The $\delta$-deformation of the Fock space

Krzysztof Kowalski and Jakub Rembieliński

Department of Theoretical Physics, University of Łódź, ul. Pomorska 149/153, 90-236 Łódź, Poland

Abstract. A deformation of the Fock space based on the finite difference replacement for the derivative is introduced. The deformation parameter is related to the dimension of the finite analogue of the Fock space.

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1. Introduction

In recent years there has been a growing interest to discretizations of quantum mechanics based on the finite difference replacement for the derivative. This is motivated by the well-known speculations that below the Planck scale the conventional notions of space and time break down and the new discrete structures are likely to emerge. This has echoes in the arguments put forward in string theory and quantum gravity. We also mention the technical reasons for the application of discrete models. Let us only recall the lattice gauge theories. As a matter of fact the connection has been shown in ref. 1 between ordinary quantum mechanics on a equidistant lattice, where the the role of the derivative is played by the forward or backward discrete derivative, and $q$-deformations utilizing the Jackson derivative, nevertheless no explicit form of the corresponding deformation of the Fock space has been provided in ref. 1. On the other hand, there are indications [2] that approaches based on the central difference operator are more adequate for discretization of quantum mechanics than those using asymmetric forward or backward discrete derivatives.

In this paper we introduce a deformation of the Fock space, such that the creation and annihilation operators are elements of the quotient field of the deformed Heisenberg algebra generated by the usual position operator and the central difference operator. The deformation parameter $\delta$ describing the fixed coordinate spacing is naturally related to the dimension of the finite-dimensional space which can be regarded as an analogue of the Fock space. In the formal limit $\delta \rightarrow 0$ we arrive at the infinite-dimensional space coinciding with the usual Fock space.
2. The $\delta$-deformation of the Heisenberg algebra

As mentioned in the introduction there are indications that discretizations of quantum mechanics should involve the central difference operator such that

$$\Delta_\delta f(x) = \frac{f(x+\delta) - f(x-\delta)}{2\delta}.\quad (2.1)$$

Furthermore, it seems to us that the most natural candidate for the position operator in any discretized version of quantum mechanics is the standard one of the form

$$\hat{x}f(x) = xf(x).\quad (2.2)$$

In order to close the algebra satisfied by the operators $\Delta_\delta$ and $\hat{x}$ we introduce the operator $I_\delta$ defined by

$$I_\delta f(x) = f(x+\delta) + f(x-\delta).\quad (2.3)$$

It follows that

$$[\Delta_\delta, \hat{x}] = I_\delta, \quad [I_\delta, \hat{x}] = \delta^2 \Delta_\delta, \quad [I_\delta, \Delta_\delta] = 0.\quad (2.4)$$

Evidently,

$$\Delta_\delta = \frac{i}{\delta} \sin \delta \hat{p}, \quad I_\delta = \cos \delta \hat{p},\quad (2.5)$$

where $\hat{p} = -i\frac{d}{dx}$ is the usual momentum operator, so the contraction of the algebra (2.4) referring to $\delta \to 0$, is the usual Heisenberg algebra

$$[\hat{x}, \hat{p}] = iI.\quad (2.6)$$

Using (2.1), (2.2) and (2.3) we find easily the following Casimir operator for the algebra (2.4):

$$I_\delta^2 - \delta^2 \Delta_\delta^2 = 1.\quad (2.7)$$

We now discuss the representations of the algebra (2.4). We first observe that (2.4) can be related to the following deformation of the $e(2)$ algebra (A.3) (see appendix):

$$[J, U_\delta] = \delta U_\delta,\quad (2.8)$$

where $U_\delta$ is unitary, by means of the relations such that

$$\hat{x} = J, \quad (2.9a)$$

$$\Delta_\delta = \frac{1}{2\delta}(U_\delta - U_\delta^\dagger), \quad (2.9b)$$

$$I_\delta = \frac{1}{2}(U_\delta + U_\delta^\dagger). \quad (2.9c)$$

Consider the representation of (2.8) spanned by eigenvectors of the Hermitian operator $J$. Taking into account (2.8) and (A.5) we find

$$J|j\delta\rangle = j\delta|j\delta\rangle.\quad (2.10)$$
Hence, with the help of (2.8) we get
\[ U_\delta |j\delta\rangle = |(j + 1)\delta\rangle, \quad U_\delta^\dagger |j\delta\rangle = |(j - 1)\delta\rangle. \] (2.11)

Equations (2.9)–(2.11) taken together yield
\[ \hat{x}|j\delta\rangle = j\delta|j\delta\rangle, \] (2.12a)
\[ \Delta_\delta|j\delta\rangle = -\frac{1}{2\pi}(|(j + 1)\delta\rangle - |(j - 1)\delta\rangle), \] (2.12b)
\[ I_\delta|j\delta\rangle = \frac{1}{2}(|(j + 1)\delta\rangle + |(j - 1)\delta\rangle). \] (2.12c)

Let us now specialize to the case with integer \( j \) (see appendix). In view of the form of eq. (2.12a) it turns out that the operator \( \hat{x} \) really describes the position of a particle on equidistant lattice with the fixed coordinate spacing \( \delta \). The completeness condition satisfied by the vectors \(|j\delta\rangle\) can be written as
\[ \sum_{j=-\infty}^{\infty} \delta|j\delta\rangle\langle j\delta| = I. \] (2.13)

The relation (2.13) leads to the realization of the abstract Hilbert space of states specified by the inner product
\[ \langle f|g \rangle = \sum_{j=-\infty}^{\infty} \langle f|j\delta\rangle \langle j\delta|g \rangle \delta = \sum_{j=-\infty}^{\infty} f^*(j\delta) g(j\delta) \delta, \] (2.14)
where \( f(j\delta) = \langle j\delta|f \rangle \). The action of operators in the representation (2.14) is of the following form:
\[ \hat{x} f(j\delta) = j\delta f(j\delta), \] (2.15a)
\[ \Delta_\delta f(j\delta) = \frac{1}{2\pi} [f((j + 1)\delta) - f((j - 1)\delta)], \] (2.15b)
\[ I_\delta f(j\delta) = \frac{1}{2} [f((j + 1)\delta) + f((j - 1)\delta)]. \] (2.15c)

We now study the representation generated by eigenvectors \(|\varphi\rangle_\delta, \varphi \in \mathbb{R}, \) of the unitary operator \( U_\delta \) such that
\[ U_\delta |\varphi\rangle_\delta = e^{-i\delta\varphi} |\varphi\rangle_\delta. \] (2.16)

It follows immediately from (2.9) and (2.16) that
\[ \Delta_\delta |\varphi\rangle_\delta = \frac{i}{\delta} \sin \delta \varphi |\varphi\rangle_\delta, \] (2.17a)
\[ I_\delta |\varphi\rangle_\delta = \cos \delta \varphi |\varphi\rangle_\delta. \] (2.17b)

The completeness of the vectors \(|\varphi\rangle_\delta\) can be expressed by
\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} |\varphi\rangle_\delta \langle \varphi| = I. \] (2.18)
The resolution of the identity (2.18) gives rise to the functional representation of vectors

$$
\langle f | g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f^*(\varphi) g(\varphi) d\varphi,
$$

(2.19)

where $f(\varphi) = \langle \varphi | f \rangle$, and we have omitted for brevity the dependence of $f(\varphi)$ on $\delta$. The operators act in the representation (2.19) as follows:

$$
\hat{x} f(\varphi) = i \frac{d}{d\varphi} f(\varphi),
$$

(2.20a)

$$
\Delta_\delta f(\varphi) = \frac{i}{\delta} \sin \delta \varphi f(\varphi),
$$

(2.20b)

$$
I_\delta f(\varphi) = \cos \delta \varphi f(\varphi).
$$

(2.20c)

Our purpose now is to analyze the contraction $\delta \to 0$ of the representations (2.14) and (2.19) introduced above. Taking into account (2.16), (2.13) and (2.11) we find that the passage from the representation spanned by the vectors $|j\delta\rangle$ and that generated by the vectors $|\varphi\rangle$ can be described by the kernel

$$
\langle j\delta | \varphi \rangle_{\delta} = e^{ij\delta \varphi}.
$$

(2.21)

Equations (2.18) and (2.21) taken together yield

$$
\langle j\delta | j'\delta \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ij(j'-j)\delta \varphi} d\varphi = \frac{\sin \pi(j - j')}{\pi(j - j') \delta}.
$$

(2.22)

Therefore

$$
\langle j\delta | j'\delta \rangle = \frac{1}{\delta} \delta_{jj'},
$$

(2.23)

whenever $\delta \neq 0$. On the other hand, defining the continuum limit as

$$
j \to \infty, \quad \delta \to 0, \quad j\delta = \text{const} = x,
$$

(2.24)

and using the well known formula on the Dirac delta function

$$
\delta(x) = \lim_{\alpha \to \infty} \frac{1}{\pi} \frac{\sin \alpha x}{x},
$$

(2.25)

we find that (2.22) takes the form

$$
\lim_{j, j' \to \infty, \delta \to 0} \langle j\delta | j'\delta \rangle = \delta(x - x').
$$

(2.26)

Hence, we get

$$
\lim_{j \to \infty, \delta \to 0} |j\delta\rangle = |x\rangle,
$$

(2.27)
where $|x\rangle$, $x \in \mathbb{R}$, are the usual normalized eigenvectors of the position operator for a quantum mechanics on a real line. This observation is consistent with the fact that for $\delta \to 0$ the sum from (2.14) is simply the integral sum for the scalar product in $L^2(\mathbb{R}, dx)$. By (2.15) and (2.24) it is also evident that in the limit $\delta \to 0$ we arrive at the Heisenberg algebra (2.6). We have thus shown that the contraction referring to $\delta \to 0$ of the representation of the algebra (2.4) given by (2.14) and (2.15) coincides with the standard coordinate $L^2$ representation of the Heisenberg algebra (2.6). Analogously, we have

$$\delta\langle \varphi | \varphi' \rangle_\delta = \sum_{j=-\infty}^{\infty} e^{-ij\delta(\varphi - \varphi')} \delta. \quad (2.28)$$

Therefore,

$$\lim_{\delta \to 0} \delta\langle \varphi | \varphi' \rangle_\delta = 2\pi\delta(\varphi - \varphi'), \quad (2.29)$$

and we can identify

$$\lim_{\delta \to 0} |\varphi\rangle_\delta = \sqrt{2\pi}|p\rangle, \quad (2.30)$$

where $p = \varphi$, and $|p\rangle$, $p \in \mathbb{R}$, are the normalized eigenvectors of the momentum operator. Further, in view of (2.20) the case $\delta \to 0$ really corresponds to the Heisenberg algebra (2.6). So the representation specified by (2.19) coincides in the limit $\delta \to 0$ with the standard momentum representation. We conclude that the introduced deformation works both on the level of the algebra and the representation.

### 3. The $\delta$-deformation of the Heisenberg-Weyl algebra

In this section we study the $\delta$-deformation of the Heisenberg-Weyl algebra satisfied by the Bose creation and annihilation operators. Let us introduce the following family of operators:

$$A(s) = \frac{1}{\sqrt{2}}[\hat{x} + (1 - \delta^2 s) \Delta_\delta I^{-1}_\delta], \quad A^\dagger(s) = \frac{1}{\sqrt{2}}[\hat{x} - (1 - \delta^2 s) \Delta_\delta I^{-1}_\delta], \quad (3.1)$$

where $s = 0, 1, \ldots$. Clearly, these operators reduce to the standard Bose creation and annihilation operators in the limit $\delta \to 0$. We point out that then $A(s)$ and $A^\dagger(s)$ do not depend on $s$. Notice that in view of (2.9) $A^\dagger(s)$ is really the Hermitian conjugate of $A(s)$. It should also be noted that in the representation (2.20) the action of the operator $I^{-1}_\delta$ is simply the multiplication by $\sec\delta \varphi$. We now seek the vectors $|s\rangle$ and functions $\alpha(s)$ and $\beta(s)$, satisfying

$$A(s)|s\rangle = \alpha(s)|s - 1\rangle, \quad A^\dagger(s)|s\rangle = \beta(s)|s + 1\rangle, \quad s = 0, 1, \ldots \quad (3.2)$$

In other words, we are looking for the $\delta$-deformation of vectors spanning the occupation number representation. Using the following form of the Casimir (2.7) which can be obtained with the help of (3.1):

$$A(s + 1)A^\dagger(s) - A^\dagger(s - 1)A(s) = (1 - \delta^2 s)I, \quad (3.3)$$
where $I$ is the unit operator, we get
\[
\alpha(s + 1)\beta(s) - \alpha(s)\beta(s - 1) = 1 - \delta^2 s.
\] (3.4)

Hence, setting $\alpha(0) = 0$ and solving the elementary recurrence (3.4) we obtain
\[
\alpha(s)\beta(s - 1) = s - \frac{\delta^2}{2} s(s - 1).
\] (3.5)

The following solution of (3.5) consistent with the limit values $\alpha(s) = \sqrt{s}$ and $\beta(s) = \sqrt{s + 1}$, corresponding to $\delta = 0$, when $|s\rangle$ span the usual occupation number representation can be guessed easily:
\[
\alpha(s) = \sqrt{s - \frac{\delta^2}{2} s(s - 1)}, \quad \beta(s) = \sqrt{s + 1 - \frac{\delta^2}{2} s(s + 1)},
\] (3.6)

so we have
\[
A(s)|s\rangle = \sqrt{s - \frac{\delta^2}{2} s(s - 1)}|s - 1\rangle, \quad A^\dagger(s)|s\rangle = \sqrt{s + 1 - \frac{\delta^2}{2} s(s + 1)}|s + 1\rangle. (3.7)
\]

Now, by virtue of
\[
\langle s | A^\dagger(s)A(s)|s\rangle = [s - \frac{\delta^2}{2} s(s - 1)] \langle s - 1 | s - 1 \rangle \geq 0,
\] (3.8)

we see that the sequence of $s$ and thus $|s\rangle$ should truncate. The only possibility left is to set
\[
\delta^2 = \frac{1}{s_{\text{max}}},
\] (3.9)

Indeed, by (3.1) we then have
\[
A(s_{\text{max}}) = A^\dagger(s_{\text{max}}) = \frac{1}{\sqrt{2}} \hat{x}.
\] (3.10)

Using this and (3.7), we find
\[
|s_{\text{max}} + 1\rangle = |s_{\text{max}} - 1\rangle,
\] (3.11)

where $|s_{\text{max}} + 1\rangle = A^\dagger(s_{\text{max}})|s_{\text{max}}\rangle$. We have thus shown that instead of $\delta$ we can use the parameter $s_{\text{max}}$ exceeding by one the dimension of the system of vectors $\{|s\rangle\}_{0 \leq s \leq s_{\text{max}}}$. Such systems for $s_{\text{max}} = 1$, $s_{\text{max}} = 2$ and so on, can be interpreted as a finite-dimensional analogues of the usual infinite-dimensional Fock space. The latter evidently refers to the case with $s_{\text{max}} = \infty$, when $\delta = 0$.

We now discuss the algebra satisfied by the operators (3.1), that is the $\delta$-deformation of the Heisenberg-Weyl algebra. Taking into account (3.7) we get
\[
A(s') = \left[1 - \frac{\delta^2(s - s')}{2(\delta^2 s - 1)}\right] A(s) + \frac{\delta^2(s - s')}{2(\delta^2 s - 1)} A^\dagger(s), \quad \delta^2 s, \quad s < s_{\text{max}}.
\] (3.12a)

\[
A^\dagger(s') = \frac{\delta^2(s - s')}{2(\delta^2 s - 1)} A(s) + \left[1 - \frac{\delta^2(s - s')}{2(\delta^2 s - 1)}\right] A^\dagger(s),
\] (3.12b)

Making use of (2.4), (3.1), (3.12) and the following form of the Casimir (2.7), which can be easily derived with the help of (3.1):
\[
\delta^2[A(s) - A^\dagger(s)]^2 = 2(1 - \delta^2 s)(1 - I^{-2}_\delta),
\] (3.13)
we arrive at the commutation relations such that

\[ [A(s), A^\dagger(s')] = (1 - \frac{\delta^2}{2}(s + s'))I_\delta^{-2}, \]
\[ [A(s), A(s')] = [A^\dagger(s'), A^\dagger(s)] = \frac{\delta^2}{2}(s' - s)I_\delta^{-2}, \quad s, s' \leq s_{\text{max}}, \]  
(3.14a)
\[ [A(s), I_\delta^{-2k}] = [A^\dagger(s), I_\delta^{-2k}] = [A(s_{\text{max}}), I_\delta^{-2k}] = k \frac{\delta^2}{1 - \delta^2 s} B_k(s), \quad s < s_{\text{max}}, \]  
(3.14b)
\[ [A(s), I_\delta^{-2l}] = [A^\dagger(s), I_\delta^{-2l}] = [B_k(s), B_l(s')] = 0, \quad s, s' \leq s_{\text{max}}, \quad k, l = 1, 2, \ldots, \]  
(3.14c)
\[ [A(s), B_k(s')] = [A^\dagger(s), B_k(s')] = 2k(1 - \delta^2 s')I_\delta^{-2k} - (2k + 1)(1 - \delta^2 s')I_\delta^{-2(k+1)}, \]  
(3.14d)
where \( B_k(s) = [A(s) - A^\dagger(s)]I_\delta^{-2k} \). We remark that due to the commutator (3.14d) the algebra (3.14) is infinite dimensional. It should also be noted that in view of the following relation:

\[ A(s) = (1 - \frac{\delta^2}{2})A(0) + \frac{\delta^2}{2} A^\dagger(0), \quad 0 \leq s \leq s_{\text{max}}, \]  
(4.1)
which is an immediate consequence of (3.1), \( A(s), A^\dagger(s) \) and \( B_k(s) \) can be regarded as a discrete curve in the algebra generated by \( A(0), A^\dagger(0), I_\delta^{-2k} \) and \( B_k(0) \) of the form

\[ [A(0), A^\dagger(0)] = I_\delta^{-2}, \]  
(4.16a)
\[ [A(0), I_\delta^{-2k}] = [A^\dagger(0), I_\delta^{-2k}] = k \delta^2 B_k(0), \]  
(4.16b)
\[ [A(0), B_k(0)] = [A^\dagger(0), B_k(0)] = 2k I_\delta^{-2k} - (2k + 1) I_\delta^{-2(k+1)}, \]  
(4.16c)
\[ [B_k(0), I_\delta^{-2l}] = [I_\delta^{-2k}, I_\delta^{-2l}] = [B_k(0), B_l(0)] = 0, \quad k, l = 1, 2, \ldots. \]  
(4.16d)
Of course, both (3.14) and (4.16) reduce to the Heisenberg-Weyl algebra in the limit \( \delta \to 0 \), that is \( s_{\text{max}} \to \infty \).

4. The \( \delta \)-deformation of the Fock space

We now discuss the \( \delta \)-deformation of the Fock space expressed by (3.7) in a more detail. We first observe that the generation of the states \( |s\rangle \), with \( s \geq 1 \), from the “vacuum vector” \( |0\rangle \) can be described with the help of the second equation of (3.7) by

\[ |s\rangle = \left( \prod_{s' = 0}^{s-1} \frac{1}{\sqrt{s' + 1 - \frac{\delta^2}{2} s'(s' + 1)}} \right) A^\dagger(s-1) \cdots A^\dagger(1) A^\dagger(0)|0\rangle, \quad 0 < s \leq s_{\text{max}}. \]  
(4.1)
The vectors \( |s\rangle \) are not orthonormal. In fact, using (3.12a) with \( s' = s + 1 \), and (3.7) we find

\[ \delta^2 \sqrt{s + 1 - \frac{\delta^2}{2} s(s + 1)} \sqrt{s - \frac{\delta^2}{2} s(s - 1)} \langle s - 1|s + 1 \rangle = -2(\delta^2 s - 1)[s + 1 - \frac{\delta^2}{2} s(s + 1)] \langle s|s \rangle \]
\[ + [\delta^2 + 2(\delta^2 s - 1)][s + 1 - \frac{\delta^2}{2} s(s + 1)] \langle s + 1|s + 1 \rangle. \]  
(4.2)
Further, calculating the expectation value of the Casimir (3.13) in the state $|s\rangle$ with the use of (3.14a) for $s = s'$, and taking into account (4.2), we obtain

\[ \delta^2 \sqrt{s + 1 - \frac{\delta^2}{2} s(s + 1)} \sqrt{s - \frac{\delta^2}{2} s(s - 1)} \langle s + 1|s - 1 \rangle = 2(\delta^2 - 1)[s - \frac{\delta^2}{2} s(s - 1)]\langle s|s \rangle + [\delta^2 - 2(\delta^2 s - 1)][s - \frac{\delta^2}{2} s(s - 1)]\langle s - 1|s - 1 \rangle. \]  

(4.3)

Equating right-hand sides of (4.2) and (4.3) we finally arrive at the following recursive formula on the squared norm of $|s\rangle$:

\[ 2(\delta^2 s - 1)(2s + 1 - \delta^2 s^2)\langle s|s \rangle + [\delta^2 - 2(\delta^2 s - 1)][s - \frac{\delta^2}{2} s(s - 1)]\langle s - 1|s - 1 \rangle - [\delta^2 + 2(\delta^2 s - 1)][s + 1 - \frac{\delta^2}{2} s(s + 1)]\langle s + 1|s + 1 \rangle = 0, \quad s \leq s_{\text{max}}. \]  

(4.4)

A straightforward calculation shows that the recurrence (4.4) can be written in a more convenient form such that

\[ \langle 1|1 \rangle = \frac{2}{2 - \delta^2} \langle 0|0 \rangle, \]  

(4.5a)

\[ \langle s|s \rangle = \frac{\delta^2}{2(\delta^2 s - 1) - \delta^2}[s - \frac{\delta^2}{2} s(s - 1)] \sum_{s' = 0}^{s - 2} (\delta^2 s' - 1)\langle s'|s \rangle \]

\[ + \left( 1 + \frac{\delta^2[\delta^2(s - 1) - 1]}{2(\delta^2 s - 1) - \delta^2}[s - \frac{\delta^2}{2} s(s - 1)] \right) \langle s - 1|s - 1 \rangle, \quad 2 \leq s \leq s_{\text{max}}. \]  

(4.5b)

Finally, eqs. (3.12a) and (3.7) taken together yield

\[ \sqrt{s - \frac{\delta^2}{2} s(s - 1)}\langle s|s' \rangle = \left[ 1 - \frac{\delta^2(s' - s + 1)}{2(\delta^2 s' - 1)} \right] \sqrt{s' - \frac{\delta^2}{2} s'(s' - 1)}\langle s - 1|s' - 1 \rangle \]

\[ + \frac{\delta^2(s' - s + 1)}{2(\delta^2 s' - 1)} \sqrt{s' + 1 - \frac{\delta^2}{2} s'(s' + 1)}\langle s - 1|s' + 1 \rangle, \quad 0 < s \leq s_{\text{max}}, 0 \leq s' < s_{\text{max}}. \]  

(4.6)

The equations (4.5) and (4.6) form the closed system which enables to calculate the inner product $\langle s|s' \rangle$ for arbitrary $s, s' \leq s_{\text{max}}$. In particular, utilizing the relation

\[ \langle s|s + 1 \rangle = 0, \quad s \leq s_{\text{max}}, \]  

(4.7)

implied by (4.6) and using recursively (4.6) we find that

\[ \langle s|s' \rangle = 0, \quad s, s' \leq s_{\text{max}}, \]  

(4.8)

where $s$ is even and $s'$ is odd.

We finally discuss the concrete realization of the introduced $\delta$-deformation of the abstract Fock space in the representation (2.19). On using (2.20) and (3.7) we arrive at the following system:

\[ \left[ \frac{d}{d\varphi} + (1 - \delta^2 s)\frac{1}{\delta} \tan \varphi \delta \right] f_s(\varphi) = -i\sqrt{2} \sqrt{s - \frac{\delta^2}{2} s(s - 1)} f_{s-1}(\varphi), \]  

(4.9a)

\[ \left[ \frac{d}{d\varphi} - (1 - \delta^2 s)\frac{1}{\delta} \tan \varphi \delta \right] f_s(\varphi) = -i\sqrt{2} \sqrt{s + 1 - \frac{\delta^2}{2} s(s + 1)} f_{s+1}(p), \]  

(4.9b)
where \( f_s(\varphi) = \langle \varphi | s \rangle \). We remark that the system (4.9) is the special case of the more general one

\[
\begin{align*}
\left[ \frac{d}{d\varphi} + k(s, \varphi) \right] f_s(\varphi) &= -i\mu(s) f_{s-1}(\varphi), \\
\left[ \frac{d}{d\varphi} - k(s, \varphi) \right] f_s(\varphi) &= -i\nu(s) f_{s+1}(\varphi).
\end{align*}
\] (4.10a)

It can be easily checked that (4.10) is equivalent to

\[
\begin{align*}
\left[ \frac{d}{dx} + k(s, x) \right] y_s(x) &= \mu(s) y_{s-1}(x), \\
\left[ -\frac{d}{dx} + k(s, x) \right] y_s(x) &= \nu(s) y_{s+1}(x).
\end{align*}
\] (4.11a)

The system (4.11) was studied by Jannussis et al.\[3\] in the context of the generalization of the Infeld-Hull method of factorization in the case of the harmonic oscillator. Analyzing the compatibility of the two second order differential equations implied by (4.11) they showed that besides the periodic solution there exists the following one:

\[
k(s, x) = a\text{ctg}(ax + \theta) s - \frac{b}{a}\text{ctg}(ax + \theta) + \frac{c}{\sin(ax + \theta)},
\] (4.12)

provided

\[
\mu(s)\nu(s - 1) = -a^2 s(s - 1) + 2bs + \lambda,
\] (4.13)

where \( a, b, c, \theta \) and \( \lambda \) are arbitrary constants. A look at (4.12), (4.13), (4.9) and (3.5) is enough to conclude that the actual treatment refers to the case with \( a = \delta, b = 1, c = 0, \theta = \pi/2 \) and \( \lambda = 0 \). We point out that within the formalism introduced herein the second order equations implied by (4.9) are simply the realization of the abstract equations

\[
\begin{align*}
A(s + 1)A^\dagger(s)|s\rangle &= \beta(s)\alpha(s + 1)|s\rangle, \\
A^\dagger(s - 1)A(s)|s\rangle &= \alpha(s)\beta(s - 1)|s\rangle.
\end{align*}
\] (4.14a)

in the representation (2.19). The compatibility of the eqs. (4.14) is ensured by the Casimir (3.3). In this sense the actual approach can be interpreted as an abstract form of the Infeld-Hull factorization method.

We now return to (4.9). Using (4.9a) and the limit

\[
\lim_{\delta \to 0} (\cos \delta \varphi)^{\frac{1}{\delta}} = e^{-\frac{\varphi^2}{2}},
\] (4.15)

we find

\[
f_0(\varphi) = \pi^{-\frac{1}{4}} (\cos \delta \varphi)^{\frac{1}{\delta}}.
\] (4.16)
Furthermore, utilizing (4.9b) and
\[
\frac{d}{d\phi} (\cos \delta \phi)^{\frac{1}{2 \pi}} = -\frac{\tan \delta \phi}{\delta} (\cos \delta \phi)^{\frac{1}{2 \pi}}, \quad \frac{d}{d\phi} \left(\frac{\tan \delta \phi}{\delta}\right) = 1 + \delta^2 \left(\frac{\tan \delta \phi}{\delta}\right)^2, \tag{4.17}
\]
we get
\[
f_s(\phi) = \frac{\pi^{-\frac{1}{4}}(-i)^s}{(\sqrt{2})^s} \left(\prod_{s'=0}^{s-1} \frac{1}{\sqrt{s' + 1 - \frac{\delta^2}{2} s'(s' + 1)}}\right) H_s^{(\delta)} \left(\frac{\tan \delta \phi}{\delta}\right) (\cos \delta \phi)^{\frac{1}{2 \pi}}, \tag{4.18}
\]
where \(1 \leq s \leq s_{\text{max}}\), and \(H_s^{(\delta)}(x)\) are the polynomials satisfying the recurrence
\[
H_{s+1}^{(\delta)}(x) = (2 - \delta^2 s) x H_s^{(\delta)}(x) - (1 + \delta^2 x^2) H_s^{(\delta)'}(x),
\]
\[
H_0^{(\delta)}(x) = 1, \tag{4.19}
\]
where the prime designates the differentiation with respect to \(x\). Of course, \(H_s^{(\delta)}(x)\) are simply the \(\delta\)-deformation of the usual Hermite polynomials refering to the limit \(\delta \to 0\), i.e. \(s_{\text{max}} \to \infty\). The first few \(\delta\)-deformed Hermite polynomials are of the form
\[
H_0^{(\delta)}(x) = 1,
\]
\[
H_1^{(\delta)}(x) = 2x,
\]
\[
H_2^{(\delta)}(x) = 4(1 - \delta^2)x^2 - 2,
\]
\[
H_3^{(\delta)}(x) = 8(1 - \delta^2)(1 - 2\delta^2)x^3 - 12(1 - \delta^2)x,
\]
\[
H_4^{(\delta)}(x) = 16(1 - \delta^2)(1 - 2\delta^2)(1 - 3\delta^2)x^4 - 48(1 - \delta^2)(1 - 2\delta^2)x^2 + 12(1 - \delta^2). \tag{4.20}
\]
As with the standard Hermite polynomials the general formula on the \(\delta\)-deformed ones can be derived such that
\[
H_0^{(\delta)}(x) = 1,
\]
\[
H_s^{(\delta)}(x) = \sum_{j=0}^{[\frac{s}{2}]} (-1)^j \frac{s!}{j!(s - 2j)!} 2^{s-2j} \left[\prod_{s'=0}^{s-j-1} (1 - \delta^2 s')\right] x^{s-2j}, \quad 1 \leq s \leq s_{\text{max}}, \tag{4.21}
\]
where \([y]\) is the biggest integer in \(y\).

We finally write down the following formula on the matrix elements \(\langle s | s' \rangle\) implied by (2.18) and (4.18):
\[
\langle s | s' \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_s^*(\varphi) f_{s'}(\varphi) d\varphi, \tag{4.22}
\]
where \(f_s(\varphi)\) is given by (4.16) and (4.18). The calculation of the integral from (4.22) for arbitrary \(s, s'\) seems to be more complicated than the solution of the recurrences (4.5) and (4.6). It should be noted however that (4.22) enables to calculate the squared norm.
of the “vacuum vector” \( |0\rangle \) parametrizing solutions of (4.5) and (4.6). Namely, we find

\[
\langle 0|0 \rangle = \frac{1}{2\pi^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos \varphi) \pi^2 d\varphi = \sqrt{\frac{s_{\text{max}}}{\pi}} \frac{(2s_{\text{max}} - 1)!!}{(2s_{\text{max}})!!} = \sqrt{s_{\text{max}} \frac{\pi}{2}} \frac{\Gamma(s_{\text{max}} + \frac{1}{2})}{\Gamma(s_{\text{max}} + 1)},
\]

(4.23)

where \( \delta^2 s_{\text{max}} = 1 \) and \( \Gamma(x) \) is the gamma function.

5. Conclusion

We have introduced in this work the deformation of the Fock space based on the utilization of the central difference operator instead of the usual derivative. It should be mentioned that there exist alternative approaches for discretization of quantum mechanics relying on finite difference representations of the usual Heisenberg [4] or Heisenberg-Weyl algebra [5]. Nevertheless, the general problem with them is the interpretation of the nonequivalence of the obtained representations of the canonical commutation relations and the standard Schrödinger one. Some problems with the spectrum of operators within such approaches have been also reported [4]. We also recall the discretization of the harmonic oscillator introduced in [6] relying on the replacement of the Hermite polynomials with the Kravchuk polynomials in a discrete variable as well as the finite-dimensional counterpart of the Fock space spanned by the eigenvectors of the phase operator discussed in [7]. In analogy with the actual treatment in both approaches taken up in [6] and [7] the standard infinite-dimensional Fock space refers to the formal limit \( N \to \infty \), where \( N \) is dimension of the finite-dimensional discrete version of the Fock space. Moreover, in the case with the discretization described in [6] one can recognize a counterpart of the parameter \( \delta \) specified by (3.9) such that \( \delta \simeq N^{-\frac{1}{2}} \). Nevertheless, besides of those similarities we have also serious differences. For example, in opposition to the operators (3.1) the generalizations of the Bose operators introduced in [6] do not depend on the index labelling the basis of the finite-dimensional analogue of the Fock space. On the other hand, the alternatives to the number states discussed in [7] form the orthonormal set. This is not the case for the states \( |s\rangle \) described herein. Last but not least we point out that besides of quantum mechanics the results of this paper would be of importance in the theory of differential equations. We only recall the abstract form of the Infeld-Hull method of factorization described by the equations (4.14) and (3.3).

Here we briefly discuss the basic properties of the \( e(2) \) algebra. Consider the \( e(2) \) algebra

\[
[J, X] = iY, \quad [J, Y] = -iX, \quad [X, Y] = 0.
\]

(A.1)
The Casimir operator for (A.1) is of the following form:

\[ X^2 + Y^2 = r^2. \]  

(A.2)

Making use of (A.1) and (A.2) we arrive at the following form of the algebra (A.1):

\[ [J, U] = U, \]  

(A.3)

where

\[ U = \frac{1}{r}(X + iY) \]  

(A.4)

is unitary. Consider the eigenvalue equation

\[ J|j\rangle = j|j\rangle. \]  

(A.5)

From equations (A.3) and (A.5) it follows that the operators \( U \) and \( U^\dagger \) act on the vectors \( |j\rangle \) as the rising and lowering operator, respectively, that is

\[ U|j\rangle = |j + 1\rangle, \quad U^\dagger |j\rangle = |j - 1\rangle. \]  

(A.6)

Taking into account (A.6) we find that the whole basis \( |j\rangle \) of the Hilbert space of states can be generated from the unique “vacuum vector” \( |j_0\rangle \), where \( j_0 \in [0, 1] \). The non-equivalent irreducible representations of the commutation relations (A.3) are labelled by different \( j_0 \). We remark that the algebra (A.3) is the most natural for the study of a quantum particle on a circle [5]. In such a case \( J \) represents the angular momentum and the unitary operator \( U \) describes the position of a particle on a unit circle. We now demand the time-reversal invariance of the algebra (A.3). Having in mind the interpretation of \( J \) as the angular momentum this leads to

\[ T J T^{-1} = -J, \]  

(A.7)

\[ T U T^{-1} = U^{-1}, \]  

(A.8)

where \( T \) is the anti-unitary operator of time inversion. Using (A.5)–(A.8) we obtain

\[ T|j\rangle = |-j\rangle. \]  

(A.9)

As an immediate consequence of (A.9), we find that \( T \) is well defined on the Hilbert space of states generated by the vectors \( |j\rangle \) if and only if the spectrum of \( J \) is symmetric with respect to zero. Hence, in view of (A.6) the only possibility left is \( j_0 = 0 \) or \( j_0 = \frac{1}{2} \). Obviously, \( j_0 = 0 \) \((j_0 = \frac{1}{2})\) implies integer (half-integer) eigenvalues \( j \).

However, in this work we interpret \( J \) as the position operator for a quantum particle on a lattice. Accordingly, the operator \( T \) from (A.7) should be replaced with a unitary parity operator \( P \) and the invariance of (A.3) under parity transformation demanded. In that case the relations (A.7) and (A.8) (with \( T \) replaced by \( P \)) and their consequences (\( j \) integer or half-integer) remain unchanged.
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