Privacy on the Blockchain: Unique Ring Signatures.

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1This report is substantially the result of my own work except where explicitly indicated in the text. The report may be freely copied and distributed provided the source is explicitly acknowledged.
Abstract

Ring signatures are cryptographic protocols designed to allow any member of a group to produce a signature on behalf of the group, without revealing the individual signer’s identity. This offers group members a level of anonymity not attainable through generic digital signature schemes. We call this property ‘plausible deniability’, or anonymity with respect to an anonymity set. We concentrate in particular on implementing privacy on the blockchain, introducing a unique ring signature scheme that works with existing blockchain systems.

We implement a unique ring signature (URS) scheme using secp256k1, creating the first implementation compatible with blockchain libraries in this way, so as for easy implementation as an Ethereum smart contract.

We review the privacy and security properties offered by the scheme we have constructed, and compare its efficiency with other commonly suggested approaches to privacy on the blockchain.
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Appendices

A  Homogeneous Elliptic Curve Equations
B  Elliptic Curve Arithmetic & ECDSA
C  The Franklin and Zhang NIZK Proof
D  Completeness
E  Unforgeability
F  Anonymity
1 Introduction

1.1 Cryptocurrencies

Satoshi Nakamoto introduced the world to the proof-of-work blockchain, through the release of the bitcoin whitepaper in 2008, allowing users and interested parties to consider for the first time a trustless system, with which it is possible to securely transfer money to untrusted and unknown recipients. Since its launch, the success of bitcoin has motivated the creation of many other cryptocurrencies, both those built upon bitcoin’s underlying structure and those built entirely independently.

Cryptocurrencies are most simply described as ‘blockchains’ with a corresponding token or coin, with which you can create transactions that are then verified and stored in a block on the underlying blockchain.

Figure 1: An illustration of a typical blockchain.

Blocks themselves are formed of a block header, which includes metadata such as the block reference number, timestamp, and a link back to the previous block; and the block content, where the list of validated transactions is stored, each containing the transaction value, sender and recipient address. The blockchain is simply the chain of blocks that have been verified and accepted as valid by the consensus protocol in play.

Before bitcoin and the blockchain, in 1983, Chaum constructed e-cash, an untraceable electronic alternative to cash. Chaum’s e-cash differs from the schemes we will explore in that it assumes existence of a central bank, from which anonymous coins can be withdrawn using blind signatures. These coins are produced with a bank’s secret key, meaning that the scheme is based on assumptions both of an honest central bank, and a central secret.

The cryptocurrencies that we explore are intended to be analogous to digital cash, with the same fungibility and anonymity guarantees that are implicit when using fiat currencies such as the pound, euro, or dollar, but without the necessity of a central, issuing authority.

1.2 Bitcoin

Bitcoin is the most popular and successful cryptocurrency, with a total market capitalisation of almost $10 billion, as of August 2016.

Bitcoin was formed by combining Adam Back’s HashCash puzzles with the established public key infrastructure, enabling the confirmation of integrity of transactions published to the bitcoin
blockchain through digital signatures. HashCash puzzles are used as a Proof of Work (PoW) task to be completed by miners, a connected network of scheme participants, in order to make the cost of subverting or reversing transactions prohibitively expensive [65].

Figure 2: Market Capitalisation of bitcoin (in USD) [9]

Due to the nature of a PoW algorithm, in order to make a change to a previous transaction, an adversary would require control of an amount of power equal to the power of all those trying to contribute to the network honestly. For this reason, it is essential that finding a solution to the given PoW algorithm is very computationally expensive. In order for blocks to be verified quickly, it is also generally required that given a solution, it is very efficient to verify that it does satisfy the PoW algorithm in question.

For bitcoin[^3], the PoW puzzle is formed, with $||$ representing concatenation:

$$\text{MSB}_k[\text{SHA-256}(\text{SHA-256}(\text{nonce}||\text{block contents})))] = 0$$

Miners try to find a nonce such that when combined with the block header of the previous block, and SHA-256 (taken from the NIST SHA-2 suite [72]) is performed on the product twice, the output is lower than a dynamically adjusted target (above, the target is $2^{256-k}$) [65].

The proof of work algorithm was an attempt to implement a ‘1 CPU, 1 vote’ distribution of power in the consensus [65], and enables the system to operate without a political or reputational requirement to be met before a potential miner is able to contribute to the consensus algorithm – a miner must simply control a sufficient amount of computational power.

The hash target is chosen in proportion to the total amount of hashing power at work across the network, so that a new block is published approximately every 10 minutes [15]. The target is adjusted every 2016 blocks (approximately every two weeks) so that mining the next 2016 blocks should take exactly two weeks, if the total network power does not change [7].

Miners are incentivised to perform these computationally expensive actions through a reward of 12.5 bitcoins per block mined (this reward halves approximately every 4 years – the most recent halving was July 9th, 2016). In addition to this, transactions can include optional transaction fees to be rewarded to the miner that includes the given transaction in a block, as additional motivation for miners to process certain transactions as quickly as possible [65]. A graph of transaction fees as a percentage of total transaction values is shown in Figure 4 below.

Bitcoin transactions simply grant the recipient the right to spend some currently ‘unspent’ bitcoin belonging to the sender. This is done through unspent transaction outputs, or UTXOs [65]. This right is transferred through ECDSA digital signatures, requiring use of a private key to send bitcoin,

[^3]: We call both the blockchain system and the unit of currency bitcoin, with no capital letter, unless starting a sentence.
and a public key to receive bitcoin. The recipient’s public address, formed through taking a hash of the public key, can also be used in the creation of transactions. The relationship between the three pieces of information for each given user is shown in Figure 5. We have, as usual, private key $x$, and public key $y = g^x$, with $g$ a generator of the group in which we are working.

The arrows along the bottom of the image are labelled with known computationally hard problems. If both of these hard problems are broken, users will be able to calculate all pieces of information with knowledge of any one. If ECDLP is broken, in particular, users will be able to calculate others’ private keys from public keys. Public addresses are used as an additional layer of security in case of this.

Users are referenced in transactions by their bitcoin addresses, of which users can have many, generally managed by a ‘wallet’. Addresses act as pseudonyms for users. Transactions are issued by a digital signature on the hash of certain data concerning the current transaction, and this digital signature and additional transaction data is then stored on the blockchain to be referenced in future transactions [42].

1.3 Blockchains

The blockchain can be described as an immutable, cumulative ledger, with consensus protocol working to maintain this ledger of all valid transactions on every node in the network. Every transaction
There are two commonly discussed problems which a blockchain consensus protocol must defend against:

**Byzantine Fault Tolerance.** Consensus must hold in the presence of many faulty or badly connected nodes in the network [30].

**Sybil Attacks.** Consensus must hold even in a network with many nodes with faked or forged identities – generally assumed to be owned by adversarial individuals [65].

Bitcoin’s protocol works towards achieving consensus under these conditions through the slow and expensive PoW algorithm, and under the additional requirement of an honest majority. Honest, here, means that the actor or actors in question follow the protocol as specified. Alternative protocols, such as a Proof of Stake rather than PoW algorithm, are currently both implemented [12], [58], and being researched in industry as a way to reach consensus more quickly and with higher security guarantees [78].

At a very high level, Proof of Stake in its current form generally works with validators instead of PoW’s miners, committing (or ‘staking’) money in the form of bets on certain blocks being accepted. These stakes replace the money miners spend on exertion of electricity in the PoW algorithm, leading to the analogy of staking as ‘virtual mining’. An often cited problem with several possible Proof of Stake algorithms is the issue of ‘nothing at stake’, meaning parties can act in a way where they can subvert the consensus algorithm at no cost to themselves.

The bitcoin blockchain is equipped with a restricted programming language, which is used to enable transactions such as transfer of funds from several input accounts to several output accounts, including multi-signature accounts and transactions, where several signatures are required before funds can be released from an account. The scripting language is expressive enough to allow cryptographic protocols such as secure multi-party computations to be constructed [14].

As bitcoin is primarily a transaction only blockchain, it is wise to restrict the scripting language to a few well-examined opcodes, as this prevents risks of unknown attacks [4].

There are, however, many other issues contributing to the hesitation of widespread adoption of bitcoin and other public blockchains, including:

**The threat of attacks by an anonymous majority or powerful individual.** This could take place in the form of theft or subversion of the consensus algorithm. Blockchain history can be rewritten if a dishonest party is in control of 51% or more of the network [65]. Other attacks

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4Without restrictions or formal verification, scripting languages (such as Ethereum’s Solidity) can lead to unexpected attacks, such as the $50 million stolen from the DAO, due to a formally unknown recursive call exploit [32].
Figure 6: Transactions per bitcoin block over time [9].

on the system can be carried out with control of around 25% of the network hashing power [58], or by groups of any size [45].

Wastefulness and lack of sustainability. In order to make double spending and other blockchain manipulation prohibitively expensive, the PoW algorithm is very computationally expensive, and so maintaining the blockchain consumes very large amounts of energy.

Low transaction rate. Bitcoin’s peak transaction rate is currently 7 transactions per second (tps) [31]. For comparison, VISA handles an average of around 2000 tps, with peak capacity of 24,000 [11].

Lack of control over the monetary supply, and extreme volatility in currency value. Cryptocurrency value is determined by market forces, and uncertainty in the new technology often leads to high volatility. Financial institutions that would otherwise find blockchain technology attractive are dissuaded by the lack of control over the monetary creation [43]. This problem is resolved somewhat through the use of private or consortium blockchains.

1.4 Ethereum

In response to the limitations of bitcoin’s restricted scripting language, the Ethereum platform was created, offering an almost Turing-complete distributed virtual machine atop the Ethereum blockchain [31], along with a currency called Ether. The increased scripting ability of the system enables developers to create ‘smart contracts’ on the blockchain, programs with rich functionality and the ability to operate on the blockchain state. The blockchain state records current ownership of money and of the local, persistent storage offered by Ethereum. Smart contracts are limited only by the amount of gas they consume. Gas is a sub-currency of the Ethereum system, existing to impose a limit on the amount of computational time an individual contract can use [77].

Users specify an upper limit on the amount of gas they are willing to spend, and the price, in Ether per gas, that they are willing to pay [77]. An overall block gas limit determines how many computations can be completed per block, and although initially set as a constant, miners have the

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5 At $0.125/kWh (US average), hash rate of 100 Mhash/s, power use of 7W, and total bitcoin hash rate of 1.626,365 TH/s, if all miners were using this FPGA [1] in the US to mine, $ would be spent on mining bitcoin every second. Realistically, there are FPGAs more efficient than this, and ASICs are 10-100 times more efficient [4], power consumption and expenditure may be considerably lower.

6 Though not exactly - executions in the Ethereum virtual machine are guaranteed to terminate, as there is an upper limit imposed on the execution time allowed before a program must halt.
ability to adjust the block limit by adding or subtracting the variable $\frac{1}{1024} \times$ previous block limit to each new block’s limit [31].

There are two types of account in the Ethereum blockchain system: ‘externally owned’ accounts, controlled with a private key (like all accounts in the bitcoin system), and ‘contracts’, controlled by the code that resides in the specific address in question. Contracts have immutable code stored at the contract address, and additional storage which can be read from and written to by the contract.

An Ethereum transaction contains the destination address, optional data, the gas limit, the sequence number and signature authorising the transaction. If the destination address corresponds to a contract, the contract code is then executed, subject to the gas limit defined above, which allows a certain number of computational steps before halting.

The ability to form smart contracts is one potential method of addressing the lack of privacy and corresponding potential lack of fungibility of coins in the Ethereum system. Although cryptocurrencies provide some privacy with the absence of identity related checks required to buy, mine, or spend coins, the full transaction history is public, enabling any motivated individual to track and link users’ purchases. This concept heavily decreases the fungibility of cryptocurrencies, allowing very revealing taint analysis of coins to be performed, and leading to suggestions of blacklisting coins which were once flagged as stolen [13].

1.5 Privacy

The blockchain privacy model is succinctly explained in the bitcoin whitepaper, and contrasted with the traditional financial security model as follows. Although transactions are public, as long as public keys do not become associated with individuals’ off-chain identities, users remain ‘anonymous’. This unlinking of off-chain identities with virtual addresses and transactions offers blockchain users a property known as pseudonymity [65].

There are certain advantages offered by a public blockchain - for example, it is possible to track assets through the blockchain with ‘coloured coins’ [6], and certificates published on the blockchain enable collectors to verify that they are acquiring a legitimate product.

Others have seen the public nature of transactions as a threat to fungibility and a contradiction to the otherwise ‘pseudonymous’ nature of bitcoin. In reaction to this, the cryptocurrencies Darkcoin (now known as Dash [54]), Monero [67], and ShadowCash [3] have been released, and many new protocols have been suggested as privacy enhancing overlays to the existing bitcoin system, most notably Bitcoin Core Developer Greg Maxwell’s CoinJoin [61], and the Zerocoin protocol [64].

We explore these schemes and the cryptographic protocols they involve, considering the practicality and security of each.

1.6 Contributions

We produce a practical implementation of a unique ring signature scheme[7] and provide detailed proofs and intuitive explanations of what security guarantees are provided by the resulting scheme. We analyse cryptographic protocols and primitives that are currently implemented in blockchain systems, and review the strengths, vulnerabilities, and the practicality of implementing suggested improvements. We explore hashing into elliptic curve groups, specifically implementing a scheme to hash into secp256k1. We work towards constructing and implementing a scheme that enables the existence of a blockchain system that is transparent enough to guarantee that there are no forged coins in the system, remains auditable as a whole, but enables privacy for the individual.

[7] All code related to this project can be found at https://github.com/rebekah93/secp256k1-urs/
2 Background

2.1 Cryptographic Primitives

Group signatures, ring signatures and their variants are constructed primarily from a few well known cryptographic primitives. We will first describe and discuss these primitives, so we can more easily describe the protocols and requirements of the schemes we deal with further on in this paper.

2.1.1 Elliptic Curve Cryptography

Until the introduction of Elliptic Curve (EC) cryptography and the Elliptic Curve Discrete Logarithm Problem (ECDLP) by Koblitz and Miller in 1985, cryptographic protocols were defined over multiplicative groups of finite fields.

The advantages of using EC groups rather than finite fields include the efficiency and speed of EC arithmetic, and the absence of any sub-exponential time algorithms with which to find the discrete logarithm in an EC group. The ECDLP in the group $E(F_q)$ is believed to be strictly more difficult than the DLP in finite fields of size $q$ \[^68\].

An elliptic curve, labelled $E(F_q)$, (with characteristic $\neq 2, 3$) consists of all the points $(x, y) \in F_q \times F_q$ that satisfy the following equation:

$$y^2 = x^3 + ax + b,$$  \hspace{1cm} $a, b \in F_q$. \hspace{1cm} (1)

$F_q$ is known as the base field of the group, and the cardinality, or order, of the EC group, represented $n = \#E(F_q)$, is defined as the total number of points $(x, y) \in E(F_q)$.

The order of the curve also is the smallest number such that for a generator $g \in E(F_q)$, we have $g^n = 1$.

Arithmetic in elliptic curve groups does not occur as one may naively assume, with $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$. Instead, point addition takes place through the formulae given below – the equations used to perform point doubling and scalar multiplication follow from the point addition formula, and are given in Appendix B. A full description can be found in \[^63\].

**Point Addition** For simple point addition, $(x_1, y_1) + (x_2, y_2) = (x_3, y_3)$, with $(x_1 \neq x_2)$, we form $(x_3, y_3)$ through the following:

For $\lambda = \frac{y_2 - y_1}{x_2 - x_1}$, \hspace{1cm} (2)

$$x_3 = \lambda^2 - x_1 - x_2,$$  \hspace{1cm} (3)

$$y_3 = \lambda(x_1 - x_3) - y_1.$$  \hspace{1cm} (4)

It is clear here that $\lambda$ is formed by taking the gradient of the line joining the two points. Point doubling replaces $\lambda$ with the tangential gradient at the point, as explained in Appendix B. Scalar multiplication occurs through repeated point doubling.

It is useful to note that is EC groups are mathematical groups, they have only one operation – EC groups are additive groups, meaning although multiplication by a scalar can be computed through repeated point doublings, there is no multiplication of two points, no exponentiation and no division. As the schemes we implement are using multiplicative notation, it is important to be aware of this. Multiplication in suggested algorithms will become point addition, and exponentiation will become scalar multiplication.

\[^6\]Characteristic is defined as the number if times the identity element must be added to itself to return the zero element.
ECDSA Transactions in blockchain systems such as bitcoin and Ethereum use the Elliptic Curve Digital Signature Algorithm (ECDSA) as authentication of a transaction [63]. ECDSA signatures are formed of pairs, \((r, s)\), constructed as shown in Appendix B, and explained in full detail in [57]. At a high level, these digital signatures fit into the scheme as shown in Figure 3 on page 8.

### 2.1.2 Assumed Hard Elliptic Curve Problems

As we have seen above, elliptic curve points are additive groups, in contrast to finite fields which are generally defined over multiplicative groups. This means the notation used when implementing and discussing schemes over elliptic curve groups differs from the notation given in the papers suggesting the algorithms. We will informally define the most commonly used cryptographic assumptions using the additive notation of elliptic curve arithmetic, and then formally explore the assumption that the scheme we use depends on. All definitions are defined with respect to an adversary \(A\).

**Definition 2.1. Elliptic Curve Discrete Logarithm Problem (ECDLP)**

\(A\) has no advantage in solving following:

Given \(Y, G \in E(\mathbb{F}_q)\), find \(x \in \mathbb{Z}\) such that \(Y = x \cdot G\).

**Definition 2.2. Computational Diffie-Hellman Assumption (EC-CDH)**

\(A\) has no advantage in the following:

Given \(a \cdot G, b \cdot G \in E(\mathbb{F}_q)\), \(a, b \in \mathbb{Z}\), compute \(ab \cdot G\).

**Definition 2.3. Decisional Diffie-Hellman Assumption (EC-DDH)**

\(A\) has no advantage in the following:

Given \(a \cdot G, b \cdot G, c \cdot G \in E(\mathbb{F}_q)\), with \(a, b, c \in \mathbb{Z}\), decide whether \(c \cdot G = ab \cdot G\).

### 2.1.3 Zero-Knowledge Proofs

Goldwasser et al. introduced ‘zero-knowledge proof of knowledge’ schemes in 1985 [52]. The purpose of the Zero-Knowledge Proof (ZKP) of knowledge is for a party to prove to a verifier that they know some secret information (represented below as \(x\)) without revealing anything about the secret in the process.

**Goldwasser’s Scheme:** Prover and verifier both know \((g, h, y_1, y_2)\), with \(g, h \neq 1, g, h \in \mathbb{G}\), \(y_1 = g^x, y_2 = h^x\), for exponent \(x \in \mathbb{Z}_q\). The scheme runs as follows [52]:

1. Prover chooses \(r \leftarrow R \mathbb{Z}_q\), sends \(a \leftarrow g^r, b \leftarrow h^r\) to the verifier.
2. Verifier responds with challenge \(c \leftarrow R \mathbb{Z}_q\) for the prover.
3. Prover responds, sending \(t \leftarrow r - cx \mod q\) to the verifier.
4. Verifier accepts if and only if \(a = g^t y_1^c\) and \(b = h^t y_2^c\)

For \(g\) and \(h\) in a group \(\mathbb{G}\) where the discrete logarithm problem is assumed hard, the scheme above is both sound and honest-verifier zero-knowledge.

A zero knowledge property would guarantee that no malicious verifier can extract additional information from the prover. Honest verifier zero knowledge, a weaker property, guarantees that if the verifier follows the protocol honestly, and chooses the challenge randomly, the zero knowledge property follows.

Proof of knowledge, as achieved in this scheme, is a property stronger than soundness. Proof of knowledge guarantees not only that if the verifier is convinced a witness exists (above, \(r\) is the
witness), but also that the prover knows such a witness \cite{22}. The property of soundness, in contrast, simply guarantees that it is impossible to prove a false statement.

Unique ring signatures rely heavily on Non-Interactive Zero Knowledge (NIZK) proofs. In contrast to the original zero-knowledge proofs, NIZK proofs require only that the two parties have access to a randomly generated ‘common reference string’.

The Fiat-Shamir transform converts a three round ZKP with interaction and public randomness needed to determine the challenge into a one round, non-interactive zero-knowledge proof of knowledge, with a hash function modelled as a random oracle.

2.1.4 Random Oracle Model

A random oracle (RO) is generally viewed as a ‘black box’ accepting inputs and generating truly random outputs for them. The random oracle records the outputs corresponding to each input queried, so that the same input will always return the same output. The random oracle model is an assumption that there exists a random oracle producing a truly random output.

Random oracles are typically modelled with hash functions in practice, leading to some criticism – hash functions have a deterministic output and so are not truly random, security proofs in the ROM may not translate to security in a practical environment \cite{53}.

The security properties of all linkable, traceable, and unique ring signatures are proved in the random oracle model, including the one we have implemented \cite{9}.

The ROM is at the foundation of the security of our scheme, and many others. As an example, the security of bitcoin mining is dependent on hash functions acting as random oracles – unpredictable, well distributed, (pseudo) random functions. Similarly, NIZKs are dependent on the common reference string being modelled as a RO – otherwise a dishonest prover would have a non-negligible chance of correctly predicting the challenge and forge a proof without having knowledge of the secret.

2.2 Group Signatures

Group signatures were introduced by Chaum and van Heyst in 1991 \cite{38}. They were conceived to allow any member of a group to produce a signature on behalf of the group, enabling users to sign with the authority of the group, without revealing the specific signer’s identity.

The concept of a group signature is that a trusted group master or manager is responsible for setting up a ‘group’ of users, (not related to the abstract mathematical structure of groups – we simply mean a collection of users of the scheme), who can then each sign messages on behalf of the whole group, without revealing their individual identity. The group master holds a master key with the ability to reveal the signer of any signature generated by a group member in the past. As a result of this, group signatures offer the participants anonymity only under the condition that the group master does not choose to reveal the signer’s identity \cite{34}.

A group signature scheme must satisfy several essential properties:

**Anonymity** An adversary has no more than a negligible advantage of correctly identifying the individual that produced the signature.

**Unforgeability** An adversary without a key has no more than a negligible probability of producing a signature that verifies correctly.

**Collusion resistance** Dishonest participants in the group cannot collude to produce a signature which will verify as another’s signature, and the scheme must offer soundness and correctness under the signature verification algorithm.

\footnote{There are, however, some group and ring signatures that exist without the random oracle model \cite{36}.}
Consider the definition of anonymity with respect to a two-party group. Less than a negligible advantage would mean here that an adversary would guess correctly which individual produced the signature with probability $\frac{1}{2} + \varepsilon$, for some negligible $\varepsilon$. This means that although the adversary has negligible advantage, the anonymity set is small and so the definition of anonymity does not align with our intuitive sense of the word. Therefore we call this property plausible deniability. Each signer can deny that they produced the signature, but the property of ‘anonymity’ that is offered does not agree with one’s intuitive definition of the word.

Although we will avoid detailing a group signature scheme (thorough explanations can be found in [38], [34], [44]), we will describe possible uses of group signature schemes. For example, a bank could allow its employees to authorise transactions by signing via a group signature protocol. This offers the employees privacy from eavesdropping third parties, and reveals nothing to the recipient other than that it was a bank employee that authorised the transaction. However, for auditability or in case of a dispute, the group master can reveal the identity of the signer in each disputed or audited transaction.

Another example is of Direct Anonymous Attestation (DAA), which uses a variant of group signatures, to enable a server to authenticate a trusted platform running remotely on an authorised user’s laptop, without compromising the individual user’s privacy by requiring their identity [33].

However, group signatures have several limitations. For example, it is impossible for third parties to know whether or not a series of messages have been signed by the same person – the only way to link messages would be through the group manager, who would have to revoke the anonymity of the signer in the process. The linking of messages is an essential property in any mixing scheme, as it can be used to provide confirmation of whether or not a group signer has already withdrawn the coins that they have rightful access to. There is also a large overhead when attempting to add or remove members from an established group, with the group manager or all group members required to perform a computationally expensive task. The presence of a trusted group manager also restricts possible use of the protocol.

2.3 Ring Signatures

Ring signatures were first suggested by Rivest et al, who introduced the RST scheme in their 2001 paper, ‘How to Leak a Secret’ [70]. Ring signatures were created in response to the limitations of group signatures, and in particular they offer honestly participating users with ‘unconditional anonymity’, and are formed without a complex setup procedure or the requirement for a group manager. They simply require users to be part of an existing public key infrastructure [16].

Ring Signature Mixing Schemes (RSMSs) allow different sets of blockchain users to generate groups and signatures on the fly, without requiring any additional trust, at the cost of little added computational time.

Ring signatures are constructed in a way that the ring can only be ‘completed’, and therefore verify correctly, if the signer has knowledge of some secret information, most commonly a private key corresponding to one of the public keys in the ring. In the signature generation algorithm, a number is generated at random for each of the other public keys in the ring, and then the signer uses the knowledge of their own private key, or some other ‘trapdoor information’, to close the ring.

Ring signatures offer users a type of anonymity by hiding transactions within a set of others’ transactions. If there are many users contributing very similar amounts to the ring, then the ring is said to have good liquidity, meaning the transactions can occur quickly, and also that transactions can be effectively mixed, with a high resistance to attempted mixing analysis attacks.

2.4 Linkable Ring Signatures

Linkable ring signature algorithms provide a scheme that allows users to sign on behalf of a group, again without revealing the individual signer’s identity, but with the additional property that any
signatures produced by the same signer, whether signing the same message or different messages, have an identifier, called a tag, linking the signatures. With this tag, third parties can efficiently verify that the signatures were produced by the same signer, without learning who that signer is.

2.4.1 Unique Ring Signatures

Unique ring signatures have a tag that links signatures if and only if the signer, message, and ring are the same across the two signatures. This tag is constructed using the signer’s private key, message, and description of the ring (most commonly a list of public keys), and enables both other ring members and third parties to observe whether or not two identical messages have been signed by the same ring member.

Possible use cases for linkable ring signatures are restricted access archives, for example a journalist may pay for access to one query from each of a range of topics. By creating a unique ring signature with the topic as message, this would enable the archive server to allow the journalist appropriate access, without compromising the individual’s privacy. Extending this example, we would create ‘$k$-times anonymous access’ [74]. Other potential uses for unique ring signatures including e-voting schemes without the need for central authorities, and other e-token systems [48].

Unique ring signatures were introduced by Franklin and Zhang in 2012 [48]. The security properties of this scheme are stated as unforgeability, secure linkability, and restricted anonymity – a fully anonymous linkable ring signature scheme would be impossible, due to the linkability itself. These security properties are guaranteed by the collision resistance of the hash functions involved, and the soundness and completeness of the zero-knowledge proof.

The unique ring signature (URS) scheme results in the “most efficient linkable ring signature in the random oracle model, for a given level of provable security” [48]. Specifically, this scheme is tightly reduced to the DDH problem [10].

Franklin and Zhang build the security proof of their suggested scheme upon security proofs of other, well-established schemes, starting from a generic digital signature scheme, and ending with an abstract unique ring signature over bilinear pairings [48].

3 Motivation

The motivation for increased privacy in cryptocurrencies is self-evident. We wish to enable cryptocurrencies to be used with a guarantee of the same level of anonymity that users take for granted with cash – it can be withdrawn from a bank without the user needing to reveal their intent, and can be spent without the merchant learning the payer’s identity. This is not yet offered by any cryptocurrency or e-cash scheme, and certainly is not offered by credit card purchases.

The most commonly considered options for implementing privacy on the blockchain include the following:

**Homomorphic Encryption (HE)** All transactions are encrypted in a way that allows transactions to be easily audited, without revealing the actual value of such transactions. Fully HE is currently very inefficient, with a billion factor overhead and keys up to 25 GB in size [50].

Confidential Transactions (CT), used to hide the value of a given transaction, are implemented using additively homomorphic encryption, which has less of a computational overhead [67].

**Mixing services** Coins can be mixed using third party servers. An example mixing scheme is shown in Figure 7. The CoinJoin scheme has been implemented many times, as it is one of the only mixing solutions available to use without a modification to the bitcoin protocol. An example mixing service, based on the CoinJoin protocol, is SharedCoin, which was hosted by blockchain.info

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10 Tight in this setting means that the probability of solving the hard problem on which the security of the scheme relies is roughly equal to the probability of solving the DDH problem in the same given time period.
until June 2014 [4], when it was found to be broken [2]. Alternatively, all participating parties can execute part of the mixing contract, eliminating the need for trusted servers. An example of this is JoinMarket, where requests for mixing orders are broadcast over Internet Relay Chat (IRC) [5]. In July 2015, it was discovered that JoinMarket was incorrectly implemented, enabling curious parties to perform analysis to link input and output address pairs [10].

Figure 7: A simple mixing contract, used to improve the fungibility of coins.

Cryptocurrencies with privacy by design This could include, for example, an obligation to enter a mix when performing any transaction – the cryptocurrency may not recognise transactions with only one input. Such a cryptocurrency could also produce all transactions in ‘zero-knowledge’, such as ZCash [55].

Access structures We can increase privacy on the blockchain by concealing metadata, or having transactions only readable to certain parties determined through access control mechanisms.

Traitor tracing This could be implemented in order to punish parties who act to decrease anonymity of others, in turn increasing the anonymity of the scheme as parties are less likely to act adversarially.

Broadcast encryption This provides a high level of anonymity for receivers, as every user in the group receives the encrypted message, although only users with the correct permission or key can decrypt. Broadcast encryption is also perfectly collusion resistant [27].

Secure hardware This can act to limit what the user can do, for example the hardware could enforce mixing before outputting transactions, or could add randomness such as that used in blind signatures [37].

Intermediaries For example, depositing and withdrawing money from an exchange removes the ‘traceability’ property.

Secure Multi-Party Computation (SMPC) This enables parties to act together in a way that no single one of them has access to all of the data, and hence no one can leak any secret information. However, inefficiencies are significant and parties are required to behave honestly [79].
Off-chain storage
This would increase privacy on the blockchain by storing sensitive data off-chain and simply accessing it when needed [79]. Either the storage hosts are trusted, or data is fragmented and stored across many nodes.

3.1 What could go wrong?
Mixes provide only plausible deniability – the transaction, sender and recipient addresses are all still public, but are no longer obviously linked, and rather exist as if ‘hidden in a crowd’. The size of the crowd, or anonymity set, depends on parameters chosen either by the mixing scheme itself [5], or by the user selecting an anonymity level they are comfortable with.

Due to the definition of anonymity, ring mixes with group size 2 have been proven to provide the user with ‘anonymity’, as long as the underlying ring signature protocol is provably secure [24]. However, in a blockchain system, this may not be an adequate level of privacy – additional information is often available and could be used to construct a persona or perform other revealing analysis on each member of the ring, enabling a motivated adversary to trace the signers as with the pseudonymity that bitcoin and Ethereum provide.

It is important to note that mixed based protocols rely on a large number of honestly participating users, in order to offer a desirable level of anonymity. We define anonymity here with respect to an anonymity set, with a higher number being more desirable and offering ‘stronger’ properties of anonymity.

In the two-party ring, the anonymity set is of size two. If a user wanted a higher level of anonymity, they could enter into a chain of mixes, each of which with a size two, to reach the preferred level of anonymity. For example, entering into eight two-party ring mixing schemes would give the user an anonymity set of $2^8$, as an adversary would have to deduce which of the two parties is the individual in question, for each of the eight rings.

Ring signatures are not produced instantaneously, and so it is preferable to allow users to enter into one large ring, rather than a chain of smaller rings.

3.1.1 The Threat of Sybil Attacks
It is not unreasonable to suggest a motivated attacker may chose to create many accounts and flood many mixing contracts with the intent to publish all of their public and private keys pairs after the legitimate party has withdrawn their funds, hence revealing the honest input output address pair. This attack would be very straight-forward to execute, and a simple overview is shown in the figure below.

Although ring signatures offer an increased level of anonymity to users, dishonest participants can remove themselves from the set of possible signers of a formerly signed message by simply revealing their own private key or tag based on the specific message and ring in question. This would allow third parties to trivially observe that the given user is not the signer of the original message.

If all but one users in a ring choose to act adversarially and reveal their private key or tag, then the remaining user is left only with the ‘pseudonymity’ they would receive using bitcoin or a similar cryptocurrency.

For a ring signature scheme, a straight-forward way to prevent an adversarial ring from revoking the exculpability of any honest member would be to implement a scheme with blinded signatures. Chaum’s blinded signatures, first suggested in 1983 [37], work as follows.

For a signer with private key $x$ and public key $Y = x \cdot G$, the message is signed with key $Y + P$, with $P = p \cdot G$, for some randomly generated $p \in \mathbb{Z}_q$. The signer also sends a zero knowledge proof of knowledge of $p$.

To produce the ring signature, the signer produces a random $r \overset{\$}{\leftarrow} \mathbb{Z}_q$ for each public key in the ring, and uses these to form $pk_i + R_i$, with $R_i = r_i \cdot G$ as the new public keys to be used to form the ring.
Zero knowledge proofs of each of these random blinding constants are also produced and sent with the signature to the verifier.

If this is done correctly, the non-signing parties will not be able to generate tags with which to compare and prove that they are not the signer in question, as they will not be aware of the $R_i$ used in conjunction with their public key. If the signer also forgets the blinding constants, they will not be able to incriminate themselves even if asked to. However, it is unclear whether this scheme would work with linkable or unique ring signatures.

### 3.1.2 Failed Mixing Attempts

Dash and SharedCoin both relied on third parties to host implementations of the CoinJoin mixing protocol. For Dash, then known as DarkCoin, nodes were not willing to host the ‘DarkSend’ mixing feature, and so five ‘masternodes’ were chosen to provide mixing services to the whole network, and in turn these nodes were promised a reward. These five nodes form an easy and attractive target for attacks, and could have chosen to collude and act adversarially, logging or publishing all the information collected about the input-output pairs in each mix. The service is now called PrivateSend, and all nodes on the network are required to run the mixing protocol. In exchange for this service, 45% of all block rewards go to the nodes that provided the mixing.

SharedCoin was a service hosted by blockchain.info, which has also been broken. In this case, an eavesdropping adversary could easily trace input-output pairs, based on transaction values and analysis explained in CoinJoin Sudoku [2].

In JoinMarket [5], all users wishing to join the mix broadcast their intentions over an IRC channel, and then all act as the mixing server. There is, however, a security problem with these mixing schemes, under the threat model of an active adversary who sets up rings and simply collects messages and input-output address pairs, without ever executing the mixing protocol. This adversary can collect valuable information about transactions while never executing the mixing protocol, meaning the information is collected at no cost [10].
3.1.3 Low Power Problems

It is easy to see that the more legitimate hashing power contributing to a given network, the more computationally expensive an attack would be. If this expense is large enough, an attacker would be crippled trying to gain a sufficient amount of hashing power, and so the honest majority will cause the blockchain to grow legitimately.

However, for some smaller alt-coins, the expenditure required to gain a sufficient amount of hashing power is not large enough to prohibit a motivated attacker from taking over the network. This was a high risk in 2014 for Litecoin and Dogecoin, which use the same hashing algorithm. If either system’s miners had decided to collude and attack the other blockchain, they could do so with devastating affects [39].

On July 20th, 2016, the Ethereum blockchain successfully completed a hard fork [32]. ETC (Ethereum Classic – the chain that did not fork) has around 16% of the hashing power of Ethereum, which leads to a vulnerability in the ETC system. If the Ethereum miners wanted to attack the ETC network, they could do so very easily.

The security that comes with the network effect and higher hashing power of existing cryptocurrencies is the main reason why creating our own private cryptocurrency is not the best choice – we choose instead to leverage the power of existing blockchain platforms.

3.2 Anonymous Alt-Coin

3.2.1 ZCash and zk-SNARKs

Zero-Knowledge Succinct Non-Interactive ARguments of Knowledge (zk-SNARKs) give cryptocurrency users the ability to hide all transaction data [21]. Zk-SNARKs require the sender to produce a proof, in zero-knowledge, of the ability to spend an amount greater than or equal to the value of the transaction they are submitting. Zk-SNARKs satisfy perfect completeness and computational soundness properties, in addition to the property of succinctness, which simply means that the proof is polynomial in the security parameter.

The proofs are currently computationally expensive to produce [23], taking several minutes to generate a key pair, and several more to generate a zero knowledge proof, but this is an active research area. The proofs are, however, very computationally efficient to verify, and the signatures are very small in size – 322 bytes for the signature, and “a few milliseconds” for the verification [75].

ZCash is a cryptocurrency which attempts to address the lack of privacy in blockchain transaction through use of zk-SNARKs. ZCash uses zk-SNARKs over a merkle tree that contains all previous transactions. Used naively, this could leak information about the latest transaction, as the transaction would be in the most recent set but not the previous one. To prevent this, the merkle tree is of constant size, $2^{64}$ [55] – for scale, if ZCash transactions occur at an average rate of 1000 transactions per second, it would take around 292 million years for the merkle tree to be filled.

ZCash is based on Zerocash which itself is an improvement of a protocol called Zerocoin, designed by a team of researchers at John Hopkins University in 2013 [64]. Zerocoin was designed to work atop the bitcoin blockchain system, with transaction origin hidden through a process involving minting and pouring of coins off the bitcoin blockchain and into zerocoins to obfuscate the otherwise transparent bitcoin transactions.

Zerocoin was implemented as an extension to bitcoin, but was not commonly used, due to efficiency problems and imperfections of the scheme [20]. Most importantly, using Zerocoin requires a 25kb proof to be produced for every coin spent, resulting in a total transaction size of 49kB. The transactions also...

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11 Formerly it was thought that an attack required control of 51% of the network [28], more recently it has been shown that adversaries can negatively influence the system with 34% or less [39].

12 A change to the accepted blockchain protocol – in this case, it was an irregular state change in order to return stolen funds to their original account.
only exist in single-denomination values, meaning many large proofs must be computed, and multiple transactions sent, in order to transfer a large number of zerocoins.

In 2014, the scheme was extended and renamed Zerocash, a full cryptocurrency that used zk-SNARKs to protect the transaction value, sender and recipient address [20].

ZCash has since been developed as an independent cryptocurrency with a further refined version of the Zerocash protocol [55]. The properties that each scheme offers, in addition to the properties offered by “Zoe”, or ZCash on Ethereum, are summarised in Table 1 below.

| Year Proposed | Zerocoin [64] | Zerocash [20] | ZCash [55] | ZoE |
|---------------|---------------|---------------|------------|-----|
| Hides origin  | ✓             | ✓             | ✓          | ✓   |
| Hides recipient | ×             | ✓             | ✓          | ✓   |
| Hides value   | ×             | ×             | ✓          | ✓   |
| Blockchain    | Bitcoin [14]  | Zerocash      | ZCash      | Ethereum |

The property of succinctness allows these schemes to produce proofs that are ‘small’ and quickly verified in comparison with other zero-knowledge proofs [20]. However, when compared with other blockchain transactions, the proofs and resulting transactions are very large, and there are stages which take a considerable amount of time. In Table 2 we compare performance and cost of the zk-SNARK based cryptocurrencies with Ethereum and the original performance of SNARKs.

| SNARKs for C [21] | Zerocoin [64] | Zerocash [20] | Ethereum |
|-------------------|---------------|---------------|----------|
| Txn size (bytes)  | 322           | ~49,000       | 996      | 65     |
| Security level (bit) | N/A       | 80/128        | 128      | 128    |
| Key gen time (mins) | 20          | ?             | 5        | 0      |
| Prove time (mins) | 22           | ‘a few’       | 1        | 0      |
| Verify time       | 4.68 secs    | 450 ms        | 5.4 ms   | ?      |

All of the schemes based on zk-SNARKs at present require a trusted setup stage – without this, the secret information used in the construction of the system can be used dishonestly to fake transactions and create new, counterfeit coins. However, even with knowledge of the secret information, user anonymity cannot be compromised under any threat model, due to the zero-knowledge property of zk-SNARKs. A multi-party setup is being considered as a way to mitigate against the necessity of a trusted setup stage [22]. Using N parties, rather than one trusted party, would offer the scheme the stronger property of security under just one-of-N of the involved parties acting honestly.

3.2.2 Monero and CryptoNote

Monero describes itself as an anonymous cryptocurrency, and uses the CryptoNote protocol to implement mixing through “Multi-layered Linkable Spontaneous Anonymous Group Signatures (M-LSAGS)” [76].

As of August 2016, Monero has a total market cap of $57,000,000, and a daily transaction volume of $8,000,000, almost double that of Ethereum, at $4,500,000 [5], both of which are only a fraction of bitcoin’s $58,000,000.

The Zerocoin operation time does not include the time required to compute the accumulator used, explained in [64].
At present, Monero has implemented a bespoke scheme to provide users with increased privacy, achieved through use of **stealth addresses** to hide transaction data, and **key images** to prevent double spending [76]. Key images are the result of a one-way function being performed on the given user's private key.

Monero use ring signatures in a ‘passive’ mixing protocol [67]. Each transaction is signed using Monero’s ring signature scheme, which produces a key image, containing information letting third parties know that the transaction has been formed correctly and is not an attempt to double spend.

Ring signatures are combined with stealth addresses, one time use addresses which are not associated with any user. The recipient of the coins can then identify where they are stored by using a private ‘viewkey’. They can then be spent by this recipient by him forming a ring signature with his private ‘spending key’ [76].

Stealth addresses were suggested by Bitcoin developer Peter Todd, and are widely used in bitcoin. The idea is as follows – instead of having one address, which leads to easy tracing, or having many addresses, which means a user must store and remember multiple different private keys, stealth addresses allow user to use multiple addresses as if they were just one – a server handles the multiple accounts to improve user experience, coin fungibility and user privacy.

Ring signatures are combined with Greg Maxwell’s **Confidential Transaction** (CT), forming ‘Ring Confidential Transactions’ [62]. CTs are used to hide the value of the transaction, but not the sender or recipient address. This is achieved through additively homomorphic encryption, resulting in transactions around 5kB, and proof size 2.5kB [62].

Monero uses an original method in order to hash to an elliptic curve, one that doesn’t not appear in any research papers, although the Monero team claim it is ‘a secure hash function’ [66]. However, there is no analysis of whether the function output is distributed uniformly at random, leading to it being accepted as indistinguishable from a random oracle, and whether the implementation is truly one way. Monero have chosen to use an Edwards curve to base their EC cryptography, due to the higher speed and, under some definitions, higher level of security offered by Curve25519 [25].

### 3.3 An Anonymous Cryptocurrency Comparison

The cryptocurrencies discussed above are addressing the privacy problem with different approaches, and so are appropriate for use in different situations.

For example, ZCash requires a trusted set up stage, but after that the system is entirely anonymous. Due to the nature of the system and its use of zero-knowledge proofs, after the first transaction involving a coin, all coins are entirely anonymous and the blockchain is ‘opaque’, revealing nothing about senders, recipients, or transaction values [55].

Although an essential property for a cryptocurrency with total anonymity, this opaqueness makes the scheme impossible to audit. It is also impossible to see whether forged coins have entered into the blockchain system, and so parties involved in the set up process must be trusted to act honestly for all time.

Ring signature mixes (RMSs) do not offer this same level of anonymity, which results in the underlying blockchain system remaining auditable. The ‘plausible deniability’ or reduced level of anonymity provided is sufficient to make statistical analysis cumbersome and unenlightening, but the scheme as a whole is still possible to audit as a unit. RMSs do not require any additional trust, which lends well to use within cryptocurrency systems.

RSMSs have much faster generation and verification stages than ZCash style zero-knowledge proofs. ZCash proofs, although fairly small at 322 bytes, and very fast to verify, take ‘one to two minutes’ to produce [55].

Linkable ring signatures have applications beyond the blockchain, for example enabling spam prevention on anonymous online forums or chat rooms, through rate limiting or blocking known spammer signatures. Another possible use case is that of random authentication from within a group - for example an attacker would find it much more difficult to perform a DoS attack or otherwise.
change the behaviour of a group of servers if the server requests and responses were signed by a randomly selected server producing a ring signature, rather than each server providing authentication by producing a digital signature.

4 Implementation & Methodology

4.1 Our Mixing Contract at a High Level

Assuming that the blockchain is equipped with an adequate scripting language, for example Ethereum’s Solidity, or Rootstock for bitcoin [59], the unique ring signature based mix scheme is implemented as follows:

1. A contract is made to verify ring signatures, receive and distribute coins on the blockchain. Parameters for the specific mix are entered into the contract.

2. Users generate public and private elliptic curve key pairs. These are not the public and private keys associated with the accounts that the coins are being sent to or from. They are from an off-chain public key infrastructure. We use randomly generated elliptic curve key-pairs (explored in more detail in Section 2.1.1)

3. Users wishing to participate in the ring mix send their public key and the agreed denomination of the cryptocurrency, for example 1 Ether, to the contract. When a sufficient number of users have sent their public keys to the contract, with sufficient defined in respect to the original contract parameters, the contract publishes the list of public keys which together form the ring.

4. The intended recipients of the coins from the mix are the holders of the secret keys corresponding to the public keys submitted to the contract. The recipient can be the same user as the sender, in which case that person can generate the key pair, or the recipient can be different to the sender, which requires the recipient of the funds to generate the key pair, and send only the public key to the sender of the funds.

5. Intended recipients send the signature to the contract. The signature includes a tag, which is unique to each signer, message, and ring.

6. The contract verifies that the tag is formed correctly, corresponding to one of the public keys in the ring. The signature and tag will only verify if:

   • The message signed is the correct message,
   • The ring in question is correct,
   • The tag is correctly formed,
   • The tag has not been seen before.

7. Funds are released to each sender of a verified signature and tag.

4.2 The Franklin-Zhang URS Scheme

The unique ring signature scheme is as follows, with the signer as the \( i^{th} \) user in the ring. We use the notation \( \leftarrow_R \), to indicate choosing an element at random from a set, for example \( t_j \leftarrow_R \mathbb{Z}_q \) shows \( t_j \) chosen at random from \( \mathbb{Z}_q \).

- **Setup(\( \lambda \))**: For \( \lambda \) the security parameter, choose multiplicative group \( \mathbb{G} \) with prime order \( q \), and randomly chosen generator \( g \) of \( \mathbb{G} \). Choose also two hash functions \( H \) and \( H' \) such that:
- $H : \{0,1\}^* \rightarrow \mathbb{G}$
- $H' : \{0,1\}^* \rightarrow \mathbb{Z}_q$

Output public parameters $\mathbf{pp} = (\lambda, q, \mathbb{G}, H, H')$.

- **RingGen($1^\lambda, \mathbf{pp}$):** Key generation algorithm for user $i$:
  - $x_i \leftarrow_R \mathbb{Z}_q$
  - $y_i \leftarrow g^{x_i}$

Output public key $\mathbf{pk}_i = (\mathbf{pp}, y_i)$, secret key $\mathbf{sk}_i = (\mathbf{pp}, x_i)$.

- **RingSig($\mathbf{sk}_i, R, m$):** The below is the protocol to sign message $m$ in the ring with description $R = (\mathbf{pk}_1, \ldots, \mathbf{pk}_n)$.

  1. For $j \in [n]$, $j \neq i$, we have:
     - $t_j, c_j \leftarrow_R \mathbb{Z}_q$
     - $a_j \leftarrow g^{t_j} y_j^{c_j}$
     - $b_j \leftarrow H(m||R)^{t_j} (H(m||R)^{x_i})^{c_j}$.

  2. For $j = i$, we have:
     - $r_i \leftarrow_R \mathbb{Z}_q$
     - $a_i \leftarrow g^{t_i} y_j^{c_j}$
     - $b_i \leftarrow H(m||R)^{t_i}$.

  3. Calculate $c_i \leftarrow [H'(m, R, \{a_j, b_j\}_1^n) - \sum_{j \neq i} c_j] \mod q$,
     - $t_i \leftarrow r_i - c_i x_i \mod q$.

  4. Return $(R, m, H(m||R)^{x_i}, c_1, t_1, \ldots, c_n, t_n)$.

- **RingVer($R, m, \sigma$):**

  Parsing the output of **RingSig**, and using the notation $H(m||R)^{x_i} = \tau$, we perform the comparison:

  $\sum_1^n c_j = H'(m, R, \{g^{t_i} y_j^{c_j}, H(m||R)^{t_i} \tau^{c_j}\}_1^n)$.

There is an important line given in this scheme, with important implications to note that when implementing this scheme over ECs. Starting from an assignment given in the scheme, we have:

$t_i \leftarrow r_i - c_i x_i \mod q, \implies r_i = t_i + c_i x_i \implies g^{r_i} = g^{t_i} g^{c_i x_i} \implies g^{r_i} = g^{t_i} (g^{x_i})^{c_i} \implies g^{r_i} = g^{t_i} y_i^{c_i}$.

Although these implications hold unconditionally over the integers, when defined over elliptic curves we must instead define them modulo the order of the generator of the EC in question, rather than the generator of the base field.

In our implementation, the generator order is labelled $n$ – when instead, the exponents of the generator are expressed modulo $q$, equalities such as $g^{x_i} = (g^{x_i})^{c_i}$, which are essential for correct, public verification of the scheme, as the precomputed $g^{x_i}$ is a public constant, but $x_i$ itself is not.

### 4.2.1 The Franklin-Zhang URS Construction

Franklin and Zhang’s unique ring signature scheme relies on Blum, Feldman and Micali’s transformation to the original zero knowledge proof scheme [45]. Blum, Feldman, and Micali (BFM) applied a transformation to the original zero-knowledge proof scheme, producing a non-interactive zero-knowledge proof (NIZK) scheme, improving greatly on the efficiency of the formerly interactive scheme [20].
With a hash function $H(\cdot)$ modelled as a random oracle, it is a known result that setting $F = H^x$ constructs a pseudo-random function $F$. As the security of our unique ring signature scheme relies on the ‘random oracle’ property of this hash function, we must be careful to use a well-distributed hash function.

Proof of membership is explained most simply as a proof of knowledge of (at least) one of a specific set of numbers. The notable change is that rather than constructing a random challenge, the challenge takes into account the randomness that was incorporated the scheme. In the case of the URS, this randomness corresponds to each of the other public keys in the ring. is included in the scheme as a way of ‘blinding’ the true signer, as probabilistically, each of the numbers are equally likely to have been generated at random.

In our scheme, we have:

$$a_i = g^{r_i} = g^{t_i} y^{c_i} = g^{t_i} g^{x_i c_i} \quad (5)$$

$$b_i = H(m||R)^{r_i} = H(m||R)^{t_i} (H(m||R)^x)^{c_i} \quad (6)$$

The exponent $t_i$ is applied to $H(m||R)$, which acts as a random oracle and is uniformly distributed in accordance with Franklin and Zhang. The scheme is congruent to a typical zero knowledge proof, with the following substitutions:

- Random witness: $W = g^{r_i}$.
- Random challenge: $c_i = H'(m, R, \{a_j, b_j\}_{1}^{n}) - \Sigma c_j \mod q$.
- Response: With $t_i = r_i - c_i x_i \mod q$, $a_i$ and $b_i$ are formed as shown in equations (5) and (6) above.
- Verification: $a_i = g^{r_i} = g^{t_i} y^{c_i} = g^{t_i} g^{c_i x_i}$, and $b_i = H(m||R)^{r_i}$, similarly.

We use the two generators to ensure that the equality holds for both the specific ring and message in question, and in the general case with the group generator.

An explanation of this proof is given in Appendix D, and is followed by proofs of completeness, unforgeability and anonymity, which are give in Appendices D, E, and F, respectively. The proofs of unforgeability and anonymity are explored through games, and adapted from the proofs given in the Franklin and Zhang paper [48]. The proof of completeness and the exploration of the NIZK scheme used are new, but were constructed in a straight-forward manner from the URS scheme.

It is also important to note here that as we are constructing these signatures within EC groups, there are some technicalities which must be addressed for the scheme to work correctly.

### 4.3 Hashing to Elliptic Curves

The Franklin-Zhang URS scheme requires us to construct a hash of the form:

$$H: \{0,1\}^* \rightarrow G.$$  

We have chosen to implement the system over EC groups for both efficiency of implementation and security, rather than the alternative of multiplicative groups formed by finite prime fields. Hence, we require a hash function of the form:

$$H: \{0,1\}^* \rightarrow E(\mathbb{F}_q).$$

More specifically, we have chosen to implement the scheme over the EC used in both bitcoin and Ethereum, secp256k1, which is defined in Section 4.3.1.

There are several algorithms that can potentially be used to hash from an arbitrary length string to an EC group. The simplest methods are “try and increment”, and simply using an existing, secure
hash function, and multiplying the output by a generator of the EC group in question. Both of these options are explored below. Other hashing to EC algorithms can be placed into two broad categories: “Icart-like” functions, based on the algorithms proposed by Icart in his 2009 ‘How to Hash into Elliptic Curves’ [56], and the more general Shallue-Woestijne-Ulas (SWU) algorithm, proposed in 2006 in the paper titled ‘Construction of rational points on elliptic curves over finite fields’ [71]. In order for either scheme to produce a reasonable output, certain criteria must be met by the underlying field over which the EC group in question is defined. Some common EC choices and their parameters are displayed in Table 3.

Table 3: A comparison of some common EC parameters.

| Source    | EC       | Security | Parameters                                      |
|-----------|----------|----------|-------------------------------------------------|
| CNSS 2012 | p256     | 128      | \(a = -3, q = 2^{256} - 2^{224} + 2^{192} + 2^{96} - 1\) |
| CNSS 2015 | p384     | 192      | \(a = -3, q = 2^{384} - 2^{128} - 2^{96} + 2^{32} - 1\) |
| CNSS 2015 | p521     | 260      | \(a = -3, q = 2^{384} - 2^{128} - 2^{96} + 2^{32} - 1\) |
| bitcoin   | secp256k1| \(\approx 128\) | \(a = 0, b = 7, q = 2^{256} - 2^{32} - 977\) |
| Bernstein | Curve25519| \(\approx 128\) | \(y^2 = x^3 + 486662x^2 + x, q = 2^{255} - 19\) |

4.3.1 secp256k1

Bitcoin and Ethereum both use the Koblitz curve secp256k1, recommended by SECG \(^{15}\) in 2000 \(^{25}\), over which to perform ECDSA to produce the signatures on transactions \(^{65}\). secp256k1 was chosen over the random secp256r1 in order to avoid having to trust the ‘randomness’ used to generate the curves parameters, and to avoid the possibility of a backdoor being included in this randomness. Koblitz curves have a particular structure which allows very fast performance when implementing EC point addition and multiplication by a scalar \(^{49}\).

The increased speed in implementation of general elliptic curve arithmetic over secp256k1 also corresponds to increased efficiency of the fastest known attacks on the ECDLP. As a result of this, secp256k1 is ‘several bits’ less secure than one would normally expect a 256 bit elliptic curve to be (which is \(\sim 128\) \(^{25}\)). The use of secp256k1 to secure all money in both the bitcoin and Ethereum systems has drawn criticism from some researchers \(^{40}\), but generally the elliptic curve choice is not thought of as weak.

secp256k1 consists of all the rational points \((x, y) \in E(\mathbb{F}_q)\) that satisfy\(^{16}\)

\[
y^2 = x^3 + 7.
\]

(7)

4.3.2 Simple Methods

**Generator Multiplication**  Shadowcash is an example of a cryptocurrency that tried to use this straight-forward but naive approach to ‘hash’ to an elliptic curve point. With \(G\) a group generator, ShadowCash achieves an EC hash output through the construction \(^{66}\):

\[
H = \text{SHA3}(y_i) \cdot G.
\]

As usual, we have that the public key is formed \(y_i = x_i \cdot G\), with \(x_i\) the randomly generated secret key. Although this seems like a reasonable approach to producing random EC points, using this method removes all privacy enhancing properties of the scheme, as identifying information can be revealed without the discrete logarithm problem being solved. Instead, the public key of the

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\(^{15}\) SECG: Standards for Efficient Cryptography Group

\(^{16}\) A brief overview of the EC cryptography leading up to how this equation is formed is given in Appendix A.
signer is trivially revealed, allowing the input and output addresses to be linked in the same way as a transaction not involving ring signatures.

ShadowCash uses key images in place of Franklin-Zhang’s tags, although they are constructed very similarly – a ShadowCash key image, \( \tau \), is formed:

\[
\tau = x_i \cdot H_p(y_i).
\]

Rearranging this equation and including the construction of the hash function \( H_p \):

\[
\tau = x_i \cdot \text{SHA3}(y_i) \cdot G = \text{SHA3}(y_i) \cdot (x_i \cdot G) = \text{SHA3}(y_i) \cdot y_i.
\]

It is clear that \( \tau \) can be produced using only public information – all we require is access to the SHA3 function, and the public key of each individual in the ring. We can produce \( \tau \) corresponding to each public key and compare against the tag in each signature, resulting in the signers having only the pseudonymity they achieve using bitcoin or a similarly unblinded cryptocurrency.

In our case, the identification tag is \( H(m || R^x) \), which corresponds to \( x_i \cdot H(m || R) \) when defined over an elliptic curve group. An analogous attack could be carried out by an adversary with knowledge of the message being signed, and all public keys belonging to the ring.

**Try and Increment** This method is applicable to all ECs, however, is probabilistic, so vulnerable to timing based side channel attacks and will not find a point with certainty. From Icart’s ‘How to Hash into Elliptic Curves’, with security parameter \( k \), the try and increment scheme is given by Algorithm 1.

Algorithm 1 Try and Increment

*Input* \( u \).

for \( i = 0 \) to \( k - 1 \) do

\[ x = u + i \]

if \( x^3 + ax + b \) is a quadratic residue in \( \mathbb{F}_q \) then

return \( Q = (x, (x^3 + ax + b)^{1/2}) \)

end if

end for

return \( \bot \)

We can use the try and increment method without compromising the security of our scheme as the input information does not need to be kept secret. So timing or other side channel attacks made possible through the non-constant time function would reveal nothing besides public information to the adversary.

There are several well-tested schemes for finding rational points on elliptic curves, such as [71], [69], [46]. However, as these schemes are not for ‘hashing’ to an elliptic curve, there is no consideration given to the distribution of the points produced, and the ‘indifferentiability’ of the scheme compared to with the random oracle model, which is essential for our scheme.

### 4.3.3 Schemes from Literature

As mentioned above, EC parameters generally have to satisfy some criteria before a known algorithm can be used with provable security guarantees. Icart-like functions build hash functions of the form [56]:

\[
H(m) = f(H'(m)),
\]
Table 4: A comparison of several ECs, with typical hash-to-EC function requirements.

| Curve     | q mod 3 | q mod 4 | q mod 12 |
|-----------|---------|---------|----------|
| p224      | 1       | 1       | 1        |
| p256      | 1       | 3       | 7        |
| p384      | 2       | 3       | 11       |
| p521      | 1       | 3       | 7        |
| secp256k1 | 1       | 3       | 7        |

with \( H' \) modelled as a random oracle, and practically implemented with an existing, secure hash function, and \( f \) a hash encoding first suggested, and detailed in full, in Icart’s ‘How to Hash into Elliptic Curves’ [56]. These actually cannot be used within our specific scheme, as Ethereum’s Virtual Machine (the EVM) must be purely deterministic, and so the randomness used within these functions is unavailable. Nevertheless, we will review possibilities in case of future relevance.

With \( q \) the order of the underlying field for the elliptic curve (so the \( \mathbb{F}_q \) used to construct \( E(\mathbb{F}_q) \)), Icart’s function requires both that \( q \equiv 3 \mod 4 \) [56], and also that \( q \equiv 2 \mod 3 \). secp256k1 satisfies the former condition, but has \( q \equiv 1 \mod 3 \), and so neither of these schemes are suitable.

The SWU scheme again requires \( q \equiv 3 \mod 4 \), which secp256k1 satisfies, but also, for an EC defined \( y^2 = x^3 + ax + b \), the scheme requires that \( a, b \neq 0 \). As secp256k1 is defined with equation \( y^2 = x^3 + 7, a = 0 \), and this function cannot be used.

Another scheme, suggested by Brier et al., takes the form \( H(m) = f(h_1(m)) + f(h_2(m)) \) with \( h_1, h_2 \) independent random oracles with values in \( \mathbb{F}_q \), \( f \) as Icart’s encoding [29]. However, this also requires \( q \equiv 2 \mod 3 \).

A comparison of some typical choices of ECs, and whether they satisfy the necessary criteria, is shown in Table 4 below.

In elliptic curve groups where the equivalence \( q \equiv 3 \mod 4 \) holds, square roots modulo \( q \) can be calculated efficiently and deterministically through use of the Tonelli-Shanks algorithm, given below. A critical review of this algorithm can be found in [60], and notably the algorithm does not run in constant time, and rather the timing follows a normal distribution.

**Definition 4.1.** Tonelli-Shanks Algorithm For \( q \) an odd prime, with \( q \equiv 3 \mod 4 \), and \( n \in \mathbb{F}_q \), the two square roots of \( n \) are \( u \) and \( -u \), with \( u \) calculated as given:

\[
u = n^{\frac{q+1}{4}} \mod q.
\]

This can be shown to hold through a very straight-forward deduction, starting with Euler’s criterion.

**Definition 4.2.** Euler’s Criterion

1. If \( n \) is a square modulo \( q \), then \( n^{\frac{q+1}{4}} \equiv 1 \mod q \).
2. If \( n \) is not square modulo \( q \), then \( n^{\frac{q+1}{4}} \equiv -1 \mod q \).

Another scheme, suggested by Brier et al., takes the form \( H(m) = f(h_1(m)) + f(h_2(m)) \) with \( h_1, h_2 \) independent random oracles with values in \( \mathbb{F}_q \), \( f \) as Icart’s encoding [29]. However, this also requires \( q \equiv 2 \mod 3 \).

**4.4 secp256k1 – not quite a Barreto-Naehrig curve**

The anonymity property of the URS scheme we have implemented relies on the hardness of the DDH problem in the group we are working in [48]. Due to this, if there exists a group pairing, from secp256k1
to a field in which the DDH problem is trivial to break, even the weakest adversaries would have the
ability to remove the additional anonymity given by the ring signature based mixing scheme.

A pairing-friendly curve is one with an embedding degree that is very low, which leads to the
finite field being mapped to having a small size, and hence low bitwise security. The embedding factor
is the factor by which the storage costs grow when storing the field that the elliptic curve has been
mapped to. With embedding degree $d$, the field size (of the new field in which we can break the DDH
problem) is $2^{256d}$.

With the underlying finite field $\mathbb{F}_q$ defined such that $q \equiv 1 \mod 3$, Barreto-Naehrig curves are
defined with the equation:

$$y^2 = x^3 + b, \quad b \in \mathbb{F}_q.$$ 

It is trivial to see that secp256k1 has this form, and it can be shown that the condition on $q$ is also
satisfied, as is given in Table 4.

However, Barreto-Naehrig curves have an embedding degree of $d = 12$ [19], and secp256k1 specifically has an embedding degree of $256$:

$$d = 19298\ 68153955\ 26992372\ 61830834\ 78131797\ 54729273\ 79845817\ 39710086\ 05235863\ 60249056.$$  (8)

This means a single pairing value would require $256 \cdot d$ bits of storage – approximately $5 \times 10^{78}$. Even
with the more optimal $d = 12$, the finite field with which the elliptic curve group is paired grows to an
extent that the discrete logarithm problem in the finite field requires more computational time than
Pollard’s Rho in the elliptic curve group [19]. Clearly, a pairing-based ECDLP break is not a threat
at present.

4.5 Hashing to Barreto-Naehrig Curves

The scheme we choose to implement works as following. Although technically not a Barreto-Naehrig
curve, secp256k1 satisfies all of the requirements for this hashing to elliptic curve algorithm to work.
This scheme is elegant, deterministic and executes in constant time, eliminating the possibility of side
channel attacks.

The scheme used and its security properties are given in full in Fouque and Tibouchi’s ‘Indif-
erentiable Hashing to Barreto-Naehrig Curves’ [47]. Importantly, it is proven that the scheme is
indifferentiable from a random oracle. With $\#E(\mathbb{F}_q)$ defined as the number of rational points on
the elliptic curve in question, the image of the hashing function is approximately $\frac{9}{16} \cdot \#E(\mathbb{F}_q)$. For
comparison, Icart’s function produces an image of $\frac{5}{8} \cdot \#E(\mathbb{F}_q)$ if $a \neq 0$, and $\frac{2}{3} \cdot \#E(\mathbb{F}_q)$ if $a = 0$ [51].

As we need to use the hashing function to generate the tags and the zero-knowledge proof of
membership, as constructed $H(m||R)^w$, with $w = x_i, r_i, t_i$, we need the hash to act as a random
oracle in order for the $H^w$ to act as pseudo-random functions as defined in the Franklin and Zhang
paper [48]. The hashing scheme we have implemented is shown in Algorithm 2, with $t$ as input, and
$b$ as defined in the EC equation $y^2 = x^3 + b$, with $(x, y) \in \mathbb{F}_q \times \mathbb{F}_q$, taken from [17].
Algorithm 2 Hashing to a Barreto-Naehrig Curve

input \( t \)

\[
w \leftarrow \frac{\sqrt{3} \cdot t}{t + b + t^2} \]

\[
x_1 \leftarrow \frac{-1 + \sqrt{-3}}{2} - tw \]

\[
x_2 \leftarrow -1 - x_1 \]

\[
x_3 \leftarrow 1 + \frac{1}{w^2} \]

\[
r_1, r_2, r_3 \leftarrow R \cdot F_q^* \]

\[
\alpha \leftarrow \chi_q(r_1^2 \cdot (x_3^2 + b)) \]

\[
\beta \leftarrow \chi_q(r_2^2 \cdot (x_3^2 + b)) \]

\[
i \leftarrow [(\alpha - 1) \cdot \beta \mod 3] + 1 \]

return \((x_i, \chi_q(r_3^2 \cdot t) \cdot \sqrt{x_1^2 + b})\).

\( \chi \) is a function defined in the Foque and Tibouchi paper such that:

\[
\chi(a) = \begin{cases} 
0, & \text{if } a = 0, \\
1, & \text{if } a \text{ is a square}, \\
-1, & \text{otherwise}.
\end{cases}
\]

We construct the \( \chi_q \) function for use within the hashing algorithm using Euler’s Criterion as previously defined, with \( q \) the order of the desired output EC group. This is possible due to secp256k1 satisfying \( q \equiv 3 \mod 4 \), as explained previously.

Using this scheme as \( H \), and SHA-256 as \( H' \), we can very easily and efficiently implement the full unique ring signature algorithm, using only either Ethereum or bitcoin’s existing dependencies.

5 Evaluation

5.1 Privacy

There is a real risk of all but one of the participating parties in the mixing contract, either through a Sybil attack or collusion, revealing their tags or secret keys, and so eliminating themselves from the anonymity group, and therefore linking the honest user’s input and output addresses.

There are several ways to mitigate against this. Chaum blinding could be used to prevent this attack from occurring, or we could punish users who reveal their signatures, in order to keep the scheme auditable. The more suitable option would be decided based on the use case – for example, in a financial environment or in a consortium use case, simply punishing users may be the easier and preferable option. Alternatively, we could randomly allocate the mix that users get entered into, which would make coordinating such a de-anonymising attack much more difficult.

Otherwise, the privacy of the scheme relies on ECDDH, which itself is closely related to ECDLP, which has been described as ‘the hardest math problem ever’ [68]. Although currently less efficient than even a brute force attack, index calculus style attacks are being developed against certain EC groups, for example p-224. As p-224 also does not satisfy the requirements for any known hashing to EC function that is indifferentiable from a random oracle, it is not a wise choice for use in this scheme.

Although transactions can be split across multiple mixes and the total transaction value can be obfuscated in this way, we have not directly incorporated hiding of transaction value within our scheme.
5.2 Scheme Security

The scheme we have implemented is secure under the EC-DDH assumption in the random oracle model. Before proving soundness, completeness and anonymity, we must formally define them in relation to our scheme. Franklin and Zhang start the proof that the zero-knowledge proof scheme is sound, but do not finish the proof or extend to the unique ring signature protocol.

The authors specifically say that it is more efficient in both the signing and verification algorithms than those suggested in ‘Linkable Spontaneous Anonymous Group Signatures in Ad Hoc Groups’, by Liu, Wei, and Wong [2004].

Security is generally described through ‘games’ - interactions between a challenger and a computationally bound adversary, with restrictions based on the threat model of the scheme. This is how the security level of the protocol is produced, as shown by the security proofs detailed in Appendices D, E, and F.

5.3 Efficiency

Although the scheme is noted as being sub-linear in the number of users in the ring, it holds rather that the tag is sub-linear (in fact, constant size), but the full signature, as we have seen before, is constructed as:

\[
\sigma = \tau, m, R, \{c_i, t_i\}_{1}^{n} \\
= \tau_x, \tau_y, m, R, \{c_i, t_i\}_{1}^{n}
\]

Depending on the construction of the contract which uses this ring signature scheme, it may be possible to simply send \(\tau\) and \(\{c_i, t_i\}_{1}^{n}\). However, as each \(c_i\) and \(t_i\) are 32 bytes, and taking into account \(\tau\), this signature is still \(64(n+1)\) bytes in length. Contrasting with ZCash, where each proof is 322 bytes, we achieve a shorter signature length for rings with size \(n < 4\), but a zk-SNARK is smaller for \(n \geq 5\).

However, ZCash is currently running on a ‘test-net’ only, and the fully functional version is due for release in October 2016 [55]. Our ring signature scheme, although expensive to implement on Ethereum, is ready to be used. The compromise between anonymity guarantees and scheme usability is at the discretion of the user, but zk-SNARKs would required a large improvement in key and proof generations times until they are comparable to URS-based mixing schemes in this aspect.

5.4 Usability

It may be possible to implement the scheme in a more user-friendly manner in the future. For example, more of the mixing contract may be automated, for example setting up rings in an ad-hoc way, and removing the necessity of sender and recipient interaction.

For transactions of large value or if there is not a lot of liquidity in the scheme, we could dynamically adjust the timings of the coin mixes so as to increase anonymity as much as possible. For example, if a large transaction is sent, rather than hoping there are other transactions of the same value for the scheme to mix among, we will split the transaction into denominations with high liquidity and mix. We may be able to further improve the anonymity of this scheme by not including all of these segments of the transaction in the same block, unlinking the input and output addresses of the full, larger transaction so as to make the large transaction less conspicuous even under careful blockchain analysis. We could do this automatically, although we would need to implement the scheme in a completely different way, as the senders and recipients may no longer be able to interact with just one contract.

Another idea for consideration, if sub-linear linkable ring signatures were available, would be to include all previously used public keys in the ring. We could do this up to a predefined maximum,
and select the public keys to be used at random, so that analysis the ring of one transaction and the ring of the previous transaction would not give away the second signer, as theirs would be the only public key not common to the two sets. We could also look into hashing the set of public keys, rather than appending them in the signature, to reduce signature size.

6 Conclusions, Summary, Further Work

6.1 Conclusions

This research has examined the existing solutions to the lack of inherent privacy for individuals in current blockchain systems. The research showed that there are schemes that offer anonymity on the blockchain, and schemes that offer a form of anonymity that we refer to as plausible deniability. Both of these have their possible limitations – for the truly anonymous blockchain systems, there is an ‘opaqueness’ to the once transparent ledger of transactions, and there is additional trust required to construct such a scheme. For the platforms offering ‘plausible deniability’, there are often vulnerabilities introduced to the scheme privacy due to incorrect implementation or practical limitations such as small anonymity groups, and there are ways for a motivated adversary to subvert the privacy of existing schemes functioning in this way.

There are no schemes or systems currently available or proposed that enable a blockchain platform to retain some of its transparency, while enabling individual users to remain truly anonymous.

6.2 Summary

We have explored cryptographic protocols, specifically concentrating on Franklin and Zhang’s unique ring signature protocol. We have implemented this scheme over secp256k1, enabling compatibility with both bitcoin and Ethereum’s EC libraries.

We have clarified an essential condition for a URS to be constructed correctly over an EC group - all variables given in the Franklin-Zhang algorithm that are intended to be used as exponents, or rather as a scalar multiplier of a point in the EC group, must be defined modulo the EC group generator order, rather than the order of underlying finite field.

We have evaluated the security of the scheme, the privacy offered by the scheme, and the common vulnerabilities introduced by imperfect implementations of similar protocols.

6.3 Further Work

There are many open problems in this general area, ranging from constructing a blinding system that works alongside linkable or unique ring signatures, to making existing linkable ring signature schemes more efficient and scalable. Sub-linear unique ring signatures exist, although they are based on bilinear pairings of ECs, and so implementing them is a much larger project than this one.

We could use a tree structure to produce a proof of knowledge of one public key of the many potentially contained on the leaves of the tree – this structure would allow proofs to be verified in logarithmic space and time \( \mathcal{O} \log n \), and can be instantiated as a zero-knowledge proof.

There are large improvements to be made in the efficiency of the Ethereum Virtual Machine, where our final scheme will be implemented. At present, it has ‘precompiles’ for verifying ECDSA, querying RIPE160 and SHA256, but there are only basic implementations of arbitrary precision big integer arithmetic, and no elliptic curve arithmetic other than ECDSA verification.

We could also approach the privacy problem with zk-SNARKs, exploring options to implement them in such a way that the blockchain platform as a whole remains a publicly verifiable system. As each potential solution is made up of many cryptographic protocols, there are many areas where efficiencies can still be gained, or security of schemes can be improved.
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Appendices

A Homogeneous Elliptic Curve Equations

Elliptic curves are defined as the coordinate points that satisfy the following (Weierstrass) equation. Formally, for a finite field \( \mathbb{F}_q \), with \( q \) a prime power, and algebraic closure of \( \mathbb{F}_q \) (which is the set \( \bigcup_{m \geq 1} \mathbb{F}_q^m \)), we have the equation below defining an elliptic curve:

\[
Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_5Z^3, \tag{9}
\]

with \( a_1, a_2, a_3, a_4, a_5 \in \bigcup_{m \geq 1} \mathbb{F}_q^m \). We call \( \mathbb{F}_q \) the base field for the elliptic curve group.

The equation shown in (9) is called the homogeneous or projective Weierstrass equation, and is defined up to equivalence in the projective plane, \( \mathbb{P}^2(\mathbb{F}_q) \). The equivalence is a simple linear relation acting on \( (\mathbb{F}_q)^3 \backslash \{(0,0,0)\} \) with \( (x_1, y_1, z_1) \sim (x_2, y_2, z_2) \) if and only if there exists \( u \in \mathbb{F}_q^* \) such that \( x_1 = ux_2, y_1 = uy_2, \) and \( z_1 = uz_2 \). The points \( (X, Y, Z) \) satisfying equation (9), above form an equivalence class of projective points \( (X: Y: Z) \).

An elliptic curve, formally, is the set of all solutions in \( \mathbb{P} \) to (9). To define smooth in this context, we first rearrange equation (9), to form:

\[
F(X, Y, Z) = Y^2Z + a_1XYZ + a_3YZ^2 - X^3 - a_2X^2Z - a_4XZ^2 - a_5Z^3 = 0.\tag{10}
\]

A smooth curve is one which has no points at which all of \( \frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \) and \( \frac{\partial F}{\partial Z} \) vanish. In other words, at least one partial derivative of \( F \) must be non-zero at each point satisfying \( F(X, Y, Z) \). The point with \( Z = 0, (0:1:0) \) acts as the ‘point at infinity’, and in affine coordinates, is represented by a line at plus and minus infinity on the \( y \) axis. This point at infinity acts as the additive zero element.

As the curve in homogeneous form is a linear equivalence class, we can divide each of \( X, Y \) and \( Z \) by \( Z \), thus giving \( Z = 1 \), a constant that we needn’t write down, and with \( x = \frac{X}{Z} \) and \( y = \frac{Y}{Z} \) we produce the affine coordinates of the curve.

B Elliptic Curve Arithmetic & ECDSA

Arithmetic in elliptic curve groups does not occur as one may naively assume, with \( (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \). Instead, point addition, point doubling and scalar point multiplication take place through the formulae given below.

**Point Addition** For simple point addition, \( (x_1, y_1) + (x_2, y_2) = (x_3, y_3) \), with \( x_1 \neq x_2 \), we form \( (x_3, y_3) \) through the following:

\[
\text{For } \lambda = \frac{y_2 - y_1}{x_2 - x_1}, \tag{11}
\]

\[
x_3 = \lambda^2 - x_1 - x_2, \tag{12}
\]

\[
y_3 = \lambda(x_1 - x_3) - y_1. \tag{13}
\]

It is clear here that \( \lambda \) is formed by taking the gradient between the two points, and addition formed in accordance with the elliptic curve equation.

**Point Doubling** For \( y_1 \neq 0 \), we set \( \lambda = \frac{3x_1^2 + a}{2y_1} \), and use the formulae for \( x_3 \) and \( y_3 \) as given above. We see that \( \lambda \) here is the tangential gradient of the point we are doubling, formed through differentiation of the elliptic curve equation. Special cases, such as when \( y_1 = 0 \), are explained in detail in Menezes’ Elliptic Curve Public Key Cryptosystems [63].
Scalar Multiplication Scalar multiplication occurs through repeated point doubling, much as ex-ponentiation over the integers $g^x$ can be calculated through multiplying $g$ by itself $x$ times.

ECDSA ECDSA signatures are formed of pairs, $(r,s)$, constructed as follows.

With $k$ a random nonce, $r$ is first formed $r = (k \cdot g)_x$ (the $x$ coordinate of the elliptic curve point given by $k \cdot g$). $s$ is then constructed $s = \frac{z + rd}{k}$, with $d$ the private key, $z$ the hash of the signed message authorising the transaction.

If $k$ is known, for example due to low entropy sources of randomness being used, we will be able to calculate the private key, as all other information is public. Reusing $k$ and $d$ together also gives adversaries all information needed to calculate the private key. This has happened several times in the history of bitcoin, leading to losses of up to $58$ million $[40]$. RFC6979 is suggested as a deterministic source of randomness, both to increase security against low entropy randomness, and against the potential for backdoors to be introduced through manipulated sources of randomness $[40]$.

C The Franklin and Zhang NIZK Proof

The Franklin and Zhang paper gives the assumption (without loss of generality) that $\log_{H(m||R)} \tau = \log_y y_i$ $[48]$. This assumption is critical in the proof. The explanation for its formation is as follows, starting from construction:

$$\tau = H(m||R)^{x_i}, \; y_i = g^{x_i}$$

$$\Rightarrow x_i = \log_{H(m||R)}(\tau) = \log_y(y_i)$$

$$\Rightarrow \log_{H(m||R)}(\tau) = \log_y(y_i)$$

From this assumption, the following proof system is constructed $[48]$:

1. For $j \in [n]$ and $j \neq i$, prover selects $c_j, t_j \leftarrow_R \mathbb{Z}_q$, and computes $a_j \leftarrow g^{c_j} y_j^{c_j}$ and $b_j \leftarrow H(m)^{t_j}(H(m)^{x_i})^{c_j}$; for $j = i$, prover selects $r_i \leftarrow_R \mathbb{Z}_q$ and computes $a_i \leftarrow g^{x_i}$ and $b_i \leftarrow H(m)^{r_i}$. Prover then sends the set \{a_j, b_j\}_1^n to the verifier.

2. Verifier sends challenge, formed $c \leftarrow_R \mathbb{Z}_q$, to the prover.

3. Prover computes $c_i \leftarrow c - \sum_{j \neq i} c_j$ and $t \leftarrow r - c_i x_i \mod q$, sends the pairs $c_1, t_1, \ldots, c_n, t_n$ to the verifier.

4. Verifier accepts if and only if $a_j = g^{c_j} y_j^{c_j}$ and $b_j = H(m)^{x_i} \tau^{c_j}$, for all $j \in [n]$.

D Completeness

We need only prove that a correctly formed signature always verifies. Therefore, our proof seeks to show:

$$\sum_{i=1}^n c_j \overset{?}{=} H'(m, R, \{g^{c_j} y_j^{c_j}, H(m||R)^{x_i} \tau^{c_j} \}_1^n).$$

By definition, we have $j \neq i$, $c_j \leftarrow_R \mathbb{Z}_q$, $c_i \leftarrow H'(m, R, \{a_j, b_j\}_1^n) - [\sum_{j \neq i} c_j] \mod q$, which gives:
\[
\sum_{j=1}^{n} c_j = \sum_{j \neq i} c_j + H'(m, R, \{a_j, b_j\}) - \sum_{j \neq i} c_j,
\]
\[
\implies \sum_{j=1}^{n} c_j = H'(m, R, \{a_j, b_j\}) - H'(m, R, \{a_i, b_i\})
\]
\[
\iff \{a_j, b_j\} = \{a_i, b_i\}
\]

For \(j \neq i\), by construction we have \(a_j = g^{i_j} y_{j}^{c_j}\), and \(b_j = H(m||R)^{\tau_i}(H(m||R)^{\tau_i})^{c_j}\).
Using the definition \(\tau = H(m||R)^{\tau_i}\) (constructed with a valid secret key \(x_i\)), we see that the verification equation holds unconditionally by definition for \(j \neq i\).

For the case \(j = i\), we need to prove the equivalence,
\[
\{a_i, b_i\} = \{g^{\tau_i}, H(m||R)^{\tau_i}\} = \{g^{\tau_i} y_{i}^{c_i}, H(m||R)^{\tau_i} \tau_i^{c_i}\}.
\]

We here must use the construction of \(t_i\) in order to show the sides of the equation are equivalent. Again this assumes knowledge of a valid secret key \(x_i\), as \(t_i\) is defined \(t_i = r_i - c_i x_i\).
Taking the first element, \(a_i\), and using the knowledge that \(y_i = g^{\tau_i}\), we see that:
\[
g^{\tau_i} = g^{t_i} y_{i}^{x_i c_i} \implies g^{\tau_i} = g^{t_i - c_i x_i} g^{x_i c_i} \implies g^{\tau_i} = g^{t_i}
\]

The proof of equivalence on the other element works similarly, replacing \(g\) with \(H(m||R)\). \(\blacksquare\)

## E Unforgeability

This proof is completed through a series of games, suggested and briefly explored in [48]. We let \(W_i\) be the event that the adversary, \(A\), succeeds in Game \(i\).

### Game 0

The original unforgeability experiment between challenger and adversary, \(A\), is described as follows:

Let \((R^*, m^*, \sigma^*)\) be the output of adversary \(A\), let \(W_0\) be the event that \(A\) succeeds in producing a verifiable triple, under the conditions that \(R^* \subseteq T\backslash CU\) (with \(T\) defined below in the key generation stage), and \(CU\) as the set of keys of corrupted users, which the adversary has access to.

In short, this means that the adversary must produce a signature with a key pair which they have not been explicitly given. We also have the condition that \(A\) has never queried \(RS(\cdot, \cdot, \cdot)\) with \((\cdot, R^*, m^*)\), and obviously we must have \(Ver(R^*, m^*, \sigma^*) = 1\).

By definition, with \(Adv^{u_1}_{RS}(A)\) meaning the advantage of adversary \(A\) in relation to the unforgeability property on the ring signing algorithm, and \(Pr[w_0]\) as the probability of success in the game above, we have:
\[
Adv^{u_1}_{RS}(A) = Pr[W_0].
\]

### Generation of \(n\) Public Keys

Challenger chooses \(x, y \leftarrow_R \mathbb{Z}_q\), and computes DDH triple (defined in section 1) \((g, X, Y, Z)\), such that \(X = g^x, Y = g^y,\) and \(Z = g^{xy}\). For all \(i \in [n]\), challenger randomly selects \(x_i \leftarrow_R \mathbb{Z}_q\) and calculates \(pk_i \leftarrow X \cdot g^{x_i}\), and gives \(T = \{pk_i\}_{i=1}^n\) to the adversary \(A\).

We have that:
\[
Adv^{u_1}_{RS}(A) = Pr[W_0].
\]
Game 1

Verification  The final forgery $(R^*, m^*, (\tau^*, \pi^*))$ checks $\exists i \in [n]$ such that $pk_i \in R^*$, and $(g, pk_i, H(m^* || R^*)^{pk_i}, \tau^*)$ is a DDH triple. This means that given $g, pk_i$ and either $H(m^* || R^*)^{pk_i}$ or $\tau^*$, it is impossible to determine which of the latter two elements you have received.

With $A_1$ an adversary that attacks the adaptive soundness property of the underlying NIZK proof system, we have that:

$$Pr[\mathcal{W}_0] - Pr[\mathcal{W}_1] \leq \text{Adv}_{\text{sound}}^{\text{sound}}(P,V)(A_1).$$

This probability is bounded in real terms by $(q_h + 1)/q$, with 1 occurring only if the adversary did not query the $H'$ oracle during its attempted forgery.

Game 2

Game 2 builds on Game 1 to include a simulator $S$ to simulate the NIZK proof of queried signatures. We have here that, for an adversary that attacks the adaptive security of the NIZK proof in Franklin and Zhang’s ‘Framework for Unique Ring Signatures’:

$$Pr[\mathcal{W}_1] - Pr[\mathcal{W}_2] \leq \text{Adv}_{\text{zk}}^{\text{zk}}(P,V)(A_2).$$

Game 3

Game 3 is an adaptation of Game 2, with the DDH triple provided replaced with a random tuple. Any adversary that can distinguish between the DDH and random triple can be converted into an adversary, here called $A_3$, who can solve the DDH problem. That is,

$$Pr[\mathcal{W}_2] - Pr[\mathcal{W}_3] \leq \text{Adv}_{\text{ddh}}^{\text{ddh}}(A_3).$$

Due to this, we have here that $Pr[\mathcal{W}_3] \leq n/q$.

The following result holds, concluding the unforgeability proof:

$$\text{Adv}_{\text{RS}}^{\text{uf}}(A) \leq \text{Adv}_{\text{ddh}}^{\text{ddh}}(A_3) + (2q_h + n + 1)/q.$$
Queries to $H$  Challenger maintains set $V$, constructed $(m, R, h, u)$, initially empty. Responses to hash queries (of the form $(m_j, R_j)$) are not as straightforward as simply hashing the input — instead, the hash queries are formed like queries to a random oracle constructed in the following way:

- Challenger randomly selects $d \leftarrow R \mathbb{Z}_q$.
- In response to a hash query on $(m_j, R_j)$, challenger checks if $\exists (m_j, R_j, h_j, u_j) \in V$ for any $h_j, u_j$.
- If so, challenger returns $h_j$.
- If not, challenger randomly selects $u_j \leftarrow R \mathbb{Z}_q$, constructs $h_j \leftarrow Y^d \cdot g^{u_j}$, returns $h_j$ and adds $(m_j, R_j, h_j, u_j)$ to set $V$.

Queries to $H'$ Challenger maintains set $V'$, constructed $(m, R, \{a_j, b_j\}_1^n, c)$, initially empty. Responses to hash queries (of the form $(m', R', \{a'_j, b'_j\}_1^n, c')$) work as follows:

- Challenger checks if there exists $(m', R', \{a'_j, b'_j\}_1^n, c')$ in $V'$.
- If so, challenger returns $c'$.
- If not, challenger picks random $c' \leftarrow R \mathbb{Z}_q$, returns this $c'$, and adds $(m', R', \{a'_j, b'_j\}_1^n, c')$ to $V'$.

Signing Queries Signing queries consider two encapsulated adversarial parties, as follows.

- Adversary $A$ queries the signing oracle with an input of the form $(j, R, m)$.
- Adversary $B$ queries $H$ and receives $h$ constructed $h \leftarrow Y^d \cdot g^u$.
- Challenger computes $\tau \leftarrow Z^d \cdot X^u \cdot Y^{dx_j} \cdot g^{\tau_j \cdot u}$, computes corresponding NIZK proof $\pi$, using secret key of user $j$.
- Challenger returns $(m, R, \tau, \pi)$ to $A$.

Challenge Adversary $A$ requests challenge $(i_0, i_1, R^*, m^*)$, with $m^*$ to be signed with respect to ring $R^*$, and $i_0, i_1 \in [n]$ indices such that $pk_{i_0}, pk_{i_1} \in T \cap R^*$. Define $T$ in the Anonymity games so that it’s known already here. Or define here. Challenger randomly chooses bit $b \leftarrow R \{0, 1\}$, returns challenge signature $\text{RS}(sk_{i_b}, R^*, m^*)$ to $A$. $A$ cannot query $\text{RS}(\cdot, \cdot)$ with $(i_0, R^*, m^*)$ or $(i_1, R^*, m^*)$.

Output Adversary $A$ outputs $b'$ as its guess.

**Game 1**

Game 1 is similar to Game 0, but when responding to a signing query $(j, m, R)$, the challenger simply simulates the proof. $c_1, t_1, \ldots, c_n, t_n$ are chosen randomly from $\mathbb{Z}_q$, and $a_j = g^{t_j} y_j^c$ and $b_j = H(m||R^|\tau_j)$ are computed for each $j \in [n]$. We let $W_1$ be the event of $A$’s success at Game 1. It is obvious to deduce that if there exists an adversary $A_1$ that can attack the adaptive NIZK property of the fundamental NIZK proof system $(P, V)$, the following holds:

$$Pr[W_0] - Pr[W_1] \leq \text{Adv}_{(P, V)}(A_1).$$
Game 2

In Game 2, the signing oracle is modified, such that the DDH triple is replaced by a random triple. Adversary $A$ can only notice the difference with negligible probability, under the DDH assumption. Specifically, the challenger simply replaces $Z$ with some $\hat{Z}$ where $\hat{Z} = g^c$ and $c \leftarrow_R \mathbb{Z}_q$. Let $W_2$ be the event that $A$ succeeds in Game 2. We can show that there exists an adversary $A_2$ such that:

$$\Pr[W_1] - \Pr[W_2] \leq \text{Adv}^{\text{ddh}}_G(A_2), \quad \Pr[W_2] = 0.5.$$  

Finally, we can combine the equations produced for each individual game, to give the result:

$$\text{Adv}^{\text{anon}}_{RS}(A) \leq \text{Adv}^{\text{ddh}}_G(A_2) + q_h/q.$$