Form factors of the finite quantum XY-chain

Nikolai Iorgov

Bogolyubov Institute for Theoretical Physics, Kiev 03680, Ukraine

E-mail: iorgov@bitp.kiev.ua

Received 7 April 2011, in final form 24 June 2011
Published 27 July 2011
Online at stacks.iop.org/JPhysA/44/335005

Abstract

Explicit factorized formulas for the matrix elements (form factors) of the spin operators $\sigma^x$ and $\sigma^y$ between the eigenvectors of the Hamiltonian of the finite quantum periodic XY-chain in a transverse field were derived. The derivation is based on the relations between three models: the model of quantum XY-chain, Ising model on 2D lattice and $N=2$ Baxter–Bazhanov–Stroganov $\tau^{(2)}$-model. Due to these relations we transfer the formulas for the form factors of the latter model recently obtained by the use of separation of variables method to the model of quantum XY-chain. Hopefully, the formulas for the form factors will help in analysis of multipoint dynamic correlation functions at a finite temperature. As an example, we re-derive the asymptotics of the two-point correlation function in the disordered phase without the use of the Toeplitz determinants and the Wiener–Hopf factorization method.

PACS numbers: 75.10.Jm, 75.10.Pq, 05.50+q, 02.30.Ik

1. Introduction

The quantum XY-chain is one of the simplest models which is rich enough from the point of view of physics and at the same time permits strict mathematical analysis. The study of this model was started in [1] where it was rewritten in terms of fermionic operators by means of the Jordan–Wigner transformation. Now, this relation is a standard means to study different properties (the spectrum of the Hamiltonian [1, 2], the correlation functions [3–7], the emptiness formation probability [8] and the entanglement entropy [9–13]), quantum quenches [14] in the XY-chain. Although the Hamiltonian of the model is equivalent to the Hamiltonian of a free fermionic system, the spin operators $\sigma^x$ and $\sigma^y$ are expressed in terms of fermionic operators in a non-local way. Thus, the study of correlation functions of such operators is a non-trivial problem. For example, the correlation function $\langle \sigma_i^x \sigma_j^y \rangle$ can be written through Toeplitz determinant of size $d$ and the derivation of the asymptotics $d \to \infty$ requires [3] the use of the Szegő theorem and the Wiener–Hopf factorization method.

In this paper, we propose an alternative way to study correlation functions of the XY-model: we derive the formulas for the matrix elements of spin operators $\sigma^x$ and $\sigma^y$ between the eigenvectors of the Hamiltonian of the finite quantum XY-chain in a transverse field. These
formulas allow us to obtain at least a formal expression for multipoint dynamic correlation functions at a finite temperature. For this aim, it is enough to insert the resolution of an identity operator as a sum of projectors to the eigenspaces of the Hamiltonian. Hopefully, the correlation functions in terms of these sums will be more easily analyzed. As an application of the formulas for form factors, we re-derive the asymptotics of the correlation function \( \langle \sigma_x^0 \sigma_x^d \rangle \) at \( d \to \infty \).

The idea of derivation of form factors of the quantum finite XY-chain is to use the relations between three models: the model of quantum XY-chain, the Ising model on 2D lattice and the \( N = 2 \) Baxter–Bazhanov–Stroganov (BBS) model [15, 16]. The relation between the first and the second model was observed in [17] (relation (16) in this paper between the energies of fermionic excitations of these two models seems to be new); the relation between the second and the third model was found in [18, 19]. The parameters of the models are \( (h, \kappa), (K_x, K_y) \) and \( (a, b) \), respectively. Due to these relations we transfer the formulas for the form factors of \( N = 2 \) BBS model, recently obtained [20, 21] by the use of the separation of variables method, to the model of quantum XY-chain. The main formulas are (31), (32) together with (20), (29), (16).

The separation of variables method for the quantum integrable systems (with the basic example being the Toda chain) was introduced by Sklyanin [22] and further developed by Kharchev and Lebedev [23]. In [19], this method was adapted for the BBS model (a \( \mathbb{Z}_N \)-symmetric quantum spin system) to obtain the eigenvectors of this model. At \( N = 2 \) and special values of parameters, the BBS model reduces [18, 19, 24] to the Ising model. The eigenvectors of the transfer matrix of the Ising model obtained by the separation of variables method allowed [20, 21] us to prove the conjectural formula [25, 26] for the matrix elements of the spin operator for a finite Ising model. This derivation had provided the first proof of the formula. A summarizing overview of the results on separation of variables for the BBS model is given in [24]. It is interesting that factorized formulas for the matrix elements of spin operators exist also for the superintegrable \( \mathbb{Z}_N \)-symmetric chiral Potts quantum chain [27, 28].

In section 2, we recall the definition of the finite quantum XY-chain in a transverse field, its phase diagram and eigenvalues of the Hamiltonian and give general comments on the matrix elements of spin operators between the eigenvectors of the Hamiltonian. Section 3 is devoted to the description of relations between three models: the model of quantum XY-chain, the Ising model on 2D lattice and \( N = 2 \) BBS model. Using these relations, in section 4 we derive formulas for the matrix elements (form factors) of the spin operators \( \sigma^\pm \) between the eigenvectors of the Hamiltonian of the finite quantum XY-chain. In section 5, these formulas are rewritten as the case of the chain of infinite length. In section 6, as an application of the formulas for form factors, we re-derive the asymptotics [3] of the correlation function \( \langle \sigma_x^0 \sigma_x^d \rangle \) at \( d \to \infty \) without the use of the Toeplitz determinants and Wiener–Hopf factorization method.

2. Definition of the finite quantum XY-chain in a transverse field

2.1. The Hamiltonian and phase diagram

The Hamiltonian of the XY-chain of length \( n \) in a transverse field \( h \) is [1, 2]

\[
\mathcal{H} = -\frac{1}{2} \sum_{k=1}^{n} \left( \frac{1 + \kappa}{2} \sigma_k^x \sigma_{k+1}^y + \frac{1 - \kappa}{2} \sigma_k^y \sigma_{k+1}^x + h \sigma_k^z \right),
\]

(1)

where \( \sigma_k^i \) are Pauli matrices and \( \kappa \) is the anisotropy. In the case \( \kappa = 0 \), we obtain the XX-chain (isotropic case). The value \( \kappa = 1 \) corresponds to the quantum Ising chain in a transverse


field. In what follows we restrict ourselves to the case \( x > 0, h \geq 0 \). Other signs of \( x \) and \( h \) can be obtained using automorphisms of the algebra of Pauli matrices. Also we will suppose the periodic boundary condition \( \sigma_i^z = \sigma_{i+n}^z \). In [29], it is shown that the formulas for the matrix elements of spin operators obtained in section 4 are also applicable for the antiperiodic boundary condition \( \sigma_i^z = -\sigma_i^z, \sigma_{i+n}^z = -\sigma_{i+n}^z \) and \( \sigma_{i+n}^y = \sigma_i^y \).

Now, about the values of \( h \). Due to the relation of the XY-chain with a 2D Ising model, which will be discussed in the next section, the coupling constant \( h \) plays the role of a temperature-like variable. The value \( h > 1 \) corresponds to the paramagnetic (disordered) phase. The value \( 0 \leq h < 1 \) corresponds to the ferromagnetic (ordered) phase. If \( 0 \leq h < (1 - x^2)^{1/2} \), it is an oscillatory region (because of the oscillatory behavior of the two-point correlation function). Another peculiarity related to this region is the following. At fixed \( x \), \( 0 < x \leq 1 \), in the region where \( h > (1 - x^2)^{1/2} \) the NS-vacuum energy is lower than \( R \)-vacuum energy (asymptotically, at \( n \to \infty \), they become coinciding). In the region \( 0 \leq h \leq (1 - x^2)^{1/2} \), there are intersections at special values of \( h \) of these vacuum levels even at finite \( n \). The number of these intersections grows with \( n \). For a detailed analysis of the oscillatory region see [3, 30].

In this paper, we derive the formulas for the matrix elements in the paramagnetic phase \( (h > 1, 0 < x < 1) \) and add comments on the modification of the formulas for other values of parameters.

### 2.2. Eigenvalues and eigenvectors of the Hamiltonian of the XY-chain

Using Jordan–Wigner and Bogoliubov transformations, the Hamiltonian \( \mathcal{H} \) of the XY-chain can be rewritten as the Hamiltonian of the system of free fermions and diagonalized [1, 2]. The relation between energies \( \epsilon(q) \) and momenta \( q \) of the fermionic excitations is

\[
\epsilon(q) = ((h - \cos q)^2 + x^2 \sin^2 q)^{1/2}, \quad q \neq 0, \pi,
\]

\[
\epsilon(0) = h - 1, \quad \epsilon(\pi) = h + 1.
\]

The Hamiltonian \( \mathcal{H} \) commutes with the operator \( V = \sigma_1^z \sigma_2^z \cdots \sigma_n^z \). Since \( V^2 = 1 \), the eigenvectors are separated into two sectors with respect to the eigenvalue of \( V \). Below the sign +/− in front of \( \epsilon(q) \) in the expression for energies \( E \) corresponds to the absence/presence of the fermionic excitation with the momentum \( q \). Each such excitation carries the energy \( \epsilon(q) \).

- **NS-sector**: \( V \to +1 \), the fermionic excitations have ‘half-integer’ quasi-momenta

\[
q \in \text{NS} = \left\{ \frac{2\pi}{n} (j + 1/2), \ j \in \mathbb{Z}_n \right\} \quad \Rightarrow \quad E = -\frac{1}{2} \sum_{q \in \text{NS}} \pm \epsilon(q).
\]

This sector includes the states only with an even number of excitations.

- **R-sector**: \( V \to -1 \), the fermionic excitations have ‘integer’ quasi-momenta

\[
q \in \mathbb{R} = \left\{ \frac{2\pi}{n} j, \ j \in \mathbb{Z}_n \right\} \quad \Rightarrow \quad E = -\frac{1}{2} \sum_{q \in \mathbb{R}} \pm \epsilon(q).
\]

In the paramagnetic phase, this sector includes the states only with an odd number of excitations. In the ferromagnetic phase \( 0 \leq h < 1 \), it is natural to re-define the energy of zero-momentum excitation as \( \epsilon(0) = 1 - h \) to be positive. From formula (3) for energy \( E \), this change of the sign of \( \epsilon(0) \) in the ferromagnetic phase leads to a formal change between the absence/presence of zero-momentum excitation in the labeling of eigenstates. Thus, although the analytical expressions for energies \( E \) in terms of \( h \) and \( x \) are the same in both phases, because of the redefinition of \( \epsilon(0) \) in the case of \( 0 \leq h < 1 \) the number of excitations in the ferromagnetic phase is even.
We will denote the eigenstates $|\Phi\rangle_{\text{model}}$ by the set of values of the excited quasi-momenta $\Phi = \{q_1, q_2, \ldots, q_L\}$, the label of the sector $\alpha = NS$ or $R$ and the label of the model. For example, the state in R-sector of the quantum XY-chain with $n = 3$ sites in the paramagnetic phase with all possible quasi-momenta $\Phi = \{0, 2\pi/3, -2\pi/3\}$ excited is $|0, 2\pi/3, -2\pi/3\rangle^R_{XY}$. It has the energy

$$E = -\frac{1}{2} \left( 1 - h - \epsilon(2\pi/3) - \epsilon(-2\pi/3) \right).$$

The same formula for the energy in the ferromagnetic phase corresponds to the state $|2\pi/3, -2\pi/3\rangle^R_{XY}$.

2.3. Matrix elements of spin operators

Formally, in order to calculate any correlation function for the $XY$-chain it is sufficient to find the matrix elements of spin operators $\sigma^x_k$, $\sigma^y_k$ and $\sigma^z_k$ between the eigenstates of the Hamiltonian $H$. 

- **Matrix elements of $\sigma^x_k$.** The operator $\sigma^x_k$ commutes with $V = \sigma^x_1 \sigma^x_2 \cdots \sigma^x_n$. Therefore, the action of $\sigma^x_k$ does not change thesector. In fact, the operator $\sigma^x_k$ can be presented as a bilinear combination of operators of creation and annihilation of the fermionic excitations. Hence, the matrix elements of $\sigma^x_k$ between eigenvectors of $H$ can be calculated easily (most of them are 0). We will not consider such matrix elements in this paper.

- **Matrix elements of $\sigma^y_k$ and $\sigma^z_k$.** The operators $\sigma^z_k$ and $\sigma^y_k$ anticommute with $V = \sigma^y_1 \sigma^y_2 \cdots \sigma^y_n$. Therefore, their action changes the sector. The operators $\sigma^z_k$ and $\sigma^y_k$ cannot be presented in terms of fermionic operators in a local way. All the matrix elements of them between the eigenvectors of $H$ from different sectors are non-zero!

The aim of this paper is to derive explicit factorized formula for the matrix elements of $\sigma^x_k$ and $\sigma^y_k$. The idea is to relate three models: the quantum $XY$-chain in a transverse field, the Ising model on 2D lattice and $N = 2$ BBS model. The relation between the first and the second model is based on the observation by Suzuki [17]. The relation between the second and the third model is based on [18]. The latter relation together with the results on separation of variables for the BBS model allowed [21] us to prove the formulas for the matrix elements of spin operator of the Ising model found by Bugrij and Lisovyy [25, 26]. In this paper, we transfer these results on the matrix elements to the case of the $XY$-chain. The parameters of these three models are $(h, \chi, (K_x, K_y))$ and $(a, b)$, respectively.

3. Relation between three models

3.1. Relation between the quantum $XY$-chain and the Ising model on a lattice

The row-to-row transfer matrix of the two-dimensional Ising model with parameters $K_x$ and $K_y$ can be chosen as

$$t_{XY} := T_1^{1/2} T_2^{1/2} = \exp \left( \sum_{k=1}^n K^y_k \sigma^x_k \right) \exp \left( \sum_{k=1}^n K_k \sigma^x_k \sigma^x_{k+1} \right) \exp \left( \sum_{k=1}^n K^y_k \sigma^y_k \right).$$

(4)

where the spin configurations of the rows are chosen to be labeled by the eigenvectors of the operators $\sigma^x_k$, the parameter $K^y_k$ is dual to $K_y$, that is $\tanh K_y = \exp(-2K^y)$, and

$$T_1 = \exp \left( \sum_{k=1}^n K^y_k \sigma^x_k \right), \quad T_2 = \exp \left( \sum_{k=1}^n K_k \sigma^x_k \sigma^x_{k+1} \right).$$

(5)
In [17], Suzuki observed that if we choose $K_x$ and $K^*_y$ such that
\[
\tanh 2K_x = \frac{\sqrt{1 - \kappa^2}}{h}, \quad \cosh 2K^*_y = \frac{1}{\kappa},
\]
then the Hamiltonian (1) of the $XY$-chain will commute with the transfer matrix of the 2D Ising model (4) and these two operators have a common set of eigenvectors.

### 3.2. $N = 2$ BBS model and its relation to the Ising model

To define the $N = 2$ BBS model, we use the following $L$-operator\footnote{In comparison with [21], we interchanged $\sigma^z_k$ and $\sigma^x_k$ in the $L$-operator (7). It is just another representation of the Weyl algebra entering the definition of the $L$-operator.} [16, 31]:
\[
L_k(\lambda) = \begin{pmatrix} 1 + \lambda \sigma^z_k & \lambda \sigma^y_k (a - b \sigma^z_k) \\ \sigma^z_k (a - b \sigma^z_k) & \lambda a^2 + \sigma^z_k b^2 \end{pmatrix},
\]
depending on the parameters $a$, $b$ and spectral parameter $\lambda$. It satisfies the Yang–Baxter equation with the (twisted) quantum trigonometric $R$-matrix. In particular, it means that the eigenvectors of the transfer matrix $t(\lambda) = \text{tr} \ L_1(\lambda) L_2(\lambda) \cdots L_n(\lambda)$ built from such $L$-operators are independent of $\lambda$.

Fixing the spectral parameter to the value $\lambda = b/a$, the $L$-operator (7) degenerates
\[
L_k(b/a) = \begin{pmatrix} 1 + \sigma^z_k (a - b/a) & 1, b \sigma^z_k \\ 0, b \sigma^z_k \\ 1 + \sigma^z_k (a - b/a) & \sigma^z_k b^2 \end{pmatrix},
\]
and the transfer matrix $t(\lambda)$ can be put into a nonsymmetric Ising form
\[
t(b/a) = \prod_{k=1}^{n} (1 + \sigma^z_k \cdot b/a) \cdot \prod_{k=1}^{n} (1 + \sigma^z_k \sigma^z_{k+1} \cdot ab) = (\cosh K_x \cosh K^*_y)^{-n} T_1 T_2,
\]
\[
t(b/a) \sim T_1 T_2 = \exp \left( \sum_{k=1}^{n} K^*_y \sigma^z_k \right) \exp \left( \sum_{k=1}^{n} K_x \sigma^x_k \sigma^z_{k+1} \right),
\]
if we use periodic boundary condition $\sigma^i_{n+k} = \sigma^i_k$ and identify
\[
e^{-2K_x} = \tanh K^*_y = b/a, \quad \tanh K_y = ab.
\]
Thus at $\lambda = b/a$, we obtain the transfer matrix of the Ising model. If we do not fix the spectral parameter to this special value, we shall talk of the ‘generalized Ising model’. However, transfer matrix eigenstates are independent of the choice of $\lambda$. In [19–21], the eigenvectors for the nonsymmetric transfer matrix (8) and the matrix elements of $\sigma^x_k$ between these eigenvectors were derived using the method of separation of variables.

Comparing (6) and (9), we obtain the following simple relations for the parameters of the $XY$-model and special BBS model with $L$-operator (7):
\[
\kappa = \frac{a^2 - b^2}{a^2 + b^2}, \quad h = \frac{1 + a^2 b^2}{a^2 + b^2}.
\]

### 3.3. Relation between the energies of excitations for the Ising model and $XY$-chain

In the previous subsections we have shown how the quantum $XY$-chain in a transverse field, the Ising model on a 2D lattice and $N = 2$ BBS model are related. The parameters of the models are $(h, \kappa)$, $(K_x, K_y)$ and $(a, b)$, respectively. The relations between these pairs of parameters are given by (6), (9) and (10).
Since the transfer matrices \( t_{XY} = T_1^{1/2}T_2T_1^{1/2}, \ t_{Is} = T_2^{1/2}T_1T_2^{1/2}, \ T_1T_2 \) of the 2D Ising model are related by similarity transformations, the enumeration of the eigenstates of all these transfer matrices is the same as described in section 2.2 for the quantum XY-chain. They will be denoted, respectively, by \( |\Phi\rangle_{XY}^{\alpha}, |\Phi\rangle_{Is}^{\alpha}, |\Phi\rangle^\alpha \), where \( \Phi = \{p_1, p_2, \ldots, p_L\} \) is the set of values of the excited quasi-momenta and \( \alpha = NS \) or \( R \) is the label of the sector. Their eigenvalues \( e^{-\gamma p} \) are the same:

\[
\begin{align*}
|\Phi\rangle_{XY}^\alpha &= e^{-\gamma p}|\Phi\rangle_{XY}^\alpha, \\
|\Phi\rangle_{Is}^\alpha &= e^{-\gamma p}|\Phi\rangle_{Is}^\alpha, \\
T_1T_2|\Phi\rangle^\alpha &= e^{-\gamma p}|\Phi\rangle^\alpha, \\
\gamma &= \sum_{l=1}^{L} \gamma(p_l) - \frac{1}{2} \sum_{p \in a} \gamma(p), \\
\cosh \gamma(p) &= \frac{(t_x + t_x^{-1})(t_y + t_y^{-1})}{2(t_x^{-1} - t_x)} - \frac{t_x^{-1} - t_x}{t_y^{-1} - t_y} \cos p, \\
t_x &= \tanh K_x, \quad t_y = \tanh K_y.
\end{align*}
\]

The eigenvalues of the transfer matrix \( t(\lambda) \) of the BBS model with \( L \)-operator (7) are proportional to \( \prod_{\rho}(\lambda \pm s_{\rho}) \) (see formula (68) of [20]),

\[
s_{\rho} = \left( \frac{b^4 - 2b^2 \cos p + 1}{a^4 - 2a^2 \cos p + 1} \right)^{1/2},
\]

where the sign +/- in front of \( s_{\rho} \) in the expression for the eigenvalues of \( t(\lambda) \) corresponds to the absence/presence of the fermionic excitation with the momentum \( p \). The momentum \( p \) runs over the same sets as in the case of the quantum XY-chain. Due to (8), (11) and (12) we have the relation between \( \gamma(p) \) and \( s_{\rho} \) (see [21]):

\[
\begin{align*}
e^{\gamma(p)} &= \frac{as_{\rho} + b}{as_{\rho} - b}, \\
\sinh \gamma(p) &= \frac{2ab s_{\rho}}{a^2 s_{\rho}^2 - b^2} = \frac{2ab}{(a^2 - b^2)(1 - a^2 b^2)} \sqrt{(b^4 - 2b^2 \cos p + 1)(a^4 - 2a^2 \cos p + 1)} \\
&= \frac{2ab(a^2 + b^2)}{(a^2 - b^2)(1 - a^2 b^2)} \epsilon(p) = \frac{\sqrt{1 - \kappa^2}}{\kappa \sqrt{\kappa^2 + h^2} - 1} \epsilon(p).
\end{align*}
\]

The existence of the relation between \( \gamma(p) \) and \( \epsilon(p) \) is surprising because the commutativity of the Hamiltonian (1) of the XY-chain and the transfer matrix (4) of the 2D Ising model does not imply \textit{a priori} any relation between their eigenvalues.

### 3.4. Uniformization of the dispersion relation (13)

We use a parametrization of the dispersion relation (13) of the 2D Ising model in terms of an elliptic function at \( h > 1, 0 < \kappa < 1 \) which corresponds to the paramagnetic phase of the Ising model. This parametrization is a modification of parametrization from [32] given for the ferromagnetic phase of the Ising model and corresponding to \( 0 < h < 1, (1 - h^2)^{1/2} < \kappa < 1 \) for the XY-chain.

We introduce the modulus of elliptic curve \( k^{-1} \) by

\[
k^{-1} = \sinh 2K_x \sinh 2K_y = \kappa / \sqrt{\kappa^2 + h^2 - 1} = (a^2 - b^2)/(1 - a^2 b^2).
\]
In the paramagnetic phase, we have $0 \leq k^{-1} < 1$. Complete elliptic integrals for $k^{-1}$ and for the supplementary modulus are $K = K(k^{-1})$ and $K' = K((1 - k^{-2})^{1/2})$, respectively. We define the real parameter $a$, $0 < a < K'/2$, by one of the equivalent relations

$$1/ \sinh 2K_x = ik/\sin(2ia, k^{-1}), \quad 1/ \sinh 2K_y = -i \sin(2ia, k^{-1}).$$

The two elliptic functions

$$\lambda(u) = \sinh(u - ia, k^{-1})/\sinh(u + ia, k^{-1}), \quad z(u) = k^{-1} \sinh(u - ia, k^{-1}) \sinh(u + ia, k^{-1})$$

satisfy the relation

$$\sinh 2K_x (z + z^{-1}) + \sinh 2K_y (\lambda + \lambda^{-1}) = 2 \cosh 2K_x \cosh 2K_y. \quad (17)$$

To prove it we note that the left-hand side of (17) is an elliptic function without poles (the poles at $\pm a$ and $\pm a + iK'$ are canceled) and therefore it is a constant. Thus, it is sufficient to establish validity of (17) at $u = 0$. For this end we use $\lambda(0) = -1$,

$$z(0) = -k^{-1} \sin^2(ia, k^{-1}) = -k \left(1 - \text{dn}(2ia, k^{-1})\right)^{-1} \cosh 2, \quad z^{-1}(0) = -k \left(1 + \text{dn}(2ia, k^{-1})\right) \cosh 2,$$

which follow from the formulas of example 6, section 22.21 of [33], and

$$\cosh 2K_x = \text{dn}(2ia, k^{-1}), \quad \cosh 2K_y = i \operatorname{cn}(2ia, k^{-1})/\sinh(2ia, k^{-1}).$$

Relation (17) coincides with the dispersion relation (13) if one identifies $z(u) = e^{-i\tilde{\eta}}$ and $\lambda(u) = e^{-\gamma(p)}$. The parameter $u$ on the elliptic curve is an analog of rapidity. Now, if $p$ runs from $-\pi$ to $\pi$, then $u$ runs along the segment from $iK'/2$ to $2K + iK'/2$.

There is another dispersion relation corresponding to the evolution in the transverse direction on the Ising lattice:

$$\cosh \tilde{\gamma}(\tilde{p}) = \frac{(t_x + t_x^{-1})(t_y + t_y^{-1})}{2(t_y^{-1} - t_y)} - \frac{t_x^{-1} - t_x}{t_y^{-1} - t_y} \cos \tilde{p}. \quad (18)$$

It is uniformized by $\lambda(u) = e^{\tilde{\gamma}p}$ and $z(u) = e^{-\gamma(p)}$. Now, if $\tilde{p}$ runs from $-\pi$ to $\pi$, then $u$ runs along the segment from 0 to $2K$.

From (10), we have

$$a^2 = \frac{h - \sqrt{h^2 + x^2} - 1}{1 - x}, \quad b^2 = \frac{h - \sqrt{h^2 + x^2} - 1}{1 + x}. \quad (19)$$

The points with $z = a^{\pm 2}$ and $z = b^{\pm 2}$ are the branching points of the spectral curve (17) considered as $\lambda(z)$. The parameters $a^2, b^2$ correspond, respectively, to $\lambda_2, \lambda_1^{-1}$ of [3] and to $\alpha_1^{-1}, \alpha_2^{-1}$ of [32]. We have also $a^2 = e^{-\gamma(0)}, b^2 = e^{-\gamma(\pi)}$.

4. Formulas for the matrix elements of spin operators

In this section, we will derive the formulas for the matrix elements of spin operators for the quantum XY-chain of finite length. The derivation for the basic region of parameters $h > 1, 0 < x < 1$ is given in section 4.1. The formulas for other values of parameters can be obtained by analytic continuation. The details of the continuation are given in the following subsections.
4.1. Paramagnetic phase: $h > 1$, $0 < x < 1$

We use the Bugrij–Lisovyy formula (40) of [26]) for the matrix element of a spin operator between the eigenvectors $|\Phi_0\rangle_{1k} = |q_1, q_2, \ldots, q_K\rangle^N_{1k}$ and $|\Phi_1\rangle_{1k} = |p_1, p_2, \ldots, p_L\rangle^R_{1k}$ of the symmetric transfer matrix $t_{1k} = T_{1/2} T_{1} T_{2}^{1/2}$ for the finite 2D Ising model (the states are labeled by the momenta of excited fermions as is explained in sections 2.2 and 3.3):

$$
\Xi_{\Phi_0, \Phi_1} = |\langle q_1, q_2, \ldots, q_K | \sigma_m^n | p_1, p_2, \ldots, p_L \rangle|^{1/2}_{1k} \\
= \xi \prod_{k=1}^{K} \prod_{q_{k,p_k}}^{N} \sinh \frac{\gamma(q_k)p_k}{2} \prod_{l=1}^{L} \prod_{q_{l,p_l}}^{R} \sinh \frac{\gamma(p_l)q_l}{2} \left( \frac{t_y - t^{-1}_y}{t_x - t^{-1}_x} \right)^{K-L/2} \\
\times \prod_{k<l}^{R} \frac{\sin^2 \frac{\gamma(q_k)p_k}{2} \sinh \frac{\gamma(q_k)p_k}{2}}{\sinh^2 \frac{\gamma(q_k)p_k}{2}} \prod_{l<k}^{L} \frac{\sin^2 \frac{\gamma(p_l)q_l}{2} \sinh \frac{\gamma(p_l)q_l}{2}}{\sinh^2 \frac{\gamma(p_l)q_l}{2}}.
$$

\( \xi = k^2 - 1 = (\sinh 2K_x \sinh 2K_y)^{-2} - 1 = \frac{\hbar^2 - 1}{\kappa^2} = \frac{(a^2 - b^2)}{(a^2 - b^2)^2} \),

\( \xi_T = \frac{\prod_{q}^{NS} \prod_{q}^{R} \sinh \frac{\gamma(q)p}{2} \prod_{p}^{R} \sinh \frac{\gamma(p)q}{2}}{\prod_{q}^{NS} \prod_{q}^{R} \sinh \frac{\gamma(q)p}{2} \prod_{p}^{R} \sinh \frac{\gamma(p)q}{2}}. \)

$$
t_x = \tanh K_x = ab, \quad t_y = \tanh K_y = \frac{a - b}{a + b},
$$

\[ \frac{t_y - t^{-1}_y}{t_x - t^{-1}_x} = \frac{4a^2b^2}{(a^2 - b^2)(1 - a^2b^2)} = \frac{1 - \kappa^2}{\kappa \sqrt{\kappa^2 + h^2}}. \]

where we used (6), (9) and (10) to write equivalent expressions in terms of different parameters.

Note that the Bugrij–Lisovyy formula (20) is given for the normalized eigenvectors $|\Phi\rangle_{1k}$ of the transfer matrix $t_{1k} = T_{1/2} T_{1} T_{2}^{1/2}$ which differs from the transfer matrix (4). The eigenvectors $|\Phi_{XY}\rangle$ of the Hamiltonian (1) of the $XY$-chain and (4) are in one-to-one correspondence with the eigenvectors $|\Phi_{1k}\rangle$ and the eigenvectors $|\Phi\rangle$ of the BBS model. All the eigenvectors with the same $\Phi$ (the same set of excited fermion excitations) are related by similarity transformations.

In [19, 20], the left and right eigenvectors $|\Phi\rangle$ and $|\Phi\rangle$ of $T_1 T_2$ were found. They are related to $|\Phi_{1k}\rangle$ by the action of operator $T_{1/2}$ and its inverse. Since these operators commute with $\sigma_m^n$, we have the relation

$$
\Xi_{\Phi_0, \Phi_1} = \frac{\langle \Phi_0 | \sigma_m^n | \Phi_1 \rangle \langle \Phi_1 | \sigma_m^n | \Phi_0 \rangle}{\langle \Phi_0 | \Phi_0 \rangle \langle \Phi_1 | \Phi_1 \rangle} \tag{24}
$$

expressing the matrix elements of the 2D Ising model in terms of the matrix elements of the BBS model found in [21] and used to prove (20).

Here, we want to use the matrix elements of the BBS model to derive the matrix elements of spin operators between the eigenstates of the $XY$-quantum chain. Since the Hamiltonian of the $XY$-chain and the transfer matrix $t_{1k}$ are Hermitian matrices, there is a natural way to relate the left and right eigenvectors and to normalize them. In the case of a nonsymmetric transfer matrix $T_1 T_2$ arising in the BBS model, the left and right eigenvectors $|\Phi\rangle$ and $|\Phi\rangle$ are unrelated but we will relate them to normalized eigenvectors of the $XY$-chain by $a^k_{XY} |\Phi\rangle = |\Phi_{1/2}\rangle$ and $a^k_{XY} |\Phi\rangle = |T_{1/2}| \Phi\rangle$. We fix $a^k_{XY} = \| T_{1/2} \Phi\rangle > 0$; then the coefficient $a^k_{XY}$ is determined from the requirement $XY \langle \Phi\rangle = \langle \Phi_{1/2}\rangle$. We also have $a^k_{XY} a^k_{XY} = \langle \Phi | \Phi\rangle$. Since
\[ \langle \Phi|\Phi \rangle \] is real at all \( a \) and \( b \), both \( \alpha_x^a \) and \( \alpha_x^b \) are real too. The matrix elements of the BBS model are related to the matrix elements of the \( XY \)-chain by

\[ \langle \Phi_0|\sigma^a_m|\Phi_1 \rangle = \alpha_x^a \alpha_x^b \cdot e^{\gamma_{b-a}} \langle \Phi_0|T_{1}^{1/2}\sigma^a_m T_{1}^{-1/2} \rangle \langle \Phi_1 \rangle_{XY} \]

(25)

\[ \langle \Phi_0|\sigma^a_m|\Phi_1 \rangle = \alpha_x^a \alpha_x^b \cdot e^{\gamma_{b-a}} \langle \Phi_0|T_{1}^{1/2}\sigma^a_m T_{1}^{-1/2} \rangle \langle \Phi_1 \rangle_{XY} . \]

(26)

where the last relation follows from the facts that \( |\Phi_0 \rangle_{XY} \) is an eigenvector of \( T_1^{1/2}T_2T_1^{-1/2} \) with the eigenvalue \( e^{\gamma_{2-2}} \) (see (11)) and \( T_2 \) commutes with \( \sigma^a_m \). The complex conjugation of (26) together with (25) with interchanged \( \Phi_0 \) and \( \Phi_1 \) gives

\[ \langle \Phi_0|\sigma^a_m|\Phi_1 \rangle = \alpha_x^a \alpha_x^b \cdot e^{\gamma_{b-a}} \langle \Phi_0|T_{1}^{1/2}\sigma^a_m T_{1}^{-1/2} \rangle \langle \Phi_1 \rangle_{XY} . \]

(27)

From the other side, formulas (56) and (57) of [21] give the following factorized presentations for the matrix element between eigenstates of the transfer matrix \( T_1T_2 \) (see footnote 1):

\[ \langle \Phi_0|\sigma^a_m|\Phi_1 \rangle = f_1(b)f_2(\tilde{b}_2), \quad \langle \Phi_1|\sigma^a_m|\Phi_0 \rangle = f_1(-b)f_2(\tilde{b}_2), \]

where \( f_1(b) \) is real (we suppose that \( a \) and \( b \) are real) and \( f_2(\tilde{b}_2) \) is complex but invariant with respect to \( b \to -b \). Therefore,

\[ \frac{\langle \Phi_0|\sigma^a_m|\Phi_1 \rangle}{\langle \Phi_1|\sigma^a_m|\Phi_0 \rangle} = f_1(b) \frac{f_1(-b)}{f_1(-b)} = C_{\Phi_0,\Phi_1}. \]

(28)

It is easy to prove using explicit formulas for \( f_1(b) \) from [21] that

\[ C_{\Phi_0,\Phi_1} = \prod_{p \in R} e^{\gamma(p)/2} \prod_{q \in NS} e^{\gamma(q)/2} \prod_{l=1}^K e^{\gamma(p_l)} = e^{\gamma_{b-a}} \]

(29)

for \( |\Phi_0 \rangle = |q_1, q_2, \ldots, q_K \rangle^NS \) and \( |\Phi_1 \rangle = |p_1, p_2, \ldots, p_L \rangle^R \). Let us consider, for example, the case of odd \( n \) (the length of the chain) and \( \alpha_0 = \sigma_\pi \) (\( \sigma_\pi = 0/\sigma_\pi \) \( = 1 \) corresponds to the absence/presence of fermion excitation with momentum \( q \)). From equation (56) and the discussion in section 6.1 of [21] we can choose

\[ f_1(b) = ((-1)^{n_0} - ab) \prod_{k \in \tilde{D}} ((-1)^{b} + s_{nk/n}) \prod_{k \in \tilde{D}} ((-1)^{b} - s_{nk/n}), \]

where \( s_p \) is given by (14) and the set \( \tilde{D} \) (resp. \( \tilde{D} \)) consists of such \( k \) from \( \{1, 2, \ldots, n - 1\} \) for which the fermions with both momenta \( \pm \pi k/n \) are excited (resp. not excited) in the states \( |\Phi_0 \rangle \) and \( |\Phi_1 \rangle \). Using (15) we obtain

\[ C_{\Phi_0,\Phi_1} = \frac{f_1(b)}{f_1(-b)} = ((-1)^{n_0} - ab) \prod_{p \in R, p \neq 0} e^{\gamma(p)/2} \prod_{q \in NS, q \neq 0} e^{\gamma(q)/2} \prod_{l=1}^K e^{\gamma(p_l)} \]

Thus, it remains to verify that the first fraction also fits (29) as contribution of the momenta 0 and \( \pi \). For this end, we need just to take into account (14), (15) and

\[ s_0 = \frac{h^2 - 1}{a^2 - 1}, \quad s_\pi = \frac{h^2 + 1}{a^2 + 1}, \quad e^{\gamma(0) - \gamma(\pi)/2} = 1 - \frac{ab}{1 + ab}. \]

The correct sign of the latter formula can be fixed from the limit \( h = 0 \) (the quantum Ising chain limit) and then taking limit \( a \to 0 \) (it corresponds to the limit of strong external field \( h \to \infty \)). All the other three cases of odd (even) \( n \) and \( \alpha_0 = \sigma_\pi \) (\( \sigma_0 \neq \sigma_\pi \)) can be analyzed similarly. It proves (29).

Taking into account (24) and (28), we obtain

\[ \frac{\langle \Phi_0|\sigma^a_m|\Phi_1 \rangle}{\langle \Phi_0|\Phi_0 \rangle \langle \Phi_1|\Phi_1 \rangle} = e^{i\theta_{b_0,\pi_1}(C_{\Phi_0,\Phi_1,\Xi_{\Phi_0,\Phi_1})} \]

(30)
where \( \delta_{\Phi_0, \Phi_1} \) is a phase related to a particular normalization of eigenvectors. To relate these matrix elements to the matrix elements of the \( XY \)-chain we observe that (27) and (28) imply \( a_{\Phi_0}^R a_{\Phi_1}^R = a_{\Phi_0}^L a_{\Phi_1}^L \). Since \( a_{\Phi_0}^L a_{\Phi_0}^R a_{\Phi_1}^R a_{\Phi_1}^L = \langle \Phi_0 | \Phi_0 \rangle \langle \Phi_1 | \Phi_1 \rangle \), we obtain \( a_{\Phi_0}^L a_{\Phi_1}^L = ((\Phi_0 | \Phi_0 \rangle \langle \Phi_1 | \Phi_1 \rangle)^{1/2} \). Thus, from (25), (26), (29) and (30) we derive

\[
\begin{align*}
XY(\Phi_0|T_{\Phi_1}^{-1/2}\sigma_m^+ T_{\Phi_1}^{1/2}|\Phi_1)_{XY} &= \mathcal{X}_n(\Phi_0)|\Phi_1\rangle
= e^{i\delta_{\Phi_0, \Phi_1}} C_{\Phi_0, \Phi_1} \Xi_{\Phi_0, \Phi_1} |\Phi_1\rangle,
\end{align*}
\]

\[
\begin{align*}
XY(\Phi_0|T_{\Phi_1}^{1/2}\sigma_m^- T_{\Phi_1}^{-1/2}|\Phi_1)_{XY} &= \mathcal{X}_n(\Phi_0)|\Phi_1\rangle
= e^{i\delta_{\Phi_0, \Phi_1}} C_{\Phi_0, \Phi_1} \Xi_{\Phi_0, \Phi_1} |\Phi_1\rangle.
\end{align*}
\]

Finally, taking appropriate linear combinations of these two formulas, we obtain the main result of the paper: the matrix elements of spin operators between the eigenvectors \( |\Phi_0\rangle_{XY} = |q_1, q_2, \ldots, q_K\rangle_{NS} \) from the \( NS \)-sector and \( |\Phi_1\rangle_{XY} = |p_1, p_2, \ldots, p_L\rangle_{R} \) from the \( R \)-sector of the Hamiltonian (1) of the \( XY \)-chain are

\[
|XY(\Phi_0|\sigma_m^+|\Phi_1)_{XY}\rangle^2 = \frac{\alpha}{2(1 + \alpha)} (C_{\Phi_0, \Phi_1} + C_{\Phi_0, \Phi_1}^{-1})^2 \Xi_{\Phi_0, \Phi_1},
\]

\[
|XY(\Phi_0|\sigma_m^-|\Phi_1)_{XY}\rangle^2 = \frac{\alpha}{2(1 - \alpha)} (C_{\Phi_0, \Phi_1} - C_{\Phi_0, \Phi_1}^{-1})^2 \Xi_{\Phi_0, \Phi_1},
\]

where \( \Xi_{\Phi_0, \Phi_1}, C_{\Phi_0, \Phi_1} \) and \( \gamma_\Phi \) are given by (20), (29), (12) and (16). We also used

\[
\sinh K^x = \frac{b}{\sqrt{a^2 - b^2}} = \sqrt{1 - \kappa^2}, \quad \cosh K^x = \frac{a}{\sqrt{a^2 - b^2}} = \sqrt{\frac{1 + \kappa}{2\kappa}}.
\]

4.2. Ferromagnetic phase: \( 0 < \kappa < 1, \sqrt{1 - \kappa^2} < h < 1 \)

Let us give a general comment on the continuation of the formulas from the region \( h > 1 \) to region \( 0 \leq h < 1 \). Formally, all the formulas for matrix elements of spin operators are correct for the paramagnetic phase where \( h > 1 \) and for the ferromagnetic phase where \( 0 \leq h < 1 \). But for the case \( 0 < h < 1 \), it is natural to change the sign of \( \varepsilon(0) \) (and also of \( \gamma(0) \)) of zero-momentum excitation to be positive: \( \varepsilon(0) = 1 - h \). From (3), this change of the sign of \( \varepsilon(0) \) (and of \( \gamma(0) \)) in the ferromagnetic phase leads to a formal change between the absence/presence of zero-momentum excitation in the labeling of eigenstates. Therefore, the number of the excitations in each sector (\( NS \) and \( R \)) becomes even. Direct calculation shows that the change of the sign of \( \gamma(0) \) in (31) and (32) can be absorbed to obtain formally the same formulas (31) and (32) but with new \( \gamma(0) \), even \( L \) (the number of the excitations in the \( R \)-sector) and new \( \kappa = (1 - \kappa^2)^{1/4} = (1 - h^2)^{1/4} \).

The explicit formulas for the matrix elements for the region of parameters \( 0 < \kappa < 1, \sqrt{1 - \kappa^2} < h < 1 \) are given [38].
4.3. Region $\kappa > 1$

From relation (16) between the energies of the XY-chain and Ising model excitations it follows that the energies $\gamma(p)$ of the Ising model excitations become complex and it is useful to introduce $\gamma(p) = i\tilde{\gamma}(p)$ such that

$$\sin \tilde{\gamma}(p) = \frac{\sqrt{x^2 - 1}}{x\sqrt{x^2 + h^2} - 1} \varepsilon(p).$$

Here, $\tilde{\gamma}(p)$ should be chosen to be a monotonically increasing function of $p$ when $p$ runs from 0 to $\pi$. If $h \geq x^2 - 1$, the energy $\varepsilon(p)$ is a monotonically increasing function of $p$ with minimum at $p = 0$ and maximum at $p = \pi$. In this case $0 < \tilde{\gamma}(p) \leq \pi/2$. If $0 \leq h < x^2 - 1$, the energy $\varepsilon(p)$ is a non-monotonical function having an additional extremum (maximum) at $p = p_c$, $\cos p_c = h/(1 - x^2)$, $\tilde{\gamma}(p_c) = \pi/2$. In this case $0 < \tilde{\gamma}(p) < \pi/2$ for $0 \leq p < p_c$ and $\tilde{\gamma}(p) > \pi/2$ for $p_c < p \leq \pi$.

We continue analytically with formulas (31) and (32) and write them in terms of $\tilde{\gamma}(p)$. All the changes in the final formulas are as follows:

$$|_{XY}(\Phi_0 | \sigma_m^x | \Phi_1)_{XY}|^2 = \frac{2x}{x+1} \cos^2 \frac{\tilde{\gamma}_{\Phi_0} - \tilde{\gamma}_{\Phi_1}}{2} \tilde{\Xi}_{\Phi_0, \Phi_1},$$

$$|_{XY}(\Phi_0 | \sigma_m^y | \Phi_1)_{XY}|^2 = \frac{2x}{x-1} \sin^2 \frac{\tilde{\gamma}_{\Phi_0} - \tilde{\gamma}_{\Phi_1}}{2} \tilde{\Xi}_{\Phi_0, \Phi_1},$$

where $\tilde{\Xi}_{\Phi_0, \Phi_1}$ is given by (20) with substitutions

$$\sin \left( \frac{\gamma(p) + \gamma(q)}{2} \right) \rightarrow \sin \left( \frac{\tilde{\gamma}(p) + \tilde{\gamma}(q)}{2} \right),$$

$$t_x - t^{-1}_\tau \rightarrow \frac{x^2 - 1}{x\sqrt{x^2 + h^2} - 1},$$

and $\tilde{\gamma}_\Phi$ is given by (12) with $\gamma(p) \rightarrow \tilde{\gamma}(p)$. Also, for $0 \leq h < 1$, one has to take into account the modifications related to the zero mode described in section 4.2.

4.4. Oscillatory region $0 < \kappa < 1$, $0 \leq h < \sqrt{1-x^2}$

Similar to the region $\kappa > 1$, the energies $\gamma(p)$ of the Ising model excitations become complex and it is useful to rewrite the matrix elements of spin operators in terms of $\tilde{\gamma}(p) = \gamma(p) + i\pi/2$:

$$\cosh \tilde{\gamma}(p) = \frac{\sqrt{1-x^2}}{x\sqrt{1-x^2} + h^2} \varepsilon(p).$$

Here, $\tilde{\gamma}(p)$ should be chosen to be a monotonically increasing function of $p$ when $p$ runs from 0 to $\pi$. If $1 - x^2 \leq h < \sqrt{1-x^2}$, the energy $\varepsilon(p)$ is a monotonically increasing function of $p$ with minimum at $p = 0$ and maximum at $p = \pi$. In this case $\tilde{\gamma}(p) \geq 0$. If $0 \leq h < 1 - x^2$, the energy $\varepsilon(p)$ is a non-monotonical function having an additional extremum (minimum) at $p = p_c$, $\cos p_c = h/(1 - x^2)$, $\tilde{\gamma}(p_c) = 0$. In this case $\tilde{\gamma}(p) < 0$ for $0 \leq p < p_c$ and $\tilde{\gamma}(p) > 0$ for $p_c < p \leq \pi$. We continue analytically with formulas (31) and (32) and write them in terms of $\tilde{\gamma}(p)$. All the changes in the final formulas are as follows:

$$|_{XY}(\Phi_0 | \sigma_m^x | \Phi_1)_{XY}|^2 = \frac{x}{2(1+x)} \left( e^{\frac{\tilde{\gamma}_{\Phi_0} - \gamma_\Phi}{2}} + (-1)^{\kappa L} e^{\frac{\gamma_\Phi - \tilde{\gamma}_{\Phi_1}}{2}} \right)^2 \tilde{\Xi}_{\Phi_0, \Phi_1},$$

$$|_{XY}(\Phi_0 | \sigma_m^y | \Phi_1)_{XY}|^2 = \frac{x}{2(1-x)} \left( e^{\frac{\tilde{\gamma}_{\Phi_0} - \gamma_\Phi}{2}} - (-1)^{\kappa L} e^{\frac{\gamma_\Phi - \tilde{\gamma}_{\Phi_1}}{2}} \right)^2 \tilde{\Xi}_{\Phi_0, \Phi_1},$$

where $\tilde{\Xi}_{\Phi_0, \Phi_1}$ is given by (20) with $\xi = (1 - k^2)^{1/4} = ((1-h^2)/(x^2))^{1/4}$ (since it is the ferromagnetic phase) and with the replacements

$$\sin \left( \frac{\gamma(p) + \gamma(q)}{2} \right) \rightarrow \cosh \left( \frac{\tilde{\gamma}(p) + \tilde{\gamma}(q)}{2} \right),$$

$$t_x - t^{-1}_\tau \rightarrow \frac{1 - x^2}{x\sqrt{1-x^2} - h^2}.$$
and \( \tilde{\sigma}_k \) is given by (12) with \( \gamma(p) \rightarrow \tilde{\gamma}(p) \). Also one has to take into account the modifications related to the zero mode described in section 4.2.

4.5. Other values of parameters

4.5.1. Quantum Ising chain: \( \kappa = 1 \). In the case of the quantum Ising chain (\( \kappa = 1 \)), the formula for the matrix elements of the spin operator \( \sigma^k_m \) can be derived by a limiting procedure. The final formula was derived in [21, 34] and it is expressed in terms of the energies of excitations \( \epsilon(q) \). In the case of the general XY-chain we were not able to find an analogous formula for the matrix elements in terms of \( \epsilon(q) \).

4.5.2. Boundary of the oscillator region: \( \kappa^2 + h^2 = 1 \). One of the peculiarities of the XY-chain when the parameters belong to the curve \( \kappa^2 + h^2 = 1 \) is that the ground states \( |\rangle_{XY} \) and \( \tilde{|\rangle}_{XY} \) are degenerate and each of them can be presented as a sum of two pure tensors [35, 36]. In [3], for such values of parameters, the two-point correlation function was found explicitly. Here, we give some comments on the matrix elements of spin operators. They can be derived from the general formulas for the region \( 0 < \kappa < 1 \), \( \sqrt{1-\kappa^2} < h < 1 \) by a limiting procedure. From (16), denoting \( \xi = \sqrt{\kappa^2 + h^2 - 1} \) we obtain in the limit \( \xi \rightarrow 0 \)

\[
\begin{align*}
\epsilon\gamma(p) &= \frac{2h}{\xi\sqrt{1 - h^2}} \epsilon(p), \quad \epsilon(p) = 1 - h \cos p, \\
\sinh \frac{\gamma(p) + \gamma(q)}{2} &= \frac{e^{(\gamma(p)+\gamma(q))/2}}{2}, \quad \xi = 1, \quad \xi_T = 1.
\end{align*}
\]

Using these formulas, it is easy to take the limit \( \xi \rightarrow 0 \) in the general formulas for the matrix elements. We obtain, in particular, that the matrix elements are non-zero if and only if \( K = L \) or \( K - L = \pm 2 \).

5. Asymptotics of form factors in the limit of the infinite chain

In this section, we analyze the asymptotics of different parts of form factors in the limit of infinite length (\( n \rightarrow \infty \)) of the XY-chain. To this end, it is helpful to use the following integral representations for different parts of form factors at finite \( n \) [26]. For

\[
\Lambda^{-1} = \frac{1}{2} \left( \sum_{q \in N_S} \gamma(q) - \sum_{p \in R} \gamma(p) \right), \quad \eta(q) = \frac{1}{\pi} \int_0^{\pi} dp \log \coth \frac{n\tilde{\gamma}(p)}{2},
\]

and \( \xi_T \) (see (22)), we have

\[
\Lambda^{-1} = \frac{1}{\pi} \int_0^{\pi} dp \log \coth \frac{n\tilde{\gamma}(p)}{2},
\]

\[
\eta(q) = \frac{1}{\pi} \int_0^{\pi} dp \frac{\cos p - e^{-\gamma(q)}}{\cosh \gamma(q) - \cos p} \log \coth \frac{n\tilde{\gamma}(p)}{2},
\]

\[
\xi_T = \frac{n^2}{2\pi^2} \int_0^{\pi} dp \int_0^{\pi} dq \frac{\tilde{\gamma}'(p)\tilde{\gamma}'(q)}{\sinh n\tilde{\gamma}(p) \sinh n\tilde{\gamma}(p)} \log \left[ \frac{\sin((p + q)/2)}{\sin((p - q)/2)} \right].
\]

Let us show that \( \Lambda^{-1} \rightarrow 0 \) if \( n \rightarrow \infty \). In fact, we have the following asymptotics:

\[
\Lambda^{-1} \rightarrow 2 \frac{1}{\pi} \int_0^{\pi} dp \frac{e^{-\gamma(p)}}{\pi} \int_0^{\pi} dp \frac{e^{-\gamma(q)}}{\pi} \rightarrow 2 \frac{2}{n\pi \tilde{\gamma}'(0)} \rightarrow 0,
\]

\[
\simeq \frac{e^{-\gamma(0)}}{\pi} \int_{-\infty}^{\infty} dp \frac{e^{-(\gamma(0)/p)^2/2}}{\sqrt{n\pi \tilde{\gamma}'(0)}}\rightarrow 0.
\]
where we used the fact that \( \tilde{\phi}(p) > 0 \) (there is a gap in the spectrum for the non-critical parameters). Similarly, we obtain \( \eta(q) \to 0, \xi_l \to 1 \) at \( n \to \infty \). For the derivation of these formulas together with the more precise asymptotics at \( n \to \infty \), see [37]. Another way to obtain the asymptotics in the limit of infinite length of the chain is given in [38].

In the limit of the infinite XY-chain formulas (31), (32) for the matrix elements of spin operators between the eigenstates |\( \Phi_0 \rangle_{XY} = |q_1, q_2, \ldots, q_L \rangle_{NS} \rangle_{N} and |\( \Phi_1 \rangle_{XY} = |p_1, p_2, \ldots, p_L \rangle_{R} \rangle_{N} from the NS-sector and the \( R \)-sector, and (20) become

\[
|XY\langle \Phi_0 | \sigma^z_n | \Phi_1 \rangle_{XY} |^2 = \Xi_{\Phi_0, \Phi_1} \frac{2\alpha}{1 + \alpha} \cos^2 \frac{\gamma(q_k) - \gamma(p_l)}{2},
\]

\[
|XY\langle \Phi_0 | \sigma^z_n | \Phi_1 \rangle_{XY} |^2 = \Xi_{\Phi_0, \Phi_1} \frac{2\alpha}{1 - \alpha} \sinh^2 \frac{\gamma(q_k) - \gamma(p_l)}{2},
\]

\[
\Xi_{\Phi_0, \Phi_1} = \frac{\xi}{K} \prod_{k=1}^{K} \frac{1}{n \sinh \gamma(q_k)} \prod_{l=1}^{L} \frac{1}{n \sinh \gamma(p_l)} \cdot \left( \frac{t_q - t^{-1}_q}{t_q - t^{-1}_q} \right)^{(K-L)/2} \prod_{k<k}^{K} \frac{\sin^2 \frac{\gamma(q_k) - \gamma(q_l)}{2}}{\sinh^2 \gamma(q_k) \gamma(q_l)} \prod_{l<l}^{L} \frac{\sin^2 \frac{\gamma(p_l) - \gamma(p_l)}{2}}{\sinh^2 \gamma(p_l) \gamma(p_l)} \prod_{i<j}^{K \times K} \frac{\sin^2 \frac{\gamma(q_i) - \gamma(q_j)}{2}}{\sinh^2 \gamma(q_i) \gamma(q_j)},
\]

where relation (16) between \( \gamma(q) \) and the energy of excitation \( \epsilon(q) \) in the XY-chain and relation (23) are also to be used.

In the ferromagnetic phase \( (0 \leq h < 1) \) after an appropriate modification as explained in section 4.2, the thermodynamic limit formulas (34) and (35) at \( K = L = 0 \) allow us to re-obtain [38] formulas for the spontaneous magnetization found in [17]. Indeed, in the thermodynamic limit \( n \to \infty \), the energies of the vacuum states |\( \Phi_0 \rangle_{XY} = |\rangle_{NS} \rangle_{XY} and |\( \Phi_1 \rangle_{XY} = |\rangle_{R} \rangle_{XY} of different sectors asymptotically coincide giving the degeneration of the ground state of the Hamiltonian. It leads to a non-zero value of vacuum Bogoliubov quasi-average of \( \sigma^z \) which is spontaneous magnetization:

\[
\langle \sigma_x \rangle_{XY} = \frac{R}{XY} |\langle \rho \rangle_{XY} |_{NS} \rangle_{XY}, \quad \langle \sigma^z \rangle_{XY} = \sqrt{2} \left( \frac{x^2(1 - h^2)}{(1 + \alpha)^4} \right)^{1/8} \langle \sigma^z \rangle_{XY} = 0.
\]

The matrix element \( \frac{R}{XY} |\langle \rho \rangle_{XY} |_{NS} \rangle_{XY} gives spontaneous magnetization because it appears as a zero-particle contribution from the \( R \)-sector in the long-distance expansion of the two-point correlation function \( |\langle \sigma^z_0 \sigma^z_k \rangle_{XY} | \) in the ferromagnetic phase.

6. Asymptotics of the two-point correlation function of the infinite XY-chain

We are interested in the asymptotics of the two-point correlation function when the distance \( d \) between the correlating spins \( \sigma_x^z \) and \( \sigma_y^z \) is large (and, of course, the length \( n \) is much larger). We show how to re-derive the formula from [3] without the use of the Szegő theorem and Wiener–Hopf method for finding the asymptotics of Toeplitz determinant.

For definiteness we consider the paramagnetic phase. In this case, the intermediate states between two spin operators are odd-number particle states from the \( R \)-sector. Let us show that the main contribution is due to one-particle states (the contribution of many particle states is exponentially suppressed for large \( d \)). We have\(^2\) in this section, all the eigenstates of the XY-chain Hamiltonian are labeled only by \( NS \) or \( R \) depending on the sector.
\[ |\sigma_0 \sigma_d^j| = \sum_{p \in \mathbb{R}} |\langle \sigma_0 | p \rangle_{\mathbb{R}} \cdot \mathcal{R} (p | \sigma_d^j) |_{\text{NS}} \]
\[ + \sum_{\{p_1 < p_2 < p_3\} \in \mathbb{R}} |\langle \sigma_0 | p_1, p_2, p_3 \rangle_{\mathbb{R}} \cdot \mathcal{R} (p_1, p_2, p_3 | \sigma_d^j) |_{\text{NS}} + \cdots \]
\[ = \sum_{p \in \mathbb{R}} e^{ipd} |\langle \sigma_0 | p \rangle_{\mathbb{R}}|^2 + \sum_{\{p_1 < p_2 < p_3\} \in \mathbb{R}} e^{ip_1 + p_2 + p_3 d} |\langle \sigma_0 | p_1, p_2, p_3 \rangle_{\mathbb{R}}|^2 + \cdots, \]

where \(|\_\rangle_{\text{NS}}\) is the vacuum state. The idea is to make a transformation which corresponds to the change of the direction of evolution to the transverse direction in the lattice formulation of the Ising model. Let us estimate the \(L\)-particle contribution (in the limit \(n \to \infty\) we use integrals instead of sums):
\[
\cosh^2 K^+ \sum_{\{p_1 < \cdots < p_L\} \in \mathbb{R}} e^{i(p_1 + \cdots + p_L d)} |\langle \sigma_0 | p_1, \ldots, p_L \rangle_{\mathbb{R}}|^2 = \xi \left( \frac{t_y - t_{y^{-1}}}{t_x - t_{x^{-1}}} \right)^{L/2}
\times \frac{1}{(2\pi)^L L!} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \prod_{l=1}^{L} \frac{dp_l}{\sinh \gamma(p_l)} e^{idL \sum_{l=1}^{L} p_l}
\times \cosh^2 \sum_{l=1}^{L} \gamma(p_l) \prod_{l \neq \ell} \frac{\sin^2 \frac{p_{\ell} - p_{\ell^{-1}}}{2}}{\sinh^2 \frac{\gamma(p_{\ell}) + \gamma(p_{\ell^{-1}})}{2}}. \tag{37} \]

Now, we change the variables in the integrals from the momenta \(\{p_l\}\) to parameters \(\{u_l\}\) on the elliptic curve by means of relations from section 3.4 and from [32]:
\[
du_l = \frac{t_y^{-1} - t_y}{4} \frac{dp_l}{\sinh \gamma(p_l)}, \quad \frac{\sin \frac{u_{\ell} - u_{\ell^{-1}}}{2}}{\sinh \frac{\gamma(p_{\ell}) + \gamma(p_{\ell^{-1}})}{2}} = \frac{t_y^{-1} - t_y}{2k} \sinh(u_l - u_{l^{-1}}, k^{-1}).
\]

Thus, the \(L\)-particle contribution is
\[
\xi \left( \frac{t_y - t_{y^{-1}}}{t_x - t_{x^{-1}}} \right)^{L/2} \frac{1}{(2\pi)^L L!} \frac{4^{L-1}}{(t_y^{-1} - t_y)^L} \left( \frac{t_x^{-1} - t_x}{2k} \right)^{L(L-1)} \int_0^{2K} \cdots \int_0^{2K} \left( \prod_{l \neq \ell} \frac{u_{\ell} - u_{\ell^{-1}}}{2k} \right)^2 \prod_{l=1}^{L} \frac{\sin^2(u_l - u_{l^{-1}}, k^{-1})}{2k}.
\]

Shifting the contours of the integrations by \(-iK'/2\) we obtain
\[
\xi \left( \frac{t_y - t_{y^{-1}}}{t_x - t_{x^{-1}}} \right)^{L/2} \frac{1}{(2\pi)^L L!} \frac{4^{L-1}}{(t_y^{-1} - t_y)^L} \left( \frac{t_x^{-1} - t_x}{2k} \right)^{L(L-1)} \int_0^{2K} \cdots \int_0^{2K} \left( \prod_{l \neq \ell} \frac{u_{\ell} - u_{\ell^{-1}}}{2k} \right)^2 \prod_{l=1}^{L} \sin^2(u_l - u_{l^{-1}}, k^{-1}).
\]

With the use of the relations \(\lambda(u_l) = e^{ip_l}, z(u_l) = e^{-ip}(p_l),\)
\[
du_l = \frac{t_y^{-1} - t_y}{4} \frac{dp_l}{\sinh \gamma(p_l)}, \quad \frac{\sin \frac{u_{\ell} - u_{\ell^{-1}}}{2}}{\sinh \frac{\gamma(p_{\ell}) + \gamma(p_{\ell^{-1}})}{2}} = \frac{t_y^{-1} - t_y}{2k} \sinh(u_l - u_{l^{-1}}, k^{-1}),
\]
the $L$-particle contribution to the two-point correlation function can be rewritten in the terms of energies and momenta corresponding to the transverse direction of the Ising lattice:

$$
\xi \left( \frac{t_z - t_z^{-1}}{t_y - t_y^{-1}} \right)^{L/2} \frac{1}{(2\pi)^L L!} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{dp_1}{\sinh \tilde{\gamma}(p_1)} \cdots \frac{dp_L}{\sinh \tilde{\gamma}(p_L)}
$$

$$
\times e^{-d \sum_{l=1}^L \tilde{\gamma}(p_l) \cos^2 \frac{\sum_{l=1}^L p_l}{2} \sin^2 \frac{p_y - p_y}{2} \sin^2 \frac{p_z - p_z}{2}}. \tag{38}
$$

In the limit $d \to \infty$, the main contribution of these integrals is given by the neighborhood of zero momenta, where $\tilde{\gamma}(p)$ has a minimum. When the parameters of the model are non-critical, we have $0 < \exp(-\tilde{\gamma}(0)) = a^2 < 1$ and the asymptotics of the $L$-particle contribution to the two-point correlation function is proportional to $e^{-d L \tilde{\gamma}(0)}$. Thus, the one-particle contribution dominates in the paramagnetic phase.

From (37) for the one-particle contribution, we have

$$
cosh^2 K_x^2 \langle \sigma_0^x \sigma_L^2 \rangle \simeq \frac{\xi}{2\pi} \left( \frac{t_z - t_z^{-1}}{t_y - t_y^{-1}} \right)^{1/2} \int_{-\pi}^{\pi} dp \, e^{ipd} \coth \frac{\gamma(p)}{2}. \tag{39}
$$

Now, we take into account (14), (15) in order to rewrite the formula in terms of $a$ and $b$. With $z = e^{ip}$, we have

$$
\coth \frac{\gamma(p)}{2} = \frac{a}{b} s_p = \frac{a}{b} \left( \frac{(b^2 - z)(b^2 - z^{-1})}{(a^2 - z)(a^2 - z^{-1})} \right)^{1/2} = \sqrt{\frac{(b^2 - z)(b^2 - z)}{(a^2 - z)(a^2 - z)}}.
$$

We rewrite the integral as a contour integral:

$$
I(d) = \frac{1}{2\pi i} \int_{|z|=1} dz \, z^{-d-1} \sqrt{\frac{(b^2 - z)(b^2 - z)}{(a^2 - z)(a^2 - z)}}.
$$

To fix a branch of function under the integral we make two cuts: from $b^2$ to $a^2$ ($0 < b^2 < a^2 < 1$) and from $a^{-2}$ to $b^{-2}$ ($1 < a^{-2} < b^{-2}$). We move the contour to be along the cut from $b^2$ to $a^2$:

$$
I(d) = \frac{1}{\pi} \int_{b^2}^{a^2} ds \, s^{d-1} \frac{(s - b^2)(b^2 - s)}{(a^2 - s)(a^2 - s)}.
$$

Since we consider the limit $d \to \infty$, the main contribution to the integral is given by the neighborhood of $a^2$ due to the factor $s^{d-1}$. We change the variable of integration to $t = a^2 - s$:

$$
I(d) = \frac{1}{\pi} \int_0^{a^2 - b^2} dt \, (a^2 - t)^{d-1} \sqrt{\frac{(a^2 - b^2 - t)(b^2 - a^2 + t)}{a^2 - a^2 + t}}
$$

and use the expansion at $d \to \infty$ and $t \to 0$

$$(a^2 - t)^{d-1} = e^{(d-1)\log a^2 + \log(1 - t/a^2)} \sim a^{2(d-1)} e^{-(d-1)t/a^2}$$

to obtain the leading asymptotics by the steepest descent method:

$$
I(d) \sim a^{2(d-1)} (d - 1)^{-1/2} \sqrt{\frac{a^2(b^2 - b^2)(b^2 - a^2)}{\pi}(a^2 - a^2)}.
$$

Subsequent terms of the asymptotics for the one-particle contributions are given by (4.22) of [3]. Finally, the leading term of asymptotics of the two-point correlation function is

$$
\langle \sigma_0^x \sigma_L^2 \rangle \sim \frac{\xi}{2} \left( \frac{t_z - t_z^{-1}}{t_y - t_y^{-1}} \right)^{1/2} \frac{I(d)}{\cosh^2 K_x^2} \sim a^{2d} d^{-1/2} \pi^{-1/2} \left( \frac{1}{1 - b^2} \right)^{1/4}, \tag{39}
$$

15
where we used (21), (23) and (33). It coincides up to factor $(-1)^d/4$ with (4.25) of [3] after identification $\lambda_1 = b^{-2}$, $\lambda_2 = a^2$. The origin of this discrepancy factor is as follows. In [3] the spin operators are $1/2$ of Pauli matrices. It gives factor $1/4$ for the two-point correlation functions. Also in [3], the coefficients at $\sigma_i^x \sigma_{i+1}^x$ and $\sigma_i^y \sigma_{i+1}^y$ in the Hamiltonian are positive. In this paper, they are negative. To change the signs we make automorphism of the algebra of spin operators: $\sigma_i^x \rightarrow (-1)^d \sigma_i^x$, $\sigma_i^y \rightarrow (-1)^d \sigma_i^y$, $\sigma_i^z \rightarrow \sigma_i^z$. This map changes the signs of the mentioned coefficients and leads to the factor $(-1)^d$ in the two-point correlation function. To rewrite the correlation function (39) in terms of $h$ and $\kappa$ one needs (19).

Another way to calculate the two-point correlation function is to use the integral representation for the $L$-particle contribution (38) in terms of energies in the transverse direction on the Ising lattice:

$$\cosh^2 K_y^* \langle \sigma_i^x \sigma_{i+j}^x \rangle \sim \frac{\xi}{2\pi} \frac{t_y^{-1} - t_x}{t_y^{-1} - t_y} \int_{-\pi}^{\pi} dp \frac{e^{-d\tilde{\gamma}(p)}}{\sinh \tilde{\gamma}(p)} \cos^2 \frac{p}{2}.$$

$$\approx \frac{\xi}{2\pi} \frac{t_y^{-1} - t_x}{t_y^{-1} - t_y} \cdot \frac{1}{\sinh \tilde{\gamma}(0)} \int_{-\pi}^{\pi} dp \frac{e^{-d\tilde{\gamma}(0) + \tilde{\gamma}(0)p^2/2}}{\gamma(\tilde{\gamma}(p))}.$$

$$\approx \frac{\xi}{2\pi} \frac{t_y^{-1} - t_x}{t_y^{-1} - t_y} \cdot \frac{e^{-d\tilde{\gamma}(0)}}{d^{1/2} \sinh \tilde{\gamma}(0)} \sqrt{\frac{2\pi}{\gamma(\tilde{\gamma}(0))}}.$$

We also use (18) and (23)

$$\tilde{\gamma}(0) = \frac{t_x^{-1} - t_x}{t_y^{-1} - t_y}, \quad \frac{1}{\sinh \tilde{\gamma}(0)} = e^{-\tilde{\gamma}(0)} = a^2, \quad \sinh \tilde{\gamma}(0) = \frac{a^{-2} - a^2}{2}$$

to obtain by the steepest descent method the same asymptotics of the two-point correlation function (39).

To obtain more precise asymptotics, we change the variable of integration $s = \exp(-\tilde{\gamma}(p))$, $-ds/s = d\tilde{\gamma}(p)$ and use the dispersion relation (18) to obtain

$$\frac{dp}{\sinh \tilde{\gamma}(p)} = \frac{d\tilde{\gamma}}{\sin p} \cdot \frac{t_y^{-1} - t_x}{t_x^{-1} - t_x} = \frac{-ds}{s \sin \tilde{\gamma}(p)} \cdot \frac{t_y^{-1} - t_x}{t_x^{-1} - t_x}.$$

It gives

$$\frac{t_x^{-1} - t_x}{t_y^{-1} - t_y} \int_{-\pi}^{\pi} dp \frac{e^{-d\tilde{\gamma}(p) d}}{\sinh \tilde{\gamma}(p)} \cos^2 \frac{p}{2} = \int_{d/2}^{a^2} ds \frac{d^{1/2} \sinh \tilde{\gamma}(0)}{\sqrt{\frac{2\pi}{\gamma(\tilde{\gamma}(0))}}}.$$

From (18), we have

$$\cot \frac{p}{2} = \sqrt{\frac{(s - b^2)(b^2 - s)}{(a^2 - s)(a^2 - s)}}.$$

Substituting this expression into the integral we obtain the formula coinciding with formula (4.21) from [3]. It can be expanded as described there to obtain the subsequent terms of the asymptotics.

7. Discussion

Using the results of [21] on the separation of variables method for the Baxter–Bazhanov–Stroganov model, in this paper we derived matrix elements (form factors) of the spin operators $\sigma_i^x$ and $\sigma_i^y$ between the eigenvectors of the Hamiltonian (1) of finite (length $n$) XY-chain. The final formulas are (31), (32) and (20). In the limit of infinite chain the formulas are reduced
to (34), (35) and (36) and allow us to re-derive the asymptotics of the correlation function $\langle \sigma_x^0 \sigma_x^d \rangle$ at $d \to \infty$ in a simple way. It is interesting to derive analytically the long-time and long-distance asymptotic behavior of the $\sigma^z$ spin correlations at finite temperature observed recently in [39] by numeric analysis. Other asymptotic expansions relevant for equal-time pair correlations in the $XY$-model at a zero transverse field $h$ were found in [40].

Usually, the standard efficient way to study the Ising model and $XY$-chain is to use the language of fermions. Recently, the factorized formulas for the matrix elements of spin operators were re-derived for the quantum Ising chain [34], for the Ising model on the two-dimensional lattice [41], for the general free fermion model on the two-dimensional lattice and the quantum $XY$-chain [29] using the algebra of fermion operators.

Acknowledgments

The author thanks A Bugrij, A Klümper, O Lisovyy and V Shadura for helpful discussions. This work was partially supported by the Program of Fundamental Research of the Physics and Astronomy Division of the NAS of Ukraine, by Joint Ukrainian-Russian SFFR-RFBR project F40.2/108, by French-Ukrainian joint project PICS of CNRS and NAS of Ukraine.

References

[1] Lieb E, Schultz T and Mattis D 1961 Two soluble models of an antiferromagnetic chain Ann. Phys. 16 407–66
[2] Katsura S 1962 Statistical mechanics of the anisotropic linear Heisenberg model Phys. Rev. 127 1508–18
[3] Barouch E and McCoy B M 1971 Statistical mechanics of the XY model: II. Spin-correlation functions Phys. Rev. A 3 786–804
[4] Barouch E and McCoy B M 1971 Statistical mechanics of the XY model: III Phys. Rev. A 3 2137–40
[5] Barouch E, McCoy B M and Abraham D B 1971 Statistical mechanics of the XY model: IV. Time-dependent spin-correlation functions Phys. Rev. A 4 2331–41
[6] Izergin A G, Kapitnov V S and Kitanin N A 2000 Equal-time temperature correlators of the one-dimensional Heisenberg XY chain J. Math. Sci. 100 2120–40
[7] Kapitnov V S and Pronko A G 2003 Time-dependent temperature correlators of local spins of the one-dimensional Heisenberg XY chain J. Math. Sci. 115 2009–32
[8] Franchini F and Abanov A G 2005 Asymptotics of Toeplitz determinants and the emptiness formation probability for the XY spin chain J. Phys. A: Math. Gen. 38 5069–96
[9] Vidal G, Latorre J I, Rico E and Kitaev A 2003 Entanglement in quantum critical phenomena Phys. Rev. Lett. 90 227902
[10] Peschel I 2004 On the entanglement entropy for an XY spin chain J. Stat. Mech. P12005
[11] Its A R, Jin B-Q and Korepin V E 2005 Entanglement in the XY spin chain J. Phys. A: Math. Gen. 38 2975–90
[12] Franchini F, Its A R and Korepin V E 2008 Renyi entropy of the XY spin chain J. Phys. A: Math. Theor. 41 025302
[13] Franchini F, Its A R, Korepin V E and Takhtajan L A 2011 Spectrum of the density matrix of a large block of spins of the XY model in one dimension Quantum Inf. Process. 10 325–41
[14] Guo H L, Liu Z, Fan H and Chen S 2011 Correlation properties of anisotropic XY model with a sudden quench Eur. Phys. J. B 79 503–7
[15] Baxter R J 1989 Superintegrable chiral Potts model: thermodynamic properties, an ‘Inverse’ model, and a simple associated Hamiltonian J. Stat. Phys. 57 1–39
[16] Bazhanov V V and Stroganov Y G 1990 Chiral Potts model as a descendant of the six-vertex model J. Stat. Phys. 59 709–817
[17] Suzuki M 1971 Equivalence of the two-dimensional Ising model to the ground state of the linear XY-model Phys. Lett. A 34 94–5
[18] Bugrij A I, Iorgov N Z and Shadura V N 2005 Alternative method of calculating the eigenvalues of the transfer matrix of the $t_J$ model for $N = 2$ JETP Lett. 82 311–5
[19] von Gehlen G, Iorgov N, Pakuliak S and Shadura V 2006 The Baxter–Bazhanov–Stroganov model: separation of variables and the Baxter equation J. Phys. A: Math. Gen. 39 7257–82
[20] von Gehlen G, Iorgov N, Pakuliak S, Shadura V and Sykh Y 2007 Form-factors in the Baxter–Bazhanov–Stroganov model I: norms and matrix elements J. Phys. A: Math. Theor. 40 14117–38 (arXiv:0708.4342)
[21] von Gehlen G, Iorgov N, Pakuliak S, Shadura V and Tykhyy Y 2008 Form-factors in the Baxter–Bazhanov–Stroganov model: II. Ising model on the finite lattice J. Phys. A: Math. Theor. 41 095003 (arXiv:0711.0457)
[22] Sklyanin E 1985 The quantum Toda chain Lecture Notes Phys. 226 196–233
[23] Kharchev S and Lebedev D 2000 Eigenfunctions of $GL(N, \mathbb{R})$ Toda chain: the Mellin–Barnes representation JETP Lett. 71 235–8
[24] von Gehlen G, Iorgov N, Pakuliak S and Shadura V 2009 Factorized finite-size Ising model spin matrix elements from separation of variables J. Phys. A: Math. Theor. 42 304026 (arXiv:0904.2265)
[25] Bugrij A and Lisovyy O 2003 Spin matrix elements in 2D Ising model on the finite lattice Phys. Lett. A 319 390 (arXiv:0708.3625)
[26] Bugrij A and Lisovyy O 2004 Correlation function of the two-dimensional Ising model on a finite lattice: II JETP Lett. 79 48–51 (arXiv:0708.3643)
[27] Baxter R J 2010 Spontaneous magnetization of the superintegrable chiral Potts model: calculation of the determinant $D_{eq}$ J. Phys. A: Math. Theor. 43 145002 (arXiv:0912.4549)
[28] Iorgov N, Pakuliak S, Shadura V, Tykhyy Y and von Gehlen G 2010 Spin operator matrix elements in the superintegrable chiral Potts quantum chain J. Stat. Phys. 139 743–68 (arXiv:0912.4466)
[29] Iorgov N and Lisovyy O 2011 Finite-lattice form factors in free-fermion models J. Stat. Mech. P04011 (arXiv:1102.2145)
[30] Hoeger C, von Gehlen G and Rittenberg V 1985 Finite-size scaling for quantum chains with an oscillatory energy gap J. Phys. A: Math. Gen. 18 1813–26
[31] Korepanov I G 1987 Hidden symmetries in the 6-vertex model Archive VINITI No 1472-V87 Chelyabinsk Polytechnical Institute (in Russian)
Korepanov I G 1994 Hidden symmetries in the 6-vertex model of statistical physics Zap. Nauchn. Sem. POMI 215 163–77 (arXix:hep-th/9410066)
[32] Palmer J 2007 Planar Ising Correlations (Progress in Mathematics and Physics vol 49) (Boston, MA: Birkhäuser)
[33] Whittaker E T and Watson G N 1927 A Course of Modern Analysis (Cambridge: Cambridge University Press)
[34] Iorgov N, Shadura V and Tykhyy Y 2011 Spin operator matrix elements in the quantum Ising chain: fermion approach J. Stat. Mech. P02028 (arXiv:1011.2603)
[35] Kurmann J, Thomas H and Müller G 1982 Antiferromagnetic long-range order in the anisotropic quantum spin chain Physica A 112 235–55
[36] Müller G and Shoreck R E 1985 Implications of direct-product ground states in the one-dimensional quantum XYZ and XY spin chains Phys. Rev. B 32 5845–50
[37] Bugrij A I 2001 Correlation function of the two-dimensional Ising model on a finite lattice: I Theor. Math. Phys. 127 528–48
[38] Iorgov N 2010 Spontaneous magnetization of quantum XY-chain from finite chain form-factors Uke. J. Phys. 55 116–20 (http://www.ujp.bitp.kiev.ua)
[39] Krones J and Stolze J 2011 Exponential asymptotic spin correlations in XY chains arXiv:1102.4943
[40] Perk J H H and Au-Yang H 2009 New results for the correlation functions of the Ising model and the transverse Ising chain J. Stat. Phys. 135 599–619 (arXiv:0901.1931)
[41] Iorgov N and Lisovyy O 2011 Ising correlations and elliptic determinants J. Stat. Phys. 143 33–59 (arXiv:1012.2856)