Multidimensional necklaces and measurable colorings of $\mathbb{R}^n$

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November 22, 2011

Abstract

A well known generalization of Alon’s "splitting necklace theorem" by Longueville and Živaljević states that every $k$-colored $n$-dimensional cube can be fairly split using only $k$ cuts in each dimension. Here we prove that for every $t$ there exist a finite coloring (with at least $(t+4)^d - (t+3)^d + (t+2)^d - 2^d + d(t+2) + 3$ different colors) of $\mathbb{R}^n$ such that no $n$-dimensional cube can be fairly split using at most $t$ cuts in each dimension. In particular there is a finite coloring of $\mathbb{R}^n$ such that no two disjoint $n$-dimensional cubes have the same measure of each color.

1 Introduction

In 2009 Alon, Grytczuk, Lasoń and Michałek [3] proved that there is a measurable $t+3$-coloring of the real line such that no interval has a fair $t$-splitting. As one of the corollaries they proved that there is a $5$-coloring of the real line such that no two intervals have the same measure of each of the $5$ colors. In the present paper we present a multidimensional generalization of their result - there is a $(t+4)^d - (t+3)^d + (t+2)^d - 2^d + d(t+2) + 3$-coloring of the $d$-dimensional Euclidean space such that no cube has a fair $d$-dimensional $t$-splitting.

*This paper is supported by joint programme SSDNM and is written under supervision of Prof. Jarosław Grytczuk.
By a $d$-dimensional spitting of a cube $[0, 1]^d$ of size at most $t$ in each dimension we mean a choice, for $i = 1, \ldots, d$ of $t_i$, $0 \leq t_i \leq t$, of nonnegative numbers $0 \leq z_i^1 \leq \ldots \leq z_i^{t_i} \leq 1$ in the unit interval $[0, 1]$ such that $\sum t_i \geq 1$. The hyperplanes defined by $z_i^j$ (i.e. $I \times \ldots \times \{z_i^j\} \times \ldots \times I \subseteq \mathbb{R}^d$) cut the cube $[0, 1]^d$ into at most $\prod (t_i + 1)$ smaller cubes. Whenever we speak of a splitting of exact size we mean that the number of $t_i$ for each $i$ is given and we assume that the inequalities between $z_i^j$'s, 0 and 1 are sharp.

By a $k$-coloring of $A \subseteq \mathbb{R}^d$ we mean a function $\phi : A \rightarrow \{1, \ldots, k\}$ such that each set $f^{-1}(i)$ is Lebesgue measurable.

**Theorem 1.1.** For every $k \geq (t + 4)^d - (t + 3)^d + (t + 2)^d - 2^d + d(t + 2) + 3$ there is a $k$-coloring of $\mathbb{R}^d$ such that no $d$-dimensional cube has a splitting of size at most $t$.

The proof is similar in nature to that of [3] and uses Baire category argument applied to the space of all measurable colorings. The same methods with only minor changes in proofs lead to a different version of the result:

**Theorem 1.2.** There is a $2^{2d+1} - 2 \cdot 3^d + 4d + 4$-coloring of $\mathbb{R}^d$ such that no two (disjoint) cubes have the same measure of each color.

## 2 Proof of theorem 1.1

Recall that a set in a metric space is nowhere dense if the interior of its closure is empty. A set is said to be of first category if it can be represented as a countable union of nowhere dense sets. In the proof of theorem 1.1 we apply the Baire category theorem.

**Theorem 2.1** (Baire). If $X$ is a complete metric space and $A$ is a set of first category in $X$ then $X \setminus A$ is dense in $X$ (and in particular is nonempty).

Our plan is to mimic the argument of [3] and construct a suitable metric space of colorings of $\mathbb{R}^d$ and demonstrate that the subset of bad colorings is of first category.

### 2.1 The setting.

Let $k$ be a fixed positive integer and let $\{1, 2, \ldots, k\}$ be the set of colors. Let $f$ and $g$ be two measurable colorings of $\mathbb{R}^d$ ($d \geq 1$). For a positive integer $n$
we set
\[ D_n(f, g) = \{ x \in [-n, n]^d : f(x) \neq g(x) \}. \]

Clearly \( D_n(f, g) \) is Lebesgue measurable so we may define the normalized
distance between \( f \) and \( g \) on \([-n, n]^d\) by
\[ d_n(f, g) = \frac{\lambda(D_n(f, g))}{n^d}, \]
where \( \lambda \) is the \( d \)-dimensional Lebesgue measure. Since \( d_n(f, g) \) is bounded
from above by \( 2^d \) we may define the distance between two measurable color-
ings \( f \) and \( g \) by
\[ d(f, g) = \sum_{n=1}^{\infty} \frac{d_n(f, g)}{2^{n+1}}. \]

Identifying colorings whose distance is zero gives a metric space \( M \) of equiva-
rence classes of all measurable \( k \)-colorings. Note that the splitting properties
are preserved by equivalent colorings.

**Lemma 2.2.** The space \( M \) is a complete metric.

We omit the proof since this is a simple generalization (see [3] or [3]) of
a result stating that sets of finite measure in any metric space form a complete
metric space with symmetric difference as the distance function.

Let \( t \geq 1 \) be a fixed integer. Let \( D_t \) be the subspace of \( M \) consisting of
those \( k \)-colorings that avoid intervals having a \( d \)-dimensional splitting of size
at most \( t \) in each dimension. We will show that \( D_t \) is not empty provided
that \( k \geq (t + 4)^d - (t + 3)^d + (t + 2)^d - 2^d + d(t + 2) + 3 \). By granularity
of a splitting we mean the length of the shortest subinterval \([z_j^i, z_{j+1}^i]\) in the
splitting. For \( n \geq 1 \) and \( r_1, \ldots, r_n \) let \( B_n^{(r_i)} \) be the set of those colorings from
\( M \) for which there exists at least one \( d \)-dimensional cube in \([-n, n]^d\) having
a \( d \)-dimensional splitting of size exactly \( r_i \) in the \( i \)-th dimension for each \( i \)
and granularity at least \( 1/n \). Finally let us denote all the bad colorings by
\[ B_n(t) = \bigcup_{r_i \leq t} B_n^{(r_i)}. \]

Clearly we have
\[ D_t = M \setminus \bigcup_{n=1}^{\infty} B_n(t). \]
Now our aim is to apply Baire category theorem to show that the sets $B_n(t)$ are nowhere dense, provided that $k \geq (t + 4)^d - (t + 3)^d + (t + 2)^d - 2^d + d(t + 2) + 3$.

### 2.2 The sets $B_n(t)$.

We show that each set $B_n^{(r_i)}$ is a closed subset of $M$. Since $B_n(t)$ is a finite union of these sets, it must be closed too.

**Theorem 2.3.** The set $B_n^{(r_i)}$ for every $r_i \geq 1$ and $n \geq 1$ is a closed subset of $M$.

**Proof.** Let $\{f_m\}$ be a sequence of colorings converging in $M$ to $f$. For each $m$ let $C_m$ denote a $d$-dimensional cube in $[-n, n]^d$ of granularity $\geq 1/n$ and having a fair splitting into exactly $r_i$ points in the $i$-th dimension. Let us denote by $\phi_m: [r_1] \times \ldots \times [r_d] \to \{1, 2\}$ the labeling function defining the two families from the fair splitting of $C_m$. Since $[-n, n]^d$ is compact we may assume that vertices of the sliced cube $C_m$ converge to vertices of some cube $C$ and since there is finite number of labeling functions we may assume that $\phi_m = \phi$ for every $m$. Now it is easy to see that $\phi$ gives a fair splitting for $C$. \hfill $\square$

Next we prove that each $B_n(t)$ has empty interior provided the number of colors $k$ satisfies $k > (t + 4)^d - (t + 3)^d + (t + 2)^d - 2^d + d(t + 2) + 2$. For this purpose let us call $f \in M$ a cube coloring on $[-n, n]^d$ if there is a partition of $[-n, n]^d$ into some number of (half open) $d$-dimensional cubes of equal size in each dimension, each filled with only one color. Let $I_n$ denote the set of all colorings from $M$ that are cube colorings on $[-n, n]^d$.

**Lemma 2.4** (comp. [3]). Let $f \in M$ be a $k$-coloring. Then for every $\epsilon > 0$ and $n \in \mathbb{N}$ there exists a coloring $g \in I_n$ such that $d(f, g) < \epsilon$.

**Proof.** Let $C_i = f^{-1}(i) \cap [-n, n]^d$ and let $C_i^* \subseteq [-n, n]^d$ be a finite union of intervals such that

$$\lambda((C_i^* \setminus C_i) \cup (C_i \setminus C_i^*)) < \frac{\epsilon}{2k^2}$$

for each $i = 1, 2, \ldots, k$. Define coloring $h$ so that for each $i = 1, 2, \ldots, k$ the set $C_i^* \setminus (C_i^* \cup \ldots \cup C_{i+1}^*)$ is filled with color $i$, the rest of the cube $[-n, n]^d$ is filled with any of these colors. Moreover we set $h$ to be equal $f$ outside $[-n, n]^d$. Note that $d(f, h) < \epsilon/2$ and $h^{-1}(i) \cap [-n, n]^d$ is a finite union of
cubes. Let $A_1, A_2, \ldots, A_N$ be the whole family of these cubes. Now split the cube $[-n, n]^d$ into $M^d$ cubes $B_1, \ldots, B_{M^d}$ equally spaced in $[-n, n]^d$. We define $g$ to be equal $h$ on $A_i$ whenever $A_i \subseteq B_j$ for some $j$ and $g(A_i)$ is of any color otherwise. Note that $g$ differs from $h$ on a set of $d$-dimensional measure at most $t((2n+4n/M)^d - (2n)^d)$ so that for sufficiently large $M$ $d(g, h) < \epsilon/2$ and we get $d(f, g) < \epsilon$.

In order to state the next lemma we will use the following notation:

$$D(d) := \sum_{i=1}^{d} \binom{d}{i} (t+2)^i (2^{d-i} - 1) = (t+4)^d + 1 - (t+3)^d - 2^d$$

**Lemma 2.5.** If $k > (t+2)^d + d(t+2) + 1 + D(d)$ then each $B_n(t)$ has empty interior.

**Proof.** Let $f \in B_n(t)$ be any bad coloring. Let $U(f, \epsilon)$ be the open $\epsilon$-neighborhood of $f$ in the space $\mathcal{M}$. Assume the assertion of the lemma is false: there is some $\epsilon > 0$ for witch $U(f, \epsilon) \subseteq B_n(t)$. By lemma 2.4 there is a coloring $g \in I_n$ such that $d(f, g) < \epsilon/2$, so that $U(g, \epsilon/2) \subseteq B_n(t)$. The idea is to modify slightly the cube coloring $g$ so that the new coloring will still be close to $g$, but there will be no cube in $[-n, n]^d$ possessing a fair splitting of size at most $t$ and granularity at least $1/n$. Without loss of generality we may assume that there are equally spaced cubes $C_{i_1, \ldots, i_d}$ for $i_1, \ldots, i_d \in \{1, 2, 3, \ldots, N\}$ in $[-n, n]^d$ such that $1 > 6n^2/N$ each cube is filled with a unique color in the cube coloring $g$. Let $\delta > 0$ be a real number satisfying

$$\delta < \min\{\sqrt[d]{\frac{\sqrt{\epsilon/2N}}{2N}}, \frac{2n}{N^2}\}.$$ 

Choose a color (which we will call from now on “white”).

Let $W'_{i_1, \ldots, i_d}$ where $i_1, \ldots, i_d \in \{1, 2, \ldots, N\}$ be a cube $[0, 2\delta]^d$ colored as follows: choose a countable set

$$\{m^j_{i_1, \ldots, i_d}\}_{j=1, \ldots, k; i_1, \ldots, i_d \in \{1, 2, 3, \ldots, N\}}$$

of linearly independent over $\mathbb{Q}$ real numbers such that $0 < m^j_{i_1, \ldots, i_d} < (\delta/k)^d$. We color $W'_{i_1, \ldots, i_d}$ white except for small cubes

$$V^n_{i_1, \ldots, i_d} = \left(\frac{2\eta - 1}{k}, \ldots, \frac{2\eta - 1}{k} \delta\right) + \prod_{j=1}^{d}[-\sqrt[m^j_{i_1, \ldots, i_d}]{\delta}, \sqrt[m^j_{i_1, \ldots, i_d}]{\delta}]$$

We get $d(f, g) < \epsilon$. 

In order to state the next lemma we will use the following notation:
colored using color \( \eta \) for \( \eta = 1, 2, \ldots, k \) (compare Pic. 1 for the 2-dimensional case). Note that the \( d \)-dimensional Lebesgue measure of \( V_{i_1, \ldots, i_d}^\eta \) is equal \( 2^d m_{i_1, \ldots, i_d}^2 \) hence measures of these cubes are linearly independent over \( \mathbb{Q} \).

Now modify the coloring \( g \) to get a coloring \( h \) outside \( B_n(t) \). The coloring \( h \) is equal to \( g \) outside \( [-n, n]^d \). Inside \( C_{i_1, \ldots, i_d} \) the coloring \( h \) is equal \( g \) except in

\[
W_{i_1, \ldots, i_d} = (\left( i_1 - \frac{1}{2} - \delta \right) \frac{2n}{N} - n, \ldots, (i_d - \frac{1}{2} - \delta \frac{2n}{N} - n) + W_{i_1, \ldots, i_d}'
\]

where \( h \) is defined by the coloring of \( W_{i_1, \ldots, i_d}' \).

Note that \( d(g, h) < \epsilon/2 \) so that there exists a \( d \)-dimensional cube \( C \) in \( [-n, n]^d \) with granularity at least \( 1/n \) such that there is a fair splitting of size at least \( t \). The fair splitting divides \( C \) into at most \( (t + 1)^d \) cubes hence we obtain a \( d \)-dimensional cell complex in \( [-n, n]^d \) (which we will also denote by \( C \)).

Let us denote by \( A \) the measure of \( C_{i_1, \ldots, i_d} \setminus W_{i_1, \ldots, i_d} \) (note that \( A \) does not depend on the set of indexes chosen and we may assume it is linearly independent with the \( m_{i_1, \ldots, i_d}^2 \) chosen before).

By the determinant of \( C_{i_1, \ldots, i_d} \) in \( C \) (den. by \( \det_C C_{i_1, \ldots, i_d} \)) we mean the lowest dimension of cells \( C \) that intersect \( C_{i_1, \ldots, i_d} \) (there is only one cell reaching the minimum – denoted by \( d_C(C_{i_1, \ldots, i_d}) \)). If \( C_{i_1, \ldots, i_d} \) lays outside \( C \) we set \( \det_C C_{i_1, \ldots, i_d} = d \). Note that cells of \( C \) divide each cube \( C_{i_1, \ldots, i_d} \) into \( 2^{\text{codim} \det_C C_{i_1, \ldots, i_d}} \) cubes of measures

\[
\alpha_1(d_C(C_{i_1, \ldots, i_d})), \alpha_2(d_C(C_{i_1, \ldots, i_d})), \ldots, \alpha_{2^{\text{codim} \det_C C_{i_1, \ldots, i_d}}}(d_C(C_{i_1, \ldots, i_d}))
\]

and their sum is equal to \( A \). In fact (up to indexing) \( \alpha_i(d_C(C_{i_1, \ldots, i_d})) \) does not depend on \( d_C(C_{i_1, \ldots, i_d}) \) but on the \( \det_C(C_{i_1, \ldots, i_d}) \)-dimensional subspace of \( \mathbb{R}^d \) spanned by it. The subspace can be identified by a suitable choice of \( \text{codim} \det_C C_{i_1, \ldots, i_d} \) slices (or ends) of \( C \) on some of the dimensions. Hence we get that \( \alpha_i(d_C(C_{i_1, \ldots, i_d})) = \alpha_i(t_1, \ldots, t_s) \) for \( t_1, \ldots, t_s \in \{0, 1, 2, \ldots, t+1\} \) and \( s = 0, 1, 2, \ldots, d \). Of course \( \alpha_1(\emptyset) = A \).

Note that the dimension of the space spanned by \( \alpha_i(t_1, \ldots, t_s) \) where \( s > 0 \) is no greater than \( D(d) \).

Now note that all the vertices of \( C \) are colored at most by \( (t + 2)^d \) colors. Moreover cells of dimensions \( d - 1 \) of \( C \) intersect at most one of the cubes \( V_{i_1, \ldots, i_d}^\eta \subseteq C_{i_1, \ldots, i_d} \) and two such cell intersect the cubes of the same color if they span the same subspace of \( \mathbb{R}^d \). Since there are at most \( d(t + 2) \)
different subspaces of $\mathbb{R}^d$ obtained in such a way then if $C$ intersects one of $V_{i_1,\ldots,i_d} \subseteq C_{i_1,\ldots,i_d}$ then $d-1$ of $C$ also does and it has one of $d(t+2)$ colors.

Summing up let us consider a color $c$ different from white and the $(t+2)^d + d(t+2)$ colors mentioned before. Since our splitting is fair $d$-dimensional cells colored partially by $c$ can be divided into two families having equal measure of $c$. Hence the measure satisfies equality of the form:

$$T(0)A + \sum \epsilon(0)_{i_1,\ldots,i_d} 2^d m_{i_1,\ldots,i_d} + \sum S(0)_{i_1,\ldots,i_d} \alpha_i(t_1,\ldots,t_s)$$

$$- T(0)A - \sum \epsilon(0)_{i_1,\ldots,i_d} 2^d m_{i_1,\ldots,i_d} - \sum S(0)_{i_1,\ldots,i_d} \alpha_i(t_1,\ldots,t_s) = 0$$

where $T(0), T(1) \in \mathbb{N}$, $\epsilon(0)_{i_1,\ldots,i_d}, \epsilon(1)_{i_1,\ldots,i_d} \in \{0, 1\}$, $S(0), S(1) \in \mathbb{N}$ and not all $\epsilon(0)_{i_1,\ldots,i_d}$ are equal to 0. Note that for each color the numbers

$$T(0)A + \sum \epsilon(0)_{i_1,\ldots,i_d} 2^d m_{i_1,\ldots,i_d} - T(0)A - \sum \epsilon(0)_{i_1,\ldots,i_d} 2^d m_{i_1,\ldots,i_d}$$

are independent over $\mathbb{Q}$. On the other hand they can be generated over $\mathbb{Q}$ by $\alpha_i(t_1,\ldots,t_s)$ so they lie in $D(d)$-dimensional space. Since the number of remaining colors is greater than $D(d)$ we get a contradiction that ends the proof.

3 Generalizations and open problems

Note that when $t = 1$ and $d = 1$ we get that for $k > 7$ there is a $k$-coloring of $\mathbb{R}$ such that no consecutive intervals have the same measure of every color. This result is weaker than the one obtained in [3] where it was proved for $k = 4$, but our approach has a higher-dimensional generalizations. Moreover, even in our approach, the number of colors can be easily reduced to 4 in the case of the real line and one slice (or to $t+3$ in the case of the real line and $t$ slices).

Nevertheless the difference between the positive answer, i.e. $k$-colored $d$-dimensional cube can be fairly split using $kd$ cuts (comp. [4]), and our result is surprisingly big. Hence we hope that our result can be further improved.

Note also that a small modification of our result gives the following

**Theorem 3.1.** There is a $2^{2d+1} - 2 \cdot 3^d + 4d + 4$-coloring of $\mathbb{R}^d$ such that no two (disjoint) cubes have the same measure of each color.
Once again our approach can be improved in the case of the real line to match the result of [3] – there is a 5-coloring of the real line such that no two disjoint intervals have the same measure of each color.

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