ON A DISCRETE JOHN-TYPE THEOREM

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ABSTRACT. As a discrete counterpart to the classical John theorem on the approximation of (symmetric) $n$-dimensional convex bodies $K$ by ellipsoids, Tao and Vu introduced so called generalized arithmetic progressions $P(A, b) \subset \mathbb{Z}^n$ in order to cover (many of) the lattice points inside a convex body by a simple geometric structure. Among others, they proved that there exists a generalized arithmetic progressions $P(A, b)$ such that $P(A, b) \subset K \cap \mathbb{Z}^n \subset P(A, O(n)^{3n/2})$. Here we show that this bound can be lowered to $n^{O(\log n)}$ and study some general properties of so called unimodular generalized arithmetic progressions.

1. Introduction

Let $K^n$ be the set of all $o$-symmetric convex bodies in $\mathbb{R}^n$, i.e., $K \in K^n$ is a compact convex set in $\mathbb{R}^n$ with non-empty interior and $K = -K$. By $B_n \in K^n$ we denote the $n$-dimensional unit ball, i.e., $B_n = \{ x \in \mathbb{R}^n : \langle x, x \rangle \leq 1 \}$, where $\langle \cdot, \cdot \rangle$ is the standard inner product.

For the family $K^n$ of $o$-symmetric convex bodies in $\mathbb{R}^n$, John’s (ellipsoid) theorem states that there exists an ellipsoid $E \in K^n$ such that (see, e.g., [AAGM15, Theorem 2.1.3], [Sch14, Theorem 10.12.2])

$$\mathcal{E} \subseteq K \subseteq \sqrt{n} \mathcal{E}. \quad (1.1)$$

It turns out that the volume maximal ellipsoid contained in $K$ gives the desired approximation, and in the non-symmetric (or general) case the factor $\sqrt{n}$ has to be replaced by $n$ (after a suitable translation of $K$).

This theorem has numerous applications in Convex Geometry or in the local theory of Banach spaces (see, e.g., [AAGM15], [Sch14]), as it allows to get a first quick estimate on the value $f(K)$ of any homogenous and monotone functional $f$ on $K^n$ by the value of the functional at ellipsoids. For instance, if $\text{vol}(\cdot)$ denotes the $n$-dimensional volume, i.e., $n$-dimensional Lebesgue measure, than (1.1) implies that for $K \in K^n$ there exists an ellipsoid $\mathcal{E}$ such that

$$\text{vol}(\mathcal{E}) \leq \text{vol}(K) \leq n^{\frac{3}{2}} \text{vol}(\mathcal{E}). \quad (1.2)$$

In particular, the volume of an ellipsoid can easily be evaluated as $\mathcal{E} = AB_n$ for some $A \in \text{GL}(n, \mathbb{R})$, and thus $\text{vol}(\mathcal{E}) = |\det A| \text{vol}(B_n)$.

In [TV06], Tao and Vu started to study a discrete version of John’s theorem where the aim of the approximation is the set of lattice points in $K$, i.e., the set $K \cap \mathbb{Z}^n$. The approximation itself is carried out not by lattice
points in ellipsoids, which are hard to control or to compute, but by a so called symmetric generalized arithmetic progression (GAP for short)

\[ P(A, b) = \{Az : z \in \mathbb{Z}^n, |z_i| \leq b, 1 \leq i \leq n \}, \]

where \( A \in \mathbb{Z}^{n \times n} \), \( \det A \neq 0 \), and \( b \in \mathbb{R}^n \). Hence, \( P(A, b) \) are the lattice points of the lattice \( AZ^n \) in the parallelepiped \( \sum_{i=1}^{n} \text{conv} \{-b_i a_i, b_i a_i\} \), where \( a_i \) is the \( i \)th column of \( A \) and \( \text{conv} \) denotes the convex hull.

By improving on an earlier result from [TV06, Lemma 3.36], Tao and Vu proved in [TV08]

**Theorem ([TV08, Theorem 1.6]).** Let \( K \in \mathcal{K}_s^n \). Then there exists a GAP \( P(A, b) \subset K \) such that

i) \( K \cap \mathbb{Z}^n \subset P\left(A, O(n^{3n/2})b\right) \),

ii) \( |K \cap \mathbb{Z}^n| < O(n^{7n/2}|P(A, b)|) \).

Here, for a finite set \( C \) we denote by \( |C| \) its cardinality, and observe that \( |P(a, b)| = \prod_{i=1}^{n} (2|b_i| + 1) \) can be easily computed. Obviously, i) and ii) of the theorem above may be regarded as discrete counterparts to (1.1) and (1.2).

A first qualitative version of such a theorem, i.e., without mentioning explicit constants, is contained in the paper [BV92, Theorem 3]. Here we show

**Theorem 1.1.** Let \( K \in \mathcal{K}_s^n \).

i) There exists a GAP \( P(A, b) \subset K \) such that

\[ K \cap \mathbb{Z}^n \subset P\left(A, O(n^{(\ln n)|b|}) \right) \).

ii) There exists a GAP \( P(A, b) \subset K \) such that

\[ |K \cap \mathbb{Z}^n| < O(n^{2n/\ln n}|P(A, b)|). \]

In comparison to the volume case (John’s ellipsoid) a GAP contained in \( K \in \mathcal{K}_s^n \) which is optimal for the cardinality bound (1.4), i.e., covering most of the lattice point in \( K \), does not need to be optimal for the inclusion bound (1.3) as well. We will give an example of this occurence in Proposition 2.1. In fact, also the two GAPs leading to the bounds in (1.3) and (1.4) are different (in general).

Regarding a GAP \( P(A, b) \) which is simultaneously good with respect to inclusion and cardinality we have the following slight improvement on the above theorem of Tao and Vu.

**Theorem 1.2.** Let \( K \in \mathcal{K}_s^n \). Then there exists a GAP \( P(A, b) \subset K \) such that

i) \( K \cap \mathbb{Z}^n \subset P\left(A, O(n^{2n/\ln n})b\right) \),

ii) \( |K \cap \mathbb{Z}^n| < O(n^{2n}|P(A, b)|). \)

For unconditional convex bodies \( K \in \mathcal{K}_s^n \), i.e., \( K \) is symmetric to all coordinate hyperlanes, the inclusion bound can be made linear.
Proposition 1.3. Let $K \in K^n_{(s)}$ be an unconditional convex body. Then there exists a GAP $P(A, b) \subset K$ such that
\begin{enumerate} 
\item $K \cap \mathbb{Z}^n \subseteq P(A, n b)$,
\item $|K \cap \mathbb{Z}^n| < O(n^n |P(A, b)|)$.
\end{enumerate}

As we will point out in Proposition 3.4, the linear inclusion bound in Proposition 1.3 is essentially best possible, and it might be even true that the general bound of order $n^{O(\ln n)}$ in (1.3) can be replaced by a linear or polynomial bound in $n$.

The paper is organized as follows. In the second section we introduce and collect some basic properties of GAPs approximating the lattice points in symmetric convex bodies. In turns out that GAPs where the columns of $A$ form a lattice basis of $\mathbb{Z}^n$ are of particular interest and we study them in Section 3. Finally, Section 4 contains the proof of the theorems and of the proposition above.

2. Preliminaries and GAPs

For the proof of Theorem 1.1 it is more convenient to introduce GAPs for general lattices $\Lambda \subset \mathbb{R}^n$, i.e., $\Lambda = B \mathbb{Z}^n$, $B \in \mathbb{R}^{n \times n}$ with $\det B \neq 0$. Let $\mathcal{L}^n$ be the set of all these lattices. Following [TV08], and adapting their definition to our special geometric situation, we call for a matrix $A \in \mathbb{R}^{n \times n}$ with columns $a_i \in \Lambda$, $1 \leq i \leq n$, and for $b \in \mathbb{R}^n > 0$ the set of lattice points in $\Lambda$ given by

$$P(A, b) = \{Az : -b \leq z \leq b, z \in \mathbb{Z}^n\}$$

a generalized symmetric arithmetic progression with respect to $\Lambda$, for short, just GAP.

Actually, Tao and Vu defined GAPs more generally, namely, for general $n \times m$ matrices $A$. In our geometric setting, however, this would make the inclusion bound needless as $A$ may consist of all (up to $\pm$) lattice points in $K \in K^n_{(s)}$ and by letting $b = (1 - \epsilon)1$, where $1$ is the appropriate all 1-vector and $\epsilon$ an arbitrary positive number less than 1, we get

$$\{0\} = P(A, b) \subset K \cap \mathbb{Z}^n \subset P(A, (1 - \epsilon)^{-1}b)$$

Moreover, Tao and Vu were mainly interested in so called infinitely proper GAPs which here means $m = \text{rank}(A)$, and so we restrict the definition to the case $A \in \mathbb{R}^{n \times n}$, $\det A \neq 0$.

The size or cardinality of a GAP $P(A, b)$ is given

$$|P(A, b)| = \prod_{i=1}^{n} (2\lfloor b_i \rfloor + 1),$$

where $\lfloor \cdot \rfloor$ denotes the floor function. In general, for a vector $b \in \mathbb{R}^n$ we denote by $\lfloor b \rfloor = ([b_1], \ldots, [b_n])^T$ its integral part. With $P_\mathbb{R}(A, b)$ we denote the parallelepiped

$$P_\mathbb{R}(A, b) = \{Ax : -b \leq x \leq b, x \in \mathbb{R}^n\} = \sum_{i=1}^{n} \text{conv} \{-b_i a_i, b_i a_i\}$$
associated to the GAP $P(A, b)$. Observe that

$$P_R(A, [b]) = \text{conv } P(A, b).$$

Whenever we are interested in a GAP $P(A, b)$ covering most of the lattice points in a convex body, i.e., a GAP which is optimal with respect to the cardinality bound, then it suffices to assume $b \in \mathbb{N}^n$. However, for an optimal GAP with respect to the inclusion bound it might be essential to consider non-integral vectors $b \in \mathbb{R}_+^n$. This is also reflected by the next example showing that those GAPs yielding an optimal cardinality bound can be different from those leading to an optimal inclusion bound.

**Proposition 2.1.** Let $n \geq 2$. There exists a $K \in K^n_{(s)}$ such that any GAP $P(A, b) \subseteq K$ covering most of the lattice points of $K$ is not an optimal GAP with respect to inclusions, i.e., there exists another GAP $P(\overline{A}, \overline{b}) \subseteq K$ such that for any $t > 1$ with $K \cap \mathbb{Z}^n \subseteq P(A, tb)$ there exists a $\overline{t} < t$ with $K \cap \mathbb{Z}^n \subseteq P(\overline{A}, \overline{t}\overline{b})$.

**Proof.** We start with dimension 2, and let $K = \text{conv } \{\pm(3,0)^\top, \pm(-3,1)^\top, \pm(-1,1)^\top\}$ (see Figure 2.1).

![Figure 2.1. Different optimal GAPs](image.png)

We will argue that an optimal cardinality GAP $P(A, b) \subseteq K$ contains 9 out the 13 lattice points in $K$. To this end we may assume that the columns $a_i$ of $A$ belong to $K$, i.e., $a_i \in K$ and $b \geq 1$. Otherwise, we could only cover lattice points on a line which would be at most 7. Since for all $x \in K$ we have $|x_2| \leq 1$, and since also the sum $a_1 + a_2$ has to belong to $K$, there is at most one column $a_i$ of $A$ having a non-zero last coordinate.

If there would be none, then again only the 7 points with last coordinate 0 could be covered.

Next assume that $a_2$ is the vector having last coordinate non-zero and let $a_1$ be the vector with last coordinate 0. The only possibility so that $a_1 \pm a_2$ belong to $K$ (up to sign) is the one depicted in the left figure, i.e., $a_1 = (1,0)^\top$ and $a_2 = (-2,1)^\top$, and for any $b$ with $1 \leq b_i < 2$, $i = 1, 2$, the GAP $P(A, b)$ covers 9 out of the 13 lattice points of $K$. Hence, the GAPs covering the maximal amount of lattice points of $K$ are – up to ± and permutation of the columns of $A$ – given by $P(A, b)$ for any $b$ with $1 \leq b_i < 2$, $i = 1, 2$. Since $(3,0)^\top \in K$, we observe that in order to cover all the points of $K \cap \mathbb{Z}^2$ by $P(A, tb)$ we must have $t > 3/2$.

On the other hand, if we take for the columns of $A$ the vectors $(1,0)^\top$ and $(0,1)^\top$ and setting $\overline{b} = (3,1-\epsilon)$ we get $|P(A, \overline{b})| = 7$, but $K \cap \mathbb{Z}^2 \subseteq P(\overline{A}, (1-\epsilon)^{-1}\overline{b})$ for any $\epsilon \in (0,1)$ (cf. right hand side picture in Figure 2.1).
This verifies the assertion in the plane. By building successively prisms over Q the example can be extended to all dimensions.

\[ \square \]

\section{Unimodular GAPs}

Without loss of generality we consider here only the case \( \Lambda = \mathbb{Z}^n \). The group of all unimodular matrices, i.e., integral \( n \times n \)-matrices of determinant \( \pm 1 \), is denoted by \( \text{GL}(n, \mathbb{Z}) \); it consists of all lattices bases of \( \mathbb{Z}^n \). Apparently, if \( K \cap \mathbb{Z}^n \) contains a lattice basis of \( \mathbb{Z}^n \) and \( K \cap \mathbb{Z}^n \subseteq P(A, b) \) then \( A \in \text{GL}(n, \mathbb{Z}) \). This basically shows that for the inclusion bound it suffices to consider GAPs \( P(U, b) \) with \( U \in \text{GL}(n, \mathbb{Z}) \). We will call such a GAP, an unimodular GAP.

\begin{proposition}
Let \( c = c(n) \in \mathbb{R}_{>0} \) be a constant depending on \( n \). The following statements are equivalent.

\begin{itemize}
\item[i)] For every \( K \in \mathcal{K}^n_{(s)} \) there exists a GAP \( P(A, b) \subset K \) such that \( K \cap \mathbb{Z}^n \subseteq P(A, c \beta b) \).
\item[ii)] For every \( K \in \mathcal{K}^n_{(s)} \) there exists an unimodular GAP \( P(U, b) \subset K \) such that \( K \cap \mathbb{Z}^n \subseteq P(U, c \beta b) \).
\end{itemize}
\end{proposition}

\begin{proof}
Obviously, we only have to show that i) implies ii). To this end let \( l \in \mathbb{N} \) such that \( lK \) contains a basis of \( \mathbb{Z}^n \). By assumption there exists a GAP \( P(U, b) \subset lK \) such that \( lK \cap \mathbb{Z}^n \subseteq P(U, c \beta b) \) and since \( lK \) contains a basis of \( \mathbb{Z}^n \) we have \( U \in \text{GL}(n, \mathbb{Z}) \). Next we claim that

\begin{equation}
P(U, l^{-1}b) \subseteq K \cap \mathbb{Z}^n \subseteq P(U, cl^{-1}b).
\end{equation}

Let \( u \in P(U, l^{-1}b) \). Then there exists a \( z \in \mathbb{Z}^n \) with \( u = uz \) and \( -l^{-1}b \leq z \leq l^{-1}b \). Thus \( lu = Ulz \) and since \( lz \in \mathbb{Z}^n \) we get \( lu \in P(U, b) \subset lK \). Hence \( u \in K \cap \mathbb{Z}^n \) which shows the first inclusion in (3.1). For the second let \( a \in K \cap \mathbb{Z}^n \). Then \( l a \in lK \cap \mathbb{Z}^n \subseteq P(U, c \beta b) \) and so there exists a \( z \in \mathbb{Z}^n \) with \( -c \beta b \leq z \leq c \beta b \) with \( l a = uz \). Hence, \( a = U^{-1}z \) and since \( U \in \text{GL}(n, \mathbb{Z}) \) we conclude \( l^{-1}z \in \mathbb{Z}^n \) which shows \( a \in P(U, cl^{-1}b) \).
\end{proof}

Next we want to point out a relation between GAPs and approximations of a convex body by an “unimodular” parallelepiped \( P_{\mathbb{R}}(U, u) \), \( U \in \text{GL}(n, \mathbb{Z}) \).

To this we first note that

\begin{lemma}
Let \( K \in \mathcal{K}^n_{(s)} \) containing \( n \) linearly independent points \( \beta a_i \) with \( \beta \in \mathbb{R}_{>0} \) and \( a_i \in \mathbb{Z}^n \), \( 1 \leq i \leq n \). Then for any unimodular GAP \( P(U, u) \) with \( K \subseteq P_{\mathbb{R}}(U, u) \) we have \( u_i \geq \beta \), \( 1 \leq i \leq n \).
\end{lemma}

\begin{proof}
Let \( \beta a_i = Ux_i \) with \( -u \leq x_i \leq u \), \( x_i \in \mathbb{R}^n \). Since \( U \in \text{GL}(n, \mathbb{Z}) \) we get \( x_i \in \beta \mathbb{Z}^n \), which shows that for each non-zero coordinate \( j \), say, of \( x_i \) we have \( u_j \geq \beta \). Since \( x_1, \ldots, x_n \) are linearly independent for each coordinate \( k \) we can find a vector \( x_i \) whose \( k \)th coordinate is non-zero.
\end{proof}

Observe, for an unimodular GAP \( P(U, u) \) we have \( P(U, u) = P_{\mathbb{R}}(U, u) \cap \mathbb{Z}^n \).

\begin{proposition}
Let \( c = c(n) \in \mathbb{R}_{>0} \) be a constant depending on \( n \). The following statements are equivalent.
\end{proposition}
i) For every \( K \in \mathcal{K}_n^{(s)} \), there exists a GAP \( P(A, b) \subseteq K \) such that 
\[ K \cap \mathbb{Z}^n \subseteq P(A, c b). \]

ii) For every \( K \in \mathcal{K}_n^{(s)} \), there exists an unimodular GAP \( P(U, u) \subseteq K \) such that 
\[ P_\mathbb{R}(U, u) \subseteq K \subseteq P_\mathbb{R}(U, c u). \]

Proof. We start with the implication i) implies ii). Let \( \varepsilon > 0 \), and let 
\( Q \subseteq K \) be a \( \sigma \)-symmetrical rational polytope with \( K \subseteq (1 + \varepsilon)Q \) (see, e.g., [Sch14, Theorem 1.8.19]). Moreover, let \( m \in \mathbb{N} \) such i) \( mQ \) is an integral polytope, i.e., all vertices are in \( \mathbb{Z}^n \), and ii) \( mQ \) contains the scaled unit vectors \( c(1 + \varepsilon)e_i, 1 \leq i \leq n \). In view of Proposition 3.1 there exists an unimodular GAP \( P(U, u) \) such that 
\[ P(U, u) \subseteq mQ \cap \mathbb{Z}^n \subseteq P(U, c u). \]

The polytopes \( P_\mathbb{R}(U, [u]) \) and \( mQ \) are integral and so we get 
\[ P_\mathbb{R}(U, [u]) = \text{conv}(P_\mathbb{R}(U, [u])) \cap \mathbb{Z}^n = \text{conv}(P(U, [u])) \]
\[ \subseteq \text{conv}(P(U, u)) \subseteq \text{conv}(mQ \cap \mathbb{Z}^n) = mQ \subseteq mK. \]

Since \( mQ \) integral we have \( mQ \subseteq P_\mathbb{R}(U, c u) \) and due to Lemma 3.2 we know for the entries of \( u \) that \( u_i \geq 1 + \varepsilon/c, 1 \leq i \leq n \), which implies that 
\[ \frac{u_i}{|u_i|} \leq \frac{u_i}{u_i - 1} \leq 1 + \frac{\varepsilon}{c}, \]
and thus \( c u \leq (c + \varepsilon)[u] \). Hence, 
\[ mQ = \text{conv}(mQ \cap \mathbb{Z}^n) \subseteq \text{conv}(P(U, c u)) \]
\[ \subseteq P_\mathbb{R}(U, [u]) \subseteq P_\mathbb{R}(U, (c + \varepsilon)[u]), \]

and with (3.2) 
\[ P_\mathbb{R}(U, m^{-1}[u]) \subseteq K \subseteq P_\mathbb{R}(U, (1 + \varepsilon)(c + \varepsilon)m^{-1}[u]). \]

Observe, that actually \( U = U_\varepsilon, u = u_\varepsilon \) as well as \( m = m_\varepsilon \) depend on the chosen \( \varepsilon \). Now, since \( K \) is bounded and all entries of \( U \) are integral, the first inclusion shows that the sequence \( m_\varepsilon^{-1}[u_\varepsilon], \varepsilon > 0 \), has to be bounded. Therefore, we may assume that it converges to \( \overline{u} \) as \( \varepsilon \) approaches 0. Next, let us assume that a sequence of a (fixed) column vector of the unimodular matrices \( U_\varepsilon \) is unbounded. Since \( \text{vol}(P_\mathbb{R}(U_\varepsilon, 1)) = 2^n \) and since \( m_\varepsilon^{-1}[u_\varepsilon] \) is bounded this shows that the inradius of \( P_\mathbb{R}(U_\varepsilon, (1 + \varepsilon)(c + \varepsilon)m_\varepsilon^{-1}[u_\varepsilon]) \) converges to 0 as \( \varepsilon \) tends to 0. Hence, also \( U_\varepsilon \) converges to an unimodular matrix \( \overline{U} \) and so we have shown 
\[ P_\mathbb{R}(\overline{U}, \overline{u}) \subseteq K \subseteq P_\mathbb{R}(\overline{U}, c \overline{u}). \]

For the reverse implication we assume that there exists an unimodular GAP \( P(U, u) \) fulfilling ii). Then 
\[ P_\mathbb{R}(U, u) \cap \mathbb{Z}^n \subseteq K \cap \mathbb{Z}^n \subseteq P_\mathbb{R}(U, c u) \cap \mathbb{Z}^n, \]
and by the unimodularity of \( U \) we have \( P_\mathbb{R}(U, u) \cap \mathbb{Z}^n = P(U, u) \) as well as 
\[ P_\mathbb{R}(U, c u) \cap \mathbb{Z}^n = P(U, c u). \]

We close this section with lower bounds on the factors in (1.3) and (1.4) of Theorem 1.1.
Proposition 3.4.

i) Let $\tau = \tau(n) \in \mathbb{R}_{>0}$ be a constant depending on $n$ such that for every $K \in \mathcal{K}_n^{(s)}$ there exists a GAP $P(A, \mathbf{b}) \subset K$ such that $K \cap \mathbb{Z}^n \subseteq P(A, \tau \mathbf{b})$. Then $\tau \geq n!^{1/n} > \frac{1}{en}$. 

ii) Let $\nu = \nu(n) \in \mathbb{R}_{>0}$ be a constant depending on $n$ such that for every $K \in \mathcal{K}_n^{(s)}$ there exists a GAP $P(A, \mathbf{b}) \subset K$ such that $|K \cap \mathbb{Z}^n| \leq \nu |P(A, \mathbf{b})|$. Then $\nu \geq (2^n + 1)/3$.

Proof. For i) we consider for an integer $m \in \mathbb{N}$ the cross-polytope $mC_n^* = \{ \mathbf{x} \in \mathbb{R}^n : |x_1| + \cdots + |x_n| \leq m \}$ and let $P(U, \mathbf{u})$ be a GAP such that

$$(3.3) \quad P(U, \mathbf{u}) \subseteq mC_n^* \cap \mathbb{Z}^n \subseteq P(U, \tau \mathbf{u}).$$

In view of Proposition 3.1, or since $mC_n^*$ contains the unit vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ we have $U \subseteq \mathbb{R}^n$. More precisely, for a lattice $\Lambda \in \mathcal{L}_n$ with basis $\mathbf{B}$ yielding the desired lower bound.

On the other hand, the first inclusion in (3.3) implies

$$(4.1) \quad P_{\mathbb{R}}(U, [\mathbf{u}]) = \text{conv} P(U, \mathbf{u}) \subseteq mC_n^*,$$

and so

$$(3.4) \quad \text{vol} (mC_n^*) = m^n \frac{n^n}{n!} \leq \tau^n \frac{n^n}{n!} \prod_{i=1}^n u_i.$$

On the other hand, the first inclusion in (3.3) implies

$$(3.3) \quad P_{\mathbb{R}}(U, [\mathbf{u}]) = \text{conv} P(U, \mathbf{u}) \subseteq mC_n^*,$$

and so

$$(4.1) \quad P_{\mathbb{R}}(U, [\mathbf{u}]) = \text{conv} P(U, \mathbf{u}) \subseteq mC_n^*.$$

Combined with (3.4) we obtain

$$\tau \geq n!^{1/n} \left( \prod_{i=1}^n \frac{|u_i|}{u_i} \right)^{1/n}.$$

This is true for any $m \in \mathbb{N}$, and since $u_i \rightarrow \infty$ for $m \rightarrow \infty$, we find $\tau \geq n!^{1/n} > \frac{n}{en}$. In order to prove ii), let $Q$ be the o-symmetric lattice polytope given by $Q = \text{conv} (\pm(\{0,1\}^{n-1} \times \{1\}))$. Then it is easy to see that $Q \cap \mathbb{Z}^n = \pm(\{0,1\}^{n-1} \times \{1\}) \cup \{0\}$ and hence, $Q$ does not contain $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^n \setminus \{0\}$, $\mathbf{x} \neq \mathbf{y}$, and $\mathbf{x} + \mathbf{y} \in Q$. Thus for any GAP $P(A, \mathbf{b}) \subset Q$ we have $|P(A, \mathbf{b})| \leq 3$ and so

$$2^n + 1 = |Q \cap \mathbb{Z}^n| \leq \nu |P(A, \mathbf{b})| \leq 3 \nu$$

yielding the desired lower bound. \qed

4. Proofs of the Theorems

For the proof of the inclusion bound (1.3) of Theorem 1.1 we follow essentially the proof of [TV08], but we will apply a different lattice reduction taking into account also the polar lattice. More precisely, for a lattice $\Lambda \in \mathcal{L}_n$ with basis $\mathbf{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$, i.e., $\Lambda = B\mathbb{Z}^n$, we denote by

$$\Lambda^* = \{ \mathbf{y} \in \mathbb{R}^n : (\mathbf{x}, \mathbf{y}) \in \mathbb{Z} \text{ for all } \mathbf{x} \in \Lambda \} = B^{-\top} \mathbb{Z}^n$$

its polar lattice. In particular, if $B^{-\top} = (\mathbf{b}_1^*, \ldots, \mathbf{b}_n^*)$, then

$$\langle \mathbf{b}_i^*, \mathbf{b}_j \rangle = \delta_{i,j}.$$
where $\delta_{i,j}$ denotes the Kronecker-symbol. Now a basis $B$ of a lattice $\Lambda$ is called Seysen reduced if

$$S(B) = \sum_{i=1}^{n} \|b_i\|^2 \|b_i^*\|^2$$

is minimal among all bases of $\Lambda$ (cf. [Sey93]). Here, $\|\cdot\|$ denotes the Euclidean norm. Seysen proved

**Theorem 4.1 ([Sey93, Theorem 7]).** Let $\Lambda \in \mathcal{L}^n$. There exists a basis $B = (b_1, \ldots, b_n)$ of $\Lambda$ such that $S(B) \leq n^{O(\ln n)}$. In particular, for $1 \leq i \leq n$

(4.2)

$$\|b_i\| \|b_i^*\| \leq n^{O(\ln n)}.$$

For an explicit bound we refer to [Maz10] and for more information on lattice reduction and Geometry of Numbers we refer to [GL87], [Cas71]. For sake of comprehensibility we split the proof of Theorem 1.1 into two parts, one covering the inclusion bound and one the cardinality bound.

**Proof of i) of Theorem 1.1.** In view of John’s theorem (1.1) we may apply a linear transformation $T$ to $K$ such that with $\tilde{K} = T\Lambda$

(4.3)

$$B_n \subseteq \tilde{K} \subseteq \sqrt{n}B_n.$$ With $\Lambda = T\mathbb{Z}^n$ the problem is now to find a GAP $P(A, b)$ in $\Lambda$ such that $P(A, b) \subset K$ and $\tilde{K} \cap \Lambda \subset P(A, n^{O(\ln n)}b)$.

Let $B = (b_1, \ldots, b_n)$ be a Seysen reduced basis of $\Lambda$ with associated basis $B^{-1} = (b_1^*, \ldots, b_n^*)$ of the polar lattice and let $u \in \mathbb{R}^n$ be given by $u_i = (1/n)\|b_i\|^{-1}, 1 \leq i \leq n$.

First, for $x \in P(\mathbb{R}, B, u)$ we have $x = \sum_{i=1}^{n} \lambda_i b_i$ with $|\lambda_i| \leq u_i$ and by the triangle inequality we conclude $\|x\| \leq 1$. Hence, with (4.3) we certainly have $P(B, u) \subset \tilde{K}$. On the other hand, given $x = \sum_{i=1}^{n} \beta_i b_i \in \tilde{K}$ we get by Cramer’s rule and (4.3)

$$|\beta_i| = \frac{\det(x, b_1, \ldots, b_{i-1}, b_{i+1}, b_n)}{\det B} \leq \sqrt{n} \frac{\text{vol}_{n-1}(b_1, \ldots, b_{i-1}, b_{i+1}, b_n)}{\text{vol}(b_1, \ldots, b_n)},$$

where $\text{vol}_k(c_1, \ldots, c_k)$ denotes the $k$-dimensional volume of the parallelepiped $\{\sum_{i=1}^{k} \mu_i c_i : 0 \leq \mu_i \leq 1\}$. By (4.1) we find that

$$\text{vol}(b_1, \ldots, b_n) = \text{vol}_{n-1}(b_1, \ldots, b_{i-1}, b_{i+1}, b_n) \frac{\langle b_i^*, b_i \rangle}{\|b_i^*\|} = \text{vol}_{n-1}(b_1, \ldots, b_{i-1}, b_{i+1}, b_n) \frac{1}{\|b_i^*\|},$$

and thus for $1 \leq i \leq n$

(4.4)

$$|\beta_i| \leq \sqrt{n}\|b_i^*\|.$$ Together with the definition of $u_i$ and Seysen’s bound (4.2) we conclude $|\beta_i| \leq n^{3/2}n^{O(\ln n)}u_i, 1 \leq i \leq n$. Hence,

$$\tilde{K} \cap \Lambda \subset P(\mathbb{R}, b, n^{O(\ln n)}u) \cap \Lambda = P(B, n^{O(\ln n)}u),$$ since $B$ is a basis of $\Lambda$. \hfill \square
Remark 4.2. The optimal upper bound in Theorem 4.1 for a Seysen reduced basis is not known, but any improvement on this bound would immediately yield an improvement of (1.3).

For the cardinality bound (1.4) of Theorem 1.1 we need another tool from Geometry of Numbers, namely Minkowski’s successive minima \( \lambda_i(K, \Lambda) \), which for \( K \in K_n(\mathbb{S}) \), \( \Lambda \in \mathcal{L}^n \) and \( 1 \leq i \leq n \) are defined by

\[
\lambda_i(K, \Lambda) = \min \{ \lambda > 0 : \dim(\lambda K \cap \Lambda) \geq i \}.
\]

In words, \( \lambda_i(K, \Lambda) \) is the smallest dilation factor \( \Lambda \) such that \( \lambda K \) contains \( i \) linearly independent lattice points of \( \Lambda \). Minkowski’s fundamental second theorem on successive minima states that [GLS7, §9, Theorem 1]

\[
\text{vol}(K) \leq \det \lambda^{\frac{1}{n}} \prod_{i=1}^{n} \frac{2}{\lambda_i(K, \Lambda)},
\]

and here we need a discrete version of it. In [Hen02] it was shown that

(4.5) \[
|K \cap \Lambda| \leq 2^{n-1} \prod_{i=1}^{n} \left( \frac{2}{\lambda_i(K, \Lambda)} + 1 \right),
\]

and for an improvement on the constant \( 2^{n-1} \) and related results we refer to [Mal10, Mal12]. It is conjectured in [BHW93] that (4.5) holds without any additional factor in front of the product which would, in particular, imply Minkowski’s volume bound.

Proof of ii) of Theorem 1.1. Let \( \mathbf{a}_i \in \mathbb{Z}^n \), \( 1 \leq i \leq n \), be linearly independent lattice vectors corresponding to the successive minima \( \lambda_i = \lambda_i(K, \mathbb{Z}^n) \), i.e., \( \mathbf{a}_i \in \lambda_i K \), \( 1 \leq i \leq n \). Since \( \lambda_i^{-1} \mathbf{a}_i \in K \) it follows

\[
\left\{ \sum_{i=1}^{n} \mu_i \frac{1}{n \lambda_i} \mathbf{a}_i : -1 \leq \mu_i \leq 1 \right\} \subset \text{conv} \left\{ \pm \lambda_i^{-1} \mathbf{a}_i : 1 \leq i \leq n \right\} \subset K.
\]

Thus, denoting by \( A \) the matrix with columns \( \mathbf{a}_i \) and letting \( \mathbf{b} \) be the vector with entries \( b_i = (n \lambda_i)^{-1} \) we have \( P(A, \mathbf{b}) \subset K \) and

\[
|P(A, \mathbf{b})| = \prod_{i=1}^{n} \left( 2 \left\lfloor \frac{1}{n \lambda_i} \right\rfloor + 1 \right).
\]

Now it is not hard to see that

\[
2 \left\lfloor \frac{1}{n \lambda_i} \right\rfloor + 1 \geq \frac{11}{3n} \left\lfloor \frac{2}{\lambda_i} + 1 \right\rfloor
\]

and with (4.5) we get

\[
|P(A, \mathbf{b})| \geq \left( \frac{1}{3n} \right)^n \left( \frac{1}{2} \right)^{n-1} \prod_{i=1}^{n} \left( \frac{2}{\lambda_i} + 1 \right) > (6n)^{-n}|K \cap \mathbb{Z}^n|.
\]

This shows (1.4). \( \square \)

Remark 4.3. We want to point out that the columns of the matrix \( A \) of the GAP in the above proof of the cardinality bound of Theorem 1.1 do not build a basis of \( \mathbb{Z}^n \) (in general) and hence, this GAP cannot be used in order to obtain an inclusion bound.
Now the proof of Theorem 1.2 is a kind of combination of the two proofs leading to (1.3) and (1.4). Instead of a Seysen reduced basis we exploit properties of a so called Hermite-Korkine-Zolotarev (HKZ) reduced basis \(b_1, \ldots, b_n\) of the lattice \(\Lambda\). For such a basis it was shown by Mahler (see, e.g., [LLS90, Theorem 2.1]) that for \(1 \leq i \leq n\)

\[
\|b_i\| \leq \frac{\sqrt{i + 3}}{2} \lambda_i(B_n, \Lambda). \tag{4.6}
\]

Moreover, Håstad & Lagarias [HL90] pointed out that for such a HKZ-basis one has

\[
\|b_i\| \cdot \|b_i^*\| \leq \left(\frac{3}{2}\right)^n < n^{1/2} \ln n, \tag{4.7}
\]

This bound is worse than the one given in (4.2), but the advantage of a HKZ reduced basis is its close relation to the successive minima (4.6).

**Proof of Theorem 1.2.** First we may assume that \(\lambda_n(K, \mathbb{Z}^n) \leq 1\), i.e., that \(K\) contains \(n\) linearly independent lattice points. Otherwise, all lattice points of \(K\) lying in a hyperplane \(H\) and it would be sufficient to prove the theorem with respect to the \(n-1\)-dimensional convex body \(K \cap H\) and lattice \(H \cap \mathbb{Z}^n\).

Now we proceed completely analogously to the proof of i) in Theorem 1.1; we just replace the Seysen reduced basis by a HKZ-reduced basis \(B = (b_1, \ldots, b_n)\), and the GAP is given by \(P(B, u)\) with \(u_i = (1/n)\|b_i\|^{-1}, 1 \leq i \leq n\). Replacing (4.2) by (4.7) in (4.4) leads then to

\[
P(B, u) \subseteq \tilde{K} \cap \Lambda \subseteq P(B, u^{O(n/\ln n)}),
\]

where \(\tilde{K}\) was a linear image of \(K\) such that

\[
B_n \subseteq \tilde{K} \subseteq \sqrt{n} B_n. \tag{4.8}
\]

It remains to prove the cardinality bound for the GAP \(P(B, u)\) and \(\tilde{K}\). Regarding the size of \(P(B, u)\) we have

\[
|P(B, u)| = \prod_{i=1}^{n} \left( 2 \left[ \frac{1}{n\|b_i\|} \right] + 1 \right) \geq n^{-n} \prod_{i=1}^{n} \frac{1}{\|b_i\|}. \tag{4.9}
\]

On the other hand, for an upper bound on \(\tilde{K} \cap \Lambda\) we use (4.5) and since \(\lambda_n(K, \Lambda) \leq 1\) we get

\[
|K \cap \Lambda| \leq 2^{n-1} \prod_{i=1}^{n} \left( \frac{2}{\lambda_i(K, \Lambda)} + 1 \right) \leq 6^n \prod_{i=1}^{n} \frac{1}{\lambda_i(K, \Lambda)}. \tag{4.10}
\]

In view of (4.8) and (4.6) we obtain

\[
|K \cap \Lambda| \leq 6^n \prod_{i=1}^{n} \frac{1}{\lambda_i(\sqrt{n} B_n, \Lambda)} = (6\sqrt{n})^n \prod_{i=1}^{n} \frac{1}{\lambda_i(B_n, \Lambda)} \leq (6n)^n \prod_{i=1}^{n} \frac{1}{\|b_i\|}. \tag{4.11}
\]

Combined with (4.9) we get \(|K \cap \Lambda| \leq O(n^{2n}|P(B, u)|)\).

Finally, we consider unconditional bodies \(K \in K^n_{(s)}\), i.e., bodies which are symmetric to all coordinate hyperplanes. As stated in Proposition 1.3, in this special case the inclusion bound can be made linear in the dimension. In view of Proposition 3.4 this is also the optimal order within this class of
bodies as the given example used for the lower bound in Proposition 3.4 is unconditional.

**Proof of Proposition 1.3.** For \( i = 1, \ldots, n \) let \( u_i \) be the maximal entry of the \( i \)th coordinate of a point of \( K \). Then \( u_i > 0 \) and

\[
K \cap \mathbb{Z}^n \subseteq P(I_n, \mathbf{u})
\]

with \( \mathbf{u} = (u_1, \ldots, u_n)^T \) and \( I_n \) the \( n \times n \)-identity matrix. By the unconditionality of \( K \) we have \( \pm u_i e_i \in K, 1 \leq i \leq n \), and thus

\[
P_R(I_n, n^{-1}\mathbf{u}) \subset \text{conv}\{\pm u_i e_i : 1 \leq i \leq n\} \subseteq K.
\]

Hence, \( P(I_n, n^{-1}\mathbf{u}) \subseteq K \). For the remaining cardinality bound we observe that \((2u_i + 1) < (2\lfloor u_i/n \rfloor + 1)3n\) and so (4.10) implies

\[
|K \cap \mathbb{Z}^n| \leq \prod_{i=1}^{n}(2\lfloor u_i \rfloor + 1) < (3n)^n \prod_{i=1}^{n}(2\lfloor u_i/n \rfloor + 1) = (3n)^n |P(I_n, n^{-1}\mathbf{u})|.
\]

\[\Box\]

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