Towards a Tractable Analysis of Localization Fundamentals in Cellular Networks

Javier Schloemann, Harpreet S. Dhillon, and R. Michael Buehrer

Abstract

When dedicated positioning systems, such as GPS, are unavailable, mobile devices have no choice but to fall back on cellular networks for localization. These, however, are designed primarily with communications goals in mind, where ideally only one base station (BS) is hearable at a given location. Since BSs are not placed with the goal of providing favorable geometries for localization, it is appropriate to study cellular localization performance using random geometries. In the literature, however, localization performance is typically only studied analytically for deterministic (and usually favorable) geometries (e.g., using the Cramér-Rao lower bound). Random geometries are studied through simulation, as no tractable approach exists for gaining analytical insights into how system design affects localization performance in nondeterministic setups. In this paper, we develop a new tractable approach where we endow the BS locations with a distribution by modeling them as a Poisson Point Process (PPP), and use tools from stochastic geometry to obtain easy-to-use expressions for key performance metrics. In particular, we focus on the probability of decoding a given number of BSs at the device of interest, which is shown to be closely coupled with the eventual localization performance. Due to the presence of dominant interferers in the form of other BSs, this probability decreases dramatically with an increase in the desired number of decodable BSs. In order to mitigate this excessive interference, we incorporate both BS coordination and frequency reuse in the proposed framework and quantify the resulting performance gains analytically.

Index Terms

Cellular localization, E911, hearability, stochastic geometry, point process theory, base station coordination, frequency reuse.

The authors are with the Mobile and Portable Radio Research Group (MPRG), Wireless@Virginia Tech, Blacksburg, VA, USA. Email: {javier, hdhillon, buehrer}@vt.edu. This paper is submitted in part to IEEE ICC 2015 Workshop on Advances in Network Localization and Navigation (ANLN), London, UK [1]. Manuscript last updated: February 25, 2015.
I. Introduction

Geolocation (also called *positioning, localization*, and *position location*) is deeply ingrained in our daily lives and has been studied by the scientific community for many years [2]–[6]. The driving force behind much of the research is a mandate by the Federal Communications Commission (FCC) requiring cellular network operators to locate those calling 911 to within certain accuracy requirements [7], [8]. Until recently, these requirements included only outdoor location accuracies. Accordingly, the predominant way cellular network operators have met the requirements of the mandate is by relying on the Global Positioning System (GPS). In January of 2015, however, the FCC expanded its mandate to include a phase-in of indoor positioning requirements, citing that the bulk of emergency calls now originate indoors [9]. While accurate outdoor positioning using GPS is reliably available under clear sky conditions, many years of positioning study have not yet resulted in equally reliable positioning methods in *urban canyons* and indoor scenarios. Consider the classical example of locating emergency personnel (e.g., firefighters) indoors, where our current inability to provide accurate indoor positioning has recently been described as a dilemma “where people [are] literally dying within a hundred feet of safety” [10]. Despite this, economic limitations dictate that global navigation satellite systems (GNSS), such as GPS, are likely to be the only widespread dedicated location systems in the foreseeable future. This necessitates a fallback to terrestrial cellular networks for geolocation in situations where these prevalent location technologies are unavailable. Presently, however, no analytical approaches exist to study the fundamentals of localization performance in these networks. It is the objective of this paper to introduce a new tractable model for studying terrestrial geolocation using cellular networks. The model uses concepts from point process theory [11] and stochastic geometry [12] to lend tractability to the study of network localization, which is demonstrated through accurate and easy-to-use expressions characterizing the number of base stations (BSs) that are able to participate in a localization procedure, a key factor influencing the localization performance.

A. Prior Art and Motivation

Regardless of the technique, geolocation performance fundamentally depends upon three things: *(i) the geometry or locations of the BSs relative to the device being localized, (ii) the number of participating BSs, and (iii) the accuracy of the positioning observations.* When these factors
are deterministic, localization performance is typically studied analytically using the Cramér-Rao lower bound (CRLB) (e.g. [13]–[15]). Since localization systems are being discussed, the topologies considered are often favorable to location estimation (such as by using a carefully control set of scenarios to avoid poor geometries, e.g. [14]) and may not accurately depict the majority of topologies encountered in networks not catered to localization, such as cellular networks. This is because, broadly speaking, cellular BSs are deployed in locations so as to maximize coverage and rate for communication purposes, goals which are not sympathetic to localization goals, thereby potentially resulting in highly unfavorable localization scenarios. Consider that in communication systems, it is ideal for a mobile device to receive a strong signal from its serving BS and weak signals from all its neighbors, whereas localization is not possible unless a mobile device receives usable signals from these neighbors, with accuracy generally improving as more neighbors are heard. To underscore the conflicting nature of the goals, the need to cater to localization demands has a name among cellular system designers, who refer to it as the hearability problem, due to the fact that increasing hearability from neighboring cells for the purposes of positioning is contrary to the principles of cellular system design [16]. Since communication will always be the primary goal of cellular networks, a fair study of localization in cellular networks should consider random BS geometries and not just favorable BS geometries. Unfortunately, a metric such as the CRLB cannot be employed when the aforementioned factors are not deterministic, except through the simulation of large numbers of network realizations.

One field of study in which random network topologies are considered when discussing localization is that of wireless or ad-hoc sensor networks. However, these studies differ from the cellular scenario we consider in that the sensor network literature almost certainly ignores interference and propagation effects, instead opting to define coverage using some fixed detection range [14], [17], [18], though in rare cases the received signal strength may be used to define a circular coverage region when shadowing is also considered [19]. In order to apply to cellular networks, our analysis differs by not only including interference, but also by employing a model enriched with additional cellular and localization-specific parameters.

Lastly, when cellular networks are specifically considered, they are typically studied using the popular hexagonal grid model, making analytical results difficult to come by. Instead, network designers resort to complex system-level simulations using a common set of agreed-upon evaluation parameters [20], as is clear from the 3GPP standardization process [21]. This is also a common practice in the literature [22]–[24]. The work in [22] and [23] is most similar to ours, studying cell
hearability in GSM and WCDMA [22] and LTE [23]. Interference, coordination, and fractional load are considered, but in the end, one is left with only simulation results, from which it is not possible to gather general insights into how localization performance is impacted by system design parameters and propagation effects. This motivates the need for a tractable analytical model that can provide preliminary design insights and will either circumvent the need for simulations completely or limit the ranges of the simulation parameters. In [25] and [26], such tractable, yet accurate, analytical models were developed to study the coverage and rate of single-tier and multi-tier cellular communication systems, respectively. It was shown that the spatial layout of the BSs in cellular networks can be tractably and realistically modeled using Poisson point processes (PPPs), thereby allowing the use of powerful tools from stochastic geometry to derive closed form expressions for key performance metrics. Motivated by these advances, we develop a similar approach to lend tractability to the analysis of localization systems.

B. Contributions

The main contributions of this paper are as follows.

A general model for studying localization in cellular networks: In Section II, we build on the foundations of [25] and [26] and present a tractable model for studying localization in cellular networks. The model includes average network load, the ability for BSs participating in the localization procedure to coordinate transmissions (e.g., through joint scheduling in LTE [16]), and is easily adapted to frequency reuse when frequency bands are assigned randomly. From [25], it is easy to see that self-interference from the network often leads to poor coverage probabilities even when the reception of only a single signal is required. Since localization procedures require the successful reception of multiple signals and are thus even more demanding, localization systems typically necessitate the integration of signals over time in order to improve detectability. Our model assumes that this integration provides some fixed processing gain, while fast fading is excluded under the assumption that it is averaged out at the receiver. This exclusion of small-scale fading effects in our model agrees with the state-of-the-art 3GPP simulation models [20].

Definition and characterization of accuracy-related hearability metric: We define a metric in order to study the number of BSs whose signals arrive with sufficient quality to successfully participate in a localization procedure. After presenting a clear relationship between the metric and positioning accuracy, upper bounds and approximations for the distribution of this metric are derived using the proposed model, resulting in expressions that are closed-form in some special
cases of interest, while easy-to-compute integral expressions for the others. A dominant-interferer type analysis is employed [27], [28], whereby the strongest interference source is always treated exactly, leading to remarkably accurate approximations, nearly indistinguishable from truth.  

*Design insights:* Lastly, we present results and draw out some interesting observations useful in the design of localization systems. We make evident the fact that localization performance is limited by the interference from the BSs participating in the localization procedure. It is shown that mitigating this interference through BS cooperation helps up to a point, beyond which the true gains in localization performance are had with frequency reuse. The need to employ frequency reuse for accurate positioning has been observed previously, but only by first going through the process of running many complex system-level simulations. In LTE, for example, a reuse of six was deemed necessary for downlink TDOA positioning, a verdict made after many simulations.

II. **SYSTEM MODEL**

We now formally describe the proposed model. The key notation presented in this section is summarized in Table I.

A. **Spatial base station layout**

The locations of the BSs are modeled using a homogeneous PPP $\phi \in \mathbb{R}^2$ with density $\lambda$ [12]. Due to the stationarity of a homogeneous PPP, the device to be localized is assumed to be located at the origin $o$. If the interference is treated as noise at the receiver, the most appropriate metric that captures link quality is the signal-to-interference-plus-noise ratio (SINR). For the link from some BS $x \in \phi$ to the origin, the SINR can be expressed as:

$$\text{SINR}_x = \frac{PS_x \|x\|^{-\alpha}}{\sum_{y \in \phi \setminus \{x\}} PS_y \|y\|^{-\alpha} + \sigma^2},$$

where $P$ is the transmit power, $S_z$ denotes the independent shadowing affecting the signal from BS $z$ to the origin, $\alpha > 2$ is the pathloss exponent, and $\sigma^2$ is the noise variance. Note that (1) represents the SINR prior to any processing gain, yet as will be evident in the sequel, positioning systems typically have to work at lower target SINRs, thereby necessitating the need for some form of processing gain. In general, the post-processing SINR will include some multiplicative factor $\gamma$ representing the processing gain, which depends upon system parameters (e.g. integration time) and is assumed to average out the effect of small scale fading. This is the reason why the
TABLE I
SUMMARY OF KEY NOTATION

| Notation | Description |
|----------|-------------|
| $\alpha$ | Path loss exponent ($\alpha > 2$) |
| $\|\cdot\|_2$ | $\ell_2$-norm |
| $o$ | Origin (location of the typical user) |
| $\Phi$ | PPP of BS locations |
| $\lambda$ | Density of $\Phi$ |
| $x_i (\hat{x}_i)$ | Location of the $i$th closest (active) BS in $\Phi$ |
| $R_i (\hat{R}_i)$ | Distance of $x_i (\hat{x}_i)$ from the origin |
| $\gamma$ | Processing gain |
| $\beta$ | Post-processing SINR required for a successful wireless link |
| $L$ | Number of participating BSs |
| $p$ | Activity factor among $L$ participating BSs |
| $q$ | Average network load for non-participating BSs |
| $b(\theta, r)$ | A ball centered at $\theta$ with radius $r$ |
| $A^c$ | The complement of area $A$ |
| $A \setminus B$ | The set (or area) $A$ excluding $B$ |
| $\mathbb{1}(\cdot)$ | Indicator function, 1/0 when its argument is true/false |

SINR expression in (1) does not contain a fast fading term, which is consistent with current models for evaluating cellular positioning performance [20]. Those conversant with stochastic geometry-based analyses of wireless networks will recognize that the absence of fast fading on the serving link, specifically one from the exponential family of distributions (e.g., Rayleigh fading), adds to the technical challenge of a localization system analysis.

B. Selection of participating base stations

As is common in communication system analyses, serving BSs are selected according to the strongest BS association policy, measured using average signal strength, which typically includes long time-scale effects such as shadowing and pathloss. Now, consider that we desire to use a total of $L$ BSs for positioning. Thus, we assume that the $L$ BSs which provide the highest average received power make up the set of participating BSs. Their successful participation, however, is not guaranteed as there is some post-processing SINR threshold $\beta$ (or equivalently, pre-processing SINR threshold $\beta/\gamma$) above which the signals from the participating BSs must arrive in order
for them to successfully contribute to the localization procedure. In the absence of shadowing, the set of potential BSs simply corresponds to the set of the $L$ nearest BSs. When shadowing is considered and BSs are selected according to average signal strength, the effect of shadowing may be absorbed as a perturbation in the locations of the BSs provided that $\mathbb{E}\left[ S_z^{2/\alpha} \right] < \infty$ [29], [30]. Thus, without loss of generality, we define a new equivalent PPP with density $\lambda \mathbb{E}\left[ S_z^{2/\alpha} \right]$. In doing so, we ensure that the strongest BS association policy in the original PPP is equivalent to the nearest BS association policy in the transformed PPP. For notational simplicity, we will continue to represent the transformed PPP by $\Phi$ with density $\lambda$, under the assumption that if shadowing is present, it is already reflected in the density of $\Phi$. Note that the condition on the fractional moment is fairly mild and will almost always be satisfied by the distributions of interest, including the most common assumption of log-normal shadowing with finite mean and standard deviation [30]. As a final note, it has been shown that for SINR, shadowing causes even more regular network models, such as the common hexagonal lattice, to behave like a PPP model [31], [32]. This further validates the use of a PPP to model BS locations.

C. Base station coordination and network load

BS coordination and network load are modeled through two activity factors, $p$ and $q$, respectively. During a localization procedure, the $L$ participating BSs coordinate with each other and attempt to blank their own transmissions while the others are active, but are unable to do so with probability $p$ due to network traffic demands. The remaining BSs are assumed active with probability $q$, which is simply the average network load [33]. For simplicity of exposition, let us order the BSs in the now shadowing-transformed $\Phi$ in terms of increasing distance from the origin such that the location of the $k$th farthest BS from the origin is denoted by $x_k \in \Phi$. We now enrich the previous SINR expression in (1) for the signals arriving from the participating BSs (i.e., $x_k$ for $k \in \{1, \ldots, L\}$) by including BS coordination and network load as follows:

$$\text{SINR}_k(L) = \frac{P \| x_k \|^{-\alpha}}{\sum_{i=1}^{L} a_i P \| x_i \|^{-\alpha} + \sum_{j=L+1}^{\infty} b_j P \| x_j \|^{-\alpha} + \sigma^2},$$

where $a_i$ and $b_j$ are independent indicator random variables (fixed throughout the localization procedure) modeling the thinning of the interference field by taking on value 1 with probabilities $p$ and $q$, respectively. Note that $\text{SINR}_k$ is now a function of $L$ because of the potentially different activity factors for the participating BSs and the rest of the network. The number of active
participating BSs interfering with the $L^{th}$ BS is $\Omega = \sum_{i=1}^{L-1} a_i$, which is a binomial random variable with probability mass function
\[
f_\Omega(\omega) = \binom{L - 1}{\omega} p^\omega (1 - p)^{L-1-\omega}, \quad \omega \in \{0, \ldots, L - 1\}. \tag{3}
\]

\section*{D. Base station range distributions}

From (2), it is clear that the SINRs are dependent on the distances of the BSs from the origin, rather than the locations themselves. Thus, it is worthwhile to characterize these. Let $R_k = \|x_k\|$ and $\hat{R}_m = \|\hat{x}_m\|$, where $\hat{x}_m$ is now the $m^{th}$ farthest active BS. This is illustrated in Figure 1 for the closest active ($\hat{R}_1$) and $L^{th}$ closest overall ($R_L$) BSs. The distribution of $R_L$ is known to be [34]:
\[
f_{R_L}(r) = e^{-\lambda \pi r^2} \frac{2(\lambda \pi r^2)^L}{r\Gamma(L)}. \tag{4}
\]

Now, conditioned on the location (or distance) of the $L^{th}$ BS and $\Omega$, we present a useful lemma for understanding the distribution of the locations of the $\Omega$ active participating BSs.

\begin{lemma}
Conditioned on the location of the $L^{th}$ BS, the $\Omega$ active BSs closer to the origin are distributed according to a binomial point process (BPP) (i.e., in a uniformly random manner) inside the circle of radius $R_L$ centered at the origin.
\end{lemma}
Proof. See Appendix A. ■

Using Lemma 1, we obtain the following distribution for the most dominant (i.e., closest active) interferer, given the distance of the $L^{th}$ BS and $\Omega$. This is formally presented below.

**Lemma 2.** The cumulative distribution function of the closest active BS distance $\hat{R}_1$ given $R_L$ and $\Omega$ is

$$F_{\hat{R}_1|R_L}(r|R_L) = 1 - \left(\frac{R_L^2 - r^2}{R_L^2}\right)^\Omega. \quad (5)$$

Proof. See Appendix B. ■

Let $A = \mathbb{B}(o, R_L) \setminus \mathbb{B}(o, \hat{R}_1)$, where $\mathbb{B}(\theta, r)$ represents a ball of radius $r$ centered at $\theta$. We have the following lemma characterizing the distribution of the $\Omega - 1$ remaining active BSs in the annular area $A$.

**Lemma 3.** Conditioned on $\hat{R}_1$ and $R_L$, the $\Omega - 1$ active BSs located inside the annular region $\mathbb{B}(o, R_L) \setminus \mathbb{B}(o, \hat{R}_1)$ are distributed according to a BPP.

Proof. See Appendix A. ■

### III. Localization Performance

As mentioned in Section I, localization performance fundamentally depends on the number of positioning observations, their accuracies, and the locations of the participating BSs relative to the device being localized. These three factors are all highly interdependent. For instance, both the number of BSs whose signals arrive with sufficiently-high SINRs to participate in the localization procedure and the quality of their positioning observations are strongly dependent on the interference field, which is itself driven by the realized network deployment. Thus, a full characterization of localization performance would take into account all possible geometric conditionings of the BSs, a task which is extremely difficult. By taking into account even just the number of participating BSs, however, valuable insights into localization performance may be gleaned.

**A. Base station participation and geolocation performance**

A clear relationship exists between the number of BSs involved in positioning and the resulting localization accuracy. In order to see this, consider the CRLB, which provides a lower bound on
the achievable performance of an unbiased estimator, applied to time-difference-of-arrival (TDOA) positioning, commonly employed in cellular localization (e.g., observed time-difference-of-arrival, OTDOA, in LTE). Using the model described in the previous section with a BS deployment density equivalent to that of an infinite hexagonal grid with 500m intersite distances and a shadowing standard deviation of 8 dB, a localization system was simulated requiring $\beta = -6$ dB for successful signal detection after a processing gain of $\gamma = 10$ dB. A fully-loaded network was assumed with no BS coordination (i.e., $p = q = 1$). The threshold $\beta$ determines which BSs may successfully participate in the localization procedure, while the BSs’ exact SINRs determine the accuracies of their individual ranging observations, calculated using the well-known TOA ranging CRLB [35] with an assumed signal bandwidth of 2 MHz. Lastly, as is typical, the strongest BS is selected as the reference BS for TDOA. From Figure 2, which plots the range of the lower 95th percentiles of the achievable root-mean-square (RMS) positioning errors versus the number of successfully participating BSs, we see that not only does the achievable localization performance improve with an increasing number of BSs, it starts to become almost predictable. This is not altogether intuitive because as more BS transmissions arrive with SINRs $> \beta$, one would expect their individual SINRs to generally decrease since they also interfere with each other, meaning that while more ranging observations are available, their accuracies are generally poorer. Instead of considering an exact number of participating BSs, Figure 3 reveals that if a cellular network is designed such that it is able to reliably provide some minimum number of successful positioning connections, a good understanding of the expected positioning accuracies may be had, which would certainly be useful during network design for meeting the FCC E911 requirements.

Motivated by these trends, we now formally propose a metric to study the number of BSs able to successfully participate in a localization procedure for a given target SINR $\beta$ and processing gain $\gamma$.

**Definition 1 (Participation metric).** For a given BS deployment $\phi \in \Phi$, let $\Upsilon$ represent the maximum number of selectable BSs such that all successfully participate in a localization procedure. This can be mathematically defined as:

$$\Upsilon = \arg \max_\ell \ell \times \prod_{k=1}^{\ell} \mathbf{1} \left( \text{SINR}_k(\ell) \geq \frac{\beta}{\gamma} \right).$$

(6)

Note that this metric differs from a traditional metric such as the CRLB in that it is not directly tied to a specific positioning accuracy value, though it is still closely related. Its advantage lies in
its tractability for random network analyses, whereas the CRLB, on the other hand, does provide
direct insight into achievable positioning accuracy, but only for deterministic networks, quickly
becoming intractable to characterize for random networks.

B. Participation from a desired number of base stations

While generally better localization performance can be attained by increasing BS participation,
it is clear from Figure 4 that the probability of attaining some desired number of successful
BS connections decreases sharply as the number increases. The results in Figure 4, derived
from the aforementioned simulation, assume no BS coordination ($p = 1$). How the fundamental
behavior of the localization system changes for more general parameters, such as increased BS coordination \((p \leq 1)\), is exactly the objective of our study. Thus, it is desirable to understand exactly how \(\Upsilon\) is impacted by the network design parameters and propagation effects across all possible geometric conditions.

**Definition 2** \((L\)-localizability probability). For a given \(\phi \in \Phi\), a mobile device is said to be \(L\)-localizable if at least \(L\) BSs may successfully participate in the localization procedure. The probability of this occurring is simply:

\[
P_L = \mathbb{P}(\Upsilon \geq L) = \mathbb{E}\left[\prod_{k=1}^{L} \mathbb{I}\left(\text{SINR}_k(L) \geq \frac{\beta}{\gamma}\right)\right].
\]  

(7)

Note that for \(L = 1\), this simply gives the downlink coverage probability.

It may be insightful to step back and consider a specific application of \(P_L\). Note that the first objective in any location system is to make sure that the device to be located can decode positioning signals from a sufficient number of BSs (i.e., that it is *localizable*). By this, we simply mean that an estimate of the device’s location can be found without ambiguity. In the noiseless case, this means that there can only be one solution. In the noisy case, this means that there is a single global minimum to the appropriate cost function. Commonly-accepted minimum values of \(L\) for the unambiguous operation of a localization system in the \(\mathbb{R}^2\) plane are 2, 3, and 4 for AOA, TOA/RSS, and TDOA, respectively. Thus, for example, \(P_4\) can be equivalently thought of as the coverage probability of TDOA positioning.
For two edge cases of interest, that with no BS coordination \( (p = 1) \) and that with perfect BS coordination \( (p = 0) \), it is straightforward to infer from (2) that

\[
1 \cdot (\text{SINR}_k(L) \geq \beta) \geq 1 \cdot (\text{SINR}_l(L) \geq \beta)
\]

for all \( k \leq l \leq L \). This simply means that the received SINR from a BS farther from the mobile device is lower than that of a closer BS, implying that the probability of \( L \)-localizability in (7) can be equivalently expressed as

\[
P_L = \mathbb{E} \left[ 1 (\text{SINR}_L(L) \geq \beta/\gamma) \right].
\]

With partial BS coordination \( (0 < p < 1) \), the relationship in (8) does not hold in certain corner cases. These rare occurrences have a minimal impact on our analysis, and we proceed by using the expression for \( P_L \) in (9) for all \( p \). In the numerical results, we will validate that (9) is in fact an accurate approximation of \( P_L \) for all \( p \) by comparing with the true \( P_L \), which jointly considers the SINRs of all participating BSs.

C. A simple bound on \( P_L \)

We now derive an upper bound on \( L \)-coverage probability by (i) considering the closest interferer’s location exactly, (ii) placing the remaining \( \Omega - 1 \) active interior interferers at the edge of the field of participating BSs, and (iii) ignoring thermal noise and all distant BSs. We illustrate this setup in Figure 5, which applies the above considerations to the network realization of Figure 1. Note that the interference from all BSs beyond the \( L^{th} \) BS is ignored, while the active interior BSs have been pushed out to the same distance as the \( L^{th} \) BS. The dominant interferer, however, has been left untouched.

**Theorem 1** (Upper bound on \( L \)-coverage probability). The probability that a device can successfully utilize a desired \( L \) BSs for positioning is upper bounded by

\[
P_L(p, q, \alpha, \beta, \gamma, \lambda) \leq \sum_{\omega=0}^{\chi} \left( 1 - \left( \frac{\gamma}{\beta} - (\omega - 1) \right)^{-\frac{2}{\alpha}} \right)^{\omega} f_{\Omega}(\omega),
\]

where \( \chi = \min\{L - 1, \lfloor \gamma/\beta \rfloor \} \).

**Proof.** See Appendix C.

One thing we notice immediately from (10) is that the bound is independent from the BS deployment density \( \lambda \). This is in line with similar observations made in coverage analyses for
interference-limited networks (i.e., network interference dominates thermal noise) [25], [26]. We will return to this observation at the end of this section. In the special case when all participating BSs transmit simultaneously, the above bound reduces to a simple closed-form expression.

**Corollary 1.1** (The special case of \( p = 1 \)). When all participating BSs transmit simultaneously, 
\[
P_L(1, 1, \alpha, \beta, \gamma, \lambda) \leq \left( 1 - \left( \frac{\gamma}{\beta} - (L - 2) \right)^{-\frac{2}{\alpha}} \right)^{L-1},
\]
\[
\text{for } \beta < \frac{\gamma}{L-1}, \text{ and zero otherwise.}
\]

**Proof.** When \( p = 1 \), \( f_\Omega(L - 1) = 1 \) and the result follows from (10).

From this result, we see that as greater numbers of participating BSs are desired, the probability of attaining those numbers of BSs with sufficiently strong connections decreases dramatically. Note that by rearranging (11), we also obtain a useful lower bound on the processing gain required to achieve some desired \( L \)-localizability probability.

**Corollary 1.2** (Lower bound on processing gain with no BS coordination). The processing gain required to reach a desired \( L \)-coverage probability \( P_L \) when \( p = 1 \) is lower-bounded by
\[
\gamma \geq \beta \left( \left( 1 - P_L^{\frac{L-1}{2}} \right)^{-\frac{2}{\alpha}} + L - 2 \right).
\]

\[
\text{(12)}
\]
D. Approximations of $P_L$

Let
\[ I_1 = \sum_{i=2}^{\Omega} P\|\hat{x}_i\|^{-\alpha} \]  (13)
be the aggregate interference due to the active BSs (if any) between $\hat{x}_1$ and $x_L$. Furthermore, let
\[ I_2 = \sum_{j=L+1}^{\infty} b_j P\|x_j\|^{-\alpha} \]  (14)
be the aggregate interference due to the infinite field of BSs located further than the $L^{th}$ BS. Clearly then, when $\Omega \geq 1$,
\[ \text{SINR}_L(L) = \frac{PR_L^{-\alpha}}{P\hat{R}_1^{-\alpha} + I_1 + I_2 + \sigma^2}. \]  (15)
In this section, we will approximate (15) and in turn $P_L$ by making the following assumption.

Assumption 1. We assume that if the dominant interferer (i.e., the closest active BS) is considered exactly, the remaining interference terms in (15), namely $I_1$ and $I_2$, may be accurately approximated by their means conditioned on $\hat{R}_1$, $R_L$, and $\Omega$.

This type of dominant interferer analysis has been employed previously with desirable results [27], [28] and will yield remarkably simple, yet accurate, approximations in our analysis, as will be demonstrated later. This is because considering $\hat{R}_1$ exactly and $I_1$ using its mean (conditioned on $\hat{R}_1$), results in an accurate approximation of the total interference due to the BSs closer than $x_L$, which is typically the performance-limiting term.

In interference-limited networks, Assumption 1 allows us to replace $\text{SINR}_L(L)$ in (9) by
\[ \text{SIR}_L(L) = \frac{PR_L^{-\alpha}}{P\hat{R}_1^{-\alpha} + \mathbb{E}[I_1|\hat{R}_1, R_L, \Omega] + \mathbb{E}[I_2|R_L]^2}, \]  (16)
where SIR stands for the signal-to-interference ratio. As the deployment density grows, the assumption of interference-limited networks increases in validity, and since our work is motivated by the reuse of existing infrastructures, such as cellular networks with small cell extensions which have relatively high deployment densities, we aver that it is quite reasonable. Expressions for the above conditional means are presented in the following lemmas.

Lemma 4. The expected value of $I_1$ conditioned on $\hat{R}_1$, $R_L$, and $\Omega$ is
\[ \mathbb{E}[I_1|\hat{R}_1, R_L, \Omega] = \frac{2P(\Omega - 1)}{2 - \alpha} \cdot \frac{R_L^{2-\alpha} - \hat{R}_1^{2-\alpha}}{R_L^2 - \hat{R}_1^2}, \]  (17)
for $\Omega \geq 1$, and zero otherwise.

**Proof.** See Appendix D. ■

**Lemma 5.** The expected value of $I_2$ conditioned on $R_L$ is

$$
\mathbb{E}[I_2 | R_L] = \frac{2P\pi q\lambda}{\alpha - 2} R_L^{2-\alpha},
$$

(18)

for $\alpha > 2$, and unbounded otherwise.

**Proof.** See Appendix E. ■

By inserting the results of Lemmas 4 and 5 into (16), we obtain increasingly specialized (and simpler) approximations of the $L$-localizability probability. In order to proceed, however, we must first account for the case when $\Omega = 0$, in which case $\hat{R}_1$ in (16) has no meaning. Thus, we first derive an approximation of $P_L$ for the special $\Omega = 0$ case, which incidentally corresponds to the perfect coordination ($p = 0$) scenario.

**Proposition 1.** Under Assumption 1, the probability of $L$-localizability with perfect BS coordination in interference-limited networks is

$$
P_L(0, q, \alpha, \beta, \gamma, \lambda) = 1 - \sum_{\ell=0}^{L-1} e^{-\frac{\alpha-2}{2q^3/\gamma}} \left(\frac{\alpha-2}{2q^3/\gamma}\right)^\ell \ell!.
$$

(19)

**Proof.** See Appendix F. ■

Once again, the density of the BS deployment does not appear to affect $P_L$. With this special case out of the way, we now proceed, starting with our most general result.

**Theorem 2.** Under Assumption 1, the probability that a mobile device is able to localize itself using $L$ BSs in interference-limited networks is

$$
P_L(p, q, \alpha, \beta, \gamma, \lambda) = \left(1 - \sum_{\ell=0}^{L-1} e^{-\frac{\alpha-2}{2q^3/\gamma}} \left(\frac{\alpha-2}{2q^3/\gamma}\right)^\ell \ell!\right) f_\Omega(0) + \frac{4(\lambda\pi)^L}{(L-1)!} \sum_{\omega=1}^{L-1} f_\Omega(\omega) \int_0^\infty \int_0^{r_L} \hat{r}_1 e^{-\frac{\beta}{\gamma}} (r^2_L - r^2_1) \omega^{2(L-\omega)-1} \omega e^{-\lambda\pi r^2_L} \, d\hat{r}_1 \, dr_L.
$$

(20)

**Proof.** See Appendix G. ■
Thus, we obtain an expression which requires only double integration in order to determine $P_L$, and though the outside integral has an infinite integration bound, the expression is not difficult to evaluate numerically thanks to the decaying exponential term in the integrand. To appreciate the value of this approximation, consider the $p = 1$ scenario in which case the first term in (20) disappears and the summand need only be evaluated at $\omega = L - 1$. Though the authors did not have localization in mind, the exact expression for the $k$-coverage problem in the absence of fast fading in [36, Corr. 7] applies directly to $P_L$ for $p = q = 1$. A cursory glance at the exact expression in [36] reveals its cumbersome nature, involving sums over many multi-fold integrals even in the absence of thermal noise, which is considerably more complex in comparison to (20). Moreover, by applying a slight secondary approximation and multiplying $E[I_2|R_L]$ in (16) by $E[R_1^2]/R_1^2$, where $E[R_1^2] = \frac{1}{\pi \lambda}$ [34], (20) is further reduced, yielding a single-integral expression applicable to the general case.

**Corollary 2.1.** Under Assumption 1, the general expression for $P_L$ in (20) may be further approximated for interference-limited networks by

$$P_L(p, q, \alpha, \beta, \gamma, \lambda) = P_L(0, q, \alpha, \beta, \gamma, \lambda) f_{\Omega}(0) + 2 \sum_{\omega=1}^{L-1} f_{\Omega}(\omega) \omega \int_1^{\infty} \mathbf{1} \left( x^\alpha + \frac{2(\omega - 1)}{2 - \alpha} \cdot \frac{x^2 - x^\alpha}{x^2 - 1} + \frac{2qx^2}{\alpha - 2} \leq \frac{\gamma}{\beta} \right) \frac{(1 - x^{-2})^{\omega-1}}{x^3} dx. \quad (21)$$

**Proof.** See Appendix H. ■

This expression is instantly and accurately evaluated in a numerical computation environment such as MATLAB. We now briefly describe how the introduction of the additional terms into (16) reduces (20). The key is that after some algebraic manipulation, (16) may now be expressed as follows (see (32) in the appendix):

$$\text{SIR}_L(L) \approx \frac{1}{\left( \frac{R_L}{R_1} \right)^\alpha + \frac{2(\Omega-1)}{2-\alpha} \cdot \left( \frac{R_L}{\hat{R}_1} \right)^2 - \left( \frac{R_L}{\hat{R}_1} \right)^\alpha \cdot \frac{2q}{\alpha-2} \left( \frac{R_L}{\hat{R}_1} \right)^2}, \quad (22)$$

which contains $\hat{R}_1$ and $R_L$ only through their ratio $X = R_L/\hat{R}_1$. The cumulative distribution function of this ratio exhibits the simple form of $F_X(x) = (1 - 1/x^2)^\Omega$, which is readily obtained from (5). For the special case of $\alpha = 4$, this additional approximation needn’t be used, however, as a single integral expression is obtained directly from (20).
Corollary 2.2. Under Assumption 1 and for the special case of $\alpha = 4$ in interference-limited networks,

$$P_L(p, q, 4, \beta, \gamma, \lambda) = \sum_{\omega=0}^{\chi} f_\Omega(\omega) \int_{0}^{\sqrt{\gamma/\beta - \pi q \lambda \omega}} \left(1 - \frac{1}{\sqrt{\gamma/\beta - \pi \lambda r^2 + \frac{(\omega-1)^2}{4} - \frac{\omega-1}{2}}}\right)^\omega f_{RL}(r)dr,$$

where $\chi = \min\{L - 1, \lceil\gamma/\beta\rceil\}$.

Proof. See Appendix I.

For fully-loaded networks (i.e., $p = q = 1$), (23) simplifies further to the following expression.

Corollary 2.3. Under Assumption 1 and for the special case of $\alpha = 4$ in fully-loaded interference-limited networks,

$$P_L(1, 1, 4, \beta, \gamma, \lambda) = \int_{0}^{\sqrt{\gamma/\beta - (L-1)\pi \lambda}} \left(1 - \frac{1}{\sqrt{\gamma/\beta - \pi \lambda r^2 + \frac{(L-2)^2}{4} - \frac{L-2}{2}}}\right)^{L-1} f_{RL}(r)dr,$$

for $\beta < \frac{\gamma}{L-1}$ and zero otherwise.

For larger values of $L$ and higher values of $p$, it may be reasonable to ignore the $I_2$ term in (16) under the assumption that the interference due to the $\Omega$ active participating BSs dominates that of the rest of the network. By doing so, expressions not requiring any integration may be found for the special cases covered in Corollaries 2.2 and 2.3.

Corollary 2.4. Under Assumption 1 and when $\alpha = 4$, $P_L$ for networks limited by the interference from the nearby BSs participating in the localization procedure is

$$P_L(p, q, 4, \beta, \gamma, \lambda) = \sum_{\omega=0}^{\chi} f_\Omega(\omega) \left(1 - \frac{1}{\sqrt{\gamma/\beta + \frac{(\omega-1)^2}{4} - \frac{\omega-1}{2}}}\right)^\omega,$$

where $\chi = \min\{L - 1, \lceil\gamma/\beta\rceil\}$.

Proof. Consider the limit of (23) as $q \to 0$. The parenthetical term becomes independent of $r$ and may be pulled out of the integral, while the remaining integral covers the probability density function $f_{RL}(r)$ over the entirety of its domain, thus evaluating to 1.
**Corollary 2.5.** Under Assumption 1 and when \( \alpha = 4 \), \( P_L \) for fully-loaded networks limited by the interference from the nearby BSs participating in the localization procedure is

\[
P_L(1, 1, 4, \beta, \gamma, \lambda) = \left( 1 - \frac{1}{\sqrt{\gamma/\beta + \frac{(L-2)^2}{4} - \frac{L-2}{2}}} \right)^{L-1},
\]

for \( \beta < \frac{\gamma}{L-1} \) and zero otherwise.

Interestingly, when interference from the participating BSs is ignored in (19) and when distant network interference is ignored in (25) and (26), the resulting expressions are independent of the BS deployment density. This is termed as *scale invariance* in the literature and is known to hold for most of the metrics that are derived from SINR when the thermal noise is negligible or ignored (interference-limited scenarios), e.g., see [25], [26].

**IV. Numerical Results and Discussion**

We now study the tightness of the bound and the approximations, as well as gather insights from numerical results. For consistency, all results were gathered using a BS deployment density equivalent to that of an infinite hexagonal grid with 500m intersite distances and a shadowing standard deviation of 8 dB. While our analytical expressions were derived by focusing solely on the post-processing SINR from the \( L^{th} \) BS surpassing \( \beta \), the truth data, which is compiled through simulation, contains no such approximation and considers the SINRs from all \( L \) participating BSs jointly with each required to surpass \( \beta \). Let us first consider \( P_L \) for \( L = 4 \) in the case of \( \alpha = 4 \) and fully-loaded networks \((p = q = 1)\), the setup considered in Figure 6. To interpret the meaning of this figure, it may be helpful to provide a simple example. Consider a cellular localization system employing TDOA observations and requiring a post-processing SINR of at least \(-6\) dB for successful operation after experiencing a 10 dB processing gain due to its integration time (i.e. \( \beta/\gamma = -16 dB \)). Figure 6 tells us that when all BSs in the network are fully-loaded (i.e., actively transmitting), unambiguous TDOA positioning (i.e., using \( L = 4 \) or more BSs) is possible only 50% of the time, a result consistent with the fact that universal frequency reuse (e.g., using synchronization signals) was deemed untenable for positioning signals in LTE. Also in this figure, the truth data is compared against the closed-form bound in Corollary 1.1, the nice closed-form approximation in Corollary 2.5 which ignores distant interference, as well as the simple single-integral approximation in Corollary 2.3. We note that the bound is tight for the *high-reliability regime*, which is the range of greatest interest (localizability probabilities greater
than 0.75). For low probabilities of localizability, the bound diverges by a maximum of around 3 dB. However, this low coverage range is not very desirable for system design, anyway. In addition to the bound, the closed-form approximation performs well for $P_L > 0.6$, while the single-integral approximation provides a remarkably tight approximation which is invariably indistinguishable from truth across all SINR values. The key factor in enabling these accurate approximations is that the locations of the strongest BS (i.e., the dominant interferer) and the $L^{th}$ BS are considered exactly.

Inspired by the results in Figure 6, we now delve deeper into the approximation of Theorem 2 in (20) and its single-integral counterpart in (2.1) to examine how they perform under more general conditions, such as across the range of realistic pathloss exponents. Figure 7 shows that the former remains almost indistinguishable from truth for all $\alpha$ values, while the latter is very tight in the high-reliability regime for $\alpha > 3$ and exhibits a noticeable deviation (amounting to less than 1 dB) when $\alpha = 3$. For $P_L < 0.5$, the single-integral approximation deviates significantly, however, due to the fact that the interference from beyond the $L^{th}$ BS becomes more influential with a higher SINR requirement. This deviation is further accentuated by our intentional selection of disadvantageous $p$ and $q$ values, where $p = 2/3$ partially de-emphasizes the interference from the closest BSs and $q = 1$ is the worst case $q$ for the single-integral approximation since it maximally emphasizes the distant interference term where the additional approximation lies. Nevertheless, this single-integral approximation for the general parameter case performs well.
Fig. 7. General Theorem 2 approximations across a range of pathloss exponents for $L = 4$, $p = 2/3$, and $q = 1$.

even under these adverse conditions for the higher $P_L$ values of interest, and will only improve with more favorable values of $p$ and $q$. On the other hand, subsequent figures will make it evident that the approximation in Theorem 2 and its exact derivatives remain extremely accurate across the ranges of all network parameters values.

As discussed in Corollary 1.2, the upper-bound on coverage provides a useful closed-form lower-bound on the processing gain required to obtain a desired $L$-localizability probability, which is likely a specification set prior to system design. From Figure 8, it is clear that for $p = 1$ and high desired values of $P_L$, the bound provides a tight criterion for the minimum processing gain required. Unfortunately, a closer look reveals that the processing gains required are impractically high, highlighting the need for some sort of interference mitigation. Moreover, consider that both
Fig. 8. **The importance of interference mitigation:** Processing gains required to achieve $P_L = 0.8$ without any sort of interference mitigation ($p = 1$) for $q = 1$ and $\alpha = 4$.

bounds are very tight in the region of interest despite the fact that the interference from beyond the $L^{th}$ BS is fully present (i.e., $q = 1$), while the bound derivation essentially assumes $q = 0$. This provides more insight into the nature of the interference, revealing that the localization performance is limited by the interference from other participating BSs (which is consistent with conclusions drawn from LTE positioning simulations [21]), prompting one to consider coordination amongst the BSs as the desired interference mitigation technique. This becomes even more evident as the propagation conditions become more severe, such as indoors, where higher path loss exponents demand higher processing gains to achieve the same $P_L$ as seen in Figure 9.

Through Figure 10, we demonstrate the value of mitigating interference through BS coordination. With the nodes beyond the $L^{th}$ BS fully-loaded, we note an increasing benefit from BS coordination as the degree of the coordination is incrementally increased. The large improvement in hearability from $p = 1$ to $p = 0$ again sheds light on the issue of interference from nearby BSs. In fact, at its least detrimental point, the interference from the $L - 1$ closest BSs accounts for nearly a 10 dB drop in signal quality. That’s roughly the equivalent of going from successfully obtaining eleven to obtaining only four BSs in the network considered in Figure 11. While coordination is able to help significantly, a closer look at Figures 10 and 11 makes it clear that more is necessary in order to reach the truly high $L$-localizability probabilities at reasonable SINR values. The most prevalent technique to further reduce the interference is that of thinning the interference field by
Fig. 9. **THE IMPACT OF PATH LOSS**: Processing gains required for desired $P_L$ values grow exponentially with the pathloss exponent. ($P_L = 0.8, L = 4, p = q = 1$.)

Fig. 10. **THE BENEFIT OF BS COORDINATION**: These $L$-localizability probabilities emphasize the gains achievable through the coordinated mitigation of participating BS interference. ($L = 4, \alpha = 4, q = 1$.)

separating nearby transmitters in frequency, which is discussed next.

**A. Application to frequency reuse**

An effective technique to reduce interference from co-located transmitters is that of frequency reuse, commonly included in wireless standards [37]. If a total of $K$ frequency bands are available and we independently assign one of the bands to each $x \in \Phi$ with equal probability, we can easily incorporate frequency reuse into our model by considering the transmission activity on each band separately using independent PPPs whose densities are that of the original PPP thinned
Fig. 11. **The cost of involving more BSs:** This figure underscores the high cost of obtaining a greater desired number of BSs, particularly in the high-reliability regime. ($\alpha = 4$, $p = 1/2$, $q = 3/4$.)

by the frequency reuse factor $K$. Let us denote the number of decodable BSs in the $k^{th}$ band by $n_k$. For $L$-localizability, we thus require $\sum_{k=1}^{K} n_k \geq L$. The main technical difference between this analysis and that already presented is that we are now obliged to evaluate the probability of being able to decode exactly $n$ BSs in a given frequency band. Under the localization setup we consider, it is not difficult to show that this can be readily determined by considering the difference between the probabilities of $n$ and $n + 1$ localizability. From here, the expression for the $L$-localizability probability with frequency reuse is procured by simply considering all combinations of $\{n_1, \ldots, n_K\}$ for which $\sum_{k=1}^{K} n_k \geq L$. This is formally stated in the following theorem.

**Theorem 3** ($L$-localizability probability with random frequency reuse). The $L$-localizability probability with random frequency reuse and reuse factor $K$ is:

$$P_L^K(p, q, \alpha, \beta, \gamma, \lambda) = 1 - \sum_{l=0}^{L-1} \sum_{\{n_1, \ldots, n_K\} \atop \sum_i n_i = l} \prod_{i=1}^{K} \left( P_{n_i} \left( p, q, \alpha, \beta, \gamma, \frac{\lambda}{K} \right) - P_{n_i+1} \left( p, q, \alpha, \beta, \gamma, \frac{\lambda}{K} \right) \right).$$

(27)

This expression is easily implemented in software using a recursive function and any of the previous expressions for evaluating $P_L$ may be substituted into (27). Using the worst-case interference scenario ($p = q = 1$), the value of frequency reuse is clearly demonstrated in Figure 12 for $K = 3$ and 6, which correspond to the values used in LTE when positioning using cell-specific reference
signals (CRS) and positioning reference signals (PRS), respectively. Since random frequency reuse makes no attempt to intelligently separate interference sources, gains from planned frequency reuse are likely to be even better.

V. CONCLUSION

We have employed concepts from point process theory and stochastic geometry in order to provide a new tractable model for studying localization in cellular networks. The model is accompanied by an analysis of hearability, which is shown to be an important metric providing insight into fundamental localization performance, and easy-to-use analytical expressions for this metric are provided. This is in contrast to most previous approaches, which either provide insights specific to deterministic deployments or rely on time-consuming simulations, neither of which allow for the drawing of general insights. While more base stations participating in localization is beneficial to its accuracy, the analytical results show that obtaining an increasing number of successful base stations connections decreases exponentially. The primary culprit for this is the interference due to the participating base stations themselves. When base stations coordinate (e.g., through a network controller), more of them are able to successfully participate in the localization procedure. However, in the end, it is clear that the greatest gains in hearability are had through frequency reuse. This is consistent with the findings during the standardization of LTE positioning, where this conclusion was reached after performing many complex system-level simulations.
We have only scratched the surface of the model, which may be used to extend this work in many different ways. Besides hearability, understanding how other metrics, such as ones focused on network geometry, are affected by changes in design parameters would yield valuable contributions to the localization literature. In addition, recognizing how directional antennas and power control affect localization performance would be helpful to cellular system designers. Lastly, location information can be gathered in diverse ways, for example by simply knowing that one is able to hear a low-power femtocell, which lends itself nicely to the consideration of different classes of base stations through multi-tiered heterogeneous networks [26].

APPENDIX

A. Proof of Lemmas 1 and 3

Let \( i < j \), \( n \leq \Xi \), \( \Xi \) be the number of BSs between \( x_i \) and \( x_j \), \( A = b(o, \|x_i\|+da/2)\setminus b(o, \|x_i\|-da/2) \), \( B = b(o, \|x_j\|+db/2)\setminus b(o, \|x_j\|-db/2) \), \( C = b(o, \|x_j\|-db/2)\setminus b(o, \|x_i\|+da/2) \), \( D \subseteq C \), \( b(\theta, r) \) represent a ball of radius \( r \) centered at \( \theta \), and \( N_H \) be the number of points in area \( H \). Then,

\[
\mathbb{P}(N_D = n | x_i, x_j) = \lim_{da, db \to 0} \frac{\mathbb{P}(N_D = n | N_C = \Xi, N_B = 1, N_A = 1)}{\mathbb{P}(N_C = \Xi, N_B = 1, N_A = 1)}
\]

\[
= \lim_{da, db \to 0} \frac{\mathbb{P}(N_D = n, N_C = \Xi, N_B = 1, N_A = 1)}{\mathbb{P}(N_C = \Xi, N_B = 1, N_A = 1)}
\]

\[
= \frac{\mathbb{P}(N_D = n) \mathbb{P}(N_{C\setminus D} = \Xi - n)}{\mathbb{P}(N_C = \Xi)}
\]

\[
= \frac{(\lambda|D|)^n e^{-\lambda|D|/n!} (\lambda(|C| - |D|))^\Xi - n e^{-\lambda(|C| - |D|)}}{(\lambda|C|)^\Xi e^{-\lambda|C|/\Xi!}}
\]

\[
= \frac{\Xi!}{n!(\Xi - n)!} \left( \frac{|D|}{|C|} \right)^n \left( 1 - \frac{|D|}{|C|} \right)^{\Xi - n}
\]

which shows that the \( \Xi \) BSs inside the annular region \( \lim_{da, db \to 0} C = b(o, \|x_j\|)\setminus b(o, \|x_i\|) \) make up a BPP. By letting \( i = 0, j = L \), and defining \( x_0 \triangleq o \) (i.e., \( \|x_0\| = 0 \)), the BSs inside the circular region \( b(o, \|x_L\|) \) are shown to make up a BPP, thus proving Lemma 1. By letting \( i = 1 \) and \( j = L \), Lemma 3 is proved.
B. Proof of Lemma 2

Let \( \hat{x} \) be an active BS that is uniformly distributed inside the circular region \( \mathbb{b}(o, R_L) \). Then, for \( \Omega \geq 1 \),

\[
F_{R_1|R_L}(r|R_L, \Omega) = \mathbb{P}(\hat{R}_1 \leq r|R_L, \Omega) = 1 - \mathbb{P}(\hat{R}_1 > r|R_L, \Omega)
\]

\[
= 1 - \mathbb{P}(\min\{\hat{x} : \|\hat{x}\| < R_L\} > r|R_L, \Omega) \overset{(a)}{=} 1 - \prod_{\|\hat{x}\| < R_L} \mathbb{P}(\|\hat{x}\| > r|R_L)
\]

\[
\overset{(b)}{=} 1 - \prod_{\|\hat{x}\| < R_L} \frac{\pi R_L^2 - \pi r^2}{\pi R_L^2} = 1 - \left( \frac{R_L^2 - r^2}{R_L^2} \right)^\Omega,
\]

where (a) and (b) follow from Lemma 3.

C. Proof of Theorem 1

For \( \Omega \geq 1 \),

\[
P_L = \mathbb{P} \left( \sum_{i=1}^{\infty} \frac{PR_L^{-\alpha}}{PR_L^{-\alpha} - PR_L^{-\alpha}} \geq \frac{\beta}{\gamma} \right) \leq \mathbb{P} \left( \frac{PR_L^{-\alpha}}{\sum_{i=1}^{\infty} PR_i^{-\alpha} - PR_L^{-\alpha}} \geq \frac{\beta}{\gamma} \right)
\]

\[
\overset{(a)}{=} \mathbb{P} \left( \frac{R_L^{-\alpha}}{\hat{R}_1^{-\alpha} + \sum_{i=2}^{\Omega} \hat{R}_i^{-\alpha}} \geq \frac{\beta}{\gamma} \right) \leq \mathbb{P} \left( \frac{R_L^{-\alpha}}{\hat{R}_1^{-\alpha} + (\Omega - 1)R_L^{-\alpha}} \geq \frac{\beta}{\gamma} \right)
\]

\[
= \mathbb{P} \left( \hat{R}_1 \geq R_L \left( \frac{\gamma}{\beta} - (\Omega - 1) \right)^{-\frac{1}{\alpha}} \right) \overset{(c)}{=} \mathbb{P} \left( \min\{\|\hat{x}\| : \|\hat{x}\| < R_L\} \geq R_L \left( \frac{\gamma}{\beta} - (\Omega - 1) \right)^{-\frac{1}{\alpha}} \right)
\]

\[
\overset{(d)}{=} \mathbb{E}_\Omega \left[ \left( 1 - \left( \frac{\gamma}{\beta} - (\Omega - 1) \right)^{-\frac{2}{\alpha}} \right) \Omega \left( \frac{\beta}{\gamma} \leq \Omega^{-1} \right) \right],
\]

where (a) follows from \( \hat{R}_{\Omega+1} \triangleq R_L \) and pulling the dominant interferer out of the sum, (b) from the fact that \( \hat{R}_i \leq R_L \) for \( i \leq \Omega \), (c) since \( \hat{R}_1 \) is the smallest among all interferers, and (d) from the fact that given \( R_L \), all the BSs are uniformly distributed in the circle of radius \( R_L \), followed by some algebraic manipulations. The final result is obtained by deconditioning over \( \Omega \geq 1 \), resulting in an expression which also trivially valid for the \( \Omega = 0 \) case.

D. Proof of Lemma 4

Let \( \hat{x} \) be an active BS that is uniformly distributed inside the annular region \( \mathbb{b}(o, R_L) \setminus \mathbb{b}(o, \hat{R}_1) \). Then,

\[
\mathbb{E}[\mathbb{P}^\alpha|\hat{R}_1, R_L] \overset{(a)}{=} \int_{\hat{R}_1}^{R_L} P R^{-\alpha} \frac{2r}{R_L^2 - \hat{R}_1^2} dr = \frac{2P}{R_L^2 - \hat{R}_1^2} \int_{\hat{R}_1}^{R_L} r^{1-\alpha} dr
\]
\[
\begin{align*}
&= \frac{2P}{R_L^2 - \hat{R}_1^2} \left( \frac{2 - \alpha}{2 - \alpha} \right)_{R = \hat{R}_1} = \frac{2P}{2 - \alpha} \cdot \frac{R_L^{2 - \alpha} - \hat{R}_1^{2 - \alpha}}{R_L^2 - \hat{R}_1^2},
\end{align*}
\]

where (a) follows from the BPP condition proved in Lemma 3. Lastly, the mean of the sum of the interference from all interferers located inside the annular region \( b(o, R_L) \setminus b(o, \hat{R}_1) \) is simply the sum of their individual means, from which the result follows.

**E. Proof of Lemma 5**

In order to accommodate partial network loading, \( q \) is the fraction of the far-off nodes transmitting throughout the localization procedure, which can be modeled using an equivalent thinned PPP \( \tilde{\Phi} \) with density \( q\lambda \).

\[
\mathbb{E}[I_2^2 | R_L] = \mathbb{E} \left[ \sum_{x \in \tilde{\Phi} \setminus b(o, R_L)} P \|x\|^{-\alpha} \right] \stackrel{(a)}{=} q\lambda \int_{b(o, R_L)} P \|x\|^{-\alpha} \, dx
\]

\[
= Pq\lambda \int_0^{2\pi} \int_{R_L}^\infty r^{-\alpha} \, dr \, d\theta = 2P\pi q\lambda \int_{R_L}^\infty r^{1-\alpha} \, dr = \frac{2P\pi q\lambda}{\alpha - 2} R_L^{2-\alpha},
\]

for \( \alpha > 2 \) (and unbounded otherwise), where (a) follows from Campbell’s theorem [12].

**F. Proof of Proposition 1**

\[
P_{L|\Omega=0} = \mathbb{E}_{R_L} \left[ 1 \left( \frac{PR_L^{-\alpha}}{E[I_2^2 | R_L]} \geq \frac{\beta}{\gamma} \right) \right] = \mathbb{E}_{R_L} \left[ 1 \left( \frac{\beta}{\gamma} \geq \frac{2\pi q\lambda}{\alpha - 2} R_L^2 \right) \right]
\]

\[
= \mathbb{E}_{R_L} \left[ 1 \left( \frac{\alpha - 2}{2\pi q\lambda \beta / \gamma} \geq R_L^2 \right) \right] = \mathbb{E}_{R_L} \left[ 1 \left( R_L \leq \sqrt{\frac{\alpha - 2}{2\pi q\lambda \beta / \gamma}} \right) \right]
\]

\[
= P \left( R_L \leq \sqrt{\frac{\alpha - 2}{2\pi q\lambda \beta / \gamma}} \right) \stackrel{(a)}{=} 1 - \sum_{\ell=0}^{L-1} e^{-\frac{\alpha - 2}{2\pi q\lambda \beta / \gamma}} \frac{(\alpha - 2)^\ell}{\ell!},
\]

where (a) follows from the fact that the previous expression is simply the probability that at least \( L \) nodes lie inside the region \( b(o, \sqrt{\frac{\alpha - 2}{2\pi q\lambda \beta / \gamma}}) \).

**G. Proof of Theorem 2**

\[
P_L = P \left( \text{SIR}_L \geq \frac{\beta}{\gamma} \right) = \mathbb{E}_\Omega \left[ \mathbb{E}_{R_L} \left[ \mathbb{E}_{R_1} \left[ 1 \left( \text{SIR}_L \geq \frac{\beta}{\gamma} \right) | R_L, R_1, \Omega \right] | R_L, \Omega \right] \right]
\]

\[
= \mathbb{E}_\Omega \left[ \mathbb{E}_{R_L} \left[ \mathbb{E}_{R_1} \left[ 1 \left( \frac{PR_L^{-\alpha}}{PR_1^{-\alpha} + E[I_1^2 | R_1, R_L, \Omega] + E[I_2^2 | R_L]} \geq \frac{\beta}{\gamma} \right) | R_L, \Omega \right] \right] \right]
\]
where

\[
\hat{\text{SIR}} = \left( \frac{PR_1^{-\alpha}}{PR_1^{-\alpha} + \mathbb{E}[\mathcal{I}_1|\hat{R}_1, R_L] + \mathbb{E}[\mathcal{I}_2|R_L]} \right)
\]

The result follows by integrating over the densities of \( \hat{R}_1 \), \( R_L \), and \( \Omega \geq 1 \), and treating the \( \Omega = 0 \) case separately using Proposition 1.

### H. Proof of Corollary 2.1

Beginning with (31) and multiplying the \( \mathbb{E}[\mathcal{I}_2|R_L] \) term by \( \mathbb{E}[\hat{R}_1^2]/\hat{R}_1^2 = 1/(\pi \lambda \hat{R}_1^2) \),

\[
P_L = \mathbb{E}_\Omega \left[ \mathbb{E}_{R_L} \left[ \mathbb{E}_{\hat{R}_1} \left[ 1 \left( \frac{PR_1^{-\alpha}}{PR_1^{-\alpha} + \mathbb{E}[\mathcal{I}_1|\hat{R}_1, R_L] + \mathbb{E}[\mathcal{I}_2|R_L]} \geq \frac{\beta}{\gamma} \right) \right] \right] \right],
\]

where (a) follows from letting \( X = R_L/\hat{R}_1 \), and noting that \( F_X(x) = (1 - 1/x^2)\Omega \), which is readily obtained from (5), is no longer a function of \( \hat{R}_1 \) or \( R_L \). The final expression is obtained by integrating over the support of \( X \) and subsequently summing over the probability mass of \( \Omega \), excluding \( \Omega = 0 \) which is treated separately using Proposition 1.

### I. Proof of Corollary 2.2

\[
P_L = \mathbb{P}(\text{SIR}_L \geq \beta/\gamma) = \mathbb{E}_\Omega \left[ \mathbb{E}_{R_L} \left[ \mathbb{E}_{\hat{R}_1} \left[ 1 \left( \frac{\text{SIR}_L \geq \beta/\gamma}{\hat{R}_1, R_L, \Omega} \right) R_L, \Omega \right] \right] \right].
\]

The \( \text{SIR}_L \) term in the above expression is

\[
\text{SIR}_L = \frac{PR_1^{-\alpha}}{PR_1^{-\alpha} + \mathbb{E}[\mathcal{I}_1|\hat{R}_1, R_L] + \mathbb{E}[\mathcal{I}_2|R_L]} = \frac{PR_1^{-\alpha}}{PR_1^{-\alpha} + \frac{2P(\Omega-1)}{2-\alpha} \frac{R_L^{\alpha-\alpha}}{R_L^{\alpha-\alpha}} + \frac{2P\pi q \lambda}{\alpha-2} R_L^{-2-\alpha}}
\]

where (a) follows by defining \( X = \frac{R_L}{\hat{R}_1} \) and substituting \( \alpha = 4 \). Using (31), we now simplify \( \text{SIR}_L \geq \beta/\gamma \) which will then be used to derive \( P_L \) in (33). Defining \( Y = X^2 \) we have

\[
\text{SINR}_L \geq \frac{\beta}{\gamma} \Rightarrow Y^2 + (\Omega - 1)Y \leq \kappa^{-1} \Rightarrow \left( Y + \frac{(\Omega - 1)}{2} \right)^2 \leq \kappa^{-1} + \frac{(\Omega - 1)^2}{4}
\]

\[
\Rightarrow 0 \leq Y \leq \sqrt{\kappa^{-1} + \frac{(\Omega - 1)^2}{4}} - \frac{(\Omega - 1)}{2} \Rightarrow X^2 \leq \sqrt{\kappa^{-1} + \frac{(\Omega - 1)^2}{4}} - \frac{(\Omega - 1)}{2}
\]
\[ 1 \leq X \leq \sqrt{\frac{1}{\kappa - 1} + \frac{(\Omega - 1)^2}{4} - \frac{(\Omega - 1)}{2}} \Rightarrow \frac{R_L}{\sqrt{\frac{1}{\kappa - 1} + \frac{(\Omega - 1)^2}{4} - \frac{(\Omega - 1)}{2}}} \leq \hat{R}_1 \leq R_L, \]

where \( \kappa^{-1} = \frac{\gamma}{\beta} - \pi \lambda R_L^2 \) in (a), (b) follows from the fact that \( Y \geq 1 \), and (c) from \( Y = X^2 \). Note that (b) and (c) require \( \kappa^{-1} \geq -\frac{(\Omega - 1)^2}{4} \). Step (d) follows from \( X \geq 1 \). The earlier condition on \( \kappa^{-1} \) is replaced by a more strict condition \( \kappa^{-1} \geq \Omega \) in (d). Substituting this back in (33) and doing some algebraic manipulations, we get

\[
P_L = \mathbb{E}_\Omega \left[ \mathbb{E}_{R_L} \left[ \left( \frac{R_L^2}{\sqrt{\frac{1}{\kappa - 1} + \frac{(\Omega - 1)^2}{4} - \frac{(\Omega - 1)}{2}}} \right)^{L-1} \right] \right]
\]

from which the result follows by simply deconditioning over \( R_L \) and \( \Omega \geq 1 \) (and treating the \( \Omega = 0 \) case separately). Due to \( \kappa^{-1} \geq \Omega \), the integration limits are from 0 to \( \sqrt{\frac{\gamma/\beta - \omega}{\pi \lambda}} \).

REFERENCES

[1] J. Schloemann, H. S. Dhillon, and R. M. Buehrer, “Localization performance in cellular networks,” submitted to the IEEE ICC 2015 Workshop on Advances in Network Localization and Navigation (ANLN), London, UK, Jun. 2015.
[2] S. Stein, “Algorithms for ambiguity function processing,” IEEE Trans. Acoust. Speech, Signal Process., vol. 29, no. 3, pp. 588–599, 1981.
[3] D. J. Torrieri, “Statistical theory of passive location systems,” IEEE Trans. Aerosp. Electron. Syst., vol. AES-20, no. 2, pp. 183–198, Mar. 1984.
[4] S. Stein, “Differential delay/Doppler ML estimation with unknown signals,” IEEE Trans. Signal Process., vol. 41, no. 8, pp. 2717–2719, 1993.
[5] Y. T. Chan and K. C. Ho, “A simple and efficient estimator for hyperbolic location,” IEEE Trans. Signal Process., vol. 42, no. 8, pp. 1905–1915, 1994.
[6] R. Zekavat and R. M. Buehrer, Handbook of Position Location: Theory, Practice, and Advances. Wiley, 2012.
[7] T. S. Rappaport, “Position location using wireless communications on highways of the future,” IEEE Commun. Mag., vol. 34, no. 10, pp. 33–41, 1996.
[8] J. H. Reed and K. J. Krizman, “An overview of the challenges and progress in meeting the E-911 requirement for location service,” IEEE Commun. Mag., vol. 36, no. 4, pp. 30–37, 1998.
[9] Federal Communications Commission, “Wireless E911 location accuracy requirements,” PS Docket No. 07-114, Jan. 2015.
[10] M. Harris, “How new indoor navigation systems will protect emergency responders,” IEEE Spectr., Sep. 2013.
[11] J. F. C. Kingman, Poisson Processes. Oxford University Press, 1993.
[12] M. Haenggi, Stochastic Geometry for Wireless Networks. New York: Cambridge University Press, 2013.
[13] C. Chang and A. Sahai, “Estimation bounds for localization,” in Proc. IEEE Conf. Sens. Ad-Hoc Commun. Netw., Oct. 2004, pp. 415–424.
[14] A. Savvides, W. Garber, R. Moses, and M. Srivastava, “An analysis of error inducing parameters in multihop sensor node localization,” IEEE Trans. Mob. Comput., vol. 4, no. 6, pp. 567–577, Nov. 2005.

[15] I. Guvenc and C.-C. Chong, “A survey on TOA based wireless localization and NLOS mitigation techniques,” IEEE Commun. Surv. Tutorials, vol. 11, no. 3, pp. 107–124, 2009.

[16] Third Generation Partnership Project (3GPP), “R1-090053: Improving the hearability of LTE positioning service,” Alcatel-Lucent, 3GPP TSG-RAN WG1 #55bis, Ljubljana, Slovenia, Jan. 2009.

[17] X. Li and D. K. Hunter, “Probabilistic model of triangulation,” in Proc. ACM Symp. Solid Phys. Mod., Stony Brook, New York, USA, Jun. 2008, pp. 301–306.

[18] J. Gribben and A. Boukerche, “Probabilistic estimation of location error in wireless ad hoc networks,” in Proc. IEEE Glob. Telecommun. Conf., Dec. 2010.

[19] F. Daneshgaran, M. Laddomada, and M. Mondin, “Connection between system parameters and localization probability in network of randomly distributed nodes,” IEEE Trans. Wirel. Commun., vol. 6, no. 12, pp. 4383–4389, Dec. 2007.

[20] Third Generation Partnership Project (3GPP), “R1-091443: Evaluation parameters for positioning studies,” Alcatel-Lucent, Ericsson, Motorola, Nokia, Nokia Siemens Networks, Nortel, Qualcomm Europe, 3GPP TSG-RAN WG1 #56bis, Seoul, Korea, Mar. 2009.

[21] ——, “R1-091912: Discussions on UE positioning issues,” Nortel, 3GPP TSG-RAN WG1 #57, San Francisco, USA, May 2009.

[22] J. Yap, “Accuracy and hearability of mobile positioning in GSM and CDMA networks,” in Third Int. Conf. 3G Mob. Commun. Technol., 2002, pp. 350–354.

[23] A. Oborina, T. Hettonen, and V. Koivunen, “Cell hearability analysis in UTRAN Long Term Evolution downlink,” in Proc. Forty-Third Asilomar Conf. Signals, Syst. Comput., 2009, pp. 991–995.

[24] R. M. Vaghefi and R. M. Buehrer, “Improving positioning in LTE through collaboration,” in Proc. Work. Positioning, Navig. Commun., Mar. 2014.

[25] J. G. Andrews, F. Baccelli, and R. K. Ganti, “A tractable approach to coverage and rate in cellular networks,” IEEE Trans. Commun., vol. 59, no. 11, pp. 3122–3134, Nov. 2011.

[26] H. S. Dhillon, R. K. Ganti, F. Baccelli, and J. G. Andrews, “Modeling and analysis of K-tier downlink heterogeneous cellular networks,” IEEE J. Sel. Areas Commun., vol. 30, no. 3, pp. 550–560, Apr. 2012.

[27] R. W. Heath, M. Kountouris, and T. Bai, “Modeling heterogeneous network interference using Poisson point processes,” IEEE Trans. Signal Process., vol. 61, no. 16, pp. 4114–4126, Aug. 2013.

[28] S. Weber, J. G. Andrews, and N. Jindal, “The effect of fading, channel inversion, and threshold scheduling on ad hoc networks,” IEEE Trans. Inf. Theory, vol. 53, no. 11, pp. 4127–4149, Nov. 2007.

[29] B. Blaszczyszyn and M. K. Karray, “Quality of service in wireless cellular networks subject to log-normal shadowing,” IEEE Trans. Commun., vol. 61, no. 2, pp. 781–791, Feb. 2013.

[30] H. S. Dhillon and J. G. Andrews, “Downlink rate distribution in heterogeneous cellular networks under generalized cell selection,” IEEE Wireless Commun. Letters, vol. 3, no. 1, pp. 42 – 45, Feb. 2014.

[31] B. Blaszczyszyn, M. K. Karray, and H. P. Keeler, “Using Poisson processes to model lattice cellular networks,” in 2013 Proc. IEEE INFOCOM, Apr. 2013, pp. 773–781.

[32] H. P. Keeler, N. Ross, and A. Xia, “When do wireless network signals appear Poisson?” preprint no. 2044, WIAS, Berlin, 2014, available online: arxiv.org/abs/1411.3757.

[33] H. S. Dhillon, R. K. Ganti, and J. G. Andrews, “Load-aware modeling and analysis of heterogeneous cellular networks,” IEEE Trans. on Wireless Commun., vol. 12, no. 4, pp. 1666 – 1677, Apr. 2013.
[34] M. Haenggi, “On distances in uniformly random networks,” *IEEE Trans. Inf. Theory*, vol. 51, no. 10, pp. 3584–3586, Oct. 2005.

[35] H. Urkowitz, *Signal Theory and Random Processes*. Norwood, MA: Artech House, 1983.

[36] H. P. Keeler, B. Blaszczyszyn, and M. K. Karray, “SINR-based k-coverage probability in cellular networks with arbitrary shadowing,” in *IEEE Int. Symp. Inf. Theory*, Jul. 2013, pp. 1167–1171.

[37] Third Generation Partnership Project (3GPP), “Evolved Universal Terrestrial Radio Access Network (E-UTRAN); Stage 2 functional specification of user equipment (UE) positioning in E-UTRAN,” Mar. 2013.