1. Introduction

Let \( P \subset \mathbb{Z}^d \otimes \mathbb{Q} \) be a convex lattice polytope of dimension \( d \). Let \( L_P(k) := |kP \cap \mathbb{Z}^d| \) count the number of lattice points in dilations \( kP \), \( k \in \mathbb{Z}_{\geq 0} \). Ehrhart [9] showed that \( L_P \) can be written as a degree \( d \) polynomial

\[
L_P(k) = c_d k^d + \ldots + c_1 k + c_0
\]

which we call the Ehrhart polynomial of \( P \). The leading coefficient \( c_d \) is given by \( \text{Vol}(P)/d! \), \( c_{d-1} \) is equal to \( \text{Vol}(\partial P)/(2(d-1))! \), and \( c_0 = 1 \). Here \( \text{Vol}(\cdot) \) denotes the normalised volume, and \( \partial P \) denotes the boundary of \( P \). For example, if \( P \) is two-dimensional (that is, \( P \) is a lattice polygon) we obtain

\[
L_P(k) = \frac{\text{Vol}(P)}{2} k^2 + \frac{|\partial P \cap \mathbb{Z}^2|}{2} k + 1.
\]

Setting \( k = 1 \) in this expression recovers Pick’s Theorem [16]. The values of the Ehrhart polynomial of \( P \) form a generating function \( \text{Ehr}_P(t) := \sum_{k \geq 0} L_P(k) t^k \) called the Ehrhart series of \( P \).

When the vertices of \( P \) are rational points the situation is more interesting. Recall that a quasi-polynomial with period \( s \in \mathbb{Z}_{>0} \) is a function \( q : \mathbb{Z} \to \mathbb{Q} \) defined by polynomials \( q_0, q_1, \ldots, q_{s-1} \) such that

\[
q(k) = q_i(k) \quad \text{when } k \equiv i \mod s.
\]

The degree of \( q \) is the largest degree of the \( q_i \). The minimum period of \( q \) is called the quasi-period, and necessarily divides any other period \( s \). Ehrhart showed that \( L_P \) is given by a quasi-polynomial of degree \( d \), which we call the Ehrhart quasi-polynomial of \( P \). Let \( \pi_P \) denote the quasi-period of \( P \). The smallest positive integer \( r_P \in \mathbb{Z}_{>0} \) such that \( r_P P \) is a lattice polytope is called the denominator of \( P \). It is certainly the case that \( L_P \) is \( r_P \)-periodic, however it is perhaps surprising that the quasi-period of \( L_P \) does not always equal \( r_P \); this phenomenon is called quasi-period collapse.

Example 1.1 (Quasi-period collapse). Consider the triangle \( P := \text{conv} \{ (5, -1), (-1, -1), (-1, 1/2) \} \) with denominator \( r_P = 2 \). This has \( L_P(k) = 9/2k^2 + 9/2k + 1 \), hence \( \pi_P = 1 \).

Quasi-period collapse is poorly understood, although it occurs in many contexts. For example, de Loera–McAllister [7, 8] consider polytopes arising naturally in the study of Lie algebras (the Gel’fand–Tsetlin polytopes and the polytopes determined by the Clebsch–Gordan coefficients) that exhibit quasi-period collapse. In dimension two McAllister–Woods [15] show that there exist rational polytopes with \( r_P \) arbitrarily large but with \( \pi_P = 1 \) (see also Example 3.8). Haase–McAllister [10] give a constructive view of this phenomena in terms of \( \text{GL}_d(\mathbb{Z}) \)-scissor congruence; here a polytope is partitioned into pieces that are individually modified via \( \text{GL}_d(\mathbb{Z}) \) transformation and lattice translation, then reassembled to give a new polytope which (by construction) has equal Ehrhart quasi-polynomial but different \( r_P \).
Example 1.2 (GL$_2$(Z)-scissor congruence). The lattice triangle $Q := \text{conv}\{(2, -1), (-1, -1), (-1, 2)\}$ with Ehrhart polynomial $L_Q(k) = 9/2k^2 + 9/2k + 1$ can be partitioned into two rational triangles as depicted on the left below. Fix the bottom-most triangle, and transform the top-most triangle via the lattice automorphism $e_1 \mapsto (3, -1), e_2 \mapsto (4, -1)$. This gives the rational triangle $P$ (depicted on the right) from Example 1.1.

We give an explanation for quasi-period collapse in two dimensions for a certain class of polygons in terms of recent results in algebraic geometry arising from Mirror Symmetry. In §2 we explain how mutation — a combinatorial operation arising from the theory of cluster algebras — gives an explanation of this phenomenon, and explain how this is related to Q-Gorenstein (qG-) deformations of del Pezzo surfaces as studied by Wahl [17], Kollár–Shepherd-Barron [14], Hacking–Prokhorov [11], and others. Finally, in Corollary 3.6 we completely characterise the discrepancy between the denominator and the quasi-period for this class of polygons.

2. Mutation

In [10] Haase–McAllister propose the open problem of finding a systematic and useful technique that implements GL$_d$(Z)-scissor congruence for rational polytopes. In the case when the dual polyhedron is a lattice polytope it was observed in [2] that one such technique is given by mutation.

2.1. The combinatorics of mutation. Let $N \cong \mathbb{Z}^d$ be a rank $d$ lattice and set $N_Q := N \otimes \mathbb{Q}$. Let $P \subset N_Q$ be a lattice polytope. We require — and will assume from here onwards — that $P$ satisfies the following two conditions:

(a) $P$ is of maximum dimension in $N$, $\dim(P) = d$;
(b) the origin is contained in the strict interior of $P$, $0 \in P^\circ$.

Condition (b) is not especially stringent, and can be satisfied by any polytope with $P^\circ \cap N \neq \emptyset$ by lattice translation. It is, however, an essential requirement in what follows.

Let $M := \text{Hom}(N, \mathbb{Z}) \cong \mathbb{Z}^d$ denote the dual lattice. Given a polytope $P \subset N_Q$, the dual polyhedron is defined by

$$P^* := \{ u \in M_Q \mid u(\nu) \geq -1 \text{ for all } \nu \in P \} \subset M_Q.$$  

Condition (b) gives that $P^*$ is a (typically rational) polytope. It is on rational polytopes dual to lattice polytopes that we focus. In this section we will explain how mutation corresponds to a piecewise-GL$_d$(Z) transformation of $P^*$, and hence is an instance of GL$_d$(Z)-scissor congruence for $P^*$.

Following [2, §3], let $w \in M$ be a primitive lattice vector. Then $w : N \rightarrow \mathbb{Z}$ determines a height function (or grading) which naturally extends to $N_Q \rightarrow \mathbb{Q}$. We call $w(\nu)$ the height of $\nu \in N_Q$. We denote the set of all points of height $h$ by $H_{w,h}$, and write

$$w_h(P) := \text{conv}(H_{w,h} \cap P \cap N) \subset N_Q$$

for the (possibly empty) convex hull of lattice points in $P$ at height $h$.

**Definition 2.1.** A factor of $P \subset N_Q$ with respect to $w \in M$ is a lattice polytope $F \subset w^+$ such that for every negative integer $h \in \mathbb{Z}_{\leq 0}$ there exists a (possibly empty) lattice polytope $R_h \subset N_Q$ such that

$$H_{w,h} \cap \text{vert}(P) \subseteq R_h + |h|F \subseteq w_h(P).$$

Here ‘+’ denotes Minkowski sum, and we define $\emptyset + Q = \emptyset$ for every lattice polytope $Q$.

**Definition 2.2.** Let $P \subset N_Q$ be a lattice polytope with $w \in M$ and $F \subset N_Q$ as above. The mutation of $P$ with respect to the data $(w, F)$ is the lattice polytope

$$H_{w,F}(P) := \text{conv}\left( \bigcup_{h \in \mathbb{Z}_{\leq 0}} R_h \cup \bigcup_{h \in \mathbb{Z}_{\leq 0}} (w_h(P) + hF) \right) \subset N_Q.$$
It is shown in [2, Proposition 1] that, for fixed data \((w, F)\), any choice of \(\{R_h\}\) satisfying Definition 2.1 gives \(GL_d(\mathbb{Z})\)-equivalent mutations. Since we regard lattice polytopes as being defined only up to \(GL_d(\mathbb{Z})\)-equivalence, this means that mutation is well-defined. One can readily see that translating the factor \(F\) by some lattice point \(v \in w^+ \cap N\) gives isomorphic mutations: \(\mu_{(w,F+v)}(P) \equiv \mu_{(w,F)}(P)\). In particular if \(\text{dim}(F) = 0\) then \(\mu_{(w,F)}(P) \equiv P\). Finally, we note that mutation is always invertible [2, Lemma 2]: if \(Q := \mu_{(w,F)}(P)\) then \(P = \mu_{(-w,F)}(Q)\).

**Remark 2.3.** Informally, mutation corresponds to the following operation on slices \(w_t(P)\) of \(P\): at height \(h\) one Minkowski adds or “subtracts” \(|h|\) copies of \(F\), depending on the sign of \(h\). Definition 2.1 ensures that the concept of Minkowski subtraction makes sense.

Mutation has a natural description in terms of the dual polytope \(P'\) [2, Proposition 4 and pg. 12].

**Definition 2.4.** The inner-normal fan in \(M_Q\) of a polytope \(F \subset N_Q\) is generated by the cones
\[
\sigma_{v_F} := \{u \in M_Q \mid u(v_F) = \min\{u(v) \mid v \in F\}\}, \quad \text{for each } v_F \in \text{vert}(F).
\]
A mutation \(\mu_{(w,F)}\) induces a piecewise-\(GL_d(\mathbb{Z})\) transformation \(\varphi_{(w,F)}\) on \(M_Q\) given by
\[
\varphi_{(w,F)} : u \mapsto u - u_{\min} w, \quad \text{where } u_{\min} := \min\{u(v_F) \mid v_F \in \text{vert}(F)\}.
\]
The inner-normal fan of \(F\) determines a chamber decomposition of \(M_Q\), and \(\varphi_{(w,F)}\) acts linearly within each chamber. Let \(Q := \mu_{(w,F)}(P)\). Then \(\varphi_{(w,F)}(P') = Q'\). It is clear that the Ehrhart quasi-polynomials \(L_P\) and \(L_Q\) for the dual polytopes are equal, since the map \(\varphi_{(w,F)}\) is piecewise-linear. Hence mutation gives a systematic way to produce examples of \(GL_d(\mathbb{Z})\)-scissor congruence.

**Example 2.5 (Mutation).** Let \(P = \text{conv}\{(1,0), (0,1), (-1,1)\} \subset N_Q\) and \(w = (2, -1) \in M\). Then \(F = \text{conv}\{(0,0), (-1,-2)\} \subset w^+\) is a factor. We see that \(Q := \mu_{(w,F)}(P) = \text{conv}\{(1,0), (0,1), (-1,-4)\}\).

On the dual side we have that \(M_Q\) is divided into two chambers whose boundary is given by \(Q \cdot w\), and
\[
\varphi_{(w,F_1)} : (u_1, u_2) \mapsto \begin{cases} (u_1, u_2), & \text{if } u_1 + 2u_2 \leq 0; \\ (3u_1 + 4u_2, -u_1 - u_2), & \text{otherwise}. \end{cases}
\]

Thus we recover Example 1.2 from the view-point of mutation.

From here onwards we assume that \(P \subset N_Q\) is Fano. That is, in addition to conditions (a) and (b) above, \(P\) satisfies:

(c) the vertices \(\text{vert}(P)\) of \(P\) are primitive lattice points.

The property of being Fano is preserved under mutation [2, Proposition 2]. A Fano polytope \(P\) corresponds to a toric Fano variety \(X_P\) via the spanning fan (that is, the fan whose cones are spanned by the faces of \(P\)). See [6] for the theory of toric varieties and [13] for a survey of Fano polytopes. When \(P\) is a Fano polygon, \(X_P\) corresponds to a toric del Pezzo surface with at worst log terminal singularities. The singularity content
of $P$, which we recall in Definition 2.10 below, is a mutation-invariant of $P$ introduced in [3]. In §2.4 we remark briefly on the connection between singularity content and the qG-deformation theory of $X_P$, and how this gives a geometric explanation for the quasi-period collapse of $P^\ast$.

2.2. Quotient singularities. In order to state the definition of singularity content we first recall some of the theory of quotient or orbifold surface singularities. A cyclic quotient singularity is a surface singularity isomorphic to a quotient $\mathbb{A}^2/G$, where $G$ is a finite cyclic group acting diagonally on $\mathbb{A}^2$. Assuming that $G$ acts faithfully means that it can be expressed as a subgroup of GL$_2(\mathbb{C})$ generated by

$$\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^a \end{pmatrix}$$

where $\epsilon$ is a root of unity and $a \in \mathbb{Z}$. Suppose that $G$ has order $r$; all possible representations are obtained (non-uniquely) by letting $a$ range over $0, \ldots, r-1$. If $G$ is generated by the matrix above for $a$ a primitive $r$-th root of unity, then denote by $\frac{1}{r}(1, a)$ the singularity $\mathbb{A}^2/G$. As a quotient of affine space by an abelian group, $\frac{1}{r}(1, a)$ is an affine toric variety whose fan we now describe.

Let $N \equiv \mathbb{Z}^2$ and $M = \text{Hom}_\mathbb{Z}(N, \mathbb{Z})$ be the cocharacter and character lattices respectively of an algebraic two-torus ($C^\ast)^2$. A cone $\sigma \subset N_\mathbb{Q}$ whose rays are generated by lattice points in $N$ describes an affine toric variety $X_\sigma$. More generally, a collection of cones given by a fan $\Sigma$ describes a non-affine toric variety $X_\Sigma$. The singularity $\frac{1}{r}(1, a)$ is the affine toric variety associated to the cone

$$\sigma = \text{cone}(\epsilon z, r \epsilon^{-1} - a \epsilon z) \subset N_\mathbb{Q}.$$ 

The lattice height of such a cone – that is, the lattice distance between the origin and the line segment joining the two primitive ray generators of the cone (the edge of the cone) – is called the local index, and can be calculated to be

$$\ell_\sigma = \frac{r}{\gcd(r, a + 1)}.$$ 

The width of the cone is the number of unit-length lattice line segments along the edge of the cone or, equivalently, one less than the number of lattice points along the edge. The width is equal to $\gcd(r, a + 1)$. We will often conflate a singularity and its corresponding cone in $N_\mathbb{Q}$. An isolated cyclic quotient singularity is a T-singularity if it is smoothable by a qG-deformation.

**Lemma 2.6** ([14, Proposition 3.11]). An isolated cyclic quotient singularity is a T-singularity if and only if it takes the form

$$\frac{1}{dn^2}(1, dnc - 1)$$

for some $c$ with $\gcd(n, c) = 1$.

The cone $\sigma \subset N_\mathbb{Q}$ associated to a T-singularity $\frac{1}{dn^2}(1, dnc - 1)$ has local index $\ell = n$ and width $dn$; it is easily seen that T-singularities are characterised by having the width divisible by the local index. Suppose that $P \subset N_\mathbb{Q}$ is a Fano polygon with edge $E$ spanning $\sigma$. Let $w \in M$ be the primitive inner-normal such that $w(E) = -\ell$, and choose $F \subset w^\perp$ of lattice length $d$. The mutation $\mu_{(w,f)}(P)$ collapses the edge $E$ to a vertex, removing the cone $\sigma$. This is equivalent to a local qG-smoothing of the T-singularity.

**Example 2.7.** Consider the polytope $Q := \text{conv} \{(1, 0), (0, 1), (-1, -4)\}$ appearing in Example 2.5. The corresponding spanning fan has three two-dimensional cones, two of which are smooth and one of which, cone$\{(1, 0), (-1, -4)\}$, corresponds to a $\frac{1}{r}(1, 1)$ T-singularity.

The other relevant class of quotient singularities are the R-singularities introduced in [3].

**Definition 2.8.** A cyclic quotient singularity of local index $\ell$ and width $k$ is an R-singularity if $k < \ell$.

Let $\sigma \subset N_\mathbb{Q}$ be a cone of local index $\ell$ and width $k$. Write $k = d\ell + r$, where $d, r \in \mathbb{Z}_{\geq 0}, 0 \leq r < \ell$. If $r = 0$ then $\sigma$ is a T-singularity. Assume that $r \neq 0$ and, as before, suppose that $P \subset N_\mathbb{Q}$ is a Fano polygon with edge $E$ spanning $\sigma$. Let $w \in M$ be the corresponding inner-normal, and pick $F \subset w^\perp$ of lattice length $d$. The mutation $\mu_{(w,f)}(P)$ transforms $\sigma$ to a cone $\tau$ of width $r$ corresponding to a $\frac{1}{r}(1, rc/k - 1)$ singularity. Crucially, $\tau$ has width strictly less than the local index, and so cannot be simplified via further mutation. This is equivalent to a partial qG-smoothing of the original singularity $\sigma$, resulting in a singularity $\tau$ that is rigid under qG-deformation. The R-singularity $\tau$ is independent of the choices made [3, Proposition 2.4].
**Definition 2.9.** Let \( \sigma \subset N_Q \) be a cone corresponding to a \( \frac{1}{2}(1, c - 1) \) singularity. Let \( \ell \) be the local index and let \( k \) be the width of the cone. Write \( k = d\ell + r \), where \( d, r \in \mathbb{Z}_{\geq 0}, 0 \leq r < \ell \). The residue of \( \sigma \) is

\[
\text{res}(\sigma) = \begin{cases} 
\frac{1}{2}(1, rc/k - 1), & \text{if } r \neq 0; \\
\emptyset, & \text{otherwise.}
\end{cases}
\]

The singularity content of \( \sigma \) is the pair \((d, \text{res}(\sigma))\). The singularity content is local qG-deformation-theoretic data about \( \sigma \).

**Definition 2.10.** Let \( P \subset N_Q \) be a Fano polygon with cones \( \sigma_1, \ldots, \sigma_n \). The basket of \( P \) is the multiset

\[
\mathcal{B} := \{\text{res}(\sigma_i) \mid 1 \leq i \leq n\},
\]

where the empty residues are omitted\(^1\). The singularity content of \( P \) is the pair

\[
(d_1 + \cdots + d_n, \mathcal{B}),
\]

where the \( d_i \) are the integers appearing in the singularity content of the \( \sigma_i \). Singularity content is a qG-deformation-invariant of \( X_P \).

**2.3. Hilbert series.** Any projective toric variety \( X_P \) arising from a polytope \( P \) comes with a natural ample divisor \( D \) given by its toric boundary \( D = X_P \setminus T \), where \( T \) is the big torus inside \( X_P \). When \( P \) is Fano, \( D = -K \), the anti-canonical divisor on \( X_P \). In this case, due to the standard toric dictionary allowing one to move between lattice points in \( M \) and sections of line bundles on \( X_P \), one has that the Hilbert function of \((X_P, -K)\) equals the Ehrhart quasi-polynomial \( \text{Hilb}_{-K} \) of the rational polytope \( P^* \). Hence the generating function \( \text{Hilb}_{-K}(t) \) for the Hilbert function of \((X_P, -K)\) is equal to the Ehrhart series of \( P^* \). From here onwards we suppress \(-K\) from the notation.

The Hilbert series of an orbifold del Pezzo surface \( X \) with basket \( B \) can be written in the form [3, Corollary 3.5]:

\[
\text{Hilb}_X(t) = \frac{1 + (K^2 - 2)t + t^2}{(1 - t)^3} + \sum_{\sigma \in \mathcal{B}} Q_{\sigma} t^{d_{\sigma}},
\]

where \( Q_{\sigma} \) are orbifold correction terms given by certain rational functions with denominators \( 1 - t^{d_{\sigma}} \). For example, the orbifold correction term for the R-singularity \( \frac{1}{3}(1, 1) \) is

\[
Q_{\frac{1}{3}(1, 1)} = \frac{-t}{3(1 - t^3)} = \frac{1}{3}(t + t^4 + t^7 + \ldots)
\]

which contributes \(-1/3\) to the coefficient of \( t^d \) when \( d \equiv 1 \pmod{3} \).

The Hilbert function is a quasi-polynomial when \( X \) is an orbifold (because the anti-canonical divisor is \( \mathcal{Q} \)-Cartier rather than Cartier). The anti-canonical divisor does not correspond to a line bundle, but some integer multiple of it does. The smallest integer \( d \) such that \(-dK\) is Cartier is called the Gorenstein index of \( X \) and denoted by \( \ell_X \). In the toric setting, \(-dK\) is Cartier if and only if \( dP^* \) is a lattice polytope. Hence the Gorenstein index \( \ell_X \) of \( X_P \) equals the denominator \( r_N \) of \( P^* \).

**2.4. Algebraic geometry and the quasi-period.** Mutations were introduced in [2] as part of an ongoing program investigating Mirror Symmetry for Fano manifolds [5]. In two dimensions the picture is very well understood: see [1] for the details. In summary, if two Fano polygons \( P \) and \( Q \subset N_Q \) are related by a sequence of mutations then there exists a qG-deformation between the corresponding toric del Pezzo surfaces \( X_P \) and \( X_Q \). Such a qG-deformation preserves the anti-canonical Hilbert series, hence \( L_{P^*} = L_{Q^*} \) and so the quasi-periods of \( P^* \) and \( Q^* \) agree. However it does not in general preserve the Gorenstein index, and hence the denominators \( r_P \) and \( r_Q \) need not be equal. The cones over the edges of \( P \) correspond to the singularities of \( X_P \), and these admit partial qG-smoothenings to the qG-rigid singularities given by the basket \( B \) of residues.

Suppose that the singularity content of \( P \) is \((d, B)\). Then, by the absence of global obstructions to qG-deformations on Fano varieties, \( X_P \) is qG-deformation-equivalent to a (not necessarily toric) del Pezzo surface \( X \) with singularities \( B \) and whose non-singular locus has topological Euler number \( d \). Since \( \text{Hilb}_{X_P}(t) = \text{Hilb}_X(t) \), we have an explanation for quasi-period collapse of the dual polytope \( P^* \). Specifically, the Gorenstein index of \( X \) is equal to the quasi-period of \( P^* \).

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\(^1\)In [3] the basket is cyclically ordered. Although important from the viewpoint of classification, it is not required here.
3. Studying quasi-period collapse

The Hilbert series of orbifold del Pezzo surfaces were studied in [18] with the aim of describing the structure of the set of possible baskets $B$ of $R$-singularities on orbifold del Pezzo surfaces with a fixed Hilbert series. This is achieved by partitioning $B$ into two pieces: a reduced basket and an invisible basket. The latter, along with the $T$-singularities, is not detectable by the Hilbert series, and from our viewpoint it is this invisibility that causes quasi-period collapse.

**Definition 3.1.** A collection $\sigma_1, \ldots, \sigma_n$ of $R$-singularities is a cancelling tuple if
\[ Q_{\sigma_1} + \cdots + Q_{\sigma_n} = 0. \]
A collection of $R$-singularities is called invisible if it is a union of cancelling tuples.

**Example 3.2.** Let $\sigma$ be an $R$-singularity of local index $\ell$ and width $k$. Then there exists an $R$-singularity $\sigma'$ of local index $\ell$ and width $\ell - k$ such that $Q_\sigma + Q_{\sigma'} = 0$. Combinatorially, this is understood by the observation that the union of the two cones gives a $T$-singularity.

![Diagram of cancelling tuple](image)

**Definition 3.3.** Let $X$ be an orbifold del Pezzo surface. A maximal invisible subcollection of the basket $B$ of $X$ is called an invisible basket for $X$. Notice that such a maximal subcollection is not unique, since singularities can appear in many different cancelling tuples. Given a choice of invisible basket $IB \subset B$, the complement $RB = B \setminus IB$ is called the reduced basket for $X$ corresponding to the choice of $IB$.

Let $P \subset N_Q$ be a Fano polygon with singularity content $(d, B)$. Let $IB$ be an invisible basket of $B$ with corresponding reduced basket $RB$. Hence $B = RB \cup IB$. Denote the collection of $T$-singularities on $X_P$ by $T$ (so $|T| = d$).

**Theorem 3.4.** Let $P \subset N_Q$ be a Fano polygon. The quasi-period of $P^*$ is given by:
\[ \pi_{p^*} = \lcm\{\ell_\sigma \mid \sigma \in RB\}. \]
Furthermore, $P^*$ exhibits quasi-period collapse if and only if there exists some $\tau \in IB \cup T$ of local index not dividing $\lcm\{\ell_\sigma \mid \sigma \in RB\}$. Moreover, the quasi-period collapse is measured by $IB$:
\[ r_{p^*} = \lcm(\{\pi_{p^*}\} \cup \{\ell_\sigma \mid \sigma \in IB \cup T\}). \]

**Proof.** We have
\[ \Ehr_{p^*}(t) = \Hilb_{X_P}(t) = \text{initial term} + \sum_{\sigma \in B} Q_\sigma = \text{initial term} + \sum_{\sigma \in RB} Q_\sigma. \]
As discussed, each orbifold correction term $Q_\sigma$ contributes to the coefficients of this series as a quasi-polynomial with quasi-period $\ell_\sigma$. When $\sigma \in RB$ these terms are not cancelled and so make non-zero contributions to the coefficients of the Ehrhart series, hence its quasi-period is given by:
\[ \pi_{p^*} = \lcm\{\ell_\sigma \mid \sigma \in RB\}. \]
The Gorenstein index of $P$ is equal to $\ell_{X_P} = \lcm\{\ell_\sigma \mid \sigma \in B \cup T\}$. Hence
\[ r_{p^*} = \ell_{X_P} = \lcm(\{\ell_\sigma \mid \sigma \in RB \cup IB \cup T\}) = \lcm(\{\pi_{p^*}\} \cup \{\ell_\sigma \mid \sigma \in IB \cup T\}). \]
This is distinct from $\pi_{p^*}$ if and only if $\lcm\{\ell_\sigma \mid \sigma \in IB \cup T\}$ does not divide $\pi_{p^*}$. □

**Remark 3.5.** It follows from [18, §4] that the choice of $IB$ is irrelevant in the statement of Theorem 3.4.

As a corollary to Theorem 3.4 we immediately obtain:

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3We adopt the convention that $\lcm(\emptyset) = 1$. 


Corollary 3.6. Let $P \subset N_Q$ be a Fano polygon. The discrepancy between the quasi-period and denominator of $P^*$ is
\[
\frac{r_P}{\pi_P} = \frac{\text{lcm}\{\ell_\alpha \mid \alpha \notin IB \cup T\}}{\gcd\{\text{lcm}\{\ell_\alpha \mid \alpha \notin RB\}, \text{lcm}\{\ell_\alpha \mid \alpha \notin IB \cup T\}\}}.
\]

Example 3.7 (Detecting quasi-period collapse). Consider the polytope $Q := \text{conv}\{(1, 0), (0, 1), (-1, -4)\}$ appearing in Example 2.5. This has singularity content $(3, \emptyset)$, and $T = \{2 \times \text{smooth}, \frac{1}{2}(1, 1)\}$. Applying Corollary 3.6 we have that $r_Q = 2\pi_Q$.

We now give an example of an infinite family of Fano triangles, obtained via mutation, where the denominator $r_P$ can become arbitrarily large but where $\pi_P = 1$. Let $P \subset N_Q$ be a Fano triangle. Recall that the corresponding toric variety $X_P$ is a fake weighted projective plane [12]: a quotient of a weighted projective plane by a finite group $N/N'$ acting free in codimension one, where $N'$ is the sublattice generated by the vertices of $P$.

Example 3.8 (Mutations of $\mathbb{P}^2$). In [4, 11] the graph of mutations of $\mathbb{P}^2$ is constructed. The vertices of this graph are given by $P(a^2, b^2, c^2)$, where $(a, b, c) \in \mathbb{Z}_+^3$ is a Markov triple satisfying
\[
a^2 + b^2 + c^2 = 3abc.
\]

Let $X_P = P(a^2, b^2, c^2)$ be such a weighted projective plane, with $P \subset N_Q$ the corresponding Fano triangle. Since $X_P$ is qG-deformation-equivalent to $\mathbb{P}^2$, so $X_P$ is smoothable and its anti-canonical Hilbert function has quasi-period one. Hence $\pi_P = 1$. However, the denominator $r_P$ of $P^*$ can be arbitrarily large. To see this, note first that $a, b, c$ must be pairwise coprime: if $p \mid a$ and $p \mid b$ then $p^2 \mid 3abc = a^2 + b^2 + c^2$, and hence $p \mid c$; but then $p$ appears as a square on the left-hand side and as a cube on the right-hand side of (3.1). Let $b$ be an inverse of $b \pmod{a^2}$. Note that $c^2 b^{-1} + 1 \equiv (3abc - b^2)b^{-1} + 1 \equiv 3abc \pmod{a^2}$, and so the singularity $\frac{a^2}{\text{lcm}\{b^2, c^2\}}$ on $X_P$ has local index
\[
\frac{a^2}{\gcd\{a^2, c^2 b^{-1} + 1\}} = \begin{cases} a, & \text{if } a \equiv 0 \pmod{3}; \\ a/3, & \text{if } a \equiv 0 \pmod{3}. \end{cases}
\]

Considering equation (3.1) (mod 3) shows that no Markov numbers are divisible by three. Hence the three local indices on $X_P$ are $a$, $b$, and $c$, and so $r_P = abc$. The two triangles $P$ and $Q$ in Example 2.5 are the simplest examples, arising from the Markov triples $(1, 1, 1)$ and $(1, 1, 2)$ respectively, and corresponding to $\mathbb{P}^2$ and $P(1, 1, 4)$.

Remark 3.9. There exist Fano triangles of quasi-period one not arising from the construction in Example 3.8. For example, consider
\[
P = \text{conv}\{(3, 2), (-1, -2), (-1, -2)\} \subset N_Q.
\]

The corresponding fake weighted projective plane $X_P = P(1, 1, 2)/(\mathbb{Z}/4)$ has $2 \times \frac{1}{2}(1, 1, 3)$ and $\frac{1}{2}(1, 3)$ $T$-singularities. We see that $P^*$ has $r_P = 2$ and $\pi_P = 1$. In fact $X_P$ is qG-smoothable to the nonsingular del Pezzo surface of degree two, and hence $L_P(k) = k^2 + k + 1$.

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