A Lower Bound of the Number of Threshold Functions in Terms of Combinatorial Flags on the Boolean Cube

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Abstract

Let \( E = \{(1, b_1, \ldots, b_n) \in \mathbb{R}^{n+1} \mid b_i = \pm 1, \ i = 1, \ldots, n\} \),

\[ E^{\times n}_{\neq 0} := \{W = (w_{i_1}, \ldots, w_{i_n}) \mid w_{i_k} \in E, k = 1, \ldots, n, \dim \text{span}(w_{i_1}, \ldots, w_{i_n}) = n\}, \]

and \( q_W^l := |\text{span}(w_{i_{l+1}}, \ldots, w_{i_n}) \cap E| \). Then for any weights \( p = (p_1, \ldots, p_{2^n}), p_i \in \mathbb{R}, \sum_{i=1}^{2^n} p_i = 1 \) we have for the number of threshold functions \( P(2, n) \) the following lower bound

\[ P(2, n) \geq 2 \sum_{W \in E^{\times n}_{\neq 0}} \frac{1 - p_{i_1} - p_{i_2} - \cdots - p_{i_{q_W^n}}}{q_W^n \cdot q_W^{n-1} \cdots q_W^1}, \]

and the right side of the inequality doesn’t depend on the choice of \( p \). Here the indices used in the numerator correspond to vectors from \( \text{span}(w_{i_1}, \ldots, w_{i_n}) \cap E = \{w_{i_1}, \ldots, w_{i_n}, \ldots w_{i_{q_W^n}}\} \).

Keywords: threshold function, combinatorial flag, Möbius function.

1 Introduction.

Let \( E_2 = \{\pm 1\} \) and \( E_2^n := \underbrace{E_2 \times \cdots \times E_2}_{n} \).

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Definition 1 A function $f : E_2^n \to E_2$ is called a threshold function if there exist real numbers $\alpha_0, \alpha_1, \ldots, \alpha_n$ such that

$$f(x_1, \ldots, x_n) = 1 \text{ iff } \alpha_1 x_1 + \cdots + \alpha_n x_n + \alpha_0 \geq 0.$$  

Denote by $P(2, n)$ the number of threshold functions.

Let us note that

$$f(x_1, \ldots, x_n) = \text{sign} \langle \bar{\alpha}, (1, \bar{x}) \rangle,$$

where $(1, \bar{x}) := (1, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ and $\bar{\alpha} = (\alpha_0, \ldots, \alpha_n) \in \mathbb{R}^{n+1}$.

This observation allows us to identify a threshold function with its $(n + 1)$-weight vector $\bar{\alpha}$ which is a point in the dual space $(\mathbb{R}^{n+1})^* = \mathbb{R}^{n+1}$. It is shown in the paper [16] that $P(2, n)$ can be expressed by the number $C(E)$ of disjoint chambers obtained as a compliment in $\mathbb{R}^{n+1}$ to the arrangement of $2^n$ hyperplanes all passing through the origin with the normal vectors from the set

$$E = \{ (1, b_1, \ldots, b_n) \mid b_i \in E_2, \ i = 1, \ldots, n \}. \quad (1)$$

The upper bound of the number $C(H)$ for any central arrangement of hyperplanes with a set $H$ of normal vectors was established by L. Schl"afli in [13]. For the case $H = E$ we have the following upper bound:

$$P(2, n) = C(E) \leq 2 \sum_{i=0}^{n} \binom{2^n - 1}{i}. \quad (2)$$

It should be noted that in the early 60s of the 20th century the upper bound (2) was obtained by several authors [2], [10], [14], [15]. The detailed information of contribution of above mentioned authors can also be found in [3].

One of the first lower bound of $P(2, n)$ was established by S. Muroga in [11]:

$$P(2, n) \geq 2^{0.33048n^2}. \quad (3)$$

S. Yajima and T. Ibaraki in [17] improved the order of the logarithm of the lower bound [3] upto $n^2/2$:
\[ P(2, n) \geq 2^{n(n-1)/2+8} \quad \text{for} \quad n \geq 6. \quad (4) \]

Further significant improvements of the bound (4) were obtained basing on the paper [12] of A. M. Odlyzko. In the paper [19] it was noted that from the papers [12], [18] follows:

\[ C(E) = P(2, n) \geq 2^{n^2-10n^2/\ln n + O(n \ln n)}. \quad (5) \]

Taking into account the upper bound (2) and the inequality (5) it is easy to see that

\[ \lim_{n \to \infty} \frac{\log_2 P(2, n)}{n^2} = 1. \quad (6) \]

In the paper [6] a combination of an original geometric construction with the result from the paper [12] made it possible to improve the inequality (5) upto:

\[ P(2, n) \geq 2^{n^2(1-7/\ln n)} \cdot P\left(2, \left\lceil \frac{7(n-1)}{\log_2(n-1)} \right\rceil \right). \quad (7) \]

The generalization of the inequality (7) for the number of threshold k-logic functions was obtained in [9]. Asymptotics of logarithm of the number of polynomial threshold functions has been recently obtained in [1].

Let \( H = \{w_1, \ldots, w_T\} \) be a set of vectors in \( \mathbb{R}^{n+1} \). Let us denote by \( H^{s} \) the set of ordered collections of different \( s \) vectors, \( s = 1, \ldots, T \), from \( H \) and let \( [H]^{s, \neq 0} \subset H^{s} \) be a subset of linearly independent vectors

\[ H^{s, \neq 0} := \{ (w_{i_1}, \ldots, w_{i_s}) \mid \dim \text{span}(w_{i_1}, \ldots, w_{i_s}) = s \}. \]

For any \( W = (w_{i_1}, \ldots, w_{i_n}) \in H^{n, \neq 0} \) let

\[ q_{l_i}^W := |L_l(W) \cap H| := |\text{span}(w_{i_{l_i+1}}, \ldots, w_{i_n}) \cap H|. \quad (8) \]

**Definition 2** For any \( W \in H^{n} \) the ordered set of numbers

\[ W(H) := (q_n^W, q_{n-1}^W, \ldots, q_1^W) \]

is called a combinatorial flag of the ordered set \( W \).

If \( W \in [H]^{n, \neq 0} \), then \( W(H) \) is called a full combinatorial flag of \( W \).
For the sake of simplicity we will use the following notation:
\[ W[H] := q_n^W \cdot q_{n-1}^W \cdots q_1^W. \] (9)

The aim of this paper is to prove the following lower bound of the number of threshold functions. For any weights \( p = (p_1, \ldots, p_{2^n}) \), \( p_i \in \mathbb{R}, \sum_{i=1}^{2^n} p_i = 1 \) on the set \( E \) (see (1)) we have
\[ P(2, n) \geq 2 \sum_{W \in E} \frac{1 - p_{i_1} - p_{i_2} - \cdots - p_{i_{qW}^W}}{W[E]}, \] (10)
and the right side of the inequality doesn’t depend on the choice of \( p \). Here the indices used in the numerator correspond to vectors from \( L_n(W) \cap E = \{w_{i_1}, \ldots, w_{i_n}, \ldots w_{i_{qW}^W}\} \).

2 Preliminaries.

Let \( H^\perp \) be a finite arrangement of hyperplanes all passing through the zero in \( \mathbb{R}^{n+1} \) (central arrangement) and denote by \( H = \{w_1, \ldots, w_T\} \) the set of their normal vectors. We define a partially ordered set (poset) \( L^H \) in the following way. By definition any subspace of \( \mathbb{R}^{n+1} \) generated by some (possibly empty) subset of \( H \) is an element of the poset \( L^H \). An element \( s \in L^H \) is less than an element \( t \in L^H \) iff the subspace \( t \) contains the subspace \( s \). For any poset \( P \) we can define a simplicial complex \( \Delta_P \) in the following way. The set of vertices of \( \Delta_P \) coincides with the set of elements \( P \) and a set of vertices of \( P \) defines a simplex of \( \Delta_P \) iff this set forms a chain in \( P \). Let us denote by \( \Delta_{L^H} \) the simplicial complex of the poset
\[ (0_{L^H}, 1_{L^H}) := \{ z \in L^H \mid 0_{L^H} < z < 1_{L^H} \}, \]
where \( 0_{L^H} \) and \( 1_{L^H} \) are the elements of the poset \( L^H \) corresponding to the zero subspace of \( \mathbb{R}^{n+1} \) and the whole space \( \mathbb{R}^{n+1} \), respectively. It has been shown in [13] that the number \( C(H) \) of \((n+1)\)-dimensional regions into which \( \mathbb{R}^{n+1} \) is divided by hyperplanes from the set \( H^\perp \) can be found by the formula:
\[ C(H) = \sum_{t \in L^H} |\mu(0_{L^H}, t)|, \] (11)
where \( \mu(s, t) \) is M"obius function of the poset \( L^H \).
Möbius functions of partially ordered sets in the Zaslavsky’s formula (11) for calculation of the number of chambers $C(H)$ can be interpreted by tools of algebraic topology in the following way. First, we introduce a simplicial complex $K^H$. The set of vertices of $K^H$ coincides with the set $H$. A subset \( \{w_{i_1}, \ldots, w_{i_s}\} \) of $H$ forms a simplex of $K^H$ iff
\[
\text{span}(w_{i_1}, \ldots, w_{i_s}) \neq \mathbb{R}^{n+1}.
\]

Taking into account the results of the papers [4], [5] it is possible to show (see [8]) that the absolute value of the Möbius function \( |\mu(0_L H, u)| \) is equal to the dimension of the reduced homology group of the complex $K^H \cap u$ with coefficients in an arbitrary field $F$:
\[
|\mu(0_L H, u)| = \text{rank} \tilde{H}_{\dim u-2}(K^H \cap u; F). \tag{12}
\]

Here the set $H \cap u$ consists of all vectors $H$ belonging to the subspace $u \subset \mathbb{R}^{n+1}$ and is considered as a subset of $\mathbb{R}^{\dim u} := u$.

It follows from the definition of Möbius function that
\[
\sum_{0_L H \leq u < 1_L H} |\mu(0_L H, u)| \geq - \sum_{0_L H \leq u < 1_L H} \mu(0_L H, u) = |\mu(0_L H, 1_L H)|. \tag{13}
\]

Hence,
\[
C(H) = |\mu(0_L H, 1_L H)| + \sum_{0_L H \leq u < 1_L H} |\mu(0_L H, u)| \geq 2|\mu(0_L H, 1_L H)|. \tag{14}
\]

From (12) and (14) we have:
\[
C(H) \geq 2 \text{rank} H_{n-1}(K^H; F). \tag{15}
\]

As a consequence of (15) for the case $H = E$, we have:
\[
P(2, n) = C(E) \geq 2 \text{rank} H_{n-1}(K^E; F). \tag{16}
\]

Let us fix an arbitrary order $\pi : [T] \to H$ on the set $H$. We denote by $\Lambda^H$ the number of all collections of different vectors \( (w_{i_1}, \ldots, w_{i_s}) \), $2 \leq i_1, \ldots, i_s \leq T$ such that for any $l$, $1 \leq l \leq n$, the vector $w_{i_l}$ is the minimal
vector in the order $\pi$ among all vectors from the set $H \cap \text{span}(w_i, \ldots, w_{i_n})$. It follows from the definition of $\Lambda^H$ that

$$\dim \text{span}(w_i, \ldots, w_{i_n}) = n - l + 1,$$

i.e. the vectors $w_i, \ldots, w_{i_n}$ are linearly independent.

The theorem 7 of [8] is also true for any set $H$ generating $\mathbb{R}^{n+1}$ and here we rewrite it as follows:

$$\text{rank} H_{n-1}(K^H; \mathbb{F}) = \Lambda^H. \quad (17)$$

The description (17) of the number $\Lambda^H$ as rank of the homology group shows us that it doesn’t depend on the order $\pi$ on $H$ and we can fix the order $\pi$ as $\pi(i) := w_i, i \in [T]$. In that case $w_i <_\pi w_j \iff i < j$. We denote by $\Gamma$ the set of all orders on the set $H$. Then any order on $H$ can be defined as composition

$$[T] \xrightarrow{\gamma} [T] \xrightarrow{\pi} H$$

of a permutation $\gamma : [T] \to [T]$ with $\pi$, and

$$w_i <_\gamma w_j \iff (\pi\gamma)^{-1}(w_i) < (\pi\gamma)^{-1}(w_j) \iff \gamma^{-1}(i) < \gamma^{-1}(j).$$

Thus, $\Gamma$ can be identified with the symmetric group $\text{Sym}([T])$ and any permutation $\sigma : [T] \to [T]$ defines a basis of the homology group $H_{n-1}(K^H; \mathbb{F})$ considering as a vector space over the field $\mathbb{F}$ as a subset of collections of $n$ vectors from $H$

$$B^\sigma \subset H_{\neq 0}^\times$$

obeying the following conditions. Let $W^\sigma = (w_{\sigma(i_1)}, \ldots, w_{\sigma(i_n)})$ and $L(W^\sigma)$ be a flag of subspaces in $\mathbb{R}^{n+1}$

$$L_n(W^\sigma) \supset L_{n-1}(W^\sigma) \supset \ldots \supset L_l(W^\sigma) \supset \ldots \supset L_1(W^\sigma),$$

where

$$L_l(W^\sigma) := \text{span}(w_{\sigma(i_{n-l+1})}, \ldots, w_{\sigma(i_n)}) \quad l = 1, \ldots, n. \quad (18)$$

Then $W^\sigma \in B^\sigma$ iff

$$i_1 < i_2 < \ldots < i_n \quad (19)$$

$$w_{\sigma(1)} \notin L_n(W^\sigma) \quad (20)$$

and

$$\min\{i \in [T] \mid w_{\sigma(i)} \in H \cap L_l(W^\sigma)\} = i_{n-l+1}, \quad l = 1, \ldots, n. \quad (21)$$
3 A lower bound of threshold functions.

**Theorem 1** For any probability distribution \( p = (p_1, \ldots, p_T) \) on the set \( H \) the following equality is true:

\[
\text{rank} \ H_{n-1} \left( K^H; F \right) = \sum_{W \in H_{\neq 0}^{x_n}} \frac{1 - p_{i_1} - p_{i_2} - \cdots - p_{i_{Wn}}}{W[H]}.
\]

Here the indices used in the numerator correspond to vectors from \( L_n(W) \cap H = \{ w_{i_1}, \ldots, w_{i_n}, \ldots w_{Wn} \} \).

**Proof.** We define a probability distribution \( \tilde{p} \) on the set \( \Gamma \cong \text{Sym}([T]) \) by the formula:

\[
\tilde{p}(\gamma) = p_{\gamma(1)} \frac{1}{(T - 1)!}, \quad \gamma \in \text{Sym}([T]).
\]

For any collection \( W = (w_{i_1}, \ldots, w_{i_n}) \in H_{\neq 0}^{x_n} \) we define a random function \( I_W : \Gamma \to \mathbb{R} \) by the formula:

\[
I_W(\gamma) = \begin{cases} 
1, & \text{if } \gamma^{-1}(i_1) < \gamma^{-1}(i_2) < \cdots < \gamma^{-1}(i_n) \\
& \text{and the conditions (20), (21) hold for the collection} \\
& W^\gamma := (w_{\gamma^{-1}(i_1)}, \ldots, w_{\gamma^{-1}(i_n)}) = (w_{i_1}, \ldots, w_{i_n}); \\
& \text{in the order generated by } \gamma \\
0, & \text{in all other cases.}
\end{cases}
\]

Let

\[
I := \sum_{W \in H_{\neq 0}^{x_n}} I_W : \Gamma \to \mathbb{R}.
\]

Then for any \( \gamma \in \Gamma \)

\[
I(\gamma) = \text{const} = \text{rank} \ H_{n-1} \left( K^H; F \right).
\]

Hence the expectation of \( I \) is equal to the rank of the homology group:

\[
\mathbb{E}[I] = \text{rank} \ H_{n-1} \left( K^H; F \right). \quad (22)
\]
Additivity of expectation reduces the problem of calculation $\mathbf{E}[I]$ to counting the probability $\mathbf{Pr}(I_W = 1)$:

$$\mathbf{E}[I] = \sum_{W \in H^W_{\neq 0}} \mathbf{E}[I_W] = \sum_{W \in H^W_{\neq 0}} \mathbf{Pr}(I_W = 1).$$  \hfill (23)$$

Further we calculate the number of permutations $\gamma$ such that $I_W(\gamma) = 1$. Since $w_{\gamma(1)} \notin L_n(W)$ then $q^W_n$ vectors from $L_n(W) \cap H$ can be located in any places except the first one, i.e. $\gamma^{-1}(j) \neq 1$ for any $j \in [T]$ such that $w_j \in L_n(W) \cap H$. The arrangement of the remaining vectors from $H \setminus \{w_{\gamma(1)} \cup \{L_n(W) \cap H\}$ does not affect the fulfillment of the conditions (20) and (21). By the condition (21) the vector $w_i$ has to be located in the first place from the chosen $q^W_n$ positions for arrangement of the set $L_n(W) \cap H$, while $q^W_{n-1}$ vectors from $L_{n-1}(W) \cap H$ can be located in any of the remaining $q^W_n - 1$ places. The arrangement of the vectors from $\{L_n(W) \cap H\} \setminus \{w_{n-1} \cup \{L_{n-1}(W) \cap H\}$ in $q^W_n - q^W_{n-1} - 1$ places left after choosing $q^W_{n-1} + 1$ places for arrangement of the set $L_{n-1}(W) \cap H$ and $w_i$ doesn’t affect the fulfillment of the conditions (20) and (21). Continuing the same way, we get that in the first place from $q^W_l$ positions arranged for the vectors from $L_l(W) \cap H$ has to be located the vector $w_{\gamma(1)}$, while $q^W_{l-1}$ vectors from $L_{l-1}(W) \cap H$ can be located in any of the remaining $q^W_l - 1$ places and the positions of the vectors $\{L_l(W) \cap H\} \setminus \{w_{l-1} \cup \{L_{l-1}(W) \cap H\}$ in $q^W_l - q^W_{l-1} - 1$ places left after choosing $q^W_{l-1} + 1$ places for arrangement of the set $L_{l-1}(W) \cap H$ and $w_{\gamma(1)}$ doesn’t affect the fulfillment of the conditions (20) and (21).

Denote by $N(\gamma(1) = i)$ the number of permutations $\gamma$ with fixed value $\gamma(1) = i$ such that $w_i \notin L_n(W)$. Then

$$N(\gamma(1) = i) = \frac{(T-1)!}{q^W_1} (T - 1 - q^W_n)! \cdot \left(\frac{q^W_{n-1}}{q^W_n-1}\right) (q^W_n - q^W_{n-1} - 1)! \cdots$$

$$\cdot \left(\frac{q^W_{l-1}}{q^W_l-1}\right) (q^W_l - q^W_{l-1} - 1)! \cdots \left(\frac{q^W_1}{q^W_2-1}\right) (q^W_2 - q^W_1 - 1)! (q^W_1 - 1)! =$$

$$= \frac{(T-1)!}{q^W_n!} \cdot \frac{q^W_{n-1}!}{q^W_{n-1}!} \cdot \frac{q^W_{l-1}!}{q^W_{l-1}!} \cdots \frac{q^W_1!}{q^W_1!} (q^W_1 - 1)! =$$

$$= \frac{(T-1)!}{W[H]}$$

Then, we have

$$\mathbf{Pr}(I_W = 1) = \sum_{i \in [T] \text{ s.t. } w_i \notin L_n(W)} \frac{1}{(T-1)!} \frac{(T-1)!}{W[H]} =$$
\[ 1 - p_{i_1} - \cdots - p_{i_q W_n} \]

where \( L_n(W) \cap H = \{w_{i_1}, \ldots, w_{i_n}, \ldots, w_{i_q W_n}\} \).

Q.E.D.

**Remark 1** Since on the right side of equation of Theorem 1 is a polynomial of degree 1 then the Theorem 1 is true for any \( p_i \in \mathbf{R}, \; i = 1, \ldots, T \), such that \( \sum_{i=1}^T p_i = 1 \).

**Corollary 1** For any weights \( p = (p_1, \ldots, p_{2^n}) \), \( p_i \in \mathbf{R}, \sum_{i=1}^{2^n} p_i = 1 \) on the set \( E \) we have

\[ P(2, n) \geq 2 \sum_{W \in E \times \mathbf{R}^{\times n}} \frac{1 - p_{i_1} - p_{i_2} - \cdots - p_{i_q W_n}}{W[E]}, \]

and the right side of the inequality doesn’t depend on the choice of \( p \). Here the indices used in the numerator correspond to vectors from \( L_n(W) \cap E = \{w_{i_1}, \ldots, w_{i_n}, \ldots, w_{i_q W_n}\} \).

**Proof.** The inequality follows from (16) and Theorem 1

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