Variational Inequalities Governed By Merely Continuous and Strongly Pseudomonotone Operators

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Abstract. Qualitative and quantitative aspects for variational inequalities governed by merely continuous and strongly pseudomonotone operators are investigated in this paper. First, we establish a global error bound for the solution set of the given problem with the residual function being the normal map. Second, we will prove that the iterative sequences generated by gradient projection method (GPM) with stepsizes forming a non-summable diminishing sequence of positive real numbers converge to the unique solution of the problem with bounded constraint set. Two counter-examples are given to show the necessity of the boundedness assumption and the variation of stepsizes. A modification of GPM is proposed for unbounded case. Finally, we analyze the convergence rate of the iterative sequences generated by this method.

Keywords: Variational inequalities · Strong pseudomonotonicity · Strong monotonicity · Gradient projection method · Variable stepsizes · Error bound · Convergence · Convergence rate

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1 Introduction

Variational inequality (VI) is a powerful mathematical model which unifies the study of important concepts such as optimization problems, equilibrium problems, complementarity problems, obstacle problems and continuum problems in the mathematical sciences (see e.g. [2, 11]).

Qualitative properties of VI strongly depend on some kind of monotonicity. In particular, the existence and uniqueness of the solution to the VI can be established under strong monotonicity. In view of the natural residue of the projection, Facchinei and Pang [2] obtained an upper error...
bound for strongly monotone and Lipschitz continuous VI. In [7], Khanh and Minh introduced a sharper error bound for this class of VI and gave a counter-example to show the necessity of Lipschitz continuity. Moreover, an extended result for strongly pseudomonotone and Lipschitz continuous VI was established in [9]. Such error bound not only plays an important role in proving the convergence of algorithms but also serves as a termination criteria for iterative algorithms. A question arises: can we find an error bound that does not require the Lipschitz continuity assumption? In this paper, by using the normal map which is closely related to the natural map as the residual function, we present a new error bound for merely continuous and strongly pseudomonotone VI.

There are several algorithms solving VI with certain monotonicity and continuity assumptions. Among those methods, the GPM [2, Algorithm 12.1.1] which solves strongly monotone and Lipschitz continuous VI is one of the cheapest. A modified GPM with variable stepsizes solving strongly pseudomonotone and Lipschitz continuous VI has recently been established in [8]. They also proposed a GPM with non-summable diminishing stepsize sequence in which we do not need to know a priori constants. Following this idea, we will prove in this paper that the Lipschitz continuity can be completely omitted in the modified GPM, however the boundedness of the constraint set is required. A counter-example is given to show the necessity of this boundedness assumption. By using the new error bound to find a closed ball containing the solution, then projecting on a bounded constraint set which is the intersection of the original one and that closed ball, we can overcome the difficulty of the unbounded case. We also give a counter-example to show that the traditional GPM with constant stepsize cannot be applied when the Lipschitz continuity is omitted. When the stepsizes are sequences of terms defining the $p$-series, we can estimate the rate of convergence of modified GPM which depends on the interval containing $p$.

Following this introduction, we give some preliminaries in Section 2 in which we recall some well-known definitions and properties of the projection mapping, kinds of monotonicity as well as the natural map and the normal one. In Section 3, we establish the error bound for continuous and strongly pseudomonotone VIs. In Section 4, we recall the classical GPM and give a counter-example to show its unavailability when omitting Lipschitz continuity condition. Two modifications for this method are proposed for the given problem. Some convergence rate results are established in Section 5. Finally, concluding remarks are given in Section 6.

2 Preliminaries

Let $K \subset \mathbb{R}^n$ be a non-empty closed convex set and $F: K \rightarrow \mathbb{R}^n$ be a continuous operator. The variational inequality problem defined by $K$ and $F$, denoted by VI($K, F$), is to find $x^* \in K$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in K. \quad (1)$$

Clearly, if $x^*$ satisfies (1) and belongs to the interior of $K$ then $F(x^*) = 0$.

For each $x \in \mathbb{R}^n$, there exists a unique point in $K$ [11, Chapter 1, Lemma 2.1], denoted by $\text{Pr}_K(x)$, such that

$$\|x - \text{Pr}_K(x)\| \leq \|x - y\|, \quad \forall y \in K.$$
The point $\text{Pr}_K(x)$ is called the projection of $x$ on $K$. Some well-known properties of the projection mapping $\text{Pr}_K : \mathbb{R}^n \to K$ are recalled in the following theorem.

**Theorem 2.1.** Let $K \subset \mathbb{R}^n$ be non-empty closed convex set.

(a) For all $x \in \mathbb{R}^n$ and $y \in K$, it holds that

$$\langle x - \text{Pr}_K(x), y - \text{Pr}_K(x) \rangle \leq 0.$$  

(b) The projection mapping is non-expansive, that is

$$\|\text{Pr}_K(x) - \text{Pr}_K(y)\| \leq \|x - y\|, \quad \forall x, y \in \mathbb{R}^n.$$  

One often considers $\text{VI}(K, F)$ when $F$ possesses a certain monotonicity property.

**Definition 2.2.** (see [4] and [5]) Let $K \subset \mathbb{R}^n$ be arbitrary. The mapping $F : K \to \mathbb{R}^n$ is said to be

(a) monotone on $K$ if

$$\langle F(x) - F(y), x - y \rangle \geq 0, \quad \forall x, y \in K.$$  

(b) strongly monotone on $K$ if there exists $\gamma > 0$ such that

$$\langle F(x) - F(y), x - y \rangle \geq \gamma \|x - y\|^2, \quad \forall x, y \in K.$$  

(c) pseudomonotone on $K$ if

$$\langle F(y), x - y \rangle \geq 0 \implies \langle F(x), x - y \rangle \geq 0, \quad \forall x, y \in K.$$  

(d) strongly pseudomonotone on $K$ if there exists $\gamma > 0$ such that

$$\langle F(y), x - y \rangle \geq 0 \implies \langle F(x), x - y \rangle \geq \gamma \|x - y\|^2, \quad \forall x, y \in K.$$  

Obviously, the following relations hold: (b) $\implies$ (a) $\implies$ (c) and (b) $\implies$ (d) $\implies$ (c). The reversed implications are not true in general.

We recall the Lipschitz continuity of a mapping.

**Definition 2.3.** Let $K \subset \mathbb{R}^n$ be arbitrary. A mapping $F : K \to \mathbb{R}^n$ is said to be Lipschitz continuous on $K$ if there exists $L > 0$ such that

$$\|F(x) - F(y)\| \leq L \|x - y\|, \quad \forall x, y \in K.$$  

Now we consider two well-known mappings associated with the problem $\text{VI}(K, F)$: the natural map $F_K^{\text{nat}}$ and the normal map $F_K^{\text{nor}}$.

**Definition 2.4.** Let $K$ be a non-empty closed convex set and $F : K \to \mathbb{R}^n$ be arbitrary.
(a) The natural map $F_{\text{nat}}^K: K \to \mathbb{R}^n$ is defined as

$$F_{\text{nat}}^K(x) := x - \text{Pr}_K(x - F(x)), \quad \forall x \in K.$$ 

(b) The normal map $F_{\text{nor}}^K: \mathbb{R}^n \to \mathbb{R}^n$ is defined as

$$F_{\text{nor}}^K(x) := F(\text{Pr}_K(x)) + x - \text{Pr}_K(x), \quad \forall x \in \mathbb{R}^n.$$ 

The mappings $F_{\text{nat}}^K$ and $F_{\text{nor}}^K$ are very useful for characterizing the solution set of $\text{VI}(K, F)$ (see [2, Propositions 1.5.8 and 1.5.9]).

**Theorem 2.5.** Let $K$ be non-empty closed convex set and $F: K \to \mathbb{R}^n$ be arbitrary.

(a) $x^*$ is a solution of $\text{VI}(K, F)$ if and only if $F_{\text{nat}}^K(x^*) = 0$.

(b) $x^*$ is a solution of $\text{VI}(K, F)$ if and only if there exists $z \in \mathbb{R}^n$ such that $x^* = \text{Pr}_K(z)$ and $F_{\text{nor}}^K(z) = 0$.

3 Error bound for strongly pseudomonotone VIs

With the help of degree theory, Facchinei and Pang proved that VI associated with strongly monotone and continuous operator admits a unique solution. The following error bound is widely used in that case [2, Theorem 2.3.3].

**Theorem 3.1.** Let $K \subset \mathbb{R}^n$ be a non-empty closed convex set, $F: K \to \mathbb{R}^n$ be Lipschitz continuous with constant $L$ and strongly monotone with modulus $\gamma$, and $x^*$ be the unique solution of $\text{VI}(K, F)$. For all $x \in K$, we have

$$\|x - x^*\| \leq \frac{L + 1}{\gamma} \|x - \text{Pr}_K(x - F(x))\|.$$ 

Remind that the term $x - \text{Pr}_K(x - F(x))$ is $F_{\text{nat}}^K(x)$, thus the above inequality is written as

$$\|x - x^*\| \leq \frac{L + 1}{\gamma} \|F_{\text{nat}}^K(x)\|.$$ 

Since $F_{\text{nat}}^K$ is continuous and $F_{\text{nat}}^K(x^*) = 0$ (Theorem 2.5), $F_{\text{nat}}^K(x)$ converges to 0 as $x$ tends to $x^*$. Therefore, we can use the error bound in Theorem 3.1 as a stopping criterion in methods solving VIs for Lipschitz continuous and strongly monotone operators.

Extending [2, Theorem 2.3.3], Kim et al. proved in [9, Theorem 2.1] the solution uniqueness for strongly pseudomonotone VI. Moreover, they established an error bound for strongly pseudomonotone and Lipschitz continuous VIs [9, Theorem 4.2]. Recently, a sharper upper error bound and a new lower error bound for strongly monotone and Lipschitz continuous VIs were established in [7, Theorem 3.1]. The authors also showed in [7] that we cannot omit the Lipschitz continuity assumption in Theorem 3.1 [7, Remark 3.1]. To deal with the non-Lipschitz case, we could establish a new error bound by using the normal map.
Theorem 3.2. Let $K \subset \mathbb{R}^n$ be a non-empty closed convex set, $F: K \to \mathbb{R}^n$ be a continuous and strongly pseudomonotone map with modulus $\gamma$. For all $x \in \mathbb{R}^n$, we have

$$\|x^* - \text{Pr}_K(x)\| \leq \frac{1}{\gamma} \|F^\text{nor}_K(x)\|,$$

where $x^*$ is the unique solution of VI($K, F$).

Proof. For a given vector $x \in \mathbb{R}^n$, write $r = F^\text{nor}_K(x)$. By Theorem 2.1(a),

$$\langle x - \text{Pr}_K(x), \text{Pr}_K(x) - y \rangle \geq 0, \quad \forall y \in K.$$

Substitute $\text{Pr}_K(x) = F(\text{Pr}_K(x)) + x - r$ and choose $y = x^*$ in the above inequality, we obtain

$$\langle r - F(\text{Pr}_K(x)), \text{Pr}_K(x) - x^* \rangle \geq 0.$$

This inequality is equivalent to

$$\langle r, \text{Pr}_K(x) - x^* \rangle \geq \langle F(\text{Pr}_K(x)), \text{Pr}_K(x) - x^* \rangle.$$  \hfill (3)

Since $x^*$ is the solution of VI($K, F$), we have

$$\langle F(x^*), \text{Pr}_K(x) - x^* \rangle \geq 0.$$

By the strong pseudomonotonicity of $F$, the right-hand side of (3) is not smaller than $\gamma \|x^* - \text{Pr}_K(x)\|^2$, while the left-hand side is not greater than $\|r\| \cdot \|x^* - \text{Pr}_K(x)\|$ by Cauchy-Schwarz inequality. Therefore,

$$\|r\| \cdot \|x^* - \text{Pr}_K(x)\| \geq \gamma \|x^* - \text{Pr}_K(x)\|^2,$$

which deduces to (2). \hfill \Box \hfill \Box

Remark 3.3. If $x \in K$, $\text{Pr}_K(x) = x$. Thus $F^\text{nor}_K(x) = F(\text{Pr}_K(x)) + x - \text{Pr}_K(x) = F(x)$. It follows from (2) that

$$\|x^* - x\| \leq \frac{1}{\gamma} \|F(x)\|, \quad \forall x \in K.$$  \hfill (4)

It deduces from (4) that for an arbitrary $x \in K$, $x^*$ is always in the closed ball with center $x$ and radius $\frac{1}{\gamma}\|F(x)\|$. In case $K = \mathbb{R}^n$, $F(x^*) = 0$ since $x^*$ lies in the interior of $K$. Therefore (4) can be used as a stopping criterion for methods solving strongly pseudomonotone VIs.

4 Gradient projection method for strongly pseudomonotone VIs

We recall the classical gradient projection method solving VI($K, F$) where $F$ is Lipschitz continuous with constant $L$ and strongly monotone with modulus $\gamma$. It is well-known that the iterative sequences generated by this method converge to the unique solution of the given problem.

Algorithm 4.1. (Gradient projection algorithm with constant stepsize)
Data. Select $x_1 \in K$ and $\lambda \in \left(0, \frac{2\gamma L}{\gamma^2}\right)$.

**Step 0:** Set $k = 1$.

**Step 1:** Compute $x_{k+1} = \Pr_K(x_k - \lambda F(x_k))$.

**Step 2:** Check $x_{k+1} = x_k$. If Yes then Stop. Else set $k = k + 1$ and go to **Step 1**.

The following example shows that the iterative sequence may not converge to the solution when the Lipschitz continuity of $F$ is omitted.

**Example 4.2.** Let $K = [-1, 1]$ and $F : K \to \mathbb{R}$ be defined as

$$
F(x) = \begin{cases} 
2\sqrt{x} & \text{if } 0 \leq x \leq 1 \\
-2\sqrt{-x} & \text{if } -1 \leq x < 0 
\end{cases}.
$$

Since $F - \text{id}$ is an increasing function on $K$, $F$ is strongly monotone with modulus 1 on $K$. On the other hand, $F$ is not Lipschitz continuous on $K$ since $F(x) - F(0) = x - 0 \to \infty$ as $x \to 0^+$. Moreover, $\text{VI}(K, F)$ has a unique solution $x^* = 0$. Let $\lambda \in (0, 1)$, $x_1 \in (0, \lambda^2) \subset K$ and $\{x_k\}_{k \geq 1}$ be the iterative sequence generated by Algorithm 4.1. Observe that for an arbitrary $k$, if $0 < x_k < \lambda^2$ then $0 < x_k < x_{k+2} < \lambda^2$. Indeed, since $0 < x_k < \lambda^2 < 1$, we have

$$
x_k - \lambda F(x_k) = x_k - 2\lambda \sqrt{x_k} \in (-\lambda^2, 0) \subset (-1, 0) \subset K,
$$

thus

$$
x_{k+1} = \Pr_K(x_k - \lambda F(x_k)) = x_k - 2\lambda \sqrt{x_k} \in (-\lambda^2, 0) \subset (-1, 0).
$$

Next, we have

$$
x_{k+1} - \lambda F(x_{k+1}) = x_{k+1} + 2\lambda \sqrt{-x_k} \in (0, \lambda^2) \subset (0, 1) \subset K,
$$

then

$$
x_{k+2} = \Pr_K(x_{k+1} - \lambda F(x_{k+1})) = x_{k+1} + 2\lambda \sqrt{-x_k} \in (0, \lambda^2).
$$

It remains to show that $x_{k+2} > x_k$. We have

$$
x_{k+2} - x_k = -2\lambda \sqrt{x_k} + 2\lambda \sqrt{2\lambda \sqrt{x_k} - x_k}
$$

$$
= 4\lambda \sqrt{x_k} \cdot \frac{\lambda - \sqrt{x_k}}{\sqrt{x_k} + \sqrt{2\lambda \sqrt{x_k} - x_k}} > 0,
$$

which is true since $0 < x_k < \lambda^2$. Following this observation, since $0 < x_1 < \lambda^2$, it can be proved by induction that

$$
0 < x_{2k+1} < x_{2k+3} < \lambda^2, \quad \forall k \geq 0,
$$

which means $\{x_{2k+1}\}_{k \geq 0}$ is an increasing positive sequence. Thus $\{x_{2k+1}\}_{k \geq 0}$ is a subsequence of $\{x_k\}_{k \geq 1}$ that does not converge to 0 which implies $\{x_k\}_{k \geq 1}$ does not converge to 0.
In [2, Algorithm 12.1.4], the authors introduced a gradient projection method with variable stepsizes. If $F$ is co-coercive with constant $c$ and the stepsize sequence $\{\lambda_k\}$ satisfies

$$0 < \inf_k \lambda_k \leq \sup_k \lambda_k < 2c,$$

the algorithm will converge to a solution of $\text{VI}(K, F)$ [2, Theorem 12.1.8]. This means we can still apply Algorithm 4.1 with $\lambda < 2c$ for co-coercive operators. Note again that the co-coerciveness property implies Lipschitz continuity, thus Example 4.2 is also a counter-example for [2, Algorithm 12.1.4] in case $\{\lambda_k\}$ is a constant sequence.

We now consider the case $F$ is strongly pseudomonotone and merely continuous. The following algorithm overcomes the disadvantages of Algorithm 4.1 when omitting the Lipschitz assumption.

### Algorithm 4.3. (Gradient projection algorithm with variable stepsizes)

**Data.** Select $x_1 \in K$ and a positive sequence of stepsizes $\{\lambda_k\}$ satisfying $\sum_{k=1}^{\infty} \lambda_k = \infty$ and $\lim_{k \to \infty} \lambda_k = 0$.

**Step 0:** Set $k = 1$.

**Step 1:** Compute $x_{k+1} = \text{Pr}_K(x_k - \lambda_k F(x_k))$.

**Step 2:** Check $x_{k+1} = x_k$. If Yes then Stop. Else set $k = k + 1$ and go to Step 1.

In comparison with Algorithm 4.1, Algorithm 4.3 has two major advantages:

- Algorithm 4.3 does not require the Lipschitz continuity of $F$ and thus it can be applied for a wider class of operators.
- Algorithm 4.3 does not require the modulus of strong pseudomonotonicity to determine the stepsizes.

We will prove the iterative sequence in Algorithm 4.3 converges to the unique solution $x^*$ of $\text{VI}(K, F)$ when $K$ is a closed, bounded and convex set. First, we need the following lemma which is a special case of [3, Lemma 1.5].

**Lemma 4.4.** Let $\{\eta_k\}$ be a positive sequence satisfying $\sum_{k=1}^{\infty} \eta_k = \infty$ and $\lim_{k \to \infty} \eta_k = 0$, $\{\delta_k\}$ be a real sequence satisfying $\lim_{k \to \infty} \delta_k = 0$. Assume that $\{a_k\}$ is a non-negative sequence such that

$$a_{k+1} \leq (1 - \eta_k) a_k + \eta_k \delta_k, \quad \forall k \geq 1,$$

Then $\{a_k\}$ converges to 0.

We are ready to prove the convergence of the iterative sequence in Algorithm 4.3.

**Theorem 4.5.** Let $K \subset \mathbb{R}^n$ be non-empty closed bounded convex, $F: K \to \mathbb{R}^n$ be continuous and strongly pseudomonotone. Every sequence $\{x_k\}$ produced by Algorithm 4.3 converges to the unique solution $x^*$ of $\text{VI}(K, F)$. 

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**Proof.** Suppose that $F$ is strongly pseudomonotone with modulus $\gamma$. Since $x^*$ is the solution of $\text{VI}(K,F)$ and $x_k \in K$, we have

$$\langle F(x^*), x_k - x^* \rangle \geq 0.$$ 

The above inequality and the strong pseudomonotonicity of $F$ imply that

$$\langle F(x_k), x_k - x^* \rangle \geq \gamma\|x^* - x_k\|^2.$$ 

Multiplying $\lambda_k$ to both sides, the latter inequality is equivalent to

$$\langle x^*, \lambda_k F(x_k) \rangle \leq \lambda_k \langle F(x_k), x_k \rangle - \lambda_k \gamma\|x^* - x_k\|^2. \quad (5)$$

Since $x_{k+1} = \text{Pr}_K(x_k - \lambda_k F(x_k))$, it follows from Theorem 2.1(a) that

$$\langle x^* - x_{k+1}, x_k - \lambda_k F(x_k) - x_{k+1} \rangle \leq 0,$$

which is equivalent to

$$\langle x^*, x_k - \lambda_k F(x_k) - x_{k+1} \rangle \leq \langle x_{k+1}, x_k - \lambda_k F(x_k) - x_{k+1} \rangle. \quad (6)$$

Adding (5) and (6), we obtain

$$\langle x^*, x_k - x_{k+1} \rangle \leq \langle x_{k+1}, x_k - \lambda_k F(x_k) - x_{k+1} \rangle + \lambda_k \langle F(x_k), x_k \rangle - \lambda_k \gamma\|x^* - x_k\|^2.$$ 

This inequality can be written as

$$2\lambda_k \gamma\|x^* - x_k\|^2 \leq 2\langle x_{k+1} - x^*, x_k - x_{k+1} \rangle + 2\lambda_k \langle F(x_k), x_k - x_{k+1} \rangle. \quad (7)$$

Since

$$2\langle x_{k+1} - x^*, x_k - x_{k+1} \rangle = \|x^* - x_k\|^2 - \|x^* - x_{k+1}\|^2 - \|x_k - x_{k+1}\|^2,$$

$$2\lambda_k \langle F(x_k), x_k - x_{k+1} \rangle \leq \lambda_k^2 \|F(x_k)\|^2 + \|x_k - x_{k+1}\|^2,$$

it follows from (7) that

$$2\lambda_k \gamma\|x^* - x_k\|^2 \leq \|x^* - x_k\|^2 - \|x^* - x_{k+1}\|^2 + \lambda_k^2 \|F(x_k)\|^2,$$

which is equivalent to

$$\|x^* - x_{k+1}\|^2 \leq (1 - 2\lambda_k \gamma)\|x^* - x_k\|^2 + \lambda_k^2 \|F(x_k)\|^2. \quad (8)$$

Let $a_k = \|x^* - x_k\|^2$, $\eta_k = 2\lambda_k \gamma$ and $\delta_k = \frac{\|F(x_k)\|^2}{2\gamma} \lambda_k$. We have $\{\eta_k\}$ is a positive sequence satisfying $\sum_{k=1}^{\infty} \eta_k = \infty$ and $\lim_{k \to \infty} \eta_k = 0$. Since $F$ is continuous on compact set $K$ and $\{x_k\} \subset K$, the sequence $\{\|F(x_k)\|^2\}$ is bounded and so $\{\delta_k\}$ is a real sequence satisfying $\lim_{k \to \infty} \delta_k = 0$. Therefore, it follows from (8) and Lemma 4.4 that $\|x^* - x_k\|^2 \to 0$ which implies $x_k \to x^*$. $\square$

The next example shows that the boundedness of $K$ in Theorem 4.5 cannot be omitted.
Example 4.6. Let $K = \mathbb{R}, F(x) = 2|x|x$ and $\lambda_k = \frac{1}{k}$ for all $k \geq 1$. The operator $F$ is continuous, strongly monotone with modulus 1 (thus strongly pseudomonotone with modulus 1) and $\text{VI}(K, F)$ has a unique solution $x^* = 0$. Moreover, the sequence $\{\lambda_k\}$ satisfying $\sum_{k=1}^{\infty} \lambda_k = \infty$ and $\lim_{k \to \infty} \lambda_k = 0$.

The iterative sequence $\{x_k\}$ in Algorithm 4.3 is defined as

$$x_{k+1} = x_k \left(1 - \frac{2|x_k|}{k}\right), \quad \forall k \geq 1.$$ 

Let $x_1 = 2$. We will prove by induction that

$$|x_k| \geq 2k, \quad \forall k \geq 1.$$ 

The inequality is true for $k = 1$. Assume that $|x_k| \geq 2k$, we have

$$|x_{k+1}| = |x_k| \left|1 - \frac{2|x_k|}{k}\right| = |x_k| \left(\frac{2|x_k|}{k} - 1\right) \geq 2k \left(\frac{4k}{k} - 1\right) \geq 2(k + 1).$$

Hence $\{x_k\}$ is not bounded, which means $\{x_k\}$ does not converge to $x^*$.

This example arises a question: can we use Algorithm 4.3 for unbounded $K$?

Fortunately, the answer is affirmative by making use of the error bound in Theorem 3.2. First, select an arbitrary $x \in K$. It follows from inequality (4) that the solution $x^*$ of $\text{VI}(K, F)$ lies in the closed ball with center $x$ and radius $\frac{1}{\gamma} \|F(x)\|$ where $\gamma$ is the modulus of strong pseudomonotonicity. The set $K' = K \cap B\left(x, \frac{1}{\gamma} \|F(x)\|\right)$ is a non-empty, closed, bounded and convex subset of $K$ containing $x^*$. Moreover, $F$ is continuous and strongly pseudomonotone on $K'$. This implies $\text{VI}(K', F)$ has a unique solution which coincides with the solution $x^*$ of $\text{VI}(K, F)$. Now we can apply Algorithm 4.3 to $\text{VI}(K', F)$.

5 Rate of convergence

In this section, we will investigate the rate of convergence of Algorithm 4.3 when the stepsizes are sequences of terms defining the $p$-series, i.e, $\lambda_k = \frac{1}{k^p}$, where $p \in (0, 1]$.

First, let us note that we can always scale the given operator $F$ by $\frac{1}{\gamma}$ so that the resulting operator $F'$ is continuous, strongly pseudomonotone with modulus $\gamma' = \frac{1}{2}$ and $\text{VI}(K, F')$ admits the same solution with $\text{VI}(K, F)$. Thus, we only need to consider a strongly pseudomonotone operator with modulus $\frac{1}{2}$.
Let \( \{x_k\} \) be the iterative sequence generated by Algorithm 4.3. Recall inequality (8) in the proof of Theorem 4.5 (remind that we assumed \( \gamma = \frac{1}{2} \)):

\[
\|x^* - x_{k+1}\|^2 \leq (1 - \lambda_k) \|x^* - x_k\|^2 + \lambda_k^2 \|F(x_k)\|^2, \quad \forall k \geq 1.
\]

Since \( F \) is continuous on compact set \( K \), there exists \( M > 0 \) such that

\[
\|F(x)\| \leq M, \quad \forall x \in K.
\]

Since \( \{x_k\} \subset K \), it follows that

\[
\|x^* - x_{k+1}\|^2 \leq (1 - \lambda_k) \|x^* - x_k\|^2 + M^2 \lambda_k^2, \quad \forall k \geq 1.
\]

For simplicity, denote \( \|x^* - x_k\|^2 = a_k \). The above inequality becomes

\[
a_{k+1} \leq (1 - \lambda_k)a_k + M^2 \lambda_k^2, \quad \forall k \geq 1. \tag{9}
\]

This inequality plays an important role in determining the rate of convergence of the algorithm. We will consider three cases: \( p = 1 \), \( p \in (\frac{1}{2}, 1) \) and \( p \in (0, \frac{1}{2}] \). Let us remind that \( \{a_k\} \) converges to 0 with rate \( O(b_k) \), where \( \{b_k\} \) is a sequence known to converge to 0, if there exists a constant \( K > 0 \) such that

\[
|a_k| \leq K|b_k|, \quad \text{for sufficiently large } k
\]

(see [1, Definition 1.18]).

5.1 The case \( p = 1 \)

**Theorem 5.1.** Let \( \{x_k\} \) be the sequence generated by Algorithm 4.3 with stepsizes \( \lambda_k = \frac{1}{k} \) for all \( k \geq 1 \) and \( x^* \) be the solution of VI\((K, F)\). Then \( \|x^* - x_k\| \) converges to 0 with rate \( O\left( \sqrt{\frac{\ln k}{k}} \right) \).

**Proof.** From inequality (9), we have

\[
a_{k+1} \leq \left( 1 - \frac{1}{k} \right) a_k + M^2 \cdot \frac{1}{k^2}, \quad \forall k \geq 1.
\]

By induction, we obtain

\[
a_{k+1} \leq \frac{a_2}{k} + \frac{M^2}{k} \left( \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} \right), \quad \forall k \geq 2.
\]

By the well-known inequality

\[
\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} < \ln k, \quad \forall k \geq 2,
\]

it follows that

\[
a_{k+1} < \frac{a_2}{k} + M^2 \frac{\ln k}{k}, \quad \forall k \geq 2.
\]

Thus

\[
a_{k+1} < (a_2 + M^2) \frac{\ln k}{k} \leq 2(a_2 + M^2) \frac{\ln(k + 1)}{k + 1}, \quad \forall k \geq 3.
\]

Therefore, the sequence \( \{a_k\} \) converges to 0 with rate \( O\left( \frac{\ln k}{k} \right) \). In other words, \( \|x^* - x_k\| \) converges to 0 with rate \( O\left( \frac{\ln k}{k} \right) \).

\( \square \)
5.2 The case $p \in \left(\frac{1}{2}, 1\right)$

Theorem 5.2. Let $\{x_k\}$ be the sequence generated by Algorithm 4.3 with stepsizes $\lambda_k = \frac{1}{k^p}$ for all $k \geq 1$ where $p \in \left(\frac{1}{2}, 1\right)$ and $x^*$ be the solution of VI$(K, F)$. Then $\|x^* - x_k\|$ converges to 0 with rate $O\left(k^{\frac{1}{2} - p}\right)$.

Proof. From inequality (9), for every $k \geq 1$, we have

$$ a_{k+1} \leq \left(1 - \frac{1}{k^p}\right) a_k + M^2 \frac{1}{k^{2p}} \leq \left(1 - \frac{1}{k}\right) a_k + M^2 \frac{1}{k^{2p}}. $$

Following the proof of Theorem 5.1, we get

$$ a_{k+1} \leq \frac{a_2}{k} + \frac{M^2}{k} \left(\frac{1}{2^{2p-1}} + \frac{1}{3^{2p-1}} + \cdots + \frac{1}{k^{2p-1}}\right), \quad \forall k \geq 2. $$

Since $1 - 2p < 0$, the function $x^{1-2p}$ is decreasing on $[1, \infty)$. It follows that

$$ \frac{1}{2^{2p-1}} + \frac{1}{3^{2p-1}} + \cdots + \frac{1}{k^{2p-1}} < \int_1^k x^{1-2p} dx = \frac{k^{2-2p} - 1}{2 - 2p}, $$

then

$$ a_{k+1} \leq \frac{a_2}{k} + \frac{M^2}{2 - 2p} k^{1-2p} \leq \left(\frac{a_2}{2} + \frac{M^2}{2 - 2p}\right) k^{1-2p} \leq 2 \left(\frac{a_2}{2} + \frac{M^2}{2 - 2p}\right) (k+1)^{1-2p}, \quad \forall k \geq 1. $$

This implies the conclusion of the theorem. \qed

5.3 The case $p \in \left(0, \frac{1}{2}\right)$

Theorem 5.3. Let $\{x_k\}$ be the sequence generated by Algorithm 4.3 with stepsizes $\lambda_k = \frac{1}{k^p}$ for all $k \geq 1$ where $p \in \left(0, \frac{1}{2}\right)$ and $x^*$ be the solution of VI$(K, F)$. Then $\|x^* - x_k\|$ converges to 0 with rate $O\left(k^{-\frac{1}{2}}\right)$.

Proof. From inequality (9), we have

$$ a_{k+1} \leq \left(1 - \frac{1}{k^{p}}\right) a_k + M^2 \frac{1}{k^{2p}}, \quad \forall k \geq 1. \quad (10) $$

Let $\{u_k\}$ be defined recursively as

$$ u_2 = \frac{1}{2^p}, \quad u_k = \frac{k^p - 1}{(k-1)^p} u_{k-1} + \frac{1}{k^{p}}, \quad \forall k \geq 3. $$

Firstly, we prove by induction that

$$ a_{k+1} \leq \left(1 - \frac{1}{k^{p}}\right) \left(1 - \frac{1}{(k-1)^{p}}\right) \cdots \left(1 - \frac{1}{2^{p}}\right) a_2 + \frac{M^2}{k^{p}} u_k, \quad \forall k \geq 2. \quad (11) $$

If $k = 2$, (11) becomes

$$ a_3 \leq \left(1 - \frac{1}{2^{p}}\right) a_2 + \frac{M^2}{2^p} u_2, $$

$$ a_3 \leq \left(1 - \frac{1}{2^{p}}\right) a_2 + \frac{M^2}{2^p} \frac{1}{2^{2p}}, $$

$$ a_3 \leq \left(1 - \frac{1}{2^{p}}\right) a_2 + \frac{M^2}{2^{3p}}. $$
which is true by (10). Suppose (11) is true for \( k \). By (10) and induction hypothesis, we have

\[
\begin{align*}
    a_{k+2} & \leq \left( 1 - \frac{1}{(k+1)^p} \right) a_{k+1} + \frac{M^2}{(k+1)^{2p}} \\
    & \leq \left( 1 - \frac{1}{(k+1)^p} \right) \left[ \left( 1 - \frac{1}{k^p} \right) \left( 1 - \frac{1}{(k-1)^p} \right) \cdots \left( 1 - \frac{1}{2^p} \right) \right] a_2 + \frac{M^2}{k^p} u_k \right] + \frac{M^2}{(k+1)^{2p}} \\
    & = \left( 1 - \frac{1}{(k+1)^p} \right) \left( 1 - \frac{1}{k^p} \right) \left( 1 - \frac{1}{(k-1)^p} \right) \cdots \left( 1 - \frac{1}{2^p} \right) a_2 + \frac{M^2}{(k+1)^p} \left[ \frac{(k+1)^p - 1}{k^p} u_k + \frac{1}{(k+1)^p} \right] \\
    & = \left( 1 - \frac{1}{(k+1)^p} \right) \left( 1 - \frac{1}{k^p} \right) \left( 1 - \frac{1}{(k-1)^p} \right) \cdots \left( 1 - \frac{1}{2^p} \right) a_2 + \frac{M^2}{(k+1)^p} u_{k+1}.
\end{align*}
\]

By induction principle, (11) is true for all \( k \geq 2 \).

Secondly, we show that \( \lim_{k \to \infty} u_k = 1 \). By direct calculations, we have

\[
    u_k = 1 + (4^p - 1) \left( \frac{1}{9^p} - \frac{1}{12^p} \right) > 1.
\]

If \( u_{k-1} > 1 \) then

\[
    u_k = \frac{k^p - 1}{(k-1)^p} u_{k-1} + \frac{1}{k^p} > \frac{k^p - 1}{(k-1)^p} + \frac{1}{k^p} > \frac{k^p - 1}{k^p} + \frac{1}{k^p} = 1,
\]

thus \( u_k > 1 \) for all \( k \geq 4 \). Denote \( v_k = u_k - 1 \), then \( \{v_k\}_{k \geq 4} \) is a positive sequence and

\[
    v_k = \frac{k^p - 1}{(k-1)^p} v_{k-1} + \frac{1}{k^p} + \frac{k^p - 1}{(k-1)^p} - 1, \quad \forall k \geq 5.
\]

Let

\[
    \eta_k = 1 - \frac{(k+1)^p - 1}{k^p} \quad \text{and} \quad \delta_k = \left( \frac{1}{(k+1)^p} + \frac{(k+1)^p - 1}{k^p} - 1 \right) \frac{1}{1 - \frac{(k+1)^p - 1}{k^p}}, \quad \forall k \geq 4,
\]

then

\[
    v_{k+1} = (1 - \eta_k) v_k + \eta_k \delta_k, \quad \forall k \geq 4.
\]

It is clear that \( \lim_{k \to \infty} \eta_k = 0 \). We will prove \( \{\eta_k\} \) is a positive sequence. By Lagrange theorem, there exists \( c_k \in (k, k+1) \) such that

\[
    \eta_k = \frac{k^p - (k+1)^p + 1}{k^p} = \frac{1 - pc_k^{p-1}}{k^p} > \frac{1 - c_k^{p-1}}{k^p} > 0, \quad \forall k \geq 4.
\]

Next, we will prove that \( \lim_{k \to \infty} \delta_k = 0 \). We have

\[
    \delta_k = \frac{k^p}{(k+1)^p(k^p - (k+1)^p + 1)} - 1 = \frac{k^p}{(k+1)^p} \cdot \frac{1}{k^p - (k+1)^p + 1} - 1.
\]

Since \( \lim_{k \to \infty} \frac{k^p}{(k+1)^p} = 1 \), we only need to prove

\[
    \lim_{k \to \infty} \frac{1}{k^p - (k+1)^p + 1} = 1, \quad \text{or} \quad \lim_{k \to \infty} (k^p - (k+1)^p) = 0.
\]
By Lagrange theorem, for all $k$, there exists $c_k \in (k, k+1)$ such that
\[ k^p - (k+1)^p = -pc_k^{p-1}. \]

When $k$ tends to $\infty$, $c_k$ also tends to $\infty$. Since $p < 1$, it follows
\[ |k^p - (k+1)^p| = pc_k^{p-1} \to 0 \quad \text{as} \quad k \to \infty. \]

Therefore, $\lim_{k \to \infty} \delta_k = 0$.

We continue to prove that
\[ \eta_k = 1 - \frac{(k+1)^p - 1}{k^p} > \frac{1}{(k+1)^2p}, \quad \forall k \geq 4. \]

This inequality is equivalent to
\[ \frac{(k+1)^{2p} - 1}{(k+1)^{2p}} > \frac{(k+1)^p - 1}{k^p}, \]

or
\[ \frac{(k+1)^p + 1}{(k+1)^{2p}} > \frac{1}{k^p}. \]

We rewrite the above inequality as
\[ k^{-p} - (k+1)^{-p} < \frac{1}{(k+1)^{2p}}. \]

By Lagrange theorem, there exists $c_k \in (k, k+1)$ such that
\[ k^{-p} - (k+1)^{-p} = \frac{p}{c_k^{p+1}}. \]

Since $p + 1 > 2p$ and $c_k > k$, for all $k \geq 4$ we have
\[
\frac{p}{c_k^{p+1}} < \frac{1}{k^{p+1}} = \left(\frac{k+1}{k}\right)^{2p} \cdot \frac{1}{k^{1-p}} \cdot \frac{1}{(k+1)^{2p}} < 2^{2p} \cdot \frac{1}{4^{1-p}} \cdot \frac{1}{(k+1)^{2p}} = \frac{1}{2^{2-4p}} \cdot \frac{1}{(k+1)^{2p}} < \frac{1}{(k+1)^{2p}}.
\]

This leads to our desired inequality. Since $p < \frac{1}{2}$, it follows that $\sum_{k=4}^{\infty} \eta_k = \infty$.

By Lemma 4.4, we have $\lim_{k \to \infty} v_k = 0$. Thus $\lim_{k \to \infty} u_k = 1$ which means $\{u_k\}$ is bounded above by some $C > 0$.

Finally, following (11), for all $k \geq 2$, we have
\[
a_{k+1} \leq \left(1 - \frac{1}{k^p}\right) \left(1 - \frac{1}{(k-1)^p}\right) \cdots \left(1 - \frac{1}{2^p}\right) a_2 + \frac{M^2}{k^p} u_k
\leq \left(1 - \frac{1}{k}\right) \left(1 - \frac{1}{k-1}\right) \cdots \left(1 - \frac{1}{2}\right) a_2 + \frac{M^2 C}{k^p}
= \frac{a_2}{k} + \frac{M^2 C}{k^p}
\leq (a_2 + M^2 C) \frac{1}{k^p}
< 2(a_2 + M^2 C) \frac{1}{(k+1)^{p}}.
\]
Therefore \( \|x^* - x^k\| \) converges to 0 with rate \( O\left(k^{-\frac{p}{2}}\right) \). 

\[ \square \]

6 Concluding remarks

In this article, we obtained an error bound and proved the convergence of iterative sequences generated by modified GPM for VIs governed by strongly pseudomonotone operators. Two counter-examples were given to show the necessity of Lipschitz continuity assumption in classical GPM as well as the boundedness hypothesis in modified GPM. We proposed a method to overcome the difficulty when applying modified GPM for VIs with unbounded constraint sets. Rate of convergence was also estimated when the stepsizes are sequences of terms defining the \( p \)-series.

There are still some open questions for whom who may concern:

1. The extragradient projection method (EPM) (see [10]) is another classical method solving a wider class of VIs than the GPM, i.e., VIs with monotone and Lipschitz continuous operators. In [6], Khanh proved that modified EPM with variable stepsizes is applicable for strongly pseudomonotone and Lipschitz continuous VIs. It is natural to ask whether modified EPM could solve VIs governed by pseudomonotone operators.

2. It is also worth to consider the choice of \( p \) to optimize the speed of convergence of iterative sequences produced by Algorithm 4.3 when \( \lambda_k = \frac{1}{k^p} \) for all \( k \geq 1 \). Obviously, the optimized value of \( p \) is not the same for all cases but depends on the constraint set \( K \) and operator \( F \).

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