Note on an Inversion Formula to Determine Binary Elements by Astrometry

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Abstract

Simplified solutions to determine binary elements by astrometry were obtained in terms of elementary functions (Asada et al. 2004, PASJ, 56, L35), and therefore require neither iterative nor numerical methods. In the framework of the simplified solution, this paper discusses the remaining two parameters of the time of periastron passage and the longitude of the ascending node in order to complete the solution. We thus clarify a difference between the simplified solution and other analytical methods.

Key words: astrometry — celestial mechanics — stars: binaries: general

1. Introduction

Recently, we developed a formulation for determining orbital elements of binary with astrometric observations. The simplified solution is written in terms of elementary functions, and therefore requires neither iterative nor numerical methods (Asada et al. 2004). This solution has been generalized to a binary system in open (hyperbolic or parabolic) orbits as well as closed (elliptic) ones (Asada 2007). An extension to observational data has also been discussed (Asada et al. 2007). The solution gives an explicit form of binary elements, such as the eccentric anomaly and the major axis of elliptic orbits. Oyama et al. (2008) made an attempt to use this solution for discussing some uncertainty in binary elements because of a large scatter of their data points, when they measured the proper motions of maser sources in the galactic center with VERA.

On the other hand, the remaining parameters of the time of periastron passage and the longitude of the ascending node are not addressed in the simplified solution. Hence, the solution is rather simplified. However, these parameters are needed to make a comparison between the simplified solution and conventional ones. In addition, a lack of information on the remaining parameters apparently suggests a certain incompleteness of the simplified solution. In this brief article, therefore, we shall derive, in the framework of the simplified solution, both the time of periastron passage and the longitude of the ascending node in order to complete the solution.

Astrometry plays a fundamental role in astronomy by providing useful star catalogs based on precise measurements of the positions and movements of stars and other celestial bodies. For instance, astrometric observations provide a useful method for determining the mass of various unseen celestial objects currently, such as a massive black hole (Miyoshi et al. 1995), an extra-solar planet (Benedict et al. 2002), and two new satellites of Pluto (Weaver et al. 2006). The astrometry of Sharpless 269 with VERA detects a trigonometric parallax corresponding to a distance of 5.28 kpc, which is the smallest parallax ever measured, and puts the strongest constraint on the flatness of the outer rotation curve (Honma et al. 2007). Accordingly, astrometry has attracted renewed interest, since the Hipparcos mission successfully provided a precise catalog at the level of a milliarcsecond. In fact, there exist several projects involving space-borne astrometry while aiming at an accuracy of a few microarcseconds, such as SIM1 (Shao 2004), GAIA2 (Mignard 2004; Perryman 2004), and JASMINE3 (Gouda et al. 2007).

In this paper, we focus on an astrometric binary, for which only one of the component stars can be visually observed, but the other cannot, like a black hole or a very dim star. In this case, it is impossible to directly measure the relative vector connecting the two objects, because the secondary is not directly observed. The position of the star is repeatedly measured relative to reference stars or quasars. On the other hand, the orbit determinations of resolved double stars (visual binaries), which are a system of two visible stars, was solved first by Savary in 1827 and by many authors including Kowalsky, Thiele, and Innes (Binnendijk 1960; Aitken 1964 for a review on earlier works; for the state-of-the-art techniques, e.g., Eichhorn & Xu 1990; Catović & Olević 1992; Olević & Cvetković 2004). The relative vector from the primary star to the secondary has an elliptic motion with a focus at the primary. This relative vector is observable only for resolved double stars.

In conventional methods of orbit determination, the time of periastron passage is one of the important parameters, because it enters Kepler’s equation as

\[ t = t_0 + \frac{T}{2\pi} (E - e_K \sin E), \]

where \( t_0 \), \( T \), \( e_K \), and \( E \) denote the time of periastron passage, the orbital period, the eccentricity, and the eccentric anomaly, respectively (e.g., Danby 1988; Roy 1988; Murray & Dermott 1999; Beutler 2004). The simplified solution does not use Kepler’s equation in order to avoid treating such a transcendental equation.

This paper is organized as follows. Our notation in the simplified solution is summarized in section 2. The time
of periastron passage in the simplified solution is derived in section 3. The longitude of the ascending node is obtained in section 4.

2. Simplified Solution: Our Notation

Our notation used in the simplified solution is briefly summarized as follows. We neglect the motions of the observer and the common center in our Galaxy. Namely, we take account of only the Keplerian motion of a star around the common center of mass of a binary system. Let us define \((x, y)\) as the Cartesian coordinates on the celestial sphere, in such a way that an apparent (observed) ellipse on the celestial sphere can be expressed in the standard form as

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{2}
\]

where \(a \geq b\). The eccentricity, \(e\), is \(\sqrt{1 - b^2/a^2}\). This eccentricity may be different from \(e_\Sigma\), the eccentricity of the actual elliptic orbit, because of the inclination of the orbital plane with respect to the line of our sight. The star is located at \(P_j = (x_j, y_j)\) on the celestial sphere at time \(t_j\) for \(j = 1, \cdots, n\).

We use the fact that the law of constant-areal velocity still holds, even after a Keplerian orbit is projected onto the celestial sphere. Here, the area is swept by the line interval between the star and the projected common center of mass but not a focus of the apparent ellipse (see figure 1). This fact, is expressed as

\[
\frac{S}{T} = \frac{S(k,j)}{T(k,j)}, \tag{3}
\]

where \(S(k,j)\) and \(S\) denote the area swept during the time interval \(T(k,j) = t_k - t_j\) for \(t_k > t_j\), and the total area of the apparent ellipse, \(\pi ab\), respectively. The swept area is expressed as (Asada et al. 2004; Asada 2007)

\[
S(k,j) = \frac{1}{2}ab \left[ u_k - u_j - \frac{x_e}{a}(\sin u_k - \sin u_j) \right] + \frac{y_e}{b}(\cos u_k - \cos u_j). \tag{4}
\]

The eccentric anomaly in the apparent ellipse is given by \(u_j = \arctan[ay_j/(bx_j)]\).

The orbital elements can be expressed explicitly as elementary functions of the locations of four observed points and their time intervals (Asada et al. 2004). Let us take four observed points \((P_1, P_2, P_3, P_4)\) for \(t_1 < t_2 < t_3 < t_4\). The location \((x_e, y_e)\) of the projected common center is given by

\[
x_e = -a \frac{E_1F_2 - G_1F_2}{E_1F_2 - F_1E_2}, \tag{5}
\]

\[
y_e = b \frac{G_1E_2 - E_1G_2}{E_1F_2 - F_1E_2}, \tag{6}
\]

where \(E_i\), \(F_j\), and \(G_j\) are elementary functions of \(T(j + 2, j + 1), T(j + 1, j),\) and \(u_k\) for \(k = j, j + 1, j + 2\). The eccentric anomaly in the actual ellipse (on the orbital plane) is denoted as \(E\) [see equation (1)].

Given \(a, b, x_e,\) and \(y_e\), we can analytically determine the parameters \(e_\Sigma, i, a_\Sigma,\) and \(o\) as (Asada et al. 2004)

\[
e_\Sigma = \sqrt{\frac{x_e^2}{a^2} + \frac{y_e^2}{b^2}}. \tag{7}
\]
Fig. 2. Flow chart of our procedure of orbit determination. The thin arrow denotes purely theoretical steps, where we initially assume an actual Keplerian orbit parameterized by \((a_K, e_K, t_0, T)\). A star’s position on the orbital plane at each time \(t_j\) is projected onto the celestial sphere, defined by \(i, \omega, \Omega\). The thick arrow denotes the observational steps, where we start from measuring the star’s positions on the celestial sphere as \((t_j, x_j)\). The steps for determining \(e_K, a_K, T, (i, \omega)\) have been examined (Asada et al. 2004; Asada 2007). The remaining steps for computing \(t_0, \Omega, \) and \((t_k, x_k)\) in the simplified solution, denoted by the dashed arrow, are discussed in this paper.

3. Time of Periastron Passage

In order to determine \(a_K\) and \(e_K\) for an actual ellipse, the simplified solution requires neither the time of periastron passage, \(t_0\), nor the longitude of the ascending node, \(\Omega\) (Asada et al. 2004). If one wishes to know \(t_0, \Omega\), however, they can be determined as follows (see also figure 2). First, we discuss \(t_0\) in this section.

The projected position of the periastron on the celestial sphere, \(P_A\), is determined as

\[
P_A = \frac{1}{e_K}(x_e, y_e),
\]

because the ratio of the semimajor axis to the distance between the center and the focus of the ellipse remains unchanged, even after the projection (Asada et al. 2004). The eccentric anomaly, \(u_A\), of the periastron in the apparent ellipse is introduced as

\[
P_A = (a \cos u_A, b \sin u_A),
\]

where \(P_A\) is also given by equation (14). Thereby, we can determine \(u_A\) (mod \(2\pi\)).

By using equation (3), we obtain

\[
\frac{S(1.0)}{T(1.0)} = \frac{S(2.1)}{T(2.1)}
\]

where we can determine \(S(1,0)\), because the eccentric anomaly in the apparent ellipse at \(t_0\), denoted as \(u_0\), is nothing but \(u_A\), which has been determined by equations (14) and (15).

Therefore, equation (16) is solved for \(t_0\) as

\[
t_0 = \frac{S(2,0)}{S(2,1)} t_1 - \frac{S(1,0)}{S(2,1)} t_2,
\]

where the R.H.S. is obtained from the observed quantities.

4. Longitude of the Ascending Node

Let us consider the projected periastron at \(P_A\) on the apparent ellipse. In the simplified solution, \(P_A\) is expressed as equation (14). Here, we make a translation of \((x, y)\) in such a way that the common center of mass can be located at the origin of new coordinates \((x', y')\). Namely, the \(x'\) axis is taken to lie along the major axis of the apparent ellipse in the celestial sphere, and the \(y'\) axis is perpendicular to the \(x'\) axis in the celestial sphere (see figure 3). In the coordinates \((x', y')\), the position of the projected periastron becomes

\[
P_A = \frac{1 - e_K}{e_K}(x_e, y_e).
\]

On the other hand, by projecting the actual ellipse onto the celestial sphere, we obtain
where the coordinates \((\hat{x}, \hat{y})\) are chosen so that the ascending node can be in the \(\hat{x}\)-direction (see figure 3).

The longitude of the ascending node, which is the angle \(\Omega\) from the reference direction to the direction of the ascending node. To compute the longitude of the ascending node, therefore, the angle \(\delta\Omega\) from the reference direction to the major axis is added into the angle measured from the major axis. In short, the longitude of the ascending node is generally given by the sum of \(\Omega_0\) and \(\delta\Omega\), where \(\Omega_0\) is the angle \(\Omega\) determined by using equation (21).

\[ S(n, 1) = \frac{S(2, 1)}{T(n, 1)} \]

This is a transcendental equation for \(u_n\) on the celestial sphere. This situation seems to be similar to that Kepler’s equation is transcendental in \(E\) on the orbital plane. Here, we should note that the time of periastron passage is needed in order to treat Kepler’s equation, whereas it is not for equation (22). This is because we employ the time interval \(T(k, j)\), while Kepler’s equation needs the time itself instead of the interval.

Regarding this point, Thiele’s method for visual binaries is straightforward manner, for instance as equation (6) in Asada et al. (2007), where they denote \(\delta\Omega\) as \(\Omega\).

Tables 1 and 2 give examples to show the flow of the actual determination of all six orbital elements. This would be helpful to those readers who code computing routines in practical applications to check their programs. Table 1 lists some the elements were then reproduced from the data by using the proposed method.

Here, we discuss how to determine in the simplified solution the position of a component star at arbitrary time \(t \equiv t_n \pmod{T}\). For \(t_1\), \(t_2\), and \(t_n\), equation (3) becomes

\[ \frac{S(n, 1)}{T(n, 1)} = \frac{S(2, 1)}{T(2, 1)} \]

It should be noted that in practical applications a reference direction chosen by observers may be different from the major axis of the apparent ellipse. In such a practical case, \(\Omega\) is the angle from the reference direction to the direction of the ascending node. To compute the longitude of the ascending node, therefore, the angle \(\delta\Omega\) from the reference direction to the major axis is added into the angle measured from the major axis. In short, the longitude of the ascending node is generally given by the sum of \(\Omega_0\) and \(\delta\Omega\), where \(\Omega_0\) is the angle \(\Omega\) determined by using equation (21). The expression of \(\delta\Omega\) is obtained in a straightforward manner, for instance as equation (6) in Asada et al. (2007), where they denote \(\delta\Omega\) as \(\Omega\).

Tables 1 and 2 give examples to show the flow of the actual determination of all six orbital elements. This would be helpful to those readers who code computing routines in practical applications to check their programs. Table 1 lists some the elements were then reproduced from the data by using the proposed method.
time interval in order to delete the time of periastron passage. A crucial difference is that Thiele's method uses Kepler's equation on the orbital plane (Thiele 1883), while the simplified one uses the constant areal velocity in the apparent ellipse on the celestial sphere. In this sense, the simplified solution more respects measured quantities on the celestial sphere than conventional ones (see figure 2). It was verified numerically that the above procedure enables us to determine in the simplified solution locations of a star at arbitrary time (see figure 4 for an example).

5. Conclusion

In this paper, we describe how we obtained in the simplified solution both the time of periastron passage and the longitude of the ascending node in order to complete the solution (see figure 2). In conclusion, the simplified solution requires neither iterative nor numerical methods when we determine all of the elements, including $t_0$ and $\Omega$. It only does when we wish to determine the star’s position at arbitrary time.

Before closing this paper, it is worth mentioning that equations (17), (21), and (22) can be applied to the case of open orbits in a straightforward manner. For open orbits, expressions of $x_e$, $y_e$, $eK$, $aK$, $i$, and $\omega$ have been already derived within the framework of the simplified solution (Asada 2007).

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Fig. 4. Example of orbit determination. Here we assume $a = 1$, $e = 0.5$ as an apparent ellipse on the celestial sphere. Let a star located at $u_1 = 60^\circ$, $u_2 = 130^\circ$, $u_3 = 170^\circ$, $u_4 = 200^\circ$ at each time $t_i$ for $i = 1, 2, 3, 4$. Regarding time, we assume that the time interval $T(i + 1, i)$ is the same as unity, namely $t_j = i - 1$ (i.e., $t_i = 0$), for simplicity. The simplified solution allows for arbitrary an time interval. The observed positions of the star are denoted by the filled circle. The quantities determined in the present procedure are $eK = 0.56$, $aK = 1.2$, $i = 42^\circ$, and $\omega = 79^\circ$. We obtained $t_0 = 0.04$ by equation (17). The star’s position at $t_5$ was obtained as $u_5 = 2.3 \times 10^5$. The location is denoted by the circle. Using the determined $eK$, $aK$, $i$, and $\omega$, we determined $u_5$ at $t_5$, also by employing the conventional procedure (indicated by the thin arrow in figure 2). In the latter case, we need to take account of the longitude of the ascending node, $\Omega$. We obtained $\Omega = 15^\circ$ by equation (21). The results of $u_5$ by both methods agree with each other.