A note on unfolding manifolds of meromorphic connections on the Riemann sphere with unramified singularities

Dedicated to Prof. Yoshitsugu Takei on the occasion of his 60th birthday

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Abstract
This note explains a construction of a Poisson manifold whose symplectic foliation describes a deformation of a moduli space of meromorphic connections with unramified irregular singularities. In particular, this deformation of the moduli space corresponds to the unfolding of irregular singularities of the meromorphic connections. This is an announcement of some results in the forthcoming paper.

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Introduction

This note is an announcement of some results in the forthcoming paper [4]. Detailed descriptions of the statements without proofs in this note shall be given in [4].

For linear ordinary equations on complex domains, the confluence of singular points is a classical tool to investigate irregular singularities. For example, it is well-known that each differential equation of Airy, Hermite-Weber, and Kummer confluent hypergeometric functions is obtained from the differential equation

\[ z(1 - z) \frac{d^2}{dz^2} w + (c - (a + b + 1)z) \frac{d}{dz} w - abw = 0 \]

of the Gauss hypergeometric function by the confluence of singular points \( z = 0, 1, \infty \), and further, analytic properties of these functions can be related to that of the Gauss hypergeometric function. Similarly for the Heun differential equation which has an accessory parameter and is known as a generalization of Gauss hypergeometric differential equations, there is a well-known family of differential equations obtained by the confluence of singular points, confluent Heun, bi-confluent Heun, tri-confluent Heun, and doubly-confluent Heun equations. Furthermore, for the study of higher dimensional Painlevé equations, Kawakami-Nakamura-Sakai constructed many confluent families of differential equations with 4 accessory parameters in [5].

We may notice that these confluence families of differential equations are considered as deformations of spaces of their accessory parameters, namely, we can say that the confluence of singularities gives rise to deformations of moduli spaces of differential equations. Indeed in [2], Chekhov-Mazzocco-Rubtsov defined a deformation of Painlevé monodromy manifolds associated to the confluence of singular points of differential equations. Namely, they considered monodromy manifolds of linear ordinary differential equations on the Riemann sphere whose isomonodromic deformations give rise to Painlevé equations, and obtained explicit deformations of these manifolds which correspond to the confluence of singular points of differential equations. In case of moduli spaces of connections,
Inaba constructed in [8] a one-parameter deformation of moduli spaces of meromorphic connections with unramified irregular singular points whose Hukuhara-Turrittin-Levelt normal forms have distinct eigenvalues. A similar deformation also considered by Gaiur-Mazzocco-Rubtsov in [3], in which they defined a deformation of coadjoint orbits of a Lie algebra of polynomials which they call the Takiff algebra. Furthermore, Gaiur-Mazzocco-Rubtsov and Inaba considered in their papers [3] and [8] the confluence of isomonodromic deformation equations.

Based on these preceding works, we shall construct a deformation space of moduli space of differential equations describing the confluence of their singularities. Let us explain our main result. We focus on the opposite operation to the confluence, i.e., the unfolding of irregular singular points. Namely, we consider the procedure to unfold an unramified irregular singular points to regular singular ones. Let us consider meromorphic connections on a trivial bundle defined over the complex projective line \( \mathbb{P}^1 \) with unramified irregular singularities on a finite subset \( D \subset \mathbb{P}^1 \). We fix a collection of unramified Hukuhara-Turrittin-Levelt normal forms \( H = (H_a)_{a \in D} \) at each singular point \( a \in D \).

Then Boalch introduced in [1] the moduli space of irreducible meromorphic connections with the fixed normal forms \( H \) which we denote by \( \mathcal{M}_s^*(H) \). Firstly, we introduce the deformation \( H(c), c \in \mathbb{C}^{NH} \) of the collection of normal forms \( H \), where \( NH \) is the positive integer uniquely determined by \( H \). This is a family of unramified Hukuhara-Turrittin-Levelt normal forms describing the unfolding procedure of unramified irregular singular points to regular singular ones. Then our main theorem is the following.

**Theorem 0.1** (Theorem 2.15, Theorem 2.16). Suppose that \( \mathcal{M}_s^*(H) \neq \emptyset \). Then, there are a Zariski open subset \( \mathbb{D}(H) \subset \mathbb{C}^{NH} \), holomorphic Poisson manifold \( \mathcal{M}_s^*(H)_{\mathbb{D}(H)} \), and holomorphic map \( \theta: \mathcal{M}_s^*(H)_{\mathbb{D}(H)} \to \mathbb{D}(H) \) such that for each \( c \in \mathbb{D}(H) \) there exits an embedding

\[
\theta^{-1}(c) \hookrightarrow \mathcal{M}_s^*(H(c))
\]

onto a Zariski open subset. Namely, the complex manifold \( \mathcal{M}_s^*(H)_{\mathbb{D}(H)} \) is a deformation of the moduli space \( \mathcal{M}_s^*(H) \) and moreover each fiber \( \theta^{-1}(c) \) is isomorphic to a Zariski open subset of the moduli space of meromorphic connections \( \mathcal{M}_s^*(H(c)) \) obtained by the unfolding of \( H \).

Furthermore, the family of the fibers \( (\theta^{-1}(c))_{c \in \mathbb{D}(H)} \) is the symplectic foliation of the regular Poisson manifold \( \mathcal{M}_s^*(H)_{\mathbb{D}(H)} \) and the above embeddings are symplectic morphisms.

Our families \( \mathcal{M}_s^*(H)_{\mathbb{D}(H)} \) of moduli space cover many of known deformations of differential equations induced by the confluence of their singular points. For example, confluent families explained above, i.e., families of Gauss hypergeometric equation, Heun equation, and equations with 4 accessory parameters appearing in [5] by Kawakami-Nakamura-Sakai are all contained in our families when we focus on unramified irregular singularities. Our construction of the deformation is strongly influenced by Oshima’s work in [12]. In that paper, Oshima defined a good confluence family of linear differential equations on the Riemann sphere without accessory parameter which he called the versal family of rigid differential equations. Our result can be said to be a generalization of his versal family to non-rigid cases. For instance, Oshima’s families recovers from our deformation spaces of moduli spaces of rigid connections.

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1 Moduli space of meromorphic connections on a trivial bundle over the Riemann sphere with unramified irregular singularities

In the paper [9], Jimbo, Miwa, and Ueno developed a general study of symplectic structure of the spaces of accessory parameters of differential equations with irregular singular points. After their work, in [1], Boalch introduced a natural construction of the symplectic structure from the symplectic geometry of coadjoint orbits and the theory of symplectic reduction. Following Boalch’s method, we give a review of the symplectic structure of the moduli spaces of meromorphic connections on the trivial bundle with unramified irregular singular points. A detailed exposition of the contents in this section can also be found in [16].

1.1 Hukuhara-Turrittin-Levelt normal forms

A classification theory of local and formal differential equations is obtained by Hukuhara, Turrittin, and Levelt, see [7], [13], [10]. We give a quick review of this classification theory.

Definition 1.1. Let \( R \) be either the ring of formal power series \( \mathbb{C}[z] \) or the field of Laurent series \( \mathbb{C}((z)) \), and \( V \) be a free \( R \)-module of finite rank. Let us consider a \( \mathbb{C} \)-linear map \( \nabla: V \to V \otimes_R \mathbb{C}((z))dz \) satisfying the Leibniz rule,

\[
\nabla(fv) = v \otimes (df) + f \nabla(v), \quad f \in R, \ v \in V,
\]

which is called meromorphic connection or connection on \( V \). Here we denote the vector space of formal meromorphic 1-forms by \( \mathbb{C}((z))dz := \{ f(z)dz \mid f(z) \in \mathbb{C}((z)) \} \), and \( d: \mathbb{C}((z)) \to \mathbb{C}((z))dz \) stands for the exterior derivative defined by \( df(z) := \left( \frac{d}{dz}f(z) \right) dz \) for \( f(z) \in R \). Then we also call the pair \( (\nabla, V) \) a formal meromorphic connection over \( R \).

Let \( (\nabla, V) \) be a connection over \( \mathbb{C}((z)) \) of rank \( n \) and consider an algebraic field extension \( L \) of \( \mathbb{C}((z)) \). Then Newton-Puiseux theory tells us that there exists a positive integer \( q \in \mathbb{Z}_{>0} \) and \( L \cong \mathbb{C}((t)) \) with \( t^q = z \). Then the inclusion map \( \iota: \mathbb{C}((z)) \ni f(z) \to f(t^q) \in \mathbb{C}((t)) \) induces the map

\[
\iota_*: \mathbb{C}((z))dz \longrightarrow \mathbb{C}((t))dt
\]

\[
f(z)dz \mapsto f(t^q)qt^{q-1}dt
\]

which is compatible with the exterior derivative. Then we can define the connection \( (\nabla_t, V_t) \) over \( \mathbb{C}((t)) \) which we call the extension of \( (\nabla, V) \) to \( \mathbb{C}((t)) \) as follows. Let \( V_t := \mathbb{C}((t)) \otimes_{\mathbb{C}((z))} V \) be the extension of scalars of \( V \) with the inclusion map \( \iota_V: V \ni v \mapsto \)
1 ⊗ v ∈ V_t as \( \mathbb{C}((z)) \)-vector spaces. As well as the above, we have the induced morphism as \( \mathbb{C}((z)) \)-vector spaces,

\[
\iota_V : \quad V \otimes_{\mathbb{C}((z))} \mathbb{C}((z))dz \to V_t \otimes_{\mathbb{C}((z))} \mathbb{C}((t))dt \\
v \otimes f(z)dz \mapsto \iota_V(v) \otimes \iota_V(f(z)dz)
\]

Then we define the following \( \mathbb{C} \)-linear map \( \nabla_t : V_t \to V_t \otimes_{\mathbb{C}((t))} \mathbb{C}((t))dt \) by

\[
\nabla_t(f(t) \otimes v) := \iota_V(v) \otimes df(t) + f(t) \otimes \iota_V \nabla(v), \quad f(t) \in \mathbb{C}((t)), \ v \in V.
\]

The classification theory of connections over the algebraic closure \( \mathbb{C}((z))^{alg} \) was developed by Hukuhara and Turrittin independently and improved and simplified as in the following manner by Levelt afterward. We say that a connection \( (\nabla, V) \) over \( \mathbb{C}((t)) \) is diagonalizable when \( V \) is a direct sum of 1-dimensional \( \nabla \)-invariant subspaces \( V_i \), i.e., \( \nabla(V_i) \subset V_i \otimes \mathbb{C}((t))dt \).

**Theorem 1.2** (Hukuhara, Turrittin, Levelt, [7], [13], [10]). Let \( (\nabla, V) \) be a connection over \( \mathbb{C}((z)) \). Then there exists an algebraic field extension \( \mathbb{C}((z)) \) of \( \mathbb{C}((z)) \) of degree \( q \) such that \( \nabla_t \) has the decomposition

\[
t \nabla_t = S + N
\]

where \( S \) is a diagonalizable connection on \( V_t \), \( N \) is a nilpotent \( \mathbb{C}((t)) \)-linear map and \( S \) and \( N \) commute with each other, i.e., \([S, N] := SN - NS = 0\). The smallest integer \( q \) satisfying the above is called ramification index of the connection. If the ramification index is \( q = 1 \), we say that \( \nabla \) is unramified.

Furthermore, there exists a basis of \( V_t \) over \( \mathbb{C}((t)) \) under which \( S \) and \( N \) can be written as

\[
S = td_t - \sum_{i=0}^{k} S_i t^{-i} dt, \quad N = -N_0 dt
\]

with diagonal matrices \( S_i \in M_n(\mathbb{C}), i = 0, 1, \ldots, k \) and a nilpotent matrix \( N_0 \in M_n(\mathbb{C}) \). Here \( n \) is the rank of \( V \) and \( Adt \) for \( A \in M_n(\mathbb{C}((t))) \) denotes the \( \mathbb{C}((t)) \)-linear map \( V_t \ni v \mapsto Av dt \in V_t \otimes_{\mathbb{C}((t))} \mathbb{C}((t))dt \).

This leads to the following definition of normal forms in \( M_n(\mathbb{C}((t)))dt \) which are usual Jordan normal forms if \( k = 1 \).

**Definition 1.3** (HTL normal form). A Hukuhara-Turrittin-Levelt normal form or HTL normal form for short, is an element in \( M_n(\mathbb{C}((t)))dt \) of the form

\[
\left( \frac{S_k}{t^k} + \cdots + \frac{S_1}{t} + S_0 + N_0 \right) \frac{dt}{t}
\]

where \( S_i \in M_n(\mathbb{C}) \) are diagonal matrices commute with the nilpotent matrix \( N_0 \in M_n(\mathbb{C}) \), i.e., \([S_i, N_0] = 0\) for all \( i = 0, 1, \ldots, k - 1 \). The above theorem says that for a connection \( (V, \nabla) \) over \( \mathbb{C}((z)) \), we can find an extension field \( \mathbb{C}((t)) \) with \( t^q = z \) and a HTL normal form \( H \in M_n(\mathbb{C}((t))) \) such that

\[
\nabla_t = d - H
\]

under a suitable basis of \( V_t \) over \( \mathbb{C}((t)) \). This normal form \( H \) is called the HTL normal form of \( (V, \nabla) \). In particular, we say that \( (V, \nabla) \) is an unramified connection if we can take an HTL normal form \( H \) from \( M_n(\mathbb{C}((z)))dz \subset M_n(\mathbb{C}((t)))dt \). In this case, we also call this HTL normal form \( H \in M_n(\mathbb{C}((z)))dz \) an unramified HTL normal form.
Fix an unramified HTL normal form

\[ H = \left( \frac{S_{k-1}}{z^k} + \cdots + \frac{S_1}{z} + S_0 + N_0 \right) \frac{dz}{z} \in M_n(\mathbb{C}(z)dz) \]

and consider an unramified meromorphic connection \((\nabla_H, \mathbb{C}[z]^{\oplus n})\) over \(\mathbb{C}[z]\) defined by

\[ \nabla_H := d - H. \]

Then, a connection \((\nabla, \mathbb{C}[z]^{\oplus n})\) is isomorphic to \((\nabla_H, \mathbb{C}[z]^{\oplus n})\) if and only if there exists \(g(z) \in \text{GL}_n(\mathbb{C}[z])\) such that

\[ \nabla = d - g(z)[H], \]

where

\[ g(z)[H] := g(z)Hg(z)^{-1} + \left( \frac{d}{dz} g(z) \right) g(z)^{-1}dz. \]

Thus the isomorphic class of \((\nabla_H, \mathbb{C}[z]^{\oplus n})\) is parametrized by the orbit

\[ O_H := \{ g(z)[H] \in M_n(\mathbb{C}(z))dz | g(z) \in \text{GL}_n(\mathbb{C}[z]) \}. \]

As an analogue of this isomorphic class \(O_H\), Boalch introduced the truncated orbit of \(H\) in \([1]\).

**Definition 1.4** (truncated orbit). Let \(H\) be an HTL normal form and we regard \(H\) as an element in \(M_n(\mathbb{C}(z))/\mathbb{C}[z]dz\). Then the orbit of \(H\),

\[ \mathcal{O}_H := \{ gHg^{-1} \in M_n(\mathbb{C}(z))/\mathbb{C}[z]dz | g \in \text{GL}_n(\mathbb{C}[z]) \} \]

is called the truncated orbit of \(H\).

Note that the truncated orbit of \(\mathcal{O}_H\) is characterized by \(H\) in the following sense.

**Proposition 1.5** (Boalch \([1]\), Yamakawa \([14]\)). If an HTL normal form \(H'\) contained in \(\mathcal{O}_H\), there exists \(C \in \text{GL}_n(\mathbb{C})\) such that \(H' = CHC^{-1}\).

### 1.2 Truncated orbits and extended orbits

Let a fraktur \(q\) be a \(\mathbb{C}\)-subalgebra of \(\mathfrak{gl}_n(\mathbb{C}) = M_n(\mathbb{C})\) which can be seen as a Lie subalgebra defining the bracket by \([X,Y] = XY - YX\) for \(X, Y \in q\). Then we denote the corresponding analytic subgroup of \(\text{GL}_n(\mathbb{C})\) by the roman \(Q\).

For a positive integer \(l\), we consider a \(\mathbb{C}[z]\)-module \(\mathbb{C}[z^{-1}]_l := z^{-(l+1)}\mathbb{C}[z]/\mathbb{C}[z]\). Since the annihilator ideal of this module is \(\text{Ann}_{\mathbb{C}[z]}(\mathbb{C}[z^{-1}]_l) = \langle z^{l+1} \rangle_{\mathbb{C}[z]}\), \(\mathbb{C}[z^{-1}]_l\) can also be seen as a \(\mathbb{C}[z]_l := \mathbb{C}[z]/\langle z^{l+1} \rangle_{\mathbb{C}[z]}\)-module. Usually we fix the following basis and identifications as \(\mathbb{C}\)-vector spaces, \(\mathbb{C}[z^{-1}]_l \cong \{ \sum_{i=0}^{l} a_i z^i | a_i \in \mathbb{C} \} \cong \mathbb{C}^{l+1},\mathbb{C}[z]_l \cong \{ \sum_{i=0}^{l} a_i z^i | a_i \in \mathbb{C} \} \cong \mathbb{C}^{l+1}\).

#### 1.2.1 Truncated orbits as coadjoint orbits

We can consider the finite dimensional complex Lie group \(\text{GL}_n(\mathbb{C}[z]_l)\) with the corresponding Lie algebra \(\mathfrak{gl}_n(\mathbb{C}[z]_l) := M_n(\mathbb{C}[z]_l)\). For a \(\mathbb{C}\)-subalgebra \(q \subset \mathfrak{gl}_n(\mathbb{C})\), we define the \(\mathbb{C}\)-algebra \(q(\mathbb{C}[z]_l) := q \otimes_{\mathbb{C}} \mathbb{C}[z]_l\) which is naturally regarded as a \(\mathbb{C}\)-subalgebra of
Then the trace form
\[ \text{gl}_n(\mathbb{C}[z]) \] of the corresponding analytic subgroup of the complex Lie group \( \text{GL}_n(\mathbb{C}[z]) \) by \( Q(\mathbb{C}[z]) \).

If we regard \( M_n(\mathbb{C}[z^{-1}]k)dz \) as a subspace of \( M_n(\mathbb{C}(z)/\mathbb{C}[z])dz \) by the inclusion map \( \mathbb{C}[z^{-1}]k \to \mathbb{C}(z)/\mathbb{C}[z] \), then the truncated orbit \( O_H \) can be seen as the orbit through \( H \) in \( M_n(\mathbb{C}[z^{-1}]k)dz \) under the action of \( \text{GL}_n(\mathbb{C}[z]) \), i.e.,

\[ O_H = \{ gHg^{-1} \in M_n(\mathbb{C}[z^{-1}]k)dz \mid g \in \text{GL}_n(\mathbb{C}[z]) \}. \]

Then the trace form
\[ M_n(\mathbb{C}[z]) \times M_n(\mathbb{C}[z^{-1}]k)dz \ni (A, Bdz) \mapsto \text{res}_{z=0}(\text{tr}(AB))dz \in \mathbb{C}, \]

is non-degenerate and thus this enable us to identify the dual space \( \text{gl}_n(\mathbb{C}[z])^* \) with \( M_n(\mathbb{C}[z^{-1}]k)dz \). Thus \( O_H \) is regarded as the coadjoint orbit of \( \text{GL}_n(\mathbb{C}[z]) \) through \( H \in M_n(\mathbb{C}[z^{-1}]k)dz \cong \text{gl}_n(\mathbb{C}[z])^* \).

### 1.2.2 Extended orbits of HTL normal forms

Here we recall the extended orbit which is introduced by Boalch.

First we recall the fact that the coadjoint orbit of a complex Lie group \( G \) through an element \( \xi \) of \( \mathfrak{g}^* \) is isomorphic to a symplectic reduction of the cotangent bundle \( T^*G \) as follows. The cotangent bundle \( \theta: T^*M \to M \) of a complex manifold \( M \) has the standard holomorphic symplectic form \( \Omega_{T^*M} := d\omega_{T^*M} \), the derivation of the canonical 1-form \( \omega_{T^*M} \) on \( T^*M \).

For a holomorphic map \( f: M \to \Omega_{T^*M} \) between complex manifolds, we denote the induced maps \( T_mM \to T_{f(m)}N \) and \( T_{f(m)}N \to T_mM \) by \( (Tf)_m \) and \( (T^*f)_m \) respectively for \( m \in M \). Sometimes we write \( f_{\#,m} = (Tf)_m \) and \( f_m^* = (T^*f)_m \) for simplicity. Now we suppose \( M = G \) and denote the right translation by \( R_g: G \ni x \mapsto gx \in G \) and the left translation by \( L_g: \Omega_{T^*G} \ni x \mapsto gx \in G \) for \( g \in G \). We define the left action of \( G \) on \( T^*G \) by \( \rho(g): T^*G \ni \alpha \mapsto R_g^*(\alpha) \in T^*G \) for \( g \in G \). Then the map

\[ \mu_G: T^*G \ni \alpha \mapsto L_0^*(\alpha) \in \mathfrak{g}^* \]

is a moment map with respect to the action \( \rho \), and for the coadjoint orbit \( O_\xi \) through \( \xi \in \mathfrak{g}^* \), there exists an isomorphism

\[ O_\xi \cong G_\xi \backslash \mu_G^{-1}(\xi) \]

as holomorphic symplectic manifolds. Here \( G_\xi \) is the stabilizer subgroup of \( \xi \) in \( G \).

Now we consider the specific coadjoint orbit \( \Omega_H \) of \( \text{GL}_n(\mathbb{C}[z]) \) through the unramified HTL normal form \( H \). Note that \( \text{GL}_n(\mathbb{C}[z]) \) has the description as the semi-direct product of \( \text{GL}_n(\mathbb{C}) \) and the normal subgroup

\[ \text{GL}_n(\mathbb{C}[z]) := \{ g(z) \in \text{GL}_n(\mathbb{C}[z]) \mid g(0) = E_n \}, \]

i.e., \( \text{GL}_n(\mathbb{C}[z]) = \text{GL}_n(\mathbb{C}) \ltimes \text{GL}_n(\mathbb{C}[z]) \). The Lie algebra of this normal subgroup is denoted by \( \mathfrak{gl}_n(\mathbb{C}[z]) \). Also we denote the inclusion map of \( \text{GL}_n(\mathbb{C}) \) and \( \text{GL}_n(\mathbb{C}[z]) \) into \( \text{GL}_n(\mathbb{C}[z]) \) by \( \iota_0 \) and \( \iota_1 \), and the projection maps from \( \text{GL}_n(\mathbb{C}[z]) \) onto \( \text{GL}_n(\mathbb{C}) \) and \( \text{GL}_n(\mathbb{C}[z]) \) by \( \pi_0 \) and \( \pi_1 \) respectively.

Then we can moreover consider the intermediate reduction by considering the moment map with respect to the normal subgroup \( \text{GL}_n(\mathbb{C}[z]) \),

\[ \mu_{\text{GL}_n(\mathbb{C}[z])} := \iota_1^* \circ \mu_{\text{GL}_n(\mathbb{C}[z])}: T^*\text{GL}_n(\mathbb{C}[z]) \to (\mathfrak{gl}_n(\mathbb{C}[z])^*)^*, \]

where \( \iota_1^*: \mathfrak{gl}_n(\mathbb{C}[z])^* \to (\mathfrak{gl}_n(\mathbb{C}[z])^*)^* \) is the induced projection map by \( \iota_1 \).

---

1Here we notice that many literature adopt \(-d\omega_{T^*M}\) as the standard symplectic form.
Proposition 1.6. Let us set $H_{\text{irr}} := t^*_0(H)$ and call it the irregular type of $H$. We denote the coadjoint orbit of $\text{GL}_n(\mathbb{C}[z]_k)^1$ through $H_{\text{irr}}$ by $\Omega_{H_{\text{irr}}}$. Then there exists an isomorphism

$$\text{GL}_n(\mathbb{C}[z]_k)^1_{H_{\text{irr}}} \backslash (\mu_{\text{GL}_n(\mathbb{C}[z]_k)}^{-1}(H_{\text{irr}}) \cong T^*\text{GL}_n(\mathbb{C}) \times \Omega_{H_{\text{irr}}}.$$ as holomorphic symplectic manifolds.

Proof. This is a direct consequence of the Hamiltonian reduction by stages with respect to the semi-direct product $\text{GL}_n(\mathbb{C}[z]_k) = \text{GL}_n(\mathbb{C}) \times \text{GL}_n(\mathbb{C}[z]_k)^1$, see Theorem 10.5.1 in [11]. \qed

Here the isomorphism is induced from the map between the cotangent bundles $T^*(t_0 \times t_1) : T^*\text{GL}_n(\mathbb{C}[z]_k) \to T^*\text{GL}_n(\mathbb{C}) \times T^*\text{GL}_n(\mathbb{C}[z]_k)^1$ associated to the isomorphism given by the multiplication map $t_0 \times t_1 : \text{GL}_n(\mathbb{C}) \times \text{GL}_n(\mathbb{C}[z]_k)^1 \to \text{GL}_n(\mathbb{C}[z]_k)$. This intermediate reduction space is called the extended orbit of $H_{\text{irr}}$.

Let us explain that the truncated orbit $\Omega_{H_{\text{irr}}}$ is a symplectic reduction of this extended orbit. Since GL$_n$(C[z]$_k$)$^1$ is a normal subgroup of GL$_n$(C[z]$_k$), the subgroup GL$_n$(C) $\subset$ GL$_n$(C[z]$_k$)$^1$ acts on GL$_n$(C[z]$_k$)$^1$ by the adjoint action. Thus we can consider the induced action of GL$_n$(C) on $(\text{gl}_n(\mathbb{C}[z]_k)^1)^*$. We denote the stabilizer subgroup of $H_{\text{irr}} \in (\text{gl}_n(\mathbb{C}[z]_k)^1)^*$ in GL$_n$(C) by GL$_n$(C)$_{H_{\text{irr}}}$ and put $t_0, H_{\text{irr}} : \text{GL}_n(\mathbb{C})_{H_{\text{irr}}} \hookrightarrow \text{GL}_n(\mathbb{C}[z]_k)$, the inclusion map. Then we can define a map from $T^*\text{GL}_n(\mathbb{C}[z]_k)^1$ to $\text{gl}_n(\mathbb{C})_{H_{\text{irr}}^*}$, the dual space of the Lie algebra $\text{gl}_n(\mathbb{C})_{H_{\text{irr}}}$ of GL$_n$(C)$_{H_{\text{irr}}}$, as the composite map

$$\mu_{\text{GL}_n(\mathbb{C}[z]_k)}^{-1} : T^*\text{GL}_n(\mathbb{C}[z]_k)^1 \xrightarrow{T^*\pi_1} T^*\text{GL}_n(\mathbb{C}[z]_k) \xrightarrow{\mu_{\text{GL}_n(\mathbb{C}[z]_k)}} \text{gl}_n(\mathbb{C}[z]_k)^* \xrightarrow{t_0^* \text{irr}^{-1}} \text{gl}_n(\mathbb{C})_{H_{\text{irr}}^*}.$$ Here $T^*\pi_1 : T^*\text{GL}_n(\mathbb{C}[z]_k)^1 \to T^*\text{GL}_n(\mathbb{C}[z]_k)$ is defined by $T^*\pi_1(\alpha) := (T^*\pi_1)_{\mu(\alpha)}(\alpha)$ for $\alpha \in T^*\text{GL}_n(\mathbb{C}[z]_k)^1$. Then we can show the following.

Proposition 1.7. The restriction of $\mu_{\text{GL}_n(\mathbb{C}[z]_k)}^{-1} : \text{GL}_n(\mathbb{C}[z]_k)^1_{H_{\text{irr}}} \to \Omega_{H_{\text{irr}}} = \text{GL}_n(\mathbb{C}[z]_k)^1_{H_{\text{irr}}} \backslash (\mu_{\text{GL}_n(\mathbb{C}[z]_k)}^{-1}(H_{\text{irr}})).$ Namely, there uniquely exists the map $\text{res}_{H_{\text{irr}}} : \Omega_{H_{\text{irr}}} \to \text{gl}_n(\mathbb{C})_{H_{\text{irr}}}$ such that the diagram

$$\begin{array}{ccc} \mu_{\text{GL}_n(\mathbb{C}[z]_k)}^{-1}(H_{\text{irr}}) & \xrightarrow{q} & \text{gl}_n(\mathbb{C})_{H_{\text{irr}}} \\ \text{res}_{H_{\text{irr}}} \downarrow & & \downarrow \text{res}_{H_{\text{irr}}} \\ \Omega_{H_{\text{irr}}} & \to & \text{gl}_n(\mathbb{C})_{H_{\text{irr}}} \\
\end{array}$$

is commutative.

Proof. Since the restriction map $\mu_{\text{GL}_n(\mathbb{C}[z]_k)}^{-1} : \mu_{\text{GL}_n(\mathbb{C}[z]_k)}^{-1}(H_{\text{irr}})$ is GL$_n$(C[z]$_k$)$^1_{H_{\text{irr}}}$-invariant, this map factors through the quotient map. \qed

Let $t_{H_{\text{irr}}} : \text{GL}_n(\mathbb{C})_{H_{\text{irr}}} \hookrightarrow \text{GL}_n(\mathbb{C})$ be the inclusion map. Then Boalch showed that the truncated orbit $\Omega_H$ is a symplectic reduction of the extended orbit $T^*\text{GL}_n(\mathbb{C}) \times \Omega_{H_{\text{irr}}}$ as follows.
Theorem 1.8 (Boalch [1]). Let us consider the map

$$\mu_{\text{ext}} : T^*\GL_n(\mathbb{C}) \times \mathcal{O}_{H_{irr}} \longrightarrow \gl_n(\mathbb{C})^*_{\text{H_{irr}}} \quad (\alpha, \xi) \mapsto \iota_{H_{irr}}^* \circ \mu_{\GL_n(\mathbb{C})}(\alpha) + \text{res}_{\mathcal{O}_{H_{irr}}}(-\xi).$$

Then this map is a moment map with respect to the $\GL_n(\mathbb{C})_{\text{H_{irr}}}$-action which is induced from the natural $\GL_n(\mathbb{C})$-action on $T^*\GL_n(\mathbb{C}) \times T^*\GL_n(\mathbb{C}[z]_k)^1 \cong T^*\GL_n(\mathbb{C}[z]_k)$. Moreover setting $H_{\text{res}} := \iota_0^*(H)$, we have a symplectic isomorphism

$$\mathcal{O}_H \cong (\GL_n(\mathbb{C})_{\text{H_{irr}}})_{H_{\text{res}}} \backslash \mu_{\text{ext}}^{-1}(H_{\text{res}}).$$

1.3 Triangular factorization of $\mathcal{O}_{H_{irr}}$

We explain the coadjoint orbit $\mathcal{O}_{H_{irr}}$ has a nice affine coordinate system which is very important to define a deformation of the truncated orbit in the latter section.

For a subgroup $G \subset \GL_n(\mathbb{C}[z]_l)$, we denote the intersection $G \cap \GL_n(\mathbb{C}[z]_l)^1$ by $G^1$. Also we denote the intersection $\gl_n(\mathbb{C}[z]_l)^1$ with a Lie subalgebra $\mathfrak{g} \subset \gl_n(\mathbb{C}[z]_l)$ by $\mathfrak{g}^1$.

1.3.1 Compositions of a natural number and subgroups of $\GL_n(\mathbb{C})$

Let us recall compositions of natural numbers. Let $n$ and $k$ be strictly positive integers. Then we denote the set of weak composition of $n$ with $l$ terms by

$$\mathcal{C}(n, l) := \{ (a_1, a_2, \ldots, a_l) \in \mathbb{Z}_{\geq 0}^l \mid a_1 + a_2 + \cdots + a_l = n \}$$

where we allow some components $a_i$ to be zero. If all components in $(a_1, a_2, \ldots, a_l) \in \mathcal{C}(n, l)$ are strictly positive integers, then it is called strict composition or composition shortly and we denote the set of strict composition of $n$ with $l$ terms by $\mathcal{C}^+(n, l)$. The sets of all weak compositions and strict compositions of $n$ are denoted by

$$\mathcal{C}(n) := \bigsqcup_{l \in \mathbb{Z}_{\geq 0}} \mathcal{C}(n, l), \quad \mathcal{C}^+(n) := \bigsqcup_{l \in \mathbb{Z}_{\geq 0}} \mathcal{C}^+(n, l),$$

respectively.

Let us take $m = \{n_1, n_2, \ldots, n_m\} \subset \mathcal{C}(n)$. Let $e_i \in \mathbb{C}^n$ be the vector with 1 as the $i$-th entry and 0 as the others for each $i = 1, 2, \ldots, n$. Then we define inclusion maps

$$\iota_i : \mathbb{C}^{n_i} \longrightarrow \mathbb{C}^{n_1+n_2+\cdots+n_i} \quad (a_1, a_2, \ldots, a_{n_i}) \longmapsto \sum_{j=n_1+n_2+\cdots+n_{i-1}+1}^{n_1+n_2+\cdots+n_i} a_j e_j$$

for $i = 1, 2, \ldots, m$ and an isomorphism

$$\iota_m : \bigoplus_{i=1}^m \mathbb{C}^{n_i} \longrightarrow \mathbb{C}^n \quad (a_i)_{i=1,2,\ldots,m} \longmapsto \sum_{i=1}^m \iota_i(a_i),$$

where if $n_i = 0$, then we set $\iota_i$ as the zero map. We identify $\mathbb{C}^n$ and $\bigoplus_{i=1}^m \mathbb{C}^{n_i}$ through this isomorphism.
Let us regard $\text{Hom}_C(C^n, C^{n'})$ as a subspace of $\text{End}_C(C^n) = M_n(C)$ through the canonical projection $\text{pr}_i: C^n \to C^{n_i}$ and injection $\iota_i: C^{n_i'} \to C^n$. Then we define the following $C$-subalgebras of $M_n(C)$,

\[ l_m := \bigoplus_{i=1, \ldots, m} \text{Hom}_C(C^{n_i}, C^{n_i'}), \]

\[ n_{m}^+ := \bigoplus_{i, i' \in \{1, \ldots, m\}, i < i'} \text{Hom}_C(C^{n_i}, C^{n_i'}), \]

\[ n_{m}^- := \bigoplus_{i, i' \in \{1, \ldots, m\}, i > i'} \text{Hom}_C(C^{n_i}, C^{n_i'}), \]

and regard them as the Lie subalgebras of $\text{gl}_n(C) = M_n(C)$. Let $L_m$, $N_{m}^+$, and $N_{m}^-$ be the corresponding analytic subgroups of $\text{GL}_n(C)$.

### 1.3.2 Sequences of compositions of positive integers

Let us take weak compositions $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_m) \in C(n)$. Then we say that $\beta$ is a refinement of $\alpha$ if there exists a surjection $\phi: \{1, 2, \ldots, m\} \to \{1, 2, \ldots, l\}$ from the index set of $\beta$ to that of $\alpha$ such that equations $\alpha_i = \sum_{j \in \phi^{-1}(i)} \beta_j$ hold for all $i = 1, 2, \ldots, l$. In this case we write $\beta \preceq \alpha$ or $\beta \preceq_{\phi} \alpha$.

Emphasizing the surjection $\phi$ which we call the refinement map between $\alpha$ and $\beta$. Let $\widehat{\mathbf{m}} = (\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_k)$ be an ascending sequence of compositions of $n$, i.e., entries of $\widehat{\mathbf{m}}$ consist of the following ascending sequence of refinement,

\[ \mathbf{m}_1 \preceq_{\phi_1} \mathbf{m}_2 \preceq_{\phi_2} \cdots \preceq_{\phi_{k-1}} \mathbf{m}_k. \]

We write $\mathbf{m}_i = \{n_{i,1}, n_{i,2}, \ldots, n_{i,m(i)}\}$ and define a total ordering $\preceq_i$ on the index set $\{\langle i, 1 \rangle, \langle i, 2 \rangle, \ldots, \langle i, m(i) \rangle\}$ of $\mathbf{m}_i$ for each $i = 1, 2, \ldots, k$. We say that the collection of orderings $\{\preceq_i\}_{i=1, \ldots, k}$ is the ordering of the sequence $\widehat{\mathbf{m}}$ if

\[ \langle i, j \rangle \preceq_i \langle i, j' \rangle \text{ imply } \phi_i(\langle i, j \rangle) \preceq_{i+1} \phi_i(\langle i, j' \rangle) \]

for any $\langle i, j \rangle, \langle i, j' \rangle \in \{\langle i, 1 \rangle, \langle i, 2 \rangle, \ldots, \langle i, m(i) \rangle\}$ and $i = 1, 2, \ldots, k - 1$.

Let us fix a surjection $\phi^i: \{1, 2, \ldots, n\} \to \{1, 1, 1, 2, \ldots, 1, m(1)\}$ and define the maps $\phi^i: \{1, 2, \ldots, n\} \to \{1, 1, 1, 2, \ldots, 1, m(1)\}$ by $\phi^i := \phi_{i-1} \circ \phi_{i-2} \circ \cdots \circ \phi_1 \circ \phi^1$ for $i = 2, 3, \ldots, k$. Then we can regard the vector space $C^n$ as the subspace of $\mathbb{C}^n$ by the equation

\[ \mathbb{C}^{n_{(i,j)}} = \bigoplus_{k \in (\phi^i)^{-1}(1)} \mathbb{C}e_k, \]

for each $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, m(i)$. Then we obtain the decompositions

\[ \mathbb{C}^n = \bigoplus_{j=1}^{m(i)} C^{n_{(i,j)}} \]

for $i = 1, 2, \ldots, k$ associated to $\widehat{\mathbf{m}}$.

As we saw in the previous section, a composition of $n$ defines the block decomposition of $\text{gl}_n(C) = M_n(C)$. Therefore a pair of a sequence $\widehat{\mathbf{m}}$ of compositions and an ordering $\{\preceq_i\}_{i=1, 2, \ldots, k}$ of $\widehat{\mathbf{m}}$ defines the following Lie subalgebras of $\text{gl}_n(C)$ with inclusion relations

\[ l_{m_1} \subset l_{m_2} \subset \cdots \subset l_{m_k}, \]

\[ n_{m_k}^+ \subset \cdots \subset n_{m_2}^+ \subset n_{m_1}^+. \]

(1)
Moreover we define
\[ n_{m_i}^+ := l_{m_i} \cap m_{m_i}, \]
for pairs \((i, i')\) satisfying \(1 \leq i < i' \leq k + 1\), which give us further decompositions
\[ l_{m_i} = n_{m_i}^- \oplus l_{m_i} \oplus n_{m_i}^+. \]

Here we formally put \( l_{m_{k+1}} = gl_n(\mathbb{C}) \). Throughout this note, when we consider a sequence \( \tilde{m} \) of compositions, we always fix an ordering \( \{\leq\}_{i=1,2,...,k} \) of \( \tilde{m} \).

### 1.3.3 Spectral types of HTL-normal forms

We shall introduce spectral types of HTL normal forms. As the easiest case, let us first consider a square matrix \( X \in M_n(\mathbb{C}) \) and define the spectral type of \( X \) as follows. Take the Jordan decomposition of \( X \),
\[ X = S + N \]
with the diagonalizable matrix \( S \) and the nilpotent matrix \( N \) satisfying the relation
\[ [S, N] = 0. \]
Then we have the decomposition \( \mathbb{C}^n = \bigoplus_{i=1}^m V_i \) as the direct sum of eigenspaces of \( S \) which defines a composition of \( n \),
\[ m := (\dim_{\mathbb{C}} V_1, \dim_{\mathbb{C}} V_2, \ldots, \dim_{\mathbb{C}} V_m). \]

Let us look at the nilpotent matrix \( N \) and notice that the restriction \( N_i := N|_{V_i} \) on \( V_i \) defines a nilpotent endomorphism of \( V_i \) for each \( i = 1, 2, \ldots, m \) since \([S, N] = 0\). Then we have the partial flag \( F_{N; V_i} \),
\[ V_i \supsetneq (V_i)_1 := \text{Im} N_i \supsetneq (V_i)_2 := \text{Im} N_i^2 \supsetneq \cdots \supsetneq (V_i)_d_i := \text{Im} N_i^d_i = \{0\}, \]
and descending sequence of positive integers
\[ \dim_{\mathbb{C}} V_i > \dim_{\mathbb{C}} (V_i)_1 > \dim_{\mathbb{C}} (V_i)_2 > \cdots > \dim_{\mathbb{C}} (V_i)_{d_i} = 0. \]

We call the sequence of the integers
\[ \sigma(F_{N; V_i}) := (\dim_{\mathbb{C}} V_i, \dim_{\mathbb{C}} (V_i)_1, \dim_{\mathbb{C}} (V_i)_2, \ldots, \dim_{\mathbb{C}} (V_i)_{d_i-1}) \]
the \textit{signature} of \( F_{N; V_i} \). We denote the set of the signatures by
\[ \sigma(m) := \{\sigma(F_{N; V_i})\}_{i=1,2,...,m} \]
and call it the \textit{signature} of \( m \).

**Definition 1.9** (spectral type of a square matrix). \textit{Let} \( X \in M_n(\mathbb{C}) \) \textit{be a square matrix with the Jordan decomposition} \( X = S + N \). \textit{Then we can define the composition} \( m \) \textit{of} \( n \) \textit{and the signature} \( \sigma(m) \) \textit{of} \( m \) \textit{as above. we call the pair}
\[ (m, \sigma(m)) \]
\textit{the spectral type of} \( X \). \textit{In particular, if} \( X = S \), \textit{i.e.,} \( X \) \textit{is a semisimple matrix, then the signature} \( \sigma(m) \) \textit{is trivial. In this case, we simply denote the spectral type by} \( (m, \text{triv}) \).
Next we consider an unramified HTL-normal form \( \left( \frac{S_k}{z^k} + \cdots + \frac{S_1}{z} + S_0 + N_0 \right) \frac{dz}{z} \).

Then we obtain the decomposition \( \mathbb{C}^n = \bigoplus_{j=1}^{m(i)} V_{(i;j)} \) as the direct sum of simultaneous eigenspaces of \((S_i, S_{i+1}, \ldots, S_k)\) which defines the following composition of \( n, \)

\[
m_i := (\dim \mathbb{C}V_{(i;1)}, \dim \mathbb{C}V_{(i;2)}, \ldots, \dim \mathbb{C}V_{(i;m(i))}),
\]

for each \( i = 0, 1, \ldots, k \). Also for each \( i = 0, 1, \ldots, k - 1 \), let us define the map

\[
\phi_i : \{ \langle i, 1 \rangle, \langle i, 2 \rangle, \ldots, \langle i, m(i) \rangle \} \rightarrow \{ \langle i + 1, 1 \rangle, \langle i + 1, 2 \rangle, \ldots, \langle i + 1, m(i + 1) \rangle \}
\]

between index sets of \( m_i \) and \( m_{i+1} \) which satisfies that

\[
V_{(i+1;j)} = \bigoplus_{\mu \in \phi_i^{-1}(j)} V_{(i;\mu)}
\]

for all \( j \in \{1, 2, \ldots, m(i)\} \). Then we obtain a refinement sequence of partitions

\[
m_0 \leq \phi_0 \leq m_1 \leq \phi_1 \cdots \leq \phi_{k-1} m_k.
\]

The restrictions of nilpotent matrices \( N_j := N_0|_{V_{(0;j)}} \in \text{End}_\mathbb{C}(V_{(0;j)}) \) define the flags \( F(N_0, V_{(0;j)}) \) of \( V_{(0;j)} \),

\[
V_{(0;j)}^{\geq} (V_{(0;j)})_1 := \text{Im} N_j \supseteq (V_{(0;j)})_2 := \text{Im} N_j^2 \supseteq \cdots \supseteq (V_{(0;j)})_{d_j} := \text{Im} N_j^{d_j} = \{0\},
\]

with the signature

\[
\sigma(F(N_0, V_{(0;j)})) = (\dim \mathbb{C}(V_{(0;j)}), \dim \mathbb{C}(V_{(0;j)})_1, \dim \mathbb{C}(V_{(0;j)})_2, \ldots, \dim \mathbb{C}(V_{(0;j)})_{d_j-1}).
\]

Also we set

\[
\sigma(m_0) := (\sigma(F(N_0, V_{(0;j)})))_{j=1,2,\ldots,m(0)}.
\]

**Definition 1.10** (spectral types of HTL normal forms). The pair

\[
\text{sp}(H) := (m_0 \leq \phi_0 \leq m_1 \leq \phi_1 \cdots \leq \phi_{k-1} m_k, \sigma(m_0))
\]

of the refinement sequence of partitions of \( n \) and signature \( \sigma(m_0) \) of \( m_0 \) defined above is called the *spectral type* of the HTL-normal form

\[
H = \left( \frac{S_k}{z^k} + \cdots + \frac{S_1}{z} + S_0 + N_0 \right) \frac{dz}{z}.
\]

When \( N_0 = 0 \), the signature \( \sigma(m_0) \) is trivial. Thus we denote the spectral type by

\[
(m_0 \leq \phi_0 \leq m_1 \leq \phi_1 \cdots \leq \phi_{k-1} m_k, \text{triv})
\]

in this case.
1.3.4 Triangular factrization of $O_{H_{av}}$

Firstly, we recall a general fact about coadjoint orbits. It is well-known that a coadjoint orbit $O_\xi$ of a complex Lie group $G$ through $\xi \in g^*$ is a immersed submanifold of $g^*$ via the injective immersion

$$O_\xi \cong G/G_\xi \ni [g] \mapsto \text{Ad}^*(g)(\xi) \in g^*, $$

where $[g] \in G/G_\xi$ is the class of $g \in G$. Under the identification $O_\xi \cong G_\xi/\mu_G^{-1}(\xi)$, this injective immersion is written as follows. Let us consider the map

$$\nu_G : T^*G \ni \alpha \mapsto R_{\theta(\alpha)}(\alpha) \in g^*, $$

where $\theta : T^*G \to G$ is the natural projection. Then we note that this map is invariant under the action $\rho$ of $G$ on $T^*G$, i.e., we have $\nu_G \circ \rho(g) = \nu_G$ for all $g \in G$. Therefore the restriction map $\nu_G : \mu_G^{-1}(\xi) \to g^*$ factors through the quotient map $q : \mu_G^{-1}(\xi) \to O_\xi$ and there uniquely exists the map $O_\xi \to g^*$ such that the diagram

$$\begin{array}{ccc}
\mu_G^{-1}(\xi) & \xrightarrow{\nu_G} & g^* \\
q \downarrow & & \\
O_\xi & & \\
\end{array}$$

is commutative. Then we can check that this map $O_\xi \to g^*$ coincides with the above injective immersion.

Let us consider the subgroups $(N^-)^{(l)} := N_{m_l}^-(\mathbb{C}[z]_{l-1})_l \subset \text{GL}_n(\mathbb{C}[z]_{l-1})_l$ for $l = 2, 3, \ldots, k$. Then we recall that $\text{GL}_n(\mathbb{C}[z]_{l-1})_l$ admits the LU decompositions.

**Proposition 1.11 (LU decomposition).** Let $m$ be a composition of $n$. Then for a positive integer $j$, the multiplication gives the isomorphism as complex manifolds,

$$N_m^-(\mathbb{C}[z]_j)_1 \times L_m(\mathbb{C}[z]_j) \times N_m^+(\mathbb{C}[z]_j)_1 \ni (n^-, h, n^+) \mapsto n^- \cdot h \cdot n^+ \in \text{GL}_n(\mathbb{C}[z]_j)_1.$$

**Proof.** Since any principal minors of $g \in \text{GL}_n(\mathbb{C}[z]_j)_1$ are units in $\mathbb{C}[z]_j$, thus the usual LU decomposition algorithm is valid in this setting. \qed

Thus we obtain the projection $\text{pr}_{(N^-)^{(l)}} : \text{GL}_n(\mathbb{C}[z]_{l-1})_l \to (N^-)^{(l)}_l$ along the LU decomposition. Regarding the projection $\text{pr}_{k-1, l}^k : \mathbb{C}[z]_k \to \mathbb{C}[z]_{l-1}$ as a map of $\mathbb{C}$-vector spaces, we can define a section $s_{l-1}^k : \mathbb{C}[z]_{l-1} \ni \sum_{i=0}^{l-1} c_i z^i \mapsto \sum_{i=0}^{l-1} c_i z^i \in \mathbb{C}[z]_k$. Then we define the projection

$$\pi_{(N^-)^{(l)}} := \text{pr}_{(N^-)^{(l)}} \circ \text{pr}_{k-1, l}^k : \text{GL}_n(\mathbb{C}[z]_k)_1 \to (N^-)^{(l)}_l.$$

Also the inclusion map $i_{(N^-)^{(l)}} : (N^-)^{(l)}_l \hookrightarrow \text{GL}_n(\mathbb{C}[z]_k)_1$ defines a section

$$i_{(N^-)^{(l)}} := s_{l-1}^k \circ i_{(N^-)^{(l)}} : (N^-)^{(l)}_l \hookrightarrow \text{GL}_n(\mathbb{C}[z]_k)_1$$

of the projection $\pi_{(N^-)^{(l)}}$. By using these projection and section, we obtain a map between cotangent bundles,

$$T^*\pi_{(N^-)^{(l)}} : T^*(N^-)^{(l)}_l \ni \alpha \mapsto (T^*\pi_{1, (N^-)^{(l)}})_{i_{(N^-)^{(l)}}(\theta(\alpha))}(\alpha) \in T^*\text{GL}_n(\mathbb{C}[z]_k)_1.$$
For $H_{\text{irr}} \in (\mathfrak{gl}_n(C[z])^1)^*$, we set $\omega_{H_{\text{irr}}} := (L^*_g - 1(H_{\text{irr}}))_{g \in \text{GL}_n(C[z])^1}$, the left invariant 1-form on $\text{GL}_n(C[z])^1$ associated to $H_{\text{irr}}$. Define a map $F: \prod_{l=2}^k T^*(N^{-}(l)) \to T^*\text{GL}_n(C[z])^1$ by

$$F((\alpha_l)_{l=2,3,\ldots,k}) := \omega_{H_{\text{irr}},\theta(\alpha)^{[2;k]}} + \sum_{l=2}^k L_{(\theta(\alpha)^{[l+1,k]-1})}^* \circ R_{(\theta(\alpha)^{[2,l-1]-1})}^* \circ T^*\pi(N^{-}(l))(\alpha_l)$$

for $(\alpha_l)_{l=2,3,\ldots,k} \in \prod_{l=2}^k T^*(N^{-})(l)$, where we put

$$\theta(\alpha)^{[i,j]} := \begin{cases} \theta(\alpha_j)\theta(\alpha_{j-1})\cdots\theta(\alpha_i) & i \leq j \\ e & i > j \end{cases}$$

The next theorem shows that $T^*(N^{-})(l)$ defines an affine coordinate system of $\mathcal{O}_{H_{\text{irr}}}$.

**Theorem 1.12** (Yamakawa [15], cf. Hiroe-Yamakawa [6]). The map

$$\nu_{\text{GL}_n(C[z])^1} \circ F: \prod_{l=2}^k T^*(N^{-})(l) \to (\mathfrak{gl}_n(C[z])^1)^*$$

gives a symplectic isomorphism

$$\prod_{l=2}^k T^*(N^{-})(l) \cong \mathcal{O}_{H_{\text{irr}}}.$$ 

Let us give a description of the map $\text{res}_{\mathcal{O}_{H_{\text{irr}}}}: \mathcal{O}_{H_{\text{irr}}} \to \mathfrak{gl}_n(C)_{H_{\text{irr}}}$ under the above identification.

**Proposition 1.13.** Under the isomorphism $\prod_{l=2}^k T^*(N^{-})(l) \cong \mathcal{O}_{H_{\text{irr}}}$, the image of each $(\alpha_l)_{l=2,3,\ldots,k} \in \prod_{l=2}^k T^*(N^{-})(l)$ by the map $\text{res}_{\mathcal{O}_{H_{\text{irr}}}}: \mathcal{O}_{H_{\text{irr}}} \to \mathfrak{gl}_n(C)_{H_{\text{irr}}}$ is given by

$$\text{res}_{\mathcal{O}_{H_{\text{irr}}}}((\alpha_l)_{l=2,3,\ldots,k}) = \iota_{0,H_{\text{irr}}}^* \circ \left( \sum_{l=2}^k \text{Ad}^*(\theta(\alpha)^{[2,l-1]})_{(T^*(\pi(N^{-})(l)) \circ \pi_1)}(L^*_{(\theta(\alpha))(\alpha_l)}) \right).$$

**Proof.** Directly follows from the definition of $\text{res}_{\mathcal{O}_{H_{\text{irr}}}}$ and Theorem 1.12.  

### 1.4 Moduli space of meromorphic connections on a trivial bundle over the Riemann sphere with unramified irregular singularities

Now we are ready to define a moduli space of meromorphic connections on a trivial bundle over the Riemann sphere as a symplectic reduction of a product of truncated orbits.
1.4.1 Algebraic meromorphic connections on a trivial bundle over the Riemann sphere

Let us denote the sheaf of regular function on \( \mathbb{P}^1 \) by \( \mathcal{O}_{\mathbb{P}^1}^{\text{reg}} \). Let \( z \) be the standard coordinate on \( \mathbb{C} \subset \mathbb{P}^1 \). For \( c \in \mathbb{P}^1 \), set \( z_c := z - c \) if \( c \in \mathbb{C} \) and \( z_\infty := \frac{1}{z} \) if \( c = \infty \). Let \( \hat{\mathcal{O}}_c := \mathbb{C}[z_c] \) be the ring of formal power series and \( \hat{\mathcal{M}}_c := \mathbb{C}(z_c) \) the field of formal Laurent series around \( c \in \mathbb{P}^1 \). Also set \( \mathcal{O}_{c,i} := \mathbb{C}[z - c]_i \) and \( \mathcal{M}_{c,i} := \mathbb{C}[(z - c)^{-1}]_i \) for a positive integer \( i \).

Let us consider a finite set \( \{a_0, a_1, \ldots, a_d\} \) of points in \( \mathbb{P}^1 \) and fix an effective divisor \( D := \sum_{a \in \{a_0, a_1, \ldots, a_d\}} (k_a + 1) \cdot a \) with \( k_a \in \mathbb{Z}_{\geq 0} \). We denote the set of points by \( \{a_0, a_1, \ldots, a_d\} \). Let \( \mathcal{O}^{\text{reg}}_{\mathbb{P}^1}(\ast |D|) \) and \( \mathcal{O}^{\text{reg}}_{\mathbb{P}^1}(\ast |D|) \) denote the sheaves of rational function and of rational 1-forms with poles on \( |D| \) respectively.

**Definition 1.14.** An algebraic meromorphic connection on the trivial bundle is a differential operator, namely morphism of sheaves of \( \mathbb{C} \)-vector spaces

\[
\nabla : (\mathcal{O}_{\mathbb{P}^1}^{\text{reg}})^n \longrightarrow (\mathcal{O}_{\mathbb{P}^1}^{\text{reg}})^n \otimes_{\mathcal{O}_{\mathbb{P}^1}^{\text{reg}}} \mathcal{O}_{\mathbb{P}^1}^{\text{reg}}(\ast |D|),
\]

satisfying the Leibniz rule

\[
\nabla(fs) = df \otimes s + f \nabla(s)
\]

for all open subsets \( U \subset \mathbb{P}^1 \), \( f \in \mathcal{O}_{\mathbb{P}^1}^{\text{reg}}(U) \), and \( s \in (\mathcal{O}_{\mathbb{P}^1}^{\text{reg}})^n(U) \). On any open subset \( U \subset \mathbb{P}^1 \), \( \nabla \) can be uniquely written as the matrix form,

\[
\nabla = d - A_U(z)dz
\]

where \( A_U(z)dz \in M_n(\mathcal{O}_{\mathbb{P}^1}^{\text{reg}}(\ast |D|)(U)) \). Thus there uniquely exists the matrix 1-form \( A(z)dz \in M_n(\mathcal{O}_{\mathbb{P}^1}^{\text{reg}}(\ast |D|)(\mathbb{P}^1)) \) defined globally on \( \mathbb{P}^1 \) such that

\[
A(z)dz|_U = A_U(z)dz
\]

for any open subset \( U \subset \mathbb{P}^1 \). We call this matrix \( A(z) \) the coefficient matrix of the connection \( \nabla \).

The automorphism group of the free \( \mathcal{O}_{\mathbb{P}^1}^{\text{reg}} \)-module \( (\mathcal{O}_{\mathbb{P}^1}^{\text{reg}})^n = \mathbb{C}^n \otimes_{\mathcal{O}_{\mathbb{P}^1}^{\text{reg}}} \mathcal{O}_{\mathbb{P}^1}^{\text{reg}} \) is isomorphic to \( \text{GL}_n(\mathbb{C}) \). Thus meromorphic connections \( \nabla_1 = d - A_1(z)dz \) and \( \nabla_2 = d - A_2(z)dz \) of rank \( n \) are isomorphic if and only if there exists \( g \in \text{GL}_n(\mathbb{C}) \) such that

\[
A_2(z) = gA_1(z)g^{-1}.
\]

A connection \( \nabla \) is said to be irreducible if there is no nontrivial subspace \( W \subset \mathbb{C}^n \) such that the sub \( \mathcal{O}_{\mathbb{P}^1}^{\text{reg}} \)-module \( W \otimes_{\mathcal{O}_{\mathbb{P}^1}^{\text{reg}}} \mathcal{O}_{\mathbb{P}^1}^{\text{reg}} \subset \mathbb{C}^n \otimes_{\mathcal{O}_{\mathbb{P}^1}^{\text{reg}}} \mathcal{O}_{\mathbb{P}^1}^{\text{reg}} = (\mathcal{O}_{\mathbb{P}^1}^{\text{reg}})^n \) is \( \nabla \)-stable.

1.4.2 The space of meromorphic connections as a complex manifold

By the projective transformation on \( \mathbb{P}^1(\mathbb{C}) \), we may suppose that \( |D| \subset \mathbb{C} \). Let us consider a space of connections with coefficients in \( \mathcal{O}_{\mathbb{P}^1}^{\text{reg}}(\ast |D|) \), i.e.,

\[
\mathcal{M}_D := \{ \nabla = d - A(z)dz \mid A(z)dz \in M_n(\mathcal{O}_{\mathbb{P}^1}^{\text{reg}}(\ast |D|)(\mathbb{P}^1)) \}.
\]

For \( d - A(z)dz \in \mathcal{M}_D \), we can write

\[
A(z) = \sum_{a \in |D|} \text{prc}_a(A(z)) = \sum_{a \in |D|} \sum_{i=0}^{k_a} \frac{A_i^{(a)}}{(z - a)^i} \frac{dz}{z - a}.
\]
with the relation
\[ \sum_{a \in |D|} A_{0}^{(a)} = 0. \]

Thus we can identify \( \mathcal{M}_D \) as a space of matrices,
\[ M_D := \left\{ \left( X_{i}^{(a)} \right)_{a \in |D|, i = 0, 1, \ldots, k_a} \in \prod_{a \in |D|} \prod_{i=0}^{k_a} M_n(\mathbb{C}) \mid \sum_{a \in |D|} \sum_{i=0}^{k_a} X_{i}^{(a)} = 0 \right\}, \]

which is a union of hyperplanes in the affine space \( \prod_{a \in |D|} \prod_{i=0}^{k_a} M_n(\mathbb{C}) \), i.e., \( M_D \) is a complex manifold. The \( \text{GL}_n(\mathbb{C}) \)-action on \( M_D \) is translated into the diagonal action on \( M_D \), i.e.,
\[ \text{GL}_n(\mathbb{C}) \times M_D \ni \left( g, \left( X_{i}^{(a)} \right)_{a \in |D|, i = 0, 1, \ldots, k_a} \right) \mapsto \left( g X_{i}^{(a)} g^{-1} \right)_{a \in |D|, i = 0, 1, \ldots, k_a} \in M_D. \]

Under this identification, we regard \( \mathcal{M}_D \) as a complex manifold with holomorphic \( \text{GL}_n(\mathbb{C}) \)-action. The irreducibility of connections on \( \mathcal{M}_D \) corresponds to the following condition on \( M_D \), namely, \( \left( X_{i}^{(a)} \right)_{a \in |D|, i = 0, 1, \ldots, k_a} \in \prod_{a \in |D|} \prod_{i=0}^{k_a} M_n(\mathbb{C}) \) is said to be irreducible if it has no nontrivial simultaneous invariant subspace of \( \mathbb{C}^n \). That is to say, if a subspace \( W \subset \mathbb{C}^n \) satisfies \( X_{i}^{(a)} W \subset W \) for all \( a \in |D| \) and \( i = 0, 1, \ldots, k_a \), then \( W = \{0\} \) or \( W = \mathbb{C}^n \). Let us denote subspaces of irreducible elements in \( \mathcal{M}_D \) and \( M_D \) by \( \mathcal{M}_D^\text{irr} \) and \( M_D^\text{irr} \) respectively.

### 1.4.3 Residue maps of truncated orbits

Let \( \mathcal{O}_H \) be the truncated \( \text{GL}_n(\mathbb{C}[z]_k) \)-orbit of an unramified HTL normal form \( H \in M_n(\mathbb{C}[z^{-1}]_k) \) and recall that the map
\[ \mu_{\mathcal{O}_H} : \mathcal{O}_H \cong \text{GL}_n(\mathbb{C}[z]_k) / \text{GL}_n(\mathbb{C}[z]_k)_H \ni [g] \mapsto -\text{Ad}^*(g)(H) \in M_n(\mathbb{C}[z^{-1}]_k), \]
is a moment map with respect to the action of \( \text{GL}_n(\mathbb{C}[z]_k) \). Then the map \( \mu_{\mathcal{O}_H}^0 := \text{res}_{z=0} \circ \mu_{\mathcal{O}_H} : \mathcal{O}_H \to M_n(\mathbb{C}) \cong \mathfrak{gl}_n(\mathbb{C})^* \) becomes a moment map with respect to the action of \( \text{GL}_n(\mathbb{C}) \). Let us consider the map
\[ \nu_{\text{GL}_n(\mathbb{C})} \circ \text{pr}_1 : T^* \text{GL}_n(\mathbb{C}) \times \mathcal{O}_H \to \mathfrak{gl}_n(\mathbb{C})^* \]
and its restriction \( \nu_{\text{GL}_n(\mathbb{C})} \circ \text{pr}_1|_{\mu_{\text{ext}}^{-1}(\mathcal{O}_H)} \) on the subspace \( \mu_{\text{ext}}^{-1}(\mathcal{O}_H) \subset T^* \text{GL}_n(\mathbb{C}) \times \mathcal{O}_H \). Then since \( \nu_{\text{GL}_n(\mathbb{C})} \circ \text{pr}_1|_{\mu_{\text{ext}}^{-1}(\mathcal{O}_H)} \) is \( (\text{GL}_n(\mathbb{C})_H)_H \text{res}^{-1} \)-invariant, there uniquely exists the map \( \nu_{\mathcal{O}_H} : \mathcal{O}_H \to \mathfrak{gl}_n(\mathbb{C})^* \) such that the diagram
\[
\begin{array}{ccc}
\mu_{\text{ext}}^{-1}(\mathcal{O}_H) & \xrightarrow{\nu_{\text{GL}_n(\mathbb{C})}\circ\text{pr}_1} & \mathfrak{gl}_n(\mathbb{C})^* \\
\downarrow \quad & & \quad \downarrow \nu_{\mathcal{O}_H} \\
\mathcal{O}_H & & \mathfrak{gl}_n(\mathbb{C})^* 
\end{array}
\]
is commutative. Then by Theorem [16], we obtain the following.

**Proposition 1.15.** The map \(-\nu_{\mathcal{O}_H} : \mathcal{O}_H \to \mathfrak{gl}_n(\mathbb{C})^* \) defined above coincides with the moment map \( \mu_{\mathcal{O}_H}^0 : \mathcal{O}_H \to \mathfrak{gl}_n(\mathbb{C})^* \).
1.4.4 Moduli space of meromorphic connections on a trivial bundle with unramified irregular singularities

For each \( a \in |D| \), let us take an unramified HTL normal form

\[
H_a := \left( \frac{S_{k_a}}{z_a^{k_a-1}} + \cdots + \frac{S_1}{z_a} + \frac{S_0}{z_a} + N_0(a) \right) \frac{dz_a}{z_a},
\]

such that \( \sum_{a \in |D|} \text{res}_{z=a} \text{tr}(H_a) = 0 \). We denote the collection of these HTL normal forms by \( H := (H_a)_{a \in |D|} \).

Let us consider the map

\[
\mu_H : \prod_{a \in |D|} \mathcal{O}_H \rightarrow \mathfrak{gl}_n(C)^* (X_a) \mapsto \sum_{a \in |D|} \mu_{O_H}^0 (X_a)
\]

which is a moment map with respect to the diagonal action of \( \text{GL}_n(C) \) on the product space \( \prod_{a \in |D|} \mathcal{O}_H \).

By the injective immersion

\[
\iota \prod_{a \in |D|} \mathcal{O}_H : \prod_{a \in |D|} \mathcal{O}_H \rightarrow \prod_{a \in |D|} M_n(C[z_a^{-1}]^k_a),
\]

we regard \( \prod_{a \in |D|} \mathcal{O}_H \) as a subspace of \( \prod_{a \in |D|} \prod_{i=0}^{k_a} M_n(C) \) and we denote the space of irreducible elements in \( \prod_{a \in |D|} \mathcal{O}_H \subset \prod_{a \in |D|} \prod_{i=0}^{k_a} M_n(C) \) by \( (\prod_{a \in |D|} \mathcal{O}_H)^s \).

**Definition 1.16.** The symplectic quotient space

\[
\mathcal{M}_s^* (H) := \text{GL}_n(C) \backslash (\mu_H^s)^{-1}(0)
\]

is the moduli space of meromorphic connections on the rank \( n \) trivial bundle on \( \mathbb{P}^1 \) with respect to the unramified HTL normal forms \( H \). It is known that if \( (\mu_H^s)^{-1}(0) \neq \emptyset \), then the moduli space \( \mathcal{M}_s^* (H) \) is a holomorphic symplectic manifold.

## 2 Construction of unfolding manifold

This section is the main part of the note. Let us consider a moduli space of meromorphic connection \( \mathcal{M}_s^* (H) \) with respect to a collection of unramified HTL normal forms \( H \). First we introduce a holomorphic family of collections of unramified HTL normal forms which describes unfolding procedure of irregular singularities of HTL normal forms in \( H \). We call this family the unfolding of \( H \). Then we shall explain the construction of a Poisson manifold whose symplectic leaves are isomorphic to Zariski open subsets of the moduli spaces of meromorphic connections with respect to the collections of HTL normal forms appearing in this unfolding family.

### 2.1 Deformation of HTL normal forms and unfolding of spectral types

#### 2.1.1 An open subset of \( \mathbb{C}^{k+1} \) associated to \( H \)

Let us consider an unramified HTL normal form

\[
H = \left( \frac{S_k}{z^k} + \cdots + \frac{S_1}{z} + S_0 + N_0 \right) \frac{dz}{z} \in M_n(C[z^{-1}]_k)dz
\]
with the spectral type
\[ \text{sp}(H) = (m_0 \leq \phi_0 \leq m_1 \leq \phi_1 \cdots \leq \phi_{k-1} \leq m_k, \sigma(m_0)). \]

Let \( \mathbb{C}^n = \bigoplus_{j=1}^{m(i)} V_{i,j} \) be the direct sum of simultaneous eigenspaces of \( (S_1, S_{i+1}, \ldots, S_k) \) defined in Section 1.3.3. We denote the eigenvalues of \( S_l \) on \( V_{i,j} \) by \( s_{l(i,j)}^{(i)} \) for \( i \leq l \leq k \).

Let us introduce a distance function \( d_i(\cdot, \cdot) \) on \( \{1, 2, \ldots, m(i)\} \) for each \( i = 0, 1, \ldots, k \) as follows,
\[ d_i(j, j') := \min\{l \in \{i, i + 1, \ldots, k\} \mid \phi_i^{(j)} = \phi_i^{(j')}\} - i, \]
where \( \phi_i^{(j)} \) is the composition of refinement maps \( \phi_i \circ \cdots \circ \phi_{i+1} \circ \phi_i \) and we formally set \( \phi_i := \phi_i \) and \( \phi_k : \{1, 2, \ldots, m(k)\} \to \{1\} \). Then we note that
\[
\begin{cases}
V_{l, \phi_i^{(j)}} = V_{l, \phi_i^{(j')}} & d_i(j, j') + i \leq l \leq k, \\
V_{l, \phi_i^{(j)}} \neq V_{l, \phi_i^{(j')}} & l = d_i(j, j') + i - 1.
\end{cases}
\]

Since \( V_{l, \phi_i^{(j)}} \) is a direct summand of \( V_{l, \phi_i^{(j')}} \), this implies that eigenvalues \( s_{(i,j')}^{(l)} \) satisfy the following relations,
\[
\begin{align*}
\begin{cases}
s_{l(i,j')}^{(l)} = s_{l(i,j')}^{(l)} & d_i(j, j') + i \leq l \leq k, \\
s_{l(i,j)}^{(l)} \neq s_{l(i,j')}^{(l)} & l = d_i(j, j') + i - 1.
\end{cases}
\end{align*}
\]

Let us define polynomials
\[ \alpha(x_0, x_1, \ldots, x_k)_{i,j} := \sum_{i \leq l \leq k} s_{l(i,j)}^{(l)} \prod_{l \leq \nu \leq k} (x_i - x_\nu), \]
and hypersurfaces
\[ D_{i,j,j'} := \{ c \in \mathbb{C}^{k+1} \mid \alpha(c)_{i,j} = \alpha(c)_{i,j'} \} \]
for \( i = 0, 1, \ldots, k - 1 \) and \( j \neq j' \in \{1, 2, \ldots, m(i)\} \). Then (2) implies that
\[
\begin{align*}
\alpha(x_0, x_1, \ldots, x_k)_{i,j} - \alpha(x_0, x_1, \ldots, x_k)_{i,j'} &= \sum_{i \leq l \leq i + d_i(j,j') - 1} (s_{l(i,j)}^{(l)} - s_{l(i,j')}^{(l)}) \prod_{l \leq \nu \leq k} (x_i - x_\nu) \\
&= \prod_{d_i(j,j') + 1 \leq l \leq k} (x_i - x_\nu) \left( \sum_{i \leq l \leq i + d_i(j,j') - 1} (s_{l(i,j)}^{(l)} - s_{l(i,j')}^{(l)}) \prod_{l \leq \nu \leq i + d_i(j,j') - 1} (x_i - x_\nu) \right).
\end{align*}
\]

This leads us to define the following polynomials
\[ \alpha^*(x_0, x_1, \ldots, x_k)_{i,j,j'} := \sum_{i \leq l \leq i + d_i(j,j') - 1} (s_{l(i,j)}^{(l)} - s_{l(i,j')}^{(l)}) \prod_{l \leq \nu \leq i + d_i(j,j') - 1} (x_i - x_\nu). \]

Then we define the subset of \( D_{i,j,j'} \) by
\[ D^*_{i,j,j'} := \{ c \in \mathbb{C}^{k+1} \mid \alpha^*(c)_{i,j,j'} = 0 \} \subset D_{i,j,j'}. \]

Then we define the Zariski open subset of \( \mathbb{C}^{k+1} \) by
\[
\mathbb{D}(H) := \mathbb{C}^{k+1} - \left( \bigcup_{i=0}^{k-1} \bigcup_{j,j' \in \{1, 2, \ldots, m(i)\} \setminus \{j \leq j'\}} D^*_{i,j,j'} \right). \tag{3}
\]

We note that \( \mathbb{D}(H) \) contains the origin \( 0 \in \mathbb{C}^{k+1} \).
2.1.2 A stratification of $C^{k+1}$ associated to partitions of a finite set

Firstly we introduce a stratification of $C^{k+1}$ which is induced from partitions of the finite set $\{0, 1, 2, \ldots, k\}$. Let $I: I_1, I_2, \ldots, I_r$ be a partition of $\{0, 1, 2, \ldots, k\}$, i.e., the direct sum decomposition $I_0 \sqcup I_1 \sqcup \cdots \sqcup I_r = \{0, 1, 2, \ldots, k\}$. We may suppose that elements in $I_j = \{i[j,0], i[j,1], \ldots, i[j,k_j]\}$ are arranged in the ascending order, i.e.,

$$i[j,0] < i[j,1] < \cdots < i[j,k_j]$$

as positive integers. Also we assume that $0 \in I_0$. Along with the partition $I$, we define an embedding of $C^{r+1}$ into $C^{k+1}$,

$$\nu_I: C^{r+1} \ni a = (a_0, a_1, \ldots, a_r) \mapsto (\nu_I(a)_0, \nu_I(a)_1, \ldots, \nu_I(a)_k) \in C^{k+1}$$

by setting

$$\nu_I(a)_i := a_i \quad (i \in I_i).$$

Let us denote the configuration space of $r+1$ points in $C$ by

$$C_{r+1}(C) := \{(a_0, a_1, \ldots, a_r) \in C^{r+1} \mid a_i \neq a_j \text{ for } i \neq j\}.$$ 

Then we define a subspace of $C^{k}$ by

$$C(I) := \nu_I(C_{r+1}(C)) = \left\{ (a_0, a_1, \ldots, a_k) \in C^{k+1} \mid \begin{array}{l} a_i = a_j \quad \text{if } i, j \in I_l \text{ for some } l, \\ a_i \neq a_j \quad \text{otherwise} \end{array} \right\}.$$ 

Then we obtain the direct sum decomposition

$$C^{k+1} = \bigsqcup_{I \in P_{[k+1]}} C(I), \quad (4)$$

where $P_{[k+1]}$ is the set of all partitions of $\{0, 1, \ldots, k\}$. The set $P_{[k+1]}$ of partitions is naturally equipped with the partial order defined by the refinement of the partitions and the each direct summand satisfies the closure relation

$$\overline{C(I)} = \bigsqcup_{I' \in P_{[k+1]}, I \leq I'} C(I').$$

Thus the decomposition (4) gives a stratification of $C^{k+1}$ associated to the poset $P_{[k+1]}$.

2.1.3 Unfolding of an HTL normal form

**Definition 2.1** (Unfolding of an HTL normal form). The unfolding of the irregular type $H_{irr}$ of $H$ is the function from $C^{k+1}$ to $M_n(C(z))dz$ of the form,

$$H_{irr}(c_0, c_1, \ldots, c_k) := \left( \frac{S_k}{(z-c_1)\cdots(z-c_k)} + \frac{S_{k-1}}{(z-c_1)\cdots(z-c_{k-1})} + \cdots + \frac{S_1}{z-c_1} \right) \frac{dz}{z-c_0}.$$ 

We also define the unfolding of $H$ by

$$H(c_0, c_1, \ldots, c_k) := H_{irr}(c_0, c_1, \ldots, c_k) + (S_0 + N_0) \frac{dz}{z-c_0}$$

and call $H_{irr}(c_0, c_1, \ldots, c_k)$ the irregular part of the unfolding $H(c_0, c_1, \ldots, c_k)$.
Let $\mathbb{C}^{k+1} = \bigsqcup_{I \in \mathcal{P}_{(k+1)}} C(I)$ be the stratification of $\mathbb{C}^{k+1}$ defined in the previous section. We shall compute the spectral types of $H(c_1, c_2, \ldots, c_k)$ for the parameters $(c_0, c_1, \ldots, c_k)$ on these strata $C(I)$.

Fix a partition $I: I_0, I_1, \ldots, I_r \in \mathcal{P}_{(k+1)}$ and consider the embedding $\iota_I: \mathbb{C}^{r+1} \hookrightarrow \mathbb{C}^{k+1}$ defined previously. Then since $C(I) = \{\iota_I(c) \in \mathbb{C}^{k+1} \mid c \in C_{r+1}(\mathbb{C})\}$, we can introduce a parameterization of the unfolding of $H$ on $C(I)$ by setting

$$H_I(c) := H(\iota_I(c)) \quad (c \in C_{r+1}(\mathbb{C}))$$

Let us suppose that $(c_0, c_1, \ldots, c_r) \in C_r(\mathbb{C})$. Then $H_I(c) \in M_n(\mathbb{C}(z))dz$ has poles at $c_j$ of order at most $k_j + 1 = |I_j|$ for $j = 0, 1, \ldots, r$. Thus the partial fractional decomposition algorithm gives the description $H_I(c) = \sum_{j=0}^r \sum_{\nu=0}^{k_j} A_{\nu}^{[j]} / z_\nu c_j dz_\nu$ where $A_{\nu}^{[j]} \in M_n(\mathbb{C})$ and $z_\nu = z - c_j$. We denote each component of the sum by

$$H_I(c)_{z_\nu} := \sum_{\nu=0}^{k_j} A_{\nu}^{[j]} / z_\nu dz_\nu.$$

**Proposition 2.2** ([3]). Let us suppose that $\iota_I(c) \in \mathbb{D}(H)$. Then $H_I(c)_{z_\nu}$ become unramified HTL-normal forms with the spectral types

$$\begin{cases}
(m_{i,[j,0]} \leq \phi_{i,[j,0]}^{[j,1]} m_{i,[j,1]} \leq \phi_{i,[j,1]}^{[j,2]} \cdots \leq \phi_{i,[j,k_j]}^{[j,k_j]} m_{i,[j,k_j]}^{[j,k_j]}, \text{triv}) \quad &\text{for } j = 1, 2, \ldots, r, \\
(m_{i,[0,0]} \leq \phi_{i,[0,0]}^{[0,1]} m_{i,[0,1]} \leq \phi_{i,[0,1]}^{[0,2]} \cdots \leq \phi_{i,[0,k_0]}^{[0,k_0]} m_{i,[0,k_0]}, \sigma(m_0)) \quad &\text{for } j = 0.
\end{cases}$$

This proposition says that the each fiber of the unfolding of $H$ is a sum of unramified HTL normal forms $H_I(c)_{z_\nu}$ and spectral types of them do not depend on the parameter $c$ but only on the stratum $C(I) \subset \mathbb{C}^{k+1}$.

### 2.2 Deformation of $O_{H^{irr}}$

In the previous section, we introduced the holomorphic family $(H(c))_{c \in \mathbb{D}(H)}$ and explained that each fiber $H(c)$ is a sum of unramified HTL normal forms. Next we shall construct a deformation of the truncated orbit $O_H$ in accordance with this family $(H(c))_{c \in \mathbb{D}(H)}$. For this purpose, we first explain the construction of a deformation of the orbit $O_{H^{irr}}$ of the irregular type.

#### 2.2.1 Deformations of $\mathbb{C}[z]$ and $\mathbb{C}[z^{-1}]dz$

To a point $c = (c_0, c_1, \ldots, c_k) \in \mathbb{C}^{k+1}$, we associate an effective divisor

$$D(c) := c_0 + c_1 + \cdots + c_k$$

of $\mathbb{A}^1(\mathbb{C}) = \mathbb{C}$. Then we consider a subspace of rational 1-forms

$$\mathcal{O}^{rat}(D(c)) := \{f \in \mathbb{C}(z)dz \mid \text{ord}_P(f) \geq -n_P \text{ for all } P \in \mathbb{C}\}$$

where $n_P$ is the coefficient of $P \in \mathbb{C}$ in the formal sum $D(c) = \sum_{P \in \mathbb{C}} n_P P$, and $\text{ord}_P(f)$ is the order of $f$ at $P$. We can regard $\mathcal{O}^{rat}(D(c))$ as $\mathbb{C}[z]$-module containing $\mathbb{C}[z]dz$ as a submodule and consider a quotient module

$$\hat{\Omega}(c) := \mathcal{O}^{rat}(D(c)) / \mathbb{C}[z]dz.$$
We also define the linear system

\[ L^{\text{rat}}(-D(c)) := \{ f \in \mathbb{C}(z) \mid \text{ord}_P(f) \geq n_P \text{ for all } P \in \mathbb{C} \}. \]

Since \( D(c) \) is an effective divisor, i.e., \( n_P \geq 0 \text{ for all } P \in \mathbb{C} \), we may regard \( L^{\text{rat}}(-D(c)) \) as an ideal of \( \mathbb{C}[z] \), namely,

\[ L^{\text{rat}}(-D(c)) = \left( \prod_{i=0}^{k} (z - c_i) \right)_{\mathbb{C}[z]}, \]

and take the quotient ring

\[ \mathbb{C}[z](c) := \mathbb{C}[z]/L^{\text{rat}}(-D(c)). \]

We denote the class of \( f(z) \in \mathbb{C}[z] \) in \( \mathbb{C}[z](c) \) and that of \( g(z)dz \in \Omega^c_{\text{rat}}(D(c)) \) in \( \hat{\Omega}_c(c) \) by the same notations, if it may not cause any confusion.

**Definition 2.3** (Residue and evaluation maps). For \( l \in \{0, 1, \ldots, k\} \), we define the residue map at \( c_l \) by

\[ \text{res}_{z=c_l}([f(z)dz]) := \text{res}_{z=c_l}f(z)dz \]

for \([f(z)dz] \in \hat{\Omega}(c)\), the class of \( f(z) \in \Omega^c_{\text{rat}}(D(c)) \). Further, we define the residue at \( \infty \) by

\[ \text{res}_{z=\infty}([f(z)dz]) := -\text{res}_{w=0}f(1/w) \frac{dw}{w^2}. \]

We note the all these residue maps are well-defined in \( \hat{\Omega}(c) \).

Also we define the evaluation map \( \text{ev}_{c_j} : \mathbb{C}[z](c) \rightarrow \mathbb{C} \) at \( c_j \) by

\[ \text{ev}_{c_j} : \mathbb{C}[z](c) = \mathbb{C}[z]/\left( \prod_{\mu=0}^{k} (z - c_\mu) \right)_{\mathbb{C}[z]} \rightarrow \mathbb{C}[z]/\langle z - c_j \rangle_{\mathbb{C}[z]} = \mathbb{C}, \]

the natural projection induced by the inclusion \( \langle \prod_{\mu=0}^{k} (z - c_\mu) \rangle_{\mathbb{C}[z]} \subset \langle z - c_j \rangle_{\mathbb{C}[z]} \). Here we identify \( \mathbb{C}[z]/\langle z - c_j \rangle_{\mathbb{C}[z]} = \mathbb{C} \) by the unique \( \mathbb{C} \)-algebra isomorphism. Usually we simply write \( f(c_j) := \text{ev}_{c_j}(f(z)) \) for \( f(z) \in \mathbb{C}[z](c) \).

Note that \( \hat{\Omega}(c) \) is a \( \mathbb{C}[z](c) \)-module, since elements of \( L^{\text{rat}}(-D(c)) \subset \mathbb{C}[z] \) act trivially on \( \hat{\Omega}_c(c) \). Since we have the isomorphisms

\[ \mathbb{C}[z](0) \cong \mathbb{C}[z]_{k}, \quad \hat{\Omega}(0) \cong \mathbb{C}[z^{-1}]_k dz, \]

we can regard \( \mathbb{C}[z](c) \) and \( \hat{\Omega}(c) \) as deformations of \( \mathbb{C}[z]_{k} \) and \( \mathbb{C}[z^{-1}]_k dz \).

Let us define a filtration on \( \hat{\Omega}(c) \) and cofiltration on \( \mathbb{C}[z](c) \). Consider projection maps \( \text{pr}^l : \mathbb{C}^{k+1} \supset (x_0, x_1, \ldots, x_k) \rightarrow (x_0, x_1, \ldots, x_l) \in \mathbb{C}^{l+1} \) and define divisors \( D(c)_{\leq l} := D(\text{pr}^l(c)) \) for \( l = 0, 1, \ldots, k \). Then we obtain the following filtration

\[ \Omega^{\text{rat}}(D(c)_{\leq 0}) \subset \Omega^{\text{rat}}(D(c)_{\leq 1}) \subset \cdots \subset \Omega^{\text{rat}}(D(c)_{\leq k}) = \Omega^{\text{rat}}(D(c)). \]
Definition 2.4 (Standard filtration and basis of \( \hat{\Omega}(c) \)). Let us set
\[
\hat{\Omega}(c)_l := \Omega^{rat}(D(c)_{\leq l})/\mathbb{C}[z] dz \text{ for } l = 0, 1, \ldots, k.
\]
The filtration
\[
\hat{\Omega}(c)_0 \subset \hat{\Omega}(c)_1 \subset \cdots \subset \hat{\Omega}(c)_k = \hat{\Omega}(c)
\]
induced by the above filtration of \( \Omega^{rat}(D(c)) \) is called the standard filtration of \( \hat{\Omega}(c) \). The basis of \( \hat{\Omega}(c) \),
\[
\frac{1}{z - c_0}, \frac{1}{(z - c_0)(z - c_1)}, \ldots, \frac{1}{(z - c_0)(z - c_1) \cdots (z - c_k)}
\]
as \( \mathbb{C} \)-vector space is called the standard basis of \( \hat{\Omega}(c) \).

Let us note that \( \frac{1}{z - c_0}, \frac{1}{(z - c_0)(z - c_1)}, \ldots, \frac{1}{(z - c_0)(z - c_1) \cdots (z - c_l)} \) becomes a basis of the \( l \)-th component \( \hat{\Omega}(c)_{\leq l} \) of the standard filtration for each \( l = 0, 1, \ldots, k \).

Similarly, we can consider the filtration
\[
\langle z - c_0 \rangle_{\mathbb{C}[z]} \supset \langle (z - c_0)(z - c_1) \rangle_{\mathbb{C}[z]} \supset \cdots \supset \langle \prod_{i=0}^{k}(z - c_i) \rangle_{\mathbb{C}[z]}
\]
of ideals on \( \mathbb{C}[z] \).

Definition 2.5 (Standard cofiltration and basis of \( \overline{\mathbb{C}[z]}(c) \)). Let us set
\[
\overline{\mathbb{C}[z]}(c)_l := \mathbb{C}[z]/\langle \prod_{i=0}^{l}(z - c_i) \rangle_{\mathbb{C}[z]} \text{ for } l = 0, 1, \ldots, k.
\]
Then the sequence of projection maps
\[
\overline{\mathbb{C}[z]}(c)_0 \rightarrow \overline{\mathbb{C}[z]}(c)_1 \rightarrow \cdots \rightarrow \overline{\mathbb{C}[z]}(c)_k = \overline{\mathbb{C}[z]}(c)
\]
induced by the above filtration of ideals of \( \mathbb{C}[z] \) is called the standard cofiltration of \( \overline{\mathbb{C}[z]}(c) \). The basis of \( \overline{\mathbb{C}[z]}(c) \),
\[
1, (z - c_0), (z - c_0)(z - c_1), \ldots, (z - c_0)(z - c_1) \cdots (z - c_{k-1})
\]
as \( \mathbb{C} \)-vector space is called the standard basis of \( \overline{\mathbb{C}[z]}(c) \).

Next we introduce the pairing of \( \overline{\mathbb{C}[z]}(c)_l \) and \( \hat{\Omega}(c)_l \) as \( \mathbb{C} \)-vector spaces defined by
\[
\langle \cdot, \cdot \rangle_{c,l} : \overline{\mathbb{C}[z]}(c)_l \times \hat{\Omega}(c)_l \rightarrow \mathbb{C}
\]
\[
(f(z), g(z)dz) \mapsto -\text{res}_{z=\infty}(f(z) \cdot g(z)dz)
\]
for each \( l = 0, 1, \ldots, k \). Then we can show that the paring \( \langle \cdot, \cdot \rangle_{c,l} \) is non-degenerate and the bases
\[
1, (z - c_0), (z - c_0)(z - c_1), \ldots, (z - c_0)(z - c_1) \cdots (z - c_{l-1})
\]
and
\[
\frac{1}{z - c_0}, \frac{1}{(z - c_0)(z - c_1)}, \ldots, \frac{1}{(z - c_0)(z - c_1) \cdots (z - c_l)}
\]
are dual bases with respect to this paring.
2.2.2 Partial fraction decomposition

In Section 2.1.3 we saw that each fiber \( H_2(c) \) of the unfolding of \( H \) on a stratum \( C(\mathcal{I}) \subset \mathbb{C}^{k+1} \) decomposes into the sum of unramified HTL normal form by the partial fraction decomposition algorithm. Now we explain that on each stratum \( C(\mathcal{I}) \), the partial fraction decomposition gives an decomposition of \( \mathbb{C}[z](\mathcal{I})_l \)-module \( \widehat{\Omega}(c)_l \).

Let us take a partition \( \mathcal{I} : I_1, I_2, \ldots, I_r \) of the finite set \( \{0, 1, \ldots, k \} \). Then for each \( l = 0, 1, \ldots, k \), we define a partition \( \mathcal{I}^{(l)} : I_1^{(l)}, I_2^{(l)}, \ldots, I_r^{(l)} \) of the subset \( \{0, 1, \ldots, l \} \subset \{0, 1, \ldots, k \} \) by \( I_j^{(l)} := I_j \cap \{0, 1, \ldots, l \} \), allowing \( I_j^{(l)} = \emptyset \). Let us set \( c_j^{(l)} := |I_j^{(l)}| - 1 \). Let us take \( c = (c_0, c_1, \ldots, c_r) \in C_{r+1}(\mathcal{C}) \) so that \( \nu_{\mathcal{I}}(c) \in C(\mathcal{I})_l \). Then we have \( D(\nu_{\mathcal{I}}(c))_{\leq l} = \sum_{j=0}^r (k_j^{(l)} + 1) \cdot c_j \). The algorithm of the partial fraction decomposition of rational functions gives us the direct sum decomposition

\[
\widehat{\Omega}(\nu_{\mathcal{I}}(c))_l = \bigoplus_{j=0}^r \mathbb{C}[(z - c_j)^{-1}]_{k_j^{(l)}}
\]
as \( \mathbb{C}[z](\nu_{\mathcal{I}}(c))_l \)-modules.

We also have a similar decomposition of \( \mathbb{C}[z](\nu_{\mathcal{I}}(c))_l \). Since we have \( c_i \neq c_j \) for \( i \neq j \), it follows that

\[
\langle (z - c_i)^{k_i^{(l)}} \rangle_{\mathbb{C}[z]} + \langle (z - c_j)^{k_j^{(l)}} \rangle_{\mathbb{C}[z]} = \mathbb{C}[z], \quad i \neq j,
\]

\[
\bigcap_{i=0}^l \langle (z - c_i)^{k_i^{(l)}} \rangle_{\mathbb{C}[z]} = L^{\text{rat}}(-D(\nu_{\mathcal{I}}(c))_{\leq l}).
\]

Therefore the Chinese remainder theorem implies that the map

\[
\prod_{i=0}^l \text{pr}_{c_i} : \mathbb{C}[z](\nu_{\mathcal{I}}(c))_l \longrightarrow \prod_{i=0}^l \mathbb{C}[(z - c_i)^{-1}]_{k_i^{(l)}}
\]

is an algebra isomorphism. Here

\[
\text{pr}_{c_i} : \mathbb{C}[z](\nu_{\mathcal{I}}(c))_l \longrightarrow \mathbb{C}[z]/\langle (z - c_i)^{k_i^{(l)} + 1} \rangle_{\mathbb{C}[z]} = \mathbb{C}[z - c_i]_{k_i^{(l)}}
\]

are the natural projections for \( i = 0, 1, \ldots, l \).

If we note that the ideal \( \langle (z - c_i)^{k_i^{(l)} + 1} \rangle_{\mathbb{C}[z]} \) annihilates \( \mathbb{C}[(z - c_i)^{-1}]_{k_i^{(l)}} \), it follows that the diagrams

\[
\begin{array}{ccc}
\mathbb{C}[z](\nu_{\mathcal{I}}(c))_l \times \mathbb{C}[(z - c_i)^{-1}]_{k_i^{(l)}} & \longrightarrow & \mathbb{C}[(z - c_i)^{-1}]_{k_i^{(l)}} \\
\downarrow_{\text{pr}_{c_i} \times \text{id}} & & \downarrow_{\text{id}} \\
\mathbb{C}[z - c_i]_{k_i^{(l)}} \times \mathbb{C}[(z - c_i)^{-1}]_{k_i^{(l)}} & \longrightarrow & \mathbb{C}[(z - c_i)^{-1}]_{k_i^{(l)}}
\end{array}
\]

are commutative. Here horizontal maps are scalar multiplications on \( \mathbb{C}[z - c_i]_{k_i^{(l)}} \) as \( \mathbb{C}[z](\nu_{\mathcal{I}}(c))_l \) and \( \mathbb{C}[z - c_i]_{k_i^{(l)}} \) modules respectively. Further, we obtain the following commutative diagram,

\[
\begin{array}{ccc}
\mathbb{C}[z](\nu_{\mathcal{I}}(c))_l \times \widehat{\Omega}(\nu_{\mathcal{I}}(c))_l & \longrightarrow & \widehat{\Omega}(\nu_{\mathcal{I}}(c))_l \\
\downarrow_{\prod_{i=0}^l (\text{pr}_{c_i} \times \text{id})} & & \downarrow_{=} \\
\prod_{i=0}^r \left( \mathbb{C}[z - c_i]_{k_i^{(l)}} \times \mathbb{C}[(z - c_i)^{-1}]_{k_i^{(l)}} \right) & \oplus_{i=0}^r \mathbb{C}[(z - c_i)^{-1}]_{k_i^{(l)}}
\end{array}
\]
2.2.3 Lie groupoid as a deformation of $N_{m_i+1}^{-1}(\mathbb{C}[z]_{l-1})$

For a $\mathbb{C}$-subalgebra $\mathfrak{k} \subset M_n(\mathbb{C})$ and $c = (c_0, c_1, \ldots, c_{k+1}) \in \mathbb{D}(H)$, we define subalgebras of $M_n(\mathbb{C}[z](c)_{l-1}) := M_n(\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[z](c)_{l-1}$ by

$$\mathfrak{k}(\mathbb{C}[z](c)_{l-1}) := \left\{ X = X_0 + \sum_{i=1}^{l-1} X_i(z - c_0)(z - c_1) \cdots (z - c_{i-1}) \bigg| X_i \in \mathfrak{k} \right\},$$

$$\mathfrak{k}(\mathbb{C}[z](c)_{l-1})^1 := \{ X \in \mathfrak{k}(\mathbb{C}[z](c)_{l-1}) | X_0 = 0 \}.$$

Then we have the natural direct sum decomposition as $\mathbb{C}$-algebras,

$$\mathfrak{k}(\mathbb{C}[z](c)_{l-1}) = \mathfrak{k} \oplus \mathfrak{k}(\mathbb{C}[z](c)_{l-1})^1.$$

Moreover we define $\mathbb{C}$-vector subspaces of $M_n(\hat{\mathcal{O}}(c)_{l-1}) := M_n(\mathbb{C}) \otimes_{\mathbb{C}} \hat{\mathcal{O}}(c)_{l-1}$ by

$$\mathfrak{k}(\hat{\mathcal{O}}(c)_{l-1}) := \left\{ X = \sum_{i=0}^{l-1} \frac{X_i}{(z - c_0)(z - c_1) \cdots (z - c_i)} dz \bigg| X_i \in \mathfrak{k} \right\},$$

$$\mathfrak{k}(\hat{\mathcal{O}}(c)_{l-1})^1 := \{ X \in \mathfrak{k}(\hat{\mathcal{O}}(c)_{l-1}) | X_0 = 0 \}.$$

Especially for $n_{m_i+1}^- \subset M_n(\mathbb{C})$, we define

$$N_{m_i+1}^{-1}(\mathbb{C}[z](c)_{l-1}) := \left\{ E_n + X \bigg| X \in n_{m_i+1}^-(\mathbb{C}[z](c)_{l-1}) \right\},$$

$$N_{m_i+1}^{-1}(\mathbb{C}[z](c)_{l-1})^1 := \left\{ E_n + X \bigg| X \in n_{m_i+1}^-(\mathbb{C}[z](c)_{l-1})^1 \right\},$$

which are equipped with the Lie group structures and their corresponding Lie algebras are isomorphic to $n_{m_i+1}^-(\mathbb{C}[z](c)_{l-1})$ and $n_{m_i+1}^-(\mathbb{C}[z](c)_{l-1})^1$ respectively.

The paring

$$n_{m_i+1}^-(\mathbb{C}[z](c))_{l-1} \times n_{m_i+1}^+(\hat{\mathcal{O}}(c))_{l-1} \rightarrow \mathbb{C}$$

$$(X, Y) \mapsto -\text{tr}(\text{res}_{z=\infty}(X \cdot Y)).$$

is shown to be non-degenerate and thus gives the identifications

$$n_{m_i+1}^+(\hat{\mathcal{O}}(c)_{l-1}) \cong n_{m_i+1}^-(\mathbb{C}[z](c)_{l-1})^*, \quad n_{m_i+1}^+(\hat{\mathcal{O}}(c)_{l-1})^1 \cong (n_{m_i+1}^-(\mathbb{C}[z](c)_{l-1})^1)^*.$$

Similarly, by setting $p_{m_i+1}^+ = i_m \oplus n_{m_i+1}^+$ and $p_{m_i+1}^- = n_{m_i+1}^- \oplus i_m$, we have

$$p_{m_i+1}^+(\hat{\mathcal{O}}(c)_{l-1}) \cong (p_{m_i+1}^-(\mathbb{C}[z](c))_{l-1})^*.$$

Now let us consider the family $(N_{m_i+1}^{-1}(\mathbb{C}[z](c)_{l-1})^1_{c \in \mathbb{D}(H)})$ of complex Lie groups and equip this family with a structure of complex Lie groupoid.

**Definition 2.6** (Complex Lie groupoid). *Let us consider a complex manifold $\Gamma$ and its submanifold $\Gamma_0$ with the inclusion map $i: \Gamma_0 \hookrightarrow \Gamma$. Let us also consider holomorphic surjective submersions $s, t: \Gamma \to \Gamma_0$ satisfying $t \circ i = s \circ i = 1_{\Gamma_0}$. Further, we consider a holomorphic map $m: \Gamma^{(2)} := \{(\gamma_1, \gamma_2) \in \Gamma \times \Gamma | s(\gamma_1) = t(\gamma_2)\} \rightarrow \Gamma$ satisfying $s \circ m(g_1, g_2) = s(g_2)$ and $t \circ m(g_1, g_2) = t(g_1)$. Then if the following conditions are satisfied, the tuple $(\Gamma, \Gamma_0, s, t, m)$ is called a complex Lie groupoid.*
1. **Associativity:** \( m(m(g_1, g_2), g_3) = m(g_1, m(g_2, g_3)) \) for all \((g_1, g_2, g_3)\) satisfying \( s(g_1) = t(g_2) \) and \( s(g_2) = t(g_3) \).

2. **Units:** \( m(i \circ t(g), g) = g = m(g, i \circ s(g)) \) for all \( g \in \Gamma \).

3. **Inverses:** For all \( g \in \Gamma \), there exists \( h \in \Gamma \) such that \( s(h) = t(g) \), \( t(h) = s(g) \) and \( m(g, h), m(h, g) \in \Gamma_0 \).

The maps \( s \) and \( t \) are called the source map and target map respectively, and \( m \) is called the multiplication map. Also elements of the submanifold \( i : \Gamma_0 \hookrightarrow \Gamma \) are called units.

Let us consider the trivial vector bundle \( \theta : \left( \mathbb{C} \mathbb{N}_{m_{l+1}^i} \mathbb{D}(H) \right) \rightarrow \mathbb{D}(H) \) and introduce a Lie groupoid structure on this bundle. As both of the source and target maps, we employ the projection map \( \theta \). We regard \( \mathbb{D}(H) \) as a submanifold of \( \left( \mathbb{C} \mathbb{N}_{m_{l+1}^i} \mathbb{D}(H) \right) \) by the zero section \( i : \mathbb{D}(H) \ni c \mapsto (0, c) \in \left( \mathbb{C} \mathbb{N}_{m_{l+1}^i} \mathbb{D}(H) \right) \). Let us consider a linear isomorphism

\[
\phi_c : \left( \mathbb{C} \mathbb{N}_{m_{l+1}^i} \mathbb{D}(H) \right) \rightarrow \left( \mathbb{C} \mathbb{N}_{m_{l+1}^i} \mathbb{D}(H) \right),
\]

\[
(X_{i})_{i=0,1,...,l+1} \rightarrow \mathbb{D}(H) + \sum_{j=1}^{L+1} X_{i}(z - c_0)(z - c_1) \cdots (z - c_{i-1})
\]

Then we define the multiplication \( m(X(c), Y(c)) \) of \( X(c) : = ((X_i), c) \) and \( Y(c) : = ((Y_i), c) \) in \( \mathbb{C} \mathbb{N}_{m_{l+1}^i} \mathbb{D}(H) \) by

\[
m(X(c), Y(c)) := \phi_c^{-1}(\phi_c((X_i)) \cdot \phi_c(((Y_i))).
\]

Here \( \cdot \) in the right hand side of the equation is the multiplication in \( \mathbb{C} \mathbb{N}_{m_{l+1}^i}(\mathbb{C} \mathbb{D}[z](c))_{l-1} \).

Then the tuple

\[
\left( \mathbb{C} \mathbb{N}_{m_{l+1}^i}(\mathbb{C} \mathbb{D}[z](c))_{l-1} \right)
\]

becomes a complex Lie groupoid. Together with this Lie groupoid, we consider the subbundle \( \left( \mathbb{C} \mathbb{N}_{m_{l+1}^i} \mathbb{D}(H) \right) \rightarrow \mathbb{D}(H) \) with the inclusion

\[
\left( \mathbb{C} \mathbb{N}_{m_{l+1}^i} \mathbb{D}(H) \right) \rightarrow \left( \mathbb{C} \mathbb{N}_{m_{l+1}^i} \mathbb{D}(H) \right),
\]

\[
((X_1, X_2, \ldots, X_{l-1}), c) \rightarrow ((0, X_1, X_2, \ldots, X_{l-1}), c)
\]

Then this subbundle has the natural Lie groupoid structure whose fiber at each \( c \in \mathbb{D}(H) \) is isomorphic to \( \mathbb{C} \mathbb{N}_{m_{l+1}^i}(\mathbb{C} \mathbb{D}[z](c))_{l-1} \). We especially denote this Lie subgroupoid by \( \mathbb{C} \mathbb{N}_{m_{l+1}^i}(\mathbb{C} \mathbb{D}[z](c))_{l-1} \).

Next we introduce complex Lie algebroids.

**Definition 2.7** (Complex Lie algebroid). A complex Lie algebroid over a complex manifold \( M \) is a holomorphic vector bundle \( A \rightarrow M \), together with a Lie bracket \([\cdot, \cdot]\) on its space of holomorphic sections, such that there exists a vector bundle map \( a : A \rightarrow TM \) called the anchor map satisfying the Leibniz rule

\[
[\sigma, f \tau] = f[\sigma, \tau] + (a(\sigma)f)\tau
\]

for all global sections \( \sigma, \tau \) of \( A \rightarrow M \) and \( f \in \mathcal{O}_M(M) \).
Let $\mathfrak{k} \subset M_n(\mathbb{C})$ be a $\mathbb{C}$-subalgebra. We consider a linear isomorphism

$$\psi_c: \mathfrak{k}^\mathbb{P} \mathbb{D} \longrightarrow \mathfrak{k}(\mathbb{C}[z])(c)_{l-1} \rightarrow X_0 + \sum_{i=1}^{l-1} X_i(z - c_0)(z - c_1) \cdots (z - c_{i-1})$$

and define the Lie bracket $[X(c), Y(c)]$ of $X(c) := ((X_i), c)$ and $Y(c) := ((Y_i), c) \in \mathfrak{k}^\mathbb{P} \mathbb{D} \times \mathbb{D}(H)$ by

$$[(X(c), Y(c))] := \psi_c^{-1}([\psi_c((X_i), \psi_c((Y_i))]).$$

Then the vector bundle $\theta: \mathfrak{k}^\mathbb{P} \mathbb{D} \times \mathbb{D}(H) \rightarrow \mathbb{D}(H)$ with the Lie bracket and the zero map as the trivial anchor map becomes a Lie algebroid which we denote by $k$. As a dual of this adjoint action, we can define the coadjoint action of $(\mathfrak{k}^\mathbb{P} \mathbb{D} \times \mathbb{D}(H), \mathbb{D}(H))$ respectively. Then we define the holomorphic section of $\mathfrak{D}(H)$ by

$$\text{Ad}_{\mathbb{D}(H)}(n)(X)(c) := \psi_c^{-1} \circ \text{Ad}_{N_{m_{i+1}}(\mathbb{C}[z])(c)_{l-1}}(n(c))\psi_c(X(c)) \quad (c \in \mathbb{D}(H)).$$

As a dual of this adjoint action, we can define the coadjoint action of $(N_{m_{i+1}}^-(\mathbb{D}(H), \mathfrak{D}(H)))$, and $\mathfrak{D}(H)$, we define the section of $\mathfrak{D}(H)$ by

$$(\text{Ad}^*_{\mathbb{D}(H)}(n)(\xi))(X)(c) := \xi(c)(\text{Ad}_{N_{m_{i+1}}(\mathbb{C}[z])(c)_{l-1}}(n(c)^{-1})(X(c))) \quad (c \in \mathbb{D}(H)).$$

We denote the subbundle $(\mathfrak{D}(H))_1$ of $\mathfrak{D}(H)$ by $(n_{m_i})_{l}^{(l)}(\mathbb{D}(H))$ particularly when $\mathfrak{k} = n_{m_i}^{-1}$. The paring $[5]$ allows us to identify the fiber of the dual bundle $\mathfrak{D}(H)$ at each $c \in \mathbb{D}(H)$ with $n_{m_{i+1}}^+(\mathbb{D}(c)_{l-1})$ (resp. $p_{m_{i+1}}^+(\mathbb{D}(c)_{l-1})$) when $\mathfrak{k} = n_{m_{i+1}}^{-1}$. Thus we can regard $((n_{m_i})_{l}^{(l)}(\mathbb{D}(H))^*)$ as the subbundle of $(p_{m_{i+1}}^+)_{l-1}^*(\mathbb{D}(c))_{l-1}$ at each fiber.

### 2.2.4 Symplectic foliation as deformation of $\mathbb{O}_{H_{irr}}$

In Theorem 1.12, we saw that $\mathbb{O}_{H_{irr}}$ is isomorphic to the product of $T^*(N^-)^{(l)}$. Thus deforming $T^*(N^-)^{(l)}$, we can obtain a deformation of $\mathbb{O}_{H_{irr}}$, as follows.

Let us consider a relative cotangent bundle

$$\theta_{(N^-)^{(l)}_{D(H)}}: T^*((N^-)^{(l)}_{D(H)}) / \mathbb{D}(H) \rightarrow (N^-)^{(l)}_{D(H)},$$

i.e., the vector bundle corresponding to the quotient sheaf $\Omega_{(N^-)^{(l)}_{D(H)}} / \theta^* \Omega_{D(H)}$. Here $\theta: (N^-)^{(l)}_{D(H)} / \mathbb{D}(H)$ is the Lie groupoid defined above and $\Omega_M$ denotes the sheaf of
holomorphic differential forms on a complex manifold $M$. Set $n^{(l)} := \left(n_{m_{l}+1}^{(l)}\right)^{(l-1)}$ for simplicity. Recall that if we forget the structure of Lie groupoid from $(\theta)$ the trivial bundle of holomorphic differential forms on a complex manifold $M$ where $T$ also $D$ $T$ ifolds.

Since in particular we have $(\theta_{n}^{(l)} \circ T^{*}n^{(l)} \to \mathbb{D}(H)$ is the standard projection of the cotangent bundle. Here $T^{*}n^{(l)}$ has the Poisson structure induced from the standard symplectic form $\Omega$ and also $\mathbb{D}(H)$ can be seen as the Poisson manifold with the trivial Poisson bracket. Thus $T^{*}\left((N^{-})^{(l)}_{\mathbb{D}(H)}\right) / \mathbb{D}(H)$ becomes a Poisson manifold as the product of these Poisson manifolds.

**Definition 2.8** (Deformation of $T^{*}(N^{-})^{(l)}$). Let us set

$$\left(T^{*}(N^{-})^{(l)}\right)_{\mathbb{D}(H)} := T^{*}\left((N^{-})^{(l)}_{\mathbb{D}(H)}\right) / \mathbb{D}(H)$$

and call this Poisson manifold the deformation of $T^{*}(N^{-})^{(l)}$ with respect to the HTL normal form $H$.

Let us define the projection map $\theta_{l}^{*} : \left(T^{*}(N^{-})^{(l)}\right)_{\mathbb{D}(H)} \to \mathbb{D}(H)$ by setting $\theta_{l}^{*} := \theta_{l} \circ \theta_{(N^{-})^{(l)}_{\mathbb{D}(H)}}$. Then we note that $\theta_{l}^{*} : \left(T^{*}(N^{-})^{(l)}\right)_{\mathbb{D}(H)} \to \mathbb{D}(H)$ is still a holomorphic vector bundle regarding $T^{*}n^{(l)}$ as a $\mathbb{C}$-vector space through the isomorphism $T^{*}n^{(l)} \cong n^{(l)} \oplus (n^{(l)})^{*}$. For each fiber at $c \in \mathbb{D}(H)$, there exists the natural isomorphism

$$\left(\theta_{l}^{*}\right)^{-1}(c) \cong T^{*}N_{m_{l}+1}^{-}\left(\mathbb{C}[z]\right)_{l-1}^{1}.$$ 

Since in particular we have $\left(\theta_{l}^{*}\right)^{-1}(0) \cong T^{*}N_{m_{l}+1}^{-}\left(\mathbb{C}[z]\right)_{l-1}^{1} = T^{*}(N^{-})^{(l)}$, we can regard $\left(T^{*}(N^{-})^{(l)}\right)_{\mathbb{D}(H)}$ as a deformation of $T^{*}(N^{-})^{(l)}$. It directly follows from the definition that the family $\left(\left(\theta_{l}^{*}\right)^{-1}(c)\right)_{c \in \mathbb{D}(H)}$ is the symplectic foliation of $\left(T^{*}(N^{-})^{(l)}\right)_{\mathbb{D}(H)}$.

Since $N_{m_{l}+1}^{-}\left(\mathbb{C}[z]\right)_{l-1}^{1}$ is normalized by $L_{m_{l}} = \text{GL}_{n}(\mathbb{C})_{H_{\mathbb{C}}}$, the cotangent bundle $T^{*}N_{m_{l}+1}^{-}\left(\mathbb{C}[z]\right)_{l-1}^{1}$ has the natural $L_{m_{l}}$-action. Hence $T^{*}(N^{-})^{(l)}_{\mathbb{D}(H)}$ is equipped with the $L_{m_{l}}$-action.

Then as the direct sum of vector bundles $\theta_{l}^{*} : \left(T^{*}(N^{-})^{(l)}\right)_{\mathbb{D}(H)} \to \mathbb{D}(H)$ we define a deformation of $\mathbb{Q}_{H_{\mathbb{C}}}$ as follows.

**Definition 2.9** (Deformation of $\mathbb{Q}_{H_{\mathbb{C}}}$). We consider the direct sum of the vector bundles

$$\theta_{\mathbb{Q}_{H_{\mathbb{C}}}} := \bigoplus_{l=2}^{k} \left(T^{*}(N^{-})^{(l)}\right)_{\mathbb{D}(H)} \to \mathbb{D}(H)$$

and we denote the total space of this bundle by $\left(\mathbb{Q}_{H_{\mathbb{C}}}\right)_{\mathbb{D}(H)}$, which is called the deformation of $\mathbb{Q}_{H_{\mathbb{C}}}$. This complex manifold $\left(\mathbb{Q}_{H_{\mathbb{C}}}, \mathbb{D}(H)\right)$ is naturally equipped with the Poisson structure and $L_{m_{l}}$-action as well as $\left(T^{*}(N^{-})^{(l)}\right)_{\mathbb{D}(H)}$.

**Proposition 2.10.** The family $\left(\theta_{\mathbb{Q}_{H_{\mathbb{C}}}}^{-1}(c)\right)_{c \in \mathbb{D}(H)}$ is the symplectic foliation of $\left(\mathbb{Q}_{H_{\mathbb{C}}}, \mathbb{D}(H)\right)$. In particular, the special fiber at $c = 0$, we have the symplectic isomorphism

$$\theta_{\mathbb{Q}_{H_{\mathbb{C}}}}^{-1}(0) \cong \mathbb{Q}_{H_{\mathbb{C}}}.$$
Proof. The first assertion is obvious since the family \(((\theta^2)^{-1}(c))_{c\in D(H)}\) is the symplectic foliation of \((T^*(N^{-}(l)))_{D(H)}\) for each \(l\). The second assertion follows from Theorem 1.12.

The right trivialization

\[
N_{n_{l+1}}^-(\mathbb{C}[z](c)_{l-1})^1 \times (n_{l_{1+1}}^-) \rightarrow T^*N_{n_{l+1}}^-(\mathbb{C}[z](c)_{l-1})^1
\]

\[\rightarrow R_{n_{l+1}}^*(\xi)\]



gives the isomorphism

\[
(N^-)^{(l)}_{D(H)} \oplus ((n^-)^{(l)}_{D(H)})^* \cong (T^*(N^-)^{(l)})_{D(H)}.
\]

(6)

as vector budless. Thus we have the isomorphism

\[
(\mathbb{O}_{H_{irr}})_{D(H)} \cong \bigoplus_{l=2}^{k} ((N^-)^{(l)}_{D(H)} \oplus ((n^-)^{(l)}_{D(H)})^*)
\]

(7)

2.3 Local unfolding manifold

In the previous section, we defined a deformation of \(\mathbb{O}_{H_{irr}}\). Now we shall construct a deformation of the truncated orbit \(\mathbb{O}_H\).

2.3.1 Unfolding manifold of a truncated orbit

As we saw in Theorem 1.8, truncated orbit \(\mathbb{O}_H\) is obtained as a symplectic reduction of the extended orbit \(T^*GL_n(\mathbb{C}) \times \mathbb{O}_{H_{irr}}\). Based on this fact, we shall define a deformation of the truncated orbit as a Poisson reduction of \(T^*GL_n(\mathbb{C}) \times (\mathbb{O}_{H_{irr}})_{D(H)}\). To this purpose, we first consider a deformation of the map \(\text{res}_{\mathbb{O}_{H_{irr}}}: \mathbb{O}_{H_{irr}} \rightarrow \text{gl}_n(\mathbb{C})_{H_{irr}}^\ast\) as follows. Let \(\tau_l: ((n^-)^{(l)}_{D(H)})^* \rightarrow (p^-_{l+1})^*_\mathbb{D}(H)\) be the inclusion as subbundle. Under the identification (7), we define the map \(\text{res}_{n_{l+1}}: (\mathbb{O}_{H_{irr}})_{D(H)} \rightarrow L_{n_{l+1}}\) by setting

\[
\text{res}_{n_{l+1}}(n_l(c), X_l(c))_{l=2, 3, \ldots, k} := \pi_{(\hat{\mathfrak{p}}_{l+1}^-)^\ast\mathfrak{m}_{l+1}} \left( \sum_{l=2}^{k} -\text{res}_{x=\infty} \left( \text{Ad}_{\ast}(n_l(c))(\tau_l(X_l(c))) \right) \right).
\]

for \((n_l(c), X_l(c))_{l=2, 3, \ldots, k} \in \bigoplus_{l=2}^{k} \left( N_{n_{l+1}}^-(\mathbb{D}(H))^1 \oplus (n_{l_{1+1}}^-) \right)^*\). In the right hand side, \(\pi_{(\hat{\mathfrak{p}}_{l+1}^-)^\ast\mathfrak{m}_{l+1}}: (\hat{\mathfrak{p}}_{l+1}^-)^\ast \rightarrow L_{n_{l+1}}\) denotes the natural projection. Then we obtain the following.

Proposition 2.11. 1. The map \(\text{res}_{n_{l+1}}\) is \(L_{n_{l+1}} = GL_n(\mathbb{C})_{H_{irr}}\) equivariant.

2. The map \(\text{res}_{n_{l+1}}\) is a deformation of \(\text{res}_{\mathbb{O}_{H_{irr}}}\). Namely, under the identification \(\theta_{\mathbb{O}_{H_{irr}}}^{-1}(0) \cong \mathbb{O}_{H_{irr}}\), we have

\[
\text{res}_{n_{l+1}}|_{\theta_{\mathbb{O}_{H_{irr}}}^{-1}(0)} = \text{res}_{\mathbb{O}_{H_{irr}}}.
\]

Proof. The first assertion follows from the definition and second one from Proposition 1.8. \qed
Let us consider the map
\[
\mu_{\text{ext}, \mathcal{D}(H)} : \quad T^* \text{GL}_n(C) \times (\mathcal{O}_{H_{\text{irr}}} \mathcal{D}(H)) \to \mathfrak{gl}_n(C)^*_{H_{\text{irr}}} \quad \longleftrightarrow \quad \iota^*_{H_{\text{irr}}} \circ \mu_{\text{GL}_n(C)}(\alpha) + \text{res}_{\text{sym}}(\xi) .
\]
Then we can see that this map is a $\text{GL}_n(C)^*_{H_{\text{irr}}}$-equivariant moment map.

**Definition 2.12 (Local unfolding manifold).** The Poisson reduction
\[
(\mathcal{O}_H)_{\mathcal{D}(H)} := (\text{GL}_n(C)^*_{H_{\text{irr}}} \mathcal{O}_{H_{\text{res}}} \mathcal{D}(H))^{-1}_{\text{ext}, \mathcal{D}(H)}(H_{\text{res}})
\]
is called the local unfolding manifold of the truncated orbit $\mathcal{O}_H$. We have the projection map $\theta_{\mathcal{D}(H)} : (\mathcal{O}_H)_{\mathcal{D}(H)} \to \mathcal{D}(H)$ induced from the projection $\theta_{H_{\text{irr}} : (\mathcal{O}_{H_{\text{irr}}} \mathcal{D}(H)) \to \mathcal{D}(H)$.

It is easy to see that the $\text{GL}_n(C)^*_{H_{\text{irr}}}$-action on $T^* \text{GL}_n(C) \times (\mathcal{O}_{H_{\text{irr}}} \mathcal{D}(H))$ is proper. Moreover a similar argument as on the symplectic reduction shows that $H_{\text{res}}$ is a regular value of the map $\mu_{\text{ext}, \mathcal{D}(H)}$. Thus the local unfolding manifold $(\mathcal{O}_H)_{\mathcal{D}(H)}$ is truly a complex manifold.

**2.3.2 Symplectic foliation of $(\mathcal{O}_H)_{\mathcal{D}(H)}$**

Take a partition $\mathcal{I} : I_0, I_1, \ldots, I_r \in \mathcal{P}_{k+1}$ and the corresponding embedding $\iota_{\mathcal{I}} : \mathbb{C}^{r+1} \hookrightarrow \mathbb{C}^{k+1}$ as before. Then we can consider the deformation $H_{\mathcal{I}}(c) = H(\iota_{\mathcal{I}}(c))$ of $H$ on $C(\mathcal{I}) = \{ \iota_{\mathcal{I}}(c) \in \mathbb{C}^{k+1} | c = (c_0, c_1, \ldots, c_r) \in C_{r+1}(C) \}$. Then as we saw in Proposition 2.2, the partial fraction decomposition algorithm gives a description of $H_{\mathcal{I}}(c)$ as the sum of HLT normal forms $H_{\mathcal{I}}(c)_{z_{c_j}}$ at $z = c_j$ with spectral types
\[
\begin{cases}
(m_{i_1,0}) \leq (m_{i_1,1}) \leq \cdots \leq (m_{i_1,k_1}) \leq (m_{l,1})_{\text{triv}} & \text{for } j = 1, 2, \ldots, r, \\
(m_{i_1,0}) \leq (m_{l,0}) \leq \cdots \leq (m_{l,k_0})_{\sigma(m_0)} & \text{for } j = 0.
\end{cases}
\]
We denote the coadjoint orbit of $\text{GL}_n(C[z_{c_j},k_j])$ through $H_{\mathcal{I}}(c)_{z_{c_j}}$ by $\mathcal{O}_{H_{\mathcal{I}}(c)_{z_{c_j}}}$. As we saw in Section 2.2.2, the partial fraction decomposition of $\hat{O}(c)_I$ is compatible with the decomposition of $\mathbb{C}[z](\alpha)_I$. Thus this induces the partial fraction decomposition of the each fiber of $T^* N(\alpha)$ through the trivialization (6), and we obtain a symplectic holomorphic map
\[
\varphi_{\mathcal{I}}(c) : \theta_{\mathcal{D}(H)}^{-1}(\iota_{\mathcal{I}}(c)) \to \prod_{j=0}^r \mathcal{O}_{H_{\mathcal{I}}(c)_{z_{c_j}}}.
\]
Then we obtain the following.

**Theorem 2.13 (6).** 1. The holomorphic family $(\theta_{\mathcal{D}(H)}^{-1}(c))_{c \in \mathcal{D}(H)}$ is the symplectic foliation of $(\mathcal{O}_H)_{\mathcal{D}(H)}$.

2. Let $\mathcal{I} : I_0, I_1, \ldots, I_r \in \mathcal{P}_{k+1}$ be a partition. Let us take $c = (c_0, c_1, \ldots, c_r) \in C_{r+1}(C)$ so that $\iota_{\mathcal{I}}(c) \in \mathcal{D}(H)$. Then the partial fraction decomposition explained in Section 2.2.2 gives the holomorphic symplectic embedding
\[
\varphi_{\mathcal{I}}(c) : \theta_{\mathcal{D}(H)}^{-1}(\iota_{\mathcal{I}}(c)) \hookrightarrow \prod_{j=0}^r \mathcal{O}_{H_{\mathcal{I}}(c)_{z_{c_j}}}
\]
whose image is a Zariski open dense subset.
This shows that the symplectic foliation of the unfolding manifold $\mathcal{O}_H$ gives a deformation of the truncated orbit $\mathcal{O}_H$ along with the deformation $(H(\mathcal{C}))_{c \in \mathbb{D}(H)}$ of the HTL normal form $H$ introduced in Definition 2.1.

Let us introduce a deformation of the residue map $\mu_{\mathcal{O}_H}^0 : \mathcal{O}_H \rightarrow \mathfrak{gl}_n(\mathbb{C})^*$ of the truncated orbit defined in Section 1.4.3 as follows. Consider the map

$$
\nu_{\mathfrak{gl}_n(\mathbb{C})} \circ \text{pr}_1 : T^* \mathfrak{gl}_n(\mathbb{C}) \times (\mathcal{O}_{H_{\text{res}}})_{\mathbb{D}(H)} \rightarrow \mathfrak{gl}_n(\mathbb{C})^*.
$$

Then as well as in Section 1.4.3, we can see that its restriction $\nu_{\mathfrak{gl}_n(\mathbb{C})} \circ \text{pr}_1|_{\mu_{\mathcal{O}_H}^{-1}}(H_{\text{res}})$ on the subspace $\mu_{\mathcal{O}_H}^{-1}(H_{\text{res}})$ is $(\mathfrak{gl}_n(\mathbb{C})_{\mathbb{D}(H)})_{\mathbb{H}_{\text{res}}}$-invariant. Thus there uniquely exists the map $\nu_{\mathcal{O}_H}(\mathbb{D}(H)) : (\mathcal{O}_H)_{\mathbb{D}(H)} \rightarrow \mathfrak{gl}_n(\mathbb{C})^*$ such that the diagram

$$
\begin{CD}
\mu_{\mathcal{O}_H}^{-1}(H_{\text{res}}) @>{\nu_{\mathfrak{gl}_n(\mathbb{C})} \circ \text{pr}_1}>> \mathfrak{gl}_n(\mathbb{C})^*
\end{CD}
$$

is commutative. Then Proposition 1.15 assures that the map

$$\mu_{\mathcal{O}_H}^0 := -\nu_{\mathcal{O}_H}(\mathbb{D}(H)) : (\mathcal{O}_H)_{\mathbb{D}(H)} \rightarrow \mathfrak{gl}_n(\mathbb{C})^*
$$

is a deformation of the map $\mu_{\mathcal{O}_H}^0 : \mathcal{O}_H \rightarrow \mathfrak{gl}_n(\mathbb{C})^*$, i.e., we have the equation

$$
\nu_{\mathcal{O}_H}(\mathbb{D}(H))|_{\mu_{\mathcal{O}_H}^{-1}(0)} = \mu_{\mathcal{O}_H}^0
$$

under the isomorphism $\theta_{\mathcal{O}_H}^{-1}(0) \cong \mathcal{O}_H$.

### 2.4 Unfolding manifold of meromorphic connections

Now we are ready to define the unfolding manifold which is a Poisson manifold descending the unfolding of the unramified irregular singularities of connections in the moduli space $\mathcal{M}^*_\mathcal{A}(\mathbf{H})$.

#### 2.4.1 Stratification of the product of local unfolding manifolds

Let $\mathbf{H} = (H_a)_{a \in |D|}$ be a collection of unramified HTL normal forms,

$$
H_a = \left( \frac{S_{0}^{(a)}}{z_{a}^{k_{a} - 1}} + \cdots + \frac{S_{1}^{(a)}}{z_{a}} + S_{0}^{(a)} + N_{0}^{(a)} \right) \frac{dz_{a}}{z_{a}}
$$

with the spectral types $\text{sp}(H_a) := (m_{0}^{[a]} \leq m_{1}^{[a]} \leq m_{2}^{[a]} \cdots \leq m_{k_{a} - 1}^{[a]} m_{k_{a}}^{[a]} \sigma(m_{k_{a}}^{[a]}))$. We consider the moduli space $\mathcal{M}^*_\mathcal{A}(\mathbf{H}) = \mathfrak{gl}_n(\mathbb{C}) \setminus (\mu_{\mathbf{H}}^{-1})^{-1}(0)$ of meromorphic connections with respect to $\mathbf{H}$. Then we define a deformation of $\mathcal{M}^*_\mathcal{A}(\mathbf{H})$ as follows. Let us consider the product of the local unfolding manifolds

$$
\prod_{a \in |D|} \mathcal{O}_{H_a} \mathbb{D}(H_a)
$$

with the natural projection $\tilde{\theta}_{\prod_{a \in |D|} \mathbb{D}(H_a)} : \prod_{a \in |D|} \mathcal{O}_{H_a} \mathbb{D}(H_a) \rightarrow \prod_{a \in |D|} \mathbb{D}(H_a)$.
As well as in Section 2.1.2, we can consider a stratification

\[ \prod_{a \in |D|} C^{k_a+1} = \bigcup_{(\mathcal{I}_a)_{a \in |D|} \in \prod_{a \in |D|} \mathcal{P}_{k_a+1}} \left( \prod_{a \in |D|} C(\mathcal{I}_a) \right), \]

of \( \prod_{a \in |D|} C^{k_a+1} \) associated to collections of partitions \((\mathcal{I}_a)_{a \in |D|} \in \prod_{a \in |D|} \mathcal{P}_{k_a+1} \). This induces the following stratification of \( \prod_{a \in |D|} (\mathcal{O}_{H_a})_{\mathbb{D}(H_a)} \),

\[ \prod_{a \in |D|} (\mathcal{O}_{H_a})_{\mathbb{D}(H_a)} = \bigcup_{(\mathcal{I}_a)_{a \in |D|} \in \prod_{a \in |D|} \mathcal{P}_{k_a+1}} \prod_{a \in |D|} (\mathcal{O}_{H_a})^T_{\mathbb{D}(H_a)}, \]

where

\[ (\mathcal{O}_{H_a})^T_{\mathbb{D}(H_a)} := \bigcup_{c_a \in \mathbb{D}(H_a) \cap C(\mathcal{I}_a)} \theta^{-1}_{\mathbb{D}(H_a)}(c_a). \]

Let us look at a stratum \( \prod_{a \in |D|} (\mathcal{O}_{H_a})^T_{\mathbb{D}(H_a)} \). Then Theorem 2.13 says that each fiber \( \prod_{a \in |D|} (\mathcal{O}_{H_a})^T_{\mathbb{D}(H_a)}((\mathcal{I}_a,c_a)) \) is isomorphic to a Zariski open subset of \( \prod_{a \in |D|} \prod_{j=0}^r \mathbb{H}_{z_{a_j}}(c_a) \).

By these isomorphisms, we can define an open subspace \( \left( \prod_{a \in |D|} (\mathcal{O}_{H_a})^T_{\mathbb{D}(H_a)} \right)^s \) of the stratum as the union of the inverse images of \( \left( \prod_{a \in |D|} (\mathcal{O}_{H_a})^T_{\mathbb{D}(H_a)} \right)^s \). Further we define

\[ \left( \prod_{a \in |D|} (\mathcal{O}_{H_a})_{\mathbb{D}(H_a)} \right)^s := \bigcup_{(\mathcal{I}_a)_{a \in |D|} \in \prod_{a \in |D|} \mathcal{P}_{k_a+1}} \left( \prod_{a \in |D|} (\mathcal{O}_{H_a})^T_{\mathbb{D}(H_a)} \right)^s. \]

### 2.4.2 Unfolding manifold of meromorphic connections

Let us consider the \( GL_n(\mathbb{C}) \)-equivariant map

\[ \mu^*_H, \prod_{a \in |D|} \mathbb{D}(H_a) : \left( \prod_{a \in |D|} (\mathcal{O}_{H_a})_{\mathbb{D}(H_a)} \right)^s \rightarrow \mathfrak{g}l_n(\mathbb{C})^* \rightarrow \sum_{a \in |D|} \mu^0_{(\mathcal{O}_{H_a})_{\mathbb{D}(H_a)}}(X_a), \]

**Definition 2.14 (Unfolding manifold).** The Poisson quotient space

\[ \mathcal{M}^*_s(H)_{\prod_{a \in |D|} \mathbb{D}(H_a)} := GL_n(\mathbb{C}) \backslash \left( \mu^*_H, \prod_{a \in |D|} \mathbb{D}(H_a) \right)^{-1}(0) \]

is called the unfolding manifold of meromorphic connections on the rank \( n \) trivial bundle on \( \mathbb{P}^1 \) with respect to the unramified HTL normal forms \( H \).

Since the projection map \( \bar{\theta}_{\prod_{a \in |D|} \mathbb{D}(H_a)} : \prod_{a \in |D|} (\mathcal{O}_{H_a})_{\mathbb{D}(H_a)} \rightarrow \prod_{a \in |D|} \mathbb{D}(H_a) \) is \( GL_n(\mathbb{C}) \)-invariant, this map naturally induces the projection map

\[ \theta_{\prod_{a \in |D|} \mathbb{D}(H_a)} : \mathcal{M}^*_s(H)_{\prod_{a \in |D|} \mathbb{D}(H_a)} \rightarrow \prod_{a \in |D|} \mathbb{D}(H_a). \]

Then we can show that the closed submanifold \( (\mu^*_H, \prod_{a \in |D|} \mathbb{D}(H_a))^{-1}(0) \) of the Poisson manifold \( \left( \prod_{a \in |D|} (\mathcal{O}_{H_a})_{\mathbb{D}(H_a)} \right)^s \) has a clean intersection with both the symplectic leaves \( \bar{\theta}_{\prod_{a \in |D|} \mathbb{D}(H_a)}(c) \) of \( \left( \prod_{a \in |D|} (\mathcal{O}_{H_a})_{\mathbb{D}(H_a)} \right)^s \) and the orbits of \( GL_n(\mathbb{C}) \). Therefore we obtain the following our first main theorem.
Theorem 2.15. Suppose that the moduli space $\mathcal{M}_s^*(\mathbf{H})$ is not the empty set. Then the unfolding manifold $\mathcal{M}_s^*(\mathbf{H})_{\Pi_{\alpha \in [D]} \mathbb{D}(\mathbf{H}_\alpha)}$ is a Poisson manifold with the symplectic foliation $(\theta_{\Pi_{\alpha \in [D]} \mathbb{D}(\mathbf{H}_\alpha)}^{-1}(c))_{c \in \Pi_{\alpha \in [D]} \mathbb{D}(\mathbf{H}_\alpha)}$.

Let us define a subspace

$$\mathcal{M}_s^*(\mathbf{H})_{\Pi_{\alpha \in [D]} \mathbb{D}(\mathbf{H}_\alpha)} := \prod_{\alpha \in [D]} \bigcup_{c \in \mathbb{D}(\mathbf{H}) \cap C(\mathcal{I}_\alpha)} \theta_{\Pi_{\alpha \in [D]} \mathbb{D}(\mathbf{H}_\alpha)}^{-1}(c)$$

for each $(\mathcal{I}_\alpha)_{\alpha \in [D]} \in \prod_{\alpha \in [D]} \mathcal{P}_{k_\alpha + 1}$. Then we obtain a stratification

$$\mathcal{M}_s^*(\mathbf{H})_{\Pi_{\alpha \in [D]} \mathbb{D}(\mathbf{H}_\alpha)} = \bigcup_{(\mathcal{I}_\alpha)_{\alpha \in [D]} \in \prod_{\alpha \in [D]} \mathcal{P}_{k_\alpha + 1}} \mathcal{M}_s^*(\mathbf{H})_{\Pi_{\alpha \in [D]} \mathbb{D}(\mathbf{H}_\alpha)}$$

Let us fix a collection $(\mathcal{I}_\alpha)_{\alpha \in [D]}$ of partitions and consider the corresponding stratum $\mathcal{M}_s^*(\mathbf{H})_{\Pi_{\alpha \in [D]} \mathbb{D}(\mathbf{H}_\alpha)}$. The fibers $\theta_{\Pi_{\alpha \in [D]} \mathbb{D}(\mathbf{H}_\alpha)}^{-1}(c)$ contained in this stratum are of the forms $\theta_{\Pi_{\alpha \in [D]} \mathbb{D}(\mathbf{H}_\alpha)}^{-1}(t_{\mathcal{I}_\alpha}(c))_{\alpha \in [D]}$ with some $(c_\alpha)_{\alpha \in [D]} \in \prod_{\alpha \in [D]} \mathbb{C}^{r_{\mathcal{I}_\alpha} + 1}$. We associate the collection of HTL normal forms

$$\mathbf{H}(c_\alpha)_{\alpha \in [D]} := (H_{\mathcal{I}_\alpha}(c_\alpha))_{\alpha \in [D]}$$

for each $c_\alpha \in \mathbb{D}(\mathbf{H}_\alpha)$, and then the fiber $H_{\mathcal{I}_\alpha}(c_\alpha)$ of the unfolding manifold is embedded into a Zariski open subset of a moduli space of connections, namely, the symplectic leaves describe a deformation of the moduli space of connections and moreover this deformation corresponds to the unfolding of the irregular singularities of the associated HTL normal forms to the moduli space.

Theorem 2.16. In a stratum $\mathcal{M}_s^*(\mathbf{H})_{\Pi_{\alpha \in [D]} \mathbb{D}(\mathbf{H}_\alpha)}$ of the unfolding manifold associated to $(\mathcal{I}_\alpha)_{\alpha \in [D]}$, each fiber $\theta_{\Pi_{\alpha \in [D]} \mathbb{D}(\mathbf{H}_\alpha)}^{-1}(t_{\mathcal{I}_\alpha}(c))_{\alpha \in [D]}$ is isomorphic to a Zariski open subspace of the moduli space $\mathcal{M}_s^*(\mathbf{H}(c_\alpha)_{\alpha \in [D]})$ of meromorphic connections on $\mathbb{P}^1$.

2.4.3 Example: unfolding manifold of unramified Painlevé phase spaces

Let us consider the HTL normal form

$$H_{II} := \left(\begin{array}{c} \frac{a_3}{z^3} b_3 + \frac{a_2}{z^2} b_2 + \frac{a_1}{z} b_1 + \frac{a_0}{b_0} \end{array}\right) dz$$

Then it is known that $\mathcal{M}_s^*(H_{II})$ is the phase space of the Hamiltonian system of the Painlevé II equation. The unfolding of $H_{II}$ is

$$H_{II}(c_0, c_1, c_2, c_3) = \left(\begin{array}{c} \frac{a_3}{z - c_1} b_3 + \frac{a_2}{z - c_2} b_2 + \frac{a_1}{z - c_3} b_1 + \frac{a_0}{b_0} \end{array}\right) \frac{dz}{z - c_0}.$$
The unfolding parameter space $\mathbb{C}^4$ has the following 5 types of strata

$$C_{II} := \{(c_i)_{i=0, \ldots, 3} \in \mathbb{C}^4 \mid c_0 = c_1 = c_2 = c_3\},$$
$$C_{III}^{(i,j,k,l)} := \{(c_i)_{i=0, \ldots, 3} \in \mathbb{C}^4 \mid c_i = c_j, c_k = c_l, c_i \neq c_j\},$$
$$C_{IV}^{(i,j,k,l)} := \{(c_i)_{i=0, \ldots, 3} \in \mathbb{C}^4 \mid c_j = c_k = c_l, c_i \neq c_j\},$$
$$C_{V}^{(i,j,k,l)} := \{(c_i)_{i=0, \ldots, 3} \in \mathbb{C}^4 \mid c_k = c_l, c_i \neq c_j, c_i \neq c_k, c_j \neq c_k\},$$
$$C_{VI}^{(i,j,k,l)} := \{(c_i)_{i=0, \ldots, 3} \in \mathbb{C}^4 \mid c_s \neq c_t, s \neq t\},$$

where $\{i, j, k, l\} = \{0, 1, 2, 3\}$. The diagram below is the Hasse diagram of the closure relation of the strata $C_{II}, C_{III}^{(0,1,2,3)}, C_{IV}^{(0,1,2,3)}, C_{V}^{(0,1,2,3)}, C_{VI}^{(0,1,2,3)}$.

If $c = (c_0, \ldots, c_3) \in C_{III}^{(i,j,k,l)} \cap \mathbb{D}(H_{II})$ for example, there exist holomorphic functions $a_0^{(i)}(c), b_0^{(i)}(c)$ of $c \in C_{III}^{(i,j,k,l)} \cap \mathbb{D}(H_{II})$ such that we can write $H_{II}(c_0, c_1, c_2, c_3) = H_{III}^{(i,j,k,l)} + H_{III, c_i}$ by

$$H_{III, c_i}^{(i,j,k,l)} := \left(\frac{a_1^{(i)}(c)}{z - c_i} b_1^{(i)}(c) + \left(\frac{a_0^{(i)}(c)}{z - c_i} b_0^{(i)}(c)\right)\right) \frac{d}{z - c_i},$$

$$H_{III, c_k}^{(i,j,k,l)} := \left(\frac{a_1^{(k)}(c)}{z - c_k} b_1^{(k)}(c) + \left(\frac{a_0^{(k)}(c)}{z - c_k} b_0^{(k)}(c)\right)\right) \frac{d}{z - c_k}.$$

The condition $c \in C_{III}^{(i,j,k,l)} \cap \mathbb{D}(H_{II})$ moreover assures that the moduli space $\mathcal{M}_*^*(H_{II})$ associated to the collection $H_{II} := (H_{III, c_i}, H_{III, c_k})$ is isomorphic to the phase space of Painlevé III equation. In this case, our main theorem says the following.

**Proposition 2.17.** For each $c \in C_{III}^{(i,j,k,l)} \cap \mathbb{D}(H_{II})$, the fiber $\theta_{\mathbb{D}(H_{II})}^{-1}(c)$ of the unfolding manifold $\theta_{\mathbb{D}(H_{II})} : \mathcal{M}_*^*(H_{II}) \rightarrow \mathbb{D}(H_{II})$ is isomorphic to a Zariski open subset of $\mathcal{M}_*^*(H_{III})$, the phase space of Painlevé III equation.

The same statement holds for each above 5 strata, namely, we have the following.

**Theorem 2.18.** The symplectic foliation of $\mathcal{M}_*^*(H_{II}) \mathbb{D}(H_{II})$ consists of Zariski open subsets of Painlevé II to VI phase spaces, and the types II to VI of leaves $\theta_{\mathbb{D}(H_{II})}^{-1}(c)$ correspond to the strata $C_{II}$ to $C_{VI}^{(i,j,k,l)}$ containing the deformation parameter $c$. 
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