Aspects of (2+1) dimensional gravity: AdS$_3$ asymptotic dynamics in the framework of Fefferman-Graham-Lee theorems

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Abstract. Using the Chern-Simon formulation of (2+1) gravity, we derive, for the general asymptotic metrics given by the Fefferman-Graham-Lee theorems, the emergence of the Liouville mode associated to the boundary degrees of freedom of (2+1) dimensional anti de Sitter geometries.

Keywords: anti de Sitter, Chern-Simon, Liouville

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1 Introduction

The interest of studying (2+1) dimensional gravity has initially been emphasized in [1] and has recently been revived with the discovery of black holes in spaces with negative cosmological constant [2]. Since then, a large number of studies has been devoted to the elucidation of classical as well as to quantum (2+1) gravity.

In particular, we examined [3] stellar-like models corresponding to stationary, rotationally symmetric gravitational sources of the perfect fluid type, embedded in spaces of arbitrary cosmological constant, and showed how causality privileges anti de Sitter (AdS) backgrounds. As this part of the talk has already been published, we will not re-describe it here.

On the other hand, (2 + 1) gravity with negative cosmological constant has been proven to be equivalent to Chern-Simons (CS) theory with $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ as gauge group [4]. Assuming the boundary of the space to be a flat cylinder $\mathbb{R} \times S^1$, Coussaert, Henneaux and van Driel (CHD) [5] demonstrated the equivalence between this CS theory and a non-chiral Wess-Zumino-Witten (WZW) theory [6], and showed that the AdS$_3$ boundary conditions as defined in [7] implement the constraints that reduce the WZM model to the Liouville theory [8].

In this short note, we show that, using the less restrictive AdS boundary conditions deduced from the Fefferman-Graham-Lee theorems [9,10], the CHD analysis can
be extended and leads to the Liouville theory formulated on a 2-dimensional curved background. A detailed version of this work will be found in [12].

2 Asymptotically anti de Sitter spaces

Graham and Lee [9] proved that, under suitable topological assumptions, Euclidean Einstein spaces with negative cosmological constant $\Lambda$ are completely defined by the geometry on their boundary. Furthermore, Fefferman and Graham [10] showed that, whatever the signature, there exists a formal asymptotic expansion of the metric, which formally solves the Einstein equations with $\Lambda < 0$. The first terms of this expansion are given by even powers of a radial coordinate $r$:

$$ ds^2 \approx \frac{\ell^2}{r^2} \frac{dr^2}{r^2} + \frac{r^2}{\ell^2} \mathbf{g}^{(0)}(x^i) + \mathbf{g}^{(2)}(x^i) + \cdots. $$

On $d$-dimensional space-times, the full asymptotic expansion actually continues with terms of negative even powers of $r$ up to $r^{-(d-3)}$, with in addition a logarithmic term of the order of $r^{-(d-3)} \log r$ when $d$ is odd and larger than 3. All these terms are completely defined by the boundary geometry. They are followed by terms of all negative powers starting from $r^{-(d-3)}$; the trace-free part of the $r^{-(d-3)}$ coefficient is not fully determined by $\mathbf{g}^{(0)}$ but contains degrees of freedom [11].

It is instructive to look at the first iterations of this expansion to see the special character of 3 dimensions. We therefore write the metric in terms of forms $\Theta^\mu$ as $ds^2 = \Theta^0 \otimes \Theta^0 + \eta_{ab} \Theta^a \otimes \Theta^b$ with $\mu, \nu$ [resp. $a, b$] running from 0 to $n$ [resp. 1 to $n$] and $\eta_{ab}$ a flat $n$-dimensional minkowskian metric $\text{diag}(1, \ldots, 1, -1)$. The forms $\Theta^\mu$ read as:

$$ \Theta^0 = \ell \frac{dr}{r}, \quad \Theta^a = \frac{r}{\ell} \theta^a + \ell \sigma^a + O(r^{-3}), $$

where the forms $\theta^a$ and $\sigma^a \equiv \sigma^a_b \theta^b$ are $r$-independent. These provide the dominant and sub-dominant terms of the metric expansion:

$$ \mathbf{g}^{(0)} = \eta_{ab} \theta^a \otimes \theta^b, \quad \mathbf{g}^{(2)} = g_{ab} \theta^a \otimes \theta^b = (\sigma_{ab} + \sigma_{ba}) \theta^a \otimes \theta^b. $$

Here and in what follows, the $n$-dimensional indices and the covariant derivatives are defined with respect to the metric $\mathbf{g}^{(0)}$. Using these definitions, the components of the $(n+1)$-dimensional Riemann curvature 2-form $\mathbf{R}$ become:

$$ \mathbf{R}^{(0)}_{ab} = -\frac{1}{\ell^2} \Theta_a \wedge \Theta_b - \frac{1}{r} \left( d\gamma_a + \omega_{ab} \wedge \gamma^b \right) + O(r^{-3}), $$

$$ \mathbf{R}^{(2)}_{ab} = -\frac{1}{\ell^2} \Theta_a \wedge \Theta_b + \mathbf{R}^{(0)}_{ab} + \frac{1}{\ell^4} (\Theta_a \wedge \gamma_b + \gamma_a \wedge \Theta_b) + O(r^{-2}), $$

where $\omega_{ab}$ is the $n$-dimensional Levi-Civita connection and $\mathbf{R}^{(0)}_{ab}$ the $n$-dimensional curvature 2-form, both defined by the metric $\mathbf{g}^{(0)}$, and $\gamma_a \equiv g_{ab} \theta^b$. If we impose the
metric of the (n+1)-dimensional space to be Einsteinian, these equations, at order $r^2$, fix $\Lambda = -1/\ell^2$. Moreover, at order $1$ and $r^{-1}$, they yield:

$$\begin{align*}
(0) R_{ab} + \frac{1}{\ell^2}[(n-2) (2) g_{ab} + \eta_{ab} (2) g] &= 0, \\
(2) b_a (2) b - (2) b_a (2) b &= 0,
\end{align*}$$

where $R_{ab}$ are the components of the $n$-dimensional Ricci tensor. These equations clearly reveal the pecularity of 3-dimensional spaces. Indeed, when $n \neq 2$, eq. (6) fully specifies the metric $(2) g$ and eq. (7) becomes the Bianchi identity satisfied by the $n$-dimensional Einstein tensor. If we moreover require the space to be asymptotically AdS, the finite terms in eqs (4-5) have to vanish, thereby implying the $n$-dimensional geometry to be conformally flat.

On the contrary, when $n = 2$, only the trace of $(2) g$ is fixed by eq. (6):

$$(2) g^{c} c = 2 \sigma^{c} \equiv 2 \sigma = -\frac{\ell^2}{2} (0) R,
$$

and the other components of $(2) g$ have only to satisfy the equations:

$$(2) a b_{;a} = -\frac{\ell^2}{2} (0) R_{;b}.
$$

The subdominant metric components are thus not all determined by the asymptotic metric in 3 dimensions, but there remains one degree of freedom, which we shall explicit in the next section. Note that in 3 dimensions Einstein spaces with $\Lambda < 0$ are locally AdS and metrics on cylindrical boundaries are conformally flat; this implies the equivalence between eqs (6,7) and the vanishing of the sub-dominant terms on the right-hand side of eqs (4,5).

### 3 From Einstein-Hilbert to Liouville action

The Einstein-Hilbert (EH) (2+1) gravity action with $\Lambda < 0$ is equivalent to the difference of two CS actions $S_{CS}[A] - S_{CS}[\tilde{A}]$ with

$$S_{CS}[A] = \frac{1}{2} \int Tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A).
$$

The gauge fields $A = A_{\mu} \Theta^\mu = J_{\mu} A^\mu$, with $J_{\mu}$ generators of the $sl(2, \mathbb{R})$ algebra\(^a\), are given as functions of the 3-bein $\Theta^\mu$ and the Levi-Civita connection form $\Omega^{\mu\nu}$ by:

$$\begin{align*}
A^\mu &= \frac{1}{\ell} \Theta^\mu + \frac{1}{2} \epsilon^{\mu \nu \rho} \Theta^{\nu\rho}, \\
\tilde{A}^\mu &= -\frac{1}{\ell} \Theta^{\mu} + \frac{1}{2} \epsilon^{\mu \nu \rho} \Theta^{\nu\rho}.
\end{align*}$$

\(^a\) We use the conventions: $J_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $J_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $J_2 = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. 

\(\epsilon_{012} = 1\).
In cylindrical coordinates \( \{r, \phi, t\} \), the CS action may be written as:

\[
S_{CS}[A] = \frac{1}{2} \int_M Tr(2 A_t F_{r\phi} + A_\phi \dot{A}_r - A_r \dot{A}_\phi) \, dr \, d\phi \, dt + S_B[A]
\]  
(12)

\[
S_B[A] = -\frac{1}{2} \int_{\partial M} Tr(A_r A_\phi) \, d\phi \, dt
\]  
(13)

where \( M \) stands for bulk. The on-shell variation of this action is given by:

\[
\delta S_{CS}[A] = \frac{1}{2} \int_{\partial M} Tr(A_t \delta A_\phi - A_\phi \delta A_t) \, d\phi \, dt
\]  
(14)

The asymptotic behaviour of the fields \( A_\phi \) and \( \tilde{A}_\phi \), dictated by the AdS\(_3\) boundary conditions, are easily expressed using the null frame \( \hat{\theta}^\pm = \theta^1 \pm \theta^2 \) and its dual vectorial frame \( \hat{e}_\pm = \frac{1}{2}(e_1 \pm e_2) \). Indeed, the null components \( A_- \) and \( \tilde{A}_- \) do not contain any degrees of freedom, i.e. they only depend on the metric \( \tilde{g} \) and on its scalar curvature (see eq. \[8\]). At order \( r^{-1} \), they are equal to:

\[
A_- = \frac{1}{2} \begin{pmatrix} \omega_- & 0 \\ 0 & -\omega_- \end{pmatrix} = K_-
\]
\[
\tilde{A}_- = \frac{1}{2} \begin{pmatrix} \omega_- & 0 \\ -\omega_- & -\omega_- \end{pmatrix} = \tilde{K}_-
\]  
(15)

where we have introduced the null components of the connection 2-form \( \omega^{\phi t} = \omega_+ \hat{\theta}^t + \omega_- \hat{\theta}^r \). The other components:

\[
A_+ = \begin{pmatrix} \frac{\omega_+}{r} & \frac{\sigma_+ - \sigma_-}{2r} \\ \frac{\sigma_+}{2r} & \frac{\omega_+}{2} \end{pmatrix}
\]
\[
\tilde{A}_+ = \begin{pmatrix} \frac{\omega_+}{r} & \frac{\sigma_+ + \sigma_-}{2r} \\ \frac{\sigma_+}{2r} & \frac{\omega_+}{2} \end{pmatrix}
\]  
(16)

depend on the dynamical part of \( \tilde{g} \), which is not fixed by \( \tilde{g} \). This implies that \( \delta A_- = O(r^{-2}) = \delta \tilde{A}_- \) and \( A_- \delta A_+ = O(r^{-2}) = \tilde{A}_- \delta A_+ \). So, rewriting the variation of the action \( \ref{14} \) in terms of null components yields:

\[
\delta S_{CS}[A] = \frac{1}{2} \int_{\partial M} Tr(A_\phi \delta A_- - A_- \delta A_+) \, r \, d\phi \, dt = \int_{\partial M} O(r^{-2}) \, d\phi \, dt
\]  
(17)

with \( \theta = \theta^1 \theta_\phi - \theta^2 \theta_{\phi} \). It is thus easy to see that, owing to the boundary conditions \( \ref{15} \), the variation of the action \( S_{SC} \) vanishes, without the addition of any extra boundary term. However, as the practical implementation of the boundary condition \( \ref{16} \) is not obvious at this stage, we prefer to modify the action by the addition of the boundary term

\[
S_B'[A] = \frac{1}{2} \int_{\partial M} Tr(A_- A_+) \, r \, d\phi \, dt
\]  
(18)

which ensures that \( \delta (S_{CS}[A] + S_B'[A]) = 0 \) independently of the boundary condition \( \ref{16} \). A similar modification is applied to the \( \tilde{A} \) sector.

Furthermore, the time components \( A_t \) and \( \tilde{A}_t \) play the rôle of Lagrange multipliers and can be eliminated from the bulk action by solving the constraint equations \( F_{r\phi} = 0 \) and \( F_{r\phi} = 0 \) as \( A_i = Q_1^{-1} \partial_i Q_1 \) and \( \tilde{A}_i = Q_2^{-1} \partial_i Q_2 \), with \( i = (r, \phi) \). The asymptotic
AdS$_3$ behaviour implies that the $SL(2,\mathbb{R})/Z_2$ group elements $Q_1$ and $Q_2$ asymptotically factorize into $Q_1(r,\phi,t) = q_1(\phi,t)H(r)$ and $Q_2(r,\phi,t) = q_2(\phi,t)H(r)^{-1}$, with $H(r) = diag(\sqrt{r}/t, \sqrt{t}/r)$. On the other hand, the components $A_t$ and $\tilde{A}_t$ in the boundary action $S_B$ may be eliminated in terms of $A_\phi$, $\tilde{A}_\phi$, $K_-$ and $\tilde{K}_+$, using the boundary conditions, which can be re-expressed as:

$$
A_t = \frac{1}{e_-}(-e_\phi A_\phi + K_-) , \quad \tilde{A}_t = \frac{1}{e_+}(e_\phi \tilde{A}_\phi + \tilde{K}_+) .
$$

These relations allow to write the complete action $S = S_{SC}[\mathcal{A}] + S'[\mathcal{A}] - S_{SC}[\tilde{\mathcal{A}}] - S'[\tilde{\mathcal{A}}]$ as:

$$
S = -\Gamma[Q_1] + \frac{1}{2} \int_{\partial_\mathcal{M}} Tr[\frac{1}{e_-} q'_1 (q_1^{-1} \partial_- q_1 - k_-)] dt d\phi
+\Gamma[Q_2] - \frac{1}{2} \int_{\partial_\mathcal{M}} Tr[\frac{1}{e_+} q'_2 (q_2^{-1} \partial_+ q_2 - k_+)] dt d\phi ,
$$

where $\partial_+$ and $\partial_-$ are derivatives along the vectors $\dot{e}_+$ and $\dot{e}_-$, $q' = q^{-1} \partial_\phi q$, $k_- = HK_-.H^{-1}$, $k_+ = H^-1K_+H$ and $\Gamma[Q] = \frac{1}{3 !} \int Tr[q^{-1}dQ \wedge Q^{-1}dQ \wedge Q^{-1}dQ]$. Using the new variable $q = q_1^{-1}q_2$, the fields $q_1$ or $q_2$ can be eliminated using their equations of motion, as they only appear in quadratic expressions of their derivatives with respect to the angular variable $\phi$. The resulting action becomes a non-chiral WZW-like action containing only the field $q$:

$$
S = \Gamma[Q] - \frac{1}{2} \int_{\partial_\mathcal{M}} Tr[q^{-1} \partial_+ q q^{-1} \partial_- q + 2 \partial_+ q q^{-1} k_- - 2 q^{-1} \partial_+ q k_+] \theta dt d\phi .
$$

Let us for a moment focus on the equations of $q'_1$ and $q'_2$ as functions of $q$:

$$
q'_1 = \theta[e_+ (\partial_+ q q^{-1} - q k_+ q^{-1})] + e_+ k_-, \quad q'_2 = \theta[e_+ (q^{-1} \partial_+ q - q^{-1} k_- q)] - e_+ k_+ .
$$

Using the Gauss decomposition $q = \begin{pmatrix} e^{\Phi/2} & X \cdot e^{-\phi/2} \\ Y \cdot e^{-\phi/2} & e^{-\Phi/2} \end{pmatrix}$ for $SL(2,\mathbb{R})$ elements, eqs (22, 23) lead to 6 equations. Four of them:

$$
X = \frac{\ell}{2} \partial_+ \Phi , \quad Y = \frac{\ell}{2} \partial_- \Phi ,
$$

$$
\ell(\partial_+ + \omega_+)X + \frac{1}{2} \sigma + e^\Phi = 0 , \quad \ell(\partial_- - \omega_+)Y + \frac{1}{2} \sigma + e^\Phi = 0 ,
$$

determine $X$ and $Y$ as functions of $\Phi$ and combine to give:

$$
\Box \Phi + \frac{8}{\ell^2} e^\Phi + \frac{4}{\ell^2} \sigma = 0 .
$$

This is the Liouville equation on curved background, the curvature being given by $R_{ab}$.

The other equations yield relations between the energy-momentum tensor of the Liouville field $T_{ab}$ and the metric:

$$
g_{ab} = \frac{\ell^2}{2} T_{ab} - \eta_{ab} \frac{\ell^2}{2} R^{(0)}
$$

$$
= \frac{\ell^2}{2} \left[ \frac{1}{4} \Phi ; a \Phi ; b - \Phi ; ab + \frac{1}{4} \Phi ; c \Phi ; d + \frac{1}{4} \Phi ; c \Phi ; d + \frac{1}{2} (0) \right] .
$$
As a consequence, \( q \) can be expressed in terms of \( \Phi \) and its derivatives only. Note that the elimination of the \( X \) and \( Y \) variables in the non-chiral WZW action \( (21) \), using the constants of motion defined by eqs \( (25) \), has to be performed following the same trick as the one that leads to the Maupertuis action in classical mechanics when the energy is conserved. This yields the equivalent action defined on the boundary only (without any remaining bulk terms):

\[
S = \frac{1}{2} \int_{@M} \left[ \frac{1}{2} g^{ab} \partial_a \Phi \partial_b \Phi - \frac{8}{\ell^2} e^{\Phi} + (0) R \Phi \right] \theta \, dt \, d\phi ,
\]  

(29)

which is the Liouville action. Let us emphasize that the curvature term appearing here comes directly from its definition in terms of the asymptotic metric \( (0) g \), and not through \( \sigma \) as it is the case in eq. \( (26) \).

To conclude, we would like to emphasize several points. First, the usual EH action is divergent and equal to \( S_{CS}[A] - S_{CS}[\tilde{A}] \) plus an additional term \( \frac{\ell}{2} \int_{@M} Tr[A \wedge \tilde{A}] \), which is equal to half of the extrinsic curvature term usually added to the EH action to cancel its variation \( [13] \). However, owing to the AdS boundary conditions, this additional term does not contain any dynamical degrees of freedom and may be dropped, thereby rendering the resulting action finite. Furthermore, the developments following eq. \( (20) \), which lead from the non-chiral WZW action to the Liouville action \( (29) \), are classically valid, but have to be re-examined in the framework of quantum mechanics. Indeed, in quantum mechanics, the changes of variables leading to eq. \( (21) \) and the subsequent elimination of the \( X \) and \( Y \) variables in terms of \( \Phi \) involve functional determinants that have been completely ignored.

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