Quantum Kasner transition in a locally rotationally symmetric Bianchi II universe

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The Belinski-Khalatnikov-Lifshitz (BKL) conjecture predicts a chaotic alternation of Kasner epochs in the evolution of generic classical spacetimes towards a spacelike singularity. As a first step towards understanding the full quantum BKL scenario, we analyze a vacuum Bianchi II model with local rotational symmetry, which presents just one Kasner transition. During the Kasner epochs, the quantum state is coherent and it is thus characterized by constant values of the different quantum fluctuations, correlations and higher-order moments. By computing the constants of motion of the system we provide, for any peaked semiclassical state, the explicit analytical transition rules that relate the parametrization of the asymptotic coherent state before and after the transition. In particular, we obtain the modification of the transition rules for the classical variables due to quantum back-reaction effects. This analysis is performed by considering a high-order truncation in moments (the full computations are performed up to fifth-order, which corresponds to neglecting terms of an order $\hbar^3$), providing a solid estimate about the quantum modifications to the classical model. Finally, in order to understand the dynamics of the state during the transition, we perform some numerical simulations for an initial Gaussian state, that show that the initial and final equilibrium values of the quantum variables are connected by strong and rapid oscillations.

I. INTRODUCTION

It has been conjectured that, close to a spacelike singularity, the classical dynamics of any universe follows a chaotic behavior given by the Belinski-Khalatnikov-Lifshitz (BKL) scenario [1]. According to this, different points decouple and the dynamics of each point can be described by a Bianchi IX universe, which is characterized by an alternation of different Kasner epochs (corresponding to a Bianchi I vacuum or Kasner solution) connected by quick transitions (as compared with the duration of the epochs). The alternation among epochs implies that different (anisotropic) spatial directions are squeezed and stretched during the evolution towards the singularity, resulting in the mentioned chaotic behavior. One relevant characteristic of this conjecture is that this behavior is dominated by the vacuum solution, and the introduction of matter (besides particular cases) does not change the qualitative behavior.

The analysis of Bianchi IX models can be very complicated and it is often convenient to describe the system in terms of the Misner variables [2,3], in the context of the so-called Mixmaster Universe; that is, instead of using the scale factors, the system is characterized in terms of the spatial volume and two anisotropic shape-parameters. The dynamics of these models is then described as a free particle with a potential that drives the transitions between Kasner regimes via exponential walls [2,4,5], in the so-called “cosmological billiard” [6,7]. A simpler scenario corresponds to a Bianchi II universe. This model presents only one transition between two different Kasner epochs on its evolution towards the singularity. Nonetheless, any Bianchi IX spacetime can be understood as a succession of Bianchi II models providing transitions among different Kasner epochs [8].

This picture of the classical dynamics remains being a conjecture, although it is widely accepted and numerous numerical studies confirm the described behavior [9–11]. But let us remark that the BKL scenario takes place very close to the singularity. Therefore, at this stage, one would expect quantum effects to become relevant in the analysis. However, it is still unclear how this scenario can be modified when quantum effects are taken into account. In the literature, there are several approaches to quantum models of Bianchi IX, focused on different features and questions (as the avoidance of the singularity or the survival of the chaotic behavior) [12–23], as well as simpler approaches to Bianchi II quantum models [24–27].

In particular, one question concerns the survival of the structure of Kasner epochs connected by quick transitions, and whether they follow the same transition rules as in the classical picture. In this paper, we analyze this question within the context of a simple locally rotationally symmetric (LRS) Bianchi II model. We first analyze the canonical structure and exact solutions of the classical model, identifying the transition rules of the canonical variables of the system. Then we develop an exact quantization of the model and focus on analyzing the survival of the Kasner transition and the quantum modification of the transition rules. For such a purpose, we will perform a decomposition of the wavefunction into its infinite set of moments following the framework first presented in [28]. This formalism is very useful to understand the dynamics of peaked semiclassical states and it has already
been applied to the analysis of a variety of cosmological models; see, e.g., [29–38].

The rest of the paper is organized as follows. In Sec. II the canonical structure of the classical model is described in terms of Misner variables and the classical transition rules are obtained. The full quantum analysis is then performed in Sec. III, which is divided into four subsections. In Subsec. IIIA the quantum moments and their equations of motion are derived, whereas in Subsec. IIIB the constants of motion of the system are constructed. Making use of these constants of motion, in Subsec. IIIC the quantum transition rules for any semiclassical peaked state are obtained. Subsec. IIID completes the study with a numerical simulation that describes in detail the dynamics during the transition. The discussion of the results and possible future developments are finally presented in Sec. IV.

II. CANONICAL ANALYSIS OF THE CLASSICAL SYSTEM

The metric of a general homogeneous but anisotropic universe can be expressed as follows

\[ ds^2 = -N^2 dt^2 + \sum_{i,j,k=1}^{3} a^2_{i} l^k_i l^k_j dx_i dx_j, \]  

(1)

where \( N \) is the lapse function, \( a_k \) is the scale factor in the \( l^k_i = (l^k_1, l^k_2, l^k_3) \) spatial direction, and the \( l^k_i \) vectors (or Kasner axes) form an orthonormal triad that determines the spatial direction of the anisotropic expansion or contraction of the universe. For the Bianchi II model, these vectors verify the following conditions,

\[ \sum_{i,j=1}^{3} (l^k_{i,j} - l^k_{j,i}) l^k_{n,m} = \delta_{k,4} \epsilon_{knm}, \]

with \( \epsilon_{knm} \) being the complete antisymmetric Levi-Civita symbol.

A very convenient way to describe this spacetime is given by the variables introduced by Misner [2, 3]. The three different scale factors are written as

\[ a_k = e^{\alpha + \beta_k}, \]  

(2)

where \( \alpha \) encodes the spatial volume, \( e^\alpha = (a_1a_2a_3)^{1/3} \), and the variables \( \beta_k \) satisfy the constraint \( \beta_1 + \beta_2 + \beta_3 = 0 \). Therefore, the following two independent shape-parameters can be defined

\[ \beta_+ := -\frac{1}{2} \beta_3 = -\frac{1}{2} \ln \left( \frac{a_3}{(a_1a_2a_3)^{1/3}} \right), \]

\[ \beta_- := \frac{1}{2\sqrt{3}} (\beta_1 - \beta_2) = \frac{1}{2\sqrt{3}} \ln \left( \frac{a_1}{a_2} \right). \]  

(3)

As one can directly check, the range of \( \alpha \) and \( \beta_\pm \) corresponds to the whole real line.

In this paper we will focus on the analysis of the vacuum case. For this Bianchi II vacuum model, the Hamiltonian constraint reads

\[ C = \frac{1}{2} e^{-3\alpha} (p_\alpha^2 + p_-^2 + p_+^2) + e^\alpha U(\beta_+, \beta_-) = 0, \]  

(4)

where \( p_\alpha = -\frac{3\alpha}{N} \frac{da}{dx} \) and \( p_\pm = e^{3\alpha} \frac{d\beta_\pm}{dx} \) are the conjugate momenta of \( \alpha \) and \( \beta_\pm \), respectively, and the potential term takes the value \( U(\beta_+, \beta_-) = e^{-8\beta_3} \). Other Bianchi models are described by this very same Hamiltonian, but with a different form of the potential. In particular, the vacuum Bianchi I –or Kasner– model, which will be relevant in the subsequent discussion, is given by \( U(\beta_+, \beta_-) = 0 \).

Since it is a monotonic function, in these models it is usual to choose \( \alpha \) as the internal time. The above constraint is then deparametrized to define the physical Hamiltonian

\[ H := -p_\alpha = \left[ p_\alpha^2 + p_-^2 + 2 e^{4\alpha} U(\beta_+, \beta_-) \right]^{1/2} = \left( p_+^2 + p_-^2 + 2 e^{4\alpha - 8\beta_3} \right)^{1/2}. \]  

(5)

In order to understand and study this system analytically in detail, we will introduce a further simplifying assumption, and focus on models with a preferred spatial direction, which are known as locally rotationally symmetric (LRS) spacetimes\(^1\). In this case, by choosing the third direction as the preferred one, this corresponds to assuming \( a_1 = a_2 \) and thus the shape-parameter \( \beta_- \), as well as its conjugate momentum \( p_- \), are vanishing: \( \beta_- = 0 = p_- \). Therefore, defining for compactness \( \beta := \beta_+ \) and \( p := p_+ \), in this LRS model the Hamiltonian (5) is reduced to

\[ H = (p^2 + 2 e^{4\alpha} U)^{1/2} = (p^2 + 2 e^{4\alpha - 8\beta})^{1/2}. \]  

(6)

This is a system with just one degree of freedom \((\beta, p)\), and the equations of motion for the canonical pair can readily be obtained,

\[ \dot{\beta} = \frac{p}{H}, \]  

(7)

\[ \dot{p} = \frac{8}{H} e^{4\alpha} U = \frac{8}{H} e^{4\alpha - 8\beta}, \]  

(8)

where the dot represents the derivative with respect to the internal time \( \alpha \).

The regions where the potential term \( e^{4\alpha} U \) has a negligible contribution to the Hamiltonian \( H \), that is, where the dimensionless ratio \( r = 2 e^{4\alpha} U / p^2 \) is very small \( r \ll 1 \), are the so-called Kasner regimes or epochs. (Note that, as commented above, in the Kasner model the potential is

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\(^1\) These spacetimes have been largely studied in cosmology; check, e.g., [39–48].
Towards the singularity

2nd Kasner regime

β

Towards the singularity

1st Kasner regime

α

Towards the singularity

FIG. 1. Classical evolution of the variable β with respect to α, where we have chosen p > 0 for large values of α and β.

FIG. 2. Classical evolution of the variable p with respect to α, where we have chosen p > 0 for large values of α and β.

all along the evolution, these quantities shall be the same for both Kasner epochs. Therefore, one can write them in terms of the corresponding parameters in each Kasner epoch, before (α → +∞) and after (α → −∞) the transition, and obtain the relations c = c(c, p) and ˜p = ˜p(c, ˜p). These relations will be known as the transition rules.

The first nontrivial task is thus to find the constants of motion. Any conserved quantity R = R(α, β, p) must obey the following equation

$0 = \frac{dR}{d\alpha} = \frac{\partial R}{\partial \alpha} + \frac{\partial R}{\partial \beta} \frac{d\beta}{d\alpha} + \frac{\partial R}{\partial p} \frac{dp}{d\alpha}$

$= \frac{\partial R}{\partial \alpha} + \frac{\partial R}{\partial \beta} \frac{d\beta}{dp} - \frac{\partial R}{\partial \beta} \frac{dH}{dp} \frac{\partial H}{\partial \beta},$ (10)

where we have made use of definitions (7)-(9). In order to find the solution, it is convenient to perform the following transformation α → H, so that R = R(H, β, p), and the above equation reads

$0 = \frac{\partial R}{\partial H} \frac{dH}{d\alpha} + \frac{\partial R}{\partial \beta} \frac{d\beta}{dp} - \frac{\partial R}{\partial \beta} \frac{dH}{dp} \frac{\partial H}{\partial \beta}.$ (11)

Replacing in this expression the form of the Hamiltonian (6) one obtains,

$0 = 2(H^2 - p^2) \frac{dR}{dH} + \frac{\partial R}{\partial \beta} p + 4(H^2 - p^2) \frac{\partial R}{\partial p}.$ (12)
The general solution to this equation yields
\[ R = F \left[ 2H - p, e^{2(\alpha + \beta)}(H - p) \right], \]  
(13)
for any generic function \( F \). Consequently, we will consider the following two independent conserved quantities to analyze the transition
\[ R_1 := 2H - p, \]
(14)
\[ R_2 := e^{2(\alpha + \beta)}(H - p). \]
(15)

In the Kasner regimes, where the ratio is very small \( r \ll 1 \), the conserved quantities are approximately given by
\[ R_1 \approx 2|p| - p, \]
(16)
\[ R_2 \approx e^{2(\alpha + \beta)} \left| p \right| \left( 1 + \frac{e^{4\alpha - 8\beta}}{p^2} \right) - p \].
(17)

By considering the functional form for \( \beta \) in both asymptotic regions, and taking into account that the initial value of \( p \) is chosen as positive, \( p > 0 \), the requirement of conservation of the above quantities provides the following two equations,
\[ \bar{p} = [2\text{sign}(\bar{p}) - 1] \bar{p}, \]
(18)
\[ e^{-6\sigma} \bar{p} = \frac{e^{\alpha}}{p} \cdot e^{\beta} e^{2(1+\text{sign}(\bar{p})\alpha + 2\bar{c})} \]
\[ \times \left[ \text{sign}(p) \left( 1 + \frac{e^{4(1-2\text{sign}(\hat{p})\alpha - 8\bar{c})}}{p^2} \right) - 1 \right], \]
which relate the parameters of the first Kasner epoch \((\bar{c}, \bar{p})\) with those of the second Kasner regime \((\hat{c}, \hat{p})\). It is then straightforward to solve this system and obtain the classical transition rules
\[ \bar{p} = -\frac{1}{3} \bar{p}, \]
(20)
\[ \bar{c} = -3\bar{c} - \frac{1}{2} \ln \left( \frac{2}{3} \sigma \right). \]
(21)

III. QUANTUM ANALYSIS

Once we have provided a canonical description of the classical system and the classical results are under control, we are ready to develop the quantum analysis. The present section is divided into four subsections. In Subsec. III A we will define the quantum moments, which will be the basic variables to describe the quantum states. Their equations of motion are obtained and solved in the asymptotic Kasner regimes. In particular, it is shown that, for semiclassical states, the qualitative dynamical picture of the system is kept as in the classical case: namely, there will be two different Kasner epochs with a transition connecting them. Following the classical analysis, in Subsec. III B the conserved quantities for the quantum system will be derived. These constants of motion will be then used to obtain analytically the quantum transition rules between the two Kasner epochs in Subsec. III C. This will provide the asymptotic behavior of all the quantum moments for any initial state. Nonetheless, in order to understand better the dynamics during the transition, in Subsec. III D a numerical implementation of the system of equations is performed for the particular case of a Gaussian initial state.

A. Set up

The quantum dynamics of this model is governed by a Hamiltonian \( \hat{H} \), which is obtained by promoting the classical Hamiltonian (6) to an operator. We will analyze the dynamics by using a formalism based on a moment decomposition of the wave function [28]. More precisely, we define the quantum moments as
\[ \Delta(\beta^i p^j) := \left\langle (\hat{p} - \langle \hat{p} \rangle)^i (\hat{p} - \langle \hat{p} \rangle)^j \right\rangle_{\text{Weyl}}, \]
(22)
where \( i, j \in \mathbb{N} \), \( \left\langle \right\rangle \) indicates the expectation value and the Weyl subscript refers to a totally symmetric ordering of the operators. This infinite set of moments, together with \( \langle \hat{p} \rangle \) and \( \langle \hat{p} \rangle \), fully characterizes the quantum state. The sum of the indices \( i + j \) will be referred as the order of the corresponding moment. This order will be later used to introduce a cut-off on this infinite system.

The dynamics of the moments (22) is driven by a quantum effective Hamiltonian, that is defined as the expectation value of the operator \( \hat{H} \). By performing a Taylor expansion around the expectation values of \( \hat{p} \) and \( \hat{p} \), this effective Hamiltonian can be written as follows,
\[ H_Q = \langle \hat{H}(\hat{p}, \hat{p}) \rangle = H + \sum_{i+j=2}^{\infty} \frac{1}{i!j!} \frac{\partial^{i+j} H(\beta, p)}{\partial \beta^i \partial p^j} \Delta(\beta^i p^j), \]
(23)
where \( \beta := \langle \hat{p} \rangle \) and \( p := \langle \hat{p} \rangle \) have been defined, and \( H(\beta, p) \) is the classical Hamiltonian [6]. As it can be seen, the Hamiltonian operator has been chosen as Weyl-ordered.

The equations of motion for the variables of the system \((\beta, p, \Delta(\beta^i p^j))\) can then be obtained by computing their Poisson brackets with the effective Hamiltonian \( H_Q \):
\[ \frac{d\beta}{d\alpha} = \{ \beta, H_Q \} = \frac{\partial H_Q}{\partial p}, \]
(24)
\[ \frac{dp}{d\alpha} = \{ p, H_Q \} = -\frac{\partial H_Q}{\partial \beta}, \]
(25)
\[ \frac{d\Delta(\beta^i p^j)}{d\alpha} = \{ \Delta(\beta^i p^j), H_Q \} \]
(26)
\[ = \sum_{m,n=0}^{+\infty} \frac{1}{m!n!} \frac{\partial^{m+n} H(\beta, p)}{\partial \beta^m \partial p^n} \{ \Delta(\beta^i p^j), \Delta(\beta^m p^n) \}, \]
\{\Delta(\beta^k p^l), \Delta(\beta^m p^n)\} = ml \Delta(\beta^k p^{l-1}) \Delta(\beta^m p^{n-1}) - kn \Delta(\beta^{k-1} p^l) \Delta(\beta^m p^{n-1}) + \sum_{r=1}^{S} \left( \frac{i\hbar}{2} \right)^{r-1} K_{klmn}^r \Delta(\beta^{k-m-r} p^{l+n-r}),

where \( S := \min(k + l, m + n, k + m, l + n) \) and the coefficients \( K_{klmn}^r \) are defined as

\[ K_{klmn}^r = \sum_{s=0}^{r} (-1)^s s!(r-s)! \left( \begin{array}{c} k \\ r-s \end{array} \right) \left( \begin{array}{c} l \\ s \end{array} \right) \left( \begin{array}{c} m \\ s \end{array} \right) \left( \begin{array}{c} n \\ r-s \end{array} \right). \]

Note that, in general, (24)–(26) form an infinite system of coupled equations. Therefore, in order to analyze the system, one usually introduces a truncation by considering negligible all the moments \( \Delta(\beta^n p^m) \) with an order \( n + m \) higher than a certain cut-off \( N \). A moment \( \Delta(\beta^n p^m) \) has the dimensions of \( h^{(n+m)/2} \) and, in this sense, this kind of truncation is valid during semiclassical regimes, that is, as long as the state remains peaked on a classical trajectory.

Nonetheless, if the Hamiltonian is at most quadratic on the basic variables, the different orders decouple and one usually is able to solve the complete infinite set of equations. In fact, this is the case in the Kasner epochs analyzed above (when the ratio \( r = 2e^{\beta_0 - 3\beta_1} / p^2 \) is negligible), as then the classical Hamiltonian is linear in \( p \). In this approximation, the equations of motion take the form,

\[ \frac{d\beta}{d\alpha} \approx \text{sign}(p), \quad \frac{dp}{d\alpha} \approx 0, \quad \frac{d\Delta(\beta^k p^l)}{d\alpha} \approx 0, \]

which are immediate to solve

\[ \beta \approx \text{sign}(p)\alpha + c, \]

\[ p \approx \text{const.}, \]

\[ \Delta(\beta^k p^l) \approx \text{const.}, \]

with the integration constant \( c \). As can be seen, the dynamics during the Kasner regime is that of a coherent state. There is no quantum back-reaction, as the moments are constants of motion and evolve decoupled from the expectation values, and the behavior of the expectation values remains as in the classical case. In summary, the parameters that characterize each quantum Kasner epoch will be, as in the classical case, the constants \( c \) and \( p \), along with the infinite constant set of moments \( \Delta(\beta^k p^l) \).

Therefore, our goal is to obtain the quantum transition rules, that relate the set of parameters that describe the initial Kasner epoch \( (\pi, \bar{p}, \Delta(\beta^i p^j)) \), which will be denoted with a bar, with the parametrization of the second Kasner regime \( (\beta, \bar{p}, \Delta(\beta^i p^j)) \), denoted with a tilde. In this way, we will be able to see how the classical transition rules, described by equations (20)–(21), are modified by quantum effects. For such a purpose, let us obtain the constants of motion for the quantum system.

B. Conserved quantities

In order to obtain the constants of motion, we first note that, since we are dealing with an infinite system of equations, we will need an infinite amount of them. Nonetheless, if two independent conserved operators \( \hat{R}_1 \) and \( \hat{R}_2 \) are known, their expectation values \( \langle \hat{R}_1 \rangle \) and \( \langle \hat{R}_2 \rangle \) would be trivially conserved. In fact, one would be able to generate an infinite set of conserved quantities, just by taking the expectation values of any product between them, that is,

\[ \langle \hat{R}_i^n \hat{R}_j^m \rangle = \text{const.}, \quad \text{for } m, n \in \mathbb{N} \text{ and } i, j \in \{1, 2\}. \]

Obtaining the operators \( \hat{R}_1 \) and \( \hat{R}_2 \) might be a very complicated task, so we will proceed order by order. Up to third-order in moments, one can construct these conserved operators just by directly promoting the conserved classical quantities (14) to operators:

\[ \hat{R}_1 = 2\hat{H} - \hat{p}, \]

\[ \hat{R}_2 = e^{2(\alpha + \beta)\hat{H}}(\hat{H} - \hat{p}). \]

Up to the mentioned order, the expectation values of these operators can then be written in terms of moments by performing an expansion around the expectation values of the basic variables

\[ \langle \hat{R}_1 \rangle = R_1 + \sum_{i+j=2}^{3} \frac{\partial^{i+j} R_1}{\partial \beta^i \partial p^j} \Delta(\beta^i p^j), \]

\[ \langle \hat{R}_2 \rangle = R_2 + \sum_{i+j=2}^{3} \frac{\partial^{i+j} R_2}{\partial \beta^i \partial p^j} \Delta(\beta^i p^j). \]

Note that these expressions are valid for any ordering of the basic operators in the definitions (31)–(32). Any re-ordering of the operators would give a factor proportional
to $\hbar^2$ in the above expansions, which is of fourth-order
and thus negligible at this level of approximation. Therefore,
up to third order, one can iteratively construct all
the necessary constants of motion.

For instance, at second-order, the system is described
by the two expectation values $\langle \beta, p \rangle$ and the three
moments $\Delta(\beta^i\beta^j)$ with $i + j \leq 2$, namely the correlation $\Delta(\beta p)$ and the two fluctuations $\Delta(\beta^2)$ and $\Delta(p^2)$. Therefore,
in order to completely solve the system, one needs to
calculate five constants of motion. The expectation
values $\langle \hat{R}_1 - R_1 \rangle^2$, $\langle \hat{R}_1 - R_1 \rangle(\hat{R}_2 - R_2) + (\hat{R}_2 - R_2)(\hat{R}_1 - R_1)$,
and $\langle (\hat{R}_2 - R_2)^2 \rangle$ can be defined as,

$$
\langle (\hat{R}_1 - R_1)^2 \rangle, \tag{35}
\langle (\hat{R}_1 - R_1)(\hat{R}_2 - R_2) + (\hat{R}_2 - R_2)(\hat{R}_1 - R_1) \rangle, \tag{36}
\langle (\hat{R}_2 - R_2)^2 \rangle. \tag{37}
$$

Writing these expressions in terms of moments, as performed
in (33)–(34), and truncating the series at second-
order, one ends up with five constants of motion for five
variables and the system is thus completely determined.

Note that, for convenience, we have defined the conserved
quantities as expectation values of powers of differences,
like $(R_i - R_j)^2$, instead of expectation values of powers,
like $R_i^2$, and with a symmetric ordering of $R_1$ and $R_2$. In
this way one automatically gets pure second-order real
expressions (there is no zeroth order contributions when
expanding these expectation values).

The rationale within a third-order truncation scheme is
exactly equivalent to the one applied at second order, but
with some more variables. In this case the number of variables is nine: two expectation values $(\beta, p)$, three second-
order moments, and four third-order moments (the two pure fluctuations $\Delta(\beta^3)$ and $\Delta(p^3)$, in combination with
the two high-order correlations $\Delta(\beta^2 p^3)$, $\Delta(\beta^3 p^2)$). In
addition to the conserved quantities $\Delta(33)$–$\Delta(37)$, one can construct
another four by computing the following expectation values

$$
\langle (\hat{R}_1 - R_1)^3 \rangle, \tag{38}
\langle (\hat{R}_1 - R_1)^2(\hat{R}_2 - R_2) + (\hat{R}_2 - R_2)(\hat{R}_1 - R_1)^2 \rangle, \tag{39}
\langle (\hat{R}_1 - R_1)(\hat{R}_2 - R_2)^2 + (\hat{R}_2 - R_2)^2(\hat{R}_1 - R_1) \rangle, \tag{40}
\langle (\hat{R}_2 - R_2)^3 \rangle. \tag{41}
$$

From fourth order on, the situation gets more involved,
as the ordering of the basic operators in the definitions
$\langle 42 \rangle$–$\langle 43 \rangle$ comes into play. We find out that the expectation value of the operator $\hat{R}_1$, with a completely symmetric ordering of basic operators,

$$
\hat{R}_1 = 2\hat{H} - \hat{p} \big|_{\text{Weyl}} \tag{42}
$$

is conserved up to seventh-order in moments. Nonetheless,
this is not the case for $\hat{R}_2$: by considering a Weyl-
ordered form for $\hat{R}_2$, its time derivative turns out not to be vanishing, due to some terms proportional to $\hbar^2$. Still, we have been able to explicitly integrate these non-vanishing terms and, by subtracting the result from the expectation value of the Weyl-ordered operator version of $\hat{R}_2$, construct the corresponding conserved quantity.
This leads to the following form for the second conserved operator,

$$
\hat{R}_2 = e^{2(\alpha + \beta)}(\hat{H} - \hat{p}) + \hbar^2 e^{2(\alpha + \beta)}(\hat{H} - \hat{p})^2 \frac{15\hat{p}\hat{H}^2 + 11\hat{H}^3 + 3\hat{H}\hat{p}^2 - 4\hat{p}^3}{2\hat{H}^4\hat{R}_1^2} \big|_{\text{Weyl}}. \tag{43}
$$

In summary, up to seventh-order in moments, the two
operators $\langle 42 \rangle$–$\langle 43 \rangle$ are conserved. Therefore, one can use these expressions to generate, at each order, all
the necessary constants of motion by computing expectation
values of their products, as explained above for second-
and third-order truncations.

C. Quantum Kasner transition

At this point, in order to obtain the quantum transition
rules, one can proceed systematically. One just needs to
compute the expectation values of products of the con-
served operators $\langle 42 \rangle$–$\langle 43 \rangle$, and write them in terms of
moments by performing an expansion around the expectation
values $\beta$ and $p$. These constants of motion are then evaluated on each asymptotic Kasner regions, by consid-
ering the behavior of different variables $\langle 27 \rangle$–$\langle 29 \rangle$. The requirement of conservation of these quantities leads to
a system of equations that relates the parameters of the first Kasner epoch $(\bar{\tau}, \bar{p}, \bar{\Delta}(\beta^3 p^2))$ with those of the second
Kasner regime $(\tilde{c}, \tilde{p}, \tilde{\Delta}(\beta^5 p^4))$. This system of equations must then be solved to obtain the quantum transition rules.

For definiteness, we will complete this procedure with
a fifth-order truncation, by assuming that all moments
$\Delta(\beta^i p^j)$ with $i + j > 5 = N$ are negligible. Up to this
order, there are 20 variables and one needs to construct
the same amount of constants of motion.

Regarding the classical variables we find out that, on
the one hand, making use of the conserved quantity $\langle \hat{R}_1 \rangle$, which takes the simple form $\langle \hat{R}_1 \rangle = \langle 2|p| - p \rangle$ at every order in the Kasner regimes, one can immediately obtain the
transition rule for the momentum \( p \),

\[
\tilde{p} = \frac{1}{3} \dot{p}.
\]  

(44)

This expression keeps the same form as its classical counterpart [20] and it is not affected by the moments. On the other hand, the quantum back-reaction does affect the transition rule for the parameter \( c \), which encodes the asymptotic behavior of the shape-parameter \( \beta \), and it takes the form

\[
\tilde{c} = -3\tau - \frac{1}{2} \ln \left( \frac{2\pi^2}{3} \right) + \sum_{n=2}^{N} (-1)^n \frac{\Delta(p^n)}{np^n}.
\]  

(45)

However, note that only the initial relative pure fluctuations of the momentum \( \Delta(p^n)/p^n \) appear in this relation.

Concerning the moments, their transition rules are more involved, and they are explicitly displayed in App. A. Nevertheless, the general transition rule can be written as follows\(^3\)

\[
\frac{\Delta(\beta^n p^m)}{p^m} = (-3)^n \frac{\Delta(\beta^n p^m)}{p^m} + \sum_{r=0}^{r_{\text{max}}} \sum_{k=0}^{k_{\text{max}}} \sum_{l=m_{\text{min}}}^{l_{\text{max}}} a_{nmklr} \frac{\Delta(p^n)}{p^n} \frac{\Delta(\beta^l p^{m+l-r-k})}{p^{m+l-r-k}},
\]  

(46)

with \( b_{nmklr} = (-3)^m a_{nmklr} \). In this way, the transition rule for the relative moments does not explicitly depend on the values of the asymptotic classical parameters \( p \) or \( c \).

In the above expressions, the lower limit for \( l \) is defined as \( l_{\text{min}} = \max\{-r, -k - (n + m)\} \). This ensures that the right-hand sides of (46) and (47) are exclusively given by contributions of order higher or equal to \( n + m \). Moreover, the upper limits of two of the sums, \( r_{\text{max}} \) and \( k_{\text{max}} \), are an artifact of the truncation we are considering, and they just impose that the product of two moments inside the sum to be, at most, of order \( N \), that is, \( n + m + l + r \leq N \).

However, the upper limit in the sum in \( k \) is independent of the truncation, and thus physically meaningful. More precisely, the largest possible value for \( k \) is \( n - 1 \), that is, lower that the index in \( \beta \) of the final moment under consideration. Therefore, this means that the asymptotic form of a moment \( \Delta(\beta^n p^m) \) after the Kasner transition, depends on the initial value of the very same moment \( \Delta(\beta^p) \) and all moments \( \Delta(\beta^l p^q) \), with any \( q \geq 0 \) but with \( 0 \leq k \leq n - 1 \). This translates into the fact that for a given moment, the larger the index of the shape parameter \( n \), the more complicate is the corresponding transition rule, as more initial moments enter into play.

In particular, the relative pure fluctuations of the momentum \( p \), that is moments with \( n = 0 \), have a very simple transition rule. In fact, they have the same asymptotic value after and before the transition

\[
\frac{\Delta(p^n)}{p^n} = \frac{\Delta(p^n)}{p^n}.
\]  

(48)

This is due to the fact that, at any order, the conserved quantity \( \langle (\dot{R}_1 - \dot{\bar{R}}_1)^m \rangle \) takes the simple form \( \langle (\dot{R}_1 - \dot{\bar{R}}_1)^m \rangle = (2\text{sgn}(p) - 1)^m \Delta(p^m) \) in the Kasner regime.

For \( n = 1 \) the transition rule complicates a bit,

\[
\frac{\Delta(\beta^p)}{p} = -3 \frac{\Delta(\beta^p)}{p} + \sum_{k=m+1}^{N} (-1)^{k-m} \frac{\Delta(p^k)}{p^k} + \sum_{k=2}^{N-m} (-1)^{k-1} \frac{\Delta(p^m)}{p^m} \frac{\Delta(p^l)}{p^l},
\]  

(49)

but, as can be seen, only moments \( \Delta(\beta^m p^m) \) and \( \Delta(p^m) \) appear on the right-hand side. In order to check the

\[\text{source of the truncation we are considering, and they just impose that the product of two moments inside the sum to be, at most, of order N, that is, n + m + l + r \leq N.} \]
transition rules for higher values of \( n \), we refer the reader to App. A.

Another important property of the transition rule (47) is that it is composed by a linear \((r = 0)\) as well as quadratic \((r \neq 0)\) terms on initial moments. Nonetheless, concerning these quadratic combinations, note that one of the components is always given by pure fluctuations of the momentum \( \Delta(p^2)/p^4 \), which are conserved through the Kasner transition [48].

In expressions (46)–(47), we have chosen to explicitly write the contribution of the initial moment \( \Delta(\beta^n p^m) \) on the right-hand side, instead of including it in the sum. In this way, one can see that the transition produces two distinct effects on the moments.

On the one hand, the moment \( \Delta(\beta^n p^m) \) is modified by a multiplicative factor \((-3)^{n-m}\). Therefore, for moments with more weight on \( \beta \) than on \( p \), that is with \( n > m \), the absolute value of the moment is increased by this factor. On the contrary, for moments with a dominant contribution of \( p \) \((n < m)\), this effect produces a decrease in their absolute value. Finally, moments with \( n = m \) are not affected by this global factor. All in all, this effect produces a squeezing of the state on the phase space by compressing it in the \( p \) direction, and stretching it in the \( \beta \) direction. The compression in the \( p \) direction, given by the factor \((-3)^m\), is clearly inherited from the transition rule for the momentum [44]. This is why the relative moments \( \Delta(\beta^n p^m)/p^m \) do not suffer such compression in \( p \). The effect in the \( \beta \) direction is also related to the factor \(-3\) that appears in the transition rule (45) for the parameter \( c \) that defines the asymmetric behavior of \( \beta \). But since, in this case, there are other moments involved, it is not straightforward to absorb the factor \((-3)^n\) in the definition of certain relative moments.

On the other hand, the moment \( \Delta(\beta^n p^m) \) is modified by the presence of other moments \( \Delta(\beta^k p^l) \), with \( 0 \leq k \leq n - 1 \). This effect, which couples the final value of a given moment with the initial value of other moments, has not a definite sign, since the numerical factors \( a_{nmklr} \) can be either positive or negative. Therefore, the interpretation of this effect is not so clear as the previous one, and one has to check individual moments (and particular states) to see how each moment is affected.

For instance, if we consider an initial Gaussian state in the momentum \( p \), these effects can easily been observed. Its corresponding moments, i.e., the asymptotic moments before the transition, are given as follows

\[
\Delta(\beta^2 p^2) = 2^{-2(n+m)} \hbar^{2n} \sigma^{2(m-n)} \frac{(2n)! (2m)!}{n! m!},
\]

for \( \forall n, m \in \mathbb{N} \), and vanishing otherwise. These moments will evolve and after the transition, they will transform according to [49]. For the sake of clarity, let us focus on how the second-order moments are modified

\[
\tilde{\Delta}(\beta p) = \frac{\Delta(p^2)}{3p^3} + \frac{\Delta(p^4)}{9p^4},
\]

\[
\tilde{\Delta}(p^2) = \frac{1}{9} \Delta(p^2),
\]

\[
\tilde{\Delta}(\beta^2) = 9 \Delta(\beta^2) + \frac{\Delta(p^2)}{p^2} \left(1 - \frac{\Delta(p^2)}{4p^2}\right) + \frac{11 \Delta(p^4)}{12p^4},
\]

\( p \) being the value of \( p \) in the initial Kasner regime. First of all, from (51) we observe that \( \tilde{\Delta}(\beta p) > 0 \), which means that the transition generates a positive correlation between \( \beta \) and \( p \). Thus, since this correlation is vanishing for any Gaussian state, it is clear that the final state does not belong to this class. Moreover, from relation (52), one can see that the initial state is compressed in the \( p \) direction. However, from the transition rule (53) it is not straightforward to deduce whether the initial state is stretched or compressed in the \( \beta \) direction. Nevertheless, by considering the explicit form (50) for the initial moments, it can be shown that

\[
\frac{\tilde{\Delta}(\beta^2)}{\Delta(\beta^2)} = \frac{5\sigma^6}{14p^6} + \frac{\sigma^4}{p^4} + 9,
\]

which is clearly larger than one. Therefore, an initial Gaussian state is certainly stretched in the \( \beta \) direction.

Finally, it is also interesting to evaluate the Heisenberg uncertainty principle which, for the final state takes the form,

\[
\Delta(p^2) \Delta(\beta^2) - \left(\tilde{\Delta}(\beta p)\right)^2 = \hbar^2 = \frac{\sigma^6 (p^2 - \sigma^2)}{144p^5} \geq 0.
\]

In order to be satisfied, the relative initial fluctuation of the momentum should be small, \( \sigma/p \leq 1 \), which is consistent with our assumption of peaked semiclassical states.

\[\text{D. Analysis of the transition dynamics for a Gaussian initial state}\]

In this section we are interested in examining in detail the evolution of the state and, more specifically, the transition dynamics. In order to develop this study, we will need to perform a numerical analysis. For this purpose, let us consider an initial unsqueezed Gaussian state in the initial Kasner epoch, characterized by a Gaussian width \( \sigma = \sqrt{\hbar} \). The moments for such a state take the form given in equation (50). In addition, for the rest of the variables, we choose as initial conditions \( p = 1 \), \( \beta = 100 \) and \( \alpha = 100 \). In this way we ensure that the potential of the Hamiltonian is negligible and thus the system begins in a Kasner regime.
As expected, the numerical study shows that, evolving the system towards lower values of $\alpha$, the system follows the Kasner dynamics until it undergoes a transition, which happens near $\alpha = 0$. This transition takes place in a very small period of time and then the system reaches its final equilibrium value—the final Kasner regime—very quickly.

On the one hand, concerning the evolution of the expectation values $p$ and $\beta$, we conclude that they remain almost identical as in the classical case, that is, as depicted in Figs. 1 and 2. In fact, only in a very close zoom into the transition (near $\alpha = 0$) we can appreciate a slight modification in their evolution, which is shown in Figs. 3 and 4. In particular, from these figures one can see that quantum effects slightly increase the value of $p$ and $\beta$ around the transition.

On the other hand, in Figs. 5–8 the evolution of the moments has been plotted. As it can be seen, during the initial Kasner regime they hold a constant value, at certain point they begin to perform strong oscillations but then they quickly relax to another constant value, which characterizes the coherent state during the final Kasner regime. It is particularly interesting to observe that the moments that were vanishing in the initial Kasner epoch—due to the choice of the Gaussian state—are not vanishing in the final one. This is indeed an expected outcome according to the transition laws that we have discussed in the previous section. In particular, this means that even when we choose an initial state where there are no correlations between $\beta$ and $p$, the transition generates correlations for the final state.

In fact, all the initial vanishing moments are activated in a similar way: as they approach the transition they experience an excitation and start to grow exponentially, until they begin to oscillate. This behavior is shown in Fig. 9 where one can notice that all of them grow with the same slope in a logarithmic plot. This slope is related to the potential of the Hamiltonian and is found to be equal to $-4$. That is, between the exact Kasner regime and the transition, the moments begin to feel the presence of a nonvanishing potential and their evolution is driven by $\Delta(\beta^n p^m) \propto e^{-4\alpha}$. Furthermore, we see that the lower the order of a given moment, the sooner it is activated, which is in agreement with the semiclassical hierarchy of moments we are assuming. The same applies to the beginning of the oscillation period: lower-order moments begin their oscillation regime before higher-order ones.

The precise dynamics of the moments is quite complicated but, if we look into the transition in more detail, there are some general features that one can see. For instance, we observe that lower-order moments tend to oscillate with bigger amplitudes as higher-order ones, as can be seen in Figs. 5–8. Moreover, at each given order, the pure fluctuations of $p$ have the biggest amplitudes. And the higher the index of $p$ the earlier the moments reach their final equilibrium value.

Furthermore, in general, it can be seen that moments with at least one odd index, i.e., the initially vanishing ones, present more oscillations. In fact, as shown in Fig. 10 pure fluctuations of the shape-parameter $\Delta(\beta^n)$ with an even index $n$ do not oscillate per se; they just have a local minimum in the transition and then grow up to their final value.
FIG. 6. Evolution of the third-order moments, with the whole range of the amplitude in the first plot and with a zoom in the second. \( \Delta (p^2 \beta) \) is depicted in blue, \( \Delta (\beta^2 p) \) in purple, \( \Delta (p^3) \) in orange and \( \Delta (\beta^3) \) in green.

FIG. 7. Evolution of the fourth-order moments, with the whole range of the amplitude in the first plot and with a zoom in the second. \( \Delta (\beta^3 p) \) is depicted in blue, \( \Delta (p^2) \) in purple, \( \Delta (\beta^4) \) in orange, \( \Delta (p^3) \) in green and \( \Delta (\beta^2 p^2) \) in black.

FIG. 8. Evolution of the fifth-order moments, with the whole range of the amplitude in the first plot and with a zoom in the second. \( \Delta (\beta^5) \) is depicted in blue, \( \Delta (p^5) \) in purple, \( \Delta (\beta^4 p^2) \) in orange, \( \Delta (\beta^3 p^3) \) in green, \( \Delta (\beta^2 p^4) \) in black and \( \Delta (\beta p^5) \) in red.

FIG. 9. Logarithmic evolution of the initial vanishing moments with respect to \( \alpha \) as they approach the transition.

FIG. 10. Evolution of \( \Delta (\beta^2) \) and \( \Delta (\beta^4) \) during the transition, in the first and second plot respectively.

IV. DISCUSSION

In this paper, we have considered a quantum locally rotationally symmetric Bianchi II model in order to analyze the transition between Kasner epochs. The behavior of the classical variables through this transition is well-known, but there is an open discussion about how this behavior might be modified when considering quantum
effects that are expected to be relevant close to the singularity.

In order to describe the dynamics of this system, we have considered Misner variables, which are given by the (logarithm of the) spatial volume and two anisotropic shape-parameters. In the locally rotationally symmetric case under consideration, one of the shape-parameters is vanishing. In addition, the Hamiltonian constraint is deparametrized by choosing the (logarithm of the) spatial volume as the internal time, which leaves the system with one physical degree of freedom (β, p), β being the nonvanishing shape-parameter and p its conjugate momentum.

Once performed the quantization of the system, the whole information of the quantum state is encoded in the infinite set of moments (22). Our analysis then shows that, for peaked semiclassical states, the classical qualitative picture is maintained in the quantum regime: there are two asymptotic Kasner regions, connected by a quick transition. In particular, during the Kasner epochs, the asymptotic values of the moments are connected through quick and strong oscillations.

By considering a fifth-order truncation in the moments, which corresponds to neglecting terms of an order ℏ^5 and higher, we have been able to obtain the explicit analytic transition rules, that relate the parametrization of the state before and after the transition. These rules, schematically given by equations (44)–(46) and explicitly displayed in App. A, comprise the most relevant result of the present paper.

The transition rule for the momentum p [44] remains unchanged with respect to its classical counterpart. Nonetheless the transition rule for the parameter c [45], which characterizes the asymptotic behavior of the shape-parameter β, is slightly modified by quantum back-reaction effects. Interestingly, this back-reaction is completely encoded in the relative pure fluctuations of the momentum Δ(p^n)/ρ^n.

Concerning the moments, their transition rule [46] is much more involved, but one can notice certain general properties. In particular, the asymptotic form of a given moment Δ(β^nρ^m) after the transition is given by certain combination of the initial value of all moments Δ(β^kρ^q) with k ≤ n and any q. Therefore, the complexity of the transition rule is largely governed by the index of the shape-parameter n. Furthermore, one can see that the transition introduces two distinct deformations on the state. On the one hand, the initial value of the moment is multiplied by certain coefficient, which produces a squeezing of the state on the phase space by compressing it in the p direction and stretching it in the β direction. On the other hand, the back-reaction of other moments enters as an additive deformation in the transition rule. The physical interpretation of this second effect is though not so clear.

Finally, in order to understand more precisely the dynamics during the transition, in the last section, we have performed some numerical simulations for an initial Gaussian state. The state is deformed throughout the transition, as expected from the transition rules, and the initial vanishing moments are excited. In particular the transition generates a positive correlation of the initial vanishing moments are excited. In particular the transition generates a positive correlation of the system. In addition, we observe that the constant asymptotic values of the moments are connected through quick and strong oscillations.

Let us remark that this analysis has been carried out for a particular Kasner transition. We have developed a first approach to understand the prevalence and changes of Kasner transitions, which play a key role in the chaotic oscillatory BKL behavior, under quantum effects. A further study to a general case should be analyzed in the future to have a full understanding of the quantum survival of the BKL transitions.

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Appendix A: Quantum transition rules

In this appendix we display the complete explicit transition rules for the 20 parameters that describe the Kasner regime of the system up to fifth-order in moments, that is, p, c, and all moments Δ(β^nρ^m) with n + m ≤ 5. In order to remove all explicit dependence on p from the expressions, we provide the transition rules for the relative moments Q^{nm} := Δ(β^nρ^m)/ρ^m. As in the main text, the objects before the transition are denoted with a bar (\bar{p}, \bar{c}, \bar{Q}^{nm}), whereas after the transition they are denoted with a tilde (\tilde{p}, \tilde{c}, \tilde{Q}^{nm}):

\begin{align}
\tilde{p} &= -\frac{1}{3} p, \\
\tilde{c} &= -3c - \frac{1}{2} \ln \left( \frac{2p^2}{3} \right) + \frac{1}{2} \tilde{Q}^{02} - \frac{1}{3} \tilde{Q}^{03} + \frac{1}{4} \tilde{Q}^{04} - \frac{1}{5} \tilde{Q}^{05}.
\end{align}
\[ \bar{Q}^{11} = -3\bar{Q}^{11} - \bar{Q}^{02} + \frac{1}{2}\bar{Q}^{03} - \frac{1}{3}\bar{Q}^{04} + \frac{1}{4}\bar{Q}^{05}, \]
\[ \bar{Q}^{20} = \bar{Q}^{02} + 6\bar{Q}^{11} + 9\bar{Q}^{20} - \bar{Q}^{03} - 3\bar{Q}^{12} + 2\bar{Q}^{13} - \frac{1}{4}\left(\bar{Q}^{02}\right)^2 + \frac{11}{12}\bar{Q}^{04} + \frac{1}{3}\bar{Q}^{02}\bar{Q}^{03} - \frac{5}{6}\bar{Q}^{05} + \frac{3}{2}\bar{Q}^{14}, \]
\[ \bar{Q}^{02} = \bar{Q}^{02}, \]
\[ \bar{Q}^{30} = -\frac{9}{4}\bar{Q}^{10} - 27\bar{Q}^{20} - 9\bar{Q}^{12} - \frac{3}{2}\left(\bar{Q}^{02}\right)^2 + \frac{3}{2}\bar{Q}^{04} + 9\bar{Q}^{13} + \frac{27}{2}\bar{Q}^{02}\bar{Q}^{11} + \frac{7}{4}\bar{Q}^{05} - \frac{33}{4}\bar{Q}^{14} - 9\bar{Q}^{23} + 6\bar{Q}^{03}\bar{Q}^{11} + 9\bar{Q}^{03}\bar{Q}^{20} + \frac{5}{2}\bar{Q}^{02}\bar{Q}^{03} + \frac{9}{2}\bar{Q}^{02}\bar{Q}^{12}, \]
\[ \bar{Q}^{03} = \bar{Q}^{03}, \]
\[ \bar{Q}^{21} = 9\bar{Q}^{21} + 6\bar{Q}^{12} + \bar{Q}^{03} - \bar{Q}^{04} - 3\bar{Q}^{13} + 3\bar{Q}^{02}\bar{Q}^{11} + 2\bar{Q}^{14} + \frac{11}{12}\bar{Q}^{05} - 2\bar{Q}^{03}\bar{Q}^{11} - \frac{7}{6}\bar{Q}^{02}\bar{Q}^{03}, \]
\[ \bar{Q}^{12} = -3\bar{Q}^{12} - \bar{Q}^{03} + \frac{1}{2}\bar{Q}^{04} + \frac{1}{2}\left(\bar{Q}^{02}\right)^2 - \frac{1}{3}\bar{Q}^{05} + \frac{1}{3}\bar{Q}^{02}\bar{Q}^{03}, \]
\[ \bar{Q}^{40} = 81\bar{Q}^{40} + \bar{Q}^{04} + 12\bar{Q}^{13} + 54\bar{Q}^{22} + 108\bar{Q}^{31} - 2\bar{Q}^{05} - 18\bar{Q}^{14} - 54\bar{Q}^{23} - 54\bar{Q}^{32} + 2\bar{Q}^{02}\bar{Q}^{30} + 18\bar{Q}^{02}\bar{Q}^{12} + 54\bar{Q}^{02}\bar{Q}^{11} + 54\bar{Q}^{02}\bar{Q}^{03}, \]
\[ \bar{Q}^{04} = \bar{Q}^{04}, \]
\[ \bar{Q}^{13} = -3\bar{Q}^{13} - \bar{Q}^{04} + \frac{1}{2}\bar{Q}^{05} - \frac{1}{2}\bar{Q}^{02}\bar{Q}^{03}, \]
\[ \bar{Q}^{31} = -27\bar{Q}^{31} - 9\bar{Q}^{13} - 27\bar{Q}^{22} - \bar{Q}^{04} + \frac{3}{2}\bar{Q}^{05} + 9\bar{Q}^{14} + \frac{27}{2}\bar{Q}^{23} - \frac{3}{2}\bar{Q}^{02}\bar{Q}^{03} - 9\bar{Q}^{02}\bar{Q}^{12} - \frac{27}{2}\bar{Q}^{02}\bar{Q}^{31}, \]
\[ \bar{Q}^{22} = 9\bar{Q}^{22} + \bar{Q}^{04} + 6\bar{Q}^{13} - 3\bar{Q}^{14} - \bar{Q}^{05} + \bar{Q}^{02}\bar{Q}^{03} + 3\bar{Q}^{02}\bar{Q}^{12}, \]
\[ \bar{Q}^{50} = -243\bar{Q}^{50} - \bar{Q}^{05} - 15\bar{Q}^{14} - 90\bar{Q}^{23} - 270\bar{Q}^{32} - 405\bar{Q}^{41}, \]
\[ \bar{Q}^{05} = \bar{Q}^{05}, \]
\[ \bar{Q}^{14} = -3\bar{Q}^{14} - \bar{Q}^{05}, \]
\[ \bar{Q}^{41} = 81\bar{Q}^{41} + \bar{Q}^{05} + 12\bar{Q}^{14} + 54\bar{Q}^{23} + 108\bar{Q}^{32}, \]
\[ \bar{Q}^{23} = 9\bar{Q}^{23} + 6\bar{Q}^{14} + \bar{Q}^{05}, \]
\[ \bar{Q}^{32} = -27\bar{Q}^{32} - \bar{Q}^{05} - 9\bar{Q}^{14} - 27\bar{Q}^{23}. \]

[1] V. A. Belinskii, I. M. Khalatnikov, and E. M. Lifschitz, “A general solution of the Einstein equations with a time singularity”, Adv. Phys. 13 (1982) 639.
[2] C. W. Misner, “Mixmaster Universe”, Phys. Rev. Lett. 22 (1969) 1071.
[3] C. W. Misner, “Quantum Cosmology I”, Phys. Rev. 186 (1969) 1319.
[4] M. Bojowald, “Loop quantum cosmology”, Living Rev. Rel. 8 (2005) 11.
[5] M. A. H. MacCallum, “Mixmaster Universe Problem”, Nature Physical Science 230 (1971) 112.
[6] T. Damour, M. Henneaux, and H. Nicolai, “Cosmological billiards”, Class. Quant. Grav. 20 (2003) R145.
[7] J. M. Heinzle, C. Uggla, and N. Rohr, “The Cosmological billiard attractor”, Adv. Theor. Math. Phys. 13 (2009) 293.
[8] J. M. Heinzle and C. Uggla, “A New proof of the Bianchi type IX attractor theorem”, Class. Quant. Grav. 26
M. Bojowald and A. Skirzewski, “Effective equations of motion for quantum systems”, Rev. Math. Phys. 18 (2006) 713.

M. Bojowald and D. Ding, “Canonical description of cosmological back-reaction”, JCAP 03 (2021) 083.

A. Alonso-Serrano, M. Bojowald, and D. Brizuela, “Quantum approach to a Bianchi I singularity”, Phys. Rev. D 101 (2020) 104062.

B. Baytacak, M. Bojowald, and S. Crowe, “Effective potentials from semiclassical truncations”, Phys. Rev. A 99 (2019) 042114.

B. Baytacak and M. Bojowald, “Minisuperspace models of discrete systems”, Phys. Rev. D 95 (2017) 086007.

M. Bojowald and S. Brahma, “Minisuperspace models as infrared contributions”, Phys. Rev. D 93 (2016) 125001.

D. Brizuela, “Classical versus quantum evolution for a universe with a positive cosmological constant”, Phys. Rev. D 91 (2015) 085003.

D. Brizuela, “Classical and quantum behavior of the harmonic and the quartic oscillators”, Phys. Rev. D 90 (2014) 125018.

D. Brizuela, “Statistical moments for classical and quantum dynamics: formalism and generalized uncertainty relations”, Phys. Rev. D 90 (2014) 085027.

M. Bojowald, “Fluctuation energies in quantum cosmology”, Phys. Rev. D 89 (2014) 124031.

M. Bojowald, D. Brizuela, H. H. Hernandez, M. J. Koop, and H. A. Morales-Tecotl, “High-order quantum backreaction and quantum cosmology with a positive cosmological constant”, Phys. Rev. D 84 (2011) 043514.

G. F. R. Ellis, “Dynamics of pressure free matter in general relativity”, J. Math. Phys. 8 (1967) 1171.

H. van Elst and G. F. R. Ellis, “The Covariant approach to LRS perfect fluid space-time geometries”, Class. Quant. Grav. 13 (1996) 1099.

C. A. Clarkson and R. K. Barrett, “Covariant perturbations of Schwarzschild black holes”, Class. Quant. Grav. 20 (2003) 3855.

G. Betschart and C. A. Clarkson, “Scalar and electromagnetic perturbations on LRS class II space-times”, Class. Quant. Grav. 21 (2004) 5587.

C. Clarkson, “A Covariant approach for perturbations of rotationally symmetric spacetimes”, Phys. Rev. D 76 (2007) 104034.

S. Singh, G. F. R. Ellis, R. Goswami, and S. D. Maharaj, “New class of LRS spacetimes with simultaneous rotation and spatial twist”, Phys. Rev. D 94 (2016) 104040.

S. Khan, T. Hussain, A. H. Bokhari, and G. A. Khan, “Conformal Killing Vectors in LRS Bianchi Type V Spacetimes”, Commun. Theor. Phys. 65 (2016) 315.

N. Van den Bergh, “Rotating and twisting locally rotationally symmetric imperfect fluids”, Phys. Rev. D 96 (2017) 104056.

S. Singh, G. F. R. Ellis, R. Goswami, and S. D. Maharaj, “Rotating and twisting locally rotationally symmetric spacetimes: a general solution”, Phys. Rev. D 96 (2017) 064049.

M. Bojowald, “Homogeneous loop quantum cosmology”, Class. Quant. Grav. 20 (2003) 2595.