A general expression of the triplets of integer sided triangles with a 120° angle, in parallel with the case of a 60° angle.

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Abstract
A research for a general expression of the triplets of integer sided triangles with a 120° angle, in parallel with the case of a 60° angle.

Keywords. triplet, integer sided triangle, Eisensteinian triplet, relatively prime, equilateral triangle, primitive Eisensteinian triangle, primitive sub-Eisensteinian triangle

0. Introduction

Recently we have got interested in integer sided triangles after reading Zimhoni’s paper [1], in which Zimhoni treated the triplets of integer sided triangles with a 60° angle, showed a general expression of the triplets proved by Katayama [5], and proved another general expression by the hint of [2] and [5].

Then we have tried to find a general expression of the triplets of integer sided triangles with a 120° angle, and shall show it in this paper.

1. For the case of a 60° angle

For the integer sided triangles with a 60° angle, the triplets of the sides of which are known as Eisensteinian triplets, and Zimhoni show that any such triangle is obtained by pre-multiplication (7,8,5) or (13,15,7) by a product of five matrices [1], by the hint of Berggren [2].

Zimhoni defined a definition and proved a theorem in [1], then we show them as below:

Definition 1.
A primitive Eisensteinian triangle is an integer sided triangle with side lengths (a,b,c) such that the angle opposite the side of length a is of 60° and gcd(a,b,c)=1. Here, we will also follow the convention of ordering the triplet such that b>c.

Theorem 1.
An integer sided triangle is primitive Eisensteinian if, and only if it has side lengths (1,1,1) or it has side lengths (a,b,c) or (a,b,b−c) such that:

\[
\begin{pmatrix}
  a \\
  b \\
  c
\end{pmatrix} = M \cdot v, \quad \text{where } v \in \left\{ \begin{pmatrix} 7 \\ 8 \\ 5 \end{pmatrix}, \begin{pmatrix} 13 \\ 15 \\ 17 \end{pmatrix} \right\}
\]

And M is a finite (or empty) product of the matrices:

\[
M_1 = \begin{pmatrix} 7 & -6 & 6 \\ 8 & -7 & 7 \\ 4 & -4 & 3 \end{pmatrix}, M_2 = \begin{pmatrix} 7 & 6 & -6 \\ 8 & 7 & -7 \\ 4 & 3 & -4 \end{pmatrix}, M_3 = \begin{pmatrix} 7 & 6 & 0 \\ 8 & 7 & 0 \\ 4 & 3 & 1 \end{pmatrix},
\]

\[
M_4 = \begin{pmatrix} 7 & -6 & 6 \\ 8 & -7 & 7 \\ 4 & -4 & 3 \end{pmatrix}, M_5 = \begin{pmatrix} 7 & 6 & -6 \\ 8 & 7 & -7 \\ 4 & 3 & -4 \end{pmatrix}
\]

Any such triangle is obtained exactly once.

Zimhoni proved this theorem skillfully by using the Stern-Brocot tree [3],[4].

Next we show the following theorem proved here [5]:

**Theorem 2.**

A non-equilateral integer sided triangle is primitive einsteinian if, and only if, it is of the class of triangles whose sides are \(m^2+mn+n^2\), \(m^2+2mn\) and either \(n^2+2mn\) or \(m^2-n^2\), where m and n are positive relatively prime integers with \(m>n>0\) and \(m \equiv n (\text{mod } 3)\). Any such triangle is achieved exactly once.

Zimhoni proved Theorem 1 by the aid of above-mentioned Theorem 2.

2. For the case of a 120° angle

We originally define a definition as below:

**Definition 2.**

A primitive sub-Eisensteinian triangle is an integer sided triangle with side lengths (a,b,c) such that the angle opposite the side of length a is of 120° and gcd(a,b,c)=1.

We assume a theorem and shall prove it as follows:
Theorem 3.

There exists a one-to-one mapping between the set of non-equilateral primitive Eisensteinian triangles and the set of primitive sub-Eisensteinian triangles.

Proof.

According to Definition 1 and Theorem 1, a non-equilateral primitive Eisensteinian triangle is an integer triangle with side lengths \((a,b,c)\) such that the angle opposite the side of length \(a\) is of \(60^\circ\) and \(\gcd(a,b,c)=1\) and \(b>c\).

Now we use a \(3\times 3\) matrix

\[
S = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & -1 \\
\end{pmatrix}
\]

and present above-mentioned \((a,b,c)\) as a column, then the product of the two is:

\[
S \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & -1 \\
\end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ c \\ b - c \end{pmatrix}
\]

Now we assume that \((a,b,c)\) is an element the set of non-equilateral primitive Eisensteinian triangles.

By Definition 1, the Law of Cosines says that:

\[
(2.1) \quad a^2 = b^2 + c^2 - 2bc \cos 60^\circ = b^2 + c^2 - bc
\]

This is followed by:

\[
(2.2) \quad a^2 = c^2 + (b - c)^2 + c(b - c) = c^2 + (b - c)^2 - 2c(b - c) \cos 120^\circ
\]

According to Definition 2, \((a,c,b-c)\) is an element of the set of primitive sub-Eisensteinian triangles, because \(b-c\) is a positive integer and \(\gcd(a,c,b-c)=1\), owing to \(\gcd(a,b,c)=1\).

Inversely we assume that \((a,b,c)\) is an element the set of primitive sub-Eisensteinian triangles. Now we compute the inverse of \(S\):

\[
S^{-1} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 1 & 0 \\
\end{pmatrix}
\]
and present above-mentioned \((a,b,c)\) as a column, then the product of the two is:
\[
S^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b+c \\ b \end{pmatrix}
\]

Now we assume that \((a,b,c)\) is an element the set of primitive sub-Eisensteinian triangles.

By Definition 2, the Law of Cosines says that

\[
(2.3) \quad a^2 = b^2 + c^2 - 2bc \cos120^\circ = b^2 + c^2 + bc
\]

This is followed by

\[
(2.4) \quad a^2 = (b + c)^2 + b^2 - (b + c)b = (b + c)^2 + b^2 - 2(b + c)b \cos60^\circ
\]

According to Definition 1, \((a,b+c,b)\) is an element of the set of non-equilateral primitive Eisensteinian triangles, because \(b+c>b\) and \(\gcd(a,b+c,b)=1\), owing to \(\gcd(a,b,c)=1\).

These complete the proof.

\[
\square
\]

By using Theorem 3, we can easily see as follows:

Corollary 1.

An integer sided triangle is primitive sub-eisensteinian if, and only if, it is of the class of triangles whose sides are \(m^2+mn+n^2\), \(n^2+2mn\) and \(m^2-n^2\), where \(m\) and \(n\) are positive relatively prime integers with \(m>n>0\) and \(m\not\equiv n\pmod{3}\). Any such triangle is achieved exactly once.

Proof.

By using \(S\), we see
\[
S\begin{pmatrix} m^2+mn+n^2 \\ m^2+2mn \\ n^2+2mn \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} m^2+mn+n^2 \\ m^2+2mn \\ n^2+2mn \end{pmatrix} = \begin{pmatrix} m^2+mn+n^2 \\ n^2+2mn \\ m^2-n^2 \end{pmatrix}
\]

and
\[
S\begin{pmatrix} m^2+mn+n^2 \\ m^2+2mn \\ m^2-n^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} m^2+mn+n^2 \\ m^2+2mn \\ m^2-n^2 \end{pmatrix} = \begin{pmatrix} m^2+mn+n^2 \\ m^2-n^2 \\ n^2+2mn \end{pmatrix}.
\]
According to what Theorem 3 says, this completes Corollary 1.

\[ \square \]

**Theorem 4.**

An integer triangle is primitive sub-Eisenstein if, and only if it has side lengths \((a,b,c)\) or \((a,c,b)\) such that:

\[
\begin{pmatrix}
a \\
b \\
c
\end{pmatrix} = N \cdot v, \text{ where } v \in \left\{ \begin{pmatrix} 7 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} 13 \\ 7 \\ 8 \end{pmatrix} \right\}
\]

And \(N\) is a finite (or empty) product of the matrices:

\[
\begin{align*}
N_1 &= \begin{pmatrix} 7 & 0 & -6 \\ 4 & -1 & -4 \\ 4 & 1 & -3 \end{pmatrix}, & N_2 &= \begin{pmatrix} 7 & 0 & -6 \\ 4 & -1 & 3 \\ 4 & 1 & 4 \end{pmatrix}, & N_3 &= \begin{pmatrix} 7 & 6 & 6 \\ 4 & 4 & 3 \\ 4 & 3 & 4 \end{pmatrix}, \\
N_4 &= \begin{pmatrix} 7 & 6 & 0 \\ 4 & 4 & 1 \\ 4 & 3 & -1 \end{pmatrix}, & N_5 &= \begin{pmatrix} 7 & -6 & 0 \\ 4 & -3 & 1 \\ 4 & -4 & -1 \end{pmatrix}
\end{align*}
\]

Any such triangle is obtained exactly once.

**Proof.**

According to Theorem 1, a non-equilateral integer sided triangle is primitive Eisensteinian if, and only if it has side lengths \((a,b,c)\) or \((a,b,b-c)\) such that:

\[
\begin{pmatrix}
a \\
b \\
c
\end{pmatrix} = M \cdot v, \text{ where } v \in \left\{ \begin{pmatrix} 7 \\ 8 \\ 5 \end{pmatrix}, \begin{pmatrix} 13 \\ 15 \\ 17 \end{pmatrix} \right\}
\]

And \(M\) is a finite (or empty) product of the matrices:

\[
\begin{align*}
M_1 &= \begin{pmatrix} 7 & -6 & 6 \\ 8 & -7 & 7 \\ 4 & -4 & 3 \end{pmatrix}, & M_2 &= \begin{pmatrix} 7 & 6 & -6 \\ 8 & 7 & -7 \\ 4 & 3 & -4 \end{pmatrix}, & M_3 &= \begin{pmatrix} 7 & 6 & 0 \\ 8 & 7 & 0 \\ 4 & 3 & 1 \end{pmatrix}, \\
M_4 &= \begin{pmatrix} 7 & -6 & 6 \\ 8 & -7 & 7 \\ 4 & -4 & 3 \end{pmatrix}, & M_5 &= \begin{pmatrix} 7 & 6 & -6 \\ 8 & 7 & -7 \\ 4 & 3 & -4 \end{pmatrix}
\end{align*}
\]

and any such triangle is obtained exactly once.
We now temporally consider the case of side lengths \((a,b,c)\). Then we set

\[
(2.5) \quad M = M_{p_1}M_{p_2} \cdots M_{p_n}
\]

where for any integer \(1 \leq k \leq n\) and
\[
M_{p_k} = M_1 \text{ or } M_2 \text{ or } M_3 \text{ or } M_4 \text{ or } M_5.
\]

Moreover we obtain that:

\[
(2.6) \quad S \cdot M \cdot v = S(M_{p_1}M_{p_2} \cdots M_{p_n})v
= (SM_{p_1}S^{-1})(SM_{p_2}S^{-1}) \cdots (SM_{p_n}S^{-1})(S \cdot v)
\]

According to the proof of Theorem 2, \(S \cdot v\) is an element of the set of primitive sub-Eisensteinian triangles. And we easily see that two different elements of the set of primitive Eisensteinian triangles is respectively mapped to different elements of the set of primitive Eisensteinian triangles by \(S\).

And owing to Theorem 1, if any two certificates for \(M\) that have a different expression from each other, then their values are also different from each other.

We arrange (2.6) as follows:

\[
(2.7) \quad S \cdot M \cdot v = S(M_{p_1}M_{p_2} \cdots M_{p_n})v
= (SM_{p_1}S^{-1})(SM_{p_2}S^{-1}) \cdots (SM_{p_n}S^{-1})(S \cdot v)
= (N_{p_1})(N_{p_2}) \cdots (N_{p_n})(S \cdot v)
= N(S \cdot v).
\]

We let:

\[
(2.8) \quad N = N_{p_1}N_{p_2} \cdots N_{p_n}
\]

where for any integer \(1 \leq k \leq n\) and
\[
N_{p_k} = SM_1S^{-1} \text{ or } SM_2S^{-1} \text{ or } SM_3S^{-1} \text{ or } SM_4S^{-1} \text{ or } SM_5S^{-1}.
\]

In the above-mentioned equations, any two certificates for \(N\) that have a different expression from each other, then their values are also different from each other.

Then by computing we see:
Eisensteinian triangles, mapped by $S$ from the elements of the set of non-
triplet $[1]$. The general expression of the elements of the set of primitive sub-
Eisensteinian triangles, mapped by $S$ from the elements of the set of non-
equilateral primitive Eisensteinian triangles, in the case of side lengths
$(a,b,c)$, is (2.7).

Then next we consider the case of side lengths $(a,b,b−c)$. In this connection, the pair $(a,b,c)$ and $(a,b,b−c)$ is called the the twin Eisensteinian
triplet [1].

With the aid of Theorem 1, we obtain:

\[
S \begin{pmatrix} a \\ b \\ b−c \end{pmatrix} = S \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = S \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} (M \cdot v)
\]
\[
= S \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} S^{-1}(SMv) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} (SMv)
\]
It is evident that:

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
c
\end{pmatrix} =
\begin{pmatrix}
a \\
c \\
b
\end{pmatrix},
\]

therefore we see as follows:

If the case of side lengths \((a,b,c)\) for primitive Eisensteinian corresponds to the case of side lengths \((a,b,c)\) for primitive sub-Eisensteinian, then the case of side lengths \((a,b,b−c)\) for primitive Eisensteinian corresponds to the case of side lengths \((a,c,b)\) for primitive sub-Eisensteinian.

Thus any primitive sub-Eisensteinian triangle is also obtained by pre-multiplication \((7,5,3)\) or \((13,7,8)\) by a product of five matrices.

These complete the proof.

\[\square\]

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