Blow-up behavior of solutions to the heat equation with nonlinear boundary conditions

Junichi Harada

Department of Applied Physics, School of Science and Engineering, Waseda University, 3-4-1 Okubo, Shinjuku-ku, Tokyo 169-8555, Japan
harada-j@aoni.waseda.jp

Abstract

We study the asymptotic behavior of blow-up solutions of the heat equation with nonlinear boundary conditions. In particular, we classify the asymptotic behavior of blow-up solutions and investigate the spacial singularity of their blow-up profiles.

1 Introduction

We study positive solutions of the heat equation with nonlinear boundary conditions:

\[
\begin{cases}
\partial_t u = \Delta u & \text{in } \mathbb{R}^n_+ \times (0, T), \\
\partial_n u = u^q & \text{on } \partial \mathbb{R}^n_+ \times (0, T), \\
u(x, 0) = u_0(x) & \text{in } \mathbb{R}^n_+,
\end{cases}
\]

where \( \mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}; x_n > 0\} \), \( \partial_n = -\partial/\partial x_n \), \( 1 < q < n/(n-2) \) if \( n \geq 3 \) and

\[ u_0 \in C(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+), \quad u_0 \geq 0. \]

A solution \( u(x, t) \) is said to blow up in a finite time, if there exists \( T > 0 \) such that

\[ \limsup_{t \to T} \|u(t)\|_{L^\infty(\mathbb{R}^n_+)} = \infty. \]

It is known that a solution blows up in a finite time for any \( q > 1 \), if the initial data is positive and large enough. In particular, every positive solution blows up in a finite time even if the initial data is small for the case \( 1 < q \leq (n + 1)/n \) (\( \text{(3)} \)). In this paper, we study the asymptotic behavior of blow-up solutions of (P) near the blow-up time and their blow-up profiles. If a limit

\[ U(x) = \lim_{t \to T} u(x, t) \in [0, \infty] \]

exists for any \( x \in \mathbb{R}^n_+ \), we call \( U(x) \) a blow-up profile of \( u(x, t) \). For a one dimensional case, Fila-Quittner (see also (3)) constructed a backward self-similar blow-up solution:

\[ u_B(x, t) = (T - t)^{-1/2(q-1)} \varphi_0((T - t)^{-1/2}x), \]

where \( \varphi_0 \in BC^2(\mathbb{R}_+) \) is a positive solution of

\[
\begin{cases}
\varphi''_0 - \frac{\xi}{2} \varphi'_0 - \frac{\varphi_0}{2(q-1)} = 0 & \text{in } \xi \in \mathbb{R}_+, \\
\partial_\nu \varphi_0(0) = \varphi_0(0)^q.
\end{cases}
\]
This special blow-up solution \( u_B(x,t) \) has the following blow-up profile:
\[
U_B(x) = c_B x^{-1/(q-1)}.
\]

As for general blow-up solutions for a multidimensional case with \( 1 < q < n/(n-2) \) if \( n \geq 3 \), the author in \[11\] proved that if a positive \( x_m \)-axial symmetric solution blows up at the origin in a finite time, then its blow-up profile \( U(x) \) satisfies
\[
U(x) = c_B (1 + o(1))(\cos \theta)^{-1/(q-1)}|x|^{-1/(q-1)}
\]
along \( x_m = |x| \cos \theta \) for any fixed \( \theta \in [0, \pi/2) \). Unfortunately the expression (2) does not hold on \( \partial \mathbb{R}^n_+ \), since \( \cos \theta = 0 \) if \( x \in \partial \mathbb{R}^n_+ \). The purpose of this paper is to derive a formula on \( \partial \mathbb{R}^n_+ \) corresponding to (2).

We recall known results concerning the asymptotic behavior of blow-up solutions of the following semilinear heat equation with \( 1 < p < (n+2)/(n-2) \) if \( n \geq 3 \):
\[
\partial_t u = \Delta u + u^p \quad \text{in } \mathbb{R}^n \times (0,T) \quad (u > 0).
\]

Let \( u(x,t) \) be a solution of (F) which blows up at the origin in a finite time \( T > 0 \). Giga-Kohn (\[8\ \[9\ \[10\]) derived the following asymptotic formula for blow-up solutions:
\[
\lim_{t \to T^-} \sup_{|x|<\nu(T-t)^{1/2}} \left| (T-t)^{1/(p-1)} u(x,t) - (p-1)^{-1/(p-1)} \right| = 0 \quad \text{for } \nu > 0.
\]

This description gives the first approximation for blow-up solutions. In their paper, they introduced self-similar variables: \( x = (T-t)^{1/2} y, \ s = -\log(T-t) \) and a rescaled function defined by
\[
\varphi(y,s) = (T-t)^{1/(p-1)} u(x,t).
\]

The asymptotic formula (3) is equivalent to
\[
\lim_{s \to \infty} \varphi(y,s) = (p-1)^{-1/(p-1)} \quad \text{in } C_{\text{loc}}(\mathbb{R}^n).
\]

Filippas-Kohn \[6\] and Herrero-Velázquez \[15\] independently studied the second approximation for blow-up solutions for a one dimensional case (see also \[1\ \[7\ \[21\]). Let \( H_k(y) \) and \( \lambda_k \) \( (k = 0, 1, \cdots) \) be the \( k \)-th eigenfunction and eigenvalue of
\[
-(H'' - \frac{y}{2}H' + H) = \lambda H
\]
in \( L^2(\mathbb{R}) = \{ H \in L^2_{\text{loc}}(\mathbb{R}); \int_{-\infty}^{\infty} H(y)^2 e^{-y^2/4}dy < \infty \} \). It is well known that \( H_k(y) \) is a \( k \)-th polynomial and \( \lambda_k = (k-2)/2 \). Then one of the following cases occurs (\[8\ \[12\ \[13\]):

(i) there exists \( \kappa_p > 0 \) such that
\[
\varphi(y,s) = (p-1)^{-1/(p-1)} - \kappa_p H_2(y)s^{-1} + o(s^{-1}) \quad \text{in } L^2_{\rho}(\mathbb{R}),
\]

(ii) there exist even integer \( m \geq 4 \) and \( \kappa \neq 0 \) such that
\[
\varphi(y,s) = (p-1)^{-1/(p-1)} + \kappa H_m(y)e^{-\lambda_m s} + o(e^{-\lambda_m s}) \quad \text{in } L^2_{\rho}(\mathbb{R}).
\]

In particular, the case (i) actually occurs if the initial data is even and monotone decreasing on \( (0, \infty) \).

As a further step of this second approximation formula, the blow-up profile for solutions of (F) was derived by Herrero-Velázquez (\[12\ \[13\ \[14\ \[15\ \[21\ \[23\ \[24\ \[26\]). For a one dimensional case, Herrero-Velázquez \[13\] proved that if (i) occurs, then there exists \( \kappa_p' > 0 \) such that the blow-up profile \( U(x) \) satisfies
\[
\lim_{x \to 0} \left( \frac{x^2}{|\log x|} \right)^{1/(p-1)} U(x) = \kappa_p'.
\]
on the other hand, if (ii) occurs, then there exists \( \kappa' > 0 \) such that the blow-up profile \( U(x) \) satisfies
\[
\lim_{x \to 0} x^{m/(p-1)} U(x) = \kappa'.
\]

In this paper, we investigate the blow-up profile for solutions of (P). Following their arguments, we introduce a rescaled solution of (P):
\[
\varphi(y,s) = (T-t)^{1/2(q-1)} u((T-t)^{1/2} y, t), \quad s = -\log(T-t).
\]
This rescaled solution \( \varphi(y,s) \) solves \( (s T = -\log T) \)
\[
\begin{cases}
\partial_s \varphi = \Delta \varphi - \frac{y}{2} \cdot \nabla \varphi - \frac{\varphi}{2(q-1)} & \text{in } \mathbb{R}^n_+ \times (s T, \infty), \\
\partial_\nu \varphi = \varphi & \text{on } \partial \mathbb{R}^n_+ \times (s T, \infty).
\end{cases}
\]
Then it is shown in [3] that
\[
\lim_{s \to \infty} \varphi(y,s) = \varphi_0(y, n) \quad \text{in } C_{\text{loc}}(\mathbb{R}^n_+),
\]
where \( \varphi_0(y_n) \) is a unique bounded positive solution of (1) (see also [2, 17]). This formula is equivalent to
\[
\frac{u(x,t)}{(T-t)^{1/2(q-1)}} \left( \varphi_0((T-t)^{-1/2} x_n) + o(1) \right), \quad |x| < \nu(T-t)^{1/2}
\]
for any \( \nu > 0 \). By virtue of the asymptotic formula of \( \varphi_0(\xi) \): \( \varphi_0(\xi) \sim c_B \xi^{1/(q-1)} \) as \( \xi \to \infty \), we formally obtain
\[
U(x) = c_B (1 + o(1)) x_n^{1/(q-1)}.
\]
This formal argument was justified in [1]. As is stated above, this formula has no meaning on \( \partial \mathbb{R}^n_+ \), since the right-hand side diverges on \( \partial \mathbb{R}^n_+ \). To obtain the blow-up profile on \( \partial \mathbb{R}^n_+ \), we need to derive more precise large time behavior of a rescaled solution \( \varphi(y,s) \), that is the second approximation formula. To do that we introduce a new function
\[
v(y,s) = \varphi(y,s) - \varphi_0(y_n)
\]
and study the large time behavior of \( v(y,s) \). Then \( v(y,s) \) solves
\[
\begin{cases}
\partial_s v = \Delta v - \frac{y}{2} \cdot \nabla v - \frac{v}{2(q-1)} & \text{in } \mathbb{R}^n_+ \times (0, \infty), \\
\partial_\nu v = q \varphi_0^{q-1} v + O(v^2) & \text{on } \partial \mathbb{R}^n_+ \times (0, \infty).
\end{cases}
\]
A corresponding eigenvalue problem is given by
\[
\begin{cases}
-\left( \Delta E - \frac{y}{2} \cdot \nabla E - \frac{E}{2(q-1)} \right) = \lambda E & \text{in } \mathbb{R}^n_+, \\
\partial_\nu E = q \varphi_0^{q-1} E & \text{on } \partial \mathbb{R}^n_+.
\end{cases}
\]
Let \( E_i(y) \in L^2_\rho(\mathbb{R}^n_+) \) be the \( i \)-th eigenfunction of (5), where \( L^2_\rho(\mathbb{R}^n_+) \) is a weighted \( L^2 \)-space defined by
\[
L^2_\rho(\mathbb{R}^n_+) = \left\{ v \in L^2_{\text{loc}}(\mathbb{R}^n_+); \ |v(y)|^2 e^{-|y|^2/4} dy < \infty \right\}.
\]
The first result in this paper is a classification of the large time behavior of \( v(y,s) \).
Theorem 1.1. Let \( u(x, t) \) be a positive \( n \)-axial symmetric solution of (P) which blows up at the origin and \( v(y, s) \) be given above. Then one of two cases occurs.

(I) there exists \( \nu_0 > 0 \) such that \( \|v(s) - \nu_0 s^{-1} \mathcal{E}\|_{L^2_\rho(\mathbb{R}^n_+)} = o(s^{-1}) \),

(II) there exist \( c > 0 \) and \( \gamma > 0 \) such that \( \|v(s)\|_{L^2_\rho(\mathbb{R}^n_+)} \leq ce^{-\gamma s} \),

where \( \mathcal{E}(y) \) is the eigenfunction of \([6]\) with zero eigenvalue, which is defined in \([12]\).

Theorem 1.2. Let \( u(x, t) \) and \( v(y, s) \) be as in Theorem 1.1. Additionally we assume that

\[ x' \cdot \nabla' u_0 \leq 0. \]

If the case (I) in Theorem 1.1 occurs, then the blow-up profile \( U(x) \in C(\mathbb{R}^n_+ \setminus \{0\}) \) exists and there exist positive constants \( c_1 < c_2 \) such that

\[ c_1 \left( \frac{\log |x'|}{|x'|^2} \right)^{1/2(q-1)} \leq U(|x'|, 0) \leq c_2 \left( \frac{\log |x'|}{|x'|^2} \right)^{1/2(q-1)} \text{ for } |x'| < 1. \]

Remark 1.1. The author in \([11]\) proved that if the initial data \( u_0 \) is \( n \)-axial symmetric and satisfies

\[ x' \cdot \nabla' u_0 \leq 0 \quad (\nabla' u_0 \neq 0), \quad \partial_n u_0 \leq 0, \tag{a} \]

then \( v(y, s) \) actually behaves as the case (I) in Theorem 1.1. As for the case (F), if the initial data is radially symmetric and monotone decreasing, then those properties are preserved for \( t > 0 \). Therefore the solutions has a unique local maximum point at the origin for \( t > 0 \) and no local minimum points for \( t > 0 \). From the view point of this geometry of the solution, it is easily proved that the rescaled solution \( \varphi(y, s) \) satisfies the asymptotic formula (i). However this kind of observation can not be applicable to solutions of (P), since solutions treated here are not radially symmetric.

We explain our strategy of the proof for Theorem 1.1 and Theorem 1.2. Our argument mainly consists of three steps. In the first part, we regard \([1]\) as a dynamical system on the Hilbert space \( L^2_\rho(\mathbb{R}^n_+) \) and study the large time behavior of \( v(y, s) \) in \( L^2_\rho(\mathbb{R}^n_+) \). To do that we expand \( v(y, s) \) by using eigenfunctions of \([6]\) as follows:

\[ v(s) = \sum_{i=1}^{\infty} a_i(s) E_i \text{ in } L^2_\rho(\mathbb{R}^n_+). \]

Here we determine the large time behavior of the coefficients \( a_i(s) \). Each coefficient \( a_i(s) \) satisfies some ordinary differential equations. We will see that these ordinary differential equations are finally reduced to well understood ordinary differential inequalities discussed in \([6]\), and this proves Theorem 1.1.

In the second step, we assume that \( v(y, s) \) behaves as the case (I) in Theorem 1.1 and investigate the large time behavior of \( v(y, s) \) along \( |y| \sim s^{1/2} \) on \( \partial \mathbb{R}^n_+ \). This step is crucial and much harder than the first step. The first step provides the precise decay rate of \( v(y, s) \) on \( L^2_\rho(\mathbb{R}^n_+) \), however the convergence is taken in a local sense. Hence in the original valuable \((x, t)\), the following asymptotic formula:

\[ u(x, t) = (T - t)^{-1/2(q-1)} \varphi((T - t)^{-1/2} x, - \log(T - t)) \]

\[ = (T - t)^{-1/2(q-1)} (\varphi_0((T - t)^{-1/2} x_n) + o(1)) \]

holds only for a small region \(|x| \leq \nu(T - t)^{1/2} \) for any \( \nu > 0 \). Arguments given in \([13]\), \([15]\) revealed that the convergence in this small region is not sufficient to extract information of the blow-up profile
of solutions of (F). In particular for a one dimensional case, they proved that if an initial data is nonnegative, symmetric and nonincreasing, then the solution $u(x,t)$ satisfies

$$
\lim_{t \to T}(T-t)^{(p-1)/2}u(z(T-t)^{1/2}|\log(T-t)|^{1/2},t) = ((p-1) + K_p z^2)^{-1/(p-1)}
$$

in $C_{\text{loc}}(\mathbb{R})$. Unlike this case, we do not know whether this limit exists for solutions of (P). However we will see that there exist $\theta > 0$ and $0 < A_- < A_+$ such that

$$
A_- \leq (T-t)^{1/2(q-1)}u(z(T-t)^{1/2}|\log(T-t)|^{1/2},t)|_{\partial \mathbb{R}_+^n} \leq A_+ \quad \text{for } |z| = \theta.
$$

Finally in the third step, we determine the singularity of the blow-up profile. To do that we introduce another rescaled function:

$$
v_s(x,t) = e^{-ms}u(e^{-s/2}x + \theta \sqrt{se^{-s/2}\bar{e}},T + (t-1)e^{-s}),
$$

where $s \gg 1$ is a parameter and $\bar{e} = (1,0,\cdots,0)$. We establish a uniform pointwise estimate of $v_s(x,t)$ with respect to $s \gg 1$ and $t \in (0,1)$. As a consequence of this uniform pointwise estimate, we can derive Theorem 1.2.

The rest of this paper is organized as follows. In Section 2 we provide eigenfunctions and eigenvalues of (F) and establish the global heat kernel estimates of the linearized equation. Moreover we give a representation formula of solutions of the linearized equation. Section 3 is devoted to study the large time behavior of $v(y,s)$ in $L^2_0(\mathbb{R}_+^n)$. In particular, an exact decay rate of $v(y,s)$ in $L^2_0(\mathbb{R}_+^n)$ is derived. In Section 4 we obtain a refined asymptotic formula of $v(y,s)$. Then the large time behavior of $v(y,s)$ along $|y| \sim s^{1/2}$ on $\partial \mathbb{R}_+^n$ is discussed, which is a crucial part in this paper. Finally in Section 5 we study the blow-up profile by applying the arguments given in [13,22]. In Appendix, some inequalities and some properties of the linearized operator are discussed.

2 Preliminary

Throughout this paper, for simplicity of notations, we put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and

$$
m = 1/2(q-1).
$$

Let $u(x,t)$ be a solution of (P) which blows up at the origin in a finite time $T > 0$. To study the blow-up behavior of $u(x,t)$, we put $s_T = -\log T$ and

$$
\varphi(y,s) = e^{-ms}u(e^{-s/2}y,T-e^{-s}).
$$

Then $\varphi(y,s)$ satisfies

$$
\begin{aligned}
\partial_s \varphi &= \Delta \varphi - \frac{y}{2} \cdot \nabla \varphi - m \varphi \quad \text{in } \mathbb{R}_+^n \times (s_T,\infty), \\
\partial_{\nu} \varphi &= \varphi^q \quad \text{on } \partial \mathbb{R}_+^n \times (s_T,\infty), \\
\varphi(y,s_T) &= T^m u_0(T^{1/2}y) \quad \text{in } \mathbb{R}_+^n.
\end{aligned}
$$

(6)

It is known that if $1 < q < n/(n-2)$, (6) admits the unique bounded positive stationary solution $\varphi_0(y_n)$ depending only on $y_n$-variable, that is a positive solution of

$$
\begin{aligned}
\varphi'' - \frac{\xi}{2} \varphi' - m \varphi &= 0 \quad \text{in } \xi \in \mathbb{R}_+, \\
\partial_\nu \varphi_0(0) &= \varphi_0(0)^q.
\end{aligned}
$$

The existence and the uniqueness of this equation are shown in Lemma 3.1 of [5] and in Theorem 3.1 of [3], respectively. For the rest of this paper, we put

$$
B = \varphi_0(0).
$$

Here we recall the fact concerning the asymptotic behavior of $\varphi(y,s)$.
Theorem 2.1 ([2], Theorem 3.2 in [3]). Let $\varphi(y, s)$ be defined above. Then $\varphi(y, s)$ is uniformly bounded on $\mathbb{R}^n_+ \times (s_T, \infty)$ and $\varphi(y, s)$ converges to $\varphi_0(y_n)$ as $s \to \infty$ uniformly on any compact set in $\mathbb{R}^n_+$.

A boundedness of $\varphi(y, s)$ is the first step to study the large time behavior of $\varphi(y, s)$. Once the boundedness is assured, from the energy identity, we immediately obtain $\varphi_s(y, s) \to 0$ as $s \to \infty$. Furthermore as a consequence of a boundedness of $\varphi(y, s)$, a boundedness of spacial and time derivatives of $\varphi(y, s)$ are also derived.

Lemma 2.1. Let $\varphi(y, s)$ be as in Theorem 2.1. Then for any $\delta > 0$ there exists $c_\delta > 0$ such that

$$\sum_{|\alpha| \leq 2} |D^\alpha \varphi(y, s)| + (1 + |y|)^{-1/2}|\partial_s \varphi(y, s)| \leq c_\delta \quad \text{for } (y, s) \in \mathbb{R}^n_+ \times (s_T + \delta, \infty),$$

where $D$ represents the spacial derivatives.

Proof. The proof follows from the argument in Proposition 1’ of [3]. Since their proof relies only on the scaling argument, the argument can be applicable to (P).

To derive the second approximation for $\varphi(y, s)$ as $s \to \infty$, we set

$$v(y, s) = \varphi(y, s) - \varphi_0(y_n).$$

Then $v(y, s)$ satisfies

$$\begin{cases}
\partial_s v = \Delta v - \frac{y}{2} \cdot \nabla v - mv & \text{in } \mathbb{R}^n_+ \times (s_T, \infty), \\
\partial_t v = qB^{q-1}v + f(v) & \text{on } \partial \mathbb{R}^n_+ \times (s_T, \infty), \\
v(y, s_T) = \varphi(y, s_T) - \varphi_0(y_n) & \text{in } \mathbb{R}^n_+,
\end{cases} \tag{7}$$

where $f(v)$ is given by

$$f(v) = (v + B)q - Bq - qB^{q-1}v.$$ 

Then since $\varphi(y, s)$ is positive and uniformly bounded on $\mathbb{R}^n_+ \times (s_T, \infty)$, we easily see that

$$f(v) = \frac{q(q - 1)B^{q-2}}{2}v^2 + O(v^3), \quad |f'(v)| \leq c'|v| \tag{8}$$

for $(y, s) \in \mathbb{R}^n_+ \times (s_T, \infty)$. We define the linear operator $A$ associated with (7) and its domain by

$$Av = \left( \Delta - \frac{y}{2} \cdot \nabla - m \right) v,$$

$$D(A) = \{ v \in H^2_\rho(\mathbb{R}^n_+); \partial_\nu v = qB^{q-1}v \text{ on } \partial \mathbb{R}^n_+ \},$$

where $H^2_\rho(\mathbb{R}^n_+)$ is a weighted Sobolev space defined in Section 2.1.

2.1 Linear operator $A$

Here we consider the following eigenvalue problems:

$$\begin{cases}
- \left( \Delta - \frac{y}{2} \cdot \nabla - m \right) E = \lambda E & \text{in } \mathbb{R}^n_+, \\
\partial_\nu E = qB^{q-1}E & \text{on } \partial \mathbb{R}^n_+.
\end{cases} \tag{9}$$

We define a weight function:

$$\rho(y) = e^{-|y|^2/4},$$

6
Then it is clear that $\rho(y') = -e^{-|y'|^2/4}$ on $\partial \mathbb{R}^n_+$. Moreover we define functional spaces:

$$L^p_\rho(\mathbb{R}^n_+) = \left\{ v \in L^p_{\text{loc}}(\mathbb{R}^n_+); \int_{\mathbb{R}^n_+} |v(y)|^p \rho(y)dy < \infty \right\},$$

$$H^k_\rho(\mathbb{R}^n_+) = \left\{ v \in L^2_\rho(\mathbb{R}^n_+); D^\alpha v \in L^2_\rho(\mathbb{R}^n_+) \text{ for any } \alpha = (\alpha_1, \cdots, \alpha_n) \text{ satisfying } |\alpha| \leq k \right\},$$

$$L^p_\rho(\partial \mathbb{R}^n_+) = \left\{ v \in L^p_{\text{loc}}(\partial \mathbb{R}^n_+); \int_{\partial \mathbb{R}^n_+} |v(y')|^p \rho(y')dy' < \infty \right\}.$$

The norms are given by

$$\|v\|_{L^p_\rho(\mathbb{R}^n_+)}^p = \int_{\mathbb{R}^n_+} |v(y)|^p \rho(y)dy, \quad \|v\|_{H^2_\rho(\mathbb{R}^n_+)}^2 = \sum_{|\alpha| \leq k} \|D^\alpha v\|_{L^2_\rho(\mathbb{R}^n_+)}^2,$$

$$\|v\|_{L^p_\rho(\partial \mathbb{R}^n_+)}^p = \int_{\partial \mathbb{R}^n_+} |v(y')|^p \rho(y')dy',$$

and the inner product on $L^2_\rho(\mathbb{R}^n_+)$ is naturally defined by

$$(v_1, v_2)_\rho = \int_{\mathbb{R}^n_+} v_1(y)v_2(y)\rho(y)dy.$$ 

For simplicity, the norm of $L^2_\rho(\mathbb{R}^n_+)$ is denoted by $\| \cdot \|_\rho = \| \cdot \|_{L^2_\rho(\mathbb{R}^n_+)}$. Since the operator $A: D(A) \to L^2_\rho(\mathbb{R}^n_+)$ is self-adjoint and has a compact inverse from $L^2_\rho(\mathbb{R}^n_+)$ to $L^2_\rho(\mathbb{R}^n_+)$ (see Appendix), $L^2_\rho(\mathbb{R}^n_+)$ is spanned by the eigenfunctions of $A$. Let $\tilde{H}_k(\xi)$ be the $k$-th Hermite polynomial defined by

$$\tilde{H}_k(\xi) = (-1)^k e^{\xi^2/2} \frac{d^k}{d\xi^k} \left(e^{-\xi^2}\right) \quad (k \in \mathbb{N}_0)$$

and set $H_k(\xi) = c_k \tilde{H}_k(\xi/2)$, where $c_k$ is a normalization constant such that $\int_{-\infty}^{\infty} H_k(\xi)^2 e^{-\xi^2/4}d\xi = 1$.

From classical results, it is known that $H_k(\xi)$ satisfies

$$- \left(H_k'' - \frac{\xi}{2} H_k'\right) = \frac{k}{2} H_k \quad \text{in } \mathbb{R}.$$ 

Moreover let $I_k(\xi)$ and $\kappa_k$ $(k \in \mathbb{N})$ be the $k$-th eigenfunction with $\int_0^{\infty} I_k(\xi) e^{-\xi^2/4}d\xi = 1$ and the $k$-th eigenvalue of

$$\begin{cases} - \left(I''_n - \frac{\xi}{2} I'_n\right) = \kappa I \quad \text{in } \mathbb{R}_+, \\ \partial_\nu I(0) = qB^{n-1}I(0) \end{cases} \quad (10)$$

and $\alpha$ be a multi-index:

$$\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathcal{A} := \mathbb{N}_0^{n-1} \times \mathbb{N}.$$ 

Then the eigenfunction $E_{\alpha}(y)$ of $A$ and its eigenvalue $\lambda_{\alpha}$ are given by

$$E_{\alpha}(y) = H_{\alpha_1}(y_1) \cdots H_{\alpha_{n-1}}(y_{n-1})I_{\alpha_n}(y_n), \quad \lambda_{\alpha} = \sum_{i=1}^{n-1} \frac{\alpha_i}{2} + \kappa_{\alpha_n} + m.$$ 

Here we recall a classical result about some special functions (see Lemma 3.1 in [5]).
Lemma 2.2. Let \( b(\xi) \in L^2_\rho(\mathbb{R}_+) \) be a positive solution of
\[
- \left( b'' - \frac{\xi}{2} b' \right) = \mu b \quad \text{in } \mathbb{R}_+ \quad (\mu < 0)
\]
and set
\[
U(a, a', r) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-rt} t^{a-1}(1+t)^{a'-1}dt \quad (a, r > 0),
\]
where \( \Gamma \) is the Gamma function. Then there exists \( c_1 > 0 \) such that \( b(\xi) \) is expressed by
\[
b(\xi) = c_1 U(-\mu, 1/2, \xi^2/4).
\]
By using this formula, we compute the first and the second eigenvalue of (10).

Lemma 2.3. Let \( \kappa_i \) be the \( i \)-th eigenvalue of (10). Then it holds that
\[
\kappa_1 = -(m+1), \quad \kappa_2 > 0.
\]

Proof. Since \( \kappa_1 \) is the first eigenvalue of (10), it is characterized by
\[
\kappa_1 = \inf_{I \in H^1_\rho(\mathbb{R}_+)} \| \partial_\xi I \|_{L^2_\rho(\mathbb{R}_+)}^2 - qB^{q-1}I(0)^2 \| I \|_{L^2_\rho(\mathbb{R}_+)}^2.
\]
This implies \( \kappa_1 < 0 \). Therefore by Lemma 2.2 there exists \( c_1 > 0 \) such that the first eigenfunction \( I_1(\xi) \) is given by
\[
I_1(\xi) = c_1 U\left( -\kappa_1, 1/2, \xi^2/4 \right).
\]
From (3.8) in [5], we note that for \( a > 0 \)
\[
U(a, 1/2, 0) = \frac{\sqrt{\pi}}{\Gamma(a + 1/2)}, \quad \lim_{\xi \to 0} \partial_\xi U(a, 1/2, \xi^2/4) = -\frac{a\sqrt{\pi}}{\Gamma(a + 1)} \quad (11)
\]
Therefore we obtain
\[
\partial_\nu I_1(0)/I_1(0) = (-\kappa_1) \Gamma \left( -\kappa_1 + 1/2 \right) / \Gamma(-\kappa_1 + 1).
\]
On the other hand, we recall from the proof of Lemma 3.1 in [5] that \( \varphi_0(\xi) \) is written by
\[
\varphi_0(\xi) = \frac{m^{1/(q-1)}}{\sqrt{\pi}} \left( \frac{\Gamma(m + 1/2)^q}{\Gamma(m + 1)} \right)^{1/(q-1)} U(m, 1/2, \xi^2/4).
\]
Then by using (11), we get
\[
qB^{q-1} = q\varphi_0(0)^{q-1} = qm \Gamma \left( m + 1/2 \right) / \Gamma(m + 1).
\]
Therefore since \( \partial_\nu I_1(0) = qB^{q-1}I_1(0) \), \( \kappa_1 \) is determined by
\[
qm \Gamma \left( m + 1/2 \right) / \Gamma(m + 1) = (-\kappa_1) \Gamma \left( -\kappa_1 + 1/2 \right) / \Gamma(-\kappa_1 + 1). \quad (12)
\]
First we claim that (12) admits at most one root. We set
\[
g(\lambda) = \lambda \Gamma \left( \lambda + 1/2 \right) / \Gamma(\lambda + 1) \quad (\lambda > 0).
\]
By $\Gamma(\lambda + 1) = \lambda \Gamma(\lambda)$, it follows that

$$g(\lambda) = \Gamma\left(\lambda + \frac{1}{2}\right)/\Gamma(\lambda). \quad (13)$$

Hence from the property of the Gamma function (see p. 4 in [16]), $g(\lambda)$ is strictly increasing function for $\lambda > 0$, which shows the claim. To assure $\kappa_1 = -(m + 1)$, we substitute $\lambda = m + 1$ in (13):

$$g(m + 1) = \Gamma\left(m + \frac{3}{2}\right)/\Gamma(m + 1) = \left(m + \frac{1}{2}\right) \Gamma\left(m + \frac{1}{2}\right)/\Gamma(m + 1)
= qm \Gamma\left(m + \frac{1}{2}\right)/\Gamma(m + 1).$$

Therefore we conclude that $\kappa_1 = -(m + 1)$. Next we show that $\kappa_2 > 0$. We suppose $\kappa_2 \leq 0$. For the case $\kappa_2 < 0$, by the same way as above, the second eigenfunction $I_2(\xi)$ is given by

$$I_2(\xi) = c_1 U\left(-\kappa_2, \frac{\xi^2}{4}\right)$$
for some $c_1 \neq 0$. Hence it holds that $I_2(\xi) > 0$ on $\mathbb{R}_+$ or $I_2(\xi) < 0$ on $\mathbb{R}_+$. However this contradicts $\int_0^\infty I_1(\xi) I_2(\xi) e^{-\xi^2/4} d\xi = 0$. For the case $\kappa_2 = 0$, we easily see that

$$I_2'(\xi) = -qB^{-1} I_2(0) e^{\xi^2/4}$$

and

$$I_2(\xi) = I_2(0) - qB^{-1} I_2(0) \int_0^\xi e^{t^2/4} dt.$$ 

Hence it follows that $I_2 \notin L^2_\mu(\mathbb{R})$, which is a contradiction. Thus the proof is completed.

From the above facts, the operator $(-A)$ has two negative eigenvalues $-1$ and $-1/2$ and zero eigenvalue.

### 2.2 Linear backward heat equation with Robin type boundary conditions

Here we study the following linear parabolic equation:

$$\begin{cases}
\partial_t v = \Delta v - \frac{y}{2} \cdot \nabla v & \text{in } \mathbb{R}_n^+ \times (0, \infty), \\
\partial_\nu v = K v & \text{on } \partial \mathbb{R}_n^+ \times (0, \infty),
\end{cases}$$

where $K$ is a positive constant. When $K = qB^{-1}$, this equation coincides with the linearized equation of (6) around $\varphi_0$. Let $b_K(\xi)$ and $\mu_K < 0$ be the first positive eigenfunction with $b_K(0) = 1$ and the first eigenvalue of

$$\begin{cases}
- (b'' - \frac{\xi}{2} b') = \mu b & \text{in } \mathbb{R}_+, \\
\partial_\nu b(0) = K b(0).
\end{cases}$$

Then by Lemma 2.2 with (11), we find that $b_K(\xi)$ is written by

$$b_K(\xi) = \frac{\Gamma(-\mu_K + 1/2)}{\sqrt{\pi}} U(-\mu_K, 1/2, \xi^2/4)
= c_K \int_0^\infty e^{-t^2/4} t^{-\mu_K - 1} (1 + t)^{-1/2 + \mu_K} dt.$$
where \(c_K = \Gamma(-\mu_K + 1/2)/\sqrt{\pi}\Gamma(-\mu_K)\). To estimate the above integral we change variables \(s = \xi^2 t\) and integrate by parts, then we get
\[
b_K(\xi) = c_K \xi^{2\mu_K} \int_0^\infty e^{-s/4s-\mu_K-1} (1 + \xi^{-2}s)^{-1/2+\mu_K} ds
\]
\[
\leq c_K \xi^{2\mu_K} \int_0^\infty e^{-s/4s-\mu_K-1} ds.
\]
Here we fix \(K_0 > 1\). Then there exist two positive constants \(C_1 < C_2\) such that \(C_1 \leq -\mu_K \leq C_2\) for \(K_0^{-1} < K < K_0\). Therefore there exists \(C_0 > 0\) such that
\[
\sup_{K_0^{-1} < K < K_0} \int_0^\infty e^{-s/4s-\mu_K-1} ds \leq C_0.
\]
As a consequence, we obtain for \(K_0^{-1} \leq K \leq K_0\)
\[
b_K(\xi) \leq C_0 c_K \xi^{2\mu_K}.
\]
Furthermore since \(b_K(0) = 1\) and \(b'_K(0) = -K\), there exists \(C'_0 > 0\) such that
\[
\sup_{K_0^{-1} < K < K_0} \sup_{0 < \xi < 1} b_K(\xi) \leq C'_0. \tag{14}
\]
Combining the above estimates, we obtain the following lemma.

**Lemma 2.4.** For any \(K_0 > 1\) there exists \(c = c(K_0) > 0\) such that for \(K_0^{-1} < K < K_0\)
\[
b_K(\xi) \leq c(1 + \xi)^{2\mu_K} \quad \text{for } \xi \in \mathbb{R}^+.
\]

Next we compute \(b'_K(\xi)/b_K(\xi)\) and \(b''_K(\xi)/b_K(\xi)\). By using integration by parts, we get
\[
b'_K(\xi) = -\frac{c_K \xi}{2} \int_0^\infty e^{-\xi^2 t/4} t^{-\mu_K} (1 + t)^{-1/2+\mu_K} dt
\]
\[
= -\frac{2c_K \xi}{\xi} \int_0^\infty e^{-\xi^2 t/4} t^{-\mu_K} (1 + t)^{-1/2+\mu_K} \frac{d}{dt} dt
\]
\[
= \frac{2c_K \xi}{\xi} \int_0^\infty e^{-\xi^2 t/4} \left( \mu_K + \frac{1}{2} - \mu_K \right) t^{-\mu_K-1} (1 + t)^{-1/2+\mu_K} dt.
\]
Hence it holds that
\[
\frac{|b'_K(\xi)|}{b_K(\xi)} \leq \frac{1 + 4|\mu_K|}{\xi}.
\]
Here we again fix \(K_0 > 1\). Since \(|\mu_K| < c\) for \(K \in (0, K_0)\), there exists \(c > 0\) such that
\[
\sup_{0 < K < K_0} \sup_{\xi \geq 1} \left( \frac{|b'_K(\xi)|}{b_K(\xi)} \right) \leq c.
\]
Repeating the above argument, we see that
\[
\sup_{0 < K < K_0} \sup_{\xi \geq 1} \left( \frac{|b''_K(\xi)|}{b_K(\xi)} \right) \leq c'.
\]
Furthermore by the same way as \([14]\), we obtain
\[
\sup_{0 < K < K_0} \sup_{\xi \leq 1} \left( \frac{|b'_K(\xi)| + |b''_K(\xi)|}{b_K(\xi)} \right) \leq c''.
\]
Therefore we conclude the following lemma.
Lemma 2.5. Let $B_K(\xi) = b'_K(\xi)/b_K(\xi)$. Then for any $K_0 > 1$ there exists $c = c(K_0) > 0$ such that

$$\sup_{0 < K < K_0} \sup_{\xi \in \mathbb{R}_+} (B_K(\xi) + |B'_K(\xi)|) \leq c.$$ 

We introduce another function:

$$w_K(y, s) = e^{\mu K s} v(y, s)/b_K(y_n).$$

Then $w_K(y, s)$ satisfies

$$\begin{cases} \partial_s w_K = \Delta w_K - \frac{y}{2} \cdot \nabla w_K + 2B_K(y_n)\partial_n w_K & \text{in } \mathbb{R}^n_+ \times (0, \infty), \\ \partial_s w_K = 0 & \text{on } \partial \mathbb{R}^n_+ \times (0, \infty), \end{cases} \quad (15)$$

where the coefficient $B_K(y_n)$ is given by

$$B_K(y_n) = \left( \frac{b'_K(y_n)}{b_K(y_n)} \right).$$

First we construct a heat kernel $\Gamma_K(y, \xi, s)$ of (15) and provide time local heat kernel estimates. Let $\gamma'(y', \xi', s)$ be the heat kernel of $\partial_s \theta = \Delta' \theta - \frac{y'}{2} \cdot \nabla' \theta$ in $\mathbb{R}^{n-1} \times (0, \infty)$, which is explicitly given by (c.f. p. 141 in [15])

$$\gamma'(y', \xi', s) = \left( \frac{1}{4\pi} \right)^{(n-1)/2} \left( \frac{1}{1 - e^{-s}} \right)^{(n-1)/2} \exp \left( -\frac{|y' e^{-s/2} - \xi'|}{4(1 - e^{-s})} \right).$$

Now we construct a heat kernel $\gamma_K(z, \zeta, s)$ of

$$\begin{cases} \partial_s \vartheta = \vartheta_{zz} - \frac{z}{2} \vartheta_z + 2B_K(z)\vartheta_z & \text{in } \mathbb{R}_+ \times (0, \infty), \\ \partial_s \vartheta = 0 & \text{on } z = 0, \ s \in (0, \infty). \end{cases} \quad (16)$$

Let $\vartheta_K(z, s)$ be a solution of (16) and put

$$U_K(x, t) = \vartheta_K \left( (1 - t)^{-1/2} x, -\log(1 - t) \right) \quad (x \in \mathbb{R}_+).$$

Then $U_K(x, t)$ solves

$$\begin{cases} \partial_t U = U_{xx} + 2 \left( \frac{B_K(x, t)}{\sqrt{1 - t}} \right) U_x & \text{in } \mathbb{R}_+ \times (0, 1), \\ \partial_x U = 0 & \text{on } x = 0, \ t \in (0, 1), \end{cases} \quad (17)$$

where $B_K(x, t) = B_K(x/\sqrt{1 - t})$. Let $G_0(x, \xi, t)$ be the standard heat kernel with the Neumann boundary condition on $\mathbb{R}_+$ given by

$$G_0(x, \zeta, t) = \frac{1}{\sqrt{4\pi t}} \left( e^{-|x-\zeta|^2/4t} + e^{-|x+\zeta|^2/4t} \right).$$

It is known that by the Levi parametrix method (see pp. 356–363 in [19]), (17) admits the heat kernel $G_K(x, \zeta, t, \tau)$ which has the following form:

$$G_K(x, \zeta, t, \tau) = G_0(x, \zeta, t - \tau) + \mathcal{G}_K(x, \zeta, t, \tau),$$
where $G_K(x, \zeta, t, \tau)$ is given by
\begin{equation}
G_K(x, \zeta, t, \tau) = \int_x^t d\sigma \int_0^\infty G_0(x, \eta, t - \sigma) Q_K(\eta, \zeta, \sigma, \tau) d\eta
\end{equation}
and $Q_K(\eta, \zeta, \sigma, \tau)$ is a unique solution of the integral equation of
\begin{equation}
Q_K(\eta, \zeta, \sigma, \tau) = F_K(\eta, \zeta, \sigma, \tau) + \int_x^t d\nu \int_0^\infty F_K(\eta, \mu, \nu, \sigma) Q_K(\mu, \zeta, \nu, \tau) d\mu
\end{equation}
with\[F_K(\eta, \zeta, \sigma, \tau) = 2 \left( \frac{B_K(\eta, \sigma)}{\sqrt{1 - \sigma}} \right) \partial_x G_0(\eta, \zeta, \sigma - \tau)\]
Here we put \[G_0^0(\eta, \zeta, t) = \frac{1}{\sqrt{4\pi t}} \left( \exp \left( -\frac{(\eta - \zeta)^2}{4(1 + \epsilon)t} \right) + \exp \left( -\frac{(\eta + \zeta)^2}{4(1 + \epsilon)t} \right) \right)\]
and fix three constants \[K_0 > 1, \quad \epsilon_0 \in (0, 1), \quad \sigma_0 \in (0, 1)\].
Then by Lemma 2.3 for any $a \in (0, 1)$ there exists $c = c(a) > 0$ such that for $K \in (0, K_0)$ and $\epsilon \geq \epsilon_0$
\[|F_K(\eta, \zeta, \sigma, \tau)| \leq c \left( \frac{G_0^0(\eta, \zeta, \sigma - \tau)}{\sqrt{\sigma - \tau}} \right),\]
\[|F_K(\eta, \zeta, \sigma, \tau) - F_K(\eta', \zeta, \sigma, \tau)| \leq c \frac{|\eta - \eta'|}{(\sigma - \tau)^{a_1}} \left( G_0^0(\eta, \zeta, \sigma - \tau) + G_0^0(\eta', \zeta, \sigma - \tau) \right)\]
for $\eta, \zeta \in \mathbb{R}_+$ and $0 < \sigma < \sigma_*, \eta = a_1 = (1 + a)/2$. Hence by the same calculations as in pp.
360–363 in [19], for any $a' \in (0, 1)$ there exist $c = c(a') > 0$ such that for $K \in (0, K_0)$ and $\epsilon \geq \epsilon_0$
\[|Q_K(\eta, \zeta, \sigma, \tau)| \leq c \frac{G_0^0(\eta, \zeta, \sigma - \tau)}{\sqrt{\sigma - \tau}}\]
\[|Q_K(\eta, \zeta, \sigma, \tau) - Q_K(\eta', \zeta, \sigma, \tau)| \leq c \frac{|\eta - \eta'|}{(\sigma - \tau)^{a_1}} \left( G_0^0(\eta, \zeta, \sigma - \tau) + G_0^0(\eta', \zeta, \sigma - \tau) \right)\]
for $\eta, \zeta \in \mathbb{R}_+$ and $0 < \sigma < \sigma_*$, where $a_1 = (1 + a)/2$. Therefore by the same way as in pp.
376–378 in [19], we find that for any $t_0 \in (0, 1)$ there exists $c = c(t_0) > 0$ such that for any $K \in (0, K_0)$ and $\epsilon \geq \epsilon_0$
\[|\partial_x^i G_K(x, \zeta, t, \tau)| \leq c(t - \tau)^{(1 - i)/2} G_0^0(x, \zeta, t - \tau) \quad (i = 0, 1, 2),\]
\[|\partial_t G_K(x, \zeta, t, \tau)| \leq c(t - \tau)^{-1/2} G_0^0(x, \zeta, t - \tau)\]
for $x, \zeta \in \mathbb{R}_+$ and $0 < \tau < t < t_0$. Moreover from (18) and (19), a direct computation shows that
\[\lim_{x \to 0^+} \partial_x \partial_t G_K(x, \zeta, t, \tau) = 0 \quad \text{for } \zeta \in \mathbb{R}_+, \quad t > \tau \geq 0.\]
Therefore we find that $G_K(x, \zeta, t, \tau)$ satisfies (17) for $0 < \tau < t < 1$. Moreover from (20), it holds
that for $f \in BC(\mathbb{R}_+)$
\[\lim_{t \to 0^+} \int_0^\infty G_K(x, \zeta, t, \tau) f(\zeta) d\zeta = \lim_{t \to 0^+} \int_0^\infty G_0(x, \zeta, t - \tau) f(\zeta) d\zeta = f(x).\]
As a consequence, $G_K(x, \zeta, t, \tau)$ turns out to be the heat kernel of (17). Going back to the variable
$(z, s)$, we obtain the heat kernel of (16):
\[\gamma_K(z, s) = G_K \left( e^{-s/2}z, \zeta, 1 - e^{-s}, 0 \right).\]
Then from (20), for any \( s \) there exists \( c = c(s_*) > 0 \) such that for \( K \in (0, K_0) \) and \( \epsilon \geq \epsilon_0 \)

\[
\gamma_K(z, \zeta, s) \leq cG_{0}^\epsilon \left( e^{-s/2} z, \zeta, 1 - e^{-s}, 0 \right)
\]

\[
= \frac{c}{(1 - e^{-s})^{1/2}} \left( \exp \left( -\frac{(ze^{-s/2} - \zeta)^2}{4(1 + \epsilon)(1 - e^{-s})} \right) + \exp \left( -\frac{(ze^{-s/2} + \zeta)^2}{4(1 + \epsilon)(1 - e^{-s})} \right) \right)
\]

(22)

for \( z, \zeta \in \mathbb{R}_+ \) and \( s \in (0, s_*) \). Since the heat kernel \( \Gamma_K(y, \xi, s) \) of (15) is given by

\[
\Gamma_K(y, \xi, s) = \gamma'(y', \xi', s) \gamma_K(y_n, \xi_n, s),
\]

summarizing the above estimates, we obtain the following lemma.

**Lemma 2.6.** For any \( K_0 > 1 \), \( \epsilon_0 > 0 \) and \( s_* > 0 \) there exists \( c = c(K_0, \epsilon_0, s_*) > 0 \) such that for \( K \in (0, K_0) \) and \( \epsilon \geq \epsilon_0 \)

\[
\Gamma_K(y, \xi, s) \leq \frac{c}{(1 - e^{-s})^{n/2}} \exp \left( -\frac{|y' e^{-s/2} - \xi'|}{4(1 - e^{-s})} \right) \exp \left( -\frac{|y' e^{-s/2} + \xi'|}{4(1 + \epsilon)(1 - e^{-s})} \right) + \exp \left( -\frac{(ze^{-s/2} + \zeta)^2}{4(1 + \epsilon)(1 - e^{-s})} \right)
\]

for \( y, \xi \in \mathbb{R}_+^n \), \( s \in (0, s_*) \).

As a consequence of this heat kernel estimate, we obtain \( L^p - L^p \) type estimates. Here we define another weighted \( L^p \)-space:

\[
\|w\|_{L^p_{K, \rho} (\mathbb{R}_+^n)} = \int_{\mathbb{R}_+^n} |w(y)|^p b_K(y_n)^2 \rho(y) dy.
\]

**Lemma 2.7.** For any \( K_0 > 1 \), \( p, r \in (1, \infty) \), \( \delta \in (0, p - 1) \) and \( s_* > 0 \) there exists \( c = c(K_0, p, r, s, \delta) > 0 \) such that for \( K \in (K_0^{-1}, K_0) \) and \( s \in (\max\{0, \log\{(r - 1)/(p - 1 - \delta)\}\}, s_*) \)

\[
\begin{align*}
\|\int_{\mathbb{R}_+^n} \Gamma_K(y', \xi, s)|w_0(\xi)|d\xi\|_{L^p_{K, \rho}(\mathbb{R}_+^n)} & \leq c(1 - e^{-s})^{-n/2p} \|w_0\|_{L^p_{K, \rho}(\mathbb{R}_+^n)}, \\
\|\int_{\partial \mathbb{R}_+^n} \Gamma_K(y', \xi', s)|w_0(\xi')|d\xi'\|_{L^p_{K, \rho}(\partial \mathbb{R}_+^n)} & \leq c(1 - e^{-s})^{-((n - 1)/2p - 1/2)} \|w_0\|_{L^p_{K, \rho}(\partial \mathbb{R}_+^n)}.
\end{align*}
\]

**Proof.** Let \( y' \in \partial \mathbb{R}_+^n \) and \( p' > 1 \) be a dual exponent of \( p \). From the Hölder inequality, we easily see that

\[
\begin{align*}
\int_{\mathbb{R}_+^n} \Gamma_K(y', \xi, s)|w_0(\xi)|d\xi & \leq \|w_0\|_{L^p_{K, \rho}(\mathbb{R}_+^n)} \left( \int_{\mathbb{R}_+^n} \Gamma_K(y', \xi, s)^{p'} \left( \frac{e|\xi|^2}{b_K(\xi_n)^{2/p}} \right)^{p'} d\xi \right)^{1/p'}, \\
\int_{\partial \mathbb{R}_+^n} \Gamma_K(y', \xi', s)|w_0(\xi')|d\xi' & \leq \|w_0\|_{L^p_{K, \rho}(\partial \mathbb{R}_+^n)} \left( \int_{\mathbb{R}_+^n} \left( \Gamma_K(y', \xi', s)e(|\xi'|^2/4p) \right)^{p'} d\xi' \right)^{1/p'}.
\end{align*}
\]

Here we note that

\[
\frac{|y' e^{-s/2} - \xi'|}{4(1 - e^{-s})} = \frac{|\xi'|^2}{4p} = \frac{p - (1 - e^{-s})}{4p} \left( \xi' - \frac{pe^{-s/2}y'}{p - (1 - e^{-s})} \right)^2 - \frac{e^{-s}|y'|^2}{4(p - (1 - e^{-s}))}
\]

(24)
and
\[ \frac{\xi^2}{4(1 + \epsilon)(1 - e^{-s})} - \frac{\xi^2}{4p} = \frac{\xi^2}{4p(1 + \epsilon)(1 - e^{-s})}(p - (1 + \epsilon)(1 - e^{-s})). \]

Therefore we apply Lemma 2.6 with \( \epsilon = (p - 1)/2 \) to get
\[
\int_{\mathbb{R}^n_+} \Gamma_K(y', \xi, s)^{\nu'} \frac{e^{\|\xi\|^2/4p}}{b_K(\xi_n)^{2/p}} \, d\xi \leq \left( \frac{c}{(1 - e^{-s})^{n/2p}} \right)^{\nu'} \exp \left( \frac{p^e s |y'|^2}{4(p - (1 - e^{-s}))} \right),
\]
\[
\int_{\partial \mathbb{R}^n_+} \Gamma_K(y', \xi', s)^{\nu'} e^{\|\xi\|^2/4p} \, d\xi' \leq \left( \frac{c}{(1 - e^{-s})^{(n-1)/2p+1/2}} \right)^{\nu'} \exp \left( \frac{p^e s |y'|^2}{4(p - (1 - e^{-s}))} \right),
\]
where we used \( b_K(y_n) \leq c(1 + y_n)^2 \mu_K \) for \( K \in (K_0^{-1}, K_0) \) (see Lemma 2.4). From these estimates, we obtain
\[
\int_{\mathbb{R}^n_+} \Gamma_K(y', \xi, s)|w_0(\xi)| \, d\xi \leq \left( \frac{c}{(1 - e^{-s})^{n/2p}} \right) \|w_0\|_{L_p^p(\mathbb{R}^n_+)}^{\nu'},
\]
\[
\int_{\partial \mathbb{R}^n_+} \Gamma_K(y', \xi', s)|w_0(\xi')| \, d\xi' \leq \left( \frac{c}{(1 - e^{-s})^{(n-1)/2p+1/2}} \right) \|w_0\|_{L_p^p(\partial \mathbb{R}^n_+)}^{\nu'},
\]
Moreover we note that
\[
\frac{r e^{-s}|y'|^2}{4(p - (1 - e^{-s}))} - \frac{|y'|^2}{4} = - \left( \frac{(p - 1) - (r - 1)e^{-s}}{4(p - (1 - e^{-s}))} \right) |y'|^2.
\]
Then it holds that for \( \delta \in (0, p - 1) \)
\[
(p - 1) - (r - 1)e^{-s} \geq \delta \quad \text{if} \quad s \geq \log \left( \frac{r - 1}{p - 1 - \delta} \right).
\]
Hence it follows that
\[
\left\| \int_{\mathbb{R}^n_+} \Gamma_K(y', \xi, s)|w_0(\xi)| \, d\xi \right\|_{L_p^p(\partial \mathbb{R}^n_+)} \leq \left( \frac{c_\delta^{-(n-1)/2v}}{(1 - e^{-s})^{n/2p}} \right) \|w_0\|_{L_p^p(\mathbb{R}^n_+)},
\]
\[
\left\| \int_{\partial \mathbb{R}^n_+} \Gamma_K(y', \xi', s)|w_0(\xi')| \, d\xi' \right\|_{L_p^p(\partial \mathbb{R}^n_+)} \leq \left( \frac{c_\delta^{-(n-1)/2v}}{(1 - e^{-s})^{(n-1)/2p+1/2}} \right) \|w_0\|_{L_p^p(\partial \mathbb{R}^n_+)},
\]
for \( s \geq \log \{|(r - 1)/(p - 1 - \delta)\} \), which completes the proof. \( \square \)

Next we provide time global estimates of \( \gamma_K(z, \zeta, s) \) to establish time global estimates of \( \Gamma_K(y, \xi, s) \). We fix \( \theta \in (0, 1) \) such that \( (1 - \theta)^2/\theta^2 = 2 \) and set
\[
\Lambda(z, \zeta, s) = \frac{1}{(1 - e^{-s})^{1/2}} \left( \exp \left( -\frac{|z e^{-s/2} - \theta \zeta|^2}{4(1 - e^{-s})} \right) + \exp \left( -\frac{|ze^{-s/2} + \theta \zeta|^2}{4(1 - e^{-s})} \right) \right).
\]
Then \( \Lambda(z, \zeta, s) \) satisfies for \( \zeta \in \mathbb{R}_+ \)
\[
\begin{align*}
\partial_s \Lambda = \Lambda_{zz} - \frac{z}{2} \Lambda_z & \quad \text{in} \ \mathbb{R}_+ \times (0, \infty), \\
\Lambda_z = 0 & \quad \text{on} \ z = 0, \ s \in (0, \infty). 
\end{align*}
\]
Lemma 2.8. For any $K_0 > 1$ there exists $c_0 = c_0(K_0) > 0$ such that for $K \in (0, K_0)$

$$\gamma_K(z, \zeta, s) \leq c_0(1 - e^{-s})^{-1/2} \quad \text{for } z, \zeta, s \in \mathbb{R}_+, \ s \geq 0.$$  

Moreover for any $K_0 > 1$ there exists $c_1 = c_1(K_0) > 0$ such that for $K \in (0, K_0)$

$$\gamma_K(z, \zeta, s) \leq c_1 \Lambda(z, \zeta, s)$$

for $z \in (0, (\theta\zeta - 2)e^{s/2})$, $\zeta \in (2/\theta, \infty)$ and $s \geq 1$.

Proof. We fix $c_0 = s_* = 1$ in (22). Then there exists $c_0 > 0$ such that for $K \in (0, K_0)$

$$\gamma_K(z, \zeta, s) \leq c_0(1 - e^{-s})^{-1/2} \quad \text{for } s \in (0, 1).$$

Since a constant function $\bar{\vartheta}(z, s) := c_0(1 - e^{-1})^{-1/2}$ is a solution of (16), from a comparison argument, we obtain $\gamma_K(z, \zeta, s) \leq \bar{\vartheta}(z, s)$ for $s \geq 1$. Since $(1 - e^{-s})^{-1/2} \geq 1$ for $s \geq 0$, it holds that $\gamma_K(z, \zeta, s) \leq c_0(1 - e^{-1})^{-1/2}(1 - e^{-s})^{-1/2}$ for $s \geq 1$, which show the first statement. Next we show the second statement. Since $(1 - \theta)/\sqrt{2} = \sqrt{2}$, we note that

$$0 < \theta\zeta e^{s/2} - z \leq \theta\zeta e^{s/2} \leq \frac{\theta}{1 - \theta}(\zeta e^{s/2} - z) = \frac{1}{\sqrt{2}}(\zeta e^{s/2} - z) \quad \text{for } z \leq \theta\zeta e^{s/2}. \quad (26)$$

We fix $c_0 = s_* = 1$ in (22) again. Then by (22) and (26), we see that

$$\gamma_K(z, \zeta, 1) \leq \frac{2c}{(1 - e^{-1})^{1/2}} \exp\left(-\frac{|ze^{-1/2} - \zeta|^2}{8(1 - e^{-1})}\right) \leq \frac{2c}{(1 - e^{-1})^{1/2}} \exp\left(-\frac{|ze^{-1/2} - \theta\zeta|^2}{4(1 - e^{-1})}\right) \leq 4c\Lambda(z, \zeta, 1) \quad \text{for } z \leq \theta\zeta e^{1/2}. \quad (27)$$

By definition of $\Lambda$, there exists $a_1 > 0$ such that

$$\Lambda\left((\theta\zeta - 2)e^{s/2}, \zeta, s\right) \geq \frac{e^{-1/(1-e^{-s})}}{(1 - e^{-s})^{1/2}} \geq a_1 \quad \text{for } \zeta \in (2/\theta, \infty), \ s \geq 1.$$  

From the first statement, we can define

$$a_2 = \sup_{s \geq 1} \sup_{z, \zeta \in \mathbb{R}_+} \gamma_K(z, \zeta, s).$$

Hence we obtain

$$\gamma_K\left((\theta\zeta - 2)e^{s/2}, \zeta, s\right) \leq \left(\frac{a_2}{a_1}\right) \Lambda\left((\theta\zeta - 2)e^{s/2}, \zeta, s\right) \quad \text{for } \zeta \in (2/\theta, \infty), \ s \geq 1. \quad (28)$$

Now we claim that

$$\Lambda_z(z, \zeta, s) \geq 0 \quad (29)$$

for $z \in (0, (\theta\zeta - 2)e^{s/2})$ and $\zeta \in (2/\theta, \infty)$. By definition of $\Lambda$, we see that

$$\Lambda_z(z, \zeta, s) = \frac{e^{-s/2}}{(1 - e^{-s})} \left(\frac{\theta\zeta - ze^{-s/2}}{2\sqrt{1 - e^{-s}}}\exp\left(-\frac{|ze^{-s/2} - \theta\zeta|^2}{4(1 - e^{-s})}\right) - \frac{\theta\zeta + ze^{-s/2}}{2\sqrt{1 - e^{-s}}}\exp\left(-\frac{|ze^{-s/2} + \theta\zeta|^2}{4(1 - e^{-s})}\right)\right). \quad (30)$$
Let \( \zeta_1 = \theta \zeta - ze^{-s/2}/2\sqrt{1 - e^{-s}} \) and \( \zeta_2 = \theta \zeta + ze^{-s/2}/2\sqrt{1 - e^{-s}}, \) and set \( g(\zeta) = \zeta e^{-\zeta^2}. \) Then (29) is rewritten by
\[
\Lambda_z(z, \zeta, s) = \frac{e^{-s/2}}{1 - e^{-s}} (g(\zeta_1) - g(\zeta_2)).
\]

Then we observe that
\[
\zeta_1 = \frac{\theta \zeta - ze^{-s/2}}{2\sqrt{1 - e^{-s}}} \geq \frac{\theta \zeta - (\theta \zeta - 2)}{2\sqrt{1 - e^{-s}}} = \frac{1}{\sqrt{1 - e^{-s}}} \geq 1
\]
for \( z \in (0, (\theta \zeta - 2)e^{s/2}) \) and \( \zeta \in (2/\theta, \infty). \) Since \( g(\zeta) = \zeta e^{-\zeta^2} \) is monotone decreasing for \( \zeta \geq 1/\sqrt{2}, \) it follows from \( \zeta_1 < \zeta_2 \) that
\[
\Lambda_z(z, \zeta, s) = \frac{e^{-s/2}}{1 - e^{-s}} (g(\zeta_1) - g(\zeta_2)) > 0
\]
for \( z \in (0, (\theta \zeta - 2)e^{s/2}) \) and \( \zeta \in (2/\theta, \infty), \) which shows the claim. Therefore since \( b'_K(z) < 0, \) it holds from (29) that
\[
B_K(z)\Lambda_z(z, \zeta, s) = \left( \frac{b'_K(z)}{b_K(z)} \right) \Lambda_z(z, \zeta, s) \leq 0
\]
for \( z \in (0, (\theta \zeta - 2)e^{s/2}) \) and \( \zeta \in (2/\theta, \infty). \) Therefore since \( \Lambda(z, \zeta, s) \) satisfies (25) for \( \zeta \in \mathbb{R}_+, \) we obtain
\[
\left( \partial_s - \partial_{zz}^2 + \frac{z}{2} \partial_z - 2B_K(z)\partial_z \right) \Lambda(z, \zeta, s) \geq 0
\]
for \( z \in (0, (\theta \zeta - 2)e^{s/2}) \) and \( \zeta \in (2/\theta, \infty). \) Here we apply a comparison argument in an expanding domain \( Q = \{(z, s) ; z \in (0, (\theta \zeta - 2)e^{s/2}) \} \) for \( \zeta \in (2/\theta, \infty). \) From (27) and (28), we conclude that
\[
\gamma_K(z, \zeta, s) \leq \max \{ 4c, a_2/a_1 \} \Lambda(z, \zeta, s) \quad \text{for} \quad (z, s) \in Q,
\]
hence the proof is completed. \( \square \)

Finally we provide time global \( L^\infty_+L^2_{K,p} \) estimates of solutions of (14).

**Lemma 2.9.** For any \( K_0, R > 1 \) there exists \( c = c(K_0, R) > 0 \) such that for \( K \in (0, K_0) \)
\[
\int_{\mathbb{R}^n_+} \Gamma_K(y, \xi, s)|w_0(\xi)|d\xi \leq c \exp \left( -\frac{e^{-s}|y|^2}{4(1 + e^{-s})} \right) \| w_0 \|_{L^2_{K,p}(\mathbb{R}^n_+)}
\]
for \( y = (y', y_n) \in \mathbb{R}^{n-1} \times (0, R) \) and \( s \geq 1. \)

**Proof.** We note from (23) that
\[
\int_{\mathbb{R}^n_+} \Gamma_K(y, \xi, s)|w_0(\xi)|d\xi \leq \| w_0 \|_{L^2_{K,p}(\mathbb{R}^n_+)} \left( \int_{\mathbb{R}^n_+} \gamma(y', \xi', s)^2 \gamma_K(y_n, \xi_n, s)^2 \left( \frac{e^{\xi_n^2/8}}{b_K(\xi_n)} \right)^2 d\xi \right)^{1/2}.
\]

Then by using (24), we get
\[
\int_{\mathbb{R}^n_+} \gamma(y', \xi', s)^2 \gamma_K(y_n, \xi_n, s)^2 \left( \frac{e^{\xi_n^2/8}}{b_K(\xi_n)} \right)^2 d\xi \leq c \left( \frac{1}{1 - e^{-s}} \right)^{(n-1)/2}
\]
\[
\times \exp \left( \frac{e^{-s}|y'|^2}{2(1 + e^{-s})} \right) \int_0^\infty \gamma_K(y_n, \zeta, s)^2 \left( \frac{e^{\zeta^2/4}}{b_K(\zeta)^2} \right) d\zeta.
\]
Fix $K_0 > 1$ and let $\theta \in (0,1)$ be as in Lemma 2.8, which is defined by $(1-\theta)/\theta = \sqrt{2}$. Then since $b_K(\zeta)$ is positive on $\mathbb{R}_+^n$ with $b_K(0) = 1$, there exists $c = c(K_0) > 0$ such that

$$\sup_{K \in (0,K_0)} \sup_{\zeta \in (0,3/\theta)} b_K(\zeta)^{-1} \leq c.$$ 

Therefore from the first statement of Lemma 2.8 we obtain

$$\int_0^{3/\theta} \gamma_K(y_n,\zeta,s)^2 \left( \frac{e^{\zeta^2/4}}{b_K(\zeta)^2} \right) d\zeta \leq \frac{c}{(1-e^{-s})}.$$ 

Next we estimate the rest of integral over $(3/\theta, \infty)$. Here we note that

$$|y_n e^{-s/2} - \theta \zeta| = \theta \zeta - y_n e^{-s/2} \geq \theta \zeta - \frac{\theta \zeta}{3} = \frac{2}{3} \theta \zeta$$

for $y_n \in (0,e^{s/2})$ and $\zeta \in (3/\theta, \infty)$. Then we apply the second statement of Lemma 2.8 to obtain

$$\int_0^{3/\theta} \gamma_K(y_n,\zeta,s)^2 \left( \frac{e^{\zeta^2/4}}{b_K(\zeta)^2} \right) d\zeta \leq \frac{c}{(1-e^{-s})^{1/2}} \int_0^{3/\theta} \gamma_K(y_n,\zeta,s) \exp \left( \frac{(\theta \zeta)^2}{9(1-e^{-s})} \right) \left( \frac{e^{\zeta^2/4}}{b_K(\zeta)^2} \right) d\zeta$$

for $y_n \in (0,e^{s/2})$ and $s \geq 1$. Here applying Lemma 2.10 with $w_0(\zeta) = e^{-(\theta \zeta)^2/9} e^{\zeta^2/4}/b_K(\zeta)^2$ and combining the above estimates, we obtain the desired estimate.

**Lemma 2.10.** For any $M,R,K_0 > 1$ there exists $c = c(M, R, K_0) > 0$ such that

$$\int_0^{\infty} |w_0(\zeta)| b_K(\zeta)^2 e^{-\zeta^2/4} d\zeta \leq M,$$

then it holds that

$$\int_0^{\infty} \gamma_K(z,\zeta,s)|w_0(\zeta)| d\zeta \leq c \quad \text{for} \quad z \in (0,R), \quad s \geq 1.$$

**Proof.** We set

$$\partial_K(z,s) = \int_0^{\infty} \gamma_K(z,\zeta,s)|w_0(\zeta)| d\zeta.$$ 

Then $\partial_K(z,s)$ is a solution of (16) with the initial data $|w_0(z)|$. Multiplying (16) by $b_K(z)^2 e^{-z^2/4}$ and integrating over $(0, \infty)$, we obtain

$$\partial_s \int_0^{\infty} \partial_K(z,s) b_K(z)^2 e^{-z^2/4} dz = 0.$$ 

Hence by assumption and $\partial_K \geq 0$, it follows that

$$\int_0^{\infty} |\partial_K(z,s)| b_K(z)^2 e^{-z^2/4} dz = \int_0^{\infty} |w_0(z)| b_K(z)^2 e^{-z^2/4} dz \leq M.$$ 

As a consequence, for any $R > 1$ there exists $c = c(R) > 0$ such that

$$\sup_{s \in (0,\infty)} \int_0^{R+1} |\partial_K(z,s)| dz \leq cR.$$ 

17
Hence by a local $L^\infty$-estimate for parabolic equations, we obtain
\[
\sup_{K \in (0,K_0)} \sup_{z \in (0,R)} |\partial_K(z,s)| \leq c \sup_{K \in (0,K_0)} \int_{s-1/2}^{s+1/2} d\mu \int_{0}^{R+1} |\partial_K(z,\mu)| dz \leq cM
\]
for $s \geq 1$, which completes the proof. \qed

### 2.3 Representation formula

Here we provide a representation formula of solutions of
\[
\begin{cases}
\partial_s v = \Delta v - \frac{y}{2} \cdot \nabla v - mv + g_1(y, s) & \text{in } \mathbb{R}^n_+ \times (0, \infty), \\
\partial_s v = Kv + g_2(y', s) & \text{on } \partial \mathbb{R}^n_+ \times (0, \infty), \\
v(y,0) = v_0(y) & \text{in } \mathbb{R}^n_+,
\end{cases}
\]
where $K$ is a positive constant. Let $b_K(y_n)$ and $\mu_K$ be introduced in Section 2.2 and set $w_K(y_s) = v(y, s)/b_K(y_n)$. Then since $b_K(0) = 1$, we easily see that $w_K(y, s)$ solves
\[
\begin{cases}
\partial_s w_K = \Delta w_K - \frac{y}{2} \cdot \nabla w_K + 2b_K(y_n)\partial_n w_K - (m + \mu_K)w_K + \frac{g_1(y, s)}{b_K(y_n)} & \text{in } \mathbb{R}^n_+ \times (0, \infty), \\
\partial_s w_K = g_2(y', s) & \text{on } \partial \mathbb{R}^n_+ \times (0, \infty), \\
w_K(y,0) = v_0(y)/b_K(y_n) & \text{in } \mathbb{R}^n_+.
\end{cases}
\]

Define
\[
P_1(y, s) = \int_0^s e^{-(m+\mu_K)\rho}G_0(e^{-\rho/2}y_n,0,\mu) \left( \int_{\mathbb{R}^{n-1}} \gamma(y',\xi',\mu)g_2(\xi',s-\mu)d\xi' \right) d\mu,
\]
\[
P_2(y, s) = \int_0^s e^{-(m+\mu_K)\rho}G_K(e^{-\rho/2}y_n,0,\mu,0) \left( \int_{\mathbb{R}^{n-1}} \gamma(y',\xi',\mu)g_2(\xi',s-\mu)d\xi' \right) d\mu.
\]
Then it is known that (cf. p. 173 in [1])
\[
\lim_{y_n \to 0} \partial_n P_1(y, s) = -g_2(y', s).
\]
On the other hand, from (20) and (21), we see that
\[
\lim_{y_n \to 0} \partial_n P_2(y, s) = 0.
\]
Hence by definition of $\Gamma_K(y,\xi,s,\mu)$, we find that
\[
\lim_{y_n \to 0} \partial_n \left( \int_0^s e^{-(m+\mu_K)\rho}d\rho \int_{\partial \mathbb{R}^n_+} \Gamma_K(y,\xi',s-\mu)g_2(\xi',\mu)d\xi' \right)
= \lim_{y_n \to 0} \partial_n (P_1(y, s) + P_2(y, s)) = -g_2(y', s).
\]
Due to this fact, we obtain a representation formula of a solution of (32).
\[
w_K(y, s) = e^{-(m+\mu_K)s} \int_{\mathbb{R}^n_+} \Gamma_K(y,\xi,s) \left( \frac{v_0(\xi)}{b_K(\xi_n)} \right) d\xi
+ \int_0^s e^{-(m+\mu_K)(s-\mu)}d\mu \int_{\partial \mathbb{R}^n_+} \Gamma_K(y,\xi,s-\mu) \left( \frac{g_1(\xi,\mu)}{b_K(\xi_n)} \right) d\xi
+ \int_0^s e^{-(m+\mu_K)(s-\mu)}d\mu \int_{\partial \mathbb{R}^n_+} \Gamma_K(y,\xi',s-\mu)g_2(\xi',\mu)d\xi'.
\]
For simplicity of notations, we define
\[
S_K(s)w_0 = \int_{\mathbb{R}_+^n} \Gamma_K(\cdot, \xi, s)w_0(\xi)d\xi,
\]
\[
T_K(s, \mu)h = \int_{\partial\mathbb{R}_+^n} \Gamma_K(\cdot, \xi', s - \mu)h(\xi')d\xi'.
\]

Since \(w_K(y, s) = v(y, s)/b_K(y_n)\), a representation formula of a solution of (31) is given by
\[
\left( \frac{v(s)}{b_K} \right) = e^{-(m+\mu_K)s}S_K(s)\left( \frac{v_0}{b_K} \right) + \int_0^s e^{-(m+\mu_K)(s-\mu)}
\]
\[
\times \left( S_K(s - \mu)\left( \frac{g_1(\mu)}{b_K} \right) + T_K(s, \mu)g_2(\mu) \right) d\mu.
\]

(33)

2.3.1 Comparison argument

Here we provide a pointwise estimate of solutions of (31) by using a representation formula (33).

\[
\begin{align*}
\partial_s v &\leq \Delta v - \frac{y}{2} \cdot \nabla v - mv + g_1(y, s) & & \text{in } \mathbb{R}_+^n \times (0, \infty), \\
\partial_s v &\leq K(y', s)v + g_2(y', s) & & \text{on } \partial\mathbb{R}_+^n \times (0, \infty), \\
v(y, 0) &\leq v_0(y) & & \text{in } \mathbb{R}_+^n.
\end{align*}
\]

(34)

For the case \(K(y', s) \equiv K_0\) with some positive constant \(K_0\), by using a solution of (31) as a comparison function, we obtain from (33)
\[
\left( \frac{v(s)}{b_{K_0}} \right) \leq e^{-(m+\mu_{K_0})s}S_{K_0}(s)\left( \frac{v_0}{b_{K_0}} \right) + \int_0^s e^{-(m+\mu_{K_0})(s-\mu)}
\]
\[
\times \left( S_{K_0}(s - \mu)\left( \frac{g_1(\mu)}{b_{K_0}} \right) + T_{K_0}(s, \mu)g_2(\mu) \right) d\mu.
\]

(35)

For the case \(K(y', s) \in L^\infty(\partial\mathbb{R}_+^n \times (0, \infty))\), we use the following solution as a comparison function.
\[
\begin{align*}
\partial_s \bar{v} &\leq \Delta \bar{v} - \frac{y}{2} \cdot \nabla \bar{v} - m\bar{v} + |g_1(y, s)| & & \text{in } \mathbb{R}_+^n \times (0, \infty), \\
\partial_s \bar{v} &\leq K_0\bar{v} + |g_2(y', s)| & & \text{on } \partial\mathbb{R}_+^n \times (0, \infty), \\
\bar{v}(y, 0) &\geq v_0(y) & & \text{in } \mathbb{R}_+^n,
\end{align*}
\]

where \(K_0 = \|K(y', s)\|_{L^\infty(\partial\mathbb{R}_+^n \times (0, \infty))}\). By a comparison argument, it is easily shown that \(\bar{v} \geq 0\). Hence it follows that
\[
\partial_s (\bar{v} - v) = K(y', s)(\bar{v} - v) + (K_0 - K(y'))\bar{v} \geq K(y', s)(\bar{v} - v).
\]

Then by using a comparison argument again, we get
\[
v(y, s) \leq \bar{v}(y, s).
\]

Therefore we apply (33) to obtain
\[
\left( \frac{v(s)}{b_{K_0}} \right) \leq e^{-(m+\mu_{K_0})s}S_{K_0}(s)\left( \frac{|v_0|}{b_{K_0}} \right) + \int_0^s e^{-(m+\mu_{K_0})(s-\mu)}
\]
\[
\times \left( S_{K_0}(s - \mu)\left( \frac{|g_1(\mu)|}{b_{K_0}} \right) + T_{K_0}(s, \mu)|g_2(\mu)| \right) d\mu.
\]

(36)
3 Dynamical system approach

In this section, we study the asymptotic behavior of solutions of (7). Our argument is based on the argument in [6]. By using Lemma A.1, we slightly simplify their arguments. Let \( v(y,s) \) be a bounded solution of (7) converging to zero in \( L^2_\rho(\mathbb{R}^n_+) \) as \( s \to \infty \). We expand \( v(y,s) \) by eigenfunctions of (9):

\[
v(s) = \sum_\alpha a_\alpha(s) E_\alpha \quad \text{in} \ L^2_\rho(\mathbb{R}^n_+),
\]

where \( a_\alpha(s) = (v(s), E_\alpha)_\rho \). We denote by \( \pi_s, \pi_n \) and \( \pi_u \) projection operators onto the subspace spanned by eigenfunctions of \((-A)\) with the positive eigenvalue, the zero eigenvalue and the negative eigenvalue and set

\[
v_s = \pi_s v, \quad v_n = \pi_n v, \quad v_u = \pi_u v.
\]

The subspace \( \pi_s L^2_\rho(\mathbb{R}^n_+) \) is infinite dimension and \( \pi_n L^2_\rho(\mathbb{R}^n_+), \pi_u L^2_\rho(\mathbb{R}^n_+) \) are finite dimension. First we prepare the following elementary lemma, which is useful in this section. We put

\[
\Omega = \{ y = (y', y_n) \in \mathbb{R}^{n-1} \times (0,1) \}.
\]

**Lemma 3.1.** There exists \( c > 0 \) such that

\[
\int_{\partial \mathbb{R}^n_+} g(y') v^2 \rho dy' \leq c \int_\Omega g(y') (v^2 + |\nabla v|) \rho dy \quad \text{for} \ v \in H^1_{\rho}(\mathbb{R}^n_+).
\]

**Proof.** Let \( \eta(y_n) \) be a cut off function such that \( \eta(y_n) = 1 \) if \( y_n \in (0,1/2) \) and \( \eta(y_n) = 0 \) if \( y_n \geq 1 \). Then it is verified that

\[
\int_{\partial \mathbb{R}^n_+} g(y') v^2 \rho dy' = - \int_{\mathbb{R}^{n-1}} g(y') \rho dy' \int_0^1 \partial_n (v^2 \eta(y_n)) dy_n
\]

\[
\leq c \int_0^1 dy_n \int_{\mathbb{R}^{n-1}} g(y') (v^2 + |\nabla v|) \rho dy'.
\]

Since \( \rho(y')|_{\partial \mathbb{R}^n_+} = e^{-|y'|^2/4} \), it is clear that \( \rho(y) = e^{-y_n^2/4} \rho(y') \) for \( y \in \mathbb{R}^n_+ \). This implies that \( \rho(y') \leq e^{1/4} \rho(y) \) for \( y_n \leq 1 \). Therefore we complete the proof.

Now we state a main result in this section.

**Proposition 3.1.** One of the following two cases holds.

(I) \( \lim_{s \to \infty} \left( \frac{\|v_s(s)\|_{\rho} + \|v_u(s)\|_{\rho}}{\|v_n(s)\|_{\rho}} \right) = 0, \)

(II) \( \|v(s)\|_{\rho} \) decays to zero exponentially.

**Proof.** Multiplying (7) by \( v_s, v_n \) and \( v_u \) respectively, then we verify that

\[
\frac{1}{2} \partial_s ||v_s||_{\rho}^2 = -m ||v_s||_{\rho}^2 + \int_{\mathbb{R}^n_+} v \nabla (\rho v_s) dy + \int_{\partial \mathbb{R}^n_+} f(v)v_s \rho dy' \quad (a \in \{n, u\})
\]

\[
\frac{1}{2} \partial_s ||v_u||_{\rho}^2 = -m ||v_u||_{\rho}^2 - \int_{\mathbb{R}^n_+} \nabla v \cdot \nabla v_s \rho dy + \int_{\partial \mathbb{R}^n_+} (qB^{q-1}v_s + f(v)v_u) \rho dy'
\]

\[
= -m ||v_u||_{\rho}^2 - ||\nabla v_s||_{\rho}^2 + qB^{q-1} ||v_s||_{L^2(\partial \mathbb{R}^n_+)}^2 + \int_{\partial \mathbb{R}^n_+} f(v)v_s \rho dy'.
\]
Hence we obtain an ODE system.

\[
\begin{aligned}
\frac{1}{2} \partial_s \|v_n\|_\rho^2 &\geq \frac{1}{2} \|v_n\|_\rho^2 + \int_{\mathbb{R}_+^n} f(v) v_n \rho \, dy', \\
\frac{1}{2} \partial_s \|v_n\|_\rho^2 &= \int_{\mathbb{R}_+^n} f(v) v_n \rho \, dy', \\
\frac{1}{2} \partial_s \|s\|_\rho^2 &= -\|\nabla v_k\|_\rho^2 - m \|v_k\|_\rho^2 + qB^q-1 \|v_k\|_{L^2_{\rho}(\mathbb{R}_+^n)}^2 + \int_{\mathbb{R}_+^n} f(v) v_k \rho \, dy'.
\end{aligned}
\]

By the Schwarz inequality, we see that for \(a \in \{s, n, u\}\)

\[
\int_{\mathbb{R}_+^n} f(v) v_a \rho \, dy' \leq \int_{\mathbb{R}_+^n} \left( \epsilon (1 + |y'|^2)^{1/2} v_a^2 + \frac{1}{\epsilon} (1 + |y'|^2)^{-1/2} f(v)^2 \right) \rho \, dy'.
\]

From Lemma 3.1 and Lemma A.1, it holds that

\[
\int_{\mathbb{R}_+^n} (1 + |y'|^2)^{1/2} v_a^2 \rho \, dy' \leq c \int_{\mathbb{R}_+^n} (1 + |y'|^2) v_a^2 + \|\nabla v_a^2\| \rho \, dy \leq c \|v_a\|_{H^1_\rho(\mathbb{R}_+^n)}^2.
\]

Here we recall from [8] that \(|f(v)| \leq c v^2\). Then from Lemma 3.1 and Lemma 2.1 we get

\[
\int_{\mathbb{R}_+^n} (1 + |y'|^2)^{-1/2} f(v)^2 \rho \, dy' \leq c \int_{\mathbb{R}_+^n} (1 + |y'|^2)^{-1/2} v^4 \rho \, dy' \\
\leq c \int_{\mathbb{R}_+^n} (1 + |y'|^2)^{-1/2} \left( |v|^3 |\partial_n v| + v^4 \right) \rho \, dy \\
\leq c \int_{\mathbb{R}_+^n} (1 + |y'|^2)^{-1/2} |v|^3 \rho \, dy \leq c M(s) \|v\|_\rho^2,
\]

where \(M(s)\) is given by

\[
M(s) = \sup_{y \in \Omega} (1 + |y'|^2)^{-1/2} |v(y, s)|.
\]

As a consequence, there exists \(\epsilon_0 > 0\) such that for \(\epsilon \in (0, \epsilon_0)\)

\[
\begin{aligned}
\frac{1}{2} \partial_s \|v_n\|_\rho^2 &\geq \frac{1}{4} \|v_n\|_\rho^2 - \frac{\epsilon}{\epsilon} M(s) \|v\|_\rho^2, \\
\frac{1}{2} \partial_s \|v_n\|_\rho^2 \leq c \|v_n\|_\rho^2 + \frac{\epsilon}{\epsilon} M(s) \|v\|_\rho^2
\end{aligned}
\]

and

\[
\frac{1}{2} \partial_s \|s\|_\rho^2 \leq -(1 - \epsilon) \left( \|\nabla v_k\|_\rho^2 + m \|v_k\|_\rho^2 \right) + qB^q-1 \|v_k\|_{L^2_{\rho}(\mathbb{R}_+^n)}^2 + \frac{\epsilon}{\epsilon} M(s) \|v\|_\rho^2.
\]

Let \(\lambda_\ast > 0\) be the smallest positive eigenvalue of \((-\Delta)\), which is given by

\[
\lambda_\ast = \inf_{E \in \mathcal{H}_s} \frac{\|\nabla E\|_\rho^2 + m \|E\|_\rho^2 - qB^q-1 \|E\|_{L^2_{\rho}(\mathbb{R}_+^n)}^2}{\|E\|_\rho^2},
\]

where \(\mathcal{H}_s = H^1_\rho(\mathbb{R}_+^n) \cap \pi_qL^2_{\rho}(\mathbb{R}_+^n)\). By continuity, there exists \(\epsilon_\ast > 0\) such that for \(\epsilon \in (0, \epsilon_\ast)\)

\[
\inf_{E \in \mathcal{H}_s} (1 - \epsilon) \left( \|\nabla E\|_\rho^2 + m \|E\|_\rho^2 \right) - qB^q-1 \|E\|_{L^2_{\rho}(\mathbb{R}_+^n)}^2 \geq \frac{\lambda_\ast}{2}.
\]

Hence there exists \(\epsilon_1 \in (0, \epsilon_0)\) such that for \(\epsilon \in (0, \epsilon_1)\)

\[
\frac{1}{2} \partial_s \|v_n\|_\rho^2 \leq -\frac{\lambda_\ast}{2} \|v_n\|_\rho^2 + \frac{\epsilon}{\epsilon} M(s) \|v\|_\rho^2.
\]

Since \(\lim_{s \to \infty} M(s) = 0\), applying Lemma 3.1 in [6] to (39) and (40), we obtain the conclusion. \(\square\)
Next we study the asymptotic behavior of $\nabla' v(y, s)$. We set 
\[ ||v||^2 = ||v||^2 + ||\nabla' v||^2. \]

**Proposition 3.2.** One of the following two cases holds.

(I) \( \lim_{s \to \infty} \left( \frac{||v_s||_\rho + ||v_n||_\rho}{||v_n||_\rho} \right) = 0, \)

(II) \( ||v(s)||_\rho \) decays to zero exponentially,

**Proof.** We repeat the proof of Proposition 3.1. We set \( \pi \rho \). Then we obtain \( \pi \) satisfies
\[
\begin{cases}
\partial_s \pi = \Delta \pi - \frac{y}{2} \cdot \pi - \left( m + \frac{1}{2} \right) \pi \quad \text{in } \mathbb{R}^n_+ \times (s, \infty), \\
\partial_v \pi = qB^{q-1} \pi + f'(v) \pi \quad \text{on } \partial \mathbb{R}^n_+ \times (s, \infty).
\end{cases}
\]

The main linear part (neglecting \( f'(v) \)) is written by
\[
\partial_s \pi = \tilde{A} \pi := \left( A - \frac{1}{2} \right) \pi, \quad D(\tilde{A}) = D(A).
\]

Let \( \tilde{\pi}_s, \tilde{\pi}_n \) and \( \tilde{\pi}_0 \) be projection operators onto the subspace spanned by eigenfunctions of \( -\tilde{A} \) with the positive eigenvalue, the zero eigenvalue and the negative eigenvalue and set
\[ V_s = \tilde{\pi}_s \pi, \quad V_n = \tilde{\pi}_n \pi, \quad V_u = \tilde{\pi}_u \pi. \]

Then we obtain
\[
\begin{align*}
\frac{1}{2} \partial_s ||V_u||^2 &= \frac{1}{2} ||V_u||^2 + \int_{\partial \mathbb{R}^n_+} f'(v) V_u \rho dy', \\
\frac{1}{2} \partial_s ||V_n||^2 &= \int_{\partial \mathbb{R}^n_+} f'(v) V_n \rho dy', \\
\frac{1}{2} \partial_s ||V_s||^2 &= -||\nabla V_s||^2 - \left( m + \frac{1}{2} \right) ||V_s||^2 + qB^{q-1} ||V_s||^2_{L^2(\partial \mathbb{R}^n_+)} + \int_{\partial \mathbb{R}^n_+} f'(v) V_s \rho dy'.
\end{align*}
\]

Here we recall from (38) that \( |f'(v)| \leq c|v| \). Therefore by the same calculation as (38), we verify that
\[
\int_{\partial \mathbb{R}^n_+} |f'(v) V_s \rho dy' \leq c \int_{\partial \mathbb{R}^n_+} \left( 1 + |y|^2 \right)^{-1/2} |v V|^2 + c(1 + |y|^2)^{1/2} |V_s|^2 \rho dy' \\
\leq c \int_{\partial \mathbb{R}^n_+} (1 + |y|^2)^{-1/2} \left( |v|^2 + |V|^2 \right) \rho dy' + c \int_{\partial \mathbb{R}^n_+} \left( M_V(s) ||v||_\rho^2 + M_V(s) ||V||_\rho^2 \right) + c \int_{\partial \mathbb{R}^n_+} ||V_s||^2_{L^2(\partial \mathbb{R}^n_+)}.
\]

where
\[ M_V(s) = \sup_{y \in \Omega} (1 + |y|^2)^{-1/2} |V(y, s)|. \]

Hence there exists \( \epsilon_0 > 0 \) such that for \( \epsilon \in (0, \epsilon_0) \)
\[
\begin{align*}
\frac{1}{2} \partial_s ||V_u||^2 &\geq \frac{1}{4} ||V_u||^2 - \frac{c}{\epsilon} \left( M_V(s) ||v||_\rho^2 + M_V(s) ||V||_\rho^2 \right), \\
\frac{1}{2} \partial_s ||V_n||^2 &\leq c ||V_n||^2 + \frac{c}{\epsilon} \left( M_V(s) ||v||_\rho^2 + M_V(s) ||V||_\rho^2 \right), \\
\frac{1}{2} \partial_s ||V_s||^2 &\leq -\frac{\lambda}{2} ||\nabla V_s||^2 + \frac{c}{\epsilon} \left( M_V(s) ||v||_\rho^2 + M_V(s) ||V||_\rho^2 \right). \quad (41)
\end{align*}
\]
Therefore since \( \tilde{\pi}_a(\partial_i v) = \partial_i (\pi_a v) \) \((i = 1, \ldots, n - 1)\) for \( a \in \{s, n, u\} \), by (39), (40) and (41), we obtain

\[
\begin{aligned}
\frac{1}{2} \partial_s \|v_u\|_\rho^2 & \geq \frac{1}{4} \|v_u\|_\rho^2 - \frac{c}{\epsilon} (M(s) + M_V(s)) \|v\|_\rho^2, \\
\frac{1}{2} \partial_s \|v_n\|_\rho^2 & \leq ce \|v_n\|_\rho^2 + \frac{c}{\epsilon} (M(s) + M_V(s)) \|v\|_\rho^2, \\
\frac{1}{2} \partial_s \|v_s\|_\rho^2 & \leq -\lambda_s \|\nabla v_s\|_\rho^2 + \frac{c}{\epsilon} (M(s) + M_V(s)) \|v\|_\rho^2.
\end{aligned}
\]

Since \( \lim_{s \to \infty} M_V(s) = 0 \), applying Lemma 3.1 in [3], we complete the proof.

3.1 Case (I)

In this subsection, we study a precise asymptotic behavior for the case (I).

**Definition 3.1.** We call a function \( v(y) \) \( y_n \)-axial symmetric, if the function \( v(y) \) can be expressed by \( v(y) = v(|y'|, y_n) \).

For the rest of this paper, solutions are always assumed to be \( y_n \)-axial symmetric. Then the kernel of \( A = \pi_n L_\rho^2(\mathbb{R}_+^n) \) turns out to be one dimension under a symmetric assumption. In fact, let

\[
\mathcal{E}(y) = c(H_2(y_1) + \cdots H_2(y_{n-1}))I_1(y_n),
\]

where \( c > 0 \) is a normalization constant such that \( \|\mathcal{E}\|_\rho = 1 \). Then it holds that \( \ker A = \text{span}\{\mathcal{E}\} \).

**Proposition 3.3.** Let \( y_n \)-axial symmetric and behave as the case (I) in Proposition 3.1. Then it follows that

\[
\left\| v(s) + \frac{\nu_q}{s} \mathcal{E} \right\|_\rho = o(s^{-1}),
\]

where \( \nu_q \) is given by

\[
\nu_q^{-1} = \left( \frac{q(q-1)B^q - 2}{2} \right) \int_{\partial\mathbb{R}_+^n} \mathcal{E}^3 \rho dy' > 0.
\]

**Proof of Theorem 1.1.** Theorem 1.1 follows from Proposition 3.1 and Proposition 3.3.

First we prepare two lemmas.

**Lemma 3.2.** Let \( v(y, s) \) be as in Proposition 3.3. Then for any \( \delta > 0 \) there exist \( c_1, c_2 > 0 \) such that

\[
c_1 \|v_n(s - \delta)\|_\rho \leq \|v_n(s)\|_\rho \leq c_2 \|v_n(s + \delta)\|_\rho.
\]

**Proof.** Since \( |f(v)| \leq c v^2 \) (see (8)), by Lemma 3.1 the second equation in (37) is estimated by

\[
|\partial_s \|v_n\|_\rho^2| \leq c \int_{\partial \mathbb{R}_+^n} v^2 \|v_n\|_\rho dy' \leq c \left( \int_{\partial \mathbb{R}_+^n} v^4 \rho dy' + \|v_n\|_\rho^2 \right) \\
\leq c \left( \int_{\mathbb{R}_+^n} (|\nabla v||v|^3 + v^4) \rho dy + \|v_n\|_\rho^2 \right).
\]

Therefore since \( v(y, s) \) and \( |\nabla v(y, s)| \) are uniformly bounded (see Theorem 2.1), from Proposition 3.1 we obtain

\[
|\partial_s \|v_n\|_\rho^2| \leq c \left( \|v\|_\rho^2 + \|v_n\|_\rho^2 \right) \leq c \|v_n\|_\rho^2.
\]

Then this implies

\[
-c \leq \partial_s (\log \|v_n\|_\rho^2) \leq c.
\]

Integrating both sides, we obtain the conclusion.
Lemma 3.3. Let \( v(y, s) \) be as in Proposition 3.3. Then for \( r \in (1, \infty) \) there exists a positive continuous function \( \nu(s) \) satisfying \( \lim_{s \to \infty} \nu(s) = 0 \) such that

\[
\|(v - v_n)(s)\|_{L_r^\infty(\partial \mathbb{R}^d_+)} \leq \nu(s)\|v_n(s)\|_r.
\]

Proof. We set \( z(y, s) = v(y, s) - v_n(y, s) \). Then \( z(y, s) \) solves

\[
\begin{cases}
\partial_s z = \Delta z - \frac{y}{2} \cdot \nabla z - mz - X(s)E & \text{in } \mathbb{R}^d_+ \times (s_T, \infty), \\
\partial_{\nu} z = qB^{q-1}z + f(v) & \text{on } \partial \mathbb{R}^d_+ \times (s_T, \infty),
\end{cases}
\]

where \( X(s) \) is given by

\[
X(s) = \int_{\mathbb{R}^d_+} f(v(s))E \rho \, dy'.
\]

Since \( |f(v)| \leq cv^2 \) (see (36)), we easily see that

\[
\partial_{\nu} z \leq qB^{q-1}z + cv^2, \quad \partial_s z \geq qB^{q-1}z - cv^2 \quad \text{on } \partial \mathbb{R}^d_+ \times (s_T, \infty).
\]

Then we apply (36) in (43) with \( K_0 := qB^{q-1} \) to obtain

\[
|z(s)| \leq e^{-(m + \mu K_0)(s-s_0)} S_{K_0} (s-s_0) \left( \frac{z(s_0)}{b_{K_0}} \right) + \int_{s_0}^s e^{-(m + \mu K_0)(s-\mu)}
\]

\[
\times \left( |X(\mu)| S_{K_0} (s-\mu) \left( \frac{E}{b_{K_0}} \right) + c |T_{K_0} (s, \mu) v^2| \right) d\mu
\]

\[
=: J_1 + J_2 + J_3 \quad \text{for } y \in \partial \mathbb{R}^d_+, \ s \geq s_0.
\]

To apply Lemma 2.7 we fix \( p > \max\{2, n-1\} \) and \( \delta \in (0, 1) \) and set

\[
s_1 := s_0 + \max \left\{ 1, \log \left( \frac{r-1}{1-\delta} \right) \right\}.
\]

Then from Lemma 2.7 with \( p = 2 \), we see that

\[
\|J_1(s_1)\|_{L_r^\infty(\partial \mathbb{R}^d_+)} \leq c e^{-(m + \mu K_0)(s_1-s_0)} \left( 1 - e^{-(s_1-s_0)} \right)^{n/2} \|z(s_0)\|_r \leq c \|z(s_0)\|_r
\]

\[
= c \|(v - v_n)(s_0)\|_r.
\]

Next we estimate \( J_2 \). From the Schwarz inequality, we get

\[
|X(\mu)| \leq c \int_{\partial \mathbb{R}^d_+} v(\mu)^2 |E| \rho \, dy' \leq c \left( \int_{\partial \mathbb{R}^d_+} (1 + |y'|^2)^{-1/2} v(\mu)^4 \rho \, dy' \right)^{1/2}.
\]

Then by the same calculation as (38), we see that

\[
\int_{\partial \mathbb{R}^d_+} (1 + |y'|^2)^{-1/2} v(\mu)^4 \rho \, dy' \leq c M(\mu) \|v\|_{\rho}^2,
\]

where \( M(s) \) is the same as in (38). Furthermore since \( A E = 0 \), we find that \( S_{K_0}(s - \mu)(E/b_{K_0}) = e^{\mu K_0(s-\mu)} E/b_{K_0} \). Therefore since \( b_{K_0} \equiv 1 \) on \( \partial \mathbb{R}^d_+ \), we obtain

\[
\|J_2(s_1)\|_{L_r^\infty(\partial \mathbb{R}^d_+)} \leq c \left( \int_{s_0}^{s_1} M(\mu)^{1/2} \|v(\mu)\|_{\rho} \, d\mu \right) \|E\|_{L_r^\infty(\partial \mathbb{R}^d_+)}.
\]

\[
\leq c \left( \sup_{\mu \in (s_0, s_1)} M(\mu)^{1/2} \right) \left( \sup_{\mu \in (s_0, s_1)} \|v(\mu)\|_{\rho} \right).
\]
Finally we compute \( J_3 \). Since \( p > 2 \), by definition of \( s_1 \), we easily see that
\[
s_1 - s_0 \geq \log \left( \frac{r - 1}{1 - \delta} \right) > \log \left( \frac{r - 1}{p - 1 - \delta} \right).
\]
Hence we can apply Lemma 3.2 and obtain
\[
\| J_3(s_1) \|_{L_p^p(\partial \mathbb{R}_n^+)} \leq c e^{-(m + \mu K_0)(s_1 - s_0)} \int_{s_0}^{s_1} \frac{\| v(\mu)^2 \|_{L_{K_0,\rho}(\mathbb{R}_n^+)}^p}{(1 - e^{-(s_1 - s_0)})(n-1)/2p+1/2} d\mu.
\]
Then by the Schwarz inequality and \( |b_{K_0} (\xi_n) | \leq K_0 \) for \( \xi_n \in \mathbb{R}_+ \), we compute
\[
\| v(\mu)^2 \|_{L_{K_0,\rho}(\mathbb{R}_n^+)}^p = \int_{\mathbb{R}_n^+} |v(\xi, \mu)|^{2p} b_{K_0} (\xi_n)^2 \rho(\xi) d\xi \\
\leq c \left( \int_{\mathbb{R}_n^+} (1 + |\xi|^2)^{-1/2} |v(\xi, \mu)|^{2p} \rho(\xi) d\xi \right)^{1/2} \\
\leq c M_1(\mu) \| v(\mu) \|_{\rho},
\]
where \( M_1(\mu) = \sup_{\xi \in \mathbb{R}_n^+} (1 + |\xi|^2)^{-1/2} |v(\xi, \mu)|^{2p-1} \). Since \( v(y, \mu) \) is uniformly bounded and \( v(y, \mu) \to 0 \) uniformly on any compact set in \( \mathbb{R}_n^+ \) as \( \mu \to \infty \), we easily see that \( M_1(\mu) \to 0 \) as \( \mu \to \infty \). Therefore since \( p > n - 1 \), we obtain
\[
\| J_3(s_1) \|_{L_p^p(\partial \mathbb{R}_n^+)} \leq c \int_{s_0}^{s_1} \frac{M_1(\mu) \| v(\mu) \|_{\rho}}{(1 - e^{-(s_1 - s_0)})(n-1)/2p+1/2} d\mu \\
\leq c \left( \sup_{\mu \in (s_0, s_1)} M_1(\mu) \left( \sup_{\mu \in (s_0, s_1)} \| v(\mu) \|_{\rho} \right) \right).
\]
Put \( \nu_1(s) = M(s)^{1/2} + M_1(s) \). Then combining the above estimates and applying Proposition 3.1 and Lemma 3.2, we conclude that
\[
\| z(s_1) \|_{L_p^p(\partial \mathbb{R}_n^+)} \leq c \left( \left\| (v - v_n)(s_0) \right\|_{\rho} + \left( \sup_{\mu \in (s_0, s_1)} \nu_1(\mu) \right) \left( \sup_{\mu \in (s_0, s_1)} \| v_n(\mu) \|_{\rho} \right) \right) \\
\leq c \left( \left\| (v - v_n)(s_0) \right\|_{\rho} + \left( \sup_{\mu \in (s_0, s_1)} \nu_1(\mu) \right) \| v_n(s_1) \|_{\rho} \right).
\]
Since \( \nu_1(\mu) \to 0 \) as \( \mu \to 0 \), the proof is completed.

Proof of Proposition 3.3. Set \( a_0(s) = (v(s), \mathcal{E})_{\rho} \). Then it is verified that
\[
\dot{a}_0 = \int_{\partial \mathbb{R}_n^+} f(v) \mathcal{E} \rho dy'.
\]
Then since \( f(v) = k_q v^2 + O(v^3) \) with \( k_q = q(q - 1)Bq^{-2}/2 \) (see (5)), we get
\[
\dot{a}_0 = k_q \int_{\partial \mathbb{R}_n^+} v_n^2 \mathcal{E} \rho dy' + \int_{\partial \mathbb{R}_n^+} (k_q(v^2 - v_n^2) + O(v^3)) \mathcal{E} \rho dy'.
\]
Since \( v(y,s) \) is uniformly bounded on \( \mathbb{R}^n_+ \times (s_T, \infty) \), the second integral on the right-hand side is estimates as follows.

\[
\int_{\partial \mathbb{R}^n_+} |v^2 - v_n^2| |\mathcal{E}| \rho dy' \leq c \int_{\partial \mathbb{R}^n_+} |v + v_n||v - v_n| |\mathcal{E}| \rho dy' \\
\leq c\|v - v_n\|L^2_2(\partial \mathbb{R}^n_+)^n + \|v_n\|L^2_2(\partial \mathbb{R}^n_+)^n.
\]

\[
\int_{\partial \mathbb{R}^n_+} |v^3| |\mathcal{E}| \rho dy' \leq c \int_{\mathbb{R}^n_+} (|v(v^2 - v_n^2)| + |v v_n^2|) |\mathcal{E}| \rho dy' \\
\leq c \int_{\mathbb{R}^n_+} (|v^2 - v_n| + |v v_n^2|) |\mathcal{E}| \rho dy' \\
\leq c \left(\|v - v_n\|L^4_2(\partial \mathbb{R}^n_+) + \|v_n\|L^4_2(\partial \mathbb{R}^n_+) \right).
\]

Hence we get

\[
|\hat{a}_0 - v_q^{-1} a_0^2| \leq c \left(\frac{\|v - v_n\|L^4_2(\partial \mathbb{R}^n_+) + \|v_n\|L^4_2(\partial \mathbb{R}^n_+)}{|v_n^2|} + \|v\|L^2_2(\partial \mathbb{R}^n_+)^n \right) a_0^2 =: c v(s)a_0^2,
\]

where \( v_q^{-1} = k_q \int_{\partial \mathbb{R}^n_+} \mathcal{E}^3 \rho dy' > 0 \) (see p. 164 in [13]). Then Proposition 3.2 and Lemma 3.3 implies

\[
\lim_{s \to \infty} \nu(s) = 0.
\] (44)

From the above differential inequality, we get

\[
\left| \frac{1}{a_0(s)} - \frac{1}{a_0(s_T)} + \frac{s - s_T}{\nu_q} \right| \leq c \int_{s_T}^s \nu(\tau) d\tau.
\]

Hence it holds that

\[
\left| \frac{1}{a_0(s)} + \frac{1}{\nu_q} \right| \leq \frac{s_T}{\nu_q s} - \frac{1}{a_0(s_T)s} + \frac{c}{s} \int_{s_T}^s \nu(\tau) d\tau.
\]

Therefore combining (44), we obtain

\[
\lim_{s \to \infty} sa_0(s) = -\nu_q.
\]

As a consequence, since \( v_n = a_0(s)\mathcal{E} \), it follows that

\[
\left\| v_n(s) + \frac{\nu_q}{s} \mathcal{E} \right\|_\rho = \left(a_0(s) + \frac{\nu_q}{s}\right) \|\mathcal{E}\|_\rho = o(s^{-1}),
\]

\[
\left\| \nabla' \left(v_n(s) + \frac{\nu_q}{s} \mathcal{E} \right) \right\|_\rho = \left(a_0(s) + \frac{\nu_q}{s}\right) \|\nabla'\mathcal{E}\|_\rho = o(s^{-1}).
\]

Thus by Proposition 3.2 we conclude

\[
\left\| v(s) + \frac{\nu_q}{s} \mathcal{E} \right\|_\rho = o(s^{-1}), \quad \left\| \nabla' \left(v(s) + \frac{\nu_q}{s} \mathcal{E} \right) \right\|_\rho = o(s^{-1}),
\]

which completes the proof. \( \square \)

## 4 Estimate for a large range

Throughout this section, we assume that \( v(y,s) \) is \( y_0 \)-axial symmetric and behaves as the case (I) in Proposition 3.1. Additionally, we assume that \( v(y,s) \) satisfies a monotonicity condition:

\[
y' \cdot \nabla' v(y,s) \leq 0 \quad \text{for} \ (y,s) \in \mathbb{R}^n_+ \times (s_T, \infty).
\] (45)
In this section, following the arguments in [15] and [24], we derive pointwise estimate of $v(y', s)$ along $|y'| \sim s^{1/2}$ on $\partial \mathbb{R}^n_+$ from the asymptotic behavior $v(y, s) \sim -\nu_q s^{-1}E(y)$ (Proposition 3.3) with global heat kernel estimates given in Section 2.2. For simplicity of notations, we set

$$B_{\partial \mathbb{R}^n_+}(r) = \{y' \in \partial \mathbb{R}^n_+; |y'| < r \}.$$ 

First we show that condition (4.10) is assured if the original initial data $u_0(x)$ satisfies (15).

**Lemma 4.1.** Let $u_0(x)$ be $x_n$-axial symmetric and satisfy $x' \cdot \nabla' u_0(x) \leq 0$ for $x \in \mathbb{R}^n_+$. Then $v(y, s)$ satisfies (4.10) for $s \in (s_T, \infty)$.

**Proof.** Since $x = (T-t)^{1/2}y$, it is clear that $y' \cdot \nabla' v(y, s) \leq 0$ is equivalent to $x' \cdot \nabla' u(x, t) \leq 0$. From Lemma 2.1 in [11], it holds that $x' \cdot \nabla' u(x, t) \leq 0$ if $x' \cdot \nabla' u_0(x) \leq 0$. Hence the proof is completed. □

As a consequence of assumption (4.10), we obtain the following lemma immediately.

**Lemma 4.2.** There exists $c > 0$ such that

$$\varphi(y', s) \leq B + cs^{-1}\chi_{|y'| < \sqrt{n}}$$

for $y' \in \partial \mathbb{R}^n_+$, $s \gg 1$.

**Proof.** Since $E(y) = c(H_2(y_1) + \cdots + H_2(y_n-1)I_1(y_n))$ and $H_2(\xi) = c'(\xi^2 - 2)$ with $c, c' > 0$, $E(y)$ is explicitly expressed by

$$E(y') = cc'(|y'|^2 - 2(n - 1))I_1(0)$$

for $y' \in \partial \mathbb{R}^n_+$. Hence it follows that $E(y') < 0$ for $y' \in \mathbb{R}^n_+$ with $|y'| > \sqrt{2(n - 1)}$. By assumption, we recall that $v(y, s)$ behaves $v(y, s) \sim -\nu_q s^{-1}E$ as $s \to \infty$. Therefore we find that $v(y', s)|_{\partial \mathbb{R}^n_+} < 0$ for $|y'| = \sqrt{2n}$ and large $s > s_T$. As a consequence, by using (4.10), we see that $v(y', s)|_{\partial \mathbb{R}^n_+} < 0$ for $|y'| > \sqrt{2n}$ and large $s > s_T$. Thus we complete the proof. □

A goal of this section is to show the following pointwise estimate along $|y'| \sim s^{1/2}$ on $\partial \mathbb{R}^n_+$.

**Proposition 4.1.** There exist $\theta \in (0, 1)$ and $0 < k_1 < k_2 < 1$ such that

$$k_1B \leq \varphi(y', s)|_{\partial \mathbb{R}^n_+} \leq k_2B$$

for $|y'| = \theta\sqrt{s}$.

**Lemma 4.3.** There exists a positive continuous function $\nu(s)$ satisfying $\lim_{s \to \infty} \nu(s) = 0$ such that

$$v(y', s)|_{\partial \mathbb{R}^n_+} \geq \frac{\nu_q}{s}E(y') - \nu(s)$$

for $y' \in B_{\partial \mathbb{R}^n_+}(\sqrt{s})$.

**Proof.** We set $K = qB^{-1}$ and $D(y) = E(y)/b_K(y_n)$. Then since $f(v) \geq 0$ in (7), by using (35), we get

$$v(s) \geq e^{-(m+\mu_K)(s-s_0)}S_K(s-s_0) \left(\frac{v(s_0)}{b_K}\right)$$

$$= e^{-(m+\mu_K)(s-s_0)}S_K(s-s_0) \left(-\frac{\nu_q}{s_0}D\right) + e^{-(m+\mu_K)(s-s_0)}$$

$$\times S_K(s-s_0) \left(\frac{v(s_0) + \nu_q s_0^{-1}E}{b_K}\right)$$

for $y' \in \partial \mathbb{R}^n_+, s > s_0$.

Here since

$$\left\{\begin{array}{ll}
-\left(\Delta - \frac{y}{2} \cdot \nabla + \left(\frac{2b_K}{b_K}\right) \partial_n\right) D = -(m+\mu_K)D & \text{in } \mathbb{R}^n_+,
\partial_\nu D = 0 & \text{on } \partial \mathbb{R}^n_+,
\end{array}\right.$$
we note that
\[ S_K(s - s_0) \mathcal{D} = e^{(m+\mu_K)(s-s_0)} \mathcal{D}. \]
Therefore since \( \mathcal{D}(y') = \mathcal{E}(y') \) on \( \partial \mathbb{R}^n_+ \), it follows that
\[ e^{-(m+\mu_K)} S_K(s - s_0) \mathcal{D}\left(\frac{\nu q}{s_0} \mathcal{D}\right) = \frac{\nu q}{s_0} \mathcal{E} \quad \text{on} \quad \partial \mathbb{R}^n_+. \]
Furthermore from Lemma 2.9, we observe that
\[ y \]
for \( s \). Then we see that
\[ \text{we set} \]
\[ y \]
for \( s \). We set
\[ y \]
for \( s \). We note from Lemma 4.2 with (45) that
\[ \text{Here we recall from Lemma 2.3 that} \]
\[ \mu \]
\[ \text{Therefore since} \]
\[ \text{Thus the proof is completed.} \]

**Lemma 4.4.** For any \( R > 0 \) there exists \( c_R > 0 \) such that
\[ |\nabla' v(y,s)| \leq c_R s^{-1/2} \quad \text{for} \quad |y'| \leq R \sqrt{s}, \quad y_\nu \in (0, R). \]

**Proof.** We set \( V(y,s) = \partial_i v(y,s) \) (\( i = 1, \cdots, n-1 \)), then \( V(y,s) \) solves
\[ \begin{cases} \partial_s V = \Delta V - \frac{n}{2} \cdot \nabla V - \left( m + \frac{1}{2} \right) V \quad \text{in} \quad \mathbb{R}^n_+ \times (sT, \infty), \\ \partial_s V = q \varphi^{q-1} V \quad \text{on} \quad \partial \mathbb{R}^n_+ \times (sT, \infty). \end{cases} \]
Since \( v(s) \) behaves \( v(y,s) \sim -\nu q s^{-1} \mathcal{E} \) as \( s \to \infty \), we note from Proposition 3.3 that \( |V(y',s)| \leq c s^{-1} \) for \( y' \in B_{\partial \mathbb{R}^n_+}(\sqrt{2n}) \). Hence by Lemma 4.2, we observe that
\[ q \varphi^{q-1} V = q B^{q-1} V + q (\varphi^{q-1} - B^{q-1}) V \leq q B^{q-1} V + c s^{-2} \quad \text{for} \quad y' \in B_{\partial \mathbb{R}^n_+}(\sqrt{2n}). \]
We set
\[ K(y',s) = \begin{cases} q B^{q-1} & \text{if} \quad |y'| \leq \sqrt{2n}, \\ q \varphi(y',s)^{q-1} & \text{if} \quad |y'| \geq \sqrt{2n}. \end{cases} \]
Then we see that
\[ \partial_s V \leq K(y',s) V + c s^{-2} \quad \text{on} \quad \partial \mathbb{R}^n_+. \]
We note from Lemma 4.2 with (45) that
\[ |K(y',s)| \leq q B^{q-1} =: K_0. \]
Therefore we apply the estimate (30) to \( \pm V(y,s) \) and obtain
\[ \left( \frac{|V(s)|}{b_{K_0}} \right) \leq e^{(s-s_0)/2 S_{K_0}(s-s_0) \left( \frac{|V(s)|}{b_{K_0}} \right)} + c \int_{s_0}^s e^{(s-s')/2 \mu_{-2}} T_{K_0}(s, \mu) d\mu. \]
Here we used $\mu_{K_0} = -(m + 1)$ (Lemma 2.3). By Lemma 2.9 for any $R > 0$ there exists $c = c(R) > 0$ such that

$$
\left(\frac{|V(y, s)|}{b_{K_0}(y_n)}\right) \leq c e^{(s-s_0)/2} \left(\exp\left(\frac{e^{-(s-s_0)}|y'|^2}{4(1 + e^{-(s-s_0)})}\right) \|V(s_0)\|_\rho + \int_{s_0}^s \frac{\mu^{-2}d\mu}{\sqrt{1 - e^{-(s-s_0)}}}\right)
$$

for $y' \in \mathbb{R}^{n-1}$, $y_n \leq R$. As in the proof of Lemma 4.3 we choose $s > s_0$ such that $s = e^{s-s_0}$. Then from Proposition 3.3 we obtain

$$|V(y, s)| \leq cb_{K_0}(y_n)s^{-1/2} \quad \text{for} \quad |y'| \leq R\sqrt{s}, \quad y_n \leq R,
$$

which completes the proof.

\textbf{Lemma 4.5.} There exists $c > 0$ such that

$$
\sup_{y' \in B_{\alpha^n}(\sqrt{s})} |\partial_s \varphi(y', s)| \leq cs^{-1}.
$$

We set $Y(y, s) = \partial_s v(y, s)$, then $Y(y, s)$ solves

$$
\begin{cases}
\partial_s Y = \Delta Y - \frac{y}{2} \nabla Y - mY & \text{in } \mathbb{R}^n_+ \times (sT, \infty), \\
\partial_s Y = q \varphi^{q-1} Y & \text{on } \partial\mathbb{R}^n_+ \times (sT, \infty).
\end{cases}
$$

We expand $Y(y, s)$ by using eigenfunctions $\{E_\alpha\}_{\alpha \in A}$ of (9):

$$Y(s) = \sum_{\alpha \in A} a_\alpha(s) E_\alpha, \quad \text{in } H^1(\mathbb{R}^n_+),
$$

where $a_\alpha(s) = (v(s), E_\alpha)_\rho$. Since $v \in C^1((sT, \infty); L^2(\mathbb{R}^n_+))$, it is verified that $a_\alpha(s) = \partial_s(v(s), E_\alpha)_\rho$. Let $\lambda_\alpha$ be the eigenvalue corresponding to $E_\alpha$. Then it holds that

$$a'_\alpha = -(m + \lambda_\alpha)a_\alpha + q \int_{\partial\mathbb{R}^n_+} (\varphi^{q-1} - B^{q-1}) Y E_\alpha \rho dy'.
$$

(46)

The proof of Lemma 4.5 follows immediately from the following two lemmas.

\textbf{Lemma 4.6.} For any $\alpha \in A$ there exists $c_\alpha > 0$ such that

$$|a_\alpha(s)| \leq c_\alpha s^{-2}.
$$

\textbf{Proof.} Since $Y(y, s) = \partial_s v(y, s)$, a direct computation shows that

$$q \int_{\partial\mathbb{R}^n_+} (\varphi^{q-1} - B^{q-1}) Y E_\alpha \rho dy' = q \int_{\partial\mathbb{R}^n_+} ((v + B)^{q-1} - B^{q-1})(\partial_s v) E_\alpha \rho dy'$$

$$= \partial_s \int_{\partial\mathbb{R}^n_+} f(v) E_\alpha \rho dy'.
$$

For simplicity, we set

$$X_\alpha = \int_{\partial\mathbb{R}^n_+} f(v) E_\alpha \rho dy'.
$$

Then (46) is written by

$$\partial_s (e^{(m+\lambda_\alpha)s} a_\alpha) = e^{(m+\lambda_\alpha)s} \partial_s X_\alpha$$

$$= \partial_s \left(e^{(m+\lambda_\alpha)s} X_\alpha\right) - (m + \lambda_\alpha)e^{(m+\lambda_\alpha)s} X_\alpha.$$
We fix $s_1 > 1$. Then we integrate both sides over $(s, s_1)$ to obtain
\[ e^{(m + \lambda_\alpha)s}(a_\alpha(s) - X_\alpha(s)) - e^{(m + \lambda_\alpha)s_1}(a_\alpha(s_1) - X_\alpha(s_1)) = -(m + \lambda_\alpha) \int_{s_1}^s e^{(m + \lambda_\alpha)\mu} X_\alpha(\mu)d\mu. \] (47)

Here we recall from (45) that $|f(v)| \leq c v^2$. Then by the Hölder inequality, we see that
\[
|X_\alpha| \leq c \int_{\partial \mathbb{R}^n_+} v^2 |E_\alpha| \rho dy' \leq c \int_{\partial \mathbb{R}^n_+} (|v v_n| + |v(v - v_n)|) |E_\alpha| \rho dy'
\]
\[
\leq c \|v\| L^2_\rho(\partial \mathbb{R}^n_+) \left( \int_{\partial \mathbb{R}^n_+} v^2 |E_\alpha|^2 \rho dy' \right)^{1/2} + \|E_\alpha\| L^2_\rho(\partial \mathbb{R}^n_+) \|v - v_n\| L^2_\rho(\partial \mathbb{R}^n_+)
\]
\[
\leq c \|v\| L^2_\rho(\partial \mathbb{R}^n_+) \left( \|v_n\|_\rho + \|v - v_n\| L^2_\rho(\partial \mathbb{R}^n_+) \right).
\]

Since $\|v\| L^2_\rho(\partial \mathbb{R}^n_+) \leq \|v_n\| L^2_\rho(\partial \mathbb{R}^n_+) + \|v - v_n\| L^2_\rho(\partial \mathbb{R}^n_+)$, it follows from Lemma 3.3 and Proposition 3.3 that
\[
|X_\alpha(s)| \leq c \|v_n(s)\|_\rho^2 \leq \frac{c}{s^2}.
\]

Then the right-hand side on (47) is estimated by
\[
\int_{s_1}^s e^{(m + \lambda_\alpha)\mu} X_\alpha(\mu)d\mu \leq \left( \int_{s_1}^{(s_1 + s)/2} + \int_{(s_1 + s)/2}^s \right) e^{(m + \lambda_\alpha)\mu} X_\alpha(\mu)d\mu
\]
\[
\leq c \left( e^{(m + \lambda_\alpha)(s + s_1)/2} + e^{(m + \lambda_\alpha)s(s + s_1)^2} \right).
\]

Therefore applying this estimate in (47), we obtain the conclusion. \qed

We define $A_\lambda \subset A = \mathbb{N}_0^{n-1} \times \mathbb{N}$ by
\[ A_\lambda = \{ \alpha \in A; \lambda_\alpha < \lambda \}. \]

**Lemma 4.7.** There exists $\lambda_\alpha > 0$ such that
\[ \left\| Y(s) - \sum_{\alpha \in A_\lambda} a_\alpha(s) E_\alpha \right\|_\rho \leq c s^{-2}. \]

**Proof.** We set $P_\lambda(y, s) = Y(y, s) - \sum_{\alpha \in A_\lambda} a_\alpha(s) E_\alpha(y)$, then $P_\lambda(y, s)$ is a solution of
\[
\begin{cases}
\partial_t P_\lambda = \Delta P_\lambda - \frac{y}{2} \nabla P_\lambda - m P_\lambda - \sum_{\alpha \in A_\lambda} Q_\alpha E_\alpha & \text{in } \mathbb{R}^n_+ \times (s_T, \infty), \\
\partial_n P_\lambda = q \varphi^{q-1} P_\lambda + q (\varphi^{q-1} - B^{q-1}) \sum_{\alpha \in A_\lambda} a_\alpha E_\alpha & \text{on } \partial \mathbb{R}^n_+ \times (s_T, \infty),
\end{cases}
\]
where $Q_\alpha$ is given by
\[ Q_\alpha = q \left( \int_{\partial \mathbb{R}^n_+} (\varphi^{q-1} - B^{q-1}) Y E_\alpha \rho dy' \right). \]

Multiplying by $P_\lambda \rho$ and integrating over $\mathbb{R}^n_+$, we get
\[
\frac{1}{2} \partial_s \|P_\lambda\|_\rho^2 = -\|\nabla P_\lambda\|_\rho^2 - m \|P_\lambda\|_\rho^2 - \sum_{\alpha \in A_\lambda} Q_\alpha (P_\lambda, E_\alpha)_\rho
\]
\[
+ q \int_{\partial \mathbb{R}^n_+} \varphi^{q-1} P_\lambda^2 \rho dy' + q \sum_{\alpha \in A_\lambda} \int_{\partial \mathbb{R}^n_+} (\varphi^{q-1} - B^{q-1}) P_\lambda E_\alpha \rho dy'. \] (48)

30
Since \( \varphi(y, s) \) is positive and uniformly bounded, the third term and the last term on the right-hand side are estimated by

\[
\frac{Q_{\lambda}}{q} = \int_{\partial \Omega^+} (\varphi^{q-1} - B^{q-1}) \left( P_{\lambda} + \sum_{\beta \in A_{\lambda}} a_{\beta} E_{\beta} \right) E_{\alpha} \rho \, dy' \\
\leq c \| E_{\alpha} \|_{L^1(\partial \Omega^+)} \left( \| v \|_{L^1(\partial \Omega^+)} \| P_{\lambda} \|_{L^1(\partial \Omega^+)} + \sum_{\beta \in A_{\lambda}} a_{\beta} \| E_{\beta} \|_{L^1(\partial \Omega^+)} \| v \|_{L^2} \right),
\]

\[
\sum_{\alpha \in A_{\lambda}} a_{\alpha} \int_{\partial \Omega^+} |\varphi^{q-1} - B^{q-1}| P_{\lambda} E_{\alpha} \rho \, dy' \leq \sum_{\alpha \in A_{\lambda}} a_{\alpha} \| E_{\alpha} \|_{L^1(\partial \Omega^+)} \| v \|_{L^2} \| P_{\lambda} \|_{L^2(\partial \Omega^+)}.
\]

Here we apply Lemma 3.3 and Proposition 3.3 to obtain

\[
\| v \|_{L^1(\partial \Omega^+)} \leq \| v_n \|_{L^1(\partial \Omega^+)} + \| v - v_n \|_{L^1(\partial \Omega^+)} \leq c \| v_n \|_p \leq c s^{-1}.
\]

Therefore there exists \( c_{\lambda} > 0 \) such that

\[
\sum_{\alpha \in A_{\lambda}} |Q_{\alpha}(P_{\lambda}, E_{\alpha})| + q \sum_{\alpha \in A_{\lambda}} a_{\alpha} \int_{\partial \Omega^+} |\varphi^{q-1} - B^{q-1}| P_{\lambda} E_{\alpha} \rho \, dy' \\
\leq c_{\lambda} \left( s^{-1} \| P_{\lambda} \|_{H^1(\Omega^+)}^2 + s^{-1} \sum_{\alpha \in A_{\lambda}} a_{\alpha}^2 \right) \leq c_{\lambda} \left( s^{-1} \| P_{\lambda} \|_{H^1(\Omega^+)}^2 + s^{-5} \right),
\]

where we used Lemma 4.6 in the last inequality. Substituting this estimate into (18) and noting that \( \varphi^{q-1}|_{\partial \Omega^+} \leq B^{q-1}(1 + cs^{-1}) \) from Lemma 4.2, we obtain

\[
\frac{1}{2} \partial_s \| P_{\lambda} \|_p^2 \leq - \left( 1 - c_{\lambda}s^{-1} \right) \left( \| \nabla P_{\lambda} \|_p^2 + m \| P_{\lambda} \|_p^2 \right) + (1 + cs^{-1})q B^{q-1} \| P_{\lambda} \|_{L^2(\partial \Omega^+)}^2 + c_{\lambda}s^{-5}.
\]

Let \( \Pi_{\lambda} \) be a subspace of \( H^1(\Omega^+) \) defined by

\[
\Pi_{\lambda} = \{ E \in H^1(\Omega^+); (E, E_{\alpha}) = 0 \text{ for any } \alpha \in A_{\lambda} \}.
\]

Then it holds that

\[
\inf_{E \in \Pi_{\lambda}} \frac{\| \nabla E \|_p^2 - q B^{q-1} \| E \|_{L^2(\partial \Omega^+)}^2}{\| E \|_p^2} \geq \lambda.
\]

By continuity, there exists \( \epsilon_0 > 0 \) such that for \( \epsilon \in (0, \epsilon_0) \)

\[
\inf_{E \in \Pi_{\lambda}} \frac{(1 - \epsilon) \| \nabla E \|_p^2 - (1 + \epsilon)q B^{q-1} \| E \|_{L^2(\partial \Omega^+)}^2}{\| E \|_p^2} \geq \lambda/2.
\]

Hence there exist \( \lambda_* > 0 \) and \( s_1 > s_T \) such that for \( \lambda \geq \lambda_* \)

\[
\frac{1}{2} \partial_s \| P_{\lambda} \|_p^2 \leq - \frac{m}{2} \| P_{\lambda} \|_p^2 + c_{\lambda}s^{-5} \quad \text{for } s > s_1,
\]

which implies

\[
\| P_{\lambda}(s) \|_p^2 \leq \frac{1}{2} \| P_{\lambda} \|_p^2 + c_{\lambda} \int_{s_T}^s e^{-m(s - \mu)}(1 + \mu)^{-5} \, d\mu
\]

\[
\leq e^{-m(s - s_T)} \| P_{\lambda}(s_T) \|_p^2 + c_{\lambda}s^{-4} \quad \text{for } s > s_1.
\]

Therefore the proof is completed.

Proof of Lemma 4.5. From Lemma 4.6 and Lemma 4.7, we obtain

\[ \|Y(s)\|_\rho \leq cs^{-2}. \]

Repeating the argument in the proof of Lemma 4.4, we obtain the conclusion.

Lemma 4.8. For any \( \theta > 0 \) there exist \( l_1 \in (0, 1) \) and \( s_* > 0 \) such that

\[ v(y', s)|_{\partial \mathbb{R}^n_*} \leq -l_1 B \quad \text{for } |y'| = \theta \sqrt{s}, \ s \geq s_* . \]

Proof. We fix \( \theta = 1 \). The proof for the other case \( \theta \neq 1 \) follows from the quite same argument as \( \theta = 1 \). Since \( v(y, s) \) behaves as the case (I), we note from Lemma 4.2 that

\[ v(y', s)|_{\partial \mathbb{R}^n_*} \leq cs^{-1} \chi_{|y'| < \sqrt{2n}} \quad \text{for } s \gg 1 . \]

To derive a contradiction, we suppose that there exist sequences \( \{\tau_j\}_{j \in \mathbb{N}} \) and \( \{y'_j\}_{k \in \mathbb{N}} \subset \partial \mathbb{R}^n_* \) with \( |y'_j| = \sqrt{\tau_j} \) such that \( \tau_j \to \infty \) and

\[ \lim_{j \to \infty} v(y'_j, \tau_j) = 0. \]

Then it follows from (45) and (49) that

\[ \lim_{j \to \infty} \sup_{y' \in B_{\partial \mathbb{R}^n_*}(\sqrt{\tau_j})} |v(y', \tau_j)| = 0. \]  

(50)

Now we define \( s_j < \tau_j \) by

\[ e^{\tau_j - s_j} = \tau_j . \]

From this definition, we find that

\[ 1 - \left( \frac{s_j}{\tau_j} \right) = \left( \frac{\log \tau_j}{\tau_j} \right) . \]

Since \( B_{\partial \mathbb{R}^n_*}(\sqrt{s}) \subset B_{\partial \mathbb{R}^n_*}(\sqrt{\mu}) \) if \( s < \mu \), we apply Lemma 4.5 to get

\[ |v(y', s) - v(y', \tau_j)| \leq \int_s^{\tau_j} |\partial_s v(y', \mu)| d\mu \leq c \log \left( \frac{\tau_j}{s_j} \right) \]

(51)

for \( y' \in B_{\partial \mathbb{R}^n_*}(\sqrt{s}) \) and \( s \in (s_j, \tau_j) \). Now we set

\[ \epsilon_j = \sup_{y' \in B_{\partial \mathbb{R}^n_*}(\sqrt{s}), \ s \in (s_j, \tau_j)} |v(y', s)| . \]

Then (50) and (51) imply

\[ \lim_{j \to \infty} \epsilon_j = 0. \]

Furthermore by definition of \( \epsilon_j \) and (49), we get

\[ \partial_y v = q \left( \int_0^1 (B + \theta v)^{q-1} d\theta \right) v \]

\[ \leq \left( \int_0^1 (B + cs^{-1})^{q-1} d\theta \right) v_+ - \left( \int_0^1 (B - c\epsilon_j)^{q-1} d\theta \right) v_- \]

\[ \leq (qB^{q-1} + cs^{-1})v_+ - qB^{q-1}(1 - c\epsilon_j)v_- \quad \text{for } y' \in B_{\partial \mathbb{R}^n_*}(\sqrt{s}), \ s \in (s_j, \tau_j) . \]

Here we note from (49) that \( v_+(y, s) \leq cs^{-1} \). Therefore we obtain

\[ \partial_y v \leq qB^{q-1}(1 - c\epsilon_j)v + c(\epsilon_j + s^{-1})s^{-1} + c\chi_{|y'| > \sqrt{s}} \quad \text{for } y' \in \partial \mathbb{R}^n_*, \ s \in (s_j, \tau_j) . \]
We set \( K_j = qB^{q-1}(1-ce_j) \) and
\[
\mu_j := \mu_{K_j}, \quad S_j(s) := S_{K_j}(s), \quad T_j(s, \mu) := T_{K_j}(s, \mu), \quad b_j(y_n) := b_{K_j}(y_n).
\]
Then we apply (55) to obtain
\[
v(s)|_{\partial \mathbb{R}_+^n} \leq e^{-(m+\mu_k)(s-s_k)}S_j(s-s_j) \left( \frac{v(s_j)}{b_j} \right) + c \int_{s_j}^{s} e^{-(m+\mu_j)(s-\mu)} \times T_j(s, \mu) \left( (\epsilon_j + \mu^{-1})\mu^{-1} + \chi(|\xi'| > \sqrt{s}) \right) \, d\mu \quad \text{for } s \in (s_j, \tau_j).
\]
Since \( e^{-(\tau_j-s_j)}|y'|^2 \leq 1 \) for \( y' \in B_{\partial \mathbb{R}_+^n}(\sqrt{\tau_j}) \), it follows from Lemma 2.9 and Proposition 3.3 that
\[
S_j(\tau_j - s_j) \left( \frac{v(s_j) + \nu q s_j^{-1}E}{b_j} \right) \leq \exp \left( \frac{e^{-(\tau_j-s_j)}|y'|^2}{4(1+e^{-(\tau_j-s_j)})} \right) o(s_j^{-1}) = o(s_j^{-1}) \quad \text{for } y' \in B_{\partial \mathbb{R}_+^n}(\sqrt{\tau_j}).
\]
We set \( D_j = E/b_j \). Then by the same way as in the proof of Lemma 4.3 we see that
\[
S_j(\tau_j - s_j) D_j = e^{(m+\mu_j)(\tau_j-s_j)} D_j.
\]
Since \( K_j = qB^{q-1}(1-ce_j) < qB^{q-1} \), we note from Lemma 2.8 that
\[
\mu_j > \mu|_{K=qB^{q-1}} = -(m+1).
\]
Hence by using \( e^{\tau_j-s_j} = \tau_j \) and \( D_j = E \) on \( \partial \mathbb{R}_+^n \), we obtain
\[
e^{-(m+\mu_j)(\tau_j-s_j)} S_j(\tau_j - s_j) \left( \frac{v(s_j)}{b_j} \right) = e^{-(m+\mu_j)(\tau_j-s_j)} S_j(\tau_j - s_j) \times \left( \frac{\nu q D_j + (v(s_j) + \nu q s_j^{-1}E)}{b_j} \right)
\]
\[
= -\frac{\nu q D_j + e^{-(m+\mu_j)(\tau_j-s_j)} o(s_j^{-1})}{s_j} = -\frac{\nu q E}{s_j} + \tau_j^{-(m+\mu_j)} o(s_j^{-1})
\]
\[
= -\frac{\nu q E}{s_j} + o(1) \quad \text{for } y' \in B_{\partial \mathbb{R}_+^n}(\sqrt{\tau_j}).
\]
Furthermore by Lemma 2.8 we compute
\[
T_j \left( (\epsilon_j + \mu^{-1})\mu^{-1} + \chi(|\xi'| > \sqrt{s}) \right) \leq c \left( \frac{\epsilon_j + \mu^{-1}}{\sqrt{1-e^{-(s-\mu)}}} \right)
\]
\[
+ \int_{\mathbb{R}^{n-1}} \frac{\gamma(y', \xi', s-\mu)}{\sqrt{1-e^{-(s-\mu)}}} \chi(|\xi'| > \sqrt{s}) \, d\xi' \quad \text{for } y' \in \partial \mathbb{R}_+^n.
\]
Here since \( \tau_j \leq 9s_j/4 \) for large \( j \in \mathbb{N} \), we observe that
\[
|y'| \leq \frac{1}{2} \sqrt{\tau_j} \leq \frac{3}{4} \sqrt{s_j} \leq \frac{3}{4} |\xi'| \quad \text{for } |y'| < \frac{\sqrt{\tau_j}}{2}, \ |\xi'| > \sqrt{s_j}.
\]
This implies
\[
|y' e^{-(s-\mu)/2} - \xi'| \geq |\xi'| - |y'| \geq \frac{|\xi'|^2}{4} \quad \text{for } |y'| < \frac{\sqrt{\tau_j}}{2}, \ |\xi'| > \sqrt{s_j}.
\]
Therefore we get
\[ \int_{\mathbb{R}^{n-1}} \gamma(y', \xi', s - \mu) \chi_{\{|\xi'| > \sqrt{n}\}} |\xi'|^2 d\xi' \leq c \int_{|\xi'| \geq \sqrt{n}} \frac{\exp \left(- \frac{|\xi'|^2}{64(1 - e^{-(s-\mu)})}\right)}{1 - e^{-(s-\mu)(n-1)/2}} d\xi' \]

\[ = c \int_{|\xi'| \geq \frac{\mu}{1 - e^{-(s-\mu)}}} e^{-|\xi'|^2/64} d\xi' \leq c \int_{|\xi'| \geq \sqrt{n}} e^{-|\xi'|^2/64} d\xi' \]

\[ \leq c \mu^{(n-3)/2} e^{-\mu/64} \quad \text{for} \quad y' \in \partial \Omega^n_+(\frac{\sqrt{T_j}}{2}), \quad \mu > s_j. \]

As a consequence, it follows that

\[ \int_{s_j}^{\tau_j} e^{-(m+\mu)(\tau_j-\mu)} T_j(s, \mu) \left((\epsilon_j + \mu^{-1})\mu^{-1} + \chi_{\{|\xi'| > \sqrt{n}\}}\right) d\mu \]

\[ \leq c \left( (\epsilon_j + s_j^{-1}) s_j^{-1} + s_j^{(n-3)/2} e^{-s_j/64} \right) \left(1 + e^{-(m+\mu)(\tau_j-s_j)}\right) \]

\[ = c \left( \epsilon_j + O(s_j^{-1}) \right) \quad \text{for} \quad y' \in \partial \Omega^n_+(\frac{\sqrt{T_j}}{2}). \]

Thus finally we obtain

\[ v(y', \tau_j) \leq - \frac{\nu \mathcal{E}(y')}{s_j} + o(1) \quad \text{for} \quad y' \in \partial \Omega^n_+(\frac{\sqrt{T_j}}{2}). \]

Noting that \( \mathcal{E}(y') = c(|y'|^2 - 2(n - 1)) \) on \( \partial \Omega^n_+(c > 0) \), then we find that

\[ v(y', \tau_j)|_{\partial \Omega^n_+} \leq - \frac{c \nu q}{4} \frac{\tau_j}{s_j} + o(1) \quad \text{for} \quad |y'| = \frac{\sqrt{T_j}}{2}. \]

However since \( \tau_j/s_j \to 1 \) as \( j \to \infty \), this contradicts \( \text{(II)} \), which completes the proof. \( \square \)

**Proof of Proposition 4.1.** We recall from Lemma 4.3 that

\[ \varphi(y', s)|_{\partial \Omega^n_+} \geq B - \nu q s^{-1} \mathcal{E} + o(1) \quad \text{for} \quad y' \in \partial \Omega^n_+(\sqrt{s}). \]

Since \( \mathcal{E}(y') = c(|y'|^2 - 2(n - 1)) \) on \( \partial \Omega^n_+(c > 0) \), there exists \( \theta \in (0, 1) \) such that

\[ \varphi(y', s)|_{\partial \Omega^n_+} \geq \frac{B}{2} \quad \text{for} \quad |y'| = \theta \sqrt{s}, \quad s \gg 1. \]

On the other hand, from Lemma 4.3, there exists \( l_1 \in (0, 1) \) such that

\[ \varphi(y', s)|_{\partial \Omega^n_+} = v(y', s)|_{\partial \Omega^n_+} + B \leq (1 - l_1)B \quad \text{for} \quad |y'| = \theta \sqrt{s}, \quad s \gg 1. \]

Thus the proof is completed. \( \square \)

### 5 Spacial singularities for a blow-up profile

Here we apply methods given in \( \text{[13, 22]} \) to investigate spacial singularities of blow-up profiles. As in Section 4, \( v(y, s) \) is assumed to be \( y_n \)-axial symmetric, behave as the case (I) in Proposition 3.1 and satisfy \( \text{(III)}. \) Let \( \theta \in (0, 1) \) be given in Proposition 4.1 and set \( \vec{e} = (1, 0, \cdots, 0) \). We introduce

\[ v_s(x, t) = e^{-ms} u(e^{-s/2} x + \theta \sqrt{s} e^{-s/2} \vec{e}, T + (t - 1)e^{-s}), \]

34
where \( s > s_T \) is a parameter. Then \( v_s(x, t) \) satisfies
\[
\begin{cases}
\partial_t v = \Delta v & \text{in } \mathbb{R}^n_+ \times (0, 1), \\
\partial_r v = v' & \text{on } \partial \mathbb{R}^n_+ \times (0, 1).
\end{cases}
\]

The following proposition gives a key estimate, whose proof is given in the next subsection.

**Proposition 5.1.** There exist \( 0 < c_- < c_+ \), \( s^* \gg 1 \) and \( t_1 \in (0, 1) \) such that
\[
c_- \leq v_s(0, t) \leq c_+ \text{ for } s > s^*, \ t \in (t_1, 1).
\]

The proof of Theorem 1.2 follows directly from Proposition 5.1.

**Proof of Theorem 1.2.** Now we define an inverse function \( s(r) \) by \( \sqrt{s} e^{-s/2} = r/\theta \) (\( s > 1 \)). Then from Proposition 5.1, there exist \( r_+ \in (0, 1) \) and \( t_1 \in (0, 1) \) such that
\[
c_- e^{m s(r)} \leq u(r \bar{e}, T + (t - 1)e^{-s(r)}) \leq c_+ e^{m s(r)} \text{ for } r \in (0, r_+), \ t \in (t_1, 1).
\]

Here we take \( t = 1 \) to obtain
\[
c_- e^{m s(r)} \leq u(r \bar{e}, T) \leq c_+ e^{m s(r)} \text{ for } r \in (0, r_+).
\]

From definition of \( s(r) \), we compute
\[
s = \log s + 2|\log r| + 2 \log \theta,
\]
\[
e^{-s} = e^{-s} (\log s + 2|\log r| + 2 \log \theta) = (r/\theta)^2.
\]

Hence it follows that
\[
e^{-s} = \frac{\theta^2}{2\theta^2|\log r|} \left( \frac{2|\log r|}{\log s + 2|\log r| + 2 \log \theta} \right).
\]

Since \( s \leq c|\log r| \) for small \( r > 0 \), it is clear that
\[
\left( \frac{2|\log r|}{\log s + 2|\log r| + 2 \log \theta} \right) = 1 + o(1).
\]

Therefore there exists \( 0 < c'_- < c'_+ \) such that
\[
c'_- \left( \frac{|\log r|}{r^2} \right)^m (1 + o(1)) \leq u(r \bar{e}, T) \leq c'_+ \left( \frac{|\log r|}{r^2} \right)^m (1 + o(1)).
\]

Since \( u(x, t) \) is \( x_n \)-axial symmetric, we obtain the conclusion.

5.1 Proof of Proposition 5.1 (upper bound)

We consider a rescaled solution \( w_s(y, \tau) \) defined by \( (y \in \mathbb{R}^n_+, \ \tau \in \mathbb{R}_+) \)
\[
w_s(y, \tau) = e^{-m \tau} v_s(e^{-\tau/2} y, 1 - e^{-\tau})
\]
\[
= e^{-m(\tau + s)} u(e^{-(\tau + s)/2} y + \theta \sqrt{s} e^{-s/2} \bar{e}, T - e^{-(\tau + s)})
\]
\[
= \varphi(y + \theta \sqrt{s} e^{-s/2} \bar{e}, \tau + s).
\]

Then \( w_s(y, \tau) \) satisfies
\[
\begin{cases}
\partial_y w_s = \Delta w_s - \frac{y}{2} \cdot \nabla w_s - m w_s & \text{in } \mathbb{R}^n_+ \times (0, \infty), \\
\partial_{\tau} w_s = w_s^q & \text{on } \partial \mathbb{R}^n_+ \times (0, \infty)
\end{cases}
\]
(52)
Lemma 5.1. Let
diverges to zero uniformly on \( \mathbb{R} \). First we claim that there exists \( \xi > h \) such that
\[
\left( \frac{k_1}{2} \right) \varphi_0(y_n) \leq w_s(y_n, 0) \big|_{y' = 0} \leq \left( \frac{1 + k_2}{2} \right) \varphi_0(y_n) \quad \text{for } y_n \in (0, \delta_0).
\]

On the other hand, since \( \varphi(y, s) \to \varphi_0(y_n) \) in \( C_{\text{loc}}(\mathbb{R}^n_+) \) as \( s \to \infty \), from (45), we obtain
\[
\limsup_{s \to \infty} w_s(y_n, 0) \big|_{y' = 0} = \lim_{s \to \infty} \varphi(y_n, s) \big|_{y' = 0} = \varphi_0(y_n)
\]
uniformly on \( y_n \in [0, R] \) for any \( R > 0 \). We fix a function \( w_s(\xi) \in C^2(\mathbb{R}_+) \) such that
\[
\left( \frac{1 + k_2}{2} \right) \varphi_0(\xi) \leq w_s(\xi) < \varphi_0(\xi) \quad \text{if } \xi \in (0, \delta_0),
\]
\[
w_s(\xi) = \varphi_0(\xi) \quad \text{if } \xi \in (\delta_0, \infty).
\]

Let \( \tilde{w}(\xi, \tau) \) be a solution of
\[
\begin{aligned}
\partial_{\tau} \tilde{w} &= \tilde{w}_{\xi \xi} - \frac{\xi}{2} \tilde{w}_{\xi} - m \tilde{w} \quad \text{in } \mathbb{R}_+ \times (0, \infty), \\
\partial_{\xi} \tilde{w} &= \tilde{w}^q \quad \text{on } \xi = 0, \ \tau \in (0, \infty), \\
\tilde{w} &= w_s \quad \text{in } \mathbb{R}_+.
\end{aligned}
\]
For the case (i), by definition of $\xi_1$, it holds that $h'_0(\xi_1) - h'_0(\xi_1) = e^{\xi^2/4}g(\xi_1)/h_*(\xi_1) = g(0)/h_*(\xi_1) < 0$. However this contradicts the definition of $\xi_1$. For the case (ii), since $\varphi_0(\xi)$ is uniformly bounded on $\mathbb{R}_+$, it is verified that $h(\xi)$, $h_0(\xi)$ and their derivatives are uniformly bounded on $\mathbb{R}_+$. Hence it follows that $\lim_{\xi \to \infty} g(\xi) = 0$. However since $g(\xi) \equiv g(0) < 0$, this is a contradiction. Therefore the claim is proved. We denote by $\xi_0$ the first zero of $h_0(\xi)$. Since $h'_0(\xi_0) < 0$, by continuity, $h_*(\xi)$ has a unique zero near $\xi = \xi_0$ for small $\epsilon \in (0,1)$, which is denoted by $\xi_\epsilon$. We fix $\epsilon = \epsilon_0$ small enough. Now we construct a super-solution by using $h_{\epsilon_0}(\xi)$. We set

$$
\psi_a(\xi) = \begin{cases} 
\varphi_0(\xi) - ah_{\epsilon_0}(\xi) & \text{if } \xi \in (0, \xi_0), \\
\varphi_0(\xi) & \text{if } \xi \in (\xi_0, \infty),
\end{cases}
$$

where $a \in (0,1)$ is a parameter. Then since $(B - a)^q = B^q - qB^{q-1}a + O(a^2)$, we get

$$
\partial_\nu \psi_a = B^q - (1 - \epsilon_0)qB^{q-1}a
$$

$$
= (B - a)^q + \epsilon_0qB^{q-1}a + O(a^2)
$$

$$
\geq (B - a)^q = \psi_a^\# \quad \text{for } \xi = 0, \quad 0 < a \ll 1.
$$

Since $w_a(\xi) \leq \varphi_0(\xi)$, a comparison argument implies $\bar{w}(\xi, s) \leq \varphi_0(\xi)$. Hence, by a strong maximum principle, there exists $a_1 \in (0,1)$ such that for $a \in (0, a_1)$

$$
\bar{w}(\xi, 1) \leq \psi_a(\xi) \quad \text{for } \xi \in \mathbb{R}_+.
$$

Let $W_a(\xi, s)$ be a solution of (57) with the initial data $\psi_a(\xi)$. Then a comparison argument implies that

$$
\bar{w}(\xi, s + 1) \leq W_a(\xi, s).
$$

Now we claim that $W_a(\xi, s)$ converges to zero uniformly on $\mathbb{R}_+$ as $s \to \infty$. Since $\psi_a(\xi)$ is a super-solution, it holds that $W_a(\xi, s) \leq \psi_0(\xi)$ for $s > 0$. By the unique solvability of solutions of (57) and a comparison argument, we see that

$$
W_a(\xi, s + s') \leq W_a(\xi, s') \quad \text{for } s, s' > 0.
$$

Hence it follows that $\partial_\nu W_a(\xi, s) \leq 0$ for $s > 0$. As a consequence, $W_a(\xi, s)$ converges to some function $W_*(\xi)$ satisfying $0 \leq W_*(\xi) < \varphi_0(\xi)$ uniformly on $\mathbb{R}_+$ as $s \to \infty$. By a standard argument, we find that $W_*(\xi)$ is one of stationary solutions of (57). Since $\varphi_0(\xi)$ is the unique bounded positive solution of (57) (see Lemma 3.1 [5]), $W_*(\xi)$ must be zero, which shows the claim. Therefore from $\bar{w}(\xi, s + 1) \leq W_a(\xi, s)$, $\bar{w}(\xi, s)$ also converges to zero uniformly on $\mathbb{R}_+$ as $s \to \infty$, which completes the proof.

The function $\bar{w}(\xi, \tau)$ is naturally extended to a function $\hat{w}(y, \tau)$ defined on $\mathbb{R}_+^n \times (0, \infty)$ by

$$
\hat{w}(y, s) = \bar{w}(y_n, s).
$$

From Lemma 5.1 it is clear that $\lim_{\tau \to \infty} ||\hat{w}(\tau)||_\rho = 0$. Let $\tau_\epsilon > 0$ be the first time of

$$
||\hat{w}(\tau_\epsilon)||_\rho = \epsilon.
$$

**Lemma 5.2.** For any $\epsilon > 0$ there exist $s_\epsilon > 0$ such that

$$
||w_s(\tau_\epsilon)||_\rho \leq 2\epsilon \quad \text{for } s \geq s_\epsilon.
$$
Proof. Suppose that there exist $\epsilon_0 > 0$ and a sequence $\{s_k\}_{k \in \mathbb{N}}$ $(s_k \to \infty)$ such that
\[
\|w_{s_k}(\tau_0)\|_\rho < 2\epsilon_0.
\] (58)
Since $w_s(y, \tau)$ is defined by $w_s(y, \tau) = \varphi(y + \theta e^{\tau/2} s^{1/2} e, \tau + s)$, we apply Lemma 2.1 to obtain
\[
\sup_{s \in (s, \infty)} \sup_{(y, \tau) \in \mathbb{R}_+^n \times (0, \infty)} \left( \sum_{|\alpha| = 0}^2 |D^\alpha w_s(y, \tau)| \right) < \infty.
\] (59)
Furthermore since $w_s(y, \tau)$ satisfies
\[
\partial_\tau w_s = \Delta_y w_s - \frac{y}{2} \cdot \nabla_y w_s - mw_s \quad \text{in} \quad \mathbb{R}_+^n \times (0, \infty),
\]
it follows from (59) that
\[
\sup_{s \in (s, \infty)} \sup_{(y, \tau) \in \mathbb{R}_+^n \times (0, \infty)} \left( (1 + |y|^2)^{-1/2} |\partial_\tau w_s(y, \tau)| \right) < \infty.
\]
Hence there exist a limiting function $w_\infty(y, \tau) \in C^{2,1}((\mathbb{R}_+^n \times [0, \infty)) \cap L^\infty(\mathbb{R}_+^n \times (0, \infty))$ and a subsequence $\{w_{s_k}(y, \tau)\}_{k \in \mathbb{N}}$ which is denoted by the same symbol such that
\[
w_{s_k}(y, \tau) \to w_\infty(y, \tau)
\]
in $C_\text{loc}(\mathbb{R}_+^n \times [0, \infty)) \cap C([0, \tau'); L^2(\mathbb{R}_+^n))$ for any $\tau' > 0$. Then from (51), (55) and (56), we see that
\[
0 \leq w_\infty(y, 0) \leq w_s(y, 0).
\]
Moreover from Lemma 4.4 it follows that $\nabla' w_\infty(y, 0) = 0$. Hence we obtain
\[
0 \leq w_\infty(y, 0) \leq w_s(y, 0).
\]
As a consequence, since $w_\infty(y, \tau)$ satisfies (52), a comparison argument shows that for $\tau \geq 0$
\[
0 \leq w_\infty(y, \tau) \leq w(y, \tau).
\]
Hence by definition of $\tau_0$, it follows that $\|w_\infty(\tau_0)\|_\rho \leq \epsilon_0$. Since $w_{s_k} \to w_\infty$ in $C([0, \tau'); L^2(\mathbb{R}_+^n))$ for any $\tau' > 0$, it follows that $\|w_{s_k}(\tau_0)\|_\rho < 2\epsilon_0$ for large $k \in \mathbb{N}$. However this contradicts (58), which completes the proof. 

We prepare a local $L^\infty$-estimate which is directly derived from a standard linear parabolic theory.

Lemma 5.3. For any $R > 0$ there exists $c_R > 0$ such that
\[
\sup_{|y| < R} w_s(y, \tau) \leq c_R \sup_{\tau' \in (\tau - 4R^2, \tau)} \|w_s(\tau')\|_\rho \quad \text{for} \quad \tau > 4R^2.
\]

Proof. From (53), applying a local $L^\infty$-estimate for a linear parabolic equation to (52) (see Theorem 6.17 in [20]), we obtain
\[
\left( \sup_{|y| < R, \tau - 4R^2 < \tau' < \tau} w_s(y, \tau') \right)^2 \leq c_R^2 \int_{\tau - 4R^2}^{\tau} d\tau' \int_{|y| < 2R} w_s(y, \tau')^2 dy
\]
\[
\leq c_R^2 e^{4R^2} \int_{\tau - 4R^2}^{\tau} d\tau' \int_{|y| < 2R} w_s(y, \tau')^2 e^{-|y|^2/4} dy
\]
\[
\leq c_R^2 e^{4R^2} \left( \sup_{\tau' \in (\tau - 4R^2, \tau)} \|w_s(\tau')\|_\rho \right)^2,
\]
which completes the proof. 

38
Lemma 5.4. For any \( \delta_0 > 0 \) there exists \( \delta_1 \in (0, \delta_0) \) such that if \( \|w_s(\tau_0)\|_\rho \leq \delta_1 \) holds for some \( \tau_0 > 0 \) and \( s > s_T \), then it holds that
\[
\|w_s(\tau)\|_\rho \leq \delta_0 \quad \text{for } \tau \geq \tau_0.
\]

Proof. Multiplying (62) by \( w_s \rho \) and integrating over \( \mathbb{R}^n_+ \), we get
\[
\frac{1}{2} \partial_\tau \|w_s\|_\rho^2 = -\|\nabla w_s\|_\rho^2 - m\|w_s\|_\rho^2 + \|w_s\|_{L^{q+1}_\rho(\partial \mathbb{R}^n_+)}^q.
\]
From Lemma 3.1 and Lemma A.1 we verify that
\[
\|w_s\|_{L^{q+1}_\rho(\partial \mathbb{R}^n_+)} \leq \left( \int_{\mathbb{R}^n_+} (1 + |y'|^2)^{1/2} w_s^2 \rho \, dy \right)^{1/2} \left( \int_{\mathbb{R}^n_+} (1 + |y'|^2)^{-1/2} w_s^2 \rho \, dy \right)^{1/2} \leq cK_s(\tau) \left( \|w_s\|_\rho^2 + \|w_s\|_\rho \|\nabla w_s\|_\rho \right),
\]
where \( K_s(\tau) \) is given by
\[
K_s(\tau) = \left( \sup_{y' \in \partial \mathbb{R}^n_+} (1 + |y'|^2)^{-1/2} w_s(y', \tau)^{2(q-1)} \right)^{1/2}.
\]
Hence it holds that
\[
\frac{1}{2} \partial_\tau \|w_s\|_\rho^2 \leq -\|\nabla w_s\|_\rho^2 - m\|w_s\|_\rho^2 + cK_s(\tau) \left( \|w_s\|_\rho^2 + \|w_s\|_\rho \|\nabla w_s\|_\rho \right). \tag{60}
\]
We note from (53) that \( K_s(\tau) \) is uniformly bounded on \( s \in (s_T, \infty) \) and \( \tau \in (0, \infty) \). Therefore there exists \( \alpha_0 > 0 \) independent of \( s > s_T \) such that
\[
\partial_\tau \|w_s\|_\rho^2 \leq \alpha_0^2 \|w_s\|_\rho^2,
\]
which implies
\[
\|w_s(\tau)\|_\rho \leq e^{\alpha_0(\tau-\tau_0)} \|w_s(\tau_0)\|_\rho \quad \text{for } \tau > \tau_0. \tag{61}
\]
Let \( \delta_0 > 0 \) be a constant given in this lemma and \( \epsilon_0 > 0 \) be a small constant. Then we can fix \( 0 < \delta_1 < \delta_2 < \delta_0 \) and \( R_1 > 0 \) such that
\[
(1 + R_1^2)^{-1/2} M_0^{2(q-1)} + (c_{R_1} \delta_2)^2(q-1) < \epsilon_0, \quad \delta_1 e^{4\alpha_0 R_1^2} < \delta_2, \tag{62}
\]
where \( c_{R_1} > 0 \) is given in Lemma 5.3 and \( M_0 > 0 \) is given in (53). Here we assume that \( \|w_s(\tau_0)\|_\rho \leq \delta_1 \) for some \( \tau_0 > 0 \) and \( s > s_T \). Then we will see that \( \|w_s(\tau)\|_\rho < \delta_2 \) for \( \tau > \tau_0 \). In fact, we first define
\[
\bar{\tau} = \inf \{ \tau > \tau_0; \|w_s(\tau)\|_\rho = \delta_2 \}.
\]
To derive a contradiction, we suppose \( \bar{\tau} < \infty \). Then it follows from (61) and (62) that
\[
\bar{\tau} > \tau_0 + 4R_1^2.
\]
Furthermore we apply Lemma 5.3 with (62) to obtain
\[
\sup_{|y| < R_1} w_s(y, \tau) \leq c_{R_1} \sup_{\tau' \in (\tau-4R_1^2, \tau)} \|w_s(\tau')\|_\rho \leq c_{R_1} \sup_{\tau' \in (\tau_0, \bar{\tau})} \|w_s(\tau')\|_\rho \leq c_{R_1} \delta_2 \quad \text{for } \tau \in (\tau_0 + 4R_1^2, \bar{\tau}).
\]

39
Hence by using this estimate and \( (52) \), we get
\[
K_4(\tau)^2 = \sup_{y' \in \partial R^+_1} (1 + |y'|^2)^{-1/2} w_s(y', \tau)^2(\nu-1)
\]
\[
\leq \max \left\{ (c_{R_1}, \delta_2)^2(\nu-1), (1 + R_1^2)^{-1/2} M_0^2(\nu-1) \right\}
\]
\[
\leq \epsilon_0^2 \quad \text{for } \tau \in (\tau_0 + 4R_1^2, \bar{\tau}).
\]
Therefore substituting this estimate into \( (60) \) and taking \( \epsilon_0 > 0 \) small enough, we obtain
\[
\partial_\tau\|w_s(\tau)\|_\rho^2 < 0 \quad \text{for } \tau \in (\tau_0 + 4R_1^2, \bar{\tau}),
\]
which implies \( \|w_s(\bar{\tau})\|_\rho < \|w_s(\tau_0 + 4R_1^2)\|_\rho \). Furthermore by \( (51), \ (52) \) and \( \|w_s(\tau_0)\|_\rho \leq \delta_1 \), we see that
\[
\|w_s(\tau_0 + 4R_1^2)\|_\rho \leq e^{4\alpha_2 R_1^2} \|w_s(\tau_0)\|_\rho < \delta_2.
\]
Therefore we obtain \( \|w_s(\bar{\tau})\|_\rho < \delta_2 \). However this contradicts definition of \( \bar{\tau} \), which assures \( \bar{\tau} = \infty \). Thus the proof is completed.

Next we provide uniform decay estimates.

**Lemma 5.5.** For any \( \nu > 0 \) there exist \( c, s_1^*, \tau_1 > 0 \) depending only on \( \nu > 0 \) such that for \( s \geq s_1^* \)
\[
\|w_s(\tau)\|_\rho \leq c e^{-(m-\nu)(\tau-\tau_1)} \quad \text{for } \tau \geq \tau_1.
\]

**Proof.** From Lemma 5.2–Lemma 5.4 for any \( \nu \in (0, 1) \) there exists \( s_1^*, \tau_1 > 0 \) depending only on \( \nu > 0 \) such that
\[
\sup_{s > s_1^*, \tau > \tau_1} K_s(\tau) \leq \nu.
\]
Hence substituting this estimate into \( (60) \), we get for \( s > s_1^* \)
\[
\partial_\tau\|w_s(\tau)\|_\rho^2 \leq -2(m - c\nu)\|w_s(\tau)\|_\rho^2 \quad \text{for } \tau \geq \tau_1.
\]
As a consequence, it holds that for \( s \geq s_1^* \)
\[
\|w_s(\tau)\|_\rho^2 \leq e^{2(m - c\nu)(\tau-\tau_1)} \|w_s(\tau_1)\|_\rho^2 \quad \text{for } \tau \geq \tau_1.
\]
Here we note from \( (53) \) that \( \|w_s(\tau)\|_\rho \) is uniformly bounded on \( s > s_T \) and \( \tau > 0 \). Thus we completes the proof.

Let \( E_0 \) be a positive constant such that \( \|E_0\|_\rho = 1 \). Here we decompose \( w_s(y, \tau) \) by
\[
w_s(\tau) = w_{s0}(\tau) + w_{s-}(\tau),
\]
where \( w_{s0}(\tau) = (w_s(\tau), E_0)_\rho E_0 \).

**Lemma 5.6.** There exists \( c, \nu_0, s_2^*, \tau_2 > 0 \) such that for \( s \geq s_2^* \)
\[
\|w_{s-}(\tau)\|_\rho \leq c e^{-(1+\nu_0)m\tau}, \quad \int_{\tau_1}^{\infty} e^{2(1+\nu_0)m\tau} \|\nabla w_{s-}(\tau)\|_\rho^2 d\tau \leq c \quad \text{for } \tau \geq \tau_2.
\]
Proof. Since \((w_s^-,w_{s0})_\rho = 0\), we easily see that
\[
\frac{1}{2} \partial_\rho \|w_s-\|^2_\rho = -\|\nabla w_s-\|^2_\rho - m \|w_s-\|^2_\rho + \int_{\partial \mathbb{R}^n_+} w_s^q w_s- \rho \, dy'.
\]
Now we estimate the last term on the right-hand side.
\[
\int_{\partial \mathbb{R}^n_+} w_s^q |w_s-| \rho \, dy' = \int_{\partial \mathbb{R}^n_+} w_{s0}^q |w_s-| \rho \, dy' + \int_{\partial \mathbb{R}^n_+} (w_s^q - w_{s0}^q) |w_s-| \rho \, dy'.
\]
Then by Lemma 3.1 it holds that
\[
\int_{\partial \mathbb{R}^n_+} w_{s0}^q |w_s-| \rho \, dy' \leq c \|w_{s0}\|_{L^2(\partial \mathbb{R}^n_+)}^2 + \frac{c}{\epsilon} \|w_{s0}\|_\rho^{2q} \leq c \epsilon \|w_{s0}\|_{L^2(\partial \mathbb{R}^n_+)}^2 + \frac{c}{\epsilon} \|w_{s0}\|_\rho^{2q}.
\]
Furthermore the mean value theorem implies
\[
\int_{\partial \mathbb{R}^n_+} (w_s^q - w_{s0}^q) |w_s-| \rho \, dy' \leq q \int_{\partial \mathbb{R}^n_+} (w_s + |w_{s0}|)^{q-1} |w_s-|^2 \rho \, dy' \leq q M_s(\tau) \int_{\partial \mathbb{R}^n_+} (1 + |y'|^2)^{1/2} |w_s-|^2 \rho \, dy',
\]
where \(M_s(\tau)\) is given by
\[
M_s(\tau) = \sup_{y' \in \partial \mathbb{R}^n_+} (1 + |y'|^2)^{-1/2} (w_s(y',\tau) + |w_{s0}(y',\tau)|)^{q-1}.
\]
Hence from Lemma 3.1 and Lemma A.11 we get
\[
\int_{\partial \mathbb{R}^n_+} (w_s^q - w_{s0}^q) |w_s-| \rho \, dy' \leq c M_s(\tau) \|w_s-\|_{H^1(\mathbb{R}^n_+)}^2.
\]
Therefore since \(\|w_{s0}\|_\rho \leq \|w_s\|_\rho\), we obtain
\[
\frac{1}{2} \partial_\rho \|w_s-\|^2_\rho \leq -(1 - c \epsilon - c M_s(\tau)) (\|\nabla w_s-\|^2_\rho + m \|w_s-\|^2_\rho) + \frac{c}{\epsilon} \|w_s\|_\rho^{2q}.
\]
Since \(q > 1\), from Lemma 5.3 there exists \(s^*_1, \tau_1, \nu_1 > 0\) such that for \(s > s^*_1\)
\[
\|w_s(\tau)|\|_\rho^{2q} \leq c e^{-2(m+\nu_1)(\tau-\tau_1)} \text{ for } \tau > \tau_1.
\]
Moreover by the same estimate as \(K_s(\tau)\) in the proof of Lemma 5.3 we find that
\[
\lim_{s,\tau \to \infty} M_s(\tau) = 0.
\]
Here we note that \(E_0\) is the first eigenfunction with zero eigenvalue of
\[
- \left( \Delta - \frac{\nu}{2} \nabla \right) E = \lambda E \text{ in } \mathbb{R}^n_+, \quad \partial_\rho E = 0 \text{ on } \partial \mathbb{R}^n_+. \tag{63}
\]
Since the second eigenvalue of \((63)\) is one, it holds that \(\|\nabla w_s-\|_\rho \geq \|w_s-\|_\rho\). Hence we take \(\epsilon > 0\) small enough, then there exists \(\nu_0 \in (0, \nu_1), s_2^* > s_1^*\) and \(\tau_2 > \tau_1\) such that for \(s > s_2^*\)
\[
\partial_\tau \|w_s-\|^2_\rho \leq - \|\nabla w_s-\|^2_\rho - 2(1 + \nu_0) m \|w_s-\|^2_\rho + c e^{-2(1+\nu_1)m(\tau-\tau_1)} \text{ for } \tau > \tau_2.
\]
Multiplying by \(e^{2(1+\nu_0)\tau_2}\) and integrating both sides over \((\tau_2, \tau)\), we obtain the conclusion. \(\square\)
Lemma 5.7. There exists \( c, s^*_3, \tau_3 > 0 \) such that for \( s \geq s^*_3 \)
\[
\|w_{s0}(\tau)\|_\rho \leq ce^{-m\tau} \quad \text{for} \quad \tau \geq \tau_3.
\]

Proof. From definition of \( w_{s0} \), we easily see that
\[
\frac{1}{2} \partial_\tau \|w_{s0}\|_\rho^2 \leq -m\|w_{s0}\|^2_\rho + c\|w_{s0}\|_\rho \int_{\partial \mathbb{R}_+^n} w_{s0}^q \rho dy' \\
\leq -m\|w_{s0}\|^2_\rho + c\|w_{s0}\|_\rho \int_{\partial \mathbb{R}_+^n} (|w_{s0}|^q + |w_{s-}|^q) \rho dy'.
\]
Since \( q > 1 \), we fix \( r > 2 \) such that \( r'q > 2 \), where \( r' \) is defined by \( 1 = 1/r + 1/r' \). Then by the Hölder inequality and a boundedness of \( w_{s-}(y, \tau) \), we obtain
\[
\|w_{s0}\|_\rho \int_{\partial \mathbb{R}_+^n} |w_{s-}|^q \rho dy' \leq c\|w_{s0}\|_\rho^r + \int_{\partial \mathbb{R}_+^n} |w_{s-}|^{r'q} \rho dy' \\
\leq c\|w_{s0}\|_\rho^r + c\int_{\partial \mathbb{R}_+^n} |w_{s-}|^2 \rho dy' \\
\leq c\|w_{s0}\|_\rho^r + c\|w_{s-}\|_{H^1_\rho(\mathbb{R}_+^n)}^2.
\]
From Lemma 5.5 there exists \( s_1, \tau_1, \nu_0 > 0 \) such that for \( s > s_1 \) and \( \tau > \tau_1 \)
\[
\frac{1}{2} \partial_\tau \|w_{s0}\|_\rho^2 \leq -m\|w_{s0}\|_\rho^2 + ce^{-2(1+\nu_0)m\tau} + c\|w_{s-}\|_{H^1_\rho(\mathbb{R}_+^n)}^2.
\]
Thus multiplying by \( e^{2m\tau} \) and integrating over \((\tau_1, \tau)\), from Lemma 5.6 we obtain the conclusion. \( \square \)

Proof of Proposition 5.1 (upper bound). We apply Lemma 5.3 with \( R = 1 \) to obtain
\[
w_{s}(0, \tau) \leq c \sup_{y \in B_1} \sup_{\tau' \in (\tau - 1, \tau)} w_{s}(y, \tau') \leq c \sup_{\tau' \in (\tau - 4, \tau)} \|w_s(\tau')\|_\rho.
\]
Therefore from Lemma 5.6 and Lemma 5.7 there exists \( s^*_0, \tau_0 > 0 \) such that for \( s > s^*_0 \)
\[
w_{s}(0, \tau) \leq ce^{-m\tau} \quad \text{for} \quad \tau > \tau_0.
\]
By definition of \( w_s(y, \tau) \), we note that \( v_s(0, 1 - e^{-\tau}) = e^{m\tau} w_s(0, \tau) \). Thus we conclude that for \( s \geq s^*_0 \)
\[
v_s(0, 1 - e^{-\tau}) \leq c \quad \text{for} \quad \tau \geq \tau_1,
\]
which completes the proof. \( \square \)

5.2 Proof of Proposition 5.1 (lower bound)

Proof of Proposition 5.1 (lower bound). The proof of lower bound is much easier than that of upper bound. From (54) and \( |\nabla w_s(y, \tau)| \leq c \), there exist a nonnegative smooth function \( w_s(y) \neq 0 \) and \( s^*_1 \gg 1 \) such that for \( s > s^*_1 \)
\[
w_s(y, 0) \geq w_s(y) \quad \text{for} \quad y \in \mathbb{R}_+^n.
\]
Let \( \bar{w}(y, \tau) \) be a solution of
\[
\begin{aligned}
\partial_\tau \bar{w} &= \Delta \bar{w} - \frac{y}{2} \cdot \nabla \bar{w} - m\bar{w} \quad \text{in} \quad \mathbb{R}_+ \times (0, \infty), \\
\partial_\nu \bar{w} &= 0 \quad \text{on} \quad \partial \mathbb{R}_+^n \times (0, \infty), \\
\bar{w}(y, 0) &= w_s(y) \quad \text{in} \quad \mathbb{R}_+^n.
\end{aligned}
\]
Then a comparison argument shows that for $s > s_1^*$

$$w_s(y, \tau) \geq \tilde{w}(y, \tau).$$

As in Section 5.1, we expand $\tilde{w}(y, s)$ by using eigenfunctions of (63). Let $E_0$ be a positive constant with $\|E_0\|_\rho = 1$. Then $E_0$ turns out to be the first eigenfunction of (63). We decompose $\tilde{w}(y, \tau)$ as follows.

$$\tilde{w}(\tau) = \tilde{w}_0(\tau) + \tilde{w}_-(\tau),$$

where $\tilde{w}_0 = (\tilde{w}, E_0)_\rho E_0$. Since the second eigenvalue of (63) is one, we get

$$\tilde{w}_0(\tau) = (w_*, E_0)_\rho e^{-m\tau}E_0, \quad \|\tilde{w}_-(\tau)\|_\rho \leq \|w_*\|_\rho e^{-(m+1)\tau} \quad \text{for } \tau > 0.$$

Hence a local parabolic regularity theory shows that

$$\sup_{y \in B_1} |\tilde{w}_s-(y, \tau)| \leq ce^{-(m+1)\tau} \quad \text{for } \tau > 0.$$

As a consequence, since $a_0 := (w_*, E_0)_\rho > 0$, we obtain

$$\tilde{w}(y, \tau) = a_0 (1 + O(e^{-\tau})) e^{-m\tau}E_0 \quad \text{for } y \in B_1.$$

Thus we conclude that for $s > s_1^*$

$$w_s(0, \tau) \geq \tilde{w}(0, \tau) = a_* (1 + O(e^{-\tau})) e^{-m\tau}E_0,$$

which completes the proof. \qed

**A Appendix**

**A.1 Compact embedding inequality**

Here we provide the embedding inequality on a weighted Sobolev space.

**Lemma A.1.** It holds that for $u \in H^1_\rho(\mathbb{R}^n_+)$

$$\int_{\mathbb{R}^n_+} |y_i|^2u^2\rho dy \leq 16\|\partial_i u\|^2_\rho + 4\|u\|^2_\rho \quad (i = 1, \ldots, n).$$

In particular, it holds that for $u \in H^1_\rho(\mathbb{R}^n_+)$

$$\int_{\mathbb{R}^n_+} |y|^2u^2\rho dy \leq 16\|\nabla u\|^2_\rho + 4n\|u\|^2_\rho.$$

**Proof.** The proof is based on that of Lemma 2.1 in [23] (see also Lemma 2.1 in [18] p. 430). Since $C^\infty_c(\mathbb{R}^n_+)$ is dense in $H^1_\rho(\mathbb{R}^n_+)$, we assume that $u \in C^\infty_c(\mathbb{R}^n_+)$. We set

$$v(y) = u(y)e^{-|y|^2/8}.$$

A direct computations shows that

$$|\partial_i v|^2 = |\partial_i u|^2 e^{-|y|^2/4} + \frac{y_i^2}{16}u^2 e^{-|y|^2/4} - \frac{1}{2}y_i(\partial_i u)u e^{-|y|^2/4}$$

$$= |\partial_i u|^2 e^{-|y|^2/4} + \frac{y_i^2}{16}u^2 e^{-|y|^2/4} - \frac{1}{4}y_i(\partial_i u^2) e^{-|y|^2/4}.$$
Then integrating by parts, we see that
\[
\int_{\mathbb{R}^n_+} y_i (\partial_i u^2) e^{-|y|^2/4} dy = - \int_{\mathbb{R}^n_+} \partial_i \left( y_i e^{-|y|^2/4} \right) u^2 dy
\]
\[
= - \int_{\mathbb{R}^n_+} u^2 e^{-|y|^2/4} dy + \frac{1}{2} \int_{\mathbb{R}^n_+} |y|^2 u^2 e^{-|y|^2/4} dy.
\]
Hence it holds that
\[
||\partial_i v||^2_{L^2(\mathbb{R}^n_+)} = ||\partial_i u||^2_{p} + \frac{1}{4} ||u||^2_{p} - \frac{1}{16} \int_{\mathbb{R}^n_+} y_i^2 u^2 e^{-|y|^2/4} dy,
\]
which completes the proof.

From this inequality, we obtain a compact embedding from \( H^1_\rho(\mathbb{R}^n_+) \) to \( L^2_\rho(\mathbb{R}^n_+) \).

**Lemma A.2.** The embedding from \( H^1_\rho(\mathbb{R}^n_+) \) to \( L^2_\rho(\mathbb{R}^n_+) \) is compact.

**Proof.** Let \( \{ u_k \}_{k \in \mathbb{N}} \) be a bounded sequence in \( H^1_\rho(\mathbb{R}^n_+) \). Then there exists \( u \in H^1_\rho(\mathbb{R}^n_+) \) and a subsequence \( \{ u_{k'} \}_{k' \in \mathbb{N}} \) which is denoted by the same symbol such that \( u_{k'} \rightharpoonup u \) weakly in \( H^1_\rho(\mathbb{R}^n_+) \). Then from Lemma A.1 we verify that
\[
\int_{|y|>R} |u_{k'}(y) - u(y)|^2 \rho(y) dy \leq R^{-2} \int_{|y|>R} |y|^2 |u_{k'}(y) - u(y)|^2 \rho(y) dy
\]
\[
\leq c R^{-2} ||u_{k'} - u||^2_{H^1_\rho(\mathbb{R}^n_+)}. \]
Hence for any \( \epsilon > 0 \) there exists \( R_0 > 0 \) such that
\[
\int_{|y|>R_0} |u_{k'}(x) - u(x)|^2 \rho(y) dy \leq \epsilon/2.
\]
Since the embedding from \( H^1(B_{R_0}) \) to \( L^2(B_{R_0}) \) is compact, there exists \( k_0 \in \mathbb{N} \) such that for \( k \geq k_0 \)
\[
\int_{|y|<R_0} |u_{k'}(x) - u(x)|^2 \rho(y) dy \leq \epsilon/2.
\]
Combining these estimates, we obtain for \( k \geq k_0 \)
\[
||u_{k'} - u||^2_{\rho} \leq \epsilon,
\]
which completes the proof.

**A.2 Linear operator \( A \)**

In this subsection, we show the operator
\[
A_0 v = \left( \Delta - \frac{y}{2} \cdot \nabla \right) v
\]
with \( D(A_0) = \{ v \in H^2_\rho(\mathbb{R}^n_+); \partial_\nu v = Kv \text{ on } \partial \mathbb{R}^n_+ \} \) (\( K \in \mathbb{R} \) is a constant) is self-adjoint. From Lemma 3.1 with \( g(y') \equiv 1 \), there exists \( c > 0 \) such that for \( v \in H^1_\rho(\mathbb{R}^n_+) \)
\[
\int_{\partial \mathbb{R}^n_+} v^2 \rho dy' \leq \epsilon \| \nabla v \|^2_{\rho} + \frac{c}{\epsilon} ||v||^2_{\rho}.
\]

44
Hence there exists $\lambda_0 > 0$ such that for $v \in H^1_\rho(\mathbb{R}_+^n)$

$$K \int_{\partial_\infty \mathbb{R}_+^n} v^2 \rho \, dy' \leq \frac{1}{2} \|\nabla v\|_\rho^2 + \frac{\lambda_0}{2} \|v\|_\rho^2.$$ 

Here we show that the operator $A_{\lambda_0} = A_0 - \lambda_0$ with $D(A_{\lambda_0}) = D(A_0)$ is self-adjoint on $L^2_\rho(\mathbb{R}_+^n)$. Once this is proved, it is clear that the operator $A_0$ with $D(A_0)$ is also self-adjoint on $L^2_\rho(\mathbb{R}_+^n)$. By definition of $\lambda_0$, it is verified that $A_{\lambda_0}$ is a symmetric operator and satisfies

$$(-A_{\lambda_0} v, v)_\rho \geq 0, \quad v \in D(A_0). \quad (64)$$

Hence it is sufficient to show that for any $f \in L^2_\rho(\mathbb{R}_+^n)$ there exist $v \in D(A_0)$ such that $-A_{\lambda_0} v = f$. First we assume that $f \in C_c^\infty(\mathbb{R}_+^n)$. From (64), there exists a weak solution $v \in H^1_\rho(\mathbb{R}_+^n)$ such that for $\psi \in H^1_\rho(\mathbb{R}_+^n)$

$$\int_{\mathbb{R}_+^n} (\nabla v \cdot \nabla \psi + \lambda_0 v \psi) \rho \, dy' - K \int_{\partial_\infty \mathbb{R}_+^n} v \psi \, dy' = \int_{\mathbb{R}_+^n} f \psi \, dy.$$

By using $\psi = v$ as a test function, we obtain

$$\|v\|_{H^1_\rho(\mathbb{R}_+^n)} \leq c \|f\|_\rho. \quad (65)$$

In particular, since $f \in C_c^\infty(\mathbb{R}_+^n)$ is a smooth function, a standard elliptic regularity theory shows that $v \in C^\infty(\mathbb{R}_+^n)$. Let $\eta_k(r) \in C_c^\infty(\mathbb{R}_+)$ be a cut off function such that $\eta_k(r) = 1$ if $r \in (0, k)$, $\eta_k(r) = 0$ if $r \in (2k, \infty)$. By using $v_1 = |y|^2 \eta_k(|y|)^2$ and $v_2 = y_n^2 v \eta_k(|y|)^2$ as test functions respectively and from Lemma 3.1 and Lemma 4.1, we obtain

$$\int_{\mathbb{R}_+^n} |y|^2 |\nabla v|^2 \eta_k^2 \rho \, dy' \leq 2 \int_{\mathbb{R}_+^n} |y|^2 |f v| \eta_k^2 \rho \, dy' + c \|v\|_{H^1_\rho(\mathbb{R}_+^n)}^2,$$

$$\int_{\mathbb{R}_+^n} y_n^2 |\nabla v|^2 \eta_k^2 \rho \, dy' \leq 2 \int_{\mathbb{R}_+^n} y_n^2 |f v| \eta_k^2 \rho \, dy' + c \|v\|_{H^1_\rho(\mathbb{R}_+^n)}^2.$$ 

Next we use $\psi = \nabla'(\eta_k(|y|)^2 \nabla v)$ as a test function. Here we note that

$$\nabla\{\nabla'(\eta_k^2 \nabla v')\} = \nabla'(\eta_k^2 \nabla v) + \nabla'(\nabla(\eta_k^2) \nabla v).$$

Hence integrating by parts, we get

$$\int_{\mathbb{R}_+^n} \nabla v \cdot \nabla\{\nabla'(\eta_k^2 \nabla v')\} \rho \, dy' = -\sum_{i=1}^n \sum_{j=1}^{n-1} \int_{\mathbb{R}_+^n} (\partial_i \partial_j v)^2 \eta_k^2 \rho \, dy'$$

$$+ \frac{1}{2} \sum_{j=1}^{n-1} \int_{\mathbb{R}_+^n} y_j (\nabla v \cdot \nabla \partial_j v) \eta_k^2 \rho \, dy' + \int_{\mathbb{R}_+^n} \nabla v \cdot \nabla'(\nabla(\eta_k^2) \nabla v') \rho \, dy'.$$

Hence it follows that

$$-\int_{\mathbb{R}_+^n} \nabla v \cdot \nabla\{\nabla'(\eta_k^2 \nabla v')\} \rho \, dy' \geq \frac{1}{2} \sum_{j=1}^{n-1} \int_{\mathbb{R}_+^n} |\nabla \partial_j v|^2 \eta_k^2 \rho \, dy'$$

$$- c \int_{\mathbb{R}_+^n} |y'|^2 |\nabla v|^2 \eta_k^2 \rho \, dy' - c \|v\|_{H^1_\rho(\mathbb{R}_+^n)}^2. \quad (67)$$

The boundary integral is calculated as follows:

$$\left| \int_{\partial_\infty \mathbb{R}_+^n} v \nabla'(\eta_k^2 \nabla v') \rho \, dy' \right| = \left| - \int_{\partial_\infty \mathbb{R}_+^n} |\nabla' v|^2 \eta_k^2 \rho \, dy' + \frac{1}{2} \int_{\partial_\infty \mathbb{R}_+^n} y' \cdot (\nabla' v) \eta_k^2 \rho \, dy' \right|$$

$$\leq c \int_{\partial_\infty \mathbb{R}_+^n} (|\nabla' v|^2 \eta_k^2 + |y'|^2 v^2 \eta_k^2) \rho \, dy'.$$
We set \( V_1 = |\nabla | \eta k \) and \( V_2 = v \eta k \). Then from Lemma 3.1, it is verified that
\[
\int_{\partial R^n_+} (V_1^2 + |y'|^2 V_2^2) \rho dy' \leq c \int_{\mathbb{R}^n_+} (|V_2^2 + V_1 |\nabla V_1| + |y'|^2 (V_2^2 + |\nabla V_2|)) \rho dy
\]
\[
\leq c \sum_{j=1}^{n-1} \int_{\mathbb{R}^n_+} |\nabla' v| |\nabla \partial_j v||^2 \eta k^2 \rho dy + c \int_{\mathbb{R}^n_+} |y'|^2 |\nabla v|^2 \eta k^2 \rho dy + c \|v\|^2_{H^1_\rho(\mathbb{R}^n_+)}.
\]
Therefore we find that
\[
\left| \int_{\partial R^n_+} v \nabla' (\eta k \nabla v) \rho dy \right| \leq c \sum_{j=1}^{n-1} \int_{\mathbb{R}^n_+} |\nabla' v| |\nabla \partial_j v||^2 \eta k^2 \rho dy + c \int_{\mathbb{R}^n_+} |y'|^2 |\nabla v|^2 \eta k^2 \rho dy + c \|v\|^2_{H^1_\rho(\mathbb{R}^n_+)}.
\]
(68)

From (66), (67) and (68), we get
\[
\sum_{j=1}^{n-1} \int_{\mathbb{R}^n_+} |\nabla \partial_j v|^2 \eta k^2 \rho dy \leq c \|f\|_{\rho} \left( \int_{\mathbb{R}^n_+} |y'|^2 V_2^2 \rho dy \right)^{1/2} + c \|v\|^2_{H^1_\rho(\mathbb{R}^n_+)}.
\]
Then applying Lemma A.1 two times, we compute the first term on the right-hand side.
\[
\int_{\mathbb{R}^n_+} |y'|^2 (|y'| ||v| \eta k)^2 \rho dy \leq c \int_{\mathbb{R}^n_+} (|\nabla' (|y'| ||v| \eta k)|^2 + (|y'| ||v| \eta k)^2) \rho dy
\]
\[
\leq c \int_{\mathbb{R}^n_+} |y'|^2 (|\nabla' v| \eta k)^2 \rho dy + c \|v\|^2_{H^1_\rho(\mathbb{R}^n_+)}
\]
(69)
Thus finally we obtain
\[
\sum_{j=1}^{n-1} \int_{\mathbb{R}^n_+} |\nabla \partial_j v|^2 \eta k^2 \rho dy \leq c \|f\|^2_{\rho} + c \|v\|^2_{H^1_\rho(\mathbb{R}^n_+)}.
\]
(70)

Since \( v \) is a solution of \(-A_{\lambda_0} v = f\), it follows that
\[
\partial_n^2 v = \frac{y_n}{2} \partial_n v + F,
\]
where \( F \) is given by
\[
F = -\Delta' v + \frac{y'}{2} \cdot \nabla' v - \lambda_0 v + f.
\]
Multiplying by \((\partial_n^2 v) \rho^2 \eta k\) and integrating over \(\mathbb{R}^n_+\), we obtain
\[
\|((\partial_n^2 v) \eta k\|_{\rho}^2 \leq c \int_{\mathbb{R}^n_+} y_n^2 |\nabla v|^2 \eta k^2 \rho dy + c \|F\|^2_{\rho}.
\]
Then it follows from (66) that
\[
\|((\partial_n^2 v) \eta k\|_{\rho}^2 \leq c \|f\|_{\rho} \left( \int_{\mathbb{R}^n_+} y_n^4 v^2 \eta k^2 \rho dy \right)^{1/2} + c \|v\|^2_{H^1_\rho(\mathbb{R}^n_+)} + c \|F\|^2_{\rho}.
\]
46
By the same calculation as (69), we see that

$$\int_{\mathbb{R}^n_+} y_n^4 v^2 \eta_k \rho \, dy \leq c \| (\partial_n^2 v) \eta_k \|_{\rho}^2 + c \| v \|_{H^1_\rho(\mathbb{R}^n_+)}^2,$$

which implies

$$\| (\partial_n^2 v) \eta_k \|_{\rho}^2 \leq c \| f \|_{\rho}^2 + c \| v \|_{H^1_\rho(\mathbb{R}^n_+)}^2 + c \| F \|_{\rho}^2. \tag{71}$$

Thus combining (65), (70) and (71) and taking $k \to \infty$, we conclude that

$$\| v \|_{H^2_\rho(\mathbb{R}^n_+)} \leq c \| f \|_{\rho}.$$ 

Since $C_\infty^\infty(\mathbb{R}^n_+)$ is dense in $L^2_\rho(\mathbb{R}^n_+)$, by a density argument, we complete the proof.

**Acknowledgement**

The author would like to express his gratitude to Professor Mitsuharu Ōtani for his valuable comments, suggestions and his encouragements.

**References**

[1] J. Bebernes and S. Bricher, Final time blow-up profiles for semilinear parabolic equations via center manifold theory, SIAM J. Math. Anal. 23 (1992), 852-869

[2] M. Chlebík, M. Fila, On the blow-up rate for the heat equation with a nonlinear boundary condition, Math. Methods Appl. Sci. 23 no. 15 (2000) 1323-2330.

[3] M. Chlebík, M. Fila, Some recent results on blow-up on the boundary for the heat equation, Evolution equations: existence, regularity and singularities, Banach Center Publ. 52 (2000) 61-71.

[4] K. Deng, M. Fila, H. A. Levine, On critical exponents for a system of heat equations coupled in the boundary conditions, Acta Math. Univ. Comenian. 63 (1994) 169-192.

[5] M. Fila, P. Quittner, The blow-up rate for the heat equation with a nonlinear boundary condition, J. Math. Anal. Appl. 14 (1991) 197-205.

[6] S. Filippas, R. V. Kohn, Refined asymptotics for the blowup for $u_t - \Delta u = u^p$, Comm. Pure Appl. Math. 45 (1992) 821-869.

[7] S. Filippas, W. Liu, On the blowup of multidimensional semilinear heat equations, Ann. Inst. H. Poincaré Anal. Non Linéaire 10 no. 3 (1993) 313-344.

[8] Y. Giga, R. Kohn, Asymptotically self-similar blow-up of semilinear heat equations. Comm. Pure Appl. Math. 38 no. 3 (1985) 297-319.

[9] Y. Giga, R. Kohn, Characterizing blowup using similarity variables, Indiana Univ. Math. J. 36 no. 1 (1987) 1-40.

[10] Y. Giga, R. Kohn, Nondegeneracy of blowup for semilinear heat equations, Comm. Pure Appl. Math. 42 (1989) 845-884.

[11] J. Harada, Single point blow-up solutions to the heat equation with nonlinear boundary conditions, to appear in Differ. Equ. Appl. 5 no. 2 (2013).
[12] M. A. Herrero, J. J. L. Velázquez, Flat blow-up in one-dimensional semilinear heat equations, Differential Integral Equations 5 no. 5 (1992) 973-997.
[13] M. A. Herrero, J. J. L. Velázquez, Blow-up profiles in one-dimensional semilinear parabolic problems, Comm. Partial Differential Equations 17 no 1-2 (1992) 205-219.
[14] M. A. Herrero, J. J. L. Velázquez, Generic behaviour of one-dimensional blow up patterns, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 19 no. 3 (1992) 381-450.
[15] M. A. Herrero, J. J. L. Velázquez, Blow-up behavior of one-dimensional semilinear parabolic equations, Ann. Inst. Henri Poincare 10 no. 2 (1993) 131-189.
[16] S. Hitomatsu, S. Moriguchi, K. Udagawa, Mathematical Formula III, Iwanamishoten, 1987 (in Japanese).
[17] B. Hu, H. -M. Yin, The profile near blowup time for solution of the heat equation with a nonlinear boundary condition, Trans. Am. Math. Soc. 346 (1994) 117-135.
[18] O. Kavian, Remarks on the large time behaviour of a nonlinear diffusion equation, Ann. Inst. H. Poincaré Anal. Non Linéaire 4 no. 5 (1987) 423-452.
[19] O. A. Ladyženskaja, V.A. Solonnikov, N.N. Ural’ceva, Linear and Quasi-linear Equations of Parabolic Type, Amer. Math. Soc., Trans. Math. Monographs vol. 23, Providence, R.I., 1968.
[20] G. M. Lieberman, Second Order Parabolic Differential Equations, World Scientific Publ. Co., 2005.
[21] W. Liu, Blow-up behavior for semilinear heat equations: multi-dimensional case, Rocky Mountain J. Math. 23 no. 4 (1993) 1287-1319.
[22] F. Merle, H. Zaag, Refined uniform estimates at blow-up and application for nonlinear heat equations, Geom. Funct. Anal. 8 (1998) 1043-1085.
[23] Y. Naito, T. Suzuki, Existence of type II blowup solutions for a semilinear heat equation with critical nonlinearity, J. Differential Equations 232 (2007) 176-211.
[24] J. J. L. Velázquez, Higher dimensional blow up for semilinear parabolic equations, Commun. in Partial Differential Equations 17 no 9-10 (1992) 1567-1596.
[25] J. J. L. Velázquez, Estimates on the \((n − 1)\)-dimensional Hausdorff measure of the blow-up set for a semilinear heat equation, Indiana Univ. Math. J. 42 no. 2 (1993) 445-476.
[26] J. J. L. Velázquez, Classification of singularities for blowing up solutions in higher dimensions, Trans. Amer. Math. Soc. 338 no. 1 (1993) 441-464.