Connes-amenability of bidual and weighted semigroup algebras

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Abstract

We investigate the notion of Connes-amenability, introduced by Runde in [11], for bidual algebras and weighted semigroup algebras. We provide some simplifications to the notion of a $\sigma WC$-virtual diagonal, as introduced in [10], especially in the case of the bidual of an Arens regular Banach algebra. We apply these results to discrete, weighted, weakly cancellative semigroup algebras, showing that diagonal, as introduced in [10], especially in the case of the bidual of an Arens regular Banach algebra.

We also show that for each one of these cancellative semigroup algebras $l^1(S,\omega)$, we have that $l^1(S,\omega)$ is Connes-amenable (with respect to the canonical predual $c_0(S)$) if and only if $l^1(S,\omega)$ is amenable, which is in turn equivalent to $S$ being an amenable group. This latter point was first shown by Grønbæk in [6], but we provide a unified proof. Finally, we consider the homological notion of injectivity, and show that here, weighted semigroup algebras do not behave like C*-algebras.

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1 Introduction

We first fix some notation, following [2]. For a Banach space $E$, we let $E'$ be its dual space, and for $\mu \in E'$ and $x \in E$, we write $\langle \mu, x \rangle = \mu(x)$ for notational convenience. We then have the canonical map $\kappa_E : E \to E''$ defined by $\langle \kappa_E(x), \mu \rangle = \langle \mu, x \rangle$ for $\mu \in E', x \in E$. For Banach spaces $E$ and $F$, we write $\mathcal{B}(E, F)$ for the Banach space of bounded linear maps between $E$ and $F$. We write $\mathcal{B}(E) = \mathcal{B}(E, E)$.

Let $\mathcal{A}$ be a Banach algebra. A Banach left $\mathcal{A}$-module is a Banach space $E$ together with a bilinear map $\mathcal{A} \times E \to E; (a, x) \mapsto a \cdot x$, such that $\|a \cdot x\| \leq \|a\|\|x\|$ and $a \cdot (b \cdot x) = (ab) \cdot x$ for $a, b \in \mathcal{A}$ and $x \in E$. Similarly, we have the notion of a Banach right $\mathcal{A}$-module and a Banach $\mathcal{A}$-bimodule. If $E$ is a Banach $\mathcal{A}$-bimodule (resp. left or right module) then $\mathcal{A}'$ is a Banach $\mathcal{A}$-bimodule (resp. right or left module) with module action given by

$$\langle a \cdot \mu, x \rangle = \langle \mu, x \cdot a \rangle \quad \langle \mu \cdot a, x \rangle = \langle \mu, a \cdot x \rangle \quad (a \in \mathcal{A}, x \in E).$$

Notice that as $\mathcal{A}$ is certainly a bimodule over itself (with module action induced by the algebra product) we also have that $\mathcal{A}'$, $\mathcal{A}''$ etc. are Banach $\mathcal{A}$-bimodules. Given a Banach $\mathcal{A}$-bimodule $E$, a submodule $F$ of $E$ is a submodule if $a \cdot x, x \cdot a \in F$ for each $a \in \mathcal{A}$ and $x \in F$. For Banach $\mathcal{A}$-bimodules $E$ and $F$, $T \in \mathcal{B}(E, F)$ is an $\mathcal{A}$-bimodule homomorphism when

$$a \cdot T(x) = T(a \cdot x) \quad T(x) \cdot a = T(x \cdot a) \quad (a \in \mathcal{A}, x \in E).$$

A linear map $d : \mathcal{A} \to E$ between a Banach algebra $\mathcal{A}$ and a Banach $\mathcal{A}$-bimodule $E$ is a derivation if $d(ab) = a \cdot d(b) + d(a) \cdot b$ for $a, b \in \mathcal{A}$. For $x \in E$, we define $d_x : \mathcal{A} \to E$ by $d_x(a) = a \cdot x - x \cdot a$. Then $d_x$ is a derivation, called an inner derivation.
A Banach algebra $\mathcal{A}$ is said to be super-amenable or contractable if every bounded derivation $d : \mathcal{A} \to E$, for every Banach $\mathcal{A}$-bimodule $E$, is inner. For example, a C$^*$-algebra $\mathcal{A}$ is super-amenable if and only if $\mathcal{A}$ is finite-dimensional. It is conjectured that there are no infinite-dimensional, super-amenable Banach algebras.

If we restrict to derivations to $E'$ for Banach $\mathcal{A}$-bimodules $E$ then we arrive at the notion of amenability. For example, a C$^*$-algebra $\mathcal{A}$ is amenable if and only if $\mathcal{A}$ is nuclear; a group algebra $L^1(G)$ is amenable if and only if the locally compact group $G$ is amenable (which is the motivating example). See [13] for further discussions of amenability and related notions.

Let $E$ be a Banach space and $F$ a closed subspace of $E$. Then we naturally, isometrically, identify $F'$ with $E'/F^\circ$, where

$$F^\circ = \{ \mu \in E' : \langle \mu, x \rangle = 0 \ (x \in F) \}.$$ 

**Definition 1.1.** Let $E$ be a Banach space and $E_*$ be a closed subspace of $E'$. Let $\pi_{E_*} : E'' \to E''/E^\circ$ be the quotient map, and suppose that $\pi_{E_*} \circ \kappa_E$ is an isomorphism from $E$ to $E_*$.

Then we say that $E$ is a dual Banach space with predual $E_*$. When $\mathcal{A}$ is a dual Banach algebra with predual $\mathcal{A}_*$ which is also a submodule of $\mathcal{A}'$ we say that $\mathcal{A}$ is a dual Banach algebra.

For a dual Banach algebra $\mathcal{A}$ with predual $\mathcal{A}_*$, we henceforth identify $\mathcal{A}$ with $\mathcal{A}'_*$. Thus we get a weak$^*$-topology on $\mathcal{A}$, which we denote by $\sigma(\mathcal{A}, \mathcal{A}_*)$. It is a simple exercise to show that $\mathcal{A}$ is a dual Banach algebra if and only if $\mathcal{A}$ is a dual Banach space such that the algebra product is separately $\sigma(\mathcal{A}, \mathcal{A}_*)$-continuous (see [14]). The following lemma is standard.

**Lemma 1.2.** Let $E$ and $F$ be dual Banach spaces with preduals $E_*$ and $F_*$ respectively, and let $T \in \mathcal{B}(E, F)$. Then the following are equivalent:

1. $T$ is $\sigma(E, E_*) - \sigma(F, F_*)$ continuous;
2. $T'(\kappa_{F_*}(F_*)) \subseteq \kappa_{E_*}(E_*);$ 
3. there exists $S \in \mathcal{B}(F_*, E_*)$ such that $S' = T$. 

As noticed by Runde (see [14]), there are very few Banach algebras which are both dual and amenable. For von Neumann algebras, which are the motivating example of dual Banach algebras, there is a weaker notion of amenability, called Connes-amenability, which has a natural generalisation to the case of dual Banach algebras.

**Definition 1.3.** Let $\mathcal{A}$ be a dual Banach algebra with predual $\mathcal{A}_*$. Let $E$ be a Banach $\mathcal{A}$-bimodule. Then $E'$ is a $w^*$-Banach $\mathcal{A}$-bimodule if, for each $\mu \in E'$, the maps

$$\mathcal{A} \to E', \ a \mapsto \begin{cases} a \cdot \mu, \\ \mu \cdot a \end{cases}$$

are $\sigma(\mathcal{A}, \mathcal{A}_*) - \sigma(E', E)$ continuous.

Then $(\mathcal{A}, \mathcal{A}_*)$ is Connes-amenable if, for each $w^*$-Banach $\mathcal{A}$-bimodule $E'$, each derivation $d : \mathcal{A} \to E'$, which is $\sigma(\mathcal{A}, \mathcal{A}_*) - \sigma(E', E)$ continuous, is inner. 

Given a Banach algebra $\mathcal{A}$, we define bilinear maps $\mathcal{A}'' \times \mathcal{A}' \to \mathcal{A}'$ and $\mathcal{A}' \times \mathcal{A}'' \to \mathcal{A}$ by

$$\langle \Phi \cdot \mu, a \rangle = \langle \Phi, \mu \cdot a \rangle \quad \langle \mu \cdot \Phi, a \rangle = \langle \Phi, a \cdot \mu \rangle \quad (\Phi \in \mathcal{A}'', \mu \in \mathcal{A}', a \in \mathcal{A}).$$
We then define two bilinear maps $\boxtimes, \lozenge: \mathcal{A}'' \times \mathcal{A}'' \to \mathcal{A}''$ by
\[
(\Phi \boxtimes \Psi, \mu) = \langle \Phi, \Psi \cdot \mu \rangle, \quad (\Phi \lozenge \Psi, \mu) = \langle \Psi, \mu \cdot \Phi \rangle \quad (\Phi, \Psi \in \mathcal{A}'', \mu \in \mathcal{A}').
\]
We can check that $\boxtimes$ and $\lozenge$ are actually algebra products, called the first and second Arens products respectively. Then $\kappa_\mathcal{A}: \mathcal{A} \to \mathcal{A}''$ is a homomorphism with respect to either Arens product. When $\boxtimes = \lozenge$, we say that $\mathcal{A}$ is Arens regular. In particular, when $\mathcal{A}$ is Arens regular, we may check that $\mathcal{A}''$ is a dual Banach algebra with predual $\mathcal{A}'$.

**Theorem 1.4.** Let $\mathcal{A}$ be an Arens regular Banach algebra. When $\mathcal{A}$ is amenable, $\mathcal{A}''$ is Connes-amenable. If $\kappa_\mathcal{A}(\mathcal{A})$ is an ideal in $\mathcal{A}''$ and $\mathcal{A}''$ is Connes-amenable, then $\mathcal{A}$ is amenable.

Let $\mathcal{A}$ be a $C^*$-algebra. Then $\mathcal{A}$ is Arens regular, and $\mathcal{A}''$ is Connes-amenable if and only if $\mathcal{A}$ is amenable.

**Proof.** The first statements are [14, Corollary 4.3] and [14, Theorem 4.4]. The statement about $C^*$-algebras is detailed in [14, Chapter 6].

Another class of Connes-amenable dual Banach algebras is given by Runde in [11], where it is shown that $M(G)$, the measure algebra of a locally compact group $G$, is amenable if and only if $G$ is amenable.

The organisation of this paper is as follows. Firstly, we study intrinsic characterisations of amenability, recalling a result of Runde from [10]. We then simplify these conditions in the case of Arens regular Banach algebras. We recall the notion of an injective module, and quickly note how Connes-amenability can be phrased in this language. The final section of the paper then applies these ideas to weighted semigroup algebras. We finish with some open questions.

## 2 Characterisations of amenability

Let $E$ and $F$ be Banach spaces, and form the algebraic tensor product $E \otimes F$. We can norm $E \otimes F$ with the projective tensor norm, defined as
\[
\| u \|_\pi = \inf \left\{ \sum_{k=1}^n \| x_k \| \| y_k \| : u = \sum_{k=1}^n x_k \otimes y_k \right\} \quad (u \in E \otimes F).
\]
Then the completion of $(E \otimes F, \| \cdot \|_\pi)$ is $E \hat{\otimes} F$, the projective tensor product of $E$ and $F$.

Let $\mathcal{A}$ be a Banach algebra. Then $\mathcal{A} \hat{\otimes} \mathcal{A}$ is a Banach $\mathcal{A}$-bimodule for the module actions given by
\[
a \cdot (b \otimes c) = ab \otimes c, \quad (b \otimes c) \cdot a = b \otimes ca \quad (a \in \mathcal{A}, b \otimes c \in \mathcal{A} \hat{\otimes} \mathcal{A}).
\]
Define $\Delta_\mathcal{A}: \mathcal{A} \hat{\otimes} \mathcal{A} \to \mathcal{A}$ by $\Delta_\mathcal{A}(a \otimes b) = ab$. Then $\Delta_\mathcal{A}$ is an $\mathcal{A}$-bimodule homomorphism.

**Theorem 2.1.** Let $\mathcal{A}$ be a Banach algebra. Then the following are equivalent:

1. $\mathcal{A}$ is amenable;
2. $\mathcal{A}$ has a virtual diagonal, which is a functional $M \in (\mathcal{A} \hat{\otimes} \mathcal{A})''$ such that $a \cdot M = M \cdot a$ and $\Delta''(M) \cdot a = \kappa_\mathcal{A}(a)$ for each $a \in \mathcal{A}$.

Runde introduced, in [10], the following notion in order to prove a version of the above theorem for Connes-amenability.
Definition 2.2. Let $\mathcal{A}$ be a dual Banach algebra with predual $\mathcal{A}_*$, and let $E$ be a Banach $\mathcal{A}$-bimodule. Then $x \in \sigma_{WC}(E)$ if and only if the maps $\mathcal{A} \to E$, 

$$a \mapsto \begin{cases} a \cdot x, \\ x \cdot a \end{cases}$$

are $\sigma(\mathcal{A}, \mathcal{A}_*) - \sigma(E, E')$ continuous. \hfill $\Box$

It is clear that $\sigma_{WC}(E)$ is a closed submodule of $E$. The $\mathcal{A}$-bimodule homomorphism $\Delta_\mathcal{A}$ has adjoint $\Delta_\mathcal{A}' : \mathcal{A}' \to (\hat{\mathcal{A}} \otimes \mathcal{A})'$. In [10, Corollary 4.6] it is shown that $\Delta_\mathcal{A}'(\mathcal{A}_*) \subseteq \sigma_{WC}((\hat{\mathcal{A}} \otimes \mathcal{A})')$. Consequently, we can view $\Delta_\mathcal{A}'$ as a map $\mathcal{A}_* \to \sigma_{WC}((\hat{\mathcal{A}} \otimes \mathcal{A})')$, and hence view $\Delta_\mathcal{A}''$ as a map $\sigma_{WC}((\hat{\mathcal{A}} \otimes \mathcal{A})')' \to \mathcal{A}'_* = \mathcal{A}$, denoted by $\tilde{\Delta}_\mathcal{A}$.

Theorem 2.3. Let $\mathcal{A}$ be a dual Banach algebra with predual $\mathcal{A}_*$. Then the following are equivalent:

1. $\mathcal{A}$ is Connes-amenable;
2. $\mathcal{A}$ has a $\sigma_{WC}$-virtual diagonal, which is $M \in \sigma_{WC}((\hat{\mathcal{A}} \otimes \mathcal{A})')'$ such that $a \cdot M = M \cdot a$ and $a \tilde{\Delta}_\mathcal{A}(M) = a$ for each $a \in \mathcal{A}$.

Proof. This is [10, Theorem 4.8]. \hfill $\Box$

In particular, we see that a Connes-amenable Banach algebra is unital (which can of course be shown in an elementary fashion, as in [14, Proposition 4.1]).

3 Connes-amenability for biduals of algebras

Recall Gantmacher’s theorem, which states that a bounded linear map $T : E \to F$ between Banach spaces $E$ and $F$ is weakly-compact if and only if $T''(E'') \subseteq \kappa_F(F)$. We write $\mathcal{W}(E, F)$ for the collection of weakly-compact operators in $\mathcal{B}(E, F)$.

Lemma 3.1. Let $E$ be a dual Banach space with predual $E_*$, let $F$ be a Banach space, and let $T \in \mathcal{B}(E,F')$. Then the following are equivalent, and in particular each imply that $T$ is weakly-compact:

1. $T$ is $\sigma(E, E_*) - \sigma(F', F'')$ continuous;
2. $T''(F'') \subseteq \kappa_{E_*}(E_*);$
3. there exists $S \in \mathcal{W}(F, E_*)$ such that $S' = T$.

Proof. That (1) and (2) are equivalent is standard (compare with Lemma 1.2).

Suppose that (2) holds, so that we may define $S \in \mathcal{B}(F,E_*)$ by $\kappa_{E_*} \circ S = T' \circ \kappa_F$. Then, for $x \in E$ and $y \in F$, we have

$$\langle x, S(y) \rangle = \langle T'(\kappa_F(y)), x \rangle = \langle T(x), y \rangle,$$

so that $S' = T$. Then $S''(F'') = T''(F'') \subseteq \kappa_{E_*}(E_*)$, so that $S$ is weakly-compact, by Gantmacher’s Theorem, so that (3) holds.

Conversely, if (3) holds, as $S$ is weakly-compact, we have $\kappa_{E_*}(E_*) \supseteq S''(F'') = T''(F'')$, so that (2) holds. \hfill $\Box$
It is standard that for Banach spaces $E$ and $F$, we have $(E \widehat{\otimes} F)' = B(F, E')$ with duality defined by

$$(T, x \otimes y) = (T(y), x) \quad (T \in B(F, E'), x \otimes y \in E \widehat{\otimes} F).$$

Then we see, for $a, b, c \in A$ and $T \in (A \widehat{\otimes} A)' = B(A, A')$, that $\langle a \cdot T, b \otimes c \rangle = \langle T(ca), b \rangle$ and that $\langle T \cdot a, b \otimes c \rangle = \langle T(c), ab \rangle = \langle T(c) \cdot a, b \rangle$ so that

$$\langle a \cdot T)(c) = T(ca), \quad (T \cdot a)(c) = T(c) \cdot a \quad (a, c \in A, T : A \to A').$$

Notice that we could also have defined $(E \widehat{\otimes} F)'$ to be $B(E, F')$. This would induce a different bimodule structure on $B(A, A')$, and we shall see in Section 4 that our chosen convention seems more natural for the task at hand.

**Proposition 3.2.** Let $A$ be a dual Banach algebra with predual $A_*$. For $T \in B(A, A') = (A \widehat{\otimes} A)'$, define maps $\phi_r, \phi_l : A \widehat{\otimes} A \to A'$ by

$$\phi_r(a \otimes b) = T' \kappa_A(a) \cdot b, \quad \phi_l(a \otimes b) = a \cdot T(b) \quad (a \otimes b \in A \widehat{\otimes} A).$$

Then $T \in \sigma WC(B(A, A'))$ if and only if $\phi_r$ and $\phi_l$ are weakly-compact and have ranges contained in $\kappa_{A_*}(A_*)$.

**Proof.** For $T \in B(A, A') = (A \widehat{\otimes} A)'$, define $R_T, L_T : A \to (A \widehat{\otimes} A)'$ by $R_T(a) = a \cdot T$ and $L_T = T \cdot a$, for $a \in A$. By definition, $T \in \sigma WC(B(A, A'))$ if and only if $R_T$ and $L_T$ are $\sigma(A, A_*) - \sigma(B(A, A'), (A \widehat{\otimes} A)''$) continuous. By Lemma 3.1 this is if and only if there exist $\varphi_r, \varphi_l \in W(A \widehat{\otimes} A, A_*)$ such that $\varphi'_r = R_T$ and $\varphi'_l = L_T$.

For $a \otimes b \in A \widehat{\otimes} A$ and $c \in A$, we see that

$$\langle c, \varphi_r(a \otimes b) \rangle = \langle R_T(c), a \otimes b \rangle = \langle c \cdot T, a \otimes b \rangle = \langle T(bc), a \rangle$$

$$= \langle T' \kappa_A(a), bc \rangle = \langle T' \kappa_A(a) \cdot b, c \rangle = \langle \phi_r(a \otimes b), c \rangle.$$  

$$\langle c, \varphi_l(a \otimes b) \rangle = \langle L_T(c), a \otimes b \rangle = \langle T \cdot c, a \otimes b \rangle = \langle T(b), ca \rangle$$

$$= \langle a \cdot T(b), c \rangle = \langle \phi_l(a \otimes b), c \rangle.$$  

Thus $\kappa_{A_*} \circ \varphi_r = \phi_r$ and $\kappa_{A_*} \circ \varphi_l = \phi_l$. Consequently, we see that $T \in \sigma WC(B(A, A'))$ if and only if $\phi_r$ and $\phi_l$ are weakly-compact and take values in $\kappa_{A_*}(A_*)$. \hfill $\square$

The following definition is [10] Definition 4.1.

**Definition 3.3.** Let $A$ be a Banach algebra and let $E$ be a Banach $A$-bimodule. An element $x \in E$ is **weakly almost periodic** if the maps

$$A \to E, \quad a \mapsto \begin{cases} a \cdot x, \\ x \cdot a \end{cases}$$

are weakly-compact. The collection of weakly almost periodic elements in $E$ is denoted by $WAP(E)$.

**Lemma 3.4.** Let $A$ be a Banach algebra, and let $T \in B(A, A') = (A \widehat{\otimes} A)'$. Let $\phi_r, \phi_l : A \widehat{\otimes} A \to A'$ be as above. Then $T \in WAP(B(A, A'))$ if and only if $\phi_r$ and $\phi_l$ are weakly-compact.

**Proof.** Let $R_T, L_T : A \to B(A, A')$ be as in the above proof. By definition, $T \in WAP(B(A, A'))$ if and only if $L_T$ and $R_T$ are weakly-compact. We can verify that

$$\phi'_r \circ \kappa_A = R_T, \quad \phi'_l \circ \kappa_A = L_T, \quad R'_T \circ \kappa_{A \widehat{\otimes} A} = \phi_r, \quad L'_T \circ \kappa_{A \widehat{\otimes} A} = \phi_l,$$

which completes the proof. \hfill $\square$
Corollary 3.5. Let $\mathcal{A}$ be a unital, dual Banach algebra with predual $\mathcal{A}_*$, and let $T \in \mathcal{B}(\mathcal{A}, \mathcal{A}') = (\mathcal{A} \hat{\otimes} \mathcal{A})'$. The following are equivalent, and, in particular, each imply that $T$ is weakly-compact:

1. $T \in \sigma WC(\mathcal{B}(\mathcal{A}, \mathcal{A}'))$;
2. $T(\mathcal{A}) \subseteq \kappa_{\mathcal{A}_*}(\mathcal{A}_*), T'(\kappa_{\mathcal{A}'}(\mathcal{A})) \subseteq \kappa_{\mathcal{A}_*}(\mathcal{A}_*),$ and $T \in \sigma WC(\mathcal{B}(\mathcal{A}, \mathcal{A}'))$;
3. $T(\mathcal{A}) \subseteq \kappa_{\mathcal{A}_*}(\mathcal{A}_*), T'(\kappa_{\mathcal{A}'}(\mathcal{A})) \subseteq \kappa_{\mathcal{A}_*}(\mathcal{A}_*),$ and $T \in \text{WAP}(\mathcal{B}(\mathcal{A}, \mathcal{A}'))$.

Proof. Let $e_\mathcal{A}$ be the unit of $\mathcal{A}$, so that for $a \in \mathcal{A}$, we have $T(a) = \phi_t(e_\mathcal{A} \otimes a)$ and $T'(\kappa_{\mathcal{A}'}(a)) = \phi_t(a \otimes e_\mathcal{A})$, which shows that (1) implies (2); clearly (2) implies (1).

As $\mathcal{A}_*$ is an $\mathcal{A}$-bimodule, (2) and (3) are equivalent by an application of Lemma 3.2 and Proposition 3.2.

Theorem 3.6. Let $\mathcal{A}$ be a dual Banach algebra with predual $\mathcal{A}_*$. Then $\mathcal{A}$ is Connes-amenable if and only if $\mathcal{A}$ is unital and there exists $M \in (\mathcal{A} \hat{\otimes} \mathcal{A})''$ such that:

1. $\langle M, a \cdot T - T \cdot a \rangle = 0$ for $a \in \mathcal{A}$ and $T \in \sigma WC(\mathcal{W}(\mathcal{A}, \mathcal{A}'))$;
2. $\kappa_{\mathcal{A}_*} \Delta''(M) = e_\mathcal{A}$, where $e_\mathcal{A}$ is the unit of $\mathcal{A}$.

Proof. As $\sigma WC((\mathcal{A} \hat{\otimes} \mathcal{A})')'$ is a quotient of $(\mathcal{A} \hat{\otimes} \mathcal{A})''$, this is just a re-statement of Theorem 2.3.

When $\mathcal{A}$ is an Arens regular Banach algebra, $\mathcal{A}''$ is a dual Banach algebra with canonical predual $\mathcal{A}'$. In this case, we can make some significant simplifications in the characterisation of when $\mathcal{A}''$ is Connes-amenable.

For a Banach algebra $\mathcal{A}$, we define the map $\kappa_{\mathcal{A}_*} \otimes \kappa_{\mathcal{A}_*} : \mathcal{A} \hat{\otimes} \mathcal{A} \to \mathcal{A}'' \hat{\otimes} \mathcal{A}''$ by

$$(\kappa_{\mathcal{A}_*} \otimes \kappa_{\mathcal{A}_*})(a \otimes b) = \kappa_{\mathcal{A}_*}(a) \otimes \kappa_{\mathcal{A}_*}(b) \quad (a \otimes b \in \mathcal{A} \hat{\otimes} \mathcal{A}).$$

We turn $\mathcal{A}'' \hat{\otimes} \mathcal{A}''$ into a Banach $\mathcal{A}$-bimodule in the canonical way. Then $\kappa_{\mathcal{A}_*} \otimes \kappa_{\mathcal{A}_*}$ is an $\mathcal{A}$-bimodule homomorphism. The following is a simple verification.

Lemma 3.7. Let $\mathcal{A}$ be a Banach algebra. The map

$$\iota_{\mathcal{A}} : \mathcal{B}(\mathcal{A}, \mathcal{A}') \to \mathcal{B}(\mathcal{A}'', \mathcal{A}'') ; T \mapsto T'' ,$$

is an $\mathcal{A}$-bimodule homomorphism which is an isometry onto its range. Furthermore, we have that $(\kappa_{\mathcal{A}_*} \otimes \kappa_{\mathcal{A}_*})' \circ \iota_{\mathcal{A}} = I_{\mathcal{B}(\mathcal{A}, \mathcal{A}')}$.

Define $\rho_{\mathcal{A}} : \mathcal{A}'' \hat{\otimes} \mathcal{A}'' \to (\mathcal{A} \hat{\otimes} \mathcal{A})''$ by

$$\langle \rho_{\mathcal{A}}(\tau), T \rangle = \langle T'', \tau \rangle \quad (\tau \in \mathcal{A}'' \hat{\otimes} \mathcal{A}'', T \in \mathcal{B}(\mathcal{A}, \mathcal{A}') = (\mathcal{A} \hat{\otimes} \mathcal{A})').$$

Then $\rho_{\mathcal{A}}$ is a norm-decreasing $\mathcal{A}$-bimodule homomorphism which satisfies $\rho_{\mathcal{A}} \circ (\kappa_{\mathcal{A}_*} \otimes \kappa_{\mathcal{A}_*}) = \kappa_{\mathcal{A} \hat{\otimes} \mathcal{A}}$.

For a Banach algebra $\mathcal{A}$, it is clear that $\mathcal{W}(\mathcal{A}, \mathcal{A}')$ is a sub-$\mathcal{A}$-bimodule of $\mathcal{B}(\mathcal{A}, \mathcal{A}') = (\mathcal{A} \hat{\otimes} \mathcal{A})'$.

Theorem 3.8. Let $\mathcal{A}$ be an Arens regular Banach algebra such that $\mathcal{A}''$ is unital, and let $T \in \mathcal{B}(\mathcal{A}'', \mathcal{A}'') = (\mathcal{A}'' \hat{\otimes} \mathcal{A}'')'$. Then the following are equivalent:

1. $T \in \sigma WC(\mathcal{B}(\mathcal{A}'', \mathcal{A}''))$, where we treat $\mathcal{B}(\mathcal{A}'', \mathcal{A}'')$ as an $\mathcal{A}''$-bimodule;
2. $T = S''$ for some $S \in \text{WAP}(\mathcal{W}(\mathcal{A}, \mathcal{A}'))$, where now we treat $\mathcal{W}(\mathcal{A}, \mathcal{A}')$ as an $\mathcal{A}$-bimodule.
then there exists $T_0 \in W(A'', A')$ such that $T = \kappa_A \circ T_0$, and there exists $T_1 \in W(A'', A')$ such that $T' \circ \kappa_A'' = \kappa_A' \circ T_1$. Let $S = T_0 \circ \kappa_A \in W(A, A')$. Then, for $a \in A$ and $\Psi \in A'',$ we have

$$\langle S'(\Psi), a \rangle = \langle \Psi, T_0(\kappa_A(a)) \rangle = \langle T(\kappa_A(a)), \Psi \rangle = \langle T'(\kappa_A'(\Psi)), \kappa_A(a) \rangle$$

$$= \langle \kappa_A(a), T_1(\Psi) \rangle = \langle T_1(\Psi), a \rangle,$$

so that $S' = T_1$. Thus, for $\Phi, \Psi \in A'',$ we have

$$\langle S''(\Phi), \Psi \rangle = \langle \Phi, T_1(\Psi) \rangle = \langle T'(\kappa_A'(\Psi)), \Phi \rangle = \langle T(\Phi), \Psi \rangle,$$

so that $S'' = T$. We know that the maps $L_T, R_T : A'' \to B(A'', A''''),$ defined by $L_T(\Phi) = T \cdot \Phi$ and $R_T(\Phi) = \Phi \cdot T$ for $\Phi \in A'',$ are weakly-compact. Define $L_S, R_S : A \to B(A, A')$ is an analogous manner, using $S \in W(A, A').$ For $a \in A,$ $S \cdot a \in W(A, A'),$ so for $\Psi \in A'',$ and $b \in A,$

$$\langle (S \cdot a)'(\Psi), b \rangle = \langle \Psi, (S \cdot a)(b) \rangle = \langle \Psi, S(b) \cdot a \rangle = \langle a \cdot \Psi, S(b) \rangle = \langle S'(a \cdot \Psi), b \rangle.$$

Thus, for $a \in A$ and $\Phi, \Psi \in A'',$ we have that

$$\langle \iota_A(L_S(a))(\Phi), \Psi \rangle = \langle (S \cdot a)''(\Phi), \Psi \rangle = \langle \Phi, S'(a \cdot \Psi) \rangle = \langle S''(\Phi) \cdot a, \Psi \rangle,$$

so that $\iota_A(L_S(a))(\Phi) = S''(\Phi) \cdot a,$ and hence that $\iota_A(L_S(a)) = S'' \cdot a = T \cdot a = T \cdot \kappa_A(a) = L_T(\kappa_A(a)).$ Thus we have that $L_S = (\kappa_A \otimes \kappa_A)' \circ R_T \circ \kappa_A,$ so that $L_S$ is weakly-compact. A similar calculation shows that $R_S$ is also weakly-compact, so that $S \in WAP(W(A, A')).$ This shows that (1) implies (2).

Conversely, if (2) holds, then $L_S$ and $R_S$ are weakly-compact. As $S$ is weakly-compact, $T(A'') = S''(A'') \subseteq \kappa_A'(A')$ and $T'(\kappa_A'(A''')) = S'''(\kappa_A'(A''')) = \kappa_A'(S'(A'')) \subseteq \kappa_A'(A'),$ and $T$ is weakly-compact. Thus, to show (1), we are required to show that $L_T$ and $R_T$ are weakly-compact.

For $a, b \in A$ and $\Phi \in A',$ we have

$$\langle (a \cdot S)'(\Phi), b \rangle = \langle \Phi, S(ba) \rangle = \langle a \cdot S'(\Phi), b \rangle.$$

Then, for $\Phi, \Psi \in A''$ and $a \in A,$ we thus have

$$\langle R_S'(\rho_A(\Phi \otimes \Psi)), a \rangle = \langle (a \cdot S)''(\Phi) \otimes \Psi \rangle = \langle (a \cdot S)''(\Psi), \Phi \rangle = \langle \Psi, a \cdot S'(\Phi) \rangle$$

$$= \langle \Psi \cdot a, S'(\Phi) \rangle = \langle \Psi \otimes \kappa_A(a), S'(\Phi) \rangle = \langle S'(\Phi) \cdot a, \Psi \rangle.$$

Hence we see that $R_S'(\rho_A(\Phi \otimes \Psi)) = S'(\Phi) \cdot a.$ Let $U = R_S' \circ \rho_A : A'' \otimes A'' \to A',$ so that as $R_S$ is weakly-compact, so is $U.$ Then, for $\Phi, \Psi, \Gamma \in A'',$ we have that

$$\langle U'(\Gamma), \Phi \otimes \Psi \rangle = \langle \Gamma, S'(\Phi) \cdot \Psi \rangle = \langle \Psi \otimes \Gamma, S'(\Phi) \rangle = \langle S''(\Psi \otimes \Gamma), \Phi \rangle = \langle (\Gamma \cdot S'')(\Psi), \Phi \rangle,$$

so that $U'(\Gamma) = \Gamma \cdot T,$ that is, $U' = R_T,$ so that $R_T$ is weakly-compact. Similarly, we can show that $L_T$ is weakly-compact, completing the proof.

\begin{proof}

\begin{enumerate}
\item $\Delta_A''(M) = e_{A''},$ the unit of $A''$;
\item $\langle M, a \cdot T - T \cdot a \rangle = 0$ for each $a \in A$ and each $T \in WAP(W(A, A')).$
\end{enumerate}

\end{proof}

\textbf{Theorem 3.9.} Let $A$ be an Arens regular Banach algebra. Then $A''$ is Connes-amenable if and only if $A''$ is unital and there exists $M \in (A \widehat{\otimes} A)''$ such that:

1. $\Delta_A''(M) = e_{A''},$ the unit of $A''$;
2. $\langle M, a \cdot T - T \cdot a \rangle = 0$ for each $a \in A$ and each $T \in WAP(W(A, A')).$
By Theorem 3.8, we wish to show that the existence of such an $M$ is equivalent to the existence of $N \in (\mathcal{A}'')''$ such that:

\begin{align*}
(N1) & \quad \kappa''_{\mathcal{A}'} \Delta''_{\mathcal{A}''}(N) = e_{\mathcal{A}''}; \\
(N2) & \quad \langle N, \Phi \cdot S - S \cdot \Phi \rangle = 0 \text{ for each } \Phi \in \mathcal{A}'' \text{ and each } S \in \sigma WC(\mathcal{B}(\mathcal{A}'', \mathcal{A}'')).
\end{align*}

We can verify that $\iota_{\mathcal{A}} \circ \Delta'_{\mathcal{A}} = \Delta'_{\mathcal{A}''} \circ \kappa_{\mathcal{A}'}$, so that (N1) is equivalent to $\Delta''_{\mathcal{A}''}(N) = e_{\mathcal{A}''}$. For $S \in \sigma WC(\mathcal{B}(\mathcal{A}'', \mathcal{A}''))$, we know that $S = T''$ for some $T \in \text{WAP}(\mathcal{W}(\mathcal{A}', \mathcal{A}'))$, by Theorem 3.8. That is, the maps $\phi_r$ and $\phi_t$, formed using $T$ as in Proposition 3.2, are weakly-compact. Then, for $\Phi \in \mathcal{A}'$, $\phi_r(\Phi), \phi_t(\Phi) \in \mathcal{B}(\mathcal{A}, \mathcal{A}')$, and we can check that

\[
\phi_r(\Phi)(a) = \kappa'_{\mathcal{A}'} T''(a \cdot \Phi), \quad \phi_t(\Phi)(a) = T(a) \cdot \Phi \quad (a \in \mathcal{A}).
\]

Then $\phi_r(\Phi)'$ and $\phi_t(\Phi)'$ are the maps

\[
\phi_r(\Phi)'(\Psi) = \Phi \cdot T''(\Psi), \quad \phi_t(\Phi)'(\Psi) = T''(\Phi \square \Psi) \quad (\Psi \in \mathcal{A}''),
\]

where we remember that $T''(\mathcal{A}'') \subseteq \kappa_{\mathcal{A}'}(\mathcal{A}')$. Consequently $\phi_r(\Phi)'', \phi_t(\Phi)'' \in \mathcal{B}(\mathcal{A}'', \mathcal{A}'')$ are given by

\[
\phi_r(\Phi)''(\Psi) = T''(\Psi \square \Phi), \quad \phi_t(\Phi)''(\Psi) = T''(\Psi) \cdot \Phi \quad (\Psi \in \mathcal{A}''),
\]

where $\mathcal{A}''$ is an $\mathcal{A}''$-bimodule, as $\mathcal{A}''$ is Arens regular. That is, $\phi_r(\Phi)'' = \Phi \cdot S$ and $\phi_t(\Phi)'' = S \cdot \Phi$. Hence (N2) is equivalent to

\[
0 = \langle N, \phi_r(\Phi)'' - \phi_t(\Phi)'' \rangle = \langle N, \iota_{\mathcal{A}}(\phi_r(\Phi) - \phi_t(\Phi)) \rangle = \langle \iota'_{\mathcal{A}}(N), \phi_r(\Phi) - \phi_t(\Phi) \rangle,
\]

for each $\Phi \in \mathcal{A}'$ and $S \in \sigma WC(\mathcal{B}(\mathcal{A}'', \mathcal{A}''))$. That is, (N2) is equivalent to

\[
\phi_r''(\iota'_{\mathcal{A}}(N) - \phi_t''(\iota'_{\mathcal{A}}(N)) = 0 \quad (S \in \sigma WC(\mathcal{B}(\mathcal{A}'', \mathcal{A}''))).
\]

As $\phi_r$ and $\phi_t$ are weakly-compact, $\phi_r''$ and $\phi_t''$ take values in $\kappa_{\mathcal{A}'}(\mathcal{A}')$, and so (N2) is equivalent to

\[
0 = \langle \phi_r''(\iota'_{\mathcal{A}}(N) - \phi_t''(\iota'_{\mathcal{A}}(N)), \kappa_{\mathcal{A}'}(a) \rangle = \langle \iota'_{\mathcal{A}}(N), \phi_r'(\kappa_{\mathcal{A}}(a)) - \phi_t'(\kappa_{\mathcal{A}}(a)) \rangle,
\]

for each $a \in \mathcal{A}$ and each $S \in \sigma WC(\mathcal{B}(\mathcal{A}'', \mathcal{A}''))$. However, $\phi_r'(\kappa_{\mathcal{A}}(a)) - \phi_t'(\kappa_{\mathcal{A}}(a)) = a \cdot T - T \cdot a$, so that (N2) is equivalent to

\[
0 = \langle \iota'_{\mathcal{A}}(N), a \cdot T - T \cdot a \rangle \quad (a \in \mathcal{A}),
\]

for each $T \in \mathcal{W}(\mathcal{A}, \mathcal{A}')$ such that $\phi_r$ and $\phi_t$ are weakly-compact.

Thus we have established that (N1) holds for $N$ if and only if (1) holds for $M = \iota'_{\mathcal{A}}(N)$, and that (N2) holds for $N$ if and only if (2) holds for $M = \iota'_{\mathcal{A}}(N)$, completing the proof. \qed

We immediately see that $\mathcal{A}$ amenable implies that $\mathcal{A}''$ is Connes-amenable. Furthermore, if $\mathcal{A}$ is itself a dual Banach algebra, then Corollary 3.3 shows that if $\mathcal{A}''$ is Connes-amenable, then $\mathcal{A}$ is Connes-amenable: notice that if $e_{\mathcal{A}''}$ is the unit of $\mathcal{A}''$, then

\[
\langle \kappa'_{\mathcal{A}''}(e_{\mathcal{A}''})a, \mu \rangle = \langle e_{\mathcal{A}''} \cdot a, \kappa_{\mathcal{A}'}(\mu) \rangle = \langle \kappa_{\mathcal{A}}(a), \kappa_{\mathcal{A}''}(\mu) \rangle = \langle a, \mu \rangle \quad (a \in \mathcal{A}, \mu \in \mathcal{A}''),
\]

so that $\kappa'_{\mathcal{A}''}(e_{\mathcal{A}''})$ is the unit of $\mathcal{A}$.\hfill \blacksquare
4 Injectivity of the predual module

Let $\mathcal{A}$ be a Banach algebra, and let $E$ and $F$ be Banach left $\mathcal{A}$-modules. We write $\mathcal{A}\mathcal{B}(E, F)$ for the closed subspace of $\mathcal{B}(E, F)$ consisting of left $\mathcal{A}$-module homomorphisms, and similarly write $\mathcal{B}_\mathcal{A}(E, F)$ and $\mathcal{A}\mathcal{B}_\mathcal{A}(E, F)$ for right $\mathcal{A}$-module and $\mathcal{A}$-bimodule homomorphisms, respectively. We say that $T \in \mathcal{A}\mathcal{B}(E, F)$ is admissible if both the kernel and image of $T$ are closed, complemented subspaces of, respectively, $E$ and $F$. If $T$ is injective, this is equivalent to the existence of $S \in \mathcal{B}(F, E)$ such that $ST = I_E$.

**Definition 4.1.** Let $\mathcal{A}$ be a Banach algebra, and let $E$ be a Banach left $\mathcal{A}$-module. Then $E$ is injective if, whenever $F$ and $G$ are Banach left $\mathcal{A}$-modules, $\theta \in \mathcal{A}\mathcal{B}(F, G)$ is injective and admissible, and $\sigma \in \mathcal{A}\mathcal{B}(E, F)$, there exists $\rho \in \mathcal{A}\mathcal{B}(G, E)$ with $\rho \circ \theta = \sigma$.

We say that $E$ is left-injective when we wish to stress that we are treating $E$ as a left module. Similar definitions hold for right modules and bimodules (written right-injective and bi-injective where necessary).

Let $\mathcal{A}$ be a Banach algebra, let $E$ be a Banach left $\mathcal{A}$-module, and turn $\mathcal{B}(\mathcal{A}, E)$ into a left $\mathcal{A}$-module by setting

$$(a \cdot T)(b) = T(ba) \quad (a, b \in \mathcal{A}, T \in \mathcal{B}(\mathcal{A}, E)).$$

Then there is a canonical left $\mathcal{A}$-module homomorphism $\iota : E \to \mathcal{B}(\mathcal{A}, E)$ given by

$$\iota(x)(a) = a \cdot x \quad (a \in \mathcal{A}, x \in E).$$

Notice that if $E$ is a closed submodule of $\mathcal{A}'$, then $\mathcal{B}(\mathcal{A}, E)$ is a closed submodule of $(\mathcal{A}\hat{\otimes} \mathcal{A})' = \mathcal{B}(\mathcal{A}, \mathcal{A}')$, and $\iota$ is the restriction of $\Delta' : \mathcal{A}' \to \mathcal{B}(\mathcal{A}, \mathcal{A}')$ to $E$.

Similarly, we turn $\mathcal{B}(\mathcal{A}\hat{\otimes} \mathcal{A}, E)$ into a Banach $\mathcal{A}$-bimodule by

$$(a \cdot T)(b \otimes c) = T(ba \otimes c), \quad (T \cdot a)(b \otimes c) = T(b \otimes ac) \quad (a, b, c \in \mathcal{A}, T \in \mathcal{B}(\mathcal{A}\hat{\otimes} \mathcal{A}, E)).$$

We then define (with an abuse of notation) $\iota : E \to \mathcal{B}(\mathcal{A}\hat{\otimes} \mathcal{A}, E)$ by

$$\iota(x)(a \otimes b) = a \cdot x \cdot b \quad (x \in E, a \otimes b \in \mathcal{A}\hat{\otimes} \mathcal{A}),$$

so that $\iota$ is an $\mathcal{A}$-bimodule homomorphism.

We can also turn $\mathcal{B}(\mathcal{A}, E)$ into a right $\mathcal{A}$-module by reversing the above (in particular, we need to take the other possible choice in Section 3 leading to different module actions as compared to those in [11].)

**Proposition 4.2.** Let $\mathcal{A}$ be a Banach algebra, and let $E$ be a faithful Banach left $\mathcal{A}$-module (that is, for each non-zero $x \in E$ there exists $a \in \mathcal{A}$ with $a \cdot x \neq 0$). Then $E$ is injective if and only if there exists $\phi \in \mathcal{A}\mathcal{B}(\mathcal{B}(\mathcal{A}, E), E)$ such that $\phi \circ \iota = I_E$.

Similarly, if $E$ is a left and right faithful Banach $\mathcal{A}$-bimodule (that is, for each non-zero $x \in E$ there exists $a, b \in \mathcal{A}$ with $a \cdot x \neq 0$ and $x \cdot b \neq 0$). Then $E$ is injective if and only if there exists $\phi \in \mathcal{A}\mathcal{B}_\mathcal{A}(\mathcal{B}(\mathcal{A}\hat{\otimes} \mathcal{A}, E), E)$ such that $\phi \circ \iota = I_E$.

**Proof.** The first claim is [3] Proposition 1.7], and the second claim is an obvious generalisation. 

Again, there exists a similar characterisation for right modules.

Let $\mathcal{A}$ be a dual Banach algebra with predual $\mathcal{A}_*$. It is simple to show (see [10]) that if $\mathcal{A}_*$ is bi-injective, then $\mathcal{A}$ is Connes-amenable. Helemskii showed in [7] that for a von Neumann algebra $\mathcal{A}$, the converse is true. However, Runde (see [10]) and Tabaldyev (see [13]) have shown that $M(G)$, the measure algebra of a locally compact group $G$,
while being a dual Banach algebra with predual $C_0(G)$, has that $C_0(G)$ is a left-injective $M(G)$-module only when $G$ is finite. Of course, Runde (see [11]) has shown that $M(G)$ is Connes-amenable if and only if $G$ is amenable.

Similarly, it is simple to show (using a virtual diagonal) that if $\mathcal{A}$ is a Banach algebra with a bounded approximate identity, then $\mathcal{A}$ is amenable if and only if $\mathcal{A}'$ is bi-injective.

Let $E$ and $F$ be Banach left $\mathcal{A}$-modules, and let $\phi : E \to F$ be a left $\mathcal{A}$-module homomorphism which is bounded below. Then $\phi$ is injective if and only if this short exact sequence is admissible. If, further, we may choose $P$ to be a left $\mathcal{A}$-module homomorphism, then the short exact sequence is said to split. Similar definitions hold for right modules and bimodules.

**Proposition 4.3.** Let $\mathcal{A}$ be a Banach algebra, let $E$ be a Banach left $\mathcal{A}$-module, and consider the following admissible short exact sequence:

$$
0 \to E \xrightarrow{\phi} F \xrightarrow{\phi} F/\phi(E) \to 0.
$$

If there exists a bounded linear map $P : F \to E$ such that $P \circ \phi = I_E$, then we say that the short exact sequence is admissible. If, further, we may choose $P$ to be a left $\mathcal{A}$-module homomorphism, then the short exact sequence is said to split. Similar definitions hold for right modules and bimodules.

**Proposition 4.4.** Let $\mathcal{A}$ be a unital dual Banach algebra with predual $\mathcal{A}_*$, and consider the following admissible short exact sequence of $\mathcal{A}$-bimodules:

$$
0 \to \mathcal{A}_* \xrightarrow{\Delta'_A} \sigma WC((\hat{\mathcal{A}} \hat{\otimes} \mathcal{A})') \xrightarrow{\sigma WC((\hat{\mathcal{A}} \hat{\otimes} \mathcal{A})')/\Delta'_A(\mathcal{A}_*)} \to 0.\tag{2}
$$

Then $\mathcal{A}$ is Connes-amenable if and only if this short exact sequence splits.

**Proof.** See, for example, [13] Section 5.3. \hfill \Box

Suppose that we can choose $P$ to be an $\mathcal{A}$-bimodule homomorphism. Then let $M = P'(e_\mathcal{A})$, so that for $a \in \mathcal{A}$ and $T \in \sigma WC(\mathcal{B}(\mathcal{A}, \mathcal{A}'))$,

$$
\langle a \cdot M - M \cdot a, T \rangle = \langle e_\mathcal{A}, P(T \cdot a - a \cdot T) \rangle = \langle a - a, P(T) \rangle = 0,
$$

so that $a \cdot M - M \cdot a$. Also $\Delta'_A(M) = (P \circ \Delta'_A)'(e_\mathcal{A}) = e_\mathcal{A}$, so that $M$ is a $\sigma WC$-virtual diagonal, and hence $\mathcal{A}$ is Connes-amenable by Runde’s theorem.

Conversely, let $M$ be a $\sigma WC$-virtual diagonal and define $P : \sigma WC(\mathcal{B}(\mathcal{A}, \mathcal{A}')) \to \mathcal{A}'$ by

$$
\langle P(T), a \rangle = \langle M, a \cdot T \rangle \quad (a \in \mathcal{A}, T \in \sigma WC(\mathcal{B}(\mathcal{A}, \mathcal{A}'))).
$$

Let $(a_\alpha)$ be a bounded net in $\mathcal{A}$ which tends to $a \in \mathcal{A}$ in the $\sigma(\mathcal{A}, \mathcal{A}_*)$-topology. By definition, $a_\alpha \cdot T \to a \cdot T$ weakly, for each $T \in \sigma WC(\mathcal{B}(\mathcal{A}, \mathcal{A}'))$, so that $\langle P(T), a_\alpha \rangle \to \langle P(T), a \rangle$. This implies that $P$ maps into $\mathcal{A}_*$, as required. Then, for $\mu \in \mathcal{A}_*$,

$$
\langle a, P \Delta'_A(\mu) \rangle = \langle M, a \cdot \Delta'_A(\mu) \rangle = \langle M, \Delta'_A(a \cdot \mu) \rangle = \langle e_\mathcal{A}, a \cdot \mu \rangle = \langle a, \mu \rangle \quad (a \in \mathcal{A}),
$$

where $\Delta'_A(\sum a_\alpha \otimes e_\mathcal{A}_\beta) = \sum a_\alpha \Delta'_A(\otimes e_\mathcal{A}_\beta)$. This shows that $P$ is a left $\mathcal{A}$-module homomorphism.
so that $P\Delta'_A = I_A$. Finally, we note that
\[
\langle P(a \cdot T \cdot b), c \rangle = \langle M, ca \cdot T \cdot b \rangle = \langle b \cdot M, ca \cdot T \rangle = \langle M \cdot b, ca \cdot T \rangle = \langle P(T), bca \rangle = \langle a \cdot P(T) \cdot b, c \rangle \quad (a, b, c \in A, T \in \sigma WC(\mathcal{B}(\mathcal{A}, \mathcal{A}'))),
\]
so that $P$ is an $A$-bimodule homomorphism, as required. □

Let $A$ be an Arens regular Banach algebra. By reversing the argument Theorem 3.8, we can show that $\Delta'_A : A' \to B(A, A')$ actually maps into WAP($\mathcal{W}(A, A')$). Furthermore, if $A''$ is unital, then we may define $P : WAP(\mathcal{W}(A, A')) \to A'$ by
\[
\langle P(T), a \rangle = \langle e_{A''}, P(a) \rangle \quad (a \in A, T \in WAP(\mathcal{W}(A, A'))).
\]
Then we have that
\[
\langle P\Delta'_A(\mu), a \rangle = \langle e_{A''}, a \cdot \mu \rangle = \langle \mu, a \rangle \quad (a \in A, \mu \in A').
\]

**Proposition 4.5.** Let $A$ be an Arens regular Banach algebra such that $A''$ is unital, and consider the following admissible short exact sequence of $A$-bimodules:
\[
\begin{array}{c}
0 \longrightarrow A' \xrightarrow{\Delta'_A} WAP(\mathcal{W}(A, A')) \longrightarrow WAP(\mathcal{W}(A, A'))/\Delta'_A(A') \longrightarrow 0.
\end{array}
\]

Then $A''$ is Connes-amenable if and only if this short exact sequence splits.

**Proof.** This follows in the same manner as the above proof, using Theorem 3.9. □

## 5 Beurling algebras

Let $S$ be a discrete semigroup (we can extend the following definitions to locally compact semigroups, but for the questions we are interested in, the results for non-discrete groups are trivial). A *weight* on $S$ is a function $\omega : S \to \mathbb{R}^+$ such that
\[
\omega(st) \leq \omega(s)\omega(t) \quad (s, t \in S).
\]
Furthermore, if $S$ is unital with unit $u_S$, then we also insist that $\omega(u_S) = 1$. This last condition is simply a normalisation condition, as we can always set $\hat{\omega}(s) = \sup\{\omega(st)\omega(t)^{-1} : t \in S\}$ for each $s \in S$. For $s, t \in S$, we have that $\omega(st) \leq \hat{\omega}(s)\omega(t)$, so that
\[
\hat{\omega}(st) = \sup\{\omega(str)\omega(r)^{-1} : r \in S\} \leq \sup\{\hat{\omega}(s)\omega(tr)\omega(r)^{-1} : r \in S\} = \hat{\omega}(s)\hat{\omega}(t).
\]
Clearly $\hat{\omega}(u_S) = 1$ and $\hat{\omega}(s) \leq \omega(s)$ for each $s \in S$, while $\hat{\omega}(s) \geq \omega(s)\omega(u_S)^{-1}$, so that $\hat{\omega}$ is equivalent to $\omega$.

We form the Banach space
\[
\ell^1(S, \omega) = \left\{(a_g)_{g \in S} \subseteq \mathbb{C} : \|a_g\| := \sum_{g \in S} |a_g|\omega(g) < \infty \right\}.
\]

Then $\ell^1(S, \omega)$, with the convolution product, is a Banach algebra, called a *Beurling algebra*. See [1] and [2] for further information on Beurling algebras and, in particular, their second duals.

It will be more convenient for us to think of $\ell^1(S, \omega)$ as the Banach space $\ell^1(S)$ together with a weighted algebra product. Indeed, for $g \in S$, let $\delta_g \in \ell^1(S)$ be the standard unit vector basis element which is thought of as a point-mass at $g$. Then each $x \in \ell^1(S)$
can be written uniquely as \( x = \sum_{g \in S} x_g \delta_g \) for some family \( (x_g) \subseteq \mathbb{C} \) such that \( \|x\| = \sum_{g \in S} |x_g| < \infty \). We then define
\[
\delta_g \ast \delta_h = \delta_g \ast \delta_h = \delta_{gh} \Omega(g,h) \quad (g, h \in S),
\]
where \( \Omega(g,h) = \omega(gh) \omega(g)^{-1} \omega(h)^{-1} \), and extend \( \ast \) to \( l^1(S) \) by linearity and continuity.

For example, if \( \omega \) and \( \tilde{\omega} \) are equivalent weights on \( S \), the define \( \psi : l^1(S, \omega) \to l^1(S, \tilde{\omega}) \) by \( \psi(\delta_s) = \tilde{\omega}(s) \omega(s)^{-1} \delta_s \). As \( \omega \) and \( \tilde{\omega} \) are equivalent, \( \psi \) is an isomorphism of Banach spaces. Then \( \psi(\delta_s \ast \delta_t) = \omega(st) \omega(s)^{-1} \omega(t)^{-1} \tilde{\omega}(st) \omega(st)^{-1} \delta_{st} = \psi(\delta_s) \ast \psi(\delta_t) \), so that \( \psi \) is a homomorphism.

For a set \( I \), we define the space \( c_0(I) \) as
\[
c_0(I) = \left\{ (a_i)_{i \in I} : \forall \epsilon > 0, |\{ i \in I : |a_i| \geq \epsilon \}| < \infty \right\},
\]
where \( |\cdot| \) is the cardinality of a set. We equip \( c_0(I) \) with the supremum norm; then \( c_0(I)' = l^1(I) \). For \( i \in I \), we let \( e_i \in c_0(I) \) be the point mass at \( i \), that is, \( \langle \delta_j, e_i \rangle = \delta_{i,j} \), the Kronecker delta, for \( \delta_j \in l^1(I) \). Then \( c_0(I) \) is the closed linear span of \( \{ e_i : i \in I \} \).

We let \( l^\infty(I) \) be the Banach space of all bounded families \( (a_i)_{i \in I} \), with the supremum norm. Then \( l^1(I)' = l^\infty(I) \), we can treat \( c_0(I) \) as a subspace of \( l^\infty(I) \), and the map \( \kappa_{c_0(I)} : c_0(I) \to l^\infty(I) \) is just the inclusion map.

For a semigroup \( S \) and \( s \in S \), we define maps \( L_s, R_s : S \to S \) by
\[
L_s(t) = st, \quad R_s(t) = ts \quad (t \in S).
\]
If, for each \( s \in S \), \( L_s \) and \( R_s \) are finite-to-one maps, then we say that \( S \) is weakly cancellative. When \( L_s \) and \( R_s \) are injective for each \( s \in S \), we say that \( S \) is cancellative.

When \( S \) is abelian and cancellative, a construction going back to Grothendieck shows that \( S \) is a sub-semigroup of some abelian group. However, this can fail to hold for non-abelian semigroups.

**Proposition 5.1.** Let \( S \) be a weakly cancellative semigroup, let \( \omega \) be a weight on \( S \), and let \( \mathcal{A} = l^1(S, \omega) \). Then \( c_0(S) \subseteq l^\infty(S) = \mathcal{A}' \) is a sub-\( \mathcal{A} \)-module of \( \mathcal{A}' \), so that \( l^1(S, \omega) \) is a dual Banach algebra with predual \( c_0(S) \).

**Proof.** For \( g, h \in S \) and \( a = (a_s)_{s \in S} \in l^1(S, \omega) \), we have
\[
\langle e_g \cdot \delta_h, a \rangle = \langle e_g, \delta_h \ast a \rangle = \langle e_g, \sum_{s \in S} a_s \delta_{hs} \omega(h,s) \rangle = \sum_{\{ s \in S : hs = g \}} a_s \omega(h,s).
\]
As \( S \) is weakly cancellative, there exists at most finitely many \( s \in S \) such that \( hs = g \), so that \( e_g \cdot \delta_h \) is a member of \( c_0(S) \). Thus we see that \( c_0(S) \) is a right sub-\( \mathcal{A} \)-module of \( \mathcal{A}' \). The argument on the left follows in an analogous manner. \( \square \)

Notice that the above result will hold for some semigroups \( S \) which are not weakly cancellative, provided that the weight behaves in a certain way. However, it would appear that the later results do not easily generalise to the non-weakly cancellative case.

Following [3], Definition 2.2, we have the following definition.

**Definition 5.2.** Let \( I \) and \( J \) be non-empty infinite sets, and let \( f : I \times J \to \mathbb{C} \) be a function. Then \( f \) clusters on \( I \times J \) if
\[
\lim_{n \to \infty} \lim_{m \to \infty} f(x_m, y_n) = \lim_{m \to \infty} \lim_{n \to \infty} f(x_m, y_n),
\]
whenever \( (x_m) \subseteq I \) and \( (y_n) \subseteq J \) are sequences of distinct elements, and both iterated limits exist.

Furthermore, \( f \) 0-clusters on \( I \times J \) if \( f \) clusters on \( I \times J \), and the iterated limits are always 0, whenever they exist. \( \square \)
Theorem 5.3. Let $S$ be a discrete, weakly cancellative semigroup, and let $\omega$ be a weight on $S$. Then the following are equivalent:

1. $l^1(S, \omega)$ is Arens regular;

2. for sequences of distinct elements $(g_j)$ and $(h_k)$ in $S$, we have
$$\lim_{j \to \infty} \lim_{k \to \infty} \Omega(g_j, h_k) = 0,$$
whenever the iterated limit exists;

3. $\Omega$ 0-clusters on $S \times S$.

Proof. That (1) and (2) are equivalent for cancellative semigroups is [1, Theorem 1]. Close examination of the proof shows that this holds for weakly cancellative semigroups as well. That (1) and (3) are equivalent follows by generalising the proof of [3, Theorem 7.11], which is essentially an application of Grothendieck’s criterion for an operator to be weakly-compact. Alternatively, it follows easily that (2) and (3) are equivalent by considering the opposite semigroup to $S$ where we reverse the product. 

In [1] it is also shown that if $G$ is a discrete, uncountable group, then $l^1(G, \omega)$ is not Arens regular for any weight $\omega$. Furthermore, by [1, Theorem 2], if $G$ is a non-discrete locally compact group, then $L^1(G, \omega)$ is never Arens regular.

We shall consider both the Connes-amenability of $l^1(S, \omega)$ and $l^1(S, \omega)$ (with respect to the canonical predual $c_0(S)$) as, with reference to Corollary 3.5 and Theorem 3.8 the calculations should be similar.

Proposition 5.4. Let $I$ be a non-empty set, and let $X \subseteq l^\infty(I)$ be a subset. Then the following are equivalent:

1. $X$ is relatively weakly-compact;

2. $X$ is relatively sequentially weakly-compact;

3. the absolutely convex hull of $X$ is relatively weakly-compact;

4. if we define $f : I \times X \to \mathbb{C}$ by $f(i, x) = \langle x, \delta_i \rangle$ for $i \in I$ and $x \in X$, then $f$ clusters on $I \times X$.

Proof. That (1) and (2) are equivalent is the Eberlein-Smulian theorem; that (1) and (3) are equivalent is the Krein-Smulian theorem. That (1) and (4) are equivalent is a result of Grothendieck, detailed in, for example, [3, Theorem 2.3].

It is standard that for non-empty sets $I$ and $J$, we have that $l^1(I) \widehat{\otimes} l^1(J) = l^1(I \times J)$, where, for $i \in I$ and $j \in J$, $\delta_i \otimes \delta_j \in l^1(I) \widehat{\otimes} l^1(J)$ is identified with $\delta_{(i,j)} \in l^1(I \times J)$. Thus we have $(l^1(I) \widehat{\otimes} l^1(J))' = \mathcal{B}(l^1(I), l^\infty(J)) = l^1(I \times J)' = l^\infty(I \times J)$, where $T \in \mathcal{B}(l^1(I), l^\infty(J))$ is identified with $(T_{(i,j)}) \in l^\infty(I \times J)$, where $T_{(i,j)} = \langle T(\delta_i), \delta_j \rangle$.

Is this paragraph used? Let $S$ be a countable, discrete, unital semigroup, and let $\omega$ be a weight on $S$. Then $l^1(S \times S)$ is a Banach $l^1(S, \omega)$-bimodule, with module actions
$$\delta_k \cdot \delta_{(g,h)} = \delta_{(kg, h)} \Omega(k, g), \quad \delta_{(g,h)} \cdot \delta_k = \delta_{(g, hk)} \Omega(h, k) \quad (g, h, k \in S).$$

For a non-empty set $I$, the unit ball of $l^1(I)$ is the closure of the absolutely-convex hull of the set $\{\delta_i : i \in I\}$, so that for a Banach space $E$, by the Krein-Smulian theorem, a map $T : l^1(I) \to E$ is weakly-compact if and only if the set $\{T(\delta_i) : i \in I\}$ is relatively weakly-compact in $E$. 

From now on we shall exclude the trivial case when our (semi-)group is finite.
Proposition 5.5. Let $S$ be a weakly cancellative semigroup, let $\omega$ be a weight on $S$, and let $A = l^1(S, \omega)$. Let $T \in \mathcal{B}(A, A')$ be such that $T(A) \subseteq \kappa_{c_0(S)}(c_0(S))$ and $T'(\kappa_A(A)) \subseteq \kappa_{c_0(S)}(c_0(S))$. Then $T \in \mathcal{W}(A, A')$, and $T \in \mathcal{W}(A, A')$ if and only if, for each sequence $(k_n)$ of distinct elements of $S$, and each sequence $(g_m, h_m)$ of distinct elements of $S \times S$ such that the repeated limits

$$\lim_{m} \lim_{n} \langle T(\delta_{h_m}), \delta_{k_n,g_m} \rangle, \quad \lim_{n} \lim_{m} \Omega(k_n, g_m) \quad (4)$$
$$\lim_{m} \lim_{n} \langle T(\delta_{h_m,k_n}), \delta_{g_m} \rangle, \quad \lim_{n} \lim_{m} \Omega(h_m, k_n) \quad (5)$$

all exist, we have that at least one repeated limit in each row is zero.

Proof. That $T$ is weakly-compact follows from Gantmacher’s Theorem (compare with Corollary 3.5). To show that $T \in \mathcal{W}$, by Lemma 3.4 we are required to show that the maps $\phi_r$ and $\phi_l$ are weakly-compact. We shall show that $\phi_l$ is weakly-compact if and only if one of the repeated limits in the first line (4) is zero; the proof that $\phi_r$ is related to (3) follows in a similar way. We have that

$$\phi_l(\delta_{g,h}) = \phi_l(\delta_g \otimes \delta_h) = \delta_g \cdot T(\delta_h) \quad (g, h \in S).$$

By Proposition 5.4, $\phi_l$ is weakly-compact if and only if the function

$$S \times (S \times S) \to \mathbb{C}; \quad (k, (g, h)) \mapsto \langle \delta_g \cdot T(\delta_h), \delta_k \rangle = \langle T(\delta_h), \delta_{k\delta} \rangle \Omega(k, g) \quad (g, h, k \in S)$$

clusters on $S \times (S \times S)$. As $T$ is weakly-compact, the function

$$S \times S \to \mathbb{C}; \quad (g, h) \mapsto \langle T(\delta_g), \delta_h \rangle \quad (g, h \in S)$$

does cluster on $S \times S$.

Let $(k_n)$ be a sequence of distinct elements of $S$, and let $(g_m, h_m)$ be a sequence of distinct elements of $S \times S$ such that the iterated limits

$$\lim_{n} \lim_{m} \langle T(\delta_{h_m}), \delta_{k_n,g_m} \rangle \Omega(k_n, g_m), \quad \lim_{m} \lim_{n} \langle T(\delta_{h_m}), \delta_{k_n,g_m} \rangle \Omega(k_n, g_m) \quad (6)$$

exist. We now investigate when these iterated limits are equal.

Suppose firstly that, by moving to a subsequence if necessary, we have that $g_m = g$ for all $m$. Further, by moving to a subsequence if necessary, we may suppose that

$$\lim_{n} \Omega(k_n, g) = \alpha,$$

say, and that $(k_ng)$ is a sequence of distinct elements (as $S$ is weakly cancellative). Then

$$\lim_{n} \lim_{m} \langle T(\delta_{h_m}), \delta_{k_n,g_m} \rangle \Omega(k_n, g_m) = \lim_{n} \Omega(k_n, g) \lim_{n} \lim_{m} \langle T(\delta_{h_m}), \delta_{k_n,g} \rangle = \alpha \lim_{n} \lim_{m} \langle T(\delta_{h_m}), \delta_{k_n,g} \rangle$$

$$= \lim_{m} \lim_{n} \langle T(\delta_{h_m}), \delta_{k_n,g_m} \rangle \Omega(k_n, g_m),$$

where we can swap the order of taking limits, as $T$ is weakly-compact.

Alternatively, if we cannot move to a subsequence such that $(g_m)$ is constant, then we may move to subsequence such that $(g_m)$ is a sequence of distinct elements, and such that the iterated limits

$$\lim_{m} \lim_{n} \Omega(k_n, g_m), \quad \lim_{n} \lim_{m} \Omega(k_n, g_m),$$
$$\lim_{n} \lim_{m} \langle T(\delta_{h_m}), \delta_{k_n,g_m} \rangle, \quad \lim_{m} \lim_{n} \langle T(\delta_{h_m}), \delta_{k_n,g_m} \rangle$$
all exists. As \( T(\mathcal{A}) \subseteq \kappa_{c_0(S)}(c_0(S)) \), we have that
\[
\{ g \in S : |\langle T(\delta_h), \delta_g \rangle | \geq \epsilon \} \text{ is finite} \quad (\epsilon > 0, h \in S).
\]
Consequently, and using the fact that \( S \) is weakly cancellative, we see that
\[
\lim_m \langle T(\delta_{h_m}), \delta_{kn_g m} \rangle = 0
\]
for each \( m \). Hence the iterated limits in (6) are equal if and only if we have that at least one repeated limit in (4) is zero.

**Proposition 5.6.** Let \( S \) be a discrete, unital, weakly cancellative semigroup, and let \( \omega \) be a weight on \( S \) such that \( \mathcal{A} = l^1(S, \omega) \) is Arens regular. Then WAP(\( \mathcal{W}(\mathcal{A}, \mathcal{A}') \)) = \( \mathcal{W}(\mathcal{A}, \mathcal{A}') \).

**Proof.** Let \( T \in \mathcal{W}(\mathcal{A}, \mathcal{A}') \). We can follow the above proof through until the point at which we use the fact that \( T(\mathcal{A}) \subseteq \kappa_{c_0(S)}(c_0(S)) \). However, as \( l^1(S, \omega) \) is Arens regular, by Theorem 4.3 we have that
\[
\lim_m \Omega(k_n, g_m) = \lim_n \Omega(k_n, g_m) = 0,
\]
so that the iterated limits in (6) must be 0, implying that \( \phi_l \) is weakly-compact. In a similar manner, \( \phi_r \) is weakly-compact.

**Theorem 5.7.** Let \( S \) be a discrete weakly cancellative semigroup, and let \( \omega \) be a weight on \( S \) such that \( \mathcal{A} = l^1(S, \omega) \) is Arens regular and \( \mathcal{A}' \) is unital with unit \( e_{\mathcal{A}'} \). Then \( \mathcal{A}' \) is Connes-amenable if and only if there exists \( M \in (\mathcal{A} \hat{\otimes} \mathcal{A})" = l^\infty(S \times S)' \) such that:

1. \( \langle M, (f_{gh})_{(g,h)\in S \times S} \rangle = \langle e_{\mathcal{A}'}, f \rangle \) for each bounded family \( (f_g)_{g \in S} \);
2. \( \langle M, (f(hk, g)\Omega(h, k) - f(h, kg)\Omega(k, g))_{(g,h)\in S \times S} \rangle = 0 \) for each \( k \in S \), and each bounded function \( f : S \times S \to \mathbb{C} \) which clusters on \( S \times S \).

**Proof.** We use Theorem 3.9 and Proposition 5.6. For \( f = (f_g)_{g \in S} \in l^\infty(S) \), we have
\[
\langle \Delta'_A(f), \delta_g \otimes \delta_h \rangle = \langle f, \delta_{gh} \rangle \Omega(g, h) \quad (g, h \in S),
\]
so that \( \Delta'_A(f) = \langle (f, \delta_{gh} \rangle \Omega(g, h))_{(g,h)\in S \times S} \in l^\infty(S \times S) \). As \( f \in l^\infty(S) \) was arbitrary, we have condition (1).

For \( T \in \mathcal{B}(\mathcal{A}, \mathcal{A}') \), we treat \( T \) as being a member of \( l^\infty(S \times S) \). Then \( T \) is weakly-compact if and only if the family \( \langle (T(\delta_g), \delta_h)_{(g,h)\in S \times S} \rangle \) clusters on \( S \times S \). For \( k \in S \), we have
\[
\langle \delta_k \cdot T - T \cdot \delta_k, \delta_g \otimes \delta_h \rangle = \langle T(\delta_{hk}), \delta_g \rangle \Omega(h, k) - \langle T(\delta_h), \delta_{kg} \rangle \Omega(k, g).
\]
Thus we have condition (2).

Notice that if \( S \) is unital with unit \( u_S \), then the unit of \( \mathcal{A} \) (and hence \( \mathcal{A}' \)) is \( u_S \). In this case, condition (1) reduces to \( \langle M, (f_{gh})_{(g,h)\in S \times S} \rangle = f_{u_S} \).

**Theorem 5.8.** Let \( S \) be a discrete unital semigroup, let \( \omega \) be a weight on \( S \), and let \( \mathcal{A} = l^1(S, \omega) \). Then \( \mathcal{A} \) is amenable if and only if there exists \( M \in (\mathcal{A} \hat{\otimes} \mathcal{A})" = l^\infty(S \times S)' \) such that:

1. \( \langle M, (f_{gh})_{(g,h)\in S \times S} \rangle = f_{u_S} \), where \( u_S \in S \) is the unit of \( S \), for each bounded family \( (f_g)_{g \in S} \);
2. \( \langle M, (f(hk, g)\Omega(h, k) - f(h, kg)\Omega(k, g))_{(g,h)\in S \times S} \rangle = 0 \) for each \( k \in S \), and each bounded function \( f : S \times S \to \mathbb{C} \).
Proof. This follows from Theorem 2.1 in the same way that the above follows from Theorem 3.9. □

Notice that condition (2) of Theorem 5.8 is strictly stronger than condition (2) of Theorem 5.7.

Theorem 5.9. Let $S$ be a discrete, weakly cancellative semigroup, let $\omega$ be a weight on $S$, and let $A = l^1(S, \omega)$ be unital with unit $e_A$. Then $A$ is Connes-amenable, with respect to the predual $c_0(S)$, if and only if there exists $M \in (A \hat{\otimes} A)^{\prime\prime} = l^\infty(S \times S)'$ such that:

1. $\langle M, (f_{gh}\Omega(g,h))_{(g,h)\in S \times S}\rangle = \langle e_A, f \rangle$ for each family $(f_g)_{g \in S} \in c_0(S)$;

2. $\langle M, (f(h,k)\Omega(h,k) - f(h,kg)\Omega(k,g))_{(g,h)\in S \times S}\rangle = 0$ for each $k \in S$, and each bounded function $f : S \times S \to \mathbb{C}$ which satisfies the conclusions of Proposition 5.5.

Proof. We now use Theorem 3.9. By $f$ satisfying the conclusions of Proposition 5.5, we identify $f : S \times S \to \mathbb{C}$ with $T \in B(A, A^\prime)$ by $\langle T(\delta_g), \delta_h \rangle = f(g, h)$, for $g, h \in S$.

We shall now establish when $l^1(S, \omega)$ and $l^1(S, \omega)^{\prime\prime}$ are Connes-amenable. For a discrete group $G$, a weight $\omega$ on $G$ and $h \in G$, define $J_h \in B(l^\infty(G))$ by

$$J_h(f) = (f_{hg}\Omega(h,g)\omega(h)\Omega(g^{-1}, h^{-1})\omega(h^{-1}))_{g \in G}, \quad (f = (f_g)_{g \in G} \in l^\infty(G)).$$

Notice then that, for $f \in l^\infty(G)$, we have

$$\|J_h(f)\| = \sup_g |f_{hg}| \omega(hg)\omega(g)\omega(g^{-1}h^{-1})\omega(h^{-1}) \leq \|f\| \omega(h)\omega(h^{-1}),$$

so that $J_h$ is bounded.

Definition 5.10. Let $G$ be a discrete group, and let $\omega$ be a weight on $G$. We say that $G$ is $\omega$-amenable if there exists $N \in l^\infty(G)'$ such that:

1. $\langle N, (\Omega(g,g^{-1}))_{g \in G}\rangle = 1$, where $\Omega$ is defined by $\omega$, and hence $(\Omega(g,g^{-1}))_{g \in G}$ is a bounded family forming an element of $l^\infty(G)$;

2. $J'_h(N) = N$ for each $h \in G$. □

Notice that if $\omega$ is identically 1, then this condition reduces to the usual notion of a group being amenable (we usually require that $N$ is a mean, in that $N$ is a positive functional on $l^\infty(G)$, but by forming real and imaginary parts, and then positive and negative parts, we can easily generate a non-zero scalar multiple of a mean from a functional $N$ satisfying the definition above).

Theorem 5.11. Let $G$ be a discrete group, let $\omega$ be a weight on $G$, and let $A = l^1(G, \omega)$. Then the following are equivalent:

1. $A$ is Connes-amenable, with respect to the predual $c_0(G)$;

2. $A$ is amenable;

3. $G$ is $\omega$-amenable.

Furthermore, if $A$ is Arens regular, then these conditions are equivalent to $A^{\prime\prime}$ being Connes-amenable.
Proof. It is clear that (2) implies (1). When $\mathcal{A}$ is Arens regular, (2) implies that $\mathcal{A}''$ is Connes-amenable, and $\mathcal{A}''$ Connes-amenable implies (1). We shall thus show that (1) implies (3), and that (3) implies (2).

If (1) holds, then let $M \in l^\infty(G \times G)'$ be given as in Theorem 5.9. Define $\phi : l^\infty(G) \to l^\infty(G \times G)$ by

$$\langle \phi(f), \delta_{(g,h)} \rangle = \begin{cases} f_g : g = h^{-1}, \\ 0 : g \neq h^{-1}, \end{cases} \quad (f = (f_g)_{g \in G} \in l^\infty(G)).$$

Let $N = \phi'(M) \in l^\infty(G)'$. Then we have

$$\phi(\Omega(g,g^{-1})) = (\delta_{h,g^{-1}} \Omega(g,h))_{(g,h) \in G \times G} = (\delta_{gh,g} \Omega(g,h))_{(g,h) \in G \times G},$$

where $\delta$ is the Kronecker delta, so that

$$\langle N, \Omega(g,g^{-1}) \rangle_{g \in G} = \delta_{eg,eg} = 1,$$

by condition (1) on $M$ from Theorem 5.9, clearly $(\delta_{eg,g})_{g \in G} \in c_0(G)$.

Fix $k \in G$ and $f \in l^\infty(G)$. Define $F : G \times G \to \mathbb{C}$ by

$$F(h,g) = \delta_{gh,k} f_g \omega(k) \omega(hk^{-1}) \omega(h)^{-1} \quad (g, h \in G).$$

Then we have $|F(h,g)| \leq |f_g| |\omega(k)||\omega(hk^{-1})||\omega(h)|^{-1} \leq \|f\|_\infty \|\omega(k)||\omega(k^{-1})|$, so that $F$ is bounded. Let $T : \mathcal{A} \to \mathcal{A}'$ be the operator associated with $F$. For $g, h \in G$, we have that $F(h,g) \neq 0$ only when $gh = k$, so that $T(\mathcal{A}) \subseteq c_0(S)$ and $T'(\kappa(\mathcal{A})) \subseteq c_0(S)$.

Furthermore, if $(k_n)$ is a sequence of distinct elements in $G$, and $(g_m, h_m)$ is a sequence of distinct elements in $G \times G$, then $\lim_n \lim_m F(h_m, k_n g_m) = 0$. This follows, as for $n_0$ fixed, $k_n g_m h_m = k$ only if $g_n h_m = k_n^{-1} k$, so if this holds for all sufficiently large $m$, we have that $k_n g_m h_m \neq k$ for sufficiently large $m$ and $n \neq n_0$. Similarly, $\lim_n \lim_m F(h_n k_n, g_m) = 0$, so that $F$ satisfies the conditions of Proposition 5.9.

Notice that

$$\langle \phi(J_k(f)), \delta_{(g,h)} \rangle = \delta_{gh,eg} \langle J_k(f), \delta_g \rangle = \delta_{gh,eg} f_{kg} \omega(kg) \omega(g)^{-1} \omega(g^{-1}k^{-1}) \omega(g^{-1})^{-1}.$$

Thus we have

$$F(hk,g)\Omega(h,k) - F(h,g)\Omega(k,g) = \delta_{ghk,k} f_g \omega(k) \omega(hk^{-1}) \omega(h)^{-1} \Omega(h,k) - \delta_{kg,h} f_k \omega(k) \omega(hk^{-1}) \omega(h)^{-1} \Omega(k,g)$$

$$= \delta_{gh,eg} f_{kg} - \delta_{gh,eg} f_{kg} \omega(hk^{-1}) \omega(h)^{-1} \omega(kg) \omega(g)^{-1}$$

$$= \langle \phi(f) - \phi(J_k(f)), \delta_{(g,h)} \rangle.$$

So, by condition (2) from Theorem 5.9, we have that

$$\langle N, f - J_k(f) \rangle = 0,$$

which, as $f$ was arbitrary, shows that $N = J_k'(N)$, as required.

Now suppose that $G$ is $\omega$-amenable. We shall show that $\mathcal{A}$ is amenable, which completes the proof. Define $\psi : l^\infty(G \times G) \to l^\infty(G)$ by

$$\langle \psi(F), \delta_g \rangle = F(g, g^{-1}) \quad (F \in l^\infty(G \times G), g \in G).$$

Let $N \in l^\infty(G)'$ be as in Definition 5.10 and let $M = \psi'(N)$. Then let $(f_g)_{g \in G}$ be a bounded family in $\mathbb{C}$, so that

$$\langle M, (f_{gh} \Omega(g,h))_{(g,h) \in G \times G} \rangle = \langle N, (f_{eg} \Omega(g,g^{-1}))_{g \in G} \rangle = f_{eg}.$$
verifying condition (1) of Theorem 5.8 for \( M \).

Let \( f : G \times G \to \mathbb{C} \) be a bounded function, and let \( k \in G \). Then
\[
\psi\left((f(hk, g)\Omega(h, k) - f(h, kg)\Omega(k, g))_{(g, h) \in G \times G}\right)
= \left(f(g^{-1}k, g)\Omega(g^{-1}, k) - f(g^{-1}, kg)\Omega(k, g)\right)_{g \in G}.
\]
Define \( F : G \times G \to \mathbb{C} \) by
\[
F(g, h) = f(hk, g)\Omega(h, k) \quad (g, h \in G),
\]
so that \( F \) is bounded. For \( g \in G \), we have that
\[
\langle \psi(F) - J_k(\psi(F)), \delta_g \rangle
= f(g^{-1}k, g)\Omega(g^{-1}, k) - f((kg)^{-1}k, kg)\Omega((kg)^{-1}, k)\omega(kg)\omega(g)^{-1}\omega(g^{-1}k^{-1})\omega(g^{-1})^{-1}
= f(g^{-1}k, g)\Omega(g^{-1}, k) - f(g^{-1}, kg)\omega(k)^{-1}\omega(kg)\omega(g)^{-1}
= f(g^{-1}k, g)\Omega(g^{-1}, k) - f(g^{-1}k, kg)\Omega(k, g).
\]
Consequently, using condition (2) of Definition 5.10, we have established condition (2) of Theorem 5.8 for \( M \). This shows that \( l^1(G, \omega) \) is amenable. \( \square \)

Example 5.12. If \( S \) is a semigroup which is not cancellative, then it is possible for \( l^1(S) \) to be unital while \( S \) is not. For example, let \( S \) be \( (\mathbb{N}, \max) \) (where \( \mathbb{N} = \{1, 2, 3, \ldots\} \) say) with adjoined idempotents \( u \) and \( v \) such that \( uv = vu = 1 \) and \( un = nu = vn = nv = n \) for \( n \in \mathbb{N} \). Then \( S \) is a weakly cancellative, commutative semigroup without a unit, but \( e = \delta_u + \delta_v - \delta_1 \) is easily seen to be a unit for \( l^1(S) \). Indeed, \( S \) is seen to be a finite semilattice of groups, so by the result of [6], \( l^1(S) \) is amenable.

In [8] Theorem 2.3] it is shown that if \( l^1(S, \omega) \) is amenable for a cancellative, unital semigroup \( S \) and some weight \( \omega \), then \( S \) is actually a group. We shall now show that this holds for Connes-amenable as well.

For a cancellative, unital semigroup \( S \), with unit \( u_S \), if \( g \in S \) is invertible, then \( g \) has a unique inverse, denoted by \( g^{-1} \). Furthermore, if \( g \) has a left inverse, say \( hg = u_S \), then \( ghg = g = u_Sg \) so that \( gh = u_S \); similarly, if \( gh = u_S \) then \( hg = u_S \).

**Theorem 5.13.** Let \( S \) be a weakly cancellative semigroup, let \( \omega \) be a weight on \( S \), and let \( \mathcal{A} = l^1(S, \omega) \). Suppose that \( \mathcal{A} \) is Connes-amenable with respect to the predual \( c_0(S) \).

If \( S \) is cancellative or unital, then \( S \) is a group.

**Proof.** As \( \mathcal{A} \) is Connes-amenable, let \( M = (\mathcal{A} \hat{\otimes} \mathcal{A})' \) be as in Theorem 3.9. Then \( \mathcal{A} \) is unital, with unit \( e_A = (a_s)_{s \in S} \in l^1(S, \omega) \) say. For now, we shall not assume that \( e_A \) has norm one, as the standard renorming to ensure this will not (a priori) necessarily yield an \( l^1(S, \hat{\omega}) \) algebra for some weight \( \hat{\omega} \). Suppose that \( S \) is cancellative. Fix \( h \in S \), so that
\[
\sum_{s \in S} a_s \delta_{sh} \omega(s, h) = e_A \star \delta_h = \delta_h \star e_A = \sum_{s \in S} a_s \delta_{hs} \omega(h, s).
\]
In particular, for each \( h \in S \) there is a unique \( u_h \in S \) such that \( hu_h = h^2 \) implying that \( u_h h = h \), and we have that \( a_{u_h} \omega(u_h)^{-1} = 1 \). We also see that \( a_s = 0 \) for each \( s \in S \) such that \( s \neq h \), that is, \( s \neq u_h \). However, \( h \) was arbitrary, so that \( S \) is unital with unit \( u_S \), and \( e_A = \omega(u_S) \delta_{u_S} \), where we can now assume that \( \omega(u_S) = 1 \) by a renorming.

Now suppose that \( S \) is a unital, weakly cancellative semigroup, so that the unit of \( \mathcal{A} \) is \( \delta_{u_S} \). Suppose that \( s \in S \) has no right inverse. Define \( F : S \times S \to \mathbb{C} \) by
\[
F(h, sg) = 0, \quad F(hs, g) = \begin{cases} 
\Omega(g, hs) & : gh = u_S, \\
\Omega(g, h) & : otherwise.
\end{cases} \quad (g, h \in S).
\]
To show that this is well-defined, suppose that for \( g, h, j, k \in S \), we have that \( h = js, sg = k \) and \( kj = us \). Then \( s(gj) = kj = us \), so that \( s \) has a right inverse, a contradiction. Then \( F \) is bounded, so let \( T : \mathcal{A} \to \mathcal{A}' \) be the operator associated with \( F \). Then \( F(a, b) \neq 0 \) only when \( ba = s \), so as \( S \) is weakly cancellative, we see that \( T(\mathcal{A}) \subseteq c_0(S) \) and \( T'(\kappa_{\mathcal{A}}(\mathcal{A})) \subseteq c_0(S) \).

Suppose that for sequences of distinct elements \( (k_n) \subseteq S \) and \( (g_m, h_m) \subseteq S \times S \), we have that

\[
\lim_n \lim_m \langle T(\delta_{h_m}), \delta_{k_ng_m} \rangle = \lim_n \lim_m F(h_m, k_ng_m) \neq 0.
\]

Then, for some \( N > 0 \) and \( \epsilon > 0 \), for each \( n \geq N \), \( \lim_m F(h_m, k_ng_m) \geq \epsilon \). Hence, for \( n \geq N \), there exists \( M_n > 0 \) such that if \( m \geq M_n \), then \( k_ng_mh_m = s \) (as otherwise \( F(h_m, k_ng_m) = 0 \)). This, however, contradicts \( S \) being weakly cancellative. Similarly, if \( \lim_n \lim_m \langle T(\delta_{h_nk_n}), \delta_{g_m} \rangle \neq 0 \), then we need \( g_mh_mk_n = s \) for all \( n, m \) sufficiently large, which is a contradiction. Thus \( T \) satisfies all the conditions of Proposition 5.5.

Then, for \( g, h \in S \), if \( gh = us \), we have that \( \Omega(h, s)\Omega(g, hs) = \omega(h)^{-1}\omega(g)^{-1} = \Omega(g, h) \), so that

\[
F(hs, g)\Omega(h, s) - F(h, sg)\Omega(s, g) = \begin{cases} \Omega(g, h) : gh = us, \\ 0 : \text{otherwise}. \end{cases}
\]

Hence condition (2) of Theorem 5.9 implies that \( \langle M, (\delta_{gh,us}, \Omega(g, h))_{(g,h)\in S \times S} \rangle = 0 \), which contradicts condition (1) of this theorem. Hence every element of \( S \) has a right inverse.

By symmetry (or by repeating the argument on the other side) we see that every element of \( S \) has a left inverse, and hence \( S \) must be a group.

We hence have the following theorem, which shows that weighted semigroup algebras behave like \( C^* \)-algebras with regards to Connes-amenability.

**Theorem 5.14.** Let \( S \) be a discrete cancellative semigroup, and let \( \omega \) be a weight on \( S \). The following are equivalent:

1. \( l^1(S, \omega) \) is amenable;
2. \( l^1(S, \omega) \) is Connes-amenable, with respect to the predual \( c_0(S) \);

If \( l^1(S, \omega) \) is Arens regular, then these conditions are equivalent to \( l^1(S, \omega)'' \) being Connes-amenable. These equivalent conditions imply that \( S \) is a group. \( \Box \)

This result extends the result of [12], where it is shown that \( M(G) \), the measure algebra of a locally compact group \( G \), is Connes-amenable if and only if \( G \) is amenable. This follows as, for discrete groups \( G \), \( M(G) = l^1(G) \).

**Example 5.15.** Let \( \omega \) be the weight on \( \mathbb{Z} \) defined by \( \omega(n) = 1 + |n| \) for \( n \in \mathbb{Z} \). By Theorem 5.5, \( \mathcal{A} = l^1(\mathbb{Z}, \omega) \) is Arens regular. For \( m, n \in \mathbb{Z} \) and \( f = (a_k)_{k \in \mathbb{Z}} \in l^\infty(\mathbb{Z}) \), we have that

\[
\langle \delta_m \cdot f, \delta_n \rangle = \langle f, \delta_{n+m}\Omega(n, m) \rangle = f_{n+m} \frac{1 + |n + m|}{(1 + |n|)(1 + |m|)}.
\]

Suppose that \( M \diamondsuit \kappa_{\mathcal{A}}(\delta_m) = \kappa_{\mathcal{A}}(a) \) for some \( m \in \mathbb{Z} \), \( M \in l^\infty(\mathbb{Z})' \) and \( a \in \mathcal{A} \). Then \( \langle M, \delta_m \cdot f \rangle = \langle f, a \rangle \) for each \( f \in l^\infty(\mathbb{Z}) \), so by letting \( f = \kappa_{c_0(\mathbb{Z})}(e_k) \in c_0(\mathbb{Z}) \), we see that \( a = \sum_{k \in \mathbb{Z}} a_k \delta_k \), where \( a_k = \langle M, \delta_m \cdot \kappa_{c_0(\mathbb{Z})}(e_k) \rangle \). However, \( \delta_m \cdot \kappa_{c_0(\mathbb{Z})}(e_k) \in \kappa_{c_0(\mathbb{Z})}(c_0(\mathbb{Z})) \) for each \( k \in \mathbb{Z} \), so if \( M \in c_0(\mathbb{Z})' \), then \( a = 0 \).

Consequently, if \( M \diamondsuit \kappa_{\mathcal{A}}(\delta_m) \subseteq \kappa_{\mathcal{A}}(\mathcal{A}) \) for each \( m \in \mathbb{Z} \) and \( M \in l^\infty(\mathbb{Z})' \), then \( \delta_m \cdot f \in \kappa_{c_0(\mathbb{Z})}(c_0(\mathbb{Z})) \) for each \( m \in \mathbb{Z} \) and \( f \in l^\infty(\mathbb{Z}) \). However, if \( 1 \in l^\infty(\mathbb{Z}) \) is the constant 1 sequence, then

\[
\lim_n \langle \delta_m \cdot 1, \delta_n \rangle = \lim_n \frac{1 + |n + m|}{(1 + |n|)(1 + |m|)} = \frac{1}{1 + |m|}.
\]
so that $\delta_{m} \cdot 1 \not\in \kappa_{c_0(\mathbb{Z})}(c_0(\mathbb{Z}))$.

We hence conclude that $\mathcal{A}$ is not an ideal in $\mathcal{A}''$, and so we cannot apply Theorem 5.14 in this case.

Unfortunately, it is not possible for $l^1(S, \omega)$ to be both amenable and Arens regular.

**Theorem 5.16.** Let $G$ be a discrete group, and let $\omega$ be a weight on $G$. Then $l^1(G, \omega)$ is amenable if and only if $G$ is an amenable group, and $\sup\{\omega(g)\omega(g^{-1}) : g \in G\} < \infty$.

**Proof.** This is [5, Theorem 3.2].  

**Proposition 5.17.** Let $S$ be a discrete, unital semigroup, and let $\omega$ be a weight on $S$ such that $\mathcal{A} = l^1(S, \omega)$ is Arens regular. Let $K > 0$ and $B \subseteq S$ be such that for each $g \in B$, $g$ has a right inverse $g^{-1}$ (which need not be unique), and $\omega(g)\omega(g^{-1}) \leq K$. Then $B$ is finite.

**Proof.** For $g \in B$ and $h \in S$, we have

$$\omega(g)\omega(h) = \omega(g)\omega(hgg^{-1}) \leq \omega(g)\omega(hg)\omega(g^{-1}) \leq K\omega(hg),$$

so that $\Omega(h, g) \geq K^{-1}$. Suppose now that $B$ is infinite. Then we can easily construct sequences which violate condition (2) of Theorem 5.13 showing that $\mathcal{A}$ is not Arens regular. This contradiction shows that $B$ must be finite.  

**5.1 Injectivity of the predual module**

Let $S$ be a unital, weakly cancellative semigroup, let $\omega$ be a weight on $S$, and let $\mathcal{A} = l^1(S, \omega)$, $\mathcal{A}_* = c_0(S)$. Then $\mathcal{B}(\mathcal{A}, \mathcal{A}_*) = B(l^1, c_0) = l^\infty(c_0) \subseteq l^\infty(S \times S)$, where we identify $T : \mathcal{A} \to \mathcal{A}_*$ with the bounded family $((\delta_s, T(\delta_t)))_{(s,t) \in S \times S}$. Let $\phi : \mathcal{B}(\mathcal{A}, \mathcal{A}_*) \to \mathcal{A}_*$, so that $\phi$ is represented by a bounded family $(M_s)_{s \in S} \subseteq B(\mathcal{A}, \mathcal{A}_*)'$ using the relation

$$\langle \delta_s, \phi(T) \rangle = \langle M_s, T \rangle \quad (s \in S, T \in \mathcal{B}(\mathcal{A}, \mathcal{A}_*)).$$

Suppose further that $\phi$ is a left $\mathcal{A}$-module homomorphism. Then

$$\langle \delta_s, \phi(T) \rangle = \langle \delta_u, \phi(\delta_s \cdot T) \rangle = \langle M_{us}, \delta_s \cdot T \rangle = \langle M_s, T \rangle \quad (s \in S, T \in \mathcal{B}(\mathcal{A}, \mathcal{A}_*)), \tag{7}$$

so that $M_s = M_{us} \cdot \delta_s$ for each $s \in S$. We see also that $\phi$ maps into $c_0(S)$ (and not just $l^\infty(S)$) if and only if

$$\lim_{s \to \infty} \langle M_{us}, \delta_s \cdot T \rangle = 0 \quad (T \in \mathcal{B}(\mathcal{A}, \mathcal{A}_*)).$$

Conversely, if condition (7) holds, then for $s, t \in S$ and $T \in \mathcal{B}(\mathcal{A}, \mathcal{A}_*)$, we have that

$$\langle \delta_s, \phi(\delta_t \cdot T) \rangle = \langle M_s, \delta_t \cdot T \rangle = \langle M_{us}, \delta_s \cdot \delta_t \cdot T \rangle = \Omega(s, t) \langle M_{st}, T \rangle = \Omega(s, t) \langle \delta_{st}, \phi(T) \rangle = \langle \delta_s, \delta_t \cdot \phi(T) \rangle.$$

Hence $\phi$ is a left $\mathcal{A}$-module homomorphism.

Notice that $c_0(S \times S) \subseteq \mathcal{B}(\mathcal{A}, \mathcal{A}_*)$, so that $c_0(S \times S)^0 \subseteq \mathcal{B}(\mathcal{A}, \mathcal{A}_*)'$.

**Definition 5.18.** Let $G$ be a group and $\omega$ be a weight on $G$ such that for each $\epsilon > 0$, the set $\{g \in G : \omega(g)\omega(g^{-1}) < \epsilon^{-1}\}$ is finite. Then we say that the weight $\omega$ is strongly non-amenable.

**Proposition 5.19.** Let $G$ be a group, and let $\omega$ be a weight on $G$ such that $\omega$ is not strongly non-amenable, and let $\phi : \mathcal{B}(\mathcal{A}, c_0(G)) \to c_0(G)$ be a left $\mathcal{A}$-module homomorphism. If $\phi$ is represented by $(M_g)_{g \in G}$ as above, then $M_{ug} \in c_0(S \times S)^0$. 

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\textbf{Proof.} We adapt the methods of [4] to the weighted, discrete case. As \( \omega \) is not strongly non-amenable, there exists some \( K > 0 \) such that the set \( X_K = \{ g \in G : \omega(g)\omega(g^{-1}) \leq K \} \) is infinite. Let \( M = M_{\omega G} \), and suppose that \( M \not\in c_0(G \times G)^{\circ} \), so that for some \( g, h \in G \), we have that \( \delta := \langle M, e_{(g,h)} \rangle \neq 0 \). We shall henceforth treat \( e_{(g,h)} \) as a member of \( \mathcal{B}(A, c_0(G)) \), noting that for \( k \in G \),

\[
\langle \delta_s, (\delta_k \cdot e_{(g,h)})(\delta_l) \rangle = \begin{cases} 
\Omega(t, k) & : s = g, t = h k^{-1}, \\
0 & : \text{otherwise}.
\end{cases}
\]

We claim that we can find a sequence \((g_n)_{n \in \mathbb{N}}\) of distinct elements in \( G \) such that

\[
|\langle M \cdot g_{m^{-1}}^{n}, e_{(g,h)} \rangle| \leq K^{-1} 2^{-2 - |m-n|} \quad (n \neq m),
\]

\[
\omega(g_n)\omega(g_{n}^{-1}) \leq K \quad (n \in \mathbb{N}).
\]

We can do this as \( \phi \) must map into \( c_0(G) \), so that for any \( T : A \to c_0(G) \), we have \( \lim_{g \to \infty} \langle M \cdot \delta_g, T \rangle = 0 \). Explicitly, let \( g_1 \in X_K \) be arbitrary, and suppose that we have found \( g_1, \ldots, g_k \). Then notice that the sets

\[
\{ s \in G : |\langle M \cdot \delta_{g_{m^{-1}}^{n}}, e_{(g,h)} \rangle| > K^{-1} 2^{-2 - |k+1-n|} : 1 \leq m \leq k \},
\]

\[
\{ s \in G : |\langle M \cdot \delta_{g_{m^{-1}}^{n}}, e_{(g,h)} \rangle| > K^{-1} 2^{-2 - |k+1-m|} : 1 \leq n \leq k \}
\]

are finite, so as \( X_K \) is infinite, we can certainly find some \( x_{k+1} \).

Then, for \( x = (x_n) \in l^\infty(\mathbb{N}) \), define \( T_x : A \to c_0(G) \) by setting \( \langle \delta_g, T_x(\delta_{g_n^{-1}}) \rangle = x_n\Omega(h g_{n^{-1}}, g_n) \) for \( n \geq 1 \), and \( \langle \delta_s, T_x(\delta_l) \rangle = 0 \) otherwise. Then clearly \( T_x \) does map into \( c_0(G) \), and \( \|T_x\| \leq \|x\| \). Notice that for \( s, t \in G \), we have

\[
\langle \delta_s, T_x(\delta_l) \rangle = \begin{cases} 
x_n\Omega(t, g_n) & : s = g, t = h g_{n}^{-1}, \\
0 & : \text{otherwise},
\end{cases}
\]

\[
= \sum_n x_n \langle \delta_s, (\delta_{g_n^{-1}} \cdot e_{(g,h)})(\delta_l) \rangle.
\]

Define \( Q : l^\infty(\mathbb{N}) \to c_0(\mathbb{N}) \) by

\[
\langle \delta_n, Q(x) \rangle = \langle M, \delta_{g_{n^{-1}}^{-1}} \cdot T_x \rangle \quad (n \in \mathbb{N}),
\]

so that \( Q \) is bounded and linear.

Let \( n_0 \geq 1 \) and let \( x = e_{n_0} \in c_0(\mathbb{N}) \subseteq l^\infty(\mathbb{N}) \). Then, \( T_x = \delta_{g_{n_0}^{-1}} \cdot e_{(g,h)} \), so that

\[
\langle \delta_n, Q(x) \rangle = \langle M, \delta_{g_{n_0}^{-1}} \cdot T_x \rangle = \langle M, \delta_{g_{n_0}^{-1}} \cdot (\delta_{g_{n_0}^{-1}} \cdot e_{(g,h)}) \rangle
\]

\[
= \begin{cases} 
\delta \Omega(g_{n_0}^{-1}, g_{n_0}) & : n = n_0, \\
\Omega(g_{n_0}^{-1}, g_{n_0}) \langle M \cdot e_{g_{n_0}^{-1}}, e_{(g,h)} \rangle & : n \neq n_0.
\end{cases}
\]

Define \( Q_1 \in \mathcal{B}(c_0(\mathbb{N})) \) by

\[
Q_1(x) = (\Omega(g_{n_0}^{-1}, g_{n_0}) x_n)_{n \in \mathbb{N}} \quad (x = (x_n) \in c_0(\mathbb{N})).
\]

Then, as each \( g_n \in X_K \), \( Q_1 \) is an invertible operator. Let \( Q_2 \) be the restriction of \( Q \) to \( c_0(\mathbb{N}) \), so that \( Q_2 \in \mathcal{B}(c_0(\mathbb{N})) \) and \( Q_2 = \delta Q_1 + \delta Q_3 Q_1 \) for some \( Q_3 \in \mathcal{B}(c_0(\mathbb{N})) \). Thus
\[ Q_3 = \delta^{-1}Q_2Q_1^{-1} - I_{c_0(\mathbb{N})}, \] so that for \( x \in c_0(\mathbb{N}) \), we have that
\[
\|Q_3(x)\| = \sup_n |\langle \delta_n, \delta^{-1}Q_2Q_1^{-1}(x) - x \rangle| = \sup_n \left| \sum_m x_m \langle \delta_n, \delta^{-1}Q_2Q_1^{-1}(e_m) - e_m \rangle \right|
\]
\[
= \sup_n \left| \sum_{m \neq n} x_m \Omega(g^{-1}_m, g^{-1}_m) \xi(g^{-1}_m, g^{-1}_m) \langle M \cdot \delta^{-1}g^{-1}_m, e(g,h) \rangle \right|
\]
\[
\leq K^{-1} \sup_n \sum_{m \neq n} |x_m| 2^{-2|m-n|} \omega(g_m) \omega(g^{-1}_m) \leq \|x\|/2.
\]

Consequently \( Q_3 - I_{c_0(\mathbb{N})} \) is invertible, so that \( Q_2Q_1^{-1} \) is invertible, showing that \( Q_2 \) is invertible. However, this implies that \( Q_2^{-1}Q : l^\infty(\mathbb{N}) \to c_0(\mathbb{N}) \) is a projection, which is a well-known contradiction, completing the proof. \( \square \)

**Theorem 5.20.** Let \( G \) be a countable group, let \( \omega \) be a weight that is not strongly non-amenable, and let \( A = l^1(G, \omega) \). Then \( c_0(G) \) is not left-injective.

**Proof.** Suppose, towards a contradiction, that \( c_0(G) \) is left-injective, so that there exists \( M = M_{uc} \in \mathcal{B}(A, A_\omega)' \) as above, with the additional condition that
\[
\delta_g = \langle \delta_g, \phi_A(\epsilon_h) \rangle = \langle M, \delta_g \cdot \phi_A(\epsilon_h) \rangle = \Omega(hg^{-1}, g) \langle M, \phi_A(\epsilon_{hg^{-1}}) \rangle
\]
\[
= \Omega(hg^{-1}, g) \langle M, \delta_{s,t,g} \rangle_{s,t \in G \times G} (g, h \in G).
\]

This clearly reduces to
\[
\delta_{g,uc} = \langle M, \delta_{s,t,g} \rangle_{s,t \in G \times G} (g \in G).
\]

As \( G \) is countable, we can enumerate \( G \) as \( G = \{ g_n : n \in \mathbb{N} \} \). Then, for \( g_n \in G \), let \( X_{g_n} = \{ g_1, \ldots, g_n \} \subseteq G \). Define \( Q : l^\infty(G) \to \mathcal{B}(A, c_0(G)) \) by
\[
\langle s, Q(x)(\delta_t) \rangle = \Omega(s, t) \sum_{g \in X_t} x_g \delta_{s,t,g} \quad (s, t \in G, x \in l^\infty(G)).
\]

Then, for each \( t \in G \), as \( X_t \) is finite, we see that \( Q(x)(\delta_t) \in c_0(G) \), so \( Q \) is well-defined. Clearly \( Q \) is linear, and we see that for \( x \in l^\infty(G) \),
\[
\|Q(x)\| = \sup_{s,t \in G} \Omega(s, t) \sum_{g \in X_t} x_g |\delta_{s,t,g}| \leq \sup_{s,t \in G} \sum_{g \in X_t : g = st} |x_g| = \|x\|
\]
so that \( Q \) is norm-decreasing. Then, for \( h \in G \), we have that
\[
\langle s, Q(\epsilon_h)(\delta_t) \rangle = \Omega(s, t) \sum_{g \in X_t} \delta_{g,h} \delta_{s,t,g} = \begin{cases} \langle s, \phi(\epsilon_h)(\delta_t) \rangle : h \in X_t, \\ 0 : h \notin X_t. \end{cases}
\]

Let \( h = g_{n_0} \), so that \( \{ t \in G : h \notin X_t \} = \{ g_n \in G : h \notin X_{g_n} \} = \{ g_1, g_2, \ldots, g_{n_0-1} \} \). We hence see that \( Q(\epsilon_{g_0}) - \phi_A(\epsilon_{g_0}) \in c_0(G \times G) \). By the preceding proposition, we hence have that \( I_{c_0(G)} = \phi \circ \phi_A = \phi \circ (Q|_{c_0(G)}) \). However, this implies that \( \phi \circ Q : l^\infty(G) \to c_0(G) \) is a projection onto \( c_0(G) \), giving us the required contradiction. \( \square \)

**Theorem 5.21.** Let \( S \) be a discrete, weakly cancellative semigroup, let \( \omega \) be a weight on \( S \), and let \( A = l^1(S, \omega) \). When \( S \) is unital, or \( S \) is cancellative, \( c_0(G) \) is not a bi-injective \( A \)-bimodule.
Proof. Suppose, towards a contradiction, that $c_0(G)$ is bi-injective. Then $\mathcal{A}$ is Connes-amenable, so that Theorem 5.13 implies that $\mathcal{A}$ is amenable, and that $S = G$ is a group. By Theorem 5.16, we know that $\omega$ is not strongly non-amenable. Suppose that $G$ is countable, so that the above theorem shows that $c_0(G)$ is not left-injective, and that hence $c_0(G)$ is certainly not bi-injective, a contradiction.

Suppose that $G$ is not countable. Then let $H$ be some countably infinite subgroup of $G$. Let $K = \sup\{\omega(g)\omega(g^{-1}) : g \in G\} < \infty$, and let $g, h \in G$. Then

$$
\Omega(g, h) = \frac{\omega(gh)}{\omega(g)\omega(h)} = \frac{\omega(gh)}{\omega(g)\omega(g^{-1}gh)} \geq \frac{\omega(gh)}{\omega(g)\omega(g^{-1})\omega(gh)} = \frac{1}{\omega(g)\omega(g^{-1})} \geq K^{-1},
$$

so that $\Omega$ is bounded below on $G \times G$, and hence on $H \times H$.

Then we can find $X \subseteq G$ such that $G = \bigcup_{x \in X} Hx$ and $Hx \cap Hy = \emptyset$ for distinct $x, y \in X$. Notice that if $g \in Hx$ then $g^{-1} \in x^{-1}H$, so that $G = \bigcup_{x \in X} x^{-1}H$ as well.

By the proof of Theorem 5.20, we see that $c_0(H)$ is not a left-injective $l^1(H, \omega)$-module. Suppose, towards a contradiction, that we do have some left $\mathcal{A}$-module homomorphism $\phi : B(l^1(G, \omega), c_0(G)) \rightarrow c_0(G)$ with $\phi \Delta_A = I_A$. Notice that certainly $B(l^1(G, \omega), c_0(G))$ and $c_0(G)$ are Banach left $l^1(H, \omega)$-modules, by restricting the action from $l^1(G, \omega)$.

Define a map $\psi : B(l^1(H, \omega), c_0(H)) \rightarrow B(l^1(G, \omega), c_0(G))$ by, for $g, k \in G$,

$$
\langle \delta_g, \psi(T)(\delta_k) \rangle = \begin{cases} 
\frac{\omega(s)\omega(t)}{\omega(tx)\omega(x^{-1}s)} (\delta_t, T(\delta_s)) & : g = tx, k = x^{-1}s \text{ for some } x \in X, s, t \in H, \\
0 & : \text{otherwise.}
\end{cases}
$$

Certainly $\psi$ is linear, while

$$
\|\psi(T)\| \leq \|T\| \sup_{s, t \in H, x \in X} \frac{\omega(s)\omega(t)}{\omega(tx)\omega(x^{-1}s)} \leq \|T\| \sup_{s, t \in H, x \in X} \frac{\omega(s)\omega(t)}{\omega(tx)\omega(x^{-1}s)} = \|T\| \sup_{s, t \in H} \Omega(t, s)^{-1},
$$

so that $\psi$ is bounded. For $h, s, t \in H$, and $x \in X$, we have

$$
\langle \delta_{tx}, (\delta_h \cdot \psi(T))(\delta_{x^{-1}s}) \rangle = \Omega(x^{-1}s, h)\langle \delta_{tx}, \psi(T)(\delta_{x^{-1}s}) \rangle \\
= \frac{\Omega(x^{-1}s, h)\omega(sh)\omega(t)}{\omega(tx)\omega(x^{-1}s)} (\delta_t, T(\delta_s)) = \frac{\omega(sh)\omega(t)}{\omega(x^{-1}s)\omega(h)\omega(tx)} (\delta_t, T(\delta_s)) \\
= \omega(s)\omega(x^{-1}s)\omega(t)\omega(tx)^{-1} (\delta_t, (\delta_h \cdot T)(\delta_s)) = \langle \delta_{tx}, \psi(\delta_h \cdot T)(\delta_{x^{-1}s}) \rangle.
$$

Thus $\psi$ is a left $l^1(H, \omega)$-module homomorphism. For $h, s, t \in H$ and $x \in X$, we then have that

$$
\langle \delta_{tx}, \psi(\Delta_{l^1(H, \omega)}(e_h))(\delta_{x^{-1}s}) \rangle = \frac{\omega(t)\omega(s)}{\omega(x^{-1}s)\omega(tx)} (\delta_t, \delta_s, e_h) = \Omega(tx, x^{-1}s)\delta_{ts,h} \\
= \langle \delta_{tx}, \delta_{x^{-1}s} \cdot e_h \rangle = \langle \delta_{tx}, \Delta_A'(e_h)(\delta_{x^{-1}s}) \rangle.
$$

If $g, k \in G$ are such that $gk \not\in H$ then $g = tx$ and $k = y^{-1}s$ for some $s, t \in H$ and distinct $x, y \in X$. Then, for $h \in H$, we have that $gk \neq h$, so that

$$
\langle \delta_g, \Delta_A'(e_h)(\delta_k) \rangle = \Omega(g, k)\delta_{gk,h} = 0 = \langle \delta_g, \psi(\Delta_{l^1(H, \omega)}(e_h))(\delta_k) \rangle.
$$

Hence $\psi \circ \Delta_A'(e_h)$ is equal to $\Delta_A'$ restricted to $l^1(H, \omega)$.

Let $P : c_0(G) \rightarrow c_0(H)$ be the natural projection, which is obviously an $l^1(H, \omega)$-module homomorphism. Then $Q = P \circ \phi \circ \psi : B(l^1(H, \omega), c_0(H)) \rightarrow c_0(H)$ is a bounded left $l^1(H, \omega)$-module homomorphism, and $Q \circ \Delta_{l^1(H, \omega)} = I_{c_0(H)}$. This contradiction completes the proof. \qed
We note that just because $\Omega$ is bounded below does not imply that $\omega$ is bounded, so that $l^1(G, \omega)$ is not necessarily isomorphic to $l^1(G)$, and hence we cannot simply apply the results of [4].

We have not been able to establish if $c_0(S)$ can every be a left-injective $l^1(S, \omega)$-module for some semigroup $S$ and weight $\omega$.

6 Open questions

We state a few open questions of interest:

1. Let $\mathcal{A}$ be an Arens regular Banach algebra such that $\mathcal{A}''$ is Connes-amenable. Need $\mathcal{A}$ be amenable?

2. This is true for $C^*$-algebras. Can we find a “simple” proof?

3. Let $\mathcal{A}$ be a dual Banach algebra with predual $\mathcal{A}_*$, and suppose that $\mathcal{A}_*$ is bi-injective. If $\mathcal{A}$ necessarily a von Neumann algebra or the bidual of an Arens regular Banach algebra $\mathcal{B}$ such that $\mathcal{B}$ is an ideal in $\mathcal{A}$?

4. Let $S$ be a (weakly cancellative) semigroup, and let $\omega$ be a weight on $S$. Classify (up to isomorphism) the preduals of $l^1(S, \omega)$, and calculate which preduals yield a Connes-amenable Banach algebra.

5. This question was asked by Niels Grönbæk. In most of our examples, it is obvious that when $\mathcal{A}$ is a Connes-amenable dual Banach algebra, there is $\mathcal{B} \subseteq \mathcal{A}$ which is weak*-dense and amenable. Is this always true?

References

[1] I. G. Craw, N. J. Young, ‘Regularity of multiplication in weighted group and semigroup algebras’, Quart. J. Math. Oxford 25 (1974) 351–358.

[2] H. G. Dales, Banach algebras and automatic continuity, (Clarendon Press, Oxford, 2000).

[3] H. G. Dales, A. T.-M. Lau, ‘The second dual of Beurling algebras’, preprint.

[4] H. G. Dales, M. E. Polyakov, ‘Homological properties of modules over group algebras’, Proc. Lon. Math. Soc. 89 (2004) 390–426.

[5] N. Grönbæk, ‘Amenability of weighted discrete convolution algebras on cancellative semigroups’, Proc. Roy. Soc. Edinburgh Sect. A 110 (1988) 351–360.

[6] N. Grönbæk, ‘Amenability of discrete convolution algebras, the commutative case’, Pacific J. Math. 143 (1990) 243–249.

[7] A. Ya. Helemskii, ‘Homological essence of amenability in the sense of A. Connes: the injectivity of the predual bimodule.’, Math. USSR-Sb. 68 (1991) 555–566.

[8] A. Ya. Helemskii, Banach and locally convex algebras, (Oxford Science Publications, New York, 1993).

[9] A. Ya. Helemskii, ‘Some aspects of topological homology since 1995: a survey’, ‘Banach algebras and their applications’ in Contemp. Math. 363 (2004) 145–179.
[10] V. Runde, ‘Dual Banach algebras: Connes-amenability, normal, virtual diagonals, and injectivity of the predual bimodule’, Math. Scand. 95 (2004) 124–144.

[11] V. Runde, ‘Connes-amenability and normal, virtual diagonals for measure algebras. I.’, J. London Math. Soc. 67 (2003) 643–656.

[12] V. Runde, ‘Connes-amenability and normal, virtual diagonals for measure algebras, II’, Bull. Austral. Math. Soc. 68 (2003) 325–328.

[13] V. Runde, Lectures on amenability, (Springer-Verlag, Berlin, 2002).

[14] V. Runde, ‘Amenability for dual Banach algebras’, Studia Math. 148 (2001) 47–66.

[15] S. B. Tabaldyev, ‘Noninjectivity of the predual bimodule of the measure algebra of infinite discrete groups.’, Math. Notes 73 (2003) 690–696.

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