Robust wall modes in rapidly rotating Rayleigh-Bénard convection

Benjamin Favier\textsuperscript{1}† and Edgar Knobloch\textsuperscript{2}

\textsuperscript{1}Aix Marseille Univ, CNRS, Centrale Marseille, IRPHE, Marseille, France
\textsuperscript{2}Department of Physics, University of California, Berkeley, CA 94720, USA

(Received xx; revised xx; accepted xx)

We show, using direct numerical simulations with experimentally realizable boundary conditions, that wall modes in Rayleigh-Bénard convection in a rapidly rotating cylinder persist even very far from their linear onset. These nonlinear wall states survive in the presence of turbulence in the bulk and are robust with respect to changes in the shape of the boundary of the container. In this sense these modes behave much like the topologically protected states present in two-dimensional chiral systems even though rotating convection is a three-dimensional nonlinear driven dissipative system. We suggest that the robustness of this nonlinear mode may provide an explanation for the strong zonal flows observed recently in simulations and experiments on rapidly rotating convection at high Rayleigh number.

1. Introduction

In recent years there has been considerable interest in the so-called topologically protected states on account of their robustness with respect to system perturbations. These states, originally discovered in the context of two-dimensional semiconductors, take the form of a unidirectional current along the boundary. This current persists under system perturbations, including defects and changes in the boundary, its persistence guaranteed by topological arguments. Other examples include two-dimensional chiral materials (Tsai et al. 2005; Nash et al. 2015), isostatic lattices in two dimensions (Kane & Lubensky 2014) and photonic systems (Khanikaev & Shvets 2017). Recently it was observed that similar arguments apply to equatorial shallow water waves and used to confirm the presence of two types of low frequency eastward-propagating equatorially trapped waves, Kelvin and Yanai waves, separating bulk waves at higher positive and negative frequency (Delplace et al. 2017; Tauber et al. 2019). The theory predicts that both these boundary currents are robust, for example, with respect to topography.

This type of theory applies to linear dissipationless systems exemplified by the Schrödinger equation in quantum systems (Thouless et al. 1982) and the related shallow water system studied by Delplace et al. (2017) and Tauber et al. (2019). Odd viscosity can be added without changing the conclusions of the theory because this type of viscosity does not dissipate energy (Banerjee et al. 2017; Souslov et al. 2019; Tauber et al. 2020).

In the present work we present evidence that similarly robust states are present in forced dissipative hydrodynamics and in particular in nonlinear systems of this type. While the general theory does not apply to these systems (our system supports multiple boundary currents depending on the parameters used, and these currents may themselves

† Email address for correspondence: favier@irphe.univ-mrs.fr
undergo instabilities (Liu & Ecke 1999)) they nevertheless behave in the same manner as similar but topologically protected boundary currents in linear systems.

Our work is motivated in part by the zonal currents found independently by two groups in high Rayleigh number experiments and simulations on rapidly rotating convection in a right circular cylinder rotating with constant angular velocity about its vertical axis (de Wit et al. 2020; Zhang et al. 2020). While the results differ in detail both groups observe an intense zonal flow that precesses with respect to the rotating frame and is nearly indifferent to the bulk turbulent state.

In the following we argue that in rotating Rayleigh-Bénard convection the bulk behaves like a chiral fluid (Zhong et al. 1991) and that, as a result, the presence of a boundary current is not surprising. However, real fluids are dissipative and hence such a boundary current requires the presence of forcing, i.e. a finite Rayleigh number, for its appearance. States of this type were first revealed, indirectly, in experiments by Rossby (1969) and computed, within linear theory, by Goldstein et al. (1993). The resulting modes, called wall modes to distinguish them from the bulk modes that set in at higher Rayleigh numbers, are confined to the vicinity of the boundary and have been studied in experiments in cylindrical domains (Zhong et al. 1991; Ecke et al. 1992). Both modes precess in the rotating frame in a retrograde direction but the precession rate of the wall modes is in general substantially higher than that of the bulk modes. The wall modes set in supercritically (Ecke et al. 1992), and may coexist with other wall modes with different azimuthal wavenumbers; they may also undergo modulational instabilities of Eckhaus-Benjamin-Feir type as the applied Rayleigh number increases (Liu & Ecke 1999). Despite this, we show here that these modes bear all the hallmarks of topologically protected states. We first show – in a cylindrical geometry – that these modes are robust with respect to secondary instabilities and to a turbulent bulk state as in the experiments (de Wit et al. 2020; Zhang et al. 2020) by gradually increasing the Rayleigh number until the bulk becomes turbulent. Second, we show that these states persist in the presence of different types of barriers, and show that the wall modes happily follow whatever boundary geometry is provided. Finally, we show that they persist in the presence of both types of perturbations together, i.e., when barriers are present and the bulk is turbulent.

2. Formulation

We consider the evolution of a layer of incompressible fluid inside a vertical right circular cylindrical container of diameter $D$ and height $H$. The lower horizontal plate is maintained at a higher temperature than the upper plate, while the circular walls are assumed to be thermally insulating. All walls are impenetrable and no-slip. The layer rotates counterclockwise about the $z$ axis, pointing vertically upwards, with a constant angular velocity $\Omega = \Omega e_z$ while gravity points downwards: $g = -g e_z$. The kinematic viscosity $\nu$ and thermal diffusivity $\kappa$ are assumed to be constant.

In the Boussinesq approximation, using the rotation time $1/(2\Omega)$ as a unit of time and the depth $H$ of the layer as a unit of length, the dimensionless equations are

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p - e_z \times \mathbf{u} + \frac{Ra E^2}{Pr} \theta e_z + E \nabla^2 \mathbf{u},$$

$$\nabla \cdot \mathbf{u} = 0,$$

$$\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta = w + \frac{E}{Pr} \nabla^2 \theta,$$

where $\mathbf{u} \equiv (u, v, w)$ is the velocity, $p$ is the pressure and $\theta$ is the temperature fluctuation relative to the conduction profile that varies linearly with respect to the vertical
coordinate $z$. The control parameters are the Rayleigh number $Ra = \alpha g \Delta T H^3 / (\nu \kappa)$, the Ekman number $E = \nu / (2 \Omega H^2)$ and the Prandtl number $Pr = \nu / \kappa$. These dimensionless quantities involve $\alpha$, the coefficient of thermal expansion, and $\Delta T$, the imposed temperature difference between the two horizontal plates. The aspect ratio $\Gamma \equiv D/H$ specifies the geometry.

We solve equations (2.1)-(2.3) using the open-source spectral-element code Nek5000 developed by Fischer et al. (2008) at Argonne National Laboratory. The mesh is composed of up to 19968 hexahedral elements and we use a polynomial order up to $N = 16$. The mesh is refined close to the boundaries of the domain in order to properly resolve both Ekman and Stewartson layers. Numerical convergence of the results has been checked by increasing the polynomial order.

3. Nonlinear dynamics of wall modes

We are interested in the dynamics of the wall modes which are the first unstable modes in laterally bounded rotating Rayleigh-Bénard convection. The critical Rayleigh number for the appearance of wall modes can be found in Herrmann & Busse (1993) and Kuo & Cross (1993) (see Zhang & Liao (2009) for higher order corrections) and is given in our dimensionless units by

$$Ra_{c}^{\text{wall}} \approx \pi^2 (6\sqrt{3})^{1/2} E^{-1},$$

while the critical Rayleigh number for the onset of the bulk mode is given by (Chandrasekhar 1961; Clune & Knobloch 1993; Liao et al. 2006)

$$Re_{c}^{\text{bulk}} \approx 3 \left( \frac{\pi^2}{2} \right)^{2/3} E^{-4/3}.$$

These expressions are valid for low Ekman numbers and are independent of both the Prandtl number and aspect ratio provided the latter is sufficiently large. Although formally obtained for different boundary conditions at the top and bottom (stress-free or no-slip), the analysis of Clune & Knobloch (1993) explains why the leading order behavior is independent of these boundary conditions. Both expressions are shown in Figure 1 where we also report previous experimental and numerical studies of interest. Interestingly, the range of Rayleigh numbers for which only wall modes are unstable increases as the Ekman number decreases. The above asymptotic results belie the fact...
that the onset wavenumber is \( m = (\pi/2)\Gamma(2 + \sqrt{3})^{1/2} \approx 3\Gamma \) as \( E \to 0 \) (Zhang & Liao 2009). Since \( m \) is necessarily an integer each \( m \) defines a neutral stability curve and the intersection between two successive curves heralds a transition from onset wavenumber \( m \) to onset wavenumber \( m + 1 \) as \( \Gamma \) increases. Thus the critical Rayleigh number is in fact an oscillatory function of \( \Gamma \), although its minima are independent of \( \Gamma \) for large \( \Gamma \).

In this section, we fix \( E = 10^{-6} \) so that \( Ra^\text{wall}_c \approx 3.2 \times 10^7 \) while \( Ra^\text{bulk}_c \approx 8.7 \times 10^8 \). We then vary the Rayleigh number from \( Ra = 5 \times 10^7 \) to \( Ra = 2 \times 10^9 \) to explore the onset of the wall mode, its nonlinear saturation and eventually its coexistence with the bulk mode. The aspect ratio is fixed to \( \Gamma = 1.5 \) and the Prandtl number is taken to be \( Pr = 1 \) for simplicity (section 4 describes further results for parameters relevant to the recent experiments).

The simulations are initialized with small amplitude fluctuations in the temperature \( \theta \). In each case we observe exponential growth followed by nonlinear saturation. Visualizations of the vertical velocity \( w \) and the temperature fluctuation \( \theta \) in the mid-plane
z = 0.5 in the saturated state are shown in the left panel of figure 2. At $Ra = 5 \times 10^7$, i.e. relatively close to the onset of the wall mode, we find a retrograde $m = 6$ traveling wave localized near the sidewall (figure 2(a)). The traveling nature of the wall mode is reflected in the asymmetry of its instantaneous profile but is best seen by looking at spatio-temporal diagrams of the temperature fluctuation in the mid-plane $z = 0.5$ and an arbitrary location $r$ close to the boundary: $r = 0.74 < \Gamma/2 = 0.75$ (right panels in figure 2). The results at other radii and depths are very similar provided one remains sufficiently close to the outer boundary. This traveling mode is similar to that observed in experiments in a $\Gamma = 2$ cylinder rotating at lower rotation rates (Zhong et al. 1991, 1993). The structure of this mode is also similar to that of the critical mode for $\Gamma = 4$, $E = 5 \times 10^{-5}$ and $Pr = 1$ computed by Zhang & Liao (2009) for which $m = 10$ while the asymptotic wavenumber as $E \to 0$ is $m \approx 12$. In contrast, in our case, computed for $E = 10^{-6}$, the observed wavenumber is larger than the onset wavenumber $m \approx 4.5$ in the limit $E \to 0$. Evidently our choice of parameters is outside the asymptotic regime $E \to 0$ and $\Gamma \to \infty$.

As the Rayleigh number increases – while remaining below the threshold for the bulk mode – the wall mode becomes fully nonlinear and more complex dynamical behavior is seen. We first observe (see figure 2(b)) an Eckhaus-type instability (Janiaud et al. 1992; Knobloch 1994), as found experimentally at much higher Ekman number by Liu & Ecke (1999). This instability leads to the gradual coarsening or merging of adjacent cells but the flow remains laminar. The transitions between different azimuthal wavenumbers are expected to be strongly hysteretic for reasons explained in Knobloch (1994), a behavior that is illustrated in figure 2(b) where the $m = 4$ wall mode persists even when the Rayleigh number is reduced from $2 \times 10^8$ to $5 \times 10^7$, i.e., to a Rayleigh number at which $m = 6$ is the first mode that appears. At even higher Rayleigh numbers ($Ra = 5 \times 10^8$, see figure 2(c)) the $m = 4$ traveling wave pattern persists but becomes fully nonlinear and starts to emit plumes into the stable bulk resulting in short wavelength perturbations of the mode. These fluctuations tend to propagate in the opposite direction from that of the traveling wave, i.e., in a cyclonic sense. Finally, a $m = 3$ wall mode survives even for Rayleigh numbers beyond the onset of bulk convection, where it takes the form of a nonlinear wall-confined traveling wave superposed on small scale bulk turbulence (figure 2(d)).

From the spatio-temporal diagrams one can easily extract the drift frequency $\omega_d$ as a function of the Rayleigh number. To do so, we perform a temporal Fourier transform of the fluctuating temperature at a given point and extract the frequency corresponding to the maximum of the power spectrum. This frequency is shown in figure 3(a) (all values are negative so that we actually plot $-\omega_d$). The drift frequency increases continuously with increasing $Ra$ and connects to the theoretical prediction of Herrmann & Busse (1993) at onset, $\omega_d = \omega_c \approx -59E/Pr$.

These results can be corroborated by computing the azimuthally, vertically and temporally averaged zonal velocity $\langle \dot{\varphi} \rangle$ as shown in figure 3(b). Close to onset, the mean azimuthal flow is negligible (its strength varies linearly with $Ra - Ra_c^{\text{wall}}$ near onset) but with increasing $Ra$ it strengthens rapidly and develops a concentric ring structure with cyclonic flow in the outer ring and anti-cyclonic flow in the inner ring. This flow structure is typical of Stewartson layers (Stewartson 1957; Kunnen et al. 2011), with the cyclonic mean flow confined to a layer of thickness $E^{1/3}$ close to the boundary and the anti-cyclonic flow present in a broader layer of thickness $E^{1/4}$. Interestingly, one can now compute the typical Reynolds number associated with this mean shear. Based on the largest layer thickness $E^{1/4}$ and the azimuthal mean velocity difference in the profile shown in figure 3(b), one obtains $Re \approx 2.3$ at $Ra = 2 \times 10^8$ and $Re \approx 9.7$ at $Ra = 5 \times 10^8$. 


Drift frequency $\omega_d$ as a function of $Ra$ for $\Gamma = 1.5$, $E = 10^{-6}$ and $Pr = 1$. The results for the full cylinder ($\square$) and the cylinder with a barrier ($\triangle$, see section 5 below) coincide. The theoretical value $\omega_d = \omega_c \approx -59E/Pr$ predicted by Herrmann & Busse (1993) for the onset of the instability in the presence of a planar wall is also reported (open circle). Right: azimuthally, vertically and temporally averaged zonal velocity as a function of the radial coordinate $r$. Positive values correspond to cyclonic motions while negative values correspond to anti-cyclonic motions. The two vertical lines indicate the Stewartson layer scales $E^{1/3}$ and $E^{1/4}$ (Stewartson 1957).

Since the critical Reynolds number for shear barotropic instabilities is typically around $Re \approx 10 - 20$ independently of $E$ (Hide & Titman 1967; Niino & Misawa 1984; Früh & Read 1999), we conjecture that the small scale fluctuations observed at $Ra = 5 \times 10^8$ but not at $Ra = 2 \times 10^8$ are driven by a shear instability of the mean barotropic zonal flow (Busse 1968) although a complete analysis of this instability is beyond the scope of this paper. The same nested cyclonic/anti-cyclonic structure is observed even after the appearance of the bulk mode indicating that this structure is clearly a consequence of the survival of the nonlinear wall mode into this regime.

4. Link with recent experiments and simulations at high $Ra$

Recent experiments and simulations have shown the emergence of a boundary zonal flow in confined rotating Rayleigh-Bénard convection (Kunnen et al. 2011; de Wit et al. 2020; Zhang et al. 2020). However, a link between this flow and the wall modes was not explicitly made. In further support of this link, we repeated the simulation of de Wit et al. (2020) using the parameters $E = 10^{-7}$, $Pr = 5.2$ and $\Gamma = 0.2$. Instead of focusing on the dynamics of the bulk mode, as in de Wit et al. (2020), we consider Rayleigh numbers below the onset of the bulk mode (i.e., for $Ra < Ra_c^{bulk} \approx 2 \times 10^{10}$), but above the onset of the wall mode ($Ra > Ra_c^{wall} \approx 3 \times 10^8$). We show in figure 4(a) the vertical velocity in the mid-plane at $Ra = 2 \times 10^9$, close to the onset of wall modes, once the system has reached a statistically steady state. We observe an $m = 1$ traveling wave, presumably a consequence of the small value of $\Gamma$. The asymptotic onset wave number for $\Gamma = 0.2$ is indeed $m \approx 0.6$. This state is very similar to the boundary zonal flow observed by de Wit et al. (2020) (see their figure 2(b), for example) at the much higher Rayleigh number of $Ra = 5 \times 10^{10}$. Here the nonlinear dynamics of the wall mode are clearly constrained by the small aspect ratio used in these simulations compared to the cases discussed in the previous section. However, once the Rayleigh number is increased to $Ra = 5 \times 10^{10}$, we recover the emergence of the bulk mode superposed on the nonlinear wall mode (see figure 4(b), which can be directly compared with figure 2(b) of de Wit et al. (2020)). Note that de Wit et al. (2020) also measured the drift frequency as a function of $Ra$ and found the following fit: $\omega_d \approx -6 \times 10^{-10} Ra^{1.16 \pm 0.06} E$ (using our rotation units). Using
the same approach as in section 3, we measured the drift frequency in our simulations below the onset of the bulk mode and in figure 4(c) we compare the results with the fit of de Wit et al. (2020). The results clearly bridge the gap between the onset theoretical value derived by Herrmann & Busse (1993) and the highly turbulent scaling found by de Wit et al. (2020). Note also that we reproduce many of the results presented in these studies (such as the bimodal temperature distribution and the boundary zonal flow) by considering cases below the onset of the bulk mode. This provides a clear indication that the boundary dynamics observed in these studies are related to the nonlinear saturation of the wall mode and are not driven by dynamics in the bulk (which is stable in two of our cases). The emergence of the boundary zonal flow is thus best understood as the nonlinear development of linearly unstable wall modes that eventually couple to the bulk dynamics in the manner described in these studies (Kunnen et al. 2011; de Wit et al. 2020; Zhang et al. 2020).

5. Robustness of the wall modes to geometry changes

The time-dependent wall modes we have described thus far are linked to the presence of a sidewall. These states behave like topologically protected states and, as shown above, survive even when the Rayleigh number is very far from its critical value, i.e., in the strongly nonlinear regime. The robustness of these states can also be demonstrated by perturbing the geometry of the domain. This has already been done in other similar systems (Dasbiswas et al. 2018; Souslov et al. 2019) and was in fact suggested in the very early studies of wall modes in rotating convection by Liu & Ecke (1999).

In this section, we return to the parameters used in section 3. However, the cylinder is now truncated and we introduce a rectangular barrier of length $\Gamma/2$ and thickness $\Gamma/20$ along a diameter with one end attached to the circular sidewall (see figure 5). Like the rest of the sidewall, the barrier is no-slip and insulating. The barrier breaks the rotation invariance of the system and introduces four discontinuities in the curvature of the sidewall. One might therefore expect that this change in the geometry of the domain will result in a drastic change in the nature of the wall mode. This is not the case. As shown in figures 5 and 6, for $Ra = 5 \times 10^7$, $Ra = 5 \times 10^8$ and $Ra = 2 \times 10^9$, the wall mode persists, both in its weakly nonlinear and fully nonlinear forms. More surprisingly, the drift frequency is the same as in the case without the barrier for all three Rayleigh numbers, as shown in figure 3(a). This is even more surprising when
Figure 5. Vertical velocity in the mid-plane \( z = 0.5 \) for a cylinder with a barrier. The Rayleigh number increases from left to right: (a) \( Ra = 5 \times 10^7 \), (b) \( Ra = 5 \times 10^8 \) and (c) \( Ra = 2 \times 10^9 \). Parameters are \( \Gamma = 1.5 \), \( E = 10^{-6} \) and \( Pr = 1 \).

Figure 6. Left: spatio-temporal plots showing the temperature fluctuation \( \theta \) at \( z = 0.5 \) and a fixed distance \( \delta = 10^{-2} \) from the boundary at \( Ra = 5 \times 10^7 \) (top panel) and \( Ra = 5 \times 10^8 \) (bottom panel). The vertical axis represents the arclength along the boundary while the horizontal axis is time (in rotation units). The dotted lines indicate the positions of the four corners of the barrier. Parameters are \( \Gamma = 1.5 \), \( E = 10^{-6} \) and \( Pr = 1 \). Right: vertical and temporal average of the velocity component tangential to the boundary at \( Ra = 5 \times 10^8 \). We distinguish between the cylindrical boundary and the different faces of the barrier. The results are compared to the case without barrier (see figure 3(b)).

one notices that the wavelength of the wall mode is comparable with the length of the barrier and larger than its thickness. As seen in figure 5, the wall mode simply wraps itself around the obstacle, following the same retrograde drift as in the case without a barrier, although its wavelength and frequency along the barrier are both slightly larger than around the rest of the cylinder (see the left panels in figure 6). Moreover, the presence of the corners appears to be responsible for increased shedding of cyclonic disturbances. Despite this the zonal flow discussed earlier is barely modified by the presence of the barrier (see figure 6(b)) and its average in a direction tangential to the boundary exhibits the same nested cyclonic-anticyclonic structure along all boundaries. Finally, as in the case without a barrier, the wall mode survives the onset of the bulk mode and remains visible in instantaneous visualizations of the vertical velocity at the mid-plane even in the presence of bulk turbulence (figure 5(c)).

6. Conclusions

In this paper, we have explored the nonlinear dynamics of convective wall modes in a rapidly rotating right circular cylinder. These modes appear at lower Rayleigh numbers than the classical bulk modes. We adopt much lower values of the Ekman number \( (E = 10^{-6} - 10^{-7}) \) than previous studies thereby broadening the interval of Rayleigh numbers...
within which wall modes alone are present. This approach allowed us to study these modes in the strongly nonlinear regime unencumbered by convection in the bulk.

We observed very rich dynamics with increasing Rayleigh number. Close to the linear onset, we recovered the well-known retrograde wall modes, followed by a series of merging events triggered by a Benjamin-Feir-Eckhaus instability as the Rayleigh number increases. For yet larger Rayleigh numbers we observed a secondary transition associated with a shear instability of the mean zonal flow driven nonlinearly by the traveling wall state. The resulting complex chaotic structure is confined to the vicinity of the wall and survives the onset of the bulk mode at higher Rayleigh numbers and even bulk turbulence that is present at yet higher Rayleigh numbers (de Wit et al. 2020; Zhang et al. 2020).

These observations are reminiscent of recent experimental and numerical studies demonstrating the existence of a so-called boundary zonal flow in high Rayleigh number convection in a rotating cylinder. These studies did not connect the presence of this flow with the presence a wall mode. We have shown here that many of the experimental observations can in fact be explained by the nonlinear dynamics of the wall mode itself that can be isolated by a suitable choice of parameter values.

Finally, and perhaps most importantly, we have demonstrated that the properties of this nonlinear wall mode are remarkably robust with respect to even the most drastic change in the shape of the cylindrical container. Specifically, we have seen that even the introduction of a substantial radial barrier has essentially no effect on the properties of the boundary zonal flow. In this sense these modes behave like the topologically protected states in two-dimensional chiral systems even though rotating convection is a three-dimensional nonlinear driven dissipative system. This analogy deserves further study.

Acknowledgments

To the authors’ knowledge the robustness of wall modes with respect to changes in geometry was mentioned in a talk by R. E. Ecke some 20 years ago as recalled by one of us (EK). This work was supported by the France-Berkeley Fund at the University of California, Berkeley. The computations were performed using the HPC resources of GENCI-IDRIS (Grants No. A0060407543 and A0080407543). Centre de Calcul Intensif d’Aix-Marseille is acknowledged for granting access to its high performance computing resources.

Declaration of interests

The authors report no conflict of interest.

REFERENCES

Banerjee, D., Souslov, A., Abanov, A. G. & Vitelli, V. 2017 Odd viscosity in chiral active fluids. Nature Comm. 8, 1573.

Busse, F. H. 1968 Shear flow instabilities in rotating systems. J. Fluid Mech. 33, 577–589.

Chandrasekhar, S. 1961 Hydrodynamic and Hydromagnetic Stability. Oxford University Press.

Clune, T. & Knobloch, E. 1993 Pattern selection in rotating convection with experimental boundary conditions. Phys. Rev. E 47, 2536–2550.

Dasbhiswas, K., Mandadapu, K. K. & Vaikuntanathan, S. 2018 Topological localization in out-of-equilibrium dissipative systems. Proc. Nat. Acad. Sci. 115, E9031–E9040.

de Wit, X. M., Aguirre Guzman, A. J., Madonia, M., Cheng, J., Clercx, H. J. H. & Kunnen, R. P. J. 2020 Turbulent rotating convection confined in a slender cylinder: The sidewall circulation. Phys. Rev. Fluids 5, 023502.

Delplace, P., Marston, J. B. & Venaille, A. 2017 Topological origin of equatorial waves. Science 358, 1075–1077.

Ecke, R. E., Zhong, F. & Knobloch, E. 1992 Hopf bifurcation with broken reflection symmetry in rotating Rayleigh-Bénard convection. Europhys. Lett. 19, 177–182.
Fischer, P. F., Lottes, J. W. & Kerkemeier, S. G. 2008 Nek5000 web page. http://nek5000.mcs.anl.gov.

Favier and Knobloch

Früh, W.-G. & Read, P. L. 1999 Experiments on a barotropic rotating shear layer. Part 1. Instability and steady vortices. J. Fluid Mech. 383, 143–173.

Goldstein, H. F., Knobloch, E., Mercader, I. & Net, M. 1993 Convection in a rotating cylinder. Part 1: Linear theory for moderate Prandtl numbers. J. Fluid Mech. 248, 583–604.

Herrmann, J. & Busse, F. H. 1993 Asymptotic theory of wall-localized convection in a rotating fluid layer. J. Fluid Mech. 255, 183–194.

Hide, R. & Titman, C. W. 1967 Detached shear layers in a rotating fluid. J. Fluid Mech. 29, 39–60.

Janiaud, B., Pumir, A., Bensimon, D., Croquette, V., Richter, H. & Kramer, L. 1992 The Eckhaus instability for traveling waves. Physica D 55, 269–286.

Kane, C. L. & Lubensky, T. C. 2014 Topological boundary modes in isostatic lattices. Nature Phys. 10, 39–45.

Khaniakiev, A. B. & Shvets, G. 2017 Two-dimensional topological photonics. Nature Photonics 11, 763–773.

Knobloch, E. 1994 Bifurcations in rotating systems. In Lectures on Solar and Planetary Dynamos, pp. 331–372. Cambridge University Press, Cambridge.

Kunnen, R. P. J., Stevens, R. J. A. M., Overkamp, J., Sun, C., van Heijst, G. F. & Clercx, H. J. H. 2011 The role of Stewartson and Ekman layers in turbulent rotating Rayleigh-Bénard convection. J. Fluid Mech. 688, 422–442.

Kuo, E. Y. & Cross, M. C. 1993 Traveling-wave wall states in rotating Rayleigh-Bénard convection. Phys. Rev. E 47, R2245–R2248.

Liao, X., Zhang, K. & Chang, Y. 2006 On boundary-layer convection in a rotating fluid layer. J. Fluid Mech. 549, 375–384.

Liu, Y. & Ecke, R. E. 1999 Nonlinear traveling waves in rotating Rayleigh-Bénard convection: Stability boundaries and phase diffusion. Phys. Rev. E 59, 4091–4105.

Nash, L. M., Kleckner, D., Read, A., Vitelli, V., Turner, A. M. & Irvine, W. T. M. 2015 Topological mechanics of gyroscopic metamaterials. PNAS 112, 14495–14500.

Niino, H. & Misawa, N. 1984 An experimental and theoretical study of barotropic instability. Journal of the Atmospheric Sciences 41, 1992–2011.

Rossby, H. T. 1969 A study of Bénard convection with and without rotation. J. Fluid Mech. 33, 309–335.

Schel, J. D., Paul, M. R., Cross, M. C. & Fischer, P. F. 2003 Traveling waves in rotating Rayleigh-Bénard convection: Analysis of modes and mean flow. Phys. Rev. E 68, 066216.

Souslov, A., Dasbhiswas, K., Fruchart, M., Vaikuntanathan, S. & Vitelli, V. 2019 Topological waves in fluids with odd viscosity. Phys. Rev. Lett. 122, 128001.

Stewartson, K. 1957 On almost rigid rotations. J. Fluid Mech. 3, 17–26.

Tauber, C., Delplace, P. & Venaille, A. 2019 A bulk-interface correspondence for equatorial waves. J. Fluid Mech. 868, R2.

Tauber, C., Delplace, P. & Venaille, A. 2020 Anomalous bulk-edge correspondence in continuous media. Physical Review Research 2, 013147.

Thouless, D. J., Kohmoto, M., Nightingale, M. P. & Den Nijs, M. 1982 Quantized Hall conductance in a two-dimensional periodic potential. Phys. Rev. Lett. 49, 405–408.

Tsai, J.-C., Ye, F., Rodriguez, J., Gollub, J. P. & Lubensky, T. C. 2005 A chiral granular gas. Phys. Rev. Lett. 94, 214301.

Zheng, K. & Liao, X. 2009 The onset of convection in rotating circular cylinders with experimental boundary conditions. J. Fluid Mech. 622, 63–73.

Zhong, X., van Gils, D. P. M., Horn, S., Wedi, M., Zwiener, L., Ahlers, G., Ecke, R. E., Weiss, S., Bodenschatz, E. & Shishkina, O. 2020 Boundary zonal flow in rotating turbulent Rayleigh-Bénard convection. Phys. Rev. Lett. in press.

Zhong, F., Ecke, R. E. & Steinberg, V. 1991 Asymmetric modes and the transition to vortex structures in rotating Rayleigh-Bénard convection. Phys. Rev. Lett. 67, 2473–2476.

Zhong, F., Ecke, R. E. & Steinberg, V. 1993 Rotating Rayleigh-Bénard convection: asymmetric modes and vortex states. J. Fluid Mech. 249, 135–159.