Equivariant Riemann-Roch theorems for curves over perfect fields

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Abstract

We prove an equivariant Riemann-Roch formula for divisors on algebraic curves over perfect fields. By reduction to the known case of curves over algebraically closed fields, we first show a preliminary formula with coefficients in \( \mathbb{Q} \). We then prove and shed some further light on a divisibility result that yields a formula with integral coefficients. Moreover, we give variants of the main theorem for equivariant locally free sheaves of higher rank.
Introduction

Let $X$ be a smooth, projective, geometrically irreducible curve over a perfect field $k$ and let $G$ be a finite subgroup of the automorphism group $\text{Aut}(X/k)$. For any locally free $G$-sheaf $\mathcal{E}$ on $X$, we are interested in computing the equivariant Euler characteristic

$$
\chi(G, X, \mathcal{E}) := [H^0(X, \mathcal{E})] - [H^1(X, \mathcal{E})] \in K_0(G, k),
$$

considered as an element of the Grothendieck group $K_0(G, k)$ of finitely generated modules over the group ring $k[G]$. The main example of a locally free $G$-sheaf we have in mind is the sheaf $\mathcal{L}(D)$ associated with a $G$-equivariant divisor $D = \sum_{P \in X} n_P P$ (that is $n_{\sigma(P)} = n_P$ for all $\sigma \in G$ and all $P \in X$). If two $k[G]$-modules are in the same class in $K_0(G, k)$, they are not necessarily isomorphic when the characteristic of $k$ divides the order of $G$. In order to be able to determine the actual $k[G]$-isomorphism class of $H^0(X, \mathcal{E})$ or $H^1(X, \mathcal{E})$, we are therefore also interested in deriving conditions for $\chi(G, X, \mathcal{E})$ to lie in the Grothendieck group $K_0(k[G])$ of finitely generated projective $k[G]$-modules and in computing $\chi(G, X, \mathcal{E})$ within $K_0(k[G])$.

The equivariant Riemann-Roch problem goes back to Chevalley and Weil [CW], who described the $G$-structure of the space of global holomorphic differentials on a compact Riemann surface. Ellingsrud and Lønsted [EL] found a formula for the equivariant Euler characteristic of an arbitrary $G$-sheaf on a curve over an algebraically closed field of characteristic zero. Nakajima [Na] and Kani [Ka] independently generalized this to curves over arbitrary algebraically closed fields, under the assumption that the canonical morphism $X \to X/G$ be tamely ramified. These results have been revisited by Borne [Bo], who also found a formula that computes the difference between the equivariant Euler characteristics of two $G$-sheaves in the case of a wildly ramified cover $X \to X/G$. In the same setting, formulae for the equivariant Euler characteristic of a single $G$-sheaf have been found by the second author ([Kö1], [Kö2]). Using these formulae, new proofs for the results of Ellingsrud-Lønsted, Nakajima and Kani have been given [Kö1].

In this paper, we concentrate on the case where the underlying field $k$ is perfect. Our main theorem, Theorem 3.4, is an equivariant Riemann-Roch formula in $K_0(k[G])$ when the canonical morphism $X \to X/G$ is weakly ramified and $\mathcal{E} = \mathcal{L}(D)$ for some equivariant divisor $D$. By reduction to the known case of curves over algebraically closed fields, we first show a preliminary formula with coefficients in $Q$. The divisibility result needed to obtain a formula with integral coefficients is then proved in two ways: Firstly, by applying the preliminary formula to suitably chosen equivariant divisors; and secondly, in two situations, by a local argument. The following paragraphs describe the content of each section in more detail.

It is well-known that a finitely generated $k[G]$-module $M$ is projective if and only if $M \otimes k \tilde{k}$ is a projective $k[G]$-module. In Section 2 we give a variant of this fact for classes in $K_0(G, k)$ rather than for $k[G]$-modules $M$ (Corollary 2.2). This variant is much harder to prove and is an essential tool for the proof of our main result in Section 3.

The first results in Section 3 give both a sufficient condition and a necessary condition under which the equivariant Euler characteristic $\chi(G, X, \mathcal{E})$ lies in the image of the Cartan homomorphism $c : K_0(G, k) \to K_0(k[G])$. More precisely, when $\mathcal{E} = \mathcal{L}(D)$ for some equivariant divisor $D = \sum_{P \in X} n_P P$, this holds if the canonical projection $\pi : X \to X/G$ is weakly ramified and $n_P + 1$ is divisible by the wild part $e_P^w$ of the ramification index $e_P$ for all $P \in X$. When $\pi$ is weakly ramified we furthermore derive from the corresponding result in [Kö2] the existence of the so-called ramification module $N_{G, X}$, a certain projective $k[G]$-module which embodies a global relation between the (local) representations...
m_P/m_P^2 \) of the inertia group \( I_P \) for \( P \in X \). If moreover \( D \) is an equivariant divisor as above, our main result, Theorem 3.4, expresses \( \chi(G, X, \mathcal{L}(D)) \) as an integral linear combination in \( K_0(k[G]) \) of the classes of \( N_{G,X} \), the regular representation \( k[G] \) and the projective \( k[G] \)-modules \( \text{Ind}^G_{I_P}(W_{P,d}) \) (for \( P \in X \) and \( d \geq 0 \)) where the projective \( k[G_P] \)-module \( W_{P,d} \) is defined by the following isomorphism of \( k[G_P] \)-modules:

\[
\text{Ind}^G_{I_P}(\text{Cov}((m_P/m_P^2)^{(-d)})) \cong \bigoplus_{f_P} W_{P,d};
\]

here Cov means taking the \( k[I_P] \)-projective cover and \( f_P \) denotes the residual degree. Finding an equivariant Riemann-Roch formula without denominators amounts to showing that \( W_{P,d} \) exists, i.e. that the left-hand side of the above is “divisible by \( f_P \)”. To do this, we use our prototype formula with denominators, formula \( \text{(1)} \), and apply it to certain equivariant divisors \( D \). If \( \pi \) is tamely ramified, we furthermore consider two situations where we can give a local proof of the divisibility result, yielding a more concrete description of \( W_{P,d} \), see Proposition 3.5.

In Section 4, we give some variants of the main result that hold under slightly different assumptions. In particular, these variants hold for locally free \( G \)-sheaves that do not necessarily come from a divisor.

### 1 Preliminaries

The purpose of this section is to fix some notations used throughout this paper and to state some folklore results used later.

Throughout this section, let \( X \) be a scheme of finite type over a field \( k \), and let \( \bar{k} \) be an algebraic closure of \( k \). For any (closed) point \( P \in X \), let \( k(P) := \mathcal{O}_{X,P}/m_P \) denote the residue field at \( P \). Throughout this paper, let \( \bar{X} \) denote the geometric fibre \( X \times_k \bar{k} \), which is a scheme of finite type over \( \bar{k} \), and let \( p \) denote the canonical projection \( \bar{X} \to X \). Recall that \( p \) is a closed, flat morphism which is in general not of finite type. We will see later that in dimension 1, \( p \) is “unramified” in the sense that if \( Q \in \bar{X} \) and \( P = p(Q) \), then a local parameter at \( P \) is also a local parameter at \( Q \). By Galois theory and Hilbert’s Nullstellensatz, we have for every \( P \in \bar{X} \):

\[
\#p^{-1}(P) = \# \text{Hom}_k(k(P), \bar{k}) \leq [k(p) : k] < \infty,
\]

and equality holds if \( k(P)/k \) is separable.

Let now \( G \) be a finite subgroup of \( \text{Aut}(X/k) \). Since the homomorphism

\[
\text{Aut}(X/k) \to \text{Aut}(\bar{X}/\bar{k}), \sigma \mapsto \sigma \times \text{id}
\]

is injective, which is easy to check, we may view \( G \) as a subgroup of \( \text{Aut}(\bar{X}/\bar{k}) \). Since the elements of \( G \) act on the topological space of \( X \) as homeomorphisms, \( G \) also acts on \( |X| \), the set of closed points in \( X \). Analogously, \( G \) acts on the set \( |\bar{X}| \) of closed points in \( \bar{X} \).

**Definition 1.1.** A *locally free \( G \)-sheaf* (of rank \( r \)) on \( X \) is a locally free \( \mathcal{O}_X \)-module \( \mathcal{E} \) (of rank \( r \)) together with an isomorphism of \( \mathcal{O}_X \)-modules \( v_\sigma : \sigma^*\mathcal{E} \to \mathcal{E} \) for every \( \sigma \in G \), such that for all \( \sigma, \tau \in G \), the following diagram commutes:

\[
\begin{array}{ccc}
\sigma^*\mathcal{E} & \xrightarrow{v_\sigma} & \mathcal{E} \\
\downarrow{\sigma^*v_\tau} & & \\
\sigma^*(\tau^*\mathcal{E}) & = & (\tau\sigma)^*\mathcal{E}
\end{array}
\]
If $\mathcal{E}$ is a locally free $G$-sheaf of finite rank, then the cohomology groups $H^i(X, \mathcal{E})$ ($i \in \mathbb{N}_0$) are $k$-representations of $G$. If moreover $X$ is proper over $k$, then the $H^i(X, \mathcal{E})$ are finite-dimensional and vanish for $i >> 0$ (see Theorem III.5.2 in [Ha]).

We denote the Grothendieck group of all finitely generated $k[G]$-modules (i.e. finite-dimensional $k$-representations of $G$) by $K_0(G, k)$, as opposed to the notation $R_k(G)$ used by Serre in [Se2].

**Definition 1.2.** If $X$ is proper over $k$, and $\mathcal{E}$ is a locally free $G$-sheaf of finite rank, then

$$\chi(G, X, \mathcal{E}) := \sum (-1)^i [H^i(X, \mathcal{E})] \in K_0(G, k)$$

is called the *equivariant Euler characteristic* of $\mathcal{E}$ on $X$.

For $P \in |X|$ or $P \in |\bar{X}|$, the *decomposition group* $G_P$ and the *inertia group* $I_P$ are defined as follows:

$$G_P := \{ \sigma \in G | \sigma(P) = P \};$$

$$I_P := \{ \sigma \in G_P | \sigma = \text{id}_{k(P)} \} = \ker(G_P \rightarrow \text{Aut}(k(P)/k)).$$

Here $\sigma$ denotes the endomorphism that $\sigma$ induces on $k(P)$. Note that for all $Q \in |\bar{X}|$, we have $G_Q = I_Q$ and $G_Q = I_P$, where $P := p(Q) \in |X|$.

In the following lemma, we will assume for the first time that the field $k$ is *perfect*.

**Lemma 1.3.** Assume that $k$ is perfect. Let $\mathcal{F}$ be a coherent sheaf on $X$, and let $\bar{\mathcal{F}} := p^*\mathcal{F}$. Let $P$ be a point in $X$, and let $\mathcal{F}(P) := \mathcal{F}_P \otimes_{\mathcal{O}_X} k(P)$ be the fibre of $\mathcal{F}$ at $P$. Then the canonical homomorphism

$$\mathcal{F}(P) \otimes_k \bar{k} \mapsto \bigoplus_{Q \in p^{-1}(P)} \bar{\mathcal{F}}(Q)$$

is an isomorphism. In particular, the canonical homomorphism

$$k(P) \otimes_k \bar{k} \mapsto \bigoplus_{Q \in p^{-1}(P)} k(Q)$$

is an isomorphism.

**Proof.** It follows from Galois theory that for any separable finite field extension $k'/k$, the homomorphism

$$k' \otimes_k \bar{k} \rightarrow \bigoplus_{\text{Hom}_k(k', \bar{k})} \bar{k}$$

defined by

$$y \otimes z \mapsto (\varphi(y) \cdot z)_{\varphi \in \text{Hom}_k(k', \bar{k})}$$

is an isomorphism. Since $k$ is perfect, by putting $k' = k(P)$ this implies the second part of the lemma, i.e. the special case where $\mathcal{F} = \mathcal{O}_X$.

Since the lemma is a local statement on $X$, we may assume that $X$ is affine. The general case then follows from the special case together with the definitions and basic properties of coherent sheaves and fibred products. \qed
**Proposition 1.4.** Assume that $k$ is perfect. Let $\Omega_{X/k}$ be the sheaf of relative differentials of $X$ over $k$. Then for every point $P \in |X|$, the canonical map

$$m_P/m_P^2 \rightarrow \Omega_{X/k}(P)$$

is an isomorphism.

**Proof.** Let $\Omega_{k(P)/k}$ denote the module of relative differential forms of $k(P)$ over $k$. Using some basic properties of differentials and of the cotangent space in an affine setting, it follows from Corollary 6.5 in [Ku] that we have an exact sequence

$$0 \rightarrow m_P/m_P^2 \rightarrow \Omega_{X/k}(P) \rightarrow \Omega_{k(P)/k} \rightarrow 0.$$

By Corollary 5.3 in [Ku], $\Omega_{k(P)/k}$ is trivial, so the map $m_P/m_P^2 \rightarrow \Omega_{X/k}(P)$ is an isomorphism.

Note that both Corollary 6.5 and Corollary 5.3 in [Ku] require $k(P)/k$ to be separable. Both Lemma 1.3 and Proposition 1.4 can be turned into equivariant statements in the following sense. If we require $F$ to be a locally free $G$-sheaf, then for every point $P \in |X|$, we obtain an action of the inertia group $I_P$ on the fibre $F(P)$ by $k(P)$-automorphisms.

The action of $I_P$ on the fibre $\Omega_X(P)$ of the canonical sheaf corresponds to the action on the cotangent space $m_P/m_P^2$ via the isomorphism from Proposition 1.4.

By letting $I_P$ act trivially on $\bar{k}$, we can extend the action of $I_P$ on $F(P)$ to an action on the tensor product $F(P) \otimes_k \bar{k}$. On the other hand, since $I_P = I_P$ for any point $Q \in p^{-1}(P)$, $I_P$ acts on the fibre $G(Q)$ of any locally free $G$-sheaf $G$ on $\bar{X}$ for any point $Q \in p^{-1}(P)$. In particular, this holds if $G = p^*F$ for a locally free $G$-sheaf $F$ on $X$. With respect to these group actions, the isomorphism from Lemma 1.3 is an isomorphism of $\bar{k}[I_P]$-modules.

We also have an action of the decomposition group $G_P$ on any fibre $F(P)$, but $G_P$ only acts on the fibre via $k$-automorphisms, whereas $I_P$ acts via $k(P)$-automorphisms. $G_P$ does act $k(P)$-semilinearly on the fibre, that is, for any $\sigma \in G_P, a \in k(P)$ and $x, y \in F(P)$ we have $\sigma(ax + y) = (\sigma.a)(\sigma.x) + \sigma.y$, where $\sigma$ denotes the automorphism of $k(P)/k$ induced by $\sigma$.

Let now $X$ be a smooth, projective curve over a perfect field $k$. Assume further that $X$ is geometrically irreducible, i.e. that the geometric fibre $\bar{X} = X \times_k \bar{k}$ is irreducible. Then the curve $X$ itself is irreducible.

The following lemma shows that although the canonical morphism $p : \bar{X} \rightarrow X$ is usually not of finite type, it can be thought of as an “unramified” morphism in the common sense, a fact that will be used frequently throughout this paper.

**Lemma 1.5.** Let $Q \in |\bar{X}|$ be a closed point, and let $P := p(Q)$. Then every local parameter at $P$ is also a local parameter at $Q$.

**Proof.** Let $t_P$ be a local parameter at $P$. Then $t_P$ must be an element of $m_P \setminus m_P^2$, so (the equivalence class of) $t_P$ is a generator of the one-dimensional vector space $m_P/m_P^2$ over $k(P)$. Hence, $t_P \otimes 1$ is a generator of the rank-1 module $m_P/m_P^2 \otimes_k \bar{k}$ over $k(P) \otimes_k \bar{k}$.

By Lemma 1.3 and Proposition 1.4, we have a canonical isomorphism

$$m_P/m_P^2 \otimes_k \bar{k} \rightarrow \bigoplus_{Q \in p^{-1}(P)} m_Q/m_Q^2.$$
which we can view as an isomorphism of modules over $k(P) \otimes_k \bar{k} \cong \bigoplus_{Q \in \mathcal{P}^{-1}(P)} k(Q)$. Since this isomorphism must map $t_P \otimes 1$ to a generator of the right-hand side over $\bigoplus_{Q \in \mathcal{P}^{-1}(P)} k(Q)$, the image of $t_P \otimes 1$ in each component $m_Q/m_Q^2$ must be a generator of $m_Q/m_Q^2$, i.e. the image of $t_P$ under each induced homomorphism $p_Q: \mathcal{O}_{X,P} \to \mathcal{O}_{X,Q}$ must be a local parameter at $Q$. \hfill \Box

Let now $G$ be a finite subgroup of $\text{Aut}(X/k)$. It is a well-known result that the quotient scheme $Y := X/G$ is also a smooth projective curve, with function field $K(Y) = K(X)^G$. The canonical projection $X \to Y$ will be called $\pi$. Let $P \in X$ be a closed point, $R := \pi(P) \in Y$. Let $v_p$ be the unique normed valuation of the function field $K(X)$ associated to $P$, and let $v_R$ be the unique normed valuation of $K(Y)$ associated to $R$. Then $v_P$ is equivalent to a valuation extending $v_R$. For $s \geq -1$, we define the $s$-th ramification group $G_{P,s}$ at $P$ to be the $s$-th ramification group of the extension of local fields $K(X)_{v_P}/K(Y)_{v_R}$. In particular, we have $G_{P,-1} = G_P$ and $G_{P,0} = I_P$.

The canonical projection $\pi: X \to Y$ is called unramified (tamely ramified, weakly ramified) if $G_{P,s}$ is trivial for $s \geq 0$ ($s \geq 1, s \geq 2$) and for all $P \in X$. We denote the ramification index of $\pi$ at the place $P$ by $e_P$, its wild part by $e_P^w$ and its tame part by $e_P^t$. In other words, $e_P = v_P(t_{x(P)}) = |G_{P,0}|$, $e_P^w = |G_{P,1}|$ and $e_P^t = |G_{P,0}/G_{P,1}|$.

If $Q \in |\bar{X}|$ is a closed point, $P := p(Q) \in |X|$, then for every $s \geq 0$, we have $G_{Q,s} = G_{P,s}$ (by Proposition 5 in Chapter IV in [Se1] and Lemma [L5]). In particular, we have $e_P = e_Q$, $e_P^w = e_Q^w$ and $e_P^t = e_Q^t$.

2 A Cartesian diagram of Grothendieck groups

A $k[G]$-module $M$ is projective if and only if $M \otimes_k \bar{k}$ is a projective $\bar{k}[G]$-module. In this section, we will now show variants of this well-known fact for classes in $K_0(G,k)$ rather than $k[G]$-modules.

Let $K_0(k[G])$ denote the Grothendieck group of finitely generated projective $k[G]$-modules. This is a free group generated by the isomorphism classes of indecomposable projective $k[G]$-modules. The Cartan homomorphisms $c: K_0(k[G]) \to K_0(G,k)$ and $\bar{c}: K_0(\bar{k}[G]) \to K_0(G,\bar{k})$ are injective ([Se2], 16.1, Corollary 1 of Theorem 35), so $K_0(k[G])$ may be viewed as a subgroup of $K_0(G,k)$. The homomorphism

$$\beta: K_0(G,k) \to K_0(G,\bar{k})$$

defined by tensoring with $\bar{k}$ over $k$ restricts to a homomorphism

$$\alpha: K_0(k[G]) \to K_0(\bar{k}[G]).$$

By Proposition (16.22) in [CR], both homomorphisms $\beta, \alpha$ are split injections.

Proposition 2.1. The following diagram with injective arrows is Cartesian, i.e. it commutes and viewing the injections as inclusions, we have $K_0(\bar{k}[G]) \cap K_0(G,k) = K_0(k[G])$.

$$\begin{array}{ccc}
K_0(k[G]) & \xrightarrow{\alpha} & K_0(\bar{k}[G]) \\
\downarrow{c} & & \downarrow{\bar{c}} \\
K_0(G,k) & \xrightarrow{\beta} & K_0(G,\bar{k})
\end{array}$$
Proof. The commutativity is obvious. Now consider the extended diagram (with exact rows)

\[
\begin{array}{cccccc}
0 & \rightarrow & K_0(k[G]) & \rightarrow & K_0(k[G]) & \rightarrow & M & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & K_0(G,k) & \rightarrow & K_0(G,k) & \rightarrow & N & \rightarrow & 0
\end{array}
\]

where \( M = \text{cok} \alpha \), \( N = \text{cok} \beta \), and \( f \) is the homomorphism \( M \rightarrow N \) induced by \( \bar{c} \). By the Snake Lemma, there is an exact sequence of abelian groups

\[
0 \rightarrow \ker c \rightarrow \ker \bar{c} \rightarrow \ker f \rightarrow \text{cok} c,
\]

the first two modules being trivial since \( c \) and \( \bar{c} \) are injective. Since \( \alpha \) is a split injection, \( M = \text{cok} \alpha \) is free over \( \mathbb{Z} \), and therefore \( \ker f \) must also be free over \( \mathbb{Z} \). On the other hand, by Theorem (21.22) in [CR], we have \( |G| \cdot \text{cok} c = 0 \), so \( cok c \) is a torsion module. Using the exactness of the sequence above, this implies \( \ker f = 0 \). Now an easy diagram chase completes the proof. \( \square \)

Proposition 2.1 says that given a class \( \mathcal{C} \) in \( K_0(G,k) \), \( \mathcal{C} \) lies in the image of \( c \) if and only if \( \beta(\mathcal{C}) \) lies in the image of \( \bar{c} \). The following corollary appears to be only slightly different from this, yet some additional tools will be required for its proof.

**Corollary 2.2.** Let \( \mathcal{C} \) be a class in \( K_0(G,k) \). Then \( \mathcal{C} \) is the class of a projective \( k[G] \)-module if and only if \( \beta(\mathcal{C}) \) is the class of a projective \( \bar{k}[G] \)-module.

Before proving Corollary 2.2 we will need a few preliminary results on \( k[G] \)-modules. Recall that a \( k[G] \)-module is called *simple* if it is nonzero and has no proper \( k[G] \)-submodules, and *indecomposable* if it is nonzero and is not a direct sum of proper \( k[G] \)-submodules.

**Proposition 2.3.**

(a) For every simple \( k[G] \)-module \( M \), the \( \bar{k}[G] \)-module \( M \otimes_k \bar{k} \) is semisimple.

(b) Let \( \{P_1, \ldots, P_s\} \) be a set of representatives of the isomorphism classes of indecomposable projective \( k[G] \)-modules, and let

\[
P_i \otimes_k \bar{k} = \bigoplus_{j=1}^{r_i} \bar{Q}_{ij}, \quad \bar{Q}_{ij} \text{ indecomposable projective } \bar{k}[G] \text{-modules.}
\]

Then every indecomposable \( \bar{k}[G] \)-module is isomorphic to some \( \bar{Q}_{ij} \). Further \( \bar{Q}_{ij} \cong \bar{Q}_{ij'} \) implies that \( i = i' \), i.e. there is no overlap between the sets of indecomposable \( \bar{k}[G] \)-modules which come from different indecomposable \( k[G] \)-modules.

**Proof.** This proposition is a variation of Theorem 7.9 in [CR]. In [CR], the algebraic closure \( \bar{k} \) is replaced by a finite algebraic extension \( E \) of \( k \), and part (b) is stated for *simple* modules rather than for indecomposable projective modules. Using only elementary algebraic methods, it can be shown that there is a finite algebraic extension \( E/k \) such that every simple \( \bar{k}[G] \)-module can be realized as a simple \( E[G] \)-module, i.e. every simple \( \bar{k}[G] \)-module \( M \) can be written as \( M = N \otimes_E \bar{k} \) for some simple \( E[G] \)-module \( N \). This suffices to derive part (a) from the result in [CR]. Furthermore, it is well-known that mapping every projective \( k[G] \)-module \( P \) to the \( k[G] \)-module \( P/\text{rad} P \) gives a 1-1 correspondence between the isomorphism classes of indecomposable projective \( k[G] \)-modules.
and the isomorphism classes of simple $k[G]$-modules, whose inverse is given by taking $k[G]$-projective covers. We can thus deduce our proposition from the result in [CR], using that projective covers are additive (by Corollary 6.25 (ii) in [CR]) and commute with tensor products (by Corollary 6.25 (i) in [CR]). \qed

\textit{Proof of Corollary 2.3.} The “only if” direction is obvious. For the “if” direction, we note first of all that if $C$ is a class in $K_0(G, k)$ and $\beta (C)$ is the class of a projective $\hat{k}[G]$-module, then Proposition 2.1 yields that $C$ can be viewed as a class in $K_0(k[\hat{G}])$. Hence it suffices to show the “if” direction for classes $C \in K_0(k[\hat{G}])$, replacing the homomorphism $\beta$ by its restriction $\alpha$.

Let $\{P_1, \ldots, P_s\}$ be a set of representatives of the isomorphism classes of indecomposable $k[\hat{G}]$-modules. Every $C \in K_0(k[\hat{G}])$ can now be written as a $\mathbb{Z}$-linear combination of the classes $[P_i]$, and all coefficients of this linear combination are nonnegative if and only if $C$ is the class of a projective module. Using Proposition 2.3, one now easily shows that if $\alpha (C)$ is the class of a projective module in $K_0(k[\hat{G}])$, then $C$ is the class of a projective module in $K_0(k[G])$, which proves the assertion. \qed

\section{The equivariant Euler characteristic in terms of projective $k[G]$-modules}

By a theorem of Nakajima, the equivariant Euler characteristic of any locally free $G$-sheaf on $X$ lies in the image of the Cartan homomorphism $c : K_0(k[\hat{G}]) \to K_0(G, k)$, provided that the canonical projection $\pi : X \to Y = X/G$ is \textit{tamely ramified}. In this section, we will also consider the more general case where $\pi$ is \textit{weakly ramified}. We give both a necessary condition and a sufficient condition for the equivariant Euler characteristic to lie in the image of $c$, provided that the $G$-sheaf in question has rank 1 (comes from a divisor). Under this condition, we state an equivariant Riemann-Roch formula in the Grothendieck group of projective $k[G]$-modules.

We make the same assumptions and use the same notations as in section 1. In particular $p$ denotes the projection $\tilde{X} = X \times_k \hat{k} \to X$. Additionally, let $\bar{\pi}$ denote the canonical projection $\bar{X} \to \bar{Y} := \tilde{X}/G = Y \otimes_k \hat{k}$, and let $\bar{p}$ denote the projection $\bar{Y} \to Y$. We have the following commutative diagram:

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{p} & X \\
\downarrow^\pi & & \downarrow^\pi \\
\bar{Y} & \xrightarrow{\bar{p}} & Y 
\end{array}
$$

\textbf{Theorem 3.1.} If $\pi$ is tamely ramified and $E$ is a locally free $G$-sheaf on $X$, then the equivariant Euler characteristic $\chi(G, X, E)$ lies in the image of the Cartan homomorphism $c : K_0(k[\hat{G}]) \to K_0(G, k)$.

\textit{Proof.} Follows directly from Theorem 1 in [Na]. \qed

\textbf{Theorem 3.2.} Let $D = \sum_{P \in |X|} n_P P$ be a $G$-equivariant divisor on $X$.

(a) If $\pi$ is weakly ramified and $n_P \equiv -1 \mod e_P^G$ for all $P \in X$, then the equivariant Euler characteristic $\chi(G, X, \mathcal{L}(D))$ lies in the image of the Cartan homomorphism $c : K_0(k[\hat{G}]) \to K_0(G, k)$. If moreover one of the cohomology groups $H^i(X, \mathcal{L}(D))$, $i = 0, 1$, vanishes, then the other one is a projective $k[G]$-module.
(b) Let \( \deg D > 2g_X - 2 \). If the \( k[G] \)-module \( H^0(X, \mathcal{L}(D)) \) is projective, then \( \pi \) is weakly ramified and \( n_P \equiv -1 \text{ mod } e_P^w \) for all \( P \in |X| \).

**Proof.** If \( k \) is algebraically closed, the theorem coincides with Theorem 2.1 in [Kö2]. In the general case, if \( \pi \) is weakly ramified and \( D \) satisfies the congruence condition “\( n_P \equiv -1 \text{ mod } e_P^w \) for all \( P \)”, then \( \bar{\pi} : \bar{X} \to \bar{Y} \) is weakly ramified, and by Lemma 1.5 the divisor \( p^*D \) on \( \bar{X} \) also satisfies the congruence condition. By the special case, \( \chi(G, X, \mathcal{L}(p^*D)) \) then lies in the image of \( \bar{c} \). Hence by Proposition 2.1, \( \chi(G, X, \mathcal{L}(D)) \) lies in the image of \( c \). Here we have used that \( H^i(X, \mathcal{L}(D)) \otimes_k \bar{k} = H^i(\bar{X}, \mathcal{L}(p^*D)) \) for every \( i \) (cf. Proposition III.9.3 in [Ha]). This also implies the rest of part (a).

For part (b), let \( \deg D > 2g_X - 2 \) and let \( H^0(X, \mathcal{L}(D)) \) be projective. Then \( \deg p^*D > 2g_X - 2 \) and \( H^0(\bar{X}, \mathcal{L}(D)) \) is projective. Thus \( \bar{\pi} : \bar{X} \to \bar{Y} \) is weakly ramified and the congruence condition holds. But then \( \pi \) is weakly ramified also, and the congruence condition holds for \( D \), again by Lemma 1.5 \( \square \).

The following theorem generalizes Theorem 4.3 in [Kö2] and will be used in the formulation of the (main) Theorem 3.4. We refer the reader to page 1101 of the paper [Kö2] for an account of the nature, significance and history of the “ramification module” \( N_{G,X} \) and for simplifications of formulae (1) and (2) when \( \pi \) is tamely ramified.

**Theorem 3.3.** Let \( \pi \) be weakly ramified. Then there is a projective \( k[G] \)-module \( N_{G,X} \) such that

\[
\bigoplus_{\bar{c} \in \mathcal{C}} N_{G,X} \cong \bigoplus_{\bar{d} \in \mathcal{D}} \bigoplus_{\bar{e}^w_\bar{d}} \bigoplus_{\bar{e}^d_\bar{e}^w_\bar{d}} \text{Ind}_{G}(\text{Cov}(\text{(mod } \mathfrak{m}_P/\mathfrak{m}_P^2)^{\otimes d})),
\]

where \( \text{Cov} \) denotes the \( k[I_P] \)-projective cover. The class of \( N_{G,X} \) in \( K_0(G, k) \) is given by

\[
[N_{G,X}] = (1 - g_Y)[k[G]] - \chi(G, X, \mathcal{L}(E))
\]

where \( E \) denotes the \( G \)-equivariant divisor \( E := \sum_{P \in X}(e_P^w - 1) \cdot P \).

**Proof.** Theorem 4.3 in [Kö2] yields that there is a projective \( \bar{k}[G] \)-module \( N_{G,\bar{X}} \) such that

\[
\bigoplus_{\bar{c} \in \mathcal{C}} N_{G,\bar{X}} \cong \bigoplus_{\bar{d} \in \mathcal{D}} \bigoplus_{\bar{e}^w_\bar{d}} \bigoplus_{\bar{e}^d_\bar{e}^w_\bar{d}} \text{Ind}_{G}(\text{Cov}(\text{(mod } \mathfrak{m}_Q/\mathfrak{m}_Q^2)^{\otimes d})),
\]

and that the class of \( N_{G,\bar{X}} \) is given by

\[
[N_{G,\bar{X}}] = (1 - g_Y)[\bar{k}[G]] - \chi(G, X, \mathcal{L}(\bar{E}))
\]

where \( \bar{E} := \sum_{Q \in \bar{X}}(e_Q^w - 1) \cdot Q = p^*E \). Thus \( [N_{G,\bar{X}}] = \beta(\mathcal{C}) \) where

\[
\mathcal{C} := (1 - g_Y)[k[G]] - \chi(G, X, \mathcal{L}(E)) \in K_0(G, k).
\]

By Corollary 2.2, \( \mathcal{C} \) is the class of some projective \( k[G] \)-module, say \( N_{G,X} \). Using Lemma 1.3 and the injectivity of \( \beta \), one easily shows that \( N_{G,X} \) satisfies Formula (1) \( \square \).

For every point \( P \in X \), let \( f_P \) denote the residual degree \( [k(P) : k(\pi(P))]. \)
Theorem 3.4 (Equivariant Riemann-Roch formula). Let $\pi$ be weakly ramified.

(a) Let $P \in |X|$ be a closed point. For every $d \in \{0, \ldots, e_P - 1\}$, there is a unique projective $k[G_P]$-module $W_{P,d}$ such that

$$\text{Ind}_{I_P}^{G_P}((\text{Cov}((\mathfrak{m}_P/\mathfrak{m}_P^2)^{\otimes(-d)}))) \cong \bigoplus W_{P,d}$$

as $k[G_P]$-modules.

(b) Let $D = \sum_{P \in X} n_P \cdot P$ be a divisor on $X$ with $n_P \equiv -1 \mod e_P^u$ for all $P \in X$. For any $P \in X$, we write $n_P = (e_P^u - 1) + (l_P + m_P e_P) e_P$ with $l_P \in \{0, \ldots, e_P - 1\}$ and $m_P \in \mathbb{Z}$. Furthermore, for any $R \in Y$, fix a point $\tilde{R} \in \pi^{-1}(R)$. Then we have in $K_0(k[G])_Q$:

$$\chi(G, X, \mathcal{L}(D)) = -[N_{G,X}] + \sum_{R \in Y} \sum_{d=1}^{l_R} \text{Ind}_{I_R}^{G_R}((\text{Cov}((\mathfrak{m}_R/\mathfrak{m}_R^2)^{\otimes(-d)}))) + \left(1 - g_Y + \sum_{R \in Y} [k(R) : k|m_R]\right)[k[G]]. \quad (3)$$

Proof. We first show that under the preconditions of (b), the following holds in the Grothendieck group with rational coefficients $K_0(k[G])_Q$:

$$\chi(G, X, \mathcal{L}(D)) = -[N_{G,X}] + \sum_{R \in Y} \frac{1}{f_R} \sum_{d=1}^{l_R} \text{Ind}_{I_R}^{G_R}((\text{Cov}((\mathfrak{m}_R/\mathfrak{m}_R^2)^{\otimes(-d)}))) + \left(1 - g_Y + \sum_{R \in Y} [k(R) : k|m_R]\right)[k[G]] \quad (4)$$

With suitably chosen divisors $D$, Formula (4) will then be used to show part (a). Formula (4) and part (a) obviously imply part (b).

For curves over algebraically closed fields, we have $f_P = 1$ for all $P$, so Formula (4) coincides with Theorem 4.5 in [Kö2].

The injective homomorphism $\beta : K_0(G, k) \to K_0(G, \bar{k})$ maps $\chi(G, X, \mathcal{E})$ to $\chi(G, \bar{X}, p^*\mathcal{E})$, and by Theorem 3.2 both of these Euler characteristics lie in the image of the respective Cartan homomorphisms. Hence it suffices to show that $\beta$ maps every summand of the right-hand side of formula (3) (applied to $X, D$) to the corresponding summand of the right-hand side applied to $\bar{X}, p^*D$.

From the proof of Theorem 3.3 we see that $\beta([N_{G,X}]) = [N_{G,X}]$.

By Lemma 1.3, we have $l_Q = l_P$ and $m_Q = m_P$ whenever $Q \in p^{-1}(P)$. Furthermore, the number of preimages of a point $R \in Y$ under $\pi : X \to Y$ is $\frac{n}{e_R l_R}$. For any $S \in |\bar{Y}|$,
fix a point $\tilde{S} \in \pi^{-1}(S)$. Using Lemma [1,3] we see that
\[
\beta \left( \sum_{R \in Y} \frac{1}{f_R} \sum_{d=1}^{l_R} \left[ \text{Ind}_{I_R}^G(Cov((m_R/m_R^2)^{\otimes(-d)})) \right] \right)
\]
\[
= \sum_{Q \in \hat{X}} \frac{\epsilon_Q}{n} \sum_{d=1}^{l_Q} \left[ \text{Ind}_{G_Q}^F(Cov((m_Q/m_Q^2)^{\otimes(-d)})) \right]
\]
\[
= \sum_{s \in Y} \sum_{d=1}^{l_S} \left[ \text{Ind}_{G_S}^F(Cov((m_S/m_S^2)^{\otimes(-d)})) \right]
\]

Moreover, we have
\[
\beta \left( (1 - g_Y + \sum_{R \in Y} [k(R) : k|m_R]) \cdot [k[G]] \right) = (1 - g_Y + \sum_{s \in Y} m_S) \cdot [k[G]],
\]
which completes the proof of Formula (3).

We now prove part (a). Let $P \in X$ be a closed point. For $d = 0$, the statement is obvious because $(m_P/m_P^2)_{0}$ is the trivial one-dimensional $k(P)$-representation of $I_P$, so it decomposes into $f_P$ copies of the trivial one-dimensional $k(R)$-representation of $I_P$, where $R := \pi(P)$. Hence we only need to do the inductive step from $d$ to $d + 1$, for $d \in \{0, \ldots, \epsilon_P - 2\}$.

If $\pi$ is unramified at $P$, then $\epsilon_P = 1$, so there is no $d \in \{0, \ldots, \epsilon_P - 2\}$. Hence we may assume that $\pi$ is ramified at $P$. Set $H := G_P$, the decomposition group at $P$, and let $\pi'$ denote the projection $X \rightarrow X/H := Y'$. For every closed point $Q \in |X|$ and for every $s \geq -1$, let $H_{Q,s}$ be the $s$-th ramification group at $Q$ with respect to that cover, as introduced in Section [1]. Then we have $H_{Q,s} = G_P \cap G_{Q,s}$ for every $s \geq -1$ and every $Q \in |X|$. In particular, if $\pi$ is weakly ramified, then so is $\pi'$.

For $Q = P$, we get $H_{P,s} = G_{P,s}$ for all $s \geq -1$; in particular, the ramification indices and residual degrees of $\pi$ and $\pi'$ at $P$ are equal.

Let now $D := \sum_{Q \in |X|} n_Q \cdot Q$ be the $H$-equivariant divisor with coefficients
\[
n_Q = \begin{cases} 
(d + 2) \epsilon_Q - 1 & \text{if } Q = P \\
\epsilon_Q - 1 & \text{otherwise}
\end{cases}
\]

Then formula (4) applied to $H, X, D$ gives
\[
\chi(H, X, \mathcal{L}(D)) = -[N_{H,X}] + \frac{1}{f_P} \sum_{n=1}^{d} \left[ \text{Ind}_{I_P}^H(Cov((m_P/m_P^2)^{\otimes(-n)})) \right]
\]
\[
+ \frac{1}{f_P} \left[ \text{Ind}_{I_P}^H(Cov((m_P/m_P^2)^{\otimes(-(d+1))})) \right] + (1 - g_Y)[k[H]]
\]
\[
(5)
\]
in $K_0(k[H])_Q$. By the induction hypothesis, the sum from $n = 1$ to $d$ in this formula is divisible by $f_P$ in $K_0(k[H])$; hence the remaining fractional term $\frac{1}{f_P} \left[ \text{Ind}_{I_P}^H(Cov((m_P/m_P^2)^{\otimes(-(d+1))})) \right]$ must lie in $K_0(k[H])$. In other words, when writing $\text{Ind}_{I_P}^H(Cov((m_P/m_P^2)^{\otimes(-(d+1))}))$ as a direct sum of indecomposable projective $k[H]$-modules, every summand occurs with a multiplicity divisible by $f_P$. This proves the assertion. 
\[\square\]
In the proof of Theorem 3.4(a), we have used a preliminary version of the equivariant Riemann-Roch formula to show the divisibility of \( \text{Ind}_{I_P}^{G}((m_P/m_P^2)^{\otimes (-d)}) \) by \( f_P \), i.e. we have used a global argument to prove a local statement. This tells us very little about the structure of the summands \( W_{P,d} \), which leads to the question whether one could find a “local” proof for the divisibility. In two different situations, the following proposition provides such a proof, yielding a concrete description of \( W_{P,d} \).

**Proposition 3.5.** Assume that \( \pi \) is tamely ramified, let \( P \in |X| \) and \( d \in \{1, \ldots, e'_P-1\} \).

(a) If \( \text{Gal}(k(P)/k(\pi(P))) \) is abelian, then we have \( W_{P,d} \cong (m_P/m_P^2)^{\otimes (-d)} \) as \( k[G_P] \)-modules.

(b) If \( I_P \) is central in \( G_P \), then \( W_{P,d} \) is of the form \( W_{P,d} = \text{Ind}_{I_P}^{G_P}(\chi_d) \) for some \( k[I_P] \)-module \( \chi_d \). If moreover \( G_P \cong I_P \times G_P/I_P \), then \( W_{P,d} \cong (m_P/m_P^2)^{\otimes (-d)} \) as \( k[G_P] \)-modules.

Note that since every Galois extension of a finite field is cyclic, the first part of this proposition gives a “local” proof of Theorem 3.4(a) for the important case where \( \pi \) is tamely ramified and the underlying field \( k \) is finite.

Proposition 3.5 can be deduced from the following purely algebraic result. Note that, in this result, we don’t use the notations introduced earlier in this paper; when Proposition 3.6 is being applied to prove Proposition 3.5 the fields \( k \) and \( l \) become the fields \( k(\pi(P)) \) and \( k(P) \), respectively, the group \( G \) becomes \( G_P \) and \( V \) becomes \( (m_P/m_P^2)^{\otimes (-d)} \) which is viewed only as a representation of \( I_P \) (and not of \( G_P \)) in Theorem 4.6(a).

**Proposition 3.6.** Let \( l/k \) be a finite Galois extension of fields. Let \( G \) be a finite group, and let \( I \) be a cyclic normal subgroup of \( G \), such that \( G/I \cong \text{Gal}(l/k) \), i.e. we have a short exact sequence \( 1 \to I \to G \to \text{Gal}(l/k) \to 1 \).

Let \( V \) be a one-dimensional vector space over \( l \) such that \( G \) acts semilinearly on \( V \), that is, for any \( g \in G, \lambda \in l, v, w \in V \), we have \( g.(\lambda v + w) = \bar{g}(\lambda)(g.v) + g.w \), where \( \bar{g} \) denotes the image of \( g \) in \( \text{Gal}(l/k) \).

(a) If \( \text{Gal}(l/k) \) is abelian, then we have \( \text{Ind}_{I}^{G} \text{Res}_{I}^{G}(V) \cong \bigoplus (G:I) V \) as \( k[G] \)-modules.

(b) If \( I \) is central in \( G \), then there is a (non-trivial) one-dimensional \( k \)-representation \( \chi \) of \( I \) such that \( \text{Res}_{I}^{G}(V) \cong \bigoplus (G:I) \chi \) as \( k[I] \)-modules.

If moreover \( G = I \times \text{Gal}(l/k) \), then we have \( \text{Ind}_{I}^{G} \chi \cong V \) and \( \text{Ind}_{I}^{G} \text{Res}_{I}^{G}(V) \cong \bigoplus (G:I) V \) as \( k[G] \)-modules.

**Proof.** (a) We have (isomorphisms of \( k[G] \)-modules):

\[
\text{Ind}_{I}^{G} \text{Res}_{I}^{G}(V) \\
\cong V \otimes_k \text{Ind}_{I}^{G}(k) \quad \text{by Corollary 10.20 in } \text{[CR]}
\cong V \otimes_k k[G/I] \quad \text{(cf. §10A in } \text{[CR]})
\cong V \otimes_k k[\text{Gal}(l/k)] \quad \text{as } \text{Gal}(l/k) \cong G/I
\cong V \otimes_k l
\cong \bigoplus_{\sigma \in \text{Gal}(l/k)} V.
\]
The last two isomorphisms can be derived as follows. By the normal basis theorem, there is an element \( x_0 \in \ell \) such that \( \{ g(x_0) \mid g \in \text{Gal}(\ell/k) \} \) is a basis of \( \ell \) over \( k \). The resulting isomorphism

\[
\ell[\text{Gal}(\ell/k)] \to \ell \quad \text{given by} \\
[g] \mapsto g(x_0) \quad \text{for every } g \in \text{Gal}(\ell/k).
\]

is obviously \( \ell[G] \)-linear. This is the second last isomorphism. For the last one, we define

\[
\varphi : \ell \otimes_k V \to \bigoplus_{\sigma \in \text{Gal}(\ell/k)} V \quad \text{by} \\
\quad a \otimes v \mapsto (\sigma(a) \cdot v)_{\sigma \in \text{Gal}(\ell/k)} \quad \text{for every } a \in \ell, v \in V.
\]

\( \varphi \) is an isomorphism of vector spaces over \( k \), by the Galois Descent Lemma. If \( \text{Gal}(\ell/k) \) is commutative, then \( \varphi \) is also compatible with the \( G \)-action on both sides: Let \( a \in \ell, v \in V, g \in G \), then we have

\[
\varphi(g.(a \otimes v)) = \varphi(\bar{g}(a) \otimes g.v) = ((\sigma \bar{g})(a) \cdot g.v)_{\sigma \in \text{Gal}(\ell/k)} = g.((\sigma(a) \cdot v)_{\sigma \in \text{Gal}(\ell/k)}) = g.\varphi(a \otimes v).
\]

(b) Since \( I \) is cyclic, it acts by multiplication with \( e \)-th roots of unity, where \( e \) divides \( |I| \). If \( I \) is central in \( G \), then it follows that the \( e \)-th roots of unity are contained in \( k \). For if \( h \) is a generator of \( I \) and \( h.v = \zeta_e \cdot v \) for all \( v \in V \), \( \zeta_e \) an \( e \)-th root of unity, then we have for all \( g \in G \) and all \( v \in V \):

\[
\bar{g}(\zeta_e)(g.v) = g.(\zeta_e v) = (gh).v = (hg).v = \zeta_e(g.v).
\]

Hence for every \( \bar{g} \in \text{Gal}(\ell/k) \), we have \( \bar{g}(\zeta_e) = \zeta_e \), which means that \( \zeta_e \) lies in \( k \). Let now \( \{x_1, \ldots, x_f\} \) be a \( k \)-basis of \( V \), where \( f = (G : I) \). Then we have

\[
V = kx_0 \oplus \ldots \oplus kx_f
\]

not only as vector spaces over \( k \), but also as \( k[I] \)-modules, since

\[
Ix_i = \{ \zeta_i^j x_i \mid j = 0, \ldots, e - 1 \} \subseteq kx_i
\]

for every basis vector \( x_i \). Furthermore, the summands \( kx_i \) are isomorphic as \( k[I] \)-modules because \( I \) acts on each of them by multiplication with the same roots of unity in \( k \). Setting for example \( kx_1 =: \chi \), we can write

\[
\text{Res}_I^G(V) \cong \bigoplus_{\chi} f
\]

as requested.

Assume now that \( G = I \times \text{Gal}(\ell/k) \). Then by the Galois Descent Lemma, we have

\[
V \cong \ell \otimes_k V^{\text{Gal}(\ell/k)}
\]

as \( k[G] \)-modules, where \( I \) acts trivially on \( \ell \) and \( \text{Gal}(\ell/k) \) acts trivially on \( V^{\text{Gal}(\ell/k)} \). This is isomorphic to \( \ell \otimes_k \chi \), where \( \chi \) is regarded as a \( k[G] \)-module via the projection \( G = I \times \text{Gal}(\ell/k) \to I \). By the normal basis theorem, we have

\[
\ell \otimes_k \chi \cong \text{Ind}_I^G(k) \otimes \chi = \text{Ind}_I^G(\chi),
\]

so \( V \cong \text{Ind}_I^G(\chi) \) as requested. Together with what we have shown before, this implies the last identity of the proposition:

\[
\text{Ind}_I^G \text{Res}_I^G(V) = \text{Ind}_I^G(f \bigoplus \chi) = f \bigoplus V.
\]

\( \square \)


4 Some variants of the main theorem

Throughout the previous section, we have concentrated on the case where $\pi : X \to Y$ is weakly ramified and where the locally free $G$-sheaf we are considering comes from an equivariant divisor. If $\pi$ is tamely ramified, we have the following variant of Theorem 3.4 for locally free $G$-sheaves that need not come from a divisor. It generalizes Corollary 1.4(b) in [Kö1].

**Theorem 4.1.** Let $\pi : X \to Y$ be tamely ramified. Let $E$ be a locally free $G$-sheaf of rank $r$ on $X$. For every closed point $P \in |X|$ and for $i = 1, \ldots, r$, let the integers $l_{P,i} \in \{0, \ldots, e_P - 1\}$ be defined by the following isomorphism of $k(P)[I_P]$-modules:

$$
\mathcal{E}(P) \cong \bigoplus_{i=1}^{r} \left( \frac{m_P}{m_P^2} \right)^{\otimes l_{P,i}}.
$$

For every $R \in |Y|$, let $\tilde{R} \in |X|$ and $W_{\tilde{R},d}$ be defined as in Theorem 3.4. Furthermore, let $N_{G,X}$ be the ramification module from Theorem 3.3. Then we have in $K_0(k[G])$:

$$
\chi(G, X, \mathcal{E}) \equiv -r[N_{G,X}] + \sum_{R \in Y} \sum_{i=1}^{r} \sum_{d=1}^{l_{R,i}} \sum_{d=0}^{e_P-1} d \left[ \text{Ind}_{G}^{G} (W_{\tilde{R},d}) \right] \mod \mathbb{Z}[G].
$$

Moreover, one can show an equivariant Riemann-Roch formula for arbitrarily ramified covers $\pi : X \to Y$. Recall that in Theorem 3.2 we have shown that in virtually all cases where the Euler characteristic lies in the image of the Cartan homomorphism, the cover $\pi$ is weakly ramified. So in the general case, one cannot possibly find a formula in the Grothendieck group $K_0(k[G])$ of projective $k[G]$-modules. However, in the Grothendieck group $K_0(G, k)$ of all $k[G]$-modules, we have the following result, which generalizes Theorem 3.1 in [Kö2].

**Theorem 4.2.** Let $\mathcal{E}$ be a locally free $G$-sheaf. Then we have in $K_0(G, k)$:

$$
n \chi(G, X, \mathcal{E}) = C_{G, X, \mathcal{E}} [k[G]] - \sum_{P \in |X|} e_P^{e_P-1} \sum_{d=0}^{e_P} d \left[ \text{Ind}_{G}^{G} (\mathcal{E}(P) \otimes_{k(P)} (\frac{m_P}{m_P^2})^{\otimes d}) \right],
$$

where

$$
C_{G, X, \mathcal{E}} = r(1 - g_X) + \deg \mathcal{E} + \frac{r}{2} \sum_{P \in |X|} [k(P) : k] (e_P - 1).
$$

We omit the proofs of Theorem 4.1 and Theorem 4.2 due to their similarity with the proof of Theorem 3.4.

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