Some results on Best proximity in geodesic spaces

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Abstract. Eldred and Veeramani have proved the existence and uniqueness of best proximity for contraction cyclic contraction mapping in uniformly convex banach spaces. In this paper, Eldred and Veeramani study is extended to geodesic spaces particularly in CAT(0) spaces, prove the sequences \( x_{2n} \) and \( x_{2n+1} \) are Cauchy sequences and prove the existence and uniqueness of best proximity point for contraction cyclic mapping in CAT(0) spaces.

Keywords. approximation fixed sequence, best proximity, contraction cyclic mapping, CAT(0) space, fixed point, geodesic spaces.

1. Introduction

In metric space \((M,d)\), let \( N \subseteq M \) and \( A : N \rightarrow M \) is non self-mapping. If the equation \( A(x) = x \) has no solution we seek a point \( x \) in \( N \) such that it closest to the image of mapping \( A \),

\[
d(x, A(x)) = d(N, M) \quad (1.1)
\]

where \( d(N, M) = \inf\{d(a, b) : a \in N, b \in M\} \).

Best proximity consider as generalization of fixed point or is a technique to find a closest point in the domain of mapping \( A : N \rightarrow M \) to its image. When mapping \( A \) has no a fixed point in \( N \).

According to this point we can adding conditions to guarantees existing the fixed point of the mapping \( A \), Moreover fixed point can be view as corollaries of best proximity theorems.

Some results for existence and convergence of single valued mapping in Banach spaces have been introduced[1]. He discussed the best proximity for cyclic mapping in single value case and gave existence results for some contraction type, Moreover he gave an algorithm to find best proximity in the uniformly convex Banach spaces.

In this study, the best proximity is extended to CAT(0) space, and give condition to guarantees the existence and uniqueness of fixed point and prove that the sequence converges to best proximity point is a Cauchy sequence.

2. Prelimanaries

Now we give many basic definitions concerns special kind of nonlinear spaces is called geodesic spaces which enjoy important properties of normed spaces like convexity and significant spaces, CAT(0) spaces...
which consider a copy of Hilbert space because it enjoy the two important properties uniformly convex and uniqueness limit point of weak convergence.

From now on, (M.S), (G.M.S), stands for metric space, geodesic metric space.

**Definition (2.1):**[2] Let (M, d) be metric space and x, y belongs to M, an mapping α:[0, τ] → M joining x to y is called geodesic curve if α(0) = x and nd α(τ) = y such that d(x, y) = τ and for each t1, t2 ∈ [0, τ] then \(d(α(t_1), α(t_2)) = |t_1 - t_2|\). The image of α is called the geodesic segment and denoted by \([x, y]\).and define as \(z ∈ [x, y]\), \(z = α(t); \exists t ∈ [0, τ]\) such that \(d(x, z) + d(z, y) = d(x, y)\).

**Definition (2.2):** Let (M, d) be metric space, M is called geodesic space if for each x, y belongs to M there exist a geodesic curve joining them and it’s called unique geodesic space if there exist one geodesic curve joining each distinct two points.

The Definition below talk about important property in geodesic spaces, it is called Menger convexity.

**Definition (2.3):** [3] Let (M, d) be metric space, \(z ∈ M\) is called between x and y if \(z ∈ [x, y]\), we observe that \(d(x, y) = d(x, z) + d(z, y)\), this property is called betweenness and considered a generalization of convexity in linear spaces and it’s called convexity in Menger[4] sense. In unique geodesic space for each, \(t ∈ [0, τ]\) we denote the image of \(α(t) = x\) such that \(x_t ∈ [x, y]\) and we write \(x_t = (1-t)x ⊕ ty\) and the distance \(d(x, x_t) = td(x, y)\).

**Remark (2.4):** In geodesic space (M, d) every three-point \((x, y, z)\) construct geodesic triangle denoted by \(\overline{\triangle}(x, y, z)\) such that consists \((x, y, z)\) as vertices and three geodesic segments \([x, y], [x, z]\) and \([y, z]\) as edges of \(\overline{\triangle}(x, y, z)\). A comparison triangle for the triangle \(\overline{\triangle}(x, y, z)\) is a triangle \(\overline{\triangle}(\tilde{x}, \tilde{y}, \tilde{z})\) in \(R^2\) such that

\[d(x, y) = d(\tilde{x}, \tilde{y}), d(x, z) = d(\tilde{x}, \tilde{z}) \text{ and } d(y, z) = d(\tilde{y}, \tilde{z})\]

and for each point \(p ∈ [x, y]\) then there exist \(\tilde{p} ∈ [\tilde{x}, \tilde{y}]\) such that \(d(x, y) = d(\tilde{x}, \tilde{y})\), comparison points for points in \([x, z]\) and \([y, z]\) defined in the same way.

**Definition (2.5):**[2] A geodesic space (M, d) is called CAT(0) space if each geodesic triangle \(\overline{\triangle}(x, y, z)\) in (M, d) satisfying the CAT(0) comparison axiom, that is for every \(p, q ∈ \overline{\triangle}(x, y, z)\) and their comparison points \(\tilde{p}, \tilde{q} ∈ \overline{\triangle}(\tilde{x}, \tilde{y}, \tilde{z})\), respectively, then \(d(x, y) ≤ d(\tilde{x}, \tilde{y})\).

**Remark (2.6):** in [5][4] Let (M, d) be CAT(0) space satisfying the following facts.

The CN inequality of Bruhat and Tits, That is for every three points \(x, y_1, y_2 ∈ M\)

\[d(x, \frac{1}{2}(y_1 ⊕ y_2))^2 ≤ \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2 \quad (2.1)\]

Let \(p ∈ [x, y]\) in \(\overline{\triangle}(x, y, z)\) then \(d(x, p) ≤ d(\tilde{x}, \tilde{p})\) for comparison point \(\tilde{p}\) in \(\overline{\triangle}(\tilde{x}, \tilde{y}, \tilde{z})\).

Let \(N\) be closed convex subset in M and the mapping \(P_N : M → N\) defined as

\[P_N(x) = \{y ∈ N : d(x, y) = d(x, N)\}\]

is called projection map from M to N then for each \(x ∈ M\) then \(P_N(x)\) is unique such that \(d(x, P_N(x)) = d(x, N) = \inf_{y ∈ N} \{d(x, y)\}\).

For each \(y ∈ [x, P_N(x)]\) then \(P_N(y) = P_N(x)\)
In \( (x,y,z) \), let \( m_1 \) and \( m_2 \) belong to \([x,y]\) and \([x,z]\) respectively and \( t \in (0,1) \) such that \( d(x,m_1) = t(x,y) \) and \( d(x,m_2) = t(x,z) \) then \( d(m_1,m_2) \leq td(y,z) \).

**Def.(2.7)** [6]: Let \((M,d)\) be geodesic metric space, \( M \) is called hyperbolic metric space if for each \( x, y, p \in M \) and \( 0 < \alpha < 1 \), Let \( m_1 \in [p,x] \) such that \( d(p,m_1) = \alpha d(p,x) \) and \( m_2 \in [p,y] \) such that \( d(p,m_2) = \alpha d(p,y) \) implies \( d(m_1,m_2) \leq \alpha d(x,y) \).

If \( d(x,y_1) = d(x,y_2) = 1 \) and \( d(y_1,y_2) > \varepsilon \) then equation (2.1) becomes

\[
d(x,y_0)^2 \leq 1 - \frac{1}{4} \varepsilon^2 \tag{2.1}
\]

And \( d(x,y_0) \leq \sqrt{1 - \frac{1}{4} \varepsilon^2} \).

In particular, if \( d(x,y_1) \leq R \) and \( d(x,y_2) \leq R \) and \( d(y_1,y_2) > r \) then the inequality (2.1) becomes

\[
d(x,y_0) \leq \sqrt{1 - \delta \left( \frac{r}{R} \right)} R \tag{2.2}
\]

This result analogs with uniform convexity of Banach space.

**Def.(2.8)** [7]: Let \((M,d)\) be a metric space and \( N \) is a subset of \( M \), \( y \in N \) is called a best approximation of \( x \), if \( x \in M \) \[ d(x,y) = d(x,N) \],where \( d(x,N) = \inf \{d(x,y) : y \in N \} \).

**Def.(2.9)** [7]: Let \((M,d)\) be a metric space and \( N \) is a subset of \( M \). \( N \) is called proximinal (Chebyshev resp.) set if for each \( x \in M \) has at least (exactly resp.) one best approximation in \( N \).

**Theorem 1**: [8]

Let \((M,d)\) be a complete CAT(0) space and let \( N \) be a closed subset of \( M \). Then \( N \) is a proximinal set and if \( N \) is closed and convex. Then \( N \) is Chebyshev set.

**Theorem 2**: [9][10]

Let \((M,d)\) be a complete CAT(0) space and let \( N \) be a convex and closed subset of \( M \) then.

The projection mapping \( P_N : M \rightarrow N \) is nonexpansive.

For each \( x \in M \) there exist a unique \( P_N(x) \) in \( N \) such that \( d(x,P_N(x)) = d(x,N) \).

If \( y \in [x,P_N(x)] \) then \( P_N(y) = P_N(x) \).

**Theorem 3**: [8]

Let \((M,d)\) be a complete CAT(0) space and let \( N \) be a closed subset(closed and convex resp.) then \( N \) is proximal (Chebyshev resp.).

In addition, CAT(0) space shares many properties with uniformly convex Banach space the important one is descending sequence of nonempty closed, bounded and convex subsets have nonempty intersection.

Next, we extend the essential ideas into geodesic spaces as considers a general situation of nonlinear spaces, also discuss another ideas in CAT(0) space as considers corresponding uniformly convex in linear spaces.

3. **Main results**
Def.(3.1)[11]: Let \((M,d)\) be geodesic space and \(N,Q \subseteq M\), Let \(A: N \to Q\) be a mapping, \(x\) in \(N\) is called best proximity for \(A\) if \(d(x,A(x)) = d(N,Q)\) where \(d(N,Q) = \inf\{d(x,y): x \in N, y \in Q\}\).

And there are two subset \(N_0\) and \(Q_0\) from \(N\) and \(Q\) resp. defined as
\[N_0 = \{x \in N: d(x,y) = d(N,Q)\ \text{for some} \ y \in Q\}\] and
\[Q_0 = \{y \in Q: d(x,y) = d(N,Q)\ \text{for some} \ x \in N\}\]

Clear that when \(N_0 \neq \emptyset\) implies \(Q_0 \neq \emptyset\) and if either \(N_0 = \emptyset\) or \(Q_0 = \emptyset\) then there is no best proximity, [1], [12] gave some required conditions to confirm noemptiness of \(N_0\) and \(Q_0\).

Remark (3.2):
Let \((M,d)\) be a CAT(0) space, Let \(N\) and \(Q\) be two subsets of \(M\). Let \(x_0 \in N_0\) then there exist a \(y_0 \in Q_0\) such that \(y_0 = P_Q(x_0)\)

Proof: let \(x_0 \in N_0\) then there exist \(y_0 \in Q_0\) and \(d(x_0,y_0) = d(N,Q)\) in the other hand
\[d(N,Q) \leq d(x_0, P_Q(x_0)) \leq d(x_0,y_0) = d(N,Q)\]

and it implies \(d(x_0, P_Q(x_0)) = d(x_0,y_0)\).

Def.(3.4):[1]
Let \((M,d)\) be geodesic space, \(N\) and \(Q\) be nonempty subsets of \(M\). A cyclic mapping \(A: N \cup Q \to N \cup Q\) is called cyclic contraction mapping if there exist \(\alpha \in (0,1)\)
\[d(A(x),A(y)) \leq \alpha d(x,y) + (1-\alpha) d(N,Q)\]  \hspace{1cm} (3.1)

Remark (3.5):
In equation (3.1) if we substitute \(d(x,y)\) instead \(d(N,Q)\) we get
\[d(A(x),A(y)) \leq \alpha d(x,y) + (1-\alpha)d(x,y)\]
\[\leq \alpha d(x,y) + (1-\alpha)d(x,y)\]
\[\leq d(x,y)\]

And the cyclic mapping \(A\) is called relatively nonexpansive.

Remark (3.6)
In definition above if \(N \cap Q \neq \emptyset\) \(d(N,Q) = 0\), this case return to Banach fixed point theorem.

Theorem (3.7):[1]
Let \((M,d)\) be a metric space, Let \(N\) and \(Q\) be two subsets of \(M\). let \(A: N \cup Q \to N \cup Q\) be a cyclic contraction mapping then for any \(x_0 \in N \cup Q\) we define \(x_{n+1} = A(x_n)\) \(n=0,1,2,...\) we have
\[d(x_n,x_{n+1}) \to d(N,Q)\].

Theorem (3.8):
Let \((M,d)\) be a complete metric space, Let \(N\) and \(Q\) be two subsets of \(M\). let \(A: N \cup Q \to N \cup Q\) be a cyclic contraction mapping, define \(x_{n+1} = A(x_n)\) then for any \(x_0 \in N \cup Q\)
we Suppose $x_{2n}$ (resp. $x_{2n+1}$) has a convergent subsequence in $N$ (resp. $Q$) there exist $x$ in $N$ (resp. $Q$) such that $d(x, A(x)) = d(N, Q)$.

Proof: Let $d_n = d(x_n, x_{n+1})$, by theorem 4 $\lim_{n\to\infty} d(x_n, x_{n+1}) = d(N, Q)$ also $d_{2n} = d(x_{2n}, x_{2n+1})$

is a subsequence from $d_n = d(x_n, x_{n+1})$ then $\lim_{n\to\infty} d(x_{2n}, x_{2n+1}) = d(N, Q)$.

Now let $x_{2n_k}$ is convergent subsequence of $x_{2n}$ and $\lim x_{2n_k} = x$, then we get

$$\lim_{n\to\infty} d(x_{2n_k}, x_{2n_k+1}) = \lim_{n\to\infty} d(x_{2n_k}, A(x_{2n_k})) = d(x, A(x)) = d(N, Q) \blacksquare$$

Def.: let $(M,d)$ be a metric space and $N \subset M$ is called sequentially compact if each bounded sequence has convergent subsequence.

Corollary (3.9)

Let $(M,d)$ be a metric space and let $N$ is a sequentially compact subset of $M$ and $Q$ is a closed subset of $M$ then there exist $x \in N$ such that $d(x, A(x)) = d(N, Q)$.

Proof: let $x_n$ be a sequence in $N$ such that $x_{n+1} = A(x_n)$, Then $(x_n)$ contains convergent subsequence $(x_{n_k})$, It converges to $x$ in $N$, It conclude by theorem 5 above $d(x, A(x)) = d(N, Q)$ \blacksquare

Theorem (3.10): Let $(M,d)$ be a CAT(0) space, Let $N$ be nonempty convex subsets of $M$ and $Q$ be subset of $M$. Let $A : N \cup Q \to N \cup Q$ be a cyclic contraction mapping then for any $x_0 \in N \cup Q$ we define $x_{n+1} = A(x_n)$ for each $n \geq 0$ then $d(x_{2n+2}, x_{2n}) \to 0$.

Proof: Assume the contrary. Then for each $\varepsilon > 0$ there exist $k \geq 1$ there exist $n_k \geq k$ such that $d(x_{2n_k+2}, x_{n_k}) > \varepsilon$.

Let $\gamma \in (0,1)$ such that $\frac{\varepsilon}{\gamma} > d(N,Q)$ and choose $\varepsilon > 0$ and

$$\varepsilon \leq \min \left\{ \frac{\varepsilon}{\gamma} - d(N,Q), \frac{d(N,Q)\delta(\gamma)}{1-\delta(\gamma)} \right\}$$

By theorem 4 there exist $\square_1$ such that $d(x_{2n_k+2}, x_{2n_k+1}) \leq d(N,Q) + \varepsilon$ for all $n_k \geq \square_1$ and there exist $\square_2$ such that $d(x_{2n_k}, x_{2n_k+1}) \leq d(N,Q) + \varepsilon$ for all $n_k \geq \square_2$.

Let $\square = \max \{\square_1, \square_2\}$, by the convexity of $N$ we get $(\frac{1}{2}x_{2n_k+2} \oplus \frac{1}{2}x_{n_k}) \in N$ and the uniform convexity of $M$ we get

$$d(\frac{1}{2}x_{2n_k+2} \oplus \frac{1}{2}x_{n_k}, x_{2n_k+1}) \leq (1 - \delta(\frac{\varepsilon}{d(N,Q) + \varepsilon}))d(N,Q) + \varepsilon$$

But $\delta$ is increasing and $\varepsilon$ is an arbitrary constant, This leads to

$$\varepsilon \leq d(\frac{1}{2}x_{2n_k+2} \oplus \frac{1}{2}x_{n_k}, x_{2n_k+1}) \leq d(N,Q) \text{ for all } n_k \geq \square$$

And it leads to contradiction. In the same manner we prove $d(x_{2n+3}, x_{2n+1}) \to 0$.
Corollary in theorem 6 above the sequences \((x_{n+1})\) and \((x_{n+2})\) are Cauchy sequence.

**Theorem (3.11):**

Let \((M,d)\) be a CAT(0) space, Let \(N\) and \(Q\) be two nonempty closed and convex subsets of \(M\).

Let \((x_{n})\) and \((y_{n})\) be two sequences in \(N\) and let \((z_{n})\) be sequence in \(Q\) satisfying the following

\[
d(x_{n},z_{n}) \to d(N,Q)
\]

\[
d(y_{n},z_{n}) \to d(N,Q)
\]

Then \(d(x_{n},y_{n}) \to 0\).

Proof: Assume \(d(x_{n},y_{n})\) don’t converges to 0, then for each \(\varepsilon > 0\) and \(k \in \mathbb{N}\), then there exist \(n_{k},m_{k} \geq k\) such that \(d(x_{m_{k}},y_{n_{k}}) > \varepsilon_{0}\).

Let \(\gamma \in (0,1)\) such that \(\varepsilon_{0} > d(N,Q)\) and choose \(\varepsilon > 0\) and

\[
\varepsilon \leq \min \left\{ \frac{\varepsilon_{0}}{\gamma} - d(N,Q), \frac{d(N,Q)\delta(\gamma)}{1-\delta(\gamma)} \right\}.
\]

by hypothesis \(d(y_{n},z_{n}) \to d(N,Q)\) then there exist \(\varepsilon > 0\) such that for all \(m_{k},n_{k} > \varepsilon\) we get

\[
d(x_{m_{k}},z_{m_{k}}) \leq d(N,Q) + \varepsilon,
\]

Also there exist \(\varepsilon > 0\) such that

\[
d(y_{n_{k}},z_{n_{k}}) \leq d(N,Q) + \varepsilon \quad \text{for all} \; n_{k},m_{k} > \varepsilon.
\]

Choose \(\varepsilon = \max\{\varepsilon_{0},\varepsilon\}\), Moreover by convexity of \(N\), then

\[
\frac{1}{2}x_{n_{k}} + \frac{1}{2}y_{m_{k}} \in N.
\]

Therefore, CAT(0) space is uniformly convex we get.

\[
d(x_{m_{k}},y_{n_{k}}) > \varepsilon_{0}, \quad d(x_{m_{k}},z_{m_{k}}) \leq d(N,Q) + \varepsilon, \quad d(y_{n_{k}},z_{n_{k}}) \leq d(N,Q) + \varepsilon
\]

and \(\frac{1}{2}x_{m_{k}} + \frac{1}{2}y_{n_{k}} \in N\), We get

\[
d(\frac{1}{2}x_{m_{k}} + \frac{1}{2}y_{n_{k}},z_{n_{k}}) \leq (1-\delta(\frac{\varepsilon_{0}}{d(N,Q) + \varepsilon}))d(N,Q) + \varepsilon.
\]

But \(\delta\) is increasing and \(\varepsilon\) is an arbitrary constant, This leads to

\[
d(\frac{1}{2}x_{m_{k}} + \frac{1}{2}y_{n_{k}},z_{n_{k}}) \leq d(N,Q)
\]

Which is a contradiction and it concludes \(d(x_{n},y_{n}) \to 0\).

**Corollary (3.12):**

if \(x\) is best proximity of the cyclic contraction mapping \(A\) then \(d(x,A^{x}(x)) \to 0\).

Proof: \(x\) is best proximity then \(d(x,A(x)) = d(N,Q)\) and by remark 1 we get

\[
d(A^{x}(x),A(x)) \leq d(x,A(x)).
\]

By theorem 7 we get \(d(x,A^{x}(x)) \to 0\).

**Theorem (3.13):**
Let \((M,d)\) be a metric space, let \(N\) and \(Q\) be two subsets of \(M\). Let \(A : N \cup Q \to N \cup Q\) be a cyclic contraction mapping then for any \(x_0 \in N \cup Q\) we define \(x_{n+1} = A(x_n)\) \(\forall n \geq 0\) then for each \(n \geq 0\) there exist \(\bar{x} \in \bar{Q}\) such that for all \(m > n \geq \bar{x}\) \(d(x_{2m}, x_{2n+1}) \leq d(N, Q) + \varepsilon\).

Proof: assume the contrary, then \(\exists \varepsilon\) such that \(\forall k \geq 1\) there is \(m_k\) such that \(m_k > n_k \geq k\) smallest integer greater than \(n_k\) satisfying
\[
d(x_{2m_k}, x_{2n_k+1}) \geq d(N, Q) + \varepsilon \quad (3.2)
\]
And
\[
d(x_{2(m_k-1)}, x_{2n_k+1}) < d(N, Q) + \varepsilon \quad (3.3)
\]
From equation (3.2) by using triangle inequality we get
\[
d(N, Q) + \varepsilon \leq d(x_{2m_k}, x_{2n_k+1})
\]
\[
\leq d(x_{2m_k}, x_{2m_k-2}) + d(x_{2m_k-2}, x_{2n_k+1})
\]
\[
\leq d(x_{2m_k}, x_{2m_k-2}) + d(N, Q) + \varepsilon
\]
But \(d(x_{2m_k}, x_{2m_k-2}) \to 0\) it follows from theorem 7 and we get
\[
\lim_{k \to \infty} d(x_{2m_k}, x_{2n_k+1}) = d(N, Q) + \varepsilon \quad (3.4)
\]
By apply the triangle inequality we get
\[
d(N, Q) + \varepsilon \leq d(x_{2m_k}, x_{2n_k+1})
\]
\[
\leq d(x_{2m_k}, x_{2n_k+2}) + d(x_{2n_k+2}, x_{2n_k+3}) + d(x_{2n_k+3}, x_{2n_k+1})
\]
But \(d(x_{2n_k+2}, x_{2n_k+3}) = d(A(x_{2n_k+1}, A(x_{2n_k+2})) \leq d(x_{2n_k+1}, x_{2n_k+2})\) it follows from remark 2.

therefore the inequality (3.5) becomes
\[ d(x_{2m_k}, x_{2n_k+1}) \leq d(x_{2m_k}, x_{2n_k+3}) + d(x_{2n_k+1}, x_{2n_k+3}) + d(x_{2n_k+3}, x_{2n_k+1}) \]

\[ \leq d(x_{2m_k}, x_{2n_k+3}) + k^2 d(x_{2n_k+1}, x_{2n_k}) + (1-k^2)d(N, Q) + d(x_{2n_k+3}, x_{2n_k+1}) \]

\[ \leq d(x_{2m_k}, x_{2n_k+3}) + k^2 d(N, Q) + (1-k^2)d(N, Q) + d(x_{2n_k+3}, x_{2n_k+1}) \]

\[ \leq d(x_{2m_k}, x_{2n_k+3}) + k^2 d(N, Q) + d(x_{2n_k+3}, x_{2n_k+1}) \]

\[ \leq k^2 d(N, Q) \]

Because \( d(x_{2m_k}, x_{2n_k+3}) \to 0 \) also \( d(x_{2n_k+3}, x_{2n_k+1}) \to 0 \) it follows from theorem 7, and we get contradiction, therefore \( d(x_{2m}, x_{2n+1}) \leq d(N, Q) + \varepsilon \).

**Theorem (3.14):**

Let \((M, d)\) be a CAT(0) space, let \(N\) and \(Q\) be nonempty convex subsets of \(M\). Let \(A : N \cup Q \to N \cup Q\) be a cyclic contraction mapping, for \(x_0 \in N \cup Q\) we define \(x_{n+1} = A(x_n) \) \(\forall n \geq 1\) then \((x_{2n})\) and \((x_{2n+1})\) are Cauchy sequences.

Proof: let \((x_{2n})\) be not Cauchy sequence, then there exist \(\varepsilon_0 \geq 0\) such that for each \(k \geq 1\), there exist \(m_k, n_k \geq k\) such that \(d(x_{2m_k}, x_{2n_k}) \geq \varepsilon_0\).

Let \(\gamma \in (0,1)\) such that \(\varepsilon_0 > d(N, Q)\) and choose \(\varepsilon > 0\) and

\[ \varepsilon \leq \min \left\{ \frac{\varepsilon_0}{\gamma}, \frac{d(N, Q)\delta(\gamma)}{1 - \delta(\gamma)} \right\}. \]

Now by theorem 9, there exist \(\square_1\) such that

\[ d(x_{2m_k}, x_{2n_k+1}) \leq d(N, Q) + \varepsilon \quad (3.6) \]

for all \(m_k \geq \square_1\). By theorem 4, there exist \(\square_2\) such that

\[ d(x_{2n_k}, x_{2n_k+1}) \leq d(N, Q) + \varepsilon \quad (3.7) \]

for all \(n_k \geq \square_2\). Let \(\square = \max \{\square_1, \square_2\}\); By the uniform convexity of space \(M\) and from (3.6), (3.7) we get

\[ d\left(\frac{1}{2}x_{2m_k}, \frac{1}{2}x_{2n_k+1}\right) \leq (1 - \delta(\frac{\varepsilon_0}{d(N, Q) + \varepsilon}))(d(N, Q) + \varepsilon) \]
For all $m_k, n_k \geq 1$, but $\varepsilon$ is an arbitrary and $\delta$ is an increasing function, we get

$$d\left(\frac{1}{2}x_{2n+1} + \frac{1}{2}x_{2n+2}, x_{2n+1}\right) \leq d(N, Q)$$

Its contradiction and we get $(x_{2n})$ and $(x_{2n+1})$ are Cauchy sequence.

Finally, The main result in this paper, prove the uniqueness and existence of best proximity for cyclic contraction in CAT(0) space.

**Theorem (3.15)**

Let $(M, d)$ be a complete CAT(0) space, Let $N$ and $Q$ be nonempty closed and convex subsets of $M$. Let $A : N \cup Q \to N \cup Q$ be a cyclic contraction mapping. for $x_0 \in N \cup Q$ we define $x_{n+1} = A(x_n) \quad \forall \ n \geq 1$. then there exist a unique $x \in N$ such that $x_{2n} \to x$, $A^2(x) = x$ and $d(x, A(x)) = d(N, Q)$

Proof: by theorem 9 $(x_{2n})$ is a Cauchy sequence in closed subset $N$, then $x_{2n} \to x$ and by theorem 5 $x$ is a best proximity.

To prove $A^2(x) = x$, $d(x, A(x)) = d(N, Q)$ and $d(A(x), A^2(x)) \leq d(x, A(x))$ and it implies $d(A(x), A^2(x)) = d(N, Q)$, then by theorem 7 we get $A^2(x) = x$.

To prove the uniqueness of best proximity, Let $y$ a best proximity for mapping $A$, then $d(y, A(y)) = d(N, Q)$ and $A^2(y) = y$.

d$(A(x), y) = d(A(x), A^2(y)) \leq d(x, A(y))$ and $d(A(y), x) \leq d(y, A(x))$, then we get $d(A(y), x) = d(y, A(x))$.

Now to show that $d(A(y), x) = d(y, A(x)) = d(N, Q)$. Assume $d(A(y), x) < d(N, Q)$,

$$d(N, Q) < d(A(y), x) = d(A(y), A^2(x)) \leq kd(y, A(x)) + (1 - k)d(N, Q)$$

Then

$$\leq kd(N, Q) + (1 - k)d(N, Q) = d(N, Q)$$

A contradiction and get $d(A(y), x) = d(y, A(x)) = d(N, Q)$.

To prove uniqueness, it follows form convexity of $N$ that $\frac{1}{2}x \oplus \frac{1}{2}y \in N$ and from strict convexity of $M$, $d(x, A(x)) = d(y, A(x)) = d(N, Q)$ then we get

$$d\left(\frac{1}{2}x \oplus \frac{1}{2}y, A(x)\right) < d(N, Q)$$

A contradiction and we get $x = y$.

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