Acoustic sensing method based on the inverse problem of recovering the density profile and the bulk modulus of the inhomogeneous medium

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Abstract. The model of inhomogeneous acoustic medium without loss in dimensional approximation based on the Riccati equation for the impedance of the medium is considered. The inverse operator problem of recovering the coefficients of the Riccati equation is formulated. The numerical algorithm for solving the problem based on the integral equation coupling input acoustic impedance or admittance to the distributions of density or bulk modulus of the medium is proposed. The method of acoustic scanning is proposed. The numerical simulation of the proposed algorithm is carried out.

1. Introduction
In this paper we consider the problem of determining continuously varying mechanical properties of one-dimensional acoustic medium, in particular, the density profile or the bulk modulus that is relevant to NDT methods. It should be noted that most of the existing methods are based on measurement of irregularities in the form of abrupt disorders of the mechanical properties of the tested materials [1-3]. The proposed method will be tested and used on ultrasonic scanning system is being developed by our research group [4-6].

2. Mathematical model the acoustic medium in one-dimensional approximation
Propagation of acoustic oscillations in the medium is described by equations of hyperbolic type [7,8], which in a frequency domain are the system of equations:

\[ \frac{\partial p(x, j\omega)}{\partial x} + \rho(x) j\omega v(x, j\omega) = 0, \quad (1) \]

\[ \frac{\partial v(x, j\omega)}{\partial x} + \frac{j\omega}{k(x)} p(x, j\omega) = 0, \quad (2) \]
where $\rho(x)$, $k(x)$ are the density and bulk modulus of the medium [9], $p(x, j\omega)$ and $v(x, j\omega)$ are the pressure and particle velocity in the medium [9]. We make the transition to the impedance of the acoustic medium:

$$Z(x, j\omega) = \frac{p(x, j\omega)}{v(x, j\omega)},$$

and the system (1, 2) we transform to the Riccati equation [7]:

$$\frac{dZ(x, j\omega)}{dx} - \frac{j\omega}{k(x)} Z^2(x, j\omega) + j\omega \rho(x) = 0,$$

where $x \in [0, l]$.

Analogous equation can be derived for the admittance of the medium:

$$Y(x, j\omega) = \frac{v(x, j\omega)}{p(x, j\omega)},$$

$$\frac{dY(x, j\omega)}{dx} - j\omega \rho(x) Z^2(x, j\omega) + \frac{j\omega}{k(x)} = 0.$$

3. Formulation of the inverse problem

We form the equation Riccati which generalizes equations (4) and (6):

$$\frac{dH(x, j\omega)}{dx} - j\omega \gamma(x) H^2(x, j\omega) + j\omega \beta(x) = 0,$$

where $H(x, j\omega)$ is the immittance equal to impedance or admittance of the medium, $\gamma(x)$ and $\beta(x)$ for the equation (4) respectively equal $\frac{1}{k(x)}$ and $\rho(x)$, and for equation (6) equal to the same values, but in reverse order.

In this work the task of determining the coefficient $\beta(x)$ equation (7) for a known and given constant coefficient $\gamma(x)$ is assigned:

$$\gamma(x) = \gamma_0.$$

We formulate the inverse operator problem of determining $\beta(x)$ based on the equation (7).

As boundary conditions we consider the case when at the end of the line immittance equal to the definite value $H_i$:

$$H(l, j\omega) = H_i.$$

We also assume that the input immittance of the medium $H_s(j\omega)$ is measured with the standard error $\delta_s$:

$$H(0, j\omega) = H_s(j\omega).$$
We believe that the small perturbation the density of the medium $\delta \beta(x)$ leads to the corresponding small increment immittance $\delta H_s(0, j \omega)$. This relationship is represented as the Fredholm integral equation of the first kind \[10,11\]:

$$\int_0^l \exp\left(-\int_0^x j \omega' \cdot 2H(x', j \omega)dx'\right)j \omega \delta \beta(x)dx = \delta H_s(0, j \omega).$$ (11)

Solution of the integral equation Fredholm of the first kind is ill-posed problem. One of the standard methods for solving this equation is the Tikhonov regularization \[7,10,11\].

Using the boundary value problem (7), (9), by the relations (8), (10), (11) to recover the distribution of the parameter $\beta(x)$ is necessary.

4. The algorithm for solving the inverse problem and its numerical simulation

Based on the equation (4) the algorithm for determining $\beta(x)$ on the measured input immittance $H_s(j \omega)$ is constructed:

1. Setting the initial approximation $\beta_0(x)$ and starting the iteration process ($i = 0$ – number of iteration);
2. Calculation of the frequency response of immittance $H_i(x, j \omega)$ by solving the equation (7) for the model with the function $\beta_i(x)$;
3. Finding the right side of equation (11) by the formula: $\delta H_s(0, j \omega) = H_s(j \omega) - H_i(0, j \omega)$;
4. Calculation $\delta \beta_i(x)$ by solving the equation (11) using the method of Tikhonov regularization;
5. If $\int_0^l |\delta \beta_i(x)|^2 dx \leq \varepsilon^2$, where $\varepsilon$ is the set value, then $\beta_i(x)$ is the sought value, if not, then calculating the following approximation $\beta_{i+1}(x) = \beta_i(x) + \delta \beta_i(x)$ with the transition to step 3.

The reconstruction medium density $\rho(x)$ algorithm from the measured input impedance $Z_s(j \omega)$ and the algorithm for determining the bulk modulus $k(x)$ on the input acoustic admittance $Y(0, j \omega)$ was considered for the numerical realization. Main relations (7), (9) - (11) were reduced to dimensionless form: $\bar{x} = x/l, \bar{\omega} = \omega l \sqrt{K_0 \rho}, \bar{H} = H_l \omega \gamma_0, \bar{\beta} = \beta / \beta_0, \bar{\gamma} = \gamma / \gamma_0 = 1$. Riccati equation and the corresponding boundary conditions thus take a normalized form:

$$\frac{d\bar{H}}{dx} - \bar{H}^2 - \bar{\omega}^2 \bar{\beta}(x) = 0; \quad 0 < \bar{x} < 1,$$ (12)

$$\bar{H}(0, j \omega) = \bar{H}_l.$$ (13)

Further, to simplify writing the upper horizontal line will be dropped and all statement in this article will be leaded to normalized values.

For the regularized solution of Fredholm integral equation of the first kind we used the standard program TIKH 1 given in \[12\], where the method of Tikhonov regularization with the regularization parameter $\alpha_T$ selection method of the generalized discrepancy is used. In all calculations we used the first-order regularizer. We set the upper $\alpha_{T1}$ and lower $\alpha_{Tm}$ bounds of the regularization parameter varies as:
\[ \alpha_T > 0, \quad \alpha_{T_i} = \theta \alpha_{T(i-1)}, \quad i = 2, 3, ..., m, \quad 0 < \theta < 1. \]  \hspace{1cm} (14)

As the lower bound \( \alpha_m \) was selected value \( \alpha_m \approx \delta_R^2 \), where \( \delta_R \) measurement error, and upper bound is was selected value \( \alpha_r = 0.1 \). Numerical solution of the Riccati equation was based on recurrent reverse BR - algorithm [10,13]. When solving the Fredholm integral equation of the first kind (11) was set error right side \( \delta_R \):

\[ \| \tilde{H}_s - H_s \|_{L_2} \leq \delta_h, \quad \delta_h = \delta_s \sqrt{\omega_{\text{max}} - \omega_{\text{min}}}; \]  \hspace{1cm} (15)

where \( \delta_s \) is the standard error of measurement of the real and imaginary parts of immittance \( H_s \). Dependence coefficient \( \beta(x) \) from the longitudinal coordinate is modeled in the form:

\[ \beta(x) = \beta_0 \psi_F(x). \]  \hspace{1cm} (16)

As an example, the inhomogeneity function \( \psi_F(x) \) was defined by two quadratic polynomials:

\[ \psi_{F_1} = 1 + 0.1x + 0.1x^2. \]  \hspace{1cm} (17)

\[ \psi_{F_2} = 1 - 0.1x - 0.1x^2. \]  \hspace{1cm} (18)

Probing the medium was carried out from one end. Area of solving the integral equation (11) and the Riccati equation (7) was set on the mesh \( N_x \times N_y \), homogeneous frequency and spatial coordinates \( x \).

Numerical experiments was carried out on PC using the programming environment Lab VIEW 8.5. Calculations were performed with complex numbers with double precision. The dimensionless frequency were lying in two ranges \( 0.032 < \omega < 1.18 \), \( 0.032 < \omega < 1.24 \) for expressions (17) and (18) respectively. In these frequency ranges error recovery is minimal. Space-frequency mesh dimension was \( N_x \times N_y = 30 \times 100 \). Boundary value of the immittance at the end of the line is zero:

\[ H_l = 0. \]  \hspace{1cm} (19)

In all examples, the initial approximations \( \psi_F(x) \) was set constant and equal to unity.

In the Figure 1a and Figure 1b are shown respectively desired (1) and reconstructed (2) profile inhomogeneity function \( \psi_{F_1}(x) \), defined in the form (17), when the measurement error input immittance \( \sigma_s = 0.1\%, \ 1\% \). The relative error recovery amounted 0.94\%, 1.04\%. 

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Figure 1a. Desired profile inhomogeneity function $\psi_{F1}(x)$

Figure 1b. Reconstructed profile inhomogeneity function $\psi_{F1}(x)$

In the Figure 2a and Figure 2b are shown respectively results of restoration the function $\psi_{F2}(x)$ defined in the form (18) at various levels of error $\sigma_S$ (0.1%, 1%) measurements of input immittance $Z_s(j\omega)$. For this case, the error of the algorithm was amounted 1.21%, 2.09%, respectively. Calculations showed that the first 2-3 iterations provide a very rapid approach to the reversal of the desired curve, and then the process of continuous improvement solutions comes with a very low speed.

Figure 2a. Result of restoration function $\psi_{F2}(x)$ at level of error $\sigma_S=0.1\%$

Figure 2b. Result of restoration function $\psi_{F2}(x)$ at level of error $\sigma_S=1\%$

5. Conclusions
In this work the model of inhomogeneous acoustic medium without loss in dimensional approximation based on the Riccati equation for the impedance of the medium is considered. The inverse operator problem of recovering the coefficients of the Riccati equation is formulated. The numerical algorithm for solving the problem based on the integral equation coupling input acoustic impedance or admittance to the distributions of density or bulk modulus of the medium is proposed. The method of acoustic scanning is proposed.

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