EQUALITY OF ORDERS OF A SET OF INTEGERS MODULO A PRIME

OLLI JÄRVINIEMI

ABSTRACT. For finitely generated subgroups $W_1, \ldots, W_t$ of $\mathbb{Q}^\times$, integers $k_1, \ldots, k_t$, a Galois extension $F$ of $\mathbb{Q}$ and a union of conjugacy classes $C \subset \text{Gal}(F/\mathbb{Q})$, we develop methods for determining if there exists infinitely many primes $p$ such that the index of the reduction of $W_i$ modulo $p$ divides $k_i$, and such that the Artin symbol of $p$ on $F$ is contained in $C$. The results are a multivariable generalization of H.W. Lenstra’s work. As an application, we determine all integers $a_1, \ldots, a_n$ such that $\text{ord}_p(a_1) = \ldots = \text{ord}_p(a_n)$ for infinitely many primes $p$. We also discuss the set of those $p$ for which $\text{ord}_p(a_1) > \ldots > \text{ord}_p(a_n)$. The obtained results are conditional to a generalization of the Riemann hypothesis.

1 Introduction

H.W. Lenstra [6] has considered the following problem:

Let $K$ be a global field, $W$ an infinite, finitely generated subgroup of $K^\times$, $F$ a finite Galois extension of $K$, $C$ a union of conjugacy classes of $\text{Gal}(F/K)$, and $k$ a positive integer. Are there infinitely many primes $p$ such that the index of the reduction of $W$ modulo $p$ divides $k$, and the Artin symbol of $p$ on $F$ is contained in $C$?

Denote this set of primes $p$ by $M = M(K, F, C, W, k)$. Assuming a suitable generalization of the Riemann hypothesis (GRH) Lenstra determines a necessary and sufficient condition for $M$ to be infinite. One way to formulate this condition is that $M$ is finite if and only if there is an obstruction at a finite level. This is discussed in more detail later.

The results are general, and can be applied, for example, to determine when there are infinitely many $p$ such that a given integer $a$ is a primitive root modulo $p$ (under GRH). The famous Artin’s primitive conjecture is that this set is infinite for all $a$ not equal to $-1$ or a square. Furthermore, it is conjectured that this set has a positive natural density. Hooley [4] was the first to prove that the conjecture holds under GRH. While the Artin’s primitive conjecture is still a conjecture, Heath-Brown [3] has unconditionally proven results such as that there are at most two primes $a$ for which there are only finitely many desired primes $p$. For a comprehensive survey on the conjecture and related problems, see [10].

There is still some room for generalizations of Lenstra’s work. For example, a natural question on orders of integers modulo primes is “are there infinitely many primes $p$ such that $\text{ord}_p(2) = \text{ord}_p(3)$?”, and this does not directly follow from Lenstra’s results. The Schinzel-Wójcik problem asks to determine all integers $A_1, \ldots, A_n$ such that for infinitely many primes $p$ we have $\text{ord}_p(A_1) = \text{ord}_p(A_2) = \ldots = \text{ord}_p(A_n)$. Schinzel and Wójcik [11]

\[1\] In the context of the Artin conjecture, by GRH it is often meant that the zeros of the Dedekind zeta-functions of number fields having real part between 0 and 1 have real part $\frac{1}{2}$. This generalization of the Riemann hypothesis is sometimes referred to as the extended Riemann hypothesis (ERH). For $K$ a function field the required Riemann hypothesis has been proven, but for number fields it has been not.
solved the problem for \( n = 2 \). In this case, there exists infinitely many such \( p \) as long as \( |A_1|, |A_2| > 1 \). Their argument is elementary, though highly nontrivial.

Pappalardi and Susa \[8\] have proven that under GRH the density of these \( p \) exists for any \( A_i \). However, their results do not provide a condition for the infinitude of these primes. In \[8\], proposition 14, it is proven that if \( A_i \) satisfy certain properties, there exists only finitely many such \( p \). We prove that, assuming GRH, this is the only obstruction, and in other cases there are infinitely many desired \( p \). If there are infinitely many such \( p \), then they also have positive density.

While proving this type of results seems to be out of reach without assuming any unproven conjectures, GRH is not the only hypothesis whose assumption leads to progress. In \[13\] it has been proved that under the Schinzel Hypothesis H there are infinitely many desired \( p \) in the case \( n = 3 \) assuming that \( |A_i| > 1 \) and that the subgroup of \( \mathbb{Q}^\times \) generated by \( A_1, A_2 \) and \( A_3 \) does not contain \(-1\). Recently Anwar and Pappalardi \[1\] have determined a necessary and sufficient condition for the infinitude of \( p \) such that all of \( A_i \) are primitive roots modulo \( p \) under the Schinzel Hypothesis H. Matthews \[7\] has previously proven stronger density results for such \( p \) under GRH.

Our strategy to solve the Schinzel-Wójcik problem under GRH is to develop a generalization of Lenstra’s results for several groups \( W_1, \ldots, W_t \) in the place of a single group \( W \). We first present a brief overview of Lenstra’s work, after which we provide our generalization. We then apply our machinery to prove the following results.

**Theorem 1.1.** Assume GRH. Let \( A_1, \ldots, A_t \) be integers with absolute value greater than 1. There are infinitely many primes \( p \) such that \( \text{ord}_p(A_1) = \ldots = \text{ord}_p(A_t) \) if and only if at least one of the following statements is false:

1. There exist integers \( e_i \) such that \( \prod A_i^{e_i} = -1 \).
2. There exist integers \( f_i \) with an odd sum such that \( \prod A_i^{f_i} = 1 \).

Furthermore, if there are infinitely many such primes, then the density of such primes exists and is positive.

We quickly prove the necessity of the conditions of the theorem. If such integers \( e_i \) exist, and we have \( \text{ord}_p(A_i) = O \) for all \( i \), then \( O \) has to be even, as we have

\[
(-1)^O = \prod (A_i^{e_i})^O = 1 \pmod{p}.
\]

Similarly, if such integers \( f_i \) exist, we must have \( O \equiv 1 \pmod{2} \), as otherwise we would have

\[
1 = 1^{O/2} = \prod A_i^{f_i O/2} = \prod (-1)^{f_i} = -1 \pmod{p}.
\]

The conditions of Theorem 1.1 hold if \( A_i > 1 \) for all \( i \):

**Corollary 1.2.** Assume GRH. Let \( A_1, \ldots, A_t > 1 \) be integers. There are infinitely many primes \( p \) such that \( \text{ord}_p(A_1) = \ldots = \text{ord}_p(A_t) \). Furthermore, the density of such primes exists and is positive.

For positive \( A_i \) we may choose all of \( \text{ord}_p(A_i) \) to be equal to \( \left\lfloor \frac{p-1}{2} \right\rfloor \) above, and in general the index can be taken to be of the form \( 3 \cdot 2^k \) for a suitable \( k \).
Theorem 1.3. Assume GRH. Let $A_1, A_2, \ldots, A_t$ be integers which are pairwise multiplicatively independent over $\mathbb{Q}$. There are infinitely many primes $p$ such that $\text{ord}_p(A_1) > \text{ord}_p(A_2) > \ldots > \text{ord}_p(A_t)$. Furthermore, the density of such primes exists and is positive.

In other words, all $t!$ orderings of $\text{ord}_p(A_i)$ are possible under the assumption of pairwise independence. Without this assumption the statement need not hold. A couple of counterexamples are

- $\text{ord}_p(a^2) > \text{ord}_p(a)$,
- $\text{ord}_p(a) > \text{ord}_p(a^3) > \text{ord}_p(a^2)$, and
- $\text{ord}_p(a) > \text{ord}_p(b) > \text{ord}_p(b^2) > \text{ord}_p(a^2)$.

For more discussion on this problem, see Section 6. The proof of Theorem 1.3 actually gives the following stronger statement.

Theorem 1.4. Assume GRH. Let $A_1, A_2, \ldots, A_t$ be integers which are pairwise multiplicatively independent over $\mathbb{Q}$, and let $C > 1$ be a constant. There are infinitely many primes $p$ such that $\text{ord}_p(A_i) > C \text{ord}_p(A_{i+1})$ for all $1 \leq i < t$. Furthermore, the density of such primes exists and is positive.

The theorems do not require GRH for all numbers fields. It suffices to assume the GRH for number fields obtained by adjoining roots of integers and unity to $\mathbb{Q}$ (the Kummer extensions).

2 Lenstra’s work

We only cover those parts of Lenstra’s work which concern the number field case of the problem. Many of the details are omitted, and some are covered later when proving the generalization.

First, notation. Let $K, W, F, C, k$ and $M$ be as above. The letter $\ell$ will always denote a prime number. Let $q(\ell)$ be the smallest power of $\ell$ not dividing $k$. Define $L_\ell = \mathbb{Q}(\zeta_q(\ell), W^{1/q(\ell)})$, where $\zeta_q$ is a $q$th primitive root of unity and $W^{1/n}$ denotes the set $\{w^{1/n}, w \in W\}$. For squarefree $n$, define $q(n) = \prod_{\ell | n} q(\ell)$ and $L_n$ to be the compositum of $L_\ell, \ell \mid n$. Define $C_n$ to be the set of $\sigma \in \text{Gal}(FL_n/K)$ such that $\sigma|F \in C$ and $\sigma|L_n \neq \text{id}_{L_n}$ for all $\ell \mid n$. Here $\sigma|L$ denotes the restriction of $\sigma$ on $L$, and $\text{id}_{L_n}$ denotes the identity on $L$. Finally, define $a_n = \frac{|C_n|}{|\text{Gal}(FL_n/K)|}$.

Clearly for all $n|m$ we have $a_n \geq a_m \geq 0$. Therefore, the numbers $a_n$ have a limit $a = a(M)$ when $n$ ranges over the squarefree positive integers ordered by divisibility.\footnote{That is, for all $\epsilon > 0$ there exists an integer $N$ such that for all $n$ divisible by $N$ we have $|a - a_n| < \epsilon$.}

The conjecture is that the density $d(M)$ of $M$ (with respect to the set of primes of $K$) equals $a(M)$. The motivation is that for unramified $p$ we have $p \in M$ if and only if $(p|F) = C$ and $p$ splits in none of $L_\ell$, with $(p|F)$ being the Artin symbol.\footnote{If $p$ splits in $L_\ell$, then the index of $W$ is divisible by $q(\ell)$, which does not divide $k$. The other direction follows similarly. See Lenstra’s work for details.} By the Chebotarev density theorem $a_n$ equals to the density of those $p$ for which this holds for all $\ell|n$, so by taking limits one would expect to have $d(M) = a(M)$.

The case when $F = K$ and $C = \{\text{id}_K\}$ of this conjecture has been dealt before, for which Lenstra refers to [2]. This is the only step of the proof requiring GRH. Lenstra then proves the general case by reducing to the case $F = K'$ by an elementary argument.

Having proven $d(M) = a(M)$, Lenstra focuses on determining a condition for the positivity of $a(M)$. This is done in two parts. First, it is proven that if $a_n \neq 0$ for all $n$, then...
Let $L$ that the infinite product converges to a strictly positive number. This follows from the bound $[L_\ell : K] \geq \ell(\ell - 1)$, which holds for large enough $\ell$.

\section{Generalization for several groups}

The notation is similar to that of Lenstra. Let $W_1, \ldots, W_t$ be infinite, finitely generated subgroups of $Q^\times$. Let $k_1, \ldots, k_t$ be positive integers, $F$ a finite Galois extension of $Q$, and $C$ a conjugacy class of $\text{Gal}(F/Q)$. We are interested in the set $M$ of primes $p$ for which the index of the reduction of $W_i$ modulo $p$ divides $k_i$ for all $i$, and for which $(p|F) \in C$. The letters $p$ and $\ell$ always denote prime numbers.

For each $1 \leq i \leq t$, let $q_i(\ell)$ be the smallest power of $\ell$ not dividing $k_i$. For each $\ell$ define $L_\ell = \mathbb{Q}(\zeta_{\prod_{i|\ell, i \leq \ell} q_i}, W_1^{1/q_i(\ell)}, \ldots, W_t^{1/q_i(\ell)})$. Let $L_n$ be the compositum of $L_\ell$, $\ell|n$. Let $C_n$ be the set of $\sigma \in \text{Gal}(FL_n/Q)$ such that $(\sigma|F) \in C$ and such that $\sigma$ is not the identity on $\mathbb{Q}(\zeta_{q_i(\ell)}, W_i^{1/q_i(\ell)})$ for any $1 \leq i \leq t$ and $\ell|n$. Let $a_n = \frac{|C_n|}{|\text{Gal}(F L_n/Q)|}$. As with the case $t = 1$ of one group, we have $a_n \geq a_m \geq 0$ for all $n|m$, and therefore the limit $a(M)$ of $a_n$ exists when $n$ goes through the squarefree positive integers ordered by divisibility.

In the next two subsections we will prove the following theorems.

\textbf{Theorem 3.1.} Assume GRH. The density $d(M)$ of $M$ exists, and we have $d(M) = a(M)$.

\textbf{Theorem 3.2.} If $a_n \neq 0$ for all squarefree positive integers $n$, then $a(M) \neq 0$.

\subsection{Proof of Theorem 3.1}

$d(M) = a(M)$

As in Lenstra’s work (Lemma 3.2), we may reduce to the case $F = \mathbb{Q}$ and $C = \{\text{id}_\mathbb{Q}\}$. The line of attack is similar to that used by Cooke and Weinberger [2], and many of the needed results for our generalization can be found in [2]. We also borrow ideas from the work of Matthews [7] on several integers being simultaneously primitive roots modulo a prime.
Let \( R(\ell, p) \) be the statement “\( p \) splits in \( \mathbb{Q}(\zeta_{q(\ell)}, W_1^{1/q(\ell)}) \) for at least one index \( 1 \leq i \leq t' \). Let \( N(x, \delta) \) be the number of \( p \leq x \) such that \( R(\ell, p) \) is false for all \( \ell \leq \delta \). We want to prove that \( N(x, x - 1) = |M \cap [1, x]| \) tends to infinity with \( x \) with rate \( a(M) \pi(x) \), where \( \pi(x) \) denotes the number of primes \( \leq x \). Let \( P(x, k) \) be the number of \( p \leq x \) such that \( R(\ell, p) \) is true for all \( \ell | k \). By inclusion-exclusion we have

\[
N(x, \delta) = \sum_k \mu(k)P(x, k),
\]

where the sum goes through all \( k \) whose all prime divisors are \( \leq \delta \).

Let \( \xi_1 = \frac{1}{6} \log(x), \xi_2 = x^{1/2}/\log(x)^2 \), and \( \xi_3 = x^{1/2} \log(x) \). Let \( L(x, \eta_1, \eta_2) \) be the number of \( p \leq x \) such that \( R(\ell, p) \) is true for at least one prime \( \eta_1 \leq \ell \leq \eta_2 \). Now, similarly to Hooley’s [4] and Matthews’ [7] work we get the equations

\[
N(x, x - 1) = N(x, \xi_1) + O(L(x, \xi_1, x - 1)),
\]

and

\[
L(x, \xi_1, x - 1) \leq L(x, \xi_1, \xi_2) + L(x, \xi_2, \xi_3) + L(x, \xi_3, x - 1).
\]

We first prove that \( N(x, \xi_1) \) grows asymptotically as \( a(M) \pi(x) \), after which we will show that \( L(x, \xi_1, x - 1) \) is small.

Let \( \mathcal{P}(y) \) be the product of primes at most \( y \). We have

\[
N(x, \xi_1) = \sum_{k \mid \mathcal{P}(\xi_1)} \mu(k)P(x, k),
\]

For any fixed \( m \), the sum

\[
\sum_{k \mid \mathcal{P}(m)} \mu(k)P(x, k)
\]

approaches \( a_{\mathcal{P}(m)} \pi(x) \). Thus, for \( m \to \infty \) this approaches \( a(M) \pi(x) \), which is the desired claim. However, we need the error term with \( m = \xi_1 \) to be \( o(x/\log(x)) \). This indeed is the case (under GRH), as we will now show.

Let \( W_S \) be the subgroup of \( \mathbb{Q}^\times \) generated by \( W_i, i \in S \), where \( S \) is an arbitrary subset of \( \{1, 2, \ldots, t\} \). To \( W_S \) we associate the field

\[
L_{\ell, S} := \mathbb{Q}(\zeta_{\max\{q(\ell)\}, i \in S}, W_i^{1/q(\ell)}, i \in S),
\]

and denote by \( L_{n,S} \) the compositum of \( L_{\ell, S} \) with \( \ell \mid n \). We have \( L_{n,\emptyset} = \mathbb{Q} \) for all \( n \).

\( N(x, \xi_1) \) can be calculated by inclusion-exclusion as

\[
\sum_{k \mid \mathcal{P}(\xi_1)} \sum_{S \subseteq \{1, 2, \ldots, t\}} \left| \{p : p \leq x, p \text{ splits in } L_{k,S} \} \right| \mu(k)^{|S|}.
\]

By Theorem 1.4 of [2] we have

\[
\left| \{p : p \leq x, p \text{ splits in } L_{k,S} \} \right| = \frac{\text{li}(x)}{[L_{k,S} : \mathbb{Q}]} + O\left(x^{1/2} \log(x) \prod_{i \in S} q_i(k) \right)
\]

under GRH. Here \( \text{li}(x) \) denotes the logarithmic integral \( \int_0^x \frac{1}{\log(t)} \, dt \).

In the sum above we have \( 2^{\zeta(\xi_1) + 1} = O(x^{1/3}) \) summands, so the error term for the whole sum becomes \( O(x^{5/6} \log(\mathcal{P}(\xi_1)x)) \) using the fact \( q_i(\ell) = \ell \) for large enough \( \ell \) and all \( i \). By
the prime number theorem we have
\[ \log(P(\xi_1)) = \sum_{p \leq \xi_1} \log(p) = O(\xi_1) = O(\log(x)). \]

Therefore, the error term is \( o(x/\log(x)) \).

Our next goal is to prove that \( L(x, \xi_1, x-1) \leq L(x, \xi_1, \xi_2) + L(x, \xi_2, \xi_3) + L(x, \xi_3, x-1) \) is small. We do this term-by-term. Our first step in this direction is an analogue of Lemma 1.6 of [2].

**Lemma 3.3.** We have \( L(x, \xi_2, \xi_3) = o(x/\log(x)) \) and \( L(x, \xi_3, x-1) = o(x/\log(x)) \).

**Proof.** We have
\[ L(x, \xi_2, \xi_3) \leq \sum_{\xi_2 \leq \ell \leq \xi_3} P(x, \ell). \]

For \( P(x, \ell) \) we have the inequality
\[ P(x, \ell) \leq |\{ p : p \leq x, p \equiv 1 \pmod{\ell} \}|, \]
as \( p \) splitting in \( \mathbb{Q}(\zeta_{q(\ell)}, W_i^{1/q(\ell)}) \) means \( p \) splits in \( \mathbb{Q}(\zeta_{\ell}) \), and therefore \( p \equiv 1 \pmod{\ell} \). By Brun-Titchmarsh we have
\[ |\{ p : p \leq x, p \equiv 1 \pmod{\ell} \}| \leq \frac{2x}{(\ell - 1)\log(x/\ell)} = O\left(\frac{x}{\ell \log(x)}\right), \]
so
\[ \sum_{\xi_2 \leq \ell \leq \xi_3} P(x, \ell) = O\left(\frac{x}{\log(x)} \sum_{\xi_2 \leq \ell \leq \xi_3} \frac{1}{\ell}\right) = o\left(\frac{x}{\log(x)}\right). \]

For \( L(x, \xi_3, x-1) \) we note that if \( p \) is counted by \( L(x, \xi_3, x-1) \), then \( R(\ell, p) \) is true for some \( \ell \geq \xi_3 \), so \( p \) splits in some \( \mathbb{Q}(\zeta_{q(\ell)}, W_i^{1/q(\ell)}) \). This means that for any \( w \in W_i \) we have \( w^{(p-1)/q(\ell)} \equiv 1 \pmod{\ell} \). Fix some such \( w = \frac{a}{b} > 1 \). Now, any \( p \) counted by \( L(x, \xi_3, x-1) \) by the index \( i \in \{1, \ldots, t\} \) divides
\[ \prod_{m \leq (x-1)/(\xi_3-1)} (a^m - b^m), \]
so
\[ \prod_{p \text{ counted by } i} p = O\left(\prod_{m \leq x^{1/2}/\log(x)} a^m\right) \]
and thus
\[ \sum_{p \text{ counted by } i} \log(p) = O\left(\sum_{m \leq x^{1/2}/\log(x)} m\right) = O(x/\log^2(x)). \]

Since we have a fixed number of indices \( i \), this proves that the number of counted \( p \) is \( O(x/\log^2(x)) = o(x/\log(x)) \). \( \square \)

We are left with proving \( L(x, \xi_1, \xi_2) = o(x/\log(x)) \). First we note that for any \( w \in \mathbb{Q}, w \not\in \{-1, 0, 1\} \) we have \( [\mathbb{Q}(\zeta_{\ell}, w^{1/\ell}) : \mathbb{Q}] = \ell (\ell - 1) \) for all large enough \( \ell \). This is because for a prime \( \ell \) the binomial \( x^\ell - w \) is reducible over \( \mathbb{Q} \) if and only if \( w \) is a perfect \( \ell \)th power, which can only happen for small enough \( \ell \). Thus, the degree of \( \mathbb{Q}(\zeta_{q(\ell)}, W_i^{1/q(\ell)}) \) is at
equality of orders of a set of integers modulo a prime

least \( \ell(\ell - 1) \) for \( \ell \geq \xi_1 \) and any \( i \) when \( x \) is large enough. Combining this result with the GRH conditional version of Chebotarev density theorem we have for large \( x \) the estimates

\[
L(x, \xi_1, \xi_2) \leq \sum_{\xi_1 \leq \ell \leq \xi_2} P(x, \ell)
\]

\[
\leq \sum_{\xi_1 \leq \ell \leq \xi_2} \sum_{1 \leq i \leq \ell} \left( \frac{\text{li}(x)}{[Q(\zeta_{q_i(\ell)}, W_i^{1/q_i(\ell)}): Q]} + O(x^{1/2} \log(q_i(\ell)x)) \right)
\]

\[
\leq \sum_{\xi_1 \leq \ell \leq \xi_2} \sum_{1 \leq i \leq \ell} \left( \frac{\text{li}(x)}{\ell(\ell - 1)} + O(x^{1/2} \log(\ell x)) \right)
\]

\[
\leq \text{li}(x)O \left( \sum_{\xi_1 \leq \ell \leq \xi_2} \frac{1}{\ell^2} \right) + O(x^{1/2} \pi(\xi_2^2) \log(\xi_2 x))
\]

\[
= o(x) + O(x/\log(x)^2)
\]

This finishes the proof of \( \delta(M) = a(M) \).

3.2 Proof of Theorem 3.2: \( a_n \neq 0 \) for all \( n \) implies \( a(M) \neq 0 \)

Let \( W \) be the subgroup of \( \mathbb{Q}^\times \) generated by all of \( W_i \). By Lemma 5.6. of Lenstra we have \( L_\ell \) and \( L_d F \) linearly disjoint for all \( \ell \mid d \), \( \ell \) large enough, as we have \( L_\ell = \mathbb{Q}(\zeta_\ell, W^{1/\ell}) \) for \( \ell \) large. By this we get the product formula

\[
a_{dl} = a_d \frac{f(\ell)}{[L_\ell : \mathbb{Q}]}
\]

Here \( f(\ell) \) denotes the number of \( \sigma \in \text{Gal}(L_\ell/\mathbb{Q}) \) not fixing any of the subfields \( \mathbb{Q}(\zeta_{q_i(\ell)}, W_i^{1/q_i(\ell)}) \subset L_\ell \). We will now prove that the infinite product

\[
\prod_\ell \frac{f(\ell)}{[L_\ell : \mathbb{Q}]}
\]

converges to a strictly positive number.

For sufficiently large \( \ell \) we have \( q_i(\ell) = \ell - 1 \) for all \( i \), the product formula above, and also \( [L_\ell : \mathbb{Q}] = (\ell - 1)^{r^\ell} \), where \( r \) is the rank of \( W \), by Lemma 5.2 of Lenstra. Consider only those \( \ell \) large enough such that these conditions hold. The size of the Galois group \( \text{Gal}(L_\ell/\mathbb{Q}) \) is \( (\ell - 1)^{r^\ell} \), so the Galois group is a direct product of a cycle of size \( \ell - 1 \) and \( r \) cycles of sizes \( \ell \). There are \( r^\ell \) elements of the Galois group which fix \( \zeta_\ell \), while \( (\ell - 2)^{r^\ell} \) of them do not. We have thus proven \( f(\ell) \geq (\ell - 2)^{r^\ell} \). We need a sharper lower bound, and now focus on the maps of \( \text{Gal}(L_\ell/\mathbb{Q}) \) fixing \( \zeta_\ell \).

Let \( b_1, b_2, \ldots, b_r \) be a basis of \( W \). For each \( i \) choose some \( w_i \in W_i \) not equal to \( \pm 1 \). Each \( w_i \) can be expressed as a product of the form

\[
w_i = \prod_{1 \leq j \leq r} b_j^{e_{i,j}}
\]

where \( e_{i,j} \) are integers.
Since the Galois group $\text{Gal}(L_i/\mathbb{Q})$ is a direct product of cyclic groups, for each $r$-tuple of integers $x_1, \ldots, x_r$ there exists an element of $\text{Gal}(L_i/\mathbb{Q}(\zeta_i))$ sending $b_i^{1/\ell}$ to $b_i^{1/\ell} \zeta_i^x$. The number $w_i^{1/\ell}$ is sent to itself if and only if
\[
\sum_{1 \leq j \leq r} e_{i,j} x_j \equiv 0 \pmod{\ell}.
\]
For large enough $\ell$ we have $e_{i,j} \neq 0 \pmod{\ell}$ unless $e_{i,j} = 0$. Since $w_i \neq \pm 1$, some $e_{i,j}$ is nonzero and thus nonzero modulo $\ell$ for $\ell$ large, and therefore the number of $r$-tuples $(x_1, \ldots, x_r)$ sending $w_i^{1/\ell}$ to itself is exactly $\ell^{r-1}$. Now, the number of $r$-tuples not sending any of $w_i^{1/\ell}$ to itself is at least
\[
\ell^r - t \ell^{r-1}.
\]
Therefore we have the bound
\[
f(\ell) \geq (\ell - 2)\ell^r + \ell^r - t \ell^{r-1} = (\ell - 1)\ell^r - t \ell^{r-1},
\]
so the infinite product of $f(\ell)/\ell^{2\ell}$ converges to a positive value.

4 Proof of Theorem 1.1: equality of orders

The proof of the existence of the densities in Theorems 1.1, 1.3 and 1.4 is postponed to the end of Section 5.

To demonstrate the methods we first prove Corollary 1.2. For each $i$, choose $W_i = \{A_i^n, n \in \mathbb{Z}\}$ and $k_i = 2$. Let $F = \mathbb{Q}(\sqrt{A_1}, \ldots, \sqrt{A_1})$ and $C = \{\text{id}_F\}$. Now $M$ contains those primes $p$ for which the index of $W_i$ modulo $p$ divides 2 for all $i$, and for which $A_i$ is a quadratic residue modulo $p$. Therefore, $p \in M$ if and only if $\text{ord}_p(A_1) = \ldots = \text{ord}_p(A_n) = \frac{p-1}{2}$.

The statement $\delta(M) \neq 0$ is, by Theorems 3.1 and 3.2, equivalent to the statement $a_n \neq 0$. That $a_n \neq 0$ for all $n$ follows by noting that the element $\sigma$ of $\text{Gal}(FL_n/\mathbb{Q})$ which maps $a + bi$ to its complex conjugate $a - bi$ fixes the real field $F$, but does not fix any of $\mathbb{Q}(\zeta_q(t), W_i^{1/q(t)})$, as all of these fields are nonreal. This proves Corollary 1.2.

We then focus on the general case of Theorem 1.1. Let $W_i$ be the subgroup generated by $A_i$. We will choose all $k_i$ to be equal to $k$, where $k$ is an integer of the form $3 \cdot 2^s$ with $s$ an integer chosen later. Pick $F = \mathbb{Q}(\zeta_3, A_1^{1/k}, \ldots, A_i^{1/k})$ with $C = \{\text{id}_F\}$. Define $q(\ell) = q_1(\ell)$, so we have $q(2) = 2^{s+1}$, $q(3) = 9$, and $q(\ell) = \ell$ for $\ell \geq 5$.

As above, we want $a_n \neq 0$ for all $n$, that is, the existence of an isomorphism $\sigma \in \text{Gal}(FL_n/\mathbb{Q})$ not fixing any of $\mathbb{Q}(\zeta_q(t), A_i^{1/q(t)})$ with $\ell | n$. We will first prove this for $n = 2$ (for some $s$), after which we extend the results to the general case.

4.1 Finding $\sigma$ for $n = 2$

We need the following lemma (see the end of Section 5.3 in [5]).

Lemma 4.1. Let $A_1, \ldots, A_m$ be multiplicatively independent integers, let $W$ be an odd positive integer and let $\epsilon \in \{-1, 1\}$. There exists an integer $k_0$ such that for all $k \geq k_0$ and for all integers $x_1, x_2, \ldots, x_m$ divisible by $2^{k_0}$ there exists an element of the Galois group of
\[
\mathbb{Q}(\zeta_{2^{k+1}}W, A_1^{1/2^k}, \ldots, A_i^{1/2^k})/\mathbb{Q}(\zeta_{2W})
\]
which maps $\zeta_{2^{k+1}}$ to $\epsilon \zeta_{2^{k+1}}$ and $A_i^{1/2^k}$ to $A_i^{1/2^k} \zeta_x^{x_i}$ for all $i$. 
Lemma 4.2. Let $A_1, \ldots, A_m$ be nonzero integers with no product of them equal to $-1$. There exists a set $S \subset \{A_1, \ldots, A_m\}$ with the following properties:

1. The elements of $S$ are multiplicatively independent.
2. For all $i$ there exists integers $f, f_1, \ldots, f_{|S|}$ with $f$ odd such that

$$A_i^f = \prod_{1 \leq j \leq |S|} s_j^{f_j},$$

where $s_1, \ldots, s_{|S|}$ are the elements of $S$.

To prove that there exists an element $\sigma \in \text{Gal}(FL_2/\mathbb{Q})$ not fixing any of $\mathbb{Q}(\zeta_{q(2)}, A_1^{1/q(2)})$ we split into two cases.

In the first case we assume that there do not exist integers $e_i$ such that $\prod A_i^{e_i} = -1$. This allows us to apply Lemma 4.2 to the numbers $A_1, \ldots, A_t$. Let $S = \{s_1, \ldots, s_{|S|}\}$ be the obtained set. By Lemma 4.1 there exists an isomorphism of $\mathbb{Q}$ $A$ we split into two cases.

- In the second case we assume that there exists integers $s_i$ such that $\prod s_i^{e_i} = -1$. A desired $\sigma$ would fix $F$ and therefore all of $a_i^{1/2}$, and so would fix $(-1)^{1/2^s} = \zeta_{2^s+1}$. In order to have $\sigma$ not fix any of $\mathbb{Q}(\zeta_{q(2)}, a_i^{1/q(2)}) = \mathbb{Q}(\zeta_{2^s+1}, a_i^{1/2^{s+1}})$ we must map $a_i^{1/2^{s+1}}$ to $-a_i^{1/2^{s+1}}$. This is clearly impossible if there exists integers $f_i$ with odd sum such that $\prod a_i^{f_i} = 1$. We now prove that this is the only obstruction. We divide into two subcases.

In the first subcase we assume that for some $e_i$ with even sum we have $\prod A_i^{e_i} = -1$. Now, $(-1)^{1/2^{s+1}} = \zeta_{2^{s+1}}$ must be mapped to itself (note that $\zeta_{2^{s+1}} \in FL_2$), so $(-A_i)^{1/2^s}$ is mapped to $(-A)^{1/2^s}$ if and only if $A_i^{1/2^s}$ maps to $-A_i^{1/2^s}$. Our problem is now equivalent to mapping all of $|A_i|^{1/2^s}$ to $|A_i|^{1/2^s+1}$ by Lemma 4.1. We must map each of $s_i^{1/2^{s+1}}$ to $-s_i^{1/2^{s+1}}$ (which is possible by Lemma 4.1 for large $s$), and each of

$$A_i^{1/2^s+1} = \prod_{1 \leq j \leq |S|} s_j^{f_j/(2^s+1)}$$

to $-A_i^{1/2^s+1}$. The latter condition for $A_i$ is met if the sum of $f_j$ above is odd (for all $i$). Assume now that for some $i$ the representation of $A_i$ in terms of $s_j$ has an even sum of $f_j$. Now, we have

$$A_i^f \prod s_j^{-f_j} = 1,$$

and the sum of $f$ and $f_j$ is odd. Each of $s_j$ corresponds to an element $A_k$ or $-A_k$, and each $-A_k$ can be expressed as a product of an odd number of $A_1, \ldots, A_s$, since $-1$ is the product of an even number of $A_i$. Therefore, there exists an odd number of $A_i$ with product 1, as desired.
In the second subcase we assume that for some \( e_i \) with odd sum we have \( \prod A_i^{e_i} = -1 \). Working similarly as above, \((-A)^{1/2k+1}\) maps to \(-(-A)^{1/2k+1}\) if and only if \( A^{1/2k+1} \) is fixed. Our goal is to map \( A_i^{1/2k+1} \) to \(-A_i^{1/2k+1}\) when \( A_i > 0 \), and to fix \( |A_i|^{1/2k+1} \) when \( A_i < 0 \). Let \( s_j \) be a basis for \(|A_i|\) as above, and map \( s_j^{1/2k+1} \) to \( x_j s_j^{1/2k+1} \) for \( x_j \in \{-1, 1\} \). We obtain a system of linear equations over \( \mathbb{F}_2 \) which the numbers \( x_j \) must satisfy. There are two types of equations:

\[
\sum x_j \equiv 1 \pmod{2}
\]

and

\[
\sum x_j \equiv 0 \pmod{2}.
\]

If there exists a solution to this system, we are done by Lemma 4.1. Assume now that there is no solution. By linear algebra there exists a linear combination of equations of the second type, and an odd number of equations of the first type which result in the equation \( 0 \equiv 1 \pmod{2} \). This corresponds to having

\[
\prod_{A_i > 0} A_i^{e_i} \cdot \prod_{A_i < 0} |A_i|^{e_i} = 1,
\]

for some integers \( e_{i,+} \) and \( e_{i,-} \) with \( \sum e_{i,+} \equiv 1 \pmod{2} \). If \( e_{i,-} \) is even, we may drop the absolute value around \( A_i \) in the product, and if \( e_{i,-} \) is odd, we may replace \( |A_i|^{e_{i,-}} \) with \( A_i^{e_{i,-}} \cdot \prod A_j \), where the latter product contains an odd number of \( A_j \) with product equal to \(-1\). In either case, we obtain an equation of the type \( \prod A_i^{e_i} = 1 \) with an odd sum of \( e_i \), as desired.

### 4.2 Finding \( \sigma \) for general \( n \)

We will find an element of \( \text{Gal}(FL_n/\mathbb{Q}) \) which fixes \( F \), but does not fix any \( \mathbb{Q}(\zeta_{q(\ell)}, A_i^{1/q(\ell)}) \) with \( 1 \leq i \leq t \) and \( \ell \mid n \). Without loss of generality assume \( n \) is divisible by 6.

Our strategy is to extend the \( \sigma \in \text{Gal}(FL_2/\mathbb{Q}) \) obtained in the previous subsection to satisfy the conditions on \( FL_n \). We do so by proving that \( \zeta_{q(\ell)} \not\in \text{Gal}(FL_m/\mathbb{Q}) \) for \( 3 \leq \ell \not\equiv m \), which allows us to extend a suitable \( \sigma_{m, \ell} \in \text{Gal}(FL_{m, \ell}/\mathbb{Q}) \) not mapping \( \zeta_{q(\ell)} \) to itself. This guarantees that no \( \mathbb{Q}(\zeta_{q(\ell)}, A_i^{1/q(\ell)}) \) is fixed.

We use the following facts: \( \sqrt{\ell} \in \mathbb{Q}(\zeta_{\ell}) \) for \( \ell \equiv 1 \pmod{4} \), \( \sqrt{-\ell} \in \mathbb{Q}(\zeta_{\ell}) \) for \( \ell \equiv 3 \pmod{4} \), and \( \sqrt{2} \in \mathbb{Q}(\zeta_{8}) \).

The largest abelian subextension of \( FL_m \) is of the form

\[
K_m := \mathbb{Q}(\zeta_{m^*}, \sqrt{B_1}, \sqrt{B_2}, \ldots, \sqrt{B_u})
\]

with suitable \( B_i \) and \( m^* \), with \( m^* \) not divisible by \( q(\ell) \). We prove that \( \zeta_{q(\ell)} \) is not contained in any field of such form. We may without loss of generality assume that \( B_i \) are primes, \( 4 \mid m^* \), and \( B_u = \ell \). Now, \( \sqrt{B_i} \) is contained in \( \mathbb{Q}(\zeta_{q(\ell)}) \) for all \( i < u \), and therefore \( K_m \) is a subfield of \( K'_m = \mathbb{Q}(\zeta_{m^*B_1\cdots B_{u-1}}, \sqrt{B_u}) \).

Assume that \( \zeta_{q(\ell)} \in K_m \). Now, \( K'_m = \mathbb{Q}(\zeta_{m'B_1\cdots B_{u-1}}, \sqrt{B_u}) = \mathbb{Q}(\zeta_{m'B_1\cdots B_{u-1}q(\ell)}) \). The degree of \( \mathbb{Q}(\zeta_{m'B_1\cdots B_{u-1}q(\ell)}) \) is \( \phi(m'B_1\cdots B_{u-1}q(\ell)) \), and the degree of \( \mathbb{Q}(\zeta_{m'B_1\cdots B_{u-1}}) \) is (at most) \( 2\phi(m'B_1\cdots B_{u-1}) \). This is a contradiction as \( \phi(q(\ell)) > 2 \). Therefore \( \zeta_{q(\ell)} \not\in K_m \), which was to be proven.
5 Proof of Theorem 1.3: Order of orders

We begin with the following lemma.

Lemma 5.1. Assume GRH. Let \(A_1, \ldots, A_t\) be integers with absolute value greater than 1. There exists an integer \(k\) with the following property: the lower density of those \(p\) for which the index of each \(A_i\) modulo \(p\) is at most \(k\) is positive.

Proof. A much stronger statement is true. If \(S_{A,k}\) denotes the set of primes for which the index of \(A\) modulo \(p\) is exactly \(k\), then it has been proved in [12] (Section 5) that under GRH the sum \(\sum_{k=1}^{\infty} d(S_{A,k})\) equals 1 for any \(A \neq -1, 0, 1\) (and that \(d(S_{A,k})\) exists). In other words, for a density of at least \(1 - \epsilon\) of primes the index of \(A\) modulo \(p\) is at most \(k\), when \(k = k(\epsilon)\) is large enough. Applying this for some \(\epsilon < \frac{1}{t}\) gives the conclusion of the lemma. \(\square\)

By the pigeonhole principle, there exists some integers \(k_1, k_2, \ldots, k_t\) such that for a set of primes \(p\) of positive lower density we have \(\text{ord}_p(A_i) = \frac{p-1}{k_i}\) for all \(i\).

To prove Theorem 1.3 we plan to choose integers \(r_1, r_2, r_3, \ldots, r_t\) such that \(k_1 r_1 > k_2 r_2 > \ldots > k_t r_t\), and such that for a set of primes \(p\) of positive (lower) density we have \(\text{ord}_p(A_i) = \frac{p-1}{k_i r_i}\). We do this inductively by first picking \(r_2\), after which we pick \(r_3\), and so on, so that after each choice the set of desired primes is infinite.

Assume we have already chosen \(r_2, r_3, \ldots, r_{i-1}\). Let \(M_{i-1}\) be the set of \(p\) for which \(\text{ord}_p(A_j) = k_j r_j\) for \(j < i\) and \(\text{ord}_p(A_j) = k_j\) for \(j \geq i\). By assumption \(M_{i-1}\) is infinite. Therefore, if we denote

\[ F_{i-1} = \mathbb{Q}(\zeta_T, A_1^{1/(k_1 r_1)}, \ldots, A_{i-1}^{1/(k_{i-1} r_{i-1})}, A_i^{1/k_i}, \ldots, A_t^{1/k_t}), \]

where \(T\) is the least common multiple of \(k_j r_j\), \(j < i\) and \(k_j, j \geq i\), there are infinitely many primes \(p\) which split in \(F_{i-1}\), but not in any of \(\mathbb{Q}(\zeta_{q_j'(\ell)}, A_j^{1/q_j'(\ell)})\). Here \(q_j'(\ell)\) is the smallest power of \(\ell\) not dividing \(k_j r_j\) for \(j < i\), and otherwise the smallest power of \(\ell\) not dividing \(k_j\). Let \(L_i\) and \(L_n\) be the corresponding fields for these functions \(q_j\).

We claim that for any sufficiently large prime \(r_i\) the set \(M_i\) of primes for which \(\text{ord}_p(A_j) = k_j r_j\) for \(j \leq i\) and \(\text{ord}_p(A_j) = k_j\) for \(j > i\) is infinite. Fix some such \(r_i\), and define

\[ F_i = F_{i-1}(\zeta_{r_i}, A_i^{1/r_i}), \]

and let \(q_j'(\ell) = q_j(\ell)\) for \(\ell \neq r_i\), \(q_j'(r_i) = r_i^2\), \(q_j'(r_i) = r_i\) for \(j \neq i\). Let \(M_i\) be the corresponding set of primes and \(L'_i\) and \(L'_n\) be the corresponding fields.

By applying Theorems 3.1 and 3.2 we reduce the problem to proving that for any \(n\) there exists a map \(\sigma_{i,n} \in \text{Gal}(F_i L'_n/Q)\) fixing \(F_i\) and not fixing any of \(\mathbb{Q}(\zeta_{q_j'(\ell)}, A_j^{1/q_j'(\ell)})\). Without loss of generality assume \(r_i|n\). By assumption on the infinitude of \(M_{i-1}\) there exists an element \(\sigma_{i-1,n/r_i}\) of \(\text{Gal}(F_{i-1} L_{n/r_i}/Q)\) fixing \(F_{i-1}\) and not fixing any \(\mathbb{Q}(\zeta_{q_j'(\ell)}, A_j^{1/q_j'(\ell)})\) with \(\ell|n/r_i\).

By Lemma 5.6. of Lenstra the fields \(L'_i\) and \(F_{i-1} L_{n/r_i} = F_{i-1} L'_n/r_i\) are linearly disjoint for any \(n\) divisible by \(r_i\) when \(r_i\) is a large enough prime.

For large enough \(r_i\) there exists a \(\sigma_{r_i,n} \in \text{Gal}(L'_n/Q)\) fixing \(\mathbb{Q}(\zeta_{r_i}, A_i^{1/r_i})\), but not fixing any of \(A_j^{1/r_i}\) or \(\mathbb{Q}(\zeta_{r_i}, A_i^{1/r_i^2})\). This can be easily proven with the methods of Sections 5.4 and 6 of [5] with the assumption of the multiplicative independence of \(A_i\) and \(A_j\) for all \(j \neq i\).
We combine these observations: \( \sigma_r \) has the desired properties on \( L_{1/r}^i \), \( \sigma_{i-1/n/r_i} \) has the desired properties on \( F_{i-1}^i L_{n/r_i} \), and \( L_{1/r}^i \) and \( F_{i-1}^i L_{n/r_i} \) are linearly disjoint. Therefore, there exists a desired \( \sigma \in \text{Gal}(L_{1/r}^i L_{n/r_i} F_{i-1}^i / \mathbb{Q}) = \text{Gal}(L_{1/r}^i F_{i} / \mathbb{Q}) \).

To prove the existence of the densities in Theorems 1.1, 1.3 and 1.4 we first note that for any nonzero integers \( A_i \) (with \( |A_i| > 1 \)) and \( k_i \) the set \( M \) of primes \( p \) for which \( \text{ord}_p(A_i) = \frac{p-1}{k_i} \) for all \( i \) exists (under GRH). This follows from Theorem 3.1 with the choice \( F = \mathbb{Q}((\zeta_T, A_1^{1/k_1}, \ldots, A_t^{1/k_t})) \), where \( T \) is the least common multiple of \( k_i \). By the proof of Lemma 5.1 one can approximate the density of the primes satisfying Theorem 1.1 within an error of \( \epsilon \) by a finite number of such sets \( M \) by letting \( k_i \) run over all \( t \)-tuples of positive integers at most \( B \) for some \( B \). This proves the existence of the density in Theorem 1.1 and the existence follows for the other theorems similarly.

6 Discussion

A natural generalization of Theorem 1.3 would be determining all integers \( A_i \) such that for infinitely many \( p \) we have \( \text{ord}_p(A_1) > \ldots > \text{ord}_p(A_t) \). In principle, our methods can be applied to solve this problem. However, we argue that the necessary and sufficient condition for the infinitude of these \( p \) becomes very messy.

For the sake of discussion, assume all \( A_i \) are positive. The chain of inequalities \( \text{ord}_p(A_1) > \ldots > \text{ord}_p(A_t) \) can be seen as a collection of chains of the form \( \text{ord}_p(B_i^{e_{i1}}) > \text{ord}_p(B_i^{e_{i2}}) > \ldots > \text{ord}_p(B_i^{e_{it}}) \) interlacing with each other, where \( B_i \) are pairwise multiplicatively independent and none of \( B_i \) is a perfect power. For each \( i \) we have to decide on the divisibility of the order of \( B_i \) by various prime powers to satisfy the inequalities \( \text{ord}_p(B_i^{e_{ij}}) > \text{ord}_p(B_i^{e_{ij}+1}) \). In addition to this, in all cases it is not possible to guarantee that some \( \text{ord}_p(B_i) \) are divisible by \( d \) and some are not (for the case \( d = 2 \), see [5] – the general case is analogous), and we have to make sure that we may combine the chains \( \text{ord}_p(B_i^{e_{i1}}) > \ldots > \text{ord}_p(B_i^{e_{it}}) \). As mentioned above, in principle these issues can be dealt with, but determining all local obstructions is quite tedious.

Another problem of this type is considering the solvability of a system of equations of the form \( A_i^{x_i} \equiv B_i \pmod{p} \), \( 1 \leq i \leq t \), which is equivalent to the conditions \( \text{ord}_p(B_i) | \text{ord}_p(A_i) \), \( 1 \leq i \leq t \). This is related to the two-variable Artin conjecture (see [9], also [5]). By similar work as in the proof of Theorem 1.1 the problem can be reduced to investigating the 2-adic valuations of orders, but again, determining the local obstructions is rather unpleasant, and the necessary and sufficient condition is not very enlightening.

Let \( S \) be the set of primes \( p \) for which \( \text{ord}_p(2) > \text{ord}_p(3) \), and let \( T \) be the set of primes \( p \) for which \( \text{ord}_p(2) < \text{ord}_p(3) \). Numerical results for the first 10⁹ primes suggest that the density of \( S \) is approximately 0.368 and that of \( T \) is approximately 0.349. In particular, the densities do not appear to be the same. Inspired by the results in [9] one could ask whether the quotient of these densities is rational, and whether in the general case of \( \text{ord}_p(a) > \text{ord}_p(b) \) the corresponding density is a rational multiple of some universal constant.

We conclude by mentioning the following generalization of Corollary 1.2 Assume GRH. For any integers \( a_i \) greater than 1 and for any finite Galois extension \( F \) of \( \mathbb{Q} \), there are infinitely many primes \( p \) (with positive density) such that \( \text{ord}_p(a_1) = \ldots = \text{ord}_p(a_m) \) and such that \( p \) splits in \( F \). This can be proven by our methods.

This result can be interpreted as follows. For any non-constant polynomials \( P_1, \ldots, P_k \) with integer coefficients and for any pairs \( (a_1, b_1), \ldots, (a_n, b_n) \) of integers greater than 1,
there are infinitely many primes $p$ such that the equations $P_i(x) \equiv 0 \pmod{p}$ and $a_i^x \equiv b_i \pmod{p}$ are solvable. The polynomial equations are handled by choosing $F$ to be the compositum of the splitting fields of $P_i$, and the solvability of the exponential equations follow choosing $a_1, \ldots, a_m, b_1, \ldots, b_m$ to have the same order modulo $p$. Thus, for infinitely many primes $p$ there are integer numbers corresponding to any finite number algebraic numbers and logarithms of positive integers when doing arithmetic modulo $p$.

References

[1] Mohamed Anwar, Francesco Pappalardi, On simultaneous primitive roots, Acta Arithmetica 180 (2017), 35-43
[2] Cooke, G. and P. J. Weinberger, On the construction of division chains in algebraic number fields, with applications to $SL_2$, Comm. Alg. 3 (1975), 481–524.
[3] D. R. Heath-Brown. “Artin’s Conjecture for Primitive Roots”. The Quarterly Journal of Mathematics. 37 (1): 27–38, (1986).
[4] C. Hooley. “On Artin’s conjecture”. J. Reine Angew. Math. 225: 209–220, (1967)
[5] Olli Jarvinemi, Simultaneous insolvability of exponential congruences (to be published)
[6] Lenstra, H.W., On Artin’s conjecture and Euclid’s algorithm in global fields, Invent. Math. 42, 1977
[7] K. R. Matthews, A generalisation of Artin’s conjecture for primitive roots, Acta Arith., (1976), 113–146
[8] Pappalardi, F., & Suss, A. (2009). On a problem of Schinzel and Wójcik involving equalities between multiplicative orders. Mathematical Proceedings of the Cambridge Philosophical Society, 146(2), 303-319.
[9] Pieter Moree, Peter Stevenhagen, A Two-Variable Artin Conjecture, Journal of Number Theory, Volume 85, Issue 2, 2000, Pages 291-304
[10] Pieter Moree, Artin’s primitive root conjecture—a survey. (English summary) Integers 12 (2012), no. 6, 1305–1416
[11] Schinzel, A., & Wójcik, J. (1992). On a problem in elementary number theory. Mathematical Proceedings of the Cambridge Philosophical Society, 112(2), 225-232.
[12] S.S. Wagstaft, Jr., Pseudoprimes and a generalization of Artin’s conjecture, Acta Arith. 41 (1982), 141–150.
[13] Wójcik, J. (1996). On a problem in algebraic number theory. Mathematical Proceedings of the Cambridge Philosophical Society, 119(2), 191-200.

University of Helsinki
E-mail address: olli.jarvinemi@helsinki.fi