MSGUT : From Bloom to Doom

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Abstract

By a systematic survey of the parameter space we confirm our surmise[1] that the Minimal Supersymmetric GUT(MSGUT) based on the $210 \oplus 126 \oplus \overline{126} \oplus 10$ Higgs system is incompatible with the generic Type I and Type II seesaw mechanisms. The incompatibility of the Type II seesaw mechanism with this MSGUT is due to its generic extreme sub-dominance with respect to the Type I contribution. The Type I mechanism although dominant over Type II is itself unable to provide Neutrino masses larger than $\sim 10^{-3}$ eV anywhere in the parameter space. Our Renormalization Group based analysis shows the origin of these difficulties to lie in a conflict between baryon stability and neutrino oscillation. The MSGUT completed with a $120$-plet Higgs is the natural next to minimal candidate. We propose a scenario where the $120$-plet collaborates with the $10$-plet to fit the charged fermion masses. The freed $126$-plet couplings can then give sub-dominant contributions to charged fermion masses and enhance the Type I seesaw masses sufficiently to provide a viable seesaw mechanism. We give formulae required to verify this scenario.

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1 Introduction

SO(10) GUTs accommodate complete fermion families together with the superheavy right handed neutrinos (required by the seesaw mechanism \[2,3\]) in a 16-plet spinorial irreps. They also provide very natural Higgs multiplets (126) (or \(\overline{16}_H \times \overline{16}_H\)) to generate the \((U(1)_{B-L}\) breaking) right handed neutrino Majorana masses and the \((SU(2)_L \times U(1)\) breaking) triplet vevs required to implement the Type I[2] and Type II[3] mechanisms respectively. Thus models which incorporate high scale breaking of the \(SU(2)_R \times U(1)_{B-L}\) components of the SO(10) gauge symmetry provide an elegant and natural context to understand the clear indication of a seesaw connection between neutrino mass and GUT mass scales provided by the discovery of neutrino masses in the 50 milli-eV range \((\Delta m^2_\nu \sim 2.2 \times 10^{-3}eV^2 \sim ((10^2 GeV)^2/(10^{15} GeV))^2)\). In supersymmetric models with left right symmetric \((SU(3) \times SU(2)_L \times SU(2)_R \times U(1)_{B-L} \subset G)\) gauge groups, R-parity \((R_p = (-)^{3(B-L)+2S})\) is a part of the Gauge symmetry and is naturally preserved till low energies in models using only B-L even vevs for the seesaw. They thus predict a stable LSP which is welcome as cosmological dark matter. With such a complement of virtues Supersymmetric SO(10) GUTs of some variety or the other are considered leading contenders by a majority of workers. The Supersymmetric SO(10) GUT based on the \(126 \oplus \overline{126} \oplus 210\) Higgs multiplets proposed long ago \[11,12\] shares all these manifest virtues of LR supersymmetric models. It has the additional virtue of maximal simplicity from a parameter counting and representation economy point of view \[11,12,13,10\]. It thus lays claim to the role of contributing the correct GUT symmetry breaking sector (AM Higgs) for a fully realistic Minimal Supersymmetric GUT (MSGUT). The other crucial component required to fully define a MSGUT is however the complement of fermion mass (FM)Higgs used to fit the charged fermion masses and mixings and the neutrino oscillation mass and mixing parameters which constitute the actual ‘data’ that a GUT must confront. SO(10) permits an FM Higgs system based on \(10 \oplus 120 \oplus \overline{126}\) irreps. The first and third of these have Yukawa couplings to the matter 16-plets which are symmetric under interchange of family indices and have been extensively considered. The 120-plet with its antisymmetric Yukawa couplings has so far been conceded much less importance, acting either as a perturbation to the main structure determined by the 10 and \(\overline{126}\) plets \[17,24\] or else somewhat cursorily \[14,16,18,19\]. In this paper we shall develop a line of argument that culminates in a suggestion that a central role for the 120-plet is natural.

In 1992, with LEP data in hand, Babu and Mohapatra \[13\] proposed that if only the \(10 \oplus \overline{126}\) irreps were used to fit the charged fermion masses and mixing then the matter fermion Yukawa couplings of the GUT would be completely determined. Hence the model would become predictive in the neutrino sector. The compatibility of this scenario with the (now better known) neutrino mass-mixing data has been the subject of an accelerating succession of works since their proposal \[20,15,21,22,24,25\]. These works have demonstrated the compatibility of the data with the generic (Georgi-Jarlskog \[26\] plus Type I and Type II seesaw) structural form of the fermion mass formulae dictated by SO(10) klebsches and restricted FM Higgs content. Symmetric Yukawa couplings alone are quite successful but the possibility of improving the fits, particularly as regards the CKM CP phase, by using a subdominant contribution of the 120-plet has also been demonstrated \[24,17\]. However the
generic fitting procedure does not fix the over all mass scale of the neutrino masses or the relative strength of the Type I and Type II masses. On the other hand these quantities are fixed in a fully specified GUT model and it is thus crucial to investigate whether any given GUT model of this general type is compatible with both the generic fits and the actual neutrino mass data. In the 1990’s the construction of natural and fully consistent Minimal Left Right Supersymmetric models (MSLRMs) was accomplished\[5, 6, 7\]. Moreover these analyses showed clearly\[28, 9\] what was already noted at the beginning\[11, 27\] of the study of multi-scale Susy GUTs: that in LR Susy GUTs there are light multiplets which violate the conventional wisdom of the “survival principle”. Thus it was clear\[9, 28\] that Susy GUTs required the use of calculated rather than merely (“survival principle”) estimated masses for RG analyses. The RG analysis carried out in this work already indicated that the use of calculated spectra forces together the various possible intermediate scales into a narrow range close to the GUT scale resulting in an effective “SU(5) conspiracy” i.e the necessity for single step breaking of SO(10).

A calculation of the full GUT spectrum and couplings in various SO(10) GUT models required complete knowledge of the ”Clebsches” relating \(SO(10)\) and \(G_{SM} \times U(1)_R\) group labels on \(SO(10)\) fields - at least for tensorial irreps - before progress in the Renormalization Group (RG) analysis could be made. Furthermore the compatibility of fermion fits with GUTs could also be investigated only if Clebsches were also available for the spinorial \((16\)-plet) irrep. The requisite computations (based on decomposition of SO(10) labels into those of the \(SU(4) \times SU(2)_L \times SU(2)_R\) (“Pati-Salam”) sub-group ) were presented by us in \[30\]. Using these Clebsches first the mass matrix of the MSSM type doublets \([1,2,\pm 1]\) and classic GUT \(\Delta B \neq 0\) process mediating triplets \([3,1,\pm \frac{2}{3}]\) \[30\] and then a complete calculation\[31\], of all the couplings and mass matrices of the MSGUT was given by us. With similar motivations two calculations, one in parallel and cross checking with ours\[32\], and another \[33\] quite separate, which both used the same somewhat abstract\[34\] method (but different phase conventions) to calculate the “Clebsches” of tensor (but not spinor) multiplets, appeared. In \[30, 31\] we provided the complete (chiral and gauge) spectra, neutrino mass matrices, gauge and chiral couplings and the effective \(d = 5\) operators for Baryon violation in terms of GUT parameters and MSSM fields. Thus the stage was laid both for a completely explicit RG based analysis of the MSGUT and a completely specified investigation of the compatibility of Type I and Type II seesaw mechanisms. The calculation of threshold effects based on these spectra was described in\[31, 1\] and shown to controvert the conventional wisdom\[35\] which doubted the stability of the successful unification (and \(m_t, \sin^2 \theta_W\) prediction\[36\]) of the one-loop MSSM coupling evolution\[36, 37, 38\].

The question of the fitting of the charged fermion masses, the relative size of the neutrino mass splittings and the quark and lepton mixing matrices had been extensively analyzed\[13, 20, 14, 21, 22, 24, 25\] using the generic formulae valid in SO(10) GUTs with only\[13\] 10, 126 FM Higgs representations. However this procedure does not constrain the relative size of the Type I and Type II contributions or the overall scale of the neutrino masses. Within a fully specified theory like the MSGUT, however, both these parameters are specified. In \[23\] the question of assuring Type II (over Type I) dominance on the basis of the Yukawa values found in the successful generic Type II fits\[21, 22\] was considered. The authors
concluded on the basis of order of magnitude estimates that the Type I mass would emerge as too large unless the $B - L$ breaking scale was raised. Furthermore they found that the mass of the $SU(2)_L$ triplet field whose tadpole vev gives rise to the Type II seesaw must be lowered to enhance Type II seesaw masses. To ensure these they considered extending the model by modifying the GUT scale breaking to be via $SU(5)$ (this required introducing a $54$-plet) or else by perturbing it by introducing a $120$-plet.

Our point of view, program and results on these questions are related but still quite distinct from the estimates and scenarios of [23]. In [1] we took up the question of the compatibility of the MSGUT with the successful generic $10 + \bar{T}_{26}$ FM Higgs fits. We surveyed the variation of the relative strength of Type I and Type II seesaw mechanisms, as well as their absolute magnitudes while keeping in view the viability of the relevant regions of the GUT parameter space i.e with the stability of the basic unification scenario. Our graphical survey yielded the striking observations that:

- For generic as well as special values of MSGUT couplings and FM Yukawas taken from the generic Type II seesaw fit, the Type II seesaw itself implies that it is highly subdominant to Type I seesaw in the MSGUT unless the relative strength of Type I versus Type II is adjusted to a very small value. The primary reason for this relative dominance of Type I is not the value of the symmetry breaking scale but rather the value of the effective coupling in the Type I seesaw formula implied by the Yukawas found in the Type II generic fit (which simply assumes Type II is dominant)

- The maximal values of the Type I seesaw masses attainable in the MSGUT are at least one order of magnitude short of those required by atmospheric neutrino oscillations.

Note that on both accounts our conclusions are at variance with the [23] although they also find difficulties in ensuring Type II dominance over Type I.

In the present paper we present a detailed survey of the parameter space to definitely confirm the above conclusions and thus pose a stringent challenge to the viability of the original proposal of Babu and Mohapatra [13] in the context of the MSGUT. Already in [31] we noted that the variation of the unification stability monitoring parameters (USMPs)(i.e $\Delta(\log M_X), \Delta(sin^2 \theta_W), \Delta(\alpha_G^{-1})$) with the fast control parameter $\xi$ of the MSGUT exhibited very striking sharp peaks and dips (with the characteristic appearance of poles of a function of $\xi$). In many cases these spikes were at the values of points of increased symmetry uncovered by the analytic solution of the AM spontaneous symmetry breaking problem in the MSGUT [10, 31, 32]. In addition there were other spikes (not obviously related to points of extended symmetry) whose investigation was called for by the “tomogram” of the MSGUT parameter space that we had provided [31]. In [10, 31, 32] an analytic parametrization of the SSB in the MSGUT in terms of the solutions of a cubic equation(linear in a fast control parameter $\xi$) was achieved. In particular in [10, 32, 43] the parameter $\xi$ was eliminated -using the cubic equation- for the solution values $x$ and thus a very direct, elegant and unified parametrization of the pole and zero structures characterizing the MSGUT AM-SSB was achieved in terms of the variation of a single complex variable $x$. This allowed the identification of additional special points. In particular, motivated by our observations [1], of a set of points associated with possible growth of the Type I and Type II seesaw contributions [10, 32, 43]. Such growth had already been previously observed at
some points in [1] but found to give unviable USMPs. Thus to complete the program of [1] we have now investigated the behaviour of the Type I and Type II coefficient functions defined by the precise clebsches and fermion mass matrices of the MSGUT [31, 1] over the complex \( x \) (equivalently to \( \xi \) ) plane: including all the special points described above. The loopholes offered [1, 43] by the special points are illusory rather than real: typically some or all of the USMPs explode due to the very growth which is being invoked to strengthen one or the other seesaw. Our conclusions are thus unchanged from those of [1]: the MSGUT has suffered a failure of its most critical features and is thus now defunct.

A way out of this impasse appears by allowing in the 120-plet which was earlier discriminated against on grounds of convenience and simplicity alone. ‘Lamppost logic’ may have failed here as so often before. Nature does not heed the wishful thinking behind such logic: the morning often reveals that the keys sought for all night lay in the gutter just beyond the lamppost! Another alternative is that SSB in the MSGUT may actually be controlled [39, 40] by the explosive growth of the gauge coupling above \( M_X \) due to gaugino condensation in the coset \( SO(10)/G_{123} \) which drives [40] chiral condensates of the AM Higgs fields whose value may be calculable due to the power of supersymmetric holomorphy. In such a scenario the crucial composition of the zero mode Higgs doublets must be evaluated without using the “cubic” solution mentioned above.

In Section 2, referring the reader to the original papers [30, 31, 1] for all derivations, we present the basic formulae [31, 1] governing the Type I and II seesaw mechanisms in the MSGUT and identify the Seesaw Monitoring Parameters (SMPs) \( (R, F_I, F_{II}) \) that we will use for our graphical presentation of the seesaw relevant topography of the parameter space. We list the obvious points of enhanced gauge symmetry. This is accompanied by an Appendix where these formulae are also evaluated in the (Rational) function (of \( x \)) form advocated by [43]. This allows the identification of an additional list of exceptional points requiring special treatment. In Section 3 we describe the analytic and graphical investigation of the behaviour of the USMPs and SMPs as we scan over the complex \( x \) plane for generic and for exceptionally favourable values of the ‘slow’ AM Higgs parameters [31] i.e. \( x \) (equivalently \( \xi \) ). By covering the behaviour at the enhanced symmetry points and exceptional seesaw points (collectively called ESPs) and scanning the whole \( x \)-plane we eliminate all escape routes and pin down the MSGUT to its doom. In Section 4 we propose that the inclusion of the 120-plet in a very specific role can solve the problem with the small neutrino masses found for the MSGUT and give the fermion mass formulae for the NMSGUT along with the doublet Higgs mass matrix \( \mathcal{H} \). This specifies the composition of the massless doublets once \( \text{det}\mathcal{H} = 0 \) is imposed. Thus the stage is set for the accomplishment of a program analogous to that accomplished for the MSGUT [11, 12, 30, 10, 31, 32, 33, 44, 42, 1, 19], which can be implemented without ambiguity. We conclude with a discussion of the outlook for future work.
2 Seesaw formulae and SMPs for the MSGUT

2.1 MSGUT couplings, vevs and masses

The MSGUT is the renormalizable globally supersymmetric $SO(10)$ GUT whose Higgs chiral supermultiplets consist of Adjoint Multiplet (AM) type totally antisymmetric tensors: \( 210(\Phi_{ijkl}), \mathbf{T}_{26}(\Sigma_{ijklm}), 126(\Sigma_{ijklm})(i,j = 1...10) \) which break the GUT symmetry to the MSSM, together with Fermion mass (FM) Higgs $10$-plet($H_i$). The $\mathbf{T}_{26}$ plays a dual or AM-FM role since it also enables the generation of realistic charged fermion and neutrino masses and mixings (via the Type I and/or Type II mechanisms); three $16$-plets $\Psi_A(\ A = 1,2,3)$ contain the matter including the three conjugate neutrinos ($\bar{\nu}_L^A$).

The superpotential (see\cite{10,30,31,32} for comprehensive details) contains the mass parameters

\[
\begin{align*}
  m : 210^2 & ; & M : 126 \cdot \mathbf{T}_{26}; & M_H : 10^2 \\
\end{align*}
\]

and trilinear couplings

\[
\begin{align*}
\lambda : 210^3 & ; & \eta : 210 \cdot 126 \cdot \mathbf{T}_{26}; & \gamma \oplus \bar{\gamma} : 10 \cdot 210 \cdot (126 \oplus \mathbf{T}_{26}) \\
\end{align*}
\]

The GUT scale vevs that break the gauge symmetry down to the SM symmetry (in the notation of\cite{30}) are\cite{11,12}

\[
\begin{align*}
  \langle (15, 1, 1)\rangle_{210} : & \langle \phi_{abcd} \rangle = \frac{a}{2} \epsilon_{abcdef} \epsilon_{ef} \\
  \langle (15, 1, 3)\rangle_{210} : & \langle \phi_{ab\tilde{d}\tilde{\beta}} \rangle = \omega \epsilon_{ab\tilde{d}\tilde{\beta}} \\
  \langle (1, 1, 1)\rangle_{210} : & \langle \phi_{\tilde{a}\tilde{b}\tilde{\beta}\tilde{\delta}} \rangle = p \epsilon_{\tilde{a}\tilde{b}\tilde{\beta}\tilde{\delta}} \\
  \langle (10, 1, 3)\rangle_{\mathbf{T}_{26}} : & \langle \Sigma_{13580} \rangle = \bar{\sigma} \\
  \langle (10, 1, 3)\rangle_{126} : & \langle \Sigma_{24679} \rangle = \sigma. \\
\end{align*}
\]

The vanishing of the D-terms of the $SO(10)$ gauge sector potential imposes only the condition $|\sigma| = |\bar{\sigma}|$. Except for the simpler cases corresponding to enhanced unbroken gauge symmetry ($SU(5) \times U(1), SU(5), G_{3,2,2,B-L}, G_{3,2,R,B-L}$ etc)\cite{10,32}, this system of equations is essentially cubic and can be reduced to the single equation\cite{10} for a variable $x = -\lambda \omega / m$, in terms of which the vevs $a, \omega, p, \sigma, \bar{\sigma}$ are specified :

\[
8x^3 - 15x^2 + 14x - 3 = -\xi(1 - x)^2
\]

where $\xi = \frac{\lambda M}{\eta m}$. Then the dimensionless vevs in units of $(m/\lambda)$ are $\bar{\omega} = -x$\cite{10} and

\[
\begin{align*}
\tilde{a} = & \frac{(x^2 + 2x - 1)}{1 - x} ; & \tilde{p} = & \frac{x(5x^2 - 1)}{(1 - x)^2} ; & \tilde{\sigma} = & \frac{2\lambda x(1 - 3x)(1 + x^2)}{\eta} \\
\end{align*}
\]

This exhibits the crucial importance of the parameters $\xi, x$. Note that one can trade\cite{10,32,43} the parameter $\xi$ for $x$ with advantage (using equation (8)) since $\xi$ is uniquely fixed given $x$. By a survey of the behaviour of the theory as a function of the complex parameter $x$ we are simultaneously covering the behaviour of the three different solutions possible for
each complex value of $\xi$. We thus change our graphical presentation in terms of three plots versus $\xi$ to a single plot versus $x$ for each USMP or SMP. Moreover, as emphasized by analysis of the spikes observed in is also facilitated by the use of the parameter $x$. Using the above and the methods of we calculated the complete gauge and chiral multiplet GUT scale spectra and couplings for the 52 different MSSM multiplet sets falling into 26 different MSSM multiplet types (prompting a natural alphabetization of their naming convention) of which 18 are unmixed while the other 8 types occur in multiple copies. The spectra may be found in and equivalent results (with slightly differing conventions) are presented in. A related calculation with very different conventions has been reported in and the initially controversial relation between the overlapping parts of these papers was discussed and resolved in.

Among the mass matrices is the all important $4 \times 4$ Higgs doublet mass matrix

$$H = \begin{pmatrix}
-M_H & +\sqrt{3}(\omega - a) & -\sqrt{3}(\omega + a) & -\gamma \\
-\sqrt{3}(\omega + a) & 0 & -(2M + 4\eta(a + \omega)) & 0 \\
\sqrt{3}(\omega - a) & -(2M + 4\eta(a - \omega)) & 0 & -2\sqrt{3} \\
-\sigma & -2\sqrt{3} & 0 & -2m + 6\lambda(\omega - a)
\end{pmatrix}$$

(10)

$H$ can be diagonalized by a bi-unitary transformation from the 4 pairs of Higgs doublets $h^{(i)}$, $\bar{h}^{(i)}$ arising from the SO(10) fields to a new set $H^{(i)}$, $\bar{H}^{(i)}$ of fields in terms of which the doublet mass terms are diagonal.

$$U^T H U = \text{Diag}(m_H^{(1)}, m_H^{(2)}, \ldots)$$

$$h^{(i)} = U_{ij} H^{(j)}; \quad \bar{h}^{(i)} = \bar{U}_{ij} \bar{H}^{(j)}$$

(11)

To keep one pair of these doublets light one tunes $M_H$ so that $\text{Det}H = 0$. This matrix can then be diagonalized by a bi-unitary transformation yielding thereby the coefficients describing the proportion of the doublet fields in the GUT multiplets present in the light doublets: which proportions are important for many phenomena. In the effective theory at low energies the GUT Higgs doublets $h^{(i)}$, $\bar{h}^{(i)}$ are present in the massless doublets $H^{(1)}$, $\bar{H}^{(1)}$ in a proportion determined by the first columns of the matrices $U$, $\bar{U}$:

$$E << M_X \quad ; \quad h^{(i)} \rightarrow \alpha_i H^{(1)} \quad ; \quad \alpha_i = U_{i1}$$

$$\bar{h}^{(i)} \rightarrow \bar{\alpha}_i \bar{H}^{(1)} \quad ; \quad \bar{\alpha}_i = \bar{U}_{i1}$$

(12)

The all important normalized 4-tuples $\alpha, \bar{\alpha}$ can be easily determined by solving the zero mode conditions: $H\alpha = 0$; $\bar{\alpha}^T H = 0$.

### 2.2 RG Analysis and USMPs

In we discussed at length the use of plots of the USMPs versus the fast parameter $\xi$ to investigate the question:

*Are the one loop values of $\alpha_G(M_X)$, $\text{Sin}^2\theta_W$ and $M_X$ generically stable against superheavy threshold corrections?*

We followed the approach of Hall in which the mass $M_X = \frac{m_X}{\lambda} \sqrt{4|\tilde{a} + \tilde{w}|^2 + 2|\tilde{p} + \tilde{\omega}|^2}$ of the baryon number violating superheavy ($[3, 2, \pm \frac{\lambda}{3}]$ or X-type) gauge bosons is chosen.
as the transition scale between the effective MSSM and the full SO(10) GUT with all superheavy fields retained. Thus $M_X$ is used as common physical superheavy matching point ($M_i = M_V = M_X$) in the equations relating the MSSM couplings to the SO(10) coupling:

$$\frac{1}{\alpha_i(M_S)} = \frac{1}{\alpha_G(M_X)} + 8\pi b_i \ln \frac{M_X}{M_S} + 4\pi \sum_j b_{ij} \ln X_j - 4\pi \lambda_i(M_X) \tag{13}$$

See [31, 1] for a detailed discussion. We find the corrections invalidate the neglect of one-loop effects in the couplings $\lambda, \eta, \gamma$, implement these by the requirements that the USMPs obey perturbative after threshold and two loop correction and

$$\alpha$$

point ($M_{\log SO(10)}$ only to leading order. Thus we compute the corrections to the three USMPs $\log_{10} M_X, \sin^2 \theta_W(M_S), \alpha_G^{-1}(M_X)$ as a function of the MSGUT parameters and the answer to the question of stability of the perturbative unification is determined by the ranges of GUT parameters where these corrections are consistent with the known or surmised data on $\log_{10} M_X, \sin^2 \theta_W(M_S), \alpha_G$. The consistency requirement that the SO(10) theory remain perturbative after threshold and two loop correction and $\alpha_G$ not decrease so much as to invalidate the neglect of one-loop effects in the couplings $\lambda, \eta, \gamma, \tilde{\gamma}$ is also appropriate. We implement these by the requirements that the USMPs obey

$$|\Delta_G| = |\Delta(\alpha_G^{-1}(M_X))| \leq 10$$

$$\Delta_X = \Delta(\log_{10} M_X) \geq -1$$

$$\Delta_W = |\Delta(\sin^2 \theta_W(M_S))| < .02 \tag{14}$$

See [31, 11] for a detailed discussion. We find the corrections

$$\Delta^{(th)}(\log_{10} M_X) = .0217 + .0167(5b'_1 + 3b'_2 - 8b'_3) \log_{10} \frac{M'}{M_X}$$

$$\Delta^{(th)}(\sin^2 \theta_W(M_S)) = .00004 - .00024(4b'_1 - 9.6b'_2 + 5.6b'_3) \log_{10} \frac{M'}{M_X}$$

$$\Delta^{(th)}(\alpha_G^{-1}(M_X)) = .1565 + .01832(5b'_1 + 3b'_2 + 12b'_3) \log_{10} \frac{M'}{M_X} \tag{15}$$

Where $b'_i = 16\pi^2 b'_i$ are 1-loop $\beta$ function coefficients ($\beta_i = b_i g_i^2$) for multiplets with mass $M'$ (a sum over representations is implicit). Note a minor, but crucial correction, in the above equations relative to [31, 11]: there and here, we actually used $\frac{M'}{M_X}$ as the argument of the logarithms in these formulae. This is the value given by the algebra, which is then conventionally approximated by $\frac{M'}{M_X}$ since the difference is of order $g[15].$ However in our case we can retain the “exact” form and still calculate the threshold corrections since those depend on ratios of masses. This improves the analysis and makes it more convenient. Moreover (see below) it means that the overall scale parameter that we use, namely the coefficient $m$ of $210^2$ in the superpotential, is not fixed at the one loop unification scale $10^{16.25} GeV$ but slides with the corrections to $M_X$. These corrections are to be added to the one loop values corresponding to the successful gauge unification of the MSSM: Using the values

$$\alpha_G^0(M_X)^{-1} = 25.6 \quad ; \quad M_X^0 = 10^{16.25} GeV \quad ; \quad M_S = 1 TeV$$

$$\alpha_1^{-1}(M_S) = 57.45 \quad ; \quad \alpha_2^{-1}(M_S) = 30.8 \quad ; \quad \alpha_3^{-1}(M_S) = 11.04 \tag{16}$$
the two loop corrections are

\[ \Delta^{2-\text{loop}}(\log_{10} \frac{M_X}{M_S}) = -0.08 \quad ; \quad \Delta^{2-\text{loop}}(\sin^2 \theta_W(M_S)) = 0.0026 \]

\[ \Delta^{2-\text{loop}} \alpha^{-1}_G(M_X) = -0.546 \quad (17) \]

We see that in comparison with the large threshold effects that might be expected in view of the large number (504) of heavy fields the 2 loop corrections are quite small. The striking new result of was thus the explicit demonstration that the conjecture concerning the “Futility of high precision SO(10) calculations” was in fact false and that a proper calculation based on calculated spectra yields well defined and consistent results for significant regions of the MSGUT parameter space (see below).

The parameter \( \xi = \lambda M/\eta m \) is the only numerical parameter that enters into the cubic eqn. that determines the parameter \( x \) in terms of which all the superheavy vevs are given. It is thus the most crucial determinant of the mass spectrum. The dependence of the threshold corrections on the parameters \( \lambda, \eta, \gamma, \bar{\gamma} \) is comparatively mild except when coherent e.g when many masses are lowered together leading to \( \alpha_G \) explosion, \( \log M_X \) collapse or large changes in \( \sin^2 \theta_W(M_S) \).

The typical behaviours of the USMPs can be seen in Figs 1-6.

The parameter ratio \( m/\lambda \) can be extracted as the overall scale of the vevs. Since the threshold corrections we calculate are dependent only on (logarithms of) ratios of masses, the parameter \( m \) is fixed in terms of the mass \( M_V = M_X \) of the lightest superheavy vector particles mediating proton decay by :

\[ \Delta_X = \Delta(\log_{10} \frac{M_X}{1 GeV}) \]

\[ |m| = 10^{16.25+\Delta_X} \frac{\lambda}{g \sqrt{4|\hat{a} + \hat{w}|^2 + 2|\hat{p} + \hat{\omega}|^2}} GeV \quad (18) \]

The presence of the factor \( 10^{\Delta_X} \) in the formulæ for the neutrino masses will play a crucial role as one of the hands of the “scissor” operated by Baryon Decay constraints and neutrino oscillation data. The lowering of the “slow diagonal” parameters \( \lambda, \eta \) tends to make fields light and thus give large negative corrections to \( \Delta_G \) (as can be seen in the expansion of the darker areas in Fig.4 relative to Fig.3). Such growth of the gauge coupling can invalidate the self consistency of the perturbative approximation used throughout. Conversely when \( \alpha_G \) decreases too much the neglect of Yukawa loops becomes moot. Thus \( \lambda, \eta \) are effectively restricted to magnitudes of order 1. Note, however, that \( \Delta_W, \Delta_X \) are insensitive to the precise values - whether real or complex - of \( \lambda, \eta \) as long as their magnitudes are \( \sim 1 \) : as can be seen in the pairs Fig.1 and Fig. 6 and Fig. 2 and Fig. 5. In addition there are the “slow off diagonal” parameters \( \gamma, \bar{\gamma} \) whose effect is quite mild, and can be taken to be any values \( \sim .1 \) to 1., without any appreciable change in the numerical plots.

### 2.3 Fermion mass formulæ and SMPs

The Dirac masses of matter fermions in the MSGUT at scale \( M_X \) are

\[ m^u = v(\hat{h} + \hat{f}) \]
$\sin^2 \theta_w$ Contour Plot on the complex $x$ plane. Contours at $\Delta w = -.02, -.01, .01, .02$. The shading progresses from black ($< -.02$) to white $> .02$.

$$
m^\nu = v(\hat{h} - 3\hat{f})
\qquad m^d = v(r_1\hat{h} + r_2\hat{f})
\qquad m^l = v(r_1\hat{h} - 3r_2\hat{f})$$

Here we have extracted $v = 174 GeV$. Note the characteristic (Georgi-Jarlskog) [26] factors of (-3) in the lepton masses relative to the quark masses. Here [30, 31, 11]

$$
\hat{h} = 2\sqrt{2}h\alpha_1 \sin \beta; \quad \hat{f} = -4\sqrt{2}f\alpha_2 \sin \beta
\qquad r_1 = \frac{\alpha_1}{\alpha_2} \cot \beta; \quad r_2 = \frac{\alpha_2}{\alpha_2} \cot \beta$$

and $(h, f)$ are the couplings in the MSGUT superpotential. They are $3 \times 3$ symmetric matrices with rows and columns indexed by the SO(10) family indices $A, B = 1, 2, 3$ [10]
Figure 2: Contour Plot of the threshold corrections $\Delta_X$ to $\log_{10}(M_X/1\,\text{GeV})$ on the complex $x$ plane. Contours at $\Delta_X = -1, -0.5, 0.5, 1$. The shading progresses from black ($< -1$) to white $> 1$.

Notice that since $|\alpha_i| \leq 1$ by unitarity, and $|h, f| < .5$ by perturbativity, the inclusion of $\alpha_1, \alpha_2$ in the definitions can yield subtleties only at zeros of $\alpha_1, \alpha_2$. The Majorana mass parameters $M_\nu, M_\bar{\nu}$ of the left and right handed neutrinos (defined as the coefficients of $\nu\nu, \bar{\nu}\bar{\nu}$ in the superpotential are

$$M_\bar{\nu} = \hat{f} \hat{\sigma}$$
$$M_\nu = r_3 \hat{f}$$

where

$$\hat{\sigma} = \frac{i \bar{\sigma} \sqrt{3}}{\alpha_2 \sin \beta}$$
$$M_O = 2(M + \eta(3a - p))$$

(21) (22)
Figure 3: Contour Plot of the threshold corrections $\Delta_G$ to $\alpha_G^{-1}(M_X)$ on the complex $x$ plane. Contours at $\Delta_G = -10, -5, 5, 10$. The shading progresses from black ($< -10$) to white $> 10$.

\[ r_3 = -2i\sqrt{3}(\alpha_1 \gamma + 2\sqrt{3}\eta \alpha_2)(\frac{\alpha_4}{\alpha_2})(\frac{v}{M_O}) \sin \beta \]

From these one obtains the Type I and Type II seesaw Majorana masses (in the conventional normalization $W = M_\nu \nu \nu / 2 + ...$) of the low energy neutrinos:

\[ M_I^\nu = vr_4 \hat{n} \quad ; \quad M_{II}^\nu = 2vr_3 \hat{f} \]

\[ r_4 = \frac{i\alpha_2 \sin \beta v}{2\sqrt{3} \bar{\sigma}} \]

\[ \hat{n} = (\hat{h} - 3\hat{f}) \hat{f}^{-1}(\hat{h} - 3\hat{f}) \]  

(23)

For investigation of the Type I seesaw we will find it useful to define functions $F_I, F_{II}$ by

\[ M_I^\nu = (1.70 \times 10^{-3} eV) \sin \beta F_I \hat{n} \]  

\[ M_{II}^\nu = (1.70 \times 10^{-3} eV) \sin \beta F_{II} \hat{n} \]  

\[ (1.70 \times 10^{-3} eV) = v^2 / M_X^0 \]  

and we have omitted irrelevant phases.
Figure 4: Contour Plot of the threshold corrections to $\alpha_G^{-1}(M_X)$ on the complex $x$ plane illustrating the drastic effect of decreasing the value of the diagonal parameter $\lambda$. Contours at $\Delta_G = -10, -5, 5, 10$ the shading progresses from black ($<-10$) to white $>10$.

\begin{align*}
M_{\nu}^{II} &= (1.70 \times 10^{-3} \text{eV}) \sin \beta F_{II} \hat{f} \\
\hat{\sigma} &= \sqrt{x(1-3x)(1+x^2)} / (1-x)^2 \\
F_I &= 10^{-\Delta x} \frac{i\alpha_2}{2\sqrt{6}} \frac{g\sqrt{\eta}}{\Lambda} \sqrt{4|\tilde{a} + \tilde{\omega}|^2 + 2|\tilde{p} + \tilde{\omega}|^2} \\
F_{II} &= 10^{-\Delta x} (-2i\sqrt{3}g\alpha_4) \frac{(\alpha_1\gamma + 2\sqrt{3}\alpha_2\eta)}{\alpha_2\eta} \sqrt{4|\tilde{a} + \tilde{\omega}|^2 + 2|\tilde{p} + \tilde{\omega}|^2} \frac{\xi + 3\tilde{a} - \tilde{p}}{(\xi + 3\tilde{a} - \tilde{p})}
\end{align*}

(24)
Figure 5: Contour Plot of the threshold corrections $\Delta X$ to $\log_{10}(M_X/1\text{GeV})$ on the complex $x$ plane showing the minor effect of reducing the value of the diagonal parameter $\lambda$. Contours at $\Delta X = -1, -.5, .5, 1$ the shading progresses from black ($< -1$) to white.

In these formulae the scale parameter $m$ has been eliminated in favour of $\Delta X$ using eqn(18). Explicit forms for $F_I, F_{II}$ in terms of the parameter $x$ are given in the Appendix. We also define the ratio $R = |F_I/F_{II}|$ to capture the relative strength of the two seesaw contributions. A crucial point is that if one wishes to maintain perturbativity and work with GUT Yukawa couplings $\leq 1$ then the large $\tan \beta$ scenario is necessary. The Type I and Type II fits of the fermion mass-mixing data are carried out [20, 14, 22, 24, 25] assuming only the characteristic form of eqns.[19, 21] to derive “sum rules” [13, 20, 14, 21] among the different mass matrices at the GUT scale. In these generic fits, the freedom to choose the coefficients $r_1, r_2, r_3, r_4$ is assumed to be compatible with the theory in which the mechanisms are embedded. The fits then yield the neutrino masses only up to an overall scale and the required relative strength of Type I and Type II is simply assumed. In [1]
we showed that these assumptions are unjustified for the MSGUT which seems to impose extreme dominance of Type I over Type II fits and moreover is incapable of achieving neutrino masses as large as .05eV as required by experiment. Our argument was based on the following observations:

- When a pure Type II fit is assumed one finds that the maximal value of $\hat{f}$ eigenvalues is $\sim 10^{-2}$ while the corresponding values for $\hat{h}$ are about $10^2$ times larger. This is a structural feature due to the approximate equality of $m_b(M_X)$ and $m_\tau(M_X)$ which ensures that no Georgi-Jarlskog type contribution can be important for the third generation. As a result $\hat{n} \sim 10^2 \hat{f}$. This implies that the ratio $R$ defined above must obey $R \leq 10^{-3}$ for the pure Type II not to be overwhelmed by the Type I values it implies. Our initial survey of the MSGUT parameter space showed that, generically, this did not happen and pure Type II could not work. Even near exceptional points.
no escape was allowed by the need to maintain viable USMPs. In the next section we
describe a much more exhaustive survey of the MSGUT to support this conclusion.
Note that this is the crux of the reason for the failure of the Type II seesaw and it
applies equally to “mixed Type I- Type II ” fits[25].

- The Type I fits [15, 25] yield, typically, \( \hat{n} \sim 5\hat{f} \sim .3 \) and thus we see that the
magnitude of the function \( F_I \) will need to be greater than about \( 10^2 \) in order that
neutrino masses as large as the .05 eV required for compatibility with neutrino mass
squared splittings as large as \( \Delta(m^\nu)^2_{23} \sim .002eV^2 \) be achievable. This was seen to be
generically un-achievable in[1]. In the next section we prove that this requirement
combined with that of viable USMPs makes it impossible for the Type I fit to reach
viability. This will be interpreted as indicating a missing piece to the puzzle, the
omission of the 120-plet.

The estimate of the magnitude of \( \hat{n} \) chosen is obviously critical to the disqualification
of the MSGUT Type I masses. A value value \( \hat{n} \sim 10 \) would drastically modify our
conclusions. The 3 generation fits performed so far are all simply numerical-although
guided by analytic insight into where in the parameter space a fit might occur [15, 25].
Therefore every fit will have its own value of the eigenvalues of \( \hat{n}\hat{n}^\dagger \) which might in
principle have widely different magnitudes. However no such exceptional Type I fit
has been found so far. Although an analytical proof of this is still elusive - because
of the very complexity that obstructs the determination of an analytic fit, there
are heuristic arguments why a large \( \hat{n} \) fit is unlikely to exist. The charged fermion
fit succeeds because the second generation couplings of the 126 dominate those of the 10 so that the
Georgi- Jarskog mechanism can operate. Although the opposite (\( h_{33} >> f_{33} \)) is true for the third generation so that no such mechanism is effective
(as it should not since at \( M_X \) the b, \( \tau \) masses are roughly equal) yet the the values of
\( \hat{f} \) in the 2 – 3 sector are not so small that the eigenvalues of \( \hat{n} \) become larger than 1.
The Type I neutrino masses are found to obey a normal hierarchy where the second
and third generation neutrino masses are much larger than the first generation so
that the 2 – 3 block of \( \hat{n} \) is dominant over the rest of \( \hat{n} \). The large neutrino mixing in
the 2 – 3 sector implies that the matrix elements \( \hat{n} \) in that block are roughly of the
same order of magnitude. These elements are generically found to be of magnitude
less than 1 : the inverse hierarchical structure of \( \hat{f} \) ensures that the normal hierarchy
in \( \hat{h} – 3\hat{f} \) does not get amplified by the \( \hat{f}^{-1} \) factor in the Type I seesaw formula to
the extent of increasing the maximal eigenvalue of \( \hat{n} \) beyond .5 or so. An analytic
proof of this is still lacking. Thus in principle there is a loophole which is still
unclosed : exceptional Type I fits with maximal eigenvalue greater than 10 , if found,
will need to be reanalyzed separately. We emphasize that the question is not one of
finding improved or optimal fits but only of the order of magnitude of the maximal
eigenvalue of \( \hat{n} \). **We shall assume that such exceptional fits do not exist.**
In other words we assume that the Type I fits of [15, 25] are generic
and representative as regards the order of magnitude of \( \hat{n} \). It is important
to build a data base or other more compact description of all passable achieved fits
that may be used to survey all the possibilities. Thus a naive estimate for the typical
maximal magnitude of \( \hat{n} \): Max \( \hat{n} \sim \frac{h_3^3}{f_{33}} \gg 1 \) is actually far off the mark. As explained above the characteristic off-diagonal structure of the successful Type I fits in the generic Babu-Mohapatra proposal, seems to be intimately bound up with this difference between the naive and actual magnitudes of (the eigenvalues of) \( \hat{n} \). All these remarks are illustrated by the values taken from the published example in [25] (which is, however, not a completely satisfactory fit since the electron mass is fit only to an accuracy of about 10%).

\[
\hat{h} = \begin{pmatrix}
-0.00016 - 0.0013i & -0.00456 - 0.0035i & -0.00642 + 0.002i \\
-0.0046 - 0.0035i & -0.0584 - 0.00101 & 0.03142 \\
-0.00642 + 0.002i & 0.03142 & -0.5633 + 0.95531i
\end{pmatrix}
\]  

(26)

\[
\hat{f} = \begin{pmatrix}
0.00023 + 0.00018i & 0.00018 + 0.00017i & 0.00285 - 0.00053i \\
0.00018 + 0.00017i & 0.00697 + 0.00164i & -0.01347 - 0.00158i \\
0.00285 - 0.00053i & -0.01347 - 0.00158i & 0.00053 - 0.037858i
\end{pmatrix}
\]  

(27)

\[
\hat{f}^{-1} = \begin{pmatrix}
2997.76 + 347.393i & 115.886 - 418.543i & 125.607 - 173.883i \\
115.886 - 418.543i & 51.4218 + 16.1651i & -40.8494 + 15.3566i \\
125.607 - 173.883i & -40.8494 + 15.3566i & 18.5738 + 4.48526i
\end{pmatrix}
\]  

(28)

\[
\hat{n}_{33}^{est} = (\hat{h}_{33} - 3\hat{f}_{33})^2/\hat{f}_{33} = 6.37485 + 6.98329i
\]  

(29)

\[
\hat{n} = \begin{pmatrix}
0.0278 + 0.0284i & 0.0712 + 0.0287i & 0.02697 + 0.0393i \\
0.0712 + 0.0287i & 0.09225 + 0.02520i & -0.12975 - 0.08165i \\
0.02697 + 0.0393i & -0.12975 - 0.08165i & 0.00256 + 0.15193i
\end{pmatrix}
\]  

(30)

\[(\text{Eigenvalues}[\hat{n}, \hat{n}^\dagger])^\dagger = \{0.2771, 0.06298, 0.0188\}
\]  

(31)

The magnitude of the maximum eigenvalue of \( \hat{n} \) is seen to be almost 35 times smaller than the “generic argument” one! Thus generic arguments for this magnitude need to be taken with a large lump of salt !.

3 USMP and SMP plots : Gloomy Pictures ?

We wish to survey the USMP and SMP variation over the complex \( x \) plane. We will use generic but fixed values with magnitudes \( \sim 1 \) for the ‘diagonal’ slow parameters\[31 \[1\] \{\lambda, \eta\}. We have checked that for any such (real or complex) values there is no significant modification of our conclusions. If one lowers \( \lambda, \eta \) below about .2 or so the USMPs exhibit pathologies due to mass scale lowering. Specifically, an increase of \( \Delta_G \) beyond a value of 10 or a decrease below (say) \( -10 \) makes the perturbative formulae used throughout unreliable. Similarly varying the ‘non-diagonal’ slow parameters \( \gamma, \bar{\gamma} \) to small values, though not forbidden by USMPs\[31 \[1\] is also infructuous for purposes of viable SMPs.
| $x$       | $\xi$        | Symmetry/remarks                                                                 |
|-----------|--------------|---------------------------------------------------------------------------------|
| $\frac{1}{2}$ | -5           | $SU(5)$, zero of $p_2$                                                           |
| -1        | 10           | $SU(5)$ flipped, zero of $p_2$                                                  |
| 0         | 3            | $SU(3) \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$                       |
| $\frac{1}{3}$ | -2/3         | Flipped $SU(5) \times U(1)$                                                   |
| $\pm i$   | $3 \mp 6i$   | $SU(3) \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$                       |
| 0.123765  | 1.93008      | zero of $p_3$                                                                   |
| 0.646451 - 0.505389 i | 4.86831 + 3.29252 i | zero of $p_3$                                                                   |
| 0.646451 + 0.505389 i | 4.86831 - 3.29252 i | zero of $p_3$                                                                   |
| $(3 \pm i\sqrt{7})/8$ | $\frac{3}{2}(1 \mp i\sqrt{7})$ | zero of $q_2$                                                                   |
| -3.46301  | 28.2958      | zero of $p_5$                                                                   |
| 0.262212 ± 0.388123 i | 2.13469 + 3.38025 i | zero of $p_5$                                                                   |
| 0.358184 ± 0.133971 i | -0.49266 + 2.30098 i | zero of $p_5$                                                                   |
| 0.1984    | 1.167        | zero of $q_3'$                                                                  |
| -0.099 ± 2.24 i | 1.596 + 15.572 | zero of $q_3'$                                                                  |

Table 1: Special points with extended gauge symmetry and/or candidate seesaw enhancement.

Besides the survey across generic regions of the complex $x$ (equivalently the $3\xi$) plane(s) we must also examine the possibility \[10, 31, 32, 1, 43\] that the behaviour at certain exceptional points may invalidate the generic trend of seesaw exclusion. For this purpose we provide in Table I a list of the exceptional points. These consist on the one hand of a list of points of enhanced gauge symmetry where the USMPs typically show spikes\[31\] and on the other of special points where the SMPs may exhibit spikes\[1, 43\]. The complete analytic\[10, 32, 43\] identification of the latter set of points using the convenient parametrization of vevs in terms of Rational functions of the variable $x$ alone (i.e with $\xi$ eliminated in favour of $x$) is related in the Appendix.

### 3.1 The fatal weakness of Type II

Let us begin with a discussion of the viability of the Type II mechanism. The SMP $R$ we proposed\[1\] to monitor the strength of Type I relative to the Type II seesaw is particularly simple to analyze. It is the magnitude of a rational analytic function (the ratio of a polynomial of degree 10 to a polynomial of degree 9) of $x$ and is dependent on no other parameters.

$$R = \left| \frac{F_I}{F_{II}} \right| = \frac{(x - 1)(3x - 1)q_2q_3^2}{4(4x - 1)(x^2 + 1)q_3^2} = \left| \frac{P(x)}{Q(x)} \right|$$  \hspace{1cm} (32)

(the polynomials $q_i, q'_i$ are defined in Table II in the appendix)

Therefore $R$ is bounded from below by 0 and its minima are isolated: they are the zeros of $P(x)$. As $|x| \to \infty$ it grows as $|x|$ (with coefficient $3/64$). For $|x| \geq 3 \Rightarrow R \geq 10^{-1}$ or so. This is clearly visible in the 3D plot shown in Fig. 7. Thus only for $|x| \leq 3$ do we need to even consider the possibility of dominant Type II seesaw. This is in fact the region
which contains the zeros of $R$ namely $x = \{1/3, 1, \{(3 \pm i\sqrt{7})/8\}, \{0.198437, -0.0992186 \pm 2.24266i\}\}$ of which the latter two sets are the zeros of $q_2$ and $q'_3$ respectively. A picture of the variation of $R$ in this region of the $x$ plane is given by the 3D plots of $R^{-1}$ in Fig. 8 which clearly shows that, as expected, the zeros of the numerator dominate the behaviour.

Figure 7: 3D Plot of the relative strength $R$ of Type I and Type II seesaw mechanisms on the complex $x$ plane showing that the region $|x| > 3$ is irrelevant for promoting Type II dominance.

The real zeros of $R$ show up as “cliffs” (instead of narrow peaks) due to periodicity in $\text{Arg} \, x$ which slices them vertically, while “ridges” observed in these graphs are associated with the complex zeros of $q_2, q'_3$ (one needs to magnify the graph to see the ridge form of the zeros of $q_2$). There is a strong preference for a narrow range of $\text{Arg} \, x$ for minimizing $R$ (as also seen directly in the plots of $R$ in the neighbourhood of these zeros discussed below). Thus the growth of $R^{-1}$ is seen to be well localized to the regions around the zeros of $R$. From this survey as well as the theory of analytic functions it is clear that the behaviour of $R$ and the USMPs in the neighbourhood of its zeros will be sufficient to decide whether any Type II solutions are viable. We now discuss this in greater detail.

Since Type II dominance desires as small a value as possible of $R$ we need to approach as closely as viable to the zeros of $P(x)$. Among the zeros of $R$, however, are also the zeros of $\alpha_2: \{x = \{1/3, 1, \{0.1984, -0.099 \pm 2.24i\}\}$ (zeros of $q'_3$). The decrease of $|\alpha_2|$ is limited by the requirement of maintaining perturbativity in the superpotential Yukawa coupling $\sim 3\hat{f}/\alpha_2 < 1$. Type II dominant fits yield $\hat{f}$ about $10^{-2}$ [22, 24, 25, 1]. Even if
Figure 8: 3D Plot of the relative strength $R^{-1}$ of Type II versus Type I seesaw mechanisms on the complex $x$ plane showing the zeros of $R$ in the region $|x| > 3$.

we are generous with the coefficients ($\sim 3$) that occur in the relation, $\alpha_2$ cannot decrease below $10^{-2}$: so we must maintain at least such distance from the zeros of $\alpha_2$ so that: $|\alpha_2| > 10^{-2}$. The remaining zeros of $R$ are just those of $q_2$. We shall see below that approach to these zeros is also limited by the USMPs. Thus $R$ cannot viably decrease below $\bar{R}_{\text{min}} = \text{Min}\{\bar{R}(x_i), i = 1, \ldots, 7\}$ the values of $R$ at closest permissible approach to its zeros $x_i$.

No further survey of the parameter space is thus required than to evaluate $\bar{R}_{\text{min}}$. To estimate $\bar{R}_{\text{min}}(x_i)$ is straightforward given our established USMP-SMP numerical techniques\cite{31,11}. We allow an approach to the zeros of $R$ as closely as will permit $\alpha_2 \geq 10^{-2}$. This limits the radius of the circle of closest approach to the zeros $x = \{1/3, 1, \{0.198437, -0.0992186 \pm 2.24266i\}\}$ of $\alpha_2$, for the 3 real zeros and the pair of complex zeros respectively, to be $\epsilon_i = 10^{-2}\{.7, 3.9, 4., .53\}^{-1} = \{.014, .0025, .0025, .02\}$. We observe values of

\[
\{x_i, \epsilon_i, \bar{R}_{\text{min}}(x_i), \{USMPs\}\} = \{1, .014, .007, \{\Delta_G \sim -14, \Delta_W \sim -.11\}\}
= \{.3333, .0025, .015, \{USMPs OK\}\}
= \{.198437, .0025, 6 \times 10^{-5}, \{\Delta_X < -2.9\}\}
= \{.198437, .025, .016, \{\Delta_X < -2.4\}\}
= \{-.099219 + 2.2426i, .02, 2 \times 10^{-5},...
\]
The closest one comes to the required values of $R \sim 10^{-4}$ or smaller is in the case of the complex zeros of $q'_3$. However even in this case the value of $F_{II}$ is simply too small and yields a maximal neutrino mass of about $4 \times 10^{-5} \text{eV}$ (since $\tilde{f} \sim 10^{-2}$). To illustrate the behaviour in this most favourable but still unviable case we exhibit the variation of $R, \Delta_X, F_{II}$ on a circle of radius .02 around a complex zero of $q_3$ as Figs. 9,10,11.

Finally we turn to the zeros of $q_2$. They are not zeros of $\alpha_2$ so there is no constraint on approach to the zero from perturbativity but the conclusions are equally negative:

$$\{x, \epsilon, R_{\min}(x_i), \{USMPs\} =$$
$$\{(3 + i\sqrt{7})/8, .06, .015, \{\Delta_X < -0.9, F_{II} \sim 80\}\}$$
$$\{(3 + i\sqrt{7})/8, .005, .0014, \{\Delta_X < -1.3, F_{II} \sim 2100\}\}$$

The rapid fall of $\Delta_X$ (which also promotes $F_{II}$ growth) as one approaches the zeros excludes them long before the required small values of $R$ are reached. The strong decrease of $\Delta_X$ for values of $x$ near $x = 0$ is obvious from Fig. 2 and Fig. 5.

To sum up: the zeros of $R$ are ringed by zones of exclusion that ensure the parameters of the MSGUT can never be chosen to ensure Type II domination while maintaining viable USMPs.
Figure 10: Plot of $\Delta X$ on a circle of radius .02 around $x_0 = -0.099219 + 2.2426i$

Figure 11: Plot of $F_{II}$ on a circle of radius .02 around $x_0 = -0.099219 + 2.2426i$
3.2 Type I fails: Collapse on the final lap.

We next turn to a consideration of the overall magnitude of the Type I seesaw masses which we have shown to dominate the Type II contribution almost everywhere. The Type I mass formula reads - up to an irrelevant overall phase -

\[ M_\nu^I = 1.7 \times 10^{-3} eV F_I \hat{n} \sim \beta \]

\[ F_I = \frac{\gamma g}{2\sqrt{2}\eta\lambda} \left( \frac{1 - 3x}{x(x^2 + 1)} \right) \left( \frac{|p_2p_3|\sqrt{z_2}}{\sqrt{z_{16}}} \right) 10^{-\Delta x} q'_3 \]

\[ = \hat{F}_I 10^{-\Delta x} \]  

The complicated effect of the RG thresholds and evolution is all contained inside \( \Delta_X \). However this quantity is itself limited to lie above \(-1\) by Baryon lifetime > \(10^{33}\) yrs! Before actually inspecting the effect of \( \Delta_X \) in detail we should therefore inspect whether the almost completely known function \( \hat{F}_I \) could ever achieve the magnitude of 10 or so it would require to yield neutrino masses of around .05 eV (recall \( \hat{n} \sim .3 \) and \( \Delta_X \) > \(-1\)) in the most favourable case. This seems a rather mild requirement but we shall show that it cannot be satisfied viably anywhere in the parameter space of the MSGUT!

The limiting behaviour of \( |\hat{F}_I|\) is

\[ |x| \to \infty \Rightarrow |\hat{F}_I| \to \sim .02 \]

\[ |x| \to \{0, \pm i\} \Rightarrow |\hat{F}_I| \to \infty \]  

The approach to the asymptotic value is rather rapid while the singularities as \( x \sim \{0, \pm i\} \) are quite mild : \( \sim |\Delta x|^{-\frac{1}{2}} \). Figs 12, 13 which are 3D plots of \( |\hat{F}_I| \) over the complex \( x \) plane for \( |x| < 10 \) (more than sufficient to reach the limiting form) and \( |x| < 1.5 \) (to show the singularities at \( x = 0, \pm i \) more clearly).

The dependence of \( \hat{F}_I \) on the ‘slow AM’ parameters \( \{\gamma, \bar{\gamma}, \eta, \lambda\} \) is rather marginal - as long as we do not consider pathological cases where \( \eta \) or \( \lambda \) are very small (forbidden by the USMP variation [1]) or pathological cases where perturbativity in any of of these couplings is lost. We have verified this by examining 3-dimensional plots of \( F_I \) over the complex \( x \) plane for random complex values of these couplings without finding any visible variation. Apart from the constraint that there magnitudes are greater than \(.2\) their values can be chosen randomly without any appreciable effect.

In Fig. 12 and Fig. 13 the rapid approach to the asymptotic value of \( |\hat{F}_I| \) beyond \( |x| = 1.2 \), the ridge associated with \( x = 0 \) (due to its indeterminate phase), and the pointed ridges associated with \( x = \pm i \), as well as the boundedness by about \(.1\) except right close to the singularities, are the relevant and crucial features. Plots which have very different coupling values are almost indistinguishable from Figs. 12, 13. This is easy to understand in terms of the expression for \( Z_{16} \) (Table II) since that is where the dependence on these parameters enters. It is evident that the variation of the slow parameters has no appreciable effect at all. So they can be set to have any values with magnitudes \( \sim 1 \)
without any loss of generality. The behaviour of \( \hat{F} \) implies that only very close to the singular points \( x = \{0, \pm i\} \) is there any hope that \(|\hat{F}_I|\) will achieve the required values \( \geq 10 \). Elsewhere it is manifestly bounded by \( .1 \) or so and is thus a factor of \( 10^2 \) short of the required magnitude (in the most favourable case of \( \Delta_X = -1 \); which value is itself only borderline compatible with Super-Kamiokande data on baryon lifetime).

In Fig. 14, 15 we exhibit contour plots of \( \hat{F}_I \) and \( F_I \) respectively on the complex \( x \)-plane. Evidently only around \( x = 0, \pm i \) is there any chance of large enough \( \hat{F}_I \) However neither region gives a viable result. Firstly, it is evident from Fig. 2 that the region around \( x = 0 \) is forbidden since \( \Delta_X < -1 \). A ‘scissors’ operates between the exploding effect of \( \Delta_X \) as one approaches \( x = 0 \) and the rapidly decreasing \(|\hat{F}_I|\) as one recedes from it.

Around \( x = \pm i, \Delta_X \) is > 1 and hence will suppress rather than enhance the favourable value of \( \hat{F}_I \). This is evident in Fig. 15 where \( x = \pm i \) are out of the running altogether. One is again caught in a scissors. At this point (of enhanced \( G_{3211} \) symmetry) \( \Delta_X \) is large and \textit{positive} at the singularity. So it completely overwhelms the growth of \( \hat{F}_I \) due to the singularity.

\textbf{Thus we have shown that even via Type I seesaw there is no possibility of achieving neutrino masses larger than about} \( 5 \times 10^{-3} \text{eV} \) \textit{in the MSGUT} !

Although the argument given above is complete we also studied the behaviour at other special points such as \( x = \{-1, 1/3, .5\} \) and the zeros of \( \{q_2(x), q_3, p_3(x), p_5(x)\} \), some
of which have been claimed (and some not!) to represent candidate points for Type I enhancement \cite{13} but are not in fact so. We did not find any appreciable growth of $F_I$ with tolerable values of $\Delta_X$ at any such point : exactly as expected from our arguments given above. Some growth is observed for the $x$ values (digits truncated ) 0.1237 (real zero of $p_3$), 0.33333 (enhanced symmetry point with symmetry $SU(5)_{\text{flipped}} \times U(1)$) and .198 (real zero of $q'_3$ ). In all three cases it is due to catastrophic fall in $\Delta_X$. The contour plot of $F_I$ neatly summarizes all this behaviour since only the forbidden region around $x = 0$ ever gives large enough $F_I$.

To trace the reason for the divergence in our candidates from those of \cite{13} is easy : if we translate the expressions of \cite{13} which are written in terms of $m_l, m_d - m_l$ into our expressions written in terms of $\hat{h}, \hat{f}$ we immediately obtain that the functions $f_I, f_{II}$ of \cite{13} need to be multiplied by (in our notation) $\{r_1^2/(4r_2), r_2\} \sim \{(p_3p_4)/(p_5p_2), (p_5p_2)/(p_3p_4)\}$ respectively and this then removes the aforementioned points from the list of candidates , leaving , after all repeated factors are cancelled, only $\{0, \pm i\}$ as candidates for strong Type I seesaw and the zeros of $R$ namely $x = \{1/3, 1\}$ and the zeros of $q_2$ and $q'_3$.

4 The Next MSGUT

We have seen that the Type II seesaw mechanism does not work when only the $10$ and $\overline{126}$ FM Higgs irreps are employed and even the Type I mechanism fails even though not as badly
as Type II. It is natural then to look for a role for the remaining FM Higgs multiplet type allowed by SO(10): the long neglected 120-plet irrep. This irrep has a Yukawa coupling matrix that obeys $g_{AB} = -g_{BA}$ i.e. is family index antisymmetric. As such it introduces novel properties and problems into the fitting matrix problem. It has so far been cast in a minor supporting role\cite{17, 24} in the story of the seesaw. Only a few authors\cite{14, 16, 18, 19} have considered the 120plet in a large role and even they did not analyse the fitting problem comprehensively. The 10 and 120 irreps may together successfully explain the charged fermion spectrum, particularly since the 120-plet also contains, besides a $(1,2,2)$ sub-representation, a $(15,2,2)$ subrep (w.r.t the Pati-Salam subgroup). Thus it can also be used to implement -albeit in a novel way due to its antisymmetric Yukawa couplings-a Georgi-Jarlskog\cite{26} mechanism to explain the approximate equality of $m_s$ and $m_\mu$ at $M_S$ in the MSSM. This relieves the 126 of the multiple loads it has shouldered so far and which, finally, the present papers show it tends to cast down on the last lap of tests.

Figure 14: Contour Plot of $|\hat{F}_I|$ on the $x$-plane with contours at $|\hat{F}_I| = .1,.9$. Lower values are shaded darker. The small white islands are around $x = 0, \pm i$. 
Figure 15: Contour Plot of $|F_1|$ on the $x$-plane with contour at $|F_1| = 10$ lower values are shaded darker. Note that the required large values can only lie right close to the forbidden region around $x = 0$ where $\Delta X < -1$.

Once the $\mathbf{126}$-plet coupling $f_{AB}$ is relieved of the necessity of being comparable to the couplings of the $\mathbf{10}$-plet the Type I seesaw mechanism can receive the required further 10-100 fold boost in magnitude since the Type I seesaw masses are proportional to $f^{-1}$. It should be emphasized that the smallness of $f$ in this scenario must be considered as a structurally defining condition rather than as the smallness of a perturbation because it leads to light right handed neutrinos. A similar scenario was considered in [16] but analyzed by making various somewhat arbitrary assumptions. Thus no generic analysis like that for the $\mathbf{10} - \mathbf{126}$ [21, 29, 22, 24] system is yet available to compare with.

In this way we propose that our results provide a hint of the direction to proceed: every member of the SO(10) FM Higgs family will receive its characteristic function, role and gainful employment. It seems that the Good God may be just at least to the different breeds of Higgs particles!
To implement our (long felt\cite{48}) intuition of a proper share and role for each Higgs type in the intricately wrought elegance of SO(10) unification we have calculated\cite{46} the complete set of couplings and masses of the Next to Minimal Supersymmetric GUT or NMSGUT defined by the inclusion of a 120-plet Higgs into the MSGUT Higgs complement. Here we restrict ourselves to a brief survey of the essential features.

In the notation of\cite{30} the real vector indices of the upper left block embedding of SO(6) in SO(10) are denoted by a, b = 1, 2, 3, 4 and of the lower right block embedding of SO(4) in SO(10) by ū, v = 7, 8, 9, 10. The doublet indices of SU(2)L(SU(2)R) are denoted by α, β = 1, 2(ū, v = 1, 2). The index of the fundamental 4-plet of SU(4) is denoted by a, b = 1, 2, 3, 4 and its upper-left block SU(3) subgroup indices are ť, ū = 1, 2, 3.

The decomposition of the 120-plet w.r.t \(SU(4) \times SU(2)_L \times SU(2)_R\) is as follows.

\[
O_{ijk}(120) = O_{abc}(10 + \overline{10}, 1, 1) + O_{a\bar{a}b\bar{b}}(15, 2, 2) + O_{a\bar{a}b\bar{b}}((6, 1, 3) + (6, 3, 1)) + O_{\bar{a}\bar{b}\bar{b}}(1, 2, 2)
\]

= \(O_{\mu}^{(s)}(10, 1, 1) + O_{\mu}^{(a)}(1, 2, 2)\)

\[
+ O_{\mu\bar{a}\bar{b}}(6, 1, 3) + O_{\mu\bar{a}\bar{b}}(6, 3, 1) + O_{\bar{a}\bar{b}}(1, 2, 2)
\]

(37)

(where we have used the superscripts \(^{(s)},^{(a)}\) to discriminate the symmetric 10-plet from the antisymmetric 6-plet).

The decomposition of its coupling to 16-plets of SO(10) is \cite{30}

\[
\frac{1}{(3!)} \psi C_{2}^{(5)} \gamma_i \gamma_j \gamma_k \chi O_{ijk} = -2(O_{\mu}^{(s)} \psi_{\mu} \chi_{\alpha}) + O_{\mu}^{(a)} \psi_{\mu} \chi_{\alpha} + 2\sqrt{2}O_{\mu}^{\mu \alpha \dot{\alpha}} \psi_{\mu} \chi_{\alpha} + 2(O_{\mu}^{(a)} \psi_{\mu} \chi_{\alpha}) + 2 \sqrt{2}O^{\alpha \dot{\alpha}} \psi_{\mu} \chi_{\alpha} - \sqrt{2}O^{\alpha \dot{\alpha}} \psi_{\mu} \chi_{\alpha} + \sqrt{2}O^{\alpha \dot{\alpha}} \psi_{\mu} \chi_{\alpha}
\]

(38)

The \(SU(4) \times SU(2)_L \times SU(2)_R\) labels in the above\cite{30} equations are trivially decomposable to SM labels and thus such expressions – available for every coupling in the MSGUT and NMSGUT–constitute a complete solution for the Clebsches to these MSGUTs.

The 120-plet contributes two pairs \(h_5^\alpha = O_1^\alpha, \tilde{h}_6^\alpha = O_2^\alpha, h_5^\alpha = O_1^{(15)}^\alpha, \tilde{h}_6^\alpha = O_2^{(15)}^\alpha\) of MSSM \([1,2,1] \oplus [1,2,-1]\) doublets. Their Yukawa couplings to MSSM fermions follow immediately from the above unitary decompositions:

\[
\begin{align*}
m^u & = v(\hat{h} + \hat{f} + \hat{g}); \quad r_1 = \frac{\bar{a}_1}{a_1} \cot \beta; \quad r_2 = \frac{\bar{a}_2}{a_2} \cot \beta \\
m^d & = v(r_1 \hat{h} + r_2 \hat{f} + r_3 \hat{g}); \quad r_5 = \frac{4i\sqrt{3} \bar{a}_5}{a_6 + i\sqrt{3} \bar{a}_5} \\
m^l & = v(r_1 \hat{h} - 3r_2 \hat{f} + r_3 \hat{g}); \quad r_5 = \frac{4i\sqrt{3} \bar{a}_5}{a_6 + i\sqrt{3} \bar{a}_5} \cot \beta
\end{align*}
\]
\[ \hat{g} = 2i g \sqrt{\frac{2}{3}} (\alpha_6 + i \sqrt{3} \alpha_5) \sin \beta ; \quad r_7 = (\bar{r}_5 - 3r_6) \] (39)

which clearly exhibits the analogy with the Georgi-Jarlskog type couplings in the MSGUT. We propose to fit the charged fermion masses using light doublet components of predominantly 10 and 120-plet origin. The additional terms in the superpotential do not modify the AM SSB and moreover still contain only the 26 MSSM multiplet types that we described\cite{31} for the MSGUT. The corresponding mass matrices - some of whose dimensions, like the doublet mass matrix given below, are raised relative to the MSGUT - will be given, in our notation, and along with the corresponding RG and USMP-SMP analysis in\cite{10}. A corresponding calculation of mass matrices only - with very different conventions - is already available\cite{44}.

The extra terms contributed by the 120-plet to the superpotential are:

\[
W_{120} = \frac{M_0}{2(3!)} O_{ijk} O_{ijk} + \frac{k}{3!} H_i O_{jkl} \Phi_{ijkl} + \frac{\rho}{4!} O_{ijm} O_{klm} \Phi_{ijkl} \nonumber \]

\[
+ \frac{\zeta}{2(3!)} + O_{ijk} \Sigma_{ijklmn} \Phi_{klmn} + \frac{\bar{\zeta}}{2(3!)} O_{ijk} \Sigma_{ijklmn} \Phi_{klmn} \tag{40}
\]

The coupling of the \( P([3, 3, \pm 2/3]) \) and \( K([3, 1, \pm 8/3]) \)-type color triplets contained in the 120-plet leads to quite novel, family index antisymmetric and even \( SU(2)_L \) triplet mediated, contributions to the effective LLLL and RRRR type \( \Delta B \neq 0 \) violating \( d=5 \) operators in the low energy effective theory\cite{10}. These may lend additional texture to the emerging story of the deep connections between neutrino mass and Baryon violation\cite{49, 31}.

The composition of the MSSM \([1, 2, \pm 1]\) doublet sector can be deduced from the left and right null eigenvectors of the 6 \( \times \) 6 mass matrix of the NMSGUT analogously to those defined for the MSGUT in Section II.

\[
\begin{pmatrix}
-M_H & \gamma \sqrt{3}(\omega - a) & -\gamma \sqrt{3}(\omega + a) & -\gamma \bar{\sigma} & kp & -\sqrt{3}k \omega \\
-\gamma \sqrt{3}(\omega + a) & 0 & -(2M + 4\eta(a + \omega)) & 0 & -\sqrt{3}\zeta \omega & i(p + 2\omega) \zeta \\
\gamma \sqrt{3}(\omega - a) & -(2M + 4\eta(a - \omega)) & 0 & -2\eta \sqrt{3} & \sqrt{3} \zeta \omega & -i(p - 2\omega) \zeta \\
-\sigma \bar{\gamma} & -2\eta \sqrt{3} & 0 & -2m + 6\lambda(\omega - a) & \zeta \sigma & \sqrt{3} \zeta \sigma \\
kp & \sqrt{3} \omega & -\sqrt{3} \omega & \zeta \theta & -M_o & \frac{\sqrt{3} \omega}{\zeta} \\
\sqrt{3}k \omega & i(p - 2\omega) \zeta & -i(p + 2\omega) \zeta & -\sqrt{3} \zeta \sigma & -\frac{\sqrt{3} \omega}{\zeta} & -M_o - \frac{4\sqrt{3}}{3} a
\end{pmatrix}
\]

The left and right null eigenvectors are calculated after imposing the light doublet condition \( \text{Det} \mathcal{H} = 0 \).

The relative strength of the contributions, of the 10 and 120-plets vis a vis the 126-plet, to the charged fermion mass matrices can be studied by examining the variation of \( \alpha_2/\alpha_{1,5,6}, \bar{\alpha}_2/\bar{\alpha}_{1,5,6} \) in exactly the same manner as already done by us in this paper. The pure form of this scenario can thus be examined directly in terms of fitting the charged fermion masses and mixings in terms of 10 and 120-plet contributions only. If this succeeds to a within about 10%, the effect of including the 126-plet contributions can be examined\cite{10}. Naturally, the investigation is best begun with the 2 \( \times \) 2 case for the 10 + \( \bar{126} \) system\cite{21, 29, 19}. This is reported in\cite{47}.

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5 Conclusions and Outlook

The Supersymmetric GUT based on the the $210 \oplus 126 \oplus T_{26}$ AM Higgs system is the simplest Supersymmetric GUT that elegantly (almost, v.supral!) realizes the classic program of Grand Unification. Its symmetry breaking structure is so simple as to permit an explicit analysis of its mass spectrum at the GUT scale and an evaluation therefrom of the threshold corrections and mixing matrices relevant to various important quantities. It implements in a fascinating and elegant way an intimate (“ouroborotic”) connection between the physics of Lepton number and Baryon number violation. Since it has the least number of parameters of any theory that accomplishes as much this theory merited the name of the minimal supersymmetric GUT or MSGUT. The same simplicity and analyzability of GUT scale structure also applies to the theory with an additional $120$-plet, since it contains no standard model singlets, and thus justifies calling it the Next to Minimal (or New Minimal) Susy GUT (NMSGUT). The small number of Yukawa couplings of the MSGUT makes the fit to the now well characterized fermion mass spectra very tight. In[1] we initially observed and in this paper we have shown that the combined constraints of the seesaw fit and the preservation of MSSM one loop gauge unification are enough to rule out the MSGUT.

We now argue that the very natural inclusion of the third possible SO(10)FM Higgs type, i.e. the $120$-plet, which was somewhat arbitrarily[9]–but very fruitfully[13, 20, 15, 14, 21, 29, 22, 17, 24, 25] –excluded from a leading role in the ‘SO(10)party on the GUT scene’, offers a natural resolution of the difficulty of the MSGUT in achieving large enough Type I seesaw masses. In this scenario the $10$-plet and $120$-plet are primarily responsible for the charged fermion mass fit due to a relatively suppressed contribution of the $T_{26}$-plet to the charged fermion masses due to a small value of the Yukawa coupling $16 \cdot 16 \cdot T_{26}$. This very suppression implies that the Type I seesaw will become enhanced essentially due to lowering of the righthanded neutrino masses. Thus far from being a perturbation the role of the $T_{26}$ is rather to define a very specific class of Grand Desert in which the right handed neutrinos have masses considerably less than the generically expected GUT scale masses. Thus this proposal, which was thrust upon us by detailed analysis of compatibility between GUT and Neutrino mass scale hierarchies, is distinct from previous scenarios and, moreover, has very clear and distinct phenomenological consequences. Furthermore the $T_{26}$-plet Yukawa coupling is itself released from the Type I charged fermion constraints and is subject only to the relatively mild constraints coming from the right handed Majorana neutrino mass limits from cosmology. If this does not conflict with the requirement to achieve the small-large /quark-lepton mixing duality then the one degree of magnitude failure of the seesaw in the MSGUT should be quite surmountable. In a slogan:

The MSGUT is dead! Long live the (Next)MSGUT!
6 Acknowledgments

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7 Appendix

In this Appendix we express our formulae for neutrino masses etc in terms of the variable \( x \) which parameterizes the fast variation of AM scale mass matrices in an elegant and simplifying manner, as has been particularly emphasized by. This allows the easy analytic localization of putative points of seesaw growth. It enhances confidence in the completeness of the exclusion by survey of any possible seesaw fit. The use of the variable \( x \) also significantly simplifies the presentation of data since considering variation over the \( x \)-plane unifies consideration of the three solutions for \( x \) corresponding to a given value of \( \xi \) while retaining complete clarity due to the rational (3 to one) map from the \( x \)-plane to the \( \xi \) plane. The process of translation merely makes explicit in the mass matrices what was already done when parameterizing the AM vevs in terms of \( x \).

If we substitute for the AM vevs in terms of \( x \), the equations for the Null eigen-vectors of \( \mathcal{H} \) - the doublet mass matrix of the MSGUT - subject to the condition \( \text{Det}\mathcal{H} = 0 \) become:

\[
A = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} = N\{\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4\} = N\hat{A}
\]

\[
\hat{A} = \{\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4\} = \overline{N}\{\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4\} = \overline{N}\hat{A}
\]

\[
\hat{A} = \left\{1, \left(\frac{\sqrt{3} \gamma p_4 (1-x)}{2 \eta p_5}\right), \left(\frac{\sqrt{3} \gamma p_2 (1-x)}{2 \eta p_3}\right), \frac{\gamma q_3 \bar{\sigma} (x-1)}{2 \sqrt{\eta \lambda}}\right\}
\]

\[
\hat{A} = \left\{1, \left(\frac{\sqrt{3} \gamma p_4 (1-x)}{2 \eta p_5}\right), \left(\frac{\sqrt{3} \gamma p_2 (1-x)}{2 \eta p_3}\right), \frac{\gamma q_3 \bar{\sigma} (x-1)}{2 \sqrt{\eta \lambda}}\right\}
\]

\[
N = \frac{|p_3 p_5|}{\sqrt{z_{16}}}
\]

\[
\overline{N} = \frac{|p_3 p_5|}{\sqrt{z_{16}}}
\]

The crucial functions \( F_I, F_{II}, R \) used as SMPs in our survey now become (up to irrelevant overall parameter phases)

\[
(41)
\]
\[
F_I = \frac{10^{-\Delta X}}{2\sqrt{2}} \frac{\gamma g}{\sqrt{\eta \lambda}} |p_2 p_3 p_5| \sqrt{\frac{z_2}{z_{16}}} \left( \frac{1 - 3x}{x(1 + x^2)} \right) p_3 \\
F_{II} = \frac{10^{-\Delta X}}{2\sqrt{2}} \frac{2\sqrt{2} \gamma g}{\sqrt{\eta \lambda}} |p_2 p_3 p_5| \sqrt{\frac{z_2}{z_{16}}} \left( \frac{(x^2 + 1)(4x - 1)q_2^2}{x(1 - 3x)} \right) q_3 q_2 p_5 \\
R = \frac{|F_I|}{|F_{II}|} = \left| \frac{(x - 1)(3x - 1)q_2 q_3^2}{8(4x - 1)(x^2 + 1)q_3^3} \right|
\]

(42)

The various polynomials defined in [32, 43] together with some additional definitions are given in Table II.

A number of remarks are in order

• The asymptotic behaviour of \{\hat{F}_I, \hat{F}_{II}, R\} as \(|x| \rightarrow \infty\) is \{|x|^0, |x|^{-1}, |x|\} and as \(x \rightarrow 0\) is \{|x|^{-\frac{1}{2}}, |x|^{-\frac{1}{2}}, |x|^0\} so it is obvious that only a bounded range of (say) \(|x| < 10\) need be considered to check if \(R\) is small or \(\hat{F}_I\) is large.

• For Type II not to be dominated by the Type I implies the ratio \(R\) should be very small (\(\sim 10^{-4}\)). So it would seem that the enhanced symmetry points \(x = 0, 1/3\) and the additional points \(x = \left\{ \frac{3 \pm i\sqrt{7}}{8}, 0.198437, -0.0992186 \pm 2.24266 i \right\}\), which are the zeros of \(q_2, q_3^3\) may help to strengthen the Type II seesaw and should be added to our list of exceptional points. If one extracts a factor \(\alpha_2^{-2}\) to define \(R' = R\alpha_2^{-2}\) one sees that only the the zeros of \(q_2\) are of any relevance to strengthening Type II seesaw. Our different parametrization of the problem of dominance together with the extraction of the hidden dependence on the threshold corrections to \(M_X\) which is coded in \(\Delta X\) makes our analysis different from [43]. Differences from [43] in the points relevant for Type II have emerged: we exclude \(x = 0\) and the zeros of \(p_5, p_2\) since they cancel with the normalizations (which we have not left implicit as in [32]). Moreover we have used expressions in terms of the actual Yukawa couplings rather than mass eigenvalues which can conceal compensating growth of Yukawas and coefficients \(\alpha_i\) necessitating separate consideration of whether one has ventured into strong Yukawa coupling regions. Nevertheless we have included all such putative special points points in our survey and checked that they do not provide the required exceptional behaviour.

• The functions \(z_2, z_{16}\) have no zeros even when \(\gamma, \bar{\gamma}\) are both zero. When \(\gamma, \bar{\gamma}\) are non zero the Normalizing factors never diverge.

• The points \(x = 0, \pm i\) are the only remaining candidates to strengthen Type I and are already included in the list of enhanced symmetry points.

**Note Added:** Our surmise was announced in May/June 2005 (PLANCK05,Trieste, May 2005 and [hep-ph/0506291]) and our proof appeared as [hep-ph/0512224v1]). In May 2006 a calculation appeared [50] confirming that optimal FM fits found by the “downhill simplex” method are not viable in the MSGUT, for exactly the reasons given by us.
| $q_2$ | $4x^2 - 3x + 1$ |
| $q_3$ | $4x^3 - 9x^2 + 9x - 2$ |
| $q_4'$ | $p_4/(3x - 1) = x^3 + 5x - 1$ |
| $p_2$ | $(2x - 1)(x + 1)$ |
| $p_3$ | $12x^2 - 17x^2 + 10x - 1$ |
| $p_4$ | $(3x - 1)(x^3 + 5x - 1)$ |
| $p_5$ | $9x^5 + 20x^4 - 32x^3 + 21x^2 - 7x + 1$ |
| $z_2$ | $2x^2 + [1 - x]^2$ |
| $z_{16}$ | $|p_3p_5|^2 + \frac{3}{4} \gamma(x-1)p_3p_5|2 + \frac{3}{4} \gamma(x-1)p_2p_4|2 + \frac{1}{2} |\gamma(x-1)p_3q_3\sqrt{x(1-3x)(1+x^2)|2}$ |
| $\bar{z}_{16}$ | $|p_3p_5|^2 + \frac{3}{4} \gamma(x-1)p_3p_4|2 + \frac{3}{4} \gamma(x-1)p_2p_4|2 + \frac{1}{2} |\gamma(x-1)p_3q_3\sqrt{x(1-3x)(1+x^2)|2}$ |
| $N$ | $|p_3p_5|/\sqrt{z_{16}}$ |
| $\bar{N}$ | $|p_3p_5|/\sqrt{\bar{z}_{16}}$ |

**Table 2: Functions of $x$ entering the expressions for the USMPs and SMPs.**

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