A NOTE ON PAIRS OF PROJECTIONS

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Abstract.
We give a brief proof of a recent result of Avron, Seiler and Simon.

In [1], it is proved that if \( P, Q \) are (not necessarily self-adjoint) projections on a Hilbert space and \( (P - Q)^n \) is trace-class (i.e. nuclear) for some odd integer \( n \) then \( \text{tr} (P - Q)^n \) is an integer and in fact, if \( P \) and \( Q \) are self-adjoint, \( \text{tr} (P - Q)^n = \dim E_{10} - \dim E_{01} \) where \( E_{ab} = \{ x : Px = ax, Qx = bx \} \); (see also [2]). The proof given in [1] uses the structure of the spectrum of \( P - Q \) and Lidskii’s theorem; it is therefore not applicable to more general Banach spaces. The purpose of this note is to give a very brief proof of the same result which involves only simple algebraic identities and is valid in any Banach space with a well-defined trace (i.e. with the approximation property). We use \([A, B]\) to denote the commutator \( AB - BA \).

The basic material about operators on Banach spaces which we use can be found in the book of Pietsch [3]. We summarize the two most important ingredients.

We will need the following basic result from Fredholm theory. Suppose \( X \) is a Banach space and \( A : X \rightarrow X \) is an operator such that for some \( m, A^m \) is compact. Let \( S = I - A \); then \( F = \cup_{k \geq 1} S^{-k}(0) \) is finite-dimensional and if \( Y = \cap_{k \geq 1} S^k(X) \) then \( Y \) is closed and \( X \) can be decomposed as a direct sum \( X = F \oplus Y \). Furthermore \( F \) and \( Y \) are invariant for \( S \) and \( S \) is invertible on \( Y \). We refer to [3] 3.2.9 (p. 141-142) for a slightly more general result.

We will also need the following properties of nuclear operators and the trace. If \( X \) is a Banach space then an operator \( T : X \rightarrow X \) is called nuclear if it can written as a series \( T = \sum_{n=1}^{\infty} A_n \) where each \( A_n \) has rank one and \( \sum_{n=1}^{\infty} \|A_n\| < \infty \). The nuclear operators form an ideal in the space of bounded operators. When \( X \) has the approximation property, one can then define the trace of \( T \) unambiguously by \( \text{tr} T = \sum_{n=1}^{\infty} \text{tr} A_n \) (where the trace of a rank one operator \( A = x^* \otimes x \) is defined in

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the usual way by \( \text{tr} A = x^*(x) \). The trace is then a linear functional on the ideal of nuclear operators and has the property that \( \text{tr} [A, T] = 0 \) if \( A \) is bounded and \( T \) is nuclear. See Chapter 4 of [3] and particularly Theorem 4.7.2.

**Lemma.** Let \( X \) be a Banach space and suppose \( P \) and \( Q \) are two projections on \( X \). Let \( M = P - Q, \ U = (I - Q)(I - P) +QP, \ V = (I - P)(I - Q) + PQ \) and suppose \( T \) is any operator which commutes with both \( P \) and \( Q \). Then

1. \( M^2 \) commutes with both \( P \) and \( Q \).
2. \( [(I - 2Q)TM, PV] = TM(I - M^2) \).
3. If \( I - M^2 \) is invertible \( [(I - 2Q)TM(I - M^2)^{-1}, PV] = TM \).

**Proof.** (1) was first observed by Dixmier, Kadison and Mackey as remarked in [1]. For (2) observe that \( QU = UP \) and \( UV = VU = I - M^2 \). Hence \( M(I - M^2) = PUV - QUV = PVU - UPV = \) \([I, PV] = [I - U, PV] = [(I - 2Q)M, PV] \). If \( T \) commutes with \( P \) and \( Q \) then (2) follows. Note that (3) is immediate from (2), replacing \( T \) by \( T(I - M^2)^{-1} \). \( \square \)

**Theorem.** Let \( X \) be a Banach space with the approximation property, and suppose \( n \) is an odd integer. If \( P, Q \) are two projections on \( X \) so that \( (P - Q)^n \) is nuclear, then \( \text{tr} (P - Q)^n = \dim E_{10} - \dim \tilde{E}_{01} = \dim \tilde{E}_{10} - \dim E_{01} \), where \( E_{ab} = \{x \in X : Px = ax, \ Qx = bx \} \) and \( \tilde{E}_{ab} = \{x^* \in X^* : P^* x^* = ax^*, \ Q^* x^* = bx^* \} \).

**Remark.** If \( X \) is a Hilbert space and \( P \) and \( Q \) are self-adjoint this is equivalent to the result of Avron, Seiler and Simon.

**Proof.** We use the notation of the lemma. If \( M^n \) is nuclear then some power of \( M^2 \) is compact. Let \( S = I - M^2 \) and let \( F = \bigcup_{k \geq 1} S^{-k}(0) \) and \( Y = \cap_{k \geq 1} S^k(X) \). Then as noted above we have that \( \dim F < \infty, X = F \oplus Y \) and \( S \) is invertible on \( Y \). Since \( P \) and \( Q \) commute with \( S \) both \( F \) and \( Y \) are invariant for \( P \) and \( Q \). We denote the restriction of an operator \( T \) to \( F \) or \( Y \) by \( T_F \) or \( T_Y \).

Now \( (I - M_F^n) \) is invertible on \( Y \) so that (3) of the lemma expresses \( M_Y^n \) as the commutator of a nuclear operator and a bounded operator. Hence \( \text{tr} M_Y^n = 0 \).

On the other hand, by (2) of the Lemma, \( M_F - M_F^n \) is a commutator on \( F \) which is finite-dimensional so that \( \text{tr} M_F^n = \text{tr} M_F = \text{tr} P_F - \text{tr} Q_F \in \mathbb{Z} \).

It is easy to see from elementary computations that \( \text{tr} P_F - \text{tr} Q_F = \dim F - \text{rank} (I - P_F) - \text{rank} Q_F = \dim F - \text{dim} ((I - P)F + Q(F)) - \dim E_{01} \). Now \( \dim F - \text{dim} ((I - P)F + Q(F)) \) is the dimension of the subspace of \( F^* \) of all \( f^* \) such that \( P_F f^* = f^* \) and \( Q_F f^* = 0 \); if we identify \( F^* \) with \( Y^\perp \) via the direct sum decomposition this space coincides with \( \tilde{E}_{10} \). Now, it follows easily from the properties of the trace that \( \text{tr} M^n = \text{tr} M_P^n + \text{tr} M_Y^n \). This gives the second formula for \( \text{tr} M^n \). The other formula is similar. \( \square \)
References

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