Quantum Theory of Tensionless Noncommutative $p$-Branes

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The quantum theory involving noncommutative tensionless $p$-branes is studied following path integral methods. Our procedure allow a simple treatment for generally covariant noncommutative extended systems and it contains, as a particular case, the thermodynamics and the quantum tensionless string theory. The effect induced by noncommutativity in the field space is to produce a confinement among pairing of null $p$-branes.

I. INTRODUCTION

Tensionless strings are extended objects discovered by Schild many years ago [1] and it corresponds –formally– to a set of infinite massless relativistic particles satisfying the constraint

$$\mathcal{H}_1 = p_\mu x'\mu,$$

where $x^\mu$ transform as a scalar on the volume world and the spacetime index $\mu$ runs over $0, 1, 2, ..., D - 1$ with $D$, the spacetime dimension [23].

Physically speaking this problem is related to the behavior of string theory at very high energy [8], also known as the strong coupling limit. More exactly, when the Regge slope goes to infinite, the spectrum of the string theory is massless; the situation is similar to what occurs in the standard model before the gauge symmetry is broken.

In string theory, however, the situation is quite involved because the “gauge group” corresponds to diffeomorphisms and the gauge group –in this case is infinite dimensional. As in the standard string theory, the critical dimensions for space time will be 26 (or 10) depending on the bosonic (or fermionic)character [4] as in the tensionful case.

The general case (that is $p$-branes), concerning to the critical dimensions, only partial results are known and, probably, they are not definitive [3].

The fact that null strings exist at very high energies, in the sense previously explained, together with the possibility that the Lorentz invariance could be deformed or even broken at such energies [1, 3, 4, 5, 6, 7, 8] rise the question of how to study the effect of such deformation in the null string scenario.

There are several proposals to formulate such Lorentz invariance deformations. They can be classified in two groups depending on the existence of a preferred reference frame. For such a case there is a proposal [11, 11] based in the deformation of the commutators between fields which is appropriated to discuss the question previously formulated.

However, one could argue that such deformation should not have any impact in the sense of a measurable consequence. That is, the deformation at the level of just one string or $p$-brane, could be washed out. Then, there should be an amplifier mechanism.

Because of this, instead of considering just one $p$-brane, turns out to be more interesting to consider a gas of such objects and to study the thermodynamics of such system. Then, one can introduce the noncommutative fields and explore the amplified consequences of it.

The purpose of the present paper is to study the properties of these extended object described by noncommutative fields and as well as some statistical mechanics of null $p$-branes issues that include the null string is as a particular case.

The paper is organized as follow: in section II we consider relativistic particles in a noncommutative space where several quantum statistical mechanics considerations are studied. In section III, the previous results are extended non-commutative $p$-dimensional null branes and its quantum statistical mechanics properties described. In section IV, we discuss the previous results as a possible interaction mechanism and also we present our main conclusions. An appendix including the statistical mechanics for free relativistic particles is considered and the nonexistence of Matsubara modes for this system is established.

II. NONCOMMUTATIVE RELATIVISTIC QUANTUM MECHANICS

In this section we will construct noncommutative versions of generally covariant systems. We will start considering, firstly, the relativistic particle on a $D$-dimensional spacetime and later– in the next section – we will extend our results to tensionless strings and membranes.

A. Relativistic free particle and the proper-time gauge

There are many approaches to discuss relativistic quantum mechanics of a free particle. One of them is the so called proper-time method, which was used in the early 50th in connection with quantum electrodynamics [12]. The idea is to consider a particle in a $D+1$-dimensional Euclidean spacetime.
The diffusion equation for such system is
\[-\frac{1}{2} \Box \varphi(x, s) = \frac{\partial \varphi}{\partial s}, \tag{1}\]
where \(\Box\) is the \(D\)-dimensional Laplacian.

Then, using the ansatz
\[\varphi(x, s) = e^{-\frac{m^2}{2} s} \phi(x), \tag{2}\]
one finds that \(\phi(x)\) satisfies the Klein-Gordon equation if \(m\) is the mass of the particle.

In this approach, the propagation amplitude is given by the Laplace transform
\[G[x, x' ; m^2] = \int_0^\infty ds e^{-\frac{m^2}{2} s} G[x, x' ; s], \tag{3}\]
where
\[G[x, x' ; s] = \int D x \ e^{-\int_0^1 dt \frac{s^2}{2^D}}, \tag{4}\]
\[= s^{-D/2} e^{-\frac{\langle s \theta^2 \rangle}{2}}. \tag{5}\]

From this one obtains the partition function for a gas of \(N\) free relativistic particles
\[Z_s = \left(\text{Tr} \left[ e^{-\frac{m^2}{2} s} G[x, x' ; s] \right] \right)^N, \tag{6}\]
or equivalently
\[\ln Z = N \left[ -\frac{m^2}{2} s - \frac{D}{2} \ln s + \ln \mathcal{V} \right], \tag{7}\]
where \(\mathcal{V} = V \times \text{const.}\) is the \(D\)-dimensional spacetime, \(V\) is the \(D\) \(-1\)-dimensional ordinary spatial volume and \(s\) plays the role of \(\beta = 1/kT\).

B. The relativistic particle in a noncommutative space

Equation (1) suggests a simple way to extend the problem to a gas of relativistic particles on a noncommutative space.

Indeed, from (1) we see that the Hamiltonian for a relativistic particle is
\[\hat{H} = \frac{1}{2} p^2. \tag{8}\]

Once (7) is given, noncommutativity is implemented through the deformed algebra
\[[x_\mu, x_\nu] = i \theta_{\mu\nu}, \quad [p_\mu, p_\nu] = i B_{\mu\nu}, \tag{9}\]
where \(\theta_{\mu\nu}\) and \(B_{\mu\nu}\) are the deformation parameters in the phase space.

For convenience we choose
\[\theta_{i0} = 0, \quad \theta_{ij} = \epsilon_{ij} \theta, \quad B_{i0} = 0, \quad B_{ij} = \epsilon_{ij} B. \tag{10}\]

Therefore, the equation of motion for this particle is
\[\dot{x}_\mu = p_\mu, \quad \dot{p}_i = \epsilon_{ij} B p_j. \tag{11}\]

These equations can be integrated directly by using (10) and (11). Indeed, one of the equations is trivial, namely, the energy conservation condition \((\dot{p}_0 = 0)\). Note that the symmetric gauge we have chosen, implies that non-commutativity is realized only for the first two momenta and coordinates components. The other components are treated as usual. In principle, we could extend this hypothesis taking also other pairs of momenta and coordinates components, but this is not essential for our discussion.

Keeping this in mind, the remaining equations have the solution
\[p_1 = \frac{1}{2} (\alpha e^{-i B t} + \alpha^\dagger e^{i B t}), \tag{12}\]
\[p_2 = \frac{1}{2i} (\alpha e^{-i B t} - \alpha^\dagger e^{i B t}), \tag{13}\]
where \(\alpha\)'s are constant operators.

The coordinates \(x_{1,2}\) are obtained in a similar way using (12), i.e.
\[x_1 = \frac{1}{2i B} (\alpha^\dagger e^{i B t} - \alpha e^{-i B t}) + x_{01}, \tag{14}\]
\[x_2 = \frac{1}{2i B} (\alpha e^{-i B t} + \alpha^\dagger e^{i B t}) + x_{02}. \tag{15}\]

From the commutation relation of \(p\)'s, we see that it is possible to define operators \(a\) and \(a^\dagger\) satisfying the algebra
\[[a, a^\dagger] = 0 = [a^\dagger, a^\dagger], \tag{16}\]
\[[a, a] = 1, \tag{17}\]
where
\[\alpha \rightarrow \sqrt{B} a, \quad \alpha^\dagger \rightarrow \sqrt{B} a^\dagger. \tag{18}\]

The equations of motion –as a second order equation system– are
\[\dot{x}_\mu = B_{\mu\nu} \dot{x}_\nu, \tag{19}\]
which can be solved by the Ansatz \(x_\mu = a_\mu e^{i \omega s}\).

The last equation is
\[(i \omega \delta_{\mu\nu} - B_{\mu\nu}) a_\nu = 0. \tag{20}\]

Therefore, the dispersion relation for this system is
\[\omega_\pm = \left\{ \pm B \right\}; \tag{21}\]
and since one of the eigenvalues vanishes the Hamiltonian spectrum is degenerated.

Thus, the Hamiltonian for a relativistic particle living on a noncommutative space is

$$H = \frac{B}{2} \left( a^2 a + \frac{1}{2} \right) + \frac{1}{2} \sum_{n=1}^{D-3} (p^2)_n.$$  \hspace{1cm} (17)

Finally, the statistical mechanics for a gas of $N$ relativistic particles on a noncommutative space, in the symmetric gauge, is obtained from the partition function

$$Z_s = \left( s^{-\frac{D-3}{2}} e^{-\frac{m^2}{2} \sum_{n=0}^{\infty} g_0 e^{-s \frac{Q}{2} (n+\frac{1}{2})}} \right)^N = \left[ g_0 e^{-\frac{m^2}{2} s} s^{-\frac{D-3}{2}} \sinh \left( \frac{Q s}{2} \right) \right]^N, \hspace{1cm} (18)$$

where $g_0$ is the degeneracy factor due to the zero eigenvalue of the Hamiltonian.

The thermodynamic properties of this system can be computed directly from (18).

III. THE STRONG COUPLING REGIME FOR MEMBRANES IN NONCOMMUTATIVE SPACES

In this section we will discuss the extension of the previous problem to membranes moving on a noncommutative space in the strong coupling regime.

A relativistic membrane is a $p$-dimensional object embedded on a $D$-dimensional flat spacetime and described by the lagrangean density

$$\mathcal{L} = \frac{T}{2} \sqrt{g^{(p+1)} g_{\alpha \beta} G_{\mu \nu} \partial^\alpha x^\mu \partial^\beta x^\nu - (p-1)},$$

where $g^{(p+1)}_{\alpha \beta}$ is a metric tensor on the world-volume, $a G^{\mu \nu}$ is the metric tensor where the $p$-brane is embedded with $\mu, \nu = 0, 1, 2, ..., D$ and $T$ the superficial tension.

The Hamiltonian analysis yields to the following constraints

$$H_\perp = \frac{1}{2} p^2, \hspace{1cm}$$

$$H_i = p_i \partial_i x^\mu, \hspace{1cm}$$

where $g^{(p)}$ is the spatial metric determinant and $T$ is the superficial tension.

The strong coupling regime corresponds to $T \to 0$ and, in this limit the constraints are

$$H_\perp = \frac{1}{2} p^2, \hspace{1cm}$$

$$H_i = p_i \partial_i x^\mu, \hspace{1cm}$$

and the membrane becomes an infinite set of free massless relativistic particles moving perpendicularly to the $p$-dimensional surface.

In the special case of the tensionless string ($p = 1$), each point of the string is associated with a massless relativistic particle and, as a consequence, all the points of the string are causally disconnected.

In this tensionless string approach the field $x^{\mu}(\sigma, \tau)$ is replaced by $x^{\mu}_i(\tau)$, where $i = 1, 2, ...,$, is an infinite countable set labeling each point of the tensionless string.

Using this philosophy, we will start constructing tensionless strings.

A. Tensionless strings from particles

Let us start by noticing that a tensionless string is made up of infinite massless relativistic particles causally disconnected and, therefore, instead of (11) one have

$$-\frac{1}{2} \square \varphi_1(x, s_1) = \frac{\partial \varphi_1}{\partial s_1},$$

$$-\frac{1}{2} \square \varphi_2(x, s_2) = \frac{\partial \varphi_2}{\partial s_2},$$

$$\vdots,$$

$$-\frac{1}{2} \square \varphi_k(x, s_k) = \frac{\partial \varphi_k}{\partial s_k}.$$  \hspace{1cm} (23)

These equations can be solved by generalizing the Ansatz (2), i.e

$$\varphi(x_1, \ldots, x_k; s_1, \ldots, s_k) = \prod_{i=1}^{\infty} e^{-\frac{m^2}{2} s_i} \phi(x_i),$$

where $m^2$ is an infrared regulator that will vanish at the end of the calculation.

The limit of an infinite number of particles is delicate but here – formally – one can take this limit, simply, assuming that in the continuous limit one can replace the set $\{i\}$ by an integral in $\sigma$ and, as a consequence, the propagation amplitude can be written as:

$$G[x(\sigma), x'(\sigma)] = \int_0^{\infty} \mathcal{D}s(\sigma) e^{-\frac{m^2}{2} \int ds(\sigma) G[x(\sigma), x'(\sigma); s(\sigma)]},$$

where $G[x(\sigma), x'(\sigma); s(\sigma)]$ is given by

$$G[x(\sigma), x'(\sigma); s(\sigma)] = s^{-D/2} e^{-\int ds(\frac{d^2 x(\sigma)}{d \sigma})^2}. \hspace{1cm} (26)$$

The formula (26) generalizes the proper-time method to the tensionless string case. Probably this approach to string theory was first used by Eguchi in (3).

Using (26) and (27), the partition function of an $N$ tensionless string gas is

$$Z[s(\sigma)] = \left[ \int \mathcal{D}x(\sigma) G[x(\sigma), x(\sigma); s(\sigma)] \right]^N = \left( s^{-D/2} e^{-\int ds(\frac{d^2 x}{d \sigma})^2} \right)^N. \hspace{1cm} (27)$$
This partition function reproduces correctly the results for the thermodynamics of a tensionless string gas\cite{13}. Indeed, from\cite{24}, the Helmholtz free energy is
\[
F[s] = \frac{N}{s(\sigma)} \left[\frac{D}{2} \ln(s(\sigma)) + \frac{m^2}{2} \int d\sigma s(\sigma) + \ln(V)\right].
\]

As 1/s is the temperature, then from the limit \(m^2 \to 0\) we see that \(F/T \sim \ln(T)\), again in agreement with other null string calculations\cite{13,14}.

From the last equation one obtain that
\[
P[s(\sigma)] V = \frac{N}{s(\sigma)},
\]
(28)
is the state equation for an ideal tensionless string gas.

B. Tensionless membranes from tensionless strings

In order to construct tensionless membranes, we begin by considering a membrane as an infinite collection of tensionless strings. Thus, if the membrane is a p-dimensional object, with local coordinates \((\sigma_1, \ldots, \sigma_p)\), then the propagation amplitude, formally, corresponds to\cite{25}, with the substitution
\[
\sigma \to (\sigma_1, \ldots, \sigma_p).
\]
Therefore, the partition function for a gas of \(N\) tensionless membranes is
\[
Z[s(\sigma)] = \left[\lim_{n \to \infty} \left(s(\sigma)^{-D/2} e^{-\frac{n^2}{2} \int d\sigma s(\sigma)}\right)^n\right]^N,
\]
(29)
where \(n\) is the number of tensionless strings.

One should note here that the expression
\[
\left(s(\sigma)^{-D/2} e^{-\frac{n^2}{2} \int d\sigma s(\sigma)}\right)^n,
\]
formally emphasizes that a tensionless p-branes is made-up of \(n\) tensionless strings.

However, this last expression was computed in\cite{25}, and in our case is
\[
\prod_{i=1}^{p} [s(\sigma_i)^{-D/2} e^{-\frac{n^2}{2} \int d\sigma s(\sigma_i)}],
\]
then, the total partition function for an ideal gas of \(N\) tensionless p-branes is given by
\[
Z = \prod_{i=1}^{p} \left(s(\sigma_i)^{-D/2} e^{-\frac{n^2}{2} \int d\sigma s(\sigma_i)}\right)^N.
\]
In order to compute the state equation we proceed as follow: firstly one chooses \(s(\sigma_1) = s(\sigma_2) = \ldots = s(\sigma)\) and one put also \(m_1 = m_2 = \ldots = m\), then
\[
P[s(\sigma)] V = \frac{N}{s(\sigma)}.
\]
(30)
The Helmholtz free energy, compared to the tensionless string case, has a different behavior. Indeed, the Helmholtz free energy becomes
\[
F[s] = \frac{pN}{s(\sigma)} \left[\frac{D}{2} \ln(p s(\sigma)) + \frac{m^2}{2} \int d\sigma s(\sigma) + \ln(V)\right].
\]
and for \(s \to \infty\), one has that the quantity \(sF \sim \frac{D}{2} \ln[p s]\) is similar to the string case but, in this case \(p\) could smooth out the behaviour of \(sF\).

C. Including noncommutativity in Tensionless p-branes

Using the previous results, we can generalize our arguments in order to include noncommutativity in tensionless p-branes. In order to do that, one start considering a tensionless p-brane described by the field \(x_i^p(\tau)\) with \(i\) labeling the dependence in \((\sigma_1, \sigma_2, \ldots, \sigma_p)\). This field transforms as a scalar on the world-volume but as a vector in the space where the p-brane is embedded.

Let us suppose that the components –we say \(x_i^{D-1}\) and \(x_i^{D-2}\)– do not commute, then in such case the Green function can be written as
\[
G[x(\sigma), x'(\sigma); s(\sigma)] = \int_0^\infty ds e^{-\frac{s}{2} \int d^D x_i e^{-\frac{1}{2s} \int d\tau x_i^\mu}}
\]
\[
\times \int D x_i^{(D-2)} D x_i^{(D-1)} e^{-\int d\tau x_i^\mu} (\dot{x}_i^{(D-2)})^2 + (\dot{x}_i^{(D-1)})^2.
\]
(31)
The integral in the second line in the RHS, corresponds formally to a non-relativistic particle with mass \((s^{-1})\) moving in plane in the presence of a constant perpendicular magnetic field \(B\). In the first line in the RHS, however, the integral formally correspond to the Green function for a set of \(p\) free relativistic particles moving in \((D-3)\)-dimensional spacetime.

Thus, the calculation of these integral is straightforward. Indeed,
\[
G[x(\sigma), x'(\sigma)] = \int_0^\infty ds e^{-\frac{s}{2} \int d^D x_i e^{-\frac{1}{2s} \int d\tau x_i^\mu}}
\]
\[
\times H. O.,
\]
where \(H. O.\) means the harmonic oscillator calculation for the two-dimensional relativistic Landau problem.

The partition function for this gas of \(N\)-tensionless \(p\) branes
\[
Z[s(\sigma)] = \text{Tr} [G[x(\sigma), x'(\sigma); s(\sigma)]]
\]
\[
= \left(s(\sigma)^{-D/2} e^{-\frac{m^2}{2s} \int d\sigma s(\sigma)} \sum_{n=0}^{\infty} G_0 e^{-\frac{n}{2s} \int d^D x_i e^{-\frac{1}{2s} \int d\tau x_i^\mu}}\right)^N.
\]
\[
\left[ \frac{g_0[s(\sigma)]}{\sinh \left( \frac{2\theta B}{T} \int ds s(\sigma) \right)} \right]^N.
\]

Therefore, if we assume pairing interaction, then noncommutativity induces a motion for a tensionless p-branes confined via a harmonic potential oscillator.

### IV. Interactions via Noncommutativity in the Phase Space

In the previous section we argued how to construct noncommutative extended objects. In this section we would like to give an insight in a different physical context and to investigate the possibility of a possible interaction by means noncommutativity. This procedure, is simple extension of the noncommutative field.

From the non-relativistic point of view, apparently there is no problem with nonlocal communication \[18\]. Indeed, let us suppose two non-relativistic particles in one dimension, labeled by coordinates \( x_1 \) and \( y_1 \) and canonical momenta \( p_1 \) and \( p_2 \) respectively. Note that the index refers now to the particles involved.

The Hamiltonian for this system is

\[
H = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2.
\]

Although naively the particles in \[18\] are free, they can interact if we posit the commutator

\[
[p_1, p_2] = iB,
\]

where \( B \) measures the strength of this interaction which can play –or not–the role of a magnetic field.

The exact equivalence between this system and the Landau problem is a subtle point because by considering only a noncommutative phase space with noncommutative parameters \( \theta \) and \( B \), one can show that noncommutative quantum mechanics and the Landau problem coincide if the relation \( \theta B = 1 \) is fulfilled, i.e. if we have just the magnetic length \[18\]. From this example, one extract as conclusion that the equivalence between a physical system such as the Landau problem and noncommutative quantum mechanics only occurs for the critical point \( \theta B = 1 \). For other values of \( \theta B \), noncommutative quantum mechanics describes a physics completely different from the Landau problem.

The above example can be generalized for more particles; for instance, let us consider two free particles moving in a commutative plane.

The Hamiltonian is

\[
H = \frac{1}{2}(p_{1x}^2 + p_{1y}^2) + \frac{1}{2}(p_{2x}^2 + p_{2y}^2).
\]

Then, let us assume that the interaction is given by

\[
[p_{1x}, p_{2x}] = iB, \quad [p_{1y}, p_{2y}] = iB.
\]

Then, as in the previous case, the Hamiltonian is

\[
H = \frac{1}{2}(p_{1x}^2 + p_{1y}^2) + \frac{1}{2}(p_{2x}^2 + p_{2y}^2).
\]

Thus, the commutator \[36\] and the hamiltonian \[37\] describe a couple of particles living on a plane and interacting formally with a magnetic field perpendicular to the plane.

We would like to remark that our procedure, of course, has generated a nonlocal interaction between both particles.

In the general case for \( N \) particles moving on a \( D \) dimensional commutative space, the generalization is straightforward.

Indeed, the Hamiltonian is

\[
H = \frac{1}{2}(p_{1x}^2 + p_{1y}^2 + \cdots) + \frac{1}{2}(p_{2x}^2 + p_{2y}^2 + \cdots) + \cdots,
\]

then the interaction can be written as

\[
[p_i^a, p_j^b] = i\delta_{ij}\epsilon^{ab}B,
\]

where \( a, b \) run on \( 1, \ldots, N \) labeling the different species of particles and the indexes \( i, j, \ldots \) select the vectorial component of \( \mathbf{x} \) \[27\].

If we rewrite the Hamiltonian as

\[
H = \frac{1}{2}(p_{1x}^2 + p_{1y}^2 + \cdots) + \frac{1}{2}(p_{2x}^2 + p_{2y}^2 + \cdots) + \cdots
\]

Thus, in the critical point, this generalized system is related to the quantum Hall effect as has been proposed using a different argument by \[17\].

Thus, in our context, one could conclude that if two particles interact via nonlocal communication, the phase space could be noncommutative. However, this fact does not exclude other possible mechanisms as a source of nonlocal interactions.

### V. Conclusions

In conclusion, we have constructed the statistical mechanics of generally covariant systems such as \( p \)-branes assuming that for each point of the world-volume one define a noncommutative field. From these results, we have studied the quantum statistical mechanics of tensionless \( p \)-branes gas that is a qualitatively different system in comparison with the commutative one.

In addition, we have discussed a possible mechanism for to implement no-local interactions by means of noncommutativity that could be useful in the quantum Hall effect or other systems.

The possible cosmological implications of these results as well as other results are also studied in \[22\].
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APPENDIX A: STATISTICAL MECHANICS FOR A FREE RELATIVISTICS PARTICLES GAS

In this appendix we will study the statistical mechanics for a free relativistics particles gas. One start considering the partition function

\[ Z = Tr \, G[x_2, x_1], \quad (A1) \]

where \( G[x_2, x_1; m^2] \) is the Green function for a free particle in the sense discussed in section II and it is given by

\[ G[x_2, x_1; m^2] = s^{-D/2} e^{-\frac{(\Delta x)^2}{2s}}, \quad (A2) \]

where \( D \) is the dimension of the spacetime.

In the euclidean space \([11]\) is computed using periodic boundary conditions but instead of one use

\[ x_1^0(0) = x_2^0(T) + 2n\pi R, \quad (A3) \]

where \( n = 1, 2, ..., D - 1 \), \( n = 0, \pm 1, \pm 2, ... \) are the Matsubara frequencies and \( R \) is the compactification radius.

Using this fact, one find that the total partition function is

\[ Z = \sum_{n=1}^{\infty} Z^{(n)}, \quad (A5) \]

\[ = \sum_{n=1}^{\infty} e^{i\vartheta n \beta} Z^{(n)}, \quad (A6) \]

\[ = s^{-D/2} \vartheta_3 \left( e^{-\frac{\pi^2 n^2}{D}} \right) e^{i\vartheta}, \quad (A7) \]

where \( \vartheta_3 \) is the Jacobi function and \( \vartheta \) is phase factor that plays the analogous role of the magnetic flux in the Aharonov-Bohm effect. Since \( s \) play an analog role of \( \beta \) in statistical mechanics, in the high temperatures limit \( \vartheta_3 \to 1 \) and the logarithm of the partition function in this case is

\[ \ln Z = N \left[ -\frac{m^2}{2} s - \frac{D}{2} \ln s + \ln V \right]. \quad (A8) \]

Thus, one find that in the spinless case and in the high temperature region, there are no Matsubara modes and the statistics is, of course, the Maxwell-Boltzmann one.

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Along this paper we assume that the particles are spinless. The reader should note also that we are assuming the Maxwell-Boltzmann statistics, for a justification about this see appendix.

Although this factor can be computed by using a regularization prescription, here this factor is absorbed as a normalization constant.

Of course this is a simplification because we are assuming that the noncommutative parameters are the same.

The components of the antisymmetric density tensor $\epsilon^{ab}$ are defined as $+1$ if $a > b$. 

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