Large spin limit of $AdS_5 \times S^5$ string theory and low energy expansion of ferromagnetic spin chains

M. Kruczenski$^{1,*}$, A.V. Ryzhov$^{1,†}$ and A.A. Tseytlin$^{2,‡}$

$^1$ Department of Physics, Brandeis University
Waltham, MA 02454, USA

$^2$ Department of Physics, The Ohio State University
Columbus, OH 43210-1106, USA

Abstract

By considering $AdS_5 \times S^5$ string states with large angular momenta in $S^5$ one is able to provide non-trivial quantitative checks of the AdS/CFT duality. A string rotating in $S^5$ with two angular momenta $J_1, J_2$ is dual to an operator in $\mathcal{N} = 4$ SYM theory whose conformal dimension can be computed by diagonalizing a (generalization of) spin 1/2 Heisenberg chain Hamiltonian. It was recently argued and verified to lowest order in a large $J = J_1 + J_2$ expansion, that the Heisenberg chain can be described using a non-relativistic low energy effective 2-d action for a unit vector field $n_i$ which exactly matches the corresponding large $J$ limit of the classical $AdS_5 \times S^5$ string action. In this paper we show that this agreement extends to the next order and develop a systematic procedure to compute higher orders in such large angular momentum expansion. This involves several non-trivial steps. On the string side, we need to choose a special gauge with a non-diagonal world-sheet metric which insures that the angular momentum is uniformly distributed along the string, as indeed is the case on the spin chain side. We need also to implement an order by order redefinition of the field $n_i$ to get an action linear in the time derivative. On the spin chain side, it turns out to be crucial to include the effects of integrating out short wave-length modes. In this way we gain a better understanding of how (a subsector of) the string sigma model emerges from the dual gauge theory, allowing us to demonstrate the duality beyond comparing particular examples of states with large $J$.

$^*E$-mail: martink@brandeis.edu
$^†E$-mail: ryzhovav@brandeis.edu
$^‡$Also at Imperial College London and Lebedev Institute, Moscow
1 Introduction

Understanding AdS/CFT duality beyond the BPS or near BPS limit remains an important challenge. It was suggested in [2] that concentrating on string states with large quantum numbers, like angular momentum in AdS5, one finds a qualitative (modulo interpolating function of ‘t Hooft coupling $\lambda$) agreement between the AdS5 string energies and anomalous dimensions of the corresponding gauge theory operators (see also [3] [4] [5]). About a year ago, it was observed [6] that semiclassical string states with several non-zero angular momenta (with large total $S^5$ momentum $J$) have a remarkable property that their energy admits an analytic expansion in $\tilde{\lambda} \equiv \lambda J^2$ at large $J$. * It was proposed, therefore, that the coefficients of such an expansion can be matched precisely with the perturbative anomalous dimensions of the corresponding scalar SYM operators computed in the same $J \to \infty, \tilde{\lambda} < 1$ limit [6]. That would provide the first quantitative check of AdS/CFT duality far from the BPS limit. The reason for this expectation was that for such special solutions all string $\alpha' \sim 1/\sqrt{\lambda}$ corrections might be suppressed in the large $J$ limit (as was explicitly checked for a particular case in [9]; see also [10] for a review). Then, the classical string energy would represent an exact string theory prediction in this limit. This proposal received a spectacular confirmation in [11] [12] where the one-loop anomalous dimensions of the relevant scalar SYM operators were computed utilizing a remarkable Heisenberg spin chain interpretation of the one-loop anomalous dimension in the scalar sector [13] and taking the thermodynamic $J \to \infty$ limit of the Bethe ansatz solution for the eigenvalues. The detailed agreement of the functional dependence of the leading “one-loop” coefficient on the ratio of spins for “inhomogeneous” folded and circular rotating string solutions was further demonstrated in [14] [15] [12]. Same was found also for the “homogeneous” [16] two-spin circular solutions [17] and for particular three-spin states [18] [19]. This agreement was extended to the “two-loop” level using integrable model/Bethe ansatz techniques [20] [17].

One would obviously like to achieve a better understanding of how and why this correspondence between the string theory and gauge theory works, e.g., the general rules of how particular string states are mapped onto particular SYM states. It would be interesting to see how string sigma model world-sheet action emerges on the gauge theory side (cf. [11]), allowing one to go beyond discussion of matching of individual states. An important suggestion in this direction was made recently in [21], and our aim here will be to further clarify and extend it beyond the leading (“one-loop”) order.

In more detail, here we would like to try to understand in general the correspondence between (“semiclassical”) string states with two large angular momenta $J_1, J_2$ in $S^5$ and single-trace SYM operators $O_{J_1, J_2} = \text{tr} (\Phi_1^{J_1} \Phi_2^{J_2} + \ldots)$ ($\Phi_1, \Phi_2$ are two complex combinations of SYM scalars). The main assumption is that the limit

\begin{equation}
J \equiv J_1 + J_2 \gg 1, \quad \tilde{\lambda} \equiv \frac{\lambda}{J^2} = \text{fixed} < 1, \quad (1.1)
\end{equation}

* Earlier examples of similar solutions were found in [3] [7] [8].
i.e. the expansion in powers of $\frac{1}{J}$ and $\tilde{\lambda}$ is well-defined on both the string and the SYM sides of the duality. The classical energy of such rotating string solutions admits the following expansion \[6\]

$$E = J \, F\left(\frac{J_2}{J}, \tilde{\lambda}\right), \quad F = 1 + c_1 \tilde{\lambda} + c_2 \tilde{\lambda}^2 + \ldots,$$

(1.2)

where $c_i$ depend on ratios of spins (and other parameters like winding numbers). If quantum string corrections to $c_i$ are suppressed by extra powers of $\frac{1}{J}$ \[9\], the classical string energy (1.2) should represent the exact string result in the large $J$ limit. Then the AdS/CFT duality implies that one should be able to match (1.2) with dimensions of the corresponding SYM operators found in the same limit. Indeed, it was demonstrated in \[11, 14, 15, 12\] and \[20, 17\] that energies of a particular classical rotating 2-spin string solutions agree precisely with anomalous dimensions of the corresponding SYM operators at the first two – “one-loop” and “two-loop” – orders in expansion in $\tilde{\lambda}$ at large $J$. There is also matching at the level of integrable structures \[22, 18\] clarified and established in general in \[17\].

The one-loop anomalous dimension matrix in the sector of 2-spin operators $O_{J_1, J_2}$ happens to be equivalent to the ferromagnetic Heisenberg XXX$_{1/2}$ ($SU(2)$) spin chain Hamiltonian $H$ \[13\]. With the 2-loop \[26\] and 3-loop \[26, 27, 28\] corrections included $H$ may be interpreted as a generalized spin chain Hamiltonian containing further next to nearest neighbor interactions. To find the eigenvalues of $H$ in the one-loop approximation one is able to apply the Bethe ansatz techniques with crucial simplification of thermodynamic limit $J \rightarrow \infty$ \[11, 12, 17\]. Furthermore, one is able to extend this to the two-loop level by embedding \[20\] the anomalous dimension operator into a particular integrable spin chain system and then again utilize the thermodynamic limit of the Bethe ansatz \[20, 17\]. Using the Bethe ansatz for the spin chain on the gauge side and integrability of the classical sigma model on the string side ref.\[17\] managed to prove the one-loop and two-loop matching for generic solutions.

Our approach below will be simpler and in a sense complementary to the one of \[17\] based on integrability. Following \[21\], we would like to establish a more direct way of comparing the string and SYM results without the need to go through detailed analysis of particular solutions (or their general classification) and use of complicated Bethe ansatz techniques which seem hard to extend beyond few leading orders without knowledge of integrability of the exact spin chain Hamiltonian. The idea is to extract an effective 2-d action describing low-energy states of the ferromagnetic Heisenberg spin chain in the limit $J \rightarrow \infty$, $\tilde{\lambda} = \text{fixed}$ and then to show that it agrees with the string sigma model action expanded in the same limit. On the string side that would correspond to “gauging away” a collective coordinate associated with the total (orbital and internal) $S^5$ spin $J$ (which may be viewed as a non-trivial generalization of light-cone gauge fixing in the BMN case \[11\]).\[†\] An agreement between the resulting effective spin chain and string actions would then

\[†\]This amounts to a reorganization of the classical string action in the large $J$ sector so that the expansion effectively goes in powers of $\lambda \sim \tilde{\lambda}$ and thus corresponds to a “near zero-tension” \[20\].
imply an agreement between energies of particular solutions/states (as well as matching of underlying integrable structures). As a consequence, one can directly relate the target space spinning string configurations to configurations of 1-d spins \(21\), a non-trivial connection which emerged also in the Bethe ansatz approach \(11, 17\).

Once the configurations are related, many questions, as for example the agreement between Bethe-Ansatz and sigma model calculations, between integrable structures etc. become questions regarding the spin chain Hamiltonian and not the AdS/CFT correspondence. By that we mean that the agreement is between two different ways of describing a spin chain, a low energy description in terms of an effective action or an exact description using the integrability properties of the model. The AdS/CFT correspondence simply establishes that one of those two ways is directly related to (a limit of) the action of a string moving in a specific target space.

By starting with the one-loop expression for the dilatation operator \(13\)

\[
H = \frac{\lambda}{(4\pi)^2} \sum_{a=1}^{J} (1 - \sigma_a \cdot \sigma_{a+1}) , \quad \sigma_a^i \sigma_a^j = \delta^{ij} + i\epsilon^{ijk} \sigma_a^k , \tag{1.3}
\]

one finds \(21\) (see, e.g., \(23, 24\) and refs. there) that the corresponding action in coherent state \(25\) \(\langle n | \sigma_a^i | n \rangle = n_a^i, (n_i)^2 = 1, i = 1, 2, 3\) path integral is

\[
S = \int dt \left[ \sum_{a=1}^{J} L_{\text{WZ}}(n_a) - \langle n | H | n \rangle \right] , \tag{1.4}
\]

\[
\langle n | H | n \rangle = \frac{\lambda}{2(4\pi)^2} \sum_{a=1}^{J} (n_{a+1} - n_a)^2 , \tag{1.5}
\]

where \(L_{\text{WZ}}(n_a)\) depends on \(n_a\) only at a given site \(a\) and is linear in \(\partial_t n_a\).\(^\dagger\) Since we are interested in the limit of \(J \rightarrow \infty\) with \(\tilde{\lambda}\) fixed this suggests to take the continuum limit, introducing \(n(\sigma), 0 < \sigma \leq 2\pi\), with \(n_a = n(\frac{2\pi a}{J})\). We then finish with

\[
S = J \int dt \int_0^{2\pi} \frac{d\sigma}{2\pi} L , \quad L = C_t(n) - \frac{1}{8} \tilde{\lambda}(\partial_t n)^2 , \quad \partial_t \equiv \partial_\sigma , \tag{1.6}
\]

where we set \(L_{\text{WZ}}(n) \equiv C_t(n)\) to indicate that this term is linear in \(\partial_t n^i\). The equation of motion corresponding to \(1.6\) are the classical ferromagnet or Landau-Lifshitz (LL) equation\(^\S\)

\[
\partial_t n_i = \frac{1}{2} \tilde{\lambda} \epsilon_{ijk} n_j \partial_1^2 n_k . \tag{1.7}
\]

\(^\dagger\)\(L_{\text{WZ}}\) may be viewed as an analog of the usual “\(p \dot{q}\)” term, i.e. this non-relativistic action may be interpreted as a phase space action with \(n_i\) describing both coordinates and momenta. This ensures proper commutation relations if we reverse the logic and promote \(n_i\) to quantum operators: \(\dot{n}_i \equiv \hat{n}_i = \sigma_i, [\hat{n}_i, \hat{n}_j] = 2i\epsilon_{ijk} \hat{n}_k\).

\(^\S\)The LL equation describes evolution in time of magnetization vector of a (one-dimensional in the present context) macroscopic ferromagnet (see, e.g., \(28, 60\)).
We have omitted higher-derivative terms coming from \((n_{a+1} - n_a)^2\) since they are suppressed by powers of \(\frac{1}{J}\). We further observe that since \(J\) is the coefficient in front of the action, it plays the role of the inverse Planck constant in the spin chain path integral. Then in the limit when \(J \to \infty\) and \(\tilde{\lambda} = \text{fixed}\) one should be able to ignore quantum corrections and thus to treat \((1.6)\) as a classical action that should be matched to the string action expanded in the same limit. This matching was indeed demonstrated in [21] and will be further discussed and extended to higher orders in \(\tilde{\lambda}\) below.

We should remark that at higher loops, the operators of interest are still described by a spin \(s = \frac{1}{2}\) chain and only the Hamiltonian gets modified. At each loop interactions involving larger number of neighbors are introduced. However, at least at the next order, the interaction is still ferromagnetic (for small \(\tilde{\lambda}\)) and therefore, at low energy, the description of the system in terms of long-wave length spin waves should be valid. The low energy effective action governing these modes contains, a priori, all possible terms compatible with the symmetries. Our task is then to compute the corresponding coefficients which should include the effects of integrating out the high momentum modes.

Thus, with higher-loop corrections included in \(H\) in \((1.3)\), we expect to find a low-energy effective action analog of \((1.6),(1.7)\) with higher powers of \(\tilde{\lambda}\) multiplying higher derivatives of \(n_i\)

\[
L = C_t(n) - H(\partial_1 n, \partial_1^2 n, \partial_1^3 n, \ldots) ,
\]

\[
H = a_0 \tilde{\lambda}(\partial_1 n)^2 + \tilde{\lambda}^2 \left[a_1(\partial_1^2 n)^2 + a_2(\partial_1 n)^4\right]
+ \tilde{\lambda}^3 \left[a_3(\partial_1^3 n)^2 + a_4(\partial_1 n)^2(\partial_1^2 n)^2 + a_5(\partial_1 n \partial_1^2 n)^2 + a_6(\partial_1 n)^6\right] + O(\tilde{\lambda}^3) .
\]

We have written down all possible local structures with 4 and 6 spatial derivatives built out of a unit vector field \(n_i(t,\sigma)\) (modulo integration by parts). The coefficients \(a_0, a_1, a_2, \ldots\) are determined by detailed microscopics of the spin chain.\footnote{In principle, one could use the knowledge of a few families of microscopic solutions to fix these coefficients, but it turns out one can work with general field configurations, as we will show in Section 4.5}

It is useful to note that since \((1.7)\) and \(L\) in \((1.6)\) or \((1.8)\) are linear in time derivatives, the overall factor of \(\tilde{\lambda}\) in \(H\) can be absorbed into a rescaling of the time coordinate (this does not change the coefficient in front of the action \((1.6)\))

\[
t \to \tilde{\lambda}^{-1}t , \quad L \rightarrow L = C_t(n) - a_0(\partial_1 n)^2 - \tilde{\lambda}[a_1(\partial_1^2 n)^2 + a_2(\partial_1 n)^4] + \ldots .
\]

Our first aim below will be to show, generalizing the suggestion of [21], how such an action with a single time derivative (in \(C_t(n)\) term) but containing all orders of derivatives in \(\sigma\), emerges from the usual second-time-derivative \(AdS_5 \times S^5\) sigma model action expanded in the limit \(J \to \infty, \tilde{\lambda} < 1\). In the 2-spin sector the classical string equations in \(AdS_5 \times S^5\) expanded for large spin reduce to a higher-derivative
generalization of the LL equation for a unit 3-vector \( n_i \) describing a shape of a string rotating in two orthogonal planes. It is remarkable that such a non-relativistic “classical ferromagnetic” action containing all orders in spatial derivatives but only first order in time derivative happens to be a particular “re-expansion” (in the \( \tilde{\lambda} \to 0 \) limit) of the usual relativistic string sigma model action.

Next, we will compare the coefficients in \( H (1.9) \) appearing from the (quantum) spin chain with the corresponding coefficients coming out of the string sigma model action. We shall conclude, in full agreement with the previous results based on Bethe ansatz technique \([20, 17]\), that the correspondence does extend to the next \( \tilde{\lambda}^2 \) (“two-loop”) order. We shall find that beyond the leading \( \tilde{\lambda} \) order one is not able to ignore quantum corrections on the spin chain side: before taking the continuum limit one should compute a quantum effective action analog of \( (1.4) \). While the terms quadratic in \( n_i \) in \( (1.9) \) are indeed correctly found by simply taking the continuum limit of the coherent state expectation value of the spin chain Hamiltonian equal to the SYM dilatation operator, to reproduce the \( n^4 \) and higher order terms one needs to take into account quantum corrections. We shall also discuss how the matching should work at the “three-loop” \( \tilde{\lambda}^3 \) order but detailed quantum computations on the spin chain side (beyond the evaluation of the corresponding coherent state expectation value \( \langle n|H|n \rangle \)) will not be described here.

The rest of this paper is organized as follows. In section 2 we discuss the large angular momentum limit of the string action. We first consider the conformal gauge (section 2.2) and obtain the expansion of the action in powers of \( \tilde{\lambda} \). Although this approach happens to be enough up to second order in \( \tilde{\lambda} \), at higher orders the action contains non-local terms. For that reason in section 2.3 we develop a more systematic expansion by finding an appropriate “uniform” gauge and applying an order by order field redefinition to eliminate terms which are non-linear in time derivatives. In section 3 we take the continuum limit of the coherent state expectation value \( \langle n|H|n \rangle \) of the spin chain Hamiltonian representing the SYM dilatation operator with two-loop and three-loop corrections included. We show, however, that this naive approach does not reproduce the full string results at \( \tilde{\lambda}^2 \) and \( \tilde{\lambda}^3 \) orders. In section 4 we compute quantum corrections to the effective action of the spin chain coming from integrating out high energy modes. They turn out to contribute starting with \( \tilde{\lambda}^2 \) order and after including them we find perfect agreement with the string side at this order. In section 5 we perform some checks of the \( \tilde{\lambda}^2 \) result for particular two-spin string configurations. Concluding remarks are made in section 6. Appendices contain some technical details and useful relations.

2 Generalized “classical magnetic” action from string sigma model on \( R \times S^3 \)

Our first task will be to show how the action \( (1.8) \) appears from the standard string sigma model action on \( R \times S^3 \). Here \( R \) factor represents the time direction in \( AdS_5 \) and
$S^3$ factor is from $S^5$: we shall consider only string configurations that are located in the center of $AdS_5$ and belong to $S^3 \subset S^5$. They may carry two out of three possible independent angular momenta and should describe string states corresponding to eigenstates of closed $SU(2)$ subsector of anomalous dimension matrix or spin chain Hamiltonian. There exists a generalization of the procedure described below to the 3-spin ($SU(3)$) sector but it will not be discussed here.

2.1 Parametrization of $R \times S^3$

The metric of $R \times S^3$ space-time can be parametrized as follows

$$ds^2 = -dt^2 + |dX_1|^2 + |dX_2|^2 , \quad |X_1|^2 + |X_2|^2 = 1 , \quad (2.1)$$

$$X_1 \equiv X_1 + iX_2 = \cos \psi \ e^{i\varphi_1} , \quad X_2 \equiv X_3 + iX_4 = \sin \psi \ e^{i\varphi_2} . \quad (2.2)$$

In this parametrization string states that carry two angular momenta $J_1, J_2$ should be rotating in the two orthogonal planes $(X_1, X_2)$ and $(X_3, X_4)$ (see [6, 10] for details).

To consider the limit of large total spin $J = J_1 + J_2$ we would like to isolate the corresponding collective coordinate, i.e. the common phase of $X_1$ and $X_2$. In the familiar case of fast motion of the center of mass the role of $J$ is played by linear momentum or $p^+$. Here, however, $J$ represents the sum of “orbital” as well as “internal” angular momentum and thus does not correspond simply to the center of mass motion. This is thus a generalization of the limit considered in [1]: we are interested in “large” extended string configurations and not in a nearly point-like strings. Let us thus set

$$X_1 = U_1 \ e^{i\alpha} , \quad X_2 = U_2 \ e^{i\alpha} , \quad |U_1|^2 + |U_2|^2 = 1 , \quad (2.3)$$

i.e. parametrize $S^3$ in terms of $CP^1$ coordinates $U_i$ and an angle $\alpha$ (Hopf $S^1$ fibration of $S^3$). The angle $\alpha$ representing simultaneous rotation in the two planes will be the collective coordinate corresponding to $J$. In terms of standard $S^3$ angles

$$U_1 = \cos \psi \ e^{i\beta} , \quad U_2 = \sin \psi \ e^{-i\beta} , \quad \alpha = \frac{1}{2}(\varphi_1 + \varphi_2) , \quad \beta = \frac{1}{2}(\varphi_1 - \varphi_2) . \quad (2.4)$$

Then

$$(ds^2)_{S^3} = (d\alpha - iU_r^*dU_r)^2 + dU_r^*dU_r + (U_r^*dU_r)^2 , \quad r = 1, 2 , \quad (2.5)$$

i.e.

$$D\alpha \equiv d\alpha + C , \quad DU_r = dU_r - iCU_r , \quad C \equiv -iU_r^*dU_r . \quad (2.6)$$

It is useful to replace $U_r$ by a unit vector $n_i$ representing $CP^1$

$$n_i \equiv U^\dagger \sigma_i U , \quad U = (U_1, U_2) , \quad (2.7)$$

where $\sigma_i$ are Pauli matrices. Then

$$(ds^2)_{S^3} = (D\alpha)^2 + \frac{1}{4}dn_idn_i , \quad D\alpha = d\alpha + C(n) , \quad (2.8)$$
where $C(n)$ has a non-local WZ-type representation $C = -\frac{1}{2} \int_0^1 d\xi \ \epsilon_{ijk} n_i \partial_k n_j \partial_n k$. In terms of $S^3$ angles one has

$$n_i = (\sin 2\psi \ \cos 2\beta; \ \sin 2\psi \ \sin 2\beta; \ \cos 2\psi) \ , \quad (2.9)$$

$$(ds^2)_{S^3} = (d\alpha + C)^2 + d\psi^2 + \sin^2 2\psi \ d\beta^2 , \quad C = \cos 2\psi \ d\beta \ . \quad (2.10)$$

It is interesting to note a direct analogy between (2.7) relating $n_i$ and $U_1, U_2$ and the coherent state basis on the spin chain side.

The string action is then (we use signature $(- +)$)

$$I = \sqrt{\lambda} \int dt \int_0^{2\pi} d\sigma \ L \ , \quad (2.11)$$

$$L = -\frac{1}{2} \sqrt{-g} g^{\alpha \beta} \left( - \partial_\alpha \partial_\beta t + D_\alpha \partial_\beta \alpha + \frac{1}{4} \delta_{\alpha \beta} n_i \partial_\alpha n_i \right) \ , \quad (2.12)$$

where

$$D_\alpha \alpha = \partial_\alpha \alpha + C_\alpha (n) , \quad C_\alpha = -\frac{1}{2} \int_0^1 d\xi \epsilon_{ijk} n_i \partial_\xi n_j \partial_\xi n_k , \quad (2.13)$$

$$n_i(\tau, \sigma, \xi = 1) = (n_i)_0 , \quad n_i(\tau, \sigma, \xi = 0) = n_i(\tau, \sigma) ,$$

$$\partial_\alpha C_\alpha - \partial_\xi C_\xi = \frac{1}{2} \epsilon_{ijk} n_i \partial_\alpha n_j \partial_\alpha n_k , \quad \delta C_\alpha = \frac{1}{2} \epsilon_{ijk} n_i \partial_\xi n_j \partial_\xi n_k + \partial_\alpha \chi \ . \quad (2.14)$$

The crucial point is that one should view $t$ and $\alpha$ as “longitudinal” coordinates that reflect the redundancy of the reparametrization-invariant string description: they are not “seen” on the gauge theory side, and should be gauged away (or eliminated using the constraints). At the same time, the unit vector $n_i$ should be interpreted as a “transverse” or physical coordinate which should thus have a counterpart on the $SU(2)$ spin chain side (with an obvious candidate being a vector parametrizing the coherent state). To put (2.12) into first-time-derivative form like (1.8) one will need to properly expand the action and make a field redefinition of $n_i$.

The conserved charges corresponding to translations in time, rotations of $\alpha$ and $SO(3)$ rotations of $n_i$ are

$$(E, J, S_i) = \sqrt{\lambda} \ \langle \mathcal{E}, \mathcal{J}, \mathcal{S}_i \rangle , \quad \mathcal{E} = -\int_0^{2\pi} d\sigma \ \frac{d\alpha}{2\pi} \sqrt{-g} g^{\alpha \beta} \partial_\alpha t \ , \quad (2.15)$$

*One can check this relation using $\sigma_i^{\alpha \beta \sigma_i^{\gamma \delta}} = -\delta^{\gamma \delta} \delta^{\alpha \beta} + 2\delta^{\alpha \beta} \delta^{\gamma \delta}$. $C$ may be interpreted as a vector potential of a Dirac monopole at the origin.

†In the spin chain case we have complex scalars $\Phi_1, \Phi_2$ in the operator $\text{tr}(\Phi_1^I \Phi_2^J + \ldots)$ representing spins up and down. In coherent state basis $\langle n | \sigma^I | n \rangle = n^I$ where total $|n\rangle$ is a product of doublet coherent states at each node. We may thus view $U_r$ as “radial” coordinates directly corresponding to the two complex scalars $\Phi_1, \Phi_2$ on the SYM side, with $U^I \sigma^I U = n^I$ being the “classical” analog of $\langle n | \sigma^I | n \rangle = n^I$. The $U(1)$ phase $\alpha$ corresponds to rotating the chiral superfields by the same overall phase.

‡Note that $\sqrt{\lambda}$ is the effective string tension in (2.11) and so all classical string charges are proportional to $\sqrt{\lambda}$.
\[ J = -\int_0^{2\pi} \frac{d\sigma}{2\pi} \sqrt{-g} g^{0p} D_p \alpha \, , \quad S^i = -\int_0^{2\pi} \frac{d\sigma}{2\pi} \sqrt{-g} g^{0p} q_p^i \, , \] (2.16)

where the local current \( q_p^i \) is
\[ q_p^i \equiv D_p \alpha n_i + \frac{1}{2} \epsilon_{ijk} n_j \partial_p n_k \, . \] (2.17)

Given a generic string configuration, one can apply a global rotation to put \( S_i \) in the “canonical” form \((0, 0, S_3)\), where \( S_3 \) will then correspond to the difference \( J_1 - J_2 \) of the two \( S^3 \) spins whose sum is \( J \) (the corresponding angular coordinate is \( \beta \) in (2.4)).

Note that the effective coupling constant \( \tilde{\lambda} \) in (1.1) is directly related to the (rescaled) charge \( J \) in (2.16)
\[ \tilde{\lambda} \equiv \frac{\lambda}{J^2} = \frac{1}{\sqrt{\lambda}} \, , \quad \text{i.e.} \quad J = \frac{1}{\sqrt{\lambda}} . \] (2.18)

Thus expansion in powers of \( \frac{1}{J^2} \) is the same as expansion in powers of \( \tilde{\lambda} \).

### 2.2 Conformal gauge choice

While the final expressions for the physical quantities for specific string solutions like the energy as a function of spins should not of course depend on a particular choice of reparametrization gauge, the simplicity of the “off-shell” correspondence between the string and spin chain 2-d actions is sensitive to a choice of world-sheet coordinates. It turns out that the simplest conformal gauge choice fails to be the adequate one beyond the leading order in \( \tilde{\lambda} \) expansion: one needs to choose instead a non-trivial (“non-diagonal”) gauge. This may be viewed as a technical complication, but it actually highlights the importance of a suitable reparametrization gauge choice in understanding how string action originates from gauge theory.

Nevertheless, it is still instructive to start with the discussion of the standard conformal gauge \( \sqrt{-g} g^{ab} = \eta^{ab} = \text{diag}(-1, +1) \) (this gauge was used in [21]). In conformal gauge \( t \) satisfies the free equation of motion, and we can fix the residual conformal diffeomorphism freedom by the usual condition
\[ t = \kappa \tau \, , \]
relating the world-sheet and the target space energies. Then (2.15) implies
\[ E = \kappa \, , \quad J = \int_0^{2\pi} \frac{d\sigma}{2\pi} D_0 \alpha . \] (2.19)

The problem with the conformal gauge choice turns out to be related to the fact that here the energy is homogeneously distributed along \( \sigma \) while the angular momentum \( J \) is not, while the situation on the spin chain side is just the opposite. As a result, the comparison of the expressions for the actions and the energies becomes complicated;
in particular, the expression for $E$ in terms of $J$ contains “non-local” contributions 
(given by multiple integrals over $\sigma$).

The equations of motion for $\alpha$ and $n_i$ are found to be

$$\partial^p D_p \alpha = 0 , \quad \partial^p q^i_p = 0 , \quad (2.20)$$

where the latter may be explicitly written as

$$D^p \alpha \epsilon_{ijk} n_j \partial_p n_k - \frac{1}{2} (\partial^p \partial_p n_i)_\perp = 0 , \quad (2.21)$$

$$(m_i)_\perp \equiv m_i - (n_k m_k) n_i . \quad (2.22)$$

Equivalently, (2.21) may be written as

$$D^p \alpha \partial_p n_i = \frac{1}{2} \epsilon_{ijk} n_j \partial^p \partial_p n_k . \quad (2.23)$$

The conformal gauge constraints

$$(\partial_0 t)^2 + (\partial_1 t)^2 = (q_0^i)^2 + (q_1^i)^2 , \quad \partial_0 t \partial_1 t = q_0^i q_1^i \quad (2.24)$$

are easily solved expressing $D_p \alpha$ in terms of $n_i$ * $D_\pm \alpha = \pm \sqrt{\kappa^2 - \frac{1}{4}(\partial_\pm n_i)^2}$, where

$$D_0 \alpha = \frac{1}{2} \left[ \sqrt{\kappa^2 - \frac{1}{4}(\partial_+ n_i)^2} + \sqrt{\kappa^2 - \frac{1}{4}(\partial_- n_i)^2} \right] = \kappa - \frac{1}{8\kappa} (\partial_1 n_i)^2 + \ldots , \quad (2.25)$$

$$D_1 \alpha = \frac{1}{2} \left[ \sqrt{\kappa^2 - \frac{1}{4}(\partial_+ n_i)^2} - \sqrt{\kappa^2 - \frac{1}{4}(\partial_- n_i)^2} \right] = -\frac{1}{4\kappa} \partial_1 n_i \partial_0 n_i + \ldots . \quad (2.26)$$

We expanded in large $\kappa$ which is related to expansion in small $\lambda = \frac{\lambda}{\kappa^2}$. Indeed,

$$\mathcal{J} = \int_0^{2\pi} \frac{d\sigma}{2\pi} D_0 \alpha = \kappa - \frac{1}{8\kappa} \int_0^{2\pi} \frac{d\sigma}{2\pi} (\partial_1 n_i)^2 + O\left(\frac{1}{\kappa^2}\right) , \quad (2.27)$$

i.e. (see (2.18))

$$\bar{\lambda} = \frac{1}{\kappa^2} + \frac{1}{4\kappa^4} \int_0^{2\pi} \frac{d\sigma}{2\pi} (\partial_1 n_i)^2 + O\left(\frac{1}{\kappa^6}\right) . \quad (2.28)$$

Thus in the conformal gauge the natural expansion is in powers of $\frac{1}{\kappa}$, while on the spin chain side it is the expansion in powers of $\bar{\lambda}$. The two coincide at the leading order, but it is clear that beyond the leading order (and keeping $n_i$ general) the two expansions are related in an indirect way which is effectively non-local in $\sigma$.

*We choose particular signs in the solution to ensure regularity of large $\kappa$ or large $\mathcal{J}$ expansion.
Eliminating $D_p \alpha$ from (2.23) using (2.25), (2.26) we get an equation for $n_i$.\footnote{One can check that the equation for $\alpha$ in (2.20) is then identically satisfied as a consequence of the equation for $n_i$.} Expanding in large $\kappa$ we get

$$\kappa \partial_0 n_i = \frac{1}{2} \epsilon_{ijk} n_j (\partial_i^2 - \partial_0^2)n_k + O\left(\frac{1}{\kappa}\right).$$

(2.29)

If we first assume that all derivatives stay finite at large $\kappa$ then the leading-order equation is simply $\partial_0 n_k = 0$. Then also $\partial_1 n_i \perp = 0$; or $\partial_1^2 n_i = -(\partial_1 n)^2 n_i$. Multiplying this by $\partial_1 n_i$ we get also $\text{tr} \partial_1 n_i = 0$, or $\partial_1^2 n_i = -\text{tr} \partial_1 n$. Multiplying this by $\partial_1 n_i$ we get also $\text{tr} \partial_1 n_i = 4m^2 = \text{const}$, and thus $n_i$ has $\cos 2m\sigma$ and $\sin 2m\sigma$ as its two non-zero components. Then $E = \sqrt{J^2 + m^2}$ and $S_i = \kappa \int_0^{2\pi} \frac{d\sigma}{2\pi} n_i = 0$. This corresponds to the simplest circular string solution with two equal angular momenta $J_1 = J_2$ (this is, in fact, an exact solution of the full string equations [6], see also below). Thus expanding in $\kappa$ in this way corresponds to expanding near this special circular solution.

Instead, it is more natural to follow [21] and assume that $\kappa \partial_0 n_i$ and $\partial_1 n_i$ stay finite in the limit, i.e. to rescale the time coordinate (cf. (1.10))

$$\tau \to \kappa \tau, \quad \kappa \partial_0 \to \partial_0, \quad \text{i.e.} \quad t \to \kappa^2 t \approx \tilde{\lambda}^{-1} t.$$ (2.30)

Here we used that $t = \kappa \tau$; note that this is the same rescaling of time $t$ that was needed in (1.17) or (1.6) to eliminate the $\tilde{\lambda}$ dependence. Then terms with more time derivatives will be suppressed by higher powers of $\frac{1}{\kappa}$. Observing that $(\partial_\pm n_i)^2$ in (2.25), (2.26) is now equal to $(\partial_1 n_i \pm \frac{1}{\kappa} \partial_0 n_i)^2$ we find that (2.29) becomes

$$\partial_0 n_i = \frac{1}{2} \epsilon_{ijk} n_j \partial_1^2 n_k + \frac{1}{\kappa^2} G_i^{(1)} + O\left(\frac{1}{\kappa^4}\right),$$

(2.31)

$$G_i^{(1)} = -\frac{1}{2} \epsilon_{ijk} n_j \partial_0^2 n_k - \frac{1}{4} (\partial_0 n \partial_1 n) \partial_1 n_i + \frac{1}{8} (\partial_1 n \partial_1 n) \partial_0 n_i.$$ (2.32)

The leading term thus takes the form of the LL equation linear in time derivative which is equivalent to (1.7). Solutions of this leading-order equation include [21] non-trivial folded and circular string configurations (which are large $\kappa$ limits of the corresponding exact solutions in [6, 14, 15]), i.e. this is a natural starting point of an expansion that should describe generic string states with large $J$.

Given the remarkable fact that this leading-order equation is linear in the time derivative, solving (2.31) perturbatively in $\frac{1}{\kappa}$ one is able to eliminate all time derivatives from the correction terms in favor of spatial derivatives and thus to convert (2.31) into a local equation with only spatial derivatives appearing on the r.h.s., i.e. into an equation of the type expected on the spin chain side (following from (1.8)).

It is important to stress that this “re-expansion” of the original $R \times S^3$ sigma model equations of motion effectively selects a subclass of solutions with large $J$ or large $\kappa$. That eq. (2.31) does not as a result describe all possible solutions of $R \times S^3$ sigma model is just as well since other solutions which do not carry large angular
momentum \( J \) are not dual to SYM operators from the \( SU(2) \) sector and thus should not be related to eigenstates of spin chain Hamiltonian.

Another remark is that this large \( \kappa \sim J \) conversion of the original second-order equations into a first-order one is similar to the usual large friction case or non-relativistic large mass limit. The role of large \( \kappa \) or \( J = (\sqrt{\lambda})^{-1} \) is indeed analogous to that of large light-cone momentum \( p^+ \) in other similar cases like BMN one where one expands near a point-like configuration. Again, the novelty of the present expansion is that it isolates a sector of non-trivial extended solitonic string states which are far from being point-like.

The leading term in (2.31) corresponds to the following action for \( n_i \)

\[
I = J \int d\tau \int_0^{2\pi} \frac{d\sigma}{2\pi} L, \quad L = C_0 - \frac{1}{8}(\partial_1 n_i)^2, \tag{2.33}
\]

where we took into account the above rescaling of \( \tau \) (\( C_0 \sim \partial_\tau n \) refers to the new \( \tau \)) to combine the string tension factor \( \sqrt{\lambda} \) with the \( \kappa \) factor coming from \( d\tau \) into \( J \) (which is the same as \( \sqrt{\lambda} \kappa \) to leading order in \( \frac{1}{\kappa} \)). This action is equivalent to (1.6). The explicit form of \( L \) in terms of independent angular coordinates in (2.9) is

\[
L = \cos 2\psi \partial_0 \beta - \frac{1}{2}[(\partial_1 \psi)^2 + \sin^2 2\psi (\partial_1 \beta)^2]. \tag{2.34}
\]

To see the emergence of the WZ term \( C_0 \) \(^\dagger\) directly at the level of the action we may “boost” \( \alpha \), i.e. introduce instead a new “light-cone” coordinate \( u \)

\[
u \equiv \alpha - t. \tag{2.35}
\]

Then the original sigma model Lagrangian may be written in conformal gauge as

\[
L = -\partial^\sigma t D_\rho u - \frac{1}{2}(D_\rho u)^2 - \frac{1}{8}(\partial_\rho n_i)^2. \tag{2.36}
\]

Using that \( t = \kappa \tau \) and dropping a total derivative term we get

\[
L = \kappa C_0 - \frac{1}{2}(D_\rho u)^2 - \frac{1}{8}(\partial_\rho n_i)^2, \tag{2.37}
\]

or, after the rescaling (2.30) of the time coordinate

\[
L = C_0 - \frac{1}{8}(\partial_1 n_i)^2 - \frac{1}{2}(D_1 u)^2 + \frac{1}{2\kappa^2}[(D_0 u)^2 + \frac{1}{4}(\partial_0 n_i)^2]. \tag{2.38}
\]

Observing that according to (2.25) (after the rescaling (2.30))

\[
D_0 u = D_0 \alpha - 1 = -\frac{1}{8}(\partial_1 n_i)^2 - \frac{1}{2\kappa^2} \left[ \frac{1}{64} (\partial_1 n_i)^4 + \frac{1}{4}(\partial_0 n_i)^2 \right] + O(\frac{1}{\kappa^4}), \tag{2.39}
\]

\(^\dagger\)Note that this is not the “usual” covariant sigma model WZ term which contains both \( \tau \) and \( \sigma \) derivatives: \( C_0 \) term is local in \( \sigma \) (we did not have a \( B_{mn} \)-term in the original sigma model action).
\[ D_1 u = D_1 \alpha = -\frac{1}{4 \kappa^2} \partial_0 n_i \partial_1 n_i + O\left(\frac{1}{\kappa^4}\right), \quad (2.40) \]

and assuming that we can use the constraints to eliminate \( u \) from the action (which, in general, requires a justification but here does lead to the correct result for the first subleading term) we get

\[ L = C_0 - \frac{1}{8}(\partial_1 n_i)^2 + \frac{1}{8 \kappa^2}([\partial_0 n_i]^2 + \frac{1}{16}(\partial_1 n_i)^4] + O\left(\frac{1}{\kappa^4}\right). \quad (2.41) \]

The final step is to make a field redefinition to eliminate the time derivative term from the correction. This amounts to the use of the leading-order equation in (2.31) implying

\[ (\partial_0 n_i)^2 = \frac{1}{4}[(\partial_1^2 n_i)^2 - (\partial_1 n_i)^4] + O\left(\frac{1}{\kappa^2}\right). \quad (2.42) \]

As a result, we get (in terms of redefined field \( n_i \))

\[ L = C_0 - \frac{1}{8}(\partial_1 n_i)^2 + \frac{1}{32 \kappa^2}[(\partial_1^2 n_i)^2 - \frac{3}{4}(\partial_1 n_i)^4] + O\left(\frac{1}{\kappa^4}\right). \quad (2.43) \]

We shall obtain the same action (2.43) from a more systematic approach of the next subsection, and later in Sections 3 and 4 we will check that the coefficients of the 4-derivative terms in (2.43) match the ones appearing in the effective action (1.8) of the spin chain. Similar procedure can be applied at higher orders, converting the original sigma model action into the first-time-derivative action of the type (1.8).

As already mentioned, the problem of the conformal gauge we used above is that here the expansion goes in powers of \( \frac{1}{\kappa^2} \) instead of powers of \( \frac{1}{\sqrt{J}} = \tilde{\lambda} \) and thus the comparison with the spin chain side is indirect beyond the leading order: converting the \( \frac{1}{\kappa^2} \) expansion into \( \frac{1}{\sqrt{J}} \) one, using the expression for \( J \) (2.27), brings in non-local terms given by powers of integrals over \( \sigma \). For that reason we shall now describe an alternative gauge fixing procedure which is better suited for establishing the correspondence between the spin chain and the string sigma model effective actions.

### 2.3 Non-diagonal “uniform” gauge

Let us start with rewriting generic string sigma model action in first-order form and then discuss gauge fixing. Given

\[ L = -\frac{1}{2} \sqrt{-g} g^{\mu \nu} G_{\mu \nu}(x) \partial_\mu x^\mu \partial_\nu x^\nu \quad (2.44) \]

and introducing the momenta

\[ p_\mu = -G_{\mu \nu}(A \partial_0 x^\mu + B \partial_1 x^\mu), \quad A \equiv \sqrt{-g} g^{00}, \quad B \equiv \sqrt{-g} g^{01}, \quad (2.45) \]

we can rewrite \( L \) in the first-order form with respect to the time derivatives

\[ L = p_\mu \partial_0 x^\mu + \frac{1}{2} A^{-1} [G^{\mu \nu} p_\mu p_\nu + G_{\mu \nu}(x) \partial_1 x^\mu \partial_1 x^\nu] + B A^{-1} p_\mu \partial_1 x^\mu. \quad (2.46) \]
Here $A^{-1}$ and $BA^{-1}$ play the role of Lagrange multipliers for the constraints.

The action for (2.46) is reparametrization invariant, and then as in [3] (in flat space) and in [33] (in $AdS$ space) one may fix a gauge, e.g., on a combination of coordinates $x^\mu$ and momenta $p_\mu$. The result is of course equivalent to the corresponding gauge fixing directly in the Polyakov’s action (cf. [33, 34]) but starting with (2.46) may have some conceptual advantages since, in the presence of isometries, some momenta are related to densities of conserved currents.

In the case of our interest $L$ in (2.44) is given by (2.12) and so its first-order version (2.46) has the following explicit form

$$L = p_t \partial_0 t + p_\alpha D_0 \alpha + p_i \partial_0 n_i$$

$$+ \frac{1}{2} A^{-1}[- p_t^2 + p_\alpha^2 + 4 p_i^2 + (\partial_1 t)^2 + (D_1 \alpha)^2 + \frac{1}{4} (\partial_1 n)^2]$$

$$+ BA^{-1}(p_t \partial_1 t + p_\alpha D_1 \alpha + p_i \partial_1 n_i) .$$

Here we have chosen to couple $p_\alpha$ to $D_0 \alpha$ and not to $\partial_0 \alpha$. The equations of motion for $p_t, p_\alpha$ are equivalent to the definitions of the momenta in terms of $\partial_0 x^\mu$ and $A, B$ in (2.45),

$$p_t = A \partial_0 t + B \partial_1 t , \quad p_\alpha = -AD_0 \alpha - BD_1 \alpha .$$

Note also that we have defined $p_i$ so that they are different from the usual canonical momenta for $n_i$: $p_i$ does not include the contribution of the $\partial_0 n_i$-dependence of $C_0$ in $D_q \alpha$, i.e.

$$p_i = -\frac{1}{4} (A \partial_0 n_i + B \partial_1 n_i) .$$

The equations for the metric components $A$ and $B$ give the constraints

$$- p_t^2 + p_\alpha^2 + 4 p_i^2 + (\partial_1 t)^2 + (D_1 \alpha)^2 + \frac{1}{4} (\partial_1 n)^2 = 0 ,$$

$$p_t \partial_1 t + p_\alpha D_1 \alpha + p_i \partial_1 n_i = 0 .$$

The equations for $t, \alpha$ and $n_i$ are found to be

$$\partial_0 p_t + \partial_1 [A^{-1}(\partial_1 t + Bp_t)] = 0 ,$$

$$\partial_0 p_\alpha + \partial_1 [A^{-1}(D_1 \alpha + Bp_\alpha)] = 0 ,$$

$$-\frac{1}{2} \epsilon_{ijk} n_j [p_\alpha \partial_0 n_k + A^{-1}(D_1 \alpha + Bp_\alpha) \partial_1 n_k] + \left( \partial_0 p_t + \partial_1 [A^{-1}(\frac{1}{4} \partial_1 n_i + Bp_i)] \right)_\perp = 0 .$$

$n_i^2 = 1$ implies that $n_i \partial_0 n_i = 0, p_i n_i = 0$ and that the variation over $n_i$ should be orthogonal to $n_i$. The first $\epsilon_{ijk}$-term in (2.54) comes from the variation of $C_0$ in $D_q \alpha$ (an additional gradient variation term is proportional to the equation (2.53) for $\alpha$ and thus is ignored).
The conserved charges corresponding to the invariances under translation in time, translation in \( \alpha \) and \( O(3) \) rotations (\( \delta n_i = \epsilon_{ijk}\beta_j n_k \)) of \( n_i \) are (see (2.15), (2.16), (2.17))

\[
\mathcal{E} \equiv P^t = \int_0^{2\pi} \frac{d\sigma}{2\pi} \mathcal{H} , \quad \mathcal{J} = \int_0^{2\pi} \frac{d\sigma}{2\pi} p_\alpha , \quad \mathcal{S}_i = \int_0^{2\pi} \frac{d\sigma}{2\pi} q_i ,
\]

(2.55)

\[
\mathcal{H} \equiv -p_t , \quad q_i = p_\alpha n_i + 2\epsilon_{ijk} n_j p_k .
\]

(2.56)

Let us now choose a gauge. The gauge used in the previous section was the orthogonal one \( A = 1, B = 0 \) and \( t = \kappa \tau \). Having in mind comparison with the spin chain it is natural to request that translations in time in the target space and on the world sheet should indeed be related. Also, we should ensure that the angular momentum \( \mathcal{J} \) is homogeneously distributed along the string, i.e. let us require*

\[
(i) \quad t = \tau , \quad (ii) \quad p_\alpha = \mathcal{J} = \text{const} .
\]

(2.57)

Here we have chosen to set a proportionality coefficient between \( t \) and \( \tau \) to 1 but later we will need to rescale \( \tau \) as in the above conformal gauge discussion. In this case (2.48) and (2.49) imply

\[
A = -\mathcal{H} , \quad \mathcal{J} = \mathcal{H} D_0 \alpha - BD_1 \alpha , \quad p_i = \frac{1}{4}(\mathcal{H}\partial_0 n_i - B\partial_1 n_i) ,
\]

(2.58)

and the constraints (2.51), (2.50) give

\[
D_1 \alpha = -\frac{1}{\mathcal{J}} p_i \partial_1 n_i ,
\]

(2.59)

\[
\mathcal{H} \equiv \mathcal{H}(n,p) = \sqrt{\mathcal{J}^2 + 4p_i^2 + \frac{1}{\mathcal{J}^2} (p_i \partial_1 n_i)^2 + \frac{1}{4}(\partial_1 n)^2} .
\]

(2.60)

Using the constraints, the resulting “reduced” Lagrangian for the independent variables \( n_i \) and \( p_i \) is found from (2.47) to be \( L = \mathcal{J} D_0 \alpha + p_i \partial_0 n_i - \mathcal{H} \). Omitting the total derivative \( \mathcal{J} \partial_0 \alpha \) term, we get

\[
L(n,p) = \mathcal{J} C_0 + p_i \partial_0 n_i - \mathcal{H}(n,p) ,
\]

(2.61)

where \( C_0 \) is again the same WZ term as in (2.37) or (2.13). Thus both the first and the second terms here depend on \( \partial_0 n_i \), reflecting the fact that \( p_i \) was defined not to include a contribution from \( C_0 \) in \( D_0 \alpha \).

Next, we should check that this Lagrangian (2.61) does indeed lead to the correct equations for \( n_i, p_i \), i.e. that here it is legitimate to use the gauge conditions and the

*This is another example illustrating that in curved space it is often natural to use a non-conformal gauge. In fact, use of such a more general gauge (or a special choice of \( \tau \) and \( \sigma \) variables) may be “required” by duality to gauge theory. Again, it is true of course that all final physical observables (like values of energies on particular solutions) should be gauge-independent. But an “off-shell” comparison of string and gauge theory may be greatly facilitated by the right choice of world-sheet coordinates.
constraints directly in the action. With the above gauge choice the general equations (2.52) – (2.54) become

\[ \partial_0 H - \partial_1 B = 0 , \quad \partial_1 [H^{-1}(B - \frac{1}{\mathcal{J}^2} p_i \partial_1 n_i)] = 0 , \]  

\[ -\frac{1}{2} \mathcal{J} \epsilon_{ijk} n_j \left[ \partial_0 n_k - H^{-1}(B - \frac{1}{\mathcal{J}^2} p_i \partial_1 n_i) \partial_1 n_k \right] + \left( \partial_0 p_i - \partial_1 [H^{-1}(\frac{1}{4} \partial_1 n_i + B p_i)] \right)_\perp = 0 , \]  

where according to (2.58) \( p_i = \frac{1}{4} (\mathcal{H} \partial_0 n_i - B \partial_1 n_i) \). We are to compare (2.54), (2.49) with the equations for \( n_i \) and \( p_i \) that follow directly from (2.61)

\[ \partial_0 n_i - H^{-1}[4p_i + \frac{1}{\mathcal{J}^2} (p \partial_1 n) \partial_1 n_i] = 0 , \]  

\[ -\frac{1}{2} \mathcal{J} \epsilon_{ijk} n_j \partial_0 n_k + \left( \partial_0 p_i - \partial_1 [H^{-1}(\frac{1}{4} \partial_1 n_i + \frac{1}{\mathcal{J}^2} (p \partial_1 n) p_i)] \right)_\perp = 0 . \]  

We conclude that they agree provided

\[ B = \frac{1}{\mathcal{J}^2} (p \partial_1 n) , \quad \text{i.e.} \quad D_1 \alpha = -\mathcal{J} B . \]  

We are still to check that this does not contradict the first equation in (2.62), i.e.

\[ \partial_0 H = \frac{1}{\mathcal{J}^2} \partial_1 (p \partial_1 n) . \]  

This, indeed, follows from the equations of motion (2.64), (2.65) for \( p_i \) and \( n_i \).† Thus our procedure is consistent: the extra condition (2.66) is a solution of our equations. We could, in fact, impose (2.66) and \( p_\alpha = \mathcal{J} \) in (2.57) as two reparametrization gauge conditions; then an extra choice of \( t = \tau \) in (2.57) would correspond to fixing a residual reparametrization freedom, as in the conformal gauge.

Since \( p_i \) enters (2.61) only algebraically, we can now solve for it and get a (non-polynomial) action that depends only on \( n_i \) and its first derivatives \( \partial_0 n_i \) and \( \partial_1 n_i \). We find from (2.64)

\[ p_i = \frac{1}{4} \mathcal{H} [\partial_0 n_i - \left( \frac{\partial_0 n \partial_1 n_i}{4 \mathcal{J}^2 + (\partial_1 n)^2} \partial_1 n_i \right) ] , \quad p_i n_i = 0 , \]  

† In general, given the Lagrangian which is first-order in \( \partial_0 n_i \) one finds that (omitting \( \perp \) sign)

\[ \partial_0 p_i - \frac{1}{2} \mathcal{J} \epsilon_{ijk} n_j \partial_0 n_k = \partial_1 \frac{\partial \mathcal{H}}{\partial \partial_1 n_i} , \quad \partial_0 n_i = \frac{\partial \mathcal{H}}{\partial p_i} , \]  

from where it follows that (WZ term does not contribute)

\[ \partial_0 \mathcal{H} = \frac{\partial \mathcal{H}}{\partial \partial_1 n} \partial_0 \partial_1 n + \frac{\partial \mathcal{H}}{\partial p} \partial_0 p = \left( - \partial_1 \frac{\partial \mathcal{H}}{\partial \partial_1 n} \partial_0 n + \frac{\partial \mathcal{H}}{\partial p} \partial_0 p \right) + \partial_1 \left( \frac{\partial \mathcal{H}}{\partial \partial_1 n} \partial_0 n \right) . \]  

The first bracket vanishes on the equations of motion, so \( \mathcal{H} \) is conserved. Computing \( \frac{\partial \mathcal{H}}{\partial \partial_1 n} \partial_0 n \) using (2.63) one finds indeed that it is equal to \( B \), i.e. to \( \frac{1}{\mathcal{J}^2} (p \partial_1 n) \).
\[ p_i \partial_1 n_i = \frac{1}{4} \mathcal{H} \frac{(\partial_0 n \partial_1 n)}{1 + \frac{1}{4J^2} (\partial_1 n)^2} = J^2 B . \] (2.69)

Substituting \( p_i \) into \( \mathcal{H} \) gives

\[ \mathcal{H} = J \frac{1 + \frac{1}{4J^2} (\partial_1 n)^2}{\sqrt{[1 + \frac{1}{4J^2} (\partial_1 n)^2][1 - \frac{1}{4} (\partial_0 n)^2] + \frac{1}{16J^2} (\partial_0 n \partial_1 n)^2}} , \] (2.70)

and eliminating \( p_i \) from the Lagrangian (2.61), we get\(^\dagger\)

\[ L(n) = JC_0 - \mathcal{H}_n , \quad \mathcal{H}_n \equiv \mathcal{H} - p_i \partial_0 n_i , \] (2.71)

\[ \mathcal{H}_n = J \sqrt{[1 + \frac{1}{4J^2} (\partial_1 n)^2][1 - \frac{1}{4} (\partial_0 n)^2] + \frac{1}{16J^2} (\partial_0 n \partial_1 n)^2} , \] (2.72)

Equivalently,

\[ \mathcal{H}_n = \sqrt{-\det\gamma_{pq}} , \quad \gamma_{pq} = \tilde{\eta}_{pq} + \frac{1}{4} \partial_p n_i \partial_q n_i , \quad \tilde{\eta}_{ab} \equiv \text{diag}(-1, J^2) . \] (2.73)

Remarkably, if not for the WZ term \( C_0 \), the expression for (2.71) is reminiscent of the Nambu Lagrangian in a static gauge (suggesting that there may be a more direct way of deriving this action).

We have thus managed to recast the original sigma model action in the form resembling (1.8). The remaining steps are:

(i) to define a consistent \( \frac{1}{J} \) expansion, and then

(ii) to eliminate time derivatives in \( \mathcal{H}_n \) by field redefinitions order by order in \( \frac{1}{J} \).

To define the large \( J \) expansion near the same “classical ferromagnet” limit as in the conformal gauge (cf. (2.43)) we need to rescale (cf. (1.10), (2.30))

\[ \tau \to J^2 \tau = \tilde{\lambda}^{-1} \tau , \quad \text{i.e.} \quad t \to \tilde{\lambda}^{-1} t , \quad \partial_0 \to \frac{1}{J^2} \partial_0 , \] (2.74)

thus getting the string action in the form

\[ I = J \int d\tau \int_0^{2\pi} \frac{d\sigma}{2\pi} L , \quad L = C_0 - \bar{\mathcal{H}} , \quad \bar{\mathcal{H}} \equiv J \mathcal{H}_n \] (2.75)

\[ \bar{\mathcal{H}} = J^2 \sqrt{[1 + \frac{1}{4J^2} (\partial_1 n)^2][1 - \frac{1}{4} (\partial_0 n)^2] + \frac{1}{16J^2} (\partial_0 n \partial_1 n)^2} , \] (2.76)

i.e. (omitting constant term in \( \bar{\mathcal{H}} \))

\[ L = C_0 - \frac{1}{8} (\partial_1 n_i)^2 + \frac{1}{8J^2} [(\partial_0 n_i)^2 + \frac{1}{16} (\partial_1 n_i)^4] \]

\( \dagger \)We also find that \( D_0 \alpha = \frac{1 - \frac{1}{4} (\partial_0 n)^2}{\sqrt{1 + \frac{1}{4J^2} (\partial_1 n)^2 - \frac{1}{4} (\partial_0 n)^2 + \frac{1}{16J^2} [(\partial_0 n \partial_1 n)^2 - (\partial_1 n)^2(\partial_0 n)]}} \).
The second and fourth derivative terms here are the same as in the conformal gauge expression (2.31), and thus the equations of motion to this order also have the same form (2.31), (2.32).

Note that it is the energy density (2.70), or, in terms of redefined time derivative

\[ \mathcal{H} = \mathcal{J} \frac{1 + \frac{1}{4\mathcal{J}^2}(\partial_1 n)^2}{\sqrt{1 + \frac{1}{4\mathcal{J}^2}(\partial_1 n)^2}[1 - \frac{1}{4\mathcal{J}^2}(\partial_0 n)^2]} + \frac{1}{16\mathcal{J}^4}(\partial_0 n\partial_1 n)^2 \]  

that is conserved on the equations of motion. It is only after a field redefinition done in \( \mathcal{J} \) part of \( L \) in (2.74), i.e. that puts \( L \) into the form

\[ L = C_0(\tilde{n}) - H(\partial_1 \tilde{n}, \partial_1^2 \tilde{n}, \partial_1^3 \tilde{n}, \ldots) \]  

that \( H \) should be conserved.\(^6\) On general grounds, since (the integral of) \( \mathcal{H} \) generates translations in \( t \) and (the integral of) \( H \) generates translations in \( \tau \), and \( t \sim \tau \) according to the above choice, the two must be equivalent on-shell (after the same field redefinition done in \( \mathcal{H} \), modulo a total \( \partial_t \) derivative), i.e.

\[ [\mathcal{H}(\partial_0 n, \partial_1 n)]_{\text{field restr. on-shell}} = \mathcal{J}^{-1} H(\partial_1 \tilde{n}, \partial_1^2 \tilde{n}, \ldots) + \partial_1 F . \]  

This is what is required for correspondence with the spin chain results. In contrast to the conformal gauge case, the relation between the space-time and 2-d energies here does not involve non-local (multiple \( \sigma \) integral) terms.

The form of field redefinition which is needed to put \( L \) in (2.75) into the form (2.80) is suggested by the comparison of the \( O(3) \) conserved charges which should also match: for (2.80) we get instead of the expression in (2.15), (2.16)

\[ S_i = \mathcal{J} \int_0^{2\pi} \frac{d\sigma}{2\pi} \tilde{n}_i, \]  

implying that the required field redefinition should be (we rescale \( q_i \) in (2.56) by factor of \( \mathcal{J}^4 \))

\[ \tilde{n}_i = q_i(n, \partial_0 n, \partial_1 n) + \partial_1 f = n_i + O\left(\frac{1}{\mathcal{J}^2}\right), \]  

\(^6\)Indeed, if \( H \) depends only on spatial derivatives the equations of motion following from take the form \( \sum_i \epsilon_{ijk} \partial_0 n_j \partial_0 n_k = (\partial_1 [\frac{\partial H}{\partial \partial_1 n_i} + \partial_1 [\frac{\partial H}{\partial \partial_1^2 n_i} + \ldots ]]) \perp = K_i \), and then \( K_i \partial_1 n_i = 0 \) so that

\[ \partial_0 H = \partial_0 \partial_1 n_i \frac{\partial H}{\partial \partial_1 n_i} + \partial_0 \partial_1^2 n_i \frac{\partial H}{\partial \partial_1^2 n_i} + \ldots = \partial_1 (\partial_0 n_i \frac{\partial H}{\partial \partial_1 n_i} + \partial_0 \partial_1 n_i \frac{\partial H}{\partial \partial_1^2 n_i} + \ldots) , \]  

and the integral of \( \partial_0 H \) vanishes.

\(^4\)Once the action is put into the form (2.80), the density of the \( O(3) \) rotational current is determined simply by the WZ term \( C_0 \), i.e. it should be equal (up to a total derivative) to \( \tilde{n}_i \).
where a total derivative term $\partial_1 f$ may be needed to ensure that $\tilde{n}_i^2 = 1$.

Let us demonstrate how this works to the two leading orders in large $J$ expansion (see also Appendix A). Starting with (2.77) let us do a field redefinition that converts the $(\partial_0 n_i)^2$ term into its “on-shell” value (2.42), i.e. $[ (\partial_0 n_i)^2 ]_{\text{on-shell}} = \frac{1}{4}((\partial_1^2 n_i)_\perp)^2 = \frac{1}{4}((\partial_1^2 n_i)_\perp - (\partial_1 n_i)^4]$. Observing that the variation of the first two leading terms in (2.77) is proportional to the leading-order equations (omitting total spatial derivative)

$$\delta[C_0 - \frac{1}{8}(\partial_1 n_i)^2] = \left[ \frac{1}{2} \epsilon_{ijk} n_j \partial_0 n_k + \frac{1}{4}((\partial_1 n_i)_\perp)^2 \right] \delta n_i,$$

we conclude that we need

$$\delta n_i = \frac{1}{2J^2} \left[ -\frac{1}{2} \epsilon_{ijk} n_j \partial_0 n_k + \frac{1}{4}(\partial_1^2 n_i)_\perp \right], \quad [\delta n_i]_{\text{on-shell}} = \frac{1}{4J^2} (\partial_1^2 n_i)_\perp \quad (2.84)$$

This redefinition effectively replaces $\frac{1}{8J^2}(\partial_0 n_i)^2$ term with $\frac{1}{32J^2}((\partial_1^2 n_i)_\perp)^2$, so that in terms of redefined field $\tilde{n}_i = n_i - \delta n_i$ we get (2.81) with (omitting tilda on $n_i$)

$$H = H_0 + H_1 + H_2 + O(\tilde{\lambda}^3), \quad H_0 = \frac{1}{8}(\partial_1 n_i)^2, \quad (2.85)$$

$$H_1 = -\frac{\tilde{\lambda}}{32} \left[ ((\partial_1^2 n_i)_\perp - \frac{3}{4}(\partial_1 n_i)^4 \right], \quad (2.86)$$

where we used (2.41) to express $J$ in terms of $\tilde{\lambda}$. This happens to be the same result as in the conformal gauge (2.43) with $\kappa \rightarrow J$. Since the two actions – (2.77) and (2.80) – are related by a field redefinition, they have equivalent equations of motion, i.e. (2.81) with $H$ given by (2.80) is guaranteed to reproduce the same string solutions.

The field redefinition we have used is indeed equivalent, on-shell, and to the leading order in $\frac{1}{J}$, to the one in (2.83) that transforms $n_i$ into the charge density $q_i$

$$n_i = q_i - \frac{1}{2J^2} \epsilon_{ijk} q_j \partial_0 q_k + O(\frac{1}{J^4}) \rightarrow q_i + \frac{1}{4J^2} (\partial_1^2 q_i)_\perp + O(\frac{1}{J^4}), \quad \tilde{n}_i \equiv q_i \quad (2.87)$$

Let us now show that the energy density (2.48) becomes indeed equal (up to an overall factor of $J$) to $H$ in (2.86) after the above field redefinition and evaluation of the result on the equations of motion. It is important to stress that while the use of equations of motion in the action is equivalent to field redefinitions (at least, to leading order), this is not so in the energy: field redefinition and evaluation on-shell are two different steps. Notice that compared to $\tilde{H}$ in (2.77) the sign of the $(\partial_0 n_i)^2$ term in $\tilde{H}$ is the opposite, so if we would simply evaluate $\tilde{H}$ on-shell we would not match $H$ in (2.80). Instead, we are instructed to do the field redefinition (2.84) first, and it gives $(\partial_1 n_i)^2 \rightarrow -\partial_1^2 n_i \delta n_i$. Evaluating the result on the leading-order equations of motion we end up with (up to a total derivative, subtracting the constant term $J$, rescaling by factor of $J$ and omitting tilda on $n_i$)

$$J(\tilde{H} - J) = \frac{1}{8}(\partial_1 n_i)^2 - \frac{1}{8J^2} [2\partial_1^2 n_i((\partial_1^2 n_i)_\perp + (\partial_0 n_i)^2 - \frac{1}{16}(\partial_1 n_i)^4 + O(\frac{1}{J^4})$$

$$= \frac{1}{8}(\partial_1 n_i)^2 - \frac{\tilde{\lambda}}{32} [((\partial_1^2 n_i)_\perp - \frac{3}{4}(\partial_1 n_i)^4 + O(\tilde{\lambda}^2). \quad (2.88)$$
The role of the field redefinition is thus to invert the sign of the \( (\partial_0 n)^2 \) term in \( \mathcal{H} \); then, upon evaluation on-shell, \( \mathcal{H} \) matches \( H \) in (2.80).

It is clear that the same field redefinition procedure should apply at higher orders of expansion in \( \frac{1}{J} \). We demonstrate this explicitly at the next \( \frac{1}{J^2} \) order in Appendix A. Here we will just quote the final result for the corresponding term in \( H \) in (2.80), (2.85) (again omitting tilde on \( n_i \) and using (2.18))\
\[
H_2 = \frac{\lambda^2}{64} \left[ (\partial_1^2 n)^2 - \frac{7}{4} (\partial_1 n)^2 (\partial_1^2 n)^2 - \frac{25}{2} (\partial_1 n \partial_1^2 n)^2 + \frac{13}{16} (\partial_1 n)^6 \right].
\]

(2.89)

Below we shall compare these results with the corresponding expressions on the spin chain side (1.8).

It would be interesting to determine terms in \( H \), to a given order in an expansion in powers of \( n_i \), but to all orders in spatial derivatives. This is possible to do for all quadratic terms by expanding near the “vacuum” configuration \( n_i = (0, 0, 1) \) which essentially corresponds to the BMN limit (small fluctuations above the BPS vacuum). Then starting with (2.76) where all higher than quadratic terms should be omitted and solving for the frequency in terms of the spatial momentum one finds that all quadratic terms in the effective Hamiltonian can be written as follows (modulo integration by parts)\[
H = \frac{1}{4} \lambda^{-1} n_i \left( \sqrt{1 - \lambda \partial_1^2} - 1 \right) n_i + O(n^4).
\]

(2.90)

The expansion of the BMN square root here is in agreement with the coefficients of the quadratic terms in (2.80), (2.85), (2.89).

2.4 Some special solutions

Let us mention how some known two-spin solutions of the \( R \times S^3 \) sigma model fit into the above discussion. A class of folded and circular string solutions with non-constant radii \[14, 15\] (having \( \phi_k = w_k \tau, \psi_\sigma = \psi(\sigma) \) in (2.2)) have \( n_i \) (1.2) satisfying\[
\partial_0 n_i \partial_1 n_i = 0.
\]

(2.91)

As follows from (2.70), (2.78) in this case**\[
\mathcal{H} = \mathcal{J} \left[ \frac{1 + \frac{1}{4 \mathcal{J}^2} (\partial_1 n)^2}{1 - \frac{1}{4 \mathcal{J}^2} (\partial_0 n)^2} \right], \quad \bar{\mathcal{H}} = \mathcal{J}^2 \sqrt{\left[ 1 + \frac{1}{4 \mathcal{J}^2} (\partial_1 n)^2 \right] \left[ 1 - \frac{1}{4 \mathcal{J}^2} (\partial_0 n)^2 \right]}.
\]

(2.92)

**The same expression for the 6-derivative term in \( H \) was found by A. Dymarsky and I. Klebanov by reconstructing the \( C_0 - H \) Lagrangian from the condition that it correctly reproduces the energies of the folded \[14\] and circular \[16\] solutions expanded to the given order in \( \lambda \).**

**Here \( p_i = \frac{1}{\mathcal{J}} \mathcal{J} \partial_0 n_i, p_i \partial_0 n_i = 0 \) and \( B \) and \( D_0 \alpha \) vanish so the “uniform” gauge is diagonal (see (2.49), (2.66)). Also, \( D_0 \alpha \) is conserved, i.e. it depends only on \( \sigma \), so the “uniform” gauge may be reached from the conformal gauge by an additional redefinition of \( \sigma \).**
Another class of solutions are circular solutions with constant radii \[16\] for which the angles in (2.2) are \(\psi = \psi_0 = \text{const}, \varphi_k = m_k \sigma + w_k \tau\), (i.e. \(\beta = \frac{1}{2} (m_1 - m_2) \sigma + \frac{1}{2} (w_1 - w_2) \tau\)), where (in conformal gauge) \(w_k = \sqrt{m_k^2 + \nu^2}, \quad \mathcal{E}^2 = 2(w_1 J_1 + w_2 J_2) - \nu^2\) and \(\nu\) is a solution of \(\frac{2}{w_1} + \frac{2}{w_2} = 1\), \(m_1 J_1 + m_2 J_2 = 0\). These are homogeneous solutions: all invariants built out of \(n_i\), in particular, \(\partial_0 n_i \partial_1 n_i\) are constant.

In the special case of circular solution with equal spins \[6\] (\(J_1 = J_2 = \frac{1}{2} J\), \(m_1 = -m_2 = m\)) one has \(\psi_0 = \frac{\pi}{4}, \beta = m \sigma, n_i = (\cos 2m \sigma, \sin 2m \sigma, 0)\) and \(\partial_0 n_i = 0\). This is, in fact, the general static solution of the leading-order LL equations (1.7) or (2.31)

\[
\partial_0 n_i = 0, \quad (\partial^2 n_i)_\perp = 0, \quad (\partial_1 n)^2 = \text{const} = 4m^2,
\]

and is also an exact solution of the full system to all orders in \(\frac{1}{J}\). Here we get \(S_i = 0\) and

\[
\mathcal{H} = J^{-1} \mathcal{H} = \sqrt{J^2 + \frac{1}{4}(\partial_1 n)^2} = \sqrt{J^2 + m^2}.
\]

It is easy to check that eqs. (2.83), (2.84), (2.89) are indeed in agreement with the expansion of (2.94). We shall also check the second-order expression (2.86) against the energy of the folded string solution \[14\] in section 5.

### 3 Expectation value of dilatation operator in coherent state and “naive” continuum limit

Let us now consider the higher order (higher-loop) corrections (1.9) on the SYM, i.e. the spin chain, side. In general, one is supposed to find eigenvalues of the SYM dilatation operator and compare them to the AdS\(_5\) energy of the corresponding string states. In the large \(J\) limit (1.1), this problem happened to be essentially semiclassical at the leading order in \(\lambda\) (cf. (1.6)). One may expect that the same may be true also at higher loop orders. In this case to compare to string theory it would be sufficient to know the analogue of the action (1.3), (1.6) in the case when higher order corrections are included in the spin chain Hamiltonian (1.3). Our first task, therefore, will be to compute the action (1.3) that appears in the coherent state path integral of the quantum spin chain with the Hamiltonian given by the 3-loop SYM dilatation operator in the \(SU(2)\) sector. We will then address the issue of taking continuum limit and whether one is actually able to ignore quantum corrections beyond the leading (one-loop) order. We will describe how to consistently include quantum corrections in the next section.

The one [13], two [26] and three [26, 27, 28] loop dilatation operator of the \(\mathcal{N}=4\) SYM theory in the \(SU(2)\) (2 chiral scalar operator) sector has the form

\[
D = \sum_{r=0}^{\infty} \frac{\lambda^r}{(4\pi)^2r} D_{2r}, \quad D_{2r} = \sum_{a=1}^{J} D_{2r}(a),
\]

\[
D_0 = I, \quad D_2 = 2(I - P_{a,a+1}),
\]

and
\[ \mathcal{D}_4 = -8I + 12P_{a,a+1} - 2(P_{a,a+1}P_{a+1,a+2} + P_{a+1,a+2}P_{a,a+1}) , \quad (3.3) \]

\[ \mathcal{D}_6 = 60I - 104P_{a,a+1} + 24(P_{a,a+1}P_{a+1,a+2} + P_{a+1,a+2}P_{a,a+1}) \\
+ 4P_{a,a+1}P_{a+2,a+3} - 4(P_{a,a+1}P_{a+1,a+2} + P_{a+2,a+3}P_{a,a+1}) . \quad (3.4) \]

Here the projection operator is

\[ P_{a,b} = \frac{1}{2}(I + 4S_a \cdot S_b) , \quad S^i = \frac{1}{2}\sigma^i , \quad (3.5) \]

and \( \sigma^i \) are Pauli matrices. Since the 3-loop term \( \mathcal{D}_6 \) was not yet explicitly derived from SYM theory but was fixed using indirect (based, in particular, on integrability [26] or superconformal symmetry [25]) considerations, let us mention that \( \mathcal{D}_6 \) in (3.4) is the \( \alpha_1 = 0 \) member of a one-parameter class of operators in [27] (we set \( \alpha_2 \) in [27] to be zero) that all have similar properties like a consistent BMN limit:

\[ \mathcal{D}_6(\alpha_1) = (60 + 6\alpha_1)I - (104 + 14\alpha_1)P_{a,a+1} \\
+ (24 + 2\alpha_1)(P_{a,a+1}P_{a+1,a+2} + P_{a+1,a+2}P_{a,a+1}) + (4 + 6\alpha_1)P_{a,a+1}P_{a+2,a+3} \\
- 4(P_{a,a+1}P_{a,a+1}P_{a+2,a+3} + P_{a+2,a+3}P_{a,a+1}P_{a,a+1}) \\
- \alpha_1(P_{a,a+1}P_{a+2,a+3}P_{a+1,a+2} + P_{a+1,a+2}P_{a,a+1}P_{a+2,a+3}) . \quad (3.6) \]

Using that for each site index \( a \) one has \( \sigma^i_a \sigma^j_a = \delta^{ij} + i\epsilon^{ijk}\sigma^k_a \) it is straightforward to show that \( \mathcal{D}_2, \mathcal{D}_4, \mathcal{D}_6 \) can be written in the following equivalent forms\(^\dagger\)

\[ \mathcal{D}_2 = 2Q_{a,a+1} , \quad \mathcal{D}_4 = -2(4Q_{a,a+1} - Q_{a,a+2}) , \quad (3.7) \]

\[ \mathcal{D}_6 = 4(15Q_{a,a+1} - 6Q_{a,a+2} + Q_{a,a+3}) + 4(Q_{a,a+2}Q_{a+1,a+3} - Q_{a,a+3}Q_{a+1,a+2}) , \quad (3.8) \]

where

\[ Q_{a,b} \equiv I - P_{a,b} = \frac{1}{2}(1 - 4S_a \cdot S_b) = \frac{1}{2}(1 - \sigma_a \cdot \sigma_b) . \quad (3.9) \]

In general, for [3.6] we get instead of [3.8]

\[ \mathcal{D}_6(\alpha_1) = 4(15Q_{a,a+1} - 6Q_{a,a+2} + Q_{a,a+3}) \\
+ (4 - \alpha_1)(Q_{a,a+2}Q_{a+1,a+3} - Q_{a,a+3}Q_{a+1,a+2}) + 5\alpha_1Q_{a,a+1}Q_{a+2,a+3} . \quad (3.10) \]

Note that \( D \) [3.1] interpreted as a generalized spin chain Hamiltonian has a ferromagnetic nature: the coefficient multiplying each linear \( Q_{a,a+c} = \frac{1}{4}(\sigma_a - \sigma_{a+c})^2 \) term is positive (assuming that perturbation theory in \( \lambda < 1 \) applies).

\(^*\)We correct a misprint in eq.(8) of [27]: the sign of the 14\( \alpha_1 \) term there should be minus (we thank N. Beisert for pointing this out to us). Strangely, the \( \alpha_1 = 2 \) operator (apparently not consistent with the integrability) appears to be special in what concerns comparison to string theory, see discussion in [27].

\(^\dagger\)Here we use that for the case of a periodic spin chain which we are interested in one can shift the summation index and thus combine terms which are equivalent under the summation.
Consider now a coherent state $|n_a\rangle$ defined as

$$|\bar{n}_a\rangle = R(n_a)|\uparrow\rangle,$$

(3.11)

where $|\uparrow\rangle$ is a spin up state and $R(n_a)$ is the rotation

$$R(n) = e^{-i\phi S_3}e^{-i\theta S_2}, \quad n_i = (\sin \theta \cos \phi; \sin \theta \sin \phi; \cos \theta),$$

(3.12)

where we use $\theta$ and $\phi$ (instead of $2\psi$ and $2\beta$ in (2.3)) as polar coordinates for the unit vector $n$. From such coherent states at each site we can build a “coherent” state of the whole chain as $\prod_a |n_a\rangle = |n_1, \ldots, n_a, \ldots\rangle$ in which case

$$\langle n| \hat{S}_a|n\rangle = \frac{1}{2}n_a^i, \quad n_a \cdot n_a = 1,$$

(3.13)

$$\langle n| \hat{Q}_{a,b}|n\rangle = \frac{1}{2}N_{a,b}, \quad N_{a,b} = 1 - n_a \cdot n_b = \frac{1}{2}(n_a - n_b)^2.$$

(3.14)

We then end up with

$$\langle n| \mathcal{D}_2|n\rangle = N_{a,a+1}, \quad \langle n| \mathcal{D}_4|n\rangle = -4N_{a,a+1} + N_{a,a+2},$$

(3.15)

$$\langle n| \mathcal{D}_6|n\rangle = 30N_{a,a+1} - 12N_{a,a+2} + 2N_{a,a+3} + N_{a,a+2}N_{a+1,a+3} - N_{a,a+3}N_{a+1,a+2}.$$  

(3.16)

Now, $\langle n| \mathcal{H}|n\rangle$ in (1.4) should be replaced by

$$\langle n| \mathcal{H}|n\rangle = \sum_{a=1}^{J} \left[ \frac{\lambda}{(4\pi)^2} \langle n| \mathcal{D}_2(a)|n\rangle + \frac{\lambda^2}{(4\pi)^4} \langle n| \mathcal{D}_4(a)|n\rangle + \frac{\lambda^3}{(4\pi)^6} \langle n| \mathcal{D}_6(a)|n\rangle + \ldots \right].$$

(3.17)

Next, let us take a continuum limit as in (1.6) by introducing a spatial coordinate $0 < \sigma < 2\pi$, and $n(\sigma_a) = n(\frac{2\pi a}{J})$, so that $n_{a+1} - n_a = \frac{2\pi}{J}\partial_\sigma n + \ldots$, etc. Then we find (using Taylor expansion and dropping a total derivative over $\sigma$)

$$\frac{\lambda}{(4\pi)^2} \langle n| \mathcal{D}_2|n\rangle \to \frac{\lambda^2}{8} \left[ (\partial_1 n)^2 + O(\frac{1}{J^2}\partial_1^2 n) \right], \quad \frac{\lambda^2}{(4\pi)^4} \langle n| \mathcal{D}_4|n\rangle \to -\frac{\lambda^2}{32} \left[ (\partial_1^2 n)^2 + O(\frac{1}{J^2}\partial_1^3 n) \right],$$

(3.18)

$$\frac{\lambda^3}{(4\pi)^6} \langle n| \mathcal{D}_6|n\rangle \to \frac{\lambda^3}{64} \left[ \frac{7}{4(2\pi)^2} J^2(\partial_1 n)^4 + (\partial_1^2 n)^2 - \frac{19}{24}(\partial_1 n)^2(\partial_1^2 n)^2 - \frac{115}{12}(\partial_1 n\partial_1^2 n)^2 + O(\frac{1}{J^2}\partial_1^3 n) \right].$$

(3.19)

$\text{Published Work}$

$I$Ironically, instead of “coherent”, a better name for such state would be “decoherent” as it is simply a direct product of independent states at each site.
In general, starting with (3.16) one finds
\[
\frac{\lambda^3}{(4\pi)^6} \langle n | D_6(\alpha_1) | n \rangle \rightarrow \frac{\lambda^3}{64} \left[ \frac{1}{8(2\pi)^2} (14 - \alpha_1) \mathcal{J}^2(\partial_1 n)^4 + (\partial_1 n)^2 - \frac{1}{48} (38 - 7\alpha_1)(\partial_1 n)^2(\partial_1^2 n)^2 - \frac{5}{24} (46 + \alpha_1)(\partial_1 n \partial_1^2 n)^2 + O\left( \frac{1}{J^2} \partial_1^8 n \right) \right]. \quad (3.21)
\]
Here we used the identity (A.14) (for some useful relations see Appendix B).

We see that if \( \lambda \) is fixed in the large \( J \) limit we may ignore higher derivative terms in one-loop (3.18) and two-loop (3.19) terms. It is crucial for the consistency of our limit that the subleading \( \mathcal{O}(\lambda_1)^2 \) terms cancel out in the higher-order \( \langle n | D_{2r+2} | n \rangle \) terms.\(^\S\) We see explicitly that this is indeed the case for \( \langle n | D_4 | n \rangle \) (as was first noticed in [21]) and for \( \langle n | D_6 | n \rangle \).

However, the presence of “subleading” \( \mathcal{O}(\lambda_1)^4 \) term in (3.21) that blows up in the \( J \rightarrow \infty \) limit implies a problem with taking the continuum limit directly in \( \langle n | D_6 | n \rangle \). Our conjecture is that this singular term in (3.20) will be canceled out once quantum corrections are properly included. As we shall see in the next section, one needs indeed to include order \( n^4 \) quantum corrections, i.e. to compute first an effective spin chain action and only then take the continuum limit.

Assuming that the “scaling-violating” \( \mathcal{O}(n)^4 \) term in (3.21) can indeed be omitted, we then finish with the following generalization of the semiclassical spin chain action (1.6)
\[
S = J \int dt \int_0^{2\pi} d\sigma \frac{d\sigma}{2\pi} \left[ C_t(n) - \langle H \rangle \right], \quad (3.22)
\]
\[
\langle H \rangle = \frac{\lambda}{8} (\partial_1 n)^2 - \frac{\lambda^2}{32} (\partial_1^2 n)^2 + \frac{\lambda^3}{64} \left[ (\partial_1^3 n)^2 - \frac{19}{24} (\partial_1 n)^2(\partial_1^2 n)^2 - \frac{115}{12} (\partial_1 n \partial_1^2 n)^2 \right] + O(\lambda^4). \quad (3.23)
\]
where we used the notation \( \langle H \rangle \) instead of \( \langle n | H | n \rangle \) to indicate that some quantum correction was taken into account.

To compare (3.22) to a similar action (2.85), (2.86), (2.89) obtained from the string sigma model we need to rescale the time \( t \rightarrow \lambda^{-1} t \) as in (1.11) to absorb one (overall) power of \( \lambda = \frac{1}{J^2} \). We then find perfect agreement of coefficients of all quadratic \( \mathcal{O}(\partial_1^2 n)^2 \) (\( r = 1, 2, 3 \)) terms.\(^\S\) However, there is no detailed agreement at \( n^4 \) and \( n^6 \) level. In particular, (3.23) is missing \( \lambda (\partial_1 n)^4 \) term present on the string side in (2.86). The coefficients of the quartic 6-derivative terms in (3.23) are different from the ones in (2.89); also, (3.23) does not contain \( (\partial_1 n)^6 \) term. As we shall demonstrate below, quantum corrections on the spin chain side at order \( \lambda^2 \) induce the term \( (\partial_1 n)^4 \) with precisely the same relative coefficient \( -\frac{3}{4} \) as appearing on the string side in (2.89).

\(^\S\)This property of \( D_{2r+2} \) should be a consequence of supersymmetry of underlying SYM theory that restricts the structure of the dilatation operator.

\(^\S\)The agreement at quadratic order in \( n \) is related to having the correct BMN limit (see section 4).
Similarly, we expect that a systematic account of quantum corrections will be necessary to verify the agreement of the $n^4$ and $n^6$ terms at the next 6-derivative order. The inclusion of quantum corrections may turn out to be effectively equivalent to modifying the original dilatation operator by apparently non-local terms like $D_2^2, D_2 D_4, D_2^3, \ldots$ As we shall show in Appendix C, their coherent state expectation values contain local parts very similar to the ones in (3.15) (3.16).

Let us note also that the procedures of including quantum corrections and taking continuum limit may not commute already at the first subleading order. One may wonder that if one first takes the continuum limit, then one gets the $J$-factor in front of the action (3.22) and thus all quantum corrections would then be expected to be suppressed in the limit of $J \to \infty$. This, however, ignores a possibility of potential UV divergences that are regularized away in the discrete spin chain but may appear in the continuum limit. Since the effective short-distance cutoff here is essentially $\frac{1}{J}$, there may be additional finite contributions coming from divergent quantum corrections (due to cancellation of the $\frac{1}{J}$ suppression factor against the divergent cutoff factor). Instead of trying to sort out such contributions in a continuum version, here we shall consider directly the quantum version of the discrete theory, derive a spin chain analog of the quantum effective action, and then take the continuum limit.

As we shall show in Appendix C, second and third powers of $D_2$ operator have coherent state expectation values which look similar to those of $D_4$ and $D_6$. In particular, to quadratic order in $n$ in the integrands one observes that

$$\langle n|D_4|n\rangle = \frac{1}{2} \langle n|(D_2)^2|n\rangle + O(n^4), \quad \langle n|D_6|n\rangle = \frac{1}{2} \langle n|(D_2)^3|n\rangle + O(n^4). \quad (3.24)$$

We can make a conjecture similar to the one in [26]† and assume that for small fluctuations above the ferromagnetic ground state the exact dilatation operator has the form (cf. (3.1))

$$D = J + \left[ \sqrt{1 + 2 - \frac{\lambda}{(4\pi)^2} D_2} - 1 \right] = J + \frac{\lambda}{(4\pi)^2} D_2 - \frac{1}{2} \left( \frac{\lambda}{(4\pi)^2} D_2 \right)^2 + \frac{1}{2} \left( \frac{\lambda}{(4\pi)^2} D_2 \right)^3 + \ldots , \quad (3.25)$$

where 1 indicates the identity operator. Then, we take the coherent state expectation value of (3.25), keeping only the local terms (which are also the ones quadratic in $n$). In the continuum limit this becomes:

$$\langle n|D|n\rangle = J \int_0^{2\pi} \frac{d\sigma}{2\pi} \left[ 1 + \frac{1}{8} \tilde{\lambda}(\partial_1 n)^2 - \frac{1}{32} \tilde{\lambda}^2 (\partial_1^2 n)^2 + \frac{1}{64} \tilde{\lambda}^3 (\partial_1^3 n)^2 + \ldots \right]. \quad (3.26)$$

This is in precise agreement with the classical string expression (2.90) that sums up all terms in the effective action that are quadratic in $n$.

†The conjecture in [26] was that, restricted to two-impurity BMN states, the exact dilatation operator has the form $D = J + 2\sqrt{1 + \frac{\lambda}{(4\pi)^2} D_2}$.
4 Energy of spin chain at order $\lambda^2$: quantum corrections to effective action

In this section we compute the energy of the spin chain at order $\lambda^2$ including quantum corrections. This gives the correct conformal dimension of the corresponding operators at the same order and therefore we now expect to reproduce what we obtained before from the string calculation, namely (2.85). In the same spirit of [21], the states we are interested in are represented by spin waves with wavelength of order the size of the chain $J$. These waves are going to be described by classical solutions of a low energy effective action. Such a (“Wilsonian”) action can, and will, get contributions from integrating out the large momentum modes. The calculation is therefore more complicated than at the leading 1-loop order [21] but the end result is that, after including the quantum corrections and then taking the continuum limit, we reproduce the string theory result (2.80).

4.1 Spin chain Hamiltonian and first excited states

Our starting point is the Hamiltonian of the spin chain proportional to the dilatation operator at order $\lambda^2$ [26]:

\[
H = \tilde{\lambda}^{-1} (D_2 + D_4) = -J^2 \left( \lambda_1 \sum_{a=1}^{J} \left[ S_a^x S_{a+1}^x - \frac{1}{4} \right] + \lambda_2 \sum_{a=1}^{J} \left[ S_a^y S_{a+2}^y - \frac{1}{4} \right] \right),
\]

where

\[
\lambda_1 \equiv \frac{1}{4\pi^2} - \frac{1}{16\pi^4} \lambda, \quad \lambda_2 \equiv \frac{1}{64\pi^4} \lambda.
\]

This is the same expression as in (3.1) and (3.7) after one uses (3.9). We included the factor $\tilde{\lambda} = \lambda/J^2$ between $H$ and the dilatation operator $D = D_2 + D_4$ to account for the fact that we shall be assuming that time $t$ is rescaled by $\tilde{\lambda}^{-1}$ as in (1.10). As a result, we get the factors $J^2$ in front of $H$ which seem unconventional, but, as we will see below, in the end make the perturbative expansion in $\tilde{\lambda}$ more transparent.

The Hamiltonian (4.1) describes a 1-dimensional spin $\frac{1}{2}$ ferromagnetic* chain with first and second neighbor interactions. The ferromagnetic ground state is

\[
|0\rangle = \prod_{a=1}^{J} | \uparrow \rangle_a
\]

with all spins parallel and therefore the total third spin projection is $S_z = \frac{1}{2}J$. The total spin is $S = \frac{1}{2}J$ and so there are actually $2S + 1 = J + 1$ degenerate ground states. The energy of the ground state is zero in agreement with the fact that the ground state describes a protected (chiral primary) operator $\text{tr}\Phi^J$ whose conformal dimension has no corrections to any order in perturbation theory.

*Note that the Hamiltonian is positive for $\lambda_1 > 0, \lambda_2 > 0$ (which is the case assuming $\lambda < 1$).
Before proceeding, it is useful to find the first excited states of the Hamiltonian (4.1). These are spin waves where one spin is down and all the others up. We can find these eigenstates exactly: they are just given by momentum eigenstates:

\[ |k⟩ = \frac{1}{\sqrt{J}} \sum_{a=1}^{J} e^{ika} |↑↑ ... ↓_{a} ... ↑↑⟩ \]  

(4.4)

where we denote with \(|↑↑ ... ↓_{a} ... ↑↑⟩\) a configuration with all spins up except at site \(a\) where it is down. It is easy to see that this is an eigenstate of the Hamiltonian with eigenvalue

\[ \epsilon(k) = J^2 [\lambda_1(1 - \cos k) + \lambda_2(1 - \cos 2k)] , \]  

(4.5)

which is positive for \(\lambda_1, \lambda_2 > 0\).

The next excitations correspond to two spins down and can be found as superpositions of two spin waves. For large \(J\) we can use a dilute gas approximation and write those eigenstates as

\[ |kk'⟩ = \frac{\sqrt{2}}{J} \sum_{a > a' = 1}^{J} e^{ika+ik'a} |↑↑ ... ↓_{a} ... ↑↑ ... ↓_{a'} ... ↑⟩ , \]  

(4.6)

\[ H|kk'⟩ \simeq [\epsilon(k) + \epsilon(k')] |kk'⟩ , \]  

(4.7)

where the error is of order \(1/J^4\). This approximation is good as long as we assume that the number of spins down is much smaller than \(J\).

4.2 Defining the effective action

If we include a large number of spins down, which corresponds to taking \(J_1\) and \(J_2\) to be of the same order, i.e. of the same order as \(J = J_1 + J_2\), the correct description is in terms of an effective action for low energy modes with momenta \(\sim 1/J\). To compute it we should divide the spin fields \(S_a\) into “slow” and “fast” parts, with the slow modes being described by a unit vector \(\vec{n}\) such that, when we take the lattice spacing \(2\pi/J\) to zero, the derivative \(\partial\vec{n}\) remains finite (\(\partial\vec{n} \sim 1\)). Taking into account the rotational invariance, the effective action for \(\vec{n}\) up to four derivatives should have the same form as in (1.10) (in this section \(\partial \equiv \partial_\sigma = \partial_1\)):

\[ S = S_{WZ} - J \int dt \int_0^{2\pi} d\sigma \left\{ \frac{1}{8} (\partial \vec{n})^2 + \tilde{\lambda} \left[ a_1 (\partial^2 \vec{n})^2 + a_2 (\partial \vec{n})^4 \right] + O(\partial^6 n) \right\} . \]  

(4.8)

The first term is the Wess-Zumino term which, upon quantization ensures that we have a spin \(\frac{1}{2}\) at each site (\(i.e.\) only two states). The overall factor of \(J\) comes from the spin chain length and \(J^2\) factor in (4.1) gets absorbed into the definition of the quadratic derivative term (the coefficient of \((\partial \vec{n})^2\) was fixed in [21]). We anticipated that the 4-derivative terms are of order \(\tilde{\lambda}\) but this should be an outcome of the calculation, namely we should get the coefficients \(a_1, a_2 \sim 1\).

\[ \footnote{Also, \(|kk'⟩\) is normalized up to factors suppressed by \(1/J\).} \]
Actually, the coefficients of all the terms quadratic in \( \vec{n} \) (i.e. \( (\partial \vec{n})^2 \), \( (\partial^2 \vec{n})^2 \), etc.) can be fixed by expanding \( \epsilon(k) \) in (4.5) for small \( k \) and comparing with the energy we get from the effective action for small oscillations around \( \vec{n} = (0, 0, 1) \). This corresponds in operator language to the BMN limit. This small oscillation analysis does not, however, allow one to fix the non-trivial coefficient \( a_2 \) so we prefer to do a direct computation of the effective action (4.8).

The effective action can be computed in various equivalent ways. One usual method is to use the path integral formalism to integrate out the high energy modes. This leads to a diagrammatic expansion. However, this approach presumes that one can separate the action into a free and an interacting part with the free part being quadratic in the fields. In our case we can use \( S^+ = S_x + iS_y \) and \( S^- = S_x - iS_y \) as our fields and replace \( S_z \) through the identity

\[
S_z = \frac{1}{2} - S^- S^+ ,
\]

valid for spin \( \frac{1}{2} \) (as can be seen by using the relation of \( \vec{S} \) to Pauli matrices or by acting on the up and down basis: \( \{ | \uparrow \rangle, | \downarrow \rangle \} \)). The problem is that we get quartic interactions in the Hamiltonian with no small coupling constant in front. Instead, what is usually done for this system is to consider a spin chain with arbitrary spin \( s \) at each site. Near the state of maximum \( S_z \) projection, we can expand \( S_z \) as

\[
S_z = \left[ \sqrt{s(s+1)} - (S_x^2 + S_y^2) \right]^{1/2} \approx s \left[ 1 - \frac{1}{4s^2} (S^+ S^- + S^- S^+) + \ldots \right] .
\]

Now we get vertices of all orders in the fields \( S^\pm \) but they are suppressed in the limit \( s \to \infty \). It follows then that in this limit we can treat these interactions perturbatively which is the content of the Holstein-Primakoff expansion [36].

This is, however, not possible here since we definitely have to assume \( s = \frac{1}{2} \) for the spin chain Hamiltonian to represent the SYM dilatation operator. Furthermore, such an approach actually obscures the simple nature of the system which is obvious from (4.3) and (4.5) where we found the ground state and first excited states with no difficulty. Nevertheless, as we will see at the end of this section, the final result seems to be compatible with a large \( s \) limit result which suggest that an effective parameter of expansion is \( J_s \).

We will leave further investigation of this issue for the future and here will concentrate on an alternative approach which is suggested by the fact that the effective action is the minimal value of the energy of the state with fixed expectation values for the fields.

Due to the condition \( S^2 = \frac{3}{4} \) there are only two independent fields at each site so we need to fix two conditions on the expectation values. It is natural to introduce a unit vector at each site \( \vec{n}_a \) and look for the lowest energy state such that the mean value of the spin at each site \( a \) points in the direction of \( \vec{n}_a \). That is, we describe the low energy wave as oscillations of the direction in which the spins point. This gives an effective action for \( \vec{n}_a \) which we can then minimize by solving the classical
equations of motion. The field $\vec{n}_a$ is going to be considered as slowly varying in time and therefore to be a static background for the fast modes. Formally, what we want to find is the lowest energy state $|\psi(\vec{n}_a)\rangle$ such that

$$\langle \psi(\vec{n})|\vec{S}_a|\psi(\vec{n})\rangle_\perp = 0, \quad a = 1, \ldots, J,$$

where $\perp$ indicate the component of the vector in the direction perpendicular to $\vec{n}_a$.

It is not possible to find such a state exactly, so we need to resort to a perturbation theory. Since we are interested in the limit of large $J$ (long chains) it is natural to use $1/J$ as a small parameter with a requirement of keeping terms of order $\tilde{\lambda} = \lambda J^2$ since we are interested in the limit (1.1) when this quantity remains finite. It is natural to consider the lattice spacing to be $2\pi/J$ so that the length of the chain is fixed in the limit. The field $\vec{n}$ represent modes whose wave-length is fixed with respect to the length of the chain (but grows to infinity in units of the lattice spacing). More precisely, we are to keep $\partial \vec{n} \sim 1$ in the limit. This means that the vectors $\vec{n}_a$, $\vec{n}_{a+1}$, at neighboring sites are almost parallel. Thus, if we consider a state where, at each site, the spin is aligned (i.e. has maximum projection) along $\vec{n}$, its energy will not differ much from that of the vacuum. We can actually estimate the energy to be of order $E \sim J \int (\partial \vec{n})^2 \sim J$. Such a state, constructed out of coherent states at each site (3.11), is a candidate state to be the one of smallest energy such that $\langle \vec{S} \rangle \parallel \vec{n}$ and is actually the one that was used in (3.13) (here we denote it $|\psi\rangle_0$ instead of $|\vec{n}\rangle$):

$$|\psi\rangle_0 = \prod_a |\vec{n}_a\rangle.$$  

(4.12)

We can correct this state using perturbation theory in $\tilde{\lambda}$ to obtain

$$|\psi(\vec{n}_a)\rangle = |\psi\rangle_0 + |\psi\rangle_1 + \ldots .$$

(4.13)

In this paper we are going to compute only the first correction. The effective action for $\vec{n}$ is then given by

$$S = S_{\text{WZ}} - \langle \psi(\vec{n}_a)|H|\psi(\vec{n}_a)\rangle.$$  

(4.14)

As was already mentioned, the WZ term provides the correct quantization for $\vec{n}$ (just $(2s + 1)$ states at each site where here $s = \frac{1}{2}$) and therefore, its coefficient cannot be renormalized when integrating out the higher momentum modes.\(^\dagger\)

### 4.3 Computing the effective action

Thus, to the lowest order in the perturbative expansion (i.e. with $|\psi\rangle \to |\psi\rangle_0$) the non-trivial part of the action is thus determined by the coherent-state expectation

\(^\dagger\)We remind the reader that we rescaled the time by $\tilde{\lambda}^{-1}$ so without rescaling this energy is actually of order $J\tilde{\lambda} = \frac{1}{J}$.

\(^\ddagger\)Alternatively, in the path integral approach a topological argument also implies that the coefficient of $S_{\text{WZ}}$ is quantized in half integer units and therefore cannot be renormalized.
\[ 0\langle \psi | H | \psi \rangle_0 = -\frac{1}{4} J^2 \left[ \lambda_1 \sum_{a=1}^J (\bar{n}_a \bar{n}_{a+1} - 1) + \lambda_2 \sum_{a=1}^J (\bar{n}_a \bar{n}_{a+2} - 1) \right]. \] (4.15)

This happens to be simply the result found by replacing \( \vec{S} \rightarrow \frac{1}{2} \vec{n} \) in (4.11) (see (3.13), (3.14)). It is this expression that we have already discussed in the previous section and which matched only partially the string result: it did not contain the \( (\partial \vec{n})^4 \) term (cf. (3.19)).

Now we should compute the correction induced by the high energy modes and see if it contains the missing terms. Since the coherent states (3.11) are defined as

\[ |\vec{n}\rangle = R(\vec{n}) |\uparrow\rangle, \] (4.16)

where \( R(\vec{n}) \) is the rotation operator in (3.12) it turns out to be useful to define new operators at each site by

\[ \tilde{S}^i_a = R(\vec{n}) S^i_a R^{-1}(\vec{n}) = R_{ji}(\vec{n}) S^j_a, \] (4.17)

i.e. \( \tilde{S}^i_a \) are rotated with respect to the original operators \( S^i_a \) with the rotation matrix depending on the “background field” \( \vec{n} \). \( \tilde{S}^i_a \) obey the same commutation relations, and the zero-order state reads

\[ |\psi\rangle_0 = \prod_a |\uparrow\rangle_a. \] (4.18)

Hence \( |\psi\rangle_0 \) (which is the product of coherent states (4.12)) now looks similar to the vacuum state (4.3). The condition (4.11) now reads

\[ \langle \psi(\vec{n}) | \tilde{S}^+_a | \psi(\vec{n}) \rangle = \langle \psi(\vec{n}) | \tilde{S}^-_a | \psi(\vec{n}) \rangle = 0, \quad a = 1, \ldots, J, \] (4.19)

which we should now impose order by order in the perturbative expansion in a similar way as we impose the normalization condition:

\[ \langle \psi(\vec{n}) | \psi(\vec{n}) \rangle = 1. \]

Everything now looks simple except for the Hamiltonian that when written in terms of \( \tilde{S}^j_a \) reads

\[ H = H_0 + V, \] (4.20)

\[ H_0 = -J^2 \sum_{q=1}^2 \sum_{a=1}^J [\tilde{S}^i_a \tilde{S}^j_{a+q} - \frac{1}{4}], \] (4.21)

\[ V = -J^2 \sum_{q=1}^2 \sum_{a=1}^J A^{a,a+q}_{ij} \tilde{S}^i_a \tilde{S}^j_{a+q}, \] (4.22)

\( \dagger \)This step may be viewed as an analog of a covariant background-quantum field split in the sigma model.
where we introduced the $3 \times 3$ matrices
\[
A_{ij}^{a,a+q} = R_{li}(\vec{n}_a)R_{lj}(\vec{n}_{a+q}) - \delta_{ij} .
\] (4.23)

These are small if the variations of $\vec{n}_a$ from site to site are small (i.e. $\frac{1}{J}\partial \vec{n}$ are small) as we are going to assume below.

The crucial point is that while the Hamiltonian now is more complicated it has naturally separated into “large” and “small” parts. Moreover, the state $|\psi\rangle_0$ is an eigenstate of $H_0$. It is then easy to see that minimizing $\langle \psi| H |\psi\rangle$ with respect to arbitrary corrections to the “ground state” gives the usual perturbation theory result where $V$ is considered a perturbation of $H_0$.

Let us briefly summarize how that happens. We consider a state
\[
|\psi\rangle = (1 + c_0)|\psi\rangle_0 + \sum_{|p\rangle \neq |0\rangle} c_p|p\rangle + \ldots ,
\] (4.24)
where $|\psi\rangle_0 \equiv |0\rangle$ and $|p\rangle$ are eigenstates of $H_0$ with energy $\epsilon_p$. The mean value of the energy is, to first order,
\[
\langle \psi| H |\psi\rangle = \epsilon_0 + (c_0 + c_0^*)\epsilon_0 + \langle 0| V |0\rangle + \sum_p c_p\langle 0| V |p\rangle + \sum_p c_p^*\langle p| V |0\rangle + \sum_p |c_p|^2 \epsilon_p + \ldots ,
\] (4.25)
where in our case $\epsilon_0 = 0$. Expressing $c_0$ in terms of $c_p$ by taking into account that
\[
\langle \psi| \psi\rangle = 1 \Rightarrow c_0 + c_0^* = -\sum_p |c_p|^2 ,
\] (4.26)
and minimizing with respect to $c_p^*$ we get that
\[
|\psi\rangle = |\psi\rangle_0 - \sum_p \frac{\langle p| V |0\rangle}{\epsilon_p - \epsilon_0} |p\rangle - \frac{1}{2} \sum_p \frac{|\langle p| V |0\rangle|^2}{(\epsilon_p - \epsilon_0)^2} |0\rangle + \ldots .
\] (4.27)

From here we get for the energy
\[
\epsilon \equiv \langle \psi| H |\psi\rangle = \epsilon_0 + \langle 0| V |0\rangle - \sum_p \frac{\langle 0| V |p\rangle \langle p| V |0\rangle}{\epsilon_p - \epsilon_0} + \ldots .
\] (4.28)
which is the standard perturbation theory expression. The only difference is that in our case we should be careful to include in the sum over $|p\rangle$ only states such that $|\psi\rangle$ satisfies the condition (4.19).

Next, it is important to notice that since $V$ in (4.22) is quadratic in the spin operators we only need to consider states which differ from the ground state by just one or two spin flips. Those are precisely the states that we already discussed above in subsection 4.1, so that, up to corrections of order $1/J$ we have
\[
|\psi\rangle = |\psi\rangle_0 + |\psi\rangle_1 + \ldots ,
\] (4.29)
\[
|\psi\rangle_1 = \sum_k \alpha_k |k\rangle + \sum_{k,k'} \alpha_{kk'} |kk'\rangle .
\] (4.30)
Then, to first order in the coefficients $\alpha_k, \alpha_{kk'}$, we can write condition (4.19) as
\[
\langle \psi | \tilde{S}^+_{a} | \tilde{S}^-_{a+k} \rangle = \frac{1}{\sqrt{J}} \sum_k e^{ika} \alpha_k + ... = 0 , \quad a = 1, \ldots, J .
\tag{4.31}
\]
This implies that $\alpha_k = 0$, i.e. we only need to consider states with two spin flips. Evaluating the energy (4.28) we get (remembering that here $\epsilon_0 = 0$)
\[
\epsilon = -\frac{J^2}{4} \sum_{q=1}^2 \lambda_q \sum_{a=1}^J A_{33}^{a,a+q} \\
- J^4 \frac{2}{16} \sum_{a,a'=1}^J \sum_{kk'} \sum_{qq'} \lambda_q \lambda_{q'} A_{33}^{a,a+q} A_{33}^{a',a'+q'} \langle 0 | \tilde{S}^+_{a} \tilde{S}^+_{a+q} | kk' \rangle \langle kk' \tilde{S}^-_{a+q} \tilde{S}^-_{a+q'} | 0 \rangle \frac{\epsilon(k) + \epsilon(k')}{2} \tag{4.32}
\]
where the factor $1/16$ came from the fact that the indices $+, -$ contract with coefficient $1/2$ and the factor $2/J^2$ – from the normalization of the states $| kk' \rangle$ in (4.6).

**4.4 Taking the continuum limit**

Since the field $\vec{n}$ is slowly varying we can now take the continuum limit and expand the mean energy (4.32) or effective action in derivatives of $\vec{n}$. In Appendix D we give the expansion of $A_{ij}^{a,a+q}$ in powers of $q \partial \vec{n}$ (this makes sense since $q = 1, 2$ and $\vec{n}$ varies little between the neighboring sites).

The first term with $A_{33}$ is the same as the naive coherent state expectation value (and is thus quadratic in $n$), while the leading contribution to $A_{++} A_{--}$ is already quartic in derivatives. If we ignore the quantity $\epsilon(k) - \epsilon(k')$ in the denominator of the last term in (4.32), the sum over $k + k'$ gives a delta function $J \delta(a - a')$. If instead we expand it in powers of $(k + k')$, non-zero powers will give rise to derivatives of the coefficients $A$ and therefore to higher than fourth order derivatives. In view of that we can consider just intermediate states with $k = -k'$ and ignore the contribution of the rest of the states. Using the relations from Appendix D
\[
A_{++} = \frac{1}{2} \tilde{A}_{++} + \ldots , \quad | \tilde{A}_{++} |^2 = \left( \frac{2\pi q}{J} \partial \vec{n} \right)^4 + \ldots ,
\tag{4.33}
\]
\[
A_{33} = -\frac{1}{2} \left( \frac{2\pi q}{J} \right)^2 \left( \partial \vec{n} \right)^2 + \frac{1}{24} \left( \frac{2\pi q}{J} \right)^4 \left( \partial^2 \vec{n} \right)^2 + \ldots ,
\tag{4.34}
\]
we get
\[
\epsilon = J \int_0^{2\pi} \frac{d\sigma}{2\pi} \left[ d_0 (\partial \vec{n})^2 + d_1 (\partial^2 \vec{n})^2 + d_2 (\partial \vec{n})^4 + \ldots \right] ,
\tag{4.35}
\]
32
where
\[
\begin{align*}
d_0 &= \frac{\pi^2}{2} \sum_{q=1}^{2} \lambda_q q^2, \\
d_1 &= \frac{\pi^4}{6J^2} \sum_{q=1}^{2} \lambda_q q^4, \quad (4.36) \\
d_2 &= -\frac{\pi^3}{8} \sum_{q,q'} \lambda_q \lambda_{q'} q^2 q'^2 \int_0^{2\pi} dke^{ik(q'-q)} / \epsilon(k'). \quad (4.37)
\end{align*}
\]

Here we have replaced the sums over \(k = \frac{2\pi n}{J}, n = 1, \ldots, J\) and over \(a = 1, \ldots, J\) by the integrals:
\[
\sum_k \rightarrow J \int_0^{2\pi} \frac{dk}{2\pi}, \quad \sum_a \rightarrow J \int_0^{2\pi} \frac{d\sigma}{2\pi}. \quad (4.38)
\]

As follows from (4.2),
\[
d_0 = \frac{1}{8}, \quad d_1 = \frac{\pi^4}{6J^2} \left( \frac{1}{4\pi^2} - \frac{3}{16\pi^4} \lambda \right) = \frac{\pi^2}{24J^2} - \frac{\tilde{\lambda}}{32}. \quad (4.39)
\]

Omitting the first (subleading at large \(J\)) term in \(d_1\), we thus get the same coefficients as in (3.18).

Using (4.5) and expanding in powers of \(\lambda\) we get for the third remaining coefficient (which, up to \(\tilde{\lambda}\) factor, should be equal to the \(a_2\) coefficient in (4.8))
\[
d_2 \equiv \tilde{\lambda} a_2 = -\frac{\pi^3}{8J^2} \int_0^{2\pi} \frac{dk}{1 - \cos k} \frac{\lambda_1^2 + 8\lambda_1 \lambda_2 \cos k + 16\lambda_2^2}{\lambda_1(1 - \cos k) + \lambda_2(1 - \cos 2k)} \quad (4.40)
\]
\[
= -\frac{\pi}{32J^2} \int_0^{2\pi} \frac{dk}{1 - \cos k} + \frac{3}{2\pi} \lambda \int_0^{2\pi} \frac{dk}{1 - \cos k} - \frac{1}{211\pi^3} \lambda^2 J^2 \tilde{\lambda}^2 \int_0^{2\pi} dk (3 \cos k + 5) + \ldots,
\]

where we substituted the values of \(\lambda_1\) and \(\lambda_2\) from (4.2).

Now few comments are in order. The first integral over \(k\) here is divergent. This is related to the fact that at this order the interaction \(V\) mixes two ground states and therefore produces an IR divergence. We could correct this using perturbation to a degenerate level, but, in any case this contribution is of order \(1/J^2\) so it is subleading in the limit \(J \to \infty\). The second finite term gives precisely the same coefficient
\[
a_2 = \frac{1}{32} \times \frac{3}{4}, \quad (4.41)
\]

as required for the agreement with the string result in (2.86).

One may worry though about the third and higher terms in the above expression that seem to dominate in the large \(J\) limit. However, since we did not include higher order (three-loop, etc.) terms in \(\tilde{\lambda}\) in the Hamiltonian (4.11) the above computation does not properly account for the terms of order \(\tilde{\lambda}^2\). All such singular terms in \(d_2\) should cancel after we include all the terms \(D_n\)'s \((n \geq 3)\) in the dilatation operator.
In fact, as we saw in the previous section 3, there is a similar singular \((\partial \vec{n})^4\) contribution \((3.20)\) coming from the naive continuum limit of the 3-loop operator \(D_6\). However, to carefully check the cancellation of such singular terms we should also include other quantum corrections coming from \(D_6\). Assuming the required cancellations occur to all orders so that the coefficient of \((\partial \vec{n})^4\) is simply given by \((4.41)\), we end up with the following expression for the effective action to the 4-derivative order

\[
S = S_{WZ} - J \int \frac{dt}{2\pi} \int_0^{2\pi} \frac{d\sigma}{2\pi} \left\{ \frac{1}{8}(\partial \vec{n})^2 - \frac{\lambda}{32} \left[ (\partial^2 \vec{n})^2 - \frac{3}{4}(\partial \vec{n})^4 \right] + O(\partial^6 \vec{n}) \right\} , \tag{4.42}
\]

which is in perfect agreement with \((2.75),(2.86),(2.88)\) or \((2.43)\).

As a final comment let us note that if we would have done the above calculation for a spin chain with an arbitrary spin \(s\) representation for the \(SU(2)\) generators the coefficient \(a_1\) would have been multiplied by \(s^2\) and the coefficient \(a_2\) – by \(s\). This suggests that the \(a_2(\partial \vec{n})^4\) term we computed can be interpreted as a \(1/s\) correction in the large \(s\) expansion. It will be interesting to pursue this issue further since it might indicate that the same calculation can be done directly in a (regularized) continuum sigma model set up.

### 5 Folded string solution: a check

As a check of the action \((4.42)\) we have derived both from string theory and the spin chain let us show that it indeed correctly reproduces the second-order \((\lambda^2/J^3)\) correction in the expansion of the classical energy for the folded string solution \([14,12]\). On the spin chain side the same “two-loop” term was found using Bethe ansatz technique in \([20]\).

At lowest order this check was performed in \([21]\) where it was shown that the corresponding solution of the LL equation of motion \((1.7)\) following from the first and second derivative terms in \((4.42)\) was given by

\[
\phi = \omega t , \quad \theta = \theta(\sigma) , \tag{5.1}
\]

with

\[
\partial_x \theta = \sqrt{a_0 + b_0 \cos \theta} , \tag{5.2}
\]

where \(\theta\) and \(\phi\) are polar coordinates for the unit vector \(\vec{n}\) (see \((3.12)\)). The integration constants \(a_0\) and \(b_0\) should be expressed in terms of the angular momenta \(J = J_1 + J_2\) and \(S_3 = (J_2 - J_1)/2\), through

\[
J = J_1 + J_2 = 4 \int_0^{\theta_0} \frac{d\theta}{\sqrt{a_0 + b_0 \cos \theta}} \tag{5.3}
\]

\[
S_3 = \frac{J_1 - J_2}{2} = -2 \int_0^{\theta_0} \frac{\cos \theta d\theta}{\sqrt{a_0 + b_0 \cos \theta}} . \tag{5.4}
\]
Here \( \theta_0 = \arccos(-a_0/b_0) \) (we assume \( b_0 > |a_0| \)). After that, the energy is computed by substituting the values of \( a_0 \) and \( b_0 \) into
\[
E = \frac{\lambda}{8\pi^2} \int_0^{\theta_0} \sqrt{a_0 + b_0 \cos \theta} \, d\theta .
\]
(5.5)

All these integrals can be done in terms of elliptic integrals and reproduce the term of order \( \lambda/J \) as obtained by expanding the energy of the exact folded rotating string solution [12].

To extend this to the next order we shall start with the action (4.42). Here it is useful to rescale back the time \( t \) and to reintroduce \( \lambda = J^2 \lambda \). We also define the coordinate \( x = \frac{L}{2\pi} \sigma \). The length of the chain is now \( J \). In this way we get an action
\[
S = S_{WZ} - \int dt \int_0^J dx \left\{ \frac{\lambda}{32\pi^2} (\partial \vec{n})^2 - \frac{\lambda^2}{512\pi^4} \left[ (\partial^2 \vec{n})^2 - \frac{3}{4} (\partial \vec{n})^4 \right] \right\} ,
\]
(5.6)

which leads to the modified Landau-Lifshitz equations:
\[
\partial_t \vec{n} = -\frac{\lambda}{16\pi^2} \vec{n} \times \partial_x^2 \vec{n} + \frac{\lambda^2}{128\pi^4} \vec{n} \times \partial_x^4 \vec{n}
+ \frac{3\lambda^2}{256\pi^4} \left\{ (\partial_x \vec{n})^2 \vec{n} \times \partial_x^2 \vec{n} + 2[(\partial_x \vec{n}) \partial_x^2 \vec{n}] \vec{n} \times \partial_x \vec{n} \right\} .
\]
(5.7)

It is straightforward to check that the same ansatz (5.1) satisfies these equations of motion provided
\[
\omega \sin \theta + \frac{\lambda}{16\pi^2} \partial_x^2 \theta - \frac{3\lambda^2}{256\pi^4} (\partial_x \theta)^2 \partial_x^2 \theta + \frac{\lambda^2}{128\pi^4} \partial_x^4 \theta = 0 .
\]
(5.8)

We can simplify the equation by using \( \theta \) as an independent variable and introducing a new function \( u(\theta) = (\partial_x \theta)^2 \). The resulting equation can be integrated once resulting in
\[
-256\pi^4 \omega \cos \theta + 8\lambda \pi^2 u - \frac{3}{4} \lambda^2 u^2 + \lambda^2 (uu'' - \frac{1}{4} u'^2) = \tilde{a}
\]
(5.9)

where we used primes to indicate derivatives with respect to \( \theta \) and \( \tilde{a} \) is an integration constant. In spite of being simpler this equation cannot be integrated exactly. Doing a perturbative expansion in \( \lambda \) (which is here equivalent to expansion in \( \tilde{\lambda} \)) we get, to lowest order
\[
u = \frac{\tilde{a}_0}{8\pi^2} + 32\pi^2 \omega_0 \cos \theta + \frac{3\tilde{a}_0^2 \lambda}{2048\pi^6} + 32\pi^2 \omega_0^2 + \frac{5\tilde{a}_0 \lambda \omega_0}{4\pi^2} \cos \theta + 192\lambda \pi^2 \omega_0^2 \cos^2 \theta + \ldots
\]
(5.10)

Here we defined \( \tilde{a} = \lambda \tilde{a}_0 \) and \( \omega = \lambda \omega_0 \) to reflect the fact that \( \omega \) and \( a \) are of order \( \lambda \). Eq.(5.10) is required only to know the relation between the parameters of the solution and the angular velocity \( \omega \). For all other purposes we can simply write the solution, at this order, as
\[
u(\theta) = (\partial_x \theta)^2 = a + b \cos \theta + c \cos^2 \theta ,
\]
(5.11)
with
\[ c = \frac{3\lambda}{16\pi^2} b^2 . \]  
(5.12)

The constants \(a\) and \(b\) follow from the condition that \(J\) and \(S_3\) are fixed. We can compute them in an expansion
\[ a = a_0 + a_1 + \ldots, \quad b = b_0 + b_1 + \ldots, \quad c = c_1 + \ldots \]
where \(a_0, b_0\) are the constants in (5.2). The other constant \(c\) is already of order \(\lambda^2\) since there is no \(\cos^2 \theta\) term at lowest order.

The fact that we perturb the solution keeping \(J\) and \(S_3\) fixed implies that
\[
\int_{\theta_0}^{\theta_1} \frac{d\theta}{\sqrt{a + b \cos \theta + c \cos^2 \theta}} = \int_{\theta_0}^{\theta_0} \frac{d\theta}{\sqrt{a_0 + b_0 \cos \theta}}
\]
(5.13)

where, on the left hand side, \(\theta_1\) is a zero of the denominator,
\[ a + b \cos \theta_1 + c \cos^2 \theta_1 = 0. \]

A straightforward but lengthy calculation that we describe in Appendix E shows that this actually implies that at order \(\lambda^2\) one also has
\[
\int_{\theta_1}^{\theta_0} \sqrt{a + b \cos \theta + c \cos^2 \theta} d\theta \sim \int_{\theta_0}^{\theta_0} \sqrt{a_0 + b_0 \cos \theta} d\theta .
\]
(5.15)

This means that the energy evaluated from the term with two derivatives (i.e. \(\sim (\partial_x \vec{n})^2\)) is, at order \(\lambda^2\), the same for the lowest order solution \((\partial_x \vec{n})^2 = a_0 + b_0 \cos \theta\) as it is for the corrected solution \((\partial_x \vec{n})^2 = a + b \cos \theta + c \cos^2 \theta\). Therefore, the only non-vanishing \(\lambda^2\) contribution comes from the evaluation of the term quartic in derivatives on the leading-order solution. This gives for the corrected energy
\[
\epsilon_2 = -\frac{\lambda}{512\pi^4} \int_0^J dx \left[ (\partial_x^2 \theta)^2 + \frac{1}{4} (\partial_x \theta)^4 \right] ,
\]
(5.16)

which, when evaluated on the unperturbed solution, becomes
\[
\epsilon_2 = -\frac{\lambda}{512\pi^4} \int_0^J \frac{d\theta}{\partial_x \theta} \left[ (\partial_x^2 \theta)^2 + \frac{1}{4} (\partial_x \theta)^4 \right] (5.17)
\]
\[ = -\frac{\lambda^2}{512\pi^4} \int_0^{\theta_0} \frac{a_0^2 + b_0^2 + 2a_0 b_0 \cos \theta}{\sqrt{a_0 + b_0 \cos \theta}} d\theta \]
(5.18)
\[ = \frac{2 \lambda^2}{\pi^4} \int_0^3 K_0 \left\{ (1 - 2x_0)E_0 + (1 - x_0)^2 K_0 \right\} \]
(5.19)

where \(E_0 = E(x_0)\) and \(K_0 = K(x_0)\) are standard elliptic integrals and
\[
x_0 = \frac{a_0 + b_0}{2b_0} .
\]
(5.20)

This is in perfect agreement with the results obtained from expanding the energy of the exact rotating string solution to this order [14, 12, 20].

*Similar result confirming the action (4.42) by comparing to the energy of folded string solution was independently obtained by A. Dymarsky and I. Klebanov (unpublished).
6 Concluding remarks

In this paper we have shown that, up to order $\tilde{\lambda}^2$, the anomalous dimensions of "long" two-scalar operators in SYM theory can be obtained from a semiclassical "string" action that precisely agrees with the expansion, to the same order, of the string sigma model action in $R \times S^3 \subset AdS_5 \times S^5$. Furthermore, we have shown that, if one is able to compute the dilatation operator to all loops, then one can use a systematic procedure to reconstruct a string action from gauge theory.

Although this suggestion was already made in [21] based on order $\tilde{\lambda}$ calculation and previous results of [1, 6, 12], the general procedure turns out to involve some novel and non-trivial steps which are crucial to obtain agreement between gauge theory and string theory predictions already at the first subleading order $\tilde{\lambda}^2$.

The main idea is that comparing the actions gives a map between configurations on both sides of the AdS/CFT duality and therefore contains much more information than the comparison of energies of particular solutions. Since one has to match not only the conserved charges (total energy and angular momenta) but to actually map the variables on one side to the variables on the other side, gauge choices and field redefinitions turn out to play a significant role. Our gauge choice on the string side is motivated by the fact that the angular momentum is uniformly distributed along the spin chain. Moreover, comparing other $SO(3)$ conserved charges also suggests the required field redefinition which one can make in a systematic order by order way. On the spin chain side, we found that a naive semiclassical limit was not sufficient since high energy modes contribute to the low energy effective action starting at order $\tilde{\lambda}^2$ (producing terms of quartic and higher order in the field).

Given the non-trivial agreement between the spin chain and string effective actions at the two leading orders it seems natural to conjecture that, if one could sum the whole perturbative series on the gauge theory side, the resulting effective action would agree with the usual classical string action in $R \times S^3$ sector of $AdS_5 \times S^5$.

On the other hand, the simplicity of the original "unexpanded" Polyakov’s string action in $R \times S^3$ seems to suggest that there may be other more efficient methods of extracting relevant "string" information from the gauge theory side. In particular, let us mention that the idea of a low energy effective action for the Heisenberg-type spin chain is much more powerful than what we actually used here. For example, using the BMN result on the gauge theory side [1, 35] one can immediately fix all (higher-derivative) terms quadratic in $\vec{n}$ in the action. One may expect that the coefficient of the leading quartic term $(\partial \vec{n})^4$ we have computed here may be implicitly determining the coefficients of all other higher derivative terms quartic in $\vec{n}$.

Leaving aside the possibility of an all loop summation, there are many directions one can investigate to put these ideas on firmer ground. In particular, one would like to understand the role of subleading $1/J$ corrections. They should correspond to quantum $\alpha'$ corrections on the string side (for which fermionic terms become important [9]). The fact that the classical actions agree does not a priori guarantee the agreement of these subleading terms. It would also be interesting to understand the
map between string configurations and operators at higher orders. Let us recall how it worked at lowest order \[21\]: given a string configuration \(n(\sigma)\) on the string side (in the coordinates of section 2) one constructs a spin chain \(\prod_a |n_a\rangle\) with coherent states (by discretizing \(n(\sigma) \rightarrow n_a\)). Then one simply writes this state in the basis \(|\uparrow\rangle\) and \(|\downarrow\rangle\) of spins and identifies \(|\uparrow\rangle \rightarrow X, |\downarrow\rangle \rightarrow Y\). This gives a particular operator in the SYM theory that corresponds to the given string configuration. At the next order, that we analyze in the present paper, the map is already more involved since there is a field redefinition we need to do to put the string action into the form required to match the spin chain effective action. The field redefinition is suggestive of an identification between the mean value of the spin and the charge density associated to rotations on the string side. Presumably, a better understanding of the map at higher loops would require making this identification precise.

7 Acknowledgments

We are grateful to G. Arutyunov, N. Beisert, S. Deser, A. Dymarsky, S. Frolov, I. Klebanov, J. Maldacena, A. Marshakov, J. Minahan, A. Parnachev, J. Russo, M. Staudacher and K. Zarembo for useful discussions and/or e-mail correspondence. The work of M.K. was supported in part by NSF grants PHY-0331516, PHY99-73935 and DOE grant DE-FG02-92ER40706 and that of A.R. in part by NSF grant PHY99-73935. The work of A.T. was supported by DOE grant DE-FG02-91ER40690 and the INTAS contract 03-51-6346.
Appendix A Redefining away time derivatives

Let us explain the general procedure of elimination of time derivatives from the action by local field redefinitions order by order in $\frac{1}{J}$, including the first two orders in expansion. The key point is that the variation of the leading term in the action is proportional to $\partial_0 n$. Let us consider a general Lagrangian of field $n_i$ of the following structure

$$L = L_0 + \epsilon L_1 + \epsilon^2 L_2 + \ldots$$

where terms with $L_0' \equiv \delta L_0 / \delta n$ are proportional to leading-order equations of motion (in our case these are the terms containing $\partial_0 n_i$). The notation $(\ldots)_0$ is used for terms obtained by applying leading-order equations of motion to eliminate $\partial_0 n$ in terms of spatial derivatives. $b_n$ may depend on $n$ and its derivatives. $\epsilon$ is a small expansion parameter ($\frac{1}{J}$ in our case).

The idea is to make a field redefinition

$$n \to n + \epsilon m_1 + \epsilon^2 m_2 + \ldots$$

so that to eliminate the above $L_0'$-terms. Expanding in $\epsilon$

$$L = \left[ L_0 + \epsilon L_0' m_1 + \epsilon^2 L_0'' m_2 + \frac{1}{2} \epsilon^2 L_0''' m_1 m_1 + O(\epsilon^3) \right]$$

$$+ \epsilon [b_1 L_0' + (L_1)_0 + \epsilon b_1 L_0'' m_1 + \epsilon b_1' m_1 L_0' + \epsilon (\epsilon (L_1)_0)' m_1 + O(\epsilon^2)]$$

$$+ \epsilon^2 [b_2 L_0' + (L_2)_0 + O(\epsilon)] ,$$

and requiring $m_1 = -b_1$ we get

$$L = L_0 + \epsilon (L_1)_0 + \epsilon^2 \hat{L}_2 + O(\epsilon^3) ,$$

$$\hat{L}_2 = L_0' (b_2 + m_2 - m_1' m_1) - \frac{1}{2} \epsilon L_0'' m_1 m_1 + ((L_1)_0)' m_1 + (L_2)_0 .$$

The remaining $L_0'$ term can be canceled by a proper choice of $m_2$ term in the redefinition. In the case we are interested in, $L_1$ is quadratic in time derivative, i.e. contains a term quadratic in $L_0'$. Also, $L_0''$ may contain (up to integration by parts, and up to extra spatial derivatives) a term proportional to $L_0'$, i.e.

$$m_1 = -b_1 = c_1 L_0' + (m_1)_0 , \quad L_0'' = e_1 L_0' + (L_0'')_0 ,$$

where $(\ldots)_0$ again means part obtained by using the leading-order equations of motion. Then (A.5) becomes

$$\hat{L}_2 = L_0' \left[ b_2 + m_2 - m_1' m_1 \right]$$
\[ -\epsilon L_0''c_1(m_1)_0 - \frac{1}{2}\epsilon L_0''L_0'c_1c_1 + ((L_1)_0)'c_1 - \frac{1}{2}e_1(m_1)_0(m_1)_0 \]

\[ - \frac{1}{2}(L_0')_0(m_1)_0(m_1)_0 + ((L_1)_0)'(m_1)_0 + (L_2)_0 . \]  
(A.6)

Choosing \( b_2 \) to cancel the first bracket proportional to \( L_0' \) we are left with the following expression for the second correction after the required field redefinition.

\[ \hat{L}_2 = (L_2)_0 + ((L_1)_0)'(m_1)_0 - \frac{1}{2}(L_0')_0(m_1)_0(m_1)_0 . \]  
(A.7)

It is thus not enough to drop terms in \( L_2 \) proportional to leading order equations of motion (i.e. to consider its \((L_2)_0\) part) and also to include a variation of the first-order correction \((L_1')\) term: one also needs a term with second variation of \( L_0 \) (with the opposite sign to the one that would correspond to the expansion of \( L_0 \) term in \( L \)). In our case of \( L_0 \).

\[ L_0 = C_0 - \frac{1}{8}(\partial_1 n)^2 , \]
\[ L_1 = \frac{1}{8}(\partial_0 n_i)^2 + \frac{1}{128}(\partial_1 n_i)^4 , \]  
(A.8)

\[ L_2 = -\frac{1}{32}[(\partial_0 n_i\partial_1 n_i)^2 - \frac{1}{2}(\partial_0 n_i)^2(\partial_1 n_k)^2 + \frac{1}{32}(\partial_1 n_i\partial_1 n_i)^3] , \]  
(A.9)

\[ (L_0')_i = \frac{1}{2}\epsilon_{ijk}n_j\partial_0 n_k + \frac{1}{4}(\partial_1^2 n_i)_{\perp} . \]  
(A.10)

To eliminate time derivatives to first order we need to do a field redefinition with

\[ m_1 = -\frac{1}{2}[\frac{1}{2}\epsilon_{ijk}n_j\partial_0 n_k - \frac{1}{4}(\partial_1^2 n_i)_{\perp}] , \quad (m_1)_0 = \frac{1}{4}(\partial_1^2 n_i)_{\perp} . \]  
(A.11)

That gives \( L_1 \rightarrow (L_1)_0 \), i.e. the expression in \( 2.86 \) (again omitting tilda on redefined fields)

\[ (L_1)_0 = -H_1 = \frac{1}{32}(n'''' - \frac{3}{4}n'^4) , \quad n_i' = \partial_1 n_i . \]  
(A.12)

Below we shall need the relations following from \( n^2 = 1 \) (prime on \( n \) will stand for \( \partial_1 \)):

\[ n_i'' = -n_i' n' , \quad n_i''' = -3n_i' n'' , \quad (\epsilon_{ijk}n_i n_j n_k')^2 = n^2 n'^2 - (n'n'')^2 - n_i^6 , \]  
(A.13)

as well as

\[ n_i^2(n'n'') = -2(n'n'')^2 - n_i^2(n''')^2 - (n^2 n''')' . \]  
(A.14)

Using the leading-order equation of motion we have

\[ (\partial_0 n)^2 n_i^2 = \frac{1}{4}(n'' - n'^4)n_i^2 , \quad (n'\partial_0 n)^2 = \frac{1}{4}[n^2 n'' - (n'n'')^2 - n_i^6] , \]  
(A.15)

so that

\[ (L_2)_0 = -\frac{1}{64}[\frac{1}{4}n^2 n'' - \frac{1}{2}(n'n'')^2 - \frac{3}{16}n_i^6] . \]  
(A.16)
We also find by direct computation (using integration by parts)

\[
((L_1)_0)'(m_1)_0 \equiv \frac{\delta(L_1)_0}{\delta n}(m_1)_0 = -\frac{1}{64}[n''m^2 - \frac{5}{2}n'^2n''^2 - 11(n'n'')^2 + \frac{3}{2}n''^2]. \tag{A.17}
\]

Finally, we need to compute

\[
-\frac{1}{2}(L_0'')_0(m_1)_0(m_1)_0 \equiv -\frac{1}{2}(\delta^2L_0_0)(m_1)_0(m_1)_0 \\
\hspace{1cm} = -\frac{1}{64}\left[\left(\epsilon_{ijk}(n_i''n_i'\partial_0n_k'')_0 + \frac{1}{2}\left[((n_i'')_n)^2(n_i'')_n + n'^2((n_i'')_n)^2\right]\right), \tag{A.18}
\]

where \((...)_0\) again means that we are allowed to use the leading-order equations to eliminate time derivatives in favor of spatial ones. We then find, using (A.14) and ignoring total derivatives

\[
-\frac{1}{2}(L_0'')_0(m_1)_0(m_1)_0 = -\frac{1}{64}\left[\frac{1}{2}n'^2n'^2 - (n'n'')^2 - \frac{1}{2}n''^2\right]. \tag{A.19}
\]

Combining the above three contributions together we end up with the following expression for \(\hat{L}_2\) in (A.7)

\[
\hat{L}_2 = -H_2 = -\frac{1}{64}\left[n''m^2 - \frac{7}{4}n'^2n''^2 + \frac{25}{2}(n'n'')^2 + \frac{13}{16}n''^2\right]. \tag{A.20}
\]

**Appendix B  Some useful relations**

To simplify the expression for \(D_6\) in (3.4), (3.6) (i.e. to eliminate terms with repeated spins) we used the fact that spins at different sites commute, and also the special relation between the spin 1/2 \(SU(2)\) generators \(S_i = \frac{1}{2}\sigma_i\)

\[
2S_a^iS_a^j = [S_a^i, S_a^j] + \{S_a^i, S_a^j\} = \epsilon^{ijk}S_a^k + \frac{1}{2}\delta^{ij}. \tag{B.1}
\]

This implies

\[
(S_a \cdot S_b)(S_b \cdot S_c) = \frac{1}{4}S_a \cdot S_c + \frac{i}{2}\epsilon^{ijk}S_a^iS_b^jS_c^k, \tag{B.2}
\]

\[
(S_a \cdot S_b)(S_b \cdot S_c)(S_c \cdot S_d) = \frac{1}{16}S_a \cdot S_d + \frac{1}{4}(S_a \cdot S_d)(S_b \cdot S_c) - \frac{1}{4}(S_a \cdot S_c)(S_b \cdot S_d) \\
\hspace{1cm} + \frac{i}{8}\epsilon^{ijk}S_a^iS_b^jS_c^kS_d^k, \tag{B.3}
\]

and similar relations.

In taking the continuum limit we noted that

\[
N_{a+p,a+q} \equiv \frac{1}{2}(n_{a+p} - n_{a+q})^2 \rightarrow \sum_{m=l=0}^{\infty} \sum_{m=l}^{\infty} \frac{p^l q^{m-l}}{l!(m-l)!} \partial^l n \cdot \partial^{m-l} n
\]
\[ \frac{(p-q)^2}{2} (\partial n)^2 + \frac{(p+q)(p-q)^2}{2} (\partial n \cdot \partial^2 n) + \frac{(p-q)^2(p+q)}{8} (\partial^2 n)^2 \]
\[ + \frac{(p-q)^2(p^2 + pq + q^2)}{6} (\partial n \cdot \partial^3 n) + O(\partial^6 n) \]
\[ = \frac{(p-q)^2}{2!} (\partial n)^2 - \frac{(p-q)^4}{4!} (\partial^2 n)^2 + \frac{(p-q)^6}{6!} (\partial^3 n)^2 + O(\partial^8 n) \], \quad (B.4)

where \( \partial \) is derivative over \( x = \frac{j}{2\pi} \sigma \). We omitted total derivatives and used various identities following from \( n^2 = 1 \). From this we also find (again, integrating by parts)

\[ N_{a+p,a+q} N_{a+r,a+s} \rightarrow \frac{1}{4} (p-q)^2(r-s)^2(\partial n)^4 \]
\[ + \frac{1}{4} (p-q)^2(r-s)^2(p+q)(r+s)(\partial n \cdot \partial^2 n)^2 \]
\[ + \frac{1}{12} (p-q)^2(r-s)^2[(p+q)^2 + (r+s)^2 - pq - rs](\partial n)^2(\partial n \cdot \partial^3 n) \]
\[ + \frac{1}{16} (p-q)^2(r-s)^2[(p+q)^2 + (r+s)^2](\partial n)^2(\partial^2 n)^2 + O(\partial^8 n) \]. \quad (B.5)

Appendix C  Continuum limit of coherent state values of product operators

Let us repeat the computations done in section 3 for the following expectation values of the products of \( D_2 \) and \( D_4 \) operators in (3.1).* Let us define (subtracting disconnected parts of correlators)

\[ \tilde{D}_{2r} \equiv D_{2r} - \langle n | D_{2r} | n \rangle \], \quad (C.1)

and consider

\[ K_4 \equiv \langle n | \tilde{D}^2 | n \rangle = \langle n | (D_2)^2 | n \rangle - (\langle n | D_2 | n \rangle)^2 \], \quad (C.2)
\[ K_6 \equiv \frac{1}{2} \langle n | (\tilde{D}_2 \tilde{D}_4 + \tilde{D}_4 \tilde{D}_2) | n \rangle \]
\[ = \frac{1}{2} \langle n | (D_2 D_4 + D_4 D_2) | n \rangle - \langle n | D_2 | n \rangle \langle n | D_4 | n \rangle \], \quad (C.3)
\[ \tilde{K}_6 \equiv \langle n | (\tilde{D}_2)^3 | n \rangle \]
\[ = \langle n | (D_2)^3 | n \rangle - 3 \langle n | (D_2)^2 | n \rangle \langle n | D_2 | n \rangle + 2 (\langle n | D_2 | n \rangle)^3 \]. \quad (C.4)

One could expect a priori that since each \( D_{2r} \) factor involves a sum over sites, contributions of their products will be non-local in the continuum limit. However, the

*Although outside the main line of development of this paper, we include these results partially motivated by analogous remarks in [26, 20].
above “connected” parts turn out to contain local terms which are very similar to the ones present in $\langle n|D_4|n \rangle$ \textcolor{red}{(3.16)} and $\langle n|D_6|n \rangle$ \textcolor{red}{(3.20)}.

Let us start with $K_4$, i.e. (cf. \textcolor{red}{(3.7)}, \textcolor{red}{(3.9)})

$$K_4 = \sum_{a=1}^J \sum_{b=1}^J \langle n|(1 - \sigma_a \cdot \sigma_{a+1})(1 - \sigma_b \cdot \sigma_{b+1})|n \rangle - \left( \sum_{b=1}^J \langle n|(1 - \sigma_a \cdot \sigma_{a+1})|n \rangle \right)^2 . \quad \text{(C.5)}$$

To evaluate this we split the sum over $b$ in the first term into $3+1$ terms: with $b = a, b = a+1, b = a-1$ (whose sum will be denoted as $K_4^{(1)}$) and the rest (denoted $K_4^{(2)}$). The latter is given by (cf. \textcolor{red}{(3.15)})

$$K_4^{(2)} = \sum_{a=1}^J \sum_{b \neq a,a+1}^J N_{a,a+1} N_{a,a+1} - \left( \sum_{a=1}^J N_{a,a+1} \right)^2$$

$$= - \sum_{a=1}^J N_{a,a+1}(N_{a,a+1} + N_{a-1,a} + N_{a+1,a+2}) = - \sum_{a=1}^J N_{a,a+1}(N_{a,a+1} + 2N_{a-1,a}) . \quad \text{(C.6)}$$

The sum of the first three terms can be written as

$$K_4^{(1)} = \sum_{a=1}^J \langle n|(1 - \sigma_a \cdot \sigma_{a+1}) \left[ (1 - \sigma_a \cdot \sigma_{a+1}) + 2(1 - \sigma_{a-1} \cdot \sigma_a) \right]|n \rangle , \quad \text{(C.7)}$$

where we combined the terms with $b = a-1$ and $b = a+1$ by changing the summation variable from $a$ to $a + 1$ (in view of the periodicity of the chain). Simplifying this using \textcolor{red}{(B.1)}, \textcolor{red}{(B.2)} we get (note that $\epsilon^{ijk}n_i^a n_{a-1}^j n_{a+1}^k$ gives vanishing contribution)

$$K_4^{(1)} = \sum_{a=1}^J \langle n|(6 - 8\sigma_a \cdot \sigma_{a+1} + 2\sigma_a \cdot \sigma_{a+2})|n \rangle = 2 \sum_{a=1}^J (4N_{a,a+1} - N_{a,a+2}) . \quad \text{(C.8)}$$

Comparing this to \textcolor{red}{(3.7)} we conclude that

$$\langle n|(D_2)^2|n \rangle - (\langle n|D_2|n \rangle)^2 = -2 \langle n|D_4|n \rangle + K_4^{(2)} . \quad \text{(C.9)}$$

The continuum limit is then found to be (including the $\lambda^2$ factor and separating the overall power of $J$ with the integral over $\sigma$ as \textcolor{red}{(3.22)})

$$\frac{\lambda^2}{(4\pi)^2} K^{(2)} \rightarrow J \int_0^{2\pi} \frac{d\sigma}{2\pi} \left[ - \frac{3}{64} \lambda^2 (\partial_1 n)^4 + O\left( \frac{1}{J^2} \partial_1^6 n \right) \right] , \quad \text{(C.10)}$$

i.e. (cf. \textcolor{red}{(3.15)})

$$\frac{\lambda^2}{(4\pi)^2} \left[ (\langle n|(D_2)^2|n \rangle - (\langle n|D_2|n \rangle)^2 \right] \rightarrow J \int_0^{2\pi} \frac{d\sigma}{2\pi} \frac{1}{16} \lambda^2 (\partial_1^2 n)^2 - \frac{3}{4} (\partial_1 n)^4 . \quad \text{(C.11)}$$

It is a surprising coincidence that the local part of the coherent state expectation value of the operator $-\frac{1}{2} \frac{\lambda^2}{(4\pi)^2} (D_2)^2$ is thus the same as required to match the string
theory expression in (2.86). Note also that as in \langle n|D_4|n \rangle in (3.15) here there is no "subleading" \((\partial_1^2 n)^2\) term that would spoil the scaling limit.

Similarly, we find

\[
K_6 = \sum_{a=1}^{J} \left[ -30N_{a,a+1} + 12N_{a,a+2} - 2N_{a,a+3} - (N_{a,a+2} - 4N_{a,a+1})(N_{a,a+1} + N_{a+1,a+2} + N_{a-1,a}) - N_{a,a+2}N_{a+2,a+3} \right],
\]

which has similar structure to (minus) \langle n|D_6|n \rangle in (3.16). In the continuum limit then

\[
\frac{\lambda^3}{(4\pi)^6} K_6 \rightarrow J \int_0^{2\pi} d\sigma \frac{\lambda^3}{64} \left[ -\frac{1}{(2\pi)^2} J^2 (\partial_1 n)^4 \right.
\]

\[
- (\partial_1^2 n)^2 + \frac{7}{6} (\partial_1 n)^2 (\partial_1^2 n)^2 + \frac{25}{3} (\partial_1 n\partial_1^2 n)^2 + O\left(\frac{1}{J^2 \partial_1^8 n}\right). \]

\[\text{(C.12)}\]

A much more involved computation gives

\[
\frac{\lambda^3}{(4\pi)^6} \tilde{K}_6 \rightarrow J \int_0^{2\pi} d\sigma \frac{\lambda^3}{64} \left[ \frac{1}{(2\pi)^2} J^2 (\partial_1 n)^4 \right.
\]

\[
+ 2(\partial_1^2 n)^2 - \frac{31}{6} (\partial_1 n)^2 (\partial_1^2 n)^2 - \frac{55}{3} (\partial_1 n\partial_1^2 n)^2 + \frac{5}{2} (\partial_1 n)^6 + O\left(\frac{1}{J^2 \partial_1^8 n}\right). \]

\[\text{(C.13)}\]

The linear combination \(K'_6 = \frac{1}{2}(\bar{D} \bar{D} + \bar{D} \bar{D}) + (\bar{D}_2)^3\) thus has the coherent state expectation value that does not contain the "scaling-violating" \(J^2(\partial_1 n)^4\) term. Note that it contains \((\partial_1 n)^6\) term (that was absent in the expectation value of \(D_6\) in (3.21)).

The above discussion may serve as an indication that a proper account of quantum corrections may produce an effective spin chain action whose continuum limit will match the string theory result at the six-derivative order (2.89).

Appendix D Rotation Matrices

Here we compute the matrices \(A_{ij}\) that appear in section 4 and were useful to calculate the quantum corrections to the action. By definition they are

\[
A_{ij} = R_{li}(\vec{n}_a)R_{lj}(\vec{n}_{a+q}) - \delta_{ij} = \left(R^{-1}(\vec{n}_a)R(\vec{n}_{a+q})\right)_{ij} - \delta_{ij},
\]

\[\text{(D.1)}\]

where \(R\) is the matrix of the rotation

\[R_{lj}(\vec{n}_a) = e^{-i\phi_a S_z} e^{-i\theta_a S_y}\]

\[\text{(D.2)}\]
in the vector representation. The angles \(\phi_a\) and \(\theta_a\) are polar coordinates of \(\vec{n}_a\) (see (3.12)). The \(3 \times 3\) matrix \(A_{ij}\) can be evaluated as:

\[
A = e^{i\theta S_y} e^{-i\Delta \phi S_z} e^{-i\theta S_y}
\]

\[\text{(D.3)}\]
resulting in

\begin{align*}
A_{11} &= \cos \theta \cos \theta' \cos \Delta \phi + \sin \theta \sin \theta' - 1, \quad A_{12} = -\cos \theta \sin \Delta \phi, \\
A_{13} &= \cos \theta \sin \theta' \cos \Delta \phi - \sin \theta \cos \theta', \\
A_{21} &= \sin \Delta \phi \cos \theta', \quad A_{22} = \cos \Delta \phi - 1, \quad A_{23} = \sin \Delta \phi \sin \theta', \\
A_{31} &= \sin \theta \cos \theta' \cos \Delta \phi - \cos \theta \sin \theta' \cos \Delta \phi + \cos \theta \cos \theta' - 1.
\end{align*}

where for brevity we defined \( \theta = \theta_a \), \( \theta' = \theta_{a+q} \) and \( \Delta \phi = \phi_{a+q} - \phi_a \). Since the background field \( \vec{n}_a \) is slowly varying we can expand it in derivatives. The components relevant for the calculations in section 4 are:

\begin{align}
A_{++} &= A_{-}^* = (A_{11} - A_{22}) - i(A_{12} + A_{21}) \simeq \frac{1}{2} \tilde{A}_{++} + \ldots \\
A_{33} &= \tilde{n}_a \tilde{n}_{a+q} - 1 \simeq -\frac{1}{2} \left( \frac{2\pi q}{J} \right)^2 (\partial_\sigma \tilde{n})^2 + \frac{1}{24} \left( \frac{2\pi q}{J} \right)^4 (\partial^2_\sigma \tilde{n})^2 + \ldots 
\end{align}

with

\[ \tilde{A}_{++} = \left( \frac{2\pi q}{J} \right)^2 \left[ - (\partial_\sigma \theta)^2 + \sin^2 \theta (\partial_\sigma \phi)^2 + 2i \sin \theta \partial_\sigma \theta \partial_\sigma \phi \right] . \]

In \( A_{33} \) we omitted total derivatives since \( A_{33} \) appears in the action integrated over \( \sigma \). Notice also that

\[ |\tilde{A}_{++}|^2 = \left( \frac{2\pi q}{J} \right)^4 (\partial_\sigma \tilde{n})^4 . \]

**Appendix E  Integrals for the folded string solution**

Here we want to prove the fact, used in section 5, that the energy at order \( \lambda^2 \) does not receive any corrections from the term in the action which is quadratic in derivatives. This is tantamount to showing that eqs. (5.13), (5.14) imply also that at order \( \lambda^2 \)

\[
\int_{\theta_0}^{\theta_1} \sqrt{a + b \cos \theta + c \cos^2 \theta} \ d\theta \simeq \int_{0}^{\theta_0} \sqrt{a_0 + b_0 \cos \theta} \ d\theta .
\]

On both sides the upper limit of integration is a zero of the corresponding function under the square root. To check this fact we need to evaluate the following integrals\(^*\) for small \( c \):

\[
\int_{0}^{\theta_1} \frac{d\theta}{\sqrt{a + b \cos \theta + c \cos^2 \theta}} \simeq \sqrt{\frac{2}{b}} K + \frac{(b^2 - 2a^2 + ab)K + 2(2a^2 - b^2)E}{\sqrt{2b^2(b^2 - a^2)}} c + \ldots
\]

\(^*\)These integrals can be evaluated exactly in terms of standard elliptic functions \( E(x) \) and \( K(x) \) and then expanded for small \( c \).


\[ \int_{0}^{\theta_1} \frac{\cos \theta \, d\theta}{\sqrt{a + b \cos \theta + c \cos^2 \theta}} \approx \sqrt{\frac{2}{b}}(2E - K) + \]
\[ + \frac{\sqrt{2}}{3} \frac{(b^2 - 4a^2)K + (8a^2 + 2ab - 3b^2)E}{b^2(a + b)}c - \frac{(b^2 - 2a^2 + ab)K + 2(2a^2 - b^2)E}{\sqrt{2b^2}(b^2 - a^2)}c + \ldots \]
\[ \int_{0}^{\theta_1} \sqrt{a + b \cos \theta + c \cos^2 \theta} \, d\theta \approx \sqrt{\frac{2}{b}}(aK + b(2E - K)) + \frac{(2a + b)K - 4aE}{3\sqrt{2b^2}}c + \ldots \]

where we defined

\[ K = K \left( \frac{a + b}{2b} \right), \quad E = E \left( \frac{a + b}{2b} \right), \] (E.3)

with \( K(x) \) and \( E(x) \) being standard elliptic integrals. In these expressions we should replace \( a = a_0 + a_1, b = b_0 + b_1 \) and expand the result at first order in \( a_1, b_1 \). Demanding that this first order terms in \( J \) and \( S_3 \) cancel against the contributions proportional to \( c \) gives

\[ \frac{a_1}{c} = \frac{(b_0^2 + a_0b_0 - 2a_0^2)K_0^2 + (6b_0^2 - 8a_0^2)E_0^2 + (8a_0^2 - 2a_0b_0^2 - 6b_0^2)K_0E_0}{3b_0((b_0 - a_0)K_0^2 + 2(b_0 - a_0)K_0E_0 + 2b_0E_0^2)} \] (E.4)
\[ \frac{b_1}{c} = \frac{2(b_0^2 + a_0b_0 - 2a_0^2)K_0^2 + 2a_0b_0E_0^2 + (4a_0^2 - 2b_0^2 - 2a_0b_0)K_0E_0}{3b_0((b_0 - a_0)K_0^2 + 2(b_0 - a_0)K_0E_0 + 2b_0E_0^2)} \] (E.5)

where now

\[ K_0 = K \left( \frac{a_0 + b_0}{2b_0} \right), \quad E_0 = E \left( \frac{a_0 + b_0}{2b_0} \right), \] (E.6)

Applying the same procedure to the energy and using the values of \( a_1 \) and \( b_1 \) we have just obtained it is easy to show that the first correction also cancels. This means that the term quadratic in derivatives gives no correction to the energy at order \( \lambda^2 \).

References

[1] D. Berenstein, J. M. Maldacena and H. Nastase, “Strings in flat space and pp waves from N =4 super Yang Mills,” JHEP 0204, 013 (2002) [hep-th/0202021].
[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “A semi-classical limit of the gauge/string correspondence,” Nucl. Phys. B 636, 99 (2002) [hep-th/0204051].
[3] S. Frolov and A. A. Tseytlin, “Semiclassical quantization of rotating superstring in AdS_5 × S^5,” JHEP 0206, 007 (2002) [hep-th/0204226].
[4] A. A. Tseytlin, “Semiclassical quantization of superstrings: AdS_5 × S^5 and beyond,” Int. J. Mod. Phys. A 18, 981 (2003) [hep-th/0209116].
[5] M. Kruczenski, “A note on twist two operators in N = 4 SYM and Wilson loops in Minkowski signature,” JHEP 0212, 024 (2002) [hep-th/0210115].
[6] S. Frolov and A. A. Tseytlin, “Multi-spin string solutions in \( AdS_5 \times S^5 \),” Nucl. Phys. B 668, 77 (2003) [hep-th/0304255].

[7] J. G. Russo, “Anomalous dimensions in gauge theories from rotating strings in \( AdS_5 \times S^5 \),” JHEP 0206, 038 (2002) [hep-th/0205244].

[8] J. A. Minahan, “Circular Semiclassical String Solutions on \( AdS_5 \times S^5 \),” Nucl. Phys. B 648, 203 (2003) [hep-th/0209047].

[9] S. Frolov and A. A. Tseytlin, “Quantizing three-spin string solution in \( AdS_5 \times S^5 \),” JHEP 0307, 016 (2003) [hep-th/0306130].

[10] A. A. Tseytlin, “Spinning strings and AdS/CFT duality,” [hep-th/0311139].

[11] N. Beisert, J. A. Minahan, M. Staudacher and K. Zarembo, “Stringing spins and spinning strings,” JHEP 0309, 010 (2003) [hep-th/0306139].

[12] N. Beisert, S. Frolov, M. Staudacher and A. A. Tseytlin, “Precision spectroscopy of AdS/CFT,” JHEP 0310, 037 (2003) [hep-th/0308117].

[13] J. A. Minahan and K. Zarembo, “The Bethe-ansatz for \( N = 4 \) super Yang-Mills,” JHEP 0303, 013 (2003) [hep-th/0212208].

[14] S. Frolov and A. A. Tseytlin, “Rotating string solutions: AdS/CFT duality in non-supersymmetric sectors,” Phys. Lett. B 570, 96 (2003) [hep-th/0306143].

[15] G. Arutyunov, S. Frolov, J. Russo and A. A. Tseytlin, “Spinning strings in \( AdS_5 \times S^5 \) and integrable systems,” Nucl. Phys. B 671, 3 (2003) [hep-th/0307191].

[16] G. Arutyunov, J. Russo and A. A. Tseytlin, “Spinning strings in \( AdS_5 \times S^5 \) \new integrable system relations,” [hep-th/0311004].

[17] V. A. Kazakov, A. Marshakov, J. A. Minahan and K. Zarembo, “Classical/quantum integrability in AdS/CFT,” [hep-th/0402207].

[18] J. Engquist, J. A. Minahan and K. Zarembo, “Yang-Mills duals for semiclassical strings on \( AdS_5 \times S^5 \),” JHEP 0311, 063 (2003) [hep-th/0310188].

[19] C. Kristjansen, “Three-spin strings on \( AdS_5 \times S^5 \) from \( N = 4 \) SYM,” [hep-th/0402033].

[20] D. Serban and M. Staudacher, “Planar \( N = 4 \) gauge theory and the Inozemtsev long range spin chain,” [hep-th/0401057].

[21] M. Kruczenski, “Spin chains and string theory,” [hep-th/0311203].
[22] G. Arutyunov and M. Staudacher, “Matching higher conserved charges for strings and spins,” JHEP 0403, 004 (2004) [hep-th/0310182]. “Two-loop commuting charges and the string/gauge duality,” [hep-th/0403077].

[23] E. H. Fradkin, “Field Theories Of Condensed Matter Systems,” Redwood City, USA: Addison-Wesley (1991) 350 p. (Frontiers in physics, 82). I. Affleck, “Quantum spin chains and the Haldane gap,” J. Phys C 1 (1989), 3047

[24] S. Randjbar-Daemi, A. Salam and J. Strathdee, “Generalized spin systems and sigma models,” Phys. Rev. B 48, 3190 (1993) [hep-th/9210145].

[25] A. Perelomov, ”Generalized Coherent States and Their Applications”, Berlin, Germany: Springer (1986) 320 p.

[26] N. Beisert, C. Kristjansen and M. Staudacher, “The dilatation operator of N = 4 super Yang-Mills theory,” Nucl. Phys. B 664, 131 (2003) [hep-th/0303060].

[27] N. Beisert, “Higher loops, integrability and the near BMN limit,” JHEP 0309, 062 (2003) [hep-th/0308074].

[28] N. Beisert, “The su(2|3) dynamic spin chain,” [hep-th/0310252]

[29] D. Mateos, T. Mateos and P. K. Townsend, “Supersymmetry of tensionless rotating strings in AdS5 × S5 , and nearly-BPS operators,” [hep-th/0309114]. “More on supersymmetric tensionless rotating strings in AdS5 × S5 ,” [hep-th/0401058]

[30] A. Mikhailov, “Speeding strings,” [hep-th/0311019]. “Slow evolution of nearly-degenerate extremal surfaces,” [hep-th/0402067].

[31] A. M. Kosevich, B. A. Ivanov and A. S. Kovalev, “Magnetic Solitons”, Physics Reports 194,117 (1990).

[32] P. Goddard, J. Goldstone, C. Rebbi and C. B. Thorn, “Quantum Dynamics Of A Massless Relativistic String,” Nucl. Phys. B 56, 109 (1973).

[33] R. R. Metsaev, C. B. Thorn and A. A. Tseytlin, “Light-cone superstring in AdS space-time,” Nucl. Phys. B 596, 151 (2001) [hep-th/0009171].

[34] R. R. Metsaev and A. A. Tseytlin, “Superstring action in AdS5 × S5 : kappa-symmetry light cone gauge,” Phys. Rev. D 63, 046002 (2001) [hep-th/0007036].

[35] A. Santambrogio and D. Zanon, “Exact anomalous dimensions of N = 4 Yang-Mills operators with large R charge,” Phys. Lett. B 545, 425 (2002) [hep-th/0206079].
[36] T. Holstein and H. Primakoff, “Field Dependence of the Intrinsic Domain Magnetization of a Ferromagnet”, Phys. Rev. 58, 1098 (1940); D. Polder, Phil. Mag. 40, 99 (1949); M. Klein and R. Smith, “A Note on the Classical Spin-Wave Theory of Heller and Kramers”, Phys. Rev. 80, 1111 (1951); P.W. Anderson, “An Approximate Quantum Theory of the Antiferromagnetic Ground State”, Phys. Rev. 86, 694, (1952).