One-loop renormalization of general noncommutative Yang-Mills field model coupled to scalar and spinor fields

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Abstract

We study the theory of noncommutative $U(N)$ Yang-Mills field interacting with scalar and spinor fields in the fundamental and the adjoint representations. We include in the action both the terms describing interaction between the gauge and the matter fields and the terms which describe interaction among the matter fields only. Some of these interaction terms have not been considered previously in the context of noncommutative field theory. We find all counterterms for the theory to be finite in the one-loop approximation. It is shown that these counterterms allow to absorb all the divergencies by renormalization of the fields and the coupling constants, so the theory turns out to be multiplicatively renormalizable. In case of 1PI gauge field functions the result may easily be generalized on an arbitrary number of the matter fields. To generalize the results for the other 1PI functions it is necessary for the matter coupling constants to be adapted in the proper way. In some simple cases this generalization for a part of these 1PI functions is considered.
1 Introduction

Noncommutative field theories have been attracting great attention for the past few years. Interest in these theories began with the discovery of their relation to string theory (see [1] and references therein). Apart from the string theory interest they are interesting on their own as a sufficiently consistent non-local quantum field model. (see reviews [2, 3, 4]).

Noncommutativity has some important consequences. Two main consequences are a restriction on the gauge group [5, 6] and charge quantization [4, 13]. One of the consequences at quantum level is the so-called UV/IR mixing. Although the limit $\theta^{\mu\nu} \rightarrow 0$ (1) ($\theta^{\mu\nu}$ are the noncommutativity parameters) reduces a classical noncommutative theory to its commutative counterpart, at the quantum level this is not the case due to the UV/IR mixing [7, 8, 9, 10, 11]. This phenomenon of mixing of UV and IR singularities appears in the so-called nonplanar diagrams: some of the UV singularities of a commutative theory convert in IR singularities in its noncommutative counterpart. So, contributions of the nonplanar diagrams to the effective action are singular in $\theta^{\mu\nu}p^\nu$ ($p$ is external momenta). These divergencies are interpreted as IR ones [7] and UV singularities of the noncommutative theories are not the same as in their commutative counterparts. As a consequence, it may violate renormalizability of the noncommutative field theories. Although there is a general statement that a noncommutative field theory should be renormalizable if its commutative counterpart is renormalizable (see e.g. [12] and the reviews [2, 3, 4]) we need an explicit check to support this statement in each new concrete model (see the discussion of this point in review [3]). By now, as far as the nonsupersymmetric field theories are concerned, it has been checked by direct calculations two-loop renormalizability of $\phi_4^4$ theory [13, 14] and one loop renormalizability both pure noncommutative $U(N)$ gauge theory [15, 16, 17] and noncommutative $U(N)$ gauge theory interacting with the fermionic field in the fundamental representation [19] and the bosonic field in the adjoint representation [20] separately. We are going to consider here renormalizability of a general theory of a noncommutative $U(N)$ gauge field interacting with matter fields. But in contrast to the previous works where Yang-Mills field interacts with only a single kind of matter field we consider a most general action and include the scalar and the spinor fields both in the fundamental and in the adjoint representations. The action also contains terms which describe interaction among the matter fields and some of them have not been considered previously.

Also we point out the activity concerning the supersymmetric field theories. There exist two approaches: the fermionic coordinates of a superspace may be endowed with noncomutativity [24] or not [25]. The second approach is more usual one. Different quantum properties of matter and gauge fields have been investigated both for $\mathcal{N} = 1$ (see e.g. [23, 27, 18]) and extended supersymmetric theories (see e.g. [28, 30, 31, 32]).

This paper is organized as follows. In next section we briefly review basic properties of noncommutative field theories and also fix the notation and the action to be studied. The action contains the scalar and the spinor fields both in the fundamental and the adjoint representations and terms describing interaction among the fields. In section 3 we find all counterterms needed to cancel the divergencies of the theory in the one-loop approximation. It is shown that these counterterms allow us to carry out renormalization of the fields and the coupling constants of the theory. Thus, the theory is multiplicatively renormalizable in the one-loop approximation. We also discuss the generalization of the

\[1\text{We do not take into account gauge groups which are only constructed perturbatively in the noncommutativity parameter. Discussion of such field theories see i.e. in [33, 34, 35, 36, 37].}\]
theory for an arbitrary number of the matter fields. In the Appendix we write out the propagators and the vertices of the theory. The calculations are given using the dimensional regularization and standard methods of quantum field theory. We do not consider the details of the calculations and present only the final results.

2 The Model

We start this section with a brief formulation of some basic properties of noncommutative field theories. As it well known that a noncommutative field theory may be constructed from commutative field theory by replacing the usual product of the fields by the star one

\[ f \cdot g \rightarrow (f \star g)(x) = \exp\left(\frac{i}{2} \theta^{\mu \nu} \partial^\mu \partial^\nu\right) f(x + u) g(x + v) \bigg|_{u=v=0} \neq (g \star f)(x), \quad (1) \]

where the constants \( \theta^{\mu \nu} \) are the noncommutativity parameters.

As was shown in \[5,6\], the only possible gauge group admitting simple noncommutative extension (all pointwise products are replaced by the star one) for a noncommutative gauge field theory is \( U(N) \). Matter fields may transform or in the fundamental representation

\[ \phi'_i(x) = U^i_j(x) \star \phi_j(x), \quad i, j = 1, \ldots, N, \]

or in the adjoint representation

\[ \Phi^a_{ij}(x) = U^k_j(x) \star \Phi^a_k(x) \star U^{+ i}_{+ m}(x), \quad U^k_j \star U^{+ i k} = U^{+ k j} \star U^i_k = \delta^i_j. \]

The covariant derivatives are defined as follows

\[
\begin{align*}
D_\mu \phi_i &= \partial_\mu \phi_i - ig A^{\mu \star}_{\mu i} \star \phi_j, \\
D_\mu \Phi^a_{ij} &= \partial_\mu \Phi^a_{ij} - ig A^{\mu \star}_{\mu k} \Phi^a_{kj} + ig \Phi^a_{ik} \star A^{\mu \star}_{\mu k} \\
&= \partial_\mu \Phi^a_{ij} - ig [A_\mu, \Phi]^a_{ij}
\end{align*}
\]

for the fundamental and the adjoint representations respectively. Under the gauge transformation these covariant derivatives transform as

\[
\begin{align*}
D'_\mu \phi'_i &= U^j_i \star D_\mu \phi_j, \\
D'_\mu \Phi'^a_{ij} &= U^k_j \star D_\mu \Phi^a_{kj} \star U^{+ i}_{+ m}
\end{align*}
\]

if the gauge field \( A \) has the transformation law

\[ A'^{\mu i}_{\mu j} = U^k_j \star A^{\mu \star}_{\mu k} \star U^{+ i}_{+ m} - \frac{i}{g} \partial_\mu U^k_j \star U^{+ i k}. \quad (2) \]

As a consequence, the field strength takes the form

\[ F^{\mu \nu}_{\mu \nu} = \partial_\mu A^{\mu \star}_{\nu j} - \partial_\nu A^{\mu \star}_{\mu j} - ig A^{\mu \star}_{\mu j} \star A^{\mu \star}_{\nu k} + ig A^{\mu \star}_{\nu j} \star A^{\mu \star}_{\mu k} \]

and has the following transformation law

\[ F'_{\mu \nu} = U \star F^{\mu \nu} \star U^+. \]

\[2\]In principle, the matter fields can also belong to antifundamental representation \[19\]. However we do not consider this case here.
Hereafter we shall often omit matrix indices. From the transformation law (2) we see $A$ may be restricted to be selfconjugated $(A^i_\mu)^* = A^j_\mu$.

Now we can write down the action of the theory which we are going to study

$$S_{cl} = \int d^4x \left[ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi} i\gamma^\mu D_\mu \Psi - M_1 \bar{\Psi} \Psi \right]$$

+ $\bar{\psi} i\gamma^\mu D_\mu \psi - m_1 \bar{\psi} \psi$

+ $D_\mu \phi^+ D^\mu \phi - \frac{\lambda_1}{4!} \phi^+ \phi^+ \phi^+ \phi - M^2 \phi^+ \phi$

+ $\frac{\lambda_2}{4!} \text{tr} \left[ \phi^+ \phi^+ \phi^+ \phi \right] - \frac{\lambda_3}{4!} \text{tr} \left[ \phi^+ \phi^+ \phi^+ \phi^+ \right]$

+ $- f_\alpha \phi^+ \phi^+ \phi - f_\beta \phi^+ \phi^+ \phi^+ - h \bar{\psi} \Psi \phi^+ \phi$.

Here $\Psi$ is a fermionic field in the adjoint representation, $\psi$ is a fermionic field in the fundamental representation, $\Phi$ is a bosonic field in the adjoint representation and $\phi$ is a bosonic field in the fundamental representation. In comparison with the works [13, 20, 21] we have included in the action (3) scalar and spinor fields both in the fundamental and the adjoint representations. We have also included in the action (3) terms which describe interaction among the matter fields allowed by symmetry and reality conditions. Since in the literature only one matter field has been studied to be coupled to a gauge field, the terms with $f_\alpha$, $f_\beta$, and $h$ have never been considered. We use the couplings $f_\alpha$, $f_\beta$, and $\lambda_{2\alpha}$, $\lambda_{2\beta}$ as independent in contrast to the works [3, 9, 20]. Of course, we could include in the action (3) some more interaction terms (for example $\phi^+ \phi^3 + c.c.$) but in order to preserve multiplicative renormalizability it is necessary to consider in the action (3) mass-like terms containing $\phi^2 + c.c.$ which would complicate a consideration.

The infinitesimal symmetry transformations of the action have the form

$$U = \exp igT(x) = 1 + igT(x) + \frac{1}{2} igT(x) \times igT(x) + \ldots$$

$$\delta \psi = igT \psi, \qquad \delta \psi^+ = -ig\bar{\psi}^+ T, \qquad T^+ = T,$$

$$\delta \Psi = ig[T, \Psi], \qquad \delta \bar{\Psi} = ig[T, \bar{\Psi}],$$

$$\delta \phi = igT \phi, \qquad \delta \phi^+ = -ig\phi^+ \times T,$$

$$\delta \Phi = ig[T, \Phi], \qquad \delta \Phi^+ = ig[T, \Phi^+].$$

$$\delta A_\mu = \partial_\mu T(x) - ig[A_\mu, T],$$

$$\delta F_{\mu\nu} = ig[T, F_{\mu\nu}].$$

For any field $f$ one has

$$(\delta T_1 \delta T_2 - \delta T_2 \delta T_1) f = \delta T_3 f,$$

$T_3 = ig[T_1, T_2].$

We quantize this theory using the Faddeev-Popov method, by introducing the ghost field $C$ and antighost field $\bar{C}$, adding the ghost action and the gauge-fixing term (we use the Lorentz gauge) to the initial action (3). Then the action we quantize reads

$$S = S_{cl} + S_{GF+FP},$$

$$S_{GF+FP} = - \int d^4x \text{tr} \left( \frac{1}{2\alpha} (\partial^\mu A_\mu)^2 + \bar{C} \partial^\mu D_\mu C \right).$$
The aim of our analysis below is to calculate all one-loop divergencies and to check the multiplicative renormalizability of the theory in the one-loop approximation.

## 3 Renormalization of the one-loop effective action

Let $\Phi^A$ denote all the fields in the theory $\Phi^A = (\phi, \phi^+, \Phi, \Phi^+, A, C, \bar{C}, \bar{C}, \psi, \bar{\psi}, \Psi, \bar{\Psi})$, let bosonic part of these fields be $\varphi^i = (\phi, \phi^+, \Phi, \Phi^+, A, C)$ and fermionic part be $\theta^\alpha = (\bar{C}, \psi, \bar{\psi}, \Psi, \bar{\Psi})$. ($A$, $i$ and $\alpha$ are condensed indices which include discrete indices and space-time coordinates. Both summing and integration over repeated indices are assumed.) We use the background field method and split the action (4) into two parts $S_0$ and $V$, where $S_0$ is quadratic in its fields and $V$ is the rest of the total action (4) both depend on arbitrary background fields $\bar{\Phi}$ and quantum fields $\Phi$. Then, we have (up to a constant)

$$e^{i\Gamma_1} = \int D\Phi^A e^{\frac{i}{2} S_{AB}(\bar{\Phi})\Phi^B\Phi^A},$$

where $\Gamma_1$ is the one-loop effective action (EA) and all derivatives in fields are right $S_{AB}(\bar{\Phi}) = \frac{\delta^r}{\delta \bar{\Phi}^B} \delta^r S(\bar{\Phi})$. We rewrite $S_{AB}\Phi^B\Phi^A$ as

$$\frac{1}{2} S_{AB}(\bar{\Phi})\Phi^B\Phi^A = \frac{1}{2} S_{(ij)}(\bar{\Phi})\varphi^i\varphi^j + \frac{1}{2} S_{[\beta\alpha]}(\bar{\Phi})\theta^\alpha\theta^\beta + S_{(i\alpha)}(\bar{\Phi})\theta^\alpha \varphi^i = \frac{1}{2} S_{(ij)}(\bar{\Phi})\tilde{\varphi}^i\tilde{\varphi}^j + \frac{1}{2} \tilde{S}_{[\beta\alpha]}(\bar{\Phi})\theta^\alpha\theta^\beta,$$

where

$$\tilde{\varphi}^i = \varphi^i + G^{ik} S_{ko}\theta^\alpha, \quad S_{ij} G^{jk} = \delta^k_i.$$ $S_{ij}$ and $S_{ij}$ are condensed indices which include discrete indices and space-time coordinates. Here $G^{ij}$, $S_{i\alpha}$ and $S_{ij}$ depend on background fields $\bar{\Phi}$. After these redefinitions we get Gaussian functional integral and can integrate over bosonic and fermionic fields respectively. As a result we have (up to a constant)

$$\Gamma_1 = \int \frac{i}{2} \text{Tr} \left[ \ln S_{ij}(\bar{\Phi}) - \ln S_{0ij} \right] - \frac{i}{2} \text{Tr} \left[ \ln \bar{S}_{[\beta\alpha]}(\bar{\Phi}) - \ln \bar{S}_{(0\beta\alpha)} \right]. \quad (5)$$

Let us consider the first term in the rhs (5) and do the following transformations

$$\frac{i}{2} \text{Tr} \left[ \ln S_{ij}(\bar{\Phi}) - \ln S_{0ij} \right] = \frac{i}{2} \text{Tr} \left[ \ln S_{0mn}(\delta^m_j) + G_{0}^{mk} V_{kj}(\bar{\Phi}) \right] - \frac{i}{2} \text{Tr} \ln(\delta^i + G^{ik} V_{kj}(\bar{\Phi})) = - \frac{i}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{Tr} \left( G_{0}^{ik} V_{kj}(\bar{\Phi}) \right)^n. \quad (6)$$

Here $G_{0}^{ik}$ are propagators for the bosonic fields $S_{0ij} G_{0}^{ik} = \delta^k_i$. Then one can show by dimensional analysis, the divergences in (5) may be originated only in the first four terms

$$\frac{i}{2} G_{0}^{ik} V_{ki}, \quad - \frac{i}{4} G_{0}^{ik} V_{kj} G_{0}^{jn} V_{ni}, \quad \frac{i}{6} G_{0}^{ik} V_{kj} G_{0}^{jn} V_{nm} G_{0}^{ml} V_{li}, \quad - \frac{i}{8} G_{0}^{ik} V_{kj} G_{0}^{jn} V_{nm} G_{0}^{ml} V_{lp} V_{pq} G_{0}^{rs} V_{si},$$
where \( V_{ij} \) depends on background fields \( \Phi \). Doing similar procedure for the second term in the rhs of (3) we find that divergences may be originated in the following terms

\[
-\frac{i}{2} G_0^{\alpha\beta} W_{\beta\alpha}, \quad \frac{i}{4} G_0^{\alpha\beta} W_{\beta\gamma} G_0^{\gamma\delta} W_{\delta\alpha}, \\
-\frac{i}{6} G_0^{\alpha\beta} W_{\beta\gamma} G_0^{\gamma\delta} W_{\delta\epsilon} G_0^{\epsilon\sigma} W_{\sigma\alpha}, \\
-\frac{i}{8} G_0^{\alpha\beta} W_{\beta\gamma} G_0^{\gamma\delta} W_{\delta\epsilon} G_0^{\epsilon\sigma} W_{\sigma\tau} G_0^{\tau\rho} W_{\rho\alpha}.
\]

Here \( G_0^{\alpha\beta} \) are propagators for the fermionic fields \( S_{0\alpha\beta} G_0^{\beta\gamma} = \delta^{\gamma}_{\alpha} \) and \( W_{\beta\alpha} = V_{[\beta\alpha]} + G^{ij} V_{i[\alpha} V_{\beta]j} \) depending on background fields \( \Phi \). To simplify calculations we perform the Fourier transformation of the propagators and the vertices

\[
G(x, x') = \int \left( \frac{dp}{2\pi} \right)^d \left( \frac{dp'}{2\pi} \right)^d e^{ipx + ip'x'} G(p, p') \equiv \int_{pp'} e^{ipx + ip'x'} G(p, p'), \\
V(x, x') = \int_{pp'} e^{-ipx - ip'x'} V(p, p').
\]

Resulting propagators \( G_0^{AB'} \) and vertices \( V_{AB'} \) have been written out in Appendix A.

Let us review some more properties of the noncommutative field theories. As it may easily be seen from the definition of the star product (1), there is the following identity

\[
\int d^d x (\phi_1 \star \phi_2)(x) = \int d^d x \phi_1(x) \phi_2(x), \quad (7)
\]

which is proved with the help of integration by parts and the assumption that the functions \( \phi_1(x) \) and \( \phi_2(x) \) have the proper asymptotic conditions. From this identity follows that the quadratic part of the action for any noncommutative theory is the same as that in its commutative counterpart. And as a consequence the propagators of the noncommutative theory and the commutative one coincide. The only thing which is modified is the interaction. After Fourier transform of the fields

\[
\phi(x) = \int \left( \frac{dp}{2\pi} \right)^d e^{ipx} \tilde{\phi}(p) \equiv \int_p e^{ipx} \tilde{\phi}(p)
\]

any interaction term gets an additional momentum dependence \( V \)

\[
\int d^d x (\phi_1 \star \phi_2 \star \ldots \star \phi_n)(x) = \int_{p_1 \ldots p_n} \tilde{\delta}(p_1 + p_2 + \ldots + p_n) \tilde{\phi}(p_1) \ldots \tilde{\phi}(p_n) V(p_1, \ldots, p_n)
\]

\[
V(p_1, \ldots, p_n) = e^{-\frac{i}{4} \sum_{j=1}^n p_1^\mu p_2^\nu} \equiv p_\mu \theta^{\mu\nu} k_\nu.
\]

Due to this factor some diagrams become finite. Consider a simple example of a one-loop graph. Let it contains two vertices with three fields in each one

\[
\int d^d x \phi_{i_{1a}} \phi_{i_{2a}} \phi_{i_{3a}}(x) \quad \text{and} \quad \int d^d x' \phi_{j_{1a}} \phi_{j_{2a}} \phi_{j_{3a}}(x').
\]

Here \( \phi \) are some fields and \( i \) and \( j \) are the group indices. To get a one-loop graph from these vertices we need to contract two fields from one vertex with two fields from another one. With the help of the cyclic property of the star product

\[
\int d^d x \phi_1 \star \phi_2 \star \ldots \star \phi_n = \int d^d x \phi_2 \star \ldots \star \phi_n \star \phi_1
\]
which follows from (7), the first contraction may always be done between the last field of the first vertex and the first field of the second one. In momentum space this reads

\[
\int_{p_1, p_2, k_1, k_2} \delta(p_1 + p_2 + p_3) \delta(k_1 + k_2 + k_3) \phi^{i_1}_{1i_3}(p_1) \phi^{i_2}_{2i_1}(p_2) \times \\
\times < \phi^{i_3}_{3i_2}(p_3) \phi^{j_1}_{4j_3}(k_1) > \phi^{j_2}_{5j_1}(k_2) \phi^{j_3}_{6j_2}(k_3) V(p_1, p_2, p_3) V(k_1, k_2, k_3).
\]

Any contraction of the fields has the following form

\[
< \phi^{i_3}_{3i_2}(p_3) \phi^{j_1}_{4j_3}(k_1) >= \delta^{i_3}_{j_3} \delta^{j_2}_{i_3} \delta(p_3 + k_1) G(p_3).
\]

Integrating over \(k_1\) and replacing \(p_3 \to p\) one gets

\[
\int_{p_1, p_2, k_1, k_3} \delta(p_1 + p_2 + p) \delta(-p + k_2 + k_3) \phi^{i_1}_{1i_3}(p_1) \phi^{i_2}_{2i_1}(p_2) \phi^{j_2}_{5j_2}(k_2) \phi^{i_3}_{6i_3}(k_3) \times \\
\times V(p_1, p_2, p) V(-p, k_2, k_3) G(p) = \\
= \int_{p_1, p_2, k_3} \delta(p_1 + p_2 + k_3) \phi^{i_1}_{1i_3}(p_1) \phi^{i_2}_{2i_1}(p_2) \phi^{j_2}_{5j_2}(k_2) \phi^{i_3}_{6i_3}(k_3) V(p_1, p_2, k_2, k_3) \times \\
\times \int_{p} \delta(p_1 + p_2 + p) G(p) \tag{9}
\]

Here we use the equality \(V(p_1, \ldots, p_{n-1}, p) V(-p, k_2, \ldots, k_n) = V(p_1, \ldots, p_{n-1}, k_2, \ldots, k_n)\) which follows from the definition of \(V\) (8) and the delta functions \(\delta(p_1 + \ldots + p_{n-1} + p), \delta(-p + k_2 + \ldots + k_n)\). Note that the group indices in (8) are contracted as a trace of product of all the fields.

To get a one-loop graph which is sufficient for our purpose, we need one more contraction of the fields. This may be done by several ways. If we contract neighbor fields (i.e. 2 and 5 or 1 and 6 assuming the cyclic property of the trace over the group indices) we will get so-called ”planar” diagram which has the UV divergences and its trace is over the product of all the fields of the graph. Consider for example contraction of fields 1 and 6. One has

\[
< \phi^{i_1}_{1i_3}(p_1) \phi^{i_2}_{6i_3}(k_3) >= \delta^{i_1}_{i_3} \delta^{i_2}_{i_3} \delta(p_1 + k_3) G'(p_1).
\]

and after integration over \(k_3\) and replacement \(p_1 \to p'\)

\[
\delta^{i_3}_{i_3} \int_{p'p_2k_2} \delta(p_2 + k_2) \phi^{i_2}_{2i_1}(p_2) \phi^{j_2}_{5j_2}(k_2) V(p', p_2, k_2, -p') \times \\
\times \int_{p} \delta(p_2 + p + p') G(p) G'(p') = \\
= N \int_{p_2k_2} \delta(p_2 + k_2) \phi^{i_2}_{2i_1}(p_2) \phi^{j_2}_{5j_2}(k_2) V(p_2, k_2, ) \times \\
\times \int_{p'p'} \delta(p_2 + p + p') G(p) G'(p') \tag{10}
\]

The last line of (10) is a usual one-loop UV divergent integral.

Another variant of contraction of the fields are 1 and 5 or 2 and 6. In these cases we get so-called ”nonplanar” diagrams. Let us contract the fields 2 and 6. One has

\[
< \phi^{i_2}_{2i_1}(p_2) \phi^{i_3}_{6i_3}(k_3) >= \delta^{i_2}_{i_4} \delta^{i_3}_{i_4} \delta(p_2 + k_3) G''(p_2).
\]
After integration over $k_3$ and replacement $p_2 \to p'$ we have
\[
\int_{p_1 k_2} \tilde{\delta}(p_1 + k_2) \tilde{\phi}_{i_1 i_2}^i(p_1) \tilde{\phi}_{5i_2}^j(k_2) V(p_1, p', k_2, -p') \times \\
\times \int_{p} \tilde{\delta}(p_1 + p + p') G(p) G''(p') = \\
= \int_{p_1 k_2} \tilde{\delta}(p_1 + k_2) \tilde{\phi}_{i_1 i_2}^i(p_1) \tilde{\phi}_{5i_2}^j(k_2) V(p_1, k_2, ) \times \\
\times \int_{pp'} \tilde{\delta}(p_1 + p + p') G(p) G''(p') e^{ip'/\theta k_2} 
\]
(11)

Here we see two features differing planar diagrams (10) from nonplanar ones (11). The first feature is the presence of the exponential factor $e^{ip'/\theta k_2}$ in (11). Namely this factor makes nonplanar diagrams finite. And the second feature is that in the nonplanar diagrams the group indices trace is not over the product of all the fields of the diagram. Due to this we shall have for example equations like (14,18,19). More detailed information on this subject may be found i.g. in refs. [7, 12, 23].

3.1 Two-point gauge field function

The diagrams which give the one-loop correction to the gauge field self-energy are shown in Figure 1. Note that we may generalize the consideration of the two-point gauge field function to an arbitrary number of the matter fields. Let $n_f$ be the number of the fermionic fields in the fundamental representation, $n_F$ be the number of the fermionic fields in the adjoint representation, $n_b$ be the number of the bosonic fields in the fundamental representation, $n_B$ be the number of the bosonic fields in the adjoint representation. The tadpole diagram with a gauge field loop (Fig.1b) has no UV divergence. Using the minimal substraction scheme and the dimensional regularization we find that the other diagrams give the following contributions to the one-loop counterterm
\[
S_{1A^2} = \frac{1}{(4\pi)^2} \int d^d x \text{tr} \left[ (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \right] \times \\
n_a + c d e f + g h + i \times \left[ N(3\alpha - 13) + 4n_f + 8Nn_F + n_b + 2Nn_B \right]. \tag{12}
\]

As a consequence, the renormalizations of $A$ and $\alpha$ are easily found
\[
\tilde{A}_\mu = Z_A A_\mu \quad Z_A = 1 + \frac{1}{(4\pi)^2} \frac{1}{d - 4} [n_b + 4n_f + N(3\alpha - 13 + 2n_B + 8n_F)], \tag{13}
\]
\[
\alpha = Z_\alpha \quad Z_\alpha = Z_A^2.
\]

More precisely, $n_f$ is the number of multiplets (N fields in each) of the fermionic fields in the fundamental representation. Using QCD terminology, $n_f$ is the number of flavours, $N$ is the number of colours. For the other fields situation is similar.
Here the bare quantities are labeled with \( \circ \) mark. From (13) we see that the renormalization of \( SU(N) \) part of the gauge fields is the same as in commutative \( SU(N) \) gauge theory with the same matter field content.

Also note here that the non-planar contributions of these diagrams have the following structure

\[
\int_{k_1 k_2 } \delta(k_1 + k_2) \, \text{tr}A_\alpha(k_1) \, \text{tr}A_\beta(k_2) \int \mathcal{F}^{\alpha \beta}(k, k_1, k_2), \tag{14}
\]

where \( \mathcal{F}^{\alpha \beta}(k, k_1, k_2) \) are some functions. If we denote \( T_0 \) and \( T_a \) to be generators of \( U(1) \) and \( SU(N) \) groups respectively (\( U(N) = U(1) \times SU(N) \)), then we will see that only \( U(1) \) part (and not \( SU(N) \) part) of \( U(N) \) group contributes to (14) due to tracelessness of \( T_a \). So term (14) with UV/IR mixing depends on \( U(1) \) part of the gauge fields only.

### 3.2 Three- and four-point gauge field functions

The diagrams which have UV divergent contributions to the three- and four-point gauge field functions are shown in Figures 2 and 3 respectively. As in the case of the two-point

\[
\begin{align*}
S_{1A^3} &= \frac{1}{(4\pi)^2 d - 4} \int d^d x \text{tr} (\partial^\mu A^\nu \ast [A_\mu, A_\nu]) \times \\
&\quad (a + b + c) \quad d \quad e \quad f \quad g \\
&\quad \times [N(9\alpha - 17) + 8n_f + 16Nn_F + 2n_b + 4Nn_B]
\end{align*}
\tag{15}
\]

for the counterterm proportional to \( A^3 \) and

\[
\begin{align*}
S_{1A^4} &= \frac{1}{(4\pi)^2 d - 4} \int d^d x \text{tr} (A_\mu \ast A_\nu \ast [A^\mu, A^\nu]) \times \\
&\quad (a + b + c + d) \quad e \quad f \quad g + h + i \quad j + k + l \\
&\quad \times [N(6\alpha - 4) + 4n_f + 8Nn_F + n_b + 2Nn_B]
\end{align*}
\tag{16}
\]
for the four-point counterterm. From (13) we get the renormalization of the gauge field coupling constant (we keep all the renormalized coupling constants to be dimensionless) which is the same as in commutative $SU(N)$ gauge theory with the same matter field content

$$\hat{g} = \mu^\frac{4-d}{2} Z_g g = 1 + \frac{1}{(4\pi)^2} \frac{g^2}{d-4} \left[ N(22 - 2n_B - 8n_F) - n_b - 4n_f \right],$$ (17)

where $\mu$ is an arbitrary parameter with dimension of mass. As far as the counterterm (16) is concerned it is absorbed by the renormalization of the gauge field (13) and the gauge coupling constant (17).

Note here that the structure of the non-planar contributions to the three-point 1PI gauge field function has the form

$$\int_{k_1, k_2, k_3} \tilde{\delta}(k_1 + k_2 + k_3) \operatorname{tr}(\tilde{A}_\alpha(k_1)\tilde{A}_\beta(k_2)) \operatorname{tr}\tilde{A}_\gamma(k_3) \int_k f_1^{\alpha\beta\gamma}(k, k_1, k_2, k_3)$$ (18)

$$+ \int_{k_1, k_2, k_3} \tilde{\delta}(k_1 + k_2 + k_3) \operatorname{tr}\tilde{A}_\alpha(k_1) \operatorname{tr}\tilde{A}_\beta(k_2) \operatorname{tr}\tilde{A}_\gamma(k_3) \int_k f_2^{\alpha\beta\gamma}(k, k_1, k_2, k_3)$$ (19)

($f_1^{\alpha\beta\gamma}$ are some functions) and as in the case of the two-point 1PI gauge field function there is no pure $SU(N)$ contribution here (18,19) due to tracelessness of generators $T_a$ of $SU(N)$ group.

Also note that the quantity

$$Z_g Z_A = 1 + \frac{1}{(4\pi)^2} \frac{g^2 N}{d-4} (\alpha + 3)$$ (20)

is independent of the presence of the matter fields (as well as in the commutative $SU(N)$ gauge field theory).

### 3.3 1PI functions with ghost field external lines

There is only one diagram contributing in the two-point 1PI ghost field function. It is shown in Figure 4 and the counterterm which results from it has the form

$$S_{1C^2} = -\frac{1}{(4\pi)^2} \frac{g^2 N}{d-4} (3 - \alpha) \int d^d x \operatorname{tr}(C \partial^2 C).$$ (21)

The renormalization of the ghost fields is easily found from (21)

$$\hat{C} = Z_C C \quad Z_C = 1 + \frac{1}{(4\pi)^2} \frac{g^2 N}{d-4} \frac{\alpha - 3}{2}.$$ (22)

The number of all the fields in the adjoint representation (including the ghost fields) is greater by one in comparison with its number in commutative $SU(N)$ gauge field theory.
due to the existence one more field corresponding $U(1)$ generator of $U(N)$ group. The renormalization of $SU(N)$ part of the ghost fields in the noncommutative case is the same as in the commutative $SU(N)$ gauge field theory.

Turning to the three-point function of the ghost field coupled to the gauge field. The relevant diagrams are shown in Figure 5 which result in the following counterterm

$$S_{1C^2A} = \frac{1}{(4\pi)^2} \frac{2i\alpha g^3 N}{d-4} \int d^d x \, \text{tr} \left( \tilde{C} \star \partial^\mu [A_\mu, C] \right).$$

This counterterm is absorbed by the renormalization of the fields and the gauge coupling constant $(\alpha, 22)$ and does not violate multiplicative renormalizability of the theory and $U(N)$ gauge invariance at the one-loop level.

### 3.4 1PI functions with gauge field and fermion external lines

Let us first consider 1PI two-point function of the fermion in the fundamental representation. There are only two diagrams contributing to this function (Fig.6), which have no non-planar contributions (and as a consequence there is no UV/IR mixing here). They lead to the counterterms

$$S_{1\psi^2} = \frac{1}{(4\pi)^2} \frac{N}{d-4} \left(2\alpha g^2 - |h|^2\right) \int d^d x \, \bar{\psi} i\gamma^\mu \partial_\mu \psi $$

$$+ \frac{1}{(4\pi)^2} \frac{-2N}{d-4} \left(m_1(\alpha + 3)g^2 + M_1|h|^2\right) \int d^d x \, \bar{\psi} \psi.$$

which have the same structure as in commutative $SU(N)$ theory but differ by the numerical coefficients.

Now we try to generalize the result to the case of an arbitrary number of the matter fields. The relevant part of the classical action should have the following form

$$\int d^d x \left( \sum_{A=1}^{n_f} \bar{\psi}_A \star i\gamma^\mu D_\mu \psi_A - \sum_{A,A'=1}^{n_f} \bar{\psi}_A m_{1AA'} \psi_{A'} \right)$$

$$- \sum_{A=1}^{n_f} \sum_{B=1}^{n_f} \sum_{C=1}^{n_f} \int d^d x \left( h_{ABC} \bar{\psi}_A \star \Psi_B \star \phi_C + h^*_{ABC} \phi^*_C \star \bar{\Psi}_B \star \psi_A \right). \quad (23)$$
In general case the mass matrix $m_{1AA'}$ is a constant hermitian matrix.

Let us briefly describe the general structure of renormalization of the field $\bar{\psi}_A$ and its mass matrix. After calculating the one-loop counterterms the relevant part of the classical action plus the counterterms have the form

$$
\int d^d x \sum_{A,A'=1}^{n_f} \left\{ \bar{\psi}_A \left[ \delta_{AA'} + \frac{1}{d-4} E_{AA'} \right] i\gamma^\mu \partial_\mu \psi_{A'} - \bar{\psi}_A \left[ m_{1AA'} + \frac{1}{d-4} M_{AA'} \right] \psi_{A'} \right\}.
$$

(24)

Here $E_{AA'}$ and $M_{AA'}$ are some constant hermitian matrices $E_{AA'} = E_{A'A}$, $M_{AA'} = M_{A'A}$ generated by the divergences. From the first term of (24) we get the renormalization of the field

$$
\bar{\psi}_A = \sum_{A'=1}^{n_f} \left( \delta_{AA'} + \frac{1}{d-4} E_{AA'} \right) \psi_{A'}.
$$

Having expressed the renormalized field $\psi_A$ from the bare one $\bar{\psi}_A$ we substitute $\psi_A$ to (24) and get the mass term in the form

$$
- \sum_{A,A',A''=1}^{n_f} \int d^d x \bar{\psi}_A \left[ m_{1AA'} + \frac{1}{d-4} \left( M_{AA'} - \frac{1}{2} E_{AA'} m_{1A'A'} - \frac{1}{2} m_{1AA''} E_{AA''} \right) \right] \psi_{A''}
$$

From this expression we see that the renormalization of the mass matrix looks like

$$
m_{1AA'} = m_{1AA'} + \frac{1}{d-4} \left[ M_{AA'} - \frac{1}{2} \sum_{A''=1}^{n_f} \left( E_{AA'} m_{1A'A'} + m_{1AA''} E_{AA''} \right) \right].
$$

Note that if we assume that the renormalized mass matrix is diagonal $m_{1AA'} = m_{1A} \delta_{AA'}$, then in general case the bare mass matrix $\bar{m}_{1AA'}$ can’t be diagonal since neither $M_{AA'}$ nor $E_{AA'}$ must be diagonal. It should also be noted that the above general structure of the renormalizations of the field and the mass matrix is independent of whether the theory is noncommutative or not.

Further we will discuss mainly the features associated with relationship between renormalizations of commutative and noncommutative theories. To avoid the unessential complications and tedious relations and understand how mass renormalization is organized in noncommutative models we consider a special situation when the bare mass matrix is diagonal $\bar{m}_{1AA'} = m_{1A} \delta_{AA'}$ and the corresponding renormalized matrix is also diagonal $m_{1AA'} = m_{1A} \delta_{AA'}$.

Taking into account the above assumption we get the relevant part of the classical action plus the counterterms in the form (24) where

$$
E_{AA'} = \frac{N}{(4\pi)^2} \left( 2\alpha g^2 \delta_{AA'} - \sum_{B=1}^{n_F} \sum_{C=1}^{n_b} h_{ABC} h_{A'BC}^* \right),
$$

$$
M_{AA'} = \frac{2N}{(4\pi)^2} \left( m_{1AG}^2 (3 + \alpha) \delta_{AA'} + \sum_{B=1}^{n_F} \sum_{C=1}^{n_b} M_{1B} h_{ABC} h_{A'BC}^* \right).
$$

(25)

From (25) we see that $E_{AA'}$ and $M_{AA'}$ are not diagonal and if we demand the bare mass matrix $\bar{m}_{1AA'}$ to be diagonal then we must impose the following restrictions on the parameters of the theory

$$
\sum_{B=1}^{n_F} \sum_{C=1}^{n_b} \left( 4M_{1B} + m_{1A} + m_{1A'} \right) h_{ABC} h_{A'BC}^* = C_A \delta_{AA'},
$$

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with $C_A$ being some quantities. Since all the diagrams contributing to this 1PI functions are planar, these restrictions are the same as in the corresponding commutative theory. Therefore the interaction (23) should be adapted in the proper way or be discarded completely.

In the case of a single fermion field in the fundamental representation we have

\[
\bar{\psi} = Z_\psi \psi \quad Z_\psi = 1 + \frac{1}{(4\pi)^2 d - 4} N \left( \alpha g^2 - \frac{1}{2} \sum_{B=1}^{n_F} \sum_{C=1}^{n_h} |h_{BC}|^2 \right), \tag{26}
\]

\[
\bar{m}_1 = m_1 + \frac{1}{(4\pi)^2 d - 4} N \left[ 6g^2 m_1 + \sum_{B=1}^{n_F} \sum_{C=1}^{n_h} (m_1 + 2M_{1B}) |h_{BC}|^2 \right]. \tag{27}
\]

Note here that the renormalization of the fermionic field in the fundamental representation and its mass have the same structure as in the commutative theory and the same numerical coefficients due to the absence of nonplanar diagrams contributing to the 1PI function in this case. It is interesting to point out that although the interaction Lagrangians in commutative and noncommutative theories differ, the renormalization relations (26,27) under the above restrictions, turned out to be the same in both cases.

Let us examine the fermion-gauge field vertex. The relevant diagrams are shown in Figure 7. Their contributions in the case of a single fermion field in the fundamental representation is

\[
S_{1\psi^2 A} = \frac{1}{(4\pi)^2 d - 4} gN \left( 3g^2(\alpha + 1) - \sum_{B=1}^{n_F} \sum_{C=1}^{n_h} |h_{BC}|^2 \right) \int d^d x \bar{\psi} \gamma^\mu A_\mu \psi.
\]

Since the renormalization of the fields $\psi$ and $A_\mu$ and the gauge coupling constant $g$ have already been done, in the general case this counterterm may break the multiplicative renormalizability of the theory. But this does not happen due to the preservation of $U(N)$ gauge invariance at the one-loop level and it is absorbed by the renormalization of the spinor and the gauge fields and the renormalization of the gauge coupling constant (21,27).

Note here that nonplanar contributions to this three-point 1PI function are independent of $SU(N)$ part of the gauge fields.

Similar situation arises for the fermion field in the adjoint representation. The diagrams are shown in Figure 8 and 9. The counterterms coming from these diagrams in case of one fermion field in the adjoint representation are

\[
\frac{1}{(4\pi)^2 d - 4} \left( 4\alpha g^2 N - \sum_{A=1}^{n_f} \sum_{C=1}^{n_h} |h_{AC}|^2 \right) \int d^d x \text{tr} \left( \bar{\Psi} i\gamma^\mu \partial_\mu \Psi \right) \tag{28}
\]

\[
+ \frac{-1}{(4\pi)^2 d - 4} \left( 4(\alpha + 3) g^2 N M_1 + 2 \sum_{A=1}^{n_f} \sum_{C=1}^{n_h} m_{1A} |h_{AC}|^2 \right) \int d^d x \text{tr} \left( \bar{\Psi} \Psi \right) \tag{29}
\]

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Figure 8: Diagrams contributing to the two-point function of a fermion field in the adjoint representation

Figure 9: Diagrams contributing to the three-point function of a fermion field in the adjoint representation coupling to gauge field

\[ + \frac{1}{(4\pi)^2 d - 4} \left( (3 + 5\alpha)g^2N - \sum_{A=1}^{n_f} \sum_{C=1}^{n_b} |h_{AC}|^2 \right) \int d^d x \text{tr} \left( \bar{\Psi} \gamma^\mu [A_\mu, \Psi] \right). \]  

(30)

From (28) we have the renormalization of the field \( \Psi \)

\[ \hat{\Psi} = Z_\Psi \Psi \quad Z_\Psi = 1 + \frac{1}{(4\pi)^2 d - 4} \left( 2\alpha g^2N - \frac{1}{2} \sum_{A=1}^{n_f} \sum_{C=1}^{n_b} |h_{AC}|^2 \right), \]  

(31)

and from (29) we have the renormalization of the mass \( M_1 \)

\[ \hat{M}_1 = M_1 + \frac{1}{(4\pi)^2 d - 4} \left[ 12g^2NM_1 + \sum_{A=1}^{n_f} \sum_{C=1}^{n_b} (2m_{1A} + M_1) |h_{AC}|^2 \right]. \]  

(32)

Thus, we see, the fermion masses (27, 32) are mixed with each other only in the presence of the boson-fermion interaction (23). It should be also noted that renormalization of the \( SU(N) \) part of the fermionic field in the adjoint representation and its mass are the same as in the commutative case. Counterterm (30), as it may easily be checked, is absorbed by the renormalization (20, 31).

As far as the nonplanar contributions to these 1PI functions are concerned their structure is similar to (14) and (18, 19) for the cases of the two- and three-point functions respectively. And, as a consequence, the nonplanar contribution to the two-point function depends on the \( U(1) \) parts of the fields only and nonplanar contribution to the three-point function has no pure \( SU(N) \) field dependence.

### 3.5 1PI functions with gauge field and boson external lines

For the boson field in the fundamental representation the diagrams corresponding to its 1PI two-point function are shown in Figure 10, which result in the counterterms

\[ + \frac{1}{(4\pi)^2 d - 4} \left( \frac{2N}{2} \left( g^2(\alpha - 3) + 2|h|^2 \right) \int d^d x \partial^\mu \phi^+ \partial_\mu \phi \right. \]

\[ + \frac{1}{(4\pi)^2 d - 4} \left( \frac{1}{3!} \frac{\lambda_1}{m_2^2} (N + 1) + 2N(f_a + f_b)M_2^2 - 2\alpha g^2Nm_2^2 \right. \]

\[ - 8N|h|^2(m_1^2 + m_1M_1 + M_1^2) \left) \int d^d x \phi^+ \phi. \]  

13
has the following form. First of all we must write down the relevant part of the classical action. It
This 1PI function as well as 1PI function of the fermionic field in the fundamental repre-
and then the renormalization of its mass matrix
Here
Doing similar calculations as in case of the field $\psi$ and the bosonic field in the fundamental representation
run from 1 to $n_F$, C run from 1 to $n_b$, D run from 1 to $n_B$. In expression \( f_{AC_1D_1C_2} \) and \( f_{BC_1D_1C_2} \) are real constants and $\lambda_1$ has the symmetry
which follows from the reality condition of the action and properties of the star-product.
The structure of renormalization of the field $\phi_C$ and its mass matrix is similar to that in case of the fermionic field in the fundamental representation which was described earlier. It is also independent of whether the theory is noncommutative or not. To simplify the calculations we assume like in section 3.4 that both bare and renormalized mass matrices are diagonal $\hat{m}_{2CC'} = \hat{m}_{2CC'}^2$, $m_{2CC'} = m_{2CC'}^2$. Then the relevant part of the classical action plus the counterterms have the form
\[
\int d^dx \left\{ D\mu \phi_C^+ D\mu \phi_C - \phi_C^+ m_{2CC'}^2 \phi_{C'} - \frac{\lambda_{1C_1C_2C_3C_4}}{4!} \phi_C^+ \phi_{C_2} \phi_{C_3} \phi_{C_4} - f_{AC_1D_1C_2} \phi_C^+ \Phi_D^+ \phi_{C_2} - f_{BC_1D_1C_2} \phi_C^+ \Phi_D^+ \phi_{C_2} - h_{ABC} \bar{\psi}_A \Psi_B \phi_C - h^*_{ABC} \bar{\phi}_C^+ \Psi_B \phi_A \right\}.
\]
Hereafter summing over repeated indices $A,B,C,D$ is assumed. Indices $A$ run from 1 to $n_f$, B run from 1 to $n_f$, C run from 1 to $n_b$, D run from 1 to $n_B$. In expression \( f_{AC_1D_1C_2} \) and \( f_{BC_1D_1C_2} \) are real constants and $\lambda_1$ has the symmetry
\[
\lambda_{1C_1C_2C_3C_4} = \lambda_{1C_1C_2C_1C_4} = \lambda_{1C_1C_3C_1C_4} = \lambda_{1C_1C_3C_2C_1},
\]

This 1PI function as well as 1PI function of the fermionic field in the fundamental representation has no nonplanar contribution.

Let us try to generalize these counterterms to the case of an arbitrary number of the matter fields. First of all we must write down the relevant part of the classical action. It has the following form
\[
\int d^dx \left\{ D\mu \phi_C^+ D\mu \phi_C - \phi_C^+ m_{2CC'}^2 \phi_{C'} - \frac{\lambda_{1C_1C_2C_3C_4}}{4!} \phi_C^+ \phi_{C_2} \phi_{C_3} \phi_{C_4} - f_{AC_1D_1C_2} \phi_C^+ \Phi_D^+ \phi_{C_2} - f_{BC_1D_1C_2} \phi_C^+ \Phi_D^+ \phi_{C_2} - h_{ABC} \bar{\psi}_A \Psi_B \phi_C - h^*_{ABC} \bar{\phi}_C^+ \Psi_B \phi_A \right\}.
\]

\[
\int d^dx \partial^\mu \phi_C^+ \partial^\mu \phi_{C_2} \left( \phi_C^+ \phi_{C_2} + \frac{1}{d-4} E_{C_1C_2} \right) - \phi_C^+ \phi_{C_2} \left( m_{2C_1C_2}^2 \delta_{C_1C_2} + \frac{1}{d-4} M_{C_1C_2} \right).
\]

Here
\[
E_{C_1C_2} = \frac{2N}{(4\pi)^2} \left( g^2 (3 - \pi) \delta_{C_1C_2} + 2h^*_{ABC} h_{ABC_2} \right),
\]
\[
M_{C_1C_2} = \frac{1}{(4\pi)^2} \left( 2g^2 N m_{2C_1C_2}^2 \delta_{C_1C_2} + 8Nh^*_{ABC} h_{ABC_2} (m_{1A}^2 + m_{1A} M_{1A} + M_{1A}^2) \right)
\]
\[
-2NM_{2D}^2 (f_{AC_1D_1C_2} + f_{BC_1D_1C_2}) - \frac{m_{2C}^2}{3!} (N \lambda_{1CC_1C_2} + \lambda_{1C_1CC_2_2}).
\]

Doing similar calculations as in case of the field $\psi_A$ we at first get the renormalization of the bosonic field in the fundamental representation
\[
\circ \phi_C = \left( \delta_{CC'} + \frac{1}{d-4} \frac{1}{2} E_{CC'} \right) \phi_{C'}.
\]

and then the renormalization of its mass matrix
\[
\hat{m}_{2CC'} = m_{2CC'}^2 \delta_{CC'} + \frac{1}{d-4} \left[ M_{CC'} - \frac{1}{2} m_{2C}^2 E_{CC'} - \frac{1}{2} E_{CC'} m_{2C'}^2 \right].
\]
If we demand the bare mass matrix \( \tilde{m}^2 \) to be also diagonal we must impose the following restrictions on the parameters of the theory

\[
2Nh_{ABC}h_{ABC} \left( 4m_{1A}^2 + 4m_{1A}M_{1B} + 4M_{1B}^2 - m_{2C1}^2 - m_{2C2}^2 \right) \\
-2NM_{2D}^2 \left( f_{aC} + f_{bC} \right) \frac{m_{2C}^2}{3!} \left( N\lambda_{1CCC1C2} + \lambda_{1C1C2} \right) = A\delta_{CC'},
\]

with \( A_C \) being some quantities and summing over indices A, B, C and D is assumed. This situation is completely analogous to that in the commutative theory. Since there are no nonplanar diagrams contributing to the 1PI function under consideration then the same restriction arise in the corresponding commutative theory.

In the following we shall not discuss the generalization of the theory to an arbitrary number of the matter fields. In the case of one field of each type we have the renormalization of \( \phi \) and \( m_2 \)

\[
\tilde{\phi} = Z_\phi \phi, \quad Z_\phi = 1 + \frac{1}{(4\pi)^2} \frac{N}{d-4} \left[ (\alpha - 3)g^2 + 2|h|^2 \right],
\]

\[
\tilde{m}_2^2 = m_2^2 + \frac{1}{(4\pi)^2} \frac{1}{d-4} \left[ 4N|h|^2(2m_1^2 + 2m_1M_1 + 2M_1^2 - m_2^2) + 6g^2Nm_2^2 \right. \\
\left. -2N(f_a + f_b)M_2^2 - \frac{\lambda_1}{3!}(N + 1)m_2^2 \right].
\]

The renormalization of the bosonic field in the fundamental representation and its mass are the same as in the corresponding commutative theory due to the absence of the nonplanar diagrams contributing to the relevant 1PI function.

Since we have already renormalized the gauge field coupling, the gauge field and the bosonic field in the fundamental representation we should check that these renormalization relations absorb the divergencies of the three- and four-point 1PI functions. The divergent diagrams corresponding to three-point function are shown in Figure 11. Summing up the contributions of these diagrams we find the counterterm

\[
\frac{1}{(4\pi)^2} \frac{igN}{d-4} \left( \begin{array}{cccc}
a + b & c & d \\
-3g^2 & +3\alpha g^2 & +4|h|^2 \end{array} \right) \int d^d x \left[ \phi^+ \ast A_\mu \ast \partial^\mu \phi - \partial^\mu \phi^+ \ast A_\mu \ast \phi \right],
\]

which is absorbed by the renormalization of the fields and the gauge coupling constant \(^{(24, 34)}\).

The diagrams contributing to the four-point function are shown in Figure 12.\(^4\) Diagnostics which are not shown in Figure 12 are non-planar.
g and h, j and k, i and l, cancel each other. The others give the following contribution to the counterterm

\[ \frac{1}{16 \pi^2} \left( \frac{g^2 N}{d-4} \right) \int d^4 x \phi^+ \ast A_\mu \ast A^\mu \ast \phi \ast \right. 
\times \left( \begin{array}{cccc}
-a & b & c & d \\
-\frac{3}{2}g^2(3 + \alpha^2) & -\frac{1}{2}g^2(3 + \alpha) & + \frac{3}{2}g^2(4 + \alpha + \alpha^2) & +4|h|^2 & +3\alpha g^2 & \\
\end{array} \right), \]

which is also absorbed by the renormalization of the fields and the gauge coupling constant \( (24,34) \).

For the case of the boson field in the adjoint representation we have diagrams in Figure 13 for the two-point 1PI function. The resulting counterterm and renormalization

relations of the field \( \Phi \) and the mass \( M_2 \) are

\[ \frac{1}{16 \pi^2} \left( \frac{4g^2 N}{d-4} \right) \int d^4 x \text{tr} \left( \partial^\mu \Phi^+ \partial_\mu \Phi \right) \]

\[ + \frac{1}{16 \pi^2} \left( \frac{1}{d-4} \right) \left[ 2m_2^2(f_a + f_b) + NM_2^2 \left( \frac{1}{3!} (2\lambda_2 a + \lambda_2 b) - 4\alpha g^2 \right) \right] \int d^4 x \text{tr} \left( \Phi^+ \Phi \right), \]

\[ Z_\Phi = Z_\Phi \Phi \]

\[ Z_\Phi = 1 + \frac{1}{16 \pi^2} \left( \frac{2g^2 N}{d-4} \right) (\alpha - 3), \]

\[ M_2^2 = M_2^2 + \frac{1}{16 \pi^2} \left( \frac{1}{d-4} \right) \left[ 12g^2 NM_2^2 - \frac{1}{3!} N(2\lambda_2 a + \lambda_2 b)M_2^2 - 2(f_a + f_b)m_2^2 \right]. \]

The renormalization of \( SU(N) \) part of the bosonic field in the adjoint representation is the same as in the commutative \( SU(N) \) gauge field theory with the same matter field content.
Since the three- and four-point 1PI functions depend only on the gauge coupling constant, the gauge field and the bosonic field in the adjoint representation for which renormalization has already been done we need to check that the divergences of these 1PI functions are absorbed by the fields and the gauge coupling constant renormalization. Corresponding diagrams are shown in Figure 14 for the three-point function and Figure 15 for the four-point function. Calculating these diagrams one obtains

\[
S_{1Φ^2A} = \frac{1}{(4\pi)^2} \frac{ig^3N}{d-4} \int d^4x \text{tr} \left( [Φ^+, A_μ] * \partial^μ Φ - \partial^μ Φ^+ * [A_μ, Φ] \right) \times \\
\text{a + b + c + d + e + f + g + h + i + j + k + l + m + n + o + p + q + r + s + t + u + v + w + x + y + z}
\]

For the four-point function contributions of h and i, j and o, k and l, m and n diagrams cancel each other. From these expressions for the counterterms we see that these counter-

\[
S_{1Φ^2A^2} = \frac{1}{(4\pi)^2} \frac{6g^4N}{d-4} (1 - α) \int d^4x \text{tr} \left( [A^μ, Φ^+] * [A_μ, Φ] \right)
\]

\[
(a) = \frac{1}{(4\pi)^2} \frac{g^4N}{d-4} \frac{9 + 3α^2}{2} \int d^4x \text{tr} \left( [A^μ, Φ^+] * [A_μ, Φ] - 2Φ^+ * A_μ * Φ * A^μ \right)
\]

\[
(b) = + \frac{1}{(4\pi)^2} \frac{g^4N}{d-4} \frac{3 + 3α^2}{2} \int d^4x \text{tr} \left( 5[A^μ, Φ^+] * [A_μ, Φ] - 2Φ^+ * A_μ * Φ * A^μ \right)
\]

\[
(c) = + \frac{1}{(4\pi)^2} \frac{g^4N}{d-4} \frac{-12 - 3α - 3α^2}{2} \int d^4x \text{tr} \left( [A^μ, Φ^+] * [A_μ, Φ] - 2Φ^+ * A_μ * Φ * A^μ \right)
\]

\[
(d + e(= d)) = + \frac{1}{(4\pi)^2} \frac{g^4N}{d-4} (-3α) \int d^4x \text{tr} \left( [A^μ, Φ^+] * [A_μ, Φ] + 2Φ^+ * A_μ * Φ * A^μ \right)
\]

\[
(f + g(= f)) = + \frac{1}{(4\pi)^2} \frac{g^4N}{d-4} (-4α) \int d^4x \text{tr} \left( [A^μ, Φ^+] * [A_μ, Φ] - Φ^+ * A_μ * Φ * A^μ \right)
\]
terms are absorbed by the renormalization of the fields and the gauge coupling constant \( \alpha \).

The other 1PI functions can’t destroy multiplicative renormalization of the theory, since they can be absorbed by the renormalization of the coupling constants of the matter fields for which renormalization has not been done yet and we may always absorb the divergences by their renormalization. The rest of the counterterms are

\[
1 \left( \frac{2\lambda_1}{4!} \right)^2 + N(f_a^2 + f_b^2) - \frac{\alpha g^2 N\lambda_1}{3!} \\
+ 3g^4N - 4|h|^4N \right] \int d^4 x \phi^+ \star \phi \star \phi^+ \star \phi
\]

\[
+ \frac{1}{(4\pi)^2 d - 4} \left[ 6g^4N + f_a^2 + f_b^2 + \frac{2}{4!^2}N(4\lambda_{2a}^2 + \lambda_{2b}^2) \\
- \frac{1}{3}\alpha g^2 N\lambda_{2a} \right] \int d^4 x \text{tr} \left[ \Phi^+ \star \Phi \star \Phi^+ \star \Phi \right]
\]

\[
+ \frac{1}{(4\pi)^2 d - 4} \left[ 6g^4N + 2f_a f_b + \frac{2}{4!^2}N\lambda_{2a} (4\lambda_{2a} + \lambda_{2b}) - \frac{1}{3}\alpha g^2 N\lambda_{2b} \right] \\
+ 2N \left( \frac{|\lambda_3|}{3!} \right)^2 \int d^4 x \text{tr} \left[ \Phi^+ \star \Phi^+ \star \Phi \star \Phi \right]
\]

\[
+ \frac{1}{(4\pi)^2 d - 4} \left[ 6g^4N - 6\alpha f_a g^2N + \frac{1}{3!}\lambda_1 f_a + 2f_a^2N \\
+ \frac{2}{4!} N(2f_a\lambda_{2a} + f_b\lambda_{2b}) \right] \int d^4 x \phi^+ \star \Phi \star \Phi^+ \star \phi
\]

\[
+ \frac{1}{(4\pi)^2 d - 4} \left[ 6g^4N - 6\alpha f_b g^2N + \frac{1}{3!}\lambda_1 f_b + 2f_b^2N \\
+ \frac{2}{4!} N(2f_b\lambda_{2a} + f_a\lambda_{2b}) \right] \int d^4 x \phi^+ \star \Phi^+ \star \Phi \star \phi
\]

\[
+ \frac{1}{(4\pi)^2 d - 4} \frac{2N}{3!} \left( \frac{\lambda_{2b}}{4!} - \frac{\alpha g^2}{4!} \right) \int d^4 x \text{tr} \left[ \Phi \star \Phi \star \Phi \star \Phi \right]
\]

\[
+ \frac{1}{(4\pi)^2 d - 4} \frac{2N}{3!} \left( \frac{\lambda_{2a}}{4!} - \frac{\alpha g^2}{4!} \right) \int d^4 x \text{tr} \left[ \Phi^+ \star \Phi^+ \star \Phi^+ \star \Phi^+ \right]
\]

\[
+ \frac{1}{(4\pi)^2 d - 4} \frac{g^2 Nh}{(6 + 4\alpha)} \int d^4 x \bar{\psi} \star \Psi \star \phi
\]

\[
+ \frac{1}{(4\pi)^2 d - 4} \frac{g^2 Nh^*}{(6 + 4\alpha)} \int d^4 x \phi^+ \star \bar{\Psi} \star \psi.
\]

These counterterms lead to the following renormalization of the coupling constants of the matter fields

\[
\mu^{d-4} \left( 1 \right)^{\phi} = \frac{\lambda_1}{4!} + \frac{1}{(4\pi)^2 d - 4} \left[ 4|h|^4N - 3g^4N + \frac{1}{2}g^2 N\lambda_1 - \frac{1}{3}|h|^2 N\lambda_1 - N(f_a^2 + f_b^2) \\
- (N + 1) \left( \frac{2\lambda_1}{4!} \right)^2 \right],
\]

\[18\]
\[
\mu^{d-4} \frac{\lambda_{2a}}{4!} = \frac{\lambda_{2a}}{4!} + \frac{1}{(4\pi)^2} \frac{1}{d-4} \left[ g^2 N \lambda_{2a} - \frac{2}{(4\pi)^2} N \left(4\lambda_{2a}^2 + \lambda_{2b}^2\right) - f_a^2 - f_b^2 - 6g^4 N \right],
\]
\[
\mu^{d-4} \frac{\lambda_{2b}}{4!} = \frac{\lambda_{2b}}{4!} + \frac{1}{(4\pi)^2} \frac{1}{d-4} \left[ g^2 N \lambda_{2b} - \frac{2}{(4\pi)^2} N \lambda_{2b} \left(4\lambda_{2a} + \lambda_{2b}\right) - 2f_a f_b \right.
\]
\[
-6g^4 N - 2N \left(\frac{1}{3!}\right)^2 \left(\lambda_3\right)^2 \right],
\]
\[
\mu^{d-4} f_a = f_a + \frac{1}{(4\pi)^2} \frac{1}{d-4} \left[ 18g^2 N f_a - 6g^4 N - \frac{\lambda_1}{3!} f_a - 2f_a^2 N - 4|h|^2 N f_a \right.
\]
\[
- \frac{2}{4!} N (2f_a \lambda_{2a} + f_b \lambda_{2b}) \right],
\]
\[
\mu^{d-4} f_b = f_b + \frac{1}{(4\pi)^2} \frac{1}{d-4} \left[ 18g^2 N f_b - 6g^4 N - \frac{\lambda_1}{3!} f_b - 2f_b^2 N - 4|h|^2 N f_b \right.
\]
\[
- \frac{2}{4!} N (2f_b \lambda_{2a} + f_a \lambda_{2b}) \right],
\]
\[
h = \mu^{d-4} Z_h h \quad Z_h = 1 + \frac{1}{(4\pi)^2} \frac{1}{d-4} \left[ \frac{1}{3} |h|^2 (1 - 3N) - g^2 N (3 + 8\alpha) \right], \quad (39)
\]
\[
\lambda_3 = \mu^{d-4} Z_{\lambda_3} \lambda_3 \quad Z_{\lambda_3} = 1 + \frac{1}{(4\pi)^2} \frac{N}{d-4} \left[ 24g^2 - \frac{1}{3} \lambda_{2b} \right]. \quad (40)
\]

As a result we see that the theory under consideration is multiplicatively renormalizable in the one-loop approximation. If we consider \( f_a, f_b \) and \( \lambda_{2a}, \lambda_{2b} \) to be not independent [3, 4, 20],
\[
f_a \rightarrow f_{a_1}, \quad f_b \rightarrow f_{b_1}, \quad a_1 + b_1 = 1,
\]
\[
\lambda_{2a} \rightarrow \lambda_{2a_2}, \quad \lambda_{2b} \rightarrow \lambda_{2b_2}, \quad a_2 + b_2 = 1,
\]

where \( a \) and \( b \) are real numbers (which are not renormalized), then the theory will be renormalizable if we put the following restrictions on these numbers
\[
a_1 = b_1, \quad a_2 = b_2, \quad \lambda_3 = 0.
\]

From the above formulae (39, 40) we see that if we would like to reduce the number of interactions without breaking multiplicative renormalizability we could neglect only \( h \) and \( \lambda_3 \) couplings. It should be noted that these formulae of renormalization of the matter fields coupling constants have never been written out in explicit form in the literature. For example in the works [3, 20] only the structure of the divergencies (37, 38) was discussed.

### 4 Summary

We have studied the one-loop renormalizability of the general noncommutative Yang-Mills field coupled to different kinds of matter fields interacting among themselves.

Unlike all the previous works we have included in the action the scalar and the spinor matter fields both in the fundamental and in the adjoint representations. The action also contains some new terms describing interaction among the matter fields which have not been considered previously in the context of the noncommutative field theories. Naturally, inclusion of any new term in the action may influence on renormalizability of the
theory. To prove the theory is one-loop multiplicatively renormalizable we computed all counterterms needed to cancel the one-loop divergences of the effective action. The formal structure of the counterterms of the noncommutative theory is the same as the formal structure of the corresponding counterterms of the commutative theory but pointwise multiplication of the fields is replaced by the star product. One more distinctive feature of the counterterms is the numerical factors in the renormalization constants which differ from the corresponding factors of the commutative theory due to the appearance of nonplanar diagrams. The number of diagrams contributing to a given 1PI function is the same as in a commutative theory and its noncommutative counterpart. But some of the diagrams of the noncommutative theory are nonplanar and so have no UV divergences. This leads to difference of the numerical factors in the renormalization constants. Since the numerical factors are changed multiplicative renormalizability of the theory may be destroyed but it does not happen due to the preservation of $U(N)$ gauge invariance of the model at the one-loop level.

We have also shown that the result for pure gauge field 1PI function may be generalized to the case of an arbitrary number of the matter fields. All results concerning the renormalization of the fields and coupling constants agree with the previous results in the literature and include them as a partial case.

Our calculations, in the framework of a general model confirm the specific features of noncommutative field theories which were found within the various simple models. The number of UV divergent diagrams is reduced due to the appearance of the nonplanar diagrams which are considered to be UV finite.

On the whole, we have established the one-loop multiplicative renormalizability of general noncommutative Yang-Mills field model interacting with the matter fields. At different values of its parameters, this model is reduced to a number of various concrete models. Therefore the results obtained here allow us to find the one-loop counterterms for many concrete noncommutative field theories.

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**A  Feynman rules**

**Propagators**

\[
A_{\mu j}^i(k) \sim \sim \sim \sim A_{\nu j'}^{i'}(k') = G_0(A_{\mu j}^i(k); A_{\nu j'}^{i'}(k')) \\
= \tilde{\delta}(k + k')\delta_{j j'}(\cdots) [-g_{\mu\nu} \frac{k^2}{k^4} + (1 - \alpha) \frac{k_{\mu} k_{\nu}}{k^4}]
\]

\[
\bar{C}_{\alpha s_1}^{r_1}(l_1) \sim \sim \sim \sim \sim C_{\alpha s_2}^{r_2}(l_2) = G_0(\bar{C}_{\alpha s_1}^{r_1}(l_1); C_{\alpha s_2}^{r_2}(l_2)) = \frac{\tilde{\delta}(l_1 + l_2)\delta_{s_1 s_2}^{r_1 r_2}}{l_1^2}
\]

\[
\bar{\Psi}_{l_1}^{bk_1}(Q_1) - - \rightarrow \rightarrow - - \Psi_{l_2}^{bk_2}(Q_2) = G_0(\bar{\Psi}_{l_1}^{bk_1}(Q_1); \Psi_{l_2}^{bk_2}(Q_2))
\]
Here we have kept only terms which give contributions in the planar diagrams.

\[ \Phi^m_{n_1}(P_1) \quad \Phi^m_{n_2}(P_2) = \frac{\delta(P_1 + P_2)\delta^m_{n_1}\delta^m_{n_2}}{P_1^2 - m_2^2} \]

\[ \phi^m(p_1) \quad \phi_n(p_2) = \frac{\delta(p_1 + p_2)\delta^m_n}{p_1^2 - m_2^2} \]

\[ \tilde{\delta}(k + p) = (2\pi)^d\delta(k + p) \]

**Vertices**

Here we have kept only terms which give contributions in the planar diagrams.

\[ A_{\alpha j}^i(k) \quad A_{\beta j'}^{i'}(k') = V(A_{\alpha j}^i(k); A_{\beta j'}^{i'}(k')) \]

\[ = g \int_{k_1 + k_2} \delta(k + k + k_1) \left[ \delta^j_i \tilde{A}^j_{\alpha i'}(k_1)e^{\frac{ik\theta k_1}{2}} - \delta^j_i \tilde{A}^j_{\beta i'}(k_1)e^{-\frac{ik\theta k_1}{2}} \right] \]

\[ \times \left[ g_{\alpha\beta}(k^\mu - k'^\mu) + \delta^\mu_{\alpha'}(k_{1\beta} - k_{1\beta}) + \delta^\mu_{\beta'}(k_{1\alpha} - k_{1\alpha}) \right] + g^2 \int_{k_1 + k_2} \delta(k + k + k_1 + k_2) \left\{ e^{\frac{ik\theta (k_1 + k_2)}{2}} - e^{\frac{ik\theta (k_1)}{2}} \delta^j_i \right\} \]

\[ \times \left[ 2\tilde{A}^m_{\alpha i'}(k_1)\tilde{A}^m_{\beta m}(k_2) - g_{\alpha\beta}\tilde{A}^m_{\alpha i'}(k_1)\tilde{A}^m_{\beta m}(k_2) \right] \]

\[ - g_{\alpha\beta}\tilde{A}^m_{\alpha i'}(k_1)\tilde{A}^m_{\beta m}(k_2) - \tilde{A}^m_{\alpha i'}(k_1)\tilde{A}^m_{\beta m}(k_2) \]

\[ + g^2 \int_{p_1 + p_2} \delta(k + k + p_1 + p_2)g_{\alpha\beta}e^{\frac{ip_1\theta p_2}{2}} \]

\[ \left[ \delta^j_i \tilde{\phi}^j_{i'}(p_1)\tilde{\phi}^j_{i'}(p_2)e^{\frac{ip_1\theta (p_1 + p_2)}{2}} + \delta^j_i \tilde{\phi}^j_{i'}(p_1)\tilde{\phi}^j_{i'}(p_2)e^{-\frac{ip_1\theta (p_1 + p_2)}{2}} \right] \]

\[ + g^2 \int_{p_1 + p_2} \delta(k + k + p_1 + p_2)g_{\alpha\beta} \left\{ e^{\frac{ip_1\theta p_2}{2}} - e^{\frac{ip_1\theta p_2}{2}} \delta^j_i \right\} \]

\[ \times \left[ \tilde{A}^m_{\alpha i'}(P_1)\tilde{A}^m_{\beta m}(P_2)e^{-\frac{iP_1\theta P_2}{2}} + \tilde{A}^m_{\alpha i'}(P_1)\tilde{A}^m_{\beta m}(P_2)e^{\frac{iP_1\theta P_2}{2}} \right] \]

\[ + e^{-\frac{iP_1\theta (P_1 + P_2)}{2}} \delta^j_i \]

\[ \times \left[ \tilde{A}^m_{\alpha i'}(P_1)\tilde{A}^m_{\beta m}(P_2)e^{-\frac{iP_1\theta P_2}{2}} + \tilde{A}^m_{\alpha i'}(P_1)\tilde{A}^m_{\beta m}(P_2)e^{\frac{iP_1\theta P_2}{2}} \right] \]

\[ A_{\mu j}^i(k) \quad - \tilde{\psi}^{bk}(q_1) = V(A_{\mu j}^i(k), \tilde{\psi}^{bk}(q_1)) \]

\[ = -g \int_{q_2} \delta(q_1 + q_2 + k) \gamma^b_{\alpha}\delta^j_i \tilde{\psi}_{\alpha i}(q_2)e^{\frac{iq_1\theta q_2}{2}} \]
\[ A_{\mu j}^i(k) \sim \psi_{al}(q_2) = V(A_{\mu j}^i(k), \psi_{al}(q_2)) \]
\[ = g \int_{q_1} \delta(q_1 + q_2 + k) \gamma_{\mu b}^i \delta_{l_2}^i \bar{\psi}_{bj}(q_1) e^{i\theta\theta_2} \]
\[ A_{\mu j}^i(k) \sim \Psi_{l_1}^{b k_1}(Q_1) = V(A_{\mu j}^i(k), \bar{\Psi}_{l_1}^{b k_1}(Q_1)) \]
\[ = q \int_{Q_2} \delta(Q_1 + Q_2 + k) \gamma_{\mu b}^i \left[ \delta_{l_2}^i \bar{\Psi}_{l_1}(Q_2) e^{-iQ_1\theta Q_2} - \delta_{k_1}^i \bar{\Psi}_{l_1}(Q_2) e^{iQ_1\theta Q_2} \right] \]
\[ A_{\mu j}^i(k) \sim \Psi_{l_2}^{k_2}(Q_2) = V(A_{\mu j}^i(k), \Psi_{l_2}^{k_2}(Q_2)) \]
\[ = g \int_{Q_1} \delta(Q_1 + Q_2 + k) \gamma_{\mu b}^i \left[ \delta_{l_2}^i \bar{\Psi}_{l_2}(Q_1) e^{iQ_1\theta Q_2} - \delta_{k_2}^i \bar{\Psi}_{l_2}(Q_1) e^{-iQ_1\theta Q_2} \right] \]
\[ A_{\mu j}^i(k) \sim \phi^m(p_1) = V(A_{\mu j}^i(k), \phi^m(p_1)) \]
\[ = g \int_{p_2} \delta(p_1 + p_2 + k) \delta_m \tilde{\phi}(p_2) (p_{1}^\mu - p_{2}^\mu) e^{i\theta p_1 p_2} \]
\[ + g^2 \int_{p_2 k_1} \delta(p_1 + p_2 + k + k_1) \delta_m \tilde{\phi}(p_2) \tilde{A}^{\mu_1}(k_1) \tilde{\phi}_{m_1}(p_2) e^{-i\theta(k_1 + p_1) - i\theta p_1 k_1} \]
\[ A_{\mu j}^i(k) \sim \phi_n(p_2) = V(A_{\mu j}^i(k), \phi_n(p_2)) \]
\[ = g \int_{p_1} \delta(p_1 + p_2 + k) \delta_n \tilde{\phi}(p_1) (p_{1}^\mu - p_{2}^\mu) e^{i\theta p_1 p_2} \]
\[ + g^2 \int_{p_1 k_1} \delta(p_1 + p_2 + k + k_1) \delta_n \tilde{\phi}(p_1) \tilde{A}^{\mu_1}(k_1) \tilde{\phi}_{n_1}(p_2) e^{i\theta(k_1 + p_1) - i\theta p_1 k_1} \]
\[ A_{\mu j}^i(k) \sim \Phi_{l_1}^{m_1}(P_1) = V(A_{\mu j}^i(k), \Phi_{l_1}^{m_1}(P_1)) \]
\[ = g \int_{P_2} \delta(P_1 + P_2 + k) \left( P_{1}^\mu - P_{2}^\mu \right) \left[ \delta_{m_1} \Phi_{l_1}(P_2) e^{i\theta P_1 P_2} - \delta_{n_1} \Phi_{l_1}(P_2) e^{-i\theta P_1 P_2} \right] \]
\[ + g^2 \int_{P_2 k_1} \delta(P_1 + P_2 + k + k_1) \left[ e^{i\theta(k_1 + P_1) \delta_{m_1}} \right. \]
\[ \times \left[ \tilde{F}_{m_1}(P_2) \tilde{A}^{\mu_1}(k_1) e^{-i\theta P_2 k_1} - 2 \tilde{A}^{\mu_1}(k_1) \tilde{F}_{l_1}(P_2) e^{i\theta P_2 k_1} \right] \]
\[ + e^{-i\theta(k_1 + P_1) \delta_{m_1}} \times \left[ \tilde{F}_{l_1}(P_2) \tilde{A}^{\mu_1}(k_1) e^{-i\theta P_2 k_1} - 2 \tilde{A}^{\mu_1}(k_1) \tilde{F}_{n_1}(P_2) e^{i\theta P_2 k_1} \right] \}
\[ A_{\mu j}^i(k) \sim \Phi_{l_2}^{m_2}(P_2) = V(A_{\mu j}^i(k), \Phi_{l_2}^{m_2}(P_2)) \]
\[ = g \int_{P_2} \delta(P_1 + P_2 + k) \left( P_{1}^\mu - P_{2}^\mu \right) \left[ \delta_{m_1} \Phi_{l_2}(P_1) e^{i\theta P_1 P_2} - \delta_{n_1} \Phi_{l_2}(P_1) e^{-i\theta P_1 P_2} \right] \]
\[ + g^2 \int_{P_2 k_1} \delta(P_1 + P_2 + k + k_1) \left[ e^{-i\theta(k_1 + P_1) \delta_{m_1}} \right. \]
\[ \times \left[ \tilde{F}_{m_2}(P_1) \tilde{A}^{\mu_2}(k_1) e^{-i\theta P_2 k_1} - 2 \tilde{A}^{\mu_2}(k_1) \tilde{F}_{l_1}(P_1) e^{i\theta P_2 k_1} \right] \]
\[ + e^{i\theta(k_1 + P_1) \delta_{m_2}} \times \left[ \tilde{F}_{l_2}(P_1) \tilde{A}^{\mu_2}(k_1) e^{-i\theta P_2 k_1} - 2 \tilde{A}^{\mu_2}(k_1) \tilde{F}_{n_2}(P_1) e^{i\theta P_2 k_1} \right] \}
\[ C_{s_1}^{r_1}(l_1) \sim \cdots \cdots \sim C_{s_2}^{r_2}(l_2) = V(C_{s_1}^{r_1}(l_1), C_{s_2}^{r_2}(l_2)) = \frac{\delta R}{\delta C_{s_2}^{r_2}(l_2) \delta C_{s_1}^{r_1}(l_1)} \]
\[ = g \int_k \delta(l_1 + l_2 + k) \left[ \delta_{s_2} \tilde{A}_{s_1}(k) e^{-i\theta l_2} - \delta_{s_1} \tilde{A}_{s_2}(k) e^{i\theta l_2} \right] \]
\[
\psi_{al}(q_2) \leftrightarrow - \psi^{bk}(q_1) = V(\psi_{al}(q_2), \psi^{bk}(q_1)) = \frac{\delta_R}{\delta \psi^{bk}(q_1)} \delta_R V \psi_{al}(q_2)
\]

\[
\psi_{al}(q_2) \leftrightarrow - \bar{\psi}^{bk}_l(Q_1) = V(\bar{\psi}^{bk}_l(Q_1), \psi_{al}(q_2))
\]

\[
= h^* \int \bar{\delta}(p_1 + Q_1 + q_2) \delta^a_\mu \delta^l_\nu \delta_{al}^l(p_1)e^{-\frac{1}{2}q_1 \theta k}
\]

\[
\phi^+(p_1) \leftrightarrow - \psi^{bk}(q_1) = V(\phi^+(p_1), \psi^{bk}(q_1))
\]

\[
= h^* \int Q_1 \bar{\delta}(p_1 + Q_1 + q_2) \bar{\psi}_{al}(Q_1)e^{\frac{1}{2}q_1 \theta k}
\]

\[
\phi_n(p_2) \leftrightarrow - \bar{\psi}^{bk}_l(Q_2) = V(\bar{\psi}^{bk}_l(Q_2), \phi_n(p_2))
\]

\[
= h \int \bar{\delta}(q_1 + Q_2 + p_2) \bar{\psi}_{al}(Q_2)e^{\frac{1}{2}q_1 \theta k}
\]

\[
\psi_{al}^{k_2}(Q_2) \leftrightarrow - \bar{\psi}^{bk}_l(Q_1) = V(\bar{\psi}^{bk}_l(Q_1), \psi_{al}^{k_2}(Q_2))
\]

\[
= g \int k \bar{\delta}(Q_1 + Q_2 + k) \gamma^\mu \gamma^l \bar{\psi}^{a\mu \nu \lambda}_l(k)e^{-\frac{1}{2}Q_1 \theta k}
\]

\[
\phi^+(p_1) \leftrightarrow - \bar{\psi}^{bk}_l(Q_1) = V(\phi^+(p_1), \bar{\psi}^{bk}_l(Q_1))
\]

\[
= h \int q_1 \bar{\delta}(p_1 + Q_1 + q_2) \bar{\psi}_{al}(Q_2)e^{\frac{1}{2}q_1 \theta k}
\]

\[
\phi^+(p_1) \leftrightarrow \Phi^{m_2}_{n_2}(P_2) = V(\phi^+(p_1), \Phi^{m_2}_{n_2}(P_2))
\]

\[
= -f_a \int_{p_2 P_1} \delta^m \delta^a \delta_{m a}^l (P_1) \hat{n}_{m a}^l (P_2) e^{\frac{1}{2}P_1 \theta (p_2 + P_1) - \frac{1}{2}P_1 \theta P_2}
\]

\[
\phi^+(p_1) \leftrightarrow \Phi^{m_1}_{n_1}(P_1) = V(\phi^+(p_1), \Phi^{m_1}_{n_1}(P_1))
\]

\[
= -f_b \int_{p_2 P_1} \delta^m \delta_{m a}^l (P_1) \hat{n}_{m a}^l (P_2) e^{\frac{1}{2}P_1 \theta (p_2 + P_1) + \frac{1}{2}P_1 \theta P_2}
\]

\[
\phi^n(p_2) \leftrightarrow \phi^+(p_1) = V(\phi^n(p_2), \phi^+(p_1))
\]

\[
= g \int k \bar{\delta}(p_1 + Q_1 + k) \psi^a_\mu \hat{A}_{\mu k}(k)e^{-\frac{1}{2}q_1 \theta k}
\]

\[
+ g^n \int k_1 k_2 \bar{\delta}(p_1 + Q_1 + k_1 + k_2) \hat{A}^{\mu}_{\mu k}(k_1) \hat{A}^{\mu}_{\mu k}(k_2)e^{-\frac{1}{2}q_1 \theta (k_1 + k_2) - \frac{1}{2}k_1 \theta k_2}
\]

\[-\frac{2}{4!} \int_{p_3 P_4} \delta^m \delta_{m a}^l (P_3) \hat{n}_{m a}^l (P_4) e^{\frac{1}{2}P_1 \theta (p_3 + P_4) - \frac{1}{2}P_3 \theta P_4}
\]
\[
\begin{align*}
\Phi_{m2}(P_2) &= V(\phi_n(p_2), \Phi_{m2}^n(P_2)) \\
\phi_n(p_2) \rightarrow \Phi_{m2}^{n_1}(P_1) &= V(\phi_n(p_2), \Phi_{m2}^{n_1}(P_1)) \\
\Phi_{n2}^{m2}(P_2) &\rightarrow \Phi_{n3}^{m2}(P_4) = V(\Phi_{m2}^{n_1}(P_2), \Phi_{n4}^{m2}(P_4)) \\
\Phi_{n2}^{m1}(P_1) &\rightarrow \Phi_{n3}^{m1}(P_3) = V(\Phi_{m2}^{n_1}(P_2), \Phi_{n3}^{n_1}(P_3))
\end{align*}
\]
\[-\frac{\lambda^*}{3!} \int_{P_5 P_7} \tilde{\delta}(P_1 + P_3 + P_5 + P_7) \delta_{m_3}^{n_3} \tilde{\Phi}_{* m_1}^{k} (P_5) \tilde{\Phi}_{* n_1}^{k} (P_7) e^{\frac{i}{2} P_1 \theta (P_5 + P_7) - \frac{i}{2} P_5 \theta P_7} \]

\[-\frac{\lambda^*}{3!} \int_{P_5 P_7} \tilde{\delta}(P_1 + P_3 + P_5 + P_7) \delta_{m_3}^{n_3} \tilde{\Phi}_{* m_1}^{k} (P_5) \tilde{\Phi}_{* n_1}^{k} (P_7) e^{-\frac{i}{2} P_1 \theta (P_5 + P_7) - \frac{i}{2} P_5 \theta P_7} \]

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