Random matrix approach to ‘nonuniversal’ conductance

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Recent experiments on the conductance of high quality quantum wires have revealed an unexpected feature: the quantization step of the conductance is apparently system dependent. We provide the understanding of this behaviour using the appropriately extended random matrix approach. A single additional parameter governs the size of the conductance quantization steps. In effect the behaviour seems to remain ‘universal’, generic for the conductance of a class of mesoscopic systems.

The conductance of mesoscopic devices, the so called quantum dots or quantum wires, exhibits a number of universal features such as the quantization of the average conductance or the magnitude of the conductance fluctuations. For the ideal one-dimensional (1D) quantum wire the dc conductance is quantized in units of $G_0 = 2e^2/h$ (per channel in the wire, the factor 2 corresponding to the electron spin). $G = G_0 M_1$ with $M_1$ being equal to the number of transverse modes supported by the wire. Thus the dimensionless conductance $G/G_0$ changes by integer steps when $M_1$ increases. Similarly, the average conductance of a quantum dot coupled to the outside world by two leads each of which supports $M_1$ open channels is given by

$$G = G_0 \frac{M_1^2}{2M_1 - 1 + 2/\beta}$$

where $\beta = 1$ for time reversal invariant systems and $\beta = 2$ when this symmetry is broken strongly. These results are readily obtained within the random matrix theory (RMT) approach, recently reviewed in detail by Beenakker.

Experiments, carried out recently on high quality wires revealed quite surprisingly smaller quantization steps of the height $g < 1$, with $g$ varying from sample to sample and reaching 0.75 at low temperatures. It has been pointed out that such a behaviour may be an evidence of a coherent backscattering between the 1D wire and the 2D leads. In such a case the conductance becomes $G = G_0 T$ where $T$ is a $M_1$-dependent transmission coefficient. In the same work three different theoretical possibilities for the explanation of the data are discussed The difficulties with the RMT approach and the Luttinger liquid theory are pointed out. The authors give their own explanation in terms of the competition between the scattering from 2D into the edge modes.

The purpose of this communication is to show that the experimental results may be, however, reproduced by the appropriate RMT model of scattering by a slight modification of the approach which yielded in the past many successful predictions for the transport properties in the mesoscopic media. The universal parameter determining the height of the conductance steps is defined.

To this end let us assume that the almost ideal 1D wire is coupled to 2-dimensional (2D) leads as realized in the recent experiment. We consider a standard Heidelberg scattering matrix approach expressing the scattering matrix $S$ as

$$S = 1 - 2\pi i W^\dagger (E_F - H + i\pi W W^\dagger)^{-1}W,$$

where $H$ is the internal Hamiltonian of the system represented by a matrix of rank $N$ while $W$ is a $N \times M$ matrix representing the coupling between the $N$ internal states and $M$ scattering channels in the leads. Assuming two identical leads one gets $M = 2M_1$.

In the application to a chaotic cavity scattering one assumes that the number of internal states, $N$, around the Fermi energy, $E_F$, is much larger than $M$. Taking typical RMT assumptions about the statistical properties of $H$ and $W$ one may then derive a number of predictions concerning the statistical properties of $S$ and of the measurable observables. As shown by Brouwer, such an approach is equivalent (for $M \ll N$), to making RMT assumptions concerning directly the unitary $S$ matrix itself. For example, if $H$ pertains to the Gaussian Orthogonal Ensemble (GOE) and $W$ are composed of real random vectors (the situation appropriate for time reversal invariant systems), then the $M \times M$ matrix $S$ belongs to the corresponding circular orthogonal ensemble of unitary matrices (COE) in the limit $N \to \infty$. Similarly, if time reversal symmetry is broken and $H$ pertains to Gaussian Unitary Ensemble (GUE), the corresponding $S$ matrix shows statistical properties typical for the Circular Unitary Ensemble (CUE). Thus it is justifiable to derive transport properties by making statistical predictions for $S$ matrices themselves. Such an approach yields, e.g., Eq. (2). The advantage of the former, Heidelberg approach is that it allows also to calculate energy dependent quantities such as correlators or time delays,
while the direct RMT approach to $S$ matrices says nothing about the dependence on the scattering energy, $E_F$.

Consider now the experimental system of [1]. The 1D almost ideal wire placed between 2D leads takes the place of the internal scattering system in the Heidelberg approach with $N$ being now the number of states in the internal wire around $E_F$ or the number ‘internal channels’. Note that really the Hamiltonian describing the internal wire supports an infinite number of states. Most of them does not contribute to conductance being vanishingly small (evanecent) on the left or right side of the 1D wire. The important $N$ “states” are the $N$ scattering channels through the 1D wire if it were coupled incoherently to leads. Thus $N$ can be even only. Moreover there is no ground to assume that the internal matrix $H$ pertains to GOE. Rather, since the wire is almost ideal, one may assume that the motion of the electrons through the wire is ballistic. Since the leads are assumed to be two dimensional, $M = 2M_l$ should be much larger than $N$. Note that the limit $N \ll M$ is the opposite to that taken in the standard transport theory [3].

The structure of the $S$ matrix, Eq.(3), indicates that $N$ out of $M$ of its eigenphases may be nontrivial and different from 0 (i.e. the remaining $M - N$ eigenvalues of $S$ are equal to unity). This is due to the fact that the part coupling the channels to the internal states has at most the rank $N$. Writing $S$ as

\[
S = \begin{pmatrix} r & t \\ t' & r' \end{pmatrix}
\]

one realizes that the dimensionless conductance $G/G_0 = Trt t^\dagger$ (the Landauer formula) may depend only on the parameter $c = N/M$. Further we shall expect that the average conductance increases in steps when $N$ changes. The size of the steps may depend on $c$.

To test this qualitative picture we have simulated the conductance of the system by averaging the transmission obtained over several random realizations of $H$ and $W$. In all the simulations $E_F = 0$ while we have varied $N$, $M$, as well as assumed different statistical properties of $H$. Specifically, we shall assume either GOE case or the situation where the eigenvalues of $H$ are uncorrelated. The latter case we shall call the Poissonian ensemble (PE) since the nearest neighbour statistics takes then a Poissonian form. The $N \times M$ coupling matrix $W$ was composed of $N$ mutually orthonormal random vectors of length $M$. The average is obtained by taking 1000 different realizations of a given system. Fig. 1(a) shows the average transmission (dimensionless conductance) obtained keeping a fixed value of $c = N/M$ and increasing $N$ by two. Observe that regardless of the properties of the internal matrix $H$ the qualitative behaviour of the conductance is quite similar, it increases in steps smaller than unity, the value of the step being dependent on $c$ and to a much lesser extend on the statistical properties of $H$. Panel (b) shows the behaviour of the system while keeping fixed the number of ‘internal channels’ $N$ and increasing $M$. Observe that the conductance steps actually decrease with $M$ for $M$ large. It is the number of ‘internal channels’, $N$, which limits the conductance value. The dependence on $M$ is much weaker and indicates that for larger $M$ the backscattering plays a larger role leading to decrease of the conductance steps.

Note that it is the backscattering on the abrupt transition between the 1D wire and the 2D leads that is solely responsible for the size of the conductance steps. Were the transition from wire to leads a smooth one, no backscattering would occur and the conductance steps would be equal to unity as follows from Levinson study [10].

Fig. 1 shows already that the experimental observations of [1] may be at least qualitatively explained by the simple RMT model. To exemplify this point further we have assumed that the density of states changes according to a triangular potential well (as in the experiment) when the applied voltage is varied. After choosing the free parameter in the model, i.e., $c$, the conductance dependence on the applied voltage reproduces fairly accurately the Fig. 2 of [1] (see Fig. 2).

Let us point out that the results obtained are very weakly dependent on the statistical properties of the internal Hamiltonian $H$. For a given $c$, the conductance quantization step, observed when $N$ is varied, increases slightly as the statistical properties of $H$ change from PE to GOE or the picket fence spectrum corresponding to the levels repulsion parameter $\beta \to \infty$. The quantization step size remains practically unaffected (within the statistical significance of our data) if we consider the case of broken time reversal symmetry, i.e., with $H$ belonging to GUE.

Our simple model cannot account for the changes of the conductance steps with the temperature, $T$ (as observed in [1] for larger $T$ the conductance step size increases and approaches unity). Such temperature changes are indicators of the importance of the electron-electron (e-e) interactions [4]. It seems thus quite intuitive to blame this interaction also for the non-integer conductance steps. In this respect the fact that our model, being a single particle approach, also yields $c$ dependent conductance steps is quite surprizing. Apparently, the step size can be reconstructed from RMT, i.e. a single particle approach (where at least a part of e-e interaction may be in principle included via Hartree or mean field approach). The increase of the step size with temperature will then point out to the increasing importance of the genuine many particle effects.

In effect, our model is appropriate, strictly speaking, for low temperatures only. Still qualitatively, one may explain the increase of the conductance with $T$ in terms of decreasing backscattering as mentioned in [1]. The point is that when temperature increases, the internal Hamiltonian $H$, described for low temperatures by PE, has to be - due to the increasing role of the temperature dependent disorder - replaced by a GOE matrix, and this leads to an increase in the conductance step.

Accepting that the model presented yields reasonable
predictions concerning the conductance steps one can ask whether in the studied, \( N \ll M \) case similar predictions may be obtained using RMT assumptions directly for the \( S \) matrix. Naturally, the standard approach has to be modified since the \( S \) matrix, must have \( M - N \) unity eigenvalues.

We are thus going to mimic the scattering matrix by \( S = UDU^\dagger \), where \( D \) is a diagonal matrix consisting of \( M - N \) unit eigenvalues and \( N \) eigenvalues \( \exp(i\varphi_i) \). The nontrivial eigenphases \( \varphi_i \) are distributed according to some assumed joint probability distribution \( P_\beta(\varphi_1, \ldots, \varphi_N) \), characterized by the level repulsion parameter \( \beta \in [0, \infty] \). Random unitary rotation matrix \( U \) is drawn uniformly with respect to the Haar measure on \( M \) dimensional unitary space and pertains to CUE. Such an assumption concerning \( U \) is fully correct for a broken time-reversal invariance, the situation not realized in the experiment. We know, however, from the standard RMT of scattering (in the \( M \ll N \) limit) that the dependence of the conductance on the symmetry is relatively small for disordered wires and appears only on the level of weak localization corrections through the eigenphases repulsion parameter \( \beta \). Thus the obtained should only weakly depend on detailed properties of \( U \). This assumption is even more justified by the numerical results, mentioned above, that revealed that the conductance step size is not sensitive to the change from GOE to GUE within the Heidelberg model.

The total conductivity in the system is given by a sum of the individual transmission coefficients \( G/G_0 = \sum_{l=m/2+1}^{M} |S_{lm}|^2 \). There exist \( M^2/4 \) elements of the matrix \( S \), contributing to the total conductance. Each element of this sum can be written as \( S_{lm} = \sum_{k=1}^{N} U_{lk}^* U_{km} e^{i\varphi_k} - 1 \), with \( l \neq m \). The double average \( \langle |S_{lm}|^2 \rangle_U \) over \( N \) random phases of the diagonal matrix \( D \) and over random rotation matrix \( U \), consists of \( N \) diagonal and \( N(N - 1) \) off-diagonal terms. The averages over unitary matrices \( U \) distributed according to the Haar measure are known\(^{[1]} \). They allow us to compute the average conductance at least for the two limiting cases, namely uncorrelated eigenphases (\( \beta = 0 \) - Poissonian case), and picket fence (\( \beta \rightarrow \infty \) - equally spaced) distribution. We get

\[
(G/G_0)_P = \frac{MN}{4(M^2 - 1)} (2M - N - 1), \tag{4}
\]

for the Poissonian case and

\[
(G/G_0)_C = \frac{MN}{4(M^2 - 1)} (2M - N), \tag{5}
\]

for the most rigid crystalline, picket fence case. Clearly, the results for other ensembles should lie between these two limits.

Using the above formulae one may calculate the conductance steps for fixed \( c = N/M \). In the limit of large \( N \), the step \( g \) (when \( N \) increases by two) is equal to \( g_P = 1 - 1/c \) for the former and \( g_C = 1 - 1/2c \) for the latter distribution. Clearly, also the model constructed to mimic the \( S \) matrix directly is capable to yield the prediction for the conductance step size smaller than unity.

While both the approaches, the Heidelberg method and the direct modelling of the \( S \) matrix properties yield similar predictions for \( N \ll M \), i.e. give conductance steps smaller than unity, the models seem not to be equivalent (as in the opposite case of \( M \ll N \) - see\(^{[1]} \)). For example, assuming in the former approach that the internal \( H \) pertains to GOE does not assure that the nontrivial eigenphases of the \( S \) matrix obey the appropriate COE statistics (as has been checked numerically). Still, as shown in Fig.\(^{[1]} \) for \( N \ll M \) and fixed \( c \) both models yield quite similar prediction for the average conductance (and thus the size of the quantization steps). This robustness of the nonuniversal step size to the details of the random model assumed suggests strongly that the phenomenon is quite general and occurs whenever the number of open channels \( M \) exceeds the number of internal states.

Apart from the pure wire case discussed in\(^{[1]} \) one can envision a chaotic quantum dot (with many thousands of levels) coupled by two almost ideal 1D leads to the broad connectors. Provided the coherence length exceeds the length of 1D leads we expect coherent backscattering on the border between the leads and the connectors. Then the number of original channels in the 1D leads determines the number of extended states, \( N \), in the system: quantum dot + leads. This number may be quite small. All other levels of the quantum dot remain localized and do not contribute to the conductance. As the number of channels in the connectors, \( M \), is large, the situation \( N \ll M \) is recovered and we predict that the conductance quantization steps will be smaller than unity and will depend on the ratio \( c = N/M \).

To summarize, we have presented simple, Random Matrix Theory based models, which predict that the conductance steps may be smaller than unity in agreement with the recent experiment. The origin of the deviation from unit steps comes from the coherent backscattering on the border between the quantum dot and the leads. The effect is quite general, does not depend on the details of the model, the only requirement being that the number of internal states, \( N \), is much less than the number of open channels, \( M \). Since the RMT model is essentially a single particle model our results indicate that electron-electron interactions can not be a sole origin of the “nonuniversal” conductance steps.

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FIG. 1. Panel (a) displays mean dimensionless conductance $G/G_0$ evaluated for $c=0.5$ (thick lines) and 0.3 (thin lines). The full line corresponds to the Poissonian case, broken lines represent results obtained for GOE. Panel (b) shows $G/G_0$ as a function of the number of channels $M$. The number of internal states $N=20$. Thick (thin) line corresponds to GOE (PE) case, respectively.

FIG. 2. Dependence of the dimensionless mean conductance on the applied voltage $V$. Filled dots connected by the line (to guide the eye) represent the results obtained in our model calculations with Poissonian internal matrix. The number of the channels $M$ and internal states $N$ depends on the voltage $V$ as $M = \left\lfloor (a - 3.7/V) \right\rfloor$; $N = \left\lfloor -3.7/V \right\rfloor$ with $a = 8.8$. Here $\lfloor x \rfloor$ represents even number being most close to $x$. Diamonds correspond to the experimental results obtained in [4].

FIG. 3. Mean conductance obtained for $c=0.3$ in the Poissonian case is ploted as a function of $N$ (full line) and compared with the Eq. (4) (crosses).
Figure 1b
Figure 2

[Graph showing the relationship between voltage and \( G/G_0 \).]
Figure 3

The graph shows a step function with $G/G_0$ on the y-axis and $N$ on the x-axis. The function increases in steps of 1, starting from $G/G_0 = 0$ at $N = 0$ and reaching $G/G_0 = 15$ at $N = 30$. The steps are equally spaced along the x-axis.