Operational Resource Theory of Coherence

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(Dated: 16 January 2016)

We establish an operational theory of coherence (or of superposition) in quantum systems, by focusing on the optimal rate of performance of certain tasks. Namely, we introduce the two basic concepts — “coherence distillation” and “coherence cost” in the processing quantum states under so-called incoherent operations [Baumgratz/Cramer/Plenio, Phys. Rev. Lett. 113:140401 (2014)]. We then show that in the asymptotic limit of many copies of a state, both are given by simple single-letter formulas: the distillable coherence is given by the relative entropy of coherence (in other words, we give the relative entropy of coherence its operational interpretation), and the coherence cost by the coherence of formation, which is an optimization over convex decompositions of the state. An immediate corollary is that there exists no bound coherent state in the sense that one would need to consume coherence to create the state but no coherence could be distilled from it. Further we demonstrate that the coherence theory is generically an irreversible theory by a simple criterion that completely characterizes all reversible states.

PACS numbers: 03.65.Aa, 03.67.Mn

Introduction.— The universality of the superposition principle is the fundamental non-classical characteristic of quantum mechanics: Given any configuration space $X$, its elements $x$ label an orthogonal basis $|x\rangle$ of a Hilbert space, and we have all superpositions $\sum_x \psi_x |x\rangle$ as the possible states of the system. In particular, we could choose a completely different orthonormal basis as an equally valid computational basis, in which to express the superpositions. However, often a basis is distinguished, be it the eigenbasis of an observable or of the system’s Hamiltonian, so that conservation laws or even superselection rules apply. In such a case, the eigenstates $|x\rangle$ are distinguished as “simple” and superpositions are “complex”. Indeed, in the presence of conservation laws, structured superpositions of eigenstates can serve as so-called “reference frames” which are resources to overcome the conservation laws. Based on this idea, Aberg [4] and more recently Baumgratz et al. [5] have proposed to consider any non-trivial superposition as a resource, and to create a theory in which computational basis states and their probabilistic mixtures are for free (or worthless), and operations preserving these “incoherent” states are free as well. This suggests that coherence theory can be regarded as a resource theory.

Let us briefly recall the general structure of a quantum resource theory (QRT) and basic questions that be asked in a QRT through entanglement theory (ET), which is a well-known QRT. For our purposes, a QRT has three ingredients: (1) free states (separable state in ET), (2) resource states (entangled state in ET), (3) the restricted or free operations (LOCC in ET). A necessary requirement for a consistent QRT is that no resource state can be created from any free state under any free operation. The QRT is then the study of interconversion between resource states under the restricted operations. The pure resource states play a special role and are much more preferable because usually they are used to circumvent the restriction on operations (Bell states are used in teleportation to overcome LOCC operations, for instance). So the conversions between pure resource states and mixed ones are a major focus of QRTs. A standard unit resource measure can be constructed if the conversions between pure states are asymptotically reversible (entropy of entanglement of a pure entangled state in ET, with a Bell state as the unit). Then there are two basic transformation processes that are well-motivated: one is the so-called resource distillation, that is the transformation from a mixed resource state to the unit resource, and the other is resource formation, that is the reverse transformation from the unit resource measure can be constructed if the conversions between pure states are asymptotically reversible (entropy of entanglement of a pure entangled state in ET, with a Bell state as the unit). Then there are two basic transformation processes that are well-motivated: one is the so-called resource distillation, that is the transformation from a mixed resource state to the unit resource, and the other is resource formation, that is the reverse transformation from the unit resource to a mixed state (entanglement distillation and entanglement formation in ET). Because of the reversibility in the pure state conversion we need not worry about what kind of pure state is the target, as they are equivalent up to the transformation rate between them. Thus, two well-motivated quantities arise from the two basic processes, distillable resource and resource cost, which have a clear operational interpretation (distillable entanglement and entanglement cost in ET). The principal objective of the theory is the characterization of these two quantities. This is often a highly complex problem, but resource monotones yield various limits on possible transformations and achievable rates. Another basic question in any QRT is to ask whether the theory is irreversible or not. If the conversion between pure states is reversible, then the reversibility problem is reduced to the question
whether or not the optimal conversion rate in the formation process is equal to that in the distillation process. If a QRT is reversible, a unique resource measure exits, quantifying the conversion rate between different states, so that everything about possible resource transformations becomes clear and simple. However, if a QRT is irreversible, the phenomena are ample and further interesting questions can be asked, for example whether there exist so-called bound resource states as an analogue of bound entangled states in ET, in the sense that from them no resource can be distilled, but for which, in order to create them, nonzero resource is required. Several QRTs were constructed along these lines, some them indeed irreversible: entanglement theory (with respect to LOCC), thermodynamics (w.r.t. energy-conserving operations and thermal states), reference frames (w.r.t. group-covariant operations), etc.

In this Letter, we establish an operational coherence theory in the framework proposed in [8] and [9]. Namely, first we show that the conversion between the pure coherent states is asymptotically reversible, so that the standard unit coherence measure exists. Then we introduce the basic transformation processes—coherence distillation and coherence formation—from which two basic coherent coherence measures naturally arise: distillable coherence and coherence cost, with operational interpretations. Remarkably, both are given by information theoretic single-letter formulas that hence make these quantities computable. These results in turn allow us to formulate a simple criterion to decide whether a given state is irreversible or not and to show that there is no bound coherence. Although the main results are in the asymptotic setting, we also get the single copy conversion of pure states along the way. In the following, we state and discuss our results carefully, while all proofs are found in Appendix B.

Coherence as a resource theory.— We follow the framework of coherence theory by Baumgratz et al. [8]. Let \{\\{i\\}\} be a fixed basis in the finite dimensional Hilbert state. The free states called incoherent states are those whose density matrices are diagonal in the basis, being of the form \(\sum_i p_i |i⟩⟨i|\) where \(p_i\) is a probability distribution and the set of incoherent states is denoted as \(\Delta\). The resource states called coherent states are those not of this form. Quantum operations are specified by a set of Kraus operators \(\{K_ℓ\}\) satisfying \(\sum_ℓ K_ℓ^† K_ℓ = 1\); a quantum operation can have many different Kraus representations. The free operations, called incoherent operations (IC) are those for which there exists a Kraus representation \(\{K_ℓ\}\) such that \(\frac{1}{Tr \rho K_ℓ^† K_ℓ} K_ℓ^† K_ℓ \in \Delta\) for all \(ℓ\) and all \(\rho \in \Delta\). Such restriction guarantees that even if one has access to individual measurement outcomes \(ℓ\) of the instrument \(\{K_ℓ\}\), one cannot generate coherent states from an incoherent state. Under this restriction, each Kraus operator is easily seen to be of the form \(K_ℓ = \sum_i c(ℓ) |i⟩⟨i|\) where \(j(i)\) is a function from the index set of the computational basis, and \(c(ℓ)\) are coefficients; we call such Kraus operators incoherent, too. If not only \(K_ℓ\) but also \(K_ℓ^†\) is incoherent, we call it strictly incoherent, and the corresponding quantum operation a strictly incoherent operation. Strict incoherence can be characterized by the function \(j(i)\) above being one-to-one. Another equivalent form of a general incoherent Kraus operator is \(\mathcal{R} = \sum_j |j(γ_j)⟩⟨γ_j|\) with \(|γ_j⟩\in\text{span}\{|i⟩: i \in S_j\}\) for a partition \([d] = \bigcup S_j\).

We will be mainly concerned with state transformations, for which we introduce some notation: \(\rho \xrightarrow{\text{IC}} \sigma\) means that there exists an incoherent operation \(T\) such that \(\sigma = T(\rho) = \sum_ℓ K_ℓ^† \rho K_ℓ^†\); if the operation is strictly incoherent, we write \(\rho \xrightarrow{\text{IC}} \sigma\). If the transformation is obtained probabilistically, i.e. if \(\sigma \propto K_ℓ^† \rho K_ℓ^†\) and \(Tr K_ℓ^† \rho K_ℓ^† \neq 0\) for some \(ℓ\), we write \(\rho \xrightarrow{\text{IC}} \sigma\) and \(\rho \xrightarrow{\text{IC}} \sigma\), for probabilistic incoherent and probabilistic strictly incoherent transformations, respectively.

When we consider composite systems, we simply declare as incoherent the tensor product basis of the local computational bases; the incoherent operations are then defined with respect to the tensor product basis. Notice that there are several special transformations that are incoherent: phase and permutation unitaries. In particular, in a composite system of two qubits, CNOT: \(|i⟩|j⟩\mapsto|i⟩|i+j\ mod\ d⟩\) is an incoherent operation, as it is simply a permutation of the tensor product basis vectors [13, 14]. In Appendix C we discuss a slightly more general and flexible model.

We define the decohering operation \(Δ(\rho) = \sum_i (|i⟩⟨i|)\sum |i⟩⟨i|\); i.e. the diagonal part of the density matrix. This makes the incoherent states \(Δ\) the image of the map \(Δ\), thus justifying the slight abuse of notation.

Pure state transformations.— We start by developing the theory of pure states transformations; the main result in the present context is that in the asymptotic setting of many copies this becomes reversible, the rates governed by the entropy of the decohered state, a quantity we dub entropy of coherence. First however, we review the situation for exact single-copy transformations.

We start with a simple observation on ranks: Namely, let \(ϕ\) be transformed to \(ψ\) by an incoherent, or more generally a probabilistic incoherent operation, \(|ψ⟩\propto K|ϕ⟩\neq 0\), for an incoherent Kraus operator \(K = \sum_i c_i |j(i)⟩⟨i|\). The rank \(r\) of \(Δ(ϕ)\) is precisely the number of nonzero diagonal entries of \(Δ(ϕ)\), which is the number of nonzero terms in \(|ϕ⟩ = \sum_{i∈R}ϕ_i |i⟩\); \(|R| = r\). Thus,

\[|ψ⟩\propto K|ϕ⟩ = \sum_{i∈R}ϕ_i c_i |j(i)⟩\]

has at most \(r = |R|\) terms. This proves the following.
Lemma 1 If $\varphi \mathcal{IC} \psi$, then $\text{rank} \Delta(\psi) \leq \text{rank} \Delta(\varphi)$, i.e. the rank of the diagonal part of pure states cannot increase under incoherent operations.

Theorem 2 (Cf. Du/Bai/Guo [15]) For two pure states $\psi = |\psi\rangle\langle\psi|$ and $\varphi = |\varphi\rangle\langle\varphi|$, if $\Delta(\psi)$ majorizes $\Delta(\varphi)$, $\Delta(\psi) \succ \Delta(\varphi)$, then there is an incoherent (in fact, a strictly incoherent) operation transforming $\varphi$ to $\psi$: $\varphi \mathcal{IC} \psi$.

Conversely, if $\varphi \mathcal{IC} \psi$, or if $\varphi \mathcal{IC} \psi$ and in addition $\Delta(\varphi) = \text{rank} \Delta(\psi)$, then $\Delta(\psi) \succ \Delta(\varphi)$.

Here, the majorization relation for matrices, $\rho \succ \sigma$, means that the spectra $\text{spec}(\rho) = (p_1 \geq \ldots \geq p_d)$ and $\text{spec}(\sigma) = (q_1 \geq \ldots \geq q_d)$ are in majorization order [16–18]:

$$\forall t < d \sum_{i=1}^t p_i \geq \sum_{i=1}^t q_i, \text{ and } \sum_{i=1}^d p_i = \sum_{i=1}^d q_i.$$ 

As a consequence of Theorem 2 just as for pure state entanglement [18], there is catalysis for pure state incoherent transformations, cf. [14, 20] examples such that initial and final states have equal rank are given. An immediate corollary of Theorem 2 is the following.

Corollary 3 (Baumgratz et al. [3]) Let $\rho$ be an arbitrary state in $\mathbb{C}^d$, and $\Phi_d = |\Phi_d\rangle\langle\Phi_d| = \frac{1}{d} \sum_{ij=0}^{d-1} |i\rangle\langle j|$. Then, there is an incoherent operation transforming $\Phi_d$ to $\rho$.

This motivates the name maximally coherent state for $\Phi_d$. In addition to enabling the creation of arbitrary $d$-dimensional coherent superpositions by incoherent means, $\Phi_d$ also allows the implementation of arbitrary unitaries $U \in \text{SU}(d)$ [3]. Then, fixing the qubit maximally coherent pure state $\Phi_d = \frac{1}{d} \sum_{ij=0}^{d-1} |i\rangle\langle j|$ as a unit reference, we are ready to consider asymptotic pure states transformations with vanishing error as the number of copies goes to infinity. Special cases of this are coherence concentration, the transformation from a non-maximally coherent pure state to the unit coherent state, and coherence dilution from unit coherent state to the non-maximally coherent pure one. As in information theory, entanglement theory and other similar cases (cf. [21, 22]), this simplifies the picture dramatically. To express our result, we introduce the entropy of coherence for pure states as

$$C(\psi) = S(\Delta(\psi)).$$

Theorem 4 (Yuan/Zhou/Cao/Ma [23]) For two pure states $\psi$ and $\varphi$ and a rate $R \geq 0$, the asymptotic incoherent transformation

$$\psi \otimes n \mathcal{IC} \frac{1}{\epsilon} \approx \varphi \otimes n R \quad \text{as } n \to \infty, \quad \epsilon \to 0,$$

is possible if $R < C(\varphi|\psi)$ and impossible if $R > C(\varphi|\psi)$.

In particular, $\psi$ can be asymptotically reversibly transformed into $\Phi_2$, and vice versa, at optimal rate $C(\psi)$.

Here, $\rho \approx \sigma$ signifies that the two states have high fidelity: $F(\rho, \sigma) \geq 1 - \epsilon$ (see Appendix A for details).

Now we are ready to introduce two fundamental tasks for arbitrary mixed states, namely asymptotic distillation of $\rho^n$ to $\Phi_2^n R$ and the reverse process of formation $\rho^n$ from $\Phi_2^n R$. Note in this respect the fundamental importance of Theorem 1 which shows that we could equivalently elect any pure state $\psi$ (that is coherent) as unit reference for formation and distillation, and all rates would change by the same factor $\frac{1}{C(\psi/\psi)}$. It turns out that both quantities have single-letter, additive expressions: the former is given by the relative entropy of coherence, the latter by the coherence of formation; both are additive. This is in marked contrast to other resource theories, perhaps most prominently that of entanglement under LOCC, in which the basic operational tasks are only characterized by regularized formulas, and the fundamental quantities, such as entanglement of formation [24], relative entropy of entanglement [25], etc. are not additive [24, 25].

Distillable coherence.— The distillation process is the process that extracts pure coherence from a mixed state by incoherent operations. The distillable coherence of a state is the maximal rate at which $\Phi_2$ can be obtained from the given state. The precise definition is the following.

Definition 5 The distillable coherence of a state $\rho$ is defined as

$$C_d(\rho) = \sup_R, s.t. \rho \otimes n \mathcal{IC} \Phi_2^n R \quad \text{as } n \to \infty, \epsilon \to 0.$$ 

We look at the asymptotic setting to get rid of the error $\epsilon$ motivated by general information theoretic practice [21, 22] and the success of this point of view in the pure state case (Theorem 4). By definition, $C_d$ naturally has an operational meaning as the optimal rate performance at a natural task. Theorem 4 shows that the distillable coherence is given by a closed form expression.

Theorem 6 For any state $\rho$, the distillable coherence is given by the relative entropy of coherence:

$$C_d(\rho) = C_\rho(\rho) = \min_{\sigma \in A} S(\rho||\sigma).$$
The relative entropy of coherence is introduced and studied in detail in [4, 5]. Here, $S(\rho|\sigma) = \text{Tr} \rho \log \rho - \log \sigma$ is the quantum relative entropy. In [4, 5], it is shown that
\[
C_r(\rho) = S(\Delta(\rho)) - S(\rho).
\] (1)

Furthermore, that the relative entropy of coherence is convex in the state, and a coherence monotone, meaning that for every incoherent transformation $T(\rho) = \sum_{\ell} K_{\ell} \rho K_{\ell}^\dagger$, $C_r(\rho) \geq C_r(T(\rho))$. In fact, it is even strongly monotonic [4].

\[
C_r(\rho) \geq \sum_{\ell} p_\ell C_r(\rho_\ell), \text{ where } p_\ell K_{\ell} = K_{\ell} p_\ell K_{\ell}^\dagger.
\]

But it is only due to Theorem 4 that we can give it a clear operational interpretation in the distillation process. Note that in concurrent independent work, Singh et al. [28] show that the same quantity, $C_r(\rho)$, by virtue of it being equal to the entropy difference $S(\Delta(\rho)) - S(\rho)$, arises also as the minimum amount of incoherent noise that has to be applied to the state to decohere it.

**Coherence cost.**— The formation process is that which prepares a mixed state by consuming pure coherent states under incoherent operations. The coherence cost is the minimal rate at which $\Phi_2$ has to be consumed for preparing the given state.

**Definition 7** The coherence cost of a state $\rho$ is defined as
\[
C_c(\rho) = \inf R, \text{ s.t. } \Phi_2^{\otimes nR} \xrightarrow{IC} \approx^{1-\epsilon} \rho \otimes \rho \text{ as } n \to \infty, \epsilon \to 0.
\]

The next result shows that the coherence cost has a single-letter formula, involving a simple entropy optimization.

**Theorem 8** For any state $\rho$, the coherence cost is given by the coherence of formation,
\[
C_c(\rho) = C_f(\rho),
\]
where the coherence of formation is given by
\[
C_f(\rho) = \min \sum_i p_i S(\Delta(\psi_i)) \text{ s.t. } \rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|.
\]

The coherence of formation is introduced as a monotone in [4], where its convexity and monotonicity under incoherent operations were observed (see Lemma 12 in Appendix A), however its additivity was not remarked. A priori, one might have expected the cost to be given by the regularization of the coherence of formation, which would have involved infinitely many optimization problems so that the evaluation of the operational cost had become infeasible. Indeed it is additivity that makes the single-letter formulas in Theorem 4 and Theorem 5 available, and because of its importance we record it as a theorem.

**Theorem 9** Both $C_f$ and $C_r$ are additive under tensor products,
\[
C_f(\rho \otimes \sigma) = C_f(\rho) + C_f(\sigma),
C_r(\rho \otimes \sigma) = C_r(\rho) + C_r(\sigma).
\]

Theorem 9 means that also the distillable coherence and the coherence cost are additive, and so there are no super-additivity or activation phenomena in the resource theory of coherence, unlike the theory of entanglement [24, 27], that of communication via channels [25, 26], and many other resource theories, where the yield (cost) of two resources together may be strictly larger (smaller) than the sum of the yields (costs) of the resources processed individually.

Based on the formulas for the distillable coherence and the coherence cost, we are now ready to characterize precisely the (ir-)reversible states.

*(Ir-)reversibility.*— From the definitions of $C_d$ and $C_c$, it is immediate that $C_d(\rho) \leq C_c(\rho)$. A state is reversible if the equality holds, otherwise it is called irreversible. From Theorem 3, we see that all pure states are asymptotically reversible, the optimal transformation rate given by the ratio of their entropies of coherence. In the mixed state case, Theorem 10 provides a simple criterion to decide whether the given state is reversible or not that completely characterizes all the reversible states. From this, we conclude that the mixed states are generically irreversible and the coherence theory is an irreversible resource theory. However in contrast to the irreversibility in entanglement theory [5, 7, 30], there is no “bound coherence” (as an analogue of “bound entanglement” [5, 7]), from which no coherence could be distilled, but for which, in order to create it, nonzero coherence would be required.

**Theorem 10** A mixed state is reversible if and only if its eigenvectors are supported on the orthogonal subspaces spanned by a partition of the incoherent basis. That is,
\[
\rho = \bigoplus_j P_j |\phi_j\rangle \langle \phi_j|,
\]
and each $|\phi_j\rangle \in \mathcal{H}_j = \text{span}\{|i\rangle : i \in S_j\}$, such that $S_j \cap S_k = \emptyset$ for all $j \neq k$ and $\mathcal{H} = \bigoplus_j \mathcal{H}_j$. In other words, $\rho$ is reversible if and only if there exists an incoherent projective measurement, consisting in fact of the projectors $P_j$ onto $\mathcal{H}_j$, s.t. $\rho = \sum_j P_j \rho P_j$, and $P_j \rho P_j$ is proportional to a pure state.

Note that the criterion is easy to check, and contrast this with the entanglement irreversibility of two-qubit maximally correlated states considered in [31], where Wootters’ formula [32] for calculating the entanglement of formation is used: Here we do not need the explicit formula of the coherence of formation, which involves an
optimization problem itself, and indeed we do not know a closed-form expression for it in high dimension. However, here the equality constraint is so severe that we can learn the structure of the state.

**Theorem 11** There is no bound coherence: $C_d(\rho) = 0$ implies $C_r(\rho) = 0$. In other words, every state with any coherence (non-zero off-diagonal part) is distillable.

Discussion.— We have shown that the incoherent operations proposed by Baumgratz et al. [2] give rise to a well-behaved operational resource theory of coherence. Remarkably, almost all basic questions in this resource theory have simple answers. We saw that it is a theory without bound coherence, but exhibiting generic irreversibility for transformations between pure and mixed states. This should be contrasted with the general abstract framework of Brandão and Gour [33], which applies to the present theory of coherence, but rather than the incoherent operations considered here, requires all cptp maps $E$ such that the weaker condition $E(\Delta) \subset \Delta$ holds. Both the distillable coherence and the coherence cost under this relaxed premise become $C_r(\rho)$, meaning that while we cannot distill more efficiently with this broader class of operations, formation becomes cheaper.

One curious observation, for which we would like to have a strictly operational interpretation, is that the resulting theory of coherence resembles so closely the theory of maximally correlated entangled states. Indeed, under the correspondence

$$
\rho = \sum_{ij} \rho_{ij} |i>|j| \leftrightarrow \tilde{\rho} = \sum_{ij} \rho_{ij} |ii>|jj|,
$$

$C(\psi)$ is identified with $E(\tilde{\psi})$, the entropic measure of pure entanglement, $C_r(\rho) = C_f(\rho)$ is identified with the entanglement cost (which equals the entanglement of formation for these states), $E_r(\tilde{\rho}) = E_f(\tilde{\rho})$, and $C_d(\rho) = C_r(\rho)$ is identified with the distillable entanglement (which equals the relative entropy of entanglement for these states), $E_d(\tilde{\rho}) = E_f(\tilde{\rho})$ [34]. Indeed, this answers all the basic asymptotic questions in the theory, which is much simpler than general entanglement theory.

What is missing to elevate this correspondence from an observation to a theoretical explanation (cf. Streltsov et al. [14]) is a matching correspondence between incoherent operations and LOCC operations. That would truly show that the two theories are equivalent, relieving us of the need to prove any of the optimal conversion rates reported in this Letter, which now we have to do by manually adapting the entanglement manipulation protocols.

A notable gap in the above correspondence is the non-asymptotic theory of pure states: The diagonal entries of a pure state correspond to the Schmidt coefficients of the associated pure entangled state, and by Nielsen’s theorem [18] a pure state can be transformed by LOCC into another one if and only if the Schmidt vectors are in majorization relation. In the case of incoherent operations on pure states, we only know that majorization is sufficient for transformability but not whether it is necessary.

To study this correspondence further, it might be worth investigating the optimal conversion rates $R$ of incoherent transformations

$$
\rho^{\otimes n} \xrightarrow{IC} \approx_{\epsilon} \sigma^{\otimes n R}
$$

for general (mixed) states, and which for the moment we can only bound using the existing monotones:

$$
\frac{C_r(\rho)}{C_f(\sigma)} \leq R \leq \min \left\{ \frac{C_r(\rho)}{C_f(\sigma)}, \frac{C_f(\rho)}{C_f(\sigma)} \right\}
$$

Given the close resemblance to entanglement theory, we may expect that one-shot coding theorems as well as finite block length analyses can be carried out, but we have refrained from entering this domain to maintain the simplicity of the asymptotic picture. As in one-shot information theory [35], we expect that min- and max-entropies and relative entropies and Rényi (relative) entropies govern the optimal rates, which now would carry an explicit dependence on the protocol error.

It remains to be seen whether similarly complete theories of asymptotic operational transformations can be carried out for other resource theories, such as that of reference frames [12]. Observe that as reference frame theories are built on group actions under which the free states are precisely the invariant ones, the present theory of coherence may be viewed as the special case of the group of the diagonal phase unitaries.

Acknowledgments.— We thank Swapan Rana for spurring our interest in this project, Gerardo Adesso, Lidia del Rio, Jonathan Oppenheim and Alexander Streltsov for enlightening discussion on different ways of building a resource theory of coherence, and on resource theories in general, and Sandu Popescu for a conversation on relations between the resource theory of coherence and the measurement of time. Furthermore Arun Pati and his co-authors for sharing a preliminary draft of their paper [28].

The present work was initiated when the authors were attending the programme “Mathematical Challenges in Quantum Information” (MQI) at the Isaac Newton Institute in Cambridge, whose hospitality is gratefully acknowledged, and where DY was supported by a Microsoft Visiting Fellowship. The authors’ work was supported by the European Commission (STREP “RAQUEL”), the ERC (Advanced Grant “IRQUAT”), the Spanish MINECO (grant numbers FIS2008-01236 and FIS2013-40627-P) with the support of FEDER funds, as well as by the Generalitat de Catalunya CIRIT, project 2014-SGR-966. DY furthermore acknowledges support from the NSFC, grant no. 11375165.
APPENDICES

A. Miscellaneous facts and lemmas

In this appendix, we collect standard facts about various functionals we use and show some lemmas that we need in the proofs of the main results.

We use the Bures distance $B$ based on the fidelity $F$, but it is essentially equivalent to the trace norm:

$$B(\rho, \sigma) = \sqrt{2} \sqrt{1 - F(\rho, \sigma)},$$

$$F(\rho, \sigma) = \text{Tr} \sqrt{\rho^{1/2} \sigma \rho^{1/2}}.$$

Namely

$$\frac{1}{2} B(\rho, \sigma)^2 \leq \frac{1}{2} \|\rho - \sigma\|_1 \leq B(\rho, \sigma).$$

In the proof of Theorem 6 below, we need the asymptotic continuity of $C_r$, namely

$$|C_r(\rho) - C_r(\sigma)| \leq \epsilon \log d + 2h(\epsilon/2),$$

where $d$ is the dimension of the supporting Hilbert space, and $h(x) = -x \log x - (1-x) \log (1-x)$ is the binary entropy function.

**Proof.** The proof is straightforward because of Eq. (1): From $\|\rho - \sigma\|_1 \leq \epsilon$, we get $\|\Delta(\rho) - \Delta(\sigma)\|_1 \leq \epsilon$ by the monotonicity of the trace distance under the dephasing operation $\Delta$ which is a cptp map. By Audenaert’s improvement of Fannes’ inequality, which states for two states $\rho$ and $\sigma$ on a $d$-dimensional Hilbert space such that $\frac{1}{2} \|\rho - \sigma\|_1 \leq \eta$, then $|S(\rho) - S(\sigma)| \leq \eta \log d + h(\eta)$, we have

$$|C_r(\rho) - C_r(\sigma)| = |S(\Delta(\rho)) - S(\rho) - (S(\Delta(\sigma)) - S(\sigma))|,$$

$$\leq |S(\Delta(\rho)) - S(\Delta(\sigma))| + |S(\rho) - S(\sigma)|,$$

$$\leq \epsilon \log d + 2h(\epsilon/2),$$

concluding the proof. □

In the proof of Theorem 8 below, we need the monotonicity of the coherence of formation,

$$C_f(\rho) = \min \sum_i p_i S(\Delta(\psi_i)) \text{ s.t. } \rho = \sum_i p_i \psi_i,$$

under incoherent operations.

**Lemma 13** (Åberg [4]) The coherence of formation is convex and a strong coherence monotone: Indeed, for

$$\rho \xrightarrow{\text{IC}} \sigma = \sum_{i} K_i \rho K_i^\dagger = \sum_{i} p_i \sigma_i,$$

it follows that

$$C_f(\rho) \geq \sum_{i} p_i C_f(\sigma_i) \geq C_f(\sigma).$$

**Proof.** As a convex roof (aka convex hull), $C_f$ is automatically convex. Furthermore, because of the convex roof property, we need to prove the strong monotonicity only for pure states. Consider a pure state $\psi$ and an incoherent operation $E$ whose Kraus operator set is $\{K_\ell\}$, where each $K_\ell$ is incoherent operator, satisfying $\sum K_\ell^\dagger K_\ell = 1$. Suppose $K_\ell \psi = \sqrt{p_\ell} \psi_\ell$; from this we construct a new incoherent operation $\tilde{E}$ whose Kraus operator set is $\{|\ell\rangle \otimes K_\ell\}$ where $|\ell\rangle$ are the basis states of an ancillary system. From the monotonicity of the relative entropy of coherence under the incoherent operation $\tilde{E}$, we obtain

$$C_r(\psi) = C_r(\tilde{E}(\psi)) = \sum_{\ell} p_\ell C_r(\psi_\ell) \geq C_f(\tilde{E}(\psi)),$$

and we are done. □

Our proof for the asymptotic continuity of coherence of formation comes from that of entanglement of formation.

**Lemma 14** (Nielsen [39], Winter [40]) For two bipartite states $\rho$ and $\sigma$ supported on a $d \times d$-dimensional Hilbert space with Bures distance $B(\rho, \sigma) \leq \epsilon \leq 1$, then

$$|E_f(\rho) - E_f(\sigma)| \leq \epsilon \log d + (1+\epsilon) h\left(\frac{\epsilon}{1+\epsilon}\right).$$

In the proof of Theorem 8 we need the asymptotic continuity of $C_f$. 

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Lemma 15 The coherence of formation is asymptotically continuous, i.e. for $B(\rho, \sigma) \leq \epsilon \leq 1$,

$$|C_f(\rho) - C_f(\sigma)| \leq \epsilon \log d + (1 + \epsilon) h \left( \frac{\epsilon}{1 + \epsilon} \right),$$

where $d$ is the dimension of the supporting Hilbert space.

Proof. Notice two facts: One is that the bipartite states on the right hand side being supported on a $d \times d$-dimensional Hilbert space, and the other is that $C_f(\rho) = E_f(CNOT(\rho \otimes |0\rangle\langle 0|)CNOT^\dagger)$.

From these and Lemma 14, the claim follows. \qed

In the proof of Theorem 10 we need the following slight modification of \cite{41}, Prop. 2.4.

Lemma 16 (Cf. Devetak/Winter \cite{41}, Prop. 2.4)]

For a family $\{W_x\}_{x \in X}$ of quantum states on a $d$-dimensional Hilbert space $\mathcal{H}$, and the type class $\mathcal{T}_P = \{W^n_x\}$ w.r.t. an empirical type $P$ of length-$n$-sequences over $X$, let $(Y_1, \ldots, Y_S)$ be a random tuple whose elements are sampled from $\mathcal{T}_P$ uniformly without replacement. Define the average state

$$\sigma(P) = \frac{1}{|\mathcal{T}_P|} \sum_{x^n \in \mathcal{T}_P} W^n_x.$$

Then, for every $0 < \epsilon, \delta < 1$ and sufficiently large $n$,

$$\Pr \left\{ \left\| \frac{1}{S} \sum_{j=1}^{S} Y_j - \sigma(P) \right\|_1 \geq \epsilon \right\} \leq 2d^n \exp \left( -\frac{\epsilon^2}{288 \ln 2} \right),$$

where $s = 2^{-H(X;W) + \delta}$ and $I(X : W) = S(X) + S(W) - S(XW) = S(\sum_i P_i W_i) - \sum_i P_i S(W_i)$ is the mutual information evaluated on the state $\omega = \sum_i P_i |x\rangle \langle x| \otimes W_x$.

Proof. Prop. 2.4 in \cite{41} follows directly from the matrix tail bound in \cite{42} when the matrices are sampled uniformly with replacement, that is, $Y_j$ are i.i.d. random variables. As pointed out in \cite{13}, the matrix tail bound in \cite{42} still holds when the $Y_j$ are sampled uniformly without replacement \cite{14}. So the modified \cite{41}, Prop. 2.4 also holds. \qed

In the proof of Theorem 10 we need the following lemma, which comes essentially from the structure of the state with vanishing conditional mutual information \cite{13}.

Lemma 17 (Winter/Yang \cite{46}) A state $\rho_{AB}$ in the finite dimensional Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ satisfying

$$S(B) - S(AB) = E_f(AB)$$

is of the form

$$\rho_{AB} = \bigoplus_i p_i \rho_i^{B^i} \otimes \phi_i^{AB^i},$$

where $\phi_i^{AB^i}$ are pure states and system $B$ is decomposed as a direct sum of tensor products: $\mathcal{H}_B = \bigoplus \mathcal{H}_{B^i} \otimes \mathcal{H}_{B^{\bar{i}}}$.

B. Proofs

Here we provide the detailed proofs of the results in the main text.

Proof of Theorem 2. Assume $\spec(\psi) = \vec{p} \succ \vec{q} = \spec(\varphi)$, and without loss of generality,

$$|\psi\rangle = \sum_i \sqrt{p_i} |i\rangle, \quad |\varphi\rangle = \sum_i \sqrt{q_i} |i\rangle,$$

since the diagonal unitaries required to adjust the phases are incoherent operations.

It is well-known \cite{16} that majorization implies that there is a probability distribution $\lambda_\pi$ over permutations $\pi \in S_d$ such that

$$\vec{q} = \sum_\pi \lambda_\pi \vec{p}_\pi,$$

where $\vec{p}_\pi$ is the vector $\vec{p}$ with indices permuted according to $\pi$: $\vec{p}_\pi(i) = p_{\pi(i)}$. Observe that we may without loss of generality assume the $p_i$ and $q_i$ to be ordered non-increasingly, and also w.l.o.g. that all $q_i > 0$, otherwise we reduce the Hilbert space dimension $d$.

Now, define the Kraus operators

$$K_{\pi} := \sum_i \sqrt{p_{\pi(i)}} \begin{cases} q_i & |i\rangle \langle i| \\ 0 & \text{otherwise} \end{cases},$$

which are evidently incoherent, and define a cptp map:

$$\sum_{\pi} K_{\pi}^\dagger K_{\pi} = \sum_{\pi} \lambda_\pi \frac{p_{\pi(i)}}{q_i} |i\rangle \langle i| = I.$$

Furthermore, it effects the desired transformation:

$$K_{\pi} |\varphi\rangle = \sum_i \sqrt{\lambda_\pi \frac{p_{\pi(i)}}{q_i}} |\varphi\rangle = \sqrt{\lambda_\pi} |\psi\rangle,$$

for every permutation $\pi$.

Conversely, suppose $\varphi \mapsto \psi$ by the incoherent operation with Kraus operators $\{K_{\pi}\}$, then $K_{\pi}|\varphi\rangle \propto |\psi\rangle$ for every $\ell$. Since rank $\Delta(\psi) = \text{rank } \Delta(\varphi)$, we arrive at
Without loss of generality, we be obtained from $\Phi$.

Proof of Corollary 3. For the moment, Theorem 2 is the best assumptions. For the moment, Theorem 2 is the best general incoherent transformation performs the permutation transformation that majorization $\Delta(\psi)$. In [15], it was claimed by a different approach to $|\psi\rangle$. Now we show that $|\psi\rangle$ can be converted to $|\psi\rangle$ by an LOCC operation; from [13], we get that $\Delta(\psi) > \Delta(\varphi)$. □

Remark In [15], it was claimed by a different approach that majorization $\Delta(\psi) > \Delta(\varphi)$ follows already from a general incoherent transformation $\varphi \rightarrow \psi$. However, it seems that the proof has a gap, by way of some unjustified assumptions. For the moment, Theorem 2 is the best available statement.

Proof of Corollary 3. If $\rho$ is pure, then $\Phi_d \rightarrow IC \rho$ follows from Theorem 2.

Any mixed state is a convex combination of pure states, $\psi_i$ with probability weight $p_i$, each of which can be obtained from $\Phi_d$ by an incoherent map $E_i$. Then, $\mathcal{E} = \sum_i p_i E_i$ is also incoherent and $\rho = \mathcal{E}(\Phi_d)$. □

Proof of Theorem 3. Without loss of generality, we suppose a general pure state in $d$-dimensional Hilbert space in the form $|\psi\rangle = \sum_{i=1}^d \sqrt{q_i} |i\rangle$ with a probability distribution $(q_i)$. The state of $n$ copies of $|\psi\rangle$ is

$$|\psi\rangle^\otimes n = \sum_{i_1, \ldots, i_n} \sqrt{q_{i_1} \cdots q_{i_n}} |i_1 \cdots i_n\rangle.$$}

We shall use information theory abbreviations for a generic sequence of length $n$, $i^n = i_1 \cdots i_n$, the probability $q_i^n = q_{i_1} \cdots q_{i_n}$ and the state $|i^n\rangle = |i_1 \cdots i_n\rangle$ [21].

In analogy to entanglement concentration and dilution for bipartite pure state, we establish the conversion rate between the state $|\psi\rangle$ and the unit coherence state $|\Phi_2\rangle$.

Concentration: Consider the coefficients and the basis sequences of $n$ copies of the state $|\psi\rangle$, we perform the type measurement $\{M_P, P \in \mathcal{P}_n\}$, where $\mathcal{P}_n$ is the set of types of sequences with length $n$, and

$$M_P = \sum_{i^n \in T(P)} |i_1 \cdots i_n\rangle |i_1 \cdots i_n\rangle,$$

with $T(P)$ the type class of $P$, i.e. the set of sequences $i^n$ in which the relative frequency of each letter $i$ equals $P(i)$. By the law of large numbers, with probability converging to 1, we get a type $P \approx Q$ (these are called typical), which hence has the property that $H(P) \approx H(Q)$, where $H(P) = -\sum_p P(i) \log P(i)$ is the Shannon entropy of the distribution $P$. Notice that this measurement is an incoherent operation, and that the output state after application of $M_P$ is the maximally coherent state on the subspace spanned by the type class $T(P)$, which has dimensionality

$$2^n H(P) \geq |T(P)| \geq (n + 1)^{-d} 2^{n H(P)},$$

so by relabelling the indices of the state – which can be realized by an incoherent unitary transformation –, the state can be transformed to $|\Phi_2\rangle^\otimes n R$ with $H(P) \geq R \geq H(P) - \frac{\log(n+1)}{n}$, which tends to $H(P) \approx H(Q) = S(\Delta(\psi))$ when $n \rightarrow \infty$.

Dilution: Consider the coefficients and the basis sequences of $n$ copies of the state $|\psi\rangle$. We decompose the state into two terms

$$|\psi\rangle^\otimes n = \sqrt{\Pr(T_{Q,\delta}^n)} |\text{typ}\rangle + \sqrt{1 - \Pr(T_{Q,\delta}^n)} |\text{atyp}\rangle,$$

with the typical part of the state,

$$|\text{typ}\rangle = \frac{1}{\sqrt{\Pr(T_{Q,\delta}^n)}} \sum \sqrt{q_{i_1} \cdots q_{i_n}} |i_1 \cdots i_n\rangle,$$

summing over all $i_1 \cdots i_n \in T_{Q,\delta}^n$, and an atypical rest $|\text{atyp}\rangle$. Here, $T_{Q,\delta}^n$ the set of (entropy) typical sequences defined as

$$T_{Q,\delta}^n := \left\{ i^n = i_1 \cdots i_n : \left| -\frac{1}{n} \log(q_{i_1} \cdots q_{i_n}) - H(Q) \right| \leq \delta \right\}.$$}

By the law of large numbers, for any $\epsilon > 0$ and sufficiently large $n$, $\Pr(T_{Q,\delta}^n) \geq 1 - \epsilon$. Furthermore, $|T_{Q,\delta}^n| \leq 2^n (H(Q) + \delta)$. The dilution protocol uses then $n(H(Q) + \delta)$ copies of $|\Phi_2\rangle$, whose coefficients are uniform in dimension $2^n (H(Q) + \delta)$ and by Theorem 2 this state can be deterministically transformed by an incoherent operation to any other pure state in dimension $2^n (H(Q) + \delta)$. So we can prepare the state $|\text{typ}\rangle$ satisfying $F(|\text{typ}\rangle |\text{typ}\rangle, |\psi\rangle^\otimes n) \geq \sqrt{1 - \epsilon}$, and the required rate of qubit maximally coherent states $|\Phi_2\rangle$ tends to $H(Q) = S(\Delta(\psi))$.

Optimality: We show both at the same time. Namely, consider a pure state $|\psi\rangle$ and asymptotic transformations $|\psi\rangle^\otimes n \rightarrow IC |\Phi_2\rangle^\otimes n R$ and $|\Phi_2\rangle^\otimes n R \rightarrow IC |\psi\rangle^\otimes n$. We know already that $R = C(|\psi\rangle) = R'$ are achievable. Assume
by contradiction that one of the two can be improved, i.e. \( R > C(\psi) \) achievable for concentration or \( R' < C(\psi) \)
achieved for dilution. Thus, we can have \( \tilde{R} = \frac{R + R'}{2} > 1 \). Then, by composing the dilution and concentration protocols, we get an incoherent transformation

\[
\Phi_{2^n} \overset{IC}{\longrightarrow} \Phi_{2^n}^{R-o(n)}
\]

for large \( n \). But by Lemma 1, each Kraus operator \( K_i \) of this transformation must produce a state \( |\psi_i\rangle \) with rank \( \Delta(\psi) \leq 2^n \). From this one can readily derive

\[
F^2(\psi, \Phi_{2^n}^{R-o(n)}) = \text{Tr} \psi \Phi_{2^n}^{R-o(n)} \leq 2^{-n(R-R')-o(n)},
\]

and by convex combination, the output of any incoherent operation on \( \Phi_{2^n}^{R-o(n)} \) has exponentially small fidelity \( \leq 2^{-n(R-R')/2-o(n)} \) with \( \Phi_{2^n}^{R-o(n)} \). This contradiction shows that necessarily \( R \leq C(\psi) \leq R' \).

**Proof of Theorem 6.** The proof consists of two parts, the direct part showing that the claimed rate is achievable, and the converse part showing that no more can be distilled. In the direct part, we use the typicality technique and a covering lemma on operators. For the converse part, we follow the standard argument where we need the monotonicity, asymptotic continuity, and additivity of relative entropy of coherence.

Consider the purification

\[
|\phi\rangle^{AE} = \sum_i \sqrt{p_i} |i\rangle^A |\phi_i\rangle^E
\]

of \( \rho \), i.e. \( \text{Tr}_E |\phi\rangle\langle\phi|^{AE} = \rho^A \). We perform the type measurement \( \{M_P, P \in \mathcal{P}_n\} \), where \( \mathcal{P}_n \) is the set of types of sequences with length \( n \), and

\[
M_P = \sum_{i^* \in T(P)} |i_1 \ldots i_n\rangle |i_1 \ldots i_n\rangle,
\]

where \( T(P) \) the type class of \( P \) (see the proof of Theorem 4).

By the law of large numbers, with probability converging to 1, we get as outcome a typical type \( Q \approx P \), hence with the property \( H(Q) \approx H(P) \). Then the bipartite state we are left with is

\[
|\phi(Q)\rangle^{AE} = \frac{1}{\sqrt{|T(Q)|}} \sum_{i^* \in T(Q)} |i^*\rangle^A |\phi_{i^*}\rangle^E.
\]

From Lemma 10, we conclude that there exists a partition of the type class \( T(Q) \) into \( |T(Q)|/S := M \) subsets \( \{S_m\} \) with \( |S_m| = S \) such that at least \( M(1-\epsilon) \) of these subsets of \( \{S_m\} \) are “good” in the sense that the average states of system \( E \) over the subsets is almost the same as a fixed state, the average state over \( T(Q) \),

\[
\frac{1}{S} \sum_{i^* \in S_m} \phi_{i^*} \approx \frac{1}{|T(P)|} \sum_{i^* \in T(P)} \phi_{i^*}.
\]

The other, “bad”, subsets are few. Relabelling the indices \( i^* \leftrightarrow (m, s) \), we can write Eq. 2 as follows.

\[
|\phi(Q)\rangle^{AE} = \frac{1}{\sqrt{M}} \sum_{m=1}^{M} |m\rangle \otimes \frac{1}{\sqrt{S}} \sum_{s=1}^{S} |s\rangle |\phi_{ms}\rangle.
\]

Now introduce

\[
|\phi(Q)_m\rangle := \frac{1}{\sqrt{S}} \sum_{s=1}^{S} |s\rangle |\phi_{ms}\rangle;
\]

by Uhlmann’s theorem 13, for “good” \( m \), there exists a unitary \( U_m \) on \( A \) such that \( (U_m \otimes 1)|\phi(Q)_m\rangle \approx |\phi(Q)_0\rangle \).

Now we perform the measurement \( \{P_{\text{good}}, P_{\text{bad}}\} \), and with probability \( 1-\epsilon \), we get the state

\[
\frac{1}{\sqrt{M(1-\epsilon)}} \sum_{\text{good } m} |m\rangle |\phi(Q)_m\rangle.
\]

Then we construct the incoherent operation whose Kraus operators are of the form

\[
K_s = \sum_{\text{good } m} |m\rangle\langle m| \otimes |0\rangle\langle s| U_m.
\]

This gives

\[
K_s |\phi(Q)\rangle^{AE} \approx \frac{1}{\sqrt{M(1-\epsilon)}} \sum_{\text{good } m} |m\rangle |0\rangle |\phi(Q)_0\rangle,
\]

which shows that we obtain an approximation to the coherent state \( \frac{1}{\sqrt{M}} \sum |m\rangle |0\rangle \).

It remains to estimate the quantity \( M \) in the above protocol. Observe \( 2^{nH(Q)} \geq |T(Q)| \geq (n+1)^{-d} 2^{nH(Q)} \) and \( S = 2^{n(A,E)'} \), where the mutual information is calculated with respect to the state \( \omega = \sum_i q_i |i\rangle |i\rangle^A \otimes |\phi_i\rangle |\phi_i\rangle^E \).

Since \( Q \) is a typical type, we have \( H(Q) \approx H(P) = S(\Delta(\rho)) \), and

\[
I(A : E) = S(\omega^E) \approx S \left( \sum_i p_i |\phi_i\rangle \right) = S(\rho^A).
\]

Thus, we get \( \frac{1}{n} \log M \to S(\Delta(\rho)) - S(\rho) \) when \( n \to \infty \).

Conversely, for any protocol \( L_n \) such that \( \| L_n(\rho^{\otimes n}) - \Phi_{2^n}^{R-o(n)} \|_1 \leq \epsilon \), we have

\[
n C_\epsilon(\rho^{\otimes n}) = C_\epsilon(\rho^{\otimes n}) \geq C_\epsilon(\Phi_{2^n}^{R-o(n)}) \geq C_\epsilon(\Phi_{2^n}^{R-o(n)}) - n\epsilon \log d - 2h(\epsilon/2) \approx nR - n\epsilon \log d - 2h(\epsilon/2),
\]

where Eqs. 3 and 4 come from additivity and Ineq. 1 due to monotonicity and Ineq. 2 due to asymptotic continuity. So \( R \leq C_\epsilon(\rho) + \epsilon \log d + 2h(\epsilon/2) \). When \( \epsilon \to 0 \) and \( R \to C_d(\rho) \) as \( n \to \infty \), we obtain \( C_d(\rho) \leq C_\epsilon(\rho) \).
Proof of Theorem\[8\]. The proof consists of two parts, the direct part that shows we can prepare the state at the claimed rate, and the converse part that says that to prepare the state we have to consume \(\Phi_2\) at least at that rate. The proof is very similar to the entanglement cost\[11\], except that here we have additivity which makes it simpler. In the direct part, we just prepare the typical part of the state and for the converse part, we still have the standard argument where we need the monotonicity, asymptotic continuity, and additivity of coherence of formation.

For an optimal convex decomposition, \(\rho = \sum_i p_i \psi_i\), where the indices \(i\) range over an alphabet \(\Omega\) (w.l.o.g. of cardinality \(|\Omega| \leq d^2\), by Caratheodory’s Theorem), we have
\[
\rho^{\otimes n} = \sum_i p_i^n \psi_i^n,
\]
with \(i^n = i_1 \ldots i_n\), \(p_i^n = p_{i_1} \cdots p_{i_n}\) and \(\psi_i^n = \psi_{i_1} \otimes \cdots \otimes \psi_{i_n}\).

The set of (frequency-)typical sequences,
\[
\mathcal{T} = \{i^n : \forall j \quad f_j(i^n) - p_j \leq \delta_1\},
\]
with \(f_j(i^n) = \frac{1}{n!} \sum t : i_t = j\},\) has the property that for large enough \(n\),
\[
\Pr(T) \geq 1 - \epsilon_1.
\]

Now we rewrite \(\rho^{\otimes n} = \rho(T) + \rho_0\), with the sub-normalized state
\[
\rho(T) = \sum_{i^n \in \mathcal{T}} p_i^n |\psi_{i^n}\rangle \langle \psi_{i^n}|,
\]
and a rest \(\rho_0\). The protocol is now to sample an element \(i^n \in \mathcal{T}\) according to the conditional distribution \(\frac{1}{p^n(T)} p^{\otimes n} |\_T\) of \(p^{\otimes n}\) restricted to \(\mathcal{T}\). The number of occurrences of \(j\) in each typical sequence \(i^n \in \mathcal{T}\) is \(N(j|i^n) \leq n(p_j + \delta_1)\). By the coherence dilution protocol we can prepare a state
\[
\rho_j^{(N(j|i^n))} = \frac{1 - \epsilon_2}{\epsilon_2} \rho_j^{\otimes n} \approx \psi_j^n,
\]
by consuming at most \(n(p_j + \delta_1)(S(\Delta(\psi_j)) + \delta_2)\) copies of \(\Phi_2\). So we can prepare
\[
\bigotimes_j \rho_j^{(N(j|i^n))} = \rho^{\otimes n} \approx \psi^n,
\]
using at most \(\sum_j n(p_j + \delta_1)(S(\Delta(\psi_j)) + \delta_2)\) copies of \(\Phi_2\). Now we prepare a mixed state as the convex combination of these \(\rho_i^n\), i.e.
\[
\rho^{(n)} = \frac{1}{p^{\otimes n}(T)} \sum_{i^n \in \mathcal{T}} p_i^n \rho_i^n.
\]

By the joint concavity of the fidelity\[17, 48\], we get
\[
F(\rho^{\otimes n}, \rho^{(n)}) \geq p^{\otimes n}(T) F\left(\frac{\rho(T)}{p^{\otimes n}(T)}, \rho^{(n)}\right) \geq (1 - \epsilon_1)(1 - \epsilon_2)^{\Omega}.
\]

Since \(|\Omega|\) is bounded, when \(n \to \infty\), \(\epsilon_1, \epsilon_2 \to 0\) and \(F(\rho^{\otimes n}, \rho^{(n)}) \to 1\) for \(\delta_1, \delta_2 \to 0\. The required rate of \(\Phi_2\) tends to \(\sum_j p_j S(\Delta(\psi_j)) = C_f(\rho)\). This shows that the rate \(C_f(\rho)\) is asymptotically achievable.

For the optimality, consider an incoherent protocol that produces \(\rho^{(n)} = L(\Phi_2^{\otimes n})\), which is close to \(\rho^{\otimes n}\) up to error \(\epsilon\), i.e. \(B(\rho^{(n)}, \rho^{\otimes n}) \leq \epsilon\). From the monotonicity, asymptotic continuity and additivity of \(C_f(\rho)\), we get
\[
m = C_f(\Phi_2^{\otimes n}) \geq C_f(\rho^{(n)}) \geq C_f(\rho) - n \epsilon \log d - (1 + \epsilon)h\left(\frac{\epsilon}{1 + \epsilon}\right) \geq n C_f(\rho) - n \epsilon \log d - (1 + \epsilon)h\left(\frac{\epsilon}{1 + \epsilon}\right),
\]
where Eqs.\[7\] and \[10\] come from the additivity, Ineq.\[8\] from the monotonicity and Ineq.\[9\] from the asymptotic continuity. Thus,
\[
\frac{m}{n} \geq C_f(\rho) - \epsilon \log d - (1 + \epsilon)h\left(\frac{\epsilon}{1 + \epsilon}\right).
\]

Letting \(\epsilon \to 0\) as \(n \to \infty\), we get \(C_c(\rho) \geq C_f(\rho)\), as advertised. \(\square\)

Proof of Theorem\[9\]. We observe that the coherence of formation \(C_f(\rho)\) is equal to the entanglement of formation of the bipartite state \(\text{CNOT}(\rho \otimes |0\rangle|0\rangle)\text{CNOT}^\dagger\). The latter is a maximally correlated state in the sense that in any decomposition into pure states ensemble, the pure states have the same Schmidt basis. In \[31\], the entanglement of formation for this class of states is proved to be additive. So the coherence of formation is additive.

Additivity of \(C_f\) comes directly from Eq.\[11\] because \(S(\rho \otimes \sigma) = S(\rho) + S(\sigma)\) and \(S(\Delta(\rho \otimes \sigma)) = S(\Delta(\rho)) + S(\Delta(\sigma))\). \(\square\)

Proof of Theorem\[10\]. Given a mixed state \(\rho^A\), we construct the bipartite state \(\sigma^{AB} = \text{CNOT}(\rho^A \otimes |0\rangle|0\rangle)\text{CNOT}^\dagger\) which is a maximally correlated state of the form \(\sum_{i,j} \rho_{ij} |i\rangle|j\rangle\). Now the reversibility implies that \(S(\sigma^B) - S(\sigma^{AB}) = E_f(\sigma^{AB})\). Applying Lemma\[17\] to this and noticing that the \(\sigma^A = \sigma^B\) we get
\[
\sigma^{AB} = \bigoplus_j p_j |\phi_j\rangle^A |\phi_j\rangle^B,
\]
where \(\phi_j^B \perp \phi_j^A\) when \(j \neq k\). Then using the CNOT again on \(\sigma^{AB}\) we recover the original state \(\rho^A\) of the form
\[
\rho^A = \bigoplus_j p_j |\phi_j\rangle |\phi_j\rangle^A,
\]
where each $|\phi_j\rangle^A$ is in the subspace $\mathcal{H}_j = \text{span}\{ |i\rangle : i \in S_j \}$ and these subspaces are orthogonal to each other. □

**Proof of Theorem 11.** This follows from Theorem 4, $C_0(\rho) = C_r(\rho)$ and the fact that $C_r(\rho)$ is a faithful coherence measure in the sense that $C_r(\rho) = 0$ if and only if $\rho$ is incoherent. □

C. Generalized model

While we restricted our treatment for simplicity of notation to the case that the incoherent states are precisely a fixed orthogonal basis of the Hilbert space and convex combinations, we observe that our results carry over unchanged to the case of a general decomposition into subspaces $\mathcal{H} = \bigoplus_i \mathcal{H}_i$.

In generalization of the picture in the main body of the present work, with an orthogonal basis and its superpositions, and inspired by Åberg [4], let us consider an orthogonal decomposition of the Hilbert space $\mathcal{H}$ into eigenspaces of some observable

$$O = \sum_i E_i P_i,$$

with an orthogonal direct sum, and $P_i$ is the projector onto the subspace $\mathcal{H}_i$ of $\mathcal{H}$. With respect to this decomposition, we declare states as *incoherent* that respect the direct sum decomposition:

$$\Delta := \left\{ \rho = \sum_i q_i \rho_i : P_i \rho_i P_i = \rho_i \right\},$$

where the $\rho_i$ are states (supported on $\mathcal{H}_i$) and $(q_i)$ is a probability distribution. For the case of rank-one projectors $P_i = |i\rangle\langle i|$ we recover the notion of Baumgratz et al. [2].

Slightly abusing notation, we also introduce the decohering projection map

$$\Delta(A) = \sum_i P_i A P_i,$$

such that $\Delta = \Delta(S(\mathcal{H}))$ is the image of the state space under the decohering map; the image of the set of all operators on $\mathcal{H}$ we denote by

$$\mathcal{D} := \Delta(B(\mathcal{H})) = \bigoplus_i B(\mathcal{H}_i).$$

We call an operator $K$ acting on $\mathcal{H} = \bigoplus_i \mathcal{H}_i$, or more generally mapping $\mathcal{H}$ to $K = \bigoplus_j K_j$, *incoherent (IC)* if $K \mathcal{H}_i \subset K \mathcal{H}_j$ for every $i$ and a function $i \mapsto j(i)$. If $j(i)$ is injective – which for operators mapping $\mathcal{H}$ to itself means that it is a permutation –, we call the operator *strictly incoherent (strictly IC)*. The operator $K$ is strictly incoherent if and only if $K$ as well as $K^\dagger$ are incoherent. When composing systems, each with its own orthogonal decomposition, $\mathcal{H} = \bigoplus_i \mathcal{H}_i$ and $\mathcal{H}' = \bigoplus_j \mathcal{H}'_j$, we consider the tensor product space with the decomposition $\mathcal{H} \otimes \mathcal{H}' = \bigoplus_{i,j} \mathcal{H}_i \otimes \mathcal{H}'_j$. In this way, the decohering map on the composite space becomes the tensor product $\Delta \otimes \Delta'$.

With this, following [2], we can introduce the sets of incoherent and strictly incoherent operations as special ctp maps:

$$\mathcal{IC} := \left\{ T \text{ ctp}: T(\rho) = \sum_\alpha K_\alpha \rho K_\alpha^\dagger, \forall \alpha K_\alpha \text{ IC} \right\},$$

$$\mathcal{IC}_0 := \left\{ T \text{ ctp}: T(\rho) = \sum_\alpha K_\alpha \rho K_\alpha^\dagger, \forall \alpha K_\alpha \text{ strictly IC} \right\}.$$

By slight abuse of notation, we will write for a incoherent (strictly incoherent) Kraus operator $K$, that $K \in \mathcal{IC}$ ($K \in \mathcal{IC}_0$).

Finally, the non-coherence-generating maps according to Brandão and Gour [33] are

$$\widetilde{\mathcal{IC}} := \{ T \text{ ctp}: T(\Delta) \subset \Delta \},$$

so that $\mathcal{IC}_0 \subseteq \mathcal{IC} \subseteq \widetilde{\mathcal{IC}}$.

Our main results, Theorems 2, 4, 6, 8, and the additivity Theorem 9 remain unchanged, as one can check.

**Remark** All of the transformations or asymptotic transformations contained in the above mentioned theorems can be effected by strictly incoherent operations ($\mathcal{IC}_0$), except the distillation of mixed states. Indeed, it is not clear whether or not the rate $C_r(\rho)$ is attainable with strictly incoherent operations. Our protocol for Theorem 6 at any rate, uses $\mathcal{IC}$ in a non-trivial way.