Quantum synchronization and correlations of two qutrits in a non-Markovian bath

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We investigate quantum synchronization and correlations of two qutrits in one non-Markovian environment using the hierarchy equation method. There is no direct interaction between two qutrits and each qutrit interacts with the same non-Markovian environment. The influence of the temperature of the bath, correlation time, and coupling strength between qutrits and bath on the quantum synchronization and correlations of two qutrits are studied without the Markovian, Born, and rotating wave approximations. We also discuss the influence of dissipation and dephasing on the synchronization of two qutrits. In the presence of dissipation, the phase locking between two qutrits without any direct interaction can be achieved when each qutrit interacts with the common bath. Two qutrits within one common bath can not be synchronized in the purely dephasing case. In addition, the Arnold tongue can be significantly broadened by decreasing the correlation time of two qutrits and bath.

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I. INTRODUCTION

Synchronization which describes the adjustment of rhythms of self-sustained oscillators due to an interaction is a fundamental phenomenon of nonlinear sciences. This phenomenon has been observed in physical, chemical, biological, and social systems [1]. In recent years, many efforts have been devoted to extend the concept of synchronization to quantum systems such as Van der Pol oscillators [2–4], atomic ensembles [5, 6], trapped ions [7], and cavity optomechanics [8–11].

In general, a quantum system can be either continuous or discrete. In the previous studies [2–11], most authors have considered the quantum synchronization of continuous-variable systems with classical analogs since they can be described by quasiprobability distributions in phase space such as the Wigner function. For example, in Refs. [8–11], the authors have investigated the quantum synchronization of optomechanical systems formed by optical and mechanical modes [12–15]. The measures of complete and phase synchronization of continuous-variable quantum systems have been proposed [10].

For discrete-variable systems without classical analogue, the measure of synchronization proposed in Ref. [16] is not valid. In this case, the Pearson product-moment correlation coefficient can be used to measure the degree the synchronization of spin systems [17]. The authors have investigated the synchronization of two qubits in a common environment using the Bloch-Redfield master equation and found that two qubits can not be synchronized for purely dephasing case. The Markovian and Born approximation were employed in Ref. [17]. Recently, the measure of quantum synchronization using the Husimi Q representation and the concept of spin coherent states has been suggested by Roulet and Bruder [18]. This measure can be used to study the synchronization of discrete-variable systems including qubits and qutrits. The authors have pointed out that qubits can not be synchronized since they lack a valid limit cycle and a spin 1 could be phase-locked to a weak external driving [18]. Later, the authors investigated the quantum synchronization and entanglement generation of two qutrits using the Lindblad master equation [19]. Very recently, the quantum synchronization of two quantum oscillators within one common dissipative environment at zero temperature was investigated with the help of a path integral formalism [20].

In the previous works [17–19], the Markovian and Born approximations were employed. Note that the rotating wave approximation was used in Ref. [20]. In addition, the temperature of bath was assumed to be zero in Refs. [18, 20]. Thus, the influence of the temperature of the bath or the non-Markovian effects were not taken into account in the above works. In the present paper, we study the quantum synchronization and correlations of two qutrits within one common bath using the hierarchy equation method [21–24]. The two qutrits have no direct interaction. In particular, in the derivation of hierarchy equations, the Markovian, Born, and rotating wave approximations are not used. The hierarchy equation method is a high-performance method and is suitable for strong- and ultrastrong-coupling systems like chemical and biophysical systems [25, 26]. Our results show that the measures of quantum synchronization and correlations could increase with the increase of the coupling strength between each qutrit and the common bath. The influence of the temperature of the bath depends heavily on the detuning of two qutrits. If the detuning is much smaller than the frequencies of two qutrits, then the maximal value of the measure of quantum synchronization increases with the increase of the temperature of the bath. However, if the detuning is not much smaller than...
the frequencies of two qutrits, the temperature of the bath could play a destructive role in the synchronization of two qutrits. In addition, the correlation time of the qutrits and bath plays an important role in the generation of quantum synchronization and correlations. The phase locking between two qutrits without direct interaction can be achieved if they are put into one bath and the dissipation is taken into account. In particular, two qutrits can not be synchronized in the purely dephasing case. The Arnold tongue of synchronization and correlations of two qutrits is not used. The hierarchy equation method is an exact method which is also suitable for strong- and ultrastrong-coupling systems. The density matrix of two qutrits at the frequencies of two qutrits, the temperature of the bath is

$$J(\omega) = \frac{2\lambda\gamma}{\gamma^2 + \omega^2},$$

where $\lambda$ is the coupling strength between qutrits and bath, $\gamma$ represents the width of the spectral distribution of the bath mode. The quantity $1/\gamma$ represents the correlation time of the bath. Particularly, if $\gamma$ is much larger than any other frequency scale, the Markovian approximation is valid.

The hierarchy equation is as follows \cite{21}

$$\dot{\rho}^n(t) = -(iH_s^x + \sum_{\mu=1,2} \sum_{k=0}^M n_{\mu k} V_k) \rho^n(t)$$

$$- \sum_{\mu=1,2} \sum_{k=0}^M \left( \frac{2\lambda}{\beta\gamma} - i\lambda \right) V_\mu^X V_\mu^X \rho^n(t)$$

$$-i \sum_{\mu=1,2} \sum_{k=0}^M n_{\mu k} [c_k V_\mu \rho^{\mu-\mu k}(t) - c_k^* \rho^{\mu-\mu k}(t)V_\mu]$$

$$-i \sum_{\mu=1,2} \sum_{k=0}^M V_\mu^X \rho^{\mu-\mu k}(t),$$

where $\nu_0 = \gamma$, $\nu_k = 2\pi k/\beta$, $c_0 = \lambda\gamma [-i + \cot(\beta\gamma)/2]$, $c_k = 4\lambda\gamma \nu_k (\nu_k^2 - \gamma^2)/\beta$, and $\beta = 1/(kT)$ is the inverse temperature of the thermal bath. Note that $\rho^{\mu k} = \rho^{\mu-\mu k+1}$ (in the $\mu k$th component of the multi-index. It is worth noting that in the derivation of the above equation, the Markovian, Born, and rotating wave approximations are not used. The hierarchy equation method is an exact method which is also suitable for strong- and ultrastrong-coupling systems. The density matrix of two qutrits at arbitrary time can be obtained from the initial state of the system and the above hierarchy equation of motion.

### II. MODEL AND HIERARCHY EQUATION METHOD

In this section, we introduce the model and hierarchy equation method used in the present work. We consider a system formed by two qutrits with no direct interaction and the free Hamiltonian is (set $\hbar = 1$)

$$H_S = \omega_1 J_1^z + \omega_2 J_2^z,$$

where $\omega_1$ and $\omega_2$ are frequencies of qutrit 1 and qutrit 2, respectively. The detuning between two qutrits is $\Delta = \omega_2 - \omega_1$. We assume two qutrits are put into a common thermal bath. The free Hamiltonian of the thermal bath is

$$H_B = \sum_k \omega_k b_k^\dagger b_k,$$

where $\omega_k$ is the frequency of the $k$th mode of the thermal bath. The interaction Hamiltonian of two qutrits and bath is

$$H_I = \sum_k g_k V(b_k^\dagger + b_k),$$

where $g_k$ is the coupling strength between the qutrits and the $k$th mode of the bath. Here, $b_k^\dagger$ and $b_k$ are the creation and annihilation operators of the thermal bath; $V$ is the system operator coupled to the bath. Without loss of generality, we suppose

$$V = (1 + h)(J_1^z + J_2^z) + (1 - h)(J_1^x + J_2^x),$$

where $h$ is an anisotropy coefficient with $-1 \leq h \leq 1$.

Similar to Refs. \cite{21, 24}, we choose the Drude-Lorentz spectrum

$$J(\omega) = \frac{2\lambda\gamma}{\gamma^2 + \omega^2}.$$
The measure of quantum synchronization proposed by Roulet and Bruder is defined as \[ |S_r(\phi)| = \int_0^{2\pi} d\phi_2 \int_0^{2\pi} d\theta_2 \frac{1}{2\pi} \times Q(\theta_1, \theta_2, \phi + \phi_2, \phi_2) - \frac{1}{2\pi}, \]

where

\[ Q(\theta_1, \theta_2, \phi + \phi_2, \phi_2) = \frac{9}{16\pi^2} \frac{\rho((\theta_1, \phi + \phi_2) \otimes (\theta_2, \phi_2))}{\rho(\theta_1, \phi + \phi_2 \otimes (\theta_2, \phi_2))}. \]

Here, \( Q(\theta_1, \theta_2, \phi + \phi_2, \phi_2) \) is the Husimi Q function and \( \phi = \phi_1 - \phi_2 \) is the relative phase of two spins. It can be viewed as a phase-space distribution of density matrix \( \rho \) based on spin coherent states. Note that \( S_r(\phi) \) depends upon the relative phase \( \phi \) explicitly. Physically, it can be used to estimate whether two spins have tendency towards phase locking [19]. If \( S_r(\phi) \) is always zero, then there is no fixed phase relation of two spins, i.e., no phase locking of two spins. Using Eqs. (9)-(11), we obtain the measure of synchronization of two spins 1 as

\[ S_r(\phi) = \frac{(32\xi + 9\pi^2\eta)}{256\pi}, \]

where \( \xi = e^{2i\phi}\rho_{33} + e^{-2i\phi}\rho_{33}, \)

\[ \eta = e^{i\phi}(\rho_{24} + \rho_{35} + \rho_{57} + \rho_{68}) + e^{-i\phi}(\rho_{42} + \rho_{53} + \rho_{75} + \rho_{86}), \]

where \( \rho_{jk} \) is the element of density matrix \( \rho \).

In order to measure the entanglement of two spins, we employ the logarithmic negativity which is defined by [30]

\[ E(\rho) \equiv \log_2 (1 + 2N) = \log_2 ||\rho^T||, \]

with \( \rho^T \) being the partial transpose of density matrix \( \rho \). Here, \( ||\rho^T|| \) is the trace norm of \( \rho^T \) and \( N \) is negativity defined by [29, 30]

\[ N \equiv \frac{||\rho^T|| - 1}{2}. \]

\( N \) is the absolute value of the sum of the negative eigenvalues of \( \rho^T \).

Now, we consider the quantum mutual information \( I \) as a measure of all quantum correlations between two subsystems [19, 31]

\[ I = S(\rho_1) + S(\rho_2) - S(\rho), \]

with \( \rho_1 = Tr_2(\rho) \) and \( \rho_2 = Tr_1(\rho) \). Note that \( S(\rho) = -Tr[\rho \ln(\rho)] \) is the Von Neumann entropy of density matrix \( \rho \). In Ref. [31], the authors have proposed mutual information as an order parameter for quantum synchronization of a quantum system.

**IV. DISCUSSIONS**

**A. Influence of coupling strength \( \lambda \)**

In Fig. 1, we plot \( S_r(\phi) \) as a function of the relative phase \( \phi \) of two spins for different values of coupling strength \( \lambda \). From Fig. 1, one can find that if the coupling constant is zero, then \( S_r(\phi) \) is always zero and there is no fixed phase relation between two spins. This implies two spins can not be synchronized in the case of \( \lambda = 0 \). Physically, in the case of \( \lambda = 0 \), there is no direct or indirect interaction between two qutrits. It is obvious that two qubits can not be synchronized without any interaction. On the other hand, the maximal value of \( S_r(\phi) \) increases with the increase of the coupling strength \( \lambda \). For example, the maximum of \( S_r(\phi) \) can be about 0.037 if \( \lambda = 0.05 \). Therefore, two spins can be synchronized in the presence of the interaction of spins and the common bath.

**B. Influence of anisotropy coefficient \( h \)**

We now turn to discuss the influence of the anisotropy coefficient \( h \) on the synchronization of two spins. In Ref. [17], the authors have investigated the synchronization of two qubits within a common environment and found that two qubits can not be synchronized for purely dephasing case. It is worth noting that they have used the Bloch-Redfield master equation approach to study the system. The Markovian and Born approximation were employed in Ref. [17]. In the following, we will show that two spins can not be synchronized in purely dephasing case without using the Markovian and Born approximation.

In Fig. 2, we plot \( S_r(\phi) \) as a function of \( \phi \) for different values of \( h \) with \( \gamma = 0.2\omega_1 \) (upper panel) and \( \gamma = 4\omega_1 \) (lower panel). One can clearly see that the maximal value of \( S_r(\phi) \) decreases with the increase of the parameter \( h \). In particular, the values of \( S_r(\phi) \) for \( \gamma = 0.2\omega_1 \) (upper
is much smaller than the frequencies of spins (∆
\[ \Delta \]
are
\[ \approx \]
the maximal value of
\[ S_r(\phi) \]
and
\[ r = 4 \omega_1 \]
(lower panel) and
\[ \gamma = 4 \omega_1 \]
(upper panel) and
\[ \gamma = 4 \omega_1 \]
(lower panel) are always zero if
\[ h = 1 \]
and two spins can not be synchronized for the purely dephasing case. Note that, in Ref. [17], the authors have assumed that
\[ \gamma \gg \omega_1 \]
and
\[ \gamma \gg \omega_2 \]
in order to ensure the validity the Markovian approximation. However, in the present work, we use the hierarchy equation method to investigate the present system without the Markovian and Born approximations. More precisely, it is not necessary to assume
\[ \gamma \gg \omega_1 \]
and
\[ \gamma \gg \omega_2 \]
in our work. We extend the result of Ref. [17] to the case of non-Markovian bath, i.e., two spins without direct interaction can not be synchronized in the purely dephasing case. We find dissipation is indispensable for the synchronization of two spins in Markovian or non-Markovian environment.

C. Influence of temperature

In Refs. [18, 19], the authors have studied the synchronization of one or two spins with the help of the Lindblad master equation. The temperature of the baths was assumed to zero. In this section, we investigate the influence of the temperature of the bath. Comparing the upper panel and lower panel of Fig. 3, we see the effects of the temperature of the bath depends crucially on the detuning of two spins. On the one hand, if the detuning is much smaller than the frequencies of spins (∆ \ll \omega), the maximal value of
\[ S_r(\phi) \]
increases with the increase of the temperature as one can see from the upper panel of Fig. 3. On the other hand, the maximum of
\[ S_r(\phi) \]
decreases with the increase of the temperature if ∆ = 0.1\omega_1 as one can see from the lower panel of Fig. 3.

D. Arnold tongue

In Figs. 4 and 5, we plot the quantum mutual information
\[ I(\phi) \]
(left panel) and maximum of
\[ S_r(\phi) \]
(right panel)
as functions of the detuning $\Delta$ and coupling strength $\lambda$.

The Arnold tongue which is the characteristic property of synchronization can be observed in these figures. We calculate the logarithmic negativity of two spins for many different parameters and find that there is no steady state entanglement even in the presence of synchronization. This result is similar to Ref. [31, 32]. Consequently, the authors of Ref. [31] have used mutual information as an order parameter for quantum synchronization. In Ref. [19], the authors considered two spins with direct interaction and found two spins could be entangled in the steady state. In the present work, we assume there is no direct interaction between two spins and find they could not be entangled in the steady state. Therefore, we plot the mutual information of two spins in Figs. 4 and 5. Comparing Fig. 4 and Fig. 5, we find the Arnold tongue could be adjusted by the parameter $\gamma$. Particularly, the Arnold tongue in Fig. 4 is very narrow and it is usually very difficult observe synchronization of two spins experimentally [1]. If we increase the parameter $\gamma$, then the Arnold tongue could be broadened significantly as one can see from Fig. 5. Therefore, the synchronization of two spins could be observed in experiments more easily if we increase the parameter $\gamma$.

V. CONCLUSIONS

In the present work, we have studied the quantum synchronization and correlations of two qutrits in one non-Markovian environment with the help of the hierarchy equation method. There is no direct interaction between two qutrits. Each qutrit interacts with the common non-Markovian bath. In order to measure quantum synchronization of discrete systems, we adopted the measure $S_r(\phi)$ proposed in Refs. [18, 19]. This measure is based on the Husimi Q representation and spin coherent states. We have investigated the influence of the temperature, correlation time, and coupling strength between qutrits and bath on the quantum synchronization and correlations of two qutrits without using the Markovian, Born, and rotating wave approximations. The influence of dissipation and dephasing on the synchronization of two qutrits was also discussed.

We first discussed the influence of the coupling strength of qutrits and bath on the quantum synchronization of two qutrits. If there is no interaction between each qutrit and the common bath, then they do not interact with each other at all. Obviously, they can not be synchronized in this case. If we increase the coupling strength of qutrits and bath, they can be synchronized when dissipation is taken into account. Particularly, we found that two spins without direct interaction in a non-Markovian bath can not be synchronized for purely dephasing case which is a generalization of Ref. [17]. In other words, dissipation is indispensable for the quantum synchronization of two spins in non-Markovian or Markovian bath.

Then, we studied the influence of the temperature of the common bath on the quantum synchronization of two spins. Our results show that the influence of the temperature of the common bath depends heavily on the detuning between two spins. If the detuning is much smaller than the frequencies of two spins, the maximal value of $S_r(\phi)$ increases with the increase of the temperature. However, when the detuning is not much smaller than the frequencies of two spins, the maximal value of $S_r(\phi)$ decreases with the increase of the temperature.

Finally, we plot the maximal value of $S_r(\phi)$ as a function of the detuning $\Delta$ and coupling strength $\lambda$. The Arnold tongue which is the characteristic property of synchronization can be observed in the present model. The logarithmic negativity of two spins for many different parameters was also calculated. We find that there is no steady state entanglement even in the presence of synchronization which is similar to Refs. [31, 32]. Therefore, we plot the mutual information of two spins. The Arnold tongue could be adjusted by the parameter $\gamma$ significantly. Particularly, the Arnold tongue is very narrow in the non-Markovian case $\gamma < \omega_i$ ($i = 1, 2$). Thus, it is usually very difficult observe synchronization of two spins experimentally in the non-Markovian case [1]. If we increase the parameter $\gamma$, then the Arnold tongue could be broadened significantly. Therefore, the synchronization of two spins could be observed in experiments more easily if they are put into a Markovian environment.

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