A novel expansion — which generalizes Magnus expansion — of the evolution operator associated with a (in general, time-dependent) perturbed Hamiltonian is introduced. It is shown that it has a wide range of possible solutions that can be fitted according to computational convenience. The time-independent and the adiabatic case are studied in detail.

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1 Introduction

The explicit determination of the evolution operator associated with a quantum system is a 'touchy business'. If the Hamiltonian of the system does not depend on time and has the form of the sum of a solvable unperturbed Hamiltonian plus an analytic perturbation, one can use the tools of standard perturbation theory for linear operators, based on the expansion of the resolvent, in order to achieve approximate expressions of the evolutor, or apply a suitable unitary operator perturbative approach.

If the Hamiltonian is time-dependent (i.e. it describes a non-isolated quantum system), the problem is in general even more radical. In fact, it is well known that, whenever the values of the Hamiltonian at different times do not commute, the evolutor does not admit a simple formal expression.

In two fundamental papers in the history of quantum electrodynamics, Dyson developed an expansion of the evolution operator that has been adopted extensively in any field of physics. Dyson expansion has a transparent physical interpretation in terms of time ordered elementary processes which makes its application particularly appealing, especially in quantum field theory. On the other hand, for many applications, Dyson expansion has severe drawbacks, as a low convergence rate and the lack of unitarity of its truncations.

Later, Magnus introduced an expansion of the evolution operator such that each of its truncations retains the property of being unitary. Magnus expansion has been 'rediscovered' and re-elaborated several times (see for instance ref.), applied successfully to several problems — nuclear magnetic resonance, atomic collision theory, molecular systems in intense laser fields, neutrino oscillations in matter, to quote just a small sample — and its convergence properties have been studied.

On our opinion, Magnus expansion, rather than Dyson expansion, should be regarded as the most natural generalization of the expression of the evolutor associated with a time-independent Hamiltonian. In fact, it is written in the form of the exponential of the expansion of a suitable time-dependent anti-hermitian operator which can be deduced, order by order, from the Hamiltonian of the system. Now, precisely for this reason, just like for the evolutor generated by a time-independent Hamiltonian, the problem of computing explicitly the action of (any truncation of) the Magnus expansion on the state vectors is non-trivial. Expanding the exponential would lead to non-unitary truncations, thus to the loss of the most important feature of Magnus expansion. Then, the issue of finding a generalization of Magnus expansion retaining the property of having unitary truncations, but allowing simpler explicit solutions, arises in a natural way.

In the present paper, we have tried to achieve this result. Our basic idea is simple: to combine the Magnus expansion with the passage to a suitable interaction picture. Precisely, given a perturbed Hamiltonian, after the usual passage to the interaction picture that decouples the unperturbed dynamics (which is supposed to be known explicitly), one switches to a further interaction picture, depending on the perturbative parameter, in order to achieve computational
advantages. We stress that the idea of ‘adapting’ Magnus expansion is not new. It appears in a paper by Casas et al. [15] in which the authors introduce the Floquet-Magnus expansion for the evolution operator associated with a (interaction picture) Hamiltonian depending periodically on time. Our approach generalizes the one proposed by Casas et al. even in the case when the interaction picture Hamiltonian is periodic on time.

We have made the choice of skipping mathematical complications. For instance, it is known that even a simple passage to an interaction picture can be mathematically tricky (see, for example, ref. [16]). Our choice is motivated by various reasons. First of all, we believe that heuristic investigation should always precede rigorous re-elaboration. Once that it is clear what the basic ‘rules of the game’ are, one can adopt the most appropriate mathematical tools. Moreover, we avoid the risk of hiding in a cloud of technicalities the main ideas and of discouraging those physicists who may want to apply our method for solving problems. It should be also observed that a recent trend in quantum mechanics is to focus on systems which can be described by effective Hamiltonians in finite-dimensional Hilbert spaces (consider, in particular, the huge research area related to quantum computation and quantum information theory; see ref. [17] and the rich bibliography therein). The study of these systems is not affected by all the technicalities associated with the infinite-dimensional spaces but retains all the most intriguing features of quantum physics.

The paper is organized as follows. In section 2 we introduce the basic decomposition of the evolution operator which will allow us to obtain a perturbative expansion. Two important cases — the time-independent case and the adiabatic case — will be considered in sections 3 and 4. In section 5 we will study the general time-dependent case and show how the solutions obtained correspond, in the time-independent case, to the ones obtained in section 3.

2 Basic assumptions and strategy

Let us consider a time-dependent perturbed Hamiltonian \( H(\lambda; t) \), namely a selfadjoint linear operator of the form

\[
H(\lambda; t) = H_0(t) + H_\delta(\lambda; t),
\]

where \( H_0(t) \) is a selfadjoint (and, in general, time-dependent) operator — the ‘unperturbed component’ — and \( H_\delta(\lambda; t) \) is a time-dependent perturbation; precisely, we will assume that \( \lambda \mapsto H_\delta(\lambda; t) \) is (for the perturbative parameter \( \lambda \) in a certain neighborhood of zero and for any \( t \)) a real analytic, selfadjoint, bounded operator-valued function, with \( H_\delta(0; t) = 0 \). A real analytic function can be extended to a domain in the complex plane. Keeping this fact in mind, we will specify that a given property holds for \( \lambda \) real. For instance, the analytic function \( \lambda \mapsto H_\delta(\lambda; t) \) will take values in the selfadjoint operators for \( \lambda \) real only.

Let \( U(\lambda; t, t_0) \) be the evolution operator associated with \( H(\lambda; t) \), with initial
time $t_0$; namely ($\hbar = 1$):

$$i \dot{U}(\lambda; t, t_0) = H(\lambda; t) U(\lambda; t, t_0), \quad U(\lambda; t, t_0) = \text{Id},$$

(2)

where the dot denotes the time derivative. Then, we have that

$$U(\lambda; t, t_0) = U_0(t, t_0) T(\lambda; t, t_0),$$

(3)

where $U_0(t, t_0)$ and $T(\lambda; t, t_0)$ are respectively the evolution operator associated with the unperturbed component $H_0(t)$ (evolution operator which, if the unperturbed Hamiltonian is time-independent, $H_0(t) \equiv H_0$, is obviously given by $e^{-iH_0(t-t_0)}$) and the evolution operator associated with the interaction picture Hamiltonian

$$\tilde{H}(\lambda; t, t_0) := U_0(t_0, t) H(\lambda; t) U_0(t, t_0).$$

(4)

Let us notice explicitly that, since $\tilde{H}(0; t, t_0) = 0$, we have:

$$T(0; t, t_0) = \text{Id}.\quad (5)$$

We will suppose that the unperturbed evolution $U_0(t, t_0)$ is explicitly known. Then the problem is to determine perturbative expressions of $T(\lambda; t, t_0)$. To this aim, the central point of the paper is the assumption that $T(\lambda; t, t_0)$ has the following general form:

$$T(\lambda; t, t_0) = \exp \left( -i Z(\lambda; t, t_0) \right) \exp \left( -i \int_{t_0}^{t} C(\lambda; t, t_0) \, dt \right) \exp \left( i Z(\lambda; t_0, t_0) \right),$$

(6)

where $(\lambda; t) \mapsto Z(\lambda; t, t_0)$, $(\lambda; t) \mapsto C(\lambda; t, t_0)$ are operator-valued functions which depend analytically on the perturbative parameter $\lambda$; in agreement with condition (5), we set:

$$Z(0; t, t_0) = 0, \quad C(0; t, t_0) = 0, \quad \forall t.\quad (7)$$

We stress that the presence of the term $\exp(i Z(\lambda; t_0, t_0))$ in formula (6) ensures that $T(\lambda; t_0, t_0) = \text{Id}$, allowing the possibility that $Z(\lambda; t_0, t_0) \neq 0$.

It will be seen that decomposition (6) has a wide range of solutions and that a possible choice for fixing a certain class of solutions is given by imposing the condition $C(\lambda; t, t_0) = C(\lambda)$, i.e. assuming that the function $(\lambda; t) \mapsto C(\lambda; t, t_0)$ does not depend on time. This decomposition includes, as particular cases, two decompositions of the evolution operator that have been considered in the literature:

- the decomposition that is obtained setting
  $$Z(\lambda; t, t_0) = 0, \quad \forall t,$$

  in formula (6), decomposition which is at the root of the Magnus expansion of the evolution operator [3];
• the classical Floquet decomposition that holds in the case where the interaction picture Hamiltonian depends periodically on time (let us denote the period by \( T \)) — decomposition which is obtained setting

\[
C(\lambda; t, t_0) = C(\lambda), \quad Z(\lambda; t_0, t_0) = 0,
\]

and assuming that \((\lambda, t) \mapsto Z(\lambda; t, t_0)\) is periodic with respect to time with period \( T \) — and that is at the root of the Floquet-Magnus expansion of the evolution operator \([15]\).

From this point onwards, for notational convenience, we will fix \( t_0 = 0 \). Then, decomposition \([9]\) can be rewritten as

\[
T(\lambda; t) = \exp (-iZ(\lambda; t)) \exp \left( -i \int_0^t C(\lambda; t) \, dt \right) \exp (iZ(\lambda)), \quad (8)
\]

where we have set:

\[
T(\lambda; t) = T(\lambda; t_0), \quad Z(\lambda; t) = Z(\lambda; t_0), \quad Z(\lambda) = Z(\lambda; 0), \quad C(\lambda; t) = C(\lambda; t_0).
\]

We are now ready to obtain a perturbative expansion of \( T(\lambda; t) \). In fact, if we require the interaction picture evolution operator to satisfy the Schrödinger equation, we get:

\[
\hat{H}(\lambda; t) T(\lambda; t) = i \hat{T}(\lambda; t)
\]

\[
= e^{-iZ(\lambda; t)} \int_0^t \left( e^{isZ(\lambda; t)} \hat{Z}(\lambda; t) e^{-isZ(\lambda; t)} \right) ds \ e^{-i\int_0^t C(\lambda; t) \, dt} e^{iZ(\lambda)} +
\]

\[
+ e^{-iZ(\lambda; t)} \int_0^t \left( e^{-is\int_0^t C(\lambda; t) \, dt} C(\lambda; t) e^{is\int_0^t C(\lambda; t) \, dt} \right) ds \ e^{-i\int_0^t C(\lambda; t) \, dt} e^{iZ(\lambda)}, \quad (9)
\]

where we have used the remarkable formula (see, for instance, ref. \([15]\))

\[
\frac{d}{dt} e^F = e^F \int_0^1 \left( e^{-sF} \hat{F} e^{sF} \right) ds = \int_0^1 \left( e^{sF} \hat{F} e^{-sF} \right) ds \ e^F, \quad (10)
\]

which extends to an operator-valued function \( t \mapsto F(t) \) the standard formula for the derivative of the exponential of an ordinary function. Next, let us apply to each member of equation \([9]\) the operator \( e^{iZ(\lambda; t)} \) on the left and the operator \( e^{-iZ(\lambda)} e^{tC(\lambda)} \) on the right; we find:

\[
\text{Ad}_{\exp(iZ(\lambda; t))} \hat{H}(\lambda; t) = \int_0^1 \left( \text{Ad}_{\exp(isZ(\lambda; t))} \hat{Z}(\lambda; t) \right) ds
\]

\[
+ \int_0^1 \left( \text{Ad}_{\exp(-is\int_0^t C(\lambda; t) \, dt)} C(\lambda; t) \right) ds, \quad (11)
\]

where we recall that, given linear operators \( X, Y \), with \( X \) invertible,

\[
\text{Ad}_X Y := X Y X^{-1}. \quad (12)
\]
Then, since $X$ is of the form $e^X$, we can use the well known relation

$$\text{Ad}_{\exp(X)} Y = \exp(\text{ad}_X) Y = \sum_{k=0}^{\infty} \frac{1}{k!} \text{ad}_X^k Y, \quad (13)$$

with $\text{ad}_X^k$ denoting the $k$-th power ($\text{ad}_X^0 \equiv \text{Id}$) of the superoperator $\text{ad}_X$ defined by

$$\text{ad}_X Y := [X, Y]. \quad (14)$$

Eventually, applying formula (13) to equation (11) and performing the integrations, we obtain:

$$\sum_{k=0}^{\infty} \frac{i^k}{k!} \text{ad}^k_{Z(\lambda; t)} \dot{H}(\lambda; t) = \sum_{k=0}^{\infty} \frac{i^k}{(k+1)!} \text{ad}^k_{Z(\lambda; t)} \ddot{Z}(\lambda; t)$$

$$+ \sum_{k=0}^{\infty} \frac{(-i)^k}{(k+1)!} \text{ad}^k_{\int_0^t C(\lambda; t) dt} C(\lambda; t). \quad (15)$$

This equation will be the starting point for the determination of the operator-valued functions $(\lambda, t) \mapsto Z(\lambda; t)$ and $(\lambda, t) \mapsto C(\lambda; t)$ at each perturbative order in $\lambda$, task that will be pursued systematically in the next sections.

3 The time-independent case and the perturbative adiabatic approximation

In this section, we will consider two important cases:

1. the case where the Hamiltonian (11) does not depend on time;
2. the case where the Hamiltonian (11) is slowly varying with respect to time (so that the adiabatic approximation can be applied).

As we will show later on, the second case, within the adiabatic approximation, can be treated by a method analogous to the one adopted in the first case. In both cases it will be convenient to set

$$Z(\lambda; t) := U_0(t) Z(\lambda; t) U_0(t)^\dagger, \quad Z(\lambda) \equiv Z(\lambda; 0) = Z(\lambda), \quad (16)$$

and re-express equation (15) in terms of the transformed operator $Z(\lambda; t)$. To this aim, let us first notice that

$$\dot{Z}(\lambda; t) = \text{Ad}_{U_0(t)^\dagger} \left( \dot{Z}(\lambda; t) - i \text{ad}_{Z(\lambda; t)} H_0 \right). \quad (17)$$

Besides, given linear operators $X$, $X$ and $Y$, with $X$ invertible, one can show inductively that

$$\text{ad}_{\text{Ad}_X}^k \text{Ad}_X Y = \text{Ad}_X \text{ad}_X^k Y, \quad k = 0, 1, 2, \ldots . \quad (18)$$
Then, since $Z(\lambda; t) = \text{Ad}_{U_0(t)} Z(\lambda; t)$, $\dot{H}(\lambda; t) = \text{Ad}_{U_0(t)} H_0(\lambda; t)$ and relation (17) holds, using formula (18), from equation (15) we obtain:

$$\text{Ad}_{U_0(t)} \sum_{k=0}^{\infty} \frac{i^k}{k!} \text{ad}_{Z(\lambda; t)} H_0(\lambda; t) = \text{Ad}_{U_0(t)} \sum_{k=0}^{\infty} \frac{i^k}{(k+1)!} \text{ad}_{Z(\lambda; t)} \dot{Z}(\lambda; t)$$

$$- \sum_{k=0}^{\infty} \frac{(-i)^k}{(k+1)!} \text{ad}_{Z(\lambda; t)} H_0$$

$$+ \sum_{k=0}^{\infty} \frac{i^k}{k!} \text{ad}_{Z(\lambda; t)} H_0.$$  \hspace{2cm} (19)

Next, applying the superoperator $\text{Ad}_{U_0(t)}$ to each member of this equation and rearranging the terms, we get

$$\sum_{k=1}^{\infty} \frac{i^k}{k!} \text{ad}_{Z(\lambda; t)} (H_0(t) + H_0(\lambda; t)) + H_0(\lambda; t) = \text{Ad}_{U_0(t)} \left(C(\lambda; t) \right)$$

$$+ \sum_{k=1}^{\infty} \frac{(-i)^k}{(k+1)!} \text{ad}_{Z(\lambda; t)} C(\lambda; t)$$

$$+ \sum_{k=0}^{\infty} \frac{i^k}{(k+1)!} \text{ad}_{Z(\lambda; t)} \dot{Z}(\lambda; t).$$  \hspace{2cm} (20)

### 3.1 The time-independent case

In the time-independent case, we have that $H_0(t) \equiv H_0$, $H_0(\lambda; t) \equiv H_0(\lambda)$, and it is natural to set:

$$C(\lambda; t) = C(\lambda), \quad Z(\lambda; t) = Z(\lambda; 0) = Z(\lambda) \equiv Z(\lambda).$$  \hspace{2cm} (21)

Then, equation (20) assumes a much simpler form:

$$\sum_{k=1}^{\infty} \frac{i^k}{k!} \text{ad}_{Z(\lambda)} (H_0 + H_0(\lambda)) + H_0(\lambda) = e^{-iH_0 t} C(\lambda) e^{iH_0 t}.$$  \hspace{2cm} (22)

Now, observe that the first member of this equation does not depend on time, hence the function $t \mapsto e^{-iH_0 t} C(\lambda) e^{iH_0 t}$ must be constant. It follows that, if we want equation (22) to be consistent, we have to assume that

$$[C(\lambda), H_0] = 0,$$  \hspace{2cm} (23)

i.e. that $C(\lambda)$ is a constant of the motion for the unperturbed evolution generated by $H_0$. Eventually, we obtain the following fundamental formula:

$$\sum_{k=1}^{\infty} \frac{i^k}{k!} \text{ad}_{Z(\lambda)} (H_0 + H_0(\lambda)) + H_0(\lambda) = C(\lambda).$$  \hspace{2cm} (24)
At this point, we are ready to obtain perturbative expansions of the operators \( C(\lambda) \) and \( Z(\lambda) \) (hence, of the interaction picture evolution operator \( T(\lambda; t) \)). We will assume that the unperturbed Hamiltonian \( H_0 \) has a pure point spectrum, while the case where this hypothesis is not satisfied is a particular case of the general treatment developed in section 5 (indeed, in this case the formulae obtained in this section, which involve eigenvalues and eigenprojectors, make no sense). We will denote by \( E_1, E_2, \ldots \) the (possibly degenerate) eigenvalues of \( H_0 \) and by \( P_1, P_2, \ldots \) the associated eigenprojectors. Since the functions \( \lambda \mapsto H_\lambda \), \( \lambda \mapsto C(\lambda) \) and \( \lambda \mapsto Z(\lambda) \) are analytic and \( H_\lambda(0) = C(0) = Z(0) = 0 \), we can write:

\[
H_\lambda(\lambda) = \sum_{n=1}^{\infty} \lambda^n H_n, \quad C(\lambda) = \sum_{n=1}^{\infty} \lambda^n C_n, \quad Z(\lambda) = \sum_{n=1}^{\infty} \lambda^n Z_n. \tag{25}
\]

Now, in order to determine the operators \( \{C_n\}_{n \in \mathbb{N}} \) and \( \{Z_n\}_{n \in \mathbb{N}} \), let us substitute the power expansions (25) in equation (24); in correspondence to the various orders in the perturbative parameter \( \lambda \), we get the following set of conditions:

\[
C_1 - i [Z_1, H_0] - H_1 = 0, \quad [C_1, H_0] = 0 \tag{26}
\]

\[
C_2 - i [Z_2, H_0] + \frac{1}{2} [Z_1, [Z_1, H_0]] - i [Z_1, H_1] - H_2 = 0, \quad [C_2, H_0] = 0 \tag{27}
\]

\[\vdots\]

where we have taken into account the additional constraint \([C(\lambda), H_0] = 0\). This infinite set of equations can be solved recursively and the solution — as it should be expected (we will clarify this point soon) — is not unique. The first equation, together with the first constraint, determines \( Z_1 \) up to an operator commuting with \( H_0 \) and \( C_1 \) uniquely. Indeed, since

\[
[C_1, H_0] = 0 \implies C_1 = \sum_m P_m C_1 P_m \tag{28}
\]

and

\[
[Z_1, H_0] = \sum_{j \neq l} (E_l - E_j) P_j Z_1 P_l, \tag{29}
\]

we conclude that

\[
C_1 = \sum_m P_m H_1 P_m \tag{30}
\]

and

\[
Z_1 = \sum_m P_m Z_1 P_m + i \sum_{j \neq l} (E_l - E_j)^{-1} P_j H_1 P_l. \tag{31}
\]

This last equation admits a minimal solution which is obtained by imposing a further condition, namely

\[
P_m Z_1 P_m = 0 \quad m = 1, 2, \ldots .
\]
For $n > 1$, we will adopt an analogous reasoning. Indeed, given an operator $X$, let us set

$$
G_n(X; Z_1, \ldots, Z_n) := \sum_{m=1}^{n} \frac{j^m}{m!} \sum_{k_1 + \cdots + k_m = n} \text{ad}_{Z_{k_1}} \cdots \text{ad}_{Z_{k_m}} X,
$$

with $n \geq 1$. Then we can define the operator function

$$
G_n(H_0, \ldots, H_n; Z_1, \ldots, Z_{n-1}) := \sum_{m=0}^{n-1} G_{n-m}(H_m; Z_1, \ldots, Z_{n-m}) - i[Z_n, H_0] + H_n \quad n \geq 2.
$$

At this point, one can show that the sequence of equations generated by formula (24) is given by

$$
C_1 - i[Z_1, H_0] = H_1, \quad [C_1, H_0] = 0
$$

$$
\vdots
$$

$$
C_n - i[Z_n, H_0] = G_n(H_0, \ldots, H_n; Z_1, \ldots, Z_{n-1}), \quad [C_n, H_0] = 0 \quad n \geq 2
$$

In order to write the general solution of this sequence of equations, it will be convenient to introduce a shorthand notation: given a linear operator $X$, we set:

$$
\langle| X \rangle_{H_0} := \sum_m P_m X P_m := \langle| X \rangle_{\{P_m\}},
$$

$$
|X\rangle_{H_0} := X - \langle| X \rangle_{H_0} = \sum_{j \neq l} P_j X P_l := |X\rangle_{\{P_m\}},
$$

$$
||X||_{H_0} := i \sum_{j \neq l} (E_l - E_j)^{-1} P_j X P_l.
$$

Notice that for the superoperators $\langle | \cdot | \rangle_{H_0}$ and $| \cdot \rangle \langle | \cdot |_{H_0}$, which differ from the superoperator $| \cdot \rangle \langle |$ (which do not depend on the eigenvalues of $H_0$, we have introduced the respective alternative symbols $\langle | \cdot \rangle_{\{P_m\}}$ and $| \cdot \rangle \langle |_{\{P_m\}}$ that will be used in the following whenever a certain set of spectral projections $\{P_m\}$ (in general, not eigenprojections) of a selfadjoint operator are involved.

Now, assume that the first $n$ equations have been solved. Then, the operator $G_{n+1}(H_0, \ldots, H_{n+1}; Z_1, \ldots, Z_n)$ is known explicitly and hence

$$
C_{n+1} = \langle| G_{n+1}(H_0, \ldots, H_{n+1}; Z_1, \ldots, Z_n) \rangle_{H_0},
$$

$$
[Z_{n+1}, H_0] = i \langle| G_{n+1}(H_0, \ldots, H_{n+1}; Z_1, \ldots, Z_n) \rangle_{H_0}.
$$

Again, this last equation determines $Z_{n+1}$ up to an arbitrary operator $\langle| Z_{n+1} \rangle_{H_0}$ commuting with $H_0$; in fact, we have:

$$
Z_{n+1} = \langle| Z_{n+1} \rangle_{H_0} + \langle| G_{n+1}(H_0, \ldots, H_{n+1}; Z_1, \ldots, Z_n) \rangle_{H_0}.
$$
We stress that, in general, the choice of a particular solution for $Z_n$ will also influence the form of $C_{n+1}, Z_{n+2}, \ldots$.

Thus, we conclude that the sequence of equations defined above admits infinite solutions (even in the case where $H_0$ has a nondegenerate spectrum). However, there is a unique minimal solution $\{c_n, z_n\}_{n \in \mathbb{N}}$ which fulfills the following additional condition:

$$\langle |c_n\rangle |_{H_0} = 0, \quad n = 1, 2, \ldots.$$

(41)

In order to clarify the link of our approach with standard perturbation theory for linear operators, let us recall a few facts (see [2] [19]). It is possible to show that, under certain technical conditions, there exist positive constants $r_1, r_2, \ldots$ and a simply connected neighborhood $I$ of zero in $\mathbb{C}$ such that, for any $\lambda \in I$ and $m = 1, 2, \ldots$, one has that:

1) the following contour integral on the complex plane

$$P_m(\lambda) = \frac{1}{2\pi i} \oint_{\Gamma_m} dz \ (z - H(\lambda))^{-1}$$

(42)

— where $\Gamma_m$ is the anticlockwise oriented circle $[0, 2\pi] \ni \theta \mapsto E_m + r_m e^{i\theta}$ around the eigenvalue $E_m$ — defines a projection ($P_m(\lambda)^2 = P_m(\lambda)$), which is an orthogonal projection for $\lambda \in I \cap \mathbb{R}$, with $P_m(0) = P_m$, and $I \ni \lambda \mapsto P_m(\lambda)$ is an analytic operator-valued function;

2) the range of the projection $P_m(\lambda)$ is an invariant subspace for $H(\lambda)$ (but, if the range of $P_m$ is not 1-dimensional, in general not an eigenspace), hence

$$H(\lambda) P_m(\lambda) = P_m(\lambda) H(\lambda) P_m(\lambda);$$

(43)

3) there exists a (non-unique) analytic family $\lambda \mapsto W(\lambda)$ of invertible operators such that

$$P_m = W(\lambda)^{-1} P_m(\lambda) W(\lambda), \quad W(0) = \text{Id}$$

(44)

— with $W(\lambda)$ unitary for $\lambda$ real — which is solution of a Cauchy problem of the type

$$i W'(\lambda) = J(\lambda) W(\lambda), \quad W(0) = \text{Id},$$

(45)

where the apex denotes the derivative with respect to the perturbative parameter and $\lambda \mapsto J(\lambda)$ is any analytic family of operators — selfadjoint for $\lambda$ real — such that

$$|J(\lambda)\rangle |\{P_m(\lambda)\} = i \sum_m P_m'(\lambda) P_m(\lambda) \ast P_m(\lambda) P_m'(\lambda) P_m(\lambda) = 0 \ast.$$  

(46)
In standard (Rayleigh-Schrödinger-Kato) perturbation theory, one can obtain the perturbative corrections to unperturbed eigenvalues and eigenvectors exploiting (see, for instance, ref. [20]) formula (13) and the expansion of the resolvent operator \((z - H(\lambda))^{-1}\), namely
\[
(z - H_0 - H_0(\lambda))^{-1} = (z - H_0)^{-1} - \lambda (z - H_0)^{-1} H_1 (z - H_0)^{-1} - \lambda^2 (z - H_0)^{-1} H_2 (z - H_0)^{-1}
\]
\[+ (z - H_0)^{-1} H_1 (z - H_0)^{-1} H_1 (z - H_0)^{-1} + \ldots.
\]
(47)

In our approach we use, instead, properties (2) and (3). Indeed, let us define the operator \(\Pi(\lambda)\) by
\[
\Pi(\lambda) := W(\lambda)^{-1} H(\lambda) W(\lambda),
\]
(48)
which, for real \(\lambda\), is unitarily equivalent to \(H(\lambda)\). Using relations (48) and (49), we find
\[
\Pi(\lambda) P_m = W(\lambda)^{-1} H(\lambda) P_m(\lambda) W(\lambda) = W(\lambda)^{-1} P_m(\lambda) H(\lambda) P_m(\lambda) W(\lambda)
\]
(49)
and hence:
\[
\Pi(\lambda) P_m = P_m \Pi(\lambda) P_m \quad m = 1, 2, \ldots.
\]
(50)

It follows that
\[
[N \Pi(\lambda), H_0] = 0
\]
(51)
and then we obtain the following important relation:
\[
[W(\lambda)^{-1} H(\lambda) W(\lambda) - H_0, H_0] = 0.
\]
(52)

Thus, if we set
\[
W(\lambda)^{-1} H(\lambda) W(\lambda) - H_0 = C(\lambda), \quad W(\lambda) = \exp(-iZ(\lambda)),
\]
(53)
and we apply relation (48), we find precisely formula (24).

Concluding our treatment of the time-independent case, it is worth stressing that, due to conditions (21), for the overall evolution operator we have:
\[
U(\lambda;t) = e^{-iH_0 t} e^{-iZ(\lambda)t} e^{-iC(\lambda)t} e^{iZ(\lambda)}
\]
\[
* Z(\lambda;t) = e^{iH_0 t} Z(\lambda) e^{-iC(\lambda)t} e^{iZ(\lambda)}
\]
\[
* [C(\lambda), H_0] = 0 \quad e^{-iZ(\lambda)t} e^{-i(H_0 + C(\lambda))t} e^{iZ(\lambda)}
\]
(54)
or, more explicitly,
\[
U(\lambda;t) = e^{-iZ(\lambda;t)} \sum_m e^{-i(E_m + C(\lambda))t} P_m e^{iZ(\lambda)}
\]
\[= e^{-iZ(\lambda;t)} \sum_m \exp(-i(E_m + [C]_m(\lambda))t) P_m e^{iZ(\lambda)},
\]
(55)
where we have introduced the ‘reduced rank operators’
\[
[C]_m(\lambda) := C(\lambda) P_m = P_m C(\lambda) P_m, \quad m = 1, 2, \ldots.
\]
(56)
3.2 The adiabatic approximation

Let us now consider the case where the perturbative adiabatic approximation can be applied. This approximation consists essentially in partially neglecting the last term in the r.h.s. of equation (20) — the one involving the time derivative of \((\lambda, t) \mapsto Z(\lambda; t)\) — under suitable conditions.

Precisely, we will assume that the unperturbed Hamiltonian has (instantaneously) a pure point spectrum with time-independent eigenprojectors. Namely, we will suppose that there exists a set of orthogonal projectors \(\{P_m\}_{m=1,2,...}\) forming a resolution of the identity — \(\text{Id} = \sum_m P_m\) — such that it coincides with the set of eigenprojectors of \(H_0(t)\), for any \(t\).

This hypothesis prevents the possibility of occurrence of ‘level crossings’ in the spectrum of the unperturbed Hamiltonian. Indeed, it implies that there exist real functions \(t \mapsto E_1(t), t \mapsto E_2(t), \ldots\) such that, for any \(t\), \(\{E_m(t)\}_{m=1,2,...}\) is the set of the eigenvalues of \(H_0(t)\) — specifically: \(H_0(t) P_m = E_m(t) P_m\) — and hence

\[
E_1(t) \neq E_2(t) \neq \ldots \quad \forall t.
\]

We will further assume that the functions \(\{t \mapsto E_m(t)\}_{m=1,2,...}\) belong to \(C^1(\mathbb{R})\).

Moreover, notice that — since \([H_0(t), H_0(t')] = 0\), \(\forall t, t'\) — the unperturbed evolution operator will be given by:

\[
U_0(t) = e^{-i \int_0^t H_0(t') dt'} = \sum_m e^{-i \int_0^t E_m(t') dt} P_m.
\]

Then, we will consider a solution \(\{t \mapsto \tilde{C}(\lambda; t), t \mapsto \tilde{Z}(\lambda; t)\}\) of the equation

\[
\sum_{k=0}^{\infty} \frac{i^k}{k!} \text{ad}^k_{\tilde{Z}(\lambda; t)} (H_0(t) + H_\varepsilon(\lambda; t)) = H_0(t) + \tilde{C}(\lambda; t)
\]

\[
+ \sum_{k=1}^{\infty} \frac{(-i)^k}{(k+1)!} \text{ad}^k_{\tilde{C}(\lambda; t)} dt \tilde{C}(\lambda; t)
\]

\[
+ \left\langle \sum_{k=0}^{\infty} \frac{i^k}{(k+1)!} \text{ad}^k_{\tilde{Z}(\lambda; t)} \dot{\tilde{Z}}(\lambda; t) \right\rangle_{H_0(t)}.
\]

subject to the condition that

\[
[\tilde{C}(\lambda; t), P_m] = 0 \quad \forall t, \forall m.
\]

and

\[
\left\langle \sum_{k=0}^{\infty} \frac{i^k}{(k+1)!} \text{ad}^k_{\tilde{Z}(\lambda; t)} \dot{\tilde{Z}}(\lambda; t) \right\rangle_{H_0(t)} \simeq 0.
\]

We observe explicitly that such a solution may not exist. In more detail, one can show that a solution of equation (59) verifying condition (60) alone always exists (and, as in the time-independent case, it is not unique), but, in general, not a solution verifying condition (61) too; we do not insist on this point here
since it will be clarified afterwards. We will then say that the perturbative adiabatic approximation is applicable if a solution of equation (59) verifying both conditions (60) and (61) does exist.

Condition (60) is analogous to condition (23) which has been assumed in the time-independent case. Indeed, it is equivalent to the condition that

$$[\tilde{C}(\lambda; t), H_0(t')] = 0 \quad \forall t, \forall t'.$$

(62)

Integrating with respect to time relation (60), we find also that

$$\left[ \int_0^t \tilde{C}(\lambda; t) \, dt, P_m \right] = 0 \quad \forall t, \forall m.$$

(63)

Then, as a consequence of relation (60), recalling formula (58), we have that

$$\left[ \tilde{C}(\lambda; t), U_0(t') \right] = 0,$$

$$\left[ \int_0^t \tilde{C}(\lambda; t) \, dt, U_0(t') \right] = 0 \quad \forall t, t';$$

(64)

therefore:

$$\left[ U_0(t), \tilde{C}(\lambda; t) + \sum_{k=1}^{\infty} \frac{(-i)^k}{(k+1)!} \text{ad}_{U_0(\lambda; t)}^k \tilde{C}(\lambda; t) \right] = 0 \quad \forall t.$$  

(65)

Now, observe that if there exists a solution \( \{ t \mapsto \tilde{C}(\lambda; t), t \mapsto \tilde{Z}(\lambda; t) \} \) of equation (59) verifying condition (60) (hence relation (65)) and condition (61), then equation (20) will be approximately satisfied setting \( C(\lambda; t) = \tilde{C}(\lambda; t) \) and \( Z(\lambda; t) = \tilde{Z}(\lambda; t) \). Thus equation (59) will be the starting point for obtaining a perturbative expansion of the evolution operator when the perturbative adiabatic approximation is applicable. This task will be pursued in the next section. Since at this stage the approximation considered may appear as a mere ad hoc computational expedient, we will devote the last part of this section to showing what is its meaning; in particular, why we call it ‘adiabatic’.

We will first show that the evolution operator \( \tilde{U}(\lambda; t) \) associated with the solution \( \{ t \mapsto \tilde{C}(\lambda; t), t \mapsto \tilde{Z}(\lambda; t) \} \) behaves like an adiabatic evolutor. To this aim, let us set

$$W(\lambda; t) := \exp \left(-i \tilde{Z}(\lambda; t)\right),$$

(66)

and let us define the projection

$$P_m(\lambda; t) := W(\lambda; t) P_m W(\lambda; t)^{-1}$$

(67)

and the selfadjoint operators

$$\tilde{E}(\lambda; t) := \tilde{C}(\lambda; t) + \sum_{k=1}^{\infty} \frac{(-i)^k}{(k+1)!} \text{ad}_{U_0(\lambda; t)}^k \tilde{C}(\lambda; t),$$

(68)

$$K(\lambda; t) := i W(\lambda; t)^{-1} \dot{W}(\lambda; t) = \sum_{k=0}^{\infty} \frac{i^k}{(k+1)!} \text{ad}_{\dot{Z}(\lambda; t)}^k \dot{Z}(\lambda; t) = \tilde{K}(\lambda; t) + \hat{K}(\lambda; t).$$

(69)
with
\[
\left\{ \begin{array}{l}
\hat{K}(\lambda; t) := \left\langle \left( \sum_{k=0}^{\infty} \frac{i^k}{(k+1)!} \text{ad}_{\hat{z}(\lambda; t)} \hat{Z}(\lambda; t) \right) \right\rangle_{H_0(t)}, \\
\tilde{K}(\lambda; t) := \left\langle \left( \sum_{k=0}^{\infty} \frac{i^k}{(k+1)!} \text{ad}_{\hat{z}(\lambda; t)} \hat{Z}(\lambda; t) \right) \right\rangle_{H_0(t)}.
\end{array} \right.
\]
(70)

The operator \( \tilde{\mathcal{C}}(\lambda; t) \) (by virtue of relations (60) and (63)) and the operator \( \hat{K}(\lambda; t) \) (by definition) satisfy:
\[
[\tilde{\mathcal{C}}(\lambda; t), P_m] = 0, \quad [\tilde{K}(\lambda; t), P_m] = 0, \quad \forall t, \forall m.
\]
(71)

Then, equation (59) can be rewritten as
\[
H(\lambda; t) = W(\lambda; t) \left( H_0(t) + \tilde{\mathcal{C}}(\lambda; t) + \tilde{K}(\lambda; t) \right) W(\lambda; t)^{-1}
\]
(72)
and, by this equation, we have:
\[
H(\lambda; t) P_m(\lambda; t) = W(\lambda; t) \left( H_0(t) + \tilde{\mathcal{C}}(\lambda; t) + \tilde{K}(\lambda; t) \right) P_m W(\lambda; t)^{-1}
\]
* relations (71) *
\[
= W(\lambda; t) P_m \left( H_0(t) + \tilde{\mathcal{C}}(\lambda; t) + \tilde{K}(\lambda; t) \right) P_m W(\lambda; t)^{-1}
\]
\[
= P_m(\lambda; t) H(\lambda; t) P_m(\lambda; t),
\]
(73)
i.e. the range of the projection \( P_m(\lambda; t) \) is an invariant subspace for \( H(\lambda; t) \). At this point, it is convenient to make a detour.

We remark that as in the time-independent case — instead of assuming that decomposition (72) holds — one can prove that, under suitable conditions, there exists a complete set \( \{ P_m(\lambda; t) \}_{m=1,2,...} \) of spectral projections of \( H(\lambda; t) \) such that the functions \( \lambda \mapsto P_m(\lambda; t) \) and \( t \mapsto P_m(\lambda; t) \) are analytic and
\[
P_m(0; 0) = P_m, \quad m = 1, 2, \ldots
\]
(74)
Then, there is a (non-unique) unitary operator \( W(\lambda; t) \) such that
\[
P_m(\lambda; t) = W(\lambda; t) P_m W(\lambda; t)^{-1}, \quad \forall t, \forall m.
\]
(75)
The operator \( W(\lambda; t) \) can be decomposed as
\[
W(\lambda; t) = A(\lambda; t) W(\lambda),
\]
(76)
where \( \lambda \mapsto W(\lambda) \) and \( (\lambda, t) \mapsto A(\lambda; t) \) are unitary operator-valued functions such that
\[
P_m(\lambda; 0) = W(\lambda) P_m W(\lambda)^{-1},
\]
(77)
\[
P_m(\lambda; t) = A(\lambda; t) P_m(\lambda; 0) A(\lambda; t), \quad \forall t, \forall m.
\]
(78)
One can show that these functions are solutions of Cauchy problems of the type

\[ i W'(\lambda) = J(\lambda) W(\lambda), \quad W(0) = \text{Id}, \quad (79) \]

\[ i \dot{A}(\lambda; t) = K(\lambda; t) A(\lambda; t), \quad A(\lambda; 0) = \text{Id}, \quad (80) \]

with \( J(\lambda) \) and \( K(\lambda; t) \) selfadjoint operators satisfying the following conditions:

\[ \| J(\lambda) \|_{\mathcal{H}(\lambda; \lambda)} = i \sum_{m} P_m'(\lambda; 0) P_m(\lambda; 0) \]

\[ * 0 = \frac{d}{d\lambda} \text{Id} = \frac{d}{d\lambda} \sum_{m} P_m(\lambda; t)^2 * = -i \sum_{m} P_m(\lambda; 0) P_m'(\lambda; 0), \quad (81) \]

\[ \tilde{K}(\lambda; t) := \| K(\lambda; t) \|_{\mathcal{H}(\lambda; \lambda)} = i \sum_{m} \dot{P}_m(\lambda; t) P_m(\lambda; t) \]

\[ * 0 = \frac{d}{dt} \text{Id} = \frac{d}{dt} \sum_{m} P_m(\lambda; t)^2 * = -i \sum_{m} P_m(\lambda; t) \dot{P}_m(\lambda; t). \quad (82) \]

Conversely, if \( \lambda \mapsto W(\lambda) \) and \( (\lambda, t) \mapsto A(\lambda; t) \) are operator-valued functions which are solutions of the Cauchy problems (79) and (80) (with \( J(\lambda) \) and \( K(\lambda; t) \) selfadjoint operators subject to conditions (81) and (82)), then they will satisfy equations (77) and (85). In fact, suppose that relation (78) is satisfied (we will only argue for \( A(\lambda; t) \) since the argument for \( W(\lambda) \) is analogous). Then, setting \( K(\lambda; t) := i \dot{A}(\lambda; t) A(\lambda; t)^{-1} \), we have:

\[ i \dot{P}_m(\lambda; t) = [K(\lambda; t), P_m(\lambda; t)], \quad \forall t, \forall m. \quad (83) \]

Conversely, if this relation holds, we have:

\[ A(\lambda; t)^{-1} P_m(\lambda; t) A(\lambda; t) = A(\lambda; t)^{-1} \dot{P}_m(\lambda; t) A(\lambda; t) + A(\lambda; t)^{-1} P_m(\lambda; t) \dot{A}(\lambda; t) - A(\lambda; t)^{-1} \dot{A}(\lambda; t) A(\lambda; t)^{-1} P_m(\lambda; t) A(\lambda; t) = 0, \quad \forall t, \forall m, \quad (84) \]

where we have used the fact that \( A(\lambda; t)^{-1} = -\dot{A}(\lambda; t) \); hence:

\[ A(\lambda; t)^{-1} P_m(\lambda; t) A(\lambda; t) = A(\lambda; 0)^{-1} P_m(\lambda; 0) A(\lambda; 0) = P_m(\lambda; 0), \quad \forall t, \forall m. \quad (85) \]

Thus, relation (78) holds if and only if \( K(\lambda; t) \) satisfies equation (85). At this point, using the fact that \( P_m(\lambda; t) \dot{P}_m(\lambda; t) P_m(\lambda; t) = 0, \forall m \), one can check easily that

\[ K(\lambda; t) = \tilde{K}(\lambda; t) + \dot{\tilde{K}}(\lambda; t), \quad (86) \]

— with \( \tilde{K}(\lambda; t) := \| K(\lambda; t) \|_{H(\lambda; t)} \) and \( \dot{\tilde{K}}(\lambda; t) := \| K(\lambda; t) \|_{H(\lambda; t)} \) — solves equation (85) if and only if:

\[ \tilde{K}(\lambda; t) = i \sum_{m} \dot{P}_m(\lambda; t) P_m(\lambda; t). \quad (87) \]
In conclusion, it is proven that, if there exists a complete set of spectral projections \( \{P_m(\lambda; t)\}_{m=1,2,...} \) of \( H(\lambda; t) \) with the properties specified above, then equation (56) is satisfied by only and all the unitary operator-valued functions \( (\lambda, t) \rightarrow W(\lambda; t) \) such that \( W(\lambda) = W(\lambda; 0) \) and \( A(\lambda; t) = W(\lambda)^{-1}W(\lambda; t) \) are solutions of the Cauchy problems (79) and (80), with conditions (81) and (82). Moreover, if one sets

\[
\bar{\Pi}(\lambda; t) := W(\lambda; t)^{-1}H(\lambda; t)W(\lambda; t),
\]

the following relation holds:

\[
\bar{\Pi}(\lambda; t) P_m = P_m \bar{\Pi}(\lambda; t) P_m, \quad \forall t, \forall m.
\]

Then, setting

\[
\begin{align*}
\tilde{\mathcal{C}}(\lambda; t) &= i \left( \left[ W(\lambda; t)^{-1}W(\lambda; t) \right]_{H_0(t)} \right), \\
\tilde{\mathcal{C}}(\lambda; t) &= \bar{\Pi}(\lambda; t) - H_0(t) - \tilde{\mathcal{C}}(\lambda; t),
\end{align*}
\]

one finds that

\[
[\tilde{\mathcal{C}}(\lambda; t), P_m] = 0, \quad \forall t, \forall m.
\]

In the next section, it will be shown that equation (88) allows to recover the operator \( \tilde{\mathcal{C}}(\lambda; t) \) from \( \bar{\Pi}(\lambda; t) \). Thus one can actually define \( \tilde{\mathcal{C}}(\lambda; t) \) through formula (88). One can also prove that if relation (92) holds then, as a consequence, \( [\tilde{\mathcal{C}}(\lambda; t), P_m] = 0, \forall t, \forall m \). Hence, setting \( W(\lambda; t) = \exp(-i \tilde{\mathcal{C}}(\lambda; t)) \), one reobtains equation (57) with condition (59) automatically satisfied.

Let us now come back to our original purpose of investigating the behavior of the evolution operator \( \bar{U}(\lambda; t) \) that approximates \( U(\lambda; t) \):

\[
\begin{align*}
U(\lambda; t) &= U_0(t) e^{-iZ(\lambda; t)} e^{-i\int_0^t \mathcal{C}(\lambda; t) dt} e^{iZ(\lambda)} \\
* Z(\lambda; t) &= \text{Ad}_{U_0(t)} \mathcal{Z}(\lambda; t) * = e^{-iZ(\lambda; t)} U_0(t) e^{-i\int_0^t \mathcal{C}(\lambda; t) dt} e^{iZ(\lambda)} \\
* \text{adiabatic approximation} * \simeq e^{-i\tilde{Z}(\lambda; t)} U_0(t) e^{-i\int_0^t \tilde{\mathcal{C}}(\lambda; t) dt} e^{i\tilde{Z}(\lambda)} =: \tilde{U}(\lambda; t).
\end{align*}
\]

Recalling formula (88) and relation (92), the \textit{adiabatic evoluter} \( \tilde{U}(\lambda; t) \) can be written as

\[
\tilde{U}(\lambda; t) = e^{-i\tilde{Z}(\lambda; t)} \sum_m e^{-i \int_0^t \left( E_m(t) + \tilde{\mathcal{C}}(\lambda; t) \right) dt} P_m e^{i\tilde{Z}(\lambda)},
\]

or — introducing the reduced rank operator

\[
[\tilde{\mathcal{C}}]_m(\lambda; t) := \tilde{\mathcal{C}}(\lambda; t) P_m = P_m \tilde{\mathcal{C}}(\lambda; t) P_m, \quad m = 1, 2, \ldots
\]

— in the more expressive form:

\[
\tilde{U}(\lambda; t) = e^{-i\tilde{Z}(\lambda; t)} \sum_m e^{-i \int_0^t \left( E_m(t) + [\tilde{\mathcal{C}}]_m(\lambda; t) \right) dt} P_m e^{i\tilde{Z}(\lambda)}.
\]
To show that $\tilde{U}(\lambda; t)$ behaves indeed as an adiabatic evolutor, let us observe that

$$\tilde{U}(\lambda; t) P_j(\lambda; 0) = e^{-i\tilde{Z}(\lambda; t)} U_0(t) e^{-i \int_0^t \tilde{C}(\lambda; t) \, dt} e^{i \tilde{Z}(\lambda)} P_j(\lambda; 0)$$

* $P_j(\lambda; 0) = \text{Ad}_{e^{-i\tilde{Z}(\lambda)}} P_j$ *

$$= e^{-i\tilde{Z}(\lambda; t)} \sum_m e^{-i \int_0^t (E_m(t) + \tilde{C}(\lambda; t)) \, dt} P_m P_j e^{i \tilde{Z}(\lambda)}$$

$$= e^{-i\tilde{Z}(\lambda; t)} P_j U_0(t) e^{-i \int_0^t \tilde{C}(\lambda; t) \, dt} e^{i \tilde{Z}(\lambda)}$$

* definition (67) *

$$= P_j(\lambda; t) \tilde{U}(\lambda; t), \quad j = 1, 2, \ldots ;$$

(97)

namely, $\tilde{U}(\lambda; t)$, as it should, intertwines the projection $P_j(\lambda; 0)$ with the projection $P_j(\lambda; t)$. Let us do the following observations:

1. The evolutor $\tilde{U}(\lambda; t)$ intertwines spectral projections, that in general, for $\lambda \neq 0$, are not eigenprojections. This situation is more general than the one considered originally by Kato [21] and earlier by Born and Fock in their seminal paper [22]. Nevertheless, due to its importance in several applications, this situation has been studied in later times by other authors (see, for instance, Nenciu’s paper [23]). Considering this more general situation is in our case unavoidable, since in presence of the perturbation (i.e. for $\lambda \neq 0$) the unperturbed eigenvalues (‘energy levels’) can ‘split’.

2. For $\lambda = 0$, the adiabatic evolutor reduces to the unperturbed evolution operator:

$$\tilde{U}(0; t) = U_0(t).$$

(98)

This fact justifies the term ‘perturbative adiabatic approximation’.

Anyway, in order to better highlight the typical structure of an adiabatic evolutor, it is convenient to rewrite the expression of $\tilde{U}(\lambda; t)$ as follows:

$$\tilde{U}(\lambda; t) = A(\lambda; t) \sum_m e^{-i \int_0^t (E_m(t) + \Omega_m(\lambda; t)) \, dt} P_m(\lambda; 0),$$

(99)

where

$$A(\lambda; t) := W(\lambda; t) W(\lambda; 0)^{-1},$$

$$\Omega_m(\lambda; t) := W(\lambda; 0) \tilde{C}(\lambda; t) W(\lambda; 0)^{-1} P_m(\lambda; 0)$$

(100)

$$= W(\lambda; 0) [\tilde{C}]_m(\lambda; t) W(\lambda; 0)^{-1}.$$

Assume that the time-dependence of the Hamiltonian $H(\lambda; t)$ is characterized by a time scale $T > 0$, i.e.

$$H_0(t) = \mathcal{H}_0(\lambda; t/T), \quad H_0(\lambda; t) = \mathcal{H}_0(\lambda; t/T), \quad t \in [0, T],$$

(101)

with $[0, 1] \ni s \mapsto \mathcal{H}_0(s)$ and $[0, 1] \ni s \mapsto \mathcal{H}_0(\lambda; s)$ given operator-valued functions of the scaled time $s$. From the physicist’s point of view, the parameter $T$ measures the ‘slowness’ with which the non-isolated quantum system described
by the Hamiltonian $H(\lambda; t)$ is influenced by the external world. Then, for the spectral projections \( \{P_m(\lambda; t)\}_{m=1,2,\ldots} \) of the Hamiltonian $H(\lambda; t)$ we have:

$$P_m(\lambda; t) = P_m(\lambda; t/T), \quad \forall t \in [0, T], \forall m,$$

where $P_m(\lambda; s)$ is a projection-valued function of the scaled time.

Recall now that the operator $A(\lambda; t)$ is solution of an initial value problem of the form

$$i \dot{A}(\lambda; t) = K(\lambda; t) A(\lambda; t), \quad A(\lambda; 0) = \text{Id},$$

where $K(\lambda; t)$ has to satisfy the following condition:

$$\langle K(\lambda; t) | \{P_m(\lambda; t)\} \rangle \equiv i \sum_m \dot{P}_m(\lambda; t) P_m(\lambda; t)$$

More specifically, using the fact that

$$\langle \text{Ad}_W(\lambda; t)^{-1} \dot{\hat{K}}(\lambda; t) | H_0(t) \rangle = \langle \text{Ad}_W(\lambda; t)^{-1} | \{P_m\} \rangle \cdot \langle |\{P_m(\lambda; t)\}| \rangle,$$

we have:

$$\hat{K}(\lambda; t) = W(\lambda; t)^{-1} \hat{K}(\lambda; t) W(\lambda; t) = O\left(\frac{1}{T}\right).$$

This observation ‘completes the picture’. Indeed, it turns out that, in the adiabatic limit, the contribution of the operator $\hat{K}(\lambda; t)$ in equation (20) can be neglected, namely that approximation (61) is justified as it is equivalent to the standard adiabatic approximation.

### 4 Solution with the perturbative adiabatic approximation

Let us now face the task of providing the perturbative solutions of equation (59), which we recall is subject to condition (60) and to what we can at this point legitimately call ‘adiabatic approximation’, i.e. condition (61).

It will be convenient to express equation (59) in terms of the operator $\hat{C}(\lambda; t)$ in place of $\hat{C}(\lambda; t)$. In fact, as it will be shown later on in this section, one can develop a simple perturbative procedure which allows to compute, order
by order, the operator $\mathcal{C}(\lambda; t)$ from $\mathcal{C}(\lambda; t)$. Then, the equation that has to be solved perturbatively is the following:

$$
\sum_{k=1}^{\infty} \frac{i^k}{k!} \text{ad}^k \mathcal{Z}(\lambda; t) (H_0(t) + H_0(\lambda; t)) = \mathcal{C}(\lambda; t) - H_0(\lambda; t)
$$

$$
+ \left\langle \left[ \sum_{k=0}^{\infty} \frac{i^k}{(k+1)!} \text{ad}^k \mathcal{Z}(\lambda; t) \right] H_0(t) \right\rangle, \quad (108)
$$

where the operator $\mathcal{C}(\lambda; t)$ is subject to the constraint

$$
[\mathcal{C}(\lambda; t), H_0(t)] = 0, \quad \forall t. \quad (109)
$$

It will be also shown that condition (109) is actually equivalent to condition (60) of which it is a straightforward consequence. This fact is not immediately evident from the definition of $\mathcal{C}(\lambda; t)$. We will postpone the problem of checking the validity of the adiabatic approximation (condition (61)) to later analysis.

Given linear operators $X, X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$, let us set

$$
\mathcal{R}^+_n(X; Y_1, \ldots, Y_n) := \sum_{m=1}^{n} \frac{(-i)^m}{m!} \sum_{k_1 + \cdots + k_m = n} \text{ad}_{Y_{k_1}} \cdots \text{ad}_{Y_{k_m}} X, \quad n \geq 1. \quad (110)
$$

Then, for $n \geq 2$, we can define the operator function

$$
\mathcal{R}^+_n(X_1, \ldots, X_{n-1}; Y_1, \ldots, Y_{n-1}) := \pm \sum_{m=1}^{n-1} \mathcal{R}^+_m(X_m; Y_1, \ldots, Y_{n-m}). \quad (111)
$$

Noting the analogy of equation (108) with equation (24) obtained in the time-independent case, we conclude that, given the power expansions

$$
H_0(\lambda; t) = \sum_{n=1}^{\infty} \lambda^n H_n(t), \quad \mathcal{C}(\lambda; t) = \sum_{n=1}^{\infty} \lambda^n \mathcal{C}_n(t), \quad \mathcal{Z}(\lambda; t) = \sum_{n=1}^{\infty} \lambda^n Z_n(t), \quad (112)
$$

the sequence of equations generated by formula (108), with condition (109), has the following form:

$$
\mathcal{C}_1(t) - i \left[ Z_1(t), H_0(t) \right] = H_1(t) - \left\langle \mathcal{Z}_1(t) \right\rangle_{H_0(t)},
$$

$$
\left[ \mathcal{C}_1(t), H_0(t) \right] = 0,
$$

$$
\vdots
$$

$$
\mathcal{C}_n(t) - i \left[ Z_n(t), H_0(t) \right] = \mathcal{G}_n(H_0(t), \ldots, H_n(t); Z_1(t), \ldots, Z_{n-1}(t))
$$

$$
- \left\langle \left[ \mathcal{R}^+_n(\mathcal{Z}_1(t), \ldots, \mathcal{Z}_{n-1}(t); \mathcal{Z}_1(t), \ldots, \mathcal{Z}_{n-1}(t)) \right] \right\rangle_{H_0(t)}
$$

$$
- \left\langle \mathcal{Z}_n(t) \right\rangle_{H_0(t)},
$$

$$
\left[ \mathcal{C}_n(t), H_0(t) \right] = 0 \quad n \geq 2,
$$

$$
\vdots
$$

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Exactly as in the time-independent case, this system of equations can be solved recursively. Indeed, we have that

\[
\bar{\mathcal{C}}_1(t) = \langle | H_1(t) \rangle | H_0(t) - \frac{d}{dt} \langle | \bar{Z}_1(t) \rangle | H_0(t),
\]

(115)

\[
\bar{Z}_1(t) = \langle | \bar{Z}_1(t) \rangle | H_0(t) + \langle | H_1(t) \rangle | H_0(t),
\]

(116)

— where \( \langle | \bar{Z}_1(t) \rangle | H_0(t) \) is an arbitrary operator commuting with \( H_0(t) \) (notice that \( \frac{d}{dt} \langle | \bar{Z}_1(t) \rangle | H_0(t) = \langle | \dot{Z}_1(t) \rangle | H_0(t) \) — and, assuming that the first \( n \) equations have been solved, so that the operator functions

\[
G_{n+1}(\ldots, H_{n+1}(t); \ldots, \bar{Z}_n(t)) \quad \text{and} \quad R^+_{n+1}(\ldots, \bar{Z}_n(t); \ldots, \bar{Z}_n(t))
\]

are known explicitly, the solution of the \((n+1)\)-th equation is given by

\[
\bar{\mathcal{C}}_{n+1}(t) = \left\langle \left[ G_{n+1}(H_0(t), \ldots, H_{n+1}(t); \bar{Z}_1(t), \ldots, \bar{Z}_n(t)) \right] \right\rangle_{H_0(t)}
\]

\[-\left\langle \left[ R^+_{n+1}(\bar{Z}_1(t), \ldots, \bar{Z}_n(t); \bar{Z}_1(t), \ldots, \bar{Z}_n(t)) \right] \right\rangle_{H_0(t)}
\]

\[-\frac{d}{dt} \left\langle \left[ \bar{Z}_{n+1}(t) \right] \right\rangle_{H_0(t)},
\]

(117)

\[
\left[ \bar{Z}_{n+1}(t), H_0(t) \right] = i \left\langle \left[ G_{n+1}(H_0(t), \ldots, H_{n+1}(t); \bar{Z}_1(t), \ldots, \bar{Z}_n(t)) \right] \right\rangle_{H_0(t)}.
\]

(118)

Again, equations (117) and (118) determine \( \bar{\mathcal{C}}_{n+1}(t) \) and \( \bar{Z}_{n+1}(t) \) up to an operator \( \langle | \bar{Z}_{n+1}(t) \rangle | H_0(t) \) commuting with \( H_0(t) \) and we have:

\[
\bar{Z}_{n+1}(t) = \left\langle \left[ \bar{Z}_{n+1}(t) \right] \right\rangle_{H_0(t)}
\]

\[+ \left\langle \left[ G_{n+1}(H_0(t), \ldots, H_{n+1}(t); \bar{Z}_1(t), \ldots, \bar{Z}_n(t)) \right] \right\rangle_{H_0(t)},
\]

(119)

Eventually, one has to check that condition (61) is satisfied. Explicitly, one has to check that the solution obtained \( \{ t \mapsto \bar{\mathcal{C}}_n(\lambda; t), t \mapsto \bar{Z}_n(\lambda; t) \}_{n \in \mathbb{N}} \) is such that

\[
\left\langle \left[ \bar{Z}_1(t) \right] \right\rangle_{H_0(t)} \simeq 0,
\]

(120)

\[\vdots\]

\[
\left\langle \left[ R^+_n(\bar{Z}_1(t), \ldots, \bar{Z}_{n-1}(t); \bar{Z}_1(t), \ldots, \bar{Z}_{n-1}(t)) - \bar{Z}_n(t) \right] \right\rangle_{H_0(t)} \simeq 0, \quad n \geq 2,
\]

(120)

\[\vdots\]

At this point, we have to show how the operator \( \bar{\mathcal{C}}(\lambda; t) \) can be recovered, at each perturbative order, from the operator \( \bar{\mathcal{C}}(\lambda; t) \). To this aim, let us recall that

\[
\bar{\mathcal{C}}(\lambda; t) = \bar{\mathcal{C}}(\lambda; t) - \sum_{k=1}^{\infty} \frac{(-i)^k}{(k+1)!} \text{ad}_{\bar{\mathcal{C}}(\lambda; t)} \frac{d}{dt} \bar{\mathcal{C}}(\lambda; t).
\]

(121)
Now, notice that, if we substitute in equation (121) the power expansions
\[ \tilde{C}(\lambda; t) = \sum_{n=1}^{\infty} \lambda^n \tilde{C}_n(t) \] and \[ \tilde{C}(\lambda; t) = \sum_{n=1}^{\infty} \lambda^n \tilde{C}_n(t), \]
and we single out the various perturbative orders, we conclude that the \( n \)-th order, which on the l.h.s. is given simply by \( \lambda^n \tilde{C}_n(t) \), consists on the r.h.s. of \( \lambda^n \tilde{C}_n(t) \) plus a function of \( \tilde{C}_1(t), \ldots, \tilde{C}_n(t) \) and \( \int_0^t \tilde{C}_1(t) \, dt, \ldots, \int_0^t \tilde{C}_{n-1}(t) \, dt \). Thus, exploiting this fact, we can achieve an order by order solution. Indeed, recalling definition (111), one finds out that \( \tilde{C}(\lambda; t) \) can be obtained order by order from \( \tilde{C}(\lambda; t) \) by means of the following recursive process:

\[ \tilde{C}_1(t) = \tilde{C}_1(t), \]
\[ \vdots \]
\[ \tilde{C}_n(t) = R_n^{-1}(\tilde{C}_1(t), \ldots, \tilde{C}_{n-1}(t); \int_0^t \tilde{C}_1(t) \, dt, \ldots, \int_0^t \tilde{C}_{n-1}(t) \, dt) + \tilde{C}_n(t), \quad n \geq 2, \quad (122) \]
\[ \vdots \]

This recursive process allows to easily prove by induction that the commutation relation \( [\tilde{C}(\lambda; t), H_0(t)] = 0 \) (or, equivalently, \( [\tilde{C}(\lambda; t), P_m] = 0, \forall m \)) implies that \( [\tilde{C}(\lambda; t), H_0(t)] = 0 \). Thus equation (108), with condition (109), is indeed equivalent to equation (59), with condition (60).

We want to show now that there is also another recursive process allowing to recover \( \tilde{C}(\lambda; t) \) from \( \tilde{C}(\lambda; t) \) which less expensive from the computational point of view.

To this aim, let us define the function \( \text{avxp} : \mathbb{R} \to \mathbb{R}^+ \) in the following way:
\[ \text{avxp}(x) = \frac{1}{x} \int_0^x e^t \, dt = \frac{e^x - 1}{x} \quad \text{for} \quad x \neq 0, \quad \text{avxp}(0) = 1. \quad (123) \]

This function extends to an entire holomorphic function on the complex plane which is given by
\[ \text{avxp}(z) = \sum_{k=0}^{\infty} \frac{1}{(k + 1)!} z^k. \quad (124) \]

We recall that the function \( \text{avxp} \) is fundamental in the theory of Lie groups since it is strictly related to the differential of the exponential map, usually denoted by \( \text{dexp} \) (see ref. [21]). In fact, given a Lie group \( G \) with Lie algebra \( \mathfrak{g} \), identifying the tangent spaces at any point of \( G \) and of \( \mathfrak{g} \) with \( \mathfrak{g} \) itself, one has:
\[ \text{dexp}(X) Y = \text{avxp}(-X) Y, \quad X, Y \in \mathfrak{g}. \quad (125) \]

Besides, it is well known (see, for instance, ref. [25]) that the the meromorphic function \( 1/\text{avxp} \) admits the following expansion in the open disk of radius \( 2\pi \) centered at zero:
\[ \text{avxp}(z)^{-1} = \sum_{k=0}^{\infty} \beta_k \frac{z^k}{k!}, \quad (126) \]
where \( \{ \beta_0, \beta_1, \ldots \} \) are the Bernoulli numbers, namely the rational numbers defined recursively by

\[
\beta_0 = 1, \quad \left( \begin{array}{c} k + 1 \\ 0 \end{array} \right) \beta_0 + \cdots + \left( \begin{array}{c} k + 1 \\ k \end{array} \right) \beta_k = 0 \quad k = 1, 2, \ldots . \quad (127)
\]

We recall also that

\[
\beta_{2k+1} = 0, \quad \left\lvert \beta_{2k} \right\rvert = (-1)^{k+1}, \quad k = 1, 2, \ldots . \quad (128)
\]

Now, notice that, according to definition (68) and formula (124), we have:

\[
\tilde{C}(\lambda; t) = \text{avxp}\left(-i \text{ad}_{\lambda t} \tilde{C}(\lambda, t) \right) \tilde{C}(\lambda; t).
\]

Then, by means of formula (126), we can write

\[
\tilde{C}(\lambda; t) = \text{avxp}\left(-i \text{ad}_{\lambda t} \tilde{C}(\lambda, t) \right)^{-1} \tilde{C}(\lambda; t) = \sum_{k=0}^{\infty} \frac{(-i)^k \beta_k}{k!} \text{ad}_{\lambda t}^k \tilde{C}(\lambda, t) \tilde{C}(\lambda; t).
\]

Let us observe that, as in the previous case, if we substitute in this equation the power expansions \( \tilde{C}(\lambda; t) = \sum_{n=1}^{\infty} \lambda^n \tilde{C}_n(t) \) and \( \tilde{E}(\lambda; t) = \sum_{n=1}^{\infty} \lambda^n \tilde{E}_n(t) \), and we single out the various perturbative orders, we find that the \( n \)-th order, which on the l.h.s. is given simply by \( \lambda^n \tilde{C}_n(t) \), consists on the r.h.s. of a function of \( \tilde{C}_1(t), \ldots, \tilde{C}_n(t) \) and \( \int_0^t \tilde{C}_1(t) \, dt, \ldots, \int_0^t \tilde{C}_{n-1}(t) \, dt \). Thus, again, we can achieve an order by order solution.

To this aim, given linear operators \( X \) and \( Y_1, \ldots, Y_n, n \geq 1 \), let us set

\[
\mathcal{B}^\pm_n(X; Y_1, \ldots, Y_n) := \pm \sum_{m=1}^{n} \frac{(-i)^m \beta_m}{m!} \sum_{k_1 + \cdots + k_m = n} \text{ad}_{Y_{k_1}} \cdots \text{ad}_{Y_{k_m}} X; \quad (131)
\]

namely, we have that \( \mathcal{B}^\pm_1(X; Y_1) = \frac{i}{2} \text{ad}_{Y_1} X \) (since \( \beta_1 = -1/2 \)) and, by virtue of relations (128),

\[
\mathcal{B}^\pm_n(X; Y_1, \ldots, Y_n) = \frac{i}{2} \text{ad}_{Y_n} X \pm \sum_{m=1}^{p(n)/2} \frac{|\beta_{2m}|}{2m!} \sum_{k_1 + \cdots + k_{2m} = n} \text{ad}_{Y_{k_1}} \cdots \text{ad}_{Y_{k_{2m}}} X, \quad (132)
\]

for \( n \geq 2 \), where \( p(n) \) is equal to \( n \) if \( n \) is even and to \( n - 1 \) otherwise. Next, for \( n \geq 2 \), given linear operators \( X_1, \ldots, X_{n-1} \) and \( Y_1, \ldots, Y_{n-1} \), we can define

\[
\mathcal{B}^\pm_n(X_1, \ldots, X_{n-1}; Y_1, \ldots, Y_{n-1}) := \sum_{m=1}^{n-1} \mathcal{B}^\pm_{n-m}(X_m; Y_1, \ldots, Y_{n-m}), \quad (133)
\]
where the definition of $B^+_n(X_1, \ldots, X_{n-1}; Y_1, \ldots, Y_{n-1})$ will be used in the next section. Then, one finds out that $\hat{C}(\lambda; t)$ can be obtained order by order from $\hat{C}(\lambda; t)$ by means of the following recursive process:

$$
\hat{C}_1(t) = \hat{C}_1(t),
$$

$$
\vdots
$$

$$
\hat{C}_n(t) = B_n\left(\hat{C}_1(t), \ldots, \hat{C}_{n-1}(t); \int_0^t \hat{C}_1(t) \, dt, \ldots, \int_0^t \hat{C}_{n-1}(t) \, dt\right) + \hat{C}_n(t), \quad n \geq 2, (134)
$$

One can verify that this recursive process is computationally convenient with respect to the recursive process (122). Specifically, it turns out that, from the 4-th perturbative order on, each step involves a smaller number of terms in comparison with process (122), with a gain which increases step after step.

### 5 The general case

In this section we will consider equation (135) in its full generality. This equation, taking into account definition (68), can be rewritten as

$$
\sum_{k=0}^{\infty} \frac{i^k}{k!} \left[ H(\lambda; t) - \frac{1}{k+1} \right] Z(\lambda; t) = \mathcal{C}(\lambda; t), (135)
$$

where the operator $\mathcal{C}(\lambda; t)$ is linked to the operator $C(\lambda; t)$, which appears in decomposition (3) of the interaction picture evolution operator, by the same relation that links the operators $\hat{C}(\lambda; t)$ and $\hat{C}(\lambda; t)$ introduced for studying the adiabatic approximation; we rewrite it here for the sake of clarity:

$$
\mathcal{C}(\lambda; t) := \exp\left(-i \int_0^t C(\lambda; t) \, dt\right) C(\lambda; t)
$$

$$
= C(\lambda; t) + \sum_{k=1}^{\infty} \frac{(-i)^k}{(k+1)!} \int_0^t C(\lambda; t) \, dt C(\lambda; t). (136)
$$

We already know (see section 5) that the operator $C(\lambda; t)$ can be recovered from the operator $\mathcal{C}(\lambda; t)$ by means of an order by order procedures. Thus, we can solve equation (135) for $\mathcal{C}(\lambda; t)$ up to a given perturbative order and obtain the perturbative expansion of $C(\lambda; t)$ truncated at the same order parallelly.

First of all, we want to provide an interpretation of decomposition (3) which sheds light on its meaning. To this scope, notice that this decomposition can be rewritten as

$$
T_C(\lambda; t) = T_Z(\lambda; t) \, T(\lambda; t) \, T_Z(\lambda; 0), (137)
$$
where

\[ T_C(\lambda; t) := \exp \left( -i \int_0^t C(\lambda; t) \, dt \right) \quad \text{and} \quad T_Z(\lambda; t) := \exp(-i Z(\lambda; t)). \quad (138) \]

Now, formula (137) can be regarded as a passage to a further (generalized) interaction picture performed on the Hamiltonian \( \tilde{H}(\lambda; t) \). Indeed, assuming that \( T_Z(\lambda; t) \) satisfies the equation

\[ i \dot{T}_Z(\lambda; t) = T_Z(\lambda; t) \mathfrak{A}(\lambda; t), \quad (139) \]

one finds that

\[ i \dot{T}_C(\lambda; t) = \left( T_Z(\lambda; t) \right)^\dagger \tilde{H}(\lambda; t) T_Z(\lambda; t) - \mathfrak{A}(\lambda; t) \right) T_C(\lambda; t). \quad (140) \]

Then, since

\[ \mathfrak{A}(\lambda; t) = \sum_{k=0}^{\infty} \frac{i^k}{(k+1)!} \text{ad}_{Z(\lambda; t)}^k \dot{Z}(\lambda; t), \quad (141) \]

equation (135) expresses precisely the fact that \( \mathfrak{C}(\lambda; t) \) is the transformed Hamiltonian obtained by switching to this new interaction picture; namely:

\[ \mathfrak{C}(\lambda; t) = T_Z(\lambda; t) \tilde{H}(\lambda; t) T_Z(\lambda; t) - \mathfrak{A}(\lambda; t). \quad (142) \]

It follows that

\[ T_C(\lambda; t) = \exp \left( -i \sum_{n=1}^{\infty} \lambda^n \int_0^t C_n(t) \, dt \right) \quad (143) \]

is nothing but the Magnus expansion of evolution operator associated with the new interaction picture Hamiltonian \( \mathfrak{C}(\lambda; t) \).

Let us now investigate the perturbative solutions of equation (135). Substituting the power expansions

\[ \tilde{H}(\lambda; t) = \sum_{n=1}^{\infty} \lambda^n \tilde{H}_n(t), \quad \mathfrak{C}(\lambda; t) = \sum_{n=1}^{\infty} \lambda^n \mathfrak{C}_n(t), \quad Z(\lambda; t) = \sum_{n=1}^{\infty} \lambda^n Z_n(t), \quad (144) \]

in equation (135), we obtain an infinite set of coupled equations that allows to compute the operators \( \{ \mathfrak{C}_n(t) \}_{n \in \mathbb{N}}, \{ Z_n(t) \}_{n \in \mathbb{N}} \). In fact, in analogy with the time-independent case (compare with definition (32)), given linear operators \( X \) and \( Y \), for \( n \geq 1 \), let us set

\[ \mathcal{G}_n(X, Y; Z_1, \ldots, Z_n) := \sum_{m=1}^{n} \frac{i^m}{m!} \sum_{k_1 + \cdots + k_m = n} \text{ad}_{Z_{k_1}} \cdots \text{ad}_{Z_{k_m}} \left( X - \frac{Y}{m+1} \right). \quad (145) \]

Then we can define \( \mathcal{G}_n(\tilde{H}_1(t), \ldots, \tilde{H}_n(t); Z_1(t), \ldots, Z_{n-1}(t); \dot{Z}_1(t), \ldots, \dot{Z}_{n-1}(t)) \) as

\[ \sum_{m=1}^{n-1} \mathcal{G}_{n-m}(\tilde{H}_m(t), \dot{Z}_m(t); Z_1(t), \ldots, Z_{n-m}(t)) + \tilde{H}_n(t), \quad n \geq 2. \quad (146) \]
With these notations, one can write the sequence of coupled equations which gives a perturbative solution of equation (135) as follows:

\begin{align*}
\dot{Z}_1(t) &= \tilde{H}_1(t) - C_1(t), \\
\vdots \\
\dot{Z}_n(t) &= \tilde{G}_n(\tilde{H}_1(t), \ldots, \tilde{H}_n(t); Z_1(t), \ldots, Z_{n-1}(t); \dot{Z}_1(t), \ldots, \dot{Z}_{n-1}(t)) \\
&\quad - C_n(t), \quad n \geq 2,
\end{align*}

(147)

It is clear that, as in the time-independent case, this infinite set of equations can be solved recursively. Moreover, recalling the recursive process (134) that allows to obtain, at each perturbative order, the expression of \( C(\lambda; t) \), one can calculate order by order both the operators \( \{C_n(t)\}_{n \in \mathbb{N}} \) and \( \{Z_n(t)\}_{n \in \mathbb{N}} \). Indeed, integrating with respect to time each equation in the sequence (147) and combining the new sequence of equations so obtained with the recursive process (134) (or (122)), we find:

\begin{align*}
Z_1(t) &= \int_0^t (\tilde{H}_1(t) - C_1(t)) \, dt + Z_1, \\
C_1(t) &= C_1(t), \\
\vdots \\
Z_n(t) &= \int_0^t \left( \tilde{G}_n(\tilde{H}_1(t), \ldots, \tilde{H}_n(t); Z_1(t), \ldots, Z_{n-1}(t)) - C_n(t) \right) \, dt + Z_n, \\
C_n(t) &= B_n - \left( \ldots - C_{n-1}(t); \ldots, \int_0^t C_{n-1}(t) \, dt \right) + C_n(t), \quad n \geq 2,
\end{align*}

(148)

This time, differently from the equations obtained in section 3 for the time-independent case, at each perturbative order we have a couple of equations. The solution of the first couple of equations is obtained by simply choosing the arbitrary operator-valued function \( t \mapsto C_1(t) \) and the arbitrary operator \( Z_1 \); similarly, the solution of the \( n \)-th couple of equations, for \( n \geq 2 \), involves the previously computed functions \( t \mapsto Z_1(t), \ldots, t \mapsto Z_{n-1}(t) \) and requires only the choice of the arbitrary operator-valued function \( t \mapsto C_n(t) \) and of the arbitrary operator \( Z_n \). This choice can be fitted according to computational convenience. Notice that, in particular, the choice of the operators \( \{Z_n\}_{n \in \mathbb{N}} \) determines the initial condition

\[ Z(\lambda) \equiv Z(\lambda; 0) = \sum_{n=1}^{\infty} \lambda^n Z_n. \]

(149)

We will see soon that one can set quite natural conditions which fix a unique choice of these arbitrary quantities uniquely. Before doing this, we want to
show that it is possible to write a sequence of equations which is equivalent and structurally similar to the one given above but such that the solution of the \( n \)-th couple of equations — the one determining \( C_n(t) \) and \( Z_n(t) \) — does not involve explicitly the operators \( \dot{Z}_1(t), \ldots, \dot{Z}_{n-1}(t) \).

The first step is to rewrite equation (153) as

\[
\text{avxp}(i \text{ad}_{Z(\lambda,t)}) \dot{Z}(\lambda; t) = \exp(i \text{ad}_{Z(\lambda,t)}) \dot{H}(\lambda; t) - C(\lambda).
\] (150)

Next, using the relation

\[
\text{avxp}(z)^{-1} \exp(z) = \text{avxp}(-z)^{-1},
\] (151)

we have:

\[
\dot{Z}(\lambda; t) = \text{avxp}(i \text{ad}_{Z(\lambda,t)})^{-1} \left( \exp(i \text{ad}_{Z(\lambda,t)}) \dot{H}(\lambda; t) - C(\lambda) \right) = \sum_{k=0}^{\infty} \frac{i^k \beta_k}{k!} \text{ad}_{Z(\lambda,t)}^k \left( (-1)^k \dot{H}(\lambda; t) - C(\lambda) \right).
\] (152)

Recalling the fact that \( \beta_0 = 1, \beta_1 = -1/2 \) and relations (128), we can further simplify equation (152):

\[
\dot{Z}(\lambda; t) = \ddot{H}(\lambda; t) - C(\lambda) + \frac{i}{2} \text{ad}_{Z(\lambda,t)} \left( \ddot{H}(\lambda; t) + C(\lambda) \right) + \sum_{k=1}^{\infty} (-1)^k \frac{\beta_{2k}}{2k!} \text{ad}_{Z(\lambda,t)}^{2k} \left( \ddot{H}(\lambda; t) - C(\lambda) \right) + \left( 1 - \sum_{n=1}^{\infty} \frac{|\beta_{2k}|}{2k!} \text{ad}_{Z(\lambda,t)}^{2k} \left( \ddot{H}(\lambda; t) - C(\lambda) \right) \right).
\] (153)

Notice that in the time-independent case (\( H_0(\lambda; t) = H_0(\lambda) \)), assuming as in section 3 that \( [C(\lambda), H_0] = 0 \) and \( Z(\lambda; t) = \exp(iH_0t) Z(\lambda) \exp(-iH_0t) \), hence

\[
\dot{Z}(\lambda; t) = -i \text{Ad}_{\exp(iH_0t)} \text{ad}_{Z(\lambda)} H_0,
\] (154)

from equation (153), using relation (18), we find:

\[
-i \text{ad}_{Z(\lambda)} H_0 = H_0(\lambda) - C(\lambda) + \frac{i}{2} \text{ad}_{Z(\lambda,t)}(H_0(\lambda) + C(\lambda)) - \sum_{k=1}^{\infty} \frac{\beta_{2k}}{2k!} \text{ad}_{Z(\lambda,t)}^{2k} \left( H_0(\lambda) - C(\lambda) \right).
\] (155)

This equation yields another order by order solution procedure with respect to the one described in section 3. Anyway, we will not insist on this point and we leave the details to the reader.

Returning to the general time-dependent case, from equation (153), we can obtain an infinite set of equations which allows to give perturbative expressions
of $Z(\lambda; t)$, $C(\lambda)$ and can be solved recursively. To this aim, given linear operators $X$, $Y$ and $Z_1, \ldots, Z_n$, let us set

$$B_n(X, Y; Z_1, \ldots, Z_n) := \sum_{m=1}^{n} \left( \frac{i^m \beta_m}{m!} \right) \sum_{k_1 + \cdots + k_m = n} \ad_{Z_{k_1}} \cdots \ad_{Z_{k_m}} ((-1)^m X - Y),$$

(156)

for $n \geq 1$; namely, $B_1(X, Y; Z_1) = \frac{i}{2} \ad_{Z_1}(X + Y)$ and, recalling definition (152),

$$B_n(X, Y; Z_1, \ldots, Z_n) := \frac{i}{2} \ad_{Z_n}(X + Y) \right.$$  

$$+ \sum_{m=1}^{p(n)/2} \left( \frac{|\beta_{2m}|}{2m!} \right) \sum_{k_1 + \cdots + k_{2m} = n} \ad_{Z_{k_1}} \cdots \ad_{Z_{k_{2m}}}(Y - X)$$

$$= B_n^-(X; Z_1, \ldots, Z_n) + B_n^+(Y; Z_1, \ldots, Z_n),$$

(157)

for $n \geq 2$, where $p(n)$ is equal to $n$ if $n$ is even and to $n - 1$ otherwise. Then we can define, for $n \geq 2$,

$$B_n = B_n(\bar{H}_1(t), \ldots, \bar{H}_{n-1}(t); C_1(t), \ldots, C_{n-1}(t); Z_1(t), \ldots, Z_{n-1}(t))$$

as

$$B_n := \sum_{m=1}^{n-1} B_{n-m}(\bar{H}_m(t), C_m(t); Z_1(t), \ldots, Z_{n-m}(t))$$

$$= B_n^- (\ldots, \bar{H}_{n-1}(t); \ldots, Z_{n-1}(t)) + B_n^+ (\ldots, C_{n-1}(t); \ldots, Z_{n-1}(t)).$$

(158)

At this point, inserting the power expansions in equation (152) and equating the terms of the same order in the perturbative parameter $\lambda$, we obtain the following sequence of equations:

$$\dot{Z}_1(t) = \bar{H}_1(t) - C_1(\lambda; t),$$

$$C_1(t) = C_1(t),$$

$$\vdots$$

$$\dot{Z}_n(t) = B_n(\bar{H}_1(t), \ldots, \bar{H}_{n-1}(t); C_1(t), \ldots, C_{n-1}(t); Z_1(t), \ldots, Z_{n-1}(t))$$

$$+ \bar{H}_n(t) - C_n(t),$$

$$C_n(t) = B_n^-(C_1(t), \ldots, C_{n-1}(t); \int_0^t C_1(t) \, dt, \ldots, \int_0^t C_{n-1}(t) \, dt)$$

$$+ C_n(t), \quad n \geq 2,$$

(159)

$$\vdots$$

Again, this sequence of equations can be solved recursively and the solution of the $n$-th couple of equations requires simply the choice of an arbitrary operator-valued function $(t \mapsto C_n(t))$ and of an arbitrary operator $(Z_n)$.

An important class of solutions is determined by the condition

$$C_1(t) = C_1(0) \equiv C_1, \ldots, C_n(t) = C_n(0) \equiv C_n, \quad \forall t.$$  

(160)
This condition is equivalent to the following:
\[ C_1(t) = C_1(0) \equiv C_1, \ldots, C_n(t) = C_n(0) \equiv C_n, \quad \forall t. \quad (161) \]
Moreover, if this condition holds, we have:
\[ C_1 = \mathcal{C}_1, \ldots, C_n = \mathcal{C}_n, \ldots. \quad (162) \]
Then the solution of the first equation, namely
\[
Z_1(\{C_1, Z_1\}; t) = \int_0^t \tilde{H}_1(t) \, dt - t C_1 + Z_1,
\]
is fixed by the choice of the ‘arbitrary constants’ \( C_1 \) and \( Z_1 \). Inductively, the solution of the \( n \)-th equation is obtained by substituting the previously chosen arbitrary constants \( C_1, \ldots, C_{n-1} \) and the solutions
\[
t \mapsto Z_1(\{C_1, Z_1\}, t), \ldots, t \mapsto Z_{n-1}(\{C_k, Z_k\}_{k=1}^{n-1}; t)
\]
of the first \( n - 1 \) equations — that are fixed by the choice of the additional arbitrary constants \( Z_1, \ldots, Z_{n-1} \) — in the formula
\[
Z_n(\{C_k, Z_k\}_{k=1}^{n}; t) = \int_0^t B_n(\ldots, \tilde{H}_{n-1}(t); \ldots, C_{n-1}; \ldots, Z_{n-1}(\{C_k, Z_k\}_{k=1}^{n-1}; t) dt + \int_0^t \tilde{H}_n(t) dt - t C_n + Z_n,
\]
which involves the \( n \)-th order arbitrary constants \( C_n \) and \( Z_n \).

Now, as anticipated, we will give suitable conditions that fix the arbitrary constants \( \{C_n, Z_n\}_{n \in \mathbb{N}} \) — hence, a solution of the system \((159)\) — uniquely up to a certain perturbative order \( N \in \mathbb{N} \).

To this aim, it will be convenient to introduce the following notations. First, for the sake of brevity, let us define
\[
J_n(\ldots) := B_n(\ldots, \tilde{H}_{n-1}(t); \ldots, C_{n-1}; \ldots, Z_{n-1}(t)) + \tilde{H}_n(t), \quad n \geq 2. \quad (165)
\]
Next, given an analytic function \( \lambda \mapsto f(\lambda) = \sum_{n=0}^{\infty} \lambda^n f_n \), we will set
\[
f_{[N]}(\lambda) := \sum_{n=0}^{N} \lambda^n f_n. \quad (166)
\]
Moreover, given another analytic function \( \lambda \mapsto h(\lambda) \), we will set:
\[
f(\lambda) \overset{\lambda^N}{\approx} h(\lambda) \quad \overset{\text{def}}{\iff} \quad f_{[N]}(\lambda) = h_{[N]}(\lambda). \quad (167)
\]
Finally, given \( \bar{t} \in (0, \infty] \), we will say that a function \( t \mapsto F(t) \) has zero average over the time span \([0, \bar{t}]\) if
\[
\langle F(\cdot) \rangle_{\bar{t}} := \lim_{t \to \bar{t}} \frac{1}{\bar{t}} \int_0^\bar{t} F(t) \, dt = 0,
\]
28
where, obviously, the limit is essential in the previous definition only in the case where \( t = \infty \). For \( t < \infty \), equation (168) expresses the fact that the function \( t \mapsto F(t) \) has a ‘purely oscillatory behavior’ in the interval \([0, t]\).

Then, fixed a certain perturbative order \( N \) and \( \tau \in \]0, \infty[, we set the following conditions:

**C1** the following relation holds:

\[
Z_n(\tau) = Z_n, \quad n = 1, \ldots, N; \tag{169}
\]

**C2** the operator-valued function \( t \mapsto Z_n(\lambda; t), n = 1, \ldots, N, \) has zero average over the time span \([0, \tau]\).

Condition **C1** is equivalent to the following:

\[
Z_{[N]}(\lambda; \tau) = Z_{[N]}(\lambda). \tag{170}
\]

Thus, since

\[
T(\lambda; \tau) \overset{\lambda^N}{\approx} \exp(-i Z_{[N]}(\lambda; \tau)) \exp(-i C_{[N]}(\lambda) \tau) \exp(i Z_{[N]}(\lambda)) ,
\]

condition **C1** implies that

\[
T(\lambda; \tau) \overset{\lambda^N}{\approx} \exp(-i Z_{[N]}(\lambda)) \exp(-i C_{[N]}(\lambda) \tau) \exp(i Z_{[N]}(\lambda)) ,
\]

\[
\overset{\lambda^N}{\approx} \exp(-i Z(\lambda)) \exp(-i C(\lambda) \tau) \exp(i Z(\lambda)) . \tag{172}
\]

If we are able to show that condition **C1** can indeed be satisfied, this result has the following interpretation. There exists a hermitian operator \( \mathcal{H}(\lambda) \), depending analytically on the perturbative parameter \( \lambda \), such that the 1-parameter group of unitary operators generated by it interpolates, up to the \( N \)-th perturbative order, the evolutor \( T(\lambda; t) \) at \( t = \tau \); namely:

\[
T(\lambda; \tau) \overset{\lambda^N}{\approx} \exp(-i \mathcal{H}(\lambda) \tau) . \tag{173}
\]

Indeed, this relation is satisfied if we set

\[
\mathcal{H}(\lambda) = \exp(-i Z(\lambda)) C(\lambda) \exp(i Z(\lambda)). \tag{174}
\]

Now, observe that, denoted by \( \{C_n, Z_n\}_{n=1}^N \) an item of the first \( N \) arbitrary operator constants satisfying conditions **C1** and **C2**, applying condition **C1** to formula (164) yields \( C_1 = (\mathcal{H}(\cdot))_\tau \), and

\[
C_n = \frac{1}{\tau} \int_0^\tau J_n(\ldots, \tilde{H}_n(t); \ldots, \tilde{C}_{n-1}; \ldots, \tilde{Z}_{n-1} \{C_k, Z_k\}_{k=1}^{n-1} ; t) \, dt, \tag{175}
\]

for \( n = 2, \ldots, N \). Thus condition **C1** determines the operators \( \{C_n\}_{n=1}^N \) uniquely for a fixed \( N \)-tuple \( \{Z_n\}_{n=1}^N \). Suppose that \( t \mapsto \tilde{H}(\lambda; t) \) is periodic up
to the \( N \)-th perturbative order, with period \( T \) — \( \tilde{H}_{n}(\lambda; t) = \tilde{H}_{n}(\lambda; t + T) \), \( \forall t \) — or equivalently

\[
\tilde{H}_{n}(t) = \tilde{H}_{n}(t + T), \quad \forall t, \ \forall n \in \{1, \ldots, N\}.
\] (176)

Then the functions \( t \mapsto Z_{1}(C_{1}, Z_{1}; t), \ldots, t \mapsto Z_{N}(\{C_{k}, Z_{k}\}_{k=1}^{N-1}; t) \) are periodic, with period \( T \), if and only if \( C_{1} = \tau C_{1}, \ldots, C_{N} = \tau C_{N} \), with \( \tau = m_{0}T \), for some \( m_{0} \in \mathbb{N} \). In fact, we have:

\[
Z_{1}(C_{1}, Z_{1}; t + T) = \int_{0}^{T} \tilde{H}_{1}(t) \, dt + \int_{t}^{t+T} \tilde{H}_{1}(t) \, dt - (t + T) C_{1} + Z_{1}
\]

* \( \tilde{H}_{1}(\cdot) \) periodic * \( = \int_{0}^{T} \tilde{H}_{1}(t) \, dt - t C_{1} + Z_{1} + \left( \int_{0}^{T} \tilde{H}_{1}(t) \, dt - T C_{1} \right). \) (177)

Hence, the function \( t \mapsto Z_{1}(C_{1}, Z_{1}; t) \) is periodic if and only if

\[
C_{1} = \langle \tilde{H}_{1}(\cdot) \rangle_{\tau} = \tau C_{1}, \quad \text{for} \quad \tau = m_{0}T, \ m_{0} \in \mathbb{N}.
\] (178)

Next, reasoning by induction and using the fact that the function

\[
t \mapsto J_{n} \left( \tilde{H}_{1}(t); \ldots, \tilde{H}_{n}(t); \ldots, C_{n-1}; Z_{1}(C_{1}, Z_{1}; t); \ldots, Z_{n-1}(\{C_{k}, Z_{k}\}_{k=1}^{n-1}; t) \right)
\]

is periodic, with period \( T \), if the functions

\[
\tilde{H}_{1}(\cdot), \ldots, \tilde{H}_{n}(\cdot); Z_{1}(C_{1}, Z_{1}; \cdot), \ldots, Z_{n-1}(\{C_{k}, Z_{k}\}_{k=1}^{n-1}; \cdot)
\]

are periodic, with period \( T \), one proves the claim.

It follows that, if \( t \mapsto \tilde{H}(\lambda; t) \) is periodic up to the \( N \)-th order, then, setting \( \tau = m_{0}T \) in condition \( \textbf{C1} \) for some nonzero positive integer \( m_{0} \), we have:

\[
T(\lambda; mT) \cong \exp(-i S_{\lambda}(\lambda) mT), \quad m = 1, 2, \ldots
\] (179)

Condition \( \textbf{C2} \) is equivalent to the condition that the operator-valued function \( t \mapsto Z_{n}(\lambda; t) \) has zero average over the time span \( [0, \tau] \). Besides, applying condition \( \textbf{C2} \), namely

\[
\frac{1}{\tau} \int_{0}^{\tau} Z_{n}(t) \, dt = 0, \quad n = 1, \ldots, N,
\] (180)

to formula (164) yields \( \tau Z_{1} = -\langle \int_{0}^{\tau} \tilde{H}_{1}(t) \, dt \rangle_{\tau} + \frac{1}{2} \tau \tau C_{1} \) and

\[
\tau Z_{n} = -\frac{1}{\tau} \int_{0}^{\tau} \left( \int_{0}^{t} J_{n}(\ldots, \tilde{H}_{n}(t); \ldots, \tau C_{n-1}; \ldots, Z_{n-1}(\{\tau C_{k}, \tau Z_{k}\}_{k=1}^{n-1}; t)) \, dt \right) \, dt
\]

\[
+ \frac{1}{2} \tau \tau C_{n}, \quad n = 2, \ldots, N.
\] (181)

Hence, as claimed before, there is a unique \( N \)-tuple \( \{C_{n}, \tau Z_{n}\}_{n=1}^{N} \), determined by formulæ (175) and (181), satisfying conditions \( \textbf{C1} \) and \( \textbf{C2} \).
Now, let us suppose that the following limits exist:

\[ \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \dot{H}_1(t) \, dt, \]

\[ \lim_{\tau \to \infty} \left( -\frac{1}{\tau} \int_0^\tau \left( \int_0^t \dot{H}_1(t) \, dt \right) \, dt + \frac{1}{2} \tau \right)^\infty C_1, \]

\[ \vdots \]

\[ \lim_{\tau \to \infty} \left( -\frac{1}{\tau} \int_0^\tau \left( \int_0^t \dot{H}_1(t) \, dt \right) \, dt + \frac{1}{2} \tau \right)^\infty C_N. \]  

Then, the following relations hold:

\[ \lim_{t \to \infty} \frac{1}{t} Z_n(\{C_k, Z_k\}_{k=1}^n; t) = 0, \quad n = 1, \ldots, N, \]  

(183)

\[ \lim_{t \to \infty} \frac{1}{t} \int_0^t Z_n(\{C_k, Z_k\}_{k=1}^n; t) \, dt = 0, \quad n = 1, \ldots, N, \]  

(184)

where \( C_1 = \infty C_1, Z_1 = \infty Z_1, \ldots, C_N = \infty C_N, Z_N = \infty Z_N. \) Conversely, if relations (183) and (184) hold for some operators \( \{C_k, Z_k\}_{k=1}^n, \) then the limits (182) exist and \( C_1 = \infty C_1, Z_1 = \infty Z_1, \ldots, C_N = \infty C_N, Z_N = \infty Z_N. \) Thus we can include the set \( \{\infty C_k, \infty Z_k\}_{k=1}^N \) among the sets of operator constants determined by conditions \( \text{C1} \) and \( \text{C2} \) if we allow \( \tau = \infty \) and we rewrite condition \( \text{C1} \):

\text{C1}  \quad \text{the following relation holds:}

\[ \lim_{t \to \infty} \frac{1}{t} Z_n(t) = \lim_{t \to \tau} \frac{1}{t} Z_n, \quad n = 1, \ldots, N, \]  

(185)

while condition \( \text{C2} \) remains unchanged. Indeed, relation (183) reduces to equation (189) if \( \tau < \infty \) and to relation (184) if \( \tau = \infty. \) Moreover, relation (183) expresses the fact that the function \( t \mapsto Z_n(\lambda; t), \) \( n = 1, \ldots, N, \) has zero average over the time span \([0, \infty].\)

At this point, considering the time-independent case — \( H_0(t) \equiv H_0 \) and \( H_0(\lambda; t) \equiv H_0(\lambda) \) — it is natural to ask what is the relation between the solution associated with the arbitrary constants \( \{\infty C_n, \infty Z_n\}_{n \in \mathbb{N}} \) (if they exist) discussed in this section and the solutions obtained in section 3. As we have done in that section, we will assume that the unperturbed Hamiltonian \( H_0 \) has a pure point spectrum \( E_1, E_2, \ldots \) and we will denote by \( P_1, P_2, \ldots \) the associated eigenprojectors. Then, we want to prove that:

1. for any \( N \in \mathbb{N}, \) the limits (182) — thus the set of operator constants \( \{\infty C_n, \infty Z_n\}_{n \in \mathbb{N}} \) — exist;
2. the minimal solution \( \{C_n, \ Z_n\}_{n\in\mathbb{N}} \) of the sequence of equations (34), i.e. the solution obtained imposing condition (41), satisfies the relation
\[
\forall n\in\mathbb{N}; \quad C_n = \infty C_n, \quad Z_n = \infty Z_n.
\] (186)

3. the operator-valued function \( t \mapsto Z(\lambda; t) = \sum_{n=1}^{\infty} \lambda^n Z_n(t) \) which verifies conditions \( C_1 \) and \( C_2 \), with \( \tau = \infty \), is such that
\[
Z_n(t) \equiv Z_n(\{\infty C_k, \ Z_k\}_{k=1}^{n-1}; t) = e^{iH_0 t} \infty Z_n e^{-iH_0 t}, \quad \forall n \in \mathbb{N}. \tag{187}
\]

In fact, we know that \( \{C_n(t) = \infty C_n, \ Z_n(t) = e^{-iH_0 t} \infty Z_n e^{iH_0 t}\}_{n\in\mathbb{N}} \) is a solution of the sequence of equations (159) with condition (161). Observe that
\[
\lim_{t\to\infty} \frac{1}{t} \infty Z_n(t) = 0, \quad \forall n \in \mathbb{N}, \tag{188}
\]
thus condition \( C_1 \), with \( \tau = \infty \) and \( \mathbb{N} \) arbitrary, is satisfied. Moreover, given a linear operator \( X \), we have:
\[
X(t) := e^{iH_0 t} X e^{-iH_0 t}
\]
* since \([\langle X \rangle_{H_0}, H_0] = 0 \)
\[= (\langle X \rangle_{H_0} + e^{iH_0 t}) X \langle H_0, e^{-iH_0 t} \rangle\]
\[= (\langle X \rangle_{H_0} + \sum_{j\neq l} e^{i(E_j - E_l) t} P_j X P_l); \quad (189)
\]

hence:
\[
\langle X(\cdot) \rangle_{\infty} = \langle X \rangle_{H_0}. \tag{190}
\]

By condition (41), it follows that \( \langle \infty Z_n(\cdot) \rangle_{\infty} = 0, \quad n = 1, 2, \ldots \), thus condition \( C_2 \), with \( \tau = \infty \) and \( \mathbb{N} \) arbitrary, is satisfied. Then, by the uniqueness of the solution satisfying conditions \( C_1 \) and \( C_2 \), for a given \( \tau \in [0, \infty] \) and up to a certain perturbative order \( \mathbb{N} \), we must conclude that
\[
\forall n \in \mathbb{N}; \quad C_n = \infty C_n \quad \text{and} \quad Z_n(t) = Z_n(\{\infty C_k, \ Z_k\}_{k=1}^{n-1}; t), \quad \forall n \in \mathbb{N}. \tag{191}
\]

Consequently, for any \( n \in \mathbb{N} \), we have:
\[
\infty Z_n = \infty Z_n(0) = Z_n(\{\infty C_k, \ Z_k\}_{k=1}^{n-1}; 0) = \infty Z_n \tag{192}
\]

and
\[
Z_n(\{\infty C_k, \ Z_k\}_{k=1}^{n-1}; t) = \infty Z_n(t)
= e^{-iH_0 t} \infty Z_n e^{iH_0 t}
= e^{-iH_0 t} \infty Z_n e^{iH_0 t}. \tag{193}
\]

This completes our proof.
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