The Dynamics of Relativistic Membranes
II: Nonlinear Waves and Covariantly Reduced
Membrane Equations

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Abstract

By explicitly eliminating all gauge degrees of freedom in the 3 + 1-gauge description of a classical relativistic (open) membrane moving in \( \mathbb{R}^3 \) we derive a 2 + 1-dimensional nonlinear wave equation of Born-Infeld type for the graph \( z(t, x, y) \) which is invariant under the Poincaré group in four dimensions. Alternatively, we determine the world-volume of a membrane in a covariant way by the zeroes of a scalar field \( u(t, x, y, z) \) obeying a homogeneous Poincaré-invariant nonlinear wave-equation. This approach also gives a simple derivation of the nonlinear gas dynamic equation obtained in the light-cone gauge.
1. Generalizing the Nambu-Goto action to describe higher-dimensional relativistically invariant minimal surfaces in $D$-dimensional Minkowski space, the classical equations of motion constitute a set of $D$ coupled, highly nonlinear, partial differential equations for $D$ functions $x^\mu(\varphi^0, \ldots, \varphi^M)$ of $M + 1$ variables that parametrize the minimal surface. The solution of these equations is hindered by the large, non-explicit, redundancy resulting from the gauge degrees of freedom that reflect the invariance under the diffeomorphism group reparametrizing $(\varphi^0, \ldots, \varphi^M)$. Previous attempts to solve the equations have, so far, mainly been confined to the light cone description, due to the fact that one of the unknown functions can then be eliminated. In a recent paper [1] the constraints arising from this elimination were solved for the case $D = 4, M = 2$, and the theory was shown to be equivalent to a 2+1-dimensional inviscid irrotational isentropic gas whose pressure is inversely proportional to minus the gas density.

In this letter, we would like to consider the following, slightly more geometrical situation: choosing $\varphi^0 = x^0$, the time dependent shape of the $M$-dimensional surface, as seen by an observer with time $t = x^0$, is given by $x^\mu(t, \varphi^1, \ldots, \varphi^M)$, $i = 1, \ldots, D - 1$. A further gauge choice (noted already in [2]) allows one to assume that the $D - 1$-dimensional velocity vector $\dot{x}^i(t, \vec{\varphi})$ is normal to the surface at $(t, \vec{\varphi})$ and that the purely spatial part of the induced metric equals $1 - \dot{x}^2$ for all $t$. We shall refer to this kind of gauge (cf. [2]) as the “orthonormal 3+1-gauge”.

In the case $D = 4, M = 2$ we arrive at a first order vector equation for the spatial part $\vec{x}$ which in particular implies that $\dot{x}$ is proportional to the surface normal in $\mathbb{R}^3$ (if this dynamical proportionality factor was equal to $|\partial_1 \vec{x} \times \partial_2 \vec{x}|$ this equation would coincide with the $SU(\infty)$-Nahm-equation which was shown to be linearizable by Ward (cf. [3])). The vector equation may now be reduced by a hodograph technique similar to the one we applied to the equations of motion in the light cone gauge ([1]); finally we deduce a second order equation for $z = x^3$ as a function of $t$ and $(x^1, x^2)$ which can be derived from the Born-Infeld type Lagrangean $\sqrt{1 - \partial^\alpha z \partial_\alpha z}$. Despite its appearance as a covariant 2 + 1-dimensional equation it actually does have the full 3 + 1-dimensional Poincaré invariance.

In a second part of this letter we show that the equation for $z$ can be derived in a more direct way from a manifestly covariant nonlinear 3 + 1-dimensional wave equation for a scalar function $u(t, x^1, x^2, x^3)$ whose hypersurfaces $u = \text{const}$ are all minimal hypersurfaces, i.e. solutions of the membrane equation. The Lagrangean for $u$ is simply given by the homogeneous functional $\sqrt{(\partial u)^2}$ times the space-time volume. The derivation is valid for a curved background.

Finally, we show that the second order equation for the velocity potential $p$ coming from the reduction of the membrane equation in the light cone gauge (see [1]) can also be derived from the above-mentioned covariant equation for $u$. The Lagrangean for $p$ is given by $\sqrt{\dot{p} + \frac{1}{4}(\nabla p)^2}$. 
2. Let $\Sigma$ be an $M$-dimensional manifold with co-ordinates $(\varphi^1, \ldots, \varphi^M)$ and $(\mathcal{M}, \eta)$ be a $D$-dimensional (possibly curved) Lorentz manifold, and the action for the world-volume $x = (x^\mu): \mathbb{R} \times \Sigma \to \mathcal{M}$ of the surface $\Sigma$ moving in $\mathcal{M}$ be given by the $M+1$-dimensional volume swept out in space-time:

$$S[x] = \int d\varphi^0 d^M\varphi \sqrt{G};$$  \hspace{1cm} (1)

$G$ is $(-1)^M$ times the determinant of the induced world-volume metric $G_{\alpha\beta}$ (which we assume to be nondegenerate throughout this paper):

$$G_{\alpha\beta} \overset{\text{def}}{=} x^{\mu,\alpha} x^{\nu,\beta} \eta_{\mu\nu}(x).$$  \hspace{1cm} (2)

The index notation will always be $(\alpha, \beta, \gamma, \ldots = 0, 1, \ldots, M$ and $\lambda, \mu, \nu, \ldots = 0, 1, \ldots, D - 1)$, and as usual, a comma in front of an index denotes partial derivative with respect to the corresponding variable. The resulting field equations are

$$\frac{1}{\sqrt{G}}(\sqrt{G}G^{\alpha\beta}x_{\mu,\alpha})_{,\beta} + G^{\alpha\beta}x_{\nu,\alpha}x_{\rho,\beta}\Gamma_{\nu\rho}^\mu(x) = 0,$$  \hspace{1cm} (3)

where $\Gamma_{\nu\rho}^\mu$ denote the Christoffel symbols of the metric $\eta$ (which vanish if $\mathcal{M}$ is flat Minkowski space and $x^\mu$ are the standard co-ordinates). The following alternative formulation of the field equations turns out to be useful later: Let

$$(\Pi_{\perp})^\mu_{\nu}(x) \overset{\text{def}}{=} \delta^\mu_{\nu} - x^{\mu,\alpha} x^{\nu,\beta} G_{\alpha\beta}(x) \eta_{\mu\nu}(x)$$  \hspace{1cm} (4)

be the orthogonal projection on the subspace orthogonal to the tangent space of the world-volume in $\mathcal{M}$. Eqn (3) may then be written as (cf. e.g. [3], p. 178):

$$(\Pi_{\perp})^\mu_{\nu}(x)(x_{\nu,\alpha\beta} + \Gamma_{\rho\sigma}^\nu(x)x^{\rho,\alpha}x^{\sigma,\beta})G^{\alpha\beta} = 0,$$  \hspace{1cm} (5)

showing that only the equations along the orthogonal components are truly dynamical (in particular, there would be no such equation if $M + 1 = D$). Both (3) and (5) are invariant under both arbitrary reparametrizations $\mathbb{R} \times \Sigma \to \mathbb{R} \times \Sigma$ of the world-volume and isometries $\mathcal{M} \to \mathcal{M}$.

3. Let us now reduce the dynamical equations (3) for a two-dimensional membrane moving in four-dimensional Minkowski space $\mathbb{R}^{(1,3)}$ by first choosing $\varphi^0 = x^0 \overset{\text{def}}{=} t$ ("3+1-gauge"). Denoting by $\vec{x}$ the spatial part of the four-vector $x$ we get $(r, s = 1, 2)$

$$G_{\alpha\beta} = \begin{pmatrix}
1 - \dot{\vec{x}}^2 & -\vec{x} \cdot \vec{x}_r \\
-\vec{x} \cdot \vec{x}_r & -\vec{x}_r \cdot \vec{x}_s
\end{pmatrix}$$  \hspace{1cm} (6)

and

$$G = (\vec{x}_1 \times \vec{x}_2)^2 - (\dot{\vec{x}} \cdot (\vec{x}_1 \times \vec{x}_2))^2.$$  \hspace{1cm} (7)
One then observes that the spatial part of the field equations (3)\(\mu = i = 1, 2, 3\) is orthogonal to both tangent vectors \(\vec{x}, 1\) and \(\vec{x}, 2\), and that its component normal to the embedded surface coincides with the temporal part of eqn (3)\(\mu = 0\), which in the gauge (3, p. 149)

\[
\dot{x} \cdot \vec{x}, r = 0 \quad \text{for} \quad r = 1, 2 \tag{8}
\]

reads

\[
\frac{\partial}{\partial t} \left( \sqrt{\frac{g}{1 - \dot{x}^2}} \right) = 0 \tag{9}
\]

where \(g\) denotes the determinant of the spatial metric \(g_{rs} \overset{\text{def}}{=} \vec{x}, r \cdot \vec{x}, s,\)

\[
g = (\vec{x}, 1 \times \vec{x}, 2)^2 \tag{10}
\]

As the gauge choice (8) still leaves the freedom of performing time-independent reparametrisations \(\Sigma \to \Sigma\), one may choose \(g\) (at a particular time) to be actually equal to \(1 - \dot{x}^2\) (3). Equation (8) then implies that this choice is preserved in time, thus eliminating the arbitrary time-independent function \(\Sigma \to \mathbb{R}\) allowed by (8). Combining this choice and the gauge (8) to a single first order vector equation one gets

\[
\dot{\vec{x}} = \sqrt{\frac{1}{(\vec{x}, 1 \times \vec{x}, 2)^2}} - 1 (\vec{x}, 1 \times \vec{x}, 2) \tag{11}
\]

Written in the form

\[
\dot{x}_i = \frac{\gamma}{2} \epsilon_{ijk} \{x^j, x^k\} \tag{12}
\]

(where \(\gamma = \sqrt{\frac{1}{g}} - 1\) and \(\{,\}\) is the Poisson bracket with respect to the variables \(\varphi^1\) and \(\varphi^2\) on \(\Sigma\)) this equation would be equal to the \(su(\infty)\)-Nahm equation (known to be integrable [5]) if the prefactor \(\frac{\gamma}{2}\) was 1.

Equations (12) are still invariant under area-preserving reparametrizations \(\Sigma \to \Sigma\) (i.e. reparametrizations whose Jacobi determinant is equal to 1). In order to remove this residual degree of gauge freedom let us give a description of the two-dimensional surface at a given time in terms of the graph \((x^1, x^2, z(t, x^1, x^2))\), rather than in terms of the three Cartesian co-ordinates as functions of the “fictitious” parameters \((\varphi_1, \varphi_2)\). We therefore perform the following change of variables (which had already turned out to be useful in the light-cone description (cf. [3])):

\[
(\varphi^0 = t, \varphi^1, \varphi^2) \mapsto (t = x^0, x^1(t, \varphi), x^2(t, \varphi)) \tag{13}
\]

where we have set \(\varphi = (\varphi^1, \varphi^2)\). Observing that the Jacobi determinant \(J\) of this transformation is equal to the Poisson bracket \{\(x^1, x^2\)\} and that (due to
eqn (12), \( i = 1, 2 \) the “old” time derivative \( \frac{\partial}{\partial t} \) becomes \( \frac{\partial}{\partial x_0} - \gamma J \nabla z \cdot \nabla \), the \( i = 3 \) part of eqn (12) becomes (where ‘ from now on denotes the “new” time derivative and \( \nabla \) stands for the gradient with respect to \( x^1 \) and \( x^2 \)):

\[
\dot{z} = \gamma J (1 + (\nabla z)^2),
\]

(14)

while the two equations corresponding to \( i = 1, 2 \) in eqn (12) imply

\[
\dot{J} = -J^2 \nabla \cdot (\gamma \nabla z);
\]

(15)

in the new variables \( \gamma \) can be expressed as

\[
\gamma = \sqrt{J^{-2}(1 + (\nabla z)^2)^{-1} - 1}.
\]

(16)

Rewriting eqn (15) in a form involving only \( z \), its derivatives, and \( \gamma J \) (for which we can use eqn (14)) we get the following second order equation for \( z \):

\[
\ddot{z} - \nabla^2 z = (\nabla z)^2 (-\dot{z} + \nabla^2 z) - \frac{1}{2} \nabla z \cdot \nabla (\nabla z)^2 + \nabla \dot{z}^2 \cdot \nabla z - \dot{z}^2 \nabla^2 z.
\]

(17)

This is nothing but the Euler-Lagrange equation for a 2+1-dimensional scalar field theory described by a Lagrangean density

\[
\mathcal{L} \overset{\text{def}}{=} -\sqrt{1 - \dot{z}^2 + (\nabla z)^2} = -\sqrt{1 - z_{\alpha} z^{\alpha}};
\]

(18)

here the \( \alpha \) indices range over 3-dimensional Minkowski space \( \mathbb{R}^{(1,2)} \). The corresponding Hamiltonian is given by

\[
H = \int d^2x \sqrt{1 + \pi^2} \sqrt{1 + (\nabla z)^2}
\]

(19)

where \( \pi = \delta \mathcal{L}/\delta \dot{z} \) is the momentum conjugate to the field \( z \). Note that an expansion of the square root in eqn (19) will give a free field theory in lowest nonconstant order. Long ago, such nonlinear field theories had been investigated by Born and Infeld (3) and Heisenberg (7). Later on, in 8, the integrability of the 1+1-dimensional theory (\( z(t, x) \)) was noted (which is clear by the hodograph trick, compare 3 or 10, p. 617), and a quantum theory was developed (cf. 8). While the \( SO(1,2) \)-invariance of the Lagrangean (18) and the field equations (17) is obvious,

\[
(1 - z^{\alpha} z_{\alpha}) z^{\beta}_{,\beta} + z^{\alpha} z^{\beta} z_{,\alpha \beta} = 0,
\]

(20)

it was apparently not noticed that the theory has a much wider dimensional invariance group. In order to see the invariance under the Lorentz group \( SO(1,3) \) one remembers that \( z \) is the 3-component of a four-vector,
\[ x^\mu = (t, x^1, x^2, z(t, x^1, x^2))^T, \]

and transforms this four-vector in the usual way by a Lorentz transformation \( \Lambda^\mu_\nu \), i. e.

\[
\tilde{x}^\mu = (\tilde{t}, \tilde{x}^1, \tilde{x}^2, \tilde{z}(\tilde{t}, \tilde{x}^1, \tilde{x}^2))^T = \Lambda^\mu_\nu x^\nu ,
\] (21)

implicitly determining the function \( \tilde{z} \). The infinitesimal action of the Lie algebra of the Lorentz group (obtained by setting \( \Lambda = e^{\epsilon K} \) and differentiating eqn (21) w. r. t. \( \epsilon \) at \( \epsilon = 0 \)) reads:

\[
(\delta_K z)(t, x^1, x^2) = -\partial_\alpha z(t, x^1, x^2)(K_\alpha^\beta x^\beta + K_3^\alpha z(t, x^1, x^2)) + K_3^\beta x^\beta. \] (22)

The infinitesimal boost in the \( z \)-direction, for instance, results in

\[
(\delta_K z)(t, x^1, x^2) = -\partial_t z(t, x^1, x^2) + t \text{ whereas an infinitesimal rotation in the } x^1 - z\text{-plane gives}
\]

\[
(\delta_K z)(t, x^1, x^2) = -\partial_1 z(t, x^1, x^2) x^1. \]

In order to see that the transformation (22) is a symmetry of eqn (20) one linearizes the l. h. s. of (20) getting a linear equation for \( (\delta_K z)(t, x^1, x^2) \) with coefficients depending on \( z \) which gives zero after inserting (22) and using the fact that \( z \) solves (20).

It is also not difficult to write down the corresponding conserved charges, e. g. (for the boost in the \( z \)-direction):

\[
Q_{tz} \overset{\text{def}}{=} t \int d^2 x \pi - \int d^2 x \ z \sqrt{1 + (\nabla z)^2} \sqrt{1 + \pi^2}.
\] (23)

Along these lines one can verify that

\[
M^2 \overset{\text{def}}{=} H^2 - \left( \int d^2 x \pi \nabla z \right)^2 - \left( \int d^2 x \pi \right)^2
\] (24)

is Poincaré-invariant.

4. We shall now deduce eqn (20) from a single four-dimensional scalar equation:

The crucial observation is that the world-volume of a two-dimensional membrane moving through four-dimensional Minkowski space is a minimal hypersurface (i. e. of codimension 1) and that hypersurfaces may alternatively be described by the zero set of a smooth function (an idea which is already mentioned, though not pursued, in Dirac’s paper \( \text{[1]} \)). Considering again the case of a curved Lorentz manifold \( (M, \eta) \) of arbitrary dimension, suppose that there is a smooth function \( u : M \rightarrow \mathbb{R} \) such that its zero set \( \{ r \in M | u(r) = 0 \} \) is a minimal hypersurface. Parametrizing this hypersurface by \( (\varphi^0, \ldots, \varphi^M) \) means that

\[
u(x(\varphi)) = 0 \quad ,
\] (25)

which by differentiating with respect to the parameters \( \varphi^\alpha \) implies:

\[
u_{\mu}(x)x^\mu,\alpha = 0 \quad .
\] (26)

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Assuming that $\eta^\mu\nu(x)u_{\mu}(x)u_{\nu}(x) \neq 0$ we immediately get a formula for the projector $[\Pi]_\nu^\mu(x) = \delta_\nu^\mu - x^\rho,\alpha x^\rho,\beta G^{\alpha\beta} \eta_{\mu\nu}(x) = \frac{\eta^\mu\nu(x)u_{\rho}(x)u_{\nu}(x)}{\eta^\kappa\lambda(x)u_{\kappa}(x)u_{\lambda}(x)}$.

(27)

Now we can reformulate the field equations in its projector form (28):

\begin{align*}
0 &= u,_{\kappa}(x)(\eta^\mu\nu(x) - (\Pi)_{\mu\nu}(x))
\end{align*}

(28)

where the semicolon $;$ denotes the covariant derivative with respect to the metric $\eta$. Using eqn (27) this finally results in

\begin{align*}
0 &= (\partial u)^2 \Box u - w^{\kappa\lambda}u,_{\kappa\lambda}
\end{align*}

(28)

where $(\partial u)^2$ stands for $\eta^\mu\nu u_{\mu,\nu}$. A priori, this equation holds on the world-volume $u = 0$ only.

However, it can be shown that locally one gets no restrictions when eqn (28) is allowed to hold outside the world-volume: Firstly, we easily see that this equation is then invariant under the group of isometries of $(M, \eta)$. Moreover, the above derivation shows that every hypersurface $u = const$ is a minimal hypersurface if $(\partial u)^2 \neq 0$, because eqn (24) is still valid. Consequently, every solution of (28) regarded on all of $M$ results in a foliation of $M$ into minimal hypersurfaces. In fact, such a foliation exists as a local extension of any given local minimal hypersurface at least for homogeneous $M$ (such as e. g. Minkowski space). Suppose one is given a small piece of a minimal hypersurface in $M$. Then (after shrinking the piece if necessary) there exists a Killing field of the metric $\eta$ which is nowhere tangent to this surface patch. The translates by the flow of the Killing field of the surface patch will all be minimal hypersurfaces, hence a small open neighbourhood of the initial surface patch is indeed foliated into minimal hypersurfaces. Any (locally defined) function $u$ "counting" the leaves of this foliation satisfies eqn (28).

A rather elementary example is provided by the world-volume $(t, x, y) \mapsto (t, x, y, 0)$ of the open $\mathbb{R}^2$-type membrane. Clearly, this is the zero set of the linear function $u(x) = z$ which obviously satisfies the nonlinear wave equation (28). But also the affine hyperplanes $u = z_0 \neq 0$ are minimal world-volumes (being translates along the $z$-direction).

Note that eqn (28) is also invariant under redefinitions $u \mapsto s(u)$ where $s$ is any smooth real-valued function of $\mathbb{R}$.

As can easily be computed, eqn (28) can be derived from the Lagrangean

\[ L \overset{\text{def}}{=} \sqrt{\det(\eta^\mu\nu)\eta^\kappa\lambda u_{\kappa\lambda}}. \]
This Lagrangean is homogeneous of weight one and (for flat \(\mathcal{M}\)) therefore falls into the class of Lagrangeans considered by Fairlie et al (\cite{12}). This means in particular that any solution of the field equation (28) is automatically a solution of the 4-dimensional universal field equation

\[ u^{\mu\nu} (\partial^2 u)^{-1}_{\mu\nu} = 0 \]  

(30)

whose general solution is known (again by a hodograph transformation, cf. \cite{13}).

In order to see how equation (20) can be derived from eqn (28) consider again four-dimensional Minkowski space \(\mathbb{R}^{(1,3)}\). Take a solution \(u\) of the above field equation (28) and consider

\[ 0 = u(t, x^1, x^2, z(t, x^1, x^2)) \]  

(31)

which implicitly defines the scalar function \(z(t, x^1, x^2)\). Differentiating this equation once with respect to \((t, x^1, x^2)\) one can express the first derivatives of \(z\) and \(u\) in terms of first derivatives of \(u\):

\[ z,\alpha = -u,\alpha / u,3 \]  

while differentiating twice one can also express second derivatives of \(z\) by first and second derivatives of \(u\). After some calculation one finds that the above-defined function \(z\) indeed satisfies the field equations (20) as a consequence of eqn (28).

5. In \cite{1} we had derived equations of the following isentropic irrotational inviscid gas-dynamic type using the light-cone gauge: \(\dot{q} + \nabla \cdot (q \nabla p) = 0\) and \(\dot{p} + \frac{1}{2} (\nabla p)^2 - F(q) = 0\) where \(q\) and \(p\) are functions of time and \(M\) spatial variables and \(F\) is a monotonous function \(\mathbb{R} \to \mathbb{R}\) such that the pressure \(P\) depends via \(P'(q) = -qF'(q)\) on the gas density \(q\). In our case of four-dimensional Minkowski space we had \(M = 2\) and \(F(q) = \frac{1}{2}q^{-2}\). Solving the second equation for \(q\) and inserting the result in the first one gets a second order equation for the velocity potential \(p\) alone which is apparently called “Steichen equation” (cf. \cite{4}, p. 45). A little computation reveals that this equation can be derived from the Lagrangean \(\mathcal{L} \overset{\text{def}}{=} h(\dot{p} + \frac{1}{2}(\nabla p)^2)\) where the real-valued function \(h\) of one real variable is such that its derivative equals the inverse function of \(F\). In particular, for an adiabatic relation like \(P(q) = aq^\gamma\) with a nonzero \(a\) and \(0 \neq \gamma \neq -1\) one would arrive at \(\mathcal{L} = (\dot{p} + \frac{1}{2}(\nabla p)^2)^{\frac{1}{1-\gamma}}\) up to a constant prefactor. For the membrane one has \(\gamma = -1\), hence

\[ \mathcal{L} = \sqrt{\dot{p} + \frac{1}{2}(\nabla p)^2} \]  

(32)

This leads to the following equation for the membrane:

\[ \ddot{p} + 2\nabla p \cdot \dot{\nabla p} + \nabla p \cdot (\nabla p \cdot \nabla) \nabla p) - 2(\dot{p} + \frac{1}{2}(\nabla p)^2)\nabla^2 p = 0 \]  

(33)

This equation can also be derived from the covariant scalar field equation (28): set

\[ 0 = u(t + \frac{p(t, x^1, x^2)}{2}, x^1, x^2, t - \frac{p(t, x^1, x^2)}{2}) \]  

(34)
Again the first and second derivatives of $p$ can be expressed in terms of first and second derivatives of $u$ by differentiating this equation with respect to $(t, x^1, x^2)$ getting for instance $\dot{p} = -2(\partial_{0}u + \partial_{3}u)/(\partial_{0}u - \partial_{3}u)$, $p, i = -2u, i / (\partial_{0}u - \partial_{3}u)$.

Note that – as in the case for the function $z$ – one can deduce the Lorentz symmetry of eqn (33) (cp. [1], eqs (44)–(47) for the Hamiltonian treatment) by transforming the four-vector $(t + \frac{p(t, x^1, x^2)}{2}, x^1, x^2, t - \frac{p(t, x^1, x^2)}{2})^\tau$ by an arbitrary Lorentz transformation $\Lambda_{\mu}^{\nu}$. The Poincaré invariance, and the scaling symmetries of eqn (33), has also been obtained by computer algebraic calculations (cf. [14]).

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