Perfect partial reconstructions for multiple simultaneous sources

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Abstract. In this paper we show that the signal apparition method for encoding multiple sources excited during seismic acquisition results in optimally large regions in the frequency-wavenumber space where exact separation of sources is achieved. These regions are diamond-shaped and we prove that using any other method of source encoding results in strictly smaller regions of exact separation. The results are valid for arbitrary number of sources. A numerical example demonstrates that these diamond-shaped regions are twice as large as the regions of exact separation obtained by using random dithering.

1. Introduction

Methods for simultaneous source separation have been a major focus in the seismic industry over the last two decades [9]. Acquiring seismic data without having to wait for the response of one source to be recorded before exciting one or more sources at other shot points promises to radically increase productivity. This can be essential for instance to make complex wide azimuth seismic surveys cost-effective. Other constraints such as completing a survey within time-share agreements or in between fish spawning seasons can also greatly benefit from a significant increase in productivity.

The simultaneous source problem is fundamentally an ill-posed problem above a certain frequency and to solve the problem it is necessary to introduce additional constraints [5]. A popular method used in industry is based on the science of compressive sensing. By using random time dithers when exciting sources relative to other sources being excited, it is possible to invert the recorded seismic data for individual source responses under assumptions such as coherency and sparseness of seismic data (see [1, 8, 10, 11, 12, 13, 14, 15, 16, 18] and references therein). The recently introduced concept of signal apparition offers a fundamentally different approach to solve the source separation problem [17]. Instead of using random dithers, deterministic periodic variations of excitation times of a source relative to other sources result in the mapping of data into multiple signal cones away from the usual signal cone centered at wavenumber zero and bounded by the propagation velocity of the recording medium (1500 m/s in the case of marine seismic data). Andersson et al. [5] showed that for two sources simultaneously acquiring data along two lines over a general 3D heterogeneous sub-surface, the apparition-style acquisition strategy results in a region in the frequency-wavenumber space where the separation of sources is exact which is twice as large as what is possible to achieve using random dithers for instance. Andersson et al. [5] referred to these regions as “flawless diamonds” and demonstrated that exact separation of sources in these regions is unique to signal apparition. Outside the flawless diamonds, the simultaneous source problem is solved using additional constraints to tackle the otherwise ill-posed problem (e.g., [4, 6]).
In this paper we generalize the findings of [5] to that of $M$ simultaneous sources [2, 3]. We show that the method of signal apparition results in optimally large regions in the frequency-wavenumber space for exact separation of sources. Moreover, we show that all other methods for source encoding results in regions of exact separation which are smaller than that obtained by encoding the sources using signal apparition. The main part of the paper contains the proofs of two theorems that demonstrate that signal apparition is unique and optimal in the sense of exactly separating the response of $M$ sources. The theorems are supported by lemmas included in the appendix. Following the theory section we present a numerical example.

2. Theory

We will consider simultaneous source acquisition in a marine environment. However, similar arguments can of course also be extended to other situations such as land seismic data acquisition. We consider $M$ simultaneous sources acquired along a single line over a complex 3D sub-surface (source modulation function along a line). Note that source modulation along a line by no means limits the application to 2D data. However, source modulation in both dimensions along the source coordinates in the acquisition plane can further increase simultaneous source efficiency gains. The generalization of the results presented in this paper for source modulation along a line to source modulation in a plane is straight forward.

First, consider an experiment where a source is fired on equidistant shot positions, spaced a distance $\triangle x$ apart, along a line (of infinite length) and recorded on a stationary receiver. The Nyquist wavenumber corresponding to this spatial sampling frequency is $k_S = 1/(2\triangle x)$. The slowest possible apparent velocity in the recordings is identical to the sound speed in water resulting in a maximum frequency (Nyquist frequency) below which all energy is unaliased given the spatial sampling interval $\triangle x$. As a result, after a temporal and spatial Fourier transform ($\omega k_x$), all signal energy resides inside a “signal cone” bounded by the propagation velocity of the recording medium. Note that large parts of the $\omega k_x$-spectrum inside the Nyquist frequency and wavenumber are therefore zero.

Let us now consider what happens in the case of $M$ sources being fired simultaneously in a way that is varying between shot locations. If an amplitude variation $a_n$ and a shift variation $\tau_n$ is applied to the $n$th source, the recorded data will be of the form

$$d(t,j) = \sum_{n=1}^{M} a_n(j) f_n(t + \tau_n(j), \Delta_x j), \quad j = -Mm, \ldots, Mm - 1,$$

with each $f_n$ representing seismic data recorded at a certain depth. We may without loss of generality assume that $\tau_1 \equiv 0$. As explained above, if $\mathcal{F}(f_n)$ denotes the (continuous) temporal and spatial Fourier transform of $f_n$, the support of each $\mathcal{F}(f_n)$
will be contained in the conic set
\[ \mathcal{C} = \{ (\omega, k_x) : |\omega| > c_0 \cdot |k_x| \} \]
where \( c_0 \approx 1500 \text{m/s} \) assuming a marine environment. Introduce the diamond shaped set
\[ \mathcal{D} = \mathcal{C} \setminus \{ (\omega, k_x) : |\omega| \geq c_0 \cdot |k_x| \pm 1/(M \Delta_x) \} , \]
and define
\[ \omega_0 = \frac{c_0}{2M \Delta_x} . \]
Then \( 0 < \omega < 2\omega_0 \) when \((\omega, k_x) \in \mathcal{D} \), and all energy is unaliased when \( 0 < \omega < \omega_0 \).

The consequence of the first theorem is that the only way that \( F(f_n) \) to be perfectly reconstructed in \( \mathcal{D} \), see [2, 3]. Supposing that the \( M \) sources \( f_n \) are sampled in \( 2Mm \) points for some integer \( m \), we define the semi-discrete Fourier transform of \( f_n \) as
\[ \hat{f}_n(\omega, k) = \sum_j \exp \left( \frac{-2\pi i k j}{2Mm} \right) \int_{-\infty}^{\infty} f_n(t, \Delta_x, j) \exp (-2\pi i t \omega) dt. \]
Using the Poisson summation formula and the condition \( \text{supp} F(f_n) \subset \mathcal{C} \), it is straightforward to check that
\[ \hat{f}_n(\omega, k) = \frac{1}{\Delta_x} F(f_n) \left( \omega : \frac{k}{2Mm \Delta_x} \right) \]
for \((\omega, k)\) such that \( |\omega| < 2\omega_0 \) and \(-Mm \leq k \leq Mm - 1\). This provides a relation between \( \hat{f}_n \) and \( F(f_n) \) in the domain of interest.

We will now present the main results of the paper in the following two theorems. The consequence of the first theorem is that the only way that \( F(f_n)(\omega, k_x) \) to be uniquely determined when \((\omega, k_x) \in \mathcal{D} \), it is required that \( a_n \) and \( \tau_n \) are periodic of period \( M \), i.e.,
\[ a_n(j \text{ mod } M) = a(j), \quad \tau_n(j \text{ mod } M) = \tau_n(j) \]
for \( 1 \leq n \leq M \).

**Proof.** Applying a one-dimensional Fourier transform with respect to \( t \) to (2.1) gives
\[ F_t(d)(\omega, j) = \sum_{n=1}^{M} a_n(j) \exp \left( -2\pi i \tau_n(j) \omega \right) F_t(f_n)(\omega, \Delta_x) \]
for \( j = -Mm, \ldots, Mm - 1 \). For fixed \( \omega \), let \( w_n^\omega(k) \) be the discrete Fourier transform of \( j \mapsto a_n(j) \exp (2\pi i \tau_n(j) \omega) \) evaluated at \( k \), i.e.,
\[ w_n^\omega(k) = \sum_j a_n(j) \exp \left( -2\pi i \left( \tau_n(j) \omega + \frac{j k}{2Mm} \right) \right). \]
If we apply a discrete Fourier transform to (2.5), we obtain

$$\hat{d}(\omega, k) = \sum_{n=1}^{M} w_n^* \hat{f}_n(\omega, k),$$

where the (discrete) convolution acts on the second variable. Let us now consider a fixed $\omega$ such that

$$\frac{2m - 1}{m} \cdot \omega_0 \leq \omega < 2\omega_0.$$ 

In view of (2.3) and the support condition $\text{supp} \mathcal{F}(f_n) \subset \mathcal{C}$, this implies that the function $k \mapsto \hat{f}_n(\omega, k)$ has support contained in $[-2m + 1, 2m - 1]$. Similarly, the assumption that $\mathcal{F}(f_n)(\omega, k_x)$ can be uniquely determined when $(\omega, k_x) \in \mathcal{D}$ turns into a condition of the type given in Lemma 3. Hence, applying the lemma we conclude that for each $\omega$ satisfying (2.7), it holds that $w_n^*(l) = 0$ unless $l = 2ml'$ where $l'$ is an integer. In particular, the Fourier inversion formula gives

$$a_n(j) \exp(2\pi i \tau_n(j) \omega) = \frac{1}{2Mm} \sum_{l=-M}^{Mm-1} \exp\left(\frac{2\pi ilj}{2Mm}\right) w_n^*(l)$$

if $M = 2M'$ is even, and

$$a_n(j) \exp(2\pi i \tau_n(j) \omega) = \frac{1}{2Mm} \sum_{l'=-M'}^{M'-1} \exp\left(\frac{2\pi il'j}{M'}\right) w_n^*(2ml')$$

if $M = 2M' + 1$ is odd. In either case the expressions above clearly have period $M$, so

$$a_n(j + M) \exp(2\pi i \tau_n(j + M) \omega) = a_n(j) \exp(2\pi i \tau_n(j) \omega).$$

Taking absolute values we immediately infer that all $a_n(j)$ have period $M$. Taking logarithms and dividing by $2\pi i$ we then get

$$(\tau_n(j + M) - \tau_n(j)) \omega = \kappa(\omega)$$

for some integer-valued function $\kappa : \omega \mapsto \kappa(\omega) \in \mathbb{Z}$. However, according to (2.7) this has to hold for a continuous range of values $\omega$. Since the left-hand side is a continuous function of $\omega$ it follows that $\kappa$ is constant, and varying $\omega$ slightly shows that the only possible choice is $\kappa(\omega) \equiv 0$. Therefore $\tau_n$ is periodic with period $M$. \qed

We now present the second main result of the paper. In view of Theorem [4] the result shows that signal apparition is optimal in the sense that it maximizes the region of exact separation of multiple sources excited during seismic acquisition. In fact, using any other type of sampling method results in a strictly smaller domain of exact separation.

**Theorem 2.** Suppose that data is given by (2.1), with $\text{supp} \mathcal{F}(f_n) \subset \mathcal{C}$ for $1 \leq n \leq M$ Let $\mathcal{E} \subset \mathcal{C}$ and suppose that $\mathcal{F}(f_n)(\omega, k_x)$ is uniquely determined when $(\omega, k_x) \in \mathcal{E}$. Then the area of $\mathcal{E}$ cannot be larger than the area of $\mathcal{D}$, and if the areas are equal then $\mathcal{E} = \mathcal{D}$. 

Proof. As in the proof of Theorem 1 we apply a semi-discrete Fourier transform to the data (2.1) and obtain
\[
\hat{d}(\omega, k) = \sum_{n=1}^{M} w_{\omega}^{n} * \hat{f}_{n}(\omega, k),
\]
where the (discrete) convolution acts on the second variable, and \(w_{\omega}^{n}(k)\) for fixed \(\omega\) is the discrete Fourier transform of \(j \mapsto a_{n}(j) \exp(2\pi i \tau_{n}(j) \omega)\) evaluated at \(k\). We then fix \(\omega\) such that
\[
(2.8) \quad m + l \cdot \omega_{0} \leq \omega < m + l + 1 \cdot \omega_{0},
\]
where \(0 \leq l \leq m - 1\) is arbitrary. Using (2.3) we now translate the assumption that \(\mathcal{F}(f_{n})(\omega, k_{x})\) can be uniquely determined when \((\omega, k_{x}) \in \mathcal{E}\) into a similar condition for the function \(k \mapsto \hat{f}_{n}(\omega, k)\) via
\[
(\omega, k/(2Mm\Delta_{x})) \in \mathcal{E} \iff k \in J_{\text{good}}
\]
for some subset of integers \(J_{\text{good}} \subset [-m - l, m + l]\). One easily verifies that the conditions of Lemma 4 are satisfied, so the number \(N_{\text{good}}(l)\) of \(k\) for which \(\hat{f}_{n}(\omega, k)\) is uniquely determined satisfies
\[
N_{\text{good}}(l) \leq 2m - 2l - 1.
\]
It is easy to see that for \(\omega\) satisfying (2.8) we have \((\omega, k/(2Mm\Delta_{x})) \in \mathcal{D}\) if and only if \(-(m - l - 1) \leq k \leq m - l - 1\), so \(N_{\text{good}}(l)\) is not larger than the number of \(k\) such that \((\omega, k/(2Mm\Delta_{x})) \in \mathcal{D}\). Since \(l\) was arbitrary this proves that \(\mathcal{E}\) cannot have larger area than \(\mathcal{D}\). The same argument shows that if the areas of \(\mathcal{E}\) and \(\mathcal{D}\) are the same, it is required that \(N_{\text{good}}(l) = 2m - 2l - 1\) for each \(0 \leq l \leq m - 1\). In particular, when \(l = m - 1\) we have \(N_{\text{good}}(m - 1) = 1\), which by Lemma 5 can only happen if \(J_{\text{good}} = \{0\}\). An application of Lemma 3 then completely determines the support of all \(w_{\omega}^{n}\), and a repetition of the arguments at the end of the proof of Theorem 1 shows that (2.4) holds. Clearly, an apparition style sampling has then been used, so the region \(\mathcal{E}\) in which perfect reconstruction is obtained must be equal to the diamond-shaped set \(\mathcal{D}\). \(\Box\)

3. Results

We use the formalism described above to separate the data corresponding to three simultaneous source experiments. In one experiment the data are encoded using conventional random dithers, and in the other experiment the same data are encoded using purely periodic modulation functions using time shifts. The same inversion methodology will be used to separate the sources.

As an example we have tested the methods on a synthetic data set generated using an acoustic 3D finite-difference solver and a model based on salt-structures in the sub-surface and a free-surface bounding the top of the water layer. A common-receiver gather located in the middle of the model was simulated using this model in which a vessel acquires two shotlines with an inline shot spacing of 25 m. The vessel tows the three sources at 150, 175, and 200 m cross-line offset from the receiver location. The source wavelet comprises a Ricker wavelet with a maximum frequency of 30 Hz. For the apparition setup, a time delay of \(\pm 24\) ms has been applied to the data set, while for the random dithering case, time delays of \(\pm 120\) ms have been used. The results of the experiments are displayed in Figures 2 and 3. Both methods reconstruct data well for lower frequencies, but while the random dithering method does not do well
at all in the upper part of the diamond region, the apparition method performs perfectly within the entire diamond.

It should be stressed that better results can often be obtained by incorporating additional reconstruction constraints, i.e., using regularization or sparseness. Constraints can be incorporated in many ways and the reconstruction quality will depend on the data. The purpose of this example is to clarify what information can be recovered directly, without using additional constraints. This exercise is illustrative since the reconstruction quality will not be data dependent; in the regions where it will work, it will provide exact results. Moreover, these exact results are not sensitive to random noise appearing in the periodic time-shifts [19], and they can subsequently be used to recover the remaining parts of the data by more elaborate de-aliasing methods, for instance as presented in [4], [6] and [7].

4. DISCUSSION AND CONCLUSIONS

In this paper we have presented a formal proof showing that encoding $M$ simultaneously excited sources in a seismic survey using the method of signal apparition results in optimally large regions in the frequency-wavenumber space for exact separation of sources. We also presented a proof that all other methods for source encoding results in regions of exact separation which are smaller than that obtained by encoding the sources using signal apparition.

Through a synthetic example we demonstrated the exact separation of the response from three simultaneous sources within the optimally large “flawless diamonds” in the frequency-wavenumber space. We note that methods based on random dither encoding can only recover an area half the size of that of signal apparition.
Encoding of sources using signal apparition also has other advantages. The small time-shifts (10’s of ms for typical exploration surveys) being used enable the acquisition of excellent low-frequency content in the data which is often compromised using the larger time-dithers common for random dithering methods (100’s of ms). Time shifts on the order of 10’s of ms also result in significantly reduced peak-amplitude in source signatures and reduce output energy in the range 100–1000 Hz thus reducing potential negative impact on marine mammals.

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APPENDIX A.

In this appendix we establish a framework in which we prove auxiliary results used in the proofs of Theorems 1 and 2. If \( M \) is the number of sources, and \( m \) a positive integer, we shall henceforth assume that any given vector \( \mathbf{f} \in \mathbb{R}^{2Mm} \) is indexed on the range \([-Mm, Mm - 1] \), so that
\[
\mathbf{f} = (f(-Mm), \ldots, f(-1), f(0), f(1), \ldots, f(Mm - 1)).
\]
In other words, a function \( f : [-Mm, Mm - 1] \rightarrow \mathbb{R} \) is identified with its range, which is a vector in \( \mathbb{R}^{2Mm} \) denoted by \( \mathbf{f} \). We let \( \mathbf{e}_{-Mm}, \ldots, \mathbf{e}_{Mm-1} \) be the canonical basis of \( \mathbb{R}^{2Mm} \).

Throughout the appendix we let \( l \) denote an integer \( 0 \leq l \leq m - 1 \), and \( J_{\text{good}} \) some subset of integers \( J_{\text{good}} \subset [-m - l, m + l] \). Define \( J_{\text{bad}} \) by
\[
[-m - l, m + l] = J_{\text{good}} \bigcup J_{\text{bad}},
\]
and let \( N_{\text{good}} \) and \( N_{\text{bad}} \) denote the cardinality (number of elements) of \( J_{\text{good}} \) and \( J_{\text{bad}} \), respectively. We then make the following standing assumption.

**Assumption A.** The vectors \( \mathbf{w}_2, \ldots, \mathbf{w}_M \in \mathbb{R}^{2Mm} \) have the property that for any \( \mathbf{g}_n \in \mathbb{R}^{2Mm}, \ 1 \leq n \leq M, \) with support contained in \( [-m - l, m + l] \), it holds that the values \( \mathbf{g}_n(j) \) for \( j \in J_{\text{good}} \) are uniquely determined by
\[
\mathbf{b} = \mathbf{g}_1 + \mathbf{g}_2 \ast \mathbf{w}_2 + \ldots + \mathbf{g}_M \ast \mathbf{w}_M
\]
for \( 1 \leq n \leq M \).

Note that for vectors \( \mathbf{g}_n \) as in the statement of Assumption A, we may write
\[
\mathbf{g}_n \ast \mathbf{w}_n(k) = \sum_{j=-m-l}^{m+l} \mathbf{g}_n(j) \mathbf{w}_n(k-j).
\]
We identify each such \( \mathbf{g}_n \) with a vector in \( \mathbb{R}^{2m+2l+1} \) and introduce a linear map
\[
F : \mathbb{R}^{2m+2l+1} \times \ldots \times \mathbb{R}^{2m+2l+1} \rightarrow \mathbb{R}^{2Mm}
\]
given by \( F(\mathbf{g}_1, \ldots, \mathbf{g}_M) = \mathbf{b} \). Let \( U \) be the linear subspace of the domain of \( F \) describing the values \( \mathbf{g}_n(j) \) for \( j \in J_{\text{good}} \) and \( 1 \leq n \leq M \), i.e., the values which are uniquely determined by \( \mathbf{b} \). Using the identification above we permit us to let \( \mathbf{e}_{-m-l}, \ldots, \mathbf{e}_{m+l} \) also denote the canonical basis of \( \mathbb{R}^{2m+2l+1} \). For notational purposes, write \( \mathbf{w}_1 = \mathbf{e}_0 \), so that \( \mathbf{b} = \sum \mathbf{g}_n \ast \mathbf{w}_n \). By Lemma A.1 in [5] there exists a linear subspace \( V = F(U) \) of \( \mathbb{R}^{2Mm} \) with
\[
\dim(V) = \dim(U) = N_{\text{good}} \cdot M
\]
such that if $(g_1, \ldots, g_M) \in U^\perp$ then $P_V(\sum g_n * w_n) = 0$, where $P_V$ is the orthogonal projection onto $V$. Then all the $M$-tuples $(0, \ldots, 0, e_k, 0, \ldots, 0)$ with $e_k$ at position $i$ belong to $U^\perp$ for $k \in J_{\text{bad}}$ and $1 \leq i \leq M$. Likewise, the $M$-tuples $(0, \ldots, 0, e_k, 0, \ldots, 0)$ with $e_k$ at position $i$ belong to $U$ for $k \in J_{\text{good}}$ and $1 \leq i \leq M$.

This implies that

- $w_n * e_k \in V$ for each $n$ if $k \in J_{\text{good}},$
- $w_n * e_k \in V^\perp$ for each $n$ if $k \in J_{\text{bad}},$ and
- $V$ is spanned by $\{w_n * e_k : k \in J_{\text{good}}, 1 \leq n \leq M\}.$

Moreover, for $1 \leq n_1, n_2 \leq M$ we have

(A.3) \[ \langle w_{n_1} * e_{k_1}, w_{n_2} * e_{k_2} \rangle = 0, \quad k_1 \in J_{\text{good}}, \; k_2 \in J_{\text{bad}}. \]

Finally, let $v_1, \ldots, v_M$ be an orthonormal basis of $\text{span}\{w_1, \ldots, w_M\}$ obtained from $w_1, \ldots, w_M$ by a Gram-Schmidt procedure, with $v_1 = w_1 = e_0.$ It is straightforward to check that

(A.4) \[ V = \text{span}\{v_n * e_k : k \in J_{\text{good}}, 1 \leq n \leq M\}. \]

Indeed, since $\dim(V) = N_{\text{good}} \cdot M$ it suffices to show that the set is linearly independent, but this is immediate consequence of the third bullet above. Also, using (A.3) it is easy to see that

(A.5) \[ \langle v_{n_1} * e_{k_1}, v_{n_2} * e_{k_2} \rangle = 0, \quad k_1 \in J_{\text{good}}, \; k_2 \in J_{\text{bad}} \]

for each $1 \leq n_1, n_2 \leq M$.

The following lemma is used in Theorem 1 to show that only a signal apparition style sampling allows for perfect reconstruction in the diamond-shaped set $D$.

**Lemma 3.** Let $l = m - 1$ and suppose Assumption A holds. If $J_{\text{good}} = \{0\}$, then $\text{supp}(w_n)$, $1 \leq n \leq M$, is contained in the discrete set

- \[ \{-Mm, -(M-2)m, \ldots, (M-2)m\} \quad \text{if } M \text{ is even}, \]
- \[ \{-M-1)m, -(M-3)m, \ldots, (M-1)m\} \quad \text{if } M \text{ is odd}. \]

In other words, $w_n(k) = 0$ unless $k = 2ml'$ for some integer $l'$.

**Proof.** Clearly, it suffices to prove that each $v_n$ has the desired support described in the statement of the lemma. With $l = m - 1$ and $J_{\text{good}} = \{0\},$ the discussion above implies that $v_n \in V$ for each $n$, that $V$ is spanned by $\{v_n\}_{n=1}^M$, and that

(A.6) \[ \langle v_{n_1} * e_k, v_{n_2} \rangle = 0, \quad k = \pm 1, \ldots, \pm (2m-1) \]

for each $1 \leq n_1, n_2 \leq M$. In particular we can take $n_1 = n_2 = 1$. Since we also have $\langle v_n, v_1 \rangle = 0$ by orthogonality, this means that

(A.7) \[ v_n(j) = 0 \quad \text{for } j \in [-2m+1, 2m-1] \text{ and } n = 2, \ldots, M. \]

We now claim that the collection

(A.8) \[ \{v_n * e_k : n = 1, \ldots, M, k = 1, \ldots, 2m - 1\} \]

is a basis for $V^\perp$. Indeed, the cardinality of the set is $M(2m - 1) = \dim(V^\perp)$ so the claim follows if we prove that the set is linearly independent. Arguing by contradiction, suppose that the set not linearly independent. Then for some indices $i, j$ and constants $c_{nk}$ we have

$$v_i * e_j = \sum_{(n,k) \neq (i,j)} c_{nk} v_n * e_k$$
with the convention that the sum is taken over \(1 \leq n \leq M, 1 \leq k \leq 2m - 1\). Convolving both sides with \(e_{-j}\), taking the scalar product with \(v_i\) and using \((A.6)\) we get

\[
1 = \langle v_i, v_i \rangle = \sum_{(n,k) \neq (i,j)} c_{nk} \langle v_i, v_n \ast e_{k-j} \rangle = 0,
\]
a contradiction.

Next, we claim that \(e_{2m} \in V\). To see this, recall that \(\mathbb{R}^{2Mm} = V \oplus V^\perp\). By the definition of direct sum, \(e_{2m}\) has a unique representation \(e_{2m} = a_1 + a_2\) with \(a_1 \in V\) and \(a_2 \in V^\perp\). In view of \((A.8)\) we can write

\[
a_2 = \sum_{n=1}^M \sum_{k=1}^{2m-1} c_{nk} v_n \ast e_k.
\]

But \(v_n(j) = 0\) for \(j = 1, \ldots, 2m - 1\) by \((A.7)\), so

\[
a_2(2m) = \sum_{n=1}^M \sum_{k=1}^{2m-1} c_{nk} v_n(2m - k) = 0.
\]

Hence, \(a_1(2m) = 1\), and \(a_1(j) = -a_2(j)\) for \(j \neq 2m\). By orthogonality we have

\[
0 = \langle a_1, a_2 \rangle = -\sum_{j \neq 2m} a_2(j)^2
\]

showing that \(a_2 \equiv 0\), which proves the claim.

Since \(e_{2m} \in V\) it follows that each \(v_n, 2 \leq n \leq M\), satisfies

\[
v_n(2m + k) = \langle v_n, e_{2m+k} \rangle = \langle v_n \ast e_{-k}, e_{2m} \rangle = 0, \quad k = 1, \ldots, 2m - 1,
\]

where the last identity is a consequence of \((A.6)\). But this means that we may now repeat the arguments in the preceding paragraph to conclude that \(e_{4m} \in V\), so that \(v_n(4m + k) = 0\) for \(k = 1, \ldots, 2m - 1\). Iterating we find that \(e_{2ml'} \in V\) for all integers \(l'\) and that \(v_n(2ml' + k) = 0\) for \(k = 1, \ldots, 2m - 1\). Thus each \(v_n\) has the desired support, which completes the proof.

The next two lemmas are used in Theorem 2 to show that the method of signal apparition maximizes the area of the domain in which perfect reconstruction is possible.

**Lemma 4.** Suppose that Assumption A holds. Then the cardinality \(N_{\text{good}}\) of \(J_{\text{good}}\) is not greater than the cardinality of the discrete set \([- (m - l - 1), (m - l - 1)]\), i.e., \(N_{\text{good}} \leq 2m - 2l - 1\).

**Proof.** We begin by proving the lemma for the case of \(M = 2\) sources where the ideas are easy to convey; the proof for \(M \geq 3\) sources is done in the same spirit but the arguments are more intricate then. Let \(N_{\text{bad}}\) be the cardinality of \(J_{\text{bad}}\). By the second bullet on page 8 we have \(\dim(V^\perp) \geq N_{\text{bad}}\). Since \(N_{\text{bad}} = 2m + 2l + 1 - N_{\text{good}}\), this together with \((A.1)\) and \((A.2)\) implies that

\[
2m + 2l + 1 - N_{\text{good}} \leq \dim(V^\perp) = 4m - \dim(V) = 4m - 2N_{\text{good}}.
\]

Thus, \(N_{\text{good}} \leq 2m - 2l - 1\), which completes the proof when \(M = 2\).

We now turn to the case of general \(M \geq 2\) where we will use the same ideas, namely that if there are too many elements in \(J_{\text{good}}\) then \(\dim(V)\) will as a result be too big, forcing \(\dim(V^\perp)\) to be too small. However, since the cardinality of \(J_{\text{bad}}\) does not increase with \(M\) while \(\dim(V^\perp)\) does, the proof requires more finesse. We stress that it is unknown whether \(0 \in J_{\text{good}}\).
First note that
\[ \dim(V^\perp) = 2Mm - \dim(V) = M(2m - N_{\text{good}}) = M(N_{\text{bad}} - (2l + 1)). \]
Write \( N_0 = N_{\text{bad}} - (2l + 1) \) so that \( \dim(V^\perp) = MN_0 \). Assume to reach a contradiction that \( N_{\text{good}} > 2m - 2l - 1 \). Then \( N_{\text{bad}} = 2m + 2l + 1 - N_{\text{good}} < 2(2l + 1) \) so \( 2N_0 + 1 \leq N_{\text{bad}} \).

It is easy to see that this implies the existence of \( N_0 + 1 \) consecutive integers
\[ j_1 < j_2 < \ldots < j_{N_0 + 1}, \quad j_k \in J_{\text{bad}}, \]
such that for some \( j_0 \in \mathbb{Z} \), either \( j_1 + j_0 \) or \( j_{N_0 + 1} + j_0 \) belongs to \( J_{\text{good}} \) and all perturbed elements \( j_k + j_0 \) belong to \([-m - l, m + l]\). We treat the case when \( j_{N_0 + 1} + j_0 \in J_{\text{good}} \), the proof of the other case is similar. Consider the set
\[ Z = \text{span}\{e_{jk} \ast w_n : 1 \leq n \leq M, \ 1 \leq k \leq N_0\} \subset V^\perp. \]
Assume first that \( Z \) is linearly independent. Since the cardinality of \( Z \) is equal to \( \dim(V^\perp) \), \( Z \) then constitutes a basis of \( V^\perp \). Hence, there are constants \( c_{nk} \) (not all zero) such that
\[ e_{j_{N_0} + 1} = \sum_{n=1}^{M} \sum_{k=1}^{N_0} c_{nk} e_{jk} \ast w_n. \]
After convolution with \( e_{j_0} \) we get
\[ e_{j_0 + j_{N_0} + 1} = \sum_{n=1}^{M} \sum_{k=1}^{N_0} c_{nk} e_{j_0 + jk} \ast w_n. \]
By assumption, all terms on the right satisfy \( j_0 + j_k \in [-m - l, m + l] \). But then we can choose \( g_n \) so that
\[ e_{j_0 + j_{N_0} + 1} = \sum_{n=1}^{M} \sum_{k=1}^{N_0} g_n \ast w_n = F(g). \]
The left-hand side is also equal to \( F(g') \) with \( g' = (e_{j_0 + j_{N_0} + 1}, 0, \ldots, 0) \), and since \( j_0 + j_{N_0 + 1} \in J_{\text{good}} \), this contradicts Assumption A.

It remains to consider the case when \( Z \) is linearly dependent. Then there are constants \( c_{nk} \) (not all zero) such that
\[ 0 = \sum_{n=1}^{M} \sum_{k=1}^{N_0} c_{nk} e_{jk} \ast w_n. \]
Let \( q \) be the largest integer such that \( 1 \leq q \leq N_0 \) and at least one \( c_{nk} \) is nonzero for \( k = q \). Convolving with \( e_{j_0 + j_{N_0 + 1} - j_q} \) we obtain
\[ 0 = \sum_{n=1}^{M} \sum_{k=1}^{q} c_{nk} e_{jk + j_0 + j_{N_0 + 1} - j_q} \ast w_n. \]
Note that for \( 1 \leq k \leq q \), each integer \( j_k + j_0 + j_{N_0 + 1} - j_q \) satisfies
\[ -m - l \leq j_k + j_0 \leq j_k + (j_0 + j_{N_0 + 1} - j_q) \leq j_0 + j_{N_0 + 1} \leq m + l, \]
and when \( q = k \) we have \( e_{j_k + j_0 + j_{N_0 + 1} - j_k} = e_{j_0 + j_{N_0 + 1}} \in J_{\text{good}} \). But then we can choose \( g_n \) so that
\[ 0 = \sum_{n=1}^{M} \sum_{k=1}^{q} g_n \ast w_n = F(g), \]
where \( \mathcal{P}_v(F(g)) \neq 0 \). Clearly, this is a contradiction since the projection of the left-hand side onto \( V \) is 0. This completes the proof. \( \square \)

**Lemma 5.** Let \( l = m - 1 \) and suppose that Assumption A holds. If the cardinality of \( J_{\text{good}} \) is equal to 1, then \( J_{\text{good}} = \{0\} \).

**Proof.** Let \( j_0 \) denote the single element in \( J_{\text{good}} \), and assume to reach a contradiction that \( j_0 \neq 0 \). By symmetry we may without loss of generality assume that \( 1 \leq j_0 \leq 2m - 1 \). Using \( (A.5) \) it is straightforward to check that

\[
(A.9) \quad v_n(j) = 0, \quad 1 \leq |j| \leq j_0 + 2m - 1, \quad n = 1, \ldots, M.
\]

(Note that compared to signal apparition, this is a larger range of values where each \( v_n \) vanishes. In particular, \( v_n(\pm 2m) = 0 \).) We will now follow the strategy in the proof of Lemma 3 and

i) determine a basis for \( V^\perp \),

ii) show that \( e_{2m+2j_0} \in V \),

iii) conclude that \( v_n(2m + j_0 + k) = 0 \) for \( k = 1, \ldots, 2m - 1 \).

As we shall see, steps ii) and iii) may then be repeated, and iteration will lead to a contradiction showing that \( J_{\text{good}} = \{0\} \).

We begin with i) and claim that

\[
(A.10) \quad \{ v_n * e_{j_0+k} : n = 1, \ldots, M, \quad k = 1, \ldots, 2m - 1 \}
\]

is a basis for \( V^\perp \). Inspecting the proof of \( (A.8) \) in Lemma 3 we see that the same arguments can be repeated verbatim as long as we establish that

\[
(A.11) \quad v_n * e_{j_0+k} \in V^\perp, \quad \text{for } n = 1, \ldots, M, \quad k = 1, \ldots, 2m - 1.
\]

For \( k \geq 1 \) this is clear as long as \( j_0 + k \leq 2m - 1 \), but since \( j_0 \geq 1 \) the upper range needs to be checked. To this end, let \( u \in V \) be arbitrary. Using \( (A.4) \) with \( J_{\text{good}} = \{j_0\} \) we can write

\[
\langle u, v_n * e_{j_0+k} \rangle = \sum_{n'} c_{n'} \langle v_{n'} * e_{j_0}, v_n * e_{j_0+k} \rangle = \sum_{n'} c_{n'} \langle v_{n'} * e_{j_0-k}, v_n * e_{j_0} \rangle.
\]

By \( (A.5) \) the right-hand side is zero when \( j_0 - k \in J_{\text{bad}}, \) which holds when \( k = 1, \ldots, 2m - 1 \). Hence, \( (A.11) \) is valid, so \( (A.10) \) is a basis for \( V^\perp \). (The computation even shows that \( (A.11) \) is true for \( k = 1, \ldots, 2m + j_0 - 1 \).)

Next, we prove ii). By the definition of direct sum we can write \( e_{2m+2j_0} = a_1 + a_2 \) uniquely with \( a_1 \in V \) and \( a_2 \in V^\perp \). By \( (A.10) \) and \( (A.9) \) we have

\[
a_2(2m+2j_0) = \sum_{n=1}^M c_{nk} v_n * e_{j_0+k}(2m+2j_0) = \sum_{n=1}^M c_{nk} v_n(2m+j_0-k) = 0.
\]

Hence, \( a_1(2m+2j_0) = 1 \) and \( a_2(j) = -a_1(j) \) for all other \( j \). But then

\[
0 = \langle a_1, a_2 \rangle = - \sum_{j \neq 2m+2j_0} a_2(j)^2
\]

by orthogonality, so \( a_2 \equiv 0 \) which proves ii).

We now turn to iii). Since \( e_{2m+2j_0} \in V \) it follows that each \( v_n, \ 2 \leq n \leq M \), satisfies

\[
v_n(2m + j_0 + k) = \langle v_n, e_{2m+j_0+k} \rangle = \langle v_n * e_{j_0-k}, e_{2m+2j_0} \rangle = 0
\]

when \( j_0 - k \in J_{\text{bad}} \). In particular,

\[
v_n(j) = 0, \quad 2m + j_0 + 1 \leq j \leq 4m + 2j_0 - 1,
\]
which proves iii). As mentioned above, we may now repeat these arguments and conclude that $e_{m+3j_0} \in V$, so that $v_n(4m + 2j_0 + k) = 0$ for $k = 1, \ldots, 2m - 1$. Iterating we find that $e_j \in V$ when $j = j_0 + (2m + j_0)j' \mod 2Mm$ for nonnegative integers $j'$ which shows that $v_n(j) = 0$ unless $j = (2m + j_0)j' \mod 2Mm$. However, taking (A.9) into account, the number of such points in $[-Mm, Mm - 1]$ is strictly less than $M$. This means that each $v_n$ is a linear combination of fewer than $M$ number of elements $e_j, j = (2m + j_0)j' \mod 2Mm$. Clearly, this contradicts the fact that $\dim(V) = M$, which proves that $J_{\text{good}} = \{0\}$.

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