Quantum state transfer on integral oriented circulant graphs

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Abstract

An oriented circulant graph is called integral if all eigenvalues of its Hermitian adjacency matrix are integers. The main purpose of this paper is to investigate the existence of perfect state transfer (PST for short) and multiple state transfer (MST for short) on integral oriented circulant graphs. Specifically, a characterization of PST (or MST) on integral oriented circulant graphs is provided. As an application, we also obtain a closed-form expression for the number of integral oriented circulant graphs with fixed order having PST (or MST).

Keywords: Oriented circulant graphs; integral graphs; perfect state transfer; multiple state transfer.

AMS Classification: 05C50; 15A18; 81P45; 81P68

1 Introduction

All graphs considered in this paper have neither loops nor multiple edges. Let $\Gamma = (V, E, A)$ be a mixed graph (introduced by Harary and Palmer [20]) with vertex set $V$, undirected edge set $E$, and arc (directed edge) set $A$. In particular, we say that $\Gamma$ is oriented (resp. undirected) if it contains only directed (resp. undirected) edges. The Hermitian adjacency matrix of $\Gamma$, introduced by Liu and Li [26], and independently by Guo and Mohar [19], is defined as $H_\Gamma = (h_{uv})_{u,v \in V}$, where

$$h_{uv} = \begin{cases} 1, & \text{if } (u, v) \in E, \\ i, & \text{if } (u, v) \in A, \\ -i, & \text{if } (v, u) \in A, \\ 0, & \text{otherwise}. \end{cases}$$

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Here $i = \sqrt{-1}$. Since $H_\Gamma$ is a Hermitian matrix, all its eigenvalues called the Hermitian eigenvalues of $\Gamma$ are real. The multiset of Hermitian eigenvalues of $\Gamma$ is called the Hermitian spectrum of $\Gamma$, and denoted by $\text{Sp}_H(\Gamma)$. For more results on Hermitian eigenvalues of mixed graphs, we refer the reader to [28, 33].

A mixed graph is called integral if all its Hermitian eigenvalues are integers. The problem of characterizing integral (undirected) graphs was proposed by Harary and Schwenk [21] in 1974. It has been discovered that integral graphs can play a role in the so-called perfect state transfer (defined below) in quantum spin networks. See [2] for a survey on integral graphs.

The concept of perfect state transfer was introduced by Bose [6] in 2003. Let $\Gamma$ be a mixed graph, and let $H_\Gamma$ be the Hermitian adjacency matrix of $\Gamma$. We say that $\Gamma$ has perfect state transfer (PST for short) from $u$ to $v$ if there exists a time $t \in \mathbb{R}$ and a complex unimodular scalar $\gamma$ such that

$$U(t)e_u = \gamma e_v,$$

(1)

where $U(t) = \exp(itH_\Gamma)$ is the transition matrix of $H_\Gamma$, and $i = \sqrt{-1}$. Here $\Gamma$ is called phase of PST. In particular, if $u = v$ in Eq. (1), we say that $\Gamma$ is periodic at vertex $u$. Furthermore, if $U(t)$ is a scalar multiple of the identity matrix, then $\Gamma$ is periodic. In recent years, the study of PST on graphs has aroused a great deal of interest, and it is well known that graphs having PST are rare. In 2011–2012, Godsil [15–17] provided some basic properties for periodicity and perfect state transfer of graphs. For more details on PST, we refer the reader to [13].

In quantum informatics and quantum computing, there has been tremendous interest in PST on Cayley graphs. Let $G$ be a finite group with identity element $e$, and let $S$ be a subset of $G \setminus \{e\}$. The Cayley graph $\text{Cay}(G, S)$ is defined as the graph with vertex set $G$ and arc set $E = \{(x, y) \in G \times G : yx^{-1} \in S\}$. In particular, if $S = S^{-1}$ then $\text{Cay}(G, S)$ is an undirected graph, and if $S \cap S^{-1} = \emptyset$ then $\text{Cay}(G, S)$ is an oriented graph. In [9–11, 32], Cao et al. investigated PST on Cayley graphs over abelian groups or dihedral groups. For a comprehensive survey about PST on Cayley graphs, we refer the reader to [27, Chapter 9].

A circulant graph is a Cayley graph over a cyclic group. Let $\mathbb{Z}_n$ be the additive group of integers module $n$, and let $\mathcal{C}$ be a subset of $\mathbb{Z}_n \setminus \{0\}$. The circulant graph $G(\mathbb{Z}_n, \mathcal{C})$ has vertex set $\mathbb{Z}_n$ and arc set $\{(a, b) : b - a \in \mathcal{C}, a, b \in \mathbb{Z}_n\}$. Here $\mathcal{C}$ is called the symbol of $G(\mathbb{Z}_n, \mathcal{C})$. In 2003, So [31] characterized all integral undirected circulant graphs. Based on this result, Bašić [3] gave a characterization for PST on integral undirected circulant graphs. For more results about PST on integral undirected circulant graphs, see [1, 4, 5, 29, 30]. With regard to oriented circulant graphs, it is natural to ask the following question.

**Question 1.** Which oriented circulant graphs have PST?

According to Godsil and Lato [18], if an oriented graph has PST, then all its Hermitian eigenvalues are integers or integer multiples of $\sqrt{\Delta}$, where $\Delta$ is a square-free integer. In this paper, our first goal is to give an answer to Question 1 for integral oriented circulant graphs.

Very recently, Kadyan and Bhattacharjya [23] provided a characterization for integral oriented circulant graphs.

**Theorem 1.1.** (See [23, Theorem 5.3]) Let $\Gamma = G(\mathbb{Z}_n, \mathcal{C})$ be an oriented circulant graph.

(i) If $n \not\equiv 0 \pmod{4}$, then $\Gamma$ is integral if and only if $\mathcal{C} = \emptyset$. 

The following two statements are equivalent:

(i) If \( n \equiv 0 \pmod{4} \), then \( \Gamma \) is integral if and only if \( \mathcal{C} = \bigcup_{d \in \mathbb{D}} S_n(d) \), where \( \mathbb{D} \subseteq \{ d : d \mid n/4 \} \), and \( S_n(d) = G_n^1(d) \) or \( S_n(d) = G_n^3(d) \).

Here \( G_n^r(d) = \{ dk : k \equiv r \pmod{4}, \gcd(dk, n) = d \} = dG_{n/d}^r(1) \).

Let \( G(\mathbb{Z}_n, \mathcal{C}) \) be a non-empty integral oriented circulant graph. By Theorem 1.1, we see that \( n \equiv 0 \pmod{4} \), and the symbol \( \mathcal{C} = \bigcup_{d \in \mathbb{D}} S_n(d) \) corresponds to the mapping \( \sigma : \mathbb{D} \rightarrow \{ 1, -1 \} \) where \( \sigma(d) = 1 \) if \( S_n(d) = G_n^1(d) \) and \( \sigma(d) = -1 \) if \( S_n(d) = G_n^3(d) \). Therefore, each non-empty integral oriented circulant graph \( G(\mathbb{Z}_n, \mathcal{C}) \) is determined by its order \( n \ (n \equiv 0 \pmod{4}) \), a set \( \mathbb{D} \) of positive divisors of \( \frac{n}{4} \), and a mapping from \( \mathbb{D} \) to \( \{ 1, -1 \} \).

For this reason, we use \( \text{IOCG}_n(\mathbb{D}, \sigma) \) instead of \( G(\mathbb{Z}_n, \mathcal{C}) \) in what follows. For example, if \( G(\mathbb{Z}_8, \mathcal{C}) = \text{IOCG}_8(\{1, 2\}, \{1, -1\}) \), then \( G_8^1(1) = \{ k : k \equiv 1 \pmod{4}, \gcd(k, 8) = 1 \} = \{1, 5\} \), \( G_8^3(2) = 2G_8^2(1) = 2\{k : k \equiv 3 \pmod{4}, \gcd(k, 4) = 1 \} = 2\{3\} = \{6\} \), and hence \( \mathcal{C} = G_8^1(1) \cup G_8^3(2) = \{1, 5, 6\} \).

Let \( \vartheta_2(n) \) be the largest positive integer \( \alpha \) such that \( 2^\alpha \mid n \). We define \( \mathbb{D}_i = \{ d \in \mathbb{D} \mid \vartheta_2(n/d) = i, 0 \leq i \leq \vartheta_2(n) \} \), where \( \mathbb{D} \subseteq \mathbb{D}_n = \{ d : d \mid n, 1 \leq d < n \} \). The first result of this paper is as below.

**Theorem 1.2.** Let \( \Gamma = \text{IOCG}_n(\mathbb{D}, \sigma) \) be an integral oriented circulant graph. Then the following two statements are equivalent:

1. \( \Gamma \) has PST between vertices \( b + n/2 \) and \( b \), for all \( b \in \mathbb{Z}_n \);
2. \( n \in 4 \mathbb{N} \) and \( \mathbb{D}_2 = \{n/4\} \).

Up to now, there are few results about PST on oriented graphs. For undirected graphs, Kay [24] proved that PST (whose adjacency matrix is real symmetric) occurs only between two non-disjoint pairs of vertices. Unlike undirected graphs, some oriented graphs having PST between multiple vertices were found. Here, we introduce some results in recent years. We say that a graph admits universal state transfer (UST for short) if there exists PST between every pair of vertices. Cameron et al. [8] proposed this definition and showed that only the complete graphs \( K_2 \) and \( K_3 \) with complex Hermitian adjacency matrices which have UST. Connelly et al. [12] conjectured that \( K_3 \) is the only nontrivial unweighted oriented graph with UST. It is clear that graphs having UST are more rare than PST. For this reason, Godsil and Lato [18] proposed the concept of multiple state transfer. A graph is said to have multiple state transfer (MST for short) if it contains a vertex subset \( S \) such that PST occurs between each pair of vertices in \( S \). Also, they gave some examples of graphs having MST. For some results of PST and MST on oriented graphs, we refer the reader to [25]. In [25], Lato asked the following question.

**Question 2.** Can we build infinite families of graphs with MST?

In the second part of this paper, we focus on studying Question 2, and obtain a characterization for integral oriented circulant graphs having MST.

**Theorem 1.3.** Let \( \Gamma = \text{IOCG}_n(\mathbb{D}, \sigma) \) be an integral oriented circulant graph. Then the following two statements are equivalent:
(i) $\Gamma$ has MST between vertices $b$, $b + n/4$, $b + n/2$, $b + 3n/4$, for all $b \in \mathbb{Z}_n$;

(ii) $n \in 8\mathbb{N}$, $D_2 = \{n/4\}$, and $D_3 = \{n/8\}$.

The paper is organized as follows. In Section 2, we provide an exact formula of the Ramanujan’s sine sum, and give an expression for the eigenvalues of integral oriented circulant graphs. In Section 3, we give the proof of Theorem 1.2. In Section 4, we give the proof of Theorem 1.3.

2 The eigenvalues of integral oriented circulant graphs

In the section, we will give a characterization of the eigenvalues of integral oriented circulant graphs. For further research, we need to introduce the concepts of Ramanujan’s sum and Ramanujan’s sine sum.

Let $n$ be a positive integer, and let $d$ be a divisor of $n$. Recall that $G_n(d) = \{k : 1 \leq k \leq n - 1, \gcd(k, n) = d\}$ and $G_n^r(d) = \{dk : k \equiv r \pmod{4}, \gcd(dk, n) = d\} = dG_n^{r/d}(1)$.

**Definition 2.1.** (See [22, p. 308]) Let $q$ and $n$ be positive integers. Define

$$c_n(q) = \sum_{\substack{1 \leq a \leq n \\
\gcd(a, n) = 1}} e^{2\pi i \frac{aq}{n}} = \sum_{a \in G_n(1)} \cos \frac{2\pi aq}{n}. \quad (2)$$

The expression for $c_n(q)$ is known as Ramanujan’s sum.

**Proposition 2.2.** Let $c_n(q)$ be the Ramanujan’s sum. Then

(i) $c_1(q) = 1$, for all positive integers $q$;

(ii) if $n$ is a prime number,

$$c_n(q) = \begin{cases} -1, & \text{if } n \nmid q, \\ n - 1, & \text{if } n \mid q. \end{cases}$$

A set $C \subseteq G_n(1)$ is called skew-symmetric if $n - a \in C^{-1}$ for all $a \in C$, where $C^{-1} = G_n(1) \setminus C$. Now, the Ramanujan’s sum Eq. (2) can also be written as

$$c_n(q) = \sum_{a \in G_n(1)} w_n^{aq} = \sum_{a \in C} 2 \cos \frac{2\pi aq}{n}, \quad (3)$$

where $\omega_n = \exp(i 2\pi/n)$ is the $n$-th root of unity.

Note that the Ramanujan’s sum $c_n(q)$ is an integer, for any $q, n \in \mathbb{N}$. We are replacing cosine with sine in Eq. (3), obtain

$$s_n^C = \sum_{a \in C} \frac{w^aq - w^{-aq}}{1} = \sum_{a \in C} 2 \sin \frac{2\pi aq}{n},$$

where $\omega_n = \exp(i 2\pi/n)$ is the $n$-th root of unity.

In [23], Kadyan and Bhattacharjya proved that, $s_n^C$ is still an integer, for any $q, n \equiv 0 \pmod{4} \in \mathbb{N}$ if and only if $C = G_n^1(1)$ or $G_n^3(1)$. To match sign $\sigma$, we adjust the sign in [23] to redefine the following.
Definition 2.3. (See [23]) Let $q$ and $n \equiv 0 \pmod{4}$ be positive integers. Define

$$s_n^\sigma(q) = i \sum_{a \in S_n(1)} (\omega_n^{aq} - \omega_n^{-aq}) = - \sum_{a \in S_n(1)} 2 \sin \frac{2\pi a q}{n},$$

where $\omega_n = \exp(i 2\pi / n)$, if $S_n(1) = G_n^1(1)$ then $\sigma = 1$ or if $S_n(1) = G_n^3(1)$ then $\sigma = -1$. The expression for $s_n^\sigma(q)$ is called as Ramanujan’s sine sum.

Note that $s_n^1(q) = -s_n^{-1}(q)$, for any $q, n \equiv 0 \pmod{4}; n \in \mathbb{N}$. Without loss of generality, we only consider $s_n^1(q)$, for short denoted by $s_n(q)$. Now, we have $s_n^\sigma(q) = \sigma s_n(q)$.

In the following, we will give a characterization of Ramanujan’s sine sum $s_n(q)$. First, we will prove some lemma.

Lemma 2.4. Let $n = 4m$ and with $m$ being an odd positive integer. Then

$$m + 4r \in \begin{cases} G_{4m}^1(1), & \text{if } m \equiv 1 \pmod{4}, \\ G_{4m}^3(1), & \text{if } m \equiv 3 \pmod{4}, \end{cases}$$

for some $r \in G_m(1)$.

Proof. Assume that $m \equiv 1 \pmod{4}$. Let $r \in G_m(1)$. Then $\gcd(r, m) = 1$, which implies that $\gcd(m + 4r, 4m) = 1$, and $m + 4r \equiv 1 \pmod{4}$. Thus, $m + 4r \in G_{4m}^1(1)$. Similarly, if $m \equiv 3 \pmod{4}$, then $m + 4r \in G_{4m}^3(1)$.

Lemma 2.5. Let $n = 4m$ and with $m$ being an odd positive integer. Then

$$s_n(q) = \begin{cases} (-1)^{(m-1)/2}(-1)^{(q+1)/2}2c_m(q), & \text{if } q \text{ is odd}, \\ 0, & \text{otherwise}. \end{cases}$$

(4)

Proof. By Lemma 2.4, it follows that

$$s_n(q) = i \sum_{a \in G_m^1(1)} (\omega_n^{aq} - \omega_n^{-aq})$$

$$= \begin{cases} i \sum_{a \in G_m^1(1)} (\omega_n^{aq} - \omega_n^{-aq}), & m \equiv 1 \pmod{4} \\ -i \sum_{a \in G_m^3(1)} (\omega_n^{aq} - \omega_n^{-aq}), & m \equiv 3 \pmod{4} \end{cases}$$

$$= \begin{cases} i \sum_{m+4r \in G_{4m}^1(1)} (\omega_n^{(m+4r)q} - \omega_n^{-(m+4r)q}), & m \equiv 1 \pmod{4} \\ -i \sum_{m+4r \in G_{4m}^3(1)} (\omega_n^{(m+4r)q} - \omega_n^{-(m+4r)q}), & m \equiv 3 \pmod{4} \end{cases}$$

$$= \begin{cases} i \sum_{r \in G_m(1)} (\omega_n^q \omega_n^{r q} - \omega_n^{-q} \omega_n^{r q}), & m \equiv 1 \pmod{4} \\ -i \sum_{r \in G_m(1)} (\omega_n^q \omega_n^{-r q} - \omega_n^{-q} \omega_n^{-r q}), & m \equiv 3 \pmod{4} \end{cases}$$

$$= (-1)^{(m-1)/2} i \sum_{r \in G_m(1)} (\omega_n^q \omega_n^{r q} - \omega_n^{-q} \omega_n^{r q})$$

$$= (-1)^{(m-1)/2} \left(i^{q+1} c_m(q) - i^{q+1} c_m(-q)\right)$$

$$= (-1)^{(m-1)/2} (-1)^q c_m(q)$$

$$= (-1)^{(m-1)/2} (-1)^{(q+1)/2} (1 - (-1)^q) c_m(q)$$

$$= \begin{cases} (-1)^{(m-1)/2} (-1)^{(q+1)/2} 2c_m(q), & \text{if } q \text{ is odd}, \\ 0, & \text{otherwise.} \end{cases}$$

This proves Lemma 2.5. □
Lemma 2.6. Let \( n = 2^t m \) and with \( m \) being an odd positive integer and \( t \geq 2 \). Then
\[
s_n(q) = 2^{t-2}s_{4m}(q/2^{t-2}),
\]
for some \( q/2^{t-2} \in \mathbb{N} \).

**Proof.** We prove the lemma by using induction on \( t \). It is clear that the identity holds for the case \( t = 2 \). Let \( t \geq 3 \) and assume that the identity holds for each \( t = k \). Now let \( n' = 2^{k+1}m \). Since
\[
G_{2^{t+1}m}(1) = G_{2^tm}(1) \cup (2^tm + G_{2^tm}(1)),
\]
we have
\[
s_{n'}(q) = i \sum_{a \in G_{2^km}^{'}} (\omega_{2^{k+1}m}^a - \omega_{2^{k+1}m}^{-a})
\]
\[
= i \sum_{a \in G_{2^km}^{'}} (\omega_{2^{k+1}m}^a - \omega_{2^{k+1}m}^{-a}) + i \sum_{a \in G_{2^{k+1}m}^{'}} (\omega_{2^{k+1}m}^a - \omega_{2^{k+1}m}^{-a})
\]
\[
= i \sum_{a \in G_{2^km}^{'}} (\omega_{2^{k+1}m}^a - \omega_{2^{k+1}m}^{-a}) + i \sum_{a \in G_{2^{k+1}m}^{'}} (\omega_{2^{k+1}m}^a - \omega_{2^{k+1}m}^{-a})
\]
\[
= i \sum_{a \in G_{2^{k+1}m}^{'}} ((1 + (-1)^q)\omega_{2^{k+1}m}^a - (1 + (-1)^{-q})\omega_{2^{k+1}m}^{-a})
\]
\[
= \frac{2i}{2^t} \sum_{a \in G_{2^km}^{'}} \left( \omega_{2^{k+1}m}^{aq/2} - \omega_{2^{k+1}m}^{-aq/2} \right)
\]
\[
= 2s_{2^km}(q/2)
\]
\[
= 2^{k-1}s_{4m}(q/2^{k-1}),
\]
for some \( q/2^{k-1} \in \mathbb{N} \). By the induction hypothesis the identity holds. This proves Lemma 2.6. \( \square \)

Let \( q' = q/2^{t-2} \in \mathbb{N} \). By substituting Eq. (4) into Eq. (5), we can obtain Theorem 2.7.

**Theorem 2.7.** Let \( n = 2^t m \) and with \( m \) being an odd positive integer and \( t \geq 2 \). Then
\[
s_n(q) = \begin{cases} (-1)^{(m-1)/2}(-1)^{(q+1)/2}2^{t-1}c_m(q'), & \text{if } q' \text{ is odd}, \\ 0, & \text{otherwise}, \end{cases}
\]
where \( q' = q/2^{t-2} \in \mathbb{N} \).
In the remainder of this section, we will characterize the eigenvalues of integral oriented circulant graphs using Ramanujan’s sine sum.

Let $\Gamma = G(\mathbb{Z}_n, C)$ be an integral oriented circulant graph, and let $H$ be the Hermitian adjacency matrix of $\Gamma$. According to [23], the eigenvalues and eigenvectors of $H$ are, respectively, given by

$$
\mu_j = i \sum_{k \in C} (\omega_n^{jk} - \omega_n^{-jk}), \quad v_j = [1 \omega_n^k \omega_n^{2k} \cdots \omega_n^{(n-1)k}],$$

(6)

for $0 \leq j \leq n-1$, where $\omega_n = \exp(i 2\pi/n)$ is the $n$-th root of unity.

**Theorem 2.8.** Let $\Gamma = \text{IOCG}_n(D, \sigma)$ be an integral oriented circulant graph. Then the eigenvalues of $\Gamma$ are

$$
\mu_j = \begin{cases} 
\sum_{d \in D} \sigma(d)(-1)^{1/2}(1/2^{i-1}2^{i-2})c_n(j/2^i-2), & \text{if } j/2^i-2 \text{ is odd,} \\
0, & \text{otherwise,}
\end{cases}
$$

for $2 \leq i \leq \vartheta_2(n)$, $0 \leq j \leq n-1$, where $D = \bigcup_{i=2}^{\vartheta_2(n)} D_i \subseteq \{d : d \mid n/4\}$.

**Proof.** Let $\Gamma = \text{IOCG}_n(D, \sigma)$ be an integral oriented circulant graph, and let $H$ be the Hermitian adjacency matrix of $\Gamma$. Then the eigenvalues $\mu_j$ of $H$ can be expressed in terms of Ramanujan’s sine sum as follows

$$
\mu_j = i \sum_{k \in \bigcup_{d \in D} S_n(d)} (\omega_n^{jk} - \omega_n^{-jk})
$$

$$
= \sum_{d \in D} i \sum_{k \in S_n(d)} (\omega_n^{jk} - \omega_n^{-jk})
$$

$$
= \sum_{d \in D} i \sum_{k \in d S_n(1)} (\omega_n^{jk} - \omega_n^{-jk})
$$

$$
= \sum_{d \in D} i \sum_{k \in S_n(1)} (\omega_n^{jk} - \omega_n^{-jk})
$$

$$
= \sum_{d \in D} \sigma(d) s_{n/d}(j)
$$

(7)

$$
= \sum_{d \in D} \sigma(d) s_{n/d}(j),
$$

for $0 \leq j \leq n-1$, where $D \subseteq \{d : d \mid n/4\}$. For the Ramanujan’s sine sum $s_{n/d}(j)$, let $n/d = 2^i m$ and $j = 2^{i-2} j'$, with $m$ and $j'$ be an odd positive integer. If we fix the integer $i$, then $n/d$ and $j$ are determined by $i$. Let $D = \bigcup_{i=2}^{\vartheta_2(n)} D_i \subseteq \{d : d \mid n/4\}$ for $2 \leq i \leq \vartheta_2(n)$.

For Eq. (7), if $j$ is fixed, then $i$ is determined by $j$, and it follows that $d \in D_i$. Thus, we have

$$
\mu_j = \sum_{d \in D_i} \sigma(d) s_{n/d}(j).
$$

(8)

By Theorem 2.7 and Eq. (8), the result follows. \qed
3 Proof of Theorem 1.2

In the section, we will extend some results in [3, 5] to integral oriented circulant graphs. We first introduce some related concepts and theorems.

Let $H$ be a Hermitian matrix, let $\mu_0, \ldots, \mu_{n-1}$ be all eigenvalues (not necessarily distinct) of $H$, and let $u_0, \ldots, u_{n-1}$ be the corresponding normalized eigenvectors. By spectral decomposition (see [14, Theorem 5.5.1]), we have $H = \sum_{r=0}^{n-1} \mu_r u_r u_r^*$. Furthermore, the transition matrix $U(t)$ of $H$ can be written as

$$U(t) = \sum_{r=0}^{n-1} \exp(it \mu_r) u_r u_r^*.$$  

Now, let $\Gamma = \text{IOCG}_n(D, \sigma)$ be an integral oriented circulant graph, and let $H$ be the Hermitian adjacency matrix of $\Gamma$. By Eq. (6), we see that $u_k = v_k/\sqrt{n}$. Then the transition matrix $U(t)$ of $H$ becomes

$$U(t) = \frac{1}{n} \sum_{r=0}^{n-1} \exp(i \mu_r t) v_r v_r^*.$$

In particular, by Eq. (6) and Eq. (9), we have

$$U(t)_{ab} = \frac{1}{n} \sum_{r=0}^{n-1} \exp(i \mu_r t) \omega_r^{(a-b)} = \frac{1}{n} \sum_{r=0}^{n-1} \exp \left(i \mu_r t + \frac{2\pi r(a-b)}{n}\right).$$  

This expression is given in [30, Proposition 1]. Finally, our aim is to investigate whether there exist distinct integers $a, b \in \mathbb{Z}_n$ and a positive real number $t$ such that $|U(t)_{ab}| = 1$. Let $M_r = \imath(\mu_r t + 2\pi r(a-b)/n)$, for all $1 \leq r \leq n-1$. Obviously, $|U(t)_{ab}| \leq 1$, and equality holds if and only if for all $1 \leq r \leq n-1$, the exponents $\exp(M_r)$ are equal in Eq. (10), or equivalently, $(M_{r+1} - M_r)/(2\pi) \in \mathbb{Z}$. Let $t' = t/(2\pi)$. Then we have

$$\frac{M_{r+1} - M_r}{2\pi} = (\mu_{r+1} - \mu_r)t' + \frac{a-b}{n} \in \mathbb{Z},$$

for all $r = 0, \ldots, n-1$. Since $\mu_r \in \mathbb{Z}$ for all $r = 0, \ldots, n-1$, we have $t' \in \mathbb{Q}$. At this point, we can obtain a necessary and sufficient condition as follows.

**Theorem 3.1.** Let $\Gamma = \text{IOCG}_n(D, \sigma)$ be an integral oriented circulant graph. Then for distinct $a, b \in \mathbb{Z}_n$, $\Gamma$ has PST between vertices $a$ and $b$ if and only if there are integers $p$ and $q$ such that $\gcd(p, q) = 1$ and

$$\frac{p}{q} (\mu_{j+1} - \mu_j) + \frac{a-b}{n} \in \mathbb{Z},$$  

for all $j = 0, \ldots, n-1$.

By Theorem 3.1, if there exists PST on integral oriented circulant graphs, then we can easily deduce the following corollary.
Corollary 3.2. Let $\Gamma = \text{IOCG}_n(D, \sigma)$ be an integral oriented circulant graph. Then for distinct $a, b \in \mathbb{Z}_n$ and $1 \leq k \leq n$, $\Gamma$ has PST between vertices $a$ and $b$ if and only if there are integers $p$ and $q$ such that $\gcd(p, q) = 1$ and

$$\frac{p}{q} (\mu_{j+k} - \mu_j) + \frac{k(a - b)}{n} \in \mathbb{Z},$$

for all $j = 0, \ldots, n - 1$.

In general, if an integral oriented circulant graph has PST between vertices $a$ and $b$, then the order of $a - b$ is two. Furthermore, the order-two element is unique, that is, $a - b = n/2$. By Theorem 3.1, we give a necessary and sufficient condition for the existence of PST on between vertices $b + n/2$ and $b$ on integral oriented circulant graphs.

Lemma 3.3. Let $\Gamma = \text{IOCG}_n(D, \sigma)$ be an integral oriented circulant graph. Then for all $b \in \mathbb{Z}_n$, $\Gamma$ has PST between vertices $b + n/2$ and $b$ if and only if there exists a number $m \in \mathbb{N}$ such that

$$\vartheta_2(\mu_{j+1} - \mu_j) = m,$$

for all $j = 0, 1, \ldots, n - 1$.

Proof. Let $\mu_{j+1} - \mu_j = 2^{\alpha_j} m_j$ where $\alpha_j = \vartheta_2(\mu_{j+1} - \mu_j) \geq 0$ and $m_j$ is an odd integer for each $j = 0, 1, \ldots, n - 1$.

($\Rightarrow$) Suppose that $\Gamma$ has PST between vertices $b + n/2$ and $b$. According to the Theorem 3.1, there exist relatively prime integers $p, q$ such that

$$\frac{p}{q} (\mu_{j+1} - \mu_j) + \frac{1}{2} \in \mathbb{Z},$$

for all $j = 0, 1, \ldots, n - 1$. Rewrite Eq. (14) in the following form:

$$\frac{2^{\alpha_j+1} q m_j + 1}{2} \in \mathbb{Z}.$$  

From the last expression we can conclude that $2^{\alpha_j+1} q m_j$ are odd, for each $j = 0, 1, \ldots, n - 1$. Since $\gcd(p, q) = 1$, $p$ is odd, we can obtain $q = 2^{\alpha_j+1} m_q$ and $m_q \mid m_j$, $m_j$ is an odd integer for each $j = 0, 1, \ldots, n - 1$. Therefore, $\vartheta_2(\mu_{j+1} - \mu_j) = \alpha_j = \vartheta_2(q) - 1$, for each $j = 0, 1, \ldots, n - 1$. Let $m = \vartheta_2(q) - 1 \in \mathbb{N}$. Then we can obtain $\vartheta_2(\mu_{j+1} - \mu_j) = m$.

($\Leftarrow$) Now suppose $\vartheta_2(\mu_{j+1} - \mu_j) = m$. Put $q = 2^{m+1}$ and $p = 1$. Then

$$\frac{p (\mu_{j+1} - \mu_j) + 1}{2} = \frac{m_j + 1}{2} \in \mathbb{Z},$$

for all $j = 0, 1, \ldots, n - 1$. Therefore, $\Gamma$ has PST.

Lemma 3.4. Let $n$ be a positive integer and let $D$ be an odd positive integer set. Then for all odd integer $1 \leq q \leq n$ and each positive integers $k$, $\sum_{d \in D} (-1)^k c_d(q)$ have the same parity if and only if $D = \{1\}$. 


Proof. Since addition and subtraction do not affect parity. Without loss of generality, we consider \( \sum_{d \in D} c_d(q) \). (\( \Leftarrow \)) For set \( D = \{1\} \), since \( c_1(q) = 1 \), we can derive \( \sum_{d \in D} c_d(q) \) have the same parity for all \( q \in \mathbb{N} \). (\( \Rightarrow \)) For set \( D \neq \{1\} \). Suppose that there exists an odd prime set \( D' = \{d_1, d_2\} \neq \{1\} \) such that \( \sum_{d \in D'} c_d(q) \) have same parity for odd integer number \( 1 \leq q \leq n \). By Proposition 2.2. If \( q = d_1 \), then \( \sum_{d \in D'} c_d(q) = c_{d_1}(d_1) + c_{d_2}(d_1) = d_1 - 1 - 1 = d_1 - 2 \in 2\mathbb{N} + 1 \). If \( q = d' \), \( d_1 \nmid d' \) and \( d_2 \nmid d' \), then \( \sum_{d \in D'} c_d(q) = c_{d_1}(d') + c_{d_2}(d') = -1 - 1 = -2 \in 2\mathbb{N} \), which leads to a contradiction. \( \square \)

Let \( \Gamma = \text{IOCG}_n(D, \sigma) \) be an integral oriented circulant graph. By Theorem 2.8, the eigenvalues of \( \Gamma \) are \( \mu_j = 2 \sum_{d \in D} \sigma(d)(-1)^{(\frac{n}{d} - 1)/2}(-1)^{(j+1)/2}c_{\frac{n}{d}}(j) \) for \( j \in 2\mathbb{N} + 1 \). By Lemma 3.4, we obtain Lemma 3.5.

Lemma 3.5. Let \( n \equiv 0 \pmod{4} \). Then \( \mu_j/2 \) have the same parity for \( j \in 2\mathbb{N} + 1 \) if and only if \( D_2 = \{n/4\} \).

For \( D_2 = \{n/4\} \), by Theorem 2.8, we extract the main features of the eigenvalues to obtain Lemma 3.6.

Lemma 3.6. Let \( \Gamma = \text{IOCG}_n(D, \sigma) \) be an integral oriented circulant graph. If \( D_2 = \{n/4\} \), then

\[
\mu_j \in \begin{cases} 
4\mathbb{Z}, & \text{if } j/2^{i-2} \text{ is odd and } i \geq 3, \\
2\sigma(n/4)(-1)^{(j+1)/2}, & \text{if } j \text{ is odd,} \\
0, & \text{otherwise,}
\end{cases}
\]

for \( 2 \leq i \leq \vartheta_2(n) \), \( 0 \leq j \leq n - 1 \), where \( D = \bigcup_{i=2}^{\vartheta_2(n)} D_i \subseteq \{d : d \mid n/4\} \).

Proof of Theorem 1.2. (i) \( \Rightarrow \) (ii) Suppose that \( D_2 \neq \{n/4\} \). By Lemma 3.5, \( \mu_j/2 \) have no the same parity for all \( j \in 2\mathbb{N} + 1 \). This implies that \( (\mu_{j+1} - \mu_j)/2 \) have no the same parity for all \( j \in 2\mathbb{N} + 1 \). At this point, Eq. (13) does not hold. By Lemma 3.3, \( \Gamma \) has no PST, a contradiction.

(i) \( \Leftarrow \) (ii) If \( D_2 = \{n/4\} \), by Lemma 3.6, then \( |\mu_{j+1} - \mu_j| \in 4\mathbb{Z} \pm 2 = 2(2\mathbb{Z} \pm 1) \) for all \( 0 \leq j \leq n - 1 \). This implies that \( \vartheta_2(\mu_{j+1} - \mu_j) = 1 \) for all \( 0 \leq j \leq n - 1 \). By Lemma 3.3, \( \Gamma \) has PST between vertices \( b + n/2 \) and \( b \). \( \square \)

From the above characterization, we can calculate the number of integral oriented circulant graphs of a given order having PST.

Corollary 3.7. Let \( \Gamma = \text{IOCG}_n(D, \sigma) \) be an integral oriented circulant graph. Then the number of \( \Gamma \) having PST is

\[
|\Gamma| = 2 \times 3^{\tau(\frac{n}{4}) - \tau(\frac{n}{2\vartheta_2(n)})}, \quad n \in 4\mathbb{N},
\]

where \( \tau(n) \) denotes the number of the divisors of \( n \).

Proof. Based on mapping \( \sigma \). For \( n \in 4\mathbb{N} \), \( d = n/4 \) have two choices, the cardinality of the set \( \tilde{D} = \{d : d \mid n/4, \; n/d \in 4\mathbb{N}\} \setminus D_2 \) is equal to \( \tau(n/4) - \tau(n/2\vartheta_2(n)) \), and each \( d \) in \( \tilde{D} \) have two choices. According to the Binomial Theorem (see [7, Theorem 5.2.2]), we can obtain the result. \( \square \)
4 Proof of Theorem 1.3

In the previous section, we prove that there exists PST on integral oriented circulant graphs between vertices \( b + n/2 \) and \( b \), and a sufficient necessary condition is also obtained. In this section, we will find MST on integral oriented circulant graphs.

**Lemma 4.1.** Let \( \Gamma = \text{IOCG}_n(D, \sigma) \) be an integral oriented circulant graph. For any two distinct vertices \( a, b \in \mathbb{Z}_n \), if \( \Gamma \) has PST between \( a \) and \( b \), then \( a = b + kn/4 \) for some \( k \in \{1, 2, 3\} \).

**Proof.** Suppose that \( \Gamma \) has PST between vertices \( a \) and \( b \). By Corollary 3.2, there are integers \( p \) and \( q \) such that \( \gcd(p, q) = 1 \)

\[
\frac{p}{q}(\mu_{j+4} - \mu_j) + \frac{4(a-b)}{n} \in \mathbb{Z},
\]

for all \( j = 0, \ldots, n-1 \). By Theorem 1.2 and Lemma 3.6, we have \( \mu_{j+4} - \mu_j = 0 \) whenever \( j \) is odd, and hence \( 4(a-b)/n \in \mathbb{Z} \). Therefore, \( a - b \in \{n/4, n/2, 3n/4\} \), and the result follows.

For all \( b \in \mathbb{Z}_n \), if \( \Gamma \) has PST between vertices \( b + n/4 \) and \( b \), by Corollary 3.2, then Eq. (12) there are integers \( p \) and \( q \) such that \( \gcd(p, q) = 1 \)

\[
\frac{p}{q}(\mu_{j+2} - \mu_j) + \frac{2(b+n/4-b)}{n} = \frac{p}{q}(\mu_{j+2} - \mu_j) + \frac{1}{2} \in \mathbb{Z},
\]

for all \( j = 0, \ldots, n-1 \). We can easily obtain

\[
\frac{p}{q}(\mu_{j+2} - \mu_j) + \frac{3}{2} = \frac{p}{q}(\mu_{j+2} - \mu_j) + \frac{2(b+3n/4-b)}{n} \in \mathbb{Z},
\]

for all \( j = 0, \ldots, n-1 \). Therefore, \( \Gamma \) has PST between vertices \( b + 3n/4 \) and \( b \). We can find that \( \Gamma \) has PST between vertices \( b + n/4 \) and \( b \) and \( \Gamma \) has PST between vertices \( b + 3n/4 \) and \( b \) are equivalent. At this point, we can obtain the following lemma.

**Lemma 4.2.** Let \( \Gamma = \text{IOCG}_n(D, \sigma) \) be an integral oriented circulant graph. Then for all \( b \in \mathbb{Z}_n \), \( \Gamma \) has PST between vertices \( b + n/4 \) and \( b \) and between vertices \( b + n/2 \) and \( b \) if and only if \( \Gamma \) has MST between vertices \( b, b + n/4, b + n/2, b + 3n/4 \).

Suppose that \( \Gamma \) has PST between vertices \( b + n/2 \) and \( b \). By Lemma 4.2, if we want to determine the existence of MST in \( \Gamma \), then we only need to satisfy that \( \Gamma \) has PST between vertices \( b + n/4 \) and \( b \). Similar to the proof of Lemma 3.3, we can obtain Lemma 4.3.

**Lemma 4.3.** Let \( \Gamma = \text{IOCG}_n(D, \sigma) \) be an integral oriented circulant graph. Suppose that for all \( b \in \mathbb{Z}_n \), \( \Gamma \) has PST between vertices \( b + n/2 \) and \( b \). Then for all \( b \in \mathbb{Z}_n \), \( \Gamma \) has PST between vertices \( b + n/4 \) and \( b \) if and only if there exists a number \( m' \in \mathbb{N} \) such that

\[
\vartheta_2(\mu_{j+2} - \mu_j) = m',
\]

for all \( j = 0, 1, \ldots, n-1 \).
Lemma 4.4. Let \( \Gamma = \text{IOCG}_n(D, \sigma) \) be an integral oriented circulant graph. Then for all \( b \in \mathbb{Z}_n \), \( \Gamma \) has MST between vertices \( b, b + n/4, b + n/2, b + 3n/4 \) if and only if

\[
\vartheta_2(\mu_{j+1} - \mu_j) = 1 \quad \text{and} \quad \vartheta_2(\mu_{j+2} - \mu_j) = 2,
\]

for all \( j = 0, 1, \ldots, n - 1 \).

**Proof.** (\( \Rightarrow \)) Since \( \Gamma \) has PST between vertices \( b + n/2 \) and \( b \), according to the proof of Theorem 1.2, we have \( \vartheta_2(\mu_{j+1} - \mu_j) = 1 \) for all \( j = 0, 1, \ldots, n - 1 \). It remains to prove \( \vartheta_2(\mu_{j+2} - \mu_j) = 2 \) for all \( j = 0, 1, \ldots, n - 1 \). By Theorem 1.2, we obtain \( n \in 4\mathbb{N} \) and \( D_2 = \{n/4\} \). Furthermore, by Lemma 3.6, we have \( |\mu_{j+2} - \mu_j| = 4 \) whenever \( j \) is odd, and hence \( \vartheta_2(\mu_{j+2} - \mu_j) = 2 \) whenever \( j \) is odd. Since \( \Gamma \) also has PST between vertices \( b + n/4 \) and \( b \), by Lemma 4.3, we conclude that \( \vartheta_2(\mu_{j+2} - \mu_j) = 2 \) for all \( j = 0, 1, \ldots, n - 1 \), as desired.

(\( \Leftarrow \)) If \( \vartheta_2(\mu_{j+1} - \mu_j) = 1 \) and \( \vartheta_2(\mu_{j+2} - \mu_j) = 2 \) for all \( j = 0, 1, \ldots, n - 1 \), by Lemma 3.3 and Lemma 4.3, then \( \Gamma \) has PST between vertices \( b + n/2 \) and \( b \), and between vertices \( b + n/4 \) and \( b \). Then it follows from Lemma 4.2 that \( \Gamma \) has MST between vertices \( b, b + n/4, b + n/2, b + 3n/4 \).

Let \( \Gamma = \text{IOCG}_n(D, \sigma) \) be an integral oriented circulant graph. By Theorem 2.8, the eigenvalues of \( \Gamma \) are \( \mu_j = 4 \sum_{d \in D_3} \sigma(d) (-1)^{(j + 1)/2} c_{\frac{n}{d}} (j/2) \) for \( j/2 \in 2\mathbb{N} + 1 \). By Lemma 3.4, we can obtain Lemma 3.5.

**Lemma 4.5.** Let \( n \equiv 0 \pmod{4} \). Then \( \mu_j/4 \) have the same parity for \( j/2 \in 2\mathbb{N} + 1 \) if and only if \( D_3 = \{n/8\} \).

For \( D_2 = \{n/4\} \) and \( D_3 = \{n/8\} \), by Theorem 2.8, we extract the main features of the eigenvalues to obtain Lemma 4.6.

**Lemma 4.6.** Let \( \Gamma = \text{IOCG}_n(D, \sigma) \) be an integral oriented circulant graph. If \( D_2 = \{n/4\} \) and \( D_3 = \{n/8\} \), then we have

\[
\mu_j \in \begin{cases} 
8 \mathbb{Z}, & \text{if } j/2^{i-2} \text{ is odd and } i \geq 4, \\
4\sigma(n/8)(-1)^{(j/2+1)/2} & \text{if } j/2 \text{ is odd,} \\
2\sigma(n/4)(-1)^{(j+1)/2} & \text{if } j \text{ is odd,} \\
0, & \text{otherwise,}
\end{cases}
\]

for \( 2 \leq i \leq \vartheta_2(n), 0 \leq j \leq n - 1 \), where \( D = \bigcup_{i=2}^{\vartheta_2(n)} D_i \subset \{d : d \mid n/4\} \).

**Proof of Theorem 1.3.** By Lemma 4.2 and Theorem 1.2, \( D_2 = \{n/4\} \) is a necessary condition for the existence of MST on integral oriented circulant graphs. Next, let \( D_2 = \{n/4\} \). We prove that \( D_3 = \{n/8\} \). (i) \( \Rightarrow \) (ii) Suppose that \( D_3 \neq \{n/8\} \). By Lemma 4.5, then \( \mu_j/4 \) have no the same parity for all \( j/2 \in 2\mathbb{N} + 1 \). This implies that \( (\mu_{j+1} - \mu_j)/4 \) have no the same parity for all \( j/2 \in 2\mathbb{N} + 1 \). At this point, Eq. (16) does not hold. By Lemma 4.3 and 4.4, \( \Gamma \) has no MST, a contradiction.
If $D_3 = \{n/8\}$, then we have $n \in 8\mathbb{N}$. By Lemma 4.6, then $|\mu_{j+2} - \mu_j| = 4$ for all $j$ is odd, and

$$|\mu_{j+2} - \mu_j| = \begin{cases} 4(2\mathbb{Z} \pm 1), & \text{if } j/2^{i-2} \text{ is odd}, \\ 4, & \text{otherwise}, \end{cases}$$

for all $j$ is even. This implies that $\vartheta_2(\mu_{j+2} - \mu_j) = 2$ for all $0 \leq j \leq n - 1$. By Lemma 4.4, $\Gamma$ has MST between vertices $b, b + n/4, b + n/2, b + 3n/4$.

By Theorem 1.3, we can obtain that MST on integral oriented circulant graphs only occurs between four vertices and $n \in 8\mathbb{N}$. Therefore, we have the following corollary.

**Corollary 4.7.** Let $\Gamma = \text{IOCG}_n(D, \sigma)$ be an integral oriented circulant graph. Then $\Gamma$ has no UST.

From the above characterization, we can calculate the number of integral oriented circulant graphs of a given order having MST.

**Corollary 4.8.** Let $\Gamma = \text{IOCG}_n(D, \sigma)$ be an integral oriented circulant graph. Then the number of $\Gamma$ having MST is

$$|\Gamma| = 2 \times 2 \times 3^{\tau(\frac{n}{4})} - 2^{\tau(\frac{n}{2^n(2^{i-2})})}, \quad n \in 8\mathbb{N},$$

where $\tau(n)$ denotes the number of the divisors of $n$.

**Proof.** Based on mapping $\sigma$. For $n \in 8\mathbb{N}$, $d = n/4$ and $d = n/8$ have two choices respectively, the cardinality of the set $\tilde{D} = \{d : d \parallel n/4, \ n/d \in 8\mathbb{N}\} \setminus \{D_2, D_3\}$ is equal to $\tau(n/4) - 2^{\tau(n/2^{i-2}(n))}$, and each $d$ in $\tilde{D}$ have two choices. According to the Binomial Theorem (see [7, Theorem 5.2.2]), we can obtain the result.

**5 Conclusion**

This work focuses on the study of existence problem for PST and MST on integral oriented circulant graphs. We find that there are some nice PST and MST properties on integral oriented circulant graphs. PST and MST determined by its order $n$ and the set of divisors $D$, not related to the selection of mapping $\sigma$. For PST (or MST), we obtain necessary and sufficient condition for the existence of PST (or MST) on integral oriented circulant graphs. For MST, we prove that there only exists MST for four vertices on integral oriented circulant graphs.

The following question now arises naturally:

(i) Determine non-integer mixed (or oriented) Cayley (or circulant) graphs having PST and MST.

(ii) Determine integral (weighted) mixed circulant graphs having PST and MST. According to the calculation, we find that there are many PST on integral mixed circulant graphs.
(iii) Determine the mixed Cayley graphs with PST for some kinds of groups.

(iv) So far, we only find that MST can occur between three vertices and four vertices. Is there a MST with more than four vertices?

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