THE TROPICAL COMMUTING VARIETY

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Abstract. We study tropical commuting matrices from two viewpoints: linear algebra and algebraic geometry. In classical linear algebra, there exist various criteria to test whether two square matrices commute. Similarly, in the realm of tropical linear algebra, we determine conditions for two tropical matrices that are Kleene stars to commute. Shifting to an algebro-geometric perspective, we explicitly compute the tropicalization of the classical variety of commuting matrices in dimensions 2 and 3.

1. Introduction

There are various ways to study the pairs of $n \times n$ matrices $X$ and $Y$ over a field $k$ that commute under matrix multiplication. Linear algebraically, one can ask for criteria to determine commutativity. Algebro-geometrically, one can study the commuting variety, which is generated by the $n^2$ equations $(XY)_{ij} - (YX)_{ij} = 0$. These perspectives and many other variants have been studied in the classical setting [OCV, §5]. This paper considers similar questions for tropical and tropicalized matrices.

The tropical semiring $(\mathbb{R}, \oplus, \odot)$ is defined by $\mathbb{R} = \mathbb{R} \cup \{\infty\}$, $a \oplus b = \min(a, b)$, $a \odot b = a + b$. It is also known as the min-plus algebra. This algebra has the structure of a semiring, which means it has all the structure of a ring except that it does not in general contain additive inverses. A pair of $n \times n$ tropical matrices $A = (a_{ij})$ and $B = (b_{ij})$ commutes if $A \odot B = B \odot A$, where matrix multiplication takes place in the min-plus algebra. Explicitly, this means that for all $1 \leq i, j \leq n$,

$$(A \odot B)_{ij} := \min_{s=1,\ldots,n} (a_{is} + b_{sj}) = \min_{s=1,\ldots,n} (b_{is} + a_{sj}) =: (B \odot A)_{ij}.$$  

Tropical linear algebra has extensive applications to discrete event systems [BCOQ], scheduling [Bu], pairwise ranking [Tr], and economics [BK, YS2], amongst others. Various authors have approached tropical matrices using group and semigroup theory [IJK, YS1]. However, the tropical analogues of many fundamental results in classical linear algebra remain unknown or unproven. Finding geometric characterizations of commuting tropical matrices is one such example.

Classically, if $A, B \in \mathbb{C}^{n \times n}$ and $A$ has $n$ distinct eigenvalues, then $AB = BA$ if and only if $B$ can be written as a polynomial in $A$ [OCV, §5]. Moreover, if $B$ has $n$ distinct eigenvalues, then $AB = BA$ if and only if $A$ and $B$ are simultaneously diagonalizable. In a similar spirit, Theorem 1.1 gives criterion for a special class of matrices called Kleene stars to commute tropically, illustrated in Figure 1.

Theorem 1.1. Suppose $A, B \in \mathbb{R}^{n \times n}$ are Kleene stars. If $A \oplus B$ is a Kleene star, then $A$ and $B$ commute, that is, $A \odot B = B \odot A$. If $A \odot B = B \odot A$, then $(A \oplus B)^{\odot 2}$ is the Kleene
star of $A \oplus B$. In particular, only for $n = 2$ and $n = 3$ we have: $A \odot B = B \odot A$ if and only if $A \oplus B = A \odot B = B \odot A$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{An illustration of Theorem 1.1 for two $n \times n$ matrices, with proper containment if and only if $n > 3$.}
\end{figure}

Kleene stars are also known as strongly transitive closures [Bu, §1.6.2.1]. They are projectors in the min-plus semiring [KSS, SSB], in the sense that if $A$ is a Kleene star, then $A \odot A = A$. Their image sets are polytopes [JK], so-called because such a set is both a tropical polytope and an ordinary polytope. They appear as a special subclass of normal matrices studied in [LP2]. They play a fundamental role in tropical spectral theory, and thus have been extensively studied [KSS, SSB, Bu, BCOQ]. To the best of our knowledge, this is the first necessary and sufficient characterization of commutativity beyond $A \odot B = B \odot A$ of Kleene stars for $n < 4$.

The second half of our paper considers tropical commuting matrices from the viewpoint of tropical algebraic geometry. This is a relatively young field bridging combinatorics and algebraic geometry. It has many applications, ranging from curve counting [Mi] and number theory [Gu] to phylogenetics [PS]; see the monograph [MS] for more details. We study the relationships between three spaces of pairs of matrices which all ‘commute tropically’ in different senses: the tropical commuting set $\mathcal{T}S_n$, the set of all pairs of tropical commuting matrices; the tropical commuting variety $\mathcal{T}C_n$, the tropicalization of the classical commuting variety; and the tropical commuting prevariety $\mathcal{T}P_n$, the intersection of the tropical hypersurfaces corresponding to the natural generators of the classical commuting variety. Their precise definitions are given in Section 2. Our main result concerns the relationship between these three spaces, illustrated in Figure 2.

**Theorem 1.2.** For $n = 2$, we have
\[ \mathcal{T}P_2 = \mathcal{T}C_2 = \mathcal{T}S_2 \cap \{a_{12} + b_{21} = a_{21} + b_{12}\}. \]
In particular, $\mathcal{T}P_2 = \mathcal{T}C_2 \subset \mathcal{T}S_2$. For $n \geq 3$, $\mathcal{T}C_n \subset \mathcal{T}P_n \cap \mathcal{T}S_n$, and neither $\mathcal{T}P_n$ nor $\mathcal{T}S_n$ are contained in one another.

From the viewpoint of tropical algebraic geometry, the tropical commuting variety $\mathcal{T}C_n$ is perhaps the most natural of the three objects. One way to describe such a tropical variety is to explicitly give its tropical basis. For most varieties, this is a difficult challenge. This is one of the two major open problems in our paper. We present some computational results on the case $n = 3$ in Section 6.2.

**Organization.** In Section 2 we present background, notation, and results in tropical linear algebra. We also illustrate the geometry of commuting Kleene stars, using the characterization of the pre-image of a Kleene star (cf. Theorem 2.4). We use these results to prove...
Figure 2. An illustration of Theorem 1.2 on the relationship between the three spaces of tropicalized commuting matrices.

Theorem 1.1 in Section 3. In Section 4 we review basic concepts in tropical algebraic geometry, and define the three spaces of interest. We compute their homogeneity space and prove Theorem 1.2 in Section 5. In Section 6, we gather some computational results using the software gfan [Je] on the $2 \times 2$ and $3 \times 3$ tropical commuting varieties. A complete description of our computations, including input files, commands and output files, can be found at the public GitHub repository http://github.com/princengoc/tropicalCommutingVariety. We conclude with open problems in Section 7.

2. Background results from tropical linear algebra

We begin with some notation and basic facts in tropical linear algebra that we use in this paper. For a more thorough introduction to tropical linear algebra, see [Bu, §1-3], [MS, §1, §5], or [BCOQ, §3].

For $n$ a positive integer, let $[n] = \{1, 2, \ldots, n\}$. Let $I$ denote the tropical identity matrix, with 0 entries on the diagonal and $\infty$ entries elsewhere. Let $TP^{n-1} := \mathbb{R}^n / \mathbb{R}(1, \ldots, 1)$ be the tropical torus. We will identify $TP^{n-1}$ with $\mathbb{R}^{n-1}$ by the function $(x_1, \ldots, x_n) \mapsto (x_2 - x_1, \ldots, x_n - x_1)$, which has inverse $(x_1, \ldots, x_n) \mapsto (0, x_1, \ldots, x_n)$. We use this function to draw sets in $TP^{n-1}$ in various examples. We define tropical scalar multiplication entrywise: for $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, and $a \in \mathbb{R}$, the vector $a \odot x \in \mathbb{R}^n$ is given by

$$(a \odot x)_i = a + x_i \text{ for all } i \in [n],$$

and the matrix $a \odot A \in \mathbb{R}^{n \times n}$ is given by

$$(a \odot A)_{ij} = a + A_{ij} \text{ for all } i, j \in [n].$$

2.1. Tropical polytopes. Say that a set $C \subset \mathbb{R}^n$ is closed under scalar tropical multiplication if $x \in C$ implies $a \odot x \in C$. Such a set can also be regarded as a subset of $TP^{n-1}$. An example is the tropical convex hull between two points $x, y \in \mathbb{R}^n$, defined as

$$[x, y] = \{a \odot x \oplus b \odot y : a, b \in \mathbb{R}\}.$$ 

It is also called the tropical line segment between $x$ and $y$. A tropical polytope, also known as a tropical semi-module, is the tropical convex hull of finitely many points. An important example is the image of an $n \times n$ matrix $A$:

$$\text{im}(A) := \{y \in \mathbb{R}^n : A \odot x = y \text{ for some } x \in \mathbb{R}^n\}.$$
As a set in $\mathbb{T}P^{n-1}$, $\text{im}(A)$ is a tropical polytope with at most $n$ distinct vertices in $\mathbb{T}P^{n-1}$. See [DS], the first paper on tropical polytopes, for beautiful figures and detailed examples.

2.2. Polytopes and Kleene stars. Another example of a tropical polytope is the tropical eigenspace of $A$, denoted $\text{eigen}(A)$. This is the set of all eigenvectors of $A$. A pair $(\lambda, x) \in \mathbb{R} \times \mathbb{R}^n$ is called a tropical eigenvalue-eigenvector pair of a matrix $A \in \mathbb{R}^{n \times n}$ if $A \odot x = \lambda \odot x$. Unlike in classical linear algebra, a matrix only has one tropical eigenvalue, denoted $\lambda(A)$. As a set in $\mathbb{T}P^{n-1}$, $\text{eigen}(A)$ is both a tropical polytope and an ordinary polytope. Such a set is called a polytrope [JK], or alcoved polytope of type $A$ [LP1]. As an ordinary polytope, a polytrope has the form $\{ x \in \mathbb{R}^n : x_i - x_j \leq A_{ij} \}$. It follows from [Bu, §2] that every polytrope in $\mathbb{T}P^{n-1}$ is the tropical eigenspace of some $n \times n$ matrix.

Many operations such as scalar multiplication or permutation of rows and columns preserve the tropical eigenspace of a matrix. Thus, for a given polytrope $P \subset \mathbb{T}P^{n-1}$, there are infinitely many matrices $A \in \mathbb{R}^{n \times n}$ such that $\text{eigen}(A) = P$. However, one can associate with $P$ a unique matrix $A$ such that $A_{ii} = 0$ for all $i \in [n]$, $\lambda(A) = 0$ and $\text{eigen}(A) = \text{im}(A) = P$ [JK, Tr2]. Such matrices are known as Kleene stars in the literature. This is not how Kleene stars are usually defined, however. We state below the definition from [Bu, §1.6.2.1].

**Definition 2.1.** Let $A \in \mathbb{R}^{n \times n}$ be a matrix with eigenvalue $\lambda(A) = 0$. The Kleene star of a matrix $A \in \mathbb{R}^{n \times n}$, denoted $A^*$, is

$$A^* = I \oplus \bigoplus_{i=1}^{\infty} A^{\odot i}.$$  

A matrix $B \in \mathbb{R}^{n \times n}$ is called a Kleene star if $B^* = B$.

There are many equivalent characterizations of Kleene stars. For ease of reference, we collect the characterizations used in this paper below. See [Bu, §1.6] for proofs.

**Lemma 2.2.** Let $A \in \mathbb{R}^{n \times n}$ with $\lambda(A) = 0$. The following are equivalent.

- $A_{ii} = 0$ for all $i \in [n]$, and $A_{ij} \leq A_{ik} + A_{kj}$ for all $i, j, k \in [n]$.
- $A_{ii} = 0$ for all $i \in [n]$, and $A = A^{\odot 2}$.
- $A = A^*$ (that is, $A$ is a Kleene star by Definition 2.1)
- $A_{ii} = 0$ for all $i \in [n]$, and $\text{eigen}(A) = \text{im}(A)$ is the tropical convex hull of the columns of $A$ (that is, $\text{im}(A)$ is a polytrope).

While our main theorem concerns commuting Kleene stars only, our proof utilizes a number of properties of Kleene stars that apply more generally to $n \times n$ tropical matrices with zero diagonals and eigenvalue zero. Note that such matrices are more general than the class of normal matrices considered in [LP2]. We collect some facts about them here. Only the last two statements are new, and they are directly needed for the proof of Theorem 1.1. Therefore, we only prove those statements. Proofs of the first four statements are direct computations, and can be found in [Bu, §1-3].

**Lemma 2.3.** Suppose $A, B \in \mathbb{R}^{n \times n}$ are matrices with $\lambda(A) = \lambda(B) = 0$, and $A_{ii} = B_{ii} = 0$ for all $i \in [n]$. Then the following hold:

1. $A \odot B \leq A \oplus B$.
2. $A^{\odot (n-1)} = A^*$.
3. $A^{\odot 2} = A$ if and only if $A = A^*$.
4. $A \odot x = x$ if and only if $x$ is in the image of $A^*$.
(5) $\text{im}((A \circ B)^*) = \text{im}((A \oplus B)^*) = \text{im}(A^*) \cap \text{im}(B^*)$.

(6) $(A \circ B)^* = (B \circ A)^* = (A \oplus B)^*$.

**Proof.** Let us prove (5). By (4), we know that $A \circ x = x$ if and only $x$ is in the image of $A^*$, and so

$$\text{im}(A^*) = \{x \in TP^{n-1} : x_i - x_j \leq A_{ij}, i, j \in [n]\}$$

$$\text{im}(B^*) = \{x \in TP^{n-1} : x_i - x_j \leq B_{ij}, i, j \in [n]\}.$$

This implies

$$\text{im}(A^*) \cap \text{im}(B^*) = \{x \in TP^{n-1} : x_i - x_j = \min(A_{ij}, B_{ij}), i, j \in [n]\} = \text{im}((A \oplus B)^*).$$

Consider the first claimed equality, namely that $\text{im}((A \circ B)^*) = \text{im}((A \oplus B)^*)$. As before,

$$\text{im}((A \circ B)^*) = \{x \in TP^{n-1} : x_i - x_j \leq (A \circ B)_{ij}, i, j \in [n]\}.$$

Now, for each $i, j \in [n],$

$$(A \circ B)_{ij} = \min\{A_{ik} + B_{kj} : k \in [n]\} = \min\{A_{ij}, B_{ij}, \min\{A_{ik} + B_{kj} : k \neq i, j\}\}.$$

The second equality follows since $A_{ii} = B_{ii} = 0$ for all $i \in [n]$. Thus

$$\text{im}((A \circ B)^*) \subseteq \text{im}((A \oplus B)^*).$$

Conversely, suppose that $x \in \text{im}(A^*) \cap \text{im}(B^*)$. By the statement (4) of the lemma,

$$A \circ B \circ x = B \circ A \circ x = x,$$

and therefore $x \in \text{im}((A \circ B)^*)$. It follows that $\text{im}(A^*) \cap \text{im}(B^*) \subseteq \text{im}((A \circ B)^*)$. This proves the desired equality.

Now we prove (6). By (5) we know that the polytropes $\text{im}((A \circ B)^*)$ and $\text{im}((A \oplus B)^*)$ are equal to one another. As discussed before Definition 2.1, there is a unique Kleene star whose image is a given polytrope. Since $(A \circ B)^*$ and $(A \oplus B)^*$ are both Kleene stars, it follows that

$$(A \circ B)^* = (A \oplus B)^* = (B \circ A)^* = (A \circ B)^*.$$

This proves (6). \hfill $\square$

### 2.3. The pre-image of a Kleene star.

For a matrix $A \in \mathbb{R}^{n \times n}$ and a vector $b \in \mathbb{R}^n$ in the image of $A$, we can consider the preimage of $b$ under $A$. This is the set of $x \in \mathbb{R}^n$ such that $A \circ x = b$ [Bu, §3.1-3.2]. If $A$ is a Kleene star, then $A \circ 2 = A$. That is, $A$ is a projection of $TP^{n-1}$ onto its image, which is the tropical convex hull of its column vectors. The following theorem gives a simple and explicit formula for this projection. One can view Kleene stars as a closest-point map in the sense studied in [SSB, Kr]. This theorem is a special case of [Bu, Theorem 3.1.1], attributed to Cunningham-Green (1960) and Zimmerman (1976). It was also independently re-discovered by Krivulin [Kr]. The theorem gives a visual way to check if two Kleene stars commute, especially in the case $n = 3$, as illustrated below.

**Theorem 2.4.** Let $A \in \mathbb{R}^{n \times n}$ be a Kleene star. For some $1 \leq k \leq n$, let $I = \{i_1, \ldots, i_k\} \subset [n]$ be a subset of distinct indices from 1 to $n$. Let $I^c = [n] \setminus I$ be the set of indices not in $I$. Suppose that $b \in \text{im}(A) \subset \mathbb{R}^n$ has the form

$$b = \bigoplus_{i \in I} a_i \circ A_i = a_1 \circ A_{i_1} \oplus \ldots \oplus a_k \circ A_{i_k},$$

where
where \( A_i = (A_{i1}, \ldots, A_{in}) \) is the \( i \)-th column of \( A \). Then \( A \odot x = b \) if and only if

\[
x = b + \sum_{j \in I} t_j e_j,
\]

where \( t_j \geq 0 \), and \( e_j \in \mathbb{R}^n \) is the unit vector on the \( j \)-th coordinate. Note that the operations in (2.2) are in the classical algebra.

**Figure 3.** Theorem 2.4 illustrated for a 3 \( \times \) 3 Kleene star \( A \). The figure shows how \( A \) maps points in \( \mathbb{T} \mathbb{P}^2 \) onto its image set.

See Figure 3 for a geometric illustration of Theorem 2.4 for \( n = 3 \). Here we have a Kleene star acting on \( \mathbb{T} \mathbb{P}^2 \). The columns of the Kleene star in \( \mathbb{T} \mathbb{P}^2 \) are represented as dots and their tropical convex hull, a polytrope, is drawn in gray. The three columns of the matrix are fixed by the action, as is their tropical convex hull. The remainder of the plane except for three rays is divided into three regions that are mapped in the directions \((0, -1), (-1, 0), \) or \((1, 1)\). This maps each point to an upside-down tropical line with center at one of the three columns. The rays of these lines which are not in the tropical convex hull are mapped to the point at the center of the tropical line.

Theorem 2.4 gives a geometric check if two Kleene stars commute: for each \( i = 1, \ldots, n \), one just needs to check that \( A \) maps the \( i \)-th column of \( B \) to the same point where \( B \) maps the \( i \)-th column of \( A \). We give some explicit examples below for \( n = 3 \), using three Kleene stars whose images are illustrated in the first picture in Figure 4.

**Example 2.5.** The matrices \( A \) and \( B \) (with circles and boxes) commute. Consider \( \text{im}(A) \cap \text{im}(B) \), which is a hexagon. The vertices of the hexagon are the vertices of \( \text{im}(A \odot B) \); to see this, simply map the columns of \( B \) to \( \text{im}(A) \) as prescribed by Theorem 2.4 and Figure 3. Similarly, the vertices of the hexagon are the vertices of \( \text{im}(B \odot A) \). These vertices are illustrated as dots in the second picture in Figure 4. It follows that \( A \odot B = B \odot A \).

**Example 2.6.** The matrices \( A \) and \( C \) (with circles and crosses) do not commute. The pentagon \( \text{im}(A) \cap \text{im}(C) \) is not \( \text{im}(A \odot C) \) (or \( \text{im}(C \odot A) \)). For instance, the upper-right cross vertex is not mapped to this intersection by the action of \( A \). This is illustrated by dots in the third picture in Figure 4. This means that \( A \) and \( C \) do not commute.
Figure 4. The images of three Kleene stars for $n = 3$, which are three polytropes in $\mathbb{T}P^2$. Let $A$, $B$, and $C$ have the images of their columns labeled by circles, boxes, and crosses, respectively. The middle figure shows that $A$ and $B$ commute. The last figure shows that $A$ and $C$ do not commute.

3. Proof of Theorem 1.1

Since the statement of Theorem 1.1 contains several assertions, we split the proof into three parts for clarity. First, we prove the first two statements. Then, we reiterate the last statement for the case $n = 3$ and prove that separately. Finally, we provide counterexamples for $n > 3$ to complete the proof of the last statement.

Proof of Theorem 1.1, first two statements. For the first statement, we need to show that if $A \oplus B = (A \oplus B)^*$, then $A$ and $B$ commute. By Lemma 2.3, $A \oplus B = (A \oplus B)^2$. We have

$$A \oplus B = A^{\odot 2} \oplus B^{\odot 2} \oplus A \odot B \oplus B \odot A = A \oplus B \odot A \oplus B \odot A.$$  

This implies $A \oplus B \leq A \odot B, B \odot A$. By Lemma 2.3, $A \odot B, B \odot A \leq A \oplus B$. So we must have

$$A \odot B = A \oplus B, \quad B \odot A = A \oplus B,$$

which implies $A \odot B = B \odot A$, and so $A$ and $B$ commute as claimed. We now prove the second statement. Suppose that $A \odot B = B \odot A$. We need to show that $(A \oplus B)^{\odot 2} = (A \oplus B)^*$.

Note that for any $m \geq 2$,

$$(A \oplus B)^{\odot m} = \bigoplus_{k=1}^{m} A^{\odot k} \odot B^{\odot (m-k)}.$$

But since $A$ and $B$ are Kleene stars, $A^{\odot k} = A$ and $B^{\odot (m-k)} = B$ for all $k = 1, \ldots, m$. Thus,

$$(A \oplus B)^{\odot m} = \bigoplus_{k=1}^{m} A \odot B = A \oplus B \oplus A \odot B = A \odot B$$

Therefore, $(A \oplus B)^{\odot 2} = (A \oplus B)^*$.

Statement of Theorem 1.1, the case $n = 3$. For $n = 3$, $A \odot B = B \odot A$ if and only if $A \oplus B = (A \oplus B)^*$. 

□
Proof of Theorem 1.1, the case $n = 3$. The previous proof covers the ‘if’ direction. For the converse, note that $A \odot B = B \odot A$ implies
\[
(A \oplus B)^2 = A \oplus B \oplus A \odot B = A \odot B.
\]
Now, suppose for the sake of contradiction that $A \odot B$ is strictly smaller than $A \oplus B$ at some coordinate, say, $(1, 2)$. That is,
\[
(A \odot B)_{12} = \min \{A_{11} + B_{12}, A_{12} + B_{22}, A_{13} + B_{32}\}
\]
But $A$ and $B$ have zero diagonals, and so
\[
(A \odot B)_{12} = \min \{B_{12}, A_{12}, A_{13} + B_{32}\}.
\]
For strict inequality to occur, we necessarily have $(A \odot B)_{12} = A_{13} + B_{32}$. But $A$ and $B$ are Kleene stars, so
\[
A_{12} \leq A_{13} + A_{32}, B_{12} \leq B_{13} + B_{32}.
\]
Therefore,
\[
A_{32} > B_{32}, \quad A_{13} < B_{13}.
\]
On the other hand, $(A \odot B)_{12} = (B \odot A)_{12}$, and by the same argument, we necessarily have
\[
(B \odot A)_{12} = B_{13} + A_{32} < B_{12} < B_{13} + B_{32},
\]
which implies $A_{32} < B_{32}$, a contradiction. Hence there is no coordinate $(i, j) \in [3] \times [3]$ such that $(A \odot B)_{ij} < A_{ij} \oplus B_{ij}$. In other words, $A \odot B = A \oplus B$, which then implies $A \oplus B = (A \oplus B)^2$.

We have shown the set inclusion
\[
\{(A, B) : A \oplus B = (A \oplus B)^*\} \subseteq \{(A, B) : A \odot B = B \odot A\} \subseteq \{(A, B) : (A \oplus B)^2 = (A \odot B)^*\}.
\]
For $n = 3$, we have shown that
\[
\{(A, B) : A \oplus B = (A \oplus B)^*\} = \{(A, B) : A \odot B = B \odot A\}
\]
\[
\subseteq \{(A, B) : (A \oplus B)^2 = (A \odot B)^*\} = \mathbb{R}^{2n^2}.
\]
To complete the proof of Theorem 1.1, it remains to show that these inclusions are strict for $n \geq 4$. We prove this by providing explicit examples.

The following is an example for $n = 3$ that shows that it is not sufficient to have $A \odot B = A \oplus B$: one needs $A \oplus B = (A \odot B) \oplus (B \odot A)$ for $A$ and $B$ to commute.

Example 3.1. Let
\[
A = \begin{bmatrix}
0.00 & 6.4 & 6.10 \\
3.01 & 0.0 & 0.54 \\
5.41 & 2.4 & 0.00
\end{bmatrix}, \quad B = \begin{bmatrix}
0.00 & 2.25 & 5.04 \\
6.81 & 0.00 & 2.79 \\
4.02 & 6.27 & 0.00
\end{bmatrix}.
\]

In this case, $A \odot B = A \oplus B$, but $B \odot A \neq A \oplus B$. These two matrices differ in the $(1, 3)$ coordinate
\[
(B \odot A)_{13} = 2.79 < (A \oplus B)_{13} = 5.04,
\]
so in particular, $A \odot B \neq B \odot A$.

The following two examples prove the strict set inclusions for $n = 4$. Corresponding examples for $n > 4$ generalize trivially by inflating the current examples with zeros to meet the dimension requirement. Together with Example 3.1, they complete the proof of Theorem 1.1.
Example 3.2 \((A \odot B = B \odot A \text{ but } A \oplus B > (A \oplus B)^*)\). Let

\[
A = \begin{bmatrix}
0.00 & 4.10 & 3.43 & 0.95 \\
4.94 & 0.00 & 1.20 & 5.89 \\
3.74 & 4.44 & 0.00 & 4.69 \\
3.39 & 6.92 & 2.48 & 0.00
\end{bmatrix}, \quad B = \begin{bmatrix}
0.00 & 1.11 & 8.21 & 9.02 \\
6.74 & 0.00 & 7.61 & 9.82 \\
9.96 & 9.56 & 0.00 & 9.77 \\
1.03 & 2.14 & 1.36 & 0.00
\end{bmatrix}.
\]

One can check that \(A \odot B = B \odot A\), but \(A \oplus B\) differs from \((A \oplus B)^2\) in the \((1, 2)\) entry:

\[
(A \oplus B)_{13} = 3.43 > (A \oplus B)^2_{13} = 2.31.
\]

Example 3.3 \(((A \oplus B)^2 = (A \oplus B)^* \text{ but } A \odot B \neq B \odot A\). Let

\[
A = \begin{bmatrix}
0.00 & 1.09 & 4.02 & 3.33 \\
6.77 & 0.00 & 2.93 & 3.47 \\
7.77 & 8.00 & 0.00 & 6.20 \\
3.30 & 1.85 & 1.39 & 0.00
\end{bmatrix}, \quad B = \begin{bmatrix}
0.00 & 5.02 & 1.45 & 2.58 \\
3.53 & 0.00 & 2.01 & 2.12 \\
7.10 & 3.57 & 0.00 & 1.13 \\
7.71 & 6.04 & 2.47 & 0.00
\end{bmatrix}.
\]

Direct computation again shows this is our desired example.

4. Tropicalization of the classical commuting variety

In the previous section we studied tropical commuting matrices from a linear algebraic viewpoint. We will now consider them from a more geometric viewpoint, using tropical geometry. We first review three ways to define tropical geometry: in terms of polynomials over the tropical semiring, in terms of varieties over non-Archimedean fields, and in terms of initial ideals. See [MS] for more background.

4.1. Background on tropical geometry. Consider polynomials over our tropical semiring \((\mathbb{R}, \odot, \oplus)\). A polynomial in \(n\) variables over this semiring can be interpreted as a function from \(\mathbb{R}^n\) to \(\mathbb{R}\) given by the minimum of linear forms, one for each monomial of the polynomial.

Definition 4.1. Let \(f\) be a tropical polynomial in \(n\) variables. The tropical hypersurface \(V(f)\) is the set of all points in \(\mathbb{R}^n\) where the minimum in \(f\) is attained at least twice.

Example 4.2. The tropical polynomial \(f\) given by

\[
f(x, y) := 1 \odot x \odot^2 \oplus x \odot y \oplus 1 \odot y \odot^2 \oplus x \oplus y \oplus 1
\]

is a function from \(\mathbb{R}^2\) to \(\mathbb{R}\), sending \((x, y)\) to \(\min\{1+2x, x+y, 1+2y, x, y, 1\}\). Here \(V(f)\) is the tropical plane curve pictured in Figure 5. This tropical curve divides \(\mathbb{R}^2\) into six unbounded regions, one for each of the six monomials in the polynomial achieving the minimum. The tropical curve consists of four vertices and nine one-dimensional cells (three bounded edges and six rays), and so it has \((4 \cdot 9)\) for its \(f\)-vector, which records the number of cells of each dimension a polyhedral complex contains.

Definition 4.3. A tropical prevariety is the intersection of a finite number of tropical hypersurfaces.

Now we define tropical varieties via the tropicalization of classical varieties over non-Archimedean fields. Let \(k\) be an algebraically closed field with non-trivial non-Archimedean valuation \(\text{val} : k^* \to \mathbb{R}\). One such example is \(k = \mathbb{C}\{\{t\}\}\), the field of Puiseux series over the
complex numbers with indeterminate $t$. This is the algebraic closure of the field of Laurent series over $\mathbb{C}$, and can be defined as

$$\mathbb{C}\{t\} = \left\{ \sum_{i=\ell}^{\infty} a_i t^{i/n} : a_i \in \mathbb{C}, \ell \in \mathbb{Z}, n \in \mathbb{N} \right\},$$

with $\text{val}\left( \sum_{i=\ell}^{\infty} a_i t^{i/n} \right) = \ell/n$ if $a_\ell \neq 0$. In particular, $\text{val}(t) = 1$.

The tropicalization map $\text{trop} : (k^*)^n \to \mathbb{R}^n$ sends points in the $n$-dimensional torus over $k$ into Euclidean space under coordinate-wise valuation:

$$\text{trop} : (a_1, \ldots, a_n) \mapsto (\text{val}(a_1), \ldots, \text{val}(a_n)).$$

**Definition 4.4.** For a variety $X \subset (k^*)^n$, the tropicalization of $X$, denoted $\text{Trop}(X)$, is the closure under the Euclidean topology on $\mathbb{R}^n$ of the image of $X$ under the tropicalization map. A tropical variety in $\mathbb{R}^n$ is the tropicalization of some variety in $(k^*)^n$.

There is another definition for tropical varieties. Let $\omega \in \mathbb{R}^n$. In the classical algebra, the initial form $\text{in}_\omega(f)$ of a polynomial $f = \sum u c_u x^u \in k[x_1, \ldots, x_n]$ is the sum of terms $c_u x^u$, where $\omega \cdot u$ is maximal amongst all monomials of $f$. Let $I \subset k[x_1, \ldots, x_n]$ be an ideal. The initial ideal of $I$ is

$$\text{in}_\omega(I) := \langle \text{in}_\omega(f) : f \in I \rangle.$$ 

**Definition 4.5.** The tropical variety $\mathcal{T}(I)$ defined by $I$ is the set of $\omega \in \mathbb{R}^n$ such that $\text{in}_\omega(I)$ does not contain a monomial.

The tropical variety $\mathcal{T}(I)$ is a polyhedral subcomplex of the Gröbner fan of $I$ [MS, §3].

Finally, there is a third definition of tropical varieties. For a classical polynomial $f = \sum u c_u x^u \in k[x_1, \ldots, x_n]$, let the tropicalization of $f$ be the tropical polynomial $\text{trop}(f) := \bigoplus u \text{val}(c_u) \odot x^u$. Let $I \subset k[x_1, \ldots, x_n]$ be an ideal.

**Definition 4.6.** The tropical variety defined by $I$ is $\bigcap_{f \in I} V(\text{trop}(f))$.

The following theorem says that the three definitions of tropical varieties above coincide. This result was proven in an unpublished manuscript by Mikhail Kapranov in the case of hypersurfaces. See [MS, Theorem 3.2.5] for a proof of the general result.
**Theorem 4.7** (The Fundamental Theorem of Tropical Geometry). Let $I \subset k[x_1, \ldots, x_n]$ be an ideal, and let $X = V(I) \cap (K^*)^n$. Then

\[
\text{Trop}(X) = \bigcap_{f \in I} V(\text{trop}(f)) = \mathcal{T}(I).
\]

If the intersection $\bigcap_{f \in I} V(\text{trop}(f))$ as in Theorem 4.7 can be taken over a particular finite subset of $I$, we call that finite collection of polynomials a *tropical basis* for $	ext{Trop}(X)$. Every tropical variety has a tropical basis, although as we will see in Section 6.2, it may require more elements than a minimal generating set for the classical ideal $I$. (The existence of a tropical basis implies that a tropical variety is indeed a tropical prevariety as defined above.)

### 4.2. The three spaces of tropical commuting matrices

We are now ready to bring the tools of tropical geometry to bear on our commuting matrices. First we need a classical variety describing them. Let $k$ be an algebraically closed non-Archimedean field with non-trivial valuation, and fix an integer $n \geq 2$. Let $S_n = k[x_{ij}, y_{ij}]_{i,j \in \{1, \ldots, n\}}$.

**Definition 4.8.** The commuting ideal $I_n \subset S_n$ is the ideal generated by the $n^2$ classical polynomials

\[
\sum_{i=1}^{n} x_{\ell i} y_{im} - \sum_{j=1}^{n} x_{jm} y_{\ell j}
\]

where $\ell, m \in \{1, \ldots, n\}$. We call the variety $V(I_n)$ the $n \times n$ commuting variety over $k$.

It is irreducible and has dimension $n^2 + n$ [GS, MT]. The points in the variety correspond to pairs of matrices $X, Y \in k^{n \times n}$ that commute with one another.

There are three natural analogous tropical spaces to consider:

- The *tropical commuting set* $\mathcal{T}S_n$, which is the collection of all pairs of $n \times n$ tropical commuting matrices in $\mathbb{R}^{2n^2}$

\[
\mathcal{T}S_n = \{(A, B) \in \mathbb{R}^{2n^2} : A \circ B = B \circ A\}.
\]

- The *tropical commuting variety* $\mathcal{T}C_n$, which is the tropicalization of the commuting variety

\[
\mathcal{T}C_n = \mathcal{T}(I_n) = \bigcap_{f \in I_n} V(\text{trop}(f))
\]

- The *tropical commuting prevariety* $\mathcal{T}P_n$, which is the tropical prevariety defined by the tropicalizations of the $n^2$ equations in (4.1)

\[
\mathcal{T}P_n = \bigcap_{f \text{ of the form (4.1)}} V(\text{trop}(f)).
\]

The classical analogs of these three spaces are all identical; that is, we may identify pairs of commuting matrices with points on the commuting variety, which is precisely the set defined by the equations in (4.1). We will show that, tropically, all of them are different. The difference between the first two illustrates the discrepancy between *tropical commuting matrices* and *tropicalizations of commuting matrices*. 
5. Main results on the three tropical spaces

5.1. Characterization of the homogeneity space. Our first result concerns the homogeneity space of the tropicalization of the commuting ideal $T(I_n)$, denoted $\text{homog}(I_n)$. This is the set of $\omega \in \mathbb{R}^{2n^2}$ such that $in_\omega(I_n) = I_n$. In our case, this set is a subspace of dimension $n + 1$, which coincides with the lineality space of the Gröbner fan of $I_n$ [St]. (Recall that the lineality space $S$ of a fan $F$ is a subspace such that $x + s \in F$ for all $s \in S$.) This homogeneity space is the lineality space of the three sets $TS_n, TC_n$ and $TP_n$ above.

When studying a polyhedral complex like a tropical variety, it is natural to consider such a space modulo its lineality space, since the lineality space does not contribute to any interesting combinatorics or geometry. In this way we decrease the dimension while preserving geometry, giving us a more hands-on object to study. The following result tells us just how much of the dimension comes from the homogeneity space.

Proposition 5.1. Suppose $n \geq 3$. For $\omega = (\omega^x, \omega^y) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n}$, $\omega \in \text{homog}(I_n)$ if and only if there exist $a, b \in \mathbb{R}$ and $c \in \mathbb{R}^n$ such that for all $i, j \in [n]$

$$\omega^x_{i j} = a, \omega^y_{i j} = b, \omega^x_{i j} = \omega^y_{i j} - a + b, \text{ and } \omega^y_{i j} = c_i - c_j + a. \tag{5.1}$$

In particular, $\text{homog}(I_n)$ has dimension $n + 1$ for $n \geq 3$. For $n = 2$, $\omega \in \text{homog}(I_2)$ if and only if there exist $a, b \in \mathbb{R}$ such that

$$\omega^x_{11} = \omega^x_{22} = a, \omega^y_{11} = \omega^y_{22} = b, \omega^y_{12} = \omega^y_{12} - a + b, \text{ and } \omega^y_{21} = \omega^y_{21} - a + b. \tag{5.2}$$

In particular, $\text{homog}(I_2)$ has dimension 4.

Proof. We will show that $\omega$ satisfies Equation (5.1) if and only if $\text{in}_\omega(g_{i j}) = g_{i j}$ for all $i, j \in [n]$. Since the $g_{i j}$’s generate $I_n$, this then implies $
_{\omega}(I_n) = \text{in}(I_n)$.

Suppose $\omega$ is such that $\text{in}_\omega(g_{i j}) = g_{i j}$. For each fixed $i, j \in [n]$, the monomials $x_{i i}y_{i j}$ and $y_{i j}x_{i j}$ have equal weights. Thus $\omega^x_{i i} = \omega^x_{i j} = a$ for all $i, j \in [n]$. Similarly, $\omega^y_{i i} = \omega^y_{i j} = b$. Now, $x_{i i}y_{i j}$ and $x_{i j}y_{i j}$ have equal weights. Thus

$$\omega^x_{i j} = \omega^y_{i j} - \omega^x_{i i} + \omega^y_{i j} = \omega^x_{i j} - a + b \tag{5.3}$$

Consider a triple $i, j, k \in [n]$ of distinct indices. The monomials $x_{i k}y_{k j}$ and $x_{i j}y_{i j}$ have equal weights. Thus

$$0 = \omega^x_{i k} + \omega^y_{k j} - (\omega^x_{i j} + b) = \omega^x_{i k} + (\omega^x_{k j} - a + b) - (\omega^x_{i j} + b) = \omega^x_{i k} + \omega^x_{k j} - a - \omega^x_{i j}. \tag{5.4}$$

Since this holds for all triples $i, j, k \in [n]$, we necessarily have

$$\omega^x_{i j} = c_i - c_j + a \tag{5.5}$$

for some $c \in \mathbb{R}^n$. Thus, $\omega$ is of the form given in Equation (5.1).

Finally, for $n = 2$, Equation (5.3) still holds. So we have Equation (5.2). Define $\omega^x_{12} = c$, $\omega^x_{21} = d$, we see that $\text{homog}(I_2)$ is a linear subspace of $\mathbb{R}^8$ of dimension 4, parametrized by four parameters $a, b, c, d$. $\square$

We remark that the lineality space comes from the transformation $(A, B)$ goes to $(cD^{-1}AD, D^{-1}BD)$, where $D$ is a tropically diagonal matrix.
5.2. **Proof of Theorem 1.2.** For ease of reference, we split the proof into two parts: the case \( n = 2 \), and the case \( n = 3 \). The case of general \( n \) follows by immediate generalization of the examples given for \( n = 3 \).

**Proof of Theorem 1.2 for \( n = 2 \).** Computation with \texttt{gfan} shows that the tropical prevariety equals the tropical variety, that is, \( T\mathcal{P}_2 = T\mathcal{C}_2 \). In other words, the three polynomials

\[
\begin{align*}
g_{11} &= x_{12}y_{21} - y_{12}x_{21}, \\
g_{12} &= x_{11}y_{12} + x_{12}y_{22} - y_{11}x_{12} - y_{12}x_{22}, \text{ and} \\
g_{21} &= x_{21}y_{11} + x_{22}y_{21} - y_{21}x_{11} - y_{22}x_{21}
\end{align*}
\]

form a tropical basis for \( T(I_2) \). It remains to prove the claimed relation to \( T\mathcal{S}_2 \). Consider a point \((A, B) \in T\mathcal{S}_2\). By the definition of \( T\mathcal{S}_2 \), we know

\[
\min\{a_{11} + b_{12}, a_{12} + b_{22}\} = \min\{b_{11} + a_{12}, b_{12} + a_{22}\}, \text{ and}
\min\{a_{21} + b_{11}, a_{22} + b_{21}\} = \min\{b_{21} + a_{11}, b_{22} + a_{21}\}.
\]

So two of the generators of our tropical basis, namely \( g_{12} \) and \( g_{21} \), are tropically satisfied. Therefore, \((A, B) \in T\mathcal{C}_2\) if and only if \( g_{11}\) is also tropically satisfied, which means

\[
a_{12} + b_{21} = a_{21} + b_{12}.
\]

It follows that \( T\mathcal{S}_2 \cap \{a_{12} + b_{21} = a_{21} + b_{12}\} = T\mathcal{C}_2 \), as claimed. \(\square\)

**Proof of Theorem 1.2 for \( n = 3 \).** To see that each region in Figure 2b is really nonempty, consider the following examples.

(a) A pair in \((T\mathcal{P}_3 \cap T\mathcal{S}_3) \setminus T\mathcal{C}_3\). Consider the pair

\[
A = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 8 \\ 0 & 4 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 12 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 6 \end{bmatrix}.
\]

Direct computation shows that \((A, B) \in T\mathcal{P}_3 \cap T\mathcal{S}_3\). Computations with \texttt{gfan} show that the initial monomial ideal with this weight vector contains the monomial \( x_{31}y_{12}y_{31}y_{21} \). Thus, \((A, B) \notin T\mathcal{C}_3\). To see how this is possible, note that the polynomial with the leading term \( x_{31}y_{12}y_{31}y_{21} \) is given by

\[
(XY - YX)_{31}y_{32}y_{21} - (XY - YX)_{32}y_{31}y_{21} - (XY - YX)_{21}y_{31}y_{32}.
\]

Each of the three terms \((XY - YX)_{31}, (XY - YX)_{32}\) and \((XY - YX)_{21}\) is a sum of six monomials, two of which are initial monomials. This gives 18 monomials in total with 6 initial monomials. However, the six initial monomials come in three pairs, which are cancelled out by the signs. So (5.6) has 12 monomials, and the weights are such that there is a unique leading term.

(b) A pair in \( T\mathcal{S}_3 \setminus T\mathcal{P}_3 \). Consider the pair

\[
C = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 0 & 4 \\ 4 & 4 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 2 & 4 \\ 1 & 0 & 4 \\ 4 & 4 & 0 \end{bmatrix}
\]

Direct computation shows that these matrices commute. The containment in \( T\mathcal{P}_3 \) fails in the \((1,1)\) and the \((2,2)\) entries of the products.
(c) A pair in $TP_3 \setminus TS_3$. Consider the pair

$$E = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 1 \\ 0 & 3 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}.$$ 

Direct computation shows that matrices fail to commute in the $(3,3)$ entry of the products, and that $(E,F) \in TP_3$.

In summary, we have $(A,B) \in (TP_3 \cap TS_3) \setminus TC_3$, $(C,D) \in TS_3 \setminus TP_3$, and $(E,F) \in TP_3 \setminus TS_3$. □

6. Computational results on the tropical commuting variety

6.1. The $2 \times 2$ Tropical Commuting Variety. Using Theorem 1.2, or using gfan [Je], one finds that the tropical variety $TC_2$ lives in an 8-dimensional ambient space, corresponding to the four $x_{ij}$ and the four $y_{ij}$ coordinates. It is 6-dimensional, with a 4-dimensional lineality space, which is the homogeneity space given in Proposition 5.1. Modding out by this lineality space gives a 2-dimensional fan with f-vector $(1, 4, 6)$, meaning there are four rays and six 2-dimensional cones, as illustrated in Figure 6, with the rays meeting at the origin. The tropical variety is simplicial and pure.

![Figure 6. The tropical variety $TC_2$ modulo its lineality space](image)

Example 6.1. Let $k = \mathbb{C}\{t\}$ be the field of Puiseux series over $\mathbb{C}$ with the usual valuation. Theorem 1.2 tells us when commuting $2 \times 2$ tropical matrices with entries in $\text{val}(k)$ can be lifted to commuting matrices over $k$. Since the pair of matrices

$$\left( \begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 1 & -1 \end{bmatrix} \right)$$

satisfies $4 + 1 = 2 + 3$, so by Theorem 1.2, it can be lifted, for instance to the pair of commuting matrices

$$\left( \begin{bmatrix} 1 + t & t^4 \\ t^2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & t^3 \\ t & t^{-1} \end{bmatrix} \right).$$

Example 6.2. Consider the pair of matrices $\left( \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right)$. These commute under tropical matrix multiplication, but do not tropically satisfy the polynomial $g_{11} = x_{12}y_{21} - y_{12}x_{21}$. Thus, this pair of matrices is in $TS_2$, but not in $TP_2$. 
6.2. Towards a tropical basis: the gap between the tropical prevariety and the tropical variety. Every tropical variety has a tropical basis. However, few natural tropical varieties have had their bases explicitly characterized. This is an important and challenging computational problem in tropical geometry. For the tropical commuting variety, we computed the basis for $TC_2$ in Theorem 1.2. We now gather some preliminary computational results for $TC_3$. We hope that these results will give insights towards computing its tropical basis, and perhaps for the general case $TC_n$.

The tropical variety $TC_3$ lives in an 18-dimensional ambient space, corresponding to the nine $x_{ij}$ and the nine $y_{ij}$ coordinates. It is a 12-dimensional space with a 4-dimensional lineality space, which is the homogeneity space of $T(I_3)$ given in Proposition 5.1. Modding out gives us an 8-dimensional space. The f-vector, as calculated using gfan, is

$$ (1 \ 1658 \ 23755 \ 143835 \ 481835 \ 972387 \ 1186489 \ 808218 \ 235038), $$

which ranges from the 1658 rays to the 235,038 8-dimensional cones. The tropical variety is pure, but not simplicial.

By comparison, the tropical prevariety is much bigger than the tropical variety. The prevariety is neither pure nor simplicial. Modulo the lineality space, its largest cones are of dimension 10. Its $f$-vector is

$$ (1 \ 146 \ 2290 \ 16322 \ 66193 \ 162886 \ 241476 \ 199030 \ 71766 \ 2397 \ 58). $$

We now consider what polynomials might be added to the usual generating set to obtain a tropical basis. We do this by adding in polynomials in the classical ideal which, when considered tropically, give a smaller and smaller tropical prevariety. In doing so we may assume they are part of a tropical basis. As shown in the proof of Theorem 1.2, apart from the generators of the pre-variety, a tropical basis for $TC_3$ can contain the polynomial

$$(XY - YX)_{31}Y_{32}Y_{21} - (XY - YX)_{32}Y_{31}Y_{21} - (XY - YX)_{21}Y_{31}Y_{32}.$$ 

It can also contain all of its permutations under the action of $S_3 \times S_2$ on the space of pairs of matrices $\mathbb{R}^{n\times n\times 2}$, by permuting the rows and columns of the matrices simultaneously and swapping the two matrices. By a similar argument, another set of polynomials in a tropical basis are all permutations of

$$(XY - YX)_{12}Y_{21} - (XY - YX)_{21}Y_{12}.$$ 

However, these two sets of polynomials cannot alone account for the gap in the dimension of the maximal cones between $TC_3$ and $TP_3$, as shown by computations with gfan. A full tropical basis of $TC_3$ contains more (but finitely many more) polynomials. Computing such a basis explicitly is an interesting open question.

6.3. The symmetric commuting pre-variety. As a first step to computing a tropical basis of $TC_3$, we study the analogue of $TP_3$ and $TC_3$ for pairs of symmetric matrices, that is, commuting pairs $(X,Y)$ with $X = X^T, Y = Y^T$. These two varieties live in a 12-dimensional ambient space, corresponding to six $x_{ij}$ and six $y_{ij}$ coordinates. The ideal $I_3^{sym}$ is generated by the following three polynomials

$$(XY)_{12} - (YX)_{12} = x_{11}y_{12} - y_{11}x_{12} + x_{12}y_{22} - y_{12}x_{22} + x_{13}y_{23} - y_{13}x_{23}$$

$$(XY)_{13} - (YX)_{13} = x_{11}y_{13} - y_{11}x_{13} + x_{12}y_{23} - y_{12}x_{23} + x_{13}y_{33} - y_{13}x_{33}$$

$$(XY)_{23} - (YX)_{23} = x_{12}y_{13} - y_{12}x_{13} + x_{22}y_{23} - y_{22}x_{23} + x_{23}y_{33} - y_{23}x_{33}$$
The symmetric tropical commuting variety is 9-dimensional, with a 2-dimensional lineality space. Its f-vector is 
\[
\begin{pmatrix}
1 & 66 & 705 & 3246 & 7932 & 10878 & 8184 & 2745 
\end{pmatrix}.
\]

By comparison, the symmetric tropical commuting prevariety is only one dimension bigger. It has dimension 10, also with a 2-dimensional lineality space. Its f-vector is 
\[
\begin{pmatrix}
1 & 39 & 375 & 1716 & 4359 & 6366 & 5136 & 1869 & 6 
\end{pmatrix}.
\]

We investigated the orbit of the six cones full-dimensional cones of the symmetric tropical commuting prevariety under the action of $\mathbb{S}_3 \times \mathbb{S}_2$. We found that these six cones form three orbits. We name them type I, II and III. Type I has orbit size 1, with initial monomials 
\[
x_{13}y_{23} - x_{23}y_{13}, \quad x_{12}y_{23} - x_{23}y_{12}, \quad x_{12}y_{13} - x_{13}y_{12}
\]

Type II has orbit size 2, with initial monomials 
\[
x_{12}y_{11} - x_{12}y_{22}, \quad x_{13}y_{11} - x_{13}y_{33}, \quad x_{23}y_{22} - x_{23}y_{33}
\]

Type III has orbit size 3, with initial monomials 
\[
x_{11}y_{12} - x_{12}y_{11}, \quad x_{11}y_{13} - x_{13}y_{11}, \quad x_{12}y_{13} - x_{13}y_{12}.
\]

We have not yet understood the constraints that lead to these three orbits. In theory, since there are three generators with six terms, there can be at most $\binom{6}{2}^3 = 15^3$ possible cones of the symmetric tropical commuting prevariety with maximal dimension. It remains a mystery to us why only the above six cones are full-dimensional. Unlocking this mystery may be the key towards finding a tropical basis for the symmetric tropical commuting variety.

7. Summary and Future Directions

As has occurred in other works [BJMS, Ch], we find a discrepancy between a tropicalization and a natural tropical analog: the tropicalization of the commuting variety does not contain all pairs of matrices that commute tropically. This is not to say that either object is the wrong object to study: the tropicalization of the commuting variety has nicer geometric properties and is more relevant for lifting to a non-Archimedean field, while the tropical commuting set is more natural when working with min-plus linear algebra. They are both relevant spaces in their own right, and an important goal is to understand the difference between them, as with many objects in tropical geometry. In particular, two major open problems remain in dimensions $n \geq 3$: to give necessary and sufficient conditions for two tropical matrices to commute; and to determine a description for a tropical basis for the tropical commuting variety.

Another research direction is to consider triples of pairwise-commuting $n \times n$ matrices. It was shown in [Ge, GS] that the variety of triples of commuting $4 \times 4$ matrices is irreducible for $n \leq 4$ but reducible for $n \geq 32$. More generally, one can study the space $\mathcal{C}(d,n)$ of commuting $d$-tuples of $n \times n$ matrices. For $d \geq 4$ and $n \geq 4$, this variety is reducible [Ge]. Studying the tropical analogues of these spaces would be a natural generalization of the work we have done here.

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