Abstract

We follow Jacquet-Shalika [JS90], Matringe [Mat14] and Cogdell-Matringe [CM15] to define exterior square gamma factors for irreducible cuspidal representations of $\text{GL}_n(F_q)$. These exterior square gamma factors are expressed in terms of Bessel functions, or in terms of the regular characters associated with the cuspidal representations. We also relate our exterior square gamma factors over finite fields to those over local fields through level zero representations.

1. Introduction

Let $F$ be a $p$-adic local field of characteristic zero with residue field $\mathfrak{f}$. Fix a non-trivial additive character $\psi$ of $F$. In their work [JS90], Jacquet and Shalika define important integrals which we call local Jacquet-Shalika integrals, see [JS90] Sections 7, 9.3. These Jacquet-Shalika integrals enable them to introduce integral representations for the exterior square $L$-function $L(s, \pi, \wedge^2)$ of a (generic) representation $\pi$ of $\text{GL}_n(F)$. Later, Matringe [Mat14] and Cogdell-Matringe [CM15] prove local functional equations for these local exterior square $L$-functions, in which local factors $\gamma(s, \pi, \wedge^2, \psi)$ and $\epsilon(s, \pi, \wedge^2, \psi)$ play an important role. These local factors are related via the following equation:

$$\gamma(s, \pi, \wedge^2, \psi) = \epsilon(s, \pi, \wedge^2, \psi) L(1-s, \overline{\pi}, \wedge^2) L(s, \pi, \wedge^2). \quad (1.1)$$

If $\pi$ is an irreducible supercuspidal representation of $\text{GL}_n(F)$, then it is generic, and the local factors $\gamma(s, \pi, \wedge^2, \psi)$ and $\epsilon(s, \pi, \wedge^2, \psi)$ are defined. By type theory of Bushnell and Kutzko [BK93], $\pi$ is constructed from some maximal simple type $(J, \lambda)$, see [BK93] Section 6] for details. Partially, $\lambda$ comes from some irreducible cuspidal representation $\pi_0$ of $\text{GL}_m(\mathfrak{e})$, where $\mathfrak{e}$ is some finite extension of $\mathfrak{f}$. In this paper, we are interested in defining the exterior square gamma factor $\gamma(\pi_0, \wedge^2, \psi_0)$ for some non-trivial additive character $\psi_0$ of $\mathfrak{e}$, and we relate $\pi$ to $\pi_0$ via their exterior square gamma factors, in the case where $\pi$ is of level zero representations, regular characters, exponential sums.

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zero. To be concrete, our main result of this paper is that if $\pi$ is a level zero representation constructed from $\pi_0$, a cuspidal representation of $GL_n(f)$ which does not admit a Shalika vector, then $\gamma(s, \pi, \wedge^2, \psi) = \gamma(\pi_0, \wedge^2, \psi_0)$.

In Section 2, we work with a general finite field $F$ and a non-trivial additive character $\psi$. For a generic representation $\pi$ of $GL_n(F)$ where $n = 2m$ or $n = 2m + 1$, we follow [JS90] to define the Jacquet-Shalika integral $J_{\pi, \psi}(W, \phi)$ and its dual $\tilde{J}_{\pi, \psi}(W, \phi)$, where $W$ is a Whittaker function of $\pi$ and $\phi$ is a complex valued function on $F^m$. Using ideas from [Mat14, CM15], we define the exterior square gamma factor $\gamma(\pi, \wedge^2, \psi)$ of $\pi$ in

**Theorem.** Let $\pi$ be an irreducible cuspidal representation of $GL_n(F)$ that does not admit a Shalika vector. Then there exists a non-zero constant $\gamma(\pi, \wedge^2, \psi)$ such that

$$\tilde{J}_{\pi, \psi}(W, \phi) = \gamma(\pi, \wedge^2, \psi) \cdot J_{\pi, \psi}(W, \phi),$$

for any Whittaker function $W \in \mathcal{W}(\pi, \psi)$ and any complex valued function $\phi$ on $F^m$. Here $n = 2m$ or $n = 2m + 1$.

This theorem is presented as Theorem 2.18 in Section 2. Its proof is separated into the even and odd cases, since the Jacquet-Shalika integrals differ by these cases. The proof relies heavily on some multiplicity one theorems, whose details can be found in appendix A. If $\pi$ admits a Shalika vector, we need to borrow results from the local field case in order to define the exterior square gamma factor, which is discussed in Theorem 3.15. By choosing some suitable $W$ and $\phi$, we are able to express $\gamma(\pi, \wedge^2, \psi)$ as a summation of the Bessel function $B_{\pi, \psi}$ over some tori of $GL_n(F)$, see Theorem 2.31 and Theorem 2.32. The ability to write local factors closely related to the exterior square gamma factor in terms of integrals of partial Bessel functions over tori as in [CST17, Proposition 4.6], is one of the key ingredients that Cogdell, Shahidi and Tsai use in order to prove the stability of exterior square gamma factors for local fields.

In Section 2.5, we perform some computations using the correspondence between irreducible cuspidal representations of $GL_n(F)$ and equivalence classes of regular multiplicative characters of $F^*_n$, where $F_n/F$ is a field extension of degree $n$. This correspondence can be seen as an analog of the Langlands correspondence for finite fields. Let $\pi$ be an irreducible cuspidal representation of $GL_n(F)$ associated with a regular character $\theta$ of $F^*_n$, as in [Gel70, Section 6]. We are able to express $\gamma(\pi, \wedge^2, \psi)$ explicitly in terms of $\theta$ and $\psi$ only, for $n = 2, 3, 4$.

In Section 3, we assume that $\pi$ is a level zero representation of $GL_n(F)$, where $F$ is a $p$-adic local field of characteristic zero with residue field $\mathfrak{f}$. Let $\psi$ be a non-trivial additive character of $F$, which descends to a non-trivial additive character $\psi_0$ of $\mathfrak{f}$. By type theory of Bushnell and Kutzko, $\pi$ is constructed from a maximal simple type $(GL_n(\mathfrak{o}), \pi_0)$, where $\mathfrak{o}$ is the ring of integers of $F$ and $\pi_0$ is an irreducible cuspidal representation of $GL_n(f)$. Our main result is in Theorem 3.16 which states that if $\pi_0$ does not admit a Shalika vector, then

$$\gamma(s, \pi, \wedge^2, \psi) = \gamma(\pi_0, \wedge^2, \psi_0),$$

$\gamma(\pi_0, \wedge^2, \psi_0)$ is a non-zero constant, thus the above equation implies that $\gamma(s, \pi, \wedge^2, \psi)$ is a non-zero constant independent of $s$. The case where $\pi_0$ admits a Shalika vector is also treated.
in Theorem \(3.17\). We are also able to specify the local exterior square factors \(L(s, \pi, \wedge^2)\) and \(\epsilon(s, \pi, \wedge^2, \psi)\) explicitly along with the equalities of the exterior square gamma factors.

This paper grows out of part of a thesis project of the first author, and the master’s thesis \([\text{Zel17}]\) of the second author.

2. The Jacquet-Shalika integral over a finite field

In this section, we define analogs of the local Jacquet-Shalika integrals \([\text{JS90}, \text{Section 7}]\), \([\text{CM15}, \text{Section 3.2}]\) over a finite field. We then prove that they satisfy a functional equation, which defines an important invariant, the exterior square gamma factor. This functional equation is valid only under some assumption on the involved representation, which we restate in several equivalent ways. We then express the exterior square gamma factor in terms of the Bessel function of Gelfand \([\text{Gel70}]\). Finally, we express the exterior square gamma factor in terms of the multiplicative regular character associated with the representation for \(n = 2, 3, 4\).

2.1. Preliminaries and notations.

2.1.1. Notations. Let \(\mathbb{F}\) be a finite field and denote \(q = |\mathbb{F}|\). Fix an algebraic closure \(\overline{\mathbb{F}}\) of \(\mathbb{F}\). For every positive integer \(m\), we denote by \(\mathbb{F}_m\) the unique field extension of \(\mathbb{F}\) in \(\overline{\mathbb{F}}\) of degree \(m\).

Let \(\psi : \mathbb{F} \to \mathbb{C}^*\) be a non-trivial additive character.

For a non-negative integer \(m\), we denote \(\mathcal{S}(\mathbb{F}^m) = \{f : \mathbb{F}^m \to \mathbb{C}\}\), the space of complex valued functions on \(\mathbb{F}^m\).

If \(\phi \in \mathcal{S}(\mathbb{F}^m)\), we define its Fourier transform with respect to \(\psi\) by the formula
\[
\mathcal{F}_\psi \phi (y) = q^{-\frac{m}{2}} \sum_{x \in \mathbb{F}^m} f(x) \psi (\langle x, y \rangle),
\]
where \(\langle x, y \rangle\) is the standard bilinear form on \(\mathbb{F}^m\). We have the following Fourier inversion formulas:
\[
\mathcal{F}_\psi \mathcal{F}_\psi \phi (x) = \phi (-x),
\]
\[
\mathcal{F}_\psi^{-1} \mathcal{F}_\psi \phi (x) = \phi (x).
\]

For an irreducible generic representation \(\pi\) of \(\text{GL}_n(\mathbb{F})\), we denote by \(\mathcal{W}(\pi, \psi)\) the Whittaker model of \(\pi\) with respect to the character \(\psi\).

For \(W \in \mathcal{W}(\pi, \psi)\), we define \(\widetilde{W} \in \mathcal{W}(\overline{\pi}, \psi^{-1})\) by \(\widetilde{W}(g) = W(w_n g^t)\), where \(g^t = t g^{-1}\) and \(w_n = \begin{pmatrix} 1 & \cdots & 1 \\ & \ddots & \end{pmatrix}\).
Let $n_1, \ldots, n_r \geq 1$ with $n_1 + \cdots + n_r = n$ and $g_i \in \GL_{n_i}(\F)$ for every $i$. We denote
\[
antidiag(g_1, \ldots, g_r) = \begin{pmatrix}
g_1 \\
g_2 \\
\vdots \\
g_r
\end{pmatrix} \in \GL_n(\F).
\]

2.1.2. The Bessel function. Let $(\pi, V_\pi)$ be a irreducible generic representation of $\GL_n(\F)$. By [Gel70, Propositions 4.2, 4.3] there exists a unique element $B_{\pi, \psi} \in \mathcal{W}(\pi, \psi)$ satisfying $B_{\pi, \psi}(I_n) = 1$ and $B_{\pi, \psi}(u_1 g u_2) = \psi(u_1) \psi(u_2) B_{\pi, \psi}(g)$, for every $g \in \GL_n(\F)$ and $u_1, u_2 \in N_n$, where $N_n$ is the upper triangular unipotent subgroup of $\GL_n(\F)$. We gather some properties of $B_{\pi, \psi}$.

**Theorem 2.1** ([Gel70, Proposition 4.9]). Let $w, d \in \GL_n(\F)$, where $w$ is a permutation matrix and $d$ is a diagonal matrix. Suppose that $B_{\pi, \psi}(d w) \neq 0$. Then $d w = \antidiag(\lambda_1 I_{n_1}, \ldots, \lambda_r I_{n_r})$, where $n_1 + \cdots + n_r = n$ and $\lambda_1, \ldots, \lambda_r \in \F^*$.

**Theorem 2.2** ([Gel70, Proposition 4.5]).
\[
B_{\pi, \psi}(g) = \frac{1}{|N_n|} \sum_{u \in N_n} \tr(\pi(g u)) \psi^{-1}(u).
\]

**Proposition 2.3** ([Nie14, Proposition 2.15] or [Rod10, Proposition 6.1.2]).
\[
B_{\pi, \psi}(g^{-1}) = B_{\pi, \psi}(g).
\]

Since representations of finite groups are unitary, we get the following

**Corollary 2.4** ([Nie14, Proposition 3.5]).
\[
B_{\pi, \psi}(g) = B_{\pi, \psi^{-1}}(g).
\]

2.2. The Jacquet-Shalika integral. In this section, we define the Jacquet-Shalika integral analog over the finite field $\F$.

2.2.1. The even case. Let $(\pi, V_\pi)$ be an irreducible generic representation of $\GL_{2m}(\F)$. For $W \in \mathcal{W}(\pi, \psi)$ and $\phi \in \mathcal{S}(\mathbb{F}^m)$ we define the Jacquet-Shalika integral as
\[
J_{\pi, \psi}(W, \phi) = \frac{1}{[G : N][M : B]} \sum_{g \in N \backslash G} \sum_{X \in B \backslash M} W(\sigma_{2m}(\begin{pmatrix} I_m & X \\ I_m & g \end{pmatrix})) \psi(-\tr X) \phi(\varepsilon g).
\]

Here $G = \GL_m(\F)$, $N \leq G$ is the upper triangular unipotent subgroup, $M = M_m(\F)$, $B \leq M$ is the upper triangular matrix subspace, $\varepsilon = \varepsilon_m = (0 \cdots 0 1)$. $\sigma_{2m}$ is the column permutation matrix corresponding to the permutation
\[
\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & m & m + 1 & m + 2 & \cdots & 2m \\ 1 & 3 & 5 & \cdots & 2m - 1 & 2 & 4 & \cdots & 2m \end{pmatrix}.
\]
We also define the dual Jacquet-Shalika integral as
\[
\tilde{J}_{\pi,\psi} (W, \phi) = J_{\tilde{\pi},\psi^{-1}} \left( \tilde{\pi} \left( I_m \atop I_m \right) \tilde{W}, F_\psi \phi \right).
\]
It is immediate from the definition that

**Proposition 2.5** (Double-duality).
\[
\tilde{J}_{\pi,\psi} \left( \tilde{\pi} \left( I_m \atop I_m \right) \tilde{W}, F_\psi \phi \right) = J_{\pi,\psi} (W, \phi).
\]

One can show the following

**Proposition 2.6.**
\[
\tilde{J}_{\pi,\psi} (W, \phi) = \frac{1}{[G : N] [M : B]} \sum_{g \in N \backslash G} \sum_{X \in B \backslash M} W \left( \sigma_{2m} \left( I_m \atop I_m \right) \left( \begin{array}{cc} g & X \\ I_m & g \end{array} \right) \right) \psi (-\text{tr} X) F_\psi \phi (\varepsilon_1 g'),
\]
where \( \varepsilon_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} \).

The following proposition is a simple exercise that follows from Theorem 2.1. It will also follow as a special case of Lemma 2.33.

**Proposition 2.7.** Let \( W = [G : N] [M : B] \pi (\sigma_{2m}^{-1}) B_{\pi,\psi} \), and let \( \phi (x) = \delta_\varepsilon (x) = \begin{cases} 1 & x = \varepsilon \\ 0 & \text{else} \end{cases} \).

Then \( J_{\pi,\psi} (W, \phi) = 1 \).

**Definition 2.8.** The Shalika subgroup \( S_{2m} \leq \text{GL}_{2m} (F) \) is defined as
\[
S_{2m} = \left\{ \begin{pmatrix} g & X \\ g & g \end{pmatrix} \mid g \in \text{GL}_m (F), X \in M_m (F) \right\}.
\]

\( S_{2m} \) acts on \( S (F^m) \) by
\[
\rho \begin{pmatrix} g & X \\ g & g \end{pmatrix} \phi (y) = \phi (yg).
\]
We define a character \( \Psi : S_{2m} \to \mathbb{C}^* \) by
\[
\Psi \begin{pmatrix} g & X \\ g & g \end{pmatrix} = \psi \left( \text{tr} \left( X g^{-1} \right) \right).
\]

One easily checks that

**Proposition 2.9** (Equivariance property). Let \( s \in S_{2m} \) and \( B \in \left\{ J_{\pi,\psi}, \tilde{J}_{\pi,\psi} \right\} \), we have
\[
B \left( \pi (s) W, \rho (s) \phi \right) = \Psi (s) B (W, \phi).
\]

**Definition 2.10.** A vector \( 0 \neq v \in V_\pi \) is called a Shalika vector, if for every \( s \in S_{2m} \) we have
\[
\pi (s) v = \Psi (s) v.
\]

**Remark 2.11.** Suppose that \( \pi \) admits a Shalika vector \( v \). Then for every \( a \in F^* \) we have \( aI_{2m} \in S_{2m} \) and \( \omega_\pi (a) v = \pi (aI_{2m}) v = v \), and therefore \( \pi \) has trivial central character.
2.2.2. The odd case. Let \((\pi, V_\pi)\) be an irreducible generic representation of \(\text{GL}_{2m+1}(\mathbb{F})\). For \(W \in \mathcal{W}(\pi, \psi)\) and \(\phi \in \mathcal{S}(\mathbb{F}^m)\) we define the Jacquet-Shalika integral as

\[
J_{\pi, \psi}(W, \phi) = \frac{1}{|G : N| |M : \mathcal{B}| |M_{1 \times m}(\mathbb{F})|} \sum_{g \in N \setminus G} \sum_{X \in \mathcal{B} \setminus M} \sum_{Z \in M_{1 \times m}(\mathbb{F})} W \left( \sigma_{2m+1} \left( \begin{array}{cc} I_m & X \\ I_m & 1 \end{array} \right) \left( \begin{array}{cc} g & 1 \\ g & 1 \end{array} \right) \left( \begin{array}{cc} I_m & -^t Z \\ Z & 1 \end{array} \right) \right) \psi(-\text{tr}X) \phi(Z).
\]

Here the notations are the same as in the even case, except for \(\sigma_{2m+1}\) which is the column permutation matrix corresponding to the permutation

\[
\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & m & m + 1 & m + 2 & \cdots & 2m & 2m + 1 \\ 1 & 3 & 5 & \cdots & 2m - 1 & 2 & 4 & \cdots & 2m & 2m + 1 \end{pmatrix}.
\]

We also define the dual Jacquet-Shalika integral as

\[
\tilde{J}_{\pi, \psi}(W, \phi) = J_{\tilde{\pi}, \psi^{-1}} \left( \tilde{\pi} \left( \begin{array}{cc} I_m & 1 \\ \end{array} \right) \widetilde{W}, \mathcal{F}_\psi \phi \right).
\]

Again, we get immediately from the definition the following analog of Proposition 2.5.

**Proposition 2.12** (Double-duality).

\[
\tilde{J}_{\pi, \psi}^{-1} \left( \pi \left( \begin{array}{cc} I_m & 1 \\ \end{array} \right) \widetilde{W}, \mathcal{F}_\psi \phi \right) = J_{\pi, \psi}(W, \phi).
\]

Similarly to Proposition 2.6 one can show the following

**Proposition 2.13.**

\[
\tilde{J}_{\pi, \psi}(W, \phi) = \frac{1}{|G : N| |M : \mathcal{B}| |M_{1 \times m}(\mathbb{F})|} \sum_{g \in N \setminus G} \sum_{X \in \mathcal{B} \setminus M} \sum_{Z \in M_{1 \times m}(\mathbb{F})} W \left( \begin{array}{cc} I_{2m} & 1 \\ \end{array} \right) \sigma_{2m+1} \left( \begin{array}{cc} I_m & X \\ I_m & 1 \end{array} \right) \left( \begin{array}{cc} g & 1 \\ g & 1 \end{array} \right) \left( \begin{array}{cc} I_m & -^t Z \\ Z & 1 \end{array} \right) \psi(-\text{tr}X) \mathcal{F}_\psi \phi(Z).
\]

Similarly to Proposition 2.7 one can check the following

**Proposition 2.14.** Let \(W = [G : N] [M : \mathcal{B}] [M_{1 \times m}(\mathbb{F})] \pi(\sigma_{2m+1}^{-1}) \mathcal{B}_{\pi, \psi}\), and \(\phi = \delta_0\) be the indicator function of \(0 \in \mathbb{F}^m\). Then \(J_{\pi, \psi}(W, \phi) = 1\).

**Definition 2.15.** The Shalika subgroup \(S_{2m+1} \leq \text{GL}_{2m+1}(\mathbb{F})\) is defined as

\[
S_{2m+1} = \left\{ \begin{pmatrix} g & X & Y \\ g & 1 \\ Z & 1 \end{pmatrix} \mid g \in \text{GL}_m(\mathbb{F}), X \in M_m(\mathbb{F}), Y \in M_{m \times 1}(\mathbb{F}), Z \in M_{1 \times m}(\mathbb{F}) \right\}.
\]
Let $P_{2m+1} \leq \text{GL}_{2m+1}(\mathbb{F})$ be the Mirabolic subgroup, i.e., the subgroup of matrices having $\varepsilon_{2m+1} = (0 \cdots 0 1)$ as their last row. We define a character $\Psi : S_{2m+1} \cap P_{2m+1} \to \mathbb{C}^*$ by

$$\Psi \begin{pmatrix} g & X & Y \\ g & g \\ 1 \end{pmatrix} = \psi \left( \text{tr} \left( Xg^{-1} \right) \right).$$

The Shalika subgroup acts on $S(\mathbb{F}^m)$ by the following relations [CM15, Proposition 3.1]

- $\rho \begin{pmatrix} g \\ g \\ 1 \end{pmatrix} \phi (x) = \phi (xg)$.
- $\rho \begin{pmatrix} I_m & z_0 \\ I_m & 1 \end{pmatrix} \phi (x) = \psi (-\text{tr} z_0) \phi (x)$.
- $\rho \begin{pmatrix} I_m & y_0 \\ I_m & 1 \end{pmatrix} \phi (x) = \psi (\langle x, y_0 \rangle) \phi (x)$.
- $\rho \begin{pmatrix} I_m \\ I_m \\ x_0 \end{pmatrix} \phi (x) = \phi (x + x_0)$.

As in [CM15, Proposition 3.1], one has

**Proposition 2.16.** The action of $S_{2m+1}$ on $S(\mathbb{F}^m)$ is equivalent to $\text{Ind}_{P_{2m+1} \cap S_{2m+1}}^{S_{2m+1}}(\Psi^{-1})$ given by mapping $f \in \text{Ind}_{P_{2m+1} \cap S_{2m+1}}^{S_{2m+1}}(\Psi^{-1})$ to $\phi \in S(\mathbb{F}^m)$, defined as $\phi (x) = f \begin{pmatrix} I_m \\ I_m \\ x \end{pmatrix}$.

As in [CM15, Lemma 3.2, 3.3], one easily checks that we have the following analog of Proposition 2.9

**Proposition 2.17 (Equivariance property).** Let $B \in \left\{ J_{\pi, \psi}, \tilde{J}_{\pi, \psi} \right\}$. Then for every $s \in S_{2m+1}$,

$$B (\pi (s) W, \rho (s) \phi) = B (W, \phi).$$

Since in the odd case, the character $\Psi$ is defined only on the subgroup $S_{2m+1} \cap P_{2m+1}$, we don’t have a definition for a Shalika vector in this case.

### 2.3. The functional equation

In this section, we prove the functional equation satisfied by the Jacquet-Shalika integral. This allows us to define the exterior square gamma factor of an irreducible cuspidal representation of $\text{GL}_n(\mathbb{F})$.

**Theorem 2.18 (The functional equation).** Let $n = 2m$ or $n = 2m + 1$, and let $\pi$ be an irreducible cuspidal representation of $\text{GL}_n(\mathbb{F})$. If $n$ is even, suppose that $\pi$ does not admit a Shalika vector. Then there exists a non-zero constant $\gamma (\pi, \wedge^2, \psi) \in \mathbb{C}^*$, such that for every $W \in \mathcal{W}(\pi, \psi)$ and $\phi \in S(\mathbb{F}^m)$, we have

$$J_{\pi, \psi} (W, \phi) = \gamma (\pi, \wedge^2, \psi) \cdot J_{\pi, \psi} (W, \phi).$$
The proof of this theorem is based on the proofs for the local field case of the local Jacquet-Shalika integrals [Mat14, Section 4], [CM15, Section 3.3]. The Shalika subgroup $S_n$ plays an important role in the proof. We treat the even case and the odd case separately. In both cases, the main idea is to show that the space of $S_n$ equivariant bilinear forms

$$B : \mathcal{W}(\pi, \psi) \times \mathcal{S}(\mathbb{F}^m) \to \mathbb{C}$$

is at most one dimensional. Since $J_{\pi,\psi}$ and $\tilde{J}_{\pi,\psi}$ define non-zero elements of this space (see Proposition 2.7, Proposition 2.9 for the even case, and Proposition 2.14, Proposition 2.17 for the odd case), the theorem follows.

**Definition 2.19.** We call $\gamma(\pi, \wedge^2, \psi)$ in the above theorem the **exterior square gamma factor** of $\pi$ with respect to the character $\psi$.

**Remark 2.20.** By double duality (Proposition 2.5 and Proposition 2.12), we have that

$$\gamma(\pi, \wedge^2, \psi) \cdot \gamma(\tilde{\pi}, \wedge^2, \psi^{-1}) = 1.$$ 

Substituting in the functional equation the functions from Proposition 2.7 and Proposition 2.14 in the even case and the odd case respectively, and using the fact that $B_{\pi,\psi} = B_{\tilde{\pi},\psi}$, we get that $\gamma(\pi, \wedge^2, \psi) = \gamma(\tilde{\pi}, \wedge^2, \psi^{-1})$, and therefore $|\gamma(\pi, \wedge^2, \psi)| = 1$.

### 2.3.1. The even case.

The proof of the functional equation in the even case relies on the following lemma.

**Lemma 2.21.** Let $(\pi, V_\pi)$ be an irreducible cuspidal representation of $GL_{2m}(\mathbb{F})$. Then

$$\dim_{\mathbb{C}} \text{Hom}_{S_{2m} \cap P_{2m}}(\pi, \Psi) \leq 1.$$ 

Here $P_{2m} \leq GL_{2m}(\mathbb{F})$ is the mirabolic subgroup.

**Proof.** We define a homomorphism

$$\Lambda : \text{Hom}_{S_{2m} \cap P_{2m}}(\pi, \Psi) \to \text{Hom}_{M_{m,m} \cap P_{2m}}(\pi, 1)$$

by

$$\Lambda(L)(v) = \frac{1}{|GL_m(\mathbb{F})|} \sum_{g \in GL_m(\mathbb{F})} L\left(\pi\left(\begin{pmatrix} g & \hfill 0 \\ \hfill 0 & I_m \end{pmatrix} \right) v\right),$$

where $M_{m,m} \leq GL_{2m}(\mathbb{F})$ is the Levi subgroup corresponding to the partition $(m,m)$. We claim that $\Lambda$ is injective: suppose that $L \neq 0$ and that $v \in V_\pi$ satisfies $L(v) \neq 0$. We define for a function $\Phi : M_{m}(\mathbb{F}) \to \mathbb{C}$, a vector $v_{\Phi} \in V_{\pi}$ by

$$v_{\Phi} = \frac{1}{\sqrt{|M_{m}(\mathbb{F})|}} \sum_{X \in M_{m}(\mathbb{F})} \Phi(X) \pi\left(\begin{pmatrix} I_m \\ X \end{pmatrix}\right) v.$$ 

Then

$$\Lambda(L)v_{\Phi} = \frac{1}{|GL_m(\mathbb{F})|} \sum_{g \in GL_m(\mathbb{F})} F_{\psi} \Phi(g) L\left(\pi\left(\begin{pmatrix} g & \hfill 0 \\ \hfill 0 & I_m \end{pmatrix} \right) v\right).$$

Choosing $\Phi$ such that $F_{\psi} \Phi = \delta_{I_m}$, we get that $\Lambda(L)v_{\Phi} = \frac{1}{|GL_m(\mathbb{F})|} v \neq 0$, and therefore $\Lambda(L) \neq 0$. Thus, $\Lambda$ is injective. The lemma then follows from the below multiplicity one theorem.

q.e.d.
Theorem 2.22. Let $(\pi, V_\pi)$ be an irreducible cuspidal representation of $GL_{2m}(F)$. Then
\[ \dim \operatorname{Hom}_{M_{m,m} \cap P_{2m}}(\pi, 1) \leq 1. \]
Here $M_{m,m} \leq GL_{2m}(F)$ is the Levi subgroup corresponding to the partition $(m, m)$.

The proof of Theorem 2.22 will be given in the appendix, as it detours from the main line of the paper.

Proof of Theorem 2.18. $n = 2m$. The idea is to show that $\dim \operatorname{Hom}_{S_{2m}}(\pi \otimes S(F^m), \Psi) \leq 1$, and since $J_{\pi,\psi}$ and $\check{J}_{\pi,\psi}$ define non-zero elements of $\operatorname{Hom}_{S_{2m}}(\pi \otimes S(F^m), \Psi)$, it follows that such a constant exists.

We first claim that the restriction map
\[ \operatorname{Hom}_{S_{2m}}(\pi \otimes S(F^m), \Psi) \to \operatorname{Hom}_{S_{2m}}(\pi \otimes S(F^m - \{0\}), \Psi) \]
is injective, where $S(F^m - \{0\})$ is realized as a subspace of $S(F^m)$ by the set of elements of $S(F^m)$, vanishing at zero.

Suppose that this restriction map is not injective. Then there exists $0 \neq B : V_\pi \times S(F^m) \to C$ satisfying
\[ B(\pi(s)v, \rho(s)\phi) = \Psi(s)B(v, \phi) \]
for every $s \in S_{2m}$, such that $b \in \overline{V_\pi}$ defined by $b(v) = B(v, \delta_0)$ is not the zero functional. Let $(\cdot, \cdot)$ be an inner product on $V_\pi$ with respect to which $\pi$ is unitary, and let $0 \neq v_0 \in V_\pi$, such that $b(v) = (v, v_0)$. Then since $\rho(s)\delta_0 = \delta_0$ for every $s \in S_{2m}$, it follows from the equivariance property of $B$ that
\[ b(\pi(s)v) = B(\pi(s)v, \delta_0) = B(\pi(s)v, \rho(s)\delta_0) = \Psi(s)B(v, \delta_0) = \Psi(s)b(v). \]
Thus, for any $s \in S_{2m}$ and $v \in V_\pi$, we have
\[ (v, \pi(s)v_0) = (\pi(s^{-1})v, v_0) = b(\pi(s^{-1})v) = \Psi(s^{-1})b(v) = \overline{\Psi(s)}(v, v_0) = (v, \Psi(s)v_0). \]
This shows that $v_0$ is a Shalika vector, which is a contradiction.

Next we write a sequence of isomorphisms from which it follows that
\[ \dim \operatorname{Hom}_{S_{2m}}(\pi \otimes S(F^m - \{0\}), \Psi) \leq 1. \]
Since $(S_{2m} \cap P_{2m}) \setminus S_{2m} \cong F^m - \{0\}$ by the map $(g \frac{x}{y}) \mapsto \varepsilon g$, we have that $S(F^m - \{0\}) \cong \operatorname{Ind}_{S_{2m} \cap P_{2m}}(1)$ and we have the following isomorphisms:
\[ \operatorname{Hom}_{S_{2m}}(\pi \otimes \operatorname{Ind}_{S_{2m} \cap P_{2m}}(1), \Psi) \cong \operatorname{Hom}_{S_{2m}}(\Psi^{-1} \otimes \pi, \operatorname{Ind}_{S_{2m} \cap P_{2m}}(1)) \]
\[ \cong \operatorname{Hom}_{S_{2m}}(\Psi^{-1} \otimes \pi, \operatorname{Ind}_{S_{2m} \cap P_{2m}}(1)). \]

Thanks to Frobenius reciprocity, we get that
\[ \operatorname{Hom}_{S_{2m}}(\Psi^{-1} \otimes \pi, \operatorname{Ind}_{S_{2m} \cap P_{2m}}(1)) \cong \operatorname{Hom}_{S_{2m} \cap P_{2m}}(\Psi^{-1} \otimes \pi, 1) \cong \operatorname{Hom}_{S_{2m} \cap P_{2m}}(\pi, \Psi), \]
and by Lemma [2.22] the last space has dimension \( \leq 1 \).

**Remark 2.23.** As seen in the proof, the proof fails if \( \pi \) admits a Shalika vector. In this case, a modified functional equation is valid. This is discussed in Theorem 3.15.

### 2.3.2. The odd case.

Similarly to the even case, the proof relies on the following lemma:

**Lemma 2.24.** Let \((\pi, V_\pi)\) be an irreducible cuspidal representation of \( GL_{2m+1}(F) \). Then
\[
\dim \mathbb{C} \text{Hom}_{S_{2m+1} \cap P_{2m+1}}(\pi, \Psi) \leq 1,
\]
where \( P_{2m+1} \) is the mirabolic subgroup.

**Proof.** Let \( L_{2m+1} \) be the following maximal (non-standard) Levi subgroup corresponding to the partition \((m + 1, m)\):
\[
L_{2m+1} = \left\{ \begin{pmatrix} g_1 & u \\ g_2 & v \\ \lambda \\ \end{pmatrix} \mid u \in M_{m \times 1}(F), v \in M_{1 \times m}(F), \lambda \in F, g_1, g_2 \in GL_m(F) \right\} \cap GL_{2m+1}(F).
\]

We define a homomorphism
\[
\Lambda : \text{Hom}_{S_{2m+1} \cap P_{2m+1}}(\pi, \Psi) \to \text{Hom}_{L_{2m+1} \cap P_{2m+1}}(\pi, 1),
\]
by
\[
\Lambda(L)(v) = \frac{1}{|GL_m(F)|} \sum_{g \in GL_m(F)} L\left(\pi \left( g \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \right) v \right).
\]

We claim that \( \Lambda \) is injective. Let \( L \neq 0 \) and let \( v \in V_\pi \) such that \( L(v) \neq 0 \). For a function \( \Phi : M_m(F) \to \mathbb{C} \), we define
\[
v_\Phi = \frac{1}{\sqrt{|M_m(F)|}} \sum_{X \in M_m(F)} \Phi(X) \pi \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} 1 v.
\]

Then
\[
\Lambda(L)(v_\Phi) = \frac{1}{|GL_m(F)|} \sum_{g \in GL_m(F)} F_\psi \Phi(g) L\left(\pi \left( g \begin{pmatrix} I_m & X \\ & I_m \end{pmatrix} \right) v \right).
\]

As in the even case, choosing \( \Phi \) with \( F_\psi \Phi = \delta_{I_m} \), we get that \( \Lambda(L)(v) \neq 0 \), thus \( \Lambda \) is injective. The lemma now follows from the following multiplicity one theorem. q.e.d.

**Theorem 2.25.** Let \((\pi, V_\pi)\) be an irreducible cuspidal representation of \( GL_{2m+1}(F) \). Then
\[
\dim \mathbb{C} \text{Hom}_{L_{2m+1} \cap P_{2m+1}}(\pi, 1) \leq 1.
\]

As Theorem 2.22, the proof of Theorem 2.25 will be given in the appendix, as it detours from the main line of the paper.

**Proof of Theorem 2.18.** \( n = 2m + 1 \). By Proposition 2.29 \( J_{\pi, \psi}, \tilde{J}_{\pi, \psi} \) define non-zero elements of \( \text{Hom}_{S_{2m+1}}(\pi \otimes S(F^m), 1) \). Therefore, to conclude the theorem, it suffices to prove
\[
\dim \text{Hom}_{S_{2m+1}}(\pi \otimes S(F^m), 1) \leq 1.
\]
By Proposition \ref{2.16}, the action of $S_{2m+1}$ on $\mathcal{S}(\mathbb{R}^m)$ is equivalent to the representation $\text{Ind}_{S_{2m+1} \cap P_{2m+1}}^{S_{2m+1}} (\Psi^{-1})$.

Using the same steps as in the even case, we have

$$\text{Hom}_{S_{2m+1}} \left( \pi \otimes \text{Ind}_{S_{2m+1} \cap P_{2m+1}}^{S_{2m+1}} (\Psi^{-1}), 1 \right) \cong \text{Hom}_{S_{2m+1}} \left( \pi, \text{Ind}_{S_{2m+1} \cap P_{2m+1}}^{S_{2m+1}} (\Psi^{-1}) \right) \cong \text{Hom}_{S_{2m+1}} \left( \pi, \text{Ind}_{S_{2m+1} \cap P_{2m+1}}^{S_{2m+1}} (\widetilde{\psi}^{-1}) \right).$$

By Frobenius reciprocity, we have that the last space is isomorphic to $\text{Hom}_{S_{2m+1} \cap P_{2m+1}} (\pi, \Psi)$, and therefore by Lemma \ref{2.24} the dimension of this space is at most 1.

\begin{flushright}
q.e.d.
\end{flushright}

2.3.3. Equivalent conditions for admitting a Shalika vector. Let $(\pi, V_\pi)$ be an irreducible cuspidal representation of $\text{GL}_{2m}(\mathbb{F})$. In this section we state equivalent conditions for $\pi$ to admit a Shalika vector.

**Proposition 2.26.** $\pi$ admits a Shalika vector if and only if there exists $W \in \mathcal{W}(\pi, \psi)$ such that $J_{\pi, \psi} (W, 1) \neq 0$, where 1 denotes the constant function valued 1 on $\mathbb{R}^m$.

**Proof.** Suppose that there exists $W \in \mathcal{W}(\pi, \psi)$ such that $J_{\pi, \psi} (W, 1) \neq 0$. Define $W_0 \in \mathcal{W}(\pi, \psi)$ by

$$W_0 (h) = \frac{1}{|G : N|} \frac{1}{|M : B|} \sum_{g \in N \backslash G} \sum_{X \in B \backslash M} W \left( h \left( \begin{array}{cc} I_m & X \\
 & g \end{array} \right) \right) \psi (-\text{tr}X).$$

Then $W_0 \neq 0$ as $W_0 (\sigma_{2m}) = J_{\pi, \psi} (W, 1)$. By changing variables, we can show that

$$W_0 (hs) = \Psi (s) W_0 (h),$$

for every $s \in S_{2m}$, thus $W_0$ is a Shalika vector.

For the other direction, assume $\pi$ admits a non-zero Shalika vector $W_0 \in \mathcal{W}(\pi, \psi)$. Choose an inner product $(\cdot, \cdot)$ on $\mathcal{W}(\pi, \psi)$, with respect to $\pi$ is unitary. Then $W_0$ defines a non-zero element $T_{W_0} \in \text{Hom}_{S_{2m}} (\pi, \Psi)$ by $T_{W_0} (W') = (W', W_0)$. We have that

$$\text{Hom}_{S_{2m}} (\pi, \Psi) \subseteq \text{Hom}_{S_{2m} \cap P_{2m}} (\pi, \Psi).$$

By Lemma \ref{2.24}, we have in this case that $0 \neq \text{Hom}_{S_{2m}} (\pi, \Psi) = \text{Hom}_{S_{2m} \cap P_{2m}} (\pi, \Psi)$. We define $W \in \mathcal{W}(\pi, \psi)$ by

$$W (h) = \sum_{g \in N \backslash P} \sum_{X \in B \backslash M} B_{\pi, \psi} \left( h \left( \begin{array}{cc} I_m & X \\
 & g \end{array} \right) \right) \sigma_{2m}^{-1} \psi (-\text{tr}X),$$

where $P = P_{m}(\mathbb{F}) = \{ g \in \text{GL}_m(\mathbb{F}) \mid \varepsilon_m g = \varepsilon_m \}$ is the mirabolic subgroup. We can show that for $s \in S_{2m} \cap P_{2m}$,

$$W (hs) = \Psi (s) W (h).$$

Therefore, $W$ defines an element $T_W \in \text{Hom}_{S_{2m} \cap P_{2m}} (\pi, \Psi)$ by $T_W (W') = (W', W)$. Thus $T_W \in \text{Hom}_{S_{2m}} (\pi, \Psi)$. This implies that $W$ is a Shalika vector. We can also show that $J_{\pi, \psi} (W, 1) = W (\sigma_{2m})$. By Proposition \ref{2.7} we have that $W (\sigma_{2m}) = 1$, hence $J_{\pi, \psi} (W, 1) = W (\sigma_{2m}) \neq 0$.

\begin{flushright}
q.e.d.
\end{flushright}
Corollary 2.27. If \( \pi \) admits a Shalika vector, then the functional equation in Theorem 2.18 does not hold.

Proof. Let \( W \in \mathcal{W}(\pi, \psi) \), such that \( J_{\pi,\psi}(W, 1) = 1 \). Then since \( F_\psi 1 = q^{\frac{m}{2}} \cdot \delta_0 \), we have that \( \tilde{J}_{\pi,\psi}(W, 1) = 0 \), and therefore no such non-zero constant exists. q.e.d.

Suppose that \( \pi \) is associated with the regular character \( \theta : \mathbb{F}^*_{2m} \to \mathbb{C}^* \). We now recall the work of Prasad \([Pra00]\) in order to classify when \( \pi \) admits a Shalika vector, in terms of \( \theta \).

Let \( N_{m,m} \) be the unipotent radical of \( GL_{2m}(\mathbb{F}) \) corresponding to the partition \((m, m)\). Denote by \( V_{\pi,N_{m,m},\Psi} \) the twisted Jacquet submodule of \( N_{m,m} \) with respect to the character \( \Psi \):

\[
V_{\pi,N_{m,m},\Psi} = \left\{ v \in V_{\pi} \mid \forall X \in M_m(\mathbb{F}), \pi(X) v = \psi(\text{tr}X) v \right\}.
\]

Then \( V_{\pi,N_{m,m},\Psi} \) is a subspace invariant under the action \( \pi_{N_{m,m},\Psi} \) of \( GL_m(\mathbb{F}) \), defined by \( \pi_{N_{m,m},\Psi}(g) v = \pi(g \cdot g) v \), where \( g \in GL_m(\mathbb{F}) \), \( v \in V_{\pi,N_{m,m},\Psi} \). Prasad shows

Theorem 2.28 (\([Pra00]\) Theorem 1). \( \pi_{N_{m,m},\Psi} \cong \text{Ind}_{\mathbb{F}^*_{m}}^{GL_m(\mathbb{F})} (\theta \mid_{\mathbb{F}^*_{m}}) \).

From the definition of a Shalika vector and the definition of the representation \( \pi_{N_{m,m},\Psi} \), we see that \( \pi \) admits a Shalika vector if and only if \( \text{Hom}_{GL_m(\mathbb{F})}(1, \pi_{N_{m,m},\Psi}) \neq 0 \). From Frobenius reciprocity we have that \( \text{Hom}_{GL_m(\mathbb{F})}(1, \pi_{N_{m,m},\Psi}) \cong \text{Hom}_{\mathbb{F}^*_{m}}(1, \theta \mid_{\mathbb{F}^*_{m}}) \), and the last space is non-zero if and only if \( \theta \mid_{\mathbb{F}^*_{m}} = 1 \), and then it is one dimensional. Therefore we get the following

Corollary 2.29. \( \pi \) admits a Shalika vector if and only if \( \theta \mid_{\mathbb{F}^*_{m}} = 1 \), and in this case, the space of Shalika vectors is one dimensional.

We conclude this section in a theorem.

Theorem 2.30 (Equivalent conditions for admitting a Shalika vector). Suppose \( \pi \) that is an irreducible cuspidal representation of \( GL_{2m}(\mathbb{F}) \) associated with the regular character

\[
\theta : \mathbb{F}^*_{2m} \to \mathbb{C}^*.
\]

The following are equivalent.

1) \( \pi \) admits a Shalika vector.
2) There exists \( W \in \mathcal{W}(\pi, \psi) \), such that \( J_{\pi,\psi}(W, 1) \neq 0 \).
3) \( \theta \mid_{\mathbb{F}^*_{m}} = 1 \).

Moreover, in these cases, the space of Shalika vectors is one-dimensional.
2.4. An expression for the exterior square gamma factor. Let $\pi$ be an irreducible cuspidal representation of $\text{GL}_n(F)$. If $n$ is even, suppose that $\pi$ does not admit a Shalika vector.

In this section we express the exterior square gamma factor of $\pi$ in terms of the Bessel function. One may use Proposition 2.7 and Proposition 2.14 in order to get such an expression, but the Jacquet-Shalika integral runs over too many elements, some of which are not in the support of the Bessel function. We find a more accurate expression which involves only elements of the form of Theorem 2.1.

Our main results are the following theorems:

**Theorem 2.31.** If $n = 2m$, the exterior square gamma factor is given by the formula

$$
\gamma (\pi, \wedge^2, \psi) = q^{-\frac{m^2}{2} + 2\binom{m}{2}} \sum_{m_1, \ldots, m_r \geq 1 \atop m_1 + \cdots + m_r = m} \sum_{\lambda_1, \ldots, \lambda_r \in F^*} q^{-\sum_{i=1}^r 2\binom{m_i}{2}} \cdot B_{\pi, \psi} \left( \text{antidiag} (\lambda_1 I_{2m_1}, \ldots, \lambda_r I_{2m_r})^{-1} \right) \cdot \psi (\lambda_r \cdot \delta_{m_r, 1}),
$$

where $\delta_{m_r, 1} = \begin{cases} 1 & \text{if } m_r = 1, \\ 0 & \text{if } m_r \neq 1. \end{cases}$

**Theorem 2.32.** If $n = 2m + 1$, the exterior square gamma factor is given by the formula

$$
\gamma (\pi, \wedge^2, \psi) = q^{\frac{m^2}{2} + 2\binom{m}{2}} \sum_{m_1, \ldots, m_r \geq 1 \atop m_1 + \cdots + m_r = m} \sum_{\lambda_1, \ldots, \lambda_r \in F^*} q^{-\sum_{i=1}^r 2\binom{m_i}{2}} \cdot B_{\pi, \psi} \left( \text{antidiag} (\lambda_1 I_{2m_1}, \ldots, \lambda_r I_{2m_r}, 1I_1)^{-1} \right).
$$

The following lemma indicates which representatives for $g$ and $X$ in the Jacquet-Shalika integral contribute to the sum. It is a key to the proofs of Theorem 2.31 and Theorem 2.32.

**Lemma 2.33.** Let $g \in \text{GL}_m(F)$ and $X \in \mathcal{N}_m^- (F)$ a lower triangular nilpotent matrix (i.e. a lower triangular matrix with zeros on its diagonal). Suppose that $g = wdu$, where $u \in N_m(F)$, $w$ is a permutation matrix and $d$ is a diagonal matrix. Denote by $\tau$ the permutation defined by the columns of $w$. Write $X = (x_{ij})$. Suppose that

$$
\sigma_{2m} \left( \begin{pmatrix} I_m & X \\ I_m \end{pmatrix} \right) \left( \begin{pmatrix} g \\ g \end{pmatrix} \right) \sigma_{2m}^{-1} \in N_{2m} \text{antidiag} (\lambda_1 I_{n_1}, \ldots, \lambda_r I_{n_r}) N_{2m}, \quad (2.1)
$$

where $\lambda_1, \ldots, \lambda_r \in F^*$, and $n_1 + \cdots + n_r = 2m$. Then

1) $wd = \text{antidiag} (\lambda_1 I_{m_1}, \ldots, \lambda_r I_{m_r})$, where $n_i = 2m_i$, for every $1 \leq i \leq r$ (and therefore $m_1 + \cdots + m_r = m$).
2) $x_{ij} = 0$ for every $(i, j)$ satisfying $j < i$ and $\tau^{-1} (j) < \tau^{-1} (i)$.

Furthermore, in this case, $\sigma_{2m} \left( \begin{pmatrix} I_m & X \\ I_m \end{pmatrix} \right) \left( \begin{pmatrix} g \\ g \end{pmatrix} \right) \sigma_{2m}^{-1} = \text{antidiag} (\lambda_1 I_{2m_1}, \ldots, \lambda_r I_{2m_r}) \cdot v$, where $v \in N_{2m}(F)$ is an upper triangular unipotent matrix with zeros right above its diagonal.
Proof. Since $\sigma_{2m}(u_u)\sigma_{2m}^{-1} \in N_{2m}$, we have by eq. (2.1) that

$$\sigma_{2m}\begin{pmatrix} I_m & X \\ I_m & I_m \end{pmatrix}\begin{pmatrix} wd \\ wd \end{pmatrix}\sigma_{2m}^{-1} \in N_{2m} \text{ antidiag } (\lambda_1 I_{n_1}, \ldots, \lambda_r I_{n_r}) N_{2m}.$$ 

Denote $w' = \sigma_{2m}(w)\sigma_{2m}^{-1}$. Then $w'$ is a column permutation matrix of the permutation $\rho'$, where $\rho'(2j) = 2\tau(j)$, and $\rho'(2j-1) = 2\tau(j) - 1$, for every $1 \leq j \leq m$, therefore

$$w' = \begin{pmatrix} e_{2\tau(1)-1} & e_{2\tau(1)} & \cdots & e_{2\tau(m)-1} & e_{2\tau(m)} \end{pmatrix},$$

where $e_i$ is the $i$-th standard column vector in $\mathbb{F}^{2m}$.

Denote $d = \text{diag}(a_1, \ldots, a_m)$ and $d' = \sigma_{2m}(d_d)\sigma_{2m}^{-1}$. Then $d' = \text{diag}(a_1, a_2, a_2, \ldots, a_m, a_m)$.

Denote $U_X = \sigma_{2m}(I_m X I_m)\sigma_{2m}^{-1}$. Since $X = (x_{ij})$ is a lower triangular nilpotent matrix, $U_X$ is a lower triangular unipotent matrix, and its columns are given by $\text{column}_{2j-1}(U_X) = e_{2j-1}$ and

$$\text{column}_{2j}(U_X) = e_{2j} + \sum_{i=j+1}^{m} x_{ij} e_{2i-1}.$$ 

Denote $Z = \sigma_{2m}(I_m X I_m)(wd \ \ wd)\sigma_{2m}^{-1} = U_X w'd'$. We get that $\text{column}_{2j-1}(Z) = a_j e_{2\tau(j)-1}$ and

$$\text{column}_{2j}(Z) = a_j e_{2\tau(j)} + \sum_{i=\tau(j)+1}^{m} a_j x_{ir(j)} e_{2i-1}.$$ 

We claim that $Z$ has the form

$$Z = \begin{pmatrix} 0 & 0 & \cdots & 0 & \lambda_1 I_{2m_1} \\ 0 & 0 & \cdots & \lambda_2 I_{2m_2} & * \\ 0 & 0 & \cdots & * & * \\ 0 & \lambda_{r-1} I_{2m_{r-1}} & \cdots & * & * \\ \lambda_r I_{2m_r} & * & \cdots & * & * \end{pmatrix},$$

where $2m_i = n_i$, for every $1 \leq i \leq r$.

The proof of the claim is by applying row and column reduction carefully, in order to obtain the diagonal and permutation matrices that are involved in the Bruhat decomposition of $Z$. Every $h \in \text{GL}_n(\mathbb{F})$ has a Brahat decomposition $h = u_1 w_h d_h u_2$, where $u_1, u_2 \in N_n(\mathbb{F})$, $w_h$ is a permutation matrix and $d_h$ is a diagonal matrix. Such $w_h$ and $d_h$ are unique. Throughout the text, we refer to the uniqueness of $w_h$ and $d_h$ by the uniqueness of the Bruhat decomposition. We will first show that $Z$ is of the form

$$Z = \begin{pmatrix} 0 \\ \lambda_r I_{2m_r} \\ * \end{pmatrix}.$$ 

By the above description of the columns of $Z$, we have that the first column of $Z$ is $a_1 \cdot e_{2\tau(1)-1}$. Denote $2m - 2\tau(1) + 2 = 2m_r$. We show by induction that for every $1 \leq l \leq 2m_r$, $\text{column}_{l}(Z) = \lambda_r e_{2m-r+2m-r+l}$. For $l = 1$, we have $\text{column}_1(Z) = a_1 e_{2m-2m_r+1}$. By assumption $Z \in N_{2m} \text{ antidiag } (\lambda_1 I_{n_1}, \ldots, \lambda_r I_{n_r}) N_{2m}$ and therefore by the uniqueness of the
Bruhat decomposition of $Z$, we must have $a_1 = \lambda_r$. Assume that the claim is true for all columns before $l$, that is to say, column$_i$($Z$) = $\lambda_re_{2m-2m_r+i}$ for $1 \leq i < l$. Since column$_{l-1}$($Z$) = $\lambda_re_{2m-2m_r+l-1}$, we expect column$_l$($Z$) = $\lambda_re_{2m-2m_r+l}$.

If $l = 2j - 1$ is odd, then since column$_{2j-1}$($Z$) = $a_je_{2r(j)-1}$, in order for $Z$ to have the required Bruhat decomposition $Z = u_1 \text{antidiag} (\lambda_1I_{n_1}, \ldots, \lambda_rI_{n_r}) u_2$ for some $u_1, u_2 \in N_{2m}(\mathbb{F})$, we must have that $a_j = \lambda_r$ and $2r(j)-1 = 2m - 2m_r + 2(j-1)$, i.e. $\tau(j) = m - m_r + j$.

If $l = 2j$ is even, then we have that

$$\text{column}_2j(Z) = \lambda_re_{2m-2m_r+2j} + \sum_{i=m-m_r+j+1}^{m} \lambda_rx_{i,m-m_r+j}e_{2i-1}.$$ 

If any of the $x_{i,m-m_r+j}$ is non-zero, then by applying row reduction (by multiplying from the left by an upper triangular unipotent matrix that annihilates all the elements above the lowest element of column$_2j(Z)$), we get that the Bruhat decomposition of $Z$ is not of the required form. Therefore we get that column$_2j(Z) = \lambda_re_{2m-2m_r+2j}$, as required.

We have shown that for every $1 \leq l \leq 2m_r$, column$_l$($Z$) = $\lambda_re_{2m-2m_r+l}$. By the uniqueness of the Bruhat decomposition of $Z$, we must have $n_r = 2m_r$.

We now have that

$$Z = \begin{pmatrix} 0 & Z' \\ \lambda_rI_{2m_r} & A \end{pmatrix},$$

where $Z' \in \text{GL}_{2m-2m_r}(\mathbb{F})$ and $A \in M_{2m_r \times (2m-2m_r)}(\mathbb{F})$. We have that

$$Z \cdot \begin{pmatrix} I_{2m_r} & -\lambda_r^{-1}A \\ 0 & I_{2m-2m_r} \end{pmatrix} = \begin{pmatrix} 0 & Z' \\ \lambda_rI_{2m_r} & 0 \end{pmatrix}.$$ 

By uniqueness of the Bruhat decomposition of $Z$, we must have that $Z' \in N_{2m-2m_r} \text{antidiag} (\lambda_1I_{n_1}, \ldots, \lambda_{n_r-1}I_{n_r-1}) N_{2m-2m_r}$. Since $Z'$ inherits an analogous column description from $Z$ but is of smaller size than that of $Z$, we get by induction on the size of the matrix, i.e. by repeating the above steps applied to $Z$, that

$$Z' = \begin{pmatrix} 0 & 0 & \ldots & 0 & \lambda_1I_{2m_1} \\ 0 & 0 & \ldots & \lambda_2I_{2m_2} & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \lambda_{r-2}I_{2m_{r-2}} & \ldots & * & * \\ \lambda_{r-1}I_{2m_{r-1}} & \ldots & \ldots & * & * \end{pmatrix},$$

and that $n_i = 2m_i$ for every $1 \leq i \leq r - 1$, and therefore $Z$ has the desired form.

By the above claim, we get that the Bruhat decomposition of $Z$ is $Z = w_Zd_Zu_Z$, where $w_Zd_Z = \text{antidiag} (\lambda_1I_{2m_1}, \ldots, \lambda_rI_{2m_r})$ ($w_Z$ is a permutation matrix, $d_Z$ is a diagonal matrix, and $u_Z$ is a unipotent matrix). We conclude from $\sigma_{2m}(wd_{\varphi})\sigma_{2m}^{-1} = w_Zd_Z\mathbb{T}$ that $wd = \text{antidiag} (\lambda_1I_{m_1}, \ldots, \lambda_rI_{m_r})$, which is the first part of the lemma. We also conclude from the induction process of the above claim that $x_{\tau(i)\tau(j)} = 0$ for every $i > j$ with
This finishes the second part of the lemma.

Finally, we write $Z = w'd' (d'^{-1} w'^{-1} U_X w'd')$. We claim that $w'^{-1} U_X w'$ is an upper triangular matrix with zeros right above its diagonal: the only non-zero non-diagonal components of $U_X$ are located in the positions of the form $(2i - 1, 2j)$ with values $x_{ij}$ for $j < i$, and these move in the conjugation $w'^{-1} U_X w'$ to $(\tau'^{-1} (2i - 1), \tau'^{-1} (2j)) = (2\tau^{-1} (i) - 1, 2\tau^{-1} (j))$. If $2\tau^{-1} (j) < 2\tau^{-1} (i) - 1$, then we get that $x_{ij} = 0$, and therefore $d'^{-1} w'^{-1} U_X w'd'$ is an upper unipotent matrix with zeros right above its diagonal. Since

$$\sigma_{2m} \left( \begin{pmatrix} I_m & X \\ I_m \end{pmatrix} \right) \left( \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \right)^{-1} \sigma_{2m}^{-1} = Z \sigma_{2m} \left( \begin{pmatrix} u \\ u \end{pmatrix} \right)^{-1} \sigma_{2m}^{-1},$$

and since $\sigma_{2m} \left( \begin{pmatrix} u \\ u \end{pmatrix} \right) \sigma_{2m}^{-1}$ is an upper triangular unipotent matrix with zeros right above its diagonal, it follows that $\sigma_{2m} \left( \begin{pmatrix} I_m & X \\ I_m \end{pmatrix} \right) \left( \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \right)^{-1} = \text{antidiag}(\lambda_1 I_{2m_1}, \ldots, \lambda_r I_{2m_r}) \cdot v$, for some upper triangular unipotent matrix $v \in N_{2m}(\mathbb{F})$ with zeros right above its diagonal.

q.e.d.

We need one more lemma, regarding the number of elements involved in the Jacquet-Shalika integral, such that $g$ has a given Bruhat decomposition. As we’ll see in the proofs of Theorem 2.31 and Theorem 2.32 in order for a coset $g \in N\backslash G$ to contribute to the Jacquet-Shalika integral, we must have $g \in N\text{antidiag}(\lambda_1 I_{m_1}, \ldots, \lambda_r I_{m_r}) N$. Let $wd = \text{antidiag}(\lambda_1 I_{m_1}, \ldots, \lambda_r I_{m_r})$. Given $g \in N wd N$ and $X \in N^-$ as in Lemma 2.33 we’ll see in the following proofs that the summand of the Jacquet-Shalika integral on a special choice of functions, depends only on $wd$. To evaluate the Jacquet-Shalika integral, we should count the number of cosets in the set $\{N wd u \mid u \in N\} \subseteq N\backslash G$ and the number of options for a matrix $X \in N^-$ satisfying the condition of Lemma 2.33.

**Lemma 2.34.** Let $g = wd = \text{antidiag}(\lambda_1 I_{m_1}, \ldots, \lambda_r I_{m_r})$, where $w = \text{antidiag}(I_{m_1}, \ldots, I_{m_r})$ is a permutation matrix, $d = \text{diag}(\lambda_r I_{m_r}, \ldots, \lambda_1 I_{m_1})$ is a diagonal matrix, $m_1 + \cdots + m_r = m$, $\lambda_1, \ldots, \lambda_r \in \mathbb{F}^*$. Then $|N g|$ is of size $q^{\binom{m}{2}} - \sum_{i=1}^r \binom{m_i}{2}$.

1) Consider the right action of the upper triangular unipotent subgroup $N = N_m$ on $N\backslash G$, where $G = \text{GL}_m(\mathbb{F})$. Then the orbit of $N g$ is of size $q^{\binom{m}{2}} - \sum_{i=1}^r \binom{m_i}{2}$.

2) Let $\tau$ be the permutation corresponding to columns of $w$. Then the set

$$\{X \in N^-_m \mid x_{ij} = 0, \forall 1 \leq j < i \leq m, \text{ s.t. } \tau^{-1} (j) < \tau^{-1} (i)\}$$

is of cardinality $q^{\binom{m}{2}} - \sum_{i=1}^r \binom{m_i}{2}$.

**Proof.** By the orbit-stabilizer theorem, we have that the orbit of $N wd$ is of size $[N : \text{stab}_N(N wd)]$. We have that

$$\text{stab}_N(N wd) = N \cap w^{-1} N w = \{(u_{ij}) \in N \mid u_{ij} = 0, \forall i < j \text{ s.t. } \tau^{-1} (i) > \tau^{-1} (j)\}.$$

Therefore

$$\log_q |\text{stab}_N(N wd)| = \left| \left\{(i, j) \mid i < j \text{ and } \tau^{-1} (i) < \tau^{-1} (j) \right\} \right| = \sum_{i=1}^r \binom{m_i}{2}.$$
and the first part is proved. The second part follows from the fact that
\[ \left| \{(i, j) \mid j < i \text{ and } r^{-1}(j) < r^{-1}(i)\} \right| = \sum_{i=1}^{r} \binom{m_i}{2}. \]

q.e.d.

We are now ready to prove Theorem 2.31 and Theorem 2.32.

Proof of Theorem 2.31. We have that \( F_{\psi} \delta_{-\varepsilon_1}(x) = q^{-\frac{m}{2}} \psi(-x, \varepsilon_1) \) and therefore by the Fourier inversion formula, if \( \phi(x) = \psi(-x_1) \), then \( F_{\psi} \phi = q^{\frac{m}{2}} \delta_{\varepsilon_1} \).

We compute \( \tilde{J}_{\pi, \psi}(W, \phi) \), for \( W = [G : N] [M : B] \pi (\sigma_{2m}^{-1}) B_{\pi, \psi} \) and \( \phi(x) = \psi(-x_1) \). We have from Proposition 2.6 that
\[ q^{-\frac{m}{2}} \tilde{J}_{\pi, \psi}(W, \phi) = \sum_{X \in B \setminus M} \sum_{g \in N \setminus G} B_{\pi, \psi} \left( \sigma_{2m} \left( I_m X I_m \right) \left( g \ g \right) \sigma_{2m}^{-1} \right) \psi(-\text{tr} X) \delta_{\varepsilon_1} (\varepsilon_1 g'). \]

Notice that \( \varepsilon_1 g' = \varepsilon_1 \) if and only if \( g \) has \( \varepsilon_1 = \varepsilon_1 \) as its first column. It follows now (similarly to the proof of Proposition 2.7) that \( \tilde{J}_{\pi, \psi}(W, \phi) = q^{\frac{m}{2}} \), and therefore
\[ q^{\frac{m}{2}} \gamma(\pi, \Lambda^2, \psi)^{-1} = J_{\pi, \psi}(W, \phi) = \sum_{g \in N \setminus G} \sum_{X \in B \setminus M} B_{\pi, \psi} \left( \sigma_{2m} \left( I_m X I_m \right) \left( g \ g \right) \sigma_{2m}^{-1} \right) \psi(-\text{tr} X) \psi(-g_{m1}). \]

(2.2)

Since the Jacquet-Shalika integral runs over cosets of the form \( g \in N \setminus G \) and is constant on these), it follows from the Bruhat decomposition that it suffices to consider elements of the form \( g = wd' \), where \( w \) is a permutation matrix, \( d' \) is a diagonal matrix and \( u \) is an upper triangular unipotent matrix. By Lemma 2.33 we only need to consider \( w, d' \) such that \( wd = \text{antidiag}(\lambda_1 I_{m_1}, \ldots, \lambda_r I_{m_r}) \).

We get from Lemma 2.33 that
\[ B_{\pi, \psi} \left( \sigma_{2m} \left( I_m X I_m \right) \left( wdu \ wdu \right) \sigma_{2m}^{-1} \right) = B_{\pi, \psi} (\text{antidiag}(\lambda_1 I_{2m_1}, \ldots, \lambda_r I_{2m_r})). \]

Implicitly, we see that it does not depend on \( X \) and \( u \). By Lemma 2.34 given such \( wd \), we have \( q^{\binom{m}{2} - \frac{m_1}{2} - \cdots - \frac{m_r}{2}} \) options for \( u \in N \), and the same number of options for \( X \in N^- \).

Therefore we get the following formula for the sum
\[ J_{\pi, \psi}(W, \phi) = \sum_{m_1, \ldots, m_r \geq 1, m_1 + \cdots + m_r = m} q^{2(\binom{m}{2} - \frac{m_1}{2} - \cdots - \frac{m_r}{2})} B_{\pi, \psi} (\text{antidiag}(\lambda_1 I_{2m_1}, \ldots, \lambda_r I_{2m_r})) \psi(-\lambda_r \delta_{m_r, 1}). \]

The theorem now follows from eq. (2.2), Proposition 2.3 and Remark 2.20.

q.e.d.
Remark 2.35. Denote
\[ S_0 = \sum_{m_0 > 1, m_1, \ldots, m_r = 1} q^{-2 \sum_{i=0}^{r-1} \binom{m_i}{2}} \cdot B_{\pi, \psi} \left( \text{antidiag} \left( \lambda_0 I_{2m_0}, \lambda_1 I_{2m_1}, \ldots, \lambda_r I_{2m_r} \right) \right) \]

Then for every \( a \in F^* \), \( S_0 = \omega_{\pi}(a) \cdot S_0 \), and therefore if \( \pi \) has a non-trivial central character, then \( S_0 = 0 \).

Also denote
\[ S_1 = \sum_{m_1, \ldots, m_r \geq 1, m_1 + \cdots + m_r = m} q^{-2 \sum_{i=1}^{r-1} \binom{m_i}{2}} \cdot B_{\pi, \psi} \left( \text{antidiag} \left( 1I_2, \lambda_1 I_{2m_1}, \ldots, \lambda_r I_{2m_r} \right) \right). \]

Then by a change of variables, we have that
\[ \gamma(\pi, \Lambda^2, \psi) = q^{-\frac{m}{2} + 2 \binom{m}{2}} \left( S_0 + S_1 \cdot \sum_{a \in F^*} \omega_{\pi}(a^{-1}) \psi(a) \right). \]

We now move to prove the odd case of the theorem:

Proof of Theorem 2.32. Choose \( \phi = \delta_0 \) and \( W = [G : N] \cdot [M : B] \cdot |M_{1 \times m}(F)| \cdot \pi(\sigma_{2m+1}) B_{\pi, \psi} \), as in Proposition 2.14. Then \( J_{\pi, \psi}(W, \phi) = 1 \), and \( F_{\psi, \phi} = q^{-\frac{m}{2}} \). By Proposition 2.13, we have that
\[ \gamma(\pi, \Lambda^2, \psi) = q^{\frac{m}{2}} \sum_{g \in N \backslash G} \sum_{X \in B \backslash M} \sum_{Z \in M_{1 \times m}(F)} B_{\pi, \psi} \left( \left( \begin{array}{c} 1 \\ I_{2m} \\ \end{array} \right) \sigma_{2m+1} \left( \begin{array}{c} I_m \\ X \\ I_m \end{array} \right) \right) \left( \begin{array}{c} g \\ g \\ 1 \end{array} \right) \left( \begin{array}{c} I_m \\ I_m \\ 1 \\ \end{array} \right) \left( \begin{array}{c} I_m \\ I_m \\ \end{array} \right) \sigma_{2m+1}^{-1} \right) \psi(-\text{tr}X). \]

Since \( \sigma_{2m+1} \left( \begin{array}{cc} I_m & -Z \\ I_m & \end{array} \right) \sigma_{2m+1}^{-1} \) is an upper triangular unipotent matrix with zeros right above its diagonal, we can evaluate the summation over \( Z \) (using the right translation \( N_{2m+1} \)-equivariance property of the Bessel function), and we are left with the expression
\[ \gamma(\pi, \Lambda^2, \psi) = q^{\frac{m}{2}} \sum_{g \in N \backslash G} \sum_{X \in B \backslash M} B_{\pi, \psi} \left( \left( \begin{array}{c} 1 \\ I_{2m} \\ \end{array} \right) \sigma_{2m+1} \left( \begin{array}{c} I_m \\ X \\ I_m \end{array} \right) \right) \left( \begin{array}{c} g \\ g \\ 1 \end{array} \right) \sigma_{2m+1}^{-1} \right) \psi(-\text{tr}X). \]

Since \( \sigma_{2m+1} = \sigma_{2m} \), we get that the argument of Bessel function in the summand is \( \left( \begin{array}{c} Y \\ 1 \end{array} \right) \), where \( Y = \sigma_{2m} \left( \begin{array}{cc} I_m & X \\ I_m & \end{array} \right) \sigma_{2m}^{-1} \). Let \( g \in G \), \( X \in M \), and denote \( Y = \sigma_{2m} \left( \begin{array}{cc} I_m & X \\ I_m & \end{array} \right) \sigma_{2m}^{-1} \). In order for \( g \), \( X \) to contribute to the sum, we must have that
\[ \left( \begin{array}{c} Y \\ 1 \end{array} \right) \in N_{2m+1} \text{ antidiag} (\lambda_0 I_{m_0}, \lambda_1 I_{m_1} \ldots \lambda_r I_{m_r}) N_{2m+1}, \]
where \( n_0 + \cdots + n_r = 2m + 1 \) and \( \lambda_0, \ldots, \lambda_r \in \mathbb{F}^* \). Let \( u_1, u_2 \in N_{2m} \), such that \( u_1 Y u_2 = w'd' \), where \( w' \) is a permutation matrix and \( d' \) is a diagonal matrix. Then

\[
\begin{pmatrix} 1 & u_1 \\ Y & 1 \end{pmatrix} \begin{pmatrix} u_2 \\ 1 \end{pmatrix} = \begin{pmatrix} w'd' & 1 \end{pmatrix}.
\]

By the uniqueness of the Bruhat decomposition, we get that \( n_0 = 1, \lambda_0 = 1, \) and

\[
u_1 Y u_2 = \text{antidiag} (\lambda_1 I_{n_1}, \ldots, \lambda_r I_{n_r}), \]

i.e. \( \sigma_{2m} \left( I_{m} X I_{m} \right) \sigma^{-1}_{2m} \in N_{2m} \) \text{antidiag} (\lambda_1 I_{n_1}, \ldots, \lambda_r I_{n_r}) N_{2m} \). As in the even case, since \( g \) runs on cosets of \( N \setminus G \), using the Bruhat decomposition, we may assume its representatives are in the form \( g = w d u \), where \( w \) is a permutation matrix, \( d \) is a diagonal matrix, and \( u \in N_{2m} \). By Lemma \[2.33\] we have that \( w d = \text{antidiag} (\lambda_1 I_{m_1}, \ldots, \lambda_r I_{m_r}) \) and then \( Y = \text{antidiag} (\lambda_1 I_{m_1}, \ldots, \lambda_r I_{m_r}) v \), where \( n_i = 2m_i \), for every \( 1 \leq i \leq r \), and \( v \in N_{2m} \) has zeros right above its diagonal. Since \( \begin{pmatrix} Y^{-1} & 1 \end{pmatrix} \begin{pmatrix} w & 1 \end{pmatrix} = \begin{pmatrix} Y & 1 \end{pmatrix} \), and the matrix \( \begin{pmatrix} w & 1 \end{pmatrix} \) has zeros right above its diagonal, we get that \( \mathcal{B}_{\pi, \psi} (\begin{pmatrix} Y^{-1} & 1 \end{pmatrix}) = \mathcal{B}_{\pi, \psi} (\text{antidiag} (I_1, \lambda_1 I_{2m_1}, \ldots, \lambda_r I_{2m_r})) \). Using Lemma \[2.34\] to count how many such \( u \) and \( X \) correspond to a given choice of \( w d = \text{antidiag} (\lambda_1 I_{m_1}, \ldots, \lambda_r I_{m_r}) \), we get that

\[
\gamma (\pi, \Lambda^2, \psi) = q \frac{m}{2} + 2 \binom{m}{2} \sum_{\substack{m_1, \ldots, m_r \geq 1 \\text{such that } m_1 + \cdots + m_r = m \\text{and } \lambda_1, \ldots, \lambda_r \in \mathbb{F}^*}} q^{-\sum r_i \binom{m_i}{2}} \mathcal{B}_{\pi, \psi} (\text{antidiag} (I_1, \lambda_1 I_{2m_1}, \ldots, \lambda_r I_{2m_r})),
\]

as required. \( \Box \)

### 2.5. Computations for small \( n \)

In this section, we use the correspondence between irreducible cuspidal representations of \( \text{GL}_n (\mathbb{F}) \) and equivalence classes of regular characters of \( \mathbb{F}^n \) in order to find explicit expressions for the exterior gamma factor.

Recall that a character \( \theta : \mathbb{F}^n\rightarrow \mathbb{C}^* \) is called regular if its Galois orbit is of size \( n \), i.e. for every \( 1 \leq k \leq n-1, \theta^k \neq \theta \). We define an equivalence relation on the set of regular characters of \( \mathbb{F}^n \), where \( \theta \sim \theta' \) if they are in the same Galois orbit, i.e. if \( \theta^k = \theta' \) for some \( k \). By \[\text{Gel70} \text{ Section 6]}\], there exists a bijection between equivalence classes of regular characters of \( \mathbb{F}^n \) and irreducible representations of \( \text{GL}_n (\mathbb{F}) \). Furthermore, if \( \pi \) is an irreducible representation that corresponds under this bijection to the equivalence class of the regular character \( \theta \), then there exists a formula that expresses the character \( \chi_\pi (g) = \text{tr} (\pi (g)) \) in terms of \( \theta \). See also \[\text{Nie17} \text{ Section 3.1]}\] for more details.

Suppose that \( \pi \) is an irreducible cuspidal representation of \( \text{GL}_n (\mathbb{F}) \) corresponding under the above bijection to the equivalence class of the regular character \( \theta : \mathbb{F}^n\rightarrow \mathbb{C}^* \). If \( n = 2m \) is even, assume that \( \pi \) does not admit a Shalika vector. By Theorem \[2.30\] this is equivalent to the assumption that \( \theta \mid \mathbb{F}_m \) is non-trivial. By Theorem \[2.2\] the Bessel function \( \mathcal{B}_{\pi, \psi} \) can be expressed in terms of \( \chi_\pi \), the character of the representation \( \pi \), and \( \psi \) only. Therefore, by Section \[2.4\] the exterior square gamma factor \( \gamma (\pi, \Lambda^2, \psi) \) is expressible in terms of \( \theta \) and \( \psi \) only. It is desirable to find such a precise expression for \( \gamma (\pi, \Lambda^2, \psi) \). A similar computation has been done by Nien in \[\text{Nie17 Theorem 1.1}]\] for the gamma factor \( \gamma (\pi \times \tau, \psi) \), where \( \pi \) and \( \tau \) are irreducible representations of \( \text{GL}_n (\mathbb{F}) \) and \( \text{GL}_1 (\mathbb{F}) \) respectively.
We were not able to achieve this for general \( n \). We specify the results for \( n = 2, 3, 4 \). These computations demonstrate that the solution for this problem is not straightforward. Furthermore, since by Remark 2.20, \( \gamma (\pi, \wedge^2, \psi) \) has absolute value one, our computations actually lead to some special exponential sums having absolute value one, which is a rare situation.

**Remark 2.36.** Towards these computations, we will encounter sums of the form

\[
\sum_{\lambda \in \mathbb{F}^*} \sum_{\xi \in \mathbb{F}_n^* \atop N_{\mathbb{F}_n/\mathbb{F}}(\xi) = \lambda^2} J(\xi, \lambda),
\]

where \( J \) is some complex valued function. One can show that \( N_{\mathbb{F}_n/\mathbb{F}}(\xi) \) is a square in \( \mathbb{F}_n^* \) if and only if \( \xi \in \mathbb{F}_n^* \) is a square. We therefore have that

\[
\left\{ (\xi, \lambda) \in \mathbb{F}_n^* \times \mathbb{F}^* \mid N_{\mathbb{F}_n/\mathbb{F}}(\xi) = \lambda^2 \right\} = \left\{ (\xi^2, \pm N_{\mathbb{F}_n/\mathbb{F}}(\xi')) \mid \xi' \in \mathbb{F}_n^* \right\}.
\]

Since the map \( \xi \mapsto \xi^2 \) is two-to-one for fields with characteristic \( \neq 2 \), we get that for these fields

\[
\sum_{\lambda \in \mathbb{F}^* \atop N_{\mathbb{F}_n/\mathbb{F}}(\xi) = \lambda^2} \sum_{\xi \in \mathbb{F}_n^*} J(\xi, \lambda) = \frac{1}{2} \sum_{\xi \in \mathbb{F}_n^*} \left( J(\xi^2, N_{\mathbb{F}_n/\mathbb{F}}(\xi)) + J(\xi^2, -N_{\mathbb{F}_n/\mathbb{F}}(\xi)) \right).
\]

This formula is also true for fields with characteristic 2, as in this case the map \( \xi \mapsto \xi^2 \) is a bijection, and \( N_{\mathbb{F}_n/\mathbb{F}}(\xi) = -N_{\mathbb{F}_n/\mathbb{F}}(\xi) \).

If \( n \) is odd, then \( N_{\mathbb{F}_n/\mathbb{F}}(-1) = -1 \), and then by replacing \( \xi \) with \( -\xi \) in the second sum we get

\[
\sum_{\lambda \in \mathbb{F}^* \atop N_{\mathbb{F}_n/\mathbb{F}}(\xi) = \lambda^2} \sum_{\xi \in \mathbb{F}_n^*} J(\xi, \lambda) = \sum_{\xi \in \mathbb{F}_n^*} J(\xi^2, N_{\mathbb{F}_n/\mathbb{F}}(\xi)).
\]

### 2.5.1. Computation for \( GL_2(\mathbb{F}) \)

Since \( \theta |_{\mathbb{F}^*} = \omega_\pi \), we have that \( \pi \) has a non-trivial central character. We have in Remark 2.35 that \( S_0 = 0, S_1 = 1 \), and therefore we get the following

**Theorem 2.37.** Let \((\pi, V_\pi)\) be an irreducible cuspidal representation of \( GL_2(\mathbb{F}) \), with a non-trivial central character. Then

\[
\gamma(\pi, \wedge^2, \psi) = q^{-\frac{1}{2}} \sum_{a \in \mathbb{F}^*} \omega_\pi(a^{-1}) \psi(a).
\]

### 2.5.2. Computation for \( GL_3(\mathbb{F}) \)

By Theorem 2.32 we have that

\[
\gamma(\pi, \wedge^2, \psi) = q^{\frac{3}{2}} \sum_{a \in \mathbb{F}^*} B_{\pi, \psi} \begin{pmatrix} 0 & 1 \\ aI_2 & 0 \end{pmatrix}.
\]

In order to proceed, we use the formula for the value of the Bessel function from [Gel70, Section 7].
\[ B_{\pi,\psi} (\text{antidiag} (\lambda_1 I_1, \lambda_2 I_2)) = q^{-2} \sum_{\xi \in \mathbf{F}_3^* \atop N_{\mathbf{F}_3/\mathbf{F}} (\xi) = \lambda_1 \cdot \lambda_2^2} \psi (-\lambda_2^{-1} \text{Tr}_{\mathbf{F}_3/\mathbf{F}} (\xi)) \theta (\xi). \]

Using Remark 2.36, we get

**Theorem 2.38.** Let \( \pi \) be an irreducible cuspidal representation of \( \text{GL}_3 (\mathbf{F}) \) associated with the regular multiplicative character \( \theta : \mathbf{F}_3^* \rightarrow \mathbb{C}^* \). Then

\[ \gamma (\pi, \wedge^2, \psi) = q^{-\frac{4}{3}} \sum_{\xi \in \mathbf{F}_3^*} \psi \left( -\frac{\text{Tr}_{\mathbf{F}_3/\mathbf{F}} (\xi^2)}{N_{\mathbf{F}_3/\mathbf{F}} (\xi)} \right) \theta (\xi^2). \]

2.5.3. **Computation for** \( \text{GL}_4 (\mathbf{F}) \). By Remark 2.35, we have that

\[ \gamma (\pi, \wedge^2, \psi) = q \left( S_0 + S_1 \cdot \sum_{a \in \mathbf{F}_*} \omega_\pi (a^{-1}) \psi (a) \right), \]

where in our case \( S_0 = q^{-2} \sum_{\lambda \in \mathbf{F}_*} B_{\pi,\psi} (\lambda I_4) = \left\{ \begin{array}{ll} q^{-1} - q^{-2} & \text{\(\omega_\pi\) is trivial} \\ 0 & \text{otherwise} \end{array} \right. \), and \( S_1 = \sum_{\lambda \in \mathbf{F}_*} B_{\pi,\psi} (\lambda_2 I_2 \ 0). \)

In order to proceed, we use the formulas for the Bessel function of \( \text{GL}_4 (\mathbf{F}) \). These are computed by Deriziotis and Gotsis. We have by [DG98, Page 103] that for \( w = w_6 = \begin{pmatrix} 0 \\ I_2 \\ 0 \\ 0 \end{pmatrix} \) and \( t = \begin{pmatrix} \mu I_2 \\ \nu I_2 \end{pmatrix} \), the value \( B_{\pi,\psi} (tw) \) is given by the formula

\[ B_{\pi,\psi} (tw) = \sum_{\xi \in \mathbf{F}_4^* \atop N_{\mathbf{F}_4/\mathbf{F}} (\xi) = \det t} F_6 (\xi, t) \theta (\xi), \]

where

\[ F_6 (\xi, t) = -q^{-4} \left( F'_6 (\xi, t) + \sum_{\beta \in \mathbf{F}_*} \psi \left( -\beta + \frac{a_1 (\xi) + a_3 (\xi) \mu \nu}{\beta \mu \nu^2} \right) \right), \]

where \( \sum_{i=0}^3 a_i (\xi) t^i = \prod_{i=0}^3 (t - \xi^q) \) and

\[ F'_6 (\xi, t) = \left\{ \begin{array}{ll} -q & \xi \in \mathbf{F}_2 = \mathbf{F} \text{ and } \mu \nu = -N_{\mathbf{F}_2/\mathbf{F}} (\xi) \\ 0 & \text{otherwise} \end{array} \right. \]

Denote

\[ I (\xi, \lambda, \beta) = \theta (\xi) \psi \left( -\beta + \frac{a_1 (\xi) + a_3 (\xi) \lambda}{\beta \lambda^2} \right) \quad \xi \in \mathbf{F}_4^*; \ \beta, \lambda \in \mathbf{F}_*. \]

We therefore have that

\[ S_1 = -q^{-4} \sum_{\lambda \in \mathbf{F}_*} \left( \sum_{\xi \in \mathbf{F}_4^* \atop N_{\mathbf{F}_4/\mathbf{F}} (\xi) = \lambda^2} \sum_{\beta \in \mathbf{F}_*} I (\xi, \lambda, \beta) - q \sum_{\xi \in \mathbf{F}_4^* \atop N_{\mathbf{F}_4/\mathbf{F}} (\xi) = -\lambda} \theta (\xi) \right). \]
Regarding the second summand, since \(-\lambda\) runs over all the possible norms of elements of \(\mathbb{F}_2\), we have that
\[
\sum_{\lambda \in F^*} \sum_{\xi \in F_2 - F} \theta(\xi) = \sum_{\xi \in F_2} \theta(\xi) - \sum_{\xi \in F^*} \theta(\xi).
\]

Since \(\theta \mid F_2\) is non-trivial, and since \(\theta \mid F^* = \omega_\pi\) we have that
\[
\sum_{\lambda \in F^*} \sum_{\xi \in F_2 - F} \theta(\xi) = -\sum_{\xi \in F^*} \omega_\pi(\xi) = \begin{cases} 1 - q & \omega_\pi \text{ is trivial} \\ 0 & \text{otherwise} \end{cases}.
\]

Note that if \(\omega_\pi\) is trivial, \(\sum_{a \in F^*} \omega_\pi(a^{-1}) \psi(a) = -1\), and therefore we have that
\[
\gamma(\pi, \Lambda^2, \psi) = q^{-3} q T_0 - S_1^{'},
\]
where \(T_0 = \begin{cases} q^2 - 1 & \omega_\pi \text{ is trivial} \\ 0 & \text{otherwise} \end{cases}\), and
\[
S_1^{' \pm} = \sum_{\xi \in F_4} \theta(\xi^2) K_{\psi} \left(1, \frac{\Tr_{F_4/F}(\xi^2)}{\N_{F_4/F}(\xi)^2} \pm \N_{F_4/F}(\xi) \Tr_{F_4/F}\left(\frac{1}{\xi^2}\right)\right),
\]
where
\[
K_{\psi}(a, b) = \sum_{x \in F^*} \psi(ax) \psi\left(\frac{b}{x}\right)
\]
is the Kloosterman sum.

**Theorem 2.39.** Let \((\pi, V_\pi)\) be an irreducible cuspidal representation of \(GL_4(F)\), associated with the regular multiplicative character \(\theta : F^*_4 \to \mathbb{C}^*\), satisfying that \(\theta \mid F_2^*\) is non-trivial. Then
\[
\gamma(\pi, \Lambda^2, \psi) = q^{-2} T_0 - \frac{1}{2} q^{-3} G_\psi(\theta) \sum_{\xi \in F_4^*} \theta(\xi^2) K_{\psi} \left(1, \frac{\Tr_{F_4/F}(\xi^2)}{\N_{F_4/F}(\xi)^2} + \N_{F_4/F}(\xi) \Tr_{F_4/F}\left(\frac{1}{\xi^2}\right)\right)
- \frac{1}{2} q^{-3} G_\psi(\theta) \sum_{\xi \in F_4^*} \theta(\xi^2) K_{\psi} \left(1, \frac{\Tr_{F_4/F}(\xi^2)}{\N_{F_4/F}(\xi)^2} - \N_{F_4/F}(\xi) \Tr_{F_4/F}\left(\frac{1}{\xi^2}\right)\right),
\]
where \(G_\psi(\theta) = \sum_{a \in F^*} \theta(a^{-1}) \psi(a)\) is the Gauss sum, \(K_{\psi}(a, b) = \sum_{x \in F^*} \psi(ax) \psi(bx^{-1})\) is the Kloosterman sum, and \(T_0 = \begin{cases} q^2 - 1 & \omega_\pi \text{ is trivial} \\ 0 & \text{otherwise} \end{cases}\).
3. The relation with the Jacquet-Shalika integral over a local field

In this section we relate between our theory of the Jacquet-Shalika integral over a finite field, to the theory of the Jacquet-Shalika integral over a local non-archimedean field, via level zero representations.

3.1. Preliminaries and notations.

3.1.1. Notations. Let $F$ be a local non-archimedean field. Denote by $\mathfrak{o}$ the ring of integers of $F$, by $\mathfrak{p}$ the unique prime ideal of $\mathfrak{o}$. Let $\varpi$ be a uniformizer of $F$, a generator of $\mathfrak{p}$. Denote by $\mathfrak{f} = \mathfrak{o}/\mathfrak{p}$ the residue field of $F$ and by $q = |\mathfrak{f}|$. We use the standard normalization for the absolute value, so that $|\varpi| = \frac{1}{q}$.

Denote by $\nu : \mathfrak{o} \to \mathfrak{f}$ the quotient map. We continue to denote by $\nu$ the maps induced by $\nu$ from $\mathfrak{o}^m \to \mathfrak{f}^m$, $M_m(\mathfrak{o}) \to M_m(\mathfrak{f})$, $GL_m(\mathfrak{o}) \to GL_m(\mathfrak{f})$, etc.

Let $\psi : F \to \mathbb{C}^*$ be a non-trivial additive character, with conductor $\mathfrak{p}$, i.e., $\psi$ is trivial on $\mathfrak{p}$ but not on $\mathfrak{o}$. Then $\psi$ defines a non-trivial additive character $\psi_0 : \mathfrak{f} \to \mathbb{C}^*$ by $\psi_0 \circ \nu = \psi |_{\mathfrak{o}}$.

We denote by $S(F^m)$ the space of Schwartz functions $f : F^m \to \mathbb{C}$ - locally constant and compactly supported functions.

We choose the standard normalizations for the Haar measures on $F$ and $F^*$, i.e.

$$\int_{\mathfrak{o}} dx = 1,$$

$$\int_{\mathfrak{o}^*} d^* x = 1.$$

For $f \in S(F^m)$, we denote its Fourier transform

$$\mathcal{F}_\psi f (y) = q^{\frac{m}{2}} \int_{F^m} f(x) \cdot \psi (\langle x, y \rangle) dx,$$

where $\langle x, y \rangle$ is the standard bilinear form on $F^m$. The Fourier inversion formulas for this normalization are given by

$$\mathcal{F}_{\psi^{-1}} \mathcal{F}_\psi f (x) = f (x),$$

$$\mathcal{F}_\psi \mathcal{F}_\psi f (x) = f (-x).$$

If $(\pi, V_\pi)$ is an irreducible generic representation of $GL_n(F)$, we denote by $W(\pi, \psi)$, as in Section 2.1, its Whittaker model with respect to the character $\psi$. We also denote for an element $W \in W(\pi, \psi)$, an element $\tilde{W} \in W(\tilde{\pi}, \psi^{-1})$, defined by $\tilde{W}(g) = W(w_n g^t)$, where $g^t = g^{-1}$ and $w_n = \begin{pmatrix} 1 & \cdots & 1 \\ & \ddots & \\ & & 1 \end{pmatrix}$. 
3.1.2. The local Jacquet-Shalika integral. We briefly review the theory of the local Jacquet-Shalika integral.

Let \((\pi, V_\pi)\) be a generic irreducible representation of \(GL_n(F)\). We first give the formal formulas for the Jacquet-Shalika integral and its dual. These should initially be treated as formal expressions. Theorem 3.1 and the discussion afterwards explain how to interpret these integrals for general \(s \in \mathbb{C}\).

Suppose \(n = 2m\). The Jacquet-Shalika integral of \(\pi\) with respect to the character \(\psi\) is defined as follows: for every \(s \in \mathbb{C}\), \(W \in \mathcal{W}(\pi, \psi)\), \(\phi \in \mathcal{S}(F^m)\)

\[
J_{\pi, \psi}(s, W, \phi) = \int_{N \setminus G} \int_{B \setminus M} W \left( \sigma_{2m} \left( \begin{array}{cc} I_m & X \\ I_m & g \end{array} \right) \right) \psi(-\text{tr} X) dX \cdot |\det g|^s \phi(\varepsilon g) d^x g,
\]

where the notations are the same as in Section 2.2, this time defined over \(F\).

We define the dual Jacquet-Shalika integral as

\[
\tilde{J}_{\pi, \psi}(s, W, \phi) = J_{\tilde{\pi}, \psi^{-1}} \left( 1 - s, \tilde{\pi} \left( \begin{array}{cc} I_m \\ \end{array} \right) \right) \tilde{W}, \tilde{F}_\psi \phi).
\]

Now suppose \(n = 2m + 1\). In this case, the Jacquet-Shalika integral of \(\pi\) with respect to the character \(\psi\) is defined as

\[
J_{\pi, \psi}(s, W, \phi) = \int_{N \setminus G} \int_{B \setminus M} \int_{M \times n(F)} W \left( \sigma_{2m+1} \left( \begin{array}{cc} I_m & X \\ I_m & g \end{array} \right) \left( \begin{array}{cc} I_m \\ I_m \\ Z \end{array} \right) \right) \psi(-\text{tr} X) \phi(Z) |\det g|^{s-1} dZ dX d^x g.
\]

In this case, we define the dual Jacquet-Shalika integral as

\[
\tilde{J}_{\pi, \psi}(s, W, \phi) = J_{\tilde{\pi}, \psi^{-1}} \left( 1 - s, \tilde{\pi} \left( \begin{array}{cc} I_m \\ \end{array} \right) \right) \tilde{W}, \tilde{F}_\psi \phi).
\]

These Jacquet-Shalika integrals converge in suitable half planes. From [JS90], we have the following theorems (for \(n = 2m\) or \(n = 2m + 1\)) regarding their convergence:

**Theorem 3.1** ([JS90] Section 7, Proposition 1; Section 9, Proposition 3). There exists \(r_{\pi, \lambda^2} \in \mathbb{R}\), such that for every \(s \in \mathbb{C}\) with \(\text{Re}(s) > r_{\pi, \lambda^2}\), the integral \(J_{\pi, \psi}(s, W, \phi)\) converges, for every \(W \in \mathcal{W}(\pi, \psi)\) and \(\phi \in \mathcal{S}(F^m)\).

Correspondingly, the dual Jacquet-Shalika integrals \(\tilde{J}_{\pi, \psi}(s, W, \phi)\) converge in a left half plane \((\text{Re}(s) < 1 - r_{\tilde{\pi}, \lambda^2})\).

By [Kew11] Proposition 2.3, [CM15] Lemma 3.1, for fixed \(W\) and \(\phi\), \(J_{\pi, \psi}(s, W, \phi)\) results in its convergence domain in an element of \(\mathbb{C}(q^{-s})\) (a rational function in \(q^{-s}\)) and...
therefore has an meromorphic continuation to the entire complex plane, which we continue to denote $J_{\pi,\psi}(s, W, \phi)$. Similarly, we continue to denote the meromorphic continuation of $\tilde{J}_{\pi,\psi}(s, W, \phi)$ with the same notation. Moreover, the set

$$I = \text{span}_\mathbb{C} \{ J_{\pi,\psi}(s, W, \phi) \mid W \in W(\pi, \psi), \phi \in S(F^m) \}$$

is a fractional ideal of $\mathbb{C}[q^s, q^{-s}]$. There exists a unique polynomial $p \in \mathbb{C}[Z]$, such that $p(0) = 1$ and $I = \frac{1}{p(q^{-s})} \mathbb{C}[q^{-s}, q^s]$. We denote $L(s, \pi, \wedge^2) = \frac{1}{p(q^{-s})}$.

Similarly to Proposition 2.6 and Proposition 2.13, we can express $\tilde{J}_{\pi,\psi}(s, W, \phi)$ as in the following

**Proposition 3.2.**  
1) For $n = 2m$,

$$\tilde{J}_{\pi,\psi}(s, W, \phi) = \int_{N \setminus G} \int_{B \setminus M} W \left( \sigma_{2m} \begin{pmatrix} I_m & X \\ I_m & g \end{pmatrix} \right) \psi(-\text{tr}X) dX \cdot |\det g|^{s-1}. \mathcal{F}_{\psi,\phi}(\varepsilon_1 g^t) d^\times g.$$

2) For $n = 2m + 1$,

$$\tilde{J}_{\pi,\psi}(s, W, \phi) = \int_{N \setminus G} \int_{B \setminus M} \int_{M_{1 \times m}(F)} W \left( \begin{pmatrix} 1 \\ (I_{2m} \ 1) \sigma_{2m+1} \begin{pmatrix} I_m & X \\ I_m & 1 \end{pmatrix} \begin{pmatrix} g & \ 1 \\ g & 1 \end{pmatrix} \begin{pmatrix} I_m & -Z \\ I_m & 1 \end{pmatrix} \right) \right) \cdot \psi(-\text{tr}X) \mathcal{F}_{\psi,\phi}(Z) |\det g|^s dZ dX d^\times g.$$

We now introduce the important local exterior square factors $\gamma(s, \pi, \wedge^2, \psi)$ and $\epsilon(s, \pi, \wedge^2, \psi)$, relating the Jacquet-Shalika integral $J_{\pi,\psi}(s, W, \phi)$ to its dual $\tilde{J}_{\pi,\psi}(s, W, \phi)$.

**Theorem 3.3** ([Mat14 Theorem 4.1], [CM15 Theorem 3.1]). There exists an element $\gamma(s, \pi, \wedge^2, \psi) \in \mathbb{C}(q^{-s})$, such that for every $W \in W(\pi, \psi)$ and $\phi \in S(F^m)$,

$$\tilde{J}_{\pi,\psi}(s, W, \phi) = \gamma(s, \pi, \wedge^2, \psi) \cdot J_{\pi,\psi}(s, W, \phi).$$

Furthermore

$$\gamma(s, \pi, \wedge^2, \psi) = \epsilon(s, \pi, \wedge^2, \psi) \cdot \frac{L(1-s, \tilde{\pi}, \wedge^2)}{L(s, \pi, \wedge^2)},$$

where $\epsilon(s, \pi, \wedge^2, \psi)$ is an invertible element of $\mathbb{C}[q^{-s}, q^s]$.

In the case where $\pi$ is supercuspidal representation, the following result regrading $L(s, \pi, \wedge^2)$ is known

**Theorem 3.4** ([Jo18 Theorem 6.2]).  
1) If $n = 2m + 1$, then $L(s, \pi, \wedge^2) = 1$.

2) If $n = 2m$, then $L(s, \pi, \wedge^2) = \frac{1}{p(q^{-s})}$, where $p(Z) \in \mathbb{C}[Z]$ is a polynomial dividing $1 - \omega_\pi(\varpi) Z^m$, satisfying $p(0) = 1$.

Actually a more precise version of this theorem is known, in which $L(s, \pi, \wedge^2)$ is expressed in terms of twisted Shalika functionals in the even case. This is discussed in Section 3.4.1.
3.1.3. Level zero supercuspidal representations. Let \((\pi_0, V_{\pi_0})\) be an irreducible cuspidal representation of \(GL_n(f)\). Let \(\chi : F^* \to C^*\) be a multiplicative character, such that \(\chi \mid_{O^*} = \omega_{\pi_0} \circ \nu \mid_{O^*}\). Let \((\pi_0 \cdot \chi, V_{\pi_0})\) be the representation of the group \(F^* \cdot GL_n(o)\), defined by the formula \(\pi_0 \cdot \chi(zk) = \chi(z) \pi_0(\nu(k))\), where \(z \in F^*\) and \(k \in GL_n(o)\).

**Theorem 3.5** ([PR08 Theorem 6.2]). Let \(\pi = \text{ind}_{F^* \cdot GL_n(o)}^{GL_n(F)}(\pi_0 \cdot \chi)\). Then \(\pi\) is an irreducible supercuspidal representation of \(GL_n(F)\), with central character \(\chi\).

In fact, this theorem is a special case of the construction of supercuspidal representations using type theory, due to Bushnell and Kutzko. See [BK93 Chapter 6] for details.

We say that \(\pi\) is a level zero supercuspidal representation of \(GL_n(F)\) constructed from the representation \(\pi_0\), with respect to the central character \(\chi\), or simply a level zero supercuspidal representation constructed from the representation \(\pi_0\) (as we can recover \(\chi\) via the central character \(\omega_{\pi}\)).

3.1.4. Lifts. In order to relate Jacquet-Shalika integrals of cuspidal representations of \(GL_n(f)\) to the local Jacquet-Shalika integrals of their corresponding level-zero representations, we need to be able to lift Schwartz functions and Whittaker functions to corresponding functions defined over the local field. We describe here briefly the process of doing so, leaving the details to the reader.

**Lifts of Schwartz functions.** Denote for a function \(\phi_0 \in S(f^m)\) a lift \(L(\phi_0) \in S(F^m)\) defined by \(L(\phi_0)(x) = \begin{cases} \phi_0(\nu(x)) & x \in o^m, \\ 0 & \text{otherwise}. \end{cases}\)

It is easy to verify the following relation between the Fourier transforms and the lifts

**Proposition 3.6.** Let \(\phi_0 \in S(f^m)\). Then

\[ F_{\psi}L(\phi_0) = L(F_{\psi_0}\phi_0). \]

**Lifts of Whittaker functions.** Let \((\pi_0, V_{\pi_0})\) be an irreducible cuspidal representation of \(GL_n(f)\). Let \(\pi\) be a level zero supercuspidal constructed from \(\pi_0\). Let \(0 \neq T_0 \in \text{Hom}_{N_n(f)}(\pi_0, \psi_0)\) be a Whittaker functional of \(\pi_0\). The following proposition explains how to lift \(T_0\) to a Whittaker functional of \(\pi\).

**Proposition 3.7** ([Zel17 Theorem 4.3]). Denote by \(T : V_{\pi} \to C\) the functional

\[ \langle T, f \rangle = \int_{N_n(o) \setminus N_n(F)} \langle T_0, f(u) \psi^{-1}(u) \rangle d^nxu, \]

where \(f \in V_{\pi}\) (recall from Theorem 3.5 that \(f : GL_n(F) \to V_{\pi_0}\)). Then this integral converges and \(0 \neq T \in \text{Hom}_{N_n(F)}(\pi, \psi)\) is a Whittaker functional.

Using the lifted Whittaker functional \(T\), we are now able to define a lift of a Whittaker function.
Definition 3.8. Let $W_0 \in \mathcal{W}(\pi_0, \psi_0)$. Let $v_{W_0} \in V_{\pi_0}$ be the unique vector such that $W_0(g) = \langle T_0, \pi_0(g) v_{W_0} \rangle$, for every $g \in \text{GL}_n(f)$. Let $f_{W_0} \in V_{\pi}$ be defined as

$$f_{W_0}(g) = \begin{cases} \omega_{\pi}(z) \pi_0(\nu(k)) v_{W_0} & g = z k, z \in F^*, k \in \text{GL}_n(o) \\ 0 & \text{otherwise} \end{cases}$$

We define $\mathcal{L}(W_0) \in \mathcal{W}(\pi, \psi)$ by $\mathcal{L}(W_0)(g) = \langle T, \pi(g) f_{W_0} \rangle$ for $g \in \text{GL}_n(F)$.

One easily checks that $\mathcal{L}(W_0)$ is given by a simple formula:

**Proposition 3.9** ([Zeil17, Proposition 4.4]). $\mathcal{L}(W_0)$ is supported on $N_n(F) \cdot F^* \cdot \text{GL}_n(o)$ and

$$\mathcal{L}(W_0)(u_0zk) = \psi(u_0) \cdot \omega_{\pi}(z) \cdot W_0(\nu(k)),$$

for any $u_0 \in N_n(F)$, $z \in F^*$, $k \in \text{GL}_n(o)$.

### 3.2. A relation between the Jacquet-Shalika integrals.

In this section, we find a relation between local Jacquet-Shalika integrals of level zero representations and their corresponding residue field Jacquet-Shalika integrals. Our main result is the following:

**Theorem 3.10.** Let $n = 2m$ or $n = 2m + 1$. Let $(\pi_0, V_{\pi_0})$ be an irreducible cuspidal representation of $\text{GL}_n(f)$. If $n$ is even, suppose that $\pi_0$ does not admit a Shalika vector. Let $\pi$ be a level zero supercuspidal (irreducible) representation constructed from $\pi_0$. Then for every $W_0 \in \mathcal{W}(\pi_0, \psi_0)$, $\phi_0 \in \mathcal{S}(\mathfrak{p}^m)$, $s \in \mathbb{C}$,

$$J_{\pi, \psi}(s, \mathcal{L}(W_0), \mathcal{L}(\phi_0)) = J_{\pi_0, \psi_0}(W_0, \phi_0),$$

$$\tilde{J}_{\pi, \psi}(s, \mathcal{L}(W_0), \mathcal{L}(\phi_0)) = \tilde{J}_{\pi_0, \psi_0}(W_0, \phi_0).$$

Theorem 3.10 implies that if $\pi_0$ does not admit a Shalika vector, the Jacquet-Shalika integrals of its corresponding level zero representation for fixed lifted functions result in constant elements of $\mathbb{C}(q^{-s})$, i.e. they are independent of $s$. In the case where $n$ is even, we have a modified theorem that also handles the case in which the representation $\pi_0$ admits a Shalika vector:

**Theorem 3.11.** Let $(\pi_0, V_{\pi_0})$ be an irreducible cuspidal representation of $\text{GL}_{2m}(f)$. Then for every $W_0 \in \mathcal{W}(\pi_0, \psi_0)$, $\phi_0 \in \mathcal{S}(\mathfrak{p}^m)$, $s \in \mathbb{C}$,

$$J_{\pi, \psi}(s, \mathcal{L}(W_0), \mathcal{L}(\phi_0)) = J_{\pi_0, \psi_0}(W_0, \phi_0) + q^{-ms} \omega_{\pi}(\varpi) \phi_0(0) L(m, s, \omega_{\pi}) J_{\pi_0, \psi_0}(W_0, 1),$$

$$\tilde{J}_{\pi, \psi}(s, \mathcal{L}(W_0), \mathcal{L}(\phi_0)) = \tilde{J}_{\pi_0, \psi_0}(W_0, \phi_0) +$$

$$+ q^{-m(1-s)} \omega_{\pi}^{-1}(\varpi) F_{\psi_0} \phi_0(0) L(m(1-s), \omega_{\pi}^{-1}) J_{\pi_0, \psi_0}(W_0, 1).$$

By Theorem 2.30 we have that Theorem 3.11 implies Theorem 3.10 in the even case.

As usual, we will treat the even case and the odd case separately. Before the proof, we recall the following lemmas from [JS90], which will be useful throughout the proof.

**Lemma 3.12** ([JS90, Section 5, Proposition 4]). Let $Z \in \mathcal{N}_m^-(F)$ be a lower triangular nilpotent matrix. Denote $u_Z = \sigma_{2m} \left( \begin{smallmatrix} 1_m & Z \\ I_m & 1_m \end{smallmatrix} \right) \sigma_{2m}^{-1}$ - a lower triangular unipotent matrix. Let
\[ u_Z = n_Z \cdot t_Z \cdot k_Z \] be an Iwasawa decomposition of \( u_Z \), where \( n_Z \in N_{2m} \), \( k_Z \in \text{GL}_{2m}(\mathfrak{o}) \) and \( t_Z \) is a diagonal matrix. Write \( t_Z = \text{diag}(t_1, \ldots, t_{2m}) \). Then \( |t_1| = |t_{2m}| = 1 \), and \( |t_i| \geq 1 \) for odd \( i \), and \( |t_i| \leq 1 \) for even \( i \).

**Lemma 3.13** ([JS90], Section 5, Proposition 5). For the same notations as in Lemma 3.12, we get that there exists some \( \alpha > 0 \), such that \( \|Z\|^{1/2} \leq \prod_{i \text{ is odd}} |t_i| \), where \( \|Z\| = \max_{1 \leq i,j \leq m} |z_{ij}| \).

The following lemma will be useful for the proofs of Theorem 3.10 and Theorem 3.11. Let \( A_m \) be the diagonal subgroup of \( \text{GL}_m(F) \).

**Lemma 3.14.** Suppose that \( \sigma_{2m}(I_m X (I_m) (a) = \lambda \cdot u \cdot k \), where \( a = \text{diag}(a_1, \ldots, a_m) \in A_m \) is an invertible diagonal matrix, \( X \in N_m^-(F) \) is a lower triangular nilpotent matrix, \( \lambda \in F^* \), \( u \in N_{2m} \) and \( k \in K_{2m} = \text{GL}_{2m}(\mathfrak{o}) \). Then

1) \( |a_1| = \cdots = |a_m| = |\lambda| \).

2) \( X \in M_m(\mathfrak{o}) \).

Proof. Denote \( Z = a^{-1} X a \), and \( u_Z = \sigma_{2m}(I_m Z (I_m) \sigma_{2m}^{-1} = \text{diag}(a_1, a_1, a_2, a_2, \ldots, a_m, a_m) \). Then we have that \( bu_Z b \sigma_{2m} = \lambda u k \). Writing \( u_Z = n_Z t_Z k_Z \) as in Lemma 3.12 we get that \( \lambda^{-1} b t_Z = (b n_Z b^{-1} u) \cdot (k \sigma_{2m}^{-1} k \sigma_{2m}^{-1}) \). Since \( A_{2m} \cap (N_{2m} \cdot K_{2m}) = A_{2m} \cap K_{2m} = (\mathfrak{o})^{2m} \), we get that \( \lambda^{-1} b t_Z \) is a diagonal matrix having units on its diagonal. Writing \( t_Z = \text{diag}(t_1, \ldots, t_{2m}) \), we have that \( |t_{2i-1}| = |t_{2i}| = |\lambda|^{-1} \), for every \( 1 \leq i \leq m \). By Lemma 3.12 we get that \( |t_i| = 1 \) for every \( 1 \leq i \leq 2m \), and therefore \( |a_i| = |\lambda| \), for every \( 1 \leq i \leq m \). Finally, by Lemma 3.13 we get that \( Z \in M_m(\mathfrak{o}) \). Since \( |a_i| = |\lambda| \), for every \( 1 \leq i \leq m \), we get that \( X \in M_m(\mathfrak{o}) \), as required.

We now move to prove Theorem 3.10 and Theorem 3.11. Throughout both proofs, we will use the following symbols:

\[ W = \mathcal{L}(W_0), \quad \phi = \mathcal{L}(\phi_0). \quad A = A_m \text{ is the diagonal subgroup of } G. \quad K = \text{GL}_m(\mathfrak{o}). \quad \mathcal{N}^- = \mathcal{N}_m^-(F) \subseteq M_m(F) \text{ is the subspace consisting of lower triangular nilpotent matrices.} \]

Let \( X \in \mathcal{N}^- \). Also let \( g = a k \), where \( a = \text{diag}(a_1, \ldots, a_m) \in A, \ k \in K \). Then by the Iwasawa decomposition of \( \text{GL}_m(F) \), we have \( d^* g = \delta_{B_m}^{-1}(a) \prod_{i=1}^m d^* a_i \cdot d^* k \), where \( \delta_{B_m}^{-1}(a) = \prod_{1 \leq i < j \leq m} \left| \frac{a_i}{a_j} \right| \).

**Proof of Theorem 3.11.** We prove only the equation regarding \( J_{\pi,\psi} \). The proof of the equation regarding \( J_{\pi,\psi} \) is similar. It can also be deduced from the first equation.

The proof consists of two parts. In the first part we find the support of \( g, \ X \) in the Jacquet-Shalika integral. In the second part, we evaluate the integral on this domain.

Suppose that \( \sigma_{2m}(I_m X (I_m) (a) (k) \in \text{supp}(W) \subseteq F^* N_{2m} K_{2m} \) (see Proposition 3.12). Then by Lemma 3.14 we have that \( |a_i| = |a_m| \), for every \( 1 \leq i \leq m \), and \( X \in M_m(\mathfrak{o}) \).
Therefore, we get that \( a_i = u_i \cdot a_m \), where \( u_i \in \mathfrak{o}^* \), \( d^x u_i = d^x a_i \), for every \( 1 \leq i \leq m - 1 \). We also get \( \delta_{B_m}^{-1}(a) = \prod_{1 \leq i < j \leq m} \left| \frac{a_j}{a_i} \right| = 1 \).

Therefore, the Jacquet Shalika integral is integrated on \( X \in \mathcal{N}_m^- (\mathfrak{o}) \), \( g = ak \), where \( k \in K \), \( a = a_m \cdot \text{diag} (u_1, \ldots, u_{m-1}, 1) \), \( u_i \in \mathfrak{o}^* \) for every \( 1 \leq i \leq m - 1 \), \( d^x g = d^x a_m \cdot \prod_{i=1}^{m-1} d^x u_i \cdot d^x k \), and by replacing the variable \( k \) with \( \text{diag} (u_1, \ldots, u_{m-1}, 1)^{-1} \cdot k \), we have that \( J_{\pi, \psi} (s, W, \phi) \) is given by

\[
\int_{F^*} \int_{K} \int_{\mathcal{N}_m^- (\mathfrak{o})} W \left( \sigma_{2m} \left( \begin{array}{cc} I_m & X \\ I_m & I_m \end{array} \right) \left( \begin{array}{c} k \\ k \end{array} \right) \right) \phi (\varepsilon k a_m) |a_m|^{ms} \omega_\pi (a_m) dX d^x k d^x a_m.
\]

Since \( \phi = L (\phi_0) \) is a lift of the Schwartz function \( \phi_0 \), we have that for a fixed \( k \), \( \phi (\varepsilon k a_m) = 0 \) for \( |a_m| > 1 \) and \( \phi (\varepsilon k a_m) = \phi_0 (0) \) for \( |a_m| < 1 \). Therefore

\[
\int_{F^*} \phi (\varepsilon k a_m) |a_m|^{ms} \omega_\pi (a_m) d^x a_m = \int_{\mathfrak{o}^*} \phi (\varepsilon k a_m) \omega_\pi (a_m) d^x a_m + \phi_0 (0) \cdot \sum_{i=1}^{\infty} \int_{\mathfrak{o}^*} |a_m|^{ms} \omega_\pi (a_m) d^x a_m.
\]

We have that

\[
I (s) = \sum_{i=1}^{\infty} \int_{\mathfrak{o}^*} |a_m|^{ms} \omega_\pi (a_m) d^x a_m = \int_{\mathfrak{o}^*} \omega_\pi (a_m) d^x a_m \cdot \sum_{i=1}^{\infty} q^{-im^s} \omega_\pi (\mathfrak{o}^*)^{mi},
\]

and therefore \( I (s) = q^{-ms} \omega_\pi (\mathfrak{o}) L (ms, \omega_\pi) \) if \( \omega_\pi \) is unramified (i.e. \( \omega_\pi |_{\mathfrak{o}^*} \) trivial, which happens if and only if \( \omega_{\pi_0} \) is trivial), and otherwise \( \int_{\mathfrak{o}^*} \omega_\pi (a_m) d^x a_m = 0 \), so \( I (s) = 0 \).

We are left to evaluate

\[
\int_{\mathfrak{o}^*} \int_{K} \int_{\mathcal{N}_m^- (\mathfrak{o})} W \left( \sigma_{2m} \left( \begin{array}{cc} I_m & X \\ I_m & I_m \end{array} \right) \left( \begin{array}{c} k \\ k \end{array} \right) \right) \omega_\pi (a_m) \cdot \phi (\varepsilon k a_m) + \phi_0 (0) I (s) \right) dX d^x k d^x a_m.
\]

Since \( W, \phi \) and \( \omega_\pi \) are lifts of \( W_0, \phi_0 \) and \( \omega_{\pi_0} \) respectively, and the expression is constant on the quotient spaces \( \text{GL}_m (\mathfrak{o}) / 1 + \mathfrak{o} M_m (\mathfrak{o}) \cong \text{GL}_m (\mathfrak{f}), \mathcal{N}^- (\mathfrak{o}) / \mathcal{N}^- (\mathfrak{p}) \cong \mathcal{N}^- (\mathfrak{f}) \) and \( \mathfrak{o}^* / 1 + \mathfrak{p} \cong \mathfrak{f}^* \), we get that this sum equals

\[
\frac{1}{|\mathfrak{f}^*| \cdot |\text{GL}_m (\mathfrak{f})| \cdot |\mathcal{N}^- (\mathfrak{f})|} \sum_{a_m \in \mathfrak{o}^*} \sum_{k \in \text{GL}_m (\mathfrak{f})} \sum_{X \in \mathcal{N}^- (\mathfrak{f})} W_0 \left( \sigma_{2m} \left( \begin{array}{cc} I_m & X \\ I_m & I_m \end{array} \right) \left( \begin{array}{c} k \\ k \end{array} \right) \right) \cdot \omega_{\pi_0} (a_m) \cdot (\phi_0 (\varepsilon k a_m) + \phi_0 (0)) I (s) \right).
\]

By replacing the variable \( k \) with \( k = k' a_m^{-1} \), we get

\[
\frac{1}{|\text{GL}_m (\mathfrak{f})| \cdot |\mathcal{N}^- (\mathfrak{f})|} \sum_{k' \in \text{GL}_m (\mathfrak{f})} \sum_{X \in \mathcal{N}^- (\mathfrak{f})} W_0 \left( \sigma_{2m} \left( \begin{array}{cc} I_m & X \\ I_m & I_m \end{array} \right) \left( \begin{array}{c} k' \\ k' \end{array} \right) \right) (\phi_0 (\varepsilon k') + \phi_0 (0)) I (s) \right).
\]

Since this expression is constant on cosets of \( k' \in \mathcal{N}_m (\mathfrak{f}) \setminus \text{GL}_m (\mathfrak{f}) \) and since \( \mathcal{B}_m (\mathfrak{f}) \setminus M_m (\mathfrak{f}) \cong \mathcal{N}^- (\mathfrak{f}) \), we get that

\[
J_{\pi, \psi} (s, W, \phi) = J_{\pi, \psi_0} (W_0, \phi_0) + \phi_0 (0) I (s) J_{\pi, \psi_0} (W_0, 1).
\]

Finally, we claim that

\[
\phi_0 (0) I (s) J_{\pi, \psi_0} (W_0, 1) = \phi_0 (0) q^{-ms} \omega_\pi (\mathfrak{o}) L (ms, \omega_\pi) J_{\pi, \psi_0} (W_0, 1).
\]
If $\omega_{\pi_0}$ is trivial, we already saw that this is true. If $\omega_{\pi_0}$ is non-trivial, $\pi_0$ does not admit a Shalika vector, and we get from Theorem 2.30 that $J_{\pi_0, \psi_0}(W, 1) = 0$, and therefore the result follows.

**Proof of Theorem 3.10.** We are left to prove the odd case, $n = 2m + 1$. Again, we only prove the equation regarding $J_{\pi, \psi}$. Let $Z \in M_{1 \times m}(F)$. Suppose that $Z \in \text{supp}(\phi) \subseteq \mathfrak{o}^m$. Therefore $Z$ is integrated on $M_{1 \times m}(\mathfrak{o})$. Suppose that $\sigma_{2m+1} \left( \begin{pmatrix} I_m & X \\ I_m & 1 \end{pmatrix} \begin{pmatrix} a & k \\ a & 1 \end{pmatrix} \begin{pmatrix} I_m & X \\ I_m & 1 \end{pmatrix} \begin{pmatrix} I_m & I_m \\ Z & 1 \end{pmatrix} \right) \in \text{supp}(W)$.

Since $\text{supp}(W) \subseteq N_{2m+1} \cdot F^* \cdot K_{2m+1}$, we get that $\sigma_{2m+1} \left( \begin{pmatrix} I_m & X \\ I_m & 1 \end{pmatrix} \begin{pmatrix} a & k \\ a & 1 \end{pmatrix} = u' \cdot \lambda \cdot k'$, where $u' \in N_{2m+1}$, $\lambda \in F^*$, $k' \in K_{2m+1}$. Writing $\sigma_{2m+1} = (\sigma_{2m}^{-1})$, comparing the last row of the matrices, and modifying $\lambda$ by a unit, we get that $k' = (k''_0 \ Y')$, where $k'' \in K_m$ and $Y \in M_{2m \times 1}(\mathfrak{o})$. Replacing $u'$ by $u'(\begin{smallpmatrix} 2m \ Y_0 \\ 0 \ 1 \end{smallpmatrix})$, we may assume that $k' = (k''_1)$, which implies that $u'' \in N_{2m}$. We also get $\lambda = 1$. Therefore we have that $\sigma_{2m} \left( \begin{pmatrix} I_m & X \\ I_m & 1 \end{pmatrix} \begin{pmatrix} a \\ 1 \end{pmatrix} = 1 \cdot u'' \cdot k''$. By Lemma 3.11, this implies that $X \in M_m(\mathfrak{o})$ and $|a_i| = 1$, for every $1 \leq i \leq m$, which implies that $\delta_{B_m}^{-1}(a) = 1$. Therefore, $J_{\pi, \psi}(s, W, \phi)$ is given by

$$
\int_K \int_{N_m(\mathfrak{o})} \int_{M_{1 \times m}(\mathfrak{o})} W \left( \sigma_{2m+1} \left( \begin{pmatrix} I_m & X \\ I_m & 1 \end{pmatrix} \begin{pmatrix} k \\ 1 \end{pmatrix} \begin{pmatrix} I_m & I_m \\ Z & 1 \end{pmatrix} \phi (Z) dZ dX d^k k. 
\right.
$$

Since the integrand is constant on $k \in K/1 + \mathfrak{o} M_m(\mathfrak{o}) \cong \text{GL}_m(\mathfrak{f})$, $X \in N_m(\mathfrak{o}) \subset \text{GL}_m(\mathfrak{f})$ and $Z \in M_{1 \times m}(\mathfrak{o})/M_{1 \times m}(\mathfrak{f}) = M_{1 \times m}(\mathfrak{f})$, and since $W$ and $\phi$ are lifts of $W_0$ and $\phi_0$, respectively, we have that this expression equals

$$
\frac{1}{|\text{GL}_m(\mathfrak{f})| |N_m(\mathfrak{f})| |M_{1 \times m}(\mathfrak{f})|} \sum_{k \in \text{GL}_m(\mathfrak{f})} \sum_{X \in N_m(\mathfrak{f})} \sum_{Z \in M_{1 \times m}(\mathfrak{f})} W_0 \left( \sigma_{2m+1} \left( \begin{pmatrix} I_m & X \\ I_m & 1 \end{pmatrix} \begin{pmatrix} k \\ 1 \end{pmatrix} \begin{pmatrix} I_m & I_m \\ Z & 1 \end{pmatrix} \phi (Z) dZ dX d^k k. 
\right.
$$

and it equals $J_{\pi_0, \psi_0}(W_0, \phi_0)$, as the summand is constant on the cosets of $N_m(\mathfrak{f}) \setminus \text{GL}_m(\mathfrak{f})$, and since $N_m(\mathfrak{f}) \cong B_m(\mathfrak{f}) \setminus M_m(\mathfrak{f})$. Hence, we proved that equality

$$
J_{\pi, \psi}(s, L(W_0), L(\phi_0)) = J_{\pi_0, \psi_0}(W_0, \phi_0).
$$

q.e.d.

**3.3. The modified functional equation in the even case.** As a result of Theorem 3.11 we get the following modified functional equation, a generalization of Theorem 2.18, this time valid for all irreducible cuspidal representations of $\text{GL}_{2m}(\mathbb{F})$, regardless whether they admit a Shalika vector.
**Theorem 3.15** (The modified functional equation). Let \( \mathbb{F} \) be a finite field with \( |\mathbb{F}| = q \). Let \( \psi_0 : \mathbb{F} \to \mathbb{C}^* \) be a non-trivial (additive) character. Let \((\pi_0, V_{\pi_0})\) be an irreducible cuspidal representation of \( \text{GL}_m(\mathbb{F}) \). Then there exists \( \gamma(s, \pi_0, \lambda^2, \psi_0) \in \mathbb{C}(q^{-1}) \), such that for every \( W_0 \in \mathcal{W}(\pi_0, \psi_0) \) and \( \phi_0 \in \mathcal{S}(\mathbb{F}^m) \), we have

\[
\tilde{J}_{\pi_0, \psi_0}(W_0, \phi_0) + q^{-m(1-s)} \mathcal{F}_{\psi_0} \phi_0(0) L(m(1-s), 1) J_{\pi_0, \psi_0}(W_0, 1) = \\
\gamma(s, \pi_0, \lambda^2, \psi_0) \cdot (J_{\pi_0, \psi_0}(W_0, \phi_0) + q^{-ms} \phi_0(0) L(ms, 1) J_{\pi_0, \psi_0}(W_0, 1)).
\]

**Proof.** If \( \pi_0 \) does not admit a Shalika vector, then from Theorem 2.30 for every \( W_0 \in \mathcal{W}(\pi_0, \psi_0) \), we have that \( J_{\pi_0, \psi_0}(W_0, 1) = 0 \), and therefore we get the same functional equation as in Theorem 2.18.

Suppose that \( \pi_0 \) admits a Shalika vector. Then by Remark 2.11 \( \pi_0 \) has a trivial central character. Choose any local field \( F \) with \( \mathbb{F} \) as its residue field, and \( \psi : F \to \mathbb{C}^* \), an additive character, such that \( \psi |_{\mathbb{F}} = \psi_0 \circ \nu \). Let \((\pi, V_{\pi})\) be the level zero supercuspidal representation constructed from \( \pi_0 \), with respect to the trivial central character. The statement now follows from Theorem 3.14 and Theorem 3.3. q.e.d.

### 3.4. Exterior square gamma factors for level zero supercuspidal representations.

Let \((\pi_0, V_{\pi_0})\) be an irreducible cuspidal representation of \( \text{GL}_n(\mathbb{F}) \), and let \((\pi, V_{\pi})\) be a level zero representation of \( \text{GL}_n(\mathbb{F}) \) constructed from \( \pi_0 \).

As a corollary of Theorem 3.10 we obtain the following main theorems of the paper.

**Theorem 3.16.** If \( \pi_0 \) does not admit a Shalika vector, then \( \gamma(s, \pi, \lambda^2, \psi) \) is an invertible constant (i.e., independent of \( s \)), and \( \gamma(s, \pi, \lambda^2, \psi) = \gamma(\pi_0, \lambda^2, \psi_0) \). Furthermore, \( L(s, \pi, \lambda^2) = 1, \epsilon(s, \pi, \lambda^2, \psi) = \gamma(\pi_0, \lambda^2, \psi_0) \).

**Proof.** We can choose \( W_0 \in \mathcal{W}(\pi_0, \psi_0) \) and \( \phi_0 \in \mathcal{S}(\mathbb{F}^m) \), such that \( J_{\pi_0, \psi_0}(W_0, \phi_0) \neq 0 \) (for instance take the functions in Proposition 2.7, Proposition 2.11). By Theorem 3.10 we have that \( J_{\pi, \psi}(s, L(W_0), L(\phi_0)) = J_{\pi_0, \psi_0}(W_0, \phi_0) \) and \( J_{\pi, \psi}(s, L(W_0), L(\phi_0)) = J_{\pi_0, \psi_0}(W_0, \phi_0) \). We get from the functional equations in Theorem 2.18 and Theorem 3.3 that

\[
\gamma(s, \pi, \lambda^2, \psi) = \frac{J_{\pi_0, \psi_0}(W_0, \phi_0)}{J_{\pi_0, \psi_0}(W_0, \phi_0)} = \frac{J_{\pi_0, \psi_0}(W_0, \phi_0)}{J_{\pi_0, \psi_0}(W_0, \phi_0)} = \gamma(\pi_0, \lambda^2, \psi_0).
\]

This proves the result regarding the gamma factors.

If \( n = 2m + 1 \), we have from Theorem 3.4 that \( L(s, \pi, \lambda^2) = 1 \). Suppose \( n = 2m \), and denote \( L(s, \pi, \lambda^2) = \frac{1}{p_1(q^{-s})} \), \( L(s, \tilde{\pi}, \lambda^2) = \frac{1}{p_2(q^{-s})} \), where \( p_1(Z), p_2(Z) \in \mathbb{C}[Z] \) are polynomials with \( p_1(0) = p_2(0) = 1 \). Since \( \gamma(s, \pi, \lambda^2, \psi) \) is a constant, by Theorem 3.3 we must have that

\[
\frac{p_1(q^{-s})}{p_2(q^{-s})} = \frac{L(1-s, \tilde{\pi}, \lambda^2)}{L(s, \pi, \lambda^2)} = c \cdot q^{ks},
\]

where \( k \in \mathbb{Z} \) and \( c \in \mathbb{C}^* \). This implies that \( p_1(Z) \) and \( p_2(q^{-1}Z^{-1}) \) have the same non-zero roots. By Theorem 3.4, we have that \( p_1(Z) \) divides \( 1 - \omega_\pi(\varpi)Z^m \) and \( p_2 \) divides \( 1 - \omega_\pi(\varpi)^{-1}Z^m \), and therefore \( p_1(Z) \) and \( p_2(q^{-1}Z^{-1}) \) can't have mutual roots, as roots \( r \) of
Moreover, these theorems demonstrate a close connection between the existence of Shalika \( \pi \) equation in Theorem 3.3. q.e.d.

**Theorem 3.17.** If \( n = 2m \) and \( \pi_0 \) admits a Shalika vector, then
\[
\gamma \left( s, \pi, \Lambda^2, \psi \right) = \frac{q^{m s}}{q^r \omega_\pi (\varpi)} \cdot \frac{L (m (1 - s), \omega_\pi^{-1})}{L (m s, \omega_\pi)}. 
\]
Furthermore, \( L (s, \pi, \Lambda^2) = L (ms, \omega_\pi), \) and \( \epsilon (s, \pi, \Lambda^2, \psi) = \frac{q^{ms}}{q^r \omega_\pi (\varpi)}. \)

Also in this case \( \gamma (s, \pi, \Lambda^2, \psi) = \gamma (s - s_0, \pi_0, \Lambda^2, \psi_0), \) where \( q^{ms_0} = \omega_\pi (\varpi) \) (see also Theorem 3.15).

**Proof.** By Remark 2.11 since \( \pi_0 \) admits a Shalika vector, the central character \( \omega_{\pi_0} \) is trivial. Thus, the central character \( \omega_\pi \) is unramified. Therefore,
\[
L (s, \omega_\pi) = \frac{1}{1 - \omega_\pi (\varpi) q^{-s}}.
\]

By Theorem 2.30 there exists \( W_0 \in W (\pi_0, \psi_0) \), such that \( J_{\pi_0, \psi_0} (W_0, 1) = 1. \) We substitute in Theorem 2.30 such \( W_0 \) and \( \phi_0 = 1, \) \( J_{\psi_\psi} \phi_0 = q^m \delta_0 \) to get
\[
J_{\pi, \psi} (s, L (W_0), L (\phi_0)) = 1 + q^{ms} \omega_\pi (\varpi) L (ms, \omega_\pi) = L (ms, \omega_\pi), \tag{3.1}
\]
and
\[
\tilde{J}_{\pi, \psi} (s, L (W_0), L (\phi_0)) = \frac{q^m}{q^r} q^{-m (1 - s)} \omega_\pi (\varpi)^{-1} L (m (1 - s), \omega_\pi^{-1}).
\]

The result regarding \( \gamma (s, \pi, \Lambda^2, \psi) \) follows as \( \gamma (s, \pi, \Lambda^2, \psi) = \frac{\tilde{J}_{\pi, \psi} (s, L (W_0), L (\phi_0))}{J_{\pi, \psi} (s, L (W_0), L (\phi_0))}. \)

Regarding \( L (s, \pi, \Lambda^2), \) denote \( L (s, \pi, \Lambda^2) = \frac{1}{p (q^{-s})}, \) where \( p (Z) \in \mathbb{C} [Z] \) is a polynomial with \( p (0) = 1. \) By Theorem 3.4 we have that \( p (Z) \mid 1 - \omega_\pi (\varpi) Z^m. \) From eq. (3.1), we have that
\[
J_{\pi, \psi} (s, L (W_0), L (\phi_0)) = L (ms, \omega_\pi) = \frac{1}{1 - \omega_\pi (\varpi) q^{-ms}} \in \frac{1}{p (q^{-s})} \mathbb{C} [q^s, q^{-s}],
\]
so \( 1 - \omega_\pi (\varpi) Z^m \mid p (Z). \) Therefore we must have \( p (Z) = 1 - \omega_\pi (\varpi) Z^m, \) and the result \( L (s, \pi, \Lambda^2) = L (ms, \omega_\pi) \) follows. The result regarding \( \epsilon (s, \pi, \Lambda^2, \psi) \) now follows from the equation in Theorem 3.3. q.e.d.

Theorem 3.16 and Theorem 3.17 establish a connection between a cuspidal representation \( \pi_0 \) and its corresponding level zero representation \( \pi \) via the local exterior square factors of \( \pi. \) Moreover, these theorems demonstrate a close connection between the existence of Shalika vectors and the existence of poles of the local exterior square \( L \)-function.

**Corollary 3.18.** Let \( \pi \) be a level zero representation constructed from an irreducible cuspidal representation \( \pi_0. \) Then \( \pi_0 \) admits a Shalika vector if and only if \( L (s, \pi, \Lambda^2) \) has a pole.
Proof. On one hand, if \( \pi_0 \) admits a Shalika vector, then by Theorem 3.17 \( L(s, \pi, \wedge^2) = L(ms, \omega_\pi) \) has a pole.

On the other hand, if \( \pi_0 \) does not admit a Shalika vector, then by Theorem 3.16 \( L(s, \pi, \wedge^2) = 1 \) does not have any poles.

3.4.1. Relation with Shalika functionals. Shalika vectors in \( V_{\pi_0} \) are closely related to poles of \( L(s, \pi, \wedge^2) \), which in turn are closely akin to (twisted) Shalika functionals on \( \mathcal{W}(\pi, \psi) \). After we relate Shalika vectors and Shalika periods, we will give another explanation for the computation of \( L(s, \pi, \wedge^2) \) in the case where \( \pi_0 \) admits a Shalika vector. We begin with the introduction of twisted Shalika functionals \( \Lambda_s \) on \( \mathcal{W}(\pi, \psi) \).

**Definition 3.19.** Let \((\pi, V_\pi)\) be a representation with an unramified central character. For any \( s \in \mathbb{C} \) satisfying \( q^m s = \omega_\pi(\varpi) \), the twisted Shalika period \( \Lambda_s \) on \( \mathcal{W}(\pi, \psi) \) is defined to be the following linear functional \( \Lambda_s : \mathcal{W}(\pi, \psi) \to \mathbb{C} \)

\[
\Lambda_s(W) = \int_{F^* N \backslash G} \int_{B \backslash M} W \left( \sigma_{2m} \left( \begin{array}{cc} I_m & X \\ g & g \end{array} \right) \right) \psi(-\text{tr}X) dX \cdot |\det g|^s dx g.
\]

For any \( s \in \mathbb{C} \), we set \( \nu^s \) to be the one dimensional representation of \( \text{GL}_{2m}(F) \) given by \( \nu^s(g) = |\det g|^s \) for any \( g \in \text{GL}_{2m}(F) \). Applying [Kew11 Lemma 4.2] to the representation \( \pi \otimes \nu^\frac{s}{2} \), we know that \( \Lambda_s \) converges absolutely. \( \Lambda_s \) is a twisted Shalika functional in the sense that for any \( h \in S_{2m} \) and any \( W \in \mathcal{W}(\pi, \psi) \), we have

\[
\Lambda_s(\pi(h)W) = |\det h|^{-s} \psi(h) \Lambda_s(W). \tag{3.2}
\]

By eq. (3.2), we have

\[
\Lambda_s \in \text{Hom}_{S_{2m}(F)} \left( \pi \otimes \nu^{\frac{s}{2}}, \Psi \right).
\]

By [Jo18], we have

**Theorem 3.20** ([Jo18 Theorem 6.2, Theorem 6.7]). Let \((\pi, V_\pi)\) be an irreducible supercuspidal representation of \( \text{GL}_{2m}(F) \), with an unramified central character. Then

\[
L(s, \pi, \wedge^2) = \prod_\alpha (1 - \alpha q^{-s})^{-1},
\]

where the product runs over all \( \alpha = q^m s \), such that \( \alpha^m = \omega_\pi(\varpi) \) and \( \Lambda_{s_0} \neq 0 \). Equivalently, the product runs over all \( \alpha = q^m s \) such that \( \text{Hom}_{S_{2m}(F)} \left( \pi \otimes \nu^{\frac{s}{2}}, \Psi \right) \neq 0 \).

As before, let \((\pi_0, V_{\pi_0})\) be an irreducible representation of \( \text{GL}_{2m}(F) \) and let \((\pi, V_\pi)\) be a level zero representation constructed from \( \pi_0 \). We recall that the central character \( \omega_{\pi_0} \) is trivial if and only if the central character \( \omega_\pi \) is unramified. Similarly to the proof of Theorem 3.11 one can show

**Proposition 3.21.** Suppose \( \omega_{\pi_0} \) is trivial and equivalently \( \omega_\pi \) is unramified. For any \( W_0 \in \mathcal{W}(\pi_0, \psi_0) \) and any \( s \in \mathbb{C} \), such that \( q^m s = \omega_\pi(\varpi) \), we have

\[
\Lambda_s(L(W_0)) = J_{\pi_0, \psi_0}(W_0, 1).
\]
We remark that the right hand side of the equation in the above proposition is independent of \( s \in \mathbb{C} \).

We can now use Theorem 3.20 and Proposition 3.21 to give another explanation why in the case that \( \pi_0 \) admits a Shalika vector, \( L(s, \pi, \wedge^2) = L(ms, \omega_\pi) \): Since \( \pi_0 \) admits a Shalika vector, we have by Remark 2.11 that the central character in the case that \( \pi \), with \( q^m s = \omega_\pi(w) \), \( \Lambda_s \neq 0 \). Therefore, the condition \( \alpha = q^m s \) with \( \Lambda_s \neq 0 \) is always valid. Since \( \alpha \) in the product in Theorem 3.20 runs over the \( m \)-th roots of \( \omega_\pi(w) \), we get
\[
L(s, \pi, \wedge^2) = \prod_{\alpha} (1 - \alpha q^{-s})^{-1} = (1 - \omega_\pi(w)q^{-ms})^{-1} = L(ms, \omega_\pi).
\]

We conclude this section by giving a characterization of the existence of Shalika vectors in terms of the existence of (twisted) Shalika functionals.

**Proposition 3.22.** \( \text{Hom}_{S_{2m}(\mathfrak{f})}(\pi_0, \Psi_0) \neq 0 \) if and only if \( \text{Hom}_{S_{2m}(\mathfrak{f})}(\pi \otimes \nu^{s/2}, \Psi) \neq 0 \) for some \( s \in \mathbb{C} \). Here \( \Psi_0 : S_{2m}(\mathfrak{f}) \to \mathbb{C}^* \) is the character on Shalika subgroup, defined by the character \( \psi_0 \) (see Definition 2.8).

**Proof.** With a choice of inner product on \( V_{\pi_0} \), with respect to which \( \pi_0 \) is unitary, we can show that the existence of a Shalika vector is equivalent to \( \text{Hom}_{S_{2m}(\mathfrak{f})}(\pi_0, \Psi_0) \neq 0 \). By Theorem 3.20 \( L(s, \pi, \wedge^2) \) has a pole at \( s \in \mathbb{C} \) if and only if \( \text{Hom}_{S_{2m}(\mathfrak{f})}(\pi \otimes \nu^{s/2}, \Psi) \neq 0 \). Therefore, the proposition now follows from Corollary 3.18 q.e.d.

**Appendix A. Multiplicity one theorems**

In the appendix, we follow [Mat12] in order to prove the following multiplicity one theorems, which are keys to the proofs of the functional equation for the finite field case (Theorem 2.18).

Let \( \mathbb{F} \) be a finite field.

**Theorem A.1.** Let \( (\pi, V_\pi) \) be an irreducible cuspidal representation of \( \text{GL}_{2m}(\mathbb{F}) \). Then
\[
\dim_{\mathbb{C}} \text{Hom}_{M_{m,m} \cap P_{2m}}(\pi, 1) \leq 1,
\]
where \( M_{m,m} \) is the Levi subgroup corresponding to the partition \( (m, m) \), and \( P_{2m} \) is the mirabolic subgroup.

**Theorem A.2.** Let \( (\pi, V_\pi) \) be an irreducible cuspidal representation of \( \text{GL}_{2m+1}(\mathbb{F}) \). Then
\[
\dim_{\mathbb{C}} \text{Hom}_{L_{2m+1} \cap P_{2m+1}}(\pi, 1) \leq 1,
\]
where \( P_{2m+1} \) is the mirabolic subgroup and \( L_{2m+1} \) is the following maximal (non-standard) Levi subgroup of \( \text{GL}_{2m+1}(\mathbb{F}) \) corresponding to the partition \( (m + 1, m) \):
\[
L_{2m+1} = \left\{ \begin{pmatrix} g_1 & u \\ g_2 & \lambda \\ v & \end{pmatrix} \middle| g_1, g_2 \in \text{GL}_m(\mathbb{F}) \ , u \in M_{m \times 1}(\mathbb{F}) \ , v \in M_{1 \times m}(\mathbb{F}) \ , \lambda \in \mathbb{F} \right\} \cap \text{GL}_{2m+1}(\mathbb{F}).
\]
The proofs of these theorems require some preparations.

For a finite group $G$ and a vector space $V$ over $\mathbb{C}$, denote
$$S(G,V) = \{ f : G \to V \}.$$ Denote by $\rho$ and $\lambda$ the right and left actions of $G$ on $S(G,V)$ respectively: $(\rho(g_0)f)(g) = f(gg_0), (\lambda(g_0)f)(g) = f(g_0^{-1}g)$.

Let $\psi : \mathbb{F} \to \mathbb{C}$ be a non-trivial additive character of $\mathbb{F}$.

For every positive integer $n$, we denote $G_n = \text{GL}_n(\mathbb{F})$. In the following, we use the convention that for $k < n$, $G_k$ is embedded in $G_n$ by mapping $g \mapsto (g \, I_{n-k})$. Suppose that $p + q = n$, where $p \geq q \geq 0$. Let $\sigma_{p,q}$ be the following permutation:

$$
\begin{pmatrix}
1 & 2 & \cdots & p - q & p - q + 1 & p - q + 2 & \cdots & p & p + 1 & p + 2 & \cdots & p + q \\
1 & 2 & \cdots & p - q & p - q + 1 & p - q + 3 & \cdots & p + q - 1 & p - q + 2 & p - q + 4 & \cdots & p + q
\end{pmatrix}.
$$

Let $w_{p,q}$ be the column permutation matrix corresponding to $\sigma_{p,q}$.

Let $H_{p,q}^{(n)} = w_{p,q}M_{p,q}w_{p,q}^{-1}$, where $M_{p,q}$ is the Levi subgroup corresponding to the partition $(p,q)$. If $q \geq 1$, denote
$$H_{p,q-1}^{(n)} = w_{p,q} \left\{ \begin{pmatrix} m \\ 1 \end{pmatrix} \mid m \in M_{p,q-1} \right\} w_{p,q}^{-1},$$

where $M_{p,q-1}$ is the Levi subgroup of $\text{GL}_{n-1}(\mathbb{F})$ corresponding to the partition $(p,q-1)$. Note that since $\sigma_{p,q}(n) = n$, we have that $H_{p,q-1}^{(n)}$ is a subgroup of $G_{n-1}$. If $q \geq 1$, also denote
$$H_{p-1,q}^{(n-1)} = \left\{ \begin{pmatrix} h \\ I_2 \end{pmatrix} \mid h \in H_{p-1,q-1}^{(n-2)} \right\}.$$

Let
$$U_n = \left\{ \begin{pmatrix} I_{n-1} & x \\ 0 & 1 \end{pmatrix} \mid x \in M_{n-1 \times 1}(\mathbb{F}) \right\},$$
be the unipotent radical of $G_n$, corresponding to the partition $(n-1,1)$. Let $P_n \leq G_n$ be the mirabolic subgroup. We have that $P_n = U_n \rtimes G_{n-1}$. $\psi$ defines a character on $U_n$ by $\psi(u) = \psi(u_{n-1,n})$.

Recall the definition of the following Bernstein-Zelevinsky derivative: for a representation $\pi$ of $P_{n-1}$, we define $\Phi^+ (\pi) = \text{Ind}_{P_{n-1}}^{G_n} (\pi \otimes \psi)$, where $(\pi \otimes \psi)(pu) = \psi(u) \cdot \pi(p)$, for $p \in P_{n-1}$ and $u \in U_n$. We have the following relation between this functor and irreducible cuspidal representations of $G_n$:

**Theorem A.3** ([Gel70 Theorem 2.3]). Let $\pi$ be an irreducible cuspidal representation of $\text{GL}_n(\mathbb{F})$. Then the restriction of $\pi$ to the mirabolic subgroup $P_n$ is isomorphic to the representation $(\Phi^+)^{n-1}(1)$.

We start with the following lemmas from [Mat12]. These are purely algebraic statements, which are proved exactly as in [Mat12].
Lemma A.4 ([Mat12 Lemma 3.1]). Let \( p \geq q \geq 1 \) with \( p + q = n \), and let
\[
S^{(n)}_{p,q} = \{ g \in G_{n-1} \mid \psi(gu)^{-1} = 1, \forall u \in U_n \cap H^{(n)}_{p,q} \}.
\]
Then \( S^{(n)}_{p,q} = P_{n-1} \cdot H^{(n)}_{p,q-1} \).

Lemma A.5 ([Mat12 Lemma 3.2]). Let \( p \geq q \geq 1 \) with \( p + q = n \), and let
\[
S^{(n)}_{p,q-1} = \{ g \in G_{n-2} \mid \psi(gu)^{-1} = 1, \forall u \in U_{n-1} \cap H^{(n)}_{p,q-1} \}.
\]
Then \( S^{(n)}_{p,q-1} = P_{n-2} \cdot H^{(n)}_{p-1,q-1} \).

The proofs of both multiplicity one theorems rely on the following propositions:

Proposition A.6. Suppose \( p \geq q \geq 1 \) with \( p + q = n \). Let \((\sigma, V_\sigma)\) be a representation of \( P_{n-1} \). Then there exists an embedding
\[
\text{Hom}_{P_n \cap H^{(n)}_{p,q}}(\Phi^+(\sigma), 1) \hookrightarrow \text{Hom}_{P_{n-1} \cap H^{(n)}_{p,q-1}}(\sigma, 1).
\]

Proposition A.7. Suppose \( p \geq q \geq 1 \) with \( p + q = n \). Let \((\sigma, V_\sigma)\) be a representation of \( P_{n-2} \). Then there exists an embedding
\[
\text{Hom}_{P_{n-1} \cap H^{(n)}_{p,q-1}}(\Phi^+(\sigma), 1) \hookrightarrow \text{Hom}_{P_{n-2} \cap H^{(n)}_{p-1,q-1}}(\sigma, 1).
\]

We prove only Proposition A.6. The proof of Proposition A.7 is similar.

Proof of Proposition A.6. Denote \( W = \Phi^+(\sigma) = \text{Ind}_{P_{n-1}U_n}^{P_n}(\sigma') \), where \( \sigma' = \sigma \otimes \psi \).

Let \( A : \mathcal{S}(P_n, V_\sigma) \to W \) be the projection operator
\[
(Af) (p) = \frac{1}{|P_{n-1}U_n|} \sum_{y \in P_{n-1}U_n} \sigma'(y)^{-1} f(yp).
\]

Let \( L \in \text{Hom}_{P_n \cap H^{(n)}_{p,q}}(\Phi^+(\sigma), 1) \). We can define using \( A \) and \( L \) a distribution \( T = L \circ A : \mathcal{S}(P_n, V_\sigma) \to \mathbb{C} \). One easily checks that this distribution satisfies
\[
\langle T, p(h)f \rangle = \langle T, f \rangle \quad \forall h \in P_n \cap H^{(n)}_{p,q} \tag{A.1}
\]
\[
\langle T, \lambda(y)f \rangle = \langle T, \sigma'(y_0)^{-1} f \rangle \quad \forall y_0 \in P_{n-1}U_n \tag{A.2}
\]

Since \( A \) is onto, we have that the map \( L \mapsto L \circ A \) is an injection from \( \text{Hom}_{P_n \cap H^{(n)}_{p,q}}(\Phi^+(\sigma), 1) \) to the space of all distributions satisfying eq. (A.1) and eq. (A.2).

Let \( \Psi : P_n \to \mathbb{C} \) be the function defined by \( \Psi(ug) = \psi(u) \), where \( u \in U_n \) and \( g \in G_{n-1} \). It is well defined, since \( U_n \cap G_{n-1} = \{I_n\} \). One has \( \Psi(up) = \psi(u) \Psi(p) \) for \( u \in U_n \) and \( p \in P_n \).
Let $T$ be a distribution on $S(P_n, V_σ)$, satisfying eq. (A.1) and eq. (A.2). We denote by $Ψ · T$ the distribution defined by $⟨Ψ · T, f⟩ = ⟨T, Ψ · f⟩$. One easily checks using eq. (A.2) that
\[
⟨λ (u) (Ψ · T), f⟩ = ⟨Ψ · T, f⟩,
\]
for every $u ∈ U_n$. Define for $f : P_n → V_σ$, $f' : G_{n-1} → V_σ$ by
\[
f'(g) = \sum_{u ∈ U_n} f(ug),
\]
and define a distribution $S$ on $S(G_{n-1}, V_σ)$ by defining it on the basis by $⟨S, δ_{g_0}⟩ = ⟨Ψ · T, δ_{g_0}⟩$, for every $g_0 ∈ G_{n-1}$. Then one easily checks using eq. (A.3) that $⟨S, f'⟩ = ⟨Ψ · T, f⟩$, for every $f ∈ S(G_{n-1}, V_σ)$. It follows that $⟨Ψ · T, ρ (u_0) f⟩ = ⟨Ψ · T, f⟩$, for every $f ∈ S(P_n, V_σ)$ and $u_0 ∈ U_n$.

Suppose that $g_0 ∈ G_{n-1}$ is in the support of $S$, i.e. $⟨S, δ_{g_0}⟩ ≠ 0$. Then for every $u_0 ∈ U_n ∩ H_{p,q}^{(n)}$, we have
\[
Ψ (g_0) ⟨T, δ_{g_0}⟩ = ⟨Ψ · T, δ_{g_0}⟩ = ⟨Ψ · T, ρ (u_0) δ_{g_0}⟩ = Ψ (g_0u_0^{-1}) ⟨T, δ_{g_0}⟩.
\]
Since $⟨S, δ_{g_0}⟩ ≠ 0$, we get that $⟨T, δ_{g_0}⟩ ≠ 0$, and therefore $Ψ (g_0u_0^{-1}) = Ψ (g_0)$, which implies that $ψ (g_0u_0^{-1}g_0^{-1}) = 1$. Thus we have that $S ⊆ S_{p,q}^{(n)}$, and by Lemma A.4, $suppS ⊆ P_{n-1} · H_{p,q-1}^{(n)}$, and hence $suppT ⊆ U_n P_{n-1} H_{p,q-1}^{(n)} = P_{n-1} U_n H_{p,q-1}^{(n)}$. Hence the restriction map $T → T' |_{S(P_{n-1} U_n H_{p,q-1}^{(n)}, V_σ)}$ from the space of distributions on $S(P_n, V_σ)$ satisfying eq. (A.1) and eq. (A.2) to the space on distributions of $S(P_{n-1} U_n H_{p,q-1}^{(n)}, V_σ)$ satisfying eq. (A.1), eq. (A.2) is injective.

Consider the projection $B : S(P_{n-1} U_n × H_{p,q-1}^{(n)}, V_σ) → S(P_{n-1} U_n H_{p,q-1}^{(n)}, V_σ)$ given by
\[
(Bf) (y^{-1}h) = \frac{1}{|P_{n-1} ∩ H_{p,q-1}^{(n)}|} \sum_{a ∈ P_{n-1} ∩ H_{p,q-1}^{(n)}} f(ay, ah),
\]
for $y ∈ P_{n-1} U_n$, $h ∈ H_{p,q-1}^{(n)}$. This is well defined, and $B$ is a projection in the sense that if $f (y, h) = g (y^{-1}h)$ for $g ∈ S(P_{n-1} U_n H_{p,q-1}^{(n)}, V_σ)$, then $Bf = g$. In particular $B$ is onto.

Consider the linear isomorphism from $S(P_{n-1} U_n × H_{p,q-1}^{(n)}, V_σ)$ to itself, $φ → ˜φ$, given by $˜φ (y, h) = σ' (y)^{-1} φ (y, h)$. Let $y_0 ∈ P_{n-1} U_n$, $h_0 ∈ H_{p,q-1}^{(n)}$, $φ ∈ S(P_{n-1} U_n × H_{p,q-1}^{(n)}, V_σ)$ and denote $φ_1 = ρ (y_0, h_0) φ$. One checks that $˜φ_1 = σ' (y_0) ρ (y_0, h_0) ˜φ$ and that $B (˜φ_1) = ρ (h_0) λ (y_0) σ' (y_0) B (˜φ)$, which implies for a distribution $T$ on $S(P_{n-1} U_n H_{p,q-1}^{(n)}, V_σ)$ satisfying eq. (A.1), eq. (A.2), we have $⟨T, B (˜φ)⟩ = ⟨T, B (˜φ_1)⟩$.

For $T$ as above, we define a distribution $DT$ on the space $S(P_{n-1} U_n × H_{p,q-1}^{(n)}, V_σ)$ by $⟨DT, φ⟩ = ⟨T, B (˜φ)⟩$. The map $T → DT$ is injective, as $B$ is surjective and $φ → ˜φ$ is an
isomorphism. The above discussion shows that \( \rho(y_0, h_0) D_T = D_T \), for every \( y_0 \in P_{n-1} U_n, h_0 \in H_{p,q-1}^{(n)} \). This means that \( D_T \) is determined by the functional \( \xi_T : V_\sigma \to \mathbb{C} \), defined by 
\[
\langle \xi_T, v \rangle = \langle D_T, v \cdot \delta_{(I_n, I_n)} \rangle,
\]
and is given by the formula
\[
\langle D_T, \phi \rangle = \sum_{y \in P_{n-1} U_n} \langle \xi_T, \phi(y, h) \rangle.
\]  
(A.4)

We now show that \( \xi_T \in \text{Hom}_{P_{n-1} \cap H_{p,q-1}^{(n)}}(\pi, 1) \). Let \( \phi \in \mathcal{S}(P_{n-1} U_n \times H_{p,q-1}^{(n)}) \), let \( b \in P_{n-1} \cap H_{p,q-1}^{(n)} \), and let \( \phi_1 = \lambda(b, b) \phi \). Then an easy computation shows that \( \lambda(b, b) \sigma'(b)^{-1} \phi = \tilde{\phi}_1 \). Since \( B(\lambda(b, b) f) = B(f) \), for every \( f \in \mathcal{S}(P_{n-1} U_n \times H_{p,q-1}^{(n)}, V_\sigma) \), we get that 
\[
\langle T, \lambda(b, b) \phi \rangle = \langle T, \sigma'(b)^{-1} \phi \rangle.
\]
It follows from eq. (A.4) that \( \langle T, \lambda(b, b) \phi \rangle = \langle T, \phi \rangle \). Therefore \( \langle \xi_T, v \rangle = \langle \xi_T, \sigma'(b) v \rangle \), for every \( v \in V_\sigma \) and every \( b \in P_{n-1} \cap H_{p,q-1}^{(n)} \), as required.

q.e.d.

We are now able to prove Theorem A.1 and Theorem A.2.

Proof of Theorem A.1 Since \( H_{m,m}^{(2m)} = w_{m,m} M_{m,m} w_{m,m}^{-1} \) and \( P_{2m} = w_{m,m} P_{2m} w_{m,m}^{-1} \), we get that
\[
\text{Hom}_{P_{2m} \cap M_{m,m}}(\pi, 1) \cong \text{Hom}_{P_{2m} \cap H_{m,m}^{(2m)}}(\pi, 1)
\]
by mapping \( L \in \text{Hom}_{P_{2m} \cap H_{m,m}^{(2m)}}(\pi, 1) \) to \( L \tau(w_{m,m}) \in \text{Hom}_{P_{2m} \cap M_{m,m}}(\pi, 1) \). Therefore it suffices to prove that
\[
\dim_\mathbb{C} \text{Hom}_{P_{2m} \cap H_{m,m}^{(2m)}}(\pi, 1) \leq 1.
\]
Since \( \pi \) is an irreducible cuspidal representation, we have from Theorem A.3 that \( \pi \mid_{P_{2m}} \cong (\Phi^+)^{2m-1}(1) \). Using Proposition A.6 and Proposition A.7 repeatedly, and using the fact that \( H_{m-1,m-1}^{(2m)} \cong H_{m-1,m-1}^{(2m-2)} \), one gets an embedding
\[
\text{Hom}_{P_{2m} \cap H_{m,m}^{(2m)}}((\Phi^+)^{2m-1}(1), 1) \hookrightarrow \text{Hom}_{P_{2m} \cap H_{m,m}^{(2m)}}(1, 1).
\]
The last space is one dimensional, and therefore we get the required result. q.e.d.

Proof of Theorem A.2 Let \( \tau \) be the permutation
\[
\begin{pmatrix}
1 & 2 & 3 & \ldots & m & m+1 & m+2 & m+3 & \ldots & 2m+1 \\
1 & 2 & 3 & \ldots & m & 2m+1 & m+1 & m+2 & \ldots & 2m
\end{pmatrix},
\]
and let \( w_\tau \) be the column permutation matrix corresponding to \( \tau \). Then \( L_{2m+1} = w_\tau M_{m+1,m} w_\tau^{-1} \), where \( M_{m+1,m} \) is the standard Levi subgroup corresponding to the partition \( (m+1, m) \). A simple calculation shows that
\[
L_{2m+1} \cap P_{2m+1} = w_\tau \left\{ \begin{pmatrix} p \\ g \end{pmatrix} \mid p \in P_{m+1}, g \in \text{GL}_m(\mathbb{F}) \right\} w_\tau^{-1}.
\]
Similarly, an easy calculation shows that
\[ H_{m+1,m}^{(2m+2)} \cap P_{2m+1} = \left\{ \begin{pmatrix} p & g \\ 1 & 1 \end{pmatrix} \mid p \in P_{m+1}, g \in \text{GL}_m(\mathbb{F}) \right\} \wr_{m+1,m+1} \).

Therefore \( P_{2m+1} \cap H_{m+1,m}^{(2m+2)} \) and \( P_{2m+1} \cap L_{2m+1} \) are conjugate as subgroups of \( \text{GL}_{2m+1}(\mathbb{F}) \) (actually even as subgroups of \( P_{2m+1} \)), which implies that
\[
\text{Hom}_{P_{2m+1} \cap L_{2m+1}}(\pi, 1) \cong \text{Hom}_{P_{2m+1} \cap H_{m+1,m}^{(2m+2)}}(\pi, 1).
\]

Thus it suffices to prove that
\[
\dim \text{Hom}_{P_{2m+1} \cap H_{m+1,m}^{(2m+2)}}(\pi, 1) \leq 1.
\]

As in the previous proof, by Theorem A.3 and by using Proposition A.6 and Proposition A.7 repeatedly, we get an embedding
\[
\text{Hom}_{P_{2m+1} \cap H_{m+1,m}^{(2m+2)}}(\pi, 1) \hookrightarrow \text{Hom}_{P_{2m+1} \cap H_1}(1, 1),
\]
and the statement follows. q.e.d.

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