LARGE DEVIATIONS FOR THE TWO-TIME-SCALE STOCHASTIC CONVECTIVE BRINKMAN-FORCHHEIMER EQUATIONS

MANIL T. MOHAN

Abstract. The convective Brinkman-Forchheimer (CBF) equations characterize the motion of incompressible fluid flows in a saturated porous medium. The small noise asymptotic for the two-time-scale stochastic convective Brinkman-Forchheimer (SCBF) equations in two and three dimensional bounded domains is carried out in this work. More precisely, we establish a Wentzell-Freidlin type large deviation principle for stochastic partial differential equations with slow and fast time-scales, where the slow component is the SCBF equations in two and three dimensions perturbed by small multiplicative Gaussian noise and the fast component is a stochastic reaction-diffusion equation with damping. The results are obtained by using a variational method (based on weak convergence approach) developed by Budhiraja and Dupuis, Khasminskii’s time discretization approach and stopping time arguments. In particular, the results obtained from this work are true for two dimensional stochastic Navier-Stokes equations also.

1. Introduction

The convective Brinkman-Forchheimer (CBF) equations describe the motion of incompressible viscous fluid through a rigid, homogeneous, isotropic, porous medium. Let us first provide a mathematical formulation of the CBF equations. Let $\mathcal{O} \subset \mathbb{R}^n \ (n = 2, 3)$ be a bounded domain with a smooth boundary $\partial \mathcal{O}$. Let $X_t(x) \in \mathbb{R}^n$ denotes the velocity field at time $t \in [0, T]$ and position $x \in \mathcal{O}$, $p_t(x) \in \mathbb{R}$ represents the pressure field, $f_t(x) \in \mathbb{R}^n$ stands for an external forcing. The CBF equations are given by

$$
\begin{align*}
\frac{\partial X_t}{\partial t} - \mu \Delta X_t + (X_t \cdot \nabla)X_t + \alpha X_t + \beta |X_t|^{-1}X_t + \nabla p_t &= f_t, \quad \text{in } \mathcal{O} \times (0, T), \\
\nabla \cdot X_t &= 0, \quad \text{in } \mathcal{O} \times (0, T), \\
X_t &= 0, \quad \text{on } \partial \mathcal{O} \times (0, T), \\
X_0 &= x, \quad \text{in } \mathcal{O},
\end{align*}
$$

(1.1)

the constants $\mu > 0$ represents the Brinkman coefficient (effective viscosity), $\alpha > 0$ stands for the Darcy (permeability of porous medium) coefficient and $\beta > 0$ denotes the Forchheimer (proportional to the porosity of the material) coefficient. In order to obtain the uniqueness of the pressure $p$, we can impose the condition $\int_{\mathcal{O}} p_t(x) dx = 0$, for $t \in (0, T)$ also. The
absorption exponent $r \in [1, \infty)$ and the case $r = 3$ is known as the critical exponent. Note that for the case $\alpha = \beta = 0$, we get the classical 3D Navier-Stokes equations (see [36, 48, 66, 76, 77], etc). The works [1, 27, 43, 59], etc discuss the global solvability results (existence and uniqueness of weak as well as strong solutions) for the deterministic CBF equations in bounded domains. As in the case of classical 3D Navier-Stokes equations, the existence of a unique global strong solution for the CBF equations (1.1) for $n = 3$ and $r \in [1,3)$ is an open problem.

In the stochastic counterpart, the existence and uniqueness of strong solutions to the stochastic 3D tamed Navier-Stokes equations on bounded domains with Dirichlet boundary conditions is established in [71]. The authors in [52] obtained the existence of martingale solutions for the stochastic 3D Navier-Stokes equations with nonlinear damping. The existence of a pathwise unique strong solution satisfying the energy equality (Itô’s formula) to the stochastic convective Brinkman-Forchheimer (SCBF) equations perturbed by multiplicative Gaussian noise is proved in [60]. The author exploited the monotonicity and hemicontinuity properties of the linear and nonlinear operators as well as a stochastic generalization of the Minty-Browder technique in the proofs. The Itô formula (energy equality) is established by using the fact that there are functions that can approximate functions defined on smooth bounded domains by elements of eigenspaces of linear operators (e.g., the Laplacian or the Stokes operator) in such a way that the approximations are bounded and converge in both Sobolev and Lebesgue spaces simultaneously (such a construction is available in [27]). Making use of the exponential stability of strong solutions, the existence of a unique ergodic and strongly mixing invariant measure for the SCBF equations subject to multiplicative Gaussian noise is also established in [60]. The works [4, 8, 54, 55, 69, 70], etc discuss various results on the stochastic tamed 3D Navier-Stokes equations and related models on periodic domains as well as on the whole space.

The multiscale systems involve slow and fast components in mathematical models, and they are having wide range of applications in the areas like signal processing, climate dynamics, material science, molecular dynamics, mathematical finance, fluid dynamics, etc. The theory of averaging principle for multiscale systems is well developed for the past several years and it has extensive applications in science, technology and engineering (cf. [3, 26, 30, 41, 57, 58, 83], etc and references therein). Averaging principle proposes that the slow component of a slow-fast system may be approximated by a simpler system obtained by averaging over the fast motion. For the deterministic systems, an averaging principle was first investigated by Bogoliubov and Mitropolsky in [5] and for the stochastic differential equations, an averaging principle was first studied by Khasminskii in [45]. Several works are available in the literature for the theory of averaging principle for the stochastic partial differential equations (cf. [6, 7, 12, 13, 14, 15, 24, 31, 32, 33, 34, 81, 84], etc and references therein). An averaging principle for the slow component as two dimensional stochastic Navier-Stokes equations and the fast component as stochastic reaction-diffusion equations by using the classical Khasminskii approach based on time discretization is established in [50]. The authors in [53] established a strong averaging principle for the slow-fast stochastic partial differential equations with locally monotone coefficients, which includes the systems like stochastic porous medium equation, the stochastic Burgers type equation, the stochastic $p$-Laplace equation and the stochastic 2D Navier-Stokes equation, etc. A strong averaging principle for the stochastic convective Brinkman-Forchheimer (SCBF) equations
perturbed by Gaussian noise, in which the fast time scale component is governed by a stochastic reaction-diffusion equation with damping driven by multiplicative Gaussian noise, has been obtained in [62].

The large deviations theory, which concerns the asymptotic behavior of remote tails of sequences of probability distributions (cf. [29], [78]), is one of the classical areas in probability theory with numerous deep developments and variety of applications in the fields like queuing theory, statistics, finance, engineering, etc. The theory of large deviations explains the probabilities of rare events that are exponentially small as a function of some parameter. In the case of stochastic differential equations, this parameter can be regarded as the amplitude of the noise perturbing a dynamical system. The Wentzell-Freidlin type large deviation estimates for a class of infinite dimensional stochastic differential equations is developed in the works [9], [16], [18], [44], etc. Large deviation principles for the 2D stochastic Navier-Stokes equations driven by Gaussian noise are established in the works [17], [38], [74], etc. A Wentzell-Freidlin type large deviation principle for the stochastic tamed 3D Navier-Stokes equations driven by multiplicative Gaussian noise in the whole space or on a torus is established in [70]. Small time large deviations principles for the stochastic 3D tamed Navier-Stokes equations in bounded domains is established in the work [67]. Large deviation principle for the 3D tamed Navier-Stokes equations driven by multiplicative Lévy noise in periodic domains is established in [37]. The author in [61] obtained the Wentzell-Freidlin large deviation principle for the two and three dimensional SCBF equations in bounded domains using a weak convergence approach developed by Budhiraja and Dupuis (see, [9], [10]). The large deviations for short time as well as the exponential estimates on certain exit times associated with the solution trajectory of the SCBF equations are also established in the same work. It seems to the author that some of the LDP results available in the literature for the 3D stochastic tamed Navier-Stokes equations in bounded domains are not valid due to the technical difficulty described in the works [27], [43], [59], etc.

The large deviation theory for multi-scale systems are also well studied in the literature (see for instance, [25], [29], [46], [47], [49], [51], [64], [65], [73], [79], etc and references therein). A large deviation principle for a class of stochastic reaction-diffusion partial differential equations with slow-fast components is derived in the work [82]. The authors in [42] studied a large deviation principle for a system of stochastic reaction-diffusion equations with a separation of fast and slow components, and small noise in the slow component, by using the weak convergence method in infinite dimensions. A Wentzell-Freidlin type large deviation principle for stochastic partial differential equations with slow and fast time-scales, where the slow component is a one-dimensional stochastic Burgers’ equation with small noise and the fast component is a stochastic reaction-diffusion equation is established in [76]. In this work, we establish a Wentzell-Freidlin type large deviation principle for the two-time-scale stochastic partial differential equations, where the slow component is the SCBF equations in two and three dimensional bounded domains ($r \in [1, \infty)$, for $n = 2$ and $r \in [3, \infty)$, for $n = 3$ with $2\beta\mu > 1$ for $r = 3$) perturbed by small multiplicative Gaussian noise and the fast component is a stochastic reaction-diffusion equation with damping. We use a variational method (based on weak convergence approach) developed by Budhiraja and Dupuis (see, [9], [10]), Khasminkii’s time discretization approach and stopping time arguments to obtain the LDP for the two-time-scale SCBF equations. Furthermore, we remark that the results obtained in this work are true for two dimensional stochastic Navier-Stokes equations also.
Our main aim of this work is to study a Wentzell-Freidlin type large deviation principle (LDP) for the following two-time-scale stochastic convective Brinkman-Forchheimer (SCBF) equations in two and three dimensional bounded domains. The two-time-scale SCBF equations are given by

$$\begin{cases}
\frac{dX_t^{\varepsilon,\delta}}{dt} = [\mu\Delta X_t^{\varepsilon,\delta} - (X_t^{\varepsilon,\delta} \cdot \nabla)X_t^{\varepsilon,\delta} - \alpha X_t^{\varepsilon,\delta} - \beta|X_t^{\varepsilon,\delta}|^{-1}X_t^{\varepsilon,\delta} + F(X_t^{\varepsilon,\delta}, Y_t^{\varepsilon,\delta})]dt \\
- \nabla p_t^{\varepsilon,\delta}dt + \sqrt{\varepsilon}\sigma_1(X_t^{\varepsilon,\delta})Q_1^{1/2}dW_t, \\
\frac{dY_t^{\varepsilon,\delta}}{dt} = -\frac{1}{\delta}[\mu\Delta Y_t^{\varepsilon,\delta} - \alpha Y_t^{\varepsilon,\delta} - \beta|Y_t^{\varepsilon,\delta}|^{-1}Y_t^{\varepsilon,\delta} + G(Y_t^{\varepsilon,\delta})]dt \\
+ \frac{1}{\sqrt{\varepsilon}}\sigma_2(X_t^{\varepsilon,\delta}, Y_t^{\varepsilon,\delta})Q_2^{1/2}dW_t, \\
\nabla \cdot X_t^{\varepsilon,\delta} = 0, \quad \nabla \cdot Y_t^{\varepsilon,\delta} = 0, \\
X_t^{\varepsilon,\delta}|_{\partial\Omega} = Y_t^{\varepsilon,\delta}|_{\partial\Omega} = 0, \\
X_0^{\varepsilon,\delta} = x, \quad Y_0^{\varepsilon,\delta} = y,
\end{cases}$$

(1.2)

for $t \in (0, T)$, where $\varepsilon > 0$, $\delta = \delta(\varepsilon) > 0$ (with $\delta \to 0$ and $\frac{\delta}{\varepsilon} \to 0$ as $\varepsilon \to 0$) are small parameters describing the ratio of the time scales of the slow component $X_t^{\varepsilon,\delta}$ and the fast component $Y_t^{\varepsilon,\delta}$, $F$, $G$, $\sigma_1, \sigma_2$ are appropriate functions, and $W_t$ is a Hilbert space valued standard cylindrical Wiener process on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_t\}_{t\geq 0}$. The system (1.2) can be considered as stochastic convective Brinkman-Forchheimer equations, whose drift coefficient is coupled with a stochastic perturbation $Y_t^{\varepsilon,\delta}$, which can be considered as the dramatically varying temperature (with damping) in the system.

The rest of the paper is organized as follows. In the next section, we provide some functional spaces as well as the hypothesis satisfied by the functions $F, G, \sigma_1, \sigma_2$ needed to obtain the global solvability of the system (1.2). An abstract formulation for the two-time-scale SCBF system (1.2) is formulated in section 3 and we discuss about the existence and uniqueness of a pathwise strong solution to the system (1.2) (Theorem 3.3). A Wentzell-Freidlin type large deviation principle for the two-time-scale SCBF system is established in 4 (Theorem 4.11). The results are obtained by using a variational method (based on weak convergence approach) developed by Budhiraja and Dupuis, Khasminskii’s time discretization approach and stopping time arguments (Theorems 4.14 and 4.21). Moreover, we deduce that the results obtained this work are true for two dimensional stochastic Navier-Stokes equations also (Remark 4.22).

2. Mathematical Formulation

In this section, we describe the necessary function spaces and the hypothesis satisfied by the functions $F, G$ and the noise coefficients $\sigma_1, \sigma_2$ needed to obtain the global solvability results for the coupled SCBF equations (1.2).

2.1. Function spaces. Let $C^\infty_0(\Omega; \mathbb{R}^n)$ denotes the space of all infinitely differentiable functions ($\mathbb{R}^n$-valued) with compact support in $\Omega \subset \mathbb{R}^n$. We define

$$\mathcal{V} := \{X \in C^\infty_0(\Omega; \mathbb{R}^n) : \nabla \cdot X = 0\},$$

$$\mathcal{H} := \text{the closure of } \mathcal{V} \text{ in the Lebesgue space } L^2(\Omega) = L^2(\Omega; \mathbb{R}^n),$$

$$\mathcal{V} := \text{the closure of } \mathcal{V} \text{ in the Sobolev space } H^1_0(\Omega) = H^1_0(\Omega; \mathbb{R}^n),$$

$$\mathcal{H} := \text{the closure of } \mathcal{V} \text{ in the Sobolev space } H^1(\Omega) = H^1(\Omega; \mathbb{R}^n).$$
\[ \tilde{L}^p := \text{the closure of } V \text{ in the Lebesgue space } L^p(\mathcal{O}) = L^p(\mathcal{O}; \mathbb{R}^n), \]

for \( p \in (2, \infty) \). Then under some smoothness assumptions on the boundary, we characterize the spaces \( \mathbb{H}_0, V \) and \( \tilde{L}^p \) as \( \mathbb{H}_0 = \{ X \in L^2(\mathcal{O}) : \nabla \cdot X = 0, X \cdot n|_{\partial \mathcal{O}} = 0 \} \), with norm \( \|X\|_\mathbb{H}_0 := \int_\mathcal{O} |X(x)|^2dx \), where \( n \) is the outward normal to \( \partial \mathcal{O} \), \( V = \{ X \in H^1_0(\mathcal{O}) : \nabla \cdot X = 0 \} \), with norm \( \|X\|_V := \int_\mathcal{O} |\nabla X(x)|^2dx \), and \( \tilde{L}^p = \{ X \in L^p(\mathcal{O}) : \nabla \cdot X = 0, X \cdot n|_{\partial \mathcal{O}} = 0 \} \), with norm \( \|X\|_\tilde{L}^p := \int_\mathcal{O} |X(x)|^pdx \), respectively. Let \( \langle \cdot, \cdot \rangle \) denotes the inner product in the Hilbert space \( \mathbb{H}_0 \) and \( \langle \cdot, \cdot \rangle \) denotes the induced duality between the spaces \( V \) and its dual \( V' \) as well as \( \tilde{L}^p \) and its dual \( \tilde{L}' \), where \( \frac{1}{p} + \frac{1}{p'} = 1 \). Note that \( \mathbb{H}_0 \) can be identified with its dual \( \mathbb{H}' \). We endow the space \( V \cap \tilde{L}^p \) with the norm \( \|X\|_V + \|X\|_{\tilde{L}^p} \), for \( X \in V \cap \tilde{L}^p \) and its dual \( V' + \tilde{L}' \) with the norm

\[
\inf \left\{ \max \left( \|Y_1\|_{V'}, \|Y_1\|_{\tilde{L}'_p} \right) : Y = Y_1 + Y_2, Y_1 \in V', Y_2 \in \tilde{L}'_p \right\}.
\]

Furthermore, we have the continuous embedding \( V \cap \tilde{L}^p \hookrightarrow H \hookrightarrow V' + \tilde{L}' \).

2.2. Linear operator. Let us define

\[
\begin{align*}
AX & := -P_\mathbb{H}\Delta X, \quad X \in D(A), \\
D(A) & := V \cap H^2(\mathcal{O}).
\end{align*}
\]

It can be easily seen that the operator \( A \) is a non-negative self-adjoint operator in \( H \) with \( V = D(A^{1/2}) \) and

\[
\langle AX, X \rangle = \|X\|_V^2, \quad \text{for all } X \in V, \quad \text{so that } \|AX\|_{V'} \leq \|X\|_V.
\]

For a bounded domain \( \mathcal{O} \), the operator \( A \) is invertible and its inverse \( A^{-1} \) is bounded, self-adjoint and compact in \( H \). Thus, using the spectral theorem, the spectrum of \( A \) consists of an infinite sequence \( 0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots \), with \( \lambda_k \to \infty \) as \( k \to \infty \) of eigenvalues. Moreover, there exists an orthogonal basis \( \{e_k\}_{k=1}^\infty \) of \( H \) consisting of eigenvectors of \( A \) such that \( Ae_k = \lambda_k e_k \), for all \( k \in \mathbb{N} \). We know that \( X \) can be expressed as \( X = \sum_{k=1}^\infty \langle X, e_k \rangle e_k \) and \( AX = \sum_{k=1}^\infty \lambda_k \langle X, e_k \rangle e_k \), for all \( X \in D(A) \). Thus, it is immediate that

\[
\|\nabla X\|_H^2 = \langle AX, X \rangle = \sum_{k=1}^\infty \lambda_k |\langle X, e_k \rangle|^2 \geq \lambda_1 \sum_{k=1}^\infty |\langle X, e_k \rangle|^2 = \lambda_1 \|X\|_H^2,
\]

which is the Poincaré inequality.

2.3. Bilinear operator. Let us define the trilinear form \( b(\cdot, \cdot, \cdot) : V \times V \times V \to \mathbb{R} \) by

\[
b(X, Y, Z) = \int_\mathcal{O} (X(x) \cdot \nabla)Y(x) : Z(x)dx = \sum_{i,j=1}^n \int_\mathcal{O} X_i(x) \frac{\partial Y_j(x)}{\partial x_i} Z_j(x)dx.
\]

If \( X, Y \) are such that the linear map \( b(X, Y, \cdot) \) is continuous on \( V \), the corresponding element of \( V' \) is denoted by \( B(X, Y) \). We also denote \( B(X) = B(X, X) = P_\mathbb{H}(X \cdot \nabla)X \). An integration by parts yields

\[
\begin{align*}
b(X, Y, Y) & = 0, \quad \text{for all } X, Y \in V, \\
b(X, Y, Z) & = -b(X, Z, Y), \quad \text{for all } X, Y, Z \in V.
\end{align*}
\]

In the trilinear form, an application of Hölder’s inequality yields

\[
|b(X, Y, Z)| = |b(X, Z, Y)| \leq \|X\|_{\tilde{L}^{r+1}} \|Y\|_{\tilde{L}^{r+1}} \|Z\|_V,
\]
for all $X \in \mathbb{V} \cap \tilde{\mathbb{L}}^{r+1}$, $Y \in \mathbb{V} \cap \tilde{\mathbb{L}}^{2(r+1)}$, and $Z \in \mathbb{V}$, so that we get

$$
\|B(X, Y)\|_{\mathbb{V}'} \leq \|X\|_{\tilde{\mathbb{L}}^{r+1}} \|Y\|_{\tilde{\mathbb{L}}^{2(r+1)}}. \tag{2.4}
$$

Hence, the trilinear map $b : \mathbb{V} \times \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ has a unique extension to a bounded trilinear map from $(\mathbb{V} \cap \tilde{\mathbb{L}}^{r+1}) \times (\mathbb{V} \cap \tilde{\mathbb{L}}^{2(r+1)}) \times \mathbb{V}$ to \(\mathbb{R}\). It can also be seen that $B$ maps $\mathbb{V} \cap \tilde{\mathbb{L}}^{r+1}$ into $\mathbb{V} + \tilde{\mathbb{L}}^{r+1}$ and using interpolation inequality, we get

$$
|\langle B(X, X), Y \rangle| = |b(X, Y, X)| \leq \|X\|_{\tilde{\mathbb{L}}^{r+1}} \|X\|_{\tilde{\mathbb{L}}^{2(r+1)}} \|Y\|_{\mathbb{V}} \leq \|X\|^{\frac{r+1}{2(r+1)}}_{\tilde{\mathbb{L}}^{r+1}} \|X\|^{\frac{r-3}{2}}_{\tilde{\mathbb{L}}^{r+1}} \|Y\|_{\mathbb{V}}, \tag{2.5}
$$

for all $Y \in \mathbb{V} \cap \tilde{\mathbb{L}}^{r+1}$. Thus, we have

$$
\|B(X)\|_{\mathbb{V}' + \tilde{\mathbb{L}}^{r+1}} \leq \|X\|^{\frac{r+1}{2(r+1)}}_{\tilde{\mathbb{L}}^{r+1}} \|X\|^{\frac{r-3}{2(r+1)}}_{\tilde{\mathbb{L}}^{r+1}}, \tag{2.6}
$$

for $r \geq 3$.

For $n = 2$ and $r \in [1, 3]$, using Hölder’s and Ladyzhenskaya’s inequalities, we obtain

$$
|\langle B(X, Y), Z \rangle| = |b(X, Y, Z)| \leq \|X\|_{\tilde{\mathbb{L}}^{r+1}} \|Y\|_{\tilde{\mathbb{L}}^{r+1}} \|Z\|_{\mathbb{V}},
$$

for all $X, Y \in \tilde{\mathbb{L}}^{4}$ and $Z \in \mathbb{V}$, so that we get $\|B(X, Y)\|_{\mathbb{V}'} \leq \|X\|_{\tilde{\mathbb{L}}^{r+1}} \|Y\|_{\tilde{\mathbb{L}}^{r+1}}$. Furthermore, we have

$$
\|B(X, X)\|_{\mathbb{V}} \leq \|X\|_{\tilde{\mathbb{L}}^{r+1}}^2 \leq \sqrt{2} \|X\|_{\mathbb{V}} \|X\|_{\mathbb{V}} \leq \sqrt{\frac{2}{\lambda_1}} \|X\|_{\tilde{\mathbb{L}}^{r+1}}^2,
$$

for all $X \in \mathbb{V}$.

2.4. **Nonlinear operator.** Let us now consider the operator $C(X) := \text{P}_{\mathbb{H}}(|X|^{1-r}X)$. It is immediate that $\langle C(X), X \rangle = \|X\|^{r+1}_{\tilde{\mathbb{L}}^{r+1}}$. For any $r \in [1, \infty)$, we have

$$
\langle \text{P}_{\mathbb{H}}(|X|^{1-r}) - \text{P}_{\mathbb{H}}(|Y|^{1-r}), X - Y \rangle
\begin{align*}
&= \int_{\mathbb{O}} (X(x)|X(x)|^{r-1} - Y(x)|Y(x)|^{r-1}) \cdot (X(x) - Y(x)) \, dx \\
&= \int_{\mathbb{O}} (|X(x)|^{r+1} - |X(x)|^{r-1}X(x) \cdot Y(x) - |Y(x)|^{r-1}X(x) \cdot Y(x) + |Y(x)|^{r+1}) \, dx \\
&\geq \int_{\mathbb{O}} (|X(x)|^{r+1} - |X(x)|^{r-1}|Y(x)| - |Y(x)|^{r-1}|X(x)| + |Y(x)|^{r+1}) \, dx \\
&= \int_{\mathbb{O}}(|X(x)|^{r} - |Y(x)|^{r})(|X(x)| - |Y(x)|) \, dx \geq 0. \tag{2.7}
\end{align*}
$$

Furthermore, we find

$$
\langle \text{P}_{\mathbb{H}}(|X|^{1-r}) - \text{P}_{\mathbb{H}}(|Y|^{1-r}), X - Y \rangle
\begin{align*}
&= \langle |X|^{1-r}, |X - Y|^2 \rangle + \langle |Y|^{1-r}, |X - Y|^2 \rangle + \langle Y|X|^{1-r} - Y|Y|^{1-r}, X - Y \rangle \\
&= \|X|^{\frac{r-1}{2}}(X - Y)\|^2_{\mathbb{H}} + \|Y|^{\frac{r-1}{2}}(X - Y)\|^2_{\mathbb{H}} \\
&\quad + \langle X \cdot Y, |X|^{1-r} + |Y|^{1-r} \rangle - \langle |X|^2, |Y|^{1-r} \rangle - \langle |Y|^2, |X|^{1-r} \rangle. \tag{2.8}
\end{align*}
$$

But, we know that

$$
\langle X \cdot Y, |X|^{1-r} + |Y|^{1-r} \rangle - \langle |X|^2, |Y|^{1-r} \rangle - \langle |Y|^2, |X|^{1-r} \rangle
\begin{align*}
&= -\frac{1}{2}\|X|^{\frac{r-1}{2}}(X - Y)\|^2_{\mathbb{H}} - \frac{1}{2}\|Y|^{\frac{r-1}{2}}(X - Y)\|^2_{\mathbb{H}} + \frac{1}{2}\langle (|X|^{1-r} - |Y|^{1-r}), (|X|^2 - |Y|^2) \rangle
\end{align*}
$$
From (2.8), we finally have 
Lemma 2.3 (Theorem 2.3, [59])

\[ G(r) \]

for Definition 2.1 
for the hemicontinuity properties of the linear and nonlinear operators.

Monotonicity.

Combining (2.9) and (2.10), we obtain

\[ \langle P_\mathcal{H}(X|X|^{-1}) - P_\mathcal{H}(Y|Y|^{-1}), X - Y \rangle \geq \frac{1}{2} \|X|^{-\frac{1}{r}}(X - Y)\|_\mathcal{H}^2 + \frac{1}{2} \|Y|^{-\frac{1}{r}}(X - Y)\|_\mathcal{H}^2 \geq 0, \]

for \( r \geq 1 \). It is important to note that

\[ \|X - Y\|_{L^r+1} = \int_\mathcal{O} |X(x) - Y(x)|^{-1} |X(x) - Y(x)|^2 \, dx \]

\[ \leq 2^{r-2} \int_\mathcal{O} (|X(x)|^{-1} + |Y(x)|^{-1}) |X(x) - Y(x)|^2 \, dx \]

\[ \leq 2^{r-2} \|X|^{-\frac{1}{r}}(X - Y)\|_{L^2}^2 + 2^{r-2} \|Y|^{-\frac{1}{r}}(X - Y)\|_{L^2}^2. \]

(2.10)

Combining (2.9) and (2.10), we obtain

\[ \langle C(X) - C(Y), X - Y \rangle \geq \frac{1}{2^{r-1}} \|X - Y\|_{L^r+1}^{r+1}, \]

for \( r \geq 1 \).

2.5. Monotonicity. In this subsection, we discuss about the monotonicity as well as the hemicontinuity properties of the linear and nonlinear operators.

Definition 2.1 ([2]). Let \( \mathcal{X} \) be a Banach space and let \( \mathcal{X}' \) be its topological dual. An operator \( G : D \to \mathcal{X}' \), \( D = D(G) \subset \mathcal{X} \) is said to be monotone if

\[ \langle G(x) - G(y), x - y \rangle \geq 0, \quad \text{for all } x, y \in D. \]

The operator \( G(\cdot) \) is said to be hemicontinuous, if for all \( x, y \in \mathcal{X} \) and \( w \in \mathcal{X}' \),

\[ \lim_{\lambda \to 0} \langle G(x + \lambda y), w \rangle = \langle G(x), w \rangle. \]

The operator \( G(\cdot) \) is called demicontinuous, if for all \( x \in D \) and \( y \in \mathcal{X} \), the functional \( x \mapsto \langle G(x), y \rangle \) is continuous, or in other words, \( x_k \to x \) in \( \mathcal{X} \) implies \( G(x_k) \rightharpoonup G(x) \) in \( \mathcal{X}' \). Clearly demicontinuity implies hemicontinuity.

Lemma 2.2 (Theorem 2.2, [59]). Let \( X, Y \in \mathcal{V} \cap \mathcal{L}^{r+1} \), for \( r > 3 \). Then, for the operator \( G(X) = \mu AX + B(X) + \beta C(X) \), we have

\[ \langle (G(X) - G(Y), X - Y) + \eta \|X - Y\|_{\mathcal{H}}^2 \geq 0, \]  

(2.12)

where \( \eta = \frac{2^{r-2}}{2\mu(r-1)} \left( \frac{2^{r-1}}{2\mu(r-1)} \right)^{\frac{2}{r}}. \) That is, the operator \( G + \eta I \) is a monotone operator from \( \mathcal{V} \cap \mathcal{L}^{r+1} \) to \( \mathcal{V}' + \mathcal{L}^{\frac{2}{r+1}} \).

Lemma 2.3 (Theorem 2.3, [59]). For the critical case \( r = 3 \) with \( 2\beta \mu \geq 1 \), the operator \( G(\cdot) : \mathcal{V} \cap \mathcal{L}^{r+1} \to \mathcal{V}' + \mathcal{L}^{\frac{2}{r+1}} \) is globally monotone, that is, for all \( X, Y \in \mathcal{V} \), we have

\[ \langle G(X) - G(Y), X - Y \rangle \geq 0. \]

(2.13)
Lemma 2.4 (Remark 2.4, [39]). Let \( n = 2, r \in [1, 3] \) and \( X, Y \in \mathbb{V} \). Then, for the operator \( G(X) = \mu AX + B(X) + \beta C(X) \), we have

\[
\langle (G(X) - G(Y), X - Y) + \frac{27}{32\mu^2} N^4 \|X - Y\|_H^2 \rangle \geq 0,
\]

for all \( Y \in \mathbb{B}_N \), where \( \mathbb{B}_N \) is an \( \tilde{L}^4 \)-ball of radius \( N \), that is, \( \mathbb{B}_N := \{ z \in \tilde{L}^4 : \|z\|_{\tilde{L}^4} \leq N \} \).

Lemma 2.5 (Lemma 2.5, [59]). The operator \( G : \mathbb{V} \cap \tilde{L}^{r+1} \to \mathbb{V}' + \tilde{L}^{r+1} \) is demicontinuous.

3. Stochastic Coupled Convective Brinkman-Forchheimer Equations

Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a complete probability space equipped with an increasing family of sub-sigma fields \( \{\mathcal{F}_t\}_{0 \leq t \leq T} \) of \( \mathcal{F} \) satisfying:

(i) \( \mathcal{F}_0 \) contains all elements \( F \in \mathcal{F} \) with \( \mathbb{P}(F) = 0 \),

(ii) \( \mathcal{F}_t = \mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s \), for \( 0 \leq t \leq T \).

In this section, we consider the stochastic coupled convective Brinkman-Forchheimer equations perturbed by multiplicative Gaussian noise. On taking orthogonal projection \( P_H \) onto the first two equations in (1.2), we obtain

\[
\begin{align*}
\frac{dX_t^{\varepsilon, \delta}}{dt} &= -[\mu AX_t^{\varepsilon, \delta} + B(X_t^{\varepsilon, \delta}) + \alpha X_t^{\varepsilon, \delta} + \beta C(X_t^{\varepsilon, \delta})]dt + \sqrt{\varepsilon} \sigma_1(X_t^{\varepsilon, \delta})Q_1^{1/2} dW_t, \\
\frac{dY_t^{\varepsilon, \delta}}{dt} &= -[\mu AY_t^{\varepsilon, \delta} + \alpha Y_t^{\varepsilon, \delta} + \beta C(Y_t^{\varepsilon, \delta})]dt + \frac{1}{\sqrt{\delta}} \sigma_2(X_t^{\varepsilon, \delta}, Y_t^{\varepsilon, \delta})Q_2^{1/2} dW_t, \\
X_0^{\varepsilon, \delta} &= x, \quad Y_0^{\varepsilon, \delta} = y,
\end{align*}
\]

(3.1)

where \( Q_i's \), for \( i = 1, 2 \) are positive symmetric trace class operators in \( H \) and \( W_t \) is an \( H \)-valued standard cylindrical Wiener process. Strictly speaking, one has to use \( P_H F, P_H G, P_H \sigma_1 \) and \( P_H \sigma_2 \) instead of \( F, G, \sigma_1 \) and \( \sigma_2 \) in (3.1), and for simplicity of notations we are keeping them as it is. Since \( Q_1 \) and \( Q_2 \) are trace class operators, the embedding of \( Q_1^{1/2} H \) in \( H \) is Hilbert-Schmidt, for \( i = 1, 2 \). Let \( \mathcal{L}(H; H) \) denotes the space of all bounded linear operators on \( H \) and \( \mathcal{L}_2(H; H) \) denotes the space of all Hilbert-Schmidt operators from \( H \) to \( H \). The space \( \mathcal{L}_2(H; H) \) is a Hilbert space equipped with the norm \( \|\Psi\|^2_{\mathcal{L}_2} = \text{Tr}(\Psi\Psi^*) = \sum_{k=1}^{\infty} \|\Psi^*e_k\|^2_H \) and inner product \( (\Psi, \Phi)_{\mathcal{L}_2} = \text{Tr}(\Psi\Phi^*) = \sum_{k=1}^{\infty} (\Phi^*e_k, \Psi^*e_k) \). For more details, the interested readers are referred to see [13].

We need the following Assumptions on \( F, G, \sigma_1 \) and \( \sigma_2 \) to obtain our main results (see [62, 75] also).

Assumption 3.1. The functions \( F, G : H \times H \to H, \sigma_1 Q_1^{1/2} : H \to \mathcal{L}_2(H; H) \) and \( \sigma_2 Q_2^{1/2} : H \times H \to \mathcal{L}_2(H; H) \) satisfy the following Assumptions:

(A1) The functions \( F, G, \sigma_1, \sigma_2 \) are Lipschitz continuous, that is, there exist positive constants \( C, L_G \) and \( L_{\sigma_2} \) such that for any \( x_1, x_2, y_1, y_2 \in H \), we have

\[
\|F(x_1, y_1) - F(x_2, y_2)\|_H \leq C(\|x_1 - x_2\|_H + \|y_1 - y_2\|_H),
\]

\[
\|G(x_1, y_1) - G(x_2, y_2)\|_H \leq C\|x_1 - x_2\|_H + L_G\|y_1 - y_2\|_H,
\]

\[
\|\sigma_1(x_1) - \sigma_1(x_2)\| Q_1^{1/2} \leq C\|x_1 - x_2\|_H,
\]

\[
\|\sigma_2(x_1, y_1) - \sigma_2(x_2, y_2)\| Q_2^{1/2} \leq C\|x_1 - x_2\|_H + L_{\sigma_2}\|y_1 - y_2\|_H.
\]
(A2) The function $\sigma_2$ grows linearly in $x$, but is bounded in $y$, that is, there exists a constant $C > 0$ such that
\[
\sup_{y \in H} \|\sigma_2(x, y)Q_2^{1/2}\|_{L_2} \leq C(1 + \|x\|_H), \quad \text{for all } x \in H.
\]

(A3) The Brinkman coefficient $\mu > 0$, Darcy coefficient $\alpha > 0$, the smallest eigenvalue $\lambda_1$ of the Stokes operator and the Lipschitz constants $L_G$ and $L_{\sigma_2}$ satisfy
\[
\mu \lambda_1 + 2\alpha - 2L_G - 2L_{\sigma_2}^2 > 0.
\]

(A4) $\lim_{\varepsilon, \delta \to 0} \delta(\varepsilon) = 0$ and $\lim_{\varepsilon, \delta \to 0} \delta = 0$.

Let us now provide the definition of a unique global strong solution in the probabilistic sense to the system (3.1).

**Definition 3.2** (Global strong solution). Let $(x, y) \in H \times H$ be given. An $H \times H$-valued $(\mathcal{F}_t)_{t \geq 0}$-adapted stochastic process $(X_t^{\varepsilon, \delta}, Y_t^{\varepsilon, \delta})$ is called a strong solution to the system (3.1) if the following conditions are satisfied:

(i) the process
\[
(X_t^{\varepsilon, \delta}, Y_t^{\varepsilon, \delta}) \in L^2(\Omega; L^\infty(0, T; H) \cap L^2(0, T; V)) \cap L^{r+1}(\Omega; L^{r+1}(0, T; \tilde{L}^{r+1}))
\]
\[
\quad \times L^2(\Omega; L^\infty(0, T; H) \cap L^2(0, T; V)) \cap L^{r+1}(\Omega; L^{r+1}(0, T; \tilde{L}^{r+1}))
\]
and $(X_t^{\varepsilon, \delta}, Y_t^{\varepsilon, \delta})$ has a $(V \cap \tilde{L}^{r+1}) \times (V \cap \tilde{L}^{r+1})$-valued modification, which is progressively measurable with continuous paths in $H \times H$ and
\[
(X_t^{\varepsilon, \delta}, Y_t^{\varepsilon, \delta}) \in C([0, T]; H) \cap L^2(0, T; V) \cap L^{r+1}(0, T; \tilde{L}^{r+1})
\]
\[
\quad \times C([0, T]; H) \cap L^2(0, T; V) \cap L^{r+1}(0, T; \tilde{L}^{r+1}), \quad \mathbb{P}\text{-a.s.}
\]

(ii) the following equality holds for every $t \in [0, T]$, as an element of $(V' + \tilde{L}^{r+1}) \times (V' + \tilde{L}^{r+1})$, $\mathbb{P}$-a.s.:

\[
\begin{aligned}
(X_t^{\varepsilon, \delta}) &= x - \int_0^t [\mu A X_s^{\varepsilon, \delta} + \alpha X_s^{\varepsilon, \delta} + B(X_s^{\varepsilon, \delta}) + \beta C(X_s^{\varepsilon, \delta}) - F(X_s^{\varepsilon, \delta}, Y_s^{\varepsilon, \delta})]ds \\
&\quad + \sqrt{\varepsilon} \int_0^t \sigma_1(X_s^{\varepsilon, \delta})Q_1^{1/2}dW_s,
\end{aligned}
\]

\[
\begin{aligned}
(Y_t^{\varepsilon, \delta}) &= y - \frac{1}{\delta} \int_0^t [\mu A Y_s^{\varepsilon, \delta} + \alpha Y_s^{\varepsilon, \delta} + \beta C(Y_s^{\varepsilon, \delta}) - G(X_s^{\varepsilon, \delta}, Y_s^{\varepsilon, \delta})]ds \\
&\quad + \frac{1}{\sqrt{\delta}} \int_0^t \sigma_2(X_s^{\varepsilon, \delta}, Y_s^{\varepsilon, \delta})Q_2^{1/2}dW_s,
\end{aligned}
\]

(iii) the following Itô formula (energy equality) holds true:
\[
\begin{aligned}
\|X_t^{\varepsilon, \delta}\|_H^2 + &\|Y_t^{\varepsilon, \delta}\|_H^2 + 2\mu \int_0^t \left(\|X_s^{\varepsilon, \delta}\|_V^2 + \frac{1}{\delta} \|Y_s^{\varepsilon, \delta}\|_V^2\right)ds + 2\alpha \int_0^t \left(\|X_s^{\varepsilon, \delta}\|_H^2 + \frac{1}{\delta} \|Y_s^{\varepsilon, \delta}\|_H^2\right)ds \\
&+ 2\beta \int_0^t \left(\|X_s^{\varepsilon, \delta}\|_{L^{r+1}}^2 + \frac{1}{\delta} \|Y_s^{\varepsilon, \delta}\|_{L^{r+1}}^2\right)ds \\
= &\|x\|_H^2 + \|y\|_H^2 + 2 \int_0^t (F(X_s^{\varepsilon, \delta}, Y_s^{\varepsilon, \delta}), X_s^{\varepsilon, \delta})ds + \frac{2}{\delta} \int_0^t (G(X_s^{\varepsilon, \delta}, Y_s^{\varepsilon, \delta}), X_s^{\varepsilon, \delta})ds
\end{aligned}
\]
The inner product and norm on this Hilbert space by strong solution to the system (3.1). For convenience, we make use of the following simplified notations. Let us define

Global strong solution.

Definition 3.3. A strong solution \((X_t^{\varepsilon,\delta}, \tilde{Y}_t^{\varepsilon,\delta})\) to (3.1) is called a pathwise unique strong solution if \((\tilde{X}_t^{\varepsilon,\delta}, \tilde{Y}_t^{\varepsilon,\delta})\) is an another strong solution, then

\[
P\left\{ \omega \in \Omega : (X_t^{\varepsilon,\delta}, Y_t^{\varepsilon,\delta}) = (\tilde{X}_t^{\varepsilon,\delta}, \tilde{Y}_t^{\varepsilon,\delta}), \text{ for all } t \in [0, T] \right\} = 1.
\]

Remark 3.4. For \(n = 2\) and \(r \in [1, 3]\), making use of the Gagliardo-Nirenberg interpolation inequality, we know that \(C([0, T]; H) \cap L^2(0, T; \mathbb{V}) \subset L^{r+1}(0, T; \tilde{L}^{r+1})\), and hence we get \(C([0, T]; H) \cap L^2(0, T; \mathbb{V}) \cap L^{r+1}(0, T; \tilde{L}^{r+1}) = C([0, T]; H) \cap L^2(0, T; \mathbb{V})\).

3.1. Global strong solution. In this section, we discuss the existence and uniqueness of strong solution to the system (3.1). For convenience, we make use of the following simplified notations. Let us define \(\mathcal{H} := \mathbb{H} \times \mathbb{H}\). For any \(U = (x_1, x_2), V = (y_1, y_2) \in \mathcal{H}\), we denote the inner product and norm on this Hilbert space by

\[
(U, V) = (x_1, y_1) + (x_2, y_2), \quad \|U\|_{\mathcal{H}} = \sqrt{(U, U)} = \sqrt{\|x_1\|_H^2 + \|x_2\|_H^2}. \quad (3.4)
\]

In a similar way, we define \(\mathcal{V} := \mathbb{V} \times \mathbb{V}\). The inner product and norm on this Hilbert space is defined by

\[
(U, V)_{\mathcal{V}} = (\nabla x_1, \nabla y_1) + (\nabla x_2, \nabla y_2), \quad \|U\|_{\mathcal{V}} = \sqrt{(U, U)_{\mathcal{V}}} = \sqrt{\|
abla x_1\|_{H^1}^2 + \|
abla x_2\|_{H^1}^2}. \quad (3.5)
\]

for all \(U, V \in \mathcal{V}\). We denote \(\mathcal{V}'\) as the dual of \(\mathcal{V}\). We define the space \(\tilde{\mathcal{V}}^{r+1} := \tilde{L}^{r+1} \times \tilde{L}^{r+1}\) with the norm given by

\[
\|U\|_{\tilde{\mathcal{V}}^{r+1}} = \left\{ \|x_1\|_{L^{r+1}}^{r+1} + \|x_2\|_{L^{r+1}}^{r+1} \right\}^{\frac{1}{r+1}},
\]

for all \(U \in \tilde{\mathcal{V}}^{r+1}\). We represent the duality pairing between \(\mathcal{V}\) and its dual \(\mathcal{V}'\), \(\mathcal{L}^{r+1}\) and its dual \(\mathcal{L}^{r+1}\), and \(\mathcal{V} \cap \mathcal{L}^{r+1}\) and its dual \(\mathcal{V}' + \mathcal{L}^{r+1}\) as \((\cdot, \cdot)\). Note that we have the Gelfand
triple $\mathcal{V} \cap \Sigma^{r+1} \subset \mathcal{H} \subset \mathcal{V}' + \Sigma^{r+1}$. Let $Q = (Q_1, Q_2)$ be a positive symmetric trace class operator on $\mathcal{H}$. Let us now rewrite the system (3.1) for $Z^{\varepsilon, \delta}_t = (X^{\varepsilon, \delta}_t, Y^{\varepsilon, \delta}_t)$ as

$$
\begin{aligned}
&\quad dZ^{\varepsilon, \delta}_t = -\left[\mu A Z^{\varepsilon, \delta}_t + F(Z^{\varepsilon, \delta}_t)\right]dt + \tilde{\sigma}(Z^{\varepsilon, \delta}_t)Q^{1/2}dW_t, \\
&Z^{\varepsilon, \delta}_0 = (x, y) \in \mathcal{H},
\end{aligned}
$$

(3.6)

where

$$
\begin{aligned}
\tilde{A}Z^{\varepsilon, \delta} &= \left(AX^{\varepsilon, \delta} - \frac{1}{\delta}AY^{\varepsilon, \delta}\right), \\
\tilde{F}(Z^{\varepsilon, \delta}) &= \left(B(X^{\varepsilon, \delta}) + \alpha X^{\varepsilon, \delta} + \beta C(X^{\varepsilon, \delta}) - F(X^{\varepsilon, \delta}, Y^{\varepsilon, \delta}), \frac{\alpha}{\delta}Y^{\varepsilon, \delta} + \frac{\beta}{\delta}C(Y^{\varepsilon, \delta}) - \frac{1}{\delta}G(X^{\varepsilon, \delta}, Y^{\varepsilon, \delta})\right), \\
\tilde{\sigma}(Z^{\varepsilon, \delta}) &= \left(\sqrt{\tilde{\sigma}_1(X^{\varepsilon, \delta}), \frac{1}{\delta}\tilde{\sigma}_2(X^{\varepsilon, \delta}, Y^{\varepsilon, \delta})}\right).
\end{aligned}
$$

Note that the mappings $\tilde{A} : \mathcal{V} \to \mathcal{V}'$ and $\tilde{F} : \mathcal{V} \cap \Sigma^{r+1} \to \mathcal{V}' + \Sigma^{r+1}$ are well defined. It can be easily seen that the operator $\tilde{\sigma}Q^{1/2} : \mathcal{H} \to L_2(\mathcal{H}; \mathcal{H})$, where $L_2(\mathcal{H}; \mathcal{H})$ is the space of all Hilbert-Schmidt operators from $\mathcal{H}$ to $\mathcal{H}$ with the norm

$$
\|\tilde{\sigma}(z)Q^{1/2}\|_{L_2} = \sqrt{\|\tilde{\sigma}_1(x)Q_1^{1/2}\|^2_{L_2} + \|\tilde{\sigma}_2(x, y)Q_2^{1/2}\|^2_{L_2}}, \text{ for } z = (x, y) \in \mathcal{H}.
$$

(3.7)

The following Theorem on the existence and uniqueness of strong solution to the system (3.6) can be proved in a similar way as in Theorem 3.4, [62].

**Theorem 3.5 (Theorem 3.4, [62]).** Let $(x, y) \in \mathcal{H}$ be given. Then for $n = 2, r \in [1, \infty)$ and $n = 3, r \in [3, \infty)$ ($2\beta \mu \geq 1$, for $r = 3$), there exists a pathwise unique strong solution $Z^{r, \delta}$ to the system (3.6) such that

$$
Z^{r, \delta} \in L_2(\Omega; L_2(0, T; \mathcal{H}) \cap L_2^r(0, T; \mathcal{V})) \cap L^{r+1}(\Omega; L^{r+1}(0, T; \Sigma^{r+1})),
$$

and a continuous modification with trajectories in $\mathcal{H}$ and $Z^{r, \delta} \in C([0, T]; \mathcal{H}) \cap L_2^r(0, T; \mathcal{V}) \cap L^{r+1}(0, T; \Sigma^{r+1}), \mathbb{P}$-a.s.

4. Large Deviation Principle

In this section, we establish a Wentzell-Freidlin (see [29]) type large deviation principle for the two-time-scale SCBF equations (3.1) using the well known results of Varadhan as well as Bryc (see [78, 23]) and Budhiraja-Dupuis (see [9]). A Wentzell-Freidlin type large deviation principle for the two-time-scale one-dimensional stochastic Burgers equation is established in [75]. Interested readers are referred to see [61] (LDP for two and three dimensional SCBF equations), [74] (LDP for the 2D stochastic Navier-Stokes equations), [17] (LDP for some 2D hydrodynamic systems), [63] (LDP for the 2D Oldroyd fluids), [61] (LDP for the 2D and 3D SCBF equations) for application of such methods to various hydrodynamic models.
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with an increasing family $\{\mathcal{F}_t\}_{t \geq 0}$ of the sub $\sigma$-fields of $\mathcal{F}$ satisfying the usual conditions. We consider the following two-time-scale SCBF system:

\[
\begin{cases}
    dX_t^{\varepsilon, \delta} = -[\mu A X_t^{\varepsilon, \delta} + B(X_t^{\varepsilon, \delta}) + \alpha X_t^{\varepsilon, \delta} + \beta C(X_t^{\varepsilon, \delta}) - F(X_t^{\varepsilon, \delta}, Y_t^{\varepsilon, \delta})]dt + \sqrt{\varepsilon} \sigma_1(X_t^{\varepsilon, \delta}, Y_t^{\varepsilon, \delta})Q_{1}^{1/2}dW_t, \\
    dY_t^{\varepsilon, \delta} = -[\frac{1}{\delta} \mu A Y_t^{\varepsilon, \delta} + \alpha Y_t^{\varepsilon, \delta} + \beta C(Y_t^{\varepsilon, \delta}) - G(X_t^{\varepsilon, \delta}, Y_t^{\varepsilon, \delta})]dt + \frac{1}{\sqrt{\delta}} \sigma_2(X_t^{\varepsilon, \delta}, Y_t^{\varepsilon, \delta})Q_{2}^{1/2}dW_t,
\end{cases}
\]

\[X_0^{\varepsilon, \delta} = x, \quad Y_0^{\varepsilon, \delta} = y, \quad (4.1)\]

for some fixed point $(x, y)$ in $\mathbb{H} \times \mathbb{H}$. From Theorem 3.3 (see Theorem 3.4, [59] also), it is known that the system (4.1) has a unique pathwise strong solution $(X_t^{\varepsilon, \delta}, Y_t^{\varepsilon, \delta})$ with $\mathcal{F}_t$-adapted paths (that is, for any $t \in [0, T]$ and $x \in \mathcal{O}$, $(X_t^{\varepsilon, \delta}(x), Y_t^{\varepsilon, \delta}(x))$ is $\mathcal{F}_t$-measurable) in $C([0, T]; \mathbb{H}) \cap L^2(0, T; \mathcal{V}) \cap L^{r+1}(0, T; \mathbb{L}^{r+1}) \times C([0, T]; \mathbb{H}) \cap L^2(0, T; \mathcal{V}) \cap L^{r+1}(0, T; \mathbb{L}^{r+1})$, $\mathbb{P}$-a.s. Moreover, such a strong solution satisfies the energy equality (Itô’s formula) given in (3.3).

As the parameter $\varepsilon \downarrow 0$, the slow component $X_t^{\varepsilon, \delta}$ of (4.1) tends to the solution of the following deterministic averaged system:

\[
\begin{cases}
    d\bar{X}_t = -[\mu A \bar{X}_t + B(\bar{X}_t) + \alpha \bar{X}_t + \beta C(\bar{X}_t)]dt + \bar{F}(\bar{X}_t)dt, \\
    \bar{X}_0 = x,
\end{cases}
\]

(4.2)

with the average

\[\bar{F}(x) = \int_{\mathbb{H}} F(x, y) \nu^x(dy), \quad x \in \mathbb{H},\]

and $\nu^x$ is the unique invariant distribution of the transition semigroup for the frozen system:

\[
\begin{cases}
    d\bar{Y}_t = -[\mu A \bar{Y}_t + \alpha \bar{Y}_t + \beta C(\bar{Y}_t) - G(x, Y_t)]dt + \sigma_2(x, Y_t)Q_{2}^{1/2}d\bar{W}_t, \\
    \bar{Y}_0 = y,
\end{cases}
\]

(4.3)

where $\bar{W}_t$ is a standard cylindrical Wiener process, which is independent of $W_t$. Note that the system (4.2) is a Lipschitz perturbation of the CBF equations (see (4.19) below). Using similar techniques as in Theorem 3.4, [59] (see [27] also), one can show that the system (4.2) has a unique weak solution in the Leray-Hopf sense, satisfying the energy equality:

\[
\begin{align*}
    \|\bar{X}_t\|_{\mathbb{H}}^2 &+ 2\mu \int_0^t \|\bar{X}_s\|_{\mathbb{H}}^2 ds + 2\alpha \int_0^t \|\bar{X}_s\|_{\mathbb{H}}^2 ds + 2\beta \int_0^t \|\bar{X}_s\|_{\mathbb{L}^{r+1}}^{r+1} ds \\
    &= \|x\|_{\mathbb{H}}^2 + 2 \int_0^t (\bar{F}(\bar{X}_s), \bar{X}_s) ds,
\end{align*}
\]

for all $t \in [0, T]$ in the Polish space $C([0, T]; \mathbb{H}) \cap L^2(0, T; \mathcal{V}) \cap L^{r+1}(0, T; \mathbb{L}^{r+1})$. The strong averaging principle states that (Theorem 1.1, [62]) for any initial values $x, y \in \mathbb{H}$, $p \geq 1$ and $T > 0$, we have

\[\lim_{\varepsilon \to 0} \mathbb{E} \left( \sup_{t \in [0, T]} \|X_t^{\varepsilon, \delta} - \bar{X}_t\|_{\mathbb{H}}^{2p} \right) = 0, \quad (4.4)\]

In this section, we investigate the large deviations of $X_t^{\varepsilon, \delta}$ from the deterministic solution $\bar{X}_t$, as $\varepsilon \downarrow 0$. 

4.1. **Frozen equation.** The frozen equation associated with the fast motion for fixed slow component \( x \in \mathbb{H} \) is given by

\[
\begin{aligned}
dY_t &= -[\mu AY_t + \alpha Y_t + \beta C(Y_t) - G(x, Y_t)]dt + \sigma_2(x, Y_t)Q_2^{1/2}d\tilde{W}_t, \\
Y_0 &= y,
\end{aligned}
\tag{4.5}
\]

where \( \tilde{W}_t \) is a cylindrical Wiener process in \( \mathbb{H} \), which is independent of \( W_t \). From the Assumption 3.1, we know that \( G(x, \cdot) \) and \( \sigma_2(x, \cdot) \) are Lipschitz continuous. Thus, one can show that for any fixed \( x \in \mathbb{H} \) and any initial data \( y \in \mathbb{H} \), there exists a unique strong solution \( Y^{x,y}_t \in L^2(\Omega; L^\infty(0, T; \mathbb{H}) \cap L^2(0, T; \mathcal{V})) \cap L^{r+1}(\Omega; L^{r+1}(0, T; \mathbb{L}^{r+1})) \) to the system (4.5) with a continuous modification with paths in \( C([0, T]; \mathbb{H}) \cap L^2(0, T; \mathcal{V}) \cap L^{r+1}(0, T; \mathbb{L}^{r+1}) \), \( \mathbb{P} \)-a.s. A proof of this result can be obtained in the same way as in Theorem 3.7, [60] by making use of the monotonicity property of the linear and nonlinear operators (see Lemmas 2.2, 2.3) as well as a stochastic generalization of the Minty-Browder technique (localized version for the case \( n = 2 \) and \( r \in [1, 3] \)). Furthermore, the strong solution satisfies the following infinite dimensional Itô formula (energy equality):

\[
\begin{aligned}
\|Y^{x,y}_t\|_H^2 + 2\mu \int_0^t |Y_s|_{\mathcal{V}}^2 ds + 2\alpha \int_0^t |Y_s|_{\mathbb{H}}^2 ds + 2\beta \int_0^t \|Y_s\|_{L^{r+1}}^{r+1} ds \\
= \|y\|_H^2 - 2\int_0^t (G(x, Y_s), Y_s) ds + \int_0^t |\sigma_2(x, Y_s)Q_2^{1/2}|_{L^2}^2 ds + 2\int_0^t (\sigma_2(x, Y_s)Q_2^{1/2}d\tilde{W}_s, Y_s),
\end{aligned}
\tag{4.6}
\]

\( \mathbb{P} \)-a.s., for all \( t \in [0, T] \). Let \( P^x_t \) be the transition semigroup associated with the process \( Y^{x,y}_t \), that is, for any bounded measurable function \( \varphi \) on \( \mathbb{H} \), we have

\[
P^x_t \varphi(y) = \mathbb{E}[\varphi(Y^{x,y}_t)], \quad y \in \mathbb{H} \quad \text{and} \quad t > 0.
\tag{4.7}
\]

For the system (4.5), the following result is available in the work [62].

**Proposition 4.1** (Proposition 4.4, [62]). For any given \( x, y \in \mathbb{H} \), there exists a unique invariant measure for the system (4.5). Furthermore, there exists a constant \( C_{\mu,\alpha,\lambda_1,L_G} > 0 \) and \( \zeta > 0 \) such that for any Lipschitz function \( \varphi : \mathbb{H} \to \mathbb{R} \), we have

\[
\left| P^x_t \varphi(y) - \int_{\mathbb{H}} \varphi(z) \nu(z) dz \right| \leq C_{\mu,\alpha,\lambda_1,L_G} (1 + \|x\|_H + \|y\|_H) e^{-t\zeta} \|\varphi\|_{\text{Lip}(\mathbb{H})},
\tag{4.8}
\]

where \( \zeta = 2\mu\lambda_1 + 2\alpha - 2L_G - L_{\sigma_2}^2 > 0 \) and \( \|\varphi\|_{\text{Lip}(\mathbb{H})} = \sup_{x,y \in \mathbb{H}} \frac{\|\varphi(x) - \varphi(y)\|}{\|x - y\|_H} \).

The interested readers are referred to see [19, 22, 28, 39], etc for more details on the invariant measures and ergodicity for the infinite dimensional dynamical systems and stochastic Navier-Stokes equations. We need the following lemma in the sequel.

**Lemma 4.2.** There exists a constant \( C > 0 \) such that for any \( x_1, x_2, y \in \mathbb{H} \), we have

\[
\sup_{t \geq 0} \mathbb{E} \left[ \|Y^{x_1,y}_t - Y^{x_2,y}_t\|_{\mathbb{H}}^2 \right] \leq C_{\mu,\alpha,\lambda_1,L_G,\sigma_2} \|x_1 - x_2\|_H^2.
\tag{4.9}
\]
Proof. We know that $W_t := Y_t^{x_1,y} - Y_t^{x_2,y}$ satisfies the following system:

$$
\begin{align*}
\begin{cases}
\quad \text{d}W_t = -[\mu A W_t + \alpha W_t + \beta (C(Y_{t}^{x_1,y}) - C(Y_{t}^{x_2,y}))] dt + [G(x_1, Y_t^{x_1,y}) - G(x_2, Y_t^{x_2,y})] dt \\
\quad + [\sigma_2(x_1, Y_t^{x_1,y}) - \sigma_2(x_2, Y_t^{x_2,y})] Q_2^{1/2} d\hat{W}_t, \\
W_0 = 0.
\end{cases}
\end{align*}
$$

(4.10)

Applying the infinite dimensional Itô formula to the process $\|W_t\|_{\mathbb{H}}^2$, we find

$$
\begin{align*}
\|W_t\|_{\mathbb{H}}^2 &+ 2\mu \int_0^t \|W_s\|_{\mathbb{H}}^2 ds + 2\alpha \int_0^t \|W_s\|_{\mathbb{H}}^2 ds + 2\beta \int_0^t \langle C(Y_{s}^{x_1,y}) - C(Y_{s}^{x_2,y}), W_s \rangle ds \\
&\leq 2 \int_0^t ([G(x_1, Y_{s}^{x_1,y}) - G(x_2, Y_{s}^{x_2,y})], W_s) ds \\
&\quad + \int_0^t \|\sigma_2(x_1, Y_{s}^{x_1,y}) - \sigma_2(x_2, Y_{s}^{x_2,y})\| Q_2^{1/2} \|_{\mathbb{L}_2}^2 ds \\
&\quad + 2 \int_0^t ([\sigma_2(x_1, Y_{s}^{x_1,y}) - \sigma_2(x_2, Y_{s}^{x_2,y})] Q_2^{1/2}, W_s), \mathbb{P}\text{-a.s.,}
\end{align*}
$$

(4.11)

for all $t \in [0, T]$. Taking expectation in (4.11) and using the fact the final term appearing the right hand side of the equality (4.11) is a martingale, we get

$$
\begin{align*}
\mathbb{E}[\|W_t\|_{\mathbb{H}}^2] &+ 2\mu \mathbb{E}\left[\int_0^t \|W_s\|_{\mathbb{H}}^2 ds\right] + 2\alpha \mathbb{E}\left[\int_0^t \|W_s\|_{\mathbb{H}}^2 ds\right] \\
&= -2\beta \mathbb{E}\left[\int_0^t \langle C(Y_{s}^{x_1,y}) - C(Y_{s}^{x_2,y}), W_s \rangle ds\right] \\
&\quad + 2\mathbb{E}\left[\int_0^t ([G(x_1, Y_{s}^{x_1,y}) - G(x_2, Y_{s}^{x_2,y})], W_s) ds\right] \\
&\quad + \mathbb{E}\left[\int_0^t \|\sigma_2(x_1, Y_{s}^{x_1,y}) - \sigma_2(x_2, Y_{s}^{x_2,y})\| Q_2^{1/2} \|_{\mathbb{L}_2}^2 ds\right],
\end{align*}
$$

(4.12)

so that using the Assumption 3.1 (A1) and (2.11), we have

$$
\begin{align*}
\frac{d}{dt}\mathbb{E}[\|W_t\|_{\mathbb{H}}^2] &\leq -(2\mu \lambda_1 + 2\alpha) \mathbb{E}[\|W_t\|_{\mathbb{H}}^2] - \frac{\beta}{2^{r-2}} \mathbb{E}\left[\|W_t\|_{\mathbb{H}}^2 \right]^{r+1} \\
&\quad + 2\mathbb{E}\left[C \|x_1 - x_2\|_{\mathbb{H}}^2 \|W_t\|_{\mathbb{H}} + L_G \|W_t\|_{\mathbb{H}}^2\right] \\
&\quad + \mathbb{E}\left[C \|x_1 - x_2\|_{\mathbb{H}}^2 + L_G \|W_t\|_{\mathbb{H}}^2\right] \\
&\leq -(\mu \lambda_1 + 2\alpha - 2L_G - 2L^2 \alpha) \mathbb{E}[\|W_t\|_{\mathbb{H}}^2] + C\left(1 + \frac{1}{\mu \lambda_1}\right) \|x_1 - x_2\|_{\mathbb{H}}^2,
\end{align*}
$$

(4.13)

for a.e. $t \in [0, T]$. Using the variation of constants formula, we further have

$$
\mathbb{E}[\|W_t\|_{\mathbb{H}}^2] \leq C\left(1 + \frac{1}{\mu \lambda_1}\right) \|x_1 - x_2\|_{\mathbb{H}}^2 \int_0^t e^{-\xi(t-s)} ds \leq \frac{C}{\xi} \left(1 + \frac{1}{\mu \lambda_1}\right) \|x_1 - x_2\|_{\mathbb{H}}^2,
$$

(4.14)
for any $t > 0$, where $\xi = \mu \lambda_1 + 2\alpha - 2L_G - 2L_{\sigma_2}^2 > 0$ and the estimate (4.9) follows. □

4.2. Preliminaries. In this subsection, we provide some preliminaries regarding the Large deviation principle (LDP). Let us denote by $\mathcal{E}$, a complete separable metric space (Polish space) with the Borel $\sigma$-field $\mathcal{B}(\mathcal{E})$.

Definition 4.3. A function $I : \mathcal{E} \to [0, \infty]$ is called a rate function if $I$ is lower semicontinuous. A rate function $I$ is called a good rate function, if for arbitrary $M \in [0, \infty)$, the level set $K_M = \{x \in \mathcal{E} : I(x) \leq M\}$ is compact in $\mathcal{E}$.

Definition 4.4 (Large deviation principle). Let $I$ be a rate function defined on $\mathcal{E}$. A family $\{X^\varepsilon : \varepsilon > 0\}$ of $\mathcal{E}$-valued random elements is said to satisfy the large deviation principle on $\mathcal{E}$ with rate function $I$, if the following two conditions hold:

(i) (Large deviation upper bound) For each closed set $F \subset \mathcal{E}:
\limsup_{\varepsilon \to 0}\varepsilon \log P(X^\varepsilon \in F) \leq -\inf_{x \in F} I(x)$.

(ii) (Large deviation lower bound) For each open set $G \subset \mathcal{E}:
\liminf_{\varepsilon \to 0}\varepsilon \log P(X^\varepsilon \in G) \geq -\inf_{x \in G} I(x)$.

Definition 4.5. Let $I$ be a rate function on $\mathcal{E}$. A family $\{X^\varepsilon : \varepsilon > 0\}$ of $\mathcal{E}$-valued random elements is said to satisfy the Laplace principle on $\mathcal{E}$ with rate function $I$, if for each real-valued, bounded and continuous function $h$ defined on $\mathcal{E}$, that is, for $h \in C_b(\mathcal{E})$,

$$\lim_{\varepsilon \to 0}\varepsilon \log \mathbb{E}\left\{\exp\left[-\frac{1}{\varepsilon} h(X^\varepsilon)\right]\right\} = -\inf_{x \in \mathcal{E}} \{h(x) + I(x)\}. \quad (4.15)$$

Lemma 4.6 (Varadhan’s Lemma, [78]). Let $\mathcal{E}$ be a Polish space and $\{X^\varepsilon : \varepsilon > 0\}$ be a family of $\mathcal{E}$-valued random elements satisfying LDP with rate function $I$. Then $\{X^\varepsilon : \varepsilon > 0\}$ satisfies the Laplace principle on $\mathcal{E}$ with the same rate function $I$.

Lemma 4.7 (Bryc’s Lemma, [23]). The Laplace principle implies the LDP with the same rate function.

It should be noted that Varadhan’s Lemma together with Bryc’s converse of Varadhan’s Lemma state that for Polish space valued random elements, the Laplace principle and the large deviation principle are equivalent.

4.3. Functional setting and Budhiraja-Dupuis LDP. In this subsection, the notation and terminology are built in order to state the large deviations result of Budhiraja and Dupuis [9] for Polish space valued random elements. Let us define

$$\mathcal{A} := \left\{\mathbb{H}\text{-valued } \{\mathcal{F}_t\}\text{-predictable processes } h \text{ such that } \int_0^T \|h(s)\|^2_{H} ds < +\infty, \mathbb{P}\text{-a.s.}\right\},$$

and

$$\mathcal{S}_M := \left\{h \in L^2(0, T; \mathbb{H}) : \int_0^T \|h(s)\|^2_{H} ds \leq M\right\}.$$ 

It is known from [10] that the space $\mathcal{S}_M$ is a compact metric space under the metric $d(h, v) = \sum_{j=1}^{\infty} \frac{1}{2^j} \left|\int_0^T (h(s) - v(s), \overline{e}_j(s))ds\right|$, where $\{\overline{e}_j\}_{j=1}^{\infty}$ are orthonormal basis of $L^2(0, T; \mathbb{H})$. Since
every compact metric space is complete, the set $S_M$ endowed with the weak topology obtained from the metric $\bar{d}$ is a Polish space. Let us now define

$$A_M = \{ h \in A : h(\omega) \in S_M, \ \mathbb{P}\text{-a.s.} \}.$$ 

Next, we state an important lemma regarding the convergence of the sequence $\int_0^t h_n(s)\,ds$, which is useful in proving compactness as well as weak convergence results.

**Lemma 4.8** (Lemma 3.2, [9]). Let $\{h_n\}$ be a sequence of elements from $A_M$, for some $0 < M < +\infty$. Let the sequence $\{h_n\}$ converges in distribution to $h$ with respect to the weak topology on $L^2(0,T;\mathbb{H})$. Then $\int_0^t h_n(s)\,ds$ converges in distribution as $C([0,T];\mathbb{H})$-valued processes to $\int_0^t h(s)\,ds$ as $n \to \infty$.

Let $\mathcal{E}$ denote a Polish space, and for $\varepsilon > 0$, let $\mathcal{G}^\varepsilon : C([0,T];\mathbb{H}) \to \mathcal{E}$ be a measurable map. Let us define

$$X^\varepsilon = \mathcal{G}^\varepsilon(W(\cdot)).$$

We are interested in the large deviation principle for $X^\varepsilon$ as $\varepsilon \to 0$.

**Hypothesis 4.9.** There exists a measurable map $\mathcal{G}^0 : C([0,T];\mathbb{H}) \to \mathcal{E}$ such that the following hold:

(i) Let $\{h^\varepsilon : \varepsilon > 0\} \subset A_M$, for some $M < +\infty$. Let $h^\varepsilon$ converge in distribution as an $S_M$-valued random element to $h$ as $\varepsilon \to 0$. Then $\mathcal{G}^\varepsilon(W(\cdot) + \frac{1}{\sqrt{\varepsilon}} \int_0^t h^\varepsilon(s)\,ds)$ converges in distribution to $\mathcal{G}^0(\int_0^t h(s)\,ds)$ as $\varepsilon \to 0$.

(ii) For every $M < +\infty$, the set

$$K_M = \left\{ \mathcal{G}^0\left(\int_0^T h(s)\,ds\right) : h \in S_M \right\}$$

is a compact subset of $\mathcal{E}$.

For each $f \in \mathcal{E}$, we define

$$I(f) := \inf\left\{ h \in L^2(0,T;\mathbb{H}) : f = \mathcal{G}^0(\int_0^T h(s)\,ds) \right\},$$

where infimum over an empty set is taken as $+\infty$. Next, we state an important result due to Budhiraja and Dupuis [9].

**Theorem 4.10** (Budhiraja-Dupuis principle, Theorem 4.4, [9]). Let $X^\varepsilon = \mathcal{G}^\varepsilon(W(\cdot))$. If $\{\mathcal{G}^\varepsilon\}$ satisfies the Hypothesis 4.9, then the family $\{X^\varepsilon : \varepsilon > 0\}$ satisfies the Laplace principle in $\mathcal{E}$ with rate function $I$ given by (4.16).

It should be noted that Hypothesis 4.9 (i) is a statement on the weak convergence of a certain family of random variables and is at the core of weak convergence approach to the study of large deviations. Hypothesis 4.9 (ii) says that the level sets of the rate function are compact.

### 4.4. LDP for SCBF equations

Let us recall that the system (4.1) has an $\mathcal{F}_t$-adapted pathwise unique strong solution $(X^\varepsilon_t, Y^\varepsilon_t)$ in the Polish space $C([0,T];\mathbb{H}) \cap L^2(0,T;\mathbb{V}) \cap L^{r+1}(0,T;\mathbb{V}) \times C([0,T];\mathbb{H}) \cap L^2(0,T;\mathbb{V}) \cap L^{r+1}(0,T;\mathbb{V})$, $\mathbb{P}$-a.s.

The solution to the first equation in (4.1) denoted by $X^\varepsilon_t$ can be written as $\mathcal{G}^\varepsilon(W(\cdot))$, for a Borel measurable function $\mathcal{G}^\varepsilon : C([0,T];\mathbb{H}) \to \mathcal{E}$, where $\mathcal{E} = C([0,T];\mathbb{H}) \cap L^2(0,T;\mathbb{V}) \cap \mathbb{P}$-a.s.
The existence and uniqueness of weak solution in the Leray-Hopf sense (satisfying the energy
Thus, it is immediate that the system (4.17) is a Lipschitz perturbation of the CBF equations.

Our main goal is to verify that such a \( G^e \) satisfies the Hypothesis 4.9. Then, applying the
Theorem 4.10, the LDP for \( \{X^{\varepsilon, \delta} : \varepsilon > 0\} \) in \( \mathcal{E} \) can be established. Let us now state
our main result on the Wentzell-Freidlin type large deviation principle for the system (3.1)
\((r \in [1, \infty]), \text{for } n = 2 \text{ and } r \in [3, \infty), \text{for } n = 3 \text{ with } 2\beta \mu > 1 \text{ for } r = 3\).

**Theorem 4.11.** Under the Assumption 3.1, \( \{X^{\varepsilon, \delta} : \varepsilon > 0\} \) obeys an LDP on \( C([0, T]; \mathbb{H}) \cap L^2(0, T; \mathbb{V}) \cap L^{r+1}(0, T; \mathbb{L}^{r+1}) \) with the rate function \( I \) defined in (4.16).

The LDP for \( \{X^{\varepsilon, \delta} : \varepsilon > 0\} \) in \( \mathcal{E} \) (Theorem 4.11) is proved in the following way. We show
the well-posedness of certain controlled deterministic and controlled stochastic equations in \( \mathcal{E} \).
These results help us to prove the two main results on the compactness of the level sets
and weak convergence of the stochastic controlled equation, which verifies the Hypothesis
4.9. We exploit the classical Khasminskii approach based on time discretization in the weak
convergence part of the Hypothesis 4.9.

4.5. **Compactness.** Let us first verify the Hypothesis 4.9 (ii) on compactness.

**Theorem 4.12.** Let \( h \in L^2(0, T; \mathbb{H}) \) and the Assumption 3.1 be satisfied. Then the following
deterministic control system:

\[
\begin{aligned}
d\tilde{X}_t^h &= -[\mu A\tilde{X}_t^h + B(\tilde{X}_t^h) + \alpha\tilde{X}_t^h + \beta C(\tilde{X}_t^h)]dt + \bar{F}(\tilde{X}_t^h)dt + \sigma_1(\tilde{X}_t^h)Q_1^{1/2}h dt, \\
\tilde{X}_0^h &= x,
\end{aligned}
\]

has a unique weak solution in \( C([0, T]; \mathbb{H}) \cap L^2(0, T; \mathbb{V}) \cap L^{r+1}(0, T; \mathbb{L}^{r+1}) \), and

\[
\sup_{h \in \mathcal{S}_M} \left\{ \sup_{t \in [0, T]} \|\tilde{X}_t^h\|^2_H + \mu \int_0^T \|\tilde{X}_t^h\|^2_H dt + \alpha \int_0^T \|\tilde{X}_t^h\|^2_{\mathbb{V}} dt + \beta \int_0^T \|\tilde{X}_t^h\|^2_{\mathbb{L}^{r+1}} dt \right\}
\leq C_{\mu, \alpha, \lambda_1, L_G, L_{\alpha_2}, M, T} \left( 1 + \|x\|^2_H \right).
\]

**Proof.** From the Assumption 3.1, we know that the operator \( \sigma_1(\cdot)Q_1^{1/2} \) is Lipschitz. Since
\( F(\cdot, \cdot) \) is Lipschitz, one can show that \( \bar{F}(\cdot) \) is Lipschitz in the following way. Using the
Assumption 3.1 (A1), (1.8) and (4.9), we have

\[
\left\| \bar{F}(x_1) - \bar{F}(x_2) \right\|_H = \left\| \int_{\mathbb{H}} F(x_1, z)\mu^{x_1}(dz) - \int_{\mathbb{H}} F(x_1, z)\mu^{x_2}(dz) \right\|_H \\
\leq \left\| \int_{\mathbb{H}} F(x_1, z)\mu^{x_1}(dz) - \mathbb{E}[F(x_1, Y_t^{x_1})] \right\|_H + \left\| \mathbb{E}[F(x_2, Y_t^{x_2})] - \int_{\mathbb{H}} F(x_1, z)\mu^{x_1}(dz) \right\|_H \\
+ \left\| \mathbb{E}[F(x_1, Y_t^{x_1})] - \mathbb{E}[F(x_2, Y_t^{x_2})] \right\|_H \\
\leq C_{\mu, \alpha, L_G}(1 + \|x_1\|_H + \|x_2\|_H + \|y\|_H) e^{-\xi T} + C(\|x_1 - x_2\|_H + \mathbb{E}[\|Y_t^{x_1} - Y_t^{x_2}\|_H]) \\
\leq C_{\mu, \alpha, L_G}(1 + \|x_1\|_H + \|x_2\|_H + \|y\|_H) e^{-\xi T} + C_{\mu, \alpha, L_G}(1 + \|x_1 - x_2\|_H).
\]

Taking \( t \to \infty \), we get

\[
\left\| \bar{F}(x_1) - \bar{F}(x_2) \right\|_H \leq C_{\mu, \alpha, L_G}(1 + \|x_1 - x_2\|_H).
\]

Thus, it is immediate that the system (4.17) is a Lipschitz perturbation of the CBF equations.
The existence and uniqueness of weak solution in the Leray-Hopf sense (satisfying the energy
equality) of the system (4.17) can be proved using the monotonicity as well as demicontinuous properties of the linear and nonlinear operators and the Minty-Browder technique as in Theorem 3.4, [59]. Thus, we need to show (4.18) only. Taking inner product with \( \bar{X}_t^h \) to the first equation in (4.17), we find

\[
\frac{1}{2} \frac{d}{dt} \| X_t^h \|_{H^2}^2 + \mu \| X_t^h \|_{V^2}^2 + \alpha \| \bar{X}_t^h \|_{L^2}^2 + \beta \| X_t^h \|_{E_{r+1}^1}^2 = -(\bar{F}(X_t^h) - \sigma_1(X_t^h)Q_1^{1/2}h_t, X_t^h). \tag{4.20}
\]

since \( B(X_t^h, \bar{X}_t^h) = 0 \). Applying Gronwall’s inequality in (4.23), we get

\[
\| X_t^h \|_{H^2}^2 \leq \left( \| x \|_{H^2}^2 + C_{\mu, \alpha, \lambda_1, L_G, L_{s_2}} t + C \int_0^t \| h_s \|_{H^2}^2 ds + C_{\mu, \alpha, \lambda_1, L_G, L_{s_2}} \right) \exp \left( C_{\mu, \alpha, \lambda_1, L_G, L_{s_2}} T + C \int_0^T \| h_t \|_{H^2}^2 dt \right), \tag{4.24}
\]

for all \( t \in [0, T] \). Thus, taking \( h \in S_M \), we finally obtain (4.18).

declarationlemmanum

Lemma 4.13. For any \( x, y \in H \), \( T > 0 \), \( \varepsilon, \Delta > 0 \) small enough, there exists a constant \( C_{\mu, \alpha, \lambda_1, L_G, T} > 0 \) such that

\[
\sup_{h \in S_M} \int_0^T \| \bar{X}_t^h - \bar{X}_{t(\Delta)}^h \|_{H^2}^2 dt \leq C_{\mu, \alpha, \beta, \lambda_1, L_G, L_{s_2}, M, T} \Delta (1 + \| x \|_{H^2}^2), \tag{4.25}
\]

where \( \ell = 3 \), for \( n = 2 \), \( r \in [1, 3) \), and \( \ell = 2 \), for \( n = 2, 3 \), \( r \in [3, \infty) \) \( (2\beta \mu > 1, \text{ for } n = r = 3) \). Here, \( t(\Delta) := \left\lfloor \frac{t}{\Delta} \right\rfloor \Delta \) and \( \lfloor \cdot \rfloor \) stands for the largest integer which is less than or equal to.

declarationproof

Proof. Using (4.18), it can be easily seen that

\[
\int_0^T \| \bar{X}_t^h - \bar{X}_{t(\Delta)}^h \|_{H^2}^2 dt \leq \int_0^\Delta \| \bar{X}_t^h - x \|_{H^2}^2 dt + \int_\Delta^T \| \bar{X}_t^h - \bar{X}_{t(\Delta)}^h \|_{H^2}^2 dt.
\]
Let us first estimate the second term from the right hand side of the equality (4.26). Note that $X^h_t - X^h_{t-\Delta}$, for $r \in [t - \Delta, t]$ satisfies the following energy equality:

$$
\|X^h_t - X^h_{t-\Delta}\|^2 = -2\mu \int_{t-\Delta}^t \langle AX^h_s, X^h_s - X^h_{t-\Delta}\rangle ds - 2\alpha \int_{t-\Delta}^t (\dot{X}^h_s, X^h_s - X^h_{t-\Delta}) ds \\
- 2\int_{t-\Delta}^t \langle B(\dot{X}^h_s), X^h_s - X^h_{t-\Delta}\rangle ds - 2\beta \int_{t-\Delta}^t \langle C(\dot{X}^h_s), \dot{X}^h_s - X^h_{t-\Delta}\rangle ds \\
- 2\int_{t-\Delta}^t (F(\dot{X}^h_s), X^h_s - X^h_{t-\Delta}) ds - 2\int_{t-\Delta}^t (\sigma_1(\dot{X}^h_s)Q_1^{1/2}h_s, X^h_s - X^h_{t-\Delta}) ds \\
=: \sum_{i=1}^6 I_i(t).
$$

(4.27)

Making use of an integration by parts, Hölder’s inequality, Fubini’s Theorem and (4.18), we estimate $\int_\Delta^T |I_1(t)| dt$ as

$$
\int_\Delta^T |I_1(t)| dt \leq 2\mu \int_\Delta^T \int_{t-\Delta}^t \|X^h_s\_V \|X^h_s - X^h_{t-\Delta}\_V ds dt \\
\leq 2\mu \left( \int_\Delta^T \int_{t-\Delta}^t \|X^h_s\_V\|^2 ds dt \right)^{1/2} \left( \int_\Delta^T \int_{t-\Delta}^t \|X^h_s - X^h_{t-\Delta}\_V\|^2 ds dt \right)^{1/2} \\
\leq 2\mu \left( \Delta \int_0^T \|X^h_t\_V\|^2 dt \right)^{1/2} \left( 2\Delta \int_0^T \|X^h_t\_V\|^2 dt \right)^{1/2} \\
\leq C_{\mu,\alpha,\lambda_1,L_G,L_{\sigma_2},M,T}(1 + \|x\|_H^2). (4.28)
$$

Similarly, we estimate the term $\int_\Delta^T |I_2(t)| dt$ as

$$
\int_\Delta^T |I_2(t)| dt \leq C\alpha T \Delta \sup_{t \in [0,T]} \|X^h_t\_H^2 \leq C_{\mu,\alpha,\lambda_1,L_G,L_{\sigma_2},M,T}(1 + \|x\|_H^2 + \|y\|_H^2). (4.29)
$$

For $n = 2$ and $n \in [1, 3]$, using Hölder’s and Ladyzhenskaya’s inequalities, Fubini’s Theorem and (4.18), we estimate $\int_\Delta^T |I_3(t)| dt$ as

$$
\int_\Delta^T |I_3(t)| dt \leq 2\mu \int_\Delta^T \int_{t-\Delta}^t \|X^h_s\_L^2 \|X^h_s - X^h_{t-\Delta}\_L \_V ds dt \\
\leq 2\sqrt{2} \left( \int_\Delta^T \int_{t-\Delta}^t \|X^h_s\_L^2 \|X^h_s - X^h_{t-\Delta}\_V ds dt \right)^{1/2} \left( \int_\Delta^T \int_{t-\Delta}^t \|X^h_s - X^h_{t-\Delta}\_V ds dt \right)^{1/2} \\
\leq 2\sqrt{2} \left( \Delta \sup_{t \in [0,T]} \|X^h_t\_H^2 \int_0^T \|X^h_t\_V dt \right)^{1/2} \left( 2\Delta \int_0^T \|X^h_t\_V dt \right)^{1/2} \\
\leq C_{\mu,\alpha,\lambda_1,L_G,L_{\sigma_2},M,T}(1 + \|x\|_H^3). (4.30)
$$
For $n = 2, 3$ and $r \geq 3$ (take $2\beta \mu > 1$, for $r = 3$), we estimate $\int_\Delta^T |I_3(t)| \, dt$ using Hölder’s inequality, interpolation inequality and (4.18) as

$$
\int_\Delta^T |I_3(t)| \, dt \leq 2 \int_\Delta^T \int_{t-\Delta}^t \left\| \dot{X}_s^h \right\|_{L^{r+1}} \left\| \ddot{X}_s^h \right\|_{L^{3(r+1)}} \left\| \dddot{X}_s^h - \dddot{X}_{t-\Delta}^h \right\|_Y \, ds \, dt
$$

Once again using Hölder’s inequality, Fubini’s Theorem and (4.18), we estimate the term $\int_\Delta^T |I_4(t)| \, dt$ as

$$
\int_\Delta^T |I_4(t)| \, dt \leq 2 \int_\Delta^T \int_{t-\Delta}^t \left\| F \right\|_{L^{3(r+1)}} \left\| \dddot{X}_s^h - \dddot{X}_{t-\Delta}^h \right\|_{Y} \, ds \, dt
$$

We estimate $\int_\Delta^T |I_5(t)| \, dt$ using the Assumption 3.1 (A1) and (4.18) as

$$
\int_\Delta^T |I_5(t)| \, dt \leq 2 \int_\Delta^T \int_{t-\Delta}^t \left\| \dot{F}(\dddot{X}_s^h) \right\| \left\| \dddot{X}_s^h - \dddot{X}_{t-\Delta}^h \right\|_H \, ds \, dt
$$

Similarly, we estimate the term $\int_\Delta^T |I_6(t)| \, dt$ as

$$
\int_\Delta^T |I_6(t)| \, dt \leq 2 \int_\Delta^T \int_{t-\Delta}^t \left\| \sigma_1(\dddot{X}_s^h)Q^{1/2}h_s \right\| \left\| \dddot{X}_s^h - \dddot{X}_{t-\Delta}^h \right\|_H \, ds \, dt
$$
LARGE DEVIATIONS FOR THE TWO-TIME-SCALE SCBF EQUATIONS 21

\[ \leq C \left( \Delta \sup_{t \in [0,T]} (1 + \|X^h_t\|_{L^2}) \int_0^T \|h_t\|^2 dt \right)^{1/2} \left( 2\Delta \int_0^T \|X^h_t\|_{L^2}^2 dt \right)^{1/2} \]

\[ \leq C_{\mu,\alpha,L_G,L_{\sigma_2},M,\Delta}(1 + \|x\|_{L^2}), \quad (4.34) \]

since \( h \in S_M \). Combing (4.27)-(4.34), we obtain the required result (4.25). \( \square \)

We are now in a position to verify the Hypothesis 4.9 (ii).

\textbf{Theorem 4.14 (Compactness).} Let \( M < +\infty \) be a fixed positive number. Let

\[ \mathcal{K}_M := \{ \bar{X}^h \in C([0,T]; \mathbb{H}) \cap L^2(0,T; V) \cap L^{r+1}(0,T; \tilde{L}^{r+1}) : h \in S_M \}, \]

where \( \bar{X}^h \) is the unique Leray-Hopf weak solution of the deterministic controlled equation (4.17), and \( \bar{X}^h_0 = x \in \mathbb{H} \) in \( \mathcal{E} = C([0,T]; \mathbb{H}) \cap L^2(0,T; V) \cap L^{r+1}(0,T; \tilde{L}^{r+1}) \). Then \( \mathcal{K}_M \) is compact in \( \mathcal{E} \).

\textbf{Proof.} Let us consider a sequence \( \{ \bar{X}^{h_n} \} \) in \( \mathcal{K}_M \), where \( \bar{X}^{h_n} \) corresponds to the solution of (4.17) with control \( h_n \in S_M \) in place of \( h \), that is,

\[ \begin{aligned}
&d\bar{X}^{h_n} = -[\mu A\bar{X}^{h_n} + B(\bar{X}^{h_n}) + \alpha \bar{X}^{h_n} + \beta C(\bar{X}^{h_n})]dt + \bar{F}(\bar{X}^{h_n})dt + \sigma_1(\bar{X}^{h_n})Q_1^{1/2}h_n dt, \\
&\bar{X}^{h_n}_0 = x \in \mathbb{H}.
\end{aligned} \]

Then, by using the weak compactness of \( S_M \), there exists a subsequence of \( \{h_n\} \), (still denoted by \( \{h_n\} \)), which converges weakly to \( h \in S_M \) in \( L^2(0,T; \mathbb{H}) \). Using the estimates (4.18), we obtain

\[ \begin{aligned}
&\bar{X}^{h_n} \xrightarrow{w^*} \bar{X}^h \text{ in } L^\infty(0,T; \mathbb{H}), \\
&\bar{X}^{h_n} \xrightarrow{w} \bar{X}^h \text{ in } L^2(0,T; V), \\
&\bar{X}^{h_n} \xrightarrow{w} \bar{X}^h \text{ in } L^{r+1}(0,T; \tilde{L}^{r+1}).
\end{aligned} \]

Using the monotonicity property of linear and nonlinear operators, and Minty-Browder technique (see [59]), one can establish that \( \bar{X}^h \) is the unique weak solution of the system (4.17). In order to prove \( \mathcal{K}_M \) is compact, we need to show that \( \bar{X}^{h_n} \rightarrow \bar{X}^h \) in \( \mathcal{E} \) as \( n \rightarrow \infty \). In other words, it is required to show that

\[ \sup_{t \in [0,T]} \|\bar{X}^{h_n} - \bar{X}^h\|_{L^2}^2 + \int_0^T \|\bar{X}^{h_n} - \bar{X}^h\|_{L^2}^2 dt + \int_0^T \|\bar{X}^{h_n} - \bar{X}^h\|_{L^{r+1}}^2 dt \rightarrow 0, \quad (4.37) \]

as \( n \rightarrow \infty \). Recall that for the system (4.17), the energy estimate given in (4.18) holds true. Let us now define \( W^{h_n,h}_t := \bar{X}^{h_n} - \bar{X}^h \), so that \( W^{h_n,h}_t \) satisfies:

\[ \begin{aligned}
&dW^{h_n,h}_t = -[\mu AW^{h_n,h}_t + B(\bar{X}^{h_n}) - B(\bar{X}^h)] + \alpha W^{h_n,h}_t + \beta (C(\bar{X}^{h_n}) - C(\bar{X}^h)) dt \\
&\quad + [\bar{F}(\bar{X}^{h_n}) - \bar{F}(\bar{X}^h)] dt + \left[ \sigma_1(\bar{X}^{h_n})Q_1^{1/2}h_n - \sigma_1(\bar{X}^h)Q_1^{1/2}h_t \right] dt, \\
&W^{h_n,h}_0 = 0.
\end{aligned} \]

Taking inner product with \( W^{h_n,h}_t \) to the first equation in (4.38), we get

\[ \|W^{h_n,h}_t\|^2_{L^2} + 2\mu \int_0^t \|W^{h_n,h}_s\|^2 ds + 2\alpha \int_0^t \|W^{h_n,h}_s\|^2_{L^2} ds \]
\[ \begin{align*}
&+2\beta \int_0^t \langle C(\bar{X}^{h_n}) - C(\bar{X}^h), W_{h_n, h} \rangle \, ds \\
&= -2 \int_0^t \langle B(\bar{X}^{h_n}) - B(\bar{X}^h), W_{h_n, h} \rangle \, ds + 2 \int_0^t \langle \bar{F}(\bar{X}^{h_n}) - \bar{F}(\bar{X}^h), W_{h_n, h} \rangle \, ds \\
&\quad + 2 \int_0^t (\sigma_1(\bar{X}^{h_n})Q_1^{1/2}h_n - \sigma_1(\bar{X}^h)Q_1^{1/2}h_s, W_{h_n, h}) \, ds. 
\end{align*} \] (4.39)

Note that \( \langle B(\bar{X}^{h_n}, X^{h_n} - \bar{X}^h), \bar{X}^{h_n} - \bar{X}^h \rangle = 0 \) and it implies that
\[ \begin{align*}
\langle B(\bar{X}^{h_n}) - B(\bar{X}^h), \bar{X}^{h_n} - \bar{X}^h \rangle &= \langle B(\bar{X}^{h_n} - \bar{X}^h), \bar{X}^{h_n} - \bar{X}^h \rangle + \langle B(\bar{X}^{h_n} - \bar{X}^h, \bar{X}^{h_n} - \bar{X}^h) \\
&= -\langle B(\bar{X}^{h_n} - \bar{X}^h), \bar{X}^{h_n} - \bar{X}^h \rangle.
\end{align*} \] (4.40)

For \( n = 2 \) and \( r \in [1, 3] \), making use of Hölder’s, Ladyzhenskaya’s and Young’s inequalities, we estimate \( 2|\langle B(\bar{X}^{h_n}) - B(\bar{X}^h), W_{h_n, h} \rangle| \) as
\[ 2|\langle B(\bar{X}^{h_n}) - B(\bar{X}^h), W_{h_n, h} \rangle| = 2|\langle B(W_{h_n, h}, \bar{X}^h), W_{h_n, h} \rangle| \leq 2\|\bar{X}^h\|_V \|W_{h_n, h}\|_{L^4}^2 \]
\[ \leq 2\sqrt{2}\|\bar{X}^h\|_V \|W_{h_n, h}\|_H \|W_{h_n, h}\|_V \]
\[ \leq \mu\|W_{h_n, h}\|^2 + \frac{2}{\mu}\|\bar{X}^h\|^2 \|W_{h_n, h}\|^2. \] (4.41)

Using (2.9) and (2.10), we know that
\[ -2\beta\langle C(\bar{X}^{h_n}) - C(\bar{X}^h), W_{h_n, h} \rangle \leq -\frac{\beta}{2^{r-2}}\|W_{h_n, h}\|_{r+1}^{r+1}, \] (4.42)

for \( r \in [1, \infty) \). Using (4.41), Hölder’s and Young’s inequalities, we estimate \( 2|\langle \bar{F}(\bar{X}^{h_n}) - \bar{F}(\bar{X}^h), W_{h_n, h} \rangle| \) as
\[ 2|\langle \bar{F}(\bar{X}^{h_n}) - \bar{F}(\bar{X}^h), W_{h_n, h} \rangle| \leq 2\|\bar{F}(\bar{X}^{h_n}) - \bar{F}(\bar{X}^h)\|_H \|W_{h_n, h}\|_H \leq C_{\mu, \alpha, \lambda_1, L_G, L_{\sigma_2}} \|W_{h_n, h}\|^2. \] (4.43)

Combining (4.41)-(4.43) and substituting it in (4.39), we obtain
\[ \begin{align*}
\|W_{h_n, h}\|_{L^4}^2 + \mu \int_0^t \|W_{h_n, h}\|_V^2 \, ds + 2\alpha \int_0^t \|W_{h_n, h}\|_H^2 \, ds + \frac{\beta}{2^{r-2}} \int_0^t \|W_{h_n, h}\|_{r+1} \, ds \\
\leq \frac{2}{\mu} \int_0^t \|\bar{X}^h\|_V^2 \|W_{h_n, h}\|_H^2 \, ds + C_{\mu, \alpha, \lambda_1, L_G, L_{\sigma_2}} \int_0^t \|W_{h_n, h}\|_H^2 \, ds + I_1 + I_2,
\end{align*} \] (4.44)

where
\[ I_1 = 2 \int_0^t (\sigma_1(\bar{X}^{h_n}) - \sigma_1(\bar{X}^h))Q_1^{1/2}h_n, W_{h_n, h}) \, ds, \]
\[ I_2 = 2 \int_0^t (\sigma_1(\bar{X}^h)Q_1^{1/2}(h_s - h_n), W_{h_n, h}) \, ds. \]

Using the Cauchy-Schwarz inequality, Hölder’s and Young’s inequalities, and the Assumption 3.1 (A1), we estimate \( I_1 \) as
\[ I_1 \leq 2 \int_0^t \|((\sigma_1(\bar{X}^{h_n}) - \sigma_1(\bar{X}^h))Q_1^{1/2}h_n, W_{h_n, h})\| \, ds. \]
\[ \leq 2 \int_0^t \|(\sigma_1(\bar{X}_s^h) - \sigma_1(\bar{X}_s^h))Q_1^{1/2}\|_{L_2} \|h_s^h\|_{\mathbb{H}} \|W_s^{h_n,h}\|_{\mathbb{H}}ds \leq C \int_0^t \|h_s^n\|_{\mathbb{H}} \|W_s^{h_n,h}\|_{\mathbb{H}}^2 ds \]
\[ \leq C \int_0^t \|W_s^{h_n,h}\|^2_{\mathbb{H}} ds + C \int_0^t \|h_s^n\|^2_{\mathbb{H}} \|W_s^{h_n,h}\|_{\mathbb{H}}^2 ds. \]  
(4.45)

Making use of the Cauchy-Schwarz inequality and Young’s inequality, we estimate \( I_2 \) as

\[ \begin{align*}
I_2 & \leq 2 \int_0^t \|\sigma_1(\bar{X}_s^h)Q_1^{1/2}(h_s^n - h_s)\|_{\mathbb{H}} \|W_s^{h_n,h}\|_{\mathbb{H}} ds \\
& \leq \int_0^t \|W_s^{h_n,h}\|^2_{\mathbb{H}} ds + \int_0^t \|\sigma_1(\bar{X}_s^h)Q_1^{1/2}(h_s^n - h_s)\|_{\mathbb{H}}^2 ds. 
\end{align*} \]
(4.46)

Thus, it is immediate from (4.44) that

\[ \begin{align*}
\|W_t^{h_n,h}\|^2_{\mathbb{H}} + \mu \int_0^t \|W_s^{h_n,h}\|^2_{\mathbb{H}} ds + 2\alpha \int_0^t \|W_s^{h_n,h}\|_{\mathbb{H}}^2 ds + \frac{\beta}{2r-2} \int_0^t \|W_s^{h_n,h}\|_{L_{r+1}}^2 ds \\
\leq 2 \mu \int_0^t \|\bar{X}_s^h\|^2 \|W_s^{h_n,h}\|^2_{\mathbb{H}} ds + C_{\mu,\alpha,\lambda_1,\lambda_2} \int_0^t \|W_s^{h_n,h}\|_{\mathbb{H}}^2 ds + C \int_0^t \|h_s^n\|^2_{\mathbb{H}} \|W_s^{h_n,h}\|_{\mathbb{H}}^2 ds \\
+ \int_0^t \|\sigma_1(\bar{X}_s^h)Q_1^{1/2}(h_s^n - h_s)\|_{\mathbb{H}}^2 ds,
\end{align*} \]
(4.47)

for all \( t \in [0, T] \). An application of Gronwall’s inequality in (4.47) gives

\[ \begin{align*}
\sup_{t \in [0, T]} \|W_t^{h_n,h}\|^2_{\mathbb{H}} + \mu \int_0^T \|W_t^{h_n,h}\|^2_{\mathbb{H}} dt + 2\alpha \int_0^T \|W_t^{h_n,h}\|_{\mathbb{H}}^2 dt + \frac{\beta}{2r-2} \int_0^T \|W_t^{h_n,h}\|_{L_{r+1}}^2 dt \\
\leq \left( \int_0^T \|\sigma_1(\bar{X}_t^h)Q_1^{1/2}(h_t^n - h_t)\|^2_{\mathbb{H}} dt \right) \exp \left\{ \frac{2}{\mu} \int_0^T \|\bar{X}_t^h\|^2 dt + \int_0^T \|h_t^n\|^2_{\mathbb{H}} dt \right\} e^{C_{\mu,\alpha,\lambda_1,\lambda_2} T} \\
\leq C_{\mu,\alpha,\lambda_1,\lambda_2,\mathbb{H}, T} \left( \int_0^T \|\sigma_1(\bar{X}_t^h)Q_1^{1/2}(h_t^n - h_t)\|^2_{\mathbb{H}} dt \right),
\end{align*} \]
(4.48)

since \( \{h_n\} \in S_M \) and the process \( \bar{X}_t^h \) satisfies the energy estimate (4.18). It should be noted that the operator \( \sigma_1(\cdot)Q_1^{1/2} \) is Hilbert–Schmidt in \( \mathbb{H} \), and hence it is a compact operator on \( \mathbb{H} \). Furthermore, we know that compact operator maps weakly convergent sequences into strongly convergent sequences. Since \( \{h_n\} \) converges weakly to \( h \in S_M \) in \( L^2(0, T; \mathbb{H}) \), we infer that

\[ \int_0^T \|\sigma_1(\bar{X}_t^h)Q_1^{1/2}(h_t^n - h_t)\|^2_{\mathbb{H}} dt \to 0 \quad \text{as} \quad n \to \infty. \]

Thus, from (4.48), we obtain

\[ \sup_{t \in [0, T]} \|W_t^{h_n,h}\|^2_{\mathbb{H}} + \mu \int_0^T \|W_t^{h_n,h}\|^2_{\mathbb{H}} dt + \frac{\beta}{2r-2} \int_0^T \|W_t^{h_n,h}\|_{L_{r+1}}^2 dt \to 0 \quad \text{as} \quad n \to \infty, \]
(4.49)

which concludes the proof for \( n = 2 \) and \( r \in [1, 3] \).

For \( n = 2, 3 \) and \( r \in (3, \infty) \), from (2.37), we easily have

\[ \beta \langle C(\bar{X}^{h_n}) - C(\bar{X}^h), \bar{X}^{h_n} - \bar{X}^h \rangle \geq \beta \frac{\|\bar{X}^{h_n} - \bar{X}^h\|^2_{\mathbb{H}}}{2} + \frac{\beta}{2} \|\bar{X}^h - \bar{X}^{h_n} - \bar{X}^h\|^2_{\mathbb{H}}. \]
(4.50)
Using (4.40), Hölder’s and Young’s inequalities, we estimate \(|\langle B(\mathbf{X}^{h_n}) - B(\mathbf{X}^h), \mathbf{X}^{h_n} - \mathbf{X}^h\rangle|\) as
\[
|\langle B(\mathbf{X}^{h_n}) - B(\mathbf{X}^h), \mathbf{X}^{h_n} - \mathbf{X}^h\rangle| = |\langle B(\mathbf{X}^{h_n} - \mathbf{X}^h, \mathbf{X}^{h_n} - \mathbf{X}^h), \mathbf{X}^h\rangle| \\
\leq \|\mathbf{X}^{h_n} - \mathbf{X}^h\|_V \|\mathbf{X}^h(\mathbf{X}^{h_n} - \mathbf{X}^h)\|_H \\
\leq \frac{\mu}{2} \|\mathbf{X}^{h_n} - \mathbf{X}^h\|_V^2 + \frac{1}{2\mu} \|\mathbf{X}^h(\mathbf{X}^{h_n} - \mathbf{X}^h)\|_H^2. \tag{4.51}
\]
We take the term \(\|\mathbf{X}^h(\mathbf{X}^{h_n} - \mathbf{X}^h)\|_H^2\) from (4.51) and use Hölder’s and Young’s inequalities to estimate it as (see [40] also)
\[
\int_\mathcal{O} |\mathbf{X}^h(x)|^2 |\mathbf{X}^{h_n}(x) - \mathbf{X}^h(x)|^2 \, dx \\
= \int_\mathcal{O} |\mathbf{X}^h(x)|^2 |\mathbf{X}^{h_n}(x) - \mathbf{X}^h(x)|^{\frac{r}{r-1}} |\mathbf{X}^{h_n}(x) - \mathbf{X}^h(x)|^{\frac{2(r-3)}{r-1}} \, dx \\
\leq \left( \int_\mathcal{O} |\mathbf{X}^h(x)|^{r-1} |\mathbf{X}^{h_n}(x) - \mathbf{X}^h(x)|^2 \, dx \right)^{\frac{2}{r-1}} \left( \int_\mathcal{O} |\mathbf{X}^{h_n}(x) - \mathbf{X}^h(x)|^2 \, dx \right)^{\frac{r}{r-1}} \\
\leq \frac{\beta \mu}{2} \left( \int_\mathcal{O} |\mathbf{X}^h(x)|^{r-1} |\mathbf{X}^{h_n}(x) - \mathbf{X}^h(x)|^2 \, dx \right) + r - 3 \left( \frac{4}{\beta \mu (r-1)} \right)^{\frac{2}{r-2}} \left( \int_\mathcal{O} |\mathbf{X}^{h_n}(x) - \mathbf{X}^h(x)|^2 \, dx \right), \tag{4.52}
\]
for \(r > 3\). Combining (4.50) and (4.52), we find
\[
\beta (C(\mathbf{X}^{h_n}) - C(\mathbf{X}^h), \mathbf{X}^{h_n} - \mathbf{X}^h) + \langle B(\mathbf{X}^{h_n}) - B(\mathbf{X}^h), \mathbf{X}^{h_n} - \mathbf{X}^h\rangle \\
\geq \frac{\beta}{4} \|\mathbf{X}^{h_n} - \mathbf{X}^h\|_{L^{r+1}} \left( \frac{r-1}{2} \right)^{\frac{2}{r-3}} (\|\mathbf{X}^{h_n} - \mathbf{X}^h\|_{L^r}^2 + \frac{\beta}{2} \|\mathbf{X}^h(\mathbf{X}^{h_n} - \mathbf{X}^h)\|_H^2 \right) \\
- \frac{r-3}{2\mu (r-1)} \left( \frac{4}{\beta \mu (r-1)} \right)^{\frac{2}{r-2}} \left( \int_\mathcal{O} |\mathbf{X}^{h_n}(x) - \mathbf{X}^h(x)|^2 \, dx \right) - \frac{\mu}{2} \|\mathbf{X}^{h_n} - \mathbf{X}^h\|_V^2. \tag{4.53}
\]
Using (2.10), we obtain
\[
\frac{2^{2-r} \beta}{4} \|\mathbf{X}^{h_n} - \mathbf{X}^h\|_{L^{r+1}} \leq \frac{\beta}{4} \|\mathbf{X}^{h_n} - \mathbf{X}^h\|_{L^r}^{r-1} (\|\mathbf{X}^{h_n} - \mathbf{X}^h\|_{L^r}^2 + \frac{\beta}{4} \|\mathbf{X}^h(\mathbf{X}^{h_n} - \mathbf{X}^h)\|_H^2 \right) \tag{4.54}
\]
Thus, from (4.53), it is immediate that
\[
\beta (C(\mathbf{X}^{h_n}) - C(\mathbf{X}^h), \mathbf{X}^{h_n} - \mathbf{X}^h) + \langle B(\mathbf{X}^{h_n}) - B(\mathbf{X}^h), \mathbf{X}^{h_n} - \mathbf{X}^h\rangle \\
\geq \frac{\beta}{2r} \|\mathbf{X}^{h_n} - \mathbf{X}^h\|_{L^{r+1}}^2 - \frac{C}{2} \|\mathbf{X}^{h_n} - \mathbf{X}^h\|_{L^r}^2 - \frac{\mu}{2} \|\mathbf{X}^{h_n} - \mathbf{X}^h\|_V^2 \tag{4.55}
\]
where
\[
\tilde{\eta} = \frac{r-3}{\mu (r-1)} \left( \frac{4}{\beta \mu (r-1)} \right)^{\frac{2}{r-2}}. \tag{4.56}
\]
Using (4.43) and (4.55) in (4.39), we get
\[
\frac{\|\mathbf{X}^{h_n,h}\|_H}{\mathbf{m}} + \mu \int_0^t \|\mathbf{X}^{h_n,h}\|_V^2 \, ds + 2\alpha \int_0^t \|\mathbf{X}^{h_n,h}\|_H^2 \, ds + \frac{\beta}{2r-1} \int_0^t \|\mathbf{X}^{h_n,h}\|_{L^{r+1}}^2 \, ds
\]
Combining (4.58) and (4.59), we obtain
\[
\beta \mu > H
\]
Hence, for \(t \in [0, T]\). Then, proceeding similarly as in the previous case, we arrive at the required result (4.37).

For the case \(n = r = 3\), from (2.9), we find
\[
\beta \langle C(\bar{X}^h), \bar{X}^h - \bar{X}^h \rangle \geq \frac{\beta}{2} \| \bar{X}^h(\bar{X}^h - \bar{X}^h) \|_H^2 + \frac{\beta}{2} \| \bar{X}^h(\bar{X}^h - \bar{X}^h) \|_H^2.
\]
We estimate \(\|B(\bar{X}^h) - B(\bar{X}^h), \bar{X}^h - \bar{X}^h\|\) using Hölder’s and Young’s inequalities as
\[
\|B(\bar{X}^h) - B(\bar{X}^h), \bar{X}^h - \bar{X}^h\| \leq \|\bar{X}^h(\bar{X}^h - \bar{X}^h)\|_H \|\bar{X}^h - \bar{X}^h\|_V \leq \frac{\beta \mu}{2} \|\bar{X}^h - \bar{X}^h\|_V^2 + \frac{1}{2\beta \mu} \|\bar{X}^h(\bar{X}^h - \bar{X}^h)\|_H^2.
\]
Combining (4.58) and (4.59), we obtain
\[
\mu \langle A(\bar{X}^h - \bar{X}^h), \bar{X}^h - \bar{X}^h \rangle + \beta \langle C(\bar{X}^h) - C(\bar{X}^h), \bar{X}^h - \bar{X}^h \rangle
\]
\[
\geq \mu \left( 1 - \frac{1}{2} \right) \|\bar{X}^h - \bar{X}^h\|_V^2 + \frac{\beta}{2} \|\bar{X}^h(\bar{X}^h - \bar{X}^h)\|_H^2 + \frac{1}{2} \left( \beta - \frac{1}{\beta \mu} \right) \|\bar{X}^h(\bar{X}^h - \bar{X}^h)\|_H^2,
\]
for \(\frac{1}{\beta \mu} < \theta < 2\). Thus, we infer that
\[
\|W_t^{n,h}\|_H^2 + \mu(2 - \theta) \int_0^t \|W_s^{n,h}\|_V^2 ds + \left( \beta - \frac{1}{\beta \mu} \right) \int_0^t \|W_s^{n,h}\|_L^4 ds
\]
\[
\leq C_{\mu, \alpha \lambda, L_G, L_s} \int_0^t \|W_s^{n,h}\|_H^2 ds + C \int_0^t \|h_s^n\|_H^2 \|W_s^{n,h}\|_E^2 ds + \int_0^t \|\sigma_1(\bar{X}^h)\|_H^{1/2} (h_s^n - h_s) \|_H^2 ds.
\]
Hence, for \(2\beta \mu > 1\), arguing similarly as in the case of \(n = 2, 3\) and \(r \in [1, 3]\), we finally obtain the required result (4.37).

4.6. Weak convergence. Let us now verify the Hypothesis (4.9) (i) on weak convergence. We first establish the existence and uniqueness result of the following stochastic controlled SCBF equations.
Theorem 4.15. For any $h \in A_M$, $0 < M < +\infty$, under the Assumption [3.7], the stochastic control problem:
\[
\begin{align*}
\text{d}X_t^{\varepsilon,\delta,h} &= -[\mu AX_t^{\varepsilon,\delta,h} + B(X_t^{\varepsilon,\delta,h}) + \alpha X_t^{\varepsilon,\delta,h} + \beta C(X_t^{\varepsilon,\delta,h}) - F(X_t^{\varepsilon,\delta,h}, Y_t^{\varepsilon,\delta,h})]dt \\
&\quad + \sigma_1(X_t^{\varepsilon,\delta,h})Q_1^{1/2}h_t dt + \sqrt{\varepsilon}\sigma_1(X_t^{\varepsilon,\delta,h})Q_1^{1/2}dW_t,
\end{align*}
\]
\[
\begin{align*}
\text{d}Y_t^{\varepsilon,\delta,h} &= -\frac{1}{\delta}[\mu AY_t^{\varepsilon,\delta,h} + \alpha Y_t^{\varepsilon,\delta,h} + \beta C(Y_t^{\varepsilon,\delta}) - G(X_t^{\varepsilon,\delta,h}, Y_t^{\varepsilon,\delta,h})]dt \\
&\quad + \frac{1}{\sqrt{\delta\varepsilon}}\sigma_2(X_t^{\varepsilon,\delta,h}, Y_t^{\varepsilon,\delta,h})Q_2^{1/2}h_t dt + \frac{1}{\sqrt{\delta}}\sigma_2(X_t^{\varepsilon,\delta,h}, Y_t^{\varepsilon,\delta,h})Q_2^{1/2}dW_t,
\end{align*}
\]
\[
X_0^{\varepsilon,\delta,h} = x, \quad Y_0^{\varepsilon,\delta,h} = y,
\]
has a pathwise unique strong solution $(X_t^{\varepsilon,\delta,h}, Y_t^{\varepsilon,\delta,h}) \in L^2(\Omega; \mathcal{F}_t) \times L^2(\Omega; \mathcal{F}_t)$, where $\mathcal{F}_t = C([0, T]; \mathbb{H}) \cap L^2(0, T; \mathcal{V}) \cap L^{r+1}(0, T; \mathcal{L}^{r+1})$ with $\mathcal{F}_t$-adapted paths in $\mathcal{F}_t \times \mathcal{F}_t$, $\mathbb{P}$-a.s. Furthermore, for all $(\varepsilon, \delta) \in (0, 1)$ with $\frac{\delta}{\varepsilon} \leq \frac{1}{C_{\mu, \alpha, \lambda_1, L_G, M}}$, we have
\[
\mathbb{E}\left[ \sup_{t \in [0, T]} \|X_t^{\varepsilon,\delta,h}\|_{\mathbb{H}}^2 + \mu \int_0^T \|X_t^{\varepsilon,\delta,h}\|_{\mathbb{V}}^2 dt + \beta \int_0^T \|X_t^{\varepsilon,\delta,h}\|_{\mathbb{W}}^{r+1} dt \right] 
\leq C_{\mu, \alpha, \lambda_1, L_G, M} (1 + \|x\|^2_{\mathbb{H}} + \|y\|^2_{\mathbb{H}}),
\]
and
\[
\mathbb{E}\left[ \int_0^T \|Y_t^{\varepsilon,\delta,h}\|_{\mathbb{H}}^2 dt \right] \leq C_{\mu, \alpha, \lambda_1, L_G, M} (1 + \|x\|^2_{\mathbb{H}} + \|y\|^2_{\mathbb{H}}).
\]

Proof. The existence and uniqueness of pathwise strong solution satisfying the energy equality to the system (4.62) can be obtained similarly as in Theorem 3.4, [62] (see, Theorem 3.7, [60] also), by using the monotonicity as well as demicontinuous properties of the linear and nonlinear operators and a stochastic generalization of the Minty-Browder technique.

Let us now prove the uniform energy estimates (4.63) and (4.64). An application of the infinite dimensional Itô formula to the process $\|Y_t^{\varepsilon,\delta,h}\|^2_{\mathbb{H}}$ yields (cf. [60])
\[
\|Y_t^{\varepsilon,\delta,h}\|^2_{\mathbb{H}} = \|y\|^2_{\mathbb{H}} - \frac{2\mu}{\delta} \int_0^t \|Y_s^{\varepsilon,\delta,h}\|^2_{\mathbb{V}} ds - \frac{2\alpha}{\delta} \int_0^t \|Y_s^{\varepsilon,\delta,h}\|^2_{\mathbb{H}} ds - \frac{2\beta}{\delta} \int_0^t \|Y_s^{\varepsilon,\delta,h}\|^{r+1}_{\mathbb{W}} ds \\
&\quad + \frac{2}{\delta} \int_0^t \langle G(Y_s^{\varepsilon,\delta,h}, Y_s^{\varepsilon,\delta,h}), Y_s^{\varepsilon,\delta,h} \rangle ds \\
&\quad + \frac{2}{\sqrt{\delta\varepsilon}} \int_0^t \langle \sigma_2(X_s^{\varepsilon,\delta,h}, Y_s^{\varepsilon,\delta,h})Q_2^{1/2}h_s, Y_s^{\varepsilon,\delta,h} \rangle ds \\
&\quad + \frac{1}{\sqrt{\delta}} \int_0^t \langle \sigma_2(X_s^{\varepsilon,\delta,h}, Y_s^{\varepsilon,\delta,h}) Q_2^{1/2}dW_s, Y_s^{\varepsilon,\delta,h} \rangle,
\]
for all $t \in [0, T]$, $\mathbb{P}$-a.s. Taking expectation in (4.65) and using the fact that the final term appearing in (4.63) is a martingale, we obtain
\[
\mathbb{E}\left[ \|Y_t^{\varepsilon,\delta,h}\|^2_{\mathbb{H}} \right] 
\leq \|y\|^2_{\mathbb{H}} - \frac{2\mu}{\delta} \mathbb{E}\left[ \int_0^t \|Y_s^{\varepsilon,\delta,h}\|^2_{\mathbb{V}} ds \right] - \frac{2\alpha}{\delta} \mathbb{E}\left[ \int_0^t \|Y_s^{\varepsilon,\delta,h}\|^2_{\mathbb{H}} ds \right] - \frac{2\beta}{\delta} \mathbb{E}\left[ \int_0^t \|Y_s^{\varepsilon,\delta,h}\|^{r+1}_{\mathbb{W}} ds \right] \\
&\quad + \frac{2}{\delta} \mathbb{E}\left[ \int_0^t \langle G(Y_s^{\varepsilon,\delta,h}, Y_s^{\varepsilon,\delta,h}), Y_s^{\varepsilon,\delta,h} \rangle ds \right] \\
&\quad + \frac{2}{\sqrt{\delta\varepsilon}} \mathbb{E}\left[ \int_0^t \langle \sigma_2(X_s^{\varepsilon,\delta,h}, Y_s^{\varepsilon,\delta,h})Q_2^{1/2}h_s, Y_s^{\varepsilon,\delta,h} \rangle ds \right] \\
&\quad + \frac{1}{\sqrt{\delta}} \mathbb{E}\left[ \int_0^t \langle \sigma_2(X_s^{\varepsilon,\delta,h}, Y_s^{\varepsilon,\delta,h}) Q_2^{1/2}dW_s, Y_s^{\varepsilon,\delta,h} \rangle \right].
\]
\[
\frac{1}{\delta} \mathbb{E} \left[ \int_0^t \| \sigma_2(X_s^{\varepsilon,\delta,h}, Y_s^{\varepsilon,\delta,h}) Q_2^{1/2} \|^2 \| L_2 \right],
\]  
(4.66)
for all \( t \in [0, T] \). Thus, it is immediate that
\[
\frac{d}{dt} \mathbb{E} \left[ \| Y_t^{\varepsilon,\delta,h} \|^2 \| H \right] = -\frac{2\mu}{\delta} \mathbb{E} \left[ \| Y_t^{\varepsilon,\delta,h} \|^2 \| V \right] - \frac{2\alpha}{\delta} \mathbb{E} \left[ \| Y_t^{\varepsilon,\delta,h} \|^2 \| H \right] - \frac{2\beta}{\delta} \mathbb{E} \left[ \| Y_t^{\varepsilon,\delta,h} \|^2 \| L_{r+1} \right] 
+ \frac{2}{\sqrt{\delta}} \mathbb{E} \left[ G(X_t^{\varepsilon,\delta,h}, Y_t^{\varepsilon,\delta,h}), Y_t^{\varepsilon,\delta,h} \right] + \frac{2}{\sqrt{\delta}} \mathbb{E} \left[ (\sigma_2(X_t^{\varepsilon,\delta,h}, Y_t^{\varepsilon,\delta,h}) Q_2^{1/2}, h_t, Y_t^{\varepsilon,\delta,h}) \right] 
+ \frac{1}{\delta} \mathbb{E} \left[ \| \sigma_2(X_t^{\varepsilon,\delta,h}, Y_t^{\varepsilon,\delta,h}) Q_2^{1/2} \|^2 \| L_2 \right],
\]  
(4.67)
for a.e. \( t \in [0, T] \). Using the Assumption 3.1 (A1), the Cauchy-Schwarz inequality and Young’s inequality, we get
\[
\frac{2}{\delta} (G(X^{\varepsilon,\delta,h}, Y^{\varepsilon,\delta,h}), Y^{\varepsilon,\delta,h}) \leq \frac{2}{\delta} G(0,0)_{\| H \|} + C \| X^{\varepsilon,\delta,h} \|_{\| H \|} + L_G || Y^{\varepsilon,\delta,h} \|_{\| H \|} \| Y^{\varepsilon,\delta,h} \|_{\| H \|} 
\leq \left( \frac{\mu \lambda_1}{2\delta} + \frac{2L_G}{\delta} \right) \| Y^{\varepsilon,\delta,h} \|_{\| H \|}^2 + \frac{C}{\mu \lambda_1 \delta} || X^{\varepsilon,\delta,h} \|_{\| H \|}^2 + \frac{C}{\mu \lambda_1 \delta} \]  
(4.68)
Using the Cauchy-Schwarz inequality, Assumption 3.1 (A2) and Young’s inequality, we estimate the term \( \frac{2}{\sqrt{\delta \varepsilon}} (\sigma_2(X^{\varepsilon,\delta,h}, Y^{\varepsilon,\delta,h}) h, Y^{\varepsilon,\delta,h}) \) as
\[
\frac{2}{\sqrt{\delta \varepsilon}} (\sigma_2(X^{\varepsilon,\delta,h}, Y^{\varepsilon,\delta,h}) Q_2^{1/2}, h, Y^{\varepsilon,\delta,h}) \leq \frac{2}{\sqrt{\delta \varepsilon}} \sigma_2(X^{\varepsilon,\delta,h}, Y^{\varepsilon,\delta,h}) Q_2^{1/2} \| h \|_{\| H \|} \| Y^{\varepsilon,\delta,h} \|_{\| H \|} 
\leq \frac{C}{\sqrt{\delta \varepsilon}} \left( 1 + \| X^{\varepsilon,\delta,h} \|_{\| H \|} \right) \| h \|_{\| H \|} \| Y^{\varepsilon,\delta,h} \|_{\| H \|} 
\leq \frac{\mu \lambda_1}{4\delta} \| Y^{\varepsilon,\delta,h} \|_{\| H \|}^2 + \frac{C}{\mu \lambda_1 \varepsilon} \left( 1 + \| X^{\varepsilon,\delta,h} \|_{\| H \|}^2 \right) \| h \|_{\| H \|}^2. \]  
(4.69)
Once again, using the Assumption 3.1 (A2) and Young’s inequality, we obtain
\[
\frac{1}{\delta} \| \sigma_2(X^{\varepsilon,\delta,h}, Y^{\varepsilon,\delta,h}) Q_2^{1/2} \|^2 \| L_2 \leq \frac{C}{\delta} \left( 1 + \| X^{\varepsilon,\delta,h} \|_{\| H \|}^2 \right) \leq \frac{\mu \lambda_1}{4\delta} \| Y^{\varepsilon,\delta,h} \|_{\| H \|}^2 + \frac{C}{\mu \lambda_1 \delta} \| X^{\varepsilon,\delta,h} \|_{\| H \|}^2 + \frac{C}{\mu \lambda_1 \delta} \]  
(4.70)
Combining (4.68)-(4.70) and using it in (4.67), we find
\[
\frac{d}{dt} \mathbb{E} \left[ \| Y_t^{\varepsilon,\delta,h} \|^2 \| H \right] = -\frac{1}{\delta} (\mu \lambda_1 + 2\alpha - 2L_G) \mathbb{E} \left[ \| Y_t^{\varepsilon,\delta,h} \|^2 \| H \right] + \frac{C}{\mu \lambda_1 \delta} \mathbb{E} \left[ \| X_t^{\varepsilon,\delta,h} \|^2 \| H \right] + \frac{C}{\mu \lambda_1 \delta} \mathbb{E} \left[ \left( 1 + \| X_s^{\varepsilon,\delta,h} \|_{\| H \|}^2 \right) \| h_t \|_{\| H \|} \right],
\]  
(4.71)
for a.e. \( t \in [0, T] \). By the Assumption 3.1 (A3), we know that \( \mu \lambda_1 + 2\alpha > 2L_G \) and an application of variation of constants formula gives
\[
\mathbb{E} \left[ \| Y_t^{\varepsilon,\delta,h} \|^2 \| H \right] \leq \| Y_0 \|^2 e^{-\frac{\mu \lambda_1 t}{\delta}} + \frac{C}{\mu \lambda_1 \delta} \int_0^t e^{-\frac{\mu \lambda_1 s}{\delta}} \left( 1 + \mathbb{E} \left[ \| X_s^{\varepsilon,\delta,h} \|_{\| H \|}^2 \right] \right) ds 
+ \frac{C}{\mu \lambda_1 \delta} \int_0^t e^{-\frac{\mu \lambda_1 s}{\delta}} \mathbb{E} \left[ \left( 1 + \| X_s^{\varepsilon,\delta,h} \|_{\| H \|}^2 \right) \| h_s \|_{\| H \|}^2 \right] ds, \]  
(4.72)
for all \( t \in [0, T] \), where \( \gamma = (\mu \lambda_1 + 2\alpha - 2L_G) \). Integrating the above inequality from 0 to \( T \), performing a change of order of integration and using Fubini’s theorem, we also get

\[
\mathbb{E} \left[ \int_0^T \| Y^{\varepsilon, \delta, h}_t \|_{\mathbb{H}_2}^2 dt \right] \leq \| y \|_{\mathbb{H}}^2 + \frac{C}{\mu \lambda_1} \int_0^T \| X_t^{\varepsilon, \delta, h} \|_{\mathbb{H}}^2 dt + \frac{C}{\mu \lambda_1 \gamma} \mathbb{E} \left[ \sup_{t \in [0, T]} \left( 1 + \| X_t^{\varepsilon, \delta, h} \|_{\mathbb{H}}^2 \right) \right] \int_0^T \| h_s \|_{\mathbb{H}}^2 ds dt
\]

\[
\leq \frac{\delta}{\gamma} \| y \|_{\mathbb{H}}^2 + \frac{C}{\mu \lambda_1 \gamma} \mathbb{E} \left[ \sup_{t \in [0, T]} \left( 1 + \| X_t^{\varepsilon, \delta, h} \|_{\mathbb{H}}^2 \right) \right] \int_0^T \| h_t \|_{\mathbb{H}}^2 dt
\]

\[
\leq C_{\mu, \alpha, \lambda_1, L_G, T} (1 + \| y \|_{\mathbb{H}}^2) + C_{\mu, \alpha, \lambda_1, L_G, T} \mathbb{E} \left[ \int_0^T \| X_t^{\varepsilon, \delta, h} \|_{\mathbb{H}}^2 dt \right]
\]

\[
+ C_{\mu, \alpha, \lambda_1, L_G, M} \left( \frac{\delta}{\varepsilon} \right) \left[ T + \mathbb{E} \left( \sup_{t \in [0, T]} \| X_t^{\varepsilon, \delta, h} \|_{\mathbb{H}}^2 \right) \right], \quad (4.73)
\]

since \( \delta \in (0, 1) \).

Let us now obtain the energy estimates satisfied by the process \( X^{\varepsilon, \delta, h}_t \). Applying the infinite dimensional Itô formula to the process \( \| X^{\varepsilon, \delta, h}_t \|_{\mathbb{H}}^2 \) (see [60]), we find

\[
\| X^{\varepsilon, \delta, h}_t \|_{\mathbb{H}}^2 = \| x \|_{\mathbb{H}}^2 - 2\mu \int_0^t \| X^{\varepsilon, \delta, h}_s \|_{\mathbb{V}}^2 ds - 2\alpha \int_0^t \| X^{\varepsilon, \delta, h}_s \|_{\mathbb{H}}^2 ds - 2\beta \int_0^t \| X^{\varepsilon, \delta, h}_s \|_{2, r+1}^2 ds
\]

\[
+ 2 \int_0^t (F(X^{\varepsilon, \delta, h}_s, Y^{\varepsilon, \delta, h}_s, X^{\varepsilon, \delta, h}_s)) ds + 2 \int_0^t (\sigma_1(X^{\varepsilon, \delta, h}_s)Q_{1/2} h_s, X^{\varepsilon, \delta, h}_s) ds
\]

\[
+ \varepsilon \int_0^t \| \sigma_1(X^{\varepsilon, \delta, h}_s)Q_{1/2}^2 \|_{2, 2}^2 ds + 2\sqrt{\varepsilon} \int_0^t (\sigma_1(X^{\varepsilon, \delta, h}_s)Q_{1/2} dW_s, X^{\varepsilon, \delta, h}_s), \quad (4.74)
\]

for all \( t \in [0, T] \), \( \mathbb{P} \)-a.s. Taking supremum over \([0, T]\) and then taking expectation in (4.74), we obtain

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \| X^{\varepsilon, \delta, h}_t \|_{\mathbb{H}}^2 \right] \leq \| x \|_{\mathbb{H}}^2 + 2\mu \int_0^T \| X^{\varepsilon, \delta, h}_t \|_{\mathbb{V}}^2 dt + 2\alpha \int_0^T \| X^{\varepsilon, \delta, h}_t \|_{\mathbb{H}}^2 dt + 2\beta \int_0^T \| X^{\varepsilon, \delta, h}_t \|_{2, r+1}^2 dt
\]

\[
\leq \| x \|_{\mathbb{H}}^2 + C T + C \mathbb{E} \left[ \int_0^T \| X^{\varepsilon, \delta, h}_t \|_{\mathbb{H}}^2 dt \right] + C \mathbb{E} \left[ \int_0^T \| Y^{\varepsilon, \delta, h}_t \|_{\mathbb{H}}^2 dt \right]
\]

\[
+ 2\sqrt{\varepsilon} \mathbb{E} \left[ \left( \sigma_1(X^{\varepsilon, \delta, h}_t)Q_{1/2} dW_t, X^{\varepsilon, \delta, h}_t \right) \right] \]
where we used calculations similar to (4.68)-(4.70). We estimate the penultimate term from the right hand side of the inequality (4.75) using the Cauchy-Schwarz inequality, Hölder’s and Young’s inequalities as

\[
2E \left[ \int_0^T |(\sigma_1(X_t^{\varepsilon,\delta,h})Q_1^{1/2} h_t, X_t^{\varepsilon,\delta,h})| \, dt \right] \\
\leq 2E \left[ \int_0^T \|\sigma_1(X_t^{\varepsilon,\delta,h})Q_1^{1/2}\|_{L_2} \|h_t\|_{H} \|X_t^{\varepsilon,\delta,h}\|_{H} \, dt \right] \\
\leq 2E \left[ \sup_{t \in [0,T]} \|X_t^{\varepsilon,\delta,h}\|_{H} \int_0^T \|\sigma_1(X_t^{\varepsilon,\delta,h})Q_1^{1/2}\|_{L_2} \|h_t\|_{H} \, dt \right] \\
\leq \frac{1}{4} E \left[ \sup_{t \in [0,T]} \|X_t^{\varepsilon,\delta,h}\|_{H}^2 \right] + 4E \left( \int_0^T \|\sigma_1(X_t^{\varepsilon,\delta,h})Q_1^{1/2}\|_{L_2} \|h_t\|_{H} \, dt \right)^2 \\
\leq \frac{1}{4} E \left[ \sup_{t \in [0,T]} \|X_t^{\varepsilon,\delta,h}\|_{H}^2 \right] + 4E \left( \int_0^T \|\sigma_1(X_t^{\varepsilon,\delta,h})Q_1^{1/2}\|_{L_2}^2 \, dt \right) \left( \int_0^T \|h_t\|_{H}^2 \, dt \right) \\
\leq \frac{1}{4} E \left[ \sup_{t \in [0,T]} \|X_t^{\varepsilon,\delta,h}\|_{H}^2 \right] + 4M \left[ \int_0^T \|\sigma_1(X_t^{\varepsilon,\delta,h})Q_1^{1/2}\|_{L_2}^2 \, dt \right] \\
\leq \frac{1}{4} E \left[ \sup_{t \in [0,T]} \|X_t^{\varepsilon,\delta,h}\|_{H}^2 \right] + CM + CM \left[ \int_0^T \|X_t^{\varepsilon,\delta,h}\|_{H}^2 \, dt \right], \quad (4.76)
\]

where we used the fact that $h \in \mathcal{A}_M$. Using the Burkholder-Davis-Gundy inequality (see Theorem 1, [21] for the Burkholder-Davis-Gundy inequality for the case $p = 1$ and Theorem 1.1, [11] for the best constant, [56] for BDG inequality in infinite dimensions), we estimate the final term from the right hand side of the inequality (4.75) as

\[
2 \sqrt{E} \left[ \sup_{t \in [0,T]} \left| \int_0^t (\sigma_1(X_s^{\varepsilon,\delta,h})Q_1^{1/2} dW_s, X_s^{\varepsilon,\delta,h}) \right| \right] \\
\leq C \sqrt{E} \left[ \int_0^T \|\sigma_1(X_t^{\varepsilon,\delta,h})Q_1^{1/2}\|_{L_2} \|X_t^{\varepsilon,\delta,h}\|_{H}^2 \, dt \right]^{1/2} \\
\leq C \sqrt{E} \left[ \sup_{t \in [0,T]} \|X_t^{\varepsilon,\delta,h}\|_{H}^2 \left( \int_0^T \|\sigma_1(X_t^{\varepsilon,\delta,h})Q_1^{1/2}\|_{L_2}^2 \, dt \right)^{1/2} \right] \\
\leq \frac{1}{4} E \left[ \sup_{t \in [0,T]} \|X_t^{\varepsilon,\delta,h}\|_{H}^2 \right] + C \varepsilon \left[ \int_0^T \|\sigma_1(X_t^{\varepsilon,\delta,h})Q_1^{1/2}\|_{L_2}^2 \, dt \right] \\
\leq \frac{1}{4} E \left[ \sup_{t \in [0,T]} \|X_t^{\varepsilon,\delta,h}\|_{H}^2 \right] + C \varepsilon \left[ \int_0^T \|X_t^{\varepsilon,\delta,h}\|_{H}^{2p} \, dt \right] + C \varepsilon T, \quad (4.77)
\]

where we used the Assumption 3.1 (A1). Using (4.72), (4.76) and (4.77) in (4.75), we deduce that

\[
E \left[ \sup_{t \in [0,T]} \|X_t^{\varepsilon,\delta,h}\|_{H}^2 + 4\mu \int_0^T \|X_t^{\varepsilon,\delta,h}\|_{V}^2 \, dt + 4\alpha \int_0^T \|X_t^{\varepsilon,\delta,h}\|_{V}^2 \, dt + 4\beta \int_0^T \|X_t^{\varepsilon,\delta,h}\|_{L_{r+1}}^{r+1} \, dt \right]
\]
\[
\leq 2\|x\|_H^2 + C(1 + M + \varepsilon)T + C(1 + M + \varepsilon)E\left[\int_0^T \|X_t^{\varepsilon,\delta,h}\|_H^2 dt\right] + CE\left[\int_0^T \|Y_t^{\varepsilon,\delta,h}\|_H^2 dt\right]
\]
\[
\leq 2\|x\|_H^2 + C\mu,\alpha,\lambda_1L_GT(1 + \|y\|_H^2) + CM + CME\left[\int_0^T \|X_t^{\varepsilon,\delta,h}\|_H^2 dt\right]
\]
\[
+ C\mu,\alpha,\lambda_1L_GT\mathbb{E}\left[\int_0^T \|X_t^{\varepsilon,\delta,h}\|_H^2 dt\right] + C\mu,\alpha,\lambda_1L_GM\left(\frac{\delta}{\varepsilon}\right) T + \mathbb{E}\left(\sup_{t\in[0,T]} \|X_t^{\varepsilon,\delta,h}\|_H^2\right),
\] (4.78)
since \(\varepsilon, \delta \in (0, 1)\). By the Assumption 3.1 (A4), one can choose \(\frac{\delta}{\varepsilon} < \frac{1}{C\mu,\alpha,\lambda_1L_G,M}\), so that
\[
\mathbb{E}\left[\sup_{t\in[0,T]} \|X_t^{\varepsilon,\delta,h}\|_H^2\right] \leq C\mu,\alpha,\lambda_1L_G,M,T\left\{1 + \|x\|_H^2 + \|y\|_H^2 + \mathbb{E}\left[\int_0^T \|X_t^{\varepsilon,\delta,h}\|_H^2 dt\right]\right\}. (4.79)
\]
An application of Gronwall’s inequality in (4.79) implies
\[
\mathbb{E}\left[\sup_{t\in[0,T]} \|X_t^{\varepsilon,\delta,h}\|_H^2\right] \leq C\mu,\alpha,\lambda_1L_G,M,T(1 + \|x\|_H^2 + \|y\|_H^2). (4.80)
\]
Substitution of (4.80) in (4.78) yields the estimate (4.63). Using (4.80) in (4.73), we finally obtain (4.64).

Let us now take \((X^{\varepsilon,\delta,h}, Y^{\varepsilon,\delta,h})\) solves the following stochastic control system:
\[
\begin{aligned}
dX_t^{\varepsilon,\delta,h} &= -[\mu AX_t^{\varepsilon,\delta,h} + B(X_t^{\varepsilon,\delta,h}) + \alpha X_t^{\varepsilon,\delta,h} + \beta C(X_t^{\varepsilon,\delta,h}) - F(X_t^{\varepsilon,\delta,h}, Y_t^{\varepsilon,\delta,h})]dt \\
&\quad + \sigma_1(X_t^{\varepsilon,\delta,h})\mathbb{Q}_1^{1/2}h_t^{\varepsilon}dt + \sqrt{\varepsilon}\sigma_1(X_t^{\varepsilon,\delta,h})\mathbb{Q}_1^{1/2}dW_t,
\end{aligned}
\]
\[
\begin{aligned}
dY_t^{\varepsilon,\delta,h} &= -\frac{1}{\delta}[\mu AY_t^{\varepsilon,\delta,h} + \alpha Y_t^{\varepsilon,\delta,h} + \beta C(Y_t^{\varepsilon,\delta}) - G(X_t^{\varepsilon,\delta,h}, Y_t^{\varepsilon,\delta,h})]dt \\
&\quad + \frac{1}{\sqrt{\varepsilon}}\sigma_2(X_t^{\varepsilon,\delta,h}, Y_t^{\varepsilon,\delta,h})\mathbb{Q}_2^{1/2}h_t^{\varepsilon}dt + \frac{1}{\sqrt{\delta}}\sigma_2(X_t^{\varepsilon,\delta,h}, Y_t^{\varepsilon,\delta,h})\mathbb{Q}_2^{1/2}dW_t,
\end{aligned}
\]
\[
X_0^{\varepsilon,\delta,h} = x, \quad Y_0^{\varepsilon,\delta,h} = y.
\] (4.81)

Using Theorem 4.15, we know that the system (4.81) has a pathwise unique strong solution \((X^{\varepsilon,\delta,h}, Y^{\varepsilon,\delta,h})\) with paths in \(\mathcal{E} \times \mathcal{E}, \mathbb{P}\)-a.s., where \(\mathcal{E} = C([0, T]; \mathbb{H}) \cap L^2(0, T; \mathbb{V}) \cap L^{r+1}(0, T; \mathbb{L}^{r+1})\). Since,
\[
\mathbb{E}\left(\exp\left\{-\frac{1}{\sqrt{\varepsilon}} \int_0^T (h_t^{\varepsilon}, dW_t) - \frac{1}{2\varepsilon} \int_0^T \|h_t^{\varepsilon}\|_H^2 dt\right\}\right) = 1,
\]
the measure \(\hat{\mathbb{P}}\) defined by
\[
d\hat{\mathbb{P}}(\omega) = \exp\left\{-\frac{1}{\sqrt{\varepsilon}} \int_0^T (h_t^{\varepsilon}, dW_t) - \frac{1}{2\varepsilon} \int_0^T \|h_t^{\varepsilon}\|_H^2 dt\right\}d\mathbb{P}(\omega)
\]
is a probability measure on \((\Omega, \mathcal{F}, \mathbb{P})\). Moreover, \(\hat{\mathbb{P}}(\omega)\) is mutually absolutely continuous with respect to \(\mathbb{P}(\omega)\) and by using Girsanov’s Theorem (Theorem 10.14, [23], Appendix, [20]), we have the process
\[
\hat{W}_t := W_t + \frac{1}{\sqrt{\varepsilon}} \int_0^t h_s^{\varepsilon}ds, \quad t \in [0, T],
\]
is a cylindrical Wiener process with respect to \( \{ \mathcal{F}_t \}_{t \geq 0} \) on the probability space \( (\Omega, \mathcal{F}, \hat{P}) \). Thus, we know that (4.11) is the unique strong solution of (4.1) with \( W \) replaced by \( \hat{W} \), on \( (\Omega, \mathcal{F}, \{ \mathcal{F}_t \}_{t \geq 0}, \hat{P}) \). Moreover, the system (4.11) is the same as the system (4.81), and since \( \hat{P} \) and \( P \) are mutually absolutely continuous, we further find that \( (X^\varepsilon_{\delta,h^\varepsilon}, Y^\varepsilon_{\delta,h^\varepsilon}) \) is the unique strong solution of (4.81) on \( (\Omega, \mathcal{F}, \{ \mathcal{F}_t \}_{t \geq 0}, P) \). Thus, the solution of the first equation in (4.81) is represented as

\[
X^\varepsilon_{\delta,h^\varepsilon} = G^\varepsilon \left( W + \frac{1}{\sqrt{\varepsilon}} \int_0^t h_s^\varepsilon \mathrm{d}s \right).
\]

Since our approach is based on the Khasminskii time discretization, we need the following lemma. Similar result for the stochastic 2D Navier-Stokes equation is obtained in [50], for SCBF equations is proved in [62]. Since the system (4.81) is a controlled SCBF equations, we provide a proof here.

Let us first define a sequence of stopping times \( \{ \tau^\varepsilon_t \} \) as

\[
\tau^\varepsilon_t := \inf_{t \geq 0} \left\{ t : \| X^\varepsilon_{\delta,h^\varepsilon} \|_H > R \right\},
\]

for any \( \varepsilon, R > 0 \).

**Lemma 4.16.** For any \( x, y \in \mathbb{H}, T > 0, \varepsilon, \Delta > 0 \) small enough, there exists a constant \( C_{\mu, \alpha, \beta, \lambda_1, L_G, M, T, R} > 0 \) such that

\[
\mathbb{E} \left[ \int_0^{T^\varepsilon_{\Delta}} \| X^\varepsilon_{\delta,h^\varepsilon} - X^\varepsilon_{\delta,h^\varepsilon}(\Delta) \|_H^2 \mathrm{d}t \right] \leq C_{\mu, \alpha, \beta, \lambda_1, L_G, M, T, R} \Delta^{1/2} (1 + \| x \|^2_\mathbb{H} + \| y \|^2_\mathbb{H}).
\]

where \( t(\Delta) := \left\lfloor \frac{T}{\Delta} \right\rfloor \Delta \) and \( \lfloor s \rfloor \) stands for the largest integer which is less than or equal \( s \).

**Proof.** A straightforward calculation gives

\[
\mathbb{E} \left[ \int_0^{T^\varepsilon_{\Delta}} \| X^\varepsilon_{\delta,h^\varepsilon} - X^\varepsilon_{\delta,h^\varepsilon}(\Delta) \|_H^2 \mathrm{d}t \right]
\]

\[
\leq \mathbb{E} \left[ \int_0^\Delta \| X^\varepsilon_{\delta,h^\varepsilon} - x \|_H^2 \chi_{\{ t \leq \tau^\varepsilon_t \}} \mathrm{d}t \right] + \mathbb{E} \left[ \int_\Delta^T \| X^\varepsilon_{\delta,h^\varepsilon} - X^\varepsilon_{\delta,h^\varepsilon}(\Delta) \|_H^2 \chi_{\{ t \leq \tau^\varepsilon_t \}} \mathrm{d}t \right]
\]

\[
\leq C_R (1 + \| x \|^2_\mathbb{H}) \Delta + 2 \mathbb{E} \left[ \int_\Delta^T \| X^\varepsilon_{\delta,h^\varepsilon} - X^\varepsilon_{\delta,h^\varepsilon}(\Delta) \|_H^2 \chi_{\{ t \leq \tau^\varepsilon_t \}} \mathrm{d}t \right]
\]

\[
+ 2 \mathbb{E} \left[ \int_\Delta^T \| X^\varepsilon_{\delta,h^\varepsilon} - X^\varepsilon_{\delta,h^\varepsilon}(\Delta) \|_H^2 \chi_{\{ t \leq \tau^\varepsilon_t \}} \mathrm{d}t \right].
\]

Let us first estimate the second term from the right hand side of the inequality (4.81). Using the infinite dimensional Itô formula applied to the process \( Z_t = \| X^\varepsilon_{\delta,h^\varepsilon} - X^\varepsilon_{\delta,h^\varepsilon}(\Delta) \|_H^2 \) over the interval \( [t - \Delta, t] \), we find

\[
\| X^\varepsilon_{\delta,h^\varepsilon} - X^\varepsilon_{\delta,h^\varepsilon}(\Delta) \|_H^2 = -2\mu \int_{t-\Delta}^t \langle A X^\varepsilon_{\delta,h^\varepsilon}, X^\varepsilon_{\delta,h^\varepsilon} - X^\varepsilon_{\delta,h^\varepsilon}(\Delta) \rangle \mathrm{d}s - 2\alpha \int_{t-\Delta}^t \langle X^\varepsilon_{\delta,h^\varepsilon}, X^\varepsilon_{\delta,h^\varepsilon} - X^\varepsilon_{\delta,h^\varepsilon}(\Delta) \rangle \mathrm{d}s
\]
Similarly, we estimate

\[ E \left( \int_{\Delta}^T (B(X_s^{\varepsilon,\delta,h^\varepsilon}, X_s^{\varepsilon,\delta,h^\varepsilon} - X_{t-\Delta}^{\varepsilon,\delta,h^\varepsilon}) ds - 2\beta \int_{t-\Delta}^t (C(X_s^{\varepsilon,\delta,h^\varepsilon}, X_s^{\varepsilon,\delta,h^\varepsilon} - X_{t-\Delta}^{\varepsilon,\delta,h^\varepsilon}) ds \right) \\
+ 2 \int_{t-\Delta}^t (F(X_s^{\varepsilon,\delta,h^\varepsilon}, Y_s^{\varepsilon,\delta,h^\varepsilon}, X_{t-\Delta}^{\varepsilon,\delta,h^\varepsilon}) ds) \\
+ \varepsilon \int_{t-\Delta}^t \|\sigma_1(X_s^{\varepsilon,\delta,h^\varepsilon})Q^{1/2}_1 \|_{L^2}^2 ds + 2\sqrt{\varepsilon} \int_{t-\Delta}^t (\sigma_1(X_s^{\varepsilon,\delta,h^\varepsilon})Q^{1/2}_1 dW_s, X_s^{\varepsilon,\delta,h^\varepsilon} - X_{t-\Delta}^{\varepsilon,\delta,h^\varepsilon}) ds \\
=: \sum_{k=1}^8 I_k(t). \quad (4.85) \]

Using an integration by parts, Hölder’s inequality, Fubini’s Theorem and (4.83), we estimate

\[ \mathbb{E} \left( \int_{\Delta}^T |I_1(t)| \chi(t \leq \tau_{\varepsilon_k}) dt \right) \] (see (4.28) also)

\[
\begin{align*}
\mathbb{E} \left( \int_{\Delta}^T |I_1(t)| \chi(t \leq \tau_{\varepsilon_k}) dt \right) & \leq 2\mu \left[ \mathbb{E} \left( \int_{\Delta}^T \int_{t-\Delta}^t \|X_s^{\varepsilon,\delta,h^\varepsilon}\|_{\mathbb{H}}^2 \chi(t \leq \tau_{\varepsilon_k}) ds dt \right) \right]^{1/2} \left[ \mathbb{E} \left( \int_{\Delta}^T \int_{t-\Delta}^t \|X_s^{\varepsilon,\delta,h^\varepsilon} - X_{t-\Delta}^{\varepsilon,\delta,h^\varepsilon}\|_{\mathbb{H}}^2 \chi(t \leq \tau_{\varepsilon_k}) ds dt \right) \right]^{1/2} \\
& \leq 2\mu \left[ \Delta \mathbb{E} \left( \int_{0}^T \|X_t^{\varepsilon,\delta,h^\varepsilon}\|_{\mathbb{H}}^2 dt \right) \right]^{1/2} \left[ 2\Delta \mathbb{E} \left( \int_{0}^T \|X_t^{\varepsilon,\delta,h^\varepsilon}\|_{\mathbb{H}}^2 dt \right) \right]^{1/2} \\
& \leq C_{\mu,\alpha,\lambda_1,\mathbb{L},T} \Delta (1 + \|x\|_{\mathbb{H}}^2 + \|y\|_{\mathbb{H}}^2). \quad (4.86)
\end{align*}
\]

Similarly, we estimate \( \mathbb{E} \left( \int_{\Delta}^T |I_2(t)| \chi(t \leq \tau_{\varepsilon_k}) dt \right) \) as

\[
\begin{align*}
\mathbb{E} \left( \int_{\Delta}^T |I_2(t)| \chi(t \leq \tau_{\varepsilon_k}) dt \right) & \leq C_{\alpha} \Delta \mathbb{E} \left( \sup_{t \in [0,T]} \|X_t^{\varepsilon,\delta,h^\varepsilon}\|_{\mathbb{H}}^2 \right) \leq C_{\mu,\alpha,\lambda_1,\mathbb{L},T} \Delta (1 + \|x\|_{\mathbb{H}}^2 + \|y\|_{\mathbb{H}}^2). \quad (4.87)
\end{align*}
\]

For \( n = 2 \) and \( r \in [1,3) \), using Hölder’s and Ladyzhenskaya’s inequalities, Fubini’s Theorem and (4.63), we estimate \( \mathbb{E} \left( \int_{\Delta}^T |I_3(t)| \chi(t \leq \tau_{\varepsilon_k}) dt \right) \) as (see (4.30) also)

\[
\begin{align*}
\mathbb{E} \left( \int_{\Delta}^T |I_3(t)| \chi(t \leq \tau_{\varepsilon_k}) dt \right) & \leq 2\sqrt{2} \left[ \mathbb{E} \left( \int_{\Delta}^T \int_{t-\Delta}^t \|X_s^{\varepsilon,\delta,h^\varepsilon}\|_{\mathbb{H}}^2 \|X_s^{\varepsilon,\delta,h^\varepsilon}\|_{\mathbb{H}}^2 \chi(t \leq \tau_{\varepsilon_k}) ds dt \right) \right]^{1/2} \\
& \times \left[ \mathbb{E} \left( \int_{\Delta}^T \int_{t-\Delta}^t \|X_s^{\varepsilon,\delta,h^\varepsilon} - X_{t-\Delta}^{\varepsilon,\delta,h^\varepsilon}\|_{\mathbb{H}}^2 \chi(t \leq \tau_{\varepsilon_k}) ds dt \right) \right]^{1/2} \\
& \leq 2\sqrt{2} \left[ \Delta \mathbb{E} \left( \int_{0}^T \|X_t^{\varepsilon,\delta,h^\varepsilon}\|_{\mathbb{H}}^2 dt \right) \right]^{1/2} \left[ 2\Delta \mathbb{E} \left( \int_{0}^T \|X_t^{\varepsilon,\delta,h^\varepsilon}\|_{\mathbb{H}}^2 dt \right) \right]^{1/2} \\
& \leq C_{\mu,\alpha,\lambda_1,\mathbb{L},T} \Delta (1 + \|x\|_{\mathbb{H}}^2 + \|y\|_{\mathbb{H}}^2). \quad (4.88)
\end{align*}
\]
For \( n = 2, 3 \) and \( r \geq 3 \) (take \( 2\beta\mu > 1 \), for \( r = 3 \)), we estimate \( \mathbb{E}\left( \int_\Delta^T |I_3(t)| \chi(t \leq \tau_{\varepsilon_k}) \right) \) using Hölder’s inequality, interpolation inequality and (4.63) as (see (4.31) also)

\[
\mathbb{E}\left( \int_\Delta^T |I_3(t)| \chi(t \leq \tau_{\varepsilon_k}) \right)
\leq 2 \left[ \mathbb{E}\left( \int_\Delta^T \int_{t-\Delta}^t \|X_s^{\varepsilon,\delta,h^k}\|_{\mathbb{H}}^{2(r-1)} \|X_s^{\varepsilon,\delta,h^k}\|_{\mathbb{E}^{r+1}}^{2(r+1)} \chi(t \leq \tau_{\varepsilon_k}) \right) \right]^{1/2}
\times \left[ \mathbb{E}\left( \int_\Delta^T \int_{t-\Delta}^t \|X_s^{\varepsilon,\delta,h^k} - X_{t-\Delta}^{\varepsilon,\delta,h^k}\|^2 \chi(t \leq \tau_{\varepsilon_k}) \right) \right]^{1/2}
\leq 2 \left[ \Delta \mathbb{E}\left( \int_0^T \|X_t^{\varepsilon,\delta,h^k}\|_{\mathbb{H}}^{2(r-1)} \|X_t^{\varepsilon,\delta,h^k}\|_{\mathbb{E}^{r+1}}^{2(r+1)} \chi(t \leq \tau_{\varepsilon_k}) \right) \right]^{1/2}
\times 2 \left[ \Delta \mathbb{E}\left( \int_0^T \|X_t^{\varepsilon,\delta,h^k}\|^2 \chi(t \leq \tau_{\varepsilon_k}) \right) \right]^{1/2}
\leq C_{\mu,\alpha,\lambda_1,L_G,T} \Delta \left( 1 + \|x\|_\mathbb{H}^2 + \|y\|_\mathbb{H}^2 \right)^{\frac{1}{2} \frac{1}{r-1}}.
\]

(4.89) since \( \frac{r+1}{2(r-1)} \leq 1 \), for all \( r \geq 3 \). Once again using Hölder’s inequality, Fubini’s Theorem and (4.63), we estimate \( \mathbb{E}\left( \int_\Delta^T |I_4(t)| \chi(t \leq \tau_{\varepsilon_k}) \right) \) as (see (4.32) also)

\[
\mathbb{E}\left( \int_\Delta^T |I_4(t)| \chi(t \leq \tau_{\varepsilon_k}) \right)
\leq 2\beta \left[ \mathbb{E}\left( \int_\Delta^T \int_{t-\Delta}^t \|X_s^{\varepsilon,\delta,h^k}\|_{\mathbb{E}^{r+1}}^{r} \|X_s^{\varepsilon,\delta,h^k} - X_{t-\Delta}^{\varepsilon,\delta,h^k}\|_{\mathbb{E}^{r+1}}^{r+1} \chi(t \leq \tau_{\varepsilon_k}) \right) \right]^{1/2}
\leq 2\beta \Delta \mathbb{E}\left( \int_0^T \|X_t^{\varepsilon,\delta,h^k}\|_{\mathbb{E}^{r+1}}^{r+1} \chi(t \leq \tau_{\varepsilon_k}) \right) \left[ \Delta \mathbb{E}\left( \int_0^T \|X_t^{\varepsilon,\delta,h^k}\|_{\mathbb{H}}^2 \chi(t \leq \tau_{\varepsilon_k}) \right) \right]^{1/2}
\leq C_{\mu,\alpha,\lambda_1,L_G,T} \Delta \left( 1 + \|x\|_\mathbb{H}^2 + \|y\|_\mathbb{H}^2 \right).
\]

(4.90)

We estimate \( \mathbb{E}\left( \int_\Delta^T |I_5(t)| \chi(t \leq \tau_{\varepsilon_k}) \right) \) using the Assumption 3.1 (A1), (4.63) and (4.64) as (see (4.33) also)

\[
\mathbb{E}\left( \int_\Delta^T |I_5(t)| \chi(t \leq \tau_{\varepsilon_k}) \right)
\leq 2 \left[ \mathbb{E}\left( \int_\Delta^T \int_{t-\Delta}^t \|F(X_s^{\varepsilon,\delta,h^k}, Y_s^{\varepsilon,\delta,h^k})\|_{\mathbb{H}}^2 \chi(t \leq \tau_{\varepsilon_k}) \right) \right]^{1/2}
\times \left[ \mathbb{E}\left( \int_\Delta^T \int_{t-\Delta}^t \|X_s^{\varepsilon,\delta,h^k} - X_{t-\Delta}^{\varepsilon,\delta,h^k}\|_{\mathbb{H}}^2 \chi(t \leq \tau_{\varepsilon_k}) \right) \right]^{1/2}
\leq C \Delta \mathbb{E}\left( \int_0^T (1 + \|X_t^{\varepsilon,\delta,h^k}\|_{\mathbb{H}}^2 + \|Y_t^{\varepsilon,\delta,h^k}\|_{\mathbb{H}}^2) \chi(t \leq \tau_{\varepsilon_k}) \right) \left[ \Delta \mathbb{E}\left( \int_0^T \|X_t^{\varepsilon,\delta,h^k}\|_{\mathbb{H}}^2 \chi(t \leq \tau_{\varepsilon_k}) \right) \right]^{1/2}
\leq C_{\mu,\alpha,\lambda_1,L_G,T} \Delta \left( 1 + \|x\|_\mathbb{H}^2 + \|y\|_\mathbb{H}^2 \right).
\]

(4.91)
The term $\mathbb{E}\left(\int_{\Delta}^{T} |I_6(t)|\chi_{(t \leq \tau^\varepsilon_R)} dt\right)$ can be estimated using the Assumption 3.1 (A1) and (4.63) as (see (4.84) also)

$$\mathbb{E}\left(\int_{\Delta}^{T} |I_6(t)|\chi_{(t \leq \tau^\varepsilon_R)} dt\right) \leq 2 \left[ \mathbb{E}\left(\int_{\Delta}^{T} \int_{t-\Delta}^{t} \|\sigma_1(X^\varepsilon,\delta,h^\varepsilon_s)Q^1_{2,1}\| h^\varepsilon_{t-\Delta}^2 \chi_{(t \leq \tau^\varepsilon_R)} dsdt \right)^{1/2}\right] \leq C \left[ \Delta \mathbb{E}\left(\int_{0}^{T} (1 + \|X^\varepsilon,h^\varepsilon_t\|_H^2) h^\varepsilon_{t-\Delta}^2 \chi_{(t \leq \tau^\varepsilon_R)} dt \right)^{1/2} \right] \leq C_{\mu,\alpha,R_1,L_0,M,T,R} \Delta (1 + \|x\|_H^2 + \|y\|_H^2).$$

(4.92)

Once again using the Assumption 3.1 (A1) and (4.63), we estimate $\mathbb{E}\left(\int_{\Delta}^{T} |I_7(t)|\chi_{(t \leq \tau^\varepsilon_R)} dt\right)$ as

$$\mathbb{E}\left(\int_{\Delta}^{T} |I_7(t)|\chi_{(t \leq \tau^\varepsilon_R)} dt\right) \leq C \mathbb{E}\left(\int_{\Delta}^{T} \int_{t-\Delta}^{t} (1 + \|X^\varepsilon,\delta,h^\varepsilon_s\|_H^2) \chi_{(t \leq \tau^\varepsilon_R)} dsdt \right) \leq C \Delta \mathbb{E}\left[ 1 + \sup_{t \in [0,T]} \|X^\varepsilon,\delta,h^\varepsilon_t\|_H^2 \right] \leq C_{\mu,\alpha,R_1,L_0,T,R} \Delta (1 + \|x\|_H^2 + \|y\|_H^2),$$

(4.93)

since $\varepsilon \in (0,1)$. Finally, using the Burkholder-Davis-Gundy inequality, the Assumption 3.1 (A1), Fubini’s theorem and (4.63), we estimate $\mathbb{E}\left(\int_{\Delta}^{T} |I_8(t)|\chi_{(t \leq \tau^\varepsilon_R)} dt\right)$ as

$$\mathbb{E}\left(\int_{\Delta}^{T} |I_8(t)|\chi_{(t \leq \tau^\varepsilon_R)} dt\right) \leq C \int_{\Delta}^{T} \mathbb{E}\left[ \left(\int_{t-\Delta}^{t} \|\sigma_1(X^\varepsilon,\delta,h^\varepsilon_s)Q^1_{2,1}\| h^\varepsilon_{t-\Delta}^2 \chi_{(t \leq \tau^\varepsilon_R)} ds \right)^{1/2} \right] dt \leq CT^{1/2} \left[ \mathbb{E}\left(\int_{\Delta}^{T} \int_{t-\Delta}^{t} (1 + \|X^\varepsilon,\delta,h^\varepsilon_s\|_H^2) \chi_{(t \leq \tau^\varepsilon_R)} dsdt \right)^{1/2} \right] \leq C_{\mu,\alpha,R_1,L_0,T,R} \Delta^{1/2} (1 + \|x\|_H^2 + \|y\|_H^2).$$

(4.94)

Combining (4.86)-(4.94), we deduce that

$$\mathbb{E}\left[\int_{\Delta}^{T} \|X^\varepsilon,\delta,h^\varepsilon_t - X^\varepsilon,\delta,h^\varepsilon_{t-\Delta}\|_H^2 \chi_{(t \leq \tau^\varepsilon_R)} dt\right] \leq C_{\mu,\alpha,R_1,L_0,M,T,R} \Delta^{1/2} (1 + \|x\|_H^2 + \|y\|_H^2).$$

(4.95)
A similar argument leads to
\[
\mathbb{E} \left[ \int_{\Delta}^{T} \| X_{t(\Delta)}^{\varepsilon, \delta, h_{\varepsilon}} - X_{t-\Delta}^{\varepsilon, \delta, h_{\varepsilon}} \|^2_H \chi_{(t \leq \tau_{\Delta})} dt \right] \leq C_{\mu, \alpha, \lambda_1, L_G, M, T} \Delta^{1/2} (1 + \| x \|^2_H + \| y \|^2_H). 
\] (4.96)

Combining (4.84), (4.95) and (4.96), we obtain the required result (4.83). □

4.7. Estimates of auxiliary process \( \hat{Y}^{\varepsilon, \delta}_t \). We use the method proposed by Khasminskii, in [45] to obtain the estimates for an auxiliary process. We introduce the auxiliary process \( \hat{Y}^{\varepsilon, \delta}_t \in \mathbb{H} \) (see (4.97) below) and divide the interval \([0, T]\) into subintervals of size \( \Delta \), where \( \Delta \) is a fixed positive number, which depends on \( \delta \) and it will be chosen later. Let us construct the process \( \hat{Y}^{\varepsilon, \delta}_t \) with the initial value \( \hat{Y}^{\varepsilon, \delta}_0 = Y^{\varepsilon, \delta}_0 = y \), and for any \( k \in \mathbb{N} \) and \( t \in [k\Delta, \min\{(k + 1)\Delta, T\}] \) as
\[
\hat{Y}^{\varepsilon, \delta}_t = \hat{Y}^{\varepsilon, \delta}_{k\Delta} - \frac{\mu}{\delta} \int_{k\Delta}^{t} \hat{A}_{s} \hat{Y}^{\varepsilon, \delta}_s ds - \frac{\alpha}{\delta} \int_{k\Delta}^{t} \hat{C}_{s} \hat{Y}^{\varepsilon, \delta}_s ds - \frac{\beta}{\delta} \int_{k\Delta}^{t} C(\hat{Y}^{\varepsilon, \delta}_s) ds + \frac{1}{\sqrt{\delta}} \int_{k\Delta}^{t} \sigma_2(X_{k\Delta}^{\varepsilon, \delta, h_{\varepsilon}}, \hat{Y}^{\varepsilon, \delta}_s) Q_2^{1/2} dW_s, \quad \mathbb{P}\text{-a.s.},
\] (4.97)
which is equivalent to
\[
\begin{cases}
    d\hat{Y}^{\varepsilon, \delta}_t = -\frac{1}{\delta} \left[ \mu A \hat{Y}^{\varepsilon, \delta}_t + \alpha \hat{C} \hat{Y}^{\varepsilon, \delta}_t + \beta C(\hat{Y}^{\varepsilon, \delta}_t) - G(X_{t(\Delta)}^{\varepsilon, \delta, h_{\varepsilon}}, \hat{Y}^{\varepsilon, \delta}_t) \right] dt \\
    \hat{Y}^{\varepsilon, \delta}_0 = y.
\end{cases}
\] (4.98)

The following energy estimate satisfied by \( \hat{Y}^{\varepsilon, \delta}_t \) can be proved in a similar way as in Theorem 4.15 (see Lemma 4.2, [62] also).

**Lemma 4.17.** For any \( x, y \in \mathbb{H}, T > 0 \) and \( \varepsilon \in (0, 1) \), there exists a constant \( C_{\mu, \alpha, \lambda_1, L_G, M, T} > 0 \) such that the strong solution \( Y^{\varepsilon, \delta}_t \) to the system (4.98) satisfies:
\[
\sup_{t \in [0, T]} \mathbb{E} \left[ \| \hat{Y}^{\varepsilon, \delta}_t \|^2_H \right] \leq C_{\mu, \alpha, \lambda_1, L_G, M, T} \left( 1 + \| x \|^2_H + \| y \|^2_H \right). 
\] (4.99)

Our next aim is to establish an estimate on the difference between the processes \( Y^{\varepsilon, \delta, h_{\varepsilon}}_t \) and \( \hat{Y}^{\varepsilon, \delta}_t \).

**Lemma 4.18.** For any \( x, y \in \mathbb{H}, T > 0 \) and \( \varepsilon, \delta \in (0, 1) \), there exists a constant \( C_{\mu, \alpha, \lambda_1, L_G, T} > 0 \) such that
\[
\mathbb{E} \left( \int_{0}^{T} \| Y^{\varepsilon, \delta, h_{\varepsilon}}_t - \hat{Y}^{\varepsilon, \delta}_t \|^2_H dt \right) \leq C_{\mu, \alpha, \lambda_1, L_G, T} \left( 1 + \| x \|^2_H + \| y \|^2_H \right) \left( \frac{\delta}{\varepsilon} \right) + \Delta^{1/2}. 
\] (4.100)
Proof. Let us define $U_t^\varepsilon := Y_t^{\varepsilon,\delta,h^\varepsilon} - \tilde{Y}_t^{\varepsilon,\delta}$. Then $U_t^\varepsilon$ satisfies the following Itô stochastic differential:

$$
\begin{align*}
    dU_t^\varepsilon &= -\frac{1}{\delta} \left[ \mu A U_t^\varepsilon + \alpha U_t^\varepsilon + \beta (C(Y_t^{\varepsilon,\delta,h^\varepsilon}) - C(\tilde{Y}_t^{\varepsilon,\delta})) \right] dt \\
    &+ \frac{1}{\delta} \left[ (G(X_t^{\varepsilon,\delta,h^\varepsilon}, Y_t^{\varepsilon,\delta,h^\varepsilon}) - G(X_t^{\varepsilon,\delta,h^\varepsilon}, \tilde{Y}_t^{\varepsilon,\delta})) \right] dt \\
    &+ \frac{1}{\sqrt{\delta}} \sigma_2(X_t^{\varepsilon,\delta,h^\varepsilon}, Y_t^{\varepsilon,\delta,h^\varepsilon}) Q_2^{1/2} h_t^\varepsilon dt \\
    &+ \frac{1}{\sqrt{\delta}} \left[ \sigma_2(X_t^{\varepsilon,\delta,h^\varepsilon}, Y_t^{\varepsilon,\delta,h^\varepsilon}) - \sigma_2(X_t^{\varepsilon,\delta,h^\varepsilon}, \tilde{Y}_t^{\varepsilon,\delta}) \right] Q_2^{1/2} dW_t,
\end{align*}
$$

(4.101)

An application of the infinite dimensional Itô formula to the process $\|U_t^\varepsilon\|_H^2$ yields

$$
\|U_t^\varepsilon\|_H^2 = -\frac{2\mu}{\delta} \int_0^t \|U_s^\varepsilon\|^2 \, ds - \frac{2\alpha}{\delta} \int_0^t \|U_s^\varepsilon\|_{H_0^2}^2 ds - \frac{2\beta}{\delta} \int_0^t \langle C(Y_s^{\varepsilon,\delta,h^\varepsilon}) - C(\tilde{Y}_s^{\varepsilon,\delta}), U_s^\varepsilon \rangle \, ds \\
+ \frac{2}{\delta} \int_0^t (G(X_s^{\varepsilon,\delta,h^\varepsilon}, Y_s^{\varepsilon,\delta,h^\varepsilon}) - G(X_s^{\varepsilon,\delta,h^\varepsilon}, \tilde{Y}_s^{\varepsilon,\delta}), U_s^\varepsilon) \, ds \\
+ \frac{2}{\sqrt{\delta}} \int_0^t \langle \sigma_2(X_s^{\varepsilon,\delta,h^\varepsilon}, Y_s^{\varepsilon,\delta,h^\varepsilon}) Q_2^{1/2} h_s^\varepsilon, U_s^\varepsilon \rangle ds \\
+ \frac{1}{\sqrt{\delta}} \int_0^t \|\sigma_2(X_s^{\varepsilon,\delta,h^\varepsilon}, Y_s^{\varepsilon,\delta,h^\varepsilon}) - \sigma_2(X_s^{\varepsilon,\delta,h^\varepsilon}, \tilde{Y}_s^{\varepsilon,\delta})\|_{L_2^2}^2 ds, \text{ P-a.s.,}
$$

(4.102)

for all $t \in [0, T]$. Taking expectation in (4.102), we obtain

$$
\begin{align*}
    \mathbb{E}[\|U_t^\varepsilon\|_H^2] &= -\frac{2\mu}{\delta} \int_0^t \mathbb{E}[\|U_s^\varepsilon\|^2] \, ds - \frac{2\alpha}{\delta} \int_0^t \mathbb{E}[\|U_s^\varepsilon\|_{H_0^2}^2] ds \\
    &- \frac{2\beta}{\delta} \mathbb{E} \left[ \int_0^t \langle C(Y_s^{\varepsilon,\delta,h^\varepsilon}) - C(\tilde{Y}_s^{\varepsilon,\delta}), U_s^\varepsilon \rangle \, ds \right] \\
    &+ \frac{2}{\delta} \int_0^t \mathbb{E} \left[ (G(X_s^{\varepsilon,\delta,h^\varepsilon}, Y_s^{\varepsilon,\delta,h^\varepsilon}) - G(X_s^{\varepsilon,\delta,h^\varepsilon}, \tilde{Y}_s^{\varepsilon,\delta}), U_s^\varepsilon) \right] ds \\
    &+ \frac{2}{\sqrt{\delta}} \int_0^t \mathbb{E} \left[ \langle \sigma_2(X_s^{\varepsilon,\delta,h^\varepsilon}, Y_s^{\varepsilon,\delta,h^\varepsilon}) Q_2^{1/2} h_s^\varepsilon, U_s^\varepsilon \rangle \right] ds \\
    &+ \frac{1}{\sqrt{\delta}} \int_0^t \mathbb{E} \left[ \|\sigma_2(X_s^{\varepsilon,\delta,h^\varepsilon}, Y_s^{\varepsilon,\delta,h^\varepsilon}) - \sigma_2(X_s^{\varepsilon,\delta,h^\varepsilon}, \tilde{Y}_s^{\varepsilon,\delta})\|_{L_2^2}^2 ds \right].
\end{align*}
$$

(4.103)

Thus, it is immediate that

$$
\begin{align*}
    \frac{d}{dt} \mathbb{E}[\|U_t^\varepsilon\|_H^2] &= -\frac{2\mu}{\delta} \mathbb{E}[\|U_t^\varepsilon\|^2] - \frac{2\alpha}{\delta} \mathbb{E}[\|U_t^\varepsilon\|_{H_0^2}^2] - \frac{2\beta}{\delta} \mathbb{E} \left[ \langle C(Y_t^{\varepsilon,\delta,h^\varepsilon}) - C(\tilde{Y}_t^{\varepsilon,\delta}), U_t^\varepsilon \rangle \right] \\
    &+ \frac{2}{\delta} \mathbb{E} \left[ (G(X_t^{\varepsilon,\delta,h^\varepsilon}, Y_t^{\varepsilon,\delta,h^\varepsilon}) - G(X_t^{\varepsilon,\delta,h^\varepsilon}, \tilde{Y}_t^{\varepsilon,\delta}), U_t^\varepsilon) \right] \\
    &+ \frac{2}{\sqrt{\delta}} \mathbb{E} \left[ \langle \sigma_2(X_t^{\varepsilon,\delta,h^\varepsilon}, Y_t^{\varepsilon,\delta,h^\varepsilon}) Q_2^{1/2} h_t^\varepsilon, U_t^\varepsilon \rangle \right].
\end{align*}
$$
Similarly, using the Assumption 3.1 (A1), we obtain

$$+rac{1}{\delta} \mathbb{E} \left[ \left\| \sigma_2(X_t^{\varepsilon, \delta, h^\varepsilon}, Y_t^{\varepsilon, \delta, h^\varepsilon}) - \sigma_2(X_{t(\Delta)}^{\varepsilon, \delta, h^\varepsilon}, \bar{Y}_t^{\varepsilon, \delta}) \right\|_{L_2}^{1/2} \right], \quad (4.104)$$

for a.e. $t \in [0, T]$. Applying (2.11), we find

$$- \frac{2\beta}{\varepsilon}(C(Y_t^{\varepsilon, \delta, h^\varepsilon}) - C(\bar{Y}_t^{\varepsilon, \delta}), U_t^{\varepsilon}) \leq - \frac{\beta}{2\varepsilon^{p-2}} \|U_t^{\varepsilon}\|_{L_\varepsilon^{p+1}}^{p+1}. \quad (4.105)$$

Using the Assumption 3.1 (A1), we get

$$\frac{2}{\delta}(G(X_t^{\varepsilon, \delta, h^\varepsilon}, Y_t^{\varepsilon, \delta, h^\varepsilon}) - G(X_{t(\Delta)}^{\varepsilon, \delta, h^\varepsilon}, \bar{Y}_t^{\varepsilon, \delta}), U_t^{\varepsilon})$$

$$\leq \frac{2}{\delta}\|G(X_t^{\varepsilon, \delta, h^\varepsilon}, Y_t^{\varepsilon, \delta, h^\varepsilon}) - G(X_{t(\Delta)}^{\varepsilon, \delta, h^\varepsilon}, \bar{Y}_t^{\varepsilon, \delta})\|_{H} \|U_t^{\varepsilon}\|_{H}$$

$$\leq \frac{C}{\delta}\|X_t^{\varepsilon, \delta, h^\varepsilon} - X_{t(\Delta)}^{\varepsilon, \delta, h^\varepsilon}\|_{H} \|U_t^{\varepsilon}\|_{H} + \frac{2L_G}{\delta} \|U_t^{\varepsilon}\|_{H}^2$$

$$\leq \left( \frac{\mu_1}{2\delta} + \frac{2L_G}{\delta} \right) \|U_t^{\varepsilon}\|_{H}^2 + \frac{C}{\mu_1\delta}\|X_t^{\varepsilon, \delta, h^\varepsilon} - X_{t(\Delta)}^{\varepsilon, \delta, h^\varepsilon}\|_{H}^2 \quad (4.106)$$

Making use of the Assumption 3.1 (A2), H"older’s and Young’s inequalities, we have

$$\frac{2}{\sqrt{\delta}}(\sigma_2(X_t^{\varepsilon, \delta, h^\varepsilon}, Y_t^{\varepsilon, \delta, h^\varepsilon})Q_2^{1/2}h_t^{\varepsilon}, U_t^{\varepsilon})$$

$$\leq \frac{2}{\sqrt{\delta}}\|\sigma_2(X_t^{\varepsilon, \delta, h^\varepsilon}, Y_t^{\varepsilon, \delta, h^\varepsilon})Q_2^{1/2}\|_{L_2} \|h_t^{\varepsilon}\|_{H} \|U_t^{\varepsilon}\|_{H}$$

$$\leq \frac{\mu_1}{2\delta}\|U_t^{\varepsilon}\|_{H}^2 + \frac{C}{\mu_1\delta}\|\sigma_2(X_t^{\varepsilon, \delta, h^\varepsilon}, Y_t^{\varepsilon, \delta, h^\varepsilon})Q_2^{1/2}\|_{L_2}^2 \|h_t^{\varepsilon}\|_{H}^2$$

$$\leq \frac{\mu_1}{2\delta}\|U_t^{\varepsilon}\|_{H}^2 + \frac{C}{\mu_1\delta}(1 + \|X_t^{\varepsilon, \delta, h^\varepsilon}\|_{H}^2) \|h_t^{\varepsilon}\|_{H}^2 \quad (4.107)$$

Similarly, using the Assumption 3.1 (A1), we obtain

$$\frac{1}{\delta}\|\sigma_2(X_t^{\varepsilon, \delta, h^\varepsilon}, Y_t^{\varepsilon, \delta, h^\varepsilon}) - \sigma_2(X_{t(\Delta)}^{\varepsilon, \delta, h^\varepsilon}, \bar{Y}_t^{\varepsilon, \delta})\|_{L_2}^{1/2} \leq \frac{C}{\delta}\|X_t^{\varepsilon, \delta, h^\varepsilon} - X_{t(\Delta)}^{\varepsilon, \delta, h^\varepsilon}\|_{H}^2 + \frac{2L_G^2}{\delta} \|U_t^{\varepsilon}\|_{H}^2 \quad (4.108)$$

Combining (4.105)-(4.108) and substituting it in (4.104), we deduce that

$$\frac{d}{dt}\mathbb{E}[\|U_t^{\varepsilon}\|_{H}^2] \leq - \frac{1}{\delta}(\mu_1\lambda_1 + 2\lambda_2 - 2L_G - 2L_{G_2}) \mathbb{E}[\|U_t^{\varepsilon}\|_{H}^2] + \frac{C}{\mu_1\lambda_1}\mathbb{E}[\|X_t^{\varepsilon, \delta, h^\varepsilon}\|_{H}^2] \|h_t^{\varepsilon}\|_{H}^2$$

$$+ \frac{C}{\delta}(1 + \frac{1}{\mu_1\lambda_1}) \mathbb{E}[\|X_t^{\varepsilon, \delta, h^\varepsilon} - X_{t(\Delta)}^{\varepsilon, \delta, h^\varepsilon}\|_{H}^2]. \quad (4.109)$$

Using the Assumption 3.1 (A3) and variation of constants formula, from (4.109), we infer that

$$\mathbb{E}[\|U_t^{\varepsilon}\|_{H}^2] \leq \frac{C}{\mu_1\lambda_1}\int_0^t e^{-\frac{\theta}{\Delta}(t-s)} \mathbb{E}[\|X_s^{\varepsilon, \delta, h^\varepsilon}\|_{H}^2] \|h_s^{\varepsilon}\|_{H}^2] ds$$

$$+ \frac{C}{\delta}(1 + \frac{1}{\mu_1\lambda_1}) \int_0^t e^{-\frac{\theta}{\Delta}(t-s)} \mathbb{E}[\|X_s^{\varepsilon, \delta, h^\varepsilon} - X_{s(\Delta)}^{\varepsilon, \delta, h^\varepsilon}\|_{H}^2] ds, \quad (4.110)$$
where \( \kappa = \mu \lambda_1 + 2\alpha - 2L_G - 2L^2_{\sigma_2} > 0 \). Applying Fubini’s theorem, for any \( T > 0 \), we have

\[
\mathbb{E} \left[ \int_0^T \| U_t \|^2_{\mathbb{H}} dt \right] \leq \frac{C}{\mu \lambda_1 \varepsilon} \int_0^T \int_0^t e^{-\frac{\kappa}{2}(t-s)} \mathbb{E} \left[ \left( 1 + \| X_{s}^{\varepsilon,\delta,h} \|^2_{\mathbb{H}} \| h_s \|^2_{\mathbb{V}} \right) ds \right] dt
\]

\[
+ \frac{C}{\delta} \left( 1 + \frac{1}{\mu \lambda_1} \right) \int_0^T \int_0^t e^{-\frac{\kappa}{2}(t-s)} \mathbb{E} \left[ \| X_{s}^{\varepsilon,\delta,h} - X_{s(\Delta)}^{\varepsilon,\delta,h} \|^2_{\mathbb{H}} \right] ds
\]

\[
= \frac{C}{\mu \lambda_1 \varepsilon} \mathbb{E} \left[ \int_0^T \left( 1 + \| X_{t(\Delta)}^{\varepsilon,\delta,h} \|^2_{\mathbb{H}} \right) \| h_t \|^2_{\mathbb{V}} \int_s^T e^{-\frac{\kappa}{2}(t-s)} dt ds \right]
\]

\[
+ \frac{C}{\delta} \left( 1 + \frac{1}{\mu \lambda_1} \right) \mathbb{E} \left[ \int_0^T \| X_{t(\Delta)}^{\varepsilon,\delta,h} - X_{t(\Delta)}^{\varepsilon,\delta,h} \|^2_{\mathbb{H}} \left( \int_s^T e^{-\frac{\kappa}{2}(t-s)} dt \right) ds \right]
\]

\[
\leq \frac{C}{\mu \lambda_1 \varepsilon} \left( \frac{\delta}{\varepsilon} \right) \mathbb{E} \left[ \sup_{t \in [0,T]} \left( 1 + \| X_{t(\Delta)}^{\varepsilon,\delta,h} \|^2_{\mathbb{H}} \right) \int_0^T \| h_t \|^2_{\mathbb{V}} dt \right]
\]

\[
+ \frac{C}{\kappa} \left( 1 + \frac{1}{\mu \lambda_1} \right) \mathbb{E} \left[ \int_0^T \| X_{t(\Delta)}^{\varepsilon,\delta,h} - X_{t(\Delta)}^{\varepsilon,\delta,h} \|^2_{\mathbb{H}} \int_s^T e^{-\frac{\kappa}{2}(t-s)} dt ds \right]
\]

\[
\leq C_{\mu,\alpha,\beta,1},L,G,M,T(1 + \| \mathbf{x} \|^2_{\mathbb{H}} + \| \mathbf{y} \|^2_{\mathbb{H}}) \left( \frac{\delta}{\varepsilon} + \Delta^{1/2} \right), \tag{4.111}
\]

using (4.63) and Lemma 4.16 (see (4.83)), which completes the proof. \( \square \)

**Remark 4.19.** It can be easily seen from the stochastic differential equations corresponding to \( \mathbf{Y}_{t}^{\varepsilon,\delta,h} \) and \( \tilde{\mathbf{Y}}_{t}^{\varepsilon,\delta} \) (see (4.81) and (4.98)) that the control term involving \( h_{\varepsilon} \) in \( \mathbf{Y}_{t}^{\varepsilon,\delta,h} \) vanishes in \( \tilde{\mathbf{Y}}_{t}^{\varepsilon,\delta} \). From Lemma 4.16, we infer that the additional control term takes no effect as \( \varepsilon \to 0 \) due to the Assumption 3.3 (A4).

Our next aim is to establish an estimate for the process \( X_{t}^{\varepsilon,\delta,h} - X_{t}^{h} \). For \( n = 2 \) and \( r \in [1, 3] \), we need to construct another stopping time to obtain an estimate for \( \| X_{t}^{\varepsilon,\delta,h} - X_{t}^{h} \|^2_{\mathbb{H}} \). For fixed \( \varepsilon \in (0, 1) \) and \( R > 0 \), we define

\[
\tilde{\tau}_R^\varepsilon := \inf \{ t : \| X_{s}^{\varepsilon,\delta,h} \|^2_{\mathbb{H}} + \int_0^t \| X_{s}^{\varepsilon,\delta,h} \|^2_{\mathbb{V}} ds > R \} \tag{4.112}
\]

It is clear that \( \tilde{\tau}_R^\varepsilon(\omega) \leq \tau_R^\varepsilon(\omega) \), for all \( \omega \in \Omega \). We see in the next Lemma that such a stopping time is not needed for \( r \in (3, \infty) \) for \( n = 2 \) and \( r \in [3, \infty) \) for \( n = 3 \) (\( 2\beta \mu > 1 \), for \( r = n = 3 \)).

**Lemma 4.20.** For any \( \mathbf{x}, \mathbf{y} \in \mathbb{H} \), \( T > 0 \) and \( \varepsilon \in (0, 1) \), the following estimate holds:

\[
\mathbb{E} \left[ \sup_{t \in [0,T \wedge \tilde{\tau}_R^\varepsilon]} \| Z_{t}^\varepsilon \|^2_{\mathbb{H}} + \mu \int_0^{T \wedge \tilde{\tau}_R^\varepsilon} \| Z_{t}^\varepsilon \|^2_{\mathbb{V}} dt + \frac{\beta}{2r-2} \int_0^{T \wedge \tilde{\tau}_R^\varepsilon} \| Z_{t}^\varepsilon \|_{\mathbb{V}^{r+1}} dt \right]
\]

\[
\leq C_{\mu,\alpha,\beta,\lambda_1,L,\lambda_2,M,T,R}(1 + \| \mathbf{x} \|^3_{\mathbb{H}} + \| \mathbf{y} \|^3_{\mathbb{H}}) \left( \frac{\delta}{\varepsilon} + \Delta^{1/8} \right)
\]

\[
+ \mathbb{E} \left[ \int_0^T \| \sigma_1(\mathbf{X}_{t}^{h})Q_{1/2}(h_{t} - h_t) \|_{\mathbb{H}}^2 dt \right], \tag{4.113}
\]

for \( n = 2 \) and \( r \in [1, 3] \), and

\[
\mathbb{E} \left[ \sup_{t \in [0,T \wedge \tilde{\tau}_R^\varepsilon]} \| Z_{t}^\varepsilon \|^2_{\mathbb{H}} + \mu \int_0^{T \wedge \tilde{\tau}_R^\varepsilon} \| Z_{t}^\varepsilon \|^2_{\mathbb{V}} dt + \frac{\beta}{2r-2} \int_0^{T \wedge \tilde{\tau}_R^\varepsilon} \| Z_{t}^\varepsilon \|_{\mathbb{V}^{r+1}} dt \right]
\]
\[ \begin{align*}
&\leq C_{\mu, \alpha, \beta, \lambda_1, \lambda_2, M, T, R}(1 + \|x\|_{\mathbb{H}}^2 + \|y\|_{\mathbb{H}}^2) \left[ \varepsilon^2 + \left( \frac{\delta}{\varepsilon} \right)^2 + \delta^{1/8} \right] \\
&\quad + \mathbb{E} \left[ \int_0^T \|\sigma_1(X_t^h)Q_1^{1/2}(h_t^r - h_t)\|^2_{\mathbb{H}} dt \right], \quad (4.114)
\end{align*} \]

for \( n = 2, r \in (3, \infty) \) and \( n = 3, r \in (3, \infty) \) (2\( \beta \mu > 1 \), for \( r = 3 \)). Here, \( \tau^\varepsilon_R \) and \( \bar{\tau}^\varepsilon_R \) are stopping times defined in (4.82) and (11.12), respectively.

**Proof.** Let us denote \( Z_t^\varepsilon := X_t^{\varepsilon, \delta, h^\varepsilon} - Y_t^{\varepsilon, \delta, h^\varepsilon} \), where \((X_t^{\varepsilon, \delta, h^\varepsilon}, Y_t^{\varepsilon, \delta, h^\varepsilon})\) is the unique strong solution of the system (4.67) and \( \hat{X}_t^h \) is the unique weak solution of the system (4.17). Then \( Z_t^\varepsilon \) satisfies the following Itô stochastic differential:

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{dZ_t^\varepsilon}{\varepsilon} = -[\mu A Z_t^\varepsilon + (B(X_t^{\varepsilon, \delta, h^\varepsilon}) - B(X_t^h))]dt \\
\quad + [F(X_t^{\varepsilon, \delta, h^\varepsilon}, Y_t^{\varepsilon, \delta, h^\varepsilon}) - \hat{F}(X_t^h)]dt + [\sigma_1(X_t^{\varepsilon, \delta, h^\varepsilon})Q_1^{1/2}h_t^r - \sigma_1(X_t^h)Q_1^{1/2}h_t]dt \\
Z_0^\varepsilon = 0.
\end{array} \right.
\end{align*}
\]

**Case 1:** \( n = 2 \) and \( r \in [1, 3] \). We first consider the case \( n = 2 \) and \( r \in [1, 3] \). An application of the infinite dimensional Itô formula to the process \( \|Z_t^\varepsilon\|^2_{\mathbb{H}} \) yields

\[
\begin{align*}
\|Z_t^\varepsilon\|^2_{\mathbb{H}} &= -2\mu \int_0^t \|Z_s^\varepsilon\|^2_{\mathbb{H}} ds - 2\alpha \int_0^t \|Z_s^\varepsilon\|^2_{\mathbb{H}} ds - 2 \int_0^t \langle (B(X_s^{\varepsilon, \delta, h^\varepsilon}) - B(X_s^h)), Z_s^\varepsilon \rangle ds \\
&\quad - 2\beta \int_0^t \langle C(X_s^{\varepsilon, \delta, h^\varepsilon}) - C(X_s^h), Z_s^\varepsilon \rangle ds + 2 \int_0^t \langle (F(X_s^{\varepsilon, \delta, h^\varepsilon}, Y_s^{\varepsilon, \delta, h^\varepsilon}) - \hat{F}(X_s^h), Z_s^\varepsilon \rangle ds \\
&\quad + 2 \int_0^t [\sigma_1(X_s^{\varepsilon, \delta, h^\varepsilon})Q_1^{1/2}h_s^r - \sigma_1(X_s^h)Q_1^{1/2}h_s]ds + \varepsilon \int_0^t \|\sigma_1(X_s^{\varepsilon, \delta, h^\varepsilon})Q_1^{1/2}\|^2_{L_2} ds \\
&\quad + 2\sqrt{\varepsilon} \int_0^t \langle \sigma_1(X_s^{\varepsilon, \delta, h^\varepsilon})Q_1^{1/2}dW_s, Z_s^\varepsilon \rangle \\
&= -2\mu \int_0^t \|Z_s^\varepsilon\|^2_{\mathbb{H}} ds - 2\alpha \int_0^t \|Z_s^\varepsilon\|^2_{\mathbb{H}} ds - 2 \int_0^t \langle (B(X_s^{\varepsilon, \delta, h^\varepsilon}) - B(X_s^h)), Z_s^\varepsilon \rangle ds \\
&\quad - 2\beta \int_0^t \langle C(X_s^{\varepsilon, \delta, h^\varepsilon}) - C(X_s^h), Z_s^\varepsilon \rangle ds + 2 \int_0^t \langle (F(X_s^{\varepsilon, \delta, h^\varepsilon}, Y_s^{\varepsilon, \delta, h^\varepsilon}) - \hat{F}(X_s^h), Z_s^\varepsilon \rangle ds \\
&\quad + 2 \int_0^t \langle (F(X_s^{\varepsilon, \delta, h^\varepsilon}, Y_s^{\varepsilon, \delta, h^\varepsilon}) - \hat{F}(X_s^{\varepsilon, \delta, h^\varepsilon}), \hat{Y}_s^{\varepsilon, \delta}) + \hat{F}(X_s^{\varepsilon, \delta, h^\varepsilon}, Z_s^\varepsilon \rangle ds \\
&\quad + 2 \int_0^t \langle (F(X_s^{\varepsilon, \delta, h^\varepsilon}, Y_s^{\varepsilon, \delta, h^\varepsilon}) - \hat{F}(X_s^{\varepsilon, \delta, h^\varepsilon}), Z_s^{\varepsilon, \delta}) ds \\
&\quad + 2 \int_0^t [\sigma_1(X_s^{\varepsilon, \delta, h^\varepsilon})Q_1^{1/2}h_s^r - \sigma_1(X_s^h)Q_1^{1/2}h_s]ds + \varepsilon \int_0^t \|\sigma_1(X_s^{\varepsilon, \delta, h^\varepsilon})Q_1^{1/2}\|^2_{L_2} ds \\
&\quad + 2\sqrt{\varepsilon} \int_0^t \langle \sigma_1(X_s^{\varepsilon, \delta, h^\varepsilon})Q_1^{1/2}dW_s, Z_s^\varepsilon \rangle, \quad \mathbb{P}\text{-a.s.}, \quad (4.116)
\end{align*}
\]
for all $t \in [0, T]$. Using Hölder's, Ladyzhenskaya's and Young's inequalities, we estimate
\[ 2|\langle (B(X_s^{\varepsilon, \delta, h^\varepsilon}) - B(X_s^h), Z_s^\varepsilon) \rangle| \leq 2\|X_s^{\varepsilon, \delta, h^\varepsilon}\|_V\|Z_s^\varepsilon\|_V^2 \leq 2\sqrt{2}\|X_s^{\varepsilon, \delta, h^\varepsilon}\|_V\|Z_s^\varepsilon\|_H\|Z_s^\varepsilon\|_V \leq \mu\|Z_s^\varepsilon\|_V^2 + \frac{\mu}{2}\|X_s^{\varepsilon, \delta, h^\varepsilon}\|_V^2\|Z_s^\varepsilon\|_H^2. \]
(4.117)

Using (2.9) and (2.10), we know that
\[ -2\beta\langle C(X_s^{\varepsilon, \delta, h^\varepsilon}) - C(X_s^h), Z_s^\varepsilon \rangle \leq -\frac{\beta}{2\varepsilon - 2}\|Z_s^\varepsilon\|_{L^{r+1}}. \]
(4.118)

for $r \in [1, \infty)$. Using (4.19), Hölder’s and Young’s inequalities, we estimate $2\langle \bar{F}(X_s^{\varepsilon, \delta, h^\varepsilon}) - \bar{F}(X_s^h), Z_s^\varepsilon \rangle$ as
\[ 2\langle \bar{F}(X_s^{\varepsilon, \delta, h^\varepsilon}) - \bar{F}(X_s^h), Z_s^\varepsilon \rangle \leq 2\|\bar{F}(X_s^{\varepsilon, \delta, h^\varepsilon}) - \bar{F}(X_s^h)\|_H\|Z_s^\varepsilon\|_H \leq C_{\mu, \alpha, \lambda_1, L_G, L_G_2}\|Z_s^\varepsilon\|_H^2. \]
(4.119)

Similarly, we estimate
\[ 2\langle F(X_s^{\varepsilon, \delta, h^\varepsilon}, Y_s^{\varepsilon, \delta, h^\varepsilon}) - \bar{F}(X_s^{\varepsilon, \delta, h^\varepsilon}), \bar{Y}_s^{\varepsilon, \delta}, Z_s^\varepsilon \rangle \]
\[ \leq 2\langle F(X_s^{\varepsilon, \delta, h^\varepsilon}, Y_s^{\varepsilon, \delta, h^\varepsilon}) - F(X_s^{\varepsilon, \delta, h^\varepsilon}, \bar{Y}_s^{\varepsilon, \delta}), Z_s^\varepsilon \rangle \]
\[ \leq 2\langle F(X_s^{\varepsilon, \delta, h^\varepsilon}, Y_s^{\varepsilon, \delta, h^\varepsilon}) - F(X_s^{\varepsilon, \delta, h^\varepsilon}, \bar{Y}_s^{\varepsilon, \delta}), \|Z_s^\varepsilon\|_H \rangle \|Z_s^\varepsilon\|_H \]
\[ \leq \|Z_s^\varepsilon\|_H^2 + C_{\mu, \alpha, \lambda_1, L_G, L_G_2}\|X_s^{\varepsilon, \delta, h^\varepsilon} - X_s^{\varepsilon, \delta, h^\varepsilon}\|_H^2 + \|Y_s^{\varepsilon, \delta, h^\varepsilon} - \bar{Y}_s^{\varepsilon, \delta}\|_H^2. \]
(4.120)

Using the Assumption 3.1 (A1), Hölder’s and Young’s inequalities, we estimate the term
\[ 2\int_0^t \langle F(X_s^{\varepsilon, \delta, h^\varepsilon}, \bar{Y}_s^{\varepsilon, \delta}), Z_s^\varepsilon - Z_s^{\varepsilon, h^\varepsilon}\rangle ds \]
\[ \leq 2\int_0^t \langle F(X_s^{\varepsilon, \delta, h^\varepsilon}, \bar{Y}_s^{\varepsilon, \delta}), Z_s^\varepsilon - Z_s^{\varepsilon, h^\varepsilon}\rangle ds \]
\[ \leq 4\left( \int_0^t \|F(X_s^{\varepsilon, \delta, h^\varepsilon}, \bar{Y}_s^{\varepsilon, \delta})\|_H^2 + \|F(X_s^{\varepsilon, \delta, h^\varepsilon})\|_H^2 \right)^{1/2} \]
\[ \times \left( \int_0^t \|X_s^{\varepsilon, \delta, h^\varepsilon} - X_s^{\varepsilon, \delta, h^\varepsilon}\|_H^2 + \|X_s^{\varepsilon, \delta, h^\varepsilon} - X_s^{\varepsilon, \delta, h^\varepsilon}\|_H^2 \right)^{1/2} \]
\[ \leq C\left( \int_0^t (1 + \|X_s^{\varepsilon, \delta, h^\varepsilon}\|_H^2 + \|\bar{Y}_s^{\varepsilon, \delta}\|_H^2) ds \right)^{1/2} \left( \int_0^t (\|X_s^{\varepsilon, \delta, h^\varepsilon} - X_s^{\varepsilon, \delta, h^\varepsilon}\|_H^2 + \|X_s^{\varepsilon, \delta, h^\varepsilon} - X_s^{\varepsilon, \delta, h^\varepsilon}\|_H^2) ds \right)^{1/2}. \]
(4.121)

We estimate the term $2\int_0^t [\sigma_1(X_s^{\varepsilon, \delta, h^\varepsilon}Q_1^{1/2}h_s^\varepsilon - \sigma_1(X_s^h)Q_1^{1/2}h_s^\varepsilon] Z_s^\varepsilon ds$ from the right hand side of the inequality (4.116) using the Cauchy-Schwarz inequality and Young’s inequality, and the Assumption 3.1 (A1) as
\[ 2\int_0^t [\sigma_1(X_s^{\varepsilon, \delta, h^\varepsilon}Q_1^{1/2}h_s^\varepsilon - \sigma_1(X_s^h)Q_1^{1/2}h_s^\varepsilon] Z_s^\varepsilon ds \]
\[ \leq 2\int_0^t |(\sigma_1(X_s^{\varepsilon, \delta, h^\varepsilon} - \sigma_1(X_s^h))Q_1^{1/2}h_s^\varepsilon| Z_s^\varepsilon ds + 2\int_0^t |(\sigma_1(X_s^h)Q_1^{1/2}(h_s^\varepsilon - h_s^\varepsilon))| Z_s^\varepsilon ds \]
Combining (4.117)-(4.123), and then substituting it in (4.116), we obtain

\[
\leq 2\int_0^t \left\| \sigma_1(X_s^{\varepsilon,\delta,h^\varepsilon}) - \sigma_1(\hat{X}_s^{h^\varepsilon}) \right\| Q_{1}^{1/2} \left\| h_s^\varepsilon \right\|_{\mathbb{H}} \left\| Z_s^{\varepsilon} \right\|_{\mathbb{H}} ds \\
+ 2\int_0^t \left\| \sigma_1(\hat{X}_s^{h^\varepsilon}) Q_{1}^{1/2} (h_s^\varepsilon - h_s) \right\|_{\mathbb{H}} \left\| Z_s^{\varepsilon} \right\|_{\mathbb{H}} ds \\
\leq C \int_0^t (1 + \left\| h_s^\varepsilon \right\|_{\mathbb{H}}^2) \left\| Z_s^{\varepsilon} \right\|_{\mathbb{H}}^2 ds + \int_0^t \left\| Z_s^{\varepsilon} \right\|_{\mathbb{H}}^2 ds + \int_0^t \sigma_1(\hat{X}_s^{h^\varepsilon}) Q_{1}^{1/2} (h_s^\varepsilon - h_s) \right\|_{\mathbb{H}}^2 ds.
\]  

(4.122)

Making use of the Assumption 3.1 (A1), it can be easily seen that

\[
\int_0^t \left\| \sigma_1(X_s^{\varepsilon,\delta,h^\varepsilon}) Q_{1}^{1/2} \right\|_{\mathbb{H}}^2 ds \leq 2\int_0^t \left\| \sigma_1(X_s^{\varepsilon,\delta,h^\varepsilon}) - \sigma_1(\hat{X}_s^{h^\varepsilon}) \right\| Q_{1}^{1/2} \left\| h_s^\varepsilon \right\|_{\mathbb{H}}^2 ds + 2\int_0^t \left\| \sigma_1(\hat{X}_s^{h^\varepsilon}) Q_{1}^{1/2} \right\|_{\mathbb{H}}^2 ds \\
\leq \frac{C}{\varepsilon} \int_0^t \left\| Z_s^{\varepsilon} \right\|_{\mathbb{H}}^2 ds + C \varepsilon \int_0^t (1 + \left\| h_s^\varepsilon \right\|_{\mathbb{H}}^2) ds.
\]  

(4.123)

Combining (4.117)-(4.123), and then substituting it in (4.116), we obtain

\[
\left\| Z_t^{\varepsilon} \right\|_{\mathbb{H}}^2 + \mu \int_0^t \left\| Z_s^{\varepsilon} \right\|_{\mathbb{V}}^2 ds + \frac{\beta}{2^{r-2}} \int_0^t \left\| Z_s^{\varepsilon} \right\|_{\mathbb{L}^{r+1}}^2 ds \\
\leq \frac{2}{\mu} \int_0^t \left\| X_s^{\varepsilon,\delta,h^\varepsilon} - X_{s(\Delta)} \right\|_{\mathbb{V}}^2 ds + C_{\mu,\alpha,\lambda_1,L_G,L_{s_2}} \int_0^t \left\| Z_s^{\varepsilon} \right\|_{\mathbb{H}}^2 ds \\
+ C_{\mu,\alpha,\lambda_1,L_G,L_{s_2}} \int_0^t \left\| X_s^{\varepsilon,\delta,h^\varepsilon} - X_{s(\Delta)} \right\|_{\mathbb{H}}^2 ds + C \mu \int_0^t \left\| Y_s^{\varepsilon,\delta,h^\varepsilon} - \tilde{Y}_s^{\varepsilon,\delta} \right\|_{\mathbb{H}}^2 ds \\
+ C \int_0^t \left\| h_s^\varepsilon \right\|_{\mathbb{H}}^2 \left\| Z_s^{\varepsilon} \right\|_{\mathbb{H}}^2 ds + \frac{\beta}{2^{r-2}} \int_0^t \left\| Z_s^{\varepsilon} \right\|_{\mathbb{L}^{r+1}}^2 ds \\
+ C \left( \int_0^t (1 + \left\| X_s^{\varepsilon,\delta,h^\varepsilon} \right\|_{\mathbb{H}}^2 + \left\| Y_s^{\varepsilon,\delta,h^\varepsilon} \right\|_{\mathbb{H}}^2) ds \right)^{1/2} \left( \int_0^t \left( \left\| X_s^{\varepsilon,\delta,h^\varepsilon} - X_{s(\Delta)} \right\|_{\mathbb{H}}^2 + \left\| \tilde{X}_s^{h^\varepsilon} - X_{s(\Delta)} \right\|_{\mathbb{H}}^2 \right) ds \right)^{1/2} \\
+ 2 \int_0^t \left( F(X_s^{\varepsilon,\delta,h^\varepsilon}, \hat{Y}_s^{\varepsilon,\delta}) - \tilde{F}(X_{s(\Delta)}, \hat{Y}_s^{\varepsilon,\delta}), Z_s^{\varepsilon} \right) ds \\\n+ 2\sqrt{\varepsilon} \int_0^t \left( [\sigma_1(X_s^{\varepsilon,\delta,h^\varepsilon}) - \sigma_1(\hat{X}_s^{h^\varepsilon})] Q_1^{1/2} dW_s, Z_s^{\varepsilon} \right), \mathbb{P}\text{-a.s.},
\]  

(4.124)

for all \( t \in [0,T] \). An application of the Gronwall inequality in (4.124) gives

\[
\sup_{t \in [0,T \land \mathbb{T}_R]} \left\| Z_t^{\varepsilon} \right\|_{\mathbb{H}}^2 + \mu \int_0^{T \land \mathbb{T}_R} \left\| Z_s^{\varepsilon} \right\|_{\mathbb{V}}^2 ds + 2\alpha \int_0^{T \land \mathbb{T}_R} \left\| Z_s^{\varepsilon} \right\|_{\mathbb{H}}^2 ds + \frac{\beta}{2^{r-2}} \int_0^{T \land \mathbb{T}_R} \left\| Z_t^{\varepsilon} \right\|_{\mathbb{L}^{r+1}}^2 dt \\
\leq C_{\mu,\alpha,\lambda_1,L_G,L_{s_2},T} \left\{ \int_0^{T \land \mathbb{T}_R} \left\| X_s^{\varepsilon,\delta,h^\varepsilon} - X_{s(\Delta)} \right\|_{\mathbb{H}}^2 ds + \int_0^{T \land \mathbb{T}_R} \left\| Y_s^{\varepsilon,\delta,h^\varepsilon} - \tilde{Y}_s^{\varepsilon,\delta} \right\|_{\mathbb{H}}^2 ds \\
+ \int_0^{T \land \mathbb{T}_R} \left( \left\| \sigma_1(\hat{X}_s^{h^\varepsilon}) Q_1^{1/2} (h_s^\varepsilon - h_t) \right\|_{\mathbb{H}}^2 dt + \varepsilon^2 \int_0^{T \land \mathbb{T}_R} \left( 1 + \left\| \hat{X}_s^{h^\varepsilon} \right\|_{\mathbb{H}}^2 \right) dt \\
+ \left( \int_0^{T \land \mathbb{T}_R} \left( 1 + \left\| X_s^{\varepsilon,\delta,h^\varepsilon} \right\|_{\mathbb{H}}^2 + \left\| \tilde{Y}_s^{\varepsilon,\delta} \right\|_{\mathbb{H}}^2 \right) ds \right)^{1/2} \right\} \\
\times \left( \int_0^{T \land \mathbb{T}_R} \left( \left\| X_s^{\varepsilon,\delta,h^\varepsilon} - X_{s(\Delta)} \right\|_{\mathbb{H}}^2 + \left\| \tilde{X}_s^{h^\varepsilon} - X_{s(\Delta)} \right\|_{\mathbb{H}}^2 \right) ds \right)^{1/2}
\]  

\]  

(4.124)
\[
\begin{align*}
+ \sup_{t \in [0,T]} \left| \int_0^t \left( F(\mathbf{X}^{\varepsilon,\delta,\tilde{h}}_{s(\Delta)}, \mathbf{Y}^{\varepsilon,\delta}_{s(\Delta)}), \mathbf{Z}^{\varepsilon}_{s(\Delta)} \right) - \tilde{F}(\mathbf{X}^{\varepsilon,\delta,\tilde{h}}_{s(\Delta)}, \mathbf{Z}^{\varepsilon}_{s(\Delta)}) \right| ds \\
+ \sqrt{\varepsilon} \sup_{t \in [0,T]} \left| \int_0^t \left( [\sigma_1(\mathbf{X}^{\varepsilon,\delta,\tilde{h}}_{s(\Delta)}) - \sigma_1(\mathbf{X}^{\tilde{h}}_{s(\Delta)})] \right) Q_1^{1/2} d\mathbf{W}_s, \mathbf{Z}^{\varepsilon}_{s(\Delta)} \right| \right) \\
\times \exp \left( C \int_0^{T \wedge \tau^*_R} \| h_t^{\varepsilon} \|^2_{\mathbb{H}} dt \right) \exp \left( \frac{2}{\mu} \int_0^{T \wedge \tau^*_R} \| X_t^{\varepsilon,\delta,\tilde{h}} \|^2_{\mathcal{V}} dt \right)
\leq C_{\mu,\alpha,\lambda_1,L_G,L_{a_2},M,T,R} \left\{ \int_0^{T \wedge \tau^*_R} \| X_s^{\varepsilon,\delta,\tilde{h}} - X_s^{\varepsilon,\delta,\tilde{h}} \|^2_{\mathbb{H}} ds + \int_0^T \| Y_s^{\varepsilon,\delta,\tilde{h}} - \mathbf{Y}^{\varepsilon,\delta} \|^2_{\|_{\mathbb{H}}} ds \\
+ \int_0^T \| \sigma_1(\mathbf{X}^{\tilde{h}}_{s(\Delta)}) Q_1^{1/2} (h_t^{\varepsilon} - h_t) \|^2_{\mathbb{H}} dt + \varepsilon^2 \int_0^T \left( 1 + \| X_t^{\varepsilon,\delta,\tilde{h}} \|^2_{\mathbb{H}} \right) dt \\
+ \left( \int_0^{T \wedge \tau^*_R} \| X_s^{\varepsilon,\delta,\tilde{h}} - X_s^{\varepsilon,\delta,\tilde{h}} \|^2_{\mathbb{H}} ds + \int_0^T \| X_t^{\varepsilon,\delta,\tilde{h}} - \mathbf{X}^{\varepsilon,\delta,\tilde{h}} \|^2_{\mathbb{H}} ds \right)^{1/2} \right) \\
+ \sup_{t \in [0,T]} \left| \int_0^t \left( F(\mathbf{X}^{\varepsilon,\delta,\tilde{h}}_{s(\Delta)}, \mathbf{Y}^{\varepsilon,\delta}_{s(\Delta)}), \mathbf{Z}^{\varepsilon}_{s(\Delta)} \right) ds \right| \\
+ \sqrt{\varepsilon} \sup_{t \in [0,T]} \left| \int_0^t \left( [\sigma_1(\mathbf{X}^{\varepsilon,\delta,\tilde{h}}_{s(\Delta)}) - \sigma_1(\mathbf{X}^{\tilde{h}}_{s(\Delta)})] Q_1^{1/2} d\mathbf{W}_s, \mathbf{Z}^{\varepsilon}_{s(\Delta)} \right) \right| \right) \right, \quad \mathbb{P}\text{-a.s.}, \tag{4.125}
\end{align*}
\]

since $h^{\varepsilon} \in \mathcal{A}_M$, where we used the definition of stopping time given in (4.82) also. Taking expectation on both sides of (4.125) and then using Theorem 4.12, Lemmas 4.13, 4.16 (see (4.18), (4.23), (4.83) and (4.100)), we obtain

\[
\begin{align*}
\mathbb{E} \left[ \sup_{t \in [0,T]} \| Z_t^{\varepsilon} \|^2_{\mathbb{H}} + \mu \int_0^{T \wedge \tau^*_R} \| Z_t^{\varepsilon} \|^2_{\mathcal{V}} dt + \frac{\beta}{2r-2} \int_0^{T \wedge \tau^*_R} \| Z_t^{\varepsilon} \|^{r+1}_{\mathbb{L}^{r+1}} dt \right] \\
\leq C_{\mu,\alpha,\beta,\lambda_1,L_G,L_{a_2},M,T,R} \left\{ (1 + \| x \|^3_{\mathbb{H}} + \| y \|^3_{\mathbb{H}}) \left[ \left( \frac{\alpha}{\varepsilon} \right) + \Delta^{1/4} \right] \\
+ \mathbb{E} \left[ \int_0^T \| \sigma_1(\mathbf{X}^{\tilde{h}}_{s(\Delta)}) Q_1^{1/2} (h_t^{\varepsilon} - h_t) \|^2_{\mathbb{H}} dt \right] + \varepsilon^2 T \left( 1 + \sup_{t \in [0,T]} \| X_t^{\varepsilon,\delta,\tilde{h}} \|^2_{\mathbb{H}} \right) \\
+ \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_0^t \left( F(\mathbf{X}^{\varepsilon,\delta,\tilde{h}}_{s(\Delta)}, \mathbf{Y}^{\varepsilon,\delta}_{s(\Delta)}), \mathbf{Z}^{\varepsilon}_{s(\Delta)} \right) ds \right| \right] \\
+ \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_0^t \left( [\sigma_1(\mathbf{X}^{\varepsilon,\delta,\tilde{h}}_{s(\Delta)}) - \sigma_1(\mathbf{X}^{\tilde{h}}_{s(\Delta)})] Q_1^{1/2} d\mathbf{W}_s, \mathbf{Z}^{\varepsilon}_{s(\Delta)} \right) \right| \right] \right). \tag{4.126}
\end{align*}
\]

Using the Burkholder-Davis-Gundy inequality and Assumption 3.1 (A1), we estimate the final term from the right hand side of the inequality (4.126) as

\[
C_{\mu,\alpha,\beta,\lambda_1,L_G,L_{a_2},M,T,R} \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_0^t \left( [\sigma_1(\mathbf{X}^{\varepsilon,\delta,\tilde{h}}_{s(\Delta)}) - \sigma_1(\mathbf{X}^{\tilde{h}}_{s(\Delta)})] Q_1^{1/2} d\mathbf{W}_s, \mathbf{Z}^{\varepsilon}_{s(\Delta)} \right) \right| \right]
\]
Using (4.127) in (4.126), we get
from the right hand side of the inequality (4.129). We follow similar arguments given in

An application of Gronwall’s inequality in (4.128) yields

An application of Gronwall’s inequality in (4.129) yields

where \( I(t) = \int_0^t (F(X^{\varepsilon, h^\varepsilon}_s, \tilde{X}^{\varepsilon, \delta}_s) - \tilde{F}(X^{\varepsilon, h^\varepsilon}_s, \tilde{X}^{\varepsilon, \delta}_s, Z^\varepsilon_s) ds \). Let us now estimate the final term from the right hand side of the inequality (4.129). We follow similar arguments given in Lemma 3.8, [50] and Lemma 4.6, [62] to obtain the required result. For the sake of completeness, we provide a proof here. Note that

\[
|I(t)| = \sum_{k=0}^{[t/\Delta]-1} \int_{k\Delta}^{(k+1)\Delta} (F(X^{\varepsilon, h^\varepsilon}_s, \tilde{X}^{\varepsilon, \delta}_s) - \tilde{F}(X^{\varepsilon, h^\varepsilon}_s, \tilde{X}^{\varepsilon, \delta}_s, X^{\varepsilon, h^\varepsilon}_s - \tilde{X}^{h}_s) ds + \int_{t(\Delta)}^{t} (F(X^{\varepsilon, h^\varepsilon}_s, \tilde{X}^{\varepsilon, \delta}_s) - \tilde{F}(X^{\varepsilon, h^\varepsilon}_s, \tilde{X}^{\varepsilon, \delta}_s, X^{\varepsilon, h^\varepsilon}_s - \tilde{X}^{h}_s) ds =: |I_1(t)| + |I_2(t)|.
\]
Using the Assumption 3.11 (A1), we estimate \( \mathbb{E} \left[ \sup_{t \in [0, T]} |I_2(t)| \right] \) as

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} |I_2(t)| \right] \leq C \left[ \mathbb{E} \left( \sup_{t \in [0, T]} \left\| X_t^{\varepsilon, \delta, h^c} - X_t^{h} \right\|_{\mathbb{H}}^2 \right) \right]^{1/2} \mathbb{E} \left( \sup_{t \in [0, T]} \int_{t(\Delta)}^{t} \left( 1 + \left\| X_s^{\varepsilon, \delta, h^c} \right\|_{\mathbb{H}} + \left\| \widehat{Y}_s^{\varepsilon, \delta} \right\|_{\mathbb{H}} \right)^2 ds \right)^{1/2}
\]

\[
\leq C \Delta^{1/2} \left[ \mathbb{E} \left( \sup_{t \in [0, T]} \left( \left\| X_t^{\varepsilon, \delta, h^c} \right\|_{\mathbb{H}}^2 + \left\| X_t^{h} \right\|_{\mathbb{H}}^2 \right) \right) \right]^{1/2} \mathbb{E} \left( \int_{0}^{T} \left( 1 + \left\| X_s^{\varepsilon, \delta, h^c} \right\|_{\mathbb{H}} + \left\| \widehat{Y}_s^{\varepsilon, \delta} \right\|_{\mathbb{H}} \right)^2 ds \right)^{1/2}
\]

\[
\leq C_{\mu, \alpha, \lambda_1, L_G, L_{o_2}, M, T} (1 + \|x\|_{\mathbb{H}} + \|y\|_{\mathbb{H}}) \Delta^{1/2},
\]  

(4.131)

where we used (4.108) and (4.63). Next, we estimate the term \( \mathbb{E} \left[ \sup_{t \in [0, T]} |I_1(t)| \right] \) as

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} |I_1(t)| \right] \leq \mathbb{E} \left[ \sum_{k=0}^{[T/\Delta]-1} \int_{k\Delta}^{(k+1)\Delta} \left( F(X_{k\Delta}^{\varepsilon, \delta, h^c}, \widehat{Y}_{s}^{\varepsilon, \delta}) - \bar{F}(X_{k\Delta}^{\varepsilon, \delta, h^c}), X_{k\Delta}^{\varepsilon, \delta, h^c} - X_{k\Delta}^{h} \right) ds \right]
\]

\[
\leq \left[ \frac{T}{\Delta} \right] \max_{0 \leq k \leq [T/\Delta]-1} \mathbb{E} \left[ \int_{k\Delta}^{(k+1)\Delta} \left( F(X_{k\Delta}^{\varepsilon, \delta, h^c}, \widehat{Y}_{s}^{\varepsilon, \delta}) - \bar{F}(X_{k\Delta}^{\varepsilon, \delta, h^c}), X_{k\Delta}^{\varepsilon, \delta, h^c} - X_{k\Delta}^{h} \right) ds \right]
\]

\[
\leq C_{T} \max_{0 \leq k \leq [T/\Delta]-1} \left[ \mathbb{E} \left( \left\| X_{k\Delta}^{\varepsilon, \delta, h^c} - X_{k\Delta}^{h} \right\|_{\mathbb{H}}^2 \right) \right]^{1/2} \mathbb{E} \left( \int_{k\Delta}^{(k+1)\Delta} \left\| F(X_{k\Delta}^{\varepsilon, \delta, h^c}, \widehat{Y}_{s}^{\varepsilon, \delta}) - \bar{F}(X_{k\Delta}^{\varepsilon, \delta, h^c}) \right\|_{\mathbb{H}}^2 ds \right)^{1/2}
\]

\[
\leq \frac{C_{T}}{\Delta} \max_{0 \leq k \leq [T/\Delta]-1} \left[ \mathbb{E} \left( \left\| X_{k\Delta}^{\varepsilon, \delta, h^c} - X_{k\Delta}^{h} \right\|_{\mathbb{H}}^2 \right) \right]^{1/2} \mathbb{E} \left( \int_{0}^{\Delta} \left\| F(X_{k\Delta}^{\varepsilon, \delta, h^c}, \widehat{Y}_{s}^{\varepsilon, \delta}) \right\|_{\mathbb{H}}^2 ds \right)^{1/2}
\]

\[
\leq \frac{C_{\mu, \alpha, \lambda_1, L_G, M, T}}{\Delta} (1 + \|x\|_{\mathbb{H}} + \|y\|_{\mathbb{H}}) \delta \max_{0 \leq k \leq [T/\Delta]-1} \left[ \int_{0}^{\Delta} \int_{0}^{\Delta} \Phi_k(s, r) ds dr \right]^{1/2},
\]  

(4.132)

where for any \( 0 \leq r \leq s \leq \frac{\Delta}{\delta} \),

\[
\Phi_k(s, r) := \mathbb{E} \left[ \left( F(X_{k\Delta}^{\varepsilon, \delta, h^c}, \widehat{Y}_{s}^{\varepsilon, \delta}) - \bar{F}(X_{k\Delta}^{\varepsilon, \delta, h^c}), F(X_{k\Delta}^{\varepsilon, \delta, h^c}, \widehat{Y}_{r}^{\varepsilon, \delta}) - \bar{F}(X_{k\Delta}^{\varepsilon, \delta, h^c}) \right) \right].
\]  

(4.133)
Moreover, from the definition of the process \( \tilde{Y}_t \), for any \( \varepsilon > 0, s > 0 \) and \( \mathcal{F}_s \)-measurable \( \mathbb{H} \)-valued random variables \( X \) and \( Y \), let \( \{ Y_t^{\varepsilon, k, x, y} \}_{t \geq s} \) be the unique strong solution of the following Itô stochastic differential equation:

\[
\begin{align*}
\left\{ \begin{array}{ll}
d\tilde{Y}_t &= -\frac{1}{\delta}[\mu \tilde{Y}_t + \alpha \tilde{Y}_t + \beta C(\tilde{Y}_t) - G(X, \tilde{Y}_t)]dt + \frac{1}{\sqrt{\delta}}\sigma_2(X, \tilde{Y}_t)Q_2^{1/2}dW_t, \\
\tilde{Y}_s &= Y.
\end{array} \right.
\end{align*}
\]

(4.134)

Then, from the construction of the process \( \tilde{Y}_t^{\varepsilon, \delta} \) (see (4.97)), for any \( t \in [k\Delta, (k+1)\Delta] \) with \( k \in \mathbb{N} \), we have

\[
\tilde{Y}_t^{\varepsilon, \delta} = \tilde{Y}_t^{\varepsilon, k\Delta, x_k^{\varepsilon, \delta}, y_k^{\varepsilon, \delta}}, \quad \mathbb{P}\text{-a.s.}
\]

(4.135)

Using this fact in (4.133), we infer that

\[
\Phi_k(s, r) = \mathbb{E} \left[ (F(X_{k\Delta}^{\varepsilon, \delta, h^\varepsilon}, Y_{s_{k\Delta} + k\Delta}^{\varepsilon, \delta, h^\varepsilon}), \tilde{Y}_k^{\varepsilon, \delta}) - \tilde{F}(X_{k\Delta}^{\varepsilon, \delta, h^\varepsilon}), F(X_{r_{k\Delta}}^{\varepsilon, \delta, h^\varepsilon}, Y_{s_{k\Delta} + k\Delta}^{\varepsilon, \delta, h^\varepsilon}, \tilde{Y}_k^{\varepsilon, \delta}) - \tilde{F}(X_{k\Delta}^{\varepsilon, \delta, h^\varepsilon})) \right]
\]

\[
= \int \mathbb{E} \left[ \left( F(X_{k\Delta}^{\varepsilon, \delta, h^\varepsilon}, Y_{s_{k\Delta} + k\Delta}^{\varepsilon, \delta, h^\varepsilon}, \tilde{Y}_k^{\varepsilon, \delta}) - \tilde{F}(X_{k\Delta}^{\varepsilon, \delta, h^\varepsilon}), \left| \mathcal{F}_k \right| d\mathbb{P}(\omega) \right) \left( F(X_{r_{k\Delta}}^{\varepsilon, \delta, h^\varepsilon}(\omega), Y_{s_{k\Delta} + k\Delta}^{\varepsilon, \delta, h^\varepsilon}(\omega), \tilde{Y}_k^{\varepsilon, \delta}(\omega)) - \tilde{F}(X_{k\Delta}^{\varepsilon, \delta, h^\varepsilon}(\omega)) \right) \right] d\mathbb{P}(d\omega),
\]

(4.136)

where in the final step we used the fact the processes \( X_{k\Delta}^{\varepsilon, \delta, h^\varepsilon} \) and \( \tilde{Y}_{k\Delta}^{\varepsilon, \delta} \) are \( \mathcal{F}_{k\Delta} \)-measurable, whereas the process \( \{ \tilde{Y}_{s_{k\Delta} + k\Delta}^{\varepsilon, \delta, x, y} \}_{s \geq 0} \) is independent of \( \mathcal{F}_{k\Delta} \), for any fixed \( (x, y) \in \mathbb{H} \times \mathbb{H} \). Moreover, from the definition of the process \( \tilde{Y}_{s_{k\Delta} + k\Delta}^{\varepsilon, \delta, x, y} \), we obtain

\[
\tilde{Y}_{s_{k\Delta} + k\Delta}^{\varepsilon, \delta, x, y} = y - \frac{\mu}{\delta} \int_{k\Delta}^{s_{k\Delta} + k\Delta} A\tilde{Y}_{r_{k\Delta}}^{\varepsilon, \delta, x, y} dr - \frac{\alpha}{\delta} \int_{k\Delta}^{s_{k\Delta} + k\Delta} \tilde{Y}_{r_{k\Delta}}^{\varepsilon, \delta, x, y} dr + \frac{1}{\sqrt{\delta}} \int_{k\Delta}^{s_{k\Delta} + k\Delta} \sigma_2(x, \tilde{Y}_{r_{k\Delta}}^{\varepsilon, \delta, x, y})Q_2^{1/2}dW_r
\]

\[
= y - \frac{\mu}{\delta} \int_{0}^{s} A\tilde{Y}_{r_{k\Delta} + k\Delta}^{\varepsilon, \delta, x, y} dr - \frac{\alpha}{\delta} \int_{0}^{s} \tilde{Y}_{r_{k\Delta} + k\Delta}^{\varepsilon, \delta, x, y} dr + \frac{1}{\sqrt{\delta}} \int_{0}^{s} \sigma_2(x, \tilde{Y}_{r_{k\Delta} + k\Delta}^{\varepsilon, \delta, x, y})Q_2^{1/2}dW_r
\]

\[
= y - \mu \int_{0}^{s} A\tilde{Y}_{r_{k\Delta} + k\Delta}^{\varepsilon, \delta, x, y} dr - \alpha \int_{0}^{s} \tilde{Y}_{r_{k\Delta} + k\Delta}^{\varepsilon, \delta, x, y} dr - \beta \int_{0}^{s} \tilde{F}(Y_{r_{k\Delta} + k\Delta}^{\varepsilon, \delta, x, y}) dr + \frac{1}{\sqrt{\delta}} \int_{0}^{s} \sigma_2(x, \tilde{Y}_{r_{k\Delta} + k\Delta}^{\varepsilon, \delta, x, y})Q_2^{1/2}dW_r
\]

\[
= y - \mu \int_{0}^{s} A\tilde{Y}_{r_{k\Delta} + k\Delta}^{\varepsilon, \delta, x, y} dr - \alpha \int_{0}^{s} \tilde{Y}_{r_{k\Delta} + k\Delta}^{\varepsilon, \delta, x, y} dr + \frac{1}{\sqrt{\delta}} \int_{0}^{s} \sigma_2(x, \tilde{Y}_{r_{k\Delta} + k\Delta}^{\varepsilon, \delta, x, y})Q_2^{1/2}dW_r + \mathbb{E} \left[ \tilde{F}(Y_{r_{k\Delta} + k\Delta}^{\varepsilon, \delta, x, y}) \right].
\]

(4.137)
where \( \{W_{r+\Delta}^k := W_{r+\Delta} - W_r \}_{r \geq 0} \) is the shift version of \( W_r \) and \( \{\overline{W}_{r+\Delta}^k := \frac{1}{\sqrt{n}} W_{r+\Delta} \}_{r \geq 0} \). Using the uniqueness of the strong solutions of (4.135) and (4.137), we infer that

\[
\mathcal{L} \left( \left\{ \tilde{Y}_{s\Delta + k\Delta}^{\xi, \delta, h^\xi} \right\}_{0 \leq s \leq \frac{t}{\Delta}} \right) = \mathcal{L} \left( \left\{ Y_{s\Delta}^{\xi, \delta, h^\xi} \right\}_{0 \leq s \leq \frac{t}{\Delta}} \right),
\]

where \( \mathcal{L}(\cdot) \) denotes the law of the distribution. Using Markov’s property, Proposition 4.11, the estimates (4.163) and (4.99), we estimate \( \Phi_k(\cdot, \cdot) \) as

\[
\Phi_k(s, r) = \int_0^t \mathbb{E} \left[ \left( \mathcal{F}(X_{s\Delta}^{\xi, \delta, h^\xi}(\omega), Y_{s\Delta}^{\xi, \delta, h^\xi}(\omega) - \overline{F}(X_{s\Delta}^{\xi, \delta, h^\xi}(\omega)), \mathcal{F}(X_{s\Delta}^{\xi, \delta, h^\xi}(\omega), Y_{s\Delta}^{\xi, \delta, h^\xi}(\omega) - \overline{F}(X_{s\Delta}^{\xi, \delta, h^\xi}(\omega))) \right) \mathbb{P}(d\omega) \right] \mathbb{E} \left[ \left( \mathcal{F}(X_{s\Delta}^{\xi, \delta, h^\xi}(\omega), Y_{s\Delta}^{\xi, \delta, h^\xi}(\omega) - \overline{F}(X_{s\Delta}^{\xi, \delta, h^\xi}(\omega))) \right) \mathbb{P}(d\omega) \right]
\]

\[
= \int_0^t \int_0^t \left\| \mathbb{E} \left[ \left( \mathcal{F}(X_{s\Delta}^{\xi, \delta, h^\xi}(\omega), Y_{s\Delta}^{\xi, \delta, h^\xi}(\omega) - \overline{F}(X_{s\Delta}^{\xi, \delta, h^\xi}(\omega))) \right) \mathbb{P}(d\omega) \right] \mathbb{E} \left[ \left( \mathcal{F}(X_{s\Delta}^{\xi, \delta, h^\xi}(\omega), Y_{s\Delta}^{\xi, \delta, h^\xi}(\omega) - \overline{F}(X_{s\Delta}^{\xi, \delta, h^\xi}(\omega))) \right) \mathbb{P}(d\omega) \right] \right\|_{\mathbb{H}} \times \left( 1 + \left\| X_{s\Delta}^{\xi, \delta, h^\xi}(\omega) \right\|_{\mathbb{H}} \right)^2 + \left\| Y_{s\Delta}^{\xi, \delta, h^\xi}(\omega) \right\|_{\mathbb{H}}^2 \right) \mathbb{P}(d\omega) \mathbb{P}(d\omega)
\]

\[
\leq C_{\mu, \alpha, \lambda_1, L_G} \int_0^t \left( 1 + \left\| X_{s\Delta}^{\xi, \delta, h^\xi}(\omega) \right\|_{\mathbb{H}} \right)^2 + \left\| Y_{s\Delta}^{\xi, \delta, h^\xi}(\omega) \right\|_{\mathbb{H}}^2 \right) \mathbb{P}(d\omega) \mathbb{P}(d\omega)
\]
Using (4.141) in (4.129), we finally find
\[
\mathbb{E}\left[ \sup_{t \in [0, T]} \| Z_t^\varepsilon \|_{\mathbb{H}}^2 + \mu \int_0^T \| Z_t^\varepsilon \|_{\mathbb{V}}^2 \, dt + \frac{\beta t-\varepsilon}{2r-2} \int_0^T \| Z_t^\varepsilon \|_{\mathbb{H}^{r+1}}^2 \, dt \right]
\leq C_{\mu, \alpha, \beta, \lambda_1, L_\mathbb{H}, L_{\mathbb{V}}, M, T, R} (1 + \| x \|_{\mathbb{H}}^3 + \| y \|_{\mathbb{H}}^3) \left[ \left( \frac{\delta}{\varepsilon} \right)^{1/2} + \Delta^{1/4} \right]
+ \mathbb{E}\left[ \int_0^T \| \sigma_1(\bar{X}_t^\delta)Q_1^{1/2}(h_t^\varepsilon - h_t) \|_{\mathbb{H}}^2 \, dt \right] + \varepsilon^2 T \left( 1 + \sup_{t \in [0, T]} \| \bar{X}_t^\delta \|_{\mathbb{H}}^2 \right)
+ C_{\mu, \alpha, \beta, \lambda_1, L_\mathbb{H}, L_{\mathbb{V}}, M, T, R} (1 + \| x \|_{\mathbb{H}}^3 + \| y \|_{\mathbb{H}}^3) \left[ \varepsilon^2 + \left( \frac{\delta}{\varepsilon} \right)^{1/2} + \left( \frac{\delta}{\Delta} \right)^{1/2} + \Delta^{1/4} \right]
\leq C_{R, \mu, \alpha, \beta, \lambda_1, L_\mathbb{H}, L_{\mathbb{V}}, M, T, R} (1 + \| x \|_{\mathbb{H}}^3 + \| y \|_{\mathbb{H}}^3) \mathbb{E}\left[ \int_0^T \| \sigma_1(\bar{X}_t^\delta)Q_1^{1/2}(h_t^\varepsilon - h_t) \|_{\mathbb{H}}^2 \, dt \right],
\] (4.142)
and by choosing \( \Delta = \delta^{1/2} \), we obtain (4.143).

**Case 2:** \( n = 2, 3 \) and \( r \in (3, \infty) \). Let us now discuss the case \( n = 3 \) and \( r \in (3, \infty) \). We just need to estimate the term \( 2\langle (B(X^{\varepsilon, \delta, h^\varepsilon}) - B(\bar{X}^h)), Z^\varepsilon) \rangle \) only to get the required result. From the estimate (2.9), we easily have
\[
-2\beta \langle C(X^{\varepsilon, \delta, h^\varepsilon}) - C(\bar{X}^h), Z^\varepsilon) \rangle \leq -\beta \| X^{\varepsilon, \delta, h^\varepsilon} \|_{\mathbb{H}}^2 Z^\varepsilon \|_{\mathbb{H}}^2 - \beta \| X^{h} \|_{\mathbb{H}} + \frac{1}{\mu} \| X^{\varepsilon, \delta, h^\varepsilon} Z^\varepsilon \|_{\mathbb{H}}^2.
\] (4.143)

Using Hölder’s and Young’s inequalities, we estimate the term \( 2\langle (B(X^{\varepsilon, \delta, h^\varepsilon}) - B(\bar{X}^h)), Z^\varepsilon) \rangle = 2\langle B(Z^\varepsilon, X^{\varepsilon, \delta, h^\varepsilon}), Z^\varepsilon) \rangle \) as
\[
2\langle B(Z^\varepsilon, X^{\varepsilon, \delta, h^\varepsilon}), Z^\varepsilon) \rangle \leq 2 \| Z^\varepsilon \|_{\mathbb{V}} \| X^{\varepsilon, \delta, h^\varepsilon} Z^\varepsilon \|_{\mathbb{H}} \leq \mu \| Z^\varepsilon \|_{\mathbb{V}}^2 + \frac{1}{\mu} \| X^{\varepsilon, \delta, h^\varepsilon} Z^\varepsilon \|_{\mathbb{H}}^2.
\] (4.144)

We take the term \( \| X^{\varepsilon, \delta, h^\varepsilon} Z^\varepsilon \|_{\mathbb{H}}^2 \) from (4.144) and perform a similar calculation in (4.52) to deduce
\[
\| X^{\varepsilon, \delta, h^\varepsilon} Z^\varepsilon \|_{\mathbb{H}}^2 \leq \frac{\beta \mu}{2} \int_0^T \| X^{\varepsilon, \delta, h^\varepsilon}(x) \|_{\mathbb{H}}^2 \| Z^\varepsilon(x) \|_{\mathbb{H}}^2 \, dx + \frac{\beta \mu}{2} \int_0^T \| X^{h} \|_{\mathbb{H}} \| Z^\varepsilon(x) \|_{\mathbb{H}}^2 \, dx.
\] (4.145)

for \( r > 3 \). Combining (4.143), (4.144) and (4.145), we obtain
\[
-2\beta \langle C(X^{\varepsilon, \delta, h^\varepsilon}) - C(\bar{X}^h), Z^\varepsilon) \rangle = 2\langle B(Z^\varepsilon, X^{\varepsilon, \delta, h^\varepsilon}), Z^\varepsilon) \rangle
\leq -\beta \| X^{\varepsilon, \delta, h^\varepsilon} \|_{\mathbb{H}} \| Z^\varepsilon \|_{\mathbb{H}}^2 - \beta \| X^{h} \|_{\mathbb{H}} \| Z^\varepsilon \|_{\mathbb{H}}^2
+ \frac{r - 3}{\beta \mu^2 (r - 1)} \int_0^T \| Z^\varepsilon(x) \|_{\mathbb{H}}^2 \, dx.
\] (4.146)
Thus a calculation similar to the estimate (4.123) yields
\[
\| Z_t^\varepsilon \|_{\mathbb{H}}^2 + \mu \int_0^t \| Z_s^\varepsilon \|_{\mathbb{V}}^2 \, ds + \frac{\beta t-\varepsilon}{2r-2} \int_0^t \| Z_s^\varepsilon \|_{\mathbb{H}^{r+1}}^2 \, ds
\leq \left[ \frac{r - 3}{\mu^2 (r - 1)} \right] \int_0^t \| Z_s^\varepsilon \|_{\mathbb{H}}^2 \, ds + C_{\mu, \alpha, \beta, \lambda_1, L_\mathbb{H}, L_{\mathbb{V}}}
\int_0^t \| Z_s^\varepsilon \|_{\mathbb{H}}^2 \, ds.
\]
\[ + C_{\mu,\alpha\lambda_1,L_G,L_{\sigma_2}} \int_0^t \| X_{s(\Delta)}^{\varepsilon,\delta,h^\varepsilon} - X_{s(\Delta)} \|_{\mathbb{H}}^2 \, ds + C_{\mu,\alpha\lambda_1,L_G,L_{\sigma_2}} \int_0^t \| Y_{s(\Delta)}^{\varepsilon,\delta,h^\varepsilon} - \bar{Y}_{s(\Delta)} \|_{\mathbb{H}}^2 \, ds \\
+ C \int_0^t \| h_s^\varepsilon \|^2 \| Z_s^\varepsilon \|^2 \, ds + \int_0^t \| \sigma_1(\bar{X}_s^h)Q_1^{1/2}(h_s^\varepsilon - h_s) \|^2 \, ds + C \varepsilon^2 \int_0^t (1 + \| \bar{X}_s^h \|^2) \, ds \\
+ C \left( \int_0^t (1 + \| X_{s(\Delta)}^{\varepsilon,\delta,h^\varepsilon} \|^2 + \| \bar{Y}_{s(\Delta)} \|^2) \right) \left( \int_0^t (\| X_{s(\Delta)}^{\varepsilon,\delta,h^\varepsilon} - X_{s(\Delta)} \|^2 + \| X_s^h - X_{s(\Delta)}^h \|^2) \right) \right)^{1/2} \\
+ 2 \int_0^t \langle F(X_{s(\Delta)}^{\varepsilon,\delta,h^\varepsilon}, \bar{Y}_{s,\delta}) - F(X_{s(\Delta)}^{\varepsilon,\delta,h^\varepsilon}), Z_{s(\Delta)}^{\varepsilon,\delta} \rangle \, ds \\
+ 2\sqrt{\varepsilon} \int_0^t \left| \langle \sigma_1(X_{s(\Delta)}^{\varepsilon,\delta,h^\varepsilon}) - \sigma_1(X_s^h) \rangle \right| Q_1^{1/2} \, dW_s, Z_s^\varepsilon \right) \right) \\
(4.147) \]

for all \( t \in [0, T] \), \( \mathbb{P} \)-a.s. Applying Gronwall’s inequality and then using a calculation similar to (4.142) yields the estimate (4.144) for the case \( n = 2, 3 \) and \( r \in (3, \infty) \).

**Case 3:** \( n = r = 3 \) and \( 2\beta \mu > 1 \). For \( n = r = 3 \), from (2.9), we have
\[
-2\beta \langle C(X_{s(\Delta)}^{\varepsilon,\delta,h^\varepsilon}) - C(\bar{X}_s^h), Z_s^\varepsilon \rangle \leq -\beta \| X_{s(\Delta)}^{\varepsilon,\delta,h^\varepsilon} \|^2 + \| \bar{X}_s^h \| Z_s^\varepsilon \|^2, \tag{4.148}
\]
and a calculation similar to (4.144) gives
\[
2 \left| \langle B(Z_s^\varepsilon, X_{s(\Delta)}^{\varepsilon,\delta,h^\varepsilon}), Z_s^\varepsilon \rangle \right| \leq 2 \| Z_s^\varepsilon \| \| X_{s(\Delta)}^{\varepsilon,\delta,h^\varepsilon} \|_{\mathbb{H}} \leq \| \mu \| Z_s^\varepsilon \|^2 + 1 \| X_{s(\Delta)}^{\varepsilon,\delta,h^\varepsilon} \|^2, \tag{4.149}
\]
Combining (4.148) and (4.149), we obtain
\[
-2\mu \langle AZ_s^\varepsilon, Z_s^\varepsilon \rangle - 2\beta \langle C(X_{s(\Delta)}^{\varepsilon,\delta,h^\varepsilon}) - C(\bar{X}_s^h), Z_s^\varepsilon \rangle - 2 \langle B(Z_s^\varepsilon, X_{s(\Delta)}^{\varepsilon,\delta,h^\varepsilon}), Z_s^\varepsilon \rangle \leq (2 - \theta) \mu \| Z_s^\varepsilon \|^2 - \left( \beta - \frac{1}{\theta \mu} \right) \| X_{s(\Delta)}^{\varepsilon,\delta,h^\varepsilon} \|^2, \tag{4.150}
\]
for \( \frac{1}{\beta \mu} < \theta < 2 \) and the hence estimate (4.144) follows for \( 2\beta \mu > 1 \).

Note that by making use of the estimates (4.146) and (4.150), we don’t need the stopping time defined in (4.112) to get the required estimate (4.114).

Let us now establish the weak convergence result. The well known Skorokhod’s representation theorem (see [72]) states that if \( \mu_n, \nu_n, \ldots, \mu_0 \) are probability measures on complete separable metric space (Polish space) such that \( \mu_n \overset{w}{\rightarrow} \mu, \) as \( n \rightarrow \infty, \) then there exist a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and a sequence of measurable random elements \( X_n \) such that \( X_n \rightarrow X, \mathbb{P}\)-a.s., and \( X_n \) has the distribution function \( \mu_n, \, n = 0, 1, 2, \ldots (X_n \sim \mu_n) \), that is, the law of \( X_n \) is \( \mu_n \). We use Skorokhod’s representation theorem in the next theorem.

**Theorem 4.21** (Weak convergence). Let \( \{ h^\varepsilon : \varepsilon > 0 \} \subset A_M \) converges in distribution to \( h \) with respect to the weak topology on \( L^2(0, T; \mathbb{H}) \). Then \( \mathcal{G}^\varepsilon \left( W + \frac{1}{\sqrt{\varepsilon}} \int_0^t h_s^\varepsilon \, ds \right) \) converges in distribution to \( \mathcal{G}^0 \left( \int_0^t h_s \, ds \right) \) in \( \mathcal{G} \), as \( \varepsilon \rightarrow 0 \).

**Proof.** Let \( \{ h^\varepsilon \} \) converges to \( h \) in distribution as random elements taking values in \( S_M \), where \( S_M \) is equipped with the weak topology. Since \( A_M \) is Polish (see section 4.3 and [10]) and \( \{ h^\varepsilon : \varepsilon > 0 \} \subset A_M \) converges in distribution to \( h \) with respect to the weak topology on \( L^2(0, T; \mathbb{H}) \), the Skorokhod representation theorem can be used to construct a probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})\) and processes \( (\tilde{h}^\varepsilon, \tilde{h}, \tilde{W}^\varepsilon) \) such that the distribution
of \((\tilde{h}^\varepsilon, \tilde{W}^\varepsilon)\) is same as that of \((h^\varepsilon, W^\varepsilon)\), and \(\tilde{h}^\varepsilon \to \tilde{h}\), \(\tilde{P}\)-a.s., in the weak topology of \(\mathcal{S}_M\). Thus \(\int_0^t \tilde{h}^\varepsilon(s)ds \to \int_0^t \tilde{h}(s)ds\) weakly in \(\mathbb{H}\), \(\tilde{P}\)-a.s., for all \(t \in [0,T]\). In the following sequel, without loss of generality, we write \((\Omega, \tilde{\mathcal{F}}, \tilde{P})\) as the probability space and \((h^\varepsilon, h, W)\) as processes, though strictly speaking, one should write \((\Omega, \tilde{\mathcal{F}}, \tilde{P})\) and \((\tilde{h}^\varepsilon, \tilde{h}, \tilde{W}^\varepsilon)\), respectively for probability space and processes.

Let us define \(Z^\varepsilon_t := X^\varepsilon_t - X^\varepsilon_t\), where \(Z^\varepsilon_t\) satisfies the stochastic differential given in (4.115). We first prove the Theorem for \(n = 2\) and \(r \in [1, 3]\). Let \(\tau^*_R\) be the stopping time defined in (4.112). Then for any \(\eta > 0\), by using Markov’s inequality, we have

\[
\mathbb{P}\left\{ \left( \sup_{t \in [0,T]} \|Z^\varepsilon_t\|_H^2 + \int_0^T \|Z^\varepsilon_t\|_Y^2 dt + \int_0^T \|Z^\varepsilon_t\|_{L_r+1} \, dt \right)^{1/2} > \eta \right\} 
\]

\[
\leq \frac{1}{\eta} \mathbb{E}\left[ \left( \sup_{t \in [0,T]} \|Z^\varepsilon_t\|_H^2 + \int_0^T \|Z^\varepsilon_t\|_Y^2 dt + \int_0^T \|Z^\varepsilon_t\|_{L_r+1} \, dt \right)^{1/2} \right] \chi_{\{T \leq \tau^*_R\}} 
\]

\[
+ \frac{1}{\eta} \mathbb{E}\left[ \left( \sup_{t \in [0,T]} \|Z^\varepsilon_t\|_H^2 + \int_0^T \|Z^\varepsilon_t\|_Y^2 dt + \int_0^T \|Z^\varepsilon_t\|_{L_r+1} \, dt \right)^{1/2} \chi_{\{T > \tau^*_R\}} \right],
\]

(4.151)

where \(\chi_t\) is the indicator function. From Lemma 4.20 (see (4.113)), we obtain

\[
\mathbb{E}\left[ \left( \sup_{t \in [0,T]} \|Z^\varepsilon_t\|_H^2 + \int_0^T \|Z^\varepsilon_t\|_Y^2 dt + \int_0^T \|Z^\varepsilon_t\|_{L_r+1} \, dt \right)^{1/2} \chi_{\{T \leq \tau^*_R\}} \right] 
\]

\[
\leq \mathbb{E}\left[ \left( \sup_{t \in [0,T]} \|Z^\varepsilon_t\|_H^2 + \int_0^T \|Z^\varepsilon_t\|_Y^2 dt + \int_0^T \|Z^\varepsilon_t\|_{L_r+1} \, dt \right)^{1/2} \chi_{\{T \leq \tau^*_R\}} \right] 
\]

\[
\leq \left\{ C_{\mu, \alpha, \beta, \lambda_1, L_G, M, T, R} \left( 1 + \|\mathbf{x}\|_H^3 + \|\mathbf{y}\|_H^3 \right) \varepsilon^2 + \left( \frac{\delta}{\varepsilon} \right) + \delta^{1/8} \right\}^{1/2} 
\]

\[
+ \mathbb{E}\left[ \int_0^T \|\sigma_1(X^h_t)\|_{Q_1}^{1/2} (h^\varepsilon_t - h_t)\|\|_H^2 dt \right]^{1/2}
\]

(4.152)

Using Hölder’s and Markov’s inequalities, (4.18) and (4.63), we estimate the second term from the right hand side of the inequality (4.151) as

\[
\mathbb{E}\left[ \left( \sup_{t \in [0,T]} \|Z^\varepsilon_t\|_H^2 + \int_0^T \|Z^\varepsilon_t\|_Y^2 dt + \int_0^T \|Z^\varepsilon_t\|_{L_r+1} \, dt \right)^{1/2} \chi_{\{T > \tau^*_R\}} \right] 
\]

\[
\leq \left( \mathbb{E}\left[ \left( \sup_{t \in [0,T]} \|Z^\varepsilon_t\|_H^2 + \int_0^T \|Z^\varepsilon_t\|_Y^2 dt + \int_0^T \|Z^\varepsilon_t\|_{L_r+1} \, dt \right) \right] \right)^{1/2} \mathbb{P}(T > \tau^*_R)^{1/2} 
\]

\[
\leq C_{\mu, \alpha, \beta, \lambda_1, L_G, M, T, R} \left( 1 + \|\mathbf{x}\|_H^3 + \|\mathbf{y}\|_H^3 \right)^{1/2} \left[ \mathbb{E}\left( \int_0^T \|X^\varepsilon, h^\varepsilon\|_Y^2 ds \right) \right]^{1/2}
\]
Combining (4.152)-(4.153) and substitute it in (4.151), we find
\[
\mathbb{P} \left\{ \left( \sup_{t \in [0,T]} \| \mathbf{Z}^\varepsilon_t \|_{\mathbb{H}}^2 + \int_0^T \| \mathbf{Z}^\varepsilon_t \|_{\mathbb{V}}^2 dt + \int_0^T \| \mathbf{Z}^\varepsilon_t \|_{\mathbb{E}_{r+1}}^2 dt \right)^{1/2} > \eta \right\}
\leq \frac{1}{\eta} \left\{ C_{\mu,\alpha,\lambda_1,\lambda_2,\mu,\alpha,\lambda_1,\lambda_2} (1 + \| \mathbf{x} \|_{\mathbb{H}}^2 + \| \mathbf{y} \|_{\mathbb{H}}^2) \left[ \varepsilon^2 + \left( \frac{\delta}{\varepsilon} \right) + \delta^{1/8} \right] + \mathbb{E} \left[ \int_0^T \| \sigma_1 (\mathbf{X}^\varepsilon_t) Q_1^{1/2} (h^\varepsilon_t - h_t) \|_{\mathbb{H}}^2 dt \right] \right\}^{1/2} + \frac{C_{\mu,\alpha,\lambda_2,\mu,\alpha,\lambda_2,\mu,\alpha,\lambda_2}}{\eta \sqrt{R}}.
\] (4.154)

Once again, we use the fact that compact operators maps weakly convergent sequences into strongly convergent sequences. Since \( \sigma_1 (\cdot) \) is compact and \( \mathcal{A}_M \) converges in distribution to \( h \) with respect to the weak topology on \( L^2(0,T;\mathbb{H}) \), we get
\[
\int_0^T \| \sigma_1 (\mathbf{X}^\varepsilon_t) Q_1^{1/2} (h^\varepsilon_t - h_t) \|_{\mathbb{H}}^2 dt \to 0, \quad \text{as } \varepsilon \to 0, \quad \mathbb{P} \text{-a.s.}
\]
Furthermore, we have
\[
\mathbb{E} \left[ \int_0^T \| \sigma_1 (\mathbf{X}^\varepsilon_t) Q_1^{1/2} (h^\varepsilon_t - h_t) \|_{\mathbb{H}}^2 dt \right] \leq C \sup_{t \in [0,T]} (1 + \| \mathbf{X}^\varepsilon_t \|_{\mathbb{H}}^2) \mathbb{E} \left[ \int_0^T \| h^\varepsilon_t \|_{\mathbb{H}}^2 dt + \int_0^T \| h_t \|_{\mathbb{H}}^2 dt \right] \leq C_{\mu,\alpha,\lambda_1,\lambda_2,\mu,\alpha,\lambda_1,\lambda_2} (1 + \| \mathbf{X}^\varepsilon_t \|_{\mathbb{H}}^2),
\]
where we used (4.18) and the fact that \( h^\varepsilon \in \mathcal{A}_M \) and \( h \in \mathcal{S}_M \). Thus, an application of the dominated convergence theorem gives
\[
\mathbb{E} \left[ \int_0^T \| \sigma_1 (\mathbf{X}^\varepsilon_t) Q_1^{1/2} (h^\varepsilon_t - h_t) \|_{\mathbb{H}}^2 dt \right] \to 0, \quad \text{as } \varepsilon \to 0.
\] (4.155)
Letting \( \varepsilon \to 0 \) and then \( R \to \infty \) in (4.154), we obtain
\[
\lim_{\varepsilon \to 0} \mathbb{P} \left\{ \left( \sup_{t \in [0,T]} \| \mathbf{Z}^\varepsilon_t \|_{\mathbb{H}}^2 + \int_0^T \| \mathbf{Z}^\varepsilon_t \|_{\mathbb{V}}^2 dt + \int_0^T \| \mathbf{Z}^\varepsilon_t \|_{\mathbb{E}_{r+1}}^2 dt \right)^{1/2} > \eta \right\} = 0,
\] (4.156)
for all \( \eta > 0 \), which completes the proof for the case \( n = 2 \) and \( r \in [1,3] \).

For \( n = 2, r \in (3,\infty) \) and \( n = 3, r \in [3,\infty) \) \((2\beta_\mu > 1, \text{ for } r = 3)\), one has to use the stopping time \( \tau^\varepsilon_R \) defined in (4.82) for the estimate in (4.151) and then use the estimate (4.114) to obtain the convergence given in (4.156) \( \square \).

**Remark 4.22.** Note that LDP for \( \{ \mathbf{X}^{\varepsilon,\beta} \} \) proved in Theorem 4.11 holds true for the case \( n = 2 \) and \( r \in [1,3] \) with \( \alpha = \beta = 0 \). Therefore, the results obtained in this work is valid for 2D Navier-Stokes equations also. In this case, the state space becomes \( \mathcal{E} = C([0,T];\mathbb{H}) \cap L^2(0,T;\mathbb{V}) \) (see Remark 3.4).
Acknowledgments: M. T. Mohan would like to thank the Department of Science and Technology (DST), India for Innovation in Science Pursuit for Inspired Research (INSPIRE) Faculty Award (IFA17-MA110).

References

[1] S.N. Antontsev and H.B. de Oliveira, The Navier–Stokes problem modified by an absorption term, Applicable Analysis, 89(12), 2010, 1805–1825.
[2] V. Barbu, Analysis and control of nonlinear infinite dimensional systems, Academic Press, Boston, 1993.
[3] R. Bertram and J.E. Rubin, Multi-timescale systems and fast-slow analysis, Math. Biosci. 287 (2017) 105–121.
[4] H. Bessaih and A. Millet, On stochastic modified 3D Navier–Stokes equations with anisotropic viscosity, Journal of Mathematical Analysis and Applications, 462 (2018), 915–956.
[5] N.N. Bogoliubov, Y.A. Mitropolsky, Asymptotic Methods in the Theory of Non-linear Oscillations, Gordon and Breach Science Publishers, New York (1961).
[6] C.E. Bréhier, Strong and weak orders in averaging for SPDEs, Stochastic Process. Appl., 122 (2012), 2553–2593.
[7] C.E. Bréhier, Orders of convergence in the averaging principle for SPDEs: the case of a stochastically forced slow component, Stochastic Process. Appl. 130 (6) (2020), 3325–3368.
[8] Z. Brzeźniak and Gaurav Dhariwal, Stochastic tamed Navier-Stokes equations on $\mathbb{R}^3$: the existence and the uniqueness of solutions and the existence of an invariant measure, https://arxiv.org/pdf/1904.13295.pdf.
[9] A. Budhiraja and P. Dupuis, A variational representation for positive functionals of infinite dimensional Brownian motion, Probab. and Math. Stat., 20 (2000) 39–61.
[10] A. Budhiraja, P. Dupuis, V. Maroulas, Large deviations for infinite dimensional stochastic dynamical systems, Ann. Probab. 36 (2008), 1390–1420.
[11] D. L. Burkholder, The best constant in the Davis inequality for the expectation of the martingale square function, Transactions of the American Mathematical Society 354 (1), 1970, 91–105.
[12] S. Cerrai, A Khasminskii type averaging principle for stochastic reaction-diffusion equations, Ann. Appl. Probab., 19 (2009), 899–948.
[13] S. Cerrai, Averaging principle for systems of reaction-diffusion equations with polynomial nonlinearities perturbed by multiplicative noise, SIAM J. Math. Anal. 43 (2011) 2482–2518.
[14] S. Cerrai, M. Freidlin, Averaging principle for stochastic reaction-diffusion equations, Probab. Theory Related Fields, 144 (2009), 137–177.
[15] S. Cerrai, A. Lunardi, Averaging principle for nonautonomous slow-fast systems of stochastic reaction-diffusion equations: the almost periodic case, SIAM J. Math. Anal. 49 (2017) 2843-2884.
[16] P.-L. Chow, Stochastic partial differential equations, Chapman & Hall/CRC, New York, 2007.
[17] I. Chueshov and A. Millet, Stochastic 2D hydrodynamical type systems: Well posedness and Large Deviations, Applied Mathematics and Optimization, 61 (2010), 379–420.
[18] G. Da Prato and J. Zabczyk, Stochastic Equations in Infinite Dimensions, Cambridge University Press, 1992.
[19] G. Da Prato and J. Zabczyk, Ergodicity for Infinite Dimensional Systems, London Mathematical Society Lecture Notes, 229, Cambridge University Press, 1996.
[20] G. Da Prato, F. Flandoli, E. Priola and M. Röckner, Strong uniqueness for stochastic evolution equation in Hilbert spaces perturbed by a bounded measurable drift, The Annals of Probability, 41(5) (2013), 3306–3344
[21] B. Davis, On the integrability of the martingale square function, Israel Journal of Mathematics 8(2) (1970), 187–190.
[22] A. Debussche, Ergodicity results for the stochastic Navier-Stokes equations: An introduction, Topics in Mathematical Fluid Mechanics, Volume 2073 of the series Lecture Notes in Mathematics, Springer, 23–108, 2013.
[23] A. Dembo, and O. Zeitouni, Large Deviations Techniques and Applications, Springer-Verlag, New York, 2000.
[24] Z. Dong, X. Sun, H. Xiao, J. Zhai, Averaging principle for one dimensional stochastic Burgers equation, *J. Differential Equations*, **265** (2018), 4749–4797.

[25] P. Dupuis and K. Spiliopoulos, Large deviations for multiscale diffusion via weak convergence methods, *Stochastic Process. Appl.*, **122**(4) (2012), 1947–1987.

[26] W. E, B. Engquist, Multiscale modeling and computations, *Notice of AMS*, **50** (2003) 1062-1070.

[27] C. L. Fefferman, K. W. Hajduk and J. C. Robinson, Simultaneous approximation in Lebesgue and Sobolev norms via eigenspaces, https://arxiv.org/abs/1904.03337.

[28] F. Flandoli and B. Maslowski, Ergodicity of the 2-D Navier-Stokes equation under random perturbations, *Communications in Mathematical Physics*, **172** (1995), 119–141.

[29] M. I. Freidlin, and A. D. Wentzell, *Random Perturbations of Dynamical Systems*, Springer-Verlag, New York, 1984.

[30] H. Fu and J. Duan, An averaging principle for two-scale stochastic partial differential equations, *Stoch. Dyn.*, **11**(2011), 353–367.

[31] H. Fu and J. Liu, Strong convergence in stochastic averaging principle for two time-scales stochastic partial differential equations, *J. Math. Anal. Appl.* 384 (2011) 70-86.

[32] H. Fu, L. Wan and J. Liu, Strong convergence in averaging principle for stochastic hyperbolic-parabolic equations with two time-scales, *Stochastic Process. Appl.*, **125** (2015), 3255–3279.

[33] H. Fu, L. Wan, Y. Wang and J. Liu, Strong convergence rate in averaging principle for stochastic FitzHugh-Nagumo system with two time-scales, *J. Math. Anal. Appl.*, **416**(2) (2014), 609–628.

[34] H. Fu, L. Wan, Y. Wang and J. Liu, Strong convergence rate in averaging principle for stochastic FitzHugh-Nagumo system with two time-scales, *J. Math. Anal. Appl.*, **416**, (2014) 609–628.

[35] G. P. Galdi, An introduction to the Navier–Stokes initial-boundary value problem. pp. 11-70 in *Fundamental directions in mathematical fluid mechanics*, Adv. Math. Fluid Mech. Birkhäuser, Basel 2000.

[36] H. Gao, and H. Liu, Well-posedness and invariant measures for a class of stochastic 3D Navier-Stokes equations with damping driven by jump noise, *Journal of Differential Equations*, **267** (2019), 5938–5975.

[37] M. Gourcy, A large deviation principle for 2D stochastic Navier–Stokes equation, *Stochastic Processes and their Applications*, **117**(7) (2007), 904–927.

[38] M. Hairer, J.C. Mattingly, Ergodicity of the 2D Navier-Stokes equations with degenerate stochastic forcing, *Annals of Mathematics*, **164** (2006), 993–1032.

[39] K. W. Hajduk and J. C. Robinson, Energy equality for the 3D critical convective Brinkman-Forchheimer equations, *Journal of Differential Equations*, **263** (2017), 7141–7161.

[40] E. Harvey, V. Kirk, M. Wechselberger and J. Sneyd, Multiple timescales, mixed mode oscillations and canards in models of intracellular calcium dynamics, *J. Nonlinear Sci*. **21** (2011) 639–683.

[41] W. Hu, M. Salins and K. Spiliopoulos, Large deviations and averaging for systems of slow-fast stochastic reaction-diffusion equations, *Stoch PDE: Anal. Comp.*, **7**(4) (2019), 808–874.

[42] V. K. Kalantarov and S. Zelik, Smooth attractors for the Brinkman-Forchheimer equations with fast growing nonlinearities, *Commun. Pure Appl. Anal.*, **11** (2012) 2037–2054.

[43] G. Kallianpur, and J. Xiong, *Stochastic Differential Equations in Infinite Dimensional Spaces*, Institute of Math. Stat, 1996.

[44] R.Z. Khasminskii, On the principle of averaging the Itô’s stochastic differential equations, *Kybernetica* **4** (1968), 260–279.

[45] R. Liptser, Large deviations for two scaled diffusions, *Probab. Theory Related Fields*, **106** (1996), 71–104.

[46] O. A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach, New York, 1969.

[47] R. Kumar and L. Popovic, Large deviations for multi-scale jump-diffusion processes, *Stoch. Process. Appl.*, **127** (2017) 1297–1320.

[48] O. A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach, New York, 1969.
[52] H. Liu and H. Gao, Stochastic 3D Navier–Stokes equations with nonlinear damping: martingale solution, strong solution and small time LDP, Chapter 2 in Interdisciplinary Mathematical Sciences: Stochastic PDEs and Modelling of Multiscale Complex System, 9–36, 2019.

[53] W. Liu, M. Röckner, X. Sun, Y. Xie Strong averaging principle for slow-fast stochastic partial differential equations with locally monotone coefficients, https://arxiv.org/abs/1907.03260.

[54] W. Liu and M. Röckner, Local and global well-posedness of SPDE with generalized coercivity conditions, Journal of Differential Equations, 254 (2013), 725–755.

[55] W. Liu, Well-posedness of stochastic partial differential equations with Lyapunov condition, Journal of Differential Equations, 255 (2013), 572–592.

[56] C. Marinelli and M. Röckner, On the maximal inequalities of Burkholder, Davis and Gundy, Expositiones Mathematicae, 34(1)(2016), 1–26.

[57] E.A. Mastny, E.L. Haseltine and J.B. Rawlings, Two classes of quasi-steady-state model reductions for stochastic kinetics, J. Chem. Phys, 127 (2007) 094106.

[58] M. Mikikian, M. Cavarroc, L. Couedel, Y. Tessier, L. Boufendi, Mixed-mode oscillations in complex-plasma instabilities, Phys. Rev. Lett. 100 (2008) 225005.

[59] M. T. Mohan, On the convective Brinkman-Forchheimer equations, Submitted.

[60] M. T. Mohan, Stochastic convective Brinkman-Forchheimer equations, Submitted.

[61] M. T. Mohan, Wentzell-Freidlin large deviation principle for the stochastic convective Brinkman-Forchheimer equations, Submitted.

[62] M. T. Mohan, Averaging principle for the stochastic convective Brinkman-Forchheimer equations, Submitted.

[63] M.T. Mohan, Well posedness, large deviations and ergodicity of the stochastic 2D Oldroyd model of order one, Stochastic Processes and their Applications, 130(8) (2020), 4513–4562.

[64] L. Popovic, Large deviations of Markov chains with multiple time-scales, Stochastic Processes and their Applications, 129(9) (2019), 3319–3359.

[65] A. A. Puhalskii, On large deviations of coupled diffusions with time scale separation, Ann. Probab., 44(4) (2016), 3111–3186.

[66] J. C. Robinson and W. Sadowski, A local smoothness criterion for solutions of the 3D Navier-Stokes equations, Rendiconti del Seminario Matematico della Università di Padova 131 (2014), 159–178.

[67] M. Röckner, F.-Y. Wang and L. Wu, Large deviations for stochastic generalized porous media equations, Stochastic Processes and their Applications, 116(12) (2006), 1677–1689.

[68] M. Röckner, B. Schmuland and X. Zhang, Yamada-Watanabe theorem for stochastic evolution equations in infinite dimensions, Condensed Matter Physics, 11(2) (2008), 247–259.

[69] M. Röckner and X. Zhang, Stochastic tamed 3D Navier-Stokes equation: existence, uniqueness and ergodicity, Probability Theory and Related Fields, 145 (2009) 211–267.

[70] M. Röckner, T. Zhang and X. Zhang, Large deviations for stochastic tamed 3D Navier-Stokes equations, Applied Mathematics and Optimization, 61 (2010), 267–285.

[71] M. Röckner and T. Zhang, Stochastic 3D tamed Navier-Stokes equations: Existence, uniqueness and small time large deviations principles, Journal of Differential Equations, 252 (2012), 716–744.

[72] A. V. Skorokhod, Limit theorems for stochastic processes, Theory of Probability & Its Applications, 1(3), (1956), 261–290.

[73] K. Spiliopoulos, Large deviations and importance sampling for systems of slow-fast motion, Appl. Math. Optim., 67 (2013), 123–161.

[74] S. S. Sritharan and P. Sundar, Large deviations for the two-dimensional Navier-Stokes equations with multiplicative noise, Stochastic Processes and their Applications, 116 (2006), 1636–1659.

[75] X. Sun, R. Wang, L. Xu and X. Yang, Large deviations for two-time-scale stochastic Burgers equation, Submitted, https://arxiv.org/pdf/1811.00290.pdf.

[76] R. Temam, Navier-Stokes Equations, Theory and Numerical Analysis, North-Holland, Amsterdam, 1984.

[77] R. Temam, Navier-Stokes Equations and Nonlinear Functional Analysis, Second Edition, CBMS-NSF Regional Conference Series in Applied Mathematics, 1995.

[78] S. R. S. Varadhan, Large deviations and Applications, 46, CBMS-NSF Series in Applied Mathematics, SIAM, Philadelphia, 1984.
[79] A.Yu. Veretennikov, On large deviations for SDEs with small diffusion and averaging, *Stochastic Process. Appl.*, **89**(1) (2000), 69–79.

[80] M. I. Visik, and A.V. Fursikov, *Mathematical Problems of Statistical Hydromechanics*, Kluwer, Dordrecht, 1980.

[81] W. Wang and A.J. Roberts, Average and deviation for slow-fast stochastic partial differential equations, *J. Differential Equations*, **253** (2012), 1265–1286.

[82] W. Wang, A.J. Roberts and J. Duan, Large deviations and approximations for slow-fast stochastic reaction-diffusion equations, *J. Differential Equations*, **253** (2012), 3501–3522.

[83] F. Wu, T. Tian, J.B. Rawlings, G. Yin, Approximate method for stochastic chemical kinetics with two-time scales by chemical Langevin equations, *J. Chem. Phys*, **144** (2016) 174112.

[84] J. Xu, J. Liu, Y. Miao, Strong averaging principle for two-time-scale SDEs with non-Lipschitz coefficients, *J. Math. Anal. Appl.*, **468** (2018), 116–140.