Rank-2 syzygy bundles on Fermat curves and an application to Hilbert–Kunz functions

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Abstract In this paper we describe the Frobenius pull-backs of the syzygy bundles \( \text{Syz}_C(X^a, Y^a, Z^a) \), \( a \geq 1 \), on the projective Fermat curve \( C \) of degree \( n \) in characteristics coprime to \( n \), either by giving their strong Harder–Narasimhan filtration if \( \text{Syz}_C(X^a, Y^a, Z^a) \) is not strongly semistable or in the strongly semistable case by their periodicity behavior. Moreover, we apply these results to Hilbert–Kunz functions, to find Frobenius periodicities of the restricted cotangent bundle \( \Omega_{\mathbb{P}^2|_C} \) of arbitrary length and a problem of Brenner regarding primes with strongly semistable reduction.

Keywords Fermat curve · Hilbert–Kunz function · Vector bundle · Strongly semistable · Frobenius periodicity · Hilbert-series · Syzygy module · Projective dimension

Mathematics Subject Classification 13A35 · 13D02 · 13D40 · 14H45 · 14H60

1 Introduction

In this paper we study rank-2 syzygy bundles for certain monomial ideals, namely the ideals \( (X^a, Y^a, Z^a) \) for \( a \geq 1 \) on a smooth projective Fermat curve \( C := \text{Proj} (k[X, Y, Z]/(X^n + Y^n + Z^n)) \). We focus mainly on positive characteristic, where semistability is not preserved by the Frobenius morphism. The first example is the
restricted cotangent bundle $\Omega_{\mathbb{P}^2|C}$ from the projective plane which can be identified with the syzygy bundle $\text{Syz}_C(X, Y, Z)$. For this bundle semistability is well known (if $n \geq 2$) but no complete answer is known regarding strong semistability, i.e., for what characteristics and curve degrees is $F^e_*(\Omega_{\mathbb{P}^2|C})$ semistable for all $e \geq 0$.

The bundles $\text{Syz}_C(X^a, Y^a, Z^a)$ on Fermat curves exhibit important arithmetic properties of vector bundles in general. For instance, Brenner proved in Brenner (2005b) with this setup that there exists no restriction theorem for strong semistability of Bogomolov type. In Brenner (2005a) he used the syzygy bundle $\text{Syz}_C(X^2, Y^2, Z^2)$ on the Fermat quintic to show that strong semistability is not an open property in arithmetic deformations which was conjectured by N. I. Shepherd–Barron. In Brenner and Kaid (2008) the authors disproved a conjecture of Joshi, which is related to the Grothendieck $p$-curvature conjecture, exploiting the arithmetic properties of Fermat curves and syzygy bundles. Despite these papers, there has been no complete treatment of the semistability properties of the bundles $\text{Syz}_C(X^a, Y^a, Z^a)$ on Fermat curves.

Fermat rings are also popular examples for characteristic $p$ methods in commutative algebra, in particular Hilbert–Kunz theory and the theory of tight closure. Monsky and Han exploited Fermat rings to obtain explicit computations for the Hilbert–Kunz multiplicity and the Hilbert–Kunz function (see Han and Monsky 1993; Han 1991; Monsky 2006a, b), which we partially complement in this paper. Due to work of Brenner (2006) and Trivedi (2005) there is a strong relationship to strongly semistable vector bundles which is our main tool to compute Hilbert–Kunz functions in this paper. We remark that for the arithmetic behavior of tight closure the ideals $(X^a, Y^a, Z^a)$ which are topic of this paper are useful examples (see Brenner and Katzman 2006; Singh 1998).

In Sect. 2 we recall the necessary notations of Hilbert–Kunz theory and vector bundles and discuss their relationship mentioned above for syzygy bundles of rank 2. Afterwards, we turn our attention to syzygy gaps and how they appear in Han’s and Monsky’s computations (Han and Monsky (1993)) of Hilbert–Kunz multiplicities of ideals generated by a fixed (positive) power of the variables in two-dimensional Fermat rings.

Section 3 is devoted to the case where the bundle $\text{Syz}_C(X^a, Y^a, Z^a)$, $a \geq 1$, is strongly semistable. We will show how work of Kustin et al. (2012) can be applied in this situation to obtain a periodicity up to twist of the Frobenius pull-backs of these bundles as well as a sharp bound for the length of this periodicity (cf. Theorem 3.9). This periodic behavior enables us to compute the Hilbert–Kunz functions of the ideals $(X^a, Y^a, Z^a)$, $a \geq 1$, in the Fermat rings $k[X, Y, Z]/(X^n + Y^n + Z^n)$ (cf. Theorem 3.12) and allows us to generalize a theorem of Brenner and the second author (cf. Theorem 3.14).

In Sect. 4 we will use the methods developed in Sect. 3 to construct for every odd prime $p$ and every natural number $l$ an $n$ such that the Frobenius pull-backs of $\Omega_{\mathbb{P}^2|C}$ admit up to twist a periodicity of length $l$, where $C$ is the projective Fermat curve of degree $n$ (see Example 4.1). Moreover, we will see in Example 4.2 that in characteristic two, the Frobenius pull-backs of the restricted cotangent bundle admit a periodicity up to twist if and only if $n = 3$.

In Sect. 5 we investigate the case where $\text{Syz}_C(X^a, Y^a, Z^a)$ is not strongly semistable. In this case we can explicitly describe the minimal $e_0$ such that the $e_0$th
Frobenius pull-back of this bundle has a strong Harder–Narasimhan filtration. Moreover, this filtration is explicitly computed (cf. Theorem 5.4). This result is used to complete the computations of Hilbert–Kunz functions from Sect. 3 (cf. Corollary 5.7). The explicit computation of the entire Hilbert–Kunz function for ideals \((X^a, Y^a, Z^a)\) with \(a > 1\) is new (at least to the best knowledge of the authors of this paper).

Finally, in Sect. 6 we deal with a special instance of the Miyaoka problem proposed by Brenner, i.e., the question how the property of \(S := \text{Syz}_C(X^a, Y^a, Z^a)\) being strongly semistable depends on the parameter \(a\), the characteristic \(p\) and the degree \(n\) of the projective Fermat curve \(C\). Theorem 6.2 will show that \(S\) is semistable in characteristic zero if and only if it is strongly semistable in all characteristics \(p \equiv \pm 1\) modulo \(2n\). From this result we will deduce that the Harder–Narasimhan filtration of \(S\) in characteristic zero can be computed via reduction modulo \(p\) (cf. Theorem 6.5).

We remark that most of the results in this paper can easily be translated into algorithms that are suitable for a computer algebra system (e.g., Abbott et al. 2015).

We would like to thank Holger Brenner for many useful discussions and comments. We also thank Axel Stäbler for his corrections to this paper. Furthermore, we thank the referee for many useful comments which helped to raise the readability of this paper. The results of Sects. 3 and 4 are contained in Chapter 7 of the first author’s PhD thesis Brinkmann (2013) and those of Sects. 5 and 6 are contained in Chapter 4 of the PhD thesis Kaid (2009) of the second author.

2 Preliminaries

Let \((R, m)\) be a local Noetherian ring of characteristic \(p > 0\) and dimension \(d\). Let \(I = (f_1, \ldots, f_m)\) be an \(m\)-primary ideal. Let \(\lambda_R\) denote the length function for \(R\)-modules. The function

\[
\text{HK}(I, p^e) : \mathbb{N} \rightarrow \mathbb{N}, \ e \mapsto \lambda_R \left( R/ \left( f_1^{p^e}, \ldots, f_m^{p^e} \right) \right)
\]

is called the Hilbert–Kunz function of \(I\). We call the limit

\[
\lim_{e \rightarrow \infty} \frac{\text{HK}(I, p^e)}{p^{ed}},
\]

whose existence was proven by Monsky in Monsky (1983), the Hilbert–Kunz multiplicity of \(I\). We denote this limit by \(e_{HK}(I)\) and call \(e_{HK}(R) := e_{HK}(m)\) the Hilbert–Kunz multiplicity of \(R\). Sometimes we will use the symbol \((f_1, \ldots, f_m)^{[p^e]}\) to denote the ideal \((f_1^{p^e}, \ldots, f_m^{p^e})\).

Throughout this paper we are mainly interested in the graded situation, where \(k\) is an algebraically closed field and \(R\) a standard-graded normal \(k\)-algebra of dimension 2. Moreover, we consider homogeneous elements \(f_1, f_2, f_3 \in R\) with \(\deg(f_1) = \deg(f_2) = \deg(f_3) = a \geq 1\) such that the ideal \(I := (f_1, f_2, f_3) \subset R\) is \(R_+\)-primary. These elements give rise to the short exact (presenting) sequences
0 → \text{Syz}_C(f_1, f_2, f_3)(m) → \bigoplus_{i=1}^{3} \mathcal{O}_C(m - a)^{f_1 \cdot f_2 \cdot f_3} \mathcal{O}_C(m) → 0,

for all \( m ∈ \mathbb{Z} \) on the smooth projective curve \( C := \text{Proj}(R) \). The kernel sheaf \( \text{Syz}_C(f_1, f_2, f_3) \) is locally free of rank 2 and is called the \textit{syzygy bundle} for \( f_1, f_2, f_3 \).

In the case \( \text{char}(k) = p > 0 \) the Hilbert–Kunz function of the ideal \( I \) can be computed via the formula

\[
\dim_k \left( R/ \left( f_1^{p^e}, f_2^{p^e}, f_3^{p^e} \right) \right)_m = h^0(C, \mathcal{O}_C(m)) - \sum_{i=1}^{3} h^0(C, \mathcal{O}_C(m - p^e a)) + h^0\left( C, \text{Syz}_C(f_1^{p^e}, f_2^{p^e}, f_3^{p^e})(m) \right)
\]

and summation over all \( m ∈ \mathbb{Z} \) (see for instance Brenner 2006).

If \( S \) is a vector bundle on a smooth projective curve over an algebraically closed field \( k \) we define its \textit{slope} by the quotient \( \mu(S) := \frac{\deg(S)}{\text{rank}(S)} \) with \( \deg(S) := \deg(\bigwedge^{\text{rank}(S)} S) \). We recall that \( E \) is \textit{semistable} if for every non-trivial locally free subsheaf \( F \) the inequality \( \mu(F) ≤ \mu(E) \) holds. The bundle \( E \) is \textit{stable} if this inequality is always strict.

Hence the rank-2 syzygy bundle \( S := \text{Syz}_C(f_1, f_2, f_3) \) is not semistable if and only if there exists a line bundle \( 0 ≠ L ⊂ E \) with \( \deg(L) > \mu(S) = \deg(S)/2 = -3an/2 \), where \( n = \deg(C) \). For such a line bundle of maximal degree this filtration constitutes the so-called \textit{Harder–Narasimhan filtration} (or \textit{HN-filtration}) of \( S \) and the line bundle \( L \) is the (maximal) destabilizing subbundle. We often write the HN-filtration of a rank-2 vector bundle as a short exact sequence \( 0 → L → S → M → 0 \), where the quotient \( M \) is a line bundle with \( \deg(M) < \deg(L) \). Note that concept and existence of a HN-filtration is trivial for bundles of rank 2 but for bundles of higher rank it is much more complicated (see Harder and Narasimham 1975).

In positive characteristic \( p \) we consider the \textit{absolute Frobenius morphism} \( F : C → C \) which is the identity on the curve \( C \) and the \( p \)-th power map on the structure sheaf \( \mathcal{O}_C \). It is well-known that the Frobenius pull-back \( F^*(S) \) of a semistable vector bundle \( S \) is in general not semistable (see for instance Hartshorne 1971, Example 3.2 for Serre’s counter example). If \( F^e(S) \) is semistable for all \( e ≥ 0 \) then \( S \) is called \textit{strongly semistable}. This notion is due to Miyaoka (cf. Miyaoka 1987, Section 5). In the case of a rank-2 vector bundle the HN-filtration \( 0 → L → F^e(S) → M → 0 \) of a (non-semistable) Frobenius pull-back \( F^e(S) \) is called the \textit{strong HN-filtration} of the bundle \( S \). We indicate that, as for the HN-filtration itself, the concept and existence of a strong HN-filtration is highly non-trivial for vector bundles of higher rank (see Langer 2004 for a detailed account).

Due to the work of Brenner (2006) and Trivedi (2005) strongly semistable vector bundles are closely related to Hilbert–Kunz theory. We only state this connection for the rank-2 vector bundles which we consider in this paper.

\textbf{Theorem 2.1} Let \( C = V_+(G) ⊂ \mathbb{P}^2 \) denote a smooth plane curve of degree \( n \) over an algebraically closed field \( k \) of positive characteristic \( p \). Let \( R = k[X, Y, Z]/(G) \) be its homogeneous coordinate ring and let \( I = (f_1, f_2, f_3) \) denote a homo-
geneous $R_+\text{-primary ideal such that } \deg(f_1) = \deg(f_2) = \deg(f_3) = a.$ Let $\mathcal{S} := \text{Syz}_C(f_1, f_2, f_3).$ Then the following hold.

(1) The bundle $\mathcal{S}$ is strongly semistable if and only if $e_{\text{HK}}(I) = \frac{3n}{4}a^2$.

(2) If $\mathcal{S}$ is not strongly semistable with strong Harder–Narasimhan filtration $0 \to \mathcal{L} \to \mathcal{F}^e(\mathcal{S}) \to \mathcal{M} \to 0$, then

$$e_{\text{HK}}(I) = n \left( \left( \frac{\deg(L)}{np^e} + \frac{3a}{2} \right)^2 + \frac{3a^2}{4} \right).$$

Moreover, we have

$$\deg(L) = -\frac{3anp^e}{2} + np^e \cdot \sqrt{-\frac{3a^2}{4} + \frac{e_{\text{HK}}(I)}{n}} \in \left(-\frac{3anp^e}{2}, -anp^e\right].$$

(3) If $\mathcal{S}$ is not semistable, then the Hilbert–Kunz multiplicity of $I$ equals

$$e_{\text{HK}}(I) = \frac{3n}{4}a^2 + \frac{\ell^2}{4n},$$

where $0 < \ell \leq na$ is an integer with $\ell \equiv an \mod 2$.

(4) If the bundle $\mathcal{S}$ is semistable, but not strongly semistable, then the Hilbert–Kunz multiplicity of $I$ equals

$$e_{\text{HK}}(I) = \frac{3n}{4}a^2 + \frac{\ell^2}{4np^{2e}},$$

where $e \geq 1$ is the number such that $\mathcal{F}^{e-1}(\mathcal{S})$ is semistable and $\mathcal{F}^e(\mathcal{S})$ is not semistable and $0 < \ell \leq n(n - 3)$ is an integer with $\ell \equiv pna \mod 2$.

**Proof** For the proof of (1) and (2) see Brenner (2006), proof of Corollary 4.6. For the proof of (3) and (4) see Trivedi (2005), proof of Theorem 5.3. In all three cases the quoted proofs only consider the case $a = 1$ but they are easy to generalize. For an explicit proof see Kaid (2009), proofs of Corollary 1.4.9 and Proposition 1.4.11.

**Remark 2.2** Since bundles of the form $\mathcal{F}^e(\mathcal{S})$ as in Theorem 2.1(2) will appear several times during this paper, we remark that one gets

$$\mathcal{F}^e(\text{Syz}_C(f_1, f_2, f_3)) = \text{Syz}_C\left(f_1^{p^e}, f_2^{p^e}, f_3^{p^e}\right)$$

from the presenting sequence of the syzygy bundle.

Next, we discuss syzygy gaps and how they appear in the computation of Hilbert–Kunz multiplicities. The final result of Han and Monsky (cf. Theorem 2.5) will combine the case by case description of the Hilbert–Kunz multiplicities of the ideals $(X^a, Y^a, Z^a)$ from the last theorem. Moreover, Theorem 2.5 will give us a numerical criterion to check the strong semistability of the sheafs $\text{Syz}_C(X^a, Y^a, Z^a)$. 

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Definition 2.3 Let $R := k[X, Y]$ and let $f_1, f_2, f_3 \in R$ be non-zero and homogeneous. By Hilbert’s Syzygy Theorem we have a splitting $\text{Syz}_R(f_1, f_2, f_3) \cong R(-\alpha) \oplus R(-\beta)$ for some $\beta \leq \alpha \in \mathbb{N}$. We call the difference $\alpha - \beta$ the syzygy gap of $f_1, f_2, f_3$ and denote it by $\text{syzgap}(f_1, f_2, f_3)$. In the special case $f_1 = X^a, f_2 = Y^b, f_3 = (X + Y)^c$ for some positive integers $a, b, c$, we denote $\text{syzgap}(X^a, Y^b, (X + Y)^c)$ by $\delta(a, b, c)$.

One can show that $\delta$ extends to a unique continuous function $\delta^*$ on $[0, \infty)^3$ with the property $|\delta(t) - \delta(s)| \leq \|t - s\|_1$. Moreover, if the underlying polynomial ring is defined over a field of positive characteristic $p$, the extension of $\delta$ satisfies $\delta^* \left( \frac{t}{p} \right) = p^{-1} \cdot \delta^*(t)$. In what follows we will not distinguish between $\delta$ and $\delta^*$. Our next goal is to explain how $\delta$ can be computed. The reference is Han’s thesis (Han 1991) resp. Monsky (2006b) for an alternative proof.

We are interested in the taxicab distance of elements of the form $\frac{t}{p^s}$ to the set

$$L_{\text{odd}} := \left\{ u \in \mathbb{N}^3 \mid u_1 + u_2 + u_3 \text{ is odd} \right\},$$

where $s$ is an integer and $t$ a three-tuple of non-negative real numbers. Note that for given $\frac{t}{p^s}$ there is at most one $u \in L_{\text{odd}}$ satisfying $\| \frac{t}{p^s} - u \|_1 < 1$. Moreover, the only candidates for $u_i$ are the rounding ups and rounding downs of $\frac{t_i}{p^s}$.

Theorem 2.4 (Han) Let $t = (t_1, t_2, t_3) \in [0, \infty)^3$. If the $t_i$ do not satisfy the strict triangle inequality (w.l.o.g. $t_1 \geq t_2 + t_3$), we have $\delta(t) = t_1 - t_2 - t_3$. If the $t_i$ satisfy the strict triangle inequality and there are $s \in \mathbb{Z}, u \in L_{\text{odd}}$ with $\| p^s t - u \|_1 < 1$, then there is such a pair $(s, u)$ with minimal $s$ and with this pair $(s, u)$ we get

$$\delta(t) = \frac{1}{p^s} \cdot \left( 1 - \| p^s t - u \|_1 \right).$$

Otherwise, one has $\delta(t) = 0$.

With the help of the $\delta$-function we can state the following theorem.

Theorem 2.5 (Han, Monsky) The Hilbert–Kunz multiplicity of an ideal $I := (X^a, Y^a, Z^a), a \geq 1$, of the Fermat ring $k[X, Y, Z]/(X^n + Y^n + Z^n)$ equals

$$\frac{3a^2n}{4} + \frac{n^3}{4} \cdot \delta \left( \frac{a}{n}, \frac{a}{n}, \frac{a}{n} \right)^2.$$

Proof The case $a = 1$ is due to Han (cf. Han 1991, Theorem 2.30) and was generalized by Monsky (cf. Monsky 2006a, Theorem 2.3).

Combining Theorems 2.1(1) and 2.5 one obtains the following numerical criterion for strong semistability.

Corollary 2.6 Let $\gcd(p, n) = 1$. The bundle $\text{Syz}_{V_n(X^a + Y^a + Z^n)}(X^a, Y^a, Z^a)$ is strongly semistable if and only if $\delta \left( \frac{a}{n}, \frac{a}{n}, \frac{a}{n} \right) = 0$. 
Example 2.7 In Brenner (2005a, Remark 2) Brenner asks whether the syzygy bundle \( \text{Syz}_C(X^2, Y^2, Z^2) \) is strongly semistable on the Fermat curve \( C \) of degree \( n \) in characteristics \( p \equiv \pm 1 \mod n \). Using Corollary 2.6 we are able to answer this question positively for the interesting cases \( n \geq 3 \). Since \( 2p^s \equiv \pm 2 \mod 2n \) for every \( s \geq 0 \), we see that the distance of \( \frac{2p^s}{n} \) to the next odd integer is \( \frac{n-2}{n} \) and \( 3 \frac{n-2}{n} = 3 - \frac{6}{n} \geq 1 \). Thus, \( \delta(n^2, \frac{2}{n}, \frac{2}{n}) = 0 \) and \( e_{\text{HK}}((X^2, Y^2, Z^2)) = 3n \), which is equivalent to the strong semistability of the syzygy bundle \( \text{Syz}_C(X^2, Y^2, Z^2) \). In the case \( n = 1 \) we have \( \text{Syz}_C(X^2, Y^2, Z^2) \cong \mathcal{O}_{\mathbb{P}^1}(-3) \oplus \mathcal{O}_{\mathbb{P}^1}(-3) \) and this bundle is obviously strongly semistable. For \( n = 2 \) the curve equation \( X^2 + Y^2 + Z^2 = 0 \) yields a non-trivial global section of total degree 2. Since \( \text{deg}(\text{Syz}_C(X^2, Y^2, Z^2)(2)) = -4 \), the bundle is not semistable.

3 The strongly semistable case

We start by fixing the notations which will be used for the rest of this paper.

Situation 3.1 Let \( k \) be an algebraically closed field of characteristic \( p > 0 \) and let \( R := k[X, Y, Z]/(X^n + Y^n + Z^n) \) for \( n \geq 1 \). We denote the projective Fermat curve of degree \( n \) by \( C := \text{Proj}(R) \). We assume \( \gcd(p, n) = 1 \), hence \( C \) is a smooth curve. We will consider the ideal \( I := (X^a, Y^a, Z^a), a \geq 1 \), as well as its first syzygy module \( \text{Syz}_R(I) \) and the bundle \( \text{Syz}_C(I) \).

An important special case in the situation above is \( a = 1 \), since \( \text{Syz}_C(X, Y, Z) \) is isomorphic to the restricted cotangent bundle \( \Omega_{\mathbb{P}^2}_{|C} \) which can be seen by using the Euler sequence. Note that the restricted cotangent bundle \( \Omega_{\mathbb{P}^2}_{|C} \) is (at least) semistable on every smooth plane curve of degree \( \geq 2 \) (cf. Trivedi 2005, Corollary 3.5).

In this section we deal with the case where the syzygy bundle \( \text{Syz}_C(X^a, Y^a, Z^a) \) is strongly semistable. For this purpose we will use results of Kustin et al. (2012) where the authors study the minimal projective resolution of the quotients \( R/(X^a, Y^a, Z^a) \) depending on \( a \) and \( n \).

We start with the special case where some Frobenius pull-back of \( \text{Syz}_C(X^a, Y^a, Z^a) \) becomes a direct sum of twisted copies of the structure sheaf. Hence we have an isomorphism

\[
\text{Syz}_C(X^{aq}, Y^{aq}, Z^{aq}) \cong \mathcal{O}_C(I)^2
\]

for some \( q = p^e \) and some \( l \in \mathbb{Z} \) due to the semistability of the pull-back. Note that this may happen only if \( p = 2 \) or if \( a \) is even as one sees by comparing the degrees of the bundles above.

By Miller (2000, Theorem 2.1) the Hilbert–Kunz function of the ideal \( I \) is given by \( e_{\text{HK}}(I^{[p^e]}) \cdot p^{2(e-e_0)} \) for all \( e \geq e_0 \), where \( e_0 \) is (minimal) such that \( R/I^{[p^e]} \) has finite projective dimension.

The minimal \( e \) such that \( \text{Syz}_C(X^{ap^e}, Y^{ap^e}, Z^{ap^e}) \) is a direct sum of twisted copies of the structure sheaf can be computed with the following theorem.
Theorem 3.2 (Kustin, Rahmati, Vraciu) In Situation 3.1, the quotient $R/I$ has finite projective dimension as an $R$-module if and only if one of the following conditions is satisfied

1. $n|a$,
2. $p = 2$ and $n \leq a$,
3. $p$ is odd and there exist positive integers $J$ and $e$ with $J$ odd such that

$$|Jp^e - a| < \begin{cases} 3^{e-1} & \text{if } p = 3, \\ p^e - 1 & \text{if } p^e \equiv 1 \mod 3, \\ \frac{p^e + 1}{3} & \text{if } p^e \equiv 2 \mod 3. \end{cases}$$

Proof See Kustin et al. (2012, Theorem 6.3).

In Li (2013) the author uses this theorem to study the interaction between strong semistability of $\text{Syz}_C(I)$, the projective dimensions of $R/I^{[q]}$ and the diagonal F-threshold of $I$.

In the case where $\text{Syz}_C(X^a, Y^a, Z^a)$ is strongly semistable but no Frobenius pull-back of it splits as a direct sum of twisted copies of the structure sheaf, we have that all quotients $R/(X^{aq}, Y^{aq}, Z^{aq})$ have infinite projective dimension as $R$-module, because the first syzygy modules $\text{Syz}_R(X^{aq}, Y^{aq}, Z^{aq})$ are not free. In this case we can use a result of Kustin, Rahmati and Vraciu to obtain a twisted Frobenius periodicity of $\text{Syz}_C(X^a, Y^a, Z^a)$ in the following sense.

Definition 3.3 Let $S$ be a normal, standard-graded $k$-domain of dimension two, where $k$ is an algebraically closed field of characteristic $p > 0$. Let $E$ be a vector bundle over the smooth projective curve $Y := \text{Proj}(S)$. Assume there are $0 \leq s < t \in \mathbb{N}$ such that the Frobenius pull-backs $F^e(E)$ of $E$ (and all their twists) are pairwise non-isomorphic for $0 \leq e \leq t - 1$ and $F^e(E) \cong F^e(E)(m)$ holds for some $m \in \mathbb{Z}$. We say that $E$ admits a twisted $(s, t)$-Frobenius periodicity. The bundle $E$ admits a twisted Frobenius periodicity if there are $0 \leq s < t \in \mathbb{N}$ such that $E$ admits a twisted $(s, t)$-Frobenius periodicity.

To state the necessary result of Kustin et al. we need one more definition.

Definition 3.4 Let $r$ and $s$ be positive integers with $r + s = n$. Then we define

$$\phi_{r,s} := \begin{pmatrix} 0 & Z^r & -Y^r & X^s \\ -Z^r & 0 & X^r & Y^s \\ Y^s & -X^r & 0 & Z^s \\ -X^s & -Y^s & -Z^s & 0 \end{pmatrix}.$$

Theorem 3.5 (Kustin, Rahmati, Vraciu) In Situation 3.1, let $a = \theta \cdot n + r$ with $\theta \in \mathbb{N}$ and $r \in \{1, \ldots, n - 1\}$. Assume that $Q := R/I$ has infinite projective dimension. If $\theta = 2 \cdot \eta - 1$, then the homogenous minimal free resolution of $Q$ is given by

$$\cdots \xrightarrow{\phi_{r,n-r}} F_1(-n) \xrightarrow{\phi_{n-r,r}} F_2 \xrightarrow{\phi_{r,n-r}} F_1 \longrightarrow R(-a)^3 \longrightarrow R.$$
with free graded modules

\[ F_1 := R(-3\eta n + n - r)^3 \oplus R(-3\eta n + 2n - 3r), \]
\[ F_2 := R(-3\eta n + n - 2r)^3 \oplus R(-3\eta n). \]

If \( \theta = 2 \cdot \eta \), then the homogenous minimal free resolution of \( Q \) is given by

\[ \ldots \xrightarrow{\phi_{n-r,r}} F_1(-n) \xrightarrow{\phi_{r,n-r}} F_2 \xrightarrow{\phi_{n-r,r}} F_1 \xrightarrow{\phi_{n-r,n-r}} R(-a)^3 \xrightarrow{\phi_{n-r,n-r}} R, \]

where the free graded modules are defined as

\[ F_1 := R(-3\eta n - 2r)^3 \oplus R(-3\eta n - n), \]
\[ F_2 := R(-3\eta n - n - r)^3 \oplus R(-3\eta n - 3r). \]

**Proof** The statement follows from Kustin et al. (2012, Theorem 3.5) combined with Kustin et al. (2012, Theorems 5.14 and 6.1). \( \square \)

**Corollary 3.6** Under the hypothesis of Theorem 3.5, we have

\[ \text{Syz}_R(X^a, Y^a, Z^a)(m) \cong \begin{cases} \text{coker}(\phi_{r,n-r} : F_2 \to F_1) & \text{if } \theta \text{ is odd,} \\ \text{coker}(\phi_{n-r,r} : F_2 \to F_1) & \text{if } \theta \text{ is even,} \end{cases} \]

\[ \cong \begin{cases} \text{Syz}_R(X^{n-r}, Y^{n-r}, Z^{n-r}) & \text{if } \theta \text{ is odd} \\ \text{Syz}_R(X^r, Y^r, Z^r) & \text{if } \theta \text{ is even,} \end{cases} \]

for some \( m \in \mathbb{Z} \).

**Proof** The first isomorphism is clear from the free resolution. The second isomorphism follows from the free resolution by considering the case \( a = r \) and the dual case \( a = n - r \). \( \square \)

At this point it is not clear whether the modules \( M_r = \text{Syz}_R(X^r, Y^r, Z^r) \) for \( 1 \leq r \leq n - 1 \) are pairwise non-isomorphic or not. A criterion to decide this is given by the Hilbert-series, which can be computed with the help of the next theorem, which is a slight but useful improvement of Brenner (2005a, Lemma 1.1). Geometrically spoken, it considers a smooth projective curve

\[ D := V(D_{n} - F(X, Y)) \subset \mathbb{P}^2 = \text{Proj } k[X, Y, Z], \]

where \( F(X, Y) \in k[X, Y] \) denotes a homogeneous polynomial of degree \( n \), and relates the sheaves \( \text{Syz}_D(X^a, Y^b, Z^c) \) to the sheaves \( \text{Syz}_D(X^a, Y^b, F(X, Y)^i) \) which come from \( \mathbb{P}^1 \) via the Noetherian normalization \( D \to \mathbb{P}^1 = \text{Proj } k[X, Y] \). We will use the following result several times in the sequel of this paper.

\[ \odot \text{ Springer} \]
Theorem 3.7 Let $k$ be a field, $S := k[X, Y, Z]/(Z^n - F(X, Y))$ and $F \in k[X, Y]$ homogeneous of degree $n \geq 2$. Let $a, b, c \geq 1$ and write $c = n \cdot q + r$ with $0 \leq r \leq n-1$ and $q \in \mathbb{N}$. For all $s \in \mathbb{Z}$ we have a short exact sequence

$$0 \rightarrow \text{Syz}_S(X^a, Y^b, Z^{c+n-2r})(s-r) \rightarrow \phi \text{Syz}_S(X^a, Y^b, F^q)(s-r) \oplus \text{Syz}_S(X^a, Y^b, F^{q+1})(s) \rightarrow \phi \text{Syz}_S(X^a, Y^b, Z^c)(s) \rightarrow 0,$$

where the maps are defined via

$$\psi(h_1, h_2, h_3) := ((Z^{n-r} \cdot h_1, h_2, h_3), (-h_1, -Z^r \cdot h_2, -Z^r \cdot h_3))$$

$$\phi((f_1, f_2, f_3), (g_1, g_2, g_3)) := (f_1 + Z^{n-r} \cdot g_1, Z^r \cdot f_2 + g_2, Z^r \cdot f_3 + g_3).$$

Proof The injectivity of $\psi$ is clear and the exactness at the middle spot is straightforward. The proof that $\phi$ is surjective can be found in Brenner and Kaid (2013, Lemma 2.1). See also Brinkmann (2013, Chapter 4) for a detailed proof and a generalization.

\[ \square \]

Theorem 3.8 The notations are the same as in Theorem 3.7. For $l \in \mathbb{N}$ we use the abbreviation $S_l := \text{Syz}_S(X^a, Y^b, Z^l)$. Then the Hilbert-series of $S_c = \text{Syz}_S(X^a, Y^b, Z^c)$ is given by

$$H_{S_c}(t) = \frac{(t^r - t^n) \cdot H_{S_{c-r}}(t) + (1 - t^r) H_{S_{c+n-r}}(t)}{1 - t^n}.$$

Proof ***If $r = 0$ there is nothing to show. Assuming $r \geq 1$, set $c' := c + n - 2r = nq + n - r$ and $r' := n - r$. Since $c' + n - 2r' = c$, Theorem 3.7 yields

$$H_{S_{c'}}(t) = t^r H_{S_{c-r}}(t) + H_{S_{c+n-r}}(t) - t^r H_{S_c}(t) \quad \text{and} \quad H_{S_{c'}}(t) = t^r H_{S_{c-r}}(t) + H_{S_{c+n-r}}(t) - t^r H_{S_c}(t).$$

Substituting $H_{S_{c'}}(t)$ in the first formula and solving for $H_{S_c}(t)$ gives the result. \[ \square \]

Turning back to the question of computing the Hilbert-series of the $R$-modules $M_r = \text{Syz}_R(X^r, Y^r, Z^r)$, $1 \leq r \leq n-1$, we obtain via Theorem 3.8

$$H_{M_r}(t) = \frac{(t^r - t^n) \cdot H_{\text{Syz}_R(1, Y^r, Z^r)}(t) + (1 - t^r) H_{\text{Syz}_R(-Y^n-Z^n, Y^r, Z^r)}(t)}{1 - t^n}$$

$$= \frac{2t^r(t^r - t^n) + (1 - t^r)(t^n + t^{2r})}{(1 - t)^3}$$

$$= \frac{t^n + 3t^{2r} - 3t^{n+r} - t^{3r}}{(1 - t)^3}. \quad (3.1)$$
Since the Hilbert-series of $M_r$ and $M_s$ for $1 \leq r < s \leq n - 1$ are not multiples of each other, the modules $(M_r)_{1 \leq r \leq n - 1}$ are pairwise non-isomorphic.

Now we are able to prove the following.

**Theorem 3.9** Assume we are in Situation 3.1. The bundle $\text{Syz}_C(X^a, Y^a, Z^a)$ admits a twisted Frobenius periodicity if and only if $\delta \left( \frac{a}{n}, \frac{a}{n}, \frac{a}{n} \right) = 0$. Moreover, the length of this periodicity is bounded from above by the order of $p$ in $\mathbb{Z}/(2n)$.

**Proof** This follows from Corollary 2.6 and because the isomorphism class of the module $\text{Syz}_R(X^{aq}, Y^{aq}, Z^{aq})$ depends on $r \equiv aq (n)$ and the parity of $\frac{aq-r}{n}$. $\square$

**Remark 3.10** One should mention that one already knew that strongly semistable bundles admit a twisted Frobenius periodicity. This is because the coefficients of the equation $X^n + Y^n + Z^n = 0$ lie in a finite field and therefore all moduli spaces are finite dimensional varieties over $\mathbb{F}_p$. Hence, the number of $\mathbb{F}_p$-rational points gives an upper bound for the length of the periodicity but it’s very rough and there is no hint how to compute it explicitly.

The next example shows that the upper bound for the length of the periodicity from Theorem 3.9 is the best possible.

**Example 3.11** Let $p = 37$ and $n = 14$. Then $p^e = 14 \cdot \theta + r$ with

\[
(\theta, r) = \begin{cases} 
  (\text{even}, 9) & \text{if } e \equiv 1 (3), \\
  (\text{odd}, 11) & \text{if } e \equiv 2 (3), \\
  (\text{even}, 1) & \text{if } e \equiv 0 (3). 
\end{cases}
\]

From this computation it is easy to see that $\delta(\frac{1}{14}, \frac{1}{14}, \frac{1}{14}) = 0$, hence $\Omega_{\mathbb{P}^2}|_C$ is strongly semistable and we get a twisted $(0, 3)$-Frobenius periodicity

\[
F^{3*}(\Omega_{\mathbb{P}^2}|_C) \cong \Omega_{\mathbb{P}^2}|_C \left( -\frac{3}{2} \cdot (q - 1) \right).
\]

Theorem 3.9 enables us to compute the Hilbert–Kunz function of $(X^a, Y^a, Z^a)$, if the bundle $\text{Syz}_C(X^a, Y^a, Z^a)$ is strongly semistable and none of its Frobenius pull-backs split as a direct sum of twisted copies of the structure sheaf.

**Theorem 3.12** Assume we are in Situation 3.1 and that $\text{Syz}_C(I)$ is strongly semistable and none of its Frobenius pull-backs are a direct sum of twisted copies of the structure sheaf. Let $aq = ap^e = n\theta + r$ with $\theta \in \mathbb{N}$ and $0 \leq r < n$. Then

\[
\text{HK}(I, q) = \begin{cases} 
  \frac{3a^2n}{4} \cdot q^2 - \frac{3n}{4} r^2 + r^3 & \text{if } \theta \text{ is even,} \\
  \frac{3a^2n}{4} \cdot q^2 - \frac{3n}{4} (n-r)^2 + (n-r)^3 & \text{if } \theta \text{ is odd.} 
\end{cases}
\]

**Proof** Under our assumptions on $\text{Syz}_C(I)$, the quotients $R/(X^{aq}, Y^{aq}, Z^{aq})$ have infinite projective dimension and their resolutions are given by Theorem 3.5. By Corollary
we have an isomorphism $\text{Syz}_R(I^{[q]}) \cong \text{Syz}_R(J)(l)$, where depending on $\theta$, we use $J$ and $b$ to denote either $(X^r, Y^r, Z^r)$ and $r$ or $(X^{n-r}, Y^{n-r}, Z^{n-r})$ and $n-r$. We obtain a diagram with exact rows

$$
\begin{array}{c}
0 \longrightarrow \text{Syz}_C(I^{[q]})(m) \longrightarrow \mathcal{O}_C(m - ap^e)^3 \longrightarrow \mathcal{O}_C(m) \longrightarrow 0 \\
\downarrow \cong \\
0 \longrightarrow \text{Syz}_C(J)(m + l) \longrightarrow \mathcal{O}_C(m + l - b)^3 \longrightarrow \mathcal{O}_C(m + l) \longrightarrow 0.
\end{array}
$$

Taking global sections as in Eq. (2.1) the claim follows by using

$$
\text{h}_0\left( C, \text{Syz}_C\left( X^{ap^e}, Y^{ap^e}, Z^{ap^e} \right)(m) \right) = \text{h}_0\left( C, \text{Syz}_C\left( X^b, Y^b, Z^b \right)(m + l) \right)
$$

and a computation similar to that in the proof of Brenner and Kaid (2013, Corollary 4.1).

**Example 3.13** Let $p = 37$ and $n = 14$. By Example 3.11 we obtain the Hilbert–Kunz function

$$
\text{HK}(R, 37^e) = \left\{ \begin{array}{ll}
21 \cdot 37^{2e} - \frac{243}{2} & \text{if } e \equiv 1 (3), \\
21 \cdot 37^{2e} - \frac{135}{2} & \text{if } e \equiv 2 (3), \\
21 \cdot 37^{2e} - \frac{19}{2} & \text{if } e \equiv 0 (3).
\end{array} \right.
$$

In Brenner and Kaid (2013) the authors proved that $\Omega_{p^n}|_C$ admits a twisted $(0, 1)$-Frobenius periodicity if $p \equiv -1 (2n)$. The authors tried to adopt their proof to the case $p \equiv 1 (2n)$ but failed (cf. Brenner and Kaid 2013, Remark 3.5).

**Theorem 3.14** Assume $p \equiv \pm 1 (2n)$ in Situation 3.1. Then $\Omega_{p^n}|_C$ is strongly semistable with twisted $(0, 1)$-Frobenius periodicity

$$
F^{e*}(\Omega_{p^n}|_C) \cong \Omega_{p^n}|_C \left( -\frac{3}{2} \cdot (p^e - 1) \right).
$$

Moreover, the Hilbert–Kunz function of $R$ is given by

$$
\text{HK}(R, p^e) = \frac{3n}{4} \cdot p^{2e} + 1 - \frac{3n}{4}.
$$

**Proof** As $\delta(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}) = 0$, the bundle $\Omega_{p^n}|_C$ is strongly semistable. Since $p$ has to be odd by assumption and $a = 1$, the Frobenius pull-backs of $\Omega_{p^n}|_C$ are not of the form $\mathcal{O}_C(l)^2$ as mentioned at the beginning of this section. If $p \equiv 1 (2n)$ then all powers $p^e$ can be written in the form even-$n + 1$ and if $p \equiv -1 (2n)$ all powers $p^{2e}$ can be written as even-$n + 1$ and the powers $p^{2e+1}$ can be written as odd-$n + n - 1$. The
periodicity of $\text{Syz}_C(X, Y, Z)$ follows from Corollary 3.6 and the statement about the Hilbert–Kunz function is due to Theorem 3.12. □

**Remark 3.15** Note that one can construct $(s, t)$-Frobenius periodicities in the classical sense, e.g. of degree zero bundles, from the twisted $(s, t)$-Frobenius periodicities obtained from Theorem 3.9 as follows:

If $a$ is even, one can consider the bundle $\text{Syz}_C(X^a, Y^a, Z^a)(3a)$. If $a$ is odd, one obtains a $(s, t)$-Frobenius periodicity of $\text{Syz}_D(U^{2a}, V^{2a}, W^{2a})(3a)$ on the Fermat curve $D$ of degree $2n$ as it was done in Brenner and Kaid (2013, Example 5.1).

Recall that due to Lange and Stuhler the vector bundles of degree zero admitting a $(0, t)$-Frobenius periodicity are exactly those that are étale trivializable (cf. Lange and Stuhler 1977). For the syzygy bundles which only admit a $(s, t)$-Frobenius periodicity with $s \geq 1$ there is only a finite trivialization, namely the composition of the Frobenius morphism and a suitable étale covering.

By Theorem 3.14 and this remark, the syzygy bundle $\text{Syz}_D(U^2, V^2, W^2)(3)$ admits a $(0, 1)$-Frobenius periodicity on the projective Fermat curve $D$ of degree $2n$, provided $p \equiv \pm 1 \pmod{2n}$. Stäbler computed the étale map trivializing this bundle in characteristics $p \equiv -1 \pmod{2n}$ explicitly in Stäbler (2011).

### 4 Which Frobenius periodicities can be achieved?

The next examples deal with the question which twisted $(0, t)$-Frobenius periodicities of the bundles $\Omega_{\mathbb{P}^2}|_C$ can be achieved. A sufficient condition for having a twisted Frobenius periodicity is (cf. Theorem 3.9) $\delta \left( \frac{1}{n}, \frac{1}{n}, \frac{1}{n} \right) = 0$, which is equivalent to the condition that the distances of all triples $v_e := (\frac{p^e}{n}, \frac{p^e}{n}, \frac{p^e}{n})$ to $L_{\text{odd}}$ are at least one. Let $p^e = \theta_e \cdot n + r_e$ with $\theta_e, r_e \in \mathbb{N}$ and $0 \leq r_e < n$. The nearest element to $v_e$ in $L_{\text{odd}}$, which potentially has a taxicab distance $< 1$ from $v_e$, is given by the component-wise rounding ups of $v_e$ if $\theta_e$ is even and by the component-wise rounding downs of $v_e$ if $\theta_e$ is odd. This leads to the (sufficient) conditions

$$
\begin{align*}
3 \cdot \left( 1 - \frac{r_e}{n} \right) & \geq 1 \quad \text{if } \theta_e \text{ is even,} \\
3 \cdot \frac{r_e}{n} & \geq 1 \quad \text{if } \theta_e \text{ is odd.}
\end{align*}
$$

**Example 4.1** Let $p$ be odd, $l \in \mathbb{N}$ and $n = \frac{p^{l+1}+1}{2}$. For $0 \leq e \leq 2l + 2$ we have

$$
p^e = \begin{cases} 
0 \cdot \cdot n + p^e & \text{if } 0 \leq e \leq l, \\
(2 \cdot p^{e-l-1} - 1) \cdot n + n - p^{e-l-1} & \text{if } l + 1 \leq e \leq 2l + 1, \\
(2 \cdot p^{l+1} - 2) \cdot n + 1 & \text{if } e = 2l + 2.
\end{cases}
$$

(4.1)

This shows that $p^e$ is of the form $n \cdot p^e$ or $\text{odd} \cdot n + n - p^{e'}$ for some $0 \leq e' \leq l$. Since $2n = p^{l+1} + 1 \geq 3p^{e'}$ for all $0 \leq e' \leq l$, we see that $\delta \left( \frac{1}{n}, \frac{1}{n}, \frac{1}{n} \right) = 0$. By Corollary 3.6 we find that $F^{e*}(\Omega_{\mathbb{P}^2}|_C)$ is isomorphic to

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\[ F^{e'}(\Omega_{P^2|C})(-\frac{3}{2}(p^e - p^{e'})), \]

where \( e' \equiv e \mod l + 1 \) with \( e' \in \{0, \ldots, l\} \). With Equation (3.1) we find a twisted \((0, l + 1)\)-Frobenius periodicity. The Hilbert–Kunz function is given by

\[ HK(R, p^e) = \frac{3n}{4} \cdot \left( p^{2e} - p^{2e'} \right) + p^{3e'}, \]

where again \( e' \equiv e \mod l + 1 \) with \( e' \in \{0, \ldots, l\} \).

**Example 4.2** Let \( p = 2, 1 \leq l \in \mathbb{N} \) and \( n \) odd with \( 2l < n < 2l+1 \), hence \( n = 2l + x \) with \( 0 < x < 2l \). We want to show that \( \Omega_{\mathcal{P}^2|C} \) admits a twisted Frobenius periodicity if and only if \( n = 3 \). The condition \( \delta\left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}\right) = 0 \) forces

\[ 2n = 2l + 1 + 2x \geq 3 \cdot 2l + 3 \cdot 2e \]

for all \( 0 \leq e \leq l \). This is equivalent to \( x \geq 3 \cdot 2l - 1 \). Since \( 2l + 1 = n = 2l + x \) with \( 0 < 2l - x < 2l < n \), we need that the inequality \( 3 \cdot (2l - x) \geq n \) holds. This is equivalent to \( x \leq 2l - 1 \). This shows that a twisted Frobenius periodicity might appear only in the case \( x = 2l - 1 \) resp. \( n = 3 \cdot 2l - 1 \). Since \( n \) is odd, we obtain the single possibility \( n = 3 \). Now let \( n = 3 \). Then \( \Omega_{\mathcal{P}^2|C} \) is strongly semistable since \( \delta\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = 0 \). We remark that this is well-known: since \( C \) is an elliptic curve and \( \Omega_{\mathcal{P}^2|C} \) is semistable the strong semistability follows also from Mehta and Ramanathan (1982, Theorem 2.1).

5 The non-strongly semistable case

In this section we want to compute the Hilbert–Kunz function of \( I = (X^a, Y^a, Z^a) \) under the condition that the syzygy bundle \( \text{Syz}_C(X^a, Y^a, Z^a) \) is not strongly semistable on the Fermat curve \( C \). We do this by an explicit computation of the strong HN-filtration of these bundles.

**Lemma 5.1** Consider Situation 3.1 over a field of arbitrary characteristic. Let \( b \) denote a natural number and write \( b = nl + r \) with \( 0 \leq r \leq n - 1 \).

1. If \( l \) is even and \( r > \frac{2n}{3} \), then \( \text{Syz}_C(X^b, Y^b, Z^b) \) is not semistable on \( C \). Moreover, one has

\[ \Gamma(C, \text{Syz}_C(X^b, Y^b, Z^b))(n(l + 1 + \frac{l}{2})) \neq 0. \]

2. If \( l \) is odd and \( r < \frac{n}{3} \), then \( \text{Syz}_C(X^b, Y^b, Z^b) \) is not semistable on \( C \). Moreover, one has

\[ \Gamma(C, \text{Syz}_C(X^b, Y^b, Z^b))(n(l + \lfloor \frac{l}{2} \rfloor + 3r)) \neq 0. \]
Proof For the proof of (1) see Brenner (2005a, Proposition 1). For the proof of part (2) which is rather similar to (1) see Kaid (2009, Lemma 4.2.8 (2)). □

We list the following two neat consequences of the previous lemma.

Corollary 5.2 In Situation 3.1 assume \( p \equiv n \pm 1 \mod 2n \) for an even natural number \( n \geq 4 \). Then \( F^\bullet (\Omega^1_{\mathbb{P}^2}) \) is not semistable on \( C \).

Proof For \( p \equiv n + 1 \mod 2n \) this follows from part (1) of Lemma 5.1 and for the case \( p \equiv n - 1 \mod 2n \) one applies part (2). □

Corollary 5.3 If in Situation 3.1 we have \( a \equiv n \mod 2n \), then \( \text{Syz}_C(X^a, Y^a, Z^a) \) is not semistable on \( C \).

Proof This follows immediately from part (2) of Lemma 5.1. □

Now, using Lemma 5.1 we are able to prove that the \( s \)th Frobenius pull-back (where \( s \) is the integer in Han’s Theorem 2.4) is not semistable. Moreover, we explicitly compute a strong Harder-Narasimhan filtration of \( \text{Syz}_C(X^a, Y^a, Z^a) \).

Theorem 5.4 In Situation 3.1 assume \( p > \frac{3a}{2n} \). If \( \delta \left( \frac{a}{n}, \frac{a}{n}, \frac{a}{n} \right) \neq 0 \) and \( s \) is the integer of Han’s Theorem 2.4, then \( F^{s*}(\text{Syz}_C(X^a, Y^a, Z^a)) \) is not semistable. Let \( aps = nl + r \) with \( l \in \mathbb{N} \) and \( 0 \leq r < n - 1 \). Then the Frobenius pull-back \( F^{s*}(\text{Syz}_C(X^a, Y^a, Z^a)) \) has a strong Harder-Narasimhan filtration given by

\[
0 \to \mathcal{O}_C(-m) \to F^{s*}(\text{Syz}_C(X^a, Y^a, Z^a)) \to \mathcal{O}_C(m - 3aps) \to 0,
\]

with \( m = n(l + 1 + \frac{l}{2}) \) if \( l \) is even and \( m = n(l + \lfloor \frac{l}{2} \rfloor) + 3r \) if \( l \) is odd. Moreover, this filtration is minimal for \( p \geq n - 3 \) and \( s \geq 1 \) in the sense that \( F^{(s-1)*}(\text{Syz}_C(X^a, Y^a, Z^a)) \) is semistable.

Proof Since \( S := \text{Syz}_C(X^a, Y^a, Z^a) \) is not strongly semistable, we know by Han’s Theorem 2.4 and Corollary 2.6 that the taxicab distance from \( \left( \frac{aps}{n}, \frac{aps}{n}, \frac{aps}{n} \right) \) to the nearest element in \( L_{\text{odd}} \) is < 1. We have \( s \geq 0 \) due to the assumption \( p > \frac{3a}{2n} \). We compute the taxicab distance in dependence on \( l \). First, we consider the case where \( l \) is even. So the distance from \( \frac{aps}{n} \) to the nearest odd integer is \( \frac{n-r}{n} \) and the taxicab distance to the closest element in \( L_{\text{odd}} \) is \( 3 \frac{n-r}{n} = 3 - \frac{3r}{n} \), which is by assumption < 1. Hence, we obtain \( r > \frac{2n}{3} \). Now we apply Lemma 5.1(1) to \( b = aps \) and see that \( F^{s*}(S) \cong \text{Syz}_C(X^{aps}, Y^{aps}, Z^{aps}) \) is not semistable. Moreover, we have a non-trivial mapping \( \mathcal{O}_C(-n(l + 1 + \frac{l}{2})) \longrightarrow F^{s*}(S) \). We want to prove that this mapping constitutes the HN-filtration of the \( s \)th Frobenius pull-back; i.e., the mapping has no zeros on \( C \). First, we compute the Hilbert–Kunz multiplicity of the ideal \( I := (X^a, Y^a, Z^a) \) in the ring \( R \). Since

\[
\delta \left( \frac{a}{n}, \frac{a}{n}, \frac{a}{n} \right) = \frac{1}{p^s} \left( 1 - \left( \frac{3r}{n} \right) \right) = \frac{1}{p^s} \left( \frac{3r - 2n}{n} \right)
\]
we obtain
\[ e_{HK}(I) = \frac{3n}{4}a^2 + \frac{n^3}{4} \left( \frac{1}{p^s} \left( \frac{3r - 2n}{n} \right) \right)^2 = \frac{3n}{4}a^2 + \frac{(3r - 2n)^2}{4p^{2s}} \cdot n \]

via Theorem 2.5. Now we use the Hilbert–Kunz multiplicity \( e_{HK}(I) \) to read off the degree of the destabilizing invertible sheaf \( \mathcal{L} \subset F_{s*}(S) \). Theorem 2.1(2) yields

\[ \deg(\mathcal{L}) = -np^s \left( \frac{3a}{2} - \sqrt{\frac{(3r - 2n)^2}{4p^{2s}}} \right) = -n^2(l + 1 + \frac{l}{2}). \]

Thus \( \deg(\mathcal{L} \otimes \mathcal{O}_C(n(l + 1 + \frac{l}{2}))) = 0 \). Since this line bundle does have a non-trivial section, we obtain \( \mathcal{L} \cong \mathcal{O}_C(-n(l + 1 + \frac{l}{2})) \) and the HN-filtration is indeed

\[ 0 \subset \mathcal{O}_C(-n(l + 1 + \frac{l}{2})) \subset F_{s*}(S). \]

The assertion on the quotient line bundle is clear since the determinant bundle is additive on short exact sequences.

The case that \( l \) is odd follows essentially in the same way and we omit it here.

Finally, we prove the assertion about the minimality of the HN-filtration. Assume the minimal integer \( e \) such that \( F_{e*}(S) \) is not semistable is strictly smaller than \( s \). If \( l \) is even, we obtain the equality \( \ell = n(3r - 2n)p^{e-s} \), where the integer \( \ell \) is defined as in Theorem 2.1(4). Remark that the representation of \( \ell \) is well-defined, since we know from above that \( r > \frac{2n}{3} \). But this equality can only hold for prime numbers \( p \leq n - 3 \) since \( 0 < 3r - 2n \leq n - 3 \) (we have \( r \leq n - 1 \) and \( p \nmid n \)).

If \( l \) is odd, we have \( \ell = n(n - 3r)p^{e-s} \). If \( r \geq 1 \), we can conclude as above. In case of \( r = 0 \), we see that \( p^s \) has to divide \( l \) and we can conclude that \( a = nb \) with \( b \) odd. But this means that the minimal \( s \) of Han’s Theorem is actually \( s = 0 \) which contradicts the assumption \( s \geq 1 \).

\[ \square \]

If the characteristic is sufficiently large, Theorem 5.4 also yields a numerical criterion for semistability of the syzygy bundle \( \text{Syz}_C(X^a, Y^a, Z^a) \). Semistability of these bundles in characteristic 0 is part of Sect. 6.

**Corollary 5.5** Assume the situation of Theorem 5.4. If \( \text{char}(k) \geq n - 3 \), then \( \text{Syz}_C(X^a, Y^a, Z^a) \) is semistable if and only if \( s > 0 \).

**Proof** This is an immediate consequence of Theorem 5.4. \( \square \)

**Example 5.6** Let \( p \neq 3 \) and consider the Fermat cubic \( C(n = 3) \), which is an elliptic curve. We recall that by Mehta and Ramanathan (1982, Theorem 2.1) the syzygy bundle \( \text{Syz}_C(X^a, Y^a, Z^a) \) is semistable on \( C \) if and only if it is strongly semistable, which is by Corollary 2.6 equivalent to \( \delta \left( \frac{a}{n}, \frac{a}{n}, \frac{a}{n} \right) = 0 \). If \( p > \frac{a}{2} \), it follows from Theorem 5.4 that this bundle is not semistable on \( C \) if and only if \( a \equiv 3l \) for some odd integer \( l \). This description for non-semistability does not hold when \( p \leq \frac{a}{2} \). For
instance, let $p = 2$ and $a = 3 \cdot 2 + 1 = 7$. Then the taxicab distance from $(\frac{7}{5}, \frac{7}{5}, \frac{7}{5})$ to $(1, 1, 1) \in L_{\text{odd}}$ equals $\frac{3}{5} < 1$ and hence $\delta\left(\frac{7}{5}, \frac{7}{5}, \frac{7}{5}\right) \neq 0$.

Combining Theorem 5.4 with Miller (2000, Theorem 2.1), one obtains the Hilbert–Kunz functions of the ideals $(X^a, Y^a, Z^a)$ for $q = p^e \gg 0$. Note that the term $e_{\text{HK}}(I)$ can be explicitly computed via Theorem 2.5.

**Corollary 5.7** In the situation of Theorem 5.4 one has

$$\text{HK}(I, p^e) = e_{\text{HK}}(I) p^{2e} \text{ for all } e \gg 0. \quad (5.1)$$

**Proof** By Theorem 5.4 we have a short exact sequence of the form

$$0 \to \mathcal{O}_C(\alpha) \to F^{s \ast}(\text{Syz}_C(X^a, Y^a, Z^a)) \to \mathcal{O}_C(\beta) \to 0.$$ 

Since this is a strong Harder–Narasimhan filtration of $F^{s \ast}(\text{Syz}_C(X^a, Y^a, Z^a))$, we obtain strong Harder–Narasimhan filtrations

$$0 \to \mathcal{O}_C(\alpha \cdot p^t) \to F^{(s+t) \ast}(\text{Syz}_C(X^a, Y^a, Z^a)) \to \mathcal{O}_C(\beta \cdot p^t) \to 0$$

for all $t \geq 0$. These sequences split whenever

$$\text{Ext}^1(\mathcal{O}_C(\beta \cdot p^t), \mathcal{O}_C(\alpha \cdot p^t)) \cong H^1(C, \mathcal{O}_C((\alpha - \beta) \cdot p^t))$$

vanishes. Using Serre duality and $\omega_C \cong \mathcal{O}_C(n - 3)$ (cf. Hartshorne 1987, Example II. 8.20.3), we have to show that

$$\mathcal{O}_C((2n - 3r) \cdot p^t + n - 3) \quad \text{if } l \text{ is even or}$$

$$\mathcal{O}_C((3r - n) \cdot p^t + n - 3) \quad \text{if } l \text{ is odd}$$

has no global sections. This is true if $(2n - 3r) \cdot p^t + n - 3$ resp. $(3r - n) \cdot p^t + n - 3$ is negative. This happens for $t$ sufficiently large since we have $2n - 3r < 0$ resp. $3r - n < 0$, which can be seen by an easy calculation. In particular, we must have $t \geq 1$. Note that $t = 1$ is enough in the cases $p > n - 3$. Finally, we use Miller (2000, Theorem 2.1) to compute $\text{HK}(I, p^e)$ for $e \gg 0$. $\square$

**Example 5.8** Let $p = 3$ and $n = 7$. Then $\delta\left(\frac{1}{7}, \frac{1}{7}, \frac{1}{7}\right) = \frac{1}{63}$ with $s = 2$. By Theorem 5.4 we have the strong Harder–Narasimhan filtration

$$0 \to \mathcal{O}_C(-13) \to F^{2 \ast}(\text{Syz}_C(X, Y, Z)) \to \mathcal{O}_C(-14) \to 0.$$ 

Since $-3^t + 4 < 0 \Leftrightarrow t \geq 2$, the $t$th Frobenius pull-backs of $\text{Syz}_C(X, Y, Z)$ split as

$$\mathcal{O}_C\left(-13 \cdot p^{e-2}\right) \oplus \mathcal{O}_C\left(-14 \cdot p^{e-2}\right)$$

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for \( e \geq 4 \). This gives
\[
\text{HK}(R, 3^e) = e_{\text{HK}}(R) \cdot 3^{2e} = \frac{427}{81} \cdot 3^{2e}
\]
for \( e \geq 4 \). The other values are \( \text{HK}(R, 3^0) = 1 \), \( \text{HK}(R, 3^1) = 27 \), \( \text{HK}(R, 3^2) = 419 \) and \( \text{HK}(R, 3^3) = 3843 \) by an explicit computation with Abbott et al. (2015).

Moreover, another explicit computation shows
\[
\text{Syz}_R(X^9, Y^9, Z^9) \cong \text{Syz}_R(X^5, Y^5, Z^5)(-6) \quad \text{and} \quad \text{Syz}_R(X^{27}, Y^{27}, Z^{27}) \cong R(-39) \oplus R(-42).
\]

**Remark 5.9** We want to relate our results on the Hilbert–Kunz functions of the ideals \((X^a, Y^a, Z^a)\) with those of other authors. As a consequence of a result of Brenner (2007, Theorem 6.1), the Hilbert–Kunz function of an \((X, Y, Z)\)-primary ideal \(I\) of the ring \( R = k[X, Y, Z]/(X^n + Y^n + Z^n) \) has the form
\[
e \mapsto e_{\text{HK}}(I) \cdot p^{2e} + \phi(p, e),
\]
where \( \phi(p, \_\_) \) is an eventually periodic function. By Eq. (5.1), we see that \( \phi(p, e) = 0 \) for all large \( e \) and all \( p \) coprime to \( n \) if some Frobenius pull-back of \( \text{Syz}_C(I) \) splits as a direct sum of twisted structure sheaves. If this is not the case, we obtain from Theorem 3.12 that the shape of \( \phi(p, \_\_) \) does only depend on the residue class of \( p \) modulo \( 2n \). All in all, we have seen that for every fixed \( n \) there are only finitely many possibilities for \( \phi(p, \_\_) \).

## 6 Strongly semistable reduction on the relative Fermat curve

In this section we deal with a problem proposed by Brenner in (Brenner 2005b, Problem 5) which contains a special case of Miyaoka’s problem (Miyaoka 1987, Problem 5.4).

**Problem 6.1** (Brenner) How does the strong semistability of \( \text{Syz}_C(X^a, Y^a, Z^a) \) on the Fermat curve \( C \) of degree \( n \) depend on the characteristic \( p \), the degree \( n \) and the integer \( a \)? In particular, for fixed \( n \) and \( a \), is the set of prime numbers such that \( \text{Syz}_C(X^a, Y^a, Z^a) \) is strongly semistable finite, infinite or does it contain almost all prime numbers?

We have already answered the first part by Theorem 2.5 and Corollary 2.6 in Sect. 2 (cf. also Example 2.7). The following theorem gives a numerical criterion for semistability of the syzygy bundle \( \text{Syz}_C(X^a, Y^a, Z^a) \) on a Fermat curve in characteristic 0. It also shows that if the syzygy bundle is semistable in characteristic 0, then it has strongly semistable reduction for infinitely many prime numbers. Before we state the theorem, we recall some notation for a relative curve \( \mathcal{C} \to \text{Spec } \mathbb{Z} \). For a prime number \( p \) we denote by \( \mathcal{C}_p := \mathcal{C} \times_{\text{Spec } \mathbb{Z}} \text{Spec } \mathbb{F}_p \) the special fiber over the closed point \((p) \in \text{Spec } \mathbb{Z}\) and by \( \mathcal{C}_0 := \mathcal{C} \times_{\text{Spec } \mathbb{Z}} \text{Spec } \mathbb{Q} \) the generic fiber over the generic point \((0) \in \text{Spec } \mathbb{Z}\). Finally, we recall that by a theorem of Dirichlet there exist infinitely many prime numbers in any arithmetic progression.
Theorem 6.2 Let $a \geq 1$ be an integer, and consider the smooth projective relative
Fermat curve $\mathcal{C} := \text{Proj}(\mathbb{Z}_n[X, Y, Z]/(X^n + Y^n + Z^n)) \to \text{Spec} \mathbb{Z}_n$. Write $a = nl + r$
with $0 \leq r < n$ and let $\tilde{r} \equiv a \mod 2n$ ($0 \leq \tilde{r} < 2n$). Then the following conditions
are equivalent:

(1) One has $r \leq \frac{2n}{3}$ if $l$ is even and $r \geq \frac{n}{3}$ if $l$ is odd.
(2) One has $\tilde{r} \leq \frac{2n}{3}$ if $\tilde{r} < n$ and $\tilde{r} \geq \frac{4n}{3}$ if $\tilde{r} \geq n$.
(3) For all prime numbers $p > \max\{n - 3, \frac{3a}{2n}\}$ the integer $s$ of Han’s Theorem 2.4
is either $\geq 1$ or $\delta(\frac{a}{n}, \frac{a}{n}, \frac{a}{n}) = 0$.
(4) The syzygy bundle $\text{Syz}_{\mathcal{C}_p}(X^a, Y^a, Z^a)$ is strongly semistable on the special fiber
$\mathcal{C}_p$ for all prime numbers $p \equiv \pm 1 \mod 2n$ with $p > \frac{3a}{2n}$.
(5) The syzygy bundle $\text{Syz}_{\mathcal{C}_0}(X^a, Y^a, Z^a)$ is semistable on the generic fiber $\mathcal{C}_0$.

Proof (1) $\Leftrightarrow$ (2). This is obvious.

(1) $\Rightarrow$ (3). Assume there is a prime number $p > \frac{3a}{2n}$ such that $s = 0$. Then
$\text{Syz}_{\mathcal{C}_p}(X^a, Y^a, Z^a)$ is not semistable on the fiber $\mathcal{C}_p$ by Theorem 5.4. The proof of
Theorem 5.4 shows that we have either $r > \frac{2n}{3}$ or $r < \frac{n}{3}$ depending on $l$. But this
contradicts the assumption.

(3) $\Rightarrow$ (1). Suppose the assumption on $r$ in (1) does not hold. Then the bundle
$\text{Syz}_{\mathcal{C}_p}(X^a, Y^a, Z^a)$ is non-semistable on every fiber $\mathcal{C}_p$ by Lemma 5.1. But this con-
tradicts Theorem 5.4 since $s \geq 1$ and $F^{s*}(\text{Syz}_{\mathcal{C}_p}(X^a, Y^a, Z^a))$ is the first non-semistable
Frobenius pull-back in characteristics $p \geq n - 3$.

(2) $\Rightarrow$ (4). We have $p^s a \equiv \pm \tilde{r} \mod 2n$ for every prime number $p \equiv \pm 1 \mod 2n$
and every $s \geq 0$. If $\tilde{r} < n$, then the distance from $\frac{p^s a}{n}$ to the nearest odd integer is
$n - \frac{n}{n}$. Hence the taxicab distance from $(\frac{p^s a}{n}, \frac{p^s a}{n}, \frac{p^s a}{n})$ to the nearest element in $L_{\text{odd}}$ is
\[
3 \frac{n - \tilde{r}}{n} = 3 - \frac{3\tilde{r}}{n} \geq 3 - \frac{2n}{n} = 1.
\]

Similarly, if $\tilde{r} \geq n$, then the distance from $\frac{p^s a}{n}$ to the nearest odd integer is $\frac{\tilde{r} - n}{n}$, and
thus the taxicab distance to the nearest element in $L_{\text{odd}}$ is
\[
3 \frac{n - \tilde{r}}{n} = 3 - \frac{3\tilde{r}}{n} \geq 3 - \frac{4n}{n} = 1.
\]

Hence $\delta(\frac{a}{n}, \frac{a}{n}, \frac{a}{n}) = 0$ in characteristics $p \equiv \pm 1 \mod 2n$ (with $p > \frac{3a}{2n}$) and the
syzygy bundle $\text{Syz}_{\mathcal{C}_p}(X^a, Y^a, Z^a)$ is strongly semistable by Corollary 2.6.

(4) $\Rightarrow$ (5). This follows from the openness of semistability (see Miyaoka 1987, paragraph after Proposition 5.2).

(5) $\Rightarrow$ (1). This follows immediately from Lemma 5.1. \qed

Remark 6.3 To determine the semistability of the bundles $\text{Syz}_{\mathcal{C}}(X^a, Y^a, Z^a)$ one
might also consider the application of restriction theorems. It is well-known that the
vector bundle $S := \text{Syz}_{\mathbb{P}^2}(X^a, Y^a, Z^a)$ is stable on the projective plane with Chern
classes $c_1(S) = -3a$ and $c_2(S) = 3a^2$ (see for instance Brenner 2008a, Corollary
3.2). Hence its discriminant equals $\Delta(S) = 4c_2(S) - c_1(S)^2 = 3a^2$. So the restriction
theorem of Langer (2009, Theorem 2.19) tells us that $S|_C$ is stable on every smooth curve $C$ of degree $n > \frac{3a^2+1}{2}$. But this bound grows quadratically with $a$ and therefore becomes expeditiously high.

It was shown in Brenner (2005a, Corollary 2) that the set of primes where the bundle $\text{Syz}_C(X^a, Y^a, Z^a)$ has strongly semistable reduction contains in general not almost all prime numbers disproving a stronger version of Miyaoka (1987, Problem 5.4) which was conjectured by Shepherd-Barron (1997). That is, strong semistability is not an open property in arithmetic deformations. The following example shows that this phenomenon depends on the pair $(a, n)$. But using Han’s Theorem 2.4 and Corollary 2.6 one can exhibit also this property explicitly.

**Example 6.4** Let $n = 5$ and $p$ be an odd prime number with $\gcd(p, 5) = 1$. For all $e \in \mathbb{N}$ we can write $p^e = 10 \cdot l + r$ for some $l \in \mathbb{N}$ and $r \in \{1, 3, 7, 9\}$. Then $p^e_5$ is $2l + \frac{4}{5}, 2l + \frac{3}{5}, 2l + 1 + \frac{2}{5}$ resp. $2l + 1 + \frac{4}{5}$ and hence the nearest element in $L_{\text{odd}}$ to $(p^e_5, p^e_5, p^e_5)$ is given by $(2l + 1, 2l + 2l), (2l + 1, 2l + 1, 2l + 1), (2l + 1, 2l + 1, 2l + 1)$ resp. $(2l + 1, 2l + 2l, 2l + 2l)$. Hence, the taxicab distance of $(p^e_5, p^e_5, p^e_5)$ to $L_{\text{odd}}$ is $\frac{6}{5}$ in any case, showing $\delta \left( \frac{4}{5}, \frac{3}{5}, \frac{1}{5} \right) = 0$. Therefore, $\Omega p_2|_C$ is strongly semistable in almost all characteristics and semistable in characteristic zero by Theorem 6.2. The only exceptional prime numbers are 2 and 5.

If $\text{Syz}(X^a, Y^a, Z^a)$ is not semistable on the Fermat curve in characteristic 0 then Theorem 6.2 allows to compute its Harder-Narasimhan filtration via reduction to positive characteristic. It turns out that the HN-filtration in characteristic 0 coincides with the HN-filtration in positive characteristic $p$ (see Theorem 5.4) for almost all prime numbers.

**Theorem 6.5** Let $k$ be a field of characteristic 0 and denote by

$$C := \text{Proj}(k[X, Y, Z]/(X^n + Y^n + Z^n))$$

the Fermat curve of degree $n$ over $k$. Further, let $a \geq 1$ be an integer and write $a = nl + r$ with $0 \leq r < n$. Set $S := \text{Syz}_C(X^a, Y^a, Z^a)$. If $l$ is even and $r > \frac{2n}{3}$ or $l$ is odd and $r < \frac{n}{3}$, then $S$ is not semistable and its HN-filtration is given by the short exact sequence

$$0 \rightarrow \mathcal{O}_C(-m) \rightarrow S \rightarrow \mathcal{O}_C(m-3a) \rightarrow 0,$$

where $m$ is defined as in Theorem 5.4 in dependence of $l$.

**Proof** By Lemma 5.1 we have a non-trivial mapping $\mathcal{O}_C(-m) \rightarrow S$. We have to show that this mapping has no zeros on $C$. Since $C$ is already defined over $\mathbb{Z}$ we can reduce to the case $k = \mathbb{Q}$. Assume that we have a factorization

$$\mathcal{O}_C(-m) \rightarrow \mathcal{L} \rightarrow \text{Syz}_C(X^a, Y^a, Z^a),$$
where $L$ is the maximal destabilizing subbundle. As in Theorem 6.2 we consider $C$ as the generic fiber of the relative curve

$$C = \text{Proj}(\mathbb{Z}_n[X, Y, Z] / (X^n + Y^n + Z^n)) \longrightarrow \text{Spec} \mathbb{Z}_n.$$ 

On every special fiber $C_p$ satisfying $p > \max \{n - 3, \frac{2n}{27}\}$ the HN-filtration of the restriction equals $0 \subset O_{C_p}(-m) \subset \text{Syz}_{C_p}(X^a, Y^a, Z^a)$ by Theorem 5.4. Since the HN-filtration of $\text{Syz}_C(X^a, Y^a, Z^a)$ extends to an open subset of $\text{Spec} \mathbb{Z}_n$, we obtain $L \cong O_{C_a}(-m)$. \hfill \Box

**Example 6.6** In this example we provide an application of our results to the theory of *tight closure* which is related to Hilbert–Kunz theory and strongly connected to the theory of (strongly) semistable vector bundles due to the work of Brenner (see Brenner 2008b). Via this geometric approach Problem 6.1 is related to Hochster’s question Hochster (1994, Question 13). In the homogeneous coordinate ring of the Fermat septic ($n = 7$), Brenner and Katzman have shown in (Brenner and Katzman 2006, Theorem 4.1) by tedious computations that $X^3 Y^3 \in (X^4, Y^4, Z^4)^*$ (the tight closure) for prime numbers $p \equiv 3 \mod 7$ and $X^3 Y^3 \not\in (X^4, Y^4, Z^4)^*$ for $p \equiv 2 \mod 7$. That is, the relative curve $C : \text{Proj}(\mathbb{Z}[X, Y, Z] / (X^7 + Y^7 + Z^7)) \rightarrow \text{Spec} \mathbb{Z}$ illustrates that tight closure does not behave uniformly in the fibers of an arithmetic deformation. A similar reasoning as in Examples 2.7 and 6.4 shows that $\text{Syz}_C(X^4, Y^4, Z^4)$ is strongly semistable on the Fermat septic if and only if $p \equiv \pm 1 \mod 7$. In particular, the monomial $X^3 Y^3$ of degree 6 belongs to the tight closure of the ideal $(X^4, Y^4, Z^4)$ in $k[X, Y, Z] / (X^7 + Y^7 + Z^7)$ in these characteristics by Brenner (2008b, Theorem 6.4). Hence, we easily get infinitely many prime numbers where the tight closure inclusion does hold. Furthermore, the question for which prime numbers the syzygy bundle $\text{Syz}(X^4, Y^4, Z^4)$ has strongly semistable reduction, raised in Brenner and Katzman (2006, paragraph before Corollary 4.3), is now answered.

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