FINSLEROID–FINSLER PARALLELISM

G.S. ASANOV

Division of Theoretical Physics, Moscow State University
119992 Moscow, Russia
(e-mail: asanov@newmail.ru)
Abstract

The Finsleroid–induced scalar product, and hence the angle, proves to remain unchanged under the Finsleroid–type parallel transportation of involved vectors in the Landsberg case. The two–vector extension of the Finsleroid metric tensor is proposed. 

Key words: Finsler geometry, metric spaces, angle, scalar product, parallelism, relativity.
1. Introduction and synopsis

The principal position of the Riemannian geometry is the phenomenon that the angle between vectors does not change under the parallel transportation of the vectors. The theory of connection in the Riemannian geometry is developed, and taught to students, subject to this observation. Can the phenomenon be transgressed from the Riemannian geometry to the Finsler geometry? No transparent and constructive answer is suggested by the content of current literature devoted to Finsler spaces (see the books [1–3]). This notwithstanding, quite certain positive answer proves to be a truth in the domain of the Finsleroid–Finsler geometry outlined in [4–8]. The answer is gained in the following succession of steps. Firstly, we use the Finsleroid–produced scalar product \( \langle y_1, y_2 \rangle_x \) obtained on attentive studying the equations of geodesics in tangent spaces. Secondly, we define the parallel displacement (1.1) of vector \( y^i \) with the respective spray–induced coefficients \( \bar{G}^i_m \). Thirdly, we apply the Landsberg–case spray coefficients. Lastly, we verify by straightforward calculations (which are short and easy) that such a procedure does not change the value of \( \langle y_1, y_2 \rangle_x \) and, hence, the Finsleroid angle.

Therefore, fixing the Landsberg case, we are entitled to conclude that the Finsleroid approach proves to overcome the vague opinion that in the Finsler geometry scientists may be “in principle equipped with only a family of Minkowski norms”, so that “yardsticks are assigned, but protractors are not”. They can be equipped also with a convenient family of the two–vector products \( \langle y_1, y_2 \rangle_x \), thereby with “protractors”!

Sometimes the lack of two–vector angle is even lifted “to a high level of the principle of vintage”. The author of the present paper (and not he alone) has heard and read many times of “the specific grounds that conclusively erase the concept of two–vector angle from the Finsler geometry, at least in the dimensions \( N > 2 \)”, and even of the deep–wisdom advises “better to forget of two–vector angle when opening the door to enter the Finsler geometry!”! Secs. 1.6 and 1.7 of H.Rund’s book [1] are tortured, much and much, with ephemeral definitions of trigonometric functions and angles...

Let us try only to be self–consistent! In the dimension \( N = 2 \), the two–vector angle \textit{does} enter the Finsler geometry, namely being the Landsberg angle. The obvious definition is the advantage of this angle, with quite a similar significance as in the two–dimensional Riemannian geometry, however in general there exists no possibility to represent the angle in the form of an explicit algebraic function of two vectors. So, in the dimension \( N = 2 \), the problem with angle is of analytical, not conceptual, nature, and we are to conclude that the two–dimensional Finsler geometry is a geometry!

Let us move in the dimensions \( N > 2 \). The Finslerian indicatrix, — the extension of the Euclidean unit sphere, — is at our disposal. Nobody prevents us from measuring angle between any two common–origin vectors by means of the length of the respective arcs cut by vectors (or their continuations) from unit circles located on the indicatrix, — just in compliance with the known Euclidean school methods. Again, the problem of getting the result may be only of analytical nature.

This circumstance thrusts forth new questions fundamental to the very Realm of the nowaday Finsler geometry: should we consider that geometry “old–fashioned” from the new advantageous standpoint that is proposed by the Finsleroid–induced geometry? The vantage–ground answer is “No” in many principle aspects, particularly the concepts of the Finslerian metric function and metric tensor, the Cartan tensor, the geodesics and spray coefficients, the significance and geometry of indicatrix, the nonlinear covariant derivative, the connection and curvature on the tangent bundle, the flag curvature, etc., are keeping fine. Simultaneously, the answer is decisively “Yes” in numerous new respects, including
the occurrence of the scalar product $\langle y_1, y_2 \rangle_x$ between two vectors from which many new categories of the cardinal geometrical nature proper are stemming up.

Among such categories, “the transportation preserving angle between two vectors” is notable, — and can be consistently and explicitly tractable. Indeed, we may define the Finsleroid–Finsler covariant differential $\delta y$ of a vector $y$ along (horizontal) $dx$ in the natural way

$$
\delta y^i := dy^i + \bar{G}^i_{k}(x, y)dx^k,
$$

(1.1)

where $\bar{G}^i_{k} = \frac{1}{2} \partial G^i / \partial y^k$ and $G^i := \gamma^i_{mn}y^my^n$ are the respective spray coefficients, with $\gamma^i_{mn}$ standing for the Finslerian Christoffel symbols constructed from the Finsleroid–Finsler metric function $K$. The vector $y$ is said to undergo the parallel transportation along $dx$ if

$$
\delta y = 0.
$$

(1.2)

Then analytically the condition for the scalar product $\langle y_1, y_2 \rangle_x$ to be unchanged under such a transportation of vectors $y_1, y_2$ reads

$$
\frac{\partial \langle y_1, y_2 \rangle_x}{\partial x^k} - \bar{G}^m_{k}(x, y_1) \frac{\partial \langle y_1, y_2 \rangle_x}{\partial y^m_1} - \bar{G}^m_{k}(x, y_2) \frac{\partial \langle y_1, y_2 \rangle_x}{\partial y^m_2} = 0.
$$

(1.3)

We claim

Parallel Transportation Theorem. In the Landsberg case of the Finsleroid–Finsler space, the condition (1.3) holds fine.

The proof is arrived at after direct calculations which are not lengthy, as will be demonstrated in Appendix A. Thus, both the scalar product (angle) of pair of vectors as well the parallel transportation retaining the product (angle) can nicely be transgressed from the Riemannian geometry to the Finsleroid–Finsler geometry in a simple analytical way. This parallelism in the Finsleroid domain comes to play replacing the Levi-Civita parallelism functioned conventionally in the Riemannian geometry.

In Finsler geometry, we have two concepts of vector length. Namely, we can use the Finslerian metric function $F = \sqrt{g_{ij}(x, y)y^iy^j}$ to assign the absolute length

$$
||y||_x = \sqrt{g_{ij}(x, y)y^iy^j}
$$

(1.4)

to the vector $y \in T_x M$. Simultaneously, taken another vector $\tilde{y}$ in the same tangent space, such that $\tilde{y}, y \in T_x M$, the Finsler geometry theory [1] provides us with the relative length

$$
||y||_{\tilde{y}} = \sqrt{g_{ij}(x, \tilde{y})\tilde{y}^i\tilde{y}^j}
$$

(1.5)

which measures the vector $y$ relative to a supporting vector $\tilde{y}$.

It is natural to wonder whether the length definitions (1.4) and (1.5) can be extended to give us respective notions of angles. The second case is extending in quite an obvious and traditional way as follows: at any fixed point $x$, we have the relative scalar product

$$
\langle y_1, y_2 \rangle_y = g_{ij}(x, y)y^i_1y^j_2
$$

(1.6)

in any Finsler space.

No possibility to extend properly the absolute case (1.4) is proposed in the books [1–3] (and in the current literature). However, the Finsleroid–Finsler geometry is wonderful in that it provides us with the following absolute scalar product:

$$
\langle y_1, y_2 \rangle_x = G_{ij}(x, y_1, y_2)y^i_1y^j_2
$$

(1.7)

in the Finsleroid–Finsler space.
The occurrence of the scalar product $\langle y_1, y_2 \rangle_x$ suggests naturally proposing two–vector extensions $Y_1(x, y_1, y_2)$, $Y_2(x, y_1, y_2)$, $G_{ij}(x, y_1, y_2)$, $A_{a;ijk}(x, y_1, y_2)$, $a = 1, 2$, of the ordinary Finslerian definitions of the covariant vector $y_i = K \partial K/\partial y^i$, the metric tensor $g_{ij} = g_{ij}(x, y)$, and the Cartan tensor $A_{ijk}(x, y)$. The explicit components of these extensions are found in Section 2. By their use, direct calculations (shown in Appendix A) reveal the validity of the following theorem.

**Match Theorem.** In the Finsleroid–Finsler space under study, the following limits are fulfilled:

\[
\lim_{y_2 \to y_1 = y} Y_{1i} = \lim_{y_2 \to y_1 = y} Y_{2i} = y_i
\]

and

\[
\lim_{y_2 \to y_1 = y} G_{ij} = g_{ij},
\]

together with

\[
\lim_{y_2 \to y_1 = y} A_{a;ijk} = A_{ijk}.
\]

Therefore, we are to expect that the Finsleroid geometry theory is not of primarily complete nature and should be regarded as a limiting ($y_1 = y_2$)–case of the respective two–vector extended theory to be developed in future. The important nature of the tensor $G_{ij}$ can be seen in the equality (1.7) which expresses the scalar product $\langle y_1, y_2 \rangle_x$ by means of the tensor.

Whether the lengths and scalar products (1.4)–(1.7) remain unchanged under the parallel transportation of the involved vectors? Due answers will be formulated in Section 3, yielding the **Total Category of Parallelism**, in which all the distant–parallelism concepts are meaningful as well as representable in an explicit and simple way, applying the Landsberg case.

Throughout the paper, the notation is the same as in the previous work [4–8], in which we have introduced the Finsleroid–Finsler space $\mathcal{FF}^F_D$ under the condition that the norm $||b||$ of the Finsleroid–axis 1-form

\[
b = b_i y^i
\]

is equal to 1:

\[
a_{ij}(x)b^i(x)b^j(x) = 1,
\]

where $a_{ij}$ stands for the metric tensor of the associated Riemannian space metricized by the function $S(x, y) = \sqrt{a_{ij}(x)y^i y^j}$ of point $x$ and tangent vector $y$. We shall normalize the fundamental Finsleroid–Finsler metric function $K$ to fulfill the condition

\[
g_{ij}(x, b(x)) = a_{ij}(x)
\]

which in turn entails

\[
K(x, b(x)) = 1.
\]

Thus, the Finsleroid–geometry properties come to play when the tangent vector $y^i$ begins deviating from the vector $b^i$. The conditions (1.13) and (1.14) assign actually the **correspondence principle** to make comparison between the Finsleroid–Finsler space and the associated Riemannian space. We have also

\[
g_{ij}(x, y)|_{g=0} = a_{ij}(x), \quad K(x, y)|_{g=0} = S(x, y).
\]
The equalities (1.12)–(1.15) are essential in developing our subject.

When proceeding in this direction, should the recent Finslerian theory of connection and curvature be recapitulated anew to comply strictly with the Levi–Civita idea? May the nonlinear methods of construction of covariant derivatives start coming to play significantly? All these questions are important and open to make deep inquiry.

The limitation of our parallel transportation theorem is that we use the spray coefficients $\bar{G}^{i}$ of the Landsberg case, — and the present author has not succeeded as yet in answering the troublesome question whether the conclusion can be extended to more general cases; some auxiliary calculations are presented in Appendix B.

The present paper deals everywhere with the positive–definite case. However, all the conclusions made can directly be re–formulated to apply to the relativistic pseudo–Finsleroid–Finsler space.

2. Scalar product and two–vector tensors

Below, we use a pair $y_1, y_2 \in T_x M$ of tangent vectors supported by a fixed point $x \in M$ of the background $N$–dimensional manifold $M$.

If a Finsler space involves a scalar product $\langle y_1, y_2 \rangle_x$ which possesses the homogeneity

$$\langle ky_1, y_2 \rangle_x = k \langle y_1, y_2 \rangle_x, \quad \langle y_1, ky_2 \rangle_x = k \langle y_1, y_2 \rangle_x, \quad k > 0, \forall y_1, y_2,$$

(2.1)

then it is attractive to explicate the two–vector covariant vectors

$$Y_{1i}(x, y_1, y_2) := \frac{\partial \langle y_1, y_2 \rangle_x}{\partial y_1^i}, \quad Y_{2j}(x, y_1, y_2) := \frac{\partial \langle y_1, y_2 \rangle_x}{\partial y_2^j}$$

(2.2)

and the two–vector metric tensor

$$G_{ij}(x, y_1, y_2) := \frac{\partial^2 \langle y_1, y_2 \rangle_x}{\partial y_1^i \partial y_2^j}$$

(2.3)

together with the following two–vector extension of the Cartan tensor:

$$A_{a;ijk}(x, y_1, y_2) = \langle y_1, y_2 \rangle_x C_{a;ijk}(x, y_1, y_2), \quad a = 1, 2,$$

(2.4)

with

$$C_{1,ki}(x, y_1, y_2) := \frac{\partial G_{ij}(x, y_1, y_2)}{\partial y_1^i}, \quad C_{2,ijk}(x, y_1, y_2) := \frac{\partial G_{ij}(x, y_1, y_2)}{\partial y_2^k}.$$

(2.5)

The homogeneity (2.1) entails obviously the identities

$$G_{ij}(x, y_1, y_2)y_1^i y_2^j = y_1^i Y_{1i}(x, y_1, y_2) = Y_{2i}(x, y_1, y_2)y_2^i = \langle y_1, y_2 \rangle_x$$

(2.6)

and

$$y_1^i G_{ij}(x, y_1, y_2) = Y_{2j}(x, y_1, y_2), \quad G_{ij}(x, y_1, y_2)y_2^i = Y_{1i}(x, y_1, y_2).$$

(2.7)

The generalized symmetry

$$G_{ij}(x, y_1, y_2) = G_{ji}(x, y_2, y_1)$$

(2.8)

is valid.
Also,
\[ y_1^k C_{1;kl}(x, y_1, y_2) = C_{2;lk}(x, y_1, y_2) y_2^k = 0. \] (2.9)

We shall mark quantities by the subscript ‘1’ if they are taken at the value \( y = y_1 \), resp. by the subscript ‘2’ at the value \( y = y_2 \), as exemplified by
\[ A_1 = A(x, y_1), A_2 = A(x, y_2), B_1 = B(x, y_1), B_2 = B(x, y_2), K_1 = K(x, y_1), K_2 = K(x, y_2). \] (2.10)

The Finsleroid–Finsler scalar product was presented explicitly by the formulas (2.33) and (2.34) in [7], such that
\[ \langle y_1, y_2 \rangle_x = K_1 K_2 \cos \alpha \] (2.11)
and
\[ \alpha_x = \frac{1}{h} \arccos \lambda, \] (2.12)
where
\[ \lambda = \frac{A_1 A_2 + h^2 r_{ij} y_1^i y_2^j}{\sqrt{B_1} \sqrt{B_2}}. \] (2.13)

Introducing the notation
\[ \gamma = \frac{\sin \alpha}{\sqrt{1 - \lambda^2}} \equiv \frac{\sin \alpha}{\sin(h \alpha)} \] (2.14)
(and avoiding indication of the subscript \( x \) for \( \alpha \)), we deduce the explicit components
\[ Y_{1i} = \frac{\partial K_1}{\partial y_1^i} K_2 \cos \alpha + \frac{\gamma}{h} K_1 K_2 \frac{\partial \lambda}{\partial y_1^i}, \quad Y_{2i} = K_1 \frac{\partial K_2}{\partial y_2^i} \cos \alpha + \frac{\gamma}{h} K_1 K_2 \frac{\partial \lambda}{\partial y_2^i}, \] (2.15)
and
\[ G_{ij} = \frac{1}{K_1} \frac{\partial K_1}{\partial y_1^i} Y_{2j} + \frac{\gamma}{h} K_1 \frac{\partial K_2}{\partial y_2^j} \frac{\partial \lambda}{\partial y_1^i} + \frac{\gamma}{h} K_1 K_2 \frac{\partial^2 \lambda}{\partial y_1^i \partial y_2^j} \] (2.16)
or
\[ G_{ij} = \frac{1}{K_1} \frac{\partial K_1}{\partial y_1^i} Y_{2j} + \frac{1}{K_2} \frac{\partial K_2}{\partial y_2^j} Y_{1i} - \frac{\partial K_1}{\partial y_1^i} \frac{\partial K_2}{\partial y_2^j} \cos \alpha + \frac{\gamma}{h} K_1 K_2 \frac{\partial^2 \lambda}{\partial y_1^i \partial y_2^j} + \frac{1}{h} K_1 K_2 \frac{\partial \gamma}{\partial \lambda} \frac{\partial \lambda}{\partial y_1^i} \frac{\partial \lambda}{\partial y_2^j}. \] (2.17)

### 3. Parallel transportation

Let us introduce the parallel transportation of a vector \( X \in T_x M \) along an infinitesimal (horizontal) displacement \( dx \) by following the known method described in Section 6.4 of [1]. Below, all the components \( g_{ij} \) and \( \bar{G}^h, \bar{G}^h_{ij}, \bar{G}^h_{ki}, \bar{G}^h_{kij} \) are implied to depend on the argument \((x, X)\). The notation \( \delta X \) features the covariant differential (1.1), so that
\[ \delta X^h = dX^h + \bar{G}^h_{k} dx^k \equiv dX^h + \bar{G}^h_{km} X^m dx^k. \] (3.1)

For the covariant vector
\[ X_h = g_{hk} X^k \] (3.2)
we take
\[ \delta X_h = dX_h - X_l \bar{G}^l_{kh} dx^k, \] (3.3)
such that
\[ \delta(X_h X^h) = d(X_h X^h). \] (3.4)
Proceeding in this way, we introduce the covariant differential of the Finslerian metric tensor

\[ \delta g_{ij} = \frac{\partial g_{ij}}{\partial x^k} dx^k + \frac{\partial g_{ij}}{\partial X^h} dX^h - g_{ih}\bar{G}^h_{jk} dx^k - g_{jh}\bar{G}^h_{ik} dx^k, \]  

(3.5)

which can also be written as

\[ \delta g_{ij} = \frac{\delta g_{ij}}{\delta x^k} dx^k + \frac{\partial g_{ij}}{\partial X^h} \delta X^h \]  

(3.6)

with

\[ \frac{\delta g_{ij}}{\delta x^k} = \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{ij}}{\partial X^h} \bar{G}^h_{ik} - g_{ih}\bar{G}^h_{jk} - g_{jh}\bar{G}^h_{ik}. \]  

(3.7)

Here the right–hand part can be transformed to yield

\[ \frac{\delta g_{ij}}{\delta x^k} = X_h \bar{G}^h_{kij}, \]  

(3.8)

so that

\[ \delta g_{ij} = \frac{\partial g_{ij}}{\partial X^h} \delta X^h + X_h \bar{G}^h_{kij} dx^k. \]  

(3.9)

It is well–known that

\[ \bar{G}^h_{kij} X^i = 0 \quad \text{in any Finsler space} \]  

(3.10)

and

\[ X_h \bar{G}^h_{kij} = 0 \quad \text{in the Landsberg case of Finsler space.} \]  

(3.11)

Accordingly, we introduce

DEFINITION. A vector \( X \) is said to be parallel under the displacement, if \( \delta X = 0 \).

Also, the metric tensor \( g_{ij} = 0 \) behaves parallel, if \( \delta g_{ij} = 0 \) when \( \delta X = 0 \).

NOTE. The equalities (3.7) and (3.8) are well–known from the book [1], in which they were discussed as “Berwald covariant derivative of the Finslerian metric tensor”, with the coefficients \( \bar{G}^h_{ik} \) being treated as the “Berwald connection coefficients” (see (3.10) of Section 3.3 in [1]; the coefficients were denoted in [1] to read simply \( G^h_{ik} \)). In our case, the above formulas (3.9) and (3.5) are tantamount to, respectively, Eqs. (4.17) and (4.16) of Section 6.4 of [1]; that Section was devoted to the nonlinear connection, so that we may qualify (3.1) by the status of the nonlinear covariant differential of vector.

By comparing (3.9) and (3.10) with (1.4) and (1.6), we just conclude that

\[ \delta |y|_x = 0 \quad \text{under} \quad \delta y = 0, \quad \text{in any Finsler space} \]  

(3.12)

and

\[ \langle y_1, y \rangle_y = 0 \quad \text{under} \quad \delta y = \delta y_1 = 0, \quad \text{in any Finsler space}, \]  

(3.13)

where

\[ \langle y_1, y \rangle_y = g_{ij}(x, y)y_i^1 y^j \equiv y_i^1 y_i. \]  

(3.14)

However, the assertions of the type (3.12) and (3.13) are not applicable to the full scalar products. The reason is that the products involve the Finslerian metric tensor \( g_{ij}(x, y) \) which, in contrast to the Riemannian metric tensor proper, depends on the transported vector \( y \). The parallelism property may be a truth in the particular case when the tensor \( g_{ij}(x, y) \) itself is unchanged under the parallel transportation of the argument
vector y. Let a set of vectors \( \tilde{y}, y_1, y_2 \) be supported by same point \( x \). In view of the nullification (3.11), the property said occurs as follows:

\[
\delta g_{ij} = 0 \quad \text{under} \quad \delta y = 0, \quad \text{in the Landsberg case of Finsler space.} \quad (3.15)
\]

This directly entails the assertion

\[
||y||_\tilde{y} = 0 \quad \text{if} \quad \delta \tilde{y} = \delta y = 0, \quad \text{in the Landsberg case of Finsler space.} \quad (3.16)
\]

This chain is continuing as follows:

\[
\delta \langle y_1, y_2 \rangle_y = 0 \quad \text{under} \quad \delta y = \delta y_1 = \delta y_2 = 0, \quad \text{in the Landsberg case of Finsler space,} \quad (3.17)
\]

and

\[
\delta \langle y_1, y_2 \rangle_x = 0 \quad \text{if} \quad \delta y_1 = \delta y_2 = 0, \quad \text{in the Landsberg case of the Finsleroid–Finsler space.} \quad (3.18)
\]

If we consider the relative angle

\[
\alpha_y(y_1, y_2) = \arccos \frac{\langle y_1, y_2 \rangle_y}{||y_1||_y ||y_2||_y} \quad \text{in any Finsler space} \quad (3.19)
\]

and the absolute angle

\[
\alpha_x(y_1, y_2) = \arccos \frac{\langle y_1, y_2 \rangle_x}{||y_1||_x ||y_2||_x} \quad \text{in the Finsleroid–Finsler space,} \quad (3.20)
\]

from the above we are entitled to conclude that

\[
\delta \alpha_y(y_1, y_2) = 0 \quad \text{under} \quad \delta y = \delta y_1 = \delta y_2 = 0, \quad \text{in the Landsberg case of Finsler space,} \quad (3.21)
\]

and

\[
\delta \alpha_x(y_1, y_2) = 0 \quad \text{if} \quad \delta y_1 = \delta y_2 = 0, \quad \text{in the Landsberg case of the Finsleroid–Finsler space.} \quad (3.22)
\]

Assuming the Landsberg case is essential.

Under conditions of the previous assertion (3.22), the two–vector objects (2.2) and (2.3) are also parallel:

\[
\delta G_{ij} = 0 \quad (3.23)
\]

and

\[
\delta Y_{1i} = \delta Y_{2i} = 0. \quad (3.24)
\]

Appendix A. Two–vector limits and parallelism condition

Let us apply the formulas (2.11)–(2.13) to verify the Match Theorem of Section 1. Using the derivatives

\[
\frac{\partial b}{\partial b_i} = y^i, \quad \frac{\partial q}{\partial b_i} = -\frac{b}{q} y^i, \quad \frac{\partial A}{\partial b_i} = \frac{q - \frac{1}{2} gb}{q} y^i, \quad \frac{\partial B}{\partial b_i} = \frac{g^2 - b^2}{q} y^i, \quad (A.1)
\]
we obtain
\[ \frac{\partial \lambda}{\partial b_i} = c_1 y_1^i + c_2 y_2^i \] (A.2)
with the coefficient
\[ c_1 = \frac{1}{q_1} \left[ A_2 \left( \frac{q_1}{2} - gb_1 \right) - h^2 q_1 b_2 \right] - \frac{g \lambda}{2B_1} (q_1^2 - b_1^2) \] (A.3)
which can be simplified to read
\[ c_1 = \frac{g}{2q_1} \left[ q_1 q_2 - b_1 b_2 + \frac{1}{2} g (q_1 b_2 - q_2 b_1) \right] - \frac{\lambda}{B_1} (q_1^2 - b_1^2) \] (A.4)
The quantity \( c_2 \) is obtainable from \( c_1 \) by performing the subscript interchange \( 1 \leftrightarrow 2 \).
With the help of the notation
\[ t_{Ai} = a_{in} y_A^n + \frac{1}{2} g (q_A b_i + \frac{b_A}{q_A} v_{Ai}) \]
we find that
\[ \frac{\partial \lambda}{\partial y_1^i} = \frac{A_2}{\sqrt{B_1} \sqrt{B_2}} - \lambda t_{1i} \]
\[ \frac{\partial \lambda}{\partial y_2^i} = \frac{A_1}{\sqrt{B_1} \sqrt{B_2}} - \lambda t_{2i} \] (A.6)
Contractions show that
\[ \frac{\partial \lambda}{\partial y_1^i} y_1^i = \frac{\partial \lambda}{\partial y_2^i} y_2^i = 0 \] (A.7)
and
\[ \frac{\partial \lambda}{\partial y_1^i} b_i = \frac{A_2}{\sqrt{B_1} \sqrt{B_2}} - \lambda A_1 \]
\[ \frac{\partial \lambda}{\partial y_2^i} b_i = \frac{A_1}{\sqrt{B_1} \sqrt{B_2}} - \lambda A_2 \] (A.8)
Appropriate differentiation yields
\[ \frac{\partial^2 \lambda}{\partial y_1^i \partial y_2^j} = \frac{b_{1i} b_{2j} + h^2 r_{ij}}{\sqrt{B_1} \sqrt{B_2}} - \frac{b_{1i} A_2 + h^2 v_{2j}}{B_2 \sqrt{B_1} \sqrt{B_2}} t_{2j} - \frac{1}{B_1} t_{1i} \left( \frac{b_{2j} A_1 + h^2 v_{1j}}{\sqrt{B_1} \sqrt{B_2}} - \frac{\lambda}{B_2} t_{2j} \right) \] (A.9)
We may observe the properties
\[ \frac{\partial \lambda}{\partial y_1^i} \big|_{y_2 = y_1} = \frac{\partial \lambda}{\partial y_2^i} \big|_{y_2 = y_1} = 0 \] (A.10)
and
\[ \frac{\partial^2 \lambda}{\partial y_1^i \partial y_2^j} \big|_{y_2 = y_1 = y} = \left( \frac{b_i + \frac{1}{2} g v_i}{q} \right) \left( \frac{b_j + \frac{1}{2} g v_j}{q} \right) + h^2 r_{ij} \frac{t_i t_j}{B^2} \] (A.11)
where \( t_i = a_{in} y^n + \frac{1}{2} g (q b_i + \frac{b_i}{q} v_i) \). If we compare the right-hand part of (A.11) with the
structure of the Finsleroid angular metric tensor \( h_{ij} \) (see \([5,7]\)), we obtain the simple equality
\[
\frac{\partial^2 \lambda}{\partial y_1 \partial y_2} \bigg|_{y_2 = y_1 = y} = \frac{h^2}{K^2 h_{ij}}. \tag{A.12}
\]

Taking into account (A.11) and the nullifications (A.7), together with the limits
\[
\lim_{\lambda \to 1} \frac{\sin \alpha}{\sqrt{1 - \lambda^2}} = \frac{1}{h}, \quad \lim_{\lambda \to 1} \frac{\partial \sin \alpha}{\partial \lambda} = \frac{1 - h^2}{h^3}, \tag{A.13}
\]
we are entitled to conclude from the formulas (2.15)–(2.17) that the claimed limits (1.8)–(1.10) of the theorem are valid.

Now we turn to the Parallel Transportation Theorem of Section 1. Let us take two vectors \( y_1, y_2 \in T_x M \). To establish the vanishing (1.3), we must apply accurate calculations to verify that
\[
\frac{\partial \lambda}{\partial x^k} - \frac{1}{2} G^k_m(x, y_1) \frac{\partial \lambda}{\partial y^l_1} - \frac{1}{2} G^k_m(x, y_2) \frac{\partial \lambda}{\partial y^l_2} = 0 \tag{A.14}
\]
with the function \( \lambda \) given by (2.13), and with the Landsberg–case spray–induced coefficients
\[
G^i_k = \frac{g^k_1}{q} \left[ (u_k - b b_k) v^i + q^2 (\delta^i_k - b^i_k) \right] + 2 a^i_{km} y^m \tag{A.15}
\]
(these coefficients can be found in \([5–8]\)). Denoting
\[
G^i_{1,k} = G^i_k(x, y_1), \quad G^i_{2,k} = G^i_k(x, y_2),
\]
we obtain
\[
\frac{\partial \lambda}{\partial y^l_1} G^i_{1,k} = \frac{g^k_1}{q_1} \frac{\partial \lambda}{\partial y^l_1} \left[ (q_1)^2 \delta^i_k - b^i m_{1,k} \right] + \Delta = g k q_1 \frac{\partial \lambda}{\partial y^l_1} - \frac{g k}{q_1} m_{1,k} \frac{\partial \lambda}{\partial y^l_1} b^i + \Delta, \tag{A.16}
\]
where \( m_{1,k} = b_1 v_{1,k} + (q_1)^2 b_k \) and \( \Delta \) symbolizes the summary of the terms which involve partial derivatives of the input Riemannian metric tensor \( a_{ij} \) with respect to the coordinate variables \( x^k \). On simplifying and applying (A.6), the right–hand part in (A.16) becomes
\[
g k \left[ \frac{q_1 \frac{\partial \lambda}{\partial y^l_1} + \frac{1}{q_1} m_{1,k} \frac{\lambda A_1}{B_1} - \frac{1}{q_1} m_{1,k} \frac{A_2}{B_1 \sqrt{B_2}}}{\sqrt{B_1 \sqrt{B_2}}} \right] + \Delta = g k \left[ \frac{b_k + \frac{1}{2} g \frac{v_{1,k}}{q_1} A_2 + h^2 v_{2,k}}{\sqrt{B_1 \sqrt{B_2}}} - \frac{q_1 \lambda A_1}{B_1} - \frac{1}{q_1} m_{1,k} \frac{A_2}{\sqrt{B_1 \sqrt{B_2}}} \right] + \Delta.
\]
Here, all the terms proportional to \( b_k \) are cancelled, leaving us with
\[
g k \left[ \frac{1}{2} \frac{g q_1 A_2 v_{1,k} + h^2 (q_1)^2 v_{2,k}}{\sqrt{B_1 \sqrt{B_2}}} - \frac{q_1 \lambda A_1}{B_1} + \frac{1}{q_1} m_{1,k} \frac{A_2}{\sqrt{B_1 \sqrt{B_2}}} \right] + \Delta.
\]
Eventually,
\[
\frac{\partial \lambda}{\partial y^l_1} G^i_{1,k} = \frac{g k}{q_1} \left[ \frac{1}{2} g q_1 A_2 v_{1,k} + h^2 (q_1)^2 v_{2,k} - \frac{q_1 \lambda (q_1 + \frac{1}{2} g b_1) v_{1,k}}{B_1} + \frac{\lambda A_1 b_1 v_{1,k}}{B_1} - \frac{A_2 b_1 v_{1,k}}{\sqrt{B_1 \sqrt{B_2}}} \right] + \Delta. \tag{A.17}
\]
Interchanging here \(i \leftrightarrow 2\) yields the quantity \(\frac{\partial \lambda}{\partial y^2} G^i_k\). Now, using the characteristic Landsberg condition \(\nabla_i b_j = k(a_{ij} - b_ib_j)\), we get

\[
\frac{\partial \lambda}{\partial x^k} = k(c_1 v_{1k} + c_2 v_{2k}) + \Delta,
\]

where (A.4) should be used. With the formulas (A.17) and (A.18), the validity of the vanishing (A.14) can readily be seen.

**Appendix B. Use of full spray coefficients**

Suppressing the Landsberg condition, the full spray coefficients are given by the representation (A.48) of [7] which yields

\[
G^i_k = g P_{1k} v^i_1 + g Q_1 (\delta^i_k - b^i b_k) - \frac{v^i_k}{q_1} f^i_1 - g q_1 f^i_1 + 2 a^i_{km} y^m_1,
\]

where

\[
P_{1k} = -\frac{1}{q_1} v_{1k} y^i_1 \nabla^i b_h + \frac{1}{q_1} y^i_1 (\nabla^i b_h + \nabla^k b_k) + g b^i \nabla^i b_h
\]

and

\[
Q_1 = \frac{1}{q_1} y^i_1 y^h \nabla^i b_h + g y^h b^i \nabla^i b_h.
\]

We use the notation

\[
f^i = f^i_n y^n, \quad f^i_n = a^{ik} f_{kn}, \quad f_{mn} = \nabla_m b_n - \nabla_n b_m = \frac{\partial b_n}{\partial x^m} - \frac{\partial b_m}{\partial x^n},
\]

where the nabla means the covariant derivative in terms of the associated Riemannian space. We obtain

\[
\frac{\partial \lambda}{\partial y^i_1} G^i_k = g P_{1k} v^i_1 \frac{\partial \lambda}{\partial y^i_1} + g Q_1 (\delta^i_k - b^i b_k) \frac{\partial \lambda}{\partial y^i_1} - \frac{v^i_k}{q_1} f^i_1 \frac{\partial \lambda}{\partial y^i_1} - g q_1 f^i_1 \frac{\partial \lambda}{\partial y^i_1} + \Delta,
\]

or

\[
\frac{\partial \lambda}{\partial y^i_1} G^i_k = g Q_1 \frac{\partial \lambda}{\partial y^k} + g P_{1k} \frac{\partial \lambda}{\partial y^i_1} v^i_1 - g Q_1 b^i_k \frac{\partial \lambda}{\partial y^i_1} b^j - \frac{v^i_k}{q_1} \frac{\partial \lambda}{\partial y^i_1} f^i_1 - g q_1 \frac{\partial \lambda}{\partial y^i_1} f^i_1 + \Delta.
\]

The representation (B.6) extends the formula (A.16) of the preceding Appendix A.

Now we start calculating in the straightforward way:

\[
\frac{\partial \lambda}{\partial y^i_1} G^i_k = g Q_1 \frac{\partial \lambda}{\partial y^k} + g P_{1k} \frac{\partial \lambda}{\partial y^i_1} v^i_1 - g Q_1 b^i_k \frac{\partial \lambda}{\partial y^i_1} b^j - \frac{v^i_k}{q_1} \frac{\partial \lambda}{\partial y^i_1} f^i_1 - g q_1 \frac{\partial \lambda}{\partial y^i_1} f^i_1 + \Delta,
\]

or after required insertions

\[
\frac{\partial \lambda}{\partial y^i_1} G^i_k = g Q_1 \left( \frac{b_{1k} A_2 + h^2 v_{2k}}{\sqrt{B_1} \sqrt{B_2^2}} - \frac{\lambda}{B_1} t_{1k} \right) + g P_{1k} \left( \frac{b_{1i} A_2 + h^2 v_{2i}}{\sqrt{B_1} \sqrt{B_2^2}} - \frac{\lambda}{B_1} t_{1i} \right) v^i_1
\]

\[
- g Q_1 b^i_k \left( \frac{b_{1i} A_2 + h^2 v_{2i}}{\sqrt{B_1} \sqrt{B_2^2}} - \frac{\lambda}{B_1} t_{1i} \right) b^j - \frac{v^i_k}{q_1} \left( \frac{b_{1i} A_2 + h^2 v_{2i}}{\sqrt{B_1} \sqrt{B_2^2}} - \frac{\lambda}{B_1} t_{1i} \right) f^i_1
\]
\[-g q_1 \left( \frac{b_{1i} A_2 + h^2 v_{2i}}{\sqrt{B_1 \sqrt{B_2}}} - \frac{\lambda}{B_1} t_{1i} \right) f^i_k + \Delta, \]

where the notation (A.5) has been applied. Simplifying yields

\[
\frac{\partial \lambda}{\partial y_i} G_{1}^i k = g Q_1 \left\{ \frac{1}{2} \frac{g v_{1k}}{q_1} A_2 + h^2 v_{2k} - \frac{\lambda}{B_1} \left( 1 + \frac{1}{2} g \frac{b_{1i}}{q_1} \right) v_{1k} \right\} + g P_{1k} \left\{ \frac{1}{2} \frac{g v_{1i}}{q_1} A_2 + h^2 v_{2i} - \frac{\lambda}{B_1} \left( v_{1i} + \frac{1}{2} g \frac{b_{1i}}{q_1} v_{1i} \right) \right\} v^i_i
\]

or

\[
\frac{\partial \lambda}{\partial y_i} G_{1}^i k = g Q_1 \left[ \frac{1}{2} \frac{g v_{1k}}{q_1} A_2 + h^2 v_{2k} - \frac{\lambda}{B_1} \left( 1 + \frac{1}{2} g \frac{b_{1i}}{q_1} \right) v_{1k} \right] + g P_{1k} \left[ \frac{(A_1 - b_{1i}) A_2 + h^2 r_{ij} y_{1j} y_{2j} - h^2 b_{1i} b_{2j}}{\sqrt{B_1 \sqrt{B_2}}} - \frac{\lambda}{B_1} \left( q_{1i}^2 + \frac{1}{2} g b_{1i} q_{1j} \right) \right] f^i k
\]

Finally,
\[
\frac{\partial \lambda}{\partial y_1^i} G_{1}^{i} = g Q_1 \left[ \frac{1}{2} g \frac{v_{1k}}{q_1} A_2 + h^2 v_{2k} \right] \frac{1}{\sqrt{B_1} \sqrt{B_2}} - \frac{\lambda}{B_1} \left( 1 + \frac{1}{2} g \frac{b_1}{q_1} \right) v_{1k} \\
+ g b_1 P_{1k} \left[ - \frac{A_2 + h^2 b_2}{\sqrt{B_1} \sqrt{B_2}} + \frac{\lambda}{B_1} A_1 \right] - g \frac{v_{1k}}{q_1} \frac{h^2}{\sqrt{B_1} \sqrt{B_2}} f^i_{u_2i} \\
- g \frac{v_{1k}}{q_1} \left[ \left( 1 - \frac{1}{2} g \frac{b_1}{q_1} \right) A_2 - h^2 b_2 \right] \frac{1}{\sqrt{B_1} \sqrt{B_2}} - \frac{\lambda}{B_1} \frac{1}{2} g \left( q_1 - \frac{b_1}{q_1} \right) f^i_{b_1} \\
+ g \left[ q_2 + \frac{1}{2} g b_2 \right] \frac{\lambda}{B_1} \left( q_1 + \frac{1}{2} g b_1 \right) f_{ik} - g \left[ \frac{(q_1 - \frac{1}{2} g b_1) A_2 - h^2 b_2 q_1}{\sqrt{B_1} \sqrt{B_2}} - \frac{\lambda}{B_1} \frac{1}{2} g \left( q_1^2 + b_1^2 \right) \right] f^i_{b_1} + \Delta.
\]

(B.9)

REFERENCES

[1] H. Rund: *The Differential Geometry of Finsler Spaces*, Springer, Berlin 1959.
[2] G.S. Asanov: *Finsler Geometry, Relativity and Gauge Theories*, D. Reidel Publ. Comp., Dordrecht 1985.
[3] D. Bao, S.S. Chern, and Z. Shen: *An Introduction to Riemann-Finsler Geometry*, Springer, N.Y., Berlin 2000.
[4] G.S. Asanov: Finsleroid space with angle and scalar product, *Publ. Math. Debrecen* 67 (2005), 209-252.
[5] G.S. Asanov: Finsleroid–Finsler space with Berwald and Landsberg conditions, *arXiv:math.DG/0603472* (2006).
[6] G.S. Asanov: Finsleroid–Finsler space and spray coefficients, *arXiv:math.DG/0604526* (2006).
[7] G.S. Asanov: Finsleroid–Finsler spaces of positive–definite and relativistic types, *Rep. Math. Phys.* 58 (2006), 275–300.
[8] G.S. Asanov: Finsleroid–Finsler space and geodesic spray coefficients, *Publ. Math. Debrecen* 70 (2006) (to appear).