An Algorithmic Approach to the Extensibility of Association Schemes

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Abstract
An association scheme which is associated to a height \( t \) presuperscheme is said to be extensible to height \( t \). Smith (1994, 2007) showed that an association scheme \( \mathcal{X} = (Q, \Gamma) \) of order \( d := |Q| \) is Schurian iff \( \mathcal{X} \) is extensible to height \( (d - 2) \). In this work, we formalize the maximal height \( t_{\text{max}}(\mathcal{X}) \) of an association scheme \( \mathcal{X} \) as the largest number \( t \in \mathbb{N} \) such that \( \mathcal{X} \) is extensible to height \( t \) (we also include the possibility \( t_{\text{max}}(\mathcal{X}) = \infty \), which is equivalent to \( t_{\text{max}}(\mathcal{X}) \geq (d - 2) \)). Intuitively, the maximal height provides a natural measure of how close an association scheme is to being Schurian.

For the purpose of computing the maximal height, we introduce the association scheme extension algorithm. On input an association scheme \( \mathcal{X} = (Q, \Gamma) \) of order \( d := |Q| \) and a number \( t \in \mathbb{N} \) such that \( 1 \leq t \leq (d - 2) \), the association scheme extension algorithm decides in time \( O(t) \) if the scheme \( \mathcal{X} \) is extensible to height \( t \). In particular, if \( t \) is a fixed constant, then the running time of the association scheme extension algorithm is polynomial in the order of \( \mathcal{X} \).

The association scheme extension algorithm is used to show that all non-Schurian association schemes up to order 26 are completely inextensible, i.e. they are not extensible to any positive height \( t \in \mathbb{N}_{>0} \). Via the tensor product of association schemes, the latter result gives rise to a multitude of examples of infinite families of completely inextensible association schemes.

Keywords: algebraic computation, association scheme, \( t \)-extension, height \( t \) presuperscheme, maximal height, Schurity, tensor.

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1. Introduction

An association scheme \( \mathcal{X} = (Q, \Gamma) \) is composed of a finite, nonempty set \( Q \) and a partition \( \Gamma \) of the direct square \( Q^2 \), satisfying a certain set of combinatorial conditions (see [3, 27]). For \( t \in \mathbb{N} \), a height \( t \) presuperscheme \( (Q, \Gamma^*) \) (short: \( t \)-prescheme) is composed of a family of sets \( \{\Gamma^s\}_{0 \leq s \leq t} \), where each set \( \Gamma^s \) is a partition of the
direct power $Q^{t+2}$, satisfying a set of higher-dimensional variants of the combinatorial conditions which hold for association schemes (see [24,25,26]). Especially, if $(Q, \Gamma^*)$ is a $t$-prescheme, then $(Q, \Gamma^0)$ constitutes an association scheme. We say that $(Q, \Gamma^0)$ is associated to $(Q, \Gamma^*)$.

An association scheme which is associated to a $t$-prescheme (for some $t \in \mathbb{N}$) is said to be extensible to height $t$. Naturally, if an association scheme is extensible to height $t$, then it is also extensible to any height $t' \in \mathbb{N}$ with $0 \leq t' \leq t$. A fundamental result by Smith [20,21] states that an association scheme $S = (Q, \Gamma)$ of order $d := |Q|$ is Schurian iff $S$ is extensible to height $(d-2)$. A natural question arises: Given an arbitrary association scheme $X = (Q, \Gamma)$, what is the maximal height $t_{\text{max}}(X) \in \mathbb{N}$ which $X$ can be extended to? (Note that we include the possibility $t_{\text{max}}(X) = \infty$, which is equivalent to $t_{\text{max}}(X) \geq (d-2)$). The number $t_{\text{max}}(X)$ may provide an intuitive measure of how close the scheme $X$ is to being Schurian.

In this paper, we introduce the association scheme extension algorithm, which on input an association scheme $X = (Q, \Gamma)$ of order $d := |Q|$ and a number $t \in \mathbb{N}$ such that $1 \leq t \leq (d-2)$, decides in time $d^{O(t)}$ if the scheme $X$ is extensible to height $t$. Furthermore, if $X$ is extensible to height $t$, then the algorithm outputs its unique coarsest $t$-extension $X_t$, which represents the most ‘basic’ way in which $X$ can be extended to a $t$-prescheme. Observe that for a fixed constant $t$, the running time of the association scheme extension algorithm is polynomial in the order of $X$.

The association scheme extension algorithm can be used to compute the maximal height $t_{\text{max}}(X)$ of a given association scheme $X$; in the worst case (i.e. if $t_{\text{max}}(X)$ is large), this may take time exponential in the order of $X$. However, heuristics suggest that association schemes of constant maximal height are far more common than association schemes of large maximal height (in a similar sense as heuristics suggest non-Schurian schemes to be far more common than Schurian schemes). Since the algorithm computes the maximal height $t_{\text{max}}(X)$ of an association scheme $X$ in polynomial time if $t_{\text{max}}(X)$ is constant, it can be quite efficient in practice.

We use the association scheme extension algorithm to prove that all non-Schurian association schemes $X = (Q, \Gamma)$ of order $|Q| \leq 26$ cannot be extended to a positive height $t \in \mathbb{N}_{>0}$ (schemes of the latter type are called completely inextensible). Drawing on this result, the tensor product of association schemes yields a multitude of examples of infinite families of completely inextensible association schemes.

### 1.1. Related Notions

The focus of this work is on the notion of association schemes and their relation to $t$-preschemes. Other notions of combinatorial schemes, which include cellular algebras (Weisfeiler et al. [22]), coherent configurations (Higman [16]), Krasner algebras (Krasner [18]), superschemes (Smith [20,21]) and $m$-schemes (Ivanyos, Karpinski & Saxena [17]) are closely related and in some cases differ only slightly, or just in notation. Moreover, the concept of $t$-preschemes and the association scheme extension algorithm seem to be closely connected to the notion of stable partitions [3] and the $k$-dimensional Weisfeiler-Lehman algorithm [6,23]. Note that we do not provide an exact description of the nature of these connections, or attempt to unify the various notions, as this is beyond the scope of this text.

We remark that the notion of extensibility of association schemes has gained
interest in connection with recent scheme-theoretic approaches to the computational problem of factoring polynomials over finite fields \([2, 17]\). For this line of research, it is of particular interest to gain a more thorough understanding of the combinatorial properties possessed by association schemes which are extensible to a certain height.

1.2. Organization

Section 2 provides an introduction to the theory of combinatorial schemes. Section 2.1 defines the notion of association schemes and the concept of Schurity. Section 2.2 introduces the notion of \(t\)-preschemes and defines the concept of extensibility of association schemes. In Section 3, we introduce adjacency tensors of \(t\)-preschemes and delineate in which sense they express a central combinatorial property of \(t\)-preschemes (see Theorem 3.3). In Section 4, we give a description of the association scheme extension algorithm. Moreover, we list the computational results obtained through the application of the algorithm (see Section 4.3).

2. Definitions

In this section, we introduce the necessary background from the theory of combinatorial schemes to understand the context of this work. We give a survey of association schemes (see Section 2.1) and height \(t\) presuperschemes (short: \(t\)-preschemes, see Section 2.2). Moreover, we formalize the concept of extensibility of association schemes.

2.1. Association Schemes

**Definition 2.1 (Association Scheme).** Let \(Q\) be a finite nonempty set. Then an association scheme \(\mathfrak{X} = (Q, \Gamma)\) on \(Q\) is a partition \(\Gamma = \{C_1, ..., C_s\}\) of the direct square \(Q \times Q\), such that:

(A1) (Identity Relation) \(C_1 := \{(x, x) \mid x \in Q\}\);

(A2) (Transposition) \(\forall C_i \in \Gamma, \; C_i^* := \{(y, x) \mid (x, y) \in C_i\} \in \Gamma\);

(A3) (Intersection) \(\forall C_i \in \Gamma, \; \forall C_j \in \Gamma, \; \forall C_k \in \Gamma, \; \exists c(i, j, k) \in \mathbb{N}. \; \forall (x, y) \in C_k, \; \{\{z \in Q \mid (x, z) \in C_i, \; (z, y) \in C_j\} = c(i, j, k)\}.

We refer to the numbers \(c(i, j, k)\) as the intersection numbers of \(\mathfrak{X}\). Moreover, we call \(|Q|\) the order of \(\mathfrak{X}\).

A classical example of association schemes is provided by Schurian association schemes, which are defined below. Let \(Q\) be a finite nonempty set and let \(G\) be a transitive permutation group on \(Q\). Let \(\Gamma := \{C_1, ..., C_s\}\) denote the set of orbits of \(Q \times Q\) under the diagonal action of \(G\), where \(C_1 := \{(x, x) \mid x \in Q\}\) is the trivial orbit. Then \((Q, \Gamma)\) is an association scheme. We call schemes that arise from the action of a permutation group in the above-described manner Schurian association schemes.

Schurian schemes provide copious examples of association schemes, but they do not cover all association schemes. A list of non-Schurian association schemes of small order can be found in Hanaki and Miyamoto’s work \([14]\). Examples of infinite families of non-Schurian association schemes can for instance be found in \([10, 11]\).
Finding a polynomial-time algorithm which decides whether a given association scheme is Schurian or non-Schurian is a long-standing open problem. The methods introduced in [3, 4] yield subexponential-time algorithm for testing Schurinity of association schemes; this is currently the best known. Recently, Ponomarenko [19] devised an algorithm which decides the Schurity problem for antisymmetric association schemes in polynomial time (note that an association scheme \(X = (Q, \Gamma)\) is called antisymmetric if for all \(C_1 \neq C_i \in \Gamma, C_i^* = \{(y, x) \mid (x, y) \in C_i\} \neq C_i\).

2.2. Height \(t\) Presuperschemes (short: \(t\)-Preschemes)

Below, we introduce the notion of height \(t\) presuperschemes (short: \(t\)-preschemes), which may be regarded as a higher-dimensional analog of the notion of association schemes. In the following, let \(Q\) be a finite nonempty set. For each \(n \in \mathbb{N}_{>1}\), define a projection

\[
pr_n: Q^n \rightarrow Q^{n-1}
\]

\[
(x_1, \ldots, x_{n-1}, x_n) \mapsto (x_1, \ldots, x_{n-1})
\]

(the projection \(pr_n\) eliminates the last coordinate from tuples in \(Q^n\)). The inverse image of a set \(C \subseteq Q^{n-1}\) under \(pr_n\) is denoted by \(pr^{-1}_n(C)\). Throughout this work, we omit the index \(n\) (we assume it is clear from context) and just write \(pr\) instead of \(pr_n\). For each \(n \in \mathbb{N}\), observe that the symmetric group on \(n\) elements \(\text{Symm}_n\) acts on the set of tuples \(Q^n\) by permuting the coordinates. For all \(\bar{u} := (u_1, \ldots, u_n) \in Q^n\) and \(\tau \in \text{Symm}_n\), define

\[
\bar{u}^\tau := (u_{\tau(1)}, \ldots, u_{\tau(n)}).
\]

Furthermore, we fix the following convention:

\[
N_t := \{ n \in \mathbb{N} \mid n \leq t \}, \quad N^2_t := \{ (m, n) \in N^2 \mid m + n \leq t \}.
\]

Note that the definition of height \(t\) presuperschemes given below is equivalent to the definition given by Wojdylo [24, 25, 26].

**Definition 2.2** (Height \(t\) Presuperscheme). Let \(Q\) be a finite nonempty set and let \(t \in \mathbb{N}\). A height \(t\) presuperscheme \((Q, \Gamma^*)\) on \(Q\) is a family of sets \(\{\Gamma^n\}_{n \in N_t}\), where each set \(\Gamma^n = \{C_1^n, \ldots, C_{s_n}^n\}\) is a partition of the direct power \(Q^{n+2}\) (note that all \(C^n_i\) are assumed to be nonempty), such that:

1. **(Identity Relation)** \(C^n_0 := \{(x, x) \mid x \in Q\};
2. **(Projection)** \(\forall n \in N_t - \{0\}, \forall C^n_j \in \Gamma^n,\)

\[
pr(C^n_j) := \{pr(\bar{u}) \mid \bar{u} \in C^n_j\} \subseteq \Gamma^{n-1};
\]
3. **(Invariance)** \(\forall n \in N_t, \forall C^n_j \in \Gamma^n, \forall \tau \in \text{Symm}_{n+2},\)

\[
(C^n_j)^\tau := \{\bar{u}^\tau \mid \bar{u} \in C^n_j\} \subseteq \Gamma^n;
\]
4. **(Intersection)** \(\forall (m, n) \in N^2_t, \forall C^n_i \in \Gamma^m, \forall C^n_j \in \Gamma^n, \forall C^{m+n}_k \in \Gamma^{m+n},\)

\[
\exists c(i, j, k; m, n) \in \mathbb{N}. \ \forall (x_0, \ldots, x_m, y_0, \ldots, y_n) \in C^{m+n}_k,\]

\[
\left| \{ z \in Q \mid (x_0, \ldots, x_m, z) \in C^n_i, (z, y_0, \ldots, y_m) \in C^n_j \} \right| = c(i, j, k; m, n).
\]
For brevity, we refer to height \(t\) presuperstructures simply as \(t\)-preschemes. We call the elements of \(\Gamma^n\) \((0 \leq n \leq t)\) the relations at height \(n\). We refer to the numbers \(c(i, j, k; m, n)\) as the intersection numbers of \((Q, \Gamma^*)\).

Property (P2) interrelates the different layers \(\{\Gamma^n\}_{n \in \mathbb{N}_t}\) of a \(t\)-prescheme, while Properties (P3), (P4) may be regarded as higher-dimensional analogs of Properties (A2), (A3) of association schemes, respectively. From Definition 2.2 it is clear that a 0-prescheme and an association scheme constitute the exact same notion.

If \((Q, \Gamma^*)\) is a \(t\)-prescheme, then \((Q, \Gamma^0)\) is an association scheme. We say that the association scheme \((Q, \Gamma^0)\) is associated to the \(t\)-prescheme \((Q, \Gamma^*)\). If an association scheme \(X\) is associated to a \(t\)-prescheme \((Q, \Gamma^*)\), we call \(X\) extensible to height \(t\). In this case, we refer to the \(t\)-prescheme partitions \(\{\Gamma^n\}_{1 \leq n \leq t}\) as a \(t\)-extension of \(X\). Note that by definition, every association scheme is extensible to height 0.

We define the maximal height \(t_{\text{max}}(X)\) of an association scheme \(X\) as the largest number \(t \in \mathbb{N}\) such that \(X\) is extensible to height \(t\). If \(X\) is extensible to arbitrary heights (meaning that for all \(t \in \mathbb{N}\), \(X\) is extensible to height \(t\)), we say that \(X\) has maximal height \(\infty\). In case \(t_{\text{max}}(X) = 0\), we say that \(X\) is completely inextensible.

For an association scheme \(X = (Q, \Gamma)\) of order \(d := |Q|\), it is easily proven that \(t_{\text{max}}(X) = \infty\) iff \(X\) is extensible to height \((d - 2)\). A fundamental result by Smith connects the concept of extensibility to the notion of Schurinity of association schemes.

**Theorem 2.3** (Smith [20, 21]). An association scheme \(X = (Q, \Gamma)\) of order \(d := |Q|\) is Schurian iff \(t_{\text{max}}(X) = \infty\).

Note that Theorem 2.3 may also be phrased as follows: An association scheme \(X\) is Schurian iff \(t_{\text{max}}(X) = \infty\). Moreover, observe that if an association scheme \(X = (Q, \Gamma)\) of order \(d := |Q|\) is non-Schurian, then \(0 \leq t_{\text{max}}(X) < (d - 2)\).

3. Adjacency Tensors

In this section, we introduce the notion of adjacency tensors. The concept of adjacency tensors of \(t\)-preschemes naturally generalizes the notion of adjacency matrices of association schemes (see [6, 27]). Analogously, adjacency tensors describe the intersection property of \(t\)-preschemes in simple algebraic terms (see Theorem 3.3). We apply the notion of adjacency tensors in Section 4 when we describe the association scheme extension algorithm.

### 3.1. \(k\)-Tensors

In the following, we introduce tensors of order \(k\) (short: \(k\)-tensors) and define certain natural operations associated with this notion. Note that \(k\)-tensors constitute a natural generalization of the concept of square matrices.

**Definition 3.1** (\(k\)-Tensor). For \(k \geq 2\), a \(k\)-tensor with entries in \(\mathbb{Z}\) is a function

\[ T : \{1, \ldots, d\}^k \rightarrow \mathbb{Z}. \]

We refer to the number \(k\) as the order of the tensor \(T\). We denote by \(T_{i_1, \ldots, i_k}\) the image of \((i_1, \ldots, i_k)\) under \(T\). We call \(T_{i_1, \ldots, i_k}\) the \((i_1, \ldots, i_k)\)-entry of \(T\).
In this work, tensors are regarded simply as multidimensional arrays. For \( k = 2 \),
the notion of \( k \)-tensors with entries in \( \mathbb{Z} \) coincides with the notion of \( d \times d \) matrices
with entries in \( \mathbb{Z} \). For a more general (algebraic) treatment of tensors, the reader is
referred to [7, 8].

In the following, we define some basic operations for \( k \)-tensors. These operations
naturally generalize the standard matrix operations from linear algebra. First, for
two \( k \)-tensors \( S, T : \{1, \ldots, d\}^k \to \mathbb{Z} \), we define their \textit{sum} \( U = S + T \) as the \( k \)-tensor
\( U : \{1, \ldots, d\}^k \to \mathbb{Z} \) with entries
\[
U_{i_1 \ldots i_k} = S_{i_1 \ldots i_k} + T_{i_1 \ldots i_k}.
\]
Next, for an element \( c \in \mathbb{Z} \) and a \( k \)-tensor \( S : \{1, \ldots, d\}^k \to \mathbb{Z} \), we define their
\textit{scalar product} \( V = c \cdot S \) as the \( k \)-tensor \( V : \{1, \ldots, d\}^k \to \mathbb{Z} \) with entries
\[
V_{i_1 \ldots i_k} = c \cdot S_{i_1 \ldots i_k}.
\]
Moreover, for a \( m \)-tensor \( E : \{1, \ldots, d\}^m \to \mathbb{Z} \) and a \( n \)-tensor \( F : \{1, \ldots, d\}^n \to \mathbb{Z} \),
we define their \textit{inner product} \( W = EF \) as the rank \((m + n - 2)\) tensor
\( W : \{1, \ldots, d\}^{(m+n-2)} \to \mathbb{Z} \) with entries
\[
W_{i_1 \ldots i_{m+n-2}} = \sum_{j=1}^d E_{i_1 \ldots i_{m-1} j} \cdot F_{j i_{m+1} \ldots i_{m+n-2}}.
\]
The above operations generalize the standard addition, scalar multiplication and in-
ner multiplication of matrices. It is easily verified that addition and inner multiplication
of tensors are associative, distributive and compatible with scalar multiplication.

3.2. Adjacency Tensors of \( t \)-Preschemes

In the following, we define the notion of adjacency tensors, boolean tensors which
indicate membership to subsets of direct powers of \( Q := \{1, \ldots, d\} \).

**Definition 3.2** (Adjacency Tensor). Let \( Q := \{1, \ldots, d\} \) and let \( C \subseteq Q^n \), where
\( n \geq 2 \). We define the adjacency tensor corresponding to the subset \( C \) as the \( n \)-tensor
\( A(C) : \{1, \ldots, d\}^n \to \mathbb{Z} \) such that the component \( [A(C)]_{x_1 \ldots x_n} \) is 1 if \((x_1, \ldots, x_n) \in C \)
and 0 otherwise.

Let \((Q, \Gamma^*)\) be a \( t \)-prescheme on \( Q := \{1, \ldots, d\} \). We denote the \textit{adjacency tensor}
of a relation \( C^m_i \in \Gamma^m \) (\( m \in \mathbb{N}_t \)) as the \((m+2)\)-tensor \( A^m_i : \{1, \ldots, d\}^{m+2} \to \mathbb{Z} \),
where \( (A^m_i)_{x_1 \ldots x_{m+2}} \) is 1 if \((x_1, \ldots, x_{m+2}) \in C^m_i \) and 0 otherwise. Adjacency tensors
can be used to express the intersection property of \( t \)-preschemes in algebraic terms
(analogously to \textit{adjacency matrices} in the case of association schemes, see [5, 27]).

**Theorem 3.3.** Let \((Q, \Gamma^*)\) be a \( t \)-prescheme on the set \( Q := \{1, \ldots, d\} \). Then for all
\((m, n) \in \mathbb{N}_t^2, C^m_i \in \Gamma^m \) and \( C^n_j \in \Gamma^n \), it holds that
\[
A^m_i A^n_j = \sum_{k=1}^{s_{m+n}} c(i, j, k; m, n) A^{m+n}_k,
\]
where \( A^m_i, A^n_j \) and \( A^{m+n}_k \) denote the adjacency tensors of \( C^m_i, C^n_j \) and \( C^{m+n}_k \in \Gamma^{m+n} \),
respectively, and \( c(i, j, k; m, n) \in \mathbb{N} \) denote the intersection numbers. Moreover,
the above statement is equivalent to the intersection property of \( t \)-preschemes
(see Definition 2.2 (P4)).
**Proof.** The intersection property of $t$-preschemes states that for all $(m, n) \in \mathbb{N}_1^2$, $C_i^m \in \Gamma^m$, $C_j^n \in \Gamma^n$, $C_k^{m+n} \in \Gamma^{m+n}$ and $(x_0, \ldots, x_m, y_0, \ldots, y_m) \in C_k^{m+n}$, it holds that

$$c(i, j, k; m, n) = \left| \{ z \in Q \mid (x_0, \ldots, x_m, z) \in C_i^m, (z, y_0, \ldots, y_m) \in C_j^n \} \right|.$$ 

Note that the above equation can also be written as

$$c(i, j, k; m, n) = \sum_{z=1}^{d} (A^n_i x_0 \cdots x_m z) (A^n_j y_0 \cdots y_m),$$

where the right-hand side is $(A^n_i A^n_j) x_0 \cdots x_m y_0 \cdots y_m$ by the definition of the inner product of tensors. From this the assertion follows immediately. 

\[\square\]

4. The Association Scheme Extension Algorithm

In this section, we describe the association scheme extension algorithm. On input an association scheme $\mathcal{X} = (Q, \Gamma)$ of order $d := |Q|$ and a number $t \in \mathbb{N}$ such that $1 \leq t \leq (d-2)$, the association scheme extension algorithm decides in time $d^{O(t)}$ if $\mathcal{X}$ is extensible to height $t$. Furthermore, if $\mathcal{X}$ is extensible to height $t$, then the algorithm outputs its unique coarsest $t$-extension $\mathcal{X}_t$, which represents the most ‘basic’ way in which $\mathcal{X}$ can be extended to a $t$-prescheme. We apply the association scheme extension algorithm to determine that all non-Schurian association schemes up to order 26 are completely inextensible (see Theorem 4.4). Via the tensor product of association schemes, the latter result gives rise to a multitude of examples of infinite families of completely inextensible association schemes (see Section 4.3).

4.1. Description of the Algorithm

We now describe the association scheme extension algorithm. On input an association scheme $\mathcal{X} = (Q, \Gamma)$ on $Q := \{1, \ldots, d\}$ and a number $t \in \mathbb{N}$ such that $1 \leq t \leq (d-2)$, the algorithm begins with trivial partitions $\Gamma^s := \{Q^{s+2}\} \ (1 \leq s \leq t)$ and then gradually refines these partitions according to a set of rules derived from the properties of $t$-extensions (see Definition 2.2). Via this refinement process, the partitions $\Gamma^s \ (1 \leq s \leq t)$ either turn into a $t$-extension of $\mathcal{X}$, or they provide combinatorial justification for the conclusion that $\mathcal{X}$ cannot be extended to height $t$.

**Input:** An association scheme $\mathcal{X} = (Q, \Gamma)$ on $Q := \{1, \ldots, d\}$, and a number $t \in \mathbb{N}$ such that $1 \leq t \leq (d-2)$.

**Output:** A $t$-extension $\{\Gamma^s\}_{1 \leq s \leq t}$ of $\mathcal{X}$, or the decision that $\mathcal{X}$ is not extensible to height $t$.

**Initialization.** For each $1 \leq s \leq t$, let $\Gamma^s := \{Q^{s+2}\}$ be the trivial partition of $Q^{s+2}$.

**Step 1.** For each $1 \leq s \leq t$, refine the partition $\Gamma^s$ of $Q^{s+2}$ according to the projection property of $t$-preschemes (see Definition 2.2 (P2)). That is, for each $C \in \Gamma^s$, determine if the set $\text{pr}(C)$ can be written as a union of relations in $\Gamma^{s-1}$, i.e. if

$$\text{pr}(C) = C_{i_1}^{s-1} \cup \cdots \cup C_{i_k}^{s-1}$$

for some $C_{i_1}^{s-1}, \ldots, C_{i_k}^{s-1} \in \Gamma^{s-1}$.
If YES. Replace in \( \Gamma^s \) the set \( C \in \Gamma^s \) with the pairwise disjoint sets
\[
C \cap \text{pr}^{-1}(C_{i_1}^{s-1}), ..., C \cap \text{pr}^{-1}(C_{i_k}^{s-1}).
\]

ELSE. Distinguish between the following two cases:

(a) If \( s > 1 \). Replace in \( \Gamma^{s-1} \) each set \( C' \in \Gamma^{s-1} \) such that \( C' \cap \text{pr}(C) \neq \emptyset \) with the two disjoint sets \( C' \cap \text{pr}(C) \) and \( C' \setminus \text{pr}(C) \).

(b) If \( s = 1 \). Terminate the algorithm and output: \( \mathcal{X} \) is not extensible to height \( t \).

**Step 2.** For each \( 1 \leq s \leq t \), refine the partition \( \Gamma^s \) of \( Q^{s+2} \) according to the invariance property of \( t \)-preschemes (see Definition 2.2 (P3)). That is, for each \( C \in \Gamma^s \) and each \( \tau \in \text{Symm}_{s+2} \), replace in \( \Gamma^s \) each set \( C' \in \Gamma^s \) such that \( C' \cap C^\tau \neq \emptyset \) with the two disjoint sets \( C' \cap C^\tau \) and \( C' \setminus C^\tau \).

**Step 3.** For each \( 1 \leq s \leq t \), refine the partition \( \Gamma^s \) of \( Q^{s+2} \) according to the intersection property of \( t \)-preschemes (see Theorem 3.3). That is, for each \( m, n \in \mathbb{N} \) such that \( s = (m + n) \), and each pair of sets \( C^m_i \in \Gamma^m \) and \( C^n_j \in \Gamma^n \), compute the inner product
\[
P := A^m_i A^n_j,
\]
where \( A^m_i, A^n_j \) denote the adjacency tensors of \( C^m_i, C^n_j \), respectively (see Section 3).

The entries of \( P \) are integers in the range from 0 to \( d \). For each \( r = 0, ..., d \) define
\[
P^{-1}(r) := \{(i_1, ..., i_{s+2}) \in Q^{s+2} | P_{i_1...i_{s+2}} = r \}
\]
and replace in \( \Gamma^s \) each set \( C \in \Gamma^s \) such that \( C \cap (P^{-1}(r)) \neq \emptyset \) with the two disjoint sets \( C \cap (P^{-1}(r)) \) and \( C \setminus (P^{-1}(r)) \).

Repeat Steps 1-3. If none of them yields any further refinement of the partitions \( \Gamma^s \) (\( 1 \leq s \leq t \)), then terminate the algorithm and output \( \{\Gamma^s\}_{1 \leq s \leq t} \).

### 4.2. Correctness of the Algorithm

We now prove the correctness of the association scheme extension algorithm. We start with a preliminary lemma.

**Lemma 4.1.** Let \( \mathcal{X} = (Q, \Gamma) \) be an association scheme on \( Q := \{1, ..., d\} \) and let \( t \in \mathbb{N} \) be such that \( 1 \leq t \leq (d - 2) \). The following holds:

1. On input \( \mathcal{X} \) and \( t \), the association scheme extension algorithm terminates after at most \( d^{O(t)} \) steps.
2. On input \( \mathcal{X} \) and \( t \), if the association scheme extension algorithm outputs a set of partitions \( \{\Gamma^s\}_{1 \leq s \leq t} \), then these partitions constitute a \( t \)-extension of \( \mathcal{X} \).

**Proof.**

1. Note that the algorithm can make at most \( (d^3 + ... + d^{t+2}) \) refinements to the partitions \( \{\Gamma^s\}_{1 \leq s \leq t} \) before it must terminate. Moreover, observe that the algorithm goes through at most \( d^{O(t)} \) elementary operations in between two refinements. From this the assertion follows immediately.

2. Note that the algorithm outputs a set of partitions \( \{\Gamma^s\}_{1 \leq s \leq t} \) only if Steps 1-3 of the algorithm do not yield any further refinement of \( \{\Gamma^s\}_{1 \leq s \leq t} \). The
latter condition implies that Definition 2.2 \((P2)-(P4)\) hold for \(X\) and \(\{\Gamma^s\}_{1 \leq s \leq t}\) (see Theorem 3.3). This in turn implies that the partitions \(\{\Gamma^s\}_{1 \leq s \leq t}\) constitute a \(t\)-extension of \(X\).

Let us fix some terminology. Let \(X\) be a finite, nonempty set and let \(\mathcal{P}, \mathcal{R}\) be partitions of \(X\). If for each \(P \in \mathcal{P}\) there exist sets \(R_1, \ldots, R_n \in \mathcal{R}\) such that \(P = \bigcup_{i=1}^{n} R_i\), then we call \(\mathcal{P}\) a fusion of \(\mathcal{R}\). We use this convention in the proof of correctness of the association scheme extension algorithm given below.

**Theorem 4.2.** The association scheme extension algorithm works correctly. The running time of the association scheme extension algorithm is \(d^{O(t)}\).

**Proof.** Let \(X = (Q, \Gamma)\) be an association scheme on \(Q := \{1, \ldots, d\}\) and let \(t \in \mathbb{N}\) be such that \(1 \leq t \leq (d-2)\). First, assume \(X\) is not extensible to height \(t\). Then by Lemma 4.1 (1), (2) it follows that on input \(X\) and \(t\), the algorithm correctly outputs the decision that \(X\) is not extensible to height \(t\), in time \(d^{O(t)}\).

Now consider the converse: Assume we are given as input an association scheme \(X = (Q, \Gamma)\) on \(Q := \{1, \ldots, d\}\) and a number \(t \in \mathbb{N}\) with \(1 \leq t \leq (d-2)\) such that \(X\) is extensible to height \(t\). Choose an arbitrary \(t\)-extension \(\{\tilde{\Gamma}^s\}_{1 \leq s \leq t}\) of \(X\). Observe the following facts about the partitions \(\{\Gamma^s\}_{1 \leq s \leq t}\) which appear in the algorithm:

(i) For each \(1 \leq s \leq t\), the partition \(\Gamma^s\) is trivially a fusion of \(\tilde{\Gamma}^s\) at the initialization step.

(ii) For each \(1 \leq s \leq t\), the partition \(\Gamma^s\) remains a fusion of \(\tilde{\Gamma}^s\) over the whole course of the algorithm (this follows from Properties \((P2), (P3), (P4)\) of Definition 2.2 applied on \(X\) and \(\{\tilde{\Gamma}^s\}_{1 \leq s \leq t}\)). Especially, the algorithm never terminates during the execution of Step 1.

By statement (ii) and Lemma 4.1 (1), we conclude that on input \(X\) and \(t\), the algorithm outputs a set of partitions \(\{\Gamma^s\}_{1 \leq s \leq t}\). By Lemma 4.1 (2), the output \(\{\Gamma^s\}_{1 \leq s \leq t}\) constitutes a \(t\)-extension of \(X\).

Recall that in the proof of Theorem 4.2 the \(t\)-extension \(\{\tilde{\Gamma}^s\}_{1 \leq s \leq t}\) of \(X\) was chosen arbitrarily. Hence we obtain the following corollary:

**Corollary 4.3.** On input an association scheme \(X = (Q, \Gamma)\) and a number \(t \in \mathbb{N}\) with \(1 \leq t \leq (d-2)\) such that \(X\) is extensible to height \(t\), the association scheme extension algorithm outputs the unique coarsest \(t\)-extension \(X_t := \{\Gamma^s\}_{1 \leq s \leq t}\) of \(X\). That is, for any \(t\)-extension \(\{\tilde{\Gamma}^s\}_{1 \leq s \leq t}\) of \(X\), for each \(1 \leq s \leq t\), the partition \(\Gamma^s\) is a fusion of \(\tilde{\Gamma}^s\).

**4.3. Computational Results**

We employ the association scheme extension algorithm to determine the extensibility properties of all non-Schurian association schemes up to order 26. Note that there are exactly 142 non-Schurian schemes of order less or equal to 26 (see [12]).

**Theorem 4.4.** All non-Schurian association schemes \(X = (Q, \Gamma)\) of order \(|Q| \leq 26\) are completely inextensible.
Proof. We created a program of the association scheme extension algorithm with fixed parameter $t = 1$ in the input, written in “C”. We applied our program to all non-Schurian association schemes of order less or equal to 26; for this we relied on the classification of non-Schurian association schemes of small order by Hanaki and Miyamoto [12, 13, 14, 15]. The reader can download an organized version of the C-programs and their output online [1].

Let us fix some convention. For an association scheme $X = (Q, \Gamma)$, we denote the equivalence relation on $Q \times Q$ corresponding to the partition $\Gamma$ by $\equiv_{\Gamma}$. Recall the definition of the tensor product of association schemes. For two association schemes $X_1 = (Q_1, \Gamma_1)$ and $X_2 = (Q_2, \Gamma_2)$, the tensor product $X_1 \otimes X_2$ is defined as the association scheme $(Q_1 \times Q_2, \Gamma_1 \otimes \Gamma_2)$ such that for all $x_1, x_1', y_1, y_1' \in Q_1$ and $x_2, x_2', y_2, y_2' \in Q_2$,

$$(x_1, x_2), (x_1', x_2') \equiv_{\Gamma_1 \otimes \Gamma_2} (y_1, y_2), (y_1', y_2')$$

$$\iff (x_1, x_1') \equiv_{\Gamma_1} (y_1, y_1') \text{ and } (x_2, x_2') \equiv_{\Gamma_2} (y_2, y_2').$$

Given a number $t \in \mathbb{N}$, it is easily seen that the tensor product $X_1 \otimes X_2$ is extensible to height $t$ iff both $X_1$ and $X_2$ are extensible to height $t$. Via the above construction, Theorem 4.4 gives rise to a multitude of examples of infinite families of completely inextensible association schemes. Especially, we have the following corollary.

**Corollary 4.5.** There exist infinitely many completely inextensible association schemes.

5. Conclusion and Open Questions

For an association scheme $X = (Q, \Gamma)$, we defined the notion of extensibility to height $t$ (for a given $t \in \mathbb{N}$), the notion of the maximal height $t_{\text{max}}(X)$ and - assuming that $X$ is extensible to height $t$ - the concept of the unique coarsest $t$-extension $X_t$.

We delineated in which sense the maximal height may be regarded as an intuitive measure of how close an association scheme is to being Schurian. Furthermore, we described the association scheme extension algorithm, which on input an association scheme $X = (Q, \Gamma)$ of order $d := |Q|$ and a number $t \in \mathbb{N}$ such that $1 \leq t \leq (d - 2)$, decides in time $O(t)$ if $X$ is extensible to height $t$. We used the association scheme extension algorithm to determine that all non-Schurian association schemes up to order 26 are completely inextensible, i.e. they have maximal height 0.

Computing the maximal height of an association scheme $X = (Q, \Gamma)$ with the association scheme extension algorithm may require time exponential in $|Q|$ in the worst case. A central open question is whether there exists an algorithm for computing the maximal height which achieves a better worst-case running time (for instance, a running time in the subexponential range). A relaxation of this question would be to ask whether there exist ‘thresholds’ $t(d) \in \mathbb{N}$ such that for all association schemes $X = (Q, \Gamma)$ of order $d := |Q|$, deciding if $X$ is extensible to height $t(d)$ can be done more efficiently than using the association scheme extension algorithm. Apart from this, we remark that it is currently an open problem to identify the smallest order $d \in \mathbb{N}$ for which there exists a non-Schurian association scheme of positive maximal height. We leave the above questions to future research.
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