Singularities from the Topology and Differentiable Structure of Asymptotically Flat Spacetimes

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We prove that certain asymptotically flat initial data sets with nontrivial topology and/or differentiable structure collapse to form singularities. The class of such initial data sets is characterized by a new smooth invariant, the maximal Yamabe invariant, defined through smooth compactification of the asymptotically flat manifold. Our singularity theorem applies to spacetimes admitting a Cauchy surface of nonpositive maximal Yamabe invariant with initial data that satisfies the dominant energy condition. This class of spacetimes includes simply connected spacetimes with a single asymptotic region, a class not covered by prior singularity theorems for topological structures. The maximal Yamabe invariant can be related to other invariants including, in 4 dimensions, the $\hat{A}$-genus and the Seiberg-Witten invariants. In particular, 5-dimensional spacetimes with asymptotically flat Cauchy surfaces with non-trivial Seiberg-Witten invariants are singular. This singularity is due to the differentiable structure of the manifold.

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I. INTRODUCTION

In 1965, Penrose proved the first singularity theorem; under certain physically reasonable conditions, a spacetime must have an inextendible null geodesic, that is it must be singular \cite{1}. Precisely

Theorem (Penrose 1965). Spacetime $M$ with metric $\gamma_{ab}$ cannot be null geodesically complete if 1) The null convergence condition, $R_{ab}W^aW^b \geq 0$ for all null vectors $W^a$, holds; 2) there is a non-compact Cauchy surface $\Sigma$ in $M$; 3) there is a closed trapped surface $T$ in $M$.

This profound result demonstrated that singularities exhibited by known exact solutions such as Schwarzschild spacetime were not a consequence of their high symmetry but rather a general feature of gravitational collapse. Generalizations of this theorem demonstrated the existence of inextendible timelike geodesics under suitable energy conditions and extended its application to a variety of other physical situations \cite{2–4}. In general, singularity theorems require the existence of a trapped surface or an equivalent condition that indicates the initiation of gravitational collapse. Hence spacetimes without these structures, such as Minkowski spacetime and static star solutions, are nonsingular. In particular, spacetimes with Cauchy surfaces of $\mathbb{R}^3$ topology can be either singular or nonsingular depending on whether or not a trapped surface is present.

This is not the case if the Cauchy surface has nontrivial topology and certain asymptotic behavior. Gannon showed that any physically reasonable asymptotically flat spacetime with a non-simply connected 3-dimensional Cauchy surface must be singular \cite{5}, namely

Theorem (Gannon 1975). Let $M^4$ be a spacetime which satisfies the null convergence condition and admits a Cauchy surface $\Sigma^3$ which is regular near infinity. If $\Sigma^3$ is non-simply connected, then $M^4$ is not null geodesically complete.

Subsequent generalizations of this result extended its conclusions to a broader class of physical situations in 4 dimensions \cite{6–8}. Furthermore, the topological censorship theorem of Friedman, Schleich and Witt proved that the topology of physically reasonable, asymptotically flat spacetimes could not be actively probed by distant observers \cite{9}; all non-simply connected topological structures are behind horizons. These results were extended to the locally asymptotically anti-de Sitter case in \cite{10}. As the topology of 3-manifolds are characterized by their fundamental group,\(^1\) these results apply to all isolated topological structures in asymptotically flat 4-dimensional spacetime. In fact, topological censorship completely characterizes the topology of asymptotically flat 4-dimensional spacetime exterior to the horizons; this region is simply connected \cite{14}.

\(^1\) The recent proof of the Poincaré conjecture by Perelman removes the possibility of a homotopy 3-sphere noted in some older papers on topological censorship \cite{11,12}.
Although the singularity theorems were initially proven for 4-dimensional spacetimes, their results immediately generalize to higher dimensions; Gannon’s singularity theorem can be generalized to higher dimensional spacetimes with non-simply connected, asymptotically flat Cauchy surfaces. The topological censorship theorems also hold in higher dimensions. However, in 5 or more spacetime dimensions, the topology of the Cauchy surface is no longer completely characterized by its fundamental group. For example, all simply connected 4-manifolds are connected sums of $S^4$, $S^2 \times S^2$, $\mathbb{C}P^2$, $\mathbb{C}P^2$, and $E_8$ factors. Puncturing any such smooth manifold results in a noncompact smooth 4-manifold. This manifold can be taken to be the Cauchy surface of some globally hyperbolic 5-dimensional spacetime as it admits asymptotically flat initial data satisfying the dominant energy condition. Consequently, there are 5-dimensional spacetimes with nontrivial topological structures that evade the conditions of Gannon’s theorem and the topological censorship theorem. This is also true in 6 or more spacetime dimensions. Therefore these theorems leave open the issue of whether or not all topological structures collapse to form singularities in 5 or more dimensions.

This paper addresses this issue: we show that a certain class of topological structures in 5 or more spacetime dimensions collapse to form singularities. Specifically, we prove a new singularity theorem, Theorem 10 for spacetimes with Cauchy surfaces of topology and/or differentiable structure in a specified class with asymptotically flat initial data that satisfies the dominant energy condition. This class is defined through a natural extension of the Yamabe invariant for compact manifolds to the asymptotically flat case. An asymptotically flat $n$-manifold is related to a closed $n$-manifold by attaching $n$-balls to each asymptotic region via smooth attaching maps. The Yamabe invariant of the asymptotically flat manifold is defined to be that of the resulting closed manifold. This definition, in general dimension, depends on the choice of attaching maps. To remove this dependence, the maximal Yamabe invariant is defined as the supremum over all possible attaching maps. Theorem 10 applies to this class of spacetimes, that is ones whose Cauchy surfaces have nonpositive maximal Yamabe invariant. Included in this class are simply connected Cauchy surfaces with nontrivial topology in 5 or more spacetime dimensions. Consequently, Theorem 10 applies to a class of spacetimes not addressed by the generalization of Gannon’s theorem to higher dimensions.

Our approach to proving the singularity theorem is to demonstrate that the Cauchy surface must exhibit one or more apparent horizons. To do so involves two key results. We first prove Theorem 7, an asymptotically flat $n$-manifold with nonnegative scalar curvature has positive maximal Yamabe invariant. Next, we prove Theorem 9 if an asymptotically flat initial data set satisfying the dominant energy condition has a global solution to the Jang equation, then the Cauchy surface admits an asymptotically flat metric with zero scalar curvature.

The singularity theorem, Theorem 10 then follows from these two theorems and the existence of solutions to the Jang equation. Shoen and Yau proved the existence of solutions to the Jang equation in as part of their proof of the positive energy theorem. Furthermore, obstructions to a global solution imply that the initial data set contains apparent horizons. Although explicitly treats only the case of 3 dimensional Cauchy surfaces, these results can be extended through 7 dimensions using and . Theorem 10 follows by contradiction: Assume that there is a global solution to the Jang equation on the Cauchy surface with nonpositive maximal Yamabe invariant. Theorem 7 then implies that the Cauchy surface admits an asymptotically flat metric of zero scalar curvature. But this implies that the maximal Yamabe invariant is positive by Theorem 4 in contradiction. It follows that there is not a global solution to the Jang equation; therefore the initial data set contained one or more apparent horizons. Hence the spacetime is singular.

This approach is similar in spirit to that used in the generalization of Gannon’s theorem by Galloway; however, Galloway’s result uses a result of Meeks, Simon and Yau on the existence of minimal surfaces that applies only to 3-manifolds. Our singularity theorem for noncompact Cauchy surfaces applies to any $d$-dimensional spacetime, $3 \leq d \leq 8$, with nonpositive maximal Yamabe invariant that admits asymptotically flat initial data satisfying the dominant energy condition. The dominant energy condition is more restrictive than the null convergence condition; hence our theorem applies to a more restrictive set of spacetimes than Gannon’s singularity theorem and the topological censorship theorem. However the class of structures covered by Theorem 10 contains a set of simply connected Cauchy surfaces with a single asymptotic region - topologies that other topological singularity theorems do not address. Consequently, our theorem establishes that there is a class of simply connected spacetimes in 5 or more dimensions that collapse to form singularities. In particular, there are an infinite number of 4-dimensional simply connected asymptotically flat Cauchy surfaces with nonpositive maximal Yamabe invariant. In 4 dimensions, a nonpositive maximal Yamabe invariant is related to nonvanishing $A$-genus and to nonvanishing Seiberg-Witten invariants. As the Seiberg-Witten invariants can be used to characterize exotic differentiable structures on non-spin 4-manifolds, 5-dimensional simply connected spacetimes can collapse due to either their topology or their differentiable structure.

The roadmap of the paper is as follows: Section 11 provides a summary of basic definitions and theorems. Section

2 Note that $E_8$ does not admit a differentiable structure; however certain connected sums containing it do. In particular $K3$, a smooth manifold, is homeomorphic to the connected sum of two $E_8$ and three $S^2 \times S^2$ factors.
III defines the maximal Yamabe invariant for asymptotically flat $n$-manifolds. The required result on the scalar curvature of the compactified manifold, Theorem 11, is proven in Section VI. Theorem 9 and the singularity theorem are proven in Section VII. Section V relates nonpositive maximal Yamabe invariant to the $\hat{A}$-genus and to the Seiberg-Witten invariants and gives families of examples of such smooth, simply connected asymptotically flat 4-manifolds.

We conclude with a discussion in Section VII.

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II. PRELIMINARIES

We begin by giving some basic definitions needed in the statement and proof of the theorems in subsequent sections. An initial data set for the Cauchy problem in general relativity consists of a $n$-manifold $\Sigma^n$ that is geodesically complete with respect to riemannian metric $g_{ab}$, a symmetric tensor $p_{ab}$, energy density $\mu$, and momentum density $J^a$. These fields satisfy the Hamiltonian and momentum constraints

\begin{align}
R - p_{ab}p^{ab} + p^2 &= 2\mu \\
D_b(p^{ab} - pg^{ab}) &= J^a
\end{align}

where $R$ is the scalar curvature of the metric $g_{ab}$, $D_b$ is the covariant derivative defined with respect to $g_{ab}$, and $p = g^{ab}p_{ab}$. In addition, the fields are required to be sufficiently regular; for convenience, they will assumed to be smooth ($C^\infty$), though the results of this paper readily generalize to sufficiently differentiable fields.

Physically reasonable initial data obeys a local energy condition, the dominant energy condition (DEC), namely, $\mu \geq \sqrt{|J_a J^a|}$. When the energy and momentum densities correspond to vacuum or classical, nondissipative matter sources, local existence theorems show that the initial data evolves under the Einstein or coupled Einstein-matter equations into a globally hyperbolic spacetime with topology $\mathbb{R} \times \Sigma^n$. In this spacetime, $\Sigma^n$ is a Cauchy surface, a spacelike hypersurface such that every non-spacelike curve intersects it exactly once; $p_{ab}$ and $g_{ab}$ are now identified with the extrinsic curvature and induced metric of this Cauchy surface. From this point on, initial data sets will be assumed to be physically reasonable initial data sets.

The topology of the manifold $\Sigma^n$ is not restricted in the definition of an initial data set. However, there are two cases of particular interest: initial data sets on closed manifolds, describing cosmological models, and those on asymptotically flat manifolds, describing isolated gravitational systems. The asymptotically flat case is the focus of this paper.

Precisely, $\Sigma^n$ is an asymptotically flat $n$-manifold if, for some compact smooth submanifold with boundary $N^n \subset \Sigma^n$, $\Sigma^n - N^n$ consists of a finite number of disconnected components, each of which is diffeomorphic to $\mathbb{R}^n$ minus a $n$-ball, $\mathbb{R}^n - B^n$. Furthermore, $\partial N^n$ is a finite disjoint union of smooth $(n-1)$-spheres, $\partial N^n = \coprod S^{n-1}$.

This definition of an asymptotically flat $n$-manifold is given only in terms of the properties of differentiable manifolds and does not require any additional structure. It does not, in itself, restrict the metric, connection, or any other geometric structure on $\Sigma^n$ in any way. All asymptotically flat $n$-manifolds are, up to diffeomorphism, closed, smooth $n$-manifolds with points removed: every $\Sigma^n$ can be obtained from a closed smooth $n$-manifold $\Sigma^n$ via removing a finite number of points. Each removed point has a neighborhood diffeomorphic to $\mathbb{R}^n - B^n$; these neighborhoods are asymptotic regions. Conversely, $\Sigma^n$ can be smoothly compactified to a smooth manifold $\tilde{\Sigma}^n$ by adding a finite set of isolated points to compactify each asymptotic region. For fewer than three dimensions, all topological manifolds are smoothable and any two which are homeomorphic are diffeomorphic. However, this is no longer the case in four or

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3 A $n$-manifold is closed if it is compact and has no boundary.

4 $B^n = \{x \in \mathbb{R}^n | ||x|| \leq 1\}$. 

more dimensions; there exist topological $n$-manifolds that are not smooth; i.e., they do not admit a differentiable structure. Furthermore, topological $n$-manifolds that admit a differentiable structure may admit additional differentiable structures that are not diffeomorphic to each other. Thus, in four or more dimensions, the process of compactification of an asymptotically flat manifold requires careful specification. The details of this will be given in section III.

A metric is an *asymptotically flat metric* if the pullback of $g_{ab}$, say $\hat{g}_{ab}$, from $\Sigma^n - N^n$ onto each neighborhood $\mathbb{R}^n - B^n$ satisfies $\hat{g}_{ab} - \delta_{ab} = O\left(\frac{1}{r^{n-2}}\right)$, $\partial_r \hat{g}_{ab} = O\left(\frac{1}{r^{n-1}}\right)$, $\partial^a \partial_b \hat{g}_{ab} = O\left(\frac{1}{r^{n}}\right)$ as $r \to \infty$ where $\delta_{ab}$ is the flat metric. An initial data set is an *asymptotically flat initial data set* if $\Sigma^n$ is an asymptotically flat $n$-manifold with asymptotically flat metric $g_{ab}$ and the pullback of $p_{ab}$, $\hat{p}_{ab}$, from $\Sigma^n$ onto each $\mathbb{R}^n - B^n$ satisfies $\hat{p}_{ab} = O\left(\frac{1}{r^{n+1}}\right)$, $\partial_r \hat{p}_{ab} = O\left(\frac{1}{r^{n+2}}\right)$ as $r \to \infty$.

The pullbacks of the energy and momentum densities, $\hat{\mu}$ and $\hat{J}^a$ respectively, satisfy fall-off conditions as required to satisfy the constraints. The convention adopted in this paper is that initial data on asymptotically flat $n$-manifolds is asymptotically flat initial data unless stated otherwise.

As first shown in [16], one can construct asymptotically flat initial data sets which obey the dominant energy condition on any asymptotically flat manifold $\Sigma^n$. Therefore, the Einstein equations place no restriction on the choice of the topology of $\Sigma^n$.

We next outline the proof of a theorem, needed for later results, that demonstrates the existence of conformally related metrics with zero scalar curvature on asymptotically flat manifolds. Various forms of this result have been proven in three dimensions by several authors elsewhere [17, 22, 28]. The treatment below, generalized to $n$ dimensions, is given in detail in three dimensions in [17].

First, the conformal Laplacian operator $L$ in $n$ dimensions is given by

$$L = -a_n D^2 + R \quad \text{(3)}$$
$$a_n = \frac{4(n-1)}{n-2} \quad \text{(4)}$$

where $D^2 = D_a D^a$, and $D_a$ and $R$ are, respectively, the covariant derivative and scalar curvature of the metric $g_{ab}$.

**Lemma 1.** Let $\Sigma^n$ be an asymptotically flat manifold with smooth asymptotically flat metric $g_{ab}$. If $L$ is positive on smooth functions with compact support, then there is a smooth positive solution of $L\phi = 0$ such that $\phi \to 1$ with asymptotic fall-off as $r \to \infty$ in every asymptotic region of $\Sigma^n$.

**Proof.** On smooth functions with compact support, $\phi, \psi \in C_0^\infty(\Sigma^n)$, define

$$(\psi, \phi)_L = \int_{\Sigma^n} d\mu_g \left(a_n D_a \psi D^a \phi + R \psi \phi\right)$$

As $L$ is positive for $\phi \in C_0^\infty(\Sigma^n)$, it follows that $(\phi, \phi)_L > 0$ and $(\phi, \psi)_L = 0$ if and only if $\phi = 0$. Hence, this is an inner product on $C_0^\infty(\Sigma^n)$. The completion of this inner product yields a Hilbert space $\mathcal{H}_L$.

Consider the equation

$$-a_n D^2 \psi + \bar{R} \psi = -\bar{R} \quad \text{(5)}$$
on $\mathcal{H}_L$. Define the functional $F: \mathcal{H}_L \to \mathbb{R}$

$$F(\phi) = -\int_{\Sigma^n} d\mu_g R\phi .$$

Note that there is a constant $K > 0$ such that $||\phi|| < K ||\phi||_L$ where

$$||\phi|| = \left(\int_{\Sigma^n} d\mu_g (D_a \phi D^a \phi + \phi^2)\right)^{\frac{1}{2}}$$
is the usual norm on $\mathcal{H}_L$ and $||\phi||^2 = (\phi, \phi)_L$. Consequently $||\phi||_2 = (\int_{\Sigma^n} d\mu_g \phi^2)^{\frac{1}{2}} \leq ||\phi|| \leq K ||\phi||_L$. This implies that $F(\phi)$ is a bounded functional,

$$|F(\phi)| \leq \int_{\Sigma^n} d\mu_g |R\phi| \leq ||R||_2 ||\phi||_2 \leq C ||\phi||_L .$$

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5 See also the generalization to the vacuum case in [24].
using Holder’s inequality, \( \int_{\Sigma} d\mu_g |R\phi| \leq \|R\|_p \|\phi\|_p \), with \( p = q = 2 \) and the relationships of the norms \( \|\phi\| \) and \( \|\phi\|_L \).

Therefore, the Riesz representation theorem (See, for example \[27\]) implies that there is a unique \( \psi \in \mathcal{H}_L \) such that

\[
(\phi, \psi)_L = F(\phi)
\]

for all \( \phi \in \mathcal{H}_L \). In particular, this is true for all \( \phi \in C_0^\infty(\Sigma^n) \); consequently \( \psi \) is a weak solution to (4). Moreover, the solution is smooth by regularity theorems for second order elliptic operators (See, for example \[27\]). In addition, the pullback of \( \psi \) to \( \mathbb{R}^n - B^n \) vanishes with fall-off \( \psi = O\left(\frac{1}{r^2}\right) \) as \( r \to \infty \) in each asymptotic region because of the asymptotic behavior of \( g_{ab} \) and consequently that of \( L \) and \( R \). Consequently, the function \( \phi = 1 + \psi \) solves

\[
-a_n D^2 \phi + R \phi = 0
\]

and \( \phi \to 1 \) with asymptotic fall-off of its pullback \( \phi = 1 + O\left(\frac{1}{r^2}\right) \) as \( r \to \infty \) in each asymptotic region. Furthermore one can show \( \phi > 0 \) by smoothness and application of the maximum principle \[28\].

**Theorem 2.** Let \( \Sigma^n \) be an asymptotically flat manifold with smooth asymptotically flat metric \( g_{ab} \). If \( L \) is positive on smooth functions with compact support, then there is an asymptotically flat metric \( g_{ab}' \) conformally related to \( g_{ab} \) on \( \Sigma^n \) with vanishing scalar curvature.

**Proof.** The conformally related metric \( g_{ab}' = \phi^\frac{4}{n-2} g_{ab} \) has scalar curvature

\[
R' = \phi^{-\frac{n+2}{n-2}} \left( -a_n D^2 \phi + R \phi \right)
\]

where \( D^2 = D_a D^a \), and \( D_a \) and \( R \) are respectively the covariant derivative and scalar curvature of \( g_{ab} \). This curvature vanishes, \( R' = 0 \), if there is a smooth positive solution of \( L \phi = 0 \) where \( L \) is the conformally invariant laplacian operator \[3\]. As \( L \) is positive, such a solution exists by Lemma \[1\]. Furthermore, as \( \phi > 0 \) and has asymptotic fall-off \( \phi = 1 + O\left(\frac{1}{r^2}\right) \) as \( r \to \infty \), the conformally related metric \( g_{ab}' \) is asymptotically flat. \( \square \)

### III. A YAMABE INVARIANT FOR ASYMPTOTICALLY FLAT MANIFOLDS

A well known characterization of the allowed scalar curvature of geometries on a closed \( n \)-manifold is given by the Yamabe invariant:

**Definition 1.** The Yamabe invariant \( \sigma(M^n) \) for a closed \( n \)-manifold \( M^n, \ n \geq 2 \), is

\[
\sigma(M^n) = \sup_{g \in \text{Riem}(M^n)} \mathcal{Y}(g) \tag{6}
\]

where \( \mathcal{Y}(g) = \inf_{f \in C_0^\infty(M^n)} \mathcal{E}(e^{2f} g) \), \( \mathcal{E}(g) = \frac{\int_{M^n} R_g d\mu_g}{\left(\int_{M^n} d\mu_g\right)^{\frac{2}{n}}} \), and \( \text{Riem}(M^n) \) is the space of smooth riemannian metrics on \( M^n \).

For 2-manifolds, the Yamabe invariant is simply proportional to the Euler characteristic, \( \sigma(N^2) = 4\pi \chi(M^2) \). All closed 2-manifolds admit a metric of constant curvature by the uniformization theorem; thus the Yamabe invariant, or equivalently the Euler characteristic, fixes the sign of the curvature. Therefore, a closed 2-manifold admitting a metric of constant curvature of one sign can not admit one of a different sign; there is a topological obstruction.

This is no longer the case in three or more dimensions. Again any closed manifold \( M^n, \ n \geq 3 \), admits a metric or metrics with constant scalar curvature. However, if \( M^n \) admits a metric with positive scalar curvature, then it also admits metrics of constant scalar curvature of all signs. Consequently, if \( \sigma(M^n) > 0 \), there is no obstruction to a metric of constant scalar curvature of any sign. If \( M^n \) admits a metric with zero constant scalar curvature but not one with positive constant scalar curvature, then \( \sigma(M^n) = 0 \). Consequently, \( M^n \) will admit metrics with zero or negative constant scalar curvature, but not ones with positive constant scalar curvature. Finally, if \( M^n \) admits only metrics with negative constant scalar curvature, then \( \sigma(M^n) \leq 0 \). We now generalize the Yamabe invariant to asymptotically flat manifolds. The process is to first construct a closed \( n \)-manifold by a smooth compactification of the asymptotically flat \( n \)-manifold. The Yamabe invariant of this closed manifold is simply \[49\]. The generalized

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\[6\] Zero Yamabe invariant may occur as the supremum need not be attained in the class of metrics.
Yamabe invariant for asymptotically flat manifolds is then defined in terms of the supremum of the Yamabe invariant of the closed manifolds obtained by all possible smooth compactifications of the asymptotically flat manifold.

Before proceeding, it is useful to motivate this definition with a simple example. It is well known that the one point compactification of $\mathbb{R}^n$ is homeomorphic to $S^n$ for any $n$. It is also well known that for $n \geq 7$, $n$-spheres that are homeomorphic are not necessarily diffeomorphic; the 7-sphere has 28 inequivalent differentiable structures, the 8-sphere has 2 and the 9-sphere has 8. Furthermore, there are obstructions to positive scalar curvature on certain exotic 9-spheres and 10-spheres [29–31]. Therefore, the one point compactification of a manifold does not always provide sufficient information about the possible differentiable structures on the compactified manifold needed for the Yamabe invariant even in this simple case. In other words, the Yamabe invariant for a closed manifold is not invariant under homeomorphisms of the manifold in all dimensions; it depends on the differentiable structure. Additionally, restricting the compactification in some way so that it yields a unique differentiable structure may itself restrict the Yamabe invariant in some unknown way. Therefore, a robust generalization of the Yamabe invariant to asymptotically flat manifolds using compactification should recognize that there is more than one way to smoothly compactify each asymptotic region. This process precisely characterizes possible smooth structures on the compactification.

To begin, recall the standard definition of attaching two manifolds with boundary in which one smoothly glues the boundaries of the manifolds obtained by all possible smooth compactifications of the asymptotically flat manifold.

Theorem 3. Let $f_0 : \partial Q_0^n \to \partial P^n$ and $f_1 : \partial Q_1^n \to \partial P^n$ be diffeomorphisms. Suppose that the diffeomorphism $f_1^{-1} f_0 : \partial Q_0^n \to \partial Q_1^n$ extends to a diffeomorphism $h : Q_0^n \to Q_1^n$. Then the two $n$-manifolds $P^n \cup_{f_0} Q_0^n$ and $P^n \cup_{f_1} Q_1^n$ are diffeomorphic.

Corollary 4. Let $P^n$ and $Q^n$ be two smooth $n$-manifolds with respective boundaries $\partial P^n$ and $\partial Q^n$. If $f : \partial P^n \to \partial Q^n$ and $g : \partial P^n \to \partial Q^n$ are isotopic diffeomorphisms, then the two $n$-manifolds $P^n \cup_f Q^n$ and $P^n \cup_g Q^n$ are diffeomorphic.

Given the above, we can define the smooth compactification of an asymptotically flat manifold:

Definition 2. Given two smooth $n$-manifolds $P^n$ and $Q^n$ with boundaries $\partial P^n$ and $\partial Q^n$ and a diffeomorphism $f : \partial P^n \to \partial Q^n$, the smooth adjunction space $W^n_f$ is

$$W^n_f = P^n \cup_f Q^n \equiv \frac{P^n \coprod Q^n}{\sim}$$

where $P^n \coprod Q^n$ is the disjoint union of $P^n$ and $Q^n$ and the equivalence relation $\sim$ in the identification is given by $x \sim f(x) \forall x \in \partial Q^n$.

Two basic properties of the smooth adjunction space and its dependence on the diffeomorphism $f$ are given by the following theorem and its corollary [32]:

Theorem 3. Let $f_0 : \partial Q_0^n \to \partial P^n$ and $f_1 : \partial Q_1^n \to \partial P^n$ be diffeomorphisms. Suppose that the diffeomorphism $f_1^{-1} f_0 : \partial Q_0^n \to \partial Q_1^n$ extends to a diffeomorphism $h : Q_0^n \to Q_1^n$. Then the two $n$-manifolds $P^n \cup_{f_0} Q_0^n$ and $P^n \cup_{f_1} Q_1^n$ are diffeomorphic.

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Given the above, we can define the smooth compactification of an asymptotically flat manifold:

Definition 3. The smooth compactification $\Sigma^n_\Phi$ of an asymptotically flat manifold $\Sigma^n$ is

$$\Sigma^n_\Phi = N^n \cup_\Phi I_0 \equiv \frac{N^n \coprod I_0}{\sim}$$

where $I_0$ is a disjoint union of $n$-balls $B^n_i$’s and $\Phi$ is a finite set of diffeomorphisms $\{\phi_i\}$ where $\phi_i : S^{n-1}_i = \partial B^n_i \to \partial N^n$, indexed by $i$. $N^n \coprod I_0$ is the disjoint union of $N^n$ and $I_0$ and the equivalence relation for identification is given by $x \sim \phi_i(x), \forall x \in S^{n-1}_i$.

Corollary 3 implies that $\Sigma^n_\Phi$ is determined, up to diffeomorphism, by the isotopy classes of diffeomorphisms $\phi_i \in \Phi$. It immediately follows that as the boundary $\partial N^n$ is just the disjoint union of spheres $S^{n-1}$, the isotopy classes are given by $\pi_0 Diff(S^{n-1}) = Diff(S^{n-1})/Diff_{id}(S^{n-1})$ for the smooth compactification of an asymptotically flat $n$-manifold.

Representing the sphere $S^{n-1}$ by its embedding in Euclidean space, $S^{n-1} = \{(x^1, x^2, \ldots, x^n) \in \mathbb{R}^n | \sum_{i=1}^{n} (x^i)^2 = 1\}$, the map $P : (x^1, x^2, \ldots, x^n) \to (-x^1, x^2, \ldots, x^n)$ for points on $S^{n-1}$ is an orientation reversing $O(n)$ isometry. This isometry trivially extends to the interior of the $n$-ball $B^n$. Its composition with any diffeomorphism in the identity component of the diffeomorphism group, also extends trivially. Consequently, by Theorem 3 the smooth adjunction space formed by attaching an $n$-ball with $f$ is diffeomorphic to that formed by doing so with $P f'$ for any $f, f' \in Diff_{id}(S^{n-1})$. Hence two smooth adjunction spaces formed by attaching with diffeomorphisms $g$ and $g'$ respectively such that $g^{-1} g' = f$ or $g^{-1} g' = P f$ are diffeomorphic. In other words, attaching with diffeomorphisms in two isotopy classes equivalent under the orientation reversing map result in diffeomorphic adjunction spaces. Therefore, it suffices to consider $\phi_i \in \pi_0 Diff^+ (S^{n-1})$, the isotopy classes of orientation preserving diffeomorphisms.
Definition 4. Let $\Sigma^n$ be an asymptotically flat $n$-manifold. The asymptotically flat Yamabe invariant, $\sigma(\Sigma^n, \Phi)$, is defined by $\sigma(\Sigma^n, \Phi) = \sigma(\tilde{\Sigma}_\Phi^n)$ where $\tilde{\Sigma}_\Phi^n$ is the smooth compactification of the asymptotically flat manifold $\Sigma^n$ obtained using the finite set of attaching maps $\Phi$ and $\sigma(\tilde{\Sigma}_\Phi^n)$ is the usual Yamabe invariant for the resulting closed $n$-manifold.

It is useful to define an additional constant independent of the attaching map:

Definition 5. Let $\Sigma^n$ be an asymptotically flat $n$-manifold. The asymptotically flat maximal Yamabe invariant, $\hat{\sigma}(\Sigma^n)$, is defined by

$$\hat{\sigma}(\Sigma^n) = \sup_{\Phi \in \pi_0 \text{Diff}^+(S^{n-1})} \sigma(\Sigma^n, \Phi).$$

Note that the supremum is taken over all possible choices of the isotopy class of each $\phi_i$ in $\Phi$. Whether or not $\hat{\sigma}(\Sigma^n, \Phi)$ actually depends on $\Phi$ changes with dimension. For $n = 1, 2, 3, 4$ and $6$, there is only one isotopy class, $\pi_0 \text{Diff}^+(S^{n-1}) = 1$, and smooth compactification results in a $\Sigma^n$ that is unique up to diffeomorphism. Thus $\hat{\sigma}(\Sigma^n, \Phi)$ is unique for $n = 1, 2, 3, 4$ and $6$. However, for $n \geq 7$ the isotopy classes are not trivial: $\pi_0 \text{Diff}^+(S^6)$ has 28 elements, $\pi_0 \text{Diff}^+(S^7)$ has 2 and $\pi_0 \text{Diff}^+(S^8)$ has 8. Thus $\hat{\sigma}(\Sigma^n, \Phi)$ may now depend on the choice of the isotopy class of each $\phi_i$ in $\Phi$.

In other words, different choices of $\Phi$ may result in compactified manifolds with the same topology but inequivalent differentiable structures.\footnote{This is why $S^7$, $S^8$ and $S^9$ have their respective number of inequivalent differentiable structures.}

The dimension of $\pi_0 \text{Diff}^+(S^4)$ is unknown, so whether or not $\hat{\sigma}(\Sigma^5, \Phi)$ is unique is not determined by this construction. However, its uniqueness can be established by an alternate realization of the compactification. Observe that every homotopy $n$-sphere with $n \geq 5$ is obtained from attaching two closed $n$-balls along $\partial B^n = S^{n-1}$ via a diffeomorphism in $\text{Diff}^+(S^{n-1})$. Hence,

$$\tilde{\Sigma}_\Phi^n = \tilde{\Sigma}_{\text{id}}^{n} \# F_1^n \# F_2^n \ldots \# F_k^n$$

where $\tilde{\Sigma}_{\text{id}}^{n}$ is the compactification of $\Sigma^n$ with all attaching maps in the identity of $\text{Diff}^+(S^{n-1})$, $F_i^n \in \Theta_n$, the group of homotopy $n$-spheres, and $k$ is the number of asymptotic regions in $\Sigma^n$. Now, $\Theta_n = 1$ for $n = 5, 6$; it follows that $\hat{\sigma}(\Sigma^n, \Phi)$ is unique for $n \leq 6$. Moreover, for $n \geq 7$, this construction yields an alternate expression for $\hat{\sigma}(\Sigma^n)$ in terms of a connected sum with homotopy $n$-spheres.

Theorem 5. Let $\Sigma^n$ be an asymptotically flat $n$-manifold. For $n \leq 6$, $\hat{\sigma}(\Sigma^n, \Phi)$ is unique and $\hat{\sigma}(\Sigma^n) = \hat{\sigma}(\Sigma^n, \Phi)$. For $n \geq 7$,

$$\hat{\sigma}(\Sigma^n) = \sup_{\{F_i^n\} \in \Theta_n} \sigma(\tilde{\Sigma}_{\text{id}}^{n} \# F_1^n \# F_2^n \ldots \# F_k^n)$$

where $\tilde{\Sigma}_{\text{id}}^{n}$ is the compactification of $\Sigma^n$ with all attaching maps in the identity of $\text{Diff}^+(S^{n-1})$, $k$ the number of asymptotic regions and the supremum is over all choices of $\{F_i^n\}$, a set of $k$ elements of $\Theta_n$, the group of homotopy $n$-spheres.

Using this result, it is easy to see that the maximal Yamabe invariant is always positive for $\mathbb{R}^n$ in any dimension. However, as discussed in detail in Section IV for the case of 4 dimensions, manifolds of more complicated topology and/or differentiable structure will have nonpositive maximal Yamabe invariant.

IV. A COMPACTIFICATION THEOREM

We now prove a compactification theorem for asymptotically flat manifolds with asymptotically flat metrics of non-negative scalar curvature; this result is needed for the proof of the singularity theorem. For clarity of notation, we first do so for the case of one asymptotic region.

Theorem 6. Given an asymptotically flat manifold $\Sigma^n$ with one asymptotic region and asymptotically flat metric $g_{ab}$ with $R \geq 0$, then $\hat{\sigma}(\Sigma^n) > 0$. 
Proof. If \( R \geq 0 \), then \( L \), the conformal laplacian operator, is positive on smooth functions of compact support. Theorem 2 implies that there exists an asymptotically flat, conformally related metric \( g_{ab} = \frac{1}{\phi} g_{ab} \) with zero scalar curvature on \( \Sigma^n \). For simplicity of notation, let \( g_{ab} \) denote this conformally rescaled metric from this point onward.

Let \( t > 0 \) be a parameter chosen such that \( r = t \) is a smooth \((n-1)\)-sphere entirely contained in the asymptotic region \( \mathbb{R}^n - B^n \). Define a new family of metrics:

\[
g_{ab}(t) = \begin{cases} 
    g_{ab} & \text{\( \Sigma^n \) interior to } r < 2t \\
    \alpha_t(r) \delta_{ab} + (1 - \alpha_t(r)) g_{ab} & 2t < r < 3t \\
    \delta_{ab} & r \geq 3t
\end{cases}
\]

where \( \alpha_t \) is a family of smooth functions equal to 1 for \( r > 3t \) and 0 for \( r < 2t \). Additionally, choose \( \alpha_t \) to obey the fall-off conditions \( |\alpha'_t(r)| \leq A/r^{n-2} \) and \( |\alpha''_t(r)| \leq A/r^{n-1} \) where \( ' \) denotes the derivative with respect to \( r \) and \( A \) is a constant independent of \( t \). This choice can always be made.

The family of metrics \( g_{ab}(t) \) is asymptotically flat and complete on \( \Sigma^n \). For \( r > 3t \), the metric is, in fact, flat. The scalar curvature of \( g_{ab}(t) \), \( R_t \), is only nonzero in the deformation region \( 2t < r < 3t \). In fact, \( R_t \) curvature will typically be negative in the region of deformation. Thus the conformally invariant Laplacian operator for this family of metrics,

\[
L_t \phi \equiv -a_n D_t^2 \phi + R_t \phi ,
\]

is no longer manifestly positive. However, it is possible to show that it is positive for sufficiently small negative curvature in the deformation region. This is done by appropriately choosing \( t \). We now show that there is a \( t_0 \) such that for \( t > t_0 \), \( L_t \) is positive for \( \phi \in C_0^\infty(\Sigma^n) \). First

\[
\int_{\Sigma^n} d\mu_{g_t} \phi L_t \phi = \int_{\Sigma^n} d\mu_{g_t} (a_n (D_t \phi)^2 + R_t \phi^2)
\]

for \( \phi \in C_0^\infty(\Sigma^n) \). Next observe that

\[
\int_{\Sigma^n} d\mu_{g_t} R_t \phi^2 = \int_{\text{pos}} d\mu_{g_t} R_t \phi^2 - \int_{\text{neg}} d\mu_{g_t} |R_t \phi^2| \geq -\int_{\Sigma^n} d\mu_{g_t} |R_t \phi^2|
\]

where the domains of integration \text{pos} and \text{neg} are the support of \( R_t \geq 0 \) and \( R < 0 \) in \( \Sigma^n \) respectively. It follows that

\[
\int_{\Sigma^n} d\mu_{g_t} \phi L_t \phi \geq \int_{\Sigma^n} d\mu_{g_t} (a_n (D_t \phi)^2 - \int_{\Sigma^n} d\mu_{g_t} |R_t \phi^2|).
\]

The Sobolev inequality

\[
\left( \int_{\Sigma^n} d\mu_{g_t} |\phi|^p \right)^{\frac{2}{p}} \leq K \left( \int_{\Sigma^n} d\mu_{g_t} D_a \phi D^a \phi \right)
\]

where \( \phi \in C_0^\infty(\Sigma^n) \) and \( p = \frac{2n}{n-2} \) applied to the first term on the right hand side and Holder’s inequality

\[
\int_{\Sigma^n} d\mu_{g_t} |R_t \phi^2| \leq ||R_t||_q ||\phi^2||_\tilde{p}
\]

where \( \frac{1}{q} + \frac{1}{\tilde{p}} = 1 \) applied to the second term on the right hand side yields

\[
\int_{\Sigma^n} d\mu_{g_t} \phi L_t \phi \geq \frac{a_n}{K} ||\phi||_p^2 - ||R_t||_q ||\phi^2||_\tilde{p}.
\]

Now

\[
||\phi^2||_\tilde{p} = \left( \int_{\Sigma^n} d\mu_{g_t} |\phi^2|^{\tilde{p}} \right)^{\frac{1}{\tilde{p}}} = \left( \int_{\Sigma^n} d\mu_{g_t} |\phi|^{2\tilde{p}} \right)^{\frac{1}{2\tilde{p}}}
\]

so the choice \( \tilde{p} = \frac{p}{2} = \frac{n}{n-2} \) yields

\[
\int_{\Sigma^n} d\mu_{g_t} \phi L_t \phi \geq \left( \frac{a_n}{K} - ||R_t||_q \right) ||\phi||_p^2
\]
with \( q = \frac{9}{8} \). Hence, the operator \( L_t \) is positive if \( (\frac{9}{8} - ||R_t||_q) > 0 \). Observe that for large enough \( t \), the metric \( g_{ab} \) in the deformation region \( 2t < r < 3t \) is asymptotically flat. Thus \( g_{ab}(t) = \delta_{ab} + (1 - \alpha_t(r))h_{ab} \) where \( h_{ab} = O(\frac{1}{r^9}) \) in this region. Consequently, \( |R_t| \leq \frac{B}{t} \) in \( 2t < r < 3t \) for some constant \( \tilde{A} \) independent of \( t \). Thus

\[
||R_t||_q = \left( \int_{\Sigma^n} d\mu_g |R_t|^q \right)^{\frac{1}{q}} \leq \left( \int_{2t}^{3t} d\mu_g |R_t|^q \right)^{\frac{1}{q}} \leq \tilde{A} \left( \int_{2t}^{3t} \frac{1}{r^{q+1}} |r|_{g_{ab}}^{q-1} dr \right)^{\frac{1}{q}} \leq \frac{B}{t^{\frac{q+1}{q}}}
\]

where the constants \( \tilde{A} \) and \( B \) are independent of \( t \). Therefore, there is some \( t_0 \) such that for \( t > t_0 \), \( L_t \) is a positive operator on \( \phi \in C_0^\infty(\Sigma^n) \). Hence Lemma 1 implies there exists a smooth positive solution \( G_t \) to

\[
L_t G_t = -a_n D_t^2 G_t + R_t G_t = 0 \tag{8}
\]

for any \( t > t_0 \). From this point on, \( t \) will be taken to have one fixed value above its lower bound.

Note that, as \( g_{ab}(t) \) is flat for \( r > 3t \), the solution \( G_t \) of (8) asymptotically has the standard expansion for a harmonic function on flat space:

\[
G_t = C_n \left( 1 + \frac{A_0}{r^{2}} + \sum_{l=1}^{\infty} \sum_{\kappa(l)} A_{\kappa l} Y_{\kappa l}(\Omega) \right)
\]

where \( C_n \) is a dimension dependent constant determined by the normalization of the \( n \)-dimensional delta function and \( \kappa(l) \) is a set of integers that index the complete set of \( n \)-spherical harmonics \( Y_{\kappa l}(\Omega) \) with principal Casimir indexed by \( l \).

Next, the manifold \( \Sigma^n \) is compactified both topologically and geometrically. This compactification is motivated by the observation that stereographic projection of the round \( n \)-sphere results in \( \mathbb{R}^n \) with flat metric and that \( g_{ab}(t) \) is the flat metric for \( r > 3t \). Inverting this procedure in the asymptotic region results in the closed manifold \( \tilde{\Sigma}^n \) with metric \( \tilde{g}_{ab}(t) \).

Precisely, topologically compactify \( \Sigma^n \) by attaching an \( n \)-ball to the asymptotic region using the trivial attaching map to form the closed manifold \( \tilde{\Sigma}^n \). Define a smooth metric on \( \tilde{\Sigma}^n \) by \( \tilde{g}_{ab}(t) = \phi^{\frac{4}{n-2}} g_{ab}(t) \) where the conformal factor
\[
\phi = \gamma + \frac{(1 - \gamma)}{r^{n-2}}
\]

and \( \gamma(r) \) is a smooth bump function\(^8\) which is 0 for \( r > 5t \) and 1 for \( r < 4t \). Note that the form of the conformal factor yields \( \tilde{g}_{ab}(t) = \frac{1}{\phi} \delta_{ab} \) for \( r > 5t \). The coordinate transformation \( \tilde{r} = \frac{1}{r} \) in this region yields the flat metric \( ds^2 = d\tilde{r}^2 + \tilde{r}^2 d\Omega_{n-1}^2 \); thus \( \tilde{g}_{ab}(t) \) is smooth everywhere in the neighborhood of \( \tilde{r} = 0 \). Denote this point \( i \).

As \( \tilde{g}_{ab}(t) \) is conformal to \( g_{ab}(t) \), it follows from (3) that \( \tilde{G}_t = \phi^{-1} G_t \) with \( \phi \) given by (8) is a positive solution of

\[
-a_n \tilde{D}^2 \tilde{G} + \tilde{R} \tilde{G} = 0
\]

on \( \tilde{\Sigma}^n - i \). Note that \( \tilde{D}_a \) and \( \tilde{R} \) are with respect to \( \tilde{g}_{ab}(t) \). Furthermore, the expansion of \( \tilde{G} \) around \( i \) in the coordinate \( \tilde{r} = \frac{1}{r} \) is given by

\[
\tilde{G} = C_n \left( \frac{1}{\tilde{r}^{n-2}} + A_0 + \sum_{l=1}^{\infty} \sum_{\kappa(l)} A_{\kappa l} Y_{\kappa l}(\Omega) \tilde{r}^l \right)
\]

It is manifestly apparent that \( \tilde{G} \) is the Green’s function for the conformally invariant laplacian with respect to \( \tilde{g}_{ab}(t) \), \( \tilde{L} \), on \( \tilde{\Sigma}^n \), that is \( \tilde{L} \tilde{G} = \delta_i \).

\(^8\) For example,

\[
\gamma(r) = 1 - \frac{\int_{-\infty}^{r} dx' f(x' - 4t)f(5t - x')}{\int_{-\infty}^{\infty} dx' f(x' - 4t)f(5t - x')}
\]

where \( f \) is the smooth function
\[
f(x) = \begin{cases} 0 & x \leq 0 \\ \exp(-\frac{1}{x}) & x > 0 \end{cases}
\]

is such a bump function.
The final step is to show that there is a metric conformally related to \( \tilde{g}_{ab}(t) \) on \( \tilde{\Sigma}^n \) with everywhere positive scalar curvature. Let \( \psi_0 > 0 \) be the smooth positive solution (i.e. the ground state) to

\[
-a_n \tilde{\Delta}^2 \psi_0 + \tilde{R} \psi_0 = \lambda_0 \psi_0 \tag{10}
\]
on \( \tilde{\Sigma}^n \). Such a solution exists for smooth \( \tilde{L} \) on a closed manifold [17]. Next, using \( \psi_0 \) as the conformal factor, construct the metric \( \tilde{g}_{ab} = \psi_0^{-\frac{n+2}{2}} \tilde{g}_{ab}(t) \). The scalar curvature of \( \tilde{g}_{ab} \) is

\[
\tilde{R} = \psi_0^{-\frac{n+2}{2}} \left( -a_n \tilde{\Delta}^2 \psi_0 + \tilde{R} \psi_0 \right) = \lambda_0 \psi_0 \left( 1 - \frac{n+2}{2} \right). \tag{11}
\]

Integration of the left hand side of (10) against \( \tilde{G} \)

\[
\int_{\tilde{\Sigma}^n} d\mu_{\tilde{G}} \tilde{L} \psi_0 = \langle \tilde{G}, \tilde{L} \psi_0 \rangle = \lambda_0 \langle \tilde{G}, \psi_0 \rangle.
\]

However,

\[
\langle \tilde{G}, \tilde{L} \psi_0 \rangle = \langle \tilde{L} \tilde{G}, \psi_0 \rangle = \langle \delta_i, \psi_0 \rangle > 0.
\]

Thus \( \lambda_0 \langle \tilde{G}, \psi_0 \rangle > 0 \) and, as \( \langle \tilde{G}, \psi_0 \rangle \) is positive, it follows that \( \lambda_0 > 0 \). Hence, by (11), \( \tilde{R} > 0 \) everywhere on \( \tilde{\Sigma}^n \). Therefore, the Yamabe invariant is positive for \( \tilde{\Sigma}^n \). By definition 4, the asymptotically flat Yamabe invariant for \( \Sigma^n \) is positive for the trivial compactification. Hence, the maximal Yamabe invariant for \( \Sigma^n \) is also positive, \( \hat{\sigma}(\Sigma^n) > 0 \). □

Theorem 6 readily generalizes to asymptotically flat manifolds with multiple asymptotic regions:

**Theorem 7.** Given an asymptotically flat manifold \( \Sigma^n \) with one or more asymptotic regions and asymptotically flat metric \( g_{ab} \) with \( R \geq 0 \), then \( \hat{\sigma}(\Sigma^n) > 0 \).

The proof of this theorem directly parallels that of Theorem 6 so will not be repeated here. The key difference is that quantities associated with the asymptotic regions will now be indexed and manipulations involving them typically involve sums. In particular, the parameter \( t \) used in the deformation of the metric to one of zero scalar curvature becomes an indexed set of parameters \( t_k \), one for each asymptotic region. The bound on the norm of the deformed curvature [7] then becomes a sum over the contributions from all asymptotic regions. Again, there will be a choice of \( t_0 \) such that if all \( t_k > t_0 \), then the conformal laplacian will be a positive operator. Secondly, when \( \Sigma^n \) is compactified, the added \( n \)-balls, behavior of the conformal factor on each asymptotic region and the poles of the Green’s function will also be indexed by \( k \). In particular, \( \tilde{G} \) will now be the Green’s function for poles at \( k \) points, \( \tilde{L} \tilde{G} = \sum_k \delta_{ik} \), rather than one. This replacement does not change either the result or conclusions.

An immediate corollary of Theorem 7 is that the maximal Yamabe invariant characterizes an obstruction to asymptotically flat initial data with a maximal slice on an asymptotically flat \( n \)-manifold:

**Corollary 8.** Let \( \Sigma^n \) with metric \( g_{ab} \) and extrinsic curvature \( p_{ab} \) be an asymptotically flat initial data set with sources \( \mu \) and \( J^a \) that obey the dominant energy condition. If there is a maximal slice, then \( \hat{\sigma}(\Sigma^n) > 0 \).

**Proof.** As \( p = 0 \), the Hamiltonian constraint (11) implies

\[
R = 2\mu + p_{ab} p^{ab} \geq 0
\]
as \( \mu \geq 0 \) by the dominant energy condition. Theorem 7 now directly implies \( \hat{\sigma}(\Sigma^n) > 0 \). □

Hence, an asymptotically flat manifold \( \Sigma^n \) with \( \hat{\sigma}(\Sigma^n) \leq 0 \) does not admit asymptotically flat initial data with a maximal slice.

**V. OBSTRUCTIONS TO GLOBAL SOLUTIONS OF THE JANG EQUATION AND A NEW SINGULARITY THEOREM FOR \((n + 1)\)-DIMENSIONAL SPACETIMES**

We now prove the main results of this paper. First we prove that if an asymptotically flat initial data set satisfying the dominant energy condition has a global solution to the Jang equation, then the Cauchy surface admits an
asymptotically flat metric with zero scalar curvature. We then use this result to prove a new singularity theorem, that topological structures with nonpositive maximal Yamabe invariant collapse to form singularities.

Let \( \Sigma^n \) with metric \( g_{ab} \) and extrinsic curvature \( p_{ab} \) be an asymptotically flat initial data set and let \( \mu \) and \( J^a \) be the corresponding sources. Form the new manifold \( \Sigma^n \times \mathbb{R} \) with Riemannian metric given by line element \( ds^2 = g_{ab}dx^a dx^b + dr^2 \) where the coordinate \( \tau \) is along \( \mathbb{R} \). The tensors \( p_{ab}, \mu \) and \( J^a \) are trivially extended so they are independent along parallel lines \( \tau \). Let \( \mathcal{G}_f^n \subset \Sigma^n \times \mathbb{R} \) be the graph of a function \( f : \Sigma^n \to \mathbb{R} \) where \( \mathcal{G}_f^n = \{(x, f(x)) | x \in \Sigma^n \} \). The induced metric on \( \mathcal{G}_f^n \) and its inverse are

\[
\bar{g}_{ab} = g_{ab} + D_a f D_b f
\]
\[
\bar{g}^{ab} = g^{ab} - \frac{D^a f D^b f}{1 + |Df|^2}.
\]

By construction, the mean curvature of the graph \( \mathcal{G}_f^n \) is \( H(f) = \bar{g}^{ab} \frac{D_a D_b f}{\sqrt{1 + |Df|^2}} \) and the trace of \( p_{ab} \) restricted to the graph is \( P(f) = \bar{g}^{ab} p_{ab} \). The Jang equation is \( H(f) = P(f) \); it can alternately be written as

\[
\left( g^{ab} - \frac{D^a f D^b f}{1 + |Df|^2} \right) \left( \frac{D_a D_b f}{\sqrt{1 + |Df|^2}} - p_{ab} \right) = 0.
\]

By definition, when \( f : \Sigma^n \to \mathbb{R} \) is a solution to the Jang equation, the graph \( \mathcal{G}_f^n \) has mean curvature as prescribed by the trace of \( p_{ab} \).

Schoen and Yau proved the existence of solutions to the Jang equation [19]. There are obstructions to finding global solutions, that is solutions with everywhere bounded \( f \). If an obstruction occurs, then the initial data set contains apparent horizons, namely, closed manifolds \( T^{n-1} \subset \Sigma^n \) with \( H_{T^{n-1}} - P_{T^{n-1}} = 0 \). In other words, the obstructions are closed marginally outer trapped surfaces \( T^{n-1} \). Although [19] explicitly treats only the \( n = 3 \) case, these results can be extended to dimensions 4 and 5 using the techniques found in [20]. The results of Eichmair allow the further extension of these results through dimension 7 and imply a distributional solution in dimensions higher than 7 [21].

As we assume the existence of a regular global solution to the Jang equation in the our proof of Theorem 9, it applies to Cauchy surfaces of dimension \( n \leq 7 \) though extension to higher dimensions may be possible.

We now prove that asymptotically flat initial data sets that have a global solution to the Jang equation must admit a metric of zero scalar curvature.

**Theorem 9.** Let \( \Sigma^n \) with metric \( g_{ab} \) and extrinsic curvature \( p_{ab} \) be an asymptotically flat initial data set with sources \( \mu \) and \( J^a \) that obey the dominant energy condition. If there is a global solution to the Jang equation, then there exists an asymptotically flat metric on \( \Sigma^n \) with \( R = 0 \).

**Proof.** Let \( p_{ab} \) and \( g_{ab} \) satisfy the constraint equations [1] and [2] and let \( f \) be a function \( f : \Sigma^n \to \mathbb{R} \). Define the new quantity \( K = 1 + D_a f D^a f \); let

\[
\bar{p}_{ab} = p_{ab} - \frac{D_a D_b f}{K^{1/2}}
\]
\[
\bar{g}^{ab} = g^{ab} - \frac{D^a f D^b f}{K}.
\]

In these variables, the Jang equation takes the form

\[
\bar{p}_{ab} \bar{g}^{ab} = 0 \tag{12}
\]

If \( f \) is a global solution to the Jang equation, then the initial data \( g_{ab}, p_{ab} \) can be deformed into new data \( \bar{g}_{ab}, \bar{p}_{ab} \) on \( \Sigma^n \) with zero prescribed mean curvature. The new data does not necessarily satisfy the constraints, but does satisfy the related equation [19], first derived in [19]; an alternate derivation follows below.

First note that the covariant derivative with respect to \( \bar{g}_{ab} \) is related to that of \( g_{ab} \) by

\[
(\bar{D}_b - D_b) v^c = C^c_{bc} v^c
\]

for any vector \( v^c \) in \( \Sigma^n \) where

\[
C^a_{bc} = \frac{1}{K} D^a f D_b D_c f
\]
The Ricci curvature of $\bar{g}_{ab}$ is related to that of $g_{ab}$ by
\[ \bar{R}_{ab} = R_{ab} + Q_{ab} \]
\[ \bar{R} = R + \bar{g}^{ab}Q_{ab} - \frac{D^a f D^b f}{K} R_{ab} \]
\[ Q_{ab} = D_a C_c - D_a C_c^d + C_d C_c^d - C_d C_c^d. \]

Next, observe that
\[ p^2 = \bar{p}_{cd} \bar{g}^{cd} \left( \bar{p}_{ab} \bar{g}^{ab} + 2 \frac{D^a f D^b f}{K} \bar{p}_{ab} \right) + \left( \frac{D^a f D^b f}{K} \bar{p}_{ab} \right)^2 \]
\[ + 2 \frac{D^2 f \bar{p}_{cd} g^{cd}}{K \frac{K}{K}} + \left( \frac{D^2 f}{K} \right)^2 \]
\[ p_{ab} p_{cd} g^{ab} g^{cd} = \bar{p}_{ab} \bar{p}_{cd} \bar{g}^{ac} \bar{g}^{bd} + \frac{2 \bar{g}^{ab} D_c f \bar{p}_{ca} D_d f \bar{p}_{db}}{K} + \left( \frac{D^a f D^b f}{K} \bar{p}_{ab} \right)^2 \]
\[ + 2 \frac{D^a f D^b f}{K} \frac{D_a f D^a f}{K} \]

Substituting these relations into the hamiltonian constraint yields
\[ \bar{R} = \bar{p}_{ab} \bar{p}_{cd} \bar{g}^{ac} \bar{g}^{bd} - \frac{2 \bar{g}^{ab} D_c f \bar{p}_{ca} D_d f \bar{p}_{db}}{K} + \bar{p}_{cd} \bar{g}^{cd} \left( \bar{p}_{ab} \bar{g}^{ab} + 2 \frac{D^a f D^b f}{K} \bar{p}_{ab} \right) \]
\[ + 2 \frac{D^2 f \bar{p}_{cd} g^{cd}}{K \frac{K}{K}} - 2 \frac{D^a f D^b f}{K} \bar{p}_{ab} \]
\[ - \bar{g}^{ab} Q_{ab} + \frac{D^a f D^b f}{K} R_{ab} - \frac{D_a f D_a f D^a f}{K} + \left( \frac{D^2 f}{K} \right)^2 \]
\[ = 2\mu \quad (13) \]

Next note that
\[ \frac{D^a f D^b f}{K} = D_a \left( \frac{g^{ac} D^b f \bar{p}_{bc}}{K \frac{K}{K}} \right) + \frac{D^c f D_a f D^b f g^{ac} \bar{p}_{bc}}{K \frac{K}{K}} - \frac{D^b f}{K \frac{K}{K}} D_a (g^{ac} \bar{p}_{bc}) \]
\[ = D_a \left( \frac{D^b f}{K \frac{K}{K}} g^{ac} \bar{p}_{bc} \right) - \frac{D^b f}{K \frac{K}{K}} D_a (g^{ac} \bar{p}_{bc}) \]
and
\[ \frac{D^2 f \bar{p}_{cd} g^{cd}}{K \frac{K}{K}} = D_a \left( \frac{D^a f \bar{p}_{cd} g^{cd}}{K \frac{K}{K}} \right) + \frac{D^b f D_a f D^a f \bar{p}_{cd} g^{cd}}{K \frac{K}{K}} - \frac{D^a f}{K \frac{K}{K}} D_a (\bar{p}_{cd} g^{cd}) \]
\[ = \bar{D}_a \left( \frac{D^a f}{K \frac{K}{K}} \bar{p}_{cd} g^{cd} \right) - \frac{D^a f}{K \frac{K}{K}} D_a (\bar{p}_{cd} g^{cd}) \]
and
\[ \frac{D^a f}{K \frac{K}{K}} D_a (\bar{p}_{cd} g^{cd}) - \frac{D^b f}{K \frac{K}{K}} D_a (g^{ac} \bar{p}_{bc}) = \frac{D_a f}{K \frac{K}{K}} D_b (p^{ab} - pg^{ab}) + \frac{D_a f}{K \frac{K}{K}} D_b \left( \frac{D^a f D^b f - D^2 f g^{ab}}{K \frac{K}{K}} \right) \]
\[ = - \frac{D_a f}{K \frac{K}{K}} D_a \left( \frac{D^a f D^b f - D^2 f g^{ab}}{K \frac{K}{K}} \right) \]
which allows the terms linear in $\bar{p}_{ab}$ to be replaced in (13) by a total divergence, terms involving the momentum $J^a$ and derivatives of $f$:
\[ \bar{R} = \bar{p}_{ab} \bar{p}_{cd} \bar{g}^{ac} \bar{g}^{bd} - \frac{2}{K} g^{bd} \frac{D^a f D^b f}{K} \bar{p}_{ab} D_c f \bar{p}_{cd} + \frac{2}{K} \bar{p}_{cd} \bar{g}^{cd} \left( \bar{g}^{cd} + 2 \frac{D^2 f D^4 f}{K} \right) \]
\[ - 2 \bar{D}_a \left( \frac{D^b f}{K \frac{K}{K}} g^{ac} \bar{p}_{bc} - \frac{D^a f}{K \frac{K}{K}} g^{bc} \bar{p}_{bc} \right) + F = 2(\mu - \frac{D_a f J^a}{K \frac{K}{K}}) \quad (14) \]
\[ F = - \bar{g}^{ab} Q_{ab} + \frac{D^a f D^b f}{K} R_{ab} - \frac{D_a f D_a f D^a f}{K} + \left( \frac{D^2 f}{K} \right)^2 \]
\[ - \frac{2}{K} D_a \left( \frac{D^a f D^b f - D^2 f g^{ab}}{K \frac{K}{K}} \right) \quad (15) \]
Next, the divergence term can be rewritten in terms of \( \bar{g}_{ab} \) instead of \( g_{ab} \) using

\[
\frac{D^b f}{K^{\frac{3}{2}}} g^{ac} \bar{p}_{bc} - \frac{D^a f}{K^{\frac{3}{2}}} g^{bc} \bar{p}_{bc} = \frac{D^b f}{K^{\frac{3}{2}}} g^{ac} \bar{p}_{bc} - \frac{D^a f}{K^{\frac{3}{2}}} g^{bc} \bar{p}_{bc}
\]  

(16)

Finally, one can show that \( F \) vanishes; first note that

\[
g^{ab} Q_{ab} = \left( \frac{D^2 f}{K} \right)^2 + \frac{D^a f (D_a D^2 f - D^2 D_a f)}{K} - \frac{D_a D_b f D^a D^b f}{K^2} + \frac{D^a f D^b D_a D_b f D^c f}{K^2} - \frac{D^a f D^b f}{K^2} \]

so that

\[
-\bar{g}^{ab} Q_{ab} = - \left( \frac{D^2 f}{K} \right)^2 - \frac{D^a f (D_a D^2 f - D^2 D_a f)}{K} + \frac{D_a D_b f D^a D^b f}{K^2} + 2 \frac{D^a f D^b f}{K^2} D_a D_b f D^2 f
\]

(17)

Next, one finds that

\[
-2 \frac{D_a f}{K^{\frac{3}{2}}} D_b \left( \frac{D^a D^b f - D^2 f g^{ab}}{K^{\frac{1}{2}}} \right) = 2 \frac{D^a f (D_a D^2 f - D^2 D_a f)}{K} - 2 \frac{D^a f D^b f}{K^2} R_{ab} + 2 \frac{D^a f D^b f}{K^2} D_a D_b f D^2 f
\]

(18)

Hence, substituting (17) and (18) into (15) yields

\[
F = \frac{D^a f (D_a D^2 f - D^2 D_a f)}{K} + \frac{D^a f D^b f}{K} R_{ab}
\]

\[
= - \frac{D^a f D^b f}{K} R_{ab} + \frac{D^a f D^b f}{K} R_{ab} = 0
\]

Using this and (16), the transformed Hamiltonian constraint (14) can be written

\[
\bar{R} - \bar{p}_{ab} \bar{p}_{cd} \bar{g}^{ac} g^{bd} - 2 \frac{D^b f}{K} \bar{p}_{ab} D^c f \bar{p}_{cd} + \bar{p}_{ab} g^{ab} \bar{p}_{cd} \left( \bar{g}^{cd} + 2 \frac{D^c f D^d f}{K} \right)
\]

\[
- 2 \bar{D}_a \left( \frac{D^b f}{K^{\frac{3}{2}}} g^{ac} \bar{p}_{bc} - \frac{D^a f}{K^{\frac{3}{2}}} g^{cd} \bar{p}_{cd} \right) = 2 (\mu - \frac{D_a f}{K^{\frac{3}{2}}} J^a)
\]

where \( \bar{R} \) is the scalar curvature of \( \bar{g}_{ab} \), \( \bar{D}_a \) is the covariant derivative with respect to \( \bar{g}_{ab} \). As \( f \) is assumed to satisfy the Jang equation, (12), this simplifies to

\[
\bar{R} - \frac{2}{K} g^{bd} D^a f \bar{p}_{ab} D^c f \bar{p}_{cd} - 2 \bar{D}_a \left( \frac{D^b f}{K^{\frac{3}{2}}} g^{ac} \bar{p}_{bc} - \frac{D^a f}{K^{\frac{3}{2}}} g^{cd} \bar{p}_{cd} \right) = 2 (\mu - \frac{D_a f}{K^{\frac{3}{2}}} J^a) + \bar{p}_{ab} \bar{p}_{cd} g^{ac} g^{bd}
\]

(19)

where \( \bar{R} \) is the scalar curvature and \( \bar{D}_a \) is the covariant derivative with respect to \( \bar{g}_{ab} \). The right hand side is nonnegative if the matter source satisfies the dominant energy condition as \( |\frac{D_a f}{K^{\frac{3}{2}}}| \leq 1 \). Hence \( \bar{R} \) satisfies the inequality

\[
\bar{R} - \frac{2}{K} g^{bd} D^a f \bar{p}_{ab} D^c f \bar{p}_{cd} - 2 \bar{D}_a \left( \frac{D^b f}{K^{\frac{3}{2}}} g^{ac} \bar{p}_{bc} - \frac{D^a f}{K^{\frac{3}{2}}} g^{cd} \bar{p}_{cd} \right) \geq 0
\]

Multiplication by a function \( \phi^2 \) and rearrangement yields

\[
\phi^2 \bar{R} + 2 g^{bd} D_b \phi D_d \phi - 4 \phi D_a \phi \frac{D^a f}{K^{\frac{3}{2}}} g^{cd} \bar{p}_{cd}
\]

\[
- 2 \bar{D}_a \left( \phi^2 \frac{D^b f}{K^{\frac{3}{2}}} g^{ac} \bar{p}_{bc} - \phi^2 \frac{D^a f}{K^{\frac{3}{2}}} g^{cd} \bar{p}_{cd} \right) \geq 2 g^{bd} \left( \phi \frac{D^a f}{K^{\frac{3}{2}}} \bar{p}_{ab} - D_b \phi \right) \left( \phi \frac{D^c f}{K^{\frac{3}{2}}} \bar{p}_{cd} - D_d \phi \right)
\]
which, again assuming the the Jang equation is satisfied, simplifies to

\[
\phi^2 \hat{\mathcal{R}} + 2(\hat{D} \phi)^2 - 2\hat{D}_a \left( \phi^2 \frac{D^b f}{K^2} \hat{g}^{ac} \hat{p}_{bc} \right) \geq 0
\]

(20)

where \((\hat{D} \phi)^2 = \hat{g}^{bd} \hat{D}_b \phi \hat{D}_d \phi = \hat{g}^{bd} \hat{D}_b \phi \hat{D}_d \phi\). Since the solution is global, the graph \(\mathcal{G}_f^n = \Sigma^n\). Integration of (20) over the graph therefore yields

\[
\int_{\Sigma^n} d\mu_\hat{g}(\phi^2 \hat{\mathcal{R}} + 2(\hat{D} \phi)^2) - 2\int_{\Sigma^n} d\mu_3 \hat{D}_a \left( \phi^2 \frac{D^b f}{K^2} \hat{g}^{ac} \hat{p}_{bc} \right) \geq 0
\]

If \(\phi \in C^\infty(\Sigma^n)\), then the integral over the divergence vanishes as there are no boundaries interior to the asymptotic regions. Therefore,

\[
\int_{\Sigma^n} d\mu_3(\phi^2 \hat{\mathcal{R}} + 2(\hat{D} \phi)^2) \geq 0.
\]

This implies, as \(a_n = \frac{4(n-1)}{n-2} > 2\) for \(n \geq 3\),

\[
\int_{\Sigma^n} d\mu_3 \phi \hat{L} \phi = \int_{\Sigma^n} d\mu_3 (a_n (\hat{D} \phi)^2 + \phi^2 \hat{\mathcal{R}}) \geq \int_{\Sigma^n} d\mu_3 (2(\hat{D} \phi)^2 + \phi^2 \hat{\mathcal{R}}) \geq 0.
\]

Thus the conformal laplacian operator \(\hat{L}\) for metric \(\hat{g}_{ab}\) is positive on \(\phi \in C^\infty_0(\Sigma^n)\). Therefore, by Theorem 2 there is a smooth, everywhere positive solution \(\phi\) on \(\Sigma^n\) where \(\phi \to 1\) with asymptotically flat fall-off as \(r \to \infty\) in each asymptotic region. Define the metric \(\hat{g}_{ab} = \phi^{\frac{n-2}{2}} \hat{g}_{ab}\). This metric is a complete, asymptotically flat metric with zero scalar curvature everywhere.

This result directly implies a new singularity theorem for manifolds with nonpositive maximal Yamabe invariant. A spacetime satisfies the null convergence condition, also known as the null energy condition if \(R_{ab} W^a W^b \geq 0\) for all null \(W\), the weak energy condition if \(T_{ab} W^a W^b \geq 0\) for all timelike \(W\) and the dominant energy condition if the weak energy condition holds and \(T_{ab} W^b T_c^b W^c \leq 0\) \([4]\). Notice the least restrictive of these energy conditions is the null convergence condition. If the dominant energy condition is satisfied, then so is the null convergence condition by a continuity argument. The null convergence condition is the energy condition required in the singularity theorem of Penrose \([1]\). If a spacetime is the maximal evolution of an initial data set satisfying the dominant energy condition, it automatically satisfies the null convergence condition. However, for generality, the theorem below assumes the null convergence condition in the spacetime separately from the assumption of the dominant energy condition on the Cauchy surface.

**Theorem 10.** If a spacetime \(M^{n+1}\) has a Cauchy surface \(\Sigma^n\) with \(\hat{\sigma}(\Sigma^n) \leq 0\), asymptotically flat initial data and sources that obey the dominant energy condition, then \(M^{n+1}\) contains one or more apparent horizons. Hence if \(M^{n+1}\) satisfies the null convergence condition, then it is null geodesically incomplete.

**Proof.** Assume that a global solution of the Jang equation exists for the asymptotically flat initial data set on \(\Sigma^n\). By Theorem 2 it follows that \(\Sigma^n\) admits an asymptotically flat metric with zero scalar curvature. Theorem 7 then implies that \(\hat{\sigma}(\Sigma^n) > 0\), in contradiction to the assumption that \(\hat{\sigma}(\Sigma^n) \leq 0\). Consequently \(\Sigma^n\) must not admit a global solution to the Jang equation. Hence, as obstructions to a global solution are closed submanifolds \(T^{n-1} \subset \Sigma^n\) with \(H_{T^{n-1}} - P_{T^{n-1}} = 0\), the initial data set must contain apparent horizons. This immediately implies that \(M^{n+1}\) is singular by the Penrose singularity theorem.\(^9\)

The apparent horizons forming the obstruction to the Jang equation can be either future or past trapped (or both). Therefore, the singularities may be in either the past and/or future evolution.

The key feature of Theorem 10 is the proof of the existence of one or more apparent horizons. Consequently, other singularity theorems with conditions requiring an apparent horizon hold for spacetimes with Cauchy surface \(\Sigma^n\) with \(\hat{\sigma}(\Sigma^n) \leq 0\), asymptotically flat initial data and sources that obey the dominant energy condition. In particular, such spacetimes are singular if they also satisfy the strong energy condition and generic condition by the Hawking-Penrose theorem \([2, 55]\):

\(^9\) Note that it suffices that the trapped surfaces be outer trapped surfaces in the proof of the singularity theorem. See, for example \([2]\).
Theorem (Hawking-Penrose 1970). Spacetime $M$ with metric $\gamma_{ab}$ is not timelike and null geodesically complete if the chronology condition holds, the strong energy condition and the generic condition are satisfied, and there exists one of the following: 1) a compact achronal set without edge, 2) a closed trapped surface, 3) a point $p$ from which every past directed (or future directed) null geodesic has null expansion that becomes negative.

The strong energy condition is that $R^\alpha_{\beta\gamma\delta}W^\beta W^\gamma \geq 0$ for non-spacelike $W^\alpha$. The generic condition is that every non-spacelike geodesic contains a point at which $K^p_{\alpha} R^p_{\beta\gamma\delta\psi} K^\beta K^\gamma K^\delta K^\psi \neq 0$ where $K^\alpha$ is the tangent to the geodesic. The generic condition is satisfied for the case of vacuum spacetime containing gravitational radiation and in many other physical situations.

VI. ASYMPTOTICALLY FLAT SIMPLY CONNECTED 4-MANIFOLDS WITH NONPOSITIVE MAXIMAL YAMABE INVARIANT

In 5 or more spacetime dimensions, there are an infinite number of asymptotically flat spacetimes with topologically distinct, simply connected Cauchy surfaces. Consequently Theorem 10 implies collapse of a set of topological structures not addressed by prior singularity theorems. Given this, it is useful to discuss the construction and characterization of such spacetimes. We do so below, concentrating on the interesting case of 5 dimensional asymptotically flat spacetimes with Cauchy surfaces $\Sigma^4$.

Closed manifolds with obstructions to positive scalar curvature can be used to construct asymptotically flat manifolds with nonpositive maximal Yamabe invariant. Recall that smooth compactification of an asymptotically flat manifold $\Sigma^n$ results in a smooth closed manifold $\Sigma^n$. By Definition 5 if $\Sigma^n$ has nonpositive Yamabe invariant, the corresponding asymptotically flat manifold has $\hat{\sigma}(\Sigma^n) \leq 0$. Consequently, by puncturing closed manifolds $\Sigma^n$ with obstructions to positive scalar curvature, one can construct asymptotically flat manifolds $\Sigma^n$ with $\hat{\sigma}(\Sigma^n) \leq 0$. Furthermore, obstructions to positive scalar curvature are well known to be related to topological properties; therefore topological and smooth invariants of closed manifolds characterize classes of asymptotically flat manifolds with $\hat{\sigma}(\Sigma^n) \leq 0$.

In 4 dimensions, topological obstructions to positive curvature are characterized by the $\hat{A}$-genus. In addition, the differentiable structure of the manifold can also produce an obstruction to positive curvature. The Seiberg-Witten invariants characterize obstructions produced by both topology and differentiable structure.

A. Nonpositive maximal Yamabe invariant from topological obstructions in 4 dimensions

If $M^4$ is a smooth spin manifold, i.e. a spin manifold that admits a differentiable structure, the $\hat{A}$-genus can be defined in terms of the index of the Dirac operator. As the dimension is even, a complex spin bundle $S$ over $M^4$ has a natural decomposition into the sum $S_+ \oplus S_-$, the eigenspaces of the complex volume element $\omega_{C}$ of the Clifford algebra. Let $D^+$ be the restriction of the Dirac operator $D$ to $S_+$ and $D^-$ corresponding restriction to $S_-$. Note $D^+: S_+ \to S_-$. and the adjoint of $D^+$ is $D^-$. The index of $D^+$, $\text{ind}(D^+) = \dim(\ker D^+) - \dim(\ker D^-)$. This invariant is the $\hat{A}$-genus, $\text{ind}(D^+) = \hat{A}(M^4)$.

The Weitzenböck formula for the Dirac laplacian is

$$D^2\psi = -\nabla^2 \psi + \frac{1}{4} R \psi.$$

If $M^4$ admits a metric with $R > 0$, this implies, as the operator $-\nabla^2 + \frac{1}{4} R$ is positive, that $\ker D^2 = 0$. Consequently, as $\ker D^+ + \ker D^- = \ker D = \ker D^2$, $\text{ind}(D^+) = \hat{A}(M^4) = 0$; the $\hat{A}$-genus vanishes. Therefore, if a closed 4-manifold has nonvanishing $\hat{A}$-genus, it does not admit a metric of positive scalar curvature. This result is due to Lichnerowicz [33].

This result extends to smooth, asymptotically flat 4-manifolds:

Lemma 11. A closed smooth spin manifold $M^4$ with $\hat{A}(M^4) \neq 0$ has nonpositive Yamabe invariant, $\sigma(M^4) \leq 0$. Furthermore the maximal Yamabe invariant of the asymptotically flat manifold $M^4 - S$, $S$ a finite set of points, is also nonpositive, $\hat{\sigma}(M^4 - S) \leq 0$.

Proof. As $\hat{A}(M^4) \neq 0$, it follows that $R \leq 0$ for all metrics on $M^4$. Hence, $\sigma(M^4) \leq 0$. Next, observe that by Theorem 5 the compactification of $M^4 - S$ to $M^4$ is unique. Consequently, by Definition 5 $\hat{\sigma}(M^4 - S) \leq 0$. $\square$
In general, the $\hat{A}$-genus for any closed 4-manifold can be computed in terms of its signature. When $n = 4k$, $k \geq 1$ the signature of a closed $n$-manifold is that of the quadratic form

$$Q : H^{2k}(M^n; \mathbb{Z}) \otimes H^{2k}(M^n; \mathbb{Z}) \to \mathbb{Z}$$

where $Q(\alpha, \beta) = (\alpha \cup \beta)[M^n]$. Then $\hat{A}(M^n) = -\frac{1}{8}\text{sig}(M^n)$. If $M^4$ is a smooth closed 4-manifold, $\hat{A}[M^4] = -\frac{1}{8}\tau(M^4)$ where $\tau(M^4)$ is the Hirzebruch signature,

$$\tau(M^4) = \frac{1}{48\pi^2} \int_{M^4} C_{abcd} C_{abcd} d\mu_g$$

with $C_{abcd}$ being the Weyl curvature of any riemannian metric $g$ on $M^4$. Note that the Hirzebruch signature can be computed for any smooth 4-manifold, with or without spin structure.

The intersection form of simply connected 4-manifolds is particularly well understood. It is unimodular and its basic building blocks are $H = (\varphi \frac{1}{4}, 1), (1), (-1)$ and $E_8$. Furthermore, it classifies simply connected closed 4-manifolds in the following sense according to the results of Freedman [37]: For every unimodular symmetric bilinear form $Q$ there exists a simply connected closed topological 4-manifold $M$ such that $Q_M \cong Q$. If $Q$ is even, this manifold is unique up to homeomorphism. If $Q$ is odd, there are exactly two different homeomorphism types of manifolds with the given intersection form. At most one of these carries a smooth structure. Consequently, simply connected smooth 4-manifolds are determined up to homeomorphism by their intersection forms.

Secondly, the quadratic form of the connected sum of two simply connected 4-manifolds is the direct sum of their quadratic form, $Q_{M_1 \# M_2} = Q_{M_1} \oplus Q_{M_2}$. Consequently, families of simply connected 4-manifolds can be constructed by taking connected sums of a set of fundamental building blocks, $S^4, S^2 \times S^2, \mathbb{CP}^2, \overline{\mathbb{CP}^2}, K^3$ and $E_8$ with quadratic forms $Q_{S^4} = 0, Q_{S^2 \times S^2} = H, Q_{K^3} = 2(-E_8) \oplus 3H, Q_{\mathbb{CP}^2} = (1), Q_{\overline{\mathbb{CP}^2}} = (-1)$ and $Q_{E_8} = E_8$. As the quadratic forms of $S^4, S^2 \times S^2, K^3$ and $E_8$ are even, these manifolds have vanishing second Stiefel-Whitney class and admit a spin structure; as the quadratic forms of $\mathbb{CP}^2$ and $\overline{\mathbb{CP}^2}$ are odd, they do not. In addition, Rokhlin’s theorem [39] states that the signature of any closed smooth spin 4-manifold is a multiple of 16. A computation of its signature demonstrates that $E_8$ is a spin 4-manifold which admits no smooth structure. $^{11}$

We now apply these results to exhibit an infinite family of simply connected, asymptotically flat 4-manifolds with nonpositive maximal Yamabe invariant. First, as the signature of $K^3$ is 16, its $\hat{A}$-genus is nonzero. It follows by Lemma [11] that the asymptotically flat manifold obtained by removing a point $p$, $K^3 - p$, has $\hat{A}(K^3 - p) \leq 0$. Connected sums of $K^3$ with itself and $S^2 \times S^2$ produce more examples. For example, the connected sum $K^3 \# K^3$ also has nonzero $\hat{A}$-genus and consequently the asymptotically flat manifold $K^3 \# K^3 - p$ has nonpositive maximal Yamabe invariant. The manifold $K^3 \# (S^2 \times S^2) - p$ similarly also has nonpositive maximal Yamabe invariant. Furthermore, the quadratic form of a smooth simply connected spin 4-manifold $M^4$ is homeomorphic to $2kE_8 \oplus nH$ for $k, n$ integers so long as $b_2(M^4) \geq \frac{1}{16}\tau(M^4)$ where $b_2(M^4)$ is the second betti number of $M^4$. Therefore there is an infinite set of smooth simply connected spin 4-manifolds with nontrivial signature and consequently a corresponding infinite set of asymptotically flat manifolds with nonpositive maximal Yamabe invariant.

B. Nonpositive maximal Yamabe invariant and the Seiberg-Witten invariants

In contrast to higher dimensional manifolds, closed 4-manifolds can admit a countably infinite number of distinct differentiable structures. Furthermore, open 4-manifolds can admit an uncountably many distinct differentiable structures as dramatically illustrated for $\mathbb{R}^4$. Simply connected 4-manifolds also carry a countably infinite number of distinct differentiable structures; a theorem of Friedman and Morgan shows that simply connected manifolds corresponding to the intersection forms $2n(-E_8) \oplus (4n - 1)H, n \geq 1$, and $(2k - 1)(1) \oplus N(-1), k \geq 2, N \geq 10k - 1$, each carry infinitely many distinct differentiable structures [36][38]. Those with intersections forms of the first type have nonzero $\hat{A}$-genus and consequently collapse from their topology. However, those of the second type, realized by certain connected sums of $\mathbb{CP}^2$ and $\overline{\mathbb{CP}^2}$ do not. However, certain differentiable structures on such manifolds also produce an obstruction to positive curvature. These obstructions can be characterized by the existence of solutions to the Seiberg-Witten equations and are thus related to the Seiberg-Witten invariants. They can be viewed as arising

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10 In terms of the de Rham cohomology over $\mathbb{R}$, $Q(\alpha, \beta) = \int_{M^n} \alpha \wedge \beta$ for $\alpha, \beta \in H^{2k}(M^n; \mathbb{R})$.

11 Alternatively, the Hirzebruch signatures are $\tau(S^4) = 0, \tau(\mathbb{CP}^2) = 1, \tau(S^2 \times S^2) = 0$ and $\tau(K^3) = 16$. Consequently, Rokhlin’s theorem [39] implies that $\mathbb{CP}^2$ is a closed smooth 4-manifold which does not admit a spin structure.
from the topological structure of an associated bundle, the complex spin bundle, over the 4-manifold. Consequently, as discussed below, the related set of asymptotically flat manifolds will have nonpositive maximal Yamabe invariant.

A complex spin structure on a closed 4-manifold $M^4$ is given by replacing the group $Spin(4)$ with the group $Spin_c(4)$. Clearly, all 4-manifolds which admit a spin structure automatically admit a complex spin structure. Moreover, all non-spin 4-manifolds also admit a complex spin structure by a theorem of Wu. Let $D_A$ be the Dirac operator associated with the $U(1)$ connection $A$ on a given bundle of $Spin_C(n)$ spinors $W$ on $M^4$. As in the spinor case, $W$ can be decomposed into the sum of eigenspaces $W_+ \oplus W_-$ in even dimensions with $D_A^+$ the restriction of $D_A$ to $W_+$ and $D_A^-$ the corresponding restriction to $W_-$. Note $D_A^+ : W_+ \to W_-$ and the adjoint of $D_A^+$ is $D_A^-$. The Weitzenböck formula for the $Spin_C(n)$ Dirac operator is given by

$$D_A^2 \psi = -\nabla^2 \psi + \frac{1}{4} R \psi + \frac{1}{2} F_A \psi.$$ 

This expression now contains both the scalar curvature $R$ of the Riemannian metric and the curvature $F_A$ of the connection $A$. Consequently, the index of $D_A$ yields information regarding obstructions to scalar curvature; however, the existence of an obstruction now also depends on the curvature of the connection $A$.

In 4 dimensions, the Seiberg-Witten equations provide a particularly fruitful choice of connection and curvature. The generalized Seiberg-Witten equations are

$$D_A \psi = 0 \quad \text{and} \quad F_A^+ = q(\psi) + i \omega$$

where $D_A$ is the Dirac operator associated to $A$, $F_A^+$ is self-dual part of the curvature 2-form of $A$, $q$ is the map from $W_+$ to imaginary self-dual 2-forms which takes the square of the spinor $\psi$, namely $q(\psi) = \psi \otimes \psi^* - \frac{1}{2} |\psi|^2$, and $\omega$ is a real self-dual 2-form, typically chosen to be zero or harmonic. The Seiberg-Witten equations are the above equations with $\omega \equiv 0$. The case of $\omega \neq 0$ corresponds to a perturbation of the Seiberg-Witten equations needed to avoid singular points in the moduli space. Solutions to the Seiberg-Witten equations are called monopoles as these equations are the field equations of massless magnetic monopoles on the manifold $M^4$.

Given a closed 4-manifold $M^4$ with a $U(1)$ gauge field, the space $A[M^4]$ is the space of pairs $(A, \psi)$ with $A$ a $U(1)$ connection on the complex line bundle $L$ and $\psi \in W_+$. Moreover, the gauge transformations are given by smooth maps of $M^4$ into $U(1)$, $G[M^4] = C^\infty(M^4, U(1))$. The moduli space is $B[M^4] = A[M^4] / G[M^4]$ and the irreducible moduli space is $B^+ [M^4] = A^+ [M^4] / G[M^4]$ where $A^+ [M^4] \subset A[M^4]$ is the subspace of configurations with $\psi \neq 0$. Instead of using the full group of gauge transformations, one can fix a base point on $x_0 \in M^4$ and consider the group of base point fixing gauge transformations $G_0[M^4] = \{ g \in G[M^4] | g(x_0) = id \}$. Note that $G[M^4] / G_0[M^4] \cong U(1)$. The advantage of this group is that it acts freely on gauge configurations. One can define the corresponding moduli spaces $B[M^4] = A[M^4] / G_0[M^4]$ and $B^+ [M^4] = A^+ [M^4] / G_0[M^4]$.

There is a $U(1)$ bundle which relates the moduli spaces defined in terms of $G_0[M^4]$ to those defined in terms of $G[M^4]$, namely, $B[M^4] = A[M^4] / G_0[M^4]$ is the total space of a $U(1)$ bundle over $B[M^4] = A[M^4] / G[M^4]$. The same is true of $B^+ [M^4]$ and $B^+ [M^4]$.

The modulon moduli space is

$$\mathcal{M}_L[M^4] = \{(A, \psi) \in B[M^4] | D_A^+ \psi = 0, \quad F_A^+ = \sigma(\psi) \}.$$ 

Observe that $\mathcal{M}_L[M^4] \subset B[M^4]$ is a finite dimensional subspace. The moduli space depends on the choice of $L$. One problem is the moduli space $\mathcal{M}_L[M^4]$ can be singular. This may occur for several different reasons. First, the moduli space can be singular due to fixed points from the action of gauge transformations. Moreover, reducible solutions (ones with $\psi \equiv 0$) also give points for which the gauge group does not act freely. Finally, the equations themselves may not satisfy the conditions of the implicit function theorem which is needed to prove the desired properties of the moduli spaces. Therefore it is useful to define the following moduli spaces.

The generalized or the perturbed modulon moduli space is

$$\tilde{\mathcal{M}}_{L, \omega}[M^4] = \{(A, \psi) \in B[M^4] | D_A^+ \psi = 0, \quad F_A^+ = \sigma(\psi) + i \omega \}.$$ 

The modulon moduli spaces $\tilde{\mathcal{M}}_{L, \omega}[M^4]$ and $\mathcal{M}_L[M^4]$ are also related via $U(1)$ bundles. They are finite dimensional because they are solutions of nonlinear elliptic equations; however, these spaces can be shown to be compact. Moreover, they can shown to be smooth manifolds.
A hermitian metric on a complex 2-manifold can be written as
\[ c^{(1)} \]
mitad an anti-self dual Witten invariant, \[ SW \]
riemannian geometry question reduces to a question of \[ U \]
d of complex spin structure. In particular, one can show that every connection has a split in terms of self-dual and anti self-dual connection; furthermore, the Seiberg-Witten equations and has either
\[ \text{Theorem 12.} \]
\[ A \]
connection, then \[ L \]
\[ c^{(1)} \]
\[ 0 = \int_{M^4} \psi^* D_J^2 \psi \ d\mu = \int_{M^4} (-\psi^* \nabla^2 \psi + \frac{1}{4} \psi^* R \psi + \frac{1}{2} \psi^* F_A \psi) \ d\mu \]
\[ = \int_{M^4} (-\psi^* \nabla^2 \psi + \frac{1}{4} \psi^* R \psi + \frac{1}{2} \psi^* F_A \psi) \ d\mu = \int_{M^4} (|\nabla \psi|^2 + \frac{1}{4} R |\psi|^2 + \frac{1}{4} |\psi|^4) \ d\mu . \]
Hence, if \( R > 0 \), then the only possible solutions to the Seiberg-Witten equations are \( F_A^+ = 0 \) with \( \psi \equiv 0 \). Note that every connection has a split in terms of self-dual and anti self-dual connection; furthermore, \( F_A = F_A^+ + F_A^- \). Moreover, if \( \psi \) represents a cohomology class and may be chosen to be a harmonic real form. If \( F_A^+ = 0 \), then \( F_A = F_A^- \). The riemannian geometry question reduces to a question of \( U(1) \) instantons over \( M^4 \); specifically if the manifold does not admit an anti-self dual \( U(1) \) connection, then \( M^4 \) admits no metric with \( R \geq 0 \).

If \( |c_1(L)|^2[M^4] \geq 0 \), note that every \( U(1) \) bundle has unique bundle connection with harmonic curvature corresponding to the its first Chern class, \( c_1(L) = \frac{1}{2} F_A \). The first Chern class squared \( |c_1(L)|^2[M^4] \) of the line bundle \( L \) for the associated complex spin structure of \( M^4 \) can be written
\[ 0 \leq |c_1(L)|^2[M^4] = \frac{1}{4\pi^2} \int_K F_A \wedge F_A \ d\mu = \frac{1}{4\pi^2} \int_K (|F_A^+|^2 - |F_A^-|^2) \ d\mu . \]
As \( F_A^+ = 0 \), this inequality implies that \( F_A = 0 \). Thus the only possible solutions are flat connections of the \( U(1) \) gauge theory. The space of flat \( U(1) \) connections is determined by the representations of \( \pi_1(M^4) \) in \( U(1) \); however, these are trivial as the first betti number of \( M^4 \) vanishes. Consequently there are no solutions if \( |c_1(L)|^2[M^4] \geq 0 \).

Now suppose that \( |c_1(L)|^2[M^4] \) were negative; this would imply that \( b_2^+(M^4) > 0 \). Furthermore, \( b_2^+(M^4) > 0 \) by assumption. Hence, \( M^4 \) has indefinite intersection form. Given this, standard results for anti-self-dual \( U(1) \) connections imply that \( F_A = 0 \) for a generic metric [see Corollary 3.21 in 40]. Again, it follows that \( F_A = 0 \) and there are no solutions. Therefore, there are no nontrivial solutions of the Seiberg-Witten equations if \( M^4 \) admits a metric with \( R > 0 \). Consequently, \( \sigma(M^4) \leq 0 \).

The case of the asymptotically flat manifold \( (M^4 - S) \) where \( S \) is a finite set of points follows as in Lemma 11 by Theorem 3 its compactification to \( M^4 \) is unique. Therefore, \( \tilde{\sigma}(M^4 - S) \leq 0 \).

One particularly well studied class of 4-manifolds with nontrivial solutions to the Seiberg-Witten equations are Kähler 4-manifolds, complex 2-manifolds whose Kähler form is closed and nondegenerate 12. The following well-known result leads to a large class of 4-manifolds with an obstruction to positive curvature 11:

**Theorem (Witten 1994).** Given a closed Kähler 4-manifold \( K \) with \( b_1(K) = 0 \) and \( b_2^+(K) \geq 2 \), then the Seiberg-Witten invariant, \( SW_K \neq 0 \).

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12 A hermitian metric on a complex 2-manifold can be written as \( h = \sum h_{ij} dz^i \otimes d\bar{z}^j \) with complex coordinates \( \{z^i\}, i = 1, 2 \). If the 2-form \( \omega = \sum h_{ij} dz^i \wedge d\bar{z}^j \) is closed, \( d\omega = 0 \), and nondegenerate, then the 4-manifold is a Kähler 4-manifold.
Under the conditions of this theorem, the Seiberg-Witten invariant is the number of oriented solutions of the Seiberg-Witten equations. The reason for assuming Kähler manifolds in the above theorem is that the Seiberg-Witten equations simplify on these manifolds, making the calculations easier.

Now, all Kähler manifolds are symplectic manifolds. It is thus natural to ask whether or not this theorem can be generalized to this case. This question was answered in the affirmative by Taubes; he extended the above theorem to symplectic 4-manifolds \[12\]. One can think of this extension as holding because all symplectic 4-manifolds admit an almost Kähler structure; they are Kähler manifolds up to the requirement for transition functions to be holomorphic.

In the above theorems, the cohomology conditions are imposed to make the monopole moduli spaces well defined. One can extend the Witten’s theorem and Taubes’ theorem to include more general cases including \(b_1(K) = 0\), \(b_2^+(K) = 1\) and \([c_1(L)]^2[M^4] \geq 0\) using the techniques from the proof of Theorem \[12\].

Infinite families of Kähler manifolds with nontrivial Seiberg-Witten invariants can be constructed through blow-up.

We begin by summarizing the construction of the complex surface \(S\) where \(Seiberg-Witten equations. The reason for assuming Kähler manifolds in the above theorem is that the Seiberg-Witten generalizations to this case. This question was answered in the affirmative by Taubes; he extended the above theorem to differentiable structure. Therefore, in these cases, the obstruction is due to the differentiable structure, not the conditions of Theorem 12 which means that they do not admit positive scalar curvature with respect to this alternate structure. These examples include both symplectic and smooth families. The construction of infinitely many simply connected knot surgery developed by Fintushel and Stern \[48\]. These techniques were used to construct both examples and \(\tau\)-invariants, then so does \(\tau\). Kähler manifolds with topology of \(S^2\) can be constructed \[53–56\] which have nontrivial Seiberg-Witten invariants.

In the above theorems, the cohomology conditions are imposed to make the monopole moduli spaces well defined. Hence \(\pi_2(V) = 0\). Choose a neighborhood \(U\) of \(p\) that is biholomorphic to an open subset \(V\) of the origin in \(\mathbb{C}^2\) with \(p\) mapped to the origin. Then the blow-up of \(K\) at \(p\) is the space \(K’\) formed by removing \(U\) and replacing it with \(\pi_2^{-1}(V)\).

The blow-up of a Kähler manifold \(K\) produces another complex manifold \(K’\). In addition, if \(K\) is simply connected, then so is \(K’\). Consequently \(K’\) is also Kähler as a closed complex surface is Kähler if and only if its first betti number is even \[43\] (Also Theorem 10.1.4 in \[36\]). Furthermore, as \(\tau\) is diffeomorphic to \(CP^2 - p\), \(K’\) is diffeomorphic to the connected sum \(K \# CP^2\). The connected sum does not change \(b_2^+\); consequently if \(K\) has nontrivial Seiberg-Witten invariant, then so does \(K’\).

Kähler manifolds with topology of \(kCP^2 \# mCP^2\) with \(k \geq 2\) are of particular interest. These topological manifolds are smooth. Furthermore, there is no topological obstruction to positive scalar curvature on these manifolds. In fact, it is easy to see that they have a smooth structure that admits metrics of positive scalar curvature; take each \(CP^2\) and \(CP^2\) factor to have the standard differentiable structure. The Fubini-Study metric is defined with respect to this differentiable structure and has positive scalar curvature. The connected sum of two smooth manifolds can be carried out smoothly and is well known to preserve positive scalar curvature in any dimension. Therefore the standard differentiable structure on \(kCP^2 \# mCP^2\) always admits positive scalar curvature. However, for sufficiently large values of \(k\) and \(m\), the topological manifold \(kCP^2 \# mCP^2\) also admits other differentiable structures. Certain of these differentiable structures are induced by a corresponding Kähler structure \[44\]. They consequently obey the conditions of Theorem \[12\] which means that they do not admit positive scalar curvature with respect to this alternate differentiable structure. Therefore, in these cases, the obstruction is due to the differentiable structure, not the topology of the manifold.

Another set of examples is given by the complex surfaces \(S_d\). Let \(S_d \subset CP^3\) with \(d > 0\) an integer defined by \(S_d = \{(z_0, z_1, z_2, z_3) \in CP^3| z_0^d + z_1^d + z_2^d + z_3^d = 0\}\). This is a smooth manifold which is a complex surface in \(CP^3\). The Lefschetz hyperplane theorem implies that \(S_d\) is simply connected. Hence \(S_d\) is Kähler. Using the explicit definition of \(S_d\), one can find its intersection form. When \(d\) is odd, the intersection form is \(\lambda_d(1) \oplus \mu_d(-1)\) where \(\lambda_d = \frac{1}{d}(d^3 - 6d^2 + 11d - 3)\) and \(\mu_d = \frac{1}{d}(d - 1)(2d^2 - 4d + 3)\). When \(d\) is even, the intersection form is \(l_dH \oplus m_d(-E_8)\) where \(l_d = \frac{1}{d}(d^3 - 6d^2 + 11d - 3)\) and \(m_d = \frac{1}{d}d(d^2 - 4)\). The second betti number is \(b_2(S_d) = d(6 - 4d + d^2) - 2\) and the signature is \(\tau(S_d) = \frac{1}{2}(4 - d^2)d\). Since \(\tau(S_d) = b_2^+(S_d) - b_2^-(S_d)\) and \(b_2(S_d) = b_2^+(S_d) + b_2^-(S_d)\), it follows that \(\tau(S_d) + b_2^+(S_d) = 2b_2^-(S_d)\); thus a computation shows that \(b_2^+(S_d) \geq 2\) when \(d \geq 2\). Additionally, any blow-up of \(S_d\) is also a complex manifold and is Kähler. Thus the surfaces \(S_d\) for \(d \geq 4\) and blow-ups of these surfaces are an infinite set of manifolds with nontrivial Seiberg-Witten invariants.

Additional results can be derived using more powerful techniques including rational blowdown surgery \[46\] and knot surgery developed by Fintushel and Stern \[48\]. These techniques were used to construct both examples and infinite families of simply connected 4-manifolds with exotic differentiable structures, including the case of \(b_2^+ = 1\). These examples include both symplectic and smooth families. The construction of infinitely many simply connected symplectic 4-manifolds admitting nontrivial Seiberg-Witten invariants with \(c_1^2 > \frac{8}{3}\pi \chi\) \[15\]. More recent examples include exotic differentiable structures on 4-manifolds with small Euler characteristic, namely \(CP^2 \# kCP^2\) for \(k = 5, 6, 7, 8, 9\) \[44,51\] and infinite families of smooth 4-manifolds homeomorphic to each other with the same Seiberg-Witten invariants \[52\]. Further constructions of exotic differentiable structures and families of exotic differentiable structures on \(3CP^2 \# kCP^2\) for various integers \(k\) and \((2n + 2l - 1)CP^2 \# (2n + 4l - 1)CP^2, n \geq 0, l \geq 1\) have also been constructed \[53,56\] which have nontrivial Seiberg-Witten invariants.
In summary, it is clear that there are an infinite number of simply connected 4-manifolds which have nonpositive maximal Yamabe invariant. Consequently, Theorem 10 implies collapse of an infinite set of topological structures in 5-dimensional asymptotically flat spacetimes not covered by the generalization of Gannon’s theorem.

VII. DISCUSSION

Gannon’s theorem and its generalization to higher dimensional spacetimes and Theorem 10 apply to different but overlapping classes of topological structures. Gannon’s theorem is only applicable to asymptotically flat Cauchy surfaces with nontrivial fundamental group; Theorem 10 applies to asymptotically flat Cauchy surfaces with more general topology, but that exhibit an obstruction to nonpositive curvature characterized by the maximal Yamabe invariant. Open manifolds with nontrivial fundamental group can have nonpositive maximal Yamabe invariant; punctured $n$-tori are a simple example of such manifolds in all dimensions as the $n$-torus admits no metric with positive scalar curvature. However, manifolds such as $\mathbb{R}P^n$ admit positive scalar curvature, but have nontrivial fundamental group. Consequently punctured $\mathbb{R}P^n$ with asymptotically flat initial data collapses to form a singularity by Gannon’s theorem, but is not in the class of topologies covered by Theorem 10.

In 3 and 4-dimensional asymptotically flat spacetime, Gannon’s theorem is definitive as the fundamental group completely characterizes the topology in these dimensions. However, this is not true in higher dimensions; Theorem 10 now yields singularity formation in an infinite set of simply connected asymptotically flat 5-dimensional spacetimes, a set not covered by Gannon’s theorem. Furthermore, singularity formation now can occur due to either the topology or the differentiable structure of the Cauchy surface. Singularity formation from differentiable structure is a novel result.

The theorem also applies in asymptotically flat spacetimes of dimension up to 8. It may be possible to generalize Theorem 10 to spacetime dimension greater than 8 by generalizing the singularity theorems to distributional apparent horizons. In higher dimensions, obstructions to positive curvature on closed simply connected manifolds is well understood. Gromov and Lawson proved that any compact simply connected manifold that does not admit a spin structure admits a metric with positive scalar curvature $[57]$. Hitchin proved that if a spin manifold $M^n$ admits a metric of positive curvature then its $\alpha$-invariant vanishes, $\alpha(M^n) = 0$ $[13]$. Stolz proved that converse is also true; in 5 or more dimensions, any simply connected spin manifold with $\alpha(M^n) = 0$ admits a metric of positive scalar curvature $[58]$. Consequently, obstructions to positive curvature are completely characterized by nonvanishing $\alpha$-invariant.

Exotic spheres in 9 and 10 dimensions provide examples of manifolds with nonvanishing $\alpha$-invariant and consequently obstructions to positive scalar curvature $[29] [31]$. However, puncturing these spaces does not yield examples of asymptotically flat manifolds with nonpositive maximal Yamabe invariant. Puncturing an exotic $n$-sphere, $n \geq 5$, yields $\mathbb{R}^n$ with its unique differentiable structure. Conversely, the smooth compactification of an asymptotically flat manifold is not unique in these dimensions by Theorem 10. Consequently, the supremum over all possible attaching maps will yield a positive maximal Yamabe invariant. However, the less robust definition of the the asymptotically flat Yamabe invariant $\hat{\sigma}(\Sigma^n, \Phi)$ will yield an obstruction to positive scalar curvature that depends on $\Phi$. Consequently, the obstruction now depends on the asymptotically flat initial data.

Theorem 10 also differs from Gannon’s theorem another respect; if $\hat{\sigma}(\Sigma^n) \leq 0$ then there must be one or more apparent horizons in any Cauchy surface $\Sigma^n$. In contrast, Gannon’s theorem only implies the existence of an apparent horizon in the spacetime; it does not guarantee one in each Cauchy surface. In fact, one can exhibit Cauchy surfaces with nontrivial fundamental group and no apparent horizons. The $\mathbb{R}P^3$ geon spacetime is a simple example. This spacetime can be constructed from the initial data for Schwarzschild spacetime on the time symmetric slice by antipodally identifying points on the minimal 2-sphere at $r = 2M$. As this sphere is totally geodesic, the resulting initial data is smooth. Its evolution results in a spherically symmetric spacetime whose domain of outer communications is isomorphic to that of one asymptotic region of Schwarzschild spacetime. The $\mathbb{R}P^3$ geon has initial data with positive scalar curvature; Theorem 10 does not apply to this example, but as $\pi_1(\mathbb{R}P^3) = \mathbb{Z}_2$, Gannon’s theorem does. This spacetime clearly contains apparent horizons on Cauchy surfaces in the future of the time symmetric slice. But the time symmetric slice itself does not contain an apparent horizon; after antipodal identification the minimal 2-sphere at $r = 2M$ becomes a $\mathbb{R}P^2$; a nonorientable surface. It is therefore not an apparent horizon. Clearly, Gannon’s theorem cannot be improved to guarantee an apparent horizon for all topologies.

It is clearly interesting to consider whether Theorem 10 can be strengthened into a topological censorship theorem. The topological censorship theorems applied to asymptotically flat spacetimes imply that all topology associated with

$[13]$ The $\alpha$-invariant is a generalization of the $\tilde{A}$-genus to spin manifolds.
A nontrivial fundamental group is hidden behind horizons; i.e. that the domain of outer communications is simply connected. A corresponding strengthening of Theorem [19] would be that the domain of outer communications has positive maximal Yamabe invariant.

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