Non-abelian nature of systems with multiple exceptional points

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The defining characteristic of an exceptional point (EP) in the parameter space of a family of operators is that upon encircling the EP eigenstates are permuted. As permutations of three or more objects form a non-abelian group, the question arises whether one can experimentally demonstrate this non-commutative behavior. This requires at least two EPs of a family of operators that have at least 3 eigenstates. Encircling two or more EPs poses the question of how to compose and combine the permutations. We address this problem using the theory of fundamental groups which also has a simple generalization to higher dimensional degeneracy structures like exceptional lines in a three-dimensional parameter space. A concrete implementation in a recently proposed PT-symmetric waveguide system is suggested as an example of how to experimentally show the non-abelian nature of non-hermitian systems with multiple EPs.

I. INTRODUCTION

Exceptional points (EPs) are typically considered in systems with a discrete set of eigenstates. The exchange of eigenstates when traversing a closed loop around an EP is its defining characteristic (see e.g. [1]). The term ‘exceptional’ was originally used to indicate the presence of a degeneracy in the sense that two or more eigenvalues or levels coincide at an EP (cf. [2]). At an EP the characteristic polynomial of the operator has a higher order zero. Such a degeneracy could arise from a branch point, allowing for permutations of eigenvalues upon encircling. We take the latter property to define an EP as a degeneracy of a (matrix) operator family such that non-trivial permutations of eigenvalues occur upon following the eigenvalues along a closed loop around the degeneracy. The non-trivial branch structure implies that the matrix family is non-hermitian where at the EP the operator cannot be diagonalized.

Often the concept of PT symmetry [3, 4] is treated together with EPs. This has two main origins; in physics PT symmetry is often considered a replacement of hermiticity, and EPs usually mark points where the PT symmetry becomes spontaneously broken (PT phase transitions). Also, one may check that PT phase transitions share the higher order zero condition with EPs. However, it turned out that a PT symmetric system is sometimes equivalent to a hermitian system [5]. One can say that interesting aspects of PT symmetry may arise at an EP but PT symmetry is not the main framework to study EPs.

Studies of EPs started primarily with EP2s (e.g. [1]), i.e. EPs where two eigenvalues are interchanged. Their characteristics are now well-understood. Along a closed loop around an EP2 in the parameter plane two eigenvectors are exchanged with one acquiring a minus sign. This has also been verified experimentally [7]. Hence one has to encircle an EP2 at least four times to identify the eigenvectors, whereas the map of the eigenvalues only needs two turns to become the identity because the sign of the eigenvectors is then irrelevant. This resembles the characteristics of a diabolical point (DP), which is a degeneracy where the eigenvectors remain linearly independent. Upon encircling a DP the eigenvectors are mapped to minus themselves, and an EP2 can therefore be considered to be ‘half’ of a DP [8].

Recently, EPs with 3 coalescing levels (EP3s) have become of interest. They were already studied explicitly in [9], and now actual experiments are investigated (cf. [10] and refs therein). Here waveguides with gain and loss regions provide an optical system that is formally equivalent to a quantum system with a non-hermitian Hamiltonian. Such systems have already been introduced earlier to experimentally study aspects of PT symmetry [11].

In this paper we focus on the interplay of multiple EPs which naturally leads to the question of how the permutations obtained from encircling two or more EPs is related to the permutations associated with loops around the individual EPs. Seeming ambiguities have led, e.g., in [12] to the conclusion that no definite answer can be given to this question. We show that a definite answer can be obtained using based loops, which are oriented loops starting and ending at a fixed base points, and the continuous deformations of these loops as they enter the definition of fundamental groups. This will allow us in particular to study non-abelian effects which arise for systems with three or more levels that have two or more exceptional points. We illustrate an experimental implementation to test the results using a waveguide system.

The paper is divided into section [11] where we discuss the mathematical background required to deal with multiple EPs and the resulting non-abelian effects and section [13] where we discuss a wave-guide experiment which allows one to observe the non-abelian nature of multiple EPs. We end with a summary in section [14].
II. THEORETICAL ANALYSIS

A. The problem

Let us consider a finite-dimensional quantum problem given by a parameter family of $n \times n$ matrices. We do not impose any condition on the matrices, e.g., they may be non-hermitian. If the eigenvalues locally follow an $N$-sheet branch structure ($N \leq n$), the corresponding branch point is called an EPN. The branch structure is then similar to that of the $N$th complex root, revealing cyclic permutations of order $N$. Because of the cyclic property, non-abelian behavior can never occur using a single EP, whatever its order. We note that the eigenvectors may acquire a phase (cf. e.g. [1]). However, we will disregard phases and consider permutations of eigenvalues only.

We can now state our main question in a more precise fashion. Consider two EPs encircled by two oriented loops $\gamma_1$ and $\gamma_2$, respectively, as depicted in fig. 1. Suppose one has measured the permutations obtained from the loops $\gamma_1$ and $\gamma_2$. Which permutation should one obtain for a loop $\gamma_3$ encircling both EPs?

FIG. 1. Two EPs (crosses) encircled individually by loops $\gamma_1$ and $\gamma_2$, respectively, and a loop $\gamma_3$ which encircles both EPs.

B. Basepoint solution

The essential insight is that one first needs to fix a common base point for the paths $\gamma_1$, $\gamma_2$ and $\gamma_3$: the permutations should be compared for the same initial system parameters. Let us denote by $\Delta$ the discriminant set of the family of operators, i.e. $\Delta$ is the set of parameters for which two or more eigenvalues coincide, and by $X$ the complement of $\Delta$. Fixing a basepoint $x_0 \in X$ we can consider the measurement paths that start and end at $x_0$, that is, the loops based at $x_0$. Let us denote by $\text{Loop}(x_0)$ the set of oriented loops in $X$ that are based at $x_0$. As each loop in $\text{Loop}(x_0)$ is contained in $X$ the operators have distinct eigenvalues at any point on such a loop. Tracing a loop $\gamma \in \text{Loop}(x_0)$ induces a fixed change of eigenstates, in particular it induces a permutation $p_\gamma$ of the eigenvalues. Denote by

$$\Lambda(x_0) = \{p_\gamma \mid \gamma \in \text{Loop}(x_0)\} \quad (1)$$

distinct eigenvalues, and by using a labeling is isomorphic to a subgroup of $S_n$.

A group like $\Lambda(x_0)$ was already mentioned in the book by Kato [2] where the term exceptional point was used for the first time, and the group was called the $\lambda$-group. The $\lambda$-group there consists of the permutations that arise from analytically continuing the eigenvalues back to some initial point. The group $\Lambda(x_0)$ is a generalization by allowing for more general adiabatic connections; details of such a geometric connection can be found in [13]. To describe $\Lambda(x_0)$ more rigorously; by parallel transport each loop $\gamma \in \text{Loop}(x_0)$ induces a linear map on state space, which by the adiabatic assumption maps eigenstates to eigenstates. The group $\Lambda(x_0)$ is then obtained from the holonomy group at $x_0$ by restricting to the permutations of the eigenstates.

The concatenation of two loops in $\text{Loop}(x_0)$ defines a ‘product’ in $\Lambda(x_0)$ that is in general non-abelian. By the holonomy interpretation, the assignment $\gamma \mapsto p_\gamma$ preserves this product in the sense that

$$p_{\gamma_2 \gamma_1} = p_{\gamma_2} \circ p_{\gamma_1} \quad (2)$$

where in $\gamma_2 \gamma_1$ we first track $\gamma_1$ and then $\gamma_2$.

It is at this point that the distinction between $\gamma_2 \gamma_1$ and $\gamma_1 \gamma_2$ becomes interesting. This is because we are mapping the loops to a permutation group that in general is non-commutative. In fact, as (based) loops the two products $\gamma_2 \gamma_1$ and $\gamma_1 \gamma_2$ may be different in the sense that they are not homotopic relative to the basepoint. The loops $\gamma_1$ and $\gamma_2$ both start and end at the basepoint $x_0$. The concatenation $\gamma_2 \gamma_1$ is a loop that starts at $x_0$ and following $\gamma_1$ immediately comes back to $x_0$, after which $\gamma_2$ is traversed which again ends at $x_0$. A homotopic deformation of $\gamma_2 \gamma_1$ as a loop in $\text{Loop}(x_0)$ is a continuous deformation of the concatenation $\gamma_2 \gamma_1$ within $X$ that keeps the starting point of $\gamma_1$ and the end point of $\gamma_2$ fixed: the intermediate visit of $x_0$ becomes irrelevant. This applies analogously to the product $\gamma_1 \gamma_2$. In fig. 2 we show continuous deformations of $\gamma_1 \gamma_2$ and $\gamma_2 \gamma_1$.

We in particular see that one cannot deform $\gamma_2 \gamma_1$ to $\gamma_1 \gamma_2$ within $X$ if one needs to keep the basepoint fixed.

The continuous deformation of the based loops $\gamma \in \text{Loop}(x_0)$ and their concatenation leads to the definition of the fundamental group $\pi_1(X, x_0)$ whose elements are the equivalence classes $[\gamma]$ of loops that are homotopic to a representative $\gamma$ and where the group operation is defined by the product given by the concatenation of loops. The situation depicted in fig. 2 is then general. By the theory of fundamental groups, once fundamental paths are chosen, any loop can be written in terms of these. So far we used deformations to stress the non-commutativity of the product of two based loops. In the next subsection we will see that deformations are also relevant for the concrete question of calculating permutations.

Let us now come back to the question posed in section 11A; see fig. 4. First, we choose basepoints $x_1$, $x_2$ for the small loops $\gamma_1$ and $\gamma_2$, respectively, and $x_3$ for the big loop $\gamma_3$. The basepoints $x_1$ and $x_2$ are likely to be dif-
We now turn to the fact that permutations induced by loops around EPs are topological in nature, as opposed to geometric. This means that based loops that are homotopic induce the same permutation of eigenvalues. This does not assume anything on the nature of the degeneracies, i.e. whether they are EPs, DPs or yet another type. Note that this fact also allows one to pick the most convenient loop in a homotopy class, without any theoretical requirements on the quantum system.

**Lemma II.1.** If $\gamma, \hat{\gamma} \in \text{Loop}(x_0)$ are homotopic relative to $x_0$, then the induced permutations are equal, i.e. $p_\gamma = p_{\hat{\gamma}}$. In other words, the assignment $\gamma \mapsto p_\gamma$ factors as

$$\text{Loop}(x_0) \to \pi_1(X, x_0) \to \Lambda(x_0)$$

where each map preserves products.

*Proof.* We already remarked that a labeling induces an isomorphism between $\Lambda(x_0)$ and a subgroup of $S_n$. As $S_n$ is discrete, so is $\Lambda(x_0)$. The homotopy from $\gamma$ to $\hat{\gamma}$ induces a homotopy from $p_\gamma$ to $p_{\hat{\gamma}}$, which by discreteness is constant. \qed

Let us propose a procedure for checking the composition rule in the situation of a planar parameter space,
where we consider a loop that encircles $k$ EPs, each with winding number 1 which intuitively means that each EP is encircled exactly once (homotopy theory allows to extend such a procedure to higher dimensional parameter spaces in a straightforward way):

1. fix a loop $\gamma$ encircling all EPs once, and choose a base point $x_0$ on this loop,
2. write $[\gamma] = [\gamma_1] \cdots [\gamma_l]$ where each $\gamma_i \in \text{Loop}(x_0)$ encircles a single EP with winding number 1,
3. measure the permutations $p_i := p_{\gamma_i}$ and $p := p_{\gamma}$,
4. check $p$ and $p_k \cdots p_1$ for equality.

Non-abelian behavior occurs if there are two loops $\gamma_1, \gamma_2$ such that $p_{\gamma_2} \circ p_{\gamma_1} \neq p_{\gamma_1} \circ p_{\gamma_2}$. As $S_n$ is commutative for $n < 3$, a system in which this is possible should have $n \geq 3$ many levels. Note that EPs are not required to see non-abelian behaviour. Instead it is sufficient to have a system with three levels and two EP2s with permutations (12) and (23), respectively.

Another observable property is orientation dependence, e.g. by comparing a loop encircling two EPs with a figure 8 shaped partner loop, or more precisely, comparing the permutation along the loop $\gamma_2 \gamma_1$ shown in fig. 2 with, e.g., $\gamma_2^{-1} \gamma_1$. As opposed to the previous construction the present one requires EPNs with $N \geq 3$. This is due to the fact that for an EP2 the permutation is always a transposition and hence equals its own inverse.

**D. Remarks**

Let us inspect how the exposition above depends on choices such as basepoints and bridges. We remark that this is similar to the discussion that two fundamental groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ with different base points $x_0$ and $x_1$ are isomorphic by a conjugation-like construction, the conjugacy provided by a bridge between $x_0$ and $x_1$.

Concerning the basepoint, choosing one fixes a path-connected component of $X$. Within this component, bridges can be used to connect different basepoints, relating Loop-spaces by conjugation (for the eigenvalues, standard bookkeeping of the labels appears). We disregard the case where $X$ is not path-connected as it is in general not meaningful to compare levels associated with parameters in different connected components of $X$ because of the absence of a continuous dependence of the levels on the parameters.

Given two bridges $b, \tilde{b}$ between the same basepoints, the results of pull-back of a loop may very well differ. Key is the loop $\tilde{b}^{-1}b \in \text{Loop}(x_0)$, which may yield a non-trivial permutation. Indeed, $b$ and $\tilde{b}$ may pass an EP on different sides, such that we approach the final point using different sheets, where the loop will indeed reveal this EP permutation. Again, there is uniqueness up to conjugation, as made precise in the next lemma.

**Lemma II.2.** Let $x_0$ and $x_1$ be basepoints, let $b, \tilde{b}$ be bridges from $x_0$ to $x_1$. The two pull-back operations are related by conjugation with the permutation of $b^{-1}\tilde{b}$.

**Proof.** For $\gamma \in \text{Loop}(x_1)$ arbitrary, one has $\tilde{b}^{-1}\gamma \tilde{b}$ homotopic to $(b^{-1}b)^{-1}(b^{-1}\gamma)b(b^{-1}b)$, where all factors are in Loop$(x_0)$. The claim now follows. \hfill $\square$

Note that conjugation in $S_n$ leaves the cycle structure invariant, so one may think that a permutation depends only on the loop (and by the above, only the homotopy class). This holds true for the cycle structure, but one should still be careful when using concrete labeling, which varies even per basepoint.

Let us also discuss coordinate dependence. When reparametrizing parameter space, we assume that the reparametrization establishes a homeomorphism of the original parameter space. This induces a homeomorphism of the non-degeneracy space $X$, and so loops in one parameterization correspond to loops in the other. Also here a conjugation-like correspondence appears. It does supply another reason that deformations, even of the degeneracies themselves, do not change the physical aspects.

We emphasize that the exposition does not include any assumptions on the operators. However, it is well-known that hermitian systems do not allow for EPs. This is usually proven by the non-existence of a complete set of orthonormal eigenstates at an EP. Using the techniques above, we may provide a more topological proof, where we only need to look around the EP. More concretely, one may show non-existence of EPs by showing that $\Delta(x_0)$ is trivial, as done in the next proposition. The premise is satisfied for any hermitian family and also includes exact PT-symmetric systems [6].

**Proposition II.1.** Let $T(x)$ be a family of $n \times n$ matrix operators. If $T(x)$ has real eigenvalues for any $x \in X$, then $\Delta(x_0) = 0$ for all $x_0 \in X$.

**Proof.** Let $\gamma \in \text{Loop}(x_0)$ be any loop in $X$, denote by $\lambda_i(t)$ the induced path of the $i$th eigenvalue. By assumption, each $\lambda_i(t)$ moves on the real axis, and we may label eigenvalues such that $\lambda_i(0) < \lambda_j(0)$ whenever $i < j$, where being in $X$ allows for the strict inequalities.

Assume a non-trivial permutation is achieved, so we may consider the smallest eigenvalue (label $i$) that gets permuted. Observe that a bigger eigenvalue (label $j$) must take its place; that is $\lambda_i(0) < \lambda_j(0)$, yet $\lambda_j(1) > \lambda_i(1)$. By the Intermediate Value Theorem, one must have $\lambda_i(t^*) = \lambda_j(t^*)$ for some $t^* \in (0, 1)$. However, this implies a degeneracy which contradicts $\gamma$ being in $X$. \hfill $\square$

In conclusion, we have found that encircling two or more EPs requires based oriented loops to answer the question of how the resulting permutation is composed from the permutations associated with the individual EPs. We also described how to relate the results for different basepoints. In the next section we will describe a very concrete example using an experimental setup that could be used to test the results.
III. PROPOSED EXPERIMENT

A. Setup

In [11], two waveguides are considered that are placed next to each other and coupled. They are subjected to laser pumping giving rise to a $\mathcal{PT}$ symmetric system in which $\mathcal{PT}$ phase transitions could be observed. In this paper, we investigate a three waveguide system like in [14], see also the schematic picture in fig. [4]. Laser pumping induces complex refractive indices, which translates to a complex potential $V_k = N_k + iP_k$ in the $k$th channel, where $N_k = k_0 n_k$ is the real refractive index part and $P_k = k_0 \gamma_k / 2$ the effective pumping part.

![Schematic view of the experimental set-up](see text)

By placing channels next to each other (real) couplings $\kappa_{12}, \kappa_{23}$ are induced, which depend on the coupling lengths between the channels. The electric field amplitudes $E_k$ change along the propagation direction $x$ as (see [10] for experimental details)

$$
\begin{align*}
\frac{d}{dx} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} &= \begin{pmatrix} V_1 + iP_1 & -\kappa_{12} & 0 \\ -\kappa_{12} & V_2 + iP_2 & -\kappa_{23} \\ 0 & -\kappa_{23} & V_3 + iP_3 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} \\
&= \begin{pmatrix} -\kappa & 0 & 0 \\ 0 & -\kappa & 0 \\ 0 & 0 & -\kappa \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}
\end{align*}
$$

(4)

Let us redefine fields and measure relative to the central channel $2$. Setting $v_k = V_k - V_2$, $p_k = P_k - P_2$, and taking equal couplings $\kappa_{12} = \kappa_{23} = \kappa$, the scaled fields $\tilde{E}_k(x) = e^{i(V_k+iP_k)x}E_k(x)$ satisfy $i\frac{d}{dx} \tilde{E} = H \tilde{E}$, where $\tilde{E} = (\tilde{E}_1, \tilde{E}_2, \tilde{E}_3)^T$ and $H$ is the operator

$$
H = \begin{pmatrix} v_1 + ip_1 -\kappa & 0 & 0 \\ -\kappa & 0 & -\kappa \\ 0 & -\kappa & v_3 + ip_3 \end{pmatrix}
$$

(5)

which is similar to the idealized expression found in [14]. The electric field components hence satisfy a Schrödinger type equation with a non-hermitian operator where the role of time $t$ is played by the spatial direction $x$.

We will restrict ourselves to the subspace of operators that are of the form

$$
T(z, c) = \begin{pmatrix} z + 2i & -\sqrt{2} & 0 \\ -\sqrt{2} & 0 & -\sqrt{2} \\ 0 & -\sqrt{2} & cz - 2i \end{pmatrix}
$$

(6)

where $z$ is a complex and $c$ a real parameter. The cases $c = \pm 1$ were investigated in [9], where it was shown that these are normal forms for EPs appearing in 3 dimensional systems. It was found that for $c = 1$ the system has an EP3 at $z = 0$, while for $c = -1$ the system has an EP2 at $z = 0$. We note that $z$ is up to an offset the potential in channel 1, and $c$ is the ratio $(n_3-n_2)/(n_1-n_2)$. Observe that the whole $c$-axis ($z = 0$) is mapped to the same operator. This will return in pictures in the next subsection.

B. The parameter space and the discriminant set

The parameter space of the system is the space $C \times R$ with coordinates $(z, c)$. The EPs of the system are given by the parameters $(z, c)$ for which the eigenvalues of $T(z, c)$ conlase in a branch point singularity. One can find candidates for EPs by finding higher order zeros of the characteristic polynomial $p_{z,c}(\lambda) = \det(\lambda I - T(z, c))$. The parameter space thus decomposes into a degeneracy space $\Delta$ and a non-degeneracy space $X$. $\Delta$ is then given by the discriminant set

$$
\Delta = \{(z, c) \in C \times R \mid \text{disc}(p_{z,c}(\lambda), \lambda) = 0\},
$$

which forms lines in the three-dimensional parameter space. EPs can only be found on these lines, e.g. by finding higher order zeros of $p_{z,c}(\lambda)$ or by numerically tracking the eigenvalues along a closed loop. For the latter technique, we remark that deformation invariance of permutations allows one to check large pieces of $\Delta$ by just a single loop.

The lines in $\Delta$ contained in the plane $\Re(z) = 0$ are shown in fig. [5]. Here all points in $\Delta$ are EP2s, except for points on the $c$-axis which are EP3s. This was checked by numerically following the eigenvalues along loops (the phases of the electric field were not considered). Two main features appear: a tangent intersection of two lines at $(0, -1)$ and a cusp at $(-4i, -1)$. The higher order zero condition implies that EP3s are confined to the $c$-axis and the cusp.

Figure [6] shows what happens in the three-dimensional $(z, c)$-space, where all new lines consist of EP2s. We see that the cusp in the plane in fig. [5] is in fact part of a more complex structure in the three-dimensional space. Here four lines move out of the plane of which two have $\Im(z) > 0$ and the other two have $\Im(z) < 0$. Also two additional lines of EPs appear top-right in the picture close to the central line that is already present in fig. [5].

The tangent intersection at $(0, -1)$ and the more complex structure at $(-4i, -1)$ have another remarkable property from a $\mathcal{PT}$ symmetry perspective. One can define a parity operator $\mathcal{P}$ to swap the outer channels of the waveguide, and define a time operator $\mathcal{T}$ to be complex conjugation. The system is then $\mathcal{PT}$ symmetric at exactly three lines; the line $c = 0$, the line given by $c = 1$ and $z$ real, and the line given by $c = -1$ and $z$ imaginary; these were already drawn in fig. [6]. At the last line, $\mathcal{PT}$
FIG. 5. EP structure in the $\text{Re}(z) = 0$ plane.

FIG. 6. EP and $\mathcal{PT}$ structure in parameter space; the solid lines are EPs, the red blocks and blue stripes mark broken resp. exact $\mathcal{PT}$ symmetry. The picture is symmetric in $\text{Re}(z) \mapsto -\text{Re}(z)$.

phase transitions occur at $z = 0$ and $z = -4i$, that is precisely at the tangent intersection and the point where the complex structure with several lines emerge in fig. 6.

C. The measurement

To measure an EP, two methods stand out; one is directly tracking the eigenstates [7], which includes phase information. The other is only tracking the eigenvalues with no phase information, where the ‘merging path method’ is used for the identification of an EP [15, 16].

Here one starts with a (discretized) closed path in parameter space, where one measures the eigenvalues at each point, to obtain a (discretized) locus for each eigenvalue. If no EP structure is present, the eigenvalues will individually trace out closed loops. If an EP structure is encircled, one sees the locus of one eigenvalue ending at the initial value of another one showing that eigenvalues are permuted.

Tracking eigenvalues only has clear experimental advantages; one does not need to track eigenstates adiabatically, dynamical phases are irrelevant, and slight deformation of the path yields the same permutation. The disadvantage is that the phase information may go unrecorded.

In this system, one could for fixed system parameters measure the profile of the wave in each waveguide. That is, one obtains (complex) $\tilde{E}_k(x)$ for $k = 1, 2, 3$. Writing these in one vector $\tilde{E}(x)$, the profiles should follow

$$\tilde{E}(x) = e^{-iHx}\tilde{E}(x = 0)$$

in analogy to quantum mechanics. An advantage with respect to genuine quantum systems is that now gain and loss happen in space (along the $x$-axis) and not in time. By deducing the eigenstates $\hat{E}_k(x)$ (again $k = 1, 2, 3$) theoretically, one can change to eigenstate basis. In this basis one must have

$$\hat{E}_k(x) = e^{-i\lambda_k x}\hat{E}_k(x = 0).$$

In this way the eigenvalue(s) can be obtained.

D. Examples

Let us discuss suitable paths to check the mentioned phenomena. Although other regions shown in fig. 6 would suffice as well, we are particularly interested in the region near the tangent intersection that involves both EP2s and EP3s. The region is shown in fig. 7.

Let us first deal with the problem of concatenating loops. The relevant loops are shown in fig. 8. We deliberately take the basepoint equal in all cases, hence the slight variation on fig. 1. The upper EP is an EP2, the lower an EP3, taken in the plane $c = -0.9$. We note that the situation is similar for $c$ close to this value, although the distance between the EPs varies. Hence, one can vary $c$ if it is desirable for experiment, and the discussion below will still hold.

In figs. 8a and 8b we show the employed fundamental paths $\gamma_1$ resp. $\gamma_2$ and find their induced permutations. As a reference, we investigate the paths $\gamma_2\gamma_1$ and $\gamma_1\gamma_2$ in resp. figs. 8c and 8d. Thus, the first four pictures show the resulting permutations of eigenvalues $p_1, p_2, p_2p_1$ and $p_1p_2$, respectively. The big loop in fig. 8e is base homotopic to $\gamma_2\gamma_1$, as can be seen by pulling the left side of the loop through the area between the EPs. We observe that the permutation induced by this loop...
FIG. 7. Paths near \((z, c) = (0, -1)\). The two blue lines are EP lines: the line \(z = 0\) consists of EP3s and the other line consists of EP2s. The bold black circle and dashed red figure 8 are in the plane \(c = -0.9\) and can be used to experimentally verify non-abelian behaviour (see the text).

Indeed equals \(p_2p_1\), and does not equal \(p_1p_2\). This agrees with the problem discussed in fig. [9].

Turning to the figure 8 loop in fig. [8] we note that it can be deformed to \(\gamma_1^{-1}\gamma_2\), so one would expect the permutation \((23)^{-1}(132) = (23)(132) = (12)\). Note that this is the same permutation as the one from \(\gamma_1\gamma_2\) as \(\gamma_1\) induces a transposition, yet \(\gamma_1\gamma_2\) and \(\gamma_1^{-1}\gamma_2\) are not homotopic. This does not contradict our claims; depending on the system, non-homotopic paths may induce the same permutation. One can still measure orientation dependence by traversing the figure 8 in opposite direction. This loop is homotopic to \((\gamma_1^{-1}\gamma_2)^{-1} = \gamma_2^{-1}\gamma_1\), so yields the permutation \((132)^{-1}(23) = (123)(23) = (12)\). Hence this loop differs from \(\gamma_2\gamma_1\) (which yields a \((13)\)) only by orientation of the second part, and gives a different permutation.

E. The degree of an exceptional point in systems with more than two parameters

Taking a closer look at the tangent intersection, one may ask the question what its degree should be. As reported in [9], the tangent intersection may behave as an EP2. This means that traversing a circle in the \(c = -1\) plane which encircles this EP (and only this EP) yields the standard EP2 signature of swapping 2 eigenstates, as shown in fig. [9a] and as expected resembles the result of fig. [8] (using the obvious relabeling). However, the four lines arrive in a topological cross, and one may take a circle that goes through the other two quadrants. In this case, one can take a plane given by \(\text{Im}(z) = \epsilon i\) with \(\epsilon > 0\) small, and take a large circle. Interestingly, this yields the standard EP3 signature, as seen in fig. [9b].

One can now do a similar construction with the point at \((-4i, -1)\) and conclude that its degree depends on the plane. In a general parameter space of dimension \(d > 2\), any point where at least 3 EP lines meet has a variable degree (note that the degree of an EP is unambiguous on the lines).

Again fundamental groups provide an explanation. In case of a planar parameter space with an isolated EP, the fundamental group is \(\mathbb{Z}\) and one has a map \(\mathbb{Z} \to \Lambda(x_0)\). This has kernel \(\mathbb{N}\mathbb{Z}\), and \(N\) is the degree of the EP. Now, imagine 2 distinct EP structures/lines, as we want an intersection point necessarily for \(n \geq 3\). The fundamental group is then the free product \(\mathbb{Z} * \mathbb{Z}\), i.e. generated by 2 fundamental paths \(\gamma_1\) and \(\gamma_2\). Hence each \(\gamma_i\) induces a map \(\mathbb{Z} \to \Lambda(x_0)\), with kernel \(N_1\mathbb{Z}\) resp. \(N_2\mathbb{Z}\). In case \(N_1 \neq N_2\) clearly an issue arises, but even if \(N_1 = N_2\) we see that degree should be associated to a fundamental path, or equivalently some surface. At the intersection point there is simply no canonical choice.

IV. SUMMARY

We showed how one can compose the effects obtained from encircling multiple EPs, which in fact works for an arbitrary degeneracy structure. The problem of finding the correct calculation can be solved by using the theory of fundamental groups, which requires based oriented loops. A relevant result here is that permutations associated to the loops are of topological as opposed to geometric nature, hence deformation can be used for convenience in both theory and experiment.

Applications of these insights were explored in a waveguide system, of which we investigated the parameter space and identified a region where all tests could be performed. The presence of both EP2s and EP3s allows one to demonstrate the non-abelian nature of systems with multiple EPs by experimentally tracking eigenvalues.

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(a) Permutation (23).

(b) Permutation (132).

(c) Permutation (13) = (132)(23).

(d) Permutation (12) = (23)(132).

(e) Permutation (13).

(f) Permutation (12).

FIG. 8. Loops in the plane plane $c = -0.9$ of the parameter space (left panels). All loops have the same basepoint, marked by a circle. The solid part of the loop is always traversed first, so before the dashed part of the loop. The orientations of the loops are indicated by arrows. The EPs are marked by crosses, where the upper one is an EP2 and the lower one is an EP3. The complex energy planes (right panels) show the resulting paths of the three eigenvalues, each drawn with its own color and style. Labelling eigenvalues top to bottom, we can read off the induced permutation given in the individual captions.

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FIG. 9. The tangent intersection as EPs of different orders.