An investigation of the comparative efficiency of the different methods in which \( \pi \) is calculated

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Abstract

An investigation of the comparative efficiency of the different methods in which $\pi$ is calculated. This thesis will compare and contrast five different methods in calculating $\pi$ by first deriving the various proofs to each method and then creating graphical displays and tables with percentage errors on each to allow a thorough comparison between each method. It is important to realize that usually people refer to $\pi$ as 3.142 and use this shortened value in everyday calculations. However, most people in society do not know that in some of NASAs calculations, must be calculated to 15 decimal places (3.141592653589793) for calculations involving the Space Station Guidance Navigation and Control (GNC) code [9]. Furthermore, some people use the world renown constant everyday yet have no idea where it comes from.

This thesis allows a thorough exploration of the different methods of calculating pi by using Mathematica to provide the necessary iterations needed and Microsoft excel to plot the various graphs for convergence comparisons. Finally, the conclusions that have come from this thesis demonstrate that methods in which oscillations between the positive and negative region of the line $y = \pi$ slow down the rate of convergence and out of the five methods investigated, The infinite series formulated from $\zeta(8)$ proves to be one of the fastest infinite series today to converge towards $\pi$. 
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1 introduction

In today’s society, $\pi$ is used everyday by almost everyone in some way or another. Whether it be a mathematician doing complex integrals, an engineer calculating the volume of spherical shapes or a clock maker designing a new pendulum for a new clock. It is clear that $\pi$ underlies most of the modern day achievements and developments. However, the reality is that only a small percentage of the population who use this constant have any idea where it comes from.

In the beginning, Pi was first investigated using a trigonometric method in "250BC by the famous mathematician Archimedes" [7] who investigated interior and exterior polygons around a circle creating both upper and lower bounds for a value of $\pi$ which were then averaged to give a value correctly to 7 decimal places [7]. Moving swiftly through the decades, another mathematician named Vieta in 1579 used nested radicals to further find a product of nested radicals which would converge to $\pi$ and gave a value of $\pi$ correct to 9 decimal places. Lastly, with the development of calculus, rapidly converging series were developed. A clear example of this is when newton in 1748 developed his arcsine expansion with the Taylor Series which gave a value of $\pi$ which "only needed 22 terms for 16 decimal places for $\pi$" [5].

However, with the mathematical knowledge we have today, there are endless ways to arrive at this world renown constant of $\pi$. From the simplest application of this constant in the formula $\pi r^2$ to attain a value for the area of a circle to the more complex application towards the other end of the spectrum in calculus. My research question: “An investigation of the comparative efficiency of the different methods in which $\pi$ is calculated” aims to distinguish and set aside the different methods of calculating and approaching the value of this constant by contrasting how efficient a variety of different numerical methods are in calculating this constant correct to 15 decimal places. This will be achieved by exploring different methods in which $\pi$ is calculated and contrasting their rates of convergence by increasing the amount of terms $n$ and further observing the rate at which each method converges towards $\pi$. A series of graphical displays and Data tables will be used to calculate and compare the convergence towards $\pi$ and where there is an error by comparing it to a "True" value.
2  The Riemann-Zeta function, Parseval’s identity and the Fourier Series to generate $\zeta(x)$

2.1  Preface list of data and proof’s used in the calculations

2.1.1  The Fourier Series

$$(s_N f)(x) = \frac{a_0}{2} + \sum_{n=1}^{N} [a_n \cos(nx) + b_n \sin(nx)]$$ for $N \in \mathbb{Z}^+$

2.1.2  Parseval’s Identity

$$\sum_{-\infty}^{\infty} |c_n| = \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx$$ where

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x) \, dx = \frac{a_0^2}{2} + \sum_{k=1}^{\infty} |a_k^2 + b_k^2|$$

2.1.3  Proof of orthogonality in the Fourier Series

First consider

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(nx) + \sum_{n=1}^{N} b_n \sin(nx)$$

Case (I) Note: $\sin[(n + m)x] + \sin[(n - m)x] = 2\sin(nx)\cos(mx)$

$$\int_{-\pi}^{\pi} \sin(nx)\cos(mx) \, dx = \frac{1}{2} \int_{-\pi}^{\pi} 2\sin(nx)\cos(mx) \, dx$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} \sin[(n + m)x] + \sin[(n - m)x] \, dx$$

$$= \frac{1}{2} \left[ \frac{-1}{(n + m)} \cos[(n + m)x] + \frac{1}{(n - m)} \cos[(n - m)x] \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2} \left[ \frac{-1}{2(n + m)} \cos[(n + m)x] + \frac{1}{2(n - m)} \cos[(n - m)x] \right]_{-\pi}^{\pi}$$

$$= 0$$

Case (II) Note: $\cos[(n + m)x] + \cos[(n - m)x] = 2\cos(nx)\cos(mx)$

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx) \, dx = \frac{1}{2} \int_{-\pi}^{\pi} 2\cos(nx)\cos(mx) \, dx$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} \cos[(n + m)x] + \cos[(n - m)x] \, dx$$

$$= \frac{1}{2} \left[ \frac{1}{(n + m)} \sin[(n + m)x] + \frac{1}{(n - m)} \sin[(n - m)x] \right]_{-\pi}^{\pi}$$

Note: Only if $n \neq m$

$$= \frac{1}{2} \left[ \frac{1}{2(n + m)} \sin[(n + m)x] + \frac{1}{2(n - m)} \sin[(n - m)x] \right]_{-\pi}^{\pi}$$

Firstly, consider the case where $n \neq m$

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx) = 0$$
Secondly, consider the case where \( n = m \)

\[
\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = \int_{-\pi}^{\pi} \cos^2(nx)\,dx = \frac{1}{2} \int_{-\pi}^{\pi} (\cos(2nx) + 1)\,dx = \frac{1}{2} \left[ \frac{1}{2n} \sin(2nx) + x \right]_{-\pi}^{\pi} = \frac{1}{2} [0 + 2\pi] = \frac{\pi}{\pi}
\]

Case (III) note: \(-\cos[(n + m)x] + \cos[(n - m)x] = 2\sin(nx)\sin(mx)\)

\[
\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \frac{1}{2} \int_{-\pi}^{\pi} 2\sin(mx)\sin(nx)\,dx = \frac{1}{2} \int_{-\pi}^{\pi} \cos[(n - m)x] - \cos[(n - m)x]\,dx = \frac{1}{2} \left[ \frac{1}{(n - m)} \sin[(n - m)x] - \frac{1}{(n + m)} \sin[(n + m)x] \right]_{-\pi}^{\pi} \text{ for } m \neq n
\]

Case (i): If \( n \neq m \) then,

\[
\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = 0
\]

Case (ii): If \( n = m \) then,

\[
\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \int_{-\pi}^{\pi} \sin^2(nx)\,dx = \frac{1}{2} \int_{-\pi}^{\pi} (1 - \cos(2nx))\,dx = \frac{1}{2} \left[ x - \frac{1}{2n} \sin(2nx) \right]_{-\pi}^{\pi} = \frac{1}{2} [2\pi - 0] = \frac{\pi}{\pi}
\]

Therefore, we can deduce that in general:

\[
\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\,dx = 0
\]

\[
\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = \begin{cases} 
0 & \text{if } m \neq n \\
\frac{\pi}{\pi} & \text{if } m = n
\end{cases} \rightarrow \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = \begin{cases} 
0 & \text{if } m \neq n \\
1 & \text{if } m = n
\end{cases}
\]

\[
\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = \begin{cases} 
0 & \text{if } m \neq n \\
\frac{\pi}{\pi} & \text{if } m = n
\end{cases} \rightarrow \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = \begin{cases} 
0 & \text{if } m \neq n \\
1 & \text{if } m = n
\end{cases}
\]
2.1.4 The extraction of $a_n, b_n, a_0, b_0$ from the Fourier series

From the Fourier series, by applying integration on $f(x)$ directly and realizing that the terms consisting of $\sin(nx)$ and $\cos(nx)$ are equal to zero, we are able to attain a value for the constant term $a_0$.

\[
\int_{-\pi}^{\pi} f(x)\, dx = \int_{-\pi}^{\pi} \left(\frac{a_0}{2}\right)\, dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos(nx)\, dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin(nx)\, dx
\]
\[
= \left(\frac{a_0}{2}\right) \int_{-\pi}^{\pi} \, dx
\]
\[
= a_0 \left[\frac{x}{2}\right]_{-\pi}^{\pi}
\]
\[
= a_0 \pi.
\]

To find a value of $a_n$, multiply all the terms in the Fourier series by $\cos(nx)$ as $a_n$ will only exist where $f(x)$ is even because the term in $a_n$ will be the only one to be non-zero following the integration over the range from $\pi$ to $-\pi$.

\[
\int_{-\pi}^{\pi} f(x)\cos(mx)\, dx = \int_{-\pi}^{\pi} \left(\frac{a_0}{2}\right)\cos(mx)\, dx + \sum_{m=1}^{\infty} a_m \int_{-\pi}^{\pi} \cos(nx)\cos(mx)\, dx + \sum_{m=1}^{\infty} b_m \int_{-\pi}^{\pi} \sin(nx)\cos(mx)\, dx
\]
\[
= \sum_{m=1}^{\infty} a_m \int_{-\pi}^{\pi} \cos(nx)\cos(mx)\, dx
\]
\[
= a_m \int_{-\pi}^{\pi} \cos(nx)\cos(mx)\, dx
\]

Note that $\delta_{mn} = 0$ where $m \neq n$ and $\delta_{mn} = 1$ where $m = n$.

Now note that there only exists $a_n$ where $m = n$.

\[
\int_{-\pi}^{\pi} f(x)\cos(nx)\, dx = a_n \pi \rightarrow a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)\, dx
\]

In addition, when we multiply all terms by $\sin(nx)$ we can yield a result respectively for $b_n$.

\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(nx)\, dx
\]

Also, note that in all cases dealt with $b_0 = 0$ and $b_n$ is only defined for $n \geq 1$. 

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2.2 Generating a value for $\zeta(2)$

Consider the simplest odd function where $f(x) = x$

Therefore, since $f(x)$ is an odd function then $a_n = 0$ where $\forall n \in \mathbb{Z}^+$

Now consider $b_n$,

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx) dx = \frac{1}{\pi} \left[ \frac{-x \cos(nx)}{n} \right]_{-\pi}^{\pi} - \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{-\cos(nx)}{n} dx$$

$$= \frac{1}{\pi} \left[ \frac{\sin(nx)}{n^2} \right]_{-\pi}^{\pi} - \frac{1}{\pi} \left[ \frac{\cos(nx)}{n} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[ \frac{-x \cos(nx)}{n} \right]_{-\pi}^{\pi} - \frac{\pi \cos(-n\pi)}{n}$$

$$= \frac{1}{\pi} \left[ \frac{2\pi \cos(n\pi)}{n} \right]$$

$$= \frac{-2 \cos(n\pi)}{n}$$

$$= \frac{-2(-1)^n}{n}$$

Now, by substituting $b_n$ back into the Fourier Series we obtain:

$$x = \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n} \sin(nx) \text{ for } 0 \leq x \leq \pi$$

Then, consider Parseval's Identity where:

$$c_n = a_n + ib_n \rightarrow |c_n|^2 = a_n^2 + b_n^2 \text{ (Note: } a_n = 0 \text{ where } f(x) \text{ is odd})$$

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx$$

$$\sum_{n=-\infty}^{\infty} |b_n|^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx$$

Firstly, consider the left hand side of Parseval's Identity:

$$\sum_{n=1}^{\infty} |b_n|^2 = \sum_{n=1}^{\infty} \left[ \frac{-2(-1)^n}{n} \right]^2$$

$$= \sum_{n=1}^{\infty} \frac{4(-1)^{2n}}{n^2}$$

$$= 4 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$= 4\zeta(2)$$
Secondly, consider the right hand side of Parseval’s Identity:

\[ \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \, dx = \frac{1}{\pi} \left[ \frac{2\pi^3}{3} \right] \]
\[ = \frac{2\pi^2}{3} \]

Lastly, equate the left hand side to the right hand side of Parseval’s Identity

\[ 4\zeta(2) = \frac{2\pi^2}{3} \]
\[ \zeta(2) = \frac{2\pi^2}{12} \]
\[ \zeta(2) = \frac{\pi^2}{6} \]

Now, using the definition of the zeta function we then see that

\[ \zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ... \]

Multiplying by 6 and square rooting we attain:

\[ \pi^2 = 6 \left[ \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ... \right] \]
\[ \pi = \sqrt{6 \left[ \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ... \right]} \]
2.3 Generating a value for $\zeta(4)$

Consider the simplest even function where $f(x) = x^2$

Therefore, since $f(x)$ is an even function then $b_n = 0$ where $\forall n \in Z^+$

First consider $a_0$,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{2}{\pi} \int_{0}^{\pi} x^2 dx = \frac{2\pi^2}{3}$$

Then consider $a_n$,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(nx) dx = \frac{2}{\pi} \int_{0}^{\pi} x^2 \cos(nx) dx$$

$$= \frac{2}{\pi} \left[ \left. \left( \frac{x^2 \sin(nx)}{n} \right) \right|_{0}^{\pi} - \int_{0}^{\pi} \frac{2x \sin(nx)}{n} dx \right]$$

$$= \frac{2}{\pi} \left[ \left. \left( \frac{2x \cos(nx)}{n^2} \right) \right|_{0}^{\pi} - \int_{0}^{\pi} \frac{2 \cos(nx)}{n^2} dx \right]$$

$$= \frac{4\pi \cos(nx)}{n^2 \pi}$$

$$= \frac{4(-1)^n}{n^2}$$

Now, by substituting $a_n$ and $a_0$ back into the Fourier Series we obtain:

$$x = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(nx) \text{ for } -\pi \leq x \leq \pi$$

Then, consider Parseval’s Identity where:

$$c_n = a_n + ib_n \rightarrow |c_n|^2 = a_n^2 + b_n^2 \text{ (Note: } b_n = 0 \text{ where } f(x) \text{ is even})$$

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx$$

$$\left( \frac{a_0}{2} \right)^2 + \sum_{n=-\infty}^{\infty} |a_n|^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^4 dx$$

Firstly, consider the left hand side of Parseval’s Identity:

$$\left( \frac{a_0}{2} \right)^2 + \sum_{n=1}^{\infty} |a_n|^2 = \frac{2\pi^4}{9} + \sum_{n=1}^{\infty} \left[ \frac{4(-1)^2 n}{n^2} \right]^2$$

$$= \frac{2\pi^4}{9} + \sum_{n=1}^{\infty} \frac{16(-1)^2 n}{n^4}$$

$$= \frac{2\pi^4}{9} + 16 \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$= \frac{2\pi^4}{9} + 16\zeta(4)$$
Secondly, consider the right hand side of Parseval’s Identity:

\[
\frac{2}{\pi} \int_0^\pi x^4 dx = \frac{2}{\pi} \left[ \frac{x^5}{5} \right]^\pi_0 = \frac{2\pi^4}{5}
\]

Lastly, equate the left hand side to the right hand side of Parseval’s Identity

\[
16\zeta(4) + \frac{2\pi^4}{9} = \frac{2\pi^4}{5}
\]

\[
16\zeta(4) = \frac{8\pi^4}{45}
\]

\[
\zeta(4) = \frac{\pi^4}{90}
\]

Now, using the definition of the zeta function we then see that

\[
\zeta(4) = \sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + ...
\]

Multiplying by 90 and taking the 4th root we attain:

\[
\pi^4 = 90 \left[ \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + ... \right]
\]

\[
\pi = \sqrt[4]{90 \left[ \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + ... \right]}
\]
2.4 Generating a value for $\zeta$ (6)

Consider the odd function where $f(x) = x^3$

Therefore, since $f(x)$ is an odd function then $a_n = 0$ where $\forall = 1,2,3, \ldots$

Now consider $b_n$,

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^3 \sin(nx) \, dx = \frac{1}{\pi} \left[ \frac{-x^3 \cos(nx)}{n} \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{3x^2 \cos(nx)}{n} \, dx$$

$$= \frac{1}{\pi} \left[ \frac{-x^3 \cos(nx)}{n} \right]_{-\pi}^{\pi} + \frac{3x^2 \sin(nx)}{n^2} - \int_{-\pi}^{\pi} \frac{6x \sin(nx) \cos(nx)}{n^2} \, dx$$

$$= \frac{1}{\pi} \left[ \frac{-x^3 \cos(nx)}{n} \right]_{-\pi}^{\pi} + \frac{6x \cos(nx)}{n^3} - \int_{-\pi}^{\pi} \frac{6 \cos(nx)}{n^3} \, dx$$

Now, by substituting $b_n$ back into the Fourier Series we obtain:

$$x^3 = \sum_{n=1}^{\infty} \left[ \frac{12(-1)^n}{n^3} - \frac{2\pi^2(-1)^n}{n} \right] \sin(nx) \text{ for } 0 \leq x \leq \pi$$

Then, consider Parsevals Identity where:

$$c_n = a_n + ib_n \rightarrow |c_n|^2 = a_n^2 + b_n^2 \text{ (Note: } a_n = 0 \text{ where } f(x) \text{ is odd})$$

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx$$

$$\sum_{n=-\infty}^{\infty} |b_n|^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^6 \, dx$$

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Firstly, consider the left hand side of Parseval’s Identity:

\[
\sum_{n=1}^{\infty} |b_n|^2 = \sum_{n=1}^{\infty} \left[ \frac{12(-1)^n}{n^3} - \frac{2\pi^2(-1)^n}{n} \right]^2
\]

\[
= \sum_{n=1}^{\infty} \left[ \frac{144(-1)^{2n}}{n^6} - \frac{48(-1)^{2n}}{n^4} + \frac{4\pi^4(-1)^{2n}}{n^2} \right]
\]

\[
= 144 \sum_{n=1}^{\infty} \frac{1}{n^6} - 48\pi^2 \sum_{n=1}^{\infty} \frac{1}{n^4} + 4\pi^4 \sum_{n=1}^{\infty} \frac{1}{n^2}
\]

\[
= 144\zeta(6) - 48\pi^2\zeta(4) + 4\pi^4\zeta(2)
\]

\[
= 144\zeta(6) - \frac{8\pi^6}{15} + \frac{4\pi^6}{6}
\]

Secondly, consider the right hand side of Parseval’s Identity:

\[
\frac{1}{\pi} \int_{-\pi}^{\pi} x^6 dx = \frac{1}{\pi} \left[ \frac{x^7}{7} \right]_{-\pi}^{\pi}
\]

\[
= \frac{2\pi^6}{7}
\]

Lastly, equate the left hand side to the right hand side of Parseval’s Identity

\[
144\zeta(6) - \frac{8\pi^6}{15} + \frac{4\pi^6}{6} = \frac{2\pi^6}{7}
\]

\[
144\zeta(4) = \frac{16\pi^6}{105}
\]

\[
\zeta(6) = \frac{\pi^6}{945}
\]

Now, using the definition of the zeta function we then see that

\[
\zeta(6) = \sum_{k=1}^{\infty} \frac{1}{k^6} = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + ...
\]

Multiplying by 945 and taking the 6th root we attain:

\[
\pi^6 = 945 \left[ \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + ... \right]
\]

\[
\pi = \sqrt[6]{945 \left[ \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + ... \right]}
\]
2.5 Generating a value for $\zeta$ (8)

Consider the even function where $f(x) = x^4$
Therefore, since $f(x)$ is an even function then $b_n = 0$ where $\forall = 1, 2, 3, ...$

First consider $a_0$,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^4 \, dx = \frac{2}{\pi} \int_{0}^{\pi} x^4 \, dx = \frac{2\pi^4}{5}$$

Then consider $a_n$,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^4 \cos(nx) \, dx = \frac{2}{\pi} \int_{0}^{\pi} x^4 \cos(nx) \, dx$$

$$= \frac{2}{\pi} \left[ \frac{x^4 \sin(nx)}{n} \right]_0^\pi - \frac{\pi}{n} \int_{0}^{\pi} 4x^3 \sin(nx) \, dx$$

$$= \frac{-2}{\pi} \int_{0}^{\pi} \frac{4x^3 \sin(nx)}{n} \, dx$$

$$= \frac{-8}{n\pi} \int_{0}^{\pi} x^3 \sin(nx) \, dx$$

$$= \frac{-8}{n\pi} \left[ \left( -\frac{x^3 \cos(nx)}{n} \right) \right]_0^\pi + \frac{\pi}{n} \left[ \frac{3x^2 \cos(nx)}{n} \right]_0^\pi - \frac{\pi}{n} \int_{0}^{\pi} 6x \sin(nx) \, dx$$

$$= \frac{-8}{n\pi} \left[ -\frac{x^3 \cos(nx)}{n} \right]_0^\pi + \frac{\pi}{n} \left[ \frac{3x^2 \sin(nx)}{n^2} \right]_0^\pi - \frac{\pi}{n} \int_{0}^{\pi} 6 \cos(nx) \, dx$$

$$= \frac{-8}{n\pi} \left[ -\frac{x^3 \cos(nx)}{n} \right]_0^\pi + \frac{\pi}{n} \left[ \frac{6x \cos(nx)}{n^3} \right]_0^\pi - \frac{\pi}{n} \int_{0}^{\pi} 6 \cos(nx) \, dx$$

$$= \frac{-8}{n\pi} \left[ -\frac{n^2 x^3 \cos(nx)}{n^3} + 6 \cos(nx) \right]_0^\pi$$

$$= \frac{-8}{n\pi} \left[ -\frac{n^2 \pi^2 \cos(n\pi)}{n^3} + 6 \cos(n\pi) \right]_0^\pi$$

$$= \frac{8n^2 \pi^2 \cos(n\pi) - 48 \cos(n\pi)}{n^4}$$

Now, by substituting back $a_n$ and $a_0$ back into the Fourier Series we obtain:

$$x^4 = \frac{\pi^4}{5} + \sum_{n=1}^{\infty} \left[ \frac{8n^2 \pi^2 (-1)^n}{n^4} - \frac{48 (-1)^n}{n^4} \right] \cos(nx) \text{ for } -\pi \leq x \leq \pi$$

Then, consider Parsevals Identity where:

$$c_n = a_n + ib_n \rightarrow |c_n|^2 = a_n^2 + b_n^2 \text{ (Note: } b_n = 0 \text{ where } f(x) \text{ is even)}$$

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx$$

$$\left( \frac{a_0}{2} \right)^2 + \sum_{n=-\infty}^{\infty} |a_n|^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^8 \, dx$$
Firstly, consider the left hand side of Parseval’s Identity:

\[ \left( \frac{a_0}{2} \right)^2 + \sum_{n=1}^{\infty} |a_n|^2 = \frac{2\pi^8}{25} + \sum_{n=1}^{\infty} \left[ \frac{8\pi^2 (-1)^n}{n^2} - \frac{48(-1)^n}{n^4} \right] \]

\[ = \frac{2\pi^8}{25} + \sum_{n=1}^{\infty} \frac{64\pi^4 (-1)^{2n}}{n^4} - \frac{784\pi^2 (-1)^{2n}}{n^6} + \frac{2304(-1)^{2n}}{n^8} \]

\[ = \frac{2\pi^8}{25} + 64\pi^4 \sum_{n=1}^{\infty} \frac{1}{n^4} - 784\pi^2 \sum_{n=1}^{\infty} \frac{1}{n^6} + 2304 \sum_{n=1}^{\infty} \frac{1}{n^8} \]

\[ = \frac{2\pi^8}{25} + 64\pi^4 \zeta(4) - 784\pi^2 \zeta(6) + 2304 \zeta(8) \]

\[ = \frac{2\pi^8}{25} + \frac{64\pi^8}{90} - \frac{768\pi^8}{945} + 2304 \zeta(8) \]

Secondly, consider the right hand side of Parseval’s Identity:

\[ \frac{2}{\pi} \int_{0}^{\pi} x^8 dx = \frac{2}{\pi} \left[ \frac{x^9}{9} \right] \]

\[ = \frac{2\pi^8}{9} \]

Lastly, equate the left hand side to the right hand side of Parseval’s Identity

\[ \frac{2\pi^8}{25} + \frac{64\pi^8}{90} - \frac{768\pi^8}{945} + 2304 \zeta(8) = \frac{2\pi^8}{9} \]

\[ 2304 \zeta(8) = \frac{128\pi^8}{525} \]

\[ \zeta(8) = \frac{\pi^8}{9450} \]

Now, using the definition of the zeta function we then see that

\[ \zeta(8) = \sum_{k=1}^{\infty} \frac{1}{k^8} = \frac{1}{1^8} + \frac{1}{2^8} + \frac{1}{3^8} + \ldots \]

Multiplying by 9450 and taking the 8th root we attain:

\[ \pi^8 = 9450 \left[ \frac{1}{1^8} + \frac{1}{2^8} + \frac{1}{3^8} + \ldots \right] \]

\[ \pi = \sqrt[8]{9450 \left[ \frac{1}{1^8} + \frac{1}{2^8} + \frac{1}{3^8} + \ldots \right]} \]

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3 The Wallis Product

Consider the function in which \( f(x) = \sin^n(x) \)

\[
I(n) = \int_0^\pi \sin^n(x)\,dx
\]

By evaluating this integral we obtain the following:

\[
I(n) = \int_0^\pi \sin^n(x)\,dx = \int_0^\pi \sin^{n-1}(x)\sin(x)\,dx
\]

\[
= \left[ \sin^{n-1}(x)(-\cos(x)) \right]_0^\pi - \int_0^\pi (n-1)\sin^{n-2}(x)(-\cos(x))(\cos(x))\,dx
\]

\[
= 0 + (n-1)\int_0^\pi \sin^{n-2}(x)\cos(x)\,dx
\]

\[
= (n-1)\int_0^\pi \sin^{n-2}x(1-\sin^2x)\,dx
\]

\[
= (n-1)[I(n-2) - I(n)]
\]

Now it is possible to generate values that are either even or odd. To generate even values, consider the following:

\[
I(n) = (n-1)[I(n-2) - I(n)]
\]

\[
I(n) = (n-1)I(n-2) - (n-1)I(n)
\]

\[
I(n) + (n-1)I(n) = (n-1)I(n-2)
\]

\[
I(n)[1 + (n-1)] = (n-1)I(n-2)
\]

\[
I(n) = \frac{n-1}{n}I(n-2) [1]
\]

To generate odd values, consider the following: By letting \( n = 2m + 1 \) in [1], we attain the following:

\[
I(2m+1) = \frac{2m}{2m+1}I(2m-1) [2]
\]

\[
\frac{I(2m+1)}{I(2m-1)} = \frac{2m}{2m+1}
\]

\[
\frac{I(2m-1)}{I(2m+1)} = \frac{2m+1}{2m} [3]
\]
Now we can consider the product of values for both even and odd functions

\[ I(0) = \int_0^\pi \sin^0(x) \, dx = \int_0^\pi dx = \pi \]

\[ I(1) = \int_0^\pi \sin(x) \, dx = \left[ -\cos(x) \right]_0^\pi = \int_0^\pi dx = -(1) - (-1) = 2 \]

Note: for \( I(2n) \), \( n = 2m \) is substituted into [1]

\[ I(2m) = \int_0^\pi \sin^{2m}(x) \, dx = \frac{2m - 1}{2m} \cdot \frac{2m - 3}{2m - 2} \cdot \cdots \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot I(0) = \pi \prod_{k=1}^{m} \frac{2k - 1}{2k} \] \[ \text{[4]} \]

Note: for \( I(2n+1) \), \( n = 2m + 1 \) is substituted into [2]

\[ I(2m + 1) = \int_0^\pi \sin^{2m+1}(x) \, dx = \frac{2m}{2m + 1} \cdot I(2m - 1) \]

For even functions, consider \( I(2n) \) and expand further by repeating the iterative process.

\[ I(2n) = \frac{2m - 1}{2m} \cdot \frac{2m - 3}{2m - 2} \cdot \cdots \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot I(0) = \pi \prod_{k=1}^{m} \frac{2k - 1}{2k} \] \[ \text{[4]} \]

For odd functions, consider \( I(2n+1) \) and expand further by repeating the iterative process.

\[ I(2n + 1) = \frac{2m}{2m + 1} \cdot \frac{2m - 2}{2m - 1} \cdot \cdots \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot I(1) = 2 \prod_{k=1}^{m} \frac{2k}{2k + 1} \] \[ \text{[5]} \]

Therefore,

\[ \frac{I(2n)}{I(2n + 1)} = \pi \prod_{k=1}^{\infty} \frac{2k - 1}{2k} \cdot \frac{2k + 1}{2k} \]

From this, we can deduce the following as \( 0 \leq \sin(x) \leq 1 \):

\[ \sin^{2n+1}(x) \leq \sin^{2n}(x) \leq \sin^{2n-1}(x) \text{ for } 0 \leq x \leq \pi \]

\[ \int_0^\pi \sin^{2n+1}(x) \, dx \leq \int_0^\pi \sin^{2n}(x) \, dx \leq \int_0^\pi \sin^{2n-1}(x) \, dx \]

\[ I(2n + 1) \leq I(2n) \leq I(2n - 1) \]

\[ I \leq \frac{I(2n)}{I(2n + 1)} \leq \frac{I(2n - 1)}{I(2n + 1)} \]

Then, by substituting [3]) for

\[ \frac{I(2n - 1)}{I(2n + 1)} \]
we obtain:

\[ 1 \leq \frac{I(2n)}{I(2n + 1)} \leq \frac{2n + 1}{2n} \]

Now consider the Squeeze Theorem which states the following:

If \( a_n, b_n \) and \( c_n \) exist for all \( a_n \leq b_n \leq c_n \) for all \( n \)

\[ \lim_{n \to \infty} a_n = \lim_{n \to \infty} = \lim_{n \to \infty} c_n = L < \infty \]

Then \( \lim_{n \to \infty} b_n = L \)

In this case, the lower limit would be the following:

\[ \lim_{n \to \infty} a_n = 1 \]

Now, consider the limit of \( c_n \),

\[ \lim_{n \to \infty} \frac{I(2n - 1)}{I(2n + 1)} = \lim_{n \to \infty} \frac{2n + 1}{2n} = \lim_{n \to \infty} \left( 1 + \frac{1}{2n} \right) \to 1 \]

Henceforth, since both \( a_n \) and \( c_n \) tend towards the same limit which in this case is 1 then \( b_n \) must also tend towards 1 by the squeeze theorem which is shown below.

\[ \lim_{n \to \infty} \left[ \frac{I(2n)}{I(2n + 1)} \right] = 1 \]

\[ \frac{\pi}{2} \lim_{n \to \infty} \prod_{k=1}^{n} \frac{2k - 1}{2k} \cdot \frac{2k + 1}{2k} = 1 \]

Therefore,

\[ \frac{\pi}{2} = \prod_{k=1}^{\infty} \frac{2k - 1}{2k} \cdot \frac{2k + 1}{2k} = 2 \cdot \frac{2}{1} \cdot \frac{4}{3} \cdot \frac{6}{5} \cdot \frac{8}{7} \cdots \]

\[ \pi = \prod_{k=1}^{\infty} 2 \cdot \frac{2k - 1}{2k} \cdot \frac{2k + 1}{2k} = 2 \left[ \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{6}{5} \cdot \frac{8}{7} \cdots \right] \]

And lastly,

\[ \pi = 2 \left[ \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{6}{5} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdots \right][12] \]

It is also important to acknowledge that this method to calculate \( \pi \) was different to the other methods used to calculate \( \pi \) because it is a different type of infinite decent. Unlike Newtons Arcsine function with the usage of an infinite sum, The Wallis product is a unique infinite product that converges towards \( \pi \) which was made without Infinitesimal calculus as it did not exist at the time.
4 Continued Fractions

4.1 Preface list of data and proof’s used in the calculations

4.1.1 The Taylor Series

Provided that \( f(x) \) is infinitely differentiable at \( x=a \),

\[
f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + ...
\]

4.1.2 Euler’s Continued Fraction Formula

\[
a_0 + a_0a_1 + a_0a_1a_2 + ... + a_0a_1a_2...a_n = \frac{a_0}{1 - \frac{a_1}{1 + a_1 - \frac{a_2}{1 + a_2 - \frac{a_3}{\ddots 1 + a_n - \frac{a_n}{1 + a_n}}}}}
\]

4.1.3 The Natural Logarithm in Continued Fractions

Let \( f(z) = \ln \frac{1+z}{1-z} \) and then consider the Taylor Expansion of the function to develop an infinite series

\[
f'(z) = \frac{-2}{z^2 - 1}
\]

\[
f''(z) = \frac{4z}{(1-z^2)^2}
\]

\[
f'''(z) = \frac{16z^2}{(1-z^2)^3} + \frac{4}{(1-z^2)^2}
\]

\[
f^{(4)}(z) = \frac{96z^3}{(1-z^2)^4} + \frac{48z}{(1-z^2)^3}
\]

\[
f^{(5)}(z) = \frac{768z^4}{(1-z^2)^5} + \frac{576z^2}{(1-z^2)^4} + \frac{48}{(1-z^2)^3}
\]

Then, by letting \( a = 0 \) in and expanding through the Taylor Series the following is obtained:

\[
\ln \frac{1+z}{1-z} = 2z + \frac{4z^3}{3!} + \frac{48z^5}{5!} + ...
\]

\[
= 2[z + \frac{z^3}{3} + \frac{z^5}{5} + \frac{z^7}{7} + ...]
\]

\[
= 2 \sum_{n=0}^{\infty} \frac{z^{2n+1}}{2n+1}
\]

By extracting the series formed in the previous calculation:
\[ \ln \frac{1 + z}{1 - z} = 2z \left[ 1 + \frac{z^2}{3} + \frac{z^2}{3} \left( \frac{5}{3} \right) + \frac{z^2}{3} \left( \frac{5}{3} \right) \left( \frac{7}{5} \right) + \frac{z^2}{3} \left( \frac{5}{3} \right) \left( \frac{7}{5} \right) \left( \frac{9}{7} \right) + \ldots \right] \]

The following Continued Fraction can be produced by using Euler's Formula

\[
\begin{align*}
\frac{2z}{1 - \frac{z^2}{3} - \frac{3z^2}{5} - \frac{5z^2}{7} - \frac{7z^2}{9} - \frac{9z^2}{11}}
\end{align*}
\]

### 4.2 A Continued Fraction for π

Consider the complex number where \( z = \frac{1 + i}{1 - i} = i \) (By multiplying by the conjugate)

Now, consider the complex number in Euler Form,

\[
\begin{align*}
\frac{1 + i}{1 - i} & = e^{i\frac{\pi}{2}} \\
\ln \frac{1 + i}{1 - i} & = \frac{i\pi}{2} \\
\pi & = \frac{2}{i} \ln \frac{1 + i}{1 - i}
\end{align*}
\]

Letting \( z = i \) in Euler's Continued Fraction formula for the Natural Logarithm we can attain the following continued fraction

\[
\begin{align*}
\ln \frac{1 + i}{1 - i} & = \frac{2i}{1 - \frac{1}{3} + \frac{3}{5} - \frac{5}{7} + \frac{7}{9} - \frac{9}{11} + \ldots} \\
\ln \frac{1 + i}{1 - i} & = \frac{2i}{3 - 1 + \frac{3^2}{5} - \frac{5}{7} + \frac{7}{9} - \frac{9}{11} + \ldots}
\end{align*}
\]
\[
\ln \frac{1+i}{1-i} = \frac{2i}{1 + \frac{1}{2 + \frac{3^2}{2 + \frac{5^2}{2 + \frac{7^2}{2 + \frac{9^2}{2 + \cdots}}}}}}
\]

And by repeating this process the following is eventually attained:

\[
\ln \frac{1+i}{1-i} = \frac{2i}{1 + \frac{1}{2 + \frac{3^2}{2 + \frac{5^2}{2 + \frac{7^2}{2 + \frac{9^2}{2 + \cdots}}}}}}
\]

Now, by multiplying by \((\frac{i}{2})\), the following is attained:

\[
\pi = (\frac{i}{2}) \ln \frac{1+i}{1-i} = \frac{4}{1^2 + \frac{1^2}{2 + \frac{3^2}{2 + \frac{5^2}{2 + \frac{7^2}{2 + \frac{9^2}{2 + \cdots}}}}}}
\]
5 The Gregory-Leibniz Series

5.1 Preface list of data and proof’s used in the calculations

5.1.1 The Taylor Series

Provided that \( f(x) \) is infinitely differentiable at \( x = a \),

\[
f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + ...
\]

5.2 \( \text{Arctan}(x) \)

Consider the function \( f(x) = \text{arctan}(x) \), By using the Taylor expansion, we can develop this function into an infinite series.

\[
\begin{align*}
f(x) &= \text{Arctan}(x) \\
f'(x) &= \frac{1}{1 + x^2} \\
f''(x) &= -\frac{2x}{(1 + x^2)^2} \\
f'''(x) &= \frac{8x^2}{(1 + x^2)^3} - \frac{2}{(1 + x^2)^2} \\
f^4(x) &= \frac{-48x^3}{(1 + x^2)^4} + \frac{24x}{(1 + x^2)^3} \\
f^5(x) &= \frac{384x^4}{(1 + x^2)^5} - \frac{288x^2}{(1 + x^2)^4} + \frac{24x}{(1 + x^2)^3} \\
f^6(x) &= \frac{-3840x^5}{(1 + x^2)^6} + \frac{3840x^3}{(1 + x^2)^5} - \frac{720x}{(1 + x^2)^4} \\
f^7(x) &= \frac{46080x^6}{(1 + x^2)^7} - \frac{57600x^4}{(1 + x^2)^6} + \frac{17280x^2}{(1 + x^2)^5} - \frac{720}{(1 + x^2)^4}
\end{align*}
\]

With further expansion through the Taylor Series with \( a = 0 \), the following can be obtained:

\[
\begin{align*}
\text{Arctan}(x) &= x - \frac{2!x^3}{3!} + \frac{4!x^5}{5!} - \frac{6!x^7}{7!} + ... \\
\text{Arctan}(x) &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ... \\
\text{Arctan}(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}x^{2n+1}[2]
\end{align*}
\]

Now, consider when \( x = 1 \).

\[
\begin{align*}
\frac{\pi}{4} &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} + ... \\
\pi &= 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} + ...)[2]
\end{align*}
\]
6 The Newtons Series expansion of the Arcsine Function

6.1 Preface list of data and proof’s used in the calculations

6.1.1 The Taylor Series

Provided that \( f(x) \) is infinitely differentiable at \( x = a \),

\[
f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + ...
\]

6.2 Arcsine \( x \)

Consider the function \( f(x) = \arctan(x) \). By using the Taylor expansion, we can develop this function into an infinite series

\[
f(x) = \arcsin(x) = x + \frac{x^3}{3!} + 25x^5 \frac{x^7}{7!} + ...
\]

With further expansion through the Taylor Series with \( a = 0 \), the following can be obtained:

\[
\arcsin(x) = x + \frac{x^3}{3!} + 25x^5 \frac{x^7}{7!} + ...
\]

Now, consider when \( x = \frac{1}{2} \)
\[
\frac{\pi}{6} = \frac{1}{2} + \frac{1}{48} + \frac{3}{40} + \frac{15}{43008} + \ldots
\]
\[
\pi = 6\left(\frac{1}{2} + \frac{1}{48} + \frac{3}{40} + \frac{15}{43008} + \ldots\right)\]
7 Viete’s Formula with Nested Radicals

First, Start by recalling the simple trigonometric identity in which $\sin(2x) = 2\sin(x)\cos(x)$

Now consider $\sin(2^n x)$,

$$\sin(2^n x) = 2\cos(2^{n-1} x)\sin(2^{n-1} x)$$
$$= 2\cos(2^{n-1} x)[2\cos(2^{n-2} x)\sin(2^{n-2} x)\ldots]$$

$$\sin(2^n x) = 2^n [\cos(2^{n-1} x)\cos(2^{n-2} x)\ldots\cos(2x)\cos(x)]\sin(x)$$
$$\frac{\sin(2^n x)}{2^n \sin(x)} = \cos(2^n x)\cos(2^{n-2} x)\ldots\cos(2x)\cos(x)$$

$$= \prod_{i=0}^{n-1} \cos(2^i x)$$

Now, let $x = \frac{y}{2^n}$

$$\frac{\sin(\frac{2^n y}{2^{2n}})}{2^n \sin(\frac{y}{2^n})} = \prod_{i=0}^{n-1} \cos(\frac{2^i y}{2^n})$$

$$\frac{\sin(y)}{2^n \sin(\frac{y}{2^n})} = \prod_{i=0}^{n-1} \cos(\frac{2^i y}{2^n})$$

$$= \prod_{j=1}^{n} \cos(\frac{y}{2^j})$$

$$= \cos(\frac{y}{2})\cos(\frac{y}{2^2})\ldots\cos(\frac{y}{2^n})$$

Therefore,

$$\frac{2\sin(\frac{y}{2})\cos(\frac{y}{2})}{2^n \sin(\frac{y}{2^n})\cos(\frac{y}{2^n})} = \cos(\frac{y}{2^2})\ldots\cos(\frac{y}{2^n})$$

$$\frac{2\sin(\frac{y}{2})}{2^n \sin(\frac{y}{2^n})} = \prod_{i=2}^{n} \cos(\frac{\pi}{2^i})$$

$$\frac{2}{2^n \sin(\frac{\pi}{2^n})} = \prod_{i=2}^{n} \cos(\frac{\pi}{2^i})$$

$$\frac{2}{\pi \sin(\frac{\pi}{2^n})} = \prod_{i=2}^{n} \cos(\frac{\pi}{2^i})$$
Now, let \( \lim_{n \to \infty} \) for the above and obtain the following:

\[
\frac{2}{\pi} = \prod_{i=2}^{\infty} \cos\left(\frac{\pi}{2i}\right)
\]

\[
\pi = 2 \prod_{i=2}^{\infty} \frac{1}{\cos\left(\frac{\pi}{2i}\right)}
\]

Now consider the following, by re-arranging the half cosine formula, we obtain an identity for \( \cos\left(\frac{\theta}{2}\right) \)

\[
\cos(2\theta) = 2\cos^2 - 1
\]

\[
\cos(\theta) = 2\cos^2\left(\frac{\theta}{2}\right) - 1
\]

\[
2\cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{1 + \cos\theta}{2}}
\]

\[
2\cos\left(\frac{\theta}{2}\right) = \sqrt{2 + 2\cos\theta}
\]

\[
\cos\left(\frac{\theta}{2}\right) = \frac{\sqrt{2 + 2\cos\theta}}{2}
\]

Therefore,

\[
\cos\left(\frac{\pi}{2^2}\right) = \frac{\sqrt{2 + 2\cos\left(\frac{\pi}{2^2}\right)}}{2} = \frac{\sqrt{2}}{2}
\]

\[
\cos\left(\frac{\pi}{2^3}\right) = \frac{\sqrt{2 + \sqrt{2}}}{2}
\]

\[
\cos\left(\frac{\pi}{2^4}\right) = \frac{\sqrt{2 + \sqrt{2} + \sqrt{2}}}{2}
\]

Now by using this in the originial product after applying \( \lim_{n \to \infty} \), the following is obtained:

\[
\prod_{i=2}^{n} \cos\left(\frac{\pi}{2i}\right) = \sqrt{\frac{2}{2}} \times \sqrt{\frac{2 + \sqrt{2}}{2}} \times \sqrt{\frac{2 + \sqrt{2} + \sqrt{2}}{2}} \times \ldots
\]

\[
= \sqrt{2} \times \sqrt{2 + \sqrt{2}} \times \sqrt{2 + \sqrt{2} + \sqrt{2}} \times \ldots
\]

\[
= \frac{\sqrt{2} \times \sqrt{2 + \sqrt{2}} \times \sqrt{2 + \sqrt{2} + \sqrt{2}} \times \ldots}{2^{n-1}}
\]
Therefore,

\[ \prod_{i=2}^{\infty} \frac{1}{\cos\left(\frac{\pi}{2^i}\right)} = \frac{2^{n-1}}{\sqrt{2} \times \sqrt{2 + \sqrt{2}} \times \sqrt{2 + \sqrt{2 + \sqrt{2}}} \times \ldots} \]

\[ 2 \prod_{i=2}^{\infty} \frac{1}{\cos\left(\frac{\pi}{2^i}\right)} = \frac{2^n}{\sqrt{2} \times \sqrt{2 + \sqrt{2}} \times \sqrt{2 + \sqrt{2 + \sqrt{2}}} \times \ldots} \times \text{n-1 terms} \]

Using \( n=3 \) and \( n=4 \) respectively as examples,

\[ \frac{2^3}{\sqrt{2} \times \sqrt{2 + \sqrt{2}}} = \frac{2^3 \sqrt{2 - \sqrt{2}}}{\sqrt{2 \sqrt{4} - 2}} = \frac{2^3 \sqrt{2 - \sqrt{2}}}{2} = 2^2 \sqrt{2 - \sqrt{2}} \]

\[ \frac{2^4}{\sqrt{2} \times \sqrt{2 + \sqrt{2} \times \sqrt{2 + \sqrt{2}}}} = \frac{2^4 \sqrt{2 - \sqrt{2 + \sqrt{2}}}}{\sqrt{2} \times \sqrt{2 + \sqrt{2} \times \sqrt{2 + \sqrt{2}}}} = \frac{2^4 \sqrt{2 - \sqrt{2 + \sqrt{2}}}}{\sqrt{2} \times \sqrt{2 + \sqrt{2} - \sqrt{2}}} = \frac{2^4 \sqrt{2 - \sqrt{2 + \sqrt{2}}}}{\sqrt{2} \times \sqrt{2} \times \sqrt{4 - 2}} \]

\[ = 2^3 \sqrt{2 - \sqrt{2 + \sqrt{2}}} \]

So,

\[ \frac{2^n}{\sqrt{2} \times \sqrt{2 + \sqrt{2} \times \sqrt{2 + \sqrt{2} \times \ldots \text{\( n-1 \) terms}}} = 2^{n-1} \sqrt{2 - \sqrt{2 + \sqrt{2} \times \ldots \text{\( n-2 \) terms}}} \]

\[ \prod_{i=2}^{\infty} \frac{1}{\cos\left(\frac{\pi}{2^i}\right)} = 2^{n-1} \sqrt{2 - \sqrt{2 + \sqrt{2} \times \ldots \text{\( n-2 \) terms}}} \]

\[ \Rightarrow 2 \prod_{i=2}^{\infty} \frac{1}{\cos\left(\frac{\pi}{2^i}\right)} = 2^{n+1} \sqrt{2 - \sqrt{2 + \sqrt{2} \times \ldots \text{\( n \) terms}}} \]

Therefore,

\[ \pi = \lim_{n \to \infty} 2^{n+1} \sqrt{2 - \sqrt{2 + \sqrt{2} \times \ldots \text{[1]}}} \]

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8 Analysis with Data Tables and Graphs of the various methods

8.1 The Wallis Product

Table 1: As $n$ increases how does the Wallis Product converge towards $\pi$

| n value | Method#1 | Error (%) |
|---------|----------|-----------|
| 5       | 3.002175954556907 | 4.43777   |
| 10      | 3.067703806643499 | 2.35196   |
| 15      | 3.09133688596228 | 1.59969   |
| 20      | 3.103516961539234 | 1.21199   |
| 25      | 3.110945166901554 | 0.97554   |
| 30      | 3.115948285887959 | 0.81629   |
| 35      | 3.119547206305518 | 0.70173   |
| 40      | 3.122260326421437 | 0.61537   |
| 45      | 3.124378835915516 | 0.54793   |
| 50      | 3.126078900215411 | 0.49382   |
| 55      | 3.127473350412857 | 0.44943   |
| 60      | 3.128637797891591 | 0.41237   |
| 65      | 3.129624812079802 | 0.38095   |
| 70      | 3.130472076319065 | 0.35398   |
| 75      | 3.131207308587379 | 0.33058   |
| 80      | 3.131851351372613 | 0.31008   |
| 85      | 3.132420179022906 | 0.29197   |
| 90      | 3.132926240627509 | 0.27586   |
| 95      | 3.133379831619937 | 0.26144   |
| 100     | 3.133787490628162 | 0.24845   |
| 1000    | 3.140807746030395 | 0.02498   |
| 10000   | 3.141514118681922 | 0.00250   |
| 100000  | 3.141584799657247 | 0.00025   |
| 1000000 | 3.141591868192124 | 0.00008   |
| 10000000 | 3.141592575049982 | 0.00001   |

The following data tables have been generated using excel and the calculations have been done in code using wolfram mathematica. There is very high accuracy in these values because mathematica has the ability to calculate with to 31 decimal places however the accuracy of the attained values for the percentage errors decrease as the values of $n$ increase because as each method converges towards $\pi$ the percentage error becomes more susceptible to rounding errors and at large $n$ values the percentage error becomes so small that Mathematica has trouble calculating and a rounding error is produced as the percentage error is only taken to 5 decimal places. However, this error is insignificant and therefore will not effect the results drastically.
### 8.2 The Gregory-Leibniz Series

Table 2: As \( n \) increases how does the Gregory-Leibniz Series converge towards \( \pi \)

| n value | Method#2 | Error(%) |
|---------|----------|----------|
| 5       | 2.976046176046176 | 5.26951 |
| 10      | 3.232315809405593 | 2.88781 |
| 15      | 3.079153394197426 | 1.98750 |
| 20      | 3.189184782277595 | 1.51490 |
| 25      | 3.103145312886011 | 1.22382 |
| 30      | 3.173842337190749 | 1.02654 |
| 35      | 3.113820229023573 | 0.88402 |
| 40      | 3.165979272843215 | 0.77625 |
| 45      | 3.119856090062712 | 0.69110 |
| 50      | 3.161198612987056 | 0.62408 |
| 55      | 3.123736933726277 | 0.56837 |
| 60      | 3.157984995168666 | 0.52178 |
| 65      | 3.126442007766234 | 0.48226 |
| 70      | 3.155676462307475 | 0.44830 |
| 75      | 3.128435328236984 | 0.41581 |
| 80      | 3.153937862272616 | 0.37012 |
| 85      | 3.129965139593801 | 0.34978 |
| 90      | 3.152581332875124 | 0.33156 |
| 95      | 3.131176269454981 | 0.31515 |
| 100     | 3.151493401070910 | 0.31118 |
| 1000    | 3.142591654339543 | 0.00318 |
| 10000   | 3.141692643590543 | 0.00032 |
| 100000  | 3.141602653489794 | 0.00003 |
| 1000000 | 3.141591868192127 | 0.00001 |
| 10000000| 3.141592653518277 | 0.00000 |
8.3 Newton's Arcsine Expansion Series for $\pi$

Table 3: As $n$ increases how does Newton's Arcsine Expansion converge towards $\pi$

| n value | Method#3         | Error(%) |
|---------|------------------|----------|
| 5       | 3.141576715774866 | 0.00051  |
| 10      | 3.141592646875561 | 0.00000  |
| 15      | 3.141592653585951 | 0.00000  |
| 20      | 3.141592653589791 | 0.00000  |
| 25      | 3.141592653589793 | 0.00000  |
| 30      | 3.141592653589793 | 0.00000  |
| 35      | 3.141592653589793 | 0.00000  |
| 40      | 3.141592653589793 | 0.00000  |
| 45      | 3.141592653589793 | 0.00000  |
| 50      | 3.141592653589793 | 0.00000  |
| 55      | 3.141592653589793 | 0.00000  |
| 60      | 3.141592653589793 | 0.00000  |
| 65      | 3.141592653589793 | 0.00000  |
| 70      | 3.141592653589793 | 0.00000  |
| 75      | 3.141592653589793 | 0.00000  |
| 80      | 3.141592653589793 | 0.00000  |
| 85      | 3.141592653589793 | 0.00000  |
| 90      | 3.141592653589793 | 0.00000  |
| 95      | 3.141592653589793 | 0.00000  |
| 100     | 3.141592653589793 | 0.00000  |
| 1000    | 3.141592653589793 | 0.00000  |
| 10000   | 3.141592653589793 | 0.00000  |
| 100000  | 3.141592653589793 | 0.00000  |
| 1000000 | 3.141592653589793 | 0.00000  |
| 10000000| 3.141592653589793 | 0.00000  |

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8.4 Continued Fractions for $\pi$

Table 4: As $n$ increases how does the length of fractions converge towards $\pi$

| n value | Method#4 | Error(%) |
|---------|----------|----------|
| 1       | 2.666666666666667 | 15.11741 |
| 2       | 2.800000000000000 | 10.87322 |
| 3       | 3.4285714285714288 | 9.134824 |
| 4       | 2.911111111111111 | 7.33646 |
| 5       | 3.3316017316017322 | 6.04818 |
| 6       | 2.9807081807081808 | 5.12111 |
| 7       | 3.2808080808080808 | 4.43136 |
| 8       | 3.019032601385542 | 3.90121 |
| 9       | 3.250989945726788 | 3.48222 |
| 10      | 3.042842125195067 | 2.11028 |
| 15      | 3.207889026381334 | 1.58658 |
| 20      | 3.091748884841698 | 1.27069 |
| 25      | 3.181512659824787 | 1.05956 |
| 30      | 3.108305614026907 | 0.90853 |
| 35      | 3.170134928816534 | 0.79515 |
| 40      | 3.116612184235621 | 0.70692 |
| 45      | 3.163801158726882 | 0.63630 |
| 50      | 3.121602653391091 | 0.56818 |
| 55      | 3.123742628422234 | 0.53033 |
| 60      | 3.124931773880152 | 0.48956 |
| 65      | 3.156972717366711 | 0.45388 |
| 70      | 3.127333523307754 | 0.42432 |
| 75      | 3.154923023986425 | 0.39781 |
| 80      | 3.129095094977018 | 0.37442 |
| 85      | 3.153355324069958 | 0.35362 |
| 90      | 3.130483257145759 | 0.33502 |
| 95      | 3.152117511448855 | 0.31827 |
| 100     | 3.131593903583553 | 0.29872 |
8.5 Viete’s Formula of Nested Fractions for $\pi$

Table 5: As $n$ increases how does Viete's formula converge towards $\pi$

| n value | Method#5       | Error(%)  |
|---------|---------------|-----------|
| 1       | 3.061467458921242 | 2.55046   |
| 2       | 3.121445152263491 | 0.64135   |
| 3       | 3.136548490541725 | 0.16056   |
| 4       | 3.140331156952385 | 0.04015   |
| 5       | 3.141277250932773 | 0.01004   |
| 6       | 3.141513801175428 | 0.00251   |
| 7       | 3.141572940255612 | 0.00063   |
| 8       | 3.141587725373528 | 0.00016   |
| 9       | 3.141591413562714 | 0.00004   |
| 10      | 3.141592345570118 | 0.00000   |
| 15      | 3.141592653288993 | 0.00000   |
| 20      | 3.141592653589493 | 0.00000   |
| 25      | 3.141592653589793 | 0.00000   |
| 30      | 3.141592653589793 | 0.00000   |
| 35      | 3.141592653589793 | 0.00000   |
| 40      | 3.141592653589793 | 0.00000   |
| 45      | 3.141592653589793 | 0.00000   |
| 50      | 3.141592653589793 | 0.00000   |
| 55      | 3.141592653589793 | 0.00000   |
| 60      | 3.141592653589793 | 0.00000   |
| 65      | 3.141592653589793 | 0.00000   |
| 70      | 3.141592653589793 | 0.00000   |
| 75      | 3.141592653589793 | 0.00000   |
| 80      | 3.141592653589793 | 0.00000   |
| 85      | 3.141592653589793 | 0.00000   |
| 90      | 3.141592653589793 | 0.00000   |
| 95      | 3.141592653589793 | 0.00000   |
| 100     | 3.141592653589793 | 0.00000   |
8.6 The Zeta Function

Table 6: As \( n \) increases how do the different Zeta Series converge towards \( \pi \)

| \( n \) value | \( \zeta(2) \)       | \( \zeta(4) \)       | \( \zeta(6) \)       | \( \zeta(8) \)       |
|--------------|---------------------|---------------------|---------------------|---------------------|
| 5            | 3.09466952411370    | 3.14016117947426    | 3.14157300346359    | 3.14159231269578    |
| 10           | 3.04936163598207    | 3.14138462246697    | 3.14159185608168    | 3.14159264970117    |
| 15           | 3.079388982603209   | 3.14152783068467    | 3.14159253913011    | 3.1415926533235    |
| 20           | 3.0946952411370     | 3.14156460959141    | 3.14159262524305    | 3.14159265355327    |
| 25           | 3.10392339170058    | 3.14157807684660    | 3.14159264406125    | 3.14159265358185    |
| 30           | 3.11012872814126    | 3.14158413278489    | 3.14159264969505    | 3.14159265358752    |
| 35           | 3.1145786229313     | 3.14158724909022    | 3.14159265176594    | 3.14159265358901    |
| 40           | 3.11792619829938    | 3.14158901347572    | 3.14159265264583    | 3.14159265358948    |
| 45           | 3.12053546308708    | 3.14159008631043    | 3.14159265306227    | 3.14159265358966    |
| 50           | 3.1226265293373     | 3.14159077577492    | 3.14159265327654    | 3.14159265358973    |
| 55           | 3.12433980504914    | 3.14159123889581    | 3.14159265339440    | 3.14159265358976    |
| 60           | 3.12576920214052    | 3.14159156142938    | 3.14159265346284    | 3.14159265358977    |
| 65           | 3.12697987310384    | 3.14159179292185    | 3.14159265350444    | 3.14159265358978    |
| 70           | 3.12801845342065    | 3.14159196334879    | 3.14159265353070    | 3.14159265358979    |
| 75           | 3.12891920064047    | 3.14159209159432    | 3.14159265354784    | 3.14159265358979    |
| 80           | 3.12970784547462    | 3.14159218993944    | 3.14159265355935    | 3.14159265358979    |
| 85           | 3.13040408931831    | 3.14159226661411    | 3.14159265356727    | 3.14159265358979    |
| 90           | 3.13102327252367    | 3.14159232727297    | 3.14159265357284    | 3.14159265358979    |
| 95           | 3.1315751780151     | 3.14159237588858    | 3.14159265357684    | 3.14159265358979    |
| 100          | 3.13207653180911    | 3.14159241530737    | 3.14159265357975    | 3.14159265358979    |
Table 7: As $n$ increases how does the percentage error converge towards zero

| n value | $\zeta(2)$ | $\zeta(4)$ | $\zeta(6)$ | $\zeta(8)$ |
|---------|------------|------------|------------|------------|
| 5       | 1.49361    | 0.04557    | 0.00041    | 0.00001    |
| 10      | 2.93582    | 0.00662    | 0.00017    | 0.00000    |
| 15      | 1.97998    | 0.00206    | 0.00005    | 0.00000    |
| 20      | 1.49361    | 0.00089    | 0.00003    | 0.00000    |
| 25      | 1.19905    | 0.00046    | 0.00000    | 0.00000    |
| 30      | 1.00153    | 0.00027    | 0.00000    | 0.00000    |
| 35      | 0.85988    | 0.00017    | 0.00000    | 0.00000    |
| 40      | 0.75333    | 0.00012    | 0.00000    | 0.00000    |
| 45      | 0.67027    | 0.00008    | 0.00000    | 0.00000    |
| 50      | 0.60371    | 0.00006    | 0.00000    | 0.00000    |
| 55      | 0.54918    | 0.00003    | 0.00000    | 0.00000    |
| 60      | 0.50368    | 0.00001    | 0.00000    | 0.00000    |
| 65      | 0.46514    | 0.00000    | 0.00000    | 0.00000    |
| 70      | 0.43208    | 0.00000    | 0.00000    | 0.00000    |
| 75      | 0.40341    | 0.00000    | 0.00000    | 0.00000    |
| 80      | 0.37831    | 0.00000    | 0.00000    | 0.00000    |
| 85      | 0.35614    | 0.00000    | 0.00000    | 0.00000    |
| 90      | 0.33643    | 0.00000    | 0.00000    | 0.00000    |
| 95      | 0.31879    | 0.00000    | 0.00000    | 0.00000    |
| 100     | 0.30291    | 0.00000    | 0.00000    | 0.00000    |
9 Conclusions

9.1 Conclusion: The Wallis Product

The Wallis product provides a partially slow method of convergence towards $\pi$. It can be seen from the data table and percentage error that the Wallis Product does not oscillate between two values but instead converges at a slower rate shown by the shallowness of the curve of Fig.1. The Values of $\pi$ generated are always increasing therefore the graph attained can be deemed plausible and correct as it curves upward in Fig.1. The percentage error shows that even when $n = 10000000$ the actual value of $\pi$ attained contains an error at only the 7th decimal place.

In addition, the % error in Fig.2 gives rise to a curve which starts off with a steep fairly high percentage error and which decreases rapidly and then the rate at which the values attained become more accurate is slowed as the curve becomes much more shallow. On inspection of the error curve from n=5 and n=10 with a comparison to n=55 and n=60 this hypothesis can be proven correct.

An explanation of this lies within the fact that the product only has small increments of numbers for example:

Here, it can be seen that the fact that the Wallis product converges so slowly is due to the fact that it is being increased by such minor values that it would require a large amount of iterations to approach the true value of $\pi$ at 15 decimal places.

9.2 Conclusion: Eulers Continued Fractions

The Continued fraction used in this method for calculating $\pi$ is one of the slowest continued fractions formula used to calculate pi. An explanation for this inefficiency (Fig.9) lies in the fact that the values of $\pi$ per iteration alternate and oscillate between the positive and negative regions about the line $y = \pi$ very slowly and in turn this results for a very slow convergence.

By inspecting the data from the table and the graph in Fig.10, the percentage error’s clearly illustrate how inefficient and inappropriate the method is for calculating accurate values of $\pi$ as when $n = 1$ the percentage error is quite significant with a large 15.1%. Furthermore, the gradient and drop in percentage error is not as fast as the other methods which accounts for the very shallow curve of convergence towards 0%.

If we look at the actual continued fraction formula itself, the value of each section grows very slowly and therefore has a very slow impact on the value of the whole continued fraction itself. This accounts for part of the inefficiency of this method. In addition, the continued fraction that Euler developed is so slow that it requires ”roughly $3 \times 10^n$ terms to achieve n-decimal precision” [11].

9.3 Conclusion: The Gregory-Leibniz Series

The Gregory-Leibniz Series illustrates an out of date and inefficient infinite series which will converge to $\pi$ at a very slow rate. Some partial reasoning behind this is the fact that the
actual infinite sum contains \((-1)^n\) in the numerator and therefore causes the partial sums to oscillate between the positive and negative regions above and below the line \(y = \pi\). This can be seen through Fig.3 that the attained \(\pi\) values which alternate between every \(n\) value either going far much above the required value or too far below.

Both the Table of Data and the graph emphasize the inefficiency of the infinite sum. Fig.3 illustrates the slow convergence towards \(\pi\) as the graph is not very steep at the start and becomes even more shallow as the \(n\) values increase which is also supported by the fact that "Calculating \(\pi\) to 10 correct decimal places using direct summation of the series requires about 5,000,000,000 terms"[^4].

In inspecting the inefficiency, it is important to note that even though Mathematica would display a percentage error of 0.00000% at the 10000000\(th\) \(n\) term there is still a margin of error as the infinite series makes an error at only the 8th decimal point. Taking into account that it was discovered in 1670's where mathematical analysis was not yet fully developed and that it was the "first ever found infinite series for \(\pi\"[^4], it would have been a tremendous discovery by James Gregory and Gottfried Wilhelm Von Leibniz at the time. Furthermore, "In order to achieve 100 accurate decimal places, one would have to go through \(1 \times 10^{17}\) iterations of the series"[^4] which illustrates how it cannot satisfy modern needs for the computation and calculation of accurate values of \(\pi\) shown in Fig.4.

### 9.4 Conclusion: Newtons Arcsine Series Expansion

Newton’s Arcsine Expansion provides a rapidly converging infinite series which attains an accurate value of \(\pi\) between \(n=20\) and \(n=10\). An explanation for this accelerated convergence can be seen within the actual infinite function. If one pays close attention to the components of the infinite series, it can be seen that \((2n)!\) and \((n!)^2\) are present on the numerator and denominator respectively. The actual values of these factorials expand rapidly as \(n\) increases. This can be shown in the sub-table below:

| \(n\) value | \(\frac{2^n}{2n-1} \times \frac{2^n}{2n+1}\) |
|-------------|---------------------------------------|
| 3           | \(\frac{36}{35}\)                     |
| 4           | \(\frac{64}{63}\)                     |
| 5           | \(\frac{100}{99}\)                    |

Of course the value of \((2n)!\) will increase at a much more rapid rate which also converges to a limit fold which would further then explain the shape of the convergence graph as curve-
ing upwards exponentially as it approaches the value of \( \pi \) in Fig.5. Also, note that the middle binomial coefficient is present in Newt

on’s Arcsine expansion which is \( \binom{2n}{n} \) and this also has a high numerical value as \( n \) becomes large which accounts for the rapid convergence.

In addition, by inspecting both the data tables and Fig.6 for Newton’s Arcsine expansion, it can be seen that between the terms in which \( n = 5 \) and \( n = 10 \) the series assumes a percentage error of less that 0.00005% which accounts for the extremely steep slope in the graph for percentage error and the extremely fast convergence in the graph displaying the corresponding \( \pi \) values. Alternatively, Sterling’s approximation could also be used here to investigate the rate of convergence.

9.5 Conclusion: Vieta’s Nested Radical Formula

Vieta’s Formula consists of nested radicals which provide a fairly fast convergence towards \( \pi \). From the table, it can be seen that at the start with the first iteration at the first \( n \) value there is approximately 2.55% error and provides an incorrect value at the first decimal point. However it can be seen through the steepness of the percentage error in Fig.8 between \( n = 1 \) and \( n = 2 \) that the error rapidly decreases which can be seen by thorough inspection at the second percentage error where \( n = 2 \) has decreased by approximately a factor of 4. Furthermore, by considering the \( n \) value in which Vieta’s Formula provides an accurate value of \( \pi \) it can be seen that between \( n = 10 \) and \( n = 15 \) the re-iterative values reach the correct display of \( \pi \) for 15 decimal places which remains consistent for the rest of the iterations.

Furthermore, the explanation behind this lies within the function itself. The whole use of nested radicals allows the function to not fluctuate between the positive and negative regions about the line \( y = \pi \) and only use the positive region. Furthermore, if we consider the following part of Vieta’s Formula:

\[
\sqrt{\frac{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}}}{a}} = a
\]

Then, by squaring both sides of the equation,

\[
2 + a = a^2
\]

\[
a^2 - a - 2 = 0
\]

\[
(a - 2)(a + 1) = 0
\]

Since \( a \) must be positive, \( a = 2 \)

From this, if we now consider the whole form of Vieta’s Formula:

\[
2^{2n+1} \sqrt{\frac{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}}{a \to 2}}
\]

Therefore, as \( n \) increases the value of \( 2^{2n+1} \) will increase and the value of \( a \) nested roots two radicals will approach two and the value of the square root will decrease. However, it
is important to note that the speed at which $2^{2n+1}$ increases will be faster than the speed that the nested roots approach 2. This in turn explains the shape of the graph as it provides evidence that the graph should rapidly accelerate upwards because at first the rate at which $2^{2n+1}$ increases will be much greater than the rate of decrease in the nested radicals. Then, as shown in fig.7, the curve should flatten as the rates increase of both terms equal out until they are balanced at which $n$ tends towards infinity $\infty$ and the exact value of $\pi$ is obtained.

9.6 Conclusion: The Reimann-Zeta Function

The series developed by the Reimann-Zeta Function for $\zeta(2), \zeta(4), \zeta(6), \zeta(8)$ show an increase in accuracy as the the value of $n$ in the Zeta function increases. This in turn provides evidence for the attained graphs for the different $n$ values. For example, by looking at Fig.11 for $\zeta(2)$ one can notice that the gradient flattens out at a much slower rate than which accounts for a slower rate of convergence compared to Fig.13 where there is a much steeper gradient at the start and the rate at which the series converges is much faster.

Furthermore, by considering the various graphical displays for the error values, one can notice as expected that as the value of $n$ increases, the amount of error in the series decreases at a much faster rate. Take for example the error in Fig.15 with the graph of $\zeta(6)$. Here it is evident that the gradient at the start is quite steep and begins to touch close to 0% at about $n = 20$. Similarly, if one examine’s the graph of error in $\zeta(8)$ through Fig.18 it can be seen that the gradient is much steeper at the start which signifies how accurate this method is in calculating $\pi$.

The explanation behind this lies in the Zeta function itself. Firstly, consider how the nature of how $\zeta(2)$ is defined. The series is based on having a square root of the summation of series with fractions with a numerator of 1 and denominator of $n^2$. Because of this, each additional fraction added in the series will have less of an influence on the value of $\pi$ attained and in turn part of the series accounts for a slower convergence towards $\pi$ in Fig.11. Contrastingly, if one pays close attention to the series obtained by $\zeta(8)$, it is evident that here, the square root is raised to a power of 8 and each additional fraction added in the summation has a numerator of 1 and denominator of $n^8$. Because of this, each additional fraction added becomes even less influential to the value of $\pi$ attained which can be seen in the table below. Conclusively, this accounts for the shape of the error curves in the Figures 12 and 18 as Fig.12 is takes an $n$ value above $n = 100$ to drop below 0.3% error but for $\zeta(8)$, between $n = 5$ and $n = 10$ the% error already drops to 0.0% which shows a much steeper curve in Fig.18.

| $\zeta(n)$ | 3rd Fractional Order Value |
|------------|---------------------------|
| $\frac{1}{\sqrt{2}}$ | 0.11111 |
| $\frac{1}{\sqrt{4}}$ | 0.01232 |
| $\frac{1}{\sqrt{8}}$ | 0.00137 |
| $\frac{1}{\sqrt{6}}$ | 0.00002 |
10 Comparisons of the different methods

10.1 The Gregory-Leibniz Series VS Newtons Series: The Arcsine Expansion

\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \quad \text{And} \quad \sum_{n=0}^{\infty} \frac{(2n)!}{(2n+1)(n!)^2 2^{2n}} x^{2n+1}
\]

The infinite series on the left represents the Gregory-Leibniz series and the infinite series on the right represents Newton’s infinite series for the arcsine function. By paying close attention to the terms in common and only considering the terms that are not in common we obtain the following (as \(x = 1\) in the Gregory-Leibniz Series and \(x = 0.5\) in Newton’s Arcsine Expansion):

\[
\sum_{n=0}^{\infty} (-1)^n \quad \text{And} \quad \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2 2^{2n}}
\]

Considering this, it is obvious that the second infinite sum will converge much faster than the first because as previously stated in the analysis for Newton’s infinite series, the value of \(\frac{(2n)!}{2^{2n}(n!)^2}\) will converge at a much faster rate because the use of \((2n)!\) and \((n!)\) allows the series to converge towards a value much faster than the infinite series containing \((-1)^n\) because there is no fluctuation.

Furthermore, by having a close look at Fig.19, The graph of comparison clearly confirms the fact that Newton’s arcsine expansion from previous reasoning is a much more efficient way to achieve an accurate value of pi as by \(n=30\), the Newton’s arcsine expansion curve is practically linear whilst the Gregory-Leibniz oscillations are still oscillating in wide amounts.

Also, Fig.20 yet again emphasizes how accurate and efficient the arcsine expansion is compared to the Gregory-Leibniz series as the red plot is always underneath the blue plot. It is important to note that there is some percentage error and the line \(y = 0\) is an asymptote as there will always be some error however Mathematica was not able to calculate to that precision.

10.2 Vieta’s Nested Radicals VS Euler’s Continued Fraction

\[
\lim_{n \to \infty} 2^{n+1} \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}} \quad \text{And} \quad \frac{4}{1^2} + \frac{4}{2^2} + \frac{4}{3^2} + \frac{4}{5^2} + \frac{4}{7^2} + \frac{4}{9^2} + \ldots
\]
Vieta’s Nested radicals provide a much faster convergence than Euler’s Continued fraction for $\pi$ because Euler’s continued fraction provides a very slow increase in numerators of the continued fractions as they increase by the unique sequence of odd numbered squares. By considering how the nested $\sqrt{2}$ radicals work we can see that the fact that as $n = 3$ the nested fractions are already near complete convergence at 1.96157 and approaching towards 2 which proves why nested radicals are more efficient in calculating $\pi$.

Using graphical evidence, Fig.22 provides evidence for the previously mentioned hypothesis as said before Vieta’s nested radicals converges at a much faster rate than Euler’s continued fractions. While Vieta’s continued fractions have reached $n=25$, Euler’s continued fraction is still oscillating above and below the line $y = \pi$. Also, Fig.21 illustrates how there is virtually no percent error for Vieta’s nested fractions past $n=25$ but at $n=25$ Euler’s continued fractions still obtains a percentage error of 1.5%. Also, it becomes more difficult to calculate the actual values for Vieta’s square roots because each additional square root increase the likelyhood for a rounding error.

### 10.3 The Wallis Product VS Newtons Series: The Arcsine Expansion

$$\prod_{k=1}^{\infty} \left[ \frac{2k}{2k-1} \right] \left[ \frac{2k}{2k+1} \right] \text{ And } \sum_{n=0}^{\infty} \frac{(2n)!}{(2n+1)(n!)2^{2n}} x^{2n+1}$$

The product on the left represents the wallis product for $\pi$ and the infinite series on the right represents Newton’s Arcsine expansion. Here it is clear that Newton’s Arcsine expansion will give a much faster convergence rate. Take the transition between $k = 6 \rightarrow k = 7$ and $n = 6$ to $n = 7$ as examples. The value of the numerator of the wallis product will increase by 104 (See Calculation 1.00) and the denominator by a factor of 13x15 which is a minor increment compared to the numerator in the Arcsine function which increases by $8.67 \times 10^{10}$. The power of the factorial allows for a faster convergence then by multiplication hence the reason why the infinite series will converge faster than the product.

$$[1.00] \text{ change } = 2.2.2.4.4.6.6.8.8.10.10.12.12 \frac{2.14.14.}{1.3.3.5.5.7.7.9.9.11.11.13.} (13.15 - 1)$$

As shown in Fig.23, The speed of convergence for Newton’s Arcsine still remains unparalleled as it is able to converge much more rapidly than the Wallis product as the blue plot is linear and the purple plot remains as a curve. Fig.24 also illustrates the accuracy of Newton’s arcsine expansion as the percentage error values are constantly far below the percentage error values of the Wallis Product.

### 10.4 The Wallis Product VS The $\zeta(2)$ Series

$$\prod_{k=1}^{\infty} \left[ \frac{2k}{2k-1} \right] \left[ \frac{2k}{2k+1} \right] \text{ And } \sum_{n=0}^{\infty} \sqrt{6 \left[ \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ... \right]}$$

The product on the left is the Wallis Product for $\pi$ and the infinite series on the right represents the infinite series created at $\zeta(2)$ which in turn converges towards $\pi$. Here it is clear
that the series for $\zeta(2)$ converges at a slightly slower rate than the Wallis Product because of the fact that firstly the infinite series makes use of multiplication of 2 instead whilst the Wallis Product have a larger multiplication increment between each n value which is greater than 2 and secondly because of the fact that a square root is used in place of multiplication which causes the $\zeta(2)$ series to be slower.

By examining the graphical display in Fig.25, it is clear that the Wallis Product will converge at a faster rate which is shown as the graph representing the $\zeta(2)$ infinite series is always slightly below the graph of the Wallis Product. Furthermore, by taking into consideration the % error in each of the methods, the results shown in Fig.26 consolidate what was previously mentioned as at all values for n, the graph of the infinite series with $\zeta(2)$ is always above the graph of the Wallis Product which in turn shows that the rate of accuracy increases at a much larger rate in the Wallis product. Also, by taking into account the gradient of the percentage error curves, between values $n = 5$ and $n = 10$ one can see that the gradient of the line for the Wallis Product is much steeper than the gradient of the line for $\zeta(2)$ which emphasizes the fact that the rate of accuracy for the Wallis Product increases at a slightly faster pace than the rate of accuracy for the infinite series.

### 10.5 Newton’s Arcsine Expansion VS The $\zeta(8)$ Series

$$\sum_{n=0}^{\infty} \frac{(2n)!}{(2n+1)(n!)2^{2n}x^{2n+1}} \text{ And } \sum_{n=0}^{\infty} 8\sqrt{9450} \left[ \frac{1}{1^8} + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \ldots \right]$$

The infinite series on the left represents one of the most efficient ways to attain a value of $\pi$ developed by Newton and his expansion of arcsine. The series on the right is the fastest way in attaining a value of $\pi$ covered in this thesis which is the infinite series formed using $\zeta(8)$. In this case, it is clear that not only does Newton’s arcsine expansion makes use of rapidly increasing factorial values shown in $(2n)!$ and $(n!)$ but also uses high valued exponentials such as $2^{2n}$. However, these factor’s are outweighed by the infinite series developed from the Reimann-Zeta Function of value $\zeta(8)$ because here there is not only multiplication by 9450 but also makes use of the 8th square root and every proceeding term in the summation begins to affect the final product less and less at a much faster rate that the summation of Newton’s Arcsine Expansion.

Furthermore, if we consider the graphical display in Fig.27 and Fig.28 at $n = 5$, the produced value the $\zeta(8)$ series where $n = 5$ is much closer to $\pi$ than the produced value produced by Newtons Arcsine Expansion. Also, The % error where $n = 5$ for $\zeta(8)$ is 26 times smaller than the produced value than the % error given through newtons arcsine expansion which emphasizes the degree of accuracy that the infinite series that $\zeta(8)$ has over the series from Newton’s Arcsine expansion.
11 Appendix

11.1 Mathematic Iteration code: The Wallis Product

"Wallis Product"  For[i = 0, i != 100, i = i + 5, Print[i " SymbA ", NumberForm[N[2 /!(*UnderoverscriptBox[/(i - 1)/, i)])/(2 k - 1)]/*FractionBox[/(2 k/), /(2 k + 1/)], 16]]

5 SymbA 3.002175954556907
10 SymbA 3.067703806643499
15 SymbA 3.091336888596228
20 SymbA 3.103516961539234
25 SymbA 3.11094516690154
30 SymbA 3.115948285887959
35 SymbA 3.119547206305518
40 SymbA 3.122260326421437
45 SymbA 3.124378835915516
50 SymbA 3.126078900215411
55 SymbA 3.12743350412857
60 SymbA 3.128637797891591
65 SymbA 3.129624812079802
70 SymbA 3.130472076319065
75 SymbA 3.131207308587379
80 SymbA 3.13185135172613
85 SymbA 3.132420179022906
90 SymbA 3.132926240627509
95 SymbA 3.133379381619937
100 SymbA 3.133787490628162
1000 SymbA 3.140807746030395
10000 SymbA 3.141514118681902
100000 SymbA 3.141584799657247
1000000 SymbA 3.141592657504982

11.1.1 Mathematic Iteration code: The Wallis Product Percentage Error

- 1 - N[3.002175954556907/1, 15] N[0.044377713601277735*100, 15] 4.43777
- N[(3.067703806643499/1, 15] - 0.9764804495382736 0.02351955046172638*100 = 2.35196
- 3.091336888596228/1 - 0.9840030931648189 0.01599690683518107*100 = 1.59969
- 3.103516961539234/1 - 0.9878801307970174 0.01211986920298258*100 = 1.21199
- 3.110945166901554/1 - 0.9902446020004474 0.00975539799552606 0.97554
- 3.115948285887959/1 - 0.9918371442355739 100*0.00816285764426102 0.81629
- 1 - 0.992982716840716 100*0.007017283815928366 0.701728
- 3.12260326421437/1 - 0.9938463291393728 100*0.00615367086062736 0.615367
- 3.124378835915516/1 - 0.9945206716553123 100*0.005479328344687673 0.547933
- 3.126078900215411/1 - 0.995061819291169 100*0.004938181070883063 0.493818
- 3.127473350412857/1 - 0.9955056862127551 100*0.00449431378724852 0.449431
11.2 Mathematically Iteration code: The Gregory-Leibniz Series

For[i = 0, i <= 100, i = i + 5, Print[i " SymbA = ", NumberForm[N[4 / !(//*UnderoverscriptBox[/(/
\[Sum\]/, /(n = 0/), /(i/)
*/FractionBox[ SuperscriptBox[/(((-1/))/), /(n/))], /(2 n + 1)/]], 16]]

| i   | SymbA          |
|-----|----------------|
| 5   | 2.976046176046176 |
| 10  | 3.232315809405593 |
| 15  | 3.07915394197426 |
| 20  | 3.189184782275595 |
| 25  | 3.10314531288601 |
| 30  | 3.17384237190749 |
| 35  | 3.11382022903573 |
| 40  | 3.16597927843215 |
| 45  | 3.119856090062712 |
| 50  | 3.161198612987056 |
| 55  | 3.123736933726277 |
| 60  | 3.15798495168666 |
| 65  | 3.126442007766234 |
| 70  | 3.155676462307475 |
| 75  | 3.128435328236984 |
| 80  | 3.153937862272616 |
| 85  | 3.129965139593801 |
| 90  | 3.152581332875124 |
| 95  | 3.131176269454981 |
| 100 | 3.151493401070910 |
| 1000 | 3.142591654339543 |
| 10000 | 3.141692643590543 |
| 100000 | 3.141602653489794 |
| 1000000 | 3.141591868192127 |
| 10000000 | 3.141592653518272 |
11.2.1 Mathematic Iteration code: The Gregory-Leibniz Series Percentage Error

- \( 2.976046176046176 / 1 - 0.9473049195749638 \times 100\% = 0.052695080425036234 \times 100\% = 5.26951 \%
- \( 3.2321589456593 / 1 - 1.0288780774019617 \times 100\% = -0.9473049195749638 \times 100\% = -9.47305 \%
- \( 3.079153394197426 / 1 - 0.98012496449155 \times 100\% = 0.01987503355058451 \times 100\% = 1.9875 \%
- \( 3.18914782277595 / 1 - 1.0151490450658587 \times 100\% = -0.01514904506585868 \times 100\% = -1.5149 \%
- \( 3.103145312886011 / 1 - 0.9877618313565096 \times 100\% = 0.012238168643490366 \times 100\% = 1.22382 \%
- \( 3.1738237190749 / 1 - 1.0102653931164836 \times 100\% = -0.010265393116483557 \times 100\% = -1.02654 \%
- \( 3.113820229023573 / 1 - 0.991159762972786 \times 100\% = 0.00884023730274132 \times 100\% = 0.88402 \%
- \( 3.16597927243215 / 1 - 1.0077625019989642 \times 100\% = -0.007762501998964222 \times 100\% = -0.77625 \%
- \( 3.119856090062712 / 1 - 0.993081036937668 \times 100\% = 0.006918963062331973 \times 100\% = 0.691896 \%
- \( 3.16119861297056 / 1 - 1.00624077042669 \times 100\% = -0.0062407704266874 \times 100\% = -0.624077 \%
- \( 3.1237393726277 / 1 - 0.9943163478425145 \times 100\% = 0.005683652157485453 \times 100\% = 0.568365 \%
- \( 3.15798495168666 / 1 - 1.0052178443822568 \times 100\% = -0.005217844382256809 \times 100\% = -0.521784 \%

11.3 Mathematic Iteration code: Newtons Arcsine Expansion Series

For \( i = 0, i = 100, i = i + 5 \), Print \[ i \] " SymbA", NumberForm[N[6 /!!/*UnderoverscriptBox[/((Sum))//,//((n = 0)),//((i))//]/*FractionBox*/[/*UnderoverscriptBox*/[2n]]///*SuperscriptBox*/[1/2], //((2 n + 1))//]//], ///*SuperscriptBox*/[2, //((2 n))], ///*SuperscriptBox*/[1//n//, //((2 n + 1))]//], //((2 n))], //(((//n//))), //((2))], //((2 n + 1)))], 16]"

5 SymbA 3.141576715774866
10 SymbA 3.141592646875561
15 SymbA 3.141592653589793
20 SymbA 3.141592653589793
25 SymbA 3.141592653589793
30 SymbA 3.141592653589793
35 SymbA 3.141592653589793
40 SymbA 3.141592653589793
45 SymbA 3.141592653589793
50 SymbA 3.141592653589793
55 SymbA 3.141592653589793
60 SymbA 3.141592653589793
65 SymbA 3.141592653589793
70 SymbA 3.141592653589793
75 SymbA 3.141592653589793
80 SymbA 3.141592653589793
85 SymbA 3.141592653589793
90 SymbA 3.141592653589793
95 SymbA 3.141592653589793
100 SymbA 3.141592653589793

58
55 SymbA 3.141592653589793
60 SymbA 3.141592653589794
65 SymbA 3.141592653589793
70 SymbA 3.141592653589793
75 SymbA 3.141592653589794
80 SymbA 3.141592653589794
85 SymbA 3.141592653589794
90 SymbA 3.141592653589794
95 SymbA 3.141592653589793
100 SymbA 3.141592653589794
1000 SymbA 3.142591654339543
10000 SymbA 3.14169263590543
1000000 SymbA 3.14169263590543

11.3.1 Mathematic Iteration code: Newton's Arcsine Series Percentage Error

- 3.141592653589795/1 - 0.999999999998777' 100*1.2230216839270724*-12 1.22302*10*-10
- 3.14159264867556/1 - 0.99999978627935' 100*5.073164055402479*-6 0.00507316

12 Mathematic Iteration code: Continued Fractions for $\pi$

$g[n,x] = \frac{(2n+1)2}{2+x}$ NumberForm[$\frac{4}{1+g[0, g[1, g[2, g[3, g[4]]]]]}, 16$] TagBox[InterpretationBox[$\frac{3.331601731601732}{15}, \frac{3.3316017316017317}{15}$, AutoDelete->True], NumberForm[.16] NumberForm[$\frac{4}{1+g[0, g[1, g[2, g[3, g[4, g[5, g[6, g[7, g[8, g[9, g[10, g[11, g[12, g[13, g[14]]]]]]]]]]]]], 16$]

TagBox[InterpretationBox[$\frac{3.042842125195067}{15}, \frac{3.0428421251950666}{15}$, AutoDelete->True], NumberForm[.16] NumberForm[$\frac{4}{1+g[0, g[1, g[2, g[3, g[4, g[5, g[6, g[7, g[8, g[9, g[10, g[11, g[12, g[13, g[14, g[15, g[16, g[17, g[18, g[19]]]]]]]]]]]], 16$]

TagBox[InterpretationBox[$\frac{3.2078926381334}{15}, \frac{3.2078926381334}{15}$, AutoDelete->True], NumberForm[.16] NumberForm[$\frac{4}{1+g[0, g[1, g[2, g[3, g[4, g[5, g[6, g[7, g[8, g[9, g[10, g[11, g[12, g[13, g[14, g[15, g[16, g[17, g[18, g[19, g[20, g[21, g[22, g[23, g[24]]]]]]]]]]]]], 16$]

TagBox[InterpretationBox[$\frac{3.181512659824787}{15}, \frac{3.1815126598247874}{15}$, AutoDelete->True], NumberForm[.16] NumberForm[$\frac{4}{1+g[0, g[1, g[2, g[3, g[4, g[5, g[6, g[7, g[8, g[9, g[10, g[11, g[12, g[13, g[14, g[15, g[16, g[17, g[18, g[19, g[20, g[21, g[22, g[23, g[24, g[25, g[26, g[27, g[28, g[29]]]]]]]]]]]]], 16$]

TagBox[InterpretationBox[$\frac{3.108305614026907}{15}, \frac{3.108305614026907}{15}$, AutoDelete->True], NumberForm[.16] NumberForm[$\frac{4}{1+g[0, g[1, g[2, g[3, g[4, g[5, g[6, g[7, g[8, g[9, g[10, g[11, g[12, g[13, g[14, g[15, g[16, g[17, g[18, g[19, g[20, g[21, g[22, g[23, g[24, g[25, g[26, g[27, g[28, g[29]]]]]]]]]]]]], 16$]
Numbers 1-4 From the Function $n=1-4$

$\frac{4}{1 + g[0]}$, 16

TagBox[ InterpretationBox["3.154923023986425"", 3.1549230239864245], AutoDelete\$_True\], NumberForm[, 16]
12.1 Mathematic Iteration code: Continued Fractions for π

Percentage Error

• In[152]:= 2.666666666666667/ In[153]:= 1 - 0.8488263631567753
  Out[154]= 100*0.1511736368432247
  Out[154]= 15.1174

• In[155]:= 2.800000000000000/
  In[156]= 1 - 0.8912676813146139
  In[157]= 100*0.10873231868538613
  Out[157]= 10.8732

• In[158]:= 3.428571428571428/
  In[159]= 1 - 1.091348181201568
  In[160]= 100*-0.09134818120156796
  Out[160]= -9.13482

• In[161]:= 2.9111111111111111/
  In[162]= 1 - 0.9266354464416416
  In[163]= 100*0.07336455355358385
  Out[163]= 7.33646

• In[164]:= 3.331601731601732/
  In[165]= 1 - 1.0604817679958674
  In[166]= 100*-0.06048176799586735
  Out[166]= -6.04818

• In[167]:= 2.980708180708181/
  In[168]= 1 - 0.948788881748375
  In[169]= 100*0.051211118251685006
  Out[169]= 5.12111

• In[170]:= 3.280808080808081/
  In[171]= 1 - 1.044313646792881
  In[172]= 100*-0.04431364679288108
  Out[172]= -4.43136

• In[173]:= 3.019032601385542/
  In[174]= 1 - 0.9609879237321854
  In[175]:= 100*0.039012076267814555
  Out[175]= 3.90121

• In[176]:= 3.250989945726788/
  In[177]:= 1 - 1.0348222396089417
  In[178]= 100*-0.03482223960894171
  Out[178]= -3.48222

• In[179]:= 3.042842125195067/
  In[180]= 1 - 0.9685667305460856
  In[181]= 100*0.03143326945391445
  Out[181]= 3.14333

• In[182]:= 3.207889026381334/
  In[183]= 1 - 1.0211027908776735
  In[184]= 100*-0.02110279087767352
  Out[184]= -2.11028

• In[185]:= 3.091748884841698/
  In[186]= 1 - 0.9841342356428227
  In[187]= 100*0.015865764357177348
  Out[187]= 1.58658

• In[188]:= 3.181512659824787/
  In[189]= 1 - 1.0127069326411171
  In[190]= 100*-0.01270693264111708
  Out[190]= -1.27069

• In[191]:= 3.108305614026907/
  In[192]= 1 - 0.989404062253424
  In[193]= 100*0.01059593774657575
  Out[193]= 1.05956

• In[194]:= 3.170134928816534/
  In[195]= 1 - 1.009052883788504
  In[196]= 100*-0.00905288378850365
  Out[196]= -0.90529
In[197]:= 3.116612184235621/
    In[198]:= 1 - 0.9920484696430558'
    In[199]:= 100*0.00795153035694418'
    Out[199]= 0.795153

In[200]:= 3.163801158726882/
    In[201]:= 1 - 1.007069186742499'
    In[202]:= 100*-0.007069186742499012'
    Out[202]= -0.706919

In[203]:= 3.121602653391091/
    In[204]:= 1 - 0.9936369853119372'
    In[205]:= 100*0.006363014688062774'
    Out[205]= 0.636301

In[206]:= 3.12374262842234/
    In[207]:= 1 - 0.9943181605205365'
    In[208]:= 100*0.005681839479463546'
    Out[208]= 0.568184

In[209]:= 3.124931773880152/
    In[210]:= 1 - 0.9946966778628444'
    In[211]:= 100*0.005303322724097614'
    Out[211]= 0.530332

In[212]:= 3.156972717366711/
    In[213]:= 1 - 1.0048956263503301'
    In[214]:= 100*-0.00489562635033014'
    Out[214]= -0.489563

In[215]:= 3.12733523307754/
    In[216]:= 1 - 0.9954611778628444'
    In[217]:= 100*0.004538822137155618'
    Out[217]= 0.453882

In[218]:= 3.154923023986425/
    In[219]:= 1 - 1.0042431886837397'
    In[220]:= 100*-0.00424318868373974'
    Out[220]= -0.424319

In[221]:= 3.129095094977018/
    In[222]:= 1 - 0.9960219035403924'
    In[223]:= 100*0.003978096459607561'
    Out[223]= 0.39781

In[224]:= 3.15335324069958/
    In[225]:= 1 - 1.0037441743017588'
    In[226]:= 100*-0.0037441743017587736'
    Out[226]= -0.374417

In[227]:= 3.130483257145759/
    In[228]:= 1 - 0.9964637692823288'
    In[229]:= 100*0.003536230717671174'
    Out[229]= 0.353623

In[230]:= 3.152117511448855/
    In[231]:= 1 - 1.0033501663072186'
    In[232]:= 100*-0.003350166307218627'
    Out[232]= -0.335017

In[233]:= 3.131593903583553/
    In[234]:= 1 - 0.9968172990235336'
    In[235]:= 100*0.003182700976466446'
    Out[235]= 0.31827
13 Mathematic Iteration code: Viete’s Nested Radicals Formula for \( \pi \)

"Viete Formula" \( f[x] := \text{Sqrt}[2 + x]; p[n] := 2(n + 1)\text{Sqrt}[2 - \text{Nest}[f, \text{Sqrt}[2], n - 1]]; a[n] := \text{IntegerPart}[3.3n]; \) For\( i = 0, i \neq 100, i = i + 5, \) Print\( [i \text{ " SymbA " Text[Style[Row[N[p[i], a[i]]]]]}] \)
For\( i = 0, i \neq 10, i = i + 1, \) Print\( [i \text{ " SymbA " Text[Style[Row[N[p[i], a[i]]]]]}] \)

1 " SymbA " 3.061467458921242
2 " SymbA " 3.121445152263491
3 " SymbA " 3.136548490541725
4 " SymbA " 3.140331156952385
5 " SymbA " 3.141277250932773
6 " SymbA " 3.141513801175428
7 " SymbA " 3.141572940255612
8 " SymbA " 3.141587725373528
9 " SymbA " 3.141591413562714
10 " SymbA " 3.141592345570118
15 " SymbA " 3.141592653288993
20 " SymbA " 3.141592653589499
25 " SymbA " 3.141592653589793
30 " SymbA " 3.141592653589793
35 " SymbA " 3.141592653589793
40 " SymbA " 3.141592653589793
45 " SymbA " 3.141592653589793
50 " SymbA " 3.141592653589793
55 " SymbA " 3.141592653589793
60 " SymbA " 3.141592653589793
65 " SymbA " 3.141592653589793
70 " SymbA " 3.141592653589793
75 " SymbA " 3.141592653589793
80 " SymbA " 3.141592653589793
85 " SymbA " 3.141592653589793
90 " SymbA " 3.141592653589793
95 " SymbA " 3.141592653589793
100 " SymbA" 3.141592653589793

13.1 Mathematic Iteration code: Viete’s Nested Radicals Percentage Error

- 3.061467458921242 / 1 - 0.9744953584045994' 100*0.025504641595400557' 2.55046
- 3.121445152263491 / 1 - 0.993586851145937' 100*0.0064131548854062967' 0.641315
- 3.136548490541725 / 1 - 0.9983943930342769' 100*0.0016605609657230942' 0.1660561
- 3.140331156952385 / 1 - 0.9995984531489255' 100*0.00040154685107451904' 0.000401547
- 3.141277250932773 / 1 - 0.9998996042161419' 100*0.000010039578385812042' 0.0000100396

- 3.141513801175428 / 1 - 0.9999749004969581' 100*0.000025099503041858817' 0.0000250995
- 3.141572940255612 / 1 - 0.9999937250508405' 100*6.27494915950153*^-6 0.000627495
- 3.141587725373528 / 1 - 0.99999984313000415' 100*1.568699584874073*^-6 0.00015687
- 3.141591413562714 / 1 - 0.99999996052871216' 100*3.947128783281085*^-7 0.0000394713
14  Mathematic Iteration code: Zeta (2),(4),(6),(8)

\[N\sqrt{6\sum_{k=1}^{n}1/k^2}, n = 5, 10, 15, 20, 50, 100], 15 \] 3.14016117947426

\[N\sqrt{6\sum_{k=1}^{n}1/k^4}, n = 4, 10, 15, 20, 50, 100], 15 \] 3.14138462246697

\[N\sqrt{6\sum_{k=1}^{n}1/k^6}, n = 6, 10, 15, 20, 50, 100], 15 \] 3.14157300346359

\[N\sqrt{6\sum_{k=1}^{n}1/k^8}, n = 8, 10, 15, 20, 50, 100], 15 \] 3.14159231269578
### 14.1 Mathematic Iteration code: Zeta (2), (4), (6), (8)

| Expression | Value |
|-----------|-------|
| 3.0946952411372/3.141592653589793 | 0.985063903997084 |
| 3.12433980504914/3.141592653589793 | 0.9945082477446786 |
| 3.12576920214052/3.141592653589793 | 0.9949632389701472 |
| 3.12697987310384/3.141592653589793 | 0.9953486075066875 |
| 3.12801845342065/3.141592653589793 | 0.9956791978891241 |
| 3.12891206064047/3.141592653589793 | 0.9959659146341453 |
| 3.12970784547462/3.141592653589793 | 0.996216948081543 |
| 3.13040408931831/3.141592653589793 | 0.9964385693081843 |
| 3.1310232752367/3.141592653589793 | 0.9966356615158093 |
| 3.13157751780151/3.141592653589793 | 0.9968120832671165 |
| 3.13207653180911/3.141592653589793 | 0.9970506066992303 |
| 3.132625293373/3.141592653589793 | 0.9973090187000000 |
| 3.13300080591470/3.141592653589793 | 0.9974788635619809 |
| 3.13329530551100/3.141592653589793 | 0.9975809638427620 |
| 3.13350080591470/3.141592653589793 | 0.9976742105790050 |
| 3.13369530551100/3.141592653589793 | 0.9977574814472470 |
| 3.13389530551100/3.141592653589793 | 0.9978174814472470 |
| 3.13409530551100/3.141592653589793 | 0.9978574814472470 |
| 3.13429530551100/3.141592653589793 | 0.9978774814472470 |
| 3.13449530551100/3.141592653589793 | 0.9978874814472470 |
| 3.13469530551100/3.141592653589793 | 0.9978974814472470 |
| 3.13489530551100/3.141592653589793 | 0.9978974814472470 |
| 3.13509530551100/3.141592653589793 | 0.9978974814472470 |
| 3.13529530551100/3.141592653589793 | 0.9978974814472470 |
| 3.13549530551100/3.141592653589793 | 0.9978974814472470 |
| 3.13569530551100/3.141592653589793 | 0.9978974814472470 |
| 3.13589530551100/3.141592653589793 | 0.9978974814472470 |
| 3.13609530551100/3.141592653589793 | 0.9978974814472470 |
| 3.13629530551100/3.141592653589793 | 0.9978974814472470 |
| 3.13649530551100/3.141592653589793 | 0.9978974814472470 |
| 3.13669530551100/3.141592653589793 | 0.9978974814472470 |
| 3.13689530551100/3.141592653589793 | 0.9978974814472470 |
| 3.13709530551100/3.141592653589793 | 0.9978974814472470 |
| 3.13729530551100/3.141592653589793 | 0.9978974814472470 |
| 3.13749530551100/3.141592653589793 | 0.9978974814472470 |
| 3.13769530551100/3.141592653589793 | 0.9978974814472470 |
| 3.13789530551100/3.141592653589793 | 0.9978974814472470 |
| 3.13809530551100/3.141592653589793 | 0.9978974814472470 |
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