UNIVERSAL RIGID ABELIAN TENSOR CATEGORIES
AND SCHUR FINITENESS

BRUNO KAHN

Abstract. We study the construction of [4] in more detail, especially in the case of Schur-finite rigid \( \otimes \)-categories. This leads to some groundwork on the ideal structure of rigid additive and abelian \( \otimes \)-categories.

Contents

1. Introduction ........................................... 1
2. Kernels and ideals .................................... 3
3. Absolutely flat rings .................................. 4
4. More on rigid abelian \( \otimes \)-categories .......... 6
5. Prime \( \otimes \)-ideals .................................. 13
6. Applications to universal rigid abelian \( \otimes \)-categories .... 20
7. Schur finiteness ...................................... 23
8. The free rigid \( \otimes \)-category on one generator 28
References ............................................. 32

1. Introduction

This note complements the results of [4], where we showed that any additive rigid \( \otimes \)-category maps to an abelian one in a universal way. We retain its definitions and notation, namely

- A \( \otimes \)-category is an additive, symmetric, monoidal, unital category (with bilinear tensor product); a \( \otimes \)-functor between \( \otimes \)-categories is a strong symmetric, monoidal, unital additive functor.
- \( \text{Add}^\otimes \) is the 2-category of \( \otimes \)-categories, \( \otimes \)-functors and \( \otimes \)-natural isomorphisms.
- \( \text{Ex}^\otimes \) is the 2-category of abelian \( \otimes \)-categories, exact \( \otimes \)-functors and \( \otimes \)-natural isomorphisms.
- \( \text{Add}^{\text{rig}} \) and \( \text{Ex}^{\text{rig}} \) are their 1-full and 2-full sub-2-categories of rigid categories.
- For \( C \in \text{Add}^\otimes \), we write \( Z(C) := \text{End}_C(1) \) (the centre of \( C \)).

Date: March 8, 2022.
For $C \in \text{Add}^{\text{rig}}$, let $T(C) \in \text{Ex}^{\text{rig}}$ be the category of $[4, \text{Th. 5.1}]$: it defines a 2-left adjoint to the forgetful 2-functor $\text{Ex}^{\text{rig}} \to \text{Add}^{\text{rig}}$. As was observed in loc. cit., the centre of $T(C)$ is in general not a field even if that of $C$ is. Previously, categories $A \in \text{Ex}^{\text{rig}}$ had been considered mainly when $Z(A)$ is a field; studying the general case now becomes indispensable. This is one of the tasks of this paper; another is to study the tensor ideals of objects $C \in \text{Add}^{\text{rig}}$ in detail, and to relate them to those of the centre of $T(C)$.

The main results are:

1.1. Structure of rigid abelian $\otimes$-categories. Let $A \in \text{Ex}^{\text{rig}}$.

1. (Proposition 4.2) $Z(A)$ is absolutely flat $[5, \text{Ch. I, \S 2, ex. 17}]$ (von Neumann regular in another terminology, $[18, 4.2]$).

2. (Theorem 4.18, Remark 4.20 and Theorem 4.21) There is a one-to-one correspondence between the ideals of $Z(A)$ and the Serre subcategories $I$ of $A$ stable under external tensor product. Moreover, for such a Serre subcategory, the localisation functor $A \to A/I$ is full.

Item (1) was found independently by Peter O’Sullivan.

1.2. $\otimes$-ideals. Let $C \in \text{Add}^{\circ}$. In Definition 5.3, we introduce a “Zariski” topology on the set $\text{Spec}^{\otimes} C$ of prime $\otimes$-ideals of $C$; it is spectral in the sense of Hochster $[13]$. There is a spectral map (5.1) $\pi : \text{Spec}^{\otimes} C \to \text{Spec} Z(C)$. If $C \in \text{Add}^{\text{rig}}$, $\pi$ has a continuous closed section sending maximal ideals to maximal $\otimes$-ideals (Proposition 5.13); if moreover $C \in \text{Ex}^{\text{rig}}$, it has another “minimal” spectral section $\sigma$ (Proposition 5.14).

1.3. Application to universal rigid abelian $\otimes$-categories. If $C \in \text{Add}^{\text{rig}}$, the local abelian envelopes of $C$ in the sense of Coulembier $[6]$ are classified by a (possibly empty) closed subset of $\text{Spec} Z(T(C))$, where $T(C) \in \text{Ex}^{\text{rig}}$ is the universal category of $[4, \text{Th. 5.1}]$ (Corollary 6.2). Note that $\text{Spec} Z(T(C))$ is profinite by the already quoted proposition 4.2.

1.4. Schur-finite $\otimes$-categories. For $C$ as above, there is a canonical spectral map (6.1) $\text{Spec} Z(T(C)) \to \text{Spec}^{\otimes} C$. If $C$ is $\mathbb{Q}$-linear and Schur-finite, this map is a homeomorphism for the constructible topology on the right hand side (Corollary 7.10; see §5.1 for the constructible topology). We also justify the claim of $[4, \text{Rem. 6.6}]$ in Theorem 7.8 and get a refinement of $[4, \text{Prop. 8.5}]$, restricted to motives of abelian type, in Corollary 7.11.

1.5. Free $\otimes$-categories. In Propositions 8.4, 8.5 and in Theorem 8.7, we describe $T(L_{\mathbb{Q}})$ where $L_{\mathbb{Q}}$ is Deligne’s free additive rigid category on one generator ($[9, (1.26)]$, $[8, \S 10]$) with $\mathbb{Q}$ coefficients.
I had planned to add further results on motives as in [4], but those are meagre and limited to Example 6.4 and Corollary 7.11.

I thank Pierre Deligne for kindly explaining a misconception I had about [8, Prop. 10.17], and Ofer Gabber for suggesting that the tensor spectra of Section 5 might be spectral spaces, which clarified and simplified many of my proofs. I am especially indebted to Peter O’Sullivan, not only for his article [17] from which I have taken many results, but also for enlightening correspondence during the preparation of this work. He had the intuition that the 2-functor $T$ of [4] is analogous to the process of rendering a commutative ring absolutely flat as in [16]: this is vindicated by [4, Ex. 5.5] as well as Proposition 4.2, Proposition 6.6 and especially Theorem 7.10 of this paper.

By Example 5.4, the present theory of tensor spectra extends that from commutative algebra, but this extension is limited: there is no localisation theory (see Lemma 5.6 c) and d)), and a Noetherian theory seems uninteresting (see Remark 8.3). One thing I didn’t try is to compare with Balmer’s tensor triangular classification [3] (like here, his tt spectra are spectral spaces). It should certainly be done. See also Krause [15].

2. Kernels and ideals

This section recalls well-known facts for later reference.

Let $F : C \to D$ be an additive functor between additive categories. We write

- $\text{Ker}_m(F) = \{ f \in Ar(C) \mid F(f) = 0 \}$
- $\text{Ker}_o(F) = \{ C \in Ob(C) \mid 1_C \in \text{Ker}_m(F) \} = \{ C \in Ob(C) \mid F(C) = 0 \}$
- $\text{Ker}^*_m(F) = \{ f \in Ar(C) \mid f \text{ factors through } C \text{ for some } C \in \text{Ker}_o(F) \}$

Let $I$ be a (two-sided) ideal of $C$ [1, 1.3]. Then $I = \text{Ker}_m(C \to C/I)$; we write occasionally $I_o$ and $I^*$ for $\text{Ker}_o(C \to C/I)$ and $\text{Ker}^*_m(C \to C/I)$.

**Lemma 2.1.** Let $F : A \to B$ be an exact functor between abelian categories. Then $\text{Ker}_o(F)$ is a Serre subcategory of $A$, and $\text{Ker}^*_m(F) = \text{Ker}_m(F)$.

**Proof.** The first fact is obvious. For the second one, let $f : A \to B$ be in $\text{Ker}_m(F)$. Factor $f$ as $A \twoheadrightarrow C \hookrightarrow B$, where $C = \text{Im} f$. Then $C \in \text{Ker}_o(F)$. \(\square\)

In the situation of Lemma 2.1, we shall abbreviate $\text{Ker}_o F$ to $\text{Ker} F$.

There will be a flurry of ideals of all sorts in the sequel. To distinguish them, we shall try and follow this notation:
Ideals in commutative rings are denoted with capital italic letters.

In the special case of Boolean algebras, they are however denoted with gothic letters.

Serre $\otimes$-ideals in rigid abelian $\otimes$-categories (Definition 5.10 b)) are denoted with calligraphic letters. Serre localisations are denoted with double slashes $\sslash\sslash$, in order to distinguish them from quotients by ideals.

(Additive) $\otimes$-ideals in an rigid additive $\otimes$-category are denoted with blackboard letters.

3. Absolutely flat rings

In the sequel, we shall freely use the following equivalent properties for a commutative ring $R$ to be absolutely flat:

(1) Any principal ideal is generated by an idempotent.

(2) Any finitely generated submodule of a projective module is a direct summand.

(3) For any $x \in R$, there exists $y$ such that $xyx = x$.

Lemma 3.1. Any finitely generated ideal $\mathcal{I}$ of a Boolean algebra $B$ is principal.

Proof. 1 We first show that $\mathcal{I}$ is generated by orthogonal elements. Let $(e_1, \ldots, e_n)$ be a set of generators of $\mathcal{I}$. Assume that the statement is proven for $< e_1, \ldots, e_{n-1} >$. We may then assume that they are mutually orthogonal. Define

$$f_i = e_i e_n, \quad g_i = e_i (1 + e_n) \quad (1 \leq i \leq n - 1), \quad h = (1 + \sum e_j) e_n.$$ 

Clearly, $f_i g_j = 0$ for $i \neq j$, $f_i g_i = 0$, $g_i h = 0$ and

$$f_i h = e_i (1 + \sum e_j) e_n = (e_i + e_i) e_n = 0$$

so these elements are mutually orthogonal. Finally,

$$f_i + g_i = e_i \quad (1 \leq i \leq n - 1), \quad h + \sum f_j = e_n$$

so they generate $\mathcal{I}$.

(Thus, if we start from $n$ elements, we end up with at most $2^n - 1$ orthogonal elements.)

Now, if $e = \sum e_i$, we have $e_i = ee_i$ for all $i$, so $e$ generates $\mathcal{I}$. \hfill $\Box$

Let $R$ be a commutative ring. The set $B(R)$ of idempotents of $R$ is in one-to-one correspondence with the open-closed (clopen) subsets of (the underlying topological space to) $\text{Spec} R$. This gives $B(R)$ the

1 More directly: $B$ is absolutely flat.
structure of a Boolean algebra for the addition \( e \oplus e' = e + e' - 2ee' \) and the multiplication \( e \land e' = ee' \) corresponding to symmetric difference and intersection.

Let \( I \) be an ideal of \( R \). The set \( B(I) \) of idempotents in \( I \) is an ideal of \( B(R) \). Conversely, to any ideal \( \mathfrak{I} \) de \( B(R) \), we may associate the ideal \( I(\mathfrak{I}) \) of \( R \) generated by \( \mathfrak{I} \).

**Proposition 3.2.** The map \( I \mapsto B(I) \) is left inverse to the map \( \mathfrak{I} \mapsto I(\mathfrak{I}) \), and is a right inverse if and only if \( R \) is absolutely flat.\(^2\)

In particular, any ideal of an absolutely flat ring is generated by its idempotents.

**Proof.** We have obvious inclusions \( \mathfrak{I} \subseteq B(I(\mathfrak{I})) \) and \( I(B(I)) \subseteq I \). For an ideal \( \mathfrak{I} \) of \( B(R) \), let \( e \in B(I(\mathfrak{I})) \). Write \( e = \sum r_i e_i \) with \( r_i \in R \) and \( e_i \in \mathfrak{I} \). By Lemma 3.1, there is \( e' \in \mathfrak{I} \) such that \( e_i = f_i e' \) for all \( i \). Thus \( e = re' \) for \( r = \sum r_i f_i \); but then \( ee' = re'^2 = re' = e \), so \( e \in \mathfrak{I} \) and \( B(I(\mathfrak{I})) = \mathfrak{I} \).

Assume now that \( R \) is absolutely flat. For an ideal \( I \) of \( R \), let \( x \in I \). Then \( Rx = Re \) for some idempotent \( e \in Rx \subseteq I \). This shows that \( I(B(I)) = I \). Conversely, this equality for \( I = Rx \) with \( x \in R \) implies that \( Rx \) is generated by its idempotents. The same computation as in the beginning of the proof then shows that \( x = xe \) for some idempotent \( e \in Rx \), so that \( Rx = Re \). This implies that \( R \) is absolutely flat. \( \square \)

The following lemma will be used in the proof of Proposition 8.4.

**Lemma 3.3.** Let \( R \) be an absolutely flat \( F \)-algebra, where \( F \) is a field, and assume that the composition \( F \to R \to R/M \) is surjective for any \( M \in \text{Spec } R \). Then the rule \( a \mapsto a \pmod{M} \) yields an isomorphism

\[
\theta : R \cong \text{Cont}(\text{Spec } R, F)
\]

where \( \text{Cont} \) denotes continuous functions.

**Proof.** Write \( X = \text{Spec } R \). Let \( a \in R \). For \( f \in F \), the set

\[
\{ M \in X \mid \theta(a)(M) = f \} = \{ M \in X \mid \theta(a - f1)(M) = 0 \} = V(a - f1)
\]

is closed, hence \( \theta(a) \) is continuous and \( \theta \) is well-defined. It is injective because \( \bigcap_{M \in X} M = 0 \) since \( R \) is absolutely flat. Finally, let us show its surjectivity: Let \( \varphi \in \text{Cont}(X, F) \). Then \( \varphi(X) \) is finite since \( X \) is compact (Hausdorff), which determines a partition of \( X \) into the clopen subsets \( \varphi^{-1}(f) \) \( (f \in \varphi(X)) \). Let \( e_f \in R \) be the idempotent such that \( V(e_f) = \varphi^{-1}(f) \); then we have the “partition” \( \varphi = \theta(\sum f \in \varphi(X) f e_f) \). \( \square \)

\( ^2 \)I thank Kevin Coulembier for suggesting the “only if” part.
4. More on rigid abelian $\otimes$-categories

Let $\mathcal{A} \in \text{Ex}^{\text{rig}}$.

**Definition 4.1.** a) $\mathcal{A}$ is **connected** if $Z(\mathcal{A})$ is a field.

b) A **Serre $\otimes$-ideal** $\mathcal{I}$ of $\mathcal{A}$ is a Serre subcategory of $\mathcal{A}$ stable under external tensor product. We write $\mathcal{A} \sslash \mathcal{I}$ for the corresponding localisation ("Serre quotient"), in order to avoid confusion with the quotient by an (additive) $\otimes$-ideal.

By [14, Rem. 2.10], a) is equivalent to $\mathcal{A}$ being integral.

4.1. Structure of $Z(\mathcal{A})$. The following elaborates on [9, Rem. 1.18]:

**Proposition 4.2.** The ring $Z = Z(\mathcal{A})$ is absolutely flat; the class $\mathcal{U}$ of subobjects of $1$ is a set which is in one-to-one correspondence with the open-closed (clopen) subsets of (the underlying topological space to) $\text{Spec } Z$. This correspondence is non-decreasing (for the inclusion relation in both sets), respects intersections and exchanges union in $\text{Spec } Z$ with sum in $\mathcal{U}$.

**Remark 4.3.** This proposition justifies the terminology of Definition 4.1 a): it shows that as soon as $Z(\mathcal{A})$ is not a field, it contains a non-trivial idempotent which in turn yields a decomposition $\mathcal{A} \simeq \mathcal{A}_1 \times \mathcal{A}_2$ by [9, Rem. 1.18]. See Theorems 4.18 and 4.21 for a generalisation.

To prove Proposition 4.2, we need a lemma:

**Lemma 4.4.** a) For $U, V \in \mathcal{U}$, we have $U \otimes V = U \cap V$.

b) For $U \in \mathcal{U}$, the decomposition

\[ 1 = U \oplus U^\perp \]

of [9, Prop. 1.17] is unique.

c) For any such $U$, the canonical isomorphism $1 \simeq 1^\vee$ identifies $U$ and $U^\vee$.

d) For any $x \in Z$, we have $\text{Ker } x \oplus \text{Im } x = 1$ and $x|_{\text{Im } x}$ is invertible. In particular, $\text{Im } x = (\text{Ker } x)^\perp$.

**Proof.** a) The case $U = V$ is contained in the proof of [9, Prop. 1.17]. In general, the exactness of tensor product [9, Prop. 1.16] gives an inclusion $U \otimes V \subseteq U \cap V$, and conversely

\[ U \cap V = (U \cap V) \otimes (U \cap V) \subseteq U \otimes V. \]

b) Let $1 = U \oplus V$ be another decomposition. We have

\[
1 = (U \oplus U^\perp) \otimes (U \oplus V) = U \otimes U \oplus U \otimes V \oplus U^\perp \otimes U \oplus U^\perp \otimes V
\]

\[ = U \oplus U^\perp \cap V \]
by a). Hence $U^\perp = U^\perp \cap V = V$.

c) Dualising the sequence $1 \to U \to 1$, we get a sequence $1 \to U^\vee \to 1$. If $U = \text{Im } e$ for $e$ an idempotent of $Z(C)$, this identifies $U^\vee$ with $\text{Im } (e)$. But $e = e$ since $e \in Z(C)$.

d) By c) and [9, Prop. 17], we have

$$(\text{Im } x)^\perp = \text{Ker } (1 \to (\text{Im } x)^\vee) = \text{Ker } (1 \to \text{Im } x) = \text{Ker } x.$$

This shows that $x|_{\text{Im } x}$ is mono. The dual reasoning gives an isomorphism

$$1 \cong \text{Im } x \oplus \text{Coker } x$$

which in turn shows that $x|_{\text{Im } x}$ is epi. Thus it is an isomorphism, as claimed. \qed

Proof of Proposition 4.2. The clopen subsets of $\text{Spec } Z$ are parametrised by the idempotents of $Z$. Let $e$ be such an idempotent: we associate to it $U(e) = \text{Im } e \subseteq 1$. Conversely, if $U \subseteq 1$, let $e(U)$ be the idempotent with image $U$ given by the decomposition (4.1). Let us show that these correspondences are inverse to each other:

- $U(e(U)) = U$: this is trivial.
- $e(U(e)) = e$: this follows from Lemma 4.4 b).

Let us now show that $Z$ enjoys property (3) in the beginning of Section 3. Let $x \in Z$: Lemma 4.4 d) allows us to choose $y$ such that $y|_{\text{Ker } x} = 0$ and $y|_{\text{Im } x} = (x|_{\text{Im } x})^{-1}$. Thus $Z$ is absolutely flat.

Finally, the claims about the ordered structures is clear from this correspondence. \qed

Remark 4.5. In the sequel, the idempotent $e(U)$ associated to $U \in U$, which was used in the above proof, will play an important rôle. We record its properties: for $U, V \in U$:

- $e(U \cap V) = e(U)e(V)$;
- $e(U + V) = e(U) + e(V) - e(U)e(V)$.

4.2. The trivial part of $\mathcal{A}$.

Definition 4.6. Let $C$ be an additive category. Given $C \in \mathcal{C}$, we write $\mathcal{C}(C)$ for the smallest strictly full subcategory of $\mathcal{C}$ containing $C$ and closed under direct sums and direct summands.

Lemma 4.7. With the above notation, $\mathcal{C}(C)$ is equivalent to a full subcategory of the category $\mathcal{P}$ of finitely generated projective left $\text{End}_C(C)$-modules, with equality if $\mathcal{C}$ is pseudo-abelian.
Proof. Let $R = \text{End}_C(C)$. The preadditive subcategory of $C$ determined by $C$ is tautologically equivalent to that determined by $R$, and the full additive subcategory determined by the $C^n$ for $n \geq 0$ is equivalent to that of free finitely generated left $R$-modules. Since $\mathcal{P}$ is the pseudo-abelian hull of the latter, the conclusion follows. $lacksquare$

**Proposition 4.8.** Let $A \in \mathcal{Ex}^{\text{rig}}$. Then $A(1)$ is a Serre subcategory of $A$, split in the sense of [4, Def. 4.2]. Moreover it is a $\otimes$-subcategory of $A$, in which every object is self-dual.

Proof. We show successively:

1. Every epimorphism $f : 1^n \rightarrow B$ in $A$ has a section (in particular, $B \in A(1)$).
2. Any $B$ as in (1) is a direct sum of $n$ direct summands of 1.
3. Every short exact sequence $0 \rightarrow B' \rightarrow B \rightarrow B'' \rightarrow 0$ in $A$ with $B \in A(1)$ splits.

(1) Consider the commutative diagram

$$
\begin{array}{ccc}
1^{n-1} & \xrightarrow{g} & 1^n \\
\downarrow{i_n} & & \downarrow{f} \\
1^n & \xrightarrow{p_n} & B \\
\downarrow{h} & & \downarrow{h} \\
1 & \xrightarrow{\bar{f}} & B
\end{array}
$$

where $i_n$ is the inclusion of the first $n-1$ summands, $p_n$ is the $n$-th projection and $\bar{B} = \text{Coker} \, g$. By [9, Prop. 1.17], $f$ has a section $s$ (unique by Lemma 4.4 b), but we don’t care). Composing $s$ with a section $s_n$ of $p_n$, and then with $f$, we get a section $s'$ of $h$. Then $B = \text{Im} \, g \oplus s'(\bar{B})$. By induction on $n$, choose a section $s''$ of $g_1 : 1^{n-1} \rightarrow \text{Im} \, g$; then we get a section $\sigma$ of $f$ by $\sigma|_{\text{Im} \, g} = i_n s''$ and $\sigma|_{\text{Im} \, g} = s_n s$.

(2) follows from the proof of (1), by induction on $n$.

(3) follows formally from (1).

Item (3) shows that $A(1)$ is a Serre subcategory of $A$ and is split, by [4, Prop. 4.3 (3)]. The claim on self-duality now follows from Lemma 4.4 c). $lacksquare$

**Remark 4.9.** Conversely, the category of finitely generated projective modules over an absolutely flat commutative ring $R$ defines a split, rigid, self-dual $\otimes$-category $A(R)$ (compare property (2) at the beginning of Section 3). This shows that $Z(A)$ can be any absolutely flat
commutative ring. If we want to be fanciful, we can say that the 2-functor $R \mapsto \mathcal{A}(R)$ is 2-left adjoint to the 2-functor $\mathcal{A} \mapsto Z(\mathcal{A})$.

4.3. **The Serre $\otimes$-ideals of $\mathcal{A}(1)$**. To such an ideal $\mathcal{I}$, associate the set of idempotents $\mathfrak{I}(\mathcal{I}) = \{ e(U) \mid U \in \mathcal{I} \cap \mathfrak{U} \}$. This is an ideal of $B(Z(\mathcal{A}))$, the Boolean algebra associated to $Z(\mathcal{A})$: equivalently, $\mathcal{I} \cap \mathfrak{U}$ is closed under sums and subobjects (see Lemma 4.4 a)). Conversely, to an ideal $\mathfrak{I}$ of $B(Z(\mathcal{A}))$, associate the full additive subcategory $\mathcal{I}(\mathfrak{I})$ of $\mathcal{A}(1)$ generated by the $\text{Im} e$ for $e \in \mathfrak{I}$: it is a Serre $\otimes$-ideal of $\mathcal{A}(1)$.

**Proposition 4.10.** The maps $\mathcal{I} \mapsto \mathfrak{I}(\mathcal{I})$ and $\mathfrak{I} \mapsto \mathcal{I}(\mathfrak{I})$ are inverse to each other. They yield a bijective correspondence between the Serre $\otimes$-ideals of $\mathcal{A}(1)$ and the ideals of $B(Z(\mathcal{A}))$.

**Proof.** The inclusions $\mathcal{I} \supseteq \mathfrak{I}(\mathcal{I}(\mathcal{I}))$ and $\mathfrak{I}(\mathcal{I}(\mathfrak{I})) \supseteq \mathcal{I}$ are tautological. Let $\mathcal{I}$ be a $\otimes$-ideal of $\mathcal{A}(1)$, and let $A \in \mathcal{I}$: by item (2) of the proof of Proposition 4.8, we can write $A = \bigoplus A_i$ with $A_i \in \langle 1 \rangle^\times$, and all $A_i$ belong to $\mathcal{I}$. This shows equality in the first inclusion. Let now $\mathfrak{I}$ be an ideal of $B(Z(\mathcal{A}))$. Choose an orthogonal basis $(e_1, \ldots, e_n)$ of $\mathfrak{I}$ as in Lemma 3.1, and let $U_i = \text{Im} e_i$: then $\mathcal{A}(U_i, U_j) = 0$ if $i \neq j$, and any object $A \in \mathcal{I}(\mathfrak{I})$ has a unique decomposition of the form $A \simeq \bigoplus_{i=1}^n U_i^{n_i}$; we have $A \in \mathfrak{U}$ if and only if all $n_i$ are $\leq 1$. Let $e \in \mathcal{I}(\mathfrak{I}(\mathcal{I}))$: writing $\text{Im} e$ in this form, we find that $e = \sum n_i e_i$, hence $e \in \mathfrak{I}$ as desired. \(\square\)

**Remark 4.11.** By Propositions 3.2, 4.2 and 4.10, there is so far a bijective correspondence between

1. the ideals of $B(Z(\mathcal{A}))$;
2. the ideals of $Z(\mathcal{A})$;
3. the subsets of $\mathfrak{U}$ stable under sums and subobjects, i.e. filters for the order relation opposite to inclusion. For simplicity, we shall call the latter cofilters of $\mathfrak{U}$.
4. the Serre $\otimes$-ideals of $\mathcal{A}(1)$.

4.4. **Supports.**

**Definition 4.12.** Let $f : A \to B$ be a morphism of $\mathcal{A}$, and let $\tilde{f} : 1 \to A^\vee \otimes B$ be its adjoint. The support of $f$ is

$$\text{Supp}(f) = \text{Ker}(\tilde{f})^\perp \simeq \text{Im} \tilde{f}.$$  

For $A \in \mathcal{A}$, we define $\text{Supp}(A) = \text{Supp}(1_A)$. We set $e(f) = e(\text{Supp}(f))$ and $e(A) = e(\text{Supp}(A))$ (see Remark 4.5).

---

3Interpreting monoids as categories with one object gives their category a structure of 2-category. Given two parallel homomorphisms $f, g : M \to N$ of monoids, a natural transformation $f \Rightarrow g$ is an element $n \in N$ such that $f(m)n = ng(m)$ for all $m \in M$.
By definition, \( \text{Supp}(f) \) is the smallest subobject \( U \) of \( 1 \) such that \( f \) factors through \( U \otimes B \), and \( \text{Supp}(A) \) is the smallest subobject \( U \) of \( 1 \) such that \( U^\perp \otimes A = 0 \). Thus we also have \( \text{Supp}(f) = \text{Supp}(\text{Im}(f)) \).

**Lemma 4.13.** For any \( A \in \mathcal{A} \), one has \( A = A \otimes \text{Supp}(A) \). \( \square \)

**Lemma 4.14.**

a) \( \text{Supp}(f \otimes g) = \text{Supp}(f) \cap \text{Supp}(g) \), hence also \( e(f \otimes g) = e(f)e(g) \), for any morphisms \( f, g \).

b) \( \text{Supp}(f \circ g) \subseteq \text{Supp}(f) \cap \text{Supp}(g) \), hence also \( e(f)e(g)|e(f \circ g) \), for any composable morphisms \( f, g \).

c) \( f = f \circ e(f) \) for any \( f \) with domain \( 1 \).

*Proof.* a) follows from Lemma 4.4 a) and the fact that \( \tilde{f} \otimes g = \tilde{f} \otimes \tilde{g} \).

In b), the inclusion \( \text{Supp}(f \circ g) \subseteq \text{Supp}(f) \) is obvious, and the other inclusion can be seen dually. Finally, c) is easy. \( \square \)

Lemma 4.14 a) allows us to give the right generalisation of [4, Prop. 2.5 a) and b)]:

**Proposition 4.15.** Given two morphisms \( f, g \), one has \( f \otimes g = 0 \) if and only if \( \text{Supp}(f) \cap \text{Supp}(g) = 0 \). \( \square \)

**Proposition 4.16.** Let \( (\ast) \ 0 \to A' \to A \to A'' \to 0 \) be a short exact sequence in \( \mathcal{A} \). Then

a) \( \text{Supp}(A) = \text{Supp}(A') + \text{Supp}(A'') \).

b) If \( \text{Supp}(A') \cap \text{Supp}(A'') = 0 \), \( (\ast) \) is split.

*Proof.* a) Given a subobject \( U \) of \( 1 \), \( U^\perp \otimes A = 0 \) \( \iff \ U^\perp \otimes A' = 0 \) and \( U^\perp \otimes A'' = 0 \).

b) By the exactness of \( \otimes \), we have a short exact sequence

\[
0 \to A' \otimes \text{Supp}(A') \to A' \otimes \text{Supp}(A') \to A'' \otimes \text{Supp}(A') \to 0
\]

where, by Lemma 4.13, \( A' = A' \otimes \text{Supp}(A') \) and \( A'' \otimes \text{Supp}(A') = A'' \otimes \text{Supp}(A') \otimes \text{Supp}(A') = 0 \), the last equality by hypothesis. Thus

\[
A' \sim A \otimes \text{Supp}(A').
\]

In the same way, we have an isomorphism

\[
A \otimes \text{Supp}(A'') \sim A''.
\]

Since \( \text{Supp}(A) = \text{Supp}(A') \oplus \text{Supp}(A'') \) by a), we get an isomorphism

\[
A \simeq A' \oplus A''
\]

which splits \( (\ast) \) by construction. \( \square \)
4.5. **The Serre $\otimes$-ideals of $\mathcal{A}$**. We now want to add the latter as a fifth item to the list of Remark 4.11. In view of the above, the most convenient is to compare them with the cofilters of $U$.

Namely, to a Serre $\otimes$-ideal $\mathcal{I} \subseteq \mathcal{A}$, we associate $\Phi(\mathcal{I}) = \mathcal{I} \cap U$; this is a cofilter of $U$. Conversely, to a cofilter $\Phi$ of $U$, we associate the full subcategory $\mathcal{I}(\Phi) = \{ A \in \mathcal{A} \mid \text{Supp}(A) \in \Phi \}$: this is a Serre $\otimes$-ideal by Lemma 4.14 a) and Proposition 4.16 a).

**Lemma 4.17.** Let $\mathcal{I}$ be a Serre $\otimes$-ideal of $\mathcal{A}$. Then $A \in \mathcal{I} \iff \text{Supp}(A) \in \Phi(\mathcal{I})$.

**Proof.** If $A \in \mathcal{I}$, so do $A \otimes A^\vee$ and the image of $\eta : \mathbf{1} \to A \otimes A^\vee$. This proves $\Rightarrow$. Conversely, if $\text{Supp}(A) \in \Phi(\mathcal{I})$, then $A = A \otimes \text{Supp}(A) \in \mathcal{I}$ (Lemma 4.13). $\square$

**Theorem 4.18.** The maps $\mathcal{I} \mapsto \Phi(\mathcal{I})$ and $\Phi \mapsto \mathcal{I}(\Phi)$ are mutually inverse bijections; there is a $1-1$ correspondence between Serre $\otimes$-ideals of $\mathcal{A}$ and ideals of $Z(\mathcal{A})$.

**Proof.** For a Serre $\otimes$-ideal $\mathcal{I}$ of $\mathcal{A}$, $\mathcal{I}(\Phi(\mathcal{I})) = \mathcal{I}$ follows from Lemma 4.17. Conversely, if $\Phi$ is a cofilter of $U$, then, for $U \in U$:

$$U \in \Phi(\mathcal{I}(\Phi)) \iff U \in \mathcal{I}(\Phi) \iff \text{Supp}(U) \in \Phi$$

hence $\Phi(\mathcal{I}(\Phi)) = \Phi$ since $\text{Supp}(U) = U$. This proves the first claim, and the second one then follows from Remark 4.11. $\square$

**Example 4.19.** If $\mathcal{A}$ is connected, Theorem 4.18 says that $\mathcal{A}$ has no proper Serre $\otimes$-ideals: we recover [9, Prop. 1.19].

**Remark 4.20.** For later reference, let us specify the correspondence of Theorem 4.18: in one direction it associates to a Serre $\otimes$-ideal $\mathcal{I}$ the ideal $I(\mathcal{I}) \subseteq Z(\mathcal{A})$ generated by the idempotents $e(A)$ for $A \in \mathcal{I}$. Conversely, if $I$ is an ideal of $Z(\mathcal{A})$, the full subcategory $\mathcal{I}(I)$ of $\mathcal{A}$ formed of those $A$ such that $e(A) \in I$ is the desired Serre $\otimes$-ideal of $\mathcal{A}$.

4.6. **Centre and quotients.** Let $\mathcal{I}$ be a Serre $\otimes$-ideal of $\mathcal{A}$. By [4, Prop. 3.5], the exact localisation functor $F : \mathcal{A} \to \mathcal{A} \sslash \mathcal{I}$ descends the tensor structure of $\mathcal{A}$ to $\mathcal{A} \sslash \mathcal{I}$, giving it the structure of an abelian rigid $\otimes$-category; moreover, the induced homomorphism $Z(F) : Z(\mathcal{A}) \to Z(\mathcal{A} \sslash \mathcal{I})$ is surjective. The following is a complement to this proposition:

**Theorem 4.21.** The kernel $\text{Ker}_o(Z(F))$ of $Z(F)$ is equal to $I(\mathcal{I})$ (see Remark 4.20), and the induced functor

$$Z(\mathcal{A})/I(\mathcal{I}) \otimes_{Z(\mathcal{A})} \mathcal{A} \to \mathcal{A} \sslash \mathcal{I}$$

is an equivalence of $\otimes$-categories; in particular, $F$ is full.
\textbf{Proof.} Let \( f \in Z(\mathcal{A}) \). If \( e(f) \in I(\mathcal{I}) \), clearly \( F(f) = 0 \). Conversely, assume that \( F(f) = 0 \). By definition, there are \( \lambda_1, \ldots, \lambda_n \in I(\mathcal{I}) \) and \( g_1, \ldots, g_n \in Z(\mathcal{A}) \) such that \( f = \sum_i g_i \circ \lambda_i \). Let \( I_0 = (\lambda_1, \ldots, \lambda_n) \). As a finitely generated ideal of an absolutely flat ring, it is generated by an idempotent \( e \); thus, writing \( \lambda_i = \mu_i e \) for all \( i \), we get \( f = g \circ e \) with \( g = \sum_i g_i \circ \mu_i \). Then \( e(f) \) is a multiple of \( e \), hence belongs to \( I \).

Next, we prove the fullness of \( F \). It suffices by rigidity to show that \( \mathcal{A}(1, A) \to (\mathcal{A} \big/ \mathcal{I})(1, A) \) is surjective for any \( A \in \mathcal{A} \). Let \( f \in (\mathcal{A} \big/ \mathcal{I})(1, A) \). By \cite[III.1]{12}, \( f \) may be represented by a morphism in \( \mathcal{A} \)

\[ \tilde{f} : 1' \to A' \]

where \( 1' \) (resp. \( A' \)) is a subobject (resp. a quotient) of \( 1 \) (resp. \( A \)) such that \( 1' / 1' \in \mathcal{I} \) (resp. \( N \in \mathcal{I} \), where \( N = \text{Ker}(A \to A') \)). Let \( \Sigma = \text{Supp}(N) \). Write \( 1 = \Sigma \oplus \Sigma' \), and let \( 1'' = 1' \otimes \Sigma' \): then

\[ 1 \simeq (1 / 1') \oplus 1' \otimes \Sigma \oplus 1'' \]

where the first two summands belong to \( \mathcal{I} \), the second one because \( \Sigma \in \mathcal{I} \) by Lemma 4.17 and because \( \mathcal{I} \) is a Serre \( \otimes \)-ideal. Thus, up to replacing \( 1' \) by \( 1'' \), we may assume that \( 1' \cap \Sigma = 1' \otimes \Sigma = 0 \). By Proposition 4.16 b), the pull-back by \( \tilde{f} \) of the extension \( 0 \to N \to A \to A' \to 0 \) then splits; this yields a lift \( \tilde{f} : 1' \to A \) of \( \tilde{f} \). Since \( 1' \) is a direct summand of \( 1 \), \( \tilde{f} \) further extends to a morphism \( 1 \to A \), which still represents \( f \).

To conclude, it remains to show that, for any \( A, B \in \mathcal{A} \), \( \text{Ker}(\mathcal{A}(A,B) \to (\mathcal{A} \big/ \mathcal{I})(A,B)) \subseteq \text{Ker}(\mathcal{A}(A,B) \to Z(\mathcal{A})/I(\mathcal{I}) \otimes Z(\mathcal{A})) \mathcal{A}(A,B)) \): this follows from Lemma 2.1. \( \square \)

\textbf{Notation 4.22.} Given an ideal \( I \) of \( Z(\mathcal{A}) \), we simply write \( \mathcal{A} \big/ \mathcal{I}(I) \) for \( \mathcal{A} \big/ \mathcal{I}(I) \) (see Remark 4.20).

Note that there is a \( 1-1 \) correspondence between Serre \( \otimes \)-ideals of \( \mathcal{A} \big/ \mathcal{I}(I) \) and Serre \( \otimes \)-ideals of \( \mathcal{A} \) containing \( \mathcal{I}(I) \); by the above, we have \( \mathcal{I} \supseteq \mathcal{I}(I) \) if and only if \( I(\mathcal{I}) \supseteq I \).

\textbf{Corollary 4.23.} There exists a conservative family of \( \otimes \)-functors from \( \mathcal{A} \) to connected rigid abelian \( \otimes \)-categories \( \mathcal{A}_M \).

\textbf{Proof.} Let \( M \) run through the maximal ideals of \( Z(\mathcal{A}) \), and let \( \mathcal{A}_M := \mathcal{A} \big/ M \). By Theorem 4.21, \( Z(\mathcal{A}_M) \) is a field for all \( M \). Let \( A \in \mathcal{A} \) be such that \( A_M = 0 \) for all \( M \). Then \( e(A) \in M \) for all \( \alpha \). But, in an absolutely flat ring, the intersection of all maximal ideals is 0. Hence \( e(A) = 0 \) and \( A = 0 \). \( \square \)
Remarks 4.24. a) This corollary depends on the axiom of choice!
b) If $Z(A)$ is not Noetherian, the choice of all maximal ideals in the
proof of Corollary 4.23 is redundant. For example, let $A$ be the $\otimes$-
category of finitely generated projective modules over $R = \prod_p F_p$.
Then Spec $R$ is much larger than the set of prime numbers (ultrafilters),
but this set obviously suffices to give a conservative family.

Corollary 4.25. If $Z(A)$ is Noetherian (i.e. if Spec $Z(A)$ is finite),
the canonical functor
\[ A \to \prod_{M \in \text{Spec } Z(A)} A/\sim M \]
is an equivalence of $\otimes$-categories. \qed

Corollary 4.26. Let $F : A \to B$ be a $\otimes$-functor in $\text{Ex}^\text{rig}$, with $B$
connected. Then there is a unique maximal ideal $M$ of $Z(A)$ such
that $F$ factors through $A/\sim M$ (into a faithful $\otimes$-functor). We write
$M = M(F)$. \\
\textit{Proof.} Let $\mathcal{I} = \{A \in A \mid F(A) = 0\}$: this is a Serre $\otimes$-ideal of $A$ such that the induced $\otimes$-functor $A/\sim \mathcal{I} \to B$ is faithful. In particular,
$Z(A/\sim \mathcal{I}) \hookrightarrow Z(B)$ is a field by [4, Prop. 2.5 d)]. But $Z(A/\sim \mathcal{I}) = Z(A)/I(\mathcal{I})$ by Theorem 4.21. Let $M = I(\mathcal{I})$; Then $\mathcal{I} = \mathcal{I}(M)$ by Remark 4.20. \qed

The following strengthens Corollary 4.23 and will be used in the next
section.

Proposition 4.27. Let $X$ be a set of morphisms of $A$. Then the set
\[ \{M \in \text{Spec } Z(A) \mid f(M) \neq 0 \forall f \in X\} \]
is closed. Here $f(M)$ denotes the image of $f$ in $A/\sim M$.

\textit{Proof.} Suppose first that $X$ consists of one element $f$. Then $f(M) \neq 0 \iff \text{Im } f \notin \mathcal{I}(M) \iff e(\text{Im } f) \notin M \iff 1 - e(\text{Im } f) \in M$. The
set of those $M$ is closed. The general case follows, since an intersection
of closed subsets is closed. \qed

5. Prime $\otimes$-ideals

5.1. Spectral spaces and the constructible topology. The con-
structible topology was developed in [19, I, §7.2]. A simpler variant was
independently studied by Hochster in [13] under the name of “patch
topology”; Hochster also introduced the notion of spectral spaces and
spectral maps. Following the Stacks project https://stacks.math.columbia.edu/tag/08YF,
we take Hochster’s viewpoint but use “constructible topology” in place of “patch topology”.

**Definition 5.1** (Hochster). a) A topological space is *spectral* if it is $T_0$ and quasi-compact, the quasi-compact open subsets are closed under finite intersection and form an open basis, and every nonempty irreducible closed subset has a generic point. A continuous map of spectral spaces is *spectral* if inverse images of quasi-compact open sets are quasi-compact.

b) Let $X$ be a spectral space. The *constructible topology* $T(X)$ on $X$ is the topology which has the quasi-compact open sets and their complements as an open sub-basis.

As an example, if $f : A \to B$ is a homomorphism of commutative rings, then Spec $A$ and Spec $B$ are spectral spaces and $f^* : \text{Spec } B \to \text{Spec } A$ is a spectral map.

The following is a spectral analogue of Olivier’s theorem [16, Prop. 5]:

**Theorem 5.2** ([13, §9]). a) For any spectral space $X$, $T(X)$ is profinite (i.e. compact (Hausdorff) and totally disconnected). Any profinite space is spectral.

b) Let $S$ be the category of spectral spaces and spectral maps, and let $P$ be its full subcategory of profinite spaces. Then $T$ defines a right adjoint to the inclusion $P \hookrightarrow S$.

5.2. Zariski $\otimes$-topology. Recall that $C \in \text{Add}^\otimes$ is integral if $f \otimes g = 0 \Rightarrow f = 0$ or $g = 0$.

**Definition 5.3.** Let $C \in \text{Add}^\otimes$. A $\otimes$-ideal $P \subseteq C$ is prime if $C/P$ is integral. We write Spec$^\otimes C$ for the set of its prime $\otimes$-ideals, and provide it with the topology whose closed subsets are given by the

$$V(\mathcal{I}) = \{P \mid \mathcal{I} \subseteq P\}$$

for $\mathcal{I}$ a $\otimes$-ideal of $C$.

Since $V(\mathcal{I} \otimes \mathcal{J}) = V(\mathcal{I}) \cup V(\mathcal{J})$ and $\bigcap V(\mathcal{I}_\alpha) = V(\sum \mathcal{I}_\alpha)$, this indeed defines a topology.

**Example 5.4.** For a commutative ring $R$, let $L(R)$ be the $\otimes$-category of finitely generated free $R$-modules. There is a $1 - 1$-correspondence between $\otimes$-ideals of $L(R)$ and ideals of $R$, obtained by restricting a $\otimes$-ideal to $R = L(R)(1,1)$. In particular, Spec$^\otimes L(R)$ is canonically homeomorphic to Spec $R$.

The main result of this subsection is:
Theorem 5.5. For any $C \in \text{Add}^\otimes$, $\text{Spec}^\otimes C$ is a spectral space; for any $F \in \text{Add}^\otimes(C, D)$, $F^* : \text{Spec}^\otimes D \to \text{Spec}^\otimes C$ is a spectral map.

The proof will be given after Corollary 5.9.

Let $\mathbb{I}$ be a $\otimes$-ideal of $C$, and let $f \in C(A, B)$. We say that $f$ is contained in $\mathbb{I}$ if $f \in \mathbb{I}(A, B)$. Since an intersection of $\otimes$-ideals is a $\otimes$-ideal, there is a smallest $\otimes$-ideal $\mathbb{I}$ containing a given set $E$ of morphisms of $C$: we say that $\mathbb{I}$ is generated by $E$. We may generalise the definition of $V(\mathbb{I})$ to $V(E)$ for any such $E$, and then $V(E) = V(\mathbb{I})$ where $\mathbb{I}$ is the $\otimes$-ideal generated by $E$.

Lemma 5.6. a) Let $f \in \text{Ar}(C)$. Then the $\otimes$-ideal $(f)$ generated by $f$ is the set of compositions $h \circ (f \otimes 1_C) \circ g$ making sense, where $C \in \text{Ob}(C)$. b) A $\otimes$-ideal $I$ of $C$ is proper if and only if $1 \notin I$. c) A morphism $f : A \to B$ is not contained in any maximal $\otimes$-ideal if and only if $1$ may be factored as

$$1 \xrightarrow{h} A \otimes C \xrightarrow{f \otimes 1_C} B \otimes C \xrightarrow{g} 1$$

for some $C \in \text{Ob}(C)$. d) If $B = 1$ in c), we may choose $C = 1$ and $g = 1_1$, i.e. find a section to $f$.

Proof. a) This set is obviously closed under left and right external compositions and just as obviously under direct sums; by the usual Mac Lane diagonal/codiagonal trick, this implies that it is closed under ordinary sums. To conclude, it suffices to show that $(f \otimes 1_C) \otimes g \in (f)$ for any $C \in \text{Ob}(C)$ and any $g \in \text{Ar}(C)$. But

$$(f \otimes 1_C) \otimes g = (1 \otimes g) \circ (f \otimes 1_{C \otimes D})$$

where $D$ is the domain of $g$.

b) “If” is obvious. Conversely, if $1 \in I$ then so does $1_C = 1_1 \otimes 1_C$ for any $C \in \mathcal{C}$, hence any $f \in \text{Ar}(C)$.

c) By a), this condition is equivalent to $1 \in (f)$, and we conclude by b).

d) Use the commutation $g \circ (f \otimes 1_C) = f \otimes g = f \circ (1_C \otimes g)$. $\square$

Lemma 5.7. a) Any maximal $\otimes$-ideal is prime, and any proper $\otimes$-ideal $\mathbb{I}$ is contained in a maximal $\otimes$-ideal; in particular, $V(\mathbb{I}) \neq \emptyset$. Any prime $\otimes$-ideal contains a minimal prime $\otimes$-ideal.

b) The $D(f) := \text{Spec}^\otimes C - V(\{f\})$ for $f \in \text{Ar}(C)$ form a basis of open sets for the Zariski $\otimes$-topology.

c) One has $D(\{f_1, \ldots, f_n\}) = D(f_1 \oplus \cdots \oplus f_n)$ and $D(f \otimes g) = D(f) \cap D(g)$.

d) Let $F : \mathcal{C} \to \mathcal{D}$ be a $\otimes$-functor, with $\mathcal{C}, \mathcal{D} \in \text{Add}^\otimes$. Then $F$ induces
a continuous map $F^* : \text{Spec}^\otimes D \to \text{Spec}^\otimes C$. If $F$ is full and essentially surjective, $F^*$ is a closed immersion.
e) In d), if $F$ is the pseudo-abelian completion of $C$, $F^*$ is a homeomorphism.

Proof. Only the first statements of a) and d) deserve a proof. For a), we reduce to showing that if $C$ has no proper nonzero $\otimes$-ideals, then $C$ is integral. Let $f \in \text{Ar}(C) - \{0\}$: the set of $g$ such that $f \otimes g = 0$ is a proper $\otimes$-ideal of $C$, hence must be 0. For d), the point is that $F^{-1}(\mathbb{I})$ is a proper $\otimes$-ideal of $C$ if $\mathbb{I}$ is a proper $\otimes$-ideal of $D$, which follows from Lemma 5.6 b). Finally, e) is clear since any $\otimes$-ideal $\mathbb{I}$ of $C$ extends uniquely to a $\otimes$-ideal of $D$, prime if $\mathbb{I}$ is prime.

The following is a $\otimes$-analogue of a well-known result of commutative algebra:

**Proposition 5.8.** For any $\otimes$-ideal $\mathbb{I}$ of $C \in \text{Add}^\otimes$, the ideal $\sqrt{\mathbb{I}}$ of [1, Def. 7.4.1] is the intersection of the prime $\otimes$-ideals of $C$ containing $\mathbb{I}$.

Proof. We reduce to $\mathbb{I} = 0$. Let $f \notin \sqrt{0}$. We must find $\mathbb{P} \in \text{Spec}^\otimes C$ such that $f \notin \mathbb{P}$. We use the idea of [17, §3] (fractional closure), that we apply to the not necessarily regular morphism $f : A \to A'$. Thus, let $C[1/f]$ be the category with the same objects as $C$ and morphisms given by

$$C[1/f](C, C') = \lim_{\rightarrow n} C_{f\otimes^n}(C, C')$$

for $C, C' \in C$, where $C_{f\otimes^n}(C, C')$ is the subgroup of $C(A\otimes^n \otimes C, A'\otimes^n \otimes C')$ defined in the commutation of diagrams (3.1) in [17]. This is a rigid $\otimes$-category by the same arguments as in loc. cit. Moreover, $C[1/f] \neq 0$: to see this it suffices to see that $1_1$, the identity endomorphism of $1 \in C$, does not map to 0 in $C[1/f](1, 1)$. But this is clear, since the image of $1_1$ in $C_{f\otimes^n}(1, 1)$ is $f^{\otimes n}$. Now, the inverse image of any prime $\otimes$-ideal of $C[1/f]$ by the $\otimes$-functor $C \to C[1/f]$ is the desired prime $\otimes$-ideal $\mathbb{P}$. □

This proof is inspired by the one in commutative algebra using localisation. As O’Sullivan pointed out, there is a more elementary proof in the style of other proofs from commutative algebra as in [2, proof of Prop. 1.8].

**Corollary 5.9.** For $C \in \text{Add}^\otimes$,
a) An open subset of $\text{Spec}^\otimes C$ is quasi-compact if and only if it is of the form $D(f)$. In particular, $\text{Spec}^\otimes C$ is quasi-compact and its quasi-compact open subsets are closed under finite intersections.
b) For $Y \subseteq \text{Spec}^\otimes \mathcal{C}$, let $\mathbb{I}(Y) = \bigcap_{P \in Y} \mathbb{P}$. Then $\mathbb{I}(Y) = \sqrt{\mathbb{I}(Y)}$. Moreover, $Y$ is irreducible if and only if $\mathbb{I}(Y)$ is prime. The map $\mathbb{P} \mapsto V(\mathbb{P})$ is a bijection of $\text{Spec}^\otimes \mathcal{C}$ onto the set of irreducible closed subsets of $\text{Spec}^\otimes \mathcal{C}$.

**Proof.** As usual: see Lemma 5.7 b) and c), [5, Ch. II, §4, Prop. 11, 12 and 14] and [19, I, (1.1.4)].

Theorem 5.5 follows from Corollary 5.9, noting also that $(F^*)^{-1}(V(f)) = V(F(f))$ for any $f \in \text{Ar}(\mathcal{C})$.

**Definition 5.10.** Let $\mathcal{C} \in \text{Add}^\otimes$. A morphism $f \in \mathcal{C}$ is **quasi-invertible** if it verifies the equivalent conditions of Lemma 5.6 c).

**Lemma 5.11.** If $\mathcal{C} \in \text{Add}^{\text{rig}}$, 

a) $f : A \to B$ is quasi-invertible if and only if its right adjoint $\tilde{f} : A \otimes B^\vee \to 1$ is quasi-invertible.

b) Any quasi-invertible morphism is strongly regular in the sense of [17, §3].

**Proof.** a) follows once again from (the dual of) [1, Lemma 6.1.5]. b) by a) and its analogue for strongly regular morphisms [17, bot. p. 9], we may assume that $B = 1$ in a) and therefore, by Lemma 5.6 d), that $f$ has a section $g$. First, $f$ is regular: if $f \otimes h = 0$, then $0 = (f \otimes h) \circ (g \otimes 1) = 1 \otimes h = h$. As observed in [17, top p. 10], to prove that $f$ is strongly regular, it suffices to prove that $\mathcal{C}(1, D) \xrightarrow{f \otimes -} \mathcal{C}(1, D)$ is surjective for any $D \in \mathcal{C}$. Let $h \in \mathcal{C}(1, D)$, i.e. $h : A \to D$ makes the diagram (3.1) of [17] commute. In this special case, this means that $h$ verifies the identity

$$f \otimes h = (f \otimes h) \circ c_{A,A}$$

where $c$ is the symmetry of $\mathcal{C}$. Composing again with $g \otimes 1$ on the right, this gives

$$h = 1_1 \otimes h = (f \otimes h) \circ (g \otimes 1) = (f \otimes h) \circ c_{A,A} \circ (g \otimes 1)$$

$$= (f \otimes h) \circ (1 \otimes g) \circ c_{A,1} = f \otimes (h \circ g).$$

(The converse of Lemma 5.11 b) is false: take $\mathcal{C} = \text{Rep}_K(G_a)$ for a field $K$, and $f : V \to 1$ where $V$ is the standard 2-dimensional representation of $G_a$. Then $f$ does not have a section, but is strongly regular because it is regular and $\mathcal{C}$ is fractionally closed by [17, Lemma 3.1].)
5.3. Relationship between spectra and \(\otimes\)-spectra. Let \(C \in \text{Add}^{\otimes}\). By Example 5.4 and Corollary 5.9 d), the obvious \(\otimes\)-functor \(L(Z(C)) \rightarrow C\) induces a spectral map

\[
\pi : \text{Spec} \otimes C \rightarrow \text{Spec} Z(C).
\]

If \(I\) is a \(\otimes\)-ideal of \(C\), write more generally \(\pi(I) = I(1,1)\) for the corresponding ideal of \(Z(C)\). Conversely, if \(I\) is an ideal of \(Z(C)\), we get a \(\otimes\)-ideal \(I(I)\) of \(C\) by the formula

\[
I(I)(C, D) = I \cdot C(C, D)
\]

for the action of \(Z(C)\) on \(C\).

**Lemma 5.12.** We have \(\pi(I(I)) = I\) and \(I(\pi(I)) \subseteq I\).

**Proof.** The first point is obvious. For the inclusion, let \(C, D \in C\). We may write any \(f \in I(\pi(I))(C, D)\) as a linear combination \(\sum z_\alpha \otimes f_\alpha\) with \(z_\alpha \in I(1,1)\) and \(f_\alpha \in C(C, D)\). Thus \(f \in I(C, D)\). \(\square\)

5.4. Two continuous sections. The map \(I \mapsto I(I)\) does not send prime ideals to prime \(\otimes\)-ideals in general, so there is no map in the opposite direction to (5.1) *a priori*. This situation improves in the rigid case:

If \(C \in \text{Add}^{\text{rig}}\) and \(I \subset Z(C)\) is an ideal, we get a \(\otimes\)-ideal of \(C\) by the formula

\[
\text{tr}^*(I)(C, D) = \{ f : C \rightarrow D \mid \text{tr}(gf) \in I \ \forall \ g : D \rightarrow C \}.
\]

**Proposition 5.13.** a) We have \(I(I) \subseteq \text{tr}^*(I)\) and \(I(\text{tr}^*(I)) = I\).

b) If \(P\) is prime, so is \(\text{tr}^*(P)\). This defines a continuous and closed section \(\sigma_{\text{tr}}\) of (5.1).

c) If \(P\) is maximal, \(\sigma_{\text{tr}}(P)\) is maximal.

d) Any \(P \in \text{Spec}^{\otimes} C\) is contained in \(\sigma_{\text{tr}} \pi(P)\).

e) \(\sigma_{\text{tr}}\) is spectral if and only if the following condition holds:

For any \(D \in C\) and any \(f \in C(1, D)\), the ideal

\[
\{gf \mid g \in C(D, 1)\}
\]

of \(Z(C)\) is finitely generated.

(This is automatic if \(Z(C)\) is Noetherian.)

**Proof.** a) is obvious. b) For the first point, it suffices as usual to show that \(f \otimes g \notin \text{tr}^*(P)\) if \(f \notin \text{tr}^*(P)\) and \(g \notin \text{tr}^*(P)\) for \(f\) and \(g\) with domain \(1\). But, by hypothesis, there exist \(f', g'\) such that \(f' \circ f \notin \text{tr}^*(P)(1, 1)\) = \(P\) and \(g' \circ g \notin P\). Then \((f' \otimes g') \circ (f \otimes g) = (f' \circ f) \otimes (g' \otimes g) \notin P\).
That $\sigma_{tr}$ is a section of $\pi$ follows from a). Let $\mathbb{I}$ be a $\otimes$-ideal of $\mathcal{C}$. We claim that

$$(\text{tr}^*)^{-1}(V(\mathbb{I})) = V(\pi(\mathbb{I}))$$

which will prove that $\sigma_{tr}$ is continuous. Indeed, let $P \in \text{Spec} \ Z(C)$ be such that $I \subseteq \text{tr}^*(P)$. The $I(1,1) = I(1,1) \subseteq \text{tr}^*(P)(1,1) = P$. Conversely if this holds, then, for any $D \in \mathcal{C}$, any $f \in I(1,D)$ and any $g \in \mathcal{C}(D,1)$, $\text{tr}(gf) = gf \in I(1,1) \subseteq P$.

To see that $\sigma_{tr}$ is closed, we observe that the obvious inclusion $\text{tr}^*V(I) \subseteq V(\text{tr}^*(I))$ is an equality for all $I$, thanks to a).

c) We reduce to $P = 0$ by replacing $\mathcal{C}$ with $Z(C)/P \otimes_{Z(C)} \mathcal{C}$. Then $Z(C)$ is a field and $\text{tr}^*(0) = N_C$, which is the unique maximal proper $\otimes$-ideal of $\mathcal{C}$ by [1, Prop. 7.1.4 b)]. (One could say that $\mathcal{C}$ is $\otimes$-local in this case.)

d) More generally, $I \subseteq \text{tr}^*(\mathbb{I})$ for any $\otimes$-ideal $\mathbb{I}$ (direct check).

Finally, the ideal displayed in e) is none else than $(\text{tr}^*)^{-1}(V(f))$, and $V(g) = V(\tilde{g})$ for any $g : C \to D$ where $\tilde{g}$ is the right adjoint of $g$. $\square$

Let now $\mathcal{A} \in \text{Ex}^{rig}$. If $I$ is an ideal of $Z(\mathcal{A})$, define

$I_\oplus := \ker_m(\mathcal{A} \to \mathcal{A}/I)$

cf. Section 2 and Notation 4.22. By Lemma 2.1 and Remark 4.20, this is the $\otimes$-ideal

$$\{ f \mid e(f) \in I \}$$

(see Definition 4.12).

**Proposition 5.14.** a) We have $I_\oplus \subseteq I(I)$ and $I(I_\oplus) = I$.
b) For $P \in \text{Spec} \ Z(\mathcal{A})$, $P_\oplus$ is prime. This defines another section of (5.1):

$$(5.2) \quad \sigma : \text{Spec} \ Z(\mathcal{A}) \to \text{Spec}^\otimes \mathcal{A}$$

such that $\sigma(P) \subseteq \sigma_{tr}(P)$ for all $P$ (cf. Proposition 5.13). It is spectral.
c) Any $\mathbb{P} \in \text{Spec}^\otimes \mathcal{A}$ contains $\sigma(\mathbb{P})$.
d) For any $I$, the obvious $\otimes$-functor

$$\mathcal{A}/I_\oplus \to \mathcal{A}/I$$

is an equivalence of categories.

**Proof.** In a), the first point follows from Lemma 4.14 c) and the second one is obvious.

b) For the first claim, we use the fact that $Z(\mathcal{A}/I)$ is a field (Theorem 4.21), hence that $\mathcal{A}/I$ is integral ([4, Prop. 2.5 a]) or Proposition 4.15). This defines (5.2).
Let \( X \subset \text{Ar}(\mathcal{A}) \). By Lemma 2.1, \( P \in \text{Spec} \, Z(\mathcal{A}) \) is in the inverse image of \( V(X) \) under (5.2) if and only if \( \text{Im}(f) \in \mathcal{T}(P) \) for all \( f \in X \). By Remark 4.20, this condition amounts to \( e(\text{Im}(f)) \in P \) for all \( f \in X \), i.e. to \( P \in V(\{ e(\text{Im}(f)) \mid f \in X \}) \). This shows the spectrality of (5.2), and the fact that it is a section of (5.1) follows from a).

c) More generally we have \( (\pi(\mathbb{I}))_{\oplus} \subseteq \mathbb{I} \) for any \( \mathbb{I} \), as follows from a) and Lemma 5.12.

In d), the functor is faithful and (essentially) surjective by definition, and is full by Theorem 4.21.

Remark 5.15. Propositions 5.13 and 5.14 give the following nice picture of \( \text{Spec}^\otimes \mathcal{C} \) for \( \mathcal{C} \in \text{Add}^{\text{rig}} \): it is fibred over \( \text{Spec} \, Z(\mathcal{C}) \) by the continuous map \( \pi \) of (5.1) and, for any \( P \in \text{Spec} \, Z(\mathcal{C}) \), \( \pi^{-1}(P) \) is homeomorphic to \( \text{Spec}^\otimes(\mathcal{C}/P) \). There is a canonical “maximal” section, as well as a “minimal” 0-section (5.2) when \( \mathcal{C} \in \text{Ex}^{\text{rig}} \).

6. Applications to universal rigid abelian \( \otimes \)-categories

Let \( \mathcal{C} \in \text{Add}^{\text{rig}} \). We write \( \lambda_c : \mathcal{C} \to T(\mathcal{C}) \) for the canonical \( \otimes \)-functor to the universal abelian \( \otimes \)-category \( T(\mathcal{C}) \) from [4, Th. 5.1].

6.1. (Local) abelian \( \otimes \)-envelopes. The following corollary is a complement to [4, Prop. 6.2]:

Corollary 6.1. Suppose that \( \mathcal{C} \) admits a faithful \( \otimes \)-functor to a connected rigid abelian \( \otimes \)-category \( \mathcal{B} \). Let \( F : T(\mathcal{C}) \to \mathcal{B} \) be the induced exact \( \otimes \)-functor. If \( \mathcal{C} \) admits an abelian \( \otimes \)-envelope \( E(\mathcal{C}) \) in the sense of [4, Def. 6.1], then \( E(\mathcal{C}) = T(\mathcal{C}) \parallel M(F) \) (see Corollary 4.26).

What happens when \( \mathcal{C} \) does not admit an abelian \( \otimes \)-envelope? The answer can be seen as a more precise version of [6, Th. 5.2.2]:

Corollary 6.2. The faithful \( \otimes \)-functors to connected rigid abelian \( \otimes \)-categories are classified by those maximal ideals \( M \in \text{Spec} \, Z(T(\mathcal{C})) \) such that the composition \( \mathcal{C} \xrightarrow{\lambda_C} T(\mathcal{C}) \to T(\mathcal{C}) \parallel M \) is faithful. This set is closed, hence compact (and may be empty). We call these functors the local abelian \( \otimes \)-envelopes of \( \mathcal{C} \).

(Said otherwise: the compact set of corollary 6.2 is in \( 1 \rightarrow 1 \) correspondence with the set \( \mathcal{H}(\mathcal{C}) \) of [6, Th. 5.2.2 (1)].)

Proof. The only thing to justify is the closedness, which follows from Proposition 4.27. \( \square \)
Theorem 6.3. Let $\mathbb{I}$ be a $\otimes$-ideal of $\mathcal{C}$. Then the canonical functor $F : T(\mathcal{C}) \to T(\mathcal{C}/\mathbb{I})$ is a localisation, and its (object) kernel $\mathcal{I}$ is generated by the $\text{Supp} \lambda_{\mathcal{C}}(f)$ for $f \in \mathbb{I}$, hence the ideal $I(\mathcal{I})$ of Remark 4.20 equals $\pi(\mathbb{I}) = \mathbb{I}(1, 1)$. The induced functor

$$Z(T(\mathcal{C}))/\pi(\mathbb{I}) \otimes Z(T(\mathcal{C})) T(\mathcal{C}) \to T(\mathcal{C}/\mathbb{I})$$

is an equivalence of $\otimes$-categories.

Proof. $F$ factors as a composition of exact $\otimes$-functors

$$T(\mathcal{C}) \to T(\mathcal{C})/\mathcal{I} \xrightarrow{\bar{F}} T(\mathcal{C}/\mathbb{I})$$

where $\bar{F}$ is conservative, hence faithful. Thus a $\otimes$-functor from $\mathcal{C}$ to an abelian $\otimes$-category $\mathcal{A}$ factors through $\mathcal{C}/\mathbb{I}$ if and only if its $\otimes$-exact extension to $T(\mathcal{C})$ factors through $T(\mathcal{C})/\mathcal{I}$. Therefore $T(\mathcal{C})/\mathcal{I}$ has the same 2-universal property as $T(\mathcal{C}/\mathbb{I})$, and $\bar{F}$ is an equivalence.

If $f \in I$, then clearly $\text{Supp} \lambda_{\mathcal{C}}(f) \in \mathcal{I}$. Conversely, let $\mathcal{I}'$ be the Serre $\otimes$-ideal generated by the $\text{Supp} \lambda_{\mathcal{C}}(f)$ for $f \in I$. Then the composition

$$\mathcal{C} \xrightarrow{\lambda_{\mathcal{C}}} T(\mathcal{C}) \to T(\mathcal{C})/\mathcal{I}'$$

factors through $\mathcal{C}/I$. Hence the localisation functor $T(\mathcal{C}) \to T(\mathcal{C})/\mathcal{I}'$ factors through $T(\mathcal{C})/\mathcal{I}$, which implies $\mathcal{I}' = \mathcal{I}$. The last claim follows from Theorem 4.21.

Example 6.4. Let $k$ be a field; for $\sim$ an adequate equivalence relation on algebraic cycles, write $\mathcal{M}_{\sim}(k)$ for the category of motives modulo $\sim$ with rational coefficients, and let $T_{\sim}(k) = T(\mathcal{M}_{\sim}(k))$. Then, if $\sim \geq \sim'$, the natural functor $T_{\sim}(k) \to T_{\sim'}(k)$ is a (full) localisation, and an equivalence of categories if $T_{\sim}(k)$ is connected.

6.2. Commutation with colimits.

Proposition 6.5. Let $(C_i)_{i \in I}$ be a 2-direct system in $\text{Add}^{\text{rig}}$, and suppose that the 2-colimit $\mathcal{C} = \varinjlim_i C_i$ exists. Then so does $\varinjlim_i T(C_i)$ in $\text{Ex}^{\text{rig}}$, and the natural functor

$$\varinjlim_i T(C_i) \to T(\mathcal{C})$$

is an equivalence of $\otimes$-categories. In particular, $\varinjlim_i Z(T(C_i)) \xrightarrow{\sim} Z(T(\mathcal{C}))$.

Proof. This follows from the 2-universal property of $T$ (“a left adjoint commutes with arbitrary colimits”).
6.3. **Prime \(\otimes\)-ideals and abelian \(\otimes\)-envelopes.** Let \(\mathcal{C} \in \text{Add}^{\text{rig}}\). The map (5.2) for \(A = T(\mathcal{C})\) and the contravariance of \(\text{Spec}^{\otimes}\) define a continuous map

\[(6.1) \quad \text{Spec} Z(T(\mathcal{C})) \to \text{Spec}^{\otimes} \mathcal{C}.\]

**Proposition 6.6.** The map (6.1) is spectral. The fibre of a prime \(\otimes\)-ideal \(\mathbb{P} \in \text{Spec}^{\otimes} \mathcal{C}\) under (6.1) is in 1–1 correspondence with the set of local abelian \(\otimes\)-envelopes of \(\mathcal{C}/\mathbb{P}\).

(Considering the constructible topology on \(\text{Spec}^{\otimes} \mathcal{C}\), we recover the closedness statement of Corollary 6.2 in a more conceptual way.)

**Proof.** The first claim follows from Corollary 5.9 d) and Proposition 5.14 b). For the next one, let us first assume \(\mathbb{P} = 0\). Then the statement follows from Corollary 6.2. In general, let \(F : \mathcal{C} \to \mathcal{C}/\mathbb{P}\) be the projection, and let \(M \in \text{Spec} Z(T(\mathcal{C}))\) be in the fibre of (6.1). We then have a naturally commutative diagram

\[
\begin{array}{ccc}
\mathcal{C} & \overset{F}{\longrightarrow} & T(\mathcal{C}) \\
\downarrow & & \downarrow \text{T}(F) \\
\mathcal{C}/\mathbb{P} & \overset{G}{\longrightarrow} & T(\mathcal{C}/\mathbb{P})
\end{array}
\]

where \(G\) is a localisation as a consequence of Theorem 6.3. Thus \(M\) defines a local abelian \(\otimes\)-envelope of \(\mathcal{C}/\mathbb{P}\). Conversely, any local abelian \(\otimes\)-envelope of \(\mathcal{C}/\mathbb{P}\) gives rise to such a diagram, thus comes from an ideal in the fibre of (6.1). \(\square\)

**Corollary 6.7.** Suppose that \(Z(\mathcal{C})\) is a field and that \(S(\mathcal{C}) := (\mathcal{C}/\mathcal{N}_C)^{\natural}\) is split \([4, \text{Prop. 7.3}]\). If \(\mathcal{N}_C\) is finitely generated, the functor \(\bar{\pi} : T(\mathcal{C}) \to T(S(\mathcal{C})) = S(\mathcal{C})\) of loc. cit. (7.1) yields a canonical decomposition

\[T(\mathcal{C}) \simeq \text{Ker} \bar{\pi} \times S(\mathcal{C}).\]

**Proof.** By Proposition 6.6, the fibre of \(\mathcal{N}_C\) under (6.1) has one element; if \(\mathcal{N}_C\) is finitely generated, the open subset \(\text{Spec}^{\otimes} \mathcal{C} - \{\mathcal{N}_C\}\) of \(\text{Spec}^{\otimes} \mathcal{C}\) is quasi-compact, hence clopen in \(\text{Spec} Z(T(\mathcal{C}))\). Therefore the corresponding maximal ideal is generated by an idempotent, which induces the desired decomposition by \([4, \text{Cor. 5.2}]\). \(\square\)

**Remark 6.8.** Of course there are many cases when the finite generation hypothesis fails, e.g. from Example 5.4.
6.4. The case of an abelian category. Let \( \mathcal{A} \in \text{Ex}^{\text{rig}} \). The functor \( \lambda_A : \mathcal{A} \to T(\mathcal{A}) \) then has a canonical exact \( \otimes \)-retraction \( \rho_A : T(\mathcal{A}) \to \mathcal{A} \) [4, Cor. 5.2]. By Theorem 4.18, \( \rho_A \) factors through an exact faithful \( \otimes \)-functor
\[
(6.2) \quad \bar{\rho}_A : T(\mathcal{A}) \sslash I \to \mathcal{A}
\]
for a unique ideal \( I \) of \( Z(T(\mathcal{A})) \) (see Notation 4.22).

**Proposition 6.9.** The functor \( \bar{\rho}_A \) is an equivalence of categories, and \( I \) is maximal if and only if \( \mathcal{A} \) is connected.

**Proof.** Write \( \bar{\lambda}_A \) for the composite \( \mathcal{A} \xrightarrow{\lambda_A} T(\mathcal{A}) \sslash I \), so that \( \bar{\rho}_A \bar{\lambda}_A = \text{Id}_\mathcal{A} \). This already shows that \( \bar{\rho}_A \) is essentially surjective; to show its fullness, it suffices then to see that \( \bar{\lambda}_A \) is essentially surjective. We proceed in two steps:

1) \( \bar{\lambda}_A \) is exact: indeed, its composition with the faithful exact functor \( \bar{\rho}_A \) is exact.

2) \( \bar{\lambda}_A \) is essentially surjective. Since any object of \( T(\mathcal{A}) \), hence of \( T(\mathcal{A}) \sslash I \), is isomorphic to a subquotient of an object coming from \( \mathcal{A} \) [4, Prop. 4.4], it suffices to show that the essential image of \( \bar{\lambda}_A \) is stable by subobjects. Let \( A \in \mathcal{A} \), and let \( M \subseteq \bar{\lambda}_A(A) \). By 1), \( M' = \bar{\lambda}_A \bar{\rho}_A(M) \) is another subobject of \( \bar{\lambda}_A(A) \). Let \( M'' = M + M' \subseteq \bar{\lambda}_A(A) \); then \( \bar{\rho}_A(M) = \bar{\rho}_A(M'') = \bar{\rho}_A(M') \), hence \( M = M'' = M' \) by using again the faithful exactness of \( \bar{\rho}_A \).

The last equivalence follows from Theorem 4.21 and Corollary 4.26. (In this connected case, the proposition also follows directly from Corollary 6.1.) \[\square\]

7. Schur finiteness

7.1. Basics. Recall the following definition:

**Definition 7.1.** Let \( \mathcal{C} \in \text{Add}^{\otimes} \) be \( \mathbb{Q} \)-linear. We say that an object \( C \) of \( \mathcal{C} \) is **Schur-finite** if there exists a Schur functor \( S \) [7] such that \( S(C) = 0 \), and that \( \mathcal{C} \) is **Schur-finite** if all its objects are Schur finite.

Schur-finiteness has the following stability properties:

**Proposition 7.2.** Let \( \mathcal{C} \in \text{Add}^{\otimes} \) be \( \mathbb{Q} \)-linear.

a) \( 1 \) is Schur-finite.

b) Let \( C, C' \in \mathcal{C} \). If \( C \) and \( C' \) are Schur-finite, then \( C \oplus C' \) and \( C \otimes C' \) are Schur-finite. So is any direct summand of \( C \). If \( C \) is Schur-finite and dualisable, its dual is Schur-finite.

b) If the tensor structure of \( \mathcal{C} \) respects monomorphisms (resp. epimorphisms), any subobject (resp. quotient) of a Schur-finite object is
Schur-finite.
c) Suppose that $\mathcal{C}$ is abelian and that its tensor structure is right exact. Let $C' \to C \to C'' \to 0$ be an exact sequence in $\mathcal{C}$. Then for any Schur functor $S$, $S(C' \oplus C'') = 0$ implies $S(C) = 0$.
d) If $\mathcal{C}$ is abelian and its tensor structure is exact, the full subcategory of Schur-finite objects is a Serre subcategory of $\mathcal{C}$, stable under (internal) tensor product.

Proof. a) $\Lambda^2(1) = 0$. a) is [7, Cor. 1.13 and 1.18]. b) is obvious. c) follows from the proof of [14, Prop. 2.17]. d) follows from a), b) and c) (see also [7, Prop. 1.19]).

Proposition 7.3. In the situation of [4, Prop. 3.2], $\mathcal{A}$ is Schur-finite if and only if $\mathcal{I}$ and $\mathcal{A} \parallel \mathcal{I}$ are.

Proof. “Only if” is trivial. For “if”, let $A \in \mathcal{A}$, and suppose that its image in $\mathcal{A} \parallel \mathcal{I}$ is killed by a Schur functor $S$. Then $S(A) \in \mathcal{I}$, hence there exists another Schur functor $S'$ such that $S'(S(A)) = 0$. By [11, Ex. 6.17], this is a nontrivial direct sum of objects of the form $S''(A)$, so $A$ is Schur-finite.

We also recall the following theorems of Deligne and O’Sullivan:

Theorem 7.4. a) (Deligne) Let $\mathcal{A} \in \mathbf{Ex}^{\text{rig}}$ be $\mathbf{Q}$-linear, Schur-finite and connected. Then there exists an extension $L/K$ and an exact (hence faithful) $\otimes$-functor $\omega : \mathcal{A} \to \mathbf{Vec}_L^\pm$, where the latter category is that of $\mathbf{Z}/2$-graded finite-dimensional $L$-vector spaces (with the commutativity constraint given by the Koszul rule).
b) (O’Sullivan) Let $\mathcal{C} \in \mathbf{Ex}^{\text{rig}}$ be integral and Schur-finite. Then $\mathcal{C}$ admits a faithful $\otimes$-functor to a category $\mathcal{A} \in \mathbf{Ex}^{\text{rig}}$ as in a), and even an initial one.

Proof. a) follows easily from [7, Prop. 2.1] (for details, see [14, Ex. 2.9 b])). b) is [17, Th. 10.10]; see also Theorem 7.8 below. □

Definition 7.5. For $\mathcal{A} \in \mathbf{Add}^\otimes$, a weak fibre functor is a $\otimes$-functor $\mathcal{A} \to \mathbf{Vec}_L^\pm$.

Proposition 7.6. A $\mathbf{Q}$-linear category $\mathcal{A} \in \mathbf{Ex}^{\text{rig}}$ is Schur-finite if and only if it admits a conservative system of weak fibre functors.

Proof. Combine Theorem 7.4 a) and Corollary 4.23. □
7.2. Classification.

**Proposition 7.7.** Let $C \in \text{Add}^{\text{rig}}$. Then $C$ Schur-finite $\Rightarrow T(C)$ Schur-finite; the converse is true if $\text{Ker}(C \to T(C))$ is a nilideal of $C$.

**Proof.** As observed in [4, Lemma 4.1], any object of $\text{Ab}(C)$ is isomorphic to a subquotient of an object of the form $\iota_C(C)$ for $C \in C$. This carries over to its localisation $T(C)$. The first statement then follows from Proposition 7.2 b) since the tensor structure of $T(C)$ is exact. Conversely, suppose $T(C)$ Schur-finite, and let $C \in C$. By hypothesis, there is a Schur functor $S$ such that $1_S(C) \in \text{Ker}(C \to T(C))$. If the latter is a nilideal, we must have $1_S(C) = 0$, i.e. $C = 0$. \hfill $\blacksquare$

**Theorem 7.8.** If $T(C)$ is Schur-finite, it is 2-universal for $\otimes$-functors from $C$ to Schur-finite rigid abelian $\otimes$-categories. In particular, O’Sullivan’s hulls of [17, Lemma 10.7 and Th. 10.10] mentioned in Theorem 7.4 b) are abelian $\otimes$-envelopes in the sense of [4, §6.1].

**Proof.** This follows from [4, Th. 5.1] and Proposition 7.2 d). \hfill $\blacksquare$

**Corollary 7.9.** Let $C \in \text{Add}^{\text{rig}}$ be $\mathbb{Q}$-linear and Schur-finite. Then a morphism $f \in C$ maps to 0 in $T(C)$ if and only if it maps to 0 via any weak fibre functor.

**Proof.** “Only if” is obvious and “if” follows from Proposition 7.6. \hfill $\blacksquare$

**Theorem 7.10.** For $C \in \text{Add}^{\text{rig}}$ Schur-finite, the map (6.1) is a homeomorphism for the constructible topology on $\text{Spec}^\otimes C$. Moreover, $\text{Ker}(\lambda_C : C \to T(C)) = \square 0$.

In particular, $Z(T(C))$ is a field if and only if $C$ has a unique prime $\otimes$-ideal. If $Z(C)$ is a field, this is equivalent to saying that $\mathcal{N}_C$ is the only prime ideal of $C$.

**Proof.** The bijectivity of (6.1) follows from Proposition 6.6 and Theorem 7.8; Proposition 6.6 also implies that it is a homeomorphism. The second (resp. third) claim then follows from [4, Th. 5.1] and Proposition 5.8 (resp. from Proposition 5.13). \hfill $\blacksquare$

As an application, we get the following partial refinement of [4, Prop. 8.5]:

**Corollary 7.11.** Let $k$ be a field and, for an adequate equivalence relation $\sim$ on algebraic cycles, let $\mathcal{M}^\text{ab}_\sim(k)$ be the thick subcategory of $\mathcal{M}_\sim(k)$ generated by Artin motives and motives of abelian varieties.
Then
a) the functor
\[ \mathcal{M}^{ab}_{\text{tnil}}(k) \to T(\mathcal{M}^{ab}_{\text{tnil}}(k)) \]
is faithful.
b) \( \mathcal{M}^{ab}_{\text{tnil}}(k) \to \mathcal{M}^{ab}_{\text{num}}(k) \) is an equivalence of categories if and only if
\[ T(\mathcal{M}^{ab}_{\text{tnil}}(k)) \]is connected.

Proof. a) Indeed, \( \mathcal{M}^{ab}_{\text{tnil}}(k) \) is a Kimura category, hence Schur-finite, and we apply Theorem 7.10. b) then follows from a) in the same way as in [4, Prop. 8.5].

Remark 7.12. In contrast to Theorem 7.10, if \( Z(C) \) is a field, the map \( \text{Spec } Z(T(C)) \to \text{Spec }^\otimes C \) obtained by composing with the section \( \sigma_{tr} \) of Proposition 5.13 b) has image the maximal \( \otimes \)-ideal \( \mathcal{N}_C \) of \( C \): this follows from the obvious naturality of this section.

7.3. Schur ideals.

Definition 7.13. Let \( C \in \text{Add}^\otimes \), and let \( X \) be a collection of objects of \( C \).
a) A \( \otimes \)-ideal \( \mathcal{I} \) of \( C \) is \( X \)-Schur if every \( C \in X \) becomes Schur-finite in \( C/\mathcal{I} \).
b) The \( X \)-Schur locus of \( \text{Spec }^\otimes C \) is
\[ S(C, X) = \{ \mathcal{P} \in \text{Spec }^\otimes C \mid \mathcal{P} \text{ is } X\text{-Schur} \} \]
If \( X = \text{Ob}(C) \), we say Schur for \( X \)-Schur and write \( S(C) \) for \( S(C, X) \). If \( X = \{ C \} \), we say \( C \)-Schur for \( X \)-Schur and write \( S(C, C) \) for \( S(C, X) \).

Let \( C \in \text{Add}^{\text{rig}} \) be integral and Schur-finite. By Theorem 7.4, there exists a faithful \( \otimes \)-functor \( \omega : C/\mathcal{P} \to \text{Vec}_K^\pm \) for some extension \( K \) of \( Q \); in other words, \( C \) is proto-tannakian in the sense of [14, Def. 2.8]. By loc. cit., Lemma 2.13, the super-dimension of \( \omega(C) \) for \( C \in C \) does not depend on the choice of \( \omega \): we call it the super-dimension of \( C \) and write it \( \dim^\pm(C) \).

More generally, suppose only that \( C \) is integral. By Proposition 7.2 a), the full subcategory \( C' \) of \( C \) formed of Schur-finite objects still belongs to \( \text{Add}^{\text{rig}} \), and it evidently integral and Schur-finite. Applying the above to \( C' \), we may define the super-dimension of any Schur-finite object of \( C \). If \( C \in C \) is not Schur-finite, we set \( \dim^\pm(C) = (\infty|\infty) \).

Definition 7.14. Let \( C \in \text{Add}^{\text{rig}} \), \( C \in C \) and \( \mathcal{P} \in \text{Spec }^\otimes C \). Letting \( \pi_\mathcal{P} : C \to C/\mathcal{P} \) be the projection functor, we let
\[ \dim^\pm_\mathcal{P}(C) = \dim^\pm(\pi_\mathcal{P}(C)) \].
Lemma 7.15. Let $C \in \mathcal{C}$.

a) For $\mathfrak{P} \in \text{Spec}^{\otimes} \mathcal{C}$, write $\chi_{\mathfrak{P}}(C) = \text{Tr}(1_{\pi_{\mathcal{C}}(C)}) \in Z(\mathcal{C}/\mathfrak{P})$. Then $\chi_{\mathfrak{P}}(C)$ is independent of $\mathfrak{P}$ if $\text{Spec}^{\otimes} \mathcal{C}$ is connected; if $\dim^{\pm}_{\mathfrak{P}}(C) = (p|q)$ with $(p|q) < (\infty|\infty)$, then $\chi_{\mathfrak{P}}(C) = p - q$.

b) Suppose $\text{Spec}^{\otimes} \mathcal{C}$ connected. If $\chi_{\mathfrak{P}}(C) / \mathfrak{P} \in \mathbb{Z}$ for some (hence all) $\mathfrak{P}$, then $S(\mathcal{C}, C) = \emptyset$.

c) If $\mathfrak{P} \subseteq \mathfrak{Q}$, $\dim^{\pm}_{\mathfrak{P}}(C) \geq \dim^{\pm}_{\mathfrak{Q}}(C)$. In particular, if $Z(\mathcal{C})$ is a field, then $S(\mathcal{C}, C) \neq \emptyset \iff \dim^{\pm}_{X}(C) < (\infty, \infty)$.

Proof. a) The first fact is obvious since the trace commutes with $\otimes$-functors. The second one follows from [14, Lemma 2.13 (5)]. b) follows from a). c) is clear. ✷

Remark 7.16. It is quite possible that $S(\mathcal{C}) = \emptyset$ even if $\chi(C) \in \mathbb{Z}$ for all $C \in \mathcal{C}$: this happens e.g. if $\mathcal{C} \in \text{Add}^{\text{rig}}$ and $T(\mathcal{C}) = 0$, by Theorem 7.4 b).

Proposition 7.17. a) If $\mathbb{I}$ and $\mathbb{J}$ are $X$-Schur, so is $\mathbb{I} \otimes \mathbb{J}$, hence also $\mathbb{I} \cap \mathbb{J}$.

b) Suppose that $\mathbb{I} \subseteq \mathbb{J}$. If $\mathbb{I}$ is $X$-Schur, so is $\mathbb{J}$; the converse is true if $\mathbb{J}/\mathbb{I}$ is nil in $\mathcal{C}/\mathbb{I}$.

c) If $C \in \text{Add}^{\text{rig}}$, $\mathbb{I}$ is $X$-Schur if and only if $\sqrt[\mathbb{I}]{\mathbb{I}}$ is.

d) $\mathbb{I}$ is $X$-Schur if and only if $\mathbb{I}^{\ast}$ is (see §2).

Proof. a) follows from [7, Prop. 1.6]. The first part of b) is clear; its second part is seen as in the proof of Proposition 7.7. c) follows from b) and [1, Lemma 7.4.2 ii)]. In c), $\mathbb{I}$ is $X$-Schur $\iff$ for all $C \in \mathcal{X}$ there exists a Schur functor $S$ such that $1_{S(\mathcal{C})} \in \mathbb{I} \iff \mathbb{I}^{\ast}$ is $X$-Schur. ✷

Let $\text{Add}^{\otimes}$ be the 1-full, 2-full subcategory of $\text{Add}^{\otimes}$ formed of Schur-finite categories, and similarly for $\text{Add}^{\text{rig}}$, $\text{Ex}^{\otimes}$ and $\text{Ex}^{\text{rig}}$. By Proposition 7.2 a) and d), the inclusion functors $\text{Add}^{\otimes} \hookrightarrow \text{Add}^{\otimes}$, etc. all have a right 2-adjoint. Similarly:

Corollary 7.18. The inclusions $\text{Add}^{\otimes} \hookrightarrow \text{Add}^{\otimes}$ and $\text{Add}^{\text{rig}} \hookrightarrow \text{Add}^{\text{rig}}$ have pro-left 2-adjoints.

Proof. This follows from Proposition 7.17 a). ✷

Let $\mathfrak{I}$ be $X$-Schur. Then $V(\mathfrak{I}) \cap \text{Spec}^{\otimes} \mathcal{C} \subseteq S(\mathcal{C}, X)$. By Proposition 7.17 a) and b), this defines a topology on $S(\mathcal{C}, X)$, coarser than the induced topology from $\text{Spec}^{\otimes} \mathcal{C}$. It would be interesting to better understand it in general: for example there may not be any finitely
generated $X$-Schur ideal, and in this case $S(\mathcal{C}, X)$ is (probably) not spectral for this topology.

8. The free rigid $\otimes$-category on one generator

8.1. Introducing the player. Consider the category $\mathcal{L} \in \text{Add}^{\text{rig}}$ with distinguished object $L$ constructed in [9, (1.26)]: for any $\mathcal{C} \in \text{Add}^{\text{rig}}$, the functor $\text{Add}^{\text{rig}}(\mathcal{L}, \mathcal{C}) \to \mathcal{C}$ sending $F$ to $F(L)$ is an equivalence of categories (where we only retain the isomorphisms in $\mathcal{C}$ since $\text{Add}^{\text{rig}}(\mathcal{L}, \mathcal{C})$ is a groupoid). We have $Z(\mathcal{L}) = \mathbb{Z}[t]$. Then $T(\mathcal{L})$ has the same universal property in $\text{Ex}^{\text{rig}}$.

Describing $T(\mathcal{L})$ seems difficult; we restrict to describing $T(\mathcal{L}_Q)$ where $\mathcal{L}_Q$ is the pseudo-abelian $\otimes$-category obtained by tensoring morphisms with $Q$ and taking the Karoubian hull: it is universal for $Q$-linear rigid $\otimes$-categories. For this, we shall use the results of [10].

8.2. The fibres of (5.1). The prime ideals $P$ of $\mathbb{Q}[t]$ may be identified to $t$ as a transcendental number over $\mathbb{Q}$ (for $P = (0)$) and to the algebraic numbers up to conjugation (for the maximal ideals). We write $P_\alpha$ for the ideal corresponding to such a number $\alpha$, $\mathbb{Q}(\alpha)$ for the quotient field of $\mathbb{Q}[t]/P_\alpha$ and $\mathcal{L}_{Q}(\alpha)$ for $((\mathbb{Q}(\alpha) \otimes \mathbb{Q}[t]) \otimes Q)$. 

Proposition 8.1. If $\alpha \notin \mathbb{Z}$, $\mathcal{L}_{Q}(\alpha)$ is abelian semi-simple, hence the fibre of $P_\alpha$ under (5.1) consists of a single prime of $\text{Spec} \otimes \mathcal{L}_Q$.

Proof. This is [8, Th. 10.5]. □

8.3. The case of integers. For $\alpha = n \in \mathbb{Z}$, the situation is more interesting. Here $\mathcal{L}_Q(n)$ is the category $\text{Rep}(\text{GL}(n), Q)$ of [8, after Def. 10.2]. We still write $L$ for the image of $L$ in $\mathcal{L}_Q(n)$.

For each pair $(p, q)$ of nonnegative integers such that $p - q = n$, with $(p, q) \neq (0, 0)$, write $P(p|q)$ for the kernel of the $\otimes$-functor

\[ F(p|q) : \mathcal{L}_Q(n) \to \text{Rep}_Q(\text{GL}(p|q)) \]

sending $L$ to $Q^{p|q}$: since the target category is integral, it is a prime $\otimes$-ideal.

Lemma 8.2. a) We have a chain of inclusions

\[ \cdots \subset \mathbb{P}(p+1|q+1) \subset \mathbb{P}(p|q) \subset \cdots \subset \mathbb{P}(n|0) \subset \mathcal{L}_Q(n) \text{ if } n \geq 0 \]

\[ \cdots \subset \mathbb{P}(p+1|q+1) \subset \mathbb{P}(p|q) \subset \cdots \subset \mathbb{P}(0|-n) \subset \mathcal{L}_Q(n) \text{ if } n \leq 0. \]

b) We have $\mathbb{P}(n|0) = \mathcal{N}$ (resp. $\mathbb{P}(0|n) = \mathcal{N}$) if $n > 0$ (resp. if $n < 0$). If $n = 0$, we set $\mathbb{P}(0|0) := \mathcal{N}$.

c) We have a fully faithful $\otimes$-functor

\[ F(\infty|\infty) : \mathcal{L}_Q(n) \to \text{Rep}_Q(\text{GL}(\infty|\infty)), \]
where \( \text{Rep}_Q(\text{GL}(\infty|\infty)) \in \text{Ex}^{\text{rig}} \) is the category constructed in \cite{10} and denoted by \( \mathcal{V}_\alpha \) in loc. cit.

d) The \( \mathbb{P}(p|q) \) are the only nonzero \( \otimes \)-ideals of \( \mathcal{L}_Q(n) \).

e) The space \( \text{Spec}^\circ \mathcal{L}_Q(n) \) is homeomorphic to \( \mathbb{N} \cup \{\infty\} \) provided with the “right order topology” (whose closed subsets are \( \mathbb{N} \cup \{\infty\} \) and the intervals \( [0, r) \)) for the map

\[
\mathbb{M} : \mathbb{N} \cup \{\infty\} \to \text{Spec}^\circ \mathcal{L}_Q(n)
\]

\[
r \mapsto \begin{cases} 
\mathbb{P}(n + r|n) & \text{if } n \geq 0 \\
\mathbb{P}(r - n + r) & \text{if } n \leq 0.
\end{cases}
\]

The associated constructible topology \( A(\mathbb{N}) \) is the Alexandrov compactification of the discrete space \( \mathbb{N} \).

**Proof.** a) follows from the existence of a \( \otimes \)-functor \( \text{Rep}_Q(\text{GL}(p+1|q+1)) \to \text{Rep}_Q(\text{GL}(p|q)) \) sending \( Q^{p+1}|q+1 \) to \( Q^{p|q} \) (the Duflo-Serganova construction, \cite{10, §7.1}); or see the reasoning to obtain \cite{17}, (5.1). b) follows from \cite{8, Th. 10.4}. c) is \cite{10, Prop. 8.1.2}. d) is \cite{17, Lemma 5.2}. The first statement of e) follows immediately from d), and the second one is obvious. \( \square \)

**Remark 8.3.** Although Lemma 8.2 shows that \( \mathcal{L}_Q(n) \) is “\( \otimes \)-Noetherian”, it also shows that the analogue of Krull’s intersection theorem fails completely in this category. Namely, \( I = I^{\otimes n} \) for any \( \otimes \)-ideal \( I \) of \( \mathcal{L}_Q(n) \) and any \( n > 0 \). Indeed, all such ideals are prime, hence \( \mathcal{L}_Q(n)/I^{\otimes n} \) is integral. But \( f^{\otimes n} = 0 \) for any \( f \in I/I^{\otimes n} \).

Since all \( \otimes \)-categories \( \text{Rep}_Q(\text{GL}(p|q)) \) (including for \( (p|q) = (\infty|\infty) \)) are abelian and rigid, the functors \( F(p|q) \) of \eqref{8.1} and Lemma 8.2 c) extend canonically to exact \( \otimes \)-functors

\[
F(p|q) : T(\mathcal{L}_Q(n)) \to \text{Rep}_Q(\text{GL}(p|q)).
\]

**Proposition 8.4.** a) The map \eqref{6.1} is a homeomorphism for the constructible topology on \( \text{Spec}^\circ \mathcal{L}_Q(n) \) and yields a ring isomorphism

\[
\theta : Z(T(\mathcal{L}_Q(n))) \cong \text{Cont}(A(\mathbb{N}), Q).
\]

For \( r \in A(\mathbb{N}) \), let \( \theta^*(r) = \{ z \in Z(T(\mathcal{L}_Q(n))) | \theta(z)(r) = 0 \} \). Then the \( \otimes \)-category \( T(\mathcal{L}_Q(n)) \otimes \theta^*(r) \) is \( \text{Rep}_Q(\text{GL}(n + r|n)) \) if \( n > 0 \), \( \text{Rep}_Q(\text{GL}(r - n + r)) \) if \( n < 0 \), or \( \text{Rep}_Q(\text{GL}(r + 1|n + 1)) \) if \( n = 0 \). (This includes the case \( r = \infty \), the accumulation point of \( A(\mathbb{N}) \).)

b) Let \( A \in T(\mathcal{L}_Q(n)) \). If \( F(\infty|\infty)(A) \neq 0 \), then \( F(p|q)(A) \neq 0 \) for all but a finite number of \( (p|q) \). The converse is not true.
c) Let $S$ be a finite subset of $\mathbb{N}$. Then there is a canonical decomposition

$$T(\mathcal{L}_Q(n)) \simeq T(S) \times \prod_{s \in S} T(\mathcal{L}_Q(n)) \parallel \theta^*(s).$$

In particular, the obvious functor

$$T(\mathcal{L}_Q(n)/\mathcal{M}(r)) \to \prod_{s \leq r} T(\mathcal{L}_Q(n)) \parallel \theta^*(s)$$

is an equivalence of $\otimes$-categories for any $r \in \mathbb{N}$.

Proof. a) We first show the bijectivity of (6.1) together with the last point. This is a mere translation of [10, Th. 2]: indeed, let $F : T(\mathcal{L}_Q(n)) \to \mathcal{B}$ be an exact $\otimes$-functor, with $\mathcal{B} \in \text{Ex}^{\text{rig}}$ connected. Let $\Sigma$ be the set of Schur functors killing $F(L)$. If $\Sigma = \emptyset$ (resp. $\Sigma \neq \emptyset$), $F \circ \lambda_{\mathcal{L}_Q(n)}$ factors through $F(\infty|\infty)$ (resp. through a unique $F(p|q)$) by part (a) (resp. (b)) of [10, Th. 2]; hence $F$ factors through the corresponding $\bar{F}(p|q)$. We now conclude with Corollary 4.26.

As in the proof of Theorem 7.10, the bijectivity of (6.1) implies that it is a homeomorphism. The second point now follows from Lemma 3.3.

b) By Theorem 4.21, $F(\infty|\infty)(A) \neq 0 \iff \theta^*(\infty) \notin V(e(A))$. But the closed subsets of $A(N)$ not containing $\infty$ are finite. Therefore, $e(A) \in \theta^*(r)$ only for finitely many $r < \infty$, and $F(p|q)(A) = 0$ only for the corresponding $(p|q)$. The converse fails because any subset of $A(N)$ containing $\infty$ is closed.

c) Let $e(S) \in Z(\mathcal{L}_Q(n))$ be the idempotent corresponding via a) to the function with value 1 at every $s \in S$ and 0 elsewhere. The decomposition of the statement is the one defined by $e$, with $T(S)$ corresponding to $	ext{Ker } e$. The description of the other factor follows from Theorem 6.3 and Corollary 4.25. The second statement is the special case $S = [0, r]$.

Here is a complement in the case $n = 0$. Let $\mathcal{C} \in \text{Add}^{\text{rig}}$ be the category of [7, 5.8]: it is $\otimes$-generated by a self-dual object $X$ of Euler characteristic 0 possessing an endomorphism $\theta$ of trace 1 and square 0. By the universal property, there is a unique $\otimes$-functor $F : \mathcal{L}_Q(0) \to \mathcal{C}$ sending the generator $L$ of $\mathcal{L}_Q(0)$ to $X$.

Proposition 8.5. a) The functor $F$ is faithful, but not full.

b) We have $\mathcal{L}_Q(0)/\mathcal{N} = \mathcal{Q}$ (the category with one object having endomorphisms $\mathcal{Q}$).

Proof. a) $F$ is not full because $\theta \notin \text{Im } F$ since $\text{End}_{\mathcal{L}_Q(0)}(L)$ is generated by $1_L$. In view of Lemma 8.2 d), to prove that it is faithful it suffices
to show that $X$ is not Schur-finite, and for this it suffices to show that $S(\theta) \neq 0$ for any Schur functor $S$. Let $r > 0$ and let $\sigma \in \mathfrak{S}_r$. Since $\theta^2 = 0$, we have by [1, Cor. 7.2.2]:

$$\text{tr}(\sigma^{-1} \circ \theta^{\otimes r}) = \begin{cases} 1 & \text{if } \sigma = 1 \\ 0 & \text{if } \sigma \neq 1. \end{cases}$$

Let $\lambda$ be a partition of $r$, and let $c_\lambda \in \mathbb{Q}[\mathfrak{S}_r]$ be the corresponding Young symmetriser [11, (4.2)]. Up to a nonzero scalar, $\text{tr}(S_\lambda(\theta))$ is the coefficient of $\sigma = 1$ in $c_\lambda$. This coefficient is 1, hence $S_\lambda(\theta) \neq 0$ as requested.

b) This is because $1_L \in \mathbb{N}$. \hfill \Box

**Remark 8.6.** The argument of a) shows, more generally, that there is no $\otimes$-functor from $\mathcal{C}$ to a Schur-finite category. (See also remark 7.16.)

### 8.4. Globalisation

Recall that $Z(\mathcal{L}_Q) = \mathbb{Q}[t]$. By Proposition 4.2, there is a canonical homomorphism

$$\mathbb{Q}[t]^{\text{abs}} \to Z(T(\mathcal{L}_Q))$$

where $\mathbb{Q}[t]^{\text{abs}}$ is the flat completion of $\mathbb{Q}[t]$ [16] (recall that $\text{Spec } \mathbb{Q}[t]^{\text{abs}} \to \text{Spec } \mathbb{Q}[t]$ is bijective on the underlying sets). For any $\alpha$, it fits in a cocartesian square

$$\begin{CD}
\mathbb{Q}[t]^{\text{abs}} @>>> Z(T(\mathcal{L}_Q)) \\
@VVV @VVV \\
\mathbb{Q}(\alpha) @>>> Z(T(\mathcal{L}_Q(\alpha))
\end{CD}$$

Collecting Propositions 8.1 and 8.4, we get:

**Theorem 8.7.** The bottom horizontal homomorphism of (8.2) is an isomorphism if $\alpha \notin \mathbb{Z}$, and is the ‘constant’ homomorphism $\mathbb{Q} \to \text{Cont}(A(\mathbb{N}), \mathbb{Q})$ if $\alpha \in \mathbb{Z}$. \hfill \Box

Unfortunately, this does not quite give an explicit description of $Z(T(\mathcal{L}_Q))$ as a $\mathbb{Q}[t]^{\text{abs}}$-algebra, nor a fortiori of the category $T(\mathcal{L}_Q)$. Nevertheless, it gives a qualitative description of the map $\text{Spec } Z(T(\mathcal{L}_Q)) \to \text{Spec } \mathbb{Q}[t]^{\text{abs}}$: it is an isomorphism away from the integers, where the fibres are all isomorphic to $A(\mathbb{N})$. 
REFERENCES

[1] Y. André, B. Kahn Nilpotence, radicaux et structures monoïdales (with an appendix by P. O’Sullivan), Rend. Sem. Mat. Univ. Padova 108 (2002), 107–291.
[2] M. F. Atiyah, I. G. MacDonald Introduction to commutative algebra, Addison-Wesley, 1969.
[3] P. Balmer A guide to tensor triangular classification, in Handbook of homotopy theory, Chapman and Hall/CRC (2019), Ch. 4, 147–164.
[4] L. Barbieri-Viale, B. Kahn A universal rigid abelian tensor category, preprint, 2021. https://arxiv.org/abs/2111.11217.
[5] N. Bourbaki Éléments de mathématiques, Algèbre commutative, ch. I-IV, Masson, 1985.
[6] K. Coulembier Homological kernels of monoidal functors, preprint, 2021, https://arxiv.org/abs/2107.02374.
[7] P. Deligne Catégories tensorielles, Moscow Math. J. 2 (2002), 227–248.
[8] P. Deligne La catégorie des représentations du groupe symétrique $S_t$, lorsque $t$ n’est pas un entier naturel, in Algebraic groups and homogeneous spaces (Tata Institute of Fundamental Research Studies in Mathematics, Mumbai, 2007), 209–273; available on his Web page.
[9] P. Deligne, J. Milne Tannakian categories, in Hodge cycles, motives, and Shimura varieties, Lect. Notes in Math. 900, Springer, 1982, 101–228.
[10] I. Entova-Aizenbud, V. Hinich, V. Serganova Deligne categories and the limit of categories $\text{Rep}(\text{GL}(m|n))$, IMRN 2020, no. 15, 4602–4666.
[11] W. Fulton, J. Harris Representation theory, a first course, Springer, 2004.
[12] P. Gabriel Des catégories abéliennes, Bull. SMF 90 (1962), 323–448.
[13] M. Hochster Prime ideal structure in commutative rings, Trans. AMS 142 (1969), 43–60.
[14] B. Kahn Exactness and faithfulness of monoidal functors, preprint, 2021, https://arxiv.org/abs/2110.09381.
[15] H. Krause Smashing subcategories and the telescope conjecture – an algebraic approach, Invent. math. 139 (2000), 99–133.
[16] J-P. Olivier Anneaux absolument plats universels et épimorphismes à buts réduits, Sém. Samuel, Algèbre commutative 2 (1967), 1–12.
[17] P. O’Sullivan Super Tannakian hulls, preprint (2020), https://arxiv.org/abs/2012.15703.
[18] C. Weibel An introduction to homological algebra, Cambridge Univ. Press, 1994.
[19] A. Grothendieck (with the collaboration of J. Dieudonné) Éléments de géométrie algébrique, I: le langage des schémas, Springer, 1971.

IMJ-PRG, CASE 247, 4 PLACE JUSSIEU, 75252 PARIS CEDEX 05, FRANCE
Email address: bruno.kahn@imj-prg.fr