A free boundary problem for a ratio-dependent predator–prey system

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Abstract. In this paper, we study a free boundary problem for a ratio-dependent predator–prey system in one space dimension, in which the free boundary is only caused by prey, representing the spreading fronts of prey. We discuss the long time behaviors of solution as \( t \to \infty \) and establish a spreading–vanishing dichotomy; namely, either the two species successfully spread to infinity as \( t \to \infty \) and survive in the new environment, or they cannot spread to the whole space and the prey will vanish eventually. Then, the criteria for spreading and vanishing are obtained. Furthermore, when spreading occurs, some estimates of the asymptotic speed of \( h(t) \) as \( t \to \infty \) are provided. Finally, some realistic and meaningful phenomena are discovered.

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1. Introduction

The expanding of an alien or a invasive species and the conservation of native species are discussed widely in biological mathematics. Many mathematical models are proposed to investigate the basis of the above problems. For example, Du [10–13], Wang [17–24] and other scholars [4,5,7,8,26,28,30] have studied a lot of biomathematics models with a free boundary and established many remarkable results. The predator–prey model in a one-dimensional habitat can be represented by the following system

\[
\begin{align*}
    u_t - u_{xx} &= \lambda u - u^2 - bf(u, v) v, \quad t > 0, \quad x > 0, \\
    v_t - dv_{xx} &= \mu v - v^2 + cf(u, v) v, \quad t > 0, \quad x > 0,
\end{align*}
\]

where \( u(t, x) \) and \( v(t, x) \) denote the population densities of prey and predator at some time \( t \) and position \( x \), respectively; \( \lambda, \mu, b, d, c \) are positive constants; the function \( f(u, v) \) represents functional response. The classical Lotka–Volterra prey–predator model assumes that \( f(u, v) = u \). As far as we know, this model displays the ‘paradox of enrichment’ raised by Hairston et al. [14]) which states that sufficient enrichment or increase in the prey-carrying capacity will destabilize the otherwise stable interior equilibrium. Another is the so-called ‘biological control paradox’ proposed by Luck [16], stating that it does not exist a both low and stable prey equilibrium density, while the ratio-dependent response function \( f(u, v) = \frac{u}{u + mv} \) \((m > 0)\) does not own both defects mentioned above. More and more evidences show that in some specific ecological environments, especially when the predators have to actively seek, share and plunder the preys, a more reasonable predator–prey model should be the ratio-dependent model [1,2,25]).

In the real world, the following phenomenons are very common to us:

(i) One kind of species (prey) inhabits in a initial region. At some time, another kind of species (the alien or invasive species, predator) invades the region.

(ii) At the initial state, a kind of pest species (prey) occupied an area (initial habitat). In order to control and eliminate such pest species, the most economical and environment-friendly approach is
to use biological control, in other words, to put a certain natural enemy of pest species (predator) in this area.

Generally, both predator and prey tend to migrate outward to get a new habitat. Significantly, the prey has a stronger tendency to avoid being hunted in both phenomena above. So it is rational that the free boundary is determined only by the prey. In this model, we suppose that the predator follows almost the same trajectory as prey and so is roughly consistent with the move curve (free boundary). Assume that the left boundary is fixed and the right boundary is free, and the spreading front speed is proportional to the prey’s population gradient at the front. Thus, the biometrics model with a free boundary in one-dimensional is established as follows:

\[
\begin{align*}
\begin{cases}
u_t - u_{xx} = \lambda u - u^2 - \frac{h uv}{u + \mu v}, & t > 0, \ 0 < x < h(t), \\
v_t - dv_{xx} = \mu v - v^2 + \frac{cuv}{u + mv}, & t > 0, \ 0 < x < h(t), \\
x_x(t, 0) = v_x(t, 0) = 0, & t \geq 0, \\
u(t, x) = v(t, x) = 0, & t \geq 0, \ x \geq h(t), \\
h'(t) = -pu_x(t, h(t)), & t \geq 0, \\
u(0, x) = u_0(x), \ v(0, x) = v_0(x), & 0 \leq x \leq h_0, \\
h(0) = h_0, 
\end{cases}
\end{align*}
\]

where \(\rho, h_0\) are given positive constants and \(x = h(t)\) is the free boundary to be solved. The initial functions \(u_0(x)\) and \(v_0(x)\) satisfy

\[
u_0, v_0 \in C^2([0, h_0]), \quad u_0(x), \ v_0(x) > 0, \ x \in [0, h_0), \quad u_0(0) = u_0(h_0) = v_0(0) = v_0(h_0) = 0.
\]

Recently, Wang and Zhang [20] investigated the same spreading mechanism of the classical Lotka–Volterra type prey–predator model \((f(u, v) = u)\). They discussed the asymptotic behaviors of two species and established the criteria for spreading and vanishing. In particular, they found some new phenomena; for instance, the sharp criteria for spreading and vanishing in regard to the initial habitat \(h_0\) are not true in some cases. Besides, Wang [18] researched a free boundary decided by a ratio of prey and predator for the classical Lotka–Volterra type prey–predator model. Wang and Zhao studied a double free boundaries problem in one space dimension in [23], and they generalized these results to higher dimension and heterogeneous environment in [27]. Moreover, Wang and Zhang considered a diffusive Lotka–Volterra type prey–predator model with different free boundaries in [21].

This paper will be devoted to researching the problem (1.1) and get some differences from the Lotka–Volterra type with the same spreading mechanism.

The organization of this paper is as follows. In Sect. 2, the global existence and uniqueness result are given. In Sect. 3, we study the long time behaviors of \((u, v)\) when \(t \to \infty\). In Sect. 4, we establish the spreading-vanishing dichotomy and provide the criteria for spreading and vanishing. Section 5 is devoted to estimating the asymptotic speed of \(h(t)\) as \(t \to \infty\). Finally, in Sect. 6 we give a brief discussion.

2. Existence and uniqueness

In this section, we first prove the following local existence and uniqueness result of problem (1.1) by the contraction mapping theorem. Following that, we present some estimates to show the global solution exists and is unique.

**Theorem 2.1.** For any \(\alpha \in (0, 1)\), there exists \(T > 0\), such that the problem (1.1) admits a unique solution \((u, v, h) \in [C^{1+\alpha, 1+\alpha}(D_T)]^3 \times C^{1+\frac{\alpha}{2}}([0, T])\), and

\[
\begin{align*}
||u||_{C^{1+\frac{\alpha}{2}, 1+\alpha}(D_T)} + ||v||_{C^{1+\frac{\alpha}{2}, 1+\alpha}(D_T)} + ||h||_{C^{1+\frac{\alpha}{2}}([0, T])} \leq C,
\end{align*}
\]
where $D_T = \{(t, x) \in \mathbb{R}^2 : t \in [0, T], \; x \in [0, h(t)]\}; \; C \; \text{and} \; T \; \text{are only depending on} \; h_0, \; \alpha, \; u_0 \; \text{and} \; v_0$.

**Proof.** We employ the method in [7,11] and straighten the free boundary firstly. Let function $\varsigma(y) \in C^3[0, \infty)$ and satisfy

$$\varsigma(y) = 1 \; \text{if} \; |y - h_0| < \frac{h_0}{4}, \; \varsigma(y) = 0 \; \text{if} \; |y - h_0| > \frac{h_0}{2}, \; |\varsigma'(y)| < \frac{6}{h_0} \; \text{for} \; \forall \; y.$$ 

Consider the transformation

$$(t, y) \rightarrow (t, x), \; \text{where} \; x = y + \varsigma(y)(h(t) - h_0), \; 0 \leq y < \infty.$$ 

So the above transformation is a diffeomorphism from $[0, \infty)$ onto $[0, \infty)$ as long as $|h(t) - h_0| \leq \frac{h_0}{8}$. And it changes the free boundary $x = h(t)$ to the fixed line $y = h_0$. Direct calculations deduce that

$$\frac{\partial y}{\partial x} = \frac{1}{1 + \varsigma'(y)(h(t) - h_0)} \equiv \sqrt{A(h(t), y)},$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\varsigma''(y)(h(t) - h_0)}{[1 + \varsigma'(y)(h(t) - h_0)]^3} \equiv B(h(t), y),$$

$$\frac{\partial y}{\partial t} = \frac{\varsigma'(y)(h(t) - h_0)}{E(h(t), h'(t), y).}$$

If we define

$$u(t, x) = u(t, y + \varsigma(y)(h(t) - h_0)) =: w(t, y),$$

$$v(t, x) = v(t, y + \varsigma(y)(h(t) - h_0)) =: z(t, y),$$

then problem (1.1) becomes

$$\begin{cases}
\begin{aligned}
&w_t - Aw_{yy} - (B - E)w_y = \lambda w - w^2 - \frac{bwz}{w + m^2}, & t > 0, \; 0 < y < h_0, \\
&z_t - dA z_{yy} - (dB - E)z_y = \mu z - z^2 + \frac{cwz}{w + m^2}, & t > 0, \; 0 < y < h_0, \\
&w(t, y) = z(t, y) = 0, & t \geq 0, \; y \geq h_0, \\
&w_y(t, 0) = z_y(t, 0) = 0, & t \geq 0, \\
&w(0, y) = u_0(y), & z(0, y) = v_0(y), \; h(0) = h_0, \; 0 \leq y \leq h_0,
\end{aligned}
\end{cases} \quad (2.1)$$

where $A = A(h(t), y)$, $B = B(h(t), y)$ and $E = E(h(t), h'(t), y)$.

Let $h^* = -pu_0'(h_0).$ For $0 < T \leq T_1 := \frac{h_0}{8(1 + h')},$ define $\Delta_T = [0, T] \times [0, h_0],$ and

$$\begin{aligned}
&\mathcal{D}_{1T} = \{w \in C(\Delta_T) : w(t, h_0) = 0, \; w(0, y) = u_0(y), \; ||w - u_0||_{C(\Delta_T)} \leq 1\}, \\
&\mathcal{D}_{2T} = \{z \in C(\Delta_T) : z(t, h_0) = 0, \; z(0, y) = v_0(y), \; ||z - v_0||_{C(\Delta_T)} \leq 1\}, \\
&\mathcal{D}_{3T} = \{h \in C^1([0, T]) : h(0) = h_0, \; h'(0) = h^*, \; ||h' - h^*||_{C([0, T])} \leq 1\}.
\end{aligned}$$

Clearly, $\mathcal{D} := \mathcal{D}_{1T} \times \mathcal{D}_{2T} \times \mathcal{D}_{3T}$ is a complete metric space with the metric

$$d((w_1, z_1, h_1), (w_2, z_2, h_2)) = ||w_1 - w_2||_{C(\Delta_T)} + ||z_1 - z_2||_{C(\Delta_T)} + ||h_1' - h_2'||_{C([0, T])}.$$ 

Note that for any given $h_1, h_2 \in \mathcal{D}_{3T},$ we have $h_1(0) = h_2(0) = h_0$ and

$$||h_1 - h_2||_{C([0, T])} \leq T||h_1' - h_2'||_{C([0, T])}.$$ 

Next, we adopt the contraction mapping theorem to prove the existence and uniqueness result. Due to the choice of $T,$ for any given $(w, z, h) \in \mathcal{D},$ we have

$$|h(t) - h_0| \leq T(1 + h^*) \leq \frac{h_0}{8},$$
So the transformation \((t, y) \rightarrow (t, x)\) is well defined. Taking advantage of standard \(L^p\) theory and Sobolev's imbedding theorem, for any \((w, z, h) \in D\), the following initial boundary value problem

\[
\begin{align*}
\dot{w}_t - A \dot{w}_{yy} - (B - E) \dot{w}_y &= \lambda w - w^2 - \frac{b w z}{w + m z}, & t > 0, \ 0 < y < h_0, \\
\dot{w}(t, h_0) &= 0, \ \dot{w}_y(t, 0) = 0, & t \geq 0, \\
\dot{w}(0, y) &= u_0(y), & 0 \leq y \leq h_0,
\end{align*}
\]

admits a unique solution \(\dot{w} \in W^{1,2}_p(\Delta_T) \cap C^\frac{1 + \alpha}{2}, 1 + \alpha(\Delta_T)\). Similarity, the equations

\[
\begin{align*}
\ddot{z} - dA \ddot{z}_y - (dB - E) \dot{z}_y &= \mu z - z^2 + \frac{c w z}{w + m z}, & t > 0, \ 0 < y < h_0, \\
\dot{z}(t, h_0) &= 0, \ \dot{z}_y(t, 0) = 0, & t \geq 0, \\
\dot{z}(0, y) &= v_0(y), & 0 \leq y \leq h_0,
\end{align*}
\]

also admit a unique solution \(\ddot{z} \in W^{1,2}_p(\Delta_T) \cap C^\frac{1 + \alpha}{2}, 1 + \alpha(\Delta_T)\). And \(\dot{w}, \ddot{z}\) satisfy

\[
||\dot{w}, \ddot{z}||_{W^{1,2}_p(\Delta_T)} \leq C_1, \ ||\dot{w}, \ddot{z}||_{C^\frac{1 + \alpha}{2}, 1 + \alpha(\Delta_T)} \leq C_1,
\]

(2.2)

where \(C_1\) is a constant dependent on \(h_0, \alpha, u_0\) and \(v_0\).

Define

\[
\tilde{h}(t) = h_0 - \int_0^t \rho \dot{w}_y(\tau, h_0) d\tau.
\]

Then, we have

\[
\tilde{h}'(t) = -\rho \dot{w}_y(t, h_0), \ \tilde{h}(0) = h_0, \ \tilde{h}'(0) = -\rho \dot{w}_y(0, h_0) = h^*.
\]

Hence, \(\tilde{h}' \in C^\frac{1}{2}([0, T])\) with

\[
||\tilde{h}'||_{C^\frac{1}{2}} \leq C_2 := \rho C_1.
\]

(2.3)

Define a mapping \(\mathcal{F} : D \to C(\Delta_T) \times C(\Delta_T) \times C^1([0, T])\) by

\[
\mathcal{F}(w, z, h) = (\tilde{w}, \ddot{z}, \tilde{h}).
\]

It is clear that \((w, z, h) \in D\) is a fixed point of \(\mathcal{F}\) if and only if it solves (2.1). By use of (2.2) and (2.3), we get

\[
\begin{align*}
||\tilde{h}' - h^*||_{C([0, T])} &\leq T^\frac{1}{2} ||\tilde{h}'||_{C^\frac{1}{2}([0, T])} \leq \rho C_1 T^\frac{1}{2}, \\
||\dot{w} - u_0||_{C(\Delta_T)} &\leq T^{\frac{1 + \alpha}{4}} ||\dot{w} - u_0||_{C^\frac{1 + \alpha}{2}, 1 + \alpha(\Delta_T)} \leq C_1 T^{\frac{1 + \alpha}{4}}, \\
||\ddot{z} - v_0||_{C(\Delta_T)} &\leq T^{\frac{1 + \alpha}{4}} ||\ddot{z} - v_0||_{C^\frac{1 + \alpha}{2}, 1 + \alpha(\Delta_T)} \leq C_1 T^{\frac{1 + \alpha}{4}}.
\end{align*}
\]

So, \(\mathcal{F}\) maps \(D\) into itself if we take \(T \leq T_2 := \min\{\rho C_1^{-\frac{2}{1 + \alpha}}, (C_1)^{-\frac{2}{1 + \alpha}}\}\).

We next claim that \(\mathcal{F}\) is a contraction mapping on \(D\) for \(T > 0\) is sufficiently small. Let \((w_i, z_i, h_i) \in D(i = 1, 2)\) and denote \((\tilde{w}_i, \ddot{z}_i, \tilde{h}_i) = \mathcal{F}(w_i, z_i, h_i)\). Then, by (2.2) and (2.3), we have

\[
||\tilde{w}_i, \ddot{z}_i||_{C^\frac{1 + \alpha}{2}, 1 + \alpha(\Delta_T)} \leq C_1, \ ||\tilde{h}_i(t)||_{C^\frac{1}{2}([0, T])} \leq C_2.
\]
Set \( U = \tilde{w}_1 - \tilde{w}_2, \ V = \tilde{z}_1 - \tilde{z}_2 \) and denote \( A_i := A(h_it, y), \ B_i := B(h_it, y), \ E_i := E(h_it, h'_it, y), \ i = 1, 2 \). We can see that \( U \) and \( V \) satisfy
\[
\begin{align*}
U_t - A_2U_{yy} - (B_2 - E_2)U &= (A_1 - A_2)\tilde{w}_{1yy} + (B_1 - B_2)\tilde{w}_{1y} - (E_1 - E_2)\tilde{w}_y + \left[ \lambda - (w_2 + w_2) + \left( \frac{mz_1 w_1}{w_1 + m_z_1} \right) (w_1 - w_2) - \left( \frac{mz_1 w_2}{w_1 + m_z_1} \right) (z_1 - z_2), \ t > 0, 0 < y < h_0, \right. \\
& \left. \quad U_y(t, 0) = 0, \ U(t, h_0) = 0, \right. \\
& \left. U(0, y) = 0, \right. \\
(V_t - dA_2 \tilde{v}_{yy} - dB_2 - E_2)\tilde{v} &= d(A_1 - A_2)\tilde{z}_{1yy} + d(B_1 - B_2)\tilde{z}_{1y} - (E_1 - E_2)\tilde{z}_y + \left[ \mu - (z_1 + z_2) + \left( \frac{cz_1 w_2}{w_2 + m_z_2} \right) (z_1 - z_2) + \left( \frac{cz_1 w_2}{w_2 + m_z_2} \right) (w_1 - w_2), \ t > 0, 0 < y < h_0, \right. \\
& \left. \quad V_y(t, 0) = 0, \ V(t, h_0) = 0, \right. \\
& \left. V(0, y) = 0, \right. 
\end{align*}
\]
respectively. By the standard \( L^p \) theory and Sobolev’s imbedding theorem, it deduces that
\[
\begin{align*}
||\tilde{w}_1 - \tilde{w}_2||_{C^\frac{1+\alpha}{2} + 1 + \alpha(\Delta_T)} & \leq C_3 (||w_1 - w_2||_{C(\Delta_T)} + ||z_1 - z_2||_{C(\Delta_T)} + ||h_1 - h_2||_{C^1([0,T])}), \\
||\tilde{z}_1 - \tilde{z}_2||_{C^\frac{1+\alpha}{2} + 1 + \alpha(\Delta_T)} & \leq C_4 (||w_1 - w_2||_{C(\Delta_T)} + ||z_1 - z_2||_{C(\Delta_T)} + ||h_1 - h_2||_{C^1([0,T])}), \\
\end{align*}
\]
where \( C_3, C_4 \) depend on \( C_1, C_2 \), and functions \( A_i, B_i, E_i (i = 1, 2) \). Obviously,
\[
||\tilde{h}_1' - \tilde{h}_2'||_{C^\frac{1+\alpha}{2}([0,T])} \leq \rho ||\tilde{w}_1y - \tilde{w}_2y||_{C^\frac{1+\alpha}{2}([0,T])}.
\]
Therefore,
\[
\begin{align*}
||\tilde{w}_1 - \tilde{w}_2||_{C^\frac{1+\alpha}{2} + 1 + \alpha(\Delta_T)} + ||\tilde{z}_1 - \tilde{z}_2||_{C^\frac{1+\alpha}{2} + 1 + \alpha(\Delta_T)} + ||\tilde{h}_1' - \tilde{h}_2'||_{C^\frac{1+\alpha}{2}([0,T])} & \leq C_5 (||w_1 - w_2||_{C(\Delta_T)} + ||z_1 - z_2||_{C(\Delta_T)} + ||h_1 - h_2||_{C^1([0,T])}), \\
\end{align*}
\]
where \( C_5 \) depends on \( C_4 \) and \( \rho \). Take
\[
T := \min \left\{ 1, \left( \frac{1}{2C_5} \right)^{\frac{1+\alpha}{2}}, T_1, T_2 \right\}.
\]
Then, we have
\[
\begin{align*}
||\tilde{w}_1 - \tilde{w}_2||_{C(\Delta_T)} + ||\tilde{z}_1 - \tilde{z}_2||_{C(\Delta_T)} + ||\tilde{h}_1' - \tilde{h}_2'||_{C([0,T])} & \leq T^{\frac{1+\alpha}{2}} (||\tilde{w}_1 - \tilde{w}_2||_{C^{\frac{1+\alpha}{2} + 1 + \alpha(\Delta_T)}} + ||\tilde{z}_1 - \tilde{z}_2||_{C^{\frac{1+\alpha}{2} + 1 + \alpha(\Delta_T)}}) + T \frac{\alpha}{2} ||\tilde{h}_1' - \tilde{h}_2'||_{C^\frac{1+\alpha}{2} ([0,T])} \\
& \leq C_5 T^{\frac{1+\alpha}{2}} (||w_1 - w_2||_{C(\Delta_T)} + ||z_1 - z_2||_{C(\Delta_T)} + ||h_1' - h_2'||_{C([0,T])}) \\
& \leq \frac{1}{2} (||w_1 - w_2||_{C(\Delta_T)} + ||z_1 - z_2||_{C(\Delta_T)} + ||h_1' - h_2'||_{C([0,T])}).
\end{align*}
\]
This shows that \( F \) is a contraction mapping on \( D \). So, \( F \) has a unique fixed point \( (w, z, h) \) in \( D \). Moreover, from the above discussion we know that \( w, z \) satisfy (2.2) and \( h \) satisfies (2.3), which implies that \( (w, z, h) \) is the unique local classical solution of problem (2.1). Hence, problem (1.1) has a unique local classical solution \( (u, v, h) \).

To show the above local solution can be extended to all \( t > 0 \), we need to estimate \( u, v, h' \) as follows.

\[
\Box
\]
Lemma 2.1. Let \((u, v, h)\) be a solution to problem (1.1), then there exist constants \(M_1, M_2\) and \(M_3\) independent of \(T\) such that

\[
\begin{align*}
0 < u(x, t) &\leq M_1, \quad 0 < t \leq T, \quad 0 < x \leq h(t), \\
0 < v(x, t) &\leq M_2, \quad 0 < t \leq T, \quad 0 < x \leq h(t), \\
0 < h'(t) &\leq M_3, \quad 0 < t \leq T.
\end{align*}
\]

(2.4)

**Proof.** Note that \(u > 0, v > 0\) in \([0, T] \times [0, h(t)]\). It is easy to see that \(u\) satisfies

\[
\begin{align*}
u_t - u_{xx} &= \lambda u - u^2 - \frac{buv}{u + mv} \leq \lambda u - u^2, \quad 0 < t \leq T, \quad 0 < x \leq h(t), \\
u_x(t, 0) &= u(t, h(t)) = 0, \quad 0 \leq t \leq T, \\
u(0, x) &= u_0(x) > 0, \quad 0 \leq x \leq h_0.
\end{align*}
\]

By the comparison principle, we have \(u \leq M_1 := \max\{\|u_0\|_{\infty}, \lambda\}\). Similarly, \(v\) satisfies

\[
\begin{align*}
v_t - dv_{xx} &= \mu v - v^2 + \frac{cuv}{u + mv} \leq (\mu + c)v - v^2, \quad 0 < t \leq T, \quad 0 < x \leq h(t), \\
v_x(t, 0) &= v(t, h(t)) = 0, \quad 0 \leq t \leq T, \\
v(0, x) &= v_0(x) > 0, \quad 0 \leq x \leq h_0.
\end{align*}
\]

So \(v \leq M_2 := \max\{\|v_0\|_{\infty}, \mu + c\}\). By the strong maximum principle, we have \(u_x(t, h(t)) < 0\). So \(h'(t) > 0\). It remains to show that there exists a constant \(M_3\) independent of \(T\) such that \(h'(t) \leq M_3\). Define

\[
\Omega_M := \{0 < t < T, \quad h(t) - M^{-1} x < x < h(t)\},
\]

and construct an auxiliary function

\[
\omega(t, x) := M_1[2M(h(t) - x) - M^2(h(t) - x)^2].
\]

We will choose a proper \(M\) so that \(\omega(t, x) \geq u(t, x)\) holds over \(\Omega_M\). By direct calculations, for \((t, x) \in \Omega_M\), we have

\[
\begin{align*}
\omega_t &= 2M_1M h'(t)[1 - M(h(t) - x)] \geq 0, \\
-\omega_{xx} &= 2M_1M^2, \quad \lambda u - u^2 - \frac{buv}{u + mv} \leq \lambda M_1.
\end{align*}
\]

If \(M^2 \geq \lambda/2\), then

\[
\omega_t - \omega_{xx} \geq 2M_1M^2 \geq \lambda M_1 \geq \lambda u - u^2 - \frac{buv}{u + mv}, \quad (t, x) \in \Omega_M.
\]

It is obvious that

\[
\omega(t, h(t) - M^{-1}) = M_1 \geq u(t, h(t) - M^{-1}), \quad \omega(t, h(t)) = 0 = u(t, h(t)).
\]

Thus, if we can choose \(M\) such that \(u_0(x) \leq \omega(0, x)\) for \(x \in [h_0 - M^{-1}, h_0]\), then \(u(t, x) \leq \omega(t, x)\) in \(\Omega_M\). So

\[
u_x(t, h(t)) \geq u_x(t, h(t)) = -2M_1M_1 h'(t) = -\rho u_x(t, h(t)) \leq M_3 := 2M_1.\]

Next we will find some \(M\) independent of \(T\) such that \(u_0(x) \leq \omega(0, x)\) for \(x \in [h_0 - M^{-1}, h_0]\). Divide this interval \([h_0 - M^{-1}, h_0]\) into \([h_0 - M^{-1}, h_0 - (2M)^{-1}]\) and \([h_0 - (2M)^{-1}, h_0]\). For \(x \in [h_0 - (2M)^{-1}, h_0]\), if we choose

\[
M = \max\{h_0^{-1}, \sqrt{\frac{\lambda}{2}}, \frac{4\|u_0\|_{C^1([0, h_0])}}{3M_1}\},
\]

then

\[
\omega_x(0, x) = -2M_1M[1 - M(h_0 - x)] \leq -M M_1 \leq u_0(x).
\]

Combining this with \(\omega(0, h_0) = u_0(h_0) = 0\), we can deduce that

\[
\omega(0, h_0) \geq u_0(h_0), \quad x \in [h_0 - (2M)^{-1}, h_0].
\]
On the other hand, for \( x \in [h_0 - M^{-1}, h_0 - (2M)^{-1}] \), if \( M \geq \frac{4\|u_0\|_{C^1([0,h_0])}}{3M_1} \), then
\[
\omega(0, x) \geq 3M_1/4, \quad u_0(x) \leq \|u_0\|_{C^1([0,h_0])}M^{-1} \leq 3M_1/4.
\]
So \( u_0(x) \leq \omega(0, x) \) in \([h_0 - M^{-1}, h_0 - (2M)^{-1}]\). Therefore, \( u_0(x) \leq \omega(0, x), \quad x \in [h_0 - M^{-1}, h_0] \). The proof is finished.

**Theorem 2.2.** The solution of problem (1.1) exists and is unique for all \( t \in (0, \infty) \).

**Proof.** Suppose \( T_{\max} \) be the maximal time that the solution exists. We know \( T_{\max} > 0 \) by Theorem 2.1. It remains to show that \( T_{\max} = \infty \). We assume that \( T_{\max} < \infty \). By Lemma 2.1, there exist \( M_1, M_2 \) and \( M_3 \) independent of \( T_{\max} \) such that
\[
0 < u(x, t) \leq M_1, \quad 0 < v(x, t) \leq M_2, \quad 0 < h'(t) \leq M_3,\]
\[
h_0 \leq h(t) \leq h_0 + M_3t, \quad t \in [0, T_{\max}), \quad x \in [0, h(t)].
\]
Fix \( \delta_0 \in [0, T_{\max}) \). By Theorem 2.1, we know that for \( t \in [\delta_0, T_{\max}) \), \( v(t, \cdot), u(t, \cdot) \in C^{1+\alpha}_{\text{loc}}([0, +\infty)) \). And there exists a constant \( C^* > 0 \) independent of \( T_{\max} \) such that
\[
\|u(t, \cdot)\|_{W^p_2([0, h(t)])} + \|v(t, \cdot)\|_{W^2_2([0, h(t)])} \leq C^*, \quad t \in [\delta_0, T_{\max}],
\]
where \( p > \frac{3}{1-\alpha} \). According to the proof of Theorem 2.1, there exists a \( \tau > 0 \) independent of \( T_{\max} \) such that the solution of problem (1.1) with initial time \( T_{\max} - \tau/2 \) can be extended uniquely to the time \( T_{\max} + \tau/2 \). But this contradicts the assumption \( T_{\max} < \infty \). This completes the proof. \( \square \)

### 3. Asymptotic behavior

In this section, we will study the asymptotic behavior of solution \((u, v)\). Note that \( x = h(t) \) is monotonically increasing. So \( \lim_{t \to \infty} h(t) = h_\infty \in (0, \infty) \). If \( h_\infty < \infty \) and \( \max_{0 \leq x \leq h(t)} u(t, \cdot) \to 0 \) as \( t \to 0 \), then the prey \( u \) fails to establish and vanishes eventually, which is called vanishing. If \( h_\infty = \infty \), then the prey \( u \) can successfully establish itself in the new environment, which is called spreading.

#### 3.1. Vanishing case \((h_\infty < \infty)\)

We first give a vital estimate as below. The proof is similar to [21, Theorem 4.1] and [20, Theorem 2.1], so we omit it.

**Proposition 3.1.** Let \((u, v, h)\) be a solution of problem (1.1). If \( h_\infty < \infty \), then there exists a positive constant \( M \) such that
\[
\|u(t, \cdot), v(t, \cdot)\|_{C^1([0, h(t)])} \leq M, \quad \forall \ t > 1 \quad (3.1)
\]
and
\[
\lim_{t \to \infty} h'(t) = 0. \quad (3.2)
\]

According to Proposition 3.1 and [18, Proposition 3.1], we obtain the following theorem directly.

**Theorem 3.1.** Let \((u, v, h)\) be the solution of problem (1.1). If \( h_\infty < \infty \), then
\[
\lim_{t \to \infty} \max_{0 \leq x \leq h(t)} u(t, x) = 0. \quad (3.3)
\]

This theorem tells us that if the prey \( u \) cannot spread to the whole space, then they will be vanished eventually.

Now we discuss the long time behaviors of \( v \) when \( h_\infty < \infty \).
Theorem 3.2. Assume that \( h_\infty < \infty \).

(i) If \( h_\infty \leq \frac{\pi}{2}\sqrt{d/\mu} \), then

\[
\lim_{t \to \infty} \max_{0 \leq x \leq h(t)} v(t, x) = 0.
\]

(ii) If \( h_\infty > \frac{\pi}{2}\sqrt{d/\mu} \), then

\[
\lim_{t \to \infty} \max_{0 \leq x \leq h(t)} |v(t, x) - V(x)| = 0,
\]

where \( V(x) \) is the unique positive solution of

\[
\begin{cases}
-dV_{xx} = V(\mu - V), & 0 < x < h_\infty, \\
V_x(0) = V(h_\infty) = 0.
\end{cases}
\]

Proof. Case (i), \( h_\infty \leq \frac{\pi}{2}\sqrt{d/\mu} \). Define

\[
Q^h_\tau := \{(t, x) : t \geq \tau, 0 \leq x \leq h(t)\}.
\]

By (3.3), for any fixed \( 0 < \delta \ll 1 \), there exists \( T^\delta > 0 \), such that when \( (t, x) \in Q^h_{T^\delta} \), we have \( u < \delta \). Let \( w_\delta(t, x) \) be the unique solution of

\[
\begin{cases}
w_t - dw_{xx} = \mu w - w^2 + \frac{\delta w}{\delta + mw}, & t > T^\delta, 0 < x < h_\infty, \\
w_x(t, 0) = w(t, h_\infty) = 0, & t \geq T^\delta, \\
w(T^\delta, x) = \varpi(x), & 0 \leq x \leq h_\infty,
\end{cases}
\]

where

\[
\varpi(x) = \begin{cases}
v(T^\delta, x), & 0 \leq x \leq h(T^\delta), \\
0, & h(T^\delta) < x \leq h_\infty.
\end{cases}
\]

Then, we have

\[
v(t, x) \leq w_\delta(t, x) \text{ in } Q^h_{T^\delta}
\]

by the comparison principle. Denote \( \lim_{\delta \to 0} T^\delta = T^0 \) and let \( \delta \to 0 \), and then, the limiting form of problem (3.7) becomes

\[
\begin{cases}
w_t - dw_{xx} = \mu w - w^2, & t > T^0, 0 < x < h_\infty, \\
w_x(t, 0) = w(t, h_\infty) = 0, & t \geq T^0, \\
w(T^0, x) = \varpi^0(x), & 0 \leq x \leq h_\infty,
\end{cases}
\]

where

\[
\varpi^0(x) = \begin{cases}
v(T^0, x), & 0 \leq x \leq h(T^0), \\
0, & h(T^0) < x \leq h_\infty.
\end{cases}
\]

We denote the solution of problem (3.9) as \( \tilde{w}(t, x) \). By the comparison principle, we have \( v(t, x) \leq \tilde{w}(t, x) \) in \( Q^h_{T^0} \) as \( \delta \to 0 \). As is well known, if \( h_\infty \leq \frac{\pi}{2}\sqrt{d/\mu} \), then problem (3.9) has no positive equilibrium, which follows that \( \lim_{t \to \infty} \tilde{w}(t, x) \to 0 \) uniformly on any compact subset of \([0, h_\infty)\). Thus, we have

\[
\limsup_{t \to \infty} v(t, x) \leq 0 \text{ uniformly on any compact subset of } [0, h_\infty).
\]

Remember \( v(t, x) \geq 0 \), so we have

\[
\lim_{t \to \infty} v(t, x) = 0 \text{ uniformly on any compact subset of } [0, h_\infty).
\]

We assert (3.4) is true. Otherwise, there exists a constant \( \sigma > 0 \) and a sequence \( \{(t_j, x_j)\}_{j=1}^\infty \) with \( 0 \leq x_j \leq h(t_j) \) and \( t_j \to \infty \) as \( j \to \infty \), such that

\[
v(t_j, x_j) \geq \sigma, \quad j = 1, 2, 3, \ldots.
\]
Since $0 \leq x_j < h_\infty$, there exist a subsequence of $\{x_j\}$, denoted by itself, and $x_0 \in [0, h_\infty]$, such that $x_j \to x_0$ as $j \to \infty$. The formulas (3.11) and (3.10) imply that $x_0 = h_\infty$, i.e., $x_j - h(t_j) \to 0$ as $j \to \infty$. By the use of (3.11) firstly and (3.12) secondly, there exists $\bar{r}_j \in (x_j, h(t_j))$ such that

$$\left| \frac{\sigma}{x_j - h(t_j)} \right| \leq \left| \frac{V(t_j, x_j) - V(t_j, h(t_j))}{x_j - h(t_j)} \right| = |v_x(t_j, \bar{r}_j)| \leq M,$$

which is a contradiction, for $x_j - h(t_j) \to 0$. Hence, (3.4) holds.

Case (ii), $h_\infty > \frac{\pi}{2} \sqrt{d/\mu}$. By [6, Corollary 3.4], when $h_\infty > \frac{\pi}{2} \sqrt{d/(\mu + c)}$, the solution $w_\delta(t, x)$ of (3.7) satisfies $\lim_{t \to \infty} w_\delta(t, x) \to V_\delta(x)$ uniformly on any compact subset of $[0, h_\infty)$, where $V_\delta(x)$ is the unique positive solution of

$$\begin{cases}
-dw_{xx} = \mu w - w^2 + \frac{c_\delta w}{\delta + mw}, & 0 < x < h_\infty, \\
w_x(0) = w(h_\infty) = 0.
\end{cases} \tag{3.12}$$

Letting $\delta \to 0$, by use of the continuity of $V_\delta(x)$ in $\delta$, it follows that $V_\delta(x) \to V(x)$, where $V(x)$ is defined by (3.6). Meanwhile, the limiting form of problem (3.12) as $\delta \to 0$ becomes problem (3.6). It is well known that if $h_\infty > \frac{\pi}{2} \sqrt{d/\mu}$, then problem (3.6) has a unique positive solution $V(x)$. Combining with (3.8), when $h_\infty > \frac{\pi}{2} \sqrt{d/\mu}> \frac{\pi}{2} \sqrt{d/\mu + c}$, we have

$$\lim_{t \to \infty} \sup_{t \in [0, \infty)} v(t, x) \leq V(x) \text{ uniformly on any compact subset of } [0, h_\infty). \tag{3.13}$$

On the other hand, take $\varepsilon > 0$ so small that $h_\infty - \varepsilon = \max\{\frac{\pi}{2} \sqrt{d/\mu}, h_0\}$. So there exists $T > 0$ such that $h(t) > h_\infty - \varepsilon$ for all $t > T$. Let $V_\varepsilon(t, x)$ be the unique positive solution of

$$\begin{cases}
V_t - dV_{xx} = V(\mu - V), & t > T, 0 < x < h_\infty - \varepsilon, \\
V_x(t, 0) = V(t, h_\infty - \varepsilon) = 0, & t > T, \\
V(T, x) = v(T, x), & 0 \leq x \leq h_\infty - \varepsilon.
\end{cases}$$

By the comparison principle, we have $v(t, x) \geq V_\varepsilon(t, x)$ for $t > T$ and $x \in [0, h_\infty - \varepsilon]$. Since $h_\infty - \varepsilon > \frac{\pi}{2} \sqrt{d/\mu}$, it is well known that $\lim_{t \to \infty} V_\varepsilon(t, x) = V_\varepsilon(x)$ uniformly on $[0, h_\infty - \varepsilon]$, where $V_\varepsilon(x)$ is the unique positive solution of

$$\begin{cases}
-dV_{xx} = V(\mu - V), & 0 < x < h_\infty - \varepsilon, \\
V_x(0) = V(h_\infty - \varepsilon) = 0.
\end{cases}$$

Therefore, we have $\lim_{t \to \infty} \sup_{t \in [0, \infty)} v(t, x) \geq V_\varepsilon(x)$ uniformly on $[0, h_\infty - \varepsilon]$. By using the continuity of $V_\varepsilon$ in $\varepsilon$ and letting $\varepsilon \to 0$, we get

$$\lim_{t \to \infty} \inf_{t \in [0, \infty)} v(t, x) \geq V(x) \text{ uniformly on any compact subset of } [0, h_\infty), \tag{3.14}$$

where $V(x)$ is defined by (3.6). Combining (3.14) with (3.13) deduces that

$$\lim_{t \to \infty} v(t, x) = V(x) \text{ uniformly on any compact subset of } [0, h_\infty). \tag{3.15}$$

Now we prove (3.5). On the contrary, we assume that (3.5) is not true. Then, there exist a constant $\sigma > 0$ and a sequence $\{(t_j, x_j)\}_{j=1}^\infty$ with $0 \leq x_j < h(t_j)$ and $t_j \to \infty$ as $j \to \infty$, such that

$$|v(t_j, x_j) - V(x_j)| \geq 2\sigma, \quad j = 1, 2, \ldots \tag{3.16}$$

For $0 \leq x_j < h_\infty$, there exists a subsequence of $\{x_j\}$, denoted by itself, and $x_0 \in [0, h_\infty]$, such that $x_j \to x_0$ as $j \to \infty$. It follows from (3.15) and (3.16) that $x_0 = h_\infty$, i.e., $x_j - h(t_j) \to 0$ as $j \to \infty$. Since $V(x)$ is continuous in $[0, h_\infty]$, $V(h_\infty) = 0$ and $x_j \to \infty$, by (3.16) we have $|v(t_j, x_j)| \geq \sigma$ for all $j > 1$. Similar to the proof of (3.4), we can get a contradiction. \(\Box\)
3.2. Spreading case ($h_\infty = \infty$)

We first state a proposition as follows which can be proved similar to [23, Proposition 8.1] and the details will be omitted. Let $d$, $\beta$ be positive constants and $f(s)$ be a positive $C^1$ function for $s > 0$.

**Proposition 3.2.** For any given $\varepsilon > 0$ and $L > 0$, there exist $l_\varepsilon > \max\{L, \frac{\pi}{2}\sqrt{d/|\beta f(0)|}\}$ and $T_\varepsilon > 0$ such that when the continuous function $w(t, x) \geq 0$ satisfies

\[
\begin{cases}
    w_t - dw_{xx} \geq (\leq)w f(w)(\beta - w), & t > 0, \ 0 < x < l_\varepsilon, \\
    w_x(t, 0) = 0, & t > 0, \\
    w(0, x) > 0, & 0 < x < l_\varepsilon,
\end{cases}
\]

and $w(t, l_\varepsilon) \geq k$ if $k = 0$, while $w(t, l_\varepsilon) \leq k$ if $k > 0$ for $t > 0$, then we have

\[w(t, x) > \beta - \varepsilon \ (w(t, x) < \beta + \varepsilon), \ \forall \ t > T_\varepsilon, \ x \in [0, L].\]

Furthermore,

\[\liminf_{t \to \infty} w(t, x) > \beta - \varepsilon \ (\limsup_{t \to \infty} w(t, x) < \beta + \varepsilon)\]

uniformly on $[0, L]$.

Next, we study the long time behaviors of $u$ and $v$.

**Theorem 3.3.** Assume that $m\lambda - b \geq \max\{2b - m^2\mu, 0\}$ and $h_\infty = \infty$. Then, the solution $(u, v)$ of problem (1.1) satisfies

\[
\begin{cases}
    \lambda - u + \frac{b v}{u + m v} = 0, \\
    \mu - v + \frac{m (\lambda - u)}{u + m v} = 0.
\end{cases}
\]

(3.17)

Furthermore, if $m\lambda - b < b\mu/c$, then

\[
\lim_{t \to \infty} u(t, x) = u^* := \frac{A + \sqrt{\Delta_1}}{2(b + cm^2)}, \ \lim_{t \to \infty} v(t, x) = v^* := \frac{u^*(\lambda - u^*)}{b - m(\lambda - u^*)},
\]

where $A = \lambda(2cm^2 + b) - mb(\mu + 2c)$, $\Delta_1 = A^2 + 4(b + cm^2)[b(\mu + c) - m\lambda](m\lambda - b)$.

**Proof.** The proof uses the iteration method. Let $M_1$, $M_2$ be defined as in Lemma 2.1.

Step 1. For any fixed $L \gg 1$ and $0 < \varepsilon \ll 1$, let $l_\varepsilon$ be given by Proposition 3.2 with $d = 1$, $\beta = \lambda$ and $f(u) \equiv 1$. Taking account to $h_\infty = \infty$, we can find $T_1 > 0$ such that $h(t) \geq l_\varepsilon$ when $t > T_1$. In view of $u \geq 0, v \geq 0$, we have

\[
\begin{cases}
    u_t - u_{xx} \leq u(\lambda - u), & t > T_1, \ 0 < x < l_\varepsilon, \\
    u_x(t, 0) = 0, \ u(t, l_\varepsilon) \leq M_1, \ t > T_1.
\end{cases}
\]

Since $u(T_1, x) > 0$ for $[0, l_\varepsilon]$, by using Proposition 3.2, we have

\[\limsup_{t \to \infty} u(t, x) \leq \lambda + \varepsilon \text{ uniformly on } [0, L].\]

The arbitrariness of $\varepsilon$ and $L$ implies that

\[\limsup_{t \to \infty} u(t, x) \leq \lambda := \nu_1\]

(3.18)

uniformly on any compact subset of $[0, \infty)$. 

Step 2. For any fixed $L \gg 1$, $0 < \delta \ll 1$ and $0 < \varepsilon \ll 1$, let $l_\varepsilon$ be given by Proposition 3.2 with $\beta = v^\delta_1$ and $f(v) = m(v - v^\delta_2)/(\|\pi_1 + \delta\| + mv)$. Thanks to (3.18), there exists $T_2 > T_1$ such that $u(t, x) < \pi_1 + \delta$ for $t > T_2$. Thus,

$$v_t - dv_{xx} \leq \mu v - v^2 + c(\pi_1 + \delta)v/((\pi_1 + \delta) + mv)$$

$$= -v\{mv^2 - [m\mu - (\pi_1 + \delta) - (\mu + c)(\pi_1 + \delta)]\}/(\pi_1 + \delta) + mv$$

and

$$v_x(t, 0) = 0, \quad v(t, l_\varepsilon) \leq M_2, \quad \forall \ t > T_2,$$

where

$$v^\delta_1 = \frac{m\mu - (\pi_1 + \delta) + \sqrt{m\mu - (\pi_1 + \delta)^2 + 4m(\mu + c)\pi_1}}{2m} > 0,$$

$$v^\delta_2 = \frac{m\mu - (\pi_1 + \delta) - \sqrt{m\mu - (\pi_1 + \delta)^2 + 4m(\mu + c)\pi_1}}{2m} < 0.$$

According to $v(T_2, x) > 0$ for $x \in [0, l_\varepsilon]$ and Proposition 3.2, we have $v(t, x) < v^\delta_1 + \varepsilon$ uniformly for $t > T_2$ and $x \in [0, L]$. The arbitrariness of $\varepsilon, \delta$ and $L$ implies that

$$\limsup_{t \to \infty} v(t, x) \leq v^0_1 := \pi_1$$

uniformly on any compact subset of $[0, \infty)$, where

$$\pi_1 = \frac{m\mu - \pi_1 + \sqrt{m\mu - \pi_1^2 + 4m(\mu + c)\pi_1}}{2m} > 0.$$  

Step 3. For given $L \gg 1$, $0 < \delta \ll 1$ and $0 < \varepsilon \ll 1$, let $l_\varepsilon$ be given by Proposition 3.2 with $d = 1, f(u) = (u - u^\delta_2)/[u + m(\pi_1 + \delta)]$ and $\beta = u^\delta_1$. By (3.19), there exists $T_3 > T_2$ such that $v(t, x) < \pi_1 + \delta$. Thus,

$$u_t - u_{xx} \geq \lambda u - u^2 - bu(\pi_1 + \delta)/[u + m(\pi_1 + \delta)]$$

$$= -u\{u^2 - [\lambda - m(\pi_1 + \delta)]u - (m\lambda - b)(\pi_1 + \delta)\}/[u + m(\pi_1 + \delta)]$$

and

$$u_x(t, 0) = 0, \quad u(t, l_\varepsilon) \geq 0, \quad t > T_3,$$

where

$$u^\delta_1 = \frac{\lambda - m(\pi_1 + \delta) + \sqrt{[\lambda - m(\pi_1 + \delta)]^2 + 4m(\lambda - b)(\pi_1 + \delta)}}{2} > 0,$$

$$u^\delta_2 = \frac{\lambda - m(\pi_1 + \delta) - \sqrt{[\lambda - m(\pi_1 + \delta)]^2 + 4m(\lambda - b)(\pi_1 + \delta)}}{2} < 0.$$

According to $u(T_3, x) > 0$ for $x \in [0, l_\varepsilon]$ and Proposition 3.2, it follows that $u(t, x) > u^\delta_1 - \varepsilon$ uniformly for $t > T_3$ and $x \in [0, L]$. The arbitrariness of $\varepsilon, \delta$ and $L$ implies that

$$\liminf_{t \to \infty} u(t, x) \geq u^0_1 := \underline{u}_1 > 0.$$  

uniformly on any compact subset of $[0, \infty)$, where

$$\underline{u}_1 = \frac{\lambda - m\pi_1 + \sqrt{(\lambda - m\pi_1)^2 + 4(m\lambda - b)\pi_1}}{2}.$$
Step 4. For given \( L \gg 1, 0 < \delta \ll 1 \) and \( 0 < \varepsilon \ll 1 \), let \( l_\varepsilon \) be given by Proposition 3.2 with \( \beta = v_3^\delta \) and \( f(v) = m(v - v_3^\delta)/[(u_1 - \delta) + mv] \). By the use of (3.22), there exists \( T_4 > T_3 \) such that \( u(t, x) > u_1 - \delta \) for \( t > T_4 \). Thus,

\[
v_t - dv_{xx} \geq \mu v - v^2 + c(u_1 - \delta)v/[(u_1 - \delta) + mv] \\
= -v\{mv^2 - [m\mu - (u_1 - \delta)]v - (\mu + c)(u_1 - \delta)/[(u_1 - \delta) + mv]
= mv(v - v_3^\delta)(v_3^\delta - v)/[(u_1 - \delta) + mv],
\]

and

\[v_x(t, 0) = 0, \ v(t, l_\varepsilon) \geq 0, \ \forall \ t > T_4,\]

where

\[
v_3^\delta = \frac{m\mu - (u_1 - \delta) + \sqrt{[m\mu - (u_1 - \delta)]^2 + 4m(\mu + c)(u_1 - \delta)}}{2m} > 0,
\]

\[
v_4^\delta = \frac{m\mu - (u_1 - \delta) - \sqrt{[m\mu - (u_1 - \delta)]^2 + 4m(\mu + c)(u_1 - \delta)}}{2m} < 0.
\]

Similar to Step 3, we have

\[
\liminf_{t \to \infty} v(t, x) \geq v_3^0 := \varphi_1
\]

uniformly on any compact subset of \([0, \infty)\), where

\[
\varphi_1 = \frac{m\mu - u_1 + \sqrt{(m\mu - u_1)^2 + 4m(\mu + c)u_1}}{2m} > 0.
\]

Step 5. For given \( L \gg 1, 0 < \delta \ll 1 \) and \( 0 < \varepsilon \ll 1 \), let \( l_\varepsilon \) be given by Proposition 3.2 with \( \beta = u_4^\delta \) and \( f(u) = (u - u_4^\delta)/[u + m(u_1 - \delta)] \). By (3.21), there exists \( T_5 > T_4 \) such that \( u(t, x) > u_1 - \delta \) for \( t > T_5 \). Thus,

\[
u_t - u_{xx} \geq \lambda u - u^2 - bu(u_1 - \delta)/[u + m(u_1 - \delta)] \\
= -u\{u^2 - [\lambda - m(u_1 - \delta)]u - (m\lambda - b)(u_1 - \delta)/[u + m(u_1 - \delta)]
= u(u - u_4^\delta)(u_4^\delta - u)/[u + m(u_1 - \delta)]
\]

and

\[u_x(t, 0) = 0, \ u(t, l_\varepsilon) \geq 0, \ t > T_5,\]

where

\[
u_3^\delta = \frac{\lambda - m(u_1 - \delta) + \sqrt{[\lambda - m(u_1 - \delta)]^2 + 4(m\lambda - b)(u_1 - \delta)}}{2} > 0,
\]

\[
u_4^\delta = \frac{\lambda - m(u_1 - \delta) - \sqrt{[\lambda - m(u_1 - \delta)]^2 + 4(m\lambda - b)(u_1 - \delta)}}{2} < 0.
\]

Similar to Step 2, we have

\[
\limsup_{t \to \infty} u(t, x) \leq u_3^0 := \varphi_2 > 0
\]

uniformly on any compact subset of \([0, \infty)\), where

\[
\varphi_2 = \frac{\lambda - m\varphi_1 + \sqrt{(\lambda - m\varphi_1)^2 + 4(m\lambda - b)\varphi_1}}{2}.
\]
Repeating the above processes, we can find four sequences \( \{u_i\}, \{v_i\}, \{\overline{u}_i\}, \{\overline{v}_i\} \) such that for all \( i \),
\[
\begin{align*}
\liminf_{t \to \infty} u(t, x) & \leq \limsup_{t \to \infty} u(t, x) \leq \overline{u}_i, \\
\liminf_{t \to \infty} v(t, x) & \leq \limsup_{t \to \infty} v(t, x) \leq \overline{v}_i
\end{align*}
\]
uniformly on any compact subset of \([0, \infty)\). Moreover, for \( s > 0 \), we denote
\[
\phi(s) = \frac{\lambda - m s + \sqrt{(\lambda - m s)^2 + 4(m \lambda - b)s}}{2},
\]
\[
\psi(s) = \frac{m \mu - s + \sqrt{(m \mu - s)^2 + 4m(\mu + c)s}}{2m},
\]
so
\[
\overline{u}_i = \psi(\overline{u}_i), \quad u_i = \phi(u_i), \quad \overline{v}_i = \psi(v_i), \quad \overline{u}_{i+1} = \phi(\overline{v}_i), \quad i = 1, 2, 3, \ldots
\]
And the sequences \( \{\overline{u}_i\}, \{\overline{v}_i\}, \{u_i\}, \{v_i\} \) satisfy
\[
\begin{align*}
\liminf_{t \to \infty} u(t, x) & \leq \limsup_{t \to \infty} u(t, x) \leq \overline{u}_i, \\
\liminf_{t \to \infty} v(t, x) & \leq \limsup_{t \to \infty} v(t, x) \leq \overline{v}_i
\end{align*}
\]
which indicates that \( \{\overline{u}_i\} \) and \( \{\overline{v}_i\} \) are monotonically non-increasing with lower boundaries and \( \{u_i\} \) and \( \{v_i\} \) are monotonically non-decreasing with upper boundaries. Then, \( \{\overline{u}_i\}, \{\overline{v}_i\}, \{u_i\}, \{v_i\} \) have limits denoted by \( \overline{u}, \overline{v}, u, v \), respectively, as \( i \to \infty \). Furthermore, they satisfy
\[
\overline{v} = \psi(\overline{u}), \quad u = \phi(u), \quad \overline{u} = \phi(\overline{v}),
\]
which follows that
\[
\begin{align}
\lambda - \overline{u} - \frac{b \overline{v}}{\overline{u} + m \overline{v}} &= 0, \\
\mu - \overline{v} + \frac{c \overline{u}}{\overline{u} + m \overline{v}} &= 0.
\end{align}
\]
By (3.23), we have
\[
(\lambda - \overline{u} - \overline{v}) - (\overline{u}^2 - \overline{v}^2) - (m \lambda - b)(\overline{v} - \overline{u}) - m(\overline{v} \overline{u} - \overline{u} \overline{v}) = 0.
\]
By (3.24), we get
\[
\begin{align}
\overline{v} &= \frac{(m \mu - \overline{u}) + F(\overline{u})}{2m}, \\
v &= \frac{(m \mu - u) + F(u)}{2m},
\end{align}
\]
where \( F(s) = \sqrt{(m \mu - s^2) + 4m(\mu + c)s} \). So
\[
\overline{v} - v = -\left(\frac{1}{2m} - I\right)(\overline{u} - u),
\]
\[
\overline{u} v - u \overline{v} = \left(\frac{\mu}{2} + II\right)(\overline{u} - u),
\]
where
\[
I = \frac{(\overline{u} + u) + 2m(\mu + 2c)}{2m(F(\overline{u}) + F(u))}, \quad II = \frac{m \mu^2 (\overline{u} + u) + 2m(\mu + 2c) \overline{u} u}{2(\pi F(u) + F(\overline{u}))}.
\]
Substitute equations (3.26) and (3.27) into equation (3.25) to obtain
\[
(\overline{u} - u)G(\overline{u}, u) = 0,
\]
where
\[
G(\overline{u}, u) = \lambda - (\overline{u} + u) + \frac{m \lambda - b}{2m} - \frac{m \mu}{2} - (m \lambda - b) \cdot I - m \cdot II.
\]
If $\overline{u} \neq u$, then $G(\overline{u}, u) = 0$. It is easy to verify that $I > 0$ and $II > 0$. So

$$\lambda - (\pi + u) + \frac{m\lambda - b}{2m} - \frac{m\mu}{2} > 0,$$

which contradicts $m\lambda - b \geq 2b - m^2\mu$. Thus, $\overline{u} = u$. Then, by (3.25), we obtain $\overline{v} = v$.

By the first equation of (3.17), we have

$$v = \frac{u(\lambda - u)}{b - m(\lambda - u)}.$$  

(3.28)

Substituting (3.28) into the second equation of (3.17), we have

$$(b + cm^2)u^2 - [\lambda (2cm^2 + b) - mb(\mu + 2c)]u - [b(\mu + c) - mc\lambda](m\lambda - b) = 0.$$ 

The assumption $m\lambda - b < b\mu/c$ implies $b(\mu + c) - mc\lambda > 0$. Then,

$$u_1 = \frac{A + \sqrt{\Delta_1}}{2(b + cm^2)} > 0, \quad u_2 = \frac{A - \sqrt{\Delta_1}}{2(b + cm^2)} < 0.$$

So $u^* = u_1$ and $v^* = \frac{u^*(\lambda - u^*)}{b - m(\lambda - u^*)}$. Thus, when $0 < m\lambda - b < b\mu/c$, we get

$$(u^*, v^*) := \left(\frac{A + \sqrt{\Delta_1}}{2(b + cm^2)}, \frac{u^*(\lambda - u^*)}{b - m(\lambda - u^*)}\right).$$

$\square$

## 4. The spreading–vanishing dichotomy and the criteria of spreading and vanishing

In this section, we want to establish the spreading–vanishing dichotomy and study the criteria of spreading and vanishing of problem (1.1). Before that, we first give some comparison principles with a free boundary.

**Proposition 4.1.** ([23]) Suppose that $T > 0$, $\overline{u}, \overline{v} \in C^{1,2}(O_T)$, $\overline{h} \in C^1([0, T])$, where $O_T = \{(t, x) \in \mathbb{R}^2 : t > 0, \ 0 < x < \overline{h}(t)\}$, and $(\overline{u}, \overline{v}, \overline{h})$ satisfies

$$\begin{aligned}
\overline{u}_t - \overline{u}_{xx} &\geq (\alpha - \overline{u})\overline{u}, \quad t > 0, \ 0 < x < \overline{h}(t), \\
\overline{v}_t - d\overline{v}_{xx} &\geq (\beta - \overline{v})\overline{v}, \quad t > 0, \ 0 < x < \overline{h}(t), \\
\overline{u}_x(t, 0) &\leq 0, \ \overline{v}_x(t, 0) \leq 0, \ t > 0, \\
\overline{u}(t, \overline{h}(t)) &= \overline{v}(t, \overline{h}(t)) = 0, \ t > 0, \\
\overline{h}(t) &\geq -\rho\overline{u}_x(t, \overline{h}(t)), \quad t > 0.
\end{aligned}$$

Let $\alpha = \lambda$ and $\beta = \mu + c$. Moreover, if $\overline{u}(0, x) \geq u(0, x)$, $\overline{v}(0, x) \geq v(0, x)$, $\overline{h}(0) \geq h_0$ for $x \in [0, \overline{h}(0)]$, then the solution $(u, v, h)$ of the problem (1.1) satisfies

$$\begin{aligned}
u(t, x) &\leq \overline{u}(t, x), \ v(t, x) \leq \overline{v}(t, x), \ (t, x) \in (0, T] \times (0, h(t)), \\
h(t) &\leq \overline{h}(t), \ t \in (0, T].
\end{aligned}$$

**Proposition 4.2.** $(\overline{u}, \overline{v}, \overline{h})$ above is called an upper solution of the problem (1.1). We also can define upper solutions $(\overline{u}, \overline{h})$ and $(\overline{v}, \overline{h})$ like above. In like manner, we can defined lower solutions $(\underline{u}, \underline{v}, \underline{h})$, $(\underline{u}, \overline{h})$ and $(\underline{v}, \underline{h})$ by letting $\alpha = \lambda - b/m$, $\beta = \mu$ and reversing all the inequalities above.

We then study an eigenvalue problem. For any given $\theta$, $l > 0$, let $\sigma_1(\theta, l)$ be the first eigenvalue of

$$\begin{aligned}
-\phi_{xx} - \theta \phi &= \sigma \phi, \ 0 < x < l, \\
\phi_x(0) &= \phi(l) = 0.
\end{aligned}$$  

(4.1)

**Lemma 4.1.** Assume that $m\lambda > b$. If $h_\infty < \infty$, then $\sigma_1(\lambda - \frac{b}{m}, h_\infty) \geq 0$.  

**(Proof)**
Proof. We assume $\sigma_1(\lambda - \frac{b}{m}, h_\infty) < 0$ to get a contradiction. By the continuities of $\sigma_1(\theta, l) < 0$ in $\theta$ and $l$, and of $h(t)$ in $t$, there exists $T \geq 1$ such that $\sigma_1(\lambda - \frac{b}{m}, h(T)) < 0$. Choose $\varepsilon$ so small that $\sigma_1(\lambda - \frac{b}{m} - \varepsilon, h(T)) < 0$. Suppose $w_\varepsilon(t, x)$ be the unique solution of

$$
\begin{align*}
\left\{ \begin{array}{ll}
w_t - w_{xx} = w(\lambda - \frac{b}{m} - \varepsilon - w), & t > T, \; 0 < x < h(T), \\
w_x(t, 0) = w(t, h(T)) = 0, & t \geq T, \\
w(T, x) = u(T, x), & 0 \leq x \leq h(T).
\end{array} \right.
\end{align*}
$$

By using the comparison principle, we have

$$w_\varepsilon(t, x) \leq u(t, x), \; t \geq T, \; 0 \leq x \leq h(T).$$

In view of $\sigma_1(\lambda - \frac{b}{m} - \varepsilon, h(T)) < 0$, it is well known that $w_\varepsilon(t, x) \to w^*_\varepsilon(x)$ uniformly on any compact subset of $[0, h(T))$ as $t \to \infty$, where $w^*_\varepsilon$ is the unique positive solution of

$$
\begin{align*}
\left\{ \begin{array}{ll}
-w_{xx} = w(\lambda - \frac{b}{m} - \varepsilon - w), & 0 < x < h(T), \\
w_x(0) = w(h(T)) = 0.
\end{array} \right.
\end{align*}
(4.2)
$$

Thus, $\lim \inf_{t \to \infty} u(t, x) \geq \lim_{t \to \infty} w_\varepsilon(t, x) = w^*_\varepsilon(x) > 0$, which contradicts (3.3). The proof is completed. \hfill \Box

Define

$$\Lambda := \frac{\pi}{2} \sqrt{\frac{m}{m\lambda - b}}.$$

**Theorem 4.1.** Assume that $m\lambda > b$.

(i) If $h_\infty < \infty$, then $h_\infty \leq \Lambda$;

(ii) If $h_0 \geq \Lambda$, then $h_\infty = \infty$ for all $\rho > 0$.

Proof. We assert that (i) is true. Otherwise, $\Lambda < h_\infty < \infty$. It is well known that problem (4.1) can be solved explicitly in terms of exponentials by the principle eigenvalue $\sigma_1(\lambda - \frac{b}{m}, l) = -(\lambda - \frac{b}{m}) + (\frac{\pi}{m})^2$, and in this case $\phi_1(\lambda - \frac{b}{m}, l) = \cos \frac{\pi t}{T}$. It is follows from our assumption $h_\infty > \Lambda$ that $\sigma_1(\lambda - \frac{b}{m}, h_\infty) < 0$, which contradicts Lemma 4.1. In consequence, we get either $h_\infty \leq \Lambda$ or $h_\infty = \infty$.

Accordingly, $h_0 \geq \Lambda$ implies $h_\infty = \infty$ for all $\rho > 0$. Thus, (ii) is proved. \hfill \Box

To emphasize the dependence of $h$ on $\rho$ and $h_0$, we substitute $h(t)$ with $h(\rho, h_0; t)$. By Theorem 4.1 and [20, Lemma 3.2], we can deduce the following conclusion immediately.

**Lemma 4.2.** Suppose $m\lambda > b$. For any given $h_0 > 0$, there exists $\rho^0$ depending on $u_0$, $v_0$ and $h_0$, such that $h(\rho, h_0; \infty) = \infty$ for all $\rho > \rho^0$.

Define

$$\mathcal{E} = \{ k > 0 : h(\rho, h_0; \infty) = \infty, \; \forall \; h_0 \geq k, \forall \; \rho > 0 \},$$

$$\mathcal{F} = \{ k > 0 : \text{for any } 0 < h_0 < k, \exists \rho_0 > 0, \text{ such that } h(\rho, h_0; \infty) < \infty, \; \forall \; 0 < \rho \leq \rho_0 \}.$$

Assume $m\lambda > b$ and set

$$h^* := \inf \mathcal{E}, \; h_* := \sup \mathcal{F}.$$

It is natural that $h_* \leq h^*$. Theorem 4.1 implies that $[\Lambda, \infty) \subset \mathcal{E}$. Thus, we have $h^* \leq \Lambda$. In addition, we draw an important conclusion as the following lemma shows.
Lemma 4.3. \( h_s \geq \frac{\pi}{2} \lambda^{-\frac{1}{2}} \).

Proof. We assert that if \( h_0 < \frac{\pi}{2} \lambda^{-\frac{1}{2}} \), then there exists \( \rho_0 > 0 \) such that \( h(\rho, h_0; \infty) < \infty \) for all \( 0 < \rho \leq \rho_0 \). Thus, we have \( \frac{\pi}{2} \lambda^{-\frac{1}{2}} \in \mathcal{F} \), which implies that \( h_s \geq \frac{\pi}{2} \lambda^{-\frac{1}{2}} \). We use the method in [9] to construct a suitable upper solution of (1.1). Define
\[
\sigma(t) = h_0 \left( 1 + \delta - \frac{\delta}{2} e^{-\gamma t} \right), \quad t \geq 0; \quad V(y) = \cos \frac{\pi y}{2}, \quad 0 \leq y \leq 1,
\]
and
\[
w(t, x) = Ce^{-\alpha t} \frac{V(x)}{\sigma(t)}, \quad t \geq 0, \quad 0 \leq x \leq \sigma(t),
\]
where \( \delta, \gamma, \alpha, C \) are positive constants to be determined. We choose \( C \) sufficiently large such that \( u_0(x) \leq C \cos \left( \frac{\pi x}{2 h_0(1+\sigma/2)} \right) \). Besides, it is easy to verify that
\[
u_0(x) \leq w(0, x), \quad h_0 \left( 1 + \frac{\delta}{2} h_0 \right) = \sigma(0), \quad 0 < x < \sigma(t),
\]
\[
w_x(t, 0) = w(t, \sigma(t)) = 0, \quad t > 0.
\]
Note that \( \sigma(t) < h_0 (1 + \delta) \) for all \( t > 0 \), direct calculations deduce that
\[
w_t - w_{xx} - w(\lambda - w) = w \left[ -\alpha + \frac{\pi}{2} \sigma^{-2} \tan \left( \frac{\pi}{2} \frac{x}{\sigma(t)} \right) + \frac{\pi}{2} \sigma^{-2} (t) - \lambda + w \right] \geq w \left[ -\alpha + \frac{\pi}{2} \sigma^{-2} (1 + \delta)^{-2} h_0^{-2} - \lambda + w \right].
\]
Remember \( \lambda < \left( \frac{\pi}{2} \right)^2 h_0^{-2} \). We can find \( \delta > 0 \) such that
\[
\left( \frac{\pi}{2} \right)^2 (1 + \delta)^{-2} h_0^{-2} - \lambda = \frac{1}{2} \left( \left( \frac{\pi}{2} \right)^2 h_0^{-2} - \lambda \right).
\]
Take \( \alpha = \frac{1}{2} \left( \left( \frac{\pi}{2} \right)^2 h_0^{-2} - \lambda \right) \), and we have \( w_t - w_{xx} - w(\lambda - w) \geq 0 \) for \( t > 0 \) and \( 0 < x < \sigma(t) \). Let \( \rho_0 = \rho_0(C) := \frac{\delta \sigma h_0^2}{2 \pi} \) and \( \alpha = \gamma \), then for any \( 0 < \rho \leq \rho_0 \), we have
\[
\sigma'(t) \geq -\rho w_x(t, \sigma(t)), \quad t \geq 0.
\]
By applying Proposition 4.2, we have \( u(t, x) \leq w(t, x) \) and \( h(t) \leq \sigma(t) \) for \( t > 0 \) and \( 0 \leq x \leq \sigma(t) \). Thus, \( \sigma(t) \rightarrow h_0(1 + \delta) \) as \( t \rightarrow \infty \), implying that \( h(\rho, h_0; \infty) < \infty \). The proof is finished.

From the above discussions, we have the spreading-vanishing dichotomy and the criteria for spreading and vanishing.

Theorem 4.2. Assume that \( m \lambda > b \). Then, either spreading \( (h_\infty = \infty) \) or vanishing \( (h_\infty \leq \Lambda) \) holds. To be more precise,
(i) \( h(\rho, h_0; \infty) = \infty \) for all \( h_0 > h^* \) and \( \rho > 0 \);
(ii) for any given \( h_0 > 0 \), there exists \( \rho_0 > 0 \), which depends on \( u_0, v_0, h_0 \), such that \( h(\rho, h_0; \infty) = \infty \) for all \( \rho > \rho_0 \);
(iii) for any given \( 0 < h_0 < h_\ast \), there exists \( \rho_0 > 0 \), which also depends on \( u_0, v_0, h_0 \), such that \( h(\rho, h_0; \infty) \leq \Lambda \) for all \( 0 < \rho \leq \rho_0 \).

Remark 4.1. The estimates \( \frac{\pi}{2} \lambda^{-\frac{1}{2}} \leq h_s \leq h^* \leq \frac{\pi}{2} \sqrt{\frac{m}{m \lambda - b}} \) are obtained. But we cannot prove \( h^* = h_s \). That is why the sharp criteria for spreading and vanishing with respect to the initial \( h_0 \) cannot obtained.
5. Asymptotic speed of $h(t)$

In this section, we mainly give some estimates of $h(t)/t$ as $t \to \infty$. In the following, we always assume that $m\lambda > b$.

**Definition 5.1.** Assume that $u(t, x)$ is a nonnegative function for $x > 0$, $t > 0$. We call $c_*$ as the asymptotic speed of $u(t, x)$ if

(a) $\lim_{t \to \infty} \sup_{x > (c_* + \epsilon)t} u(t, x) = 0$ for any given $\epsilon > 0$,

(b) $\lim_{t \to \infty} \inf_{0 < x < (c_* - \epsilon)t} u(t, x) > 0$ for any given $\epsilon \in (0, c_*)$.

For the following diffusive logistic problem,

$$
\begin{aligned}
&w_t - dw_{xx} = w(a - bw), \quad t > 0, \quad x > 0, \\
&w_x(t, 0) = 0, \quad t \geq 0, \\
&w(0, x) = w_0 > 0, \quad x \geq 0.
\end{aligned}
$$

(5.1)

**Proposition 5.1.** ([3]) It is well known that the asymptotic speed of problem (5.1) $c_* = 2\sqrt{ad}$ and

$$
\liminf_{t \to \infty} \inf_{0 < x < (c_* - \epsilon)t} w(t, x) = \frac{a}{b}, \quad \limsup_{t \to \infty} \sup_{x > (c_* + \epsilon)t} w(t, x) = 0
$$

for any small $\epsilon \in (0, c_*)$.

For the classical logistic problem with a free boundary problem

$$
\begin{aligned}
&w_t - dw_{xx} = w(a - bw), \quad t > 0, \quad 0 < x < s(t), \\
&s'(t) = -pw_x(t, s(t)), \quad t \geq 0, \\
&w_x(t, 0) = w(t, s(t)) = 0, \quad t \geq 0, \\
&w(0, x) = w_0, \quad s(0) = s_0, \quad 0 \leq x \leq s_0.
\end{aligned}
$$

(5.2)

It has proved that the expanding front $s(t)$ moves at a constant speed for large time (see [4,11]), i.e.,

$$
s(t) = (c_0 + o(1))t \quad \text{as} \quad t \to \infty.
$$

And $c_0$ is determined by the following auxiliary elliptic problem

$$
\begin{aligned}
&dq''' - c q'' + q(a - bq) = 0, \quad 0 < y < \infty, \\
&q(0) = 0, \quad q'(0) = c/r, \quad q(\infty) = \frac{a}{b}, \\
&c \in (0, 2\sqrt{ad}); \quad q'(y) > 0, \quad 0 < y < \infty,
\end{aligned}
$$

(5.3)

where $\rho, d, a, b$ are positive constants.

**Proposition 5.2.** [4] The problem (5.3) has a unique solution $(q(y), c)$ and $c(\rho, d, a, b)$ is strictly increasing in $\rho$ and $a$, respectively. Moreover,

$$
\lim_{y \to \infty} \frac{c(\rho, d, a, b)}{\sqrt{ad}} = 2, \quad \lim_{y \to 0} \frac{c(\rho, d, a, b)}{\sqrt{ad}} \frac{bd}{\rho} = \frac{1}{\sqrt{3}}.
$$

(5.4)

For $t \geq 0$ and $x \geq 0$, the following inequalities are natural,

$$
\begin{aligned}
&(\lambda - b/m)u - u^2 \leq \lambda u - u^2 - \frac{bnu}{u + mv} \leq \lambda u - u^2, \\
&\mu v - v^2 \leq \mu v - v^2 + \frac{cuv}{u + mv} \leq (\mu + c)v - v^2.
\end{aligned}
$$

(5.5)

Define

$$
c_1 = 2\sqrt{\lambda - b/m}, \quad c_2 = 2\sqrt{\lambda}, \quad c_3 = 2\sqrt{d\mu}, \quad c_4 = 2\sqrt{d(\mu + c)}.
$$
Theorem 5.1. For any given $0 < \varepsilon \ll 1$, the following conclusions hold,
\[
\limsup_{\rho \to \infty} \limsup_{t \to \infty} \frac{h(t)}{t} \leq c_2, \quad \liminf_{\rho \to \infty} \liminf_{t \to \infty} \frac{h(t)}{t} \geq c_1. \tag{5.6}
\]
Moreover, when $\rho \to \infty$,
\[
\limsup_{t \to \infty} \sup_{x \geq (c_2 + \varepsilon)t} u(t, x) = 0, \tag{5.7}
\]
\[
\liminf_{t \to \infty} \inf_{0 < x < (c_1 - \varepsilon)t} u(t, x) \geq \lambda - \frac{b}{m}, \tag{5.8}
\]
\[
\limsup_{t \to \infty} \sup_{x \geq (c_4 + \varepsilon)t} v(t, x) = 0, \tag{5.9}
\]
\[
\liminf_{t \to \infty} \inf_{0 < x < (c_3 - \varepsilon)t} v(t, x) \geq \mu. \tag{5.10}
\]

Proof. It follows from the first limit of (5.4) that
\[
\lim_{\rho \to \infty} c(\rho, 1, \lambda - b/m, 1) = c_1, \quad \lim_{\rho \to \infty} c(\rho, 1, \lambda, 1) = c_2.
\]
In view of (5.5) and Proposition 4.2, it can be deduced that for any given $\varepsilon > 0$, there exists $\rho_\varepsilon \gg 1$ such that, for all $\rho > \rho_\varepsilon$,
\[
c_1 - \frac{\varepsilon}{2} < c(\rho, 1, \lambda - b/m, 1) \leq \liminf_{t \to \infty} \frac{h(t)}{t}, \quad \limsup_{t \to \infty} \frac{h(t)}{t} \leq c(\rho, 1, \lambda, 1) < c_2 + \frac{\varepsilon}{2}.
\]
By letting $\varepsilon \to 0$, (5.6) holds.

Besides, we can find $\tau_1 \gg 1$ such that for all $t \geq \tau_1$ and $\rho \gg \rho_\varepsilon$,
\[
(c_1 - \varepsilon)t < h(t) < (c_2 + \varepsilon)t. \tag{5.11}
\]
Obviously, (5.7) holds.

The proof of (5.8). Let $(u, h)$ be the unique solution of (5.2) with $(\rho, d, a, b) = (\rho, 1, \lambda - b/m, 1)$. Then, $h(t) \geq \underline{h}(t)$, $u(t, x) \geq \overline{u}(t, x)$ for $t > 0$ and $0 < x \leq \underline{h}(t)$ by Proposition 4.2. Making use of [29, Theorem 3.1], we get
\[
\lim_{t \to \infty} \frac{h(t)}{\overline{u}(t, x) - q^*(c^*t + H - x)} = 0,
\]
where $(q^*(y), c^*)$ is the unique solution of (5.3) with $(\rho, d, a, b) = (\rho, 1, \lambda - b/m, 1)$, i.e., $c^* = c(\rho, 1, \lambda - b/m, 1) > c_1 - \frac{\varepsilon}{2}$. Obviously, $\min_{\overline{u}(t, x) - q^*(c^*t + H - x)} \to \infty$ as $t \to \infty$. Owing to $q^*(y) > \lambda - b/m$ as $y \to \infty$, we have $\min_{\overline{u}(t, x) - q^*(c^*t + H - x)} \to \lambda - b/m$ as $t \to \infty$. So $\min_{\overline{u}(t, x) - q^*(c^*t + H - x)} \to \lambda - b/m$ as $t \to \infty$. Thus, (5.8) holds due to $u(t, x) \geq \overline{u}(t, x)$ for $t > 0$ and $0 < x \leq \underline{h}(t)$.

By Proposition 5.1 and the comparison principle, it is easy to obtain (5.9) and (5.10). The proofs are finished.

Intuitively, in order to survive, the prey $u$ should spread faster than the predator $v$. Before the predator occupies a new habitat, the prey has colonized the habitat. Besides, both the prey and predator live in a region enclosed by free boundary caused only by the prey, which decides that the spreading speed of prey is no less than that of predator. Thus, it is rational to assume that $c_1 \geq c_4$.

Theorem 5.2. Assume that $d(\mu + c) \leq \lambda - b/m$. Then, there exists a constant $c_5 \in (c_1, c_2)$ such that
\[
\limsup_{\rho \to \infty} \lim_{t \to \infty} \frac{h(t)}{t} \geq c_5, \tag{5.12}
\]
and
\[
\liminf_{t \to \infty} \inf_{0 < x < (c_5 - \varepsilon)t} u(t, x) > 0. \tag{5.13}
\]
Proof. The assumption \(d(\mu + c) \leq \lambda - b/m\) implies \(c_4 \leq c_1\). We can choose a small \(\varepsilon > 0\) such that \(c_4 + \varepsilon < c_1 - \varepsilon\). Thus, by (5.11), we have \((c_4 + \varepsilon)t < (c_1 - \varepsilon)t < h(t)\) for all \(t \geq \tau_1\) and \(\rho \geq \rho_{\varepsilon}\). Denote \(s := \frac{2M}{\lambda - b/m}\), where \(M_2\) is given by Lemma 2.1. Recall that \(v \leq M_2\) and \(\liminf_{t \to \infty} \inf_{0 \leq x < (c_1 - \varepsilon)t} u(t, x) \geq \lambda - b/m\), and then, there exists a large \(\tau_2 > 0\) such that \(v(t, x) \leq su(t, x)\) for all \(t \geq \tau_2\) and \(0 < x < (c_1 - \varepsilon)t\). Besides, it follows from (5.11) that \(v(t, x) = 0\) when \(x \geq (c_1 - \varepsilon)t\). Let \(\psi\) be defined by

\[
\begin{align*}
\psi_x(t, x) &= (\lambda - \frac{bs}{1+ms})\psi - \psi^2, & t > \tau_2, \quad 0 < x < g(t), \\
\psi_x(t, 0) &= \psi(t, g(t)), & t \leq \tau_2, \\
g'(t) &= -\rho \psi_x(t, g(t)), & t \geq \tau_2, \\
\psi(\tau_2, x) &= u(\tau_2, x), & 0 \leq x \leq g(\tau_2).
\end{align*}
\]

By Proposition 4.2, we have \(u(t, x) \geq \psi(t, x)\) and \(h(t) \geq g(t)\) for \(t > \tau_2\) and \(0 < x < g(t)\). According to Proposition 5.2, we get \(\lim_{t \to \infty} \frac{h(t)}{t} = c_5\). Thus,

\[
\liminf_{\rho \to \infty} \liminf_{t \to \infty} \frac{h(t)}{t} \geq \lim_{\rho \to \infty} \lim_{t \to \infty} \frac{g(t)}{t} = c_5.
\]

So (5.12) holds.

Similar to the proof of (5.8), we can show that

\[
\liminf_{t \to \infty} \inf_{0 < x < (c_5 - \varepsilon)t} u(t, x) \geq \lambda - \frac{bs}{1+ms} > \lambda - \frac{b}{m} > 0,
\]

which implies that (5.13) holds. \(\square\)

**Theorem 5.3.** Assume \(0 < m\lambda - b < b\mu/c\). For any given \(\varepsilon \in (0, c_3)\), we have

\[
\begin{align*}
\liminf_{t \to \infty} \inf_{0 < x < (c_1 - \varepsilon)t} u(t, x) &= \limsup_{t \to \infty} \sup_{0 < x < (c_1 - \varepsilon)t} u(t, x) = u^*, \\
\liminf_{t \to \infty} \inf_{0 < x < (c_1 - \varepsilon)t} v(t, x) &= \limsup_{t \to \infty} \sup_{0 < x < (c_1 - \varepsilon)t} v(t, x) = v^*,
\end{align*}
\]

where \((u^*, v^*)\) is defined by Theorem 3.3.

**Proof.** Define

\[
C_{[0, M]} := \{w \in C^2(D_\infty) : 0 \leq w \leq M\},
\]

where \(D_\infty := \{(x, t) \in \mathbb{R}^2 : t \in (0, \infty), x \in [0, h(t)]\}\). Then, \(C_{[0, M]}\) is an invariant region of the solution \(u(t, x), v(t, x)\) for problem (1.1). Thus, we have

\[
\lim_{t \to \infty} \inf_{0 < x < (c_3 - \varepsilon)t} \{u(t, x), v(t, x)\} \geq 0, \quad \lim_{t \to \infty} \inf_{0 < x < (c_3 - \varepsilon)t} \{u(t, x), v(t, x)\} \geq 0
\]

by Theorem 5.1. Then, the conclusion can be proved by the same way as that of [15, Lemma 4.6]. \(\square\)

6. Discussion

This paper is concerned with a ratio-dependent predator–prey system with a Neumann boundary condition on the left side indicating that the left boundary is fixed and a free boundary \(x = h(t)\) determined only by prey which describes the process of movement for prey species. We prove a spreading–vanishing dichotomy. In the following, the following alternate holds: either

(i) (The spreading case) The two species can spread successfully into \([0, \infty)\) (i.e., \(h_\infty = \infty\)) and the solution \((u, v)\) will stabilize at a positive equilibrium state; or
(ii) (The vanishing case) The two species cannot spread to the whole space (i.e., \( h_\infty \leq \Lambda \)) and the prey will vanish eventually.

Meanwhile, we found a critical value

\[
\Lambda = \frac{\pi}{2} \sqrt{\frac{m}{m\lambda - b}}
\]

which can be called a “spreading barrier” such that the prey will spread and successfully establish itself if it can break through this barrier \( \Lambda \), or will vanish and never break through this barrier. In addition, we obtain the criteria for spreading and vanishing:

(i) Spreading happens if the size of initial habitat \( h_0 \) is more than or equal to \( h^* \) (Theorem 4.2(i)), or the moving parameter \( \rho \) is large enough (\( \rho > \rho^0 \)) regardless of the initial habitat’s size (Theorem 4.2(ii)).

(ii) Vanishing happens if the size of initial habitat \( h_0 \) is less than \( h^* \) and the moving parameter \( \rho \) is less than \( \rho_0 \) (Theorem 4.2(iii)).

Finally, we give some estimates of asymptotic speed of \( h(t) \) as \( t \to \infty \). We want to draw a comparison with the uncoupled case. If problem (1.1) is uncoupled (i.e., \( b = c = 0 \)), then the prey satisfies

\[
\begin{align*}
\phi_t - \phi_{xx} &= \phi(\lambda - \phi), & t > 0, & 0 < x < \varsigma(t), \\
\phi_x(t,0) &= \phi(t,\varsigma(t)) = 0, & t \geq 0, \\
\varsigma'(t) &= -\rho \phi_x(t,\varsigma(t)), & t \geq 0, \\
\phi(0,x) &= u_0(x), & 0 \leq x \leq \varsigma(0).
\end{align*}
\]

By Proposition 5.2, we have

\[
\lim_{\rho \to \infty} \lim_{t \to \infty} \frac{\varsigma(t)}{t} = 2\sqrt{\lambda}. \quad (6.1)
\]

And more notably, Theorems 5.1 and 5.2 show that when the spreading speed of prey is no lower than that of predator (\( d(\mu + c) \leq \lambda - b/m \)), then we have

\[
2\sqrt{\lambda - \frac{bs}{1 + ms}} \leq \lim_{\rho \to \infty} \lim_{t \to \infty} \frac{h(t)}{t} \leq 2\sqrt{\lambda}. \quad (6.2)
\]

The formulas (6.1) and (6.2) illustrate that the predator could decrease the prey’s asymptotic propagation in the setting model studied in this paper. Furthermore, when we move to the right at a fixed speed less than \( 2\sqrt{d\mu} \), we will observe that the two species will stabilize at the unique positive equilibrium state; when we do this with a fixed speed in \( (2\sqrt{d(\mu + c)}, 2\sqrt{\lambda - \frac{bs}{1 + ms}}) \), we can only see the prey; when we do this with a fixed speed more than \( 2\sqrt{\lambda} \), we could see neither for the two species are out of sight.

This paper has the same boundary conditions as paper [20], but it gets different conclusions from it. For the classical Lotka–Volterra type [20], the sharp criteria with respect to the initial habitat \( h_0 \) can be obtained in some cases, While in this paper the predator does not impact the critical value \( \Lambda \) and we cannot get the sharp criteria for spreading and vanishing with respect to the initial habitat \( h_0 \) in any case. Besides, we provide an estimate of the asymptotic speed of the free boundary and explain some realistic and significant spreading phenomena.

Some results are instructive in real life. On the one hand, we confirm that alien species can have serious impacts on the native species: two species can coexist (the spreading case), or more seriously, the invasive species (predator) may upset the ecological balance and wipe out the local species (prey) (the vanishing case). On the other hand, in order to control and eliminate the pest species (prey), we must take both approaches at the same time: (i) reduce the size of the initial habitat of the prey, (ii) decrease the coefficient of the free boundary. Significantly, it is useful to introduce its natural enemies (predator) when the initial habitat of the pest species is not very large.
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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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