Creating desired potentials by embedding small inhomogeneities

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Abstract

The governing equation is \[ \nabla^2 + k^2 - q(x) \] \( u = 0 \) in \( \mathbb{R}^3 \). It is shown that any desired potential \( q(x) \), vanishing outside a bounded domain \( D \), can be obtained if one embeds into \( D \) many small scatterers \( q_m(x) \), vanishing outside balls \( B_m := \{ x : |x - x_m| < a \} \), such that \( q_m = A_m \) in \( B_m \), \( q_m = 0 \) outside \( B_m \), \( 1 \leq m \leq M \), \( M = M(a) \). It is proved that if the number of small scatterers in any subdomain \( \Delta \) is defined as \( N(\Delta) := \sum_{x_m \in \Delta} 1 \) and is given by the formula \( N(\Delta) = |V(a)|^{-1} \int_\Delta n(x) dx [1 + o(1)] \) as \( a \to 0 \), where \( V(a) = 4\pi a^3/3 \), then the limit of the function \( u_M(x) \), \( \lim_{a \to 0} U_M = u_e(x) \) exists and solves the equation \[ \nabla^2 + k^2 - q(x) \] \( u = 0 \) in \( \mathbb{R}^3 \), where \( q(x) = n(x) A(x) \), and \( A(x_m) = A_m \). The total number \( M \) of small inhomogeneities is equal to \( N(D) \) and is of the order \( O(a^{-3}) \) as \( a \to 0 \).

A similar result is derived in the one-dimensional case.

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1 Introduction

Consider the scattering problem:

\[ \nabla^2 + k^2 - q(x) \] \( u = 0 \) in \( \mathbb{R}^3 \), \( k = \text{const} > 0 \), \( u = e^{ikx \cdot x} + A(\beta, \alpha, k) \frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right) \), \( r := |x| \to \infty \), \( \beta = \frac{x}{r} \), \( \alpha \in S^2 \), \( (1) \)

where \( S^2 \) is the unit sphere in \( \mathbb{R}^3 \), and \( A(\beta, \alpha, k) = A_q(\beta, \alpha, k) \) is the scattering amplitude corresponding to the potential \( q(x) \), \( \alpha \) is the direction of the incident plane wave, \( \beta \) is a direction of the scattered wave, and \( k^2 \) is the energy.

Let us assume that \( p = p_M(x) \) is a real-valued compactly supported bounded function, which is a sum of small inhomogeneities: \( p = \sum_{m=1}^M q_m(x) \), where...
$q_m(x)$ vanishes outside the ball $B_m := \{ x : |x - x_m| < a \}$ and $q_m = A_m$ inside $B_m$, $1 \leq m \leq M$, $M = M(a)$.

The problem, we are studying in this paper, is:

Problem P: Under what conditions the field $u_M$, which solves the Schroedinger equation with the potential $p_M(x)$, has a limit $u_e(x)$ as $a \to 0$, and this limit $u_e(x)$ solves the Schroedinger equation with a desired potential $q(x)$?

We give a complete answer to this question. Theorem 1 (see below) is our basic result.

Our answer is, basically, as follows:

Given an arbitrary potential $q(x)$, vanishing outside of an arbitrary large but finite domain $D$, one can find a function $A(x)$ and a function $n(x)$ ≥ 0, such that $A(x_m) = A_m$, $A(x)n(x) = q(x)$, and the limit $u_e(x)$ of $u_M(x)$ as $a \to 0$ does exist, and solves problem \[1 - \delta \].

The notation $u_e(x)$ stands for the effective field, which is the limiting field in the medium.

The field $u_M$ is the unique solution to the integral equation:

$$u_M(x) = u_0(x) - \sum_{m=1}^{M} \int_D g(x, y, k) q_m(y) u_M(y) dy, \quad g(x, y, k) = \frac{e^{-ik|x-y|}}{4\pi|x-y|},$$

where $u_0(x)$ is the incident field, which one may take as the plane wave, for example, $u_0 = e^{ik\alpha \cdot x}$, where $\alpha \in S^2$ is the direction of the propagation of the incident wave.

We assume that the scatterers are small in the sense $ka << 1$. Parameter $k > 0$ is assumed fixed, so the limits below are designated as limits $a \to 0$, and condition $ka << 1$ is valid as $a \to 0$.

If $ka << 1$, then the following transformation of \[M\] is valid:

$$u_M(x) = u_0(x) - \sum_{m=1}^{M} \frac{e^{ik|x-x_m|}}{4\pi} A_m(u_M(x_m)) \int_{|y-x_m|<a} dy \frac{dy}{|x-y|} [1 + o(1)]. \quad (4)$$

In \[M\] we have used the following simple estimates:

$$|x-x_m| - a \leq |x-y| \leq |x-x_m| + a, \quad |y-x_m| \leq a.$$

These estimates imply that $\frac{e^{ik|x-y|}}{4\pi|x-y|} \approx e^{ik|x-x_m|}[1 + o(1)]$ if $|y-x_m| < a$ and $a \to 0$.

We want to prove that the sum in \[M\] has a limit as $a \to 0$, and to calculate this limit assuming that the distribution of small inhomogeneities or, equivalently, the points $x_m$, is given by formula \[M\], see below, and $M = N(D)$, where $N(\Delta)$ is defined in \[M\] for any subdomain $\Delta \subset D$, and $N(D)$ is $N(\Delta)$ for $D = \Delta$.

Our basic new tool is the following lemma.

**Lemma 1.** If the points $x_m$ are distributed in a bounded domain $D \subset \mathbb{R}^3$ so that their number in any subdomain $\Delta \subset D$ is given by the formula

$$N(\Delta) = |V(a)|^{-1} \int_{\Delta} n(x) dx [1 + o(1)] \quad a \to 0, \quad (5)$$

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where \( V(a) = 4\pi a^3/3 \), and \( n(x) \geq 0 \) is an arbitrary given continuous in \( D \) function, and if \( f(x) \) is an arbitrary given continuous in \( D \) function, then the following limit exists:

\[
\lim_{a \to 0} \sum_{m=1}^{M} f(x_m)V(a) = \int_{D} f(x)n(x)dx.
\] (6)

Let us state our basic result.

**Theorem 1.** If the small inhomogeneities are distributed so that (5) holds, and \( q_m(x) = 0 \) if \( x \notin B_m \), \( q_m(x) = A_m \) if \( x \in B_m \) where \( B_m = \{ x : |x-x_m| < a \} \), \( A_m := A(x_m) \), and \( A(x) \) is a given continuous in \( D \) function, then the limit

\[
\lim_{a \to 0} u_M(x) = u_e(x)
\] (7)

does exist and solves problem (1)-(2) with

\[
q(x) = A(x)n(x).
\] (8)

There is a large literature on wave scattering by small inhomogeneities. A recent paper is [1]. Our approach is new. Some of the ideas of this approach were earlier applied by the author to scattering by small particles embedded in an inhomogeneous medium ([2]-[8]).

In Section 2 proofs are given and the one-dimensional version of the result is formulated and proved.

## 2 Proofs

**Proof of Lemma 1.** Let \( \{\Delta_p\}_{p=1}^{P} \) be a partition of \( D \) into a union of small cubes \( \Delta_p \) with centers \( y_p \), without common interior points, and

\[
\lim_{a \to 0} \max_p \text{diam} \Delta_p = 0
\] (9)

One has:

\[
\sum_{m=1}^{M} f(x_m)V(a) = \sum_{p=1}^{P} f(y_p)V(a) \sum_{x_m \in \Delta_p} 1[1 + o(1)].
\] (10)

We use formula (5) and the assumption (9) and get

\[
\sum_{x_m \in \Delta_p} 1 = V(a)n(y_p)|\Delta_p|[1 + o(1)],
\] (11)

where \( |\Delta_p| \) is the volume of the cube \( \Delta_p \).

It follows from (10) and (11) that

\[
\sum_{m=1}^{M} f(x_m)V(a) = \sum_{p=1}^{P} f(y_p)n(y_p)|\Delta_p|[1 + o(1)],
\] (12)
which is the Riemannian sum for the integral in the right-hand side of (6), and the assumption (9) allows one to write

\[ f(x_m) = f(y_p)[1 + o(1)] \quad \forall x_m \in \Delta_p, \]

(13)

if \( f \) is continuous.

The Riemannian sum in (12) converges to the integral in the right-hand side of (6) provided that the function \( f(x) \) is continuous, or, more generally, it is bounded and its set of discontinuity points is of Lebesgue measure zero.

Lemma 1 is proved.

**Proof of Theorem 1.** We apply Lemma 1 to the sum in (4), in which we choose \( A_m := A(x_m) \), where \( A(x) \) is an arbitrary continuous in \( D \) function which we may choose as we wish. A simple calculation yields the following formula:

\[
\int_{|y-x_m|<a} |x-y|^{-1} dy = V(a)|x-x_m|^{-1}, \quad |x-x_m| \geq a, \quad (14)
\]

and

\[
\int_{|y-x_m|<a} |x-y|^{-1} dy = 2\pi(a^2 - \frac{|x-x_m|^2}{3}), \quad |x-x_m| \leq a. \quad (15)
\]

Therefore, the sum in (10) is of the form (6) with

\[ f(x_m) = e^{ik|x-x_m|}A(x_m)u_M(x_m)[1 + o(1)]. \]

Applying Lemma 1, one concludes that the limit \( u_e(x) \) in (7) does exist and solves the integral equation

\[
u_e(x) = u_0(x) - \int_D \frac{e^{ik|x-y|}}{4\pi|x-y|}q(y)u_e(y)dy, \]

(16)

where \( q(x) \) is defined by formula (8).

Applying the operator \( \nabla^2 + k^2 \) to (16), one verifies that the function \( u_e(x) \) solves problem (11)-(2).

Theorem 1 is proved.

**Remark 1.** Our method can be applied to the one-dimensional scattering problem. The role of the balls \( B_m \) is now played by the segments: \( B_m := \{ x : x \in \mathbb{R}^1, |x-x_m| < a \} \), the role of \( D \) is played by an interval \((c, d)\), the \( V(a) = 2a \) in the one-dimensional case, an analog of formula (5) for the number of small inhomogeneities \( N(\Delta) = \sum_{x_m \in \Delta} 1 \) is:

\[ N(\Delta) = (2a)^{-1} \int_{\Delta} n(x)dx[1 + o(1)], \]

(17)

and \( \Delta \) is now any interval on the line. The total number \( M \) of small inhomogeneities is now of the order of \( O(a^{-1}) \).
In the one-dimensional case an analog of the function $g(x, y, k)$ is

$$g(x, y, k) = -e^{ik|x-y|}.$$  \hspace{1cm} (18)

An analog of the potential $q_m$ is $q_m(x) = A_m$ inside the interval $B_m$, $q_m(x) = 0$ outside $B_m$, and we assume that $A_m = A(x_m)$, where $A(x)$ is a continuous function which we can choose at will. With these notations one can use equation (4) without any change, but remember that $g(x, y, k)$ is now defined as in (18).

An analog of (4) now is:

$$u_M(x) = u_0(x) + \sum_{m=1}^{M} \frac{e^{ik|x-x_m|}}{2ik} A(x_m) u_M(x_m) 2a[1 + o(1)].$$  \hspace{1cm} (19)

An analog of Theorem 1 can be stated as follows:

**Theorem 2.** If the small inhomogeneities are distributed so that (5) holds, and $q_m(x) = 0$ if $x \notin B_m$, $q_m(x) = A_m$ if $x \in B_m$ where $B_m = \{x : |x-x_m| < a$, $A_m := A(x_m)$, and $A(x)$ is a given continuous in $D$ function, then the limit $u_e(x)$ in (7) does exist and solves problem (1)-(2) with $q(x)$ defined in (8), $\nabla^2 u$ replaced by $u''$, and the radiation condition (2) modified to fit the one-dimensional problem.
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