Factorization method and general second order linear difference equation

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Abstract This paper addresses an investigation on a factorization method for difference equations. It is proved that some classes of second order linear difference operators, acting in Hilbert spaces, can be factorized using a pair of mutually adjoint first order difference operators. These classes encompass equations of hypergeometric type describing classical orthogonal polynomials of a discrete variable.

1 Introduction

The description of many problems in physics and mathematics, especially in probability, gives rise to difference equations. Difference equations relate to differential equations as discrete mathematics relates to continuous mathematics. The study of differential equations shows that even supposedly elementary examples can be hard to solve. By contrast, elementary difference equations are relatively easy to deal with. In general, the interest in difference equations can be justified for a number of reasons. Difference equations frequently arise when modelling real life situations. Since difference equations are readily handled by numerical methods, a standard approach to solving a nasty differential equation is to convert it to an approximately equivalent difference equation.

A peculiar question in the field of differential or difference equations then remains to find appropriate analytical methods for their exact solvability. For differential or difference equations having polynomial solutions, it is well known that
their solvability is closely related to the factorizability of their associated operators, (see [3] and references therein).

In the last few decades it was given a more prominent place in the discussion of operator factorization methods for solving second order differential or difference equations, the concept of which goes back to Darboux [6]. Later the method was rediscovered many times, in particular by the founders of quantum mechanics, (see Dirac [7], Schrödinger [25]), while solving the Schrödinger equation to study the angular momentum or the harmonic oscillator. In the work [17], which is now considered to be fundamental, Infeld and Hull summarized the quantum mechanical applications of the method. Later this technique was extended, see [19, 20, 14]. Some results were obtained also for $q$-difference and more general difference equations [1, 2, 5, 4, 11, 12, 10, 16, 22, 23]. In addition, special cases such as the factorization of Jacobi operators were also investigated [15]. If the operator in a second order linear ordinary differential or difference equation can be factorized, the problem of solving the equation is reduced to solving two first order linear equations; the latter can readily be solved.

Therefore, a nodal point in the application of this method consists in the existence of a pair of first order differential or difference operators, which the second order differential or difference operator decomposes into as their product, (see (9) in this work).

Using this method, we are here able to find the explicit solutions (18), via (17), to the eigenvalue problem (14) in a simple way. For additional readings, see monographs [13, 18, 21].

This work is an extension of a previous work [10]. Some results obtained in [1, 8, 9, 16] are used, and adapted to our context.

The paper is organized as follows. In Section 2, a detailed investigation of the factorization method applied to second order difference operators is given. In Section 3, our main results are described. Under given assumptions, the problem of operator factorization is solved.

2 Basic tools

In this section, in the beginning, we introduce some notations and recall some basic facts about the factorization method. Let $\ell_k(\mathbb{Z}, \mathbb{R})$ and $\ell_k(\mathbb{Z}, \mathbb{C})$, $k \in \mathbb{N} \cup \{0\}$, be the sets of real-valued and complex-valued sequences $\{x(n)\}_{n \in \mathbb{Z}}$, respectively. We define the scalar product on $\ell_k(\mathbb{Z}, \mathbb{C})$ as follows:

$$\langle x|y \rangle_k := \sum_{n=a}^{b} x(n)y(n)\rho_k(n), \quad (1)$$

where $a, b \in \mathbb{Z}$, $(a < b)$, and $\rho_k$ is a weight function. We assume that the weight sequence satisfies the Pearson difference equation.
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\[ \Delta (b_k(n)\rho_k(n)) = (c_k(n) - b_k(n))\rho_k(n), \] (2)

and the recursion relation

\[ \rho_{k-1}(n) = c_k(n)\rho_k(n), \] (3)

where \( \{b_k\} \) and \( \{c_k\} \) are some real-valued sequences. Moreover, the function \( \rho_k \) fulfills the boundary conditions

\[ b_k(a)\rho_k(a) = b_k(b+1)\rho_k(b+1) = 0. \] (4)

The forward and backward difference operators are defined by

\[ \triangle x(n) := (S^+ - 1)x(n) = x(n+1) - x(n), \] (5)

\[ \triangledown x(n) := (1 - S^-)x(n) = x(n) - x(n-1), \] (6)

where the shift operators

\[ S^\pm x(n) := x(n \pm 1). \] (7)

We want to apply the factorization method to the second order difference operators \( \mathbf{H}_k : \ell_k(\mathbb{Z}, \mathbb{C}) \rightarrow \ell_k(\mathbb{Z}, \mathbb{C}) \) given by

\[ \mathbf{H}_k := z_k(n)S^+ + w_k(n)S^- + v_k(n), \] (8)

where \( \{z_k\}, \{w_k\} \) and \( \{v_k\} \) are real-valued sequences, \( k \in \mathbb{N} \cup \{0\} \). Introducing the annihilation \( \mathbf{A}_k : \ell_k(\mathbb{Z}, \mathbb{C}) \rightarrow \ell_{k-1}(\mathbb{Z}, \mathbb{C}) \), and creation operators \( \mathbf{A}_k^* : \ell_{k-1}(\mathbb{Z}, \mathbb{C}) \rightarrow \ell_k(\mathbb{Z}, \mathbb{C}) \) (also called lowering and raising operators, respectively), we rewrite the above operators \( \mathbf{H}_k \) in the form

\[ \mathbf{H}_k := \mathbf{A}_k^*\mathbf{A}_k + \alpha_k = \mathbf{A}_{k+1}\mathbf{A}_{k+1}^* + \alpha_{k+1}, \] (9)

where \( \alpha_k \) are real constants. We construct the annihilation operator as

\[ \mathbf{A}_k := \triangle + f_k(n) = S^+ + f_k(n) - 1, \] (10)

where \( \{f_k\} \in \ell_k(\mathbb{Z}, \mathbb{R}) \).

We seek the adjoint operator \( \mathbf{A}_k^* \) of \( \mathbf{A}_k \), obeying:

\[ \langle \mathbf{A}_k^*x_{k-1}|y_k \rangle_k = \langle x_{k-1}|\mathbf{A}_k y_k \rangle_{k-1}. \] (11)

A simple computation using (11) yields

\[
\langle x_{k-1}|\mathbf{A}_k y_k \rangle_{k-1} = \sum_{n=a}^b x_{k-1}(n)y_k(n+1)\rho_{k-1}(n) \\
+ \sum_{n=a}^b x_{k-1}(n)(f_k(n) - 1)y_k(n)\rho_{k-1}(n)
\]
where we applied the formulas (2)-(4). Finally, we obtain the explicit expression for the adjoint operator (also called creation operator)

\[
A^*_k = -b_k(n) \nabla + b_k(n) + (f_k(n) - 1) c_k(n) \\
= b_k(n) S^- + (f_k(n) - 1) c_k(n).
\]

(13)

This type of factorization was presented in detail in papers [16], [11], [8] for \(\tau\)-, \(q\)- and \((q,h)\)-cases, respectively. Moreover, different cases, when the sequence \(b_k\) does not depend on parameter \(k\), were considered in [10].

The operator \(H_k = A_k^* A_k + \alpha_k\) is selfadjoint on \(\ell_k(Z, C)\). Its eigenvalue equation reads:

\[
H_k x^l_k(n) = \lambda^l_k x^l_k(n). 
\]

(14)

It is well known that the factorization gives us the eigenfunctions and corresponding eigenvalues. Indeed, the eigenvalue problem for the chain of operators (9) is equivalent to the two following equations:

\[
A_k^* A_k x^l_k(n) = \left(\lambda^l_k - \alpha_k\right) x^l_k(n), \\
A_{k+1}^* A_{k+1} x^l_k(n) = \left(\lambda^l_k - \alpha_{k+1}\right) x^l_k(n).
\]

(15)  (16)

Solving the first order homogeneous linear equation

\[
A_k x^0_k(n) = 0,
\]

(17)

we observe that (15), (16) and (17) imply that the functions

\[
x^{k-p}_k(n) = A_k^* A_{k-1}^* \ldots A_{p+1}^* x^0_{p}(n)
\]

(18)

are solutions of the eigenvalue problem (14) for the eigenvalues \(\lambda^l_k = \alpha_p\).

### 3 Factorization of operators

In this section, we solve the factorization problem (9) under some assumptions. Finding a general solution remains a cumbersome task.

Comparing the coefficients of \(S^-, S^+\) and \(1\) on both sides of the expression (9), we obtain the necessary and sufficient conditions for the existence of a factorizing pair of first order difference operators, \((A_k^*, A_k)\), as follows:
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\[ f_{k+1}(n) - 1 = \frac{b_k(n)}{b_{k+1}(n)} (f_k(n-1) - 1), \tag{19} \]

\[ c_{k+1}(n) = \frac{b_{k+1}(n)}{b_k(n)} c_k(n-1), \tag{20} \]

\[ b_k(n) - b_{k+1}(n+1) = \alpha_{k+1} - \alpha_k + \frac{b_k(n)}{b_{k+1}(n)} (f_k(n-1) - 1)^2 c_k(n-1) \]

\[ - (f_k(n) - 1)^2 c_k(n). \tag{21} \]

The conditions (19) and (20) give us the transformation formulas for the sequences \( \{f_k\} \) and \( \{c_k\} \) as below:

\[ f_k(n) = \prod_{i=1}^{k} \frac{b_{k-i}(n-i+1)}{b_{k+1-i}(n-i+1)} \left( f_0(n-k) - 1 \right) + 1 \tag{22} \]

and

\[ c_k(n) = \prod_{i=1}^{k} \frac{b_{k-i+1}(n-i+1)}{b_{k-i}(n-i+1)} c_0(n-k). \tag{23} \]

### 3.1 Example 1

We assume that \( b_{k+1}(n) = b_k(n) \), i.e. the sequence \( \{b_k\} \) does not depend on parameters \( k \), see [10]. We show that, under this assumption, we can find a general solution to the factorization problem (9), i.e. we can solve the conditions (19)-(21).

We have:

\[ \begin{cases} f_{k+1}(n) = f_k(n-1) \\ c_{k+1}(n) = c_k(n-1) \end{cases} \tag{24} \]

yielding

\[ \begin{cases} f_k(n) = f_0(n-k) \\ c_k(n) = c_0(n-k). \end{cases} \tag{25} \]

Now, let us solve the third condition. The requirement (21), using the substitution \( G_k(n) = (f_k(n) - 1)^2 c_k(n) - b_0(n+1), \) is equivalent to the equation

\[ G_k(n) = G_k(n-1) + \alpha_{k+1} - \alpha_k. \tag{26} \]

By iterating we find

\[ G_k(n) = G_k(0) + n (\alpha_{k+1} - \alpha_k). \tag{27} \]

This gives us a formula for the sequence \( \{b_0\} \):

\[ b(n+1) = (f_0(n-k) - 1)^2 c_0(n-k) - G_k(0) - n (\alpha_{k+1} - \alpha_k). \tag{28} \]

But the left-hand side of the expression (28) does not depend on the parameter \( k \). Then, we obtain the following sequence of conditions on the sequences \( \{f_0\} \) and
\{(c_0) : \\
\begin{equation}
(f_0(n) - 1)^2 c_0(n) - G_0(0) - n (\alpha_1 - \alpha_0) =
(f_0(n - k) - 1)^2 c_0(n - k) - G_k(0) - n (\alpha_{k+1} - \alpha_k),
\end{equation}
\end{align*}
for all \(k \in \mathbb{N}\). By introducing \(F(n) = (f_0(n) - 1)^2 c_0(n)\), we can write the above equation in the form:
\begin{equation}
F(n) = F(n - k) + G_0(0) - G_k(0) - n (\alpha_{k+1} - \alpha_k - \alpha_1 + \alpha_0).
\end{equation}
For \(k = 1\), we get
\begin{equation}
F(n) = F(n - 1) + G_0(0) - G_1(0) - n (\alpha_2 - 2\alpha_1 + \alpha_0).
\end{equation}
Next, by iteration we find
\begin{equation}
F(n) = F(0) + n (G_0(0) - G_1(0)) - \frac{n(n+1)}{2} (\alpha_2 - 2\alpha_1 + \alpha_0).
\end{equation}
We then arrive at a relationship between the sequences \{\(f_0\)\} and \{\(c_0\)\}:
\begin{equation}
(f_0(n) - 1)^2 c_0(n) = F(0) + n (G_0(0) - G_1(0)) - \frac{n(n+1)}{2} (\alpha_2 - 2\alpha_1 + \alpha_0).
\end{equation}
In addition, substituting (32) to (29) (because it is valid for all \(k \in \mathbb{N}\)), we find a recurrence relation on the constants \(\alpha_k\):
\begin{equation}
\alpha_{k+1} = \alpha_k + \alpha_1 - \alpha_0 + k (\alpha_2 - 2\alpha_1 + \alpha_0)
\end{equation}
and the form of the constant
\begin{equation}
G_k(0) = G_0(0) + k (G_1(0) - G_0(0)) - \frac{k(k-1)}{2} (\alpha_2 - 2\alpha_1 + \alpha_0).
\end{equation}
A straightforward calculation affords
\begin{equation}
\alpha_k = \alpha_0 + k (\alpha_1 - \alpha_0) + \frac{k(k-1)}{2} (\alpha_2 - 2\alpha_1 + \alpha_0).
\end{equation}
To sum up, the construction presented in (9) provides the chain of operators \(H_k\) parametrized by the freely chosen sequence \{\(c_0\)\} and real parameters \(\alpha_0, \alpha_1, \alpha_2, F(0), G_0(0)\) and \(G_1(0)\).

3.2 Example 2

Let us consider a case when \(f_k(n) \equiv 0\). Then, the conditions (19)-(21) can be rewritten in the form:
\begin{align}
b_{k+1}(n) &= b_k(n) = b_0(n), \\
c_k(n) &= c_0(n-k), \\
b_0(n) - b_0(n+1) &= \alpha_{k+1} - \alpha_k + c_0(n-k-1) - c_0(n-k). 
\end{align}

This is a special case of Example 1. We find \( b_0(n) \) by induction in the following form:

\begin{equation}
b_0(n) = b_0(0) - n(\alpha_{k+1} - \alpha_k) - c_0(-1-k) + c_0(n-1-k).
\end{equation}

From (33) we obtain that the sequence \{c_0\} is a polynomial of degree two:

\begin{equation}
c_0(n) = c_0(0) + n(G_0(0) - G_1(0)) - \frac{n(n+1)}{2} (\alpha_2 - 2\alpha_1 + \alpha_0).
\end{equation}

Then, the relation \( H_k x_k^l(n) = \lambda_k^l x_k^l(n) \) is equivalent to

\begin{equation}
\left( - b_0(n) \nabla + b_0(n) - c_0(n-k) \right) \Delta x_k^l(n) = (\lambda_k^l - \alpha_k) x_k^l(n),
\end{equation}

and the eigenvalue problem (14) is reduced to the difference equation of hypergeometric type:

\begin{equation}
-b_0(n) \nabla \triangle x_k^l(n) - (c_0(n-k) - b_0(n)) \triangle x_k^l(n) + \left( \alpha_k - \lambda_k^l \right) x_k^l(n) = 0.
\end{equation}

It is not difficult to see that \( b_0 \) is a second degree polynomial while the difference \( c_0(n-k) - b_0(n) \) is a first degree polynomial. From equation (17), we find that the ground state is a constant sequence \( \{x_0^0(n) = 1\} \) (normalized to one) with the sequence of eigenvalues \( \{\lambda_k^0 = \alpha_k = \alpha_0 + k(\alpha_1 - \alpha_0) + \frac{k(k-1)}{2}(\alpha_2 - 2\alpha_1 + \alpha_0)\} \). Expression (18) gives us a formula for polynomials

\begin{equation}
P_l(n) = x_k^l(n) = \prod_{i=0}^{l-1} \left( b_0(n) - c_0(n-k+i) \right)
\end{equation}

corresponding to eigenvalues \( \lambda_k^l = \alpha_{k-l} \). Using the identity \( \nabla \triangle = \triangle \nabla \) we transform the above equation (33) into the standard form

\begin{equation}
\sigma(n) \nabla \triangle x_k^l(n) + \tau(n) \triangle x_k^l(n) + \lambda_k^l x_k^l(n) = 0,
\end{equation}

where
\[\sigma(n) = -b_0(n) = \frac{1}{2} (\alpha_2 - 2\alpha_1 + \alpha_0) n^2 + n (G_1(0) - G_0(0)) + \alpha_1 - \alpha_0\]
\[= \frac{1}{2} (\alpha_2 - 2\alpha_1 + \alpha_0) - b_0(0),\] (46)

\[\tau(n) = b_0(n) - c_0(n-k) = (\alpha_0 - \alpha_1 + (1-k) (\alpha_2 - 2\alpha_1 + \alpha_0)) n\]
\[+ b_0(0) - c_0(0) + k (G_0(0) - G_1(0)) + \frac{k(k-1)}{2} (\alpha_2 - 2\alpha_1 + \alpha_0),\] (47)

\[\lambda = \alpha_k - \lambda' = \alpha_k - \lambda_{k-l} = l (\alpha_k - \alpha_0)\]
\[+ \left( kl - \frac{l(l+1)}{2} \right) (\alpha_2 - 2\alpha_1 + \alpha_0) = -l \left( \tau'(n) + \frac{l-1}{2} \sigma''(n) \right).\] (48)

It is well known that the above equation describes classical orthogonal polynomials of a discrete variable such as the Charlier, Meixner, Kravchuk, Hahn polynomials. See [24, 2] for more details.

**3.3 Example 3**

We assume that \(b_{k+1}(n) := \gamma_k b_k(n)\), where \(\gamma_k\) is some constant different from zero and one. Then, we get:

\[f_k(n) - 1 = \prod_{i=1}^k \gamma_{k-i} (f_0(n-k) - 1),\] (49)

\[c_k(n) = \prod_{i=1}^k \gamma_{k-i} c_0(n-k),\] (50)

and

\[b_k(n) - \gamma_k b_k(n+1) = \alpha_{k+1} - \alpha_k + \gamma_k^{-1} (f_k(n-1) - 1)^2 c_k(n-1)\]
\[- (f_k(n) - 1)^2 c_k(n).\] (51)

Using previous results, (see Example 1), we solve by analogy the above difference equation. By introducing \(R_k(n) = (f_k(n) - 1)^2 c_k(n) - \gamma_k b_k(n+1)\), we can write this equation in the form

\[R_k(n) = \gamma_k^{-1} R_k(n-1) + \alpha_{k+1} - \alpha_k.\] (52)

By iterating we find

\[R_k(n) = \gamma_k^{-n} R_k(0) + \frac{1 - \gamma_k^{-n}}{1 - \gamma_k^{-1}} (\alpha_{k+1} - \alpha_k).\] (53)

From here, expressing everything by the initial data we obtain
By iteration,

\[ b_k(n+1) = \gamma_k^{-1} \gamma_{k-1}^{-1} \cdots \gamma_0^{-1} (f_0(n-k) - 1)^2 c_0(n-k) - \gamma_k^{-n-1} R_k(0) \]

\[ - \gamma_k^{-1} \frac{1 - \gamma_k^{-n}}{1 - \gamma_k^{-1}} (\alpha_{k+1} - \alpha_k). \]  

(54)

This must be consistent with the initial assumption, i.e. \( b_{k+1}(n) := \gamma_k b_k(n) \). Then, we get the condition on the sequences \( \{f_0\} \) and \( \{c_0\} \):

\[ \gamma_{k+1}^{-1} \gamma_{k+2}^{-1} \cdots \gamma_0^{-1} (f_0(n-k-1) - 1)^2 c_0(n-k-1) - \gamma_{k+1}^{-n-1} R_{k+1}(0) \]

\[ - \gamma_{k+1}^{-1} \frac{1 - \gamma_{k+1}^{-n}}{1 - \gamma_{k+1}^{-1}} (\alpha_{k+2} - \alpha_{k+1}) = \gamma_k^{-1} \cdots \gamma_0^{-1} (f_0(n-k) - 1)^2 c_0(n-k) - \gamma_k^{-n} R_k(0) \]

\[ - \frac{1 - \gamma_k^{-n}}{1 - \gamma_k^{-1}} (\alpha_{k+1} - \alpha_k). \]  

(55)

Again, by entering the following auxiliary function:

\[ S_k(n) = \gamma_{k-1}^{-1} \cdots \gamma_0^{-1} (f_0(n-k) - 1)^2 c_0(n-k), \]

we get a recursion relation for \( S_k \):

\[ S_k(n) = \gamma_{k+1}^{-1} \gamma_k^{-1} S_k(n-1) - \gamma_k^{-n-1} R_{k+1}(0) + \gamma_k^{-n} R_k(0) \]

\[ - \gamma_{k+1}^{-1} \frac{1 - \gamma_{k+1}^{-n}}{1 - \gamma_{k+1}^{-1}} (\alpha_{k+2} - \alpha_{k+1}) + \frac{1 - \gamma_k^{-n}}{1 - \gamma_k^{-1}} (\alpha_{k+1} - \alpha_k). \]  

(56)

By iteration,

\[ S_k(n) = \left( \gamma_{k+1}^{-1} \gamma_k^{-1} \right)^n S_k(0) - \gamma_k^{-n-1} \frac{1 - \gamma_k^{-n}}{1 - \gamma_k^{-1}} R_{k+1}(0) + \gamma_k^{-n} \frac{1 - \gamma_{k+1}^{-n}}{1 - \gamma_{k+1}^{-1}} R_k(0) \]

\[ - \gamma_{k+1}^{-1} (\alpha_{k+2} - \alpha_{k+1}) \sum_{i=0}^{n-1} \left( \gamma_{k+1}^{-1} \gamma_k^{-1} \right)^i \frac{1 - \gamma_{k+1}^{-n+i}}{1 - \gamma_{k+1}^{-1}} \]

\[ + (\alpha_{k+1} - \alpha_k) \sum_{i=0}^{n-1} \left( \gamma_{k+1}^{-1} \gamma_k^{-1} \right)^i \frac{1 - \gamma_{k+1}^{-n+i}}{1 - \gamma_k^{-1}}. \]  

(57)

4 Concluding remarks

In this work, we have investigated a factorization method for difference equations, adapting and extending previous results known in the literature. We have showed that some classes of second order linear difference operators, acting in Hilbert spaces, are factorizable using a pair of mutually adjoint first order difference operators. These classes encompass equations of hypergeometric type describing classical orthogonal polynomials of a discrete variable. Other classes of difference equations are still under consideration, and will be in the core of our forthcoming papers.
An interesting outlook on which we are also working is the extension of this scheme to classes of higher order difference equations. It is in particular expected that this method for fourth order equations may allow to derive what one can call Krall–Laguerre–Hahn polynomials.

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References

1. Álvarez-Nodarse, R., Atakishiyev, N.M., Costas-Santos, R.S.: Factorization of the hypergeometric-type difference equation on non-uniform lattices: dynamical algebra. J. Phys. A: Math. Gen. 38, 153–174 (2005)
2. Álvarez-Nodarse, R., Atakishiyev, N.M., Costas-Santos, R.S.: Factorization of the hypergeometric-type difference equation on the uniform lattice. ETNA Electronic Transactions on Numerical Analysis, 27, 34–50 (2007)
3. Bangerezako, G., Hounkonnou, M.N.: The transformation of polynomial eigenfunctions of linear second order difference operators: a special case of Meixner polynomials. J. Phys. A: Math. Gen. 34, 1–14 (2001)
4. Bangerezako, G., Hounkonnou, M.N.: The transformation of polynomial eigenfunctions of linear second-order q-difference operators: a special case of q-Jacobi polynomials. in Contemporary Problems in Mathematical Physics, Proceedings of the Second International Workshop, World Scientific Publishing, Vol.2, 427–439 (2002)
5. Bangerezako, G., Hounkonnou, M.N.: The Factorization method for the general second order q-difference equation and the Laguerre-Hahn polynomials on the general q-lattice. J. Phys. A: Math. Gen. 36, 765–773 (2003)
6. Darboux, G.: Sur une proposition relative aux equations lineaires. C. R. Acad. Sci. Paris 94, 1456–1459 (1882)
7. Dirac, P.A.M.: The Principles of Quantum Mechanics. Oxford: Clarendon Press (1947)
8. Dobrogowska, A., Filipuk, G.: Factorization method applied to second-order \((q,h)\)-difference operators. Int. J. Difference Equ. 11, no.1, 3–17 (2016).
9. Dobrogowska, A., Jakimowicz, G.: Factorization method for \((q,h)\)-Hahn orthogonal polynomials. Geometric Methods in Physics, Part of the series Trends in Mathematics, Springer International Publishing Switzerland, Birkhäuser Basel, 237–246 (2015)
10. Dobrogowska, A., Jakimowicz, G.: Factorization method applied to the second order difference equations. Appl. Math. Lett. 74, 161–166 (2017)
11. Dobrogowska, A., Odzijewicz, A.: Second order \(q\)-difference equations solvable by factorization method. J. Comput. Appl. Math. 193, no.1, 319–346 (2006)
12. Dobrogowska, A., Odzijewicz, A.: Solutions of the \(q\)-deformed Schrödinger equation for special potentials. J. Phys. A: Math. Theor. 40, no.9, 2023–2036 (2007)
13. Dong, S.-H.: Factorization Method in Quantum Mechanics. Kluwer Academic Press, Springer (2007)
14. Fernández, D.J.: New hydrogen-like potentials. Lett. Math. Phys. 8, 337–343 (1984)
15. Gesztesy, F., Teschl, G.: Commutation methods for Jacobi operators. J. Differential Equations 128, 252–299 (1996)
16. Goliński, T., Odzijewicz, A.: Factorization method for second order functional equations. J. Comput. Appl. Math. 176 (2), 331–355 (2005)
17. Infeld, L., Hull, T.E.: The Factorization Method. Rev. Mod. Phys. 23, 21–68 (1951)
18. de Lange, O.L., Raab, R.E.: Operator Methods in Quantum Mechanics. Claredon Press - Oxford (1991)
19. Mielnik, B.: Factorization method and new potentials with the oscillator spectrum. J. Math. Phys. 25, 3387 (1984)
20. Mielnik, B., Nieto, L.M., Rosas-Ortiz, O.: The finite difference algorithm for higher order supersymmetry. Phys. Lett. A 269(2), 70–78 (2000)
21. Mielnik, B., Rosas-Ortiz, O.: Factorization: little or great algorithm?. J. Phys. A: Math. Gen. 37, 10007 (2004)
22. Miller, W., Jr.: Lie theory and difference equations. I, J. Math. Anal. Appl. 28, 383–399 (1969)
23. Miller, W., Jr.: Lie theory and q-difference equations. SIAM J. Math. Anal. 1 (2), 171–188 (1970)
24. Nikiforov, A.F., Suslov, S.K., Uvarov, V.B.: Classical orthogonal polynomials of a discrete variable. Springer Series in Computational Physics, Springer-Verlag, Berlin (1991)
25. Schrödinger, E., A method of determining quantum-mechanical eigenvalues and eigenfunctions. Proc. Roy Irish Acad. Sect. A 46, 9–16 (1940)