Conditions for Bifurcations in a Non-Autonomous Scalar Differential Equation

Sang-Mun Kim, Hyong-Chol O, Chol Kim and Gyong-Chol Kim

Faculty of Mathematics, Kim Il Sung University, Pyongyang, D.P.R Korea
e-mail address: sangmunkim@yahoo.com

Abstract

In this paper is provided a sufficient condition to occur saddle-node and transcritical bifurcations for a non-autonomous scalar differential equation.

Keywords: non-autonomous scalar differential equation, saddle-node bifurcation, forwards attracting, pullback attracting

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1 Introduction

The concept of non-autonomous dynamical systems can be said to have been made from the study on skew product flows and random dynamical systems in 1990s in the viewpoint of topological dynamics.

A lot of developments have been made together with considering problems of various concepts of attractiveness, existence and uniqueness of attracting sets and etc. [2]-[19]. In [13] they obtained sufficient conditions to occur transcritical, pitchfork and saddle-node bifurcations in a special type of non-autonomous differential equation generalized from a canonical form of autonomous differential equation where transcritical, pitchfork and saddle-node bifurcations occur. And then using them, they studied the conditions for similar bifurcations in the general scalar non-autonomous equation

\[ \dot{x} = f(x, t, \lambda), \]

where \( \lambda \) is a parameter. By imposing conditions on the Taylor coefficients in the expansion of \( f \) near \( x = \lambda = 0 \), they proved various general theorems guaranteeing transcritical, pitchfork, and saddle-node bifurcations.

In [16] they obtained a sufficient condition to occur transcritical bifurcation in non-autonomous differential equation

\[ \dot{x} = a(t, \alpha)x + b(t, \alpha)x^2 + r(t, x, \alpha) \]

and a sufficient condition to occur pitchfork bifurcation in non-autonomous differential equation

\[ \dot{x} = a(t, \alpha)x + b(t, \alpha)x^3 + r(t, x, \alpha). \]

Some particular examples have been analyzed in various settings. In [18] using the framework of skew product flows has been considered a generalized notion of a
Hopf bifurcation and in [19] studied almost periodic scalar non-autonomous differential equations. In [10] has been analyzed transcritical and pitchfork bifurcations in an almost periodic equation. [4] has considered a non-autonomous ‘two-step bifurcation’ and [11] gave a nice discussion of the general problem in the context of skew product flows.

On the other hand, conditions for bifurcations to occur in one-dimensional autonomous dynamical systems have been studied using higher order derivatives. In [20] sufficient conditions for transcritical, pitchfork, saddle-node and period doubling bifurcations to occur in one-dimensional maps with one parameter have been studied using higher order derivatives. In [1] sufficient conditions for cusp and period doubling bifurcations to occur in one-dimensional maps with two parameters have been studied using higher order derivatives.

In this paper we consider some non-autonomous differential equations generalized from autonomous dynamical systems in [1, 20]. First we try to obtain a sufficient condition to occur saddle-node and transcritical bifurcations in the equations

$$\dot{x} = \mu^{2m-1}f(t) - g(t)x^{2n}, \ m, n \in \mathbb{N} \quad (1)$$

where the sufficient condition of [13] does not satisfied. And then we try to obtain sufficient conditions to occur saddle-node and transcritical bifurcations in more general equations

$$\dot{x} = G(x, t, \lambda), \ (\lambda \text{ is a parameter})$$

which include (1).

2 Preliminaries

We consider the following initial value problem of non-autonomous differential equation

$$\dot{x} = f(t, x), x(s) = x_0 \quad (2)$$

defined on a domain $D \subset \mathbb{R}^m$ of $x$. Let denote the solution to (2) by

$$x(t, s; x_0) = S(t, s)x_0.$$

Then $\{S(t, s)\}_{t \geq s}$ becomes a two-parameter family of transformations of $D$ satisfying the following properties [6, 16]:

1) $\forall t \in \mathbb{R}, S(t, t)$ is the identity of $D$.

2) $S(t, \tau)S(\tau, s) = S(t, s) \ (\forall t, \tau, s \in \mathbb{R})$.

3) $S(t, s)x_0$ is continuous on $t, s, x_0$.

Through the whole paper, we assume that $\{S(t, s) : D \to D\}_{t \geq s}$ preserves order [1]. For some basic concepts including complete orbits, time-varying family of sets, invariant sets, Hausdorff semi distance between sets, (local or global) forwards attracting sets, (local or global) pullback attracting sets, pullback Lyapunov stability, pullback Lyapunov instability, asymptotic instability and unstable sets of invariant sets, (local) pullback repelling sets, pullback attracting sets and the definitions on several types of bifurcations in non-autonomous differential equations, we refer to [2].

We note the following well-known facts as a remark:

1) If $\Sigma(\cdot)$ is forwards attracting in $D$, then it is locally forwards attracting in $D$. 
2) If $D$ is a bounded set, any pullback attracting sets in $D$ locally pullback attracting in $D$. If $D$ is an unbounded set, global pullback attracting sets in $D$ might not be locally pullback attracting in $D$.

3) If an invariant set $\Sigma(\cdot)$ is pullback attracting set in $D$ and there is a $T$ such that $\bigcup_{t \leq T} \Sigma(t)$ is bounded, then $\Sigma(\cdot)$ is locally pullback attracting.\[2\]

4) If $x^\ast(\cdot)$ is a complete orbit and locally pullback attracting, then it is pullback Lyapunov stable.

5) If $\Sigma(\cdot)$ is asymptotically instable, then it is pullback Lyapunov instable but it cannot be locally pullback attracting.\[13\]

The following fact provides some information about attracting sets:

Let $\{K(t)\}_{t \in \mathbb{R}}$ be a family of non-empty compact sets and for all $t_0$ and compact set $B \subset D$, \[3\]

$\exists T = T(t_0, B) : \forall s \leq T, S(t_0, s)B \subset K(t_0)$.

Then there exists a pullback attracting set $A(t)$ which is a connect set for every $t \in \mathbb{R}$.\[15\]

3 Main Results

3.1 Saddle node Bifurcation

First we consider a concrete example.

Theorem 1. Let consider the following non-autonomous differential equation

$\dot{x} = \mu^{2^{m-1}}f(t) - g(t)x^{2n}, \, m, n \in \mathbb{N}$ \[3\]

Let assume that $f(t)$ and $g(t)$ satisfy

$\int_{-\infty}^{t} f(s)ds = \int_{t}^{+\infty} f(s)ds = +\infty, \quad \lim_{t \rightarrow \pm\infty} \inf \ g(t) > 0, \, 0 < l \leq \lim_{t \rightarrow \pm\infty} \frac{f(t)}{g(t)} \leq M. \quad \[4\]

Then we have the following facts:

1) When $\mu \leq 0$, non-zero bounded complete orbits do not exist. When $\mu < 0$, for any fixed $x_s$, we have

$\exists \sigma : s \leq \sigma, \exists t^\ast(s) < +\infty : \lim_{t \rightarrow t^\ast(s)} x(t, s; x_s) = -\infty$

and for any fixed $t$, we have

$\exists s^\ast(t) > -\infty : \lim_{s \rightarrow s^\ast(t)} x(t, s; x_s) = -\infty$.

2) When $\mu = 0$, the zero solution is locally pullback and forwards attracting in $[0, \infty)$. For the solution with initial value in $(-\infty, 0)$, we have the same conclusions with the case of $\mu < 0$.

3) When $\mu > 0$, there exist two orbits $x^\ast(t)$ and $y^\ast(t)$ such that $x^\ast(t)$ is pullback and forwards attracting and $y^\ast(t)$ is pullback repelling and asymptotically instable.
That is, we have

\[
\lim_{s \to -\infty} S(t, s)x_0 = x^*(t), \quad x_0 > -\sqrt[2n]{\mu^{2n-1}},
\]

\[
\lim_{t \to +\infty} \text{dist } [S(t, s)x_0, x^*(t)] = 0, \quad x_0 > -\sqrt[2n]{\mu^{2n-1}},
\]

\[
\lim_{s \to +\infty} S(t, s)x_0 = y^*(t), \quad x_0 < \sqrt[2n]{\mu^{2n-1}},
\]

\[
\lim_{t \to -\infty} \text{dist } [S(t, s)x_0, y^*(t)] = 0, \quad x_0 < -\sqrt[2n]{\mu^{2n-1}}.
\]

**Proof.** In the case of \( \mu < 0 \), by (4) we have

\[
\exists T : \forall t \leq T \Rightarrow f(t) > 0, \quad g(t) > 0.
\]

In the case of \( x_s < 0 \), by the above expression we have \( \forall t \leq T, \; \dot{x} \leq -g(t)x^{2n} \) and

\[
\int_s^t \frac{\dot{x}}{x^{2n}}dr \leq -\int_s^t g(r)dr \Rightarrow \int_s^{x(s)} \frac{1}{x^{2n}}dx \leq -\int_s^t g(r)dr
\]

\[
\Rightarrow \frac{-1}{(2n-1)}x^{-(2n-1)} \bigg|_{x=x(s)} \leq -\int_s^t g(r)dr
\]

\[
\Rightarrow \frac{-1}{(2n-1)}x(t)^{-2n-1} \leq \left(x(s)^{-2n-1} - \int_s^t g(r)dr\right)^{-1}
\]

\[
x(t) \leq \left(x(s)^{-2n-1} + (2n-1) \int_s^t g(r)dr\right)^{-1}.
\]

For fixed \( t, x_s^{-2n-1} < 0 \) and \( g \) satisfies

\[
\exists T^*: \forall s \leq T^* \Rightarrow \int_s^t g(r)dr > 0.
\]

Thus we have

\[
\exists s^*(t) > -\infty : \lim_{s \to s^*(t)} x(t, s; x_s) = -\infty.
\]

For fixed \( x_s \), we have

\[
\exists \sigma(t) : s \leq \sigma(t), \exists \sigma^*(s) < +\infty : \lim_{t \to \sigma^*(s)} x(t, s; x_s) = -\infty.
\]

Let consider the case of \( x_s < 0 \). Then for \( x_s = -1 \), we have

\[
\exists \sigma_1 : s \leq \sigma_1 \Rightarrow \exists \sigma^*(s) < +\infty, \lim_{t \to \sigma^*(s)} x(t, s; -1) = -\infty. \quad (6)
\]

If \( t \leq T \), then \( \dot{x} \leq \mu^{(2n-1)}f(t) < 0 \) and thus we have

\[
x(t, s; x_s) \leq x_s + \mu^{(2n-1)} \int_s^t f(r)dr, \; t \leq T. \quad (7)
\]

Using \( \mu < 0 \), (4) and (7), we have

\[
\exists \sigma_2 : s \leq \sigma_2 \Rightarrow t \leq \sigma_1, x(t, s; x_s) \leq -1.
\]
Thus we have

$$\forall \tau > t, \ x(\tau, t; x(t, s; x_s)) \leq x(\tau, t; -1).$$

When \( \tau \to t^+(s) \), we have \( x(\tau, t; -1) \to -\infty \) and thus \( x(\tau, s; x_s) \to -\infty \). On the other hand, for fixed \( t \), when \( s \to s_1(t) > -\infty \), we have \( x(t, s; -1) \to -\infty \).

Using (2), we have \( \exists s_2 : s \leq s_2, x(t, s; x_s) \leq -1 \) and from property of order perservation, we have \( x(t, s; x_s) \leq x(t, s; -1) \). When \( s \to s_1(s_2) \), we have \( x(t, s; x_s) \to -\infty \).

Next consider the case of \( \mu = 0 \). Then the solution of (3) is as follows:

$$x(t, s; x_s) = \frac{1}{x_s^{(2n-1)} + (2n - 1) \int_s^t g(r)dr}.$$ 

If \( x_s \geq 0 \), then \( x_s^{(2n-1)} \geq 0 \) and from (4) we have \( s \to -\infty (t \to +\infty) \). Then

\[
(2n - 1) \int_s^t g(r)dr \to +\infty, \quad \left[ x_s^{(2n-1)} + (2n - 1) \int_s^t g(r)dr \right]^{\frac{1}{2n-1}} \to +\infty.
\]

Thus we have

$$x(t, s; x_s) \to 0 (t \to +\infty, s \to -\infty).$$

In order to show that the zero solution is locally pullback attracting, we must prove

$$\left[ x_s^{(2n-1)} + (2n - 1) \int_s^t g(r)dr \right]^{\frac{1}{2n-1}} > 0, \quad \forall \tau \in [s, t].$$

If

$$x_s < \frac{1}{\sup_{r \in [r-t]} \left( \left[ (2n - 1) \int_r^t g(r)dr \right]^{\frac{1}{2n-1}} \right)} = \delta(t),$$

then the above expression holds. Thus the zero solution is locally pullback attracting. It is similar to prove that the zero solution is locally forwards attracting.

If \( x_s < 0 \), then we have the same result with the case when \( \mu < 0 \) and \( x_s < 0 \).

Let consider the case of \( \mu > 0 \). From the condition (5) we have

$$\exists T^- < 0, \exists T^+ > 0; \forall t \leq T^-, \forall t \geq T^+$$

$$\Rightarrow \dot{x} \leq \mu^{(2m-1)} M g(t) - g(t) x_{2n} = g(t) \left[ M \mu^{(2m-1)} - x_{2n} \right],$$

$$\Rightarrow \dot{x} \geq \mu^{(2m-1)} g(t) - g(t) x_{2n} = g(t) \left[ l \mu^{(2m-1)} - x_{2n} \right].$$

Thus we have

$$\dot{x} \leq g(t) \sum_{k=1}^{n} \left( 2 \sqrt{M \mu^{2m-1}} \right)^{2(n-k)} x_{2(k-1)} \left[ \left( 2 \sqrt{M \mu^{2m-1}} \right)^{2} - x_{2} \right],$$

$$\dot{x} \geq g(t) \sum_{k=1}^{n} \left( 2 \sqrt{l \mu^{2m-1}} \right)^{2(n-k)} x_{2(k-1)} \left[ \left( 2 \sqrt{l \mu^{2m-1}} \right)^{2} - x_{2} \right].$$
Thus we have 

\[ \forall \text{the above arguments, we have the following conclusion:} \]

Then \( g_1(t) \) and \( g_2(t) \) satisfy the condition (5) on \( g(t) \). Therefore we have

\[ \dot{x} \leq g_1(t) \left[ 2\sqrt{\mu}^{2m-1} + x \right] \left[ 2\sqrt{\mu}^{2m-1} - x \right], \]

\[ \dot{x} \geq g_2(t) \left[ 2\sqrt{\mu}^{2m-1} + x \right] \left[ 2\sqrt{\mu}^{2m-1} - x \right]. \]

If \( x_0 > -2\sqrt{\mu}^{2m-1} \), then

\[ 2\sqrt{\mu}^{2m-1} = \lim_{t \to -\infty} x(t, s; x_0) \leq 2\sqrt{\mu}^{2m-1}. \]

Now let \( x_1(t) \) and \( x_2(t) \) be two different solutions of (3) and \( z(t) = x_1(t) - x_2(t) \). Then

\[ \dot{x}_1(t) = \mu^{2m-1} f(t) - g(t)x_4^n(t), \quad \dot{x}_2(t) = \mu^{2m-1} f(t) - g(t)x_2^n(t). \]

Thus we have

\[ \dot{z}(t) = -g(t) \left[ x_1^n - x_2^n \right] = -g(t) \left[ \sum_{k=1}^{n} x_1^{2(n-k)}(t)x_2^{2(k-1)}(t) \right] [x_1 + x_2]z(t). \quad (8) \]

Since \( \forall t \leq T^- \) or \( \forall t \geq T^+ \), \( g(t) \left( \sum_{k=1}^{n} x_1^{2(n-k)}(t)x_2^{2(k-1)}(t) \right) > 0 \) and \( x_1(t), x_2(t) \geq 2\sqrt{\mu}^{2m-1} \), thus we have \( \forall t \leq T^- \) or \( \forall t \geq T^+ \), \( x_1(t) = x_2(t) \). Therefore there exists a positive solution \( x^*(t) \) such that it (pullback, forwards) attracts all orbits with initial data greater than \( -2\sqrt{\mu}^{2m-1} \). Now if \( x_0 < -2\sqrt{\mu}^{2m-1} \), then the solutions go to \( -\infty \) (pullback, forwards).

If \( x_0 < -2\sqrt{\mu}^{2m-1} \), then

\[ -2\sqrt{\mu}^{2m-1} \leq \lim_{t \to -\infty} x(t, s; x_0) \leq -2\sqrt{\mu}^{2m-1} \]

and for the two different solutions \( x_1(t) \) and \( x_2(t) \) of (3), we have (8). Repeating the above arguments, we have the following conclusion:

If \( \forall t \leq T^- \), \( \forall t \geq T^+ \), \( x_1(t), x_2(t) \leq -2\sqrt{\mu}^{2m-1} \), then \( x_1(t) = x_2(t) \).

Thus there exists a negative solution \( y^*(t) \) such that it (pullback, forwards) attracts all orbits with initial data less than \( 2\sqrt{\mu}^{2m-1} \) in the meaning of time inverse. That is, \( y^*(t) \) is pullback repelling.

\[ \lim_{t \to -\infty} x(t, s; x_s) = y^*(t), \quad x_s < 2\sqrt{\mu}^{2m-1}. \]
Now we consider general equations
\[ \dot{x} = G(t, x, \mu). \tag{9} \]
Assume that \( G \) is sufficiently smooth. The following is Taylor expansion of \( G \) at \((t, 0, 0)\).

\[
G(t, x, \mu) = G(t, 0, 0) + G_x(t, 0, 0)x + G_\mu(t, 0, 0)\mu + \frac{1}{2}G_{xx}(t, 0, 0)x^2
\]
\[ + G_{xx}(t, 0, 0)x + \frac{1}{2}G_{xx}(t, 0, 0)x^2 + \frac{1}{6}G_{xxx}(t, 0, 0)x^3 + \frac{1}{2}G_{x\mu}(t, 0, 0)x^2\mu \]
\[ + \frac{1}{2}G_{x\mu}(t, 0, 0)x^2 + \frac{1}{6}G_{\mu\mu}(t, 0, 0)\mu^2 + \cdots + \frac{1}{(2n)!} \left[ \frac{\partial^{2n}}{\partial x^{2n}\partial \mu} G(t, 0, 0) \right] x^{2n} \]
\[ + C_{2n+1}^1 \frac{\partial^{2n+1}}{\partial x^{2n}\partial \mu} G(t, 0, 0) x^{2n-1} \mu + \cdots + C_{2n+1}^{2n+1} \frac{\partial^{2n+1}}{\partial x^{2n}\partial \mu} G(t, 0, 0) x^{2n}. \]

Here \( n \in \mathbb{N} \).

Now assume that \( G \) satisfies the following conditions:

(i) \( G(t, 0, 0) = 0, \forall t \in \mathbb{R} \),

(ii) \( \frac{\partial}{\partial x} G(t, 0, 0) = \frac{\partial^2}{\partial x^2} G(t, 0, 0) = \cdots = \frac{\partial^{2n-1}}{\partial x^{2n-1}} G(t, 0, 0) = 0. \)

Then \( G \) is provided as follows:

\[
G(t, x, \mu) = \mu \left[ G_\mu(t, 0, 0) + G_{x\mu}(t, 0, 0)x + \frac{1}{2}G_{\mu\mu}(t, 0, 0)\mu + \frac{1}{3}G_{x\mu}(t, 0, 0)x^2 \right]
\]
\[ + \frac{1}{6}G_{x\mu}(t, 0, 0)x^2 + \frac{1}{3}G_{x\mu}(t, 0, 0)x\mu + \cdots + \frac{1}{(2n)!} C_{2n}^1 \frac{\partial^{2n}}{\partial x^{2n-1}\partial \mu} G(t, 0, 0) x^{2n-1} \mu + \cdots \]
\[ + \frac{1}{(2n)!} \frac{\partial^{2n}}{\partial x^{2n}\partial \mu} G(t, 0, 0) \mu^{2n-1} + \frac{1}{(2n+1)!} C_{2n+1}^1 \frac{\partial^{2n+1}}{\partial x^{2n+1}\partial \mu} G(t, 0, 0) x^{2n+1} \]
\[ + \frac{1}{(2n+1)!} \frac{\partial^{2n+1}}{\partial x^{2n+1}\partial \mu} G(t, 0, 0) x^{2n} + \cdots \]

Let denote \( f(t) := G_\mu(t, 0, 0), \ g(t) := -\frac{1}{(2n)!} \frac{\partial^{2n}}{\partial x^{2n}} G(t, 0, 0). \) Then can be rewritten as follows:

\[ \dot{x} = \mu[f(t) + \phi(t, x, \mu)] - x^{2n}[g(t) + \psi(t, x)]. \tag{10} \]

Here \( \phi(t, 0, 0) = 0, \ \psi(t, 0) = 0. \)
Theorem 2. Assume that

\[
\lim_{t \to \pm \infty} \inf g(t) > 0, \tag{11}
\]

\[
0 < m = \lim_{t \to \pm \infty} \inf \frac{f(t)}{g(t)} \leq \lim_{t \to \pm \infty} \sup \frac{f(t)}{g(t)} = M < +\infty, \tag{12}
\]

and there exists a positive valued function \( h(t) \) such that

\[
|\phi(t, x, \mu) \leq h(t)(|x| + |\mu|), \quad |\phi_x(t, x, \mu) \leq h(t), \tag{13}
\]

\[
|\psi_x(t, x) \leq h(t), \tag{14}
\]

\[
\lim_{t \to \pm \infty} \sup \frac{h(t)}{g(t)} \leq k. \tag{15}
\]

Then there occurs local saddle-node bifurcation when \( \mu \) passes through 0. Furthermore, when \( \mu > 0 \), locally attracting orbit \( x_\mu(t) \) is forwards attracting in \((0, \varepsilon)\) and unstable orbits are pullback repelling in \((-\varepsilon, \delta)\).

The main idea of the proof is similar to Theorem 1 and omitted.

Now assume that \( G \) satisfies the following conditions:

(iii) \( G(t, x, \mu) = G(t, x, 0) + c(x)G(0, 0, \mu), \)

(iv) \( G(t, 0, 0) = 0, \)

(v) \( \frac{\partial}{\partial x}G(t, 0, 0) = \frac{\partial^2}{\partial x^2}G(t, 0, 0) = \cdots = \frac{\partial^{2n-1}}{\partial x^{2n-1}}G(t, 0, 0) = 0, \)

(vi) \( \frac{\partial}{\partial \mu}G(t, 0, 0) = \frac{\partial^2}{\partial \mu^2}G(t, 0, 0) = \cdots = \frac{\partial^{2m-2}}{\partial \mu^{2m-2}}G(t, 0, 0) = 0. \)

Here \( n, m \in \mathbb{N} \). Then \( G \) is provided as follows:

\[
G(t, x, \mu) = \mu^{2m-1} \left[ x^{2n-2}\frac{\partial^2}{\partial x\partial \mu}G(t, 0, 0) + \frac{1}{2} x^2 \mu^{-(2m-2)} \frac{\partial^3}{\partial x^2 \partial \mu}G(t, 0, 0) + \frac{1}{2} x^{2m-3}\frac{\partial^3}{\partial x^2 \partial \mu}G(t, 0, 0) + \cdots + \right.
\]

\[
\left. + \frac{1}{(2m)!} x^{2m-2} \mu^{-(2m-3)} \frac{\partial^{2m}}{\partial x^{2m-1} \partial \mu}G(t, 0, 0) + \frac{1}{(2m)!} x^{2m-2} \mu^{-(2m-3)} \frac{\partial^{2m}}{\partial x^{2m-1} \partial \mu}G(t, 0, 0) + \cdots \right].
\]

Let denote \( f(t) := \frac{1}{2m-1} \frac{\partial^{2m-1}}{\partial x^{2m-1}}G(t, 0, 0), \; g(t) := -\frac{1}{2m} \frac{\partial^{2m}}{\partial x^{2m}}G(t, 0, 0). \) Then we can rewritten (15) as follows:

\[
\dot{x} = \mu^{2m-1}[f(t) + \phi(t, x, \mu)] - x^{2n}[g(t) + \psi(t, x)].
\]

Here

\[
\phi(t, 0, 0) = 0, \; \psi(t, 0) = 0.
\]
Theorem 3. Assume that
\[
\lim_{t \to \pm \infty} \inf g(t) > 0, \quad 0 < m = \lim_{t \to \pm \infty} \inf \frac{f(t)}{g(t)} \leq \lim_{t \to \pm \infty} \sup \frac{f(t)}{g(t)} = M < +\infty,
\]
\[
|\phi(t, x, \mu)| \leq h(t)|x| + |\mu|^{-\left(2m-2\right)}, \quad |\phi_x(t, x, \mu)| \leq h(t),
\]
\[
|\psi_x(t, x)| \leq h(t), \quad \lim_{t \to \pm \infty} \sup \frac{h(t)}{g(t)} \leq k.
\]
Then there occurs local saddle-node bifurcation when \( \mu \) passes through 0. Furthermore, when \( \mu > 0 \), locally attracting orbit \( x_\mu(t) \) is forwards attracting in \( (0, \varepsilon) \) and unstable orbits are pullback repelling in \( (-\varepsilon, \delta) \).

The proof is omitted.

Example 1. In the equation \( \dot{x} = \mu^3 t^2 - 2t^2 x^4 \), saddle node bifurcation occurs when \( \mu = 0 \).

3.2 Transcritical Bifurcation

First we consider a concrete example.

Theorem 4. Let consider the non-autonomous differential equation \( (3) \).
1) Let assume that \( f(t) \) and \( g(t) \) satisfy
\[
\forall t \in \mathbb{R}, \quad \int_{-\infty}^{t} f(s)ds = +\infty, \quad (16)
\]
\[
\exists T^- \in \mathbb{R}: \forall t \leq T^-, \quad g(t) \geq r^- > 0, \quad (17)
\]
\[
\exists \mu_0(>0): \forall \mu(-\mu_0 < \mu \leq 0), \forall t \in \mathbb{R}, \quad \lim_{s \to -\infty} \inf \frac{e^{\mu(2m-1)F(s)}}{\left(2n-1\right) \int_{s}^{t} g(r)e^{(2n-1)\mu(2m-1)F(r)}dr} \geq m_\mu > 0, \quad (18)
\]
\[
\forall \mu(0 < \mu < \mu_0), \forall t \in \mathbb{R}, \quad 0 < m_\mu \leq \epsilon_\mu(t) = \frac{e^{\mu(2m-1)F(t)}}{\left(2n-1\right) \int_{-\infty}^{t} g(r)e^{(2n-1)\mu(2m-1)F(r)}dr} \leq M_\mu. \quad (19)
\]

Here \( F \) is an antiderivative of \( f \). Then we have the following facts:
- When \( -\mu_0 < \mu \leq 0 \), the zero solution to \( (3) \) is locally pullback attracting in \( \mathbb{R} \).
- When \( \mu = 0 \), the zero solution to \( (3) \) is asymptotically unstable but locally pullback attracting in \( \mathbb{R}^+ \).
- When \( 0 < \mu < \mu_0 \), the zero solution to \( (3) \) is asymptotically unstable and the orbit \( x_\mu(t) \) is locally pullback attracting in \( \mathbb{R}^+ \).
- Furthermore
\[
\forall t \in \mathbb{R}, \quad x_\mu(t) \to 0 \quad (\mu \to 0).
\]

2) Let assume that \( f(t) \) and \( g(t) \) satisfy
\[
\exists T^+: \forall t \geq T^+, g(t) \geq r^+ > 0, \quad \int_{t}^{+\infty} f(s)ds = +\infty. \quad (20)
\]
Then there exists a \( \mu_0 > 0 \) such that the zero solution to (10) is forwards attracting for \(-\mu_0 < \mu \leq 0\) and the orbit \( x_\mu(t) \) is forwards attracting for \( 0 < \mu < \mu_0 \). Furthermore the additional condition

\[
\forall \mu < 0, \forall t \in \mathbb{R}, 0 < m_\mu \leq x_\mu(t) e^{\mu(2m-1)F(t)} \leq M_\mu \quad (21)
\]

is satisfied, then the orbit \( x_\mu(t) \) is asymptotically instable and pullback repelling when \(-\mu_0 < \mu \leq 0\). And we have

\[
\forall t \in \mathbb{R}, x_\mu(t) \to 0 \quad (\mu \to 0).
\]

The proof is omitted.

Now we consider general equations \( \dot{x} = G(t, x, \mu) \). Assume that \( G \) is sufficiently smooth. Then we obtain the following Taylor expansion of \( G \) at \( (t, 0, 0) \) as the above.

\[
G(t, x, \mu) = G(t, 0, 0) + G_x(t, 0, 0)x + G_\mu(t, 0, 0)\mu + \frac{1}{2} G_{xx}(t, 0, 0)x^2 \\
+ G_{x\mu}(t, 0, 0)x\mu + \frac{1}{2} G_{\mu\mu}(t, 0, 0)\mu^2 + \frac{1}{6} G_{xxx}(t, 0, 0)x^3 + \frac{1}{2} G_{x\mu\mu}(t, 0, 0)x^2\mu \\
+ \frac{1}{2} G_{x\mu\mu}(t, 0, 0)x\mu^2 + \frac{1}{6} G_{\mu\mu\mu}(t, 0, 0)\mu^3 + \cdots + \frac{1}{(2m-1)!} G_{x^{2m-1}\mu^{m-1}}(t, 0, 0)x^{2m-2}\mu + \cdots \\
+ C_{2m-1}^{m-1} \frac{\partial^{2m-1}}{\partial x^{2m-1}\partial \mu} G(t, 0, 0)x^{2m-2} + C_{2m-2}^{m-1} \frac{\partial^{2m-1}}{\partial x^{2m-1}\partial \mu^2} G(t, 0, 0)x^{2m-3}\mu + \cdots \\
+ C_{2m-1}^{m} \frac{\partial^{2m}}{\partial x^{2m}\partial \mu} G(t, 0, 0)x^{2m-1} + C_{2m}^{m} \frac{\partial^{2m}}{\partial x^{2m}\partial \mu^2} G(t, 0, 0)x^{2m-1}\mu + \cdots \\
+ \frac{1}{(2m)!} \frac{\partial^{2m}}{\partial x^{2m}} G(t, 0, 0)x^{2m} + \frac{1}{(2m+1)!} \frac{\partial^{2m+1}}{\partial x^{2m+1}} G(t, 0, 0)x^{2m+1} \mu + \cdots \\
+ \frac{1}{(2m+1)!} \frac{\partial^{2m+1}}{\partial x^{2m+1}\partial \mu} G(t, 0, 0)x^{2m+1} + \frac{1}{(2m+2)!} \frac{\partial^{2m+2}}{\partial x^{2m+2}\partial \mu^2} G(t, 0, 0)x^{2m+2} \mu + \cdots \\
+ \frac{1}{(2m+2)!} \frac{\partial^{2m+2}}{\partial x^{2m+2}\partial \mu^2} G(t, 0, 0)x^{2m+2} \mu^2 + \cdots.
\]

Here \( m \in \mathbb{N} \).

Now assume that \( G \) satisfies the following conditions:

(i) \( G(t, 0, \mu) = 0, \forall t, \mu \in \mathbb{R} \),

(ii) \( G_x(t, 0, 0) = 0 \),

(iii) \( \frac{\partial^2}{\partial x \partial \mu} G(t, 0, 0) = \frac{\partial^3}{\partial x \partial \mu^2} G(t, 0, 0) = \cdots = \frac{\partial^{2m-1}}{\partial x \partial \mu^{2m-2}} G(t, 0, 0) = 0 \).

From (i) and (ii) we have \( \frac{\partial^k}{\partial \mu^k} G(t, 0, 0) = 0, \forall t \in \mathbb{R}, \forall k \in \mathbb{Z}_+ \) and thus \( G \) is provided
more, when \( \mu < 0 \). Then there occurs local transcritical bifurcation when
\( \mu = 0 \).

Let denote
\( f \) can be rewritten as follows:
\[
G(t, x, \mu) = \mu^{2m-1} \left[ \frac{1}{(2m)!} C_{2m}^{2m-1} \frac{\partial^{2m}}{\partial x \partial \mu^{2m-1}} G(t, 0, 0) + \frac{1}{(2m + 1)!} \right] x + \left[ \frac{1}{2} G_{xx}(t, 0, 0) + \frac{1}{6} G_{xxx}(t, 0, 0) \right] x^3 + \cdots
\]
\[
+ \frac{1}{2} G_{x\mu}(t, 0, 0) \mu + \cdots + \frac{1}{(2m - 1)!} \frac{\partial^{2m-1}}{\partial x^{2m-1} \partial \mu} G(t, 0, 0) x^{2m-3} + \cdots
\]
\[
+ \frac{1}{(2m)!} C_{2m}^{1} \frac{\partial^{2m}}{\partial x^{2m-1} \partial \mu} G(t, 0, 0) x^{2m-2} + \cdots
\]
\[
\lim_{t \to \pm \infty} \inf_{t \in [-\epsilon, \epsilon]} g(t) > 0,
\]
\[
0 < m = \lim_{t \to \pm \infty} \inf_{t \in [-\epsilon, \epsilon]} \frac{f(t)}{g(t)} \leq \lim_{t \to \pm \infty} \sup_{t \in [-\epsilon, \epsilon]} \frac{f(t)}{g(t)} = M < +\infty,
\]
and there exists a positive valued function \( h(t) \) such that
\[
|\phi(t, \mu)| \leq h(t), \quad |r_{\mu}(t, x, \mu)| \leq h(t), \quad |r_{x}(t, x, \mu)| \leq h(t),
\]
\[
\lim_{t \to \pm \infty} \sup_{t \in [-\epsilon, \epsilon]} \frac{h(t)}{g(t)} \leq k.
\]
Then there occurs local transcritical bifurcation when \( \mu \) passes through 0. Furthermore, when \( \mu < 0 \), a complete orbit \( x_{\mu}(t) \) is pullback repelling in \( (-\epsilon, 0) \); when \( \mu = 0 \), the zero solution is locally forwards attracting in \( \mathbb{R}^+ \) and when \( \mu > 0 \), pullback attracting orbit \( x_{\mu}(t) \) is forwards attracting in \( (0, \epsilon) \).

The proof is omitted.

Now assume that \( G \) satisfies the following conditions:

(iv) \( G(t, x, \mu) = c(\mu) \cdot G(t, x, 0) + x \cdot \frac{\partial}{\partial x} G(0, 0, \mu), \)

(v) \( G(t, 0, 0) = 0, \)

(vi) \( \frac{\partial}{\partial x} G(t, 0, 0) = \frac{\partial^2}{\partial x^2} G(t, 0) = \cdots = \frac{\partial^{2n-1}}{\partial x^{2n-1}} G(t, 0, 0) = 0, \)

(vii) \( \frac{\partial}{\partial x \partial \mu} G(t, 0, 0) = \frac{\partial^3}{\partial x \partial \mu^2} G(t, 0, 0) = \cdots = \frac{\partial^{2m-1}}{\partial x \partial \mu^{2m-2}} G(t, 0, 0) = 0. \)
Here \( n, m \in \mathbb{N} \). Then \( G \) is provided as follows:

\[
G(t, x, \mu) = \mu^{2m-1} \left[ \frac{1}{(2m)!} C_{2m}^{2m-1} \frac{\partial^{2m}}{\partial x \partial \mu^{2m}} G(t, 0, 0) + \frac{1}{(2m + 1)!} \right] x^{2m} \frac{\partial^{2m+1}}{\partial x \partial \mu^{2m+1}} G(t, 0, 0) \\
+ \cdots + \frac{1}{(2m - 2)!} C_{2m-2}^{1} x^{-(2n-2m+3)} \mu^{2m-2} \frac{\partial^{2m-2}}{\partial x \partial \mu^{2m-2}} G(t, 0, 0) \\
+ \cdots + \frac{1}{(2n)!} \frac{\partial^{2n}}{\partial x^{2n}} G(t, 0, 0) + \frac{1}{(2n)!} C_{2n}^{1} x^{-(2n-2m-1)} \mu^{2n} \frac{\partial^{2n}}{\partial x^{2n} \partial \mu^{2n}} G(t, 0, 0) \\
+ \frac{1}{(2n)!} C_{2n}^{2} x^{2n-2} \mu^{2} \frac{\partial^{2n}}{\partial x^{2n-2} \partial \mu^{2}} G(t, 0, 0) + \cdots \right] x^{2n}, \quad m, n \in \mathbb{N}.
\]

Let denote

\[
f(t) := \frac{1}{(2m)!} C_{2m}^{2m-1} \frac{\partial^{2m}}{\partial x \partial \mu^{2m}} G(t, 0, 0),
\]

\[
g(t) := -\frac{1}{(2n)!} \frac{\partial^{2n}}{\partial x^{2n}} G(t, 0, 0).
\]

Then we can rewritten \( 9 \) as follows:

\[
\dot{x} = \mu^{2m-1} [f(t) + \mu \phi(t, \mu)] x - [g(t) + r(t, x, \mu)] x^{2n}.
\]

Here \( \phi(t, \mu) = \frac{1}{(2m+1)!} C_{2m+1}^{2m+1} \frac{\partial^{2m+1}}{\partial x \partial \mu^{2m+1}} G(t, 0, 0) \).

**Theorem 6.** Assume that

\[
r(t, 0, 0) = 0, \quad \lim_{t \to \pm \infty} \inf g(t) > 0,
\]

\[
0 < m = \lim_{t \to \pm \infty} \inf \frac{f(t)}{g(t)} \leq \lim_{t \to \pm \infty} \sup \frac{f(t)}{g(t)} = M < +\infty,
\]

\[
|\phi(t, \mu)| \leq h(t), \quad |r(t, x, \mu)| \leq h(t) \left[ |x|^{-(2n-2)} + |\mu| \right],
\]

\[
\lim_{t \to \pm \infty} \sup \frac{h(t)}{g(t)} \leq k.
\]

Then we have the same conclusions with the Theorem [3]

The proof is omitted.

**Example 2.** In the equation \( \dot{x} = \mu^{3} t^{2} x - 2t^{2} x^{6} \), transcritical bifurcation occurs when \( \mu = 0 \).

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