Asymptotic Behavior of Non-oscillatory Solutions of Second order Integro-Dynamic Equations on Time Scales

Said R Grace, Mohamed A EI-Beltagy and Sarah A Deif

Department of Engineering Mathematics, Faculty of Engineering, Cairo University, Orman, Giza 12221, Egypt

Abstract

In this paper, we investigate some new criteria on the asymptotic behavior of non-oscillatory solutions of second order integro-dynamic equations on time-scales. We also provide some numerical examples to illustrate the relevance of the obtained results.

Keywords: Asymptotic behavior; Non-oscillatory; Integro-dynamic equations; Time scales

Main Results

We shall employ the following lemma.

Lemma 4.1: If X and Y are nonnegative [6], then

\[ X^\Delta + (\lambda - 1)Y^\Delta - \lambda XY^{-1} \leq 0, \quad \lambda > 1 \]

and

\[ X^\Delta - (1 - \lambda)Y^\Delta - \lambda XY^{-1} \leq 0, \quad \lambda < 1 \]

where equality hold if and only if X = Y.

We define \( R(t, t_0) = \int_{t_0}^{t} \frac{\Delta s}{a(s)} \) for all \( t_0 \geq 0 \).

Here is our first result.

Theorem 4.1: Let conditions (i)-(iii) hold with \( \gamma = 1 \) and \( \beta > 1 \) and suppose

\[
\lim_{t \to \infty} R(t, t_0) \leq \int_{a(t)}^{1} 1 \, dt < \infty
\]

for all \( t_0 \geq 0 \). If x is a non-oscillatory solution of equation (1.1) for t, then

\[ x(t) = o(R(t, t_0)) \quad \text{as} \quad t \to \infty. \]
for all \( t \geq t_1 \). Let 
\[
m = \max \left\{ \|F(t,x(t))\| : t \in [0,t_1] \right\}.
\]
By assumption (i), we have 
\[
\int_{t}^{\infty} k(t,s)F(t,x(s))\Delta t \leq m\int_{t_0}^{\infty} k(t,s)\Delta t
\]
Hence from (2.5) and (2.6), we get 
\[
(a(t)x^+(t))^\lambda \leq b + \int k(t,s)\left[ p_2(s)x(s) - p_1(s)x^-(s) \right] \Delta s \quad (2.7)
\]
If we apply (2.1) with 
\[
\lambda = \beta, \quad X = p_1^{1/\gamma} x \quad \text{and} \quad Y = \left( \frac{1}{\gamma} p_1^{1/\gamma} \right)^{1/\beta - 1}
\]
we obtain 
\[
p_1(t)x(t) - p_1(t)x^+(t) \leq (\beta - 1)\beta^\gamma p_1^{\beta \gamma}(s)p_1^{\gamma}(s) \Delta t \quad (2.8)
\]
Substituting (2.8) into (2.7) gives 
\[
(a(t)x^+(t))^\lambda \leq b + (\beta - 1)\beta^\gamma \int k(t,s)p_1^{\beta \gamma}(s)p_1^{\gamma}(s) \Delta s
\]
Integrating this equality from \( t_1 \) to \( t \), we have 
\[
x^+(t) \leq \frac{a(t)}{a(t_1)} x^+(t_1) + \int_{t_1}^{t} \left[ (\beta - 1)\beta^\gamma p_1^{\beta \gamma}(s)p_1^{\gamma}(s) \right] \Delta s
\]
Where 
\[
c = \frac{a(t)}{a(t_1)} x^+(t_1) + b.
\]
Integrating this equality from \( t_1 \) to \( t \) we get, 
\[
|x(t)| + c \int_{t_1}^{t} \left[ (\beta - 1)\beta^\gamma p_1^{\beta \gamma}(s)p_1^{\gamma}(s) \right] \Delta s
\]
\[
= |x(t_1)| + c \int_{t_1}^{t} \left[ (\beta - 1)\beta^\gamma p_1^{\beta \gamma}(s)p_1^{\gamma}(s) \right] \Delta s
\]
Now assume \( x(t) \) is eventually negative, say \( x(t)<0 \) for some \( t \geq t_2 \). From equation (1.1) we find 
\[
(a(t)x^+(t))^\lambda \leq \int k(t,s)f_2(s,x(s))\Delta s \leq \int_{t_0}^{\infty} k(t,s)f_2(s,x(s))\Delta s
\]
Using (2.6) in the above inequality, we obtain 
\[
(a(t)x^+(t))^\lambda \leq b \quad \text{for} \quad t \geq t_1
\]
The rest of the proof is similar to that of Theorem 2.1 and hence is omitted.

The following corollary is immediate.

**Corollary 4.1:** Let conditions (i) and (ii) hold with \( f_2=0 \) and 
\[
x_f(t,x) > 0 \quad \text{for} \quad x \neq 0 \quad \text{and} \quad t \geq 0,
\]
\[
\int_{t_0}^{\infty} \frac{s}{a(s)} \Delta s \leq \infty \quad \text{for any} \quad t_0 \geq 0.
\]
If \( x \) is a non-oscillatory solution of equation (1.1) then \( x(t) \) \( \leq \infty \) is bounded.

**Theorem 4.3:** Let conditions (i)-(iii) hold with \( \beta = 1 \) and \( \gamma = 1 \) and suppose 
\[
\lim_{t \to \infty} \frac{1}{RT(t,t_0)} \int_{t_0}^{t} \left[ \int k(u,s) p_1^{\gamma}(s)p_1^{\gamma}(s) \Delta s \right] \Delta u \leq \infty
\]
for all \( t_0 \geq 0 \). If \( x \) is a non-oscillatory solution of equation (1.1), then 
(2.4) holds.

**Proof:** Let \( x \) be a non-oscillatory solution of equation (1.1). First assume that \( x \) is eventually positive.

Fix \( t_0 \). Assume \( x(t) > 0 \) for \( t \geq t_0 \) for some \( t_0 \geq t_0 \). Using condition (ii) and (iii) with \( \beta = 1 \) and \( \gamma < 1 \) in equation (1.1) we have 
\[
(a(t)x^+(t))^\lambda \leq \int k(t,s)F(x(s))\Delta s + \int k(t,s) \left[ p_1(s)x^+(s) - p_1(s)x(s) \right] \Delta s
\]
(2.11)
By applying (2.2) with \( \lambda = \gamma, \quad P = \frac{1}{\gamma} x \quad \text{and} \quad Y = \left( \frac{1}{\gamma} p_1^{1/\gamma} \right)^{1/\beta - 1}
\]
we have 
\[
p_1(t)x^+(t) - p_1(t)x(t) \leq (1 - \gamma)\gamma^\gamma p_1^{\gamma}(t)p_1^{\gamma}(t) \quad \text{for} \quad t \geq 0
\]
(2.12)
Using (2.12) in (2.11) we have 
\[
(a(t)x^+(t))^\lambda \leq b + (1 - \gamma)\gamma^\gamma \int k(t,s)p_1^{\gamma}(s)p_1^{\gamma}(s) \Delta s
\]
where \( b \) is as in (2.6).

The rest of the proof is similar to that of Theorem 4.1 and hence is omitted.

Finally, we present the following results with different nonlinearities i.e. with \( \beta > 1 \) and \( \gamma < 1 \).

**Theorem 4.4:** Let conditions (i)-(iii) hold with \( \beta > 1 \) and \( \gamma < 1 \) and assume that there exists a rd-continuous function \( \xi : T \to T \) such that 
\[
\lim_{t \to \infty} \frac{1}{RT(t,t_0)} \int_{t_0}^{t} \left[ \int k(u,s) \xi_2(u) + p_1^{\gamma}(u) + c_0^{\gamma}(u) \Delta s \right] \Delta u \leq \infty
\]
for all \( t_0 \geq 0 \), where \( c_0 = (\beta - 1)\beta^\gamma \) and \( c_1 = (1 - \gamma)\gamma^\gamma \) for all \( t_0 \geq 0 \). If \( x \) is a non-oscillatory solution of equation (1.1), then (2.4) holds.

**Proof:** Let \( x \) be a non-oscillatory solution of equation (1.1). First assume \( x \) is eventually positive. Fix \( t_0 \). Assume \( x(t) > 0 \) for \( t \geq t_0 \) for some \( t_0 \geq t_0 \). Using conditions (ii) and (iii) in equation (1.1) we obtain
\[ (a(t)x'(t))^2 = -\int_{t}^{\bar{t}} k(t,s)F(x(s),x(s))\Delta s + \int_{t}^{\bar{t}} \left[ \int_{t}^{s} (\xi(s) - p_i(s)x''(s))\Delta s \right] \Delta t. \]

As in the proof of Theorems 2.1 and 2.3, we can easily find

\[ (a(t)x'(t))^2 \leq b + \int_{t}^{\bar{t}} k(t,s)\left[ \beta^{(1-\beta)}\gamma^2 g_{\beta}(t,s)p_i(s)^2 + (1-\gamma^{1-\beta})\gamma^2 p_i(s)^2 \right] \Delta s. \]

The rest of the proof is similar to that of Theorem 4.1 and hence is omitted. Theorem 4.4 can be re-stated as follows:

**Theorem 4.5:** Let conditions (i)-(iii) hold with \( \beta > 1 \) and \( \gamma < 1 \) and assume that there exists a positive rd-continuous function \( \xi : T \to T \) such that

\[
\lim_{r \to 0} \frac{1}{R(t,r)} \int_{t-r}^{t} \left[ \int_{s}^{\bar{t}} k(t,u) \left[ \frac{2}{\beta} (p_i(u) \xi(t,u)) \right] \Delta u \Delta s \right] \Delta t < \infty
\]

and

\[
\lim_{r \to 0} \frac{1}{R(t,r)} \int_{t-r}^{t} \left[ \int_{s}^{\bar{t}} k(t,u) \left[ \frac{2}{\beta} (p_i(u) \xi(t,u)) \right] \Delta u \Delta s \right] \Delta t < \infty
\]

for all \( t \geq 0 \). If \( a \) is a non-oscillatory solution of equation (1.1), then (2.4) holds.

For the case of forced integro-differential equation

\[(r(t)(x'(t)))' + \int_{0}^{t} a(s,t)F(x(s))\Delta s = e(t)\]

Where \( e : T \to R \). Now, if in addition to the hypotheses of all the results presented above, we assume that \( a \) is rd-continuous function.

\[
\lim_{r \to 0} \frac{1}{R(t,r)} \int_{t-r}^{t} \left[ \int_{s}^{\bar{t}} k(t,u) \xi(t,u) \Delta u \right] \Delta t < \infty
\]

then the conclusion of these results hold for equation (2.15).

**Numerical Examples**

As we already mentioned that the results of the present paper are new even for the cases when \( T = R \) i.e., the continuous case or when \( T = Z \), i.e., the discrete case.

As a numerical illustration of Theorems 4.3, 4.4 and 4.1 respectively, let us consider the following equation

\[
(a(t)x'(t)) + \frac{1}{(r'(s) + \beta)}(x'(s) - x''(s))ds = 0 \quad ; t \geq 0 \quad (3.1)
\]

with initial conditions \( x(t_0) = x_0 \) and \( x'(t_0) = x_0' \). Equation (3.1) can be converted to two simultaneous first order ordinary differential equations by substituting \( x' = y \). This will lead to the following system:

\[
x'(t) = \frac{y(t)}{t} \quad ; x(t_0) = x_0
\]

\[
y'(t) = -\int_{t_0}^{t} \frac{1}{(r'(s) + \beta)}(x'(s) - x''(s))ds \quad ; y(t_0) = t_0'x_0
\]

Many numerical techniques can be used to solve (3.2). In the current work, the second order, accurate modified Euler technique is considered. The time interval \([t_1, T] \) will be divided into \( N \) equal subdivisions with \( \Delta t \) for each one. The prediction and correction steps of the modified Euler technique will be:

\[
X_{i+1} = x_i + \Delta t f(t_i, x_i, y_i)
\]

\[
y_{i+1} = y_i + \Delta t g(t_i, x_i, y_i)
\]

\[
X_{i+1} = x_i + \Delta t f(t_i, x_i, y_i)
\]

\[
y_{i+1} = y_i + \Delta t g(t_i, x_i, y_i)
\]

\[
X_{i+1} = x_i + \Delta t f(t_i, x_i, y_i)
\]

\[
y_{i+1} = y_i + \Delta t g(t_i, x_i, y_i)
\]

\[
X_{i+1} = x_i + \Delta t f(t_i, x_i, y_i)
\]

\[
y_{i+1} = y_i + \Delta t g(t_i, x_i, y_i)
\]

\[
X_{i+1} = x_i + \Delta t f(t_i, x_i, y_i)
\]

\[
y_{i+1} = y_i + \Delta t g(t_i, x_i, y_i)
\]

\[
X_{i+1} = x_i + \Delta t f(t_i, x_i, y_i)
\]

\[
y_{i+1} = y_i + \Delta t g(t_i, x_i, y_i)
\]

\[
X_{i+1} = x_i + \Delta t f(t_i, x_i, y_i)
\]

\[
y_{i+1} = y_i + \Delta t g(t_i, x_i, y_i)
\]

The integral (3.5) can be approximated numerically at each time instant \( t \) using the trapezoidal rule which has accuracy of \( O(\Delta t^2) \) where \( \Delta t \) is the subdivision width when dividing the interval \([t_i, t_{i+1}] \) into \( N_i \) subdivisions.

Let \( t_i = 1, T = 100, x(t_0) = 1 \) and \( x'(t_0) \) in equation (3.4) for different values of \( \gamma \) and \( \beta \). Figure 1 shows the asymptotic behavior of \( x(t) \) with the time for \( \gamma = 1/3 \) and \( \beta = 1 \) [7]. The solution \( x(t) \) asymptotes to a straight line of slope 3.73 approximately after \( t=70 \). Increasing the value of \( \beta \) to be equal 3, the line slope will increase to 74.5 approximately after \( t=90 \), as shown in Figure 2. Increasing \( \gamma \) to be equal 1 with \( \beta = 3 \), will have negligible effect on the solution behavior, as shown in Figure 3. In this case the solution asymptotes, approximately, to the same straight line in Figure 2.

**General Remarks**

We conclude by presenting several remarks and extensions of the results given above

i) The results presented in this paper are new for \( T=R \) and \( T=Z \).

ii) The results of this paper are presented in a form which is essentially new for equation (1.1) with different nonlinearities. Corollaries similar to Corollary 2.1 can be obtained. Here we omit the details.

iii) The results of this paper will remain the same if we replace (1.2) of assumption (i) by

\[
\sup_{0 \leq t \leq t_0} k(t, s) = k < \infty
\]

with \( k = k \cdot t_0 \) and \( t_0 > 0 \).
or, we replace condition (1.2) of assumption (i) by:

there exist rd-continuous functions \( k, m : T \rightarrow \mathbb{R}^+ \) such that

\[
\sup_{t \geq s} k(t) = k_s < \infty
\]

and

\[
\sup_{t \geq s} \int_s^t m(s) \Delta s = k_s < \infty.
\]

with \( k = k_0 \). Details are left to the reader.

iv) The technique offered in this paper can be employed to Volterra integral equations on timescales of the form

\[
x(t) + \int_0^t k(t, s)F(s, x(s)) \Delta s = 0.
\]

As example, we reformulate Theorem 2.1 and find

Theorem 4.1 Let conditions (ii) and (iii) hold with \( \beta = 1 \) and \( \gamma = 1 \) and suppose

\[
\lim_{t \to \infty} \int_0^t k(t, s) p_{1/\beta} \left( s \right) p_{1/\beta} (s) \Delta s < \infty,
\]

for all \( t \geq 0 \). If \( x \) is a non-oscillatory solution of equation (4.3), then \( x \) is bounded. Similar results can be obtained and the details are left to the reader.

In addition to the hypotheses of Theorems 2.1-2.6, assume

\[
\lim_{t \to \infty} R(t, t_0) < \infty.
\]

If \( x \) is a non-oscillatory solution of equation (1.1), then \( x \) is bounded

v) The results of this paper can be extended easily to delay integro-dynamic equations of the form

\[
(a(t)x^\gamma(t))^\gamma + \int_0^t k(t, s)F(s, x(g(s))) \Delta s = 0
\]

where \( g : T \rightarrow T \) is rd-continuous function, \( g(t) \leq t, g^\gamma(t) \geq 0 \) for \( \lim_{t \to \infty} g(t) = \infty \). The formulation of the results is left to the reader.

We note that we can reformulate the obtained results for the time scales \( T = \mathbb{R} \) (the continuous case), \( T = \mathbb{Z} \) (the discrete case), \( T = \mathbb{N}_0, T = h\mathbb{Z} \) with \( h > 0 \); etc. see [1]. The details are left to the reader.

References
1. Bohner M, Peterson A (2001) Dynamic Equations on Time-Scales: An Introduction with Applications, Boston, Birkhauser.
2. Karakostas G, Stavroulakis IP, Wu Y (1993) Oscillation of Volterra integral equations. Tohoku Math J 45: 583-605.
3. Onose H (1990) On Oscillation of Volterra integral equations and first order functional differential equations. Hiroshima Math J 20: 223-229.
4. Parhi N, Misra N (1983) On oscillatory and non-oscillatory behavior of solutions of Volterra integral equations. J Math Anal Appl 94: 137-149.
5. Singh B (1995) On the oscillation of Volterra integral equation. Czech Math J 45: 699-707.
6. Hardy GH, Littlewood IE, Polya G (1959) Inequalities, University Press, Cambridge.
7. Bohner M, Stvic S (2007) Asymptotic behavior of second order dynamic equations. Applied Math Comput 188: 1503-1512.