Quasinormal Modes and Greybody Bounds of Rotating Black Holes in a Dark Matter Halo

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Quasinormal mode (QNM) is a characteristic “sound” for a black hole which can provide us with a new method to verify black holes in our universe. This characteristic “sound” can be represented by a complex frequency, that is, quasinormal frequency (QNF). On the other hand, the study of greybody factors can provide us with important clues for the quantum structure of black holes. Based on these interesting physical background, we study quasinormal modes (QNMs) and greybody factors (GFs) of rotating black holes in a dark matter halo, and make comparisons with the Kerr black hole. The main results we obtained are as follows: QNFs are directly related to the rotation parameter $a$ of black holes. The oscillation frequencies (real part) of black holes both in a dark matter halo and the Kerr spacetime decrease with the increasing of rotation parameter $a$. The decay rates (imaginary part) of QNF increase with the increasing of the rotation parameter $a$. On the other hand, QNFs of black holes increase with the increasing of the angular quantum number $l$ and the magnetic quantum number $m$, respectively. The rotation parameters $a$ and separation constant $L$ of black holes both in a dark matter halo and the Kerr spacetime have a positive contribution to the greybody bounds. The value of greybody bounds decreases with increasing of rotation parameter $a$ and separation constant $L$, respectively. Besides, QNFs of black holes fitted by Pöschl-Teller potential approximation and sixth-order WKB method are in good agreement.

I. INTRODUCTION

Now, there is a large number of observational data indicating the existence of dark matter (DM), such as cosmic microwave background radiation (CMB), galaxy rotation curve (RC) and the large-scale structure of the universe. Based on these observational data, astronomers have proposed lots of dark matter models. Among all the dark matter models, the cold dark matter (CDM) model has received much attention. In these dark matter models, the most important part is to obtain the spatial density distribution function of dark matter. At present, the distribution of dark matter in the large-scale structure of galaxies is clear. However, the distribution of dark matter around supermassive black holes (BHs) is unclear. Therefore, studying the distribution of dark matter around black holes is a very significant project. It is generally believed that the distribution of dark matter around black holes is a big “spike” [5–7]. Fortunately, with our efforts, we have derived black hole (BH) spacetime metrics both in a dark matter halo [8] and dark matter spike [9], respectively, and generalized them to the case of rotation. In addition, through the black hole photos released by the Event Horizon Telescope (EHT) [10] and the observation of gravitational waves by the Laser Interferometer Gravitational Wave Observatory (LIGO) [11–14], the existence of black holes can be basically determined.

On the other hand, the term “black hole” was originally coined by Wheeler [15]. For a black hole, he proposed the famous “No-hair theorem” [16]. The object described by the No-hair theorem is a static isolated black hole. However, in our universe, it is hard to find an isolated black hole, suggesting that there may be various complex matter fields around the black hole, causing the black hole to be in a perturbed state. When a black hole is perturbed by the matter field, its initial perturbation can be represented by a complex frequency of excited oscillation mode, which is so-called “quasinormal mode” (QNM) [17]. This mode is divided into three stages, the second stage is the main mode. This main mode contains the oscillation frequency of a black hole. The real part of this frequency represents the oscillation frequency of the black hole when perturbed, while the imaginary part represents the rate of oscillation, also known as damping [18]. There are different methods in calculating QNM, such as WKB method [19, 20], Pöschl-Teller potential approximation [21–23] and continued fraction method [24, 25]. The quasinormal mode is a characteristic “sound” of a black hole which can provide us with a new method to verify a black hole in our universe [18]. In general, a perturbed black hole emits thermal radiation at its event horizon. This thermal radiation belongs to black-body radiation and it may carry some inherent information of the black hole inside [26]. Therefore, we can study the quantum structure of black holes based on this information. Outside the event horizon is an effective potential which can act like a filter [27]. Some waves can pass through the effective potential and transmit to infinity. Other waves will be reflected back
by the effective potential\cite{28}. In fact, for an observer at infinity, they can receive only a little part of the emitted thermal radiation. It is shown that the radiation emitted from the event horizon is different from the radiation received by the observer. This phenomenon occurs in Hawking radiation and is called “greybody radiation” or “greybody factors” (GFs)\cite{29, 30}. At present, there are many methods for calculating the greybody factors, such as greybody bounds method\cite{31, 32}, the WKB method\cite{33} and the matching technique\cite{34, 35} and so on.

Now, the quasinormal modes of black holes\cite{36–47} and the greybody factors\cite{48–55} have been extensively studied. In the nearest reasrch, the following questions are interesting and important. In Ref.\cite{27}, they study the quasinormal mode and greybody bounds of axisymmetric black holes in the Bumblebee gravitational model. In Ref.\cite{25}, they study the instabilities of rotating black holes in scalar fields. In Ref.\cite{56}, they study in detail the quasinormal modes and greybody bounds of non-rotating black holes in nonlinear electrodynamics.

Besides, with the release of black hole photos, people are also increasingly concerned about the interaction of other celestial bodies (or matter) around the black hole. C. Zhang et al. are interested in the physics related to dark matter around a black hole\cite{57–59}. Cardoso et al. introduced an exact solution for a black hole immersed in a galactic-like distribution of matter and used gravitational perturbations to study the quasinormal modes of this black hole\cite{60}. Based on this, Konoplya studies the matter field perturbations, the graybody factors and the Unruh temperature\cite{61} of the exact solution of this black hole. These works of them are interesting and important. In this paper, we mainly study the scalar field perturbation and greybody bounds of rotating black holes in a dark matter halo, and make comparisons with the Kerr black hole. Research on Kerr black holes can refer to these Refs.\cite{48, 62–64}. The quasinormal mode is a characteristic “sound” for a black hole which can provide us with a new method to verify black holes in our universe. In addition, studies related to the greybody factors can give us important clues about the quantum structure of black holes. This work is an in-depth study based on our previous work\cite{65–67}. Although these previous works are on spherically symmetric black holes, these valuable experiences give us enough confidence to solve the case of axisymmetric black holes.

The paper is organized as follows. In Section II, we introduce an axisymmetric metric of a black hole in a dark matter halo. In Section III, we discuss the scalar field perturbation of this rotating black hole in a dark matter halo, and derive an analytical expression for the effective potential. In Section IV, we introduce the Pöschl-Teller potential approximation and sixth-order WKB method and use it to fit the oscillation frequencies of a rotating black hole under a scalar field. Besides, we make a brief description of the case of extremal black holes. In Section V, we investigate the bounds of greybody factors of rotating black holes in a dark matter halo. Finally, Section VI is our conclusion and outlook. In this paper we use mostly the units ($G = c = 1$).

\section{The Spacetime of Rotating BHS in a Dark Matter Halo}

In this section, we should review rotating black hole metrics we obtained in a dark matter halo\cite{8}. Both of them have the following form in four-dimensional coordinates,

\begin{equation}
    ds^2 = -\left(1 - \frac{r^2 + 2Mr - r^2f(r)}{\Sigma^2}\right)dt^2 + \frac{\Sigma^2}{\Delta}dr^2 + \Sigma^2d\theta^2 + \frac{A\sin^2\theta}{\Sigma^2}d\phi^2 - \frac{2(r^2 + 2Mr - r^2f(r))a\sin^2\theta}{\Sigma^2}d\phi dt,
\end{equation}

and

\begin{equation}
    \Delta = r^2f(r) - 2Mr + a^2 = (r - r_+)(r - r_-), \quad \Sigma^2 = r^2 + a^2\cos^2(\theta), \quad A = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta,
\end{equation}

where, $f(r)$ represents the factor term for considering dark matter.

For cold dark matter (CDM) model, $f(r)$ has the form of the following,

\begin{equation}
    f_c(r) = \left(1 + \frac{r}{R_c}\right)^{-\frac{8\rho_0R_s^2}{\pi}} ,
\end{equation}

and for scalar field dark matter (SFDM) model, $f(r)$ has the form of the following,

\begin{equation}
    f_s(r) = \exp\left(-\frac{8\rho_s R_s^2 \sin(\pi r/R_s)}{\pi \rho_s R_s}ight) ,
\end{equation}

here, $M$ is the total mass of a black hole. $a$ is the rotation parameter of a black hole, and its value is equal to the per unit mass of the angular momentum $J$, that is, $a = J/M$. And $r_+$ and $r_-$ are the outer and inner horizon of a rotating black hole, respectively. The parameter $\rho$ is the density of the universe when the dark matter halo collapses, and $R$ means its characteristic radius in this
halo. If the effect of dark matter on black holes is not considered, that is, $\rho = 0$, then the dark matter term will become $f(r) = 1$. The black hole metric in a dark matter halo will degenerate into the Kerr metric we all know. If the rotation parameter $a = 0$ at this time, we can obtain the famous Schwarzschild black hole metric.

On the other hand, Eq.(2) reveals the distribution of singularities and energy layers of a rotating black hole in the halo. When $\Delta = 0$ and $\Sigma = 0$. It shows a ring-shaped singularity in the equatorial plane at the center of a rotating black hole of radius $a$.

To make our expression more compact, we continue to use $f(r)$ to represent the dark matter term. Then, we can obtain a covariant metric tensor from the Eq.(1),

$$\Psi(t, r, \theta, \phi) = e^{-i\omega t} e^{im\phi} R(r) S(\theta),$$

where, $m$ is the magnetic quantum number and $\omega$ is the energy of the particle. Therefore, the part of the radial singularities and energy layers of a rotating black hole in the halo. When $\Delta = 0$ and $\Sigma = 0$. It shows a ring-shaped singularity in the equatorial plane at the center of a rotating black hole of radius $a$.

III. SCALAR PERTURBATION OF ROTATING BHS IN A DARK MATTER HALO

In this section, we will study the scalar perturbation of rotating black holes (BHs) and derive the effective potential of scalar particles in a dark matter halo. We know that in a curved spacetime, the equation of the motion can be written as the following, 

$$g_{\mu\nu} = \begin{pmatrix} - (1 - \frac{r^2 + 2Mr - r^2f(r)}{\Sigma^2}) & 0 & \frac{\mu}{\Sigma} & 0 \\
0 & \frac{\mu}{\Sigma} & 0 & 0 \\
\frac{a \sin^2(\theta)(r^2 + 2Mr - r^2f(r))}{\Sigma^2} & 0 & A \sin^2(\theta) & 0 \\
0 & 0 & 0 & \frac{A \sin^2(\theta)}{\Sigma^2} \end{pmatrix}. \tag{5}$$

With the Eq.(4), we can calculate the determinant of this metric,

$$g = \det(g_{\mu\nu}) = -\Sigma^4 \sin(\theta)^2. \tag{6}$$

From Eqs.(4) and (5), we get the contravariant form of the metric,

$$g^{\mu\nu} = \begin{pmatrix} - \frac{A}{\Delta \Sigma^2} & 0 & \frac{\Delta}{\Sigma^2} & 0 \\
0 & \frac{\Delta}{\Sigma^2} & 0 & 0 \\
0 & 0 & \frac{1}{\Delta \Sigma^2} & 0 \\
0 & 0 & 0 & \frac{1}{\Delta \Sigma^2 \sin^2(\theta)} \end{pmatrix}. \tag{7}$$

of scalar particles can be described by the Klein-Gordon equation,

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \Psi) = \mu_0^2 \Psi, \tag{8}$$

where $\mu_0$ is the mass of the scalar particle. From Eqs.(6) and (7), we can get the following form,

$$- \frac{A}{\Delta \Sigma^2} \partial_t^2 \Psi + \frac{ar(r(r - 2rf) - 2M - r)}{\Delta \Sigma^2} \partial_\theta \partial_\phi \Psi + \frac{1}{\Sigma^2 \sin(\theta)} \partial_\theta (\sin(\theta) \partial_\theta) \Psi + \frac{1}{\Sigma^2} \partial_r (\Delta \partial_r) \Psi + \frac{-2r^2f(r) - 2M - r + \Sigma^2 + r^2f(r)}{\Delta \Sigma^2 \sin^2(\theta)} \partial_\phi^2 \Psi = \mu_0^2 \Psi. \tag{9}$$

Eq.(9) is a complex second-order partial differential equation, and this equation can be separated by the following substitution,

$$\Psi(t, r, \theta, \phi) = e^{-i\omega t} e^{im\phi} R(r) S(\theta), \tag{10}$$

where, $m$ is the magnetic quantum number and $\omega$ is the energy of the particle. Therefore, the part of the radial equation can be written as the following,

$$\frac{1}{R(r)} \frac{d}{dr} (\Delta \frac{dR(r)}{dr}) + \frac{\omega^2(r^2 + a^2)^2 + m^2a^2}{\Delta} - \frac{4amMr\omega}{\Delta} - \frac{2amr^2 \omega(1 - f(r))}{\Delta} - \omega^2a^2 - \mu_0^2 r^2, \tag{11}$$
and the angular equation can be written as
\[
\frac{1}{S(\theta) \sin(\theta)} \frac{d}{d\theta}(\sin(\theta) \frac{dS(\theta)}{d\theta}) - \frac{m^2}{\sin(\theta)^2} + a^2(\omega^2 - \mu_0^2) \cos(\theta)^2.
\] (12)

In general, we usually use the method of variable separation to deal with these two equations. Therefore, the value of Eqs.(11) and (12) can be written as a constant whose absolute value is equal. In order to make our calculation more convenient, we set this constant to \(L\). This constant also represents an important physical quantity, the angular quantum number \(l\). In Ref.[51], a new way is to the separation constant \(L\) as a function of rotation parameter \(a\) and \(\omega\), which can be solved by the Heun function. Here, for simplicity, we only set \(L = l(l + 1)\). Then, Eq.(11) can be written as
\[
\Delta \frac{d}{dr}(\Delta \frac{dR(r)}{dr}) + |\omega^2(r^2 + a^2)^2 + m^2a^2 - 4amMr\omega - 2amr^2\omega(1 - f(r)) - (\omega^2a^2 + \mu_0^2r^2 + L)\Delta]R(r) = 0.
\] (13)

For the perturbation theory of the black hole, the essence of the radial equation can be simplified as a wave equation. The solution of this equation is generally related to the oscillating mode of the matter field, that is, the quasinormal mode(QNM). The quasinormal mode is a mode with complex frequencies. Its real part is the oscillation frequency of a black hole and the imaginary part is the decay rate of this oscillation.

Next, we will derive the wave equation from the radial equation. Before that, one more transformation is required, we can set
\[
R(r) = \frac{U(r)}{\sqrt{r^2 + a^2}}.
\] (14)

At the same time, we introduce tortoise coordinates \(r_*\),
\[
dr_* = \frac{r^2 + a^2}{\Delta} dr.
\] (15)

Thus, we get the one-dimensional wave equation, which has the form of the Schrödinger-like equation,
\[
\frac{d^2U(r)}{dr_*^2} + \omega^2 - V_{\text{eff}}(r) = 0,
\] (16)
where, \(V_{\text{eff}}\) is the effective potential. For CDM model, it has the following form,
\[
V_{\text{eff}} = \frac{\Delta}{(a^2 + r^2)^2} \left( \frac{r}{a^2 + r^2} + \frac{\Delta}{a^2 + r^2} - \frac{3r^2 \Delta}{(a^2 + r^2)^2} \right) - \frac{m^2a^2}{\Delta} + \frac{4amMr\omega}{\Delta} + \frac{1 - f_c(r)}{\Delta/2am\omega r^2} + \left( \mu_0^2r^2 + \omega^2a^2 + L \right),
\] (17)

and for SFDM model, the effective potential reads,
\[
V_{\text{eff}} = \frac{\Delta}{(a^2 + r^2)^2} \left( \frac{r}{a^2 + r^2} + \frac{\Delta}{a^2 + r^2} - \frac{3r^2 \Delta}{(a^2 + r^2)^2} \right) - \frac{m^2a^2}{\Delta} + \frac{4amMr\omega}{\Delta} + \frac{1 - f_c(r)}{\Delta/2am\omega r^2} + \left( \mu_0^2r^2 + \omega^2a^2 + L \right),
\] (18)

where, \(\Delta'\) denotes \(d\Delta/dr\). And \(f(r)\) is the factor term of the dark matter of the corresponding dark matter models. In Eqs.(17) and (18), it is not difficult to find that when \(f(r) = 1\), black holes in a dark matter halo can degenerate into Kerr black holes.

The behavior of the effective potentials in scalar field is shown in Fig.1. It shows that there is a significant increase in the potential peak when \(L\) increases. Fig.2 shows that the effective potential decrease with the rotation parameter \(a\) increase. In addition, we also compare the magnitude of the effective potential of black holes in a dark matter halo and Kerr black hole under the same set of parameters, as shown in Figs.3 and 4. The results show that in the evolution of the effective potential and \(r\), the value of the effective potential of the Kerr black hole is slightly lower than that of dark matter halo models. And the effective potential of a black hole in SFDM model is large then that of the CDM model. The corresponding dark matter parameters we used are selected from the galaxy ESO 1200211 in these Refs.[68, 69].

On the other hand, there is a special case in rotating black holes where the rotating black hole transitions to the extremal black hole. An extremal black hole can be obtained by continuously increasing the value of the rotation parameter \(a\)[70]. Extremal black holes appear mathematically as the coincidence of their inner and outer horizons, and thermodynamically their Hawking temperature is zero[64]. In the next section, we will also introduce extremal black holes in a dark matter halo.
FIG. 1. The effective potentials of a black hole in CDM model (Left panel), SFDM model and Kerr spacetime (Right panel) vary with different \( L \) respectively. And the calculation parameters are \( m = M = 1, R_c = 5.7, \rho_c = 0.00245, R_s = 2.92, \rho_s = 0.01366, a = 0.3, \omega = 15, \mu = 0.5 \).

FIG. 2. The effective potentials of a black hole in CDM model (Left panel), SFDM model and Kerr spacetime (Right panel) vary with different \( a \) respectively. And the calculation parameters are \( m = M = 1, R_c = 5.7, \rho_c = 0.00245, R_s = 2.92, \rho_s = 0.01366, a = 0.3, \omega = 15, L = 12, \mu = 0.5 \).

FIG. 3. Comparison of black holes effective potentials both in a dark matter halo and Kerr spacetime with \( a = 0.2 \) (Left panel), \( a = 0.3 \) (Right panel) . And the calculation parameters are \( m = M = 1, R_c = 5.7, \rho_c = 0.00245, R_s = 2.92, \rho_s = 0.01366, L = 12, \omega = 15, \mu = 0.5 \).

IV. THE METHODS FOR CALCULATING QNFS AND EXTREMAL BLACK HOLES IN A DARK MATTER HALO

In this section, we will introduce the methods for calculating the frequencies of the quasinormal modes (QNMs) of rotating black holes (BHs) in a dark matter halo. The rotating black hole is different from the general spherically symmetric black hole because its effective potential contains the frequency \( \omega \) and the rotation parameter \( a \).

Next, we will choose two methods to obtain the quasinormal frequencies (QNFs) of black holes. In order to understand this process, we discuss the quasinormal mode frequencies of rotating black holes by the Pöschl-Teller potential approximation and sixth-order WKB method. The results of these two methods can be used to check the accuracy of the frequencies of rotating black holes in a dark matter halo. Besides, we also briefly introduce the transition of a black hole to an extremal black hole in a dark matter halo in this section.
Thus, Eq.(19) can be rewritten in the following form,

$$\frac{d^2\psi}{dr^2_*} + (\omega^2 - V(r_*,\alpha))\psi = 0,$$  \hspace{1cm} (19)

and its boundary conditions are

$$\psi = e^{\pm i\omega r_*}, r_* \to \pm \infty.$$

Then, we introduce the following transformation,

$$r_* \to -ir_*, \quad p \to p'$$  \hspace{1cm} (21)

For simplicity, we set $p = (V_0,\alpha)$ and $p' = (V_0, i\alpha)$. Under this transformation, it can ensure the effective potential unchanged,

$$V(r_*,\alpha) = V(-ir_*,\alpha').$$  \hspace{1cm} (22)

Then, we can obtain the following form,

$$\psi(r_*,\alpha) = \Psi(-ir_*,\alpha'), \quad \omega(\alpha) = \Omega(\alpha').$$  \hspace{1cm} (23)

Thus, Eq.(19) can be rewritten in the following form,

$$\frac{d^2\Psi}{dr^2_*} + (-\Omega^2 + V)\Psi = 0,$$  \hspace{1cm} (24)

Then the quasinormal frequencies can be obtained from \(\Omega_n(V_0,\alpha)\). The corresponding frequencies are given by the following form,

$$\omega_n(V_0,\alpha) = \Omega_n(V_0, i\alpha) = \alpha[\pm \sqrt{\frac{V_0}{\alpha^2} - \frac{1}{4}(n + \frac{1}{2})^2}],$$  \hspace{1cm} (28)

where, \(n\) is the overtone number and \(n = 0, 1, 2, \ldots\). Eq.(28) may be an analytical formula for the quasinormal frequencies of black holes in a dark matter halo. However, this analytical form seems to be general, and it is also used for rotating Kerr black holes[22] and Myers-Perry-de Sitter Black Holes[47].

A. The Pöschl-Teller potential approximation

In this part, we introduce the Pöschl-Teller approximation method. This method was first used by Bahram Mashhoon for calculating the QNFs of the Schwarzschild black holes and later it was used to calculate the case of rotation for Kerr black holes[22]. Next, we will use this method documented in Refs.[18, 22], and we start from the wave equation first,

$$\frac{d^2\psi}{dr^2_*} + (\omega^2 - V(r_*,\alpha))\psi = 0,$$  \hspace{1cm} (19)

and its boundary conditions are

$$\psi = e^{\pm i\omega r_*}, r_* \to \pm \infty.$$  \hspace{1cm} (20)

Then, we introduce the following transformation,

$$r_* \to -ir_*, \quad p \to p'$$  \hspace{1cm} (21)

For simplicity, we set $p = (V_0,\alpha)$ and $p' = (V_0, i\alpha)$. Under this transformation, it can ensure the effective potential unchanged,

$$V(r_*,\alpha) = V(-ir_*,\alpha').$$  \hspace{1cm} (22)

Then, we can obtain the following form,

$$\psi(r_*,\alpha) = \Psi(-ir_*,\alpha'), \quad \omega(\alpha) = \Omega(\alpha').$$  \hspace{1cm} (23)

Thus, Eq.(19) can be rewritten in the following form,

$$\frac{d^2\Psi}{dr^2_*} + (-\Omega^2 + V)\Psi = 0,$$  \hspace{1cm} (24)

Thus, Eq.(19) can be rewritten in the following form,

$$\frac{d^2\Psi}{dr^2_*} + (-\Omega^2 + V)\Psi = 0.$$

B. Sixth-order WKB method

To further verify the accuracy of our results, we will use sixth-order WKB method. It has the following form,

$$\frac{i(\omega^2 - V_0)}{\sqrt{-2V_0}} - \sum_{i=2}^{6} A_i = n + \frac{1}{2}, \quad (n = 0, 1, 2, \ldots).$$  \hspace{1cm} (29)
Among them, the correction term $\Lambda_i$ can be obtained in Refs. [19, 20]. $V_0$ is the maximum effective potential of $V$ at the tortoise coordinate $r_*$ and $n$ is the overtone number. The WKB method was first obtained from Schutz and Will [19]. They have extended the WKB method to 3rd [19] and 6th order [20]. In Ref. [71], they developed a formula for the 13th order WKB method.

C. The extremal black holes in a dark matter halo

In this part, we will discuss an interesting case: the extremal black holes in a dark matter halo. Extremal black holes can be achieved by continuously increasing the rotation parameter $a$ [64]. For a black hole, it appears as the coincidence of the inner and outer horizons, and thermodynamically, it appears as Hawking temperature equals to zero [70]. Here, we will give our conclusion by series expansion. First, let’s discuss the case of the CDM model. With Eq. (2), the inner and outer horizons have the following form after series expansion:

$$ r_\pm = e^{8\pi \rho R^2} \left( M \pm \sqrt{M^2 - a^2 e^{-8\pi \rho R^2}} \right). \quad (30) $$

For an extremal black hole in the CDM model, $M = a e^{-4\pi \rho R^2}$, the inner and outer horizons are coincide,

$$ r_+ = r_- = r_h = Me^{8\pi \rho R^2}. \quad (31) $$

The angular velocity of an extremal black hole horizon in CDM model is

$$ \Omega_C = \frac{a}{a^2 + r_+^2} = \frac{1}{a \left(e^{8\pi \rho R^2} + 1\right)}. \quad (32) $$

On the other hand, for a case of SFDM model,

$$ r_\pm = e^{\frac{8\pi \rho R^2}{a}} \left( M \pm \sqrt{M^2 - a^2 e^{-8\pi \rho R^2}} \right). \quad (33) $$

For an extremal black hole in the SFDM model, $M = a e^{-\frac{4\pi \rho R^2}{a}}$, the inner and outer horizons are coincide,

$$ r_+ = r_- = r_h = Me^{\frac{8\pi \rho R^2}{a}}. \quad (34) $$

The angular velocity of an extremal black hole horizon in the SFDM model is

$$ \Omega_S = \frac{a}{a^2 + r_+^2} = \frac{1}{a \left(e^{8\pi \rho R^2} + 1\right)}. \quad (35) $$

As can be seen from Eqs. (32) and (35), when the density of the dark matter halo equals to zero, the results will degenerate into an extremal Kerr black hole [64]. On the other hand, according to the superradiance theory, the superradiation frequencies of black holes in the CDM model and the SFDM model should be limited to the following interval

$$ \mu < \omega < \omega_0 = m\Omega, \quad (36) $$

where, $\omega, \mu, m$ are the energy, rest mass and magnetic quantum number of the particle, respectively.

On the other hand, from Eqs. (17) and (18), we can obtain the approximate behavior of the effective potential of extremal black holes in the dark matter halo. When $f(r) = 1$ in the above equation, it is the effective potential corresponding to the extremal Kerr black hole. We add the effective potentials of the extremal black holes in a dark matter halo in the following figure.

It can be seen from Fig. 5 that the effective potential of extremal black holes is different from that of general black holes in Figs. 1 and 2. With the increase of the angular quantum number $L$, the negative peak of the effective potential of extremal the black hole gradually disappears. On the other hand, in extremal black holes, the peak value of the effective potential of the Kerr black hole is larger than that of the SFDM model, and the values of SFDM model is larger than CDM model.

Next, we present the extreme black hole cases in Tables I-VI and V, and we denote these results by the symbol “ * ”. Our results are based on a black hole both in a dark matter halo and Kerr spacetime with equal mass $(M = 1)$. Under this premise, we calculated that the rotation parameter $a$ of the extremal black hole in a dark matter halo is greater than that of the extremal Kerr black hole ($a = 1$), and the extremal black hole in CDM model ($a = 2.71907$) is greater than that of the SFDM model $(a = 1.15986)$. Based on these parameters, we calculated Quasinormal frequencies (QNFs) of the modes $(l = 1, m = 0), (l = 2, m = 0), (l = 1, m = 1), (l = 2, m = 2)$ and the results are organized into the following tables (Tables I-VI).

To ensure the accuracy of our method, we first need to check the QNFs of Kerr black hole in scalar field. Our results are in good agreement with the data recorded in Refs. [63, 72]. This shows that our methods can provide an important guarantee for the calculation of QNFs of black holes in the dark matter halo.

Now, let’s return to the discussion of QNFs for black holes. QNFs is directly related to the rotation parameter $a$ of a black hole. It can be seen from the data of Tables I-VI that the oscillation frequencies (real part) of black holes both in a dark matter halo and the Kerr spacetime decrease with the increasing of the rotation parameter $a$. The decay rates (imaginary part) of QNF increase with the increasing of the rotation parameter $a$. QNFs of black holes in a dark matter halo are lower than that of the Kerr black hole because of the interaction of dark matter with a black hole. And QNFs of a black hole in cold dark matter (CDM) model are lower than scalar field dark matter (SFDM) model. Besides, our data also show that the decay rate of the Kerr black hole is smaller than that of a black hole in a dark matter halo. This shows that a black hole in a dark matter halo will decay faster than the general black hole in the process of perturbation, which may be one of the reasons why it is difficult to be detected the dark matter at present. Besides, QNFs of black holes both in a dark matter halo
and the Kerr spacetime increase with the increasing of the angular quantum number $l$ and the magnetic quantum number $m$, respectively.

On the other hand, from the trends of data of QNFs, the QNFs are to a certain extent related to the limit of the rotation parameter $a$ of the black hole. As can be seen from Tables I-VI, the QNFs decreases with increasing of rotation parameter $a$, and the decay rate increases with increasing of $a$. For the QNFs in CDM model and SFDM model, we find that the larger the limit of the rotation parameter $a$, the smaller its frequency and the faster the decay rate. Therefore, due to the limit of the rotation parameter $a$ of the Kerr black hole is smaller than that of a dark matter halo, its QNFs should be greater than that of black holes in a dark matter halo. In fact, we got the same result. Besides, the QNFs of black holes...
Table V. The frequencies of quasinormal modes in massless scalar field in SFDM model with different \(a\) by Pöschl-Teller potential approximation and sixth-order WKB method.

\[
\begin{array}{c|cc|cc}
\hline
\text{a} & \text{Pöschl-Teller} & \text{WKB method} & \text{Pöschl-Teller} & \text{WKB method} \\
\hline
0.05 & 0.251721 - 0.074021i & 0.247929 - 0.072238i & 0.416499 - 0.072396i & 0.414828 - 0.071761i \\
0.10 & 0.251293 - 0.073790i & 0.247543 - 0.072120i & 0.415799 - 0.072173i & 0.414344 - 0.071554i \\
0.30 & 0.246792 - 0.071384i & 0.243466 - 0.069944i & 0.408437 - 0.069583i & 0.406601 - 0.069403i \\
0.50 & 0.235545 - 0.065833i & 0.235545 - 0.065833i & 0.394282 - 0.065523i & 0.393346 - 0.065343i \\
0.70 & 0.225985 - 0.060979i & 0.224208 - 0.060239i & 0.374356 - 0.059734i & 0.374439 - 0.059829i \\
0.90 & 0.211132 - 0.054308i & 0.210039 - 0.053780i & 0.349970 - 0.053172i & 0.350927 - 0.053468i \\
1.00 & 0.202972 - 0.050924i & 0.202113 - 0.050449i & 0.336550 - 0.049824i & 0.337827 - 0.050181i \\
1.15 & 0.190132 - 0.046021i & 0.189467 - 0.045578i & 0.315399 - 0.044954i & 0.316980 - 0.045359i \\
1.15986^* & 0.189270 - 0.045710i & 0.188612 - 0.045269i & 0.313978 - 0.044654i & 0.315572 - 0.045048i \\
\hline
\end{array}
\]

Table VI. The frequencies of quasinormal modes in massless scalar field in SFDM model with different \(a\) by Pöschl-Teller potential approximation and sixth-order WKB method.

\[
\begin{array}{c|cc|cc}
\hline
\text{a} & \text{Pöschl-Teller} & \text{WKB method} & \text{Pöschl-Teller} & \text{WKB method} \\
\hline
0.05 & 0.251643 - 0.074505i & 0.252282 - 0.072448i & 0.416336 - 0.072438i & 0.422657 - 0.071916i \\
0.10 & 0.250984 - 0.073903i & 0.256343 - 0.072351i & 0.415152 - 0.072340i & 0.430823 - 0.071853i \\
0.15 & 0.249892 - 0.073659i & 0.260226 - 0.072102i & 0.413193 - 0.072176i & 0.438767 - 0.071635i \\
0.20 & 0.248379 - 0.073319i & 0.263918 - 0.071697i & 0.410486 - 0.071943i & 0.446465 - 0.071259i \\
\hline
\end{array}
\]

FIG. 5. The effective potentials of the nearly extremal black holes in the CDM model (Left panel), the SFDM model and the Kerr spacetime (Right panel). And the calculation parameters are \(m = M = 1, \rho_c = 5.7, \rho_s = 0.00245, a_c = 2.71907, \omega_c = 0.0438173, R_h = 2.92, \rho_h = 0.01366, a_s = 1.15986, \omega_s = 0.367625, a_h = 1, \omega_h = 0.5\) and \(\mu_0 = 0\).

both in a dark matter halo and Kerr spacetime fitted by the Pöschl-Teller potential approximation and sixth-order WKB method are in good agreement.

V. GREYBODY BOUNDS OF ROTATING BHs IN A DARK MATTER HALO

In this section, we will calculate the bounds of the greybody factors (GFs) of rotating black holes in a dark matter halo. The study of greybody factors can provide us with important clues for the quantum structure of black holes. Next, we will strictly follow the method documented in Refs.[31, 32]. In general, the bounds of greybody factors \(T_h\) can be written as

\[
T_h \geq \text{sech}^2 \left( \int_{-\infty}^{+\infty} vdr_*, \right),
\]

where \(v\) is a function

\[
v = \sqrt{h'(r_*) + (\omega^2 - V(r_*) - h^2(r_*))^2} \times 
\]

\[
2h(r_*). \tag{38}
\]

In Ref.[32], \(h\) is a positive function of \(r_*\). In order to make our equations more universal, we will take the same initial parameters of the first case discussed in page 2 in Ref.[32]. So, we set \(h\) in the Eq.(38) as \(h(r_*) = \omega\), and the boundary conditions are satisfied with \(h(-\infty) = h(\infty) = \omega\).

On the other hand, \(dr_*\) in Eq.(37) is the tortoise coordinate. The function of this coordinate is that the event horizon \(r_h\) can be moved to \((r_* \to -\infty)\). Therefore, the integral lower bound of the greybody factors in Eq.(37) can be replaced by \(r_h\). Then we obtain

\[
T_h \geq \text{sech}^2 \left( \int_{r_h}^{+\infty} \frac{V_{eff}}{2\omega} dr_* \right). \tag{39}
\]
Next, we will calculate the bounds of greybody factors in the CDM model. Using Eq.(15) into Eq.(39), it means that using \( dr \) replace the \( dr_\ast \), we can obtain the following form,

\[
T_b \geq \text{sech}^2\left(\frac{1}{2\omega} \int_{r_h}^{\infty} \frac{1}{a^2 + r^2} \left[ \frac{\Delta r}{a^2 + r^2} + \frac{\Delta}{a^2 + r^2} - \frac{3r^2\Delta}{(a^2 + r^2)^2} \right] \right.
\]

\[
- \frac{m^2a^2}{\Delta} + \frac{4amMr\omega}{r_h^2} + \frac{2am\omega(r^2 - r^2(1 + \frac{r}{R}))}{\Delta} \left( \frac{\pi r_{\ast}^2}{r_h^2} \right) + \left( \mu^2 + \omega^2a^2 + L \right)dr).
\]

After expanding the integral of Eq.(40) using series and considering the case of massless \( \mu = 0 \), we obtain

\[
T_b \geq \text{sech}^2\left(\frac{1}{2\omega} \left( \frac{a^2\omega^2 + L}{r_h} + \frac{8\pi am\rho_s R^3_{\ast}\omega}{r_h} \log (r_h) \right) \right.
\]

\[
+ \frac{2amMr\omega}{r_h^2} + \frac{4\pi am\rho_s R^3_{\ast}\omega - 8\pi am\rho_s R^3_{\ast}\omega \log (R_c)}{r_h^2}
\]

\[
+ \frac{4\pi \rho_s R^3_{\ast}\log (r_h)}{r_h^2} - \frac{4\pi \rho_s R^3_{\ast}\log (R_c)}{r_h^2} \right). \quad (40)
\]

Similarly, the bounds of the greybody factors in the SFDM model have the following form,

\[
T_b \geq \text{sech}^2\left(\frac{1}{2\omega} \left( \frac{-3a^2\omega^2 - 6am\omega e^{\frac{\omega}{a}} + 6am\omega - 3L}{3r_h} \right)
\]

\[
- \frac{6am\omega e^{\frac{16\rho_s R^2}{a}}}{3r_h^2} - \frac{a^4\omega^2}{3} - \frac{2a^3m\omega}{3r_h^2} - \frac{a^2L}{3} + a^2m^2e^{\frac{8\rho_s R^2}{a}} - \frac{3a^2e^{\frac{8\rho_s R^2}{a}}}{3r_h^3} + \frac{2a^2}{3r_h^3}
\]

\[
- \frac{8amM^2\omega e^{\frac{24\rho_s R^2}{a}}}{3r_h^4} \right) \} \right). \quad (41)
\]

When \( \rho = 0 \) in Eqs.(41) and (42), that is, without the interaction of dark matter, the greybody factors \( T_b \) degenerates to the case of Kerr spacetime. In Figs.6 and 7, we present the behavior of the greybody factors of a rotating black hole in a dark matter halo. To the right of each figure is the greybody factors behavior of the corresponding the Kerr black hole. The behavior of the greybody factors of these three black holes is roughly the same. From the figures, we can see that as the parameter \( L \) increases, the value of \( T_b \) decreases significantly. Similarly, for rotation parameter \( a \), the conclusion is the same as above. The difference is that the change trend of the graybody factors is not as obvious as in Fig.6. On the other hand, from the characteristics shown in these figures, when the energy \( \omega \) of the particle is smaller, the value of the greybody bounds will be lower. This means that the smaller energy, the less energy the excited particle can acquire to complete the tunneling effect, and will be reflected back by the effective potential. As the energy increases, some particles can pass through the effective potential and transmit to infinity.

VI. CONCLUSIONS AND DISCUSSIONS

The quasinormal mode is a characteristic “sound” for a black hole which can provide us with a new method to identify black holes in our universe. When a black hole is perturbed by a matter field, the perturbation process can be described by a wave equation. And the quasinormal mode is the solution of the wave equation for excited particles in the scalar field. On the other hand, studies related to the greybody factors can provide us with important clues about the quantum structure of black holes.

In this paper, we mainly study the quasinormal modes and greybody factors bounds of black holes in a dark matter halo, and make comparisons with the Kerr black hole. Our first work was to solve the equation of motion of black holes under a dark matter halo in the scalar field, and we present an analytical expression of the effective potential. Then, we give their quasinormal frequencies by using the Pöschl-Teller approximation method and the WKB method. Finally, we calculate the bounds of the greybody factors for black holes in a dark matter halo and the Kerr black hole. The conclusions we obtain are as follows:

1. On the whole, the peak value of a black hole effective potential in a dark matter halo is larger than that of the Kerr black hole, and the effective potential increases with the increase of the separation constant \( L \) in Fig.1. The difference is that in the comparison of rotation parameter \( a \), the peak value of the effective potential decreases with the increase of \( a \) in Fig.2. For the case of extremal black holes, the peak value of the effective potential of the extremal Kerr black hole is larger than that of SFDM model, and the SFDM model is larger than the CDM model, which is opposite to the result of nonextremal black holes. On the other hand, the negative peak of the effective potential of extremal black holes will slowly disappear as the angular quantum number \( L \) increases, leaving only one main peak in the end in Fig.5.

2. Quasinormal frequencies (QNFs) are directly related to the rotation parameter \( a \) of a black hole. The oscillation frequencies (real part) of black holes both in a dark matter halo and the Kerr spacetime decrease with the increasing of the rotation parameter \( a \). Black holes in a dark matter halo oscillate less frequently than the Kerr black hole. On the other hand, the decay rates (imaginary part) of QNF increase with the increasing of the rotation parameter \( a \). The quasinormal frequencies (QNFs) of black holes in a dark matter halo are lower than that of the Kerr black hole because of the interaction of dark matter with a black hole. And QNFs of a black hole in
At last, QNFs of black holes fitted by the Pöschl-Teller potential approximation and sixth-order WKB method are in good agreement. At last, QNFs of black holes both in a dark matter halo and the Kerr spacetime increase with the increasing of the angular quantum number $l$ and the magnetic quantum number $m$, respectively. At last, QNFs of black holes fitted by the Pöschl-Teller potential approximation and sixth-order WKB method are in good agreement.

3. The rotation parameter $a$ and separation constant $L$ of black holes both in a dark matter halo and the Kerr spacetime have a positive contribution to the greybody bounds. The values of greybody bounds decrease with increasing rotation parameter $a$ and separation constant $L$ in Figs. 6 and 7. The difference is that the change trend of the graybody factors is not as obvious as in Fig. 6. On the other hand, from the characteristics shown in these figures, when the energy $\omega$ of the particle is smaller, the value of the greybody factors bounds will be lower. This means that the smaller energy, the less energy the excited particle can acquire to complete the tunneling effect, and then will be reflected back by the effective potential. As the energy increases to a certain value, some particles can pass through the effective potential and transmit to infinity.

At last, it is also worth mentioning that we are very interested in the exploration of the relevant physical processes of black holes around the dark matter. Based on this premise, we choose to study the quasinormal mode of a rotating black hole in a dark matter halo. The quasinormal mode of a black hole is a characteristic “sound” which can provide us with a new method to verify black holes in the universe. On the other hand, in some recent studies, we found some discussion about echoes[73–78]. These studies show that the echoes appear after the quasinormal mode. It is a quantum correction of the event horizon of a black hole. Echoes are one of the important means currently used to test gravitational waves. Our next plans are to study the physics of rotating black holes associated with echoes. About the study of echoes, we have already made some preliminary attempts[79]. To sum up, we also hope that our work can form a complete research system in the direction of interaction between dark matter and black holes.

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