E DGE-C LIQUE G RAPH S OF COCKTAIL P ARTIES H AVE U NB OUNDED R ANKWIDTH

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Abstract. In an attempt to find a polynomial-time algorithm for the edge-clique cover problem on cographs we tried to prove that the edge-clique graphs of cographs have bounded rankwidth. However, this is not the case. In this note we show that the edge-clique graphs of cocktail party graphs have unbounded rankwidth.

1 Introduction

Let $G = (V, E)$ be an undirected graph with vertex set $V$ and edge set $E$. A clique is a complete subgraph of $G$.

Definition 1. A edge-clique covering of $G$ is a family of complete subgraphs such that each edge of $G$ is in at least one member of the family.

The minimal cardinality of such a family is the edge-clique covering number, and we denote it by $\theta_e(G)$.

The problem of deciding if $\theta_e(G) \leq k$, for a given natural number $k$, is NP-complete \cite{24,30,21}. The problem remains NP-complete when restricted to graphs with maximum degree at most six \cite{22}. Hoover \cite{22} gives a polynomial time algorithm for graphs with maximum degree at most five. For graphs with maximum degree less than five, this was already done by Pullman \cite{32}. Also for linegraphs the problem can be solved in polynomial time \cite{30,32}.

In \cite{24} it is shown that approximating the clique covering number within a constant factor smaller than two remains NP-complete.

Gyárfás \cite{20} showed the following interesting lowerbound. Two vertices $x$ and $y$ are equivalent if they are adjacent and have the same closed neighborhood.

Theorem 1. If a graph $G$ has $n$ vertices and contains neither isolated nor equivalent vertices then $\theta_e(G) \geq \log_2(n + 1)$.
Gyárfás result implies that the edge-clique cover problem is fixed-parameter tractable (see also [18]). Cygan et al showed that, under the assumption of the exponential time hypothesis, there is no polynomial-time algorithm which reduces the parameterized problem \((\theta_e(G), k)\) to a kernel of size bounded by \(2^{o(k)}\). In their proof the authors make use of the fact that \(\theta_e(\text{cp}(2^\ell))\) is a [sic] “hard instance for the edge-clique cover problem, at least from a point of view of the currently known algorithms.” Note that, in contrast, the parameterized edge-clique partition problem can be reduced to a kernel with at most \(k^2\) vertices [28]. (Mujuni and Rosamond also mention that the edge-clique cover problem probably has no polynomial kernel.)

2 Rankwidth of edge-clique graphs of cocktail parties

Definition 2. The cocktail party graph \( \text{cp}(n) \) is the complement of a matching with \(2n\) vertices.

Notice that a cocktail party graph has no equivalent vertices. Thus, by Theorem 1

\[ \theta_e(\text{cp}(n)) \geq \log_2(2n + 1). \]

For the cocktail party graph an exact formula for \( \theta_e(\text{cp}(n)) \) is given in [19]. In that paper Gregory and Pullman prove that

\[ \lim_{n \to \infty} \frac{\theta_e(\text{cp}(n))}{\log_2(n)} = 1. \]

Definition 3. Let \( G = (V, E) \) be a graph. The edge-clique graph \( K_e(G) \) has as its vertices the edges of \( G \) and two vertices of \( K_e(G) \) are adjacent when the corresponding edges in \( G \) are contained in a clique.

Albertson and Collins prove that there is a 1-1 correspondence between the maximal cliques in \( G \) and \( K_e(G) \). The same holds true for the intersections of maximal cliques in \( G \) and in \( K_e(G) \).

For a graph \( G \) we denote the vertex-clique cover number of \( G \) by \( \kappa(G) \). Thus

\[ \kappa(G) = \chi(\overline{G}). \]

Notice that, for a graph \( G \),

\[ \theta_e(G) = \kappa(K_e(G)). \]

Albertson and Collins mention the following result (due to Shearer) for the graphs \( K_e^r(\text{cp}(n)) \), defined inductively by \( K_e^r(\text{cp}(n)) = K_e(K_e^{r-1}(\text{cp}(n))) \).

\[ \alpha(K_e^r(\text{cp}(n))) \leq 3 \cdot 2^r! \]

Thus, for \( r = 1 \), \( \alpha(K_e(\text{cp}(n))) \leq 6 \). However, the following is easily checked.
Lemma 1. For $n \geq 2$

\[ \alpha(K_e(cp(n))) = 4. \]

Proof. Let $G$ be the complement of a matching $\{x_i, y_i\}$, for $i \in \{1, \ldots, n\}$. Let $K = K_e(G)$. Obviously, every pair of edges in the matching induces an independent set with four vertices in $K$.

Consider an edge $e = \{x_i, x_j\}$ in $G$. The only edges in $G$ that are not adjacent to $e$ in $K$, must have endpoints in $y_j$ or in $y_i$. Consider an edge $f = \{y_i, y_k\}$ for some $k \notin \{i, j\}$. The only other edge incident with $y_i$, which is not adjacent in $K$ to $f$ nor to $e$ is $\{y_i, x_k\}$.

The only edge incident with $y_j$ which is not adjacent to $e$ nor $f$ is $\{y_i, x_i\}$. This proves the lemma.

Definition 4. A class of graphs $\mathcal{G}$ is $\chi$-bounded if there exists a function $f$ such that for every graph $G \in \mathcal{G}$,

\[ \chi(G) \leq f(\omega(G)). \]

Dvořák and Král proved that the class of graphs with rankwidth at most $k$ is $\chi$-bounded [15].

We now easily obtain our result.

Theorem 2. The class of edge-clique graphs of cocktail parties has unbounded rankwidth.

Proof. It is easy to see that the rankwidth of any graph is at most one more than the rankwidth of its complement [31]. Assume that there is a constant $k$ such that the rankwidth of $K_e(G)$ is at most $k$ whenever $G$ is a cocktail party graph. Let

\[ \mathcal{K} = \{ K_e(G) \mid G \cong cp(n), \ n \in \mathbb{N} \}. \]

Then the rankwidth of graphs in $\mathcal{K}$ is uniformly bounded by $k + 1$. By the result of Dvořák and Král, there exists a function $f$ such that

\[ \kappa(K_e(G)) \leq f(\alpha(K_e(G))) \]

for every cocktail party graph $G$. This contradicts Lemma 1 and Theorem 1.

3 Concluding remark

As far as we know, the recognition of edge-clique graphs is an open problem.

Conjecture 1. The edge-clique cover problem is NP-complete for cographs.
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