New solutions in pure gravity with degenerate tetrads

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Abstract

In first order formulation of pure gravity, we find a new class of solutions to the equations of motion represented by degenerate four-geometries. These configurations are described by non-invertible tetrads with two zero eigenvalues and admit non-vanishing torsion. The homogeneous ones among these infinitely many degenerate solutions admit a geometric classification provided by the three fundamental geometries that a closed two-surface can accommodate, namely, $E^2$, $S^2$ and $H^2$.
I. INTRODUCTION

Classical theory of gravity admits two different descriptions. The usual one is based on the metric (second order) formulation, where the action functional is written in terms of the metric fields $g_{\mu\nu}(x)$:

$$S[g] = \frac{1}{2\kappa^2} \int d^4x \sqrt{g} g^{\mu\nu} R_{\mu\nu}(g)$$  (1)

The equations of motion resulting from the variation of this action leads to the theory of Einstein gravity. There exists another formulation, known as the first order theory, where Lorentzian (Euclidean) gravity can manifestly be given a local $SO(3,1)$ ($SO(4)$) gauge theoretic interpretation. The corresponding action functional depends on two sets of fields, namely, tetrad $e^I_\mu(x)$ and connection $\omega_{\mu}^{IJ}(x)$:

$$S[e, \omega] = \frac{1}{8\kappa^2} \int d^4x \varepsilon^{\mu\nu\alpha\beta} \varepsilon_{IJKL} e^I_\mu e^J_\nu R_{\alpha\beta}^{KL}(\omega)$$  (2)

where $R_{\mu\nu}^{IJ}(\omega) = \partial_{[\mu} \omega_{\nu]}^{IJ} + \omega_{[\mu}^{IK} \omega_{\nu]}^{KJ}$ is the field strength of the gauge connection $\omega_{\mu}^{IJ}$ of the gauge group. These two descriptions of gravity theory are equivalent only when the tetrad (or metric) is invertible, implying $\det e^I_\mu \neq 0$. This is so because the second order action (1) requires the inverse metric $g^{\mu\nu}$ in its construction, whereas the first order action (2) does not. In other words, the first order theory can admit solutions of equations of motion corresponding to degenerate tetrads, which are not perceived at all by the metric formulation.

There has been quite a lot of discussion on degenerate spacetime geometries in the literature, both in the context of metric [1, 2] and tetrad [3–8] formulations. In first order gravity, explicit examples of degenerate tetrad solutions of the equations of motion were first constructed by Tseytlin [3]. In a more recent work [8], a general framework to obtain all possible solutions with degenerate tetrads having one zero eigenvalue was set up. There, it was also shown that this space of solutions contains a special class that corresponds to the eight fundamental homogeneous three-geometries as classified by Thurston [9].

However, there exists another nontrivial case that deserves special attention, namely, first order gravity theory for degenerate tetrads with two zero eigenvalues. This is what we focus on here. As elucidated in the next few sections, the solution space for this theory possesses a rich structure and the essential details are qualitatively different from the case with one zero
eigenvalue in quite a few respects. The only other case of possible interest, corresponding to tetrads with three null eigenvalues, is also discussed briefly.

As emphasized in [8], it is important to realize that these degenerate tetrad solutions in pure gravity correspond to an unusual causal structure of spacetime [10]. These could also mediate topology change [11]. Further, each of these solutions provides a saddle point of the quantum path integral in first order gravity. Hence, it is important to obtain all such solutions and analyze their properties to unravel their potential role in classical as well as quantum gravity.

II. DEGENERATE TETRAD WITH TWO ZERO EIGENVALUES

Variations of the first order action functional for four-dimensional Euclidean gravity (2) with respect to the fields $\omega^{KL}_\beta$ and $e^I_\beta$ result in the following equations of motion respectively:

$$e^I_\mu D_\nu (\omega) e^J_\alpha = 0 \quad (3)$$
$$e^I_\mu R_{\nu\alpha}^{JK} (\omega) = 0 \quad (4)$$

Let us now consider degenerate tetrad fields of the form:

$$e^I_\mu = \begin{pmatrix} e^1_x & e^1_y & 0 & 0 \\ e^2_x & e^2_y & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (5)$$

with $\mu \equiv (x, y, z, \tau)$ and $I \equiv (1, 2, 3, 4)$. The two null eigenvalues of the tetrad have been chosen to lie along the $z$ and $\tau$ directions, whereas the remaining coordinates $x, y$ span a (reduced) two dimensional subspace with a non-degenerate two-metric. Non-zero determinant of the corresponding diad is $e \equiv e^1_x e^2_y - e^2_x e^1_y = \frac{1}{2} \epsilon^{ab} e_{ij} e^i_a e^j_b \ (a \equiv x, y; \ i \equiv 1, 2)$. We shall denote the inverse of this diad by $e^a_i$: $e^a_i e^i_b = \delta^a_b$ and $e^a_i e^i_b = \delta^a_b$. It is always possible to reduce any arbitrary tetrad with two zero eigenvalues to the form (5) above using local orthogonal rotations and general coordinate transformations.

Twenty four connection equations of motion in (3) can be separated into (6+6+12) equations as:

$$e^I_\mu D_a (\omega) e^J_b = 0 \quad (6)$$
\[ e^{[I}_a D_a(\omega) e_{b]}^j = 0 \tag{7} \]
\[ e^{[I}_i D_i(\omega) e_{a]}^j = 0 \tag{8} \]

A similar decomposition of 16 tetrad equations of motion in (4) leads to \((4+4+8)\) equations:
\[ e^{[I}_a R_{ab}^{JK} (\omega) = 0 \tag{9} \]
\[ e^{[I}_i R_{ab}^{JK} (\omega) = 0 \tag{10} \]
\[ e^{[I}_a R_{ab}^{JK} (\omega) = 0 \tag{11} \]

In the following section, we find the most general solutions to these equations of motion.

III. GENERAL SOLUTIONS

Let us begin our discussion with equation (6), the first among the set of connection equations of motion. The \( I = i, J = 3 \) components of these equations lead to:
\[ e^{[i}_a e_{b]}_3 \omega_{3j} = 0, \] which implies: \( \omega_{3j} = 0 \). Similarly, the \( I = i, J = 4 \) components of Eq.(6) imply that \( \omega_{4j} = 0 \). Next, for \( I = i, J = j \), we obtain:
\[ e^{[i}_a D_z e_{b]}^j - e^i_b D_z e_{a]^j = 0 , \]
or, \[ e_a^i \partial_z e_i^a = 0 \]
or, \[ \partial_z e = 0 \]
which implies that the 2-determinant \( e \) is \( z \)-independent. The only remaining equation from (6), given by \( I = 3, J = 4 \), is satisfied identically and hence does not lead to any new constraint. Altogether, Eqn.(6) implies five constraints:
\[ \omega_{3j} = 0, \quad \omega_{4j} = 0, \quad \partial_z e = 0 . \tag{12} \]

Proceeding similarly with six equations in (7) where again one equation is trivial for tetrads (5), we obtain five constraints from the other nontrivial equations:
\[ \omega_{3j} = 0, \quad \omega_{4j} = 0, \quad \partial_r e = 0 . \tag{13} \]

Thus from 12 equations in (6) and (7), we obtain ten constraints listed in Eqns.(12) and (13). The last of the connection equations of motion (8) is satisfied identically for the tetrad fields (5). This completes our discussion of the connection equations of motion (6-8). Eight
connection components, $\omega_{3i}^3$, $\omega_{4i}^3$, $\omega_{3i}^4$ and $\omega_{4i}^4$, are fixed as in Eqns. (12) and (13). Of the rest 16 connection components, as we shall see below, some will get fixed by the tetrad equations of motion (9-11) which we study next.

It is straightforward to check that the four tetrad equations of motion in (9), for the choices $(I = i, J = 3, K = 4)$, $(I = i, J = j, K = 3)$ and $(I = i, J = j, K = 4)$, are respectively equivalent to the following set of $(2 + 1 + 1)$ equations:

$$R_{za}^{34} = 0, \quad e_i^a R_{za}^{3i} = 0, \quad e_i^a R_{za}^{4i} = 0 \quad \text{(14)}$$

The first subset of two equations here leads to:

$$\frac{\partial}{\partial z} \omega_z^{34} = 0$$

These can be solved for the connection components $\omega_z^{34}$ and $\omega_a^{34}$, implying that these are just pure gauge:

$$\omega_z^{34} = \partial_z \Omega(x, y, z, \tau), \quad \omega_a^{34} = \partial_a \Omega(x, y, z, \tau) \quad \text{(15)}$$

In order to solve the last two equations of motion in (10), let us introduce a parametrization of the four components of $\omega_a^{j3}$ ($\omega_a^{j4}$) in terms of a symmetric matrix $M^{kl}$ ($\bar{M}^{kl}$) with three independent components and a scalar $\phi$ ($\bar{\phi}$):

$$\omega_a^{j3} = e_i^j M^{jl} + e_i^j e_i^l \phi, \quad \omega_a^{j4} = e_i^j \bar{M}^{jl} + e_i^j e_i^l \bar{\phi} \quad \text{(16)}$$

Since both sets of fields ($M^{kl}$, $\phi$) and ($\bar{M}^{kl}$, $\bar{\phi}$) and are arbitrary, this parametrization does not imply any loss of generality. Using these in the last two equations of (14), the most general solution for the two connection components $\omega_z^{34}$ and $\omega_z^{12}$ are found to be:

$$\omega_z^{34} = e_i^j \left[ \frac{\phi \partial_z M_a^j}{\phi M^{kk} + \phi M^{kk}} - \bar{\phi} \partial_z \bar{M}_a^j \right], \quad e_{ij} \omega_z^{ij} = \frac{e_i^a \left[ M^{kk} \partial_z M_a^j + M^{kk} \partial_z \bar{M}_a^j \right]}{\phi M^{kk} + \phi M^{kk}} + e_i^a e_i^a \partial_z e_i^j \quad \text{(17)}$$

where $M_a^j \equiv M^{jl} e_i^j$ and $\bar{M}_a^j \equiv \bar{M}^{jl} e_i^j$. Comparing Eq. (15) with (17), we note that the gauge function $\Omega(x, y, z, \tau)$ is not independent, but rather is related to $M^{kl}$, $\bar{M}^{kl}$, $\phi$ and $\bar{\phi}$ as:

$$\Omega(x, y, z, \tau) = \int^z \frac{dz}{\phi M^{kk} + \phi M^{kk}} \left[ \phi \partial_z \bar{M}_a^j - \bar{\phi} \partial_z M_a^j \right] \quad \text{(18)}$$

We now turn to solve the four equations in (10), which can be recast in terms of an equivalent set of $(2 + 1 + 1)$ equations as:

$$R_{za}^{34} = 0, \quad e_i^a R_{za}^{3i} = 0, \quad e_i^a R_{za}^{4i} = 0 \quad \text{(19)}$$
The first subset of two equations here implies that $\omega^{34}_\tau$ is pure gauge, given by the same gauge function $\Omega(x, y, z, \tau)$ introduced in Eq. (13):

$$\omega^{34}_\tau = \partial_\tau \Omega(x, y, z, \tau).$$

(20)

Using the parametrization introduced in (16), the remaining last two equations in (19) can be solved for the two connection components $\omega^{34}_\tau$ and $\omega^{12}_\tau$ as:

$$\omega^{34}_\tau = e^{a}_j \left[ \phi \partial_\tau M^j_a - \bar{\phi} \partial_\tau M^j_a \right] / \phi M^{kk} + \bar{\phi} M^{kk},$$

$$\epsilon_{ij} \omega^{ij}_\tau = \frac{e^{a}_j \left[ M^{kk} \partial_\tau M^j_a + \bar{M}^{kk} \partial_\tau \bar{M}^j_a \right]}{\phi M^{kk} + \bar{\phi} M^{kk}} + \epsilon_{ij} e^{a}_i \partial_\tau e^{a}_j$$

(21)

Finally, Eqns. (20) and (21) along with Eqn. (18) imply:

$$\Omega(x, y, z, \tau) = \int^\tau d\tau \frac{e^{a}_j \left[ \phi \partial_\tau M^j_a - \bar{\phi} \partial_\tau M^j_a \right]}{\phi M^{kk} + \bar{\phi} M^{kk}} = \int^z dz \frac{e^{a}_j \left[ \phi \partial_\tau M^j_a - \bar{\phi} \partial_\tau M^j_a \right]}{\phi M^{kk} + \bar{\phi} M^{kk}}$$

(22)

With these, the remaining set of eight equations of motion contained in (11) reduce to identities, without leading to any further constraints. This completes our analysis of the tetrad equations of motion (9)(11).

Of twenty four components of the connection $\omega_{\mu}^{IJ}$, connections equations of motion (6-8) fix eight, $\omega_z^{3i}, \omega_z^{4i}, \omega_\tau^{3i}$ and $\omega_\tau^{4i}$, to be zero as in Eqns. (12) and (13). Of rest 16 components, tetrad equations of motion (9)(11) fix fourteen, $\omega^{34}_\tau, \omega^{34}_\sigma, \omega^{4i}_\sigma, \omega^{a}_\tau, \omega^{a}_\sigma, \omega^{i}_{\sigma}$ and $\omega^{ij}_\sigma$, which are completely characterized in terms of eight fields represented by two symmetric $M^{ij}$ and $\bar{M}^{ij}$ and two scalars $\phi$ and $\bar{\phi}$ through Eqns. (15)(17) and (20)(22). Although we have analyzed all the equations of motion, two connection components, $\omega^{ij}_\sigma (a \equiv x, y; i \equiv 1, 2)$, are still left undetermined. This follows from the fact that the possible terms containing the connection components $\omega^{ij}_a$ in the equations of motion (6)(11) are zero for the tetrads as represented in (5). This represents an important difference from the case where the tetrads have one zero eigenvalue (8). There, for tetrads with one null direction, three of the nine components of $\omega^{ij}_a (a \equiv x, y, z; i \equiv 1, 2, 3)$ are determined (zero) where as six are left undetermined. Now, in the present case with tetrads of two zero eigenvalues, the two arbitrary components of $\omega^{ij}_a$ may be parametrized in terms of an arbitrary vector field $N_a \equiv [N(x, y, z, \tau), \bar{N}(x, y, z, \tau)]$ as:

$$\omega^{ij}_a = \bar{\omega}^{ij}_a (e) + \epsilon^{ij} N_a \quad [a \equiv x, y; i \equiv 1, 2]$$

(23)

where $\bar{\omega}^{ij}_a (e) = \frac{1}{2} \left[ e^{b}_j \partial_{[a} e^{c}_b] - e^{b}_j \partial_{[a} e^{c}_{b]} - e^{c}_b \epsilon^{e}_j e^{f}_a \partial_{[a} e^{b]} \right]$ is the torsion-free spin-connection completely determined by the diads $e^i_a$. The ten independent fields $N$ and $\bar{N}$, $(M^{kl}, \phi)$ and
(\(M^{kl}, \bar{\phi}\)), represent contortion. Thus, in general, torsion is non vanishing for the solutions discussed here.

Since \(R_{z\tau}^{34}(\omega)\) is zero for all the above degenerate spacetimes, the action (2) is also zero for these solutions of equations of motion.

The analysis presented in this article can be easily extended to study the solution space for tetrads with three zero eigenvalues, which is the only remaining case of possible interest. Such tetrad fields \(e^I_\mu\) can be organized by local orthogonal rotations and general coordinate transformation to have just one non-zero component, say \(e_x^1\). For these, the connection equations of motion (6-8) are identically satisfied. Thus these yield no constraints. On other hand equations of motion (9-11) yield the constraints:

\[R_{y2}^{23} = R_{y2}^{24} = R_{y2}^{34} = R_{y2}^{23} = R_{y3}^{23} = R_{y3}^{24} = R_{y3}^{34} = 0.\]

While these can be solved for the connection components involved in these field strength components, other connection components are left undetermined. Notably, there are no analogue of the contortion fields \(N_a\) in this case and hence the solution space hardly exhibits any nontrivial structure.

This completes our discussion of the most general solutions of first order gravity theory in four dimensions with degenerate tetrads having two or more zero eigenvalues. For the case with two null eigenvalues, there are an infinite number of such configurations in general, parametrized by the diads \(e_a^I\) and the set of six arbitrary fields, \(M^{kl}, \bar{M}^{kl}, \phi, \bar{\phi}, N\) and \(\bar{N}\). An interesting subset of these degenerate space-time solutions consists of those based on the types of fundamental homogeneous two-geometries represented by the diads \(e_a^I\). It is well-known that there are only three independent homogeneous geometric structures that any closed two-surface can admit. These are given by the Euclidean plane \(E^2\), the two sphere \(S^2\) and the hyperbolic plane \(H^2\). In the next section, we construct explicit four-dimensional torsional solutions (of first-order gravity theory) corresponding to these three two-geometries. Solutions based on more nontrivial two-geometries (e.g. non-orientable ones) can also be constructed by making appropriate choices of boundary conditions on the basic fields \(e^I_\mu\) and \(\omega^{IJ}_\mu\). We do not discuss such examples here.
IV. EXPLICIT SOLUTIONS CORRESPONDING TO THREE MODEL TWO-GEOMETRIES

(i) $E^2$ geometry:

Let us consider a case where the non degenerate two-subspace described by the diads $e_a^i$ in (5) is the Euclidean plane. The corresponding metric reads $ds^2_{(4)} = dx^2 + dy^2 + 0 + 0$. The torsion-free spin-connection $\bar{\omega}_{ij}^a(e)$ vanishes and hence the associated two-curvature $\bar{R}_{ab}^{ij}(\bar{\omega}) \equiv \bar{\partial}_{[a} \bar{\omega}_{b]}^{ij} + \bar{\omega}_{[a}^{ik} \bar{\omega}_{b]}^{kj} = 0$ and scalar two-curvature $\bar{R}(\bar{\omega}) := e_i^a e_j^b \bar{R}_{ab}^{ij}(\bar{\omega}) = 0$.

In general, the six fields $M_{kl}$ and $\bar{M}_{kl}$ in Eqn. (16) are arbitrary functions of the coordinates $(x, y, z, \tau)$. As an example, let us make a simple choice for these:

\[
M_a^k = \lambda e_a^k, \quad \bar{M}_a^k = \bar{\lambda} e_a^k
\]  

where $\lambda, \bar{\lambda}$ are arbitrary constants. This choice, when inserted into the general solutions above, leads to:

\[
\Omega(x, y, z, \tau) = 0, \quad \omega_{z12} = 0 = \omega_{r12}, \quad \omega_{z34} = 0 = \omega_{r34}, \quad \omega_{a34} = 0.
\]

Finally, using the parametrization (16) and (23) for the contorsion fields, all the connection one-forms are given by:

\[
\omega^{13} = \lambda dx + \phi dy, \quad \omega^{23} = -\phi dx + \lambda dy,
\]

\[
\omega^{14} = \bar{\lambda} dx + \bar{\phi} dy, \quad \omega^{24} = -\bar{\phi} dx + \bar{\lambda} dy,
\]

\[
\omega^{12} = N dx + \bar{N} dy, \quad \omega^{34} = 0
\]

where $\phi, \bar{\phi}, N$ and $\bar{N}$ are arbitrary functions of the spacetime coordinates $(x, y, z, \tau)$.

(ii) $S^2$ geometry:

Our next example is based on a degenerate four-metric, which contains a non degenerate $S^2$ subspace:

\[
ds^2_{(4)} = \ell^2 \left[d\theta^2 + \sin^2 \theta d\chi^2 \right] + 0 + 0
\]

where $\theta$ and $\chi$ are two angular coordinates on the two-sphere. The only non-vanishing component of the torsion-free spin-connection is given by $\bar{\omega}_{\chi}^{12}(e) = -\cos \theta$ and the associated
field strength also has only one non-vanishing component: \( \bar{R}^{12}_{\chi} = \sin \theta \). The two-metric characterizing the \( S^2 \) space has constant positive scalar curvature: \( \bar{R}(\bar{\omega}) \equiv e_i^a e_j^b \bar{R}^{ij}_{ab}(\bar{\omega}) = \frac{2}{\ell^2} \).

For the simple choice \( M^k_a = \lambda e^k_a, \bar{M}^k_a = \bar{\lambda} e^k_a \) with \( \lambda \) and \( \bar{\lambda} \) as constants, the solution of four dimensional gravity theory is given by the following connection one-forms:

\[
\omega^{13} = \ell [\lambda d\theta + \phi \sin \theta d\chi], \quad \omega^{23} = \ell [-\phi d\theta + \lambda \sin \theta d\chi], \\
\omega^{14} = \ell [\bar{\lambda} d\theta + \bar{\phi} \sin \theta d\chi], \quad \omega^{24} = \ell [-\bar{\phi} d\theta + \bar{\lambda} \sin \theta d\chi], \\
\omega^{12} = N d\theta + [\bar{N} - \cos \theta] d\chi, \quad \omega^{34} = 0.
\]

(iii) \( H^2 \) geometry:

The last of the three fundamental two-geometries is represented by the hyperbolic plane, nested within the four dimensional spacetime through a degenerate metric of the form:

\[
ds^2_{(\bar{\omega})} = \frac{\ell^2}{y^2} (dx^2 + dy^2) + 0 + 0, \quad y > 0
\]

The torsion-free connection has only one non-vanishing component: \( \bar{\omega}^{12}_{\chi}(e) = -\frac{1}{y} \). The associated two-curvature also has only one non-zero component, \( \bar{R}^{12}_{xy}(\bar{\omega}) = -\frac{1}{y^2} \), and the scalar two-curvature is a negative constant: \( \bar{R}(\bar{\omega}) = -\frac{2}{\ell^2} \). For the choice \( M^i_a = \lambda e^i_a, \bar{M}^i_a = \bar{\lambda} e^i_a \) and proceeding as in the previous two examples, we obtain the connection one-forms:

\[
\omega^{13} = \frac{\ell}{y} [\lambda dx + \phi dy], \quad \omega^{23} = \frac{\ell}{y} [-\phi dx + \lambda dy], \\
\omega^{14} = \frac{\ell}{y} [\bar{\lambda} dx + \bar{\phi} dy], \quad \omega^{24} = \frac{\ell}{y} [-\bar{\phi} dx + \bar{\lambda} dy], \\
\omega^{12} = \left[ N - \frac{1}{y} \right] dx + \bar{N} dy, \quad \omega^{34} = 0.
\]

V. CONCLUSION

We have constructed a complete space of solutions of the first order theory of gravity for degenerate tetrads with two zero eigenvalues. This space is spanned by an infinite number of degenerate four-geometries. Associated connection fields generically contain torsion and hence these configurations can be viewed as geometric sources of torsion in pure gravity. It should be emphasized that even though it is always possible to find (locally) a two-dimensional reduced subspace encoded by the diads \( e^i_a \) for each configuration, these are four
dimensional solutions described by the set of arbitrary fields $M^{kl}$, $\bar{M}^{kl}$, $\phi$, $\bar{\phi}$, $N$ and $\bar{N}$ which depend on all the four spacetime coordinates. The space of solutions is shown to contain a special class associated with the three independent homogeneous two-geometries of constant curvature, namely, $E^2$, $S^2$ and $H^2$. This provides a geometric classification of this family of infinitely many degenerate spacetime solutions of first order gravity in four dimensions.

The present study complements the earlier analysis for tetrads with one null eigenvalue [8]. Altogether, these configurations described by tetrads with one, two and three zero eigenvalues constitute all possible degenerate solutions of the equations of motion in first order gravity.

Let us note that one could also include the cosmological constant in this analysis. However, for the case of tetrads with two or more zero eigenvalues as discussed here, its inclusion does not affect any of the connection or tetrad equations of motion. This is in stark contrast to the case with one null eigenvalue, where the addition of the cosmological constant does affect the tetrad equations of motion, and hence also the solutions.

The degenerate spacetime solutions constructed here, as well as those found earlier [8], are potential mediators of topology (signature) change. As saddle points in the quantum path integral, these could also encode nontrivial quantum effects. In particular, these might underlie hitherto unnoticed but interesting instanton phenomena. These questions need to be explored for a deeper understanding of quantum gravity.

**ACKNOWLEDGMENTS**

S.S. thanks Sayan Kar for useful discussions. R.K.K. acknowledges the support of Department of Science and Technology, Government of India, through a J.C. Bose National Fellowship.

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