CYLINDRICALLY SYMMETRIC SOLITONS
WITH NONLINEAR SELF-GRAVITATING SCALAR FIELDS

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Static, cylindrically symmetric solutions to nonlinear scalar-Einstein equations are considered. Regularity conditions on the symmetry axis and flat or string asymptotic conditions are formulated in order to select soliton-like solutions. Some non-existence theorems are proved, in particular, theorems asserting (i) the absence of black-hole and wormhole-like cylindrically symmetric solutions for any static scalar fields minimally coupled to gravity and (ii) the absence of solutions with a regular axis for scalar fields with the Lagrangian $L = F(I)$, $I = \varphi^a \varphi^b$, for any function $F(I)$ possessing a correct weak field limit. Exact solutions for scalar fields with an arbitrary potential function $V(\varphi)$ are obtained by quadratures and are expressed in a parametric form in a few ways, where the parameter may be either the coordinate $x$, or the $\varphi$ field itself, or one of the metric coefficients. It is shown that soliton-like solutions exist only with $V(\varphi)$ having a variable sign. Some explicit examples of the solutions (including a soliton-like one) and their flat-space limit are discussed.

1. Introduction

The concept of solitons and particle-like configurations in nonlinear field theory has appeared as one of the approaches aimed at avoiding the well-known difficulties of the theories describing particles as mathematical points. In this approach, a hope to create a divergence-free particle theory was connected with a search for and studies of exact, regular, localized solutions to classical nonlinear field equations, able to describe the complicated spatial structure of particles observed in the experiment [1]. It was evident that nonlinearity should be necessarily included in the field equations in order to describe field interaction, irrespective of the divergence problem. In other words, nonlinearity is not only one of possible ways of removing the difficulties of the theory, but a reflection of real field properties.

Nowadays the problem of infinities is mostly discussed and solved in the context of numerous versions of string theory, which is known to be most promising on the main trend of modern theoretical physics, unification of the four interactions. Meanwhile, it is string theory (along with gauge field theory) that has created a new burst of interest in classical nonlinear field theories. On the one hand, there naturally appear various scalar fields with nonlinear potentials, on the other, some models of string theory create in their low-energy limits such theories as the Born-Infeld nonlinear electrodynamics or its non-Abelian modifications [2]. And, as previously, solitonic solutions are of utmost importance in any such theory.

Many papers, devoted to soliton-like solutions to nonlinear field equations, disregard the self-gravity of the field system under study, although its inclusion is of great interest since the gravitational field is intrinsically nonlinear, universal and cannot be shielded; moreover, the inclusion of self-gravity can drastically change the properties of solutions to nonlinear field equations and even their existence conditions [3].

Of greatest physical interest are evidently spherically symmetric (or more general axially symmetric) solutions able to describe localized objects in real three-dimensional space. However, some problems necessitate studies of two-dimensional, or cylindrically symmetric solutions, localized in a neighbourhood of the symmetry axis, the so-called vortices or string-like solutions [4][5]. Such solutions can both describe certain realistic objects like superconducting fibres (fluxons) [6] or light beams [7] and serve as reasonable approximations for toroidal structures when a torus of large radius is replaced by a closed string [8]. In the case of self-gravitating configurations, a natural application of soliton-like structures is the description of cosmic strings beyond the approximation treating them as simple conical singularities [9][10].

In this paper we discuss static, cylindrically symmetric, soliton-like configurations of nonlinear scalar fields with various Lagrangians in general relativity. Such scalar fields can be of any origin: Higgs fields, dilatons, inflatons, etc. The term “soliton-like” will here mean a globally isolated regular field configuration seen by a distant observer as a gravitating cylinder or a cosmic string.
In Sec. 2 we formulate the regularity and asymptotic conditions to be satisfied by the sought solutions. In Sec. 3 we prove some statements showing which kinds of soliton-like solutions cannot exist. It turns out, in particular, that configurations of scalar fields with Lagrangians of the form \( L = F(I) \), \( I = \varphi^a \varphi_a \) (for which the field equations are solved by quadratures) cannot have a regular axis, whatever is the function \( F(I) \). It is also shown that nonlinear scalar fields cannot lead to cylindrically symmetric analogues of black-hole and wormhole solutions. Sec. 4 is devoted to scalar fields with nonlinearities in the form of an arbitrary potential function \( V(\varphi) \). It is shown, in particular, that soliton-like solutions with this kind of nonlinearity can exist only if \( V(\varphi) \) has a variable sign. Treating the potential \( V(\varphi) \) as one of unknown functions, so that the set of field equations is underdetermined, we describe four ways of obtaining exact solutions by quadratures. The first way requires specifying the function \( \alpha(x) \), the second — \( V(\alpha) \) (where \( e^{2\alpha} \) is one of the metric coefficients and \( x \) is the radial coordinate). In the third approach one should specify the function \( \varphi(\alpha) \) while the fourth one starts with a given \( \alpha(\varphi) \). The nontrivial nature of the flat-space limit of these solutions is discussed in Sec. 5: when the gravitational constant \( \alpha \to 0 \), this is only a necessary rather than sufficient condition for passing to a flat-space solution. In Sec. 6 we consider three examples of solutions with different \( V(\varphi) \). The first one is the Liouville nonlinearity for which exact solutions are obtained directly, but among them there is no soliton-like one. The second one illustrates a connection between self-gravitating and flat-space solutions. The third one is an example of a soliton-like solution, which is given in a parametric form, in terms of elliptic functions.

2. Regularity and asymptotic conditions

Let us write down the metric without fixing the radial coordinate \( x \):

\[
ds^2 = e^{2\gamma}dt^2 - e^{2\alpha}dx^2 - e^{2\beta}dz^2 - e^{2\delta}d\phi^2
\]

where \( \alpha, \beta, \gamma, \xi \) are functions of \( x \); \( z \in \mathbb{R} \) and \( \phi \in [0, 2\pi) \) are the longitudinal and azimuthal coordinates, respectively.

We will try to select soliton-like configurations from the whole set of solutions. As mentioned above, this will imply two requirements: the existence of a spatial asymptotic from which our system is seen as an isolated cylindrically symmetric source of gravity or a cosmic string, and global regularity of the space-time and the fields. If the \((x, \varphi)\) surfaces are simply connected, the global regularity condition actually reduces to that of regularity on the symmetry axis. Another opportunity, a wormhole-like topology of the \((x, \phi)\) surfaces, will be discussed in Sec. 3.

2.1. Regularity on the axis

The regularity conditions on an axis, i.e. at a value \( x_{ax} \) of \( x \) such that \( e^\beta \to 0 \), include the finiteness requirement for the algebraic curvature invariants and the condition

\[
|\beta'| e^{\beta - \alpha} \to 1
\]

(2)

(where the prime denotes \( d/dx \), expressing a correct relation between infinitesimal circumferences and radii, in other words, the absence of a conical singularity. Among the curvature invariants it is sufficient to deal with the Kretschmann scalar \( K = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \) which, for the metric (1), is a sum of squared components of all nonzero Riemann tensor components \( R^{\mu\nu}_{\ \ \ \ \rho\sigma} \):

\[
K = 4 \sum_{i=1}^{6} K_i^2;
\]

\[
K_1 = R^{01}_{01} = -e^{-\alpha - \gamma} (\gamma' e^{\gamma - \alpha})', \quad K_2 = R^{02}_{02} = -e^{-2\alpha} \gamma' \xi', \quad K_3 = R^{03}_{03} = -e^{-2\alpha} \beta' \gamma', \quad K_4 = R^{12}_{12} = -e^{-\alpha - \xi} (\xi' e^{\xi - \alpha})', \quad K_5 = R^{13}_{13} = -e^{-\alpha - \beta} (\beta' e^{\beta - \alpha})', \quad K_6 = R^{23}_{23} = -e^{-2\alpha} \beta' \xi'.
\]

(3)

For \( K < \infty \) it is thus necessary and sufficient that all \( |K_i| < \infty \), and this in turn guarantees that all algebraic invariants of the Riemann tensor will be finite. Note that all \( K_i \), as well as the condition (2), are invariant under reparametrisation of \( x \).

From (2) and \( \beta \to -\infty \) it follows that \( K_3 \) and \( K_6 \) are finite if and only if

\[
\gamma' e^{-\alpha} = O(e^\beta), \quad \xi' e^{-\alpha} = O(e^\beta),
\]

(4)
and consequently $K_2 = O(e^{2\beta}) \to 0$. Here and henceforth the symbol $O(f)$ denotes a quantity either of the same order of magnitude as $f$ in a certain limit, or smaller, while the symbol $\sim$ connects quantities of the same order of magnitude.

It is easily shown that the conditions (1) can only hold if $\gamma$ and $\xi$ take finite values on the axis.

The remaining quantities $K_1$, $K_4$, $K_5$ are better dealt with using specific coordinates. Let us choose the harmonic $x$ coordinate, such that

$$\alpha = \beta + \gamma + \xi.$$  

Then, as $\beta \to -\infty$, $\alpha = \beta + O(1)$, and, since by (2) $\beta' \sim 1$, it is evident that a regular axis can only occur at $x = x_{\text{ax}} = \pm \infty$. Choosing $x_{\text{ax}} = -\infty$, one can write:

$$\beta = cx(1 + o(1)), \quad c = e^\gamma + \xi\Bigg|_{x \to -\infty} = \text{const} > 0,$$

and, as follows from (1),

$$\gamma' = O(e^{2\beta x}), \quad \xi' = O(e^{2\beta x}).$$  

One can now check that under these conditions $K_1$ and $K_4$ are finite on the axis, while the finiteness of $K_5$ requires $\beta'' = O(e^{2\beta x})$. This means that the condition (2) should be strengthened, and in a reparametrization-invariant form we have

$$|\beta'| e^{\beta - \alpha} = 1 + O(e^{2\beta}).$$  

Thus to provide a regular axis it is necessary and sufficient to require the validity of (1) and (8) as $x \to x_{\text{ax}}$.

The same conditions in terms of the conventional radial coordinate $r = e^\beta$ read

$$e^\alpha = 1 + O(r^2); \quad \gamma = \gamma_{\text{ax}} + O(r^2); \quad \xi = \xi_{\text{ax}} + O(r^2) \quad \text{as } r \to 0.$$  

Another useful necessary condition for regularity follows from the Einstein equations. At points of a regular axis, as at any regular space-time point, the curvature invariants $R$ and $R_{\mu\nu} R^{\mu\nu}$ should be finite. Since the Ricci tensor for the metric (1) is diagonal, the invariant $R_{\mu\nu} R^{\mu\nu} \equiv R_{\alpha\beta} R^\alpha_\beta$ is a sum of squares, hence each component $R_{\mu\nu}^\alpha$ (no summing) is finite at a regular space-time point. Then, by virtue of the Einstein equations, each component of the EMT $T^\nu_\mu$ is finite as well:

$$|T^\nu_\mu| < \infty.$$  

Thus, requiring only the regularity of the geometry, we obtain, as its necessary condition, the finiteness of all EMT components. This is true not only for the present case, but always when $R_{\mu\nu}^\alpha$ is diagonal.

### 2.2. Regular (flat and string) asymptotics

We will be only concerned with isolated cylindrically symmetric configurations and therefore do not consider solutions having asymptotics of cosmological nature, such as closed models like the Melvin magnetic universe or those with (anti-)de Sitter asymptotics which should appear where the EMT behaves like a cosmological constant.

We shall instead require the existence of a spatial infinity, i.e., such $x = x_{\infty}$ that $\beta \to \infty$, where the metric is either flat, or corresponds to the gravitational field of a cosmic string.

This means that, first, as $x \to x_{\infty}$, a correct behavior of clocks and rulers requires $|\gamma| < \infty$ and $|\xi| < \infty$ as $x \to x_{\infty}$, or, choosing proper scales along the $t$ and $z$ axes, one can write

$$\gamma \to 0, \quad \xi \to 0 \quad \text{as } x \to x_{\infty}.$$  

Second, at the asymptotic the condition (2) should be replaced by a more general one,

$$|\beta'| e^{\beta - \alpha} \to 1 - \mu, \quad \mu = \text{const} < 1 \quad \text{as } x \to x_{\infty},$$  

so that the circumference to radius ratio for the circles $x = \text{const}$, $z = \text{const}$ tends to $2\pi(1 - \mu)$ instead of $2\pi$. In this case the space-time is locally flat but behaves asymptotically as if it were flat everywhere but on the axis, where a conical singularity is located, creating the angular defect $\mu$. In other words, under the asymptotic conditions (1), (12), $\mu > 0$, a soliton-like solution can simulate a cosmic string. A flat asymptotic takes place if $\mu = 0$.

In what follows we will use the words “regular asymptotic” in the sense “flat or string asymptotic”.

Third, the curvature tensor should vanish at the asymptotic, and, by virtue of the Einstein equations, all the EMT components must decay quickly enough. It can be easily checked that the conditions (1) and (12)
automatically imply that all $K_i = o(e^{-2\beta})$ where $K_i$ are defined in (3). Consequently the same decay rate at a regular asymptotic takes place in all components of $T^\nu_\mu$, and one can verify, in particular, that the total material field energy per unit length along the $z$ axis is finite:

$$\int T^0_0 \sqrt{-g} \, d^3x = \int T^0_0 \, e^{\alpha + \beta + \xi} \, dx \, dz \, d\phi < \infty$$

where integration in $z$ covers a unit interval. A similar condition in flat-space field theory is used as a criterion of field energy being localized around the symmetry axis, which is one of the requirements to solitonic solutions. The set of asymptotic regularity requirements (11), (12) for self-gravitating solutions is thus much stronger than (13) and contains the latter as a corollary.

It should also be noted that even the vacuum cylindrically symmetric solution has in general no regular asymptotic; this is, physically, due to an infinite total mass of an infinitely long source cylinder. The only vacuum solution with a regular asymptotic is described by flat space-time metric, maybe with a conical singularity on the axis. So our requirement means that the soliton-like solutions sought for should behave asymptotically just as this particular vacuum solution.

A static, linear, massless, minimally coupled scalar field also cannot provide a regular asymptotic. Indeed, e.g., in the coordinates (5) the field equation reads $\varphi'' = 0$, and its nontrivial solution, $\varphi' = \text{const} \neq 0$, leads to $T^0_0 \sim e^{-2\alpha} \sim e^{-2\beta}$ as $x \to \infty$ if one requires (11) and (12). Then the integral (13) diverges at $x \to \infty$. Thus, unlike spherical symmetry (where even linear fields vanish quickly enough), a regular cylindrically symmetric asymptotic is only possible due to an essentially nonlinear behavior of material fields.

### 3. Field equations and non-existence theorems

#### 3.1. Einstein equations and regularity conditions

For the metric (1), under the coordinate condition (3), we can write down the Einstein equations in the form

$$\begin{align*}
\beta'' + \xi'' - U &= -\alpha T^0_0 \, e^{2\alpha}, \\
U &= -\alpha T^1_1 \, e^{2\alpha}, \\
\gamma'' + \xi'' - U &= -\alpha T^2_2 \, e^{2\alpha}, \\
\beta'' + \gamma'' - U &= -\alpha T^3_3 \, e^{2\alpha}
\end{align*}$$

(14)

where $U \overset{\text{def}}{=} \beta' \gamma' + \beta' \xi' + \gamma' \xi'$, and $T^\nu_\mu$ is the energy-momentum tensor (EMT). For any static scalar fields minimally coupled to gravity it has the property of importance

$$T^0_0 = T^2_2 = T^3_3. \quad (15)$$

Therefore Eqs. (14) combine to give

$$\beta'' = \gamma'' = \xi'' = \frac{1}{3} \alpha''$$

(16)

where the last equality is due to (3), whence

$$\begin{align*}
\xi &= \frac{1}{3} (\alpha - Ax), \\
\gamma &= \frac{1}{3} (\alpha - Bx), \\
\beta &= \frac{1}{3} (\alpha + Ax + Bx)
\end{align*}$$

(17)

where $A$ and $B$ are integration constants and other two constants are ruled out by a proper choice of the origin of $x$ and the scale along the $z$ axis.

The function $\beta(x)$ determines the nature of the static space. In particular, as discussed above, $\beta \to -\infty$ corresponds to an axis, if any; at an asymptotic $\beta \to \infty$.

In the coordinates (3) the conditions (3) or (12) can only hold at $x \to \pm \infty$. It is evident from (17) that if one requires either a regular axis (say, at $x \to -\infty$) or a regular asymptotic (at $x = +\infty$), the constants should satisfy the requirement

$$A = B = N > 0.$$  

(18)

So the regular axis and regular asymptotic requirements lead to the same relation (18) for the integration constants, i.e., are compatible; this is favourable for the existence of soliton-like solutions.
Suppose there is a soliton-like solution with a regular axis and a regular asymptotic. Then at both ends one has
\[ \alpha \approx \beta \approx N x, \]  
(19)

with the same constant \( N \).

Under the conditions (11) at a regular asymptotic, the constant \( N \) has a clear geometric meaning. Indeed, according to (12)
\[ N = 1 - \mu, \]  
(20)

where \( \mu \) is the angular defect at a string asymptotic.

On the other hand, comparing (19) and (11), one sees that \( c = N \), so that
\[ N = e^{2\mu} = e^{2\xi_{\text{ss}}}. \]  
(21)

Thus the angular defect is directly related to the values of \( g_{tt} = e^{2\gamma} \) and \( g_{zz} = e^{2\xi} \) on the axis. In particular, \( g_{00} \) and its gradient determine the course of clocks and the gravitational forces applied to test particles at rest, respectively. So one can conclude that solitons with a string asymptotic \((\mu > 0)\) have (at least on the average) an attracting gravitational field, and photons coming from the axis are redshifted, whereas for solitons with a flat asymptotic \((\mu = 0)\) both redshifts and forces are averaged to zero on the way from the axis to spatial infinity.

Eq. (19) can be further refined in a way similar to (11) using (7) and (8), namely, near the axis \((x \to -\infty)\)
\[ \alpha = N x + O(e^{2N x}), \quad \beta = N x + O(e^{2N x}). \]  
(22)

At a regular asymptotic \((x \to \infty)\) one has, according to (17) and (19),
\[ \alpha = N x + o(1), \quad \beta = N x + o(1). \]  
(23)

### 3.2. Non-existence of black-hole, wormhole and hornlike solutions

It can be shown that certain types of behavior of the solutions are incompatible with scalar fields as sources of geometry.

An opportunity of interest is the existence of cylindrically symmetric configurations similar to black holes ("black strings"), i.e., those with a cylinder \( x = x_{\text{hor}} \) having the properties of a horizon. Some necessary conditions for that are: (i) this surface is regular, so that all \( K_i \) defined in (3) are finite, (ii) \( e^\gamma(x_{\text{hor}}) = 0 \) (a Killing horizon for the timelike Killing vector) and (iii) \( \xi(x_{\text{hor}}) \) and \( \beta(x_{\text{hor}}) \) are finite.

One can easily show that in the coordinates (3) these conditions are feasible only as \( x \to \pm \infty \); assuming that \( x = +\infty \) is the asymptotic, we are left with \( x_{\text{hor}} = -\infty \). Then all \( K_i \) are finite only if in (17) \( A = -B/2 \neq 0 \). This is clearly in contrast to (18) if we require that the same solution has a regular asymptotic. We have to conclude:

**Proposition 1.** Static, cylindrically symmetric black holes with a regular asymptotic cannot exist in general relativity with matter whose EMT satisfies Eq. (13).

A nonsingular cylindrically symmetric solution does not necessarily have a regular axis; it may contain no axis at all, so that the circularly symmetric \((x, \phi)\) surfaces have the topology of a cylinder. Such possible cases are:

- (i) a wormhole-like configuration, which, by definition, possesses two spatial infinities connected by a neck, i.e., a regular minimum of the function \( \beta(x) \);
- (ii) a hornlike configuration, where \( \beta(x) \) monotonically approaches \( \beta_{\text{min}} \) as \( x \) tends to a certain limiting value \( x^* \); the space-time is nonsingular if, as \( x \to x^* \), the metric coefficients \( e^{2\beta(x)}, e^{2\gamma(x)} \) and \( e^{2\xi(x)} \) have finite limits while the integral \( l = \int e^\phi du \) diverges. In other words, each \((x, \phi)\) surface ends with a regular infinitely long tube of finite radius.

Let us discuss these opportunities for nonlinear scalar fields in general relativity.

Suppose first that there are two regular spatial asymptotics. As before, one of them is at \( x \to +\infty \). At this asymptotic \( \alpha \approx \beta \) and Eq. (18) holds; from (17) one easily finds that \( \alpha \approx \beta \approx N x \). Another regular asymptotic might occur at \( x \to -\infty \); however, since the relation for the integration constants \( A = B = N \) still holds, if we assume that \( \gamma \) and \( \xi \) are finite there, we arrive again at \( \alpha \sim \beta \sim N x \), but now it means that \( \beta \to -\infty \), that is, an axis (which can in principle be regular); another spatial infinity cannot exist. We arrive at the following result:
Proposition 2. Static, cylindrically symmetric wormholes with two regular asymptotics do not exist in general relativity with matter whose EMT satisfies Eq. (15).

If we deny the asymptotic regularity condition but require symmetry of a wormhole-like configuration with respect to its neck, then at such a neck \( \beta' = \gamma' = \xi' = 0 \), hence \( U = 0 \). With \( U = 0 \), Eqs. (14) can be combined to give

\[
\beta'' = -e^{2\alpha} (T_0^0 + T_2^2 - T_3^3) \quad (24)
\]

On the other hand, a minimum of \( \beta \) implies \( \beta'' > 0 \) on the neck and in its certain neighbourhood where the first-order derivatives are small compared with the second-order ones and \( U = 0 \) remains a valid approximation. Assuming \( T_2^2 = T_3^3 \) (which is true for scalar fields), Eq. (24) then means that in the same neighbourhood \( T_0^0 < 0 \) (negative energy density). The result is:

Proposition 3. A static, cylindrically symmetric wormhole, symmetric with respect to its neck, cannot exist in general relativity with matter whose EMT satisfies the conditions \( T_2^2 = T_3^3 \) and \( T_0^0 \geq 0 \).

Remark. The proof of Prop. 3 may be readily refined to include the case that, at the minimum of \( \beta \), \( \beta'' = 0 \) but the lowest nonzero derivative is even-order and positive. The result will be the same. One can also observe that, to have \( U = 0 \) on the neck, it is sufficient to require only \( \gamma' = 0 \) or \( \xi' = 0 \) rather than both. The conditions of Prop. 3 may be accordingly weakened.

Suppose now that there is a hornlike solution with a regular asymptotic. If the latter occurs at \( x \to \infty \), then the “horn” \( x^* = -\infty \), since, with \( \alpha = \beta + \gamma + \xi \) finite, the integral \( I \) can diverge only at infinite \( x \). One then has to require in Eq. (17) that \( A = B = 0 \), whereas a regular asymptotic requires the validity of (18). This proves the following:

Proposition 4. Static, cylindrically symmetric hornlike solutions with a regular asymptotic do not exist in general relativity with matter whose EMT satisfies Eq. (15).

These restrictions should be taken into account in the further analysis. In particular, we shall not seek black-hole, wormhole or hornlike solutions. Consequently, in what follows a soliton-like configuration will mean a configuration with a regular axis and a regular asymptotic.

3.3. Self-gravitating scalar field with the nonlinearity \( L = F(I), \; I = \varphi^\alpha \varphi_\alpha \)

Consider a nonlinear scalar field in general relativity, described by the total Lagrangian

\[
L = \frac{R}{2\alpha} + F(I), \quad I = \varphi^\alpha \varphi_\alpha, \quad (25)
\]

where \( R \) is the scalar curvature. It is assumed that for weak fields (\( I \to 0 \)) the scalar field Lagrangian \( F(I) \) behaves like that of a linear field: \( F = \frac{1}{2}I + o(I) \) (a linear weak field limit). The corresponding EMT is

\[
T_{\mu\nu} = 2\frac{dF}{dI} \varphi_{,\mu} \varphi_{,\nu} - \delta^\nu_\mu F = 2I \frac{dF}{dI} \delta_{\mu\nu} \delta^\nu_\nu - F(I) \delta_{\mu\nu};
I = -\varphi^2 e^{-2\alpha} < 0. \quad (26)
\]

We will first obtain the general static, cylindrically symmetric solution and then show that it cannot be soliton-like.

The scalar field equation has the form

\[
\frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \varphi \frac{dF}{dI} \right) = 0 \quad (27)
\]

which, for \( \varphi = \varphi(x) \), under the coordinate condition \( (\bar{3}) \) is integrated to give

\[
\frac{dF}{dI} \varphi'(x) = C \quad \text{const.} \quad (28)
\]

Assuming that there is a known explicit expression for \( F(I) \), one can find \( I \) and \( \varphi' \) as functions of \( \alpha \).

The \( (\bar{1}) \) component of Eqs. (14) can be written as

\[
\alpha^2 - N^2 = 3\alpha e^{2\alpha} \left( 2I \frac{dF}{dI} - F \right), \quad N^2 = \frac{1}{3} (A^2 + AB + B^2) > 0. \quad (29)
\]
Its integration gives
\[ x = \pm \int d\alpha \left[ 3\alpha e^{2\alpha} \left( F - 2I \frac{dF}{dI} \right) + N^2 \right]^{-1/2}; \] (30)

Reversing (if possible, explicitly) the dependence (30), we obtain all unknowns as functions of \( x \).

Now, the following result is easily proved.

**Proposition 5.** The system (29) does not admit a static, cylindrically symmetric solution with a regular axis if the scalar field Lagrangian \( F(I) \) has a linear weak field limit.

Indeed, let us use the regularity condition (10); by (26) this means that both \( |F(I)| \) and \( |IF_I| \) are finite on the axis \( x = x_{ax} \) (\( F_I \) \( \overset{\text{def}}{=\frac{dF}{dI}} \)). Since \( \gamma \) and \( \xi \) should be finite while \( \beta \to -\infty \), we have \( e^\alpha \sim e^\beta \to 0 \). Meanwhile, it follows from (28) that \( -IF_I^2 = C^2 e^{-2\alpha} \to \infty \). Thus simultaneously
\[ |IF_I| < \infty \quad \text{and} \quad IF_I^2 \to \infty, \]
as \( x \to x_{ax} \), which is possible only if \( I \to 0 \) and \( F_I \to \infty \). But this contradicts the assumed asymptotic linearity of the field theory which implies \( F_I \to 1/2 \) as \( I \to 0 \).

We conclude that the Lagrangian (25) is unable to provide soliton-like solutions.

The above proof is quite similar to the one in \[3\] (see also \[19\]) where it was shown that solutions with a regular center cannot exist for nonlinear electrodynamics in general relativity in the spherically symmetric case. As in \[19\], this proof is of local nature and does not depend on spatial asymptotics. Therefore, in particular, Proposition 5 is readily generalized to general relativity with a cosmological constant.

4. **Self-gravitating scalar field with the potential \( V(\varphi) \)**

Consider now a nonlinear field system with the Lagrangian
\[ L = \frac{R}{2ae} + \frac{1}{2} \varphi^\alpha \varphi_{,\alpha} - V(\varphi) \] (31)
where \( V(\varphi) \) is an arbitrary function. For the metric (1) and \( \varphi = \varphi(x) \), under the coordinate condition (1) the Einstein equations take the form
\begin{align*}
T^\nu_\mu &= \varphi_{,\nu} \varphi_{,\mu} - \delta^\nu_\mu \left[ \frac{1}{2} \varphi_{,\alpha} \varphi_{,\alpha} - V(\varphi) \right] \\
&= \frac{1}{2} \varphi^2 e^{-2\alpha} \text{diag}(1, -1, 1, 1) + V(\varphi) \delta^\nu_\mu. \tag{32}
\end{align*}
The scalar field equation is
\[ \frac{1}{\sqrt{-g}} \partial_\mu \left[ \sqrt{-gg^{\mu\nu}} \partial_\nu \varphi \right] + \frac{dV}{d\varphi} = 0. \tag{33} \]
Since Eq. (14) holds as before, in the coordinates (1) we again obtain Eqs. (17), reducing the behavior of the metric to one unknown \( \alpha(x) \). The \( \{1\} \) component of Eqs. (14) now gives
\[ \alpha'^2 - N^2 = \frac{3}{2} \pi \varphi'^2 - 3\pi e^{2\alpha}, \quad N^2 \overset{\text{def}}{=} \frac{1}{2} (A^2 + AB + B^2) > 0 \] (34)
(we take \( N > 0 \) in agreement with Sec. 2). On the other hand, a sum of Eqs. (14) with the EMT (32) and Eq. (33) give
\[ \alpha'' + 3\pi V(\varphi) e^{2\alpha} = 0, \tag{35} \]
\[ \varphi'' - (dV/d\varphi) e^{2\alpha} = 0. \tag{36} \]
Thus the original set of equations has been reduced to (34), (35) and (36), where Eq. (34) is a first integral of the other two.

The following observation can be made directly from Eq. (35):

**Proposition 6.** In a soliton-like cylindrically symmetric solution to the field equations due to (31), the potential \( V \) satisfies the condition
\[ \int_{-\infty}^{+\infty} V(\varphi(x)) e^{2\alpha} dx = 0. \tag{37} \]
Indeed, according to Sec. 3.1, in a soliton-like solution one has \( \alpha' \to N \) for both \( x \to -\infty \) and \( x \to +\infty \), therefore integration of (35) over \( \mathbb{R} \) leads to (37).

Proposition 6 means that soliton-like solutions can only be obtained with potentials having a variable sign.

Let us now show a few ways of solving Eqs. (34)–(36) by quadratures. The problem of solving the field equations with given \( V(\varphi) \) is hard even in flat space — see Eq. (21). The general solution can be obtained in a few ways by specifying other functions involved.

In practice, the quadratures and/or inverse functions, needed to express all quantities in a convenient way, are not available explicitly in most specific cases. Therefore, being concerned with particular problems, it is useful to have various forms of the general solution at one’s disposal.

**General solution I: \( x \)-parametrization**

The simplest parametrization of the general solution to Eqs. (34)–(36) is obtained by specifying the function \( \alpha(x) \). Indeed, from (35) one then finds \( V(\varphi(x)) \) and after that \( \varphi'(x) \) from (34), which yields \( \varphi(x) \) by quadrature, so that the function \( V(\varphi) \) is obtained in a parametric form. It is made explicit if one resolves \( \varphi(x) \) with respect to \( x \).

Regularity of the solution on the axis is provided by \( \alpha(x) \) satisfying the condition (22), while to have a regular asymptotic one should fulfil Eq. (23).

**General solution II: \( \alpha \)-parametrization**

Introducing the notations

\[
U(\alpha) = 3aeV(\varphi)e^{2\alpha}, \quad y(\alpha) = \alpha^2, \tag{38}
\]

we can bring Eqs. (34) and (35) to the form

\[
\frac{3ae}{2}\varphi_{\alpha}^2 = 1 - \frac{1}{y}(N^2 - U), \tag{39}
\]

\[
y_{\alpha} = -2U(\alpha), \tag{40}
\]

where the subscript \( \alpha \) means \( d/d\alpha \). Eq. (36) holds as their consequence.

Now, if the function \( U(\alpha) \) is specified, \( y(\alpha) \) is found by quadrature from (40) and then \( \varphi(\alpha) \) from (39). Furthermore, according to (38), \( x(\alpha) \) is determined as follows:

\[
x = \pm \int da / \sqrt{y(\alpha)}. \tag{41}
\]

Thus all unknowns are expressed in terms of \( \alpha \), and it remains to resolve the dependence \( x(\alpha) \) with respect to \( \alpha \) in order to express them in terms of \( x \). To find \( V(\varphi) \) it is also necessary to resolve \( \varphi(\alpha) \) with respect to \( \alpha \).

Consider now the regularity conditions. On a regular axis \( x \to -\infty \), in addition to (22), one obtains

\[
\varphi = \varphi_{ax} + O(e^{Nz}); \quad U = O(e^{2Nz}), \tag{42}
\]

where \( \varphi_{ax} = \text{const} \). These conditions imply the finiteness of both \( \varphi \) and \( V(\varphi) \) on the axis. The local flatness on the axis is provided, as before, by Eq. (21).

To have a regular asymptotic, one should necessarily provide \( V = o(e^{-2\alpha}) \) and hence \( U(\alpha) = o(1) \) as \( \alpha \to \infty \).

The necessary condition (37) for a soliton-like nature of the solution has an analogue in terms of \( \alpha \), also obtained from (38):

\[
\int_{-\infty}^{\infty} U(\alpha)d\alpha = 0. \tag{43}
\]

If (43) holds, one can adjust the emerging integration constants to satisfy (21) and (22) with given \( \mu \).
General solution III: $\alpha$-parametrization

Substituting $U(\alpha)$ (defined in (38)) from (35) into (33) and treating $\varphi(\alpha)$ as a known function, one obtains a linear first-order differential equation for the unknown $y(\alpha) \equiv \alpha^2$:

$$y_\alpha - y(2 - 3\alpha^2) = -2N^2. \quad (44)$$

Its solution and the respective expression for $U(\alpha)$ are

$$y(\alpha) = -2N^2 e^{2\alpha - \Psi} \int e^{\Psi - 2\alpha} d\alpha, \quad (45)$$

$$U(\alpha) = N^2 \left[ 1 + (2 - 3\alpha^2) e^{2\alpha - \Psi} \int e^{\Psi - 2\alpha} d\alpha \right], \quad (46)$$

where $\Psi(\alpha) = 3\alpha \int \varphi^2 d\alpha$. Eq. (45) expresses $\alpha'$ in terms of $\alpha$, whence one finds by integration $x = x(\alpha)$ and consequently all unknowns as functions of $x$. As in solution I, to obtain the function $V(\varphi)$ it is necessary to resolve the dependence $x(\alpha)$ with respect to $\alpha$.

As is easily verified, regularity on the axis takes place if and only if

$$\varphi_\alpha = O(e^{\alpha}) \Rightarrow \Psi = \text{const} + O(e^{2\alpha}) \quad \text{as} \quad \alpha \to -\infty, \quad (47)$$

and in this case

$$y = \alpha'^2 = N^2 + O(e^{2\alpha}), \quad (48)$$

which describes $\alpha(x)$ near the axis more precisely than (42).

On the other hand, a necessary condition of a regular asymptotic is that, as $\alpha \to \infty$,

$$\varphi' = o(1/x), \quad \text{so that} \quad \Psi \to \text{const}. \quad (49)$$

General solution IV: $\varphi$-parametrization

Returning to Eqs. (34)–(36), let us now put $\alpha = \alpha(\varphi)$. Then (34) gives

$$\varphi^2 = \frac{N^2 - 3\alpha V(\varphi) e^{2\alpha}}{2\alpha^2 - 3x/2}. \quad (50)$$

where the subscript $\varphi$ denotes $d/d\varphi$. Expressing $\varphi''$ from (50) and comparing it with (36), we arrive at a linear equation with respect to $V(\varphi)$:

$$V_\varphi + P(\varphi)V + Q(\varphi) = 0,$$

$$P(\varphi) = \frac{3\alpha}{2\alpha^2 - 3x/2} \left( \frac{1}{\alpha^2} - \frac{\alpha \varphi'}{\alpha^2 - 3x/2} \right), \quad Q(\varphi) = \frac{N^2 \alpha \varphi e^{-2\alpha}}{2\alpha^2 - 3x/2}. \quad (51)$$

Its solution for given $\alpha(\varphi)$ is

$$V(\varphi) = -e^{\int P(\varphi)} \left( Q(\varphi) e^{\int P(\varphi)} \right) d\varphi, \quad (52)$$

$$e^{\int P(\varphi)} = \left( \frac{3\alpha}{2\alpha^2 - 3x/2} - 1 \right) e^{-3\alpha \int d\varphi/\alpha_x}. \quad (53)$$

With known $\alpha(\varphi)$ and $V(\varphi)$, from (50) one finds $x(\varphi)$ and, reversing it, determines all unknowns as functions of $x$, thus completing the solution.

Regularity conditions for $\alpha(\varphi)$ are found from those for $\varphi(\alpha)$ in (47), (49). Thus, a regular axis may be provided by

$$e^\alpha \sim |\varphi - \varphi_a|^k, \quad k \leq 1, \quad (54)$$

where $\varphi_a$ is the (finite) value of $\varphi$ on the axis, while a similar condition for a regular asymptotic is

$$e^{-\alpha} \sim |\varphi - \varphi_\infty|^k, \quad k < 1. \quad (55)$$
5. Flat-space limit

One can naturally expect that a solution to the nonlinear scalar field equation in Minkowski space will be obtained from a solution for a self-gravitating field in the limit $\alpha \to 0$. This is indeed the case, but the transition should be performed with certain care. Indeed, as $\alpha \to 0$, matter decouples from gravity, therefore, generically, the metric from a solution with self-gravitating matter should tend to a vacuum solution of general relativity with the corresponding symmetry and only under additional assumptions it will tend to flat metric. Moreover, when it happens, the matter in the same limit may acquire different forms depending on how the parameters of the original solution (e.g., the integration constants) depend on $\alpha$, and this makes a separate assumption, therefore the same self-gravitating solution can pass to different flat-space ones. It thus seems instructive to follow this limit for our solutions I–IV.

Let us require that, as $\alpha \to 0$, the metric tend to the Minkowski metric in cylindrical coordinates satisfying (5), namely,

$$ ds^2 = dt^2 - e^{2\alpha} d\alpha^2 - e^{2\alpha} d\phi^2. $$

The conventional form of the Minkowski metric in cylindrical coordinates is recovered by putting $e^\alpha = r$.

The scalar field and the potential $V$ should, in the same limit, satisfy the flat-space equation

$$ \phi'' - (dV/d\phi) e^{2\alpha} = 0. $$

Let us put for simplicity $A = B = N = 1$ in the self-gravitating solutions themselves, although one might, in general, only require $A(\alpha) \to 1$, $B(\alpha) \to 1$, $N(\alpha) \to 1$ as $\alpha \to 0$.

Solution I. Flat space is obtained by specifying $N = 1$ and $\alpha \equiv x$, to which one can proceed from a given function $\alpha(x)$ along any sequence of functions, parametrized by $\alpha \to 0$. Then Eqs. (34) and (35) are evidently satisfied for $\alpha = 0$, while (36) takes the form (57).

Solution II, (39)–(41). Denoting, in accordance with (38),

$$ \int U(\alpha) d\alpha = 3 \alpha X(\alpha) - C $$

where $X(\alpha)$ is an $\alpha$-independent function and $C$ is an integration constant, one can rewrite Eq. (40) in the form

$$ \varphi^2 = \frac{2}{3\alpha} \left( \frac{2C - 1 - 6\alpha X + 3\alpha V e^{2\alpha}}{2(C - 3\alpha X)} \right) $$

whence it follows that to have a proper limit we must put $C = 1/2$ and consequently, in the limit $\alpha \to 0$,

$$ \varphi^2 = 2(V e^{2\alpha} - 2X(\alpha)). $$

In the metric $\alpha \equiv x$, and it is directly verified that, under the substitution $\alpha = x$, Eq. (60) is a first integral of (57).

It remains to provide a proper transition in the metric. To this end, it is sufficient to put in (17) $A = B = 1$, so that $N^2 = 1$, and to verify the coincidence of $x$ and $\alpha$ in (41) for $\alpha = 0$. With (58) and $C = 1/2$, Eq. (41) is (omitting $\pm$) rewritten as

$$ x = \int d\alpha [1 - 6\alpha X(\alpha)]^{-1/2}, $$

so that for $\alpha = 0$ one obtains $x = \alpha$ up to an inessential additive constant.

Solution III, (45)–(46). By definition, $\Psi \sim \alpha$ as $\alpha \to 0$, and from (45) one finds up to $O(\alpha)$:

$$ y \equiv \alpha^2 = 1 - \Psi - 2 e^{2\alpha} \int e^{-2\alpha} \Psi d\alpha, $$

whence it follows that, first, $\alpha = x + O(\alpha)$ (up to an additive constant) and, second, according to (17),

$$ U \equiv 3\alpha e^{2\alpha} V = \frac{1}{2} \Psi_\alpha + \Psi + 2 e^{2\alpha} \int e^{-2\alpha} \Psi d\alpha \\
= 3\alpha e^{2\alpha} \int e^{-2\alpha} \varphi_\alpha \varphi_\alpha d\alpha. $$

(64)
The expression (64) is obtained from (63) by twice integrating by parts.

Now, the flat-space equation (57) leads to the following expression for \( V(\varphi) \) for given \( \varphi(x) \):

\[
V = \int e^{-2x} \varphi' \varphi'' \, dx.
\]

(65)

Evidently (64) yields (65) since in the same limit \( \alpha \) coincides with \( x \).

One should note that, given the same functional dependence \( \varphi(\alpha) \) in (66) and \( \varphi(x) \) in (65), the resulting functions \( V(\varphi) \) will be, in general, different.

**Solution IV, (62)–(63).** Suppose that \( \alpha(\varphi) \) is \( x \)-independent and \( V(\varphi) \) does not blow up as \( x \to 0 \). Then, in this limit, Eq. (50) turns into the equality \( \alpha^2 \varphi'' = 1 \), whence it follows \( \alpha = x \) (leading to the metric (56)) for a proper choice of the sign and origin of \( x \).

Furthermore, in (51) \( P(\varphi) \to 0 \) as \( x \to 0 \), while the limiting form of \( Q(\varphi) \), with \( N = 1 \) and \( \alpha = x \), is

\[
Q(\varphi) = x \varphi e^{-2x} / x^3 \quad (x \to 0).
\]

(66)

Therefore the solution of (51), namely, \( V(\varphi) = -\int Q(\varphi) \, d\varphi \), coincides with that of (65) with respect to \( V(\varphi) \) for known \( x(\varphi) \). As in solution II, the limiting function \( V(\varphi) \) is different from the one in Eq. (53).

Thus correct transitions from self-gravitating to flat-space solutions have been obtained for all four forms of the general solution.

6. Examples

1. The first example concerns the choice of \( V(\varphi) \) when Eqs. (54)–(58) are integrated directly — the Liouville potential. Let us denote

\[
\alpha V = 2W(\psi), \quad \sqrt{\alpha / 2} \varphi = \psi
\]

and choose

\[
W(\psi) = W_0 e^{\lambda \psi}, \quad W_0, \lambda = \text{const.}
\]

(67)

(68)

Then Eqs. (57), (58) combine to give

\[
(6\psi + \lambda \alpha)'' = 0, \quad 6\psi' + \lambda \alpha' = C,
\]

(69)

where the integration constant \( C \) should be equal to \( \lambda N \) with \( N > 0 \) introduced in (58) if we require a regular axis at \( x \to -\infty \) (where \( \psi' \to 0 \) while \( \alpha' \to N \)). Then (54) takes the form

\[
\alpha' - \frac{\lambda^2}{12} (N - \alpha')^2 = N^2 - 6W_0 e^{\lambda^2 N x / 6 + 2\alpha(1-\lambda^2/12)}.
\]

(70)

In case \( \lambda^2 = 12 \) its integration gives

\[
\alpha = N x - \left(3W_0 / 2N^2\right) e^{2N x} + \text{const.}
\]

(71)

For \( \lambda^2 \neq 12 \) one obtains

\[
e^{k \alpha} = e^{(k+1)N x} \frac{\sqrt{6|k| W_0}}{2N} \left[ e^{N(x+x_0)} - \varepsilon e^{-N(x+x_0)} \right],
\]

\[
k \equiv \lambda^2 / 12 - 1, \quad \varepsilon \equiv \text{sign}(kW_0).
\]

(72)

This completes the integration. It is easily confirmed that the asymptotic regularity condition \( \alpha \sim N x \) as \( x \to \infty \) cannot be fulfilled, in agreement with the above Proposition 6.

2. In the general case the potential \( V(\varphi) \) may be \( x \)-dependent, and this opportunity can be used for obtaining specific solutions for self-gravitating scalar fields corresponding to known flat-space solutions. In particular, if, for given \( V(\varphi) \) in flat space, \( \varphi(x) \) and hence \( V(x) = V(\varphi(x)) \) are known, the same dependence ascribed to \( V(\alpha) \) should lead to a self-gravitating solution in \( \alpha \)-parametrization, with a certain function \( V(\varphi, \alpha) \) which tends to the original potential \( V(\varphi) \) as \( x \to 0 \). A reason is that, under the coordinate condition (63), \( \alpha = x \) in the flat-space
metric. But, as was already clear from our general consideration, such a transition only occurs under special assumptions on the $\varphi$-dependence of the integration constants.

Consider, e.g., in flat space-time

$$V(\varphi) = \lambda \varphi^{2n}, \quad \lambda = \text{const} > 0, \quad n = \text{const} \neq 1.$$

Then the flat-space scalar field equation (57) reads

$$\varphi'' = 2n\lambda \varphi^{2n-1}e^{2x}$$

and has a special solution of the form

$$\varphi = \varphi_0 e^{\nu x}, \quad \nu = \frac{1}{1-n}, \quad \varphi_0 = \left(\frac{2n\lambda}{\nu^2}\right)^{\nu/2}.$$

The potential $V(\varphi)$ is expressed in terms of $x$ as

$$V(\varphi) = \lambda \varphi_0^{2n}e^{2nx/(1-n)}.$$

Now, we can seek a self-gravitating solution with a proper flat-space limit assuming

$$V(\varphi) = \lambda \varphi_0^{2n}e^{2\alpha/(1-n)},$$

with some constants $n$ and $\varphi_0$. Integrating according to (38)-(41), we obtain:

$$e^{\alpha/(1-n)} = \frac{\sqrt{C}}{\cosh[C_1(x - x_0)]},$$

where the constant $C$ arises from integration in (40) and $C_1 = 3\varphi_0 \sqrt{\lambda}$. Further integration gives

$$e^{\alpha/(1-n)} = \sqrt{C} \sin \left[ \frac{1}{n} \sqrt{\frac{3\varphi_0}{2}} (\varphi - \varphi_1) \right]$$

with one more integration constant $\varphi_1$, so that the potential takes the form

$$V(\varphi) = \lambda \varphi_0^{2n} \left\{ \sqrt{C} \sin \left[ \frac{1}{n} \sqrt{\frac{3\varphi_0}{2}} (\varphi - \varphi_1) \right] \right\}^{2n}$$

which resembles the well-known sine-Gordon nonlinearity. This function tends to (73) as $\varphi \to 0$ if and only if the constants $\varphi_1$ and $C$ depend on $\varphi$ in such a way that $\varphi \to 0$ and $\varphi_0 \sqrt{3\varphi C/2} \to n$.

3. Let us try to construct a soliton-like configuration, again on the basis of general solution II, Eqs. (38)-(41). Putting, in accordance with the regularity conditions,

$$y(\alpha) = N^2 \left(1 + \frac{A}{\cosh^2 n\alpha} \right),$$

where $A > -1$ and $n > 0$ are constants, one obtains from (40)

$$U(\alpha) = N^2 \frac{\sinh n\alpha}{\cosh^3 n\alpha}.$$

It is an odd function, manifestly satisfying (13); it vanishes sufficiently rapidly as $\alpha \to \pm \infty$ if $n \geq 1$.

The integral in (41) is easily found, leading to the following relation between $x$ and $\alpha$:

$$\sinh n\alpha = \frac{1}{\sqrt{A+1}} \sinh(Nnx + \ln \sqrt{A+1}),$$

with a correct behavior at large $\alpha$ and $x$. It remains to find $\varphi$. Let us take for simplicity $N = 1$ (a flat rather than string asymptotic). Eq. (40) gives

$$\frac{3\varphi_0^2}{2} \varphi_0^2 = \frac{A(\cosh n\alpha + n \sinh n\alpha)}{\cosh n\alpha (A + \cosh^2 n\alpha)}.$$
The r.h.s. is positive for all $\alpha \in \mathbb{R}$ under the conditions $A > 0$, $n \leq 1$. Recalling that $n \geq 1$ by the regularity requirement, we are left with $n = 1$. Thus we should put $n = 1$ in (81)–(83), and the following expression for $\varphi$ is obtained:

$$\varphi = \sqrt{\frac{3A}{2}} \int e^{\alpha/2} \cosh \alpha (A + \cosh^2 \alpha)^{-1/2} d\alpha,$$

which can be written in terms of elliptic functions. Indeed, putting $e^{2\alpha} = u$, one obtains

$$\varphi = \sqrt{\frac{3A}{2}} \int \frac{du}{(u + 1)^{1/2} [u^2 + 2A(A + 1)u + 1]^{1/2}},$$

whence (integrating from finite $u$ to infinity)

$$\varphi = \frac{2}{\sqrt{3A}} \left( \frac{A}{A + 1} \right)^{1/4} F \left( \sqrt{\frac{4B}{\cosh^2 2A + 2B}}, \sqrt{\frac{A + B}{B^2}} \right),$$

where $F$ is the elliptic integral of the first kind and $B = \sqrt{A(A + 1)}$. It is easy to verify that all the regularity conditions are satisfied; in particular, $\varphi \sim e^{-\alpha}$ as $\alpha \to \infty$ and $\varphi$ tends to a finite limit as $\alpha \to -\infty$.

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**References**

[1] J.K. Thirring and T.H.R. Skyrme, *Nucl. Phys.* 31, 550–555 (1962).

[2] N. Seiberg and E. Witten, “String theory and noncommutative geometry”, JHEP 9909, 032 (1999), [hep-th/9908142].

[3] A.A. Tseytlin, “Born-Infeld action, supersymmetry and string theory”, [hep-th/9908105].

[4] K.A. Bronnikov and G.N. Shikin, “Self-gravitating particle models with classical fields and their stability”. In: “Itogi Nauki i Tekhniki” (“Results of Science and Technology”), v.2, “Gravitation and Cosmology”, VINITI, Moscow, 1991, pp. 4–55 (in Russian).

[5] H.B. Nielsen and P. Olesen, *Nucl. Phys.* B61, 45–61 (1973).

[6] Ya.P. Terletsky, *Dokl. AN SSSR* 236, 4, 828–829 (1977).

[7] A.A. Abrikosov, *Zh. Eksp. Teor. Fiz.* 32, 6, 1442–1452 (1957).

[8] V.E. Zakharov, V.V. Sobolev and V.S. Synakh, *Zh. Eksp. Teor. Fiz.* 60, 1, 136–145 (1971).

[9] H.J. De Vega, *Phys. Rev.* D18, 8, 2945–2951 (1978).

[10] B. Linet, *J. Math. Phys.* 27, 7, 1817–1818 (1976); *Phys. Lett.* A124, 240–242 (1987).

[11] D. Garfinkle, *Phys. Rev.* D32, 6, 1323–1329 (1985).

[12] P. Amderamski and P. Laguna-Castillo, *Phys. Rev.* D37, 4, 877–884 (1988).

[13] A. Vilenkin, *Phys. Rev.* D23, 4, 852–857 (1981); *Phys. Rev. Lett.* 78, 12, 2288–2291 (1996).

[14] C. Thompson, *Phys. Rev.* D37, 2, 283–297 (1988).

[15] Ch.T. Hill, H.M. Hodges and M.S. Turner, *Phys. Rev.* D37, 2, 263–282 (1988).

[16] V.A. Fock, “Theory of Space, Time and Gravity”, Fizmatgiz, Moscow, 1961 (in Russian).

[17] G.N. Shikin, “Interacting scalar and electromagnetic fields: static cylindrically symmetric solutions with gravitation”, in: “Problems in Gravitation Theory and Particle Theory”, 14th issue. Energoatomizdat, Moscow, 1984, p. 85–97 (in Russian).

[18] G.N. Shikin, “Nonlinear scalar field of Born-Infeld type: self-consistent static cylindrically symmetric soliton solutions”. In: Proc. Sir Arthur Eddington Centenary Symp., v.2: “On Relativity Theory”. World Scientific, 1985, p. 130–137.

[19] G.N. Shikin, “Fundamentals of Soliton Theory in General Relativity”, URSS Publishers, Moscow, 1995 (in Russian).

[20] K.A. Bronnikov, *Phys. Rev.* D63, 044005 (2001).