CMB Broad-Band Power Spectrum Estimation

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Abstract

The natural outcome of theoretical calculations of microwave background anisotropy is the angular power spectrum $C_\ell$ as a function of multipole number $\ell$. Experimental $C_\ell$’s are needed for direct comparison. Estimation procedures using statistics linear in the pixel amplitudes as well as the conventional but less useful quadratic combinations are described. For most current experiments, a single broad-band power amplitude is all that one can get with accuracy. Results are given for the Capri-meeting detections. Mapping experiments, sensitive to many base-lines, can also give spectral “colour” information, either with a series of contiguous narrow-band powers or as parameterized by a local “colour” index $n_{\Delta T}$ (scale invariant is -2, white noise is 0). Bayesian analyses of the full first year DMR and FIRS maps give very similar band-powers (e.g., $Q_{\text{rms,PS}} = 17.9 \pm 2.9 \mu K$ c.f. $18.6 \pm 4.7 \mu K$ for $n_{\Delta T} = -2$) and colour indices (with 1 and 2 sigma error bars) $n_{\Delta T} + 3 = 2.0^{+0.4}_{-0.4}$ and $1.8^{+0.6}_{-0.8}$ (c.f. the value 1.15 for a “standard” scale invariant CDM model). The 53 and 90 GHz DMR maps, as well as the FIRS map, have residual short-distance noise which steepens $n_{\Delta T}$. This residual has so far been modelled by allowing the pixel error bars to increase, absorbing much of the effect, but further exploration is needed to see if a second residual evident in the data — which is, in part, responsible for the high $n_{\Delta T}$ — is from systematic errors or is physical.
“Un-cleaned” data with inadequate “dirt” models do not yet give us confidence that what is primary anisotropy has been well-separated from secondary backgrounds, foregrounds and instrumental systematics in any experiment. But we will. In the meantime, we need a phenomenology to display experimental results that (1) allows us to be unafraid of presenting data with the inevitable residual contaminants, (2) gives a $\Delta T/T$ estimator which is not sensitive to the specific experimental configuration (thus not $\text{rms}$ anisotropies), (3) allows a meaningful comparison among experiments and (4) among theories. The angular $\Delta T/T$ power in broad bands that cover the multipole range explored by a given experiment satisfies these requirements. Whether the data allows one or many bands to be well-estimated depends upon the details of the experiment.

Gaussian theories are completely characterized by the power spectrum, $\mathcal{C}_{T\ell} \equiv \ell(\ell + 1)\langle |a_{\ell m}|^2 \rangle / (2\pi)$, where $\Delta T_0(q) = \sum_{\ell m} a_{\ell m} Y_{\ell m}(q)$ defines the $a_{\ell m}$. The signal $(\Delta T/T)_p$ from the $p$th pixel of a CMB anisotropy experiment can be expressed in terms of linear filters $F_{p,\ell m}$ acting on the $a_{\ell m}$: $(\Delta T/T)_p = \sum_{\ell m} F_{p,\ell m} a_{\ell m}$. The associated pixel-pixel correlation function can be expressed in terms of a quadratic $N_{\text{pix}} \times N_{\text{pix}}$ filter matrix $W_{pp'}$, while its trace defines the average filter $\overline{W}_\ell$ (e.g., Bond 1989):

$$C_{Tpp'} \equiv \langle (\Delta T/T)_p (\Delta T/T)_{p'} \rangle = \mathcal{I}[W_{pp'}, \mathcal{C}_{T\ell}] \quad W_{pp', \ell} \equiv \frac{4\pi}{2\ell + 1} \sum_{\ell m} F_{p,\ell m} F_{p',\ell m}^* ; \quad (1)$$

$$\left(\frac{\Delta T}{T}\right)_{\text{rms}}^2 \equiv \frac{1}{N_{\text{pix}}} \sum_{p=1}^{N_{\text{pix}}} C_{Tpp} \equiv \mathcal{I}[W, \mathcal{C}_{T\ell}] \quad \mathcal{W}_\ell \equiv \frac{1}{N_{\text{pix}}} \sum_{p=1}^{N_{\text{pix}}} W_{pp, \ell} ; \quad (2)$$

$$\mathcal{I}(f) \equiv \sum_{\ell} f_\ell \ell (\ell + 1) . \quad (3)$$

The “discrete-logarithmic integral” $\mathcal{I}$ of a function function $f_\ell$ is defined by eq. 3. Although there are as many as $N_{\text{pix}}(N_{\text{pix}} + 1)/2$ different filters to probe $\mathcal{C}_{T\ell}$, in practice symmetries reduce this number to at most $\sim N_{\text{pix}}$ and often only a few are large enough to be effective probes.

1. Broad-Band Power and Angular Colour Measures

If the power spectrum changes slowly over the band that $B_\ell = \overline{W}_\ell$ stretches over, one estimate of the broad-band power is just a renormalization of the experimental $\text{rms}$:

$$\langle C_{\ell} \rangle_B \equiv (\Delta T/T)_{\text{rms}}^2 / \mathcal{I}[\overline{W}_\ell] = \mathcal{I}[W, \mathcal{C}_{\ell}] / \mathcal{I}[\overline{W}_\ell] . \quad (4)$$

For an experiment probing isolated (uncorrelated) pixels, this broad-band power is all you can get (since $C_{Tpp'} = C_{T11}\delta_{pp'}$). Current intermediate and small angle experiments have insensitive $W_{pp', \ell}$ for pixel separations beyond the first few (i.e., $C_{Tpp'}$ approaches zero rapidly): splitting the $\overline{W}$-band into $B_1, B_2, ...$ sub-bands does little good. However, for mapping experiments like DMR and FIRS the number of pixel base-lines probed (number of useful $W_{pp', \ell}$) is large so the large $\overline{W}$-band can be split into a number of contiguous sub-bands (but not too many or the error bar matrix becomes too large and complicated). “Spectral colours” can be defined as logarithmic differences of the band-powers. Alternatively one
can assume a power spectrum with a local colour index \( n_{\Delta T} \) as well as a broad-band power amplitude \( \langle C_\ell \rangle_B \):

\[
C_{B\ell} = \langle C_\ell \rangle_B (\ell + \frac{1}{2})^{2+n_{\Delta T}} U_{B\ell} \mathcal{I}[\mathcal{W}_\ell]/\mathcal{I}[\mathcal{W}_\ell(\ell + \frac{1}{2})^{2+n_{\Delta T}} U_{B\ell}] .
\]  

(5)

Here an extra input estimator shape \( U_{B\ell} \) is included to give more flexibility for statistical testing. I usually choose \( U_{B\ell} = 1 \), although I find it is always instructive to see what a matched form, \( U_{B\ell} \propto \mathcal{W}_\ell \), gives. As we learn more experimentally about the true shape, \( U_{B\ell} \) can be matched to it.

The amplitudes \( \{[(C_\ell)_{B1}]^2, [(C_\ell)_{B2}]^2, \ldots \} \) can be determined by whatever statistical method we are most enamoured with, whether Bayesian or frequentist. Their likelihood function defines a multidimensional surface, whose peak gives the best band-power estimates and whose curvature matrix about the peak defines the error bar matrix; or, better, Bayesian credible regions can be defined for error estimation.

It also turns out to be simple to transform old-style presentations of \( \Delta T/T \) detections and limits to the band-power and colour language.

The infamous \( (\Delta T/T)_c \)-curves as a function of coherence angle \( \theta_c \) can be translated into the more useful band-power estimates by noting the power spectrum for a “Gaussian correlation function” is of the form eq. 3.

\[
\text{GCF : } n_{\Delta T} = 0 \& U_{B\ell} = e^{-u^2/2} , \ u \equiv (\ell + \frac{1}{2})/(\ell_c + \frac{1}{2}) , \ \ell_c + \frac{1}{2} \equiv [2\sin(\theta_c/2)]^{-1} ;
\]

\[
\langle C_\ell \rangle_B \approx (\Delta T/T)_c^2 \mathcal{I}[u^2e^{-u^2/2\mathcal{W}_\ell}]/\mathcal{I}[\mathcal{W}_\ell] .
\]  

(6)

Another form often adopted is the low-\( \ell \) “Sachs-Wolfe” power for scalar metric perturbations (e.g., Bond and Efstathiou 1987),

\[
C_\ell = C_2 \frac{\ell(\ell + 1)}{6} \frac{(\ell - 3-n_s)!}{(\ell + 3-n_s)!} \frac{(2 + 3-n_s)!}{(2 - 3-n_s)!} , \ n_{\Delta T} \approx n_s - 3 .
\]  

(8)

It differs very little from the more natural form for phenomenology, eq. 3, with the index \( n_{\Delta T} \) related to the primordial density fluctuation index \( n_s \) as shown. Thus \( n_s = 3 \) is white noise in \( \Delta T \).

For the DMR and FIRS beams (including pixelization effects), the relation between the quadrupole power \( C_2 \) (or equivalently \( Q_{rms,PS}^2 \)) and the band-power is

\[
\left( \frac{Q_{rms,PS}}{T_0} \right)^2 \equiv \frac{5}{12} C_2 \approx \frac{5}{12} \langle C_\ell \rangle_B e^{-\alpha(n_s-1)(1+0.3(n_s-1))} , \ \alpha_{dmr} = 0.73 , \ \alpha_{firs} = 1.1 ,
\]  

(9)

where \( T_0 = 2.726 \pm 0.01 \) K. An advantage of the band-power over \( Q_{rms,PS}^2 \) is that the former is roughly independent of \( n_{\Delta T} \), while the latter is quite sensitive to it.

Fig. 1 shows estimates for the power spectrum averaged over bands in \( \ell \)-space, assuming only wavelength-independent Gaussian anisotropies in \( \Delta T/T \) are contributing to the (sometimes cleaned) signals. See the reference list for the experiments’ acronyms. In Fig 1(a), the data points denote the maximum likelihood values for the band-power and the error bars give the 16% and 84% Bayesian probability values (corresponding to \( \pm 1\sigma \) if the probability distributions were Gaussian). The horizontal location is at the average value \( \langle \ell \rangle \equiv \mathcal{I}(\ell \mathcal{W})/\mathcal{I}(\mathcal{W}) \);
except for DMR, FIRS and Python, \( \langle \ell \rangle \) is near the \( W_\ell \)-maximum. The horizontal error bars denote where the filters have fallen to \( e^{-0.5} \) of the maximum.

Proceeding from small \( \ell \), \( Qdmr \) is the quadrupole power estimated using the 53GHz maps with a Galactic cut \( b_{gcut} = 20^\circ \) (Bond 1993). The treatment of the \( dmr, firs \) and Tenerife broad-band powers are described later. The \( sp91 \) band-estimates are for the 9 point scan, the 13 point scan and for the combined statistical analysis of the 9 + 13 point scan (higher value). The offsets are for clarity. All 4 channels were simultaneously analyzed (Bond 1993). (The 4th channel of the 9 point scan gives a \( 1.5 \times 10^{-10} \) 95% credible upper limit. Including a GCF with a synchrotron slope opens up the error bars of \( sp91 \) considerably.) The two recent MAX \( (M) \) results are for the scans in GUM (upper) and Mu Pegasus (lower, with a strong dust signal removed). The dotted lines ending in triangles denote the quoted 90% confidence interval for the MSAM single \( (g2) \) and double \( (g3) \) difference configurations for that half of their data that did not have obvious large sources in it. The dashed error bar shown next is the 84% upper credible limit for the \( sp89 \) 9 point scan. The OVRO 7 point upper limit is last. For most experiments with detections, the 50% Bayesian probability (where the vertical and horizontal error bars cross) lies very close to the likelihood maximum.

Fig. 1(b) gives band-powers for other experiments reported at the Capri meeting that I have estimated from GCFs (evaluated at or near the most sensitive angle \( \theta_c \) or \( (\Delta T/T)_{\text{rms}} \), but that I have not yet done a full Bayesian analysis for. The BigPlate result is denoted by \( bp \), Python \( (py) \) is next, followed by Argo. These higher MSAM2 and MSAM3 results (than in Fig. 1(a)) are what one gets when the “source-half” of the data is also included. Finally the upper limit from the \( m = 2 \) mode analysis of the WhiteDish experiment is \( wd2 \). The \( dmr2 \) number — with the smaller error bar than the \( dmr \) one in Fig. 1(a) — is a translation of the Bennett \( et \ al. \) (1994) result for the second year DMR data, \( Q_{\text{rms, PS}} = 17.6 \pm 1.7 \mu K \) for \( n_s = 1 \), to a band-power. Given my experience that transformed band-powers are in excellent agreement with the results of full Bayesian analysis, I am confident that Fig. 1(b) gives a fair account of the implications of the new detections reported at the Capri meeting.

To show how well the transformation formulae \( \{4, 7, 9\} \) work, I use some of the numbers quoted for the double difference Tenerife experiment \( (5.6^\circ \) beam and \( 8.2^\circ \) throw) at the Capri meeting. For the 15 and 33 GHz data, we heard that \( Q_{\text{rms, PS}} = 26 \pm 6 \mu K \) for \( n_s = 1 \), hence using \( \{8\} \) gives \( \langle C_\ell \rangle_B = 2.2^{+1.1}_{-0.9} \times 10^{-10} \). They also give \( (\Delta T)_c = 54^{+14}_{-10} \mu K \) at \( \theta_c = 4^\circ \) \( (\ell_c = 13.8) \). The ratio \( I\left[W_\ell^u e^{-u^2/2}W_\ell^c/2\right]/I[\bar{W}_\ell] \) turns out to be 0.587, hence the transformation \( \{6\} \) gives \( \langle C_\ell \rangle_B = 2.3^{+1.3}_{-0.8} \times 10^{-10} \). For all the 15 GHz data (covering 700 deg²), Rebolo gave \( (\Delta T/T)_{\text{rms}} = 39 \pm 10 \mu K \). I find \( I[\bar{W}_\ell] = 0.98 \), hence \( \langle C_\ell \rangle_B = 2.1^{+1.2}_{-0.5} \times 10^{-10} \) using \( \{4\} \). Thus, all are quite consistent.

Overlaying band-powers on theoretical \( C_\ell \) curves basically shows the status of the theories in question. However, since we do not know precisely where the observed power \( \langle C_\ell \rangle_{B, obs} \) lies in the band, the direct comparison should be with the theoretical band-power \( \langle C_\ell \rangle_{B, th} = I[\bar{C}\bar{W}]/I[\bar{W}] \). These are shown in Fig 1(c) for a few representative theories to illustrate the precision of power spectrum determination we need to finely differentiate among models. Error bars of 10%, the best DMR can possibly do, and that are not over-optimistic for intermediate- and small-angle mapping experiments, are also shown. To consider what is required for this accuracy, consider an experiment with \( N_{pix} \) pixels with per-pixel error \( \sigma_{pix} \). Suppose the pixels are sufficiently separated that only \( W_\ell \) is an effective probe of \( C_\ell \). For large \( N_{pix} \), the \( \nu \)-sigma uncertainty in the experimental value of the band-power is...
\[
\langle C_\ell \rangle_{B,\text{obs}} = \langle C_\ell \rangle_{B,\text{maxL}} \pm \nu \frac{\sqrt{2/N_{\text{pix}}}}{I[\mathbf{W}_t]} \left[ \langle C_\ell \rangle_{B,\text{maxL}} + \sigma_{\text{pix}}^2 / I[\mathbf{W}_t] \right] ; \quad (10)
\]

\[
\langle C_\ell \rangle_{B,\text{maxL}} = \langle C_\ell \rangle_{B,\text{th}} \pm \nu \frac{1/N_{\text{pix}}}{I[\mathbf{W}_t]} \left[ \langle C_\ell \rangle_{B,\text{th}} + \sigma_{\text{pix}}^2 / I[\mathbf{W}_t] \right] . \quad (11)
\]

To get 10% error bars requires experimental noise to be small c.f. the signal (we are basically already there), and \( N_{\text{pix}} = 200 \), i.e., a mapping experiment. For large \( N_{\text{pix}} \), the observed maximum likelihood will fluctuate from \( \langle C_\ell \rangle_{B,\text{th}} \), the quantity we want, according to eq. (11), but the error bars of eq. (10) include these realization-to-realization fluctuations (thus \( \sqrt{2} \) appears, not 1).

2. The DMR and FIRS Colour Problem

In this section, I describe some Bayesian results on the FIRS and first year DMR maps. The methods deal with all aspects of the maps simultaneously and so are highly sensitive to all components in it, whether they are the primary signals we are interested in or the warts that Dave Wilkinson admonishes us to be aware of and beware of, for they blemish “the face” that George Smoot (and we too) wish to contemplate. As we shall see, it seems to be pimples and not warts which I have to concentrate on cleaning up. Although I think even now I am doing this moderately well — for within the statistical observing strategies I use certain natural pimple-erasers are suggested — I still believe that the acne pattern colours my high precision look at the face. For the issue is one of colour.

To make the DMR map data tractable for exploration, I used one lower resolution scale than the original maps, i.e., 5.2° rather than 2.6° pixels. The dipole and average subtractions were done after the Galactic latitude cut (taken here to be \(|b| > 25°\)), but before the lowering of the resolution. I used the Kneissl and Smoot (1993) revision of the DMR beam (which is non-Gaussian), and included corrections for digitization and pixelization, especially important in view of the 5.2° pixel size. Since COBE actually measures the difference between \( \Delta T \) values at 2 beam-smeread points 60° apart, the pixel errors are correlated. A correction linear in the off-diagonal terms similar to one proposed by Lineweaver and Smoot (1993) has been applied to take this effect into account. The unknown amplitude in the average and quadrupole of the theory have been handled by treating them as marginal variables to be integrated over in the Bayesian analysis.

The FIRS (‘MIT’) map (168 GHz) has a highly inhomogeneous weighting of each of the 1.3° pixels. For the results shown here one lower resolution level was used, with 2.6° pixels, although I have checked that 1.3° pixels gives the same answer for the \( n_{\Delta T} = -2 \) case. The average and dipole were removed in the map construction. To include the effects of pixelization and the best beam-smearing estimate, a 4.2° beam was used rather than 3.8°. (The differences in the amplitudes obtained are quite small.)

A full Bayesian analysis of maps requires frequent inversion and determinant evaluations of \( N_{\text{pix}} \times N_{\text{pix}} \) correlation matrices, the sum of all \( C_{Tpp} \) in the theoretical modelling plus the pixel-pixel observational error matrix \( C_{Dpp} \). I transform this matrix and the amplitude vector \( \mathbf{\Delta} = (\Delta_p) \) to ones that absorb the pixel ‘weighting’, \( C_T \equiv C_D^{-1/2} C_T C_D^{-1/2} \) and \( \mathbf{\bar{x}} \equiv C_D^{-1/2} \mathbf{\Delta} \), with ‘dimensions’ of \((\text{signal} - \text{to} - \text{noise})^{2,1}\), respectively. The removal of averages, dipoles, gradients, etc. are included as more ‘theoretical’ signals, whose ‘unobserved’ contributions to \( \Delta_p \) are integrated over, creating marginal likelihood functions. For their
prior distribution, very broad Gaussians are adopted (essentially the same as uniform priors, but they regularize the inversions). I rotate to the eigenvectors of $E_T$, which are fully orthogonal (uncorrelated) signal-to-noise modes for the map, with eigenvalues $1 + E_{TR,k}$ and signals $\xi_k$ for each mode $k$, where $E_{TR} \equiv R E_T R^{tr}$ and $\xi \equiv R \vec{x}$. With uniform weighting and all-sky coverage, these modes $k$ are just the independent $\Re(a_{\ell m})$ and $\Im(a_{\ell m})$ for any theory, but with Galactic cuts followed by dipole removals they are complicated and theory-dependent. Even so, these modes are quite instructive linear combinations of the pixels. Often only a fraction of the modes are highly sensitive to theories being tested. For example, pixel differences within the beam are theory-insensitive, but highly susceptible to excess noise in the experiment, and the $(S/N)$-eigenmodes can be used for effective filtering or weighting strategies.

I sort the $(S/N)$-eigenvalues, $E_{TR,k}$, in decreasing values. These are plotted as open circles in Fig. 2 for DMR and FIRS, for the cases $n_s = 1$ $(a,b)$ and $n_s = 2$ $(c,d)$; the 1-sigma cosmic variance are the two solid lines surrounding the points. $E_{TR,k}$ is normalized to $\sigma_{th} = 1$, where $\sigma_{th}^2 \equiv \langle C_{\ell} \rangle_B / 10^{-10}$; this just happens to be the amplitude that the data prefers. Although it may appear from the $E_{TR,k}$ decline that only the first few modes are effective at probing the theory in question, for uniform weighting and all-sky coverage it is basically equivalent to plotting $C_{\ell} / (\ell(\ell+1))$, which falls with $\ell$ even without beam-smearing effects. A rough way to correct the visual impression is to multiply by $k$ (which would be $(\ell + 1)^2$ for homogeneous-weighting and all-sky coverage): $kE_{TR,k}$ is then nearly constant, decaying mostly because of the beam.

The reason I didn’t plot it this way is to make evident the (almost flat) excess power at high $k$ in the data, which the beam-filtered theoretical points cannot possibly explain. The observational (solid) points in Fig. 2 are the average $(S/N)$-mode band-powers $\beta$,

$$\overline{\beta}_{S/N,\beta} \equiv \frac{1}{n_{\beta} \sum_{k \in \beta} (\xi_k^2 - 1)} , \quad n_{\beta} \equiv \sum_{k \in \beta} 1 .$$

(12)

The (sparse)-binning was chosen to make the data trends clear. The error bars are estimated directly from the data, and include a correction for large scale power as well as a pixel noise error.

The form of Fig. 2 was chosen to show a pictorial procedure for fitting the data: $\sigma_{th}$ is adjusted until the error envelope on $\sigma_{th}^2 E_{TR,k}$, which also scales with $\sigma_{th}^2$, encompasses the data points as well as possible. Clearly it will not do very well for the high $k$-bins for either the $n_{\Delta T} = -2$ or $-1$ theory. A first approximation to the residual ‘noise’, which is quite good for the high $k$ end of the FIRS data, is to assume a constant offset. This can be viewed as increasing the error bars by a factor $(1 + r)$ (so the offset is $r^2 + 2r$). Equivalently I can parameterize this residual noise by a new ‘theory’ source to be added to $\sigma_{th}^2 E_{TR,k}$, with $(S/N)$-power $\sigma_{res}^2 E_{TR} \equiv r^2 + 2r$, where $E_{TR} \equiv N^{-1}_{pix} \sum_k E_{TR,k}$, arranged so that if $\sigma_{res} = \sigma_{th}$, the power would be equal.

Of course, this pictorial procedure with quadratic combinations of the modes is not the best statistical approach to deal with the data. Once one has transformed to the $(S/N)$-eigenmode system, a Bayesian analysis on the full map is very straightforward and fast, even with average and gradient subtractions. The effect of these subtractions is hard to show in Fig. 2: the $k$-modes are no longer statistically independent. (The average affects the first bin, but the dipole has influence out to $k \sim 30$, and is responsible for some of the power loss.
Note that the observational $\beta$-powers can become negative.) The first step in the Bayesian method is the construction of a joint likelihood function in $\sigma_{th}$ and $\sigma_{res}$. The contour maps in Fig. 3 show a strong maximum in $\sigma_{th}$ at almost the same level for the DMR 53A+B, 90A+B, and FIRS maps, in spite of the large differences in the position of the maximum in $\sigma_{res}$.

The single broad-band powers for DMR and FIRS plotted in Figure 1 are derived from the marginal distributions in signal amplitude, found after integrating the joint distribution over all possible residual noise amplitudes (i.e., integrating over the $\sigma_{res}$ axis of Fig. 3). For $n_s = 1$, $Q_{rms,PS} \equiv 17.6 \mu K ((C_{\ell})_{B}/10^{-10})^2$. I shall express the results in the $Q_{rms,PS}$ terms: for the 53A + B GHz map, I get $17.9 \pm 2.9 \mu K$ ($17.6 \pm 2.8 \mu K$ if I ignore off-diagonal terms in $C_{DPP}$, $17.2 \pm 2.5 \mu K$ if I use a $7^\circ$ Gaussian beam with no pixelization correction). This compares with the DMR team’s best estimation for the first year data of $17.1 \pm 2.9 \mu K$ (Wright et al., 1994) and the new two-year result of $17.6 \pm 1.7 \mu K$. For 90A+B, I get $18.6 \pm 3.8 \mu K$ and for 31A+B, $15.8 \pm 5.8 \mu K$. For 53A-B, I get $Q_{rms,PS} = 5.3^{+5.1}_{-3.3} \mu K$, while for 90A-B, $0.0^{+0.5}_{-0.0} \mu K$: i.e., no spurious large scale power in the difference maps. (Curiously I find no clear detection in 90A; however the error bars for this map are quite large.) For $n_s = 1$ and the FIRS map, the level is $18.6^{+4.8}_{-4.6} \mu K$ — i.e., at the DMR level.

The probability for the residual noise level is found by integrating over the $\sigma_{th}$-axis. The most probable value is indicated by the horizontal dotted line joining the triangular points in Fig. 2, about at the level the eye would pick. For FIRS, zero in residual noise is excluded at more than the 10 sigma level (with $r = 0.25$ most probable, independent of $n_{\Delta T}$ (Fig. 2(c) c.f. Fig. 2(a))). However, the high plateau around $k = 200$ in Fig. 2(a,c) suggests $\sigma_{pix} = 0$-enhancement is not the whole story (in accord with failure by the FIRS team to identify the source of the residual after exhaustive checks).

For the first year DMR data, the residual depends upon the map and upon the spectral steepness. I find no evidence for a residual in the 31A+B map, but do have them in both 53A+B and 90A+B. The residuals in 53B ($r = 0.07$) and 90B ($0.09$) are larger than in 53A ($0.01$) and 90A ($0.02$). 53(A-B) has residuals about the same as 53A+B, suggesting that it is not the effect of physical sources on the sky. The numbers in brackets are for $n_{\Delta T} = 0$, 25 most probable, independent of $n_{\Delta T}$ (see Fig. 3). The colour index I get is $n_{\Delta T} = 200$ in Fig. 2(a,c) suggests $\sigma_{pix} = 0$-enhancement is not the whole story (in accord with failure by the FIRS team to identify the source of the residual after exhaustive checks).

Al Kogut and the DMR team have checked the scatter in their data before conversion to a map and confirm the trends I find in $r$, but with lower amplitudes than my $n_{\Delta T} = 2$ values, more like my $n_{\Delta T} = 1$ values, the index the data prefers. However, there are effects not usually included in map analysis which will increase the variance, e.g., the “unobserved” Galactic plane pushes up the variance in off-plane pixels through the 60° correlation. Modelling the excess in Fig. 2(c,d) by a constant is clearly not optimal. The “unobserved” Galactic plane pushes up the variance in off-plane pixels through the 60° correlation. Modelling the excess in Fig. 2(c,d) by a constant is clearly not optimal. The shape of the data points is suggestive of beam-effects not being properly included. However, even with the old standard $7^\circ$ beam (quite different than the Kneissl and Smoot 1993 beam), and with no pixelization corrections, the $E_{TR,k}$ curve still falls off faster than the data.

Spectral indices from $n_{\Delta T} = 3$ to $n_{\Delta T} = 0$ in steps of 0.1 were run to construct likelihood functions in $\{n_{\Delta T}, \sigma_{th}, \sigma_{res}\}$. Integrating first over $\sigma_{res}$ (marginalizing it) allows one to construct $n_{\Delta T} - \sigma_{th}$ contour maps, not shown here. Integrating again, over $\sigma_{th}$, gives the probability distribution for $n_{\Delta T}$. The colour index I get is $n_{\Delta T} + 3 = 2.0^{+0.4}_{-0.4}$ for the DMR53A+B map and $1.8^{+0.6}_{-0.8}$ for the FIRS map, with the second errors given denoting 2-sigma values. The reason for such steep $n_{\Delta T}$ is evident from Fig. 2: higher $n_{\Delta T}$ gives a
slower fall of the mode power with increasing \( k \). However, one cannot increase \( n_{\Delta T} \) too much or the flattening of \( \mathcal{E}_{TR,k} \) at low \( k \) becomes too much for the high amplitude data points there to contend with: \( n_{\Delta T} = -1 \) is the compromise.

It might be thought that filtering out the high \( k \) modes will allow us to avoid modelling the excess found there. However, so far I find that sharp \( k \)-filtering, keeping only a few hundred modes, does not help for FIRS. The \( k \sim 200 \) plateau keeps the preferred index high. For DMR with \( k \lesssim (16)^2 \), the residual \( r \) grows for low \( n_{\Delta T} \), stealing some of the power from \( \sigma_{th} \) (which drops by 20\% for \( n_{\Delta T} = -2 \)). However, the \( n_{\Delta T} \) value (and error bars) remain about the same as when all of the modes are included. For \( k \leq (13)^2 \), \( n_{\Delta T} \) becomes poorly determined, \( 1.0^{+1.1}_{-0.7} \). If a faulty beam structure is responsible, adding to the likelihood function a beam-smearing parameter could be used to set it by finding the most probable value. However, a 7\(^\circ\) Gaussian beam and no pixelization corrections gives \( 1.7^{+0.4}_{-0.7} \) still steep c.f. the inflation prediction. Models with three or more components have yet to be explored. A particularly interesting case is a white noise \( n_{\Delta T} = 0 \) distribution to model small scale (non-primary?) sources, plus a large scale power source with variable \( n_{\Delta T} \), plus pixel error bar enhancement.

Most people probe the data using quadratic combinations \( Q_A \) of the pixel amplitudes \( (\Delta T/T)_p \), and hence of \( \xi_k \): \( Q_A = \sum_{kk'} \gamma_{kk'}^A \xi_k \xi_{k'} \). The correlation function and various combinations for the \( C_\ell \)-spectrum are examples, as, of course, are the \( (S/N) \)-powers \( P_{S/N,\beta} \). The sum of the \( P_{S/N,k} \) linearly weighted by the \( (S/N) \)-eigenvalues, \( \sum_k \mathcal{E}_{TR,k} \xi_k^2 \), is the Boughn-Cottingham (BC) statistic. Its effect can be seen in Fig. 2: multiplying the data by \( \mathcal{E}_{TR,k} \) strongly suppresses the contribution of the deviant high \( k \)-modes to the sum. A pictorial exercise such as in Fig. 2, but with only one data point, can be done for the BC statistic. A better method is to construct a likelihood using many (signal-plus-noise) Monte Carlo realizations of the \( \xi_k \), with average and dipole subtraction. This is the map. Thus, map construction is very fast since it involves just \( N_{\text{pix}} \) random number choices and no direct summation over \( Y_{tm} \) at each pixel. This high speed helps, since to create a reasonably smooth likelihood function (rather than just using frequentist statistical measures) requires fifty thousand rather than a few thousand realizations. The eigensystem is of course also excellent for sets of quadratic statistics, e.g., for angular bins of the correlation function and \( \ell \)-bins of various prescriptions for quadratic \( \ell \)-space power spectrum estimators. One just has to rotate the pixel-pixel quadratic operators into the \( \gamma_{kk'}^A \) form in \( k \)-space.

The BC quadratic allows good recovery of the most probable Bayesian values (but it costs more in computer time since no Monte Carlo runs are required for the Bayesian calculation). With appropriate normalization, the BC statistic turns out to be the most sensitive single quadratic estimator of the band-power if there were no dipole etc. removals and no other signals in the data. For FIRS, the BC value is a little bit high. I have therefore constructed best single band-power estimators with these effects included. For FIRS, the \( \langle C_\ell \rangle_B \) result is bang-on the Bayesian one. For DMR53A+B, both the BC statistic and my modification give the same value, lower than the Bayesian value with all modes included and closer to the \( k \)-cut value described above: the two forms of filtering are not that different. (Wright et al. 1994 have also used the BC quadratic on the DMR data, and find similar amplitudes.) Unfortunately, while well-suited to \( Q_{\text{rms,PS}} \) estimation, these quadratic forms change with \( n_{\Delta T} \), so joint likelihoods should not be constructed, and, in any case, are a poor way to get at \( n_{\Delta T} \). The obvious first thing to try is Weiner-filtering (with \( \mathcal{E}_{TR,k}/(1 + \mathcal{E}_{TR,k}) \) acting on
\( \xi_k \); however, not only does it change form with \( n_{\Delta T} \), it also changes form with \( \sigma_{th} \), and so it is not useful for either \( Q_{rms,PS} \) or \( n_{\Delta T} \) statistical analysis, although it can be used to construct cleaned maps. I have also tried Bayesian analyses without a residual, but with the mode sum in the log-likelihoods suppressed by a measure \( \propto E_{\kappa R,k}^n \): \( \alpha = 1 \) gives results in agreement with the BC-style quadratic statistics, but with smaller error bars because it deals with the entire map. Unfortunately, the result is not stable to variations in \( \alpha \), especially for FIRS, since lower \( \alpha \) does not sufficiently suppress the residual.

It is impractical to construct a likelihood when there are a number of \( \gamma^A \)'s to compare with: to make smooth functions, too many Monte Carlo runs are required even for this diagonal frame. Usually some distance measure of observed from simulated \( Q_A \) is used (e.g., the simple chi\(^2\) measure for \( C(\theta) \) used by the DMR team in Smoot et al. (1992), and variants which include the effect of the strong correlation from angle-bin to angle-bin in the theory, which I have used Bond 1993).

What the \( \gamma^A \) do is filter pixel-pairs, and this can have great advantages over just linear filtering of modes. For example, although I do not know the details of the angular structure of the residual noise, the pixel-error enhancement model contributes only to the zero angle bin of the correlation function. If the residual’s effect spills over to neighbouring pixels, one can just cut out the core of the correlation function, say out to \( \theta_{fwhm} \). This would be an excellent filter if some part of the residual arises because the beam is not quite right or if there are white-noise or Poisson sources on the sky. The same effective filtering does not happen with quadratic band-power estimators. For example, white sky-noise gives a rising \( \ell^2 \) contribution to \( C_\ell \). The residual in the FIRS map obscured the interpretation of my results using quadratic band-power estimators (which differ from the Hauser-Peebles form used by Wright (1993) for DMR, which might be better). Quadratic band-power operators which exclude pixel-pairs out to \( \theta_{fwhm} \) do filter this rise, but the interpretation as power in an \( \ell \)-band is lost. Thus, I believe that the Bayesian approach is superior to using a quadratic set of band-power operators for \( C_\ell \)-estimation. And what comes out is an estimate of the ensemble-averaged power, which is what we want, and not the specific realization of the power that exists in a map, even if that map is our own sky.

The filtering of small-angle systematic or physical effects in correlation function analysis may make the \( n_{\Delta T} \) determined with \( C(\theta) \) a better indication of the angular colour of the large-angle sky than I get – at least until the nature of the high \( k \)-power is better understood. For the FIRS map, there are indications from \( C(\theta) \) for a shallower slope, \( n_{\Delta T} \approx -1.7 \) (Ganga and Page 1994); even so, it is clear from Fig. 2 that there are some anomalies in the \( S/N \) band-powers that are not favourable to \( n_{\Delta T} < -2 \). Bennett et al. (1994) find \( n_{\Delta T} + 3 = 1.59^{+0.29}_{-0.55} \) for two years of DMR data. Modifications of the Bayesian approach to model a more extended residual noise structure as well as the single-pixel level one is the next step for me to better observe the angular colour of the CMB sky.

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**FIGURE CAPTIONS**

**Figure 1.** (a) and (b) give band-power estimates for the vintage Fall 1993 data. The labels for the experiments are defined in the text. The vertical error bars are 1 sigma, the inverted triangles are 90 or 95% confidence limits. Systematic errors or non-primary sources (e.g., dust, synchrotron) may contribute to these $C_\ell$’s; the underlying primary $C_\ell$ may be lower, but could be higher because of ‘destructive interference’ among component signals. $C_\ell$ is shown for a “standard” CDM model ($\Omega = 1$, $n_s = 1$, $h = 0.5$, $\Omega_B = 0.05$) with normal recombination to illustrate a typical Doppler peak, but the early reionization (at $z > 200$) $C_\ell$ is decidedly peak-less. Whether a Doppler peak exists is unclear from the current data. (c) Theoretical band-powers with (eventually) achievable 10% error bars are displayed for both the $n_s = 1$ CDM model and a chaotic-inflation inspired $n_s = 0.95$ one – including a gravity wave contribution (Crittenden et al. 1993). The dotted $C_\ell$ has $\Omega_\Lambda \neq 0$ and high $H_0$ ($\Omega_\Lambda = 0.75$, $\Omega_{cdm} = 0.22$, $\Omega_B = 0.03$, $H_0 = 75$, $n_s = 1$). This shows the high precision we require to differentiate among models (for more details, see Bond et al. 1994a,b).

**Figure 2.** $(S/N)$ band-powers $P_{S/N,\beta}$ (eq. [12]) for the orthogonal signal-to-noise eigenmodes of the FIRS and DMR 53A+B maps, and for $n_{\Delta T} = -2, -1$ powers, as indicated. The error bars denote observational variance for the quadratics. The small open points denote the $\beta$-band averages of $E_{TR,k}$. The upper and lower curves about these give the theoretical variances. All three curves scale with $\sigma_{th}^2$. The triangles joined by the dotted curve show the most likely residual noise the Bayesian analysis finds. The observed $P_{S/N,\beta}$ should be matched within the errors by $\sigma_{th}^2 E_{TR,k} + \sigma_{res}^2 E_{TR}$ through adjustment of $\sigma_{th}$ and the residual $\sigma_{res}$. For the DMR cases, the dashed line shows $E_{TR,k}$ for a $7^\circ$ Gaussian beam with no pixelization correction. The data points also move a bit, but not enough for this beam to explain the high $k$-behaviour. Small $k$ are, of course, unaffected. Although $\sigma_{res}$ drops, $\sigma_{th}$ hardly does.

**Figure 3.** Contour maps (1-sigma to 10-sigma) of DMR and FIRS likelihood functions for the scale invariant $n_{\Delta T} + 3 = 1$ case. The heavy dot denotes the maximum. Here, $\sigma_{th} = \langle C_\ell \rangle_B/10^{-10}]^{1/2}$ and the residual noise amplitude $\sigma_{res}$ is normalized to have the same total power as in the $\sigma_{th}$-band if $\sigma_{res} = \sigma_{th}$. The 53A+B, 90A+B and FIRS maps show the same band-power detection, within the errors, and in spite of the differing residuals. The 53 GHz difference map, 53A-B, shows no large scale power, although the residual offset remains at the same level as in A+B. The same holds for 90A-B. As $n_{\Delta T}$ increases, the required residual goes down for the 53A+B map, $\sigma_{res} \sim 0.466 \sqrt{-n_{\Delta T}}$, with no need of a residual for white noise ($n_{\Delta T} = 0$), whereas it remains constant at $\sigma_{res} \approx 2.5$ for the highly inhomogeneously-sampled FIRS map.
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