Dissipation production in a closed two-level quantum system as a test of the obversibility of the dynamics

Claudia L. Clarke and Ian J. Ford

Department of Physics and Astronomy and London Centre for Nanotechnology, University College London, Gower Street, London WC1E 6BT, U.K.

Irreversible behaviour is traditionally associated with open stochastic dynamical systems, but an asymmetry in the probabilistic specification of a closed deterministic system can similarly lead to a disparity between the likelihoods of a particular forward and corresponding backward behaviour starting from a specified time. Such a comparison is a test of a property denoted obversibility, which may be quantified in terms of dissipation production as a measure of irreversibility. We here discuss the procedure needed to evaluate dissipation production in a simple, deterministic two-level quantum system described by a statistical ensemble of state vectors and then provide numerical results for illustrative situations. We consider cases that both do and do not fulfill an Evans-Searles Fluctuation Theorem for the dissipation production, and identify conditions for which the system will display time-asymmetric average behaviour as it evolves.

I. INTRODUCTION

Irreversible behaviour, which cannot be reversed or undone, is ubiquitous in everyday life. Each time an ice cube melts, reaching room temperature as it does so, we observe macroscopic irreversibility. Other examples include system mixing and spreading. Such behaviour is traditionally characterised by a monotonic rise in entropy, in accordance with the second law of thermodynamics [1]. This allows past to be distinguished from present, identifying the future as the direction in which entropy increases, leading to the emergence of an arrow of (development in) time [2].

However, given that the Newtonian laws governing macroscopic motion are time reversal symmetric, we still seek, as did Boltzmann at the advent of the study of thermodynamics, a satisfactory understanding of the emergence of macroscopic irreversibility. One proposed explanation is known as the Past Hypothesis. This posits that entropy increases globally because the universe started from a state of low entropy so it is likely that all conceivable evolutions lead to its increase [3]. The irreversibility, and the evolution of its measure, entropy, ultimately depend on the initial conditions taken by a system. The persistent impact of initial conditions may have philosophical, as well as physical consequences, which can be explored [3].

Closed, deterministically evolving systems are mechanically reversible, meaning that a protocol of manipulation can be followed which returns an evolved state to its initial state. A forward trajectory can then always be followed by a backward trajectory. Entropy production [4] is primarily intended for use as a measure of irreversibility for open systems, and is intimately connected to the development of uncertainty and loss of information [5] brought about by environmental interactions or the process of measurement in quantum systems. It explores a failure of mechanical reversibility, a distinction between the probabilities of forward and subsequent backward evolution, and hence acts as a measure of irreversibility.

Progress has been made in understanding entropy production in open classical dynamical systems, as well as in studying the average [6] and single realisation quantum entropy production in quantum systems in the weak coupling limit [7, 8]. However, for systems that are closed and dynamically deterministic, such a measure of irreversibility is not available and some other approach needs to be employed in order to quantify the irreversible behaviour that can emerge in many situations.

We therefore consider an alternative measure that, like entropy production, derives from a difference in the likelihood of forward and backward trajectories, but under deterministic, mechanically reversible dynamics. To distinguish this property from the entropy measure that tests the reversibility, we call this a test of the obversibility of the dynamics, a concept that has previously been investigated in classical situations [9, 10]. We now consider it for the first time in a quantum context. We provide a definition of obversibility and quantify its measure as dissipation production. The nomenclature chosen is intended to evoke parallels with the established related quantifier of irreversibility, entropy production, and follows from an earlier name for the measure used in previous literature: the time-integrated modified dissipation function [9], which is a generalisation of the time-integrated dissipation function coined by Evans [11]. In a study of dissipation production in a purely deterministic system without environment, it is a property unrelated to traditional notions of the dissipation of heat. Instead, dissipation production is associated with the observation of unexpected behaviour, akin to microscopic violations of the second law. The entropy production is more specifically associated with the failure of a subsequent negation of behaviour. However, if the system were coupled to an environment, then there would be traditional heat dissipation.

We evaluate dissipation production and explore some of its properties in the simplest case of a closed two-level quantum system evolving without measurement, for which entropy production is not an applicable ir-
reversibility measure. We demonstrate that dissipation production may be computed for individual realisations of the system dynamics, that its average over all possible realisations is never negative, and that in certain situations its probability density function (pdf) can satisfy a symmetry known as the Evans-Searles Fluctuation Theorem (ESFT). When the ESFT is not satisfied, the average dissipation production of the system as it evolves into the future differs from its average evolution into the past.

II. METHODS

Classically, the forward trajectory taken by a system is simply the path it follows through phase space from an initial configuration defined by a point in phase space, $\Gamma_A$, to a final configuration, $\Gamma_B$. Translating to a quantum setting, these might correspond to initial and final wavefunctions $\psi_A$ and $\psi_B$, which are defined by a collection of complex probability amplitudes $\tilde{\Gamma}_A$ and $\tilde{\Gamma}_B$ respectively, according to some basis set. In order to define a backward trajectory, we employ an inversion operator $M^T$ which has the effect of transforming the final configuration reached along the forward trajectory into an appropriate starting point from which the backward trajectory can proceed under the reversed dynamics.

In a classical situation, this transformation is velocity inversion, $v \rightarrow M^Tv = -v$ [9]. However, the equivalent reversal in the quantum case is conjugation of the wavefunction, $\psi \rightarrow M^T\psi = \psi^*$, as this sets up conditions for a time reversed solution to the Schrödinger equation [12]. We associate a set of amplitudes $\tilde{\Gamma}^*$ with each $\psi^*$ and define a mapping $M^T$ such that $M^T\tilde{\Gamma} = \tilde{\Gamma}^*$.

We make the distinction between the quantum uncertainty associated with measurement outcome, and the classical uncertainty associated with the choice of state that embodies particular probabilities for given measurement outcomes. As we are considering deterministic dynamics, we only consider the classical uncertainty and postulate that the pdf encoding the probability of finding certain quantum states has ontological reality, even though it is often combined with the quantum measurement uncertainty when using a density matrix. Nevertheless, though the pdf for states may be inaccessi-ble, it is a meaningful concept, particularly if situations are considered in which we have control over the generation of initial states, and can ascribe particular probabilities to particular quantum states; pdfs can be used to construct density matrices, even if they cannot subsequently be extracted from one. Therefore, we proceed by assigning classical probabilities to the sets of numbers $\tilde{\Gamma}$ that define quantum states.

Thus, starting from $\Gamma_B$ or in the quantum case $\tilde{\Gamma}_B$, the application of an inversion operator $M^T$ followed by reversed dynamics over a further time period $t$ returns a system to the inverted form of original state if the dynamics are mechanically reversible. However, if the dynamics are indeterminate, with the evolution represented in terms of probabilities, the degree of failure of reversibility can be quantified with the stochastic entropy production, $\Delta s_t$, a measure of the increase in uncertainty brought about by the dynamics of a system’s evolution and defined as

$$\Delta s_t = \ln \frac{f(\tilde{\Gamma}_A,0)d\tilde{\Gamma}_A T(\tilde{\Gamma}_A \rightarrow \tilde{\Gamma}_B)}{f(\tilde{\Gamma}_B,t)d\tilde{\Gamma}_B P_I(\tilde{\Gamma}_B \rightarrow \tilde{\Gamma}_B^*) T(\tilde{\Gamma}_B^* \rightarrow \tilde{\Gamma}_A^*)}, \quad (1)$$

in notation suitable for a quantum setting. Replacing $\tilde{\Gamma}_i$ with $\Gamma_i$ gives the appropriate expression for the classical stochastic entropy production. In Eq. (1), $f(\tilde{\Gamma},\tau)$ is the pdf of the configuration of quantum probability amplitudes, $\Gamma$, at time $\tau$, such that $f(\tilde{\Gamma},\tau)d\Gamma$ is the probability that the configuration lies in the region $d\Gamma$ about $\tilde{\Gamma}$.

In Eq. (1) $T(\tilde{\Gamma} \rightarrow \tilde{\Gamma}^*)$ is the probability for a transition from $\tilde{\Gamma}$ to $\tilde{\Gamma}^*$ according to the dynamics in a time interval of length $t$. The probability for the inversion $P_I(\tilde{\Gamma}_B \rightarrow \tilde{\Gamma}_B^*)$ in the denominator of Eq. (1) might be omitted since it is unity, though its presence makes more apparent the precise nature of the two processes that are being compared. The inversion operation is taken to act instantaneously. The idea of Eq. (1) is to compare the probability of a forward path from $\tilde{\Gamma}_A$ to $\tilde{\Gamma}_B$, in a time interval of length $t$, with the probability of subsequently starting from a configuration $\Gamma_B$ at time $t$, inverting it and having it return to configuration $\Gamma_A^*$ after dynamical evolution for a further time $t$. For stochastic dynamics, the ratio of initial to final increments $d\tilde{\Gamma}_A/d\tilde{\Gamma}_B$ is unity and can be omitted.

For closed systems with deterministic dynamics, $\Delta s_t$ vanishes since the transition probabilities $T$ are replaced by deterministic mappings of the state, taken with unit probability. Equivalently, the evolution of $\tilde{\Gamma}_A$ to $\tilde{\Gamma}_B$ might be represented by the operation $S_t$; the backward trajectory by $S_t^*$ and, including the inversions, the reversibility of the dynamics corresponds to $M^TS_t^*M^TS_t\tilde{\Gamma}_A = \tilde{\Gamma}_A$. By conservation of probability
we have $f(\Gamma_B, t)\,d\Gamma_B = f(\Gamma_A, 0)\,d\Gamma_A$ and hence $\Delta s_t = 0$. There is no entropy production since there is no change in the uncertainty of the state brought about by the dynamics.

So for closed, deterministic systems we therefore need a different quantity with which to measure irreversibility. A suitable quantity called the dissipation production has been employed in classical situations [9], developing earlier work by Evans [11]. Rather than comparing the likelihoods of the forward and (subsequent) backward trajectories to quantify irreversibility, as in the case of entropy production, dissipation production compares the likelihoods of the forward and obverse trajectories. In the quantum situation, the obverse trajectory takes the inverted final configuration, $\Gamma_A^*$, via the reversed dynamics, $S_t^*$, to the inverted initial configuration $\Gamma_A$, but starting the reverse evolution at time zero, rather than at time $t$, which distinguishes it from the backward trajectory, which is the subsequent reversal of the previous forward trajectory. We assume that the probability density at $t = 0$ is such that all possible final configurations are accessible. By comparing the likelihood of a system being in $\Gamma_A$ at $t = 0$ with the likelihood that it begins in $\Gamma_B$, we can quantify the failure of obversibility, a counterpart to the reversibility that is tested by the entropy production. Such failure is necessarily and sufficiently a consequence of the properties of the initial probability density over the configuration space, rather than of the dynamics.

The dissipation production is defined as $\omega_t = \ln[f(\Gamma_A, 0)\,d\Gamma_A / f(\Gamma_B, 0)\,d\Gamma_B]$, where $\Gamma_A$ and $\Gamma_B$ are related by the mapping $\Gamma_B = S_t\Gamma_A$. To make a more exact parallel with the definition of stochastic entropy production, dissipation production could also be written as

$$\omega_t = \ln \frac{f(\Gamma_A, 0)\,d\Gamma_A \, \differential \
abla \Gamma_A \rightarrow \Gamma_B}{f(\Gamma_B, 0)\,d\Gamma_B \, P_I(\Gamma_B \rightarrow \Gamma_A) \, \differential \
abla \Gamma_B \rightarrow \Gamma_A^*}, \quad (2)$$

where the appropriate transition probabilities $\nabla$ are unity since we are considering deterministic dynamics under which $\Gamma_A$ inevitably evolves into $\Gamma_B$, and $\Gamma_B$ into $\Gamma_A$ (the latter under reversed dynamics). Hence $\nabla$ is omitted, together with the inversions probability $P_I(\Gamma_B \rightarrow \Gamma_A^*)$ in the denominator. Unlike entropy production, which compares the likelihood of a system evolving forward and then subsequently evolving backward, dissipation production is evaluated using only the pdf at an initial time $t = 0$. It is a comparison of the probabilities of one event or another. The evolution under $S_t$ taking $\Gamma_A$ to $\Gamma_B$ and the evolution under $S_t^*$ taking $\Gamma_B^*$ to $\Gamma_A$ both take place in a time interval $t$. A comparison of the tests for reversibility and obversibility is made in Table [1] (3)

| concept | tested | reversibility | obversibility |
|---------|--------|---------------|---------------|
| quantifying property | entropy production | $\Delta s_t = \ln \frac{f(\Gamma_A, 0)\,d\Gamma_A \, \differential \
abla \Gamma_B \rightarrow \Gamma_A}{f(\Gamma_B, 0)\,d\Gamma_B \, P_I(\Gamma_B \rightarrow \Gamma_A) \, \differential \
abla \Gamma_B \rightarrow \Gamma_A^*}$ | dissipation production | $\omega_t = \ln \frac{f(\Gamma_A, 0)\,d\Gamma_A \, \differential \
abla \Gamma_A \rightarrow \Gamma_B}{f(\Gamma_B, 0)\,d\Gamma_B \, P_I(\Gamma_B \rightarrow \Gamma_A) \, \differential \
abla \Gamma_B \rightarrow \Gamma_A^*}$ |

Table I. Comparison of measures of irreversibility. For simplicity, and a more compact expression, we assume that the deterministic dynamics conserves increments in configuration space. $f(\Gamma, t)$ is the pdf describing an ensemble of sets of probability amplitudes that define the state vectors and $T$ is a transition probability under stochastic dynamics.

A. Bloch Sphere Representation

In the two-level quantum case we shall be considering, a general $\psi$ can be written (now using ket notation) in the form

$$|\psi\rangle = \cos(\theta/2)\,|0\rangle + e^{i\phi}\sin(\theta/2)\,|1\rangle, \quad (3)$$

where $\cos(\theta/2)$ and $e^{i\phi}\sin(\theta/2)$ are the amplitudes forming the associated configuration $\Gamma$. We can then use Cartesian co-ordinates defined as

$$x = \sin \theta \cos \phi, \quad y = \sin \theta \sin \phi, \quad z = \cos \theta \quad (4)$$

to represent configurations as points on a Bloch sphere [13]. Trajectories are then paths from an initial point on a Bloch sphere to the point representing the evolved configuration.

Unitary evolutions of a two-level quantum system, representing the mapping $S_t$, can be illustrated as continuous paths on the surface of a Bloch sphere, by analogy with a classical trajectory through coordinate phase space. Without loss of generality we consider the mapping from an initial to a final state to be a rotation with unitary

$$S_t(\mathbf{n}, \alpha_{rot}(t)) = 1\,\cos(\alpha_{rot}/2) - i\mathbf{n} \cdot \mathbf{\hat{S}} \sin(\alpha_{rot}/2), \quad (5)$$

where $\mathbf{n}$ is the (normalised) axis of rotation, $\alpha_{rot}$ is the angle of rotation and $\mathbf{\hat{S}}$ is the vector of Pauli matrices. The angle of rotation is a function of the duration of the evolution. For a time-independent Hamiltonian, the
trajectories are precisely rotations on the Bloch sphere, with elapsed time proportional to angle rotated, but the initial to final mapping (though not the intervening behaviour) can also represent the effect of a Hamiltonian that is time-dependent. As we are considering only deterministic evolution, no measurement stage is involved, as this would introduce uncertainty of outcome into the dynamics.

In the Bloch sphere representation, the inversion operation, complex conjugation, is the transformation \( \phi \rightarrow -\phi \). Furthermore, the dynamics conserve areas on the surface of the Bloch sphere, in the sense that a patch of size \( d\Gamma_A \) is mapped to an equal size patch \( d\Gamma_B \). We shall therefore be able to omit these increments in the definition of the dissipation production for this case.

The choice of basis used to specify the ensemble of quantum state vectors, is arbitrary, determining only which states are found at the poles of the Bloch sphere. As unitary transformations are rotations on the Bloch sphere, a change of basis changes the axes of the Bloch sphere, defined by the basis vectors, but not any feature such as the pdf displayed upon it. As the dissipation production is defined in terms of this pdf, the particular choice of basis will not affect the values of \( \omega_t \).

B. Mathematical Properties of Dissipation Production

1. Non-negativity of Mean Dissipation Production

It can be shown that the mean of the dissipation production will never be negative, even under reversible dynamics. This property indicates that, just like entropy production, dissipation production satisfies a second law-like relation; its average behaviour is to increase as time passes [13].

To prove the non-negativity of the mean dissipation production, we start with the expression for the mean, which is

\[
\langle \omega_t \rangle = \int f(\tilde{\Gamma}_t,0) \ln \frac{f(\tilde{\Gamma}_t,0)}{f(\Gamma_t,0)} d\tilde{\Gamma}_t,
\]

where \( \tilde{\Gamma} \) and \( \Gamma_t = S_t \tilde{\Gamma} \) are configurations and \( f(\tilde{\Gamma}, \tau) \) is the pdf over configurations at time \( \tau \). Note that this expression takes the form of a Kullback-Leibler divergence [15], or relative entropy, between the initial and shifted pdfs, and a Kullback-Leibler divergence is never negative.

Equivalently, it can be shown that the mean dissipation production is a non-negative quantity by considering the average of its negative exponential:

\[
\langle e^{-\omega_t} \rangle = \int f(\tilde{\Gamma},0) \frac{f(\tilde{\Gamma}_t,0)}{f(\Gamma,0)} d\tilde{\Gamma} = \int f(\tilde{\Gamma},0) d\tilde{\Gamma}.
\]

Since \( f(\tilde{\Gamma},0) \) is a normalised pdf, and the transformation \( \tilde{\Gamma} \rightarrow \Gamma_t \) has a Jacobian of unity, we can write \( \langle e^{-\omega_t} \rangle = 1 \). Using the expansion of \( e^z \) to establish that \( e^{-z} \geq 1 - z \), \( z \in \mathbb{R} \), it follows that \( \langle e^{-\omega_t} \rangle \geq 1 - \langle \omega_t \rangle \) such that \( 1 \geq 1 - \langle \omega_t \rangle \), allowing us to conclude that \( \langle \omega_t \rangle \geq 0 \). It should be noted that this emerges for both positive and negative \( t \), namely evolution into the future and into the past relative to the starting condition.

2. Fluctuation Relation

A fluctuation relation [10][11][16] quantifies the extent to which a property such as entropy production evolves in a direction counter to that dictated by the second law of thermodynamics. The implication of such a relation is that fluctuations that ‘break’ the second law are exponentially unlikely and are never apparent on a macroscopic scale. A negative value for the dissipation production \( \omega_t \) indicates behaviour which violates a second law-like relation.

Entropy production is known to obey a number of fluctuation relations [17]. Similarly, the dissipation production \( \omega_t \) can satisfy a result known as the Evans-Searles Fluctuation Theorem (ESFT) in certain situations, which we now explore. The requirements [9][10].
are that the probabilities of two starting points of the evolution, related by a mapping $M^R$, should be equal, and that there are trajectories yielding equal and opposite dissipation productions whose starting points are also related by $M^R$. These conditions can be expressed as:

$$f(M^R\bar{\Gamma},0) = f(\bar{\Gamma},0), \quad (8)$$

and

$$\omega_t(\bar{\Gamma}) = -\omega_t(M^R\bar{\Gamma}_t). \quad (9)$$

$M^R$ is a transformation that can be more general than the map $M^T$ used in Sec. [4]. $\bar{\Gamma}_t = S_t\bar{\Gamma}$ is the configuration to which $\bar{\Gamma}$ evolves after time $t$. Provided that these two conditions hold, the derivation of the ESFT proceeds thus. The pdf of dissipation production is

$$P(\omega) = \int d\bar{\Gamma} f(\bar{\Gamma},0)\delta(\omega_t(\bar{\Gamma}) - \omega), \quad (10)$$

and we use the definition of $\omega_t$ from Sec. [4] to write

$$P(\omega) = \int d\bar{\Gamma} f(\bar{\Gamma},0)e^{\omega_t(\bar{\Gamma})}f(\bar{\Gamma},0)\delta(\omega_t(\bar{\Gamma}) - \omega)$$

$$= e^{\omega} \int d\bar{\Gamma} f(\bar{\Gamma},0)\delta(\omega_t(\bar{\Gamma}) - \omega). \quad (11)$$

Now we use the condition given in Eq. [9] to give

$$P(\omega) = e^{\omega} \int d\bar{\Gamma}_t f(\bar{\Gamma}_t,0)\delta(-\omega_t(M^R\bar{\Gamma}_t) - \omega). \quad (12)$$

Finally, Eq. [8], and a transformation of the integration measure, give

$$P(\omega) = e^{\omega} \int d(M^R\bar{\Gamma}_t) f(M^R\bar{\Gamma}_t,0)\delta(\omega_t(M^R\bar{\Gamma}_t) + \omega)$$

$$= e^{\omega} P(-\omega). \quad (13)$$

This is the ESFT. Proofs in the literature employ a transformation that time-inverts the evolved state, namely an $M^R$ given by $M^T$, but the result can clearly hold in more general circumstances. With $M^R = M^T$, we can employ the identity $M^T S_t M^T S_t = I$ to show that condition [8] follows from [5], as long as the protocol of the dynamics is symmetric over the interval: $S_{t+}^* = S_t$. The ESFT then arises in circumstances where this holds and the initial pdf is symmetric in the time-reversal operation.

We anticipate that the ESFT emerges in more general circumstances if a relation $M^R S_t M^R S_t = I$ holds. This places a requirement on the properties of $M^R$: in our system it must be a reflection in the plane containing the axis of rotation of transformation $S_t$ that governs the dynamics of the evolution. The requirement $f(M^R\bar{\Gamma},0) = f(\bar{\Gamma},0)$ further enforces a more stringent that $M^R$ is also a reflection in the plane of symmetry of the pdf. To see this, we can associate the operations as rotations about the axis and reflections in the plane as illustrated in Figure 2.

Thus, in situations where the rotation axis representing the evolution $S_t$ lies in a plane of symmetry of the pdf of the initial state of the system, we shall observe an ESFT.

3. Symmetry in Time Evolution into Future and Past

Provided that the conditions for obtaining an ESFT are met, specifically that $M^R S_t M^R S_t = I$ and $f(M^R\bar{\Gamma},0) = f(\bar{\Gamma},0)$, we can show that the mean dissipation production is the same for evolution into the past and the future. Starting from the mean dissipation production for evolution into the past:

$$\langle \omega_{-t} \rangle = \int f(\bar{\Gamma},0) \ln \frac{f(\bar{\Gamma},0)}{f(S_{-t}\bar{\Gamma},0)} d\bar{\Gamma}, \quad (14)$$

we recast as

$$\langle \omega_{-t} \rangle = \int f(M^R\bar{\Gamma},0) \ln \frac{f(M^R\bar{\Gamma},0)}{f(S_{-t}M^R\bar{\Gamma},0)} dM^R\bar{\Gamma}, \quad (15)$$

and apply $S_{-t}M^R = M^R S_t$ and Eq. [8]:

![Figure 2. Cross-section of Bloch sphere, looking down the rotation axis of the mapping, geometrically illustrating that $M^R S_t M^R S_t = I$. To see this, consider a point $a$ on the surface of the Bloch sphere, with its position specified by angle $\theta$. Application of evolution operator $S_t$ will rotate this point to $b$: $\theta \rightarrow \theta + \alpha$. Then, $M^R$ will map $b$ to $c$: $\theta + \alpha \rightarrow 2\pi - (\theta + \alpha)$. Applying $S_t$ again sends $c$ to $d$: $2\pi - (\theta + \alpha) \rightarrow 2\pi - (\theta + \alpha) + \alpha = 2\pi - \theta$. A final application of $M^R$ returns $d$ to $a$.](image)
\[
\langle \omega_\ell \rangle _\ell = \int f(MR\tilde{\Gamma}, 0) \ln \frac{f(MR\tilde{\Gamma}, 0)}{f(MR\tilde{\Gamma}, 0)} dM R\tilde{\Gamma}
\]
\[
= \int f(\tilde{\Gamma}, 0) \ln \frac{f(\tilde{\Gamma}, 0)}{f(S\tilde{\Gamma}, 0)} d\tilde{\Gamma},
\]
which is the mean dissipation production for forward evolution, \(\langle \omega_\ell \rangle _\ell\). However, in situations in which the ESFT is violated, we do not expect this result to hold, the implication being that the initial ensemble will exhibit different mean dissipation productions into the past and the future; a time asymmetry of behaviour.

### III. RESULTS

In order to demonstrate the different possible behaviours of the dissipation production, we restrict ourselves to considering simple evolutions represented by rotations

\[
S_{jkl}(\hat{n}, t) = \begin{cases} 
\cos^2 \frac{t}{2} + \sin^2 \frac{t}{2}(2\tilde{\theta}^2 - 1), & \text{if } j = k \\
\frac{\epsilon_{jkl} \tilde{n}_i \sin t}{2} & \text{if } j \neq k
\end{cases}
\]

where \(\epsilon_{jkl}\) is the Levi-Civita symbol, the rotation angle is equal to the elapsed time, \(t\), and \((n_x, n_y, n_z)\) is the rotation axis.

We consider five probability density functions over the Bloch sphere:

**Case 1a:** \(f(\theta, \phi, t = 0) = (4\pi)^{-1}(1 + z)\) which is rotationally symmetric about the \(z\)-axis.

**Case 1b:** \(f(\theta, \phi, 0) = (4\pi)^{-1}(1 + z \cos \beta + (u_x y - u_y x) \sin \beta + (u_x x + u_y y + u_z z)(1 - \cos \beta))\) with \(\beta = \pi/3\), where \((u_x, u_y, u_z) = (1/\sqrt{2}, 1/\sqrt{2}, 0)\) is the axis about which the pdf has rotational symmetry.

**Case 2a:** \(f(\theta, \phi, 0) = (4\pi)^{-1}(1 + \cos \theta)(1 + \cos \phi)\), which is symmetric with respect to the transformation \(\phi \rightarrow -\phi\) and hence has mirror symmetry in the \(xz\)-plane.

**Case 2b:** \(f(\theta, \phi, 0) = (4\pi)^{-1}(1 + \cos \theta)(1 + \cos(\phi + \pi/4))\), which is not symmetric with respect to the transformation \(\phi \rightarrow -\phi\) but does have a plane of symmetry which passes through the \(z\)-axis.

**Case 3:** \(f(\theta, \phi, 0) = (8\pi)^{-1}(1 + \cos \theta)(2 + \cos \phi + \sin 2\phi)\), which is not symmetric with respect to the transformation \(\phi \rightarrow -\phi\) and has no planes of symmetry.

Cases 1 and 2 are illustrated in Figure 3 while the fully asymmetric Case 3 is shown in Figure 4.

In Figure 3, we show the pdfs of the dissipation production, \(\omega_\ell\), for Case 1a, using rotations about the \(z\)-axis through various angles to represent the transformation \(S_t\) at various times. The shape of the pdf in Case 1a depends on the elapsed time. These pdfs can be used to compute the logarithm of the ratio of probabilities of equal and opposite values of \(\omega_\ell\). If a plot of this quantity against \(\omega_\ell\) gives a straight line with unit gradient, then an ESFT holds. Since Case 1a has rotational symmetry
about the $z$-axis, any axis of rotation defining $S_t$ (which needn’t be restricted to the Cartesian axes) will lie in a plane of symmetry of the pdf, meeting the requirements for an ESFT described earlier. Although the axis of rotational symmetry for Case 1b is $(1/\sqrt{2}, 1/\sqrt{2}, 0)$ rather than the $z$-axis, the planes of symmetry through this axis nevertheless lead to the observation of an ESFT regardless of the choice of rotation $S_t$. The inset in Figure 4 demonstrates an ESFT for an example rotation of 2.1 radians about the $x$-axis (i.e. $t = 2.1$).

To confirm the conditions required to obtain an ESFT, we consider the more complicated Case 2a, which produces more structure in the associated pdfs. Considering an evolution consisting of a rotation angle of $2\pi/3$ about each of the Cartesian axes, we generate Figure 5, which indicates, as expected, that an ESFT holds for rotations about axes which lie in the $xz$-plane, i.e. the plane of symmetry of the pdf on the Bloch sphere.

We also consider Case 2b which is a rotated version of the pdf in Case 2a. We again see an ESFT holds for evolution corresponding to rotation about the $z$-axis and its violation for rotation about the $y$-axis, but in contrast with Case 2a, an ESFT does not hold for rotation about the $x$-axis as this axis does not lie in the plane of symmetry of the pdf on the Bloch sphere.

Whenever we observe an ESFT, we also expect time-symmetric behaviour for the mean dissipation production for evolution into the future and the past, as explained in Sec. II.B3. Figure 6, depicting $\langle \omega_t \rangle$ in Cases 2a and 2b for rotations about the $z$-axis, shows this. Note that it is non-negative as the rotation angle changes from 0 to $2\pi$. Small angles of rotation give a small mean dissipation production, as in these instances there is little difference between the two configurations being compared. The non-negativity is universal and independent of choice of axis or Bloch sphere pdf, but the time-symmetric behaviour only accompanies situations which satisfy an ESFT.

We verify this by assessing Bloch sphere pdf Case 4, which has no planes of symmetry, and hence cannot give an ESFT, regardless of evolution rotation axis. Figure 7 demonstrates the associated time-asymmetry in mean dissipation production for an example rotation about the $z$-axis.

These considerations allow us to identify initial ensembles which will exhibit time-asymmetric average behaviour even under a reversible unitary evolution.

To summarise the examined Bloch sphere pdfs and rotation processes, Table IId details which combinations lead to the emergence of an ESFT. The results confirm that an ESFT depends on the relationship between the chosen rotation axis and the symmetry of the pdf on the Bloch sphere; specifically when the rotation axis lies in a plane of symmetry of the pdf, the ESFT is upheld.
Figure 7. $\langle \omega_t \rangle$ for rotations of duration $t$ about the $z$-axis in Case 3, for which the pdf is shown as an inset. This pdf does not have a plane of symmetry passing through the evolution rotation axis and an asymmetry in $\langle \omega_t \rangle$ for evolution into the future and the past is a consequence.

Table II. Fulfillment of an ESFT for assorted cases of Bloch sphere pdfs when configurations are rotated by the dynamics $S_t$ about the Cartesian axes.

| Axis | Cases 1a and 1b | Case 2a | Case 2b | Case 3 |
|------|-----------------|---------|---------|--------|
| $x$  | yes             | yes     | no      | no     |
| $y$  | yes             | no      | no      | no     |
| $z$  | yes             | yes     | yes     | no     |

IV. CONCLUSION

In quantum systems undergoing deterministic evolution, the ensemble that represents our uncertainty with regard to the initial state vector can be used to specify the likelihood of following a particular trajectory and its reversed counterpart. This allows us to test for irreversibility of behaviour in the form of the failure of obversibility (a property distinct from, but closely related to reversibility), which we quantify with dissipation production. This first study of dissipation production in a quantum system extends the use of the concepts beyond the classical realm previously considered \[9\]. In particular, we are able to determine whether a system with a given pdf describing an initial ensemble will exhibit time-asymmetry in average behaviour under a particular process.

We have studied a simple two-level system, which is nevertheless sufficient to demonstrate the use of dissipation production for situations in which entropy production is an inappropriate irreversibility measure. Our principal aim has been to identify conditions under which dissipation production satisfies an Evans-Searles Fluctuation Theorem (ESFT), in which case it follows that it evolves on average into the past in the same way as into the future. The mapping of states on the Bloch sphere after a given time interval under deterministic dynamics can be represented by a rotation about a certain axis, and the criterion for the validity of the ESFT is that this axis should lie in a plane of symmetry of the pdf describing the initial ensemble. It is also straightforward to demonstrate that the average dissipation production can never be negative, which makes it a measure of irreversibility.

Obversibility is distinct from reversibility. The latter is upheld here owing to the deterministic unitary dynamics employed: the system is isolated. Reversibility is (essentially) the property that the effects of carrying out a process, given an ensemble of initial system configurations, can be undone by inverting velocities, carrying out a reverse process, and then inverting velocities again. Obversibility is the property that the effects of a process and a reverse process, preceded by an inversion of velocities in the latter case, but starting from a given ensemble, are statistically identical \[9\].

Dissipation production is a consequence of a failure of obversibility and plays a role that is similar to, but distinct from the entropy production that arises from a failure of reversibility. In a nonequilibrium stationary state, dissipation production and entropy production are synonymous, but not in general situations. We have therefore broadened our understanding of quantities that might characterise the arrow of development in time. Furthermore, we have been able to demonstrate that the time-asymmetric nature of this arrow for a closed quantum system can arise from certain asymmetries in the pdf over the ensemble of initial states. As dissipation production depends only on the classical probabilities of starting from particular states, even when considering quantum systems, it emphasises the distinction between classical uncertainty due to a lack of knowledge regarding, for example, the initial state, and quantum uncertainty due to a lack of predictability with regard to the outcomes of measurement; we currently exclude the latter from our considerations.

We anticipate that the methods described here are readily applicable to larger systems, such as two qubits, since the tools required to calculate dissipation production (namely pdfs of the system configuration and appropriate reversal and evolution operators) can be readily defined. Although we have illustrated our examples with the Bloch sphere, this is not an essential component. For a general complex system it is likely that the ESFT will be upheld under very special circumstances, but these will always include situations where the pdf describing the initial ensemble obeys time-reversal symmetry and the protocol of dynamics is time-symmetric about its midpoint, as envisaged by Evans \[11\].

In summary we have confirmed that the failure of obversibility can be used as an indicator of irreversibility in a closed system, where mechanical reversibility is respected. Such a failure can also be used to characterise irreversibility in systems that are open to the environment, by way of measurement and/or thermalisation. The dynamics of such systems are mechanically irreversible, and we would compute dissipation production merely by inserting appropriate transition probabilities into Eq. \[2\]. The relationship between reversibility and obversibility needs to be further developed, giving consideration to the role of initial conditions in generating subsequent time-
asymmetry of the irreversibility measures. Exploring dissipation production and obversibility in open quantum situations is hence an avenue for further research.

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