Quantitative Comparative Statics
for a Multimarket Paradox*

Tobias Harks† Philipp von Falkenhausen‡

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Abstract

Comparative statics is a well established research field where one analyzes how marginal changes in parameters of a strategic game affect the resulting equilibria. While classic comparative statics is mainly concerned with qualitative approaches (e.g., deciding whether a parameter change improves or hurts equilibrium profits or welfare), we provide a framework to expose the extend (not monotonicity) of a discrete (not marginal) parameter change, with the additional benefit that our results can even be used when there is uncertainty about the exact model instance. We apply our quantitative approach to the multimarket oligopoly model introduced by Bulow, Geanakoplos and Klemperer [6]. They describe the counterintuitive example of a positive price shock in the firm’s monopoly market resulting in a reduction of the firm’s equilibrium profit. We quantify for the first time the worst case profit reduction for multimarket oligopolies with an arbitrary number of markets exhibiting arbitrary positive price shocks. For markets with affine price functions and firms with convex cost technologies, we show that the relative loss of any firm is at most 25% no matter how many firms compete in the oligopoly. We further investigate the impact of positive price shocks on total profit of all firms as well as on consumer surplus. We find tight bounds also for these measures showing that total profit and consumer surplus decreases by at most 25% and 16.6%, respectively.

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†Maastricht University, School of Business and Economics, PO Box 616, 6200 MD, Maastricht, The Netherlands, Email: t.harks@maastrichtuniversity.nl

‡Technical University Berlin, Institute of Mathematics, Email: falkenhausen@math.tu-berlin.de
1 Introduction

Many economic settings can be modeled as strategic games $G = (N, X, u)$, where $N$ denotes a set of players, $X$ is the set of possible states of the games (all possible decisions by all players) and $u : X \to \mathbb{R}^N$ denotes the players’ payoff functions, i.e., in state $x \in X$ player $i$ receives payoff $u_i(x)$. Given an instance of such a model, there are parameters that determine the precise values for $N$, $X$, and $u$. Comparative statics is a well-established area in economics analyzing how a change in these parameters affects the equilibria of the game. Examples of such changing parameters are exposure to international trade (e.g., Krugman [22] and Melitz [25]) or a forced reduction of produced quantity (e.g., Gaudet and Salant [16]) – both affect the state space – or changes of the payoff functions via a demand shift (e.g., Quirmbach [33]), a cost shift (e.g., Fèvrier and Linnemer [15]), or the introduction of export taxes/subsidies (e.g., Brander and Spencer [5], and Eaton and Grossman [10]). The prevailing approaches in comparative statics up to date rely on using either the implicit function theorem1 (e.g., Bulow et al. [6]) or the powerful machinery of lattice theory applicable to supermodular games, see Amir [2], Edlin and Shannon [11], Kukushkin [23], Milgrom and Shannon [27], Milgrom and Roberts [26, 29], Shannon [36] and Topkis [38].

We argue in this paper that there are several important reasons (the first two were already formulated and addressed in the influential paper by Milgrom and Roberts [29]) why existing methodologies employed in comparative statics are unsatisfactory in practice. First, approaches using the implicit function theorem or standard sensitivity analysis in optimization based on first and second order conditions rely on a local analysis of parameter changes and, thus, the obtained conclusions only hold for small changes of a parameter and do not allow for meaningful conclusions if parameter changes happen to be discrete. Secondly, common to most existing approaches in comparative statics is the assumption of perfect knowledge of all elements of the model including the precise parameter change in order to carry out comparative statics analysis. In practice, the economic model including the endogenous variables and parameters may not be known precisely but only approximately by knowing some functional property of the relations of endogenous variables or the space of parameters. Third, common to all existing approaches in comparative statics addressing discrete parameter changes and possibly unknown variables and parameters (cf. Milgrom [28], Milgrom and Roberts [29] and Villas-Boas [42]) is the qualitative nature of the analysis: the aim is to identify conditions guaranteeing

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1The application of the implicit function theorem for instance in oligopoly models requires some regularity assumptions such as convexity and smoothness of cost and inverse demand functions, see the discussion in Milgrom and Roberts [29].

2An exception are the papers by Milgrom and Roberts [29] and Milgrom [28], where monotone comparative statics results are derived for classes of functions without knowing the precise instantiation.

3Milgrom and Roberts [29] refer to the set of possible instantiations of the model as a class of models, or the context "representing what the modeler knows about the economic environment".
the monotonicity of an effect caused by parameter changes. While a qualitative analysis is obviously important (e.g., it is immune to monotonic transformation of utilities), there are situations, where a qualitative assessment of an effect, e.g., in terms of how much money or welfare can possibly be lost, may it be monotone or not is vital for the modeler.

We illustrate these concerns by considering a multimarket oligopoly model as described by Bulow et al. [6 Sec. II]. Consider two markets located in separate countries with firms having convex cost technologies, one of which is a monopolist on one market. A tax decrease/increase in the country of the monopolist can be considered a positive/negative price shock, respectively. The government is now facing the problem of assessing the impact of this domestic tax change. Note that such a tax change is usually a discrete parameter change, thus, a local analysis might be a too rough estimate. Even if the change was marginal, the government usually has no precise knowledge of all elements of the market, that is, the cost technologies and inverse demand functions are not precisely known except for structural properties like market prices behave approximately affine and cost technologies are convex. Lastly and perhaps most important, the government needs to quantitatively assess the possible impact of an increase of domestic taxes in order to justify this action. In light of the uncertainty of model variables and parameters, a quantitative assessment of the possible impact of the tax change should be robust in the sense that it should not only hold on average but for any instantiation of the model, in particular, for worst case scenarios.

Quantitative Comparative Statics. We address the above concerns by advocating a quantitative approach to comparative statics. We aim at quantifying the maximum possible effect that the change of a parameter can have on a given set of games.

We adopt the framework of Milgrom and Roberts [29] and denote by \( \mathcal{G} \) a class of models or games describing the information set of what the modeler knows about the economic environment. Suppose there is an objective function \( f : \mathcal{G} \to \mathbb{R} \) (e.g., welfare of the unique equilibrium outcome) to evaluate these games and denote for any \( G \in \mathcal{G} \) by \( \Delta_G \) all parameter changes that are to be considered. We express the effect of a parameter change \( \delta \in \Delta_G \) on a game \( G \in \mathcal{G} \) (assuming for the moment \( f(G) \neq 0 \)) as the value

\[
\gamma^f(G, \delta) = \frac{f(G(\delta))}{f(G)},
\]

where \( G(\delta) \in \mathcal{G} \) denotes the changed game. We set \( \gamma^f(G, \delta) = 1 \), if \( f(G) = 0 \) and \( f(G(\delta)) \geq 0 \) and \( \gamma^f(G, \delta) = -\infty \), if \( f(G) = 0 \) and \( f(G(\delta)) < 0 \). The maximum possible effect across all games in \( \mathcal{G} \) and their respective parameter changes \( \Delta_G \) is then defined as

\[
\gamma^f(\mathcal{G}, \Delta_G) = \inf_{G \in \mathcal{G}} \inf_{\delta \in \Delta_G} \gamma^f(G, \delta).
\]
Note that if we are interested in the opposite direction of $f$ we can replace the infima with suprema in the above definition. Clearly, employing equilibrium selection for games with a multiplicity of equilibria (e.g., taking the best or worst equilibrium) might be of equal interest depending on the application at hand.

This worst case approach exhibits both

1. **significance**: are changes in a given parameter worth considering?
2. **robustness**: how sensitive is the game to changes of a parameter?

Significance is a crucial motivation of both the analysis of an effect and discussion of whether it can be put to use (à la `should a new tax be introduced?'). Robustness on the other hand is important when there is uncertainty about the values of variables and parameters and when they change over time. Our approach is robust in the sense that the actual value of $\gamma_f(G, \Delta_G)$ is independent of the actual instantiation of the model. The modeler only needs to know the functional specification of $f$ (e.g., social welfare), a compact description of the class of games $G$ (e.g., cost technologies are convex) and the space of parameter changes $\Delta_G$ (e.g., positive price shocks).

The possible impact of our methodology is of course naturally tied to successful applications. We demonstrate the usefulness of our approach by applying it to the multimarket oligopoly model introduced by Bulow, Geanakoplos and Klemperer [6].

**A Paradox in Multimarket Cournot Oligopolies.** Bulow, Geanakoplos and Klemperer [6] investigated how "changes in one market have ramifications in a second market" and discovered that a positive price shock in a firm’s monopoly market can have a negative effect on the firm’s profit by influencing competitors’ strategies in a different market. This counterintuitive phenomenon led them to the classification of markets in terms of strategic substitutes and strategic complements. Our paper is about rigorously quantifying profit effects induced by price shocks in multimarket Cournot oligopolies.

Let us recall the example of two markets $\{1, 2\}$ and two firms $\{a, b\}$ given in [6]. Firm $a$ is a monopolist on market 1 and competes with firm $b$ on market 2. Demand is infinitely elastic on market 1 for the constant price $p_1 = 50$. On market 2, there is an affine price function given by $p_2(q_{a,2} + q_{b,2}) = 200 - (q_{a,2} + q_{b,2})$, where $q_{a,2}, q_{b,2}$ denote the quantities offered by the respective firms on market 2. Production costs are symmetric and given by $c(q) = \frac{1}{2}q^2$, where $q$ is the total quantity produced by a firm. In the Cournot equilibrium, we obtain $q_{a,1} = 0$ and $q_{a,2} = q_{b,2} = 50$ and each firm earns profits 3750.

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4See Section VII in [6].
5In a market with strategic substitutes, more aggressive play by a firm leads to less aggressive play of the competitors on that market; with strategic complements, more aggressive play results in more aggressive play of the competitors.
6Bulow et al. [6] assume additional fixed cost $F > 0$ to prevent firms from setting up multiple plants. Fixed costs, however, do not change the resulting equilibria assuming that the access to markets is exogenously determined.
Suppose now that market 1 experiences a positive price shock raising its constant price by five units to 55. The Cournot equilibrium changes to \( q_{a,1} = 8 \) and \( q_{a,2} = 47 \), \( q_{b,2} = 51 \). Under this new equilibrium, firm \( b \) increases its profit to 3901.5 while firm 1 obtains after the price shock a profit of 3721.5. As noted by Bulow et al. [6], the ”positive” price shock to market 1 has hurt \( a \). The actual profit reduction for firm \( a \) amounts to 0.76%. A natural question to ask is: how much can a firm lose from a positive price shock on its monopoly market?

**Our Results and Paper Organization.** We conduct quantitative comparative statics analysis for a class of games that contains the above example. We consider positive price shocks as parameter changes and the class of games contains multimarket oligopolies with affine price functions and convex cost technologies of firms. We consider three different objective functions: the individual profit of a firm, welfare measured by summing up the firm’s profits, and social surplus defined by integrating the price functions and subtracting the firm’s costs. We find that both profit and welfare can be reduced at most 25% by a positive price shock. For profit we give the bound as a function of the number of firms, showing e.g. that in the two firm case the profit reduction is a most 6.25%. Social surplus on the other hand can only be reduced by up to 16.7% by a positive price shock. Our results give the first quantitative comparative statics result of an important paradoxical phenomenon previously only qualitatively analyzed. Each of our bounds is complemented by an example instance that actually attains the worst case ratio. While a common criticism of worst case analysis is that isolated special cases lead to unrealistic objective function values, we show that in our setting any instance exhibits this maximum effect as long as it fulfills a set of rather non-restrictive properties and, hence, such criticism does not apply.

The rest of the paper is organized as follows. After reviewing the related literature in Section 2 we introduce in Section 3 a class of multimarket oligopoly models including the specification of possible parameter changes via positive price shocks. In Section 4 we rigorously analyze the worst case profit reduction of an individual firm. In Section 5 we continue with giving tight bounds for the worst case reduction of total firm’s profits as well as social surplus.

### 2 Related Literature

Comparative statics has long been used to understand how a system is affected by changes of underlying parameters, see Dixit [9] for an overview.

Since Bulow et al. [6] introduced the concepts of strategic substitutes and strategic complements, a considerable amount of research has used comparative statics to investigate situations where strategic substitutes or complements occur. Notably, Brander and
Spencer [5] found that an export subsidy can increase welfare in Cournot competition with strategic substitutes (although they do not name the concept). Eaton and Grossman [10] extended this model to a two-stage game where first governments set policies and then firms engage in competition. Gaudet and Salant [16] looked at how a forced marginal change of production quantity for a subset of firms impacts profit in a situation with strategic substitutes. Février and Linnemer [15] gave a decomposition of price shocks in Cournot oligopoly into an average effect common to all firms and a heterogeneity effect that is firm-specific. Comparative statics has been successfully applied for supermodular games, see Athey [4], Quah [32], Milgrom and Shannon [27], Topkis [37, 38] and Vives [43, 44]. Acemoglu and Jensen [1] present a framework for comparative statics results for a superclass of Cournot oligopoly called aggregative games, see also Corchón [7]. Common to all these works is that they only consider the monotonicity of effects, i.e., whether a certain objective function increases or decreases with a parameter. However, they leave the quantification of the effect open.

Worst case perspectives have a long tradition in the analysis of performance guarantees of algorithms (cf. Williamson and Shmoys [45]). With the introduction of concepts like the ”price of anarchy” they have found their way into economics and game theory (cf., Koutsoupia and Papadimitriou [21] and Nisan et al. [30]). There are several papers quantifying the worst case efficiency loss of equilibria in models of Cournot competition (cf. Anderson and Renault [3], Farhat and Perakis [13, 14], Kluberg and Perakis [19] and Titsiklis and Xu [39]). All these works follow the ”price of anarchy” methodology, where some performance measure of an equilibrium outcome (e.g., total profit and/or consumer surplus) is compared with a corresponding optimal production maximizing the respective performance measure. There is a conceptual difference of our approach compared to these previous worst case approaches: while in approximation algorithms and price of anarchy one compares an optimum to a solution that is returned by an algorithm or that is an equilibrium, we maintain an equilibrium concept and compare the equilibrium of an instance with that of a perturbed instance. In the context of multimarket Cournot oligopoly, to the best of our knowledge, we derive for the first time a worst case quantification of a parameter change previously only qualitatively described by comparative statics.

There are works in theoretical computer science and mathematics that quantify the effects of a parameter change, albeit for different settings. Prominent examples include the analysis of the famous Braess paradox, where a limit on the decrease in a network’s performance when additional links are added is shown (cf. Korilis et al. [20], Lin et al. [24], Roughgarden [35] and Valiant and Roughgarden [41]).
3 Multimarket Cournot Oligopoly Model

In this section we introduce the specific model for which we investigate quantitative comparative statics. As outlined above, the three ingredients for such analysis are a set of games $G$, for each such game $G \in G$ as set of feasible parameter changes $\Delta G$ and an objective function $f : G \to \mathbb{R}$ to evaluate the games.

Class of games $G$. As the class of games $G$ we consider multimarket oligopolies defined as follows: In a game $G \in G$ a set $N$ of $n$ firms competes on a set of markets $M$. Each firm $i \in N$ has access to some subset $M_i \subseteq M$ of these markets. A strategy of a firm $i$ is to choose production quantities $q_{i,m}$ for all markets $m \in M_i$ that it serves. We set $q_{i,m} = 0$ for any market not served by firm $i$ and denote the total quantity of firm $i$ by $q_i := \sum_{m \in M_i} q_{i,m}$. Correspondingly, the total quantity on any market $m$ is denoted by $q_m := \sum_{i \in N} q_{i,m}$.

Assumption 1. The price on the markets is an affine function of the total quantity produced, i.e., if on market $m \in M$ a quantity $q_m$ is produced the price is $p_m - r_m q_m$ where $p_m$ is an initial positive price and the coefficient $r_m > 0$ describes how the price decreases as the demand is satisfied. We sometimes denote the price on a market simply by $p_m(q_m)$.

Assumption 2. Firm $i$’s cost for producing the quantity $q_i$ is given by the function $c_i(q_i)$ which we assume to be nondecreasing, convex and differentiable in $q_i$ with $c_i(0) = 0$.

Remark 3.1. If $\mathcal{C}$ denotes a class of cost functions satisfying Assumption $\square$ and $\mathcal{A}$ denotes a class of inverse demand functions satisfying Assumption $\blacksquare$, we can compactly represent $G$ by the tuple $(N, (M_i)_{i \in N}, \mathcal{C}, \mathcal{A})$ without specifying the precise instantiation of each game in $G$.

Given production quantities for all firms, the utility of firm $i$ is defined as $u_i(q) := \sum_{i \in M} p_m(q_m)q_{i,m} - c_i(q_i)$. In a Cournot equilibrium, firms choose their quantities so as to maximize their utility. Particularly, in a Cournot equilibrium $q$ no firm can increase its utility by unilaterally deviating to a different strategy, i.e., $u_i(q) \geq u_i(q_i', q_{-i})$ for all strategies $q_i'$ available to firm $i$. As the games introduced above involve convex and compact strategy spaces together with quasi-concave utilities, standard fixed-point arguments of type Kakutani (cf. $[8, 12, 17, 18]$) imply the existence of an equilibrium. As shown in Section $\square$ the assumptions on the price and cost functions further imply that there is a unique equilibrium.

We denote the marginal revenue of firm $i$ on market $m$ by

$$\pi_{i,m}(q_{i,m}, q_{-i,m}) = \frac{\partial}{\partial q_{i,m}} (p_m(q_{i,m} + q_{-i,m})q_{i,m}) = p_m(q_m) - r_m q_{i,m},$$
where \( q_{-i,m} \) is the quantity sold by firms \( j \neq i \). The marginal cost is \( c'_i(q_i) \), and we will often write \( \pi_{i,m}(q) \) and \( c'_i(q) \). In an equilibrium \( x \), the marginal revenue of any served market \( m \) equals the marginal cost: \( \pi_{i,m}(x) = c'_i(x) \) for all \( i \in N \) with \( x_{i,m} > 0 \).

**Parameter changes \( \Delta_G \).** The parameter changes analyzed are positive shocks to the price functions: on every market \( m \) of a game \( G \in \mathcal{G} \) the price increases by some amount \( \delta_m \geq 0 \) such that \( p^\delta_m(q_m) = p_m(q_m) + \delta_m \). We denote the set of feasible parameter changes as \( \Delta_G := \mathbb{R}_{\geq 0}^{|M|} \), where \(|M|\) is the number of markets in \( G \).

**Objective functions \( f \).** The games in \( \mathcal{G} \) have unique Cournot equilibria (as shown in Section 4), and our three objective functions are equilibrium firm profit, equilibrium welfare and equilibrium social surplus. Given a game \( G \) and a price shock \( \delta \), we denote the unique equilibrium of \( G \) by \( x \) and the unique equilibrium of \( G(\delta) \) by \( y \).

1. **firm profit:** the profit of an individual firm is \( u_i(q) = \sum_{i \in M} p_m(q_m)q_{i,m} - c_i(q_i) \) and we minimize across the firms of a game to obtain an overall measure:

\[
\gamma^u(G, \delta) := \min_{i \in N} \frac{u_i^\delta(y)}{u_i(x)}.
\]

2. **welfare:** we consider utilitarian welfare, i.e. the sum of the profits of all firms \( U(q) := \sum_{i \in N} u_i(q) \), such that

\[
\gamma^U(G, \delta) := \frac{U^\delta(y)}{U(x)}.
\]

3. **social surplus:** this measure assumes that the price function of a market expresses the marginal value that the buyers in the market have from additional quantity of the good. We denote \( S(q) := \sum_{m \in M} \int_0^{q_m} p_m(z)dz - \sum_{i \in N} c_i(q_i) \), such that

\[
\gamma^S(G, \delta) := \frac{S^\delta(y)}{S(x)}.
\]

While the first measure has been analyzed by Bulow et al. [6] the second two measures have been used among others by Anderson and Renault [3] and Ushio [40]. For each measure, we strive to provide an infimum bound across all multimarket Cournot games in \( \mathcal{G} \) and price shocks \( \Delta_G \) combined with concrete games that converge to this bound. The following example instance exhibits the basic intuition for the quantitative analysis in Sections 4 and 5.

**Example instance.** Consider a game with two markets. Market 1 is served only by a monopolist (denoted as firm \( a \)), while all firms compete in market 2. Given a positive price shock on market 1, firm \( a \) will reduce its quantity on market 2 in favor of selling more on
its more profitable monopoly market, see Figure 1a. In effect, the competitors’ marginal revenue strictly increases and leads to a new equilibrium in which these competitors increase their quantities, see Figure 1b. Markets in which a less aggressive play of one firm leads to a more aggressive play of competitors are called markets with strategic substitutes [6]. The more aggressive competition experienced by firm \( a \) in market 2 reduces its profit below the original level, or said differently, after the price shock a part of firm \( a \)’s quantity on market 2 has been substituted by the competitors, see Figure 1c.

**Negative Price Shocks.** While we restrict our analysis to positive price shocks, the results immediately extend to negative price shocks. If for example a subsidy (i.e. a positive price shock) causes the equilibrium to shift such that the firms’ profits decrease, then taking back the subsidy (i.e. a negative price shock) will restore the old equilibrium and thus increase the firm’s profits. In this sense the two effects are dual: any negative price shock can be seen as taking back a positive price shock, and the profit gain from the negative price shock is equal to the profit loss from positive price shock. This is true for any objective function \( f \) and, denoting negative price shocks to a game by \( \Delta_{-G} \) we have

\[
\gamma_f(G, \Delta_{-G}) = (\gamma_f(G, \Delta_{G}))^{-1}.
\]

### 4 Maximum Profit Loss of An Individual Firm

We use quantitative comparative statics to investigate the worst case profit loss of a firm from a positive price shock as expressed by \( \gamma^u(G, \Delta_{G}) \).

**Theorem 4.1.** Given a game \( G \) with \( n \) firms, no firm loses more than a \( \frac{(n-1)^2}{4n^2} \) fraction of its profit from a price shock \( (\delta_m)_{m \in M} \) with \( \delta_m \geq 0 \) for all \( m \in M \),

\[
\gamma^u(G, \delta) \geq 1 - \frac{(n-1)^2}{4n^2} \geq \frac{3}{4}.
\]

This is our main result. It shows that no firm loses more than 25% of current profits. This bound is robust in the sense that it holds for an entire class of games and parameters, that is, in order to arrive at this bound the modeler only needs to know that price shocks are nonnegative, inverse demand functions are affine, and cost technologies are convex.

To prove the statement, we first establish uniqueness of equilibria and that a price shock \( \delta \geq 0 \) causes the price on every market to increase, and that in this favorable setting every firm increases its total quantity. Using these insights into the effect of the price shock, for any given firm \( i \) we can identify a market where this firm suffers the relatively strongest loss and use this to bound \( \frac{\omega_i^u(y)}{u_i(x)} \). This proves the theorem, as \( \gamma^u(G, \delta) \) is the minimum of this fraction across all firms.

**Lemma 4.1 (Uniqueness of equilibria).** Let \( x \) and \( y \) be equilibria of some game \( G \). Then \( x = y \).
(a) Initial equilibrium $x$: The price on market 1 is constant at $p_1$ (thus also firm $a$’s marginal revenue). On market 2 the price is decreasing, such that firm $a$ sees marginal revenue $\pi_{a,2}(\cdot, x-a)$, given its competitors’ equilibrium quantities $x-a$. Firm $a$ produces $x_{a,2}$ on market 2 such that the marginal revenue there is equal to the marginal revenue $p_1$ on market 1. Its production on market 1 is such that the marginal cost of the aggregate quantity $x_{a,1} + x_{a,2}$ is equal to $p_1$. The competitors $i \neq a$ have marginal revenue $\pi_{i,2}(\cdot, x-i)$ on market 2. They choose their equilibrium quantity $x_{i,2}$ at the intersection of marginal cost and marginal revenue.

(b) Price shock triggers strategic substitution: The shock leads firm $a$ to shift production from market 2 to market 1. This increases the marginal revenue of competitors on market 2, causing increased production $y_{i,2} > x_{i,2}$.

(c) Effect on firm $a$’s profit: This derogates firm $a$’s marginal revenue on market 2, causing it to further withdraw from the market. The shaded area indicates the profit loss from the strategic substitution, which is partially compensated by profit gain from the increased price on market 1.
Proof. As a first step of the proof, we show that on every market \( m \), \( x_m = y_m \). Let \( M^+ := \{ m \in M : x_m < y_m \} \) and assume for a contradiction that \( M^+ \neq \emptyset \). Then, there is a firm \( i \) with \( \sum_{m \in M^+} (y_{i,m} - x_{i,m}) > 0 \) and a market \( m \in M^+ \) with \( y_{i,m} > x_{i,m} \geq 0 \). It follows from the equilibrium definition that in \( y \) firm \( i \)'s marginal cost and marginal revenue on \( m \) are equal, i.e.,

\[
c'_i(y_i) = p_m(y) - r_{m}y_{i,m} < p_m(x) - r_{m}x_{i,m} \leq c'_i(x),
\]

where we used that \( p_m(y) < p_m(x) \) because \( m \in M^+ \). From \( c'_i(y_i) < c'_i(x) \) we follow that

\[
y_i < x_i.
\]

Also, for all markets \( m' \in M \) where \( y_{i,m'} < x_{i,m'} \), we follow again from the equilibrium definition that

\[
p_{m'}(y) - r_{m'}y_{i,m'} \leq c'_i(y_i) < c'_i(x) \leq p_{m'}(x) - r_{m'}x_{i,m'},
\]

and hence, \( p_{m'}(y) < p_{m'}(x) \), i.e. \( m' \in M^+ \). Then, we find a contradiction as

\[
0 > y_i - x_i = \sum_{m \in M} (y_{i,m} - x_{i,m}) > \sum_{m \in M^+} (y_{i,m} - x_{i,m}) > 0.
\]

Here, we can limit the summation from all \( m \neq 1 \) to \( m \in M^+ \) because we found that \( m' \in M^+ \) for all markets with \( y_{i,m'} < x_{i,m'} \).

As the next step of the proof, we use \( x_m = y_m \) for all \( m \in M \) to show \( x_{i,m} = y_{i,m} \) for all firms \( i \). For a contradiction, assume there are \( i \in N \) and \( m \in M \) such that \( x_{i,m} < y_{i,m} \). Then, we can again apply \( (1) \) to obtain \( y_i < x_i \) and there must be some market \( m' \in M \) with \( y_{i,m'} < x_{i,m'} \), which leads with the same reasoning to \( x_i < y_i \), a contradiction. Altogether, \( x = y \). \( \square \)

We now show that the prices on all markets increase after the positive price shock.

**Lemma 4.2.** Let \( x \) and \( y \) be equilibria of a game \( G \) before and after a price shock \( (\delta_m)_{m \in M} \) with \( \delta_m \geq 0 \) for all \( m \in M \). Then, on all markets \( m \in M \) the price in \( y \) is higher than in \( x \), i.e. \( p_{m}(y_m) \geq p_{m}(x_m) \).

**Proof.** While in the proof of Lemma 4.1 we compared two equilibria of the same game, we now compare the equilibria before and after the price shock. The analysis remains largely unchanged: if we denote by \( M^+ := \{ m \in M : x_m + \delta_m < y_m \} \) the set of markets where the price decreases, then \( M^+ \neq \emptyset \) still implies that there is some firm \( i \) with \( \sum_{m \in M^+} (y_{i,m} - x_{i,m}) > 0 \) and \( y_i < x_i \) as in \( (2) \), leading to the same contradiction as before. We follow \( M^+ = \emptyset \), i.e. \( p_{m}(y_m) \geq p_{m}(x_m) \) for all \( m \in M \). \( \square \)
Given increasing prices, all firms increase their quantity.

**Lemma 4.3.** Let $x$ and $y$ be equilibria of a game $G$ before and after a price shock $\delta_m \in M$ with $\delta_m \geq 0$ for all $m \in M$. Then, each firm $i \in N$ produces more in $y$ than in $x$, $y_i \geq x_i$.

**Proof.** Assume there is $i \in N$ with $y_i < x_i$. Then

$$\pi_{i,m}(y) \leq c'(y) \leq c'(x) = \pi_{i,m}(x)$$

on every market $m \in M$ with $x_{i,m} > 0$. With $p_{m}(y) \geq p_{m}(x)$ as found in the previous lemma, we follow $y_{i,m} \geq x_{i,m}$ on every market with $x_{i,m} > 0$, a contradiction to $y_i < x_i$. Hence $x_i \leq y_i$ for all $i \neq a$. $\square$

We are now ready to prove the main theorem.

**Proof of Theorem 4.1.** We show that for any given firm $i$, $u_i^\delta(y) \geq u_i(x)$ with $u_i(x) \geq 1 - \frac{(n-1)^2}{4n^2}$. The theorem follows because $\gamma(G, \delta)$ is the minimum of this quantity across all firms.

Denote the set of markets where firm $i$ decreases their quantity after the price shock by $M^- := \{m \in M : y_{i,m} < x_{i,m}\}$ and similarly the set where $i$ increases their quantity after the price shock by $M^+ := M \setminus M^-$. We assume that the markets are indexed such that market 1 is a solution to

$$\arg\min_{m \in M^-} \frac{(p_{m}^\delta(y) - c'(x))x_{i,m} - r_{m}y_{i,m}(x_{i,m} - y_{i,m})}{(p_{m}(x) - c'(x))x_{i,m}}.$$ (3)

Note that the denominator of the above fraction is always positive as any market $m \in M^-$ has a non-zero quantity $x_{i,m} > 0$ and thus by the first order equilibrium condition also $p_{m}(x) > c'(x)$.

We find the following relations that will be helpful later on: The quantity added on markets in $M^+$ corresponds exactly to the quantity taken away from markets in $M^-$ and the additional quantity $y_i - x_i$, i.e.

$$\sum_{m \in M^+} y_{i,m} - x_{i,m} = \sum_{m \in M^-} x_{i,m} - y_{i,m} + y_i - x_i.$$ (4)

Also, the price on markets $m \in M^+$ in $y$ is related to the marginal cost, i.e.

$$p_{m}^\delta(y) \geq p_{m}^\delta(y) - r_{m}y_{i,m} = \pi_{i,m}^\delta(y) = c'(y)$$ (5)

which in turn is related to the price on markets $m \in M^-$, i.e.

$$c'(y) \geq \pi_{i,m}^\delta(y) = p_{m}(y) - r_{m}y_{i,m}.$$ (6)
As the cost function is convex,

\[ c_i'(y)(y_i - x_i) \geq c_i(y) - c_i(x). \quad (7) \]

We combine the above to a statement that relates the profit of quantity added to \( M^+ \) to the profit lost by reducing quantity in \( M^- \),

\[
\sum_{m \in M^+} p_m^\delta(y)(y_{i,m} - x_{i,m}) \geq \sum_{m \in M^+} c_i'(y)(y_{i,m} - x_{i,m}) \\
\geq \sum_{m \in M^-} c_i'(y)(x_{i,m} - y_{i,m}) + c_i'(y)(y_i - x_i) \\
\geq \sum_{m \in M^-} (p_m^\delta(y) - r_{m,y_{i,m}})(x_{i,m} - y_{i,m}) + c_i(y) - c_i(x). \quad (8)
\]

We further assume that \( \frac{\delta_i(y)}{u_i(x)} < 1 \), as we are interested in worst case instances and our lower bounds show that such instances exist. Note that for a fraction with value less than 1, subtracting the same amount from both numerator and denominator decreases the value of the fraction. We estimate

\[
\frac{\delta_i(y)}{u_i(x)} = \frac{\sum_{m \in M^-} p_m^\delta(y)y_{i,m} + \sum_{m \in M^+} p_m^\delta(y)y_{i,m} - c_i(y)}{\sum_{m \in M^-} p_m(x)x_{i,m} + \sum_{m \in M^+} p_m(x)x_{i,m} - c_i(x)} \\
\geq \frac{\sum_{m \in M^-} (p_m^\delta(y) - c_i'(x))x_{i,m} - r_{m,y_{i,m}}(x_{i,m} - y_{i,m}) + \sum_{m \in M^+} (p_m^\delta(y) - c_i'(x))x_{i,m}}{\sum_{m \in M^-} (p_m(x) - c_i'(x))x_{i,m} + \sum_{m \in M^+} (p_m(x) - c_i'(x))x_{i,m}} \\
\geq \frac{\sum_{m \in M^-} (p_m^\delta(y) - c_i'(x))x_{i,m} - r_{m,y_{i,m}}(x_{i,m} - y_{i,m})}{\sum_{m \in M^-} (p_m(x) - c_i'(x))x_{i,m}} \\
\geq \frac{(p_1^\delta(y) - c_i'(x))x_{i,1} - r_1y_{i,1}(x_{i,1} - y_{i,1})}{(p_1(x) - c_i'(x))x_{i,1}}. \quad (9)
\]

In (9) we use that the cost function is convex and hence \(-c_i(x) \geq \sum_{m \in M} c_i'(x)x_{i,m}\) and in (10) that the price on a market with positive quantity is at least the marginal cost, i.e. \( p_m(x) \geq c_i'(x) \) for a market with \( x_{i,m} > 0 \).

We now need to further examine the relation of \( p_1^\delta(y) \), \( p_1(x) \) and \( c_i'(x) \). For any firm \( j \neq i \) that has increased its quantity on market 1, i.e. \( y_{j,1} > x_{j,1} \),

\[ p_1(x) - r_1x_{j,1} = \pi_{j,1}(x) \leq \pi_j'(x) \leq c_j'(y) = \pi_j^\delta(y) = p_j^\delta(y) - r_jy_{j,1}, \]
that is,

\[ r_1(y_{j,1} - x_{j,1}) \leq p_1^\delta(y) - p_1(x). \]  \hspace{1cm} (12)

Then, considering \( \sum_{j:y_{j,1} > x_{j,1}} (y_{j,1} - x_{j,1}) + y_{i,1} - x_{i,1} \geq y_1 - x_1 \), we can rather precisely observe how the price on market 1 changes with the price shock.

\[ p_1^\delta(y) = p_1(x) + \delta_1 - r_1(y_1 - x_2) \geq p_1(x) + r_1(x_{i,1} - y_{i,1}) - r_1 \sum_{j:y_{j,1} > x_{j,1}} (y_{j,1} - x_{j,1}) \]

\[ \geq p_1(x) + r_1(x_{i,1} - y_{i,1}) - (n - 1)(p_1^\delta(y) - p_1(x)), \]

as there are at most \( n - 1 \) firms with \( y_{j,1} > x_{j,1} \). This can be rearranged to

\[ p_1^\delta(y) \geq p_1(x) + \frac{r_1}{n}(x_{i,1} - y_{i,1}). \]  \hspace{1cm} (13)

Observe further that \( x_{i,1} > 0 \) because market 1 is in \( M^- \) and hence

\[ p_1(x) - r_1x_{i,1} = \pi_{i,1}(x) = c_i^\prime(x). \]  \hspace{1cm} (14)

We continue the proof from (11),

\[ \frac{u_1^\delta(y)}{u_1(x)} \geq \frac{(p_1^\delta(y) - c_i^\prime(x))x_{i,1} - r_1y_{i,1}(x_{i,1} - y_{i,1})}{(p_1(x) - c_i^\prime(x))x_{i,1}} \]

\[ \geq \frac{(p_1(x) - c_i^\prime(x))x_{i,1} + \frac{n}{n'}(x_{i,1} - y_{i,1})x_{i,1} - r_1y_{i,1}(x_{i,1} - y_{i,1})}{(p_1(x) - c_i^\prime(x))x_{i,1}} \]

\[ = 1 + \frac{r_1\left(\frac{1}{n}x_{i,1} - y_{i,1}\right)(x_{i,1} - y_{i,1})}{(p_1(x) - c_i^\prime(x))x_{i,1}} \]

\[ = 1 + \frac{\left(\frac{1}{n}x_{i,1} - y_{i,1}\right)(x_{i,1} - y_{i,1})}{x_{i,1}^2} \]

\[ \geq 1 - \frac{(n - 1)^2}{4n^2} = \frac{3}{4} + \frac{2n - 1}{4n^2}. \]

\[ \square \]

4.1 Lower Bound

To show that the bound of the previous theorem is tight, we construct a simple instance with matching profit loss.

**Proposition 4.1.** For any \( n \), there is an game \( G \) with \( n \) firms where a positive price shock decreases the profit of some firm \( a \) by a factor \( \frac{(n-1)^2}{4n^2} \), that is,

\[ \gamma^n(G, \delta) = 1 - \frac{(n-1)^2}{4n^2}. \]

**Proof.** The instance has two markets \( M = \{1, 2\} \) and there are \( n \) firms. All firms serve
market 2 while market 1 is only served by some firm $a \in N$. We fix the price on market 1 before the price shock to 0, i.e. $p_1 = 0$ and $r_1 = 0$. On market 2 the price is $p_2(q_2) = 2 - q_2$, where $q_2$ is the total quantity sold in market 2. The cost of firm $a$ for any total quantity $q_a = q_{a,1} + q_{a,2}$ is 0 if $q_a \leq \frac{2}{n+1}$; for any larger quantity the cost is prohibitively high. For firms $i \neq a$, the cost is always 0.

The Cournot equilibrium $x$ of this game can be found through convex optimization. In the equilibrium, no quantity is sold on market 1 and on market 2, $x_{a,2} = x_{i,2} = \frac{2}{n+1}$. Firm $a$’s profit is $u_a(x) = (2 - \frac{2}{n+1})^2 = \frac{4}{(n+1)^2}$.

A price shock that increases the price on market 1 to $\frac{n-1}{n}$ leads firm $a$ to shift to market 1. In the new equilibrium $y$, $y_{a,1} = \frac{n-1}{n(n+1)}$, $y_{a,2} = \frac{1}{n}$, and $y_{i,2} = \frac{2n-1}{n^2}$. The profit of firm $a$ is

$$u_a(y) = \left(2 - \frac{1}{n} - (n - 1)\frac{2n-1}{n^2}\right) \frac{1}{n} + \frac{n-1}{n^2} \frac{n-1}{n(n+1)} = \frac{3n-1}{n^2(n+1)}.$$

Then, the ratio of profit before and after the price shock is

$$\gamma^u(G, \delta) = \frac{u_a(y)}{u_a(x)} = \frac{(3n-1)(n+1)}{4n^2} = 1 - \frac{(n-1)^2}{4n^2}.$$

Remark 4.1. Note that this lower bound is quite generic in the sense that such an instance can be constructed for any cost function on market 2 and any linear cost function for competitors $i \neq a$. In general, the profit loss of a firm can be large when it has a strongly convex cost function, such that a positive price shock in one market causes it to decrease quantity in another market, and when this is met by competitors with linear (or not "too convex") cost functions.

4.2 Non-convex Cost

If we abandon Assumption[2] i.e. allow non-convex cost functions, we possibly lose uniqueness of equilibria and we would have to redefine our objective function, e.g. involving equilibrium selection[7]. Moreover, one can easily construct examples where a positive price shock completely eliminates the profit of firm a firm in all equilibria. If e.g. fixed costs are allowed, in the example of Bulow et al. [6] one can set the fixed cost of the monopolist equal to their revenue after the price shock using the fact that fixed costs do not change the equilibria of the game as long as nonnegative profits are guaranteed. Similar examples

[7] Cournot equilibria continue to exist for non-convex costs if inverse demand functions satisfy rather mild assumptions, see, e.g., Novshek [31] and Amir [2] and Roberts and Sonnenschein [34].
are possible if we mix between economies of scale and diseconomies of scale among firms’ cost technologies.

4.3 Concave Inverse Demand Functions

Relaxing Assumption 1 toward concave prices reveals another counterintuitive phenomenon: Very small price shocks may decrease the profit of a firm by an arbitrary amount. Consider the class $\mathcal{H} \supseteq \mathcal{G}$ that allows for concave inverse demand functions. We obtain the following value for $\gamma_u(\mathcal{H}, \Delta \mathcal{H})$.

**Proposition 4.2.** For any $k \geq 4$, there is a game with only two markets and concave price functions such that the profit ratio of one of the firms before and after a positive price shock is less than $\frac{2}{k}$. Thus, $\gamma_u(\mathcal{H}, \Delta \mathcal{H}) = 0$.

**Proof.** Fix some $k \geq 4$. We construct a game to fulfill the above claim. The firm whose profit ratio we observe, denoted by $a$, has cost $c_a(q_a) = 0$ for quantities $q_a \leq 1$ and prohibitively high cost for larger quantities. All other firms $i \neq a$ have cost $c_i(q_i) = q_i$ for any quantity.

Market 1 is only served by firm $a$ and has constant price $p_1(q_1) \equiv 0$. Market 2 is served by all firms and has a concave price function satisfying $p_2(1) = 1$ with $p_2'(1) = -1$ and $p_2(1 + \frac{1}{k}) = 1 - \frac{1}{k}$ with $p_2'(1 + \frac{1}{k}) = -k$.

The initial equilibrium $x$ of this game is $x_{a,1} = 0$, $x_{a,2} = 1$ and $x_{j,2} = 0$ for all $j \neq a$. To verify this, observe that marginal revenue and cost of firm $a$ are all equal as $\pi_{a,1}(x) = 0$, $\pi_{a,2}(x) = 1 - 1 = 0$ and $c_a'(x) = 0$ as well as for competitors $j \neq a$ we have $\pi_{j,2}(x) = 1 = c_j'(x)$.

Let the price shock be $\delta_1 = \frac{1}{k}$ and $\delta_2 = \frac{2}{k}$. The new equilibrium is $y_{a,1} = 1 - \frac{1}{k}$, $y_{a,2} = \frac{1}{k}$ and $y_{j,2} = \frac{1}{k^2}$ for all competitors $j \neq a$. We again verify $\pi_{a,1}^\delta(y) = \frac{1}{k}$, $\pi_{a,2}^\delta(y) = 1 + \frac{1}{k} - k\frac{1}{k^2} = \frac{1}{k}$ which are equal and greater than 0. For competitors $j \neq a$, $\pi_{j,2}^\delta(y) = 1 + \frac{1}{k} - k\frac{1}{k^2} = 1 = c_j'(y)$.

We calculate the profit of firm $a$ in both equilibria: $u_a(x) = 1$ and thus

$$\gamma = u_a^\delta(y) = p_a^\delta(y)y_{a,1} + p_{a,2}^\delta(y)y_{a,2} - c_a(y)$$

$$= \frac{1}{k}(1 - \frac{1}{k}) + (1 + \frac{1}{k})\frac{1}{k} = \frac{2}{k}$$

The above bound largely depends on the number of firms in the game. For games with two forms only we show that the profit loss can be up to 25%, as opposed to the 6.25% in the case of affine price functions.
Proposition 4.3. For any value \( \gamma > \frac{3}{4} \), there is a game \( G \in \mathcal{H} \) with only two firms and concave price functions and a price shock \( \delta \in \Delta_G \) such that \( \gamma(G, \delta) = \gamma \) is the profit ratio of one of the firms before and after a positive price shock.

Proof. Fix some \( k > 4 \). We construct a game with two firms \( a \) and \( b \), such that the profit ratio \( \gamma \) of firm \( a \) before and after a price shock is dependent on \( k \).

The firm \( a \) has cost \( c_a(q_a) = (k - 1)q_a \) for quantities \( q_a \leq 1 \) and prohibitively high cost for larger quantities. Firm \( b \) has cost \( c_b(q_b) \equiv 0 \) for any quantity.

Market 1 is only served by firm \( a \) and has price \( p_1(q_1) = k - 1 - \frac{1}{k}q_1 \). Market 2 is served by both firms and has a concave price function satisfying \( p_2(k + 1 - \frac{1}{k}) = k + \frac{1}{k} \) and \( p_2(k + 1) = k \) with slopes \( p'_2(k + 1 - \frac{1}{k}) = -\frac{k + \frac{1}{k}}{k + \frac{1}{k} + \frac{1}{2}} \) and \( p'_2(k + 1) = -1 \).

The initial equilibrium \( x \) of this game is \( x_{a,1} = 0 \), \( x_{a,2} = 1 \) and \( x_{b,2} = k \). To verify this, observe that marginal revenue and cost of firm \( a \) are all equal as \( \pi_{a,1}(x) = k - 1 \), \( \pi_{a,2}(x) = k - 1 \) and \( c'_a(x) = k - 1 \) as well as for firm \( b \) we have \( \pi_{b,2}(x) = 0 \).

Let the price shock be \( \delta_1 = 1 + 2\frac{1}{k} - \frac{k + \frac{1}{k}}{2(k - \frac{1}{k} + \frac{1}{2})} \) and \( \delta_2 = 0 \). The new equilibrium is \( y_{a,1} = \frac{1}{2} \), \( y_{a,2} = \frac{1}{2} \) and \( y_{b,2} = k - \frac{1}{k} + \frac{1}{2} \). We again verify \( \pi_{a,1}^\delta(y) = k + \frac{1}{k} - \frac{k + \frac{1}{k}}{2(k - \frac{1}{k} + \frac{1}{2})} \), \( \pi_{a,2}^\delta(y) = k + \frac{1}{k} - \frac{k + \frac{1}{k}}{2(k - \frac{1}{k} + \frac{1}{2})} \) which are equal and higher than \( k - 1 \). For firm \( b \), again \( \pi_{b,2}^\delta(y) = 0 \).

We calculate the profit of firm \( a \) in both equilibria: \( u_a(x) = k - (k - 1) = 1 \) and thus

\[
\gamma = u_a^\delta(y) = p_1^\delta(y)y_{a,1} + p_2^\delta(y)y_{a,2} - c_a(y) \\
= (k + 2\frac{1}{k} - \frac{k + \frac{1}{k}}{2(k - \frac{1}{k} + \frac{1}{2})} - \frac{1}{k^{\frac{1}{2}}} \frac{1}{2} + (k + \frac{1}{k} - (k - 1) \\
= 1 + \frac{5}{4} \frac{1}{4} \frac{1}{4} k - \frac{1}{k} + \frac{1}{2} \frac{k \to \infty}{k} \frac{3}{4} .
\]

\[\square\]

5 Effect of Price Shocks on Aggregates

Theorem 4.1 shows that an individual firm can lose no more than 25% of its profit as a result of a positive price shock. The lower bound, however, had the property that one firm loses but all competitors gained in their total profits. In this section, we study effects of price shocks on aggregate measures: the welfare and the social surplus.

5.1 Welfare

Our first result showed that each firm may not lose more than 25% of current profits. By the definition of \( \gamma^u(G, \delta) \) it follows that the same holds true for aggregated firm’s
profits, that is, $\gamma^U(G, \delta) \geq \gamma^u(G, \delta) \geq 3/4$ for any game $G$. More interestingly, we show an instance, where this loss is actually attained, and, thus, $\gamma^U(G, \delta) = \gamma^u(G, \delta)$.

**Proposition 5.1.** There is a game $G \in \mathcal{G}$ with $n$ firms where a positive price shock $\delta$ decreases the equilibrium welfare by a $\left(\frac{n-1}{4(n^2+n-1)}\right)$ fraction of the original welfare, that is, by almost 25% for instances with many firms, that is

$$\gamma^U(G, \delta) = \gamma^u(G, \delta).$$

**Proof.** The game is similar to that from that proof of Lemma 4.1, except that firm $a$ can produce a quantity of $q_a \leq 1$ at cost 0 and its competitors $i \neq a$ have a per-unit cost of 1, i.e. the cost is $c_i(q_i) = q_i$.

The Cournot equilibrium $x$ of this game can be found through convex optimization. In the equilibrium, no quantity is sold on market 1 and on market 2, $x_{a,2} = 1$ and $x_{i,2} = 0$ for all $i \neq a$. The equilibrium welfare is $U(x) = 1$.

A price shock that increases the price on market 1 to $\delta = \frac{n^2-1}{2(n^2+n-1)}$ leads firm $a$ to shift to market 1. In the new equilibrium $y$, $y_{a,1} = \frac{n^2-n}{2(n^2+n-1)}$, $y_{a,2} = \frac{n^2+3n-2}{2(n^2+n-1)}$, and $y_{i,2} = \frac{n-1}{2(n^2+n-1)}$ for all $i \neq a$. The new equilibrium welfare is

$$U^\delta(y) = \delta y_{a,1} + (2 - (y_{a,2} + \sum_{i \neq a} y_{i,2}))(y_{a,2} + \sum_{i \neq a} y_{i,2}) - \sum_{i \neq a} y_{i,2}$$

$$= \frac{3n^2 + 6n - 5}{4(n^2 + n - 1)}.$$

**5.2 Aggregate Social Surplus**

We now consider the effect of price shocks on social surplus defined as

$$S(q) = \sum_{m \in M} \int_0^{q_m} p_m(z) dz - \sum_{i \in N} c_i(q_i) = \sum_{m \in M} p_m q_m - \sum_{i \in N} c_i(q_i). \quad (15)$$

The first term in the above definition of $S(q)$ measures the value that the buyers in the market have of the goods while the second term simply sums up total production cost. Social surplus has been considered before, among others, by Anderson and Renault [3] and Ushio [40]. For a given game $G$, we want to bound the ratio $\gamma^S(G, \delta) = \frac{S^\delta(y)}{S(x)}$.

**Theorem 5.1.** Given a game $G$, a positive price shock $\delta$ can decrease the social surplus by at most a factor $\frac{5}{6}$,

$$\gamma^S(G, \delta) \geq \frac{5}{6}.$$
Before we prove the theorem, we characterize $y$ with the following variational inequality.

**Lemma 5.1.** Let $y$ be the equilibrium for the game with price shock $\delta_m \geq 0, m \in M$ and let $x$ be the original equilibrium. Then, for all $i \in N$ it holds

$$\sum_{m \in M_i} \left( p_m + \delta_m - r_m y_m - r_m y_{i,m} - c'_i(y_i) \right) (x_{i,m} - y_{i,m}) \leq 0.$$ 

**Proof.** For every firm $i$, given the equilibrium quantities $y_{-i}$ of its competitors, solving $\max_{(q_{i,m})} u_i(q_i, y_{-i})$ is a convex program. Thus, at an optimal solution $(y_{i,m})_{m \in M_i}$, the gradient $\nabla u_i(y)$ only decreases along any feasible direction. In particular, $(x_{i,m} - y_{i,m})_{m \in M_i}$ is a feasible direction. \hfill $\square$

We now bound the surplus gained at $x$.

**Lemma 5.2.**

$$S(x) \geq \sum_{i \in N} \sum_{m \in M} \frac{3}{2} r_m x_{i,m}^2. \quad (16)$$

**Proof.** Note that $x$ is an equilibrium for the unperturbed game. In particular, the first order necessary optimality conditions for each firm hold:

$$p_m - r_m x_m - r_m x_{i,m} - c'_i(x_i) = 0 \text{ for all } x_{i,m} > 0. \quad (17)$$

We combine this with the definition of the surplus $[15]$, 

$$S(x) = \sum_{m \in M} p_m x_m - \frac{r_m x_m^2}{2} - \sum_{i \in N} c_i(x_i) \geq \sum_{m \in M} p_m x_m - \frac{r_m x_m^2}{2} - \sum_{i \in N} c'_i(x_i) x_i \quad (18)$$

$$= \sum_{m \in M} \frac{r_m x_m^2}{2} + \sum_{i \in N} \sum_{m \in M} r_m x_{i,m} \geq \sum_{i \in N} \sum_{m \in M} \frac{3}{2} r_m x_{i,m}^2.$$ 

Here (18) follows by the convexity of $c_i$ and $c_i(0) = 0$. Finally, $x_m^2 = \left( \sum_{i \in N} x_{i,m} \right)^2 \geq \sum_{i \in N} x_{i,m}^2$. \hfill $\square$

We now prove the theorem.

**Proof of Theorem 5.1.** We establish the difference between $S^\delta(y)$ and $S(x)$ through (15) and the fact that $c_i(y) - c_i(x) \leq c'_i(y)(y_i - x_i)$ as the cost functions are convex and $y_i \geq x_i$.
as shown in Lemma 4.3.

\[ S(x) = S^d(y) + \sum_{m \in M} \left( p_m x_m - (p_m + \delta_m) y_m - \frac{r_m}{2} (x_m^2 - y_m^2) \right) + \sum_{i \in N} \left( c_i(y_i) - c_i(x_i) \right) \]

\[ \leq S^d(y) + \sum_{m \in M} \left( (p_m + \delta_m)(x_m - y_m) - \frac{r_m}{2} (x_m^2 - y_m^2) \right) + \sum_{i \in N} \left( c'_i(y_i)(y_i - x_i) \right) \]

Subtracting the variational inequality (16) summed up across all firms \( i \in N \) allows to simplify the term.

\[ S(x) \leq S^d(y) + \sum_{m \in M} \left( -\frac{r_m}{2} (x_m^2 + y_m^2) + r_m y_m x_m + r_m \sum_{i \in N} y_{i,m}(x_{i,m} - y_{i,m}) \right) \]

\[ = S^d(y) + \sum_{m \in M} r_m \left( -\frac{1}{2} (x_m - y_m)^2 + \sum_{i \in N} y_{i,m}(x_{i,m} - y_{i,m}) \right) \]

\[ \leq S^d(y) + \sum_{m \in M} r_m \sum_{i \in N} \frac{x_{i,m}^2}{4} \leq S^d(y) + \frac{1}{6} S(x) \quad (19) \]

Here, we used for the first inequality in (19) that \( (x_m - y_m)^2 \geq 0 \) and that \( y_{i,m}(x_{i,m} - y_{i,m}) \leq \frac{x_{i,m}^2}{4} \). The final step applies the result of Lemma 5.2.

We can use the construction of Proposition 5.1 to obtain a matching lower bound.

**Proposition 5.2.** There is a game \( G \) with many firms where a positive price shock decreases the overall consumer surplus by \( 1/6 \approx 16.6\% \), that is,

\[ \gamma^S(G, \delta) \xrightarrow{n \to \infty} \frac{5}{6} \]

**Proof.** Note that \( S(q) = U(q) + \frac{1}{2} \sum_{m \in M} r_m q_m^2 \). We use the instance and equilibria from Lemma 5.1 and find \( S(x) = U(x) + \frac{1}{2} 12 = \frac{3}{2} \) and

\[ S(y) = U(y) + \frac{1}{2} (y_{a,2} + \sum_{i \neq a} y_{i,2})^2 = \frac{10n^4 + 22n^3 - 7n^2 - 24n + 11}{8(n^2 + n - 1)^2} \xrightarrow{n \to \infty} 5/4. \]

Combined, this approaches \( \gamma^S(G, \delta) \xrightarrow{n \to \infty} \frac{5}{6} \).

**6 Conclusions**

Bulow et al. [6] showed that a positive price shock can reduce a monopolist’s profit. We directed a quantitative approach at their setting to provide an understanding of the significance and robustness of this effect. Our results – a positive price shock can reduce a profit by 25% and a negative price shock can increase profit by 33% – imply that the
effect may be significant. For example, the possible 33% increase in profit may be enough for a government to consider imposing a domestic sales tax in order to force domestic companies to compete more aggressively abroad. For a market participant on the other hand, our results imply that equilibrium profit is robust in the sense that at least 75% of current profit is maintained in case of a positive price shock. We further showed that social surplus is more robust against positive price shocks: the worst case loss is bounded by 16.6%.

There are several natural extensions of our model:

- What can be said for general aggregative games with strategic substitutes as considered by Acemoglu and Jensen [1]?
- Bulow et al. show qualitatively a similar paradoxical effect for markets with strategic complements and joint economies of scale. Here, a quantification remains open.

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