The transition from a classical to a quantum world as a passage from extensive to non-extensive thermodynamics

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We study the thermodynamical properties of the quantum kicked rotator, coarsened by an external fluctuation with a weak intensity \(D\), by means of the Tsallis entropy with a changing entropic index \(q\). The genuine entropic index, corresponding to given values of \(D\) and \(\hbar\) is that making the Tsallis entropy increase linearly in time, and it is proved to become \(q < 1\) for suitably large values of \(\hbar\): This indicates a subdiffusional regime which, in turn, signals the occurrence of quantum localization. Thus the process of Anderson localization is shown to be compatible with a thermodynamical representation provided that a non-extensive form of entropy is used.

As an important benefit this entropy makes it possible to extend the KS method of numerical analysis to the fractal dynamics. The authors of Ref. \(^{11}\) generalized the KS entropy by replacing the conventional Shannon expression, formally equivalent to the structure proposed by Gibbs, with the non-extensive form of Tsallis. In the special case \(q = 1\), when Eq. (2) yields Eq. (3), the generalized KS entropy, denoted by \(K_q\), becomes identical to the conventional KS entropy.

We establish a connection between \(K_q\) and the generalized entropy of Eq. (2) with the same coarsening procedure as that originally proposed by Zurek and Paz. However, to adapt it to the special case of fractal dynamics we found to be convenient to use the intuitive picture by Zaslavsky. We adopt an initial condition compatible with classical physics so as to generate a trajectory that will explore in due time a given region of the phase space according to the prescriptions of classical mechanics. Let us call \(R(t)\) the portion of this region not yet visited at the given time \(t\) by the classical trajectory. It is evident that in general this quantity will be a decreasing function of time. For generality, on the basis of the arguments used in Ref. \(^{11}\) to generalize the KS entropy, we express the general form of this time evolution as:

\[
R(t) = \frac{R(0)}{[1 + \lambda_q(1 - q)t]^{\frac{3}{3 - q}}},
\]

where \(\lambda_q\) denotes a generalized Lyapunov coefficient. Let us imagine that at the very same moment when the initial conditions are established a weak fluctuation process...
of intensity $D$ is switched on. In the absence of fluctuations a set of trajectories, with the same initial condition, will behave as a single trajectory, even if this single trajectory will be made to look erratic-like by deterministic chaos. The action of an even extremely weak fluctuation, in the presence of chaos, will force the trajectories of this set to spread over the proper phase-space region even if all of them are assumed to be given exactly the same initial condition. This weak stochastic force serves the purpose of defining in a precise way the time at which the exploration of the phase space can be regarded as being completed. We call this time $t_{CG}$ and we define it by means of the following relation:

$$R(t_{CG}) =Dt_{CG}. \quad (4)$$

We note that the solution of this fundamental equation depends on the nature of the dynamical process under study. In the case of strong chaos [11] we have $q = 1$ and, consequently,

$$t_{CG} \approx \frac{2}{\lambda_1} \log(1/\sqrt{D}), \quad (5)$$

which coincides with the estimate made by Zurek and Paz to determine the onset time of thermodynamics. However, in the case of highly correlated dynamical processes, with $q < 1$ as in the case here under study, we obtain from Eq. (4)

$$t_{CG} \approx \frac{1}{D^{q}}, \quad (6)$$

where

$$\theta = (1 - q)/(2 - q). \quad (7)$$

This result gives the misleading impression that the thermodynamical regime is dramatically postponed in the case of dynamical processes with no time scale. It is important to stress that this postponement only concerns the onset of ordinary thermodynamics. In fact, a weak stochastic force perturbing the dynamics with long-range correlations responsible for anomalous diffusion has the key effect, as shown in Ref. [13], of producing a crossover from anomalous to ordinary diffusion at the time scale of Eq. (4). However, according to the perspective established by Tsallis [10], also the regime of anomalous diffusion is given a thermodynamical significance if the proper entropic index $q \neq 1$ is used. Therefore we expect that the coarsening stochastic force producing the crossover from anomalous to ordinary diffusion at the time scale of Eq. (4) results in the fast increase of the entropy $S_q$ of Eq. (4). This means that a transition from the regime of Eq. (4) to that of Eq. (5) (with $q \neq 1$) occurs and consequently that the entropy increases with the rate $1/\lambda_q$.

In this letter we support this conjecture in a case where the origin of long-range correlation is quantum. As pointed out by Zurek and Paz [7], the time scale of Eq. (5) has to be compared to

$$t_Q \approx \frac{1}{\lambda} \log(\frac{1}{\hbar}), \quad (8)$$

which corresponds to the onset of long-range quantum correlations. The transition from quantum to classical physics takes place when $\sqrt{D}$ is made larger than $\hbar$. Rather than focusing our attention on values of $\hbar$ so small as to ensure the classicality condition [6], we explore the transition region, and the long-range correlations of the region $\hbar > \sqrt{D}$. The case study is given by the quantum kicked rotor. This means the Hamiltonian

$$H(t) = -\frac{\hbar^2}{2T} \frac{\partial^2}{\partial \theta^2} + \epsilon \cos \theta \sum_{n=-\infty}^{\infty} \delta(t - nT), \quad (9)$$

where $I$ is the rotor moment of inertia and $\theta$ is the rotor angular coordinate. This Hamiltonian results in the quantum map

$$\langle \theta | \phi(n+1) \rangle = \exp(i \frac{\hbar T}{2} \frac{\partial^2}{\partial \theta^2}) \exp(-i \epsilon \cos \theta) \langle \theta | \phi(n) \rangle, \quad (10)$$

implying the classical control parameter $K$ [14] to be expressed as: $K = \frac{\epsilon}{\sqrt{D}}k$

Here we adopt two different numerical methods to evaluate the entropy time evolution. The former is the same as that designed by Pattanayak and Brumer, based on a Fourier transform technique [6]. The latter, henceforth referred to as a statistical ensemble (SE) method, is based on the study of the time evolution of $N$ independent systems, all of them driven by Eq. (4), with a random ingredient artificially added by replacing $\exp\left(-i \frac{\epsilon^2}{2T} \right)$ with

$$\exp\left(-i \frac{\epsilon F(n)}{2\hbar} \right),$$

where $F(n)$ is a computer generated random process. This computed generated random process is introduced to play the same coarsening role as the earlier mentioned stochastic force of the picture of Zurek and Paz [7]. For this reason the variance of this random process at $t_Q$ is called $D$, is assigned very small values and is referred to throughout as intensity of the stochastic force. The statistical density matrix is then evaluated by means of the ordinary statistical average:

$$\rho(n) = \frac{1}{N} \sum_{i} \langle \phi_i(n) | \phi_i(n) \rangle, \quad (11)$$

usually setting $N = 10$. The same method has been used in an earlier paper [13] and proved to produce results equivalent to those of the literature on this subject. It is important to stress that the ingenious Fourier method of Ref. [6] establishes a connection with the system dynamics unperturbed by the stochastic force, much in the spirit of the linear response theories. The method here adopted, on the contrary, allows us to go beyond the second-order approximation, and makes it possible to detect, as we shall see, the saturation effects determined by the steady action of the stochastic force.
In Fig. 1 we show the increase of Rényi entropy as a function of time for different values of \( h \). This calculation was done using the same method as that of Ref. [8] with a choice of parameters corresponding to the quantum condition that we plan to explore by using the SE method. We checked that the SE method yields the same results as those illustrated in Fig. 1, if the same parameters are used and we set \( \sqrt{D} = 0.002 \). We note that Pattanayak and Brumer [8] considered also values of \( h \) much smaller than the value \( h = 0.01 \) here considered. This is the reason why we depart from the regime of exponential increase, searched and found by those authors, and we rather get the power law increase: \( S(t) = \text{const} \ t^\alpha \) with \( \alpha \approx 2.5 \). This property is compatible with the numerical results obtained by Pattanayak and Brumer at their large values of \( h \). These authors, however, seemingly did not consider this behavior to be thermodynamically relevant. On the basis of our earlier remark, on the contrary, we are here in a position to disclose the thermodynamical nature of this regime, in spite of the fact that one might judge it dominated by the quantum mechanical coherence, and by the Anderson quantum localization [8] as well.

We are convinced that at \( h = 0.01 \) we are in the presence of a condition equivalent to that of fractal dynamics with no time scale, and thus corresponding to an entropic index \( q \neq 1 \). However, with increasing the intensity of the stochastic force we can provoke a transition from quantum to classical physics which, as earlier pointed out, must take place at \( \sqrt{D} = h \). Fig. 2 shows the following very remarkable property. At small values of \( D \) the curve \( S_G(t) \) is characterized by a positive second-order derivative with respect to time (convex curve). With increasing the noise intensity a transition from concavity to convexity is produced at about the value of \( D \) (\( \sqrt{D} \approx 0.01 \)) corresponding to the passage from quantum to classical physics. We assume that the transition from convexity to concavity has a thermodynamical relevance, and this leads us to find still more interesting results by plotting the Tsallis entropy as a function of time for different values of the entropic index \( q \). This is shown in Fig. 3, which illustrates the remarkable property that decreasing \( q \) has effects similar to those obtained by increasing \( D \) (see Fig. 4). The critical value of \( q \) corresponding to this transition is \( q \approx 0.33 \).

To make our finding still more impressive we plot in Fig. 4 the saturation time \( t_S \) as a function of the noise intensity \( D \). To derive the results of Fig. 4 we defined the saturation time as the time at which the curve \( S_G(t) \) reaches the 50% of its maximum value. The remarkable fact illustrated by this figure is that the full line corresponds to the theoretical prediction of Eq. (6) with the index \( \theta \) given by Eq. (7) and the index \( q \) determined by the critical value established by Fig. 3. As we have seen,
this means $q \approx 0.33$, and, through Eq. (6), $\theta \approx 0.4$, which is in fact the power index of the full curve of Fig. 4. Note that the choice made to derive Eq. (6) is to some extent arbitrary. However, the key index $\theta$ can also be derived from $\theta = 1/\alpha$, where $\alpha$ refers to the result of Fig. 4. $S(t) = \text{cons} \ t^\theta$ with $\alpha \approx 2.5$, coinciding with the earlier prediction. We also note that after the transition to the classical regime the decrease of $D$ becomes faster, as made necessary by the passage from the regime of Eq. (3) to that of Eq. (5).

On the basis of the earlier arguments it is not a surprise that a condition compatible with the generalized expression of the KS entropy is found at $q \neq 1$. One might wonder, however, why $q < 1$. This can be easily explained by adopting the entropic arguments of Tsallis illustrated in Refs. [11] and [17]. In [17] the subdiffusional character corresponding to the entropic index $q$ is proved by means of a nonlinear Fokker-Planck equation. However, there are good reasons to believe that this is a quite general property of the entropic index $q$ (see also [19]). Thus we conclude that the fundamental results illustrated by Fig. 3 and Fig. 4 signal a sort of quantum-mechanically induced subdiffusion. This conclusion agrees very well with the well known fact that the quantum mechanical kicked rotor is characterized by the phenomenon of Anderson localization [4]. It is well known that the localization process has a rate proportional to about $\hbar^2$. As a consequence we expect that the entropic index $q$ as a function of $\hbar$ tends to 1 with decreasing $\hbar$, in agreement with the numerical observation that the values $h = 0.10, 0.05$ and 0.01 yield $q = 0.28, 0.30$ and 0.33, respectively. We thus reach the interesting conclusion that even the process of Anderson localization is compatible with a thermodynamical treatment, provided that a proper thermodynamical indicator is adopted: This is the Tsallis entropy. Furthermore the adoption of the SE method of calculation makes it possible to show that if the Liouville density of Eq. (2) is coarsened by means of a weak stochastic force and the proper choice of the entropic index $q$ is made, then the time derivative of $S_q(t)$ becomes constant, as prescribed by the generalization of the KS entropy proposed by the authors of Ref. [11].

FIG. 4. The saturation time $t_S$ as a function of the noise strength $\sqrt{D}$. The full line corresponds to the prediction of Eqs. (3) and (5) with the critical entropic index $q$ determined by the numerical results illustrated in Fig. 3.

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[19] In Ref. [11] it was shown that the adoption of the Tsallis entropy yields a natural generalization of the KS entropy, and this, in turns yields the important result: $\lim_{\Delta x(0) \to 0} \frac{\Delta x(t)}{\Delta x(0)} = [1 + \lambda_q t(1 - q)]^{1/\theta}$. Note that this prescription for $q > 1$ implies a divergence at finite time, and, consequently a departure of initially very close trajectories from one another faster that an exponential departure. The case $q < 1$ yields a power law dependence on time, which can be interpreted as slower than the exponential departure. It is well known that the case $q = 1$ results in ordinary Brownian diffusion. In Ref. [18] it was shown that the case $q > 1$ leads to superdiffusion. Consequently, the case $q < 1$, with a departure of the trajectories slower than exponential, ought to result, in general, in subdiffusion, regardless whether the nonlinear diffusion equation of Ref. [17] applies or not.