Abstract

\(n\)-ary algebras have played important roles in mathematics and mathematical physics. The purpose of this paper is to construct a deformation of Virasoro-Witt \(n\)-algebra based on an oscillator realization with two independent parameters \((p, q)\) and investigate its \(n\)-Lie subalgebra.

1 Introduction

\(n\)-ary algebras have close relations with many fields of theoretical and mathematical physics. In 1973, Nambu introduced Nambu 3-algebra in the Nambu mechanics which is a generalized classical Hamiltonian mechanics [32, 34]. Nambu \(n\)-algebra is a particular but very important example of \(n\)-Lie algebra. In mathematics, \(n\)-Lie algebra is also known as Filippov \(n\)-algebra which was introduced by Filippov [22]. A structure theory of finite-dimensional \(n\)-Lie algebras over a field \(F\) of characteristic 0 was developed [21][27][28]. V. Kac. et. al classified the simple linearly compact \(n\)-Lie algebras [5][26]. With a world-volume description of multiple M2-branes using the 3-algebras, great attention has been paid to the infinite-dimensional \(n\)-Lie algebras in physics [3]. Chakrabortty et al. gave a \(w_\infty\) 3-Lie algebra by applying a double scaling limits on the generators of the \(W_\infty\) algebra [10]. Chen et al. investigated the super high-order Virasoro 3-algebra and obtained the super \(w_\infty\) 3-Lie algebra by applying the appropriate scaling limits on the generators [11]. Lately, [12] established the relation between the dispersionless KdV hierarchy and \(w_\infty\) 3-Lie algebra. As a subalgebra of \(w_\infty\) algebra, Virasoro-Witt (V-W) algebra plays an important role in physics. Using the techniques of \(su(1, 1)\) enveloping algebra,
a nontrivial 3-bracket variant of the V-W algebra was constructed [15]. The ternary algebra is a Nambu 3-algebra only when a parameter in it is chosen to be the special values. [16] putted the V-W algebra as an especial example to compare classical and quantal realizations of the ternary algebraic structures. Zhang et al. found the special V-W 3-algebra in the study of the quantum Calogero-Moser mode [26]. For the n-ary brackets in the above cases with \( n > 3 \), they are null or don’t have good property.

More attentions are also paid to find quantum deformations of the infinite-dimensional Lie algebras which is closely related to quantum groups, initially proposed by Drinfeld in [19]. The single parameter deformation of the V-W algebra has been widely investigated [6] [7] [23] [25] [29] which has important applications in field theory and integrable systems. Curtright and Zachos obtained the \( q \)-deformed V-W algebra which is closely related to the vertex model of statistical mechanics and invariants of knot theory [17] [35]. The central extension of the \( q \)-deformed V-W algebra was constructed using the Jacobi identity [1]. Recently, [18] investigated the \( q \)-deformed V-W \( n \)-algebra based on an oscillator realization and explored its intriguing features. Although the \( q \)-deformed V-W \( n \)-algebra is not an \( n \)-Lie algebra, it is a sh-\( n \)-Lie algebra and owns a beautiful expression. It should be noted that both \( n \)-Lie algebra and sh-\( n \)-Lie algebra are the generalizations of Lie algebra on \( n \)-ary bracket.

Two-parameter deformations of Lie algebras including \((p, q)\)-deformed V-W algebra have been investigated [8] [9] [14]. For the case on \( n \)-ary bracket, it has not been presented in the existing literature. The aim of this paper is to investigate two-parameter deformation of the V-W \( n \)-algebra. We find that the \((p, q)\)-deformed V-W \( n \)-algebra is a generalization of \( q \)-deformed V-W \( n \)-algebra when \( n \) is odd. The \((p, q)\)-deformed \( n \)-algebra is a sh-\( n \)-Lie algebra but not an \( n \)-Lie algebra. Since the \( n \)-Lie algebra has the close relationship with integral systems [12] [13], then we also study \( n \)-Lie subalgebras of the \((p, q)\)-deformed V-W \( n \)-algebra at last.

The paper is organized as follows. In section 2, we recall the definitions of \( n \)-Lie algebra and sh-\( n \)-Lie algebra. Section 3 is dedicated to construct the \((p, q)\)-deformed V-W \( n \)-algebra. In section 4, we investigate its \( n \)-Lie subalgebras.

2 \( n \)-Lie algebra and sh-\( n \)-Lie algebra

Lie algebra is a linear space associated with a Lie bracket which satisfies the skew-symmetry and Jacobi identity. Replacing the Lie bracket with \( n \)-bracket which satisfies a specific characteristic identity, the \( n \)-ary generalizations of Lie algebra can be derived. In this section, we review the \( n \)-Lie algebra and sh-\( n \)-Lie algebra which are two kinds of generalizations of Lie algebra. More information can be found in [20] [22].
Definition 1 An $n$-Lie algebra is a vector space $V$ endowed with a multilinear map $[\cdot, \cdot, \cdot]$ from $V^{\otimes n}$ to $V$ and satisfies the skew-symmetry condition

$$[X_1, \cdots, X_i, \cdots, X_j, \cdots, X_n] = -[X_1, \cdots, X_j, \cdots, X_i, \cdots, X_n] \quad (1)$$

and the fundamental identity (FI) or Filippov condition

$$[Y_1, \cdots, Y_{n-1}, [X_1, \cdots, X_n]] = \sum_{k=1}^{n} [X_1, \cdots, X_{k-1}, [Y_1, \cdots, Y_{n-1}, X_k], X_{k+1}, \cdots, X_n], \quad (2)$$

where $X_i, Y_j \in V$.

Obviously, for $n = 2$, FI reduces to the ordinary Jacobi identity. For the simplest $n = 3$ case, FI is

$$[[Y_1, Y_2, [X_1, X_2, X_3]] = [[Y_1, Y_2, X_1], X_2, X_3] + [X_1, [Y_1, Y_2, X_2], X_3] + [X_1, X_2, [Y_1, Y_2, X_3]] \quad (3)$$

Nambu 3-bracket is an important example of 3-Lie algebra introduced in the Nambu mechanics which is a generalization of classical Hamiltonian mechanics [32]. It is defined as

$$[f_1, f_2, f_3] = \det \left( \frac{\partial (f_1, f_2, f_3)}{\partial (x_1, x_2, x_3)} \right), \quad (4)$$

where $f_i$ is considered as a classical observable on 3-dimensional phase space $\mathbb{R}^3$.

Now let us refer to sh-$n$-Lie algebra.

Definition 2 Let $V$ be a vector space, $[\cdot, \cdot, \cdot]$ be an $n$-ary skew-symmetric product on the vector space $V$. $(V, [\cdot, \cdot, \cdot])$ is a sh-$n$-Lie algebra if it satisfies the sh-Jacobi identity

$$\sum_{\sigma \in Sh(n,n-1)} (-1)^{\epsilon(\sigma)} [[X_{\sigma(1)}, \cdots, X_{\sigma(n)}], X_{\sigma(n+1)}, \cdots, X_{\sigma(2n-1)}] = 0, \quad (5)$$

for any $X_i \in V$, $\sigma$ is a permutation of the indices $(1, 2, \cdots, 2n-1)$, $\epsilon(\sigma)$ is the parity of the permutation $\sigma$, where $Sh(n,n-1)$ is the subset of $\Sigma_{2n-1}$ defined by

$$Sh(n,n-1) = \{ \sigma \in \Sigma_{2n-1}, \sigma(1) < \cdots < \sigma(n), \sigma(n+1) < \cdots < \sigma(2n-1) \}. \quad (6)$$

In terms of the Lévi-Civită symbol, i.e.,

$$\epsilon_{j_1 \cdots j_p}^{i_1 \cdots i_p} = \det \left( \begin{array} {ccc} \delta_{j_1}^{i_1} & & \\
& \ddots & \\
& & \delta_{j_p}^{i_p} \end{array} \right), \quad (7)$$

under the skew-symmetry condition, the sh-Jacobi identity can be expressed as

$$\epsilon_{1 \cdots 2n-1}^{i_1 \cdots i_{2n-1}} \left([X_{i_1}, \cdots, X_{i_n}], X_{i_{n+1}}, \cdots, X_{i_{2n-1}} \right] = 0. \quad (8)$$
When $n = 2$, the sh-Jacobi identity (5) reduces to the Jacobi identity. When $n = 3$, the sh-Jacobi identity becomes

$$[[X_1, X_2, X_3], X_4, X_5] - [[X_1, X_2, X_4], X_3, X_5] + [[X_1, X_2, X_5], X_3, X_4]$$

$$+ [[X_1, X_3, X_4], X_2, X_5] - [[X_1, X_3, X_5], X_2, X_4] + [[X_1, X_4, X_5], X_2, X_3]$$

$$- [[X_2, X_3, X_4], X_1, X_5] + [[X_2, X_3, X_5], X_1, X_4] - [[X_2, X_4, X_5], X_1, X_3]$$

$$+ [[X_3, X_4, X_5], X_1, X_2] = 0.$$  (9)

It should be noted that an $n$-Lie algebra is a sh-$n$-Lie algebra, but the converse may be not true, for example, the $q$-deformed V-W $n$-algebra in [18].

3 (p, q)-deformed V-W n-algebra

The classical V-W algebras can be constructed by the harmonic oscillator algebra $\{a, a^\dagger, N\}$. Based on the deformed oscillator generators with a single parameter or two parameters, the deformed V-W algebra are investigated in [4][6][8][21][31]. In this section, before investigating the $(p, q)$-deformed V-W $n$-algebra, we review the case of 2-bracket.

Let us take the $(p, q)$-deformed generators

$$L_m = -p^N(a^\dagger)^{m+1}a,$$  (10)

where the $(p, q)$-deformed oscillator is deformed by the following relations

$$[N, a] = -a, \quad [N, a^\dagger] = a^\dagger,$$

$$aa^\dagger - qa^\dagger a = p^{-N},$$

$$aa^\dagger - p^{-1}a^\dagger a = q^{N}.$$  (11)

Based on an oscillator realization with two independent parameters $(p, q)$, the $(p, q)$-deformed V-W algebra is constructed by

$$[L_m, L_n]_{(q^{m-p-n}, q^{n-p-m})} := q^m p^{-n} L_m L_n - q^n p^{-m} L_n L_m = -(qp^{-1})^m [n - m]_{p,q} L_{m+n},$$  (12)

where

$$[x]_{p,q} = \frac{q^x - p^x}{q - p^{-1}}.$$  (13)

It is easy to verify that the bracket (12) is skew-symmetry and satisfies the $(p, q)$-deformed Jacobi identity

$$(q^m + p^{-m})[L_m, [L_n, L_k]_{(q^{m-p-k}, q^{k-p-m})} (q^{m-p-(n+k)}, q^{n+k})_{p-m}] + cyc.p.e.r.m.s. = 0.$$  (14)
The \((p, q)\)-deformed V-W algebra \([13]\), in the limit \(p \to q\), agrees with the \(q\)-deformed V-W algebra \([18]\)

\[
[L_m, L_n]_{(q^{m-n}, q^{n-m})} := q^{m-n}L_m L_n - q^{n-m} L_n L_m = [m - n]L_{m+n},
\]

where \([k] = \frac{q^k - q^{-k}}{q - q^{-1}}\). And the \(q\)-deformed V-W algebra satisfies the \(q\)-deformed Jacobi identity

\[
[L_m, [L_n, L_k]_{(q^{n-k}, q^{k-n})}]_{(q^{2m-(n+k)}, q^{(n+k)-2m})} + \text{cycl. perms.} = 0.
\]

When \(p \to 1, q \to 1\), the \((p, q)\)-deformed V-W algebra \([13]\) becomes the well-known Virasoro-Witt algebra

\[
[L_m, L_n] = (m - n)L_{m+n}
\]

which satisfies the Jacobi identity

\[
[L_m, [L_n, L_k]] + [L_n, [L_k, L_m]] + [L_k, [L_m, L_n]] = 0.
\]

In \([32]\), the quantum Nambu 3-bracket is introduced to be a sum of single operators multiplying commutators of the remaining two, i.e.,

\[
[A, B, C] = A[B, C] + B[C, A] + C[A, B].
\]

Based on the deformation of the above defined 3-bracket, \([18]\) constructed the \(q\)-deformed V-W \(n\)-algebra which satisfies the sh-Jacobi identity. Encouraged by the \(q\)-deformed V-W \(n\)-algebra, we consider to construct the nontrivial deformed V-W \(n\)-algebra with two independent parameters.

By means of the \((p, q)\)-deformed V-W algebra \([12]\) and the \((p, q)\)-deformed oscillator generators \([14]\), we introduce the \((p, q)\)-deformed 3-bracket as

\[
[L_m, L_n, L_k] = p^m q^{m-(n+k)}L_m \cdot [L_n, L_k]_{(q^{n-k}, q^{k-n})} + p^n q^{n-(k+m)} L_n \cdot [L_k, L_m]_{(q^{k-p}, q^{p-k})} + p^k q^{k-(m+n)} L_k \cdot [L_m, L_n]_{(q^{m-p}, q^{p-m})}.
\]

Substituting \([12]\) into \([19]\), we obtain

\[
[L_m, L_n, L_k] = -\frac{1}{q - p^{-1}}((pq^{-1})^{m-n}[2m - 2n]_{p, q} + (pq^{-1})^{n-k}[2n - 2k]_{p, q})L_{m+n+k} + (pq^{-1})^{k-m}[2k - 2m]_{p, q}L_{m+n+k}
\]

\[
= -\frac{(pq^{-1})^{m+n+k}}{(q - p^{-1})^2} \det \begin{pmatrix}
  p^{-2m} & p^{-2n} & p^{-2k} \\
  p^{-m}q^{-m} & p^{-n}q^{-n} & p^{-k}q^{-k} \\
  q^{2m} & q^{2n} & q^{2k}
\end{pmatrix} L_{m+n+k}.
\]

By direct calculation, it is easy to prove that the skew-symmetry holds and we derive it satisfies the sh-Jacobi’s identity \([3]\), but the FI \([3]\) does not hold. Thus the algebra \([20]\) is a \((p, q)\)-sh-3-Lie algebra. \([20]\) reduces to the \(q\)-sh-3-Lie algebra derived in \([18]\) in the limit \(p \to q\).
Considering the interesting result derived from (20), we extend our result to \((p, q)\)-n-bracket. Define the \((p, q)\)-n-bracket with \(n > 3\) as follows:

\[
[L_{i_1}, L_{i_2}, \ldots, L_{i_n}] = \sum_{s=1}^{n} (-1)^{s+1} (pq)^{x_{i_s}+y(i_1+\cdots+i_s)} L_{i_s}[L_{i_1}, \ldots, \hat{L}_{i_s}, \ldots, L_{i_n}],
\]

in which \((x = \frac{a-1}{2}, y = -1)\) for odd \(n \geq 5\) and \((x = \frac{a}{2}, y = 0)\) for even \(n \geq 4\). Here we denote the hat symbol \(\hat{A}\) stands for the term \(A\) omitted.

**Lemma 3** The \((p, q)\)-generators \((14)\) satisfy the following closed algebraic structure relation:

\[
[L_{i_1}, L_{i_2}, \ldots, L_{i_n}] = \frac{\text{sign}(n)}{(q - p^{-1})n-1}(pq^{-1})^{|(n-1)/2|} \left[\begin{array}{cccc}
q^{-2} & \cdots & q^{-2} \\
p^{-2} & \cdots & p^{-2} \\
\vdots & \ddots & \vdots \\
p^{-(n-3)k} & \cdots & p^{-(n-3)k}
\end{array}\right] L_{\xi_{n-1}^{\Sigma_p^{2k}}} \sum_{i_k}
\]

where \([n] = \text{Max}\{m \in \mathbb{Z} | m \leq n\}\) is the floor function.

**Proof.** We prove the lemma 4 by mathematical induction for \(n\). From \((19)\) it is satisfied for \(n = 3\). Suppose \((22)\) is satisfied for \(n = 2k - 1\), from \((21)\), we obtain

\[
[L_{i_1}, L_{i_2}, \ldots, L_{i_{2k}}] = \sum_{s=1}^{2k} (-1)^{s+1} L_{i_s} \ast [L_{i_1}, \ldots, \hat{L}_{i_s}, \ldots, L_{i_{2k}}] = 2k \sum_{s=1}^{2k} (-1)^{s+1} (pq)^{x_{i_s}+y(i_1+\cdots+i_s+\cdots+i_{2k})} \frac{\text{sign}(2k-1)}{(q - p^{-1})^{2k-2}} (pq^{-1})^{(k-1)(i_1+i_2+\cdots+i_{2k})} \\
\text{det} \left[\begin{array}{cccc}
p^{-(2k-2)i_1} & \cdots & p^{-(2k-2)i_s} & \cdots & p^{-(2k-2)i_{2k}} \\
p^{-(2k-3)i_1} & \cdots & p^{-(2k-3)i_s} & \cdots & p^{-(2k-3)i_{2k}} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
p^{-i_1} & \cdots & p^{-i_s} & \cdots & p^{-i_{2k}} \\
p^{2k-2i_1} & \cdots & p^{2k-2i_s} & \cdots & p^{2k-2i_{2k}}
\end{array}\right] \left(\begin{array}{c}
p^{2N-\Sigma_{m=1}^{2k} i_m} q(a_i^{\dagger})^{\Sigma_{m=1}^{2k} i_m} + 2 a^2 - \frac{pq}{q - p^{-1}} L_{\xi_{2k}^{\Sigma_{m=1}^{2k} i_m}} + \frac{1}{q - p^{-1}} L_{\xi_{2k}^{\Sigma_{m=1}^{2k} i_m}}
\end{array}\right)
\]

\((23)\)
For \( n \) is even, we substitute \( x = k, y = 0 \) into (23), then (24) is divided into two parts

\[
[L_{i_1}, L_{i_2}, \cdots, L_{i_{2k}}] = \frac{\text{sign}(2k-1)}{(q - p^{-1})^{2k-2}} \left( p^k q^{(2-k)} \right)^{\sum_{m=1}^{2k} i_m} q^{(2k-2)i_s} 
\]

\[
\begin{pmatrix}
\begin{array}{cccc}
p^{-(2k-2)i_1} & \cdots & p^{-(2k-2)i_s} & p^{-(2k-2)i_{2k}} \\
p^{-(2k-3)i_1} q^{i_1} & \cdots & p^{-(2k-3)i_s} q^{i_s} & p^{-(2k-3)i_{2k}} q^{i_{2k}} \\
\vdots & \vdots & \vdots & \vdots \\
p^{-i_1} q^{(2k-3)i_1} & \cdots & p^{-i_s} q^{(2k-3)i_s} & p^{-i_{2k}} q^{(2k-3)i_{2k}} \\
q^{(2k-2)i_1} & \cdots & q^{(2k-2)i_s} & q^{(2k-2)i_{2k}} 
\end{array}
\right) 
\]

(24)

The first part of (24) is vanished. Then we obtain

\[
[L_{i_1}, L_{i_2}, \cdots, L_{i_{2k}}] = \frac{\text{sign}(2k)}{(q - p^{-1})^{2k-1}} (pq)^{\sum_{m=1}^{2k} i_m} q^{(2k-1)i_s} p^{i_s} 
\]

\[
\begin{pmatrix}
\begin{array}{cccc}
p^{-(2k-2)i_1} & \cdots & p^{-(2k-2)i_s} & p^{-(2k-2)i_{2k}} \\
p^{-(2k-3)i_1} q^{i_1} & \cdots & p^{-(2k-3)i_s} q^{i_s} & p^{-(2k-3)i_{2k}} q^{i_{2k}} \\
\vdots & \vdots & \vdots & \vdots \\
p^{-i_1} q^{(2k-3)i_1} & \cdots & p^{-i_s} q^{(2k-3)i_s} & p^{-i_{2k}} q^{(2k-3)i_{2k}} \\
q^{(2k-2)i_1} & \cdots & q^{(2k-2)i_s} & q^{(2k-2)i_{2k}} 
\end{array}
\right) 
\]

(25)

So (22) is satisfied for \( n = 2k \). We can obtain (22) is satisfied for \( n = 2k-1 \) following the above strategy.

Theorem 4 When \( n \geq 3 \), the \((p, q)\)-n-bracket relation (22) is a sh-\( n \)-Lie algebra.

Proof. Firstly, we prove the \( n \)-bracket (22) satisfies the sh-Jacobi identity (8) for odd \( n \). From the Lévi-Civita symbol (7), we express (22) as

\[
[L_{i_1}, \cdots, L_{i_n}] = \frac{\text{sign}(n)}{(q - p^{-1})^{2n-1} (i_1 + \cdots + i_n)} \epsilon^{i_1 \cdots i_n}_{j_1 \cdots j_{n-1}} p^{-(n-1)j_1} p^{-(n-2)j_2} q^{i_2} \cdots p^{-j_{n-1}} q^{(n-2)j_{n-1}} q^{(n-1)j_n} L_{\sum_{m=1}^{2n-1} i_m},
\]

(26)
It is clear that the skew-symmetry holds for the $n$-bracket \((22)\) for the skew-symmetry of $\epsilon_{i_1 \cdots i_n}^{j_1 \cdots j_n}$. Substituting \((20)\) into the left hand side of \((3)\), we have

$$
\epsilon_{i_1 \cdots i_{2n-1}}^{j_1 \cdots j_{2n-1}}[[L_{i_1}, \ldots, L_{i_n}], L_{i_{n+1}}, \ldots, L_{i_{2n-1}}]
= \frac{1}{(q - p^{-1})^{2n-2}} \sum_{s=0}^{n-1} \epsilon_{i_1 \cdots i_n}^{j_1 \cdots j_n} \epsilon_{i_{n+1} \cdots i_{2n-1}}^{j_{n+1} \cdots j_{2n-1}} (pq)^s L_{\Sigma_{m=1}^{2n-1} i_m}^s
= \frac{n!(n-1)!}{(q - p^{-1})^{2n-2}} \sum_{s=0}^{n-1} \epsilon_{i_1 \cdots i_{2n-1}}^{j_1 \cdots j_{2n-1}} (pq)^s L_{\Sigma_{m=1}^{2n-1} i_m}^s,
$$

(27)

where the power $\alpha$ of $(pq)$ is given by

$$
\alpha = (s - n + 1)j_1 + (s - n + 2)j_2 + \cdots + sj_n - \frac{n-1}{2}j_{n+1} + (-\frac{n-1}{2} + 1)j_{n+2} + \cdots
+ (-\frac{n-1}{2} + s - 1)j_{n+s} + \cdots + \frac{n-1}{2}j_{2n-1}.
$$

(28)

Taking $s$ from 0 to $2n-1$, we obtain the coefficients of two different $j_\mu (\mu = 1, \ldots, 2n-1)$ should be equal by means of \((28)\). The sh-Jacobi’s identity \((3)\) equals zero because $\epsilon_{i_1 \cdots i_{2n-1}}$ is completely antisymmetric. It indicates that the sh-Jacobi’s identity is satisfied by \((22)\) with odd $n$. The theorem can be also proved for even $n$ with similar procedure.

Base on the above proof, we derive the $(p, q)$-$n$-bracket relation \((22)\) is a sh $n$-algebra for $n > 3$.

In the limit $p \to q$, the $n$-bracket reduces to the $q$-deformed V-W $n$-algebra for $n$ odd. It has been proved that the $q$-deformed V-W $n$-algebra is not an $n$-Lie algebra in \([18]\). Hence it is quite clear that the $(p, q)$-deformed $n$-algebra \((22)\) is not an $n$-Lie algebra for $n$ odd. For the case of $n$ even, choosing $Y_i = L_{n-1-i}, i = 1, 2, \ldots, n-2, Y_{n-1} = L_{\frac{n(n-1)}{2}}$ and $X_j = L_{j-1}, j = 1, 2, \ldots, n$ in \((2)\), we can find the Filippov condition \((2)\) is not satisfied. Hence the $(p, q)$-deformed V-W $n$-algebra is a sh-$n$-Lie algebra but not an $n$-Lie algebra.

4 n-Lie subalgebra

Since the $n$-Lie algebra is connected with the integral system, then it is interesting and necessary to investigate whether there exists $n$-Lie subalgebras in the $(p, q)$-deformed $n$-algebra \((22)\). Here, let us consider it.

Lemma 5 Suppose $A$ is a finite-dimensional $n$-Lie subalgebra of the $(p, q)$-deformed $n$-algebra \((22)\). If $A$ admits a basis as \{$L_{i_1}, \ldots, L_{i_k}$\}, then dim $A \leq n+1$.

Proof. Assume that $i_1 < \cdots < i_k$. If the assertion dim $A \leq n+1$ would not hold, then $k > n+1$. Since \{$L_{i_1}, \ldots, L_{i_n}$\} $\in A$, then by the Lie bracket \((22)\), $L_{i_1 + \cdots + i_n} \in A$. Hence there
exists $s_1(1 \leq s_1 \leq k)$ such that
\[ i_1 + \cdots + i_n = i_{s_1}. \]  
(29)

If $s_1 > n$, then $i_1 + \cdots + i_{n-1} > 0$. Hence $i_n > 0$ for $i_1 < \cdots < i_n$. Note that $\{L_{i_1}, \ldots, L_{i_{n-1}}, L_{i_{s_1}}\} \in V$, by the similar method above, there exists $s_2(1 \leq s_2 \leq k)$ such that
\[ i_1 + \cdots + i_{n-1} + i_{s_1} = i_{s_2} \quad \text{and} \quad L_{i_{s_2}} \in V. \]  
(30)

By $i_1 + \cdots + i_{n-1} > 0$, we get $i_{s_2} > i_{s_1}$. Continue to do this, we have $\{L_{i_{s_1}}, L_{i_{s_2}}, L_{i_{s_3}}, \ldots \in A$ and $i_{s_1} < i_{s_2} < i_{s_3} < \ldots\}$. Clearly, it is contract with $A$ is a finite dimensional algebra.

Now we have
\[ i_1 + \cdots + i_n = i_{s_1}, s_1 \leq n. \]  
(31)

From (31) and $i_1 < \cdots < i_n$, we get $i_1 < 0$. From $s_1 \geq 1$ we obtain $i_n > 0$. By (31) and $s_1 > 1$, $i_1 + \cdots + \hat{i}_{s_1} + \cdots + i_n = 0$. Note that $i_1 < 0$, we have $i_2 + \cdots + \hat{i}_{s_1} + \cdots + i_n > 0$. Since $\dim A > n + 1$ and $i_n > 0$, then
\[ i_k > i_{k-1} > i_n > 0. \]  
(32)

By the Lie bracket (22) and (32), we get
\[ L_{i_2 + \cdots + \hat{i}_{s_1} + \cdots + i_n + i_{k-1} + i_k} \in A. \]  
(33)

and
\[ i_2 + \cdots + \hat{i}_{s_1} + \cdots + i_n + i_{k-1} + i_k > i_k. \]  
(34)

It is contract with the fact that $L_{i_1}, \ldots, L_{i_k}$ are the generators of $A$. Now we can get the conclusion $\dim A \leq n + 1$.

**Theorem 6** For any finite-dimensional $n$-Lie algebra $A$ in the $(p, q)$-deformed $n$-algebra (22).

If $A$ admits the basis as $\{L_{i_1}, \ldots, L_{i_k}\}$, then $A$ is $n$-dimensional and isomorphic to the following $n$-Lie algebra
\[ [B_1, \ldots, B_n] = B_1. \]  
(35)

**Proof.** Firstly, let us give a $n$-dimensional $n$-Lie algebra in the $(p, q)$-deformed $n$-algebra (22).

If $n = 2k$, we have the subalgebra
\[ [L_{-k+1}, L_{-k+2}, \cdots, L_0, \cdots, L_k] = \frac{\text{sign}(2k)}{(q-p^{-1})^{2k-1}} (pq^{-1})^{k^2-k} M_1 L_k. \]  
(36)

If $n = 2k + 1$, we have the subalgebra
\[ [L_{-k}, L_{-k+1}, \cdots, L_0, \cdots, L_k] = \frac{\text{sign}(2k+1)}{(q-p^{-1})^{2k}} M_2 L_0. \]  
(37)
where

\[
M_1 = \det \begin{pmatrix}
    p^{-(2k-2)(-k+1)} & p^{-(2k-2)(-k+2)} & \ldots & p^{-(2k-2)k} \\
    p^{-(2k-3)(-k+1)}q^{-k+1} & p^{-(2k-3)(-k+2)}q^{-k+2} & \ldots & p^{-(2k-3)k}q^k \\
    \vdots & \vdots & \ddots & \vdots \\
    q^{-(2k-2)(-k+1)} & q^{-(2k-2)(-k+2)} & \ldots & q^{-(2k-2)k} \\
    p^{-k+1}q^{-(2k-1)(-k+1)} & p^{-k+2}q^{-(2k-1)(-k+2)} & \ldots & p^kq^{-(2k-1)k}
\end{pmatrix},
\]

\[
M_2 = \det \begin{pmatrix}
p^{2k^2} & p^{2k^2-2k} & \ldots & p^{-2k^2} \\
p^{(2k-1)k}q^{-k} & p^{(2k-1)(k-1)}q^{-k+1} & \ldots & p^{-(2k-1)k}q^k \\
\vdots & \vdots & \ddots & \vdots \\
p^{k}q^{-(2k-1)k} & p^{k-1}q^{-(2k-1)(-k+1)} & \ldots & p^kq^{-(2k-1)k} \\
q^{-2k^2} & q^{-2k^2+2k} & \ldots & q^{2k^2}
\end{pmatrix},
\]

(38)

Obviously, the \(n\)-algebra with the bracket (30) or (37) is isomorphic to (35) which is anticommutative and satisfies the FI condition. Hence there exists an \(n\)-Lie algebra which is isomorphic to (35) in the \((p, q)\)-deformed \(n\)-algebra (22).

Next we will prove that any finite-dimensional \(n\)-subalgebra in \(V\) is isomorphic to (35). Since \(A\) is a finite-dimension \(n\)-Lie subalgebra with the generators \(\{L_{i_1}, \ldots, L_{i_k}\} | i_u \neq i_v, 1 \leq u, v \leq k\}. \) From the lemma 6, we have \(k \leq n + 1\).

If \(k = n\), then the \(n\)-bracket give as (22) and \(i_1 + \ldots + i_n = i_s, 1 \leq s \leq n\). Clearly, (22) is isomorphic to (35).

If \(k = n + 1\), the Lie bracket is

\[
[L_{i_1}, \ldots, \hat{L}_{i_s}, \ldots, L_{i_{n+1}}] = H_sL_{i_1+\ldots+i_s+\ldots+i_{n+1}},
\]

(39)

where

\[
H_s = \frac{\text{sign}(n)}{(q - p^{-1})^{n-1}}(pq^{-1})^{\frac{n-1}{2}}(i_1 + \ldots + i_s + \ldots + i_{n+1}) \cdot \det \begin{pmatrix}
    H_{1,1} & \cdots & \hat{H}_{1,s} & \cdots & H_{1,n+1} \\
    H_{2,1} & \cdots & \hat{H}_{2,s} & \cdots & H_{2,n+1} \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    H_{n-1,1} & \cdots & \hat{H}_{n-1,s} & \cdots & H_{n-1,n+1} \\
    H_{n,1} & \cdots & \hat{H}_{n,s} & \cdots & H_{n,n+1}
\end{pmatrix},
\]

(40)

and

\[
H_{1,s} = q^{-2\left[\frac{n+1}{2}\right]i_s}, \quad H_{1,n+1} = q^{-2\left[\frac{n+1}{2}\right]i_{n+1}},
\]

\[
H_{2,s} = p^{-2\left[\frac{n+1}{2}\right]+1}i_s, \quad H_{2,n+1} = p^{-2\left[\frac{n+1}{2}\right]+1}i_{n+1},
\]

\[
H_{n-1,s} = p^{\left[\frac{n}{2}\right]-\left[\frac{n+1}{2}\right]-1}i_s, \quad H_{n-1,n+1} = p^{\left[\frac{n}{2}\right]-\left[\frac{n+1}{2}\right]-1}i_{n+1},
\]

(41)

\[
H_{n,s} = p^{\left[\frac{n}{2}\right]-\left[\frac{n+1}{2}\right]+1}i_s, \quad H_{n,n+1} = p^{\left[\frac{n}{2}\right]-\left[\frac{n+1}{2}\right]+1}i_{n+1}.
\]

(42)
It is not difficult to show that \( \dim A^1 > 2 \).

Taking
\[
L^j = (-1)^{n+j+1} [L_{i_1}, \cdots, \hat{L}_{i_j}, \cdots, L_{i_{n+1}}], \; j = 1, \cdots, n + 1.
\]
the Lie bracket relation is given by the matrix \( H \)
\[
(L^1, \cdots, L^{n+1}) = (L_{i_1}, \cdots, L_{i_{n+1}}) B.
\]

According to Theorem 3 in [22], (39) is an \( n \)-Lie algebra if and only if the matrix \( B \) is symmetric.

Substituting (39) into (44) and by means of (45), we obtain
\[
i_1 + \cdots + \hat{i}_s + \cdots + i_{n+1} = i_s, \; s = 1, \cdots, n + 1
\]
or there exists \( 1 \leq k_1 < k_2 \leq n + 1 \) such that
\[
H_{k_1} = (-1)^{k_2-k_1} H_{k_2}.
\]
From (46), \( i_1 = i_2 = \cdots = i_{n+1} \) which is contradict with to \( i_u \neq i_v \). And (47) is impossible for \( k_1 \neq k_2 \). Hence there is not \((n+1)\)-dimensional subalgebra. Now, we complete the proof.

Note that \( \{B_1\} \) is an ideal of the \( n \)-Lie algebra \( \mathfrak{B} \), by the above theorem, it implies that

**Corollary 7** If \( A \) is an \( n \)-Lie subalgebra defined as in Lemma 5, then \( A \) is not simple.

5 Acknowledgment

The authors thank Morningside Center of Mathematics, Chinese Academy of Sciences for providing excellent research environment. This work is partially supported by National Natural Science Foundation of China (Grant No.11547101,11575286) and the Program of Higher-level talents of Inner Mongolia University (21100-5145101).

References

[1] N. Aizawa, H. Sato, q-deformation of the Virasoro algebra with central extension, Phys. Lett. B 256 185 (1991).

[2] J. A. de Azcárregua and J. C. Pérez-Bueno, Higher-order simple Lie algebras, Commun. Math. Phys. 184 669 (1997) [arXiv:9605213]

[3] J. Bagger, N. Lambert, Gauge Symmetry and Supersymmetry of Multiple M2-Branes, Phys. Rev. D 77 065008 (2008) arXiv:0711.0955.
[4] L.C. Biederharn, The quantum group $SU_q(2)$ and a $q$-analogue of the boson operators, J. Phys. A: Math. Gen. 22 L873 (1989).

[5] N. Cantarini and V. G. Kac, Classification of simple linearly compact $n$-Lie superalgebras, arXiv:0909.3284v2.

[6] M. Chaichian, P.P. Kulish and J. Lukierski, $q$-deformed Jacobi identity, $q$-oscillators and $q$-deformed infinite-dimensional algebras, Phys. Lett. B 237 401 (1990).

[7] M. Chaichian, A. P. Isaev, J. Lukierski, Z. Popowicz and P. Presnajder, $q$-deformations of Virasoro algebra and conformal dimensions, Phys. Lett. B 262 32 (1991).

[8] R. Chakrabarti and R. Jagannathan, A $(p, q)$-oscillator realization of two-parameter quantum algebras, J. Phys. A 24 L711 (1991).

[9] R. Chakrabarti and R. Jagannathan, A two-parameter deformation of the Jaynes-Cummings model: path integral representation, J. Phys. A 25 2607 (1992).

[10] S. Chakrabortty, A. Kumar and S. Jain, $w_\infty$ 3-algebra, JHEP 09 (2008) 091 arXiv:0807.0284.

[11] Min-Ru Chen, Ke Wu and Wei-Zhong Zhao, Super $w_\infty$ 3-algebra, JHEP 09 (2011) 090 arXiv:1107.3295.

[12] M. R. Chen, S. K. Wang, K. Wu, and W. Z. Zhao, Infinite-dimensional 3-algebra and integrable system, J. High Energy Phys. 12 030 (2012).

[13] M. R. Chen, S. K. Wang, X. L. Wang, K. Wu and W. Z. Zhao, On $W_{1+\infty}$ 3-algebra and integrable system, Nucl. Phys. B 891 655 (2015).

[14] W. S. Chung, Two parameter deformation of Virasoro algebra, J. Math. Phys. 35 2490 (1994).

[15] T. L. Curtright, D. B. Fairlie and C. K. Zachos, Ternary Virasoro-Witt algebra, Phys. Lett. B 666 386 (2008).

[16] T. Curtright, D. Fairlie, X. Jin, L. Mezincescu and C. Zachos, L. Mezincescu and C. Zachos, Classical and quantal ternary algebras, Phys. Lett. B 675 387 (2009).

[17] T. Curtright and C. Zachos, Deforming maps for quantum algebras, Phys. Lett. B 243 237 (1990).
[18] L. Ding, X. Y. Jia, K. Wu, Z. W. Yan and W. Z. Zhao, On q-deformed infinite-dimensional \( n \)-algebra, Nuclear Physics B 904 (2016) 18–38, [arXiv:1404.0464v3].

[19] V.G. Drinfeld, Hamiltonian structures on Lie group, Lie algebras and the geometric meaning of classical Yang-Baxter equations, Sov. Math. Dokl. 27(1) 68 (1983).

[20] M. Goze, N. Goze and E. Remm, \( n \)-Lie algebras, African J. Math. Phys. 8 17 (2010)[arXiv:0909.1419].

[21] V. T. Filippov, On \( n \)-Lie algebras of Jacobians, Sib. Mat. Zh. 39(3) (1998), 660–669, translation in Sib. Math. J. 39(3) 573 (1998).

[22] V. T. Filippov, \( n \)-Lie algebras, Sib. Mat. Zh. J. 26 (1985) 126. (English translation:Siberian Math. J. 26 (1985), no. 6, 879-891).

[23] E. Frenkel, N. Reshetikhin, Quantum affine algebras and deformations of the Virasoro and W-algebras, Comm. Math. Phys. 178 237 (1996).

[24] T. Hayashi, \( Q \)-analogues of Clifford and Weyl algebras-Spinor and oscillator representations of quantum enveloping algebras, Commun. Math. Phys. 127 129 (1990).

[25] N. Hu, \( q \)-Witt algebras, \( q \)-Virasoro algebra, \( q \)-Lie algebras, \( q \)-holomorph structure and representations, Colloq. Alg. 6(1) 51 (1999).

[26] V. G. Kac, Classification of infinite-dimensional simple linearly compact Lie superalgebras, Adv. Math. 139 1 (1998).

[27] S. M. Kasymov On the theory of \( n \)-Lie algebras, Algebra i Logika 26(3) 277 (1987), translation in Algebra and Logic 26(3) 155 (1987).

[28] S. M. Kasymov On nil-elements and nil-subsets of \( n \)-Lie algebras, Sib. Mat. J. 32(6) 77 (1991).

[29] C. Kassel, Cyclic homology of differential operators, the Virasoro algebra and a \( q \)-analogue, Comm. Math. Phys. 146 343 (1992).

[30] M. Lashkevich and Y. Pugai, Form factors in sinh-and sine-Gordon models, deformed Virasoro algebra, Macdonald polynomials and resonance identities [arXiv:1307.0243v5].

[31] A. J. MacFarlane, On \( q \)-analogues of the quantum harmonic oscillator and the quantum group \( SU(2)_q \), J. Phys. A: Math. Gen. 22 4581 (1989).

[32] Y. Nambu, Generalized Hamiltonian mechanics, Phys. Rev. D. 7 2405 (1973).
[33] J. Shiraishi, H. Kubo, H. Awata and S. Odake, A Quantum Deformation of the Virasoro Algebra and the Macdonald Symmetric Functions, Lett. Math. Phys. 38 33 (1996)[arXiv:q-alg/9507034].

[34] L. Takhtajan, On foundation of the generalized Nambu mechanics, Comm. Math. Phys. 160 295 (1994).

[35] E. Witten, Gauge theories, vertex models and quantum groups, Nucl. Phys. B 330 285 (1990).

[36] Chun-Hong Zhang, Lu Ding, Zhao-Wen Yan, Ke Wu, Wei-Zhong Zhao, 3-Algebraic structures of the quantum Calogero-Moser model, arXiv:1409.3344.