The Complexity Classes of Hamming Distance Recoverable Robust Problems

Christoph Grüne
Department of Computer Science, RWTH Aachen University, Germany
gruene@algo.rwth-aachen.de

Abstract

The well-known complexity class \( NP \) contains combinatorial problems, whose optimization counterparts are important for many practical settings. These problems typically consider full knowledge about the input. In practical settings, however, uncertainty in the input data is a usual phenomenon, whereby this is normally not covered in optimization versions of \( NP \) problems.

One concept to model the uncertainty in the input data, is recoverable robustness. The instance of the recoverable robust version of a combinatorial problem \( P \) is split into a base scenario \( \sigma_0 \) and an uncertainty scenario set \( S \). The base scenario and all members of the uncertainty scenario set are instances of the original combinatorial problem \( P \). The task is to calculate a solution \( s_0 \) for the base scenario \( \sigma_0 \) and solutions \( s \) for all uncertainty scenarios \( \sigma \in S \) such that \( s_0 \) and \( s \) are not too far away from each other according to a distance measure, so \( s_0 \) can be easily adapted to \( s \). This paper introduces Hamming Distance Recoverable Robustness, in which solutions \( s_0 \) and \( s \) have to be calculated, such that \( s_0 \) and \( s \) may only differ in at most \( k \) elements.

We survey the complexity of Hamming distance recoverable robust versions of optimization problems, typically found in \( NP \) for different scenario encodings. The complexity is primarily situated in the lower levels of the polynomial hierarchy. The main contribution of the paper is a gadget reduction framework that shows that the recoverable robust versions of problems in a large class of combinatorial problems is \( \Sigma^p_3 \)-complete. This class includes problems such as VERTEX COVER, INDEPENDENT SET, DOMINATING SET, COLORING, HAMILTONIAN PATH or SUBSET SUM and many more. Additionally, we expand the results to \( \Sigma^p_{m+1} \)- completeness for multi-stage recoverable robust problems with \( m \in \mathbb{N} \) stages.

2012 ACM Subject Classification Theory of computation → Problems, reductions and completeness

Keywords and phrases Computational Complexity, Polynomial Hierarchy, Hamming Distance Recoverable Robust Problems, Robust Optimization, Recoverable Robust Optimization, Optimization under Uncertainty, Uncertainty

Funding This work is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) — GRK 2236/1.

Acknowledgements I would like to thank Lasse Wulf, Marc Goerigk and Stefan Lendl for helpful comments and discussions on the paper especially concerning the definitions in the preliminaries.
The Complexity Classes of Hamming Distance Recoverable Robust Problems

1 Introduction

The concept of robustness in the field of optimization problems is a collection of models that consider uncertainties in the input. These uncertainties may for example arise from faulty or inaccurate sensors or from a lack of knowledge. Robustness measures can model these types of uncertainty that occur in practical optimization instances into an uncertainty set. The goal is to find solutions that are stable over all possible uncertainties in the uncertainty set. That is, these solutions remain good but not necessarily optimal regardless what the uncertainties turn out be in reality.

One specific robustness concept is recoverable robustness, which is a recently introduced concept [30]. The input of a recoverable robust version of a problem $P$ is a base scenario $\sigma_0$, which is an instance of problem $P$, as well as a set of uncertainty scenarios $S$, whose members are again instances of $P$. The set of uncertainty scenarios $S$ is the uncertainty set of the problem. We are asked to compute a base solution $s_0$ to the base scenario $\sigma_0$ and to compute recovery solutions $s$ to all members of the uncertainty scenarios $\sigma \in S$ such that $s_0$ and all $s$ are not too far away from each other corresponding to a distance measure. The solution on the base scenario does not directly include the uncertainties but needs to include the potential to adapt the base solution $s_0$ to solutions $s$ within the given distance between the solutions. Thus, the base solution $s_0$ may be restricted by these possibly harmful scenarios. We consider mostly decision problems without cost functions, that is, we ask whether a solution exists or not without considering costs on the elements. This, however, is not a limitation because the corresponding decision problems with cost functions are equally complex.

From a worst-case-analysis point of view, we assume that the uncertainty scenarios are chosen by an adversary. The algorithm computes a base solution with the potential to adapt to all scenarios. Then, the adversary chooses the most harmful scenario based on the base solution. Finally, the algorithm computes a recovery solution to adapt to the chosen scenario.

A more general concept is multi-stage recoverable robustness, in which not only one set of scenarios is provided but $m$ sets of uncertainty scenarios. The $m$-stage recoverable robust problem asks to solve the recoverable robust problem on the individual sets of scenarios inductively. That is, a base solution $s_0$ has to be found such that one can recover from $s_0$ for the first set of scenarios $S_1$ to a solution $s_1$ such that one can recover from $s_1$ for the second set of scenarios $S_2$ and so forth such that one can recover from $s_{m-1}$ for the $m$-th set of scenarios $S_m$ to a solution $s_m$.

1.1 Related Work

Recoverable robustness is a recently introduced concept by Liebchen et al. [30] to model uncertainty. A generalized concept of recoverable robustness is multi-stage recoverable robustness, which was introduced by Cicerone et al. [17]. Recoverable Robustness is used in many practical settings such as different optimization areas in air transport [22, 20, 32]. Furthermore, recoverable robustness plays a crucial role in railway optimization, a survey on this area can be found in [31]. Considered problems in railway optimization are to be found on all stages of railway operation, such as network design [35, 12], Rolling Stock Planning [10, 11], Shunting [15] and Timetabling [16, 11, 19, 18, 25, 4].

Our focus lies on the complexity of recoverable robust problems. Goerigk et al. [26] analyzed the problems INDEPENDENT SET and TRAVELING SALESMAN and showed the $\Sigma^p_3$-hardness of these problems. All other contributions analyzed the problems only on their NP-hardness or their approximability, whereby different distance measures between
the solutions are of interest. The concept of $k$-dist recoverable robustness was introduced by Büsing [5] but was also used in [27]. A $k$-dist recoverable robust problem asks for a solution for a base scenario and solutions for one or more scenarios, from which we want to recover, by adding at most $k$ elements to the recovery scenario solution, which were not used in the base scenario solution. Besides, the $k$-dist measure there are also measures which limit the number of deleted elements [8] or exchanged [13] elements. Furthermore, combinations of these distance measures are analyzed as well in the literature [9]. Hamming distance recoverable robustness is also used in the literature [21, 26]. Among the studied recoverable robust problems is Knapsack, which is NP-hard for different distance measures between the solutions [8, 4]. Recoverable Robust versions of problems that are in PTIME are shown to be NP-complete as well such as Shortest Path, which is NP-hard for $k$-dist [6], or Matching [21]. Furthermore, the Single Machine Scheduling problem is 2-approximable [3] and the recoverable robust Traveling Salesman is 4-approximable [13]. Moreover, the recoverable robust version of Spanning Tree [27] is shown to be in PTIME.

1.2 Contribution

To the best of the author’s knowledge, recoverable robustness optimization was primarily studied on algorithms for special problems. Corresponding complexity results mainly consider only NP-hardness results. The only contribution on the complexity of recoverable robust problems within the polynomial hierarchy is from Goerigk et al. [26]. Their paper was written independently and in parallel to this paper. They, however, study the complexity of problem with uncertainties in the cost function of the elements. Our contribution focuses on problems with the uncertainty over the elements that is whether an element is included in a scenario or not. We show that the complexity of problems based on this concept are typically not only NP-hard but are situated higher in the polynomial hierarchy by constructing a reduction framework. We study this on the concept of Hamming distance recoverable robustness, which is adapted from Büsing’s $k$-dist recoverable robustness concept [5].

For the complexity analysis, different forms of encodings are of interest. If the scenarios are encoded such that an explicit encoding of the scenarios can be computed in polynomial time, the problem stays NP-complete. Furthermore, we survey two forms of succinct encodings, xor-dependencies and $\Gamma$-set scenarios. The succinct encoding with xor-dependencies or $\Gamma$-set scenarios induces a combinatorial explosion because with a linear number of uncertainty elements exponential many scenarios can be encoded. This combinatorial explosion leads to a complexity which lies higher in the polynomial hierarchy. While all results other than the results on the $\Gamma$-set scenarios were developed independently from Goerigk et al. [26], the results for the $\Gamma$-set scenarios build upon the idea of their reduction of the Robust Adjustable SAT problem.

Our main contribution is a gadget reduction framework, which uses a specific definition of combinatorial problems based on a ground set and its relations. Different gadget reduction concepts were studied for example by Agrawal et al. [1], who defined gadget reductions under $AC^0$ for NP-completeness mapping one bit of the input of one problem to a bounded number of bits in the other problem. A further form of gadget reduction was introduced by Trevisan et al. [34], who formalized constraints of a linear program to be a gadget in the reduction between linear programs. We, however, need a different concept of gadgets build upon the combinatorial elements of the combinatorial problem. Thereby, each combinatorial element is mapped to a gadget that simulates the behavior of this element in the other problem. This form of reduction, however, preserves the scenarios structurally independent of the encoding. Thus, this gadget reduction framework allows for reductions between Hamming distance
recoverable robust problems. Thus, we are able to show that the recoverable robust versions
of typical \(NP\)-complete combinatorial problems with xor-dependencies or \(\Gamma\)-set scenarios
are \(\Sigma^P_3\)-complete. Finally, we extend these results to multi-stage recoverable robustness by
showing that typical \(NP\)-complete combinatorial problems are \(\Sigma^P_{2m+1}\)-complete for \(m \in N\),
where \(m\) is the number of stages.

1.3 Paper Summary

In the second section, we build a framework for combinatorial decision problems to define
Hamming distance recoverable robust problems. In the third section, we use the framework
to survey the complexity of polynomially computable scenario encodings. Thereby, we
consider typical problems, which are in \(NP\) or \(NP\)-complete. The fourth section consists of
the analysis of succinctly encoded scenarios and their complexity. Furthermore, we look at
multi-stage recoverable robustness and its complexity. At last, we establish a whole class of
Hamming distance recoverable robust problems by using our combinatorial decision problem
framework and universe gadget reductions.

2 Preliminaries

In this paper, we define Hamming distance recoverable robustness, which is also used in
the literature \[21, 26\]. We show complexity results for a large class of Hamming distance
recoverable robust problems. For this, we use a special type of reduction, which can be used
for many combinatorial problems based on scenarios on the combinatorial elements of the
problem instance.

2.1 Combinatorial Problems

In order to state a general theorem on recoverable robust problems, we need a general
definition of combinatorial problems. We begin with a rather standard definition of all
relations \(R\) that are based on a ground set \(U\), which we call the universe. For a useful
definition of the scenarios, it is important to have a precise access on the combinatorial
elements of the base combinatorial problem. These combinatorial elements are exactly the
universe \(U\) and the relations \(R\) that can be build from \(U\).

▶ \textbf{Definition 1} (Relations over a set).

\begin{align*}
    U & \in \mathcal{R}(U) \quad (1) \\
    A & \in \mathcal{R}(U), \quad \text{if } A \subseteq B \in \mathcal{R}(U) \quad (2) \\
    \bigotimes_{i} A_{i} & \in \mathcal{R}(U), \quad \text{if for all } i, A_{i} \in \mathcal{R}(U) \quad (3)
\end{align*}

We denote the set of relational elements that include \(r \in A \in \mathcal{R}(U)\) by \(R(r)\).

The set \(\mathcal{R}(U)\) is a collection of all relations that can be built from \(U\). This general
definition aims to include all combinatorial problems. All problems that we consider in this
paper use some finite subset of these relations. For example a graph \(G = (V, E)\) consists of
the universe \(U = V\), which are the vertices, and the edges \(E \in \mathcal{R}(U)\) are a relation over \(V\).
Additionally \(k\)-partitions may be defined as a \(k\)-ary relation, which is a subset of \(V^k\) and so on.
Now, we define combinatorial decision problems. In computer science, problems are defined as a formal language which is a set of words. This formal language includes all words that are a YES-instance and the goal is to decide whether the given instance is an element of the formal language or not. In mathematical optimization, optimization problems usually are defined by a set of feasible solutions and an optimization function, which has to be minimized or maximized. Then, the input, which is usually specified by some form of mathematical program, is omitted. We need to combine these approaches for our analysis of recoverable robustness. On the one hand, recoverable robust problems are combinatorial problems which are defined over the solution as the solutions need to be within a distance of each other according to a distance measure. On the other hand, the complexity analysis is executed on decision problems which involve a definition over a formal language.

Definition 2 (Combinatorial Decision Problem).
A combinatorial decision problem $P_A$ is a set of tuples $(U, R, F(R))$ with the set of universe elements $U$, relations $R \in \mathcal{R}(U)^r$, $r \in \mathbb{N}$, and the set of feasible solutions $F(R) \subseteq \mathcal{R}(U)^s$, $s \in \mathbb{N}$. The formal language of $P_A$ is the set

$$\{R \mid (U, R, F(R)) \text{ is tuple of } P_A \text{ such that } F(R) \neq \emptyset\}.$$  

We use index set $I$ to easily address the members of the tuples $R$ and $F(R)$. In other words, 

Input: $R = (R_1, \ldots, R_r)$, $R_i \in \mathcal{R}(U)$, $1 \leq i \leq r$, and $R_1 = U$. 

Question: Is $F(R) \neq \emptyset$?

For simplicity, we may omit the dependence of the feasible solutions $F(R)$ on the relations $R$ and write $F$. For a better understanding, an example for a combinatorial decision problem is the Undirected s-t-Connectivity ($UstCon$).

Example 3 (Undirected s-t-Connectivity Problem).
The problem $UstCon$ asks if there is a $s$-$t$-path in an undirected graph. The input of $UstCon$ is a graph $G = (V, E)$ and two vertices $s, t \in V$. A feasible solution is a path from $s$ to $t$ in $G$.

This translates to the following tuple $(U, R, F)$. The universe $U$ consists of the vertices $V$ of graph $G$, that is, $U = V$. The relations in $R$ are the vertices $s$ and $t$ and edges $e \in E(G)$, that is, $R = (V, s, t, E)$. The set of feasible solutions $F$ are all $s$-$t$-paths $p \in P$ in $G$, that is, $F = P \subseteq \bigtimes_{i=0}^{|V|} V^i$.

Observe that for combinatorial problems, the encoding of the input and the solutions depends only on the universe of elements. Thus, the universe elements in $U$ build the atoms of the problem. The relations $R$ model the relations between these atoms. The feasible solutions $F$ model all possible combinations of universe elements and relations that are feasible. For this, additional feasibility relations of the combinatorial problem from $\mathcal{R}(U)$, which are not part of $R$, are part of the feasible solution set.

2.2 Scenarios for Robust Problems

Before we are able to define Hamming distance recoverable robust problems, we need to define scenarios. Scenarios are a central concept in robust optimization, which model the uncertainty. A Hamming distance recoverable robust problem $P_{A_{HDRR}}$ is based on a combinatorial problem $P_A$. A scenario is then defined as follows.

Definition 4 (Scenarios).
A scenario of the problem $P_{A_{HDRR}}$ is a problem instance of the problem $P_A$. 
We distinguish the base scenario from recovery scenarios. The base scenario \(\sigma_0\) is the instance on which the first solution \(s_0\) has to be computed. The recovery scenarios \(\sigma \in S\) are the scenarios for which the solution \(s\), that has to be adapted from \(s_0\), have to be computed. All scenarios of a problem may share universe elements or relation elements.

### 2.2.1 Encoding of Scenarios

For scenarios, we use explicit encodings, implicit encodings or succinct encodings. These types are all based on combinatorial elements of an instance. That is, we consider elemental uncertainty, for which it is uncertain whether a combinatorial element is part of a scenario or not. This is different to uncertainty over the costs of elements, where the underlying combinatorial elements remain the same for all scenarios. We focus on problems, where the uncertainty is defined over all combinatorial elements, that is universe elements and relation elements. If a combinatorial element is not part of a scenario, then all relation elements that include this combinatorial element are discarded in the scenario as well. For example, if a vertex \(v\) in a graph problem is discarded, then all edges incident to \(v\) are discarded, too. We denote this removal of combinatorial elements with \(U \setminus \{r\}\) and \(R \setminus R(r)\), whereby the removal of \(r\) removes all relations \(R(r)\) that contain \(r\).

First, we will use explicit encodings by providing the complete instance encoding over the base problem \(P_A\). Additionally, we use implicitly encodings by providing a set of all elements that are different from base scenario \(\sigma_0\). We call the elements that are part of the current scenario the active elements. In other words, active elements are those elements that are usable in a solution to the scenario, otherwise we call the elements inactive. Furthermore, we address succinct encodings of scenarios as well. These encodings usually encode an exponential number of scenarios in polynomial space. The well-known \(\Gamma\)-scenarios fall into this category as well as later defined xor-dependencies, which use logical operators between the elements to encode which element is part of a scenario.

### 2.3 Hamming Distance Recoverable Robust Problems

Now, we define Hamming distance recoverable robust problems. For this, we need a definition of the Hamming distance over a set.

**Definition 5 (Hamming Distance of Sets).** Let \(A, B\) be two sets. Then, we define the Hamming distance \(H(A, B)\) of set \(A\) and \(B\) to be

\[
H(A, B) := |A \triangle B| = |\{x \mid \text{either } x \in A \text{ or } x \in B\}|
\]

Intuitively, a Hamming distance recoverable robust problem \(P_A^{HDRR}\) is based on a normal combinatorial decision problem \(P_A\), e.g. UstCon. We not only have to find a solution for one instance, but for one base scenario \(\sigma_0\) and for all recovery scenarios \(S\). That is, we can recover from every possible scenario with a new solution to the problem. The solutions to the recovery, nonetheless, may have a Hamming distance of at most \(\kappa\) to the solution of the base scenario. Formally, we obtain the following definition.

**Definition 6 (Hamming Distance Recoverable Robust Problem).** A Hamming distance recoverable robust problem \(P_A^{HDRR}\) is a combinatorial problem based on a combinatorial problem \(P_A\). The instances are defined as tuples \((U, R, F(R))\) with

- \(U = U_0 \cup \bigcup_{\sigma \in S} U_\sigma\) is the universe. The universe is the union over all universe elements that occur in the scenarios.
In other words, we define the problem as

Input: \( R = (R_0, (R_\sigma)_{\sigma \in S}) \) are the relations. The relations are separate for each scenario.

\[
F(R) = (F_0(R_0), (F_\sigma(R_\sigma))_{\sigma \in S}, \{(s_0, (s_\sigma)_{\sigma \in S}) \mid H(s_0, s_\sigma) \leq \kappa \text{ for all } \sigma \in S\}) \text{ are the feasible solutions. The Hamming distance } H(s, s') \text{ is defined over the universe elements in the solutions } s, s'.
\]

In other words, we define the problem as

Question: Is \( F(R) \neq \emptyset \)?

The feasible solutions are included separately for each scenario in \( F \). Only the last relation covers the Hamming distance of \( \kappa \) between the solution \( s_0 \) for the base scenario \( \sigma_0 \) and the recovery solutions \( s_\sigma \) for each scenario \( \sigma \in S \).

Observe that the specifications are no restriction because every decision problem can be formulated as one base scenario and no recovery scenarios, that is \( S = \emptyset \). On the other hand, the base problem \( P_A \) is a restriction of \( P_A^{HDRR} \) by setting \( S = \emptyset \). Furthermore, the base scenario is defined by \( \sigma_0 = (U_0, R_0, F_0) \) and all uncertainty scenarios \( \sigma \in S \) are defined by \( \sigma = (U_\sigma, R_\sigma, F_\sigma) \).

Again, we provide an example for a better understanding of the definition and again, we use the UstCon problem.

\bold{Example 7 (Hamming Distance Recoverable Robust UstCon)}

Let \( G = (V, E) \) a simple graph and \( s, t \in V \). Hamming Distance Recoverable Robust UstCon, short UstCon\(^{HDRR} \), is a Hamming distance recoverable robust problem with uncertainties over universe \( V(G) \). We may include cost functions \( c_V \) on \( V(G) \) or \( c_E \) on \( E(G) \) in order to find a short \( s-t \)-path; the feasible solutions are all paths that are shorter than some \( k \).

- The input \( R \) contains the following scenarios: Each scenario \( \sigma \in S \) encodes which vertices are available in \( \sigma \). (If we search for a short undirected \( s-t \)-path, each scenario \( \sigma \in S \) also encodes the costs on the vertices or edges.)
- The feasible solutions \( F \) consists of all paths \( (p, p_\sigma)_{\sigma \in S} \in P_{\sigma_0} \times P_{\sigma \in S} \) leading from \( s \) to \( t \) such that \( H(p, p_\sigma) \leq \kappa \), for all \( \sigma \in S \).

\[
\exists p \in P_{\sigma_0} : \forall \sigma \in S : \exists p_\sigma \in P_{\sigma} : (p, p_\sigma) \in F.
\]

\section{Recoverable Robust Problems with Polynomially Computable Scenario Encodings}

3 Recoverable Robust Problems with Polynomially Computable Scenario Encodings

We now survey the problems with polynomially computable scenario encodings. That is, all scenarios can be polynomially computed into fully encoded instances. For this, we will first establish the containment in \( NP \) and after that the hardness of such problems.

\bold{Theorem 8.} Let \( P_A \in NP \). Then \( P_A^{HDRR} \in NP \), if the set of scenarios \( S \) of \( P_A^{HDRR} \) is polynomially computable.

\textbf{Proof.} We present a polynomially verifiable certificate. Firstly, the base scenario \( \sigma_0 \) and all recovery scenarios \( S \) are encoded in the certificate by encoding a list of elements that are active for all scenarios. Secondly, the solution \( s_0 \) to the base scenario and the solution to the recovery scenarios \( (s_\sigma)_{\sigma \in S} \) are encoded as a list of elements that are active in the corresponding scenario. Thus, the certificate is \( (\sigma_0, S, s_0, (s_\sigma)_{\sigma \in S}) \) all encoded as lists of elements.
Because $\sigma_0$ and $S$ encoded as sets can be polynomially computed from the input encoding and $a_0$ and $(a_\sigma)_{\sigma \in S}$ are subsets of $\sigma_0$ and $S$ correspondingly, the length of the certificate is at most polynomial in the input length. Furthermore, the certificate is verifiable in polynomial time by the following algorithm. First, check whether $a_0$ is really a solution to $\sigma_0$. Second, check whether $a_\sigma$ is a solution to $\sigma$ for all $\sigma \in S$. Third, check whether $H(a_0, a_\sigma) \leq \kappa$ for all $\sigma \in S$.

Step one and two are polynomially computable because $P_A \in \text{NP}$. Thus, all scenarios for themselves are an instance of problem $P_A$. Step three is easily polynomially computable by iterating over each scenario pair. Observe again that $|S|$ is at most polynomial in the input length. ◀

Besides, general polynomially computable scenarios, we may have so called $\Gamma$-scenarios. They are an interesting and popular robustness concept. These consist of all scenarios that deviate in at most $\Gamma$ many elements from the base instance corresponding to a set of possible activatable elements. If $\Gamma$ is constant, we use the previous Theorem 8 for the following Corollary 9.

\begin{corollary}
\textbf{Corollary 9.} Let $P_A \in \text{NP}$. Then $P_A^{\text{HDRR}} \in \text{NP}$, if the set of scenarios $S$ of $P_A^{\text{HDRR}}$ consists of all possible $\Gamma$-scenarios for a constant $\Gamma$.

Lemma 10 follows directly from by Theorem 8 by reusing the original reduction to $P_A$ and setting the scenario set $S = \emptyset$.
\end{corollary}

\begin{lemma}
\textbf{Lemma 10.} Let $P_A$ be an NP-complete problem. Then, $P_A^{\text{HDRR}}$ is NP-complete if the set of scenarios $S$ of $P_A^{\text{HDRR}}$ is polynomially computable.

\textbf{Proof.} The reduction from $P_A$ is trivial because the scenarios can be set to $S = \emptyset$ showing the hardness of $P_A^{\text{HDRR}}$. On the other hand, Theorem 8 proves the containment. ◀
\end{lemma}

### 3.1 Reduction for Undirected s-t-Connectivity

We have shown that problems with polynomial computable scenarios are in $\text{NP}$. The hardness for those problems can be established as well. For this, we use a simple problem as basis, the LOGSPACE-complete undirected s-t-Connectivity ($\text{UstCon}$).

\begin{theorem}
\textbf{Theorem 11.} There is a deterministic logarithmic space computable reduction from 3-Satisfiability to $\text{UstCon}^{\text{HDRR}}$ with one base and one recovery scenario. In short, $3\text{-Satisfiability} \leq_L \text{UstCon}^{\text{HDRR}}$.

\textbf{Proof.} First of all, there is a reduction, which is based on the reductions 3-Satisfiability $\leq_L$ Directed Hamiltonian Cycle $\leq_L$ Undirected Hamiltonian Cycle from Arora and Barak [2] from 3-Satisfiability to Undirected Hamiltonian Cycle, which is computable in logarithmic space. We use this reduction to develop the reduction for $\text{UstCon}^{\text{HDRR}}$.

We can either define the scenarios over vertices or over edges. This, however, is in this reduction realm equivalent, because we can easily introduce a vertex for every edge, such that for the deletion of such an vertex the former edge is deleted. On the other hand, we can delete all incident edges of a vertex to exclude the vertex from a possible solution. For the sake of simplicity, we use edge scenarios in the reduction. Furthermore, the Hamming distance of the solution is based on the edges.

We now provide a reduction from Undirected Hamiltonian Cycle to $\text{UstCon}^{\text{HDRR}}$. Let $G = (V, E)$ be a graph of the Undirected Hamiltonian Cycle instance. We will map $G$ to a graph $G'$, a base scenario $\sigma_0$ and a recovery scenario $\sigma_1$ as a $\text{UstCon}^{\text{HDRR}}$ instance.
A simple example instance, which we use for explaining the construction, can be found in Figure 1.

First, all \( v \in V \) are duplicated \(|V| + 3\) times to connect them to one path that has consequently \(|V| + 2\) edges. Let \( v^a_i \) and \( v^b_i \) the end vertices of a vertex path of vertex \( v_i \). In Figure 2, the duplication procedure is depicted. We call these vertex paths.

The two scenarios are first the base scenario \( \sigma_0 \) and second the recovery scenario \( \sigma_1 \). For the base scenario, we design a simple to solve instance, which forces the solution to include all edges of the vertex paths. For this, we connect the vertex paths to a simple cycle by introducing an edge connecting two vertex paths. That is, we introduce edges \( \{v^b_i, v^{(i+1 \mod |V(G)|)}_a\} \) for \( 1 \leq i \leq |V(G)| \).

The simple to find solution is the cycle. We further have to introduce two vertices \( s \) and \( t \). For this, we choose one vertex path, delete the edge in the middle of the path and designate the incident vertices of the delete edge as \( s \) and \( t \). The base scenario \( \sigma_0 \) can be found in Figure 3a.

For the recovery scenario \( \sigma_1 \), we deactivate the edges between the paths but not the paths themselves. We then set \( \kappa = |V(G)| \). This forces the recovery solution to have the vertex paths as part of the solution, as only \(|V(G)|\) edges can be altered while a vertex path has at least \(|V(G)| + 1\) (including the one with \( s \) and \( t \)). Furthermore, we map and activate the actual edges of \( G \). For this, we quadruplicate the edges of \( G \). The edge \( \{v_i, v_j\} \) is quadruplicated to \( \{v^a_i, v^a_j\}, \{v^b_i, v^b_j\}, \{v^b_i, v^a_j\} \) and \( \{v^b_i, v^b_j\} \). Thus, each vertex path can be ordered in both ways in a possible Hamiltonian cycle. This is depicted in Figure 3b.

On one hand, the construction of the base scenario \( \sigma_0 \) forces the base solution \( z_0 \) to be the cycle itself. The solution \( z_0 \) is presented in Figure 4a. On the other hand, the vertex paths force the recovery solution to go over all vertex paths because of setting \( \kappa = |V(G)| \) prevents the solution \( z_1 \) from evading these paths. A selection of possible solutions for the recovery scenario are shown in Figure 4b.

\( \triangleright \) **Claim 12.** The reduction is computable in logarithmic space.
Proof. The reduction is clearly computable in logarithmic space, because we only have to count the number of vertices in the duplication procedure. The connection to the cycle is also directly possible if the number of vertices known. At last, the introduction of the edges for the base scenario is only a copy procedure based on the original graph, which is directly computable if the number of vertices is known.

\[\triangleleft\]

Claim 13. The reduction is correct.

Proof. First of all, the only solution for the base scenario is the path from \(s\) to \(t\) over the former cycle in \(\sigma_0\). If a Hamiltonian cycle exists in the graph, then it is possible find a correspondent solution \(s_1\) for the recovery scenario. We can use the edges from the Hamiltonian cycle in \(G\) and use the edges \(\{v_i^a, v_{i+j \mod |V(G)|}^b\}\) of both of the corresponding edges in the recovery scenario. Thus, the vertex paths are connected to a Hamiltonian cycle as well.

One the other hand, if there is no Hamiltonian cycle, then there is no path of the form \((s, v_1^x, v_2^y, v_3^y, \ldots, v_{|V(G)|}^x, v_{|V(G)|}^y, v_1^y, t)\), where \(x, y \in \{a, b\}\) and \(x \neq y\). This is due to the fact that the base scenario \(\sigma_0\) in combination with the too small \(\kappa = |V(G)|\) enables the possibility to switch only away from the edges that connect the vertex paths. It is not possible to switch away completely from a vertex path as there are \(|V(G)| + 1\) edges in each vertex path (including that with \(s\) and \(t\)). Thus, at least one edge that has to be in the \(s\)-\(t\)-path would not be correctly included into the \(s\)-\(t\)-path or \(s\) and \(t\) are not connected by a path.

\[\triangleleft\]
4 Recoverable Robust Problems and the Polynomial Hierarchy

In this section, we survey the connection between multi-stage Hamming distance recoverable robust problems and the polynomial hierarchy. For this, we introduce two succinct encodings: xor-dependencies and $\Gamma$-set scenarios. We first prove that the Hamming distance recoverable robust version of problems, which are in NP, are in $\Sigma^p_3$ for both encodings. Then, we prove the hardness of the Hamming distance recoverable robust 3-SATISFIABILITY for both encodings.

Definition 14 (Hamming Distance Recoverable Robust 3-SATISFIABILITY).

The problem 3-SATISFIABILITY$^{HDRR}$ is defined as follows.

**Input:** Literals $L$, clauses $C$, base scenario $\sigma_0 \subseteq L$ and recovery scenarios $S = \{L_0 \mid L_0 \subseteq L, \sigma \in S\}$

**Question:** Are there solutions $s_0 \subseteq \sigma_0$ and $s_\sigma \subseteq \sigma$ for all $\sigma \in S$ such that $H(s_0, s_\sigma) \leq \kappa$ for all $\sigma \in S$ and setting $s_0$ and $s_\sigma$ true, all corresponding formulae of clauses $C |_{L_0}$ and $C |_{L_\sigma}$ are satisfied?

At last, we extend these results to the multistage recoverable robustness case by showing the $\Sigma^p_{2m+1}$-completeness of the Hamming distance recoverable robust 3-SATISFIABILITY with $m$ recovery stages. We begin with the xor-dependency scenarios.

Definition 15 (xor-Dependency Scenarios).

Let $S = (E', \{(E_{1,1}, E_{1,2}), \ldots, (E_{n,1}, E_{n,2})\})$ be the scenario-encoding, whereby $E$ and $E_{i,j}$ are pairwise disjoint sets of combinatorial elements for all $i, j$. Then the corresponding scenario set $S$ includes all $\sigma$ of the form $\sigma = E' \cup \{E_1, \ldots, E_n\}$ with either $(E_i = E_{i,1})$ or $(E_i = E_{i,2})$ for all $i = 1, \ldots, n$ and $E'$ is a fixed set of combinatorial elements, which are activated and deactivated in all scenarios depending on whether an element is active in the base scenario.

Observe that with a linear sized encoding, exponentially many scenarios may be encoded. We study this combinatorial explosion with the result that it introduces more complexity for Hamming Distance Recoverable Robust problems. For this, we use 3-SATISFIABILITY as base problem and show the $\Sigma^p_3$-hardness of 3-SATISFIABILITY$^{HDRR}$ with a linear number of xor-dependencies. Furthermore, we show also that if $P_A \in NP$, then $P_A^{HDRR}$ with a linear number of xor-dependencies is in $\Sigma^p_3$.

Theorem 16. If $P_A \in NP$, then $P_A^{HDRR}$ with a linear number of xor-dependencies is in $\Sigma^p_3$.

Proof. We present an $\exists \forall \exists$-ATM $M$ that solves $P_A^{HDRR}$ with a linear number of xor-dependencies in polynomial time. For this, $M$ encodes the scenario $\sigma_0$ and the solution $s_0$ to the base scenario $(\exists)$. Secondly, $M$ encodes the scenarios $\sigma$ for all $\sigma \in S \ (\forall)$. Lastly, $M$ encodes the solution $s_\sigma$ for the selected $(\exists)$. Again, the solution to the scenarios $s_0$ and $(s_\sigma)_{\sigma \in S}$ are encoded as a list of elements that are active in the corresponding scenario.

Because $\sigma_0$ and each $\sigma \in S$ encoded as sets can be polynomially computed from the input encoding and $s_0$ and $(s_\sigma)_{\sigma \in S}$ are subsets of $\sigma_0$ and $\sigma \in S$ correspondingly, the length of the input to the verifying algorithm is at most polynomial in the input length. Furthermore, the given $\exists \forall \exists$-input is verifiable in polynomial time by the following algorithm. First, check whether $s_0$ is really a solution to $\sigma_0$. Second, check whether $s_\sigma$ is a solution to $\sigma$ for all $\sigma \in S$. Third, check whether $H(s_0, s_\sigma) \leq \kappa$ for all $\sigma \in S$.

Step one and two are polynomially computable because $P_A \in NP$. Thus all scenarios for themselves are an instance of problem $P_A \in NP$. Step three is easily polynomially
computable by iterating over each scenario pair. Observe again that a scenario $\sigma \in S$ is at most polynomial in the input length.

\begin{theorem}
3-Satisfiability$^\text{HDRR}$ with a linear number of xor-dependencies is $\Sigma^p_3$-hard.
\end{theorem}

\begin{proof}
We reduce $\exists\forall$3-Satisfiability to 3-Satisfiability$^\text{HDRR}$.

Let $(X,Y,Z,C)$ be the $\exists\forall$3-Satisfiability-instance, where $\exists X \forall Y \exists Z C(X,Y,Z)$ is the formula with clauses $C(X,Y,Z)$. We denote the 3-Satisfiability$^\text{HDRR}$-instance as $I$.

**Variables** We modify the variable set as follows. The variable set $X$ remains the same.

We substitute $Y$ by $\{y'_i, y''_i \mid y_i \in Y\} =: Y'$. At last, we define $Z'$ by $Z' := Z \cup \{y'_{i,0}, y'_{i,1}, y''_{i,0}, y''_{i,1} \mid y_i \in Y\}$. We further introduce a dummy variable set $X'$ of size $|Y| + |Z'|$.

**Clauses** The clauses are then modified as follows. We add $y'_i \leftrightarrow 0$ and $y'_i \leftrightarrow 1$ to the formula. Furthermore, we add $y'_i \leftrightarrow y'_{i,1}, y''_i \leftrightarrow y''_{i,0}, y'_i \leftrightarrow y''_{i,1}$ to the formula. At last, we do the following substitutions: For every clauses $c = (a, b, y_i) \in C$ with $a, b \in X \cup Y \cup Z$ we substitute $c$ by the clauses $(a, b, y''_{i,1})$ and $(a, b, y'_{i,0})$ and for clauses $c = (a, b, y'_{i,1}) \in C$ with $a, b \in X \cup Y \cup Z$ we substitute $c$ by the clauses $(a, b, y''_{i,0})$ and $(a, b, y'_{i,1})$. This is possible in polynomial time because we have a 3-Satisfiability instance and we are introducing at most sixteen clauses per clause.

**Scenarios** The first scenario of $I$ consists of variables $X$ and the fresh $X'$. Thus, the clauses of the first scenario are $C(X,Y,Z)|_X$ and the variables of $X'$ are dummy variables that force an assignment of $|Y| + |Z'|$ variables that will not be in recovery scenarios. The recovery scenarios of $I$ are encoded as scenarios with xor-dependencies. The active variables are from sets $X$, $Y'$ and $Z$. The xor-dependencies are introduced on $y'_i$ and $y''_i$ with $y'_i \oplus y''_i$ for all $i \in \{1, \ldots, |Y|\}$. The variable sets $X$ and $Z'$ are active in every recovery scenario, $X'$ on the other hand is inactive. At last, set $\kappa = |Y| + |Z'|$.

**Polynomial Time** This transformation is computable in polynomial time because for each literal and each clause in $(X,Y,Z,C)$ a fixed amount of literals and clauses in $I$ are created. Furthermore, the formula can be transformed into CNF by substituting $a \leftrightarrow b$ with clauses $(a \lor b)$ and $(a \lor \overline{b})$.

**Correctness** For the correctness, we have to prove that the xor-dependency scenarios in the construction are logically equivalent to a for all of the $\exists\forall$3-Satisfiability formula.

First, we focus on the $\exists X$ part. A solution to the base scenario is a valid solution to $C(X,Y,Z)|_X$. Overall, $|X| + |Y| + |Z'|$ variables are assigned a value. Under those are $|X|$ variables for the solution to $C(X,Y,Z)|_X$ and $|Y| + |Z'|$ dummy variables. Because all recovery scenarios do not have those $|Y| + |Z'|$ dummy variables, a solution always switches away from those dummy variables to $|Y| + |Z'|$ different variables. Because $\kappa = |Y| + |Z'|$, the base solution forces $X$ to be a solution to $C(X,Y,Z)|_X$ for all recovery scenarios. Next, we concentrate on the $\forall Y'$ part. First, the clauses $1 \leftrightarrow y'_i$ and $0 \leftrightarrow y''_i$ for all $i \in \{1, \ldots, |Y|\}$, force all variables $y'_i$ to be always true and the variables $y''_i$ to be always false if they are active. The xor-dependencies activate only one of the two $y'_i$ and $y''_i$ for all $i \in \{1, \ldots, |Y|\}$. Furthermore, if $y'_i$ is active, then $y'_{i,0}$ evaluates to 0 and $y'_{i,1}$ evaluates to 1, and if $y''_i$ is active, then $y''_{i,0}$ evaluates to 0 and $y''_{i,1}$ evaluates to 1. If on the other hand, $y'_i$ is inactive, then both $y'_{i,0}$ and $y'_{i,1}$ can be set to 1. This allows all clauses containing $y'_{i,0}$ or $y''_{i,1}$ to be trivially fulfilled, whenever $y'_i$ is inactive. The same argument holds for $y'_i$, $y''_{i,0}$ or $y''_{i,1}$, too. Because the combinations allowed by the xor-dependencies are all $2^{|Y|}$ possible truth assignments to variables $Y$, the xor-dependency scenarios are...
equivalent to a $\forall Y$ for the variables $Y$. At last, we have the $\exists Z'$ part. This is purely a
solution to the recovery scenarios with the given $Y$.
Thus, a solution is a one to one correspondence between $X,Y,Z$ and $X,Y',Z'$.

While the other parts of the paper are developed independent from Goerigk et al. \cite{26}, the results for $\Gamma$-set scenarios are built upon it. The results based on xor-dependencies are adaptable to the $\Gamma$-set scenarios as described in this section. For the $\Gamma$-set scenarios, we use the definition over sets instead of elements as in $\Gamma$-scenarios, which is defined as follows.

\begin{definition}[$\Gamma$-set Scenarios] Let $S = (E', \{E_1, E_2, \ldots, E_n\})$ be the scenario-encoding, whereby $E$ and $E_i$ are pairwise disjoint sets of combinatorial elements for all $i$. Then, the corresponding scenario set $S$ includes all $\sigma$ of the form $\sigma = E' \cup E$ with $E \subseteq \{E_1, E_2, \ldots, E_n\}$, $|E| \leq \Gamma$, and $E'$ is a fixed set of combinatorial elements, which are activated and deactivated in all scenarios depending on whether an element is active in the base scenario.
\end{definition}

Again, with a linear sized encoding, exponentially many scenarios may be encoded. We show the $\Sigma_p^p$-hardness of 3-SAT with $\Gamma$-set scenarios. A proof on the so-called Robust Adjustable SAT was already conducted by Goerigk et al. \cite{26}. This version of 3-SAT uses uncertainties over the elements instead of the elements as in $\Sigma_p^p$-hardness of 3-SAT with $\Gamma$-set scenarios. Thus, the proof is not analogous as it is different in technicalities, nevertheless, we reuse their basic idea of introducing the $s$-variables for our proof. Furthermore, we show also that if $P_A \in \text{NP}$, then $P_A^{\Gamma_{HDRR}}$ with $\Gamma$-set scenarios is in $\Sigma_p^p$.

\begin{theorem} If $P_A \in \text{NP}$, then $P_A^{\Gamma_{HDRR}}$ with $\Gamma$-set scenarios is in $\Sigma_p^p$.
\end{theorem}

\begin{proof} This proof is analogous to the proof for xor-dependencies.
\end{proof}

\begin{theorem} 3-SAT with $\Gamma$-set scenarios is $\Sigma_p^p$-hard.
\end{theorem}

\begin{proof} We heavily reuse the transformation for the xor-dependencies. Nevertheless, we have to introduce a mechanism to accommodate the less structured $\Gamma$-set scenarios in comparison to the xor-dependencies. At last, the scenarios have to be adapted to the $\Gamma$-set scenarios.

We reduce $\exists \forall \exists$-Satisfiability to 3-SAT with $\Gamma$-set scenarios in comparison to the xor-dependencies. Let $(X,Y,Z,C)$ be the $\exists \forall \exists$-Satisfiability-instance, where $\exists X \forall Y \exists Z C(X,Y,Z)$ is the formula with clauses $C(X,Y,Z)$. We denote the 3-SAT with $\Gamma$-set scenarios as $I$.

\textbf{Variables} We modify the variable set as follows. The variable set $X$ remains the same.
We substitute $Y$ by $\{y_i^f \mid y_i \in Y\} =: Y'$. Moreover, we define $Z'$ by $Z' := Z \cup \{y_i^{f,0}, y_i^{f,1}, y_i^{l,0}, y_i^{l,1} \mid y_i \in Y\} \cup \{s, s_i \mid y_i \in Y\}$. The added variables $s_i$ for each $y_i \in Y$ and the additional variable $s$ fulfill the same function as in the proof of Goerigk, Lendl and Wulf \cite{26}. We further introduce a dummy variable set $X'$ of size $|Y| + |Z'|$.

\textbf{Clauses} The clauses are then modified as follows. We add $y_i^{f,0} \leftrightarrow 0$ and $y_i^{l,1} \leftrightarrow 1$ to the formula. Furthermore, we add $y_i^{f,1} \leftrightarrow y_i^{l,1}$, $y_i^{f,1} \leftrightarrow y_i, y_i^{l,1} \leftrightarrow y_i^{f,1}$ to the formula. Then, we do the following substitutions: For every clauses $c = (a,b,y_i)$ we substitute $y_i$ by the clauses $(a,b,y_i^{f,0})$ and $(a,b,y_i^{l,0})$ and for clauses $c = (a,b,y_i)$ we substitute $y_i$ by the clauses $(a,b,y_i^{f,1})$ and $(a,b,y_i^{l,1})$. This is possible in polynomial time because we have a 3-SAT instance and we are introducing at most sixteen clauses per clause. Moreover, we add $\overline{y}$ to all clauses $c \in C$,
such that we obtain \( s \rightarrow C(X, Y, Z) \). At last, we add \( y_i' \leftrightarrow s_i \) and \( y_i'' \leftrightarrow s_i \) as well as 

\[
s \lor \exists \gamma_1 \lor \exists \gamma_2 \lor \ldots \lor \exists \gamma_{|Y|}
\]

to the clauses.

**Scenarios** The first scenario of \( I \) consists of variables \( X \) and the fresh \( X' \). Thus, the clauses of the first scenario are \( C(X, Y, Z)_X \) and the variables of \( X' \) are dummy variables that force an assignment of \(|Y| + |Z'|\) variables that will not be in recovery scenarios. The recovery scenarios of \( I \) are encoded as \( \Gamma \)-set scenarios. The active variables are from sets \( X, Y', Z' \). The set of uncertain elements is \( E \subseteq \{y_i', y_i'' \mid y_i \in Y\} \) with \( E \leq \Gamma \). The variable sets \( X \) and \( Z' \) are active in every recovery scenario, \( X' \) on the other hand is inactive. At last, set \( \kappa = |Y| + |Z'| \) and \( \Gamma = |Y| \).

**Polynomial Time** This transformation is computable in polynomial time because for each literal and each clause in \( (X, Y, Z, C) \) a fixed amount of literals and clauses in \( I \) are created. Furthermore, the formula can be transformed into CNF by substituting \( a \leftrightarrow b \) with clauses \( (a \lor \neg b) \) and \( (\neg a \lor b) \) and using Karp’s reduction from \( 3\text{SAT} \) to \( 3\text{SAT} \) \[25\].

**Correctness** For the correctness, we have to prove that the \( \Gamma \)-set scenarios in the construction are logically equivalent to \( a \) for all of the \( \exists\forall\exists\text{-Satisfiability} \) formula. First of all, we claim that, whenever the set of uncertain elements \( E \) is smaller than \( \Gamma \), the formula is trivially satisfiable by assigning one of the \( s_i \) to 0 such that \( s \) can be chosen to 0, all clauses are fulfilled by the transformation \( s \rightarrow C(X, Y, Z) \). This also holds, whenever \( y_i' \) and \( y_i'' \) are both active because then there is a \( 1 \leq j \leq |Y| \) such that neither \( y_j' \) nor \( y_j'' \) is active such that \( s_j \) can be assigned to 0.

Next, we focus on the \( \exists X \) part. A solution to the base scenario is a valid solution to \( C(X, Y, Z)_X \). Overall, \(|X| + |Y| + |Z'|\) variables are assigned a value. Under those are \(|X|\) variables for the solution to \( C(X, Y, Z)_X \) and \(|Y| + |Z'|\) dummy variables. Because all relevant recovery scenarios do not have those \(|Y| + |Z'|\) dummy variables, a solution always switches away form those dummy variables to \(|Y| + |Z'|\) different variables. Because \( \kappa = |Y| + |Z'| \), the base solution forces \( X \) to be a solution to \( C(X, Y, Z)_X \) for all recovery scenarios. Next, we concentrate on the \( \forall Y \) part. First, the clauses \( 1 \leftrightarrow y_i' \) and \( 0 \leftrightarrow y_i'' \) for all \( i \in \{1, \ldots, |Y|\} \), force all variables \( y_i' \) to be always true and the variables \( y_i'' \) to be always false if they are active. The \( \Gamma \)-set scenarios activate only one of the two \( y_i' \) and \( y_i'' \) for all \( i \in \{1, \ldots, |Y|\} \) for the relevant cases. Furthermore, if \( y_i' \) is active, then \( y_{i,0}' \) evaluates to 0 and \( y_{i,1}' \) evaluates to 1, and if \( y_i'' \) is active, then \( y_{i,0}'' \) evaluates to 0 and \( y_{i,1}'' \) evaluates to 1. If on the other hand, \( y_i'' \) is inactive, then both \( y_{i,0}' \) and \( y_{i,1}' \) can be set to 1. This allows all clauses containing \( y_{i,0}' \) or \( y_{i,1}' \) to be trivially fulfilled, whenever \( y_i'' \) is inactive. The same argument holds for \( y_{i,0}' \), \( y_{i,0}'' \) or \( y_{i,1}'' \), too. Because the combinations allowed by the \( \Gamma \)-set scenarios are all \( 2^{|Y|} \) possible truth assignments to variables \( Y \), the \( \Gamma \)-set scenarios are equivalent to a \( \forall Y \) for the variables \( Y \). At last, we have the \( \exists Z' \) part. This is purely a solution to the recovery scenarios with the given \( Y \).

Thus, a solution is a one to one correspondence between \( X, Y, Z \) and \( X, Y', Z' \).

### 4.1 Multi-Stage Recoverable Robustness

In **multi-stage recoverable robustness**, the uncertainty is not only modeled by one set of scenarios but multiple sets that are connected inductively.

**Definition 21** (Multi-Stage Recoverable Robust Problem).
A multi-stage recoverable robust problem with \( m \) recoveries \( P^m_{\text{HDRR}} \) is inductively defined as
\[
P^0_{\text{HDRR}} := P_A \\
P^1_{\text{HDRR}} := P^0_{\text{HDRR}} \\
P^m_{\text{HDRR}} := (P^1_{\text{HDRR}})^{m-1}_{\text{HDRR}} 
\]
for \( m = 0, 1 \), and
for \( m > 1 \).

The complexity results naturally extend to the multiple recoverable robustness concept. We make use of the inductive nature of the definition by proving the following theorems by induction. For this, we reuse Theorems 16, 17, 19 and 20 as induction base.

\[\text{▶ Theorem 22.} \] 3-Satisfiability \( m \)-HDRR with a linear number of xor-dependencies is in \( \Sigma^p_{2m+1} \). 3-Satisfiability \( m \)-HDRR with \( \Gamma \)-set scenarios is in \( \Sigma^p_{2m+1} \).

\[\text{Proof.} \] We reuse the argumentation from Theorem 8, in which we proved the membership to \( \text{NP} \) for polynomially computable scenarios. Instead of a certificate we present an \( \exists \forall \exists \) \( m \)-Alternating Turing Machine \( M \) that solves \( P^m_{\text{HDRR}} \). First, \( M \) guesses:
- the solution \( s_0 \) to the base problem \( P_A \) (\( \exists \))
- for \( i = 1 \) to \( m \)
  - all scenarios \( \sigma_i \in S_i \)
  - the recovery solution \( s_{\sigma_i} \) based on the base solution \( s_0 \) and preceding recovery solutions \( (s_0, s_{\sigma_1}, \ldots, s_{\sigma_{i-1}}) \) and current scenario \( \sigma_i \).
Then \( M \) can check the necessary properties:
- Check whether \( s_0 \) is a solution to \( \sigma_0 \)
- Check whether \( s_{\sigma_i} \) is a solution to \( \sigma_j \) for \( 1 \leq j < m \)
- Check whether \( H(s_0, s_1) \) and \( H(s_{\sigma_j}, s_{\sigma_{j+1}}) \) for \( 1 \leq j < m \)

\[\text{▶ Theorem 23.} \] 3-Satisfiability \( m \)-HDRR with a linear number of xor-dependencies is \( \Sigma^p_{2m+1} \)-hard. 3-Satisfiability \( m \)-HDRR with \( \Gamma \)-set scenarios is \( \Sigma^p_{2m+1} \)-hard.

\[\text{Proof.} \] Induction over \( m \).

(IB) \( m = 0 \): 3-Satisfiability is \( \text{NP} \)-complete.

(IS) \( m \mapsto m + 1 \): We extend the argument from Theorem 17. By the induction hypothesis, we know that \( P^m_{\text{HDRR}} \) is \( \Sigma^p_{2m+1} \)-hard. More precisely, the induction hypothesis yields that \( (\exists \forall)^m \exists - 3\text{-Satisfiability} \) is reducible to \( P^m_{\text{HDRR}} \). Thus, we need to model the additional alternation for \( m + 1 \) with the additional recovery stage. For this, let
\[
X_1, Y_1, X_2, Y_2, \ldots, X_{m+1}
\]
be the variable sets of the \( (\exists \forall)^{m+1} \exists - 3\text{-Satisfiability} \)-instance, whereby
\[
\exists X_1 \forall Y_1 \exists X_2 \forall Y_2 \ldots \exists X_{m+1} \exists C(X_1, Y_1, X_2, Y_2, \ldots, X_{m+1})
\]
is the formula. By interpreting the variable sets \( X_2, Y_2, \ldots, X_{m+1} \) as the variable set \( Z \), which is not altered in any way, \( Y_1 \) as variable set \( Y \) and \( X_1 \) as \( X \), the additional alternation of the \( (\exists \forall)^{m+1} \exists - 3\text{-Satisfiability} \) formula can be modeled by one more recovery step. ◀
5 Classes of Recoverable Robust Problems

In order to establish a whole class of recoverable robust problems, we need to define a reduction that preserves the structure of the scenarios. For this, consider problems $P_A$ and $P_B$. We need to achieve that a combinatorial element $e_A$ in $P_A$ is active, if and only if the combinatorial elements $e'_B$, to which $e_A$ is mapped in $P_B$, are active. Then, we can use this one-to-many correspondence to (de)activate the corresponding elements in the instance of $P_B$.

This property is already constituted by the informal concept of gadget reductions on the universe of elements. Gadget reductions describe that each part of the problem $P_A$ is mapped to a specified part of the problem $P_B$ that inherits the behavior in problem $P_A$. We adjust this concept to combinatorial elements, that is universe elements and relation elements, for our purpose, such that a gadget is a subset of combinatorial elements in $P_B$ for every combinatorial element in $P_A$. Thereby, we preserve the (in)activeness of elements in a scenario. We call this property modularity. Furthermore, the solution size of an instance must fulfill the modularity property in order to define the Hamming distance in the reduction correctly. That is, the solution size of every gadget has to be constant for each gadget based on some combinatorial element.

5.1 Universe Gadget Reduction

Let $P_A$ be a combinatorial decision problem with instance tuples $(U_A, R_A, F_A)$ and $P_B$ a combinatorial decision problem with instance tuples $(U_B, R_B, F_B)$. A Universe Gadget Reduction $f_{\leq}^{UGR}$ that many-one-reduces $P_A$ to $P_B$ is composed of a (possibly empty) constant gadget $Y_{const}$, which is the same for every instance, and of the independent mappings for all $(i, j) \in I_A \times I_B : f_{R_A, R_B}^{i,j}$:

$$R_A^i \rightarrow 2^{R_B^i}. \quad \forall (i, j) \in I_A \times I_B : f_{R_A, R_B}^{i,j}.$$

We, then, call the substructure

$$Y_x = \bigcup_{(i, j) \in I_A \times I_B} f_{R_A, R_B}^{i,j}(x)$$

the gadget for the specific universe element or relation element $x \in \bigcup_i R_A^i$. The mappings must fulfill the following properties.

1. The pre-image of an element in $R_B$ is unique or the element in $R_B$ is part of the constant gadget: Let $y \in R_B$ for some $j \in I_B$, then either $y \in Y_{const}$ or there is exactly one $(i, j) \in I_A \times I_B$ and exactly one $x \in R_A$ such that $y \in f_{R_A, R_B}^{i,j}(x)$.

2. The gadgets are modular: If a combinatorial element $r \in R_A^i$ from $P_A$ is removed to form a new instance $P_A'$, the removal of the gadget of $r$ in $P_B$ induces a correct reduction instance $P_B'$ (possibly not the smallest possible). The gadget of $r$ is substituted by a (possibly empty) removal gadget $Y_r^{rem}$ simulating the deactivation of $r$ in $P_B$: $f(U_A' \setminus \{r\}, R_A' \setminus R_A) = (U_B' \setminus f(r), R_B' \setminus f(R(r)), Y_r^{rem})$. If the removal gadget is empty for all combinatorial elements, we call the modularity strong, otherwise weak.

This definition of a gadget reduction for combinatorial decision problems ensures that the gadgets are uniquely relatable to the generating combinatorial elements and the gadgets are modular. Note that only combinatorial elements from $P_A$ can be removed such that the new instance $P_A'$ is a validly encoded instance. That is, combinatorial elements cannot be removed in general as this may void the validity of the instance, e.g. in UstCon the universe elements $s$ and $t$ cannot be deleted.
Additionally, the solution size has to adapt to the modularity of the gadgets in the universe gadget reduction. That is, if a combinatorial element in $P_A$ is removed such that the corresponding gadget in $P_B$ is removed, the solution size of the instance of $P_B$ is defined. For the sake of simplicity and because we later only use reductions from 3-Satisfiability, we define this property only for 3-Satisfiability.

5.1.1 Solution Size

In order to correctly define the Hamming distance $\kappa$ for a reduction from a problem $P_{HDRR}^A$ to $P_{HDRR}^B$ based on a universe gadget reduction from $P_A$ to $P_B$, we need to find a solution size function. For the sake of simplicity, we only define the solution size for universe gadget reduction from 3-Satisfiability to $P_B$. For this, we introduce variable gadgets and clause gadgets. 3-Satisfiability has literals as universe elements. Furthermore, it includes the following relations not exclusively:

- literals and negated literals \( \{ (\ell, \bar{\ell}) \mid \ell \in L \} \)
- clauses \( \{ (\ell^i, \ell^j, \ell^k) \mid (\ell^i, \ell^j, \ell^k) = c \in C \subseteq L^3 \} \)
- literal and clause \( \{ (\ell, c) \mid \ell \in c \in C \} \)
- negated literal and clause \( \{ (\ell, c) \mid \ell \in c \in C \} \)

A variable gadget exists for each literal $\ell$, $\bar{\ell}$ and consists of the literal gadgets of $\ell$ and $\bar{\ell}$. Furthermore, these include the gadget for the relation of negated literals $\{ (\ell, \bar{\ell}) \mid \ell \in L \}$. A clause gadget simulates a clause. For this, all gadgets for relations that include a clause (clause, literal and clause, negated literal and clause, literals in clause, negated literals in clause) build up the clause gadget.

A Yes-instance has a solution size, which is defined by the sum of all local solutions for each gadget: the constant gadget, the variable gadgets, the clause gadgets and the removal gadgets. Each gadget contributes to the target solution size with a fixed pre-defined amount. We define the solution size as follows

▶ Definition 24 (3-Satisfiability-Reduction Solution Size Function). Let $P_B$ be a problem such that a universe gadget reduction $f$ from 3-Satisfiability to $P_B$ exists. Let $(L, C)$ be a 3-Satisfiability-instance. Thus, there is the set of gadgets $\Upsilon(L, C)$ dependent on $(L, C)$ consisting of a (possibly empty) constant gadget, variable gadgets, clause gadgets and (possibly empty) removal gadgets. These gadgets have a local solution size of $\text{size}(Y)$ for each $Y \in \Upsilon(L, C)$. The function

\[
\text{size}_f : 3\text{-SAT} \rightarrow \mathbb{N} : (L, C) \mapsto \sum_{Y \in \Upsilon(L, C)} \text{size}(Y)
\]

describes the target solution size over universe elements of $f((L, C)) = R_B$ for $R_B$ to be a YES-instance of $P_B$.

5.2 Properties of Universe Gadget Reductions

The definition of universe gadget reductions implies the following three properties, which are specifically desired as illustrated before.

▶ Lemma 25. A universe gadget reduction is total and one-to-many. The inverse to a universe gadget reduction is many-to-one.

Proof. Let $P_A$ and $P_B$ combinatorial problems with $P_A \preceq UGR P_B$. For every relation element $x \in \bigcup_i R_A^i$, the mappings $f_{R_A^i \rightarrow R_B}(x)$ map to corresponding relation elements of $P_B$. 

By definition of a universal gadget reductions every relation element of $P_B$ is based on such a mapping or is part of the constant gadget $Y_{\text{const}}$ such that universal gadget reductions are total. By the definition of the mappings and the constant gadget, universe gadget reductions are one-to-many because a relation element $y \in \bigcup_j R^j_B$ of $P_B$ can be only mapped by one mapping from a relation element $x \in \bigcup_i R^i_A$ or is part of $Y_{\text{const}}$.

Thus by definition, it is ensured that each element $y \in Y_{\text{const}} \cup \bigcup_j R^j_B$ of $P_B$ is left unique.

**Lemma 26.** Polynomial Universe gadget reductions are transitive. That is, if there are problems $P_A$, $P_B$, $P_C$ with $P_A \preceq_p UGR P_B$ and $P_B \preceq_p UGR P_C$, then $P_A \preceq_p UGR P_C$.

**Proof.** Let $P_A$ be a combinatorial decision problem with relations $R_A$, $P_B$ a combinatorial decision problem with relations $R_B$ and $P_C$ a combinatorial decision problem with relations $R_C$. Firstly, the concatenation of the mappings $f_{R_A}^r, f_{R_B}^r : R_A^r \to R_B^r$ and $f_{R_B}^r, f_{R_C}^r : R_B^r \to R_C^r$ has to preserve the following property: Let $z \in R_C^r$ for some $k \in I_C$, then either $z \in Y_{A \to C}$ or there is exactly one $(i, k) \in I_A \times I_C$ and exactly one $x \in R_A^i$ such that $z = f_{R_A}^r, f_{R_C}^r (x)$. Let $z \in R_C^r$ for some $k \in I_C$.

**Case 1** $z \in Y_{A \to C}$. Then $z$ is generated as part of the constant gadget of the reduction from $P_B$ to $P_C$. Thus, $z \in Y_{\text{const}}$.

**Case 2** $z \notin Y_{A \to C}$. There is exactly one $(j, k) \in I_B \times I_C$ and exactly one $y \in R_A^j$ such that $z = f_{R_A}^j, f_{R_C}^r (y)$. Then, $y \in R_B^j$ for some $j \in I_B$.

**Case 2.1** $y \in Y_{A \to B}$. Then $z$ is generated by exactly one element of $y \in Y_{A \to B}$. Thus, $z \in Y_{\text{const}}$.

**Case 2.2** There is exactly one $(i, j) \in I_A \times I_B$ and exactly one $x \in R_A^i$ such that $y = f_{R_A}^i, f_{R_B}^j (x)$. Thus by definition, of the universe gadget reduction, $z$ is generated by exactly on $(i, j, k) \in I_A \times I_B \times I_C$ and exactly on $x$ with $z = f_{R_A}^i, f_{R_B}^j, f_{R_C}^k (x)$.

Furthermore, the modularity of the gadgets is preserved. If a relation element $r$ in $P_A$ is deleted, its gadgets are deleted from $P_B$ in the reduction $f_{P_A, P_B}$ and the instance of $P_B$ is the correct instance. Because the elements are deleted in $P_B$ (one after another), the reduction $f_{P_A, P_B}$ deletes the corresponding gadgets in $P_C$, whereby the the instance in $P_C$ stays correct for all deletions.

Furthermore, the solution size function has to adhere to the modularity of the universe gadget reduction.

**Lemma 27.** The solution size function is modular. In other words, let $(L, C)$ and $(L', C')$ be instances of 3-Satisfiability with $L' \subseteq L$ and $C' \subseteq C$. Furthermore, let $f$ be a reduction from 3-Satisfiability to $P_B$ such that $f(L', C')$ results form $f(L, C)$ by modularity. Then,

$$
size_f(L', C') = \sum_{Y \in \Omega(L', C')} size(Y)
$$

$$
= size(Y_{\text{const}}) + \sum_{l \in L'} size(Y_l) + \sum_{c \in C'} size(Y_c) + \sum_{x \in (L \setminus L') \cup (C \setminus C')} size(Y_{x, rem}).
$$

**Proof.** This follows from the modularity of the gadgets for the instance $(L, C)$. It is possible to remove variable gadgets and clause gadgets such that both $(L', C')$ and $(L'', C'')$ are correct reductions. The solution size of all three instances is then correct by definition.
Now, we prove a general reduction from 3-Satisfiability based on the structure that an universe gadget reduction provides.

**Theorem 28.** If there is a polynomial time universe gadget reduction from 3-Satisfiability to $P_B$ with a corresponding solution size function, then there is a polynomial time reduction for the Hamming distance recoverable robust version of $P_B$ with Hamming distance over the universe elements, transforming the scenarios accordingly. That is, if 3-Satisfiability $\preceq_{P_B} P_B$, then 3-Satisfiability $^{\text{HDRR}}$ $\preceq_{P_B} P_B^{m\text{-HDRR}}$, where the Hamming distance is defined over the universe elements.

**Proof.** In the following, we prove that 3-Satisfiability $^{\text{HDRR}}$ is reducible to $P_B^{\text{HDRR}}$ by reusing the reduction of $\exists y \forall \exists$-Satisfiability to 3-Satisfiability $^{\text{HDRR}}$. This also proves both the induction basis and the induction step corresponding to the proof of Theorem 23 because the set $Z$ is able to absorb the lower levels of recovery.

Let $(L, C) = (X, X', Y, Z, C)$ be the 3-Satisfiability $^{\text{HDRR}}$ instance, where

- $X$ is the literal set that is active in all base and recovery scenarios,
- $X'$ the literal set that is active only in the base scenario,
- $Y$ is the set of xor-dependencies, whereby $Y' \cup Y' = Y$ with $|Y'| = |Y|$ such that $y_i$ xor $y_i'$ or
- $\Gamma$-set scenarios, whereby $E \subseteq Y' \cup Y' = Y$ with $|E| \leq \Gamma$ describe all scenarios and
- $Z$ is the literal set only active in all recovery scenarios.

We substitute the scenarios by $X, X', Y, Z$ for $P_B^{\text{HDRR}}$.

Now, store the results of each mapping $f_{R_i, R_B} (r)$ in a table, for every relation $R_i$ and element $r \in R_i$. This is clearly computable in polynomial time, as we just store the gadget $Y_i \subseteq \bigcup R_B$, which is computable in polynomial time as we have a polynomial time universe gadget reduction between 3-Satisfiability and $P_B$. With this, we can now compute the scenarios in polynomial time with the following principle. The idea is to activate the variable and clause gadgets, whenever the variable or clause is active. The variables of $Y$, however, have a more complex (de)activation operation. For this, we reuse the reduction from $\exists y \forall \exists$-Satisfiability to 3-Satisfiability $^{\text{HDRR}}$. For each $y_i \in Y$, we not only deactivate the variable gadget but also the corresponding clauses as in Table 1.

| xor-dependencies | $y_i = y_i'$ | $y_i = y_i'$ |
|-------------------|--------------|--------------|
| and $\Gamma$-set scenarios | $y_i' \leftrightarrow 1$ | $y_i' \leftrightarrow 0$ |
|                     | $y_i' \leftrightarrow y_i'_{1,1}$ | $y_i' \leftrightarrow y_i'_{0,0}$ |
|                     | $y_i' \leftrightarrow \overline{y}_0$ | $y_i' \leftrightarrow \overline{y}_1$ |

| $\Gamma$-set scenarios | $y_i' \leftrightarrow s_i$ | $y_i' \leftrightarrow s_i$ |

**Table 1** The clauses to (de)activate for xor-dependencies and $\Gamma$-set scenarios or $\Gamma$-set scenarios only.

This is possible because the reduction is modular, thus exactly the variable gadget of $y_i$ and the gadgets of corresponding clauses are removed (with the possible addition of removal gadgets for $y_i$ and the corresponding clauses). By the correctness of the reduction from $\exists y \forall \exists$-Satisfiability to 3-Satisfiability $^{\text{HDRR}}$ and the universe gadget reduction from 3-Satisfiability $^{\text{HDRR}}$ to $P_B^{\text{HDRR}}$, the reduction remains correct.

Overall, the scenarios are built more formally as follows. We start with the instance $(X, X', Y, Z, C)$ interpreted as 3-Satisfiability instance. That is, all literals $X, X', Y, Z$ and clauses $C$ are active. This instance is still polynomial in the input because $X, X', Y, Z$
and the clauses $C$ are part of the input. We construct the reduction instance according to the reduction 3-SATISFIABILITY $\leq_{p}^{uGR} P_B$. Based on this, we deactivate the necessary gadgets to construct the reduction instance of $P_B^{HDRR}$.

More precisely in the base scenario of 3-SATISFIABILITY-$^{HDRR}$, the 3-SATISFIABILITY-instance consists of literals from $X$ and $X'$ and the clauses including $X, X'$. Thus in $P_B^{HDRR}$, the literal gadgets of $X$ and $X'$ are active and all gadgets of relation elements that include only $X$ or $X'$ are active. On the other hand, all literal gadgets and relation element gadgets including one of the literals in $Y$ and $Z$ are inactive. Because the gadgets of the universe gadget reduction are modular, the reduction for the base scenario in itself is correct. Furthermore, we stored the gadgets for relation elements in a table and the reduction is polynomially computable, the corresponding gadgets are easily activatable and deactivatable in polynomial time.

In the recovery scenarios, the 3-SATISFIABILITY-instance consists of literals from $X, Y$ and $Z$ and the clauses including $X, Y$ and $Z$. Therefore in $P_B^{HDRR}$, the literal gadgets of $X'$ and all gadgets of relation elements containing a literal of $X'$ are inactive, while the literal gadgets of $X$ and all relation elements that do not contain elements of $X'$ remain active. Additionally, all literal gadgets of $Z$ and all gadgets of relation elements containing $X, Y$ or $Z$ are active. The clauses that are active are $C(X,Y,Z)$, that is the clauses containing only literals of the sets $X,Y,Z$. For $\Gamma$-set scenarios also the literals of $\{s,s_1,s_2,\ldots,s_{|Y'|}\}$ as well as the clauses $y_i^f \leftrightarrow s_i$ and $y_i^f \leftrightarrow s_i$ are active for all $1 \leq i \leq |Y'|$. Therefore in $P_B^{HDRR}$, the literal gadgets of $X'$ and all gadgets of relation elements containing a literal of $X'$ are inactive, while the literal gadgets of $X$ and all relation elements that do not contain elements of $X'$ remain active. Again, we stored the gadgets for relation elements in a table and the reduction is polynomially computable.

All in all, we need to transform $X'$ to $X$ by adding all literal gadgets of $X$ and all gadgets of relation elements containing at least one element from $X'$ besides elements only from $X$ to $X'$ and all $\Gamma$-set scenarios only. The set $X$ is transformed to $X'$ by adding all literal gadgets of $X$ and all relation element gadgets containing only elements from $X$ to it. The set $Y$ is transformed to $Y'$ by adding the literal gadgets of $y_i^f$ and $y_i^f$ to it for all $i \in \{1, \ldots, |Y'|\}$. Furthermore, the clause gadget according to Table 1 have to be added. At last, $Z$ is transformed to $Z$ by adding all literal gadgets of $Z$ and all clause gadgets containing $X, Y$ or $Z$ (without the ones that contain only elements from $X$ as well as the ones from Table 1). Furthermore, $Z$ includes the gadgets of $y_i^{f,0}, y_i^{f,0}, y_i^{f,1}, y_i^{f,1}$ for both xor-dependencies and $\Gamma$-set scenarios and additionally $s, s_1, \ldots, s_{|Y'|}$ for $\Gamma$-set scenarios only.

As we considered $f$ to be a modular reduction based on instance $(L,C)$, we can set $\kappa = size_f(L,C) - size_f(X,C(X))$, whereby $C(X)$ is the set of clauses containing only literals from $X$. That is, all elements but the elements generated from $X$ are exactly the elements that are switched from the base scenario solution to the recovery scenario solution. Note that, $X'$ includes the dummy variables from the $\exists \forall \exists_3$-SATISFIABILITY reduction. More precisely, the base scenario includes the elements from $X$ and $X'$ and the recovery scenario includes the elements from $X, Y$ and $Z$. Thus, the decisions made on $X$ are fixed in all recovery scenarios. By the correctness of the universe gadget reduction and the one-to-one correspondence between the activation of the literals and the corresponding gadgets, the reduction is correct.

With these structural properties in mind, we can construct a whole set of Hamming distance recoverable robust problems. Note that the transitivity of the universe gadget reduction can be used to deduce further reductions.
5.3 Gadget Reductions for Various Combinatorial Decision Problems

In this section, we survey various but not all problems that are universe gadget reducible from 3-Satisfiability. The reductions are all well-known results or modifications of well-known results. We adapt these results to the universe gadget reduction framework to indicate that Theorem 28 is a general statement. We show the following theorem by showing that a universe gadget reduction from 3-Satisfiability exists for all the problems. For this, we use the transitivity of the reductions as illustrated in Figure 5.

▶ Theorem 29. The Hamming distance recoverable robust version of the following problems are NP-complete with polynomially computable scenarios and \( \Sigma_{2m+1}^P \)-complete with a linear number of xor-dependencies or \( \Gamma \)-set-scenarios: Vertex Cover, Dominating Set, Hitting Set, Feedback Vertex Set, Independent Set, Clique, Subset Sum, Knapsack, Partition, Two Machine Scheduling, (Un)directed Hamiltonian Cycle, (Un)directed Hamiltonian Path, Traveling Salesman, 2-Disjoint Directed Path, k-Disjoint Directed Path, 3-Coloring, k-Coloring, Cliquer Cover.

▶ Lemma 30. 3-Satisfiability is universe gadget reducible to Vertex Cover. Furthermore, it exists a solution size function for Vertex Cover. It follows that Vertex Cover\( m\)-HDRR with Hamming distance over vertices is \( \Sigma_{2m+1}^P \)-complete.

▶ Example 31 (Universe Gadget Reduction 3-Satisfiability to Vertex Cover).

As an introductory example, we take a close look at a universe gadget reduction of 3-Satisfiability to Vertex Cover, which was initially developed by Garey and Johnson [23]. This example directly proves Lemma 30. For the Vertex Cover-reduction we use the very fine-grained universe gadget reduction for each combinatorial element. In the following reductions, however, we directly use variable and clause gadgets as described in Section 5.1.1 to shorten our argumentation.

▶ Lemma 30. 3-Satisfiability is universe gadget reducible to Vertex Cover. Furthermore, it exists a solution size function for Vertex Cover. It follows that Vertex Cover\( m\)-HDRR with Hamming distance over vertices is \( \Sigma_{2m+1}^P \)-complete.

▶ Example 31 (Universe Gadget Reduction 3-Satisfiability to Vertex Cover).

The problem 3-Satisfiability consists of the universe \( L \) for the literals and the relations
- \( R_{\ell,\overline{\ell}} \) that relates a literal \( \ell \) with its negation \( \overline{\ell} \),
- \( R_{\ell,c} \) that relates a literal \( \ell \) to a clause \( c \) iff \( \ell \in c \) and
- \( R_{\ell,\ell',c} \) that relates literals \( \ell \) and \( \ell' \) iff \( \ell, \ell' \in c \).

The problem Vertex Cover, on the other hand, consists of vertices \( V \) and edges \( E \) that form a graph \( G = (V,E) \). Based on these universe and relations, the gadgets as in Figure 6 can be found. Therefore, we define the mappings:

\[
\begin{align*}
f_{L,V} & \colon f_{L,E} \colon f_{R_{\ell,\overline{\ell}}V} \colon f_{R_{\ell,c}V} \colon f_{R_{\ell,\ell',c}V} \colon f_{R_{\ell,c}E} \colon f_{R_{\ell,\ell',c}E}
\end{align*}
\]

The dashed vertices indicate that these are part of a different gadget.
The Complexity Classes of Hamming Distance Recoverable Robust Problems

(a) Literal Gadget for literal \( \ell \in L \). The corresponding mappings are \( f_{L,V} : \ell \mapsto \{v_\ell\} \) and \( f_{L,E} : \ell \mapsto \emptyset \).

(b) Gadget for relation \( R_{\ell,z} \) for some literal \( \ell \in L \). The corresponding mappings are \( f_{R_{\ell,z},V} : (\ell, z) \mapsto \emptyset \) and \( f_{R_{\ell,z},E} : (\ell, z) \mapsto \{\{v_\ell, v_z\}\} \).

(c) Gadget for relation \( R_{\ell,c} \) for literal \( \ell \in L \) with \( \ell \in c \in C \). The corresponding mappings are \( f_{R_{\ell,c},V} : (\ell, c) \mapsto \{v_\ell, c\} \) and \( f_{R_{\ell,c},E} : (\ell, c) \mapsto \{\{v_\ell, v_c\}\} \).

(d) Gadget for relation \( R_{\ell',c} \) for literals \( \ell, \ell' \in L \) with \( \ell, \ell' \in c \in C \). The corresponding mappings are \( f_{R_{\ell',c},V} : (\ell, \ell', c) \mapsto \emptyset \) and \( f_{R_{\ell',c},E} : (\ell, \ell', c) \mapsto \{\{v_\ell, v_{\ell'}, v_c\}\} \).

Figure 6: The gadgets for the universe and all relations for the 3-SATISFIABILITY-VERTEX COVER reduction.

A complete overview based on the example 3-SATISFIABILITY-formula

\[
L = \{\ell_1, \ell_2, \ell_3, \bar{\ell}_1, \bar{\ell}_2, \bar{\ell}_3\}, C = \{\{\ell_1, \ell_2, \ell_3\}, \{\bar{\ell}_1, \bar{\ell}_2, \bar{\ell}_3\}\}
\]

can be found in Figure 7. On the other hand, the reduction based on variable and clause gadgets can be established. For this, the relations from above are combined in the gadgets.

- The universe \( L \) is combined with relation \( R_{\ell,z} \) to a variable gadget for variable \( x \in X \).
- The relations \( R_{\ell,c} \) and \( R_{\ell',c} \) are combined to one clause gadget that connects the corresponding variable gadget correctly to a clause \( c \in C \).

These gadgets are depicted in Figure 8 in which the dashed vertices indicate that these are part of a different gadget. Observe that the gadgets only combine the more fine-grained

(a) Variable Gadget representing literals \( \ell, \ell' \in L \). The corresponding mappings are \( f_{L,V} : \ell \mapsto \{v_\ell, v_{\ell'}\} \) and \( f_{L,E} : \ell \mapsto \{\{v_\ell, v_{\ell'}\}\} \).

(b) Gadget for clause \( c \in C \). The corresponding mappings are \( f_{C,V} : \ell \mapsto \{v_{\ell_1, c}, v_{\ell_2, c}, v_{\ell_3, c}\} \) and \( f_{C,E} : \ell \mapsto \{\{v_{\ell_1, c}, v_{\ell_2, c}, v_{\ell_3, c}\}, \{v_{\ell_1, c}, v_{\ell_2, c}, v_{\ell_3, c}\}, \{v_{\ell_1, c}, v_{\ell_2, c}, v_{\ell_3, c}\}\}. \)

Figure 8: Gadgets for universe and relations for the 3-SATISFIABILITY-VERTEX COVER reduction.
relations and the overall reduction stays the same. That is, the reduction is overall the same for both views and can be found in Figure 7 as well. The existence of a variable gadget and a clause gadget also shows the modularity of this reduction. One can easily remove the variable gadget and all clauses gadgets containing that variable or removing just one clause gadget. The resulting graph is the correct reduction graph of the corresponding 3-Sat-instance.

The solution size function for each gadget is easy to find and fulfills the necessary conditions by the modularity of the gadget reduction. A solution includes one vertex for each literal in the solution of the 3-Sat-instance and two vertices for each clause. Thus, \( \text{size}(L, C) = |L|/2 + 2|C| \).

\[\boxed{\text{Lemma 32. Vertex Cover is universe gadget reducible to Dominating Set. Furthermore, it exists a solution size function for Dominating Set. It follows that Dominating Set}^{m-HDRR} \text{ with Hamming distance over vertices is } \Sigma^P_{2m+1} \text{-complete.}}\]

**Proof.** There is a folklore reduction that is a universe gadget reduction. The reduction adds for every edge \( \{u, v\} \) between vertices \( u, v \in V \) a vertex \( uv \) together with edges \( \{uv, u\} \), \( \{uv, v\} \). The universe elements of both problems are the vertices \( V \) and the relations are the edges \( E \). Thus, there are vertex gadgets, see Figure 9a, defined by \( f_{V,V} : v \mapsto \{v\} \) and \( f_{V,E} : \{u, v\} \mapsto \emptyset \).

\[\begin{align*}
(\text{a}) \text{ Vertex Gadget for } v \in V. & \text{ The corresponding mappings are } f_{V,V} : v \mapsto \{v\} \text{ and } f_{V,E} : \{u, v\} \mapsto \emptyset. \\
(\text{b}) \text{ Edge Gadget for } v \in V. & \text{ The corresponding mappings are } f_{E,V} : \{u, v\} \mapsto \{uv\} \text{ and } f_{E,E} : \{u, v\} \mapsto \{\{u, v\}, \{u, uv\}, \{uv, v\}\}.
\end{align*}\]

It is easy to see that both properties of a universe gadget reduction is fulfilled. The gadgets are disjoint. Furthermore, removing a vertex (and its incident edges) results in removing the corresponding vertex gadget and edge gadgets and the reduction remains correct. Removing only an edge resulting in removing the edge gadget also remains correct. At last, we consider the solution size function, which remains \( \text{size}(L, C) = |L|/2 + 2|C| \) as in the Vertex Cover reduction due to the transitivity of the universe gadget reduction and because the solution of the Dominating Set and Vertex Cover build up a one-to-one correspondence.

\[\boxed{\text{Lemma 33. Vertex Cover is universe gadget reducible to Hitting Set. Furthermore, it exists a solution size function for Hitting Set. It follows that Hitting Set}^{m-HDRR} \text{ with Hamming distance over the ground set is } \Sigma^P_{2m+1} \text{-complete.}}\]

**Proof.** The reduction of Karp [28] from Vertex Cover to Hitting Set is a universe gadget reduction. Vertex Cover consists of vertices \( V \) and edges \( E \). The universe of Hitting Set is a set \( U \) and the relations are subsets \( s_i \subseteq U \) for \( 1 \leq i \leq r \). Every vertex \( v \in V \) is mapped to \( v \in U \) and every edge \( (u, v) \in E \) is mapped to a subset \( s = \{u, v\} \). By the one-to-one correspondence of vertices and elements of the universe of Hitting Set and
the edges and the subsets $s_i$, the gadgets are disjoint and uniquely traceable to their origin and the reduction is modular. At last, we consider the solution size function, which remains \( \text{size}(L,C) = |L|/2 + 2|C| \) because of the transitivity of the universe gadget reduction and the solution of Hitting Set and Vertex Cover build up a one-to-one correspondence.

**Lemma 34.** Vertex Cover is universe gadget reducible to Feedback Vertex Set. Furthermore, it exists a solution size function for Feedback Vertex Set. It follows that Feedback Vertex Set$^m$-HDRR with Hamming distance over vertices is $\Sigma^P_{2m+1}$-complete.

**Proof.** The reduction of Karp [28] is a universe gadget reduction. Again, Vertex Cover consists of vertices $V$ and edges $E$. Feedback Vertex Set consists of vertices $V'$ and arcs $A'$. The reduction maps every vertex $v \in V$ to $v \in V'$ and every edge $(u,v) \in E$ is mapped to two arcs $(u,v)$ and $(v,u)$ in $A'$. Because of the one-to-one correspondence between Vertex Cover and Feedback Vertex Set and the transitivity of universe gadget reductions, the solution size remains $\text{size}((L,C)) = |L|/2 + 2|C|$. Furthermore, the one-to-one correspondence guarantees the modularity and the pre-images of all gadgets are unique.

**5.3.2 Independent Set**

**Lemma 35.** 3-Satisfiability is universe gadget reducible to Independent Set. Furthermore, it exists a solution size function for Independent Set. It follows that Independent Set$^m$-HDRR with Hamming distance over vertices is $\Sigma^P_{2m+1}$-complete.

**Proof.** For Independent Set, we reuse the 3-Satisfiability-Vertex Cover reduction from Garey and Johnson [23]. For 3-Satisfiability, we use the literals as universe elements and the relations $R_{\ell,\overline{\ell}}$, which relates a literal and its negation, $R_{\overline{\ell},c}$, which relates a clause with the negation of the its literals, $R_{\overline{\ell},c}$, which relates the literal and its negation with the clauses the literal is in. Independent Set, on the other side, consists of vertices $V$ and edges $E$. This results in the mappings for the variable gadget, see Figure 10a

\[ f_{L,V} : f_{L,E} : f_{R_{\ell,\overline{\ell}}} : f_{R_{\overline{\ell},c}} : \]

and the clause gadget, see Figure 10b

\[ f_{R_{\ell,c}} : f_{R_{\overline{\ell},c}} : f_{R_{\ell,c}} : f_{R_{\overline{\ell},c}} : f_{R_{\ell,c}} : \]

**Figure 10** Gadgets for universe and relations for the 3-Satisfiability-Independent Set reduction

Analogously to the Vertex Cover-reduction, this reduction is a universe gadget reduction. Furthermore, the solution size function includes one vertex for each variable gadget and
one vertex for each clause gadget. Thus, \( \text{size}(L, C) = \frac{|L|}{2} + |C| \). With the same arguments as for the \textsc{Vertex Cover}-reduction, the solution size function is modular.

\[ \text{Lemma 36.} \ \text{INDEPENDENT SET is universe gadget reducible to CLIQUE. Furthermore, it exists a solution size function for CLIQUE. It follows that CLIQUE}^{m\text{-HDRR}} \ \text{with Hamming distance over vertices is } \Sigma_{2m+1}^p\text{-complete.} \]

\textbf{Proof.} For \textsc{CLIQUE}, we reuse the easy-to-see \textsc{Vertex Cover}-\textsc{Independent Set}-\textsc{CLIQUE} equivalence from Garey and Johnson [23]. The problem \textsc{Independent Set} consists of a graph with vertices \( V \) and edges \( E \). On the other hand, we define \textsc{CLIQUE} with vertices \( V' \) as universe but a different relation \( \overline{E} \subseteq V' \times V' \), the set of non-edges. This definition of \textsc{CLIQUE} allows us to use the equivalence as universe gadget reduction.

For the reduction, we map every vertex \( v \in V \) to the vertex \( v' \in V' \) and we map every edge \( e \in E \) to a non-edge \( e' \in \overline{E} \). Thus, we have a one-to-one correspondence between the vertices and the edges and non-edges. This one-to-one correspondence also holds for the solution. That is, every solution of one problem is also a solution to the other problem.

By this one-to-one correspondence, the modularity, the pre-image uniqueness and the solution size of \( \text{size}(L, C) = \frac{|L|}{2} + |C| \) remains.

\[ \text{5.3.3 Subset Sum} \]

\[ \text{Lemma 37.} \ \text{3-SATISFIABILITY is universe gadget reducible to SUBSET SUM. Furthermore, it exists a solution size function for SUBSET SUM. It follows that SUBSET SUM}^{m\text{-HDRR}} \ \text{with Hamming distance over the numbers is } \Sigma_{2m+1}^p\text{-complete.} \]

\textbf{Proof.} The reduction from \textsc{3-Satisfiability} to \textsc{Subset Sum} is based on the reduction by Sipser [33]. For \textsc{3-Satisfiability}, we use the literals as universe elements and the relations \( R_{\ell, \overline{\ell}} \), which relates a literal and its negation, the clause relation \( R_c \), which is a unary relation on the clauses, and \( R_{\ell,c} \), which relates a clause with the negation of its literals.

\textsc{Subset Sum}, on the other side, consists of binary numbers of \{0, 1\}. For the sake of simplicity, the reduction description uses non-binary numbers. Numbers that are bigger than one are easily translatable in corresponding binary numbers with an offset such that a possible carry has no influence. This results in the mappings for the variable gadget, see Figure 11

\[ f_{L,\{0,1\}^*}, \]

and the clause gadget, see both Figures 12 and 13

\[ f_{R_c,\{0,1\}^*}, f_{R_{\ell,c},\{0,1\}^*}. \]

\begin{center}
\begin{tabular}{cccccccccc}
\ell_1 & \overline{\ell}_1 & \ldots & \ell_n & \overline{\ell}_n & x_1 & x_2 & \ldots & x_n & c_1 & c_2 & \ldots & c_m \\
1 & 0 & \ldots & 0 & 0 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 & 0
\end{tabular}
\end{center}

\[ \text{Figure 11} \ \text{Variable Gadget representing literals } \ell, \overline{\ell} \in L \ (\text{here for } \ell_1 \text{ and } \overline{\ell}). \]

The target sum in \textsc{Subset Sum} plays a crucial role to simulate the satisfaction of the clause correctly. In Figure 14 the target sum is depicted.

The modularity of this problem is weak, such that a removal gadget for a literal \( \ell \in L \) has to be defined. This gadget simulates the fulfillment of the clauses that contain \( \ell \) or \( \overline{\ell} \).
The Complexity Classes of Hamming Distance Recoverable Robust Problems

\[
\begin{array}{cccccccccccc}
\ell_1 & \ell_1 & \ldots & \ell_n & \ell_n & x_1 & x_2 & \ldots & x_n & c_1 & c_2 & \ldots & c_m \\
0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 11 & 0 & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 12 & 0 & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 13 & 0 & 0 & 0 \\
\end{array}
\]

**Figure 12** Clause Gadget for \(c \in C\) (here for \(c_1\)).

\[
\begin{array}{cccccccccccc}
\ell_1 & \ell_1 & \ldots & \ell_n & \ell_n & x_1 & x_2 & \ldots & x_n & c_1 & c_2 & \ldots & c_m \\
1 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 0 & 0 & 0 \\
\end{array}
\]

**Figure 13** Literal Clause Gadget for \(\ell \in c \in C\) (here for \(\ell_1 \in c_1\) and \(\ell_1 \notin c_2, c_m\)).

\[
\sum \begin{array}{cccccccccccc}
\ell_1 & \ell_1 & \ldots & \ell_n & \ell_n & x_1 & x_2 & \ldots & x_n & c_1 & c_2 & \ldots & c_m \\
1 & 1 & \ldots & 1 & 1 & 1 & 1 & \ldots & 1 & 14 & 14 & \ldots & 14 \\
\end{array}
\]

**Figure 14** The target value \(k\) of the sum.

For this, the sum for the column of the literals in \(\ell\) and \(\overline{\ell}\) as well as the column of variable \(x\) corresponding to \(\ell\) is set to 1. Furthermore, the columns for each clause that contains \(\ell\) or \(\overline{\ell}\) is set to 14. The gadget is presented in Figure 15. The removal gadget of a clause is the addition of a number that simulates the satisfaction of that clause. This gadget is depicted in Figure 16.

\[
\begin{array}{cccccccccccc}
\ell_1 & \ell_1 & \ldots & \ell_n & \ell_n & x_1 & x_2 & \ldots & x_n & c_1 & c_2 & \ldots & c_m \\
1 & 1 & \ldots & 0 & 0 & 1 & 0 & \ldots & 0 & 11 & 0 & 0 & 0 \\
1 & 1 & \ldots & 0 & 0 & 1 & 0 & \ldots & 0 & 12 & 0 & 0 & 0 \\
1 & 1 & \ldots & 0 & 0 & 1 & 0 & \ldots & 0 & 13 & 0 & 0 & 0 \\
1 & 1 & \ldots & 0 & 0 & 1 & 0 & \ldots & 0 & 14 & 0 & 0 & 0 \\
\end{array}
\]

**Figure 15** The Removal Gadget for \(\ell \in L\) (here for \(\ell_1 \in c_1\) and \(\ell_1 \notin c_2, c_m\)).

\[
\begin{array}{cccccccccccc}
\ell_1 & \ell_1 & \ldots & \ell_n & \ell_n & x_1 & x_2 & \ldots & x_n & c_1 & c_2 & \ldots & c_m \\
0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 14 & 0 & 0 & 0 \\
\end{array}
\]

**Figure 16** The Removal Gadget for \(c_1 \in C\).

At last, we describe the solution size function over the numbers. Due to weak modularity, we use a reduction \(f\) from an instance \((L', C')\) with \(L \subseteq L'\) and \(C \subseteq C'\) such that

\[
size_f(L, C) = 2 \cdot |L|/2 + |C| + |L' \setminus L|/2 + |C' \setminus C|.
\]

Overall, we include two numbers for each variable and one number for each clause. If a variable is removed, we include the removal gadget for the variable instead of the gadgets for the literal and the literal clause relation. The removal of a clause (without also removing a variable), however, adds one element, which simulates the satisfaction by adding 14 to the column of the clause. Because the clause is removed and the column still needs to sum up to 14, we have to add one element for each removed clause.

▶ **Lemma 38.** \textit{Subset Sum is universe gadget reducible to Knapsack. Furthermore, it exists a solution size function for Knapsack. It follows that Knapsack}^{m-HDRR} \textit{with Hamming distance over the objects is \(\Sigma_{2m+1}^P\)-complete.}
Proof. The reduction is easy-to-see because \textsc{Subset Sum} is a special case of \textsc{Knapsack}. The numbers in \textsc{Subset Sum} are mapped to objects of the weight and price corresponding to the value of the number. By setting the knapsack capacity and the price threshold to the target sum of \textsc{Subset Sum}, the reduction is complete.

Overall, this is a one-to-one correspondence between all combinatorial elements and the solutions. Thus, the modularity and the solution size function of \textsc{Subset Sum} trivially applies to \textsc{Knapsack} as well.

Lemma 39. \textsc{Subset Sum} is universe gadget reducible to \textsc{Partition}. Furthermore, it exists a solution size function for \textsc{Partition}. It follows that \textsc{Partition}^m-HDRR with Hamming distance over the numbers is $\Sigma^P_{2m+1}$-complete.

Proof. The reduction from \textsc{Subset Sum} to \textsc{Partition} by Karp [28] is a universe gadget reduction. The numbers $A$ in \textsc{Subset Sum} are transferred to the \textsc{Partition} instance and remain unchanged. Furthermore, let $k$ be the target sum of \textsc{Subset Sum}, then $k + 1$ and $1 - k + \sum_{a \in A} a$ are added to the \textsc{Partition} instance as well. This builds up the constant gadget.

Overall, the numbers from \textsc{Subset Sum} and \textsc{Partition} are one-to-one correspondent as well as the solutions. Thus by transitivity, the modularity, the pre-image uniqueness and the solution function remain as in \textsc{Subset Sum}.

Lemma 40. \textsc{Subset Sum} is universe gadget reducible to \textsc{Two Machine Scheduling}. Furthermore, it exists a solution size function for \textsc{Two Machine Scheduling}. It follows that \textsc{Two Machine Scheduling}^m-HDRR with Hamming distance over the jobs is $\Sigma^P_{2m+1}$-complete.

Proof. \textsc{Partition} is a special case of \textsc{Two Machine Scheduling}. By interpreting the numbers in the \textsc{Partition} instance to be the job times in \textsc{Two Machine Scheduling} and by interpreting the sets of the partition as two identical machines, we have an easy to see one-to-one correspondence between the combinatorial elements and the solutions as well. The solution size function remains the same and the pre-image uniqueness and the modularity hold by transitivity.

5.3.4 Hamiltonian Path

Lemma 41. \textsc{3-Satisfiability} is universe gadget reducible to \textsc{Directed Hamiltonian Path}. Furthermore, it exists a solution size function for \textsc{Directed Hamiltonian Path}. It follows that \textsc{Directed Hamiltonian Path}^m-HDRR with Hamming distance over arcs is $\Sigma^P_{2m+1}$-complete.

Proof. The reduction of Arora and Barak [2] is a universe gadget reduction. For \textsc{3-Satisfiability}, we use the literals as universe elements and the relations $R_{\ell,\overline{\ell}}$, which relates a literal and its negation, and $R_{\ell,c}$, which relates a clause with the negation of its literals.

\textsc{Hamiltonian Cycle}, on the other side, consists of vertices $V$ and arcs $A$. This results in the mappings for the variable gadget, see Figure 18

\[ f_{L,V}, f_{L,A}, f_{R_{\ell,\overline{\ell}}, V}, f_{R_{\ell,\overline{\ell}}, A}, \]

and the clause gadget, see Figure 19

\[ f_{R_{\ell,c}, V}, f_{R_{\ell,c}, A}, \]
In order to connect the variable gadgets, we also need a constant gadget defined by mappings $f_{\text{const},V}$ and $f_{\text{const},E}$, see Figure 17.

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{constant_gadget.png}
\caption{Constant Gadget for the reduction.}
\end{figure}

This reduction is only weakly modular because removing a variable $x_i$ results in a disconnected graph. This problem can be easily solved, by defining the removal gadget as depicted in Figure 20. The direct removal of clause, however, is without a problem possible.

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{variable_removal_gadget.png}
\caption{Variable Removal Gadget representing literals $\ell, \bar{\ell} \in L$.}
\end{figure}

At last, we describe the solution size function over the arcs. Due to weak modularity, we use a reduction $f$ from an instance $(L', C')$ with $L \subseteq L'$ and $C \subseteq C'$ such that

$$size_f(L, C) = 2 + 4 \cdot |L|/2 \cdot |C'| + |C| - (4 \cdot |C'| - 1)|L' \setminus L|/2$$

Overall, we include 2 arcs for the constant gadget. Furthermore for each variable, we include $4 \cdot |C'|$ arcs for each variable. Whenever a clause is satisfied, two arcs are used, whereby one arc in the variable gadget is bypassed. Thus, one arc per clause has to be added. The removal of a variable results in the loss of $4|C'|$ arcs, whereby one arc is added to travel through the gadget.

\begin{lemma}
Directed Hamiltonian Path is universe gadget reducible to Directed Hamiltonian Cycle. Furthermore, it exists a solution size function for Directed Hamiltonian Cycle. It follows that Directed Hamiltonian Cycle$_{m-HDGR}$ with Hamming distance over arcs is $\Sigma_{2m+1}^P$-complete.
\end{lemma}
**Proof.** This reduction is easy to see. Adding an arc from \( t \) to \( s \) as constant gadget closes the cycle. This reduction is a universe gadget reduction because the combinatorial elements are mapped one to one such that the solutions are one to one translatable as well. This directly proves the pre-image uniqueness, the modularity and the solution size function, which remains the same with an additional term for one for the arcs \((t, s)\), to be correct. ◀

---

**Lemma 43.** **Directed Hamiltonian Path** is universe gadget reducible to **Undirected Hamiltonian Path**. Furthermore, it exists a solution size function for **Undirected Hamiltonian Path**. It follows that **Undirected Hamiltonian Path**\(^{\text{\text{-HDRR}}}\) with Hamming distance over edges is \(\Sigma^P_{2m+1}\)-complete.

**Proof.** Karp’s reduction \([28]\) is a universe gadget reduction. It triples the vertices and connects the triplets as depicted in Figure 21. Furthermore, each arc \((u, v)\) in the graph is mapped to an edge \(\{u^\text{out}, v^\text{in}\} \in E'\).

![Figure 21](image.png)

**Figure 21** The vertex gadget for the reduction from **Directed Hamiltonian Path** to **Undirected Hamiltonian Path**.

The pre-image uniqueness and the modularity remain because of the transitivity of universe gadget reductions. The solution size function needs to take the two edges from \(v^\text{in}\) over \(v^\text{middle}\) to \(v^\text{out}\) into account for every vertex. That is, the number of used edges in a solution of the variable is tripled. Furthermore, each clause needs three additional edges instead of one additional arc. At last, the number of edges for a variable removal gadget is 5 instead of 1. Overall, we get

\[
\text{size}_f(L, C) = 2 + 12|L|/2 \cdot |C'| + 3|C| - (12|C'| - 5)|L' \setminus L|/2.
\]

---

**Lemma 44.** **Directed Hamiltonian Cycle** is universe gadget reducible to **Undirected Hamiltonian Cycle**. Furthermore, it exists a solution size function for **Undirected Hamiltonian Cycle**. It follows that **Undirected Hamiltonian Cycle**\(^{\text{\text{-HDRR}}}\) with Hamming distance over edges is \(\Sigma^P_{2m+1}\)-complete.

**Proof.** This reduction is completely analogous to Karp’s reduction from **Directed Hamiltonian Path** to **Undirected Hamiltonian Path** (Section 5.3.4). ◀

**Lemma 45.** **Undirected Hamiltonian Cycle** is universe gadget reducible to **Traveling Salesman**. Furthermore, it exists a solution size function for **Traveling Salesman**. It follows that **Traveling Salesman**\(^{\text{\text{-HDRR}}}\) with Hamming distance over edges is \(\Sigma^P_{2m+1}\)-complete.

**Proof.** This reduction is easy to see. We consider **Traveling Salesman** to be defined over an undirected weighted graph. This graph does not have to be complete. Then, the graph \(G = (V, E)\) of the **Undirected Hamiltonian Cycle** instance can be mapped to a weighted graph \(G' = (V', E', w')\), whereby \(V = V'\) and \(E = E'\). The weights are set to 1 and the weight threshold is set to \(\infty\).

The reductions yields a one-to-one correspondence between the vertices and edges and thus between the solutions. It follows that the solution size function remains the same. The pre-image uniqueness and modularity hold per transitivity. ◀
5.3.5 2-Disjoint Path

Lemma 46. 3-Satisfiability is universe gadget reducible to 2-Disjoint Directed Path. Furthermore, it exists a solution size function for 2-Disjoint Directed Path. It follows that 2-Disjoint Directed Path$^{m-HDRR}$ with Hamming distance over arcs is $\Sigma^p_{2m+1}$-complete.

Proof. The reduction of Li et al. [29] is a universe gadget reduction. Let $(L, C)$ be the 3-Satisfiability instance. We begin with the variable gadget, which consists of two path of length $4|C|$, which are connected at the start over a vertex $x_{start}^i$ and at the end over a vertex $x_{end}^i$. The variable gadgets are concatenated and are part of the first path. Thereby, not using the upper path simulates the assignment of $\ell_i$ to 1 while not using the lower path simulates the lower path of $\ell_i$ to 1.

Next, the clause gadgets are concatenated and build up the second path. The clause gadget of clause $c_j \in C$ consists of two vertices, which are connected with arcs $(\ell_{4j-2}^i, \ell_{4j-1}^i)$ to the corresponding variable gadgets if and only if $\ell_i \in c_j$. Thus, not using the path of $\ell_i$ enables to travel over $(\ell_{4j-2}^i, \ell_{4j-1}^i)$ such that the satisfaction of the clause $c_j$ is simulated.

Furthermore, there is a constant gadget adding vertices $s_1$ and $t_1$ for the first path and $s_2$ and $t_2$ for the second path, see Figure 24, which are the start and end point of the “variable” path from $s_1$ to $t_1$ and the “clause” path from $s_2$ to $t_2$.

This reduction is only weakly modular because removing a variable $x_i$ results in a disconnected graph. This problem can be easily solved, by defining the removal gadget as depicted in Figure 25a. For this, remember that by removing a variable, the corresponding clauses are also removed. The removal of clause has induces the same problem, thus, we use an analogous gadget, see Figure 25b.
The pre-image uniqueness is easy to see due to the presented gadgets. At last, we describe the solution size function over the arcs. Due to weak modularity, we use a reduction \( f \) from an instance \((L', C')\) with \( L \subseteq L' \) and \( C \subseteq C' \) such that
\[
size_f(L, C) = 2 + (4|C'| + 1) \cdot |L|/2 + 4|C| + |L' \setminus L|/2 + 2 \cdot |C' \setminus C|
\]
Overall, we include 2 arcs for the constant gadget, which are the arcs from \( s_1 \) and \( s_2 \). Furthermore for each variable, we include \( 4|C'| + 1 \) arcs for each variable. Whenever a clause is satisfied, four arcs are used, whereby one arc in the variable gadget is used. Thus, one four per clause has to be added. The removal of a variable results in the loss of \( 4|C'| \) arcs, whereby one arc is added to travel through the gadget. At last, the removal of a clause results in the loss of 2 arcs.

\[\blacktriangleright\]

**Lemma 47.** \textit{2-Disjoint Directed Path} is universe gadget reducible to \textit{k-Disjoint Directed Path}. Furthermore, it exists a solution size function for \textit{k-Disjoint Directed Path}. It follows that \textit{k-Disjoint Directed Path} with Hamming distance over arcs is \( \Sigma_{2m+1}^{P} \)-complete.

**Proof.** The reduction from \textit{k-Disjoint Directed Path} to \textit{k + 1-Disjoint Directed Path} for \( k \geq 2 \) is easy to see. The reduction consists only of a constant gadget, which adds an additional path from \( s_{k+1} \) to \( t_{k+1} \) over the single arc \((s_{k+1}, t_{k+1})\). The solution size functions needs to include this additional arc. Because, the instance remains the same, we have a one-to-one correspondence between all combinatorial elements. Thus, modularity and pre-image uniqueness remain.

\[\blacktriangleright\]

**5.3.6 Coloring**

\[\blacktriangleright\]

**Lemma 48.** \textit{3-Satisfiability} is universe gadget reducible to \textit{3-Coloring}. Furthermore, it exists a solution size function for \textit{3-Coloring}. It follows that \textit{3-Coloring} with Hamming distance over vertices is \( \Sigma_{2m+1}^{P} \)-complete.

**Proof.** The reduction from \textit{3-Satisfiability} to \textit{Coloring} by Garey et al. \cite{Garey1979} is a universe gadget reduction. \textit{Coloring} has the vertices of the graph \( V \) as universe elements and the edges \( E \) as relation over the universe elements. We therefore have the following mappings \( f_{const,V}, f_{const,E}, f_{L,V}, f_{L,E}, f_{C,V}, f_{C,E} \):

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{constant_gadget.png}
\caption{Constant Gadget for the reduction. The corresponding mappings are \( f_{const,V} : \emptyset \mapsto \{B,F,T\} \) and \( f_{const,E} : \emptyset \mapsto \{\{B,F\}, \{B,T\}, \{F,T\}\} \)
\end{figure}

The constant function maps to a gadget consisting of three vertices and three edges as 3-clique, see Figure 26.
For the literals, the mapping $f_{L,V}$ maps a literal $\ell \in L$ to two vertices. The mapping $f_{L,E}$, on the other hand, maps a literal $\ell \in L$ to three edges connecting the two vertices $\ell$ and $\overline{\ell}$ and vertex $B$ of the constant gadget, which is generated by the constant mapping $f_{\text{const}}$, symbolized in Figure 27.

At last, we have the clause gadget. The mapping $f_{C,V}$ maps the clause to six vertices, which are depicted as circles in Figure 28. The mapping $f_{C,E}$ maps the clause to the edges as shown as solid edges in Figure 28. The dashed vertices are part of different literal gadgets and the vertices $F$ and $B$ and the three dashed edges are from the constant gadget.

Overall, all vertices and edges are either generated by the constant function or are attributable to exactly one literal or one clause of the 3-Satisfiability-instance. Furthermore, deleting a variable gadget or a clause gadget results in the correct reduction such that we have strong modularity. Thus, the reduction fulfills the universal gadget reduction properties.

The solution size function includes all vertices in one of the partitions (the colors). Thus, $\text{size}(L,C) = 2|L| + 6|C| + 3$ because every variable introduces two vertices and every clause introduces 6 vertices. The 3 additional vertices result from the constant gadget.

\begin{lemma}
\textbf{3-Coloring} is universe gadget reducible to k-\textbf{Coloring}. Furthermore, it exists a solution size function for k-\textbf{Coloring}. It follows that k-\textbf{Coloring}\textsuperscript{m-HDRR} with Hamming distance over vertices is $\Sigma_{2m+1}^P$-complete.
\end{lemma}

\begin{proof}
The reduction from k-\textbf{Coloring} to k+1-\textbf{Coloring} is easy to see. The graph $G = (V, E)$ for k-\textbf{Coloring} remains, whereby a vertex $v_{new}$ is added and connected to all existing vertices $V$. Thus, $v_{new}$ needs to have a different color than all existing vertices in $V$. This is a universe gadget reduction because the vertex $v_{new}$ is a constant gadget and every edge to $v_{new}$ is part of the vertex gadget of $v$ together with $v$ itself. The pre-image uniqueness and the modularity results from the one-to-two correspondence of vertex $v$ and the vertex gadget consisting of $v$ and the edge $\{v, v_{new}\}$. The solution size function needs to include the additional vertex $v_{new}$, thus 1 is added.
\end{proof}
Lemma 50. \textit{\textbf{\textit{k-Coloring} is universe gadget reducible to \textit{Clique Cover}. Furthermore, it exists a solution size function for \textit{Clique Cover}. It follows that \textit{Clique Cover}^m-HDRR with Hamming distance over vertices is $\Sigma^p_{2m+1}$-complete.}}

\textbf{Proof.} This reduction is analogous to the reduction from \textit{Independent Set} to \textit{Clique} due to the fact that a coloring of a graph is a partition into independent sets while a clique cover is a partitions into cliques.

6 Prospect

We have defined Hamming distance recoverable robust problems with elemental uncertainty and applied this concept to various well-known problems in \textit{NP}. Further, we have defined universe gadget reductions to build a framework for a large class of Hamming distance recoverable robust problems. The complexity results are that the Hamming distance recoverable robust versions of \textit{NP}-complete problems remain \textit{NP}-complete if the scenarios are polynomially computable and that the \textit{NP}-complete problems are $\Sigma^p_3$-complete for a linear number of \textit{xor}-dependencies and $\Gamma$-set scenarios if 3-Satisfiability is universe gadget reducible to them. Furthermore, multi-stage problems with $m$ stages result in $\Sigma^p_{2m+1}$-completeness if the encoding of scenarios are a linear number of \textit{xor}-dependencies or $\Gamma$-set scenarios.

Remaining interesting questions are whether there is a (light-weight) reduction framework for other adversial problems or robustness concepts, for example for interdiction problems or two-stage robust problems, to derive completeness for higher levels in the polynomial hierarchy than \textit{NP}. Furthermore, it is of interest whether this concept is adaptable to problems with cost uncertainty and for other distance measures. A more special question is, which succinct encodings also result in $\Sigma^p_3$-completeness or if there are succinct encodings which result in the \textit{NP}-completeness of the problem.
The Complexity Classes of Hamming Distance Recoverable Robust Problems

References

1. Manindra Agrawal, Eric Allender, Russell Impagliazzo, Toniann Pitassi, and Steven Rudich. Reducing the complexity of reductions. In Frank Thomson Leighton and Peter W. Shor, editors, Proceedings of the Twenty-Ninth Annual ACM Symposium on the Theory of Computing, El Paso, Texas, USA, May 4-6, 1997, pages 730–738. ACM, 1997. doi:10.1145/258533.258671

2. Sanjeev Arora and Boaz Barak. Computational Complexity - A Modern Approach. Cambridge University Press, 2009. URL: http://www.cambridge.org/catalogue/catalogue.asp?isbn=9780521424264

3. Matthew Bold and Marc Goerigk. Investigating the recoverable robust single machine scheduling problem under interval uncertainty. Discret. Appl. Math., 313-314, 2022. doi:10.1016/j.dam.2022.02.005

4. Christina Büsing. The exact subgraph recoverable robust shortest path problem. In Ravindra K. Ahuja, Rolf H. Möhring, and Christos D. Zaroliagis, editors, Robust and Online Large-Scale Optimization: Models and Techniques for Transportation Systems, volume 5868 of Lecture Notes in Computer Science, pages 231–248. Springer, 2009. doi:10.1007/978-3-642-05465-5_9

5. Christina Büsing. Recoverable robustness in combinatorial optimization. Cuvillier Verlag, 2011.

6. Christina Büsing. Recoverable robust shortest path problems. Networks, 59(1):181–189, 2012. doi:10.1002/net.20487

7. Christina Büsing, Sebastian Goderbauer, Arie M. C. A. Koster, and Manuel Kutschka. Formulations and algorithms for the recoverable $\Gamma$-robust knapsack problem. EURO J. Comput. Optim., 7(1):15–45, 2019. doi:10.1007/s13675-018-0107-9

8. Christina Büsing, Arie M. C. A. Koster, and Manuel Kutschka. Recoverable robust knapsacks: $\Gamma$-scenarios. In Julia Pahl, Torsten Reiners, and Stefan Voß, editors, Network Optimization - 5th International Conference, INOC 2011, Hamburg, Germany, June 13-16, 2011. Proceedings, volume 6701 of Lecture Notes in Computer Science, pages 583–588. Springer, 2011. doi:10.1007/978-3-642-21527-8_65

9. Christina Büsing, Arie M. C. A. Koster, and Manuel Kutschka. Recoverable robust knapsacks: the discrete scenario case. Optim. Lett., 5(3):379–392, 2011. doi:10.1007/s11590-011-0307-1

10. Valentina Cacchiani, Alberto Caprara, Laura Galli, Leo G. Kroon, and Gábor Marótí. Recoverable robustness for railway rolling stock planning. In Matteo Fischetti and Peter Widmayer, editors, ATMOS 2008 - 8th Workshop on Algorithmic Approaches for Transportation Modeling, Optimization, and Systems, Karlsruhe, Germany, September 18, 2008, volume 9 of OASIcs. Internationale Begegnungs- und Forschungszentrum fuer Informatik (IBFI), Schloss Dagstuhl, Germany, 2008. URL: http://drops.dagstuhl.de/opus/volltexte/2008/1590

11. Valentina Cacchiani, Alberto Caprara, Laura Galli, Leo G. Kroon, Gábor Marótí, and Paolo Toth. Railway rolling stock planning: Robustness against large disruptions. Transp. Sci., 46(2):217–232, 2012. doi:10.1287/trsc.1110.0388

12. Luis Cadarso and Ángel Marín. Recoverable robustness in rapid transit network design. Procedia - Social and Behavioral Sciences, 54:1288–1297, 2012. Proceedings of EWGT2012 - 15th Meeting of the EURO Working Group on Transportation, September 2012, Paris. URL: https://www.sciencedirect.com/science/article/pii/S1877042812043054 doi:https://doi.org/10.1016/j.sbspro.2012.09.843

13. André B. Chassein and Marc Goerigk. On the recoverable robust traveling salesman problem. Optim. Lett., 10(7):1479–1492, 2016. doi:10.1007/s11590-015-0949-5

14. Serafino Cicerone, Gianlorenzo D’Angelo, Gabriele Di Stefano, Daniele Frigioni, and Alfredo Navarra. Recoverable robust timetabling for single delay: Complexity and polynomial algorithms for special cases. J. Comb. Optim., 18(3):229–257, 2009. doi:10.1007/s10878-009-9247-4

15. Serafino Cicerone, Gianlorenzo D’Angelo, Gabriele Di Stefano, Daniele Frigioni, and Alfredo Navarra. Recoverable robustness for train shunting problems. Algorithmic Oper. Res., 4(2):102–116, 2009. URL: http://journals.hil.unb.ca/index.php/AOR/article/view/10471
Serafino Cicerone, Gianlorenzo D’Angelo, Gabriele Di Stefano, Daniele Frigioni, Alfredo Navarra, Michael Schachtebeck, and Anita Schöbel. Recoverable robustness in shunting and timetabling. In Ravindra K. Ahuja, Rolf H. Möhring, and Christos D. Zaroliagis, editors, Robust and Online Large-Scale Optimization: Models and Techniques for Transportation Systems, volume 5868 of Lecture Notes in Computer Science, pages 28–60. Springer, 2009. doi:10.1007/978-3-642-05465-5_2.

Serafino Cicerone, Gabriele Di Stefano, Michael Schachtebeck, and Anita Schöbel. Multi-stage recovery robustness for optimization problems: A new concept for planning under disturbances. Inf. Sci., 190:107–126, 2012. doi:10.1016/j.ins.2011.12.010.

Gianlorenzo D’Angelo, Gabriele Di Stefano, and Alfredo Navarra. Evaluation of recoverable-robust timetables on tree networks. In Jirí Fiala, Jan Kratochvíl, and Mirka Miller, editors, Combinatorial Algorithms, 20th International Workshop, IWOCA 2009, Hradec nad Moravicí, Czech Republic, June 28–July 2, 2009, Revised Selected Papers, volume 5874 of Lecture Notes in Computer Science, pages 24–35. Springer, 2009. doi:10.1007/978-3-642-10217-2_6.

Gianlorenzo D’Angelo, Gabriele Di Stefano, Alfredo Navarra, and Maria Cristina Pinotti. Recoverable robust timetables: An algorithmic approach on trees. IEEE Trans. Computers, 60(3):433–446, 2011. doi:10.1109/TC.2010.142.

Bert Dijk, Bruno Filipe Santos, and João P. Pita. The recoverable robust stand allocation problem: a GRU airport case study. OR Spectr., 41(3):615–639, 2019. doi:10.1007/s00291-018-0525-3.

Mitre Costa Dourado, Dirk Meierling, Lucia Draque Penso, Dieter Rautenbach, Fábio Protti, and Aline Ribeiro de Almeida. Robust recoverable perfect matchings. Networks, 66(3):210–213, 2015. doi:10.1002/net.21624.

Gary Freyland, Stephen J. Maher, and Cheng-Lung Wu. The recoverable robust tail assignment problem. Transp. Sci., 48(3):351–372, 2014. doi:10.1287/trsc.2013.0463.

M. R. Garey and David S. Johnson. Computers and Intractability: A Guide to the Theory of NP-Completeness. W. H. Freeman, 1979.

M. R. Garey, David S. Johnson, and Larry J. Stockmeyer. Some simplified np-complete graph problems. Theor. Comput. Sci., 1(3):237–267, 1976. doi:10.1016/0304-3975(76)90059-1.

Marc Goerigk, Sacha Heße, Matthias Müller-Hannemann, Marie Schmidt, and Anita Schöbel. Recoverable robust timetable information. In Daniele Frigioni and Sebastian Stiller, editors, 13th Workshop on Algorithmic Approaches for Transportation Modelling, Optimization, and Systems, ATMOS 2013, September 5, 2013, Sophia Antipolis, France, volume 33 of OASIcs, pages 1–14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2013. doi:10.4230/OASIcs.ATMOS.2013.1.

Marc Goerigk, Stefan Lendl, and Lasse Wulf. On the complexity of robust multi-stage problems in the polynomial hierarchy. CoRR, abs/2209.01011, 2022. arXiv:2209.01011 doi:10.48550/arXiv.2209.01011.

Mikita Gradovich, Adam Kasperski, and Pawel Zielinski. Recoverable robust spanning tree problem under interval uncertainty representations. J. Comb. Optim., 34(2):554–573, 2017. doi:10.1007/s10878-016-0089-6.

Richard M. Karp. Reducibility among combinatorial problems. In Raymond E. Miller and James W. Thatcher, editors, Proceedings of a symposium on the Complexity of Computer Computations, held March 20-22, 1972, at the IBM Thomas J. Watson Research Center, Yorktown Heights, New York, USA, The IBM Research Symposia Series, pages 85–103. Plenum Press, New York, 1972. doi:10.1007/978-1-4684-2001-2_9.

Chung-Lun Li, S. Thomas McCormick, and David Simchi-Levi. The complexity of finding two disjoint paths with min-max objective function. Discret. Appl. Math., 26(1):105–115, 1990. doi:10.1016/0166-218X(90)90024-7.

Christian Liebchen, Marco E. Lübbecke, Rolf H. Möhring, and Sebastian Stiller. The concept of recoverable robustness, linear programming recovery, and railway applications. In Ravindra K. Ahuja, Rolf H. Möhring, and Christos D. Zaroliagis, editors, Robust and Online Large-Scale
Richard Martin Lusby, Jesper Larsen, and Simon Bull. A survey on robustness in railway planning. *Eur. J. Oper. Res.*, 266(1):1–15, 2018. doi:10.1016/j.ejor.2017.07.044.

Stephen J. Maher, Guy Desaulniers, and François Soumis. Recoverable robust single day aircraft maintenance routing problem. *Comput. Oper. Res.*, 51:130–145, 2014. doi:10.1016/j.cor.2014.03.007.

Michael Sipser. *Introduction to the theory of computation*. PWS Publishing Company, 1997.

Luca Trevisan, Gregory B. Sorkin, Madhu Sudan, and David P. Williamson. Gadgets, approximation, and linear programming (extended abstract). In *37th Annual Symposium on Foundations of Computer Science, FOCS ’96, Burlington, Vermont, USA, 14-16 October, 1996*, pages 617–626. IEEE Computer Society, 1996. doi:10.1109/39.548521.

D.D. Tönissen and J.J. Arts. Economies of scale in recoverable robust maintenance location routing for rolling stock. *Transportation Research Part B: Methodological*, 117:360–377, 2018. URL: https://www.sciencedirect.com/science/article/pii/S0191261517311085. doi:https://doi.org/10.1016/j.trb.2018.09.006.