Applicability of Perturbative QCD to Pion Virtual Compton Scattering

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Abstract

We study explicitly the applicability of perturbative QCD (pQCD) to the pion virtual Compton scattering. It is found that there are central-region singularities introduced by the QCD running coupling constant, in addition to the end-point singularities generally existed in other exclusive processes such as the pion form factor. We introduce a simple technique to evaluate the contributions from these singularities, so that we can arrive at a judgement that these contributions will be unharmful to the applicability of pQCD at certain energy scale, i.e., the “work point” which is defined to determine when pQCD is applicable to exclusive processes. The applicability of pQCD for different pion distribution amplitudes are explored in detail. We show that pQCD begins to work at 10 GeV\(^2\). If we relax our constraint to a weak sense, the work point may be as low as 4 GeV\(^2\).

Key words: virtual Compton scattering, perturbative QCD, pion, work point

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The application of perturbative Quantum Chromodynamics (pQCD) to exclusive processes started since late 1970’s [1,2,3], and now the pQCD approach has been widely employed to various exclusive processes. It is generally known that pQCD is applicable at very high energy scale, and one of the challenging

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questions is whether pQCD is valid or not at present experimental accessible energy scale. A typical example is the applicability of pQCD to the pion form factor. It should be actually unknown whether pQCD is applicable or not before any affirmative judgement that the dominance of perturbative effect over the soft effect is justified. It was indicated by Isgur and Llewellyn Smith [4] that a significant fraction of contributions to the form factor is from the soft end-point regions at medium-to-high energy scale. In calculating the pion electromagnetic form factor, Huang and Shen [5] showed that the non-perturbative contribution from end-point regions can be suppressed with the end-point-suppressed distribution amplitude by taking into account the transverse momentum in the pion wave function. Li and Sterman [6] reduced the end-point contribution by Sudakov effect, i.e., to replace the end-point singularity by theoretical based resummation over soft radiative corrections. There is also an attempt [7] to explain the large discrepancy between theory and experiment by large contribution from next-to-leading-order power corrections.

One of the alternative exclusive processes, Compton scattering, which is another good laboratory to study the structure of hadrons, has been extended from real photon to virtual photon for the proton case as well as the pion case. Several independent theory groups have calculated the pion virtual Compton scattering (VCS) with very different methods, such as the chiral perturbation theory [8], the QCD sum rules [9], and pQCD approach [9,10,11,12]. One should meet the same question concerning the validity of pQCD, in similar to the case of the pion form factor. It is the purpose of this paper to study explicitly the applicability of pQCD to the pion VCS process. The early literature focused on how to eliminate the specific problem of kinematic singularities by analytic integration, but lack of a full study of the end-point effect. We will show that there are central-region singularities in the pion VCS, in additional to the end-point singularities generally existed in other exclusive processes such as the pion form factor. We will introduce a technique to evaluate the contributions from these singularities, so that we can have a criterion for the applicability of pQCD at certain energy scale.

The pion VCS process is illustrated in Fig. 1, where $p^\mu$ and $q^\mu$ are the 4-momenta of the incoming pion and virtual photon respectively, and $p'{}^\mu$ and $q'{}^\mu$ are the corresponding 4-momenta of the outgoing pion and photon. We
define the following frame invariant variables:

\[ Q^2 = -q^2, \quad S = (q + p)^2 = (q' + p')^2, \]
\[ t = (p' - p)^2, \quad u = (p' - q)^2, \]

(1)

where \( q^2 \) is the squared 4-momentum of the virtual photon, and \( S \) is the squared center-of-mass energy of the virtual photon and pion system.

\[ q^\mu q \]
\[ q'^\mu q' \]
\[ p^\mu \]
\[ p'^\mu \]

Fig. 1. Virtual Compton scattering \( \pi^* \gamma \rightarrow \pi \gamma \) shown in the center-of-mass frame.

Factorization theorem [1,2] tells us that the scattering amplitude of an exclusive process can be expressed in a convolution formula:

\[ M(Q^2) = \frac{1}{2} \int_0^1 [dx] \int_0^1 [dy] \phi^*(y_i, \tilde{Q}_y) H(x_i, y_i, \tilde{Q}_x^2) \phi(x_i, \tilde{Q}_x) [1 + O(m^2/Q^2)], \]

(2)

where in the pion VCS case \([dx] \equiv dx d\bar{x} \delta(1-x-\bar{x}), \ x \equiv 1 - x, \ \tilde{Q}_x \equiv \min(xQ, \bar{x}Q)\), \( x \) is the light-cone \( k^+ = k^0 + k^3 \) momentum fraction of the struck quark (antiquark) in the incoming pion, and \( y \) is the light-cone momentum fraction of the corresponding quark (antiquark) in the outgoing pion. The validity of Eq. (2) is that there is an initial \( Q = Q_0 \) as the factorization scale above which the hard part and the soft part could be separated safely and explicitly. We expect \( S, -t, \) and \(-u\) large to guarantee factorization legal in the VCS case. The distribution amplitude \( \phi(x, Q) \) may be chosen without higher Gegenbauer polynomials to exclude \( Q \) dependence. In this paper the asymptotic (as) [1,2] and Chernyak-Zhitnitsky (cz) [13] distribution amplitudes:
\[ \phi_{as} = \sqrt{3} f_\pi x (1 - x), \]
\[ \phi_{cz} = 5 \sqrt{3} f_\pi x (1 - x)(1 - 2x)^2, \]
\[ \phi_{bh} = 1.4706 \sqrt{3} f_\pi x (1 - x) \exp\left[ \frac{-0.07043}{x(1 - x)} \right], \]
\[ \phi_{hs} = 8.8763 \sqrt{3} f_\pi x (1 - x)(1 - 2x)^2 \exp\left[ \frac{-0.07062}{x(1 - x)} \right], \]
\[ H_{LL} = H_{RR}, \quad H_{RL} = H_{LR}, \]
\[ H_{+L} = H_{+R}, \quad H_{-R} = -v^{-1} H_{+R}, \quad H_{L} = -v^{-1} H_{+L}, \]

are used and compared in our calculations, where the pion decay constant \( f_\pi = 93 \text{ MeV} \) is adopted. We use the leading-twist factorization scheme instead of the handbag scheme. The different hard scattering amplitudes \( H \) have the following relations due to parity symmetry [12]:

\[ q^\mu = (\omega, 0, 0, p), \quad p^\mu = (p, 0, 0, -p), \]

and the corresponding 4-momenta of the outgoing particles are:

\[ q^\mu = \frac{\omega + p}{2} (1, \sin \theta, 0, \cos \theta), \quad p^\mu = \frac{\omega + p}{2} (1, -\sin \theta, 0, -\cos \theta). \]
Table 1
The hard scattering amplitudes calculated by pQCD

| diagram | $H'_{LR}$ | $H'_{RR}$ | $H'_{+R}$ |
|---------|-----------|-----------|-----------|
| a       | $\frac{20xy}{9\hbar(x-a)}$ | $\frac{20xyc^2}{9\hbar s^4(x-a)}$ | $\frac{40cxy}{9s}$ |
| b       | $\frac{20x}{9\hbar}$ | $\frac{20xc^2}{9\hbar s^2}$ | $\frac{20cx}{9\hbar s}$ |
| c       | 0 | $\frac{-20xc^2}{9\hbar s^4(x-a)}$ | $\frac{-20cx}{9\hbar s}$ |
| d       | $\frac{16x^2\bar{y}s^2}{9c^2} \left( \frac{1}{x-a} - \frac{1}{x-b} \right)$ | $\frac{16xy(1-\bar{y}x^2)}{9\hbar c^4} \left( \frac{1}{x-a} - \frac{1}{x-b} \right)$ | $\frac{-16xyc(1-2\bar{y}x^2)}{9\hbar c(1-y^2)(x-b)}$ |
| e       | $\frac{-16x^2\bar{y}s^2}{9c^2} \left( \frac{1}{x-a} - \frac{1}{x-b} \right)$ | 0 | $\frac{16cxs\bar{y}y}{9(1-y^2)(x-b)}$ |
| f       | 0 | $\frac{20y}{9\hbar s^2}$ | $\frac{20y(1-2\bar{y}x)}{9\hbar s}$ |
| g       | $\frac{-20a\bar{y}xy}{9[y(1-v^2)-v]}$ | $\frac{-20xyc^2}{9\hbar s^4[y(1-v^2)-v]}$ | $\frac{-20a\bar{c}xy}{9s[y(1-v^2)-v]}$ |
| h       | $\frac{-20y[y+y\bar{v}(1-2y^2)]}{9[y(1-v^2)-v]}$ | $\frac{-20y[y-2s^2(y+\bar{v}x)+2\bar{v}xys^4]}{9\hbar s^4[y(1-v^2)-v]}$ | $\frac{-20c^2y^2(1-2s^2)}{9\hbar s[y(1-v^2)-v]}$ |
| i       | $\frac{-16x^2\bar{y}s^2\bar{y}x y^2}{9(1-y^2)[y(1-v^2)-v](x-b)}$ | $\frac{-16x^2\bar{y}s^2}{9[y(1-v^2)-v](x-b)}$ | $\frac{-16x^2\bar{y}s^2}{9(1-y^2)[y(1-v^2)-v](x-b)}$ |
| j       | $\frac{-16x^2\bar{y}s^2[y-\bar{v}x(1-2y^2)]}{9(1-y^2)[y(1-v^2)-v](x-b)}$ | $\frac{-16x^2\bar{y}s^2[y-v\bar{v}+\bar{v}x(1-2y^2)]}{9(1-y^2)[y(1-v^2)-v](x-b)}$ | $\frac{-16cxs\bar{y}y[y-v+2\bar{v}xys^2]}{9(1-y^2)[y(1-v^2)-v](x-b)}$ |

Here we define

\[ S \equiv (q^\mu + p^\mu)^2 = (\omega + p)^2, \]
\[ c \equiv \cos \frac{\theta}{2}, \quad s \equiv \sin \frac{\theta}{2}, \]
\[ v \equiv q^2/S, \quad \bar{v} \equiv 1 - v, \quad (8) \]
\[ a \equiv \frac{v}{\bar{v}}, \quad b \equiv \frac{y^2 - v^2s^2}{\bar{v}(1-y^2)}, \]
\[ t \equiv (p'^\mu - p^\mu)^2 = -S\bar{v}s^2, \]
where \( v \) stands for the photon virtuality and should be between \(-1\) and \(0\), and \( \theta \) is the angle between the incoming virtual photon and the outgoing photon and can be obtained from the experimental variables by

\[
s^2 = \sin^2 \frac{\theta}{2} = -\frac{t}{S\bar{v}} = -\frac{t}{S - q^2}.
\]  

(9)

We let \( v = -0.8 \) as an example in our calculations to ensure \( q^2 \) large. \(-t\) is monotonously increasing with the scattering angle \( \theta \) if \( S \) and \( v \) are both fixed. For diagrams a, b, c, f, g and h (see Fig. 2 of [12]), the squared 4-momentum transfer of the exchanged gluons between the two valence partons in the pion is

\[
\tilde{Q}^2 = -\bar{v}\bar{x}\bar{y}S,
\]

(10)

where the small \( \tilde{Q}^2 \) region is also the end-point region whose effects will be highly suppressed by the distribution amplitudes. For diagrams d, e, i and j, where the incident and outgoing photons are connected to different quark lines, the corresponding squared 4-momentum transfer of the exchanged gluons is

\[
\tilde{Q}^2 = \bar{v}(1 - y s^2)(x - b)S.
\]

(11)

Because of \( 0 < a < b < 1 \), whatever the kinematical region we will choose, it is impossible to guarantee \( \tilde{Q}^2 \) large enough for making pQCD legal here. The additional singularity of \( \tilde{Q}^2 \) in the central region, i.e. \( \tilde{Q}^2 \rightarrow 0 \) when \( x \rightarrow b \), makes calculations here much more complex than that of form factor. We use principle integration formula as in [12]

\[
\lim_{\epsilon \rightarrow 0} \int_0^1 \frac{f(x)}{x - a + i\epsilon} dx = P \int_0^1 \frac{f(x)}{x - a} dx - i\pi f(a)
\]

\[
= \int_0^1 \left\{ \frac{f(x) - f(a)}{x - a} + f(a)[\log \frac{1 - a}{a} - i\pi] \right\} dx
\]

(12)

to get reliable numerical results.
Straightly, we have:

\[
\frac{d\sigma}{d\cos \theta} = \frac{1}{32\pi S(1-v^2)} \sum|M|^2 = \frac{3}{4} \frac{(2\pi)^3 \alpha_s^2(\sqrt{3}f_\pi)^2}{S^3(1-v^2)} \sum|M'|^2,
\]

where \(\Sigma|M'|^2 = |M'_{LR} + M'_{RR} + (1-1/v)M'_{+R}|^2\), with \(M'\)'s calculated from Eq. (2) with \(H'\)'s in Table 1 by including the factor \(\alpha_s(\tilde{Q}^2)\) which is not chosen as a frozen coupling constant \([12]\) in this paper. \(M^2_{RR}\) contributes dominantly over \(M^2_{LR}\) and \(M^2_{+R}\). For the applicability of pQCD, the concerned gluon should be hard, i.e., \(\tilde{Q}^2\) should be large enough to prevent mis-absorbing soft contributions. Therefore, we do not fix the QCD running coupling constant

\[
\alpha_s(\tilde{Q}^2) = \frac{4\pi}{b_0 \log \tilde{Q}^2/\Lambda^2},
\]

where \(b_0 = 11 - 2n_f/3\), \(n_f\) is the number of active quark flavors, and \(\Lambda \approx 200\) MeV. We let \(\alpha_s\) running if \(\tilde{Q}^2\) is larger than a given \(g\) GeV\(^2\); otherwise, we let \(\alpha_s = 0\), expecting that only pure pQCD contributions are considered. \(\alpha_s(g)\) and factorization scale \(Q_0\) can not be determined \(a\ priori\) by pQCD theory. So the choosing of \(g\) seems of some kind of arbitrary. However, the physics should not depend on which cutoff we make, i.e., the results should not change much if we change \(g\) slightly. In Fig. 2 we find that in the \(\phi_{as}\) case different \(g\) do not affect the result through \(10^\circ \rightarrow 80^\circ\) and deviate a little in \(80^\circ \rightarrow 170^\circ\). In the \(\phi_{bhl}\) case the two curves of different \(g\) are even more adjacent because \(\phi_{bhl}\) is just the end-point-suppressed partner of \(\phi_{as}\). \(\phi_{cz}\) has bumps near the end points so its results are very sensitive to \(g\) (Fig. 3). It can be expected that \(\phi_{hs}\) will be better than \(\phi_{cz}\) at the cost of suppressing the end-points, just as we show in Fig. 2. With the squared center-of-mass energy \(S\) increasing, the gluons have generally more tendency to be hard. Thus, the parameter \(g\) becomes less important. But \(\phi_{cz}\) is still bad even when \(S\) reaches 50 GeV\(^2\); therefore we will not consider it any more as it lacks predictive power. We calculate the cross sections by using different distribution amplitudes and varying \(g = 0.09\) (\(\alpha_{max} = 1.7\)), \(g = 0.49\) (\(\alpha_{max} = 0.56\)) and \(S = 4, 10, 20, 50\) GeV\(^2\) (Figs. 2, 4, 5, and 6). Fortunately, the cross sections are independent to \(g\) when the energy scale is large enough. This leads us to put forward “work point”, a concept analogical to “Kondo temperature”.

When the cross sections of one given distribution amplitude are no longer
Fig. 2. Cross sections at $S = 4 \text{ GeV}^2$.

Fig. 3. Cross sections of $\phi_{cz}$.

Fig. 4. Cross sections at $S = 10 \text{ GeV}^2$.

affected by $g$, we define the present energy scale as the “work point” of this distribution amplitude. Different distribution amplitudes have different work points. The end-point-suppressed one usually has lower work point. Only above
Fig. 5. Cross sections at $S = 20$ GeV$^2$.

Fig. 6. Cross sections at $S = 50$ GeV$^2$.

the work point which means that the contribution from $Q^2 < g$ GeV$^2$ region can be safely neglected and all the contribution comes mainly from $Q^2$ large region, the condition for the applicability of pQCD is satisfied and the calculations of the given process using this distribution amplitude are possibly reliable. By our work it is clear that the work points defined by $S$ are, $\phi_{as}$: $\sim 20$ GeV$^2$; $\phi_{bhl}$: $\sim 10$ GeV$^2$; $\phi_{hs}$: $\sim 20$ GeV$^2$. With the transverse momentum contributions considered, Sudakov effect [9] may reduce the work point lower. If we relax our constraint to a weak sense, i.e., demanding the cross section difference with different $g$ should be smaller than some percents, the work point of $\phi_{bhl}$ and $\phi_{as}$ will be lowered to 4 GeV$^2$ in the small angle region for Fig. 2. This is in agreement with Coriano-Li [9].

As shown in Fig. 7, we calculate $S^3 \frac{d\sigma}{d\cos \theta}$ with $S$ varying from 4 GeV$^2$ to 50 GeV$^2$. $\frac{d\sigma}{d\cos \theta}$ obeys approximately the $S^{-3}$ law at fixed angle as naive pQCD has predicted [15]. The cutoff of the running $\alpha_s$ suppresses the value of the cross section with approximate one order of magnitude, but it renders enough to survive the scaling law. It is also nontrivial that the cross sections are almost the same at small angles, especially from 10° to 80°, for different distribution amplitudes (Fig. 8), as we have no justification to demand that. On the other hand, they are so apart at large angles, so that with further experiments we can determine which one should be chosen in the VCS case. We should pay
attention that the cross sections of $\phi_{hs}$ have a minimum running from 60° to 70° while the cross sections of $\phi_{as}$ and $\phi_{bhl}$ have the same minimum always at 80°. Summing up 20° to 80° to get the “total” cross section should be meaningful as predictions, as shown in Table 2.

We should state that this is not an exact numerical estimate of the cross sections, but rather a consistency check, without any higher-twist contributions or radiative corrections. SELEX Collaboration at Fermilab [16] obtained the total forward cross section of $\pi^- e \rightarrow \pi^- e \gamma$ of 38.8 nb, in agreement with the theoretical expectation by the chiral perturbation theory method [8]. Their $S \leq M_\rho^2$, which is below the work point of any distribution amplitude we have explored. It illuminates that in the nowadays experimental available energy region the non-perturbative contribution is dominant over the pQCD contribution for the pion VCS process.

The dependence on the photon virtuality $v$ is not very sensitive, as can be seen from Fig. 9, where two different values of $v = -0.3$ and $-0.5$ are examined for $\phi_{bhl}$ at $S = 4 \text{ GeV}^2$. In other words, the squared virtual photon 4-momentum transfer $q^2 = S v$ is not a signal energy scale in virtual Compton scattering. And $S = 1$ and 2 GeV$^2$ for $\phi_{bhl}$ are also examined in Fig. 10, from which one may even arrive at a conclusion for the applicability of pQCD, in a rather weak sense, for $\phi_{bhl}$ at $S = 1 \rightarrow 2 \text{ GeV}^2$ in the region 20° to 80°.

Every distribution amplitude evolves in principle to the asymptotic form at very high energy scale. Such evolution effects are small for $\phi_{as}$ and $\phi_{bhl}$, whereas they should be significant for $\phi_{cz}$ and $\phi_{hs}$. However, there has been no numerical evaluation on the evolution of $\phi_{cz}$ and $\phi_{hs}$ in previous studies. It is reasonable to expect a lower energy scale for the applicability of pQCD
for $\phi_{cz}$ and $\phi_{hs}$ than what we claimed, after taking into the evolution effects.

In summary, we studied explicitly, for the first time, the applicability of pQCD to the pion virtual Compton scattering. As in other exclusive processes such as the pion form factor, the end-point contribution is not negligible in the pQCD calculation of the pion VCS process, as manifested by the difference between the cross sections of $\phi_{as}$ with $\phi_{bhl}$ and $\phi_{cz}$ with $\phi_{hs}$. We also noticed that there exist middle-region singularities introduced by the QCD running coupling constant of the exchanged gluons between the two valence partons of the pion. We introduced a simple technique to evaluate the contributions from these singularities, so that we can arrive at a judgement that these contributions will be unharmonious to the applicability of pQCD at certain energy scale, i.e., the work point to guarantee the safety of pQCD. The work points for different distribution amplitudes are explored in detail in this work. For the end-point-suppressed distribution amplitudes such as $\phi_{bhl}$, we showed that pQCD begins to work at 10 GeV$^2$. If we relax our constraint to a weak sense, the work point may be as low as 4 GeV$^2$.

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Fig. 8. Comparison between different distribution amplitudes ($g = 0.49$).

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Fig. 9. Cross sections with different $v$.

References

[1] G.P. Lepage, S.J. Brodsky, Phys. Lett. B 87 (1979) 359; Phys. Rev. Lett. 43 (1979) 545;
G.R. Farrar, D.R. Jackson, Phys. Rev. Lett. 43 (1979) 246;
A.V. Efremov, A.V. Radyushkin, Phys. Lett. B 94 (1980) 245;
A. Duncan, A.H. Mueller, Phys. Lett. B 90 (1980) 159.

[2] G.P. Lepage, S.J. Brodsky, Phys. Rev. D 22 (1980) 2157.

[3] S.J. Brodsky, T. Huang, G.P. Lepage, in *Particles and Fields-2*, Proceedings of the Banff Summer Institute, Banff, Alberta, 1981, edited by A.Z. Capri and A.N. Kamal (Plenum, New York, 1983), p. 143.

[4] N. Isgur, C.H. Llewellyn Smith, Phys. Rev. Lett. 52 (1984) 1080; Nucl. Phys. B 317 (1989) 526.

[5] T. Huang, Q.-X. Shen, Z. Phys. C 50 (1991) 139.

[6] H. Li, G. Sterman, Nucl. Phys. B 381 (1992) 129.

[7] T.-W. Yeh, Phys. Rev. D 65 (2002) 074016.
Fig. 10. Cross sections at lower $S$.

[8] C. Unkmeir et al., Phys. Rev. C 65 (2002) 015206.

[9] C. Coriano, H. Li, Nucl. Phys. B 434 (1995) 535.

[10] M. Tamazouzt, Phys. Lett. B 211 (1988) 477.

[11] E. Maina, R. Torasso, Phys. Lett. B 320 (1994) 337.

[12] D. Zeng, B.-Q. Ma, Phys. Lett. B 542 (2002) 55.

   It should be “$1 - s^2(y + \bar{v}x) + 2\bar{v}xy^4$” in the bracket of the numerator of the
   hard part $h$ in Table 1 of this paper.

[13] V.L. Chernyak, A.R. Zhinitsky, Nucl. Phys. B 201 (1982) 492.

[14] T. Huang, B.-Q. Ma, Q.-X. Shen, Phys. Rev. D 49 (1994) 1490.

[15] S.J. Brodsky, G.R. Farrar, Phys. Rev. Lett. 31 (1973) 1153.

[16] SELEX Collaboration, A. Ocherashvili et al., Phys. Rev. C 66 (2002) 034613.