An extension theorem for planar semimodular lattices

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Abstract We prove that every finite distributive lattice $D$ can be represented as the congruence lattice of a rectangular lattice $K$ in which all congruences are principal. We verify this result in a stronger form as an extension theorem.

Keywords Principal congruence · Order · Semimodular · Rectangular

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1 Introduction

In Grätzer and Schmidt [16], we proved that every finite distributive lattice $D$ can be represented as the congruence lattice of a sectionally complemented finite lattice $K$. In such a lattice, of course, all congruences are principal, using the notation of Grätzer [11], $\text{Con } K = \text{Princ } K$.

Since every finite distributive lattice $D$ can be represented as the congruence lattice of a planar semimodular lattice $K$ (see Grätzer et al. [15]), it is reasonable to ask whether instead of the sectional complemented lattice of the previous paragraph, we can construct a planar semimodular lattice $K$.

Grätzer and Knapp [13] proved a result stronger than the Grätzer–Lakser–Schmidt result: every finite distributive lattice $D$ can be represented as the congruence lattice of a rectangular
lattice $K$—see Sect. 2.1 for the definition. (For a new proof of this result, see Grätzer and Schmidt [19].) Keeping this in mind, we prove:

**Theorem 1.1** Every finite distributive lattice $D$ can be represented as the congruence lattice of a rectangular lattice $K$ with the property that all congruences are principal.

We prove this representation result in a much stronger form, as an extension theorem.

**Theorem 1.2** Let $L$ be a planar semimodular lattice. Then $L$ has an extension $K$ satisfying the following conditions:

(i) $K$ is a rectangular lattice;
(ii) $K$ is a congruence-preserving extension of $L$;
(iii) $K$ is a cover-preserving extension of $L$;
(iv) every congruence relation of $K$ is principal.

Observe that we only have to prove Theorem 1.2. Indeed, let Theorem 1.2 hold and let $D$ be a finite distributive lattice. By Grätzer and Knapp [13], there is a planar semimodular lattice $K_1$ whose congruence lattice is isomorphic to $D$. By Theorem 1.2, the lattice $K_1$ has a congruence-preserving extension $K$ in which every congruence relation is principal. This lattice $K$ satisfies the conditions of Theorem 1.1.

We will use the notations and concepts of lattice theory as in [8]. See [7] for a deeper coverage of finite congruence lattices. See Czédli and Grätzer [4] and Grätzer [9] for an overview of semimodular lattices, structure and congruences.

2 Background

We need some concepts and results from the literature to prove Theorem 1.2.

2.1 Rectangular lattices

Let $L$ be a planar lattice. A *left corner* (resp., *right corner*) of the lattice $L$ is a doubly-irreducible element in $L - \{0, 1\}$ on the left (resp., right) boundary of $L$. A *corner* of $L$ is an element in $L$ that is either a left or a right corner of $L$. Grätzer and Knapp [13] define a *rectangular lattice* $L$ as a planar semimodular lattice which has exactly one left corner, $lc(L)$, and exactly one right corner, $rc(L)$, and they are complementary—that is, $lc(L) \lor rc(L) = 1$ and $lc(L) \land rc(L) = 0$. In a rectangular lattice $L$, there are four boundary chains: the lower left, the lower right, the upper left, and the upper right, denoted by $C_{\text{ll}}(L)$, $C_{\text{lr}}(L)$, $C_{\text{ul}}(L)$, and $C_{\text{ur}}(L)$, respectively.

Let $A$ and $B$ be rectangular lattices. We define the *rectangular gluing* of $A$ and $B$ as the gluing of $A$ and $B$ over the ideal $I$ and filter $J$, where $I$ is the lower left boundary chain of $A$ and $J$ is the upper right boundary chain of $B$ (or symmetrically).

We recap some basic facts about rectangular lattices (Grätzer and Knapp [13] and [14], Czédli and Schmidt [5] and [6]).

**Theorem 2.1** Let $L$ be a rectangular lattice.

(i) The ideal $\downarrow lc(L)$ is the chain $C_{\text{ll}}(L)$, and symmetrically.
(ii) The filter $\uparrow lc(L)$ is the chain $C_{\text{ul}}(L)$, and symmetrically.
(iii) For every $a \leq lc(L)$, the interval $[a, rc(L) \lor a]$ is a chain, and symmetrically.
(iv) For every $a \leq lc(L)$, $L$ is a rectangular gluing of the filter $\uparrow a$ and the ideal $\downarrow rc(L) \lor a$. 

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Fig. 1  The lattices $M_3$ and $S_7$

Fig. 2  A step in inserting a fork

(v) For every prime interval $p$ of the chain $[a, rc(L) \lor a]$, there is a prime interval $q$ of the chain $C_{lr}$ so that $p$ and $q$ are perspective.

Note that it follows from (v) that
$$\text{con}(C_{ul}) = \text{con}(a, rc(L) \lor a) = \text{con}(C_{lr}).$$

2.2 Eyes

Let $L$ be a planar lattice. An interior element of an interval of length two is called an eye of $L$. We will insert and remove eyes in the obvious sense. A planar semimodular lattice $L$ is slim if it has no eyes.

2.3 Forks

We need from Czédli and Schmidt [6] the fork construction.

Let $L$ be a planar semimodular lattice. Let $L$ be slim. Inserting a fork into $L$ at the covering square $S$, firstly, replaces $S$ by a copy of $S_7$. We get three new covering squares replacing $S$ of $L$. We will name the elements of the inserted $S_7$ as in Fig. 1.

Secondly, if there is a chain $u \prec v \prec w$ such that the element $v$ has just been inserted (the element $a$ or $b$ in $S_7$ in the first step) and $T = \{x = u \land z, z, u, w = z \lor u\}$ is a covering square in the lattice $L$ (and so $u \prec v \prec w$ is not on the boundary of $L$) but $x \prec z$ at the present stage of the construction, then we insert a new element $y$ into the interval $[x, z]$ such that $x \prec y \prec z$ and $y \prec v$, see Fig. 2. We get two covering squares to replace the covering square $T$. 

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Let $K$ denote the lattice we obtain when the procedure terminates (that is, when the new element is on the boundary); see Fig. 3 for an example.

The new elements form an order, called a *fork* (the black filled elements in Fig. 3). We say that $K$ is obtained from $L$ by *inserting a fork into $L$* at the covering square $S$.

Here are some basic facts, based on Czédli and Schmidt [6], about this construction.

**Lemma 2.2** Let $L$ be a planar semimodular lattice and let $S$ be a covering square in $L$. If $L$ is slim, then inserting a fork into $L$ at $S$ we obtain a slim planar semimodular lattice $K$. If $L$ is rectangular, so is $K$.

If $y$ is an element of the fork outside of $S$, then $[y^*, y]$ is up-perspective to $[o, a]$ or $[o, b]$, where $y^*$ is the lower cover of $y$ in $K - L$.

### 2.4 Patch lattices

Let us call a rectangular lattice $L$ a *patch lattice* if $\text{lcm}(A)$ and $\text{rcm}(A)$ are dual atoms; Fig. 1 has two examples. The next lemma is a trivial application of Lemma 2.2.

**Lemma 2.3** Let $L$ be a slim patch lattice and let $S$ be a covering square in $L$. Inserting a fork into $L$ at $S$, we obtain a slim patch lattice $K$.

### 2.5 The structure theorems

Now we state the structure theorems for patch lattices and rectangular lattices of Czédli and Schmidt [6].

**Theorem 2.4** Let $L$ be a patch lattice. Then we can obtain $L$ from the four-element Boolean lattice $\mathbb{C}_2^2$ by first inserting forks, then inserting eyes.

**Theorem 2.5** Let $L$ be a rectangular lattice. Then there is a sequence of lattices

$$K_1, K_2, \ldots, K_n = L$$

such that each $K_i$, for $i = 1, 2, \ldots, n$, is either a patch lattice or it is the rectangular gluing of the lattices $K_j$ and $K_k$ for $j, k < i$.

See also Grätzer and Knapp [14] and Grätzer [10].
2.6 A congruence-preserving extension

Finally, we need the following result of Grätzer and Knapp [13].

**Theorem 2.6** Let $L$ be a planar semimodular lattice. Then there exists a rectangular, cover-preserving, and congruence-preserving extension $K$ of $L$.

3 Congruences of rectangular lattices

To prove Theorem 1.2, we need a “coordinatization” of the congruences of rectangular lattices.

**Theorem 3.1** Let $L$ be a rectangular lattice and let $\alpha$ be a congruence of $L$. Let $\alpha^l$ denote the restriction of $\alpha$ to $C_{ll}$. Let $\alpha^r$ denote the restriction of $\alpha$ to $C_{lr}$. Then the congruence $\alpha$ is determined by the pair $(\alpha^l, \alpha^r)$. In fact,

$$\alpha = \text{con}(\alpha^l \cup \alpha^r).$$

**Proof** Since $\alpha \geq \text{con}(\alpha^l \cup \alpha^r)$, it is sufficient to prove that

(P) if the prime interval $p$ of $A$ is collapsed by the congruence $\alpha$, then it is collapsed by the congruence $\text{con}(\alpha^l \cup \alpha^r)$.

First, let $L$ be a slim patch lattice. By Theorem 2.4, we obtain $L$ from the square, $C_2^2$, with a sequence of $n$ fork insertions. We induct on $n$.

If $n = 0$, then $L = C_2^2$, and the statement is trivial.

Let the statement hold for $n - 1$ and let $K$ be the patch lattice we obtain by $n - 1$ fork insertions into $C_2^2$, so that we obtain $L$ from $K$ by one fork insertion at the covering square $S$. We have three cases to consider.

- **Case 1** $p$ is a prime interval of $K$. Then the statement holds for $p$ and $\alpha|_K$, the restriction of $\alpha$ to $K$ by induction. So $p$ is collapsed by $\text{con}((\alpha^l|_K \cup \alpha^r|_K))$ in $K$. Therefore, (P) holds in $L$.

- In the next two cases, we assume that $p$ is not in $K$.

- **Case 2** $p$ is perspective to a prime interval of $K$. Same proof as in Case 1. This case includes $p = [o, a]$ and any one of the new intervals up-perspective with $[o, a]$.

- **Case 3** $p = [a, c]$ and any one of the new intervals is up-perspective with $[a, c]$. Then the fork extension defines the terminating prime interval $q = [y, z]$ on the boundary of $L$ which is up-perspective with $p$, verifying (P).

Secondly, let $L$ be a patch lattice, not necessarily slim. This case is obvious because (P) is preserved when inserting an eye.

Finally, if $L$ is not a patch lattice, we induct on $|L|$. By Theorem 2.5, $L$ is the rectangular gluing of the rectangular lattices $A$ and $B$ over the ideal $I$ and filter $J$. Let $p$ be a prime interval of $L$. Then $p$ is a prime interval of $A$ or $B$, say, of $A$. (If $p$ is a prime interval of $B$, then the argument is easier.) By induction, $p$ is collapsed by $\text{con}(\alpha|_{C_{ll}(A)} \cup \alpha|_{C_{lr}(A)})$, so it is collapsed by $\text{con}(\alpha|_{C_{ll}(A)} \cup \alpha|_{C_{lr}(A)}) = \text{con}(\alpha^l \cup \alpha^r)$.

4 Construction

Now we proceed with the construction for the planar semimodular lattice $L$ for Theorem 1.2.
Step 1 We apply Theorem 2.6 to get a rectangular, cover-preserving, and congruence-preserving extension $K_1$ of $K$.

Step 2 Let $D = C_{lr}(K_1)$. We form the lattice $D^2$, and insert eyes into the covering squares of the main diagonal, obtaining the lattice $\hat{D}$, see Fig. 4.

Now we do a rectangular gluing of $K_1$ and $\hat{D}$, obtaining the lattice $K_2$. Here is the crucial statement:

**Lemma 4.1** $K_2$ is a rectangular, cover-preserving, and congruence-preserving extension of $L$. For every join-irreducible congruence $\alpha$ of $L$, there is a prime interval $p_\alpha$ of $C = C_{ll}(K_2)$ such that $\text{con}(p_\alpha)$ in $K_2$ is the unique extension of $\alpha$ to $K_2$.

**Proof** Indeed, by Theorem 3.1, there is a prime interval $q_\alpha$ of $C_{ll}(K_1)$ or a prime interval $q'_\alpha$ of $C_{lr}(K_1)$ such that $\text{con}(q_\alpha)$ or $\text{con}(q'_\alpha)$ in $K_1$ is the unique extension of $\alpha$ to $K_1$. If we have $q_\alpha \subseteq C_{ll}(K_1)$, then we are done.

If we have $q'_\alpha \subseteq C_{lr}(K_1)$, then in $K_2$ there is a unique $q \subseteq C_{ll}(\hat{D}) \subseteq C_{ll}(K_2)$ such that in $\hat{D}$, the prime intervals $q'_\alpha$ and $q$ are connected by an M3 on the main diagonal; see Fig. 5 for an illustration.

Now clearly, we can set $p_\alpha = q$. □

*Note* Lemma 4.1 is a variant of several published results. Maybe Czédli [1, Lemma 7.2] is its closest predecessor.

Step 3 For the final step of the construction, take the chain $C = C_{ll}(K_2)$ and a congruence $\alpha$ of $L$. We can view $\alpha$ as a congruence of $K_2$ and let $\alpha = \gamma_1 \vee \cdots \vee \gamma_n$ be a join-decomposition of $\alpha$ into join-irreducible congruences. By Theorem 3.1 and (P), we can associate with each $\gamma_i$, for $i = 1, \ldots, n$, a prime interval $p_i$ of $C$ so that $\text{con}(p_i) = \gamma_i$.

We construct a rectangular lattice $C[\alpha]$ (a cousin of $\hat{D}$) as follows:
Let $C_{n+1} = \{ 0 < 1 < \cdots < n \}$. Take the direct product $C \times C_{n+1}$. We think of this direct product as consisting of $n$ columns, column 1 (the bottom one), ..., column $n$ (the top one).

In column $i$, for $1 \leq i \leq n$, we take the covering square whose upper right edge is perspective to $p_i$ and insert an eye. In the covering $M_3$ sublattice we obtain, every prime interval $p$ satisfies $\text{con}(p) = \gamma_i$. See Fig. 6 for an illustration with $n = 3$; a prime interval $p$ is labelled with $\gamma_i$ if $\text{con}(p) = \gamma_i$.

Let $b$ denote the top element of the $M_3$ we constructed for $p_n$, clearly, we have $b \in C_\text{ur}(C[\alpha])$. Take the element $a \in C_\text{ul}(C[\alpha])$ so that the interval $[a, b]$ is a chain of length $n$. Then the $n$ prime intervals $q_1, \ldots, q_n$ of $[a, b]$ satisfy

$$\text{con}(q_1) = \gamma_1, \ldots, \text{con}(q_n) = \gamma_n,$$

so $\text{con}([a, b]) = \alpha$, finding that in the lattice $C[\alpha]$, the congruence $\alpha$ is principal.

We identify $C$ with $C_\text{ur}(C[\alpha])$; note that this is a “congruence preserving” isomorphism: for a prime interval $p$ of $C$, the image $p'$ of $p$ in $C_\text{ur}(C[\alpha])$ satisfies $\text{con}(p) = \text{con}(p')$.

Now we form the rectangular gluing of $C[\alpha]$ with filter $C$ and $K_2$ with the ideal $C$ to obtain the lattice $K_2[\alpha]$. Obviously, $K_2[\alpha]$ is a rectangular lattice, it is a cover-preserving congruence-preserving extension of $K_2$ and, therefore, of $L$.

It is easy to see that $C_\text{ul}(K_2[\alpha])$ is still (congruence) isomorphic to $C$; for a rigorous treatment see the Corner Lemma and the Eye Lemma in Czédli [1] as they are used in the proof of [1, Lemma 7.2]. We can continue this expansion with all the congruences of $L$. In the last step, we get the lattice $K_3 = K$, satisfying all the conditions of Theorem 1.2.
4.1 Discussion

Let $L$ be a rectangular lattice and let $\alpha$ be a join-irreducible congruence of $L$. We call $\alpha$ \textit{left-sided}, if there a prime interval $p \subseteq C\text{ll}(L)$ such that $\text{con}(p) = \alpha$ but there is no such $p \subseteq C\text{lr}(L)$. In the symmetric case, we call $\alpha$ \textit{right-sided}. The congruence $\alpha$ is \textit{one-sided} if it is left-sided or right-sided. The congruence $\alpha$ is \textit{two-sided} if it is not one-sided.

Using these concepts, we can further analyze Theorem 3.1 and condition (P). By Theorems 2.4 and 2.5, we build a rectangular lattice from a grid (the direct product of two chains) by inserting first forks and then eyes. At the start, all join-irreducible congruences are one-sided. When we insert a fork, we introduce a two-sided congruence. When we insert an eye, we identify two congruences, resulting in a two-sided congruence.

What congruence pairs occur in Theorem 3.1? Let $\beta_l$ be a congruence of $C\text{ll}(L)$ and let $\beta_r$ be a congruence of $C\text{lr}(L)$. Under what conditions is there a congruence $\alpha$ of $L$ such that $\alpha_l = \beta_l$ and $\alpha_r = \beta_r$? Here is the condition: If $p$ is a prime interval of $C\text{ll}(L)$ collapsed by $\beta_l$ and there is a prime interval $q$ of $C\text{lr}(L)$ with $\text{con}(p) = \text{con}(q)$, then $q$ is collapsed by $\beta_r$; and symmetrically.

In Step 3 of the construction, we use the chain $C_{n+1}$. Clearly, $C_n$ would have sufficed. Can we use, in general, shorter chains?

In a finite sectionally complemented lattice, the congruences are determined around the zero element. So it is clear that for finite sectionally complemented lattices, all congruences are principal.

For a finite semimodular lattice, the congruences are scattered all over. So it is somewhat surprising that Theorem 1.1 holds.

For modular lattices, the situation is similar to the semimodular case. Schmidt [21] proved that every finite distributive lattice $D$ can be represented as the congruence lattice of a countable modular lattice $K$. (See also Grätzer and Schmidt [17] and [18].) It is an interesting question whether Theorem 1.1 holds for countable modular lattices.

The congruence structure of planar semimodular lattices is further investigated in three recent papers: Czédli [2], [3] and Grätzer [12].

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