ON SPACE-TIME PROPERTIES OF SOLUTIONS FOR
NONLINEAR EVOLUTIONARY EQUATIONS WITH RANDOM
INITIAL DATA*

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Abstract

We consider space-time properties of periodic solutions of nonlinear wave equations, nonlinear Schrödinger equations and KdV-type equations with initial data from the support of the Gibbs' measure. For the wave and Schrödinger equations we establish the best Hölder exponents. We also discuss KdV-type equations which are more difficult due to a presence of the derivative in the nonlinearity.

Keywords: space-time properties; Hölder exponents; nonlinear evolutionary equations.

Resumen

Consideramos las propiedades en espacio tiempo de las soluciones periódicas de ecuaciones de onda no lineales, ecuaciones no lineales de Schrödinger y ecuaciones de tipo KdV con datos iniciales del soporte de la medida de Gibbs. Para las ecuaciones de onda y de Schrödinger establecemos los mejores exponentes de Hölder. También discutimos las ecuaciones de tipo KdV, que son más difíciles debido a la presencia de la derivada en la no linealidad.

Palabras clave: propiedades espacio-tiempo; exponentes de Hölder; ecuaciones no lineales evolutivas.

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1 Nonlinear wave equations

Consider the ID nonlinear wave equation

\[ Q_{tt} - Q_{xx} + f(Q) = 0 \]

with periodic boundary conditions \( Q(0, t) = Q(2\pi, t) \). The equation can be written in the Hamiltonian form

\[
Q_t = \{Q, H\}, \\
P_t = \{P, H\},
\]

with \( H(Q, P) = \int_0^{2\pi} \left[ \frac{P^2}{2} + \frac{Q^2}{2} + F(Q) \right] \), \( P' = f \), and a classical bracket

\[
\{A, B\} = \int_0^{2\pi} \left[ \frac{\partial A}{\partial Q(x)} \frac{\partial B}{\partial P(x)} - \frac{\partial A}{\partial P(x)} \frac{\partial B}{\partial Q(x)} \right] dx.
\]

An invariant Gibbs’ state \( e^{-H}d^\infty Qd^\infty P \) in the space\(^1\) of pairs \((Q, P) \in H^0 \times H^{-1}\) was constructed in [6] under the assumption that \( f(Q) \) is an odd locally Lipschitz function such that \( f(Q) \geq kQ \), for some \( k > 0 \) and big \( Q \). To simplify the proof we impose an additional condition on the growth of \( f \) at infinity: \( f(Q) \leq C(\epsilon) e^{\epsilon Q^2} \), for any \( \epsilon > 0 \). The Gibbs’ state is a product measure: the \( Q \) component is \( e^{-\int Q^2/2d^\infty Q} \), a circular Brownian motion with uniformly distributed initial position multiplied by the Radon-Nikodym factor \( e^{-\int F(Q)} \) and “white noise” measure \( e^{-\int P^2/2d^\infty P} \) on the \( P \) component.

Using variation of parameters we can write the original differential equation in the integral form

\[
Q(x, t) = \sin \sqrt{-\partial_x^2} t P_0(x) + \cos \sqrt{-\partial_x^2} t Q_0(x) - \int_0^t ds \frac{1}{2} \int_{x-(t-s)}^{x+(t-s)} dy f(Q(y, s))
\]

\[ = Q_W(x, t) + N(x, t), \]

where \((Q_0, P_0)\) are the initial data. The term \( Q_W(x, t) \) corresponding to the linear wave equation satisfies the Hölder condition

\[
|Q_W(x_1, t_1) - Q_W(x_2, t_2)| \leq K (|x_1 - x_2|^{\beta_1} + |t_1 - t_2|^{\beta_2}),
\]

(1)

with \( 0 < \beta_1, \beta_2 < \frac{1}{2} \) and some random constant \( K, EK^2 < \infty \), which depends on the \( \beta \)'s. The bound \( \frac{1}{2} \) is optimal in a sense that we can not have (1) with some \( \beta_1 > \frac{1}{2} \), or \( \beta_2 > \frac{1}{2} \) and random \( K, EK^2 < \infty \). The nonlinear part \( N(x, t) \) is a differentiable function of \( x \) and \( t \). The derivatives \( \partial_x N(x, t) \) and \( \partial_t N(x, t) \) satisfy (1) with the same exponents not exceeding \( \frac{1}{2} \). These simply means that the local structure of the field \( Q(x, t) \) is completely determined by the term \( Q_W(x, t) \) corresponding to the linear wave equation. We split the proof of these facts in three different steps.

\(^1\) \( H^s \) is a standard Sobolev’s space, i.e. \( Q(x) \in H^s \) if \( (1 - \Delta^2)^{s/2} Q(x) \in L^2[0, 2\pi] \).
Step 1

In the proof of the statement concerning (1) we use Kolmogoroff’s criteria of continuity, see [4].

**Theorem 1 (A. N. Kolmogoroff)** Let $Q(x,t)$, $(x,t) \in D$ be a random field with real or complex values and $D$ is a compact domain in $\mathbb{R}^2$. Assume that there exist positive constants $\gamma, C, \alpha_1$ and $\alpha_2$ with $\alpha_1^{-1} + \alpha_2^{-1} < 1$ satisfying

$$E|Q(x_1,t_1) - Q(x_2,t_2)|^? \leq C[(x_1 - x_2)^\alpha_1 + (t_1 - t_2)^\alpha_2]$$

for every $(x_i, t_i) \in D$. Then $Q(x,t)$ has a continuous modification.

Let $\beta_1$ and $\beta_2$ be arbitrary positive numbers less than $\alpha_1c_0$ or $\alpha_2c_0$ respectively, $c_0 \equiv (1 - \alpha_1^{-1} - \alpha_2^{-1})/\gamma$. Then there exists a positive random variable $K$ with $EK^? < \infty$ such that

$$|Q(x_1,t_1) - Q(x_2,t_2)| \leq K[(x_1 - x_2)^{\beta_1} + (t_1 - t_2)^{\beta_2}].$$

Consider the wave equation $Q_{tt} - Q_{xx} = 0$ with Gaussian initial data such that $Q$ is $e^{-\int Q^2/2d\infty}Q$, and $P$ is $e^{-\int P^2/2d\infty}P$ restricted to the submanifold $\hat{Q}(0) = \hat{P}(0) = 0$.

The Fourier coefficients are independent complex isotropic Gaussian variables such that $\overline{Q(n)} = \hat{Q}(-n)$, $\overline{P(n)} = \hat{P}(-n)$ and $E|\hat{Q}(n)|^2 = n^{-2}, E|\hat{Q}(n)|^2 = 1$. Using rotation invariance of the measure and its invariance under the flow

$$E|Q(x,t) - Q(x_0,t_0)|^2 = E|Q(0,h) - Q(0,0)|^2 \quad \text{(where } h = t_2 - t_1)$$

$$= E\left|\sum_{k \neq 0} (\cos kh - 1)\hat{Q}(k)\right|^2 + E\left|\sum_{k \neq 0} \frac{\sin kh}{k}\hat{P}(k)\right|^2$$

$$= 2\sum_{k > 0} \frac{(\cos kh - 1)^2}{k^2} + 2\sum_{k > 0} \frac{\sin^2 kh}{k^2}.$$

The first term can be overestimated as

$$\leq c_1 \sum_{0 < k \leq h^{-1}} \frac{(kh)^4}{k^2} + c_2 \sum_{h^{-1} < k} \frac{1}{k^2} \leq c_3 h^4 h^{-3} + c_4 h \leq c_5 h.$$

The same estimate holds for the second term. Using the Gaussian character of the field$^2$ $Q_W(x,t)$:

$$E|Q_W(x,t) - Q_W(x_2,t_2)|^{2n} \leq c_n|t_1 - t_2|^n.$$

Likewise

$$E|Q_W(x_1,t) - Q_W(x_2,t)|^{2n} \leq c_n|t_1 - t_2|^n.$$

Now apply Kolmogoroff’s criteria and pass to the limit with $n \to \infty$.

$^2E|x|^{2n} = \frac{(2\pi)^{n}}{2^{2n}}(Ex^2)^n$ if $x$ is a Gaussian variable with zero mean.
Step 2

Optimality of the H"older exponent $\frac{1}{2}$ for the space increment is a classical result. We present an elementary proof of this fact which works in other cases as well.

Note that $E|Q(\cdot, t)|_s^2 = \sum E|\hat{Q}(n, t)|^2 (1 + n^2)^s < \infty$ if and only if $s < \frac{1}{2}$. The following fact\(^3\) implies the rest. Let $Q(x)$, $x \in [0, 2\pi]$ be a rotationally-invariant Gaussian process such that

$$|Q(x_1) - Q(x_2)| < K(x_1 - x_2)^\beta$$

with some $K, EK^2 < \infty$. Then $E|Q|_s^2 < \infty$, for all $s < \beta$.

From the assumptions made, we get:

$$E \int_0^{2\pi} |Q(x + h) - Q(x)|^2 dx \leq EK^2 h^{2\beta},$$

$$Q(x + \frac{\pi h}{4}) - Q(x - \frac{\pi h}{4}) = 2i \sum_{n \neq 0} \hat{Q}(n)e^{inx} \sin \frac{\pi hn}{4}.$$ Parsevall’s identity implies

$$E \int_0^{2\pi} |Q(x + \frac{\pi h}{4}) - Q(x - \frac{\pi h}{4})|^2 dx = 4 \sum_{n \neq 0} E|\hat{Q}(n)|^2 \sin^2 \frac{\pi hn}{4}.$$ Therefore

$$\sum_{n \neq 0} E|\hat{Q}(n)|^2 \sin^2 \frac{\pi hn}{4} \leq c_1 h^{2\beta}$$

and

$$\sum_{\frac{1}{h} \leq n < \frac{2}{h}} E|\hat{Q}(n)|^2 \leq c_2 h^{2\beta}.$$ The substitution $h \rightarrow h/2^r$ yields

$$\sum_{\frac{2^r}{h} \leq n < \frac{2^{r+1}}{h}} E|\hat{Q}(n)|^2 \leq c_3 \frac{h^{2\beta}}{4^3 r^3}.$$ Finally

$$\sum_{\frac{1}{h} \leq n} E|\hat{Q}(n)|^2 \leq \sum_{r=0}^\infty \sum_{\frac{2^r}{h} \leq n < \frac{2^{r+1}}{h}} E|\hat{Q}(n)|^2 \leq c_4 h^{2\beta},$$

and\(^4\)

$$\sum_{k \leq |n|} E|\hat{Q}(n)|^2 \leq c_4 \frac{1}{k^{2\beta}}.$$\(^3\)This is stochastic version of the classical embeding theorem, [7].\(^4\)In the proof of this estimate we borrowed the idea from [1, section 82].
It implies for positive $n$:

$$S(n) \equiv \sum_{n \leq k} |\hat{Q}(k)|^2 \leq c_4 \frac{1}{n^{2\beta}}.$$ 

By Abel’s summation formula for positive $M$ and $N$, we have

$$\sum_{M}^{N} E|\hat{Q}(n)|^2(1 + n^2)^s = S(M)(1 + M^2)^s - S(N + 1)(1 + N^2)^s$$

$$+ \sum_{M+1}^{N} S(n)[(1 + n^2)^s - (1 + (n - 1)^2)^s]$$

$$\leq S(M)(1 + M^2)^s + \sum_{M+1}^{N} S(n)[(1 + n^2)^s - (1 + (n - 1)^2)^s].$$

For big $n$

$$(1 + n^2)^s - (1 + (n - 1)^2)^s = (1 + n^2)^s \left[ \frac{2s}{n} + O\left(\frac{1}{n^2}\right) \right].$$

This together with the estimate for $S(n)$ implies

$$\sum_{M}^{+\infty} E|\hat{Q}(n)|^2(1 + n^2)^s < \infty, \text{ for } s < \beta.$$ 

Negative indexes are handled in the same way. The proof is finished.  

For any fixed $x$, $Q_W(x, \bullet)$ is a $2\pi$—periodic rotationally invariant Gaussian process such that $E|\hat{Q}_W(x,n)|^2 = n^{-2}$. The same arguments used above show optimality of the exponent in the time increment.

**Step 3**

First, we estimate H"older exponents for a solution of the nonlinear equation. Let $h = t_1 - t_2$, using invariance of the measure

$$E|Q(x_1,t) - Q(x_2,t)|^{2n} = E|Q_0(x_1) - Q_0(x_2)|^{2n} \leq c_n(x_1 - x_2)^n$$

$$E|Q(x_1,t) - Q(x_2,t)|^{2n} = E|Q(x,h) - Q_0(x)|^{2n}$$

$$\leq c_n E\left[ \sin \frac{\sqrt{-\partial_x^2} h}{\sqrt{-\partial_x^2}} P_0(x) + \cos \frac{\sqrt{-\partial_x^2} h Q_0(x) - Q_0(x)}{2n} \right]$$

$$+ c_n E\left[ \int_0^h ds \frac{1}{2} \int_{x-(h-s)}^{x+(h-s)} dy f(Q(y,s)) \right]^{2n}.$$ 

To estimate the first term replace the measure $e^{-\int F \times e^{-\int Q^2/2} d\infty Q}$ by $Ce^{-k \int Q^2 \times e^{-\int Q^2/2} d\infty Q}$ with some big $C$ and proceed like in Step 1. To estimate the second term use H"older’s inequality and $E|f(Q)|^{2n} < \infty$ for every $n$. Eventually

$$E|Q(x_1,t) - Q(x_2,t)|^{2n} \leq c_n(t_1 - t_2)^n.$$
Kolmogoroff’s criteria implies that $Q(x, t)$ satisfies (1) with the same Hölder exponents not exceeding $1/2$. The last statement concerning derivatives $\partial_x N(x, t) - \partial_t N(x, t)$ follows from the explicit formulas
\[
\begin{align*}
\partial_x N(x, t) & = -\frac{1}{2} \int_0^h ds [f(Q(x + (h - s), s)) - f(Q(x - (h - s), s))], \\
\partial_t N(x, t) & = -\frac{1}{2} \int_0^h ds [f(Q(x + (h - s), s)) + f(Q(x - (h - s), s))],
\end{align*}
\]
and locally Lipschitz character of $f$. The proof is completed. 

**Nonlinear Schrödinger equations**

The next point of the discussion is 1D nonlinear Schrödinger equation
\[
i\psi_t = -\psi_{xx} + f(|\psi|^2)\psi,
\]
where $\psi(x, t)$ is a complex function $\psi = Q + iP$ which satisfies periodic boundary conditions $\psi(0, t) = \psi(2\pi, t)$. It can be written in the Hamiltonian form
\[
\psi_t = \{\psi, H\}
\]
with the Hamiltonian $H = \frac{1}{2} \int_0^{2\pi} |\psi_x|^2 + F(|\psi|^2)dx$, $F' = f$ and a bracket
\[
\{A, B\} = 2i \int_0^{2\pi} \left[ \frac{\partial A}{\psi(x)} \frac{\partial B}{\psi(x)} - \frac{\partial A}{\psi(x)} \frac{\partial B}{\overline{\psi(x)}} \right] dx.
\]
An invariant Gibbs’ state $e^{-H}d\psi d\overline{\psi}$ was constructed in [3, 5] under the assumption that $F \geq 0$ is an even polynomial. The Gibbs’ state is a product of two independent circular Brownian motions on $Q$ and $P$ whose components are coupled together by the nonlinear factor $e^{-\int F(Q^2 + P^2)}$.

Written in the integral form the equation is
\[
\psi(x, t) = e^{i\int_0^t f(|\psi|^2)\psi(x, s)ds} = \psi_S(x, t) + N(x, t),
\]
where $\psi_0(x)$ is initial data. The solution of the free Schrödinger equation satisfies
\[
|\psi_S(x_1, t_1) - \psi_S(x_2, t_2)| \leq K \left( |x_1 - x_2|^{\beta_1} + |t_1 - t_2|^{\beta_2} \right)
\]
with $0 < \beta_1 < \frac{1}{2}, 0 < \beta_2 < \frac{1}{4}$, and random constant $K$, $EK^2 < \infty$, which depend on $\beta$’s. The exponents $\frac{1}{2}, \frac{1}{4}$ are optimal. The same can be said about $\psi(x, t)$, a solution of NLS itself. The proof of this statements is similar to the corresponding one for the nonlinear wave equation.

The nonlinear term $N(x, t)$ seems to be smoother then $\psi_S(x, t)$. This implies that the microstructure of the field $\psi(x, t)$ is determined by the linear term $\psi_S(x, t)$, but the proof is not known. Presumably, Hölder exponents for $N(x, t)$ depend on arithmetical properties of the coefficients of the polynomial $F$. There is no uniform smoothing as one can see from the following example.
Example

Let \( \Gamma(x,t) \equiv \sum_{n \neq 0} e^{inx} e^{-i(n^2 + 2n\alpha)t} \hat{\psi}_0(n) \), arbitrary \( \alpha \geq 0 \) and \( \hat{\psi}_0(n) \) are independent complex isotropic Gaussian variables, \( E|\hat{\psi}_0(n)|^2 = \frac{1}{1+n^2} \). The Gaussian field \( \Gamma(x,t), x \in [0, 2\pi], s \in \mathbb{R} \) is stationary in time and rotationally invariant; \( \Gamma(\bullet, t) \) is a complex Ornstein-Uhlenbeck process with zero mean for any \( t \). By straightforward computation

\[
N(x,t) = -i \int_0^t e^{i\partial_x^2(t-s)} \Gamma(x,s)ds
\]

\[
= -i \int_0^t \sum_{n \neq 0} e^{inx} e^{-i(n^2 + 2n\alpha)s} \hat{\psi}_0(n)ds
\]

\[
= -i \sum_{n \neq 0} e^{inx} e^{-in^2t} \hat{\psi}_0(n) \frac{e^{-in\alpha t} - 1}{-in\alpha}.
\]

We see that \( N(\bullet, t) \) gains \( \alpha \) Sobolev’s exponents in comparison with \( \Gamma(\bullet, t) \).

1.1 KdV-type equations

The last topic of the discussion to KdV-type equations

\[
Q_t = -Q_{xxx} + (f(Q))_x
\]

with periodic boundary conditions \( Q(0,t) = Q(2\pi,t) \). The equation can be written in the Hamiltonian form

\[
Q_t = \{Q, H\},
\]

with the Hamiltonian \( H = \int_0^{2\pi} \frac{Q_x^2}{2} + F(Q)dx, \ F' = f \) and a bracket

\[
\{A, B\} = \int_0^{2\pi} \frac{\partial A}{\partial Q(x)} \frac{\partial B}{\partial Q(x)} dx.
\]

An invariant Gibbs state \( e^{-H} d^\infty Q \) was constructed in [3] for particular nonlinearities \( F(Q) = Q^3/3 \) (KdV) and \( F(Q) = Q^4/4 \) (modified KdV). The measure is a circular Brownian motion \( e^{-\int Q_x^2/2 d^\infty Q} \) multiplied by the nonlinear term \( e^{-\int F(Q)} \).

The equation can be written in the integral form

\[
Q(x,t) = e^{-\partial_x^2 t} Q_0(x) + \partial_x \int_0^t e^{-\partial_x^2 (t-s)} f(Q(x,s))ds
\]

\[
= Q_A(x,t) + U[f](x,t).
\]

According to J. Bourgain (private communication) the solution \( Q(x,t) \) will be continuous in space-time. The solution of the linear Airy equation satisfies

\[
|Q_A(x_1, t_1) - Q_A(x_2, t_2)| \leq \left( |x_1 - x_2|^{\beta_1} + |t_1 - t_2|^{\beta_2} \right)
\] (3)
with the optimal bounds $0 < \beta < \frac{1}{2}, \ 0 < \beta_2 < \frac{1}{7}$, and some random constant $K, EK^2 < \infty$, which depend on $\beta$'s. Nothing is known about smoothness of the nonlinear term $U[f](x, t)$. To get some idea consider the KdV equation. In symbolic form

$$
Q = Q_A + U[Q^2]
$$

$$
= Q_A + U[(Q_A + U[Q^2])^2]
$$

$$
= Q_A + U[Q^2_A] + U[2Q_AU[Q^2] + U^2[Q^2]]
$$

$$
= Q_A + U[Q^2_A] + \ldots
$$

Now look at $U[Q^2_A]$, the first term in the “approximation”. We will prove

$$
E(U[Q^2_A](x_1, t) - U[Q^2_A](x_2, t))^2 \leq C|x_1 - x_2|,
$$

$$
E(U[Q^2_A](x_1, t) - U[Q^2_A](x_2, t))^4 \leq C(\epsilon)|x_1 - x_2|^{2-\epsilon},
$$

for any $\epsilon > 0$. This indicates that $U[Q^2_A](\cdot, t)$ is $x$-continuous due to (5) by Kolmogoroff and similar to the Brownian motion because of (4). It is possible that in this case the local structure of the field $Q(x, t)$ depends on the nonlinear term $N(x, t)$.

To prove (4) and (5) we need Wick’s theorem, see [2].

**Theorem 2 (Wick)** Let $\xi_1, \xi_2, \ldots, \xi_{2n}$ are real or complex Gaussian variables with zero mean, then

$$
E\xi_1 \times \cdots \times \xi_{2n} = \frac{1}{2^n n!} \sum_\mu E\xi_{\mu_1} \xi_{\mu_2} \times \cdots \times E\xi_{\mu_{2n-1}} \xi_{\mu_{2n}},
$$

where summation is taken over the permutation group of $2n$ elements.

Let $Q_A(x, t) = \sum_{n \neq 0} e^{inx} e^{in^3 t} \hat{Q}_0(n)$ where $\hat{Q}_0(n)$ is a Gaussian complex isotropic variable, $\hat{Q}_0(n) = \hat{Q}_0(-n)$, $E|\hat{Q}_0(n)|^2 = \frac{1}{1 + n^2}$. Then

$$
U[Q^2_A](x, t) = \partial_s \int_0^t e^{-\partial_s^2(t-s)} Q^2_A(x, s) ds
$$

$$
= \sum_{n \neq 0} e^{inx} \sum_{n_1 + n_2 = n \neq 0} \hat{Q}_0(n_1) \hat{Q}_0(n_2) \frac{e^{i(n_1^3 + n_2^3)t} - e^{in^3t}}{n_1^3 + n_2^3 - n^3}.
$$

Using the arithmetical fact $n_1^3 + n_2^3 - n^3 = -3n_1 n_2$ we obtain

$$
U[Q^2_A](x, t) = \sum_{n \neq 0} e^{inx} \sum_{n_1 + n_2 = n \neq 0} \hat{Q}_0(n_1) \hat{Q}_0(n_2) M(n_1, n_2, t),
$$

where

$$
M(n_1, n_2, t) = \frac{e^{i(n_1^3 + n_2^3)t} - e^{in^3t}}{-3n_1 n_2}.
$$

Note $|M(n_1, n_2, t)| \leq 2$ if $n_1 n_2 \neq 0$. 
First, we prove that \( E(U[Q_A^2](x,t))^2 \) is finite. Using rotational invariance of the measure

\[
E(U[Q_A^2](x,t))^2 = E(U[Q_A^2](0,t))^2 = \sum_{n_1,n_2 \neq 0} \sum_{\substack{p_1+p_2=n_1 \atop p_3+p_4=n_2 \atop p_i \neq 0}} E\hat{Q}_0(p_1)\hat{Q}_0(p_2)\hat{Q}_0(p_3)\hat{Q}_0(p_4)M(p_1,p_2)M(p_3,p_4).
\]

By Wick’s rule

\[
E\hat{Q}_0(p_1)\hat{Q}_0(p_2)\hat{Q}_0(p_3)\hat{Q}_0(p_4) = \frac{1}{2^{2}\cdot 2!} \sum_{\mu} E\hat{Q}_0(p_{\mu_1})\hat{Q}_0(p_{\mu_2}) \times E\hat{Q}_0(p_{\mu_3})\hat{Q}_0(p_{\mu_4}).
\]

The sum vanishes unless \( n_1 = -n_2 \) and \( p_1 = -p_3, \ p_2 = -p_4 \) or \( p_1 = -p_4, \ p_2 = -p_3 \). Therefore

\[
E\hat{Q}_0(p_1)\hat{Q}_0(p_2)\hat{Q}_0(p_3)\hat{Q}_0(p_4) = \frac{1}{2^{2}\cdot 2!} \sum_{\mu} E\hat{Q}_0(p_{\mu_1})\hat{Q}_0(p_{\mu_2}) \times E\hat{Q}_0(p_{\mu_3})\hat{Q}_0(p_{\mu_4})
\]

\[
= E|\hat{Q}_0(p_1)|^2 E|\hat{Q}_0(p_2)|^2 = \begin{cases} \frac{1}{1+p_1^2} \frac{1}{1+p_2^2}, & \text{if } p_1 \neq p_2 \\ \frac{2}{1+p_1^2} \frac{1}{1+p_2^2}, & \text{if } p_1 = p_2 \end{cases}
\]

and

\[
E(U[Q_A^2](x,t))^2 \leq 2 \sum_{n \neq 0} \sum_{\substack{p_1+p_2=n \atop p_i \neq 0}} \frac{1}{1+p_1^2} \frac{1}{1+p_2^2} |M(p_1,p_2)|^2
\]

\[
\leq 8 \sum_{n \neq 0} \sum_{\substack{p_1+p_2=n \atop p_i \neq 0}} \frac{1}{1+p_1^2} \frac{1}{1+p_2^2} \leq c_1 \sum_{n \neq 0} \frac{1}{n^2} < \infty.
\]

In the last estimate we used

\[
\int_{-\infty}^{+\infty} \frac{1}{1+x^2} \frac{1}{1+(n-x)^2} dx = \frac{2\pi}{n^2 + 4}.
\]

The estimate for the second moment of the increment is similar.

\[
E(U[Q_A^2](h,t) - U[Q_A^2](0,t))^2 = \sum_{n_1,n_2 \neq 0} (e^{inh} - 1)(e^{inh} - 1) \times
\]

\[
\sum_{\substack{p_1+p_2=n_1 \atop p_3+p_4=n_2 \atop p_i \neq 0}} E\hat{Q}_0(p_1)\hat{Q}_0(p_2)\hat{Q}_0(p_3)\hat{Q}_0(p_4)M(p_1,p_2)M(p_3,p_4)
\]

\[
\leq 2 \sum_{n \neq 0} (e^{inh} - 1)(-e^{inh} - 1) \sum_{\substack{p_1+p_2=n \atop p_i \neq 0}} \frac{1}{1+p_1^2} \frac{1}{1+p_2^2} |M(p_1,p_2)|^2.
\]

Finally

\[
E(U[Q_A^2](h,t) - U[Q_A^2](0,t))^2 \leq c_2 \sum_{|n|<h^{-1}} \frac{|e^{inh} - 1|}{n^2} + c_3 \sum_{h^{-1} \leq |n|} \frac{1}{n^2} \leq c_4 h.
\]

The proof of (5) can be obtained by the same methods.
References

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