NOTE ON A POSITRONIUM MODEL FROM FLOW EQUATIONS
IN FRONT-FORM DYNAMICS

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In this note we address the problem of solving for the positronium mass spectrum. We use front-form dynamics together with the method of flow equations. For a special choice of the similarity function, the calculations can be simplified by analytically integrating over the azimuthal angle. One obtains an effective Hamiltonian and we solve numerically for its spectrum. Comparing our results with different approaches we find encouraging properties concerning the cutoff dependence of the results.

1. Introduction

Solving for QCD bound states from first principles remains a yet unsolved problem. There is some hope of simplifications in the framework of front-form dynamics, because of a simpler vacuum state. Different methods have been developed over the last years to cope with the stunning problem of constructing an effective Hamiltonian out of the 'infinite' canonical QCD Hamiltonian. The method of flow equations is closely related to the similarity transformations. Both methods are based on unitary transformations to block- or band-diagonalize the Hamiltonian. So far no calculations using the fully relativistic and covariant effective light-cone Hamiltonian were performed in the manner of Refs., where first the many body part of the problem is resolved. We present numerical results in the similarity flow scheme applied to positronium, with a special choice of the similarity function. We compare to the results of Ref., and point out the specific virtues of the methods.

2. Flow Equations and Similarity Renormalization

Both the flow equations of Wegner and the similarity renormalization of Wilson and Glazek can be retrieved as special cases of the more general similarity flow framework. The argument goes as follows. We construct an effective Hamiltonian $H'$ from the canonical Hamiltonian $H$ (regularized at some scale $\Lambda$) by a similarity transformation, $H' = UHU^\dagger$. The generator of the transformation is anti-hermitian, $\eta^\dagger = -\eta$. The Hamiltonian is considered a function of the flow parameter $l$, with the bare and effective Hamiltonians being $H(l = 0)$ and $H(l \to \infty)$, respectively. Its change with respect to the flow parameter is given by the commutator with the generator $\eta$ of these unitary transformations

$$\frac{dH(l)}{dl} = [\eta(l), H(l)].$$
The goal is to choose an $\eta$ such that the transformed Hamiltonian has the form
\[
H(\lambda)_{ij} = f \left( \frac{E_i - E_j}{\lambda} \right) \hat{H}_{ij}(\lambda).
\] (2)

The flow parameter is connected to the scale $\lambda$ by $l = 1/\lambda^2$, and the free particle energies $E_i$ are defined by the free Hamiltonian
\[
H_d^{(0)}(i) = E_i |i\rangle.
\] (3)

If the so-called similarity function $f(\lambda)$ vanishes for vanishing $\lambda$, then $H(l)_{ij}$ will become a (block) diagonal operator in this limit. The generator $\eta$ of the unitary transformations is by construction the commutator of the diagonal part of the Hamiltonian with its complement, $\eta = [H_d, V]$. The two schemes of Wegner and Wilson-Glazek are defined by different choices of the generator $\eta$. Note that the similarity function $f(\lambda)$ is still arbitrary in both schemes.

3. Positronium Model

A solution of Eq. (1) was described in detail in [9], so we can be brief here. One expands the Hamiltonian and the generator $\eta$ into a power series in the bare coupling $g$. With the energies $E_i$ depending on the flow parameter only in second order, we can solve the differential equation for the Hamiltonian, Eq. (1) in the energy basis defined by Eq. (3) order by order. Using the definition
\[
V^{(2)}_{ij}(l) = f(l)\hat{V}^{(2)}_{ij}(l),
\] (4)
suggested by Eq. (2), we get the solutions in the second order
\[
H^{(2)}_{d,ij}(l) = H^{(2)}_{ij}(l = 0) + \int_0^l [\eta^{(1)}_{ij}]^{(d)}(l')dl',
\] (5)
\[
\hat{V}^{(2)}_{ij}(l) = \hat{V}^{(2)}_{ij}(l = 0) + \int_0^l f(l')[\eta^{(1)}_{ij}]^{(V)}(l')dl',
\] (6)

where the superscripts $(d)$ and $(V)$ denote the diagonal and the particle number changing part, respectively. We are supposed to take the bare cutoff $\Lambda$, which defines the terms at $l = 0$, to infinity.

To evaluate Eqs. (5) and (6) and to obtain the matrix elements of the Hamiltonian, we choose the particle number conserving part of the Hamiltonian as diagonal, and we reduce the particle number violating blocks of the Hamiltonian to zero with the flow equations. This solves en passant also the multi-particle problem. Instead of having to truncate the Fock space a l`a Tamm-Dancoff, we are now dealing with isolated blocks of definite particle number. The effective matrix elements are thus obtained by the process
\[
V_{\text{eff}} = \lim_{\lambda \to 0} \left( V^{\text{gen}} + V^{\text{PT}} \right),
\] (7)
The interaction can be read off the structure of Eqs. (5) and (6): the part without the integral ($V^{PT}$) is the one obtained by usual perturbation theory. The second one ($V^{gen}$) is generated by the flow of the Hamiltonian. In other words, by reducing the off-diagonal matrix elements, we are inducing changes on the diagonal.

To calculate the matrix elements, one evaluates the associated diagrams, applying light-cone perturbation theory \[^{11}\]. The electron [positron] momenta and currents are $l_e^\mu = (k_e - k_e')^\mu$, $l_e'^\mu = (k_e - k_e'^\mu)$, and $j(j_e')^\mu = \bar{u}(k_e')\gamma'^\mu u(k_e)[j(j_e)^\mu = \bar{u}(k_e)^\mu \gamma^\mu u(k_e)]$, respectively. Here, $k_e^\mu$ and $k_e'^\mu$ are the electron momenta before and after the interaction. For the numerical calculations we use relativistic coordinates: $x = \frac{\gamma}{\gamma'}$ is the longitudinal momentum fraction, and $\vec{k}_\perp$ is the transverse momentum. We obtain

\[
V^{gen}_\lambda = \frac{j(\lambda)c_j^e}{\Delta_1} \int_{\Delta} \frac{df_{\lambda}(\Delta)}{d\lambda} f_{\lambda'}(\Delta_2) d\lambda + (1 \leftrightarrow 2) \\
V^{PT}_\lambda = \frac{j(\lambda)c_j^e}{\Delta} f_{\lambda'}(\Delta_1) f_{\lambda'}(\Delta_2) + \frac{j^+(\lambda)c_j^e}{|x-x'|}(T^{*} - \omega). 
\] (8)

Here, $T^{*} = \frac{1}{2}(k_e + k_e'^2) + \frac{1}{2}(k_e' + k_e'^2)$ is the average kinetic energy before and after the interaction and $\omega$ is one of the (unknown) eigenvalues of the full Hamiltonian. The latter ambiguity is no problem in the formalism considered here, because the perturbative term vanishes when the scale $\lambda$ goes to zero. The energy denominator is given by

\[
\mathcal{D} = |x-x'|/(T^{*} - \omega) - \frac{1}{2}(l_e^2 + l_e'^2), 
\] (9)

and we used the definition $\Delta_1 = \tilde{\Delta}_1/(x-x')$, where

\[
\tilde{\Delta}_1 = m^2 \frac{(x-x')^2}{xx'} + \frac{x'}{x} k_\perp^2 + \frac{x}{x'} k_\perp'^2 - 2k_\perp k_\perp' \cos(\varphi - \varphi'), 
\] (10)

and $\tilde{\Delta}_2 = \tilde{\Delta}_1(x \rightarrow 1-x, x' \rightarrow 1-x')$. The similarity function $f_\lambda(\Delta)$ in the interaction is still at our disposal. In the electron-positron sector we have the integral equation

\[
0 = \left( \frac{m^2 + \tilde{k}_\perp^2}{x(1-x)} - M_n^2 \right) \psi_n(x, \vec{k}_\perp; \lambda_1, \lambda_2) + \frac{G^2}{16\pi^3} \sum_{\lambda_1, \lambda_2} \int_{\mathcal{D}} dx' d^2\vec{k}'_\perp \psi_n(x', \vec{k}'_\perp; \lambda_1, \lambda_2) \psi_n(x', \vec{k}'_\perp; \lambda_1, \lambda_2) V_{\text{eff}} |x', \vec{k}'_\perp; \lambda_1, \lambda_2| \psi_n(x', \vec{k}'_\perp; \lambda_1, \lambda_2).
\] (11)

The cutoff enters into the problem via the definition of the integration domain. It is restricted by the Lepage-Brodsky cutoff on the kinetic energies $(m^2 + \tilde{k}_\perp^2)/x(1-x) \leq \Lambda^2 + 4m^2$. To simplify the numerical calculations we integrate out the azimuthal angle in the problem $(\vec{k}_\perp = k_\perp e^{i\varphi})$ by substituting it by the discrete quantum number $J_z$, cf. Refs. \[^{12}\]. To be able to perform the corresponding integral analytically, we have to choose a special similarity function

\[
f_\lambda(\Delta) = \theta(\lambda^2 - |\Delta|). 
\] (12)
Figure 1: Positronium spectrum ($\alpha = 0.3$, $\Lambda = 1.0 m$, $N_1 = N_2 = 21$): bound state masses $M_n^2$ in units of the electron mass $m^2$ in different $J_z$ sectors.

This sharp cutoff leads to difficulties in the calculations, namely the collinear singularity is not canceled exactly anymore. The scale integrals, however, become very simple in the limit $\lambda \to 0$, yielding a theta function, $\theta(\Delta_i - \Delta_j)$.

The actual matrix elements, integrated analytically over the azimuthal angle differ only by a re-definition of the spin-dependent function $\text{Int}(n)$ from the matrix elements of the Pauli ansatz, cf. e.g. Ref. 3. This makes it possible to use the numerical techniques and the computer code of Ref. 3 to calculate the positronium spectrum. The matrix elements used in the present note have an additional singular spin-independent part. We argue that this is an artifact of the choice of the similarity function and will omit it in the numerical calculations. We comment on the justification of this step below. The interaction used here is the same as listed in Appx. F of Ref. 3, with the following changes.

\[ G_1(x,k_\perp;x',k'_\perp) \rightarrow G_1 + \frac{|a_1 - a_2|}{(x - x')^2} \text{Int}(|1 - n|)\delta_{n,1} \]
\[ G_2(x,k_\perp;x',k'_\perp) \rightarrow G_2 + \frac{|a_1 - a_2|}{(x - x')^2} \text{Int}(|n|)\delta_{n,0}, \]

and in all $G_i$ the function $\text{Int}(n)$ is replaced by

\[ \text{Int}(n) = \theta(a_1 - a_2)\text{Int}_1(n) + \theta(a_2 - a_1)\text{Int}_2(n), \]

where

\[ \text{Int}_i(n) := -\frac{\alpha}{\pi}(a_i^2 - 4k_\perp^2k_{\perp'}^2(n-1)/2)^{1/2} \left( \frac{a_i(a_i^2 - 4k_\perp^2k_{\perp'}^2)^{-1/2} - 1}{2k_\perp k_{\perp'}} \right)^n. \]
Figure 2: Cutoff dependence: (a) Eigenvalues. Below: triplet (upper curve) and singlet (lower curve) ground state. Above: first excited states \( (n=2) \). (b) Hyperfine structure coefficient \( C_{hf} \). The cutoff is given in units of the electron mass \( (\alpha = 0.3, N_1 = 25, N_2 = 21) \).

Here

\[
    a_1 = m^2 \frac{(x - x')^2}{xx'} + k^2 + k'^2 - (x - x') \left[ \frac{k^2}{x} - \frac{k'^2}{x'} \right],
\]

(15)

and \( a_2 = a_1(x \to 1-x, x' \to 1-x') \). Note that the analogue of the latter expressions in the calculations with the Pauli ansatz \( \text{or} \) is their average \( a = (a_1 + a_2)/2 \).

4. Numerical results

The Hamiltonian matrix elements were derived by applying the flow equation scheme to the positronium problem in front-form dynamics. Contrary to preceding work, at this point of the calculations we use a non-perturbative method to extract the spectrum of this Hamiltonian rather than light-cone bound state perturbation theory. We have to solve the eigenvalue problem \( H_{LC}|n\rangle = M^n_2|n\rangle \) or, equivalently, the integral equation, Eq. (11). We use the algorithm set up in Ref. \( 7 \). For details of the calculations and numerical methods applied, see there. To be able to trace possible violations of rotational invariance (a non-trivial issue in front-form dynamics), we chose to work with an unphysically large coupling constant \( \alpha = 0.3 \). This is possible, because the integral equation is an algebraic function of the coupling.

The results of the computations are compiled in Fig. 1. We get the expected Bohr spectrum, the hyperfine splitting and even the correct multiplet structure. The multiplets are (almost) degenerate. We find exponential convergence of the eigenvalues when approaching the continuum limit \( (N \to \infty) \), as in Ref. \( 4 \).

The eigenvalues depend weakly on the cutoff \( \Lambda \), cf. Fig. 2(a), except for the
From analytic arguments, we expect the eigenvalues to diverge logarithmically because we used flow equations derived up second order only. However, as we shall see, the coefficient of the hyperfine splitting is very well described at moderate values of the cutoff. We therefore compare our eigenvalues to results of equal time perturbation theory at $\Lambda = 50\, m_f$. The eigenvalues are listed in Table 1 and agree cum grano salis with the known results. Note, however, that perturbation theory might be not very reliable at this large coupling.

A test for the theory is the value of the coefficient of the hyperfine splitting, $C_{hf} = (M_{\text{triplet}} - M_{\text{singlet}})/m\alpha^4$. The plot of $C_{hf}$ versus the cutoff, Fig. 2(b), is encouraging: we obtain a smooth curve, converging for large cutoffs to a value $C_{hf}^{\Lambda \to \infty} \approx 0.2825$. From equal time perturbation theory we would expect a value between $1/3$ (order $\alpha^4$) and $0.2379$ (order $\alpha^6 \log \alpha$), which is exactly what we obtain.

Let us focus on violations of rotational invariance, cf. Ref. 4. We fitted the cutoff dependence of the discrepancy of corresponding eigenvalues of different $J_z$ to a polynomial in $\log \Lambda/m_f$. The (negative) linear coefficient is smaller than $10^{-3}$, and the constant term is ca. $10^{-4}$. This means that the discrepancy between the triplet levels is 1% of the relevant (hyperfine splitting) scale at a cutoff of one fermion mass, and rises to roughly 10% when $\Lambda = 18m_f$. This suggests that to obtain full rotational invariance, one has to go to higher orders in the derivation of the flow equations.

The original idea of the flow-equation approach implies the use of a smooth similarity function to cancel the collinear singularity completely. We had to omit the spin independent singular part by hand to be able to perform the integration of the numerical counterterms. We argue that the singular part is an artifact of the choice of the similarity function. Indeed, it can be shown that the singular part of the matrix elements vanishes for the standard flow equation similarity function. There is a danger that this choice might change the eigenfunctions significantly. We found, however, that the wavefunctions are almost identical with those of Ref. 4.

### Table 1: Positronium spectrum ($\alpha=0.3, \Lambda=50\, m, N_1=N_2=21$): present results ($M_n$), perturbation theory ($M_{PT}$), and difference ($\Delta M^2=M_n^2-M_{PT}$) in percent.

| $n$ | $M_n^2(J_z=0)$ | $M_{PT}^2$ | $\Delta M^2$ | $n$ | $M_n^2(J_z=0)$ | $M_{PT}^2$ | $\Delta M^2$ |
|-----|----------------|-------------|--------------|-----|----------------|-------------|--------------|
| 1   | 3.906908       | 3.900002    | 0.6906       | 10  | 3.990189       | 3.989675    | 0.0513       |
| 2   | 3.915533       | 3.910673    | 0.4860       | 11  | 3.990239       | 3.989825    | 0.0413       |
| 3   | 3.976824       | 3.975860    | 0.0964       | 12  | 3.990345       | 3.989875    | 0.0470       |
| 4   | 3.977998       | 3.976533    | 0.1465       | 13  | 3.990368       | 3.989875    | 0.0492       |
| 5   | 3.978166       | 3.977037    | 0.1129       | 14  | 3.990423       | 3.989945    | 0.0478       |
| 6   | 3.978694       | 3.977206    | 0.1488       | 15  | 3.990432       | 3.989945    | 0.0487       |
| 7   | 3.978705       | 3.977206    | 0.1500       | 16  | 3.990441       | 3.989955    | 0.0486       |
| 8   | 3.978936       | 3.977441    | 0.1495       | 17  | 3.990506       | 3.989955    | 0.0551       |
| 9   | 3.989893       | 3.989476    | 0.0417       | 18  | 3.990519       | 3.989984    | 0.0536       |
5. Discussion

We presented the full positronium spectrum and wavefunctions using flow equation techniques in all sectors of the angular momentum $J_z$. The results are encouraging especially when looking at their cutoff dependence. Quite in general we find a weak, logarithmic, smooth dependence on the cutoff. The comparison to results of equal time perturbation theory gives good agreement. With respect to its cutoff related properties, the flow equation scheme seems to work better than the model of Ref. 4, inspired by the method of iterated resolvents. This is expected by construction of the methods and new ideas have been proposed to deal with renormalization issues in the latter method 6. The inclusion of the annihilation channel would be a straightforward calculation, cf. 7. We found that rotational invariance is obeyed to a large extent on the numerical level and withstood therefore from implementing this channel here. It remains to be investigated how the singular terms influence the spectrum. We argued that they are relics of the chosen similarity function and omitted them. The numerical results seem to support this claim. In general, it would be interesting to calculate the effective Hamiltonian to a higher order in the bare coupling constant. This would be useful to find out about the structure of the generated (irrelevant) operators and also to test the improved cutoff dependence of the results.

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1. S.J. Brodsky, H.-C. Pauli, S.S. Pinsky, Phys. Rept. 301 (1998) 299–486.
2. F. Wegner, Ann. Physik 3 (1994) 77–91.
3. K.G. Wilson, S.D. Glazek, Phys. Rev. D48 (1993) 5863; Phys. Rev. D49 (1994) 4214.
4. U. Trittmann, H.-C. Pauli, Nucl. Phys. B Proc. Suppl., in print.
5. M. Krautgärtner, H.-C. Pauli, F. Wölz, Phys. Rev. D45 (1992) 3755–3774.
6. H.-C. Pauli, Eur. Phys. Jour. C7 (1999) 289–303.
7. U. Trittmann, H.-C. Pauli, hep-th/9704215.
8. B.D. Jones, R.J. Perry, S.D. Glazek, Phys. Rev. D55 (1997) 6561–6583.
9. E. Gubankova, F. Wegner, Phys. Rev. D58 (1998) 025012.
10. T.S. Walhout, Phys. Rev. D59 (1999) 065009.
11. G.P. Lepage, S.J. Brodsky, Phys. Rev. D22 (1980) 2157–2198.
12. U. Trittmann, hep-th/9705072.