On Matrix models of M5 branes

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We compare the (0,2) theory of the single M5 brane decoupled from gravity in the lightcone with transverse $R^4$, and a matrix model description in terms of quantum mechanics on instanton moduli space. We give some tests of the Matrix model in the case of multi fivebranes on $R^4$. We extract constraints on the operator content of the field theory of the multi-fivebrane system by analyzing the Matrix model. We also begin a study of compactifications of the (0,2) theory in this framework, arguing that for large compactification scale the (0,2) theory is described by super-quantum mechanics on appropriate instanton moduli spaces.
1. Introduction.

By taking $k$ 5-branes of M theory in 11 dimensions approaching each other with the eleven dimensional Planck length, $l_p$ going to zero, one finds an interacting theory decoupled from gravity $[1][2]$. This theory has a moduli space of vacua $(R^5)^k/S_k$, and it has $(0,2)$ supersymmetry $[3]$. How to incorporate 5-branes in Matrix Theory $[4]$ was approached in $[5]$ by modifying the quantum mechanics of zero branes. Identification of the 5-brane charge and a relation to instantons was established in $[6][7]$ and developed in the context of Matrix black holes in $[8]$. A concrete proposal for describing the six dimensional theory of 5-branes by quantum mechanics on moduli space of instantons on $R^4$ was made in $[9]$, and extended to IIA 5-branes in $[10]$. We will mostly restrict the discussion to M5 branes. The $0+1$ dimensional quantum mechanics of $[5]$ has $U(N)$ gauge symmetry, and $k$ hypermultiplets in the fundamental of $U(N)$. Six dimensional Lorentz invariance is expected at large $N$. Following $[11]$ one expects that the finite $N$ theory describes the $(0,2)$ theory compactified with radius $R$ on a lightlike direction $X^-$, with momentum $P_-=N/R$.

As was discussed in $[12][13]$, M-theory compactified on a large light-like circle $R$ is related to $\tilde{M}$-theory compactified on a small space-like circle $R_{11}$ by a large boost in the eleventh direction with $\gamma = R/R_{11}$. The quantum mechanics of $N$ heavy D0 branes in IIA string theory in the presence of $k$ 4-branes is then mapped to the theory of 5-branes in M-theory compactified along a lightlike direction. Following $[12][13]$, we demand that $R/R_{11} = \frac{R_{11}}{l_p}$. The coupling constant of the quantum mechanics $g_{qm}^2 = \frac{R_{11}^3}{l_p^3}$. In the limit $l_p \to 0$, $R_{11} \to 0$ keeping $R_{11}^3/l_p^3$ fixed we have the quantum mechanics with hypermultiplets of $[8]$. Distances in the transverse dimensions also must be rescaled $\frac{R_i}{l_p} = \frac{\tilde{R}_i}{l_p}$. Then we take the limit $g_{qm} \to \infty$ (equivalently we consider energies that are small compared to the energy $g_{qm}^{2/3}$) to decouple the Higgs branch from the Coulomb branch. For $k \neq 1$, quantum mechanics on the Higgs branch contains the physics of the interacting $(0,2)$ conformal theory $[4]$.

The $(0,2)$ theory of $k$ 5-branes of M theory has a moduli space of vacua $(R^5)^k/S_k$. At the origin of the moduli space, the theory is superconformal and has $U(k)$ gauge symmetry. The quantum mechanics on instanton moduli space describes the theory in the neighbourhood of the superconformal fixed point.
1.1. Some key points.

The emphasis in [9] was on the interacting theories, but similar arguments can be used for the single 5-brane. This will be our starting point. To set this up we show how to define the decoupled theory of a single 5-brane by starting from the action developed in [14] [15] [16].

We will make a careful identification of states obtained from the quantum mechanics with the states of a tensor multiplet, using the Spin(5) R-symmetry. This allows us to see that we have a multiplet of the (0, 2) theory in six dimensions as opposes to say (1, 1).

We present a way to deal with super-quantum mechanics on symmetric products inspired by orbifold cohomologies. Essentially we extend beyond zero-energy states the prescription which works for zero energy. In the course of this discussion we will describe, for \( k = 1 \), the construction of non-zero energy states in the quantum mechanics, which, like the ground state, are still annihilated by 8 supersymmetries (some non-linearly realized in the quantum mechanics).

We will describe, for general \( k \) how the structure of the quantum mechanics on Higgs branch allows us to see the decomposition of states into those associated with the \( U(1) \) part and those associated with the \( SU(k) \) part of the spacetime theory. This decomposition arises from the fact that the Higgs branch naturally separates into strata. These strata are easy to understand from the equations describing the supersymmetric vacua of the 0-brane gauge theory. From the point of view of the 4-brane theory they are related to subtle point-like instantons. Our treatment of super-quantum mechanics on this stratified space uses again the fact that superquantum mechanics is related to cohomologies.

We will study how group actions on the instanton moduli space, again for general \( k \), can be used to get information on the symmetries and operators of the (0, 2) theory. In particular the action of \( SU(k)/Z_k \) on the space of based instantons leads to the statement that the local operators at the conformal fixed point organize themselves into representations of the gauge group which have zero \( Z_k \) charge.

We give a short discussion of toroidal compactifications. When the torus is large compared to all the scales in the correlation function of interest, then we may expect techniques valid for the (0, 2) field theory in \( R^4 \times R^+ \times R^- \) to continue to be valid. We may thus expect that moduli spaces of instantons on \( R^{4-d} \times T^d \times R^+ \times R^- \) to be relevant. We will present some evidence in favour of this conjecture, starting with a new perspective on the derivation on the Matrix model of the (0, 2) theory. Then we discuss some aspects of the opposite limit of small compactification scale.
While work on this paper was being completed, a number of comments on related issues appeared in [17]. We have also learnt that Matrix models for compactified (0, 2) theory have been considered in [18]. The (1, 0) Matrix models have been discussed in [19] [20], and Matrix models for IIB five-branes have been given in [21] [22] [23] [24].

2. On a single 5-brane of M theory

There is an action given for the single fivebrane of M theory by [14] [16] [15], and the gauge fixing required to obtain a 6 dimensional (0, 2) field theory is done in [25]. The theory of a single decoupled 5-brane can be defined to be this theory taken to the limit where \( l_p \to 0 \).

The bosonic terms of the action looks schematically like:

\[
S = \frac{1}{l_p^6} \int d^6x (1 + H^2 + H^4 + \cdots)
\]

Here \( H \) is dimensionless. The factor \( l_p^{-6} \) is needed in front of the action because that is the tension of the 5-brane of M theory. We get rid of the \( l_p \) in front by redefining the \( H \). \( H' = \frac{H}{l_p} \). It is easy to see that now the higher terms are suppressed as \( l_p \) goes to zero \(^1\). The same argument applies for the fermions, with the scaling \( \theta \to l_p^3 \theta \), showing that the only surviving terms in the limit as \( l_p \to 0 \) are the fermion kinetic terms. Terms of the form

\[
(\theta \Gamma \partial \theta)H + (\theta \Gamma \partial \theta)^2\ldots
\]

are suppressed in the \( l_p \to 0 \) limit. This kind of rescaling is subtle if we were to consider a theory with \( U(1)^n \) gauge symmetry arising at a point of broken symmetry in a non-abelian theory \(^2\), because then a rescaling puts the coupling in the gauge transformations. But since we are here dealing with a \( U(1) \) theory only there is no problem.

We will relate this free tensor multiplet theory to quantum mechanics on the moduli space of \( U(1) \) instantons. We will show that the correct quantum mechanics on instanton moduli space which follows from \(^3\) is actually a free quantum mechanics, and we outline how to recover the spacetime theory of a free tensor multiplet. Essentially the quantum mechanics gives a worldline formulation of the tensor multiplet theory in the light-cone gauge.

\(^1\) After we had developed this argument, a very similar argument for a decoupled single M5-brane being free was given by N. Seiberg \(^2\).
3. Remarks on the super-algebra.

The action of [3] has 32 supersymmetries and a global \( \text{Spin}(5) \times \text{SO}(4) \) symmetry. The \( \text{SO}(4) \) is associated with the 4 directions transverse to the light-like directions and the 5 directions transverse to the 5-brane. When the parameters \( x_0 \) and \( \theta_0 \) associated with motion on the Coulomb branch are set to zero, as appropriate for the description of the internal dynamics of the 5-branes, there are 16 supersymmetries left. They are \( Q_\alpha^I \) and \( \tilde{Q}^{\dot{\alpha}}_I \). The \((\alpha, \dot{\alpha})\) are indices in the fundamental of the left and right \( \text{SU}(2) \) in the decomposition \( \text{SO}(4) \equiv \text{SU}(2) \times \text{SU}(2) \). The index \( I \) belongs to the spinor of \( \text{Spin}(5) \). The \( Q \) are linearly realised and the \( \tilde{Q} \) are non-linearly realised.

The above structure of the supersymmetries can be understood by noting that they are SUSY surviving the presence of the M5-brane (4-brane in IIA), and that they split into two \( \text{SO}(4) \) chiralities depending on whether they are preserved by the zero brane or not. This follows from inspection of the equations

\[
Q_L = -\Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^4 \tilde{Q}_R
Q_L = \Gamma^0 \tilde{Q}_R
\]  

which give the supersymmetries preserved by the 4-brane and zero-brane respectively. Consider \( C_4 = \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^4 \) acting on \( Q \). Combining the first equation in (3.1) with the condition for supersymmetry broken by the zero branes we get \( C_4 Q = -Q \). Combining it with the condition for supersymmetry unbroken by the zero branes we get \( C_4 Q = Q \). The presence of two chiralities under \( \text{SO}(4) \) is consistent with the theory describing the \((0,2)\) in the lightcone because the \( \text{SO}(6) \) spinors decompose into \( \text{SO}(4) \) spinors of both chiralities.

Starting from the \((0,2)\) theory we may write the superalgebra in a \( \text{Spin}(5) \times \text{SU}(2) \times \text{SU}(2) \) covariant form

\[
\{ Q_\alpha^I, Q_\beta^J \} = P_+ J_{IJ} \epsilon^{\alpha \beta}
\{ Q_\alpha^I, \tilde{Q}_{\dot{\beta}}^J \} = (\Gamma^A)^{\alpha \dot{\beta}} P_A J_{IJ}
\{ \tilde{Q}^{\dot{\alpha}}_I, \tilde{Q}^{\dot{\beta}}_J \} = P_- J_{IJ} \epsilon^{\dot{\alpha} \dot{\beta}}
\]  

(3.2)

\( J \) and \( \epsilon \) are the antisymmetric invariants of the appropriate group. The \( A \) index is an index in the fundamental of \( \text{SO}(4) \). A similar set equations is used in [27]. We have set central charges corresponding to extended objects to zero, since we are mainly interested in the simplest background of the \((0,2)\) SCFT in this paper. We can convert the algebra to a form where it is written as a set of independent creation and annihilation operators.
necessary to describe the tensor multiplet, in at least two ways. One keeps the symmetry Spin(5) × U(1) × U(1). Another keeps manifest the symmetry SU(2) × U(1) × SU(2) × SU(2), where the Spin(5) has been broken to SU(2) × U(1). Either of these forms can be recovered from the Matrix model. These will be used in subsequent sections to describe the tensor multiplet. The superalgebra for the Higgs branch Matrix model can be obtained by starting from the calculations of [7], adding the contributions of the fundamental hypermultiplets, and specializing to the Higgs branch. The last step involves setting to zero all the scalars X which transform under the Spin(5).

By inspection of the commutator of Q with ˜Q, we can identify the operators P_I which generate translations in R^4. The calculation is done in [7] in the absence of fundamental hypermultiplets, and it is easy to see that the presence of these hypermultiplets does not modify the expression for P_I [8]:

\[ P_A = tr(\partial_t X_A). \] (3.3)

This stems from the relation

\[ \tilde{Q} = tr\lambda. \] (3.4)

which involves the fermions in the adjoint hypermultiplet but not those in the vector. Suppose we are given a state in the quantum mechanics, which corresponds to a state of the (0, 2) theory \( \mathcal{O}(0)|0 > \), at a point \( x^I = 0 \) in spacetime. We can find the state corresponding to the operator at a generic point by acting with \( e^{ix^I P_I} \).

\[ \mathcal{O}(x^I)|0 > = e^{ix^I P_I} \mathcal{O}(0)|0 > \] (3.5)

On operators, we have \( e^{ix^I P_I} \mathcal{O}(t) e^{-ix^I P_I} \). Thus we can construct the transverse spatial dependence of states and operators. Similarly the Hamiltonian of the quantum mechanics is \( P_+ \) of the spacetime theory. This operator does receive extra contributions from the hypers of the form:

\[ P_+ = |DH|^2 + |D\tilde{H}|^2 + \dot{\chi}\chi. \] (3.6)

In the special case \( k = 1 \) which occupies the early sections of this paper, \( H = \tilde{H} = 0 \), and these corrections to \( P_+ \) vanish.

Note that \( P_I \) has no dependence on the \( H \) and \( \tilde{H} \) fields. The space \( M_{k,N} \) can be described by holomorphic gauge invariant coordinates, as discussed for example in [28].

\[ ^2 \text{This form of the translation operator for the Matrix model of the (0, 2) theory has also been considered by O. Ganor.} \]
Coordinates made purely from the scalars of the vector hypermultiplet, of schematic form \( H\tilde{H} \) are inert under spacetime translation. One combines the adjoint hypers into complex matrices:

\[
U = X_1 + iX_2 \\
V = X_3 + iX_4
\]  

(3.7)

Gauge invariant coordinates like \( trU^2, trUHV\tilde{H} \) transform non-trivially. So wavefuntions which are independent of \( U \) and \( V \) are inert under spacetime translation.

States in the quantum mechanics for \( N \) instantons and with energy \( E \) correspond to states in the (0, 2) SCFT with \( P_+ = E \) and \( P_- = N/R \):

\[
\mathcal{O}_{N,E}(0)|0>
\]  

(3.8)

By summing over instanton numbers we reconstruct states localized in \( x^- \) dependence, and by summing over \( P^+ \) we recover the \( x^+ \) dependence of correlators.

Many of the central charges that appear in [7] involve traces of commutators of matrices, so they vanish at finite \( N \). However at finite \( N \) one can build matrices which approach the correct matrices at large \( N \). For example in the construction of membranes we have \( X_1 = P \) and \( X_2 = Q \), where \( Q \) is diagonal and \( P \) is a cyclic permutation. The construction of such matrices when we are doing quantum mechanics on the Higgs branch is restricted by the requirement that the constraints defining the vacuum are satisfied. So for \( k = 1 \), the \( X \) are diagonal and \( H = \tilde{H} = 0 \) [10]. This is an example of the general fact that while the central charges can be simply written down as in [7], new features can be expected from the specialization to the Higgs branch.

4. Instantons on \( R^4 \) and free tensor multiplet in 6 dimensions.

The quantum mechanics with gauge group \( U(N) \) and \( k \) hypermultiplets with hypermultiplets given in [3] has a Higgs branch of vacua where the adjoint or the fundamental hypermultiplets acquire vevs. We will call this space \( \bar{M}_{k,N} \). To derive the action for superquantum mechanics on this space, we give time dependence to the coordinates on the moduli space, and plug into the action for the Higgs branch variables coming from [3]. This standard procedure for setting up the collective coordinate quantum mechanics is used in this context in [10]. The bosonic terms take the form:

\[
\int G_{ij}(Z)\partial_t Z^i \partial_t Z^j
\]  

(4.1)
There is a 1-1 correspondence between the Higgs branch of the $U(N)$ gauge theory with $k$ flavors and the moduli space of $N$ instantons in $U(k)$ gauge theory. Further, the metric in (4.1) is the same as the one on instanton moduli space

$$G_{ij}(Z) = \int d^4x \sqrt{g} \delta_i^A A^\mu(Z) \delta_j^A A^\mu(Z)$$

by a theorem of [29].

4.1. Quantum Mechanics on $S^N(R^4)$ and free 6D theory.

In the simplest case $N = k = 1$ the moduli space is just $R^4$ and the action for the moduli space quantum mechanics is just

$$S = \int (\partial X^I)^2 + \text{fermions}$$

The $I$ index runs from 1 to 4, and the fermions will be discussed in detail in later sections. This a free quantum mechanics. All the interaction terms vanish on the moduli space because they either involve commutators, or the hypermultiplets which are zero for $k = 1$ [10].

For $k = 1$ with general $N$, the moduli space $M_{1,N}$ is the symmetric product

$$S^N(R^4) = (R^4 \times R^4 \cdots R^4)/S_N$$

This is the space of $N$ indistinguishable points on $R^4$. It can be decomposed into a union of successively lower dimensional spaces as follows:

$$S^N(X) = \Pi_\nu(S^N(X))_\nu$$

where if $\nu$ is the partition $(1)^{n_1}(2)^{n_2} \cdots (s)^{n_s}$ then

$$(S^N(X))_\nu = \prod_i [C_{X,n_i}]_{S_{n_i}}.$$  

$C_{X,n_i}/S_{n_i}$ is the space of $n_i$ unlabeled separate points on $X$. In this case $X = R^4$.

This space has orbifold singularities. String theory on such spaces is generally believed to be well defined, and has recently been discussed for very similar orbifolds in the context of Matrix String theory [30] [31] [32]. There the key point is that string theory on $X^N/S_N$ has a Hilbert space which is that of $N$ strings. We will argue that a very similar prescription works for superquantum mechanics. A superparticle on the symmetric product of $X$, when the Lagrangian for motion on $X$ is free, will have states corresponding to the N-particle Hilbert space.
4.2. Spectrum

The spectrum is trivial to solve in the case \( k = N = 1 \). The eigenstates of the bosonic Hamiltonian are parametrized by a 4-vector \( k^I \). Because of supersymmetry we actually have a supermultiplet for each \( k^I \). For general \( N \), we need to obtain the spectrum for quantum mechanics on \( SN(R^4) \), given the spectrum for quantum mechanics on \( R^4 \). It is known how to relate the cohomology of \( SN(X) \) to that of \( X \). Since ground states of super-quantum mechanics (SQM) are typically related to cohomology, we have an obvious prescription for obtaining the zero energy states for QM on \( SN(X) \). A simple guess, which we will argue is correct, is that the same prescription works for arbitrary states of the superquantum mechanics.

The prescription associates oscillators \( \alpha_{-l}(h) \) to each cohomology class \( h \), with \( l > 0 \). The cohomology of the symmetric product is then parametrized by conjugacy classes of \( S_N \). For a class associated with cycles of length \( l_1, l_2, \ldots, l_s \), with \( l_1 + l_2 + \cdots + l_s = N \), we have states:

\[
a_{i_1}^\dagger(h_1)a_{i_2}^\dagger(h_2)\cdots a_{i_s}^\dagger(h_s)|0> \quad (4.6)
\]

In our case we will allow the state \( h \) to be an arbitrary state of the quantum mechanics. So \( h \) will be a supermultiplet labelled by a 4-vector \( k^I \). As we will see the quantum mechanics contains non-zero momentum states which are also BPS. So applying the prescription to the case where all the \( h \) have the same transverse momentum is as well motivated as applying it to the ground state. We will make the assumption that the obvious generalization of relaxing this constraint is correct, partly in analogy to the case of strings moving on these symmetric products [30][31][32].

4.3. Oscillator number and light-like momentum

Our interpretation of these states is that they come from the free tensor multiplet in the six dimensional theory. The subscript \( l \) of the oscillator can be interpreted as the momentum in the lightlike direction. To see this, consider for example, a state of the form

\[
a_{i_1}^\dagger(\vec{k}_1)a_{i_2}^\dagger(\vec{k}_2)|0> \quad . \quad (4.7)
\]

These are associated with subvarieties of \( SN(R^4) \) where the \( N \) points form two clumps of sizes \( l_1 \) and \( l_2 \), which add up to \( N \). By specializing the Higgs branch QM lagrangian to these configurations we can calculate the energy to be

\[
R\left\{\frac{\vec{k}_1^2}{l_1} + \frac{\vec{k}_2^2}{l_2}\right\} 
\]
as appropriate for a two particle massless state in 6 dimensions with transverse momenta \(k_1, k_2\) and longitudinal momenta \(l_1, l_2\). More generally the states with a fixed amount of lightcone momentum in a free field theory are just given by the appropriately symmetrized multiparticle states with the total lightcone momentum partitioned between the several oscillators. But this is exactly how \(N\) enters the Hilbert space of our Higgs branch quantum mechanics.

4.4. *Continuum of Normalizable States.*

So far we have been discussing wavefunctions for quantum mechanics on the Higgs branch of \(U(N)\) with 1 flavour. This is appropriate for the infinite coupling limit, where Higgs and Coulomb branch decouple. At finite coupling one has to do a quantum mechanics involving the Coulomb branch. The wavefunctions for the (04) Higgs branch states will spread over the Coulomb branch. Those that are BPS (parametrized by a 4-vector \(k_A\) and a partition of \(N\) ) will survive at generic coupling. So there should in fact be a *continuum of normalizable BPS states* on the Coulomb branch starting from the zero energy state studied in [34][35]. Testing this by weak coupling methods would provide a partial test of the prescription we are using for doing SQM on these symmetric products. We expect that the same thing should be true for the case of \(U(N)\) gauge theory for \(k\) flavours, and the density of states should be larger than the case of \(k = 1\). It should receive contributions from the \(SU(k)\) degrees of freedom of \((0, 2)\) theory as well as the \(U(1)\) part. More discussion on this decomposition is given in section 4.9.

4.5. *Symmetries and the tensor multiplet.*

Here we review how the symmetries of the quantum mechanics on the Higgs branch are related to those of the \((0, 2)\) theory in the lightcone. In particular we see how we distinguish it from a \((1, 1)\) theory in a lightcone gauge. This allows us to unambiguously identify the free particle described by the Higgs branch superquantum mechanics as a tensor multiplet of \((0, 2)\) as opposed to say a vector multiplet of \((1, 1)\).

The chiralities of the fermions surviving the introduction of 4-brane are not the same. This is compatible with the fact that we are describing a \((0, 2)\) theory in a lightcone gauge because a 4 dimensional \(SO(6)\) spinor of definite chirality decomposes under \(SO(4)\) into two spinors of both chiralities. The chiralities of the spinors one gets by considering a \((1, 1)\) or a \((0, 2)\) theory in a lightcone gauge are the same. But the \((0, 2)\) theory has a \(Spin(5)\) \(R\) symmetry which the \((1, 1)\) theory does not have. Since we are doing M5 branes as opposed
to IIA 5-branes, this symmetry remains manifest. If we describe the supermultiplets in a way that keeps this symmetry manifest, we can distinguish the tensor multiplet of the (0, 2) theory from the vector multiplet of (1, 1) theory.

The action of the (04) system as described in [5] has 32 supersymmetries of which 8 are linearly realised. When we set to zero the parameters $x_0$ and $\theta_0$, there are only 16 supersymmetries, of which 8 are linearly realised. This is what we expect for the supersymmetries of a (0, 2) theory in the lightcone gauge. The zero momentum state preserves the all the linearly realised SUSY and breaks the non-linear ones. The non-zero momentum states preserve some combination of linear and non-linear SUSY as discussed later.

We now describe in some more detail the construction of the supermultiplet of zero energy states. The broken SUSY generate the states of the supermultiplet. It is convenient to perform an $Spin(5)$ invariant quantization of the zero modes: We have fermionic oscillators obeying:

$$\{a_I, (\hat{a}_J)\} = J_{IJ} \quad (4.8)$$

$J_{IJ}$ is the spin(5) invariant tensor. They are built from the generators of the kinematic supercharges of lightcone (0, 2) theory:

$$\{\tilde{Q}^\dot{\alpha}_I, \tilde{Q}^\dot{\beta}_J\} = \epsilon^{\dot{\alpha}\dot{\beta}} J_{IJ} \quad (4.9)$$

as follows:

$$a_I = \tilde{Q}^1_I + \tilde{Q}^2_I$$

$$\hat{a}_I = \tilde{Q}^2_I - \tilde{Q}^1_I \quad (4.10)$$

In some basis $J$ has components:

$$J_{12} = -J_{21} = J_{34} = -J_{43} = 1.$$ 

A subtlety is that we do not directly see the $SO(5) \times SU(2)$ covariant form of the algebra of broken supercharges (4.9) in the Higgs branch quantum mechanics. But we do see the $SO(5) \times U(1)$ covariant form (4.8). This is related to the fact that in the pure zero brane system, one has because of the Majorana nature of the fermions, to perform an $SO(7) \times U(1)$ covariant quantization as opposed to a full $SO(9)$ covariant quantization [36] [37].
We can define operators transforming in the adjoint of the $Spin(5)$, since the symmetric tensor product of the spinor of $SO(5)$ is the adjoint, by

$$O_{IJ} = 1/2(\hat{a}_Ia_J + \hat{a}_Ja_I) \quad (4.11)$$

A representation of the superalgebra is constructed by acting with the creation operators on the state $|0>$. Clearly $|0>$ is annihilated by $Spin(5)$. The state build by four creation operators $\hat{a}$ acting on the vacuum is invariant. $J_{IJ}(\hat{a}_I)(\hat{a}_J)|0>$ is also invariant. The operators quadratic in the oscillators decompose into the vector and the trivial rep of $SO(5)$ because the antisymmetric tensor product of the two spinors decomposes as $5 + 1$. We have, therefore, one vector of $Spin(5)$ and three singlets. This corresponds to the 5 scalars and the three polarizations of the antiself-dual tensor. Without analyzing the $Spin(5)$ content of the states we cannot tell if our theory has something to do with the $(0, 2)$ in six dimensions or the $(1, 1)$ in six dimensions, because each $SO(6)$ spinor decomposes into spinors of $SO(4)$ of both chiralities. However we are able to distinguish a tensor multiplet in the lightcone frame from a vector multiplet in the lightcone frame, as long as the momentum of particle is chosen to be entirely in the plane chosen to be parametrized by the light-like coordinates in going to lightcone gauge. The $5 + 3$ split tells us that we have a $(0, 2)$ algebra tensor multiplet in 6D as opposed to a $(1, 1)$ algebra.

We digress to remark on the case of IIB 5-branes [21][23]. There the proposed Matrix model comes from zero branes moving on ALE space. The Coulomb branch SQM describes the 5-brane. In that case the SQM has an $SO(4)$ which is identified with R symmetry in the 5-brane worldvolume. And the analogous discussion above will give $4+4$ split of the bosons, as appropriate for a vector multiplet in six dimensions, where the gauge field has a 4 components and there are 4 scalars.

It has been convenient in this discussion to perform a quantization with $Spin(5)$ covariance here. We could also quantize by keeping the $SO(4)$ parallel to the 4-brane manifest. Then we have to break the $Spin(5)$ to $SU(2) \times U(1)$. But the relation between the zero momentum and the non-zero momentum states is then clearer.

4.6. States carrying momentum in the quantum mechanics

When we turn on some momentum we expect that the SUSY left unbroken by the state is some linear combination of $Q_\alpha^\rho$ and $\tilde{Q}_\rho^\alpha$, which were interpreted as the SUSY of the worldvolume theory of the 5-brane. In fact we can see, in the quantum mechanics
on Higgs branch, that there are non-static configurations corresponding to eigenstates of momentum which are SUSY. In the simplest case we just have $U(1)$ with instanton number 1 and we have QM on $R^4$. The configuration

$$X_\mu = v_\mu t,$$

(4.12)

The variation of the gluino vanishes for an appropriate combination of linear and non-linear SUSY:

$$\delta \lambda = \Gamma^\mu v_\mu \epsilon + \bar{\epsilon}$$

(4.13)

We can also see this using the lightcone superalgebra acting via a more quantum mechanical construction. We consider the operator

$$e^{ik_A X^A} \Psi_k,$$

(4.14)

where $\Psi$ is the fermionic part. For $k = 0$ the fermionic vacuum is built from the broken SUSY $\tilde{Q}_I^i$, and is annihilated by all the linear SUSY $Q_I^0$. This is consistent with the relation from the superalgebra

$$\{Q, Q\} = P_+ = 0.$$

$$\{Q, \tilde{Q}\} = 0$$

(4.15)

For non-zero $k_A$ we insert into the lightcone superalgebra of the (0, 2) theory,

$$P^+ = Rk^2/N$$

$$P_A = k^A$$

$$P_- = 1$$

(4.16)

(we have set $R = 1$) to find that $k_\mu \Gamma^\mu Q + \tilde{Q}$ can be consistently set to zero. The combination with the opposite sign can be separated into creation and annihilation operators to build a representation of the superalgebra. Requiring a correct representation of the superalgebra with $k \neq 0$ fixes the fermionic part to be of the form given below:

$$e^{ik_A(X^A + b_\alpha (\Gamma^A)_{\alpha\beta} b^{\dagger \beta})} |0 >$$

(4.17)

Here $b$ are linear combinations of the fermions $\lambda^i_I$ which appear in the SUSY quantum mechanics action. The lower case $i$ index is in the fundamental of an $SU(2)$ subgroup of the Spin(5). The state $|0 >$ is an $SO(4)$ invariant vacuum, annihilated by $b$. It is thus possible to maintain the $SO(4)$ covariance at the quantum mechanical level, when we give up the Spin(5) covariance.
4.7. Propagator

The generic state in this SQM on \((R^4)^N/S_N\) in (4.6) is a multi-particle state. The \(\vec{k}_i\) are momenta which characterize the states of the QM in \(R^4\). To make the identification with one-particle states of the tensor multiplet, we identify

\[ \partial_- B(n, \vec{k})|0 > \rightarrow a^+_n(\vec{k})|0 > \] (4.18)

By simply taking inner products in the quantum mechanics we can get the two point function:

\[ \langle \partial_- B(x^-, x^i)\partial_- B(y^-, y^i) \rangle = \delta'(x^- - y^-)\delta(x^i - y^i) \] (4.19)

The field \(B\) can be thought as one of the components of the self-dual tensor.

Recall that the actual gauge fixing to lightcone gauge may be subtle in the action given by [15] [16] since, for the choice \(a = x^-\), we have a singular action. Here \(a\) is the auxiliary field which enters the Lorentz covariant action of The matrix model gives a gauge fixed, worldline formulation of the free tensor multiplet theory in the light-cone.

4.8. Correlators

The inner product if the field theory can be mapped in our \(k = 1\) example to the natural inner product in the quantum mechanics

\[ \int \sqrt{G(Z)}\Psi^*(Z)\Psi(Z) \] (4.20)

The obvious generalization is to integrate products of several wavefunctions which should correspond to correlators in the field theory.

4.9. Subvarieties of the Higgs Branch.

There is a decomposition of the Higgs branch \(\bar{M}_{k,N}\):

\[ \bar{M}_{k,N} = M_{k,N} \prod M_{k,N-1} \times R^4 \prod \cdots M_{k,1} \times S^{N-1} R^4 \prod S^N(R^4) \] (4.21)

\(\bar{M}_{k,N}\) is a connected space. This is Higgs mechanism in the zero-brane worldvolume theory, but is not Higgs mechanism in the spacetime \((0, 2)\) theory. (The latter would correspond to giving masses to the fundamental hypermultiplets). The component containing \(M_{k,N-l}\) corresponds to the subspace where we have set \(l\) components of \(H\) and \(\tilde{H}\) to zero. This decomposition is also studied in the context of instantons [38]. The component with the
l’th power of $R^4$ corresponds to the subspace where $l$ instantons are pointlike and $N - l$ are fat.

The fact that for a given $l$, the symmetric product part has no dependence on $k$ contains an important property of pointlike instantons: they do not carry gauge indices, and cannot be described by a potential or field strength. This will turn out to be just right for the physical interpretation of this decomposition.

We will argue that the wavefunction for $\bar{M}_{k,N}$ splits into a sum over each component. The last factor would describe the decoupled centre of mass motion described by a free tensor multiplet. More generally the decomposition can be interpreted by saying that of the $N$ units of momentum, $l$ are carried by the interacting part ($SU(k)$) of the theory, and $(k - l)$ are carried by the free part ($U(1)$) of the theory.

$$H(k, N) = \bigoplus_l |s_1(l) > \otimes |s_2(N - l) >$$

(4.22)

where $|s_1 >$ is a state in the $SU(k)$ part of the theory, and $|s_2 >$ is a state in the $U(1)$ part of the theory.

The fact that this is the correct way to deal with the disjoint union can be motivated as follows. Since we are dealing with supersymmetric quantum mechanics, we have a close relation to the cohomology of the space. At the level of cohomology, the correct prescription for the disjoint union is certainly to take the direct sum of states from each component. This follows from the Mayer-Vietoris sequence for the appropriate cohomology. We have seen that there are also non-zero energy states which can be related to the zero energy states by transformations which have the interpretation of boosts parallel to the 4-brane worldvolume, and which also preserve 8 supersymmetries. So certainly for this class of states taking the direct sum of the states obtained from each component is the correct prescription. It is plausible, then, that the same prescription applies to the entire super-quantum mechanics.

The very important property that the pointlike instantons do not have a connection and field strength is crucial here. We can decompose the $trF^2$ in terms of contributions from pointlike instantons and fat instantons [29] but not the connection or field strength. This is related to the fact that we need generalize the concept of bundle to sheaves when dealing with point-like instantons, which has been emphasized in a related context in [39]. If it were possible to associate some internal ‘gauge indices’ to the point-like instantons, we would expect some dependence of the spaces of point-like instantons on $k$. So we would
have, say $(R^4)^N/S_{Nk}$, which could not be interpreted in terms of a free tensor multiplet theory describing the centre of mass at momentum $N$.

In the above we have not been too precise about exactly what kind of cohomology corresponds to the ground states of the quantum mechanics. Clarifying this will be very interesting, for example the states coming from the open subset $M_{k,N}$ will be constrained by the uniqueness of the vacuum of the $SU(k)$ theory.

5. Group Actions on Moduli spaces.

In doing quantum mechanics on a space parametrized by $Z$, if there is an action of a group $Z \to gZ$, then there is also an action on wavefunctions

$$\Psi(Z) \to \Psi(gZ)$$

If $g$ acts trivially, then the wavefunctions transform trivially.

We will see in this section how some conformal symmetries of a 6D conformal theory lead to manifest symmetries of its lightcone formulation and see how these symmetries are realized in the proposed SQM on instanton moduli space.

Then we will see how the action of the global gauge transformations on instanton moduli spaces lead to some plausible constraints on the local operator content of the $(0, 2)$ theory.

5.1. Spacetime symmetries.

For a six dimensional theory in the lightcone we expect a manifest $SO(4)$ group of rotation symmetries, as well as a group of translations. The $SO(4)$ appears as an $R$ symmetry of the quantum mechanics. The generators of the translation group were constructed in section 3.

Now we consider the conformal invariance of the six dimensional theory. Consider, first, the scale transformations of the $(0, 2)$ theory. They act as

$$x^+ \to \lambda x^+$$
$$x^- \to \lambda x^-$$
$$x^i \to \lambda x^i,$$

(5.2)

where $x^\pm = x^0 \pm x^5$ and $i$ runs from 1 to 4. A Lorentz transformation can be done to get rid of the change in $x^-$. This is a desirable thing to do because symmetries which act on
$x^-$ can only be seen when we reconstruct the $x^-$ dependence of correlation functions by summing over instanton numbers. This means that we have

$$
\begin{align*}
x^+ & \rightarrow \lambda^2 x^+ \\
x^- & \rightarrow x^- \\
x^i & \rightarrow \lambda x^i
\end{align*}
$$

(5.3)

giving us transformations of the momenta:

$$
\begin{align*}
P_+ & \rightarrow \lambda^{-2} P_+ \\
P_- & \rightarrow P_- \\
P_i & \rightarrow \lambda^{-1} P_i
\end{align*}
$$

(5.4)

We can check that these transformations are indeed obeyed by the simplest moduli spaces ($k = 1$, any $N$) which are symmetric products of $R^4$ with the metric inherited from the Euclidean metric on $R^4$. For more general cases it seems to put a conformal invariance requirement on the metric of instanton moduli space (4.2).

$$
G_{\mu\nu}^\prime (\lambda x) = G_{\mu\nu} (x)
$$

(5.5)

We can see this directly from the general definition of the metric

$$
G_{ij} = \int d^4 x \sqrt{g} \delta_i^\mu A^\mu \delta_j^\mu A^\mu
$$

(5.6)

This definition suffices to prove it is conformal. When the coordinates of the instanton moduli space transform, we can find transformations of $x$, $A(x)$ which leave the metric invariant:

$$
\begin{align*}
Z_i & \rightarrow \lambda Z_i \\
x & \rightarrow \lambda x \\
A(\lambda x, \lambda Z) & = \lambda^{-1} A(x, Z)
\end{align*}
$$

(5.7)

This is also seen from the quantum mechanics, by noting that, in units where the $X$ and $H$ have worldline dimension 1, all the terms in the action for the Higgs branch quantum mechanics are of same dimension.

We have seen then that the conformal invariance in 6 dimensions has a simple implication for the form of the quantum mechanics. We might have expected to recover in a simple way the special conformal transformations which form part of the conformal group
of $R^4$, (labelled by a 4-vector $b^I$) but that turns out not to be true. The special conformal transformations, have an action which mixes the $x^-$ (the hidden dimension dual to the instanton number), with the $x^i$. This action is of course expected to act on the correlation functions we reconstruct by summing over instanton number, but is not a simple symmetry before summing over instantons. Interestingly, [29] finds in relating group actions on ADHM data to group actions on instanton moduli space data, that the dilatation subgroup of the conformal group of $R^4$ is distinguished from the special conformal transformations.

5.2. Gauge group action

The correct instanton moduli space corresponding to the Higgs branch quantum mechanics is that of based instantons [10], where two instantons related by a gauge transformation which is not the identity at infinity are considered inequivalent. Without this definition, the space does not have a dimension which is a multiple of 4, as needed if it is to admit SQM with so 8 SUSY. Based instantons are considered gauge inequivalent if they are related by a global gauge transformation. This means that the moduli space has an action of $SU(k)$. The $U(1)$ part of $U(k)$ acts trivially because it commutes with all the fields entering the instanton solution, which are in the adjoint. It follows that the centre $Z_k$ of the $SU(k)$ part also acts trivially. This means that wavefunctions will form representations of $SU(k)/Z_k$ for a theory of $k$ 5-branes. Using the operator-states correspondence this leads us to the prediction that all the local operators of the $(0,2)$ transform under the $SU(k)$ gauge group of the interacting theory as representations which carry zero $Z_k$ charge. This includes for example states which are in the adjoint or its tensor products but not the fundamental.

This may at first seem surprising from the point of view of the zero brane worldvolume theory, since it contains fundamentals, $H$, $\tilde{H}$ under the $U(k)$ flavour symmetry. However, the vacua may be parametrized only by combinations of these variables which are invariant under the $U(N)$ gauge symmetry. These are representations of $U(k)$ with zero $Z_k$ charge. Indeed, the gauge invariants are built from polynomials in the $U, V, H, \tilde{H}$. Only the $H, \tilde{H}$ carry $U(k)$ indices. The $U, V$ have two $U(N)$ indices, so to form a gauge invariant quantity the total number of $H$ and $\tilde{H}$ is even, which means that the gauge invariant object cannot transform as a fundamental.

This constraint on the operator content of the $(0,2)$ theory is not in contradiction with the fact that one can find a string soliton [40] ending on a single 5-brane, which is charged under the $U(1)$. The soliton is not created by a local operator in the field theory.
The set of states obtained from the quantum mechanics is expected to be related to the spectrum of local operators acting on the vacuum. While the algebra of local operators only contains adjoint representations the non-local operators can transform in the fundamental representation. Analogous phenomena in two-dimensional field theories are known \[11\].

The constraint on the local operator content is plausible because as we move away from the origin we expect adjoint objects like strings. But this is a stronger statement, since it is a statement about the origin of moduli space. In the context of the 4 dimensional \( N = 4 \) theory, which is obtained by the dimensional reduction of the \((0, 2)\) theory, this is quite plausible. We can consider operators obtained by taking composites of the fields that enter the action of \( N = 4 \) Yang Mills, and ask how their correlators behave as we approach the fixed point. This way we have objects that transform in the adjoint and its tensor products. In the six dimensional case we do not have an action where the strings appear as adjoint fields so we cannot use this argument.

6. **Compactifications of the \((0, 2)\) theory**

We consider compactifications of the \((0, 2)\) superconformal theory on a torus \( T^d \) with the sides of the torus being of the same order of magnitude. We can consider the dependence of the correlation functions on the spacetime coordinates and the compactification scale \( L \):

\[
< \mathcal{O}(x_1)\mathcal{O}(x_2)\cdots\mathcal{O}(x_k) >_L = f(x_i, L) \tag{6.1}
\]

If we take \( L \) to be large compared to \(|x_i - x_j|\), for all \( i, j \) we expect an asymptotic expansion of the correlation functions to exist with leading term given by the flat space correlation functions. It seems reasonable to conjecture that this large \( L \) expansion can be reconstructed by considering quantum mechanics on instanton moduli space on \( R^{4-d} \times T^d \). We will present some arguments in favour of this.

The approach of \[9\] in deriving the conjecture that the Matrix theory of 5-branes is given by quantum mechanics on instanton moduli space starts with the worldvolume theory of the zero branes. An alternative approach which may be trusted for instantons of finite size is the following. We start with the \((0, 2)\) field theory compactified on a small circle of radius \( R_5 \), This is weakly coupled 4 + 1 Yang Mills theory. We could use \( 1/R_5 \) as an ultraviolet cutoff. We want to look at the sector with momentum along the 5 direction. These appear as solitons in 5 dimensions, obtained by embedding the instantons of 4D
gauge theory into $4 + 1$ Yang Mills. Since $R_5$ is small, these are very heavy. Their non-relativistic dynamics is governed by a quantum mechanics on the moduli space of the solitons. The simplest quantum mechanics is the supersymmetrization of the action of the form (4.1). There can be higher derivative terms but they should be suppressed by powers of $R_5$, since this is the parameter which measures the strength of quantum corrections. So the QM action is indeed the simplest one. Now following [12] we argue that the $(02)$ theory compactified on a lightlike circle is related by a boost to the $(0, 2)$ theory on a very small spatial circle. How we treat the pointlike instantons is not easy to motivate from this point of view, but at least one consistent way to do it is to mimic the SQM on the Higgs branch of the model of [5]. It might be interesting to see if the symmetries of the problem could be used to constrain, from the $4+1$ dimensional point of view, the quantum mechanics to be exactly that of the Higgs branch. The advantage of developing this line of argument, is that it starts directly with the decoupled theory as opposed to 5-branes embedded in M theory. Another attraction of this approach is that it would elevate the ADHM equivalence between self-duality equations in 4 dimensions and matrix equations in 0 dimension, to a quantum equivalence between two descriptions of $(0, 2)$ in the lightcone gauge.

Now we can put the $4 + 1$ Yang Mills theory on a manifold $\mathbb{R}^{4-d} \times T^d$. The theory still has solitons which are obtained from embedding instantons. For large compactification scale, $L_c$, 4+1 Yang Mills is valid. Again the quantum corrections are suppressed by powers of $R_5$, so we can trust the minimal quantum mechanics on the moduli space of instantons on $\mathbb{R}^{4-d} \times T^d$. Upon compactification of the $(0, 2)$ theory there will arise many sectors due to Wilson surfaces of the two-form field, and fluxes. We will not attempt to give a comprehensive discussion of all these sectors.

For generic compactification size we would expect a description in terms of a $d$ dimensional field theory as the Matrix model [12]. This $d$-dimensional theory is by construction a field theory obtained by T-dualizing the zero brane worldvolume theory. We will not go into the detailed construction of this theory (which is developed in [23] [18]) but we will make some general remarks on its symmetries and dynamics, based on what we expect from the properties of compactified $(0, 2)$ theory. We will also discuss qualitative aspects of the dynamics as viewed from the $(4 - d)$ dimensional brane worldvolume theory.
6.1. $S^1$ compactification

Instantons satisfy $F = *F$. Instantons on $R^3 \times S^1$ can be constructed by an analog of the ADHM construction \[43\]. Typically they have a non-trivial dependence on the coordinate living on the $S^1$ (the “calorons” of \[43\]). The ADHMN construction of the calorons is very similar to that of monopoles. These are special instantons which have no dependence on the $S^1$ of $R^3 \times S^1$. They obey the dimensionally reduced form of the self-dual Yang Mills equations.

$$F = D\phi,$$

where $\phi = A_4$. The metric on the moduli space of monopoles is hyperkähler so it admits the extended SUSY QM with $Spin(5)$ symmetry that we want for the description of the system at large compactification scales \[44\]. The $Spin(5)$ is a consequence of hyperkähler geometry the way $SU(2)$ is a consequence of Kahler geometry \[39\]. Moduli spaces of calorons are relevant to the $S^1$ compactified $(0, 2)$ theory in the vacuum sector. The moduli spaces of monopoles would be relevant to the sector of DLCQ $(0, 2)$ theory on an $S^1$, with a Wilson surface $B_4$.

With the $S^1$ compactification, we can perform a T-duality converting the D0 brane-D4 brane (with spatial extent in $(x^1, x^2, x^3, x^4)$) system to D1 brane (extended along the compact direction $x^4$) intersecting D3 branes on with spatial extent $(x^1, x^2, x^3)$. This system has been studied in connection with monopoles in the 3-brane worldvolume\[45\], \[46\]. Now we also have calorons which correspond to D-strings that wrap the circle an integral number of times \[47\].

The $(0, 2)$ theory on $R^3 \times S^1 \times R^+ \times R^-$ is 4+1 SYM in the lightcone frame and is described by quantum mechanics on caloron moduli space. By compactifying the $(0, 2)$ theory on a circle of small radius $R$, the coupling constant for the 4+1 SYM is $1/g_4^2 = 1/R_4$ which introduces a scale. The compactified $(0, 2)$ theory is no longer scale invariant. In the opposite limit of small radii, we do not expect quantum mechanics on moduli space to be valid. In this limit the compactification radius of the $(0, 2)$ theory is small. Equivalently we are looking, in the 4+1 dimensional theory, at energies small compared to the scale set by the compactification radius. The 4+1 SYM theory flows to a non-interacting theory at long wavelengths since $g_4^2$ looks very small compared to the scale at which we are doing QFT. In the Matrix description on the 1+1 worldvolume, this limit should be related to an equivalent 1+1 dimensional theory flowing to a free fixed point.
Momentum in the \( x^4 \) direction in the \((0,2)\) theory corresponds after the T-duality elementary string wound in the compact direction. This appears as electric flux in the worldvolume of the D1-brane. The composite of D-string with electric flux appears as a dyon in the 3+1 SYM. S-duality in the 3+1 Matrix theory therefore corresponds to transformation of momenta from the lightcone direction to the compact direction. Scattering of incoming monopoles into outgoing dyons correspond, in the \((0,2)\) theory on \( S^1 \), to a state scattering from the lightcone direction to the compact direction.

Quantizing zero modes that quantum mechanics on monopole moduli space produces the 5 scalars and the 3 vectors fields of the 4+1 SYM vector multiplet. This comes from the fact that the D3-D1 brane system in IIB has a \( Spin(5)_R \) symmetry from the transverse space-time.

6.2. \( T^2 \) compactification

By analogy to the \( R^3 \times S^1 \) case, we expect that instantons on \( R^2 \times T^2 \) will have a Nahm construction which is closely related to that for the corresponding dimensionally reduced theory. Again instantons on \( R^2 \times T^2 \) will be relevant to compactified \((0,2)\) in the vacuum sector. Reducing the self-dual Yang-Mills equations to \( R^2 \times T^2 \), we find that the solutions correspond to vortices in 2+1 dimensions. The equations are \( F = D(\phi_1 + \phi_2) \). These equations were studied in [48]. The moduli space is known to be hyper-Kahler, so it will admit supersymmetric quantum mechanics with 8 supercharges and \( Spin(5) \) R-symmetry.

For the \((40)\) system on the \( T^2 \times R^2 \) we can get some insights into the moduli space by doing a T-duality on the two circles. Now we have D2 branes in \( x^0, x^1, x^2 \) intersecting D2 branes in \( x^0, x^3, x^4 \). A D2 brane looks like a vortex on the orthogonal D2 brane. This is a singular configurations. The corresponding Laplacian has a logarithmic singularity. A similar situation was encountered in \([49]\) with D4 branes ending on NS 5-branes. There the singularity was resolved by going to M-theory where the D4 branes become just one M5-brane of topology \( R^4 \times \Sigma \), where \( \Sigma \) is a Riemann surface. We expect, in analogy, that the moduli space of solutions will correspond in general to a smoothed out intersection of the orthogonal two-branes, so that there will be essentially one smooth two-brane which is a Riemann surface equipped with a holomorphic mapping to \( R^2 \times T^2 \). The mathematical formulation of this correspondence between Riemann surfaces and the instantons could proceed using spectral surfaces \([50]\). Some related points are developed in \([51]\).

These configurations are relevant to \((0,2)\) theory compactified on a \( T^2 \). For large \( T^2 \) we expect to have quantum mechanics on these spaces. For small \( T^2 \) the theory flows to
$3 + 1$ Yang Mills, where the complex coupling constant is given by the complex structure of the torus $T^2$. In this limit we need to consider a $2 + 1$ dimensional dynamics. The $3 + 1$ dimensional Yang Mills may be thought as the theory on the worldvolume of IIB three branes.

One way to describe the dynamics of this system is in terms of a theory on a two-torus, which is obtained by T-dualizing the world-volume theory of the 0-branes. Another way would be to consider the worldvolume of the 2-brane orthogonal to the torus. From the point of view of the latter theory, strings wrapped along cycles of the $T^2$ are electrically charged particles.

If we compactify the $(0, 2)$ theory on a torus and take the limit of zero area, we eliminate any scale from the theory. This leads to the conformally invariant $3 + 1$ SYM. However, the $3+1$ SYM is still interacting for appropriate choice of complex structure of the torus. In the $2 + 1$ dimensional Matrix description, we expect that $2 + 1$ dimensional theory to flow to an interacting fixed point since it describes an interacting conformal field theory in the limit that the volume of the $T^2 \to \infty$. We can contrast this to the $4+1$ SYM. In that case, the limit of zero compactification radius, the theory flows to a non-interacting fixed point, since the coupling constant is proportional to $R^4$.

Lorentz invariance of the $(0, 2)$ theory should lead to an interesting symmetry when this system is formulated in terms of the world-volume of the 2-brane orthogonal to the directions of T-duality. An interesting property of these $2 + 1$ Matrix theories is that they should have a symmetry mixing vortices, and two kinds of W-bosons. The vortices, we have seen, are related to momentum in the 11 direction. Momenta in the directions 4 and 5 become elementary strings ending on the two-branes. Viewed from the 2-brane worldvolume orthogonal to the directions of T-duality, the latter are W-bosons. So there is a symmetry exchanging electrically charged objects with vortices.

The theory on this two-brane has an $SO(6)$ R-symmetry from the transverse space-time, whereas the 4-brane worldvolume theory had an $Spin(5)$ symmetry. By quantizing the zero modes of the quantum mechanics on the moduli space of vortices in an $SO(6)$ covariant fashion, along the lines of section 4, we produce as bosonic states, 6 scalars and 2 components of a vector field as appropriate for $N = 4$ SYM vector multiplet.
6.3. $T^4$ compactification.

In the $R^4$ case we have dealt with quantum mechanics on a space which is strictly a symmetric product. In the case of $T^4$ the appropriate moduli spaces for 1 4-brane and $N$ zero branes are birational to symmetric products \[52\]. We can try to use the prescription we used to obtain the Hilbert space in the case $k = 1$ on $R^4$ in the case of $T^4$. It is less well-motivated because the moduli spaces are not strictly symmetric products. But it seems reasonable if we believe that the single 5-brane continues to behave like a standard free tensor multiplet theory when the directions transverse to the light-cone are compactified on a torus. After a boost, the system is related to 5-brane of M theory compactified on $T^5$. The system of 4-brane, zero-brane and momentum in IIA theory is U-dual to 4-brane with momentum in two different directions within the 4-brane world-volume. This is indeed counted by a free field theory because the 4-brane worldvolume theory is a $U(1)$ gauge theory. This means that the prescription we used for the counting of states is consistent with the correct number of BPS states, even for states that are not ground states in the quantum mechanics.

In the above we have only discussed a simple class of states namely 4-branes with 0-branes and momenta. This is the sector of interest here because, after decompactification, these states are related to the local operators for the $(0, 2)$ theory in the simplest background $R^4 \times R^+ \times R^-$, without extended objects of infinite energy. It will be interesting to extend this discussion to systems with more charges, e.g. 4-brane, 2-brane and 0-brane, and compare U-duality predictions with the $l_p \to 0$ limit of the 5-brane actions of \[14\],\[16\],\[15\], and to understand the relation with the approach of \[53\].

**NOTE ADDED**: A previous version of this sub-section reported an inconsistency between the picture of the single 5-brane as a free field theory, and U-duality. This was based on an inaccuracy in tracking the appropriate transformations under a sequence of dualities. As a result the above discussion on $T^4$ compactification has been rewritten. Our conclusion on this issue is that there is no conflict between U-duality, the statement that the single M5-brane is a free field theory, and the conjecture that M5-branes are described by quantum mechanics on instanton moduli spaces.
7. Summary and comments.

We have tested the Matrix model proposal for a single 5-brane and for multi-5-branes in $R^4$. The summary of our tests is in the introduction. The conjecture works well for $R^4$, to the extent that we have tested it. We then discussed toroidal compactifications of the $(0,2)$ theory, presenting a conjecture and some evidence, that it is still described by quantum mechanics on instanton moduli spaces when the compactification scale is large compared to the scales in any correlation function of interest.

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