A NEW GENERALIZATION OF SOME INTEGRAL INEQUALITIES AND THEIR APPLICATIONS

İMDAT İŞCAN

Abstract. In this paper, a new identity for convex functions is derived. A consequence of the identity is that we can derive new estimates for the remainder term of the midpoint, trapezoid, and Simpson formulae for functions whose derivatives in absolute value at certain power are convex. Some applications to special means of real numbers are also given.

1. Introduction

Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a convex function defined on the interval \( I \) of real numbers and \( a, b \in I \) with \( a < b \). The following inequality

\[
\frac{f\left(\frac{a+b}{2}\right)}{2} \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}.
\]

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions. See [1, 2, 3, 4], the results of the generalization, improvement and extention of the famous integral inequality (1.1).

The following inequality is well known in the literature as Simpson’s inequality.

Let \( f : [a, b] \to \mathbb{R} \) be a four times continuously differentiable mapping on \((a, b)\) and \( \|f^{(4)}\|_\infty = \sup_{x \in (a, b)} |f^{(4)}(x)| < \infty \). Then the following inequality holds:

\[
\left| \frac{1}{3} \left[ \frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{1}{2880} \|f^{(4)}\|_\infty (b-a)^2.
\]

In recent years many authors have studied error estimations for Simpson’s inequality: for refinements, counterparts, generalizations and new Simpson’s type inequalities, see [5, 6, 7, 8].

In [8], Sarikaya et al. obtained inequalities for differentiable convex mapping which are connected Simpson’s inequality, and they used the following lemma to prove this.

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Lemma 1. Let \( f : I \subset \mathbb{R} \rightarrow \mathbb{R} \) be an absolutely continuous mapping on \( I^o \) such that \( f' \in L[a, b] \), where \( a, b \in I^o \) with \( a < b \). Then the following equality holds:

\[
\frac{1}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x)dx = \frac{b-a}{2} \int_0^1 \left[ \left( \frac{t}{2} \right) f' \left( \frac{1+\frac{t}{2}a + \frac{1-t}{2}b}{2} \right) + \left( \frac{1-t}{2} \right) f' \left( \frac{1+t}{2}a + \frac{1-t}{2}b \right) \right] dt.
\]

The main inequality in [3], pointed out, is as follows.

Theorem 1. Let \( f : I \subset \mathbb{R} \rightarrow \mathbb{R} \) be a differentiable mapping on \( I^o \), such that \( f' \in L[a, b] \), where \( a, b \in I^o \) with \( a < b \). If \( |f'|^q \) is convex on \( [a, b] \), \( q > 1 \), then the following inequality holds,

\[
\left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{12} \left( \frac{1+2p+1}{3(p+1)} \right)^{\frac{1}{q}} \left\{ \left( \frac{3|f'(b)|^q + |f'(a)|^q}{4} \right)^{\frac{1}{q}} + \left( \frac{3|f'(a)|^q + |f'(b)|^q}{4} \right)^{\frac{1}{q}} \right\},
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \).

In [7], Sarikaya et al. obtained a new upper bound for the right-hand side of Simpson’s inequality for convex mapping.

Corollary 1. Let \( f : I \subset [0, \infty) \rightarrow \mathbb{R} \) be a differentiable mapping on \( I^o \), such that \( f' \in L[a, b] \), where \( a, b \in I^o \) with \( a < b \). If \( |f'|^q \) is convex on \( [a, b] \), \( q > 1 \), then the following inequality holds,

\[
\left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{12} \left( \frac{1+2p+1}{3(p+1)} \right)^{\frac{1}{q}} \left\{ \left( \frac{|f'(b)|^q + |f'(a+b)|^q}{4} \right)^{\frac{1}{q}} + \left( \frac{|f'(a)|^q + |f'(b)|^q}{4} \right)^{\frac{1}{q}} \right\},
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \).

In [3], some inequalities of Hermite-Hadamard type for differentiable convex mappings were presented as follows.

Theorem 2. Let \( f : I \subset \mathbb{R} \rightarrow \mathbb{R} \) be a differentiable mapping on \( I^o \), \( a, b \in I^o \) with \( a < b \). If \( |f'| \) is convex on \( [a, b] \), then the following inequality holds,

\[
\left| f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{4} \left( \frac{|f'(a)| + |f'(b)|}{2} \right).
\]

In this paper, in order to provide a unified approach to establish midpoint inequality, trapezoid inequality and Simpson’s inequality for functions whose derivatives in absolute value at certain power are convex, we derive a general integral identity for convex functions. Finally some applications for special means of real numbers are provided.
2. Main results

In order to prove our main theorems, we need the following Lemma.

**Lemma 2.** Let \( f : I \subset \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I^o \) such that \( f' \in L[a, b] \), where \( a, b \in I \) with \( a < b \) and \( \alpha, \lambda \in [0, 1] \). Then the following equality holds:

\[
\lambda (\alpha f(a) + (1 - \alpha) f(b)) + (1 - \lambda) f(\alpha a + (1 - \alpha) b) - \frac{1}{b-a} \int_a^b f(x)dx = (b-a) \left[ \int_0^{1-\alpha} (t-\alpha\lambda) f'(tb+(1-t)a) dt + \frac{1}{1-\alpha} \int_1^{1+\lambda(1-\alpha)} (t-1+\lambda(1-\alpha)) f'(tb+(1-t)a) dt \right].
\]

**Proof.** We note that

\[
I = \int_0^{1-\alpha} (t-\alpha\lambda) f'(tb+(1-t)a) dt + \frac{1}{1-\alpha} \int_1^{1+\lambda(1-\alpha)} (t-1+\lambda(1-\alpha)) f'(tb+(1-t)a) dt
\]

integrating by parts, we get

\[
I = (t-\alpha\lambda) \left[ f(tb+(1-t)a) \right]_0^{1-\alpha} - \int_0^{1-\alpha} f'(tb+(1-t)a) dt - (t-1+\lambda(1-\alpha)) \left[ f(tb+(1-t)a) \right]_1^{1-\alpha} + \int_1^{1-\alpha} f'(tb+(1-t)a) dt
\]

\[
= (1-\alpha-\alpha\lambda) \frac{f((1-\alpha)b+\alpha a)}{b-a} + \frac{\alpha \lambda f(a)}{b-a} + \frac{(1-\alpha) \lambda f(b)}{b-a} - (-\alpha+\lambda(1-\alpha)) \frac{f((1-\alpha)b+\alpha a)}{b-a} - \int_0^1 \frac{f(tb+(1-t)a)}{b-a} dt.
\]

Setting \( x = tb + (1-t)a \), and \( dx = (b-a) dt \), we obtain

\[
(b-a) I = \lambda (\alpha f(a) + (1 - \alpha) f(b)) + (1 - \lambda) f(\alpha a + (1 - \alpha) b) - \frac{1}{b-a} \int_a^b f(x)dx
\]

which gives the desired representation (2.1). \( \square \)

**Theorem 3.** Let \( f : I \subset \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I^o \) such that \( f' \in L[a, b] \), where \( a, b \in I^o \) with \( a < b \) and \( \alpha, \lambda \in [0, 1] \). If \( |f'|^q \) is convex on \([a, b]\), where \( q \in (0, 1) \).
$q \geq 1$, then the following inequality holds:

\[
(2.2) \quad \lambda (af(a) + (1 - \alpha) f(b)) + (1 - \lambda) f(\alpha a + (1 - \alpha) b) - \frac{1}{b - a} \int_a^b f(x) dx \geq \begin{cases} 
(b - a) \left\{ \gamma_2^{1 - \frac{\lambda}{\alpha}} \left( \mu_1 |f'(b)|^q + \mu_2 |f'(a)|^q \right)^{\frac{1}{q}} + \nu_2^{1 - \frac{\lambda}{\alpha}} \left( \eta_3 |f'(b)|^q + \eta_4 |f'(a)|^q \right)^{\frac{1}{q}} \right\}, & \alpha \lambda \leq 1 - \lambda \leq 1 - \lambda (1 - \alpha) \\
(b - a) \left\{ \gamma_1^{1 - \frac{\lambda}{\alpha}} \left( \mu_1 |f'(b)|^q + \mu_2 |f'(a)|^q \right)^{\frac{1}{q}} + \nu_1^{1 - \frac{\lambda}{\alpha}} \left( \eta_1 |f'(b)|^q + \eta_2 |f'(a)|^q \right)^{\frac{1}{q}} \right\}, & \alpha \lambda \leq 1 - \lambda (1 - \alpha) \leq 1 - \alpha \\
(b - a) \left\{ \gamma_3^{1 - \frac{\lambda}{\alpha}} \left( \mu_3 |f'(b)|^q + \mu_4 |f'(a)|^q \right)^{\frac{1}{q}} + \nu_3^{1 - \frac{\lambda}{\alpha}} \left( \eta_3 |f'(b)|^q + \eta_4 |f'(a)|^q \right)^{\frac{1}{q}} \right\}, & 1 - \alpha \leq \alpha \lambda \leq 1 - \lambda (1 - \alpha) 
\end{cases}
\]

where

\[
(2.3) \quad \gamma_1 = (1 - \alpha) \left[ \alpha \lambda - \frac{(1 - \alpha)}{2} \right], \quad \gamma_2 = (\alpha \lambda)^2 - \gamma_1 ,
\]

\[
(2.4) \quad v_1 = \frac{1 - (1 - \alpha)^2}{\alpha [1 - \lambda (1 - \alpha)]}, \quad v_2 = \frac{1 + (1 - \alpha)^2}{2} - (\lambda + 1) (1 - \alpha) [1 - \lambda (1 - \alpha)],
\]

\[
(2.5) \quad \mu_1 = \frac{(\alpha \lambda)^3 (1 - \alpha)^3}{3} - \alpha \lambda \frac{(1 - \alpha)^2}{2}, \quad \mu_2 = \frac{1 + \alpha^3 (1 - \alpha \lambda)^3}{3} - \frac{(1 - \alpha \lambda)}{2} (1 + \alpha^2), \quad \mu_3 = \frac{\alpha \lambda (1 - \alpha)^2}{2} - \frac{(1 - \alpha)^3}{3}, \quad \mu_4 = \frac{(\alpha \lambda - 1) (1 - \alpha^2)}{2} + \frac{1 - \alpha^3}{3},
\]

\[
(2.6) \quad \eta_1 = \frac{1 - (1 - \alpha)^3}{3} - \frac{[1 - \lambda (1 - \alpha)]}{2} (2 - \alpha), \quad \eta_2 = \frac{\lambda (1 - \alpha) \alpha^2}{2} - \frac{\alpha^3}{3}, \quad \eta_3 = \frac{[1 - \lambda (1 - \alpha)]^3}{3} - \frac{[1 - \lambda (1 - \alpha)]}{2} (1 + (1 - \alpha)^2) + \frac{1 + (1 - \alpha)^2}{3}, \quad \eta_4 = \frac{[\lambda (1 - \alpha)]^3}{3} - \frac{\lambda (1 - \alpha) \alpha^2}{2} + \frac{\alpha^3}{3}.
\]
Proof. Suppose that \( q \geq 1 \). From Lemma 2 and using the well known power mean inequality, we have

\[
\left| \lambda (\alpha f(a) + (1 - \alpha)f(b)) + (1 - \lambda)f(\alpha a + (1 - \alpha)b) - \frac{1}{b - a} \int_a^b f(x)dx \right| \leq (b - a) \left[ \int_0^{1-\alpha} |t - \alpha\lambda||f'(tb + (1-t)a)|dt + \int_{1-\alpha}^1 |t - 1 + \lambda(1 - \alpha)||f'(tb + (1-t)a)|dt \right]
\]

\[
\leq (b - a) \left\{ \left( \int_0^{1-\alpha} |t - \alpha\lambda| dt \right)^{1-\frac{1}{q}} \left( \int_{1-\alpha}^1 |t - 1 + \lambda(1 - \alpha)||f'(tb + (1-t)a)|^q dt \right)^{\frac{1}{q}} \right\}.
\]

(2.7)

Since \( |f'|^q \) is convex on \([a, b]\), we know that for \( t \in [0, 1] \)

\[
|f'(tb + (1-t)a)|^q \leq t|f'(b)|^q + (1-t)|f'(a)|^q,
\]

hence, by simple computation

(2.8)

\[
\int_0^{1-\alpha} |t - \alpha\lambda| dt = \begin{cases} \gamma_2, & \alpha\lambda \leq 1 - \alpha \\
\gamma_1, & \alpha\lambda \geq 1 - \alpha \end{cases},
\]

\[
\gamma_1 = (1 - \alpha) \left[ \alpha\lambda - \frac{(1-\alpha)^2}{2} \right], \quad \gamma_2 = (\alpha\lambda)^2 - \gamma_1,
\]

(2.9)

\[
\int_{1-\alpha}^1 |t - 1 + \lambda(1 - \alpha)| dt = \begin{cases} v_1, & 1 - \lambda(1 - \alpha) \leq 1 - \alpha \\
v_2, & 1 - \lambda(1 - \alpha) \geq 1 - \alpha \end{cases},
\]

\[
v_1 = \frac{1 - (1-\alpha)^2}{2} - \alpha[1 - \lambda(1 - \alpha)],
\]

\[
v_2 = \frac{1 + (1-\alpha)^2}{2} - (\lambda + 1)(1 - \alpha)[1 - \lambda(1 - \alpha)],
\]

(2.10)

\[
\int_0^{1-\alpha} |t - \alpha\lambda||f'(tb + (1-t)a)|^q dt \leq \int_0^{1-\alpha} |t - \alpha\lambda|[t|f'(b)|^q + (1-t)|f'(a)|^q] dt
\]

\[
= \begin{cases} \mu_1|f'(b)|^q + \mu_2|f'(a)|^q, & \alpha\lambda \leq 1 - \alpha \\
\mu_3|f'(b)|^q + \mu_4|f'(a)|^q, & \alpha\lambda \geq 1 - \alpha \end{cases},
\]

\[
\mu_1 = \frac{(\alpha\lambda)^3 + (1-\alpha)^3}{3} - \alpha \lambda \frac{(1-\alpha)^2}{2},
\]

\[
\mu_2 = \frac{1 + \alpha^3 + (1 - \alpha\lambda)^3}{3} - \frac{(1 - \alpha\lambda)}{2} (1 + \alpha^2),
\]
Corollary 4.

and

\[
\mu_3 = \alpha \lambda \frac{(1 - \alpha)^2}{2} - \frac{(1 - \alpha)^3}{3},
\]

\[
\mu_4 = \frac{(\alpha \lambda - 1) (1 - \alpha^2)}{2} + \frac{1 - \alpha^3}{3},
\]

and

\[
\int_1^{1/\alpha} |t - 1 + \lambda (1 - \alpha)| |f'((tb + (1 - t)a)|^q dt
\]

\[
\leq \int_1^{1/\alpha} |t - 1 + \lambda (1 - \alpha)| \left( |t| f'(b)|^q + (1 - t) |f'(a)|^q \right) dt
\]

(2.11) \[
= \left\{ \begin{array}{l}
\eta_1 |f'(b)|^q + \eta_2 |f'(a)|^q, \quad 1 - \lambda (1 - \alpha) \leq 1 - \alpha \\
\eta_3 |f'(b)|^q + \eta_4 |f'(a)|^q, \quad 1 - \lambda (1 - \alpha) \geq 1 - \alpha
\end{array} \right.,
\]

where \( \eta_1, \eta_2, \eta_3 \) and \( \eta_4 \) are defined as in (2.6). Thus, using (2.8)-(2.11) in (2.7), we obtain the inequality (2.12). This completes the proof. \( \square \)

Corollary 2. Let the assumptions of Theorem 3 hold. Then for \( q = 1 \) the inequality (2.2) reduced to the following inequality

(2.12) \[
\left| \lambda (\alpha f(a) + (1 - \alpha) f(b)) + (1 - \lambda) f(\alpha a + (1 - \alpha) b) - \frac{1}{b - a} \int_a^b f(x) dx \right|
\]

\[
\leq \left\{ \begin{array}{l}
(b - a) \left\{ \mu_1 + \eta_3 \right\} |f'(b)|^q + (\mu_2 + \eta_4) |f'(a)|^q, \quad \alpha \lambda \leq 1 - \alpha \leq 1 - \lambda (1 - \alpha) \\
(b - a) \left\{ \mu_1 + \eta_3 \right\} |f'(b)|^q + (\mu_2 + \eta_4) |f'(a)|^q, \quad \alpha \lambda \leq 1 - \lambda (1 - \alpha) \leq 1 - \alpha \\
(b - a) \left\{ \mu_3 + \eta_3 \right\} |f'(b)|^q + (\mu_2 + \eta_4) |f'(a)|^q, \quad 1 - \lambda \leq \alpha \lambda \leq 1 - \lambda (1 - \alpha)
\end{array} \right.
\]

Corollary 3. Let the assumptions of Theorem 3 hold. Then for \( \alpha = \frac{1}{2} \) and \( \lambda = \frac{1}{3} \), from the inequality (2.2) we get the following Simpson type inequality

(2.13) \[
\left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{a + b}{2} \right) + f(b) \right] - \frac{1}{b - a} \int_a^b f(x) dx \right|
\]

\[
\leq (b - a) \left( \frac{5}{72} \right)^{1-\frac{q}{2}} \left\{ \left( \frac{29}{1296} |f'(b)|^q + \frac{61}{1296} |f'(a)|^q \right)^{1-\frac{q}{2}}
\]

\[
+ \left( \frac{61}{1296} |f'(b)|^q + \frac{29}{1296} |f'(a)|^q \right)^{1-\frac{q}{2}} \right\},
\]

which is the same of the inequality in [7, Theorem 10] for \( s = 1 \).

Corollary 4. Let the assumptions of Theorem 3 hold. Then for \( \alpha = \frac{1}{2} \) and \( \lambda = 0 \), from the inequality (2.2) we get the following midpoint type inequality

(2.14) \[
\left| f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(x) dx \right|
\]

\[
\leq \left( \frac{b - a}{8} \right) \left\{ \left( \frac{1}{3} |f'(b)|^q + 2 |f'(a)|^q \right)^{1-\frac{q}{2}} + \left( \frac{2}{3} |f'(b)|^q + |f'(a)|^q \right)^{1-\frac{q}{2}} \right\}.
\]
Corollary 5. In Corollary 4 if \( q = 1 \), then we have the following midpoint type inequality

\[
\left| \frac{f\left(\frac{a+b}{2}\right)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{4} \left( \frac{|f'(a)| + |f'(b)|}{2} \right),
\]

which is the same of the inequality (1.4).

Corollary 6. Let the assumptions of Theorem 3 hold. Then for \( \alpha = \frac{1}{2} \) and \( \lambda = 1 \), from the inequality (2.2) we get the following trapezoid type inequality

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \\
\leq \frac{b-a}{8} \left\{ \left( \frac{|f'(b)|^q + 5|f'(a)|^q}{6} \right)^{\frac{1}{q}} + \left( \frac{5|f'(b)|^q + |f'(a)|^q}{6} \right)^{\frac{1}{q}} \right\}
\]

Using Lemma 2 we shall give another result for convex functions as follows.

Theorem 4. Let \( f : I \subset \mathbb{R} \rightarrow \mathbb{R} \) be a differentiable mapping on \( I^o \) such that \( f' \in L[a,b] \), where \( a,b \in I^o \) with \( a < b \) and \( \alpha, \lambda \in [0,1] \). If \( |f'|^q \) is convex on \( [a,b] \), \( q > 1 \), then the following inequality holds:

\[
\left| \lambda(\alpha f(a) + (1- \alpha)f(b)) + (1- \lambda)(\alpha a + (1- \alpha)b) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq (b-a)
\]

\[
\times \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ \begin{array}{ll}
(1- \alpha)^{\frac{1}{q}} \varepsilon_1^{\frac{1}{q}} \delta_1^{\frac{1}{q}} + \alpha^\frac{1}{q} \varepsilon_3^\frac{1}{q} \delta_2^{\frac{1}{q}}, & \alpha \lambda \leq 1- \alpha \leq 1 - \lambda(1- \alpha) \\
(1- \alpha)^{\frac{1}{q}} \varepsilon_1^{\frac{1}{q}} \delta_1^{\frac{1}{q}} + \alpha^\frac{1}{q} \varepsilon_2^\frac{1}{q} \delta_2^{\frac{1}{q}}, & \alpha \lambda \leq 1 - \lambda(1- \alpha) \leq 1 - \alpha \\
(1- \alpha)^{\frac{1}{q}} \varepsilon_2^{\frac{1}{q}} \delta_1^{\frac{1}{q}} + \alpha^\frac{1}{q} \varepsilon_3^\frac{1}{q} \delta_2^{\frac{1}{q}}, & 1- \alpha \leq \alpha \lambda \leq 1 - \lambda(1- \alpha)
\end{array} \right.
\]

where

\[
\delta_1 = \frac{|f'((1- \alpha)b + \alpha a)|^q + |f'(a)|^q}{2}, \quad \delta_2 = \frac{|f'((1- \alpha)b + \alpha a)|^q + |f'(b)|^q}{2},
\]

\[
\varepsilon_1 = (\alpha \lambda)^{p+1} + (1- \alpha - \alpha \lambda)^{p+1}, \quad \varepsilon_2 = (\alpha \lambda)^{p+1} - (\alpha \lambda - 1 + \alpha)^{p+1},
\]

\[
\varepsilon_3 = [\lambda(1- \alpha)]^{p+1} + [\alpha - \lambda(1- \alpha)]^{p+1}, \quad \varepsilon_4 = [\lambda(1- \alpha)]^{p+1} - [\lambda(1- \alpha) - \alpha]^{p+1},
\]

and \( \frac{1}{p} + \frac{1}{q} = 1 \).
Proof. From Lemma 2 and by Hölder’s integral inequality, we have
\[
\lambda (\alpha f(a) + (1 - \alpha) f(b)) + (1 - \lambda) f(\alpha a + (1 - \alpha) b) - \frac{1}{b-a} \int_a^b f(x)dx
\]
\[
\leq (b-a) \left[ \int_0^{1-\alpha} |t-\alpha\lambda| |f'(tb+(1-t)a)|dt + \int_{1-\alpha}^1 |t-1+\lambda(1-\alpha)| |f'(tb+(1-t)a)|dt \right]
\]
\[
\leq (b-a) \left\{ \left( \int_0^{1-\alpha} |t-\alpha\lambda|^p dt \right)^\frac{1}{p} \left( \int_0^1 |f'(tb+(1-t)a)|^q dt \right)^\frac{1}{q} \right\}.\]

(2.18) + \left( \int_1^{1-\alpha} |t-1+\lambda(1-\alpha)|^p dt \right)^\frac{1}{p} \left( \int_1^1 |f'(tb+(1-t)a)|^q dt \right)^\frac{1}{q} \right\}

Since \(|f'|^q\) is convex on \([a,b]\), for \(\alpha \in [0,1]\) by the inequality (1.1), we get
\[
\int_0^{1-\alpha} |f'(tb+(1-t)a)|^q dt = (1-\alpha) \left[ \frac{1}{(1-\alpha)(b-a)} \int_a^b |f'(x)|^q dx \right]
\]
\[
(2.19) \leq (1-\alpha) \frac{|f'((1-\alpha)b+\alpha a)|^q + |f'((1-\alpha)b)|^q}{2}.
\]

The inequality (2.19) holds for \(\alpha = 1\) too. Similarly, for \(\alpha \in (0,1]\) by the inequality (1.1), we have
\[
\int_{1-\alpha}^1 |f'(tb+(1-t)a)|^q dt = \alpha \left[ \frac{1}{\alpha(b-a)} \int_{(1-\alpha)b+\alpha a}^b |f'(x)|^q dx \right]
\]
\[
(2.20) \leq \alpha \frac{|f'((1-\alpha)b+\alpha a)|^q + |f'(b)|^q}{2}.
\]

The inequality (2.20) holds for \(\alpha = 0\) too. By simple computation
\[
(2.21) \int_0^{1-\alpha} |t-\alpha\lambda|^p dt = \left\{ \begin{array}{ll}
\frac{(\alpha \lambda)^{p+1}+(1-\alpha-\lambda)^{p+1}}{(\alpha \lambda)^{p+1}-\lambda(1-\alpha^{p+1})}, & \alpha \lambda \leq 1-\alpha \\
\frac{(\alpha \lambda)^{p+1}+(1-\alpha-\lambda)^{p+1}}{(\alpha \lambda)^{p+1}-\lambda(1-\alpha^{p+1})}, & \alpha \lambda \geq 1-\alpha
\end{array} \right.,
\]
and
\[
(2.22) \int_{1-\alpha}^1 |t-1+\lambda(1-\alpha)|^p dt = \left\{ \begin{array}{ll}
\frac{|\lambda(1-\alpha)|^{p+1}+|\alpha-\lambda(1-\alpha)|^{p+1}}{|\lambda(1-\alpha)|^{p+1}-\lambda(1-\alpha)^{p+1}}, & 1-\alpha \leq 1-\lambda(1-\alpha) \\
\frac{|\lambda(1-\alpha)|^{p+1}+|\alpha-\lambda(1-\alpha)|^{p+1}}{|\lambda(1-\alpha)|^{p+1}-\lambda(1-\alpha)^{p+1}}, & 1-\alpha \geq 1-\lambda(1-\alpha)
\end{array} \right.,
\]
thus, using (2.19)-(2.22) in (2.18), we obtain the inequality (2.16). This completes the proof. \(\square\)
Corollary 7. Let the assumptions of Theorem 4 hold. Then for \( \alpha = \frac{1}{2} \) and \( \lambda = \frac{1}{3} \), from the inequality (2.16) we get the following trapezoid type inequality

\[
(2.23) \quad \left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{a + b}{2} \right) + f(b) \right] - \frac{1}{b - a} \int_a^b f(x)dx \right| \leq \frac{b - a}{12} \left( 1 + \frac{2^{p+1}}{3(p+1)} \right)^{\frac{1}{p}} \left\{ \left( \frac{|f' (a + \frac{b}{2})|^q + |f' (a)|^q}{2} \right) \frac{1}{p} + \left( \frac{|f' (a + \frac{b}{2})|^q + |f' (b)|^q}{2} \right) \frac{1}{q} \right\}.
\]

which is the same of the inequality (1.3).

Remark 1. We note that if we use convexity of \(|f'|^q\) in the inequality (2.23) then we obtain the inequality (1.2).

Corollary 8. Let the assumptions of Theorem 4 hold. Then for \( \alpha = \frac{1}{2} \) and \( \lambda = 0 \), from the inequality (2.16) we get the following midpoint type inequality

\[
\left| f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(x)dx \right| \leq \frac{b - a}{4} \left( 1 + \frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ \left( \frac{|f' (a + \frac{b}{2})|^q + |f' (a)|^q}{2} \right) \frac{1}{p} + \left( \frac{|f' (a + \frac{b}{2})|^q + |f' (b)|^q}{2} \right) \frac{1}{q} \right\}.
\]

Corollary 9. Let the assumptions of Theorem 4 hold. Then for \( \alpha = \frac{1}{2} \) and \( \lambda = 1 \), from the inequality (2.16) we get the following trapezoid type inequality

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x)dx \right| \leq \frac{b - a}{4} \left( 1 + \frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ \left( \frac{|f' (a + \frac{b}{2})|^q + |f' (a)|^q}{2} \right) \frac{1}{p} + \left( \frac{|f' (a + \frac{b}{2})|^q + |f' (b)|^q}{2} \right) \frac{1}{q} \right\}.
\]

Theorem 5. Let \( f : I \subset \mathbb{R} \rightarrow \mathbb{R} \) be a differentiable mapping on \( I^0 \) such that \( f' \in L[a, b] \), where \( a, b \in I^0 \) with \( a < b \) and \( \alpha, \lambda \in [0, 1] \). If \(|f'|^q\) is convex on \([a, b]\), \( q > 1 \), then the following inequality holds,

\[
(2.24) \quad \left| \lambda (\alpha f(a) + (1 - \alpha) f(b)) + (1 - \lambda) f(\alpha a + (1-\alpha) b) - \frac{1}{b - a} \int_a^b f(x)dx \right| \leq (b - a)
\]

\[
\times \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ \begin{array}{ll}
\left[ \frac{1}{2} \delta_3^\frac{1}{4} + \frac{1}{2} \delta_4^\frac{1}{4} \right], & \alpha \lambda \leq 1 - \alpha \leq 1 - \lambda (1 - \alpha) \\
\left[ \frac{1}{2} \delta_3^\frac{1}{4} + \frac{1}{2} \delta_4^\frac{1}{4} \right], & \alpha \lambda \leq 1 - \lambda (1 - \alpha) \leq 1 - \alpha \\
\left[ \frac{1}{2} \delta_3^\frac{1}{4} + \frac{1}{2} \delta_4^\frac{1}{4} \right], & 1 - \alpha \leq \alpha \lambda \leq 1 - \lambda (1 - \alpha) 
\end{array} \right\}.
\]
Let the assumptions of Theorem 5 hold. Then for Corollary 10, the proof.

Hence, from the inequality (2.16) we get the following Simpson type inequality

\[ |f'(tb + (1-t)a)|^q \leq t |f'(b)|^q + (1-t) |f'(a)|^q. \]

Hence

\[
\begin{align*}
\lambda (\alpha f(a) + (1 - \alpha) f(b)) + (1 - \lambda) f(\alpha a + (1 - \alpha) b) &- \frac{1}{b - a} \int_a^b f(x)dx \\
\leq (b - a) \left\{ \left( \int_0^{1 - \alpha} |t - \alpha \lambda|^p dt \right)^\frac{1}{p} \left( \int_0^{1 - \alpha} t |f'(b)|^q + (1 - t) |f'(a)|^q dt \right)^\frac{1}{q} \\
&+ \left( \int_{1 - \alpha}^{1} |t - 1 + \lambda (1 - \alpha)|^p dt \right)^\frac{1}{p} \left( \int_{1 - \alpha}^{1} t |f'(b)|^q + (1 - t) |f'(a)|^q dt \right)^\frac{1}{q} \right\} \\
\leq (b - a) \left\{ \left( \int_0^{1 - \alpha} |t - \alpha \lambda|^p dt \right)^\frac{1}{p} \left( \frac{|f'(b)|^q (1 - \alpha)^2 + (1 - \alpha^2) |f'(a)|^q}{2} \right)^\frac{1}{q} \\
&+ \left( \int_{1 - \alpha}^{1} |t - 1 + \lambda (1 - \alpha)|^p dt \right)^\frac{1}{p} \left( \frac{|f'(b)|^q \alpha (2 - \alpha) + \alpha^2 |f'(a)|^q}{2} \right)^\frac{1}{q} \right\}. \\
\end{align*}
\]

(2.25) + \left\{ \left( \int_0^{1 - \alpha} |t - \alpha \lambda|^p dt \right)^\frac{1}{p} \left( \frac{|f'(b)|^q (1 - \alpha)^2 + (1 - \alpha^2) |f'(a)|^q}{2} \right)^\frac{1}{q} \\
&+ \left( \int_{1 - \alpha}^{1} |t - 1 + \lambda (1 - \alpha)|^p dt \right)^\frac{1}{p} \left( \frac{|f'(b)|^q \alpha (2 - \alpha) + \alpha^2 |f'(a)|^q}{2} \right)^\frac{1}{q} \right\}.
\]

thus, using (2.21), (2.22) in (2.24), we obtain the inequality (2.24). This completes the proof. \qed

Corollary 10. Let the assumptions of Theorem 5 hold. Then for \( \alpha = \frac{1}{2} \) and \( \lambda = \frac{1}{4} \), from the inequality (2.17) we get the following Simpson type inequality

\[
\begin{align*}
\left\{ \frac{1}{6} \left[ f(a) + 4f \left( \frac{a + b}{2} \right) + f(b) \right] - \frac{1}{b - a} \int_a^b f(x)dx \right\} \\
\leq \frac{b - a}{12} \left\{ \left( \frac{3 |f'(b)|^q + |f'(a)|^q}{4} \right)^\frac{1}{q} + \left( \frac{3 |f'(b)|^q + |f'(a)|^q}{4} \right)^\frac{1}{q} \right\},
\end{align*}
\]

which is the same of the inequality (1.2).
3. Some applications for special means

Let us recall the following special means of arbitrary real numbers $a, b$ with $a \neq b$ and $\alpha \in [0, 1]$

(1) The weighted arithmetic mean

$$A_\alpha (a, b) := a\alpha + (1 - \alpha)b, \ a, b \in \mathbb{R}.$$ 

(2) The unweighted arithmetic mean

$$A (a, b) := \frac{a + b}{2}, \ a, b \in \mathbb{R}.$$ 

(3) The weighted geometric mean

$$G_\alpha (a, b) = a^\alpha b^{1-\alpha}, \ a, b > 0.$$ 

(4) The unweighted geometric mean

$$G (a, b) = \sqrt{ab}, \ a, b > 0.$$ 

(5) The weighted harmonic mean

$$H_\alpha (a, b) := \left( \frac{a}{a} + \frac{1-\alpha}{b} \right)^{-1}, \ a, b \in \mathbb{R} \setminus \{0\}.$$ 

(6) The unweighted harmonic mean

$$H (a, b) := \frac{2ab}{a+b}, \ a, b \in \mathbb{R} \setminus \{0\}.$$ 

(7) The Logarithmic mean

$$L (a, b) := \frac{b - a}{\ln |b| - \ln |a|}, \ |a| \neq |b|, \ ab \neq 0.$$ 

(8) Then n-Logarithmic mean

$$L_n (a, b) := \left( \frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)} \right)^{\frac{1}{n}}, \ n \in \mathbb{Z} \setminus \{-1, 0\}, \ a, b \in \mathbb{R}, \ a \neq b.$$ 

(9) The identric mean

$$I (a, b) = \frac{1}{e} \left( \frac{b^b}{a^a} \right)^{\frac{1}{b-a}}, \ a, b > 0, \ a \neq b.$$
Proposition 1. Let \( a, b \in \mathbb{R} \) with \( a < b \), \( 0 \notin [a, b] \) and \( n \in \mathbb{Z}, \ |n| \geq 2 \). Then, for \( \alpha, \lambda \in [0, 1] \) and \( q \geq 1 \), we have the following inequality:

\[
|\lambda A_\alpha (a^n, b^n) + (1 - \lambda) A^n_\lambda (a, b) - L^n_\alpha (a, b)| \leq \begin{cases} 
(b - a) |n| \left( \gamma_2^{1 - \frac{1}{q}} \left( \mu_1 |b|^{(n-1)q} + \mu_2 |a|^{(n-1)q} \right)^\frac{1}{q} \right) \\
+ v_2^{1 - \frac{1}{q}} \left( \eta_3 |b|^{(n-1)q} + \eta_4 |a|^{(n-1)q} \right)^\frac{1}{q}, \alpha \lambda \leq 1 - \alpha \leq 1 - (1 - \alpha) \\
(b - a) |n| \left( \gamma_1^{1 - \frac{1}{q}} \left( \mu_3 |b|^{(n-1)q} + \mu_4 |a|^{(n-1)q} \right)^\frac{1}{q} \right) \\
+ v_2^{1 - \frac{1}{q}} \left( \eta_3 |b|^{(n-1)q} + \eta_4 |a|^{(n-1)q} \right)^\frac{1}{q}, \alpha \lambda \leq 1 - (1 - \alpha) \leq 1 - \alpha \\
(b - a) |n| \left( \mu_3 |b|^{(n-1)q} + \mu_4 |a|^{(n-1)q} \right)^\frac{1}{q} \\
\end{cases}
\]

where \( \gamma_1, \gamma_2, v_1, v_2, \mu_1, \mu_2, \mu_3, \mu_4, \eta_1, \eta_2, \eta_3, \eta_4 \) numbers are defined as in (2.3)-(2.7).

Proof. The assertion follows from Theorem 3 for \( f(x) = x^n, \ x \in \mathbb{R}, \ n \in \mathbb{Z}, \ |n| \geq 2 \).

Proposition 2. Let \( a, b \in \mathbb{R} \) with \( a < b \), \( 0 \notin [a, b] \), and \( n \in \mathbb{Z}, \ |n| \geq 2 \). Then, for \( \alpha, \lambda \in [0, 1] \) and \( q > 1 \), we have the following inequality:

\[
|\lambda A_\alpha (a^n, b^n) + (1 - \lambda) A^n_\lambda (a, b) - L^n_\alpha (a, b)| \leq (b - a) \left( \frac{1}{p + 1} \right)^\frac{1}{q} |n| 
\]

\[
\times \begin{cases} 
(1 - \alpha)^{\frac{1}{q}} \varepsilon_1^{\frac{1}{q}} \theta_1 + \alpha^{\frac{1}{q}} \varepsilon_2^{\frac{1}{q}} \theta_2, \alpha \lambda \leq 1 - \alpha \leq 1 - (1 - \alpha) \\
(1 - \alpha)^{\frac{1}{q}} \varepsilon_1^{\frac{1}{q}} \theta_1 + \alpha^{\frac{1}{q}} \varepsilon_3^{\frac{1}{q}} \theta_2, \alpha \lambda \leq 1 - (1 - \alpha) \leq 1 - \alpha \\
(1 - \alpha)^{\frac{1}{q}} \varepsilon_2^{\frac{1}{q}} \theta_1 + \alpha^{\frac{1}{q}} \varepsilon_4^{\frac{1}{q}} \theta_2, 1 - \alpha \leq \alpha \lambda \leq 1 - (1 - \alpha) \\
\end{cases}
\]

where

\[
\theta_1 = A^{\frac{1}{q}} \left( |A^{(n-1)q}_\alpha (a, b)|, |a|^{(n-1)q} \right), \ 
\theta_2 = A^{\frac{1}{q}} \left( |A^{(n-1)q}_{\lambda} (a, b)|, |b|^{(n-1)q} \right), \frac{1}{p} + \frac{1}{q} = 1,
\]

and \( \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \) numbers are defined as in (2.14).

Proof. The assertion follows from Theorem 4 for \( f(x) = x^n, \ x \in \mathbb{R}, \ n \in \mathbb{Z}, \ |n| \geq 2 \).
Proposition 3. Let $a, b \in \mathbb{R}$ with $a < b$, $0 \notin [a, b]$. Then, for $\alpha, \lambda \in [0, 1]$ and $q \geq 1$, we have the following inequality:

$$\left| \lambda H_{\alpha}^{-1} (a, b) + (1 - \lambda) A_{\alpha}^{-1} (a, b) - L^{-1} (a, b) \right| \leq (b-a) \left\{ \begin{array}{l}
\gamma_1 \frac{1-q}{q} \left( \mu_1 \frac{1}{|b|^{1-q}} + \mu_2 \frac{1}{|a|^{1-q}} \right) + v_2 \frac{1-q}{q} \left( \eta_1 \frac{1}{|b|^{1-q}} + \eta_2 \frac{1}{|a|^{1-q}} \right), \\
\gamma_2 \frac{1-q}{q} \left( \mu_1 \frac{1}{|b|^{1-q}} + \mu_2 \frac{1}{|a|^{1-q}} \right) + v_2 \frac{1-q}{q} \left( \eta_1 \frac{1}{|b|^{1-q}} + \eta_2 \frac{1}{|a|^{1-q}} \right), \\
\gamma_3 \frac{1-q}{q} \left( \mu_3 \frac{1}{|b|^{1-q}} + \mu_4 \frac{1}{|a|^{1-q}} \right) + v_2 \frac{1-q}{q} \left( \eta_3 \frac{1}{|b|^{1-q}} + \eta_4 \frac{1}{|a|^{1-q}} \right)
\end{array} \right\}
\alpha \lambda \leq 1 - \alpha \leq 1 - \lambda (1 - \alpha)$$

where $\gamma_1, \gamma_2, v_1, v_2, \mu_1, \mu_2, \mu_3, \mu_4, \eta_1, \eta_2, \eta_3, \eta_4$ numbers are defined as in \eqref{2.3}, \eqref{2.7}.

Proof. The assertion follows from Theorem 3 for $f(x) = \frac{1}{x}$, $x \in \mathbb{R} \setminus \{0\}$. \hfill $\Box$

Proposition 4. Let $a, b \in \mathbb{R}$ with $0 < a < b$. Then, for $\alpha, \lambda \in [0, 1]$ and $q \geq 1$, we have the following inequality:

$$\left| \lambda H_{\alpha}^{-1} (a, b) + (1 - \lambda) A_{\alpha}^{-1} (a, b) - L^{-1} (a, b) \right| \leq (b-a) \left( \frac{1}{p+1} \right)^\frac{1}{q}$$

$$\times \left\{ \begin{array}{l}
(1 - \alpha)^{\frac{1}{p}} \varepsilon_1 \theta_3 \right. + \alpha^\frac{1}{p} \varepsilon_3 \theta_4, \\
(1 - \alpha)^{\frac{1}{p}} \varepsilon_1 \theta_3 \right. + \alpha^\frac{1}{p} \varepsilon_3 \theta_4, \\
(1 - \alpha)^{\frac{1}{p}} \varepsilon_2 \theta_3 \right. + \alpha^\frac{1}{p} \varepsilon_4 \theta_4
\end{array} \right\}
\alpha \lambda \leq 1 - \alpha \leq 1 - \lambda (1 - \alpha)$$

where

$$\theta_3 = H^{-\frac{1}{q}} (A_{\alpha}^{2q} (a, b), a^{2q}) \quad \theta_4 = H^{-\frac{1}{q}} (A_{\alpha}^{2q} (a, b), b^{2q}) \quad \frac{1}{p} + \frac{1}{q} = 1,$$

and $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ numbers are defined as in \eqref{2.17}.

Proof. The assertion follows from Theorem 4 for $f(x) = \frac{1}{x}$, $x \in \mathbb{R} \setminus \{0\}$. \hfill $\Box$
Proposition 5. Let \( a, b \in \mathbb{R} \) with \( 0 < a < b \). Then, for \( \alpha, \lambda \in [0, 1] \) and \( q \geq 1 \), we have the following inequality:

\[
|A_\lambda \left( \ln G_\alpha (a, b), \ln A_\alpha (a, b) \right) - \ln I(a, b)| \leq \begin{cases}
(b - a) \left\{ \gamma_2 \left( \frac{1}{p} \left( \mu_1 \frac{1}{p} + \mu_2 \frac{1}{p} \right) \right) \right\}^{\frac{1}{p}}, & \alpha \lambda \leq 1 - \alpha \leq 1 - \lambda (1 - \alpha) \\
+ v_2 \left( \eta_3 \frac{1}{p} + \eta_4 \frac{1}{p} \right)^{\frac{1}{p}}, & 1 - \alpha \leq \alpha \lambda \leq 1 - \lambda (1 - \alpha)
\end{cases}
\]

where \( \gamma_1, \gamma_2, v_1, v_2, \mu_1, \mu_2, \mu_3, \mu_4, \eta_1, \eta_2, \eta_3, \eta_4 \) numbers are defined as in \((2.3),(2.7)\).

Proof. The assertion follows from Theorem 3 for \( f(x) = -\ln x, \ x > 0 \). □

Proposition 6. Let \( a, b \in \mathbb{R} \) with \( 0 < a < b \). Then, for \( \alpha, \lambda \in [0, 1] \) and \( q \geq 1 \), we have the following inequality:

\[
|A_\lambda \left( \ln G_\alpha (a, b), \ln A_\alpha (a, b) \right) - \ln I(a, b)| \leq (b - a) \left( \frac{1}{p + 1} \right)^{\frac{1}{p}} \left| \begin{array}{c}
(1 - \alpha)\frac{1}{p} \varepsilon_1 \theta_3 + \alpha \frac{1}{p} \varepsilon_2 \theta_4 \\
(1 - \alpha)\frac{1}{p} \varepsilon_1 \theta_3 + \alpha \frac{1}{p} \varepsilon_2 \theta_4 \\
(1 - \alpha)\frac{1}{p} \varepsilon_1 \theta_3 + \alpha \frac{1}{p} \varepsilon_2 \theta_4
\end{array} \right|, \alpha \lambda \leq 1 - \alpha \leq 1 - \lambda (1 - \alpha)
\]

and \( \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \) numbers are defined as in \((2.17)\).

Proof. The assertion follows from Theorem 4 for \( f(x) = -\ln x, \ x > 0 \). □
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Department of Mathematics, Faculty of Arts and Sciences, Giresun University, 28100, Giresun, Turkey.

E-mail address: imdat.iscan@giresun.edu.tr