Non-asymptotic moment bounds for random variables rounded to non-uniformly spaced sets

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We study the effects of rounding on the moments of random variables. Specifically, given a random variable \( X \) and its rounded counterpart \( \text{rd}(X) \), we study \( |E[X^k] - E[\text{rd}(X)^k]| \) for non-negative integer \( k \). We consider the case that the rounding function \( \text{rd} : \mathbb{R} \rightarrow F \) corresponds either to (i) rounding to the nearest point in some discrete set \( F \) or (ii) rounding randomly to either the nearest larger or smaller point in this same set with probabilities proportional to the distances to these points. In both cases, we show, under reasonable assumptions on the density function of \( X \), how to compute a constant \( C \) such that \( |E[X^k] - E[\text{rd}(X)^k]| < C \epsilon^2 \), provided \( |\text{rd}(x) - x| \leq \epsilon E(x) \), where \( E : \mathbb{R} \rightarrow \mathbb{R}_{+} \) is some fixed positive piecewise linear function. Refined bounds for the absolute moments \( E[|X^k - \text{rd}(X)^k|] \) are also given.

KEYWORDS
measurement error, moments, rounded data, rounding error

1 INTRODUCTION

Rounded data are ubiquitous; measurements of length may be given in terms of the distance between consecutive markings on a ruler, weight may be measured to the nearest kilogram, age may be rounded to the nearest year, real numbers may be represented as floating point numbers and so forth. While it is often assumed that rounding error is small in comparison to other sources of error such as the sampling error, we are increasingly faced with settings in which there is a large amount of low precision data, making the task of understanding how the distribution of a random variable \( X \) and a rounded random variable \( \text{rd}(X) \) relate increasingly important.

Famously, Sheppard studied the moments of \( \text{rd}(X) \) obtained by rounding \( X \) to a set of uniform spacing \( h \). In particular, he showed that, under suitable conditions on the density of \( X \), \( E[\text{rd}(X)] \approx E[X] \) and \( \forall \text{rd}(X) \approx \forall [X] + h^2/12 \) (Sheppard, 1897). The setting of rounding to a uniformly spaced set has remained of interest, with more recent work focusing on providing estimates and bounds under weaker conditions or for more general cases (Bai et al., 2009; Hall, 1982; Janson, 2006; Schneeweiss et al., 2010; Tricker, 1990; Ushakov & Ushakov, 2017; Vardeman, 2005; Wilrich, 2005). However, such past work makes critical use of the uniform spacing between points and is therefore neither applicable nor easily generalizable to the analysis of rounding to non-uniformly spaced sets. The idea to consider a rounding random variable rounded to finite precision has been considered (Monahan, 1985), although the statistical properties were never studied in detail.

In this paper, we show how to obtain bounds on the moments of \( \text{rd}(X) \) for a wide range of rounding modes. The techniques we develop for our analysis differ significantly from past work in that they make very limited assumptions about the set to which we are rounding; we simply require that the distance between consecutive points is bounded locally. As a result, our techniques are applicable not only to the analysis of floating point number systems but also to sets with irregular inter-point spacing such as those which might arise from sensor arrays.

1.1 Related work

The study of rounding random variables is closely related to the study of histograms. Much of the theory on histograms focuses on the quality of the histogram density (the piecewise constant density function with mass of each bin equal to the mass of the underlying random variable over the given bin) (Chaudhuri et al., 1998; Freedman & Diaconis, 1981; Knuth, 2019). This differs slightly from the rounded random variable \( \text{rd}(X) \) studied in this paper, which is a discrete random variable supported at the midpoints of the histogram bins, although the information contained...
in the two approaches is identical. However, as with the study of rounding random variables, most analyses of histograms for density estimation studies the case of uniformly spaced histogram bins.

Finally, we remark on several areas which are broadly related to this paper and may be of interest to readers. First, there is a great deal of work on the study of distribution quantization, which seeks to find a discrete random variable to represent a continuous one (Graf & Luschgy, 2007). Second, we contrast our work with rounding error analysis which makes statistical assumptions about the rounding errors incurred by a numerical algorithm (Connolly et al., 2021; Higham & Mary, 2019; Wilkinson, 1963). We study the effect of deterministic perturbations to random variables; in fact, many of the results in this paper are derived directly from the deterministic structure of rounding errors.

2 | SET-UP

2.1 | Finite precision number systems and rounding functions

Let \( F \subset \mathbb{R} \) be a discrete set on which the rounded random variable will be supported, and for notational convenience, define \( \lfloor x \rfloor := \min\{z \in F : z \geq x\} \) and \( \lceil x \rceil := \max\{z \in F : z \leq x\} \). To ensure these quantities are well defined, we will assume that \( \inf\{\|x - y\| : x, y \in F, x \neq y\} > 0 \).

Given \( F \), we consider two rounding functions \( r_d : \mathbb{R} \to F \) defined by

round to nearest

\[
\begin{align*}
\lfloor x \rfloor, & \quad x \leq \frac{1}{2}(\lfloor x \rfloor + \lceil x \rceil) \\
\lceil x \rceil, & \quad x > \frac{1}{2}(\lfloor x \rfloor + \lceil x \rceil)
\end{align*}
\]

stochastic rounding

\[
\begin{align*}
\lfloor x \rfloor, & \quad \text{w.p. } 1 - (x - \lfloor x \rfloor) / (\lceil x \rceil - \lfloor x \rfloor) \\
\lceil x \rceil, & \quad \text{w.p. } (x - \lfloor x \rfloor) / (\lceil x \rceil - \lfloor x \rfloor)
\end{align*}
\]

The first is the standard ‘round to nearest’ scheme, which minimizes the distance between \( X \) and a random variable supported on \( F \) in many metrics; for example, ‘earth mover’ distance, \( L^p \) norm and so forth. The second is a randomized scheme which has gained popularity in recent years, particularly in machine learning (Alistarh et al., 2016; Connolly et al., 2021; Gupta et al., 2015). These schemes are illustrated in Figure 1.

Once \((F, r_d)\) has been specified, we can consider how performing operations in the finite precision number system compares to performing the operations exactly. For notational convenience, we define the error function,

\[
\text{err}(x) := r_d(x) - x,
\]

which tells us both the size and direction of rounding errors. In principle, we could use this to explicitly compute quantities such as \( \mathbb{E}[r_d(X)] \), but it would be exceedingly tedious to perform a separate analysis for every finite precision number system \( F \). As such, as is common in numerical analysis, we will use the assumption that

\[
|\text{err}(x)| = |r_d(x) - x| \leq \varepsilon \mathbb{E}(x)
\]

for some fixed function \( E : \mathbb{R} \to \mathbb{R}_{\geq 0} \). If \( E(x) = |x| \), this bound is the standard bound for rounding to floating point number systems, and if \( E(x) = 1 \), this bound is the standard bound for fixed point systems; see, for instance, Higham (2002).

Note that for a given set of numbers \( F \) and non-negative function \( E \), the value of \( \varepsilon \) required for ‘stochastic round’ to satisfy \( |\text{err}(x)| \leq \varepsilon E(x) \) is roughly twice that of ‘round to nearest’. This is visible in Figure 1 and is at the core of the trade-offs between the two approaches.

![Figure 1](image-url)  
**Figure 1**  
Error \( \text{err}(x) := r_d(x) - x \) for selected rounding functions.  
*Left: ‘round to nearest’, right: ‘stochastic round’ (darker colours represent higher probability)
2.2 | Stochastic rounding

When using stochastic rounding, rd(X) is a random variable depending on both the randomness in X and in the rounding function rd. We will assume that every time X is sampled, rd is sampled according to its definition, independently of any past samples; that is, even if different samples of X take the same value, they might be rounded differently (although, in this paper, we are concerned only with continuous random variables where the probability of two identical samples is zero).

We often require the expectation of err(x)^k and |err(x)|^k taken over the randomness in the rounding function, so for convenience we give the following lemma.

Lemma 1. If rd : R → R is ‘stochastic round’,

\[
\mathbb{E}_a[err(x)^k] = (|x| - x)^k \left(1 - \frac{x - |x|}{|x| - |x|}\right) + (|x| - x)^k \left(\frac{x - |x|}{|x| - |x|}\right).
\]

\[
\mathbb{E}_a[|err(x)|^k] = (|x| - |x|)^k \left(1 - \frac{x - |x|}{|x| - |x|}\right) + (|x| - x)^k \left(\frac{x - |x|}{|x| - |x|}\right).
\]

In particular, note that \(\mathbb{E}_a[err(x)] = 0\); that is, the rounding scheme is unbiased.

2.3 | Basic bounds

Theorem 1. Suppose \(E[|X|^k] < \infty\) for some integer \(k > 0\) and that \(E : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}\) grows at most linearly at infinity; that is, for sufficiently large \(|x|\), \(|E(x)| \leq D|x|\) for some \(D \geq 0\). Then there exists a constant \(C > 0\) such that, for all \(\varepsilon \in (0, 1)\) and \((F, rd)\) where \(rd : \mathbb{R} \rightarrow \mathbb{R}\) corresponds to ‘round to nearest’ or ‘stochastic round’ and satisfies \(|rd(x) - x| < \varepsilon E(x)\),

\[
\left| \mathbb{E} [X^k] - \mathbb{E} [rd(X)^k] \right| \leq \mathbb{E} \left[ |X^k - rd(X)^k| \right] < C\varepsilon.
\]

Proof. By assumption \(\varepsilon \in (0, 1)\) so \(\varepsilon^j \leq \varepsilon\). Moreover, because \(E\) has at most linear growth at infinity and the kth absolute moment of \(X\) exists, each of the expectations \(\mathbb{E}[|E(x)/X^{k-j}|]\) is finite. Thus, for any \(j = 1, \ldots, k\),

\[
\mathbb{E} \left[ |err(X)/X^{k-j}| \right] \leq \mathbb{E} \left[ |E(x)/X^{k-j}| \right] \cdot \varepsilon^j \leq \mathbb{E} \left[ |E(x)/X^{k-j}| \right] \cdot \varepsilon < \infty.
\]

Using that \(rd(X) = X + err(X)\), we expand

\[
rd(X)^k = X^k + \sum_{j=1}^{k} \binom{k}{j} X^{k-j} err(X)^j.
\]

Then, applying the triangle inequality,

\[
\left| \mathbb{E} [X^k] - \mathbb{E} [rd(X)^k] \right| \leq \left| \sum_{j=1}^{k} \binom{k}{j} \mathbb{E} [X^{k-j} err(X)^j] \right| \leq \left| \sum_{j=1}^{k} \binom{k}{j} \mathbb{E} [E(x)/X^{k-j}] \right| \cdot \varepsilon.
\]

3 | HIGHER ORDER MOMENT BOUNDS

If \(rd\) corresponds to rounding with either ‘round to nearest’ or ‘stochastic round’, we expect cancellation in many cases. This is illustrated in Figure 2, which depicts an error function corresponding to the ‘round to nearest’ scheme. The key observation is that the integral of the error function of any finite interval is much smaller than the integral of the corresponding bound.

FIGURE 2 The contribution of the integral of the ‘round to nearest’ error function over the interval \([a, b]\) is at most the area of the rightmost darkly shaded triangle: \(E|b^2/2| \cdot \varepsilon^2\). This is in contrast to the lightly shaded area which is of size \(\int_a^b E(x) dx \cdot \varepsilon\). Legend: err(x) (___), E(x) \cdot \varepsilon (____)
Lemma 2. Let $c \in \mathbb{R}$ and $c' \in ([c], [c])$ so that $|c - c'|$ is minimal. Then, for any odd integer $k > 0$, with $d_{rd}(k)$ as in Table 1,

(i) if $rd : \mathbb{R} \rightarrow \mathbb{F}$ is 'round to nearest',
\[
\int_c^{c'} \text{err}(x)^k \, dx \leq [d_{rd}(k)E(c)]^{k+1} \cdot \epsilon^{k+1}.
\]

(ii) if $rd : \mathbb{R} \rightarrow \mathbb{F}$ is 'stochastic round', the above holds with $\text{err}(x)^k$ replaced by $E_{rd}[\text{err}(x)^k]$.

Proof. For 'round to nearest' $c' = rd(c)$. Note that $\text{err}(x) = rd(c) - x$ does not change signs on $[\min(c, c'), \max(c, c')]$. Thus, using the assumption $|c - c'| \leq E(c)$,
\[
\int_c^{c'} \text{err}(x)^k \, dx = \int_{\min(c, c')}^{\max(c, c')} |c' - x|^k \, dx = \frac{|c' - c|^{k+1}}{k+1} \leq \left[ \frac{1}{k+1} E(c)^{k+1} \right] \cdot \epsilon^{k+1}.
\]

For 'stochastic round', $E_{rd}[\text{err}(x)^k]$ also does not change signs on $[\min(c, c'), \max(c, c')]$. Moreover, it is symmetric about $\bar{c} = ([c] + [c])/2$ on $[[c], [c]]$. Using these facts and that for 'stochastic round' $|c' - c| \leq \epsilon E(c)$,
\[
\int_c^{c'} E[\text{err}(x)^k] \, dx \leq \int_{[c]}^{[c]} E[\text{err}(x)^k] \, dx = \frac{1}{k^2 + 3k + 2} \left( [c] - [c] \right)^{k+1} \leq \left[ \frac{1}{k^2 + 3k + 2} E(c)^{k+1} \right] \cdot \epsilon^{k+1}.
\]

From Figure 2, it is clear that the contribution to integrals of $\text{err}(x)$ is due to the endpoints. This is stated precisely in the following result, which is essentially a corollary of Lemma 2.

Lemma 3. Let $a, b \in \mathbb{R}$ with $a < b$. Then, for any odd integer $k > 0$, with $d_{rd}(k)$ as in Table 1,

(i) if $rd : \mathbb{R} \rightarrow \mathbb{F}$ is 'round to nearest',
\[
\int_a^b \text{err}(x)^k \, dx \leq [d_{rd}(k) \max \{E(a)^{k+1}, E(b)^{k+1}\}] \cdot \epsilon^{k+1}.
\]

(ii) if $rd : \mathbb{R} \rightarrow \mathbb{F}$ is 'stochastic round', the above holds with $\text{err}(x)^k$ replaced by $E_{rd}[\text{err}(x)^k]$.

Proof. We prove the 'round to nearest' case, but the same proof, with $\text{err}(x)^k$ replaced by $E_{rd}[\text{err}(x)^k]$, holds for 'stochastic round'. Let $a' \in ([a], [a])$ and $b' \in ([b], [b])$ so that $|a - a'|$ and $|b - b'|$ are minimal. If $k$ is odd, then for any $c$, by symmetry, the integral of $\text{err}(x)^k$ over $[[c], [c]]$ is zero. Thus, inductively,
\[
\int_{a'}^{b'} \text{err}(x)^k \, dx = 0.
\]

Applying Lemma 2 to the endpoints, we have
\[
\int_a^b \text{err}(x)^k \, dx = \int_{a'}^{a'} \text{err}(x)^k \, dx + \int_{a'}^{b'} \text{err}(x)^k \, dx = \int_{a}^{b'} \text{err}(x)^k \, dx \leq [d_{rd}(k) \max \{E(a)^{k+1}, E(b)^{k+1}\}] \cdot \epsilon^{k+1}.
\]

Next, we argue that this same higher order bound carries over to integrals of nice functions against $\text{err}(x)^k$. First, however, we recall a basic property of the Lebesgue–Stieltjes integral: If $g$ is absolutely integrable with respect to a measure $\nu$, then
\[
\mu(A) := \int_A g \, d\nu
\]
is a (signed) measure and
\[
\int_A fg \, d\nu = \int_A f \, d\mu.
\]
Let $\mathcal{O} \subset 2^\mathbb{R}$ denote the set of all open subsets of $\mathbb{R}$. Recall that any open set $A \in \mathcal{O} \setminus \{\varnothing\}$ can be written $A = \bigcup_{i=1}^K (a_i, b_i)$ where $(a_i, b_i)$ are pairwise disjoint and $K \in \mathbb{Z}_{>0} \cup \{\infty\}$. Using this notation, we have the following lemma:

**Lemma 4.** Let $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be lower semi-continuous and $g : \mathbb{R} \to \mathbb{R}$ integrable. Suppose that $fg$ is absolutely integrable and that there exists a function $G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_{\geq 0}$ such that for any $a, b \in \mathbb{R}$,

$$\int_a^b g(x) \, dx \leq G(a, b)$$

Extend $G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_{\geq 0}$ to a function $\mu : \mathcal{O} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ on open sets by $\mu(\varnothing) = 0$ and

$$\mu(A) = \mu:\left(\bigcup_{i=1}^K (a_i, b_i)\right) = \sum_{i=1}^K G(a_i, b_i), \quad \forall A \in \mathcal{O}\setminus\{\varnothing\}$$

Then,

$$\int f(x)g(x) \, dx \leq \int_0^\infty \mu(x : f(x) > u) \, du.$$

**Proof.** Define $\nu(A) := \int_A g(x) \, dx$ and observe that by definition $\nu(A) \leq \mu(A)$ for all $A \in \mathcal{O}$. Then, by definition of the Lebesgue–Stieltjes integral,

$$\int f(x)g(x) \, dx = \int_0^\infty \nu(x : f(x) > u) \, du \leq \int_0^\infty \mu(x : f(x) > u) \, du.$$

This lemma allows us to provide lower bounds on the integral of $fg$ given lower bounds on the integral of $g$ over $[a, b]$ (simply replace $g$ with $-g$ in the above statement). This bound can therefore be naturally extended to apply to any function which has negative outputs by decomposing $f = f^+ + f^-$ where $f^+, -f^- \geq 0$ provided both $f^+$ and $-f^-$ are lower semi-continuous. Moreover, if $G(a, b)$ is of the form $G(a, b) = \int_a^b h(x) \, dx$, then

$$\int_0^\infty \mu(x : f(x) > u) \, du = \int f(x)h(x) \, dx.$$

How tight Lemma 4 is depends on how tight the bound for the integral of $g$ is. For instance, if the bound on $g$ is equality, then the proposition’s bound is equality; in fact, it is simply the Lebesgue–Stieltjes integral $\int f(x) \, dG(x)$, where $G$ is an antiderivative of $g$. On the other hand, as in the case of our subsequent applications of this proposition, if the bound on $g$ does not take into account some behaviour of $g$, then the bound will be more pessimistic as it must account for the worst case interaction between $f$ and $g$.

To facilitate the use of Lemma 4, we introduce the following definition and lemma.

**Definition 1.** A function $f : \mathbb{R} \to \mathbb{R}$ is said to have $K$ regions of local maxima if, for all $t \in \mathbb{R}$,

$$\{x : f(x) > t\} = \bigcup_{i=1}^K (a_i, b_i)$$

where $(a_i, b_i)$ are pairwise disjoint.

**Lemma 5.** Suppose $f : (a, b) \to \mathbb{R}_{\geq 0}$ is bounded and has a single region of local maxima, and that $E(x) = mx + c$ for $m \geq 0$. Let $x^* \in \mathbb{R}$ be the largest point such that $f$ is non-decreasing on $(a, x^*)$ and non-increasing on $(x^*, b)$. Then, if $k$ is odd,

(i) if $rd : \mathbb{R} \to \mathbb{R}$ is ‘round to nearest’,

$$\int f(x)\mathrm{err}(x)^k \, dx \leq \left[m(k+1)d_{\mathrm{rd}}(k)\int_{x^*}^b (mx + c)^k f(x) \, dx\right] \cdot e^{k+1},$$

(ii) if $rd : \mathbb{R} \to \mathbb{R}$ is ‘stochastic round’, the above holds with $\mathrm{err}(x)^k$ replaced by $E_{\mathrm{rd}}[\mathrm{err}(x)^k]$.

**Proof.** We prove the ‘round to nearest’ case, but the same proof, with $\mathrm{err}(x)^k$ replaced by $E_{\mathrm{rd}}[\mathrm{err}(x)^k]$, holds for the ‘stochastic round’. First, we make several notational definitions. Define $\tilde{f} : (x^*, b) \to (0, f(x^*))$ as the restriction of $f$ to $(x^*, b)$. That is, for all $x \in (x^*, b)$, $\tilde{f}(x) = f(x)$. Next, define $\tilde{f}^{-1} : (0, f(x^*)) \to (x^*, b)$ by

$$\tilde{f}^{-1}(u) := \sup\{x : f(x) > u\}.$$
Define also
\[ G(\sigma', b') := [d_{id}(k) \max \{E(\sigma')^{k+1}, E(b')^{k+1}\}] \epsilon^{k+1} = [d_{id}(k)E(\sigma')^{k+1}] \epsilon^{k+1}. \]

By assumption, \( f \) has a single local maxima (or connected region of local maxima) so
\[ \{x : f(x) > u\} = (\inf \{x : f(x) > u\}, \sup \{x : f(x) > u\}). \]

Then, in the notation of Lemma 4 with \( g(x) = \text{err}(x)^k \), for \( u < f(x^*) \),
\[ \mu(\{x : f(x) > u\}) = G \left( \inf \{x : f(x) > u\}, \sup \{x : f(x) > u\} \right) \]
\[ \leq d_{id}(k)E(\sup \{x : f(x) > u\})^{k+1} \]
\[ = d_{id}(k)E(f^{k+1}(u))^{k+1} \]

where the inequality follows from the fact \( E \) is non-decreasing. Therefore, applying Lemma 4 and Lemma 3,
\[ \int_a^b f(x) \text{err}(x)^k \, dx \leq \int_0^\infty \mu(\{x : f(x) > u\}) \, du \leq \int_0^\infty d_{id}(k) \left( \int_0^{\text{err}}(\text{err}(u))^{k+1} \, du \right) \epsilon^{k+1} \]

Since \( E \) and \( \hat{f}^{k+1} \) are non-decreasing, \( u \mapsto E(\hat{f}^{k+1}(u))^{k+1} \) is also non-decreasing. Thus, reversing the axis of integration and making a change of variables \( v = E(x)^{k+1} = (mx + c)^{k+1} \),
\[ \int_0^{E(\hat{f}^{k+1}(u))^{k+1}} \, du = \int_{E^{1/(k+1)}}^{E(\hat{f}^{k+1}(u))^{k+1}} \left( \frac{yu^{1/(k+1)} - c}{m} \right) \, dv = (k + 1)m \int_0^b (mx + c)^k \, f(x) \, dx. \]

\[ \square \]

**Theorem 2.** Suppose \( E[|X|^k] < \infty \) for some integer \( k > 0 \), that \( x \mapsto x^{-1}f_x(x) \) has finitely many regions of local maxima, and that \( E : \mathbb{R} \to \mathbb{R}_{\geq 0} \) is piecewise linear with a finite number of breakpoints.

Then there exists a constant \( C > 0 \) such that, for all \( \epsilon \in (0, 1) \) and \( (\text{rd}, \text{rd}) \) where \( \text{rd} : \mathbb{R} \to \mathbb{R} \) corresponds to ‘round to nearest’ or ‘stochastic round’ and satisfies \( |rd(x) - x| < \epsilon \ E(x) \),
\[ \left| E[X^k] - E[rd(X)^k] \right| < C \epsilon^2. \]

**Proof.** As in the proof of Lemma 1, we expand
\[ E[rd(X)^k] = E[X^k] + \sum_{j=1}^k \binom{k}{j} E[X^{k-j} \text{err}(X)^j] \]
and note that each term in the sum is of size \( O(\epsilon^j) \). It suffices to show that the \( j = 1 \) term is actually \( O(\epsilon^2) \).

Let \( K_1 \) be the number of regions of local maxima and \( K_2 \) the number of breakpoints in \( E \). Then we can partition \((-\infty, \infty)\) into \( K < K_1 + K_2 \) intervals \( (a_i, a_{i+1}) \) such that on each \( (a_i, a_{i+1}) \), \( x \mapsto x^{-1}f_x(x) \) has a single region of local maxima and \( E \) is piecewise linear. Then, applying Lemma 5 to each interval we see that for some constant \( C_1 > 0 \)
\[ \left| \int_0^{a_i} \text{err}(x)^{k+1}f_x(x) \, dx \right| \leq \sum_{i=1}^{K_1} \left| \int_0^{a_i} \text{err}(x)^{k+1}f_x(x) \, dx \right| \leq C_1 \epsilon^2. \]

The result then follows by adding \( C_1 \) to the coefficients for the bounds for the \( j > 1 \) terms, similar to as in the proof of Theorem 1. \( \square \)

## 4 | REFINED ABSOLUTE MOMENT BOUNDS

Trivially, we may bound the integral of \( \text{err}(x)^k \) by the integral of \( \epsilon^k \ E(x)^k \). However, as suggested by Figure 3, if \( \text{rd} \) corresponds to ‘round to nearest’ or ‘stochastic round’, integrals against \( \text{err}(x)^k \) should be a constant fraction smaller than integrals against \( \epsilon^k \ E(x)^k \) due to the fact that the rounding function cannot always attain the size of the worst case error. As in the proof of Lemma 3, we bound the bulk of the contribution between two numbers in \( \mathcal{P} \) and then account for tiny contributions at the endpoints.

**Lemma 6.** Suppose \( E(x) = mx + b \) on \( [[c], [c]] \). Then for any integer \( k > 0 \), with \( e_{id}(k) \) as in Table 1,
Figure 3: The contribution of the integral of the kth power of the absolute error function for 'round to nearest' over the interval \( [c, [c]] \) is at most \( 1/(k + 1) \) the area of the integral of the constant function \( ([c] - [c])/2^k \) over this interval, which is itself smaller than the integral of \( e^x|x|^k \) over this interval. Note that \( E(x) \) is always larger than the tangent at \( ([c] + [c])/2 \) and that the area under this tangent is equal to that of the constant function through the tangent point.

Legend: \( E(x)^k \cdot e^x \) ( ), \( ([c] + [c])/2^k \cdot e^x \) ( ), \( ([c] - [c])/2^k \) ( ), \( |\text{err}|(x)^k \) ( ).

Proof. We first prove the 'round to nearest' case. By direct computation, using that \( |\text{err}|([c] + [c])/2) \leq \epsilon E(([c] + [c])/2) \) followed by the fact that the tangent to \( E(x)^k \) at \( ([c] + [c])/2 \) lies entirely below \( E(x) \), we find

\[
|\text{err}|(x)^k \text{dx} = \int \left| \frac{\epsilon}{k + 1} \frac{[c] + [c]}{2} \right|^k \text{dx} \leq \left( \frac{[c] + [c]}{2} \right)^k \cdot e^x.
\]

For the 'stochastic round' case, again by direct computation, followed by the fact that \( ([c] - [c]) \leq E(x) \), we find

\[
\int \left| \frac{\epsilon}{k + 1} \frac{([c] - [c])}{2} \right|^k \text{dx} \leq \left( \frac{[c] - [c]}{2} \right)^k \cdot e^x.
\]

Lemma 7. Suppose \( E(x) = mx + b \) on \( [c, [c]] \). Then for any integer \( k \geq 0 \), with \( f_{\alpha}(k) \) as in Table 1,

(i) if \( rd : \mathbb{R} \rightarrow \mathbb{F} \) is 'round to nearest', for \( \beta = 1/(1 - mc) \),

\[
|\text{err}|(x)^k \text{dx} \leq \left| f_{\alpha}(k)\beta E(c)^k \right| \cdot e^{k+1}.
\]

(ii) if \( rd : \mathbb{R} \rightarrow \mathbb{F} \) is 'stochastic round', the above holds with \( |\text{err}|(x)^k \) replaced by \( \mathbb{E}_{\alpha}(|\text{err}|(x)^k) \) and \( \beta = 1 \).

Proof. We first prove the 'round to nearest' case. Using that \( E(x) = mx + b \), we have

\[
\frac{[c] - [c]}{2} = \left| \text{err} \left( \frac{[c] + [c]}{2} \right) \right| \leq E \left( \frac{[c] + [c]}{2} \right) \cdot e \leq \left[ E(c) + m \left( \frac{[c] - [c]}{2} \right) \right] \cdot e.
\]

We therefore find that

\[
\frac{[c] - [c]}{2} \leq \frac{e}{1 - mc} E(c).
\]

Next, using that \( \text{err}(x) = \) is symmetric about \( \bar{c} = ([c] + [c])/2 \) on \( [c, [c]] \)

\[
\int \left| \text{err}(x) \right|^k \text{dx} = \frac{2}{k + 1} \left( \frac{[c] - [c]}{2} \right)^{k+1} \leq \frac{2}{k + 1} E(c)^{k+1} \cdot \left( \frac{e}{1 - Le} \right)^{k+1}.
\]

We now prove the 'round to nearest' case. Here, we have that \( [c] - [c] \leq \epsilon E(c) \) so

\[
\int \mathbb{E}_{\alpha}(|\text{err}|(x)^k) \text{dx} = \frac{2}{k^2 + 3k + 2} \left( [c] - [c] \right)^{k+1} \leq \frac{2}{k^2 + 3k + 2} E(c)^{k+1} \cdot e^{k+1}.
\]
Combining Lemma 6 with Lemma 2, we obtain the following:

**Theorem 3.** Suppose that for all $x$, $E$ is piecewise linear on $[\lfloor x \rfloor, \lfloor x \rfloor]$ with maximum slope $m$. Then

(i) if $\text{rd} : \mathbb{R} \rightarrow F$ is 'round to nearest', for $\beta = 1/(1 - mc)$,

$$
\left| \int_a^b |\text{err}(x)|^k \, dx \right| \leq \left[ c_{\text{rd}}(k) \int_a^b E(x)^k \, dx \right] \cdot \epsilon^k + \left[ f_{\text{rd}}(k) \left( (\beta E(a))^{k+1} + (\beta E(b))^{k+1} \right) \right] \cdot \epsilon^{k+1}.
$$

(ii) if $\text{rd} : \mathbb{R} \rightarrow F$ is 'stochastic round', the above holds with $|\text{err}(x)|^k$ replaced by $E_{\text{rd}}[|\text{err}(x)|^k]$ and $\beta = 1$.

**Proof.** Similar to the proof of Lemma 3, we have

$$
\int_a^b |\text{err}(x)|^k \, dx = \int_a^b |\text{err}(x)|^k \, dx + \int_a^b |\text{err}(x)|^k \, dx + \int_a^b |\text{err}(x)|^k \, dx.
$$

By Lemma 6,

$$
\int_a^b |\text{err}(x)|^k \, dx \leq \left[ c_{\text{rd}}(k) \int_a^b E(x)^k \, dx \right] \cdot \epsilon^k \leq \left[ c_{\text{rd}}(k) \int_a^b E(x)^k \, dx \right] \cdot \epsilon^k.
$$

The result follows by applying Lemma 2 to the remaining terms, accounting for the sign of the integrand.

---

### 4.1 Two-sided bounds for uniform meshes

When $F$ contains uniformly spaced points, then we can take $E$ to be constant so Lemma 6 becomes equality. This provides something akin to a non-asymptotic version of Sheppard’s corrections.

**Theorem 4.** Suppose that $F = \{ a + 2\delta z : z \in \mathbb{Z} \}$. Then

(i) if $\text{rd} : \mathbb{R} \rightarrow F$ is 'round to nearest', $\epsilon = \delta$,

$$
\left| \int_a^b |\text{err}(x)|^k \, dx - \left[ c_{\text{rd}}(k) \int_a^b 1 \, dx \right] \cdot \epsilon^k \right| \leq \left[ 2f_{\text{rd}}(k) \right] \cdot \epsilon^{k+1}.
$$

(ii) if $\text{rd} : \mathbb{R} \rightarrow F$ is 'stochastic round', $\epsilon = 2\delta$, and the above holds with $|\text{err}(x)|^k$ replaced with $E_{\text{rd}}[|\text{err}(x)|^k]$.

**Proof.** We first prove the 'round to nearest' case. By direct computation, for any $c \in [a, b]$,

$$
\int_{\lfloor x \rfloor}^{\lceil x \rceil} |\text{err}(x)|^k \, dx = \frac{1}{k+1} \int_{\lfloor x \rfloor}^{\lceil x \rceil} \epsilon^k \, dx.
$$

The result then follows by the same approach as Theorem 3.

For the 'stochastic round' case, as with past computations, we obtain the constant $2(k^2 + 2k + 2)^{-1}$. The computation matches that of Lemma 7.

---

### 5 Example

In this example, we illustrate how our techniques can be used to provide bounds on the mean, variance and several other quantities corresponding to a rounded random variable.

Suppose $X$ is distributed according to the semicircle distribution with mean $\mu$ and radius $r$. That is $f_x(x) = 2/(\pi r^2) \sqrt{r^2 - (x - \mu)^2}$ for $-r \leq x - \mu \leq r$ and $f_x(x) = 0$ otherwise.
We consider the effect of rounding to the set \( \{ a + 2\delta z : z \in \mathbb{Z} \} \) on the quantities,

\[
|\mathbb{E}[\text{rd}(X)] - \mathbb{E}[X]|, \quad |\mathbb{V}[\text{rd}(X)] - \mathbb{V}[X]|, \quad |\mathbb{E}[\text{err}(X)]| \quad \text{and} \quad \mathbb{E}[\text{err}(X)^2].
\]

For each value of \( \delta \), we consider many values of \( a \in [0, 2\delta) \), taking the supremum over \( a \). As seen in Figure 4, even for a fixed value of \( \delta \), these quantities may vary drastically as \( a \) changes. This illustrates the advantage of bounds which depend only on limited information about the set being rounded to.

We now bound the differences of the mean and variances based on increasing amounts of information:

a. \( |\text{rd}(x) - x| \leq \delta \)

b. \( \text{rd} : \mathbb{R} \to \mathbb{F} \) is 'round to nearest'

c. \( \mathbb{F} \) has uniform spacing \( 2\delta \).

Using only that \( |\text{err}(x)| \leq \delta \), we find

\[
|\mathbb{E}[\text{err}(X)]| \leq \delta, \quad |\mathbb{E}[\text{err}(X)^2]| \leq \delta^2. \quad \text{(bound (a))}
\]

If we additionally know that \( \text{rd} \) is 'round to nearest', then we can improve our bounds to quadratic in \( \delta \). In particular, using Lemma 5,

\[
|\mathbb{E}[\text{err}(X)]| = \int_{\mathbb{R}} \text{err}(x)f_x(x)dx \leq \frac{1}{2} \left( \sup_x f_x(x) \right) \cdot \delta = \left[ \frac{1}{\pi r} \right] \cdot \delta^2. \quad \text{(bound (b))}
\]

Likewise, but noting that \( x f_x(x) \) changes sign (hence the factor of 2),

\[
|\mathbb{E}[\text{err}(X)^2]| = \int_{\mathbb{R}} \text{err}(x)^2 f_x(x)dx \leq \frac{2}{2} \left( \sup_x |x f_x(x)| \right) \cdot \delta^2 \leq \left[ \frac{1}{\pi} \right] \cdot \delta^2. \quad \text{(bound (b))}
\]

Using only \( |\text{err}(x)| \leq \delta \), we already have that \( \mathbb{E}[\text{err}(X)^2] = O(\delta^2) \). However, using Lemma 5, we can improve the constant to

\[
|\mathbb{E}[\text{err}(X)^2]| = \int_{\mathbb{R}} \text{err}(x)^2 f_x(x)dx \leq \frac{1}{2} \left( \int f_x(x)dx \right) \cdot \delta^2 + \frac{4}{3} \sup_x f_x(x) \cdot \delta^2 \leq \left[ \frac{1}{3} \right] \cdot \delta^2 + \left[ \frac{8}{3\pi r} \right] \cdot \delta^3. \quad \text{(bound (b))}
\]
Finally, if our mesh is uniform, we can use Theorem 4 to provide a two-sided bound. We note that the upper bound is the same as our previous upper bound, but we can now provide a corresponding lower bound.

$$\left| E[\text{err}(X)^2] - \frac{1}{3} \cdot \delta^2 \right| = \left| \int_{\mathbb{R}} \text{err}(X)^2 f_X(x) \, dx - \frac{1}{3} \cdot \delta^2 \right| \leq \left\lfloor \frac{4}{3} \left( \sup_x f_X(x) \right) \right\rfloor \cdot \delta^2 = \left\lfloor \frac{8}{3\pi} \right\rfloor \cdot \delta^2; \quad \text{(bound (c))}$$

These bounds are shown in Figure 4. In general, we have that

$$\forall [\text{rd}(X)] - \forall [X] = \forall [\text{err}(X)] + 2\text{Co} \forall [X, \text{err}(X)].$$

Assuming that $\mathbb{E}[X] = 0$, which we may do without loss of generality because we make no assumptions on $a$, $\text{Co} \forall [X, \text{err}(X)] = \mathbb{E}[\text{err}(X)] - \mathbb{E}[X]\mathbb{E}[\text{err}(X)] = \mathbb{E}[\text{err}(X)]$ so

$$\forall [\text{rd}(X)] - \forall [X] = \mathbb{E}[\text{err}(X)^2] - \mathbb{E}[\text{err}(X)]^2 + 2\mathbb{E}[\text{err}(X)].$$

Therefore, we obtain a bound on the absolute difference of the variances,

$$|\forall [\text{rd}(X)] - \forall [X]| \leq 2|\mathbb{E}[\text{err}(X)]| + \max\{\mathbb{E}[\text{err}(X)^2], \mathbb{E}[\text{err}(X)]^2\}.$$

Note that we could use that the error function is odd about any point in the mesh to cancel some contribution to many of the integrals. This would result in a small improvement in the bounds.

6 CONCLUSIONS

In this paper, we provide non-asymptotic analysis for the effects of rounding on the moments of random variables. Our analysis requires very limited assumptions on the actual structure of the set being rounded to and is therefore applicable to a range of settings. Moreover, because our bounds are non-asymptotic, they can be used in the parameter ranges for precision encountered in practice. Our analysis also sheds light on differences between ‘round to nearest’ and ‘stochastic round’. It is well known that ‘stochastic round’ is unbiased, and this property has been used to analyse the scheme in the deterministic setting (Connolly et al., 2021). However, we show that when rounding random variables, this unbiasedness is at the cost of slower convergence of absolute and higher moments. Indeed, for a fixed $k$, the bounds for ‘stochastic round’ are a constant (growing exponentially in $k$) factor worse than the bounds for ‘round to nearest’. This suggests that in settings where it is important to preserve not just the mean but also higher moments, ‘stochastic round’ may not always be better than ‘round to nearest’. In fact, our analysis opens the possibility for picking randomized rounding schemes based on the relative accuracy constraints for different moments.

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CONFLICT OF INTEREST

The author declares no potential conflict of interests.

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