EMBEDDING OF CERTAIN VERTEX ALGEBRAS WITHOUT VACUUM INTO VERTEX ALGEBRAS

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Abstract. We show that certain vertex algebras without vacuum vector may be embedded into vertex algebras. The result is a partial analogue of the simple classical fact that any rng can be embedded into a ring. A one-line proof of the case of a vacuum-free vertex algebra (whose vertex operator map is by definition injective) appeared in [R] using a powerful result from the representation theory of vertex algebras as algebras of mutually local weak vertex operators. Here we present a more elementary proof of a somewhat more general case. We also show that our constructions are canonical.

1. Introduction

Vertex algebras were originally mathematically defined by R. Borcherds in [B]. The closely related notion of vertex operator algebra was introduced in [FLM]. The axioms used in [FLM] were in “generating function form,” as we shall use here. The main axiom of any vertex-type algebra is the Jacobi identity, or some equivalent. As a secondary point in [R], among other things, the notion of vacuum-free vertex algebra was formalized in order to present a more detailed examination of equivalent axiom systems for vertex algebras. A vacuum-free vertex algebra retains the major axiom, the Jacobi identity, and thus much of the basic theory of the axioms of vertex operator algebras can be worked out in this spare setting. As mentioned in [R], the motivation for formalizing vacuum-free vertex algebras was not example-driven but also, as there, we refer the reader to [BD] and [HL], where a vacuum-free setting appeared. In this note, we show that certain vertex algebras without vacuum can be embedded into vertex algebras.

The vertex operator map in any vertex algebra is injective. This follows from the “creation property” and so is generally not explicitly stated as an axiom. It was pointed out in [R], the proof having appeared in “pieces” in [FHL] and [LL] (see Remark 2.2.4 in [FHL] and Proposition 3.6.7 in [LL]), that if injectivity is stated explicitly as an axiom, then the “vacuum property” and creation property axioms each follow from the other in the presence of the remaining axioms. Actually, it is further pointed out in [R] that one only needs to retain “vacuum-free skew-symmetry” in place of the Jacobi identity and the result still holds (see the introduction to Section 6 and Proposition 6.1 of [R]). It is pointed out in Remark 3.6.8 of [LL] that if injectivity is not separately stated as an axiom, then the creation property is not redundant and one loses this minor symmetry among the axioms. For this reason, the notion of vacuum-free vertex algebras in [R] retained injectivity as an axiom. Injectivity was used in only two other places in that
paper, first in the proof of Proposition 4.6, showing that “weak commutativity” and vacuum-free skew-symmetry can replace the Jacobi identity in the definition of the notion of vacuum-free vertex algebra, and second, in Remark 4.1, which we next discuss.

The relation between vertex algebras and vacuum-free vertex algebras and that between rings and rngs (rings without identity element) are, of course, loosely analogous, a point made in [R]. It is well known that rngs can be embedded into rings. The corresponding question for vacuum-free vertex algebras was raised by members of the Quantum Mathematics/Lie Group seminar at Rutgers when a preliminary version of [R] was presented there, and was again raised by the referee of [R]. An easy answer, making use of the injectivity property was given in Remark 4.1 in [R]. The proof depended on a powerful result from the representation theory of vertex algebras as algebras of mutually local weak vertex operators (see [L]; cf. [LL]). We shall reproduce this result below, which shows that any vacuum-free vertex algebra may be embedded into a vertex algebra. We then give a (much more involved but somewhat pleasing) non-representation theoretic proof of somewhat more general results. This non-representation theoretic proof relies essentially on two constructions, the first of which is universal, and while the second one is basis dependent at first, we give a basis-free characterization of it in the final section.

Addendum: After finishing this paper, I learned of a recent pre-print [LTW] of Haisheng Li, Shaobin Tan and Qing Wang. Their paper is concerned, in part, with some of the topics of this paper. What we call an vertex algebra, those authors call a Leibniz vertex algebra and what we call a $D$-vertex algebra, those authors call a vertex algebra without vacuum (in the sense of Huang and Lepowsky [HL]). Our results in Remark 3.4 and Proposition 3.2, an analog of Theorem 4.1 and a canonical property analogous to our various canonical properties are obtained in [LTW]. Also, Corollary 2.27 of [LTW], which I did not obtain, gives a complete answer to one of the issues raised here.

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2. Preliminaries

We shall write $x, y, z, t$ for commuting formal variables. In this paper, formal variables will always commute, and we will not use complex variables. All vector spaces will be over $\mathbb{C}$, although one may easily generalize many results to the case of a field of characteristic 0. Let $V$ be a vector space. We use the following:

$$V[[x, x^{-1}]] = \left\{ \sum_{n \in \mathbb{Z}} v_n x^n | v_n \in V \right\}$$

(formal Laurent series), and some of its subspaces:

$$V(\langle x \rangle) = \left\{ \sum_{n \in \mathbb{Z}} v_n x^n | v_n \in V, v_n = 0 \text{ for sufficiently negative } n \right\}$$
(truncated formal Laurent series),

\[ V[[x]] = \left\{ \sum_{n \geq 0} v_n x^n \mid v_n \in V \right\} \]

(formal power series), and

\[ V[x] = \left\{ \sum_{n \geq 0} v_n x^n \mid v_n \in V, v_n = 0 \text{ for all but finitely many } n \right\} \]

(formal polynomials).

For \( f(x) = \sum_{n \in \mathbb{Z}} a_n x^n \in V[[x, x^{-1}]] \) let \( \text{Res}_x : V[[x, x^{-1}]] \to V \) be given by

\[ \text{Res}_x f(x) = a_{-1}. \]

Further, we shall frequently use the notation \( e^w \) to refer to the formal exponential expansion, where \( w \) is any formal object for which such expansion makes sense. For instance, we have the linear operator \( e^{y \frac{d}{dx}} : \mathbb{C}[[x, x^{-1}]] \to \mathbb{C}[[x, x^{-1}]][[y]] \):

\[ e^{y \frac{d}{dx}} = \sum_{n \geq 0} \frac{y^n}{n!} \left( \frac{d}{dx} \right)^n. \]

We have (cf. (2.2.18) in [LL]), the automorphism property:

\[ e^{y \frac{d}{dx}} (p(x)q(x)) = \left( e^{y \frac{d}{dx}} p(x) \right) \left( e^{y \frac{d}{dx}} q(x) \right), \]

for all \( p(x) \in \text{End } V[x, x^{-1}] \) and \( q(x) \in \text{End } V[[x, x^{-1}]]. \) We use the binomial expansion convention, which states that

\[ (x + y)^n = \sum_{k \geq 0} \binom{n}{k} x^{n-k} y^k, \]

where we allow \( n \) to be any integer and where we define

\[ \binom{n}{k} = \frac{n(n-1)(n-2) \cdots (n-k+1)}{k!}; \]

the binomial expression is expanded in nonnegative powers of the second-listed variable.

We also have (cf. Proposition 2.2.2 in [LL]) the formal Taylor theorem:

**Proposition 2.1.** Let \( v(x) \in V[[x, x^{-1}]]. \) Then

\[ e^{y \frac{d}{dx}} v(x) = v(x + y). \]

\[ \square \]

For completeness we include a proof of the following frequently used (easy but non-vacuous) fact, which equates two different expansions.

**Proposition 2.2.** For all \( n \in \mathbb{Z}, \)

\[ (x + (y + z))^n = ((x + y) + z)^n. \]
Proof. If \(w_1\) and \(w_2\) are commuting formal objects, then \(e^{w_1+w_2} = e^{w_1}e^{w_2}\). Thus we have
\[
(x + (y + z))^n = e^{(y+z)\frac{\partial}{\partial x}}x^n = e^{y\frac{\partial}{\partial x}}(e^{z\frac{\partial}{\partial x}}x^n) = e^{y\frac{\partial}{\partial x}}(x + z)^n = ((x + y) + z)^n.
\]
\(\square\)

We note as a consequence that for all integers \(n\) (and not just nonnegative integers) we have the (non-vacuous) fact that
\[
((x + y) - y)^n = (x + (y - y))^n = x^n.
\]

We recall the definition of the formal delta function (cf. (2.1.32) in [LL])
\[
\delta(x) = \sum_{n \in \mathbb{Z}} x^n.
\]

We have the following two elementary identities (cf. Proposition 2.3.8 in [LL]):
\[
x_1^{-1}\delta \left( \frac{x_2 + x_0}{x_1} \right) - x_2^{-1}\delta \left( \frac{x_1 - x_0}{x_2} \right) = 0
\]
and
\[
x_0^{-1}\delta \left( \frac{x_1 - x_2}{x_0} \right) - x_0^{-1}\delta \left( \frac{-x_2 + x_1}{x_0} \right) - x_1^{-1}\delta \left( \frac{x_2 + x_0}{x_1} \right) = 0.
\]

Finally, we recall (see Proposition 2.3.21 and Remarks 2.3.24 and 2.3.25 in [LL]) the delta-function substitution property:

**Proposition 2.3.** For \(f(x, y, z) \in \text{End} V[[x, x^{-1}, y, y^{-1}, z, z^{-1}]]\) such that for each fixed \(v \in V\),
\[
f(x, y, z)v \in \text{End} V[[x, x^{-1}, y, y^{-1}]](x),
\]
and such that
\[
\lim_{x \to y} f(x, y, z)
\]
exists (where the “limit” is the indicated formal substitution), we have
\[
\delta \left( \frac{y+z}{x} \right) f(x, y, z) = \delta \left( \frac{y+z}{x} \right) f(y, z, y, z) = \delta \left( \frac{y+z}{x} \right) f(x, x - z, z),
\]
where, in particular, all the products exist.
\(\square\)

As in [LL], we use similarly verified substitutions below without comment.
3. Definitions and motivating results

In the spirit of rings without identity being called rngs, we shall call vertex algebras without vacuum “ertex algebras”:

**Definition 3.1.** An ertex algebra is a vector space equipped, first, with a linear map (the vertex operator map) $V \otimes V \to V[[x,x^{-1}]]$, or equivalently, a linear map

$$Y(\cdot,x) : V \to (\text{End}V)[[x,x^{-1}]]$$

$$v \mapsto Y(v,x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1}.$$  

We call $Y(v,x)$ the vertex operator associated with $v$. We assume that

$$Y(u,x)v \in V((x))$$

for all $u,v \in V$, the truncation property. Finally, we require that the Jacobi identity is satisfied:

$$x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y(u,x_1)Y(v,x_2) - x_0^{-1} \delta \left( \frac{-x_2 + x_1}{x_0} \right) Y(v,x_2)Y(u,x_1)$$

$$= x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) Y(Y(u,x_0)v,v_2).$$ (3.5)

**Definition 3.2.** An injective ertex algebra is an ertex algebra whose vertex operator map is injective.

**Remark 3.1.** Following [R] we referred to “vacuum-free vertex algebras” in our introduction. Vacuum-free vertex algebras are exactly the same as what we have just called injective ertex algebras, but in the context of this paper using the term injective ertex algebra seems clearer, so from this point on, we shall not use the term “vacuum-free vertex algebra.”

We may sometimes call an ertex algebra simply by the name of the underlying vector space, or, to be more precise, sometimes by the pair $(E,Y)$.

We note the following fact, although we shall only use it in the case of an injective ertex algebra.

**Proposition 3.1.** Let $V$ be an ertex algebra. For all $u,v \in V$, we have

$$Y(Y(u,x_0)v,v_2) = Y(Y(v,-x_0)u,v_2 + x_0).$$

**Proof.** The proof is essentially a piece of a well-known proof of skew-symmetry (cf. Proposition 3.1.19 in [LL]).

The following definition of “vertex algebra” is equivalent to the definition in [B] (cf. Proposition 3.6.6 in [LL]):
Definition 3.3. A vertex algebra is an ertex algebra \((V, Y)\) together with a distinguished element \(1\) satisfying the vacuum property

\[ Y(1, x) = 1 \]

and the creation property

\[ Y(u, x)1 \in V[[x]] \text{ and } Y(u, 0)1 = u \text{ for all } u \in V. \]

Every vertex algebra \(V\) has a linear operator given by

\[ D(v) = v - 21 \text{ for all } v \in V. \]

We shall call this the “derivative operator.”

We may sometimes refer to a vertex algebra simply by the name of the underlying vector space, but for more precision we shall also sometimes refer to it using a triple such as \((V, Y, 1)\).

Remark 3.2. The injectivity of the vertex operator map follows from the creation property.

As we observed in [R], we have:

Theorem 3.1. An injective ertex algebra can always be embedded into a vertex algebra.

Proof. The adjoint representation of any given injective ertex algebra \(E\) yields a set of mutually local vertex operators, and by Theorem 3.2.10 in [L] (cf. also Theorem 5.5.18 in [LL]), these operators generate a vertex algebra which we call \(E''\). The injectivity shows that \(E\) is embedded in \(E''\). \(\square\)

We now address the situation without using representation theory. It is tempting to simply try to directly adjoin a vacuum vector \(1\) to an ertex algebra \(E\) to get a vertex algebra \(V\), but if this were possible we would necessarily have the strong creation property (cf. (3.1.29) of [LL]), which says that for all \(v \in V\),

\[ Y(v, x)1 = e^{xD}v. \]

Therefore, before adjoining a vacuum vector we might first close up \(E\) under some sort of linear map that will become \(D\). Of course, after closing \(E\) under such an operator we may already have a vertex algebra.

As a preliminary result let us consider the case in which we have an ertex algebra which already has a linear map which has certain of the properties of a derivative operator.

Definition 3.4. A D-ertex algebra is an ertex algebra \((E', Y')\) with a linear map

\[ D : E' \to E' \]

such that

\[ Y'(e^{xD}u, x)v = Y'(u, x + z)v, \]

(3.7)
\[ e^{zD}Y'(u, x)v = Y'(u, x + z)e^{zD}v, \tag{3.8} \]

and

\[ Y'(u, x)v = e^{zD}Y'(v, -x)u, \tag{3.9} \]

for all \( u, v \in E' \).

We may sometimes refer to a \( D \)-ertex algebra simply by the name of the underlying vector space, but for more precision we shall also sometimes refer to it using a triple such as \((E', Y', D)\).

**Remark 3.3.** The properties involving \( D \) in Definition 3.4 are all well-known properties of any vertex algebra. The first one (3.7) is the global form of the \( D \)-derivative property (cf. (3.1.28) in [LL]), the second one (3.8) is the global form of the \( D \)-bracket derivative formula (cf. (3.1.35) in [LL]), and the third one (3.9) is skew-symmetry (cf. (3.1.30) in [LL]). We note that (3.7) and (3.8) are equivalent to (any two of) three “infinitesimal” properties called the \( D \)-derivative property (cf. (3.1.25) in [LL]) and \( D \)-bracket derivative formulas (cf. (3.1.32) and (3.1.33) in [LL]), which together say that \([D, Y'(v, x)], \frac{d}{dx} Y(v, x)\) and \(Y(Dv, x)\) are all equal. We have used two well-known global forms encoding the same information in the official definition because they seem convenient in our proofs.

**Remark 3.4.** In Definition 3.4 we only need (3.9) together with either one of (3.7) or (3.8). To show this we note that

\[
Y'(u, x)e^{-yD}v = e^{xD}Y'(e^{-yD}v, -x)u \\
= e^{xD}Y'(v, -x - y)u \\
= e^{xD}e^{(-x-y)D}Y'(u, x + y)v \\
= e^{-yD}Y'(u, x + y)v,
\]

where the first and third equality follow from (3.9) and the second one from (3.7). This shows that we may remove (3.8) from Definition 3.4 and cyclically switching the order of this calculation shows that we may alternatively remove (3.7) from Definition 3.4.

**Proposition 3.2.** Given any \( D \)-ertex algebra there is a canonical embedding of it into a vertex algebra which has codimension one.

**Proof.** Let \((E', Y', D)\) be a \( D \)-ertex algebra and let

\[ E'' = E' \oplus C1, \]

where \(1\) will become the vacuum vector.

Let

\[ Y''(\cdot, x) : E'' \otimes E'' \rightarrow E''[[x, x^{-1}]] \]

be the unique linear map that extends \(Y'(\cdot, x)\) on \(E' \otimes E'\) and that satisfies

\[ Y''(u, x)1 = e^{xD}u \quad \text{for all } u \in E', \tag{3.10} \]
(where we identify \( E' \) as a subspace of \( E'' \)) and

\[
Y''(1, x)u = u \quad \text{for all } u \in E''.
\]

It is clear that all we need to show is that \( E'' \) together with \( Y'' \) is a vertex algebra. We first show that \( Y'' \) is a vertex algebra. By linearity, (3.10) and (3.11), together with the truncation property on \( E' \), we get the truncation property.

By linearity and since \( E' \) is a vertex algebra, in order to check the Jacobi identity, we only have to check the case where one or more of the three vectors is \( 1 \). If \( u = 1 \) in (3.5) then the result follows from (3.11) and (2.4). If \( v = 1 \) then by (3.10) and (3.11) the Jacobi identity reduces to:

\[
x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y''(u, x_1) - x_0^{-1} \delta \left( \frac{-x_2 + x_1}{x_0} \right) Y''(u, x_1)
\]

\[
= x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) Y''(e^{x_0D}u, x_2),
\]

which in turn, by (2.3), reduces to

\[
x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) Y''(u, x_1) = x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) Y''(e^{x_0D}u, x_2),
\]

which, assuming we are not acting against \( 1 \), follows by Proposition 2.3 and (3.7). And if we are acting against \( 1 \) then by (3.10) we need

\[
x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) e^{x_1D}u = x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) e^{(x_2+x_0)D}u,
\]

which follows by Proposition 2.3.

Therefore for \( u, v \in E' \) identified as elements of \( E'' \), we need only check

\[
x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y''(u, x_1)Y''(v, x_2)1 - x_0^{-1} \delta \left( \frac{-x_2 + x_1}{x_0} \right) Y''(v, x_2)Y''(u, x_1)1
\]

\[
= x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) Y''(Y''(u, x_0)v, x_2)1,
\]

which, by (3.11), reduces to

\[
x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y'(u, x_1)e^{x_2D}v - x_0^{-1} \delta \left( \frac{-x_2 + x_1}{x_0} \right) Y'(v, x_2)e^{x_1D}u
\]

\[
= x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) e^{x_2D}Y'(u, x_0)v,
\]
which, by Proposition 2.3 and (3.8), reduces to
\[
x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) e^{x_2 D} Y'(u, x_0) v - x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) e^{x_2 D} Y'(u, x_0) v
\]
\[= x_0^{-1} \delta \left( \frac{-x_2 + x_1}{x_0} \right) Y'(v, x_2) e^{x_1 D} u,
\]
which by (2.3) (checking carefully that the left hand expression is well-defined in the sense that all coefficients of monomials are finitely computable) reduces to
\[
x_0^{-1} \delta \left( \frac{-x_2 + x_1}{x_0} \right) e^{x_2 D} Y'(u, x_0) v = x_0^{-1} \delta \left( \frac{-x_2 + x_1}{x_0} \right) Y'(v, x_2) e^{x_1 D} u,
\]
which by Proposition 2.3 and taking Res_{x_0} reduces to
\[
e^{x_2 D} Y'(u, -x_2 + x_1) v = Y'(v, x_2) e^{x_1 D} u,
\]
which by (3.7) reduces to
\[
e^{x_2 D} Y'(e^{x_1 D} u, -x_2) v = Y'(v, x_2) e^{x_1 D} u,
\]
which follows from (3.9).

The vacuum property and creation property both follow immediately from (3.10) and (3.11).

**Remark 3.5.** We note, as in the case of a rng being canonically embedded into a ring, that even if \( E' \) is already a vertex algebra, we still add a new vacuum vector so that \( E' \) indeed always has codimension 1.

**Remark 3.6.** It is easy to see by (3.10) that the map \( D \) on \( E' \) extends to the derivative operator on \( E'' \). We would like to be able to embed vertex algebras into \( D \)-vertex algebras. Of course, we have a non-canonical method using the representation theoretic proof of Theorem 3.1. We could also embed an injective vertex algebra more directly using a representation theoretic proof by closing up the weak vertex operator space under derivatives instead of generating a larger space by adding the identity. We shall do this below for interest’s sake, but we note that it is unclear whether or not the resulting \( D \)-vertex algebra is injective. As noted below in Remark 5.1 this makes fail a naive attempt at proving whether or not this construction is canonical. I leave to the reader the question of whether or not it is canonical. In the next section we give a canonical embedding of an injective vertex algebra into a \( D \)-vertex algebra using a proof which does not rely on representation theory and one of the key properties of the construction is that the \( D \)-vertex algebra of this construction is, in fact, injective.

**Corollary 3.1.** Every injective vertex algebra may be embedded into a \( D \)-vertex algebra.
Proof. The result follows immediately from Theorem 3.1, but we give a second proof.

Let \( E \) be an injective vertex algebra. Consider its adjoint representation which is faithful because of the injectivity of \( E \). Take the linear span of all the higher derivatives of the weak vertex operators in the adjoint representation and call this space \( E_r' \). We claim that this vector space is a \( D \)-vertex algebra of weak vertex operators. Of course, if the claim is true then \( E_r' \) is the smallest \( D \)-vertex algebra of weak vertex operators containing the adjoint image of \( E \) which is in \((\mathcal{E}(E), Y_E, 1_E)\), which by Proposition 5.3.9 of [LL] is a weak vertex algebra. By smallest we mean that it is contained in all other such \( D \)-vertex algebras or in other words that it is the \( D \)-vertex algebra generated by the adjoint image of \( E \).

To show this claim we first show that if \( a(x) \) and \( b(x) \) are mutually local then \( a'(x) \) and \( b'(x) \) are mutually local. There exists some \( k \geq 0 \) so that
\[
(x_1 - x_2)^k a(x_1) b(x_2) = (x_1 - x_2)^k b(x_2) a(x_1)
\]
and taking \( \frac{\partial}{\partial x_1} \) of both sides gives
\[
k(x_1 - x_2)^{k-1} a(x_1) b(x_2) + (x_1 - x_2)^k a'(x_1) b(x_2)
= k(x_1 - x_2)^{k-1} b(x_2) a(x_1) + (x_1 - x_2)^k b(x_2) a'(x_1),
\]
so that
\[
(x_1 - x_2)^{k+1} a'(x_1) b(x_2) = (x_1 - x_2)^{k+1} b(x_2) a'(x_1).
\]
Thus by induction \( E_r' \) is a mutually local subspace of \((\mathcal{E}(E), Y_E, 1_E)\). Therefore \( E_r' \) is embedded in a vertex subalgebra of \((\mathcal{E}(E), Y_E, 1_E)\) by Theorem 3.2.10 in [L] (cf. also Theorem 5.5.18 in [LL]). Moreover the \( D \)-bracket and derivative properties of \((\mathcal{E}(E), Y_E, 1_E)\) show, by induction, that \( E_r' \) is closed under multiplication. Since \( E_r' \) is also obviously closed under the derivative operator then \( E_r' \) is a \( D \)-vertex algebra.

Finally, since \( E \) is injective therefore \( E \) is embedded in \( E_r' \). \(\square\)

4. Main result

Let \( (E, Y) \) be an injective vertex algebra. We shall embed \((E, Y)\) into an injective \( D \)-vertex algebra \((E', Y', D)\). Our construction will be basis dependent in this section, but in Theorem 5.1 we give a basis-free characterization of \((E', Y', D)\). Proposition 3.2 motivates our approach. Then by invoking Proposition 3.2 we canonically embed \( E' \) and hence \( E \) into a vertex algebra.

Let \( \{e_i\} \), where \( i \) ranges over an index set \( I \), be a basis of \( E \). Let
\[
S = \{D^{[n]}e_i | i \in I, n \geq 0\}.
\]
We emphasize that the symbols \( D^{[n]}e_i \) are simply the names of a doubly-indexed set and that, though intentionally suggestive, the notation does not denote operators acting on a set. For convenience we shall abuse notation and sometimes identify \( D^{[0]}e_i \) with \( e_i \). Let \( \bar{S} \) be the complex vector space with basis \( S \).
Definition 4.1. Let $\bar{Y}(\cdot, x) : S \otimes E \to E[[x, x^{-1}]]$ be the unique linear map satisfying

$$\bar{Y}(D^n e_i, x)e_j = \left( \frac{d}{dx} \right)^n Y(e_i, x)e_j,$$

for all $n \geq 0 i, j \in I$.

Let $A$ be the collection of subsets of $S$ containing $\{D[0] e_i | i \in I\}$ satisfying the following property: if $R \in A$ and if $m$ is a nonzero linear combination of elements of $R$, regarding $m$ as an element of $\bar{S}$, we require that $\bar{Y}(m, x)e = 0$ for all $e \in E$. Since $E$ is injective, $A$ is nonempty. Let $T$ be a simply ordered (by set inclusion) sub-collection of $A$. It is easy to see that the union of elements of elements of $T$ is still an element of $A$. Therefore, by Zorn’s lemma, $A$ has a maximal element. Take such a maximal element $M$ and let its elements give a basis for a vector space that we call $E'$.

Lemma 4.1. If $D^n e_i \notin M$ then there exists a unique element in $E'$ which we shall call $\bar{D}^n e_i$, such that

$$\bar{Y}(D^n e_i - \bar{D}^n e_i, x) = 0$$

when acting on $E$.

Proof. The existence follows from the maximality of $M$ and the uniqueness follows from the restricting condition on all elements of $A$. □

Definition 4.2. Let $D$ be the unique linear operator on $E'$ which satisfies

$$D(D^n e_i) = \left\{ \begin{array}{ll}
D^{n+1} e_i & D^{n+1} e_i \in M \\
\bar{D}^{n+1} e_i & \text{otherwise}
\end{array} \right.$$ 

for all $D^n e_i \in M$.

Definition 4.3. Let $Y'(\cdot, x) : E' \otimes E' \to E'[\![x, x^{-1}]\!]$ be the unique linear map which satisfies

$$Y'(D^n e_i, x)D^m e_j = \left( \frac{d}{dx} \right)^n \left( D - \frac{d}{dx} \right)^m Y(e_i, x)e_j,$$

whenever $D^n e_i$ and $D^m e_j \in M$.

We note that (4.12) may be rewritten as

$$Y'(D^n e_i, x)D^m e_j = \text{Res}_z \text{Res}_y z^{-n-1} y^{-m-1} e^{\frac{d}{dx} z + y(D - \frac{d}{dx})} Y(e_i, x)e_j$$

$$= \text{Res}_z \text{Res}_y z^{-n-1} y^{-m-1} e^{yD} Y(e_i, x+z-y)e_j.$$ 

(4.13)

Lemma 4.2. If $Y'(u, x) = Y'(v, x)$ where $u, v \in E'$ when acting on $E$ then $Y'(u, x) = Y'(v, x)$ (when acting on $E'$).
Proof. We have
\[
Y'(u, x)D^{[m]}e_j = \left( D - \frac{d}{dx} \right)^m Y'(u, x)e_j
= \left( D - \frac{d}{dx} \right)^{m} Y'(v, x)e_j
= Y'(v, x)D^{[m]}e_j,
\]
for all \( D^{[m]}e_j \in M \) so that the result follows by linearity. \( \square \)

**Remark 4.1.** Of course, a similar argument shows that if \( Y'(u, x) = \frac{d}{dx}Y'(v, x) \) where \( u, v \in E' \) when acting on \( E \) then \( Y'(u, x) = \frac{d}{dx}Y'(v, x) \) when acting on \( E' \).

**Lemma 4.3.** If \( Y'(u, x) = 0 \) then \( u = 0 \) for all \( u \in E' \).

*Proof.* When acting on \( E \) we have \( Y'(u, x) = \bar{Y}(u, x) \). Therefore the result follows by the restrictive condition in the definition of \( A \). \( \square \)

**Lemma 4.4.** Let \( u, v \in E' \). Then \( \bar{Y}(Du, x) = \frac{d}{dx}Y(u, x) \) (when acting on \( E \)).

*Proof.* Consider \( D^{[n]}e_i \in E' \). If \( D(D^{[n]}e_i) = D^{[n+1]}e_i \) then we have
\[
\bar{Y}(D(D^{[n]}e_i), x) = \bar{Y}(D^{[n+1]}e_i, x) = \frac{d}{dx} \bar{Y}(D^{[n]}e_i, x).
\]
Otherwise \( D(D^{[n]}e_i) = \bar{D}^{[n+1]}e_i \) and we have
\[
\bar{Y}(D(D^{[n]}e_i), x) = \bar{Y}(\bar{D}^{[n+1]}e_i, x) = \bar{Y}(D^{[n+1]}e_i, x) = \frac{d}{dx} \bar{Y}(D^{[n]}e_i, x),
\]
so the result follows by linearity. \( \square \)

**Lemma 4.5.** For all \( u \in E' \), \( Y'(Du, x) = \frac{d}{dx}Y'(u, x) \) (when acting on \( E' \)).

*Proof.* Consider \( D^{[n]}e_i \in E' \). If \( D(D^{[n]}e_i) = D^{[n+1]}e_i \), we have
\[
Y'(D(D^{[n]}e_i), x) = Y'(D^{[n+1]}e_i, x) = \frac{d}{dx} Y'(D^{[n]}e_i, x).
\]
Otherwise \( D(D^{[n]}e_i) = \bar{D}^{[n+1]}e_i \) and we have
\[
Y'(D(D^{[n]}e_i), x) = Y'(\bar{D}^{[n+1]}e_i, x) = \bar{Y}(\bar{D}^{[n+1]}e_i, x) = \frac{d}{dx} \bar{Y}(D^{[n]}e_i, x) = \frac{d}{dx}Y'(D^{[n]}e_i, x)
\]
where some of these equalities hold only on when acting \( E \). Then the result follows by Lemma 4.2 together with Remark 4.1 and linearity. \( \square \)

Lemma 4.5, together with the formal Taylor theorem, gives for all \( u \in E' \)
\begin{equation}
Y'(e^{zD}u, x) = Y'(u, x + z).
\end{equation}

**Lemma 4.6.** For all \( u', v' \in E' \) we have \( Y'(u', x)v' = e^{xD}Y'(v', -x)u' \).
Proof. We have for all \( u, v \in E \) when acting on \( E \):

\[
Y'(Y(u, x_0)v, x_2) = Y(Y(u, x_0)v, x_2)
= Y(Y(v, -x_0)u, x_2 + x_0)
= e^{x_0 \frac{\partial}{\partial x}} Y Y(v, -x_0)u, x_2)
= e^{x_0 \frac{\partial}{\partial x}} Y'(Y(v, -x_0)u, x_2)
= Y'(e^{x_0 D} Y(v, -x_0)u, x_2)
\]

where the second equality follows from Proposition \ref{prop:03.1}, the third equality follows from the formal Taylor theorem and the fifth equality follows from Lemma \ref{lem:4.7}. Then Lemma \ref{lem:4.2} and Lemma \ref{lem:4.3} give

\[
Y'(u, x)v = e^{x D} Y'(v, -x)u.
\]

Further recalling \ref{eq:4.13} we get

\[
Y'(D^{[m]} e_i, x) D^{[m]} e_j = \text{Res}_{\mathbb{Z}} \text{Res}_{\mathbb{Z}} \frac{z^{n-1} y^{-m-1} e^{y D} Y'(e_i, x + z - y) e_j}{z^{n-1} y^{-m-1} e^{y D} e^{(x+z-y) D} Y'(e_j, -x - z + y) e_i}
= \text{Res}_{\mathbb{Z}} \text{Res}_{\mathbb{Z}} \frac{z^{n-1} y^{-m-1} e^{y D} e^{(x+z-y) D} Y'(e_j, -x - z + y) e_i}{z^{n-1} y^{-m-1} e^{y D} e^{(x+z+y) D} Y'(e_j, -x + y - z) e_i}
= e^{x D} \text{Res}_{\mathbb{Z}} \text{Res}_{\mathbb{Z}} \frac{z^{n-1} y^{-m-1} e^{x D} Y'(e_j, -x + y - z) e_i}{z^{n-1} y^{-m-1} e^{x D} e^{(x+z+y) D} Y'(e_j, -x - z + y) e_i}
= e^{x D} Y'(D^{[m]} e_j, -x) D^{[n]} e_i,
\]

so that the result follows by linearity.

\[\square\]

Lemma 4.7. For all \( u, v \in E' \) we have \( e^{y D} Y'(u, x) e^{-y D} v = Y'(u, x + y) v \).

Proof. See Remark \ref{rem:3.4}.

We have now shown that \( E' \) has all of the properties specified in Proposition \ref{prop:3.2} that is, assuming that \( E' \) is indeed an ertex algebra. Showing that is then essentially our last task because then we will have that \((E, Y)\) is obviously embedded in \((E', Y')\) and can invoke Proposition \ref{prop:3.2}.

Theorem 4.1. The vector space \( E' \) together with map \( Y'(\cdot, x) \) is an ertex algebra.

Proof. We first show that \( Y'(\cdot, x) \cdot \) satisfies the truncation property. We have for \( u, v \in E \)

\[
Y'(e^{y D} u, x) e^{z D} v = e^{z D} Y(u, x + y - z) v = e^{z D} e^{(y-z) \frac{\partial}{\partial y}} Y(u, x) v,
\]

where the first equality follows from \ref{eq:4.14} and Lemma \ref{lem:4.7} and the second equality follows from the formal Taylor theorem. Therefore, if we extract the coefficient for a fixed power of both \( y \) and \( z \) then only finitely many terms from the exponential terms \( e^{z D} \) and \( e^{(y-z) \frac{\partial}{\partial y}} \) need to be considered. We have that \( Y'(u, x) v \) is truncated from below in powers of \( x \). Thus the coefficient of any fixed powers of both \( y \) and \( z \) of \ref{eq:4.15}, since we must take at most a bounded number of derivatives, we find is also truncated from below in powers of \( x \). The truncation property now follows from linearity.
What remains is to verify the Jacobi identity. We have that for all \( u, v \) and \( w \in E \)
\[
x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y'(e^{yD} u, x_1) Y'(e^{zD} v, x_2) e^{tD} w
\]
\[
- x_0^{-1} \delta \left( \frac{-x_2 + x_1}{x_0} \right) Y'(e^{zD} v, x_2) Y'(e^{yD} u, x_1) e^{tD} w
\]
\[
= x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) Y'(Y'(e^{yD} u, x_0) e^{zD} v, x_2) e^{tD} w
\]
is equivalent to
\[
x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y'(e^{yD} u, x_1) e^{tD} Y(v, x_2 + z - t) w
\]
\[
- x_0^{-1} \delta \left( \frac{-x_2 + x_1}{x_0} \right) Y'(e^{zD} v, x_2) e^{tD} Y(u, x_1 + y - t) w
\]
\[
= x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) Y'(e^{zD} Y(u, x_0 + y - z) v, x_2) e^{tD} w
\]
which is the same as
\[
x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) e^{tD} Y(u, x_1 + y - t) Y(v, x_2 + z - t) w
\]
\[
- x_0^{-1} \delta \left( \frac{-x_2 + x_1}{x_0} \right) e^{tD} Y(v, x_2 + z - t) Y(u, x_1 + y - t) w
\]
\[
= x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) e^{tD} Y(Y(u, x_0 + y - z) v, x_2 + z - t) w
\]
which is equivalent to
\[
x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) e^{tD} e^{(y-z) \frac{\partial}{\partial x_1} + (z-t) \frac{\partial}{\partial x_2}} Y(u, x_1) Y(v, x_2) w
\]
\[
- x_0^{-1} \delta \left( \frac{-x_2 + x_1}{x_0} \right) e^{tD} e^{(y-t) \frac{\partial}{\partial x_1} + (z-t) \frac{\partial}{\partial x_2}} Y(v, x_2) Y(u, x_1) w
\]
\[
= x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) e^{tD} e^{(y-z) \frac{\partial}{\partial x_0} + (z-t) \frac{\partial}{\partial x_2}} Y(Y(u, x_0) v, x_2) w
\]
which is the same as
\[
e^{tD} e^{(y-t) \frac{\partial}{\partial x_1} + (z-t) \frac{\partial}{\partial x_2} x_0^{-1} \delta \left( \frac{x_1 - x_2 + z - y}{x_0} \right)} Y(u, x_1) Y(v, x_2) w
\]
\[
- e^{tD} e^{(y-t) \frac{\partial}{\partial x_1} + (z-t) \frac{\partial}{\partial x_2} x_0^{-1} \delta \left( -x_2 + x_1 + z - y \right)} y_0 \left( x_0 \right) Y(v, x_2) Y(u, x_1) w
\]
\[
= e^{tD} e^{(y-z) \frac{\partial}{\partial x_0} + (z-t) \frac{\partial}{\partial x_2} x_1^{-1} \delta \left( x_2 + x_0 + t - y \right)} Y(Y(u, x_0) v, x_2) w,
\]
which is equivalent to
\[ e^{tD} e^{(y-z)\frac{\partial}{\partial x_0} + (y-t)\frac{\partial}{\partial x_1} + (z-t)\frac{\partial}{\partial x_2}} x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) \ Y(u, x_1)Y(v, x_2) w \]
\[-e^{tD} e^{(y-z)\frac{\partial}{\partial x_0} + (y-t)\frac{\partial}{\partial x_1} + (z-t)\frac{\partial}{\partial x_2}} x_0^{-1} \delta \left( \frac{-x_2 + x_1}{x_0} \right) Y(v, x_2)Y(u, x_1) w \]
\[= e^{tD} e^{(y-z)\frac{\partial}{\partial x_0} + (y-t)\frac{\partial}{\partial x_1} + (z-t)\frac{\partial}{\partial x_2}} x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) Y(Y(u, x_0)v, x_2) w, \]
which follows from the Jacobi identity. The first two equalities follow from (4.14) and Lemma 4.7, the next two by the formal Taylor theorem, the next by (2.3) together with the formal Taylor theorem and the last because we may factor out the operator \( e^{tD} e^{(y-z)\frac{\partial}{\partial x_0} + (y-t)\frac{\partial}{\partial x_1} + (z-t)\frac{\partial}{\partial x_2}} \) from all three terms and use that \( E \) is already an ertex algebra. The result now follows by linearity.

We may now give a second proof of Theorem 3.1 which does not rely on any representation theory.

**Proof.** (Second proof of Theorem 3.1) The result follows from Lemmas 4.1, 4.6 and 4.7 (4.14) and Proposition 3.2 so that \( E \) is embedded into \( E' \) which in turn is embedded into \( E'' \).

We give a basis free characterization of the \( E'' \) in the last proof in Corollary 5.1.

5. The canonical properties

We shall show that certain of our constructions are canonical. We shall define and recall the relevant notions of homomorphism needed to state the properties of interest.

An ertex algebra homomorphism between two ertex algebras \((E_1, Y_1)\) and \((E_2, Y_2)\) is a linear map \( f : E_1 \to E_2 \) such that

\[ f(Y_1(u, x)v) = Y_2(f(u), x)f(v), \quad \text{for all } u, v \in E_1. \]

This extends to the usual definition of a vertex algebra homomorphism, which we recall now. A vertex algebra homomorphism between vertex algebras \((V_1, Y_1, 1_1)\) and \((V_2, Y_2, 1_2)\) is a linear map \( f : V_1 \to V_2 \) such that

\[ f(Y_1(u, x)v) = Y_2(f(u), x)f(v), \]

and such that

\[ f(1_1) = 1_2, \]

or in other words, a vertex algebra homomorphism is a homomorphism of vertex algebras regarded as ertex algebras which, in addition, preserves the vacuum vector.

A \( D \)-ertex algebra homomorphism between two \( D \)-ertex algebras \((E_1, Y_1, D_1)\) and \((E_2, Y_2, D_2)\) is a linear map \( f : E_1 \to E_2 \) such that

\[ f(Y_1(u, x)v) = Y_2(f(u), x)f(v), \quad \text{for all } u, v \in E_1. \]
and such that
\[ f(D_1v) = D_2 f(v) \quad \text{for all } v \in E_1, \]
or in other words, a \( D \)-ertex algebra homomorphism is a homomorphism of \( D \)-ertex algebras regarded as ertex algebras which, in addition, is compatible with the respective derivative operators.

**Proposition 5.1.** Let \( f \) be a vertex algebra homomorphism between vertex algebras \((V_1,Y_1,1_1)\) and \((V_2,Y_2,1_2)\). Let \( D_1(v) = v_{-2} 1_1 \) and \( D_2(v) = v_{-2} 1_2 \). Then \( f \) is a \( D \)-ertex algebra homomorphism between \((V_1,Y_1,D_1)\) and \((V_2,Y_2,D_2)\).

**Proof.** It is clear (see Remark 3.3) that \((V_1,Y_1,D_1)\) and \((V_2,Y_2,D_2)\) are, in fact, \( D \)-ertex algebras. Since \( f \) is a vertex algebra homomorphism we have
\[ f(Y_1(v,x)1_1) = Y_2(f(v),x)f(1_1) = Y_2(f(v),x)1_2, \]
which by the strong creation property gives
\[ f(e^{xD_1}v) = e^{xD_2}f(v) \]
and equating the coefficients of the linear term yields
\[ f(D_1v) = D_2 f(v). \]

□

Given any vertex algebras when we regard them as \( D \)-ertex algebras we always use the standard derivative operator.

The next proposition shows that the vertex algebra \( E'' \) from Proposition 3.2 satisfies a universal property.

**Proposition 5.2.** Using the notation of the statement and proof of Proposition 3.2 let \( i \) be the canonical embedding of \( E' \) into \( E'' \). Given any vertex algebra \((V,Y,V,1_V)\) such that there exists a \( D \)-ertex algebra homomorphism, \( \psi : E' \to V \), there exists a unique vertex algebra homomorphism \( \phi : E'' \to V \) such that \( \phi(i(u)) = \psi(u) \) for all \( u \in E' \), or in other words such that the diagram

\[
\begin{array}{ccc}
E' & \xrightarrow{i} & E'' \\
\downarrow{\psi} & & \downarrow{\phi} \\
V & & \\
\end{array}
\]

commutes.

Furthermore, \( E'' \) is the unique, up to canonical isomorphism, vertex algebra equipped with an embedding of \( E' \) into it which satisfies the above property.

**Proof.** It is obvious by linearity that if \( \phi \) exists then it must be unique. Indeed any element of \( E'' \) may be written as \( i(u) + a1 \) where \( u \in E' \) and \( a \in \mathbb{C} \) and we must have
\[ \phi(i(u) + a1) = \psi(u) + a1_V. \]
Then $\phi$ obviously preserves the vacuum vector and we need only show that it is indeed a vertex algebra homomorphism. We have for all $u, v \in E'$ and $a, b \in \mathbb{C}$

$$
\begin{align*}
\phi \left( Y''(i(u) + a1, x)(i(v) + b1) \right) &= \phi \left( Y''(i(u), x)(i(v)) \right) + \phi \left( Y''(i(u), x)b1 \right) \\
&= \phi \left( i(Y'(u, x)v) + \phi be^{xD}i(u) + a\psi(v) + ab1_V \right) \\
&= \psi \left( Y'(u, x)v + \psi be^{xD}u + a\psi(v) + ab1_V \right) \\
&= Y_V(\psi(u), x)\psi(v) + be^{xD_V}\psi(u) + a\psi(v) + ab1_V \\
&= Y_V(\psi(u), x)\psi(v) + Y_V(\psi(u), x)b1_V + a\psi(v) + ab1_V \\
&= Y_V(\phi(i(u) + a1), x)(\phi(i(v) + b1)),
\end{align*}
$$

which is what we needed.

In order to establish the uniqueness, consider any other vertex algebra $F$ with an embedding $j : E' \to F$ satisfying the same property as $E''$ and $i$. Then we get unique vertex algebra homomorphisms $\phi : E'' \to F$ and $\xi : F \to E''$ such that for all $u \in E'$

$$
\begin{align*}
\phi(i(u)) &= j(u) \\
\xi(j(u)) &= i(u),
\end{align*}
$$

which in turn gives

$$
\begin{align*}
\xi(\phi(i(u))) &= \xi(j(u)) = i(u) \\
\text{and} \\
\phi(\xi(j(u))) &= \phi(i(u)) = j(u).
\end{align*}
$$

There is a unique vertex algebra homomorphism $\alpha : E'' \to E''$ such that $\alpha(i(u)) = i(u)$ for all $u \in E'$ or in other words such that the diagram

$$
\begin{array}{ccc}
E' & \xrightarrow{i} & E'' \\
\downarrow{i} & & \downarrow{\alpha} \\
E'' & & 
\end{array}
$$

commutes. This unique map $\alpha$ must be the identity map on $E''$, but also must be $\xi \circ \phi$. Similarly $\phi \circ \xi$ must be the identity on $F$. Therefore $\xi$ and $\phi$ are canonically given isomorphisms respecting the embeddings. $\square$

**Lemma 5.1.** Using the notation of Section 4, we have $D^n e_i = D^n D^0 e_i$ whenever $D^n e_i \in M$. 

Proof. When acting on $E$ we have

\[
Y'(D^n D^{[0]} e_i, x) = \left( \frac{d}{dx} \right)^n Y(D^{[0]} e_i, x) = \left( \frac{d}{dx} \right)^n Y(e_i, x) = Y'(D^n e_i, x),
\]

where the first equality follows from Lemma 4.5 and the second and third equalities follow from (4.12) together with linearity. Thus the result follows by Lemma 4.2 and Lemma 4.3. □

Let $(V', Y', D')$ be a $D$-ertex algebra. Let $(V, Y)$ be a sub-ertex algebra of $V'$ regarded as merely an ertex algebra. We call $\langle V, D' \rangle$ the smallest $D$-ertex algebra contained in $V'$ and containing $V$ such that the derivative operator is a restriction of $D'$. Or in other words, $(V, D')$ is the smallest $D$-ertex subalgebra of $V'$ which contains $V$. Linearity and (4.15) immediately show that $\langle V, D' \rangle$ is the linear span of vectors of the form $D^n v$ where $n \geq 0$ and $v \in V$.

Let $(E, Y)$ be an ertex algebra and $(E', Y', D')$ a $D$-ertex algebra. Let $f : E \to E'$ be an ertex algebra homomorphism. We say that $f$ is $D$-injective if $\langle f(E), D' \rangle$ is an injective ertex algebra.

**Theorem 5.1.** Using the notation of Section 4 let $i$ be the obvious embedding of $E$ into $E'$. Given any $D$-ertex algebra $(V, Y_V, D_V)$ such that there exists a $D$-injective ertex algebra homomorphism, $\psi : E \to V$, then there exists a unique $D$-ertex algebra homomorphism $\phi : E' \to V$ such that $\phi(i(u)) = \psi(u)$ for all $u \in E$, or in other words such that the diagram

\[
\begin{array}{ccc}
E & \xrightarrow{i} & E' \\
\downarrow{\psi} & & \downarrow{\phi} \\
V & & 
\end{array}
\]

commutes.

Furthermore, $E'$ is the unique, up to canonical isomorphism, $D$-ertex algebra which has a $D$-injective embedding of $E$ into it and which satisfies the above property.

Proof. If $\phi$ exists, then we must have

\[
\phi(D^n e_i) = \phi(D^n D^{[0]} e_i) = D^n_V \phi(D^{[0]} e_i) = D^n_V \phi(i(e_i)) = D^n_V \psi(e_i),
\]

for all $D^n e_i \in M$. Thus $\phi$, if it exists must be unique. And furthermore we may use the above to define $\phi$ as the unique linear map satisfying

\[
\phi(D^n e_i) = D^n_V \psi(e_i),
\]

and such that $\phi(i(u)) = \psi(u)$ for all $u \in E$.

We shall next show that $\phi$ is compatible with the derivative operators. Consider $D^n e_i \in M$. If $D^{n+1} e_i \in M$ then

\[
\phi DD^n e_i = \phi D^{n+1} e_i = D^{n+1}_V \psi(e_i) = D_V \phi D^n e_i.
\]
And in the case that \( D^{[n+1]}e_l \notin M \). Then we may write
\[
DD^{[n]}e_l = \bar{D}^{[n+1]}e_l = \sum_{k \geq 0, l \in I} a_{k,l}D^{[k]}e_l,
\]
where each summand is scalar multiple of an element of \( M \). Then for \( j \in I \) we have
\[
Y_V(\phi DD^{[n]}e_l - D_V\phi D^{[n]}e_l, x)\psi(e_j)
\]
\[
= Y_V\left( \phi \left( \sum_{k \geq 0, l \in I} a_{k,l}D^{[k]}e_l \right) - D_V^{n+1}\psi(e_l), x \right) \psi(e_j)
\]
\[
= Y_V\left( \sum_{k \geq 0, l \in I} a_{k,l}D^k\psi(e_l) - D_V^{n+1}\psi(e_l), x \right) \psi(e_j)
\]
\[
= \sum_{k \geq 0, l \in I} a_{k,l} \left( \frac{d}{dx} \right)^k Y_V(\psi(e_l), x)\psi(e_j) - \left( \frac{d}{dx} \right)^{n+1} Y_V(\psi(e_l), x)\psi(e_j)
\]
\[
= \sum_{k \geq 0, l \in I} a_{k,l} \left( \frac{d}{dx} \right)^k \psi(Y(e_l, x)e_j) - \left( \frac{d}{dx} \right)^{n+1} \psi(Y(e_l, x)e_j)
\]
\[
= \sum_{k \geq 0, l \in I} a_{k,l} \left( \frac{d}{dx} \right)^k \phi(i(Y(e_l, x)e_j)) - \left( \frac{d}{dx} \right)^{n+1} \phi(i(Y(e_l, x)e_j))
\]
\[
= \phi \left( \sum_{k \geq 0, l \in I} a_{k,l} \left( \frac{d}{dx} \right)^k Y'(i(e_l), x)i(e_j) - \left( \frac{d}{dx} \right)^{n+1} Y'(i(e_l), x)i(e_j) \right)
\]
\[
= \phi \left( Y'( \sum_{k \geq 0, l \in I} a_{k,l}D^k i(e_l), x)i(e_j) - Y'(D^{n+1}i(e_l), x)i(e_j) \right)
\]
\[
= \phi \left( Y'( \sum_{k \geq 0, l \in I} a_{k,l}D^{[k]}e_l, x)i(e_j) - Y'(D^{[n+1]}e_l, x)i(e_j) \right)
\]
\[
= \phi \left( Y'(\bar{D}^{[n+1]}e_l, x)i(e_j) - Y'(D^{[n+1]}e_l, x)i(e_j) \right)
\]
\[
= \psi(\bar{Y}(\bar{D}^{[n+1]}e_l - D^{[n+1]}e_l, x)e_j)
\]
\[
= 0.
\]
Since \( E' \) has the \( \mathcal{D} \)-bracket derivative property and because \( \psi \) is \( D \)-injective, by linearity this gives that
\[
\phi DD^{[n]}e_l = D_V\phi D^{[n]}e_l.
\]
The linearity of \( \phi D \) thus by linearity we have that
\[
\phi Du = D_V\phi u
\]
for all \( u \in E'. \)
We next show that \( \phi \) is an ertex algebra homomorphism. For all \( l, k \in I \) we have

\[
\phi(Y'(e^{yD}i(e_k), x)e^{zD}i(e_l)) = \phi e^{zD}Y'(i(e_k), x + y - z)i(e_l)
\]
\[
= e^{zD}Y'(i(e_k), x + y - z)i(e_l)
\]
\[
= e^{zD}Y'(i(Y(e_k, x + y - z)e_l))
\]
\[
= e^{zD}Y'(i(e_k, x + y - z)e_l)
\]
\[
= e^{zD}Y_Y'(\psi(e_k), x + y - z)\psi(e_l)
\]
\[
= Y_Y'(e^{yD}\psi(e_k), x)e^{zD}\psi(e_l)
\]
\[
= Y_Y'(e^{yD}\psi(i(e_k)), x)e^{zD}Y_Y'(\psi(i(e_l)))
\]
\[
= Y_Y'(\phi(e^{yD}i(e_k)), x)\phi(e^{zD}i(e_l))
\]

so that the result, except for the uniqueness of \( E' \), follows by linearity.

To establish the uniqueness of \( E' \), we first note that since \( \langle i(E), D \rangle = E' \) and by Lemma 4.3 we have that \( i \) is a \( D \)-injective ertex algebra homomorphism. Let us say that \( (F, Y_F, D_F) \) is another such \( D \)-ertex algebra, with \( D \)-injective ertex algebra embedding of \( E \) into \( F \) given by \( j \). We get a \( D \)-ertex algebra homomorphism \( \phi : E' \to F \) and a second \( D \)-ertex algebra homomorphism \( \xi : F \to E' \). Moreover, for all \( v \in E \) we have

\[
\xi(j(v)) = i(v)
\]
\[
\phi(i(v)) = j(v),
\]

which in turn gives

\[
\xi(\phi(i(v))) = \xi(j(v)) = i(v)
\]

and

\[
\phi(\xi(j(v))) = \phi(i(v)) = j(v).
\]

There is a unique \( D \)-ertex algebra homomorphism \( \alpha : E' \to E' \) such that \( \alpha(i(v)) = i(v) \) for all \( v \in E \) or in other words such that the diagram

\[
\begin{array}{ccc}
E & \xrightarrow{i} & E' \\
\downarrow & & \downarrow \alpha \\
E' & \xrightarrow{i} & E'
\end{array}
\]

commutes. This unique map \( \alpha \) must be the identity map on \( E' \), but also must be \( \xi \circ \phi \). Similarly \( \phi \circ \xi \) must be the identity on \( F \). Therefore \( \xi \) and \( \phi \) are canonically given isomorphisms respecting the embeddings. \( \square \)

**Corollary 5.1.** Using the notation of the second proof of Theorem 3.1 at the close of Section 4, let \( i \) be the obvious embedding of \( E \) into \( E'' \). Given any vertex algebra \( (V, Y_V, 1_V) \) such that there exists a \( D \)-injective ertex algebra homomorphism, \( \psi : E \to V \), then there
exists a unique vertex algebra homomorphism $\phi : E'' \rightarrow V$ such that $\phi(i(u)) = \psi(u)$ for all $u \in E$, or in other words such that the diagram

$$
\begin{array}{c}
E \xrightarrow{i} E'' \\
\downarrow \psi \quad \downarrow \phi \\
V
\end{array}
$$

commutes.

Furthermore $E''$ is the unique, up to canonical isomorphism, vertex algebra with a $D$-injective embedding of $E$ into it which satisfies the above property.

Proof. Since every vertex algebra homomorphism is also a $D$-vertex algebra homomorphism we may argue similarly as in Theorem 5.1 to get that $\phi$ must be uniquely determined on all elements of the form $D^{[n]}e_i \oplus 0$. Further, since the vacuum must be preserved, by linearity we have that $\phi$, if it exists, must be unique.

By Theorem 5.1 we get a $D$-vertex algebra homomorphism $\xi : E' \rightarrow V$ and by Proposition 5.2 this in turn gives us the desired vertex algebra homomorphism. We next check that indeed $\phi(i(u)) = \psi(u)$. Let $j$ be the obvious embedding of $E$ into $E'$ and let $k$ be the obvious embedding of $E'$ into $E''$. Then we have $\phi(k(v)) = \xi(v)$ for all $v \in E'$ and $\xi(j(u)) = \psi(u)$ for all $u \in E$. Therefore

$$
\phi(i(u)) = \phi(k(j(u))) = \xi(j(u)) = \psi(u).
$$

In order to establish the uniqueness of $E''$ we first note that $i$ is $D$-injective as we noted in the proof of Theorem 5.1. Let us say that $F$ with a $D$-injective embedding $j : E \rightarrow F$ is another vertex algebra with the same property. Then we get two vertex algebra homomorphisms $\phi : E'' \rightarrow F$ and $\xi : F \rightarrow E''$ such that $\phi(i(v)) = j(v)$ and $\xi(j(u)) = i(u)$ for all $v \in E$. Thus

$$
\phi(\xi(j(v))) = \phi(i(v)) = j(v)
$$

and

$$
\xi(\phi(i(v))) = \xi(j(v)) = i(v).
$$

There is a unique vertex algebra homomorphism $\alpha : E'' \rightarrow E''$ such that $\alpha(i(v)) = i(v)$ for all $v \in E$ or in other words such that the diagram

$$
\begin{array}{c}
E \xrightarrow{i} E'' \\
\downarrow i \quad \downarrow \alpha \\
E''
\end{array}
$$

commutes. This unique map $\alpha$ must be the identity map on $E''$, but also must be $\xi \circ \phi$. Similarly $\phi \circ \xi$ must be the identity on $F$. Therefore $\xi$ and $\phi$ are canonically given isomorphisms respecting the embeddings. \qed
We recall the notation from Theorem 3.1 and let $i$ be the obvious embedding of $E$ into $E''_r$. The following theorem states that $E''_r$ satisfies a universal property.

**Theorem 5.2.** Using the notation of Theorem 3.1 let $i$ be the obvious embedding of $E$ into $E''_r$. Given any vertex algebra $V$ such that there exists an injective vertex algebra homomorphism $\psi : E \to V$, then there exists a unique injective vertex algebra homomorphism $\phi : E''_r \to V$ such that $\phi(i(u)) = \psi(u)$ for all $u \in E$, or in other words such that the diagram

$$
\begin{array}{ccc}
E & \xrightarrow{i} & E''_r \\
\downarrow{\psi} & & \downarrow{\phi} \\
V & & 
\end{array}
$$

commutes.

Furthermore, $E''_r$ is the unique, up to canonical isomorphism, vertex algebra equipped with an embedding of $E$ into it and which satisfies the above property.

**Proof.** If $\phi$ exists it must be unique because it is determined on a generating set of $E''_r$. Let $V$ be a vertex algebra with $\psi : E \to V$ an embedding. Then consider the adjoint representation of $V$, which we call $\xi$. Consider the vertex algebra $V_r$ of weak vertex operators generated by $\xi(\psi(E))$. It is clear that $V_r$ is isomorphic to $E''_r$ under the unique vertex algebra isomorphism $j : E''_r \to V_r$ satisfying

$$j(i(v)) = \xi(\psi(v)),$$

for all $v \in E$. Further, since $V$ is a vertex algebra, $\xi$ is faithful. Let

$$\phi(v) = \xi^{-1}(j(v)),$$

for all $v \in E''_r$. Then $\phi$ is clearly an injective vertex algebra homomorphism and

$$\phi(i(v)) = \xi^{-1}(j(i(v))) = \xi^{-1}(\xi(\psi(v))) = \psi(v),$$

for all $v \in E$.

We now establish the uniqueness of $E''_r$. Let us say that $F$, with an embedding $j : E \to F$ is another vertex algebra with the same property. Then we get two vertex algebra homomorphisms $\phi : E''_r \to F$ and $\xi : F \to E''_r$ such that $\phi(i(v)) = j(v)$ and $\xi(j(v)) = i(v)$ for all $v \in E$. Thus

$$\phi(\xi(j(v))) = \phi(i(v)) = j(v)$$

and

$$\xi(\phi(i(v))) = \xi(j(v)) = i(v).$$
There is a unique vertex algebra homomorphism $\alpha : E''_r \to E''_r$ such that $\alpha(i(v)) = i(v)$ for all $v \in E$ or in other words such that the diagram

$$
\begin{array}{ccc}
E & \xrightarrow{i} & E''_r \\
\downarrow & & \downarrow \alpha \\
E''_r & \xrightarrow{i} & E''_r
\end{array}
$$

commutes. This unique map $\alpha$ must be the identity map on $E''_r$, but also must be $\xi \circ \phi$. Similarly $\phi \circ \xi$ must be the identity on $F$. Therefore $\xi$ and $\phi$ are canonically given isomorphisms respecting the embeddings. □

**Remark 5.1.** One could try to mimic the above to formulate a similar looking property for $E'_r$ from Corollary 3.1 but because it is not clear whether $E'_r$ is injective or not it is also not clear whether or not $E'_r$ is canonical, at least from this point of view.

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