Scalar Bremsstrahlung in Gravity-Mediated Ultrarelativistic Collisions

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Classical bremsstrahlung of a massless scalar field \( \Phi \) is studied in gravity mediated ultra-relativistic collisions with impact parameter \( b \) of two massive point particles in the presence of \( d \) non-compact or toroidal extra dimensions. The spectral and angular distribution of the scalar radiation are analyzed, while the total emitted \( \Phi \)-energy is found to be strongly enhanced by a \( d \)-dependent power of the Lorentz factor \( \gamma \). The direct radiation amplitude from the accelerated particles is shown to interfere destructively (in the first two leading ultra-relativistic orders) with the one due to the \( \Phi - \Phi \) - graviton interaction in the frequency regime \( \gamma/b \lesssim \omega \lesssim \gamma^2/b \) in all dimensions.

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1. INTRODUCTION AND RESULTS

The idea of TeV scale gravity with large extra dimensions (LED) has triggered a lot of activity in particle physics and gravitation theory. One of the most interesting predictions is the possibility of black hole production in colliders. According to Thorne’s hoop conjecture (generalized to higher dimensions), for energies of colliding particles higher than the $D$-dimensional Planck mass $M_\ast$ (transplanckian regime) such black holes should be produced classically for impact parameters $b \lesssim R_S$, where $R_S$ is the Schwarzschild radius associated with the center-of-mass collision energy. To establish the creation of a black hole in the collision of ultrarelativistic particles one has to find closed trapped surfaces in the corresponding space-time. To this aim, an idea due to Penrose was put into the form of an elaborated model in $D = 4$ by D’Eath and Payne and with extra dimensions by Eardley and Giddings and further refined in (see recent reviews and references therein). In that model one is after a solution of Einstein’s equations with a special metric ansatz generalizing the Aichelburg-Sexl metric (see also ). The ansatz amounts to replacing the gravitational field of two ultrarelativistic particles before the collision by colliding shock waves, while the collision region is described by some linear differential equation for a metric function, which is amenable to construct exact or approximate solutions. The closed trapped surface emerging in such a solution for appropriate initial energies and impact parameters leads to an estimate of the produced black hole mass and its difference from the initial energy is interpreted as the amount of gravitational radiation produced (see e.g. the recent paper and references therein).

The colliding waves model (CWM) of black hole formation is certainly a very nice and perhaps the simplest possible one, designed to answer an intriguing question about the nature of transplanckian collisions. It gives the gravitational radiation loss for head-on and almost head-on collisions and also demonstrates that the black hole is indeed present in the collision region. However, we would like to discuss some subtle points concerning the applicability of this model to high energy particle scattering. In fact, various effects which were not taken into account in the simple version of CWM such as an extended nature of the colliding particles and their parton structure, have already been discussed. It was shown by Meade and Randall that taking the finite size into account leads to a substantial decrease of the cross-section predicted by the CWM. It was also emphasized that an even more critical effect is the radiative energy loss of the colliding partons before their energy is trapped inside a black hole horizon. The CWM seems to be able to clarify these issues, but in a closer look it probably cannot. In fact, the Aichelburg-Sexl solution is the limiting form of the linearized gravitational field of ultrarelativistic particle moving with a constant velocity. Therefore, presenting colliding particles as plane waves implicitly assumes that their radiation is negligibly small, otherwise the particle trajectories should be modified substantially by radiation friction (for a discussion of radiation reaction in extra dimensions see ). Furthermore, the energy mismatch between the mass of the black hole and the initial energy of the colliding particles, interpreted in CWM as radiation loss, is found to be of the order of the initial energy, which is by no means small.

Therefore, alternative methods of computing radiation losses in transplanckian collisions seem to be desirable. One such approach is the one by Amati, Ciafaloni and Veneziano relevant in four dimensions and based on the combination of string and quantum field theory techniques. Other recent work on this subject includes using various analytical classical and semi-classical approaches. In the framework of purely classical $D = 4$ general relativity, numerical simulations were also performed of collision of two scalar field balls interacting gravitationally via exact Einstein equations. Gravitational radiation was extracted imposing appropriate boundary conditions. Gravitational radiation in collisions of higher-dimensional black holes (with non-compact extra dimensions) was studied numerically in.

For the ultrarelativistic scattering in models with large extra dimensions a crucial question is whether radiation is enhanced due to the extended phase space associated with extra dimensions. It was argued by Mironov and Morozov that in the case of synchrotron radiation the expected enhancement can be damped by beaming of radiation in the forward direction, suppressing the number of excited Kaluza-Klein modes. In the case of bremsstrahlung the situation is different, namely in it was found, that the energy loss of ultrarelativistic particles under non-gravitational scattering at small angle contains an additional factor $\gamma^d$ due to the emission of light massive KK modes. Qualitatively, this can be explained as follows. For non-gravitational scattering in flat space the impact parameter $b$, the radiation frequency $\omega$ and the angle of emission $\vartheta \ll 1$ with respect to the momentum of the fast particle in the rest frame of the other are related by

$$\omega b (\vartheta^2 + \gamma^{-2}) \lesssim 1.$$  \hspace{1cm} (1.1)

Frequencies near the cut-off frequency $\omega \sim \gamma^2/b$ are emitted in the narrow $(D-1)$ dimensional cone $\vartheta \lesssim \gamma^{-1}$, while intermediate frequencies are emitted into a wider cone. The main contribution comes from the first region, and in this case the emitted momenta in directions transverse to the brane are of the order $\omega/\gamma$. The number of
such light modes is of the order \((R\omega/\gamma)^d \sim (R\gamma/b)^d\) \((R\) being the size of the compact extra dimensions), giving an extra factor \(\gamma^d\) to the radiated power. Analogous enhancement was reported for gravitational bremsstrahlung in transplanckian collisions \([24]\), compatible with the numerical study of \([21]\).

The purpose of this paper is to study scalar radiation in gravity-mediated collisions in the presence of extra dimensions. Our approach was initiated in \([24, 25]\) where we have briefly formulated purely classical perturbation theory for transplanckian scattering in ADD valid for small scattering angles, or large impact parameters. Indeed, as emphasized long ago by many authors, most notably by ‘t Hooft \([20]\), the gravitational force does not only becomes dominant at transplanckian energies, but gravity itself becomes classical, at least in some significant impact parameter regime. Thus, the applicability of classical theory in the bremsstrahlung problem at transplanckian energies seems to have solid theoretical grounds.

More specifically, our approach amounts to solving the two-body field-mediated problem iteratively. In electrodynamics this is a well known method, allowing to calculate spectral-angular distributions of bremsstrahlung in the classical range of frequencies small with respect to the particle energy \(\hbar \omega \ll E\). In general relativity this approach was suggested in \([27]\) under the name of “fast-motion approximation scheme” and was further developed and called “post-linear formalism”. It was applied to the gravitational bremsstrahlung most notably by Kovacs and Thorne \([28]\). We will use a momentum space version of this approach, developed in \([29]\), which has the advantage that it allows a fully analytical treatment of the problem. In \([23, 24]\) we extended this technique to models with extra dimensions either infinite or compact. In \([24]\) we calculated scalar bremsstrahlung radiation in the collision of two ultrarelativistic point-like particles interacting via a scalar field in flat space-time. Here we consider the situation in which the particles interact gravitationally but emit scalar radiation. This is an intermediate step towards the full treatment of the gravitational bremsstrahlung presented briefly in \([24]\). The latter case has additional complications due to the tensor nature of the radiation field and it will be presented in full detail in a future publication. Here, we will assume that only one of the colliding particles is coupled to the scalar field, so that their interaction is purely gravitational. On the other hand, we will compute only the scalar radiation emitted by the system. The main novel feature is this case is that the system becomes non-linear due to scalar-scalar-graviton vertex. As a result, the effective source of radiation field in the flat space picture becomes non-local due to contribution of the field stresses.

In four dimensions, it was shown long ago \([30]\), that the contribution from the high-frequency regime \(\gamma/b \leq \omega \leq \gamma^2/b\) is suppressed by a factor \(\gamma^{-4}\) due to destructive interference between the local and the non-local amplitudes, the latter being due to the gravitational interaction of the mediating field. The remaining radiation is beamed inside the cone with angle \(1/\gamma\), it has characteristic frequencies \(\omega \sim O(\gamma/b)\) and emitted energy of order \(E \sim \gamma^3\). There is in addition a sub-leading component of emitted radiation with frequencies \(\omega \sim O(\gamma^2/b)\), which is also beamed and has \(E \sim \gamma^2\).

In the higher dimensional case the situation is more complicated. The destructive interference is also present, but with growing \(d\) the relative contribution of \(\omega \sim \gamma^2/b\) increases faster, than that of \(\omega \sim \gamma/b\) due to competition of the angular integrals.

The powers of \(\gamma\) of the emitted radiation energy in all frequency and angular regimes in the presence of \(d\) extra dimensions are summarized in the Table below.

| \(\omega\) | \(\vartheta\) | \(\omega \ll \gamma/b\) | \(\omega \sim \gamma/b\) | \(\omega \sim \gamma^2/b\) | \(\omega \gg \gamma^2/b\) |
|---|---|---|---|---|---|
| \(\gamma^{-1}\) | negligible (phase space) | \(E_d \sim \gamma^3\) | \(E_d \sim \gamma^{d+2}\) | negligible radiation |
| \(\gamma^{1}\) | negligible (phase space) | \(E_d \sim \gamma^{d+1}\) | negligible radiation | negligible radiation |

In the special cases of \(d = 1\) or \(2\) extra dimensions there is an extra \(\ln \gamma\) in the expression for the energy emitted in the regime \((\omega \sim \gamma/b, \vartheta \sim 1/\gamma)\). The energy is measured in units of \(\kappa_D^2 m^2 f^2/b^{d+3}\), with \(m\) the mass of the target particle, \(\kappa_D\) the \(D\)-dimensional gravitational coupling, \(f\) the scalar coupling of \(m\) and \(b\) the impact parameter of the collision and the overall numerical coefficients can be found in the text.
2. THE SETUP

A. The action

The goal here is to calculate within the ADD scenario classical spin-zero bremsstrahlung in ultra-relativistic gravity-mediated scattering of two massive point particles $m$ and $m'$. The space-time is assumed to be $M_4 \times T^d$, the product of four-dimensional Minkowski space and a $d$-dimensional torus, with coordinates $x^M = (x^\mu, y^i)$, $M = 0, 1, \ldots, D - 1, \mu = 0, \ldots, 3$, $i = 1, \ldots, d$.

Particles move in $M_4$ (the brane) and interact via the gravitational field $g_{MN}$, which propagates in the whole space-time $M_4 \times T^d$. We also assume the existence of a massless bulk scalar field $\Phi(x^\ell)$, which interacts with $m$, but not with $m'$. The action of the model is symbolically of the form

$$S \equiv S_g + S_\Phi + S_m + S_{m'},$$

and explicitly, in an obvious correspondence,

$$S = \int d^Dx \sqrt{|g|} \left[ -\frac{R}{2\kappa_D^2} + \frac{1}{2} g^{MN} \partial_M \Phi \partial_N \Phi \right] - \frac{1}{2} \int \left[ e g_{MN} \dot{z}^M \dot{z}^N + \frac{(m + f \Phi)^2}{e} \right] d\tau - \frac{1}{2} \int \left[ e' g_{MN} \dot{z}'^M \dot{z}'^N + \frac{m'^2}{e'} \right] d\tau'$$

with $16\pi G_D \equiv \kappa_D^2$ relating $\kappa_D$ to Newton’s constant. Here the ein-beins $e(\tau)$ and $e'(\tau')$ are introduced, which lead to a somewhat unusual form of interaction with $\Phi$, but it reduces to the standard non-derivative interaction once they are integrated out. The constant $f$ is the scalar charge of $m$. To solve the corresponding coupled equations of motion we shall use perturbation theory with respect to the scalar and gravitational couplings.

To zeroth order, both gravitational and scalar fields are absent and the two particles move along straight lines as determined by the initial conditions. In first order, one takes into account the (non-radiative Coulomb-like) gravitational and scalar fields produced by the particles in their zeroth order trajectories. Next, one computes the first order correction to their trajectories, due to their first order gravitational interaction (the scalar mutual interaction vanishes in our set-up). The leading contribution to the scalar radiation field, of interest here, emitted by the accelerated particle $m$ is then obtained in the second order of perturbation theory. This approach allows us to compute consistently the lowest order radiation in the case of ultrarelativistic collision, when deviations from straight trajectories is small and the iterative solution of the coupled particle-field equations of motion is convergent. The resulting expression for the radiative energy loss will be therefore correct only in the leading ultrarelativistic order.

To carry out such a computation it is sufficient to restrict oneself to linearized gravity. One writes $g_{MN} = \eta_{MN} + \kappa_D h_{MN}$ and replaces the Hilbert-Einstein action by its quadratic part

$$S_g = \int \left[ -\frac{1}{4} h^{MN} \Box h_{MN} + \frac{1}{4} h \Box h + \frac{1}{2} h^{MN} \partial_M \partial_N h + \frac{1}{2} h_{MN} \partial_M \partial_P h^P \right] d^Dx,$$

(2.2)

where the Minkowski metric is $\eta_{MN} = \text{diag}(1, -1, -1, ..., -1)$, $\Box_D \equiv \eta^{MN} \partial_M \partial_N$, raising/lowering the indices is performed with $\eta_{MN}$ and $h \equiv h^M_M$. To avoid classical renormalization (which in principle can be treated along the lines of [17]) we take into account only mutual gravitational interaction. At the linearized level the total gravitational field $h_{MN}$ is the superposition of the fields $h^m_{MN}$, $h^{m'}_{MN}$ due to the particles $m$ and $m'$, respectively, and of the gravitational field generated by the bulk scalar $\Phi$. Assuming that the scalar interaction is of the same order as the gravitational one, the latter contribution to $h_{MN}$ is of higher order and will be neglected here. Therefore, to this order of approximation

$$h_{MN} = h^m_{MN} + h^{m'}_{MN},$$

(2.3)

For simplicity the superscripts $m$, $m'$ will be omitted in what follows. Thus, $h^m_{MN}$ and $h^{m'}_{MN}$ will be denoted as $h_{MN}$ and $h'_{MN}$, respectively. Ignoring the self-interaction and the associated radiation reaction problem, we will consider each of the particles $m'$ and $m$ as moving in the other’s metric

$$g_{MN} = \eta_{MN} + \kappa_D h_{MN} \quad \text{and} \quad g'_{MN} = \eta_{MN} + \kappa_D h_{MN'},$$

(2.4)

respectively. Correspondingly, the action for the particle $m$, which interacts with both gravity and $\Phi$, takes the form

$$S_m = -\frac{1}{2} \int \left[ e(\eta_{MN} + \kappa_D h_{MN}) \dot{z}^M \dot{z}^N + \frac{(m + f \Phi)^2}{e} \right] d\tau,$$

(2.5)

Similarly, the action for $m'$ is

$$S_{m'} = -\frac{1}{2} \int \left[ e'(\eta_{MN} + \kappa_D h_{MN}) \dot{z}'^M \dot{z}'^N + \frac{m'^2}{e'} \right] d\tau.$$

(2.6)
Finally, the action for the scalar field, which propagates in the gravitational field $h'_{MN}$ of the uncharged particle, expanded to linearized order of the gravitational field, is

$$S_{\Phi} = \frac{1}{2} \int \partial^M \Phi \partial^N \Phi \left[(1 + \frac{2G}{L^2} h') \eta_{MN} - \kappa_D h'_{MN}\right] d^D x. \quad (2.7)$$

In principle, $\Phi$ propagates in the full gravitational field $h_{MN} + h'_{MN}$ of both particles, but the singular product of $h_{MN}$ and $\Phi$ generated by the same point particle $m$ corresponds again to the self-action problem, which is ignored here. The products of $h_{MN}^i$ due to $m'$ and $\Phi$ due to $m$ does not lead to singularities and correctly describe the situation.

### B. Equations of motion

Varying the action with respect to $z(\tau)$ and $z'(\tau)$ one obtains the linearized geodesic equations of each mass moving in the gravitational field of the other:

$$\frac{d}{d\tau} (e' g_{MN} z^N) = \frac{e}{2} g_{LR,M} z^L z^R, \quad \frac{d}{d\tau} (e' g_{MN} z'^N) = \frac{e'}{2} g_{LR,M} z'^L z'^R. \quad (2.8)$$

Variation with respect to the einbeins gives

$$e^{-2} = \frac{g_{MN} z^N}{(m + f)^2} e' - 2 = \frac{g_{MN} z'^N}{m'^2} \quad (2.9)$$

Substituting this back into the particles' actions $2.5, 2.6$ one is led to their more familiar form

$$S_m = -\int (m + f) (g_{MN} z^M z^N)^{1/2} d\tau, \quad S_{m'} = -m' \int (g_{MN} z'^M z'^N)^{1/2} d\tau,$$

from which the scalar field equation is obtained

$$\Box_D \Phi = -\frac{2G}{L^2} h' \Box_D \Phi + \kappa_D h'_{MN} \Phi^{MN} + f \int (g_{MN} z^M z^N)^{1/2} \delta^D(x - z(\tau)) d\tau, \quad (2.10)$$

Note, once again, that only the gravitational field due to the uncharged particle $m'$ enters this equation.

Finally, the linearized Einstein equations for the metric deviation due to the two particles in the De Donder gauge

$$\partial_N h_{MN} = \frac{1}{2} \partial^M h$$

are obtained from $2.5, 2.6$:

$$\Box_D h_{MN} = -\kappa_D \left(T^{MN} - \eta^{MN} \frac{T}{D - 2}\right), \quad T^{MN} = \int e\dot{z}^M \dot{z}^N \delta^D(x - z(\tau)) d\tau, \quad (2.11)$$

where $T = T^M_{MN}$ and linearization of the metric factor is understood. Similarly,

$$\Box_D h'_{MN} = -\kappa_D \left(T'^{MN} - \eta^{MN} \frac{T'}{D - 2}\right), \quad T'^{MN} = \int e'\dot{z}'^M \dot{z}'^N \delta^D(x - z'(\tau)) d\tau. \quad (2.12)$$

To ensure that the particles move on the brane, it is enough to choose the initial conditions $y^i(0) = 0$, $y'^i(0) = 0$ and similarly for $m'$. Then, using the equations of motion, it is easy to check that the entire world-lines will be $x^\mu = z^\mu(\tau)$, $x'^\mu = z'^\mu(\tau)$ and the energy-momentum tensors will only have brane components $T^{\mu\nu}$, $T'^{\mu\nu}$. However, the metric deviations $h_{MN}$ and $h'_{MN}$ will have in addition diagonal bulk components due to the trace terms in (2.11) and (2.12).

### C. Iterative solution

Even in linearized gravity the relativistic two-body problem can not be solved exactly, so one has to use some approximation scheme. With the particle masses $m$, $m'$ taken of the same order and eventually equal, the model is characterized by three classical length parameters. Namely, the classical radius of the scalar charge $23$,

$$r_f = \left(\frac{f^2}{m}\right)^{\frac{1}{5}}, \quad (2.13)$$
the $D$-dimensional gravitational radius of the mass $m$ at rest
\[ r_g = \left( \kappa D_m^2 \right)^{\frac{1}{d+1}}, \] (2.14)
and the Schwarzschild radius of the black hole, associated with the collision energy $\sqrt{s}[24]$:
\[ r_S = \frac{1}{\sqrt{\pi}} \left[ \frac{8 \Gamma \left( \frac{d+3}{2} \right)}{d+2} \right]^{\frac{1}{d+1}} \left( \frac{G_D \sqrt{s}}{c^3} \right)^{\frac{1}{d+1}}. \] (2.15)

In the (initial) rest frame of the mass $m'$ one has $\sqrt{s} = 2 m m' \gamma$, where $\gamma = 1/\sqrt{1 - v^2}$ is the Lorentz factor of the collision, $v$ being the relative velocity of the colliding particles. So
\[ r_S \sim r_g \gamma \nu, \quad \nu = \frac{1}{2(d + 1)}. \] (2.16)

It will be assumed that the parameters $r_g$ and $r_f$ are of the same order, and both much smaller than the impact parameter $b$:
\[ r_g \sim r_f \ll b \gamma^{-2\nu}, \] (2.17)
or, in terms of $r_S[24]$:
\[ b \gg r_S \gamma^\nu. \] (2.18)

Under this condition, as will be shown below, the deviation of the metric from unity in the rest frame of $m'$ is small, i.e. $\kappa D h_{MN} z' M z' N \ll 1$, which is necessary for the validity of the present perturbative treatment.

1. The formal expansion and zeroth order equations

The next step is to solve these equations iteratively. For the particle-$m$ world-line one writes
\[ z^M = z^M_0 + z^M_1 + \ldots, \quad 0 z^M_0 = u^M \tau + b^M, \] (2.19)
where the order is denoted by a left superscript and to zeroth order the particle moves in Minkowski space-time with constant velocity $u^M$, and with $0 z^M(0) = b^M$ another constant vector. Both vectors $u^M, b^M$ will be assumed to lie on the brane, i.e. to have only $\mu$-components and, in addition, to be orthogonal $b^\mu u_\mu = 0$. It will be shown that as a consequence of the equations of motion $z^M$ also lies on the brane. However, it is convenient to keep $D$-dimensional notation in all intermediate steps.

For the particle-$m'$ one writes similarly
\[ z'^M = z'^M_0 + z'^M_1 + \ldots, \quad 0 z'^M_0 = u'^M \tau, \] (2.20)
assuming that at $\tau = 0$ the particle is at the origin. We choose to work in the rest frame of $m'$, and specify the coordinate axes on the brane so that $u^\mu = (1, 0, 0, \ldots)$, $u'^\mu = \gamma(1, 0, 0, v)$, $\gamma = 1/\sqrt{1 - v^2}$, and $b^\mu = (0, b, 0, 0)$, where $b$ is the impact parameter. When needed, one may think of the brane-localized vectors as $D$-dimensional vectors with zero bulk components, e.g. $u^M = (u^\mu, 0, \ldots, 0)$.

In a similar fashion, the bulk scalar is expanded formally as:
\[ \Phi = \Phi_0 + \Phi_1 + \ldots. \] (2.21)

Substitute in (2.10) and set $\kappa_D = 0$ to obtain for the leading contribution to $\Phi$ the equation
\[ \square_D \Phi_0 = f \int \delta^D(x - u \tau - b) d\tau. \] (2.22)

The ein-beins are also expanded
\[ e = e_0 + e_1 + \ldots, \quad e' = e'_0 + e'_1 + \ldots. \] (2.23)

According to (2.19) one obtains in zeroth order
\[ e_0 = m + f \Phi_0, \quad e'_0 = m'. \] (2.24)
Finally, for the metrics one writes
\[ h_{MN} = 0 h_{MN} + 1 h_{MN} + \ldots, \] (2.25)
and similarly for \( h'_{MN} \). The leading order contributions to the metrics are then obtained from Eqs. (2.11) with the zeroth order source on the right hand side, i.e. with
\[ 0 T^{MN} = m \int \delta^D (x - 0 z(\tau)) u^M u^N d\tau, \quad 0 T'^{MN} = m \int \delta^D (x - 0 z'(\tau)) u'^M u'^N d\tau. \] (2.26)

To calculate the leading order scalar bremsstrahlung it will be sufficient to know only the zeroth order term \( 0 h_{MN} \) of \( h_{MN} \). So, in the sequel only \( 0 h_{MN} \) will appear and to simplify the notation, its left superscript will be omitted.

2. The first order equations

To derive the equations for the first order corrections to the particle world-lines one has to collect first order terms in the expansions of the embedding functions \( z^M, z'^M \) and the einbeins \( e, e' \) and choose suitable gauge condition to fix the \( \tau, \tau' \) reparametrization symmetries. From Eqs. (2.9) one finds for the first order corrections of the einbeins:
\[ 1 e = -e_0 (\kappa_D h_{MN}' u^M u^N + 2 \dot{z}_M u^M), \quad 1 e' = -e'_0 (\kappa_D h_{MN}' u'^M u'^N + 2 \dot{z}_M u'^M). \] (2.27)
The reparametrization freedom allows us to fix \( 1 e \) and \( 1 e' \) arbitrarily. We first substitute these expansions into Eq. (2.18) collect all the first order terms and then choose the gauge fixing conditions \( 1 e = 1 e' = 0 \), that is
\[ \kappa_D h_{MN}' u^M u^N + 2 \dot{z}_M u^M = 0 \] (2.28)
in the equation for \( m \), and
\[ \kappa_D h_{MN}' u'^M u'^N + 2 \dot{z}_M u'^M = 0 \] (2.29)
in the equation for \( m' \). The resulting equations for the first corrections to the particle trajectories read
\[ \Pi^{MN} 1 z'_N = -\kappa_D \Pi^{MN} \left( h'_{N,L,R} - \frac{1}{2} h'_{L,R,N} \right) u^L u^R, \] (2.30)
\[ \Pi'^{MN} 1 z'_N = -\kappa_D \Pi'^{MN} \left( h'_{N,L,R} - \frac{1}{2} h'_{L,R,N} \right) u'^L u'^R, \] (2.31)
where the projectors onto the space transverse to the world-lines are
\[ \Pi^{MN} = \eta^{MN} - u^M u^N, \quad \Pi'^{MN} = \eta'^{MN} - u'^M u'^N, \] (2.32)
whose presence demonstrates explicitly that only the transverse perturbations of the world-lines are physical.

3. The second order equation for \( \Phi \)-radiation

The radiative component of the bulk scalar arises in the next order of iterations and is given by \( 2 \Phi \). With appropriate combination of terms on the right hand side, Eqn. (2.10) is written as:
\[ \square_D 2 \Phi (x, y) = j(x, y) \equiv \rho(x, y) + \sigma(x, y), \] (2.33)
where the first term is localized on the world-line of the radiating particle \( m \)
\[ \rho(x, y) = -f \int z^{\mu}(\tau) \partial_{\mu} \delta^D(x - u \tau - b) \delta^D(y) d\tau, \] (2.34)
while the second is the non-local current
\[ \sigma(x, y) = \kappa_D \partial_M \left( h'^{MN} \partial_N 0 \Phi - \frac{1}{2} h' \partial^M 0 \Phi \right), \] (2.35)
with both \( h^{MN} \) and \( 0 \Phi \) having support in the bulk. This current arises from non-linear terms in (2.11) due to the interaction of the bulk scalar with gravity. It has to be emphasized that it is non-zero outside the world-line not only on the brane but also in the bulk. Note that the decomposition into the local and non-local parts is ambiguous in the sense that part of the non-local term can be cast into a local form using the field equations. But the total source \( j \) never reduces to a local form as a whole.
D. The solution for $^0\Phi$, $h_{MN}$ and $^{1\gamma^M}$ in $M_4 \times T^d$

Our notation and conventions for Kaluza-Klein decomposition and Fourier transformation as applied to the ADD scenario with the transverse directions being circles with radii equal to $R$, are given in Appendix A. It is important to stress at this point that in classical perturbation theory the interaction is described as an exchange of interaction modes (the classical analogs of virtual gravitons), but contrary to the Born amplitudes, where the simple pole diagrams diverge at high transverse momenta $q^2$, here the sum over these modes contains an intrinsic cut-off. Therefore the classical elastic scattering amplitude is finite and, furthermore, it reproduces the result of the eikonal method, if the eikonal is computed in the stationary phase approximation. In other words, classical calculations in ADD correspond to non-perturbative ones in quantum theory (the eikonal method is equivalent to summation of the ladder diagrams). As will be explicitly demonstrated in the present paper, the same is true for bremsstrahlung. Specifically, it will be shown that the effective number of interaction modes is finite due to the cut-off and of order $(R/b)^d$. Also, the summation over the emission modes is cut-off to a finite effective number of order $(R\gamma/b)^d$, which leads to a large extra enhancement factor $\gamma^d$ in ultrarelativistic collisions.

Straightforward Fourier transform of (2.22) gives

$$^0\Phi^\mu(p) = -\frac{2\pi f_\delta(pu)}{p^2 - p_T^2}. \tag{2.36}$$

Similarly, for the $h_{MN}$ and $h_{MN}^\gamma$, it is enough to Fourier transform the source terms (2.26) and plug into (2.11) and (2.12). One then obtains

$$h_{MN}^\mu(p) = \frac{2\pi f_\phi(pu)}{p^2 - p_T^2} e^{-i(pb)} \left(u_M u_N - \frac{1}{D-2} h_{MN}\right), \tag{2.37}$$

where $p_T^2 = p_\mu p^\mu$, $pu = p_\mu u^\mu$, and $p_T^2 = n^i R$ is the transverse momentum vector in the transverse directions. To get $h_{MN}^\mu(p)$, one has to replace $m \rightarrow m'$, $u^M \rightarrow u'^M$, $b \rightarrow 0$. Note that these fields do not describe radiation. They simply represent the scalar and gravitational potentials of the uniformly moving particles. Formally, this follows from the presence of the delta factors $\delta(pu)$ with $pu = \gamma(p^0 - p_z v)$, from which it follows that $p_\mu p^\mu = p_T^2 (v^2 - 1) < 0$, while the mass-shell condition for the emitted wave is $p^\mu p_M^\mu = p_T^2 > 0$.

Substitution of (2.37) into (2.30) and integration of the resulting equation gives

$$^{1\gamma^M}(\tau) = \frac{im' \sqrt{x^2}}{(2\pi)^3 V} \int d^4p \frac{\delta(pu)}{(p^2 - p_T^2)(pu)} e^{-i(pb)} \left(e^{-i(pu)\tau} - 1\right) \left(\gamma u'^M - \frac{1}{d+2} u^M - \frac{\gamma^2}{2(pu)p_T^2}\right), \tag{2.38}$$

with $\gamma^2 \equiv \gamma^2 - (d+2)^{-1}$ and the $D$-dimensional vector $p^\mu = (p^0, p_T^i = l^i/R)$. It is easy to check that the gauge condition (2.28) is satisfied. To ensure this exactly one has to keep the small second term in $\gamma^2$, which, however, will be dropped in what follows in view of our interest in $\gamma \gg 1$. We have chosen the initial value $^{1\gamma^M}(0) = 0$ in order to preserve the meaning of $b^{\mu}$ as the impact parameter, namely $b^{\mu} = z^{\mu}(0) - z'^{\mu}(0)$. Note that the initial value of $^{1\gamma^M}(0)$ is non-zero and is computed from the gauge condition (2.28).

From (2.38) one can prove that the gravitational interaction does not expel the particles from the brane. Indeed, only the last $p^M$-term in the last parenthesis has non-zero components $p_T^i = l^i/R$ orthogonal to the brane. But the remaining expression is even under the reflection $l^i \rightarrow -l^i$ and the sum vanishes giving $^{1\gamma^i}(0) = 0$.

The corresponding solution for the mass $m'$ can be obtained by interchanging $u^M$ and $u'^M$, replacing $m$ by $m'$ and omitting $e^{-i(pb)}$.

E. $\Phi$-radiation - Basic formulae

Finally, $\Phi$-radiation is described by the wave equation (2.33), which in terms of Kaluza-Klein modes is

$$\left(\Box + k_T^2\right)^n \Phi^n(x) = j^n(x) \equiv \rho^n(x) + \sigma^n(x), \tag{2.39}$$

1 This is on the average true also quantum mechanically. The $\Phi$-quanta emission is symmetric on the average under reflection from the brane and the brane stays on the average at rest. However, to guarantee transverse momentum conservation in single $\Phi$ emission in the bulk one should introduce brane position collective coordinates and deal also with the brane back reaction.
where \( k_T^2 = n^i/R \), while \( \rho^n(x) \) and \( \sigma^n(x) \) are

\[
\rho^n(x) = - \int \frac{d^4\tau}{V} \partial_\mu \delta^4(x - u\tau - b) d\tau, 
\]

and

\[
\sigma^n(x) = \frac{\kappa D}{V} \sum_{i} \partial_\mu \left( h''_{\mu\nu}(x) \partial_\nu 0\Phi_{n-1}(x) - \frac{1}{2} h'_{\mu}(x) \partial_{\nu} 0\Phi_{n-1}(x) \right),
\]

respectively, with \( h''_{\mu\nu}, \Phi \) and \( h'_{\mu} \) given in (2.38), (2.36) and (2.37). The rest of this paper is devoted to the solution of (2.39) and the analysis of the spectral and angular distribution of the emitted \( \Phi \)-radiation.

Once the solution of these equations is available, one can compute the energy and momentum radiated away using the standard formulae of radiation theory. To compute the momentum loss due to scalar bremsstrahlung one considers the world tube with topology \( R^{1,3} \times T^d \), with boundary \( \partial \Omega = \Sigma_{\infty} \cup \Sigma_{-\infty} \cup B \) consisting of two space-like hypersurfaces \( \Sigma_{\pm \infty} \) in \( R^{1,3} \) at \( t = \pm \infty \) and the time-like hypersurface \( B \) at \( r \to \infty \) and integrate the difference of the fluxes through \( \Sigma_{\pm \infty} \) to obtain

\[
P^\mu = \int d^3y \left( \int_{\Sigma_{\infty}} - \int_{\Sigma_{-\infty}} \right) T^\nu_\mu d^3\Sigma^\nu.
\]

Here one makes use of the brane components of the energy-momentum tensor of the bulk scalar (it is easy to show that there is no radiation flux into the compact dimensions [23]). Start with

\[
T^{MN} = \partial^M \Phi \partial^N \Phi - \frac{1}{2} \eta^{MN} (\partial \Phi)^2,
\]

where only \( \Phi \) has to be taken into account, since \( 0\Phi \) is not related to radiation. The integral over \( B \) is zero due to fall-off conditions, so the difference of the surface integrals (2.12) can be transformed by Gauss’ theorem to the volume integral

\[
P^\mu = \int d^d \theta d^d \phi \int d^4x \int_{\Omega} d^4y \int_{\Omega} (\partial^\mu \Phi) \square_D \Phi d^4x.
\]

Using (2.33) and the retarded Green’s function of the \( D \)-dimensional D’Alembert operator to solve the wave equation

\[
G_D(x - x', y - y') = \frac{1}{(2\pi)^4 V} \int d^4k e^{-ik(x-x')} \sum_{n} \frac{e^{ikr(y-y')}}{k^2 - k_T^2 + i\epsilon},
\]

one obtains

\[
P^\mu = \frac{1}{16\pi^3 V} \sum_{n} \int \frac{d^3k}{k^0} k^\mu |j^n(k)|^2 \mid_{k^0 = \sqrt{k^2 + k_T^2}},
\]

where \( j^n(k) \) is the Fourier-transform of \( j^n(x) \) (for precise definition see Appendix A1) and will be referred to as the radiation amplitude. With the parametrization \( k = |k|(\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) \) one obtains for the spectral-angular distribution of the emitted energy \( E = P^0 \):

\[
\frac{dE}{d|k| d\Omega_2} = \frac{1}{16\pi^3 V} \sum_{n} k^2 |j^n(k)|^2, \quad d\Omega_2 = \sin \theta \, d\theta \, d\varphi.
\]

Therefore, the leading order radiation loss is determined by the Fourier-transform \( j^n(k) \) of the source in the four-dimensional wave equation (2.38) for \( 1\Phi^0 \) on the mass shell of emitted waves

\[
k_\mu k^\mu = k_T^2.
\]

If the impact parameter is small compared to the compactification radius \( b \ll R \), the summation over KK masses can be replaced by integration according to (A.9), with integration measure \( d^dk_T = k_T^{-d-1} dk_T d\Omega_{d-1} \).

\footnote{\text{It was taken into account that } T^d \text{ has no boundary.}}
Since the radiation amplitude \( j^n \) depends only on \(|k_T|\), integration over the angles is trivial and gives the volume \( \Omega_{d-1} \) of the unit \( d-1 \)-dimensional sphere. Therefore,

\[
\frac{dE}{d|k|d\Omega_2} = \frac{\Omega_{d-1}}{2(2\pi)^{D-1}} \int_0^\infty k^2 k_T^{d-1} |j^n(k)|^2 dk_T. \tag{2.49}
\]

with \( n \) in \( j^n \) expressed in terms of \( k_T \), and \( k^0 \equiv \omega = \sqrt{k^2 + k_T^2} \).

When the summation over KK emission modes is replaced by integration, one can compute the emitted energy using directly higher dimensional Minkowski coordinates. Take the coordinate system with angles on the \((D-2)\)-dimensional sphere \( \Omega_{D-2} \), with \( \theta \) the angle between the \((D-1)\)-dimensional vectors \( K = (k, k_T) \) and \( u, \phi \) the polar angle in the plane perpendicular to \( u \) varying from 0 (direction of \( b \)) to \( 2\pi \) (see Figure 1) and integrate over the angles to obtain \[23\]

\[
\frac{dE}{d\omega d\Omega_{d+2}} = \frac{\omega^{d+2}}{2(2\pi)^{d+3}} |j(k)|^2. \tag{2.50}
\]

where \( \omega \equiv (ku') = k^0 = \sqrt{k^2 + k_T^2} \) is the higher-dimensional frequency.

3. THE RADIATION AMPLITUDE

As we have seen, the radiation amplitude \( j^n(k) \) is the sum of a local \( \rho^n(k) \) and a non-local part \( \sigma^n(k) \). These two parts have different dependence on the frequency and the angle \( \theta \) of the emitted wave with respect to the direction of collision. Both have an intrinsic cut-off at some (angle-dependent) frequency, which in the ultrarelativistic case is high compared to the inverse impact parameter \( 1/b \). Typically, the two amplitudes cancel each other in some range of angles and frequencies. To obtain the correct expression for their sum one has to carefully take into account non-leading contributions.

A. The local amplitude

Fourier-transformation of \( \rho^n(k) \) gives

\[
\rho^n(k) = i e^{i(kb)} \int_{-\infty}^\infty e^{i(ku)\tau} (k'z) d\tau \tag{3.1}
\]

where the scalar products are denoted as \((ab) \equiv a_\mu b_\mu \) and for simplicity the parentheses will also be omitted when it is not ambiguous. Substitution of \( \gamma_{\mu}^\prime \) from \( \text{(2.38)} \) and integration over \( \tau \) gives \[3 \]

\[
\rho^n(k) = \frac{2\pi^2 m'\omega}{4\pi^2 V(ku)^2} \sum_{l} \left[ ku \left( \gamma_{\mu}^\prime - \frac{ku}{d+2} \right) I_l - \frac{\gamma_2^2}{2} \gamma_{\mu}^\prime I_l^\mu \right] \tag{3.2}
\]

\[3 \] Note that the unity inside the first parenthesis of \( \text{(2.38)} \) corresponds to a constant \( 1/z \) and does not contribute to radiation. Formally, its contribution to \( \rho^n \) vanishes by symmetric integration \[23\].
where the integrals $I_l$ and $I_l^\mu$ are defined by

$$I_l = \int \frac{\delta(p\mu') \delta(ku - pu) e^{-i(p\mu)}}{p^2 - p_l^2} \, d^4p, \quad I_l^\mu = \int \frac{\delta(p\mu') \delta(ku - pu) e^{-i(p\mu)}}{p^2 - p_l^2} \, p^\mu \, d^4p, \quad (3.3)$$

The sum over $l$ represents the sum over the interaction modes labeled by the set of integers $l^i$, while the dependence of the amplitude $\rho^n(k)$ on the vector index $n$, which labels the emission modes $n^i$, is hidden inside the $k^0$—component of the wave vector: $k^0 = \sqrt{k^2 + k_f^2}$. The integrals are given in [23] and lead to Macdonald functions:

$$I_l = \frac{2\pi}{\gamma v} K_0(zi), \quad I_l^\mu = -\frac{2\pi}{\gamma v b^2} \left( b z K_0(zi) \frac{\gamma u^\mu - u^\mu}{\gamma v} + i \dot{K}_1(zi) b^\mu \right), \quad (3.4)$$

with

$$z \equiv \frac{(ku)b}{\gamma v}, \quad z' \equiv \frac{(ku')b}{\gamma v}, \quad z_l \equiv (z^2 + p_l^2b^2)^{1/2}, \quad (3.5)$$

and the hatted Macdonald functions defined by $\hat{K}_\nu(x) \equiv x^\nu K_\nu(x)$ and having for $\nu \neq 0$ a finite non-zero limit as $x \to 0$. Thus, in terms of Macdonald functions the local amplitude is:

$$\rho^n(k) = -\frac{x_2^2 m f}{4\pi v V} e^{i(kb)} \sum_l \left\{ \left[ \left( 2 - \frac{\gamma_z^2}{v^2\gamma_z^2} \right) \frac{z'}{2} - \frac{2}{\gamma} \left( \frac{1}{d+2} - \frac{\gamma_z^2}{2v^2\gamma_z^2} \right) \right] K_0(zi) - i \frac{\gamma_z^2}{\gamma^2 v^2 \gamma_z^2} \hat{K}_1(zi) \right\}. \quad (3.6)$$

1. The $\gamma \to \infty$ limit. Mode and frequency cut-offs.

In the ultrarelativistic limit $\gamma \to \infty$ the leading terms of $\rho^n$ are

$$\rho^n(k) \simeq -\frac{x_2^2 m f}{4\pi v V} e^{i(kb)} \sum_l \left[ \frac{z'}{2} K_0(zi) - i \frac{\gamma_z^2}{\gamma^2 \gamma_z^2} \hat{K}_1(zi) - \frac{d+1}{d+2} \left( \frac{z'}{2} - \frac{\gamma d}{d+1} \right) K_0(zi) + "O(\gamma^{-2})" \right] \quad (3.7)$$

This is a systematic ultra-relativistic expansion in powers of $1/\gamma$, modulo the coefficients of the various Macdonals as well as the overall factor in front, which depend on the velocity $v$. However, as will become evident below, this form is adequate for the following discussion and the computation of the emitted energy to leading order.

Two important general remarks are in order here:

(a) The effective number $N_{\text{int}}$ of interaction modes. The exponential fall-off of the Macdonald functions at large values of the argument $zi$ leads to an effective cut-off $N_{\text{int}}$ of the number of interaction modes $l$ in the sum. One can estimate the radius $p_T^\text{int}$ of the sphere in the space of $l^i$, beyond which the modes can be neglected, by setting

$$(p_T^\text{int})^2 b^2 \sim 1, \quad (3.8)$$

from which

$$N_{\text{int}} \sim \left( \frac{R}{b} \right)^d \sim \frac{V}{b^d}. \quad (3.9)$$

For $N_{\text{int}} \gg 1$, which is the case of interest here, one may use (A.9) to obtain [23] ($Z > 0$)

$$\frac{1}{V} \sum_l \hat{K}_\lambda \left( \sqrt{Z^2 + p_l^2b^2} \right) \simeq \frac{1}{(2\pi)^{d/2+b^d}} \hat{K}_{\lambda+d/2}(Z) \quad (3.10)$$

and end up with

$$\rho^n(k) \simeq -\frac{x_2^2 m f}{v} \left\{ \frac{z'}{2} K_{d/2}(z) - i \frac{\gamma_z^2}{\gamma z^2} \hat{K}_{d/2+1}(z) + \frac{1}{(d+2)\gamma} \left( d - \frac{(d+1)z'}{\gamma z} \right) \hat{K}_{d/2}(z) + "O(\gamma^{-2})" \right\}, \quad (3.11)$$

where

$$\lambda \equiv \frac{x_2^2 m f}{2(2\pi)^{2+b^d}}. \quad (3.12)$$
(b) Angular and frequency characteristics. The local radiation amplitude above in the \( b \ll R \) limit is expressed solely in terms of Macdonald functions with argument \( z \). Later, it will be shown that the non-local amplitude contains also Macdonald functions but with argument \( z' \). The exponential fall-off of these functions implies the effective cut-offs \( z \sim 1 \) and \( z' \sim 1 \) in the corresponding radiation amplitudes. These, in turn, translate into angular and frequency characteristics of the corresponding radiation.

Specifically, with \( \theta, \alpha \) and \( \vartheta \) as shown in Fig. 1 define

\[
\psi \equiv 1 - v \cos \theta \cos \alpha = 1 - v \cos \vartheta, \tag{3.13}
\]

which satisfies

\[
\frac{z}{z'} = \gamma \psi, \tag{3.14}
\]

and in the ultrarelativistic limit varies in the interval \( 1/2 \gamma^2 \simeq 1 - v \leq \psi \leq 1 + v \simeq 2 \).

Consider the neighborhood of \( z \sim 1 \) which gives the dominant contribution of the local radiation amplitude. One has to distinguish various domains of the emission angles. For small emission angles \( \theta, \alpha \) one has

\[
\psi \sim \frac{1}{2}(\gamma^{-2} + \theta^2 + \alpha^2), \tag{3.15}
\]

so that (i) inside the small cone \( \theta^2 + \alpha^2 \lesssim 1/\gamma^2 \) one obtains \( \psi \sim 1/\gamma^2 \), so that the characteristic frequencies \( \omega \sim \gamma^2/b \). This angle-frequency regime will be called in the sequel the \( z \)-region. (ii) For \( \psi \sim 1 \), i.e. for \( \alpha, \theta \sim \mathcal{O}(1) \), one obtains \( z' \sim z/\gamma \), which implies a low frequency regime \( \omega \sim 1/b \), whose contribution to the emitted energy is negligible in view of the relative smallness of the phase-space factor in \( \mathcal{O}(1) \).

To summarize, the above analysis of \( \rho^n \) combined with the phase space factors in \( \mathcal{O}(1) \), leads to the conclusion that the leading contribution of the radiation due to the local amplitude is beamed, i.e. directed inside the small cone \( \theta^2 + \alpha^2 \lesssim 1/\gamma^2 \) and has high frequencies \( \omega \sim \gamma^2/b \). Radiation with these characteristics will occasionally be called \( z \)-type.

\[\text{B. The non-local amplitude}\]

The non-local amplitude obtained from \( (2.1) \) by Fourier transform is

\[
\sigma^n(k) = \frac{\gamma_n^2 m' f(ku')^2}{(2\pi)^2} e^{i(kb) J^n(k)}, \quad J^n(k) \equiv \frac{1}{V} \int \sum_{l} J^{nl}(k), \tag{3.16}
\]

with

\[
J^{nl}(k) = \int d^4 p \frac{\delta(pu')\delta(ku - pu) e^{-i(pb)}}{(p^2 - p_T^2)(k - p)^2 -(k_T - p_T)^2} \tag{3.17}
\]

\( k^0 = \sqrt{k^2 + k_T^2} \) and \( k \) is a 3-dimensional vector lying on the 3-brane, where \( k_T = n^i/R \) and \( p_T = l^i/R \) with integers \( \{n^i\}, \{l^i\} \) are 3-dimensional discrete vectors corresponding to the emission and interaction modes, respectively.

Using Feynman parametrization \( J^{nl} \) takes the form:

\[
J^{nl} = \int_{0}^{1} dx e^{-i(kb)x} \int d^4 p \frac{\delta[(pu') + (ku')x] \delta[(pu) - (1-x)(ku)] e^{-i(pb)}}{[p^2 - (k_T x - p_T)^2]^2}. \tag{3.17}
\]

Integrating over \( p^0 \) and splitting \( p \) into the longitudinal \( p_\parallel \) and transversal \( p_\perp \) parts, integrate over \( p_\parallel \). Then, introducing in \( p_\perp \) the spherical coordinates \( d^2 p_\perp = |p_\perp| d|p_\perp| d|p_\perp| \) and integrating first over the angles and then over \( |p_\perp| \), one obtains

\[
J^{nl} = \frac{\pi b^2}{\gamma v} \int_{0}^{1} dx e^{-i(kb)x} \hat{K}_{li}(\zeta_{nl}), \tag{3.18}
\]

with

\[
\zeta_{nl}(x) = z'^2 x^2 + 2 \gamma z x(1 - x) + z^2(1 - x)^2 + b^2(k_T x - p_T)^2. \tag{3.19}
\]
Again, the summation over \( l \) is performed trivially for \( b \ll R \) using (3.10) \(^4\). The result is

\[
J^n(k) = \Lambda_d \int_0^1 dx e^{-i(kb)x} K_{d/2-1}(\zeta_n); \quad \Lambda_d = \frac{\pi b^{2-d}}{(2\pi)^{d/2} \gamma_v},
\]

(3.20)

with

\[
\zeta_n^2(x) = z'^2 x^2 + 2 \gamma z' x(1-x) + z^2(1-x)^2; \quad \zeta_n(0) = z, \quad \zeta_n(1) = z'.
\]

(3.21)

Writing \( \zeta_n^2 \) successively in the form

\[
\zeta_n^2(x) = -\xi^2 x^2 + 2 \beta x + z^2 = a^2 - r^2,
\]

(3.22)

with

\[
\xi^2 \equiv 2 \gamma z' - z^2 - z'^2 = \omega b^2 \sin^2 \alpha + b^2 k_r^2 = z'^2 \gamma^2 \sin^2 \vartheta, \quad \beta \equiv \gamma z' - z^2,
\]

(3.23)

and

\[
a \equiv \sqrt{\frac{\beta^2}{\xi^2} + z^2}, \quad r \equiv \xi \left( x - \frac{\beta}{\xi^2} \right),
\]

(3.24)

and using formula f.2.16.12-4 of [36]

\[
K_{\nu-1/2} \left( \sqrt{a^2 - r^2} \right) = \frac{2^{1/2}}{\pi^{1/2}} a^{2\nu} \int_0^\infty \cosh(ry) K_{-\nu} \left( a \sqrt{y^2 + 1} \right) dy, \quad \nu > -1 \quad \text{and} \quad a > 0
\]

(3.25)

for \( \mu = -1/2, \nu = (d-1)/2 \) one may rewrite (3.20) in the form

\[
J^n(k) = \Lambda_d \frac{2^{1/2} a^{2\nu}}{\pi^{1/2}} \int_{0}^{\infty} dy K_{-\nu} \left( a \sqrt{y^2 + 1} \right) \int_{0}^{1} dx e^{-i(kb)x} \cosh(ry).
\]

(3.26)

Perform, next, the integration over \( x \) and introduce the additional angle \( \phi \), so that the generic \((D-1)\)-dimensional unit vector \( \mathbf{K}/|\mathbf{K}| \) (the normalized higher-dimensional emission vector \( \mathbf{K} \)) is decomposed as:

\[
\frac{\mathbf{K}}{|\mathbf{K}|} = \frac{\mathbf{u}}{|\mathbf{u}|} \cos \vartheta + \frac{\mathbf{b}}{|\mathbf{b}|} \sin \vartheta \cos \phi + \mathbf{m} \sin \vartheta \sin \phi,
\]

(3.27)

where \( \mathbf{m} \) is a \( D-1 \) dimensional unit vector orthogonal to the collision plane (spanned by \( \mathbf{u} \) and \( \mathbf{b} \)). Then \((k \cdot b) = -\gamma z' v \sin \vartheta \cos \phi = -\xi \cos \phi \) and \( a = \omega b \vartheta / \sin \vartheta \). Substituting this into (3.26), we have

\[
J^n(k) = \Lambda_d \frac{2^{1/2} a^{2\nu}}{\pi^{1/2}} \frac{1}{\xi} \sum_{j=0,1} (-1)^{j+1} e^{-ij(kb)} \int_{0}^{\infty} dy K_{-\nu} \left( a \sqrt{y^2 + 1} \right) \frac{y \sinh(\xi \delta_j y) - i \cos \phi \cosh(\xi \delta_j y)}{y^2 + \cos^2 \phi} \equiv J^n_{0} + J^n_{1},
\]

(3.28)

where \( \delta_j = j - \beta/\xi^2, \quad j = 0, 1. \) \(^5\) The convergence of these integrals is controlled by the competition of the exponential decay of the Macdonald function and the exponential growth of the hyperbolic functions. In all cases the first is faster, but when the difference of the two arguments is small, the main contribution to the integral over \( y \) comes from large values of \( y \).

---

\(^4\) Using (3.10) the summation is converted to integration over \( d^d \). One then shifts the integration variable \( p_T' = p_T - xk_T \) and applies (3.10). In the present case \( Z \) is not constant but depends on \( x \). However, as it can also be checked numerically, (3.10) and the subsequent treatment is a good approximation for any \( 0 \leq x \leq 1 \), because \( Z(x) \geq 1 \) in all relevant frequency regimes.

\(^5\) Direct integration of (3.20) for \((\theta = 0, k_T = 0)\) gives \( \int_0^1 K_{d/2-1}(\sqrt{2\beta x + z^2}) dx = \beta^{-1}[K_{d/2}(z) - K_{d/2}(z')] \). The constants of integration of the terms \( j = 0, 1 \) are chosen so that for \( \theta = 0 \) they satisfy \( J^n_{0}|_{(\theta=0,k_T=0)} = \Lambda_d \beta^{-1} K_{d/2}(z); \quad J^n_{1}|_{(\theta=0,k_T=0)} = -\Lambda_d \beta^{-1} K_{d/2}(z'). \)
Since \( y^2 + 1 \geq 1 \) and \( 0 \leq \sin^2 \phi \leq 1 \), one can equivalently write
\[
J^n_y(k) = (-1)^{n+1} e^{-i(kb)} \Lambda_d \frac{2^{1/2} \alpha^\nu}{\pi^{1/2}} \sum_{k=0}^\infty \sin^{2k} \phi \int_0^\infty \frac{K_\nu \left(a \sqrt{y^2 + 1} \right)}{(y^2 + 1)^{\nu/2 + 1 + k}} \left[y \sinh(\xi \delta y) - i \cos \phi \cosh(\xi \delta y)\right].
\]

The \( y \)-integration for each value of \( k \) is performed by successive applications of the identity
\[
K_\nu(z) = K_{\nu+2}(z) - \frac{2(\nu + 1)}{z} K_{\nu+1}(z)
\]
(3.29)
in combination with
\[
a^{\nu} \int_0^\infty \frac{K_{\nu+2} \left(a \sqrt{y^2 + 1} \right)}{(y^2 + 1)^{(\nu/2 + 1)}} \left\{ \frac{y \sinh(\xi \delta y)}{\cosh(\xi \delta y)} \right\} = \frac{1}{a^2} \left\{ \xi \delta \hat{K}_{\nu+1/2}(z_j) \right\}
\]
(3.30)

obtained from (3.29), with argument \( z_j = \sqrt{a^2 - \xi^2 \delta^2} \), i.e. \( z_0 = z, z_1 = z' \).

For example, using (3.29) the \( k = 0 \) term leads to the integrals
\[
a^{\nu} \int_0^\infty \left[ \frac{K_{\nu+2} \left(a \sqrt{y^2 + 1} \right)}{(y^2 + 1)^{\nu/2 + 1}} - \frac{2(\nu + 1)}{a} \frac{K_{\nu+1} \left(a \sqrt{y^2 + 1} \right)}{(y^2 + 1)^{(\nu/2 + 1)}} \right] \left\{ \frac{y \sinh(\xi \delta y)}{\cosh(\xi \delta y)} \right\}.
\]
(3.31)
The first term in the square brackets is given by (3.30). The second is computed using again (3.29), which leads to two new integrals, the first of which is
\[
a^{\nu-1} \int_0^\infty \left[ \frac{K_{\nu+3} \left(a \sqrt{y^2 + 1} \right)}{(y^2 + 1)^{(\nu/2 + 1/2)}} \right] \left\{ \frac{y \sinh(\xi \delta y)}{\cosh(\xi \delta y)} \right\} = \frac{1}{a^4} \left\{ \xi \delta \hat{K}_{\nu+3/2}(z_j) \right\},
\]
(3.32)
suppressed for \( a \gg 1 \) compared to (3.30). Similarly, the second is further suppressed by two more powers of \( a \).

Terms with increasing \( k \) are evaluated in the same way and lead to further suppression by inverse powers of \( a^2 \).

The end result for \( J^n_0(k) \) keeping terms up to \( 1/a^4 \) is then
\[
J^n_0(k) = \frac{\Lambda_d}{a \xi^2} \left[ \beta \hat{K}_{d/2}(z) - i(kb) \hat{K}_{d/2+1}(z) - \frac{(d + 1) \beta}{a^2} \hat{K}_{d/2+1}(z) + \frac{\beta \sin^2 \phi}{a^2} \hat{K}_{d/2+2}(z) \right] + R_z.
\]
(3.33)

Notice that \( J^n_0(k) \) is a series of Macdonalds with argument \( z \). Consequently, it is important mainly in the \( z \)-region, where \( a = \omega b \sigma / \sin \vartheta \sim \gamma \gg 1 \), a self-consistency check of our approximations. The coefficients of all terms in (3.33) have expansions in powers of \( \gamma^{-1} \). In the \( z \)-region the first term starts with \( O(\gamma^{-3}) \), the second with \( O(\gamma^{-5}) \), the third and fourth terms with order \( O(\gamma^{-7}) \), while the remainder \( R_z = O(\gamma^{-8}) \).

Following the same procedure \( J^n_0(k) \) is written as a series of Macdonald functions with argument \( z' \), namely
\[
J^n_0(k) \approx \frac{\Lambda_d}{a \xi^2} \left[ \beta \hat{K}_{d/2}(z') - i(kb) \hat{K}_{d/2+1}(z') - \frac{(d + 1) \beta}{a^2} \hat{K}_{d/2+1}(z') + \frac{\beta \sin^2 \phi}{a^2} \hat{K}_{d/2+2}(z') \right] + R_z',
\]
(3.34)
whose main contribution comes from the region with \( z' \sim 1 \), i.e. \( \omega \sim \gamma / b, \vartheta \sim 1 \), in which indeed \( a \sim \gamma \gg 1 \).

The condition \( a^2 \gg 1 \) is not satisfied in the region with \( \vartheta \sim 1 / \gamma \). However, in that region both the exact expression and the approximate one have negligible contribution to the amplitude. Figure 2 displays graphically the maximal difference in the real part of \( J^n_0(k) \) between the two expressions.

Incidentally, notice that there is no strong anisotropy in \( \phi \) in the \( z' \)-region: the real part of main terms of \( J^n_0(k) \) (3.34) is independent on \( \phi \), while its imaginary part depends only by the overall factor \( \cos \phi \). The same picture was obtained without any approximations in \( \Re(z) \), where scalar mediated collisions were studied.

Going back to (3.10) one sees that \( \sigma^n(k) \) is the sum of two sets of Macdonald functions, one with argument \( z \) and the other with argument \( z' \). In analogy with \( \rho^n \), the first sum contributes mainly in the \( z \)-region. Similarly, the leading contribution of the second set of Macdonalds comes from the region with \( z' = \omega b / \gamma v \sim O(1) \). In the ultrarelativistic limit this translates into angular and frequency characteristics of the emitted radiation. For generic values of the angles, this means \( \omega \sim \gamma / b \) and defines what we will call the \( z' \)-region and, correspondingly, \( z'-type \) radiation. It is unbeam radiation \( (\vartheta \sim 1) \) with characteristic frequency \( \omega \sim \gamma / b \).

The non-local pieces \( \sigma^n_0(k) \) and \( \sigma^n_1(k) \). It is convenient to separate the two kinds of contributions to the non-local amplitude by writing
\[
\sigma^n(k) = \sigma^n_0(k) + \sigma^n_1(k),
\]
(3.35)
As a consequence, the two leading powers in the ultra-relativistic expansion of the direct $\Phi$ emission amplitude $\mathcal{J}_z$, with the first (second) given by (3.10) with $J_0^n$ ($J_1^n$) on the right hand side. Thus, 

$$\sigma_0^n(k) = \lambda e^{i(kb)} \frac{\gamma v z^2}{a^2 \xi^2} \left( \beta \tilde{K}_{d/2}(z) - i(kb) \tilde{K}_{d/2+1}(z) - \frac{(d+1)\beta}{a^2} \tilde{K}_{d/2+1}(z) + \frac{\beta \sin^2 \phi}{a^2} \tilde{K}_{d/2+2}(z) \right), \quad (3.36)$$

and

$$\sigma_1^n(k) \simeq \lambda \frac{\gamma v z^2}{a^2 \xi^2} \left( (\xi^2 - \beta) \tilde{K}_{d/2}(z') + i(kb) \tilde{K}_{d/2+1}(z') \right), \quad (3.37)$$

respectively.

Correspondingly, the total radiation amplitude $j^n(k)$ is written as a sum of two terms, one function of $z$, and the other function of $z'$

$$j^n(k) \equiv j^n_z(k) + j^n_{\gamma}(k), \quad j^n_z(k) = \rho^n(k) + \sigma_0^n(k), \quad j^n_{\gamma}(k) = \sigma_1^n(k) \quad (3.38)$$

C. The part $j^n_z(k)$ of the radiation amplitude and destructive interference

1. $j^n_z$ in the frequency range $\omega \gg \gamma/b$

Consider first the regime with $\theta \sim 1$. Here $z \sim \gamma$ and from (3.30) and (3.11) one obtains that $j^n_z \sim \exp(-\gamma)$ due to the Macdonald functions.

Now take the most interesting case of $\theta \sim 1/\gamma$, in which $z \sim 1$. Add $\rho^n(k)$ and $\sigma_0^n(k)$ given in (3.11) and (3.30), respectively, and use the ultra-relativistic expansions to obtain in leading order:

$$j^n_z(k) \simeq \frac{\lambda (d+1) e^{i(kb)} \gamma \psi}{\gamma \psi} \left[ \frac{2\psi - \gamma^{-2}}{d+2} \tilde{K}_{d/2}(z) - \frac{\cos^2 \alpha}{\psi^2 \omega^2 \varphi^2} \left( \frac{\sin^2 \theta + \tan^2 \alpha}{d+1} \tilde{K}_{d/2+2}(z) \right) \right]. \quad (3.39)$$

All terms inside the square brackets are of $\mathcal{O}(\gamma^{-2})$. Given that in the $z$-region $1/\gamma \psi = z'/z \sim \mathcal{O}(\gamma)$, the leading contribution to $j^n_z$ above is of $\mathcal{O}(\gamma^{-1})$. Higher order terms have been ignored. The terms of order $\mathcal{O}(\gamma)$ and $\mathcal{O}(1)$, both present in the ultra-relativistic expansions of $\sigma_0^n(k)$ and $\rho^n(k)$, have opposite signs and cancel in the sum. This is a general phenomenon of destructive interference related to the gravitational interaction. Thus, the two leading powers in the ultra-relativistic expansion of the direct $\Phi$–emission amplitude from the accelerated charged particle, cancel the ones coming from the indirect emission due to the $\Phi - \Phi - h$ interaction. As a consequence, the $z$-type (beamed and high frequency) part of the radiation is highly suppressed in the
ultra-relativistic limit, compared to the naive expectation. One can check that destructive interference is valid also in the case of \( \Phi \)-radiation in arbitrary \( D \)-dimensional Minkowski space-time, which can be obtained as a limit of the present discussion. It was first observed for gravitational radiation in \( D = 4 \) \([30]\) (using a different approach) and it was recently generalized to arbitrary dimensions in \([24]\). For the system at hand, an alternative proof is presented in Appendix \( B \) using a different approach also suitable to the frequency range \( \omega \gg \gamma/b \).

The following comments are in order here: (a) As a check of the above series of approximations, one may consider the special case of \( \theta = 0 \), for which \( (kb) = -\xi = 0 \). In this case the exact value of \( J^n \) obtained from \( (3.20) \) coincides with the one obtained from the approximate expressions \( (3.33) \) and \( (3.34) \). (b) Furthermore, equation \( (3.39) \) can be shown to coincide with the corresponding quantity in the case of non-compactified \( D = 4 + d - \)dimensional Minkowski space. This generalizes to scattering and radiation processes in the relativistic case, the non-relativistic argument about the behavior of Newton’s potential, i.e. that at distances \( b \ll R \) a point particle generates the \( D \)-dimensional potential, while at \( b \gg R \) its potential behaves as four-dimensional \([33]\).

One may equivalently parametrize \( j^n \) using the angles \( \psi \) and \( \phi \) and write it in the form:

\[
j^n(k) = \frac{\lambda e^{i(kb)}}{\gamma \psi} \left[ \frac{d+1}{d+2} \left( 2\psi - \gamma^{-2} \right) K_{d/2}(z) - \frac{\sin \psi}{\gamma^2} \left( (d+1)K_{d/2+1}(z) - \sin^2 \phi \tilde{K}_{d/2+2}(z) \right) \right]. \tag{3.40}
\]

Note that in the computation of the emitted energy below both angles will be taken continuous; a sensible approximation for \( N_{\text{int}} \gg 1 \) assumed here.

2. \( j^n \) in the frequency range \( \omega \lesssim \gamma/b \)

For \( \omega \ll \gamma/b \) and \( \psi \sim 1/\gamma \) using \( (3.11) \) for \( \rho^n \) and \( (3.16) \) and \( (3.20) \) for \( \sigma^n \), one concludes that \( |\rho^n| \gg |\sigma^n| \) and, therefore,

\[
|j^n(k)|_{\omega \lesssim \gamma/b} \approx |\rho^n(k)| \approx -\lambda \left[ \frac{1}{\gamma \psi} \tilde{K}_{d/2}(z) + \frac{\sin \psi \cos \phi}{\gamma^2 \omega b} \tilde{K}_{d/2+1}(z) \right]. \tag{3.41}
\]

For \( \psi \sim 1 \), on the other hand, \( \rho^n, \sigma^n_0 \) and \( \sigma^n_1 \) are all of the same order, but suppressed compared to the previous case. In addition, the contribution of this regime to the emitted energy will be shown to be further suppressed by the integration measure.

More interesting is the case with \( \omega \sim \gamma/b \). If \( \psi \sim 1 \), then \( z \sim \gamma \) and using \( (3.11) \) and \( (3.35) \) one concludes that \( j^n \) is exponentially suppressed because of the Macdonald functions. However, for \( \psi \sim 1/\gamma \), one may use \( (3.11) \) and \( (3.28) \) to obtain that \( \rho^n \sim \gamma \) and \( \sigma^n_0 \sim \gamma \), respectively \( 6 \).

D. The part \( j^n_0(k) \) of the amplitude

Equation \( (3.37) \) can equivalently be written in the form:

\[
j^n_0 \simeq -\frac{\lambda}{\gamma \psi} \left[ \frac{1}{\gamma^2 \psi} - 1 \right] \tilde{K}_{d/2}(z') + \frac{\sin \psi \cos \phi}{\gamma^2 \psi} \tilde{K}_{d/2+1}(z') \tag{3.42}
\]

Furthermore, using the angles \( \psi \) and \( \phi \) it becomes:

\[
j^n_0 \simeq -\frac{\lambda}{\gamma \psi} \left[ \frac{1}{\gamma^2 \psi} - 1 \right] \tilde{K}_{d/2}(z') + \frac{\sin \psi \cos \phi}{\gamma^2 \psi} \tilde{K}_{d/2+1}(z') \tag{3.43}
\]

First, for \( \omega \gg \gamma/b \) one has \( z' \gg 1 \) and, consequently, \( j^n_0 \) in \( (3.43) \) is exponentially suppressed. Next, for \( (\omega \sim \gamma/b, \psi \sim 1) \) \( j^n_0 \) in \( (3.43) \) is dominated by its real part which is of order \( O(1/\gamma) \). For \( (\omega \ll \gamma/b, \psi \sim 1) \) one obtains \( j^n_0 \sim \sigma^n_0 \sim 1/\gamma \). As will be shown below, however, this region contributes negligibly little to the emitted energy. Similarly, for \( (\omega \ll \gamma/b, \psi \sim 1/\gamma) \) on the basis of \( (3.11) \) and \( (3.20) \) one concludes that the amplitude \( j^n_0 \sim \sigma^n_0 \ll \rho^n \sim \gamma^2 \). Finally, based on numerical study and previous results in \( D = 4 \) \([31]\) one obtains that \( (3.38) \) is valid also in the regime \( (\omega \sim \gamma/b, \psi \sim 1/\gamma) \) and gives \( j^n_0 \sim \gamma \).

\[6\] Using the formulae of Appendix \( A \) one gets in this kinematical regime \( a = z' \gamma \psi / \sin \theta \sim 1 \) and also \( \xi \sim 1 \) as well as \( \beta \sim 1 \). The integrand in \( (3.28) \) is independent of \( \gamma \). All \( \gamma \) dependence comes from the overall coefficients in \( (3.28) \) and \( (3.16) \).
Figure 3: Frequency (a) and angular (b) distribution for $d = 0$ and $\gamma = 10^5$.

E. Summary

The behavior of the local and non-local currents in all characteristic frequency and angular regimes is summarized in the following Table I.

| $\gamma$ | $\omega < \gamma/b$ | $\omega \sim \gamma/b$ | $\omega \gg \gamma/b$ |
|---------|----------------------|------------------------|----------------------|
| $\gamma^{-1}$ | no destructive interference | $j_n^z \sim \rho^n \gg \sigma_0^n \sim \sigma_1^n$ | $j_n^z = \langle 3.43 \rangle \sim \gamma^{-1}$ |
| $j_n^z \sim \rho^n \gg \sigma_0^n \sim \gamma$ | $j_n^z = \langle 3.43 \rangle \sim \gamma^{-1}$ | destructive interference |
| $j_n^z = \langle 3.43 \rangle \sim \exp(-\gamma)$ |
| $\gamma$ | no destructive interference | $j_n^z \sim \rho^n \sim j_n^z \sim j_n^z$ | $j_n^z = \langle 3.11 + 3.20 \rangle \sim \exp(-\gamma)$ |
| $j_n^z = \langle 3.11 + 3.20 \rangle \sim \exp(-\gamma)$ | $j_n^z = \langle 3.43 \rangle \sim \gamma^{-1}$ | $j_n^z = \langle 3.43 \rangle \sim \exp(-\gamma)$ |

4. THE EMITTED ENERGY - SPECTRAL AND ANGULAR DISTRIBUTION

The spectral and angular distribution of the emitted energy is obtained from (2.49) or (2.50). The integrand is the sum of three pieces proportional to $|j_n^z(k)|^2$, $|j_n^z(k)|^2$ and $j_n^z j_n^z + j_n^z j_n^z$ (the bar denotes complex conjugation), so the total radiated energy splits into three parts

$$dE = dE^z + dE^{zz'} + dE^{zz'},$$

(4.1)

which will be called $z-$, $z'$- and $zz'$-radiation, respectively. All terms contain a factor of $\lambda^2$ from the square of the amplitude, an explicit $1/2(2\pi)^{d+3}$ from the Fourier transform in (2.50), and a factor $b^{-(d+3)}$ from the corresponding power of $\omega$ in the integrand. This leads to a general expression for the total emitted energy of the form

$$E \sim \frac{1}{8(2\pi)^{2d+5}} \frac{\lambda^2 m^2 r^2}{b^{d+3}} \gamma^\#,$$

(4.2)

with an overall coefficient expected to be of order one and the power of gamma depending on the particular type of radiation under discussion and which is easily determined as follows: As argued above and in [30] and also shown for example in Figure 3 (obtained numerically), the $\Phi-$radiation is emitted predominantly in well-defined relatively narrow frequency and angular windows, and with amplitudes shown in Table I. Thus, it is straightforward to estimate the powers of $\gamma$ in the various components of its energy, using (2.50)

$$E \sim \int d\omega \int d\theta |j|^2 \omega^{d+2} \sin^{d+1} \theta$$

(4.3)
with $|j|^2$ being $|j_z^n|^2$ or $|j_{y'}^n|^2$ or $j_{x'}^n j_y^m + j_y^m j_{x'}^n$, and with the range of integration not contributing extra factors of $\gamma$. For example, the contribution of $j_z^n$, which is dominant in the regime $(\omega \sim \gamma^2/b, \vartheta \sim 1/\gamma)$ has $1/\gamma^2$ from $|j_z^n|^2$, $(1/\gamma)^{d+2}$ from the angular integration, and $(\gamma^2)^{d+3}$ from the integration over $\omega$, with the final estimate being $\gamma^{d+2}$.

The result of this computation is the content of Table II below.

| $\omega / \gamma$ | $\omega \ll \gamma/b$ | $\omega \sim \gamma/b$ | $\omega \sim \gamma^2/b$ | $\omega \gg \gamma^2/b$
|-----------------|-----------------|-----------------|-----------------|-----------------|
| $\gamma^{-1}$   | negligible (phase space) | $E_d \sim \gamma^3$, from $j_z^n$ and $j_{y'}^n$ | $E_d \sim \gamma^{d+2}$, from $j_z^n$ | negligible radiation |
| 1               | negligible (phase space) | $E_d \sim \gamma^{d+1}$, from $j_z^n$ | negligible radiation | negligible radiation |

We proceed next to the detailed study of the various components of radiation with the frequency and angular characteristics of the three most important cells of Table II.

A. The $z$-type component of radiation with $\omega \sim \gamma^2/b$

According to Table II, the $z$-type radiation (due to $|j_z^n|^2$) is always beamed inside $\vartheta \sim 1/\gamma$. Furthermore, for $d \geq 2$ it is dominant with characteristic frequency $\omega \sim \gamma^2/b$. The cases $d = 0$ and $d = 1$ will be treated separately in another subsection.

It is convenient to write the current $j_z$ in the form

$$j_z^n = \omega^{(kb)} \sum_{s=0}^{2} \gamma j_z$$

$$0_j = \frac{\lambda}{\gamma} \frac{d+1}{d+2} \left( 2 - \frac{1}{\gamma^2} \right) K_{d/2}(z)$$

$$1_j = -\lambda (d+1) \frac{\sin^2 \vartheta}{\gamma^2 z^2} K_{d/2+1}(z)$$

$$2_j = \frac{\sin^2 \vartheta}{\gamma^2 z^2} K_{d/2+2}(z)$$

Squaring and substituting into (2.50) one obtains

$$\frac{dE^z}{d\omega d\Omega_{d+2}} = \frac{\omega^{d+2}}{2(2\pi)^{d+3}} \sum_{a,b=0}^{2} a_j b_j z.$$

To integrate over frequencies it is convenient to change variable from $\omega$ to $z$ and define the quantities

$$C_{ab}^{(d)} = \int K_{d/2+a}(z) K_{d/2+b}(z) z^{d+2(\delta_{ab}+\delta_{ab}+1)} dz.$$ 

The integration over $\vartheta$ is performed using (A.11). Finally, the integration of $\sin^2 \varphi$ and $\sin^4 \varphi$ over the remaining angles of $S^{d+1}$ is $\Omega_{d+1}/2$ and $\Omega_{d+1}/8$, respectively. The angular and frequency profiles of this component of radiation in dimensions $d \geq 2$, for which it is dominant, were obtained analytically and numerically, respectively, and have the general form shown for $d = 3$ in Figure 4.

The end result for the total emitted energy in this component of radiation is

$$E^z = \frac{\lambda^2 \Omega_{d+1}}{2(2\pi)^{d+3}} \gamma^{d+2} \sum_{a,b=0}^{2} C_{ab}^{(d)} D_{ab}^{(d)}$$

Note that in $D = 4$ it is found that all three types of radiation are of equal $\mathcal{O}(\gamma^3)$. This seems to disagree with [32], where it is stated that the leading $\mathcal{O}(\gamma^3)$ is due to $z'$-type alone.
where

\[
D_{00}^{(d)} = \frac{2^{d+1}(d+1)^2 \Gamma^2\left(\frac{d+2}{2}\right)}{\Gamma(d+4)}, \quad D_{01}^{(d)} = -\frac{2^{d+3}(d+1)^2 \Gamma^2\left(\frac{d+4}{2}\right)}{(d+2)\Gamma(d+4)}, \quad D_{02}^{(d)} = -\frac{1}{2(d+1)} D_{01}^{(d)}, \quad D_{11}^{(d)} = -\frac{2^{d+4}(d+1)^2 \Gamma\left(\frac{d+6}{2}\right) \Gamma\left(\frac{d+4}{2}\right)}{\Gamma(d+5)}, \quad D_{12}^{(d)} = -\frac{1}{2(d+1)} D_{11}^{(d)}, \quad D_{22}^{(d)} = \frac{3}{8(d+1)^2} D_{11}^{(d)}. \quad (4.11)
\]

When integrating over \( z \) in (4.9) one should remember that the expansion (4.40) is accurate in the high frequency domain, around and beyond \( z \sim 1 \). However, for \( d \gg 2 \) it can be checked both analytically and numerically that the integral from 0 to 1 of the difference of the exact energy density based on (3.11) and (3.28) and the approximate one based on (4.4) is negligible. Thus, one can conveniently expand the integration region in (4.9) from 0 to \( \infty \) and evaluate \( C_{ab}^{(d)} \) using \((A.13)\).

Collecting all contributions one obtains for the energy of high frequency \( z \)-type radiation

\[
E = C_d \frac{\varkappa^2}{\Lambda^{d+3}} \gamma^{d+2} \quad (4.12)
\]

with \( C_2 = 1.42 \times 10^{-6}, C_3 = 6.02 \times 10^{-7}, C_4 = 3.45 \times 10^{-7}, C_5 = 2.67 \times 10^{-7} \) and \( C_6 = 2.76 \times 10^{-7} \).

### B. The \( z' \)-type radiation with \( \vartheta \sim 1 \)

According to Table II, wide angle radiation (\( \vartheta \sim 1 \)) is mainly \( z' \)-type (due to \(|j_0|^{2} \)) in all dimensions and has characteristic frequency \( \omega \sim \gamma/b \). Also, for \( d \geq 3 \) radiation with \( \omega \sim \gamma/b \) is predominantly emitted in wide angles.

Squaring (3.43), substituting into (2.50) and integrating over \( \omega \) from 0 to \( \infty \) and all angles except \( \vartheta \), one obtains the angular distribution

\[
dE' = \frac{\varkappa^4 m^2 f^2 \gamma^{d+1}}{b^{d+3}} \frac{\Gamma\left(\frac{3d+3}{2}\right) \Gamma\left(\frac{2d+3}{2}\right) \Gamma\left(\frac{d+3}{2}\right)}{2^d \pi^{d/2+4} \Gamma\left(\frac{d+2}{2}\right)^2 \Gamma(2d+3) \psi^2} \quad \sin^{d+1} \vartheta \quad (4.13)
\]

Formula (4.13) gives the dominant wide angle radiation in all dimensions \( d \geq 3 \). Figure 5 shows the angular and frequency profile of this component of radiation for \( d = 3 \). To compute the total energy of this type we integrate over \( \vartheta \) making use of \((A.12)\).

For \( d \geq 3 \) the emitted energy is given by

\[
E' = C_d \frac{\varkappa^4 m^2 f^2}{b^{d+3}} \gamma^{d+1}, \quad C_d = \frac{2^{d-8} \pi^{(d+2)/2} \Gamma\left(\frac{d+3}{2}\right) \Gamma\left(\frac{d+3}{2}\right) \Gamma\left(\frac{d+2}{2}\right)}{\Gamma(2d+3) \Gamma(d)} \quad (4.14)
\]

For \( d = 2 \) one obtains

\[
E' = 105 \frac{\varkappa^4 m^2 f^2}{2^{16} (2\pi)^7 b^3} \gamma^{3} \ln \gamma \quad (4.15)
\]

The cases \( d = 0,1 \) have to be considered separately since for them \( z' \)- and \( zz' \)-types of radiation are comparable and splitting the amplitude into \( j_z \) and \( j_{z'} \) is not particularly useful.
Figure 5: Frequency (a) and angular (b) distribution of $z'$-radiation for $d = 3$ and $\gamma = 10^4$. The angular distribution is actually smooth, but rises very steeply at this scale for $\theta \approx 0$.

Figure 6: (a) The $\phi$-distribution in 4D for $\gamma = 10^4$ and (b) in 6D for $\gamma = 10^5$.

C. The cases $d=0, 1$

According to Table II the emitted energy in 4D is concentrated in the region $\omega \sim \gamma/b$, $\theta \sim \gamma^{-1}$. In this case the exponent $e^{i \omega b \sin \theta \cos \varphi}$ in the stress amplitude $\sigma(k)$ does not oscillate fast and the emitted energy may be easily computed numerically. The frequency and $\theta$-distributions in this case are shown for $\gamma = 10^5$ in Figure 8 while the distribution over $\phi$ (which coincides with $\varphi$ in 4D) is presented by Figure 6.

The total emitted energy is

$$E_0 = C_0 \frac{z^4 m^2 f^2}{b^3} \gamma^3, \quad C_0 \approx 8.3 \times 10^{-5}.$$ (4.16)

The frequency distribution is non-zero at $\omega = 0$ (see Figure 3), in agreement with the analytically derived value

$$\frac{dE_0}{d\omega} \bigg|_{\omega=0} = \frac{1}{3 \times 2^6 \times \pi^4} \frac{z^4 m^2 f^2}{b^2} \gamma^2,$$

due mainly to the imaginary part of the $\rho$-amplitude.

The frequency distribution of the emitted energy $E_1$ in 5D for $\gamma = 10^4$ is shown in Figure 7. It is characterized by a long tail beyond the value $\omega \sim \gamma/b$, which as can also be argued analytically leads to a behavior $dE_1/d\omega \sim 1/\omega$ (Figure 7(b)) all the way to $\omega \sim \gamma^2/b$, beyond which it falls-off exponentially. The integral

---

8 Notice from Table I that the total amplitude satisfies $j(\omega \sim \gamma/b) \sim \gamma^2 j(\omega \sim \gamma^2/b)$. This gives for $d = 1$ the estimate $|j|^2 \omega^{d+2} \sim 1/\omega$. 

In this regime one may estimate the contribution to the interference integral using subsection. and \( \hat{E} \) order as numerically to be \( \frac{dE}{d\omega} \) of \( \gamma \) argument of \( \gamma \). Integration over all angles except \( \varpi \) Macdonalds by their values at \( \varpi \). The value of the integral over \( \varpi \). One is left with the contribution from \( \varpi \). The purpose of this subsection is to estimate the contribution of the interference part (\( \hat{E} \bar{\hat{E}} \) for \( d \)). For \( z \). One then obtains for the interference part of the energy loss \( \frac{dE}{d\omega} \sim \hat{\lambda}^2 \cos(\gamma \varpi \sin \varpi \cos \varpi) \frac{\hat{K}_{d/2+1}(z) \hat{K}_{d/2}(z') \gamma^2 \varpi \sin^2 \varpi \omega^{d+2}}{2(2\pi)^{d+2} \gamma^2 \varpi^2 z^2} \). Integration over all angles except \( \varpi \) gives \( \frac{dE}{d\varpi} \) to be \( \gamma^{d/2+3/4} \). This is negligible, compared to the other contributions in all dimensions. One is left with the contribution from \( \varpi \) and \( \varpi \). Substituting \( \gamma \sim \hat{\lambda} \) and \( \hat{\gamma} \) \( \hat{K}(z) \) one estimates the integral to be of \( \mathcal{O}(\gamma) \) in all dimensions. This is of the same dominant order as \( E \) and \( E' \) for \( d = 0 \) and \( d = 1 \) and was included in the total energy evaluated in the previous subsection.

Figure 7: Frequency distribution for \( d = 1 \) and \( \gamma = 10^4 \): (a) for \( \omega \gamma > 1 \), (b) for \( \gamma / b \gamma \sqrt{2} / b \)

of \( dE/d\omega \) over the range \( \frac{\gamma}{b} > \frac{\gamma}{b} \sqrt{2} / b \) gives an extra logarithm in the total emitted energy, which is computed numerically to be

\[
E_1 = C_1 \frac{z^2 m^2 f^2}{b^6} \gamma^3 \ln \gamma, \quad C_1 \approx 1.64 \times 10^{-5}.
\]

(4.17)

D. The estimate of the \( z' \)-interference part of radiation

The purpose of this subsection is to estimate the contribution of the interference part (\( \hat{E} \bar{\hat{E}} \) + c.c.). It will be shown that it is subleading for \( d \geq 2 \) and of the same order as \( z \)- and \( z' \)- contributions for \( d = 0, 1 \).

The interference term \( E^{zz'} \sim \int (\hat{E} \bar{\hat{E}} + c.c.) \omega^{d+2} d\omega d\Omega_{d+2} \) contains the product of Macdonald functions \( \hat{K}(z) \hat{K}(z') \). Thus, its value depends on the overlap of these functions in the domain (\( z \leq 1 \), \( z' \leq 1 \)), or equivalently (\( \omega \leq \gamma / b \), \( \varpi \leq 1 / \sqrt{\gamma} \)). The presence of the factor \( \omega^{d+2} \) implies that most of the contribution to the integral comes from the large \( \omega \) regime with \( \omega \approx \gamma / b \), in which \( z' \approx 1 \).

For \( z \approx 1 / \sqrt{\gamma} \) the integral is suppressed by the volume factor. Thus, the interesting regime of \( z \) is \( \gamma^{-1} \approx z \approx 1 \). In this regime one may estimate the contribution to the interference integral using

\[
\frac{dE}{d\omega} \sim \hat{\lambda} \frac{\gamma^2 \varpi \sin \varpi \cos \varpi \omega^{d+2}}{2(2\pi)^{d+2} \gamma^2 \varpi^2 z^2} \hat{K}_{d/2+1}(z) \hat{K}_{d/2}(z') \sin^2 \varpi \omega^{d+2}.
\]

(4.19)

Integration over all angles except \( \varpi \) gives

\[
\frac{dE}{d\varpi} \sim \hat{\lambda} \frac{\gamma^2 \sin \varpi \cos \varpi \omega^{d+2}}{2(2\pi)^{(d+3)/2} \gamma^2 \varpi^2 z^2} J_0(\omega \sin \varpi \hat{\hat{K}}_{d/2+1}(z) \hat{K}_{d/2}(z') \omega^{d+2} \sin^2 \varpi \omega^{d+3} \varpi).
\]

(4.20)

The value of the integral over \( \varpi \) and \( \omega \) is controlled by \( J_0 \). For \( \varpi \approx 1 / \sqrt{\gamma} \), \( z \) and \( z' \) are both of \( \mathcal{O}(1) \), while the argument of \( J_0 \) is of \( \mathcal{O}(\sqrt{\gamma}) \). Using then the asymptotic expansion of \( J_0 \) and approximating the hatted Macdonalds by their values at \( z \approx z' \approx 1 \), one can estimate as in previous cases the power of \( \gamma \) in \( E^{zz'} \) to be \( \gamma^{d/2+3/4} \). This is negligible, compared to the other contributions in all dimensions. One is left with the contribution from \( \varpi \approx 1 / \gamma \), where \( \omega \sin \varpi \approx 1 \) and \( z \approx 1 / \gamma \). Substituting \( J_0 \approx J_0(0) = 1 \) and \( \hat{K}(z) \approx \hat{K}(z = 0) \) one estimates the integral to be of \( \mathcal{O}(\gamma^3) \) in all dimensions. This is of the same dominant order as \( E \) and \( E' \) for \( d = 0 \) and \( d = 1 \) and was included in the total energy evaluated in the previous subsection.
5. SUMMARY OF RESULTS

Scalar bremsstrahlung radiation during the transplanckian collision of two gravitating massive point particles in arbitrary dimensions was studied classically in the laboratory frame. The main goal was to compute the powers of the Lorentz factor $\gamma$ and how they depend on the number of extra dimensions $d$.

We computed both analytically and numerically radiation into truly massless and massive (for the brane observer) modes. An essential difference with the previously considered case of scalar bremsstrahlung in flat space from particles interacting via a scalar field, is that in the latter the scalar field(s) is linear, while here the bulk scalar interacts non-linearly with gravity. Within the perturbation theory with respect to both scalar and gravitational coupling constants it was found that the radiation amplitude consists of a local and a non-local part. Furthermore, it was shown that in a certain range of angles and frequencies the leading terms of these two mutually cancel, while the remaining terms can be presented as the sum of two contributions $(j_\sigma, j_{\sigma'})$ which have frequency cut-offs at $\omega \leq \gamma^2 / b$ and $\omega \leq \gamma / b$, respectively. Their contribution to the total radiated energy $E_d$ depends on the phase-space in an intricate way, so that the resulting radiation does not have a simple universal expression. Specifically, it was found that in the absence of extra dimensions one obtains

$$E_0 = C_0 \, m \left( \frac{r_g}{b} \right)^2 \left( \frac{r_f}{b} \right) \gamma^3, \quad C_0 \approx 8.3 \times 10^{-5},$$

with the “basic” relativistic enhancement factor $\gamma^3$. For one extra dimension one has

$$E_1 = C_1 \, m \left( \frac{r_g}{b} \right)^4 \left( \frac{r_f}{b} \right)^2 \gamma^3 \ln \gamma, \quad C_1 \approx 1.64 \times 10^{-5},$$

with almost the same (up to the logarithm) enhancement factor. For $d \geq 2$ one finds

$$E_d = C_d \, m \left( \frac{r_g}{b} \right)^{2(d+1)} \left( \frac{r_f}{b} \right)^{d+1} \gamma^{d+2},$$

with $C_d$ are computed from (4.10), (4.9) and (4.11) and given above for $d = 2, \ldots, 6$. So the expected enhancement factor $\gamma^{d+2}$ is regained, with each new dimension adding one power of $\gamma$ to the radiation loss.

Another feature of interest is the spectral-angular distribution of radiation. It was shown that in the usual gravity theory without extra dimensions the partial cancelation of local and non-local amplitudes in the case of gravitational interaction can be attributed to the fact that in terms of a curved space picture the world lines of a massive ultrarelativistic radiating charge and the null geodesic of the emitted radiation stay close to each other, so that the formation length of the radiation emitted predominantly in the forward direction is $\gamma$ times stronger than in flat space. In perturbation theory on a flat background this corresponds to cancelation at high frequencies. In the presence of extra dimensions the emitted radiation is predominantly massive from the brane observer point of view, so the trajectories do not stay close together. The resulting spectral distribution then has substantial remainder at high frequencies up to $\gamma^2 / b$, as illustrated in Figs 4(a) and 7(b).

The problem considered in this paper is a simplified intermediate step towards the fully gravitational counterpart, where one is interested in gravitational bremsstrahlung in particle collisions interacting gravitationally [24]. The non-local part of the amplitude in this case is due to the three-graviton vertex. The details of this problem is the subject of a forthcoming publication.

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Appendix A: Notation

1. KK mode decomposition and Fourier transformation - Notation and conventions

The Fourier decomposition of the bulk fields $h_{MN}(x^\mu, y^i)$, $h'_{MN}(x^\mu, y^i)$, $\Phi(x^\mu, y^i)$ all with periodic conditions e.g. $h_{MN}(x, y^k + 2\pi R) = h_{MN}(x, y^k)$ is of the form

$$h_{MN}(x, y) = \frac{1}{V} \sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_d=-\infty}^{\infty} h^n_{MN}(x) \exp \left( \frac{i n_3 y^i}{R} \right) \equiv \frac{1}{V} \sum_n h^n_{MN}(x) e^{i n_3 y^k / R}.$$  (A.1)
Using the representation of the delta-function

\[
\frac{1}{V} \sum_n e^{im_n(y^k - y'^k)/R} = \delta^d(y - y'), \quad \frac{1}{V} \int e^{im_n y^k} dy^d = \prod_{k=1}^d \delta_{m_k, 0},
\]  

(A.2)

where \( V = (2\pi R)^d \) is the volume of the torus, one obtains the inverse transformation:

\[
h_{MN}^n(x) = \int_V h_{MN}(x, y) e^{-in_k y^k/R} d^4 y.
\]  

(A.3)

Four-dimensional fields \( h_{MN}^n(x) \) are then expanded as

\[
h_{MN}^n(x) = \frac{1}{(2\pi)^4} \int e^{-i(px)} h_{MN}^n(p) d^4 p,
\]  

(A.4)

where \( (px) = p_x q_x x^\mu \) is four-dimensional scalar product, and the final decomposition reads

\[
h_{MN}(x, y) = \frac{1}{(2\pi)^4} \frac{1}{V} \sum_n \int h_{MN}^n(p) e^{-i(px) + in_k y^k/R} d^4 p,
\]  

(A.5)

while the inverse transformation is

\[
h_{MN}^n(p) = \int d^4 y \int_R h_{MN}(x, y) e^{i(px) - in_k y^k/R} d^4 x.
\]  

(A.6)

Occasionally we will also use another notation for the discrete transversal momenta: \( p_T^i = n^i / R \), i.e.

\[
h_{MN}^n(p) = \int \int h_{MN}(x, y) e^{i(px) - ip_T y} d^4 x.
\]  

(A.7)

with \( p_T y = p_T^i y^i \). From the four-dimensional point of view the zero mode \( h_{MN}^0(x) \) \((n = 0 \text{ means all } n^i = 0)\) is massless, while the \( n \neq 0 \) modes are massive. Indeed, in the absence of the source term Eq. (2.11) reduces to

\[
(\Box + p_T^2) h_{MN}^n(x) = 0, \quad p_T^2 = \frac{1}{R^2} \sum_{i=1}^d (n^i)^2,
\]  

(A.8)

where \( \Box = \partial_{\mu} \partial^\mu \) is the four-dimensional D’Alembert operator, while the momenta transverse to the brane give rise to the mass term. In the standard scheme one suitably combines polarization modes to get true massive gravitons with five spin states for each mass. For our purposes it will be easier to sum over modes using the original decomposition.

When the level spacing is small (e.g. when \( R \gg b \)), one can pass from summation over \( n \) to integration over \( p_T \) using

\[
\frac{1}{V} \sum_n = \frac{1}{(2\pi)^d} \int d^d p_T.
\]  

(A.9)

Here it is implicitly assumed that both the sum and the integral converge. As pointed out in the text, and in contrast to quantum Born amplitudes, this is guaranteed in the framework of the classical perturbation approach presented here.

It is worth noting, that upon integration over modes in the case of small level spacing, one obtains the results expected in the uncompactified theory in \( D = 4 + d \)-dimensional Minkowski space.

In a similar fashion, expansion of \( h_{MN}^n(x, y) \) leads to the set of four-dimensional modes \( h_{MN}^n(x) \), and an expansion of the bulk scalar \( \Phi(x, y) \) to the set \( \Phi^n(x) \). These four-dimensional fields are further Fourier transformed to \( \Phi_{MN}^n(p) \) and \( \Phi^n(p) \), respectively.

\section{Integration over angles and frequencies}

In the main text the following integrals over the radiation angle \( \theta \) were encountered

\[
V_{m}^n = \int_{0}^{\pi} \sin^n \theta \, d\theta, \quad \psi = 1 - \nu \cos \theta
\]  

(A.10)
with integers $m, n$.

For $2m > n + 1$ one finds to leading order

$$V_m^\prime = \frac{2^{m-1}}{\Gamma(m)} \Gamma \left( \frac{n+1}{2} \right) \Gamma \left( m - \frac{n+1}{2} \right) \gamma^{2m-n-1}. \quad (A.11)$$

For $n > 2m - 1$ one obtains

$$V_n^\prime = \frac{2^{n-m} \Gamma \left( \frac{n+1}{2} \right) \Gamma \left( \frac{n+1}{2} - m \right)}{\Gamma(n-m+1)}. \quad (A.12)$$

In the case $2m = n + 1$ the leading contribution to the integral is proportional to $\ln \gamma$. For example, one case needed in the text was $V_2^3 \approx 4 \ln \gamma$.

Calculation of the integrals over the frequency or over the impact parameter involving two Macdonald functions of the same argument is performed using the formula:\[\int_0^\infty K_\mu(cz)K_\nu(cz)z^{\alpha-1}dz = \frac{2^{\alpha-3} \Gamma \left( \frac{\alpha+\mu+\nu}{2} \right) \Gamma \left( \frac{\alpha+\mu-\nu}{2} \right) \Gamma \left( \frac{\alpha-\mu+\nu}{2} \right) \Gamma \left( \frac{\alpha-\mu-\nu}{2} \right)}{c^\alpha \Gamma(\alpha)}. \quad (A.13)\]

### 3. Useful kinematical formulae

The angles in the formulae below are defined in Figure 1.

- $u^\mu = \gamma(1, 0, 0, v)$, $u^\nu = (1, 0, 0, 0)$, $\psi = 1 - v \cos \theta \cos \alpha = 1 - v \cos \theta$.
- $z' = (kv)b/\gamma v$, $z = (kv)b/\gamma v \psi = z' \gamma \psi$.
- $2\gamma zz' - z^2 - z'^2 = \omega^2 b^2 \sin^2 \theta$, $\xi = 2\gamma zz' - z^2 - z'^2 = \omega^2 b^2 \sin^2 \theta \cos^2 \alpha + b^2 k^2 = (\gamma v z' \sin \theta)^2$.
- $(kb) = \xi \cos \phi = \gamma z' \sin \psi \cos \phi = \gamma z' \cos \alpha \sin \theta \cos \varphi = \omega b \sin \theta \cos \varphi$.
- $\beta = \gamma^2 z' - z^2 = \omega^2 b^2 \cos \theta(1 - v \cos \theta) = z'^2 + z^2 - 2\gamma zz' (1 - \psi)$.

### Appendix B: Destructive interference for $\gamma/b \leq \omega \leq \gamma^2/b$

An alternative proof of the destructive interference effect of the radiation amplitude in the $z$–region but with $\theta < 1/\gamma$ in higher dimensional Minkowski space will be presented here. In the main text we followed an approximation allowing to cover the full angular range. Here destructive interference in the restricted angular range will be demonstrated rigorously.

Start with (3.20), change variable $x$ to $\zeta$ given by

$$\zeta = f(x) \, dx, \quad f(x) = (z^2 + z'^2 + \gamma zz') x + \gamma zz' - z^2, \quad (B.1)$$

and integrate by parts twice using

$$\zeta K_\nu(\zeta) = -\zeta K_{\nu+1}(\zeta). \quad (B.2)$$

The first integration gives:

$$\int_0^1 dx e^{-ix(kb)} \zeta K_{d/2-1} \left[ f(x) \right] = \frac{\eta e^{-ix(kb)}}{f(1)} \frac{\zeta K_{d/2} \left[ f(x) \right] |_{\zeta=1}}{\zeta K_{d/2} \left[ f(x) \right] |_{\zeta=0}} + \int_0^1 dx \zeta K_{d/2} \left[ f(x) \right] \partial_x \left( \frac{\eta e^{-ix(kb)}}{f(x)} \right). \quad (B.3)$$

A second integration by parts leads to

$$\sigma(k) = \lambda_d \gamma^2 z' \frac{\eta e^{ikb}}{\gamma^2 z' - \gamma^2} \left[ \zeta K_{d/2} \left( \frac{f(z)}{\gamma^2 z' - \gamma^2} \right) - \frac{1}{\gamma^2 z' - \gamma^2} \left( \zeta K_{d/2} \left( \frac{f(z')}{\gamma^2 z' - \gamma^2} \right) - i q_1 \frac{\hat{f}(1/2) \left( \frac{f(z')}{\gamma^2 z' - \gamma^2} \right) + R} {\frac{f(z')}{\gamma^2 z' - \gamma^2} - \gamma^2 z' \right] \right), \quad (B.4)$$

where

$$q_0 = \frac{k}{(kb)} - \frac{\hat{f}(z^2 + z'^2 - 2\gamma zz')}{\gamma^2 z' - \gamma^2}, \quad q_1 = \frac{k}{(kb)} - \frac{\hat{f}(z^2 + z'^2 - 2\gamma zz')}{\gamma^2 z' - \gamma^2}. \quad (B.5)$$
and

\[ R = \int_0^1 dx \tilde{K}_{d/2+1}(\zeta(x)) \left[ \left( \frac{e^{-ix/(kb)}}{f(x)} \right)' - \frac{1}{f(x)} \right] \].

(B.6)

Continuing integration by parts further, one obtains an expansion in terms of \(q_0\beta^{-1}\) and \(q_1(\beta - \xi^2)^{-1}\). As we discussed before, in the \(z\)-region of interest here \(\psi \sim 1/\gamma^2\), \(z \sim 1\), \(z' \sim \gamma\), so that \(\xi^2 \sim 2 \sim \gamma^2 \sim (\beta - \xi^2)\), \(q_0 \sim q_1 \sim \gamma\) and therefore the expansion parameters are: \(q_0\beta^{-1} \sim 1\), \(q_1(\beta - \xi^2)^{-1} \sim 1\). With this accuracy one can set \(q_0 = q_1 = (kb)\), \(\beta = \gamma z z'\) and write:

\[ \sigma(k) \simeq \frac{\lambda_0}{k^4} \left[ e^{i(kb)} \left( \frac{z'}{z} \tilde{K}_{d/2}(z) - i \frac{kb}{\gamma z^2} \tilde{K}_{d/2+1}(z) \right) + \left( \frac{z'}{z - z'} \tilde{K}_{d/2}(z') + i \frac{kb}{\gamma z^2} \tilde{K}_{d/2+1}(z') \right) \right]. \]

(B.7)

The first parenthesis in \(\sigma\) cancels for \(v = 1\) the leading terms of \(\rho(3.11)\), and the total amplitude \(j(k) = \rho(k) + \sigma(k)\) contains only the second parenthesis in (B.7) plus the subleading terms mentioned above. Thus, the series obtained by integration by parts, converges inside \(z\)-cone \(\theta < \arcsin\gamma^{-1}\) and establishes the effect of destructive interference.

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