REGULARITY FOR A LOG-CONCAVE TO LOG-CONCAVE
MASS TRANSFER PROBLEM WITH NEAR EUCLIDEAN COST

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Abstract. If the cost function is not too far from the Euclidean cost, then
the optimal map transporting Gaussians restricted to a ball will be regular.
Similarly, given any cost function which is smooth in a neighborhood of two
points on a manifold, there are small neighborhoods near each such that a
Gaussian restricted to one is transported smoothly to a Gaussian on the other.

1. Introduction

This note deals with the regularity of the optimal transportation map, when
the distributions under consideration are close to restricted Gaussians. From the
work of Ma, Trudinger and Wang, ([MTW], [TW]) regularity holds for arbitrary
smooth distributions on nice domains when the cost satisfies the MTW A3s con-
dition. It is established by Loeper [L] that without this MTW condition on the
cost function, one cannot expect regularity for arbitrary smooth distributions, and
the question of regularity is wide open. Here we show that we can find smooth
optimal transportation, at least for some very nice distributions.

We show two results. The first is that when the transportation problem involves
distributions somewhat like the standard Gaussian restricted to the unit ball, then
if the cost function is close enough to the Euclidean distance squared cost, the map
must be regular. As a corollary, given two points and any cost which is smooth
near these points, we can find very focused Gaussians, restricted to very small balls
near the points, so that the optimal transport is regular.

Our method yields a way to compute precisely how close the cost function need
be to Euclidean, or relatedly, how small the balls must be around the given points.
Recently other perturbative results for regularity of optimal transport have ap-
peared: Delanoë and Ge [DG] show regularity for certain densities on metrics near
constant curvature. Caffarelli, Gonzalez and Nguyen [CGN] present estimates,
when the cost is Euclidean distance raised to powers other than 2.

Date: July 23, 2010.
The author is partially supported by an NSF grant 0901644.
Specifically, let \( f, \bar{f} \) be functions on regions \( \Omega, \bar{\Omega} \subset \mathbb{R}^n \), satisfying on \( \Omega \)

\[
|Df| \leq 1
\]

\[
1 \leq \delta \leq D^2f \leq 2
\]

\[
|D^3f| \leq 1
\]

and similarly for \( \bar{f} \) on \( \bar{\Omega} \).

We define the following mass distributions

\[
m = e^{-f(x)}\chi_{\Omega}
\]

\[
\bar{m} = e^{-\bar{f}(\bar{x})}\chi_{\bar{\Omega}}
\]

where we may add a constant to \( f \) so that both distributions have the same total mass.

The region \( \Omega \) will be required to have a defining function \( h \) so that on \( \Omega = \{ h \leq 0 \} \), \( h \) satisfies the same three conditions (a1-3) as \( f \), as well as, along the boundary \( \partial \Omega \)

\[
|Dh| \geq 1/2,
\]

which implies the second fundamental form of the set \( \partial \Omega = \{ h = 0 \} \) is bounded by 4. Similarly define an \( \bar{h}, \bar{\Omega} \).

A solution of the optimal transportation equation for these densities and a given cost function \( c(x, \bar{x}) \) is a function \( u(x) \) which satisfies

\[
det w_{ij} = e^{-f(x)}e^{\bar{f}(T(x, Du))}|\det c_{is}(x, T(x, Du))|
\]

\[
T(x, Du)(\Omega) = \bar{\Omega}
\]

where

\[
w_{ij} = u_{ij}(x) - c_{ij}(x, T(x, Du)) = c_{i\bar{s}}T_{\bar{s}j}
\]

and \( T(x, Du) = (T^1, T^2, \ldots, T^n) \subset \bar{\Omega} \) is determined by

\[
u_i(x) = c_i(x, T(x, Du)).
\]

(Such a solution must also be \( c \)-convex. In our setting, the two notions of convexity are very close, so we won’t belabour this point here, see Lemma 2.6.) We will use the following convention: The derivatives of the cost function in the first variable \( x \) will be \( i, j, k \) etc. The second variable \( \bar{x} \) will be denoted by indices \( p, s, t, \) etc.

Also upper index denotes inverse i.e \( c^{is} = (c_{is})^{-1} \).

The cost \( c(x, \bar{x}) \) will satisfy the standard conditions (A1) and (A2) but not (A3) (see for example [MTW] section 2.) We will require further that the second derivatives of the cost satisfy the following assumptions

\[
\|(c^{is} - I)\| \leq \epsilon_0 \leq 1/20
\]

\[
C(n) (\|D^3c\| + \|D^4c\|) \leq \epsilon_0 \leq 1/20
\]
where $C(n)$ is a dimensional constant, and the derivative norms are with respect to both barred and unbarred directions. Finally we will require that the densities are somewhat close to uniform

\[(cm-a3)\quad e^{-f(x)}e^{\bar{f}(\bar{x})} \left| \det c_{x\bar{x}} \right| \in [\Lambda^{-1}, \Lambda] \]

for all $x, \bar{x} \in \Omega \times \bar{\Omega}$ with

\[(cm-a3b)\quad \Lambda \leq \left( \frac{n}{3/2} \right)^n.\]

We are now ready to state our result.

**Theorem 1.1.** Let $m, \bar{m}$ be the mass densities defined by (1.1) (1.2) with $f, \bar{f}$ satisfying assumptions (a1-3) on regions $\Omega, \bar{\Omega}$ whose defining functions also satisfy (a1-3). There exists an $\epsilon_0(n)$ such that if the cost function satisfies standard assumptions (A1) and (A2) and (c-a1,a2) and (cm-a3) hold, then the optimal map transporting $m$ to $\bar{m}$ is regular.

**Remark 1.1.** These conditions are nonvacuous. For example take $f, h, \bar{f}, \bar{h}$ all to be

\[
\frac{2}{3} |x|^2 - \frac{1}{4},
\]

and

\[c(x, \bar{x}) = -x \cdot \bar{x}.\]

One can check that all the assumptions are satisfied with plenty of room to perturb any of the problems components.

The following theorem will follow by a change of coordinates and rescaling.

**Theorem 1.2.** Let $x_0, \bar{x}_0$ be two points in manifolds $X, \bar{X}$ such that near $(x_0, \bar{x}_0)$ the cost function is smooth and satisfies standard nondegeneracy conditions (A1)(A2). Then there exists a $\lambda$ large depending on the cost function, so that the optimal map from the Gaussian (after a choice of coordinates)

\[e^{-\lambda^2|x-x_0|^2/2} \chi_{B_1(\lambda x_0)}\]

to

\[e^{-\lambda^2|\bar{x}-\bar{x}_0|^2/2} \chi_{B_1(\lambda \bar{x}_0)}\]

is smooth.

**Remark 2.** We do not attempt to obtain any sharp results, rather the convenient smallness assumptions are to minimize crunchiness of the proof. Inspection of the proof will show that our choice of assumptions are robust. There is a rather large gap between what is covered here and the counterexamples, and we have no reason to suspect that these results are near sharp.

**Remark 3.** We would like to obtain a similar result for complete Gaussians, as Caffarelli obtained in the Euclidean case in [C2]. In fact, it was an attempt to generalize the calculation in [C2] that led to this result. A limitation of our current method is that we cannot force (cm-a3) to hold on large regions.
1.1. **Proof Heuristic.** We will solve the problem by continuity, starting with Euclidean cost, obtaining second derivative estimates using the approach of Urbas \[U\] and Trudinger and Wang \[TW\], making use of the Ma, Trudinger and Wang \[MTW\] calculation together with the calculation of Caffarelli \[C2\]. Making these methods work in the absence of the MTW condition, we use the following observation: The bound \(M\) on the second derivatives will satisfy the following type of inequality

\[
\delta M^2 - tM^{n+1} - 1 \leq 0.
\]

When \(t\) is zero, this bounds \(M\), so \(M\) is initially bounded. If \(t\) is small it follows that \(M(t)\) lies either on a relatively small compact interval containing \([-1/2, 1/2]\) or on a noncompact interval. The bound \(M(t)\) is changing continuously with \(t\), thus the interval it lies in must not change, thus from the initial bound we may conclude that for all \(t\) in some interval of fixed size, \(M(t)\) is bounded.

The quadratic coefficient \(\delta\) in (1.7) (same \(\delta\) as in (a2)) arises when the target distribution is log-concave, as is the case with Gaussians. This fact is essential to the proof.

2. **Calculations**

Recall the symmetric tensor \(w\) (1.6). We use the quantities defined as follows

\[
W(x) = \sum w_{ii} \sim \max_i w_{ij} \sim |T_j^s|
\]

\[
\bar{W}(x) = \sum w_{ii} \sim 1/\min_i w_{ii}
\]

\[
C_3 \geq \|D^4 c\| C(n)
\]

\[
C_4 \geq \|D^4 c\| C(n)
\]

\[
\frac{1}{C_2} |\xi|^2 \leq -c^{s_i} \xi \xi_s \leq C_2 |\xi|^2
\]

From (cm-a3) and Newton-McLaurin inequalities, it follows that

\[
\bar{W}, W \geq n \frac{1}{\Lambda^{1/n}}
\]

\[
\bar{W} \leq \frac{1}{n^{n-2} \Lambda W^{n-1}}
\]

\[
W \leq \frac{1}{n^{n-2} \Lambda \bar{W}^{n-1}}
\]

and plugging in (cm-a3b)

\[
W, \bar{W} \geq 3/2.
\]

Notice that (a1)(a2)(ca-1)(ca-2) imply the following inequality for any vector in \(\mathbb{R}^n\)

\[
(h_{st} - c^{kp} c_{kst} h_{sp}) \xi_t \xi_s \geq \frac{9}{10} |\xi|^2.
\]

Throughout this section we will be assuming we have a smooth solution \(u\) to the equation (1.4) on \(\Omega\). Our goal is to prove second derivative estimates.

We make use of the linearized operator at a solution \(u\), from \(TW\) defined by

\[
Lv = w^{ij} v_{ij} - \left(w^{ij} c_{ij,s} c^{sk} + \bar{f}_s(T(x, Du)) c^{sk} + c^{is} c_{si,p} c^{pk}\right) v_k.
\]
The following has an immediate consequence when maximums occur on the interior, and is also crucial in the boundary estimates in Section 4. The proof is a moderately long calculation and follows by the arguments in [MTW].

**Lemma 2.1.** Suppose $u(x)$ is a solution to (1.4). Then

$$Lw_{11} =$$

$$w^{ij} \left[ 2c_{ijst} T^s_1 + c_{ijst} T^s_1 T^t_1 - 2c_{11st} T^s_1 T^t_1 \right]$$

$$- c_{11p} c^{kp} \left[ -f_k + \bar{f}_k T^k_1 + c^{is} c_{ist} T^s_1 - c_{kjs} w^{ij} T^k_1 - c_{skj} c^{ts} c_{kst} w_{ij} \right]$$

$$+ f_{st} T^s_1 T^t_1 - f_{11} + c^{is} (c_{s11} + 2c_{ist1} T^t_1 + c_{istp} T^t_1 T^p_1)$$

$$+ (c^{is} + c^{ts} T^t_1) (c_{is1} + c_{istp} T^p_1)$$

$$+ (w^{ij} c_{ijp} + \bar{f}_p + c^{ij} c_{isp} ) c^{sk} \left( c_{11s} T^s_1 - c_{k1s} T^s_1 - c_{kst} T^t_1 \right)$$

$$- w^{ij} w_{ij1}.$$ 

Applying the maximum principle,

**Corollary 2.2.** If the largest eigenvalue $W$ of $w$ is attained on the interior, it must satisfy

$$\frac{\delta}{C_2} W^2 - (C_4 + C_3 + C_5 |Df|) W^{n+1} - |D^2 f| - C(C_3, C_4) \leq 0. \quad (2.6)$$

The next computation is implicit throughout [TW] sections 2, 3 and 4. We state it for concreteness.

**Lemma 2.3.** Let $v(x) = F(x, T(x, Du))$. Then

$$Lv = w^{ij} F_{ij} + 2F_{is} c^{is} + F_{st} c^{is} c^{jt} w_{ij}$$

$$+ F_p \left( -c^{pk} f_k - c^{pk} c_{kjs} c^{js} - c_{kst} c^{pk} c^{is} c^{jt} w_{ij} \right)$$

$$- F_k \left( w^{ij} c_{ijp} + \bar{f}_p + c^{ij} c_{isp} \right) c^{sk} \left( c_{11s} T^s_1 - c_{k1s} T^s_1 - c_{kst} T^t_1 \right). \quad (2.7)$$

**Corollary 2.4.** Given conditions (c-a1) (c-a2) and (a1) (a2) on the functions $f, \bar{f}, h, and \bar{h}$, we have

$$Lh \geq \frac{9}{10} \delta W - \frac{11}{10}$$

$$L\bar{h}(T(x, Du)) \geq \frac{9}{10} \delta W - \frac{11}{10}. \quad (2.8)$$

2.1. **Obliqueness.** We follow the argument from [TW] section 2. Defining

$$\gamma = Dh$$

$$\beta = \bar{h}_s c^{si} \partial_i$$

we let

$$\chi = h_k \bar{h}_s c^{sk} = \gamma \cdot \beta.$$
From Lemma 2.3 with our assumptions we have
\[ L \chi \leq W \left( |D^3 h| + C_3 + C_4 \right) + W \left( |D^3 \bar{h}| + C_3 + C_4 \right) + C_5(n). \]

Then Corollary 2.4 gives
\[ L \left\{ \chi - \lambda h - \lambda \bar{h} \circ T(x) \right\} \leq W \left( \frac{11}{10} \frac{9}{10} \lambda + W \left( \frac{11}{10} \frac{9}{10} \lambda \right) + 2 \lambda \frac{11}{10} \lambda + C_5(n) \right) \]
which is negative for \( \lambda \) reasonably chosen. (Throughout we are using bounds (2.1) etc, and our initial assumptions.) This function will then have a minimum at the boundary, precisely at the point where \( \chi \) achieves a minimum on the boundary, and at this point we have
\[ \left\{ D\chi - \lambda D(\bar{h} \circ T) - \lambda Dh \right\} \cdot \frac{\gamma}{|\gamma|} \leq 0 \]
or
\[ (2.8) \quad D \left\{ \chi - \lambda \bar{h} \circ T \right\} = \tau \gamma \]
for some \( \tau \leq \lambda \).

Now computing (following [TW, 2.31-2.33]), using (2.5) and (1.6) with our other assumptions including (1.3), we conclude

\[ D\chi \cdot \beta = e^{t_i \bar{h}_t} \left( h_{ki} e^{s_k} \bar{h}_s + h_k \left( e^{s_k} + c_p \beta_p \right) \bar{h}_s + h_k e^{s_k} \bar{h}_r \right) \]
\[ = h_{ki} e^{s_k} + e^{t_i \bar{h}_t} h_k e^{s_k} \bar{h}_s + e^{t_i \bar{h}_t} h_k e^{s_k} \bar{h}_r \]
\[ \geq |\beta|^2 - C_2 \geq \frac{1}{5} \delta, \]

The third term in (2.9) can be expressed as an inner product \( g \) of the gradients of the functions \( h(x) \) and \( \bar{h} \circ T(x) \), which are both multiples of the outward normal, where
\[ g(\xi, \nu) = (\bar{h}_{rp} - \bar{h}_s c^m c_{mnp}) \bar{c}^k \bar{c}^a \xi_k \nu_a. \]

Thus
\[ \tau \gamma \cdot \beta = D\chi \cdot \beta - \lambda D(\bar{h} \circ T) \cdot \beta \]
\[ \geq \delta/5 - \lambda \bar{h}_s T^s \cdot \bar{h}_t \]
\[ = \delta/5 - \lambda \bar{w}_\beta. \]

Thus from \( \tau \leq \lambda \),
\[ (2.10) \quad \lambda \chi \geq \delta/5 - \lambda \bar{w}_\beta. \]

Using symmetry (replacing all quantities with barred quantities we find the problem does not change, again see [TW] and Lemma 2.6), we may assume
\[ (2.11) \quad \lambda \chi \geq \delta/5 - \lambda \bar{w}_\gamma. \]

Then, using the Urbas formula [U], [TW 2.13]
\[ (\beta \cdot \gamma)^2 = w^{ij} \gamma_i \gamma_j w_{ij} \]
or

\[ \chi^2 = w_{\gamma'\gamma}w_{\beta'\beta} \]

we have combining (2.10) (2.11) and (2.12)

\[ \chi \geq \frac{\delta}{10\lambda} = \theta. \]

**Corollary 2.5.** The following holds, regarding the angle between \( \beta \) and \( \gamma \)

\[ \angle(\beta, \gamma) \leq \Delta < \pi/2. \]

2.2. **cost-convexity.**

**Lemma 2.6.** Suppose \( u(x) \) is a solution to (1.4) on a domain in \( \mathbb{R}^n \). If \( D^2u \geq 2\epsilon_0 \), and the cost function differs from the Euclidean cost function by less than then \( \epsilon_0 \) in \( C^2 \), then \( u \) is \( c \)-convex, and the mapping \( T(x, u) \) is one to one.

**Proof.** Suffice to consider the \( c = -x \cdot y + \phi(x, y) \), where \( \phi \) is small in \( C^2(\Omega) \). At a point \( x_0 \), we have \( Du(x_0) = Dc(x_0, T(x_0, Du)) = -T(x_0, Du) + D\phi(x_0, T(x_0)) \).

At another point, \( x_1 \)

\[ (Du(x_1) - Du(x_0), x_1 - x_0) \geq 2\epsilon_0|x_1 - x_0|^2. \]

Now suppose that \( u \) is not strictly \( c \)-convex. Clearly the issue would have to be nonlocal, as locally,

\[ D^2u - D^2c \geq 2\epsilon_0 - \epsilon_0 > 0. \]

Thus we can assume that there is a point \( x_0 \) and a locally supporting cost function

\[ c_{y_0}(x) = -x \cdot T(x_0) + \phi(x, T(x_0)) \]

which contacts \( u \) from below near \( x_0 \) but touches \( u \) (possibly transversely) at a point \( x_1 \). It follows that

\[ (Dc_{y_0}(x_1) - Dc_{y_0}(x_0), x_1 - x_0) \geq (Du(x_1) - Du(x_0), x_1 - x_0) \]

that is

\[ \|D^2\phi\|_{C^{1,1}} |x_1 - x_0|^2 \geq 2\epsilon_0|x_1 - x_0|^2 \]

a contradiction. It follows that \( u \) is \( c \)-convex and \( T \) is one to one. \( \square \)

2.3. **Boundary Estimate.** Let

\[ M = \max_{|e|=1, e \in T, \Omega} w_{ee} \]

be the maximum of all eigenvalues \( W \) over all of \( \Omega \). Throughout this section we will assume that the maximum occurs on the boundary.

Recalling (2.3) and Lemma 2.3 we may choose a \( C_6 \) so that

\[ L(C_6M^{n-2/n-1}h - \tilde{h}(T(x, Du)) \geq 0. \]

Since \( h, \tilde{h} \) both vanish on the boundary, the derivatives must satisfy

\[ D_{\beta}\tilde{h} \circ T(x, Du) \leq C_6M^{n-2/n-1} \]
that is
\[ h_s T_i^v \beta_i = h_s c^{v_j} w_{ij} h_t c^{v_t} = w_{\beta\beta} \leq C_6 M^{n-2/n-1}. \]

**Lemma 2.7.** At a point \( x_0 \) on the boundary \( \partial \Omega \), suppose \( w_{ee} \leq M \) for unit directions \( e \) which are tangential to the boundary. If \( z \) is any vector in \( T_{x_0} \Omega \), then
\[ w_{zz} \leq M|\hat{z}|^2 + \frac{1}{\theta^2} (z, \nabla h)^2 w_{\beta\beta}. \]
where
\[ \hat{z} = z - \frac{\gamma \cdot z}{\gamma \cdot \beta} \beta = z - y, \]
and \( \theta \) is defined by (2.13).

**Proof.** Dotting with \( \gamma \) verifies \( \hat{z} \) is tangential, thus
\[ 0 = \partial \hat{z} h \circ T(x, Du) = h_s T_j^v \hat{z}_j = h_s c^{v_j} w_{ij} \hat{z}_j. \]
Now
\[ w_{zz} = w_{\hat{z}z} + 2w_{z\hat{y}} + w_{yy} \]
but
\[ w_{z\hat{y}} = w_{ij} \hat{z}_j h_s c^{v_s} = 0 \]
so
\[ w_{zz} \leq M|\hat{z}|^2 + \left( \frac{\gamma \cdot z}{\gamma \cdot \beta} \right)^2 w_{\beta\beta}. \]

Now suppose that the maximum tangential derivative \( w_{11} = M^T \) happens at a point \( x_0 \), where \( e_1 \) is a tangential direction. Define the function
\[ \eta = w_{11} - M^T |\hat{e}_1(x)|^2 - C_6 \frac{1}{\theta^2} (e_1, \nabla h(x))^2 M^{n-2/n-1} + C_7 (M + 1)(h + \tilde{h} \circ T) \]
where
\[ |\hat{e}_1(x)|^2 = \left| e_1 - \frac{h_1(x)}{\xi(h_s(T) c^{v_k} h_k(x, T))} \beta \right|^2 \]
with \( \xi \) a smooth function satisfying \( \xi(t) = t \) for \( t > \theta/2 \), and \( \xi(t) \geq \theta/4 \). Now computing, using Lemma 2.1 and (2.2)
\[ L\eta \geq \delta w_{11}^2 - C_4 + C_3) \bar{W} W^2 - C(n) - M|L[\hat{e}_1(x)|^2)] - C_6 \frac{1}{\theta^2} L (e_1, \nabla h(x))^2 M^{n-2/n-1} + C_7 (M + 1) \left\{ \frac{9}{10} \bar{\delta} (\bar{W} + W) - 2(1 + \mu) \right\} \]
and using (considering Lemma 2.3)
\[ |L[\hat{e}_1(x)|^2] \leq C_8 (\bar{W} + 1 + W) \]
\[ |LC_6(e_1, \nabla h(x))^2 | \leq C_8 (\bar{W} + 1 + W). \]
we may choose
\[ C_7 = C_8 + (C_4 + C_3) (\bar{M} + M) \]
so that
\[ L\eta \geq 0. \]
Next we show a lower bound on $D_\beta w_{11}(x_0)$. First, observe that due to Lemma 2.7, $\eta$ has a maximum at $x_0$. It follows from the Hopf maximum principle that

$$D\eta \cdot \beta = \nu \gamma \cdot \beta \geq 0.$$ 

Thus (recalling $h_1(x_0) = 0$)

$$D_\beta w_{11}(x_0) \geq M^T D_\beta [\hat{e}_1]^2 + D_\beta C_6(e_1, \nabla h(x))^2 M^{n-2}/n-1$$

$$- \{ C_8 + (C_4 + C_3) (\hat{M} + M) \} M (D_\beta h + D_\beta H)$$

$$\geq -C(n)M^T - \{ C_8 + (C_4 + C_3) (\hat{M} + M) \} C_6(n)(1 + M^{2n-3}/n-1).$$

Finally we will derive a relation between the maximum $M$ of all eigenvalues of $w$ and for tangential eigenvalues $M^T$. Go to the point where the maximum of all eigenvalues for $w$ happens. (Again, in this section we assume this happens along the boundary.) We diagonalize $w = diag(M, \lambda_2, \ldots, \lambda_n)$ with respect to some coordinates $e_1, \ldots, e_n$, choosing $e_1 \cdot \gamma \geq 0$. Now

$$w_{\beta \beta} = (\beta \cdot e_1)^2 M + (\beta \cdot e_2)^2 \lambda_2 + \ldots (\beta \cdot e_n)^2 \lambda_n \leq C_6(n)M^{n-2}/n-1$$

thus

$$(\beta \cdot e_1)^2 \leq C_6(n)M^{-1}/n-1.$$ 

It follows that there is a $C_{10}$ depending on $C_6(n)$ and $\Delta$, (recall Corollary 2.5) such that if $M \geq C_{10}$, then

$$\angle(\beta, e_1) - \pi/2 < \frac{1}{2}(\pi/2 - \Delta)$$

in particular

$$\angle(\gamma, e_1) \geq \frac{1}{2}(\pi/2 - \Delta).$$

Thus the length of projection of the maximum eigenvector of $w$ onto the tangent plane is at least some value $\sigma M$ depending on $\Delta$. So we may assume that either $M \leq C_{10}$, or the maximum tangential value $M^T$ satisfies $M^T \geq \sigma M$.

**Proposition 2.8.** Suppose that the global maximum for $w$ is attained along the boundary. Then if $M \geq C_{10}$, $M$ must satisfy

$$M^2 - (C_4 + C_3) M^{n+1} \leq C_{11}$$

**Proof.** Differentiating $\tilde{h} \circ T(x, Du)$ twice tangentially,

$$\partial_1 \partial_1 \tilde{h} \circ T(x, Du) = \tilde{h}_s T^s_1 + \tilde{h}_{st} T^s_1 T^t_1 = -\langle \nabla \tilde{h} \circ T, II(1, 1) \rangle$$

$$= \tilde{h}_p \left( c^{pk} w_{11,k} + c^{pk} \tilde{c}_{11,s} T^s_k - c^{pk} \tilde{c}_{11,k} T^s_1 + c^{pk} \tilde{c}_{s,k} T^s_1 T^t_1 - c^{st} c^{pk} T^t_1 T^s_1 \right)$$

$$+ \tilde{h}_{st} c^{si} w_{11,l} c^{ij} w_{ij}$$

using [MTW] 4.11. Now using $\tilde{h}_p c^{pk} w_{11,k} = w_{11,\beta}$, (2.14) and the discussion in the previous paragraph we conclude that if $M \geq C_{10}$,

$$\delta \sigma M^2 - C(n)M_T - \{ C_8 + (C_4 + C_3) (\hat{M} + M) \} C_6(n)M^{2n-3}/n-1 - C_3 W^2$$

$$\leq C_8 M^{n-2}/n-1.$$
Using Young’s inequality to clean up the expression, we have

(2.19) \[ M^2 - (C_4 + C_3) M^{n+1} \leq C_{11}. \]

\[ \square \]

3. Proof of Theorem

We now go through the alternatives and make our choice of constants, in order to bound \( w \) and consequently \( D^2u \).

First, if the maximum happens in the interior, then (2.6) (3.1)

\[ M^2 - (C_4 + C_3) M^{n+1} \leq C_{12}. \]

If not, then either (2.19) (3.2)

\[ M^2 - (C_4 + C_3) M^{n+1} \leq C_{11} \]

or (3.3)

\[ M \leq C_{10}. \]

by the discussion surrounding (2.15).

So we simply must choose \( (C_4 + C_3) \) small enough, say

\( (C_4 + C_3) \leq \varepsilon_0 \)

so that the noncompact region defined by (3.1) does not intersect the compact regions defined by (3.2) and (3.3), similarly for the noncompact region defined by (3.2). Further, in order to have \( c \)-convexity, we must assume that the conditions of Lemma 2.6 are satisfied. The upper bounds in the above alternatives provide lower bounds on the Hessian, so we choose \( C_3 \) small enough so that Lemma 2.6 is satisfied.

Now by the theory of Delanoé [D], Caffarelli [C1] and Urbas [U] we have a classical solution to the problem for distance squared

\[ \ell^0(x, y) = |x - y|^2/2 \]

in Euclidean space.

We use the method of continuity. Openness is provided by Theorem 17.6 in GT, where we set

\[ G : C^{2,\alpha}(\Omega) \times [0, 1] \rightarrow C^{0,\alpha}(\Omega) \times C^{1,\alpha}(\partial \Omega) \]

with

\[ G(u, t) = \left( \ln \det \left[ u_{ij} - \ell_{ij}^{(t)}(x, T^{(t)}(x, Du)) \right] - h(x) + \bar{h}(T^{(t)}(x, Du)) \right) - \ln \det \left[ \ell_{is}^{(t)}(x, T^{(t)}(x, Du)) \right] , \]

where the cost function is changing from Euclidean to \( c \) as

\[ \ell^{(t)} = (1 - t)\ell^0 + tc \]

and \( T^{(t)} \) defined by

\[ D\ell^{(t)}(x, T^{(t)}(x, Du)) = Du. \]
Our initial solution $u_0$ is smooth, so it satisfies the above estimates (3.1) etc with $C_3C_4 = 0$. These bounds change continuously with $t$ so $D^2u$ must stay in the compact components of (3.1) (3.2) and (3.3). As is standard for this problem, we cite [LT] to obtain the $C^{2,\alpha}$ estimates. By [GT] Theorem 17.6, we have openness in $t$, and the estimates give us closedness as long as $|D^4c(t)|, |D^3c(t)| \leq \varepsilon_0$. This completes the proof of Theorem 1.1.

4. Theorem 2

First we employ a change of coordinates so that $c(x, \bar{x}) = -I_n$.

Proof. Then, on a product of very small balls $B_{1/\lambda}(x_0) \times B_{1/\lambda}(\bar{x}_0)$ we have

$$\frac{1}{C_2} |\xi|^2 \leq -c_{s1}\xi_s \leq C_2 |\xi|^2$$

for some $C_2$ near 1, and $|D^3c|, |D^4c| \leq C$ which may be large but finite.

We now rescale and consider the following problem on $B_1(0) \times B_1(\bar{0})$: Let

$$c(\lambda) = \lambda^2 c\left(\frac{y}{\lambda}, \frac{\bar{y}}{\lambda}\right)$$

be the cost function, and let the distributions to be transported be Gaussians, satisfying (a1-3) on $B_1(0), B_1(\bar{0})$.

This cost function $c(\lambda)$ now satisfies the conditions in our first theorem, as we see that choosing $\lambda$ large enough will make the third and fourth derivatives arbitrarily small.

It follows by Theorem 1.1 that the solution to this rescaled optimal transportation problem is smooth. However, the coordinate change and ”change of currency” do not change the underlying optimal transportation problem. Thus we also have smoothness for the solution of the problem sending

$$m = e^{-\lambda^2 |x-x_0|^2/2} \chi_{B_{1/\lambda}(x_0)}$$

to

$$\bar{m} = e^{-\lambda^2 |\bar{x}-\bar{x}_0|^2/2} \chi_{B_{1/\lambda}(\bar{x}_0)}.$$ 

This completes the proof. \(\Box\)

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