NUMERICAL ANALYSIS OF A THERMAL FRICTIONAL CONTACT PROBLEM WITH LONG MEMORY

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Abstract. The objective in this paper is to study a thermal frictional contact model. The deformable body consists of a viscoelastic material and the process is assumed to be dynamic. It is assumed that the material behaves in accordance with Kelvin-Voigt constitutive law and the thermal effect is added. The variational formulation of the model leads to a coupled system including a history-dependent hemivariational inequality for the displacement field and an evolution equation for the temperature field. In study of this system, we first consider a fully discrete scheme of it and then focus on deriving error estimates for numerical solutions. Under appropriate assumptions of solution regularity, an optimal order error estimate is obtained. At the end of this manuscript, we report some numerical simulation results for the contact problem so as to verify the theoretical results.

1. Introduction. Variational inequalities, as a powerful tool in the study of various nonlinear boundary value problems arising in Mechanics, Engineering Sciences and so on, are based on monotonicity arguments and convexity theory. The corresponding mathematical theory can be found in [3, 7, 15]. Hemivariational inequalities, as a generalization of variational inequalities, use properties of the subdifferential in the sense of Clarke defined for locally Lipschitz functions as main ingredient. The representative monographs include [16, 17].

Contact problems can be seen everywhere in physics, biology and engineering applications. For example, contact phenomena are omnipresent in human skeletal systems. For instance, see [18]. The model on contact problems may include mechanical damage effects, the wear or the bonding effects on contact surface, the temperature effects and so on. The related problems attracted the attention of many researchers and there have been a large number of articles in these fields. Such references include [6, 14, 21]. The friction process such as the flight of an aircraft or a sudden braking of a car is accompanied by the generation of heat. Thermal effects in contact processes affect the composition and stiffness of the contact surfaces,

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and cause thermal stresses in the contact bodies. There have been many literatures on contact problem with thermal effects, we refer to [20] for quasistatic thermoviscoelastic contact problem while the dynamic thermoviscoelastic contact problem was considered in [1]. As far as we know, more and more researchers are interested in numerical analysis of hemivariational inequality, especially for optimal order error estimate. We refer to [11] for the first paper in this field, followed by a series of monographs, see [13, 22, 23]. However, there have been few papers considering numerical solution of the contact problem with thermal, such as [8]. Moreover, there is no paper considering numerical solution of hemivariational inequality arising in contact problem with thermal effects and in this paper, we will fill this gap.

This paper is dedicated to the study on numerical approximation of a general system of a hemivariational inequality involving history-dependent operators and a variational equation which models a dynamic frictional contact problem with thermal effects and long memory. There are three traits of novelties that we describe in what follows. First, the model we consider here involves contact condition with normal compliance and memory term. Both the deformability and the memory effects of the foundation are taken into account in this condition. Second, the constitutive law of this paper is a viscoelastic constitutive law with long memory and thermal effects. Finally, the multivalued friction law is modeled by the Clarke subdifferential of a function which is locally Lipschitz and, in general, nonconvex in its last variable. We assume that the function depends on the accumulated slip over the whole time interval. The three ingredients above lead to new and interesting mathematical model. This is the first paper to consider numerical analysis of a hemivariational inequality arising in contact problem with thermal effects. Since history-dependent operators appear at several places and the contact boundary conditions and the system are complex, it is a challenge to derive the error estimates for numerical solutions of the model.

We first introduce the following frictional contact problem. Let Ω be an open bounded subset of \( \mathbb{R}^d \) which is taken up by a viscoelastic body with \( d = 2, 3 \). Suppose that the boundary \( \Gamma \) of Ω is Lipschitz continuous and split into three mutually disjoint parts \( \Gamma_D, \Gamma_N \) and \( \Gamma_C \), the measure of \( \Gamma_D \), denoted \( m(\Gamma_D) \), is assumed to be positive. The body is clamped on \( \Gamma_D \), so the displacement field disappears there. Time-dependent surface tractions of density \( f_N \) act on \( \Gamma_N \) and time-dependent volume forces of density \( f_0 \) act in Ω. The evolutionary process of the mechanical state of the body in the time interval \((0, T)\) with \( T > 0 \) is what we are concerned about.

The notation \( \mathbf{u} = (u_i), \boldsymbol{\sigma} = (\sigma_{ij}) \) and \( \mathbf{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})) \) are used to represent the displacement vector, the stress tensor, and linearized strain tensor, respectively. Sometimes we do not explicitly indicate the dependence of the variables on the spatial variable \( \mathbf{x} \). Recall that the components of the linearized strain tensor \( \varepsilon(\mathbf{u}) \) are \( \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}) \), where \( u_{i,j} = \partial u_i / \partial x_j \). The indices \( i, j, k, l \) run between 1 and \( d \) and, unless stated otherwise, the summation convention over repeated indices is used. An index following a comma indicates a partial derivative with respect to the corresponding component of the spatial variable \( \mathbf{x} \). A superscript prime of a variable stands for the time derivative of the variable. We denote by \( \mathbf{v} \) the outward unit normal on \( \partial \Omega \). Moreover, we use the notation \( v_\nu \) and \( v_\tau \) for the normal and tangential components of \( \mathbf{v} \) on \( \partial \Omega \) given by \( v_\nu = \mathbf{v} \cdot \mathbf{\nu} \) and \( v_\tau = \mathbf{v} - v_\nu \mathbf{\nu} \). The normal and tangential components of the stress field \( \boldsymbol{\sigma} \) on the boundary are defined
by \( \sigma_\nu = (\sigma \nu) \cdot \nu \) and \( \sigma_\tau = \sigma \nu - \sigma_\nu \), respectively. The symbol \( \mathbb{S}^d \) represents the space of second order symmetric tensor of \( \mathbb{R}^d \).

The mathematical model of the contact problem is stated as follows.

**Problem 1.1.** Find a displacement field \( u : \Omega \times (0,T) \to \mathbb{R}^d \), a stress field \( \sigma : \Omega \times (0,T) \to \mathbb{S}^d \) and a temperature field \( \theta : \Omega \times (0,T) \to \mathbb{R}_+ \) such that for all \( t \in (0,T) \),

\[
\sigma(t) = \mathcal{A}\varepsilon(u'(t)) + \mathcal{B}\varepsilon(u(t)) + \int_0^t \mathcal{C}(t-s, \varepsilon(u(s)), \zeta(s)) \, ds \quad \text{in } \Omega, \quad (1.1)
\]

\[
u_u(t) = \text{Div } \sigma(t) + f_0(t) \quad \text{in } \Omega, \quad (1.2)
\]

\[u(t) = 0 \quad \text{on } \Gamma_D, \quad (1.3)
\]

\[
\sigma(t) \nu + f_N(t) \quad \text{on } \Gamma_N, \quad (1.4)
\]

\[-\sigma_\nu(t) = p(u_\nu(t)) + \int_0^t b(t-s)u_\nu(s) \, ds \quad \text{on } \Gamma_C, \quad (1.5)
\]

\[-\sigma_\tau(t) \in \partial j_\tau(\int_0^t \| u_\tau(s) \| \, ds, u_\tau'(t)) \quad \text{on } \Gamma_C, \quad (1.6)
\]

\[
\theta'(t) - \text{div}(K_c \nabla \theta(t)) = -c_{ij} \frac{\partial u_i'}{\partial x_j}(t) + q(t) \quad \text{in } \Omega, \quad (1.7)
\]

\[-k_{ij} \frac{\partial \theta}{\partial x_j}(t) n_i = k_c(\theta(t) - \theta_R) \quad \text{on } \Gamma_C, \quad (1.8)
\]

\[
\theta(t) = 0 \quad \text{on } \Gamma_D \cup \Gamma_N, \quad (1.9)
\]

\[
u(0) = u_0, \quad u'(0) = w_0, \quad \theta(0) = \theta_0 \quad \text{in } \Omega. \quad (1.10)
\]

Eq. (1.1) represents the constitutive law for viscoelastic materials with thermal effects in which \( \mathcal{A} \) represents the viscosity operator, \( \mathcal{B} \) represents the elasticity operator and \( \mathcal{C} \) is the viscoelastic long memory operator. The stress tensor is split into two parts: \( \sigma(t) = \sigma_V(t) + \sigma_R(t) \), where \( \sigma_V(t) = \mathcal{A}\varepsilon(u'(t)) \) represents the purely viscous part of the stress and

\[
\sigma_R(t) = \mathcal{B}\varepsilon(u(t)) + \int_0^t \mathcal{C}(t-s, \varepsilon(u(s)), \theta(s)) \, ds
\]

represents the elastic-viscoelastic relation with a thermal effect. The similar constitutive law can be seen in [19] where an elastic-visco-plasticity constitutive law with thermal effect was studied. Eq. (1.2) is the equation of motion. Eq. (1.3) is the clamped boundary condition on \( \Gamma_D \) and we have the surface traction boundary condition (1.4) on \( \Gamma_N \).

Relation (1.5) is the contact condition in which \( p, b \) are given functions describing the instantaneous and the memory reaction of the obstacle, respectively. We refer to paper [9] for more details about this condition. We are interested in the multivalued friction law (1.6) modeled by the Clarke subdifferential of a function which is locally Lipschitz and, in general, nonconvex in its last variable. Here, we assume the function \( j_\tau \) depends on the accumulated slip over the whole time interval \( [0,t] \). In [13], a model involves a version of the Coulomb’s law of dry friction with the friction bound depending on the total slip was proposed. There, the frictional condition leads to a variational form and now the frictional condition in this paper leads to a hemivariational form. The differential equation (1.7) describes the evolution of the temperature field, where \( K_c := (k_{ij}) \) represents the thermal conductivity tensor, \( q(t) \) represents the density of volume heat sources and \( c_{ij} \) represents the thermal
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expansion tensor. The associated temperature boundary conditions are given by
(1.8) and (1.9), where \( \theta_R \) denotes the temperature of the foundation and \( k_e \)
the heat exchange coefficient between the body and the obstacle. From condition (1.9),
we see that the temperature vanishes on \( \Gamma_D \cup \Gamma_N \).

u_0, w_0, \theta_0 \) represent the initial

The structure of the paper is as follows. In Section 2, we first introduce pre-
liminary materials and list some assumptions on the data. Then, we establish a
dynamic history-dependent hemivariational inequality and a variational equation

and the contact model. In Section 3, we first introduce a discrete
problem and then give an optimal order error estimate for finite element method.
In the last section, we give some numerical simulation results of a two-dimensional
contact problem to provide the explanation of our theoretical results.

2. Notation and assumptions. In the study of the corresponding mathematical
theory, in this section, we recall notation, basic definitions and materials. We start
with the definitions of Clarke’s directional derivative and Clarke’s subdifferential.

Let \( X \) be a Banach space, and \( X^* \) its dual. Denote by \( \langle \cdot, \cdot \rangle_{X^* \times X} \)
the duality pairing between \( X^* \) and \( X \).

Definition 2.1. Let \( \psi: X \to \mathbb{R} \) be a locally Lipschitz function. The generalized
directional derivative, in the sense of Clarke’s, of \( \psi \) at \( x \in X \) in the direction
\( v \in X \), denoted by \( \psi^0(x;v) \), is defined by

\[
\psi^0(x;v) = \lim sup_{y \to x, \lambda \downarrow 0} \frac{\psi(y + \lambda v) - \psi(y)}{\lambda}
\]

and the Clarke’s subdifferential of \( \psi \) at \( x \), denoted by \( \partial \psi(x) \), is a subset of a dual
space \( X^* \) given by

\[
\partial \psi(x) = \{ \zeta \in X^* \mid \psi^0(x;v) \geq \langle \zeta, v \rangle_{X^* \times X} \ \forall \ v \in X \}.
\]

We use the standard notation for Lebesgue and Sobolev spaces. For \( v \in H^1(\Omega; \mathbb{R}^d) \),
The same symbol \( v \) for the trace of \( v \) on \( \partial \Omega \) is used and at the same time we use
the notation \( v_\nu \) and \( v_\tau \) for its normal and tangential traces. In addition, spaces \( V \)
and \( \mathcal{H} \) are introduced as follows:

\[
V = \{ v = (v_i) \in H^1(\Omega; \mathbb{R}^d) \mid v = 0 \text{ on } \Gamma_D \},
\]

\[
\mathcal{H} = L^2(\Omega; \mathbb{S}^d),
\]

\[
H = L^2(\Omega; \mathbb{R}^d).
\]

These are real Hilbert spaces with the canonical inner products in \( \mathcal{H} \) and \( H \), and
the inner product

\[
(\mathbf{u}, \mathbf{v})_V = (\mathbf{\varepsilon}(\mathbf{u}), \mathbf{\varepsilon}(\mathbf{v}))_{\mathcal{H}}
\]

in \( V \). The associated norms are \( \| \cdot \|_V \), \( \| \cdot \|_\mathcal{H} \) and \( \| \cdot \|_H \). By the Sobolev trace
theorem,

\[
\| v \|_{L^2(\Gamma_C; \mathbb{R}^d)} \leq \| \gamma \| \| v \|_V \quad \forall \ v \in V,
\]

where \( \| \gamma \| \) represents the norm of the trace operator \( \gamma: V \to L^2(\Gamma_C; \mathbb{R}^d) \).

Next, we let

\[
E = \{ \eta \in H^1(\Omega), \ \eta = 0 \text{ on } \Gamma_D \cup \Gamma_N \},
\]

\[
F = L^2(\Omega).
\]
Note that \( V \subset H \subset V^* \) form an evolution triple of function spaces. We introduce spaces \( \mathcal{V} = L^2(0, T; V) \) and \( \mathcal{W} = \{ w \in \mathcal{V} \mid w' \in V^* \} \), where the time derivative \( w' = \partial w / \partial t \) is understood in the sense of vector-valued distributions. The dual of \( \mathcal{V} \) is \( \mathcal{V}^* = L^2(0, T; V^*) \). It is known that the space \( \mathcal{W} \) endowed with the graph norm \( \| w \|_{\mathcal{W}} = \| w \|_{\mathcal{V}} + \| w' \|_{\mathcal{V}^*} \) is a separable and reflexive Banach space. The embeddings \( \mathcal{W} \subset C([0, T]; H) \) and \( \{ w \in \mathcal{V} \mid w' \in V^* \} \subset C([0, T]; \mathcal{V}) \) are continuous, \( C([0, T]; H) \) being the space of continuous functions on \([0, T] \) with values in \( H \). The duality pairing between \( \mathcal{V}^* \) and \( \mathcal{V} \) is

\[
\langle w, v \rangle_{\mathcal{V}^* \times \mathcal{V}} = \int_0^T \langle w(t), v(t) \rangle_{\mathcal{V}^* \times \mathcal{V}} \, dt, \quad w \in \mathcal{V}^*, v \in \mathcal{V}.
\]

Now we introduce assumptions on the data in the study of Problem 1.1. For the viscosity operator \( A : \Omega \times S^d \to S^d \), we assume

\[
\begin{align*}
(a) & \text{ there exists } L_A > 0 \text{ such that for all } \varepsilon_1, \varepsilon_2 \in S^d \text{ and a.e. } x \in \Omega, \\
& \| A(x, \varepsilon_1) - A(x, \varepsilon_2) \|_{S^d} \leq L_A \| \varepsilon_1 - \varepsilon_2 \|_{S^d}; \\
(b) & \text{ there exists } m_A > 0 \text{ such that for all } \varepsilon_1, \varepsilon_2 \in S^d \text{ and a.e. } x \in \Omega, \\
& (A(x, \varepsilon_1) - A(x, \varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geq m_A \| \varepsilon_1 - \varepsilon_2 \|_{S^d}^2; \\
(c) & \text{ } A(\cdot, \varepsilon) \text{ is measurable on } \Omega, \text{ for all } \varepsilon \in S^d; \\
(d) & \text{ } A(x, 0) = 0 \text{ a.e. } x \in \Omega.
\end{align*}
\]

For the elasticity operator \( B : \Omega \times S^d \to S^d \), we assume

\[
\begin{align*}
(a) & \text{ there exists } L_B > 0 \text{ such that for all } \varepsilon_1, \varepsilon_2 \in S^d \text{ and a.e. } x \in \Omega, \\
& \| B(x, \varepsilon_1) - B(x, \varepsilon_2) \|_{S^d} \leq L_B \| \varepsilon_1 - \varepsilon_2 \|_{S^d}; \\
(b) & \text{ } B(\cdot, \varepsilon) \text{ is measurable on } \Omega \text{ for all } \varepsilon \in S^d; \\
(c) & \text{ } B(\cdot, 0) \in L^2(\Omega; S^d).
\end{align*}
\]

For the viscoelastic long memory operator \( C \), we assume

\[
\begin{align*}
(a) & \text{ there exists } L_C > 0 \text{ such that} \\
& \| C(x, \varepsilon_1, \theta_1) - C(x, \varepsilon_2, \theta_2) \|_{S^d} \leq L_C (\| \varepsilon_1 - \varepsilon_2 \|_{S^d} + |\theta_1 - \theta_2|) \\
& \text{ for all } \varepsilon_1, \varepsilon_2 \in S^d, \theta_1, \theta_2 \in \mathbb{R}, \text{ and a.e. } x \in \Omega. \\
(b) & \text{ } C(\cdot, \varepsilon, \theta) \text{ is measurable on } \Omega \text{ for all } \varepsilon \in S^d, \theta \in \mathbb{R}; \\
(c) & \text{ } C(x, 0, 0) = 0 \text{ a.e. } x \in \Omega.
\end{align*}
\]

For the potential function \( j_r : \Gamma_C \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \), we assume

\[
\begin{align*}
(a) & \text{ } j_r(\cdot, r, \xi) \text{ is measurable on } \Gamma_C \text{ for all } (r, \xi) \in \mathbb{R}_+ \times \mathbb{R}^d \text{ and} \\
& \text{ there exists } e \in L^2(\Gamma_C; \mathbb{R}^d) \text{ such that } j_r(\cdot, r, e(\cdot)) \in L^1(\Gamma_C); \\text{ for a.e. } x \in \Gamma_C \text{ and all } r \in \mathbb{R}_+; \\
(b) & \text{ } j_r(x, r, \cdot) \text{ is Lipschitz continuous on } \mathbb{R}^d \\
& \text{ for a.e. } x \in \Gamma_C; \text{ and all } (r, \xi) \in \mathbb{R}_+ \times \mathbb{R}^d \text{ with } \xi_0 \geq 0; \\
(c) & |\partial j_r(x, r, \xi)| \leq \bar{c}_0 \text{ for a.e. } x \in \Gamma_C, \text{ all } (r, \xi) \in \mathbb{R}_+ \times \mathbb{R}^d \text{ with } \bar{c}_0 \geq 0; \\
(d) & j_r(x, r_1, \xi_1; \xi_2 - \xi_1) + j_r(x, r_2, \xi_2; \xi_1 - \xi_2) \\
& \leq \bar{\beta} (\| \xi_1 - \xi_2 \|_{\mathbb{R}^d}^2 + m_r |r_1 - r_2| \| \xi_1 - \xi_2 \|_{\mathbb{R}^d}) \\
& \text{ for a.e. } x \in \Gamma_C, \text{ all } (r_i, \xi_i) \in \mathbb{R} \times \mathbb{R}^d, i = 1, 2 \text{ with } \bar{\beta}, m_r \geq 0.
\end{align*}
\]
For the memory function \( b : \Gamma_C \to \mathbb{R} \), we assume
\[
\begin{cases}
(a) \ b \in L^1(0, T; L^\infty(\Gamma_C)); \\
(b) \ b(x) \geq 0 \text{ for a.e. } x \in \Gamma_C.
\end{cases}
\tag{2.5}
\]

For the contact function \( p : \Gamma_C \times \mathbb{R} \to \mathbb{R} \), we assume
\[
\begin{cases}
(a) \ p(\cdot, r) \text{ is measurable on } \Gamma_C, \text{ for any } r \in \mathbb{R}; \\
(b) \ p(\cdot, 0) \in L^2(\Gamma_C); \\
(c) \text{ there exists } L_p > 0 \text{ such that for all } r_1, r_2 \in \mathbb{R}, \text{ a.e. } x \in \Gamma_C \quad |p(x, r_1) - p(x, r_2)| \leq L_p|r_1 - r_2|.
\end{cases}
\tag{2.6}
\]

Moreover, we assume that the densities of body forces, surface tractions have the regularity
\[
f_0 \in C([0, T]; L^2(\Omega; \mathbb{R}^d)), \quad f_N \in C([0, T]; L^2(\Gamma_N; \mathbb{R}^d)).
\tag{2.7}
\]

As for the thermal tensors and the heat sources density, we assume that
\[
c_{ij} = c_{ji} \in L^\infty(\Omega), \quad q \in H^1(0, T; L^2(\Omega)).
\tag{2.8}
\]

The boundary thermic data satisfy
\[
k_e \in L^\infty(\Omega; \mathbb{R}_+), \quad \theta_R \in H^1(0, T; L^2(\Gamma_C)).
\tag{2.9}
\]

For some \( m_K > 0 \) and for all \( (\xi_i) \in \mathbb{R}^d \),
\[
K_e = (k_{ij}), \quad k_{ij} = k_{ji} \in L^\infty(\Omega), \quad k_{ij} \xi_i \xi_j \geq m_K \xi_i \xi_i.
\tag{2.10}
\]

Finally, the initial data satisfy
\[
u_0, \ w_0 \in V, \quad \theta_0 \in E.
\tag{2.11}
\]

Define a function \( f : (0, T) \to V^* \) by
\[
\langle f(t), v \rangle_{V^* \times V} = \langle f_0(t), v \rangle_H + \langle f_N(t), v \rangle_{L^2(\Gamma_N; \mathbb{R}^d)}, \quad v \in V, \text{ a.e. } t \in (0, T).
\tag{2.12}
\]

We obtain the following variational formulation of Problem 1.1 through a list of standard derivation.

**Problem 2.2.** Find a displacement field \( u : (0, T) \to V \), \( \theta : (0, T) \to E \) such that for a.e. \( t \in (0, T) \),
\[
\mathbf{\sigma}(t) = A\varepsilon(u'(t)) + B\varepsilon(u(t)) + \int_0^t C(t - s, \varepsilon(u(s)), \theta(s)) \, ds,
\tag{2.13}
\]
\[
\langle u''(t), v \rangle_{V^* \times V} + \langle \mathbf{\sigma}(t), \varepsilon(v) \rangle_H + \int_{\Gamma_C} \left( \int_0^t b(t - s)u_\nu(s) \, ds \right) v_\nu \, d\Gamma \\
+ \int_{\Gamma_C} p(u_\nu(t))v_\nu \, d\Gamma + \int_{\Gamma_C} \lambda^0(t) \left( \int_0^t \| u_\tau(s) \| \, ds, u_\tau'(t); w_\tau \right) \, d\Gamma \\
\geq \langle f(t), v \rangle_{V^* \times V} \quad \forall v \in V,
\tag{2.14}
\]
\[
\langle \theta'(t), \eta \rangle_E + \langle K\theta(t), \eta \rangle_{E^* \times E} = \langle Ru'(t), \eta \rangle_{E^* \times E} + \langle Q(t), \eta \rangle_{E^* \times E}, \quad \forall \eta \in E,
\tag{2.15}
\]

and
\[
u(0) = u_0, \quad u'(0) = w_0, \quad \theta(0) = \theta_0.
\tag{2.16}
\]
Here the functions $K : E \to E^*$, $R : V \to E^*$, and $Q : (0,T) \to E^*$ are defined by $\forall \ w \in V, \tau \in E, \eta \in E$:

$$
\langle K \tau, \eta \rangle_{E^* \times E} = \sum_{i,j=1}^{d} \int_{\Omega} k_{ij} \frac{\partial \tau}{\partial x_i} \frac{\partial \eta}{\partial x_j} \ dx + \int_{\Gamma_C} k_c \tau \cdot \eta \ da.
$$

$$
\langle R w, \eta \rangle_{E^* \times E} = \int_{\Omega} c_{ij} \frac{\partial w_i}{\partial x_j} \eta \ dx.
$$

$$
\langle Q(t), \eta \rangle_{E^* \times E} = \int_{\Gamma_C} k_c \theta_R(t) \eta \ da + \int_{\Omega} q(t) \eta \ dx.
$$

The unique solvability of Problem 2.2 is provided in the following result.

**Theorem 2.3.** Assume (2.1)–(2.11). If

$$
m_A > \bar{\beta} \| \gamma \|^2,
$$

then Problem 2.2 has a unique solution $(u, \theta)$ with regularity $u \in H^2(0,T; V)$, $\theta \in H^1(0,T; E) \cap W^{1,\infty}(0,T; F)$.

We refer to [16] for a standard argument to show the existence of a unique solution $(u, \theta)$. The proof is based on the Banach fixed point arguments and some results for hemivariational inequality.

For the convenience of the numerical approximation, we reformulate the dynamic history-dependent hemivariational inequality in terms of the velocity variable $w = u'$.

(2.18)

By using the initial value condition (2.16), we can recover $u$ from $w$ as follows:

$$
u(t) = u_0 + (Iw)(t),
$$

where

$$
(Iw)(t) = \int_0^t w(s) \ ds.
$$

(2.20)

Then we can state Problem 2.2 equivalently as follows.

**Problem 2.4.** Find a velocity field $w \in \mathcal{W}$, $\theta : (0,T) \to E$ such that for a.e. $t \in (0,T)$,

$$
\sigma(t) = A \varepsilon(w(t)) + B \varepsilon(u(t)) + \int_0^t C(t-s, \varepsilon(u(s)), \theta(s)) \ ds,
$$

(2.21)

$$
\langle w'(t), v \rangle_{V^* \times V} + \langle \sigma(t), \varepsilon(v) \rangle_H + \int_{\Gamma_C} J_v \left( \int_0^t \| u_\tau(s) \| \ ds, w_\tau(t); v_\tau \right) \ d\Gamma
$$

$$
+ \int_{\Gamma_C} \left( \int_0^t b(t-s) u_\nu(s) \ ds \right) v_\nu \ d\Gamma + \int_{\Gamma_C} p(u_\nu(t)) v_\nu \ d\Gamma
$$

$$
\geq \langle f(t), v \rangle_{V^* \times V} \ \forall \ v \in V
$$

(2.22)

$$
\langle \theta'(t), \eta \rangle_F + \langle K \theta(t), \eta \rangle_{E^* \times E} = \langle Rw(t), \eta \rangle_{E^* \times E} + \langle Q(t), \eta \rangle_{E^* \times E}, \ \forall \eta \in E.
$$

(2.23)

and

$$
w(0) = w_0, \quad \theta(0) = \theta_0.
$$

(2.24)
3. A fully discrete scheme and error estimate. In this section, Our main purpose is to introduce a fully discrete scheme for the coupled system of history-dependent hemivariational inequality and variational equation formulated in Problem 2.4 and provide a result on error estimate. First, we recall a discrete Gronwall inequality (cf. [12, Lemma 7.26]).

**Lemma 3.1.** Let $T > 0$ be given. For a positive integer $N$, define $k = T/N$. Assume that $\{g_n\}_{n=1}^N$ and $\{e_n\}_{n=1}^N$ are two sequences of nonnegative numbers satisfying

$$e_n \leq \bar{c} g_n + \bar{c} \sum_{j=1}^n k e_j, \quad n = 1, \ldots, N$$

for a positive constant $\bar{c}$ independent of $N$ or $k$. Then, there exists a positive constant $c$, independent of $N$ or $k$, such that

$$\max_{1 \leq n \leq N} e_n \leq c \max_{1 \leq n \leq N} g_n.$$

In the later error analysis, we will also make use of the following elementary inequality with an arbitrary $\epsilon > 0$:

$$a b \leq \epsilon a^2 + c b^2, \quad a, b \in \mathbb{R},$$

(3.1)

where the constant $c > 0$ depends on $\epsilon$.

Let $V^h$ be a finite dimensional subspace of $V$, $E^h$ be a finite dimensional subspace of $E$ where $h > 0$ denotes a spatial discretization parameter. In addition, we consider an equidistant time grid with abscissae $t_n = nk$, where $n = 0, 1, \cdots, N$, $N \in \mathbb{N}$, and the constant step-size $k = T/N$. For a time continuous function $z = z(t)$, we write $z_n = z(t_n)$ for $n = 0, 1, \cdots, N$.

Let $u_0^h$, $w_0^h$, $\theta_0^h$ be the appropriate approximation of initial condition $u_0, w_0, \theta_0$. The integration operator $I$ of (2.20) will be approximated by the discrete operator $I_k$ by the form

$$(I_k w)_n = k \sum_{j=1}^n w_j$$

(3.2)

and the discrete displacement field and discrete velocity field are related by the relation

$$u_{nk}^h = u_0^h + (I_k w_{nk}^h)\big|_{n},$$

(3.3)

where $(I_k w_{nk}^h)\big|_{n} = k \sum_{j=1}^n w_{nk}^h$.

**Problem 3.2.** Find a discrete velocity field $w_{nk}^h = \{w_{nk}^h\}_{n=0}^N \subset V^h$, a discrete temperature field $\theta_{nk}^h = \{\theta_{nk}^h\}_{n=0}^N \subset E^h$ such that for $1 \leq n \leq N$,

$$\sigma_{nk}^h = A\varepsilon(w_{nk}^h) + B\varepsilon(u_{nk-1}^h) + k \sum_{j=1}^n C(t_n - t_j, -\varepsilon(u_{nk-1}^h, \theta_{nk-1}^h),$$

(3.4)

$$\left(\frac{w_{nk}^h - w_{nk-1}^h}{k}, v^h\right)_H + (\sigma_{nk}^h, \varepsilon(v^h))_H + \int_{\Gamma_C} p(u_{nk-1, t}) v_\Gamma^h d\Gamma$$

$$+ \int_{\Gamma_C} \left( k \sum_{j=1}^n b(t_n - t_j) u_{nk-1, t}^h \right) v_\Gamma^h d\Gamma + \int_{\Gamma_C} \left( k \sum_{j=1}^n \|u_{nk-1, \tau}^h, w_{nk}^h, v^h\right) d\Gamma$$

$$\geq \langle f_n, v^h \rangle_{V^\ast \times V} \quad \forall v^h \in V^h,$$

(3.5)

$$\left(\frac{\theta_{nk}^h - \theta_{nk-1}^h}{k}, \eta^h\right)_E + \langle K\theta_{nk}^h, \eta^h \rangle_{E^\ast \times E}$$
\[ \langle Rw_n^{hk}, \eta^h \rangle_{E^* \times E} + \langle Q_n, \eta^h \rangle_{E^* \times E}, \quad \forall \eta^h \in E^h, \quad (3.6) \]

and

\[ w_0^{hk} = w_0^h, \quad \theta_0^{hk} = \theta_0^h. \quad (3.7) \]

In the rest of the paper, we use \( c \) for a generic positive constant whose value may change from time to time, but it is independent of \( h \) and \( k \).

**Theorem 3.3.** Assume that (2.1)–(2.11) hold. Then, there exists a unique solution \((w^{hk}, \theta^{hk})\) of Problem 3.2.

For \( n = 1, \ldots, N \), suppose that \( u_{n-1}^{hk}, w_{n-1}^{hk}, \theta_{n-1}^{hk} \) are given, substitute (3.4) into (3.5) then \( w_n^{hk} \) is uniquely determined by (3.5), we obtain \( \theta_n^{hk} \) from (3.6). Hence, under the assumptions of Theorem 2.3, there exists a unique solution \( w^{hk} \subset V^h, \theta^{hk} \subset E^h \).

Our interest in this paper lies in the error estimates for \( w_n - w^{hk} \) and \( \theta_n - \theta^{hk} \). For an error analysis of the numerical method, we make the following additional assumptions:

\[ w \in H^1(0,T;V) \cap H^2(0,T;V^*), \quad (3.8) \]
\[ \theta \in H^2(0,T;F). \quad (3.9) \]

Moreover, the first term in (2.22) can be rewritten as

\[ (w'(t),v)_H. \quad (3.10) \]

We first set \( t = t_n \) and \( v = w_n^{hk} - w_n^h \) in (2.22) and we also replace \( v^h \) with \( w_n - w_n^{hk} \) in (3.5). By adding the resulting inequalities, we obtain that

\[ \langle \mathcal{A}e(w_n) - \mathcal{A}e(w_n^{hk}), e(w_n - w_n^{hk}) \rangle_H \]
\[ \leq \langle \mathcal{A}e(w_n) - \mathcal{A}e(w_n^{hk}), e(w_n - w_n^h) \rangle_H + \langle w_n' - \frac{w_n - w_n^{hk}}{k}, w_n - w_n^h \rangle_H 
- \frac{1}{k} (e_n - e_{n-1}, e_n) + \frac{1}{k} (e_n - e_{n-1}, w_n - w_n^h) + I_1 + I_2 + I_3, \]

where

\[ I_1 = (B\mathcal{E}(u_n) - B\mathcal{E}(u_n^{hk}), e(w_n - w_n^h))_H + \left( \int_0^{t_n} C(t_n - s, \mathcal{E}(u(s)), \theta(s)) ds \right. \]
\[ \left. - k \sum_{j=1}^n C(t_n - t_{j-1}, \mathcal{E}(u_{j-1}^{hk}), \theta_{j-1}^{hk}, \mathcal{E}(w_{j-1}^{hk} - v_{j-1}^h))_H \right. \]
\[ I_2 = \int_{\Gamma} \left[ (p(u_{n,\nu}) - p(u_{n-1,\nu}))((w_n^{hk} - v_n^h) - v_{n,\nu}) d\Gamma \right. \]
\[ \left. + \int_{\Gamma} \left( \int_0^{t_n} b(t_n - s) u(s) ds - k \sum_{j=1}^n b(t_n - t_{j-1}) u_{j-1}^{hk} \right) (w_n^{hk} - v_n^h) d\Gamma, \right. \]
\[ I_3 = \int_{\Gamma} J_{\tau} \left( \int_0^{t_n} \| u(s) \| ds, w_{n,\tau}; w_{n,\tau} - v_{n,\tau} \right) d\Gamma \]
\[ \left. + \int_{\Gamma} J_{\tau} \left( k \sum_{j=1}^n \| u_{j-1,\tau} \|, w_{n,\tau}; w_{n,\tau} - v_{n,\tau} \right) d\Gamma. \right. \]
Lemma 3.4. Let $w$ and $w^{hk}$ be solutions to Problems 2.4 and 3.2, respectively. We have the following bound for $n = 1, \ldots, N$

$$
\|(Iw)_n - (I^k w^{hk})_{n-1}\|_V \leq c k \|w\|_{H^1(0,T;V)} + k \sum_{j=1}^{n-1} \|w_j - w^{hk}_j\|_V. \quad (3.12)
$$

Proof. From the definitions of $(Iw)_n$ and $(I^k w^{hk})_n$, we have

$$
\|(Iw)_n - (I^k w^{hk})_{n-1}\|_V 
\leq k \sum_{j=1}^{n-1} \|w_j - w^{hk}_j\|_V + \left| \int_0^{t_{n-1}} w(s)ds - k \sum_{j=1}^{n-1} w_j \right|_V + \left| \int_0^{t_n} w(s)ds - \int_0^{t_{n-1}} w(s)ds \right|_V
\leq \int_{t_{n-1}}^{t_n} \|w(s)\|_V ds + \left| \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} (w(s) - w_j) ds \right|_V + k \sum_{j=1}^{n-1} \|w_j - w^{hk}_j\|_V
\leq \int_{t_{n-1}}^{t_n} \|w(s)\|_V ds + \left| \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \frac{d}{d\tau} (w(\tau)) d\tau ds \right|_V + k \sum_{j=1}^{n-1} \|w_j - w^{hk}_j\|_V
\leq k \|w\|_{L^2(0,T;V)} + k \sum_{j=1}^{n-1} \|w'(\tau)\|_V d\tau + k \sum_{j=1}^{n-1} \|w_j - w^{hk}_j\|_V,
$$

i.e., $(3.12)$ holds.

We now pay attention to the estimate for $w_n - w^{hk}_n$ and we first derive upper bounds on three history-dependent terms $I_1, I_2, I_3$.

Lemma 3.5. Let $(w, \theta)$ and $(w^{hk}, \theta^{hk})$ be solutions to Problems 2.4 and 3.2, respectively. Under the regularity assumption $b \in H^1(0,T;L^\infty(\Gamma_C))$, we have the following inequality:

$$
I_1 + I_2 + I_3
\leq c \|w^{hk}_n - u^{h}_n\|_V \left( k \|w\|_{H^1(0,T;V)} + k \|\theta\|_{H^2(0,T;F)} + k \sum_{j=1}^{n-1} \|w_j - w^{hk}_j\|_V 
+ k \sum_{j=0}^{n-1} \|\theta_j - \theta^{hk}_j\|_F + \|u_0 - u^{h}_0\|_V \right) + \tilde{\beta} \|\gamma\|^2 \|w_n - w^{hk}_n\|^2_\gamma + c \|w_n - w^{hk}_n\|_{L^2(\Gamma_C;\mathbb{R})}
$$

$$
+ c \|w_n - w^{hk}_n\|_V \left( k \|w\|_{H^1(0,T;V)} + k \sum_{j=1}^{n-1} \|w_j - w^{hk}_j\|_V + \|u_0 - u^{h}_0\|_V \right)
$$

for all $w_{n}^h \in V^h$, $n = 1, \ldots, N$. \quad (3.13)

Proof. Let us first give the bound for $I_1$. By using $(2.2)(a)$, we obtain

$$
(B\varepsilon(u_n) - B\varepsilon(u^{hk}_n)) = (B\varepsilon(u_0 + (Iw)_n) - B\varepsilon(u_0^0) + (I^k w^{hk})_{n-1})
\leq L_B(\|u_0 - u_0^0\|_V + \|(Iw)_n - (I^k w^{hk})_{n-1}\|_V).
$$

Also, for the term containing the operator $C$, we have

$$
\left\| \int_0^{t_n} C(t_n - s, \varepsilon(u(s)), \theta(s)) ds - \sum_{j=1}^{n} kC(t_n - t_{j-1}, \varepsilon(u^{hk}_{j-1}), \theta^{hk}_{j-1}) \right\|_H \quad (3.14)
$$
Consequently, there is a constant $c$ such that
\begin{align*}
\|f(t_n) - f(t_0)\| &\leq k^\omega \sum_{j=0}^{n-1} \|C(t_j) - C(t_{j-1})\| + k \|\epsilon\|_{L^\infty} \sum_{j=0}^{n-1} \|\epsilon_j\|_{L^2} \\
&\leq k^\omega \sum_{j=0}^{n-1} \|C(t_j) - C(t_{j-1})\| + k \|\epsilon\|_{L^\infty} \sum_{j=0}^{n-1} \|\epsilon_j\|_{L^2}.
\end{align*}

Next, we bound $I_2$. For the term containing function $p$, by using 2.6(c), we have
\begin{align*}
\|p(u_{n+1}) - p(u_n)\|_{L^2(\Gamma_C)} &= \|p(u_{n+1} - p(u_n) - p(u_n) - p(u_{n+1}))\|_{L^2(\Gamma_C)} \\
&\leq L_p \|u_{n+1} - u_n\|_{L^2(\Gamma_C)} + \|f(u_{n+1}) - f(u_n)\|_{L^2(\Gamma_C)} \\
&\leq L_p \|u_{n+1} - u_n\|_{L^2(\Gamma_C)} + \|f(u_{n+1}) - f(u_n)\|_{L^2(\Gamma_C)} + \|f(u_{n+1}) - f(u_n)\|_{L^2(\Gamma_C)}.
\end{align*}

For the term containing $b$, similar to the estimate for the operator $C$, we obtain
\begin{align*}
\|\sum_{j=0}^{n-1} \|C(t_j) - C(t_{j-1})\| + k \|\epsilon\|_{L^\infty} \sum_{j=0}^{n-1} \|\epsilon_j\|_{L^2}.
\end{align*}

Consequently, there is a constant $c$ such that
\begin{align*}
I_2 &\leq c \|w_n - v_n\|_{L^2(\Gamma_C)} + \|f(u_{n+1}) - f(u_n)\|_{L^2(\Gamma_C)} + \|f(u_{n+1}) - f(u_n)\|_{L^2(\Gamma_C)}.
\end{align*}

Finally, we bound $I_3$, which is expressed as
\begin{align*}
I_3 &= \int_{0}^{t_n} \int_{\Gamma_C} \sum_{j=0}^{n-1} \|u_{j+1}(s) - u_j(s)\|_{L^2(\Gamma_C)} + k \sum_{j=0}^{n-1} \|u_{j+1} - u_j\|_{L^2(\Gamma_C)} + k \sum_{j=0}^{n-1} \|u_{j+1} - u_j\|_{L^2(\Gamma_C)}.
\end{align*}
By (2.4) (c), \( j^0_\tau(x, r; \xi; \eta) \leq \overline{c}_0 |\eta| \). Using the subadditivity of the generalized directional derivative, (2.4) (d) and Cauchy-Schwarz inequality, it follows that

\[
\int_{\Gamma_C} j^0_\tau \left( \int_0^{t_n} \| u_\tau(s) \| ds, w_n, \tau; w^{h_k}_{n, \tau} - v^{h_k}_{n, \tau} \right) d\Gamma \\
+ \int_{\Gamma_C} j^0_\tau \left( k \sum_{j=1}^n \| u^{h_k}_{j-1, \tau} \|, w^{h_k}_{n, \tau}; v^{h_k}_{n, \tau} - w^{h_k}_{n, \tau} \right) d\Gamma \\
\leq \int_{\Gamma_C} j^0_\tau \left( \int_0^{t_n} \| u_\tau(s) \| ds, w_n, \tau; w^{h_k}_{n, \tau} - w_{n, \tau} \right) \\
+ \int_{\Gamma_C} j^0_\tau \left( k \sum_{j=1}^n \| u^{h_k}_{j-1, \tau} \|, w^{h_k}_{n, \tau}; v^{h_k}_{n, \tau} - w_{n, \tau} \right) \\
+ \int_{\Gamma_C} j^0_\tau \left( k \sum_{j=1}^n \| u^{h_k}_{j-1, \tau} \|, w^{h_k}_{n, \tau}; w^{h_k}_{n, \tau} - w^{h_k}_{n, \tau} \right) d\Gamma \\
\leq \beta |\gamma|^2 \| w_n - w^{h_k}_{n} \|_{\gamma}^2 + 2 \overline{c}_0 \sqrt{m(\overline{\Gamma}_C)} \| w_n - w^{h_k}_{n} \|_{L^2(\overline{\Gamma}_C; \mathbb{R}^d)} \\
+ m_\tau \int_{0}^{t_n} \| u_\tau(s) \| ds - k \sum_{j=1}^n \| u^{h_k}_{j-1, \tau} \| \| \gamma w_n - \gamma w^{h_k}_{n} \|.
\]

As for \( \int_{0}^{t_n} \| u_\tau(s) \| ds - k \sum_{j=1}^n \| u^{h_k}_{j-1, \tau} \| \), we have

\[
\left| \int_{0}^{t_n} \| u_\tau(s) \| ds - k \sum_{j=1}^n \| u^{h_k}_{j-1, \tau} \| \right| \leq c (I_4 + I_5),
\]

where

\[
I_4 = \left| \int_{0}^{t_n} \| u_\tau(s) \| ds - k \sum_{j=1}^n \| u^{h_k}_{j-1, \tau} \| \right|
\]

\[
I_5 = k \left| \sum_{j=1}^n \| u^{h_k}_{j-1, \tau} \| - \sum_{j=1}^n \| u^{h_k}_{j-1, \tau} \| \right|
\]

After some simple estimates just like [23], we obtain

\[
\left| \int_{0}^{t_n} \| u_\tau(s) \| ds - k \sum_{j=1}^n \| u^{h_k}_{j-1, \tau} \| \right|
\]

\[
\leq c \left( k \| w \|_{H^1(0, T; V)} + \sum_{j=1}^{n-1} \| w_j - w^{h_k}_{j} \|_V + \| u_0 - u^{h}_0 \|_V \right).
\]

It is enough for us to obtain

\[
\int_{\Gamma_C} j^0_\tau \left( \int_0^{t_n} \| u_\tau(s) \| ds, w_n, \tau; w^{h_k}_{n, \tau} - v^{h_k}_{n, \tau} \right) d\Gamma \\
+ \int_{\Gamma_C} j^0_\tau \left( k \sum_{j=1}^n \| u^{h_k}_{j-1, \tau} \|, w^{h_k}_{n, \tau}; v^{h_k}_{n, \tau} - w^{h_k}_{n, \tau} \right) d\Gamma \\
\leq \beta |\gamma|^2 \| w_n - w^{h_k}_{n} \|_{\gamma}^2 + c \| w_n - w^{h}_{n} \|_{L^2(\overline{\Gamma}_C; \mathbb{R}^d)}
\]

(3.17)
It is easy to note that
\[ (e_n - e_{n-1}, e_n)_H = \frac{1}{2} (\|e_n\|_H^2 - \|e_{n-1}\|_H^2 + \|e_n - e_{n-1}\|_H^2) \geq \frac{1}{2} (\|e_n\|_H^2 - \|e_{n-1}\|_H^2). \]

Thus,
\[ -\frac{1}{k} (e_n - e_{n-1}, e_n)_H \leq -\frac{1}{2k} (\|e_n\|_H^2 - \|e_{n-1}\|_H^2). \]  

Denote
\[ E_n = w_n' - \frac{w_n - w_{n-1}}{k}. \]

It is easy to note that
\[ (E_n, w_n'^h - v_n^h)_V \leq \|E_n\|_V \|w_n'^h - v_n^h\|_V. \]

For any \( c > 0 \), we apply the elementary inequality (3.1) and obtain that
\[ (E_n, w_n'^h - v_n^h)_V \leq c \|e_n\|_V + c \|w_n - v_n^h\|_V. \]  

From (2.1)(a) and applying again (3.1), we have
\[ (A e(w_n) - A e(w_n'^h), e(w_n - v_n^h))_H \leq L_A \|w_n - w_n'^h\|_V \|w_n - v_n^h\|_V \]
\[ \leq c \|w_n - w_n'^h\|_V + c \|w_n - v_n^h\|_V. \]  

Then, by using inequalities (3.13), (3.18)–(3.20) on the right side of the inequality (3.11) and taking \( c > 0 \) sufficiently small, under assumption \( m_A > \beta \|\gamma\|^2 \), we obtain the following result.
\[ k \|e_n\|_V + \|e_n\|_H^2 - \|e_{n-1}\|_H^2 \]
\[ \leq k \|w_n - v_n^h\|_V + \|w_n - v_n^h\|_{L^2(\Gamma; \mathbb{R}^d)} + \|E_n\|_V \]
\[ + c k \|e_n\|_V (k \|w\|_{H^1(0,T;V)} + k \sum_{j=1}^{n-1} \|w_j - w_j'^h\|_V + \|u_0 - u_0^h\|_V) \]
\[ + c k \|w_n'^h - v_n^h\|_V \left( k \|w\|_{H^1(0,T;V)} + k \sum_{j=1}^{n-1} \|w_j - w_j'^h\|_V \right) \]
\[ + \|u_0 - u_0^h\|_V + \|e_n - e_{n-1}, w_n - v_n^h\|_H. \]

Since \( \|w_n'^h - v_n^h\|_V \leq 2(\|w_n'^h - w_n\|_V + \|w_n - v_n^h\|_V) \), it follows that
\[ k \|e_n\|_V^2 + \|e_n\|_H^2 - \|e_{n-1}\|_H^2 \]
\[ \leq k \|w_n - v_n^h\|_V^2 + \|w_n - v_n^h\|_{L^2(\Gamma; \mathbb{R}^d)} + \|E_n\|_V \]
\[ + c k^2 \|w\|_{H^1(0,T;V)} + c \|u_0 - u_0^h\|_V + c k^2 \|e_n - e_{n-1}\|_V \]
\[ + c k^2 \sum_{j=0}^{n-1} \|\theta_j - \theta_j'^h\|_p^2 + c k^2 \sum_{j=1}^{n-1} \|w_j - w_j'^h\|_V^2. \]
We replace $n$ by $l$ in (3.21) and make a summation over $l$ from 1 to $n$,

$$k \sum_{l=1}^{n} \| e_l \|_{V}^2 + \| e_n \|_{H}^2 - \| e_0 \|_{H}^2$$

$$\leq c k \sum_{l=1}^{n} (\| w_l - w_l^h \|_{V}^2 + \| w_l - w_l^h \|_{L^2(\Gamma_C; \mathbb{R}^e)} + \| E_l \|_{V^*}^2)$$

$$+ c k^2 \| w \|_{H^1(\Omega; V)}^2 + c k^2 \| \theta \|_{H^2(\Omega; F)}^2 + c \sum_{l=1}^{n} (e_l - e_{l-1}, w_l - v_l^h)_{H}$$

$$+ c \| u_0 - u_0^h \|_{V}^2 + c k \sum_{j=1}^{n-1} \sum_{l=1}^{k} \| w_j - w_j^h \|_{V}^2 + c k \sum_{j=0}^{n-1} \| \theta_j - \theta_j^h \|_{F}^2.$$  

The next, we estimate $\theta_n - \theta_n^h$. From (2.23) and (3.6) with $\eta = \eta^h$, we have

$$(\theta_n^h - \frac{\theta_{n-1}^h - \theta_{n-1}}{k}, \eta^h)_{F} + (K \theta_n - K \theta_n^h, \eta^h)_{E^* \times E}$$

$$= (R w_n - R w_n^h, \eta^h)_{E^* \times E}, \quad \forall \eta^h \in E^h.$$  

We write

$$\theta_n^h - \frac{\theta_{n}^h - \theta_{n-1}^h}{k} = \theta_n^h - \frac{\theta_{n-1}^h}{k} + \frac{\varepsilon_n - \varepsilon_{n-1}}{k},$$

and replacing $\eta^h$ by $\eta_n^h - \theta_n + \varepsilon_n$, we obtain

$$\left( \frac{\varepsilon_n - \varepsilon_{n-1}}{k}, \varepsilon_n \right)_{F} + (K \theta_n - K \theta_n^h, \varepsilon_n)_{E^* \times E}$$

$$= (R w_n - R w_n^h, \eta_n^h - \theta_n + \varepsilon_n)_{E^* \times E} - (\theta_n^h, \frac{\theta_{n}^h - \theta_{n-1}^h}{k}, \eta_n^h - \theta_n)_{F}$$

$$- (K \theta_n - K \theta_n^h, \eta_n^h - \theta_n)_{E^* \times E} - \left( \frac{\theta_{n}^h - \theta_{n-1}^h}{k}, \varepsilon_n \right)_{F}.$$  

For the left terms of (3.24), we have

$$(\frac{\varepsilon_n - \varepsilon_{n-1}}{k}, \varepsilon_n)_{F} + (K \theta_n - K \theta_n^h, \varepsilon_n)_{E^* \times E}$$

$$\geq \frac{1}{2k} \left( \| \varepsilon_n \|_{F}^2 - \| \varepsilon_{n-1} \|_{F}^2 \right) + m_{K} \| \varepsilon_n \|_{E}^2.$$  

Then we replace $n$ by $l$ in (3.24) and make a summation from 1 to $n$ to obtain

$$\frac{1}{2k} \left( \| \varepsilon_n \|_{F}^2 - \| \varepsilon_0 \|_{F}^2 \right) + \sum_{l=1}^{n} \| \varepsilon_l \|_{E}^2$$

$$\leq c \sum_{l=1}^{n} \| \theta_l - \theta_{l-1} \|_{F}^2 + c \sum_{l=1}^{n} \| \theta_l - \theta_l^h \|_{E}^2 + c \sum_{l=1}^{n} \| e_l \|_{V}^2 + c \sum_{l=1}^{n} \left( \frac{\varepsilon_l - \varepsilon_{l-1}}{k}, \theta_l - \theta_l^h \right)_{F}.$$  

Combining (3.22) and (3.27), we have

$$\| w_n - w_n^h \|_{H}^2 + k \sum_{l=1}^{n} \| w_l - w_l^h \|_{V}^2 + \| \theta_n - \theta_n^h \|_{F}^2 + k \sum_{l=1}^{n} \| \theta_l - \theta_l^h \|_{E}^2$$

$$\leq c \left( k \sum_{l=1}^{n} \| w_l - v_l^h \|_{V}^2 + \| w_l - v_l^h \|_{L^2(\Gamma_C; \mathbb{R}^e)} + \| E_l \|_{V^*}^2 \right) + \| u_0 - u_0^h \|_{V}^2.$$
For the term $E_l$, we have

$$E_l = \frac{1}{k} \int_{t_{l-1}}^{t_l} (t - t_{l-1}) w''(t) \, dt.$$ 

It follows that

$$\|E_l\|_{V'} \leq \frac{1}{k^2} \int_{t_{l-1}}^{t_l} (t - t_{l-1})^2 \, dt \int_{t_{l-1}}^{t_l} \|w''(t)\|_{V'}^2 \, dt = \frac{k}{3} \int_{t_{l-1}}^{t_l} \|w''(t)\|_{V'}^2 \, dt.$$

And then, we have

$$k \sum_{l=1}^{n} \|E_l\|_{V'} \leq \frac{k^2}{3} \|w''\|_{L^2(0,T;V')}.  \tag{3.28}$$

Similarly, we have

$$k \sum_{l=1}^{n} \|\theta'_{l} - \frac{\theta_l - \theta_{l-1}}{k}\|_E^2 \leq \frac{k^2}{3} \|\theta''\|_{L^2(0,T;F)}. \tag{3.28}$$

For the term $\sum_{l=1}^{n} (e_l - e_{l-1}, w_l - \psi^l)_{H}$, according to [12], we have

$$\sum_{l=1}^{n} (e_l - e_{l-1}, w_l - \psi^l)_{H} \leq \frac{1}{2} (\|e_n\|_{H}^2 + \|w_n - \psi^l\|_{H}^2) + \frac{k}{2} \sum_{l=1}^{n-1} (\|e_l\|_{H}^2 + k^{-2} \|(w_l - \psi^l) - (w_{l+1} - \psi^{l+1})\|_{H}^2)$$

$$+ \frac{1}{2} (\|e_0\|_{H}^2 + \|w_1 - \psi^1\|_{H}^2).$$

At the same time, for the term $k \sum_{l=1}^{n} \left(\frac{\varepsilon_l - \varepsilon_{l-1}}{k}, \theta_l - \eta^h\right)_F$, we have

$$k \sum_{l=1}^{n} \left(\frac{\varepsilon_l - \varepsilon_{l-1}}{k}, \theta_l - \eta^h\right)_F \leq \frac{1}{2} (\|\varepsilon_n\|_{F}^2 + \|\theta_n - \eta^h\|_{F}^2) + \frac{k}{2} \sum_{l=1}^{n-1} (\|\varepsilon_l\|_{F}^2 + k^{-2} \|(\theta_l - \eta^h) - (\theta_{l+1} - \eta^h_{l+1})\|_{F}^2)$$

$$+ \frac{1}{2} (\|\varepsilon_0\|_{F}^2 + \|\theta_1 - \eta^h\|_{F}^2).$$

Finally, we have the following inequality

$$\|w_n - \psi^l\|_{H}^2 + k \sum_{l=1}^{n} \|w_l - \psi^l\|_{V}^2 + \|\theta_n - \theta^h\|_{F}^2 + k \sum_{l=1}^{n} \|\theta_l - \theta^h\|_{E}^2.
3.2, respectively. Assume (2.1)–(2.11),

Let assumption in Lemma 3.5, we have

\[ \inf \sum_{l=1}^{n} \left( \| \theta_l - \eta_l^h \|_E^2 + \| w_1 - v_1 \|_H^2 \right) + k \sum_{l=1}^{n} \left( \| \theta_l - \eta_l^h \|_E^2 + \| w_1 - v_1 \|_H^2 \right) + \| \theta_n - \eta_n^h \|_E^2 + k^{-1} \sum_{l=1}^{n-1} \left( \| \theta_l - \eta_l^h \|_E^2 + \| w_1 - v_1 \|_H^2 \right)

+ \| \theta_n - \eta_n^h \|_E^2 + k^{-1} \sum_{l=1}^{n-1} \left( \| \theta_l - \eta_l^h \|_E^2 + \| \theta_n^h - \eta_n^h \|_F^2 \right) + k^2 \| \theta_n^h \|_F^2 + k \sum_{j=1}^{l} \| \theta_j - \theta_j^h \| E_n^2 \right) + \| \varepsilon_0 \| F^2 + k \sum_{j=1}^{l} \| \theta_j - \theta_j^h \| E_n^2 \right) \}

Applying the Gronwall inequality, we have

\[ \max_{1 \leq n \leq N} \| e_n \|_F^2 + k \sum_{n=1}^{N} \| e_n \|_F^2 + \max_{1 \leq n \leq N} \| \varepsilon_n \|_E^2 + k \sum_{n=1}^{N} \| \varepsilon_n \|_E^2 \leq c k^2 \left( \| w \|_{H^1(0,T;V)}^2 + \| \varepsilon_n \|_E^2 + \| \theta_n^h \|_F^2 \right) + c \max_{1 \leq n \leq N} \tilde{R}_n \]

where

\[ \tilde{R}_n = \inf_{v_n^h \in V_n^h, \eta_n^h \in E_n^h} \left\{ \sum_{l=1}^{n} \left( \| \theta_l - \eta_l^h \|_E^2 + \| w_1 - v_1 \|_H^2 \right) + \| \theta_n - \eta_n^h \|_E^2 + k^{-1} \sum_{l=1}^{n-1} \left( \| \theta_l - \eta_l^h \|_E^2 + \| \theta_n^h - \eta_n^h \|_F^2 \right) \right\} \]

Summarizing the above arguments, we have the following theorem.

**Theorem 3.6.** Let \((w, \theta)\) and \((w^{hk}, \theta^{hk})\) be solutions to Problem 2.4 and Problem 3.2, respectively. Assume (2.1)–(2.11), \(m_A > \beta \| \gamma \|^2\). Then under the regularity assumption in Lemma 3.5, we have

\[ \max_{1 \leq n \leq N} \| e_n \|_F^2 + k \sum_{n=1}^{N} \| e_n \|_F^2 + \max_{1 \leq n \leq N} \| \varepsilon_n \|_E^2 + k \sum_{n=1}^{N} \| \varepsilon_n \|_E^2 \leq c k^2 \left( \| w \|_{H^1(0,T;V)}^2 + \| \varepsilon_n \|_E^2 + \| \theta_n^h \|_F^2 \right) + c \max_{1 \leq n \leq N} \tilde{R}_n \]

where

\[ \tilde{R}_n = \inf_{v_n^h \in V_n^h, \eta_n^h \in E_n^h} \left\{ \sum_{l=1}^{n} \left( \| \theta_l - \eta_l^h \|_E^2 + \| w_1 - v_1 \|_H^2 \right) + \| \theta_n - \eta_n^h \|_E^2 + k^{-1} \sum_{l=1}^{n-1} \left( \| \theta_l - \eta_l^h \|_E^2 + \| \theta_n^h - \eta_n^h \|_F^2 \right) \right\} \]
Then we have the following optimal order estimate:

\[ + \|w_n - v_h^n\|_H^2 + k^{-1} \sum_{l=1}^{n-1} \| (w_l - v_h^l) - (w_{l+1} - v_h^{l+1}) \|_H^2 \]

\[ + \|\theta_n - \eta_n^h\|_F^2 + k^{-1} \sum_{l=1}^{n-1} \| (\theta_l - \eta_l^h) - (\theta_{l+1} - \eta_{l+1}^h) \|_F^2 \} . \]

Theorem 3.6 is the basis of analyzing the optimal error estimate. For example, let \( \Omega \) be a polygonal or polyhedral domain and let \( T^h \) be a regular triangulation of \( \Omega \) compatible with the partition of the boundary \( \Gamma = \partial \Omega \) into \( \Gamma_D \), \( \Gamma_N \) and \( \Gamma_C \). For an element \( T \in T^h \), denote by \( P_1(T; \mathbb{R}^d) \) the space of polynomials of a total degree less than or equal to one in \( T \). Then we can use the linear element space of piecewise continuous affine functions \( V_h = \{ v_h \in C(\bar{\Omega}; \mathbb{R}^d) : v_h|_T \in P_1(T; \mathbb{R}^d) \ \forall \ T \in T^h, \ v_h = 0 \text{ on } \Gamma_D \} \),

\[ E^h = \{ \eta_h \in C(\bar{\Omega}) : \eta_h|_T \in P_1(T), \ \forall \ T \in T^h, \eta_h = 0 \text{ on } \Gamma_D \cup \Gamma_N \} . \]

Corollary 3.7. Under the assumptions stated in Theorem 3.6. Assume \( \Omega \) is a polygonal or polyhedral domain, and let \( \{ V^h \} \), \( \{ E^h \} \) be the family of linear element spaces made of continuous and piecewise affine functions defined by (3.31) and (3.32), corresponding to a regular family of finite element triangulations of \( \bar{\Omega} \) into triangles or tetrahedrons. Assume further that

\[ \omega \in C([0,T]; H^2(\Omega; \mathbb{R}^d)), \quad w|_{\Gamma_C} \in C([0,T]; H^2(\Gamma_C; \mathbb{R}^d)), \]

\[ \theta \in C([0,T]; H^2(\Omega)). \]

Then we have the following optimal order error estimate:

\[ \max_{1 \leq n \leq N} \| e_n \|_H^2 + k \sum_{n=1}^{N} \| e_n \|_V^2 + \max_{1 \leq n \leq N} \| e_n \|_F^2 + k \sum_{n=1}^{N} \| e_n \|_E^2 \leq c(k^2 + h^2) . \] (3.33)

Proof. The standard finite element interpolation error estimates ([2, 5, 10]) will be applied. Take \( v^n_h \in V^h \) to be the finite element interpolation of \( w_l \). Then, we have

\[ \| w_l - v^n_h \|_V \leq ch\|w_l\|_{H^2(\Omega; \mathbb{R}^d)} . \] (3.34)

Take \( \eta^n_h \in E^h \) to be the finite element interpolation of \( \theta_l \). Then, it follows that

\[ \| \theta_l - \eta^n_h \|_E \leq ch\|\theta_l\|_{H^2(\Omega)} . \] (3.35)

Now that \( v^n_h \) interpolates \( v_l \) on \( \Gamma_C \). According to [12], we have the following error estimate:

\[ \| w_l - v^n_h \|_{L^2(\Gamma_C; \mathbb{R}^d)} \leq ch^2\|w_l\|_{H^2(\Gamma_C; \mathbb{R}^d)} . \] (3.36)

Since that \( (v^n_l - v^n_{l+1}) \) is the finite element interpolation of \( (w_l - w_{l+1}) \), then

\[ \| (w_l - v^n_l) - (w_{l+1} - v^n_{l+1}) \|_H^2 \leq ch^2\|w_l - w_{l+1}\|_V^2 \leq ch^2k \int_{t_l}^{t_{l+1}} \|w(t)\|_V^2 dt , \]

and further, we have

\[ k^{-1} \sum_{l=1}^{n-1} \| (w_l - v^n_l) - (w_{l+1} - v^n_{l+1}) \|_H^2 \leq ch\|w\|_{H^1(0,T; H^2(\Omega; \mathbb{R}^d))}^2 , \]

\[ 1 \leq n \leq N . \] (3.37)

Similarly, we have

\[ k^{-1} \sum_{l=1}^{n-1} \| (\theta_l - \eta^n_l) - (\theta_{l+1} - \eta^n_{l+1}) \|_F^2 \leq ch\|\theta\|_{H^1(0,T; H^2(\Omega))}^2 , \]

\[ 1 \leq n \leq N . \] (3.38)
Since \( u_0^h \) is the finite element interpolant of \( u_0 \), then \( \theta_0^h \) be the finite element orthogonal projection of \( \theta_0 \) in \( L^2(\Omega) \). Then we have the error estimate for the discrete initial values,

\[
\| u_0 - u_0^h \|_V + \| \theta_0 - \theta_0^h \|_F \leq c h. \tag{3.39}
\]

Finally, we have

\[
\max_{1 \leq n \leq N} \| w_n - v_n^h \|_H \leq c h^2 \| w \|_{C([0,T]; H^2(\Omega;\mathbb{R}^d))}, \tag{3.40}
\]

\[
\max_{1 \leq n \leq N} \| \theta_n - \eta_n^h \|_F \leq c h^2 \| \theta \|_{C([0,T]; H^2(\Omega))}. \tag{3.41}
\]

Combining (3.34)–(3.41) and (3.30), we obtain the optimal order error estimate

\[
\max_{1 \leq n \leq N} \| e_n \|_H^2 + k \sum_{n=1}^N \| e_n \|_V^2 + \max_{1 \leq n \leq N} \| \varepsilon_n \|_E^2 + k \sum_{n=1}^N \| \varepsilon_n \|_E^2 \leq c(k^2 + h^2). \]

This concludes the proof of Corollary 3.7.

4. **Numerical example.** In this section, we give four numerical examples and present some numerical results to illustrate the behavior of the solution of the history-dependent thermal contact problem Problem 1.1.

\[
\int_0^t C(t-s, \varepsilon(u(s)), \theta(s)) \, ds = \int_0^t \theta(s) B \varepsilon(u(s)) \, ds.
\]

Here, the elasticity tensor \( B \) is given by

\[
(B \tau)_{\alpha\beta} = \frac{E}{1 + \kappa} \tau_{\alpha\beta} + \frac{E \kappa}{(1 - \kappa)(1 - 2\kappa)} (\tau_{11} + \tau_{22}) \delta_{\alpha\beta}, \quad 1 \leq \alpha, \beta \leq 2, \quad \forall \tau \in \mathbb{S}^2,
\]

where \( E \) and \( \kappa \) are Young’s modulus and Poisson’s ratio of the material, and \( \delta_{\alpha\beta} \) denotes the Kronecker symbol. We take \( E = 1000 N/m \) and \( \kappa = 0.3 \) in the numerical examples. At the same time, the viscosity operator \( A \) has the form

\[
(A \tau)_{\alpha\beta} = \mu_1 (\tau_{11} + \tau_{22}) \delta_{\alpha\beta} + \mu_2 \tau_{\alpha\beta}, \quad 1 \leq \alpha, \beta \leq 2, \quad \forall \tau \in \mathbb{S}^2,
\]

where \( \mu_1, \mu_2 \) are viscosity constants and we take \( \mu_1 = 25, \mu_2 = 50. \)
For the friction law (1.6), it can be reduced as follows:

\[ \|\sigma_\tau(t)\| \leq F_b(\int_0^t \|u_\tau(s)\| \, ds), \]

\[-\sigma_\tau(t) = F_b(\int_0^t \|u_\tau(s)\| \, ds) \left( (1 - a)e^{-\|u'_\tau(t)\|} + a \right) \frac{u'_\tau(t)}{\|u'_\tau(t)\|} \text{ if } u'_\tau(t) \neq 0. \] (4.2)

Then, we take

\[ F_b(z) = (c_1 - c_2)e^{-\alpha z} + c_2. \] (4.3)

where

\[ z = \int_0^t \|u_\tau(s)\| \, ds, \]

\[-\sigma_\tau(t) = \begin{cases} 
F_b(z)((a - 1)e^{-\|u'_\tau(t)\|} - a) & \text{if } u'_\tau(t) < 0, \\
[F_b(z)((a - 1)e^{-\|u'_\tau(t)\|} - a), F_b(z)((1 - a)e^{-\|u'_\tau(t)\|} + a)] & \text{if } u'_\tau(t) = 0, \\
F_b(z)((1 - a)e^{-\|u'_\tau(t)\|} + a) & \text{if } u'_\tau(t) > 0.
\end{cases} \]

We use a Primal-Dual Active Set Strategy to solve the discrete problem. For more details about this strategy, we refer to paper [4].

4.1. **First example.** We consider the physical setting shown in Figure 1. Here, the domain

\[ \Omega = (0, 2) \times (0, 1) \] and its boundary is split into:

\[ \Gamma_D = \{0\} \times (0, 1), \quad \Gamma_N = ((0, 2) \times \{1\}) \cup \{(2) \times (0, 1)), \quad \Gamma_C = (0, 2) \times \{0\}. \]

\[ T = 1s, \quad f_0 = (0, 0)N/m^2 \text{ in } \Omega, \quad f_N = (0, -t)N/m, \forall t \in [0, T] \text{ on } (0, 2) \times \{1\}. \]

\[ p_r = d_1r^+, \quad d_1 = 100N/m^2, \quad b_{ijkl} = 1 \quad u_0 = 0, \quad w_0 = 0, \quad \theta_0 = 0. \]

\[ c_{ij} = k_{ij} = k_e = 1, \quad 1 \leq i, j \leq 2, \quad q = 1. \]

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\[ p_r = d_1r^+, \quad d_1 = 100N/m^2, \quad b_{ijkl} = 1 \quad u_0 = 0, \quad w_0 = 0, \quad \theta_0 = 0. \]

\[ c_{ij} = k_{ij} = k_e = 1, \quad 1 \leq i, j \leq 2, \quad q = 1. \]
In Figure 2, we plot the deformed configuration as well as the interface forces on $\Gamma_C$ during the dynamic compression process at time $t = 0.125s, t = 0.25s, t = 0.5s, t = 1s$ for coefficient $c_1 = c_2 = 0.3, a = 0.1, \theta_R = 0$, respectively. In the case $t = 0.125s$, we note that all the contact nodes are in stick status and as the time goes, the contact nodes switch to status of slip when the compression of the domain is stronger.

In Figure 3, we plot the deformed configuration as well as the interface forces on $\Gamma_C$ during the dynamic compression process at time $t = 1s$ for coefficient $c_1 = c_2 = a = 0.1, c_1 = c_2 = a = 1$ respectively. In the case $c_1 = c_2 = a = 0.1$, we note that the contact nodes are in slip contact since, there, the friction bound is low and, therefore, is reached. In contrast, in the case $c_1 = c_2 = a = 1$, the friction bound is higher and, as a consequence, some of the contact nodes are in stick status.

4.2. Second example. As the second example, the physical setting is similar to the first test. However, we let $\Gamma_D = (\{0\} \times (0,1)) \cup (\{2\} \times (0,1))$, $\Gamma_N = (0,2) \times \{1\}$ and $\Gamma_C = (0,2) \times \{1\}$. Moreover, now $f_N$ is made to be $f_N = \begin{cases} (0,-10t)N/m, \forall t \in [0,T] & \text{on } (0,2) \times \{1\}, \\ (10t,0)N/m, \forall t \in [0,T] & \text{on } (0,2) \times \{1\}. \end{cases}$
In Figure 4, we show the influence of the different temperatures of the foundation on the temperature field of the body. We observe that for larger ground temperature, the body temperature in the neighborhood of the contact surface is larger and there is loss of symmetry on the repartition.

4.3. The third example. As the third example, the physical setting is also similar to the first test. However, $\Gamma_N$ vanishes and $\Gamma_D = (\{0\} \times (0,1)) \cup (\{2\} \times (0,1)) \cup (0,2) \times \{1\}$ and $\Gamma_C = (0,2) \times \{0\}$. Now there is no $f_N$ and $f_0 = (0, -2t) N/m^2$ in $\Omega$. In Figure 5, we plot the body’s shape after $t = 0.5s$, $t = 1s$, respectively during the compression process. As the time goes, the displacement is bigger. In Figure 6, we show the influence of the different temperatures of the foundation on the temperature field of the body. We observe that for larger ground temperature, the body temperature in the neighborhood of the contact surface is larger.

4.4. The fourth example. As the fourth example, the domain $\Omega = (0,1) \times (0,1)$ and its boundary is split into: $\Gamma_N = (\{0\} \times (0,1)) \cup ((0,1) \times \{1\}) \cup ((1) \times (0,1))$, $\Gamma_C = (0,1) \times \{0\}$. Now $f_N = (3t, 0) N/m$ on $\{1\} \times (0,1)$ and $f_0 = (0, -3t) N/m^2$ in $\Omega$. In Figure 7, we show the influence of the different temperatures of the foundation on deformation of the body. We observe that for greater temperature of the foundation, larger deformations of the body.

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Figure 7. The deformed configuration at $\theta_R = 0$ and $\theta_R = 8$

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