Simulating systems of itinerant spin-carrying particles using arrays of superconducting qubits and resonators

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Abstract
We propose possible approaches for the quantum simulation of itinerant spin-carrying particles in a superconducting qubit-resonator array. The standard Jaynes–Cummings–Hubbard setup considered in several recent studies can readily be used as a quantum simulator for a number of relevant phenomena, including the interaction with external magnetic fields and spin-orbit coupling. A more complex setup where multiple qubits and multiple resonator modes are utilized in the simulation gives a higher level of complexity, including the simulation of particles with high spin values, and it allows more direct control on processes related to spin-orbit coupling. This proposal could be implemented in state-of-the-art superconducting circuits in the near future.

Keywords: quantum simulation, Jaynes–Cummings–Hubbard model, superconducting circuits

1. Introduction

Two classic paradigms in physics, namely the Jaynes–Cummings (JC) model describing the interaction between light and matter [1] and the Hubbard model describing particle motion in a periodic structure [2], were recently combined to produce the Jaynes–Cummings–Hubbard
(JCH) model [3–8]. The JCH model deals with a regular array of harmonic oscillators (e.g., optical cavities), each containing a two-level or multilevel quantum system (e.g., an atom). The particles whose properties and dynamics are studied in such a situation are generally hybridized photonic/atomic excitations that hop between the lattice sites in a Hubbard-model-like picture. Numerous theoretical proposals have highlighted the novelty and potential applications of the model [9–23], and initial experiments that can lead to demonstrating large-scale JCH arrays have been reported recently [24–26].

The most commonly studied setup for the JCH model comprises an array of cavities, between which photons can hop, and each of which contains a two-level atom. The Hamiltonian describing the setup is given by

\[
\hat{H} = \sum_i \left( \frac{\hbar \omega_a}{2} \hat{\sigma}_z^{(i)} + \hbar \omega_0 \hat{a}_i^{\dagger} \hat{a}_i + g \hat{\sigma}_z^{(i)} \left( \hat{a}_i^{\dagger} + \hat{a}_i \right) + J \sum_{\langle i,j \rangle} \left( \hat{a}_i^{\dagger} \hat{a}_j + h.c. \right) \right),
\]

where \( \omega_a \) is the atomic resonance frequency, \( \omega_0 \) is the cavity resonance frequency, \( g \) is the atom-cavity coupling strength, and \( J \) is the intercavity coupling strength (or, in other words, the intercavity hopping strength). The operators \( \hat{\sigma}_\alpha^{(i)} \) (with \( \alpha = x, y \) or \( z \)) are the Pauli matrices operating on the state of the two-level atom, while \( \hat{a}_i \) and \( \hat{a}_i^{\dagger} \) are, respectively, the annihilation and creation operators of the cavity field; the indices \( i \) and \( j \) denote the location of the atom or cavity in the lattice, and the notation \( \langle i,j \rangle \) indicates nearest-neighbour hopping. In writing equation (1), we have assumed that the different parameters are uniform across the entire lattice, and we have assumed that there is no direct coupling between the atoms [27].

One interesting property of the JCH model is the fact that in typical setups, the number of excitations is not conserved; in most proposals, excitations are to be created by coherently driving the cavities, thus creating quantum superpositions of states with different total particle numbers. As a result, one typically needs to consider the balance between the injection of excitations into the system by the externally applied fields, which can be pulsed or continuous, and the loss of excitations through various radiative or dissipative processes [11]. As such, one deals not with an equilibrium system, but with a driven dissipative system. This situation differs drastically from commonly studied condensed-matter physics problems, where the particle number is assumed to be fixed throughout an experiment, and losses are typically treated as a limitation of a given experimental setup. This difference could be seen as a disadvantage of the proposed JCH setups when viewed as alternative quantum simulators (e.g., to replace atomic gases in optical lattices [7, 28]), for simulating electrons in solids and related systems. On the other hand, the difference between the JCH model and other, well-established paradigms with conserved particle numbers (e.g., atoms in optical lattices) presents an opportunity where the new paradigm could offer novel applications, and this possibility is an active research area [11]. Furthermore, there is no fundamental difficulty in creating states with a well-defined particle number in the JCH model; Rabi-oscillation dynamics in the atoms in principle allows the controllable creation of any desired total particle number, and the suppression of radiative losses could allow for motional thermalization of the fluid on a timescale that is short compared to the lifetime of the particles.

There have been proposals for implementing the JCH model in a variety of physical systems, including optical-frequency cavities with implanted or electromagnetically trapped atoms [3], superconducting resonators and qubits fabricated on a chip [12, 14–17, 19], nanomechanical resonators [29], and phonon cavities with impurities in silicon [30]. The
superconducting platform is particularly promising from an experimental point of view. Both superconducting qubits, which serve as artificial atoms in the model, and superconducting resonators are making rapid advances in terms of their quantum coherence and the level of control and addressability that they allow [31]. Coherent coupling between qubits and resonators has been achieved in various settings and has led to demonstrations of numerous phenomena predicted from the theory of quantum optics [32]. Furthermore, large arrays of superconducting resonators can readily be fabricated on an electronic chip with present-day technology [5, 24–26]. An implementation of the JCH model in such circuits can realistically be expected within the next few years. We shall therefore focus on this implementation of the model.

Studies on the JCH model have mainly focused on the case where only one excitation mode per site plays an active role in the representation of the particles in the lattice. The result is that one ends up dealing with a system of effectively spinless particles. Even in this case, several interesting phenomena can be obtained. Examples include the fractional quantum Hall effect [10], unusual propagation behaviour [13], novel disorder effects [22], a rich phase diagram in the case of ultrastrong atom-cavity coupling [18], topological models [20], and Dirac points [21].

The addition of a spin degree of freedom to the particles can be expected to result in a variety of new phenomena that are absent in the spinless case. In this context, it is worth mentioning the large number of phenomena encountered in the study of the Bose–Einstein condensates of spin-carrying particles [33]. In particular, novel quantum states have been predicted in relation to the motion of spin-carrying atoms in optical lattices [34] and spin-orbit coupling in atomic gases [35–38]. In this paper, we present possible routes toward the addition of a spin degree of freedom to the effective particles in the JCH model.

There have been proposals to simulate spins in a JCH-like setup in the literature [9]. However, the simulated systems described stationary spins, as opposed to itinerant spin-carrying particles. The possible use of polariton spin, photon polarization, or different modes in optical cavities to introduce additional degrees of freedom has also been discussed in the literature [4, 6, 39], and in [40] the simulation of a two-component JCH model using a four-level atom in each cavity was proposed. However, in these proposals, the number of particles in the different components was conserved, so the addition of a spin degree of freedom was not truly realized. A related setup for the simulation of Luttinger liquids and spin-charge separation corresponding to hard-core repulsion in a continuous medium was proposed in [41]. It should be noted that these proposals generally rely on atoms with specific energy-level structures, making them rather difficult to implement in a superconducting architecture.

In this paper, we analyze the possibility of constructing a superconducting quantum simulator for itinerant particles that can hop between neighbouring sites in a regular lattice, and that possess an internal (spin or pseudo-spin) degree of freedom. As will be discussed in detail in section 3, different approaches have different levels of difficulty, but also allow different levels of complexity as quantum simulators.

2. The desired hamiltonian

The quintessential Bose-Hubbard Hamiltonian for spinless particles contains two terms: hopping and on-site interaction;
\[ \hat{H}_{\text{BH}} = -t \sum_{\langle i,j \rangle} \hat{b}^+_i \hat{b}_j + U \sum_i \hat{b}^+_i \hat{b}_i \hat{b}^+_i \hat{b}_i, \]  

(2)

where \( t \) is the hopping strength, \( U \) is the on-site interaction strength, and \( \hat{b} \) and \( \hat{b}^\dagger \) are, respectively, the annihilation and creation operators of the bosonic particles in the lattice. Constructing this Hamiltonian should be relatively straightforward in a superconducting architecture. The single-site energy spectrum of the JCH model, namely the JC ladder, exhibits a series of energy-level pairs. These pairs naturally define two excitation modes: the so-called lower- and upper-branch polariton modes. In the case of large detuning between the qubits and the resonators (i.e., the so-called dispersive regime), the energy levels in either one of the two modes have a linear-plus-quadratic dependence on the number of excitation quanta [43]. In the absence of inhomogeneous trapping potentials, the linear part does not have any physical consequences and can be ignored, leaving only the quadratic term, which plays the role of the on-site interaction term in equation 2. If we also assume that the hopping strength is small compared to the qubit-resonator detuning, excitations can hop between neighbouring sites, but they cannot jump from one polariton branch to the other. As a result, if we consider a system where only one of the two branches is populated, the Hamiltonian in equation (1) reduces to an effective Hamiltonian, given by equation (2).

Our aim is to design a JCH setup where the resulting effective particles possess a spin degree of freedom. We would therefore like to engineer an effective Bose-Hubbard Hamiltonian with an additional, internal degree of freedom. A Hamiltonian that contains only hopping and on-site interaction terms (i.e., the generalization of equation (2)), would read

\[ \hat{H}_{\text{BH},s} = - \sum_{\langle i,j \rangle, \alpha, \beta} t_{\alpha, \beta} \hat{b}^+_i \hat{b}^\dagger_j + \sum_{i, \alpha, \beta, \gamma, \delta} U_{\alpha, \beta, \gamma, \delta} \hat{b}^+_i \hat{b}^\dagger_i \hat{b}_i \hat{b}_i, \]  

(3)

where \( t_{\alpha, \beta} \) is a matrix of hopping strengths that describes the process in which a particle with spin value \( \beta \) hops to a neighbouring site and spin value \( \alpha \), and \( U_{\alpha, \beta, \gamma, \delta} \) is a tensor that describes the strengths of various local interaction processes that transform a pair of particles with spin values \( \gamma \) and \( \delta \) to spin values \( \alpha \) and \( \beta \). As before, we assume that the different parameters are uniform across the entire lattice. The inclusion of nonuniform parameters, such as trapping potentials, would be quite straightforward in a superconducting architecture.

A Hamiltonian describing a system of spin-carrying particles would, in general, also contain a Zeeman term, which in the Hubbard model takes the form

\[ \hat{H}_{\text{Zeeman}} = - \sum_{i, \alpha, \beta} g S_{\alpha, \beta} \cdot \vec{B} \hat{b}^\dagger_{i, \alpha} \hat{b}_{i, \beta}, \]  

(4)

where \( g \) is the Landé \( g \) factor, \( S_{\alpha, \beta} \) is the (vector) spin operator for the spin value being simulated, and \( \vec{B} \) is the magnetic field. We should note that the hopping term in equation (3) contains spin-changing hopping terms that are relevant to spin-orbit coupling, as we shall discuss below.

The interaction term in equation (3) contains a tensor with several parameters, which means that designing a system where all these parameters are tunable can be a challenging task, and we shall not attempt to do that here. We focus on the basic Hamiltonian that contains hopping and magnetic terms:
This Hamiltonian would already exhibit several features that do not exist in the spinless case, as evidenced by the rich phase diagrams discussed in [35]. As we shall see below, a system with a spin-independent hard-core repulsion between the particles (i.e., a system where double occupancy of a single site is prohibited) occurs naturally in one of the simple limits in the JCH setup. We shall therefore have in mind a Hamiltonian with such an interaction term, \( \hat{H}_{\text{int}} \), added:

\[
\hat{H} = - \sum_{\langle i,j \rangle, \alpha, \beta} t_{i, \alpha, \beta} \hat{b}_i^{\dagger} \hat{b}_j - \sum_{i, \alpha, \beta} g S_{i, \alpha, \beta} \cdot \mathbf{B} \hat{b}_{i, \alpha}^{\dagger} \hat{b}_{i, \beta}.
\]

where \( U \) is a large coefficient that represents the strong on-site repulsion.

3. Possible routes for adding spin degrees of freedom to the JCH setup

In this section, we discuss two possible approaches that can be used to introduce internal degrees of freedom to the particles in the JCH setup. The first approach, namely the use of multiple polariton branches, is rather straightforward, but could nevertheless be used to achieve demonstrations of various relevant phenomena. The second approach, namely the inclusion of multiple qubits coupled to each resonator, would allow more complex simulations. Two further alternative (but less promising) approaches are presented in appendix A.

3.1. Polariton branches

We first consider the JCH setup described in section 1, whose Hamiltonian is given by equation (1). Although most studies consider the spinless case and implicitly focus on only one excitation mode (or branch), this setup inherently supports two excitation modes, namely the lower- and upper-polariton modes. These modes could be used to represent the \( \uparrow \) and \( \downarrow \) states of spin-1/2 particles [42]. One can therefore achieve a quantum simulation of itinerant spin-1/2 (or more accurately pseudospin-1/2) particles using the basic JCH setup.

We now consider how different terms in the Hamiltonian can be simulated, and we start with the hopping term. In this work, the hopping is assumed to take place through the resonators. When the qubit-resonator detuning is large (i.e., in the dispersive regime), one of the two polariton modes has an almost purely photonic character, while the other has a large probability that the qubit will be in its excited state. The effective hopping strength for the mostly photonic mode will then be much larger than that of the other mode, which is generally undesirable because it can create a natural separation in behaviour between the particles in the different spin states. Perhaps the simplest way to make the two hopping strengths comparable, or even equal, is to move away from the dispersive regime and consider the case of near or exact resonance between the qubits and resonators. In the case of exact resonance, the two hopping strengths are exactly equal.

To fully realize the spin degree of freedom, we need to have processes that change the spin of a particle, such as a Zeeman term in the Hamiltonian. In order to simulate the Zeeman term with magnetic fields pointing in arbitrary directions, we need to be able to induce the conversion of excitations between the lower- and upper-polariton modes. Since the two modes have different energies, one might think of achieving the conversion by driving the system at
the frequency separation between the two modes. However, the relevant matrix elements for the driving fields that can be applied easily with driving operators \( \hat{a} + \hat{a}^\dagger \) or \( \sigma^x \) vanish, because these operators change the excitation number while the excitation number is conserved in the Zeeman term, precluding the conversion between modes using one of these candidates for the driving field. The conversion could be achieved, however, using a Raman process with simultaneous driving at two frequencies, as illustrated in figure 1. Depending on the detuning between the driving field frequency difference and the actual energy level separation, one can obtain an effective magnetic field that points in any desired direction [44]. In the case where the driving fields are chosen such that the Raman process occurs with perfect energy matching (while still having the detuning with the intermediate energy level), the effective magnetic field lies in the xy plane. Note that this (artificial) magnetic field couples only to the spin degree of freedom and does not lead to phenomena related to the coupling between the orbital motion and external magnetic fields (e.g., the quantum Hall effect).

As explained in detail in [45], spin-orbit coupling can be engineered by temporally alternating a lattice system between a few different Hamiltonian settings. These settings include only hopping terms and position-dependent Zeeman terms, which are readily achievable in the superconducting architecture. For example, one can take the four-step sequence

\[
\left\{ \begin{align*}
\hat{p}_x^2/m, & \quad \hat{p}_x^2 + \hat{p}_y^2/2m + \kappa \left(\hat{x} \sigma_x - \hat{y} \sigma_y\right), \\
\hat{p}_x^2/m, & \quad \hat{p}_x^2 + \hat{p}_y^2/2m - \kappa \left(\hat{x} \sigma_x - \hat{y} \sigma_y\right), \\
\end{align*} \right. \tag{7}
\]

where \( \hat{p}_x \) and \( \hat{p}_y \) are the x and y components of the momentum operator in two dimensions and the operators \( \sigma^\alpha \) (with \( \alpha = x, y \) or \( z \)) are the Pauli operators for the effective particles. (Note that the quadratic dependence of energy on momentum can be obtained straightforwardly from the Hubbard model at the bottom of the energy band.) If one applies this sequence repeatedly with a sufficiently high frequency \( \omega \), one obtains the effective Hamiltonian

\[
\hat{H}_{\text{eff}} = \frac{\hat{p}_x^2}{2m} + \frac{\hat{p}_y^2}{2m} + \lambda_R \left(\hat{p}_x \sigma_x + \hat{p}_y \sigma_y\right) + \Omega_{\text{SO}} \hat{L}_z \sigma_z, \tag{8}
\]

where \( \lambda_R = \pi \kappa/8\ m\omega \), \( \Omega_{\text{SO}} = -(8\ m/3)\lambda_R^2 \), and \( \hat{L}_z = \hat{x} \hat{p}_y - \hat{y} \hat{p}_x \). This effective Hamiltonian is related to the standard Rashba form (i.e., equation (8) without the last term) by a unitary transformation [45].

Figure 1. Schematic diagram of a Raman process for converting excitations between the two polariton modes in the JC model. This process is used to simulate the Zeeman term in the Hamiltonian, where the two states of a spin-1/2 particle mix locally (i.e., at a single site in the JCH lattice). The states \(|\mathrm{LP}\rangle\) and \(|\mathrm{UP}\rangle\) are states with a single excitation in the lower- and upper-polariton branches, respectively. The state \(|\mathrm{vac}\rangle\) is the vacuum state with no excitations. The solid lines represent the three relevant energy levels, and the dashed line represents a virtual energy level whose location is determined by the detuning between the driving ac field and the real energy levels.
Theoretical studies predict a rich phase diagram for the mean-field ground state of a Bose gas with spin-orbit coupling, including stripe and vortex patterns [35]. These states can be prepared using local operations on the different lattice sites, making them relatively easy to prepare in the proposed JCH setup with superconducting circuits. If one prepares a mean-field state that is close to the true ground state of the system for a given particle number, the subsequent dynamics would exhibit only small amplitude changes. On the other hand, if the prepared initial state is far from the true ground state for a given set of parameters, one would expect that the state of the system would change drastically as it evolves in time. This way, one can test the theoretical predictions and possibly map out the ground-state phase diagram.

There are a few points that must be noted in this context. One of the main issues that requires attention is the following: if only the $|\uparrow\rangle$ state or the $|\downarrow\rangle$ state is occupied at any given lattice site (i.e., if only one of the two polariton branches is populated at a given site), the energy-level ladder allows multiple occupations of that site. A difficulty arises, however, when dealing with quantum states that contain particles in both states $|\uparrow\rangle$ and $|\downarrow\rangle$ at the same site. The mapping between polariton excitations and spin-carrying particles breaks down in this case, as one can see by direct inspection of the energy-level structure. For example, there are three different spin states corresponding to a doubly occupied lattice site, whereas there are only two states in the JC ladder in the two-excitation subspace. The mapping is therefore only applicable when dealing with systems where it suffices to consider, at most, a single particle per site, which generally corresponds to dilute systems. Another difficulty is encountered if we take the hopping strength, $J$, to be comparable to or larger than the single-site energy separation between the two polariton branches. In this case, the intercavity coupling dominates over the qubit-cavity coupling, and the separation between the two polariton modes becomes less clearly defined. In order to avoid this difficulty, the intercavity coupling must be kept weak compared to the polariton energy separation. Note that in this regime, it becomes natural to say that particles in different spin states are strictly prohibited from occupying the same site while same-spin particles can do so at an effective energy cost (the latter having been analyzed in the literature on the JCH setup [3]). To simplify the problem and avoid dealing with these spin-dependent interactions, one can prohibit double occupancy altogether by choosing system parameters that correspond to the Mott-insulator regime. One then has hard-core on-site repulsion. We should note that even in this parameter regime, the Mott insulator state is only realized for commensurate filling of the lattice. Away from these filling values, one has a system of itinerant particles.

To conclude this subsection, we summarize its main points. Using the basic JCH setup, one can simulate a system of spin-1/2 particles where the tunneling strength can be made spin-independent, an effective magnetic field can be engineered freely by driving the transition between the two polariton modes, and spin-orbit coupling can be obtained by periodically switching the Hamiltonian between four different settings. In the following subsection, we consider a different approach that allows additional possibilities, in particular higher spin values and spin-orbit coupling with fixed, rather than pulsed, driving settings.

### 3.2. Multiple qubits coupled to each resonator

We now consider the incorporation of multiple qubits coupled to each resonator in a JCH system, as illustrated in figure 2. Each qubit would then represent one spin state of the simulated particles, keeping in mind that the effective particles would actually be qubit-resonator hybridized excitations. It is worth emphasizing that this setup is not unrealistic since there have
been experiments both with large arrays of resonators and with multiple qubits coupled to a
single resonator.

If we consider \( N \) qubits coupled to one resonator mode, we find that there are, in fact, \( N + 1 \)
excitation modes. These modes can, in principle, by used to simulate a spin of size \( s = N/2 \). By
using identical settings at all the lattice sites, one would obtain excitation modes with excitation
frequencies that are distinct from each other, but are uniform across the lattice. The different spin
states of the simulated particles can then be identified through these frequencies. In this case,
hopping processes arise naturally, and for sufficiently weak intercavity coupling, hopping
processes do not change the spin state. However, for \( N > 2 \), the occupation probability of the
resonator cannot be equal for all the different singly excited states, which means that the hopping
strengths for the different spin states cannot be made equal. In order to avoid this difficulty, one
needs to employ multiple resonator modes. We therefore assume that we have \( N \) qubits coupled to
\( N \) resonator modes at each site. Anticipating that we will use the resonator modes as mediators of
various interactions, we generally assume that the parameters of each qubit-mode pair are in the
dispersive regime, and that the simulated particles are encoded in the qubit excitations.

Transitions between the different spin states, which are needed to simulate magnetic
fields and the Zeeman term in the Hamiltonian, can be obtained by driving the qubits using any one of
the recently realized microwave-driving-based techniques for implementing swap operations
between superconducting qubits [46]. The resonators naturally provide an effective interqubit
coupling that can be used in these conversion protocols. As discussed in section 3.1, the
simulated magnetic field can be designed to point in any direction depending on the amplitude,
phase, and detuning of the driving fields.

To simulate spin-orbit coupling, one needs to induce controlled spin-changing hopping
processes. The processes needed in the case of spin-1/2 particles are illustrated in figure 3(a).
These processes can be realized by driving Raman transitions, as shown in figure 3(b). One
point that requires some attention is that when the qubits are biased at their symmetry points,
convenient operators, such as \( \hat{a} + \hat{a}^\dagger \) and \( \hat{a}_s \), have zero matrix elements for the desired

Figure 2. Schematic diagram of a superconducting resonator lattice with multiple qubits
coupled to each resonator. The cyan areas represent superconducting material, which is
used to define the resonators, while the red dots represent the qubits. In this figure, the
resonators are arranged in a Kagome lattice, similar to the setup studied in [15]. In this
particular example, we include three qubits for each resonator, which would be the case
when simulating spin-1 particles. Note that the placement of the qubits shown in this
figure does not represent the optimal placement; when designing an actual device, qubit
placement is typically determined based on the locations where the electric field in the
transmission line resonator has a large value, ideally a maximum.
transitions. Modulating the qubit frequencies (i.e., driving the qubits using the operator $\sigma_z$), however, would induce these transitions [47]. If one wishes to achieve the maximum possible coupling with this approach, one needs to work in the Landau-Zener regime, where the qubit frequencies are driven up and down past the resonator frequencies [48]. Depending on the system parameters, this requirement could imply strong modulation of the qubit frequency. Another possibility for driving the Raman transitions without necessarily requiring strong driving on the qubits is using tunable couplers between the qubits and the resonators and modulating the coupling strength at the required frequencies [49, 50]. The effective matrix elements for the spin-changing hopping processes obtained this way can be made on the order of the hopping strength $J$, which occurs when the driven qubit-resonator transitions have strengths comparable to $J$, meaning that with the appropriate driving strengths, sufficiently large values of the spin-changing hopping matrix elements can be obtained. To obtain a controllable form of spin-orbit coupling, one needs to be able to control the amplitude and phase of the matrix elements describing the spin-changing hopping processes. For example, to obtain Rashba spin-orbit coupling in a square lattice, the two spin-changing hopping matrix elements along one of the two spatial dimensions need to have the same amplitude, but opposite signs [36]. Similarly, spin-orbit coupling in one dimension can be obtained by designing the two spin-changing processes to have opposite signs [37]. This sign difference can be achieved by adjusting the phases of the ac fields that drive the Raman transitions. This goal can be achieved with the proper choice of parameters, as we show in the following derivation.

We now present a quantitative calculation showing how the different processes can arise in an effective Hamiltonian with the proper choice of parameters and driving conditions. For the simulation of spin-1/2 particles, where we only need to consider two resonator modes, the Hamiltonian of the entire system can be expressed as

$$H = \sum_i \left( \omega_{1,i} c_{1,i}^\dagger c_{1,i} + \omega_{1,i} c_{1,i}^\dagger c_{1,i}^\dagger + \omega_{r,1} a_{1,i}^\dagger a_{1,i} + \omega_{r,2} a_{2,i}^\dagger a_{2,i} \right)$$

$$+ \sum_{s,m,i} g_{s,m,i}(t) \left( c_{s,i}^\dagger a_{m,i} + c_{s,i} a_{m,i}^\dagger \right) + \sum_m \sum_{i,j} \left( a_{m,i}^\dagger a_{m,i} + a_{m,i} a_{m,i}^\dagger \right).$$  \hspace{1cm} (9)$$

1 In this derivation, we shall not explicitly include the hat symbol for the operators, and we set $\hbar = 1$.  

Figure 3. Schematic diagrams illustrating the processes involved in a spin-orbit coupled Hubbard model with spin-1/2 particles, and how these processes could be induced in a JCH system. The red arrows describe spin-conserving hopping processes, while the purple arrows describe spin-changing hopping processes. In (b), the black solid lines describe the single-excitation qubit energy levels, the blue dashed lines describe (delocalized) single-photon energy levels in the resonators, and the green dotted lines show virtual energy levels whose locations are determined by the detunings between the ac fields and the real energy levels. The virtual energy levels are somewhat detuned from the single-photon energy levels to avoid populating the resonators with real excitations.
Here, we have introduced the annihilation and creation operators for qubit excitations, $c$ and $c^\dagger$, with subscripts $\uparrow$ and $\downarrow$ as qubit labels. As above, we use $a$ and $a^\dagger$ as the resonator annihilation and creation operators with subscripts 1 and 2 as mode labels. The different parameters in the Hamiltonian are self-explanatory.

For this calculation, we consider only two neighbouring lattice sites, and we focus on the single-excitation subspace. We also assume that the coupling strengths are modulated with sinusoidal time dependence (i.e., $g(t) = g + f \cos (\omega t + \phi)$). Ignoring terms that do not have any significant effect on the system, such as the dc component of the coupling between highly detuned subsystems, the Hamiltonian can be expressed as

$$
H = \omega_1 \left( c_{1,i} c_{1,i} + c_{1,j} c_{1,j} \right) + \omega_1 \left( c_{1,i} c_{1,i} + c_{1,j} c_{1,j} \right) + \omega_{r,1} \left( a_{1,i} a_{1,i} + a_{1,j} a_{1,j} \right) \\
+ \omega_{r,2} \left( a_{2,i} a_{2,i} + a_{2,j} a_{2,j} \right) + g_{1,i,j} \left( c_{1,i} a_{1,i} + c_{1,j} a_{1,j} \right) + g_{1,2,i,j} \left( c_{1,i} a_{2,i} + c_{1,j} a_{2,j} \right) \\
+ g_{1,2,i,j} \left( c_{1,i} a_{2,i} + c_{1,j} a_{2,j} \right) \cos (\omega_1 t + \phi_1) \left( c_{1,i} a_{1,i} + c_{1,j} a_{1,j} \right) \\
+ f_{1,1,i,j} \cos (\omega_2 t + \phi_2) \left( a_{1,i} a_{1,i} + a_{1,j} a_{1,j} \right) \\
+ f_{1,2,i,j} \cos (\omega_3 t + \phi_3) \left( a_{2,i} a_{2,i} + a_{2,j} a_{2,j} \right) \\
+ f_{1,2,i,j} \cos (\omega_4 t + \phi_4) \left( a_{2,i} a_{2,i} + a_{2,j} a_{2,j} \right) \\
+ J_1 \left( a_{1,i} a_{1,i} + a_{1,j} a_{1,j} \right) + J_2 \left( a_{2,i} a_{2,i} + a_{2,j} a_{2,j} \right),
$$

(10)

where the driving signals with frequencies $\omega_i$ (with $i = 1, 2, 3, 4$) are defined through the subscripts of the coefficients, $f_i$. For example, $\omega_1$ corresponds to the signal modulating the coupling strength between the qubit labeled $\downarrow$ and resonator mode 1 at site $i$, thus corresponding to the bottom left transition in figure 3(b).

As shown in appendix B, with the proper choice of driving-field parameters, we obtain the effective Hamiltonian

$$
H = \tilde{\omega}_1 \left( c_{1,i} c_{1,i} + c_{1,j} c_{1,j} \right) + \tilde{\omega}_1 \left( c_{1,i} c_{1,i} + c_{1,j} c_{1,j} \right) + \tilde{\omega}_{r,1} \left( a_{1,i} a_{1,i} + a_{1,j} a_{1,j} \right) \\
+ \tilde{\omega}_{r,2} \left( a_{2,i} a_{2,i} + a_{2,j} a_{2,j} \right) + \tilde{J}_1 \left( a_{1,i} a_{1,i} + a_{1,j} a_{1,j} \right) + \tilde{J}_2 \left( a_{2,i} a_{2,i} + a_{2,j} a_{2,j} \right) \\
+ \tilde{\Delta}_1 \left( c_{1,i} c_{1,j} + c_{1,j} c_{1,i} \right) + \tilde{\Delta}_2 \left( c_{1,i} c_{1,j} + c_{1,j} c_{1,i} \right) \\
+ \tilde{\Delta}_3 \left( e^{-i(\phi_1+\phi_2)} c_{1,i} c_{1,j} + e^{i(\phi_1+\phi_2)} c_{1,j} c_{1,i} \right) \\
+ \tilde{\Delta}_4 \left( e^{-i(\phi_3+\phi_4)} c_{1,i} c_{1,j} + e^{i(\phi_3+\phi_4)} c_{1,j} c_{1,i} \right),
$$

(11)

where the tildes indicate renormalized parameter values, $\tilde{\Delta}_1 = \omega_{r,1} - \omega_1$, $\tilde{\Delta}_2 = \omega_{r,2} - \omega_1$, $\delta_1 = \omega_{r,1} - \omega_1$, and $\delta_2 = \omega_{r,2} - \omega_4$. The last four terms in this Hamiltonian, along with their tunable parameters, allow us to engineer any desired type of effective spin-orbit coupling. In particular, by taking the loop ($\downarrow$, $i$) → ($\uparrow$, $j$) → ($\uparrow$, $i$) → ($\downarrow$, $j$) → ($\downarrow$, $i$), a particle
picks up a phase of $\phi_1 + \phi_2 + \phi_3 - \phi_4$. In the example of the Rashba spin-orbit coupling mentioned above, one can obtain the necessary minus signs by setting the phases to appropriate values, in particular having the combination $\phi_1 + \phi_2 + \phi_3 - \phi_4$ equal to zero for one direction and $\pi$ for the other direction in the two-dimensional square lattice. Changing the amplitudes and phases of the driving fields would lead to a continuum of different types of spin-orbit coupling.

State preparation and readout are rather straightforward when encoding the different spin states in different qubits. Microwave-controlled quantum operations driven via local antennas can be used to initialize individual qubits in their excited states, thus allowing the preparation of well-defined modes at particles in the different spin states. Similarly, the state readout of individual qubits can be readily achieved in state-of-the-art setups, and this readout would yield the positions and spin states of the simulated particles. As discussed in section 3.1, one can also investigate the ground-state phase diagram by preparing theoretically predicted mean-field ground states and monitoring their time evolutions.

4. Experimental parameters

Superconducting qubits and resonators have typical frequencies in the range of 1–20 GHz. One could, therefore, think of a resonator that has a fundamental frequency of about 4 GHz, with a few additional modes at multiples of this frequency. Qubits with tunable frequencies in the vicinity of these resonator frequencies can be fabricated in present-day state-of-the-art experiments. Qubit-resonator coupling strengths can reach hundreds of megahertz. One can therefore take 100 MHz as a typical, realistic value for the coupling strength. Resonator arrays with coupling strengths of 30 MHz have been fabricated [24], and values in the 50–100 MHz range should be possible. Effective spin-conserving hopping matrix elements on the order of 10 MHz or higher should, therefore, be achievable.

For the proposal of section 3.2, the single-transition matrix elements in the Raman processes used to drive the spin-changing hopping processes are limited by the fact that one needs to keep these transitions virtual. In other words, one needs to keep the latter’s matrix elements smaller than the detuning between the resonator frequency and the virtual energy level used in the Raman process. This detuning can be 100 MHz or more. With the single-transition matrix elements tuned to around 50 MHz, the spin-changing hopping processes can have effective matrix elements on the order of 10 MHz. The fact that both spin-conserving and spin-changing hopping matrix elements can be engineered in the same range allows one to design any desired combination of hopping processes, and therefore any desired type of spin-orbit coupling.

Some superconducting qubit designs have weak anharmonicities, and in such cases one normally has to take into consideration the higher energy levels of the qubit circuit. However, the anharmonicity is higher than 200 MHz in most, if not all experiments to date. Since we focus on the regime where double occupancy is prohibited because of other considerations and the hopping strength is typically in the range of 10–100 MHz (i.e., much smaller than the anharmonicity), the weak anharmonicity of the qubits should not create any additional concerns about the validity of the single-excitation approximation.

Decay rates of superconducting qubits and resonators are steadily improving (i.e., decreasing). Decay rates of 10–100 kHz, implying excitation lifetimes of 10–100 µs (long
compared to the hopping matrix elements), are now quite realistic. An implementation of the proposed setup could therefore be realized within the next few years.

5. Conclusion

We have considered the possibility of simulating itinerant spin-carrying particles using lattices of superconducting qubits and resonators. The basic JCH setup could be used to realize a number of relevant processes in this context, while a more complex system employing multiple qubits coupled to each resonator offers additional flexibility and could lead to more sophisticated simulations in the future. In particular, such a system would allow the simulation of large spin values with the possibility of improved control over the various spin-conserving and spin-changing hopping processes.

Experiments on the use of superconducting circuits for implementing JCH systems are in the early stages of development, but are progressing at a fast pace. They hold the promise of great controllability and measurability, two properties that are highly desirable in a quantum simulator. We expect that the ability to add internal degrees of freedom to the simulated particles, along with the ability to engineer various spin-related physical processes, will add to the power of this platform for quantum simulation.

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Appendix A. Two alternative approaches for the simulation of spin-carrying particles

In this appendix, we present two alternative approaches that one might use to achieve the desired goal of simulating itinerant spin-carrying particles. These approaches might come to mind naturally in the present context. As we will discuss below, however, we find these approaches less promising than the ones presented in the main text.

**Higher qubit states**—Superconducting qubits are, in fact, multistate quantum systems where only two states are used when representing qubit states. A recent experiment [51] made use of the additional quantum states in order to simulate a single spin larger than 1/2. Constructing a resonator array with a single one of these multi-level circuits coupled to each resonator might therefore seem to be a promising approach to obtaining the desired system. However, the energy levels in the qubit circuit are almost, but not exactly, equally spaced. A similar lack of tunability also arises when considering the matrix elements that describe the qubit-resonator coupling. Consequently, the tunneling strengths in the resulting JCH model would be constrained to follow a certain, partially regular pattern, limiting the ability of the system to probe even the basic parameter regimes of interest. Constructing the desired quantum simulator using this system is therefore not as straightforward as it might seem at first sight.

**Multiple resonator modes**—Owing to their extended structure, transmission-line resonators (TLRs) generally support a large number of modes, including the so-called fundamental mode with frequency $\omega_f$ and modes with frequencies that are close to integer-
multiples of the fundamental frequency (i.e., at frequencies close to $n_m \omega_f$, where the mode index $n_m = 2, 3, 4, \ldots$). One therefore automatically obtains multiple and potentially usable degrees of freedom in a TLR. Excitations in the different modes can be used to represent particles in the different internal states. In particular, $2s + 1$ modes are needed to simulate spin-$s$ particles. The recently demonstrated parametric coupling between two modes of a TLR [52] can be used to simulate spin-changing terms in the Hamiltonian.

There are, however, a number of difficulties associated with following this approach. The hopping strength in a system with capacitive coupling between the resonators is proportional to the capacitive energy between two resonators, which is proportional to the product of the charges accumulated across the capacitor. For a given mode $n_m$, each one of the charges is proportional to $m \times n_p$, where $n_p$ is the number of photons in mode $n_m$. As a result, the hopping strength is proportional to $n_m$. To simulate particles whose hopping strength is independent of spin, one would like to have no such dependence, or at least one would like to have tunable coupling strengths with the ability to set all of them to a single value.

Another complication that arises with the use of multiple TLR modes is the simple relation between the frequencies of the different modes. For example, if one drives the system at the fundamental frequency, the drive signal would be resonant with multiple processes, including single-mode and multi-mode processes. (Note that some of these resonances can be multi-photon resonances occurring at integer multiples of the driving frequency.) One possible method to circumvent the detrimental effects of this regular structure of frequencies is to use a combination of modes that are not integer-multiples of each other (e.g., use the modes $n_m = 2$ and $n_m = 3$ in order to simulate spin-1/2 particles). However, this solution becomes more demanding for larger spin values. Because of this and the previously mentioned difficulty, this approach is also not as straightforward as it might seem at first.

Appendix B. Derivation of the spin-orbit coupling terms given in section 3.2

In this appendix, we show the derivation of equation (11) from equation (10). For this derivation, we split the problem of obtaining an effective Hamiltonian for the qubit excitations into two separate problems: one where we consider only the effects of the time-independent qubit-resonator coupling terms, with coefficients denoted by the symbol $g$ and one where we consider only the effects of the ac terms, with coefficients denoted by the symbol $f$. By doing so, we assume no interference between these two types of terms in producing the effective Hamiltonian. We shall discuss at the end of the derivation why this approximation is justified.

We first consider the Hamiltonian without the ac terms:

$$H = \omega_1 \left( c_{i,i}^\dagger c_{i,i} + c_{i,j}^\dagger c_{i,j} \right) + \omega_1 \left( c_{j,i}^\dagger c_{j,i} + c_{j,j}^\dagger c_{j,j} \right) + \omega_{r,1} \left( a_{i,i}^\dagger a_{i,i} + a_{i,j}^\dagger a_{i,j} \right) + \omega_{r,2} \left( a_{2,i}^\dagger a_{2,i} + a_{2,j}^\dagger a_{2,j} \right) + g_{1,1} \left( c_{i,i}^\dagger a_{1,i} + c_{i,j}^\dagger a_{1,j} \right) + g_{1,2} \left( c_{j,i}^\dagger a_{2,i} + c_{j,j}^\dagger a_{2,j} \right) + g_{1,1} \left( c_{i,j}^\dagger a_{1,j} + c_{j,j}^\dagger a_{1,j} \right) + g_{1,2} \left( c_{j,j}^\dagger a_{2,j} + c_{j,j}^\dagger a_{2,j} \right) + J_1 \left( a_{1,i}^\dagger a_{1,i} + a_{1,j}^\dagger a_{1,j} \right) + J_2 \left( a_{2,i}^\dagger a_{2,i} + a_{2,j}^\dagger a_{2,j} \right). \quad (B.1)$$
We now perform an adiabatic elimination of the resonator modes via the transformation $H = U \hat{H} U^\dagger$, where

$$\hat{U} = \exp\left[ S_1 - S_1^\dagger \right]$$  \hspace{1cm} (B.2)

$$S_1 = -\frac{g_{1,1,i}}{\Delta_1} c_{1,i}^\dagger a_{1,i} - \frac{g_{1,1,j}}{\Delta_1} c_{1,j}^\dagger a_{1,j} - \frac{g_{1,1,i}}{\Delta_2} c_{1,i}^\dagger a_{2,i} - \frac{g_{1,2,j}}{\Delta_2} c_{1,j}^\dagger a_{2,j},$$  \hspace{1cm} (B.3)

$\Delta_1 = \omega_{r,1} - \omega_1$ and $\Delta_2 = \omega_{r,2} - \omega_1$. After truncation to terms that are at least of order $(g/\Delta)^2$ and ignoring nonresonant terms, this transformation results in the effective Hamiltonian

$$\hat{H} = \left( \omega_1 - \frac{g_{1,1,i}^2}{\Delta_1} \right) c_{1,i}^\dagger c_{1,i} + \left( \omega_1 - \frac{g_{1,1,j}^2}{\Delta_1} \right) c_{1,j}^\dagger c_{1,j} + \left( \omega_1 - \frac{g_{1,2,i}^2}{\Delta_2} \right) c_{1,i}^\dagger c_{1,i} + \left( \omega_1 - \frac{g_{1,2,j}^2}{\Delta_2} \right) c_{1,j}^\dagger c_{1,j}

+ \left( \omega_{r,2} + \frac{g_{1,2,i}^2}{\Delta_2} \right) a_{1,i}^\dagger a_{1,i} + \left( \omega_{r,2} + \frac{g_{1,2,j}^2}{\Delta_2} \right) a_{1,j}^\dagger a_{1,j}

+ J_1 \left[ 1 - \frac{g_{1,1,i}^2 + g_{1,1,j}^2}{2\Delta_1^2} \right] (a_{1,i}^\dagger a_{1,j} + a_{1,i} a_{1,j}^\dagger)

+ J_2 \left[ 1 - \frac{g_{1,2,i}^2 + g_{1,2,j}^2}{2\Delta_2^2} \right] (a_{2,i}^\dagger a_{2,j} + a_{2,i} a_{2,j}^\dagger)

+ J_1 \frac{g_{1,1,i} g_{1,2,i}}{\Delta_1^2} (c_{1,i}^\dagger c_{1,j} + c_{1,i} c_{1,j}^\dagger)

+ J_2 \frac{g_{1,2,i} g_{1,2,j}}{\Delta_2^2} (c_{1,i}^\dagger c_{1,j} + c_{1,i} c_{1,j}^\dagger).$$  \hspace{1cm} (B.4)

All but the last two terms in the above Hamiltonian correspond to terms in the original Hamiltonian, with some small shifts in the effective parameters from the original values. (Note that the shifts can be set to equal values, such that they do not detune previously resonant energy levels.) The last two terms in the new Hamiltonian describe processes that were not explicitly present in the original Hamiltonian, namely the hopping of $\uparrow$ and $\downarrow$ excitations between neighbouring sites. Note that these processes conserve the spin of the hopping particles. It is also important to emphasize here that the appearance of these terms was the result of applying a transformation that eliminated, to the lowest order, the qubit-resonator coupling terms from the Hamiltonian.

Next we consider the Hamiltonian with the ac terms, and without the time-independent coupling terms, whose effect has already been calculated. We shall set $\omega_1 + \omega_2 = \omega_3 - \omega_3 = \omega_1 - \omega_1$ in order to drive the desired spin-changing processes. We also set $\omega_1 = 0$ as an energy reference in order to simplify the expressions below. We now perform a rotating-frame transformation, $H' = U \hat{H} U^\dagger$, where
\[ U = \exp \left\{ -i \left[ \omega_1 \left( a_{1,i}^+ a_{1,j} + a_{1,j}^+ a_{1,i} \right) + \omega_4 \left( c_{1,i}^+ c_{1,j} + c_{1,j}^+ c_{1,i} \right) \right] \right\}. \] (B.5)

After applying the rotating-wave approximation and ignoring terms that oscillate with frequencies that are on the order of the driving ac field frequencies, we obtain the Hamiltonian

\[ H' = \delta_1 \left( a_{1,i}^+ a_{1,i} + a_{1,j}^+ a_{1,j} \right) + \delta_2 \left( a_{2,i}^+ a_{2,i} + a_{2,j}^+ a_{2,j} \right) \]
\[ + \frac{f_{1.1,i}}{2} \left( e^{-i \phi_1} c_{1,i}^+ a_{1,i} + e^{i \phi_1} c_{1,j}^+ a_{1,i} \right) + \frac{f_{1.1,j}}{2} \left( e^{i \phi_2} c_{1,j}^+ a_{1,j} + e^{-i \phi_2} c_{1,j}^+ a_{1,j} \right) \]
\[ + \frac{f_{1.2,i}}{2} \left( e^{-i \phi_1} c_{1,i}^+ a_{2,i} + e^{i \phi_1} c_{1,i}^+ a_{2,i} \right) + \frac{f_{1.2,j}}{2} \left( e^{-i \phi_2} c_{1,j}^+ a_{2,j} + e^{i \phi_2} c_{1,j}^+ a_{2,j} \right) \]
\[ + J_1 \left( a_{1,i}^+ a_{1,j} + a_{1,j}^+ a_{1,i} \right) + J_2 \left( a_{2,i}^+ a_{2,j} + a_{2,j}^+ a_{2,i} \right), \] (B.6)

where \( \delta_1 = \omega_{r,1} - \omega_1, \) \( \delta_2 = \omega_{r,2} - \omega_4. \) We now perform an adiabatic elimination of the resonator modes via the transformation \( \tilde{H}' = U' H' U'^\dagger, \)

\[ U' = \exp \left[ S_2 - S_2^\dagger \right] \] (B.7)

\[ S_2 = -\frac{f_{1.1,i}}{2 \delta_1} e^{-i \phi_1} c_{1,i}^+ a_{1,i} - \frac{f_{1.1,j}}{2 \delta_1} e^{i \phi_1} c_{1,j}^+ a_{1,j} - \frac{f_{1.2,i}}{2 \delta_2} e^{-i \phi_2} c_{1,i}^+ a_{2,i} - \frac{f_{1.2,j}}{2 \delta_2} e^{i \phi_2} c_{1,j}^+ a_{2,j}. \] (B.8)

After truncation to terms that are at least of order \((f/\delta)^2\) and ignoring nonresonant terms, we obtain the effective Hamiltonian

\[ \tilde{H}' = \left( \delta_1 + \frac{f_{1.1,i}^2}{4 \delta_1^2} \right) a_{1,i}^+ a_{1,i} + \left( \delta_1 + \frac{f_{1.1,j}^2}{4 \delta_1^2} \right) a_{1,j}^+ a_{1,j} + \left( \delta_2 + \frac{f_{1.2,i}^2}{4 \delta_2^2} \right) a_{2,i}^+ a_{2,i} \]
\[ + \left( \delta_2 + \frac{f_{1.2,j}^2}{4 \delta_2^2} \right) a_{2,j}^+ a_{2,j} - \frac{f_{1.1,i}^2}{4 \delta_1^2} c_{1,i}^+ c_{1,i} - \frac{f_{1.1,j}^2}{4 \delta_1^2} c_{1,j}^+ c_{1,j} \]
\[ - \frac{f_{1.2,i}^2}{4 \delta_2^2} c_{1,i}^+ c_{1,i} - \frac{f_{1.2,j}^2}{4 \delta_2^2} c_{1,j}^+ c_{1,j} \]
\[ + J_1 \left[ 1 - \frac{f_{1.1,i}^2 + f_{1.1,j}^2}{8 \delta_1^2} \right] \left( a_{1,i}^+ a_{1,j} + a_{1,j}^+ a_{1,i} \right) \]
\[ + J_2 \left[ 1 - \frac{f_{1.2,i}^2 + f_{1.2,j}^2}{8 \delta_2^2} \right] \left( a_{2,i}^+ a_{2,j} + a_{2,j}^+ a_{2,i} \right) \]
\[ J_1 \frac{f_{1,1,i} f_{1,1,j}}{4\delta_1^2} \left( e^{-i(\phi_1 + \phi_2)} c_{i,j}^\dagger c_{i,j} + e^{i(\phi_1 + \phi_2)} c_{i,j} c_{i,j}^\dagger \right) \]
\[ + J_2 \frac{f_{1,2,i} f_{1,2,j}}{4\delta_2^2} \left( e^{-i(\phi_1 - \phi_2)} c_{i,j}^\dagger c_{i,j} + e^{i(\phi_1 - \phi_2)} c_{i,j} c_{i,j}^\dagger \right). \]  

(B.9)

Once again, all but the last two terms describe small renormalization effects to the Hamiltonian parameters. The last two terms describe spin-changing hopping processes. Combining the results of equations (B.4) and (B.9), we obtain equation (11).

We now go back to the question of whether splitting the problems into smaller and separate parts is justified. In particular, the eight different transitions contributing to the four hopping processes could, in principle, produce additional combinations that did not appear when we divided the driving fields into two separate groups. However, if the intersite hopping strengths \( J \), are much smaller than the detunings, \( \Delta \) and \( \delta \), all of these additional combinations will describe nonresonant Raman processes that can be ignored. One must also be careful about such interference effects when generalizing the above construction to an array of lattice sites. For example, if the same frequency combinations are used to drive the transitions between sites \( i \) and \( i + 1 \) and between sites \( i + 1 \) and \( i + 2 \), then undesirable transitions will be driven as well. This problem can be avoided by using different frequencies, or in other words different virtual energy levels, for the different pairs of neighbouring lattice sites.

References

[1] Gerry C C and Knight P L 2005 *Introductory Quantum Optics* (Cambridge, UK: Cambridge University Press)
Walls D F and Milburn G J 1994 *Quantum Optics* (Berlin: Springer-Verlag)
Scully M O and Zubairy M S 1997 *Quantum Optics* (Cambridge, UK: Cambridge University Press)
[2] Hubbard J 1963 *Proc. R. Soc. London, Ser. A* 276 238
Mahan G D 2000 *Many-Particle Physics* (Berlin: Springer)
Tasaki H *arXiv:cond-mat/9512169*
[3] Hartmann M J, Brandao F G S L and Plenio M B 2006 *Nat. Phys.* 2 849
Greentree A D *et al* 2006 *Nat. Phys.* 2 856
Angelakis D G, Santos M F and Bose S 2007 *Phys. Rev. A* 76 031805(R)
[4] Deng H, Haug H and Yamamoto Y 2010 *Rev. Mod. Phys.* 82 1489
[5] Houck A A, Türeci H E and Koch J 2012 *Nat. Phys.* 8 292
Schmidt S and Koch J 2013 *Ann. Phys., Lpz.* 525 395
[6] Carusotto I and Ciuti C 2013 *Rev. Mod. Phys.* 85 299
[7] Georgescu I M, Ashhab S and Nori F 2014 *Rev. Mod. Phys.* 86 153
[8] Paraoanu G S 2014 *J. Low Temp. Phys.* 175 633
[9] Hartmann M J, Brandao F G S L and Plenio M B 2007 *Phys. Rev. Lett.* 99 160501
Cho J, Angelakis D G and Bose S 2008 *Phys. Rev. A* 78 062338
[10] Cho J, Angelakis D G and Bose S 2008 *Phys. Rev. Lett.* 101 246809
Hayward A L C, Martin A M and Greentree A D 2012 *Phys. Rev. Lett.* 108 223602
Hafezi M, Lukin M D and Taylor J M 2013 *New J. Phys.* 15 063001
[11] Carusotto I, Gerace D, Tureci H E, de Liberato S, Ciuti C and Imamoglu A 2009 *Phys. Rev. Lett.* 103 033601
Hartmann M J 2010 *Phys. Rev. Lett.* 104 113601
Tomadin A, Giovannetti V, Fazio R, Gerace D, Carusotto I, Türeci H E and Imamoglu A 2010 *Phys. Rev. A* 81 061801
Liu K, Tan L, Lv C H and Liu W M 2011 Phys. Rev. A 83 063840
Tomadin A, Diehl S and Zoller P 2011 Phys. Rev. A 83 013611
Nissen F, Schmidt S, Biondi M, Blatter G, Türeci H E and Keeling J 2012 Phys. Rev. Lett. 108 233603
Marcos D, Tomadin A, Diehl S and Rabl P 2012 New J. Phys. 14 055005
le Boité A, Orso G and Ciuti C 2013 Phys. Rev. Lett. 110 233601
Grujic T, Clark S R, Jakšch D and Angelakis D G 2013 Phys. Rev. A. 87 053846

[12] Koch J and le Hur K 2009 Phys. Rev. A 80 023811
[13] Makin M I, Cole J H, Hill C D, Greentree A D and Hollenberg L C L 2009 Phys. Rev. A 80 043842
[14] Leib M and Hartmann M J 2010 New J. Phys. 12 093031
[15] Koch J, Houck A A, Hur K L and Girvin S M 2010 Phys. Rev. A 82 043811
[16] Tsomokos D I, Ashhab S and Nori F 2010 Phys. Rev. A 82 052311
[17] Nunnenkamp A, Koch J and Girvin S M 2011 New J. Phys. 13 095008
[18] Zheng H and Takada Y 2011 Phys. Rev. A 84 043819
Schirò M, Bordyuh M, Öztöb B and Türeci H E 2012 Phys. Rev. Lett. 109 053601
[19] Yang W L, Yin Z-Q, Chen Z X, Kou S-P, Feng M and Oh C H 2012 Phys. Rev. A 86 012307
[20] Xiang Z-L, Yu T, Zhang W, Hu X and You J Q 2012 Science China physics Mechanics & Astronomy 55 1549
[21] Kim N Y, Kusudo K, Leoffler A, Høfling S, Forchel A and Yamamoto Y 2013 New J. Phys. 15 035032
[22] Kulaitis G, Kruger F, Nissen F and Keeling J 2013 Phys. Rev. A 87 013840
[23] Kapit E 2013 Phys. Rev. A 87 062336
[24] Underwood D L, Shanko W E, Koch J and Houck A A 2012 Phys. Rev. A 86 023837
[25] Raftery J, Sadri D, Schmidt S, Türeci H E and Houck A A 2014 Phys. Rev. X 4 031043
[26] as discussed in Hümmer T, Reuther G M, Hänggi P and Zueco D 2012 Phys. Rev. A 85 052320
[27] Bloch I, Dalibard J and Nascimbène S 2012 Nat. Phys. 8 267
[28] Jacobs K arXiv:1209.2499
[29] Jacobs K arXiv:1209.2499
[30] Soykal Ö O and Tahan C 2013 Phys. Rev. B 88 134511
[31] Ladd T D, Jelezko F, Laflamme R, Nakamura Y, Monroe C and O’Brien J L 2010 Nature 464 45
Buluta I, Ashhab S and Nori F 2011 Rep. Prog. Phys. 74 104401
[32] Schoelkopf R J and Girvin S M 2008 Nature 451 664
You J Q and Nori F 2011 ibid. 474 589
[33] Ho T-L 1998 Phys. Rev. Lett. 81 742
Ohmi T and Machida K 1998 J. Phys. Soc. Japan 67 1822
see also Kuklov A B and Svistunov B V 2002 Phys. Rev. Lett. 89 190430
Ashhab S and Leggett A J 2003 Phys. Rev. A 68 063612
Ashhab S 2005 J. Low Temp. Phys. 140 51
Stamper-Kurn D M and Ueda M 2013 Rev. Mod. Phys. 85 1191
Kawaguchi Y and Ueda M 2012 Phys. Rep. 520 253
[34] See e.g. Demler E and Zhou F 2002 Phys. Rev. Lett. 88 163001
Kuklov A B and Svistunov B V 2003 Phys. Rev. Lett. 90 100401
Tsuchiya S, Kuribara S and Kimura T 2004 Phys. Rev. A 70 043628
Cole W S, Zhang S, Paramekanti A and Trivedi N 2012 Phys. Rev. Lett. 109 085302
Radic J, di Ciolo A, Sun K and Galitski V 2012 Phys. Rev. Lett. 109 085303
Mobarak M and Pelster A 2013 Laser Phys. Lett. 10 115501
[35] Stanescu T D, Anderson B and Galitski V 2008 Phys. Rev. A 78 023616
Ho T-L and Zhang S 2011 Phys. Rev. Lett. 107 150403
Wu C, Mondragon-Shem I and Zhou X-F 2011 Chin. Phys. Lett. 28 097102
Hu H, Ramachandhran B, Pu H and Liu X-J 2012 Phys. Rev. Lett. 108 010402
It should be noted here that in the JCH setup, we naturally obtain effective particles with bosonic statistics. Although truly spin-1/2 particles are fermions, one can think of pseudospin-1/2 bosonic particles. Such particles have, for example, been considered extensively in cold atomic gases [33].

In addition to the dispersive regime, another commonly studied special case in the the JCH setup is one where the qubits and resonators have the same frequency. In this case, the energy levels exhibit a different dependence on the number of excitation quanta. However, if one assumes a dilute system, where it suffices to consider at most double occupancy of a given site, any on-site energy level structure can be described by the interaction term given in equation (2).

Note that the field direction is also determined by the initial-time phase of the driving fields. This phase might have no physical consequences in the single-site case, but the relation between the phases at different sites in the lattice determines the profile of the simulated external field.

[47] Porras D and Garcia-Ripoll J J 2012 Phys. Rev. Lett. 108 043602

[48] Shevchenko S N, Ashhab S and Nori F 2010 Phys. Rep. 492 1

[49] Niskanen A O, Harrabi K, Yoshihara F, Nakamura Y, Lloyd S and Tsai J S 2007 Science 316 723

[50] Ashhab S, Niskanen A O, Harrabi K, Nakamura Y, Picot T, de Groot P C, Harmans C J P M, Mooij J E and Nori F 2008 Phys. Rev. B 77 014510

[51] Neeley M et al 2009 Science 325 722

[52] Zakka-Bajjani E, Nguyen F, Lee M, Vale L R, Simmonds R W and Aumentado J 2011 Nat. Phys. 7 599