On Global Existence and Blow-up for Damped Stochastic Nonlinear Schrödinger Equation

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Abstract. In this paper, we consider the well-posedness of the weakly damped stochastic nonlinear Schrödinger (NLS) equation driven by multiplicative noise. First, we show the global existence of the unique solution for the damped stochastic NLS equation in critical case. Meanwhile, the exponential integrability of the solution is proved, which implies the continuous dependence on the initial data. Then, we analyze the effect of the damped term and noise on the blow-up phenomenon. By modifying the associated energy, momentum and variance identity, we deduce a sharp blow-up condition for damped stochastic NLS equation in supercritical case. Moreover, we show that when the damped effect is large enough, the damped effect can prevent the blow-up of the solution with high probability.

1. Introduction

The nonlinear Schrödinger equation, as one of the basic models for nonlinear waves, has many physical applications to, e.g. nonlinear optics, plasma physics and quantum field theory and so on (see e.g. [3, 5, 10, 12, 17]).

In this paper, we consider the weakly damped stochastic NLS equation driven by a linear multiplicative noise in focusing mass-(super)critical range,

\[ du = i(\Delta u + \lambda |u|^{2\sigma} u)dt - au \circ dW(t), \]
\[ u(0) = u_0, \]

where \( \frac{2}{d} \leq \sigma < \frac{2}{(d-2)+}, \lambda = 1, a \geq 0, x \in \mathbb{R}^d, t \geq 0 \) and “\( \circ \)” stands for a Stratonovich product. Here \( W = \{ W(t) : t \in [0, T]\} \) is an \( L^2(\mathbb{R}^d) \)-valued \( Q \)-Wiener process on a stochastic basis \( (\Omega, \mathcal{F}, \mathcal{F}_t, P) \), i.e., there exists an orthonormal basis \( \{ e_k \}_{k \in \mathbb{N}^+} \) of \( L^2(\mathbb{R}^d) \) and a sequence of mutually independent, real-valued Brownian motions \( \{ \beta_k \}_{k \in \mathbb{N}^+} \) such that \( W(t) = \sum_{k \in \mathbb{N}^+} Q_k e_k \beta_k(t), t \in [0, T] \). We will use frequently the equivalent Itô form of Eq. (1)

\[ du = i(\Delta u + |u|^{2\sigma} u)dt - (a + \frac{1}{2}F_Q)udt + iudW(t), \]
\[ u(0) = u_0. \]
with \( F_Q := \sum_{k \in \mathbb{N}^+} (Q^2 e_k)^2 \).

In recent twenty years, much effort has been devoted to studying the well-posedness of stochastic NLS equation, see \([1, 4, 10, 12, 14]\) and the references therein. In \([10]\) and \([12]\), the local and global existence of the mild solution of stochastic NLS equation in \( L^2 \) and in \( H^1 \) are investigated, respectively. For stochastic NLS equation on a manifold, \([4]\) considers the existence and uniqueness of the solution based on the stochastic Strichartz estimate in UMD Banach space. \([14]\) shows the local well-posedness in \( L^2 \) via stochastic Strichartz estimates and the global well-posedness in subcritical case. With the help of the rescaling transformation, \([1]\) obtains that the local well-posedness in \( H^1 \) for \( \sigma < \frac{2}{(d-2)^2} \) with some decay conditions on the noise, and that the global existence when \( \lambda = -1, \sigma < \frac{2}{(d-2)^2} \) or \( \lambda = 1, \sigma < \frac{2}{d} \). The global existence of the solution in \( H^2 \) is presented in \([8]\) for one dimensional stochastic NLS equation driven by linear multiplicative noises.

For the global existence of the solution of the NLS equation in critical case, there exist much more results in the deterministic case than those in stochastic case. For instance, in the deterministic case, \([19]\) finds a threshold \( R \) by the optimal constant of Gagliardo–Nirenberg’s inequality, and proves the global existence of the solution when \( \sigma = \frac{2}{d} \) and \( \|u_0\| < \|R\| \). After adding the noise and damped effect, we wonder whether these effect will influence the existence of a threshold in mass critical case for the damped stochastic NLS equation, which is one of the main interests in this paper.

The another main interest is to study the damped effect and the noise effect on the blow-up in the focusing mass-(super)critical case. It’s well known that the solution of deterministic NLS equation with \( a = 0 \) in the focusing mass-(super)critical case will blow up in some finite time when \( u_0 \) possesses some negative Hamiltonian, see \([3, 5, 17]\) and references therein. When \( a > 0 \), the damped term has the effect to delay the blow-up, see \([16, 17, 18]\) and references therein. For instance, for \( \sigma > \frac{2}{d} \), the blow-up may occur for small values of \( a \) (see e.g. \([18]\)) and large enough values of \( a \) can ensure the global existence of the solution for \( \sigma \geq \frac{2}{d} \) (see e.g. \([16]\)). In stochastic case, the noise also has an impact on blow-up solutions. \([13]\) shows that the noise effect can accelerate the formation of singularity, and that the solution of Eq. (1) with \( a = 0 \) in focusing supercritical case will blow up in a finite time with a positive probability when the variance of the initial datum is finite. The blow-up solution for the stochastic NLS equation driven by additive noises is considered in \([11]\). When the noise of stochastic NLS equation is non-conservative, \([2]\) shows that adding a large multiplicative Gaussian noise can prevent the blow-up in any finite time with high probability.

Throughout this paper, we assume that the local-wellposedness of the solution of Eq. (1) holds. The local solution \( u(\cdot) \) is defined on a random interval \([0, \tau^*(u_0, \omega))\), where \( \tau^*(u_0, \omega) \) is a stopping time such that

\[
\tau^*(u_0, \omega) = +\infty, \quad \text{or} \quad \lim_{t \to \tau^*(u_0, \omega)} \|u(t)\|_{H^1} = +\infty.
\]

First, the evolution of charge and energy of the local solution are introduced. By using the optimal constant of Gagliardo–Nirenberg’s inequality, we show the a priori estimation in \( H^1 \)-norm, and prove that the threshold \( R \) is unchanged when \( a \geq 0, \sigma = \frac{d}{2} \) and initial datum is deterministic. Moreover, based on the proved
exponential integrability of the solution $u$, i.e.,

$$\sup_{t \in [0, \infty)} \mathbb{E} \left[ \exp \left( \frac{\|u(t)\|_{H^1}^2}{e^{\alpha t}} \right) \right] \leq C(u_0, a, Q)$$

with $\alpha$ depending on $u_0, a$ and $Q$, we obtain the strong continuous dependence on the initial data in one dimensional case, which is not a trivial property for stochastic partial differential equation with non-global coefficients, see [6, 8] and references therein. We would like to mention that this exponential integrability is useful for studying the continuous dependence on noises, exponential tail estimate of the solution, strong and weak convergence rates of numerical approximations, see [6, 7, 8, 9, 15] and references therein.

Next we consider the influence of damped term and noise on the blow-up. For the damped stochastic NLS equation, that is, $a > 0$, the method used in [13] to get the blow-up condition is not available since the variance identity of Eq. (1) do not have a polynomial expansion. To overcome this difficulty, we modify the energy, momentum and variance identity which is similar to [18], and deduce a sharp blow-up condition. Indeed, we show that under some mild assumptions on $u_0$ and $Q$, if there exist $z \geq 4a\sigma\sigma_d - 2$ and $\bar{t}$ such that

$$\mathbb{E} \left[ V(u_0) \right] + 4t\mathbb{E} \left[ G(u_0) \right] + \left(8t^2 + \frac{8}{3}t^3\right)\mathbb{E} \left[ H(u_0) \right] + \left(\frac{4}{3}t^3 + \frac{4}{3}z^2t^4\right)\mathbb{E} \left[ \|u_0\|^2 \right] \mathbb{E} \left[ \|f_\mathbf{Q}\|_{L^\infty} \right] \leq 0,$$

where $f_\mathbf{Q} = \sum_{k \in \mathbb{N}^+} |\nabla Q^{\frac{2}{d}}e_k|^2$, then

$$\mathbb{P}(\tau^*(u_0) \leq \bar{t}) > 0.$$

This implies that no matter how large the damped effect is, the blow-up phenomenon will not disappear. We remark that the above blow-up condition can be degenerated to the blow-up condition in conservative stochastic case and in the deterministic case. On the other hand, if the noise satisfies more conditions, using the rescaling transform idea in [1], we prove that when $a \to \infty$ and $\sigma \geq \frac{2}{d}$, for any fixed time $T$, the blow-up of the solution does not happen with probability 1.

This paper is organized as follows. In Section 2, we study the evolution of charge and energy, and show the global existence of the unique solution. In Section 3, the modified variance identity is given. Based on it, we obtain a sharp blow-up condition. Furthermore, we prove that when the value of the damped coefficient $a$ becomes large enough, the solution does not blow up at any finite time with high probability. At last, We give a short conclusion in Section 4.

2. Global existence of solutions for critical stochastic NLS equations

In this section, we focus on the global existence and some properties of the solution for Eq. (1). Throughout this paper, we assume that the local well-posedness for Eq. (1) holds. For the local well-posedness for Eq. (1), we refer to [1, 12, 14] and references therein. When consider the focusing mass-(super)critical case, [11] proves that the solution of Eq. (1) blows up with any initial data for the additive case. For the stochastic NLS driven by the multiplicative noise, similar situation happens with any initial datum in the super-critical case (see e.g. [2, 13]). This phenomenon is different from the deterministic case, where the solution will blow
up in some finite time when $u_0$ possesses some negative Hamiltonian in the focusing mass-(super)critical case (see e.g. \cite{3, 5, 17}). However, it is still not clear on whether or not the solution of Eq. (1) equation globally exists in critical case. Notice that when the noise is independent of space and $a = 0$, i.e.,

$$du = i(Δu + |u|^{2σ}u)dt - \frac{1}{2}udt + iudβ(t),$$

$$u(0) = u_0,$$

the global existence and blow-up results become more clear. In this case, one can use the infinite dimensional Doss–Sussman type transformation $u(t) = \exp(iβ(t))y(t)$ to get the well-posedness and blow-up results, where $y(t)$ satisfies

$$dy = iΔydt + iy|^{2σ}ydt,$$

$$y(0) = u_0.$$

We first study the global existence of the solution of Eq. (1) in the focusing critical case. This suggests that the critical nonlinearity in multiplicative cases is different from the supercritical nonlinearity, and that the critical nonlinearity combined with the dispersion term dominates the behavior of the solution. For convenience, we assume that $u_0 ∈ H^1$ is a deterministic function and that $\sum_k ||Q^2_k e_k||^2_{H^1} + ||f_Q||_{L^∞} < ∞$ with $f_Q = \sum_k |∇Q_k^2 e_k|^2$. The solution $u(·)$ of Eq. (1) is defined on a random interval $[0, τ^∗(u_0, ω))$, where $τ^∗(u_0, ω)$ is a stopping time such that

$$τ^∗(u_0, ω) = +∞,$$

$$\lim_{t→τ^∗(u_0, ω)} \|u(t)\|_{H^1} = +∞.$$

To get a priori estimate of $u$, we first study the evolution of charge $M(u(t)) := \|u(t)\|^2$ and energy $H(u) := \frac{1}{2}\|∇u\|^2 - \frac{λ}{2σ+2}\|u\|^{2σ+2}_{L^{2σ+2}}$ in the following lemma.

**Lemma 2.1.** Assume that $u_0 ∈ H^1$ and $\sum_k ||Q_k^2 e_k||^2_{H^1}, + ||f_Q||_{L^∞} < ∞$. For any $τ < τ^∗(u_0)$, we have

$$M(u(τ)) = e^{-2ατ}M(u_0), \quad a.s.,$$

and

$$H(u(τ)) = H(u_0) - a\int_0^τ (||∇u(s)||^2 - ||u(s)||^{2σ+2}_{L^{2σ+2}})ds$$

$$-\text{Im}\int_0^τ \int_{Ω^d} \bar{u}(s)∇u(s)∇dw(s)dx$$

$$+ \frac{1}{2}\sum_{k ∈ N^+} \int_0^τ ||u(s)||^2_{H^1}||Q_k^2 e_k||^2ds, \quad a.s.$$

**Proof** For purpose of obtaining the charge and energy evolution of $u$, the truncated argument in \cite{14} is applied. In detail, let $N ∈ N^+$ and $K > 0$ and define the operators $Θ_N, N ∈ N$ by

$$Θ_Nv := F^{-1}\left(θ\left(\frac{|.|}{K}\right) * N\right),$$

where $F$ is the Fourier transform and $θ ∈ C^∞_c$ is a real-valued and nonnegative function satisfying $θ(x) = 1$ for $|x| ≤ 1$, $θ(x) = 0$ for $|x| > 2$. Using the above
notation, we have the truncated approximation, for \( m = (m_1, m_2) \in \mathbb{N}^2 \),

\[
\begin{align*}
du^m_K &= i \left( \Theta_{m_1} \Delta u^m_K + \theta \left( \frac{\|\nabla u^m_K\|_{H^1}}{K} \right) \Theta_{m_2} \left( \|u^m_K\|^{2\sigma} u^m_K - au^m_K - \frac{F_{Q_{m_2}}}{2} u^m_K \right) \right) dt \\
&\quad + iu^m_K \Theta_{m_2} dW(t),
\end{align*}
\]

where \( F_{Q_{m_2}} := \sum_{k \in \mathbb{N}^+} (\Theta_{m_2}(Q_{e_k}))^2 \). Combining with Itô formula in \([0, T]\) and taking limits as \( m \to \infty \), the evolution of the charge \( (2) \) is obtained by choosing a large enough \( K \). Similarly, using the above arguments, the energy evolution law \( (3) \) can be proved. \( \square \)

**Remark 2.1.** The truncated argument is also available for stochastic NLS equation with the homogeneous Dirichlet boundary condition. In this case, replacing \( \Theta_N \) by the projection operator \( P^N \), then the truncated Galerkin approximated equation becomes

\[
\begin{align*}
du^N_K &= i \left( \Delta u^N_K + \theta \left( \frac{\|\nabla u^N_K\|_{H^1}}{K} \right) P^N \left( \|u^N_K\|^{2\sigma} u^N_K - aP^N u^N_K - P^N \left( \frac{F_Q}{2} u^N_K \right) \right) \right) dt \\
&\quad + iP^N (u^N_K dW(t)),
\end{align*}
\]

where \( K > 0, N \in \mathbb{N}^+ \). The inverse inequality, for \( s \geq 1 \),

\[
\|u^N_K\|_{H^s} \leq C(N)\|u^N_K\|_{H^1}
\]

implies the coefficients of Eq. \( (5) \) are globally Lipschitz. Therefore, by the arguments in \( [12] \), the result of Lemma \( (2.2) \) holds.

In order to illustrate the global well-posedness result, we introduce the optimal constant for Gagliardo–Nirenberg inequality and its corresponding ground state solution (see e.g. \([19]\)).

**Lemma 2.2.** The best constant \( C_{\sigma,d} \) for Gagliardo–Nirenberg inequality

\[
\|f\|_{L^{2\sigma+2}}^{2\sigma+2} \leq C_{\sigma,d} \|\nabla f\|^{\sigma} \|f\|^{2+\sigma(2-d)}
\]

with \( f \in \mathbb{H}^1(\mathbb{R}^d) \), \( 0 < \sigma < \frac{2}{d-2} \) and \( d \geq 2 \) is given by

\[
C_{\sigma,d} = (\sigma + 1) \left( \frac{2(2 + 2\sigma - \sigma d)}{(\sigma d)^{\frac{2d}{d-2}}} \right) \frac{1}{\|R\|^{2\sigma}},
\]

where \( R \) is the positive solution (ground state solution) of \( \Delta R - R + R^{2\sigma+1} = 0 \).

Based on Lemma \( (2.1) \) and Lemma \( (2.2) \) we are in position to show the global existence of \( u \). For the sake of simplicity, the procedure about truncated arguments and taking limits is omitted in the rest of this paper.

**Theorem 2.1.** Assume that \( u_0 \in \mathbb{H}^1 \) with \( \|u_0\| < \|R\|, \sum_k \|Q_k^2 e_k\|_{L^1}^2 + \|f_Q\|_{L^\infty} < \infty \). Then there exists a unique global solution of Eq. \( (1) \) in \( \mathbb{H}^1 \), i.e., \( \tau^*(u_0) = \infty \).

**Proof** Since the local well-posedness of Eq. \( (1) \) is shown by \([12, 4, 1]\), we only need to get the uniform boundedness of \( \|u\|_{\mathbb{H}^1} \) to ensure the global existence of the solution. By the charge conservation law, Gagliardo–Nirenberg inequality and \( \sigma d = 2 \),

\[
\|\nabla u\|^2 \leq 2H(u) + \frac{1}{\sigma + 1} C_{\sigma,d} \|u\|^{2\sigma} \|\nabla u\|^2,
\]
where $\sigma^{\frac{\sigma+1}{2\sigma}}$ and $R$ is the ground state solution of $\Delta R - R + R^{2\sigma+1} = 0$. Then the energy evolution of $u$ implies that for any $T_0 > 0$, any stopping time $\tau < \inf(T_0, \tau^*(u_0))$ and any time $t \leq \tau$,

\[
\begin{aligned}
(1 - \frac{\|u(t)\|^{2\sigma}}{\|R\|^{2\sigma}}) \|\nabla u(t)\|^2 &\leq 2H(u_0) - 2a \int_0^t (\|\nabla u(s)\|^2 - \|u(s)\|^{2\sigma + 2}) ds \\
&- 2\text{Im} \int_0^t \int_\Omega \overline{u(s)} \nabla u(s) \nabla dW(s) dx \\
&+ \sum_{k \in \mathbb{N}^+} \int_0^t \int_{\sigma} |u(s)|^2 |\nabla Q^\frac{1}{2} e_k|^2 dx ds,
\end{aligned}
\]

where $H(u_0) > 0$ since $\|u_0\| < \|R\|$. After taking expectation, we have

\[
\begin{aligned}
(1 - \frac{\|u_0\|^{2\sigma}}{\|R\|^{2\sigma}}) \mathbb{E}[\|\nabla u(t)\|^2] &\leq 2H(u_0) - 2a \int_0^t \mathbb{E}[H(u(s))] ds \\
&+ \int_0^t 2a\sigma \sigma + 1 \mathbb{E}[\|u(s)\|^{2\sigma + 2}] ds \\
&+ \mathbb{E} \left( \sum_{k \in \mathbb{N}^+} \int_0^t \int_{\sigma} |u(s)|^2 |\nabla Q^\frac{1}{2} e_k|^2 dx ds \right).
\end{aligned}
\]

Then Hölder inequality and Sobolev embedding theorem yield that

\[
\begin{aligned}
(1 - \frac{\|u_0\|^{2\sigma}}{\|R\|^{2\sigma}}) \mathbb{E}[\|\nabla u(t)\|^2] &\leq 2H(u_0) - 2a \int_0^t \mathbb{E}[\|\nabla u(s)\|^2] ds + \frac{2a\sigma}{\sigma + 1} C_{\sigma,d} \int_0^t \mathbb{E}[\|\nabla u(s)\|^2 \|u(s)\|^{2\sigma}] ds \\
&+ \int_0^t \sum_{k \in \mathbb{N}^+} \|\nabla Q^\frac{1}{2} e_k\|_{L^\infty}^2 \mathbb{E}[\|u(s)\|^2] ds \\
&\leq 2H(u_0) + \frac{2a\sigma}{\sigma + 1} C_{\sigma,d} \int_0^t e^{-2\sigma s} \mathbb{E}[\|\nabla u(s)\|^2] u_0^{2\sigma} ds \\
&+ \int_0^t \sum_{k \in \mathbb{N}^+} \|\nabla Q^\frac{1}{2} e_k\|_{L^\infty}^2 e^{-2\sigma s} u_0^2 ds.
\end{aligned}
\]

Gronwall inequality implies that

\[
\sup_{t \leq \tau} \mathbb{E}[\|\nabla u(t)\|^2] \leq C(u_0, R, a, Q, T_0).
\]
Moreover, using Burkholder–Davis–Gundy inequality and Young inequality,

\[
\mathbb{E} \left[ \sup_{t \leq \tau} \| \nabla u(t) \|_2 \right] \\
\leq C(u_0, R, a, Q, T_0) + C \mathbb{E} \left[ \sup_{t \leq \tau} \int_0^t \int_{\mathbb{R}^d} \bar{u}(s) \nabla u(s) \nabla W(t) dx \right] \\
\leq C(u_0, R, a, Q, T_0) + C \mathbb{E} \left[ \left\| \int_0^{\tau} \sum_{k \in \mathbb{N}^+} \| u(s) \nabla u(s) \nabla Q^\frac{1}{2} e_k \|_2^2 dt \right\|^{\frac{1}{2}} \right] \\
\leq C(u_0, R, a, Q, T_0) + \epsilon \mathbb{E} \left[ \sup_{t \leq \tau} \| \nabla u(t) \|_2 \right] \\
+ C(\epsilon) \mathbb{E} \left[ \int_0^{\tau} \sum_{k \in \mathbb{N}^+} \| \nabla Q^\frac{1}{2} e_k \|_{L^\infty}^2 \| u(s) \|_2^2 dt \right],
\]

where \( \epsilon < \frac{1}{2} \). The last term of the above inequality is bounded by Hölder inequality and the charge evolution law (2), which in turns implies that

\[
\mathbb{E} \left[ \sup_{t \leq \tau} \| \nabla u(t) \|_2 \right] \leq C(u_0, R, a, Q, T_0).
\]

These a priori estimations combined with local well-posedness imply the global existence of the unique solution. \( \square \)

The condition \( \| u_0 \| < \| R \| \) in Theorem 2.1 is a sufficient condition for the global existence of the solution. Based on it, we get a upper bound of the probability of the blow-up with a random initial datum at any finite time.

**Corollary 2.1.** Assume that \( u_0 \) is random initial datum. Under the condition of Theorem 2.1, we have

\[
\mathbb{P}(\tau^*(u_0) < \infty) \leq \mathbb{P}(\| u_0 \| \geq \| R \|).
\]

Moreover, we can get the following exponential integrability of \( u \), which is useful for studying the continuous dependence on initial data and noise, exponential tail estimate of the solution, strong and weak convergence rates of numerical approximations (see e.g. [7, 8, 9]).

**Proposition 2.1.** Assume that the condition of Theorem 2.1 holds, then there exist constants \( C = C(u_0, R, a, Q) \) and \( \alpha = \alpha(u_0, R, a, Q) \) such that

\[
\sup_{t \in [0, T]} \mathbb{E} \left[ \exp \left( \frac{\| u \|_{H^1}^2}{e^{\alpha t}} \right) \right] \leq C.
\]

**Proof** Denote by a positive number \( c := 1 - \frac{\| u_0 \|_{H^1}^2}{\| R \|_{H^1}^2} < 1 \), \( \mu(u) = i \Delta u + i |u|^2 u - \frac{1}{2} F_Q u - a \) and \( \sigma(u) = i u Q^\frac{1}{2} \). Based on Gagliardo–Nirenberg inequality (6), we only need to show the boundedness of \( \sup_{t \in [0, T]} \mathbb{E} \left[ \exp \left( \frac{2H(u(t))}{e^{\alpha t}} \right) \right] \). Using the truncated
arguments and taking limits, we get
\[
2DH(u(s))\mu(u(s)) + \frac{1}{c}||u(s)||^2\|\nabla Q^2 e_k\|^2 - \frac{2a}{c}||\nabla u(s)||^2 + \frac{2a}{c}||u(s)||^{2\sigma+2}_{L^{2^\sigma+2}}
\]
\[+ \frac{4}{c^2e^{\alpha t}}\sum_k ||\nabla u(s)||^2 ||u(s)\nabla Q^2 e_k||^2.
\]
Again by the Gagliardo–Nirenberg inequality (6) and the Sobolev embedding theorem, we obtain
\[
2DH(u(s))\mu(u(s)) + \frac{1}{c}||u(s)||^2\|\nabla Q^2 e_k\|^2 - \frac{4a}{c}H(u(s)) + \frac{2a\sigma}{c(\sigma+1)}C_{\sigma,d}\|\nabla u(s)||^2 ||u(s)||^{2\sigma}
\]
\[+ \frac{4}{c^2e^{\alpha t}}\|\nabla u(s)||^2 ||u(s)||^2 \|\nabla Q^2 e_k\| L^\infty
\]
\[\leq \frac{1}{c}||u(s)||^2 \|f_Q\| L^\infty - \frac{4a}{c}H(u(s)) + \frac{4a\sigma}{c^2(\sigma+1)}\|u(s)||^{2\sigma} C_{\sigma,d} H(u(s))
\]
\[+ \frac{8}{c^2e^{\alpha t}}||u(s)||^2 \|f_Q\| L^\infty H(u(s)).
\]
Charge evolution law in Lemma 2.1 leads that
\[
2DH(u(s))\mu(u(s)) + \frac{1}{c}||u(s)||^2 \|f_Q\| L^\infty
\]
\[\leq \frac{e^{-2\alpha t}}{c} \|u_0\|^2 \|f_Q\| L^\infty
\]
\[+ \left(-2a + \frac{2a\sigma}{c(\sigma+1)} ||u_0||^{2\sigma} C_{\sigma,d} + \frac{4}{c^2e^{(\alpha+2\alpha)t}} ||u_0||^2 \|f_Q\| L^\infty \right) \frac{2}{c} C H(u(s)).
\]
Then Lemma 3.1 in [8] implies that
\[
\sup_{t\in[0,T]} E \left[ \exp \left( \frac{2H(u(t))}{e^{\alpha t}} \right) \right] \leq E \left[ \exp \left( \frac{2}{c} H(u_0) \right) \right] \exp \left( \int_0^T \frac{\|u_0\|^2 \|f_Q\| L^\infty}{e^{(\alpha+2\alpha)t}} dt \right),
\]
where \( \alpha \geq -2a + \frac{2a\sigma}{c(\sigma+1)} ||u_0||^{2\sigma} C_{\sigma,d} + \frac{4}{c^2} ||u_0||^2 \|f_Q\| L^\infty \). \( \square \)

Applying the above exponential integrability of exact solution, we deduce the following strongly continuous dependence on the initial data.

**Corollary 2.2.** Assume that \( d = 1, \sigma = 2, a > 0, u_0, v_0 \in H^1 \) and \( \max(||u_0||, ||v_0||) < \min \left( \frac{4a(2\alpha-1)}{e^{x_0 + 2\alpha t}}, \frac{4a(2\alpha-1)}{e^{3x_0 + 2\alpha t}} \right) \) with \( \frac{1}{3} < c < 1. \) Let \( u = \{u(t) : t \in [0, T]\} \) and \( v = \{v(t) : t \in [0, T]\} \) be the solutions of Eq. (11) with initial data \( u_0 \) and \( v_0, \) respectively. Under the condition of Proposition 2.1, then there exists a constant \( C = C(p, u_0, v_0, Q, a) \) such that
\[
\sup_{t\in[0,T]} E \left[ ||u(t) - v(t)||^2 \right] \leq C E \left[ ||u_0 - v_0||^2 \right].
\]
Proof Applying the truncated arguments, Itô formula and taking limits yield that
\[
\|u(t) - v(t)\|^2 = \|u_0 - v_0\|^2 - 2a \int_0^t \|u(s) - v(s)\|^2 ds \\
+ 2 \int_0^t \langle u(s) - v(s), i\Delta(u(s) - v(s)) \rangle ds \\
+ 2 \int_0^t \langle u(s) - v(s), i(\|u(s)\|^4 u(s) - |v(s)|^4 v(s)) \rangle ds
\]
\[
= \|u_0 - v_0\|^2 - 2a \int_0^t \|u(s) - v(s)\|^2 ds \\
+ 2 \int_0^t \langle u(s) - v(s), i(\|u(s)\|^4 u(s) - |v(s)|^4 v(s)) \rangle ds,
\]
Since for \(a, b \in \mathbb{C}, |a|^4a - |b|^4b = (|a|^4 + |b|^4)(a - b) + ab(|a|^2 + |b|^2)(\bar{a} - \bar{b}) + |a|^2|b|^2(a - b),\) combining with Young inequality and Gagliardo–Nirenberg inequality, we get
\[
\|u(t) - v(t)\|^2 = \|u_0 - v_0\|^2 - 2a \int_0^t \|u(s) - v(s)\|^2 ds \\
+ 2 \int_0^t \langle u(s) - v(s), iu(s)v(s)(\|u(s)\|^2 + |v(s)|^2)(\bar{u}(s) - \bar{v}(s)) \rangle ds
\]
\[
\leq \|u_0 - v_0\|^2 - 2a \int_0^t \|u(s) - v(s)\|^2 ds \\
+ 2 \int_0^t \left( \|u(s)\|_{L^\infty}^4 + \|v(s)\|_{L^\infty}^4 \right) \|u(s) - v(s)\|^2 ds.
\]
Gronwall inequality and Gagliardo–Nirenberg inequality lead that
\[
\|u(t) - v(t)\|^2 \leq \exp \left( \int_0^t 2(\|u(s)\|_{L^\infty}^4 + \|v(s)\|_{L^\infty}^4) ds \right) \|u_0 - v_0\|^2
\]
\[
\leq \exp \left( \int_0^t 8e^{-2as} \|u_0\|^2 \|\nabla u(s)\|^2 + 8e^{-2as} \|v_0\|^2 \|\nabla v(s)\|^2 ds \right) \|u_0 - v_0\|^2.
\]
After taking expectation, by Young inequality, we obtain
\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \|u(t) - v(t)\|^2 \right] \leq \sqrt{\mathbb{E} \left[ \exp \left( \int_0^T 16e^{-2as} \|u_0\|^2 \|\nabla u(s)\|^2 ds \right) \right]}
\times \sqrt{\mathbb{E} \left[ \exp \left( \int_0^T 16e^{-2as} \|v_0\|^2 \|\nabla v(s)\|^2 ds \right) \right]} \|u_0 - v_0\|^2
\]
(7)
It is obvious that if these two exponential moments in the above inequality are bounded, the theorem is proved. For simplicity, we take \(\mathbb{E} \left[ \exp \left( \int_0^T 16e^{-2as} \|u_0\|^2 \|\nabla u(s)\|^2 ds \right) \right]

as example. By Jensen inequality, for \( \alpha < 2a \),
\[
\mathbb{E} \left[ \exp \left( \int_0^T 16e^{-2as} \| u_0 \|^2 \| \nabla u(s) \|^2 ds \right) \right]
= \mathbb{E} \left[ \exp \left( \int_0^T e^{-(2a-\alpha)s} 16 \| u_0 \|^2 e^{-\alpha s} \| \nabla u(s) \|^2 ds \right) \right]
\leq \mathbb{E} \left[ \exp \left( \frac{16}{2a-\alpha} \| u_0 \|^2 e^{-\alpha s} \| \nabla u(s) \|^2 \right) \right].
\]
Then we take \( \alpha = -2a + \frac{2\alpha_0}{c(\alpha_0 + 1)} \| u_0 \|^{2\alpha_0} C_{\sigma,d} + \frac{4}{c} \| u_0 \|^2 \| f_Q \|_{L^\infty} \) such that \( \frac{16}{2a-\alpha} \| u_0 \|^2 \leq 1 \) and \( \alpha < 2a \). Indeed, the assumption on \( u_0 \) and \( v_0 \), together with Gagliardo–Nirenberg inequality Eq. \( \text{(6)} \), implies
\[
\| u_0 \|^2 \leq \frac{a}{2} - \frac{a}{4c} - \frac{1}{4c^2} \| u_0 \|^2 \| f_Q \|_{L^\infty}
\]
and
\[
\frac{1}{c} + \frac{1}{ac^2} \| u_0 \|^2 \| f_Q \|_{L^\infty} < 2,
\]
which ensure that \( \frac{16}{2a-\alpha} \| u_0 \|^2 \leq 1 \) and \( \alpha < 2a \). Proposition \ref{prop:well-posedness} combining the above estimations, implies the uniform boundedness of the exponential moments of Eq. \( \text{(8)} \), which completes the proof. \( \square \)

**Remark 2.2.** Due to the fact that the ground state solution \( R(x) \) in one dimension is \( \frac{1}{\sqrt{\cosh(2x)}} \) with \( \| R \|_2 = \pi \sqrt{3} \), the above strong continuous dependence result on initial data holds with \( \max(\| u_0 \|, \| v_0 \|) < \sqrt{\frac{2a}{2a-\alpha}} \) when \( a \) becomes large enough. For \( \| u_0 \| < \| R \|, a \geq 0 \), by the above arguments, we can get the continuous dependence on initial data in pathwise sense,
\[
\sup_{t \in [0,T]} \| u(t) - v(t) \| \leq C(\omega) \| u_0 - v_0 \|.
\]

### 3. Blow-up of solutions in focusing mass-(super)critical case

As shown in Section \ref{sec:well-posedness} the result about well-posedness for Eq. \( \text{(1)} \) in the critical case is similar to that in deterministic case. Notice that this phenomenon is different from the additive case with \( \sigma \geq \frac{d}{2} \) and the multiplicative case with \( \sigma > \frac{d}{2} \) (see e.g. \cite{11, 13}), where the singularity happens in any finite time with a positive probability for any initial datum.

In fact, the authors in \cite{13} show that for \( \sigma \geq \frac{d}{2} \), if \( u_0 \in L^2(\Omega; \Sigma) \cap L^{2\sigma+2}(\Omega; L^{2\sigma+2}(\mathbb{R}^d)) \),
\[
f_Q = \sum_{k \in \mathbb{N}^+} | \nabla |^2 c_k |^2 \quad \text{and for some } \tilde{t} > 0,
\]
\[
\mathbb{E} \left[ V(u_0) \right] + 4 \mathbb{E} \left[ G(u_0) \right] \tilde{t} + 8 \mathbb{E} \left[ H(u_0) \right] \tilde{t}^2 + \frac{4}{3} \tilde{t}^3 \| f_Q \|_{L^\infty} \mathbb{E} \left[ M(u_0) \right] < 0,
\]
then \( \mathbb{P}(\tau^*(u_0) \leq \tilde{t}) > 0 \). The above result implies that if the energy of \( u_0 \) is a.s. negative, then \( \mathbb{P}(\tau^*(u_0) \leq t) > 0 \) for some \( t > 0 \) provided the noise is not too strong, i.e., \( \| f_Q \|_{L^\infty} \) is small enough. The natural question is whether the damped effect can prevent the blow-up phenomenon or not in stochastic case.
To study the blow-up phenomenon, we introduce the finite variance space

\[ \Sigma = \{ v \in H^1 : |x|v \in H \} \]

deeded with the norm \( \| \cdot \|_{\Sigma} \):

\[ \|v\|_{\Sigma}^2 = \| |x|v \|^2 + \|v\|_{H^1}^2, \]

the variance

\[ V(v) = \int_{\mathbb{R}^d} |x|^2|v(x)|^2 dx, \quad v \in \Sigma \]

and the momentum

\[ G(v) = \text{Im} \int_{\mathbb{R}^d} \bar{\psi}(x) x \cdot \nabla \psi(x) dx, \quad v \in \Sigma. \]

With the help of a smoothing procedure and truncated arguments (see e.g. [13]), we can prove rigorously the evolution laws of \( V \) and \( G \) for the damped stochastic NLS equation.

**Proposition 3.1.** Assume that \( u_0 \in \Sigma \). Under the conditions of Lemma 2.1, for any stopping time \( \tau < \tau^*(u_0) \) a.s., we have

\[ V(u(\tau)) = V(u_0) + 4 \int_0^\tau G(u(s))ds - 2a \int_0^\tau V(u(s))ds, \]

and

\[ G(u(\tau)) = G(u_0) + 4 \int_0^\tau H(u(s))ds - 2a \int_0^\tau G(u(s))ds + \frac{2 - \sigma d}{\sigma + 1} \int_0^\tau \|u(s)\|_{L_{2^\sigma+2}}^{2^\sigma+2} ds \\
+ \sum_{k \in \mathbb{N}} \int_0^\tau \int_{\mathbb{R}^d} |u(s, x)|^2 x \cdot \nabla(Q^{2}e_k)(x)dx d\beta_k(s). \]

**Proof** Applying Itô formula to \( V \) and \( G \), integration by parts and taking the imaginary part of the integration, we obtain

\[ V(u(\tau)) = V(u_0) + 4 \int_0^\tau G(u(s))ds - 2a \int_0^\tau V(u(s))ds, \]
and

\[ G(u(\tau)) = G(u_0) + 2 \int_0^\tau \text{Im} \int_{\mathbb{R}^d} x \cdot \nabla u \left( -i\Delta \bar{u} - i|u|^{2\sigma} \bar{u} - aiu - \frac{1}{2} F_{\mathcal{Q}} \bar{u} \right) dx ds \]

\[ - d \int_0^\tau \text{Im} \int_{\mathbb{R}^d} \left( i\Delta u + i|u|^{2\sigma} u - aiu - \frac{1}{2} F_{\mathcal{Q}} \bar{u} \right) \bar{u} dx ds \]

\[ + 2 \int_0^\tau \text{Im} \int_{\mathbb{R}^d} -ix \nabla u \bar{u} dx ds - d \int_0^\tau \text{Im} \int_{\mathbb{R}^d} i|u|^2 dx ds \]

\[ + \int_0^\tau \text{Im} \int_{\mathbb{R}^d} x \cdot \nabla (uQ^\frac{1}{2} e_k) Q^\frac{1}{2} e_k \bar{u} dx ds \]

\[ = G(u_0) + d \int_0^\tau \left( \| \nabla u(s) \|^2 - \| u(s) \|_{L^{2\sigma+2}}^{2\sigma+2} \right) ds \]

\[ - 2 \int_0^\tau \text{Im} \int_{\mathbb{R}^d} x \cdot \nabla u i \Delta \bar{u} dx ds \]

\[ - 2 \int_0^\tau \text{Im} \int_{\mathbb{R}^d} x \cdot \nabla u \left( i|u|^{2\sigma+2} \bar{u} \right) dx ds - 2a \int_0^\tau G(u(s)) ds \]

\[ + \sum_{k \in \mathbb{N}^+} \int_0^\tau \int_{\mathbb{R}^d} |u(s, x)|^2 x \cdot \nabla (Q^\frac{1}{2} e_k) (x) dx ds \beta_k(s). \]

By the definition of \( H \) and \( \sigma d = 2 \), we get

\[ G(u_\tau) = G(u_0) + 4 \int_0^\tau H(u(s)) ds + \frac{2 - \sigma d}{\sigma + 1} \int_0^\tau \| u(s) \|_{L^{2\sigma+2}}^{2\sigma+2} ds - 2a \int_0^\tau G(u(s)) ds \]

\[ + \sum_{k \in \mathbb{N}^+} \int_0^\tau \int_{\mathbb{R}^d} |u(s, x)|^2 x \cdot \nabla (Q^\frac{1}{2} e_k) (x) dx ds \beta_k(s). \]

For the damped stochastic NLS equation, the method in [13] is not available since the damped effect will lead that the expansion of \( V \) produces many addition terms which can not be estimated directly. We introduce the modified energy, invariance and momentum as in [18], and study the evolution of these modified quantities to investigate the blow-up condition for supercritical case \( \sigma d > 2, a > 0 \).

**Lemma 3.1.** Let \( b \in \mathbb{R} \). Under the same condition of Proposition [18], for any stopping time \( \tau < \tau^*(u_0) \), we have

\[ e^{b\tau} H(u(\tau)) = H(u_0) + \int_0^\tau e^{bs} H(u(s)) ds - a \int_0^\tau e^{bs} \left( \| \nabla u(s) \|^2 - \| u(s) \|_{L^{2\sigma+2}}^{2\sigma+2} \right) ds \]

\[ - \text{Im} \int_0^\tau \int_{\mathbb{R}^d} e^{bs} \bar{u} \nabla u \nabla W dx + \frac{1}{2} \sum_{k \in \mathbb{N}^+} \int_0^\tau e^{bs} \| u \nabla Q^\frac{1}{2} e_k \|^2 ds, \]

\[ e^{b\tau} G(u(\tau)) = G(u_0) - (2a - b) \int_0^\tau e^{bs} G(u(s)) ds \]

\[ + 2 \int_0^\tau e^{bs} \left( 2H(u(s)) + \frac{2 - \sigma d}{2\sigma+2} \| u(s) \|_{L^{2\sigma+2}}^{2\sigma+2} \right) ds \]

\[ + \sum_{k \in \mathbb{N}^+} \int_0^\tau e^{bs} |u(s, x)|^2 x \cdot \nabla (Q^\frac{1}{2} e_k) (x) dx ds \beta_k(s), \]
and
\[ e^{bt} V(u(\tau)) = V(u_0) + 4 \int_0^\tau e^{bs} G(u(s)) ds - (2a - b) \int_0^\tau e^{bs} V(u(s)) ds. \]

**Proof** The proof is similar to the proof of Lemma 3.1 and Proposition 3.2 by using smoothing procedures, truncated arguments, integration by parts and Itô formula. More details, we refer to [13]. \( \square \)

Based on Lemma 3.1, we prove a preliminary result on the blow-up condition for Eq. (1) in the supercritical case.

**Proposition 3.2.** Let \( \sigma d > 2, a > 0, b < 2a \) satisfy \( \frac{4a}{\sigma d - 2} \leq 2a - b \), \( u_0 \in L^2(\Omega; \Sigma) \cap L^{2\sigma+2}(\Omega; L^{2\sigma+2}) \) and \( \sum_{k \in N^+} \| Q_k^2 E_k \|^2_{H^1} + \| f_Q \|_{L^\infty} < \infty \). Assume in addition that for some \( 0 < y \leq \frac{2\sigma d}{2\sigma - b} \) such that
\[ \mathbb{E} \left[ V(u_0) + 4y^2 H(u_0) + 16y^2 H(u_0) + 8y^2 \| u_0 \|^2 \| f_Q \|_{L^\infty} \right] < 0, \]
then for some \( \bar{t} \), we have
\[ P(\tau^*(u_0) \leq \bar{t}) > 0. \]

**Proof** We prove the assertion by contradiction. Assume that the solution \( u \) exists globally. Then for any \( t > 0 \), \( \tau^*(u_0) > t \) a.s. Then we take \( \tau = t \). The evolution law of modified energy Eq. (5), charge evolution law Eq. (2) and taking expectation leads that
\[
\begin{align*}
\mathbb{E} & \left[ e^{bt} H(u(t)) \right] \\
& = H(u_0) + b \int_0^t \mathbb{E} \left[ e^{bs} H(u(s)) \right] ds - a \int_0^t \mathbb{E} \left[ e^{bs} \left( \| \nabla u(s) \|^2 - \| u(s) \|^{2\sigma+2}_{L^{2\sigma+2}} \right) \right] ds \\
& \quad + \frac{1}{2} \sum_{k \in N^+} \int_0^t \mathbb{E} \left[ e^{bs} \| \nabla Q_k^2 E_k \|^2 \right] ds \\
& \leq H(u_0) + \left( \frac{b}{2} - a \right) \int_0^t e^{bs} \mathbb{E} \left[ \| \nabla u(s) \|^2 + \frac{2}{b - 2a} (a - \frac{b}{2\sigma + 2}) \| u(s) \|^{2\sigma+2}_{L^{2\sigma+2}} \right] ds \\
& \quad + \frac{1}{2} \int_0^t \mathbb{E} \left[ e^{(b - 2a)s} \| u_0 \|^2 \| f_Q \|_{L^\infty} \right] ds.
\end{align*}
\]

Next, we aim to show a priori estimate on \( e^{bt} G(u(t)) \). For simplicity, we denote \( \tilde{H}(u) := \| \nabla u \|^2 - \frac{\sigma d}{2\sigma - 2} \| u \|^{2\sigma+2}_{L^{2\sigma+2}} \). Applying the evolution of modified momentum Eq. (4) and taking expectation, we obtain
\[ \mathbb{E} \left[ e^{bt} G(u(t)) \right] = \mathbb{E} \left[ G(u_0) \right] + 2 \int_0^t e^{bs} \mathbb{E} \left[ \tilde{H}(u(s)) \right] ds - (2a - b) \int_0^t \mathbb{E} \left[ e^{bs} G(u(s)) \right] ds. \]

To control the second term \( \mathbb{E} \left[ \tilde{H}(u(s)) \right] \) uniformly, we take \( b \leq a \left( 2 - \frac{4\sigma}{\sigma d - 2} \right) \) such that
\[
\| \nabla u \|^2 + \frac{2}{b - 2a} (a - \frac{b}{2\sigma + 2}) \| u \|^{2\sigma+2}_{L^{2\sigma+2}} \geq 2 \tilde{H}(u) + \frac{2 - \sigma d}{2\sigma + 2} \| u \|^{2\sigma+2}_{L^{2\sigma+2}} \tilde{H}(u).)
\]
Then the fact that \( \dot{H}(u) \leq 2H(u) \) leads that
\[
E \left[ \frac{1}{2} e^{2bt} \dot{H}(u(t)) \right] + (2a - b) \int_0^t E \left[ \frac{1}{2} e^{bs} \dot{H}(u(s)) \right] ds \\
\leq H(u_0) + \frac{1 - e^{(b-2a)t}}{4a - 2b} \| u_0 \|^2 \| f_Q \|_{L^\infty}.
\]
Using Gronwall inequality, we have
\[
\int_0^t E \left[ \frac{1}{2} e^{bs} \dot{H}(u(s)) \right] ds \leq \frac{1 - e^{(b-2a)t}}{2a - b} H(u_0) + \frac{1 - e^{(b-2a)t}}{2(2a - b)^2} \| u_0 \|^2 \| f_Q \|_{L^\infty} \\
- \frac{t e^{(b-2a)t}}{4a - 2b} \| u_0 \|^2 \| f_Q \|_{L^\infty},
\]
which derives that
\[
\int_0^t \left[ e^{bs} \dot{H}(u(s)) \right] ds \leq \frac{1}{2a - b} \left( 2H(u_0) + \frac{1}{2a - b} \| u_0 \|^2 \| f_Q \|_{L^\infty} \right).
\]
The above estimation and Eq. (12) yield that
\[
E \left[ e^{bt} G(u(t)) \right] \leq E \left[ G(u_0) \right] - (2a - b) \int_0^t e^{bs} G(u(s)) ds \\
+ \frac{2}{2a - b} \left( 2H(u_0) + \frac{1}{2a - b} \| u_0 \|^2 \| f_Q \|_{L^\infty} \right).
\]
Again by Gronwall inequality,
\[
E \left[ \int_0^t e^{bs} G(u(s)) ds \right] \\
\leq \frac{1 - e^{(b-2a)t}}{2a - b} E \left[ G(u_0) + \frac{2}{2a - b} \left( 2H(u_0) + \frac{1}{2a - b} \| u_0 \|^2 \| f_Q \|_{L^\infty} \right) \right].
\]
The above inequality, Eq. (10) and the non-negativity of \( V \) yield that
\[
E \left[ e^{bt} V(u(t)) \right] \\
= E \left[ V(u_0) \right] + 4 \int_0^t e^{bs} E \left[ G(u(s)) \right] ds - (2a - b) \int_0^T e^{bs} E \left[ V(u(s)) \right] ds \\
\leq E \left[ V(u_0) \right] + \frac{4}{2a - b} (1 - e^{(b-2a)t}) E \left[ G(u_0) + \frac{2}{2a - b} \left( 2H(u_0) + \frac{1}{2a - b} \| u_0 \|^2 \| f_Q \|_{L^\infty} \right) \right].
\]
Since the assumption means that \( E \left[ \frac{2a - b}{4} V(u_0) + G(u_0) + \frac{2}{2a - b} \left( 2H(u_0) + \frac{1}{2a - b} \| u_0 \|^2 \| f_Q \|_{L^\infty} \right) \right] < 0 \), we only need to make
\[
\bar{t} \geq - \frac{1}{2a - b} \ln \left( \frac{E \left[ V(u_0) \right] + E \left[ G(u_0) + \frac{2}{2a - b} \left( 2H(u_0) + \frac{1}{2a - b} \| u_0 \|^2 \| f_Q \|_{L^\infty} \right) \right]}{E \left[ G(u_0) + \frac{2}{2a - b} \left( 2H(u_0) + \frac{1}{2a - b} \| u_0 \|^2 \| f_Q \|_{L^\infty} \right) \right]} \right).
\]
By the above inequality and the positivity of $e^{bt}$, there exists some $\bar{t}$ such that 
\[ \lim_{t \to \bar{t}} E[V(u(t))] = 0. \]
By the uncertainty principle 
\[ \|u\|_2 \leq \frac{2}{d} \|\nabla u\| \|x\|_2, \]
we get
\[ \sqrt{E[\|\nabla u(t)\|^2]} \geq \frac{d\|u_0\|^2}{2e^{2at}} \sqrt{E[\|x\|_2^2]} = \frac{d\|u_0\|^2}{2e^{(2a-b)t}} \sqrt{E[V(u(t))]} . \]
The above estimation yields that
\[ \sqrt{E[\|\nabla u(t)\|^2]} \text{ goes into } \infty \text{ when } t \text{ tends to } \bar{t}, \]
which leads to a contradiction and finishes the proof.

**Remark 3.1.** The above proposition implies that when
\[ E[V(u_0) + \frac{\sigma d - 2}{a\sigma} G(u_0) + (\frac{\sigma d - 2}{a\sigma})^2 H(u_0) + \frac{1}{8} (\frac{\sigma d - 2}{a\sigma})^3 \|u_0\|^2 \|f_Q\|_\infty] < 0, \]
the solution of Eq. (11) will blow up in a finite time with a positive probability. Clearly, if the energy of $u_0$ is negative a.s., damped effect is not strong and the noise is small enough, the blow up phenomenon of the solution always happens.

The blow-up condition Eq. (11) presents the effect of the damped term. The following result gives a sharp time-dependent blow-up condition.

**Theorem 3.1.** Let $\sigma d > 2, a \geq 0, u_0 \in L^2(\Omega; \Sigma) \cap L^{2\sigma+2}(\Omega; L^{2\sigma+2})$, and 
\[ \sum_{k \in \mathbb{N}^+} \|Q_k e_k\|^2 + \|f_Q\|_{L^\infty} < \infty. \]
Assume also that for some $z \geq \frac{4a\sigma}{\sigma d - 2}$ and $\bar{t}$ such that
\[ E[V(u_0)] + 4tE[G(u_0)] + \left(8t^2 + \frac{8}{3} z t^3\right) E[H(u_0)] + \left(\frac{4}{3} t^3 + \frac{4}{3} z t^3\right) \mathbb{E}\[\|u_0\|^2\] \|f_Q\|_{L^\infty} \leq 0, \]
then we have
\[ \mathbb{P}(\tau^+(u_0) \leq \bar{t}) > 0. \]

**Proof** The proof is similar to the proof of Proposition 3.2. Using the evolutions of the modified energy Eq. (8), the new energy $\tilde{H}$ in Proposition 3.2 leads that for $z \geq \frac{4a\sigma}{\sigma d - 2}$,
\[ \int_0^t E[e^{bs} \tilde{H}(u(s))] \, ds \leq \frac{1-e^{-zt}}{z} E[2H(u_0)] + \frac{1-(1+zt)e^{-zt}}{z^2} \mathbb{E}[\|u_0\|^2] \|f_Q\|_{L^\infty}. \]
Applying the evolution of the modified momentum Eq. (9), invariance Eq. (10), and combining with the Taylor expansion of $e^{z}$, $z \in \mathbb{R}$, we get

$$
\mathbb{E} [e^{bt}V(u(t))] 
\leq \mathbb{E} [V(u_0)] + \frac{1 - e^{-zt}}{z} \mathbb{E} [4G(u_0)] + \frac{1 - (1 + zt)e^{-zt}}{z^2} \mathbb{E} [16H(u_0)] 
+ \frac{1 - (1 + zt + \frac{1}{2}(zt)^2)e^{-zt}}{z^3} 8\mathbb{E} [\|u_0\|^2] \|f_Q\|_{L^\infty} 
\leq \mathbb{E} [V(u_0)] + 4t\mathbb{E} [G(u_0)] + (8t^2 + \frac{8}{3}zt^3)\mathbb{E} [H(u_0)] 
+ (\frac{4}{3}t^3 + \frac{4}{3}zt^4) \mathbb{E} [\|u_0\|^2] \|f_Q\|_{L^\infty} 
\leq \mathbb{E} [V(u_0)] + 4t\mathbb{E} [G(u_0)] + (8t^2 + \frac{8}{3}zt^3)\mathbb{E} [H(u_0)] 
+ (\frac{4}{3}t^3 + \frac{4}{3}zt^4) \mathbb{E} [\|u_0\|^2] \|f_Q\|_{L^\infty} 
$$

Similar arguments in Proposition 3.2 yield that Eq. (13) is the blow-up condition.

\[ \square \]

**Remark 3.2.** The above blow-up condition is sharp in the sense that if $z \to 0$, the above condition can be degenerated to the blow-up condition in the conservative case, i.e. for some $\bar{t}$

$$
\mathbb{E} [V(u_0)] + 4t\mathbb{E} [G(u_0)] + 8\mathbb{E} [H(u_0)] \bar{t}^2 + \frac{4}{3}t^3 \|f_Q\|_{L^\infty} \mathbb{E} [M(u_0)] < 0.
$$

Since the blow-up condition for Eq. (1) in supercritical case is similar to the condition of Theorem 4.1 in [13], it is possible to apply the skills and arguments in [13] to get a stronger result that $\mathbb{P}(\tau^*(u_0, a) > t) > 0$, for any $t$ and $u_0$, $u_0 \neq 0$ under some assumptions on $u_0$, $d$ and $Q^\perp$.

**Remark 3.3.** The blow-up condition in critical case can not be obtained by the method in Proposition 3.2, we only get some necessary condition in Remark 2.1.

It is our future work to study the blow-up phenomena of the solution and show the sufficient blow-up condition for damped stochastic NLS equations in critical case.

It seems that when $a$ becomes larger, the blow-up time becomes longer and that when $d$ goes to $\infty$, the blow-up condition is not satisfied. Indeed, we can show that when the damped effect is large enough, the damped effect can prevent the blow-up of the solution with high probability. The key of the proof is using the infinite dimensional Doss–Sussman type transformation in [12]. In the following theorem, we assume that the noise satisfies $\sum_{k \in \mathbb{N}^+} \|Q^\perp e_k\|_{L^2} < \infty$ and the following decay condition

$$
\lim_{x \to \infty} \eta(x)(|Q^\perp e_m(x)| + |\nabla Q^\perp e_m(x)| + |\Delta Q^\perp e_m(x)|) = 0,
$$

where $\eta(x) = 1 + |x|^2$ if $d \neq 2$, and $\eta(x) = (1 + |x|^2)(\ln(2 + |x|^2))^2$ if $d = 2$. Under these assumptions, the local well-posedness is obtained in [1]. We also remark when $\{e_k\}_{k \in \mathbb{N}^+}$ is an orthonormal basis of $\mathbb{H}$, the decay condition natural holds. In this case, $e_k$ can be chosen as the $k$-th Hermite function in $\mathbb{H}$, and meanwhile as the $k$-th eigenvector of the operator $Q^\perp$.

**Theorem 3.2.** Assume that $a > 0$, $\sigma d \geq 2$ and $\sum_k \|\Delta Q^\perp e_k\|_{L^\infty}^2 < \infty$. Then for any $u_0 \in \mathbb{H}^1$ and $0 < T < \infty$, we have

$$
\lim_{a \to \infty} \mathbb{P}(\tau^*(u_0, a) > T) = 1.
$$
and get the following random partial differential equation

\[
\frac{dv}{dt} = i \exp \left( at - iW(t) \right) \Delta u(t) dt + i \exp \left( at - iW(t) \right) |u|^{2\sigma} u dt
\]

\[
= i \left( \Delta + 2i \nabla W(t) \cdot \nabla + |\nabla W(t)|^2 + i \Delta W(t) \right) v
\]

\[+ i \exp \left( -2a\sigma t \right) |v|^{2\sigma} v dt
\]

\[= A(t) v dt + i \exp \left( -2a\sigma t \right) |v|^{2\sigma} v dt.
\]

By Lemma 2.4 in [1], the solutions of Eq. (14) and Eq. (11) are equivalent. Let 
\[r = \frac{4\sigma + 4}{\sigma},\]

such that the mapping \(G\) is a Strichartz pair. Next we recall the proof of the local well-posedness and set

\[X_R := \left\{ v \in C(0, \tau; \mathbb{H}^1) \cap L^r(0, \tau; W^{1,2\sigma+2}) \right\} \]

\[\|v\|_{C(0, \tau; \mathbb{H}^1)} + \|v\|_{L^r(0, \tau; W^{1,2\sigma+2})} \leq R \}

Considering the solution map \(G\) of Eq. (14), by the random Strichartz estimate and similar arguments in [1], we obtain

\[\|G(v)\|_{C(0, \tau; \mathbb{H}^1)} + \|G(v)\|_{L^r(0, \tau; W^{1,2\sigma+2})} \leq 2C_\tau \|u\|_{\mathbb{H}^1} + (2\sigma + 1) C_\tau \|v\|_{L^{r}(0, \tau; W^{1,2\sigma+2})}, \]

where \(v, w \in X_R\), \(C_\tau = (2\sigma + 1)D^{2\sigma}\), \(D\) is the Sobolev embedding coefficient form \(L^{2\sigma+2}\) to \(\mathbb{H}^1\), \(C_\tau\) is the random Strichartz estimate coefficient, \(q > 1\) and \(\frac{1}{q} = 1 - \frac{2\sigma}{r} > 0\). Now we take \(R = 4C_\tau \|u\|_{\mathbb{H}^1}\) and

\[
\tau(a) = \inf \left\{ t > 0 \left| \frac{2C_\tau C_t}{a\sigma} R^{2\sigma} > 1 \right. \right\}
\]

such that the mapping \(G : X_R \to X_R\) has a fixed point in the Banach space \(\left( X_R, \| \cdot \|_{C(0, \tau; \mathbb{H}^1)} + \| \cdot \|_{L^r(0, \tau; W^{1,2\sigma+2})} \right)\), which implies the local well-posedness of Eq. (14).

Now, we aim to show that \(\lim_{a \to \infty} \mathbb{P}(\tau^+(u_0, a) > T) = 1\). Based on the result in [1] Lemma 2.7 that \(C_t, t > 0\) is \(\mathcal{F}_t\) measurable, increasing and continuous, the definition of \(\tau^+(u_0, a)\) and the a.s. boundedness of \(C_t\) yield that

\[\lim_{a \to \infty} \mathbb{P}(\tau^+(u_0, a) > T) \geq \lim_{a \to \infty} \mathbb{P}(\tau(a) > T)\]

\[= \lim_{a \to \infty} \mathbb{P}\left( \frac{2C_\tau C_t}{a\sigma} R^{2\sigma} \leq 1, t \in [0, T] \right)\]

\[\geq \lim_{a \to \infty} \mathbb{P}\left( \frac{2C_\tau C_T}{a\sigma} R^{2\sigma} \leq 1 \right)\]

\[\geq \lim_{a \to \infty} \mathbb{P}\left( \frac{2C_\tau}{\sigma} R^{2\sigma} \leq a \right) = 1.\]

\[\square\]

**Remark 3.4.** If the noise is space-independent or disappears, applying the arguments in Theorem 2.7 and Theorem 3.3 one can get global existence and blow-up results of the solutions for the damped stochastic NLS equation.
4. Conclusions

In this paper, we consider the influence of both damped term and noise on the stochastic nonlinear Schrödinger equation driven by multiplicative noise. We first show the global existence of the unique solution for damped stochastic NLS equation in critical case and study the exponential integrability and the continuous dependence on the initial data of the solution. Then based on the modified variance identity, we deduce a sharp blow-up condition for damped stochastic NLS equation in supercritical case. Moreover, we prove that the large damped effect can prevent the blow-up with high probability.

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