1. Introduction.

In last years theoretical analyses of light and heavy quarkonia in relativistic models draws the great attention. It is known, that criterion for the testing of hadronization models in Quantum chromodynamics (QCD) is the leptonic constant of pion decay as scale for the dynamics of hadrons was considered in work [2]. On the one hand of constant decay acts as phoundamental parameter in local effective chiral Lagrangians, on the other hand it is bound with nonlocal potential model through wave function of bound state in zero. In [3,4], where the effective hamiltanian QCD with simultaneously fourquark interaction is used, the experimental value $F_\pi$ is failed to be reproduce.

In this paper it will be shown that self-consistent defination of leptonic decay constant which simultaneously satisfies Goldstone theorem and the normalization condition are gives the close value to the experimetal data.

Using the ”minimum” quantanization method [5], based on solving the Gauss equation and on the principle of gauge invariant the action corresponding to chromostatic approximation [5] should be written as follows:

$$S(q, \bar{q}) = \int d^4x d^4y \bar{q}(x)G_0^{-1}q(y) + \frac{1}{2} \int d^4x_1 d^4y_1 d^4x_2 d^4y_2$$

$$qD_1(y_1)\bar{q}H_1(x_1)K_{D_1H_1/D_2H_2}(x_1y_1/x_2y_2)qH_2(y_2)qD_2(x_2)$$

where $G_0^{-1}(x, y) = (i\slashed{\partial} - \slashed{m}_0)\delta^4(x - y)$,

$$K_{D_1H_1/D_2H_2}(x_1y_1/x_2y_2) = \delta_{i_1j_2}\delta_{i_2j_1}(\gamma_0)_{\alpha_1\beta_2}(\gamma_0)_{\alpha_2\beta_1}$$

$$\times \left(\frac{\lambda_a}{2}\right)^{A_1B_2} \left(\frac{\lambda_a}{2}\right)^{A_2B_1} \delta^4(x_1 - y_2)\delta^4(x_2 - y_1)\delta(x_1^0 - x_2^0)V_s(\vec{x}_1 - \vec{x}_2)$$

where $D_k$ and $H_k$ ($k = 1, 2$) – are complex of colour, flavour and Lorentz indices $(A_k, i_k, \alpha_k)$ and $(B_k, j_k, \beta_k)$ respectively, $V_s$ – interaction potential. Taking into account factorize notions $\{N_c\}$ and $\{N_c^*\}$
colour group
\[ \{N_c\} \otimes \{N_c^*\} = \{1\} \oplus \{N_c^2 - 1\} \]
the potential \( K \) can be decomposed into the colour singlet and the \( N_c^2 - 1 \) –plet components through the projective operators \( P_1 \) and \( P_{N^2_c - 1} \):
\[
\sum_{a=1}^{N^2_c - 1} \left( \frac{\lambda_a}{2} \right)^{A_1B_2} \left( \frac{\lambda_a}{2} \right)^{A_2B_1} = \frac{1}{2} \left( \frac{N_c^2 - 1}{N_c} P_1 - \frac{1}{N_c} P_{N^2_c - 1} \right)
\]
Since we shall investigate colourless objects, further, we shall consider only singlet channel, i.e. \( K = K_1 \),
where \( P_1^{A_1B_2/A_2B_1} = \delta^{A_1B_1} \delta^{A_2B_2} \)
The effective covariant relativistic covariant action for singlet channel is written as:
\[
S = \int d^4x \bar{q}(x)(i\gamma - \hat{m}_0)q(x) - \frac{1}{2} \int d^4x d^4y q\beta_2(y)q\alpha_1(x) \times \left[ K_1(x - y|X)_{\alpha_1,\alpha_2,\gamma_2} q\beta_1(x)\bar{q}\alpha_2(x) \right]
\]
with interaction kernel
\( K_1(x - y|X)_{\alpha_1,\alpha_2,\gamma_2} = \eta_{\alpha_1\alpha_2} [V(z\gamma_\delta(z \cdot \eta))] \eta_{\alpha_2\gamma_2} \) quantization axis is chosen \( \eta_\mu \sim \frac{1}{i} \partial X_\mu \), \( \eta_\mu \) – single time-like fourth-vector of time axis.
2. Schwinger-Dyson and Bethe-Salpeter equations.
The covariant-effective action (2) leads to the following equations:
SD equation over function \( \phi(p) \) and energy \( E(p) \) of quark having nonzero mass. The SD-equation define one-particle energy of flavour quark.
\[
E(p) \sin \phi(p) = m^0 + \frac{1}{2} \int (dq)V(|q - p|) \sin \phi(q) \quad \text{with } (dq) = \frac{d^3q}{(2\pi)^3}, \quad p = |\vec{p}|, \quad \hat{p} = \frac{\vec{p}}{p}
\]
where \( \hat{p} \) The bound state of quark-antiquark system is described by BS equation
\[
ML_1(p) = E_T(p)L_2(p) + \int (dq) V(|p - q|) \left\{ c_p^+ c_q^+ + \hat{p}\hat{q}s_p^+ s_q^+ \right\} L_2(q)
\]
\[
ML_2(p) = E_T(p)L_1(p) + \int (dq) V(|p - q|) \left\{ s_p^+ s_q^- + \hat{p}\hat{q}c_p^+ c_q^+ \right\} L_1(q)
\]
where $M$ – eigenvalue, $L_{1,2}(p)$ – eigenfunctions, 
$s^\mp = \sin \left[ \frac{\varphi_1(p) \mp \varphi_2(p)}{2} \right]$, $c^\mp = \cos \left[ \frac{\varphi_1(p) \mp \varphi_2(p)}{2} \right]$, 
$E_T(p) = E_1(p) + E_2(p)$, $\varphi_{1,2}(p)$ – energy and wave function (the solution of the (4) equation) of particles (quark) 1 and 2.

Eigenfunctions $L_{1,2}$ define wave function of bound state in rest frame $\psi(q) = \gamma_5 [L_1(q) + \gamma_0 L_2(q)]$, the normalization condition is

$$
\frac{4N_c}{M} \int (dq) L_1(q)L_2(q) = 1 \tag{5}
$$

In the chiral limit, when current quarks masses are equal to zero and $L_2 = 0$, the equation (4) reproduces SD equation (3), which corresponds to the Goldstone theorem for consistent pseudoscalar meson with nonzero mass [5]. Corresponding (goldstone solution) solution:

$$
L_1(q) = \frac{1}{F_\pi} \sin \left[ \frac{\varphi_1(q) + \varphi_2(q)}{2} \right] \tag{6}
$$

$\varphi_{1,2}(p)$ – the solution of the SD equation in zero current quark mass limit.

Thus, the solution of BS equation (4) must be satisfy the normalization condition (5) and (in the chiral limit) the Goldstone theorem (6).

3. The interaction potential of $q\bar{q}$ – system.

The potential phenomenology of spectroscopy of quarkonia [3,4] in the meaning of the real interaction, uses in general, the sum coulomb and the increasing potential. In the increasing potential is used the lattice calculation for description of the heavy quarkonia. The definition of the form $q\bar{q}$ – potential is one of the actual problem of the meson spectroscopy. We chose the potential in the form as sum of oscillator and Coulomb terms.

$$
V(p) = -\frac{4}{3} \left[ \frac{4\pi\alpha_s}{p^2} + (2\pi)^3 V_0^3 \Delta_p \delta^3(p) \right] \tag{7}
$$

One particle SD equation (3) for quark in dimensionless units takes a form

$$
\frac{d^2 \varphi(p)}{dp^2} + \frac{2d\varphi(p)}{dp} + \frac{\sin 2\varphi(p)}{p^2} - 2p \sin \varphi(p) + 2m_0 \cos \varphi(p) \\
+ \frac{2\alpha_s}{3\pi} \int dq \frac{q}{p} \left\{ \mathcal{R}_1(p, q) \cos \varphi(p) - \mathcal{R}_2(p, q) \sin \varphi(p) \right\} = 0 \tag{8}
$$
\[ E(p) = p \cos \varphi(p) + m^0 \sin \varphi(p) - \frac{1}{2}[\varphi'(p)]^2 + \frac{2\alpha_s}{3\pi} \int \frac{dq}{p} \{R_1(p, q) \sin \varphi(p) - R_2(p, q) \cos \varphi(p)\} \quad (9) \]

where

\[ R_1(p, q) = \ln \left| \frac{p + q}{p - q} \right| \left[ \sin \varphi(q) - \frac{m^0}{r(q)} \right], \]

\[ R_2(p, q) = \left( \frac{p^2 + q^2}{2pq} \ln \left| \frac{p + q}{p - q} \right| - 1 \right) \left[ \cos \varphi(q) - \frac{q}{r(q)} \right], \]

\[ r(q) = \sqrt{q^2 + (m^0)^2} \]

The BS equation with potential (7) in dimensionless units takes a form of the integral-differential equation

\[ \tilde{D}(-) L_1(p) + \frac{\alpha_s}{\pi} \tilde{I}(-) L_1(q) = ML_2(p) \]

\[ \tilde{D}(+) L_2(p) + \frac{\alpha_s}{\pi} \tilde{I}(+) L_2(q) = ML_1(p) \quad (10) \]

where

\[ \tilde{D}(\mp) L(p) = - \left[ \frac{d^2}{dp^2} + \frac{2}{p} \frac{d}{dp} - E_T(p) + (\varphi_\mp')^2 + \frac{2}{p} (s_\mp)^2 \right] L(p) \]

\[ \tilde{I}(\mp) L(q) = \int \frac{dq}{p} \left\{ R_1(p, q) c_q^{(\mp)} c_p^{(\mp)} + R_1(p, q) s_q^{(\mp)} s_p^{(\mp)} \right\} L(q) \]

The main equations (8), (9), (10) are solved by the numerical method which is given in the papers [6,7]. On fig.1 the solution of BS equation (10) for the different values of the coulomb potential constant \((\alpha_s)\) is given, constituent quark mass \(m^0 = m_1^0 = m_2^0\) and \(m^0 = 0.021(4V^3_0/3)^{1/3}\).

4. The pion constant decay.

The self-consistent defination of the pion leptonic decay constant \(F_\pi\) which satisfies to the Goldstone theorem (in the chiral limit) and the normalization condition simultaneously gives the close value to experimental data. According to the paper [8] \(F_\pi\) is defined as

\[ F_\pi = \frac{4N_c}{M_\pi} \int (dq) L_2(q) \sin \left[ \frac{\varphi_1(q) + \varphi_2(q)}{2} \right] \quad (11) \]
At small current quark masses BS-equation has the approximate solution

\[ L_1(p) \approx \frac{\sin \varphi(p)}{F_\pi} \tag{12} \]

\[ L_2(p) \approx \frac{m^0}{M_\pi F_\pi} \tag{13} \]

where \( m^0 = m^0_u = m^0_d \). If (13) substitute for (12), then

\[ M^2_\pi F^2_\pi = 2m^0 \left( 2N_c \int (dq) \sin \varphi(q) \right) \]

is obtained the expression in brackets is considered in paper [9], this value is called the quark condensate \(- \langle \bar{q}q \rangle \). Then it can be written

\[ M^2_\pi F^2_\pi = -2m^0 \langle \bar{q}q \rangle, \]

well-known algebra current correlation.

It is interesting to note that the approximate solutions (12) and (13), in the case of small current quarks of mass \( m^0 \), allows to make the qualitative estimation of BS equation (10) solution which is founded Fig.1. Let us consider the calculated wave functions \( L_1(p) \) and \( L_2(p) \) in the limit \( p \to 0 \). From Fig.1 you can see, that at \( \alpha_s = 0.56 \)

\[ L_1(0) = 5.826 \left( \frac{4}{3} V_0^3 \right)^{-1/3}, \]

\[ L_2(0) = 0.4566 \left( \frac{4}{3} V_0^3 \right)^{-1/3} \]

Since \( \varphi(0) = \pi/2 \) from (12) and (13) we have

\[ M_\pi = m^0 \frac{L_1(0)}{L_2(0)} \quad m^0 = M_\pi \frac{L_2(0)}{L_1(0)} \tag{14} \]

\[ F_\pi = \left( \frac{4}{3} V_0^3 \right)^{1/3} \left( \frac{L_1(0)}{L_2(0)} \right)^{1/3} \tag{15} \]

The formula (14) gives the possibility to calculate the current quark mass independence on the oscillator potential parameter it is sufficiently to know the wave function in zero. If take the experimental value of pion mass \( M_\pi = 140 MeV \), then the \( u, d \) quark mass \( m^0 = 11 MeV \) will be obtained.
The form (15) gives the possibility to fixe the constant of oscillator \( (\frac{4}{3} V_0^3)^{1/3} \), if take the experimental value \( F_\pi = 93 MeV \), then

\[
(\frac{4}{3} V_0^3)^{1/3} = F_\pi L_1(0) = 542 MeV.
\]

But such estimate was made in chiral limit. More full picture can be obtained by calculation of the leptonic decay constant \( F_\pi \) by (11) with helping of the founding eigenvalue and function of the equation (10). From the experimental data it is known that correlation \( M_\pi / F_\pi \approx 1.5 \) that allows to fixe the physical parameters of the model (the current quark mass \( m^0 \), the constant of interaction potential \( (\frac{4}{3} V_0^3)^{1/3} \) and \( \alpha_s \)).

On fig.2 the dependence of the correlation \( M/F \) on the Coulomb potential interaction constant \( \alpha_s \) which is compared with experimental correlation \( M_\pi / F_\pi \approx 1.5 \).

As it is seen from the figure that there is an agreement between the theory and experimental data at the point \( \alpha_s = 0.56 \).

5. Conclusion

The quark-antiquark interaction potential chosen in the form of oscillator and coulomb terms which allows to calculate masses spectra of the pseudoscalar mesons.

In contrast with many different approaches widely developing in literature, we used the interaction potential must qualitatively describing the spectrum of mesons without introducing additional physically meaningless parameters. As repeatedly noted we used in our developed model only physical parameters namely \( q\bar{q} \)- potential parameters \( (V_0, \alpha_s) \) and the current quark masses \( (m^0) \). The current quark masses are considered as \( m_u = m_d \), though our model allow to consider \( m_u \neq m_d \).

The criteria characterizing qualitative description of spectra and wave mesons functions is the leptonic decay constant of pion \( F_\pi \) we showed that \( F_\pi \) (as opposite value of wave function \( L_1(0) \)) play role of scale to calculate wave functions and masses spectra of mesons.

In the paper [3] it was calculated \( F_\pi \). These values \( F_\pi \) were significantly reduced with respect to experimental data. In the ref. [11] the given value of constant \( F_\pi \) approached to the experimental data, the
calculation was made on changing the oscillator potential parameter \( (\frac{4}{3}V^3_0)^{1/3} \approx 750\,\text{MeV} \). Some authors [11] by means of moving the scale toward higher values attempt to account for the reason of reduced values of the leptonic decay constant. In our view, the true reason of difficult calculations \( F_\pi \) is namely in the choice of the quark antiquark interaction potential. Our approach shows the most nicely real picture of pion without adding some fitting parameters of the model.

The masses spectrum calculated and the leptonic decay constant are used for the fixing the coulomb potential parameter \( \alpha_s \) by comparing with the experimental ratio \( M_\pi/F_\pi \). We got \( \alpha_s = 0.56 \). Choosing scale or the value of the oscillator potential \( (\frac{4}{3}V^3_0)^{1/3} = 515 \), we got \( F_\pi = 90\,\text{MeV} \).

In conclusion we may add that the spectra of mesons described by the potential model (3), (4) and (8), and it based on the solution of the combined equations of Bethe-Salpeter and Schwinger-Dyson, where the interaction potential chosen in the form of sum of oscillator and coulomb terms which gives good agreement with the experimental data. We wish to note in this approach that the constant of coulomb interaction potential lies in the field \( 0.5 \leq \alpha_s < 1 \).
References

1. J.R. Finger and J.E. Mandula. *Nucl. Phys.* **B199** (1982) 168; S.L. Adler and A.C. Davis at al. *Nucl. Phys.* **B244** (1984) 469. A.M. Badalyan, D.N. Kitoroage, D.S. Parijskiy, *Yad. Fiz.* **46**, (1987) 226. M. Hirata. *Prog. Th. Phys.* **77** (1987) 939; *Phys. Rev.* **D39** (1989) 1425; R. Alkofer and P.A. Amundsen. *Nucl. Phys.* **B306** (1988) 305; A. Trzupek. *Acta Phys. Pol.* **B20** (1989) 93; N.N. Singh and A.N. Mitra *Phys. Rev.* **D38** (1988) 1454.

2. H. Pagels *Nuov. Cim.* **A17** (1973) 1.

3. A. Le Yaouanc, L. Oliver, P. Pene and J.-C. Raynal. *Phys.Rev.* **D29** (1984) 1233; *Phys. Rev.* **D31** (1985) 137.

4. Pedro J. de A. Bicudo and Jose E.F.T. Riberio *Phys. Rev.* **D42** (1990) 1611; **D42** (1990) 1625; **D42** (1990) 1635; A. Bicudo, G. Krein, T. Riberio, E.Villate *Phys. Rev.* **D45** (1992) 1673.

5. N.S. Han, V.N. Pervushin *Mod. Phys. Lett.* **A2** (1987) N.P. Ilieva, N.S. Han, V.N. Pervushin, *Yad. Fiz.* **45** (1987) 1169 V.N. Pervushin, W. Kallis, N.S. Han, N.A. Sarikov *Preprint JINR* Dubna E2–88–78 (1988).

6. I.V. Amirkhanov, O.M. Juraev, V.N. Pervushin, I.V. Puzynin, and N.A. Sarikov *Preprint JINR* E11-91-108 (1991).

7. I.V. Amirkhanov, O.M. Juraev, V.N. Pervushin, I.V. Puzynin, and N.A. Sarikov *Preprint JINR* P11-91-111 (1991).

8. V.N. Pervushin, Yu.L. Kalinovskiy, W. Kallis and N.A. Sarikov. *Fortsch Phys.* **38** (1990) 333.

9. O.M. Juraev, *Uzbek. Phys. Journ.* **2** (1993) 24.

10. N.A. Sarikov, I.A. Amirkhanov, O.M. Juraev, V.N. Pervushin and I.V. Puzynin. *JINR Preprint*, E2-91-262, Dubna, 1991

11. R. Horvat, D. Kekez, D. Klabučar and D. Palle. *Phys. Rev.* **D44** (1991) 1585.