Bounding the energy-constrained quantum and private capacities of bosonic thermal channels

Kunal Sharma∗ Mark M. Wilde† Sushovit Adhikari∗ Masahiro Takeoka ‡

August 25, 2017

Abstract

We establish three different upper bounds on the energy-constrained quantum and private capacities of bosonic thermal channels. The first upper bound, which we call the “data-processing bound,” is the simplest and is obtained by decomposing a thermal channel as a pure-loss channel followed by a quantum-limited amplifier channel. We prove that the data-processing bound can be at most 1.45 bits larger than a known lower bound on these capacities of the thermal channel. The other two upper bounds, which we call the “ε-degradable bound” and the “ε-close-degradable bound,” are established using the notion of approximate degradability along with energy constraints. A comparison of these three bounds shows that the data-processing bound is near to a known lower bound for both low and high thermal noise and is very near to the ε-close-degradable bound only for low thermal noise. Also, we find a strong limitation on any potential superadditivity of the coherent information of the thermal channel in the low-noise regime, as the data-processing bound is very near to a known lower bound in such cases. Moreover, for certain parameter regimes, we show that the ε-degradable bound is tighter than all other bounds. All three upper bounds are very near to a known lower bound for the case of low thermal noise and high transmissivity. We also find improved achievable rates of private communication through bosonic thermal channels, by employing coding schemes that make use of displaced thermal states. We end by proving that an optimal Gaussian input state for the energy-constrained, generalized channel divergence of two particular Gaussian channels is the two-mode squeezed vacuum state that saturates the energy constraint. What remains open for several interesting channel divergences, such as the diamond norm or the Rényi channel divergence, is to determine whether, among all input states, a Gaussian state is optimal.

1 Introduction

One of the main aims of quantum information theory is to characterize the capacities of quantum communication channels [Hol12, Hay06, Wil16]. A quantum channel is a model for a communication link between two parties. The properties of a quantum channel and its coupling to environment govern the evolution of a quantum state that is sent through the channel.

∗Hearne Institute for Theoretical Physics, Department of Physics and Astronomy, Louisiana State University, Baton Rouge, Louisiana 70803, USA
†Center for Computation and Technology, Louisiana State University, Baton Rouge, Louisiana 70803, USA
‡National Institute of Information and Communications Technology, Koganei, Tokyo 184-8795, Japan
The quantum capacity $Q(\mathcal{N})$ of a quantum channel $\mathcal{N}$ is the maximum rate at which quantum information (qubits) can be reliably transmitted from a sender to a receiver by using the channel many times. The private capacity $P(\mathcal{N})$ of a quantum channel $\mathcal{N}$ is defined to be the maximum rate at which a sender can reliably communicate classical messages to a receiver by using the channel many times, such that the environment of the channel gets negligible information about the transmitted message. In general, the best known characterization of quantum or private capacity of a quantum channel is given by the optimization of regularized information quantities over an unbounded number of uses of the channel [Llo97, Sho02, CWY04, Dev05]. Since these information quantities are additive for a special class of channels called degradable channels [DS05, Smi08], the capacities of these channels can be calculated without any regularization. However, for the channels that are not degradable, these information quantities can be superadditive [DSS98, SS07, SRS08, CEM+15, ES15], and quantum capacities can be superactivated for some of these channels [SY08, SSY11]. Hence, it is difficult to determine the quantum or private capacity of channels that are not degradable, and the natural way to characterize such channels is to bound these capacities from above and below.

An important class of channels called bosonic Gaussian channels act as a good model for the transmission of light through optical fibers or free space (see, e.g., [Ser17] for a review). Within the past two decades, there have been advances in finding quantum and private capacities of bosonic channels. In particular, when there is no constraint on the energy available at the transmitter, the quantum and private capacities of single-mode quantum-limited attenuator and amplifier channels were given in [HW01, WPGG07, WHG12, QW17, WQ16]. However, the availability of infinite energy at the transmitter is not practically feasible, and it is thus natural to place energy constraints on any communication protocol. Recently, a general theory of energy-constrained quantum and private communication has been developed in [WQ16], by building on notions developed in the context of other energy-constrained information-processing tasks [Hol04]. For the particular case of bosonic Gaussian channels, formulas for the energy-constrained quantum and private capacities of the single-mode pure-loss channel were conjectured in [GSE08] and proven in [WHG12, WQ16]. Also, for a single-mode quantum-limited amplifier channel, the energy-constrained quantum and private capacities have been established in [QW17, WQ16].

What remains a pressing open question in the theory of Gaussian quantum information [Ser17] is to determine formulas for or bounds on the quantum and private capacities of non-degradable bosonic Gaussian channels. Of particular interest is the thermal channel, which serves as a variant of the pure-loss channel, incorporating environmental imperfections. In this article, we address this query by providing several bounds on the energy-constrained quantum and private capacities of the thermal channel.

To motivate the thermal channel model, consider that almost all communication systems are affected by thermal noise [Cav82]. Even though the pure-loss channel has relevance in free-space communication [YS78, Sha09], it represents an ideal situation in which the environment of the channel is prepared in a vacuum state. Instead, consideration of a thermal state with a fixed mean photon number $N_B$ as the state of the environment is more realistic, and such a channel is called a bosonic Gaussian thermal channel [Sha09, RGR+17]. Hence, quantum thermal channels model free-space communication with background thermal radiation affecting the input state in addition to transmission loss. In the context of private communication, a typical conservative model is to allow an eavesdropper access to the environment of a channel, and in particular, tampering by an eavesdropper can be modeled as the excess noise realized by a thermal channel [NH04, LDTBG05].
2 Summary of results

Some of our main contributions in this paper are upper bounds on the energy-constrained quantum capacity of thermal channels. A first upper bound is established by decomposing a thermal channel as a pure-loss channel followed by a quantum-limited amplifier channel [CGH06, GPNBL12] and using a data-processing argument. We note that the same method was employed in [KS13], in order to establish an upper bound on the classical capacity of the thermal channel. Throughout, we call this first upper bound the “data-processing bound.” We also prove that this upper bound can be at most 1.45 bits larger than a known lower bound [HW01, WHG12] on the energy-constrained quantum and private capacity of a thermal channel. Moreover, the data-processing bound is very near to a known lower bound for the case of low thermal noise and both low and high transmissivity.

Recently, the notion of approximate degradability of quantum channels was developed in [SSWR14], and upper bounds on the quantum and private capacities of approximately degradable channels were established for quantum channels with finite-dimensional input and output systems. In our paper, we establish general upper bounds on the energy-constrained quantum and private capacities of approximately degradable channels for infinite-dimensional systems. These general upper bounds can be applied to any quantum channel that is approximately degradable with energy constraints on the input and output states of the channels. In particular, we apply these general upper bounds to bosonic Gaussian thermal channels.

Our second upper bound is based on the notion of ε-degradability of thermal channels, and we call this bound the “ε-degradable bound.” In this method, we first construct a degrading channel, such that a complementary channel of the thermal channel is close in diamond distance [Kit97] to the serial concatenation of the thermal channel followed by this degrading channel. In general, it seems to be computationally hard to determine the diamond distance between two quantum channels if the optimization is over input density operators acting on an infinite-dimensional Hilbert space. However, in our setup, we address this difficulty by constructing a simulating channel, which simulates the serial concatenation of the thermal channel and the aforementioned degrading channel. Using this technique, an upper bound on the diamond distance reduces to the calculation of the quantum fidelity between the environmental states of the thermal channel and the simulating channel. Based on the fact that, for certain parameter regimes, the resulting capacity upper bound is better than all other upper bounds reported here, we believe that our aforementioned choice of a degrading channel is a good choice.

A third upper bound on the energy-constrained quantum capacity of thermal channels is established using the concept of ε-close-degradability of a thermal channel, and we call this bound the “ε-close-degradable bound.” In particular, we show that a low-noise thermal channel is ε-close degradable, given that it is close in diamond distance to a pure-loss channel. We find that the ε-close-degradable bound is very near to the data-processing bound for the case of low thermal noise.

We compare these three different upper bounds with a known lower bound on the quantum capacity of a thermal channel [HW01, WHG12]. We find that the data-processing bound is very near to a known capacity lower bound for low thermal noise and for both medium and high transmissivity. Moreover, we show that the maximum difference between the data-processing bound and a known lower bound never exceeds $1/\ln 2 \approx 1.45$ bits for all possible values of parameters, and this maximum difference is attained in the limit of infinite input mean photon number. This
result places a strong limitation on any possible superadditivity of coherent information of the thermal channel. We note here that this kind of result was suggested without proof by the heuristic developments in [SS13]. Next, we plot these three upper bounds as well as a known lower bound versus input mean photon number for different values of the channel transmissivity $\eta$ and thermal noise $N_B$. In particular, we find that the $\varepsilon$-close-degradable bound is very near to the data-processing bound for low thermal noise and for both medium and high transmissivity. Moreover, all three upper bounds are very near to a known lower bound for low thermal noise and high transmissivity. We also examine different parameter regimes where the $\varepsilon$-close-degradable bound is tighter than the $\varepsilon$-degradable bound and vice versa. In particular, we find that the $\varepsilon$-degradable bound is tighter than the $\varepsilon$-close degradable bound for the case of high thermal noise.

We find an interesting parameter regime where the $\varepsilon$-degradable bound is tighter than all other upper bounds, as it becomes closest to a known lower bound for the case of high noise and high input mean photon number. However, for the same parameter regime, if the input mean photon number is low, then the data-processing bound is tighter than the $\varepsilon$-degradable bound. This suggests that the upper bounds based on the notion of approximate degradability are good for the case of high input mean photon number. We suspect that these bounds could be further improved for the case of low input mean photon number if it were possible to compute or tightly bound the energy-constrained diamond norm [Shi17, We17].

As one of the last technical developments of our paper, we address this latter question in a very broad sense, by considering the energy-constrained, generalized channel divergence of two quantum channels, as an extension of the generalized channel divergence developed in [LKDW17]. In particular, we prove that an optimal Gaussian input state for the energy-constrained, generalized channel divergence of two particular Gaussian channels is the two-mode squeezed vacuum state that saturates the energy constraint. It is an interesting open question to determine whether the two-mode squeezed vacuum is optimal among all input states, but we leave this for future work, simply noting for now that an answer would lead to improved upper bounds on the energy-constrained quantum and private capacities of the thermal channel.

Similar to our bounds on the energy-constrained quantum capacity, we establish three different upper bounds on the energy-constrained private capacity of bosonic thermal channels. We also develop an improved lower bound on the energy-constrained private capacity of a bosonic thermal channel. In particular, we find that for certain values of the channel transmissivity, a higher private communication rate can be achieved by using displaced thermal states as information carriers instead of coherent states.

The rest of the paper is structured as follows. In Section 3, we summarize definitions and prior results relevant to our paper. We provide general upper bounds on the energy-constrained quantum and private capacities of approximately degradable channels in Section 4. We use these tools to establish three different upper bounds on the energy-constrained quantum and private capacities of a thermal channel in Sections 5 and 7, respectively. A comparison of these different upper bounds on energy-constrained quantum capacity of a thermal channel is discussed in Section 6. We present an improvement on the achievable rate of private communication through thermal channels, in Section 8. We discuss the optimization of the Gaussian energy-constrained generalized channel divergence in Section 9. Finally, we summarize our results and conclude in Section 10.
3 Preliminaries

Background on quantum information in infinite-dimensional systems is available in [Hol12] (see also [Hol04, SH08, HS10, HZ11, Shi15, Shi16]). In this section, we explain our notations and discuss prior results relevant for our paper.

Quantum states and channels. Let $\mathcal{H}$ denote a separable Hilbert space, let $\mathcal{B}(\mathcal{H})$ denote the set of bounded operators acting on $\mathcal{H}$, and let $\mathcal{P}(\mathcal{H})$ denote the subset of $\mathcal{B}(\mathcal{H})$ that consists of positive semi-definite operators. Let $\mathcal{T}(\mathcal{H})$ denote the set of trace-class operators, defined such that their trace norm is finite: $\|A\|_1 \equiv \text{Tr}\{|A|\} < \infty$, where $|A| \equiv \sqrt{A^\dagger A}$. Let $\mathcal{D}(\mathcal{H})$ denote the set of density operators (positive semi-definite with unit trace) acting on $\mathcal{H}$. A quantum channel $\mathcal{N} : \mathcal{T}(\mathcal{H}_A) \rightarrow \mathcal{T}(\mathcal{H}_B)$ is a completely positive, trace-preserving linear map. Using the Stinespring dilation theorem [Sti55], a quantum channel can be expressed in terms of a linear isometry: i.e., $\mathbf{T} \in \mathcal{T}(\mathcal{H}_A) \rightarrow \mathcal{T}(\mathcal{H}_B)$ is a completely positive, trace-preserving linear map. Using the Stinespring dilation theorem [Sti55], a quantum channel can be expressed in terms of a linear isometry: i.e., there exists another Hilbert space $\mathcal{H}_E$ and a linear isometry $U : \mathcal{H}_A \rightarrow \mathcal{H}_B \otimes \mathcal{H}_E$ such that for all $\omega_A \in \mathcal{T}(\mathcal{H}_A)$, the following equality holds: $\mathcal{N}(\omega_A) = \text{Tr}_E\{U\omega_A U^\dagger\}$. A complementary channel $\mathcal{N}_A^\perp$ of $\mathcal{N}_A$ is defined as $\mathcal{N}_A^\perp = \text{Tr}_B\{U\omega_A U^\dagger\}$. A quantum channel $\mathcal{N}_A \rightarrow B$ is degradable [DS05] if there exists a quantum channel $\mathcal{D}_B \rightarrow E$ such that $\mathcal{D}_B \rightarrow E(\mathcal{N}_A \rightarrow B(\omega_A)) = \mathcal{N}_A \rightarrow E(\omega_A)$, for all $\omega_A \in \mathcal{T}(\mathcal{H}_A)$.

Quantum entropies and information. The quantum entropy of a state $\rho \in \mathcal{D}(\mathcal{H})$ is defined as $H(\rho) \equiv -\text{Tr}\{\rho \log_2 \rho\}$. It is a non-negative, concave, lower semicontinuous function [Weh76] and not necessarily finite [BV13]. The binary entropy function is defined for $x \in [0, 1]$ as

$$h_2(x) \equiv -x \log_2 x - (1-x) \log_2(1-x).$$

(3.1)

Throughout the paper we use a function $g(x)$, which is the entropy of a bosonic thermal state with mean photon number $x \geq 0$:

$$g(x) \equiv (x + 1) \log_2(x + 1) - x \log_2 x.$$

(3.2)

By continuity, we have that $h_2(0) = \lim_{x \rightarrow 0} h_2(x) = 0$ and $g(0) = \lim_{x \rightarrow 0} g(x) = 0$. The quantum relative entropy $D(\rho||\sigma)$ of $\rho, \sigma \in \mathcal{D}(\mathcal{H})$ is defined as [Lin73]

$$D(\rho||\sigma) \equiv \sum_i \langle i | \rho \log_2 \rho - \rho \log_2 \sigma | i \rangle,$$

(3.3)

where $\{|i\rangle\}_{i=1}^\infty$ is an orthonormal basis of eigenvectors of the state $\rho$, if $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$ and $D(\rho||\sigma) = \infty$ otherwise. The quantum relative entropy $D(\rho||\sigma)$ is non-negative for $\rho, \sigma \in \mathcal{D}(\mathcal{H})$ and is monotone with respect to a quantum channel $\mathcal{N} : \mathcal{T}(\mathcal{H}_A) \rightarrow \mathcal{T}(\mathcal{H}_B)$:

$$D(\rho||\sigma) \geq D(\mathcal{N}(\rho)||\mathcal{N}(\sigma)).$$

(3.4)

The quantum mutual information $I(A; B)_\rho$ of a bipartite state $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is defined as [Lin73]

$$I(A; B)_\rho = D(\rho_{AB}||\rho_A \otimes \rho_B).$$

(3.5)

The coherent information $I(A)B)_\rho$ of $\rho_{AB}$ is defined as [SN96, HS10, Kuz11]

$$I(A)B)_\rho \equiv I(A; B)_\rho - H(A)_\rho.$$

(3.6)
when $H(A)_\rho < \infty$. This expression reduces to
\[
I(A)_\rho = H(B)_\rho - H(AB)_\rho, \tag{3.7}
\]
if $H(B)_\rho < \infty$.

**Quantum fidelity, trace distance, and diamond distance.** The fidelity of two quantum states $\rho, \sigma \in D(\mathcal{H})$ is defined as [Uhl76] $F(\rho, \sigma) \equiv \|\sqrt{\rho} \sqrt{\sigma}\|_1^2$. The trace distance between two density operators $\rho, \sigma \in D(\mathcal{H})$ is equal to $\|\rho - \sigma\|_1$. The operational interpretation of trace distance is that it is linearly related to the maximum success probability in distinguishing two quantum states. The diamond norm of a Hermiticity preserving linear map $S$ is defined as $\|S\|_\diamond \equiv \sup_{\rho_{RA} \in D(\mathcal{H}_R \otimes \mathcal{H}_A)} \|\text{id}_R \otimes S_{\mathcal{A} \rightarrow \mathcal{B}}(\rho_{RA})\|_1$, where $\text{id}_R$ is the identity map acting on Hilbert space $H_R$ of the reference system [Kit97]. It suffices to optimize with $\rho$ being a pure quantum state. The diamond-norm distance $\|N - M\|_\diamond$ is a measure of distinguishability of two quantum channels $N$ and $M$.

**Approximate degradability.** The concept of approximate degradability was introduced in [SSWR14]. The following two definitions of approximate degradability will be useful in our paper.

**Definition 1 (\(\varepsilon\)-degradable [SSWR14])** A channel $N_{\mathcal{A} \rightarrow \mathcal{B}}$ is $\varepsilon$-degradable if there exists a channel $D_{\mathcal{B} \rightarrow \mathcal{E}}$ such that $\frac{1}{2} \|N - D \circ N\|_\diamond \leq \varepsilon$, where $\tilde{N}$ denotes a complementary channel of $N$.

**Definition 2 (\(\varepsilon\)-close-degradable [SSWR14])** A channel $N_{\mathcal{A} \rightarrow \mathcal{B}}$ is $\varepsilon$-close-degradable if there exists a degradable channel $M_{\mathcal{A} \rightarrow \mathcal{B}}$ such that $\frac{1}{2} \|N - M\|_\diamond \leq \varepsilon$.

**Energy-constrained continuity bounds.** Next, we recall the definition of an energy observable and a Gibbs observable [Hol12, Win16]. We also review the uniform continuity of conditional quantum entropy with energy constraints [Win16]. When defining a Gibbs observable, we follow [Hol12, Win16].

**Definition 3 (Energy observable)** Let $G$ be a positive semi-definite operator. We assume that it has discrete spectrum and that it is bounded from below. In particular, let $\{|e_k\rangle\}_k$ be an orthonormal basis for a Hilbert space $\mathcal{H}$, and let $\{g_k\}_k$ be a sequence of non-negative real numbers bounded from below. Then
\[
G = \sum_{k=1}^{\infty} g_k |e_k\rangle \langle e_k|
\]
is a self-adjoint operator that we call an energy observable.

**Definition 4 (Extension of energy observable)** The $n$th extension $G_n$ of an energy observable $G$ is defined as
\[
G_n = \frac{1}{n} (G \otimes \mathbf{I} \otimes \cdots \otimes \mathbf{I} + \cdots + \mathbf{I} \otimes \cdots \otimes \mathbf{I} \otimes G), \tag{3.9}
\]
where $n$ is the number of factors in each tensor product above.

**Definition 5 (Gibbs Observable)** An energy observable $G$ is a Gibbs observable if for all $\beta > 0$, we have $\text{Tr}\{\exp(-\beta G)\} < \infty$, so that the partition function $\text{Tr}\{\exp(-\beta G)\}$ has a finite value and hence $\exp(-\beta G) / \text{Tr}\{\exp(-\beta G)\}$ is a well defined thermal state.
For a Gibbs observable $G$, let us consider a quantum state $\rho$ such that $\text{Tr}\{G\rho\} \leq W$. There exists a unique state that maximizes the entropy $H(\rho)$, and this unique maximizer has the Gibbs form

$$
\gamma(W) = \frac{\exp(-\beta(W)G)}{Z(\beta(W))},
$$

where $\beta(W)$ is the solution of the equation:

$$
\text{Tr}\{\exp(-\beta G)(G - W)\} = 0.
$$

In particular, for the Gibbs observable $G = \hbar \omega \hat{n}$, where $\hat{n} = \hat{a}^\dagger \hat{a}$ is the photon number operator, a thermal state (mean photon number $\bar{n}$) that saturates the energy constrained inequality $\text{Tr}\{G\rho\} \leq W$, gives the maximum value of the entropy:

$$
H(\gamma(W)) = g(\bar{n}) = (\bar{n} + 1) \log_2(\bar{n} + 1) - \bar{n} \log_2 \bar{n}.
$$

(3.10)

Here, we have fixed the ground-state energy to be equal to zero. In some parts of our paper, we take the Gibbs observable to be the number operator, and we use the terminology “mean photon number” and “energy” interchangeably.

The following lemma is a uniform continuity bound for the conditional quantum entropy with energy constraints [Win16]:

**Lemma 1 (Meta-Lemma 17, [Win16])** For a Gibbs observable $G \in \mathcal{P}(\mathcal{H}_A)$, and states $\omega_{AB}, \tau_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$, such that $\frac{1}{2} \|\omega_{AB} - \tau_{AB}\|_1 \leq \epsilon < \epsilon' \leq 1$, $\text{Tr}\{(G \otimes I_B)\omega_{AB}\}$, $\text{Tr}\{(G \otimes I_B)\tau_{AB}\} \leq W$, where $W \in [0, \infty)$ and $\delta = (\epsilon' - \epsilon)/(1 + \epsilon')$, the following inequality holds

$$
|H(A|B)_\omega - H(A|B)_\tau| \leq (2\epsilon' + 4\delta)H(\gamma(W/\delta)) + g(\epsilon') + 2h_2(\delta).
$$

(3.11)

Throughout the paper, we consider only those quantum channels that satisfy the following finite output entropy condition:

**Condition 6 (Finite output entropy)** Let $G$ be a Gibbs observable and $W \in [0, \infty)$. A quantum channel $\mathcal{N}$ satisfies the finite-output entropy condition with respect to $G$ and $W$ if

$$
\sup_{\rho : \text{Tr}\{G\rho\} \leq W} H(\mathcal{N}(\rho)) < \infty,
$$

(3.12)

**Gaussian states and channels.** We now deliver a brief review of Gaussian states and channels, and we point to [Ser17] for more details. Gaussian channels model natural physical processes such as photon loss, photon amplification, thermalizing noise, or random kicks in phase space. They satisfy Condition 6 when the Gibbs observable for $m$ modes is taken to be

$$
\hat{E}_m \equiv \sum_{j=1}^{m} \omega_j \hat{a}_j^\dagger \hat{a}_j,
$$

(3.13)

where $\omega_j > 0$ is the frequency of the $j$th mode and $\hat{a}_j$ is the photon annihilation operator for the $j$th mode, so that $\hat{a}_j^\dagger \hat{a}_j$ is the photon number operator for the $j$th mode.

Let

$$
\hat{x} \equiv [\hat{q}_1, \ldots, \hat{q}_m, \hat{p}_1, \ldots, \hat{p}_m] \equiv [\hat{x}_1, \ldots, \hat{x}_{2m}]
$$

(3.14)

denote a vector of position- and momentum-quadrature operators, satisfying the canonical commutation relations:

$$
[\hat{x}_j, \hat{x}_k] = i\Omega_{j,k}, \quad \text{where} \quad \Omega \equiv \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes I_m,
$$

(3.15)
and $I_m$ denotes the $m \times m$ identity matrix. We take the annihilation operator for the $j$th mode as
\[ \hat{a}_j = (\hat{q}_j + i\hat{p}_j)/\sqrt{2}. \] For $\xi \in \mathbb{R}^{2m}$, we define the unitary displacement operator $D(\xi) \equiv \exp(i\xi^T \Omega \hat{x})$. Displacement operators satisfy the following relation:
\[ D(\xi)^\dagger D(\xi') = D(\xi') D(\xi)^\dagger \exp(i\xi^T \Omega \xi'). \] (3.16)

Every state $\rho \in \mathcal{D}(\mathcal{H})$ has a corresponding Wigner characteristic function, defined as
\[ \chi_\rho(\xi) \equiv \text{Tr}\{D(\xi)\rho\}, \] (3.17)
and from which we can obtain the state $\rho$ as
\[ \rho = \frac{1}{(2\pi)^m} \int d^{2m}\xi \, \chi_\rho(\xi) \, D(\xi)^\dagger. \] (3.18)

A quantum state $\rho$ is Gaussian if its Wigner characteristic function has a Gaussian form as
\[ \chi_\rho(\xi) = \exp\left(-\frac{1}{4} [\xi^T V^{\rho} \Omega \xi + [\Omega \mu^{\rho}]^T \xi]\right), \] (3.19)
where $\mu^{\rho}$ is the $2m \times 1$ mean vector of $\rho$, whose entries are defined by $\mu_j^{\rho} \equiv \langle \hat{x}_j \rangle_\rho$ and $V^{\rho}$ is the $2m \times 2m$ covariance matrix of $\rho$, whose entries are defined as
\[ V_{j,k}^{\rho} \equiv \langle [\hat{x}_j - \mu_j^{\rho}, \hat{x}_k - \mu_k^{\rho}] \rangle. \] (3.20)

The following condition holds for a valid covariance matrix: $V + i\Omega \geq 0$, which is a manifestation of the uncertainty principle [SMD94].

A thermal Gaussian state $\theta_\beta$ of $m$ modes with respect to $\hat{E}_m$ from (3.13) and having inverse temperature $\beta > 0$ thus has the following form:
\[ \theta_\beta = e^{-\beta \hat{E}_m} / \text{Tr}\{e^{-\beta \hat{E}_m}\}, \] (3.21)
and has a mean vector equal to zero and a diagonal $2m \times 2m$ covariance matrix. One can calculate that the photon number in this state is equal to
\[ \sum_j \frac{1}{e^{\beta \omega_j} - 1}. \] (3.22)

A single-mode thermal state with mean photon number $\bar{n} = 1/(e^{\beta \omega} - 1)$ has the following representation in the photon number basis:
\[ \theta(\bar{n}) \equiv \frac{1}{1 + \bar{n}} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{\bar{n} + 1}\right)^n |n\rangle \langle n|. \] (3.23)

It is also well known that thermal states can be written as a Gaussian mixture of displacement operators acting on the vacuum state:
\[ \theta_\beta = \int d^{2m}\xi \, p(\xi) \, D(\xi) \, |0\rangle \langle 0| \otimes^m D^\dagger(\xi), \] (3.24)
where \( p(\xi) \) is a zero-mean, circularly symmetric Gaussian distribution. From this, it also follows that randomly displacing a thermal state in such a way leads to another thermal state of higher temperature:

\[
\theta_{\beta} = \int d^{2m} \xi \, q(\xi) \, D(\xi) \theta_{\beta'} D^\dagger(\xi),
\]

where \( \beta' \geq \beta \) and \( q(\xi) \) is a particular circularly symmetric Gaussian distribution.

In our paper, we employ the two-mode squeezed vacuum state with parameter \( \bar{n} \), which is equivalent to a purification of the thermal state in (3.23) and is defined as

\[
|\psi_{\text{TMS}}(\bar{n})\rangle \equiv \frac{1}{\sqrt{\bar{n}+1}} \sum_{n=0}^{\infty} \sqrt{\left(\frac{\bar{n}}{\bar{n}+1}\right)^n} |n\rangle_R |n\rangle_A.
\]

A \( 2m \times 2m \) matrix \( S \) is symplectic if it preserves the symplectic form: \( S \Omega S^T = \Omega \). According to Williamson’s theorem [Wil36], there is a diagonalization of the covariance matrix \( V^\rho \) of the form,

\[
V^\rho = S^\rho \left( D^\rho \oplus D^\rho \right) (S^\rho)^T,
\]

where \( S^\rho \) is a symplectic matrix and \( D^\rho \equiv \text{diag}(\nu_1, \ldots, \nu_m) \) is a diagonal matrix of symplectic eigenvalues such that \( \nu_i \geq 1 \) for all \( i \in \{1, \ldots, m\} \). Computing this decomposition is equivalent to diagonalizing the matrix \( iV^\rho \Omega \) [WTLB16, Appendix A].

The entropy \( H(\rho) \) of a quantum Gaussian state \( \rho \) is a direct function of the symplectic eigenvalues of its covariance matrix \( V^\rho \) [Ser17]:

\[
H(\rho) = \sum_{j=1}^{m} g((\nu_j - 1)/2),
\]

where \( g(\cdot) \) is defined in (3.2).

The Hilbert–Schmidt adjoint of a Gaussian quantum channel \( N_{X,Y} \) from \( l \) modes to \( m \) modes has the following effect on a displacement operator \( D(\xi) \) [Ser17]:

\[
D(\xi) \mapsto D(X^T \xi) \exp \left( -\frac{1}{2} \xi^T Y \xi + i \xi^T \Omega d \right),
\]

where \( X \) is a real \( 2m \times 2l \) matrix, \( Y \) is a real \( 2m \times 2m \) positive semi-definite matrix, and \( d \in \mathbb{R}^{2m} \), such that they satisfy

\[
Y + i \Omega - i X \Omega X^T \geq 0.
\]

The effect of the channel on the mean vector \( \mu^\rho \) and the covariance matrix \( V^\rho \) is thus as follows:

\[
\mu^\rho \mapsto X \mu^\rho + d,
\]

\[
V^\rho \mapsto XV^\rho X^T + Y.
\]

All Gaussian channels are covariant with respect to displacement operators. That is, the following relation holds

\[
N_{X,Y}(D(\xi) \rho D^\dagger(\xi)) = D(X \xi) N_{X,Y}(\rho) D^\dagger(X \xi),
\]

and note that \( D(X \xi) \) is a tensor product of local displacement operators.
Just as every quantum channel can be implemented as a unitary transformation on a larger space followed by a partial trace, so can Gaussian channels be implemented as a Gaussian unitary on a larger space with some extra modes prepared in the vacuum state, followed by a partial trace [CEGH08]. Given a Gaussian channel $\mathcal{N}_{X,Y}$ with $Z$ such that $Y = ZZ^T$ we can find two other matrices $X_E$ and $Z_E$ such that there is a symplectic matrix

$$S = \begin{bmatrix} X & Z \\ X_E & Z_E \end{bmatrix},$$

(3.34)

which corresponds to the Gaussian unitary transformation on a larger space. The complementary channel $\hat{\mathcal{N}}_{X_E,Y_E}$ from input to the environment then effects the following transformation on mean vectors and covariance matrices:

$$\mu^\rho \mapsto X_E\mu^\rho,$$

(3.35)

$$V^\rho \mapsto X_E V^\rho X_E^T + Y_E,$$

(3.36)

where $Y_E \equiv Z_E Z_E^T$.

Quantum thermal channel. A quantum thermal channel is a Gaussian channel that can be characterized by a beamsplitter of transmissivity $\eta$, coupling the signal input state with a thermal state with mean photon number $N_B$. In the Heisenberg picture, the beamsplitter transformation is given by the following Bogoliubov transformation:

$$\hat{b} = \sqrt{\eta} \hat{a} - \sqrt{1 - \eta} \hat{e},$$

(3.37)

$$\hat{e}' = \sqrt{1 - \eta} \hat{a} + \sqrt{\eta} \hat{e},$$

(3.38)

where $\hat{a}, \hat{b}, \hat{e},$ and $\hat{e}'$ are the annihilation operators representing the sender’s input mode, the receiver’s output mode, an environmental input mode, and an environmental output mode of the channel, respectively. Throughout the paper, we represent the thermal channel by $L_{\eta,N_B}$. If the mean photon number at the input of a thermal channel is no larger than $N_S$, then the total number of photons that make it through the channel to the receiver is no larger than $\eta N_S + (1 - \eta) N_B$.

Continuity of output entropy. The following theorem on continuity of output entropy for infinite-dimensional systems with finite average energy constraints is a direct consequence of [LS09, Theorem 11] and Lemma 1.

**Theorem 7** Let $\mathcal{N}_{A\rightarrow B}$ and $\mathcal{M}_{A\rightarrow B}$ be quantum channels, $G \in \mathcal{P}(\mathcal{H}_B)$ be a Gibbs observable, such that

$$\text{Tr}\{\mathcal{G}_n \mathcal{N}^{\otimes n}(\rho_{A^n})\}, \text{Tr}\{\mathcal{G}_n \mathcal{M}^{\otimes n}(\rho_{A^n})\} \leq W,$$

(3.39)

where $W \in [0, \infty)$ and $\rho_{RA^n} \in \mathcal{D}(\mathcal{H}_R \otimes \mathcal{H}_A^{\otimes n})$. If $\frac{1}{2} \|N - M\|_\diamond \leq \epsilon < \epsilon' \leq 1$ and $\delta = (\epsilon' - \epsilon)/(1 + \epsilon')$, then the following inequality holds

$$|H((\text{id}_R \otimes \mathcal{N}_{A\rightarrow B}^{\otimes n})(\rho_{RA^n})) - H((\text{id}_R \otimes \mathcal{M}_{A\rightarrow B}^{\otimes n})(\rho_{RA^n}))|$$

$$\leq n[(2\epsilon' + 4\delta)H(\gamma(W/\delta)) + g(\epsilon') + 2h_2(\delta)].$$

(3.40)
Proof. Let
\[ \rho^j = (\text{id}_R \otimes M_{A\rightarrow B}^{\otimes j} \otimes N_{A\rightarrow B}^{\otimes (n-j)})(\rho_{RA^n}), \] (3.41)
and consider the following chain of inequalities:
\[
\begin{align*}
|H(RB^n)_{\rho} - H(RB^n)_{\rho^n}| &= \left| \sum_{j=1}^{n} H(RB^n)_{\rho^j - 1} - H(RB^n)_{\rho^j} \right| \\
&\leq \sum_{j=1}^{n} |H(RB^n)_{\rho^j - 1} - H(RB^n)_{\rho^j}| \\
&= \sum_{j=1}^{n} |H(B_j|RB_1\cdots B_{j-1}B_{j+1}\cdots B_n)_{\rho^j - 1} - H(B_j|RB_1\cdots B_{j-1}B_{j+1}\cdots B_n)_{\rho^j}| \\
&\leq n[(2\varepsilon' + 4\delta) \left( \sum_{j=1}^{n} \frac{1}{n} H(\gamma(W_j/\delta)) \right) + g(\varepsilon') + 2h_2(\delta)] \\
&\leq n[(2\varepsilon' + 4\delta) H(\frac{1}{n} \sum_{j=1}^{n} \gamma(W_j/\delta)) + g(\varepsilon') + 2h_2(\delta)] \\
&\leq n[(2\varepsilon' + 4\delta) H(\gamma(W/\delta)) + g(\varepsilon') + 2h_2(\delta)].
\end{align*}
\]
(3.42)
(3.43)
(3.44)
(3.45)
(3.46)
(3.47)

The first inequality follows from the triangle inequality. The second equality follows from the fact that the states \( \rho^j \) and \( \rho^j - 1 \) are the same except for the \( j \)th output system. Let \( W_j \) denote an energy constraint on the \( j \)th output state of both the channels \( \mathcal{N} \) and \( \mathcal{M} \), i.e., \( \text{Tr}\{GN(\rho_{A_j})\} \), \( \text{Tr}\{GM(\rho_{A_j})\} \leq W_j \) and \( \frac{1}{n} \sum_j W_j \leq W \). Then the second inequality follows because \( \frac{1}{2} ||\rho^j - \rho^j - 1||_1 \leq \varepsilon \) for the given channels, and we use Lemma 1 for the \( j \)th output system. The third inequality follows from concavity of entropy. The final inequality follows because

\[
\text{Tr}\left\{ \frac{1}{n} \sum_{j=1}^{n} G \gamma(W_j/\delta) \right\} = \frac{1}{n} \sum_{j=1}^{n} \text{Tr}\{G \gamma(W_j/\delta)\} \leq W/\delta,
\]
(3.48)

and \( \gamma(W/\delta) \) is the Gibbs state that maximizes the entropy corresponding to the energy \( W/\delta \). □

Continuity of capacities for channels. The continuity of various capacities of quantum channels has been discussed in [LS09, Lemma 12]. The general form for the classical, quantum, or private capacity of a channel \( \mathcal{N} \) can be defined as \( F(\mathcal{N}) = \lim_{n \rightarrow \infty} \frac{1}{n} \sup_{P(n)} f_n(\mathcal{N}^{\otimes n}, P^{(n)}) \), where \( \{f_n\}_n \) denotes a family of functions, and \( P^{(n)} \) represents states or parameters over which an optimization is performed. Then the following lemma holds [LS09].

Lemma 2 (Lemma 12, [LS09]) If \( F(\mathcal{N}) = \lim_{n \rightarrow \infty} \frac{1}{n} \sup_{P(n)} f_n(\mathcal{N}^{\otimes n}, P^{(n)}) \) for a channel \( \mathcal{N} \) and \( \forall n, P^{(n)}, \left| f_n(\mathcal{N}^{\otimes n}, P^{(n)}) - f_n(\mathcal{M}^{\otimes n}, P^{(n)}) \right| \leq nc, \) then \( |F(\mathcal{N}) - F(\mathcal{M})| \leq c. \)

Energy-constrained quantum and private capacities. The energy-constrained quantum and private capacities of quantum channels have been defined in [WQ16, Section III]. In what follows, we
review the definition of quantum communication and private communication codes, achievable rates, and regularized formulas for energy-constrained quantum and private capacities.

**Energy-constrained quantum capacity.** An \((n, M, G, W, \varepsilon)\) code for energy-constrained quantum communication consists of an encoding channel \(E^n : \mathcal{T} (\mathcal{H}_S) \to \mathcal{T} (\mathcal{H}_A^{\otimes n})\) and a decoding channel \(D^n : \mathcal{T} (\mathcal{H}_B^{\otimes n}) \to \mathcal{T} (\mathcal{H}_S)\), where \(M = \text{dim}(\mathcal{H}_S)\). The energy constraint is such that the following bound holds for all states resulting from the output of the encoding channel \(E^n\):

\[
\text{Tr} \{ G^n E^n (\rho_S) \} \leq W, \quad (3.49)
\]

where \(\rho_S \in \mathcal{D} (\mathcal{H}_S)\). Note that

\[
\text{Tr} \{ G^n E^n (\rho_S) \} = \text{Tr} \{ G \overline{\rho}_n \}, \quad (3.50)
\]

where

\[
\overline{\rho}_n \equiv \frac{1}{n} \sum_{i=1}^n \text{Tr} A_n \{ \mathcal{E}^n (\rho_S) \}. \quad (3.51)
\]

due to the i.i.d. nature of the observable \(G_n\). Furthermore, the quantum communication code satisfies the following reliability condition such that for all pure states \(\phi_{RS} \in \mathcal{D} (\mathcal{H}_R \otimes \mathcal{H}_S)\),

\[
F (\phi_{RS}, (\text{id}_R \otimes \mathcal{N}^{\otimes n} \circ \mathcal{E}^n)) (\phi_{RS}) \geq 1 - \varepsilon, \quad (3.52)
\]

where \(\mathcal{H}_R\) is isomorphic to \(\mathcal{H}_S\). A rate \(R\) is achievable for quantum communication over \(\mathcal{N}\) subject to the energy constraint \(W\) if for all \(\varepsilon \in (0, 1)\), \(\delta > 0\), and sufficiently large \(n\), there exists an \((n, 2^n [R - \delta], G, W, \varepsilon)\) energy-constrained quantum communication code. The energy-constrained quantum capacity \(Q(\mathcal{N}, G, W)\) of \(\mathcal{N}\) is equal to the supremum of all achievable rates.

If the channel \(\mathcal{N}\) satisfies Condition 6 and \(G\) is a Gibbs observable, then the quantum capacity \(Q(\mathcal{N}, G, W)\) is equal to the regularized energy-constrained coherent information of the channel \(\mathcal{N}\) [WQ16]

\[
Q(\mathcal{N}, G, W) = \lim_{n \to \infty} \frac{1}{n} I_c (\mathcal{N}^{\otimes n}, G_n, W), \quad (3.53)
\]

where the energy-constrained coherent information of the channel is defined as [WQ16]

\[
I_c (\mathcal{N}, G, W) \equiv \sup_{\rho : \text{Tr} \{ \rho G \} \leq W} H (\mathcal{N} (\rho)) - H (\hat{\mathcal{N}} (\rho)), \quad (3.54)
\]

and \(\hat{\mathcal{N}}\) denotes a complementary channel of \(\mathcal{N}\). Note that another definition of energy-constrained quantum communication is possible, but it leads to the same value for the capacity in the asymptotic limit of many channel uses [WQ16].

**Energy-constrained private capacity.** An \((n, M, G, W, \varepsilon)\) code for private communication consists of a set \(\{ \rho_{A^n}^m \}_{m=1}^M\) of quantum states, each in \(\mathcal{D} (\mathcal{H}_A^{\otimes n})\), and a POVM \(\{ \Lambda_{B^n}^m \}_{m=1}^M\) such that

\[
\text{Tr} \{ G_n \rho_{A^n}^m \} \leq W, \quad (3.55)
\]

\[
\text{Tr} \{ \Lambda_{B^n}^m \mathcal{N}^{\otimes n} (\rho_{A^n}^m) \} \geq 1 - \varepsilon, \quad (3.56)
\]

\[
\frac{1}{2} \left\| \mathcal{N}^{\otimes n} (\rho_{A^n}^m) - \omega_{E^n} \right\|_1 \leq \varepsilon, \quad (3.57)
\]
for all \( m \in \{1, \ldots, M\} \), with \( \omega_E \) some fixed state in \( \mathcal{D}(\mathcal{H}_E^{\otimes n}) \). In the above, \( \hat{N} \) is a channel complementary to \( \mathcal{N} \). A rate \( R \) is achievable for private communication over \( \mathcal{N} \) subject to energy constraint \( W \) if for all \( \varepsilon \in (0, 1), \delta > 0 \), and sufficiently large \( n \), there exists an \((n, 2^{n[R-\delta]}, G, W, \varepsilon)\) private communication code. The energy-constrained private capacity \( P(\mathcal{N}, G, W) \) of \( \mathcal{N} \) is equal to the supremum of all achievable rates.

An upper bound on the energy-constrained private capacity of a channel has been established in [WQ16], but the lower bound still needs a detailed proof. However, the results in [WQ16] suggest the validity of the following form. If the channel \( \mathcal{N} \) satisfies Condition 6 and \( G \) is a Gibbs observable, then the energy-constrained private capacity \( P(\mathcal{N}, G, W) \) is given by the regularized energy-constrained private information of the channel:

\[
P(\mathcal{N}, G, W) = \lim_{n \to \infty} \frac{1}{n} P^{(1)}(\mathcal{N}^{\otimes n}, G_n, W),
\]

where the energy-constrained private information is defined as

\[
P^{(1)}(\mathcal{N}, G, W) \equiv \sup_{\rho_{\mathcal{A}} : \text{Tr}(G \rho_{\mathcal{A}}) \leq W} \int dx \, p_X(x) \left[ D(\mathcal{N}(\rho_{\mathcal{A}}^x) || \hat{N}(\rho_{\mathcal{E}_A})) - D(\hat{N}(\rho_{\mathcal{A}}^x) || \hat{N}(\bar{\rho}_{\mathcal{E}_A})) \right],
\]

and \( \bar{\rho}_{\mathcal{E}_A} \equiv \int dx \, p_X(x) \rho_{\mathcal{A}}^x \) is an average state of the ensemble

\[
\mathcal{E}_A \equiv \{ p_X(x), \rho_{\mathcal{A}}^x \},
\]

and \( \hat{N} \) denotes a complementary channel of \( \mathcal{N} \). Note that another definition of energy-constrained private communication is possible, but it leads to the same value for the capacity in the asymptotic limit of many channel uses [WQ16].

## 4 Bounds on energy-constrained quantum and private capacities of approximately degradable channels

In this section, we derive upper bounds on the energy-constrained quantum and private capacities of approximately degradable channels. We derive these bounds for both \( \varepsilon \)-degradable (Definition 1) and \( \varepsilon \)-close-degradable (Definition 2) channels. This general form for the upper bounds on the energy-constrained quantum and private capacities of approximately degradable channels will be directly used in establishing bounds on the capacities of quantum thermal channels.

We begin by defining the conditional entropy of degradation, which will be useful for finding upper bounds on the energy-constrained quantum and private capacities of an \( \varepsilon \)-degradable channel. A similar quantity has been defined for the finite-dimensional case in [SSWR14].

**Definition 8 (Conditional entropy of degradation)** Let \( \mathcal{N}_{A \to B} \) and \( \mathcal{D}_{B \to E} \) be quantum channels, and let \( G \in \mathcal{P}(\mathcal{H}_A) \) be a Gibbs observable. We define the conditional entropy of degradation as follows:

\[
U_{\mathcal{D}}(\mathcal{N}, G, W) = \sup_{\rho : \text{Tr}(G \rho) \leq W} \left[ H(\mathcal{N}(\rho)) - H(D \circ \mathcal{N}(\rho)) \right],
\]

where \( W \in [0, \infty) \). For a Stinespring dilation \( \mathcal{V} : \mathcal{T}(B) \to \mathcal{T}(E) \otimes \mathcal{T}(F) \) of the channel \( \mathcal{D} \),

\[
U_{\mathcal{D}}(\mathcal{N}, G, W) = \sup_{\rho : \text{Tr}(G \rho) \leq W} \left[ H(F|E \circ \mathcal{V} \circ \mathcal{N}(\rho)) \right].
\]
We note that the conditional entropy of degradation can be understood as the negative entropy gain of the channel $\mathcal{D}_{B\rightarrow E}$ [Ali04, Hol10, Hol11a, Hol11b], with the optimization over input states $\mathcal{N}(\rho)$ restricted to being in the image of $\mathcal{N}$ and obeying the energy constraint $\text{Tr}\{G\rho\} \leq W$. Next, we show that the conditional entropy of degradation in (4.2) is additive.

**Lemma 3** Let $\mathcal{N}_{A\rightarrow B}$ and $\mathcal{D}_{B\rightarrow E}$ be quantum channels, let $G \in \mathcal{P}(\mathcal{H}_A)$ be a Gibbs observable, and let $W \in [0, \infty)$. Then for all integer $n \geq 1$

$$U_{\mathcal{D}^\otimes n}(\mathcal{N}^\otimes n, \mathcal{G}_n, W) = n[U_{\mathcal{D}}(\mathcal{N}, G, W)].$$  

(4.3)

**Proof.** The following inequality

$$U_{\mathcal{D}^\otimes n}(\mathcal{N}^\otimes n, \mathcal{G}_n, W) \geq n[U_{\mathcal{D}}(\mathcal{N}, G, W)]$$

(4.4)

follows trivially because a product input state is a particular state of the form required in the optimization of $U_{\mathcal{D}^\otimes n}(\mathcal{N}^\otimes n, \mathcal{G}_n, W)$. We now prove the less trivial inequality

$$U_{\mathcal{D}^\otimes n}(\mathcal{N}^\otimes n, \mathcal{G}_n, W) \leq n[U_{\mathcal{D}}(\mathcal{N}, G, W)].$$

(4.5)

Consider the following chain of inequalities:

$$H(F^n|E^n)_{(\mathcal{V}^\otimes n \circ \mathcal{N}^\otimes n)(\rho_{A^n})} \leq \sum_{i=1}^{n} H(F_i|E_i)_{(\mathcal{V} \circ \mathcal{N})(\rho_{A})}$$

(4.6)

$$\leq n[H(F|E)_{(\mathcal{V} \circ \mathcal{N})(\rho_n)}]$$

(4.7)

$$\leq n[U_{\mathcal{D}}(\mathcal{N}, G, W)].$$

(4.8)

where $\rho_n = \frac{1}{n} \sum_{i=1}^{n} \rho_{A_i}$. The first inequality follows from several applications of strong subadditivity [LR73b, LR73a]. The second inequality follows from concavity of conditional entropy [LR73b, LR73a]. The last inequality follows because $\text{Tr}\{\mathcal{G}_n\rho_{A^n}\} = \text{Tr}\{G\rho_n\} \leq W$ and the conditional entropy of degradation $U_{\mathcal{D}}(\mathcal{N}, G, W)$ involves an optimization over all input states obeying this energy constraint. Since the chain of inequalities is true for all input states $\rho_{A^n}$ satisfying the input energy constraint, the desired result follows.

### 4.1 Bound on the energy-constrained quantum capacity of an $\varepsilon$-degradable channel

An upper bound on the quantum capacity of an $\varepsilon$-degradable channel was established as [SSWR14, Theorem 3.1(ii)] for the finite-dimensional case. Here, we prove a related bound for the infinite-dimensional case with finite average energy constraints on the input and output states of the channels.

**Theorem 9** Let $\mathcal{N}_{A\rightarrow B}$ be an $\varepsilon$-degradable channel with a degrading channel $\mathcal{D}_{B\rightarrow E'}$, and let $G \in \mathcal{P}(\mathcal{H}_A)$ and $G' \in \mathcal{P}(\mathcal{H}_{E'})$ be Gibbs observables, such that for all input states $\rho_{A^n} \in \mathcal{D}(\mathcal{H}_A^\otimes n)$ satisfying input average energy constraints $\text{Tr}\{\mathcal{G}_n\rho_{A^n}\} \leq W$, the following output average energy constraints are satisfied:

$$\text{Tr}\{\mathcal{G}_n^\otimes \mathcal{N}^\otimes n(\rho_{A^n})\}, \quad \text{Tr}\{\mathcal{G}_n^\otimes (\mathcal{D}^\otimes n \circ \mathcal{N}^\otimes n)(\rho_{A^n})\} \leq W',$$

(4.9)

where $\mathcal{N}_{A\rightarrow E}$ is a complementary channel of $\mathcal{N}$ and $E' \simeq E$. Then the energy-constrained quantum capacity $Q(\mathcal{N}, G, W)$ is bounded from above as

$$Q(\mathcal{N}, G, W) \leq U_{\mathcal{D}}(\mathcal{N}, G, W) + (2\varepsilon' + 4\delta)H(\gamma(W'/\delta)) + g(\varepsilon') + 2h_2(\delta),$$

(4.10)

with $\varepsilon' \in (\varepsilon, 1], W, W' \in [0, \infty)$, and $\delta = (\varepsilon' - \varepsilon)/(1 + \varepsilon')$. 

14
Proof. Let
\[
\sigma_{B^n} = \mathcal{N}^\otimes n(\rho_{A^n}),
\]
\[
\rho_{E^{(n-j)}}^j = (\mathcal{D}^\otimes j \circ \mathcal{N}^\otimes j) \otimes \mathcal{N}^\otimes (n-j)(\rho_{A^n}),
\]
and consider the following chain of inequalities:
\[
\begin{align*}
H(B^n)_\sigma - H(E^n)_{\rho}\rho & = H(B^n)_\sigma - H(E^n)_{\rho^n} + H(E^n)_{\rho^n} - H(E^n)_{\rho}\rho \\
& \leq U_D^\otimes n(\mathcal{N}^\otimes n, \mathcal{G}_n, W) + H(E^n)_{\rho^n} - H(E^n)_{\rho}\rho \\
& = n U_D(\mathcal{N}, G, W) \\
& \quad + \sum_{j=1}^n [H(E'_j|E_1' \ldots E_{j-1}E_{j+1} \ldots E_n)_{\rho^j} - H(E_j|E_1' \ldots E_{j-1}E_{j+1} \ldots E_n)_{\rho^j-1}] \\
& \leq n[U_D(\mathcal{N}, G, W) + (2\varepsilon' + 4\delta) \left( \sum_{j=1}^n \frac{1}{n} H(\gamma(W'_j/\delta)) + g(\varepsilon') + 2h_2(\delta) \right)] \\
& \leq n[U_D(\mathcal{N}, G, W) + (2\varepsilon' + 4\delta) H \left( \frac{1}{n} \sum_{j=1}^n \gamma(W'_j/\delta) \right) + g(\varepsilon') + 2h_2(\delta)] \\
& \leq n[U_D(\mathcal{N}, G, W) + (2\varepsilon' + 4\delta) H \left( \gamma(W'/\delta) \right) + g(\varepsilon') + 2h_2(\delta)].
\end{align*}
\]

The first inequality follows from the definition in (4.1). The second equality follows from Lemma 3 and the telescoping technique. Let $W'_j$ denote the energy constraint on the $j$th output state of both the channels $\mathcal{D} \circ \mathcal{N}$ and $\hat{\mathcal{N}}$, i.e., $\text{Tr} \{G'(\mathcal{D} \circ \mathcal{N})(\rho_{A_j})\}, \text{Tr} \{G'\hat{\mathcal{N}}(\rho_{A_j})\} \leq W'_j$ where $\frac{1}{n} \sum_j W'_j \leq W'$. Then the second inequality holds because $\frac{1}{n} \|\rho^j - \rho^{j-1}\|_1 \leq \varepsilon$ for the given channels, and we use Lemma 1 for the $j$th output system. The third inequality follows from concavity of entropy. The last inequality follows because $\text{Tr} \{\frac{1}{n} \sum_{j=1}^n G \gamma(W'_j/\delta)\} = \frac{1}{n} \sum_{j=1}^n \text{Tr} \{G \gamma(W'_j/\delta)\} \leq W'/\delta$, and $\gamma(W'/\delta)$ is the Gibbs state that maximizes the entropy corresponding to the energy $W'/\delta$. Since the chain of inequalities is true for all $\rho_{A^n}$ satisfying the input average energy constraint, from (3.54) and the above, we get that
\[
\frac{1}{n} I_c(\mathcal{N}^\otimes n, \mathcal{G}_n, W) \leq U_D(\mathcal{N}, G, W) + (2\varepsilon' + 4\delta) H \left( \gamma(W'/\delta) \right) + g(\varepsilon') + 2h_2(\delta).
\]
Since the last inequality holds for all $n$, we obtain the desired result by taking the limit $n \to \infty$ and applying (3.53).

4.2 Bound on the energy-constrained quantum capacity of an $\varepsilon$-close-degradable channel

An upper bound on the quantum capacity of an $\varepsilon$-close-degradable channel was established as [SSWR14, Proposition A.2(i)] for the finite-dimensional case. Here, we provide a bound for the infinite-dimensional case with finite average energy constraints on the input and output states of the channels.
Proof. Let $\mathcal{N}_{A\rightarrow B}$ be an $\varepsilon$-close-degradable channel, i.e., $\frac{1}{2} \| \mathcal{N} - \mathcal{M} \|_1 \leq \varepsilon < \varepsilon' \leq 1$, where $\mathcal{M}_{A\rightarrow B}$ is a degradable channel. Let $G \in \mathcal{P}(\mathcal{H}_A)$, $G' \in \mathcal{P}(\mathcal{H}_B)$ be Gibbs observables, such that for all input states $\rho_{RA^n} \in \mathcal{D}(\mathcal{H}_R \otimes \mathcal{H}_A^n)$ satisfying the input average energy constraint $\text{Tr}\{G_n \rho_{A^n}\} \leq W$, the following output average energy constraints are satisfied:

$$\text{Tr}\{G'_n \mathcal{N}^\otimes n(\rho_{A^n})\}, \text{Tr}\{G'_n \mathcal{M}^\otimes n(\rho_{A^n})\} \leq W',$$  \hspace{1cm} (4.18)

where $W, W' \in [0, \infty)$. Then the energy-constrained quantum capacity $Q(\mathcal{N}, G, W)$ is bounded from above as

$$Q(\mathcal{N}, G, W) \leq I_c(\mathcal{M}, G, W) + (4\varepsilon' + 8\delta)H(\gamma(W'/\delta)) + 2g(\varepsilon') + 4h_2(\delta),$$  \hspace{1cm} (4.19)

with $\varepsilon' \in (\varepsilon, 1]$ and $\delta = (\varepsilon' - \varepsilon)/(1 + \varepsilon')$.

Proof. Let $\omega_{RB^n} = (\text{id}_R \otimes \mathcal{N}^\otimes n)(\rho_{RA^n})$ and $\tau_{RB^n} = (\text{id}_R \otimes \mathcal{M}^\otimes n)(\rho_{RA^n})$, and consider the following chain of inequalities:

$$H(B^n)_\omega - H(RB^n)_\omega - H(B^n)_\tau + H(RB^n)_\tau = H(B^n)_\omega - H(B^n)_\tau + H(RB^n)_\tau - H(RB^n)_\omega$$  \hspace{1cm} (4.20)

$$\leq 2n[(2\varepsilon' + 4\delta)H(\gamma(W'/\delta)) + g(\varepsilon') + 2h_2(\delta)].$$  \hspace{1cm} (4.21)

The first inequality follows from applying Theorem 7 twice. Then from Lemma 2,

$$Q(\mathcal{N}, G, W) \leq Q(\mathcal{M}, G, W) + (4\varepsilon' + 8\delta)H(\gamma(W'/\delta)) + 2g(\varepsilon') + 4h_2(\delta).$$  \hspace{1cm} (4.22)

The desired result follows from the fact that the energy-constrained quantum capacity of a degradable channel is equal to the energy-constrained coherent information of the channel [WQ16].

4.3 Bound on the energy-constrained private capacity of an $\varepsilon$-degradable channel

In this section, we first derive an upper bound on the private capacity of an $\varepsilon$-degradable channel for the finite-dimensional case, which is different from any of the bounds presented in [SSWR14]. Then, we generalize this bound to the infinite-dimensional case with finite average energy constraints on the input and output states of the channels.

Theorem 11 Let $\mathcal{N}_{A\rightarrow B}$ be a finite-dimensional $\varepsilon$-degradable channel with a degrading channel $\mathcal{D}_{B\rightarrow E'}$, and let $\mathcal{N} : T(A) \rightarrow T(E)$ be a complementary channel of $\mathcal{N}$, such that $E' \simeq E$. If

$$U_{\mathcal{D}}(\mathcal{N}) = \max_{\rho \in \mathcal{D}(\mathcal{H}_A)} [H(\mathcal{N}(\rho)) - H((\mathcal{D} \circ \mathcal{N})(\rho))],$$  \hspace{1cm} (4.23)

then the private capacity $P(\mathcal{N})$ of $\mathcal{N}$ is bounded from above as

$$P(\mathcal{N}) \leq U_{\mathcal{D}}(\mathcal{N}) + 6\varepsilon \log_2 \dim(\mathcal{H}_E) + 3g(\varepsilon).$$  \hspace{1cm} (4.24)

Proof. Consider Stinespring dilations $\mathcal{U} : T(A) \rightarrow T(B) \otimes T(E)$ and $\mathcal{V} : T(B) \rightarrow T(E') \otimes T(F)$ of the channel $\mathcal{N}$ and the degrading channel $\mathcal{D}$, respectively. Let $\rho_{X^n}$ be a classical–quantum state in correspondence with an ensemble $\{p_X(x), \rho_{X^n}^x\}$:

$$\rho_{X^n} = \sum_x p_X(x) |x\rangle \langle x| \otimes \rho_{X^n}^x,$$  \hspace{1cm} (4.25)
and let 
\[ \omega_{XE^nE^nF^n} = \sum_x p_X(x) |x\rangle \langle x|_X \otimes (\text{id}_{E^n} \otimes \mathcal{Y}^{\otimes n}) \circ U^{\otimes n} (\rho_{A^n}). \]  
(4.26)

Consider the following extension of \( \omega_{XE^nE^nF^n} \):
\[ \sigma_{XYE^nE^nF^n} = \sum_{x,y} p_X(x)p_Y(y) |x\rangle \langle x|_X \otimes |y\rangle \langle y|_Y \otimes (\text{id}_{E^n} \otimes \mathcal{Y}^{\otimes n}) \circ U^{\otimes n} (\psi_{A^n}^{x,y}), \]  
where \( \psi_{A^n}^{x,y} \) is a pure state, and let \( \sigma_{E^nE^nF^n} = (\text{id}_{E^n} \otimes \mathcal{Y}^{\otimes n}) \circ U^{\otimes n} (\psi_{A^n}^{x,y}) \). Consider the following chain of inequalities:
\[ I(X; B^n)_\omega - I(X; E^n)_\omega = I(X; F^n|E^n)_\omega + I(X; E^n)_\omega - I(X; E^n)_\omega \]
\[ = I(X; F^n|E^n)_\omega + H(E^n|\omega) - H(E^n|X)_\omega - H(E^n|X)_\omega \]
\[ \leq I(X; F^n|E^n)_\omega + 2n[2\varepsilon \log_2 \text{dim}(\mathcal{H}_E) + g(\varepsilon)] \]
\[ \leq I(XY; E^n|X)_\sigma + n[4\varepsilon \log_2 \text{dim}(\mathcal{H}_E) + 2g(\varepsilon)] \]
\[ = H(F^n|E^n)_\sigma - H(E^n|XY)_\sigma + H(E^n|XY)_\sigma \]
\[ + n[4\varepsilon \log_2 \text{dim}(\mathcal{H}_E) + 2g(\varepsilon)] \]
\[ \leq n[U_D(\mathcal{N}) + 6\varepsilon \log_2 \text{dim}(\mathcal{H}_E) + 3g(\varepsilon)]. \]
(4.29)

The first two equalities follow from entropy identities. The first inequality follows by applying the telescoping technique twice and using the continuity result of the conditional quantum entropy for finite-dimensional quantum systems [Win16]. The second inequality follows from the quantum data processing inequality for conditional quantum mutual information. The last two equalities follow from entropy identities and by using that \( \sigma_{E^nE^nF^n} \) is a pure state, so that \( H(F^n|E^n)_{E^n} = H(E^n)_{E^n} \). The last inequality follows from the definition in (4.23), and additivity of \( U_D(\mathcal{N}) \) [SSWR14]. Also, we applied the telescoping technique for each \( \sigma_{x,y} \) in the summation, and used the continuity result of the conditional quantum entropy for finite-dimensional systems [Win16]. Since the chain of inequalities is true for any ensemble \( \{p_X(x), \rho_{A^n}\} \), the final result follows from the definition of private information of the channel, dividing by \( n \), taking the limit \( n \to \infty \), and noting that the regularized private information is equal to the private capacity of any channel.

Next, we derive an upper bound on the energy-constrained private capacity of an \( \varepsilon \)-degradable channel.

**Theorem 12** Let \( \mathcal{N}_{A\to B} \) be an \( \varepsilon \)-degradable channel with a degrading channel \( \mathcal{D}_{B\to E'v} \), and let \( G \in \mathcal{P}(\mathcal{H}_A), G' \in \mathcal{P}(\mathcal{H}_{E'}) \) be Gibbs observables, such that for all input states \( \rho_{A^n} \in \mathcal{D}(H_A^{\otimes n}) \) satisfying input average energy constraints \( \text{Tr}\{\mathcal{G}_n \rho_{A^n}\} \leq W \), the following output average energy constraints are satisfied:
\[ \text{Tr}\{\mathcal{G}_n \mathcal{N}_{A\to E}^{\otimes n}(\rho_{A^n})\}, \text{Tr}\{\mathcal{G}_n' (\mathcal{D}_{A\to E}^{\otimes n} \circ \mathcal{N}_{A\to E}^{\otimes n})(\rho_{A^n})\} \leq W', \]
(4.35)
where \( \mathcal{N}_{A\to E} \) is a complementary channel of \( \mathcal{N} \), and \( E' \simeq E \). Then the energy-constrained private capacity is bounded from above as
\[ P(\mathcal{N}, G, W) \leq U_D(\mathcal{N}, G, W) + (6\varepsilon' + 12\delta)H(\gamma(W'/\delta)) + 3g(\varepsilon') + 6h_2(\delta), \]
(4.36)
with \( \varepsilon' \in (\varepsilon, 1], W, W' \in [0, \infty) \), and \( \delta = (\varepsilon' - \varepsilon) / (1 + \varepsilon') \).

**Proof.** Since the proof is similar to the above one and previous ones, we just summarize it briefly below. Consider Stinespring dilations \( U : \mathcal{T}(A) \rightarrow \mathcal{T}(B) \otimes \mathcal{T}(E) \) and \( V : \mathcal{T}(B) \rightarrow \mathcal{T}(E') \otimes \mathcal{T}(F) \) of the channel \( \mathcal{N} \) and the degrading channel \( \mathcal{D} \), respectively. Then the action of \( U^\otimes n \) followed by \( V^\otimes n \) on the ensemble \( \{p_X(x), \rho_{A^n}^x\} \) leads to the following ensemble:

\[
\{p_X(x), \omega_{E^n E'n F'n}^x \equiv (\text{id}_{E^n} \otimes V^\otimes n) \circ U^\otimes n(\rho_{A^n}^x)\}. 
\]

(4.37)

Similar to the above proof, from applying the telescoping technique three times and using Lemma 1, concavity of entropy, and Lemma 3, we get the following bound:

\[
I(X; B^n)_\omega - I(X; E^n)_\omega \leq n [U_D(\mathcal{N}, G, W) + (6\varepsilon' + 12\delta)H(\gamma(W'/\delta)) + 3g(\varepsilon') + 6h_2(\delta)]. 
\]

(4.38)

The desired result follows from dividing by \( n \), taking the limit \( n \rightarrow \infty \), the definition of the energy-constrained private information of the channel, and using the fact that the regularized energy-constrained private information is an upper bound on the energy-constrained private capacity of a quantum channel [WQ16]. ■

### 4.4 Bound on the energy-constrained private capacity of an \( \varepsilon \)-close-degradable channel

An upper bound on the private capacity of an \( \varepsilon \)-close-degradable channel was established as [SSWR14, Proposition A.2(ii)] for the finite-dimensional case. Here, we provide a bound for the infinite-dimensional case with finite average energy constraints on the input and output states of the channels.

**Theorem 13** Let \( \mathcal{N}_{A \rightarrow B} \) be an \( \varepsilon \)-close-degradable channel, i.e., \( \frac{1}{2} \|\mathcal{N} - \mathcal{M}\|_\infty \leq \varepsilon < \varepsilon' \leq 1 \), where \( \mathcal{M}_{A \rightarrow B} \) is a degradable channel. Let \( G \in \mathcal{P}(\mathcal{H}_A), G' \in \mathcal{P}(\mathcal{H}_B) \) be Gibbs observables, such that for all input states \( \rho_{A^n}^x \in \mathcal{D}(\mathcal{H}_A^\otimes n) \) satisfying input average energy constraints \( \text{Tr}\{\mathcal{G}_n \rho_{A^n}^x\} \leq W \), the following output average energy constraints are satisfied:

\[
\text{Tr}\{\mathcal{G}_n \mathcal{N}_{A^n}^\otimes (\rho_{A^n}^x)\}, \text{Tr}\{\mathcal{G}_n \mathcal{M}_{A^n}^\otimes (\rho_{A^n}^x)\} \leq W', 
\]

(4.39)

where \( W, W' \in [0, \infty) \). Then

\[
P(\mathcal{N}, G, W) \leq I_e(\mathcal{M}, G, W) + (8\varepsilon' + 16\delta)H(\gamma(W'/\delta)) + 4g(\varepsilon') + 8h_2(\delta),
\]

(4.40)

with \( \varepsilon' \in (\varepsilon, 1], \) and \( \delta = (\varepsilon' - \varepsilon) / (1 + \varepsilon'). \)

**Proof.** We follow the proof of [LS09, Corollary 15] closely, but incorporate energy constraints. Consider Stinespring dilations \( U : \mathcal{T}(A) \rightarrow \mathcal{T}(B) \otimes \mathcal{T}(E) \) and \( V : \mathcal{T}(B) \rightarrow \mathcal{T}(E') \otimes \mathcal{T}(F) \) of the channels \( \mathcal{N} \) and \( \mathcal{M} \), respectively. Consider an input ensemble \( \{p_X(x), \rho_{A^n}^x\} \), which leads to the output ensembles

\[
\{p_X(x), \omega^x_{E'n F'n} \equiv U^\otimes n(\rho_{A^n}^x)\},
\]

(4.41)

\[
\{p_X(x), \tau^x_{E'n F'n} \equiv V^\otimes n(\rho_{A^n}^x)\}.
\]

(4.42)
Supposing at first that the index $x$ is discrete, from four times applying Theorem 7 and employing the same expansions as in the proof of [LS09, Corollary 15], we get

$$I(X; B^n) - I(X; E^n) - I(X; E^n) \omega \leq 4n[(2\varepsilon + 4\delta)H(\gamma(W/\delta)) + g(\varepsilon') + 2h_2(\delta)].$$ (4.43)

The upper bound is uniform and has no dependence on the particular ensemble except via the energy constraints. Thus, by approximation, the same bound applies to ensembles for which the index $x$ is continuous. Then from Lemma 2, we find that

$$P(N, G, W) \leq P(M, G, W) + (8\varepsilon' + 16\delta)H(\gamma(W'/\delta)) + 4g(\varepsilon') + 8h_2(\delta)$$ (4.44)

$$= I_c(M, G, W) + (8\varepsilon' + 16\delta)H(\gamma(W'/\delta)) + 4g(\varepsilon') + 8h_2(\delta).$$ (4.45)

The equality in the last line follows from the fact that the energy-constrained private capacity of a degradable channel is equal to the energy-constrained coherent information of the channel [WQ16].

5 Upper bounds on energy-constrained quantum capacity of bosonic thermal channels

In this section, we establish three different upper bounds on the energy-constrained quantum capacity of a thermal channel:

1. We establish a first upper bound using the theorem that any thermal channel can be decomposed as the concatenation of a pure-loss channel followed by a quantum-limited amplifier channel [CGH06, GPNBL+12]. We call this bound the data-processing bound and denote it by $Q_U^1$.

2. Next, we show that a thermal channel is an $\varepsilon$-degradable channel for a particular choice of degrading channel. Then an upper bound on the energy-constrained quantum capacity of a thermal channel directly follows from Theorem 9. We call this bound the $\varepsilon$-degradable bound and denote it by $Q_U^2$.

3. We establish a third upper bound on the energy-constrained quantum capacity of a thermal channel using the idea of $\varepsilon$-close-degradability. We show that the thermal channel is $\varepsilon$-close to a pure-loss bosonic channel for a particular choice of $\varepsilon$. Since a pure-loss bosonic channel is a degradable channel [WPWG07], the bound on the energy-constrained quantum capacity of a thermal channel follows directly from Theorem 10. We call this bound the $\varepsilon$-close-degradable bound and denote it by $Q_U^3$.

In Section 6, we compare, for different parameter regimes, the closeness of these upper bounds with a known lower bound on the quantum capacity of thermal channels.

5.1 Data-processing bound on the energy-constrained quantum capacity of bosonic thermal channels

In this section, we provide an upper bound using the theorem that any thermal channel $L_{\eta,N_B}$ can be decomposed as the concatenation of a pure-loss channel $L_{\eta',0}$ with transmissivity $\eta'$ followed
by a quantum-limited amplifier channel $\mathcal{A}_G$ with gain $G$ [CGH06, GPNBL+12], i.e.,

$$L_{\eta,N_B} = \mathcal{A}_G \circ L_{\eta',0},$$

(5.1)

where $G = (1 - \eta)N_B + 1$, and $\eta' = \eta/G$. In Theorem 22, we prove that the data-processing bound can be at most 1.45 bits larger than a known lower bound.

**Theorem 14** An upper bound on the quantum capacity of a thermal channel $L_{\eta,N_B}$ with transmissivity $\eta \in [1/2, 1]$, environment photon number $N_B$, and input mean photon number constraint $N_S$ is given by

$$Q(L_{\eta,N_B}, N_S) \leq \max\{0, Q_U(L_{\eta',N_B}, N_S)\},$$

(5.2)

$$Q_U(L_{\eta',N_B}, N_S) \equiv g(\eta'N_S) - g[(1 - \eta')N_S],$$

(5.3)

with $\eta' = \eta/((1 - \eta)N_B + 1)$.

**Proof.** An upper bound on quantum capacity can be established by using (5.1) and a data-processing argument. We find that

$$Q(L_{\eta,N_B}, N_S) = Q(A_G \circ L_{\eta',0}, N_S)$$

$$\leq Q(L_{\eta',0}, N_S)$$

$$= \max\{0, g(\eta'N_S) - g[(1 - \eta')N_S]\}.$$

(5.6)

The first inequality follows from definitions and data processing—the energy-constrained capacity of $A_G \circ L_{\eta',0}$ cannot exceed that of $L_{\eta',0}$. The second equality follows from the formula for the energy-constrained quantum capacity of a pure-loss bosonic channel with transmissivity $\eta'$ and input mean photon number $N_S$ [WHG12, WQ16].

### 5.2 $\varepsilon$-degradable bound on the energy-constrained quantum capacity of bosonic thermal channels

In this section, we provide an upper bound on the energy-constrained quantum capacity of a thermal channel using the idea of $\varepsilon$-degradability. In Theorem 9, we established a general upper bound on the energy-constrained quantum capacity of an $\varepsilon$-degradable channel. Hence, our first step is to construct the degrading channel $D$ given in (5.14), such that the concatenation of a thermal channel $L_{\eta,N_B}$ followed by $D$ is close in diamond distance to the complementary channel $\hat{L}_{\eta,N_B}$ of the thermal channel $L_{\eta,N_B}$.

We start by motivating the reason for choosing the particular degrading channel in (5.14), which is depicted in Figure 1, and then we find an upper bound on the diamond distance between $D \circ L_{\eta,N_B}$ and $\hat{L}_{\eta,N_B}$. In general, it is computationally hard to perform the optimization over an infinite dimensional space required in the calculation of the diamond distance between Gaussian channels. However, we address this problem in this particular case by introducing a channel that simulates the serial concatenation of the thermal channel and the degrading channel, and we call it the simulating channel, as given in (5.18). This allows us to bound the diamond distance between the channels from above by the trace distance between the environment states of the complementary channel and the simulating channel (Theorem 15). Next, we argue that, for a given input mean photon-number constraint $N_S$, a thermal state with mean photon number $N_S$ maximizes the conditional entropy of degradation defined in (4.2), which also appears in the general upper
bound established in Theorem 9. We finally provide an upper bound on the energy-constrained quantum capacity of a thermal channel by using all these tools and invoking Theorem 9.

We now establish an upper bound on the diamond distance between the complementary channel of the thermal channel and the concatenation of the thermal channel followed by a particular degrading channel. Let $B$ and $B'$ represent beamsplitter transformations with transmissivity $\eta$ and $(1 - \eta)/\eta$, respectively. In the Heisenberg picture, the beamsplitter transformation $B_{C_1 D_1 \rightarrow C_2 D_2}$ is given by

$$
\hat{c}_2 = \sqrt{\eta}\hat{c}_1 - \sqrt{1 - \eta}\hat{d}_1,
\hat{d}_2 = \sqrt{1 - \eta}\hat{c}_1 + \sqrt{\eta}\hat{d}_1.
$$

Similarly, the beamsplitter transformation $B'_{C_1 D_1 \rightarrow C_2 D_2}$ is given by

$$
\hat{c}_2 = \sqrt{(1 - \eta)/\eta}\hat{c}_1 + \sqrt{(2\eta - 1)/\eta}\hat{d}_1,
\hat{d}_2 = -\sqrt{(2\eta - 1)/\eta}\hat{c}_1 + \sqrt{(1 - \eta)/\eta}\hat{d}_1,
$$

where $\hat{c}_1, \hat{c}_2, \hat{d}_1,$ and $\hat{d}_2$ are annihilation operators representing various modes involved in the beamsplitter transformations. Here, $\eta \in [1/2, 1]$. It is important to stress that there is a difference in phase between $B$ and $B'$ beamsplitter transformations, which is crucial in our development.

Consider the following action of the thermal channel $L_{\eta, N_B}$ on an input state $\phi_{RA}$:

$$
(id_R \otimes L_{\eta, N_B})(\phi_{RA}) = \text{Tr}_{E_1 E_2}\{B_{AE'\rightarrow BE_2}(\phi_{RA} \otimes \psi_{\text{TMS}}(N_B)_{E'E_1})\},
$$

where $R$ is a reference system and $\psi_{\text{TMS}}(N_B)_{E'E_1}$ is a two-mode squeezed vacuum state with parameter $N_B$, as defined in (3.26).

Here and what remains in the proof, we consider the action of various transformations on the covariance matrices of the states involved, and we furthermore track only the submatrices corresponding to the position-quadrature operators of the covariance matrices. It suffices to do so because all channels involved in our discussion are phase-insensitive Gaussian channels.

The submatrix corresponding to the position-quadrature operators of the covariance matrix of $\psi_{\text{TMS}}(N_B)_{E'E_1}$ has the following form:

$$
V = \begin{bmatrix}
2N_B + 1 & 2\sqrt{N_B(1 + N_B)} \\
2\sqrt{N_B(1 + N_B)} & 2N_B + 1
\end{bmatrix}.
$$

The action of a complementary channel $\hat{L}_{\eta, N_B}$ on an input state $\phi_{RA}$ is given by

$$
(id_R \otimes \hat{L}_{\eta, N_B})(\phi_{RA}) = \text{Tr}_B\{B_{AE'\rightarrow BE_2}(\phi_{RA} \otimes \psi_{\text{TMS}}(N_B)_{E'E_1})\}.
$$

It can be understood from Figure 1 that the system $R$ is correlated with the input system $A$ for the channel, and the system $E'$ is the environment’s input. The beamsplitter transformation $B$ then leads to systems $B$ and $E_2$. Hence, the output of the thermal channel $L_{\eta, N_B}$ is system $B$, and the outputs of the complementary channel $\hat{L}_{\eta, N_B}$ are systems $E_1$ and $E_2$.

Our aim is to introduce a degrading channel $D$, such that the combined state of $R$ and the output of $D \circ L_{\eta, N_B}$ emulate the combined state of $R$, $E_1$, and $E_2$, to an extent. This will then allow us to bound the diamond distance between $D \circ L_{\eta, N_B}$ and $\hat{L}_{\eta, N_B}$ from above. For the case when there
Figure 1: The figure plots a thermal channel with transmissivity $\eta \in [1/2, 1]$ and a degrading channel as described in (5.14). $\phi_{RA}$ is an input state to the beamsplitter $B$ with transmissivity $\eta$ and $\psi_{TMS}(N_B)$ represents a two-mode squeezed vacuum state with parameter $N_B$. System $B$ is the output of the thermal channel, and systems $E_1E_2$ are the outputs of the complementary channel. The second beamsplitter $B'$ has transmissivity $(1 - \eta)/\eta$, and system $B$ acts as an input to $B'$. Systems $E'_1E'_2$ represent the output systems of the degrading channel, whose action is to tensor in the state $\psi_{TMS}(N_B)_{FE'_1}$, interact the input system $B$ with $F$ according to $B'$, and then trace over system $G$. 
is no thermal noise, i.e., \( N_B = 0 \), a thermal channel reduces to a pure-loss channel. Moreover, we know that a pure-loss channel is a degradable channel and the corresponding degrading channel can be realized by a beamsplitter with transmissivity \( (1 - \eta)/\eta \) [GSE08]. Hence, we consider a degrading channel, such that it also satisfies the conditions for the above described special case.

Consider a beamsplitter with transmissivity \( (1 - \eta)/\eta \) and the beamsplitter transformation \( B' \) from (5.9)-(5.10). As described in Figure 1, the output \( B \) of the thermal channel \( \mathcal{L}_{\eta,N_B} \) becomes an input to the beamsplitter \( B' \). We consider one mode (\( F \) in Figure 1) of the two-mode squeezed vacuum state \( \psi_{\text{TMS}}(N_B)_{FE_1} \) as an environmental input for \( B' \), so that the subsystem \( E_1' \) mimics \( E_1 \). Hence, our choice of degrading channel seems reasonable, as the combined state of system \( R \) and output systems \( E_1', E_2' \) of \( D \circ \mathcal{L}_{\eta,N_B} \) emulates the combined state of \( R, E_1, \) and \( E_2 \), to an extent. We suspect that our choice of degrading channel is a good choice because an upper bound on the energy-constrained quantum capacity of a thermal channel using this technique outperforms all other upper bounds for certain parameter regimes. We denote our choice of degrading channel by \( D_{(1-\eta)/\eta,N_B} \). More formally, \( D_{(1-\eta)/\eta,N_B} \) has the following action on the output state \( \mathcal{L}_{\eta,N_B}(\phi_{RA}) \):

\[
(id_R \otimes [D_{(1-\eta)/\eta,N_B} \circ \mathcal{L}_{\eta,N_B}])(\phi_{RA}) = \text{Tr}_G\{B'_{BF \rightarrow E_2G}(\mathcal{L}_{\eta,N_B}(\phi_{RA}) \otimes \psi_{\text{TMS}}(N_B)_{FE_1'})\}.
\]

(5.14)

Next, we provide a strategy to bound the diamond distance between \( D_{(1-\eta)/\eta,N_B} \circ \mathcal{L}_{\eta,N_B} \) and \( \mathcal{L}_{\eta,N_B} \). Consider the following submatrix corresponding to the position-quadrature operators of the covariance matrix of an input state \( \phi_{RA} \):

\[
\gamma = \begin{bmatrix}
a & c \\
c & b
\end{bmatrix}.
\]

(5.15)

where \( a, b, c \in \mathbb{R} \) are such that the above is the position-quadrature part of a legitimate covariance matrix. Let \( \xi_{RE_2'E_2E_1'G} \) denote the state after the beamsplitter transformations act on an input state \( \phi_{RA} \):

\[
\xi_{RE_2'E_2E_1'G} = B'_{BF \rightarrow E_2G}[B_{AE' \rightarrow BE_2}[\phi_{RA} \otimes \psi_{\text{TMS}}(N_B)_{E_1'}] \otimes \psi_{\text{TMS}}(N_B)_{FE_1'}].
\]

(5.16)

Then the submatrix corresponding to the position-quadrature operators of the covariance matrix of the output state in (5.14) is given by [Not]:

\[
\gamma' = \begin{bmatrix}
a & c\sqrt{1-\eta} & 0 \\
c\sqrt{1-\eta} & b + \eta(1 - b + 2N_B) & 2\sqrt{N_B(1 + N_B)}(2 - 1/\eta) \\
0 & 2\sqrt{N_B(1 + N_B)}(2 - 1/\eta) & 2N_B + 1
\end{bmatrix}.
\]

(5.17)

Now, we introduce a particular channel that simulates the action of \( D_{(1-\eta)/\eta,N_B} \circ \mathcal{L}_{\eta,N_B} \) on an input state \( \phi_{RA} \). We denote this channel by \( \Xi \), and it has the following action on an input state \( \phi_{RA} \):

\[
(id_R \otimes \Xi)(\phi_{RA}) = \text{Tr}_B\{B_{AE' \rightarrow BE_2}[\phi_{RA} \otimes \omega(N_B)_{E_1'}]\},
\]

(5.18)

where \( \omega(N_B)_{E_1'} \) represents a noisy version of a two-mode squeezed vacuum state with parameter \( N_B \) and has the following submatrix corresponding to the position-quadrature operators of the covariance matrix:

\[
\gamma' = \begin{bmatrix}
2N_B + 1 & 2\sqrt{N_B(1 + N_B)(2\eta - 1)}/\eta^2 \\
2\sqrt{[N_B(1 + N_B)(2\eta - 1)]/\eta^2} & 2N_B + 1
\end{bmatrix}.
\]

(5.19)
Figure 2: The figure plots the simulating channel $\Xi$ described in (5.18). $\phi_{RA}$ is an input state to a beamsplitter $B$ with transmissivity $\eta$ and $\omega(N_B)$ represents a noisy version of a two-mode squeezed vacuum state with parameter $N_B$ (see (5.19)), one mode of which is an input to the environment mode of the beamsplitter. The simulating channel is such that system $B$ is traced over, so that the channel outputs are $E_1$ and $E_2$. Finally, the simulating channel is exactly the same as the channel from system $A$ to systems $E'_1E'_2$ in Figure 1.
The matrix $V'$ in (5.19) is a well defined submatrix of the covariance matrix for the noisy version of a two-mode squeezed vacuum state, because $(2\eta - 1)/\eta^2 \in [0, 1]$ for $\eta \in [1/2, 1]$. The submatrix of the covariance matrix corresponding to the state in (5.18) is the same as the submatrix in (5.17) [Not]. In other words, the covariance matrix for the systems $R$, $E'_1$, and $E'_2$ in Figure 1 is exactly the same as the covariance matrix for the systems $R$, $E_1$, and $E_2$ in Figure 2. This equality of covariance matrices is sufficient to conclude that the following equivalence holds for any quantum input state $\phi_R$ (see [Ser17, Chapter 5] for a proof):

$$\left(\text{id}_R \otimes [\mathcal{D}_{(1-\eta)/\eta,N_B} \circ \mathcal{L}_{\eta,N_B}]\right)(\phi_R) = \left(\text{id}_R \otimes \Xi\right)(\phi_R).$$

(5.20)

Thus, the channels $\mathcal{D}_{(1-\eta)/\eta,N_B} \circ \mathcal{L}_{\eta,N_B}$ and $\Xi$ are indeed the same.

From (5.13), (5.18), and (5.20), the action of both $\hat{L}_{\eta,N_B}$ and $\Xi$ can be understood as tensoring the state of the environment with the input state of the channel, performing the beamsplitter transformation $B$, and then tracing out the output of the channels. Using these techniques, we now establish an upper bound on the diamond distance between the complementary channel in (5.13) and the concatenation of the thermal channel followed by the degrading channel in (5.14).

**Theorem 15** Fix $\eta \in [1/2, 1]$. Let $\mathcal{L}_{\eta,N_B}$ be a thermal channel with transmissivity $\eta$, and let $\mathcal{D}_{(1-\eta)/\eta,N_B}$ be a degrading channel as defined in (5.14). Then

$$\frac{1}{2} \left\| \hat{L}_{\eta,N_B} - \mathcal{D}_{(1-\eta)/\eta,N_B} \circ \mathcal{L}_{\eta,N_B} \right\|_1 \leq \sqrt{1 - \eta^2/\kappa(\eta,N_B)},$$

(5.21)

with

$$\kappa(\eta,N_B) = \eta^2 + N_B(N_B + 1)[1 + 3\eta^2 - 2\eta(1 + \sqrt{2\eta - 1})].$$

(5.22)

**Proof.** Consider the following chain of inequalities:

$$\left\| \left(\text{id}_R \otimes \hat{L}_{\eta,N_B}\right)(\phi_R) - \left(\text{id}_R \otimes [\mathcal{D}_{(1-\eta)/\eta,N_B} \circ \mathcal{L}_{\eta,N_B}]\right)(\phi_R) \right\|_1$$

$$= \left\| \left(\text{id}_R \otimes \hat{L}_{\eta,N_B}\right)(\phi_R) - \left(\text{id}_R \otimes \Xi\right)(\phi_R) \right\|_1$$

(5.23)

$$= \left|\text{Tr}_B\{\mathcal{B}_{AE'E_2} \circ \psi_{\text{TMS}}(N_B)_{E'E_1}\} - \mathcal{B}_{AE'E_2} \circ \psi_{\text{TMS}}(N_B)_{E'E_1}\} \right|_1$$

(5.24)

$$\leq \left\| \mathcal{B}_{AE'E_2} \circ \psi_{\text{TMS}}(N_B)_{E'E_1}\} - \mathcal{B}_{AE'E_2} \circ \psi_{\text{TMS}}(N_B)_{E'E_1}\} \right\|_1$$

(5.25)

$$= \left\| \psi_{\text{TMS}}(N_B)_{E'E_1}\} - \phi_R \otimes \omega(N_B)_{E'E_1}\} \right\|_1$$

(5.26)

$$= \left\| \psi_{\text{TMS}}(N_B)_{E'E_1}\} - \omega(N_B)_{E'E_1}\} \right\|_1$$

(5.27)

$$\leq 2\sqrt{1 - F(\psi_{\text{TMS}}(N_B)_{E'E_1},\omega(N_B)_{E'E_1})}$$

(5.28)

The first equality follows from (5.20). The second equality follows from (5.13) and (5.18). The first inequality follows from monotonicity of the trace distance. The third equality follows from the invariance of the trace distance under a unitary transformation (beamsplitter). The last inequality follows from the Powers-Störmer inequality [PS70].

Next, we compute the fidelity between $\psi_{\text{TMS}}(N_B)_{E'E_1}$ and $\omega(N_B)_{E'E_1}$ by using their respective covariance matrices in (5.12) and (5.19), in the Uhlmann fidelity formula for two-mode Gaussian states [MM12]. We find [Not]

$$F(\psi_{\text{TMS}}(N_B)_{E'E_1},\omega(N_B)_{E'E_1}) = \frac{\eta^2}{\eta^2 + N_B(N_B + 1)[1 + 3\eta^2 - 2\eta(1 + \sqrt{2\eta - 1})]}.$$
Since these inequalities hold for any input state $\phi_{RA}$, the final result follows from the definition of the diamond norm. \(\blacksquare\)

**Theorem 16** An upper bound on the quantum capacity of a thermal channel $L_{\eta,N_B}$ with transmissivity $\eta \in [1/2, 1]$, environment photon number $N_B$, and input mean photon-number constraint $N_S$ is given by

$$Q(L_{\eta,N_B}, N_S) \leq Q_{U_2}(L_{\eta,N_B}, N_S) \equiv g(\eta N_S + (1 - \eta)N_B) - g(\zeta_+) - g(\zeta_-) + (2\epsilon' + 4\delta)g([(1 - \eta)N_S + (1 + \eta)N_B]/\delta) + g(\epsilon') + 2h_2(\delta),$$

with

$$\epsilon = \sqrt{1 - \eta^2/\left(\eta^2 + N_B(N_B + 1)[1 + 3\eta^2 - 2\eta(1 + \sqrt{2\eta - 1})]\right)},$$

$$\zeta_\pm = \frac{1}{2}\left(-1 + \sqrt{[(1 + 2N_B)^2 - 2\eta + (1 + 2\eta)^2 \pm 4(\vartheta - N_B)\sqrt{1 + N_B + \vartheta^2 - \varrho}/2}\right),$$

$$\varrho = 4N_B(N_B + 1)(2\eta - 1)/\eta,$$

$$\vartheta = \eta N_B + (1 - \eta)N_S,$$

$$\epsilon' \in (\epsilon, 1], \text{ and } \delta = (\epsilon' - \epsilon)/(1 + \epsilon').$$

**Proof.** From Theorem 15, we have an upper bound on the diamond distance between the complementary channel of the thermal channel and the concatenation of the thermal channel followed by the degrading channel, i.e.,

$$\frac{1}{2}\left\|L_{\eta,N_B} - D_{(1-\eta)/\eta,N_B} \circ L_{\eta,N_B}\right\|_o \leq \sqrt{1 - \eta^2/\left(\eta^2 + N_B(N_B + 1)[1 + 3\eta^2 - 2\eta(1 + \sqrt{2\eta - 1})]\right)} < \epsilon' \leq 1.\ (5.35)$$

Due to the input mean photon number constraint $N_S$, and environment photon number $N_B$ for both $L_{\eta,N_B}$ and $D_{(1-\eta)/\eta,N_B}$, there is a total photon number constraint $(1 - \eta)N_S + (1 + \eta)N_B$ for the average output of $n$ channel uses of both $L_{\eta,N_B}$ and $D_{(1-\eta)/\eta,N_B} \circ L_{\eta,N_B}$. Using these results in Theorem 9, we find the following upper bound on the energy-constrained quantum capacity of a thermal channel:

$$Q(L_{\eta,N_B}, N_S) \leq U_{D_{(1-\eta)/\eta,N_B}}(L_{\eta,N_B}, N_S) \equiv (2\epsilon' + 4\delta)\epsilon' + g(\epsilon') + 2h_2(\delta).$$

Using Proposition 17, we find that the thermal state with mean photon number $N_S$ optimizes the conditional entropy of degradation $U_{D_{(1-\eta)/\eta,N_B}}(L_{\eta,N_B}, N_S)$. For the given thermal channel in (5.11) and the degrading channel in (5.14), we find the following analytical expression [Not]:

$$U_{D_{(1-\eta)/\eta,N_B}}(L_{\eta,N_B}, N_S) = g(\eta N_S + (1 - \eta)N_B) - g(\zeta_+) - g(\zeta_-),$$

with $\zeta_\pm$ defined as in the theorem statement. \(\blacksquare\)

**Proposition 17** Let $L_{\eta,N_B}$ be a thermal channel with transmissivity $\eta \in [1/2, 1]$, environment photon number $N_B$, and input mean photon number constraint $N_S$. Let $D_{(1-\eta)/\eta,N_B}$ be the degrading channel from (5.14). Then the thermal state with mean photon number $N_S$ optimizes the conditional entropy of degradation $U_{D_{(1-\eta)/\eta,N_B}}(L_{\eta,N_B}, N_S)$, defined from (4.2).
Proof. Consider the Stinespring dilation in (5.16) of the degrading channel $\mathcal{D}_{(1-\eta)/\eta,N_B}$ from (5.14), and denote it by $W$. Then according to (4.2),

$$U_{\mathcal{D}_{(1-\eta)/\eta,N_B}}(\mathcal{L}_{\eta,N_B},N_S) = \sup_{\rho: \text{Tr}(\hat{n}\rho) \leq N_S} H(G|E_1'E_2')_{(W \circ \mathcal{L}_{\eta,N_B})(\rho)}.$$  \hspace{1cm} (5.38)

Our aim is to find an input state $\rho$ with a certain photon number $N_t \leq N_S$, such that it maximizes the conditional entropy in (5.38). From the extremality of Gaussian states applied to the conditional entropy [EW07], it suffices to perform the optimization in (5.38) over only Gaussian states.

Now, we argue that for a given input mean photon number $N_t$, a thermal state is the optimal state for the conditional output entropy in (5.38). For a thermal channel and our choice of a degrading channel, a phase rotation on the input state is equivalent to a product of local phase rotations on the outputs. Let us denote the state after the local phase rotations on the outputs by

$$\sigma_{E_2'GE_1'}(\phi) = (e^{i\phi\hat{n}} \otimes e^{i\phi\hat{n}} \otimes e^{-i\phi\hat{n}})(W \circ \mathcal{L}_{\eta,N_B})(\rho)(e^{-i\phi\hat{n}} \otimes e^{-i\phi\hat{n}} \otimes e^{i\phi\hat{n}}),$$  \hspace{1cm} (5.39)

and let

$$\xi_{E_2'GE_1'} = \frac{1}{2\pi} \int_0^{2\pi} d\phi (W \circ \mathcal{L}_{\eta,N_B})(e^{i\phi\hat{n}} \rho e^{-i\phi\hat{n}}).$$  \hspace{1cm} (5.40)

Note that the phase covariance property mentioned above is the statement that the following equality holds for all $\phi \in [0,2\pi)$ [Not]:

$$\sigma_{E_2'GE_1'}(\phi) = (W \circ \mathcal{L}_{\eta,N_B})(e^{i\phi\hat{n}} \rho e^{-i\phi\hat{n}}).$$  \hspace{1cm} (5.41)

Consider the following chain of inequalities for a Gaussian input state $\rho$:

$$H(G|E_1'E_2')_{(W \circ \mathcal{L}_{\eta,N_B})(\rho)} = \frac{1}{2\pi} \int_0^{2\pi} d\phi H(G|E_1'E_2')_{\sigma(\phi)}$$  \hspace{1cm} (5.42)

$$= \frac{1}{2\pi} \int_0^{2\pi} d\phi H(G|E_1'E_2')_{(W \circ \mathcal{L}_{\eta,N_B})(e^{i\phi\hat{n}} \rho e^{-i\phi\hat{n}})}$$  \hspace{1cm} (5.43)

$$\leq H(G|E_1'E_2')_{\xi}$$  \hspace{1cm} (5.44)

$$= H(G|E_1'E_2')_{(W \circ \mathcal{L}_{\eta,N_B})(\theta(N_t))}.$$  \hspace{1cm} (5.45)

The first equality follows from invariance of the conditional entropy under local unitaries. The second equality follows from the phase covariance property of the channel. The inequality follows from concavity of conditional entropy. The last equality follows from linearity of the channel, and the following identity:

$$\theta(N_t) = \frac{1}{2\pi} \int_0^{2\pi} d\phi e^{i\phi\hat{n}} \rho e^{-i\phi\hat{n}}.$$  \hspace{1cm} (5.46)

In (5.46), the state after the phase averaging is diagonal in the number basis, and furthermore, the resulting state has the same photon number $N_t$ as the Gaussian state $\rho$. The thermal state $\theta(N_t)$ is the only Gaussian state of a single mode that is diagonal in the number basis with photon number equal to $N_t$. 

27
Next, we argue that, for a given photon number constraint, a thermal state that saturates the constraint is the optimal state for the conditional output entropy. Let
\[
\tau_{E_1'G'E_2'}(\alpha) = [D(\sqrt{1-\eta\alpha}) \otimes D(\sqrt{2\eta-1}\alpha) \otimes I][W \circ \mathcal{L}_{\eta,N_B}(\theta(N_t))][D(\sqrt{1-\eta\alpha}) \otimes D(\sqrt{2\eta-1}\alpha) \otimes I].
\]
(5.47)

Consider the following chain of inequalities:
\[
H(G|E_1'E_2')(W \circ \mathcal{L}_{\eta,N_B})(\theta(N_t)) = \int d^2 \alpha q_{(N_S-N_t)}(\alpha) H(G|E_1'E_2')(W \circ \mathcal{L}_{\eta,N_B})(\theta(N_t))
\]
(5.48)
\[
= \int d^2 \alpha q_{(N_S-N_t)}(\alpha) H(G|E_1'E_2') \tau(\alpha)
\]
(5.49)
\[
= \int d^2 \alpha q_{(N_S-N_t)}(\alpha) H(G|E_1'E_2')(W \circ \mathcal{L}_{\eta,N_B})(D(\alpha)\theta(N_t)D(\alpha))
\]
(5.50)
\[
\leq H(G|E_1'E_2')(W \circ \mathcal{L}_{\eta,N_B})\theta(N_S),
\]
(5.51)
where \(q_N(\alpha) = \exp\{-|\alpha|^2/N\}/\pi N\) is a complex-centered Gaussian distribution with variance \(N \geq 0\). The first equality follows by placing a probability distribution in front, and the second follows from invariance of the conditional entropy under local unitaries. The third equality follows because the channel is covariant with respect to displacement operators, as reviewed in (3.33). The last inequality follows from concavity of conditional entropy, and from the fact that a thermal state with a higher mean photon number can be realized by random Gaussian displacements of a thermal state with a lower mean photon number, as reviewed in (3.25). Hence, for a given input mean photon number constraint \(N_S\), a thermal state with mean photon number \(N_S\) optimizes the conditional entropy of degradation defined from (4.2).

Remark 18 The arguments used in the proof of Proposition 17 can be employed in more general situations beyond that which is discussed there. The main properties that we need are the following, when the channel involved takes a single-mode input to a multi-mode output:

- The channel should be phase covariant, such that a phase rotation on the input state is equivalent to a product of local phase rotations on the output.
- The channel should be covariant with respect to displacement operators, such that a displacement operator acting on the input state is equivalent to a product of local displacement operators on the output.
- The function being optimized should be invariant with respect to local unitaries and concave in the input state.

If all of the above hold, then we can conclude that the thermal-state input saturating the energy constraint is an optimal input state. We employ this reasoning again in the proof of Theorem 22.

5.3 \(\varepsilon\)-close-degradable bound on the energy-constrained quantum capacity of bosonic thermal channels

In this section, we first establish an upper bound on the diamond distance between a thermal channel and a pure-loss channel. Since a pure-loss channel is a degradable channel, an upper
bound on the energy-constrained quantum capacity of a thermal channel directly follows from Theorem 10.

**Theorem 19** If a thermal channel \( \mathcal{L}_{\eta,N_B} \) and a pure-loss bosonic channel \( \mathcal{L}_{\eta,0} \) have the same transmissivity parameter \( \eta \in [0,1] \), then

\[
\frac{1}{2} \| \mathcal{L}_{\eta,N_B} - \mathcal{L}_{\eta,0} \|_\diamond \leq \frac{N_B}{N_B + 1}.
\]

**Proof.** Let \( \mathcal{B} \) represent the beamsplitter transformation, and let \( \theta_E(N_B) \) and \( \theta_E'(0) \) denote the states of the environment for the thermal channel and pure-loss channel, respectively. For any input state \( \psi_{RA} \) to both thermal and pure-loss channels, the following inequalities hold:

\[
\left\| (\text{id}_R \otimes \mathcal{L}_{\eta,N_B})(\psi_{RA}) - (\text{id}_R \otimes \mathcal{L}_{\eta,0})(\psi_{RA}) \right\|_1 = \left\| \text{Tr}_{E'} \left\{ \mathcal{B}_{AE \rightarrow BE'}(\psi_{RA} \otimes \theta_E(N_B)) - \mathcal{B}_{AE \rightarrow BE'}(\psi_{RA} \otimes \theta_E'(0)) \right\} \right\|_1 \leq \left\| \mathcal{B}_{AE \rightarrow BE'}(\psi_{RA} \otimes \theta_E(N_B)) - \mathcal{B}_{AE \rightarrow BE'}(\psi_{RA} \otimes \theta_E'(0)) \right\|_1 \leq \left\| \theta_E(N_B) - \theta_E'(0) \right\|_1
\]

\[
= \sum_{n=0}^{\infty} \frac{(N_B)^n}{(N_B + 1)^{n+1}} \left| \langle n | - | 0 \rangle \langle 0 | \right|_1 = 2N_B \frac{1}{N_B + 1}.
\]

The first equality follows from the definition of the channel in terms of its environment and a unitary interaction (beam splitter). The first inequality follows from monotonicity of the trace distance. The second equality follows from invariance of the trace distance under a unitary operator (beamsplitter). The last equality follows from basic algebra. Since these inequalities hold for any state \( \psi_{RA} \), the final result follows from the definition of the diamond norm. \( \blacksquare \)

**Remark 20** In [TW16], it has been shown that the optimal strategy to distinguish two quantum thermal channels \( \mathcal{L}_{\eta,N^1_B} \) and \( \mathcal{L}_{\eta,N^2_B} \), each having the same transmissivity parameter \( \eta \), and thermal noises \( N^1_B \) and \( N^2_B \), respectively, is to use a highly squeezed, two-mode squeezed vacuum state \( \psi_{TMS}(N_S)_{RA} \) as input to the channels. According to [TW16, Eq. (35)],

\[
\lim_{N_S \to \infty} F(\sigma_{N^1_B}, \sigma_{N^2_B}) = F(\theta(N^1_B), \theta(N^2_B)),
\]

where \( \sigma_{N^i_B} = (\text{id}_R \otimes \mathcal{L}_{\eta,N^i_B})(\psi_{TMS}(N_S)_{RA}) \), and \( \theta(N_B) \) is a thermal state with mean photon number \( N_B \). Hence, a lower bound on the diamond distance in Theorem 19 is given by

\[
\frac{1}{2} \| \mathcal{L}_{\eta,N_B} - \mathcal{L}_{\eta,0} \|_\diamond \geq 1 - \sqrt{F(\theta(N_B), \theta(0))} = 1 - \frac{1}{\sqrt{N_B + 1}},
\]

where the inequality follows from the Powers-Stormer inequality [PS70]. We also suspect that the upper bound in Theorem 19 is achievable, but we are not aware of a method for computing the trace distance of general quantum Gaussian states, which is what it seems would be needed to verify this suspicion.
Theorem 21 An upper bound on the quantum capacity of a thermal channel $\mathcal{L}_{\eta,N_B}$ with transmissivity $\eta \in [1/2, 1]$, environment photon number $N_B$, and input mean photon number constraint $N_S$ is given by

$$Q(\mathcal{L}_{\eta,N_B}, N_S) \leq Q_{U_1}(\mathcal{L}_{\eta,N_B}, N_S) \equiv g(\eta N_S) - g((1 - \eta)N_B) + (4\varepsilon' + 8\delta)g((\eta N_S + (1 - \eta)N_B)/\delta) + 2g(\varepsilon') + 4h_2(\delta),$$

(5.61)

with $\varepsilon = N_B/(N_B + 1)$, $\varepsilon' \in (\varepsilon, 1]$ and $\delta = (\varepsilon' - \varepsilon)/(1 + \varepsilon')$.

Proof. From Theorem 19, we have that $\frac{1}{2}\|\mathcal{L}_{\eta,N_B} - \mathcal{L}_{\eta,0}\|_o \leq \frac{N_B}{N_B + 1} < \varepsilon' \leq 1$. Due to the input mean photon number constraint $N_S$ for $n$ channel uses, the output mean photon number cannot exceed $\eta N_S + (1 - \eta)N_B$ for the thermal channel and $\eta N_S$ for the pure-loss channel. Hence, there is a photon number constraint $\eta N_S + (1 - \eta)N_B$ for the output of both the thermal and pure-loss channels. Since the pure-loss channel is a degradable channel for $\eta \in [1/2, 1]$ [WPGG07, GSE08], the final result follows directly from Theorem 10. ■

6 Comparison of upper bounds on the energy-constrained quantum capacity of bosonic thermal channels

In this section, we study the closeness of the three different upper bounds when compared to a known lower bound. In particular, we use the following lower bound on the quantum capacity of a thermal channel [HW01, WHG12] and denote it by $Q_L$:

$$Q(\mathcal{L}_{\eta,N_B}, N_S) \geq Q_L(\mathcal{L}_{\eta,N_B}, N_S) \equiv g(\eta N_S + (1 - \eta)N_B) - g([D + (1 - \eta)N_S - (1 - \eta)N_B - 1]/2) - g([D - (1 - \eta)N_S + (1 - \eta)N_B - 1]/2),$$

(6.1)

where

$$D^2 \equiv [(1 + \eta)N_S + (1 - \eta)N_B + 1]^2 - 4\eta N_S(N_S + 1).$$

(6.2)

We start by discussing how close the data-processing bound $Q_{U_1}$ is to the aforementioned lower bound. In particular, we show that the data-processing bound $Q_{U_1}$ can be at most 1.45 bits larger than $Q_L$.

Theorem 22 Let $\mathcal{L}_{\eta,N_B}$ be a thermal channel with transmissivity $\eta \in [1/2, 1]$, environment photon number $N_B$, and input mean photon number constraint $N_S$. Then the following relation holds between the data-processing bound $Q_{U_1}(\mathcal{L}_{\eta,N_B}, N_S)$ in (5.3) and the lower bound $Q_L(\mathcal{L}_{\eta,N_B}, N_S)$ in (6.1) on the energy-constrained quantum capacity of a thermal channel:

$$Q_L(\mathcal{L}_{\eta,N_B}, N_S) \leq Q_{U_1}(\mathcal{L}_{\eta,N_B}, N_S) \leq Q_L(\mathcal{L}_{\eta,N_B}, N_S) + 1/\ln 2.$$

(6.3)

Proof. To prove this result, we first compute the difference between the data-processing bound in (5.3) and the lower bound in (6.1) and show that it is equal to $1/\ln 2$ as $N_S \to \infty$. Next, we prove that the difference is a monotone increasing function with respect to input mean photon number $N_S \geq 0$. Hence, the difference $Q_{U_1}(\mathcal{L}_{\eta,N_B}, N_S) - Q_L(\mathcal{L}_{\eta,N_B}, N_S)$ attains its maximum value in the limit $N_S \to \infty$. We note that a similar statement has been given in [KS13] to bound the classical capacity of a thermal channel, but the details of the approach we develop here are different and are likely to be more broadly applicable to related future questions.
For simplicity, we denote \((1 - \eta)N_B\) as \(Y\), employ the natural logarithm for \(g(x)\), and omit the prefactor \(1/\ln 2\) from all instances of \(g(x)\). We use the following property of the function \(g(x)\): For large \(x\),

\[
g(x) = \ln(x + 1) + 1 + O(1/x),
\]

so that as \(x \to \infty\), the approximation \(g(x) \approx \ln(x + 1) + 1\) holds. Using (6.4), the data-processing bound in (5.3) can be expressed as follows for large \(N_S\):

\[
\ln(Y + 1 + \eta N_S) - \ln(Y + 1 + (Y + 1 - \eta)N_S) + O(1/N_S).
\]

Similarly, the lower bound \(Q_L\) in (6.1) can be expressed as

\[
\ln(1 + \eta N_S + Y) - \ln(1 + D + (1 - \eta)N_S - Y)]/2) - \ln(1 + D - (1 - \eta)N_S + Y)]/2) + O(1/N_S) - 1. \quad (6.6)
\]

Let us denote the difference between \(Q_{U_1}\) and \(Q_L\) by \(\Delta(\mathcal{L}_{\eta,N_B}, N_S)\).

\[
\Delta(\mathcal{L}_{\eta,N_B}, N_S) = Q_{U_1}(\mathcal{L}_{\eta,N_B}, N_S) - Q_L(\mathcal{L}_{\eta,N_B}, N_S).
\]

Then the difference simplifies as

\[
\Delta(\mathcal{L}_{\eta,N_B}, N_S)
= 1 - \ln(Y + 1 + (Y + 1 - \eta)N_S) + \ln([(1 + D)^2 - ((1 - \eta)N_S - Y)^2]/4 + O(1/N_S).
\]

\[
= 1 - \ln(Y + 1 + (Y + 1 - \eta)N_S) + \ln([1 + N_S(1 - \eta + 2Y) + Y + D]/2) + O(1/N_S).
\]

\[
= 1 + \ln([1 + N_S(1 - \eta + 2Y) + Y + D]/[2(Y + 1 + (Y + 1 - \eta)N_S)]) + O(1/N_S).
\]

The second equality follows from the definition of \(D^2\). Next, we show that

\[
\ln([1 + N_S(1 - \eta + 2Y) + Y + D]/[2(Y + 1 + (Y + 1 - \eta)N_S)]) \to 0 \quad (6.11)
\]
as \(N_S \to \infty\), and hence we get the desired result. Consider the following expression and take the limit \(N_S \to \infty\):

\[
\lim_{N_S \to \infty} \frac{1 + N_S(1 - \eta + 2Y) + Y + D}{2(Y + 1 + (Y + 1 - \eta)N_S)} \quad (6.12)
\]

\[
\lim_{N_S \to \infty} \frac{1/N_S + (1 - \eta + 2Y) + Y/N_S + \sqrt{((1 + \eta) + (Y + 1)/N_S)^2 - 4\eta - 4\eta/N_S}}{2((Y + 1)/N_S + (Y + 1 - \eta))^2}
\]

\[
\to \frac{(1 - \eta + 2Y) + 1 - \eta}{2(Y + 1 - \eta)} = 1. \quad (6.14)
\]

Hence, \(\lim_{N_S \to \infty} \Delta(\mathcal{L}_{\eta,N_B}, N_S) = 1\). After incorporating the \(1/\ln 2\) factor, which was omitted earlier for simplicity, we find that the difference between the upper and lower bounds approaches \(1/\ln 2 \approx 1.45\) bits as \(N_S \to \infty\).

Now, we show that the difference \(\Delta(\mathcal{L}_{\eta,N_B}, N_S)\) is a monotone increasing function with respect to input mean photon number \(N_S \geq 0\). Let \(U_{A \to B_1 E_1}^\eta\) and \(V_{B_1 \to B_2 E_2}^G\) denote Stinespring dilations of a quantum channel \(\mathcal{L}_{\eta,0} : A \to B_1\) and a quantum limited amplifier channel \(\mathcal{A}_G : B_1 \to B_2\), respectively. For the energy-constrained quantum capacity of a pure-loss channel, the thermal state as an input is optimal for any fixed energy or input mean photon number constraint \(N_S\) [WHG12]. Moreover, the lower bound in (6.1) is obtained for a thermal state with mean photon
number $N_S$ as input to the channel. Then the action of a thermal channel $\mathcal{L}_{\eta,N_B}$ on an input state $\theta(N_S)$ can be expressed as

$$\mathcal{L}_{\eta,N_B}(\theta(N_S)) = \text{Tr}_{E_1E_2}\{(\text{id}_{E_1} \otimes \mathcal{V}_{B_1 \rightarrow B_2}^\eta) \circ \mathcal{U}^{\eta}_{A \rightarrow B_1E_1}(\theta(N_S))\}.$$  

(6.15)

Consider the following state:

$$\omega_{B_2E_1E_2} = (\text{id}_{E_1} \otimes \mathcal{V}_{B_1 \rightarrow B_2}^\eta) \circ \mathcal{U}^{\eta}_{A \rightarrow B_1E_1}(\theta(N_S)).$$  

(6.16)

Since the data-processing bound $Q_{U_1}(\mathcal{L}_{\eta,N_B}, N_S)$ is equal to the quantum capacity of a pure-loss channel with transmissivity $\eta'$, which in turn is equal to coherent information for this case, (5.3) can also be represented as

$$Q_{U_1}(\mathcal{L}_{\eta,N_B}, N_S) = H(B_2|E_2) - H(E_1).$$  

(6.17)

Similarly, the lower bound can be expressed as

$$Q_L(\mathcal{L}_{\eta,N_B}, N_S) = H(B_2) - H(E_1).$$  

(6.18)

Hence the difference between (6.17) and (6.18) is given by

$$\Delta(\mathcal{L}_{\eta,N_B}, N_S) = H(E_2|B_2) + H(E_2|E_1).$$  

(6.19)

Now, our aim is to show that the conditional entropies in (6.19) are monotone increasing functions of $N_S$. We employ displacement covariance of the channels, and note that this argument is similar to that used in the proof of Proposition 17. Let

$$\sigma_{B_2E_1E_2}(\alpha) = [D(\sqrt{\eta}G\alpha) \otimes I \otimes D(\sqrt{\eta}(G-1)\alpha)] \omega_{B_2E_1E_2} [D^\dagger(\sqrt{\eta}G\alpha) \otimes I \otimes D^\dagger(\sqrt{\eta}(G-1)\alpha)],$$  

(6.20)

$$\tau_{B_2E_1E_2}(\alpha) = [I \otimes D(\sqrt{1-\eta}\alpha) \otimes D(\sqrt{\eta}(G-1)\alpha)] \omega_{B_2E_1E_2} [I \otimes D^\dagger(\sqrt{1-\eta}\alpha) \otimes D^\dagger(\sqrt{\eta}(G-1)\alpha)].$$  

(6.21)

Let $N'_S - N_S \geq 0$, and consider the following chain of inequalities:

$$H(E_2|B_2) + H(E_2|E_1) = \int d^2\alpha q(N'_S - N_S)(\alpha) [H(E_2|B_2) + H(E_2|E_1)]$$  

(6.22)

$$= \int d^2\alpha q(N'_S - N_S)(\alpha) [H(E_2|B_2)_{\sigma(\alpha)} + H(E_2|E_1)_{\tau(\alpha)}]$$  

(6.23)

$$= \int d^2\alpha q(N'_S - N_S)(\alpha) [H(E_2|B_2)_{\langle \mathcal{V}_{G \otimes G'}(\theta(N'_S))D(\alpha)\rangle} + H(E_2|E_1)_{\langle \mathcal{V}_{G \otimes G'}(\theta(N'_S))D(\alpha)\rangle}]$$  

(6.24)

$$\leq H(E_2|B_2)_{\langle \mathcal{V}_{G \otimes G'}(\theta(N'_S))\rangle} + H(E_2|E_1)_{\langle \mathcal{V}_{G \otimes G'}(\theta(N'_S))\rangle}.$$  

(6.25)

The first equality follows by placing a probability distribution in front, and the second follows from invariance of the conditional entropy under local unitaries. The third equality follows because the channel is covariant with respect to displacement operators, as reviewed in (3.33). The last inequality follows from concavity of conditional entropy, and from the fact that a thermal
Figure 3: The figures plot the data-processing bound ($Q_{U_1}$), the $\varepsilon$-degradable bound ($Q_{U_2}$), the $\varepsilon$-close-degradable bound ($Q_{U_3}$) and the lower bound ($Q_L$) on energy-constrained quantum capacity of thermal channels. In each figure, we select certain values of $\eta$ and $N_B$, with the choices indicated above each figure. In all the cases, the data-processing bound $Q_{U_1}$ is close to the lower bound $Q_L$. In (a), for medium transmissivity and low thermal noise, the $\varepsilon$-close-degradable bound is close to the data-processing bound, and they are tighter than the $\varepsilon$-degradable bound. In (b), for medium transmissivity and high thermal noise, only the data-processing bound is close to the lower bound. Also the $\varepsilon$-degradable bound is tighter than the $\varepsilon$-close-degradable bound. In (c), for high transmissivity and low thermal noise, all upper bounds are very near to the lower bound. In (d), for high transmissivity and high noise, the $\varepsilon$-degradable bound is tighter than the $\varepsilon$-close-degradable bound.
state with a higher mean photon number can be realized by random Gaussian displacements of a thermal state with a lower mean photon number, as reviewed in (3.25).

Hence, the difference between the data-processing bound in (5.3) and the lower bound in (6.1) attains its maximum value in the limit $N_S \to \infty$. ■

Next, we perform numerical evaluations to see how close the three different upper bounds are to the lower bound $Q_L$ in (6.1). Since there is a free parameter $\varepsilon'$ in both the $\varepsilon$-degradable bound in (5.30) and the $\varepsilon$-close-degradable bound in (5.61), we optimize these bounds with respect to $\varepsilon'$ [Not]. In Figure 3, we plot the data-processing bound $Q_{U_1}$, the $\varepsilon$-degradable bound $Q_{U_2}$, the $\varepsilon$-close-degradable bound $Q_{U_3}$ and the lower bound $Q_L$ versus $N_S$ for certain values of the transmissivity $\eta$ and thermal noise $N_B$. In particular, we find that the data-processing bound is close to the lower bound $Q_L$ for both low and high thermal noise. This is related to Theorem 22, as the data-processing bound can be at most 1.45 bits larger than the lower bound $Q_L$. In Figure 3(a), we plot for medium transmissivity and low thermal noise. We find that the $\varepsilon$-close-degradable bound is very near to the data-processing bound and is tighter than the $\varepsilon$-degradable bound. In Figure 3(b), we plot for medium transmissivity and high thermal noise. We find that the $\varepsilon$-degradable bound is tighter than the $\varepsilon$-close-degradable bound. In Figure 3(d), we plot for high transmissivity and low thermal noise. In Figure 3(c), we plot for high transmissivity and low thermal noise. We find that all upper bounds are very near to the lower bound $Q_L$. From Figures 3(a) and 3(c), it is evident that in the low-noise regime, there is a strong limitation on any potential super-additivity of coherent information of a thermal channel. Similar results were obtained on quantum and private capacities of low-noise quantum channels in [LLS17]. It is important to stress that the upper bound $Q_{U_3}$ can serve as a good bound only for low values of the thermal noise $N_B$, as the technique to calculate this bound requires the closeness of a thermal channel with a pure-loss channel (discussed in Theorem 19), and the closeness parameter is equal to $N_B/(N_B + 1)$.

In Figure 4, we plot all the upper bounds and the lower bound $Q_L$ versus $N_S$, for high transmissivity and high thermal noise. In Figure 4(a), we find that the $\varepsilon$-degradable bound is tighter than all other bounds for high values of $N_S$. In Figure 4(b), we plot for the same parameter values, but for low values of $N_S$. It is evident that for low input mean photon number, the data-processing bound is tighter than the $\varepsilon$-degradable bound.

The plots suggest that our upper bounds based on the notion of approximate degradability are good for the case of high input mean photon number. We suspect that these bounds can be further improved for the case of low input mean photon number by considering the energy-constrained diamond norm [Shi17, We17]. To address this question, we consider the generalized channel divergences of quantum Gaussian channels in Section 9 and argue about their optimization.

7 Upper bounds on energy-constrained private capacity of bosonic thermal channels

In this section, we provide three different upper bounds on the energy-constrained private capacity of a thermal channel. These upper bounds are derived very similarly as in Section 5. We call these different bounds the data-processing bound, the $\varepsilon$-degradable bound, and the $\varepsilon$-close-degradable bound, and denote them by $P_{U_1}$, $P_{U_2}$, and $P_{U_3}$, respectively.
Figure 4: The figures plot the data-processing bound ($Q_U^1$), the $\varepsilon$-degradable bound ($Q_U^2$), the $\varepsilon$-close-degradable bound ($Q_U^3$) and the lower bound ($Q_L$) on energy-constrained quantum capacity of thermal channels. In each figure, we select $\eta = 0.99$ and $N_B = 0.1$. In (a), the $\varepsilon$-degradable upper bound is tighter than all other upper bounds. In (b), for low values of $N_S$, the data-processing bound is tighter than the $\varepsilon$-degradable bound.
7.1 Data-processing bound on the energy-constrained private capacity of bosonic thermal channels

**Theorem 23** An upper bound on the private capacity of a thermal channel $\mathcal{L}_{\eta,N_B}$ with transmissivity $\eta \in [1/2, 1]$, environment photon number $N_B$, and input mean photon number constraint $N_S$ is given by

$$P(\mathcal{L}_{\eta,N_B}, N_S) \leq \max \{0, P_{U_1}(\mathcal{L}_{\eta,N_B}, N_S)\}$$  \hspace{1cm} (7.1)

$$P_{U_1}(\mathcal{L}_{\eta,N_B}, N_S) \equiv g(\eta N_S) - g((1 - \eta')N_S)$$  \hspace{1cm} (7.2)

with $\eta' = \eta/((1 - \eta)N_B + 1)$.

**Proof.** A proof follows from arguments similar to those in the proof of Theorem 14. Since a pure-loss channel is a degradable channel [WPGG07, GSE08], its energy-constrained private capacity is the same as its energy-constrained quantum capacity [WQ16]. □

7.2 $\varepsilon$-degradable bound on the energy-constrained private capacity of bosonic thermal channels

**Theorem 24** An upper bound on the private capacity of a thermal channel $\mathcal{L}_{\eta,N_B}$ with transmissivity $\eta \in [1/2, 1]$, environment photon number $N_B$, and input mean photon number constraint $N_S$ is given by

$$P(\mathcal{L}_{\eta,N_B}, N_S) \leq P_{U_2}(\mathcal{L}_{\eta,N_B}, N_S) \equiv g(\eta N_S + (1 - \eta)N_B - g(\zeta_+) - g(\zeta_-) + (6\varepsilon' + 12\delta)g([1 - (\eta)N_S + (1 + \eta)N_B]/\delta) + 3g(\varepsilon') + 6h_2(\delta)$$  \hspace{1cm} (7.3)

with

$$\varepsilon = \sqrt{1 - \eta^2 / \left( \eta^2 + N_B(N_B + 1)[1 + 3\eta^2 - 2\eta(1 + \sqrt{2\eta - 1})] \right)}$$  \hspace{1cm} (7.4)

$$\zeta_+ = \frac{1}{2} \left( -1 + \sqrt{[1 + 2N_B]^2 - 2\vartheta + (1 + 2\vartheta)^2 \pm 4(\vartheta - N_B)\sqrt{[1 + N_B + \vartheta]^2 - \vartheta]/2} \right)$$  \hspace{1cm} (7.5)

$$\vartheta = 4N_B(N_B + 1)(2 - 1/\eta)$$  \hspace{1cm} (7.6)

$$\vartheta = \eta N_B + (1 - \eta)N_S$$  \hspace{1cm} (7.7)

$\varepsilon' \in (\varepsilon, 1)$, and $\delta = (\varepsilon' - \varepsilon)/(1 + \varepsilon')$.

**Proof.** A proof follows from arguments similar to those in the proof of Theorem 16. The final result is obtained using Theorem 12. □

7.3 $\varepsilon$-close-degradable bound on the energy-constrained private capacity of bosonic thermal channels

**Theorem 25** An upper bound on the private capacity of a thermal channel $\mathcal{L}_{\eta,N_B}$ with transmissivity $\eta \in [1/2, 1]$, environment photon number $N_B$, and input mean photon number constraint $N_S$ is given by

$$P(\mathcal{L}_{\eta,N_B}, N_S) \leq P_{U_3}(\mathcal{L}_{\eta,N_B}, N_S) \equiv g(\eta N_S) - g((1 - \eta)N_S) + (8\varepsilon' + 16\delta)g([\eta N_S + (1 - \eta)N_B]/\delta) + 4g(\varepsilon') + 8h_2(\delta)$$  \hspace{1cm} (7.8)

with $\varepsilon = N_B/(N_B + 1)$, $\varepsilon' \in (\varepsilon, 1)$, and $\delta = (\varepsilon' - \varepsilon)/(1 + \varepsilon')$. 

36
Proof. A proof follows from arguments similar to those in the proof of Theorem 21. The final result is obtained using Theorem 13. ■

8 Lower bound on energy-constrained private capacity of bosonic thermal channels

In this section, we establish an improvement on the best known lower bound [WHG12] on the energy-constrained private capacity of bosonic thermal channels, by using displaced thermal states as input to the channel. We note that a similar effect has been observed in [RGK05] for the finite-dimensional case.

The energy-constrained private information of a channel $\mathcal{N}$, as defined in (3.59), can also be written as

$$P^{(1)}(\mathcal{N}, G, W) \equiv \sup_{\bar{\rho}_{E_A}} \left[ H(\mathcal{N}(\bar{\rho}_{E_A})) - H(\mathcal{N}(\hat{\rho}_{E_A})) - \int dx \ p_X(x) [H(\mathcal{N}(\rho^x_A)) - H(\hat{\mathcal{N}}(\rho^x_A))] \right] ,$$

(8.1)

where $\bar{\rho}_{E_A} \equiv \int dx \ p_X(x)\rho^x_A$ is an average state of the ensemble $\mathcal{E}_A \equiv \{p_X(x), \rho^x_A\}$ and $\hat{\mathcal{N}}$ denotes a complementary channel of $\mathcal{N}$. If the energy-constrained private information is calculated for coherent-state inputs, then for each element of the ensemble, the following equality holds

$$H(\mathcal{N}(\rho^x_A)) = H(\hat{\mathcal{N}}(\rho^x_A)).$$

Hence, the entropy difference $H(\mathcal{N}(\bar{\rho}_{E_A})) - H(\hat{\mathcal{N}}(\bar{\rho}_{E_A}))$ is an achievable rate, which is the same as the energy-constrained coherent information.

However, we show that displaced thermal-state inputs provide an improved lower bound for certain values of the transmissivity $\eta$, low thermal noise $N_B$, and both low and high input mean photon number $N_S$. We start with the following ensemble of displaced thermal states,

$$\mathcal{E} \equiv \{p_{N_{S_1}^1}(\alpha), D(\alpha) \theta(N_{S_2}^2) D(-\alpha)\},$$

(8.2)

chosen according to the Gaussian probability distribution

$$p_{N_{S_1}^1}(\alpha) = \frac{1}{\pi N_{S_1}^1} \exp(-|\alpha|^2/N_{S_1}^1),$$

(8.3)

where $D(\alpha)$ denotes the displacement operator, $\theta(N_{S_2}^2)$ denotes the thermal state with mean photon number $N_{S_2}^2$, and $N_{S_1}^1$ and $N_{S_2}^2$ are chosen such that $N_{S_1}^1 + N_{S_2}^2 = N_S$, which is the mean number of photons input to the channel. By employing (3.25), the average of this ensemble is a thermal state with mean photon number $N_S$, i.e.,

$$\bar{\rho}_{E} = \int d^2 \alpha \ p_{N_{S_1}^1}(\alpha) D(\alpha) \theta(N_{S_2}^2) D(-\alpha) = \theta(N_S).$$

(8.4)

Hence, this ensemble meets the constraint that the average number of photons input to the channel is equal to $N_S$.

After the action of the channel on one of the states in the ensemble, the entropy of the output state is given by

$$H(\mathcal{L}_{\eta,N_B}(D(\alpha) \theta(N_{S_2}^2) D(-\alpha))) = H(D(\sqrt{\eta}\alpha)\mathcal{L}_{\eta,N_B}(\theta(N_{S_2}^2))D(-\sqrt{\eta}\alpha))$$

$$= H(\mathcal{L}_{\eta,N_B}(\theta(N_{S_2}^2))) ,$$

(8.5)
where the first equality follows because thermal channel is covariant with respect to displacement operators, as reviewed in (3.33). The second equality follows because $D(\sqrt{\eta} \alpha)$ is a unitary operator and entropy is invariant under the action of a unitary operator. Since $H(\mathcal{L}_{\eta,N_B}(\theta(N_S^2)))$ is independent of the Gaussian probability distribution in (8.3),

$$\int d^2 \alpha \, p_{N_S^2}(\alpha) \, H(\mathcal{L}_{\eta,N_B}(\theta(N_S^2))) = H(\mathcal{L}_{\eta,N_B}(\theta(N_S^2))). \quad (8.7)$$

Similar arguments can be made for the output states at the environment mode.

Hence, a lower bound on the energy-constrained private information in (8.1) for the bosonic thermal channel is as follows:

$$P^{(1)}(\mathcal{L}_{\eta,N_B}, N_S) \geq H(\mathcal{L}_{\eta,N_B}(\theta(N_S))) - H(\hat{\mathcal{L}}_{\eta,N_B}(\theta(N_S))) - [H(\mathcal{L}_{\eta,N_B}(\theta(N_S^2))) - H(\hat{\mathcal{L}}_{\eta,N_B}(\theta(N_S^2)))]$$

$$= I_c(\mathcal{L}_{\eta,N_B}, N_S) - I_c(\mathcal{L}_{\eta,N_B}, N_S^2) \equiv P_L(\mathcal{L}_{\eta,N_B}, N_S), \quad (8.8)$$

where $\hat{\mathcal{L}}_{\eta,N_B}$ denotes the complementary channel of $\mathcal{L}_{\eta,N_B}$, and we denote the lower bound in (8.9) on the private information by $P_L(\mathcal{L}_{\eta,N_B}, N_S)$. The first inequality follows from (3.59). Here, $I_c(\mathcal{L}_{\eta,N_B}, N_S)$ denotes the coherent information of the channel for the thermal state with mean photon number $N_S$ as input to the channel. $I_c(\mathcal{L}_{\eta,N_B}, N_S)$ has the same form as (6.1), i.e.,

$$I_c(\mathcal{L}_{\eta,N_B}, N_S) = g(\eta N_S + (1 - \eta) N_B) - g([D + (1 - \eta) N_S - (1 - \eta) N_B - 1]/2)$$

$$- g([D - (1 - \eta) N_S + (1 - \eta) N_B - 1]/2), \quad (8.10)$$

where $D^2 \equiv [(1 + \eta) N_S + (1 - \eta) N_B + 1]^2 - 4\eta N_S(N_S + 1)$. Similarly, $I_c(\mathcal{L}_{\eta,N_B}, N_S^2)$ is defined by replacing $N_S$ in (8.10) with $N_S^2$. 

Figure 5: The figures plot the optimized value of the lower bound on the private information $P_L(\mathcal{L}_{\eta,N_B}, N_S)$ (dashed line) and coherent information $I_c(\mathcal{L}_{\eta,N_B}, N_S)$ (solid line) of a thermal channel versus transmissivity parameter $\eta$. In each figure, we select certain values of thermal noise $N_B$ and input mean photon number $N_S$, with the choices indicated above each figure. In all the cases, there is an improvement in the achievable rate of private communication for certain values of the transmissivity $\eta$. 

38
We optimize the lower bound in (8.9) on the private information $P_L(\mathcal{L}_{\eta,N_B}, N_S)$ with respect to $N_S^2$ for a fixed value of $N_S$ [Not]. In Figure 5, we plot the optimized value of the lower bound in (8.9) on the private information $P_L(\mathcal{L}_{\eta,N_B}, N_S)$ (dashed line) and the coherent information in (8.10) $I_c(\mathcal{L}_{\eta,N_B}, N_S)$ (solid line) of the thermal channel versus the transmissivity parameter $\eta$, for low thermal noise $N_B$ and for both low and high input mean number of photons $N_S$. We find that a larger rate for private communication can be achieved by using displaced thermal states as input to the channel instead of coherent states, for certain values of the transmissivity $\eta$.

9 On the optimization of generalized channel divergences of quantum Gaussian channels

In this section, we address the question of computing the energy-constrained diamond norm of several channels of interest that have appeared in our paper. We provide a very general argument, based on some definitions and results in [LKDW17] and phrased in terms of the “generalized channel divergence” as a measure of the distinguishability of quantum channels. We find that, among all Gaussian input states with a fixed energy constraint, the two-mode squeezed vacuum state saturating the energy constraint is the optimal state for the energy-constrained generalized channel divergence of two particular Gaussian channels. We describe these results in more detail in what follows.

We begin by recalling some developments from [LKDW17]:

Definition 26 (Generalized divergence [SW12, WWY14]) A functional $D : \mathcal{D}(\mathcal{H}) \times \mathcal{D}(\mathcal{H}) \to \mathbb{R}$ is a generalized divergence if it satisfies the monotonicity (data processing) inequality

$$D(\rho\|\sigma) \geq D(N(\rho)\|N(\sigma)), \tag{9.1}$$

where $N$ is a quantum channel.

Particular examples of a generalized divergence are the trace distance, quantum relative entropy, and the negative root fidelity.

We say that a generalized channel divergence possesses the direct-sum property on classical–quantum states if the following equality holds:

$$D\left(\sum_x p_X(x)|x\rangle\langle x| \otimes \rho_x\right)\leq \sum_x p_X(x)D(\rho_x\|\sigma_x), \tag{9.2}$$

where $p_X$ is a probability distribution, $\{|x\rangle\}_x$ is an orthonormal basis, and $\{|\rho_x\rangle\}_x$ and $\{|\sigma_x\rangle\}_x$ are sets of states. We note that this property holds for trace distance, quantum relative entropy, and the negative root fidelity.

Definition 27 (Generalized channel divergence [LKDW17]) Given quantum channels $\mathcal{N}_{A\to B}$ and $\mathcal{M}_{A\to B}$, we define the generalized channel divergence as

$$D(\mathcal{N}\|\mathcal{M}) \equiv \sup_{\rho_{RA}} D((\text{id}_R \otimes \mathcal{N}_{A\to B})(\rho_{RA})\|((\text{id}_R \otimes \mathcal{M}_{A\to B})(\rho_{RA})). \tag{9.3}$$
In the above definition, the supremum is with respect to all mixed states and the reference system \( R \) is allowed to be arbitrarily large. However, as a consequence of purification, data processing, and the Schmidt decomposition, it follows that

\[
D(N\|M) = \sup_{\psi_{RA}} D((\text{id}_R \otimes N_{A\rightarrow B})(\psi_{RA}))((\text{id}_R \otimes M_{A\rightarrow B})(\psi_{RA})),
\]

(9.4)
such that the supremum can be restricted to be with respect to pure states and the reference system \( R \) isomorphic to the channel input system \( A \).

Particular cases of the generalized channel divergence are the diamond norm of the difference of \( N_{A\rightarrow B} \) and \( M_{A\rightarrow B} \) as well as the Rényi channel divergence from [CMW16].

Covariant quantum channels have symmetries that allow us to simplify the set of states over which we need to optimize their generalized channel divergence [Hol02]. Let \( G \) be a finite group, and for every \( g \in G \), let \( g \rightarrow U_A(g) \) and \( g \rightarrow V_B(g) \) be unitary representations acting on the input and output spaces of the channel, respectively. Then a quantum channel \( N_{A\rightarrow B} \) is covariant with respect to \( \{(U_A(g), V_B(g))\}_{g} \) if the following relation holds for all input density operators \( \rho_A \) and group elements \( g \in G \):

\[
(N_{A\rightarrow B} \circ U_A^g)(\rho_A) = (V_B^g \circ N_{A\rightarrow B})(\rho_A),
\]

(9.5)
where

\[
U_A^g(\rho_A) = U_A(g)\rho_AU_A^\dagger(g),
\]

(9.6)
\[
V_B^g(\sigma_B) = V_B(g)\sigma_BV_B^\dagger(g).
\]

(9.7)
We say that channels \( N_{A\rightarrow B} \) and \( M_{A\rightarrow B} \) are jointly covariant with respect to \( \{(U_A(g), V_B(g))\}_{g} \) if each of them is covariant with respect to \( \{(U_A(g), V_B(g))\}_{g} \) [TW16, DW17].

The following lemma was established in [LKD17]:

**Lemma 4 ([LKD17])** Let \( N_{A\rightarrow B} \) and \( M_{A\rightarrow B} \) be quantum channels, and let \( \{(U_A(g), V_B(g))\}_{g} \) denote unitary representations of a group \( G \). Let \( \rho_A \) be a density operator, and let \( \phi_{RA}^\rho \) be a purification of \( \rho_A \). Let \( \tilde{\rho}_A \) denote the group average of \( \rho_A \) according to a distribution \( p_G \), i.e.,

\[
\tilde{\rho}_A = \sum_g p_G(g) U_A^g(\rho_A),
\]

(9.8)
and let \( \phi_{RA}^\tilde{\rho} \) be a purification of \( \tilde{\rho}_A \). If the generalized divergence possesses the direct-sum property on classical–quantum states, then the following inequality holds

\[
D(N_{A\rightarrow B}(\phi_{RA}^\tilde{\rho})\|M_{A\rightarrow B}(\phi_{RA}^\tilde{\rho})) \geq \sum_g p_G(g) D\left(\left(V_B^{g\dagger} \circ N_{A\rightarrow B} \circ U_A^g\right)(\phi_{RA}^\rho)\left|\left|\left(V_B^{g\dagger} \circ M_{A\rightarrow B} \circ U_A^g\right)(\phi_{RA}^\rho)\right\right.\right).
\]

(9.9)

By approximation, the above lemma can be extended to continuous groups for several generalized channel divergences of interest:
Lemma 5 Let $N_{A\to B}$ and $M_{A\to B}$ be quantum channels, and let $\{(U_{A}(g), V_{B}(g))\}_{g \in G}$ be unitary representations of a continuous group $G$. Let $\rho_A$ be a density operator, and let $\phi^\rho_{RA}$ be a purification of $\rho_A$. Let $\bar{\rho}_A$ denote the group average of $\rho_A$ according to a measure $\mu(g)$, i.e.,

$$\bar{\rho}_A = \int d\mu(g) \ U_A^\dagger(\rho_A),$$

(9.10)

and let $\phi^\rho_{RA}$ be a purification of $\bar{\rho}_A$. If the generalized divergence possesses the direct-sum property on classical–quantum states and is a Borel function, then the following inequality holds

$$D(N_{A\to B}(\phi^\rho_{RA}) \| M_{A\to B}(\phi^\rho_{RA})) \geq \int d\mu(g) \ D\left( (Y_B^{\dagger} \circ N_{A\to B} \circ U_A^\dagger)(\phi^\rho_{RA}) \| (Y_B^{\dagger} \circ M_{A\to B} \circ U_A^\dagger)(\phi^\rho_{RA}) \right).$$

(9.11)

We can apply this lemma effectively in the context of quantum Gaussian channels. To this end, we consider an energy-constrained generalized channel divergence for $W \in [0, \infty)$ and an energy observable $G$ as follows:

$$D_{G,W}(N \| M) = \sup_{\psi_{RA} : \text{Tr}(G\psi_A) \leq W} D((id_R \otimes N_{A\to B})(\psi_{RA}) \| (id_R \otimes M_{A\to B})(\psi_{RA})).$$

(9.12)

In what follows, we specialize this measure even further to the Gaussian energy-constrained generalized channel divergence, meaning that the optimization is constrained to be with respect to Gaussian input states:

$$D_{G,W}^G(N \| M) = \sup_{\psi_{RA} : \text{Tr}(G\psi_A) \leq W, \psi_{RA} \in G} D((id_R \otimes N_{A\to B})(\psi_{RA}) \| (id_R \otimes M_{A\to B})(\psi_{RA})).$$

(9.13)

where $G$ denotes the set of Gaussian states. We then establish the following proposition:

Proposition 28 Suppose that channels $N_{A\to B}$ and $M_{A\to B}$ are Gaussian, they each take one input mode to $m$ output modes, and they have the following action on a single-mode, input covariance matrix $V$:

$$V \to XV^TX^T + Y_N,$$

(9.14)

$$V \to XV^TX^T + Y_M,$$

(9.15)

where $X$ is an $m \times 1$ matrix, $Y_N$ and $Y_M$ are $m \times m$ matrices such that $N_{A\to B}$ and $M_{A\to B}$ are legitimate Gaussian channels. Suppose furthermore they these channels are jointly phase covariant (phase-insensitive), in the sense that for all $\phi \in [0, 2\pi)$ and input density operators $\rho$, the following equality holds

$$N_{A\to B}(e^{i\phi} \rho e^{-i\phi}) = \left( \bigotimes_{i=1}^m e^{i\hat{n}_i(-1)^{a_i}\phi} \right) N_{A\to B}(\rho) \left( \bigotimes_{i=1}^m e^{-i\hat{n}_i(-1)^{a_i}\phi} \right),$$

(9.16)

$$M_{A\to B}(e^{i\phi} \rho e^{-i\phi}) = \left( \bigotimes_{i=1}^m e^{i\hat{n}_i(-1)^{a_i}\phi} \right) M_{A\to B}(\rho) \left( \bigotimes_{i=1}^m e^{-i\hat{n}_i(-1)^{a_i}\phi} \right),$$

(9.17)

where $a_i \in \{0, 1\}$ for $i \in \{1, \ldots, m\}$ and $\hat{n}_i$ is the photon number operator for the $i$th mode. Then the Gaussian energy-constrained generalized channel divergence is achieved by the two-mode squeezed vacuum state with parameter $N_S$, i.e.,

$$D_{G,N_S}^G(N \| M) = D((id_R \otimes N_{A\to B})(\psi_{\text{TMS}}(N_S)) \| (id_R \otimes M_{A\to B})(\psi_{\text{TMS}}(N_S))).$$

(9.18)
\textbf{Proof.} This result is an application of Lemma 5 and previous developments in our paper. Let $\psi_{RA}$ denote an arbitrary pure Gaussian state of two modes such that \( \text{Tr}\{\hat{n}\psi_A\} = N_1 \leq N_S \). Consider that
\[
\psi_A \equiv \frac{1}{2\pi} \int_0^{2\pi} d\phi \ e^{i\hat{n}\phi}\psi_A e^{-i\hat{n}\phi} = \sum_{n=0}^\infty |n\rangle \langle n| \psi_A |n\rangle |n\rangle,
\]
\[\tag{9.19}\]
i.e., the state after phase averaging is diagonal in the number basis, and furthermore, the resulting state has the same photon number $N_1$ as $\psi_A$. The thermal state $\theta(N_1)$ is the only Gaussian state of a single mode that is diagonal in the number basis with photon number equal to $N_1$. A purification of the thermal state $\theta(N_1)$ is the two-mode squeezed vacuum $\psi_{\text{TMS}}(N_1)$ with parameter $N_1$.

By applying Lemma 5 and the joint phase covariance relations in (9.16)–(9.17), we find that the following inequality holds
\[
\mathcal{D}(\mathcal{N}_{A\rightarrow B}(\psi_{\text{TMS}}(N_1))\mathcal{M}_{A\rightarrow B}(\psi_{\text{TMS}}(N_1))) \geq \mathcal{D}(\mathcal{N}_{A\rightarrow B}(\psi_{RA})\mathcal{M}_{A\rightarrow B}(\psi_{RA})). \tag{9.20}
\]
So this means that, for a fixed photon number $N_1$, the two-mode squeezed vacuum with parameter $N_1$ is optimal among all Gaussian states with reduced state on the channel input having the same photon number.

It remains to show that the quantity only increases as the photon number increases. To show this, consider that the following relation holds
\[
\theta(N_S) = \int d^2\alpha \ p_{\text{SN}}(\alpha) \ D(\alpha)\theta(N_1)D(-\alpha), \tag{9.21}
\]
where $D(\alpha)$ is a displacement operator and $p_{\text{SN}}(\alpha) = \exp\{-|\alpha|^2/N\}/\pi N$ is a complex-centered Gaussian distribution with variance $N \geq 0$. The fact that channels $\mathcal{N}_{A\rightarrow B}$ and $\mathcal{M}_{A\rightarrow B}$ have the same $X$ matrix as in (9.14)–(9.15) implies that they are jointly covariant with respect to displacements; i.e., for all input density operators $\rho$ and displacements $D(\alpha)$, the following equalities hold
\[
\mathcal{N}_{A\rightarrow B}(D(\alpha)\rho D(-\alpha)) = \left( \bigotimes_{i=1}^m D(f_i(X)\alpha) \right) \mathcal{N}_{A\rightarrow B}(\rho) \left( \bigotimes_{i=1}^m D(-f_i(X)\alpha) \right), \tag{9.22}
\]
\[
\mathcal{M}_{A\rightarrow B}(D(\alpha)\rho D(-\alpha)) = \left( \bigotimes_{i=1}^m D(f_i(X)\alpha) \right) \mathcal{M}_{A\rightarrow B}(\rho) \left( \bigotimes_{i=1}^m D(-f_i(X)\alpha) \right), \tag{9.23}
\]
where $f_i$ for $i \in \{1, \ldots, m\}$ are functions depending on the entries of the matrix $X$. We can then exploit the joint covariance of the channels with respect to displacements, the observation in (9.21), and Lemma 5 to conclude that
\[
\mathcal{D}(\mathcal{N}_{A\rightarrow B}(\psi_{\text{TMS}}(N_S))\mathcal{M}_{A\rightarrow B}(\psi_{\text{TMS}}(N_S))) \geq \mathcal{D}(\mathcal{N}_{A\rightarrow B}(\psi_{\text{TMS}}(N_1))\mathcal{M}_{A\rightarrow B}(\psi_{\text{TMS}}(N_1))), \tag{9.24}
\]
for all $N_1 \leq N_S$. This concludes the proof. $\blacksquare$

The above proposition applies to the various settings and channels that we have considered in this paper for $\varepsilon$-degradable and $\varepsilon$-close degradable bosonic thermal channels. Thus, we can conclude in these situations that the Gaussian energy-constrained generalized channel divergence is achieved by the two-mode squeezed vacuum state.
Particular generalized channel divergences of interest are the energy-constrained diamond norm [Shi17, We17] and the energy-constrained, channel version of the C-distance [Ras02, Ras03, GLN05, Ras06], respectively defined as

\[ \| \mathcal{N}_{A \rightarrow B} - \mathcal{M}_{A \rightarrow B} \|_{G,W} \equiv \sup_{\psi_{RA} : \text{Tr}(G\psi_{A}) \leq W} \| (\text{id}_R \otimes \mathcal{N}_{A \rightarrow B})(\psi_{RA}) - (\text{id}_R \otimes \mathcal{M}_{A \rightarrow B})(\psi_{RA}) \|_1, \]

(9.25)

\[ C_{G,W}(\mathcal{N}_{A \rightarrow B}, \mathcal{M}_{A \rightarrow B}) \equiv \sup_{\psi_{RA} : \text{Tr}(G\psi_{A}) \leq W} \sqrt{1 - F((\text{id}_R \otimes \mathcal{N}_{A \rightarrow B})(\psi_{RA}),(\text{id}_R \otimes \mathcal{M}_{A \rightarrow B})(\psi_{RA}))}, \]

(9.26)

where \( F \) denotes the quantum fidelity. Proposition 28 implies that the Gaussian-constrained versions of these quantities reduce to the following for channels satisfying the assumptions stated there:

\[ \| \mathcal{N}_{A \rightarrow B} - \mathcal{M}_{A \rightarrow B} \|_{0,G,N_{S}}^G = \| (\text{id}_R \otimes \mathcal{N}_{A \rightarrow B})(\psi_{\text{TMS}(N_{S})}) - (\text{id}_R \otimes \mathcal{M}_{A \rightarrow B})(\psi_{\text{TMS}(N_{S})}) \|_1, \]

(9.27)

\[ C_{0,N_{S}}^G(\mathcal{N}_{A \rightarrow B}, \mathcal{M}_{A \rightarrow B}) = \sqrt{1 - F((\text{id}_R \otimes \mathcal{N}_{A \rightarrow B})(\psi_{\text{TMS}(N_{S})}),(\text{id}_R \otimes \mathcal{M}_{A \rightarrow B})(\psi_{\text{TMS}(N_{S})}))}. \]

(9.28)

We note that the latter quantity is readily expressed as a closed formula in terms of the Gaussian specification of the channels \( \mathcal{N}_{A \rightarrow B} \) and \( \mathcal{M}_{A \rightarrow B} \) in (9.14)–(9.15) and the parameter \( N_{S} \) by employing the general formula for the fidelity of zero-mean Gaussian states from [PS00]. One could also employ the formulas from [SLW17] or [Che05, Kru06] to compute Gaussian, energy-constrained channel divergences based on Rényi relative entropy or quantum relative entropy, respectively.

It is a very interesting open question to determine whether, under the conditions given in the above proposition, the energy-constrained generalized channel divergence is always achieved by the two-mode squeezed vacuum state (if the restriction to Gaussian input states is lifted). Divergences of interest are the trace distance, fidelity, quantum relative entropy, and Rényi relative entropies. All of these measures lead to a very interesting suite of Gaussian optimizer questions, which we leave for future work. If there is a positive answer to this question, then we would expect to see, in the low-photon-number regime, significant improvements of the \( \varepsilon \)-degradable and \( \varepsilon \)-close degradable upper bounds on the capacities of the thermal channel.

10 Conclusion

In this paper, we established three different upper bounds on the energy-constrained quantum capacity of thermal channels. We discussed the closeness of these three upper bounds with a known lower bound. In particular, we have shown that the \( \varepsilon \)-close degradable bound works well only in the low-noise regime and that the data-processing upper bound is close to a lower bound for both low and high thermal noise. Moreover, we found that the data-processing bound can be at most 1.45 bits larger than a known lower bound. We also discussed an interesting case where the \( \varepsilon \)-degradable bound is tighter than all other upper bounds. Also, our results establish strong limitations on any potential superadditivity of coherent information of a thermal channel in the low-noise regime.

Similarly, we established three different upper bounds on the energy-constrained private capacity of thermal channels. We have also shown an improvement in the achievable rates of private
communication through quantum thermal channels by using displaced thermal states as inputs to the channel.

Since thermal noise is present in almost all communication and optical systems, our results have implications for quantum computing and quantum cryptography. The knowledge of bounds on quantum capacity can be useful to quantify the performance of distributed quantum computation between remote locations, and private communication rates are connected to the ability to generate secret key.

We finally used the generalized channel divergence from [LKDW17] to address the question of optimal input states for the energy-bounded diamond norm and other related divergences. In particular, we showed that for two Gaussian channels that are jointly phase covariant, the Gaussian energy-constrained generalized channel divergence is achieved by a two-mode squeezed vacuum state that saturates the energy constraint. It is an interesting open question to determine whether, among all input states, the two-mode squeezed vacuum is the optimal input state for several energy-constrained, generalized channel divergences of interest.

As another task for future work, it would be good to extend the results of our paper to quantum amplifier channels and additive-noise Gaussian channels, and we note that many of the methods used and developed in our paper can be applied.

Acknowledgements. We thank Kenneth Goodenough, Saikat Guha, Felix Leditzky, Iman Marvian, Ty Volkoff, Christian Weedbrook, and Andreas Winter for discussions related to this paper. We also acknowledge the catalyzing role of the open problems session at Beyond i.i.d. 2015 (Banff International Research Station, Banff, Canada) in which the question of applying approximate degradability to bosonic channel capacities was raised. KS acknowledges support from the Department of Physics and Astronomy at LSU. MMW thanks NICT for hosting him during Dec. 2015 and acknowledges support from the Office of Naval Research and the National Science Foundation. SA acknowledges support from the ARO, AFOSR, DARPA, NSF, and NGAS. MT acknowledges the Open Partnership Joint Projects of JSPS Bilateral Joint Research Projects and the ImPACT Program of Council for Science, Technology, and Innovation, Japan.

References

[Ali04] Robert Alicki. Isotropic quantum spin channels and additivity questions. February 2004. arXiv:quant-ph/0402080.

[BV13] Valentina Baccetti and Matt Visser. Infinite Shannon entropy. Journal of Statistical Mechanics: Theory and Experiment, 2013(04):P04010, 2013.

[Cav82] Carlton M. Caves. Quantum limits on noise in linear amplifiers. Physical Review D, 26(8):1817, October 1982.

[CEGH08] Filippo Caruso, Jens Eisert, Vittorio Giovannetti, and Alexander S. Holevo. Multi-mode bosonic Gaussian channels. New Journal of Physics, 10(8):083030, August 2008. arXiv:0804.0511.

[CEM+15] Toby Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García, and Sergii Strelchuk. Unbounded number of channel uses are required to see quantum capacity. Nature Communications, 6:7739, March 2015. arXiv:1408.5115.
[CGH06] Filippo Caruso, Vittorio Giovannetti, and Alexander S. Holevo. One-mode bosonic Gaussian channels: a full weak-degradability classification. *New Journal of Physics*, 8(12):310, December 2006. arXiv:quant-ph/0609013.

[Che05] Xiao-yu Chen. Gaussian relative entropy of entanglement. *Physical Review A*, 71(6):062320, June 2005. arXiv:quant-ph/0402109.

[CMW16] Tom Cooney, Milan Mosonyi, and Mark M. Wilde. Strong converse exponents for a quantum channel discrimination problem and quantum-feedback-assisted communication. *Communications in Mathematical Physics*, 344(3):797–829, June 2016. arXiv:1408.3373.

[CWY04] Ning Cai, Andreas Winter, and Raymond W. Yeung. Quantum privacy and quantum wiretap channels. *Problems of Information Transmission*, 40(4):318–336, October 2004.

[Dev05] Igor Devetak. The private classical capacity and quantum capacity of a quantum channel. *IEEE Transactions on Information Theory*, 51(1):44–55, January 2005. arXiv:quant-ph/0304127.

[DS05] Igor Devetak and Peter W. Shor. The capacity of a quantum channel for simultaneous transmission of classical and quantum information. *Communications in Mathematical Physics*, 256(2):287–303, June 2005. arXiv:quant-ph/0311131.

[DSS98] David P. DiVincenzo, Peter W. Shor, and John A. Smolin. Quantum-channel capacity of very noisy channels. *Physical Review A*, 57(2):830–839, February 1998. arXiv:quant-ph/9706061.

[DW17] Siddhartha Das and Mark M. Wilde. Quantum reading capacity: General definition and bounds. March 2017. arXiv:1703.03706.

[ES15] David Elkouss and Sergii Strelchuk. Superadditivity of private information for any number of uses of the channel. *Physical Review Letters*, 115(4):040501, July 2015. arXiv:1502.05326.

[EW07] Jens Eisert and Michael M. Wolf. *Quantum Information with Continuous Variables of Atoms and Light*, chapter Gaussian Quantum Channels, pages 23–42. World Scientific, February 2007. arXiv:quant-ph/0505151.

[GLN05] Alexei Gilchrist, Nathan K. Langford, and Michael A. Nielsen. Distance measures to compare real and ideal quantum processes. *Physical Review A*, 71(6):062310, June 2005. arXiv:quant-ph/0408063.

[GPNBL+12] Raul Garcia-Patron, Carlos Navarrete-Benloch, Seth Lloyd, Jeffrey H. Shapiro, and Nicolas J. Cerf. Majorization theory approach to the Gaussian channel minimum entropy conjecture. *Physical Review Letters*, 108(11):110505, March 2012. arXiv:1111.1986.
[GSE08] Saikat Guha, Jeffrey H. Shapiro, and Baris I. Erkmen. Capacity of the bosonic wiretap channel and the entropy photon-number inequality. In Proceedings of the IEEE International Symposium on Information Theory, pages 91–95, Toronto, Ontario, Canada, 2008. IEEE. arXiv:0801.0841.

[Hay06] Masahito Hayashi. Quantum information: An introduction. Springer, 2006.

[Hol02] Alexander S. Holevo. Remarks on the classical capacity of quantum channel. December 2002. quant-ph/0212025.

[Hol04] Alexander S. Holevo. Entanglement-assisted capacities of constrained quantum channels. Theory of Probability & Its Applications, 48(2):243–255, July 2004. arXiv:quant-ph/0211170.

[Hol10] Alexander S. Holevo. The entropy gain of infinite-dimensional quantum evolutions. Doklady Mathematics, 82(2):730–731, October 2010.

[Hol11a] Alexander S. Holevo. Entropy gain and the Choi-Jamiolkowski correspondence for infinite-dimensional quantum evolutions. Theoretical and Mathematical Physics, 166(1):123–138, January 2011.

[Hol11b] Alexander S. Holevo. The entropy gain of quantum channels. In 2011 IEEE International Symposium on Information Theory Proceedings, pages 289–292, July 2011.

[Hol12] Alexander S. Holevo. Quantum Systems, Channels, Information. de Gruyter Studies in Mathematical Physics (Book 16). de Gruyter, November 2012.

[HS10] Alexander S. Holevo and Maksim E. Shirokov. Mutual and coherent information for infinite-dimensional quantum channels. Problems of Information Transmission, 46(3):201–218, September 2010. arXiv:1004.2495.

[HW01] Alexander S. Holevo and Reinhard F. Werner. Evaluating capacities of bosonic Gaussian channels. Physical Review A, 63(3):032312, February 2001. arXiv:quant-ph/9912067.

[HZ11] Teiko Heinosaari and Márió Ziman. The mathematical language of quantum theory: from uncertainty to entanglement. Cambridge University Press, 2011.

[Kit97] Alexei Y. Kitaev. Quantum computations: algorithms and error correction. Russian Mathematical Surveys, 52(6):1191–1249, December 1997.

[Kru06] Ole Krueger. Quantum Information Theory with Gaussian Systems. PhD thesis, Technische Universität Braunschweig, April 2006. Available at https://publikationsserver.tu-braunschweig.de/receive/dbbs.mods_00020741.

[KS13] Robert König and Graeme Smith. Classical capacity of quantum thermal noise channels to within 1.45 bits. Physical Review Letters, 110(4):040501, January 2013. arXiv:1207.0256.
[Kuz11] Anna A. Kuznetsova. Conditional entropy for infinite-dimensional quantum systems. *Theory of Probability & Its Applications*, 55(4):709–717, November 2011. arXiv:1004.4519.

[LDTBG05] Jérôme Lodewyck, Thierry Debuisschert, Rosa Tualle-Brouri, and Philippe Grangier. Controlling excess noise in fiber-optics continuous-variable quantum key distribution. *Physical Review A*, 72(5):050303, November 2005. arXiv:quant-ph/0511104.

[Lin73] Göran Lindblad. Entropy, information and quantum measurements. *Communications in Mathematical Physics*, 33(4):305–322, 1973.

[Lin75] Göran Lindblad. Completely positive maps and entropy inequalities. *Communications in Mathematical Physics*, 40(2):147–151, 1975.

[LKDW17] Felix Leditzky, Eneet Kaur, Nilanjana Datta, and Mark M. Wilde. 2017. manuscript in preparation.

[Llo97] Seth Lloyd. Capacity of the noisy quantum channel. *Physical Review A*, 55(3):1613–1622, March 1997. arXiv:quant-ph/9604015.

[LLS17] Felix Leditzky, Debbie Leung, and Graeme Smith. Quantum and private capacities of low-noise channels. May 2017. arXiv:1705.04335.

[LR73a] Elliott H. Lieb and Mary Beth Ruskai. A fundamental property of quantum-mechanical entropy. *Physical Review Letters*, 30(10):434–436, March 1973.

[LR73b] Elliott H. Lieb and Mary Beth Ruskai. Proof of the strong subadditivity of quantum-mechanical entropy. *Journal of Mathematical Physics*, 14(12):1938–1941, December 1973.

[LS09] Debbie Leung and Graeme Smith. Continuity of quantum channel capacities. *Communications in Mathematical Physics*, 292(1):201–215, November 2009. arXiv:0810.4931.

[MM12] Paulina Marian and Tudor A. Marian. Uhlmann fidelity between two-mode Gaussian states. *Physical Review A*, 86(2):022340, August 2012. arXiv:1111.7067.

[NH04] Ryo Namiki and Takuya Hirano. Practical limitation for continuous-variable quantum cryptography using coherent states. *Physical Review Letters*, 92(11):117901, March 2004. arXiv:quant-ph/0403115.

[Not] Mathematica files are available in the source files of our arxiv post.

[PS70] Robert T. Powers and Erling Størmer. Free states of the canonical anticommutation relations. *Communications in Mathematical Physics*, 16(1):1–33, 1970.

[PS00] Gh.-S. Paraoanu and Horia Scutaru. Fidelity for multimode thermal squeezed states. *Physical Review A*, 61(2):022306, January 2000. arXiv:quant-ph/9907068.

[QW17] Haoyu Qi and Mark M. Wilde. Capacities of quantum amplifier channels. *Physical Review A*, 95(1):012339, January 2017. arXiv:1605.04922.
[Ras02] Alexey E. Rastegin. Relative error of state-dependent cloning. *Physical Review A*, 66(4):042304, October 2002.

[Ras03] Alexey E. Rastegin. A lower bound on the relative error of mixed-state cloning and related operations. *Journal of Optics B: Quantum and Semiclassical Optics*, 5(6):S647, December 2003. arXiv:quant-ph/0208159.

[Ras06] Alexey E. Rastegin. Sine distance for quantum states. February 2006. arXiv:quant-ph/0602112.

[RGK05] Renato Renner, Nicolas Gisin, and Barbara Kraus. Information-theoretic security proof for quantum-key-distribution protocols. *Physical Review A*, 72(1):012332, July 2005. arXiv:quant-ph/0502064.

[RGR+17] Filip Rozpedek, Kenneth Goodenough, Jeremy Ribeiro, Norbert Kalb, Valentina Caprara Vivoli, Andreas Reiserer, Ronald Hanson, Stephanie Wehner, and David Elkouss. Realistic parameter regimes for a single sequential quantum repeater. May 2017. arXiv:1705.00043.

[Ser17] Alessio Serafini. *Quantum Continuous Variables: A Primer of Theoretical Methods*. CRC Press, 2017.

[SH08] Maxim E. Shirokov and Alexander S. Holevo. On approximation of infinite-dimensional quantum channels. *Problems of Information Transmission*, 44(2):3–22, 2008. arXiv:0711.2245.

[Sha09] Jeffrey H. Shapiro. The quantum theory of optical communications. *IEEE Journal of Selected Topics in Quantum Electronics*, 15(6):1547–1569, November 2009.

[Shi15] Maksim E. Shirokov. Measures of quantum correlations in infinite-dimensional systems. *Sbornik: Mathematics*, 207(5):724, 2015. arXiv:1506.06377.

[Shi16] Maksim E. Shirokov. Squashed entanglement in infinite dimensions. *Journal of Mathematical Physics*, 57(3):032203, March 2016. arXiv:1507.08964.

[Shi17] Maksim E. Shirokov. Energy-constrained diamond norms and their use in quantum information theory. June 2017. arXiv:1706.00361.

[Sho02] Peter W. Shor. The quantum channel capacity and coherent information. In *lecture notes, MSRI Workshop on Quantum Computation*, 2002.

[SLW17] Kaushik P. Seshadreesan, Ludovico Lami, and Mark M. Wilde. Renyi relative entropies of quantum Gaussian states. June 2017. arXiv:1706.09885.

[SMD94] R. Simon, N. Mukunda, and Biswadeb Dutta. Quantum-noise matrix for multimode systems: $U(n)$ invariance, squeezing, and normal forms. *Physical Review A*, 49(3):1567–1583, March 1994.

[Smi08] Graeme Smith. Private classical capacity with a symmetric side channel and its application to quantum cryptography. *Physical Review A*, 78(2):022306, August 2008. arXiv:0705.3838.
Benjamin Schumacher and Michael A. Nielsen. Quantum data processing and error correction. *Physical Review A*, 54(4):2629–2635, October 1996. arXiv:quant-ph/9604022.

Graeme Smith, Joseph M. Renes, and John A. Smolin. Structured codes improve the Bennett-Brassard-84 quantum key rate. *Physical Review Letters*, 100(17):170502, April 2008. arXiv:quant-ph/0607018.

Graeme Smith and John A. Smolin. Degenerate quantum codes for Pauli channels. *Physical Review Letters*, 98(3):030501, January 2007. arXiv:quant-ph/0604107.

Graeme Smith and John A. Smolin. An exactly solvable model for quantum communications. *Nature*, 504:263–267, December 2013. arXiv:1211.1956.

Graeme Smith and John A. Smolin. Degenerate quantum codes for Pauli channels. *Physical Review Letters*, 98(3):030501, January 2007. arXiv:quant-ph/0604107.

David Sutter, Volkher B. Scholz, Andreas Winter, and Renato Renner. Approximate degradable quantum channels. December 2014. arXiv:1412.0980.

Graeme Smith, John A. Smolin, and Jon Yard. Quantum communication with Gaussian channels of zero quantum capacity. *Nature Photonics*, 5:624–627, August 2011. arXiv:1102.4580.

William F. Stinespring. Positive functions on C*-algebras. *Proceedings of the American Mathematical Society*, 6:211–216, 1955.

Naresh Sharma and Naqueeb Ahmad Warsi. On the strong converses for the quantum channel capacity theorems. June 2012. arXiv:1205.1712.

Graeme Smith and Jon Yard. Quantum communication with zero-capacity channels. *Science*, 321(5897):1812–1815, September 2008. arXiv:0807.4935.

Masahiro Takeoka and Mark M. Wilde. Optimal estimation and discrimination of excess noise in thermal and amplifier channels. November 2016. arXiv:1611.09165.

Armin Uhlmann. The “transition probability” in the state space of a *-algebra. *Reports on Mathematical Physics*, 9(2):273–279, 1976.

Andreas Winter and et al. On the energy bounded diamond norm. 2017. manuscript in preparation.

Alfred Wehrl. Three theorems about entropy and convergence of density matrices. *Reports on Mathematical Physics*, 10(2):159–163, October 1976.

Mark M. Wilde, Patrick Hayden, and Saikat Guha. Quantum trade-off coding for bosonic communication. *Physical Review A*, 86(6):062306, December 2012. arXiv:1105.0119.

John Williamson. On the algebraic problem concerning the normal forms of linear dynamical systems. *American Journal of Mathematics*, 58(1):141–163, January 1936.

Mark M. Wilde. From classical to quantum Shannon theory. 2016. arXiv:1106.1445v7.
[Win16] Andreas Winter. Tight uniform continuity bounds for quantum entropies: conditional entropy, relative entropy distance and energy constraints. *Communications in Mathematical Physics*, 347(1):291–313, October 2016. arXiv:1507.07775.

[WPGG07] Michael M. Wolf, David Pérez-García, and Geza Giedke. Quantum capacities of bosonic channels. *Physical Review Letters*, 98(13):130501, March 2007. arXiv:quant-ph/0606132.

[WQ16] Mark M. Wilde and Haoyu Qi. Energy-constrained private and quantum capacities of quantum channels. September 2016. arXiv:1609.01997.

[WTLB16] Mark M. Wilde, Marco Tomamichel, Seth Lloyd, and Mario Berta. Gaussian hypothesis testing and quantum illumination. August 2016. arXiv:1608.06991.

[WWY14] Mark M. Wilde, Andreas Winter, and Dong Yang. Strong converse for the classical capacity of entanglement-breaking and Hadamard channels via a sandwiched Rényi relative entropy. *Communications in Mathematical Physics*, 331(2):593–622, October 2014. arXiv:1306.1586.

[YS78] Horace Yuen and Jeffrey H. Shapiro. Optical communication with two-photon coherent states—Part I: Quantum-state propagation and quantum-noise. *IEEE Transactions on Information Theory*, 24(6):657–668, November 1978.