Inertia indices and eigenvalue inequalities for Hermitian matrices

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\textbf{ABSTRACT}

We present a characterization of eigenvalue inequalities between two Hermitian matrices by means of inertia indices. As applications, we deal with some classical eigenvalue inequalities for Hermitian matrices, including the Cauchy interlacing theorem and the Weyl inequality, in a simple and unified approach. We also give a common generalization of eigenvalue inequalities for (Hermitian) normalized Laplacian matrices of simple (signed, weighted, directed) graphs. Our approach is also suitable for Hermitian matrices of the second kind of digraphs recently introduced by Mohar.

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\section{Introduction}

A univariate polynomial is real-rooted if all of its coefficients and roots are real. Let \( f \) be a real-rooted polynomial of degree \( n \). Denote its roots by \( r_1(f) \geq r_2(f) \geq \cdots \geq r_n(f) \). For convenience, we set that \( r_i(f) = +\infty \) for \( i < 1 \) and \( r_i(f) = -\infty \) for \( i > n \). Let \( f \) and \( g \) be two real-rooted polynomials of degree \( n \) and \( m \), respectively. We say that \( f \) \textit{interlaces} \( g \), denoted by \( f \hookrightarrow g \), if \( n \leq m \leq n+1 \) and \( r_i(g) \geq r_i(f) \geq r_{i+1}(g) \) for all \( i \). We say that \( f(x) \) and \( g(x) \) are \textit{compatible}, denoted by \( f \bowtie g \), if \( |m-n| \leq 1 \) and \( r_{i-1}(g) \geq r_i(f) \geq r_{i+1}(g) \) for all \( i \). Chudnovsky and Seymour [1] introduced the concept of compatible polynomials and showed that the independence polynomial of a claw-free graph has only real roots. It is easy to see that \( f \bowtie g \) is equivalent to \( g \bowtie f \) and that \( f \hookrightarrow g \) implies \( f \bowtie g \). We refer the reader to [1–9] for further information about interlacing and compatible polynomials.

Let \( A \) and \( B \) be two Hermitian matrices. We will use the notation \( A \prec B \) and \( A \bowtie B \) if their characteristic polynomials \( \det(\lambda I - A) \prec \det(\lambda I - B) \) and \( \det(\lambda I - A) \bowtie \det(\lambda I - B) \), respectively. In matrix analysis and spectral graph theory, one frequently encounters Hermitian matrices whose characteristic polynomials are interlacing or compatible. Throughout this paper, we use \( \mathcal{H}_n \) to denote the set of \( n \times n \) Hermitian matrices. For a
Hermitian matrix $H \in \mathcal{H}_n$, we arrange its eigenvalues in a nonincreasing order: $\lambda_1(H) \geq \lambda_2(H) \geq \cdots \geq \lambda_n(H)$. Let $n_+(H)$ ($n_-(H)$, resp.) denote the positive (negative, resp.) inertia index, i.e. the number of positive (negative, resp.) eigenvalues of $H$. In this paper, we present the following characterization of eigenvalue inequalities between two Hermitian matrices by means of inertia indices and then apply it to investigate when the characteristic polynomials of Hermitian matrices interlace or are compatible.

**Theorem 1.1**: Let $A$, $B$ be two Hermitian matrices and $m \in \mathbb{Z}$. Then $\lambda_{i+m}(B) \leq \lambda_i(A)$ for all $i$ if and only if $n_+(B - rI) - n_+(A - rI) \leq m$ for every $r \in \mathbb{R}$.

**Proof**: Assume that $\lambda_{i+m}(B) \leq \lambda_i(A)$ for all $i$. Then for arbitrary $r \in \mathbb{R}$,

$$\lambda_{i+m}(B - rI) = \lambda_{i+m}(B) - r \leq \lambda_i(A) - r = \lambda_i(A - rI).$$

Let $n_+(A - rI) = p$. Then $\lambda_{p+1}(A - rI) \leq 0$, and so $\lambda_{p+m+1}(B - rI) \leq 0$ by (1). It follows that $n_+(B - rI) \leq p + m$, i.e., $n_+(B - rI) - n_+(A - rI) \leq m$.

Conversely, assume that there exists an index $i_0$ such that $\lambda_{i_0+m}(B) > \lambda_{i_0}(A)$. Take $r_0 \in (\lambda_{i_0}(A), \lambda_{i_0+m}(B))$. Then $\lambda_{i_0}(A - r_0I) < 0$ and $\lambda_{i_0+m}(B - r_0I) > 0$. Thus $n_+(A - r_0I) \leq i_0 - 1$ and $n_+(B - r_0I) \geq i_0 + m$, which implies that $n_+(B - r_0I) - n_+(A - r_0I) \geq m + 1$. In other words, if $n_+(B - rI) - n_+(A - rI) \leq m$ for every $r \in \mathbb{R}$, then $\lambda_{i+m}(B) \leq \lambda_i(A)$ for all $i$. This completes the proof of the theorem.

**Remark 1.2**: Clearly, both $n_+(A - rI)$ and $n_+(B - rI)$ are right continuous staircase functions of $r \in \mathbb{R}$, and so is their difference $n_+(A - rI) - n_+(B - rI)$. Thus, for an arbitrary real number $r_0$, we have $n_+(A - rI) - n_+(B - rI) \equiv n_+(A - r_0I) - n_+(B - r_0I)$ in a certain right neighbourhood of $r_0$. As a consequence, it is impossible that an inequality for $n_+(A - rI) - n_+(B - rI)$ is false only for finitely many real numbers $r$.

**Remark 1.3**: The following are two particularly interesting special cases of Theorem 1.1.

(i) $A \preceq B$ if and only if $0 \leq n_+(B - rI) - n_+(A - rI) \leq 1$ for every $r \in \mathbb{R}$.

(ii) $A \succeq B$ if and only if $|n_+(B - rI) - n_+(A - rI)| \leq 1$ for every $r \in \mathbb{R}$.

In Section 2, we use Theorem 1.1 to deal with some classical eigenvalue inequalities for Hermitian matrices, including the Cauchy interlacing theorem and the Weyl inequality. Various Hermitian matrices are often involved in spectral graph theory, such as (Hermitian) adjacency matrices and normalized Laplacian matrices of simple graphs and digraphs. In Section 3, we use Theorem 1.1 to give a common generalization of eigenvalue inequalities for (Hermitian) normalized Laplacian matrices of simple (signed, weighted, directed) graphs.

**2. Matrix analysis**

We first use Theorem 1.1 to give a simple proof of the following result, which is known as the inclusion principle [10, Theorem 4.3.28].

**Inclusion Principle.** Let $A = (C_{x,y}B_{x,y}) \in \mathcal{H}_n$. Then $\lambda_{n-k+1}(A) \leq \lambda_i(B) \leq \lambda_i(A)$. 

Proof: Let \( r \) be a real number such that \( \det(B - rl) \neq 0 \). Then
\[
A - rl = \begin{pmatrix} B - rl & C \\ C^* & D - rl \end{pmatrix} \cong \begin{pmatrix} B - rl & 0 \\ 0 & D - rl - C^*(B - rl)^{-1}C \end{pmatrix},
\]
where \( \cong \) denotes the congruence of matrices, and so
\[
n_+(B - rl) \leq n_+(A - rl) \leq n_+(B - rl) + n - k. \tag{2}
\]
Clearly, there are only finitely many real numbers \( r \) such that \( \det(B - rl) = 0 \). Hence (2) holds for all \( r \in \mathbb{R} \). Thus \( \lambda_{n-k+i}(A) \leq \lambda_i(B) \leq \lambda_i(A) \) by Theorem 1.1.

The Cauchy interlacing theorem is a special case of the inclusion principle.
Cauchy Interlacing Theorem. Let \( A = (\alpha B \alpha^*) \in \mathcal{H}_n \), where \( \alpha \in \mathbb{C}^{n-1} \) and \( \alpha \in \mathbb{R} \). Then \( B \prec A \).

The following result is an immediate consequence of Theorem 1.1 and the inclusion principle, which can also be proved by a direct argument (we leave the proof to the interested reader).

**Proposition 2.1:** Let \( A, B \in \mathcal{H}_n \). Then \( n_+(A + B) \leq n_+(A) + n_+(B) \).

**Proof:** The statement follows from \( (\alpha A \alpha^* \beta B \beta^*) \cong (\alpha A + B \alpha^* \beta B \beta^*) \) and (2).

**Proposition 2.2:** Let \( A, B \in \mathcal{H}_n \) and \( m \in \mathbb{Z} \).

(i) If \( n_+(B) \leq m \), then \( \lambda_{i+m}(A + B) \leq \lambda_i(A) \) for all \( i \).
(ii) If \( n_-(B) \leq m \), then \( \lambda_{i+m}(A) \leq \lambda_i(A + B) \) for all \( i \).

**Proof:** For every \( r \in \mathbb{R} \), we have \( n_+(A + B - rl) - n_+(A - rl) \leq n_+(B) \) by Proposition 2.1. Thus (i) follows from Theorem 1.1. And (ii) follows from (i) by noting that \( n_+(-B) = n_-(B) \).

**Corollary 2.3 (Monotonicity Theorem):** Suppose that \( A, B \in \mathcal{H}_n \) and \( B \) is positive semidefinite. Then \( \lambda_i(A) \leq \lambda_i(A + B) \).

**Corollary 2.4:** Let \( A, B \in \mathcal{H}_n \). If \( n_+(B) \leq p \) and \( n_-(B) \leq q \), then
\[
\lambda_{i+p}(A) \leq \lambda_i(A + B) \leq \lambda_{i-p}(A).
\]

**Corollary 2.5 (Weyl Inequality):** Let \( A, B \in \mathcal{H}_n \). Then \( \lambda_{i+j-1}(A + B) \leq \lambda_i(A) + \lambda_j(B) \).

**Proof:** Clearly, \( n_+(B - \lambda_j(B)I) \leq j - 1 \). Thus \( \lambda_{i+j-1}(A + B - \lambda_j(B)I) \leq \lambda_i(A) \) by Proposition 2.2, i.e. \( \lambda_{i+j-1}(A + B) \leq \lambda_i(A) + \lambda_j(B) \).

Corollary 2.4 is equivalent to the Weyl inequality (see [9] for instance) and is often more convenient to use. A particular interesting special case of Corollary 2.4 is the interlacing theorem [10, Corollary 4.3.9] for a rank-one Hermitian perturbation of a Hermitian matrix.
Corollary 2.6 (Interlacing Theorem): Let $A \in \mathcal{H}_n$ and $\alpha \in \mathbb{C}^n$. Then $A < (A + \alpha \alpha^*)$, i.e.

$$\lambda_i(A) \leq \lambda_i(A + \alpha \alpha^*) \leq \lambda_{i-1}(A).$$

Another useful special case of Corollary 2.4 is the following result about compatible polynomials.

Corollary 2.7: Let $A, B \in \mathcal{H}_n$. If $n_+(B) = n_-(B) = 1$, then $A \equiv (A + B)$, i.e.

$$\lambda_{i+1}(A) \leq \lambda_i(A + B) \leq \lambda_{i-1}(A).$$

The Cauchy interlacing theorem states that the spectrum of a Hermitian matrix interlaces that of its bordered Hermitian matrix. More generally, suppose that $P, A \in \mathcal{H}_n$ and define

$$f(\lambda; P, A) = \det(\lambda P - A).$$

Then $f(\lambda; I, A)$ is precisely the characteristic polynomial of $A$. For a real-rooted polynomial $f$, let $n_+(f)$, $n_-(f)$ and $n_0(f)$ denote the number of positive, negative and zero roots of $f$, respectively. Call $(n_+(f), n_-(f), n_0(f))$ the inertia index of $f$. Clearly, if $f$ is the characteristic polynomial of a Hermitian matrix $A$, then the inertia index of $f$ coincides with that of $A$. We have the following result.

Corollary 2.8: Suppose that $P, A \in \mathcal{H}_n$ and $P$ is positive definite. Then

(i) $f(\lambda; P, A)$ is a real-rooted polynomial in $\lambda$;
(ii) $f(\lambda; P, A)$ has the same inertia index as $A$; and
(iii) $f(\lambda; \overline{P}, \overline{A}) < f(\lambda; P, A)$, where $\overline{P}$ (resp., $\overline{A}$) is the matrix obtained from $P$ (resp., $A$) by deleting the last row and column.

Proof: Let $B = P^{-1/2} A P^{-1/2}$. Then $B$ is a Hermitian matrix and is congruent to $A$. Moreover, $f(\lambda; P, A) = \det(\lambda P - A) = \det(P) \det(\lambda I - B)$. So $f(\lambda; P, A)$ has the same roots as the characteristic polynomial $\det(\lambda I - B)$ of $B$. Thus (i) and (ii) follow.

Similarly, let $C = \overline{P}^{-1/2} \overline{A} \overline{P}^{-1/2}$. Then $f(\lambda; \overline{P}, \overline{A})$ has the same roots as $\det(\lambda I - C)$. So, to prove (iii), it suffices to prove that $C < B$. We prove it by Remark 1.3(i). Let $r \in \mathbb{R}$. Note that $B - rI = P^{-1/2} (A - rP) P^{-1/2}$. Hence $n_+(B - rI) = n_+(A - rP)$. Similarly, $n_+(C - rI) = n_+(\overline{A} - r\overline{P})$. Since $\overline{A} - r\overline{P}$ is the $(n - 1) \times (n - 1)$ principal submatrix of $A - rP$, we have $0 \leq n_+(A - rP) - n_+(\overline{A} - r\overline{P}) \leq 1$ by the Cauchy interlacing theorem and Remark 1.3(i). Thus $0 \leq n_+(B - rI) - n_+(C - rI) \leq 1$, and so $C < B$ again by Remark 1.3(i).}

Remark 2.9: Let $\overline{P}_i$ (resp., $\overline{A}_i$) denote the matrix obtained from $P$ (resp., $A$) by deleting the $i$th row and column and $f_i(\lambda) = f(\lambda; \overline{P}_i, \overline{A}_i)$. Then each $f_i(\lambda)$ interlaces $f(\lambda; P, A)$. It follows that $\sum_i c_i f_i(\lambda)$ is real-rooted for all $c_i \geq 0$ (see [1, Theorem 3.6] for details).

Remark 2.10: Corollary 2.8(i) and (iii) also hold when $P$ is positive semi-definite by a standard continuity argument. We refer the reader to [11] for some related results.
3. Spectral graph theory

Let $G = (V, E)$ be a simple graph with $n$ vertices $v_1, \ldots, v_n$ and edge set $E \subseteq V \times V$. The adjacency matrix $A(G) = (a_{ij})_{n \times n}$ of $G$ is defined by

$$a_{ij} = \begin{cases} 1, & \text{if } (v_i, v_j) \in E; \\ 0, & \text{otherwise.} \end{cases}$$

Let $D = \text{diag}(d_1, \ldots, d_n)$ be the degree matrix of $G$, where $d_i = \deg v_i$. The matrix $L(G) = D - A(G)$ is called the Laplacian matrix. We refer the reader to [12,13] for further information about the Laplacian matrix. The normalized Laplacian matrix of $G$ is defined as $L(G) = D^{-1/2}L(G)D^{-1/2}$ with the convention that the $i$th diagonal entry of $D^{-1}$ is 0 if $d_i = 0$. For any vertex $v$ of $G$, it is clear that $A(G - v) \prec A(G)$ by the Cauchy interlacing theorem. For any edge $e$ of $G$, Chen et al. [14] showed that $L(G - e) \prec L(G)$ and $L(G - e) \bowtie L(G)$ (see [15] for a short proof).

There are some similar results for signed graphs and weighted graphs. A signed graph $G^\sigma$ consists of a simple graph $G$ and a map $\sigma : E \rightarrow \{+1, -1\}$. The signed adjacency matrix $A^\sigma(G) = (a^\sigma_{ij})_{n \times n}$ is defined by $a^\sigma_{ij} = \sigma(v_i, v_j)a_{ij}$ and the degree matrix is still $D = \text{diag}(d_1, \ldots, d_n)$. Define the signed Laplacian $L^\sigma(G) = D - A^\sigma(G)$ and the normalizer $L^\sigma(G) = D^{-1/2}L^\sigma(G)D^{-1/2}$. Clearly, $L^\sigma = L$ if $\sigma \equiv 1$. Atay and Tunçel [16, Theorem 8] showed that $L^\sigma(G - e) \bowtie L^\sigma(G)$ for any $e \in E$. A weighted graph $(G, w)$ is a graph $G$ (possibly with loops) with a nonnegative weight function $w : V \times V \rightarrow [0, \infty)$ with $w(u, v) = w(v, u)$ and $w(u, v) > 0$ if and only if there is an edge joining $u$ and $v$. The adjacency matrix is defined by $a_{ij} = w(v_i, v_j)$. The diagonal degree matrix is defined by $d_i = \sum_{v_j \sim v_i} w(v_i, v_j)$. The Laplacian $L(G, w)$ and its normalizer $L(G, w)$ is similarly defined as above. We say that $(H, w_H)$ is a subgraph of $(G, w_G)$ if $H$ is a subgraph of $G$ and $w_H(e) \leq w_G(e)$ for all $e \in E(H)$. In this case, we define the weighted graph $G - H$ with the weight function $w_{G - H} = w_G - w_H$. Let $e \in E(G)$ and $H = \{e\}$. Butler [17] showed that $L(G - H, w_{G - H}) \bowtie L(G, w_G)$.

Recently, Yu et al. [18] considered the case of simple directed graphs. A directed graph $X$ consists of a finite set $V = \{v_1, \ldots, v_n\}$ of vertices together with a subset $E \subseteq V \times V$ of ordered pairs called arcs or directed edges. If $(u, v) \in E$ and $(v, u) \notin E$, we say that the unordered pair $[u, v]$ is a digon of $X$. Following [19,20], define the Hermitian adjacency matrix $H(X) = (h_{ij})_{n \times n}$ of $X$ by

$$h_{ij} = \begin{cases} 1, & \text{if } (v_i, v_j) \in E \text{ and } (v_j, v_i) \in E; \\ i, & \text{if } (v_i, v_j) \in E \text{ and } (v_j, v_i) \notin E; \\ -i, & \text{if } (v_i, v_j) \notin E \text{ and } (v_j, v_i) \in E; \\ 0, & \text{otherwise.} \end{cases}$$

Following [18], define the Hermitian Laplacian matrix $L(X) = D - H(X)$ and the Hermitian normalized Laplacian matrix $\tilde{L}(X) = D^{-1/2}L(X)D^{-1/2}$, where $D = \text{diag}(d_1, \ldots, d_n)$ is the degree matrix of the corresponding undirected graph. Yu et al. [18, Theorem 3.6] showed that $\tilde{L}(X - e) \bowtie \tilde{L}(X)$ for any arc or digon $e$ of $X$.

More recently, Mohar [21] introduced a new kind of Hermitian matrix for digraphs. Denote by $\omega = (1 + i\sqrt{3})/2$ the primitive sixth root of unity and let $\overline{\omega}$ be its conjugate. Following Mohar [21], define the Hermitian adjacency matrix $\tilde{H}(X) = [\tilde{h}_{ij}]_{n \times n}$ of the second
kind of $X$ by

$$
\tilde{h}_{ij} = \begin{cases} 
1, & \text{if } (v_i, v_j) \in E \text{ and } (v_j, v_i) \in E; \\
\omega, & \text{if } (v_i, v_j) \in E \text{ and } (v_j, v_i) \notin E; \\
\bar{\omega}, & \text{if } (v_i, v_j) \notin E \text{ and } (v_j, v_i) \in E; \\
0, & \text{otherwise.} 
\end{cases}
$$

Define the corresponding Hermitian Laplacian matrix and Hermitian normalized Laplacian matrix by $\tilde{L}(X) = D - \tilde{H}(X)$ and $\tilde{L}(X) = D^{-1/2}\tilde{L}(X)D^{-1/2}$, respectively.

Note that for the simple graph $G$ and $e \in E$, we have

$$
L(G) - L(G - e) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \oplus 0.
$$

We use this notation as short hand for

$$
L(G) - L(G - e) = E_{u,v} - E_{u,v} - E_{v,u} + E_{v,u},
$$

where $e = (u, v)$ and $E_{u,v}$ is the matrix with 1 at the $(u, v)$ position and 0 elsewhere.

For the signed graph $G^\sigma$ and the weighted graph $(G, w)$, we have

$$
L^\sigma(G) - L^\sigma(G - e) = \begin{pmatrix} 1 & -\sigma(e) \\ -\sigma(e) & 1 \end{pmatrix} \oplus 0
$$

and

$$
L(G, w_G) - L(G - H, w_{G-H}) = w_H(e) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \oplus 0,
$$

where $H = \{e\}$. Similarly, if $X$ is a directed graph and $e$ is a digon or a directed edge of $X$, then

$$
L(X) - L(X - e) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \oplus 0 \text{ or } \begin{pmatrix} 1 & \pm i \\ \mp i & 1 \end{pmatrix} \oplus 0,
$$

and

$$
\tilde{L}(X) - \tilde{L}(X - e) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \oplus 0 \text{ or } \begin{pmatrix} 1 & \pm \omega \\ \pm \bar{\omega} & 1 \end{pmatrix} \oplus 0.
$$

A Hermitian matrix $L = (\ell_{ij})_{n \times n}$ is called a generalized Laplacian matrix if $\ell_{ii} - \sum_{j \neq i} |\ell_{ij}|$ for $i = 1, \ldots, n$. For such a matrix, define its normalizer $L = D^{-1/2}LD^{-1/2}$, where $D$ is the diagonal matrix $\text{diag}(d_1, \ldots, d_n)$ such that $d_i = \ell_{ii}$ if $\ell_{ii} \neq 0$ and $d_i = 0$ otherwise. Clearly, $L(G), L^\sigma(G), L(G, w_G), L(X), \tilde{L}(X)$ are all generalized Laplacian matrices. The following result is a common generalization of eigenvalue inequalities for (normalized) Laplacians of simple (signed, weighted, directed) graphs.

**Proposition 3.1:** Let $L_1$ and $L_2$ be two $n \times n$ generalized Laplacian matrices such that

$$
L_1 - L_2 = w \begin{pmatrix} 1 & c \\ \bar{c} & 1 \end{pmatrix} \oplus 0,
$$

where $w > 0, c \in \mathbb{C}$ and $|c| = 1$. Then $L_2 \preceq L_1$ if and only if $L_2 \succeq L_1$. 

To prove Proposition 3.1, we first establish the following result.

**Lemma 3.2:** Let $L$ be a generalized Laplacian matrix and $\mathcal{L} = D^{-1/2}LD^{-1/2}$. Then

(i) the eigenvalues of $\mathcal{L}$ are in the interval $[0, 2]$, and
(ii) $n_+(\mathcal{L} - rI) = n_+(L - rD)$ for $r \geq 0$.

**Proof:** Clearly, if the $i$th diagonal entry of $L$ is 0, then all entries in the $i$th row (column) of $L$ are 0. Assume now that $L$ has precisely $k$ zero diagonal entries. Delete the corresponding $k$ (zero) rows and $k$ (zero) columns of $L$ ($D$, $\mathcal{L}$, resp.) to obtain $\overline{L}$ ($\overline{D}$, $\overline{\mathcal{L}}$, resp.). Then $\overline{L}$ is a generalized Laplacian matrix with nonzero diagonal entry and $\overline{\mathcal{L}} = \overline{D}^{-1/2}\overline{\mathcal{L}}\overline{D}^{-1/2}$.

(i) It suffices to consider the case that all diagonal entries of $L$ are nonzero. In this case, $\mathcal{L}$ and $D^{-1}L$ have the same eigenvalues since they are similar: $D^{-1}L = D^{-1/2}\mathcal{L}D^{1/2}$. Note that the diagonal entries of $D^{-1}L$ are all 1. Hence, the eigenvalues of $D^{-1}L$ satisfy $|\lambda - 1| \leq 1$ by the Gershgorin circle theorem (see [10, Theorem 6.1.1] for instance), i.e. $\lambda \in [0, 2]$. Thus, eigenvalues of $\mathcal{L}$ are all in $[0, 2]$.

(ii) Note that the characteristic polynomials

$$
\det(\lambda I - (\mathcal{L} - rD)) = \lambda^k \det(\lambda I - (\overline{\mathcal{L}} - rI))
$$

and

$$
\det(\lambda I - (L - rD)) = (\lambda + r)^k \det(\lambda I - (\overline{L} - rI)).
$$

Hence $n_+(\mathcal{L} - rD) = n_+(\overline{\mathcal{L}} - rI)$ and $n_+(L - rD) = n_+(\overline{L} - rI)$ for $r \geq 0$. On the other hand,

$$
\overline{\mathcal{L}} - rI = \overline{D}^{-1/2}(\mathcal{L} - rD)\overline{D}^{-1/2},
$$

and so $n_+(\overline{\mathcal{L}} - rI) = n_+(\overline{L} - rD)$. Thus, we conclude that $n_+(\mathcal{L} - rI) = n_+(L - rD)$ for $r \geq 0$. 

We are now in a position to prove Proposition 3.1.

**Proof of Proposition 3.1:** Denote $\Delta = (\begin{smallmatrix} cc^1 \\ c^c \end{smallmatrix})$. Clearly, two eigenvalues of $\Delta$ are 0 and 2. Thus, $n_+(L_1 - L_2) = n_+(\Delta) = 1$ and $n_-(L_1 - L_2) = n_-(\Delta) = 0$ by (3), and therefore, $L_2 \prec L_1$ by Corollary 2.4.

For $k = 1$ and 2, let $L_k = D_k^{-1/2}L_kD_k^{-1/2}$ be the normalized Laplacians corresponding to $L_k$. We next prove that $L_1 \asymp L_2$. By Remark 1.3(ii), it suffices to prove that

$$
|n_+(L_1 - rI) - n_+(L_2 - rI)| \leq 1
$$

(4)

for any $r \in \mathbb{R}$. Recall that eigenvalues of $L_k$ are in the interval $[0, 2]$, hence $n_+(L_k - rI) \equiv n$ for $r < 0$ and $n_+(L_k - rI) \equiv 0$ for $r > 2$, and inequality (4) is obviously true unless $r \in [0, 2]$. So it remains to consider the case $r \in [0, 2]$. By Lemma 3.2 (ii), it suffices to prove that

$$
|n_+(L_1 - rD_1) - n_+(L_2 - rD_2)| \leq 1
$$

(5)

for $r \in [0, 2].$
By condition (3), we may assume that
\[ L_k = \begin{pmatrix} X_k & Y \\ Y^* & Z \end{pmatrix}, \quad k = 1, 2, \]
where \( X_k \) are 2 \times 2 matrices and \( X_1 - X_2 = w\Delta \). Let
\[ L_k - rD_k = \begin{pmatrix} X_k(r) & Y \\ Y^* & Z(r) \end{pmatrix}. \]
Then \( X_1 - X_2 = w\Delta \) implies that \( X_1(r) - X_2(r) = w(\Delta - rI) \). When \( \det(Z(r)) \neq 0 \), we have
\[ L_k - rD_k \cong \begin{pmatrix} X_k(r) - YZ^{-1}(r)Y^* & 0 \\ 0 & Z(r) \end{pmatrix}. \]
Clearly, \( n_+(L_k - rD_k) = n_+(X_k(r) - YZ^{-1}(r)Y^*) + n_+(Z(r)) \). Thus
\[ n_+(L_1 - rD_1) - n_+(L_2 - rD_2) = n_+(X_1(r) - YZ^{-1}(r)Y^*) - n_+(X_2(r) - YZ^{-1}(r)Y^*). \]
For two 2 \times 2 matrices \( X_1(r) - YZ^{-1}(r)Y^* \) and \( X_2(r) - YZ^{-1}(r)Y^* \), their difference
\[ (X_1(r) - YZ^{-1}(r)Y^*) - (X_2(r) - YZ^{-1}(r)Y^*) = X_1(r) - X_2(r) = w(\Delta - rI), \]
which is indefinite for \( r \in [0, 2] \). It must be
\[ |n_+(X_1(r) - YZ^{-1}(r)Y^*) - n_+(X_2(r) - YZ^{-1}(r)Y^*)| \leq 1. \]
It follows from (6) that (5) holds when \( r \in [0, 2] \) and \( \det(Z(r)) \neq 0 \), and so that (4) holds when \( \det(Z(r)) \neq 0 \). Clearly, there are only finitely many real numbers \( r \in [0, 2] \) such that \( \det(Z(r)) = 0 \). Hence (4) holds for all \( r \in \mathbb{R} \) by Remark 1.2, as required. The proof is complete.

Applying Proposition 3.1 to the Hermitian Laplacian matrix and Hermitian normalized Laplacian matrix of the second kind of a digraph, we obtain the following result about Mohar’s new kind of Hermitian matrices for digraphs.

**Corollary 3.3:** Let \( e \) be an arc or a digon of a digraph \( X \). Then \( \tilde{L}(X - e) \preceq \tilde{L}(X) \) and \( \tilde{L}(X - e) \bowtie \tilde{L}(X) \).

**4. Remarks**

Let \( f(x) \) be a real-rooted polynomial. For \( r \in \mathbb{R} \), let \( n(f, r) \) be the number of roots of \( f \) in the interval \( (r, +\infty) \) and denote \( f_r(x) = f(x + r) \). Then \( n(f, r) = n_+(f_r) \). Parallel to Theorem 1.1, we have the following result.

**Proposition 4.1:** Let \( f, g \) be two real-rooted polynomials and \( m \in \mathbb{Z} \). Then \( r_{i+m}(g) \leq r_i(f) \) for all \( i \) if and only if \( n(g, r) - n(f, r) \leq m \) for any \( r \in \mathbb{R} \).

In particular, \( f \bowtie g \) if and only if \( 0 \leq n(g, r) - n(f, r) \leq 1 \) for any \( r \in \mathbb{R} \), and \( f \bowtie g \) if and only if \( |n(g, r) - n(f, r)| \leq 1 \) for any \( r \in \mathbb{R} \). There are two closely related results: \( f \bowtie g \)
or $g \preceq f$ if and only if $af(x) + bg(x)$ is real-rooted for any $a, b \in \mathbb{R}$, and $f \succ g$ if and only if $af(x) + bg(x)$ is real-rooted for any $a, b \in \mathbb{R}^+$ (see [1] for more information). Using such a characterization of interlacing polynomials, Fisk [2] gave a very short proof of the Cauchy interlacing theorem.

Fan and Pall [22, Theorem 1] established the converse of the inclusion principle: Let $f$ and $g$ be two monic real-rooted polynomials satisfying $\deg f = \deg g + p$ and $r_{i+p}(f) \leq r_i(g) \leq r_{i}(f)$. Then there is one Hermitian matrix $A = [\alpha \beta, \gamma^*]$ such that the characteristic polynomials of $A$ and $B$ are $f$ and $g$, respectively. Wang and Zheng [9, Theorem 1.3] recently established a converse of Corollary 2.4: let $f$ and $g$ be two monic real-rooted polynomials with the same degree satisfying $r_{i+q}(f) \leq r_i(g) \leq r_{i-p}(f)$. Then there exist two Hermitian matrices $A$ and $B$ whose characteristic polynomials are $f$ and $g$, respectively, such that $n_+(B - A) \leq p$ and $n_-(B - A) \leq q$. These converse results can be proved by means of Proposition 4.1.

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