Probing topologically charged black holes on brane worlds in $f(R)$ bulk

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Received: 10 October 2015 / Accepted: 4 June 2016 / Published online: 14 June 2016
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Abstract The perihelion precession, the deflection of light and the radar echo delay are classical tests of General Relativity here used to probe brane-world topologically charged black holes in a $f(R)$ bulk. Moreover, such tests are used to constrain the parameter that arises from the Shiromizu–Maeda–Sasaki procedure applied to a $f(R)$ bulk. Observational data constrain the possible values of the tidal charge parameter and the effective cosmological constant in this context. We show that the observational/experimental data for both perihelion precession and radar echo delay make the black hole parameters to be more strict than the ones for the DMPR black hole. Moreover, the deflection of light constrains the tidal charge parameter similarly as the DMPR black holes, due to a peculiarity in the equation of motion.

Keywords Brane-world scenarios · Black holes · Classical tests of general relativity · $f(R)$ gravity

1 Introduction

Brane-world models play a prominent role on high energy physics, inspired in string theory advances. This framework has cosmological and astrophysical implications...
comprehensively investigated in the literature [1–8]. Besides, 5D effects originated from the gravitational collapse have been proposed [9–12]. An interesting aspect of cosmology is that the Universe goes through a phase of the accelerated expansion, supported by recent observational data [13], what can be accounted for either dark energy or modified theories of gravity [14] as well. Although the Einstein–Hilbert action can be replaced by an arbitrary function $f(R)$ of the 4D Ricci scalar $R$ [15], a Randall–Sundrum type model with $f(R)$ as the action in bulk space is still incipient in the literature [16] (hereon we denote by $R$ the 5D Ricci scalar). Moreover, recently the 5D $f(R)$ theories of gravity have been studied [17] to address the dark matter problem, whereas a $f(R)$ model of gravity with curvature-matter coupling in a 5D bulk was established in [18]. The $f(R)$ framework has been further employed to solve the brane effective field equations for dark pressure and dark radiation to acquire black hole solutions, with parameters induced from the bulk [19].

On the other hand, General Relativity (GR) explains the deflection of light and the perihelion shift of Mercury, complying with great accuracy to the experimental/observational values in the context of the Schwarzschild geometry. Such classical tests were further employed in the framework of brane-world gravity [20]. Our aim is to study these models, encompassing $f(R)$ bulk effects, and probe black holes derived in such a context, by using the classical tests. In fact, the field equations on the brane have been recently solved, obtaining a topological brane-world black hole from a 5D $f(R)$ action [21]. Theories that take into account 4D $f(R)$ effects are natural scenarios that unify and explain both the inflationary paradigm and the dark energy problem. Hence, it is natural to go beyond and consider both the brane-world model and the modified gravity likewise. Brane-world models may explain the current acceleration of our Universe, while the 4D $f(R)$ theories can either apply to the early Universe inflation or late time acceleration, depending on specific forms chosen. Here we analyse the physical consequences of merging both frameworks. The geometry to be employed here, ruled by topologically charged $f(R)$ brane-world black holes, is more general and is led to both DMPR and Schwarzschild-de Sitter solutions for suitable limits of parameters. Randall–Sundrum like models, with $f(R)$ as the action in bulk, were presented in [22] by using a generalized Shiromizu–Maeda–Sasaki procedure [23]. Nevertheless, there is a quantity $Q_{\mu\nu}$ originated in the geometry of the bulk by the function $f(R)$ that describes matter [21]. Since $Q_{\mu\nu}$ appears in the metric of topologically charged $f(R)$ brane-world black holes, we aim to study it by the classical tests of GR. Thus, it makes it possible to constrain the bulk function $f(R)$ by experimental/observational data.

This paper is organized as follows: in Sect. 2 the effective field equations are presented in the context of $f(R)$ models. In Sect. 3 we show that brane-world $f(R)$ effects can be tested by the perihelion precession of Mercury and the radar echo delay. The black hole tidal charge is then constrained by experimental/observational values. Nevertheless, data regarding the deflection of light by the Sun are shown not be able to probe brane-world $f(R)$ effects, being in agreement the literature for Solar system scales [24]. Hence, the obtainable constraint on the black hole tidal charge is led to the constraint for the DMPR black hole [20]. We conclude and discuss our results in Sect. 4.
2 Brane field equations for \( f(R) \) gravity

The fundamental equations for the gravitational field on the brane are quite well established. By taking the brane as the source of the gravitational field and a 5D cosmological constant term \( \Lambda_5 \), the bulk Einstein field equations read:

\[
^{(5)}G_{AB} = -\Lambda_5 \ g_{AB} + \kappa_5^2 \ ^{(5)}T_{AB},
\]

where \(^{(5)}G_{AB}\) denotes the 5D Einstein tensor,

\[
^{(5)}T_{AB} = \frac{1}{f'(R)} \left( \kappa_5^2 T_{AB}^{\text{bulk}} - \left( \frac{1}{2} R f'(R) - \frac{1}{2} f(R) + \Box f'(R) \right) g_{AB} + \nabla_A \nabla_B f'(R) \right),
\]

is the effective bulk stress tensor—being \( T_{AB}^{\text{bulk}} \) the bulk stress tensor. The brane metric \( g_{\mu\nu} \) and the corresponding components of the bulk metric \(^{(5)}g_{\mu\nu}\) are related by \(^{(5)}g_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu\), where \( n_\mu \) is an unit vector, normal to the brane. Since \( g_{55} = 1 \) and \( g_{\mu5} = 0 \) in the brane-world models here studied, namely the 5D bulk metric is given by \(^{(5)}g_{AB} dx^A dx^B = g_{\mu\nu}(x^\sigma, y) dx^\mu dx^\nu + dy^2\), then the bulk indexes effectively attain \( A, B = 0, 1, 2, 3 \). Moreover, \( \kappa_5^2 \) stands for the 5D gravitational coupling. The brane is placed at \( y = 0 \), where \( y \) hereon denotes the extra dimension.

The matter content on the brane constitute the effective bulk stress tensor by \(^{(5)}T_{\mu\nu} \sim S_{\mu\nu}\delta(0)\), where the delta function \( \delta(0) \) is responsible for the localization on the brane and \( S_{\mu\nu} = -\lambda g_{\mu\nu} + \tau_{\mu\nu} \). Here \( \lambda \) denotes the brane tension and \( \tau_{\mu\nu} \) describes any additional matter on the brane. The well known fine-tuning relation among the effective 4D cosmological constant \( \Lambda \) on the brane, the bulk cosmological constant \( \Lambda_5 \), and the brane tension \( \lambda \) is provided by \( \Lambda = \frac{\kappa_5^2}{2} \left( \Lambda_5 + \frac{\kappa_5^2}{6} \lambda^2 \right) \), where the 4D coupling constant \( \kappa_4^2 = 8\pi G \)—here \( G \) denotes the 4D Newton constant—and the 5D coupling constant \( \kappa_5^2 \) are related by \( \kappa_4^2 = \frac{1}{6} \lambda \kappa_5^2 \). The effective 4D field equations are complemented by a set of equations obtained from the 5D Einstein and Bianchi equations [23]. On a \( \mathbb{Z}_2 \)-symmetric brane, induced field equations generalize the Shiromizu–Maeda–Sasaki procedure [23], hence incorporating \( f(R) \) bulk effects [22]:

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\Lambda g_{\mu\nu} + \kappa_4^2 \tau_{\mu\nu} + \frac{6\kappa_4^4}{\lambda} \pi_{\mu\nu} - E_{\mu\nu} + Q_{\mu\nu},
\]

where \( \pi_{\mu\nu} = \frac{1}{12} \tau_{\mu\nu} - \frac{1}{4} \tau_{\mu\sigma} \tau^\sigma_v + \frac{1}{24} (3 \tau_{\sigma\rho} \tau^{\sigma\rho} - \tau^2) g_{\mu\nu} \). Here \( E_{\beta\sigma} = ^{(5)}C_{\beta\rho\sigma} n_\alpha n^\rho \), where \(^{(5)}C_{\beta\rho\sigma}^\alpha\) is the bulk Weyl tensor. The term

\[
Q_{\mu\nu} = \left[ \frac{1}{4} f(R) - \frac{2}{5} f'(R) - \frac{4}{15} \Box f'(R) - \frac{R}{10} \left( f'(R) + \frac{3}{2} \right) \right] g_{\mu\nu} + \frac{2}{3} \nabla_\rho \nabla_\sigma f'(R) \left( \delta_\rho^\mu \delta_\sigma^v + n^\rho n^\sigma g_{\mu\nu} \right)
\]

encompasses \( f(R) \) bulk effects [21]. The symbol \( \Box \) stands for the 5D d’Alembertian, whereas for a conformally flat bulk the term \( Q_{\mu\nu} \) is conserved [21,22].
A static spherically symmetric solution on the brane has the form
\[ g_{\mu\nu}dx^\mu dx^\nu = -\exp(v(r))dt^2 + \exp(\lambda(r))dr^2 + r^2d\Omega^2, \] (5)
where \( d\Omega^2 \) is the line element of a 2-sphere. By considering the constant Ricci curvature scalar \( R \) and solving the field equations (3) in the vacuum (\( \tau_{\mu\nu} = 0 \)), the topologically charged brane-world black hole in \( f(R) \) gravity has geometry [21]
\[ \exp(v(r)) = \exp(-\lambda(r)) = 1 - \frac{2GM}{c^2r} + \frac{G\beta}{4\pi\epsilon_0 c^4 r^2} + \frac{\Lambda_{\text{eff}}}{3} r^2, \] (6)
where \( M \) is the effective mass of the black hole and
\[ \Lambda_{\text{eff}} = \Lambda - \frac{Q}{4} \] (7)
(where \( Q = Q_{\mu}^\rho \) as usual) plays the role of an effective cosmological constant on the brane, depending upon both the brane tension and the function \( f(R) \) as well. The parameter \( \beta \) can be interpreted as a 5D mass parameter [25]. It behaves as a tidal charge associated to the bulk Weyl tensor, that imparts the tidal charge stresses from the bulk to the brane [26]. When \( Q = 4\Lambda \), \( (\Lambda_{\text{eff}} = 0) \), the solution reduces to the DMPR black hole solution [27]. For \( Q = 0 \), the solution reduces to the topologically charged black hole solution on the brane [25].

Regarding the particular function \( f(R) \sim R^n \) in the bulk [21], Eq. (4) yields
\[ Q_{\nu}^\mu = \left[ \frac{\kappa_5^2 \Lambda_5}{2} - \frac{3}{20} \left( \frac{10\kappa_5^2 \Lambda_5}{5 - 2n} \right)^{1/n} \right] \delta_{\mu}^\nu. \] (8)
The brane effective cosmological constant hence reads
\[ \Lambda_{\text{eff}} = \frac{\kappa_5^2 \lambda_5^2}{12} + \frac{3}{20} \left( \frac{10\kappa_5^2 \Lambda_5}{5 - 2n} \right)^{1/n}, \] (9)
leading to the fine-tuning condition when \( n = 1 \). In fact, in this case the black hole reduces to Schwarzschild metric with cosmological constant and tidal charge.

On the other hand, the 4D modified models \( f(R) = R + \mu^2(n+1)/R^n \) have been proposed in Refs. [28,29]. In general, for \( R^n \gg \mu^2(n-1) \), it yields \( f(R)/R \rightarrow 1 \). Hence there is no modification depending upon \( \mu \). Notwithstanding, in the limit \( R^n \ll \mu^2(n-1) \), we have \( f(R)/R \rightarrow \mu^2(n+1) \). In this last case scalar gravity is modified, further providing stable models [15]. In order to agree with Solar system experiments, in Ref. [30] the authors obtain static spherically symmetric solutions the case of \( n = 1 \), namely, for \( f(R) = R + \mu/R \) theory, in both the weak and strong gravitational field regimes. From a 5D perspective, the model \( f(R) = R + \mu^4/R \) is able to describe the positive acceleration of the Universe [21,22]. For a large value of \( R \) it gives \( f(R) \sim R \), and the 5D Ricci scalar provides a negligible modification of the usual...
solution. However, for small values of $R$ gravity is modified. Possible values for $Q_{\mu\nu}$ read:

$$Q_{\mu\nu} = -\frac{21\mu^4}{20\left(5\kappa_5^2\Lambda_5 \pm \sqrt{21\mu^4 + 25\kappa_5^4\Lambda_5^2}\right)}\delta_{\mu\nu}. \quad (10)$$

Hence, the effective cosmological constant on the brane takes the values

$$\Lambda_{\text{eff}} = \Lambda + \frac{21\mu^4}{20\left(5\kappa_5^2\Lambda_5 \pm \sqrt{21\mu^4 + 25\kappa_5^4\Lambda_5^2}\right)}. \quad (11)$$

In the case when $\mu \sim 0$, namely, when the modification in $f(R)$ is negligible, then $\Lambda_{\text{eff}} \sim \Lambda$.

### 3 Solar system classical tests

The perihelion precession of Mercury, the deflection of light by the Sun and the radar echo delay observations are well known tests for the Schwarzschild solution of GR and for the DMPR, the Casadio–Fabbri–Mazzacurati, and the minimal geometric deformation in brane-world scenarios as well [31], among others. Brane-world effects in spherically symmetric spacetimes were studied in [20] and used in the Solar system scrutiny. The Solar system tests can analyze properties of topologically charged black holes in $f(R)$ brane-world models by constraining the parameters of $f(R)$ modifications and the tidal charge proportional to $\beta$. For topologically charged black holes in $f(R)$ brane-worlds, the metric tensor components are given by Eq. (6). When $\beta \to 0$ we recover the usual general relativistic case. In what follows we show how the Solar system tests are able to impose constraints on the $f(R)$ bulk effects, and in particular to probe topologically charged black holes in a $f(R)$ brane-world.

#### 3.1 The perihelion precession

The equation of motion for a test particle under the gravitational field provided by (5) reads

$$\dot{r}^2 + \exp(-\lambda)\frac{L^2}{r^2} = \exp(-\lambda)\left(\frac{E^2}{c^2}\exp(-\nu) - 1\right), \quad (12)$$

where the constants of motion $E$ and $L$, respectively, yield energy and the angular momentum conservation. By the usual change of variables $r = 1/u$ and $\dot{r} = Lu^2 dr/d\phi$, and by representing

$$g(u) = 1 - \exp(-\lambda), \quad (13)$$
Eq. (12) reads
\[
\left( \frac{du}{d\phi} \right)^2 + u^2 = \frac{E^2}{c^2 L^2} \exp(-\nu - \lambda) - \frac{1}{L^2} \exp(-\lambda) + g(u)u^2
\] (14)

and subsequently yields
\[
\frac{d^2 u}{d\phi^2} + u = \frac{1}{2} \frac{d}{du} \left( \frac{E^2}{c^2 L^2} \exp(-\nu - \lambda) - \frac{1}{L^2} \exp(-\lambda) + g(u)u^2 \right) \equiv g(u). \quad (15)
\]

By denoting \( \gamma(u) = \left( 1 - \left. \frac{dh}{du} \right|_{u_0} \right)^{1/2} \), a circular orbit \( u = u_0 \) is determined by the root of the fixed point equation \( u_0 = h(u_0) \), and a deviation is provided by [20] \( \delta = \delta_0 \cos(\gamma(u)\phi + \alpha) \), for \( \delta_0 \) and \( \alpha \) constants. The variation of the orbital angle with respect to successive perihelia is
\[
\phi = \frac{2\pi}{\gamma(u)} = \frac{2\pi}{1 - \sigma}, \quad (16)
\]

where \( \sigma \) is the perihelion advance, given from Eq. (16) by
\[
\sigma \sim \frac{1}{2} \left( \frac{dh}{du} \right)_{u=u_0} \quad (17)
\]

for small values of \( \left. \frac{dh}{du} \right|_{u=u_0} \). For a complete rotation the perihelion advance is \( \delta \phi \sim 2\pi \sigma \).

We consider now the perihelion precession of a planet in the \( f(R) \) brane-world black hole geometry (6). Eq. (15) is thus provided by
\[
g(u) = \frac{3GMu^2}{c^2} - \frac{G\beta u^3}{2\pi \epsilon_0 c^4} + \frac{GM}{c^2 L^2} - \frac{G\beta u}{4\pi \epsilon_0 c^4 L^2} + \frac{\Lambda_{\text{eff}}}{3u^3 L^2}. \quad (18)
\]

It makes \( u_0 \) to be obtained by the equation
\[
u_0 = 3Mu_0^2 - \frac{G\beta u_0^3}{2\pi \epsilon_0 c^4} + \frac{M}{L^2} - \frac{G\beta u_0}{4\pi \epsilon_0 c^4 L^2} + \frac{\Lambda_{\text{eff}}}{3u_0^3 L^2}, \quad (19)
\]

which, to first order, is approximated to \( u_0 \sim GM/(c^2 L^2) \) when \( \Lambda_{\text{eff}} \sim 0 \) and \( \frac{G\beta}{4\pi \epsilon_0 c^4 L^2} \ll 1 \). As \( L \) is related to the orbit parameters as \( L = 2\pi a^2 \sqrt{1 - e^2}/cT \) [20], where \( T \) denotes the period of the motion, Eq. (17) yields
\[
\delta \phi = \delta \phi_{GR} - \frac{\pi c^2}{GM} \left[ \frac{G\beta}{4\pi \epsilon_0 c^4 a(1 - e^2)} + \frac{\Lambda_{\text{eff}} a^3(1 - e^2)^3}{3} \right], \quad (20)
\]

where \( \delta \phi_{GR} = 6\pi GM/c^2 a \left( 1 - e^2 \right) \) is the well known Schwarzschild precession formula. Eq. (20) is consistent with the result in Ref. [20], when \( \Lambda_{\text{eff}} \to 0 \), since our
result incorporates \( f(R) \) bulk effects. The above second term gives the correction due to the nonlocal effects arising from the Weyl tensor in the bulk [20].

With the observed value of the precession of Mercury perihelion given by \( \delta \dot{\phi} = 43.11 \pm 0.21 \) arcsec/century, the GR formula gives \( \delta \phi_{GR} = 42.98 \) arcsec/century. The difference \( \delta \dot{\phi} - \delta \phi_{GR} = 0.13 \pm 0.21 \) arcsec/century can be ascribed to \( f(R) \) brane-world effects, putting stricter conditions on the results in [20]. When this difference results from 5D \( f(R) \) bulk effects on the DMPR geometry, the bulk tidal parameter \( \beta \) and the effective cosmological constant \( \Lambda_{\text{eff}} \) are observationally constrained by

\[
\left| \frac{G\beta}{4\pi \epsilon_0 c^4 a (1 - e^2)} + \Lambda_{\text{eff}} a^3 (1 - e^2)^3 \right| \leq \frac{GM_{\odot}}{\pi c^2} |\delta \dot{\phi} - \delta \phi_{GR}|. \tag{21}
\]

Employing the observational data [20], Eq. (21) provides the parameter space

\[
\left| \frac{G\beta}{4\pi \epsilon_0 c^4} + 0.8 \Lambda_{\text{eff}} \right| \leq (5.2 \pm 6.4) \times 10^4 \text{m}^2. \tag{22}
\]

For the case \( f(R) = R^n \), Eq. (9) provides the graphics for Eq. (22) depicted in Fig. 1 (left panel). Besides, for the case \( f(R) = R + \mu^4/R \) the effective cosmological constant is provided by Eq. (11), and the constraint (22) is illustrated in Fig. 1 (right panel).

### 3.2 The deflection of light

A similar procedure takes into account photons on a null geodesic in the absence of external forces. The equation of motion yields
\[
\left( \frac{du}{d\phi} \right)^2 + u^2 = g(u)u^2 + \frac{1}{c^2} \frac{E^2}{L^2} \exp(-\nu - \lambda) \equiv p(u)
\] (23)

implying that \(\frac{d^2 u}{d\phi^2} + u = \frac{1}{2} \frac{dp(u)}{du}\). In the lowest approximation, the solution is the line \(u = \frac{\cos \phi}{R}\), where \(R\) is the distance of the closest approach to the mass \(M\). It can be iteratively employed in the above equation, yielding

\[
\frac{d^2 u}{d\phi^2} + u = \frac{1}{2} \frac{d}{du} \left[ p \left( \frac{\cos \phi}{R} \right) \right].
\] (24)

The total deflection angle of the light ray is \(\delta = 2\varepsilon\) [20].

In the case of the geometry (6) provided by the topologically charged black hole in \(f(R)\) bulk, Eq. (13) leads to \(g(u) = \left(2GM/c^2\right)u\), resulting

\[
p(u) = \frac{2GM}{c^2} u^3 - \frac{G\beta u^4}{2\pi \epsilon_0 c^4} + \frac{E^2}{c^2 L^2} - \frac{\Lambda_{\text{eff}}}{3}.
\] (25)

Since the right hand side of Eq. (24) has a derivative with respect to \(u\), \(f(R)\) effects encrypted in the effective cosmological constant \(\Lambda_{\text{eff}}\) are not perceivable. In fact, the term \(\frac{\Lambda_{\text{eff}}}{3}\) in the above equation, that contains the correction induced by \(f(R)\) effects, does not take part on it. Hence our results are equivalent to the ones for DMPR black holes [20,27]. Clearly the total deflection of light is obtained in the same steps for the DMPR black holes [20]

\[
\delta \phi = \frac{4GM}{c^2 R} \left( 1 - \frac{3\pi \beta c^2}{16GM R} \right),
\] (26)

providing the constraint on the black hole charge \(|\beta| \leq (7.0 \pm 27.9) \times 10^8\) m\(^2\) [20].

### 3.3 Radar echo delay

The radar echo delay measures the time necessary for radar signals to travel to a planet, for instance. In fact, the time for the light to travel between two planets that are distant from the Sun is \(T_0 = \int_{-\ell_1}^{\ell_2} dx/c\), where \(\ell_1\) and \(\ell_2\) are the respectively the distances from the planets to the Sun. On the other hand, if the light travels close to the Sun, the time travel reads [20]

\[
T = \frac{1}{c} \int_{-\ell_1}^{\ell_2} \exp \left[ \frac{(\lambda(r) - \nu(r))}{2} \right] dx.
\] (27)

The time difference \(\delta T = T - T_0\) is hence given by

\[
\delta T = \frac{1}{c} \int_{-\ell_1}^{\ell_2} \left\{ e^{\left[ \lambda \left( \sqrt{x^2 + R^2} \right) - \nu \left( \sqrt{x^2 + R^2} \right) \right]/2} - 1 \right\} dx, \text{ where } r = \sqrt{x^2 + R^2}.
\] (28)
The delay can be evaluated from the integral in Eq. (28). Indeed, the above integrand is recast as:

\[
\exp \left( \frac{\lambda}{2} - \frac{v}{2} \right) \sim \left( 1 - \frac{2GM}{c^2 r} + \frac{G\beta}{4\pi\epsilon_0 c^4 R^2} + \frac{\Lambda_{\text{eff}}}{3} r^2 \right), \tag{29}
\]

in a first order approximation, based upon Eq. (6). Therefore Eq. (28) reads

\[
\delta T = \frac{2GM}{c^3} \ln \left( \frac{\sqrt{\ell_1^2 + R^2 + \ell_2}}{\sqrt{\ell_1^2 + R^2 - \ell_1}} \right) - \frac{G\beta}{4\pi\epsilon_0 c^5 R} \left[ \tan^{-1} \left( \frac{\ell_2}{R} \right) + \tan^{-1} \left( \frac{\ell_1}{R} \right) \right] - \frac{\Lambda_{\text{eff}} R^2}{3c} \left[ \ell_1 \left( 1 + \frac{\ell_1^2}{3R^2} \right) + \ell_2 \left( 1 + \frac{\ell_2^2}{3R^2} \right) \right]. \tag{30}
\]

Using the approximations \( R^2/\ell_i^2 \ll 1 \) (\( i = 1, 2 \)), the above expression reduces to

\[
\delta T \sim \frac{2GM}{c^3} \ln \left( \frac{4\ell_1 \ell_2}{R^2} \right) - \frac{G\beta}{4c^5 \epsilon_0 R} - \frac{\Lambda_{\text{eff}}}{9c} \left( \ell_1^3 + \ell_2^3 \right). \tag{31}
\]

This leads to the Schwarzschild radar delay \( \delta T_{\text{GR}} = \frac{2GM}{c^3} \ln \frac{4\ell_1 \ell_2}{R^2} \) when \( \beta = 0 \) and \( \Lambda_{\text{eff}} = 0 \), and to the classical test of radar echo delay for the DMPR black hole when \( \Lambda_{\text{eff}} = 0 \) [20]. The last term on the right hand side of the above equation imposes a more strict constraint on the class of models, in particular the ones provided by \( f(R) = R^n \) and \( f(R) = R + \mu^4/R \).

In the context of the geometry provided by the topologically charged \( f(R) \) brane-world black hole metric (6), measurements of the frequency shift of radio photons [20, 32] provide now the following constraint for the tidal charge parameter \( \beta \) and the effective cosmological constant:

\[
\left| \frac{G\beta}{4\epsilon_0 c^4} + \frac{\Lambda_{\text{eff}} R_\odot}{9} \left( \ell_1^3 + \ell_2^3 \right) \right| \lesssim (5.74 \pm 6.24) \times 10^8 \text{m}^2. \tag{32}
\]

Comparing the space of parameters for the DMPR black holes [20] and for the topologically charged \( f(R) \) brane-world black hole (32), we realize that \( f(R) \) bulk effects impose a more strict regime for the tidal charge \( \beta \).

There is no theoretical constraint that yields the value of \( Q \) in Eq.(7) to be the same order of magnitude as the 4D cosmological constant \( \Lambda \) (\( \sim 10^{-52} \text{m}^{-2} \)). Whatever the order of magnitude for the trace \( Q \) of the energy-momentum tensor in Eq. (4) is, it must satisfy the constraints (22) and (32), accordingly. In fact, the experimental constraint \( 10^{32} \text{m}^4 \lesssim R_\odot \left( \ell_1^3 + \ell_2^3 \right) \lesssim 10^{35} \text{m}^4 \) holds for the Solar system, and due to the multiplication by \( \frac{R_\odot}{9\pi} \left( \ell_1^3 + \ell_2^3 \right) \), the term \( \Lambda_{\text{eff}} \) has for the radar echo delay an upper limit of \( 10^{-27} \text{m}^{-2} \). It implies that this is the upper limit for the effective order of magnitude of \( Q \), that reflects \( f(R) \) bulk effects. We shall point out our remarks in details in the next section.
4 Concluding remarks

The phenomenology regarding brane-world models relies on the astronomical and astrophysical observations at the Solar system scale. The metric for topologically charged black holes in a \( f(R) \) brane-world provides the basic theoretical tools necessary for the agreement between the theory with the observational/experimental results. In this context, the classical tests of GR were considered for topologically charged black holes in a \( f(R) \) brane-world, and then compared to the results for the DMPR and the Schwarzschild black holes as very particular limits.

Our results encompass the DMPR black hole solution in a brane-world [27], when the parameter \( \Lambda_{\text{eff}} = 0 \). The most constrained limit we got for the parameter \( Q \)—that encodes \( f(R) \) bulk effects—came from the perihelion precession of Mercury, and gives the constraint (22). These results represent a significant restriction on tidal charge parameter [20], as the space of parameters for our model in Eq. (21), illustrated in Fig. 1 for two \( f(R) \) models, is led to Eq. (72) of Ref. [20], corresponding to the space of the parameter in the DMPR black hole.

Although the metric (6) has a Schwarzschild-AdS-like aspect when \( \beta \rightarrow 0 \), it is completely different from the Schwarzschild-AdS solution for such very particular case, as the effective cosmological constant \( \Lambda_{\text{eff}} \) is now given by (7) as the sum of the brane cosmological constant and the trace of the tensor (4). For the Schwarzschild-AdS geometry, the term due to the cosmological constant does not affect the light bending for Solar system scales [24]. Thus our results are in full compliance to the literature. Indeed, for the deflection of light, Solar system observations give the same constraint as for the DMPR black hole [20].

Finally the radar echo delay, based upon topologically charged \( f(R) \) brane-world black holes, provides a stringent constraint between the tidal charge parameter \( \beta \) and the effective cosmological constant, provided by (32). The space of parameters (21) and (32) provides a precise range for the trace of the tensor (4) that encrypts \( f(R) \) effects, through the effective cosmological constant on the brane (7). Moreover, since the topologically charged brane-world black hole in (6) presents a term containing \( \Lambda_{\text{eff}} \), its upper limit of \( 10^{-27} \text{ m}^2 \) further provides an important constraint on the black hole geometry. It is worth to emphasize that Eq.(7) further constrains the trace \( Q = Q^\mu_\mu \), that arises when the Shiromizu–Maeda–Sasaki procedure is applied to a \( f(R) \) bulk. Although in 4D the effect of the term \( R^2 \) is negligible and non observable unless the coefficient of this term is larger than \( 10^{61} \), the effect of higher dimensional terms is suppressed by powers of the Planck mass. The bounds on the coefficient of \( R^2 \) in 5D are still unknown and can be addressed, being out of the scope of our results here. We expect that it will be not so much different than that in 4D. Finally, nonlinear massive theories of gravity can be further analyzed in the framework here presented [33].

Acknowledgments The authors thank Prof. R. Venegeroles and Prof. Julio M. Hoff da Silva for valuable and fruitful discussions. A. M. K. is grateful to CAPES and “Programa Ciência sem Fronteiras” (CsF) for financial support. R. d R. thanks to FAPESP Grant No. 2015/10270-0 and CNPq Grants No. 473326/2013-2 and No. 303027/2012-6 for partial financial support.
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