Free Entropy Dimension $\leq 1$
for Some Generators of
Property $\tau$ Factors of Type II$_1$

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Free Entropy Dimension $\leq 1$ for
Some Generators of Property $T$ Factors of Type $II_1$

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Abstract: The modified free entropy dimension of certain $n$-tuples of self-adjoint operators, satisfying sequential commutation, is shown to be $\leq 1$. In particular the von Neumann algebras of type $II_1$ of the groups $SL(n, \mathbb{Z})$, $n \geq 3$ have generators with free entropy dimension $\leq 1$.

The free entropy $\chi(X_1, \ldots, X_n)$ and the modified free entropy dimension $\delta_0(X_1, \ldots, X_n)$ of an $n$-tuple of selfadjoint elements in a $II_1$-factor ([6],[7],[8]) have been the key to the solution of some old problems on von Neumann algebras ([7],[3],[4],[5]).

The natural generator of a free group factor $L(F(n))$ has $\delta_0$ equal $n$, while various conditions on the $II_1$-factor (property $\Gamma$, existence of Cartan subalgebras) imply $\delta_0 \leq 1$ for any generator.

Here we prove $\delta_0 \leq 1$ for $n$-tuples of selfadjoint elements satisfying certain sequential commutation conditions and as a corollary we get that the von Neumann algebras of type $II_1$ $L(SL(n; \mathbb{Z}))$ ($n \geq 3$) have generators with $\delta_0 \leq 1$.

We would like to point out a certain (not yet explained) similarity to an ergodic theory result in [2], which inspired this note.
At the end of the paper we formulate, what seems to us, a natural question about generators of $L(SL(n; \mathbb{Z}))$ in view of the free entropy dimension result.

Throughout, $(M, \tau)$ will denote a von Neumann algebra with a faithful normal trace-state and we refer the reader for the definitions of $\chi$, $\delta_0$ and of the sets of matricial approximants $\Gamma$ to [6] and [7]. In particular,

$$
\delta_0(X_1, \ldots, X_n) = n + \limsup_{\varepsilon \to 0} \frac{\chi(X_1 + \varepsilon S_1, \ldots, X_n + \varepsilon S_n : S_1, \ldots, S_n)}{\log \varepsilon}
$$

where $S_1, \ldots, S_n$ are (0,1)-semicircular and $\{X_1, \ldots, X_n\}, \{S_1\}, \ldots, \{S_n\}$ are free.

**Theorem 1** Let $X_j = X_j^* \in M$, $1 \leq j \leq n$. Assume $[X_k, X_{k+1}] = 0$, $1 \leq k < n$ and assume that the spectral measure of $X_k$, $1 \leq k < n$ has no atoms. Then

$$
\delta_0(X_1, \ldots, X_n) \leq 1.
$$

The proof is based on a few lemmas. By $\mathcal{M}_\|$ we will denote the $k \times k$ complex matrices and by $\| \|$ the normalized Hilbert-Schmidt norm $\|A\|^2 = k^{-1} \text{Tr}(A^*A)$. Further, $(\varepsilon_{pq})_{1 \leq p, q \leq k}$ will be the canonical matrix-units in $\mathcal{M}_\|$. The real subspace of selfadjoint elements will be denoted by $\mathcal{M}_\|^*$ and $\text{vol}$ will denote the volume corresponding to the unnormalized Hilbert-Schmidt norm (i.e., $k^2 \cdot \|\|$). We shall also use $\text{vol}$ to denote the corresponding volume on $(\mathcal{M}_\|^*)^\perp$.

**Lemma 1** Let $A = \sum_{1 \leq i \leq k} \lambda_i e_{ij} \in \mathcal{M}_\|^*$ where $\lambda_1 \leq \cdots \leq \lambda_k$. Assume moreover that $|\lambda_s - \lambda_t| \geq \delta$ if $|s - t| \geq k \alpha$ where $0 < \alpha < 1$. Let

$$
\Omega(A; \varepsilon) = \{B \in \mathcal{M}_\|^* \mid \|B\|_\varepsilon \leq \alpha, \|B, A\|_\varepsilon \leq \varepsilon\}
$$

and let $\Omega(A; \varepsilon, r)$ be the $r$-neighborhood $\{B \in \mathcal{M}_\|^* \mid \exists C \in \Omega(A; \varepsilon), \|B - C\|_\varepsilon \leq \nabla\}$. Then if $\varepsilon \delta^{-1} + r \leq 1 + r$, we have

$$
\text{vol}(\Omega(A; \varepsilon, r)) \leq c((2k\alpha+1)k)(1+r)^{k^2/(2(k\alpha)+1)} \cdot \varepsilon \delta^{-1} + r)^k k^{(k-2|k\alpha|_1)k^2/2}
$$

where $c(m) = \pi^{m/2} (\Gamma(1 + m/2))^{-1}$.

**Proof.** Let $\mathcal{M}_\|^* = \mathcal{V}_\infty \oplus \mathcal{V}_\varepsilon$ where $V_1 = \text{span} \{\varepsilon_{st} \mid |s - t| < k \alpha\}$ and $V_2 = \text{span} \{\varepsilon_{st} \mid |s - t| \geq k \alpha\}$. $V_1, V_2$ are invariant subspaces of $\text{ad} A$ and $|\text{ad} A(h)|_2 \geq \delta|h|_2$. 2
if $h \in V_2$. Thus, if $h = h_1 \oplus h_2 \in \Omega(A; \varepsilon)$ then $|h_1|_2 \leq 1$ and $|h_2|_2 \leq \varepsilon^{-1}$. Hence, if $h = h_1 \oplus h_2 \in \Omega(A; \varepsilon, r)$, then $|h_1|_2 \leq 1 + r$, $|h_2|_2 \leq \varepsilon^{-1} + r$. We remark further that

$$\dim V_1 \leq k(2[k\alpha] + 1)$$
$$\dim V_2 \leq k(k - 2[k\alpha] - 1).$$

If $\varepsilon^{-1} + r \leq 1 + r$, the volume of $\Omega(A; \varepsilon, r)$ can be majorized by the product of the volumes of a ball of dimension $k(2[k\alpha] + 1)$ and radius $(1 + r)$ and of a ball of dimension $k(k - 2[k\alpha] - 1)$ and radius $\varepsilon^{-1} + r$. \hfill \Box

**Lemma 2** Assume the spectral projections of some $A \in \mathcal{M}_{\| \cdot \|}^+$ satisfy

$$\text{Tr} \ E(A; (t, t + \delta)) < k\alpha$$

for all $t \in \mathbb{R}$. Then the conclusion of Lemma 1 about $\text{vol}(\Omega(A; \varepsilon, r))$ holds.

**Proof.** Indeed, $A$ is unitarily equivalent to a diagonal matrix satisfying the assumptions of Lemma 1. \hfill \Box

We also record as the next lemma an immediate consequence of Fubini’s theorem.

**Lemma 3** Let $\mu_j$ be measures on locally compact spaces $X_j$ ($1 \leq j \leq n$) and $\mu = \mu_1 \otimes \cdots \otimes \mu_n$. Let further $Y \subset X_1 \times \cdots \times X_n$ be an open set, such that for all $1 \leq j \leq n$ and $\xi \in \text{pr}_j Y$ we have

$$\mu_{j+1}(\text{pr}_{j+1}(\text{pr}_j^{-1}(\xi))) \leq \rho.$$ 

Then

$$\mu(Y) \leq \mu_1(\text{pr}_1 Y) \cdot \rho^{n-1}.$$ 

Next, we define some sets which will appear in the next lemma.

By $\gamma(n; k, \varepsilon, \delta, \alpha)$ we denote the set of $n$-tuples $(A_1, \ldots, A_n) \in (\mathcal{M}_{\| \cdot \|}^+)^n$ such that:

$|A_s|_2 < 1$ (1 $\leq s \leq n$), $|A_j, A_{j+1}|_2 < \varepsilon$ and $\text{Tr} \ E(A_j; [p\delta, (p + 1)\delta]) < 2^{-1}k\alpha$ for all $p \in \mathbb{Z}$ and $1 \leq j \leq n$.

If $K \subset (\mathcal{M}_{\| \cdot \|}^+)^n$ we shall denote by $D(K; r)$ the set of $n$-tuples $(B_1, \ldots, B_n) \in (\mathcal{M}_{\| \cdot \|}^+)^n$ such that $|B_j - A_j|_2 < r$ (1 $\leq j \leq n$) for some $(A_1, \ldots, A_n) \in K$. 

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Lemma 4 Assume $X_j$ ($1 \leq j \leq n$) satisfy the assumptions of Theorem 1 and assume moreover that $\|X_j\| < 1/2$ ($1 \leq j \leq n$). Let further $S_j$ be $(0, 1)$-semicircular, so that \{X_1, \ldots, X_n\}, \{S_1\}, \ldots, \{S_n\} are free. Then, given $\alpha > 0$, $\beta > 0$, there are $\varepsilon > 0$, $m \in \mathbb{N}$, $\varepsilon_1 > 0$, $\delta > 0$ such that $\varepsilon \delta^{-1} < \beta$ and

$$\Gamma_2(X_1 + \theta S_1, \ldots, X_n + \theta S_n : S_1, \ldots, S_n; m, k, \varepsilon_1) \subset D(\gamma(n; k, \varepsilon, \delta, \alpha); 2\theta)$$

for all $0 < \theta < 1/2$.

Proof. Given $\varepsilon_2 > 0$, for $\varepsilon_1 > 0$ small enough

$$\Gamma_2(X_1 + \theta S_1, \ldots, X_n + \theta S_n : S_1, \ldots, S_n; m, k, \varepsilon_1) \subset D(\Gamma_4(X_1, \ldots, X_n; m, k, \varepsilon_2); 2\theta).$$

Thus it will suffice to prove the lemma with the inclusion at the end, replaced by

$$\Gamma_4(X_1, \ldots, X_n; m, k, \varepsilon_1) \subset \gamma(n; k, \varepsilon, \delta \alpha).$$

Given $\alpha > 0$, the $X_j$'s ($1 \leq j < n$) having continuous spectral measures, there is $1 > \delta > 0$ such that

$$\tau(E(X_j; ((p - 1)\delta, (p + 2)\delta))) < 4^{-1} \alpha$$

for all $p \in \mathbb{Z}$. We then choose $\varepsilon > 0$, such that $\varepsilon \delta^{-1} < \beta$. Clearly, if $\varepsilon_1 > 0$ is small enough and $m \geq 4$ the conditions $\|A_j, A_{j+1}\|_2 < \varepsilon$ ($1 \leq j < n$) and $|A_s|_2 < 1$ ($1 \leq s \leq n$) will be satisfied if $(A_1, \ldots, A_n) \in \Gamma_4(X_1, \ldots, X_n; m, k, \varepsilon_1)$.

If $p \in \mathbb{Z}$ and $[(p - 1)\delta, (p + 2)\delta] \subset [-5, 5]$, let $P_p(t)$ be a real polynomial which is $\geq 0$ for all $q \in \mathbb{R}$ and so that $P_p(t) \geq 4/5$ if $t \in [(p\delta, (p + 1)\delta]$ and $\tau(P_p(X_j)) < 4^{-1} \alpha$, ($1 \leq j < n$). Then choosing $\varepsilon_1 > 0$ small enough and $m > \deg P_p$ for the finite set of $p$ considered, we will have

$$k^{-1} \text{Tr}(P_p(A_j)) < 3^{-1} \alpha.$$ 

This in turn will insure that

$$\frac{4}{5} k^{-1} \text{Tr}(E(A_j; [p\delta, (p + 1)\delta])] < 3^{-1} \alpha$$

which implies

$$k^{-1} \text{Tr}(E(A_j; [p\delta, (p + 1)\delta])] < 2^{-1} \alpha.$$
Proof of Theorem 1. There is no loss to prove the theorem under the additional assumption \( \|X_s\| < 1/2 \) (1 \( \leq \) s \( \leq \) n). We shall use Lemma 4 to estimate
\[
\chi(X_1 + \theta S_1, \ldots, X_n + \theta S_n : S_1, \ldots, S_n).
\]
Using 1.3 in [7] this is the same as
\[
\chi_2(X_1 + \theta S_1, \ldots, X_n + \theta S_n : S_1, \ldots, S_n).
\]
By Lemma 4 we will have to estimate \( \text{vol}(D(\gamma(n; k, \varepsilon, \delta, \alpha); 2\theta)) \). Lemma 3 gives
\[
\text{vol}(D(\gamma(n; k, \varepsilon, \delta, \alpha); 2\theta)) \leq c(k^2)(1 + 2\theta)^{k^2/2} \rho^{-n-1}
\]
if \( \rho \) is an upper bound for \( \text{vol}(\Omega(A; \varepsilon, 2\theta)) \) when \( \text{Tr}(E(A; (t, t + \delta))) < k\alpha \) for all \( t \in \mathbb{R} \) (indeed this last condition follows from
\[
\text{Tr}(E(A; [p\delta, (p + 1)\delta])) < 2^{-1}k\alpha \quad \text{for} \quad p \in \mathbb{Z}).
\]
Further, by Stirling’s formula we have
\[
n^{-1} \log c(n) = 2^{-1} \log(2\pi e n^{-1}) + O(n^{-1} \log n).
\]
By Lemma 1 we can choose \( \rho \) so that
\[
k^{-2} \log \rho \leq 2^{-1} \log(2\pi e k^{-1}) + 2\alpha \log(1 + 2\theta) + (1 - 2\alpha) \log(\beta + 2\theta) + O(k^{-1} \log k).
\]
Thus we get
\[
\chi_2(X_1 + \theta S_1, \ldots, X_n + \theta S_n : S_1, \ldots, S_n)
\leq \frac{n}{2} \log 2\pi e + (2(n - 1)\alpha + 1) \log(1 + 2\theta) + (1 - 2\alpha)(n - 1)(\log(\beta + 2\theta)).
\]
Since \( \alpha > 0, \beta > 0 \) are arbitrary we have
\[
\chi_2(X_1 + \theta S_1, \ldots, X_n + \theta S_n : S_1, \ldots, S_n) \leq \frac{n}{2} \log 2\pi e + (n - 1) \log 2\theta + \log(1 + 2\theta).
\]
Hence
\[
\delta_0(X_1, \ldots, X_n) \leq n + (n - 1) \lim_{\theta \to 0} \frac{\log 2\theta}{\log \theta} = 1.
\]
For a discrete group $G$ let $L(G)$ denote the von Neumann algebra generated by the left regular representation $\lambda$ of $L(G)$ on $\ell^2(G)$ and $\tau$ the von Neumann trace.

If $G$ is $SL(n, \mathbb{Z})$, $(n \geq 3)$, then $G$ is generated by the $g_{ij} = 1 + \epsilon_{ij} \in \mathcal{M}$, $i \neq j$ and since $g_{ij}$ and $g_{k\ell}$ commute if $i = k$ or $j = \ell$, we can form a generator $g_{i_1j_1}, \ldots, g_{i_pm}$ of $SL(n, \mathbb{Z})$ such that consecutive elements commute (repetitions are allowed) ([2]). Since $g_{i_aj}$ has infinite order, $\tau$ applied to the spectral measure of $\lambda(g_{i_aj})$ is Haar measure on the unit circle. Let log denote the Borel function on the unit circle $\log \exp(2\pi i \theta) = 2\pi i \theta$ if $0 \leq \theta < 1$ and let $h_k = (2\pi i)^{-1} \log \lambda(g_{i_aj})$. Then $h_1, \ldots, h_p$ generate $L(SL(n, \mathbb{Z}))$, are selfadjoint, have Lebesgue absolutely continuous spectral measure, and consecutive $h_k$ commute. Using Theorem 1 we have proved the following corollary.

**Corollary 1** $L(SL(n, \mathbb{Z})), n \geq 3$, has a generator of selfadjoint elements $h_1, \ldots, h_p$ such that $\delta_0(h_1, \ldots, h_p) \leq 1$.

Since results about $\delta_0$ seem to have corresponding results about the number of elements of a generator, we formulate below the natural problem suggested by the preceding corollary.

**Problem.** Let $n \geq 3$. Does $L(SL(n, \mathbb{Z}))$ have for every $\varepsilon > 0$ a generator $\{X_1, X_2\}$ such that $X_j = X_j^\ast$ (1 $\leq j \leq 2$) and $\tau(E(X_2; \{0\})) > 1 - \varepsilon$$

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