t) = \lim_{\lambda \to -\infty} \theta(t, \lambda) \), for \( t \in (a, b) \). Since \( \theta(t, \lambda) \) is strictly increasing in \( \lambda \), and \( 0 \leq \theta(t, \lambda) \leq \theta(t, 0) \) for \( \lambda < 0 \), so the limit exists.

\[
\int_a^b \lambda r(t) \sin^2 \theta(t, \lambda) dt = \theta(b, \lambda) - \alpha + \int_a^b s(t) \sin 2\theta(t, \lambda) dt
\]

\[
- \int_a^b \frac{1}{p(t)} \cos^2 \theta(t, \lambda) dt + \int_a^b q \sin^2 \theta(t, \lambda) dt.
\]

Hence

\[
\int_a^b r(t) \sin^2 \theta(t, \lambda) dt \to 0, \text{ as } \lambda \to -\infty.
\]

Let \( \{\lambda_n, n \in \mathbb{N}\} \to -\infty \) and define \( f_n(s) = r(s) \sin^2 \theta(s, \lambda_n) \), then \( |f_n(s)| \leq r(s) \) and \( f_n(s) \to r(s) \sin^2 \theta_{-\infty}(s) \) as \( n \to \infty \). Hence \( \int_a^b r(s) \sin^2 \theta_{-\infty}(s) ds = 0 \), and thus \( r(s) \sin^2 \theta_{-\infty}(s) = 0 \), a.e. and \( \theta_{-\infty}(s) = 0(\text{mod } \pi) \). For \( \lambda < 0, s < t, s, t \in [a, b] \),

\[
\theta(t, \lambda) = \theta(s, \lambda) - \int_s^t \frac{1}{p(x)} \cos^2 \theta(x, \lambda) dx + \int_s^t (\lambda r(x) - q(x)) \sin^2 \theta(x, \lambda) dx
\]

\[
- \int_s^t s(x) \sin 2\theta(x, \lambda) dx + \int_s^t \frac{1}{p(x)} dx - \int_s^t q(x) \sin^2 \theta(x, \lambda) dx - \int_s^t s(x) \sin 2\theta(x, \lambda) dx.
\]

Let \( \lambda \to -\infty \), then we have \( \theta_{-\infty}(t) \leq \theta_{-\infty}(s) + \int_s^t \frac{1}{p(x)} dx \). Then we can get \( \theta(t, \lambda) \to 0, \text{ as } \lambda \to -\infty \) by a similar proof to [25] \( \text{Theorem 4.5.3} \).

As a similar proof to the classical Sturm-Liouville problems in [31], we can easily obtain that \( \lambda \) is an eigenvalue of the problem [12, 4.6] if and only if \( \theta(b, \lambda) = \beta + n\pi \). Thus we can obtain our claims. \( \square \)
The following result describes an asymptotic form of the fundamental solutions of the equation (1.2) for sufficiently negative $\lambda$.

**Lemma 2.7.** There exists $\lambda_0 \in \mathbb{R}$, $k > 0$ and a continuous function

$$\alpha : [a, b] \times (-\infty, \lambda_0] \to [0, \infty)$$

such that $\alpha(t, \lambda)$ is decreasing in $\lambda$ for each $t \in (a, b]$, $\alpha_t(t, \lambda)$ exists a.e. on $[a, b]$ for $\lambda \in (-\infty, \lambda_0]$, $p(t)(\tanh(\alpha(t, \lambda))\alpha_t(t, \lambda) + s(t))$ is continuous on $[a, b]$ for $\lambda \in (-\infty, \lambda_0]$, and

$$\lim_{\lambda \to -\infty} \alpha(t, \lambda) = \infty, \quad \lim_{\lambda \to -\infty} p(t)(\tanh(\alpha(t, \lambda))\alpha_t(t, \lambda) + s(t)) = \infty$$

for each $t \in (a, b]$. Moreover, for the fundamental solutions $\phi_1$ and $\phi_2$ of (1.2) we have

\begin{align*}
\phi_1(t, \lambda) &= k \cosh(\alpha(t, \lambda)), \\
\phi_1^{[1]}(t, \lambda) &= k \cosh(\alpha(t, \lambda))p(t)(\tanh(\alpha(t, \lambda))\alpha_t(t, \lambda) + s(t)), \\
\phi_2(t, \lambda) &= \frac{1}{k^2}\phi_1(t, \lambda) \int_a^t \frac{\text{sech}^2(\alpha(s, \lambda))}{p(s)} \, ds, \\
\phi_2^{[1]}(t, \lambda) &= \frac{1}{k^2}\phi_1^{[1]}(t, \lambda) \int_a^t \frac{\text{sech}^2(\alpha(s, \lambda))}{p(s)} \, ds + \frac{1}{k} \text{sech}(\alpha(t, \lambda))
\end{align*}

on $[a, b] \times (-\infty, \lambda_0]$.

**Proof.** Consider the Sturm-Liouville problem consisting of (1.2) and

$$y^{[1]}(a) = y^{[1]}(b) = 0.$$

Let $\lambda_0$ be the smallest eigenvalue of this problem. Then $\phi_1(\cdot, \lambda_0)$ is an eigenfunction for $\lambda_0$. Hence $\phi_1(\cdot, \lambda_0)$ has no zero on $[a, b]$. So there exists $k > 0$ such that $\phi_1(t, \lambda_0) > k$ for each $t \in [a, b]$. Denote $\theta(t, \lambda)$ the Prüfer angle for $\phi_1(t, \lambda)$,

\begin{align*}
\phi_1(t, \lambda) &= \rho(t, \lambda) \sin \theta(t, \lambda) \\
\phi_1^{[1]}(t, \lambda) &= \rho(t, \lambda) \cos \theta(t, \lambda).
\end{align*}

For each $t \in (a, b]$ and $\lambda \in (-\infty, \lambda_0]$, $\theta(a, \lambda) = \theta(a, \lambda_0) = \pi/2$, from Lemma 2.6 we have $\theta(t, \lambda)$ is strictly increasing in $\lambda$ and $\theta(t, \lambda) \in (0, \pi)$. Since $\cot \theta(t, \lambda) = \frac{\phi_1^{[1]}(t, \lambda)}{\phi_1(t, \lambda)}$, hence we have

\begin{align*}
\frac{\phi_1^{[1]}(t, \lambda)}{\phi_1(t, \lambda)} &= \frac{\phi_1^{[1]}(t, \lambda_0)}{\phi_1(t, \lambda_0)}, \quad \text{for } t \in (a, b] \text{ a.e., } \lambda \leq \lambda_0, \\
\frac{\phi_1'(t, \lambda)}{\phi_1(t, \lambda)} &= \frac{\phi_1'(t, \lambda_0)}{\phi_1(t, \lambda_0)}, \quad \text{for } t \in (a, b] \text{ a.e., } \lambda \leq \lambda_0, \\
\left( \ln \frac{\phi_1(t, \lambda)}{\phi_1(t, \lambda_0)} \right)' &= 0, \quad \text{i.e. } \ln \frac{\phi_1(t, \lambda)}{\phi_1(t, \lambda_0)} \geq \ln \frac{\phi_1(a, \lambda)}{\phi_1(a, \lambda_0)} = 0,
\end{align*}

the above inequality implies that $\phi_1(t, \lambda) \geq \phi_1(t, \lambda_0) > k$ for each $t \in (a, b]$ and $\lambda \leq \lambda_0$. In the same way we see that $\phi_1(t, \lambda)$ is strictly decreasing in $\lambda$ on $(-\infty, \lambda_0]$ for each fixed $t \in (a, b]$.

There is a unique $\alpha : [a, b] \times (-\infty, \lambda_0] \to [0, \infty)$ determined by (2.3) which is continuous. Moreover, $\alpha(t, \lambda)$ is decreasing in $\lambda$ on $(-\infty, \lambda_0]$ for each $t \in (a, b]$, $\alpha_t(t, \lambda)$ exists a.e. on $[a, b]$ by the reduction of order formula we see that $\phi_2$ satisfies (2.5) and $\phi_1^{[1]}$, $\phi_2^{[1]}$ satisfy (2.4), (2.6), respectively.
From Lemma 2.6 we have for \( t \in (a, b] \), \( \theta(t, \lambda) \to 0 \), as \( \lambda \to -\infty \). So,

\[
\lim_{\lambda \to -\infty} \frac{\phi_1^{[1]}(t, \lambda)}{\phi_1(t, \lambda)} = \infty, \quad \text{for} \ (a, b].
\]

(2.8)

Now we show that

\[
\lim_{\lambda \to -\infty} \alpha(t, \lambda) = \infty, \quad \text{for} \ (a, b].
\]

Assume the contrary, without loss of generality, let \( \lim_{\lambda \to -\infty} \alpha(b, \lambda) = r < \infty \). Then \( \alpha(b, \lambda) \leq r \) on \((-\infty, \lambda_0]\).

From (2.8), there is \( L \leq \lambda_0 \) such that \( \frac{\phi_1^{[1]}(b, L)}{\phi_1(b, L)} > 0 \), so \( \phi_1^{[1]}(b, L) > 0 \). By the continuity of \( \phi_1^{[1]}(\cdot, L) \) we have that \( \phi_1^{[1]}(t, L) > 0 \) on \([c, b]\) for some \( c \in (a, b) \). In view of (2.7) with \( \lambda_0 \) replaced by \( L \), we see that for \( \lambda \leq L \), \( \frac{\phi_1^{[1]}(t, \lambda)}{\phi_1(t, \lambda)} > 0 \) on \([c, b]\), so \( \phi_1^{[1]}(t, \lambda) > 0 \) on \([c, b]\).

From the knowledge of the differential equation, for a solution of

\[
y'(x) = p(x)^{-1} y^{[1]}(x) - s(x) y(x),
\]

we have

\[
y(x) = e^{-S(x)} \left( \int_{x_0}^x e^{S(t)} \frac{1}{p(t)} y^{[1]}(t) dt + y(x_0) \right), \quad S(x) = \int_{x_0}^x s(t) dt.
\]

Thus we have

\[
\phi_1(b, \lambda) \geq e^{-\int_c^b s(u) du} \phi_1(t, \lambda), \quad \text{for} \ t \in [c, b] \text{ and } \lambda \leq L.
\]

So

\[
k \leq \phi_1(t, \lambda) \leq e^{\int_c^b s(u) du} \phi_1(b, \lambda) \leq e^{\int_c^b s(u) du} \phi_1(b, \lambda) \leq e^{\int_c^b s(u) du} k \cosh(\alpha(b, \lambda)) \leq C,
\]

where \( C = e^{\int_c^b s(u) du} k \cosh r \) is a constant independent of \( \lambda \) and \( t \). However, for \( \lambda \leq L \), from the equation (2.2), we have

\[
\phi_1^{[1]}(t, \lambda) = e^{\int_c^t s(u) du} \left( \int_c^t e^{\int_c^u s(v) dv} (q(u) - \lambda r(u)) \phi_1(u, \lambda) du + \phi_1^{[1]}(c, \lambda) \right),
\]

\[
\begin{align*}
\phi_1(b, \lambda) &= e^{-\int_c^b s(t) dt} \int_c^b e^{\int_c^u s(v) dv} \frac{1}{p(t)} \phi_1^{[1]}(t, \lambda) dt + e^{-\int_c^b s(t) dt} \phi_1(c, \lambda) \\
&= e^{-\int_c^b s(t) dt} \int_c^b e^{\int_c^u s(v) dv} \frac{1}{p(t)} \phi_1^{[1]}(t, \lambda) dt + e^{-\int_c^b s(t) dt} \phi_1(c, \lambda) \\
&\geq e^{-\int_c^b s(t) dt} \int_c^b e^{\int_c^u s(v) dv} \frac{1}{p(t)} \phi_1^{[1]}(c, \lambda) dt + e^{-\int_c^b s(t) dt} \phi_1(c, \lambda) \\
&\geq e^{-\int_c^b s(t) dt} \int_c^b e^{\int_c^u s(v) dv} \frac{1}{p(t)} \int_c^u e^{-\int_c^v s(v) dv} q(u) \phi_1(u, \lambda) du dt \\
&\quad - \lambda e^{-\int_c^b s(t) dt} \int_c^b e^{\int_c^u s(v) dv} \frac{1}{p(t)} \int_c^u e^{-\int_c^v s(v) dv} r(u) \phi_1(u, \lambda) du dt \\
&\to \infty, \quad \text{as} \ \lambda \to -\infty.
\end{align*}
\]

Now we reach a contradiction to obtain our claim. \( \square \)
Lemma 2.8. For $\frac{1}{p}, s \in L(J, \mathbb{R})$, $p > 0$ a.e. on $(a, b)$, then there exist $p_m \in C^\infty[a, b]$, $s_m \in C^\infty_0[(a, b), m \in \mathbb{N}$, $p_m$ and $s_m$ are real-valued, $p_m > 0$ on $[a, b]$, such that $\|p_m - \frac{1}{p}\| \to 0$, $\|s_m - s\| \to 0$, as $m \to \infty$.

Proof. For $\frac{1}{p}, s \in L(J, \mathbb{R})$, define

$$\rho(x) = \begin{cases} Ce^{\frac{|x|}{1-\epsilon}}, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$

where $C = \left( \int_{-1}^{1} e^{\frac{|x|}{1-\epsilon}} dx \right)^{-1}$, and define $\rho_m(x) = m\rho(mx)$, $m \in \mathbb{N}$, let $\tilde{p}_m(x) = \int_a^b \rho_m(x-y) \frac{1}{p(y)} dy$, it is clear that $\tilde{p}_m(x) > 0$ on $[a, b]$. From [32] Theorem 2.29, we know that

$$\tilde{p}_m(x) \in C^\infty[a, b], \quad \|\tilde{p}_m - \frac{1}{p}\| \to 0, \quad \text{as} \quad m \to \infty.$$ 

So if we let $p_m = \frac{1}{\tilde{p}_m}$, we can obtain

$$p_m \in C^\infty[a, b], \quad \|p_m - \frac{1}{p}\| \to 0, \quad \text{as} \quad m \to \infty.$$ 

For $s \in L(J, \mathbb{R})$, since $C^\infty_0(a, b)$ is dense in $L(J, \mathbb{R})$, so we can find $s_m \in C^\infty_0(a, b)$, such that $\|s_m - s\| \to 0$, as $m \to \infty$.

Remark 2.9. For $p, s$ which satisfy the condition (0.2), let $L = L(p, s)$ denote the operator generated by (1.1) with the boundary conditions (1.4). For the case of $p = p_m, s = s_m$, denote the operators $L_m = L(p_m, s_m)$. For the operators $L_m$ which satisfy the conditions (0.3), since $s_m \in C^\infty_0(a, b)$, we can assume $s_m(a) = s_m(b) = 0$.

Hence the operators $L_m = L(p_m, s_m)$ are actually generated by the expression

$$L_m y = \frac{1}{r}(-p_m y')' - (p_m s_m)' y + p_m s_m^2 y + q y$$

$$= \frac{1}{r}(-p_m y')' + (-p_m s_m)' + p_m s_m^2 + q) y$$

with the boundary conditions

$$A \begin{pmatrix} y(a) \\ (p_m y')(a) \end{pmatrix} + B \begin{pmatrix} y(b) \\ (p_m y')(b) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where the complex matrices $A$ and $B$ satisfy that the $2 \times 4$ matrix $(A|B)$ has full rank, and

$$AEA^* = BEB^*, \quad E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$ 

It is obvious that each of the operators $L_m$ is a classical self-adjoint Sturm-Liouville operator with a regular potential and a positive leading coefficient.
Remark 2.10. Note that for the case of $p = p_m$, $s = s_m$, the replacement of the boundary conditions (1.7), (0.8), (1.7), (1.8) are
\begin{align}
(2.9) \quad Y(b) &= K Y(a), \\
(2.10) \quad Y(b) &= e^{\gamma} K Y(a), \\
(2.11) \quad y(a) &= 0, \quad k_{22} y(b) - k_{12} (p_m y')(b) = 0, \\
(2.12) \quad (p_m y')(a) &= 0, \quad k_{21} y(b) - k_{11} (p_m y')(b) = 0,
\end{align}
where $Y(\cdot) = \begin{pmatrix} y(\cdot) \\ (p_m y')(\cdot) \end{pmatrix}$.

Lemma 2.11. The operators $L_m$ converge to the operator $L$ in the sense of the norm resolvent convergence, i.e. $L_m \xrightarrow{\mathcal{R}} L$ as $m \to \infty$.

Proof. This is a direct consequence of Lemma [1.9] \hfill \square

Lemma 2.12. The spectrum of the operators $L$ generated by the expression (1.11) and the self-adjoint boundary conditions (0.3) is discrete.

Proof. From the above lemma, we can find a family of classical Sturm-Liouville operators $L_m$ with regular potentials such that $L_m \xrightarrow{\mathcal{R}} L$, for $\mu \in \rho(L) \cap \rho(L_m)$, the operator $(L - \mu)^{-1}$ is compact since it is the norm limit of the compact operators $(L_m - \mu)^{-1}$, from [20], the spectrum of the operator $L$ is discrete. \hfill \square

Lemma 2.13. For the Sturm-Liouville problem consisting of (1.2) and (0.3), the geometric and algebraic multiplicity of each eigenvalue are always equal.

Proof. From [18], we know that the geometric and algebraic multiplicity of each eigenvalue of self-adjoint Sturm-Liouville operators $L_m$ with regular potentials are always equal. Thus this theorem is a direct consequence of Remark 2.4 and Lemma 1.5 \hfill \square

Remark 2.14. The eigenvalues for the self-adjoint Sturm-Liouville operators with distributional potentials discussed in this paper can be ordered to form a sequence so that the number of an eigenvalue appears in the sequence is equal to its multiplicity without distinguishing the geometric and algebraic multiplicity.

For the proof of the following theorem, we introduce
\[ D_m(\lambda) := k_{11} \phi_{2,m}^{[1]}(b, \lambda) - k_{21} \phi_{2,m}(b, \lambda) + k_{22} \phi_{1,m}(b, \lambda) - k_{12} \phi_{1,m}^{[1]}(b, \lambda), \]
where $\phi_{1,m}$ and $\phi_{2,m}$ are the pair of solutions of the equation
\[-\big(p_m [y' + s_m y] + s_m (p_m [y' + s_m y]) + q_m y = \lambda r y, \quad x \in (a, b),\]
determined by the initial conditions
\[ \phi_{1,m}(a) = \phi_{2,m}^{[1]}(a) = 1, \quad \phi_{2,m}(a) = \phi_{1,m}^{[1]}(a) = 0. \]
Together with the properties of the coefficients $p_m$ and $s_m$, we have
\[ D_m(\lambda) = k_{11} (p_m \phi_{2,m}(b, \lambda) - k_{21} \phi_{2,m}(b, \lambda) + k_{22} \phi_{1,m}(b, \lambda) - k_{12} (p_m \phi_{1,m}^{[1]}(b, \lambda), \]
where $\phi_{1,m}$ and $\phi_{2,m}$ are the pair of solutions of the equation
\[-\big(p_m y' + (p_m s_m y')' + p_m s_m^2 + q \big) y = \lambda r y, \quad x \in (a, b),\]
determined by the initial conditions
\[ \phi_{1,m}(a) = (p_m \phi_{2,m}')(a) = 1, \quad \phi_{2,m}(a) = (p_m \phi_{1,m}')(a) = 0. \]
The properties of $D_m(\lambda)$ introduced above have been investigated in [17].

**Lemma 2.15.** The lowest eigenvalues of $L_m$ are uniformly semi-bounded from below.

**Proof.** We divide our proof in four steps.

(A) First, we consider the case for the separated boundary condition. Denote the first eigenvalue of $L = L(p, s)$ and $L_m = L(p_m, s_m)$ by $\lambda_0(p, s)$ and $\lambda_0(p_m, s_m)$, respectively. It suffices to show that

$$\lambda_0(p_m, s_m) \rightarrow \lambda_0(p, s), \text{ as } m \rightarrow \infty.$$  

Since $L_m \xrightarrow{R} L$, there exist eigenvalues $\Lambda(m)$ of $L_m$ such that $\Lambda(m) \rightarrow \lambda_0(p, s)$, as $m \rightarrow \infty$, and $\Lambda(m)$ is also simple when $m$ is sufficiently large. Denote the normalized eigenfunction of $\Lambda(m)$ and $\lambda_0(p, s)$ by $w_m = w(\cdot, p_m, s_m)$ and $w = w(\cdot, p, s)$, respectively.

According to Lemma 2.13, for an arbitrary $\epsilon > 0$, there exists a number $M > 0$ such that if $m > M$,

$$|w(t, p_m, s_m) - w(t, p, s)| < \epsilon$$

and

$$|w^{[1]}(t, p_m, s_m) - w^{[1]}(t, p, s)| < \epsilon$$

uniformly for all $t \in [a, b]$.

Note that $w(t, p, s)$ does not have a zero in $(a, b)$ from Lemma 2.6. So we may assume that $w(t, p, s) > 0$ on $(a, b)$.

(i) If $w(a, p, s) = 0$, from Lemma 1.11 it is a fact that $w^{[1]}(a, p, s) > 0$. Hence there exists $\epsilon_1 > 0$ such that $w^{[1]}(t, p, s) > 0$ on the interval $[a, a + \epsilon_1]$. From the knowledge of (2.14), when $m$ is sufficiently large, $w^{[1]}(t, p_m, s_m) > 0$ on $[a, a + \epsilon_1]$. Since the boundary condition is fixed, so $w(a, p_m, s_m) = 0$. Thus by Lemma 1.11 $w(t, p_m, s_m) > 0$ on the interval $(a, a + \epsilon_1)$ when $m$ is sufficiently large.

If $w(b, p, s) = 0$, through a similar process, we can also obtain that for sufficiently large $m$, there must exist $\epsilon_2 > 0$ such that $w(t, p_m, s_m) > 0$ on the interval $(b - \epsilon_2, b)$. Since $w(t, p, s) > 0$ on $[a + \epsilon_1, b - \epsilon_2]$, combining with (2.13), one deduces that $w(t, p_m, s_m) > 0$ on $[a + \epsilon_1, b - \epsilon_2]$. When $m$ is sufficiently large, $w(t, p_m, s_m) > 0$ on $(a, b)$ when $m$ is sufficiently large.

(ii) If $w(t, p, s) > 0$ on $(a, b)$, by (2.13), $w(t, p_m, s_m) > 0$ on $(a, b)$ when $m$ is sufficiently large. Thus for sufficiently large $m$, $\lambda_0(p_m, s_m) = \Lambda(m)$. Therefore, we complete the proof of this part..

In the following steps, we will denote the $n$-th eigenvalue for the operators $L_m$ with one of the boundary conditions (2.9), (2.10), (2.11), (2.12) by $\lambda_n(K, p_m, s_m)$, $\lambda_n(\gamma, K, p_m, s_m)$, $\mu_n(K, p_m, s_m)$, $\nu_n(K, p_m, s_m)$ respectively, $n \in \mathbb{N}$, and the similar denotations apply to the operator $L$.

(B) Next, assume that the self-adjoint boundary condition is the coupled one (2.9) or (2.10), with $k_{11} > 0$, $k_{12} \leq 0$. The inequalities

$$\nu_0(K, p_m, s_m) \leq \lambda_0(K, p_m, s_m) < \lambda_0(\gamma, K, p_m, s_m) \leq \lambda_0(-K, p_m, s_m) \leq \mu_0(K, p_m, s_m)$$

for Sturm-Liouville problems with regular potentials can be found in [17]. From the step (A), we know that $\{\nu_0(K, p_m, s_m)\}_{m=1}^{\infty}$ is bounded from below, thus the
lowest eigenvalues of the operators $L_m$ with the boundary conditions considered in this step are uniformly semi-bounded from below.

(C) Finally, we consider the case where the self-adjoint boundary condition is the coupled one \((2.9)\) or \((2.10)\), with $k_{11} \leq 0$, $k_{12} < 0$.

(1) From the paper \([18]\), it is an obvious fact that $D_m(\mu_0(K,p_m,s_m)) \leq -2$.

Then according to Lemma 1.8 and the fact that $\mu_0(K,p_m,s_m) \to \mu_0(K,p,s)$ as $m \to \infty$, it follows that

$$D(\mu_0(K,p,s)) \leq -2.$$

(2) From Lemma 2.7, we obtain that as $\lambda \to -\infty$, $\phi_1(b,\lambda)$ and $\phi_2(b,\lambda)$ approach infinity. By the Bounded Convergence Theorem and the decreasing property of $\alpha$ in $\lambda$,

$$\lim_{\lambda \to -\infty} \int_a^b \frac{\text{sech}^2(\alpha(t,\lambda))}{p(s)} ds = 0.$$  

Then it can be easily seen that among the functions $\phi_1(b,\lambda)$, $\phi_2(b,\lambda)$, $\phi_1(b,\lambda)$ and $\phi_2(b,\lambda)$ grows the fastest and $\phi_2(b,\lambda)$ the slowest, as $\lambda \to -\infty$. Thus from the following equality

$$D(\lambda) = k_{11}\phi_2(b,\lambda) - k_{21}\phi_2(b,\lambda) + k_{22}\phi_1(b,\lambda) - k_{12}\phi_1(b,\lambda),$$

It is easy to see that if $k_{11} \leq 0$, $k_{12} < 0$, $D(\lambda) \to \infty$ as $\lambda \to -\infty$.

From the above results (1), (2) and Lemma 1.10, there must exist an eigenvalue $\lambda_n(K,p,s)$ of the operator $L = L(p,s)$ with the boundary condition \((0.7)\) in which $k_{11} \leq 0$, $k_{12} < 0$ such that $\lambda_n(K,p,s) < \mu_0(K,p,s)$.

For Sturm-Liouville operators with regular potentials, as was proved in \([17]\), the eigenvalue $\lambda_0(K,p_m,s_m)$ is the only eigenvalue that satisfies the inequality

$$\lambda_0(K,p_m,s_m) < \lambda_0(\gamma,K,p_m,s_m) < \lambda_0(-K,p_m,s_m) \leq \mu_0(K,p_m,s_m), \quad m \in \mathbb{N}.$$  

Together with the fact $L_m \xrightarrow{K} L$ and $\mu_0(K,p_m,s_m) \to \mu_0(K,p,s)$, as $m \to \infty$, we have

$$\lambda_0(K,p_m,s_m) \to \lambda_n_0(K,p,s), \quad m \to \infty.$$  

Thus the lowest eigenvalues of the operators $L_m$ with the boundary condition considered in this step are uniformly semi-bounded from below.

(D) If neither Part (A) nor Part (B) applies to $K$, then either Part (A) or Part (B) applies to $-K$. \qed

**Lemma 2.16.** The eigenvalues of the self-adjoint differential operators associated with the differential expressions \((0.1)\) can be ordered to form a non-decreasing sequence

$$\lambda_0, \lambda_1, \lambda_2, \lambda_3, \ldots$$

approaching $+\infty$. (Note that the eigenvalues are ordered with multiplicities without distinguishing the algebraic and geometric multiplicities.)
Proof. For the separated boundary conditions, the claim has been proved in Lemma 2.6.

From Lemma 1.6 and Lemma 2.15 it is a direct result that the eigenvalues of the self-adjoint differential operator associated with the differential expression (0.1) are bounded from below. So it remains to show that the sequence of the eigenvalues approaches $+\infty$.

As is well known in [17], for the Sturm-Liouville problems with regular potentials, if $k_{11} > 0$ and $k_{12} \leq 0$, (or $k_{11} \leq 0$, $k_{12} < 0$), for $-\pi < \gamma < 0$ or $0 < \gamma < \pi$, we have

$$\mu_n(K, p_m, s_m) \leq \lambda_{n+1}(-K, p_m, s_m) < \lambda_{n+1}(\gamma, K, p_m, s_m) < \mu_{n+1}(K, p_m, s_m), \quad n \in \mathbb{N}_0.$$ 

For $n \in \mathbb{N}_0$, from Lemma 1.6, Lemma 2.6 and Lemma 2.15 one deduces that

$$\mu_n(K, p_m, s_m) \to \mu_n(K, p, s), \quad \text{as } m \to \infty.$$ 

Hence there exists a sub-sequence $\{\lambda_{n+1}(K, p_m, s_m)\}_{j=1}^{\infty}$ such that as $j \to \infty$,

$$\lambda_{n+1}(K, p_m, s_m) \to c$$

and

$$\mu_n(K, p, s) \leq c \leq \mu_{n+1}(K, p, s),$$

where $c$ is a constant. From the fact that $L_m \Rightarrow L$, $c$ must be an eigenvalue of the operator $L$ with the boundary condition (0.7).

So the eigenvalues of the operator $L$ with the boundary condition (0.7) form a non-decreasing sequence approaching $+\infty$. The claim for the operators $L$ with other boundary conditions follows from a similar proof.

Remark 2.17. Note that from [33], it is a fact that a self-adjoint operator is bounded from below if and only if its spectrum is bounded from below. From Lemma 2.16 we obtain the self-adjoint operators with distributional potentials discussed in this paper are bounded from below which has been proved in [3] by J. Eckhardt, F. Gesztesy, R. Nichols and G. Teschl using a different approach.

Based on the fact that the geometric and algebraic multiplicity of each eigenvalue of the Sturm-Liouville problem consisting of (1.2) and (0.3) are equal, we will give a lemma on the continuity of the eigenvalues which will be used to prove the continuous dependence of the $n$-th eigenvalue on the coefficients of the differential equation in Section 7.

Lemma 2.18. Let $O$ be a subset of $\Omega$. If $\lambda_0$ is uniformly bounded from below on $O$, $\omega_0 \in \Omega$, then the restrictions of the $n$-th eigenvalue to $O$ is continuous at $\omega_0$, i.e. $\lim_{\Omega \ni \omega \to \omega_0} \lambda_n(\omega) = \lambda_n(\omega_0), \quad n \in \mathbb{N}_0$.

Proof. The proof is similar to the classical Sturm-Liouville problems with regular potentials, see [17] Theorem 1.40].

Lemma 2.19. Denote the $n$-th eigenvalue of $L_m$ and $L$ by $\lambda_n(m)$ and $\lambda_n(0)$, respectively, $n \in \mathbb{N}_0$, then the sequence of the $n$-th eigenvalues $\lambda_n(m)$ of the operators $L_m$ converges to the $n$-th eigenvalue $\lambda_n(0)$ of the operator $L$, i.e. $\lambda_n(m) \to \lambda_n(0)$, as $m \to \infty$. (Note that the eigenvalues are ordered with multiplicities without distinguishing the algebraic and geometric multiplicities.)
Proof. This claim is a direct consequence of Lemma 1.6 and Lemma 2.15. \qed

Remark 2.20. Note that Lemma 1.6 describes the continuity of the $n$–th eigenvalue with respect to the coefficients $1/p$, $q$, $s$ in the Banach space $L(J, \mathbb{R})$ when the eigenvalues are ordered with geometric multiplicity. And Lemma 2.18 describes the continuity of the $n$–th eigenvalue with respect to the coefficients in the equation (1.2) when the eigenvalues are ordered with algebraic multiplicity. However, based on Lemma 2.15 and the fact that the geometric and algebraic multiplicity of each eigenvalue of the Sturm-Liouville problem consisting of (1.2) and (1.4) are always equal, either Lemma 1.6 or Lemma 2.15 can be used to obtain Lemma 2.19.

3. Continuity region of the $n$–th eigenvalue on separated boundary conditions

Since for the operators $L_m$ and $L$, the coefficients $q$ and $r$ in the expression (0.1) are fixed, for simplicity, we introduce a simpler space. Let $\Omega = \{(\omega = (1/p, q, r, s); (0.2)) \}$ holds and $q, r$ fixed. For the topology of $\Omega$ we use a metric $d$ defined as follows:

For $\omega = (1/p, q, r, s) \in \Omega$, $\omega_0 = (\frac{1}{p_0}, q, r, s_0) \in \Omega$, define

$$d(\omega, \omega_0) = \int_a^b \left( \left| \frac{1}{p} - \frac{1}{p_0} \right| + |s - s_0| \right).$$

Theorem 3.1. Let $(\alpha, \beta) \in [0, \pi] \times (0, \pi)$, and $\lambda_n = \lambda_n(S_{\alpha, \beta})$ be the $n$–th eigenvalue of the Sturm-Liouville problems consisting of (1.2) and (0.6). Let $w_n(\ast, \alpha, \beta)$ be the normalized real valued eigenfunction associated with $\lambda_n(S_{\alpha, \beta})$. Then as a function of $(\alpha, \beta)$, $\lambda_n$ is continuous on $[0, \pi] \times (0, \pi]$. Moreover,

(a) for a fixed $\beta \in (0, \pi]$, $\lambda_n$ is continuously differentiable in $\alpha$ on $[0, \pi)$,

$$\lambda_n'(\alpha) = -\frac{d}{d\alpha} \left( w_n^{(1)}(\alpha) \right)^2 - (w_n(\alpha))' \left( (w_n(\alpha))' \right)^2;$$

(b) for a fixed $\alpha \in [0, \pi)$, $\lambda_n$ is continuously differentiable in $\beta$ on $[0, \pi)$,

$$\lambda_n'(\beta) = (w_n^{(1)}(b, \beta))^2 + (w_n(b, \beta))'.

Corollary 3.2. Let $(\alpha, \beta) \in [0, \pi] \times (0, \pi)$, and $\lambda_n = \lambda_n(S_{\alpha, \beta})$ be the $n$–th eigenvalue of the Sturm-Liouville problems consisting of (1.2) and (0.6).

(a) For a fixed $\beta \in (0, \pi)$, $\lambda_n$ is strictly decreasing in $\alpha$ on $[0, \pi)$,

$$\lim_{\alpha \to \pi^-} \lambda_0(S_{\alpha, \beta}) = -\infty, \lim_{\alpha \to \pi^-} \lambda_n(S_{\alpha, \beta}) = \lambda_{n-1}(S_{0, \beta}), \text{ for } n \in \mathbb{N}.$$  

(b) For a fixed $\alpha \in [0, \pi)$, $\lambda_n$ is strictly increasing in $\beta$ on $(0, \pi]$,

$$\lim_{\beta \to 0+} \lambda_0(S_{\alpha, \beta}) = -\infty, \lim_{\beta \to 0+} \lambda_n(S_{\alpha, \beta}) = \lambda_{n-1}(S_{\alpha, \pi}), \text{ for } n \in \mathbb{N}.$$  

Proof of Theorem 3.1. As is well known in [17], for the classical Sturm-Liouville problems with regular potentials, the assertion of the theorem is true. Let $L = L(p, s)$ denote the operator generated by (1.4) with the boundary conditions (1.6). For the case of $p = p_m$, $s = s_m$, denote the operators $L_m = L(p_m, s_m)$. For the operators $L_m$, the corresponding separated boundary condition (0.9) becomes

$$S_{\alpha, \beta} : \left\{ \begin{array}{l}
\cos \alpha y(a) - \sin \alpha (p_m y')(a) = 0, \ \alpha \in [0, \pi), \\
\cos \beta y(b) - \sin \beta (p_m y')(b) = 0, \ \beta \in (0, \pi].
\end{array} \right.$$  

From Lemma 2.19, the $n$–th eigenvalues of the operators $\{L_m\}_{m \in \mathbb{N}}$ converge to the $n$–th eigenvalue of the limit operator $L$.  


(i) In this step, we denote \( \omega_m = (\frac{1}{p_m}, q, r, s_m) \in \hat{\Omega} \), \( \omega_0 = (\frac{1}{p}, q, r, s) \in \hat{\Omega} \), the \( n \)-th eigenvalue of \( L \) and \( L_m \) by \( \lambda_n(\omega_0, (\alpha, \beta)) \) and \( \lambda_n(\omega_m, (\alpha, \beta)) \), respectively, and also let \( S = [0, \pi] \times (0, \pi) \).

Suppose \( \Lambda \) is a simple continuous eigenvalue branch through the eigenvalue \( \lambda_n(\omega_0, (\alpha, \beta_0)) \) defined on a connected neighborhood \( O \) of \( (\omega_0, (\alpha, \beta_0)) \) in \( \hat{\Omega} \times S \). Since

\[
\lambda_n(\omega_m, (\alpha_0, \beta_0)) \to \lambda_n(\omega_0, (\alpha_0, \beta_0)) \text{ as } m \to \infty,
\]

we obtain

\[
\Lambda(\omega_m, (\alpha_0, \beta_0)) = \lambda_n(\omega_m, (\alpha_0, \beta_0))
\]

when \( m \) is sufficiently large.

It is clear that for sufficiently large \( m \), \( \Lambda(\omega_m, (\alpha, \beta)) \) is a continuous eigenvalue branch through \( \lambda_n(\omega_m, (\alpha_0, \beta_0)) \) defined on a neighborhood of \( (\alpha_0, \beta_0) \) in \( [0, \pi] \times (0, \pi) \). However, as was proved in [17], \( \lambda_n(\omega_m, (\alpha, \beta)) \) is the unique continuous eigenvalue branch through \( \lambda_n(\omega_m, (\alpha_0, \beta_0)) \) defined on \( [0, \pi] \times (0, \pi) \), thus

\[
\Lambda(\omega_m, (\alpha, \beta)) = \lambda_n(\omega_m, (\alpha, \beta))
\]

for sufficiently large \( m \) and \( (\omega_m, (\alpha, \beta)) \in O \).

By Lemma 2.2 for an arbitrary \( \epsilon > 0 \), there exists a \( \delta_1 > 0 \) such that if

\[
|(\alpha, \beta) - (\alpha_0, \beta_0)| < \delta_1/2, \int_a^b \left( \left| \frac{1}{p_m} - \frac{1}{p} \right| + |s_m - s| \right) < \delta_1/2,
\]

then

\[
|\lambda_n(\omega_m, (\alpha, \beta)) - \lambda_n(\omega_0, (\alpha_0, \beta_0))| = |\Lambda(\omega_m, (\alpha, \beta)) - \lambda_n(\omega_0, (\alpha_0, \beta_0))| < \epsilon/2.
\]

Moreover, for such a \( \epsilon > 0 \), and a fixed point \( (\alpha, \beta) \in [0, \pi] \times (0, \pi) \), there exists a \( \delta_2 > 0 \) such that if

\[
\int_a^b \left( \left| \frac{1}{p_m} - \frac{1}{p} \right| + |s_m - s| \right) < \delta_2/2,
\]

then

\[
|\lambda_n(\omega_m, (\alpha, \beta)) - \lambda_n(\omega_0, (\alpha, \beta))| < \epsilon/2.
\]

Thus for an arbitrary \( \epsilon > 0 \), there exists a \( \delta = \delta_1/2 \) such that if

\[
|(\alpha, \beta) - (\alpha_0, \beta_0)| < \delta,
\]

then

\[
|\lambda_n(\omega_0, (\alpha, \beta)) - \lambda_n(\omega_0, (\alpha_0, \beta_0))| \leq |\lambda_n(\omega_0, (\alpha, \beta)) - \lambda_n(\omega_m, (\alpha, \beta))| + |\lambda_n(\omega_m, (\alpha, \beta)) - \lambda_n(\omega_0, (\alpha_0, \beta_0))| < \epsilon.
\]

So it is a direct result that the eigenvalue \( \lambda_n(S_{\alpha, \beta}) \) of the problem (1.1), (1.0) is continuous on \( [0, \pi] \times (0, \pi) \).

(ii) In this step we will show that for a fixed \( \beta \), \( \lambda_n = \lambda_n(\alpha) \) is continuously differentiable in \( \alpha \) on \( [0, \pi] \). Also we assume \( \alpha \neq \frac{\pi}{2} \), the proof for the case \( \alpha = \frac{\pi}{2} \) can be completed similarly. For sufficiently small \( h \in \mathbb{R} \), denote the normalized real valued eigenfunctions of \( \lambda_n(\alpha) \) and \( \lambda_n(\alpha + h) \) by \( w_n = w_n(\cdot, \alpha) \) and \( v_n = v_n(\cdot, \alpha + h) \), respectively. According to the Lagrange’s formula in [33], it follows that

\[
(\lambda_n(\alpha + h) - \lambda_n(\alpha)) \int_a^b v_n w_n r dt = - [w_n, v_n]_a^b,
\]
where \([w_n, v_n] := w_n v_n^{[1]} - v_n w_n^{[1]}\). From the boundary condition (0.6), we obtain
\[
(\lambda_n(\alpha + h) - \lambda_n(\alpha)) \int_a^b v_n w_n r dt = w_n(a)v_n^{[1]}(a) - v_n(a)w_n^{[1]}(a)
\]
\[
= (\tan \alpha - \tan (\alpha + h))w_n^{[1]}(a)w_n^{[1]}(a)
\]
\[
= -(\tan (\alpha + h) - \tan \alpha)w_n^{[1]}(a)w_n^{[1]}(a, \alpha + h).
\]
Since \(\lambda_n(S_{\alpha, \beta})\) is continuous on \([0, \pi] \times (0, \pi]\), which has been proved in step (i), then by Lemma 2.5 we obtain
\[
|w_n(x, \alpha) - w_n(x, \alpha + h)| \to 0, \quad \left|w_n^{[1]}(x, \alpha) - w_n^{[1]}(x, \alpha + h)\right| \to 0, \quad \text{as } h \to 0,
\]
both uniformly for \(x \in [a, b]\).

Thus we get
\[
\lambda'_n(\alpha) = -\sec^2(\alpha)w_n^{[1]}(a, \alpha)^2 = -\left(\cos \alpha \frac{d}{d\alpha} \frac{\sin^2 \alpha}{\cos \alpha} \right)^2 - \tan^2(\alpha)w_n^{[1]}(a, \alpha)^2 < 0.
\]
Hence for a fixed \(\beta\), \(\lambda_n(\alpha)\) is differentiable in \(\alpha\) on \([0, \pi]\).

The statement on the differentiability of \(\lambda_n(\beta)\) can be proved by the method analogous to that used above. \(\Box\)

**Proof of Corollary 3.2.** The strict monotonicity of \(\lambda_n\) is a direct consequence of (3.11) and (3.12).

As in the proof of Theorem 3.1 we denote the \(n\)-th eigenvalue of \(L\) and \(L_m\) with the boundary conditions (0.6) and (3.5) by \(\lambda_n(\omega_0, (\alpha, \beta))\) and \(\lambda_n(\omega_m, (\alpha, \beta))\), respectively. For a fixed \(\beta \in (0, \pi]\), as is well known from [17],
\[
\inf_{\alpha \in [0, \pi]} \lambda_n(\omega_m, (\alpha, \beta)) = \lambda_{n-1}(\omega_m, (0, \beta)), \quad n \in \mathbb{N}.
\]
Since
\[
\lim_{m \to \infty} \lambda_n(\omega_m, (\alpha, \beta)) = \lambda_n(\omega_0, (\alpha, \beta)), \quad n \in \mathbb{N}_0,
\]
so we can easily obtain for \(\alpha \in [0, \pi]\),
\[
\lambda_n(\omega_0, (\alpha, \beta)) \geq \lambda_{n-1}(\omega_0, (0, \beta)), \quad n \in \mathbb{N}.
\]
As we have known that for a fixed \(\beta \in (0, \pi]\), \(\lambda_n(\omega_0, (\alpha, \beta))\) is strictly decreasing in \(\alpha\) on \([0, \pi]\), so \(\lim_{\alpha \to \pi^-} \lambda_n(\omega_0, (\alpha, \beta))\) exists, and is equal to an eigenvalue \(\lambda(\omega_0, (0, \beta))\), thus
\[
\lambda_{n-1}(\omega_0, (0, \beta)) \leq \lambda(\omega_0, (0, \beta)) < \lambda_n(\omega_0, (0, \beta)),
\]
so
\[
\lim_{\alpha \to \pi^-} \lambda_n(\omega_0, (\alpha, \beta)) = \lambda_{n-1}(\omega_0, (0, \beta)).
\]
It is obvious that
\[
\lim_{m \to \infty} \lambda_0(\omega_m, (\alpha, \beta)) = \lambda_0(\omega_0, (\alpha, \beta)),
\]
since \(\lambda_0(\omega_0, (\alpha, \beta))\) is strictly decreasing in \(\alpha\) for a fixed \(\beta \in (0, \pi]\), suppose that
\[
\lim_{\alpha \to \pi^-} \lambda_0(\omega_0, (\alpha, \beta)) = \inf_{\alpha \in [0, \pi]} \lambda_0(\omega_0, (\alpha, \beta)) = c > -\infty,
\]
thus \(c\) must be an eigenvalue \(\lambda(\omega_0, (0, \beta))\) and \(\lambda(\omega_0, (0, \beta)) < \lambda_0(\omega_0, (0, \beta))\). This contradiction leads to the conclusion (3.3). The proof for (3.3) is similar. \(\Box\)
4. Inequalities among eigenvalues

**Theorem 4.1.** Let $K \in \text{SL}(2, \mathbb{R})$. (a) If $k_{11} > 0$ and $k_{12} \leq 0$, $-\pi < \gamma < 0$ or $0 < \gamma < \pi$, then $\lambda_0(K)$ is simple, and

$$
\nu_0 \leq \lambda_0(0, K) < \lambda_0(\gamma, K) < \lambda_0(-K) \leq \{\mu_0, \nu_1\} \\
\leq \lambda_1(-K) < \lambda_1(\gamma, K) < \lambda_1(K) \leq \{\mu_1, \nu_2\} \\
\leq \lambda_2(K) < \lambda_2(\gamma, K) < \lambda_2(-K) \leq \{\mu_2, \nu_3\} \\
\leq \lambda_3(-K) < \lambda_3(\gamma, K) < \lambda_3(K) \leq \cdots.
$$

(b) If $k_{11} \leq 0$ and $k_{12} < 0$, for $-\pi < \gamma < 0$ or $0 < \gamma < \pi$, we have

$$
\lambda_0(K) < \lambda_0(0, K) < \lambda_0(-K) \leq \{\mu_0, \nu_0\} \\
\leq \lambda_1(-K) < \lambda_1(\gamma, K) < \lambda_1(K) \leq \{\mu_1, \nu_1\} \\
\leq \lambda_2(K) < \lambda_2(\gamma, K) < \lambda_2(-K) \leq \{\mu_2, \nu_2\} \\
\leq \lambda_3(-K) < \lambda_3(\gamma, K) < \lambda_3(K) \leq \cdots.
$$

(c) For $0 < \gamma_1 < \gamma_2 < \pi$, we have

$$
\lambda_0(\gamma_1, K) < \lambda_0(\gamma_2, K) < \lambda_1(\gamma_2, K) < \lambda_1(\gamma_1, K) < \lambda_2(\gamma_1, K) < \lambda_2(\gamma_2, K) < \cdots.
$$

(d) If $K$ is not included in the case (a) and (b), then $-K$ is included in either case (a) or case (b).

**Theorem 4.2.** Recall that $\lambda_n^D$ is the $n$-th Dirichlet eigenvalue. For the $n$-th eigenvalue $\lambda_n(A, B)$ of the Sturm-Liouville problems consisting of (1.2) and (1.4), we obtain the following conclusions: (a) $\lambda_0(A, B) \leq \lambda_0^D$, $\lambda_1(A, B) \leq \lambda_1^D$, (b) $\lambda_{n-2}^D < \lambda_n(A, B) \leq \lambda_n^D$, for $n \geq 2$.

**Proof of Theorem 4.1.** (i) Let $L = L(p, s)$ denote the operator generated by (1.1) with one of the boundary conditions (0.7), (0.8), (1.7), (1.8). For the case of $p = p_m$, $s = s_m$, denote the operators $L_m = L(p_m, s_m)$. For the case of $L_m$, since $s_m(a) = s_m(b) = 0$, so the replacements of the boundary conditions (0.7), (0.8), (1.7), (1.8) are (2.9), (2.10), (2.11), (2.12).

For the operators $L_m$, the inequalities among the eigenvalues $\lambda_m(K)$, $\lambda_n(\gamma, K)$, $\mu_n$, $\nu_n$ can be found in [18]. From Lemma 2.19 the $n$-th eigenvalues of the operators $\{L_m\}_{m \in \mathbb{N}}$ converge to the $n$-th eigenvalue of the limit operator $L$. Lemma 1.10 implies that $D(\lambda_n(\pm K)) = \pm 2$, $|D(\lambda_n(\gamma, K))| < 2$, thus the inequalities in (a) and (b) can be obtained.

(ii) Now we will show the monotonicity of $\lambda_n(\gamma, K)$, for $0 < \gamma_1 < \gamma_2 < \pi$. It suffices to show that $D(\lambda)$ is not zero at values of $\lambda$ such that $|D(\lambda)| < 2$.

Let $\Phi_\lambda(x, \lambda) = (d/d\lambda)\Phi(x, \lambda)$, it follows from (1.2), (1.3) that

$$
\Phi_\lambda'(x, \lambda) = (P - \lambda W)\Phi_\lambda - W\Phi, \Phi_\lambda(a, \lambda) = 0.
$$

By the variation of parameters formula, we have

$$
\Phi_\lambda(x, \lambda) = -\int_a^x \Phi(x, \lambda) \Phi^{-1} (s, \lambda) W(s) \Phi(s, \lambda) ds.
$$
So by (1.6), we can obtain that
\[
D'(\lambda) = \text{trace} K^{-1} \Phi(\lambda) (b, \lambda)
\]
\[
= -\text{trace} \int_a^b K^{-1} \Phi(b, \lambda) \Phi^{-1}(s, \lambda) W(s) \Phi(s, \lambda) ds
\]
\[
= -\text{trace} \int_a^b \left( -D_2(\lambda) \phi_1 \phi_2 + C(\lambda) \phi_1^2 * -A(\lambda) \phi_2^2 + D_1(\lambda) \phi_1 \phi_2 \right) ds
\]
(4.3) \[
= \int_a^b [A(\lambda) \phi_2^2(s, \lambda) - B(\lambda) \phi_1(s, \lambda) \phi_2(s, \lambda) - C(\lambda) \phi_1^2(s, \lambda)] ds.
\]
Since
\[
D_1(\lambda) D_2(\lambda) - A(\lambda) C(\lambda) = \text{det} K^{-1} \Phi(b, \lambda) = 1,
\]
we have
\[
4 - D^2(\lambda) = 4 - (D_1(\lambda) + D_2(\lambda))^2
\]
\[
= 4(1 - D_1(\lambda) D_2(\lambda)) - B^2(\lambda) = -(4A(\lambda) C(\lambda) + B^2(\lambda)).
\]
Hence from (4.3) it follows that
\[
4C(\lambda) D'(\lambda) = \int_a^b [4A(\lambda) C(\lambda) \phi_2^2 - 4B(\lambda) C(\lambda) \phi_1 \phi_2 - 4C^2(\lambda) \phi_1^2] ds
\]
\[
= \int_a^b [-(2C(\lambda) \phi_1 + B(\lambda) \phi_2)^2 + (4A(\lambda) C(\lambda) + B^2(\lambda)) \phi_2^2] ds
\]
(4.4) \[
= -\int_a^b (2C(\lambda) \phi_1 + B(\lambda) \phi_2)^2 ds - (4 - D^2(\lambda)) \int_a^b \phi_2^2 ds.
\]
Thus if \(|D(\lambda)| < 2\), \(D'(\lambda) \neq 0\).

**Proof of Theorem 4.2.** First we prove the inequality for separated boundary conditions, and denote the \(n\)-eigenvalue for some \(S_{\alpha,\beta}\) by \(\lambda_n\), suppose \(\lambda_n^D < \lambda_n\) for some \(n \in \mathbb{N}_0\). Denote the eigenfunctions of \(\lambda_n^D\) and \(\lambda_n\) by \(\Psi(., \lambda_n^D)\) and \(\Psi(., \lambda_n)\), respectively. From Lemma 2.6 and Lemma 1.12, we obtain that \(\Psi(., \lambda_n)\) has at least \(n + 1\) zeros on the interval \((a, b)\), this contradicts the conclusion of Lemma 2.6 Then the general inequality
\[
\lambda_n(A, B) \leq \lambda_n^D
\]
follows from (4.1), (4.2). Of course, the general inequality \(\lambda_n(A, B) \leq \lambda_n^D\) can also be obtained by using the fact that the \(n\)-th eigenvalues of the operators \(\{L_m\}_{m \in \mathbb{N}}\) converge to the \(n\)-th eigenvalue of the limit operator \(L\). For the operators \(L_m\), this inequality has been proved in [17].

Now we denote the set of all the separated boundary conditions \(S_{\alpha,\beta}\) by \(\Gamma\).

By Corollary 3.2 for an arbitrary \(\beta \in (0, \pi]\), we obtain that
\[
\lambda_n(S_{0, \beta}) > \lim_{\gamma \rightarrow 0^+} \lambda_n(S_{0, \gamma}) = \lambda_n^{D^2} = \lambda_n^{D^2} - 1, n \in \mathbb{N},
\]
(4.6)
\[
\inf_{C \in \Gamma} \lambda_n(C) = \inf_{\alpha, \beta \leq \pi} \inf_{\alpha, \beta \leq \pi} \lambda_n(S_{\alpha, \beta}) = \inf_{\alpha, \beta \leq \pi} \lambda_n^{D^2} - 1, n \geq 2.
\]
(4.7)
It can be obtained that the infimum in (4.7) can not be achieved by using Corollary 3.2.

By Theorem 4.1 for any \(K \in \text{SL}(2, \mathbb{R})\), there exists \(0 < \beta \leq \pi\), such that
\[
\lambda_n(K) \leq \lambda_n(S_{0, \beta}) \leq \lambda_{n+1}(K), n \in \mathbb{N}_0,
\]
(4.8)
and for $0 < \gamma < \pi$ or $-\pi < \gamma < 0$,

\[(4.9) \quad \lambda_n(K) < \lambda_n(\gamma, K) < \lambda_n(-K) \text{ or } \lambda_n(-K) < \lambda_n(\gamma, K) < \lambda_n(K), n \in \mathbb{N}_0,
\]

thus by (4.5)–(4.9), the proof is completed. \qed

5. Discontinuity of $\lambda_n$ on the space of self-adjoint boundary conditions

In this section, we describe the continuity region of the $n$-th eigenvalue as a function on the space of self-adjoint boundary conditions. As a similar work on the classical Sturm-Liouville problems with regular potentials, following [24], we give some notations and results on the space of self-adjoint boundary conditions.

$M_{2 \times 4}(\mathbb{C})$ stands for the set of 2 by 4 matrices over $\mathbb{C}$ with rank 2 and let $\text{GL}(2, \mathbb{C})$ be the Lie group of invertible complex matrices in dimension 2. As mentioned in the introduction, a complex boundary condition (not necessarily self-adjoint) is just a system of two linearly independent homogeneous equations on $y(a)$, $y[1](a)$, $y(b)$ and $y[1](b)$ with complex coefficients, i.e.

\[(5.1) \quad A \begin{pmatrix} y(a) \\ y[1](a) \end{pmatrix} + B \begin{pmatrix} y(b) \\ y[1](b) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]

with the $2 \times 4$ matrix $(A|B) \in M_{2 \times 4}(\mathbb{C})$.

Following [24], we will take the quotient space

\[
\text{GL}(2, \mathbb{C}) \backslash M_{2 \times 4}(\mathbb{C})
\]

as the space $\mathcal{B}^C$ of complex boundary conditions, i.e., each boundary condition is an equivalence class of coefficient matrices (with the elements of $\text{GL}(2, \mathbb{C})$ multiplying from the left) of linear systems (5.1), and the boundary condition represented by the linear system (5.1) will be denoted by $[A|B]$. Note here, that square brackets, not parentheses, are used. Similarly, the space $\mathcal{B}^R$ of real boundary conditions is just $\text{GL}(2, \mathbb{R}) \backslash M_{2 \times 4}(\mathbb{R})$. Note that the space $\mathcal{B}^R_S$ of self-adjoint real boundary conditions consists of the separated real boundary conditions and the coupled real boundary conditions of the form $[K|-I]$ where $K \in \text{SL}(2, \mathbb{R})$. Denote all the self-adjoint complex boundary conditions by $\mathcal{B}^C_S$. In this section, we characterize the discontinuity set of $\lambda_n$ as a function on $\mathcal{B}^R_S$ or $\mathcal{B}^C_S$ and determine the behavior of $\lambda_n$ near each discontinuity point.

As a similar work on the classical Sturm-Liouville problems with regular potentials, the space $\mathcal{B}^R_S$ can be obtained by “gluing” the open sets

\[
\begin{align*}
\mathcal{O}_{1,S}^R &= \mathcal{O}_{6,s}^R = \{[K|-I] ; K \in \text{SL}(2, \mathbb{R})\}, \\
\mathcal{O}_{2,S}^R &= \left\{ \begin{bmatrix} 1 & a_{12} & 0 & a_{22} \\ 0 & a_{22} & -1 & b_{22} \end{bmatrix} ; a_{12}, a_{22}, b_{22} \in \mathbb{R} \right\}, \\
\mathcal{O}_{3,S}^R &= \left\{ \begin{bmatrix} 1 & a_{12} & -a_{22} & 0 \\ 0 & a_{22} & b_{21} & -1 \end{bmatrix} ; a_{12}, a_{22}, b_{21} \in \mathbb{R} \right\}, \\
\mathcal{O}_{4,S}^R &= \left\{ \begin{bmatrix} a_{11} & 1 & 0 & -a_{21} \\ a_{21} & 0 & -1 & b_{22} \end{bmatrix} ; a_{11}, a_{21}, b_{22} \in \mathbb{R} \right\}, \\
\mathcal{O}_{5,S}^R &= \left\{ \begin{bmatrix} a_{11} & 1 & a_{21} & 0 \\ a_{21} & 0 & b_{21} & -1 \end{bmatrix} ; a_{11}, a_{21}, b_{21} \in \mathbb{R} \right\}.
\end{align*}
\]
via the coordinate transformations among these open sets. Also, the space $\mathcal{O}_S^c$ can be obtained by “gluing” the open sets
\[
\mathcal{O}_{1,S}^c = \mathcal{O}_{6,S}^c = \left\{ [e^{i\theta} K | -I] : \theta \in [0, \pi), K \in \text{SL}(2, \mathbb{R}) \right\},
\]
\[
\mathcal{O}_{2,S}^c = \left\{ \begin{bmatrix} 1 & a_{12} & 0 \\ 0 & z & -1 \\ z & b_{22} \end{bmatrix} : a_{12}, b_{22} \in \mathbb{R}, z \in \mathbb{C} \right\},
\]
\[
\mathcal{O}_{3,S}^c = \left\{ \begin{bmatrix} 1 & a_{12} & 0 \\ 0 & z & b_{21} \\ z & b_{22} \end{bmatrix} : a_{12}, b_{21}, b_{22} \in \mathbb{R}, z \in \mathbb{C} \right\},
\]
\[
\mathcal{O}_{4,S}^c = \left\{ \begin{bmatrix} a_{11} & 1 & 0 \\ z & 0 & -1 \\ z & b_{22} \end{bmatrix} : a_{11}, b_{22} \in \mathbb{R}, z \in \mathbb{C} \right\},
\]
\[
\mathcal{O}_{5,S}^c = \left\{ \begin{bmatrix} a_{11} & 1 & \bar{z} \\ z & 0 & b_{21} \\ z & b_{22} \end{bmatrix} : a_{11}, b_{21} \in \mathbb{R}, z \in \mathbb{C} \right\}
\]
via the coordinate transformations among these open sets. Details of the above results have been described in [24].

The following are some continuity results about $\lambda_n$ on $\mathcal{O}_s$. In this context, we will use the notation
\[
\mathcal{F}_R^- = \{ [K | -I] : K \in \text{SL}(2, \mathbb{R}), k_{11}k_{12} \leq 0 \},
\]
\[
\mathcal{F}_R^+ = \{ \begin{bmatrix} a_1 & 1 & 0 & -r \\ r & 0 & -1 & b_2 \end{bmatrix} : b_2 \leq 0, a_1, r \in \mathbb{R} \},
\]
\[
\mathcal{F}_R^- = \{ \begin{bmatrix} 1 & a_2 & -r & 0 \\ 0 & r & b_1 & -1 \end{bmatrix} : a_2 \leq 0, b_1, r \in \mathbb{R} \},
\]
\[
\mathcal{F}_R^- = \{ \begin{bmatrix} 1 & a_2 & 0 & r \\ 0 & r & -1 & b_2 \end{bmatrix} : a_2, b_2 \leq 0, r \in \mathbb{R}, a_2b_2 \geq r^2 \},
\]
\[
\mathcal{F}_R^+ = \{ [K | -I] : K \in \text{SL}(2, \mathbb{R}), k_{12} = 0 \}
\]
\[
\cup \left\{ \begin{bmatrix} a_1 & a_2 & 0 & 0 \\ 0 & 0 & b_1 & b_2 \end{bmatrix} \in \mathcal{O}_s^c, a_2b_2 = 0 \right\}.
\]

Note that in this section we will still use the space $\Omega$ introduced in Section 3.

**Proposition 5.1.** Let $n \in \mathbb{N}_0$. Then as a function on the space $\mathcal{O}_s$, $\lambda_n$ is continuous at each point not in $\mathcal{F}_R^-$.

**Proof.** Denote $\omega_m = (\frac{1}{p_m}, q, r, s_m) \in \Omega$, $\omega_0 = (\frac{1}{p}, q, r, s) \in \Omega$. For every $A \in \mathcal{O}_s$, the eigenvalues $\lambda_n(\omega_0, A)$ and $\lambda_n(\omega_m, A)$ of the operators $L$ and $L_m$ are well defined, respectively.

Let us consider $A_0 \in \mathcal{O}_s^c \setminus \mathcal{F}_R^-$, if the eigenvalue $\lambda_n(\omega_0, A_0)$ is double, assume that $\lambda_n(\omega_0, A_0) = \lambda_{n+1}(\omega_0, A_0)$. Suppose $\Lambda_n$ and $\Lambda_{n+1}$ are the continuous eigenvalue branches through the eigenvalue $\lambda_n(\omega_0, A_0)$ defined on a sufficiently small connected neighborhood $O$ of $(\omega_0, A_0)$ in $\Omega \times \mathcal{O}_s$ and $\Lambda_n(\omega, A) \leq \Lambda_{n+1}(\omega, A)$ for any element $(\omega, A) \in O$. Note that the number of the eigenvalue branches is counted with its multiplicities. It can be seen from [17] Proposition 3.10 that when the neighborhood $O$ is sufficiently small, $A$ is in $\mathcal{O}_s^c \setminus \mathcal{F}_R^-$ for every $(\omega, A) \in O$.

By Lemma [2.19]
\[
\lambda_n(\omega_m, A_0) \rightarrow \lambda_n(\omega_0, A_0), \lambda_{n+1}(\omega_m, A_0) \rightarrow \lambda_{n+1}(\omega_0, A_0), \text{ as } m \rightarrow \infty.
\]
Hence the multiplicity of \( \lambda_n(\omega_0, A_0) \) implies that
\[
\Lambda_n(\omega_m, A_0) = \lambda_n(\omega_m, A_0), \Lambda_{n+1}(\omega_m, A_0) = \lambda_{n+1}(\omega_m, A_0)
\]
when \( m \) is sufficiently large. It is clear that for sufficiently large \( m \), \( \Lambda_n(\omega_m, A) \) and \( \Lambda_{n+1}(\omega_m, A) \) are continuous eigenvalue branches through \( \lambda_n(\omega_m, A_0) \) and \( \lambda_{n+1}(\omega_m, A_0) \) defined on \( O \) respectively. Thus, from the multiplicity of \( \Lambda_n \) and \( \Lambda_{n+1} \), the continuity of \( \lambda_n(\omega_m, A) \) and \( \lambda_{n+1}(\omega_m, A) \) on \( \mathbb{R}_x^2 \backslash \mathcal{X}_+^R \) as was proved in [17], one deduces that
\[
\Lambda_n(\omega_m, A) = \lambda_n(\omega_m, A), \Lambda_{n+1}(\omega_m, A) = \lambda_{n+1}(\omega_m, A)
\]
for sufficiently large \( m \) and \( (\omega_m, A) \in O \).

For an arbitrary \( \epsilon > 0 \), there exists a \( \delta_1 > 0 \) such that if
\[
\|A - A_0\| < \delta_1/2, \int_a^b \left( \frac{1}{p_m} - \frac{1}{p} \right) + |s_m - s| < \delta_1/2,
\]
then
\[
|\lambda_n(\omega_m, A) - \lambda_n(\omega_0, A_0)| = |\Lambda_n(\omega_m, A) - \lambda_n(\omega_0, A_0)| < \epsilon/2.
\]
Moreover, for such a \( \epsilon > 0 \) and a fixed point \( A \in \mathcal{R}_s^R \), there exists a \( \delta_2 > 0 \) such that if
\[
\int_a^b \left( \frac{1}{p_m} - \frac{1}{p} \right) + |s_m - s| < \delta_2/2,
\]
then
\[
|\lambda_n(\omega_m, A) - \lambda_n(\omega_0, A)| < \epsilon/2.
\]
Thus for an arbitrary \( \epsilon > 0 \), there exists a \( \delta = \delta_1/2 \) such that if
\[
\|A - A_0\| < \delta,
\]
then
\[
|\lambda_n(\omega_0, A) - \lambda_n(\omega_0, A_0)| \leq |\lambda_n(\omega, A) - \lambda_n(\omega_m, A)| + |\lambda_n(\omega_m, A) - \lambda_n(\omega_0, A_0)| < \epsilon.
\]
(5.2)
(5.3)

So it is a direct result that the eigenvalue \( \lambda_n(\omega_0, A) \) is continuous at each point not in \( \mathcal{X}_R^R \).

If the eigenvalue \( \lambda_n(\omega_0, A_0) \) is simple, the proof is similar and simpler. \( \square \)

**Proposition 5.2.** For every \( n \in \mathbb{N}_0 \), the restriction of \( \lambda_n \) to each of \( \mathcal{F}_-^R, \mathcal{F}_-^R, \mathcal{H}_-^R \) and \( \mathcal{I}_-^R \) is continuous.

**Proof.** The proof is similar to that of Proposition 5.1. \( \square \)

Note that the continuity claim in Theorem 3.1 for the eigenvalues of the separated boundary conditions is a consequence of Propositions 5.1 and 5.2.

In order to describe the discontinuity of \( \lambda_n \) on \( \mathcal{R}_s^R \), we let
\[
\mathcal{F}_+^R = \mathcal{O}_6^R \backslash \mathcal{F}_-^R, \quad \mathcal{F}_+^R = \mathcal{O}_4^R \backslash \mathcal{F}_-^R, \quad \mathcal{H}_+^R = \mathcal{O}_3^R \backslash \mathcal{H}_-^R,
\]
\[
\mathcal{F}_0^R = \mathcal{O}_2^R \backslash (\mathcal{F}_-^R \cup \mathcal{F}_+^R).
\]
Note that the coupled boundary conditions in $\mathcal{H}^R$ are all in $\mathcal{F}_S$, and

$$\mathcal{H}^R \cap \Gamma = (\mathcal{H}^R \cap \mathcal{F}_S^R \cap \Gamma) \cup (\mathcal{H}^R \cap \mathcal{H}^R_\Gamma \cap \Gamma) \cup (D)$$

where $D$ is the Dirichlet boundary condition and $\Gamma$ is the set of all the separated boundary conditions $S_{n,\beta}$.

**Theorem 5.3.** The function $\lambda_0$ on $\mathcal{F}_S^R$ is continuous on $\mathcal{F}_S^R \setminus \mathcal{H}^R$ and discontinuous at each point of $\mathcal{H}^R$. For $n \in \mathbb{N}$, the function $\lambda_n$ is continuous on $\mathcal{F}_S^R \setminus \mathcal{H}^R$ and at each coupled boundary condition in $\mathcal{H}^R$ where $\lambda_n = \lambda_{n-1}$ and discontinuous at any other point of $\mathcal{H}^R$. More precisely, for each coupled boundary condition $A \in \mathcal{H}^R$, the restriction of $\lambda_n$ to $\mathcal{F}_S^R$ is continuous at $A$ for $n \in \mathbb{N}_0$ and

$$\lim_{\mathcal{F}_S^R \ni B \to A} \lambda_0(B) = -\infty, \quad \lim_{\mathcal{F}_S^R \ni B \to A} \lambda_n(B) = \lambda_{n-1}(A) \text{ for } n \in \mathbb{N};$$

for each $A \in \mathcal{H}^R \cap \mathcal{F}_S^R \cap \Gamma$, the restriction of $\lambda_n$ to $\mathcal{F}_S^R$ is continuous at $A$ for $n \in \mathbb{N}_0$ and

$$\lim_{\mathcal{F}_S^R \ni B \to A} \lambda_0(B) = -\infty, \quad \lim_{\mathcal{F}_S^R \ni B \to A} \lambda_n(B) = \lambda_{n-1}(A) \text{ for } n \in \mathbb{N};$$

while the restriction $\lambda_n$ to $\mathcal{F}_S^R$ is continuous at the Dirichlet boundary condition $D$ for $n \in \mathbb{N}_0$ and

$$\lim_{\mathcal{F}_S^R \ni B \to D} \lambda_0(B) = \lambda_1(B) = -\infty, \quad \lim_{\mathcal{F}_S^R \ni B \to D} \lambda_n(B) = \lambda_{n-1}(D) \text{ for } n \in \mathbb{N},$$

$$\lim_{\mathcal{F}_S^R \ni B \to D} \lambda_n(B) = \lambda_{n-2}(D) \text{ for } n \geq 2.$$

**Proof.** By Proposition 5.1 and 5.2, we only need to prove (5.3) - (5.4).

Fix a $K \in \text{SL}(2, \mathbb{R})$ with $k_{11} > 0$ and $k_{12} = 0$. When $L = [L \mid -I] \in \mathcal{F}_S^R$ is sufficiently close to $K = [K \mid -I] \in \mathcal{H}^R$, we have $l_{11} > 0$ and $l_{12} > 0$. Part (a) and (b) of Theorem 4.1 implies

$$\lambda_0(L) \leq \{\mu_0(L), \nu_0(L)\},$$

$$\{\mu_2n(L), \nu_2n(L)\} \leq \lambda_{2n+1}(L) < \{\mu_{2n+1}(L), \nu_{2n+1}(L)\},$$

$$\{\mu_{2n+1}(L), \nu_{2n+1}(L)\} < \lambda_{2n+2}(L) \leq \{\mu_{2n+2}(L), \nu_{2n+2}(L)\},$$

$$\nu_0(K) \leq \lambda_0(K) < \{\mu_0(K), \nu_1(K)\},$$

$$\{\mu_{2n}(K), \nu_{2n+1}(K)\} < \lambda_{2n+1}(K) \leq \{\mu_{2n+1}(K), \nu_{2n+2}(K)\},$$

$$\{\mu_{2n+1}(K), \nu_{2n+2}(K)\} \leq \lambda_{2n+2}(K) < \{\mu_{2n+2}(K), \nu_{2n+3}(K)\},$$

where $\mu_0(L)$ and $\nu_n(L)$ are the eigenvalues for the separated boundary conditions

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & l_{21} & -l_{12} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -l_{21} & -l_{11} \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & l_{22} & -l_{12} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -l_{22} & -l_{11} \end{bmatrix}.$$
respectively. By Corollary 5.2 $\mu_0(L) \to -\infty$ and $\nu_n(L) \to \nu_n(K)$ as $L$ in $\mathcal{F}^R_+$ approaches $K$, since then $l_{12} \to 0^+$, $l_{22} \to k_{22} > 0$ and $l_{11} \to k_{11} > 0$. Thus from (5.10), it follows that

$$\lim_{\mathcal{F}^R_+ \ni L \to K} \lambda_0(L) = -\infty.$$ 

Assume $\lim_{\mathcal{F}^R_+ \ni L \to K} \lambda_n(L) = \lambda_{n-1}(K)$ is false for $n \in \mathbb{N}$, from (5.10), we can find a sequence $\{L_k\} \subset \mathcal{F}^R_+$ such that

$$\lambda_n(L_k) \to c \neq \lambda_{n-1}(K) \quad \text{as } L_k \to K,$$

where $c$ is a constant. From Lemma 1.10 Lemma 1.8 and (5.10), one deduces that $c$ is an eigenvalue for $K$ and $\nu_{n-1}(K) \leq c \leq \nu_n(K)$. This contrary implies

$$\lim_{\mathcal{F}^R_+ \ni L \to K} \lambda_n(L) = \lambda_{n-1}(K) \quad \text{for } n \in \mathbb{N}.$$  

Similarly, we prove (5.4) for $K \in \text{SL}(2, \mathbb{R})$ with $k_{11} \leq 0$ and $k_{12} = 0$.

Next, we will prove (5.5) denote $\omega_m = (\frac{1}{p_m}, q, r, s_m) \in \hat{\Omega}$, $\omega_0 = (\frac{1}{p}, q, r, s) \in \hat{\Omega}$. For every $A \in \mathcal{F}^R_+$, the eigenvalues $\lambda_n(\omega_0, A)$ and $\lambda_n(\omega_m, A)$ of $L$ and $L_m$ are well defined, respectively.

Let us consider $A \in \mathcal{F}^\mathbb{R}_+ \cap \mathcal{F}^\mathbb{R}_+ \cap \Gamma$, and an arbitrary point $B \in \mathcal{F}^\mathbb{R}_+$,

$$(5.11) \quad |\lambda_n(\omega_0, B) - \lambda_{n-1}(\omega_0, A)| \leq |\lambda_n(\omega_0, B) - \lambda_n(\omega_m, B)| + |\lambda_n(\omega_m, B) - \lambda_{n-1}(\omega_0, A)|.$$ 

By Lemma 2.6 the eigenvalues for $A$ are all simple. Suppose $\Lambda_n$ is the continuous simple eigenvalue branch through the eigenvalue $\lambda_{n-1}(\omega_0, A)$ defined on a connected neighborhood $O$ of $(\omega_0, A)$ in $\hat{\Omega} \times \{\mathcal{F}^\mathbb{R}_+ \cup \{A\}\}$.

By Lemma 2.19

$$(5.12) \quad \lambda_{n-1}(\omega_m, A) \to \lambda_{n-1}(\omega_0, A), \quad \text{as } m \to \infty.$$ 

Hence the simplicity of $\Lambda_n$ implies that $\Lambda_n(\omega_m, A) = \lambda_{n-1}(\omega_m, A)$ when $m$ is sufficiently large. It is clear that for sufficiently large $m$, $\Lambda_n(\omega_m, B)$ is a continuous eigenvalue branch through $\lambda_{n-1}(\omega_m, A)$ defined on $O$. Thus, from the simplicity of $\Lambda_n$, the continuity of $\lambda_n(\omega_m, B)$ on $\mathcal{F}^\mathbb{R}_+$ and the fact $\lim_{\mathcal{F}^\mathbb{R}_+ \ni B \to A} \lambda_n(\omega_m, B) = \lambda_{n-1}(\omega_m, A)$ for $n \in \mathbb{N}$ as was proved in [17], one deduces that

$$\Lambda_n(\omega_m, B) = \lambda_n(\omega_m, B)$$

for sufficiently large $m$ and $(\omega_m, B) \in O$.

For an arbitrary $\epsilon > 0$, there exists a $\delta_1 > 0$ such that if

$$||B - A|| < \delta_1/2, \quad \int_a^b \left( \left| \frac{1}{p_m} - \frac{1}{p} \right| + |s_m - s| \right) < \delta_1/2,$$

then

$$|\lambda_n(\omega_m, B) - \lambda_{n-1}(\omega_0, A)| = |\Lambda_n(\omega_m, B) - \lambda_{n-1}(\omega_0, A)| < \epsilon/2.$$ 

Moreover, for such a $\epsilon > 0$ and a fixed point $B \in \mathcal{F}^\mathbb{R}_+$, there exists a $\delta_2 > 0$ such that if

$$\int_a^b \left( \left| \frac{1}{p_m} - \frac{1}{p} \right| + |s_m - s| \right) < \delta_2/2,$$

then

$$|\lambda_n(\omega_m, B) - \lambda_n(\omega_0, B)| < \epsilon/2.$$
Thus for an arbitrary $\epsilon > 0$, there exists a $\delta = \delta_1/2$ such that
\[
\|B - A\| < \delta,
\]
then
\[
|\lambda_n(\omega_0, B) - \lambda_{n-1}(\omega_0, A)| \leq |\lambda_n(\omega_0, B) - \lambda_n(\omega_m, B)| + |\lambda_n(\omega_m, B) - \lambda_{n-1}(\omega_0, A)| < \epsilon.
\]
So it is a direct result that
\[
\lim_{\mathcal{F}_R \ni B \to A} \lambda_n(B) = \lambda_{n-1}(A) \text{ for } n \in \mathbb{N}.
\]
Assume that $\lim_{\mathcal{F}_R \ni B \to A} \lambda_0(B) = -\infty$ is false, then there exists a sequence $\{B_k\} \subset \mathcal{F}_R$ such that $\{\lambda_0(B_k)\}$ is bounded. Without loss of generality, assume $\lambda_0(B_k) \to c$ ($c$ is a constant) as $B_k \to A$. From Lemma 5.4 Lemma 5.5 and the fact that $\lambda_0(A)$ is a simple eigenvalue, one deduces that $c$ is an eigenvalue for $A$ and $c < \lambda_0(A)$. This contrary implies
\[
\lim_{\mathcal{F}_R \ni B \to A} \lambda_0(B) = -\infty.
\]
Similarly, one proves [5.6], [5.7], [5.8], [5.9]. \hfill \square

In order to describe the discontinuity set of $\lambda_n$ as a function on $\mathcal{F}_0$, we set
\[
\mathcal{F}_- = \{ [e^{i\theta} K | I ]; K \in \text{SL}(2, \mathbb{R}), k_1, k_1 \leq 0 \},
\]
\[
\mathcal{G}_- = \left\{\begin{bmatrix} a_1 & 1 & 0 & -\bar{z} \\ z & 0 & -1 & b_2 \end{bmatrix}; b_2 \leq 0, \ a_1 \in \mathbb{R}, z \in \mathbb{C} \right\},
\]
\[
\mathcal{H}_- = \left\{\begin{bmatrix} 1 & a_2 & -\bar{z} & 0 \\ 0 & z & b_1 & -1 \end{bmatrix}; a_2 \leq 0, \ b_1 \in \mathbb{R}, z \in \mathbb{C} \right\},
\]
\[
\mathcal{J}_- = \left\{\begin{bmatrix} 1 & a_2 & 0 & \bar{z} \\ 0 & z & b_1 & -1 \end{bmatrix}; a_1, b_2 \leq 0, \ z \in \mathbb{C}, a_2 b_2 \geq z^2 \right\},
\]
\[
\mathcal{K}_- = \{ [e^{i\theta} K | -I ]; K \in \text{SL}(2, \mathbb{R}), k_1, k_1 = 0, \theta \in [0, \pi) \}
\cup \left\{\begin{bmatrix} a_1 & a_2 & 0 & 0 \\ 0 & 0 & 0 & b_1 \end{bmatrix} \in \mathcal{F}_0; a_2 b_2 = 0 \right\}.
\]
\[
\mathcal{F}_+ = \mathcal{O}_{6,8} \setminus \mathcal{F}_- \quad \mathcal{G}_+ = \mathcal{O}_{6,8} \setminus \mathcal{G}_- \quad \mathcal{H}_+ = \mathcal{O}_{4,8} \setminus \mathcal{H}_- \quad \mathcal{J}_+ = \mathcal{O}_{4,8} \setminus \mathcal{J}_- \quad \mathcal{K}_+ = \mathcal{O}_{2,8} \setminus \mathcal{K}_-.
\]

The proofs of the following results are similar to those of Proposition 5.4, 5.5 and Theorem 5.6 so we omit them.

**Proposition 5.4.** Let $n \in \mathbb{N}_0$. Then as a function on the space $\mathcal{F}_0$, $\lambda_n$ is continuous at each point not in $\mathcal{K}_0$.

**Proposition 5.5.** For every $n \in \mathbb{N}_0$, the restriction of $\lambda_n$ to each of $\mathcal{F}_-, \mathcal{G}_-, \mathcal{H}_-$ and $\mathcal{J}_-$ is continuous.

**Theorem 5.6.** The conclusions of Theorem 5.3 still hold when the super indices $\mathbb{R}$ in them are replaced by $\mathbb{C}$.
Remark 5.7. On the basis of the lemmas and theorems we give in the previous sections of this paper, we can also use the similar methods in [17] for the Sturm-Liouville problems with regular potentials to prove the propositions and theorems on the discontinuity set of \( \lambda_n \) as a function on \( \mathcal{B}_S^0 \) or \( \mathcal{B}_c^0 \) in this section. However, in this paper, we supply a simpler method which relies heavily on Lemma 2.19.

6. Oscillation Theorems

Theorem 6.1. (a) For any \( K \in \text{SL}(2, \mathbb{R}) \), let \( \psi_n(x) \) be a real eigenfunction of \( \lambda_n(K) \), then the number of zeros of \( \psi_n(x) \) on the interval \( [a, b] \) is 0 or 1 if \( n = 0 \), and if \( n \geq 1 \), the number is \( n - 1 \), or \( n \) or \( n + 1 \).

(b) For any \( K \in \text{SL}(2, \mathbb{R}) \), \( 0 < \gamma < \pi \) or \(-\pi < \gamma < 0 \), let \( \psi_n(x) \) be an eigenfunction of \( \lambda_n(\gamma, K) \), then the number of zeros of \( \text{Re}\psi_n(x) \) on the interval \( [a, b] \) is 0 or 1 if \( n = 0 \), and if \( n \geq 1 \), the number is \( n - 1 \), or \( n \) or \( n + 1 \). This claim also holds for \( \text{Im}\psi_n(x) \). Moreover, \( \psi_n(x) \) has no zeros on \( [a, b] \).

Theorem 6.2. For any \( K \in \text{SL}(2, \mathbb{R}) \), if \( k_{11} > 0 \) and \( k_{12} = 0 \), let \( \psi_n(x) \) and \( \xi_n(x) \) be real eigenfunctions of \( \lambda_n(K) \) and \( \lambda_n(-K) \), respectively. Then we obtain the following conclusions: (a) \( \psi_0(x) \) has no zeros in \( [a, b] \). (b) \( \psi_{m+1}(x) \) and \( \psi_{m+2}(x) \) have exactly \( 2m + 2 \) zeros in \( [a, b] \). (c) \( \xi_{m}(x) \) and \( \xi_{m+1}(x) \) have exactly \( 2m + 1 \) zeros in \( [a, b] \).

Proof of Theorem 6.1. (a) Denote the eigenfunction of \( \lambda_n^0 \) by \( \Psi_n(x) \), according to Lemma 2.10 it follows that \( \Psi_n(x) \) has \( n + 2 \) zeros on the interval \([a, b]\). It suffices to prove the theorem in the following two cases.

Case 1: \( k_{12} \neq 0 \), by Theorem 4.11 there exists \( 0 < \beta < \pi \), such that
\[
\lambda_n(K) \leq \lambda_n(S_0, \beta) < \lambda_n(S_0, \pi) = \lambda_n^0, \quad n \in \mathbb{N}_0,
\]
thus from Theorem 4.12 \( \lambda_{n-2}^0 < \lambda_n(K) < \lambda_n^0 \), \( n \geq 2 \), so this theorem is now a direct consequence of Lemma 1.12.

Case 2: \( k_{12} = 0 \), now the condition 0.14 is
\[
\begin{align*}
y(b) = k_{11}y(a), \\
y^{[1]}(b) = k_{21}y(a) + k_{22}y^{[1]}(a).
\end{align*}
\]
If \( \lambda_n(K) < \lambda_n^0 \), our claims can be obtained from Lemma 1.12 and Theorem 4.12. If \( \lambda_n(K) = \lambda_n^0 \), assume \( \psi_n(x) \) has \( n + 2 \) zeros on the interval \([a, b]\), now we can easily reach a contradiction by using Lemma 1.12 no matter \( \psi_n(a) = 0 \) or not. Thus the conclusions can be obtained according to Lemma 1.12 and Theorem 4.12.

(b) It can be easily seen that \( \text{Re}\psi_n(x) \) and \( \text{Im}\psi_n(x) \) are both nontrivial solutions of equation 1.1 with \( \lambda = \lambda_n(\gamma, K) \), thus by Lemma 1.12 and Theorem 4.12 the conclusions on them can be easily obtained. Since the eigenfunction \( \psi_n(x) \) of \( \lambda_n(\gamma, K) \) can not be real, it follows that \( \text{Re}\psi_n(x) \) and \( \text{Im}\psi_n(x) \) are linearly independent solutions of the equation, thus they do not have the same zeros on \([a, b]\), so we obtain that \( \psi_n(x) \) has no zero on \([a, b]\). \( \square \)

Proof of Theorem 6.2. (a) According to Theorem 4.11 for such a \( K \in \text{SL}(2, \mathbb{R}) \),
\[
\begin{align*}
\lambda_0(K) &< \lambda_0(-K) \leq \lambda_0^D \leq \lambda_1(-K) < \lambda_1(K) \leq \lambda_1^D \\
&\leq \lambda_2(-K) \leq \lambda_2^D \leq \lambda_3(-K) < \lambda_3(K) < \cdots.
\end{align*}
\]

Denote the eigenfunction of \( \lambda_0^D \) by \( \Psi_n(x) \). Since \( \Psi_0(x) \) has no zeros on \([a, b]\), and \( \lambda_0(K) < \lambda_0^D \), it follows that \( \psi_0(x) \) has at most one zero on \([a, b]\). Without loss of
generality, we assume that \( \psi_0(x_0) = 0, \psi_0^{[1]}(x_0) > 0 \) for some \( x_0 \in (a, b) \). Hence by Lemma 1.11 and the continuity of \( \psi_0(x) \), one deduces that \( \psi_0(a) < 0, \psi_0(b) > 0 \). This leads to a contradiction since \( \psi_0(x) \) satisfies the boundary condition (6.1) with \( k_{11} > 0 \). This completes the proof of part (a). It remains to prove part (b) of the theorem.

(b) Recall that \( \lambda_{2m}^D < \lambda_{2m+1}^D(K) \leq \lambda_{2m+1}^D \), \( \Psi_{2m}(x) \) and \( \Psi_{2m+1}(x) \) has \( 2m \) and \( 2m+1 \) zeros respectively on \( (a, b) \). Thus it follows from Lemma 1.12 that \( \psi_{2m+1}(x) \) has at least \( 2m + 1 \) and at most \( 2m + 2 \) zeros on \( (a, b) \).

It suffices to show that \( \psi_{2m+1}(x) \) have an even number of zeros on \( (a, b) \). In fact, if \( \lambda_{2m+1}^D(K) = \lambda_{2m+1}^D \), assume \( \psi_{2m+1}^{[1]}(a) = 0 \), then we have \( \psi_{2m+1}^{[1]}(b) > 0 \) since \( k_{22} = 1/k_{11} > 0 \). From Lemma 1.11 and the continuity of \( \psi_{2m+1}(x) \), we obtain that \( \psi_{2m+1}(x) \) must have an even number of zeros on \( [a, b] \).

With the inequality

\[
\lambda_{2m+1}^D \leq \lambda_{2m+2}(K) < \lambda_{2m+2}^D,
\]

the result for \( \psi_{2m+2}(x) \) can be proved by the same method. This proves part (b).

Part (c) can be proved in the same way as in the proof of part (b).

In order to prove this part, we need the following fact:

\[
\lambda_0(I) \leq \lambda_0^D,
\]

\[
\lambda_{2m-1}^D < \lambda_{2m}(-K) \leq \lambda_{2m}^D \quad \text{and} \quad \lambda_{2m+1}^D \leq \lambda_{2m+1}(-K) < \lambda_{2m+1}^D, \quad m \in \mathbb{N}.
\]

The only difference is the fact that \( \xi_{2m}(x) \) and \( \xi_{2m+1}(x) \) must have an odd number of zeros on \( [a, b] \) which is implied by Lemma 1.11 and the boundary condition

\[
\begin{cases}
y(b) = -k_{11}y(a), \\
y^{[1]}(b) = -k_{21}y(a) - k_{22}y^{[1]}(a).
\end{cases}
\]

\[\square\]

7. Differentiability properties of eigenvalues

As a similar space we have introduced in Section 2, in this section we introduce a “coefficient space” with a metric. Let \( \Omega = \{ \omega = (1/p, q, r, s); (1.2) \text{ holds} \} \). For the topology of \( \Omega \) we use a metric \( d \) defined as follows:

For \( \omega = (1/p, q, r, s) \in \Omega \), \( \omega_0 = (1/p_0, q_0, r_0, s_0) \in \Omega \), define

\[
d(\omega, \omega_0) = \int_a^b \left( \frac{1}{p} - \frac{1}{p_0} \right) + |q - q_0| + |r - r_0| + |s - s_0| \right).
\]

**Theorem 7.1.** For any \( n \in \mathbb{N}_0 \), the \( n \)-th eigenvalue of the problem with a fixed boundary condition depends continuously on the coefficients of the differential equation.

**Proof.** First, we consider the case where the self-adjoint boundary condition is a separated one. Let \( \omega_0 = (1/p_0, q_0, r_0, s_0) \in \Omega \). Then \( \lambda_0(\omega_0) \) is simple. Consider the continuous eigenvalue branch \( \Lambda(\omega) \) through \( \lambda_0(\omega_0) \) defined on a neighborhood of \( \omega_0 \) in \( \Omega \). Let \( w = w(\cdot, \omega_0) \) denote a normalized eigenfunction of the eigenvalue \( \lambda(\omega_0) \).
From Lemma 2.5 there exist normalized eigenfunctions \( w = w(\cdot, \omega) \) of \( \Lambda(\omega) \) such that
\[
(7.1) \quad w(\cdot, \omega) \rightarrow w(\cdot, \omega_0), \quad w^{(l)}(\cdot, \omega) \rightarrow w^{(l)}(\cdot, \omega_0), \quad \text{as} \; \omega \rightarrow \omega_0 \; \text{in} \; \tilde{\Omega},
\]
both uniformly on the interval \([a, b]\).

Note that \( w(t, \omega_0) \) does not have a zero in \((a, b)\) from Lemma 2.6. So, we may assume that \( w(t, \omega_0) > 0 \) on \((a, b)\).

(i) If \( w(a, \omega_0) = 0 \), then \( \omega \in (\omega_0, \nu) \) applies to \( \nu \). Let \( \omega \) be sufficiently close to \( \omega_0 \) such that \( w(t, \omega_0) > 0 \) on the interval \([a, a + \epsilon_1]\). By (7.1), when \( \omega \) is sufficiently close to \( \omega_0 \), \( w(t, \omega) > 0 \) on \([a, a + \epsilon_1]\). It is a fact that \( w(a, \omega) = 0 \) since the boundary condition is fixed. Thus \( w(t, \omega) > 0 \) on the interval \((a, a + \epsilon_1)\) when \( \omega \) is sufficiently close to \( \omega_0 \).

(ii) If \( w(t, \omega_0) > 0 \) on \([a, b]\), then \( \omega \in (\omega_0, \nu) \) applies to \( \nu \). Let \( \omega \) be sufficiently close to \( \omega_0 \) such that \( w(t, \omega_0) > 0 \) on the interval \((b - \epsilon_2, b)\) when \( \omega \) is sufficiently close to \( \omega_0 \). Since \( w(t, \omega_0) > 0 \) on \([a + \epsilon_1 b - \epsilon_2]\), it follows from (7.1) that \( w(t, \omega) > 0 \) on \([a + \epsilon_1 b - \epsilon_2]\) when \( \omega \) is sufficiently close to \( \omega_0 \). Hence \( w(t, \omega) > 0 \) on \((a, b)\) when \( \omega \) is sufficiently close to \( \omega_0 \).

Thus by Lemma 2.6 when \( \omega \) is sufficiently close to \( \omega_0 \), \( \Lambda(\omega) = \lambda_0(\omega) \). According to Lemma 2.18 it follows that \( \lambda_1(\omega), \lambda_2(\omega), \cdots \) are continuous at \( \omega_0 \).

Next, assume that the self-adjoint boundary condition is the coupled one (0.7) or (0.8), with \( k_{11} > 0 \), \( k_{12} \leq 0 \). Then \( v_0(\omega, K) \) is continuous at \( \omega_0 \) by the proven case. On the other hand, by part (a) of Theorem 4.1.

\[
v_0(\omega, K) \leq \lambda_0(\omega, K) < \lambda_0(\omega, \gamma, K) < \lambda_0(\omega, -K).
\]

Thus \( \lambda_0(\omega, K), \lambda_0(\omega, \gamma, K), \lambda_0(\omega, -K) \) are uniformly bounded from below in a small neighborhood of \( \omega_0 \). Therefore, Lemma 2.18 implies that for each \( n \in N_0 \), \( \lambda_n(\omega, K), \lambda_n(\omega, \gamma, K), \lambda_n(\omega, -K) \) are continuous at \( \omega_0 \).

Finally, we consider the case where the self-adjoint boundary condition is the coupled one (0.7) or (0.8) with \( k_{11} \leq 0 \), \( k_{12} < 0 \). Fix an \( \omega_0 \in \Omega \) and consider the continuous eigenvalue branch \( \Lambda \) through \( \lambda_0(\omega, K) \) defined on a connected neighborhood \( O \) of \( \omega_0 \). By part (b) of Theorem 4.1 \( \Lambda(\omega_0) = \lambda_0(\omega_0, K) < v_0(\omega_0, K) \) and \( \Lambda(\omega) \neq v_0(\omega, K) \) for any \( \omega \in O \). Hence, we have \( \Lambda(\omega) < v_0(\omega, K) \) for any \( \omega \in O \), since both \( \Lambda \) and \( v_0 \) are continuous functions on \( O \). Therefore, \( \lambda_0(\omega, K) \) is continuous for any \( \omega \in O \) still by part (b) of Theorem 4.1 i.e., \( \lambda_0(\omega, K) \) is continuous at \( \omega_0 \). On the other hand, by part (b) of Theorem 4.1.

\[
\lambda_0(\omega, K) < \lambda_0(\omega, \gamma, K) < \lambda_0(\omega, -K) \leq v_0(\omega, K).
\]

Thus \( \lambda_0(\omega, K), \lambda_0(\omega, \gamma, K), \lambda_0(\omega, -K) \) are uniformly bounded from below in a small neighborhood of \( \omega_0 \). Therefore, by Lemma 2.18 for each \( n \in N_0 \), \( \lambda_n(\omega, K), \lambda_n(\omega, \gamma, K), \lambda_n(\omega, -K) \) are continuous at \( \omega_0 \).

Note that if neither of the above cases applies to \( K \), then either of the above cases applies to \( -K \).

In the following we show that the eigenvalues are differentiable functions of the coefficients \( 1/p, q, r, s \) in the equation. Recall the definition of the Frechet derivative:

**Definition 7.2.** Let \( X \) and \( Y \) be Banach spaces, with norms \(|-|_X\) and \(|-|_Y\) respectively. Let \( U \subset X \) be an open set, and let \( A : U \rightarrow Y \) be a map. We say that \( A \)
is Frechet differentiable at a point $x_0 \in X$ if there exists a bounded linear operator $B : X \to Y$ such that for $h \in X$,
\[ \|A(x_0 + h) - A(x_0) - B(h)\|_Y = o(\|h\|_X) \text{ as } h \to 0, \]
and denote the bounded linear operator $B$ by $A'(x_0)$.

**Remark 7.3.** For investigating the differentiability of the eigenvalue $\lambda_n$ as a function of the coefficients $1/p$ and $r$, we recall the definition of the Frechet derivative on the positive cone
\[ V = \{ f \in L(J, \mathbb{R}) \mid f \geq 0 \text{ a.e. on } J \} \]
of the Banach space $L(J, \mathbb{R})$. Considering a (nonlinear) functional $\lambda_n$ from $V$ to $\mathbb{R}$, we say that $\lambda_n$ is Frechet differentiable at a point $r$ in $V$ if there exists a bounded linear functional $f : V \to \mathbb{R}$ such that for $h \in V$,
\[ |\lambda_n (r + h) - \lambda_n (r) - f(h)| = o(\|h\|_{L(J, \mathbb{R})}) \text{ as } h \to 0, \]
and denote the bounded linear functional $f$ by $\lambda_n'(r)$.

**Theorem 7.4.** Let $\omega = (A, B, 1/p, q, r, s) \in \Omega$. Fix $A, B$. Assume that $\lambda_n$ is a simple eigenvalue of $\omega$ for some $n \in \mathbb{N}_0$ and $\omega_n$ is a normalized eigenfunction of $\lambda_n$, then there is a simple closed curve $\Gamma$ in $\mathbb{C}$ with $\lambda_n(\omega)$ in its interior and a neighborhood $O$ of $\omega$ in $\Omega$ such that for any $\rho \in O$, the Sturm-Liouville problem $\rho$ has exactly one eigenvalue in the interior of $\Gamma$ and this eigenvalue is simple.

1) Fix $q, r, s$ and consider $\lambda_n$ as a function of $1/p$, $p > 0$ a.e. on $J$. Then $\lambda_n$ is Frechet differentiable at $1/p$ in $V$ and its Frechet derivative is the bounded linear transformation given by
\[ (7.2) \quad \lambda_n'(1/p)h = -\int_a^b \left| w_n^{[1]}(\cdot, 1/p) \right|^2 h, \quad h \in L(J, \mathbb{R}); \]
2) Fix $1/p, q, r$ and consider $\lambda_n$ as a function of $s$. Then $\lambda_n$ is Frechet differentiable at $s$ in $L(J, \mathbb{R})$ and its Frechet derivative is the bounded linear transformation given by
\[ (7.3) \quad \lambda_n'(s)h = 2 \int_a^b \text{Re}(\omega_n \bar{\omega}_n^{[1]})h, \quad h \in L(J, \mathbb{R}); \]
3) Fix $1/p, s, r$ and consider $\lambda_n$ as a function of $q$. Then $\lambda_n$ is Frechet differentiable at $q$ in $L(J, \mathbb{R})$ and its Frechet derivative is the bounded linear transformation given by
\[ (7.4) \quad \lambda_n'(q)h = \int_a^b |\omega_n|^2 h, \quad h \in L(J, \mathbb{R}); \]
4) Fix $1/p, s, q$ and consider $\lambda_n$ as a function of $r$, $r > 0$ a.e. on $J$. Then $\lambda_n$ is Frechet differentiable at $r$ in $V$ and its Frechet derivative is the bounded linear transformation given by
\[ (7.5) \quad \lambda_n'(r)h = -\lambda_n(r) \int_a^b |w_n|^2 h, \quad h \in L(J, \mathbb{R}). \]

**Proof.** The conclusion that the Sturm-Liouville problem $\rho$ has exactly one eigenvalue in the interior of $\Gamma$ and this eigenvalue is simple is an obvious result.

In the following, we only prove (7.2) and (7.3). The conclusion (7.4) and (7.5) can be proved similarly.
(1) Denote \( w_n = w_n(\cdot, 1/p) \), \( v_n = w_n(\cdot, 1/p_h) \) where \( 1/p_h = 1/p + h \), \( h \in V \). Note that \( 1/p \in L(J, \mathbb{R}) \) implies that \( 1/p_h \in L(J, \mathbb{R}) \) and \( p - p_h = pp_h h \). Using (1.2) and integration by parts we obtain

\[
(\lambda_n(1/p_h) - \lambda_n(1/p)) \int_a^b w_n \bar{v}_n r = \\
\lambda_n(1/p_h) \int_a^b w_n \bar{v}_n r - \lambda_n(1/p) \int_a^b w_n \bar{v}_n r = \\
\lambda_n(1/p_h) \int_a^b w_n (-(\bar{\nu}_n^{[1]})' + s \bar{\nu}_n^{[1]} + q \bar{v}_n) - \lambda_n(1/p) \int_a^b \bar{v}_n (-(w_n^{[1]})' + sw_n^{[1]} + qw_n) = \\
\left[ -w_n \bar{v}_n^{[1]} + \bar{v}_n w_n^{[1]} \right]_a^b + \int_a^b \left( w_n' \bar{v}_n^{[1]} + s w_n \bar{v}_n^{[1]} - \bar{v}_n' w_n^{[1]} - s \bar{v}_n w_n^{[1]} \right) = \\
\left[ -w_n \bar{v}_n^{[1]} + \bar{v}_n w_n^{[1]} \right]_a^b - \int_a^b w_n^{[1]} \bar{v}_n^{[1]} h.
\]

For all boundary conditions we have that

\[
\left[ -w_n \bar{v}_n^{[1]} + \bar{v}_n w_n^{[1]} \right]_a^b = 0.
\]

Noting that \( 1/p_h \to 1/p \) as \( h \to 0 \) in \( L(J, \mathbb{R}) \) and using Lemma 2.5 we have

\[
(\lambda_n(1/p + h) - \lambda_n(1/p))(1 + o(1)) = - \int_a^b \left| w_n^{[1]} \right|^2 h + o(h),
\]

and consequently,

\[
\lambda_n(1/p + h) - \lambda_n(1/p) = \left( - \int_a^b \left| w_n^{[1]} \right|^2 h + o(h) \right) (1 + o(1))^{-1} = \left( - \int_a^b \left| w_n^{[1]} \right|^2 h + o(h) \right) (1 + o(1))^{-1}.
\]

as \( h \to 0 \) in \( L(J, \mathbb{R}) \). This completes the proof of (7.2).

(2) Denote \( w_n = w_n(\cdot, s) \), \( v_n = w_n(\cdot, s_h) \) where \( s_h = s + h, h \in L(J, \mathbb{R}) \). Note that \( s \in L(J, \mathbb{R}) \) implies that \( s_h \in L(J, \mathbb{R}) \). Using (1.2) and integration by parts we obtain

\[
(\lambda_n(s_h) - \lambda_n(s)) \int_a^b w_n \bar{v}_n r = \\
\lambda_n(s_h) \int_a^b w_n \bar{v}_n r - \lambda_n(s) \int_a^b w_n \bar{v}_n r = \\
\lambda_n(s_h) \int_a^b w_n (-(\bar{\nu}_n^{[1]})' + s_h \bar{\nu}_n^{[1]} + q \bar{v}_n) - \lambda_n(s) \int_a^b \bar{v}_n (-(w_n^{[1]})' + sw_n^{[1]} + qw_n) = \\
\left[ -w_n \bar{v}_n^{[1]} + \bar{v}_n w_n^{[1]} \right]_a^b + \int_a^b \left( w_n' \bar{v}_n^{[1]} + s_h w_n \bar{v}_n^{[1]} - \bar{v}_n' w_n^{[1]} - s \bar{v}_n w_n^{[1]} \right) = \\
\left[ -w_n \bar{v}_n^{[1]} + \bar{v}_n w_n^{[1]} \right]_a^b + \int_a^b \left( h w_n \bar{v}_n^{[1]} + h \bar{v}_n w_n^{[1]} \right).
\]
For all boundary conditions we have that
\[
\left[-w_n \vec{v}_n^{[1]} + \tilde{v}_n w_n^{[1]}\right]_a^b = 0.
\]
Noting that \(s_h \to s\) as \(h \to 0\) in \(L(J, \mathbb{R})\) and using Lemma 2.5 we have
\[
(\lambda_n(s + h) - \lambda_n(s))(1 + o(1)) = \int_a^b \left(w_n \vec{w}_n^{[1]} + \tilde{w}_n w_n^{[1]}\right) h + o(h),
\]
and consequently,
\[
\lambda_n(s + h) - \lambda_n(s) = 2 \int_a^b \operatorname{Re}(w_n \vec{w}_n^{[1]}) h + o(h),
\]
as \(h \to 0\) in \(L(J, \mathbb{R})\). This completes the proof of (7.3). \(\square\)

8. Application to a class of transmission problems

Consider the Sturm-Liouville operator

\[
Ly(x) := \frac{1}{w(x)}(-p(x)y'(x))' + q(x)y(x), \quad x \in J = (a, c) \cup (c, b),
\]
with the transmission conditions

\[
\begin{align*}
 y(c+) = y(c-), \\
 (p'y)(c+) - (p'y)(c-) = ay(c),
\end{align*}
\]
where \(1/p, q, w \in L(J, \mathbb{R}), p > 0, w > 0\) a.e. on \(J\), and \(a \in \mathbb{R}\) is a constant.

According to [31], we consider the self-adjoint boundary conditions as follows:

\[
A \left( \begin{array}{c} y(a) \\ (p'y)(a) \end{array} \right) + B \left( \begin{array}{c} y(b) \\ (p'y)(b) \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right),
\]
where \(A, B\) also satisfy the condition (0.5). As is well known, the boundary conditions (8.3) can be divided into three classes of boundary conditions as follows:

1. Seperated self-adjoint boundary conditions:

\[
S_{\alpha, \beta} : \begin{cases} 
\cos \alpha y(a) - \sin \alpha (p'y)(a) = 0, \quad \alpha \in [0, \pi), \\
\cos \beta y(b) - \sin \beta (p'y)(b) = 0, \quad \beta \in (0, \pi].
\end{cases}
\]

2. Real coupled self-adjoint boundary conditions:

\[
Y(b) = KY(a), \quad K \in \text{SL}_2(\mathbb{R}).
\]

3. Complex coupled self-adjoint boundary conditions:

\[
Y(b) = e^{\gamma} KY(a), \quad -\pi < \gamma < 0 \text{ or } 0 < \gamma < \pi, \quad K \in \text{SL}_2(\mathbb{R}),
\]
where \(Y(\cdot) = \left( \begin{array}{c} y(\cdot) \\ p'y(\cdot) \end{array} \right)\).

**Theorem 8.1.** All the conclusions we have obtained in Sections 3—6 can be applied to the self-adjoint transmission problems (8.1) — (8.3).
Proof. Denote \( C = \left( \int_a^b \frac{q(s)ds}{w(s)} \right) \), \( \tilde{q} = q - Cw \). Let \( \tilde{u}(x) = \int_a^x \tilde{q}(t)dt \), then \( \tilde{u}(a) = 0 \) and
\[
\tilde{u}(b) = \int_a^b \tilde{q}(t)dt = \int_a^b q(t)dt - C \int_a^b w(t)dt = -\alpha.
\]
Define the function
\[
u_0(x) = \begin{cases} \alpha, & x \in [c, b], \\ 0, & x \in [a, c), \end{cases}
\]
and let \( \tilde{u}(x) = u_0(x) + \tilde{u}(x) \), then define the following operators on \((a, b)\),
\[
\tilde{L}y = \frac{1}{w} - (py' + \tilde{u}y)' + \frac{\tilde{u}}{p}(py' + \tilde{u}y) - \frac{\tilde{u}^2}{p}y,
\]
\[
\mathcal{D} (\tilde{L}) = \left\{ y \in AC([a, b]) \mid y(x) = 0, x \in (a, c) \cup (c, b), \right\}
\]
It is obvious that \( \tilde{L} \) is the Sturm-Liouville operator we have mainly considered in this paper, thus the eigenvalues and eigenfunctions of it satisfy the conclusion in Sections 3–6. Moreover, since \( \tilde{u}(a) = \tilde{u}(b) = 0 \), then \( \tilde{L} \) can be written as
\[
\tilde{L}y = \frac{1}{w} - (py' + \tilde{q}y)' + \tilde{q}y, \quad y \in (a, c) \cup (c, b),
\]
\[
\mathcal{D}(\tilde{L}) = \left\{ y \in AC([a, b]) \mid y(x) = 0, x \in (a, c) \cup (c, b), \right\}
\]
In conclusion, from the relation of the Sturm-Liouville problem \((\text{S.1}) - (\text{S.3})\) and the operator \( \tilde{L} \), the proof is completed. \( \square \)

References

[1] A. M. Savchuk and A. A. Shkalikov, Sturm–Liouville operators with singular potentials, Math. Notes. 66 (1999) 741–753.
[2] A. M. Savchuk and A. A. Shkalikov, Sturm–Liouville operators with distributional potentials, Trans. Moscow Math. Soc. 64 (2003) 143–192.
[3] J. Eckhardt, F. Gesztesy, R. Nichols, and G. Teschl, Weyl–Titchmarsh theory for Sturm–Liouville operators with distributional potentials, Opuscula Math. 33 (2013) 467–563.
[4] M.-L. Baeteman and K. Chadan, The inverse scattering problem for singular oscillating potentials, Nuclear Phys. A 255 (1975) 35–44.
[5] M.-L. Baeteman and K. Chadan, Scattering theory with highly singular oscillating potentials, Ann. Inst. H. Poincaré Sect. A 24 (1976) 1–16.
[6] M. Combescure, Spectral and scattering theory for a class of strongly oscillating potentials, Commun. Math. Phys. 73 (1980) 43–62.
[7] M. Combescure and J. Ginibre, Spectral and scattering theory for the Schrödinger operator with strongly oscillating potentials, Ann. Inst. H. Poincaré 24 (1976) 17–29.
[8] D. B. Pearson, Scattering theory for a class of oscillating potentials, Helv. Phys. Acta 52 (1979) 541–554.
[9] F. S. Rofe-Beketov and E. H. Hristov, Transformation operators and scattering functions for a highly singular potential, Sov. Math. Dokl. 7 (1966) 834–837.
[10] F. S. Rofe-Beketov and E. H. Hristov, Some analytical questions and the inverse Sturm–Liouville problem for an equation with highly singular potential, Sov. Math. Dokl. 10 (1969) 432–435.
[11] J. Herczyński, On Schrödinger operators with distributional potentials, J. Operator Th. 21 (1989) 273–295.
[12] S. Albeverio, F. Gesztesy, R. Høgh-Krohn, and H. Holden, Solvable Models in Quantum Mechanics, 2nd ed., AMS Chelsea Publishing, Providence, RI, 2005.
[13] S. Albeverio and P. Kurasov, Singular Perturbations of Differential Operators, Cambridge Univ. Press, Cambridge, 2001.
[14] C. Bennewitz and W. N. Everitt, On second-order left-definite boundary value problems, in Ordinary Differential Equations and Operators, (Proceedings, Dundee, 1982), W. N. Everitt and R. T. Lewis (eds.), Lecture Notes in Math. Vol. 1032, Springer, Berlin, 1983, pp. 31–67.
[15] W. N. Everitt and L. Markus, Boundary Value Problems and Symplectic Algebra for Ordinary Differential and Quasi-Differential Operators, Math. Surv. and Monographs, Vol. 61, Amer. Math. Soc., RI, 1999.
[16] J. Eckhardt, F. Gesztesy, R. Nichols, and G. Teschl, Inverse spectral theory for Sturm-Liouville operators with distributional potentials, J. Lond. Math. Soc. 88 (2013) 801–828.
[17] Q. Kong, H. Wu, and A. Zettl, Dependence of the n-th Sturm-Liouville eigenvalue on the problem, J. Differential Equations 156 (1999) 328–354.
[18] M. S. P. Eastham, Q. Kong, H. Wu, and A. Zettl, Inequalities among eigenvalues of Sturm-Liouville problems, J. Inequal. Appl. 3 (1999) 25–43.
[19] Q. Kong and A. Zettl, Eigenvalues of regular Sturm-Liouville problems, J. Differential Equations 131 (1996) 1–19.
[20] B. Chanane, Sturm-Liouville problems with impulse effects, Appl. Math. Comput. 190 (2007) 610–626.
[21] F. Gesztesy, C. Macedo, and L. Streit, An exactly solvable periodic Schrödinger operator, J. Phys. A: Math. Gen. 18 (1985) 503–507.
[22] O. Sh. Mukhtarov and M. Kandemir, Asymptotic behavior of eigenvalues for the discontinuous boundary-value problem with functional-transmission conditions, Acta Math. Scientia 22 B (3) (2002) 335–345.
[23] O. Sh. Mukhtarov and S. Yakubov, Problems for differential equations with transmission conditions, Appl. Anal. 81 (2002) 1033–1064.
[24] Q. Kong, H. Wu, and A. Zettl, Geometric aspects of Sturm-Liouville problems. I. structures on spaces of boundary conditions, Proc. Roy. Soc. Edinburgh Sect. A 130 (2000) 561–589.
[25] A. Zettl, Sturm-Liouville Theory, Amer. Math. Soc., Providence, RI, 2005.
[26] T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag, Berlin, 1966.
[27] P. Hislop, I. Sigal, Introduction to Spectral Theory. With applications to Schrödinger operators, Springer-Verlag, NY, 1996.
[28] M. Reed and B. Simon, Methods of Modern Mathematical Physics. Vols. 1–4, Academic Press, New York, 1972.
[29] J. Dieudonne, Foundations of Modern Analysis, Academic Press, New York/London, 1969.
[30] F. V. Atkinson, Asymptotics of an eigenvalue problem involving an interior singularity, Research Program Proceedings ANL-87-26, Vol.2, pp.1-18, Argonne National Lab, Illinois, 1988.
[31] E. A. Coddington and N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill, New York, 1955.
[32] R. Adams and J. Fournier, Sobolev Spaces, Academic Press, New York, 1975.
[33] J. Weidmann, Linear Operators in Hilbert Spaces, Springer, New York, 1980.
[34] A. Wang, J. Sun, and A. Zettl, Two-interval Sturm-Liouville operators in modified Hilbert spaces. J. Math. Anal. Appl. 328 (2007) 390–399.