Merging the $A$- and $Q$-spectral theories

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Abstract

Let $G$ be a graph with adjacency matrix $A(G)$, and let $D(G)$ be the diagonal matrix of the degrees of $G$. The signless Laplacian $Q(G)$ of $G$ is defined as $Q(G) := A(G) + D(G)$.

Cvetković called the study of the adjacency matrix the $A$-spectral theory, and the study of the signless Laplacian—the $Q$-spectral theory. During the years many similarities and differences between these two theories have been established. To track the gradual change of $A(G)$ into $Q(G)$ in this paper it is suggested to study the convex linear combinations $A_\alpha(G)$ of $A(G)$ and $D(G)$ defined by

$$A_\alpha(G) := \alpha D(G) + (1-\alpha) A(G), \quad 0 \leq \alpha \leq 1.$$ 

This study sheds new light on $A(G)$ and $Q(G)$, and yields some surprises, in particular, a novel spectral Turán theorem. A number of challenging open problems are discussed.

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Keywords: signless Laplacian; adjacency matrix; spectral radius; spectral extremal problems; spectral Turán theorem.

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1 Introduction

Let $G$ be a graph with adjacency matrix $A(G)$, and let $D(G)$ be the diagonal matrix of the degrees of $G$. In this paper we study hybrids of $A(G)$ and $D(G)$ similar to the signless Laplacian $Q(G) := A(G) + D(G)$, put forth by Cvetković in [5] and extensively studied since then. For extensive coverage see [7], [8], [9], [4], and their references). The research on $Q(G)$ has shown that it is a remarkable matrix, unique in many respects. Yet, $Q(G)$ is just the sum of $A(G)$ and $D(G)$, and the study of $Q(G)$ has uncovered both similarities and differences between $Q(G)$ and $A(G)$. To understand to what extent each of the summands $A(G)$ and $D(G)$ determines the properties of $Q(G)$, we propose to study the convex linear combinations $A_\alpha(G)$ of $A(G)$ and $D(G)$ defined by

$$A_\alpha(G) := \alpha D(G) + (1 - \alpha) A(G), \quad 0 \leq \alpha \leq 1.$$  

(1)

Many facts suggest that the study of the family $A_\alpha(G)$ is long due. To begin with, obviously,

$$A(G) = A_0(G), \quad D(G) = A_1(G), \quad \text{and} \quad Q(G) = 2A_{1/2}(G).$$

Since $A_{1/2}(G)$ is essentially equivalent to $Q(G)$, in this paper we take $A_{1/2}(G)$ as an exact substitute for $Q(G)$. With this caveat, one sees that $A_\alpha(G)$ seamlessly joins $A(G)$ to $D(G)$, with $Q(G)$ being right in the middle of the range; hence, we can study the gradual changes of $A_\alpha(G)$, from $A(G)$ to $D(G)$. In this setup, the matrices $A(G)$, $Q(G)$, and $D(G)$ can be seen in a new light, and many interesting problems arise. In particular, we are compelled to investigate the hitherto uncharted territory $\alpha > 1/2$, which holds some surprises, e.g., a novel version of the spectral Turán theorem (Theorem 27 below).

Let us note the crucial identity

$$A_\alpha(G) - A_\beta(G) = (\alpha - \beta) L(G),$$

(2)

where $L(G)$ is the well-studied Laplacian of $G$, defined as $L(G) := D(G) - A(G)$. This neat relation corroborates the soundness of the definition (1).

It is worth pointing out that the family $A_\alpha(G)$ is just a small subset of the generalized adjacency matrices defined in [10] and the universal adjacency matrices defined in [16]. However, our restricted definition allows to prove stronger theorems, which are likely to fail for these more general classes.

The rest of the paper is structured as follows. In the next section we introduce some notation and recall basic facts about spectra of matrices. In Section 3 we present a few general results about the matrices $A_\alpha(G)$. Section 4 deals with the largest eigenvalue of $A_\alpha(G)$. Section 5 is dedicated to spectral extremal problems, which are at the heart of spectral graph theory. A number of topics are covered in Section 6. Finally, in Section 7, we present the $A_\alpha$-spectra of the complete graphs and the complete bipartite graphs.
2 Notation and preliminaries

Let \([n] := \{1, \ldots, n\}\). Given a real symmetric matrix \(M\), write \(\lambda_k(M)\) for the \(k\)th largest eigenvalue of \(M\). For short, we write \(\lambda(M)\) and \(\lambda_{\min}(M)\) for the largest and the smallest eigenvalues of \(M\).

Given a graph \(G\), we write:
- \(V(G)\) and \(E(G)\) for the sets of vertices and edges of \(G\), and \(v(G)\) for \(|V(G)|\);
- \(\Gamma_G(u)\) for the set of neighbors of a vertex \(u\), and \(d_G(u)\) for \(|\Gamma_G(u)|\) (the subscript \(G\) will be omitted if \(G\) is understood);
- \(\delta(G)\) and \(\Delta(G)\) for the minimum and maximum degree of \(G\);
- \(w_G(u)\) for the number of walks of length 2 starting with the vertex \(u\), i.e., \(w_G(u) = \sum_{\{v,v\} \in E(G)} d_G(v)\);
- \(G[X]\) for the subgraph of \(G\) induced by a set \(X \subset V(G)\);
- \(G - X\) for the graph obtained by deleting the vertices of a set \(X \subset V(G)\).

A coclique of \(G\) is an edgeless induced subgraph of \(G\). Further, \(K_n\) stands for the complete graph of order \(n\), and \(K_{a,b}\) stands for the complete bipartite graph with partition sets of sizes \(a\) and \(b\). In particular, \(K_{1,n-1}\) denotes the star of order \(n\). We write \(S_{n,k}\) for the graph obtained by joining each vertex of a complete graph of order \(k\) to each vertex of an independent set of order \(n-k\), that is to say, \(S_{n,k} = K_k \lor K_{n-k}\).

On many occasions we shall use Weyl’s inequalities for eigenvalues of Hermitian matrices (see, e.g. [19], p. 181). Although these fundamental inequalities have been known for almost a century, it seems that their equality case was first established by So in [29], and his work was inspired by the paper of Ikebe, Inagaki and Miyamoto [20].

For convenience we state below the complete theorem of Weyl and So:

**Theorem WS** Let \(A\) and \(B\) be Hermitian matrices of order \(n\), and let \(1 \leq i \leq n\) and \(1 \leq j \leq n\). Then

\[
\lambda_i(A) + \lambda_j(B) \leq \lambda_{i+j-n}(A+B), \text{ if } i + j \geq n + 1, \tag{3}
\]

\[
\lambda_i(A) + \lambda_j(B) \geq \lambda_{i+j-1}(A+B), \text{ if } i + j \leq n + 1. \tag{4}
\]

In either of these inequalities equality holds if and only if there exists a nonzero \(n\)-vector that is an eigenvector to each of the three eigenvalues involved.

A simplified version of (3) and (4) gives

\[
\lambda_k(A) + \lambda_{\min}(B) \leq \lambda_k(A+B) \leq \lambda_k(A) + \lambda(B). \tag{5}
\]

We shall need the following simple properties of the Laplacian:

**Proposition L** If \(G\) is a graph of order \(n\), then

\[
\lambda(L(G)) \leq n \quad \text{and} \quad \lambda_{\min}(L(G)) = 0.
\]
If $G$ is connected, then every eigenvector of $L(G)$ to the eigenvalue 0 is constant.

Recall that a real symmetric matrix $M$ is called positive semidefinite if $\lambda_{\text{min}}(M) \geq 0$. Likewise $M$ is called positive definite if $\lambda_{\text{min}}(M) > 0$.

3 Basic properties of $A_\alpha(G)$

Given a graph $G$ of order $n$, it is obvious that the system of eigenequations for the matrix $A_\alpha(G)$ is

$$\lambda x_k = ad_G(k) x_k + (1 - \alpha) \sum_{\{i,k\} \in E(G)} x_i, \quad 1 < k \leq n. \quad (6)$$

3.1 The quadratic form $\langle A_\alpha x, x \rangle$

If $G$ is a graph of order $n$ with $A_\alpha(G) = A_\alpha$, and $x := (x_1, \ldots, x_n)$ is a real vector, the quadratic form $\langle A_\alpha x, x \rangle$ can be represented in several equivalent ways, for example,

$$\langle A_\alpha x, x \rangle = \sum_{\{u,v\} \in E(G)} (a x_u^2 + 2 (1 - \alpha) x_u x_v + \alpha x_v^2), \quad (7)$$

$$\langle A_\alpha x, x \rangle = (2\alpha - 1) \sum_{u \in V(G)} x_u^2 d(u) + (1 - \alpha) \sum_{\{u,v\} \in E(G)} (x_u + x_v)^2, \quad (8)$$

$$\langle A_\alpha x, x \rangle = \alpha \sum_{u \in V(G)} x_u^2 d(u) + 2 (1 - \alpha) \sum_{\{u,v\} \in E(G)} x_u x_v. \quad (9)$$

Each of these representations can be useful in proofs.

Since $A_\alpha(G)$ is a real symmetric matrix, Rayleigh’s principle implies that

Proposition 1 If $\alpha \in [0,1]$ and $G$ is a graph of order $n$ with $A_\alpha(G) = A_\alpha$, then

$$\lambda(A_\alpha) = \max_{\|x\|_2=1} \langle A_\alpha x, x \rangle \quad \text{and} \quad \lambda_{\text{min}}(A_\alpha) = \min_{\|x\|_2=1} \langle A_\alpha x, x \rangle. \quad (10)$$

Moreover, if $x$ is a unit $n$-vector, then $\lambda(A_\alpha) = \langle A_\alpha x, x \rangle$ if and only if $x$ is an eigenvector to $\lambda(A_\alpha)$, and $\lambda_{\text{min}}(A_\alpha) = \langle A_\alpha x, x \rangle$ if and only if $x$ is an eigenvector to $\lambda_{\text{min}}(A_\alpha)$.

These relations yield the following familiar relations:

Proposition 2 If $\alpha \in [0,1]$ and $G$ is a graph with $A_\alpha(G) = A_\alpha$, then

$$\lambda(A_\alpha) = \max \{\lambda(A_\alpha(H)) : H \text{ is a component of } G\},$$

$$\lambda_{\text{min}}(A_\alpha) = \min \{\lambda_{\text{min}}(A_\alpha(H)) : H \text{ is a component of } G\}.$$
Caution: If \( G \) is disconnected, \( \lambda (A_\alpha) \) can be attained on different components of \( G \), depending on \( \alpha \). For example, let \( k \geq 2 \) be an integer and let \( G \) be the disjoint union of \( K_{3k+1,3k+1} \), \( K_{3,3k^2} \), and \( K_{1,3k^2+1} \). Calculating the largest eigenvalues of \( A_0 \), \( A_{1/2} \), and \( A_1 \) for each of the three components of \( G \), we get the following table:

|                | \( K_{3k+1,3k+1} \) | \( K_{3,3k^2} \) | \( K_{1,3k^2+1} \) | \( G \) |
|----------------|----------------------|------------------|-------------------|-------|
| \( \lambda (A_0) \) | \( 3k + 1 \)         | \( 3k \)         | \( \sqrt{3k^2 + 1} \) | \( 3k + 1 \) |
| \( \lambda (A_{1/2}) \) | \( 3k + 1 \)         | \( (3k^2 + 3) / 2 \) | \( (3k^2 + 1) / 2 \) | \( (3k^2 + 3) / 2 \) |
| \( \lambda (A_1) \) | \( 3k + 1 \)         | \( 3k^2 \)       | \( 3k^2 + 1 \)     | \( 3k^2 + 1 \) |

Hence \( \lambda (A_\alpha (G)) \) may be attained on each of the components of \( G \), depending on \( \alpha \).

3.2 Monotonicity of \( \lambda_k (A_\alpha (G)) \) in \( \alpha \)

In this subsection we shall show that \( \lambda_k (A_\alpha (G)) \) is nondecreasing in \( \alpha \) for any \( k \). For a start note that if \( G \) is a \( d \)-regular graph of order \( n \), then

\[
A_\alpha (G) = adI_n + (1 - \alpha)A (G),
\]

and so there is a linear correspondence between the spectra of \( A_\alpha (G) \) and of \( A (G) \)

\[
\lambda_k (A_\alpha (G)) = ad + (1 - \alpha)\lambda_k (A (G)), \quad 1 \leq k \leq n. \quad (11)
\]

In particular, if \( G \) is a \( d \)-regular graph, then \( \lambda (A_\alpha (G)) = d \) for any \( \alpha \in [0, 1] \). Moreover, if \( G \) is regular and connected graph of order \( n \), equations (11) imply that \( \lambda_k (A (G)) \) is increasing in \( \alpha \) for any \( 2 \leq k \leq n \). It turns out that the latter property is essentially valid for any graph:

**Proposition 3** Let \( 1 \geq \alpha > \beta \geq 0 \). If \( G \) is a graph of order \( n \) with \( A_\alpha (G) = A_\alpha \) and \( A_\beta (G) = A_\beta \), then

\[
\lambda_k (A_\alpha) - \lambda_k (A_\beta) \geq 0 \quad (12)
\]

for any \( k \in [n] \). If \( G \) is connected, then inequality (12) is strict, unless \( k = 1 \) and \( G \) is regular.

**Proof** Identity (2), inequality (5), and Proposition L imply that

\[
\lambda_k (A_\alpha) - \lambda_k (A_\beta) \geq (\alpha - \beta)\lambda_{\min} (L (G)) = 0. \quad (13)
\]

If \( G \) is connected and equality holds in (13), Theorem WS implies that \( \lambda_k (A_\beta), \lambda_k (A_\alpha), \) and \( \lambda_{\min} (L (G)) \) have a common eigenvector, which by Proposition L must be constant, say the all-ones vector \( j_n \). Now, Proposition 13 implies that \( k = 1 \), and the eigenequations (6) imply that \( G \) is regular. 

With the premises of Proposition 3, note also that

\[
\lambda_k (A_\alpha) - \lambda_k (A_\beta) \leq (\alpha - \beta)n,
\]

and so we arrive at:

\[
\lambda (A_\alpha (G)) \quad \text{may be attained on each of the components of } G, \text{ depending on } \alpha.
\]
Proposition 4 If $G$ is a graph, with $A_\alpha (G) = A_\alpha$, then the function $\lambda_k (A_\alpha)$ is Lipschitz continuous in $\alpha$ for any $k \in [n]$. Furthermore, $\lambda (A_\alpha)$ is convex in $\alpha$, and $\lambda_{\min} (A_\alpha)$ is concave in $\alpha$.

Let us note that the convexity of $\lambda (A_\alpha)$ and the concavity of $\lambda_{\min} (A_\alpha)$ follow from inequalities (5).

Question 5 If $n \geq k \geq 1$, is $f (\alpha) = \lambda_k (A_\alpha)$ differentiable in $\alpha$?

3.3 Positive semidefiniteness of $A_\alpha$

An important property of the signless Laplacian $Q (G)$ is that it is positive semidefinite. This is certainly not true for $A_\alpha (G)$ if $\alpha$ is sufficiently small, but if $\alpha \geq 1/2$, then $A_\alpha (G)$ is similar to $Q (G)$:

Proposition 6 If $\alpha > 1/2$, and $G$ is a graph, then $A_\alpha (G)$ is positive semidefinite. If $G$ has no isolated vertices, then $A_\alpha (G)$ is positive definite.

Proof Let $x := (x_1, \ldots, x_n)$ be a nonzero vector. If $\alpha > 1/2$, then for any edge $\{u, v\} \in E$, we see that

$$\langle A_\alpha (G) x, x \rangle \geq (1 - \alpha) (x_u + x_v)^2 + (2\alpha - 1) x_u^2 + (2\alpha - 1) x_v^2 \geq 0.$$  \hspace{1cm} (14)

Hence $A_\alpha (G)$ is positive semidefinite. Now, suppose that $G$ has no isolated vertices. Select a vertex $u$ with $x_u \neq 0$ and let $\{u, v\} \in E$. Then we have strict inequality in (14) and so $A_\alpha (G)$ is positive definite.

Obviously Proposition 3 implies that if $A_\alpha (G)$ is positive (semi)definite for some $\alpha$, then $A_\beta (G)$ is positive (semi)definite for any $\beta > \alpha$. This observation leads to the following problem:

Problem 7 Given a graph $G$, find the smallest $\alpha$ for which $A_\alpha (G)$ is positive semidefinite.

For example, if $G$ is the complete graph $K_n$, we have $\lambda_{\min} (A_\alpha (K_n)) = n\alpha - 1$, and so $A_\alpha (K_n)$ is positive semidefinite if and only if $\alpha \geq 1/n$. This example can be generalized as follows:

Proposition 8 Let $G$ be a regular graph with chromatic number $r$. If $\alpha < 1/r$, then $A_\alpha (G)$ is not positive semidefinite.

Proof Let $G$ be a $d$-regular graph and let $A$ be its adjacency matrix. Hoffman’s bound [18] implies that

$$\lambda_{\min} (A) \leq -\lambda (A) \frac{d}{r - 1} = -\frac{d}{r - 1}.$$  \hspace{1cm} (11)

Hence, (11) implies that

$$\lambda_{\min} (A_\alpha (G)) \leq \alpha d - (1 - \alpha) \frac{d}{r - 1} = (\alpha - 1) \frac{r d}{r - 1} < 0,$$

completing the proof.
3.4 Some degree based bounds

It is not an exaggeration to say that degree bounds are the most used bounds in spectral graph theory. We give a few such bounds for $A_{\alpha} (G)$, the first of which follows from Proposition 3.

**Proposition 9** Let $G$ is a graph of order $n$ with degrees $d (1) \geq \cdots \geq d (n)$ and with $A_{\alpha} (G) = A_{\alpha}$. If $k \in [n]$, then

$$\lambda_k (A_{\alpha}) \leq d (k).$$

In particular, $\lambda (A_{\alpha}) \leq \Delta (G)$.

Using an idea of Das [11], the bound $\lambda_{\min} (A_{\alpha}) \leq \delta (G)$ can be improved further: let $u$ be a vertex with minimum degree and define the $n$-vector $x := (x_1, \ldots, x_n)$ by letting $x_u := 1$ and zeroing the other entries. Then Proposition 1 and equation (8) imply that

$$\lambda_{\min} (A_{\alpha}) \leq \langle A_{\alpha} x, x \rangle = (2 \alpha - 1) \delta + (1 - \alpha) \delta = \alpha \delta.$$

But for $\alpha \in [0, 1)$ the vector $x$ does not satisfy the eigenequations for $\lambda_{\min} (A_{\alpha})$, so in this case

$$\lambda_{\min} (A_{\alpha}) < \alpha \delta.$$

Further, Weyl’s inequality (5) immediately implies the following bounds:

**Proposition 10** If $\alpha \in [0, 1]$ and $G$ is a graph with $A (G) = A$ and $A_{\alpha} (G) = A_{\alpha}$, then

$$\alpha \delta + (1 - \alpha) \lambda_k (A) \leq \lambda_k (A_{\alpha}) \leq \alpha \Delta + (1 - \alpha) \lambda_k (A).$$

For $\lambda (A_{\alpha})$ we give a tight lower bound, which generalizes a result of Lovász ([23], Problem 11.14):

**Proposition 11** If $G$ is a graph with $\Delta (G) = \Delta$, then

$$\lambda (A_{\alpha}) \geq \frac{1}{2} \left( \alpha (\Delta + 1) + \sqrt{\alpha^2 (\Delta + 1)^2 + 4 \Delta (1 - 2 \alpha)} \right).$$

If $G$ is connected, equality holds if and only if $G = K_{1, \Delta}$.

**Proof** Proposition 38 gives the spectral radius of $A_{\alpha}$ of a star. This result, combined with Proposition 13, yields

$$\lambda (A_{\alpha} (G)) \geq \lambda (A_{\alpha} (K_{1, \Delta})) = \frac{1}{2} \left( \alpha (\Delta + 1) + \sqrt{\alpha^2 (\Delta + 1)^2 + 4 \Delta (1 - 2 \alpha)} \right).$$

The case of equality also follows from Proposition 13. □

Some algebra can be used to prove a simpler lower bound:

**Corollary 12** Let $G$ be a graph with $\Delta (G) = \Delta$. If $\alpha \in [0, 1/2]$, then

$$\lambda (A_{\alpha} (G)) \geq \alpha (\Delta + 1).$$

If $\alpha \in [1/2, 1]$, then

$$\lambda (A_{\alpha} (G)) \geq a \Delta + 1 - \alpha.$$

8
4 The largest eigenvalue $\lambda (A_\alpha (G))$

As for the adjacency matrix and the signless Laplacian, the spectral radius $\lambda (A_\alpha (G))$ of $A_\alpha (G)$ is its most important eigenvalue, due to the fact that $A_\alpha (G)$ is nonnegative and so $\lambda (A_\alpha (G))$ has maximal modulus among all eigenvalues of $A_\alpha (G)$.

4.1 Perron-Frobenius properties of $A_\alpha (G)$

In this subsection we spell out the properties of $\lambda (A_\alpha (G))$, which follow from the Perron-Frobenius theory of nonnegative matrices. Observe that if $0 \leq \alpha < 1$ and $G$ is a graph, then $G$ is connected if and only if $A_\alpha (G)$ is irreducible, because irreducibility is not affected by the diagonal entries of $A_\alpha (G)$. Hence, the Perron-Frobenius theory of nonnegative matrices implies the following properties of $A_\alpha (G)$:

**Proposition 13** Let $\alpha \in [0, 1)$, let $G$ be a graph, and let $x$ be a nonnegative eigenvector to $\lambda (A_\alpha (G))$:

(a) If $G$ is connected, then $x$ is positive and is unique up to scaling.

(b) If $G$ is not connected and $P$ is the set of vertices with positive entries in $x$, then the subgraph induced by $P$ is a union of components $H$ of $G$ with $\lambda (A_\alpha (H)) = \lambda (A_\alpha (G))$.

(c) If $G$ is connected and $\mu$ is an eigenvalue of $A_\alpha (G)$ with a nonnegative eigenvector, then $\mu = \lambda (A_\alpha (G))$.

(d) If $G$ is connected, and $H$ is a proper subgraph of $G$, then $\lambda (A_\alpha (H)) < \lambda (A_\alpha (G))$ for any $\alpha \in [0, 1)$.

A useful corollary can be deduced for the join of two regular graphs:

**Proposition 14** Let $G_1$ be a $r_1$-regular graph of order $n_1$, and $G_2$ be a $r_2$-regular graph of order $n_2$. Then

$$\lambda (A_\alpha (G_1 \lor G_2)) = \lambda \left( \frac{r_1 + \alpha n_2}{1 + (1 - \alpha)^2 n_1 n_2} \right) \frac{r_2 + \alpha n_1}{1}$$

In turn, Proposition 14 can be extended to equitable partitions.

Another practical consequence of Proposition 13 reads as:

**Proposition 15** Let $\alpha \in [0, 1)$ and let $G$ be a graph with $A_\alpha (G) = A_\alpha$. Let $u, v, w \in V(G)$ and suppose that $\{u, v\} \in E(G)$ and $\{u, w\} \notin E(G)$. Let $H$ be the graph obtained from $G$ by deleting the edge $\{u, v\}$ and adding the edge $\{u, w\}$. If $x := (x_1, \ldots, x_n)$ is a unit eigenvector to $\lambda (A_\alpha)$ such that $x_u > 0$ and

$$\langle A_\alpha (H) x, x \rangle \geq \langle A_\alpha x, x \rangle,$$

then $\lambda (A_\alpha (H)) > \lambda (A_\alpha)$. 

9
Proof Proposition 1 implies that immediately that \( \lambda (A_a (H)) \geq \lambda (A_a) \), so our goal is to show that equality cannot hold. Assume for a contradiction that \( \lambda (A_a (H)) = \lambda (A_a) \) and set \( \lambda = \lambda (A_a) \). Proposition 1 implies that \( x \) is an eigenvector to \( H \) and therefore

\[
\lambda x_w = \alpha d_H (w) x_w + (1 - \alpha) \sum_{\{i,w\} \in E(H)} x_i
\]

\[
= \alpha (d_G (w) + 1) x_w + (1 - \alpha) x_u + \sum_{\{i,w\} \in E(G)} x_i
\]

\[
> \alpha d_G (w) x_w + \sum_{\{i,w\} \in E(G)} x_i,
\]

contradicting the fact that \( x \) is an eigenvector to \( \lambda (A_a) \) in \( G \). \( \square \)

4.2 Eigenvectors to \( \lambda (A_a (G)) \) and automorphisms

Knowing the symmetries of a graph \( G \) can be quite useful to find the spectral radius of \( \lambda (A_a (G)) \). Thus, we say that \( u \) and \( v \) are equivalent in \( G \), if there exists an automorphism \( p : G \rightarrow G \) such that \( p(u) = v \). Vertex equivalence implies very useful properties of eigenvectors to \( \lambda (A_a (G)) \):

**Proposition 16** Let \( G \) be a connected graph of order \( n \), and let \( u \) and \( v \) be equivalent vertices in \( G \). If \( (x_1, \ldots, x_n) \) is an eigenvector to \( \lambda (A_a (G)) \), then \( x_u = x_v \).

**Proof** Let \( G \) be a connected graph with \( A_a (G) = A_a \); let \( \lambda := \lambda (A_a) \) and \( x := (x_1, \ldots, x_n) \) be a unit nonnegative eigenvector to \( \lambda \). Let \( p : G \rightarrow G \) be an automorphism of \( G \) such that \( p(u) = v \). Note that \( p \) is a permutation of \( V (G) \); let \( P \) be the permutation matrix corresponding to \( p \). Since \( p \) is an automorphism, we have \( P^{-1}A_a P = A_a \); hence,

\[
P^{-1}A_a P x = \lambda x,
\]

and so \( P x \) is an eigenvector to \( A_a \). Since \( A_a \) is irreducible, \( x \) is unique, implying that \( P x = x \), and so \( x_u = x_v \). \( \square \)

Note that eigenvector entries corresponding to equivalent vertices need not be equal for disconnected graphs; for example, this not the case if \( G \) is a union of two disjoint copies of an \( r \)-regular graph. However, Proposition 16 implies the following practical statement:

**Corollary 17** If \( G \) is a connected graph and \( V (G) \) is partitioned into equivalence classes by the relation “\( u \) is equivalent to \( v \)”, then every eigenvector to \( \lambda (A_a) \) is constant within each equivalence class.
4.3 A few general bounds on $\lambda \left( A_\alpha (G) \right)$

In this section we give a few additional bounds on $\lambda \left( A_\alpha \right)$.

**Proposition 18** Let $G$ be a graph, with $\Delta (G) = \Delta$, $A (G) = A$, $D (G) = D$, and $A_\alpha (G) = A_\alpha$. The following inequalities hold for $\lambda \left( A_\alpha (G) \right)$:

\[
\begin{align*}
\lambda \left( A_\alpha \right) & \geq \lambda \left( A \right), \\
\lambda \left( A_\alpha \right) & \leq a\Delta + (1 - a) \lambda \left( A \right).
\end{align*}
\]

If equality holds in (15), then $G$ has a $\lambda \left( A \right)$-regular component. Equality in (16) holds if an only if $G$ has a $\Delta$-regular component.

**Proof** Note that inequality (15) follows from Proposition 3, but we shall give another proof to deduce the case of equality. Let $H$ be a component of $G$ such that $\lambda \left( A \right) = \lambda \left( A (H) \right)$. Write $h$ for the order of $H$, and let $(x_1, \ldots, x_h)$ be a positive unit vector to $\lambda \left( A (H) \right)$. For every edge $\{u, v\}$ of $H$, the AM-GM inequality implies that

\[
2x_ux_v = 2ax_ux_v + 2 \left( 1 - a \right) x_ux_v \leq ax_u^2 + 2 \left( 1 - a \right) x_ux_v + ax_v^2.
\]

Summing this inequality over all edges $\{u, v\} \in E (H)$, and using (7), we get

\[
\lambda \left( A \right) = \lambda \left( A (H) \right) = \langle A (H) x, x \rangle \leq \langle A_\alpha (H) x, x \rangle \leq \lambda \left( A_\alpha \right),
\]

so (15) is proved. If equality holds in (15), then $x_1 = \cdots = x_h$, hence $H$ is $\lambda \left( A \right)$-regular.

Inequality (16) follows by Weyl’s inequalities (5) because

\[
\lambda \left( A_\alpha \right) \leq \lambda \left( aD \right) + \lambda \left( (1 - a) \left( A \right) \right) = (1 - a) \lambda \left( A \right) + a\Delta,
\]

but we shall give a direct proof based on (9), since it is more appropriate for the case of equality. Let $H$ be a component of $G$ such that $\lambda \left( A_\alpha \right) = \lambda \left( A_\alpha (H) \right)$ and let $h$ be the order of $H$. Let $x := (x_1, \ldots, x_h)$ be a positive unit eigenvector to $\lambda \left( A_\alpha (H) \right)$. We have

\[
\begin{align*}
\lambda \left( A_\alpha \right) &= a \sum_{u \in V(H)} x_u^2d_G (u) + 2 \left( 1 - a \right) \sum_{\{u, v\} \in E(H)} x_ux_v \\
&\leq a\Delta \left( H \right) \sum_{u \in V(H)} x_u^2 + (1 - a) \lambda \left( A (H) \right) \\
&\leq a\Delta + (1 - a) \lambda \left( A \right),
\end{align*}
\]

proving (16). If equality holds in (16), then $H$ is $\Delta$-regular.

It is not hard to see that if $G$ has a $\Delta$-regular component, then $\lambda \left( A \right) = \Delta = \lambda \left( A_\alpha \right)$, and so equality holds in (16). \qed

Having inequality (15) in hand, every lower bound of $\lambda \left( A \right)$ gives a lower bound on $\lambda \left( A_\alpha \right)$, which, however, is never better than (15). We mention just two such bounds.
Corollary 19 Let $G$ be a graph with $A_\alpha (G) = A_\alpha$. If $G$ is of order $n$ and has $m$ edges, then

$$\lambda (A_\alpha) \geq \sqrt{\frac{1}{n} \sum_{u \in V(G)} d_G^2(u)} \text{ and } \lambda (A_\alpha) \geq \frac{2m}{n}.$$  

Equality holds in the second inequality if and only if $G$ is regular. If $\alpha > 0$, equality holds in the first inequality if and only if $G$ is regular.

Proof The only difficulty is to prove that if $\alpha > 0$, then the equality

$$\lambda (A_\alpha) = \frac{1}{\sqrt{n}} \sum_{u \in V(G)} d_G^2(u)$$  

implies that $G$ is regular. Indeed, suppose that (18) holds, which implies also that

$$\lambda (A(G)) = \frac{1}{\sqrt{n}} \sum_{u \in V(G)} d_G^2(u).$$

Let $G_1, \ldots, G_k$ be the components of $G$ and $n_1, \ldots, n_k$ be their orders. We see that

$$\sum_{u \in V(G)} d_G^2(u) = \lambda^2 (A(G)) n \geq \lambda^2 (A(G_1)) n_1 + \cdots + \lambda^2 (A(G_k)) n_k$$

$$\geq \sum_{u \in V(G_1)} d_{G_1}^2(u) + \cdots + \sum_{u \in V(G_k)} d_{G_k}^2(u) = \sum_{u \in V(G)} d_G^2(u).$$

Hence,

$$\lambda (A(G_1)) = \cdots = \lambda (A(G_k)) = \lambda (A(G)),$$

and likewise,

$$\lambda (A_\alpha (G_1)) = \cdots = \lambda (A_\alpha (G_k)) = \lambda (A_\alpha (G)).$$

Now, Proposition 3 implies that all components of $G$ are regular, completing the proof.

A very useful bound in extremal problems about $\lambda (Q)$ is the following one

$$\lambda (Q) \leq \max_{v \in V} \left\{ d(u) + \frac{1}{d(u)} \sum_{\{u,v\} \in E(G)} d(v) \right\},$$

with equality if and only if $G$ is regular or semiregular. Bound (19) goes back to Merris [24], whereas the case of equality has been established by Feng and Yu in [13]. It is not hard to modify (19) for the matrices $A_\alpha (G)$:

Proposition 20 If $G$ is a graph with no isolated vertices, then

$$\lambda (A_\alpha (G)) \leq \max_{v \in V(G)} \left\{ ad(u) + \frac{1 - \alpha}{d(u)} \sum_{\{u,v\} \in E(G)} d(v) \right\}$$

(20)
\[
\lambda(A_\alpha(G)) \geq \min_{v \in V(G)} \left\{ \alpha d(u) + \frac{1 - \alpha}{d(u)} \sum_{\{u,v\} \in E(G)} d(v) \right\}.
\]

(21)

If \( \alpha \in (1/2, 1) \) and \( G \) is connected, equality in (20) and (21) holds if and only if \( G \) is regular.

**Proof** Let \( A_\alpha(G) = A_\alpha \). Our proof of (20) and (21) uses the idea of Merris. The matrix \( D^{-1}A_\alpha D \) is similar to \( A_\alpha \) and so \( \lambda(A_\alpha) = \lambda(D^{-1}A_\alpha D) \). Since \( D^{-1}A_\alpha D \) is nonnegative, \( \lambda(D^{-1}A_\alpha D) \) is between the smallest and the largest rowsums of \( D^{-1}A_\alpha D \), implying both (20) and (21).

If \( G \) is connected, then \( A_\alpha \) is irreducible and so is \( D^{-1}A_\alpha D \). Hence, if equality holds in either (20) and (21), then all rowsums of \( D^{-1}A_\alpha D \) are equal. The remaining part of the proof uses an idea borrowed from [13]. For any vertex \( v \in V(G) \), set

\[
m(u) = \frac{1}{d(u)} \sum_{\{u,v\} \in E(G)} d(v).
\]

Fix a vertex \( u \) and let \( v \) be any neighbor of \( u \). Now, from

\[
\alpha d(u) + (1 - \alpha) m(u) = \alpha d(v) + (1 - \alpha) m(v)
\]

we see that

\[
\sum_{\{u,v\} \in E(G)} \alpha d(u) + (1 - \alpha) m(u) = \sum_{\{u,v\} \in E(G)} \alpha d(v) + (1 - \alpha) m(v).
\]

Hence

\[
\alpha d^2(u) + (1 - \alpha) d(u) m(u) = \alpha d(u) m(u) + (1 - \alpha) \sum_{\{u,v\} \in E(G)} m(v).
\]

Taking \( u \) to be a vertex with maximum degree, we see that

\[
\alpha d^2(u) + (1 - 2\alpha) d(u) m(u) = (1 - \alpha) \sum_{\{u,v\} \in E(G)} m(v) \leq (1 - \alpha) d^2(u).
\]

Hence \( m(u) \geq d(u) \), which is possible only if all neighbors of \( u \) have maximal degree as well. Since \( G \) is connected, it turns out that \( G \) is regular. \( \square \)

**Corollary 21** For any graph \( G \),

\[
\lambda(A_\alpha) \leq \max_{\{u,v\} \in E(G)} \alpha d(u) + (1 - \alpha) d(v).
\]

(22)

and

\[
\lambda(A_\alpha) \geq \min_{\{u,v\} \in E(G)} \alpha d(u) + (1 - \alpha) d(v).
\]

(23)

13
**Caution** If the right side of (22) is equal to \( M \), and is maximized for \( \{u, v\} \in E(G) \), then

\[
M = \max \{ \alpha d(u) + (1 - \alpha) d(v), \alpha d(v) + (1 - \alpha) d(u) \}.
\]

A similar remark is valid for (23) with appropriate changes.

It seems that equality in (20) and (21) holds only if \( G \) is regular, except in the cases \( \alpha = 0 \) and \( \alpha = 1/2 \). If true, this fact would need new proof techniques, so we raise the following problem.

**Problem 22** Find all cases of equality in (20), (21), (22), and (23).

The last bounds in this sections are in the spirit of (20) and (21):

**Proposition 23** Let \( \alpha \in [0, 1] \). If \( G \) be a graph of order \( n \), then

\[
\lambda^2 (A_\alpha (G)) \leq \max_{k \in V(G)} \alpha d_G^2 (k) + (1 - \alpha) w_G (k)
\]

and

\[
\lambda^2 (A_\alpha (G)) \geq \min_{k \in V(G)} \alpha d_G^2 (k) + (1 - \alpha) w_G (k).
\]

**Proof** Let \( A_\alpha := A_\alpha (G) \), \( A := A (G) \), \( D := D (G) \). First, we show that for any \( k \in [n] \), the \( k \)th rowsum of \( A^2_\alpha (G) \) is equal to

\[
\alpha d_G^2 (k) + (1 - \alpha) w_G (k).
\]

Indeed, for the square of \( A_\alpha \), we see that

\[
A^2_\alpha = \alpha^2 D^2 + (1 - \alpha)^2 A^2 + \alpha (1 - \alpha) DA + \alpha (1 - \alpha) AD.
\]

So for the \( k \)th rowsum \( r_k (A^2_\alpha) \) we find that

\[
r_k(A^2_\alpha) = \alpha^2 r_k(D^2) + (1 - \alpha)^2 r_k(A^2) + \alpha (1 - \alpha) r_k(DA) + \alpha (1 - \alpha) r_k(AD)
\]

\[
= \alpha^2 d_G^2 (k) + (1 - \alpha)^2 w_G (k) + \alpha (1 - \alpha) d_G^2 (k) + \alpha (1 - \alpha) w_G (k)
\]

\[
= \alpha d_G^2 (k) + (1 - \alpha) w_G (k).
\]

Since \( \lambda^2 (A_\alpha) = \lambda (A^2_\alpha) \), the assertions follow, because \( \lambda (A^2_\alpha) \) is between the smallest and the largest rowsums of \( A^2_\alpha \). \(\square\)
5 Some spectral extremal problems

Recall that the central problem of the classical extremal graph theory is of the following type:

**Problem A** Given a graph $F$, what is the maximum number of edges of a graph of order $n$, with no subgraph isomorphic to $F$?

Such problems are fairly well understood nowadays; see, e.g., [3] for comprehensive discussion and [25] for some newer results. During the past two decades, some subtler versions of Problem A have been investigated, namely for $\lambda(A(G))$ and $\lambda(Q(G))$. In these problems, the central questions are the following ones:

**Problem B** Given a graph $F$, what is the maximum $\lambda(A(G))$ of a graph $G$ of order $n$, with no subgraph isomorphic to $F$?

**Problem C** Given a graph $F$, what is the maximum $\lambda(Q(G))$ of a graph $G$ of order $n$, with no subgraph isomorphic to $F$?

Many instances of Problem B have been solved, see, e.g., the second part of the survey paper [25]. There is also considerable progress with Problem C: see, e.g., the papers [1], [2], [14], [15], [17], [26], [27], [28], and [30].

Now, having the family $A_\alpha(G)$, we can merge Problems B and C into one, namely:

**Problem D** Given a graph $F$, what is the maximum $\lambda(A_\alpha(G))$ of a graph $G$ of order $n$, with no subgraph isomorphic to $F$?

In this survey we shall solve Problem D when $F$ is a complete graph. Several, other cases seem particularly interesting:

**Problem 24** Solve problem D if $F$ is a path or a cycle of given order.

5.1 Chromatic number and $\lambda(A_\alpha(G))$

A graph is called $r$-chromatic (or $r$-partite) if its vertices can be partitioned into $r$ edgeless sets. An interesting topic in spectral graph theory is to find eigenvalues bounds on the chromatic number of graphs. In particular, here we are interested in the maximum $\lambda(A_\alpha(G))$ if $G$ is an $r$-partite graph of order $n$.

Let us write $T_r(n)$ for the $r$-partite Turán graph of order $n$ and recall that $T_r(n)$ is a complete $r$-partite graph of order $n$, whose partition sets are of size $\lceil n/r \rceil$ or $\lfloor n/r \rfloor$. Note that for $r = 2$ we have $T_2(n) = K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$. It is known that $T_r(n)$ has the maximum number of edges among all $r$-partite graphs of order $n$. The corresponding problem for $\lambda(A_\alpha(G))$ is not so straightforward, so for reader’s sake we shall consider the case $r = 2$ first.

**Theorem 25** Let $G$ be a bipartite graph of order $n$.

(i) If $\alpha < 1/2$, then

$$\lambda(A_\alpha(G)) < \lambda(A_\alpha(T_2(n))).$$
unless \( G = T_2(n) \).

(ii) If \( \alpha > 1/2 \), then
\[
\lambda(A_\alpha(G)) < \lambda(A_\alpha(K_{1,n-1})),
\]
unless \( G = K_{1,n-1} \).

(iii) If \( \alpha = 1/2 \), then
\[
\lambda(A_\alpha(G)) \leq n/2,
\]
with equality if and only if \( G \) is a complete bipartite graph.

**Proof** Suppose that \( G \) is a bipartite graph of order \( n \) with maximum \( \lambda(A_\alpha(G)) \) among all bipartite graphs of order \( n \). Proposition 13 implies that \( G \) is a complete bipartite graph. Suppose that the partition sets \( V_1 \) and \( V_2 \) of \( G \) are of size \( n_1 \) and \( n_2 \), where \( n_1 + n_2 = n \). Set \( \lambda = \lambda(A_\alpha(G)) \) and let \((x_1, \ldots, x_n)\) be a positive eigenvector to \( \lambda \). Proposition 16 implies that entries corresponding to vertices in the same partition set have the same value, say \( z_i \) for \( V_i, i = 1, 2 \). So the equations (6) give
\[
\begin{align*}
\lambda z_1 &= an_2z_1 + (1 - \alpha) n_2z_2, \\
\lambda z_2 &= an_1z_2 + (1 - \alpha) n_1z_1.
\end{align*}
\]
Excluding \( z_1 \) and \( z_2 \), we find that
\[
(\lambda - an_2)(\lambda - an_1) = (1 - \alpha)^2 n_1n_2
\]
and therefore,
\[
\lambda = \frac{an + \sqrt{\alpha^2n^2 + 4n_1n_2(1 - 2\alpha)}}{2}.
\]
Clearly if \( \alpha < 1/2 \), then \( \lambda \) is maximum whenever \( n_1n_2 \) is maximum; hence \( G = T_2(n) \). Likewise if \( \alpha > 1/2 \), then \( \lambda \) is maximum whenever \( n_1n_2 \) is minimum, and so \( G = K_{1,n-1} \). Finally if \( \alpha = 1/2 \), then \( \lambda = n/2 \) for every complete bipartite graph. \( \square \)

For general \( r \) the statement reads as:

**Theorem 26** Let \( r \geq 2 \) and \( G \) be an \( r \)-chromatic graph of order \( n \).

(i) If \( \alpha < 1 - 1/r \), then
\[
\lambda(A_\alpha(G)) < \lambda(A_\alpha(T_r(n))),
\]
unless \( G = T_r(n) \).

(ii) If \( \alpha > 1 - 1/r \), then
\[
\lambda(A_\alpha(G)) < \lambda(A_\alpha(S_{n,r-1})),
\]
unless \( G = S_{n,r-1} \).

(iii) If \( \alpha = 1 - 1/r \), then
\[
\lambda(A_\alpha(G)) \leq (1 - 1/r)n,
\]
with equality if and only if \( G \) is a complete \( r \)-partite graph.
Proof Suppose that $G$ is an $r$-partite graph of order $n$ with maximum $\lambda (A_r (G))$ among all $r$-partite graphs of order $n$. Proposition 13 implies that $G$ is a complete $r$-partite graph. Suppose that $V_1, \ldots, V_r$ are the partition sets of $G$, with sizes $n_1, \ldots, n_r$; obviously $n_1 + \cdots + n_r = n$. Set $\lambda := \lambda (A_r (G))$ and let $(x_1, \ldots, x_n)$ be a positive eigenvector to $\lambda$. Proposition 16 implies that the entries of $x$ corresponding to vertices in the same partition set have the same value, say $z_i$ for $V_i$, $i = 1, \ldots, r$. Hence, equations (6) reduce to $r$ equations

$$\lambda z_k = \alpha (n - n_k) z_k + (1 - \alpha) \sum_{i \in [r] \setminus \{k\}} n_i z_i, \quad 1 \leq k \leq r. \quad (24)$$

If $\alpha = 1 - 1/r$, we see that $\lambda = (1 - 1/r) n$ always is an eigenvalue with an eigenvector defined by $z_i = 1/(rn_i)$, $i = 1, \ldots, r$. This proves (iii).

Further, letting $S = n_1 z_1 + \cdots + n_r z_r$, equations (24) imply that

$$(\lambda - \alpha (n - n_k) + (1 - \alpha) n_k) n_k z_k = (1 - \alpha) n_k S, \quad 1 \leq k \leq r.$$  

After some algebra, we see that $\lambda$ satisfies the equation

$$\sum_{k \in [r]} \frac{n_k}{\lambda - \alpha n + n_k} = \frac{1}{1 - \alpha}. \quad (25)$$

If $\alpha < 1 - 1/r$, then $1/(1 - \alpha) < r$. Hence some of the summands in the right side of (25) is less than 1 and so $\lambda - \alpha n > 0$. Letting

$$f (z) := \frac{z}{\lambda - \alpha n + z} = 1 - \frac{\lambda - \alpha n}{\lambda - \alpha n + z},$$

it is easy to see that

$$f'' (z) = \frac{-2 (\lambda - \alpha n)}{(\lambda - \alpha n + z)^3} < 0$$

for $z > 0$; thus $f (z)$ is concave for $z > 0$.

Let $\lambda_T := \lambda (A_\alpha (T_r (n)))$ and let $t_1, \ldots, t_r$ be the sizes of the partition sets of $T_r (n)$, that is to say, $t_i = \lfloor n/r \rfloor$ or $t_i = \lceil n/r \rceil$ and $t_1 + \cdots + t_r = n$. In view of (25) we have

$$\sum_{k \in [r]} \frac{t_k}{\lambda_T - \alpha n + t_k} = \frac{1}{1 - \alpha}.$$  

Now the concavity of $f (z)$ implies that

$$\sum_{k \in [r]} \frac{t_k}{\lambda_T - \alpha n + t_k} = \frac{1}{1 - \alpha} \leq \sum_{k \in [r]} \frac{n_k}{\lambda - \alpha n + n_k} \leq \sum_{k \in [r]} \frac{t_k}{\lambda - \alpha n + t_k}.$$  

and so $\lambda_T \geq \lambda$, with equality if and only if $n_i = \lfloor n/r \rfloor$ or $n_i = \lceil n/r \rceil$ for all $i \in [r]$. This proves (i).
The proof of (ii) goes along the same lines. If \( \alpha > 1 - 1/r \), then \( 1/ (1 - \alpha) > r \). Hence some of the summands in the right side of (25) is greater than 1 and so \( \lambda - \alpha n < 0 \). Letting

\[
f(z) := \frac{z}{\lambda - \alpha n + z},
\]

it is easy to see that \( f''(z) > 0 \) for \( z > 0 \); thus \( f(z) \) is convex for \( z > 0 \).

Let \( \lambda_S := \lambda(A_\alpha(S_{n,r-1})) \) and let \( s_1, \ldots, s_r \) be the sizes of the partition sets of \( S_{n,r-1} \), that is to say, \( s_1 = \cdots = s_{r-1} = 1 \) and \( s_r = n - r + 1 \). In view of (25), we have

\[
\sum_{k \in [r]} \frac{s_k}{\lambda_S - \alpha n + s_k} = \frac{1}{1 - \alpha}.
\]

Now the convexity of \( f(z) \) implies that

\[
\sum_{k \in [r]} \frac{s_k}{\lambda_S - \alpha n + s_k} = \frac{1}{1 - \alpha} = \sum_{k \in [r]} \frac{n_k}{\lambda - \alpha n + n_k} \leq \sum_{k \in [r]} \frac{s_k}{\lambda - \alpha n + s_k}.
\]

and so \( \lambda_S \geq \lambda \), with equality if and only if one partition set of \( G \) is of size \( n - r + 1 \), and the rest are of size 1, that is to say \( G = S_{n,r-1} \). The proof of Theorem 26 is completed. \( \square \)

5.2 Clique number and \( \lambda(A_\alpha(G)) \)

A graph is called \( K_r \)-free if it does not contain a complete graph on \( r \) vertices. It is known (see, e.g., [25] and [17]) that if \( G \) is a \( K_{r+1} \)-free graph of order \( n \), then

\[
\lambda(A(G)) \leq \lambda(A(T_r(n))),
\]

\[
\lambda(Q(G)) \leq \lambda(Q(T_r(n))).
\]

The generalization of these results to \( \lambda(A_\alpha(G)) \) turned out to be quite unexpected, and is summarized in the following encompassing theorem:

**Theorem 27** Let \( r \geq 2 \) and \( G \) be an \( K_{r+1} \)-free graph of order \( n \).

(i) If \( 0 \leq \alpha < 1 - 1/r \), then

\[
\lambda(A_\alpha(G)) < \lambda(A_\alpha(T_r(n))),
\]

unless \( G = T_r(n) \).

(ii) If \( 1 > \alpha > 1 - 1/r \), then

\[
\lambda(A_\alpha(G)) < \lambda(A_\alpha(S_{n,r-1})),
\]

unless \( G = S_{n,r-1} \).

(iii) If \( \alpha = 1 - 1/r \), then

\[
\lambda(A_\alpha(G)) \leq (1 - 1/r) n,
\]

with equality if and only if \( G \) is a complete \( r \)-partite graph.
We shall show that Theorem 27 can be reduced to Theorem 26 via a technical lemma.

**Lemma 28** Let $\alpha \in [0,1)$ and $n \geq r \geq 2$. If $G$ is a graph with maximum $\lambda (A_{\alpha}(G))$ among all $K_{r+1}$-free graphs of order $n$, then $G$ is complete $r$-partite.

For the proof of the lemma, we introduce some notation: Let $\alpha \in [0,1)$. Given a graph $G$ of order $n$ and a vector $x := (x_1, \ldots, x_n)$, set

$$S_G(x) := \langle A_{\alpha}(G)x, x \rangle,$$

and for any $v \in V(G)$, set

$$S_G(v, x) := \alpha d_G(u) + (1 - \alpha) \sum_{\{v, i\} \in E(G)} x_i.$$

**Proof of Lemma 28** Let $G$ be a graph with maximum $\lambda (A_{\alpha}(G))$ among all $K_{r+1}$-free graphs of order $n$. For short, let $\lambda := \lambda (A_{\alpha}(G))$. Clearly $G$ is connected, so there is a positive unit eigenvector $x := (x_1, \ldots, x_n)$ to $\lambda (A_{\alpha}(G))$, and therefore,

$$\lambda = S_G(x) = \sum_{v \in V(G)} x_v S_G(v, x).$$

Note that the eigenequation (6) for any vertex $v \in V(G)$ can be written as

$$\lambda x_u = S_G(v, x).$$

(26)

To prove the lemma we need two claims.

**Claim A** There exists a coclique $W \subset G$ such that

$$G = W \vee G' ,$$

where $G' = G - V(G_1)$.

**Proof** Select a vertex $u$ with

$$S_G(u, x) := \max \{ S_G(v, x) : v \in V(G) \} ,$$

and set $U := \Gamma_G(u)$ and $W := G - U$. Remove all edges within $W$ and join each vertex in $U$ to each vertex in $W$ to form $G'$. Write $H$ for the resulting graph, which is obviously of order $n$ and is $K_{r+1}$-free. We shall show that $S_H(v, x) \geq S_G(v, x)$ for each $v \in V(G)$. This is obvious if $v \in U$, since then $\Gamma_G(v) \subset \Gamma_H(v)$, and so $S_H(v, x) \geq S_G(v, x)$. Now, let $v \in V(W)$. Note that

$$S_H(v, x) = \alpha d_G(u) x_v + (1 - \alpha) \sum_{\{u, i\} \in E(G)} x_i = \alpha d_G(u) x_v + S_G(u, x) - \alpha d_G(u) x_u .$$

Hence,

$$S_H(v, x) - S_G(v, x) = S_G(u, x) - S_G(v, x) - \alpha d_G(u) (x_u - x_v) .$$
Now, equation (26) implies that \( S_G(u, x) = \lambda x_u \) and \( S_G(v, x) = \lambda x_v \). Hence
\[
S_H(v, x) - S_G(v, x) = (\lambda (x_u - x_v) - ad_G(u) (x_u - x_v)) = (\lambda - ad_G(u) (x_u - x_v)).
\]
But Corollary 12 implies that \( \lambda \cdot ad_G(u) > 0 \), and equation (26) implies that \( x_u \geq x_v \). Hence \( S_H(v, x) \geq S_G(v, x) \) for any \( v \in V(G) \), and so
\[
\lambda (A_H(G)) \geq S_H(x) \geq S_G(x) = \lambda \geq \lambda (A_H(G)).
\]
Therefore, \( \lambda (A_H(G)) = \lambda \), implying, in particular, that \( S_H(v, x) = S_G(v, x) \) for each \( v \in U \); thus each \( v \in U \) is joined in \( G \) to each \( w \in W \), and so \( G = H = W \cup G[U] \), completing the proof of Claim A.

To finish the proof of the lemma we need another technical assertion:

**Claim B** Let \( 1 \leq k < r \). If \( F \) is an induced subgraph of \( G \) and \( W_1, \ldots, W_k \) are disjoint cocliques of \( G \) such that
\[
G = W_1 \cup \cdots \cup W_k \cup F
\]
then there is a coclique \( W_{k+1} \subset F \) such that
\[
G = W_1 \cup \cdots \cup W_{k+1} \cup F',
\]
where \( F' = F - V(W_{k+1}) \).

**Proof** Select a vertex \( v \in V(F) \) with
\[
S_G(u, x) = \max \{ S_G(v, x) : v \in V(F) \},
\]
and set \( U := \Gamma_F(u) \) and \( W := F - U \). Remove all edges within \( W \) and join each vertex in \( U \) to each vertex in \( W \). Write \( H \) for the resulting graph, which is obviously of order \( n \) and is \( K_{r+1} \)-free. We shall show that \( S_H(v, x) \geq S_G(v, x) \) for each \( v \in V(G) \). This is obvious if \( v \in V \setminus V(W) \), since then either \( \Gamma_G(v) = \Gamma_H(v) \) or \( \Gamma_G(v) \subset \Gamma_H(v) \), and so \( S_H(v, x) \geq S_G(v, x) \). Now, let \( v \in V(W) \). Exactly as in the proof of Claim A we see that
\[
S_H(v, x) - S_G(v, x) = (\lambda - ad_G(u)) (x_u - x_v).
\]
Hence, \( S_H(v, x) \geq S_G(v, x) \) and
\[
\lambda (A_H(G)) \geq S_H(x) \geq S_G(x) = \lambda \geq \lambda (A_H(G)).
\]
Therefore, \( \lambda (A_H(G)) = \lambda \), implying, in particular, that \( S_H(v, x) = S_G(v, x) \) for each \( v \in U \); thus each \( v \in U \) is joined in \( H_k \) to each \( w \in W \), and so \( F = W \cup G[U] \). Letting \( W_{k+1} := W \), the proof of Claim B is completed.

To complete the proof of the lemma, we first apply Claim A and then repeatedly apply Claim B until \( k = r - 2 \). In this way we find that
\[
G = W_1 \cup \cdots \cup W_{r-1} \cup F,
\]
where \( W_1 \cup \cdots \cup W_{r-1} \) are cocliques of \( G \) and \( F \) is an induced subgraph of \( G \). Because \( G \) is \( K_{r+1} \)-free, \( F \) must be a coclique too and so, \( G \) is a complete \( r \)-partite graph. \( \square \)
6 Miscellaneous

In this section we briefly touch a few rather different topics, some of which deserve a much more thorough investigation.

6.1 The smallest eigenvalue $\lambda_{\min}(A_\alpha(G))$

The smallest eigenvalue of the adjacency matrix, which is second in importance after the spectral radius, has numerous relations with the structure of the graph. To a great extent this is also true for $\lambda_{\min}(Q(G))$; see, e.g., [12], [21], and [22]. In particular, the smallest eigenvalues of $A(G)$ and $Q(G)$ have close relations to bipartite subgraphs of $G$. A simple relation of this type can be obtained also for $\lambda_{\min}(A_\alpha(G))$.

Let $G$ be a graph of order $n$ with $m$ edges. Let $V(G) = V_1 \cup V_2$ be a bipartition and let the $n$-vector $x := (x_1, \ldots, x_n)$ be $-1$ on $V_1$ and $1$ on $V_2$. We see that

$$\langle A_\alpha(G)x, x \rangle = 2am - 2(1-\alpha)e(V_1, V_2)$$

Hence, scaling $(x_1, \ldots, x_n)$ to unit length, we get:

**Proposition 29** If $G$ is a graph of order $n$ with $m$ edges, then

$$\lambda_{\min}(A_\alpha(G)) \leq 2\alpha \frac{m}{n} - \frac{2(1-\alpha)}{n} \text{maxcut}(G).$$

It is interesting to determine the minimum value of $\lambda_{\min}(A_\alpha(G))$ if $G$ is a graph of order $n$. For $\alpha \geq 1/2$ this is easy. Indeed, if $\alpha \geq 1/2$, the matrix $A_\alpha(G)$ is positive semidefinite, and so $\lambda_{\min}(A_\alpha(G)) \geq 0$. On the other hand, if $G$ has an isolated vertex, then $\lambda_{\min}(A_\alpha(G)) = 0$, so if $\alpha \in [1/2, 1]$, then

$$\min \{\lambda_{\min}(A_\alpha(G)) : v(G) = n\} = 0.$$ 

By contrast,

$$\min \{\lambda_{\min}(A(G)) : v(G) = n\} = -\sqrt{\lceil n/2 \rceil \lfloor n/2 \rfloor};$$

hence it is worth to raise the following problem:

**Problem 30** For any $\alpha \in (0,1/2)$ determine

$$\min \{\lambda_{\min}(A_\alpha(G)) : v(G) = n\}.$$

6.2 The second largest eigenvalue $\lambda_2(A_\alpha(G))$

In this subsection we discuss how large $\lambda_2(A_\alpha(G))$ can be if $G$ is a graph of order $n$. 


Proposition 31 Let $G$ be a graph of order $n$ with $A_\alpha(G) = A_\alpha$.
(a) If $1/2 \leq \alpha \leq 1$, then
\[ \lambda_2(A_\alpha) \leq an - 1. \]
If $\alpha > 1/2$, equality is attained if and only if $G = K_n$.
(b) If $0 \leq \alpha < 1/2$, then
\[ \lambda_2(A_\alpha) \leq \frac{n}{2} - 1. \]
If $n$ is even equality holds for the graph $G = 2K_{n/2}$.

Note that we have not determined precisely how large $\lambda_2(A_\alpha(G))$ can be if $G$ is a graph of odd order $n$. Taking $G = K_{[n/2]} \cup K_{[n/2]}$, we see that
\[ \lambda_2(A_\alpha(G)) = \frac{n-1}{2} - 1, \]
but this still leaves a margin of $1/2$ to close.

6.3 Eigenvalues of $A_\alpha(G)$ and the diameter of $G$

The following theorem can be proved using the generic idea of [6].

Proposition 32 Let $a \in [0,1)$, let $G$ be a graph with $A_\alpha(G) = A_\alpha$, and let $u$ and $v$ be two vertices of $G$ at distance $k \geq 1$. Let $l \in [k]$ and set $B := A_l^\alpha$.
(a) If $l = k$, then $b_{u,v} > 0$;
(b) If $l < k$, then $b_{u,v} = 0$.

Proof Set $A = A(G)$. If $X$ and $Y$ are matrices of the same size, write $X \succ Y$, if $x_{i,j} \geq y_{i,j}$ for all admissible $i,j$.

Proof of (a) Note that $A_\alpha \succ (1-\alpha) A$, and so $A_\alpha^k \succ (1-\alpha)^k A^k$. However, the $(u,v)$ entry of $A^k$ is positive, since there is a path of length $k$ between $u$ and $v$. Hence, $b_{u,v} > 0$, proving (a).

Proof of (b) Now suppose that $l < k$, and note that $A + nI \succ A_\alpha$. Hence, $(A + nI)^l \succ A_\alpha^l$. Since
\[ (A + nI)^l = A^l + a_{l-1}A^{l-1} + \cdots + a_0I \]
for some real $a_0, \ldots, a_{l-1}$, we see that the $(u,v)$ entry of $(A + nI)^l$ is zero, because there is no path shorter than $k$ between $u$ and $v$, and so the $(u,v)$ entry of each of the matrices $A^l, \ldots, A, I$ is zero. Hence, $b_{u,v} = 0$. \qed

Corollary 33 If $G$ is a connected graph of diameter $D$, then $A_\alpha(G)$ has at least $D + 1$ distinct eigenvalues.
6.4 Eigenvalues of $A_\alpha (G)$ and traces

In this subsection we give two explicit expressions for the sums and the sum of squares of the eigenvalues of $A_\alpha (G)$.

**Proposition 34** If $G$ is a graph of order $n$ and has $m$ edges, then

$$\sum_{i=1}^{n} \lambda_i (A_\alpha (G)) = \text{tr} A_\alpha (G) = \alpha \sum_{u \in V(G)} d_G (u) = 2\alpha m.$$  

Here is a similar formula for the sum of the squares of the $A_\alpha$-eigenvalues.

**Proposition 35** If $G$ is a graph of order $n$ and has $m$ edges, then

$$\sum_{i=1}^{n} \lambda_i^2 (A_\alpha (G)) = \text{tr} A_\alpha^2 (G) = 2 (1 - \alpha)^2 m + \alpha^2 \sum_{u \in V} d_G^2 (u).$$

**Proof** Let $A_\alpha := A_\alpha (G)$, $A := A (G)$, and $D := D (G)$. Calculating the square $A_\alpha^2$ and taking its trace, we find that

$$\text{tr} A_\alpha^2 = \text{tr} (\alpha^2 D^2 + (1 - \alpha)^2 A^2 + \alpha (1 - \alpha) DA + \alpha (1 - \alpha) AD)$$

$$= \alpha^2 \text{tr} D^2 + (1 - \alpha)^2 \text{tr} A^2 + \alpha (1 - \alpha) \text{tr} DA + \alpha (1 - \alpha) \text{tr} AD$$

$$= 2 (1 - \alpha)^2 m + \alpha^2 \sum_{u \in V} d_G^2 (u),$$

completing the proof. \qed

7 The $A_\alpha$-spectra of some graphs

Equalities (11) and the fact the eigenvalues of $A (K_n)$ are \{n − 1, −1, . . . , −1\} give the spectrum of $A_\alpha (K_n)$ as follows:

**Proposition 36** The eigenvalues of $A_\alpha (K_n)$ are

$$\lambda_1 (A_\alpha (K_n)) = n - 1 \quad \text{and} \quad \lambda_k (A_\alpha (K_n)) = \alpha n - 1 \quad \text{for} \ 2 \leq k \leq n.$$

Next, we present the $A_\alpha$-spectrum of the complete bipartite graph $K_{a,b}$, but we omit the proof.

**Proposition 37** Let $a \geq b \geq 1$. If $\alpha \in [0, 1]$, the eigenvalues of $A_\alpha (K_{a,b})$ are

$$\lambda (A_\alpha (K_{a,b})) = \frac{1}{2} \left( \alpha (a + b) + \sqrt{\alpha^2 (a + b)^2 + 4ab (1 - 2\alpha)} \right),$$

$$\lambda_{\min} (A_\alpha (K_{a,b})) = \frac{1}{2} \left( \alpha (a + b) - \sqrt{\alpha^2 (a + b)^2 + 4ab (1 - 2\alpha)} \right),$$

$$\lambda_k (A_\alpha (K_{a,b})) = \alpha a \quad \text{for} \ 1 < k \leq b,$$

$$\lambda_k (A_\alpha (K_{a,b})) = ab \quad \text{for} \ b < k < a + b.$$
In particular, the $A_\alpha$-spectrum of the star $K_{1,n-1}$ is as follows:

**Proposition 38** The eigenvalues of $A_\alpha (K_{1,n-1})$ are

\[
\lambda (A_\alpha (K_{1,n-1})) = \frac{1}{2} \left( an + \sqrt{a^2 n^2 + 4 (n - 1) (1 - 2\alpha)} \right)
\]

\[
\lambda_{\min} (A_\alpha (K_{1,n-1})) = \frac{1}{2} \left( an - \sqrt{a^2 n^2 + 4 (n - 1) (1 - 2\alpha)} \right)
\]

\[
\lambda_k (A_\alpha (K_{1,n-1})) = \alpha \text{ for } 1 < k < n.
\]

8 Concluding remarks

This survey covers just a small portion of the hundreds of results about $A(G)$ and $Q(G)$ that could be extended to $A_\alpha (G)$. This is a challenging endeavor. If nothing else, Theorems 26 and 27 show that it is worth studying $A_\alpha (G)$, for it is difficult to discover them in a different context.

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