A note on the 2-dual space of $L^p[0, 1]$

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Abstract

In the present note, we are interested in bounded 2-functionals and 2-dual spaces of $L^p[0, 1]$. The 2-dual spaces of the sequence space $l^p$ is considered in the literature. But interestingly an explicit computation of $L^p$ spaces has not been considered though n-duals of general normed spaces have been considered. We shall consider the 2-dual spaces with the usual $\|\cdot\|_p$ norm and with respect to the Gähler and the Gunawan norm. The n-dual space of $L^p[0, 1]$ can be treated in a similar manner.

Key Words : $L^p[0, 1]$, 2-normed space, 2-dual space.

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1 INTRODUCTION

Let $n$ be a non-negative integer and $X$ be a vector space over $\mathbb{R}$ of dimension $d \geq n$. An $n$-norm is a real valued function on $X^n$ satisfying the following properties.

1. $\|x_1, \ldots, x_n\| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent.

2. $\|x_1, \ldots, x_n\|$ is invariant under permutation.

3. $\|\alpha x_1, \ldots, x_n\| = |\alpha| \|x_1, \ldots, x_n\|$ for $\alpha \in \mathbb{R}$.

4. $\|x_1 + x'_1, \ldots, x_n\| \leq \|x_1, \ldots, x_n\| + \|x'_1, \ldots, x_n\|$

Then $(X, \|\cdot, \cdot\|)$ is called as an $n$-normed space. Details about $n$-normed spaces can be found in [1], [8], [9], [6], and [10]. Let $1 \leq p < \infty$ and $q$ be the conjugate exponent of $p$ that is $\frac{1}{p} + \frac{1}{q} = 1$. The sequence spaces $l^p$ have been studied in the literature. See [12], [14] and [15]. Analogously one can equip $n$-norm on the space $L^p[0, 1]$, the space of all bounded measurable real valued functions that are $p$-th power Lebesgue integrable. That is

$$L^p[0, 1] := \left\{ x : [0, 1] \rightarrow \mathbb{R} : \int_0^1 |x(t)|^p dt < \infty \right\}.$$

We consider the 2-norm. An $n$-norm can be similarly defined. The following 2-norm on the space $L^p[0, 1]$ is due to Gähler. (See [3], [4] and [5].) For $x_1, x_2 \in L^p[0, 1],$

$$\|x_1, x_2\|_p^G := \sup_{y_1, y_2 \in L^q[0, 1], \|y_1\|_q \leq 1, \|y_2\|_q \leq 1} \left| \int_0^1 x_1(u)y_1(u)du \int_0^1 x_2(u)y_2(u)du \right| \int_0^1 x_1(u)y_1(u)du \int_0^1 x_2(u)y_2(u)du$$

Here $L^q[0, 1]$ is the usual dual of the space $L^p[0, 1]$. The 2-norm introduced by Gunawan [7] and [2] on the space $L^p[0, 1]$ takes the following form

$$\|x_1, x_2\|_p^H := \left( \frac{1}{2!} \int_0^1 \int_0^1 \frac{x_1(u)x_1(v)}{x_2(u)x_2(v)} du dv \right)^{\frac{1}{p}}.$$
Any real valued function $f$ on $X^n$, is called n-functional on $X$. Further if $f$ satisfies

1. $f(x_1 + y_1, \ldots, x_n + y_n) = \sum_{h_i \in x_i, y_i, i \leq n} f(h_1, \ldots, h_n)$,

2. $f(\alpha_1 x_1, \ldots, \alpha_n x_n) = \alpha_1 \ldots \alpha_n f(x_1, \ldots, x_n),$

is called multilinear n-functional on $X$.

An n-functional on a normed space $X$ is said to be bounded on $X$ if there is a constant $K > 0$ such that

$$|f(x_1, x_2, \ldots, x_n)| \leq K\|x_1\| \ldots \|x_n\|$$

If $X$ is equipped with the n-norm then n-functional $f$ is said to be bounded if

$$|f(x_1, \ldots, x_n)| \leq K\|x_1, \ldots, x_n\|$$

Remark 1.1. Every bounded multilinear n-functional $f$ on an n-normed space $X$ satisfies

$$f(x_1, \ldots, x_n) = 0,$$

whenever $x_1, \ldots, x_n$ are linearly dependent. Also $f$ is antisymmetric, that is

$$f(x_1, \ldots, x_n) = sgn(\sigma) f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

for any $x_1, x_2, \ldots, x_n \in X$ and for any permutation $\sigma$ of $\{1, 2, \ldots, n\}$

The space of bounded multilinear n-functionals on $(X, \|\|)$ is called the n dual space of $(X, \|\|)$. The norm on this space is given by

$$\|f\|_{n,1} := \sup_{\|x\|_1, \ldots, \|x\|_n \neq 0} \frac{|f(x_1, \ldots, x_n)|}{\|x_1\| \ldots \|x_n\|}$$

whereas The space of bounded multilinear n-functionals on $(X, \|\|)$ is called the $n$-dual space of $(X, \|\|)$. The norm on this space is given by
The n-dual spaces of the sequence space was considered by Pangalela and Gunawan \[12\]. He considered and identified the n-dual spaces with respect to Gähler norm and the Gunawan norm. He worked on the more general normed spaces and studied the n-dual in \[13\]. But an explicit computation and identification of the 2-dual space of $L^p[0, 1]$ with the natural $\| \cdot \|_p$ norm, Gähler norm and Gunawan norm were not considered. In this paper, we shall consider the 2-dual space of $L^p[0, 1]$. The n-dual space computation can be done on similar lines.

## 2 THE 2-DUAL SPACES OF $L^p[0, 1]$  

We shall here identify the 2-dual space of $L^p[0, 1]$ as a normed space. We shall assume that $1 \leq p < \infty$ and $q$ is the conjugate exponent of $p$. In other words $q$ satisfies $\frac{1}{p} + \frac{1}{q} = 1$.

We introduce the following normed space $Y^q_{[0,1] \times [0,1]}$ as follows:

$$Y^q_{[0,1] \times [0,1]} := \left\{ \theta : [0, 1] \times [0, 1] \to \mathbb{R}, \text{measurable} : \frac{1}{\| \theta \|^q_{Y^q_{[0,1] \times [0,1]}}} := \sup_{\|x\|_p = 1} \left( \int_0^1 \int_0^1 |x(u)\theta(u, v)|^q dv \right)^\frac{1}{q} < \infty \right\}. $$

$\| \theta \|^q_{Y^q_{[0,1] \times [0,1]}}$ defines a norm on $Y^q_{[0,1] \times [0,1]}$. In a similar manner, one defines the space

$$Y^\infty_{[0,1] \times [0,1]} := \left\{ \theta : [0, 1] \times [0, 1] \to \mathbb{R}, \text{measurable, essentially bounded} : \| \theta \|^\infty_{Y^\infty_{[0,1] \times [0,1]}} < \infty \right\},$$

where

$$\| \theta \|^\infty_{Y^\infty_{[0,1] \times [0,1]}} := \sup_{\|x\|_p = 1} \left( \sup_{u \in [0,1]} \int_0^1 |x(u)\theta(u, v)| dv \right) < \infty.$$

We first prove the main result.
Theorem 2.1. Let $1 < p < \infty$, then the 2-dual space of $(L^p[0,1], \|\cdot\|_p)$ is isometrically bijective to $(Y^q_{[0,1] \times [0,1]}, \|\cdot\|_{Y^q_{[0,1] \times [0,1]}})$

Proof. Let $\theta \in Y^q_{[0,1] \times [0,1]}$. Define a 2-functional on $L^p[0,1] \times L^p[0,1]$ given by

$$f(x, y) := \int_0^1 \int_0^1 x(u)y(v)\theta(u, v)du dv,$$

where $x, y \in L^p[0,1]$, with $\|x\|_p = \|y\|_p = 1$. On applying Hölder’s inequality, we get

$$|f(x, y)| = \int_0^1 y(v)(\int_0^1 x(u)\theta(u, v)du)dv \leq (\int_0^1 |y(v)|^p dv)^{\frac{1}{p}}(\int_0^1 |\int_0^1 x(u)\theta(u, v)du|^q dv)^{\frac{1}{q}}$$

As a consequence

$$|f(x, y)| \leq \left(\int_0^1 \left|\int_0^1 x(u)\theta(u, v)du\right|^q dv\right)^{\frac{1}{q}} \leq \sup_{\|z\|_p = 1} \left(\int_0^1 \left|\int_0^1 z(u)\theta(u, v)du\right|^q dv\right)^{\frac{1}{q}} = \|\theta\|_{Y^q_{[0,1] \times [0,1]}}$$

Hence for $x, y \neq 0$,

$$\frac{|f(x, y)|}{\|x\|_p \|y\|_q} \leq \|\theta\|_{Y^q_{[0,1] \times [0,1]}}$$

As a result

$$\|f\|_{2,1} \leq \|\theta\|_{Y^q_{[0,1] \times [0,1]}} \quad (2.1)$$

Conversely, $f$ be a bounded 2-functional on $L^p[0,1]$. That is $f : L^p[0,1] \times L^p[0,1] \to \mathbb{R}$ is a bounded linear functional. Morever

$$|f(x, y)| \leq \|f\|_{2,1} \|x\|_p \|y\|_p.$$

Let $x \in L^p[0,1]$ with $\|x\|_p = 1$. Let

$$f_x : L^p[0,1] \to \mathbb{R}$$

be a functional given by

$$f_x(y) := f(x, y).$$
Then
\[ \frac{|f_x(y)|}{\|y\|_p} = \frac{|f(x, y)|}{\|y\|_p} \leq \|f\|_{2,1}\|x\|_p. \]

\(f_x\) is a bounded linear functional on \(L^p[0, 1]\) and
\[ \|f_x\| \leq \|f\|_{2,1}. \]

\(f_x \in L^p[0, 1]'\). By the Riesz representation theorem, there exist \(z(x) \in L^q[0, 1]\) such that
\[ f(x, y) = f_x(y) = \int_0^1 z(x)(v)y(v)dv. \]

and
\[ \|f_x\| = \|z(x)\|_q = \left( \int_0^1 |z(x)(v)|^qdv \right)^{\frac{1}{q}} \leq \|f\|_{2,1}. \]

For each \(x \in L^p[0, 1]\)
\[ \left( \int_0^1 |z(x)(v)|^qdv \right)^{\frac{1}{q}} \leq \|f\|_{2,1}. \]

\(x \to z(x)\) is a Bounded linear map from \(L^p[0, 1]\) to \(L^q[0, 1]\). Set
\[ \theta(u, v) = z(x(u))(v). \]

\[ |\int_0^1 x(u)\theta(u, v)du| \leq \left( \int_0^1 |x(u)|^pdu \right)^{\frac{1}{p}} \left( \int_0^1 |\theta(u, v)|^qdu \right)^{\frac{1}{q}}. \]

As a result
\[ |\int_0^1 x(u)\theta(u, v)du|^q \leq \|x\|_p^q \int_0^1 |\theta(u, v)|^qdu. \]

So
\[ \int_0^1 |\int_0^1 x(u)\theta(u, v)du|^qdv \leq \|x\|_p^q \int_0^1 \int_0^1 |\theta(u, v)|^qdu dv = \|x\|_p^q \int_0^1 \int_0^1 |\theta(u, v)|^qdv du \]
\[ \int_0^1 |\int_0^1 x(u)\theta(u, v)du|^qdv \leq \|x\|_p^q \int_0^1 \|\theta\|_{2,1}^q du \]

and
\[
\left( \int_0^1 \int_0^1 x(u) \theta(u,v) du \, dv \right)^{\frac{1}{q}} \leq \|x\|_p \|f\|_{2,1}
\]

\[
\|\theta\|_{Y^q_{\frac{1}{p}, 1} \times [0,1]} = \sup_{\|x\|_p=1} \left( \int_0^1 \int_0^1 x(u) \theta(u,v) du \, dv \right)^{\frac{1}{q}} \leq \|f\|_{2,1}
\]  

(2.2)

Thus

\[
\|f\|_{2,1} = \|\theta\|_{Y^q_{\frac{1}{p}, 1} \times [0,1]}
\]

from (2.1) and (2.2).

Thus \( f \to \theta \) is an isometry from the 2-dual space of \((L^p[0,1], \|\cdot\|_p)\) to \((Y^q_{\frac{1}{p}, 1} \times [0,1], \|\cdot\|_{Y^q_{\frac{1}{p}, 1} \times [0,1]})\)

In a similar manner, we can prove that the 2-dual space of \((L^1[0,1], \|\cdot\|_1)\) is identified by \((Y^\infty_{\frac{1}{p}, 1} \times [0,1], \|\cdot\|_{Y^\infty_{\frac{1}{p}, 1} \times [0,1]})\) Now we shall discuss the 2-dual space of \((L^p[0,1], \|\cdot\|_{G_p})\).

See [12]. We need the concept of g-orthogonality on \(L^p[0,1]\), where \(g\) defined on \(L^p[0,1] \times L^p[0,1]\) is given by the formula

\[
g(x, y) := \|x\|_p^{2-p} \int_0^1 |x(u)|^{p-1} \sgn(x(u)) y(u) du, \ x, y \in L^p[0,1].
\]

(See [11] and [12].) Note that

1. \(g(x, x) = \|x\|_p^{2-p} \int_0^1 |x(u)|^{p-1} \sgn(x(u)) x(u) du = \|x\|_p^{2-p} \int_0^1 |x(u)|^{p-1} |x(u)| du\)
   
   As a consequence \(g(x, x) = \|x\|_p^{2-p} \int_0^1 |x(u)|^p du = \|x\|_p^{2-p} \|x\|_p^p = \|x\|_p^2\).

2. \(g(\alpha x, \beta y) = |\alpha|^{2-p} |\alpha|^{p-1} \sgn \alpha \beta g(x, y) = \alpha \beta g(x, y)\)

3. \(g(x, x + y) = \|x\|_p^2 + g(x, y).\)

4. \(|g(x, y)| \leq \|x\|_p^{2-p} \int_0^1 |x(u)|^{p-1} |y(u)| du = \|x\|_p^{2-p} \int_0^1 |x(u)|^{\frac{p}{p'}} |y(u)| du \leq \|x\|_p^{2-p} \|x\|_p^p \|y\|_{p'}\)
   
   So \(|g(x, y)| \leq \|x\|_p \|y\|_{p'}\).

5. \(g(x, y)\) is linear in \(y\).
As \( g \) satisfies the above properties \( g \) defines a semi inner product. If \( g(x, y) = 0 \), then we say that \( x \) and \( y \) are \( g \)-orthogonal and we write \( x \perp_g y \). Let \( x \in L^p[0,1] \) and \( Y = \{ y_1, y_2 \} \subset L^p[0,1] \). Let \( \Gamma(y_1, y_2) \) denote the Gram determinant:

\[
\begin{vmatrix}
  g(y_1, y_1) & g(y_1, y_2) \\
  g(y_2, y_1) & g(y_2, y_2)
\end{vmatrix}
\]

If \( \Gamma(y_1, y_2) \neq 0 \) then the vector

\[
x_Y := -\frac{1}{\Gamma(y_1, y_2)} \begin{vmatrix}
  0 & y_1 & y_2 \\
  g(y_1, x) & g(y_1, y_1) & g(y_1, y_2) \\
  g(y_2, x) & g(y_2, y_1) & g(y_2, y_2)
\end{vmatrix}
\]

is the Gram Schmidt projection of vector \( x \) on \( Y \). If \( \{ x_1, x_2 \} \) is linearly independent then

\[
x_1^o = x_1, \quad x_2^o = x_2 - \frac{g(x_1^o, x_2)}{g(x_1^o, x_1^o)} x_1^o
\]

defines a left \( g \)-orthogonal sequence. Note that \( x_1^o \perp_g x_2^o \). Define the volume of the 2-rectangle spanned by \( x_1 \) and \( x_2 \) by

\[
V(x_1, x_2) := \|x_1^o\|\|x_2^o\|.
\]

See [10]. If \( x_1, x_2 \) are linearly dependent define \( V(x_1, x_2) = 0 \). As in [10] it can be shown that

\[
V(x_1, x_2) \leq \|x_1, x_2\|^G_p
\]

for all \( x_1, x_2 \in L^p[0,1] \). Using the above result, the result concerning the equivalence of the Gähler norm and Gunawan’s norm can be exactly proved as [15].

**Theorem 2.2.** For \( x_1, x_2 \in L^p[0,1] \) we have

\[
2^{\frac{1}{p} - 1} \|x_1, x_2\|^H_p \leq \|x_1, x_2\|^G_p \leq 2^{\frac{1}{p}} \|x_1, x_2\|^H_p
\]

Using the above result we have the following theorem which is similarly proved as in Theorem 2.3. of [12].
Theorem 2.3. A bilinear 2-functional $f$ is bounded on $(L^p[0, 1], \|\cdot\|_{p}^G)$ if and only if $f$ is antisymmetric and bounded on $(L^p[0, 1], \|\cdot\|_p)$. Furthermore, we have

$$\frac{1}{2} \|f\|_{2,1} \le \|f\|_{2,2}^G \le \|f\|_{2,1}$$

where $\|\cdot\|_{2,2}^G$ is the norm on the 2-dual space of $(L^p[0, 1], \|\cdot\|_{p}^G)$.

To identify the dual space of $(L^p[0, 1], \|\cdot\|_{p}^G)$ consider the subspace of $Y^q_{[0, 1] \times [0, 1]}$. Define a subspace $Z^q_{[0, 1] \times [0, 1]}$ to be all $\theta : [0, 1] \times [0, 1] \to \mathbb{R}$ measurable such that $\theta(u, v) = -\theta(v, u)$. $Z^q_{[0, 1] \times [0, 1]}$ can be viewed as a normed space equipped with norm inherited from $Y^q_{[0, 1] \times [0, 1]}$. We have shown that the 2-dual space of $(L^p[0, 1], \|\cdot\|_p)$ is isometrically isomorphic to $(Y^q_{[0, 1] \times [0, 1]} \times \|\cdot\|_{Y^q_{[0, 1] \times [0, 1]}})$. Hence the space of antisymmetric bounded bilinear 2-functionals on $(L^p[0, 1], \|\cdot\|_p)$ can be identified with $(Z^q_{[0, 1] \times [0, 1]} \times \|\cdot\|_{Y^q_{[0, 1] \times [0, 1]}})$. We now present the following Corollaries.

Corollary 1. The function $\|\cdot\|_{Y^q_{[0, 1] \times [0, 1]}^G}$ on $Z^q_{[0, 1] \times [0, 1]}$ defined by

$$\|\cdot\|_{Y^q_{[0, 1] \times [0, 1]}^G} := \sup_{\|x, y\|_p^G \neq 0} \frac{\left| \int_0^1 \int_0^1 x(u)y(v)\theta(u, v)\,dudv \right|}{\|x, y\|_p^G}$$

defines a norm on $Z^q_{[0, 1] \times [0, 1]}$. Further, $\|\cdot\|_{Y^q_{[0, 1] \times [0, 1]}}$ and $\|\cdot\|_{Z^q_{[0, 1] \times [0, 1]}^G}$ are equivalent norms on $Z^q_{[0, 1] \times [0, 1]}$ with

$$\frac{1}{2} \|\theta\|_{Y^q_{[0, 1] \times [0, 1]}} \le \|\theta\|_{Z^q_{[0, 1] \times [0, 1]}^G} \le \|\theta\|_{Y^q_{[0, 1] \times [0, 1]}}$$

for all $\theta \in Z^q_{[0, 1] \times [0, 1]}$.

Corollary 2. The 2-dual space of $(L^p[0, 1], \|\cdot\|_p^G)$ is isometrically isomorphic to $(Z^q_{[0, 1] \times [0, 1]} \times \|\cdot\|_{Y^q_{[0, 1] \times [0, 1]}^G})$.

Corollary 3. The function $\|\cdot\|_{Y^q_{[0, 1] \times [0, 1]}^H}$ on $Z^q_{[0, 1] \times [0, 1]}$ defined by

$$\|\cdot\|_{Y^q_{[0, 1] \times [0, 1]}^H} := \sup_{\|x, y\|_p^H \neq 0} \frac{\left| \int_0^1 \int_0^1 x(u)y(v)\theta(u, v)\,dudv \right|}{\|x, y\|_p^H}$$
defines a norm on $Z^q_{[0,1] \times [0,1]}$. Further, $\| \cdot \|_{Y^q_{[0,1] \times [0,1]}}$ and $\| \cdot \|_{Z^q_{[0,1] \times [0,1]}}$ are equivalent norms on $Z^q_{[0,1] \times [0,1]}$ with

$$2^{\frac{1}{p}-1} \| \theta \|_{G^q_{[0,1] \times [0,1]}} \leq \| \theta \|_{H^q_{[0,1] \times [0,1]}} \leq 2^{\frac{1}{p}} \| \theta \|_{G^q_{[0,1] \times [0,1]}}$$

for all $\theta \in Z^q_{[0,1] \times [0,1]}$.

**Corollary 4.** The 2-dual space of $(L^p[0, 1], \| \cdot \|_p^H)$ is isometrically isomorphic to $(Z^q_{[0,1] \times [0,1]}, \| \cdot \|_{H^q_{[0,1] \times [0,1]}})$. The identification of n-dual spaces for $L^p[0, 1]$ can be done in exactly same way.

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