Towards a theory of classification

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Abstract. The well-known difficulties arising in a classification which is not set-theoretically trivial—involving what is sometimes called a non-smooth quotient—have been overcome in a striking way in the theory of operator algebras by the use of what might be called a classification functor—the very existence of which is already a surprise. Here the notion of such a functor is developed abstractly, and a number of examples are considered (including those which have arisen for various classes of operator algebras).

1. The purpose of this note is to propose an approach to the general question of classification.

Except in the simplest cases (sets, vector spaces, finite simple groups!) it is not possible even in principle to label the isomorphism classes of a given class of mathematical objects in a reasonable way. Even when this is possible, one is often interested in more than just when two objects are isomorphic. (For instance, one might be interested in when one object is isomorphic to a subobject of another.) (This is obviously important for sets. For finite simple groups it is an open problem.)

Perhaps, rather than labels for isomorphism classes, what one really wants, given a category, is a functor—distinguishing isomorphism classes—from this category into some other, simpler, category. (In other words, still labels for objects in the given category, but with isomorphic objects no longer required to have the same

The research of the author was supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

AMS 2000 Mathematics Subject Classification. Primary: 18A22, 46L35, 46M15; Secondary: 19K14, 19K35, 20E36.
label—just isomorphic labels!) (And maps between objects reflected by maps—or at least formal arrows—between labels—but reflected faintly, in the sense that certain maps will coalesce.)

It would be natural to call such a functor a classification functor, and the codomain category a classifying category.

The case in which every object of the classifying category was isomorphic to the image of some object in the given category, and every map between the images of two objects was the image of a map between them—the case that the functor was full—, would be of particular interest. (On the other hand, the case of a faithful functor—distinct maps in the given category taken by the functor into distinct maps in the codomain category—would not be of interest, the whole point being to forget at least something!)

In this note I shall review some examples of classification functors. Naturally, the more concrete such a functor is, the more interesting and useful it is likely to be. On the other hand, one may hope that a more abstract classifying category could also be interesting. (Recall that, in fact, every category is concrete—it can be described as a subcategory of the category of sets, with the maps being those preserving certain operations on the sets.)

Before stating some positive results (Theorems 1 and 3 below), let me first be a little more specific concerning the approach of just looking at isomorphism classes, and what the difficulties with this approach are.

A given category, which it is desired to classify, may or may not have other maps than just isomorphisms. In either case, it may have a natural topology or Borel structure—typically, a Polish topology or a standard Borel structure. In this case, even if one ignores maps and just looks at objects, the quotient topology or Borel structure on the isomorphism classes will in general be singular. (This is one of the lessons of the theory of operator algebras!) (The phenomenon of non-smooth quotients was first explored by Mackey, Dixmier, and Glimm in the setting of irreducible representations of a C*-algebra or locally compact group. It was later also studied by Gabriel in the setting of indecomposable finite-dimensional representations of directed graphs.)

In a category with only isomorphisms, just looking at objects means forgetting about the number of isomorphisms between two objects, and just keeping track of whether there is one or not. Passing to isomorphism classes, then, even if it destroys a given well-behaved topology or Borel structure, may, at least in a trivial way, be thought of as passing by a functor to a quotient category.

If the given category includes homomorphisms that are not isomorphisms—the
most interesting setting—the quotient category may not even exist!

What I mean by this is that, if one identifies arbitrary homomorphisms between two objects (instead of just isomorphisms) whenever they differ by an automorphism, on either the domain or codomain side, or both, then, while this determines an equivalence relation on the morphisms of the given category, this is not in general compatible with composition of morphisms. The product of two equivalence classes of morphisms, while it is always a union of equivalence classes, may fail to be a single equivalence class. Thus, already one may no longer have a category (however well behaved a given topology or Borel structure may still be). This difficulty will persist (not to mention the possible collapse of the topology or Borel structure!) on passage to the isomorphism classes. (In other words, there is no quotient category.)

For instance, the quotient category fails to exist in this way already in the case of sets. (The product of two non-constant maps may be constant, whereas the product of the equivalence classes of two non-constant maps always contains a non-constant map.)

It is perhaps worth mentioning that this difficulty does not arise if one restricts attention to the category of injective maps between sets. The quotient construction described above yields the category of cardinal numbers, with a (unique) morphism between two cardinals whenever the second is greater than or equal to the first. This classifying category thus retains the subobject information from the original category. (Starting with the category of sets and surjections, one also obtains a classifying category by this construction—namely, ignoring automorphisms. It is interesting to note that, while the two original categories, sets with injections and sets with surjections, would not seem to be simply related, the classifying categories that we have obtained for them are related in a very simple way: one is just the dual of the other—i.e., the category with the same objects and all the arrows reversed.) (This is not the case for the original categories—as can be seen by just counting numbers of arrows, for instance when the domain set is a single point.) (It is fortunate that the constructions just described work, as, as is well known, cardinal numbers cannot just be defined as equivalence classes of sets—also the order relation is needed.)

The case of vector spaces is very similar to the case of sets (and may be essentially reduced to that case by choosing bases).

It turns out that a somewhat similar difficulty occurs with the category of finite simple groups (even with injective maps). Since there is an automorphism of the group $A_6$ taking the permutation $(123)$ into the permutation $(123)(456)$ (I am indebted to J. B. Olsson for this calculation), this does not extend to an au-
tomorphism of $A_7$ (the automorphism group of which, in contrast to the case of $A_6$, is just $S_7$). The product of the equivalence classes (modulo automorphisms) of the canonical embeddings of $A_3$ in $A_6$ (as the permutations of the first three symbols) and of $A_6$ in $A_7$ (as the permutations of the first six symbols), because of this automorphism of $A_6$, is therefore strictly larger than the equivalence class of the product, the canonical embedding of $A_3$ in $A_7$. (It also contains the equivalence class of the embedding of $A_3$ in $A_7$ with multiplicity two, i.e., taking the permutation $(123)$ into $(123)(456)$.)

Somewhat as for sets (or for vector spaces), the situation for finite simple groups—indeed, for arbitrary groups—can be salvaged—in this case, by throwing away (dividing out by) fewer than all automorphisms, namely just the inner ones. (This does not amount to anything in the commutative case; the theory proposed is a purely noncommutative one.)

In other words, if one identifies two group homomorphisms if they differ by an inner automorphism—on either the domain side or the codomain side or both—then, not only does one obtain an equivalence relation, but also, it is compatible with composition: the product of two equivalence classes is again an equivalence class. (This is because the composition of a morphism with an inner automorphism on the domain side is equal to the composition of the same morphism with another inner automorphism on the codomain side.)

Clearly, the resulting functor is a classification functor—and furthermore, a similar construction works in other settings, for instance in the category of rings (where by an inner automorphism of a ring is meant one determined by an invertible element of the ring obtained by adjoining a unit).

Let us formalize this construction.

**Theorem.** Let $C$ be a category with a notion of inner automorphism, satisfying the axiom that the composition of an arbitrary morphism with (what we shall call) an inner automorphism, on the domain side, is equal to the composition of the same morphism with another inner automorphism on the codomain side (just as recalled above for groups and for rings). Note that, given sets of inner automorphisms, in this sense, simultaneously for all objects in the category, the subgroups generated by these sets of automorphisms also satisfy the axiom, and in particular are normal subgroups. Overall, these subgroups (which we shall refer to as the inner automorphism groups) form what we might refer to as a compatible family of normal subgroups of the automorphism groups.

It follows that the category $C^{out}$, the objects of which are the same as those of $C$,
and the morphisms of which are those of \( C \) considered modulo inner automorphisms, is a classifying category for \( C \).

**Proof.** The main point is that (cf. above) \( C^{\text{out}} \) is a category. It is immediate that, if a map in \( C \) is invertible in \( C^{\text{out}} \) then it is invertible. (And so the canonical functor from \( C \) to \( C^{\text{out}} \) is a classifying functor.)

2. A particular case of Theorem 1 is the category of (non-zero) finite direct sums of matrix algebras over the complex numbers, considered as \(*\)-algebras—i.e., with \(*\)-homomorphisms as maps—with the operation of taking the adjoint, or conjugate transpose, of a matrix as the \(*\)-operation. In this case (and for that matter in the category of all \( C^*\)-algebras, of which this is a subcategory), one has the compatible family of normal subgroups of the automorphism groups consisting of the inner automorphisms, i.e., the automorphisms determined by unitary elements of the \(*\)-algebra obtained by adjoining a unit.

As pointed out by Bratteli in [1], in this case the classifying category constructed in Theorem 1 has a very simple description, combinatorial in nature. The objects (non-zero finite direct sums of matrix algebras) may be viewed as (i.e., labelled precisely by) the finite column vectors, of arbitrary (non-zero) length, the coordinates of which are strictly positive integers. The morphisms between two objects, or vectors, may then be viewed as the rectangular matrices with positive (not necessarily strictly positive) integers as entries, multiplying the first vector into either the second vector (if the map is unital), or a vector with smaller coordinates. (Here the multiplication by the matrix is understood to be on the left, and the numbers of columns and rows of the rectangular matrix must therefore be equal respectively to the numbers of coordinates of the domain and codomain vectors.) In this description, according to the computation of Bratteli, composition of \(*\)-algebra morphisms—modulo inner automorphisms—corresponds to multiplication of rectangular matrices.

In slightly different words, if the set of column vectors and rectangular matrices described above is considered with its natural structure as a category (described above), then one obtains an exact replica (up to equivalence of categories) of the classifying category of Theorem 1, in the case of the category of \(*\)-algebras under consideration. What this comes down to is that if one considers two single full matrix algebras, then there is at most one unital morphism from the first to the second, up to unitary equivalence—and this exists exactly when the order of the second matrix algebra is an integral multiple of the order of the first one (sometimes called the multiplicity of the embedding). (Similarly, a non-unital morphism is also
determined up to unitary equivalence by its multiplicity—defined by cutting down by the image of the unit and so reducing to the unital case. The multiplicity can be any positive integer the product of which with the order of the first matrix algebra is less than or equal to the order of the second matrix algebra.)

The Bratteli matrix, in the case of a general pair of algebras in the category under consideration, just keeps track of the multiplicities of what might be called the partial maps, from the individual minimal direct summands of the domain algebra to those of the codomain algebra.

3. In fact, Bratteli was interested in a larger category, namely, in the category of all C*-algebra inductive limits of sequences in the category of C*-algebras considered above (non-zero finite direct sums of matrix algebras over the complex numbers). (Equivalently, Bratteli considered the category of C*-algebras obtained as the closure of an increasing sequence of sub-C*-algebras belonging to the category of Section 2.)

While Theorem 1 is applicable to this category also, in fact the classifying category arising in this way suffers from one of the defects described in Section 1: it is singular. (The group of inner automorphisms of a general C*-algebra in this category is not a closed subgroup of the group of all automorphisms in its natural topology, and so the quotient is not Hausdorff.) (This remark applies also to many other categories, for instance, infinite groups.)

What Bratteli did instead, circumventing this difficulty, was, given a sequence of C*-algebras in the category of Section 2, to look at the sequence in the classifying category of this category given by Theorem 1. In his picture, this was a diagram consisting of a whole sequence of column vectors, each one connected to the next by a rectangular matrix, as described in Section 2. This is now called a Bratteli diagram.

What Bratteli observed, to a certain extent implicitly, was that in a natural way the Bratteli diagrams form a category, and that if, for every C*-algebra in his category (which he called the approximately finite-dimensional, or AF, C*-algebras), one just chooses a Bratteli diagram (from a particular representation of this algebra as an inductive limit), then one obtains a classification functor. (In fact, Bratteli considered only isomorphisms, but his considerations can be extended in a natural way to embrace arbitrary morphisms.)

What I propose to do here is to take Theorem 1 seriously for the larger category, and indeed also for many other categories—e.g., all separable C*-algebras, and all countable groups.
It turns out that it is possible to desingularize Theorem 1.

**Theorem.** Let $C$ be a category with a notion of inner automorphism, i.e., a compatible family of normal subgroups of the automorphism groups as described in Theorem 1. Suppose that for each pair of objects the set of morphisms between these objects is endowed with a complete metric space structure, and that the following two compatibility properties with regard to composition of morphisms hold.

First, for any three objects, composition of morphisms from the first object to the second with morphisms from the second object to the third is a (jointly) continuous map into the space of morphisms from the first object to the third. (This property pertains to the topology, not the metric itself.)

Second, for any two objects, and for a fixed inner automorphism of the second object, composition with this (on the codomain side) is an isometry from the space of all morphisms from the first object to the second onto itself.

If follows from the first axiom (continuity) that the quotient structure $C^{\text{out}}$, the objects of which are the same as those of $C$ (and as those of the category $C^{\text{out}}$ of Theorem 1), and the morphisms of which are the closures of the equivalence classes of morphisms of $C$ modulo inner automorphisms (alternatively—as by continuity the closure of an equivalence class is a union of equivalence classes—the closures of the morphisms of the category $C^{\text{out}}$—in the quotient topology in which points are not necessarily closed), and for which the product of two morphisms is defined as the closure of the product of the corresponding two closed sets of morphisms of $C$ (alternatively, as the closure of the product of the corresponding closures of single morphisms in $C^{\text{out}}$)—by continuity this is the closure of a single equivalence class of morphisms of $C$ (i.e., a single morphism of $C^{\text{out}}$, namely, the product of two morphisms generating the two point closures in question), and therefore it is a morphism—is a category. Furthermore, the quotient map is a functor.

It follows from the second axiom (or, rather, the two axioms together) that the natural functor from the category $C$ to the category $C^{\text{out}}$ (i.e., the quotient map) is a classification functor. (In other words, it distinguishes isomorphism classes.) It is in fact what might be called a strong classification functor, in the sense that isomorphisms lift to $C$ from the classifying category $C^{\text{out}}$.

**Proof.** The main point is still that $C^{\text{out}}$ is a category. (The proof that the natural functor from $C$ to $C^{\text{out}}$ distinguishes isomorphism classes is, as will be seen, not new.)

It must be checked that composition of morphisms (as defined in the statement of the theorem) is associative. (Strictly speaking, this must also be checked for $C^{\text{out}}$, the product of the equivalence classes of three morphisms in $C$, in a fixed order, of
course, but grouped in either way, is just the equivalence class of the product of the three morphisms; this may be seen immediately by, roughly speaking, just moving all inner automorphisms through to the codomain side.)

Once it is noted that composition of morphisms in $\mathcal{C}^{\text{out}}$—i.e., equivalence classes of morphisms in $\mathcal{C}$—is associative, it follows immediately by continuity of multiplication that composition of morphisms in $\overline{\mathcal{C}^{\text{out}}}$—i.e., closures of equivalence classes in $\mathcal{C}$—is associative: just as one sees (by continuity) that the closure of the product of the closures of two equivalence classes is just the closure of the product of the equivalence classes themselves (and in particular is the closure of a single equivalence class), so also one sees that the two sets involved in the law of associativity for closures of equivalence classes—with multiplication of two such closures the closure of the product—equivalently, the closure of the product of the equivalence classes themselves—are equal (each one equal to the closure of the product of all three equivalence classes in question—of course, this uses associativity of the product of equivalence classes).

It is also immediate, starting from the continuity of multiplication in $\mathcal{C}$ and the functoriality of the quotient map from $\mathcal{C}$ to $\mathcal{C}^{\text{out}}$, that the quotient map from $\mathcal{C}$ to $\overline{\mathcal{C}^{\text{out}}}$ is a functor. Indeed, functoriality from $\mathcal{C}$ to $\mathcal{C}^{\text{out}}$ just says that the equivalence class of the product of two arrows in $\mathcal{C}$ is the product of the equivalence classes, and by continuity of multiplication this implies that the closure of the equivalence class of the product of two arrows is the closure of the product of the closures of the equivalence classes, which is the desired functoriality.

It is interesting to note that, so far, besides continuity of composition of morphisms (i.e., arrows) joining a fixed triple of objects (from the first to the second and the second to the third)—and of course associativity of this composition—the only thing that has been used, to obtain that $\mathcal{C}^{\text{out}}$ and (hence) $\overline{\mathcal{C}^{\text{out}}}$ are categories and that the natural maps $\mathcal{C} \to \mathcal{C}^{\text{out}}$ and (hence) $\mathcal{C} \to \overline{\mathcal{C}^{\text{out}}}$ are functors, is that the product of two equivalence classes is again an equivalence class. Whereas to prove that $\mathcal{C} \to \mathcal{C}^{\text{out}}$ is a classification functor, also nothing more is needed, to prove that $\mathcal{C} \to \overline{\mathcal{C}^{\text{out}}}$ is a classification functor seems to require the full force of the hypotheses, i.e., that the equivalence classes derive from neglecting the so-called inner automorphisms (which of course in particular means that the product of two equivalence classes is again an equivalence class), and that the topology on the set of morphisms from each fixed object to another one derives from a metric, which is assumed to be both complete and invariant under composition with a fixed inner automorphism of the codomain object.

It may also be of some interest to note that the stronger hypotheses, crucial for
the second statement of the theorem, also have two incidental consequences which one might think related to the first statement, that one has a quotient category and a functor to it, but do not appear to be so related: First, the quotient map $C \to C^{\text{out}}$ is open (when each set of morphisms between a pair of objects in $C^{\text{out}}$ is given the quotient topology), and indeed the quotient map $C \to \overline{C^{\text{out}}}$ is open (again with respect to the quotient topology—equivalently, the quotient map $C^{\text{out}} \to \overline{C^{\text{out}}}$ is also open—in fact the saturated open sets of morphisms in $C$ with respect to the map $C \to C^{\text{out}}$ are already saturated with respect to the map $C \to \overline{C^{\text{out}}}$! Second, the closures of distinct equivalence classes of morphisms in $C$ are either equal, or disjoint. (Of course, even without the stronger hypotheses, the most that can happen if the closures of two equivalence classes are neither equal nor disjoint is that one is contained in the other, properly, but this can presumably happen. Interestingly, this does not create any difficulty in the definition of the category structure of $C^{\text{out}}$.)

The fact that the functor in question, that to each morphism of $C$ associates the morphism in $C^{\text{out}}$ consisting of the closure of the equivalence class of this morphism in $C$ modulo inner automorphisms (these automorphisms defined axiomatically in the statement of the theorem), is a classification functor (i.e., distinguishes isomorphism classes), depends on a sequential approximate intertwining argument which was developed first in [1] and [8] in the case of exact intertwinings, and in [9] in the case of approximate intertwinings (i.e., approximately commutative diagrams intertwining two sequences). This technique has been used many times since, indeed, on virtually every occasion that an isomorphism theorem for $C^*$-algebras has been established. (It might almost be omitted, so many times has it been used! The basic ingredients were already described abstractly in [9]—see Theorems 2.1 and 2.2 of the that article. Surveys of the $C^*$-algebra isomorphism results are given in [12] and in [29].) The main purpose of the present article is to suggest that it might be of serious interest beyond the setting of $C^*$-algebras. (For instance, for von Neumann algebras!) (And also, for countable (non-abelian) groups.) (Some observations in this direction are reported below, in Sections 4 and 5.)

Let $a$ and $b$ be objects in $C$, and suppose that they are isomorphic in $C^{\text{out}}$. Let $f$ be an isomorphism between $a$ and $b$ in the category $C^{\text{out}}$, and let us show that $f$ is the image of an isomorphism in $C$. (In order to prove that the functor $C \to C^{\text{out}}$ is a classification functor, it appears to be expedient to prove that it is a strong classification functor. Indeed, only in [28] (and later, in a similar way, in [15]) has a classification functor been obtained without showing that it is a strong classification functor. Recently, in [7], the functor shown to be a classification functor in [28] was
shown in fact to be a strong classification functor. In this connection, the proof of Theorem 1 shows that the functor $\mathcal{C} \to \mathcal{C}^\text{out}$ is not only a strong classification functor, in the sense that every isomorphism in the codomain (classifying) category is the image of an isomorphism in the domain category, but is what might be called super-strong, in the sense that any morphism in the domain category mapping into an isomorphism in the classifying category must in fact already be an isomorphism!

Choose a morphism $f_1$ in $\mathcal{C}$ mapping to $f$ by the functor, and a morphism $g_1$ in $\mathcal{C}$ mapping to the inverse of $f$. Consider the (non-commutative!) diagram

$$
\begin{array}{c}
a \to a \\
\downarrow \searrow \downarrow \searrow \\
b \to b
\end{array}
$$

in which all the horizontal arrows are the identity map, respectively for $a$ and for $b$, and the downwards and upwards arrows are $f_1$ and $g_1$ respectively (each repeated infinitely often). Let us modify this diagram, by changing each of the downwards and upwards arrows by inner automorphisms, to make it approximately commutative—in the natural sense described in a special case in Section 2 of [9] and, in the present abstract setting, implicitly below. For convenience of notation, let us relabel the downwards maps, $f_1, f_1, \cdots$, as $f_1, f_2, \cdots$, anticipating that they will be changed. Similarly, let us relabel the upwards maps $g_1, g_1, \cdots$ as $g_1, g_2, \cdots$ (of course, to begin with, all the same).

Consider first the upper left hand triangle in the diagram. It provides two routes from $a$ to $a$, which we might refer to as “across” and “down-up”, which agree exactly in $\mathcal{C}^\text{out}$, by hypothesis, and in other words are approximately equal in $\mathcal{C}$ modulo inner automorphisms (to within an arbitrarily close degree of approximation). Since multiplying by an inner automorphism (on the codomain side) preserves distances, in the space of morphisms from $a$ to $a$, it is possible to multiply just one of the morphisms by an inner automorphism to get one arbitrarily close to the other one. Doing this to the map “down-up”, i.e., to $f_1g_1$ (we are using category theory notation for composition of arrows, with the first on the left), we obtain that $f_1g_1h$ is at distance at most $2^{-1}$ from “across”, i.e. from $\text{id}_a$, for some inner automorphism $h$ of $a$. Replacing $g_1$ by $g_1h$, but keeping the same notation, we then have

$$
d(f_1g_1, \text{id}_a) \leq 2^{-1},$$

where $d$ denotes the invariant metric.
Similarly, considering the second triangle (the lower left hand one), with (the new) \( g_1 \) as the map “up”, and \( f_2 \) as the map “down”, noting that the two routes “across” and “up-down” are (still) exactly equal in \( \overline{C_{\text{out}}} \), and therefore approximately equal in \( C \) modulo inner automorphisms, and using that the metric on the space of morphisms from \( b \) to \( b \) in \( C \) is invariant under multiplying (on the codomain side) by inner automorphisms, we obtain that \( g_1 f_2 k \), the map “up-down”, is within distance \( \epsilon_2 \) of \( \text{id}_b \), the map “across”, for some inner automorphism \( k \) of \( b \), where \( \epsilon_2 \) is to be specified. Replacing \( f_2 \) by \( f_2 k \) (but keeping the same notation), we now have “up-down” close to “across” in the second triangle:

\[
\begin{align*}
d(g_1 f_2, \text{id}_b) &\leq \epsilon_2.
\end{align*}
\]

Continuing in this way, changing the right hand non-horizontal arrow in each triangle in turn (but not the left hand one, which was the right hand one of the previous triangle—and therefore not interfering with the approximate commutativity of any previous triangle), we arrive at a new choice of the non-horizontal arrows in the diagram, \( f_1, f_2, \ldots, \) and \( g_1, g_2, \ldots, \) agreeing with the original choice up to inner automorphisms, and such that, for each \( n = 1, 2, \ldots \),

\[
\begin{align*}
d(f_n g_n, \text{id}_a) &\leq \epsilon_{2n-1} \\
d(g_n f_{n+1}, \text{id}_b) &\leq \epsilon_{2n},
\end{align*}
\]

where \( \epsilon_1 = 2^{-1} \), and \( \epsilon_{2n-1} \) and \( \epsilon_{2n} \) are to be specified.

We wish to choose \( \epsilon_2, \epsilon_3, \ldots \) in such a way that the sequences of morphisms \( (f_n) \) and \( (g_n) \) converge, say to \( f_\infty \) and \( g_\infty \), from \( a \) to \( b \) and from \( b \) to \( a \), respectively. Of course, this will probably require that \( \epsilon_k \) tend to zero, but if we actually ensure that this holds, then, in addition, necessarily \( f_n g_n \) converges to the identity for \( a \), \( \text{id}_a \), and \( g_n f_{n+1} \) converges to \( \text{id}_b \). By continuity of multiplication, this implies that

\[
\begin{align*}
f_\infty g_\infty &= \text{id}_a, \quad g_\infty f_\infty = \text{id}_b.
\end{align*}
\]

Since \( f_\infty \) and \( g_\infty \) still map into \( f \) and \( g \) in \( \overline{C_{\text{out}}} \) (being limits of elements in the equivalence classes of \( f_1 \) and \( g_1 \) respectively in \( C \)), this shows that the given isomorphism \( f \) from \( a \) to \( b \) in \( \overline{C_{\text{out}}} \) lifts to an isomorphism in \( C \), as desired.

Consider the following choice of the sequence \( \epsilon_1, \epsilon_2, \ldots \). Keep \( \epsilon_1 = 2^{-1} \); we are embarking on what is famously known as “a \( 2^{-n} \) argument”. (“The construction
of a summably Cauchy sequence" might be a clearer description!) Choose \( \epsilon_2 \), using continuity of multiplication, small enough that

\[
d(f_1(g_1 f_2), f_1) \leq 2^{-2}.
\]

(Recall that \( d(g_1 f_2, \text{id}_b) \leq \epsilon_2 \), and that \( f_1 \) is fixed; in fact also \( g_1 \) is fixed, and only \( f_2 \) has to be chosen suitably to ensure that \( \epsilon_2 \) is small—and this is crucial in obtaining the original sequence of estimates, one for each \( \epsilon_n \)—but this is now no longer needed.) Similarly, choose \( \epsilon_3 \) small enough that

\[
d(g_1(f_2 g_2), g_1) \leq 2^{-3}.
\]

(Recall that \( d(f_2 g_2, \text{id}_a) \leq \epsilon_3 \), and that, before we came to consider the final choice of \( f_2 \) and \( g_2 \), we had already fixed on a choice of \( g_1 \); in fact, the inequality \( d(f_2 g_2, \text{id}_a) \leq \epsilon_3 \) was negotiated without changing an earlier—final!—choice of \( f_2 \), and to ensure that \( \epsilon_3 \) is small it is necessary only to modify \( g_2 \)—and, again, this is crucial in obtaining the estimates involving \( \epsilon_1, \epsilon_2, \cdots \) in the first place—which depended on the choice at each stage not affecting earlier estimates—but it is not needed now.)

Continuing step by step in this way, we obtain a sequence \((\epsilon_1, \epsilon_2, \cdots)\) such that for each \( n = 1, 2, \cdots \),

\[
d(f_n(g_n f_{n+1}), f_n) \leq 2^{-(2n-1)},
\]

and

\[
d(g_n(f_{n+1} g_{n+1}), g_n) \leq 2^{-2n}.
\]

Next, revisiting the choice of the sequence \((\epsilon_1, \epsilon_2, \cdots)\), let us revise it slightly, making each \( \epsilon_k \) in turn smaller, sufficiently small that, from \( d(f_n g_n, \text{id}_a) \leq \epsilon_{2n-1} \) and \( d(g_n f_{n+1}, \text{id}_b) \leq \epsilon_{2n} \), it follows by continuity that, for each \( n = 1, 2, \cdots \),

\[
d((f_n g_n) f_{n+1}, f_{n+1}) \leq 2^{-(2n-1)},
\]

and

\[
d((g_n f_{n+1}) g_{n+1}, g_{n+1}) \leq 2^{-2n}.
\]

This requires some comment, since, after the choice of \( \epsilon_{2n-1} \), the morphism \( f_{n+1} \) will be changed, in order to ensure that \( d(g_n f_{n+1}, \text{id}_b) \leq \epsilon_{2n} \). But this (single) change consists only in multiplying by an inner automorphism, which will not affect the first inequality above. Similarly, the modification of \( g_{n+1} \) by an inner automorphism
after the choice of $\epsilon_{2n}$, in order to ensure that $d(f_{n+1}g_{n+1}, \text{id}_a) \leq \epsilon_{2n+1}$, does not affect the second inequality above.

Matching the appropriate pairs of inequalities, we obtain

$$d(f_{n+1}, f_n) \leq 2(2^{-(2n-1)}) = 2^{-2n+2},$$

and

$$d(g_{n+1}, g_n) \leq 2(2^{-2n}) = 2^{-2n+1}.$$  

In particular, the sequences $(f_n)$ and $(g_n)$ are (summably!) Cauchy, as desired. As shown above the limits are necessarily the inverses of each other, and give rise to the map $f$ and its inverse in $\mathcal{C}^{\text{out}}$.

4. Examples

4.1. Countable groups. Consider the category of countable (discrete) groups, and for each object consider the normal subgroup of the automorphism group consisting of the inner automorphisms in the usual sense. For each object $G$ choose a numbering of its elements, i.e., a bijection $G \ni g \mapsto n(g)$ of $G$ onto either $\mathbb{N}$ or an initial segment of $\mathbb{N}$, and note that for each object $H$ the formula (involving the Kronecker delta)

$$d(\varphi, \psi) = \sum_{g \in G} 2^{-n(g)} \delta_{\varphi(g), \psi(g)}$$

defines a metric on the set $\text{Hom}(G, H)$ of group homomorphisms from $G$ to $H$.

(For each $g \in G$ the quantity $d_g(\varphi, \psi) = \delta_{\varphi(g), \psi(g)}$ is already a pseudo-metric on $\text{Hom}(G, H)$, i.e., satisfies the triangle inequality.)

The resulting family of normal subgroups is a compatible one in the sense of Theorem 3 (and Theorem 1), and the resulting family of metrics on the sets of morphisms between pairs of objects satisfies the two axioms of Theorem 3. (The underlying topology is just the topology of pointwise convergence in the discrete topology; composition of morphisms, between two fixed pairs of objects, is easily seen to be continuous in this topology. If $\varphi, \psi \in \text{Hom}(G, H)$ and $\rho$ is an automorphism of $H$—not necessarily inner—then $\delta_{\rho \circ \varphi(g), \rho \circ \psi(g)} = \delta_{\varphi(g), \psi(g)}$ and so

$$d(\rho \circ \varphi, \rho \circ \psi) = \sum_{g \in G} 2^{-n(g)} \delta_{\rho \circ \varphi(g), \rho \circ \psi(g)}$$

$$= \sum_{g \in G} 2^{-n(g)} \delta_{\varphi(g), \psi(g)}$$

$$= d(\varphi, \psi);$$
in other words, composition with \( \rho \) is an isometry.)

4.2. Countably generated algebras. Consider the category of countably generated (not necessarily unital) algebras over a fixed field, and for each object consider the normal subgroup of the automorphism groups consisting of the inner automorphisms in the usual sense (i.e., those automorphisms the unique extension of which to the algebra with unit adjoined is determined by an invertible element).

For each object \( A \) choose a generating sequence \((a_n)_{n \in \mathbb{N}}\), and note that for each object \( B \) the formula

\[
d(\phi, \psi) = \sum_{n \in \mathbb{N}} 2^{-n} \delta_{\phi(a_n), \psi(a_n)}
\]

defines a metric on the set \( \text{Hom}(A, B) \) of algebra homomorphisms from \( A \) to \( B \). (For each \( a \in A \), the quantity \( d_a(\phi, \psi) = \delta_{\phi(a), \psi(a)} \) is already a pseudo-metric on \( \text{Hom}(A, B) \), i.e., satisfies the triangle inequality.)

The resulting family of normal subgroups is a compatible one in the sense of Theorem 3, and the resulting family of metrics on the sets of morphisms between pairs of objects satisfies the two axioms of Theorem 3. (As before, the underlying topology is pointwise convergence (on the domain object) in the discrete topology (of the codomain object), and it follows immediately that composition is continuous. The isometry property holds since, as before, \( \delta_{\rho \circ \psi(a), \rho \circ \psi(a)} = \delta_{\phi(a), \psi(a)} \) for any \( \phi, \psi \in \text{Hom}(A, B) \), any automorphism \( \rho \)—inner or not—of \( B \), and any \( a \in A \).)

4.3. Separable C*-algebras. Consider the category of separable C*-algebras (not necessarily unital), and for each object consider the normal subgroup of the automorphism group consisting of the inner automorphisms in the usual sense (i.e., those automorphisms determined by a unitary element of the C*-algebra obtained by adjunction of a unit). For each object \( A \) choose a generating sequence \((a_n)_{n \in \mathbb{N}}\) of elements of norm at most one, and note that for each object \( B \) the formula

\[
d(\phi, \psi) = \sum_{n \in \mathbb{N}} 2^{-n} \|\phi(a_n) - \psi(a_n)\|
\]

defines a metric on the set \( \text{Hom}(A, B) \) of morphisms from \( A \) to \( B \) (in the usual sense of *-homomorphisms—recall that *-homomorphisms between C*-algebras are norm-decreasing, and so the sum is finite).

The resulting family of normal subgroups is a compatible one in the sense of Theorem 3, and the resulting family of metrics on the sets of morphisms between pairs of objects satisfies the two axioms of Theorem 3. (The underlying topology is again the topology of pointwise convergence, now with respect to the norm topology.
on the codomain object; composition of morphisms is again continuous, as follows from the facts that pointwise convergence implies uniform convergence on compact subsets. If \( \varphi, \psi \in \text{Hom}(A, B) \) and \( \rho \) is an automorphism of \( B \)—inner or not—then \( \| \rho \circ \varphi(a) - \rho \circ \psi(a) \| = \| \varphi(a) - \psi(a) \| \) for each \( a \in A \) (and in particular for \( a_n \) for each \( n \)) and as before it follows that composition with \( \rho \) is an isometry.

(A natural generalization of this example is the category of separable C*-algebras together with actions of a fixed locally compact group—a single object consisting of a C*-algebra together with an action of the group—with as morphisms those C*-algebra morphisms which are compatible with the actions, and with as inner automorphisms those which are inner in the sense defined above—as C*-algebra automorphisms. As metric on the space of morphisms between two objects, one may just take the metric relative to a choice of a dense sequence in the unit ball of each separable C*-algebra as defined above. It would be interesting to consider the even more general setting in which the morphisms are only required to be compatible with the group actions up to a cocycle—isomorphism of two actions in this category would then be what is known as cocycle conjugacy; work of Evans and Kishimoto and also recent work of Katsura and Matsui involves an intertwining argument in this setting.)

### 4.4. Countably generated Hilbert C*-modules with embeddings.

Consider the category of countably generated right Hilbert C*-modules over a fixed C*-algebra \( A \) (see e.g. [20]), with \( A \)-valued inner product preserving \( A \)-module maps (not necessary adjointable) as morphisms, and for each object consider the normal subgroup of the automorphism group consisting of what might be called the inner automorphisms (i.e., those automorphisms arising from unitary elements of the C*-algebra of compact endomorphisms with unit adjointed). For each object \( X \) choose a generating sequence \( (\xi_n)_{n \in \mathbb{N}} \) of elements of norm at most one, and note that for each object \( Y \) the formula

\[
d(\varphi, \psi) = \sum_{n \in \mathbb{N}} 2^{-n} \| \varphi(\xi_n) - \psi(\xi_n) \|
\]

defines a metric on the set \( \text{Hom}(X, Y) \) of morphisms from \( X \) to \( Y \) (as defined above—note in particular that morphisms are isometric and so the sum is finite).

The resulting family of normal subgroups is a compatible one in the sense of Theorem 3, and the resulting family of metrics on the sets of morphisms between pairs of objects satisfies the two axioms of Theorem 3. (The compatibility follows from the fact that compact endomorphisms of a closed submodule of a Hilbert C*-module extend canonically to compact endomorphisms of the whole module. As
in the preceding example, the topology underlying the metric is that of pointwise convergence in norm, and continuity of composition follows in the same way. The isometry property is also seen in the same way: if $\varphi, \psi \in \text{Hom}(X,Y)$ are morphisms (embeddings) as above, and $\rho$ is an automorphism of $Y$—inner or not—, then $\|\rho \circ \varphi(\xi) - \rho \circ \psi(\xi)\| = \|\varphi(\xi) - \psi(\xi)\|$ for each $\xi \in X$ (and in particular for $\xi_n$ for each $n$), and it follows that composition with $\rho$ is an isometry.

4.5. von Neumann algebras with separable predual. Consider the category of von Neumann algebras with separable predual, with a specified choice of faithful normal state for each von Neumann algebra (this might be called the category of pointed von Neumann algebras), and with morphisms the (unital) normal *-homomorphisms from one von Neumann algebra to another, taking the centre of the first into (a subalgebra of) the centre of the second (automatic in the factor case), which are compatible with the chosen pair of states in the strongest sense—i.e., also intertwining the modular automorphism groups. (In particular just simple intertwining of the states implies that the homomorphism is injective.)

For each object, consider the normal subgroup of the automorphism group—i.e., the group of *-automorphisms of the von Neumann algebra commuting with the modular automorphism group of the specified faithful normal state—consisting of those automorphisms which are inner in the usual sense—they are then determined by unitaries fixed by the modular automorphisms up to central multiples. For each object $M$—let us not mention the given faithful normal state explicitly, but denote the corresponding pre-Hilbert space norm by $\| \cdot \|_2$—choose a generating sequence $(x_n)_{n \in \mathbb{N}}$ of elements of (operator) norm at most one, and note that for each object $N$ the formula

$$d(\varphi, \psi) = \sum_{n \in \mathbb{N}} 2^{-n} \|\varphi(x_n) - \psi(x_n)\|_2$$

defines a metric on the set $\text{Hom}(M, N)$ of morphisms from $M$ to $N$ (recall that by assumption morphisms preserve the canonical norm $\| \cdot \|_2$ and hence, since $\|x_n\|_2 \leq \|x_n\| \leq 1$, the sum is finite).

The resulting family of normal subgroups is a compatible one in the sense of Theorem 3, and the resulting family of metrics on the sets of morphisms between pairs of objects satisfies the two axioms of Theorem 3. (The underlying topology is again the topology of pointwise convergence, now with respect to the strong operator topology on the codomain object—which, on sets bounded in the operator norm, is what the norm $\| \cdot \|_2$ gives rise to. However, this description is not sufficient to prove continuity of multiplication, as continuity of composition of arbitrary *-
homomorphisms in this topology presumably does not hold; it is necessary to use the invariance of the norm $\| \cdot \|_2$ under morphisms as at present defined. In this setting the proof is very much the same as before—to prove continuity of multiplication in the topology of pointwise convergence with respect to the norm $\| \cdot \|_2$, it is enough to note that the topology is sequentially determined, and that pointwise convergence of morphisms, with respect to the norm $\| \cdot \|_2$, implies uniform convergence on each subset totally bounded in the norm $\| \cdot \|_2$ (and in particular on each convergent sequence).)

The subcategory of finite factors, with the specified state the trace, is particularly simple as the morphisms are arbitrary (unital) normal *-homomorphisms, and the specified normal subgroup is the group of all inner automorphisms.

5. Concrete description of the abstract classifying category

In certain cases the abstract classifying category of Theorem 3 has a relatively simple concrete form—and sometimes the classification functor when translated into these terms can be recognized.

5.1. A good example from this point of view is the class of countable groups obtained as inductive limits of sequences of finite products of alternating groups—on five or more symbols so that they are simple, and so the maps between building blocks (the finite products) are determined up to inner automorphisms—or rather, up to automorphisms determined by permutations, not necessarily even, on each simple component—just by the multiplicities of the partial maps from the various components of the domain building block to the various components of the codomain building block. (Note that, as is easily seen by induction on the number of symbols of the domain group, any homomorphism from an alternating group on five or more symbols into a larger alternating group is determined up to a permutation, not necessarily even, by an integer greater than or equal to zero which might be called the multiplicity.)

In order to ensure that the automorphism relating two maps with the same multiplicities (for all partial maps) may be chosen to be inner, i.e., to arise from an even permutation on each alternating group component of the codomain group, we must restrict to the class of maps such that the image of the domain finite product group in each simple component group of the codomain finite product is acted on trivially by some automorphism arising from an odd permutation. Two different maps of this kind, with the same domain and codomain and the same multiplicities, which can in any case differ at most by some automorphism leaving each component of the codomain invariant and so arising there from a permutation, must then in
fact differ by an even permutation in each component—i.e., must differ by an inner automorphism. This happens for instance if one of the components of the domain group is on an odd number of symbols and its multiplicity is at least two, or also if there are at least two symbols for the codomain component group which are fixed by the image of the domain group.

There is also a way to apply Theorem 3 to inductive limits of the finite product groups under consideration without restricting to special maps in the inductive limit construction. If we enlarge the specified normal subgroups of the automorphism groups in the statement of Theorem 3—somewhat misleadingly (in the present case) referred to as the inner automorphism groups!—to include certain non-inner automorphisms arising naturally in the present case, namely, the product automorphisms considered above in the case of a finite product of alternating groups (on five or more symbols), arising from a permutation on each component, and the natural extensions of these to the inductive limit group (these automorphisms can be characterized without reference to a particular inductive limit decomposition), then the hypotheses of Theorem 3 are still satisfied. In particular, the composition of an arbitrary homomorphism with such an automorphism on the domain side is equal to the composition of this same homomorphism with another such automorphism on the codomain side—the basic algebraic property of inner automorphisms still obtains.

In short, in either setting (restricted sequences and inner automorphisms, or arbitrary sequences and generalized inner automorphisms as described above), the maps between the finite product building blocks modulo the special automorphisms considered are always determined by the multiplicities of the partial maps.

In other words the classification category for the category of building blocks, i.e., the finite products of alternating groups on five or more symbols (slightly restricted, in the case that Theorem 3 is applied with actual inner automorphisms) is exactly the same as that described in Section 2 for the category of finite direct sums of matrix algebras over the complex numbers (restricted to those of order five or more for the present comparison). (Multiplicity zero for a map between full matrix algebras means it is the zero *-algebra map, while for a map between alternating groups it means it is the trivial group map, and the analogy is also close for higher multiplicities.)

It follows that the classifying category for (sequential) inductive limits of the building block groups under consideration (finite products of alternating groups, on five or more symbols) is the same as Bratteli’s classifying category for AF algebras, described briefly at the beginning of Section 3 (Bratteli diagrams). (More
precisely, we must restrict consideration to AF algebras constructed using only matrix algebras of order five or more, but up to stable isomorphism this is everything.) (Incidentally, it might be interesting to consider whether there is an analogue for groups, at least for those in the present class, of stable isomorphism for C*-algebras.)

It is interesting to note that this category is equivalent to the category of (countable) dimension groups, i.e., (countable) unperforated order groups with the Riesz decomposition property—with a specified upward directed downward hereditary generating subset of the positive cone—sometimes called a scale. (See [8], [5] and [13].)

Another point worthy of note is that, with this identification of the classifying category (common for AF algebras and the category of groups described above), whereas the classification functor in the C*-algebra setting is a familiar one—namely, $K_0$—the corresponding functor in the group setting—mapping into the same class of ordered groups—would not seem hitherto to have been considered. (And can it even be defined directly?)

(Added November 15, 2007: After this paper was submitted for publication the author discovered the article by Y. Lavrenyuk and N. Nekrashevych, On classification of inductive limits of direct products of alternating groups, J. London Math. Soc. 75 (2007), 146–162, which shows that the groups considered above are classified up to isomorphism by their (equivalence classes of) Bratteli diagrams. To be more precise, these authors consider only the groups arising from a slightly restricted class of sequences, which would seem to be restricted in a somewhat different way from the first—restricted—class considered above. Note that the second class of sequences considered above is not restricted. A second point of difference, also minor, is that, instead of observing that mappings between alternating groups on five or more symbols are automatically diagonal, with a certain multiplicity, up to conjugacy by an inner automorphism, these authors consider only diagonal maps.)

5.2. As pointed out in [8], the classification of AF algebras—the C*-algebra inductive limits of finite-dimensional C*-algebras (i.e., finite C*-algebra direct sums of full matrix algebras over the complex numbers)—is in a certain sense equivalent to that of the corresponding *-algebra inductive limits, or even just of the corresponding algebra inductive limits. This sense can be extended to the present context as follows: while these three categories are presumably not equivalent, their classifying categories given by Theorem 3 are all equivalent—and are equivalent to
the category just described (which is sometimes referred to as the category of scaled dimension groups).

5.3. A relatively simple concrete identification of the classifying category of Theorem 3, for other subcategories of the category of separable C*-algebras considered in Section 4.3 than the category of AF algebras just discussed, would of course be interesting. To a considerable extent, this is in fact how the program of classifying (various classes of) amenable C*-algebras has proceeded so far. (One would perhaps like to consider the class of all amenable C*-algebras at once—but not only did one case take longer in the analogous setting of amenable von Neumann algebras, some amenable C*-algebras will definitely be more difficult than others—see [33] and [32].)

For instance, if one considers simple inductive limits of matrix algebras over $\mathbb{C}([0, 1])$, say assumed to be unital, then it follows from [10] that the classification category of Theorem 3 consists of the category of order-unit ordered groups arising in the (simple, unital) AF case, with the modification that each one should be paired at the same time with a Choquet simplex (arising as the simplex of tracial states of the C*-algebra), with the maps respecting this pairing.

In [16], the same invariant—$K_0$ plus traces—, augmented by $K_1$, was shown to be complete when the interval $[0, 1]$ is replaced by an arbitrary (variable) compact metric space of dimension at most three (or, in fact, any fixed number, but with no new examples appearing). However, as was shown already in [23], maps between C*-algebras are not determined by these invariants up to approximate unitary equivalence—even in the case of the circle one needs to consider, instead of the Banach algebra $K_1$-groups, the (Hausdorffized) algebraic $K_1$-group (see [23] and [11]). In the case of arbitrary compact metric spaces of dimension at most three, one must also consider the K-groups with coefficients introduced in [3] (and considered in the non-simple real rank zero case in [6] and [2]). In fact, these invariants suffice, as can be seen by study of [16] (and can be seen immediately from Theorem 8.6 and Lemma 6.9 of [22]!). (The same result also holds in the more general case considered recently by Niu in [25] (see also [17]), as follows from Theorem 6.2.3 of [25] together with Lemma 6.9 of [22].)

In other words, for the class of C*-algebras classified in [16] (or, more generally, in [25] and [17]), the abstract classification functor of Theorem 3 is equivalent (by means of an equivalence of categories) to the standard K-theoretical functor consisting of the invariants just listed. Furthermore, as shown in [34], the objects arising as the values of the functor, for the class of C*-algebras considered in [16],
can be described in simple terms (much as in the simpler cases reported above). Incidentally, if the algebraic $K_1$-group is taken to be based on invertible elements rather than unitaries, then the invariant consisting of the simplex of tracial states becomes redundant, as the larger $K_1$-group is just the direct sum of the one based on unitaries and the group of continuous affine real-valued functions on the simplex. (The affine function corresponding to a projection is seen in this picture as just the $K_1$-class of the exponential of the projection.)

There is another case in which the answer is simpler!—only the abstract $K_0$- and $K_1$-groups and $K$-groups with coefficients. This is the case of Kirchberg algebras (simple purely infinite separable amenable $C^*$-algebras) classified by Kirchberg and Phillips in [19] and [26]. The classification functor of Theorem 3 is characterized as the functor $KL$ of Rordam ([27]). (The range of this is still an abstract category, but in the case that the algebras satisfy the Universal Coefficient Theorem (possibly automatic)—see [30] and [4]—this is equivalent to the concrete category of $K$-groups with coefficients (including of course $K_0$ and $K_1$) referred to above.

5.4. Consider the category of countably generated Hilbert modules over a given $C^*$-algebra $A$, with embeddings, as in Section 4.4. (Recall that by Theorem 3.5 of [20], an $A$-module map between Hilbert $A$-modules is an embedding in the present sense—i.e., preserves the $A$-valued inner product—if and only if it is isometric.) In general, the structure of the classifying category of Theorem 3—in particular, the question when two morphisms are the same—would appear to be somewhat complicated. This can be seen already with the failure of cancellation for isomorphism classes of algebraically finitely generated projective modules.

Remarkably, in the case that $A$ has stable rank one (i.e., the ring $A$ with unit adjoined has Bass stable rank one), the structure of the classifying category becomes extremely simple: between any two objects, either there are no maps, or there is exactly one map. (In other words, any two morphisms between Hilbert $C^*$-modules are approximately equal modulo inner automorphisms; this is proved in the last paragraph of the proof of Theorem 3 of [14].) (This category is then very much like the (common) classifying category given by Theorem 1 for sets, vector spaces, or Hilbert spaces, with injective maps as morphisms (isometries in the third case), and with the whole automorphism group taken as the specified normal subgroup—namely, cardinal numbers with maps just the relations $a \leq b$.)

As a consequence (just as for sets!), a Cantor-Bernstein theorem holds for the category of countably generated Hilbert $A$-modules (with embeddings) in the case that $A$ has stable rank one. Indeed, if one has maps $a \to b$ and $b \to a$ between
objects $a$ and $b$, then these persist in the classifying category, and by uniqueness
the composed maps in the classifying category must be the identities for $a$ and $b$.
In other words, the objects $a$ and $b$ are isomorphic in the classifying category, and
hence by Theorem 3—without using any more that $A$ has stable rank one—they
are isomorphic as Hilbert $A$-modules.

5.5. Consider the category of pointed von Neumann algebras with separable
predual, as described in Section 4.5. A certain subcategory of this is of partic-
ular interest—namely, that for which the maps, in addition to intertwining the
specified states, and their modular automorphism groups—and taking the centre
into the centre (automatic in the factor case)—so that they pass to the Takesaki
two-parameter crossed products—take the centre into the centre at the level of the
crossed products. The subcategory comprising these maps admits a functor to the
Connes-Takesaki flow of weights—which is just the restriction to the centre of the
dual $\mathbb{R}$-action on the Takesaki crossed product. In the amenable case this functor
is, famously, a classification functor. (To be precise one must restrict to the case of
properly infinite and continuous von Neumann algebras—i.e., those of type $\text{II}_\infty$ or
III. For these algebras the flow determines the algebra, and furthermore, the flow
may be an arbitrary measurable flow—by which is meant an $\mathbb{R}$-action on an abelian
von Neumann algebra with separable pre-dual.)

In the general case, the flow of weights functor factors through the classification
functor of Section 4.5—as inner automorphisms belonging to the category of Section
4.5 also belong to the present subcategory, and their action on the flow of weights is
trivial—so that two morphisms in the present subcategory which are approximately
unitarily equivalent, with respect to inner automorphisms in the category under
consideration—i.e., which are equal in $\mathcal{C}^{\text{out}}$—give rise to the same morphism at the
level of the flows of weights.

Is this functor, from $\mathcal{C}^{\text{out}}$ to the category of flows, in fact an equivalence of cat-
egories in the amenable case? It would appear that this might follow from careful
inspection of [18], which shows that at least this is the case if one considers only
isomorphisms. (It remains to check that, also in the setting of homomorphisms—
between pointed von Neumann algebras—, two morphisms between two objects in
this category are approximately unitarily equivalent, with respect to inner auto-
morphisms in this category, if they agree at the level of flows of weights.)

It should be noted that a general question that arises in the setting of classifica-
tion functors, and in the setting of Theorem 3 in particular, is whether a functor
exists in the opposite direction, which is a one-sided inverse to the classification
functor. (One does not expect a two-sided inverse, and in the setting of Theorem 3, at least, it cannot exist, except in the trivial case that there are no inner automorphisms—as the classification functor kills all inner automorphisms.) In [31], remarkably, a one-sided inverse is shown to exist for the category of amenable pointed von Neumann algebras with separable predual—in the restricted setting of isomorphisms only—both at the level of the algebras and at the level of the invariant, which as observed above is the same whether it is considered as the abstract category of Theorem 3, or as the category of flows.

The following concrete consequence of Theorem 3 in the case of the category of Section 4.5 (so far, the only one!) is of modest interest: namely, restricting to finite factors (pointed with respect to the unique tracial state) one obtains immediately the Murray-von Neumann uniqueness theorem for the approximately finite-dimensional case—or at least the important special case that there is a locally finite-dimensional generating sub-*-algebra. (Any two maps from such a finite factor into an arbitrary finite factor are very easily seen to be approximately unitarily equivalent—as the calculation reduces immediately to the finite-dimensional case for the domain algebra—and, furthermore, at least one such map—constructed inductively—is easily seen to exist. One is therefore in the situation of a subcategory with a classifying category, between any two objects of which there is exactly one map! (Necessarily, of course, an isomorphism.))

References

1. O. Bratteli, *Inductive limits of finite dimensional C*-algebras*, Trans. Amer. Math. Soc. 171 (1972), 195–234.
2. M. Dadarlat and G. Gong, *A classification result for approximately homogeneous C*-algebras of real rank zero*, Geom. Funct. Anal. 7 (1997), 646–711.
3. M. Dadarlat and T. A. Loring, *Classifying C*-algebras via ordered mod-p K-theory*, Math. Ann. 305 (1996), 601–616.
4. M. Dadarlat and T. A. Loring, *A universal multicoefficient theorem for the Kasparov groups*, Duke Math. J. 24 (1996), 355–377.
5. E. G. Effros, D. E. Handelman, and C.-L. Shen, *Dimension groups and their affine representations*, Amer. J. Math 102 (1980), 385–407.
6. S. Eilers, *A complete invariant for AD algebras with real rank zero and bounded torsion in K_1*, J. Funct. Anal. 139 (1996), 325–348.
7. S. Eilers and G. Restorff, *On Rørdam’s classification of certain C*-algebras with one non-trivial ideal*, Operator Algebras: The Abel Symposium 2004
(editors, O. Bratteli, S. Neshveyev, and C. Skau), Abel Symposia, 1, Springer, Berlin, 2006, pages 87–96.

8. G. A. Elliott, *On the classification of inductive limits of sequences of semisimple finite-dimensional algebras*, J. Algebra **38** (1976), 29–44.

9. G. A. Elliott, *On the classification of C*-algebras of real rank zero*, J. Reine Angew. Math. **443** (1993), 179–219.

10. G. A. Elliott, *A classification of certain simple C*-algebras*, Quantum and Non-Commutative Analysis (editors, H. Araki et al.), Kluwer, Dordrecht, 1993, pages 373–385.

11. G. A. Elliott, *A classification of certain simple C*-algebras, II*, J. Ramanujan Math. Soc **12** (1997), 97–134.

12. G. A. Elliott, *The classification problem for amenable C*-algebras*, Proceedings of the International Congress of Mathematicians, Zürich, 1994 (editor, S. D. Chatterji), Birkhäuser, Basel, 1995, pages 922–932.

13. G. A. Elliott, *The inductive limit of a Bratteli diagram*, Lecture, MSRI, Berkeley, September 2000. (Transparencies on MSRI web site.)

14. G. A. Elliott, K. T. Coward, and C. Ivanescu, *The Cuntz semigroup as an invariant for C*-algebras*, preprint.

15. G. A. Elliott, D. E. Evans, and H. Su, *Classification of inductive limits of matrix algebras over the Toeplitz algebra*, Operator Algebras and Quantum Field Theory (editors, S. Doplicher et al.), Accademia Nazionale dei Lincei, Rome, 1998, pages 36–50.

16. G. A. Elliott, G. Gong, and L. Li, *On the classification of simple inductive limit C*-algebras II: The isomorphism theorem*, preprint, 1998 (to appear, Invent. Math.).

17. G. A. Elliott and Z. Niu, *On tracial approximation*, preprint.

18. Y. Kawahigashi, C. E. Sutherland, and M. Takesaki, *The structure of the automorphism group of an injective factor and the cocycle conjugacy of discrete abelian group actions*, Acta Math. **169** (1992), 105–130.

19. E. Kirchberg, *The classification of purely infinite C*-algebras using Kasprow’s theory*, preprint, 1994.

20. E. C. Lance, *Hilbert C*-modules. A tool kit for operator algebraists*. London Mathematical Society Lecture Note Series, **210**, Cambridge University Press, Cambridge, 1995.
21. H. Lin, *An introduction to the classification of amenable C*-algebras*, World Scientific Publishing Co., Inc., River Edge, NJ, 2001.

22. H. Lin, *Simple nuclear C*-algebras of tracial topological rank one*, preprint.

23. K. E. Nielsen and K. Thomsen, *Limits of circle algebras*, Expo. Math. **14** (1996), 17–56.

24. Z. Niu, *On the classification of TAI algebras*, C. R. Math. Acad. Sci. Soc. R. Can. **26** (2004), 18–24.

25. Z. Niu, *On the classification of certain tracially approximately subhomogeneous C*-algebras*, Ph.D. thesis, University of Toronto, 2005.

26. N. C. Phillips, *A classification theorem for nuclear purely infinite simple C*-algebras*, Documenta Math. (2000), 49–114.

27. M. Rørdam, *Classification of certain infinite simple C*-algebras*, J. Funct. Anal. **131** (1995), 415–458.

28. M. Rørdam, *Classification of extensions of certain C*-algebras by their six term exact sequences in K-theory*, Math. Ann. **308** (1997), 97–117.

29. M. Rørdam, *Classification of nuclear, simple C*-algebras*, pages 1–145 of Classification of nuclear C*-algebras. Entropy in operator algebras. Encyclopedia of the Mathematical Sciences, **126**, Springer, Berlin, 2002.

30. J. Rosenberg and C. Schochet, *The Künneth Theorem and the Universal Coefficient theorem for Kasparov’s generalized K-functor*, Duke Math. J. **55** (1987), 431–474.

31. C. Sutherland and M. Takesaki, *Right inverse of the module of approximately finite dimensional factors of type III and approximately finite ergodic principal measured groupoids*. Operator Algebras and Their Applications II (editors, P. A. Fillmore and J. A. Mingo), Fields Institute Communications, **20**, American Mathematical Society, Providence, RI, 1998, pages 149–159.

32. A. Toms, *On the classification problem for nuclear C*-algebras*, Ann. of Math., to appear.

33. J. Villadsen, *Simple C*-algebras with perforation*, J. Funct. Anal. **154** (1998), 110–116.

34. J. Villadsen, *The range of the Elliott invariant of the simple AH-algebras with slow dimension growth*, K-Theory **15** (1998), 1–12.

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