On the correlation measure of a family of commuting Hermitian operators with applications to particle densities of the quasi-free representations of the CAR and CCR

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Abstract

Let $X$ be a locally compact, second countable Hausdorff topological space. We consider a family of commuting Hermitian operators $a(\Delta)$ indexed by all measurable, relatively compact sets $\Delta$ in $X$ (a quantum stochastic process over $X$). For such a family, we introduce the notion of a correlation measure. We prove that, if the family of operators possesses a correlation measure which satisfies some condition of growth, then there exists a point process over $X$ having the same correlation measure. Furthermore, the operators $a(\Delta)$ can be realized as multiplication operators in the $L^2$-space with respect to this point process. In the proof, we utilize the notion of $\star$-positive definiteness, proposed in [12]. In particular, our result extends the criterion of existence of a point process from that paper to the case of the topological space $X$, which is a standard underlying space in the theory of point processes. As applications, we discuss particle densities of the quasi-free representations of the CAR and CCR, which lead to fermion, boson, fermion-like, and boson-like (e.g. para-fermions and para-bosons of order 2) point processes. In particular, we prove that any fermion point process corresponding to a Hermitian kernel may be derived in this way.

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1 Introduction

Let $X$ be a locally compact, second countable Hausdorff topological space. We denote by $\Gamma_X$ the space of locally finite sets (configurations) in $X$. A point process in $X$ is a
probability measure on $\Gamma_X$. From the point of view of classical statistical mechanics, point processes describe infinite (interacting) particle systems in continuum.

In the study of point processes, their correlation measures play a crucial role. Denote by $\Gamma_{X,0}$ the space of all finite subsets of $X$. One says that a measure $\rho$ on $\Gamma_{X,0}$ is the correlation measure of a point process $\mu$ if, for each measurable function $G : \Gamma_{X,0} \to [0, +\infty]$, we have (see e.g. \[12\], \[15\], \[16\]):

$$\int_{\Gamma_{X,0}} G(\eta) \rho(d\eta) = \int_{\Gamma_X} (\mathcal{K}G)(\gamma) \mu(d\gamma),$$  

where the operator $\mathcal{K}$ is given by

$$(\mathcal{K}G)(\gamma) := \sum_{\eta \subseteq \gamma} G(\eta)$$

($\eta \subseteq \gamma$ denoting that $\eta$ is a finite subset of $\gamma$). It was shown by Lenard \[14\] that, under a very mild assumption on the correlation measure, it uniquely characterizes a point process. Furthermore, Lenard \[16\] and Macchi \[18\] proposed conditions that are sufficient for a given measure $\rho$ on $\Gamma_{X,0}$ to be the correlation measure of a point process. Both Lenard and Macchi essentially demanded the local densities derived from the measure $\rho$ to be non-negative.

Kondratiev and Kuna \[12\] treated the $\mathcal{K}$-transform as an analog of the Fourier transform over the configuration space (see also \[13\]). In the case where $X$ is a smooth Riemannian manifold, they defined a $\ast$-convolution of functions on $\Gamma_{X,0}$, so that

$$\mathcal{K}(G_1 \ast G_2) = \mathcal{K}G_1 \cdot \mathcal{K}G_2,$$

introduced the notion of $\ast$-positive definiteness, and proved an analog of the Bochner theorem for point processes. A spectral approach to this construction, together with a refinement of the local bound satisfied by a measure $\rho$, was proposed in \[5\]. It should be noted that, in both papers \[12\] and \[5\], the assumption that $X$ is a smooth manifold was crucial, due to the use of the Minlos theorem in \[12\], and the projection spectral theorem in \[5\].

In the first part of this paper (Section \[2\]), we consider a family of commuting Hermitian operators $\mathcal{A} = \{a(\Delta)\}_{\Delta \in \mathcal{B}_0(X)}$ indexed by all measurable, relatively compact sets $\Delta$ in $X$. Such a family of operators may be treated as a quantum stochastic process over the space $X$. We define a class $\mathcal{S}$ of “simple” functions on $\Gamma_{X,0}$ and, having fixed the family $\mathcal{A}$, introduce corresponding operators $(Q(G))_{G \in \mathcal{S}}$ such that $Q(G_1 \ast G_2) = Q(G_1) Q(G_2)$. We then fix a vector $\Omega$ and say that the family $\mathcal{A}$ possesses a correlation measure $\rho$ if

$$(Q(G)\Omega, \Omega) = \int_{\Gamma_{X,0}} G(\eta) \rho(d\eta).$$
We prove that, if the family $A$ possesses a correlation measure $\rho$ that satisfies some condition of growth, then there exists a point process $\mu$ which has correlation measure $\rho$. Furthermore, the operators $a(\Delta)$ can be realized as multiplication operators in $L^2(\Gamma_X, \mu)$. Thus, $\mu$ can be thought of as the spectral measure of the family $A$. When proving this result, we, in particular, extend the criterion of existence of a point process, proved in [12, 5] to the case of the topological space $X$, which is a standard underlying space in the theory of point processes, see e.g. [11].

Another tremendous feature of the correlation measure is that it is deeply connected with the normal ordering as it is known in the quantum field theory. Let us heuristically explain this. Let $\Psi^*(x), \Psi(x), x \in X$, be a representation of either canonical anticommutation relations (CAR), describing fermions, or canonical commutation relations (CAR), describing bosons. Then, the operators $a(x) := \Psi^*(x)\Psi(x), x \in X$, describe the particle density and commute, see e.g. [10]. Setting, for each $\Delta \in \mathcal{B}_0(X)$, $a(\Delta) := \int_\Delta a(x) \sigma(dx)$ ($\sigma$ being a Radon, non-atomic measure on $X$), we get a family of commuting Hermitian operators. Let $G^{(n)}$ be a function from $\mathcal{S}$ such that $G^{(n)}(\eta) = 0$ if the number of points in the configuration $\eta$ is not $n$. Then, one has:

$$Q(G^{(n)}) = \frac{1}{n!} \int_{X^n} \sigma(dx_1) \cdots \sigma(dx_n) G^{(n)}(\{x_1, \ldots, x_n\}) \Psi^*(x_n) \cdots \Psi^*(x_1)\Psi(x_1) \cdots \Psi(x_n)$$

(normal ordering), so that

$$\frac{n!d\rho(\{x_1, \ldots, x_n\})}{\sigma(dx_1) \cdots \sigma(dx_n)} = (\Psi^*(x_n) \cdots \Psi^*(x_1)\Psi(x_1) \cdots \Psi(x_n)\Omega, \Omega), \quad (4)$$

the expression on the left hand side of (4) being called the $n$-th correlation function. To the best of our knowledge, heuristic arguments of such type were first given by Menikoff in [19], see also [20].

So, in the second part of this paper (Sections 3, 4), we mathematicially realize this idea in the case of fermion (determinantal), boson, fermion-like, and boson-like point processes. While fermion and boson point processes have been known since about 1973-1975, when they were introduced by Girard [9], Menikoff [20], and Macchi [18] (see also [8, 21, 22, 23] and the references therein), the fermion-like and boson-like point processes first appeared in 2003 in Shirai and Takahashi’s paper [23]. We also refer to the recent paper [22], where, in particular, the case of para-bosons and para-fermions of order 2 is discussed from the quantum mechanical point of view.

In Section 3 we start with a quasi-free representation of the CAR (CCR, respectively), see e.g. [11, 24, 27]. Such a representation is completely characterized by a linear, bounded, Hermitian operator $K$ in $L^2(X, \sigma)$ which satisfies $0 \leq K \leq 1$ in the fermion case, and $K \geq 0$ in the boson case. In the case where $X = \mathbb{R}^d$ and $K$ is a convolution operator, it has been already shown in [17] that the corresponding particle density has a fermion (boson, respectively) point process as its spectral measure. We also refer to
where a theory of quantum stochastic integration in quasi-free representations of the CAR and CCR was developed (see also the references therein).

In this paper, we treat the most general case of the space \( X \) and the operator \( K \), the only additional assumption on \( K \) being that \( K \) is locally of trace class. The main mathematical (as well as physical) challenge here is to show that all heuristic arguments coming from physics indeed have a precise mathematical meaning. This is why, at many steps, we first perform our computations at a heuristic level, and then discuss the mathematical meaning of this procedure. We observe that \( K \) automatically appears to be an integral operator, and furthermore, with our approach, we do not even have to additionally discuss the problem of the choice of a version of the kernel \( k(x, y) \) of the operator \( K \), compare with \([21, \text{Lemma 1}]\) and \([8, \text{Lemma A4}]\). Thus, we, in particular, show that any fermion process corresponding to a Hermitian operator \( K \) can be thought of as the spectral measure of the family of operators which represent the particle density of a quasi-free representation of the CAR. Though all our results hold for the complex space \( L^2(X \to \mathbb{C}, \sigma) \), for simplicity of presentation we only deal with the case of the real space \( L^2(X, \sigma) \).

Finally, in Section \( 4 \), we briefly discuss the family of operators corresponding to fermion-like and (some) boson-like point processes. For a fixed \( l \geq 2 \), we consider a representation of the CAR (CCR, respectively) which is equivalent to the standard quasi-free representation, but which is based on the orthogonal sum of \( 2l \) identical copies of the space \( L^2(X, \sigma) \) (the standard quasi-free representation being using two copies of this space). The corresponding operators \( \Psi(x), x \in X \), have the form \( \Psi(x) = \sum_{i=1}^{l} \Psi_i(x) \), and the particle density is \( a(x) = \sum_{i,j=1}^{l} \Psi_i^*(x) \Psi_j(x) \). These operators evidently lead to a fermion (boson, respectively) point process. However, we can reduce the particle density by taking only the “diagonal elements” of the double sum: \( a^{(l)}(x) := \sum_{i=1}^{l} \Psi_i^*(x) \Psi_i(x) \). These operators, in turn, lead to a family of commuting, Hermitian operators \( (a^{(l)}(\Delta))_{\Delta \in \mathcal{B}_0(X)} \), whose spectral measure is from the class of point processes discussed in \([23]\), and corresponds to the index \( \alpha = -1/l \) (\( \alpha = 1/l \), respectively) from that paper. Recall that the case \( l = 2 \) corresponds to the para-fermions (para-bosons, respectively) of order 2, see \([22]\). A physical meaning of this procedure of reduction of the particle density still needs to be clarified.

## 2 Correlation measure

Let \( X \) be a locally compact, second countable Hausdorff topological space. Recall that such a space is known to be Polish. We denote by \( \mathcal{B}(X) \) the Borel \( \sigma \)-algebra in \( X \), and by \( \mathcal{B}_0(X) \) the collection of all sets from \( \mathcal{B}(X) \) which are relatively compact.

We define the space of finite multiple configurations in \( X \) as follows:

\[
\hat{\Gamma}_{X,0} := \bigcup_{n \in \mathbb{N}_0} \hat{\Gamma}^{(n)}_X.
\]
Here, $N_0 := \{0, 1, 2, \ldots\}$, $\tilde{\Gamma}_{X}^{(0)} = \{\emptyset\}$, and for $n \in \mathbb{N}$, $\tilde{\Gamma}_{X}^{(n)}$ is the factor-space $X^n/S_n$, where $S_n$ is the group of all permutations of $\{1, \ldots, n\}$, which naturally acts on $X^n$:

$$\xi(x_1, \ldots, x_n) = (x_{\xi(1)}, \ldots, x_{\xi(n)}), \quad \xi \in S_n.$$ 

We denote by $[x_1, \ldots, x_n]$ the equivalence class in $\tilde{\Gamma}_{X}^{(n)}$ corresponding to $(x_1, \ldots, x_n) \in X^n$.

Let $\mathcal{B}(\tilde{\Gamma}_{X}^{(n)})$ denote the image of the Borel $\sigma$-algebra $\mathcal{B}(X^n)$ under the mapping

$$X^n \ni (x_1, \ldots, x_n) \mapsto [x_1, \ldots, x_n] \in \tilde{\Gamma}_{X}^{(n)}.$$ 

Then, the real-valued measurable functions on $\tilde{\Gamma}_{X}^{(n)}$ may be identified with the real-valued $\mathcal{B}_{\text{sym}}(X^n)$-measurable functions on $X^n$. Here, $\mathcal{B}_{\text{sym}}(X^n)$ denotes the $\sigma$-algebra of all sets in $\mathcal{B}(X^n)$ which are symmetric, i.e., invariant under the action of $S_n$.

For measurable functions $f_1, \ldots, f_n : X \to \mathbb{R}$, we denote by $f_1 \odot \cdots \odot f_n$ the symmetric tensor product of $f_1, \ldots, f_n$. Since $f_1 \odot \cdots \odot f_n$ is $\mathcal{B}_{\text{sym}}(X^n)$-measurable, we may consider $f_1 \odot \cdots \odot f_n$ as a measurable function on $\tilde{\Gamma}_{X}^{(n)}$.

For a function $G : \tilde{\Gamma}_{X,0} \to \mathbb{R}$, we denote by $G^{(n)}$ the restriction of $G$ to $\tilde{\Gamma}_{X}^{(n)}$. Let $\mathcal{S}$ denote the set of all real-valued functions on $\tilde{\Gamma}_{X,0}$ which satisfy the following condition: for each $G \in \mathcal{S}$, there is an $N \in \mathbb{N}$ such that $G^{(n)} = 0$ for all $n > N$ and for each $n \in \{1, \ldots, N\}$, $G^{(n)}$ is a finite linear combination of the functions of the form $\chi_{\Delta_1} \odot \cdots \odot \chi_{\Delta_n}$, where $\Delta_1, \ldots, \Delta_n \in \mathcal{B}_0(X)$ and $\chi_A$ denotes the indicator of $A$. Note that, by the polarization identity (e.g. [4] Chapter 2, formula (2.7)), in the above definition it suffices to take functions of the form $\chi_{\Delta}^{\otimes n}$, where $\Delta \in \mathcal{B}_0(X)$.

Next, we can identify any $[x_1, \ldots, x_n] \in \tilde{\Gamma}_{X,0}$ with the measure $\nu_{x_1} \cdots + \nu_{x_n}$. Here, for any $x \in X$, $\nu_x$ denotes the Dirac measure with mass at $x$. We also identify $\{\emptyset\}$ with zero measure. Through this identification, $\tilde{\Gamma}_{X,0}$ becomes the set of all finite measures on $X$ taking values in $N_0$.

Next, we introduce the space of finite configurations in $X$, denoted by $\Gamma_{X,0}$. By definition, $\Gamma_{X,0}$ is the subset of $\tilde{\Gamma}_{X,0}$ given by

$$\Gamma_{X,0} := \bigsqcup_{n \in N_0} \Gamma_{X}^{(n)},$$

where $\Gamma_{X}^{(0)} := \tilde{\Gamma}_{X}^{(0)}$ and for $n \in \mathbb{N}$, $\Gamma_{X}^{(n)}$ consists of all $[x_1, \ldots, x_n] \in \tilde{\Gamma}_{X}^{(n)}$ such that $x_1, \ldots, x_n$ are different points in $X$. Hence, each $[x_1, \ldots, x_n] \in \Gamma_{X}^{(n)}$ may be identified either with the set $\{x_1, \ldots, x_n\}$, or with the measure $\eta = \sum_{i=1}^{n} \nu_{x_i}$ satisfying $\eta(\{x\}) \leq 1$ for each $x \in X$. Evidently, $\Gamma_{X}^{(n)} \subseteq \mathcal{B}(\tilde{\Gamma}_{X}^{(n)})$ and we denote by $\mathcal{B}(\Gamma_{X}^{(n)})$ the trace $\sigma$-algebra of $\mathcal{B}(\tilde{\Gamma}_{X}^{(n)})$ on $\Gamma_{X}^{(n)}$. We also introduce the $\sigma$-algebra $\mathcal{B}(\Gamma_{X,0})$ on $\Gamma_{X,0}$, whose restriction to $\Gamma_{X}^{(n)}$ is $\mathcal{B}(\Gamma_{X}^{(n)})$ for each $n \in \mathbb{N}$, and $\{\emptyset\} \in \mathcal{B}(\Gamma_{X,0})$.

Let $\Gamma_X$ denote the configuration space over $X$:

$$\Gamma_X := \{\gamma \subseteq X : |\gamma \cap \Delta| < \infty \text{ for each } \Delta \in \mathcal{B}_0(X)\}.$$
Here, $|A|$ denotes the cardinality of a set $A$. Note the evident inclusion $\Gamma_{X,0} \subset \Gamma_X$. We identify each $\gamma \in \Gamma_X$ with the Radon measure $\gamma = \sum_{x \in \gamma} \varepsilon_x$. We introduce the vague topology on $\Gamma_X$ and denote by $\mathcal{B}(\Gamma_X)$ the Borel $\sigma$-algebra on $\Gamma_X$. Note that $\mathcal{B}(\Gamma_X)$ is the minimal $\sigma$-algebra on $\Gamma_X$ making all maps

$$\Gamma_X \ni \gamma \mapsto \gamma(\Delta) = |\gamma \cap \Delta| \in \mathbb{R}, \quad \Delta \in \mathcal{B}_0(X),$$

measurable (cf. e.g. [11]). Furthermore, the trace $\sigma$-algebra of $\mathcal{B}(\Gamma_X)$ on $\Gamma_{X,0}$ coincides with $\mathcal{B}(\Gamma_{X,0})$.

For each function $G \in \mathcal{S}$, we define a measurable function $KG : \Gamma_X \to \mathbb{R}$ by (2). Let $\mu$ be a probability measure on $(\Gamma_X, \mathcal{B}(\Gamma_X))$. We say that a measure $\rho$ on $(\Gamma_{X,0}, \mathcal{B}(\Gamma_{X,0}))$ is the correlation measure of $\mu$ if each $G \in \mathcal{S}$ is integrable with respect to $\rho$, and equality (1) holds for all $G \in \mathcal{S}$.

We now define a convolution $\ast$ as the mapping $\ast : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ defined by

$$(G_1 \ast G_2)(\eta) := \sum \left[ G_1(\xi_1 + \xi_2)G_2(\xi_2 + \xi_3), \quad \eta \in \tilde{\Gamma}_{X,0}, \right]$$

where the summation is over all $\xi_1, \xi_2, \xi_3 \in \tilde{\Gamma}_{X,0}$ such that $\xi_1 + \xi_2 + \xi_3 = \eta$. It is easy to see that the convolution $\ast$ is commutative and associative. Furthermore, we easily see that equality (3) holds for all $G_1, G_2 \in \mathcal{S}$.

Next, let $F$ be either a real, or complex Hilbert space and let $D$ be a linear subset of $F$. Let $(a(\Delta))_{\Delta \in \mathcal{B}_0(X)}$ be a family of Hermitian operators in $F$ such that:

- for each $\Delta \in \mathcal{B}_0(X)$, $\text{Dom}(a(\Delta)) = D$ and $a(\Delta)$ maps $D$ into itself;
- for any $\Delta_1, \Delta_2 \in \mathcal{B}_0(X)$, $a(\Delta_1)a(\Delta_2) = a(\Delta_2)a(\Delta_1)$;
- for any mutually disjoint $\Delta_1, \Delta_2 \in \mathcal{B}_0(X)$, we have: $a(\Delta_1 \cup \Delta_2) = a(\Delta_1) + a(\Delta_2)$.

The family $(a(\Delta))_{\Delta \in \mathcal{B}_0(X)}$ can be thought of as a (commutative) quantum stochastic process over a general topological space $X$.

We recursively define the following operators:

$$Q(\chi_{\Delta_1} \circ \cdots \circ \chi_{\Delta_{n+1}}) = \frac{1}{(n + 1)^2} \left[ \sum_{i=1}^{n+1} a(\Delta_i) Q(\chi_{\Delta_1} \circ \cdots \circ \tilde{\chi}_{\Delta_i} \circ \cdots \circ \chi_{\Delta_{n+1}}) \right] - \sum_{i=1}^{n+1} \sum_{j=1, \ldots, n+1, j \neq i} Q\left( (\chi_{\Delta_i \cap \Delta_j} \circ \chi_{\Delta_1} \circ \cdots \circ \tilde{\chi}_{\Delta_i} \circ \cdots \circ \chi_{\Delta_j} \circ \cdots \circ \chi_{\Delta_{n+1}}) \right),$$

$$\Delta_1, \ldots, \Delta_{n+1} \in \mathcal{B}_0(X), \quad n \in \mathbb{N},$$

$$Q(\chi_\Delta) = a(\Delta), \quad \Delta \in \mathcal{B}_0(X),$$

where $\tilde{\chi}_\Delta$ denotes the absence of $\chi_\Delta$. Denote by $\Xi$ the function on $\tilde{\Gamma}_{X,0}$ given by

$$\Xi^{(0)} := 1, \Xi^{(n)} := 0, \quad n \in \mathbb{N}.$$
We then uniquely define \( Q(G) \) for each \( G \in \mathcal{S} \), so that
\[
Q(a_1G_1 + a_1G_2) = a_1Q(G_1) + a_2Q(G_2), \quad a_1, a_2 \in \mathbb{R}, \quad G_1, G_2 \in \mathcal{S}.
\]
It is not hard to see (e.g., by induction) that, for any \( G_1, G_2 \in \mathcal{S} \),
\[
Q(G_1)Q(G_2) = Q(G_1 \ast G_2) \tag{8}
\]
(compare with (3)).

We fix any \( \Omega \in D \) with \( \|\Omega\|_F = 1 \). Now, by analogy with (1), we say that a measure \( \rho \) on \( (\Gamma_X, \mathcal{B}(\Gamma_X)) \) is the correlation measure of the family of commuting Hermitian operators \( (a(\Delta))_{\Delta \in \mathcal{B}_0(X)} \) (with respect to the vector \( \Omega \)) if, for all \( G \in \mathcal{S} \),
\[
(Q(G)\Omega, \Omega)_F = \int_{\Gamma_X, \partial} G(\eta) \rho(d\eta). \tag{9}
\]

Now, we assume that \( \rho \) additionally satisfies:

(LB) \textit{Local bound}: for each \( \Delta \in \mathcal{B}_0(X) \), there exists \( C_\Delta > 0 \) such that
\[
\rho(\Gamma^{(n)}_\Delta) \leq C_\Delta^n, \quad n \in \mathbb{N},
\]
where \( \Gamma^{(n)}_\Delta := \{ \eta \in \Gamma^{(n)}_X \mid \eta \subset \Delta \} \). Furthermore, for any sequence \( \{\Delta_n\}_{n \in \mathbb{N}} \subset \mathcal{B}_0(X) \) such that \( \Delta_n \downarrow \emptyset \), we have \( C_{\Delta_n} \to 0 \) as \( n \to \infty \).

Theorem 1 Assume that a family \( (a(\Delta))_{\Delta \in \mathcal{B}_0(X)} \) of commuting Hermitian operators has a correlation measure which satisfies (LB). For each \( G \in \mathcal{S} \) denote \( Q(G) := Q(G)\Omega \), and let \( \mathcal{F} \) denote the real Hilbert space obtained as the closure of the set \( \mathfrak{A} := \{ Q(G) \mid G \in \mathcal{S} \} \) in \( F \). For each \( \Delta \in \mathcal{B}_0(X) \), consider \( a(\Delta) \) as an operator in \( \mathcal{F} \) with domain \( \mathfrak{A} \). Then, the operators \( a(\Delta) \) are essentially self-adjoint and their closures, \( \tilde{a}(\Delta) \), commute in the sense of their resolutions of the identity. Furthermore, there exists a unique probability measure \( \mu \) on \( (\Gamma_X, \mathcal{B}(\Gamma_X)) \) whose correlation measure is \( \rho \), the mapping
\[
\mathfrak{A} \ni Q(G) \mapsto (IQ(G))(\gamma) := \sum_{\eta \in \gamma} G(\eta) \in L^2(\Gamma, \mu)
\]
is well-defined and extends to a unitary operator \( \mathcal{I} : \mathcal{F} \to L^2(\Gamma, \mu) \) such that, under \( \mathcal{I} \), \( \tilde{a}(\Delta) \) goes over into the operator of multiplication by \( \gamma(\Delta) \).

Proof. By (8) and (9),
\[
\mathcal{S} \times \mathcal{S} \ni (G_1, G_2) \mapsto b_\rho(G_1, G_2) := \int_{\Gamma_X, \partial} (G_1 \ast G_2)(\eta) \rho(d\eta)
\]
is a bilinear, positive form on $S$. Denote by $\tilde{S}$ the factorization of $S$ consisting of factor-classes

$$\tilde{G} = \{ G' \in S : b_\rho(G - G', G - G') = 0 \}, \quad G \in S.$$ 

Define a Hilbert space $\mathcal{H}_\rho$ as the closure of $\tilde{S}$ in the norm generated by the scalar product

$$(\tilde{G}_1, \tilde{G}_2)_{\mathcal{H}_\rho} := b_\rho(G_1, G_2).$$

For each $\Delta \in \mathcal{B}_0(X)$, we define an operator $A_\Delta$ in $\mathcal{H}_\rho$ with domain $\text{Dom}(A_\Delta) = \tilde{S}$ by

$$A_\Delta \tilde{G} := \tilde{\chi}_\Delta \ast G, \quad G \in S.$$ 

(10)

Since $b_\rho(\chi_\Delta \ast G_1, G_2) = b_\rho(G_1, \chi_\Delta \ast G_2), \quad G_1, G_2 \in S,$

by [4, Chapter 5, Section 5, subsec.2], the definition (10) is indeed correct, and the operators $A_\Delta$ are symmetric.

Analogously to [5, Lemma 2], we easily get the following

**Lemma 1** Each $\tilde{G} \in \tilde{S}$ is an analytic vector for any $A_\Delta$, $\Delta \in \mathcal{B}_0(X)$.

By Lemma 1 for each $\Delta \in \mathcal{B}_0(X)$, the closure of $A_\Delta$, denoted by $\tilde{A}_\Delta$, is a self-adjoint operator in $\mathcal{H}_\rho$. Denote by $E_\Delta$ its resolution of the identity. Then, also by Lemma 1 for any $\Delta_1, \Delta_2 \in \mathcal{B}_0(X)$, the resolutions of the identity $E_{\Delta_1}$ and $E_{\Delta_2}$ commute, see e.g. [4, Chapter 5, Theorem 1.15]. Therefore, for any $\Delta_1, \ldots, \Delta_n \in \mathcal{B}_0(X)$, we can construct the joint resolution of the identity

$$E_{\Delta_1, \ldots, \Delta_n} := E_{\Delta_1} \times \cdots \times E_{\Delta_n}$$

(see e.g. [4, Chapter 3, Section 1] for details).

Recall the definition of the function $\Xi$ on $\hat{\Gamma}_{X,0}$. Then

$$\nu_{\Delta_1, \ldots, \Delta_n}(\cdot) := (E_{\Delta_1, \ldots, \Delta_n}(\cdot)\hat{\Xi}; \hat{\Xi})_{\mathcal{H}_\rho}$$

is a probability measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Furthermore, it is clear that

$$\{ \nu_{\Delta_1, \ldots, \Delta_n} \mid \Delta_1, \ldots, \Delta_n \in \mathcal{B}_0(X), \quad n \in \mathbb{N} \}$$

(11)

is a consistent family of probability measures.

For any $\Delta \in \mathcal{B}_0(X)$, denote

$$\Gamma_\Delta := \{ \eta \in \hat{\Gamma}_{X,0} \mid \eta \subset \Delta \},$$

and let $\mathcal{B}(\Gamma_\Delta)$ be the trace $\sigma$-algebra of $\mathcal{B}(\hat{\Gamma}_{X,0})$ on $\Gamma_\Delta$. We introduce a mapping $\mathcal{K}_\Delta$, which transforms the set of all (complex-valued) functions on $\Gamma_\Delta$ into itself, as follows:

$$(\mathcal{K}_\Delta G)(\eta) := \sum_{\xi \subset \eta} G(\xi), \quad \eta \in \Gamma_\Delta.$$ 

(12)
We evidently have:
\[
(K_\Delta(G_1 \ast G_2))(\eta) = (K_\Delta G_1)(\eta)(K_\Delta G_2)(\eta)
\]
\[\text{(13)}\]
\(G_1 \ast G_2\) being given by \[(5)\). The inverse of \(K_\Delta\) is then given by
\[
(K_\Delta^{-1}G)(\eta) = \sum_{\xi \subseteq \eta} (-1)^{|\eta| - |\xi|} G(\xi), \quad \eta \in \Gamma_\Delta.
\]
\[\text{(14)}\]
For any function \(f : \Delta \to \mathbb{C}\), we define a function \(\text{Exp}_\Delta(f, \cdot) : \Gamma_\Delta \to \mathbb{C}\) by
\[
\text{Exp}_\Delta(f, \emptyset) := 1,
\]
\[
\text{Exp}_\Delta(f, \{x_1, \ldots, x_n\}) := f(x_1) \cdots f(x_n), \quad \{x_1, \ldots, x_n\} \in \Gamma_\Delta, \; n \in \mathbb{N}.
\]
By \[(14)\), for any \(\varphi : \Delta \to \mathbb{C}\), we have:
\[
(K_\Delta^{-1}\exp[(\varphi, \cdot)])(\eta) = \text{Exp}_\Delta(e^{\varphi} - 1, \eta), \quad \eta \in \Gamma_\Delta,
\]
\[\text{(15)}\]
where \(\langle \varphi, \eta \rangle := \sum_{x \in \eta} \varphi(x)\).

Let \(\Delta \in B_0(X)\) be such that
\[
C_\Delta \leq \frac{1}{12 + \delta}, \quad \delta > 0
\]
\[\text{(16)}\]
(see (LB)). Following the idea of \cite{12}, we define a set function on \(B(\Gamma_\Delta)\) by
\[
\mu_\Delta(A) := \int_{\Gamma_\Delta} (K_\Delta^{-1}\chi_A)(\eta) \rho(d\eta), \quad A \in B(\Gamma_\Delta).
\]
\[\text{(17)}\]
Since
\[
\sum_{\xi \subseteq \eta} 1 = 2^n \quad \text{if} \quad |\eta| = n,
\]
\[\text{(18)}\]
(LB) and \[(16)\] imply that \(\mu_\Delta\) is a signed measure of finite variation.

Let \(\Delta_1, \ldots, \Delta_n \in B_0(X)\) be subsets of \(\Delta\), \(n \in \mathbb{N}\), and for simplicity of notations we assume that these sets are mutually disjoint. Then, by \[(13)\)–\[(15)\] , for any \((y_1, \ldots, y_n) \in \mathbb{R}^n\),
\[
L(y_1, \ldots, y_n) := \int_{\Gamma_\Delta} \exp \left[\langle i(y_1\chi_{\Delta_1} + \cdots + y_n\chi_{\Delta_n}), \eta \rangle \right] \mu_\Delta(d\eta)
\]
\[
= \int_{\Gamma_\Delta} \text{Exp}_\Delta((e^{iy_1} - 1)\chi_{\Delta_1} + \cdots + (e^{iy_n} - 1)\chi_{\Delta_n}, \eta) \rho(d\eta).
\]
\[\text{(19)}\]
Using (LB), \[(13)\), \[(16)\], and \[(19)\], we conclude that the function \(L : \mathbb{R}^n \to \mathbb{C}\) is positive definite in the sense of the Fourier analysis on \(\mathbb{R}^n\). Hence, \(L\) is the Fourier transform of a probability measure on \(\mathbb{R}^n\). Therefore, under the mapping
\[
\Gamma_\Delta \ni \eta \mapsto (\eta(\Delta_1), \ldots, \eta(\Delta_n)) \in \mathbb{R}^n,
\]
the image of the signed measure $\mu^\Delta$ is a probability measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, which we denote by $\mu^\Delta_1,..,\mu^\Delta_n$.

Using (13), (14), and (16), for any $y^{(1)}, \ldots, y^{(k)} \in \mathbb{R}^n$, $k \in \mathbb{N}$, we have:

\[
\int_{\mathbb{R}^n} \prod_{i=1}^{k} (x, y^{(i)}) d\nu_{\Delta_1,..,\Delta_n}(x) = \left( \prod_{i=1}^{k} \sum_{j=1}^{n} A_{\Delta_j} y^{(i)}_j \right) \mu^\Delta_1,..,\mu^\Delta_n(x)
\]

\[
= \int_{\Gamma_{X,0}} \left( \sum_{j=1}^{n} y^{(1)}_j \chi_{\Delta_j} \right) \cdots \left( \sum_{j=1}^{n} y^{(k)}_j \chi_{\Delta_j} \right) \rho(d\eta)
\]

\[
= \int_{\mathbb{R}^n} \prod_{i=1}^{k} (x, y^{(i)}) d\mu_{\Delta_1,..,\Delta_n}(x).
\]  

(20)

Furthermore, it follows from the proof of Lemma 1 that there exists a constant $R > 0$ such that

\[
\left| \int_{\Gamma_{X,0}} \left( \sum_{j=1}^{n} y^{(1)}_j \chi_{\Delta_j} \right) \cdots \left( \sum_{j=1}^{n} y^{(k)}_j \chi_{\Delta_j} \right) \rho(d\eta) \right| \leq R^n n! \prod_{i=1}^{k} \|y^{(i)}\|_{\mathbb{R}^n}.
\]

Hence, by the theorem on uniqueness of the solution of a moment problem (e.g. [4, Chapter 5, Theorem 2.1 and Remark 3]), we conclude from (20) and (21) that

\[
\nu_{\Delta_1,..,\Delta_n} = \mu_{\Delta_1,..,\Delta_n}.
\]

We also observe that the sets

\[
\{ \eta \in \Gamma_{X,0} \mid (\eta(\Delta_1), \ldots, \eta(\Delta_n)) \in B_n \},
\]

\[
B_n \in \mathcal{B}(\mathbb{R}^n), \Delta_1, \ldots, \Delta_n \in \mathcal{B}_0(X), \Delta_1 \cup \cdots \cup \Delta_n \subset \Delta, n \in \mathbb{N},
\]

(22)

generate the $\sigma$-algebra $\mathcal{B}(\Gamma_\Delta)$. Hence, $\mu_\Delta$ is a probability measure on $(\Gamma_\Delta, \mathcal{B}(\Gamma_\Delta))$.

Next, let $\Delta' \in \mathcal{B}_0(X)$ be such that $\Delta' \subset \Delta$. As usual, we identity $\mathcal{B}(\Gamma_{\Delta'})$ with the sub-$\sigma$-algebra of $\mathcal{B}(\Gamma_\Delta)$ generated by the sets of the form (22) where $\Delta_1, \ldots, \Delta_n$ are subsets of $\Delta'$. Then it follows from the above that $\mu_{\Delta'}$ is the restriction of $\mu_\Delta$ to $\mathcal{B}(\Gamma_{\Delta'})$.

Now, we will show that there exists a random measure $M$ on $X$ such that, for any $\Delta_1, \ldots, \Delta_n \in \mathcal{B}_0(X), n \in \mathbb{N}$, the distribution of $(M(\Delta_1), \ldots, M(\Delta_n))$ is $\nu_{\Delta_1,..,\Delta_n}$ (see e.g. [11] for details on random measures).

By (LB), for any $x \in X$, there exists an open neighborhood of $x$, denoted by $\Delta(x)$, such that $\Delta(x) \in \mathcal{B}_0(X)$ and $C_{\Delta(x)} \leq 1/(12 + \delta)$. Therefore, for any $\Delta \in \mathcal{B}_0(X)$, there exist mutually disjoint sets $\Delta_1, \ldots, \Delta_m \in \mathcal{B}_0(X), m \in \mathbb{N}$, such that $\Delta = \Delta_1 \cup \cdots \cup \Delta_m$, $C_{\Delta_i} \leq 1/(12 + \delta), i = 1, \ldots, m$. By the proved above, $\nu_{\Delta_i}([0, +\infty)) = 1, i = 1, \ldots, m$. Hence, $\nu_{\Delta_1,..,\Delta_m}([0, +\infty)^m) = 1$. By Lemma 1 for each $A \in \mathcal{B}(\mathbb{R})$,

\[
\nu_{\Delta}(A) = \nu_{\Delta_1 \cup \cdots \cup \Delta_m}(A) = \int_{[0, +\infty)^m} \chi_A(x_1 + \cdots + x_m) d\nu_{\Delta_1,..,\Delta_m}(x_1, \ldots, x_m).
\]

(19)
Hence, \( \nu_\Delta([0, +\infty)) = 1 \), and so, for any \( \Delta_1, \ldots, \Delta_n \in \mathcal{B}_0(X), \ n \in \mathbb{N} \),

\[
\nu_{\Delta_1, \ldots, \Delta_n}([0, +\infty]^n) = 1.
\]

Next, it is also clear from Lemma \( \text{[1]} \) that, for any disjoint \( \Delta_1, \Delta_2 \in \mathcal{B}_0(X) \),

\[
\nu_{\Delta_1, \Delta_2, \Delta_1 \cup \Delta_2}\left(\{(x, y, z) \in \mathbb{R}^3 \mid x + y = z\}\right) = 1.
\]

Finally, let \( \Delta_n \in \mathcal{B}_0(X), \ n \in \mathbb{N} \), be such that \( \Delta_n \downarrow \emptyset \). We state that \( \nu_{\Delta_n} \) weakly converges to \( \varepsilon_0 \). By (A2), without loss, we may assume that \( C_{\Delta_1} \leq 1/(12 + \delta) \). Then, each \( \nu_{\Delta_n} \) is concentrated on the set \( \mathbb{N}_0 \). Hence, it is enough to show that \( \nu_{\Delta_n}(\mathbb{N}) \to 0 \) as \( n \to \infty \). But this holds since \( \nu_{\Delta_n} \) is the distribution of the random variable \( \eta(\Delta_n) \).

Now, by [11, Theorem 5.4], there indeed exists a random measure \( M \) on \( X \) as described above. Furthermore, we already know that, for any \( x \in X \), there exists an open neighborhood of \( x \), denoted by \( \Delta(x) \), such that \( \Delta(x) \in \mathcal{B}_0(X) \) and the restriction of \( M \) to \( \Delta(x) \) is concentrated on \( \Gamma_{\Delta(x)} \). Hence, the random measure \( M \) is a.s. concentrated on \( \Gamma_X \). Letting \( \mu \) denote the distribution of \( M \) on \( \Gamma_X \), we obtain a unique probability measure on \( (\Gamma_X, \mathcal{B}(\Gamma_X)) \) whose “finite-dimensional distributions” are given through the measures [11].

Let again \( \Delta \in \mathcal{B}_0(X) \) be such that [11] is satisfied. As usual, we identify \( \mathcal{B}(\Gamma_\Delta) \) as a sub-\( \sigma \)-algebra of \( \mathcal{B}(\Gamma_{X,0}) \). Then, for any \( G_1, G_2 \in \mathcal{S} \) which, restricted to \( \Gamma_{X,0} \), are \( \mathcal{B}(\Gamma_\Delta) \)-measurable, we have:

\[
\int_{\Gamma_{X,0}} (G_1 * G_2)(\eta) \rho(d\eta) = \int_{\Gamma_\Delta} (G_1 * G_2)(\eta)\rho(d\eta) = \int_{\Gamma_\Delta} (K_{\Delta}G_1)(\eta)(K_{\Delta}G_2)(\eta) \mu^\Delta(d\eta) = \int_{\Gamma_X} \left( \sum_{\eta \in \gamma} G_1(\eta) \right) \left( \sum_{\eta \in \gamma} G_2(\eta) \right) \mu(d\eta). \tag{23}
\]

Next, any \( G \in \mathcal{S} \) can be represented as \( G = \sum_{j=1}^k G_j \), where \( k \in \mathbb{N} \), each \( G_j \) belongs to \( \mathcal{S} \) and, restricted to \( \Gamma_{X,0} \), is \( \mathcal{B}(\Gamma_{\Delta_j}) \)-measurable with \( \Delta_j \in \mathcal{B}_0(X) \), \( C_{\Delta_j} \leq 1/(12 + \delta) \). Hence, by (23), for any \( G_1, G_2 \in \mathcal{S} \),

\[
\int_{\Gamma_{X,0}} (G_1 * G_2)(\eta) \rho(d\eta) = \int_{\Gamma_X} \left( \sum_{\eta \in \gamma} G_1(\eta) \right) \left( \sum_{\eta \in \gamma} G_2(\eta) \right) \mu(d\eta). \tag{24}
\]

Define the mapping

\[
\tilde{\mathcal{S}} \ni \tilde{G} \mapsto (K\tilde{G})(\gamma) := \sum_{\eta \in \gamma} G(\eta). \tag{25}
\]
Then, by (24), $K$ extends to an isometry of $\mathcal{H}_\rho$ into $L^2(\Gamma, \mu)$. Furthermore, the image of $K$ is evidently dense in $L^2(\Gamma, \mu)$, and so $K$ is a unitary operator.

For each $\Delta \in \mathcal{B}_0(X)$,

$$K(\chi_\Delta \star G)(\gamma) = \gamma(\Delta)(\hat{K} \hat{G})(\gamma), \quad G \in \mathcal{S}, \gamma \in \Gamma_X.$$ 

Therefore, $\tilde{A}_\Delta$ goes over, under $K$, into the operator of multiplication by $\gamma(\Delta)$.

Finally, we can construct a unitary operator $I : \mathfrak{F} \to \mathcal{H}_\rho$ by setting $IQ(G) := \hat{G}$. Then, from the proved above, we get the conclusion of the theorem, except for the statement about the uniqueness of a measure $\mu$, whose correlation measure is $\rho$. But the uniqueness of such $\mu$ follows from \cite{14} (in fact, the uniqueness can also be derived from the above arguments). \hfill \Box

It is clear that any correlation measure $\rho$ satisfies the following condition:

\textbf{(N)} Normalization: $\rho(\Gamma_X^{(0)}) = 1$.

It follows from \textbf{(8)} and \textbf{(9)} (or \textbf{8}) that any correlation measure $\rho$ also satisfies:

\textbf{(PD)} $\star$-positive definiteness: For each $G \in \mathcal{S}$:

$$\int_{\Gamma_X^{(0)}} (G \star G)(\eta) \rho(\eta) \geq 0.$$ 

From (the proof of) Theorem \textbf{II} we easily conclude the following criterion of existence of a point process, which generalizes \cite{12} Theorem 6.5 and \cite{5} Theorem 2.

**Corollary 1** Let $\rho$ be a measure on $(\Gamma_X^{(0)}, \mathcal{B}(\Gamma_X^{(0)}))$ satisfying \textbf{(N)}, \textbf{(PD)}, and \textbf{(LB)}. Then, there exists a unique probability measure on $(\Gamma_X, \mathcal{B}(\Gamma_X))$ which has $\rho$ as correlation measure.

### 3  Particle densities in quasi-free representations of the CAR and CCR

Let $X$ be a topological space as in Section \textbf{2}. Let $\sigma$ be a non-atomic Radon measure on $(X, \mathcal{B}(X))$. We denote by $H$ the real space $L^2(X, \sigma)$. For an integral operator $I$ in $H$, we denote by $\mathcal{N}(I)$ the kernel of $I$.

Let $K$ be a linear bounded operator in $H$ which satisfies the following assumptions:

- $K$ is symmetric and $0 \leq K \leq 1$;

- for each $\Delta \in \mathcal{B}_0(X)$, the operator $P_\Delta KP_\Delta$ is of trace class. Here, $P_\Delta$ denotes the operator of multiplication by $\chi_\Delta$. 


Denote $K_1 := \sqrt{K}$. For each $\Delta \in B_0(X)$,

$$P_\Delta K_1 (P_\Delta K_1)^* = P_\Delta K P_\Delta.$$ 

Therefore, the operator $P_\Delta K_1$ is of Hilbert–Schmidt class. Hence, $P_\Delta K_1$ is an integral operator, whose kernel $\mathcal{N}(P_\Delta K_1)$ belongs to $L^2(X^2, \sigma^2)$. This implies that $K_1$ is an integral operator, whose kernel satisfies

$$\int_{\Delta} \int_X \mathcal{N}(K_1)(x, y)^2 \sigma(dx) \sigma(dy) < \infty, \quad \Delta \in B_0(X).$$

(26)

Note also that the kernel $\mathcal{N}(K_1)$ is symmetric.

Thus, $K$ is an integral operator, whose kernel is given by

$$k(x, y) := \mathcal{N}(K)(x, y) = \int_X \mathcal{N}(K_1)(x, z)\mathcal{N}(K_1)(z, y) \sigma(dz).$$

By (26), for any $\Delta \in B_0(X)$, we get:

$$\int_{\Delta} k(x, x) \sigma(dx) = \int_{\Delta} \int_X \mathcal{N}(K_1)(x, y)\mathcal{N}(K_1)(y, x) \sigma(dy) \sigma(dx)$$

$$= \int_{\Delta} \int_X \mathcal{N}(K_1)(x, y)^2 \sigma(dx) \sigma(dy) < \infty.$$ 

Note that the kernel $\mathcal{N}(K_1)(x, y)$ is defined up to a set of $\sigma^2$-measure 0 in $X^2$, but the value $\int_X k(x, x) \sigma(dx)$ is independent of the choice of a version of $\mathcal{N}(K_1)$.

Now, for a fixed $x \in X$, we define the function $\kappa_{1,x} : X \rightarrow \mathbb{R}$ by

$$\kappa_{1,x}(y) := \mathcal{N}(K_1)(x, y), \quad y \in X.$$ 

By (26),

$$\kappa_{1,x} \in L^2(X, \sigma) \quad \text{for } \sigma\text{-a.a. } x \in X.$$ 

(27)

We also define the linear bounded operator $K_2 := (1 - K)^{1/2}$. Though $K_2$ is not an integral operator, we will heuristically use $\kappa_{2,x}$, $x \in X$, to denote the “function” $\kappa_{2,x}(y) := \mathcal{N}(K_2)(x, y)$, where $\mathcal{N}(K_2)(x, y)$ is the “kernel” of $K_2$.

For a real separable Hilbert space $\mathcal{H}$, we denote by $\mathcal{AF}(\mathcal{H})$ the antisymmetric Fock space over $\mathcal{H}$:

$$\mathcal{AF}(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{AF}^{(n)}(\mathcal{H}).$$

Here, $\mathcal{AF}^{(0)}(\mathcal{H}) := \mathbb{R}$ and for $n \in \mathbb{N}$ $\mathcal{AF}^{(n)}(\mathcal{H}) := \mathcal{H}^\wedge n$, where $\wedge$ stands for the antisymmetric tensor product and $n!$ is a normalizing factor, so that, for any $f^{(n)} \in \mathcal{AF}^{(n)}(\mathcal{H})$,

$$\|f^{(n)}\|_{\mathcal{AF}^{(n)}(\mathcal{H})}^2 = \|f^{(n)}\|_{\mathcal{H}^\wedge n}^2 n!.$$ 

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We denote by $\mathcal{AF}_{\text{fin}}(\mathcal{H})$ the subset of $\mathcal{AF}(\mathcal{H})$ consisting of all elements $f = (f^{(n)})_{n=0}^{\infty} \in \mathcal{AF}(\mathcal{H})$ for which $f^{(n)} = 0$, $n \geq N$, for some $N \in \mathbb{N}$. We endow $\mathcal{AF}_{\text{fin}}(\mathcal{H})$ with the topology of the topological direct sum of the spaces $\mathcal{AF}^{(n)}(\mathcal{H})$. Thus, the convergence in $\mathcal{AF}_{\text{fin}}(\mathcal{H})$ means uniform boundedness and coordinate-wise convergence.

For $g \in \mathcal{H}$, we denote by $\Phi(g)$ and $\Phi^*(g)$ the annihilation and creation operators in $\mathcal{AF}(\mathcal{H})$, respectively. These are linear continuous operators in $\mathcal{AF}_{\text{fin}}(\mathcal{H})$ defined through the formulas

$$
\Phi(g) h_1 \wedge \cdots \wedge h_n := \sum_{i=1}^{n} (-1)^{i+1} (g, h_i)_\mathcal{H} h_1 \wedge \cdots \wedge \hat{h}_i \wedge h_{i+1} \wedge \cdots \wedge h_n,
$$

$$
\Phi^*(g) h_1 \wedge \cdots \wedge h_n := g \wedge h_1 \wedge \cdots \wedge h_n,
$$

where $h_1, \ldots, h_n \in \mathcal{H}$.

We now set $\mathcal{H} := H_1 \oplus H_2$, where $H_1$ and $H_2$ are two copies of $H$. For $f \in H$, we denote

$$
\Phi_1(f) := \Phi(f, 0), \quad \Phi_2(f) := \Phi(0, f),
$$

and analogously $\Phi_i^*(f)$, $i = 1, 2$. We set, for each $f \in H$,

$$
\Psi(f) := \Phi_2(K_2 f) + \Phi_1^*(K_1 f),
$$

$$
\Psi^*(f) := \Phi_2^*(K_2 f) + \Phi_1(K_1 f).
$$

The operators $\{\Psi(f), \Psi^*(f) \mid f \in \mathcal{H}\}$ satisfy the CAR:

$$
[\Psi(f), \Psi(g)]_+ = [\Psi^*(f), \Psi^*(g)]_+ = 0,
$$

$$
[\Psi^*(f), \Psi(g)]_+ = (f, g)_H \mathbf{1}, \quad f, g \in H,
$$

where $[A, B]_+ := AB + BA$. This representation of the CAR is called quasi-free. The so-called $n$-point functions of this representation have the structure

$$
(\Psi^*(f_1) \cdots \Psi^*(f_l)\Psi(g_1) \cdots \Psi(g_m)\Omega, \Omega)_{\mathcal{AF}(\mathcal{H})} = \delta_{n,m} \det[(K f_i, g_j)_{\mathcal{H}}]_{i,j=1}^{n}.
$$

Here, $\Omega := (1, 0, 0, \ldots)$ is the vacuum vector in $\mathcal{AF}(\mathcal{H})$.

We have the following heuristic representation:

$$
\Psi(f) = \int_X \sigma(dx) f(x) \Psi(x)
$$

$$
= \int_X \sigma(dx) \left( (K_2 f)(x) \Phi_2(x) + (K_1 f)(x) \Phi_1^*(x) \right)
$$

$$
= \int_X \sigma(dx) \left( \Phi_2(x) \int_X \sigma(dy) N(K_2)(x, y) f(y) + \Phi_1^*(x) \int_X \sigma(dy) N(K_1)(x, y) f(y) \right)
$$

$$
= \int_X \sigma(dy) f(y) \left( \int_X \sigma(dx) N(K_2)(x, y) \Phi_2(x) + \int_X \sigma(dx) N(K_1)(x, y) \Phi_1^*(x) \right).
$$
\[
\int_X \sigma(dx) f(x) \left( \Phi_2(x_2,x) + \Phi_1(x_1,x) \right),
\]
and analogously
\[
\Psi^*(f) = \int_X \sigma(dx) f(x) \Psi^*(x)
= \int_X \sigma(dx) f(x) \left( \Phi_2^*(x_2,x) + \Phi_1(x_1,x) \right).
\]

Hence, for \( x \in X \),
\[
\Psi(x) = \Phi_2(x_2,x) + \Phi_1(x_1,x),
\Psi^*(x) = \Phi_2^*(x_2,x) + \Phi_1(x_1,x).
\]

We now heuristically define
\[
a(x) : = \Psi^*(x) \Psi(x), \quad x \in X,
a(\Delta) : = \int_\Delta \sigma(dx) a(x)
= \int_\Delta \sigma(dx) \left( \Phi_2^*(x_2,x) + \Phi_1(x_1,x) \right) \left( \Phi_2(x_2,x) + \Phi_1^*(x_1,x) \right), \quad \Delta \in \mathcal{B}_0(X).
\]

We will now show that it is, in fact, possible to realize \( a(\Delta), \Delta \in \mathcal{B}_0(X) \), as linear continuous operators on \( \mathcal{AF}_{\text{fin}}(\mathcal{H}) \). We first look at the operator
\[
a^+(\Delta) : = \int_\Delta \sigma(dx) \Phi_2^*(x_2,x) \Phi_1^*(x_1,x).
\]
We heuristically have:
\[
a^+(\Delta) h_1 \wedge \cdots \wedge h_n = \int_\Delta \sigma(dx) \Phi_2(x_2,x) \wedge \Phi_1(x_1,x) \wedge h_1 \wedge \cdots \wedge h_n.
\] (31)

Here and below, we identify \( x_{1,x} \) with \((x_{1,x},0)\) and \( x_{2,x} \) with \((0,x_{2,x})\).

To make sense out of (31), we need to show that the informal expression
\[
\int_\Delta \sigma(dx) \Phi_2(x_2,x) \otimes \Phi_1(x_1,x)
\] (32)
identifies an element of the Hilbert space \( \mathcal{H}^\otimes 2 \). So, take any \( u \in H_2 \) and \( v \in H_1 \). Then
\[
\left( \int_\Delta \sigma(dx) \Phi_2(x_2,x) \otimes \Phi_1(x_1,x), u \otimes v \right)_{\mathcal{H}^\otimes 2} = \int_\Delta \sigma(dx) \left( \Phi_2(x_2,x), u \right)_{H_2} \left( \Phi_1(x_1,x), v \right)_{H_1}
\]
i.e., a continuous operator on $H$ where the scalar product is taken in the first two "variables". Therefore, should be rigorously understood as the restriction to $AF(N)$ orthogonal projection of $(\Delta)$.

Thus, the rigorous definition of $a^+(\Delta)$ is as follows:
\[
    a^+(\Delta) h_1 \wedge \cdots \wedge h_n = \mathcal{N}(K_2 P_{\Delta} K_1)_{2,1}^\wedge h_1 \wedge \cdots \wedge h_n,
\]
i.e., $a^+(\Delta)$ is the creation by $\mathcal{N}(K_2 P_{\Delta} K_1)_{2,1}^\wedge$. The $a^+(\Delta)$ is evidently a linear continuous operator on $AF_{\text{fin}}(\mathcal{H})$.

Next, the operator
\[
a^- (\Delta) := \int_\Delta \sigma(dx) \Phi_1(x_1) \Phi_2(x_2)
\]
should be rigorously understood as the restriction to $AF_{\text{fin}}(\mathcal{H})$ of the adjoint operator $(a^+(\Delta))^\ast$. Hence
\[
a^- (\Delta) h_1 \wedge \cdots \wedge h_n = n(n-1) (\mathcal{N}(K_2 P_{\Delta} K_1)_{2,1}^\wedge h_1 \wedge \cdots \wedge h_n)_{H^\otimes 2},
\]
where the scalar product is taken in the first two "variables". Therefore,
\[
a^- (\Delta) h_1 \wedge \cdots \wedge h_n = \sum_{i,j=1, \ldots, n, i \neq j} (-1)^{i+j+\chi_{i<j}} \mathcal{N}(K_2 P_{\Delta} K_1), h_i^{(2)} \otimes h_j^{(1)})_{H^\otimes 2}
\times h_1 \wedge \cdots \wedge \tilde{h}_i \wedge \cdots \wedge \tilde{h}_j \wedge \cdots \wedge h_n.
\]
Here and below, we use the notation $h_i = (h_i^{(1)}, h_i^{(2)})$. Note also that
\[
(\mathcal{N}(K_2 P_{\Delta} K_1), h_i^{(2)} \otimes h_j^{(1)})_{H^\otimes 2} = (h_i^{(2)}, K_2 P_{\Delta} K_1 h_j^{(1)})_{H}.
\]
For the operator
\[
a_1^0 (\Delta) := \int_\Delta \sigma(dx) \Phi_1(x_1) \Phi_1^* (x_1),
\]
we have (recall (27)):
\[ a_0^0(\Delta) h_1 \wedge \cdots \wedge h_n \]
\[ = \int_\Delta \| \kappa_{1,x} \|^2 \sigma(dx) h_1 \wedge \cdots \wedge h_n \]
\[ - \sum_{i=1}^n h_1 \wedge \cdots \wedge h_{i-1} \wedge \left( \int_\Delta \sigma(dx) \left( \kappa_{1,x}, h_i \right) \right) \wedge h_{i+1} \wedge \cdots \wedge h_n. \]  
(37)

For any \( u, v \in H \), we have:
\[ \left( \int_\Delta \sigma(dx) \left( \kappa_{1,x}, u \right) \right) H \sigma \left( \kappa_{1,x}, v \right) \]
\[ = \int_\Delta \sigma(dx) \left( K_1 u \right) \left( K_1 v \right) = \left( K_1 P_1 K_1 u, v \right)_H. \]  
(38)

For any linear operator \( A \) on \( H \), we define the second quantization of \( A \), denoted by \( \Gamma(A) \), as the linear continuous operator on \( A^\infty(H) \) given by
\[ \Gamma(A) \uparrow A^\infty(0)(H) = 0, \]
\[ \Gamma(A) \uparrow A^\infty(n)(H) = A \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes A \otimes 1 \otimes \cdots \otimes 1 \]
\[ + \cdots + 1 \otimes \cdots \otimes 1 \otimes A, \quad n \in \mathbb{N}. \]

We identify the operator \( K_1 P_1 K_1 \) in \( H \) with the operator \( 0 \oplus K_1 P_1 K_1 \) in \( H \). Then, by virtue of (37) and (38), we define:
\[ a_0^0(\Delta) := \int_\Delta \| \kappa_{1,x} \|^2 \sigma(dx) 1 - \Gamma(K_1 P_1 K_1). \]

Analogously, we conclude that the operator \( a_2^0(\Delta) \), which is heuristically given by
\[ a_2^0(\Delta) = \int_\Delta \sigma(dx) \Phi_2^* (\kappa_{2,x}) \Phi_2(\kappa_{2,x}), \]
can be rigorously defined as
\[ a_2^0(\Delta) := \Gamma(K_2 P_1 K_2), \]
where we identified the operator \( K_2 P_1 K_2 \) in \( H \) with the operator \( 0 \oplus K_2 P_1 K_2 \) in \( H \).

We next define, for \( \Delta \in B_0(X) \),
\[ a^0(\Delta) := a_0^0(\Delta) + a_2^0(\Delta) \]
\[ = \int_\Delta \| \kappa_{1,x} \|^2 \sigma(dx) 1 + \Gamma((-K_1 P_1 K_1) \oplus K_2 P_1 K_2). \]  
(39)

We finally set
\[ a(\Delta) := a^+(\Delta) + a^0(\Delta) + a^-(\Delta), \quad \Delta \in B_0(X), \]  
(40)
which are linear continuous operators in \( A^\infty(H) \).
Lemma 2  The operators \( a(\Delta) \), \( \Delta \in \mathcal{B}_0(X) \), commute on \( \mathcal{AF}_{\text{fin}}(\mathcal{H}) \).

Proof. For any \( \Delta_1, \Delta_2 \in \mathcal{B}_0(X) \), we trivially have:

\[
a^+(\Delta_1)a^+(\Delta_2) = a^+(\Delta_2)a^+(\Delta_1). \tag{41}
\]

Next, we evaluate

\[
d\Gamma(K_1P_{\Delta_1}K_1)N(K_2P_{\Delta_2}K_1)_{2,1}^\wedge = ((1 \otimes K_1P_{\Delta_1}K_1)N(K_2P_{\Delta_2}K_1)_{2,1})^\wedge, \tag{42}
\]

where \( \wedge \) denotes antisymmetrization. For any \( u_1 \in H_1 \) and \( u_2 \in H_2 \), we get:

\[
((1 \otimes K_1P_{\Delta_1}K_1)N(K_2P_{\Delta_2}K_1)_{2,1}, u_2 \otimes u_1)_{\mathcal{H}^{\otimes 2}}
\]

\[
= (N(K_2P_{\Delta_2}K_1)_{2,1}, u_2 \otimes K_1P_{\Delta_1}K_1u_1)_{\mathcal{H}^{\otimes 2}}
\]

\[
= (u_2, K_2P_{\Delta_2}K_1K_1P_{\Delta_1}K_1u_1)_{H}
\]

\[
= (u_2, K_2P_{\Delta_2}K_1P_{\Delta_1}K_1u_1)_{H}.
\]

Therefore, \((1 \otimes K_1P_{\Delta_1}K_1)N(K_2P_{\Delta_2}K_1)_{2,1}\) is the kernel of the operator \( K_2P_{\Delta_2}K_1 \) realized as the element of \( H_2 \otimes H_1 \). We denote it by \( N(K_2P_{\Delta_2}K_1)_{2,1} \). Therefore, by \( (42) \),

\[
d\Gamma(K_1P_{\Delta_1}K_1)N(K_2P_{\Delta_2}K_1)_{2,1}^\wedge = N(K_2P_{\Delta_2}K_1)_{2,1}^\wedge. \tag{43}
\]

Analogously, we get, for any \( u_1 \in H_1 \), \( u_2 \in H_2 \),

\[
((K_2P_{\Delta_2}K_1 \otimes 1)N(K_2P_{\Delta_2}K_1)_{2,1}, u_2 \otimes u_1)_{\mathcal{H}^{\otimes 2}} = (u_2, K_2P_{\Delta_1}(1 - K)P_{\Delta_2}K_1u_1)_{H},
\]

and hence,

\[
d\Gamma(K_2P_{\Delta_2}K_1)N(K_2P_{\Delta_2}K_1)_{2,1}^\wedge = N(K_2P_{\Delta_1}(1 - K)P_{\Delta_2}K_1)_{2,1}^\wedge. \tag{44}
\]

By \((43)\) and \((44)\), a straightforward calculation shows that

\[
a^0(\Delta_1)a^+(\Delta_2) + a^+(\Delta_1)a^0(\Delta_2) = a^0(\Delta_2)a^+(\Delta_1) + a^+(\Delta_2)a^0(\Delta_1). \tag{45}
\]

Next, by \((36)\), we have:

\[
a^-(\Delta_1)a^+(\Delta_2)h_1 \wedge \cdots \wedge h_\rho
\]

\[
= ((N(K_2P_{\Delta_1}K_1), N(K_2P_{\Delta_2}K_1))_{\mathcal{H}^{\otimes 2}} 1 + a^+(\Delta_2)a^-(\Delta_1))h_1 \wedge \cdots \wedge h_\rho
\]

\[
- \sum_{i=1}^{\rho} h_1 \wedge \cdots \wedge h_{i-1}
\]

\[
\wedge \left( \int_X \int_X N(K_2P_{\Delta_2}K_1)(x, \cdot)h_i^{(1)}(y)N(K_2P_{\Delta_1}K_1)(x, y) \sigma(dx) \sigma(dy) \right)
\]

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Therefore,
\[
\left( \int_X \int_X \mathcal{N}(K_2 P_{\Delta_2} K_1)(x, \cdot) h_i^{(1)}(y) \mathcal{N}(K_2 P_{\Delta_1} K_1)(x, y) \sigma(dx) \sigma(dy) \right)_H
= \int_X \int_X \int_X \mathcal{N}(K_2 P_{\Delta_2} K_1)(x, z) h_i^{(1)}(y) \mathcal{N}(K_2 P_{\Delta_1} K_1)(x, y) u(z) \sigma(dx) \sigma(dy) \sigma(dz)
= \int_X \sigma(dy) \int_X \sigma(dz) h_i^{(1)}(y) u(z) \int_X \sigma(dx) \mathcal{N}(K_1 P_{\Delta_2} K_2)(z, x) \mathcal{N}(K_2 P_{\Delta_1} K_1)(x, y)
= \int_X \sigma(dy) \int_X \sigma(dz) h_i^{(1)}(y) u(z) \mathcal{N}(K_1 P_{\Delta_2}(1 - K) P_{\Delta_1} K_1)(z, y)
= (K_1 P_{\Delta_2}(1 - K) P_{\Delta_1} K_1 h_i^{(1)}, u)_H.
\]

Therefore,
\[
\int_X \int_X \mathcal{N}(K_2 P_{\Delta_2} K_1)(x, \cdot) h_i^{(1)}(y) \mathcal{N}(K_2 P_{\Delta_1} K_1)(x, y) \sigma(dx) \sigma(dy)
= K_1 P_{\Delta_2}(1 - K) P_{\Delta_1} K_1 h_i^{(1)}.
\]  
(47)

Analogously
\[
\int_X \int_X \mathcal{N}(K_2 P_{\Delta_2} K_1)(\cdot, y) h_i^{(2)}(x) \mathcal{N}(K_2 P_{\Delta_1} K_1)(x, y) \sigma(dx) \sigma(dy)
= K_2 P_{\Delta_2} K_1 P_{\Delta_1} h_i^{(2)}.
\]  
(48)

By (46)–(48),
\[
\begin{align*}
a_-(\Delta_1)a_+(\Delta_2)h_1 \wedge \ldots \wedge h_n
&= \left( (\mathcal{N}(K_2 P_{\Delta_1} K_1), \mathcal{N}(K_2 P_{\Delta_2} K_1))_{H^{\otimes n}} 1 + a_+(\Delta_2)a_-(\Delta_1) \right) h_1 \wedge \ldots \wedge h_n \\
&\quad - d\Gamma ((K_1 P_{\Delta_2}(1 - K) P_{\Delta_1} K_1) \oplus (K_2 P_{\Delta_2} K P_{\Delta_1} K_2)).
\end{align*}
\]  
(49)

Using (49), we conclude that
\[
\begin{align*}
a_+(\Delta_1)a_-(\Delta_2) + a_-(\Delta_1)a_+(\Delta_2) + a_0(\Delta_1)a_0(\Delta_2)
&= a_+(\Delta_2)a_-(\Delta_1) + a_-(\Delta_2)a_+(\Delta_1) + a_0(\Delta_2)a_0(\Delta_1).
\end{align*}
\]  
(50)

By (41), (43), (50) and the equalities obtained by taking the adjoint operators in (41), (43), we conclude the statement of the lemma. \(\Box\)
We will now show that the family \( (a(\Delta))_{\Delta \in B_0(X)} \) has a correlation measure \( \rho \) with respect to the vacuum vector \( \Omega \). Using (29), we informally compute that, for any \( \Delta_1, \ldots, \Delta_n \in B_0(X) \),

\[
Q(\chi_{\Delta_1} \circ \cdots \circ \chi_{\Delta_n}) = \frac{1}{n!} \int_{\Delta_1} \sigma(dx_1) \cdots \int_{\Delta_n} \sigma(dx_n) \Psi^*(x_n) \cdots \Psi^*(x_1) \Psi(x_1) \cdots \Psi(x_n),
\]

so that

\[
Q(\chi_{\Delta_1} \circ \cdots \circ \chi_{\Delta_n}) = \frac{1}{n!} \int_{\Delta_1} \sigma(dx_1) \cdots \int_{\Delta_n} \sigma(dx_n) \Psi^*(x_n) \cdots \Psi^*(x_1) \Phi_1^*(\chi_{1,x_1}) \cdots \Phi_1^*(\chi_{1,x_n}) \Omega.
\]

Hence, we need to make sense out of the following operators, which are heuristically given by

\[
T(\Delta_1, \ldots, \Delta_n)
= \int_{\Delta_n} \sigma(dx_n) (\Phi_2^*(\chi_{2,x_n}) + \Phi_1(\chi_{1,x_n}))
\]

\[
\times \left( \int_{\Delta_{n-1}} \sigma(dx_{n-1}) (\Phi_2^*(\chi_{2,x_{n-1}}) + \Phi_1(\chi_{1,x_{n-1}})) \times \cdots \right.
\]

\[
\left. \times \left( \int_{\Delta_1} \sigma(dx_1) (\Phi_2^*(\chi_{2,x_1}) + \Phi_1(\chi_{1,x_1})) \Phi_1^*(\chi_{1,x_1}) \cdots \Phi_1^*(\chi_{1,x_{n-1}}) \right) \Phi_1^*(\chi_{1,x_n}) \right).
\]

(52)

For Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), we denote by \( \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2) \) the Banach space of linear continuous operators from \( \mathcal{H}_1 \) into \( \mathcal{H}_2 \). Let \( R_{k,n} \in \mathcal{L}(\mathcal{H}^{^\wedge}, \mathcal{H}^{^\wedge^\wedge}) \) and let \( \Delta \in B_0(X) \).

Taking into account (33) and (34), we define an operator

\[
\int_{\Delta} \sigma(dx) \Phi_2^*(\chi_{2,x}) R_{k,n} \Phi_1^*(\chi_{1,x}) \in \mathcal{L}(\mathcal{H}^{^\wedge^\wedge(k-1)}), \mathcal{H}^{^\wedge(n+1)})
\]

as follows: for each \( f^{(k-1)} \in \mathcal{H}^{^\wedge(k-1)} \) we set

\[
\int_{\Delta} \sigma(dx) \Phi_2^*(\chi_{2,x}) R_{k,n} \Phi_1^*(\chi_{1,x}) f^{(k-1)} := \mathcal{P}_{n+1}(1 \otimes (R_{k,n} \mathcal{P}_k))(\mathcal{N}(K_2 P_{\Delta} K_1) \otimes f^{(k-1)}).
\]

Here, \( \mathcal{P}_i \) denotes the orthogonal projection of \( \mathcal{H}^{^\wedge i} \) onto \( \mathcal{H}^{^\wedge^\wedge i} \).

Next, using (26), we easily conclude that the operator-valued function

\[
X \ni x \mapsto \Phi_1(\chi_{1,x}) R_{k,n} \Phi_1^*(\chi_{1,x}) \in \mathcal{L}(\mathcal{H}^{^\wedge(k-1)}), \mathcal{H}^{^\wedge(n+1)})
\]

(51)
is strongly measurable, and Bochner-integrable over $\Delta$ (see e.g. [6] for details on Bochner integral). So, we define

$$\int_{\Delta} \sigma(dx) \Phi_1(\kappa_{1,x}) R_{k,n} \Phi_1^*(\kappa_{1,x}) \in L(\mathcal{H}^{\wedge(k-1)}, \mathcal{H}^{\wedge(n-1)})$$

as a Bochner integral.

Finally, by linearity, for any linear continuous operator $R$ in $\mathcal{AF}_{\text{fin}}(\mathcal{H})$, we define

$$\int_{\Delta} \sigma(dx) \Phi_1(\kappa_{1,x}) R \Phi_1^*(\kappa_{1,x})$$

and

$$\int_{\Delta} \sigma(dx) \Phi_1(\kappa_{1,x}) R \Phi_1^*(\kappa_{1,x})$$

as linear continuous operators in $\mathcal{AF}_{\text{fin}}(\mathcal{H})$. Hence, by induction, the operator (52) is well defined.

**Lemma 3** For each $n \in \mathbb{N}$ and any $\Delta_1, \ldots, \Delta_n \in \mathcal{B}_0(X)$, we have:

$$Q(\chi_{\Delta_1} \odot \cdots \odot \chi_{\Delta_n}) = \frac{1}{n!} T(\Delta_1, \ldots, \Delta_n) \Omega.$$

**Proof.** We first state that, for any $\Delta_1, \Delta_2 \in \mathcal{B}_0(X)$ and any linear continuous operator $R$ in $\mathcal{AF}_{\text{fin}}(\mathcal{H})$, we have

$$a(\Delta_1) \int_{\Delta_2} \sigma(dx) \left( \Phi_2^*(\kappa_{2,x}) + \Phi_1(\kappa_{1,x}) \right) R \Phi_1^*(\kappa_{1,x})$$

$$= \int_{\Delta_2} \sigma(dx) \left( \Phi_2^*(\kappa_{2,x}) + \Phi_1(\kappa_{1,x}) \right) a(\Delta_1) R \Phi_1^*(\kappa_{1,x})$$

$$- \int_{\Delta_1 \cap \Delta_2} \sigma(dx) \left( \Phi_2^*(\kappa_{2,x}) + \Phi_1(\kappa_{1,x}) \right) R \Phi_1^*(\kappa_{1,x}).$$

(53)

Indeed, to show (53) it is sufficient to consider the case where $R = R_{k,n} \in L(\mathcal{H}^{\wedge k}, \mathcal{H}^{\wedge n})$ and $R_{n,k}$ has the form

$$R_{k,n} f^{(k)} = (f^{(k)}, u_1 \wedge \cdots \wedge u_k)_{\mathcal{H}^{\wedge k}} v_1 \wedge \cdots \wedge v_n,$$

with $u_1, \ldots, u_k, v_1, \ldots, v_n \in \mathcal{H}$. But (53) with $R = R_{k,n}$ of such a form can be deduced analogously to the proof of Lemma 2. Now, by virtue of the recurrence formula (6), the statement of Lemma 3 follows from (53) by induction. □

**Remark 1** It is, in fact, possible to rigorously define the operator on the right hand side of (51), and show that equality (51) indeed holds.

**Lemma 4** The family of operators $(a(\Delta))_{\Delta \in \mathcal{B}_0(X)}$ has a correlation measure $\rho$ with respect to $\Omega$, and the restriction of $\rho$ to $(\Gamma^{(n)}_X, \mathcal{B}(\Gamma^{(n)}_X))$ is given by

$$\rho^{(n)}(dx_1, \ldots, dx_n) = \frac{1}{n!} \det [k(x_i, x_j)]_{i,j=1}^n \sigma(dx_1) \cdots \sigma(dx_n)$$

(54)

(recall that we have identified $\mathcal{B}(\tilde{\Gamma}^{(n)}_X)$ with $\mathcal{B}_{\text{sym}}(X^n)$, and $\mathcal{B}(\Gamma^{(n)}_X) \subset \mathcal{B}(\tilde{\Gamma}^{(n)}_X)$).
Proof. By (52) and Lemma 3 for each $n \in \mathbb{N}$ and any $\Delta_1, \ldots, \Delta_n \in \mathcal{B}_0(X)$, we have
\[(Q(\chi_{\Delta_1} \circ \cdots \circ \chi_{\Delta_n}), \Omega)\mathcal{A}\mathcal{F}(\mathcal{H})
= \frac{1}{n!} \left( \int_{\Delta_n} \sigma(dx_n) \Phi_1(\chi_{1,x_n}) \left( \int_{\Delta_{n-1}} \sigma(dx_{n-1}) \Phi_1(\chi_{1,x_{n-1}}) \right) \times \cdots \times \left( \int_{\Delta_1} \sigma(dx_1) \Phi_1(\chi_{1,x_1}) \Phi_1^*(\chi_{1,x_1}) \right) \Phi_1^*(\chi_{1,x_1}) \Omega, \Omega \right)_{\mathcal{A}\mathcal{F}(\mathcal{H})}
= \int_{\Delta_n} \sigma(dx_n) \cdots \int_{\Delta_1} \sigma(dx_1) \|\chi_{1,x_1} \wedge \cdots \wedge \chi_{1,x_n}\|^2_{\mathcal{H}^\wedge n}
= \frac{1}{n!} \int_{\Delta_n} \sigma(dx_n) \cdots \int_{\Delta_1} \sigma(dx_1) \det [k(x_i, x_j)]_{i,j=1}^n \sigma(dx_1) \cdots \sigma(dx_n). \tag{55}
\]
Note that, by (55), the right hand side of (54) indeed defines a measure. Hence, the statement of the lemma follows from (56) \qed

Lemma 5 The correlation measure given in (54) satisfies (LB).

Proof. For each $\Delta \in \mathcal{B}_0(X)$ and $n \in \mathbb{N}$, we evidently have
\[
\rho(\Gamma^{(n)}_\Delta) \leq \left( \int_{\Delta} \|\chi_{1,x}\|^2_{\mathcal{H}} \sigma(dx) \right)^n
= \left( \int_{\Delta} \int_X \mathcal{N}(K_1)(x, y)^2 \sigma(dx) \sigma(dy) \right)^n,
\]
from where the statement follows. \qed

By Lemmas 2, 4, 5, and Theorem 1 we get

Theorem 2 For the family $(a(\Delta))_{\Delta \in \mathcal{B}_0(X)}$ defined through formulas (35), (36), (39), and (40), the statement of Theorem 1 holds with the correlation measure given by (54).

Let us now briefly mention the boson case. About the operator $K$ we make the same assumptions as in the fermion case, apart from the assumption that $K \leq 1$. We set $K_1 := \sqrt{K}$ (just as above) and $K_2 := (1+K)^{1/2}$. We then essentially repeat the fermion case, using however the symmetric Fock space $\mathcal{S}\mathcal{F}(\mathcal{H})$ instead of the antisymmetric Fock space $\mathcal{A}\mathcal{F}(\mathcal{H})$. The operators $\Psi(f), \Psi^*(f)$ (see (28)) now satisfy the CCR (use the commutator $[A, B]_- := AB - BA$ instead of the anticommutator in (29)). The counterpart of formulas (35), (36) reads as follows:
\[(Q(\chi_{\Delta_1} \circ \cdots \circ \chi_{\Delta_n}), \Omega)\mathcal{A}\mathcal{F}(\mathcal{H})
= \int_{\Delta_n} \sigma(dx_n) \cdots \int_{\Delta_1} \sigma(dx_1) \|\chi_{1,x_1} \circ \cdots \circ \chi_{1,x_n}\|^2_{\mathcal{H}^\circ n}
= \frac{1}{n!} \int_{\Delta_n} \sigma(dx_n) \cdots \int_{\Delta_1} \sigma(dx_1) \per[k(x_i, x_j)]_{i,j=1,...,n} \sigma(dx_1) \cdots \sigma(dx_n).
\]
Thus the corresponding correlation measure is given by (54) in which the determinant is replaced by the permanent.
4 Reduced particle densities

Let the operators $K, K_1, K_2$ be as in the fermion part of Section 3. Let $l \in \mathbb{N}$, $l \geq 2$, and we now take $2l$ copies of the Hilbert space $H = L^2(X, \sigma): H_{1,i}$ and $H_{2,i}$, $i = 1, \ldots, l$. We denote $\mathcal{H}^{(l)} := \bigoplus_{i=1}^{l} (H_{1,i} \oplus H_{2,i})$. For each $f \in H$, we consider the following operators in $\mathcal{A}\mathcal{F}(\mathcal{H}^{(l)})$:

$$
\Psi(f) := \sum_{i=1}^{l} \left( \Phi_{2,i}(l^{1/2}K_2 f) + \Phi_{1,i}^*(l^{1/2}K_1 f) \right),
$$

$$
\Psi^*(f) := \sum_{i=1}^{l} \left( \Phi_{2,i}^*(l^{1/2}K_2 f) + \Phi_{1,i}(l^{1/2}K_1 f) \right)
$$

(57)

(we are using obvious notations, analogous to those of Section 3). It is easy to see that these operators satisfy the CAR (29). Furthermore, the $n$-point functions of this representation of the CAR are again given by (30). Therefore, the representation of the CAR given by (57) is unitary equivalent to the representation (28).

The particle density of the representation (57) is heuristically given by

$$
a^{(l)}(x) := \Psi^*(x)\Psi(x)
$$

$$
= \sum_{i=1}^{l} \sum_{j=1}^{l} \left( \Phi_{2,i}^*(l^{-1/2}\kappa_{2,i,x} + \Phi_{1,i}^*(l^{-1/2}\kappa_{1,i,x})
\right) 
\times \left( \Phi_{2,j}(l^{-1/2}\kappa_{2,i,x} + \Phi_{1,j}(l^{-1/2}\kappa_{1,i,x}) \right).
$$

(58)

One can rigorously construct a corresponding family of commuting Hermitian operators, $(a^{(l)}(\Delta))_{\Delta \in \mathcal{B}_0(X)}$, and show that, as expected, the family $(a^{(l)}(\Delta))_{\Delta \in \mathcal{B}_0(X)}$ has the same correlation measure (54) with respect to the vacuum vector $\Omega$.

Now, let us consider the reduced particle density

$$
R^{(l)}(x) := \sum_{i=1}^{l} \left( \Phi_{2,i}^*(l^{-1/2}\kappa_{2,i,x}) + \Phi_{1,i}(l^{-1/2}\kappa_{1,i,x}) \right)
\left( \Phi_{2,i}(l^{-1/2}\kappa_{2,i,x}) + \Phi_{1,i}^*(l^{-1/2}\kappa_{1,i,x}) \right)
$$

(59)

(i.e., we have taken only the “diagonal elements” of the double sum). Analogously to Section 3 one can rigorously realize $(R^{(l)}(\Delta))_{\Delta \in \mathcal{B}_0(X)}$ as a family of commuting Hermitian operators in $\mathcal{A}\mathcal{F}(\mathcal{H}^{(l)})$. The counterpart of formulas (55), (56) reads as follows:

$$
(Q(\chi_{\Delta_1} \otimes \cdots \otimes \chi_{\Delta_n}), \Omega)_{\mathcal{A}\mathcal{F}(\mathcal{H}^{(l)})}
$$

$$
= \sum_{i_1=1}^{l} \cdots \sum_{i_n=1}^{l} \frac{1}{n!} \left( \int_{\Delta_n} \sigma(dx_n) \Phi_{1,i_n}(l^{-1/2}\kappa_{1,i_n,x_n} \right)
$$

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$$x \cdots x \left( \int_{\Delta_1} \sigma(dx_1) \Phi_{1,i_1} \left( l^{-1/2} \mathcal{X}_{1,i_1,x_1} \right) \Phi^*_{1,i_1} \left( l^{-1/2} \mathcal{X}_{1,i_1,x_1} \right) \right)$$

$$\cdots \Phi^*_{1,i_n} \left( l^{-1/2} \mathcal{X}_{1,i_n,x_n} \right) \Omega, \Omega \right)_{\mathcal{A}_F(\mathcal{H}^{(l)})}$$

$$= \int_{\Delta_n} \sigma(dx_n) \cdots \int_{\Delta_1} \sigma(dx_1) \sum_{i_1=1}^l \cdots \sum_{i_n=1}^l \left\| \left( l^{-1/2} \mathcal{X}_{1,i_1,x_1} \right) \wedge \cdots \wedge \left( l^{-1/2} \mathcal{X}_{1,i_n,x_n} \right) \right\|^2_{(\mathcal{H}(l))^{\wedge n}}.$$

(60)

Hence, \((R^{(l)}(\Delta))_{\Delta \in B_0(\mathcal{X})}\) has the correlation measure, whose restriction to \((\Gamma^{(n)}_\mathcal{X}, B(\Gamma^{(n)}_\mathcal{X}))\)

is given by

$$\rho^{(n)}(dx_1, \ldots, dx_n) = \sum_{i_1=1}^l \cdots \sum_{i_n=1}^l \left\| \left( l^{-1/2} \mathcal{X}_{1,i_1,x_1} \right) \wedge \cdots \wedge \left( l^{-1/2} \mathcal{X}_{1,i_n,x_n} \right) \right\|^2_{(\mathcal{H}(l))^{\wedge n}} \sigma(dx_1) \cdots \sigma(dx_n).$$

A combinatoric exercise shows that the \(\rho^{(n)}\) can be written in the form

$$\rho^{(n)}(dx_1, \ldots, dx_n) = \frac{1}{n!} \det_{-1/l} \left[ l \times k(x_i, x_j) \right]_{i,j=1}^n \sigma(dx_1) \cdots \sigma(dx_n).$$

(61)

Here, for any \(\alpha \in \mathbb{R}\) and a square matrix \(A = (a_{i,j})_{i,j=1}^n\), \(\det_{\alpha} A\) denotes the Vere-Jones \(\alpha\)-determinant (see [23]):

$$\det_{\alpha} A := \sum_{\xi \in S_n} \alpha^{n-\nu(\xi)} \prod_{i=1}^n a_{i,\xi(i)},$$

where \(\nu(\xi)\) denotes the number of cycles in the permutation \(\xi\).

The correlation measure (61) satisfies (LB), and so the statement of Theorem 1 holds for the family \((R^{(l)}(\Delta))_{\Delta \in B_0(\mathcal{X})}\). Formula (61) also shows that the corresponding measure on \(\Gamma^{(n)}\), which we denote by \(\mu^{(l)}\) is the fermion-like point process considered in [23].

It is heuristically clear from (59) that the measure \(\mu^{(l)}\) is the \(l\)-fold convolution of fermion point processes corresponding to the operator \(K/l\). This, in fact, can be rigorously shown, since the correlation measure of a convolution of point processes may be easily expressed in terms of the correlation measures of the initial point processes, see also [23].

Finally, an analogous construction can be carried out in the boson case, leading to the correlation function (61), in which \(\det_{-1/l}\) is replaced by \(\det_{1/l}\).

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