Instability of Gravitating Sphalerons

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Abstract

We prove the instability of the gravitating regular sphaleron solutions of the $SU(2)$ Einstein-Yang-Mills-Higgs system with a Higgs doublet, by studying the frequency spectrum of a class of radial perturbations. With the help of a variational principle we show that there exist always unstable modes. Our method has the advantage that no detailed knowledge of the equilibrium solution is required. It does, however, not directly apply to black holes.
1 Introduction

When gravity is coupled to nonlinear field theories, such as Yang-Mills fields or nonlinear \(\sigma\)-models, interesting and surprising new types of particle-like and black hole solutions have turned out to exist. Among the regular solutions the most interesting ones are those for which gravity is essential. The first example of this type was found by Bartnik and McKinnon [1] for the Einstein-Yang-Mills (EYM) system. For the same model several authors [2] discovered later the colored black hole solutions which showed that the classical uniqueness theorem for the Abelian case does not generalize. The existence of both types of solutions which meanwhile has been established rigorously [3, 4, 5], and also because they were proven to be unstable [6, 7, 8, 9], led to a search for corresponding solutions of other related field theories. It turned out, for instance, that the Einstein-Skyrme (ES) system has black hole solutions with hair which are at least linearly stable [10, 11, 12, 13]. (For a numerical investigation of nonlinear stability, see ref. [13].) Several authors looked at other models, notably the \(SU(2)\) Einstein-Yang-Mills-Higgs (EYMH) theory with a Higgs triplet [14, 15, 16], as well as the EYM-dilaton theory [17], and found in some cases other linearly stable black hole solutions. Interesting black hole solutions have recently been found numerically for the \(SU(2)\) EYMH-theory with a Higgs doublet [18], as in the standard electroweak model. These "sphaleron black holes" were suspected to be unstable, but this question has so far not yet been analyzed.

In the present paper we show that the regular sphaleron solutions are unstable, but the stability issue for the corresponding black holes remains unsettled. (Some partial results will be mentioned in the concluding section.) Our proof proceeds along the following lines. First, we show that the frequency spectrum of a class of radial perturbations is determined by a coupled system of radial "Schrödinger equations", which will be derived in section 3 by linearizing the basic equations (given in section 2) around an equilibrium solution. Bound states of this Schrödinger problem correspond to exponentially growing modes. Using the variational principle for the ground state it is then proven in section 4 that there exist always unstable modes if the soliton is a purely magnetic solution of the
EYMH equations. We show this with the help of a judiciously choosen one–parameter family of trial perturbations. Unfortunately, this cannot be applied directly to the black hole solutions, because of problems related to the boundary conditions at the horizon. We have the suspicion that some of the black hole configurations might be (linearly) stable. The reasons for this will be mentioned in the final section 5.

The instability proof presented below is quite powerful for solitons, because no detailed knowledge of the equilibrium solution is required. It has recently been generalized by some of us [20] to the EYM system for arbitrary gauge groups.

2 Basic Formulae

Since we are interested in the stability of spherically symmetric black hole and soliton solutions of the EYMH theory, we restrict ourselves to spherically symmetric fields.

The metric is parametrized in the usual manner

\[ g = -NS^2dt^2 + N^{-1}dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\phi^2). \]

(1)

Instead of \( N \) which is only a function of \( t \) and \( r \), we use also the metric function \( m(t, r) \) (mass function), defined by \( N = 1 - 2m/r \) (we set \( G = 1 \)).

For the gauge potential \( A \) there are many gauge equivalent ways to parametrize the general spherically symmetric gauge potential. (A systematic analysis for an arbitrary gauge group can be found in [19].) A convenient representation is

\[ A = a_0\tau_r dt + a_1\tau_r dr + (\omega - 1) [\tau_\varphi d\vartheta - \tau_\vartheta \sin \vartheta d\varphi] + \tilde{\omega} [\tau_\vartheta d\vartheta - \tau_\varphi \sin \vartheta d\varphi], \]

(2)

where \( a_0, a_1, \omega \) and \( \tilde{\omega} \) are functions of \( t \) and \( r \), and \( \tau_r, \tau_\vartheta, \tau_\varphi \) are the spherical \( SU(2) \) generators, defined by \( \tau_r = \vec{\tau} \cdot \vec{e}_r, \tau_\vartheta = \vec{\tau} \cdot \vec{e}_\vartheta, \tau_\varphi = \vec{\tau} \cdot \vec{e}_\varphi \), with the normalization \( \vec{\tau} = \vec{\sigma}/2i \) (\( \vec{\sigma} \): Pauli matrices) and \( \vec{e}_r, \vec{e}_\vartheta, \vec{e}_\varphi \) the unit vectors in the directions of the coordinates \( r, \vartheta, \varphi \).

The Higgs doublet can always be represented as a \( 2 \times 2 \) matrix of the form \( \Phi = (\phi + i\vec{\sigma} \cdot \vec{\psi}), \phi \) and \( \vec{\psi} \) real, which transforms under the gauge group by left-multiplication.
A spherically symmetric Higgs field has the form
\[ \Phi = \frac{1}{\sqrt{2}} (\phi \cdot 1 + i\psi \sigma_r). \]  
(3)

It is now straightforward to compute the EYMH action for these fields. For the matter part, with the Lagrangian \( L_{\text{mat}} \), we define the effective radial-temporal Lagrangian, \( L_{\text{mat}} \), by the equation
\[ \int L_{\text{mat}} \sqrt{-g} \, d^4x = \int L_{\text{mat}} S dt \, dr. \]  
(4)

One finds quite easily
\[ L_{\text{mat}} = -\frac{r^2}{4} f_{\mu\nu} f^{\mu\nu} + \frac{1}{NS^2} T - NU - P, \]  
(5)

where \( f_{\mu\nu} (\mu, \nu = 0, 1) \) is the two-dimensional field strength
\[ f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu, \]  
(6)

and \( U, T \) and \( P \) are given by the following expressions in terms of the complex matter variables \( f = \omega + i\bar{\omega}, h = \phi + i\psi \):
\[ T = \bar{D}_0 f D_0 f + \frac{r^2}{2} \bar{D}_0 h D_0 h, \]  
(7)
\[ U = \bar{D}_1 f D_1 f + \frac{r^2}{2} \bar{D}_1 h D_1 h, \]  
(8)
\[ P = \frac{(1 - \bar{f}f)^2}{2r^2} + \frac{1}{4} \bar{h} h (1 + \bar{f}f) - \frac{1}{2} \text{Re}(\bar{f}h^2) + r^2 V(\bar{h}h). \]  
(9)

Here, \( V = \frac{\lambda}{4}(\bar{h}h - v^2)^2 \) is the Higgs potential, and the covariant derivatives are defined as \( D_\mu f = (\partial_\mu - ia_\mu) f \) and \( D_\mu h = (\partial_\mu - ia_\mu/2) h \) (\( \mu, \nu = 0, 1 \)). Note that \( P \) depends only on the matter fields and \( U \) and \( T \) only on their covariant derivatives. (We have chosen the relative weight of the Yang-Mills and Higgs parts such that annoying factors \( 4\pi \) do not enter in \( L_{\text{mat}} \). In this respect we follow the conventions of ref. [18].)

The independent Einstein equations are
\[ m' = -\frac{r^2}{4} f_{\mu\nu} f^{\mu\nu} + \frac{1}{NS^2} T + NU + P, \]  
(10)
\[ \dot{m} = 2N \text{Re} \left\{ \bar{D}_1 f D_0 f + \frac{r^2}{2} \bar{D}_1 h D_0 h \right\}, \]  
(11)
\[ (\ln S)' = \frac{2}{r} \left\{ \frac{1}{(NS)^2} T + U \right\}, \]  
(12)
and the Yang–Mills–Higgs equations reduce to

\[ \partial_\mu(r^2 S f^{\mu\nu}) = 2S \text{Im} \left\{ f \bar{D}^\nu f + \frac{r^2}{4} h \bar{D}^\nu h \right\}, \] (13)

\[ \frac{1}{S} D_\mu(SD^\mu f) = \frac{\bar{f} f - 1}{r^2} f + \frac{\bar{h} h - 1}{4} h^2, \] (14)

\[ \frac{1}{S} D_\mu(S \frac{r^2}{2} D^\mu h) = \frac{\bar{f} f + 1}{4} h - \frac{1}{2} f \bar{h} + \frac{r^2}{2} \lambda (\bar{h} h - v^2) h. \] (15)

3 The Pulsation Equations

We now proceed to a linear stability analysis of a given equilibrium solution, which is assumed to be a static, regular, asymptotically flat and purely magnetic solution of the coupled EYMH equations. We choose the temporal gauge by setting \( a_0 = 0 \). This can always be obtained with a gauge transformation of the form \( \exp(\tau r^\alpha) \). In this gauge the linearized equations will lead to the standard Schrödinger eigenvalue problem for the pulsation frequencies. (From now on, the symbols \( f, h \) etc. refer to the equilibrium solution, and \( \delta f, \delta h \) etc. denote their time dependent perturbations.) As a first step we consider the Einstein equations, and obtain the following linearized equations:

\[ \delta m' = N \delta U - \delta m \frac{2}{r} U + \delta P, \] (16)

\[ \delta \dot{m} = 2N \text{Re} \left\{ \delta \bar{f} \bar{f}' + \frac{r^2}{2} \delta \bar{h} \bar{h}' \right\}, \] (17)

\[ \delta (\ln S)' = \frac{2}{r} \delta U, \] (18)

where

\[ \delta U = 2 \text{Re} \left\{ \bar{f} \delta f' + \frac{r^2}{2} \bar{h}' \delta h' \right\} + 2 \delta a_1 \text{Im} \left\{ f \bar{f}' + \frac{r^2}{4} h \bar{h}' \right\}, \] (19)

\[ \delta P = \left\{ \frac{\bar{f} f + 1}{2} + r^2 \lambda (\bar{h} h - v^2) \right\} \text{Re}(\bar{h} \delta h)
+ \left\{ 2 \frac{\bar{f} f - 1}{r^2} + \frac{\bar{h} h}{2} \right\} \text{Re}(\bar{f} \delta f) - \text{Re}(\bar{f} h \delta h) - \frac{1}{2} \text{Re}(h^2 \delta \bar{f}). \] (20)

The perturbations of the matter fields \( f \) and \( h \) can be decomposed into ”real” and ”imaginary” parts as follows.
Assume that \( f = \omega \) and \( h = \varphi \) with \( \omega \) and \( \varphi \) real, and let us decompose the perturbations as

\[
\delta f = \delta \rho + i \delta \chi, \quad \text{(21)}
\]
\[
\delta h = \delta \sigma + i \delta \xi, \quad \text{(22)}
\]

with \( \delta \rho, \delta \sigma, \delta \chi \) and \( \delta \xi \) real. We will call \( \delta \rho \) and \( \delta \sigma \) the "real" parts of the perturbations and \( \delta \chi \) and \( \delta \xi \) their "imaginary" parts.

From the full linearized system, one can easily conclude that the real and the imaginary parts decouple. We are interested only in imaginary perturbations, because we shall find instabilities within this class. In this case the metric perturbations, \( \delta m \) and \( \delta S \), vanish identically. Indeed, from equation (17) we conclude that \( \delta m \) is a function of \( r \) alone. Using \( \delta U = 0, \delta P = 0 \) and an Einstein equation for the equilibrium solution, equation (16) leads then to the differential equation \( (\delta mS)' = 0 \). Together with the boundary conditions, \( \delta m(0) = \delta m(\infty) = 0 \), this implies \( \delta m = 0 \). \( \delta S = 0 \) follows directly from (18) and the boundary conditions \( \delta S(0) = \delta S(\infty) = 0 \).

Next we consider the linearized matter equations. For imaginary perturbations, these can now be written in operator form as follows. Let

\[
\Psi = \begin{pmatrix} \delta a_1 \\ \delta \chi \\ \delta \xi \end{pmatrix}, \quad H = \begin{pmatrix} H_{aa} & H_{a\chi} & H_{a\xi} \\ H_{\chi a} & H_{\chi\chi} & H_{\chi\xi} \\ H_{\xi a} & H_{\xi\chi} & H_{\xi\xi} \end{pmatrix},
\]

with

\[
H_{aa} = 2(NS)^2(\omega^2 + \frac{r^2}{8}\varphi^2), \quad \text{(24)}
\]
\[
H_{\chi\chi} = 2p_r^2 + 2NS^2\left\{\frac{\omega^2 - 1}{r^2} + \frac{\varphi^2}{4}\right\}, \quad \text{(25)}
\]
\[
H_{\xi\xi} = 2p_r \frac{r^2}{2} p_* + 2NS^2\left\{\frac{(\omega + 1)^2}{4} + \frac{r^2}{2}\lambda(\varphi^2 - v^2)\right\}, \quad \text{(26)}
\]
\[
H_{a\chi} = 2iNS((p_\omega - \omega p_*), \quad \text{(27)}
\]
\[
H_{\chi a} = 2i \{p_* NS\omega + NS(p_\omega)\}, \quad \text{(28)}
\]
\begin{align}
H_{a\xi} &= i \frac{r^2}{2} NS (p_\ast \varphi - \varphi p_\ast), \\
H_{\xi a} &= i p_\ast \frac{r^2}{2} NS \varphi + i \frac{r^2}{2} NS (p_\ast \varphi), \\
H_{\chi\xi} &= H_{\xi\chi} = -\varphi NS^2,
\end{align}

where \( p_\ast \) denotes the differential operator

\[ p_\ast = -iNS \frac{d}{dr}. \]

Finally, let \( A \) be the diagonal matrix

\[ A = \begin{pmatrix}
Nr^2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & r^2
\end{pmatrix}. \]

With these definitions we have the following standard form for the pulsation equations

\[ H \Psi = -A \ddot{\Psi}. \]

For a harmonic time dependence, \( \Psi(t, r) = \Psi(r)e^{i\omega t} \), this gives the eigenvalue equation

\[ H \Psi = \omega^2 A \Psi. \]

One can show that \( H \) is self-adjoint relative to the scalar product

\[ \langle \Psi | \Phi \rangle = \int_0^\infty \bar{\Psi} \Phi \frac{1}{NS} dr. \]

From (35) we obtain

\[ \omega^2 = \frac{\langle \Psi | H | \Psi \rangle}{\langle \Psi | A | \Psi \rangle}. \]

For the lowest value \( \omega_0^2 \) we have the minimum principle

\[ \omega_0^2 = \inf_{\Psi} \frac{\langle \Psi | H | \Psi \rangle}{\langle \Psi | A | \Psi \rangle}, \]

where \( \Psi \) runs over all functions in the domains of the operators.

We have so far left out the linearized Gauss constraint

\[ \left( \frac{r^2}{S} \delta a_1 \right)' = -\frac{2}{NS} \left( \omega \delta \chi + \frac{r^2}{2} \varphi \delta \zeta \right). \]

It will turn out that it is automatically satisfied for physical pulsations [21].
4 Instability of Regular EYMH–Solutions

For our system, numerical solutions are given by [18], which can be classified by the number of knots $k$ (zeros) of the gauge field $f$. By a field redefinition ($\omega \rightarrow -\omega$), these solutions can be brought to the form considered above: $f = \omega$ and $h = \varphi$

For finite energy solutions, $\omega$ and $\varphi$ obey the following boundary conditions (depending on $k$)

\begin{align*}
\text{for } k \text{ odd:} & \quad \omega(0) = -1, \quad \omega(\infty) = 1,
\varphi(0) = 0, \quad \varphi(\infty) = \pm v; \\
\text{for } k \text{ even:} & \quad \omega(0) = \omega(\infty) = 1,
\varphi(0) = \varphi(\infty) = \pm v. \quad (40)
\end{align*}

The fields approach their asymptotic values exponentially.

With the following judiciously chosen one–parameter family of field configurations

\begin{align*}
\alpha_1 & = \alpha \omega', \\
f_\alpha & = \frac{\omega + 1}{2} e^{i\alpha(\omega - 1)} + \frac{\omega - 1}{2} e^{i\alpha(\omega + 1)}, \\
h_\alpha & = \varphi e^{i\alpha \frac{\omega - 1}{2}}, \quad (41)
\end{align*}

where $\omega$ and $\varphi$ denote the equilibrium solutions, we shall show that the equilibrium solutions are unstable. Obviously the family runs through the equilibrium solution for the parameter value $\alpha = 0$.

The variations of this one–parameter family at $\alpha = 0$ are

\begin{align*}
\delta a_1 & = \omega', \\
\delta \chi & = (\omega^2 - 1), \\
\delta \xi & = \frac{\omega - 1}{2} \varphi. \quad (42)
\end{align*}

These are used as trial wave functions in the minimum principle [37].
The denominator $\langle \Psi | A | \Psi \rangle$ in (37) is immediately obtained from (33) and the variations (42), giving

$$
\langle \Psi | A | \Psi \rangle = \int \left\{ \frac{r^2 (\omega')^2}{S} + 2 \frac{(\omega^2 - 1)^2}{NS} + \frac{(\omega - 1)^2 \varphi^2 r^2}{4NS} \right\} dr.
$$

(43)

According to (40) all terms vanish exponentially as $r \to \infty$, and therefore $\langle \Psi | A | \Psi \rangle$ is finite. The special choice of the family (41) shows up especially in the last term of (43), in which the asymptotically growing factor $r^2 \varphi^2$ is damped by the coupling to the gauge field.

A direct calculation of the numerator $\langle \Psi | H | \Psi \rangle$ for the operator $H$ given by (24) - (26) is somewhat tedious. We find

$$
\langle \Psi | H | \Psi \rangle = - \int \left\{ 2N (\omega')^2 + 2 \frac{(\omega^2 - 1)^2}{r^2} + \frac{\varphi^2}{2} (\omega - 1)^2 \right\} S dr,
$$

(44)

which is clearly negative.

This shows the existence of bound states of (35) with negative eigenvalue $\omega^2$. Since all eigenstates with nonzero eigenvalue automatically fulfill the Gauss constraint (39), the instability of soliton solutions is proven (see [20] and [21]).

5 Remarks

This instability proof is very general in that it can be applied to a variety of systems, including the non-abelian Proca system ("frozen Higgs field"), and, as mentioned above, it has recently been applied to the Einstein–Yang–Mills case for arbitrary gauge groups [20]. However, it does not cover the black hole case, since $N(r_H) = 0$ for black holes, and thus $\langle \Psi | A | \Psi \rangle$ in (13) diverges. In fact, it has been shown, that even the $k = 1$ Bartnck McKinnon black hole has no imaginary direction of instability. (There remains only the "real" instability found in ref. [7].) Furthermore, we have numerically found some of the
non-abelian Proca black hole solutions to be linearly stable with respect to spherically symmetric perturbations [22], [23]. From this, we are tempted to conjecture the linear stability of some EYMH black holes with respect to spherically symmetric perturbations, although we have not yet numerically studied the problem for a dynamic Higgs field.

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