Entropy of Convex Functions on $\mathbb{R}^d$

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Abstract Let $\Omega$ be a bounded closed convex set in $\mathbb{R}^d$ with nonempty interior, and let $C_r(\Omega)$ be the class of convex functions on $\Omega$ with $L^r$-norm bounded by 1. We obtain sharp estimates of the $\varepsilon$-entropy of $C_r(\Omega)$ under $L^p(\Omega)$ metrics, $1 \leq p < r \leq \infty$. In particular, the results imply that the universal lower bound $\varepsilon^{-d/2}$ is also an upper bound for all $d$-polytopes, and the universal upper bound of $\varepsilon^{-\left(\frac{d-1}{2}\right)\frac{pr}{r-p}}$ for $p > \frac{dr}{d+(d-1)r}$ is attained by the closed unit ball. While a general convex body can be approximated by inscribed polytopes, the entropy rate does not carry over to the limiting body. Our results have applications to questions concerning rates of convergence of nonparametric estimators of high-dimensional shape-constrained functions.

Keywords Metric entropy · Bracketing entropy · Convex function · Polytope · Simplicial approximation

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1 Introduction

Given a set $T$ in a metric space $(X, \rho)$, the $\varepsilon$-covering number of $T$, denoted by $N(\varepsilon, T, \rho)$, is the minimum number of closed balls of radius $\varepsilon$ in $(X, \rho)$ needed to cover $T$. It is a measurement of the massiveness of $T$ at a fixed resolution $\varepsilon$. With a varying radius, the covering number quantitatively gauges the geometric complexity of $T$. Over half a century ago, Kolmogorov and Tihomirov [16] put the study of the logarithm of covering number with varying radius, or metric entropy, at the center stage. Since then, metric entropy has come to play an increasingly important role in a wide range of problems in mathematics, including approximation theory, probability theory, information theory, and statistics. In particular, it is now widely understood that accurate bounds for metric entropy determine optimal rates of convergence in estimation problems in statistics; see, for example [1,2,15,17,22].

The focus in this paper is on metric entropy for various classes of convex functions. In particular, we study metric entropy of the classes $C^r(\Omega)$ of all real-valued convex functions $f$ on a closed convex body $\Omega$ in $\mathbb{R}^d$ having $L^r$-norm bounded by 1. These convex functions are of special importance not only because they are basic classes of functions, but also because they appear so commonly in applications. For example, exponential functions that frequently appear in statistical density estimation are convex. In these statistical applications, one often also needs to know the so-called bracketing entropy, that is, the logarithm of the minimum number $N(\varepsilon, C^\infty(\Omega), \|\cdot\|_p)$ of $\varepsilon$-brackets

$$[f, \overline{f}] := \left\{ g \in C^\infty(\Omega) \mid f \leq g \leq \overline{f}, \|\overline{f} - f\|_p \leq \varepsilon, \right\},$$

needed to cover $C^\infty(\Omega)$. It is known and easy to see that

$$N(\varepsilon, C^\infty(\Omega), \|\cdot\|_p) \leq N(\varepsilon, C^\infty(\Omega), \|\cdot\|_p).$$

Before we present our results, let us briefly review the history of metric entropy bounds for convex functions and several recent uses of such bounds.

For the class $C$ of all compact convex subsets of a fixed bounded subset $\Omega$ of $\mathbb{R}^d$ endowed with the Hausdorff metric $h$, Bronshtein [3] obtained both upper and lower bounds of the order $\varepsilon^{-(d-1)/2}$ for the metric entropy $\log N(\varepsilon, C, h)$. In the same paper, Bronshtein also obtained bounds of the order $\varepsilon^{-d/2}$ for $\log N(\varepsilon, \mathcal{F}, \|\cdot\|_\infty)$, where $\mathcal{F}$ is the class of all convex functions $f$ defined on a fixed convex body $\Omega$ in $\mathbb{R}^d$ satisfying a (uniform) Lipschitz condition: $|f(y) - f(x)| \leq L\|y - x\|$ for all $x, y \in \Omega$; here $\|\cdot\|_\infty$ denotes the supremum norm. These bounds improved earlier results of Dudley [7] and are incorporated in [8–10].

In the case $d = 1$, Gao [11] removed the requirement for the uniform Lipschitz condition and obtained sharp bounds for $\log N(\varepsilon, C^\infty([a, b]), L_2)$, and, in fact, provided upper and lower bounds of the order $\varepsilon^{-1/k}$ for the “$k$-monotone” classes

$$\mathcal{M}_k([a, b]) := \left\{ f : [a, b] \to [-1, 1] \mid (-1)^i f^{(i)}(x) \geq 0, f(a) = 0 \quad 1 \leq i \leq k, \quad x \in [a, b] \right\}.$$
When $k = 2$, it reduces to the convex case. These results were further extended for $\log N(\varepsilon, \mathcal{M}_k([a, b]), L_p)$ and $\log N(\varepsilon, \mathcal{M}_k([a, b]), L_p)$ in Gao and Wellner [12]. The results for $\log N(\varepsilon, C_\infty([a, b]), L_p)$ were also obtained independently by Dryanov [6].

For the case $d > 1$, Guntuboyina and Sen [14] extended the result for $\log N(\varepsilon, C_\infty([a, b]^d), L_p)$. More recently, Guntuboyina [13] relaxed the restriction on the uniform norm by considering classes $C_r([a, b]^d)$. He showed that the metric entropy $\log N(\varepsilon, C_r([a, b]^d), L_p)$ are infinite for $p \geq r$ (since these classes are not precompact in $L_p$ with $p \geq r$), and are of the order $\varepsilon^{-d/2}$ if $p < r$.

In this paper, we study the metric entropy of $C_r(\Omega)$ under $L_p$-norm, $1 \leq p < r$, for all compact convex sets $\Omega$ in $\mathbb{R}^d$ with nonempty interior. It turns out that the rate of metric entropy of $C_r(\Omega)$ heavily depends on the shape of the convex domain $\Omega$. This heavy dependence on shapes makes the problem both more interesting and more challenging.

Now, we turn to the statements or our results.

We first show that $\varepsilon^{-d/2}$ is the general lower bound for the metric entropy rate, and if $\Omega$ is a closed convex polytope, then $\varepsilon^{-d/2}$ is also the upper bound. More precisely, we will prove the following theorem. Throughout the rest of the paper, $|\Omega|$ stands for the Lebesgue measure of $\Omega$.

**Theorem 1** Let $\Omega$ be a compact convex set in $\mathbb{R}^d$ with nonempty interior.

(i) There exists a constant $c_1$ depending only on $d$ such that for all $\varepsilon > 0$,

$$\log N(\varepsilon, C_r(\Omega), \| \cdot \|_p) \geq c_1 |\Omega| \frac{d}{2p} - \frac{d}{2} \varepsilon^{-d/2}.$$  

(ii) If $\Omega$ can be triangulated into $m$ simplices of dimension $d$, then for any $1 \leq p < r$, there exists a constant $C_1$ depending on $p, d, r$, such that for any $\varepsilon > 0$,

$$\log N(\varepsilon, C_r(\Omega), \| \cdot \|_p) \leq C_1 m |\Omega| \frac{d}{2p} - \frac{d}{2} \varepsilon^{-d/2}.$$  

Consequently, if $\Omega$ is a convex polytope with $v$ extreme points, then we can choose $m = O(v^{[d/2]})$. When $r = \infty$, the same inequality holds for bracketing entropy.

In view of the fact that a general compact convex set can be approximated by convex sets with finitely many extreme points, one might guess that the rate $\varepsilon^{-d/2}$ holds for general compact convex sets in $\mathbb{R}^d$ with nonempty interior. This, however, is not the case. This is because the upper bound increases as $m$ increases. This dependence on $m$ is important. It enables us to establish upper bounds for general bounded convex sets. For that, we need the following definition.

**Definition 1** Let $\Omega$ be a bounded closed convex set in $\mathbb{R}^d$ with nonempty interior. A sequence of (nondegenerate) $d$-simplices $\mathcal{D} = \{D_1, D_2, \ldots\}$ is called a simplicial approximation sequence for $\Omega$ if $D_i \subset \Omega$ for all $i \in \mathbb{N}$ and $D_i \cap D_j = \emptyset$ for all $i \neq j$ (where $D^?$ denotes the interior of $D$). For $t \in (0, 1)$, we define
Fig. 1  Left A simplicial approximation sequence for the unit disk in \( \mathbb{R}^2 \). Right The graph of the function \( S_D(t, \Omega) \) in black, compared with that of the function \( f(t) = \frac{2\pi}{\sqrt{6}} t^{-1/2} - 2 \) in blue (color figure online)

\[
S_D(t, \Omega) = \min\{ j \in \mathbb{N} : |\Omega \setminus \bigcup_{i \leq j} D_i| \leq t|\Omega| \},
\]

and we call \( S_D(t, \Omega) \) the simplicial approximation number of \( \Omega \) according to \( D \).

**Example 1** As an example to illustrate simplicial approximation sequences and simplicial approximation, we consider the case when \( \Omega \) is the closed unit disk in \( \mathbb{R}^2 \). We choose \( D_1 \) as an inscribed equilateral triangle. For each edge of \( D_1 \), we build an isosceles triangle with apex on the short arc opposite to the edge. We denote these three isosceles triangles by \( D_2, D_3, D_4 \). The union of \( D_1, \ldots, D_4 \) is a regular hexagon inscribed in the disk. Now, on each edge of the hexagon, we build an isosceles triangle with apex on the short arc opposite to the edge, and denote these six isosceles triangles by \( D_5, D_6, \ldots, D_{10} \). The union of \( D_1, D_2, \ldots, D_{10} \) is a regular 12-gon inscribed in the disk (see Fig. 1). Continuing this process, we obtain a simplicial approximation sequence, \( D = \{ D_1, D_2, D_3, \ldots \} \). It is not difficult to see that for all \( n = 0, 1, \ldots \) and \( k = 1, 2, \ldots, 3 \cdot 2^n \), \( S_D(t, \Omega) = 1 \) if \( 1 - \frac{3}{2\pi} \sin \frac{\pi}{3} \cdot 2^{n-1} \leq t \leq 1 \), and

\[
S_D(t, \Omega) = 3 \cdot 2^n - 2 + k, \quad \text{for } a_n - k b_n \leq t < a_n - (k - 1) b_n,
\]

where

\[
a_n = 1 - \frac{3 \cdot 2^{n-1}}{\pi} \sin \frac{\pi}{3} \cdot 2^{n-1}, \quad b_n = \frac{1}{\pi} \left( 1 - \cos \frac{\pi}{3} \cdot 2^n \right) \sin \frac{\pi}{3} \cdot 2^n,
\]

from which we conclude that \( S_D(t, \Omega) = O(t^{-1/2}) \).

Now we can state the following theorem.

**Theorem 2** Let \( \Omega \) be a compact convex set in \( \mathbb{R}^d \) with nonempty interior. Let \( C_r(\Omega) \) be the set of convex functions on \( \Omega \) whose \( L^r(\Omega) \)-norms are bounded by 1. Then there exists a constant \( C \) depending only on \( d, p, \) and \( r \), such that for any \( 0 < \varepsilon < 1 \) and any simplicial approximation sequence \( D \),
\[
\log N(\varepsilon, |\Omega|^\frac{1}{p-1}, C_r(\Omega), \| \cdot \|_p) \leq C \int_{\delta(\varepsilon)}^{1} \frac{S_D(t, \Omega)}{t} \, dt \\
+ C \left( \int_{\delta(\varepsilon)}^{1} \left( \frac{S_D(t, \Omega)}{t} \right)^{\beta} \, dt \right)^{1/\beta} \cdot \varepsilon^{-d/2},
\]

where \(\delta(\varepsilon) = 2^{-2} \frac{r}{r-p} \varepsilon^{\frac{r}{r-p}}\) and \(\beta = \frac{2pr}{2pr+(r-p)d}\). When \(r = \infty\), the same inequality holds for bracketing entropy.

By specifically constructing a simplicial approximation for the ball, we show that Theorem 2 implies the following theorem. Our proof of the theorem also provides a general scheme of constructing simplicial approximations for a given convex set.

**Theorem 3** If \(\Omega\) is a compact convex set contained in the closed unit ball in \(\mathbb{R}^d\), then there exists a constant \(C\) depending only on \(r, p,\) and \(d\) such that for all \(1 \leq p < r \leq \infty\) and all \(0 < \varepsilon < 1\),

\[
\log N(\varepsilon, C_r(\Omega), \| \cdot \|_p) \leq \begin{cases} 
C \varepsilon^{-(d-\frac{1}{2})} \frac{pr}{r-p} & \text{if } p > \frac{dr}{d+(d-1)r}, \\
C \varepsilon^{-d/2} |\log \varepsilon|^{1+\frac{r-p+2d}{2pr}} & \text{if } p = \frac{dr}{d+(d-1)r}, \\
C \varepsilon^{-d/2} & \text{if } p < \frac{dr}{d+(d-1)r}.
\end{cases}
\]

When \(r = \infty\), the same inequality holds for bracketing entropy.

The following theorem implies the sharpness of Theorem 3 at least for the case when \(\Omega\) is the closed unit ball in \(\mathbb{R}^d\), \(r = \infty\) and \(p \neq \frac{d}{d-1}\).

**Theorem 4** If \(\Omega\) is the closed unit ball in \(\mathbb{R}^d\), then there exists a constant \(c_2\) dependent only on \(d\) and \(p\) such that for all \(0 < \varepsilon < 1\),

\[
\log N(\varepsilon, C_\infty(\Omega), \| \cdot \|_p) \geq c_2 \varepsilon^{-\gamma},
\]

where \(\gamma = \max\{(d-1)p/2, d/2\}\).

**Remark 1** Because Theorem 3 is built upon Theorem 2, which is again based on Theorem 1 (ii), Theorem 4 indicates that in some cases, the linear dependence on \(m\) in the upper bound in Theorem 1 (ii) is optimal. A more concrete example is the regular \((m+2)\)-gon in \(\mathbb{R}^2\). By the end of this paper, we will show that for this \(\Omega\), there exists a constant \(c_2\) such that for \(0 < \varepsilon \leq \frac{1}{4} m^{-2}\),

\[
\log N(\varepsilon, C_r(\Omega), \| \cdot \|_p) \geq c_2 m \varepsilon^{-1}.
\]  

(1)

In general, however, the lower bound should depend on the geometry of the set \(\Omega\) and cannot be simply captured by the minimum number of \(d\)-simplices required in a triangulation.
2 Proofs

2.1 Scaling

In this subsection, we prove two lemmas, through which we can reduce a problem on an arbitrary closed convex set with nonempty interior to a problem on a closed convex set contained in $[0, 1]^d$ with volume at least $1/d!$.

**Lemma 1** (Boxing a Convex Set) Every compact convex set $\Omega$ in $\mathbb{R}^d$ with a nonempty interior can be enclosed in a closed rectangular box of volume $d!|\Omega|$ and contains a convex polytope of at most $2d$ vertices and volume at least $|\Omega|/d!$, where $|\Omega|$ stands for the Lebesgue measure of $\Omega$.

**Proof** We use induction on $d$ to show that we can find positive numbers $h_1, h_2, \ldots, h_d$ such that $\Omega$ is contained in a rectangular box of size $h_1 \times h_2 \times \cdots \times h_d$ and contains a convex polytope of at most $2d$ vertices with volume at least $\frac{1}{d!} \cdot h_1 \times h_2 \times \cdots \times h_d$.

The statement is trivial if $d = 1$. Suppose the statement is true for $d = k$. Consider the case $d = k + 1$. Let $h_{k+1} = \text{diam}(\Omega)$. Choose $x, y \in \Omega$ so that $\|x - y\|_2 = h_{k+1}$. Let $P_x^\perp(\Omega)$ be the projection of $\Omega$ onto the affine hyperplane that contains $x$ and is orthogonal to $x - y$. Since $P_x^\perp(\Omega) \subset \mathbb{R}^k$ is a $k$-dimensional compact convex set with nonempty interior, by the induction hypothesis, we can find positive numbers $h_1, h_2, \ldots, h_k$ such that $P_x^\perp(\Omega)$ is contained in a rectangular box $R_k$ of size $h_1 \times h_2 \times \cdots \times h_d$ and contains a convex polytope $T_k$ of at most $2k$ vertices with volume at least $\frac{1}{k!} \cdot h_1 \times h_2 \times \cdots \times h_k$. If we let $[x, y]$ be the line segment between $x$ and $y$, then $\Omega$ is clearly contained in the rectangular box $R_k \times [x, y]$ of size $h_1 \times h_2 \times \cdots \times h_{k+1}$.

To show that $\Omega$ contains a convex polytope of at most $2(k + 1)$ vertices with volume at least $\frac{1}{(k+1)!} \cdot h_1 \times h_2 \times \cdots \times h_{k+1}$, we let $u_1, u_2, \ldots, u_m, m \leq 2k$, be the vertices of the convex polytope $T_k$. Clearly, the convex hull $U$ of $\{x, y, u_1, u_2, \ldots, u_m\}$ has volume $|U| \geq \frac{1}{(k+1)!} \cdot h_1 \times h_2 \times \cdots \times h_{k+1}$.

For each $1 \leq i \leq m$, there exists $z_i \in \Omega$ such that $P_x^\perp z_i = u_i$. Because $\Omega$ is convex, it contains the convex hull of $\{x, y, z_1, z_2, \ldots, z_m\}$. Denote this convex hull by $T_{k+1}$. Then $T_{k+1}$ has at most $2(k + 1)$ vertices. Note that the volume of $T_{k+1}$ is at least as large as $|U|$. Indeed, for any unit vector $u$ perpendicular to $x - y$, consider the half-line in the direction of $u$ starting from $x$. Suppose the half-line intersects the boundary $U$ at $w(u)$. Choose $z(u) \in T_{k+1}$ such that $P_x^\perp z(u) = w(u)$. Clearly, the area of $\Delta xyz(u)$ is the same as that of $\Delta xwy(u)$, which equals $\frac{1}{2} h_{k+1} \|x - w(u)\|_2$. Let $\sigma_{k-1}$ be the $(k - 1)$-dimensional spherical measure on the $(k - 1)$-dimensional unit sphere $S^{k-1}$. By using a cylindrical system to compute the volume of $T_{k+1}$, we have

$$|T_{k+1}| \geq \int_{S^{k-1}} \text{area}(\Delta xyz(u)) \, d\sigma_{k-1}(u) = \int_{S^{k-1}} \text{area}(\Delta xwy(u)) \, d\sigma_{k-1}(u) = |U|.$$  

Hence, $|T_{k+1}| \geq \frac{1}{(k+1)!} \cdot h_1 \times h_2 \times \cdots \times h_{k+1}$. This proves the case $d = k + 1$ and thus the statement at the beginning of the proof, which implies that the volume of $\Omega$ is at least $\frac{1}{d!}$ of that of the rectangular box. Hence, the volume of the rectangular box is bounded by $d!/|\Omega|$.

$\square$
Lemma 2 (Scaling) Let $\Omega$ be a bounded closed convex set contained in a closed rectangular box $R$ with volume $|R|$, and let $T$ be any affine transform that maps $R$ onto $[0, 1]^d$. Then for all $1 \leq p < r < \infty$ and $\varepsilon > 0$,

$$N(\varepsilon, C_r(\Omega), \| \cdot \|_{L^p(\Omega)}) = N(|R|^{1/r} \cdot \frac{1}{\varepsilon}, C_r(T(\Omega)), \| \cdot \|_{L^p(T(\Omega))}).$$

(2)

Similarly, for all $1 \leq p < \infty$ and $\varepsilon > 0$,

$$N_1(\varepsilon, C_\infty(\Omega), \| \cdot \|_{L^p(\Omega)}) = N(|R|^{-\frac{1}{\varepsilon}}, C_\infty(T(\Omega)), \| \cdot \|_{L^p(T(\Omega))}).$$

(3)

Proof Let $f \in C_r(T(\Omega))$. Then $|R|^{-1/r} f \circ T \in C_r(\Omega)$, since

$$\int_{\Omega} |f|^-T \circ T d\lambda = \int_{\Omega} |f|^r d\lambda = |R| \int_{\Omega} |f|^r d\lambda \leq |R|,$$

where $\lambda$ is Lebesgue measure on $\mathbb{R}^d$. Now let $f_1, \ldots, f_N$ be an $L^p(\Omega)$ $\epsilon$-net for $C_r(\Omega)$. Then, for $f \in C_r(T(\Omega))$, we have

$$\left( \int_{T(\Omega)} |f - |R|^{1/r} f_i \circ T^{-1}|^p d\lambda \right)^{1/p}$$

$$= \left( \int_{\Omega} |f \circ T - |R|^{1/r} f_i|^p |T| d\lambda \right)^{1/p}$$

$$= \left( \int_{\Omega} |R|^{1/r} \left( |R|^{-1/r} f \circ T - f_i \right) |^p |R|^{-1} d\lambda \right)^{1/p}$$

$$= |R|^{1/r-1/p} \left( \int_{\Omega} |R|^{-1/r} f \circ T - f_i |^p d\lambda \right)^{1/p},$$

and since $|R|^{-1/r} f \circ T \in C_r(\Omega)$, for some $i \in \{1, \ldots, N\}$ the last display is bounded above by $|R|^{1/r-1/p} \varepsilon$. Thus given an $L^p(\Omega)$ $\varepsilon$-net for $C_r(\Omega)$, we have constructed an $L^p(T(\Omega))$ $|R|^{1/p-1/r} \varepsilon$-net for $C_r(T(\Omega))$. It follows that

$$N(|R|^{1/r-1/p} \varepsilon, C_r(T(\Omega)), \| \cdot \|_{L^p(T(\Omega))}) \leq N(\varepsilon, C_r(\Omega), \| \cdot \|_{L^p(\Omega)}).$$

(4)

By a similar argument, we find that

$$N(|R|^{1/r-1/p} \varepsilon, C_r(T(\Omega)), \| \cdot \|_{L^p(T(\Omega))}) \geq N(\varepsilon, C_r(\Omega), \| \cdot \|_{L^p(\Omega)}),$$

(5)

and hence the equality (2) holds.

To prove (3), first note that if $f \in C_\infty(T(\Omega))$, then $\sup_{\Omega} |f \circ T| = \sup_{T(\Omega)} |f| \leq 1$, so $f \circ T \in C_\infty(\Omega)$. Then suppose that $\left[ f_j, \overline{T_i} \right]$, $1 \leq i \leq N$, are $L^p(\Omega)$ brackets of size $\varepsilon$ for $C_\infty(\Omega)$. Then $\left[ f_j \circ T^{-1}, \overline{T_i} \circ T^{-1} \right]$, $1 \leq i \leq N$, are $L^p(T(\Omega))$ brackets of size $|R|^{-1/r} \varepsilon$ for $C_\infty(T(\Omega))$. To see this, note that for some $i \in \{1, \ldots, N\}$,
\[ f_i(x) \leq f \circ T(x) \leq \overline{f}_i(x) \quad \text{for all } x \in \Omega, \]

and hence

\[ f_i \circ T^{-1}(y) \leq f(y) \leq \overline{f}_i \circ T^{-1}(y) \quad \text{for all } y \in T(\Omega). \]

Furthermore,

\[
\int_{T(\Omega)} |\overline{f}_i \circ T^{-1}(y) - f_i \circ T^{-1}(y)|^p d\lambda = \int_{\Omega} |\overline{f}_i - f_i|^p |T| d\lambda \\
= |R|^{-1} \int_{\Omega} |\overline{f}_i - f_i|^p d\lambda \leq \left( |R|^{-1/p} \varepsilon \right)^p .
\]

Thus

\[
N[|(R)|^{-1/p} \varepsilon, C_{\infty}(T(\Omega)), \| \cdot \|_{L^p(T(\Omega))}] \leq N[\varepsilon, C_{\infty}(\Omega), \| \cdot \|_{L^p(\Omega))}].
\]

A similar argument yields the reversed inequality, and hence (3) holds.

By combining Lemma 1 and Lemma 2, we have

\[
N(\varepsilon, C_r(\Omega), \| \cdot \|_{L^p(\Omega)}) \leq N((d!)^{\frac{1}{p} - \frac{1}{p} \cdot |\Omega|^{\frac{1}{p} - \frac{1}{p} \varepsilon}, C_r(T(\Omega)), \| \cdot \|_{L^p(T(\Omega)))} )
\]

and

\[
N[1](\varepsilon, C_{\infty}(\Omega), \| \cdot \|_{L^p(\Omega)}) \leq N[1](d!)^{-\frac{1}{p} \cdot |\Omega|^{-\frac{1}{p} \varepsilon}, C_{\infty}(T(\Omega)), \| \cdot \|_{L^p(T(\Omega))}).
\]

where \( T(\Omega) \) has volume at least \( 1/d! \) and is contained in \([0, 1]^d \).

### 2.2 Under Uniform Lipschitz

In this subsection, we recall that if we assume the functions in \( C_r(\Omega) \) are bounded and uniform Lipschitz, then the metric entropy estimate would follow from the following known results of Bronshtein [3].

**Lemma 3** (Bronshtein) Let \( \mathcal{K}(\rho) \) be the set of all closed convex sets contained in the closed Euclidean ball of radius \( \rho \) in \( \mathbb{R}^{d+1}, d \geq 1 \). Let \( h \) be the Hausdorff distance on \( \mathcal{K}(\rho) \). There exists a constant \( C_0 \) depending only on \( d \), such that for any \( 0 < \varepsilon < \rho \),

\[
\log N(\varepsilon, \mathcal{K}(\rho), h) \leq C_0(\rho \varepsilon^{-1})^{d/2}.
\]

**Lemma 4** Let \( \Omega \) be a closed convex set in \([0, 1]^d \), and let \( \mathcal{F}_\alpha(\Omega) \) be the class of convex functions on \( \Omega \) that are bounded by \( M \) and have Lipschitz constant bounded by \( \alpha \). Then for all \( \varepsilon < 2^{-1-1/p} \sqrt{(1+\alpha^2)(M^2+\alpha)} \),

\[ \varepsilon \]
\[
\log N_{\epsilon}(\mathcal{F}_\alpha(\Omega), \| \cdot \|_{L^p(\Omega)}) \leq 2^{-d} C_0 \{(1 + \alpha^2)(4M^2 + d)\}^{d/4} \epsilon^{-d/2},
\]
where \( C_0 \) is the same constant as in Lemma 3.

**Remark 2** Lemma 3 can be found in [3,9], or [21]. Lemma 2.7.8, page 163. Lemma 4 is also known for regular metric entropy. For example, it would follow from [21] Corollary 2.7.10, page 164. Because we deal with bracketing entropy, we include a proof here for the convenience of the reader.

**Proof** For each \( f \in \mathcal{F}_\alpha \), since \( \Omega \) is a closed and convex set and \( f \) is convex, the epigraph \( \text{epi}(f) := \{(x,t) : f(x) \leq t \leq M, \ x \in \Omega \} \), is a closed convex set contained in the closed Euclidean ball in \( \mathbb{R}^{d+1} \) with radius \( \sqrt{d/4 + M^2} \) and center at \((1/2, 1/2, \ldots, 1/2, 0)\).

On the other hand, for any \( x \in \Omega \), \( y \in \Omega \), and \( f, g \in \mathcal{F}_\alpha \),
\[
|f(x) - g(x)| \leq |f(x) - f(y)| + |f(y) - g(x)|
\leq \alpha \|x - y\|_2 + |f(y) - g(x)|
\leq \sqrt{1 + \alpha^2} \| (x, f(x)) - (y, f(y)) \|_2.
\]
Taking the infimum on \( y \in \Omega \) followed by the supremum on \( x \in \Omega \), we find that
\[
\|f - g\|_{\infty} \leq \sqrt{1 + \alpha^2} h(\text{epi}(f), \text{epi}(g)).
\]
Thus, by Lemma 3,
\[
\log N(\eta, C_{\infty}(\Omega), \| \cdot \|_{\infty}) \leq \log N \left( (1 + \alpha^2)^{-1/2} \eta, \mathcal{K} \left( \sqrt{M^2 + d/4} \right), h \right)
\leq C_0 \left\{ \sqrt{(1 + \alpha^2)(M^2 + d/4)\eta^{-1}} \right\}^{d/2}.
\]
Thus there exist \( N \leq \exp \left( C_0 \left\{ \sqrt{(1 + \alpha^2)(M^2 + d/4)\eta^{-1}} \right\}^{d/2} \right) \) functions \( f_1, \ldots, f_N \)
defined on \( \Omega \), such that for each \( f \in \mathcal{K} (\Omega) \), there exists some \( f_i, i \in \{1, 2, \ldots, N\} \),
such that \( |f(x) - f_i(x)| \leq \eta \) for all \( x \in \Omega \). For each \( i \in \{1, \ldots, N\} \), define
\[
\overline{f}_i(x) = \sup \{ f(x) : |f(x) - f_i(x)| \leq \eta, \ f \in \mathcal{F}_\alpha \};
\]
\[
\underline{f}_i(x) = \inf \{ f(x) : |f(x) - f_i(x)| \leq \eta, \ f \in \mathcal{F}_\alpha \}
\]
for each \( x \in \Omega \). Then we have
\[
\|\overline{f}_i - \underline{f}_i\|_{\infty} \leq 2\eta.
\]
In particular, this implies that for all \( 1 \leq p < \infty \),
\[
\int_{\Omega} |\overline{f}_i(x) - \underline{f}_i(x)|^p \ d\lambda(x) \leq (2\eta)^p.
\]
Letting $\varepsilon = 2\eta$, we find that

$$\log N_{[1]}(\varepsilon, C_\infty(\Omega), \| \cdot \|_p) \leq \log N(2\eta, C_\infty(\Omega), \| \cdot \|_{\infty})$$

$$\leq C_0 \left\{ \sqrt{(1 + \alpha^2)(M^2 + d/4)(2\eta)^{-1}} \right\}^{d/2}$$

$$= 2^{-d} C_0 ((1 + \alpha^2)(4M^2 + d))^{d/4} \varepsilon^{-d/2}.$$

\[\Box\]

### 2.3 Paring the Boundary

In this subsection, we show that if we pare off the boundary of $\Omega$ by $\delta$ and consider the set

$$\Omega_\delta = \left\{ x \in \Omega \mid \text{dist}(x, \partial\Omega) \geq \delta \right\},$$

then consider functions restricted to $\Omega_\delta$, the entropy can be estimated using Lemma 4. The details are proved in the following two lemmas.

**Lemma 5** Let $\Omega$ be a compact convex set in $[0, 1]^d$ with $|\Omega| \geq 1/d!$. Then there exists a constant $\Lambda$ depending only on $d$, such that for any $1 \leq r \leq \infty$ and any $0 < \delta \leq 1$,

$$C_r(\Omega) \subset \Lambda \delta^{-d/r} \cdot C_\infty(\Omega_\delta),$$

where $\Omega_\delta$ is as defined in (8). In fact, $\Lambda = \max\{(d \Gamma(d/2)/\pi^{d/2})^{1/r}, (d!)d2^{d+2}\}$ works.

**Proof** First, we show that if $f \in C_r(\Omega)$, then on $\Omega_\delta$,

$$f \geq -(d!)^{1/r} 2^{d+2} d.$$

Let $x_0$ be a minimizer of $f$ on $\Omega_\delta$. If $f(x_0) \geq 0$, then there is nothing to prove; otherwise, the set $K := \left\{ x \in \Omega \mid f(x) \leq 0 \right\}$ is a closed convex set with $x_0$ as an interior point. We write $K_0 = K - x_0$, and define

$$K_\eta = \left\{ x \in \Omega \mid x = x_0 + y, \ y \in (1 + \eta)K_0 \setminus (1 - \eta)K_0 \right\},$$

where $0 < \eta < 1$. We show that if $x \notin K_\eta$, then $|f(x)| > \eta |f(x_0)|$. Indeed, consider a function $g$ on $\Omega$ defined so that $g(x_0) = f(x_0)$, $g(\gamma) = f(\gamma)$ for all $\gamma \in \partial K$, and $g$ is linear on the line segment

$$L_\gamma := \left\{ x \in \Omega \mid x = x_0 + t(\gamma - x_0), \ t \geq 0 \right\}.$$  

Then, by the convexity of $f$ on each $L_\gamma$, we have $|f(x)| \geq |g(x)|$ on $\Omega$. Because for all $x \notin K_\eta$, $\|x - \gamma\| \geq \eta \|x_0 - \gamma\|$, we have
\[ |g(x)| = |g(\gamma)| + \frac{\|x - \gamma\|}{\|x_0 - \gamma\|} |f(x)0| > \eta |f(x0)|. \]

Hence, on \( \Omega \setminus K_\eta \), \( |f(x)| \geq \eta |f(x0)| \).

Because the volume of \( K_\eta \) is bounded by \( [(1 + \eta)^d - (1 - \eta)^d] \cdot |K| \leq d^2 \eta |\Omega| \), we have
\[
1 \geq \int_{\Omega \setminus K_\eta} |f(x)|^r d\lambda(x) \geq (\eta |f(x0)|)^r \cdot (1 - d^2 \eta) \cdot |\Omega|.
\]

This implies that
\[ |f(x0)| \leq \eta^{-1} |\Omega|^{-1/r} (1 - d^2 \eta)^{-1/r}. \]

By choosing \( \eta = [d^2 (1 + 1/r)]^{-1} \), we obtain
\[
|f(x0)| \leq |\Omega|^{-1/r} d^2 \left( 1 + \frac{1}{r} \right) (1 + r)^{1/r} \leq (d!)^{1/r} 2^{d+2} d \leq (d!) 2^{d+2} d.
\]

This proves (9).

Next, we show that there exists a constant \( \Lambda \) depending on \( \Omega \) such that on \( \Omega_\delta \),
\[ f(x) \leq \Lambda \delta^{-d/r}. \]

Let \( z_0 \) be a maximizer of \( f \) on \( \Omega_\delta \). If \( f(z_0) \leq 0 \), there is nothing to prove. So, we assume \( f(z_0) > 0 \). Let \( V = \{ x \in \Omega \mid f(x) < f(z_0) \} \). Then \( V \) is a convex set with \( z_0 \) at its boundary. There exists a hyperplane that separates \( V \) and \( z_0 \). This hyperplane separates \( \Omega \) into two parts. On the part not containing \( V \), \( f \geq f(z_0) \). In particular, \( f \geq f(z_0) \) on the half of the ball centered at \( z_0 \) with radius \( \delta \). Calling this half of the ball \( W \), we have
\[
1 \geq \int_{W} |f(x)|^r d\lambda(x) \geq \frac{\pi^{d/2}}{d \Gamma(d/2)} \delta^d f(z_0)^r,
\]
which implies that
\[ f(z_0) \leq \left( \frac{d \Gamma(d/2)}{\pi^{d/2}} \right)^{1/r} \delta^{-d/r}. \]

Together with (9), we obtain that there exists some \( \Lambda \) depending only on \( d \) such that for all \( x \in \Omega_\delta \), \( |f(x)| \leq \Lambda \delta^{-d/r} \).

\( \square \)

**Lemma 6** Let \( \Omega \) be a closed convex set in \([0, 1]^d\). For any \( 1 \leq r \leq \infty \) and any \( 0 < \delta \leq 1 \),
\[
N_{[1]}(\varepsilon, C_r(\Omega), \| \cdot \|_{L^p(\Omega_\delta)}) \leq \exp \left( C_2 \delta^{-d/2 - d^2/r} \varepsilon^{-d/2} \right),
\]
where \( \Omega_\delta \) is as defined in (8) and \( C_2 \) is a constant depending only on \( d \), \( C_0 \) in Lemma 3, and \( \Lambda \) in Lemma 5.
Proof We show that when restricted to \( \Omega_\delta \), \( f \) has a Lipschitz constant bounded by 
\[
2^{4d/r} \Lambda \delta^{-1-d/r}.
\]
Indeed, by Lemma 5, \( f \) is bounded by \( 2^d/r \Lambda \delta^{-d/r} \) on \( \Omega_{\delta/2} \). Note that \( \Omega_\delta \subset \Omega_{\delta/2} \subset \Omega \). Thus by [21], problem 2.7.4 page 165, \( f \) is Lipschitz on \( \Omega_\delta \) with Lipschitz constant 
\[
2^{4d/r} \Lambda \delta^{-1-d/r}.
\]

Thus by Lemma 4, if \( f \) follows that

\[
\log N_1(\epsilon, C_r(\Omega), \| \cdot \|_{L^p(\Omega)}) \\
\leq 2^{-d} C_0 (1 + 2^{4d/r} \Lambda \delta^{-2-2d/r})^d / 4 \Lambda \delta^{-2d/r} \leq C_2 \delta^{-d/2 - d^2/r} s^{-d/2},
\]

for some constant \( C_2 \) depending only on \( d \) and \( r \).

\( \Box \)

2.4 Combining

In this subsection, we prove some lemmas that enables us to study metric entropy by decomposing the set \( \Omega \).

Lemma 7 (Union) If \( \Omega = \bigcup_{i=1}^k \Omega_i \), then for all \( 1 \leq p < r \leq \infty \),

\[
N(\epsilon, C_r(\Omega), \| \cdot \|_{L^p(\Omega)}) \leq \prod_{i=1}^k N_1(\delta_i, C_r(\Omega_i), \| \cdot \|_{L^p(\Omega_i)}),
\]

(11)

\[
N_1(\epsilon, C_{\infty}(\Omega), \| \cdot \|_{L^p(\Omega)}) \leq \prod_{i=1}^k N_1(\delta_i, C_{\infty}(\Omega_i), \| \cdot \|_{L^p(\Omega_i)}),
\]

(12)

where \( \epsilon = \left( \sum_{i=1}^k \delta_i^p \right)^{1/p} \). Furthermore, if \( \Omega_1, \Omega_2, \ldots, \Omega_k \) have disjoint interiors, then

\[
N(\epsilon, C_r(\Omega), \| \cdot \|_{L^p(\Omega)}) \leq 4^k \prod_{i=1}^k N(\eta_i, C_r(\Omega_i), \| \cdot \|_{L^p(\Omega_i)}),
\]

(13)

where

\[
\left( \sum_{i=1}^k \eta_i^{r-p} \right)^{r-p} \leq 2^{-1/r} \epsilon,
\]

which is stronger than (11) when \( r < \infty \).

Proof For each \( i \in \{1, 2, \ldots, k\} \), there exists a set \( N_i \) of \( N_i \) elements, where

\[
N_i := N(\delta_i, C_r(\Omega_i), \| \cdot \|_{L^p(\Omega_i)}).
\]
such that, for each \( f \in C_r(\Omega) \subset C_r(\Omega_i) \), there exists \( f_i \in \mathcal{N}_i \) satisfying
\[
\int_{\Omega_i} |f_i(x) - f(x)|^p d\lambda(x) \leq \delta_i^p.
\]
Define \( \hat{f}(x) = f_i(x) \) for \( x \in \Omega_i \setminus \bigcup_{j<i} \Omega_j \), \( 1 \leq i \leq k \). Then we have
\[
\int_{\Omega} |f(x) - \hat{f}(x)|^p d\lambda(x) \leq \sum_{i=1}^{k} \int_{\Omega_i} |f(x) - f_i(x)|^p d\lambda(x) \leq \sum_{i=1}^{k} \delta_i^p = \varepsilon^p.
\]
Since \( \hat{f} \) is determined by \( f_1, f_2, \ldots, f_k \), and each \( f_i \) has at most \( N_i \) possibilities, the total number of possibilities for \( \hat{f} \) is no more than \( N_1 N_2 \cdots N_k \). Thus, (11) follows.

The proof of (12) is similar. For each \( i \in \{1, 2, \ldots, k\} \), there exists a set \( \mathcal{N}_i \) of \( \hat{N}_i \) brackets, where
\[
\hat{N}_i := N_{i[1]}(\delta_i, \mathcal{C}_\infty(\Omega_i) \\| \cdot \|_{L^p(\Omega_i)})
\]
such that, for each \( f \in C_\infty(\Omega) \subset C_\infty(\Omega_i) \), there exists a bracket \( \{f_i, \widehat{f}_i\} \in \mathcal{N}_i \)
satisfying \( f_{\hat{j}}(x) \leq f(x) \leq \widehat{f}_j(x) \) for all \( x \in \Omega_i \), and
\[
\int_{\Omega_i} |\widehat{f}_i(x) - f_{\hat{j}}(x)|^p d\lambda(x) \leq \delta_i^p.
\]
Define \( \overline{f}(x) = \widehat{f}_i(x) \), \( f(x) = f_{\hat{j}}(x) \), \( x \in \Omega_i \setminus \bigcup_{j<i} \Omega_j \), \( 1 \leq i \leq k \). Then we have \( f(x) \leq f_{\hat{j}}(x) \leq \overline{f}(x) \) for all \( x \in \Omega_i \), and
\[
\int_{\Omega} |f(x) - \overline{f}(x)|^p d\lambda(x) \leq \sum_{i=1}^{k} \int_{\Omega_i} |f_i(x) - f_{\hat{j}}(x)|^p d\lambda(x) \leq \sum_{i=1}^{k} \delta_i^p = \varepsilon^p.
\]
That is, \( [f, \overline{f}] \) is an \( \varepsilon \)-bracket in \( L^p(\Omega) \) which contains \( f \). Because there are no more than \( \hat{N}_1 \hat{N}_2 \cdots \hat{N}_k \) possibilities for \( [f, \overline{f}],(12) \) follows.

Now we turn to the proof of (13). For any \( f \in C_r(\Omega) \) and for each \( i = 1, 2, \ldots, k \), define \( n_i(f) \) as the smallest positive integer such that
\[
n_i(f) \geq k \int_{\Omega_i} |f(x)|^r d\lambda(x).
\]
Then, \( n_i(f) < k \int_{\Omega_i} |f(x)|^r d\lambda(x) + 1 \), and using the fact that \( \sum_{i=1}^{k} \int_{\Omega_i} |f|^r d\lambda(x) \leq 1 \), we get
\[
n_1(f) + n_2(f) + \cdots + n_k(f) \leq \sum_{i=1}^{k} \left( k \int_{\Omega_i} |f(x)|^r d\lambda(x) + 1 \right) \leq 2k.
\]
Let
\[ I = \left\{ (n_1, n_2, \ldots, n_k) \in \mathbb{N}^k \mid n_1 + n_2 + \cdots + n_k \leq 2k \right\}. \]

For each \( I = (i_1, i_2, \ldots, i_k) \in \mathcal{I} \), define
\[ \mathcal{F}_I = \left\{ f \in C_r(\Omega) \mid n_j(f) = i_j, 1 \leq j \leq k \right\}. \]

Then we have \( C_r(\Omega) = \bigcup_{I \in \mathcal{I}} \mathcal{F}_I \). Thus,
\[ N(\varepsilon, C_r(\Omega), \| \cdot \|_{L^p(\Omega)}) \leq \sum_{I \in \mathcal{I}} N(\varepsilon, \mathcal{F}_I, \| \cdot \|_{L^p(\Omega)}). \]

Note that for each \( j = 1, 2, \ldots, k \), \( \mathcal{F}_I \subset (i_j/k)^{1/r} C_r(\Omega_j) \). Thus,
\[ N(\eta_j, C_r(\Omega_j), \| \cdot \|_{L^p(\Omega_j)}) = N((i_j/k)^{1/r} \eta_j, (i_j/k)^{1/r} C_r(\Omega_j), \| \cdot \|_{L^p(\Omega_j)}) \geq N((i_j/k)^{1/r} \eta_j, \mathcal{F}_I, \| \cdot \|_{L^p(\Omega_j)}). \]

Therefore, for each \( 1 \leq j \leq k \), there exists a set \( \mathcal{N}_j \) of \( Z_j := N(\eta_j, C_r(\Omega_j), \| \cdot \|_{L^p(\Omega_j)}) \) elements such that for each \( f \in \mathcal{F}_I \), there exists \( f_j \in \mathcal{N}_j \) satisfying
\[ \int_{\Omega_j} |f(x) - f_j(x)|^p d\lambda(x) \leq (i_j/k)^{p/r} \eta_j^p. \]

If we define \( \hat{f}(x) = f_j(x) \) for \( x \in \Omega_j \setminus \bigcup_{r<j} \Omega_r \), then we have
\[ \int_{\Omega} |f(x) - \hat{f}(x)|^p d\lambda(x) \leq \sum_{j=1}^k (i_j/k)^{p/r} \eta_j^p \leq \left( \sum_{j=1}^k \frac{i_j}{k} \right)^p \left( \sum_{j=1}^k \eta_j^{rp/p} \right)^{1-p/r} \leq 2^p \left( \sum_{j=1}^k \eta_j^{rp/p} \right)^{1-p/r} \leq \varepsilon^p. \]

Since there are no more than \( Z_1 Z_2 \cdots Z_k \) possibilities for \( \hat{f} \), we obtain
\[ N(\varepsilon, \mathcal{F}_I, \| \cdot \|_{L^p(\Omega)}) \leq \prod_{i=1}^k N(\eta_i, C_r(\Omega_i), \| \cdot \|_{L^p(\Omega_i)}). \]

Note that \( \mathcal{I} \) has cardinality \( \binom{2k}{k} < 4^k \), and hence (13) follows. \( \square \)
2.5 With Finitely Many Facets

In this subsection, we derive a metric entropy upper bound when \( \Omega \) has finitely many facets. The idea is as follows: First, we pare off the boundary of \( \Omega \) to get a smaller set on which the functions are uniform Lipschitz and can be taken care of by Lemma 6. Next, we decompose the pared-off part into several smaller polytopes with a bounded number of facets and handle each of the smaller polytopes by scaling and paring of the boundary, and so on. The final estimate is obtained by using Lemma 6 and Lemma 7, followed by iteration. Readers who have little interest in the specific dependence of constants on the number of facets can assume that \( \Omega \) is a \( d \)-simplex with \( d + 1 \) facets. For general polytopes, one can triangulate it into \( d \)-simplices and apply Lemma 7 to get the estimate in the next subsection. The only loss is that the constant obtained that way may be bigger than the one derived directly from the number of facets in some cases.

We first prove the upper bound with constant \( C_k \gamma \) with some \( \gamma > 1 \) for a closed convex polytope with \( k \) facets. We will use it later only for the case \( k = d + 1 \). However, since the proof is the same, we prove it for the general \( k \).

By scaling, we can assume that \( \Omega \) is contained in a unit \( d \)-cube with volume at least \( 1/2^d \). Thus, there exists a point \( O \in \Omega \) such that the distance between \( O \) and the boundary of \( \Omega \) is at least \( \delta_0 := 1/(2d!) \). This is because the boundary of \([0, 1]^d\) has \((d - 1)\)-dimensional area \( 2d \), and its projection onto \( \Omega \) is a contraction; thus, the boundary of \( \Omega \) has \((d - 1)\)-dimensional area at most \( 2d \), and by a Bonnesen-style inequality (Corollary 2, page 25 of [19]), the inradius of \( \Omega \) is at least its volume divided by the \((d - 1)\)-dimensional surface area of its boundary, i.e., the inradius is at least \( \delta_0 \).

By otherwise using a translation, we can assume that \( O \) is the origin. Let \( F_i \) be the \( i \)-th facet of \( \Omega \) for \( i = 1, \ldots, k \). Let \( V_i \) denote the convex hull of \( F_i \) and \( O \). Then, \( V_i, i \in \{1, \ldots, k\} \), form a partition of \( \Omega \). For \( \delta < \delta_0 := 1/(2d!) \), let \( D_0 := (1 - \delta/\delta_0)\Omega \). Define \( \Omega_i = V_i \setminus D_0^\circ \), where \( D_0^\circ \) denotes the interior of \( D_0 \). Then we have

\[
\Omega = D_0 \cup \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_k.
\]

Note that each \( \Omega_i \) has no more than \( k + 1 \) facets. To see this, we first observe that \( V_i \) has at most \( k \) facets. Indeed, each of the facets of \( V_i \) besides \( F_i \) is the convex hull of a \((d - 2)\)-dimensional face of \( F_i \) and \( O \). However, each \((d - 2)\)-dimensional face of \( F_i \) corresponds to the intersection of \( F_i \) and another facet of \( \Omega \). Thus, the number of \((d - 2)\)-dimensional faces of \( F_i \) is at most \( k - 1 \). Therefore, the number of facets of \( V_i \) is at most \( k \). Notice that \( \Omega_i \) has one more facet than \( V_i \). Hence, the number of facets of \( \Omega_i \) is at most \( k + 1 \). By (13) we have

\[
N(\varepsilon, \mathcal{C}_r(\Omega), \|\cdot\|_{L^p(\Omega)}) \leq 4^{k+1} N(\eta_0, \mathcal{C}_r(D_0), \|\cdot\|_{L^p(D_0)}) \prod_{i=1}^k N(\eta_i, \mathcal{C}_r(\Omega_i), \|\cdot\|_{L^p(\Omega_i)}),
\]
where \( \eta_0 = 2^{-\frac{1}{p}} \varepsilon \), and

\[
\eta_i = 2^{-\frac{1}{p}} \left( \frac{|\Omega_i|}{\sum_{i=1}^{k} |\Omega_i|} \right)^{\frac{1}{p-1}} \varepsilon.
\]

Because \( D_0 \subset \Omega_\delta \), by Lemma 6, we have

\[
\log N(\varepsilon, C_r(\Omega), \| \cdot \|_{L^p(D_0)}) \leq C_2\delta^{-\frac{d}{2} - \frac{d^2}{\tau}} (\varepsilon)^{-d/2}.
\]

On the other hand, if we let \( T_i \) be an affine transform that maps \( \Omega_i \) into \([0,1]^d\) so that the volume of \( T_i(\Omega_i) \) is at least \( 1/d! \), then by scaling (6) and using the fact that

\[
\sum_{i=1}^{k} |\Omega_i| = |\Omega \setminus (1 - \delta/\delta_0)\Omega| = [1 - (1 - \delta/\delta_0)^d] |\Omega| \leq d\delta/\delta_0,
\]

we have for each \( 1 \leq i \leq k \),

\[
N(\eta_i, C_r(\Omega_i), \| \cdot \|_{L^p(\Omega_i)}) \leq N(K\varepsilon, C_r(T_i(\Omega_i)), \| \cdot \|_{L^p(T_i(\Omega_i)))},
\]

where

\[
K = 2^{-1/p}[2d^2(d!)^2\delta]^{\frac{1}{p}-\frac{1}{p}}.
\]

Plugging into (14), we obtain

\[
\log N(\varepsilon, C_r(\Omega), \| \cdot \|_{L^p(\Omega)}) \leq (k + 1) \log 4 + C_2\delta^{-\frac{d}{2} - \frac{d^2}{\tau}} \varepsilon^{-d/2}
\]

\[
+ \sum_{i=1}^{k} \log N(K\varepsilon, C_r(T_i(\Omega_i)), \| \cdot \|_{L^p(T_i(\Omega_i)))}.
\]  \( \tag{15} \)

Now let \( \mathcal{F}_k \) consist of all closed convex sets in \([0,1]^d\) with at most \( k \) faces and with volume at least \( 1/d! \), and define

\[
g(k, \varepsilon) = \sup \left\{ \log N(\varepsilon, C_r(\Omega), \| \cdot \|_{L^p(\Omega)}) \mid \Omega \in \mathcal{F}_k \right\}.
\]

For notational simplicity, we write \( M = C_2\delta^{-\frac{d}{2} - \frac{d^2}{\tau}} \). Then (15) together with the fact that \( (k + 1) \log 4 \leq 4k - 4 \) which follows from the fact that \( k \geq 3 \) implies

\[
g(k, \varepsilon) + 4 \leq M\varepsilon^{-d/2} + k[g(k + 1, K\varepsilon) + 4],
\]  \( \tag{16} \)

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which is equivalent to

\[ [g(k, \varepsilon) + 4]\varepsilon^{d/2} \leq M + \frac{k}{K^{d/2}}[g(k + 1, K\varepsilon) + 4](K\varepsilon)^{d/2}. \]

Now, we choose \( \delta \) so that \( K\varepsilon^{d/2} = 2k \). Then

\[ M = C_2\delta^{d/2} = C_3k^{(r+2d)p/(r-p)}. \]

Thus,

\[ [g(k, \varepsilon) + 4]\varepsilon^{d/2} \leq C_3k^{(r+2d)p/(r-p)} + \frac{1}{2}[g(k + 1, (2k)^{2/d}\varepsilon) + 4]((2k)^{2/d}\varepsilon)^{d/2}. \]

Hence, for any positive integer \( m \), we have

\[ [g(k, \varepsilon) + 4]\varepsilon^{d/2} \leq C_3\sum_{j=0}^{m-1} \frac{(k + j)^{(r+2d)p}/(r-p)}{2^j} + 2^{-m}[g(k + m, L_m\varepsilon) + 4](L_m\varepsilon)^{d/2}, \]

where

\[ L_m = \prod_{j=0}^{m-1} (2k + 2j)^{2/d}. \]

In particular, if we choose \( m \) to be the smallest integer so that \( L_m\varepsilon \geq 1 \), then \( g(k + m, L_m\varepsilon) = 0 \), and we obtain

\[ g(k, \varepsilon) \leq C_4k^{(r+2d)p/(r-p)}\varepsilon^{-d/2}. \]

This finishes the proof of the upper bound with constant of the order \( k^\gamma \) with \( \gamma = (r+2d)p/(r-p) \).

\[ \square \]

2.6 Upper Bound for Polytopes: Theorem 1 (ii)

In this subsection, we obtain metric entropy upper bound for the case when \( \Omega \) is a convex polytope. Our method is to triangulate \( \Omega \) into simplices and use results in the last section and Lemma 7.

Note that if \( \Omega \) is a convex polytope with \( v \) extreme points, then it has no more than \( 2v \lfloor d/2 \rfloor \) facets; see [18], Propositions 5.5.2 and 5.5.3, page 100. Therefore, we immediately obtain the upper bound with constant of the order \( v^{\gamma \lfloor d/2 \rfloor} \). We show that this estimate can be improved to \( v^{\lfloor d/2 \rfloor} \). Indeed, if \( \Omega \) has \( v \) vertices, then it is known that \( \Omega \) can be triangulated into \( m = O(v^{\lfloor d/2 \rfloor}) \) many \( d \)-simplices; this is Corollary 2.3 of [20]; see also [5]. Thus, we can write \( \Omega = \bigcup_{i=1}^{m} D_i \), where \( D_i \) are \( d \)-simplices.
Because each $D_i$ has only $(d+1)$-facets, by what we have proved above, it follows that

$$\log N(\eta_i, C_r(D_i), \| \cdot \|_{L^p(D_i)}) \leq C_5 |D_i|^\frac{d}{2p} \eta_i^{-d/2},$$

where $C_5$ is a constant depending only on $p, r, d$. Now applying (13), with

$$\eta_i = 2^{-\frac{r-p}{p}} \left( \frac{|D_i|}{|\Omega|} \right)^{\frac{1}{p} - \frac{1}{r}} \epsilon,$$

we immediately obtain

$$\log N(\epsilon, C_r(\Omega), \| \cdot \|_{L^p(\Omega)}) \leq \sum_{i=1}^m \log N(\eta_i, C_r(D_i), \| \cdot \|_{L^p(D_i)})$$

$$\leq C_6 m |\Omega|^\frac{d}{2p} - \frac{d}{2} \epsilon - \frac{d}{2} \leq C_7 v^{[d/2]} \epsilon - \frac{d}{2}.$$

This proves Part (ii) of Theorem 1. The proof for the statement of the bracketing entropy when $r = \infty$ is similar, and the details are thus omitted.

**2.7 General Upper Bound: Theorem 2**

In this subsection, we establish an upper general compact convex set with nonempty interior by simplicial approximation.

Fix $0 < \epsilon < 1$; we choose the smallest integer $s$ so that $2^{-s} |\Omega| \leq [2^{-1/p} \epsilon]^{r/p} |\Omega|$. By the definition of $S_D(t, \Omega)$, $\Omega$ contains $m_1 \leq S_D(1/2, \Omega) d$-simplices $D_{1,i}, 1 \leq i \leq m_1$, so that the volume of $\Omega \setminus \bigcup_{i=1}^{m_1} D_{1,i}$ is at most $2^{-1} |\Omega|$, and the set $\Omega \setminus \bigcup_{i=1}^{m_1} D_{1,i}$ contains $m_2 = S_D(1/4, \Omega) - m_1 < S_D(1/4, \Omega) d$-simplices $D_{2,j}, 1 \leq j \leq m_2$, so that the volume of

$$\Omega \setminus \bigcup_{i=1}^2 \bigcup_{j=1}^{m_i} D_{i,j}$$

is at most $2^{-2} |\Omega|$. Continuing this way, we obtain a sequence of $d$-simplices $D_{i,j}, 1 \leq j \leq m_i, 1 \leq i \leq s$, that are packed in $\Omega$ so that the uncovered volume of $\Omega$ is at most $2^{-s} |\Omega|$. If we write

$$\hat{\Omega}_i = \bigcup_{k=1}^i \bigcup_{j=1}^{m_k} D_{k,j},$$

then for all $f \in C_r(\Omega)$,

$$\int_{\Omega \setminus \hat{\Omega}_s} |f|^p d\lambda \leq |\Omega \setminus \hat{\Omega}_s|^{1-p} \leq \frac{\epsilon p}{2} |\Omega|^{1-p}. $$
Hence,

\[ N(\varepsilon|\Omega|^{\frac{1}{p} - \frac{1}{r}}, C_r(\Omega), \| \cdot \|_{L^p(\Omega)}) \leq N(2^{-1/p} \varepsilon|\Omega|^{\frac{1}{p} - \frac{1}{r}}, C_r(\widehat{\Omega}_s), \| \cdot \|_{L^p(\widehat{\Omega}_s)}). \quad (17) \]

Next, we choose

\[ \eta_{i,j} = 2^{-1/p} \left( \frac{|D_{i,j}|}{\sum_{j=1}^{m_i} |D_{i,j}| \cdot \frac{\alpha_i}{\sum_{k=1}^{s} \alpha_k}} \right)^{\frac{1}{p} - \frac{1}{r}} \cdot \frac{\varepsilon}{2 |\Omega|^{\frac{1}{p} - \frac{1}{r}}}, \]

where

\[ \alpha_i := (2^{-i}|\Omega|)^{1-\beta}[S(2^{-i}, \Omega)]^\beta, \quad \beta := \frac{2pr}{2pr + (r - p)d}. \]

Using the fact that \( \sum_{j=1}^{m_i} |D_{i,j}| \leq 2^{-(i-1)}|\Omega| \), we have

\[ \eta_{i,j}|D_{i,j}|^{\frac{1}{p} - \frac{1}{r}} \geq 2^{-1/p} \left( \frac{\alpha_i}{2^{-(i-1)}|\Omega| \sum_{k=1}^{s} \alpha_k} \right)^{\frac{1}{p} - \frac{1}{r}} \cdot \frac{\varepsilon}{2 |\Omega|^{\frac{1}{p} - \frac{1}{r}}}. \]

Thus, together with the fact that \( m_i \leq S_D(2^{-i}, \Omega) \), we have

\[
\sum_{j=1}^{m_i} \log N(\eta_{i,j}, C_r(D_{i,j}), \| \cdot \|_{L^p(D_{i,j})})
\leq S_D(2^{-i}, \Omega) \cdot c \left[ 2^{-1/p} \left( \frac{\alpha_i}{2^{-(i-1)}|\Omega| \sum_{k=1}^{s} \alpha_k} \right)^{\frac{1}{p} - \frac{1}{r}} \cdot \frac{\varepsilon}{2 |\Omega|^{\frac{1}{p} - \frac{1}{r}}} \right]^{-d/2}
\]
\[ = c 2^{\frac{d}{2} + \frac{d}{2p}} \left( \sum_{k=1}^{s} \alpha_k \right)^{(r-p)d}{2pr} \alpha_i \cdot [\varepsilon |\Omega|^{\frac{1}{p} - \frac{1}{r}}]^{-d/2}. \]

Therefore, by (13) and (17), we have

\[
\log N(\varepsilon|\Omega|^{\frac{1}{p} - \frac{1}{r}}, C_r(\Omega), \| \cdot \|_{L^p(\Omega)})
\leq \log 4 \sum_{i=1}^{s} S_D(2^{-i}, \Omega) + c 2^{\frac{d}{2} + \frac{d}{2p}} \left( \sum_{k=1}^{s} \alpha_k \right)^{1/\beta} \cdot [\varepsilon |\Omega|^{\frac{1}{p} - \frac{1}{r}}]^{-d/2}. \]
Let \( \gamma := rp/(r - p) \). Note that \( 2^{-s} \leq [2^{-1/p} \varepsilon \gamma] r \leq 2^{-(s-1)} \), and \( S_D(t, \Omega) \geq S(2^{-i}, \Omega) \) for \( t \in [2^{-(i+1)}, 2^{-i}] \). Thus it follows that

\[
\sum_{i=1}^{s} \alpha_i = \sum_{i=1}^{s} (2^{-i} |\Omega|) \frac{1-\beta}{\gamma} S(2^{-i}, \Omega)^{\beta} \\
= |\Omega|^{1-\beta} \sum_{i=1}^{s} 2^{-i} \left( \frac{S(2^{-i}, \Omega)}{2^{-i}} \right)^{\beta} \\
= 2|\Omega|^{1-\beta} \sum_{i=1}^{s} \int_{2^{-i-1}}^{2^{-i}} \left( \frac{S(2^{-i}, \Omega)}{2^{-i}} \right)^{\beta} \, dt \\
\leq 2|\Omega|^{1-\beta} \sum_{i=1}^{s} \int_{2^{-i-1}}^{2^{-i}} \left( \frac{S(t, \Omega)}{t} \right)^{\beta} \, dt \\
= 2|\Omega|^{1-\beta} \int_{2^{-(s+1)}}^{1} \left( \frac{S(t, \Omega)}{t} \right)^{\beta} \, dt \\
\leq 2|\Omega|^{1-\beta} \int_{2^{-[2^{-(1/p)} \varepsilon \gamma]}}^{1} \left( \frac{S(t, \Omega)}{t} \right)^{\beta} \, dt.
\]

Hence,

\[
c 2^{\frac{d}{p} + \frac{d}{p}} \left( \sum_{k=1}^{s} \alpha_k \right)^{1/\beta} \cdot |\varepsilon| |\Omega|^{\frac{1}{p} - 1} \gamma - d/2 \leq c 2^{\frac{d}{p} + \frac{d}{p} + \frac{1}{p}} \left( \int_{\delta(\varepsilon)}^{1} \left( \frac{S_D(t, \Omega)}{t} \right)^{\beta} \, dr \right)^{1/\beta} \cdot \varepsilon^{-d/2},
\]

where \( \delta(\varepsilon) = 2^{-2} \cdot [2^{-1/p} \varepsilon \gamma] r \).

Similarly,

\[
\sum_{i=1}^{s} S(2^{-i}, \Omega) \leq 2 \int_{2^{-[2^{-(1/p)} \varepsilon \gamma]}}^{1} \frac{S_D(t, \Omega)}{t} \, dt.
\]

Hence, we obtain

\[
\log N(\varepsilon |\Omega|^{\frac{1}{p} - 1} \gamma, C_r(\Omega), \| \cdot \|_{L^p(\Omega)}) \leq C \int_{\delta(\varepsilon)}^{1} \frac{S_D(t, \Omega)}{t} \, dt + C \left( \int_{\delta(\varepsilon)}^{1} \left( \frac{S_D(t, \Omega)}{t} \right)^{\beta} \, dr \right)^{1/\beta} \cdot \varepsilon^{-d/2},
\]

with \( C = \max \{ 2 \log 4, c 2^{\frac{d}{p} + \frac{d}{p} + \frac{1}{p}} \} \).

### 2.8 Worst Case Upper Bound: Theorem 3

In this subsection, we will compute a concrete general metric entropy upper bound using Theorem 2.
From Example 1 in the introduction, we know that when $\Omega$ is the closed unit ball $B_d(0, 1)$ in $\mathbb{R}^d$ with $d = 2$, then there exists a simplicial approximation sequence $\mathcal{D} = \{D_1, D_2, \ldots\}$ such that $S_{\mathcal{D}}(t, \Omega) = O(t^{-\frac{d-1}{2}})$ for all $0 < t < 1$.

Now, we assume that $\Omega \subset B_d(0, 1)$ is a general closed convex set with nonempty interior, where $B_d(z, r)$ is the closed ball in $\mathbb{R}^d$ with radius $r$ and center at $z$. Instead of constructing each simplex in the sequence individually, we will construct a sequence of inscribed polytopes, and triangulate them into simplices. To construct these polytopes, we do not work on $\Omega$ directly. Instead, we first let $\tilde{\Omega} = \Omega + B_d(0, 1)$, and for any $0 < t < 1$, we use a known result. (See, e.g., the proof of Lemma 8.4.14 of [10]; or the Lemma 10 of [4] when $\Omega$ is only of positive reach. Actually, [10] proved for $\Omega + B_d(0, 2)$ instead of $\tilde{\Omega} = \Omega + B_d(0, 1)$, but with a slightly different construction, the statement also holds.)

Known result: There exists a simplicial sphere $\tilde{P}$ (an inscribed convex polytope in $\tilde{\Omega}$ whose facets are $(d - 1)$-simplices) with $O(t^{-(d-1)/2})$ facets such that $\tilde{\Omega} \subset \tilde{P} + B_d(0, t)$.

Then, we use $\tilde{P}$ to construct a simplicial sphere $P$ in $\Omega$ with $O(t^{-(d-1)/2})$ facets such that $\tilde{\Omega} \subset P + B_d(0, t)$. Finally, we triangulate a sequence of such simplicial spheres to construct a simplicial approximation sequence $\mathcal{D}$ for $\Omega$, such that $S_{\mathcal{D}}(t, \Omega) = O(t^{-\frac{d-1}{2}})$ for all $0 < t < 1$.

For the convenience of readers who are interested in knowing how the simplicial spheres are constructed, we provide a proof for the aforementioned known result: Since $\Omega \subset B_d(0, 1)$, we have $\tilde{\Omega} \subset [-2, 2]^d$. For any integer $n > 1$, we divide each facet of $[-2, 2]^d$ into $(4n)^{d-1}$ closed $(d-1)$-cubes of side-length $1/n$. Each of these small $(d-1)$-cubes can be triangulated into no more than $d!$ closed $(d - 1)$-simplices. Thus, the boundary of $[-2, 2]^d$ can be triangulated into $m_k = (4n)^{d-1}d!$ closed $(d - 1)$-simplices, each of which has diameter at most $\sqrt{d}/n$. Let $K_i, 1 \leq i \leq m_k$, be these simplices. Clearly, the set of all vertices of these simplices forms a $\sqrt{d}/n$-net of the boundary of $[-2, 2]^d$. Each $K_i$ has $d$ vertices. The projections of these vertices onto $\tilde{\Omega}$ form a $(d - 1)$-simplex with vertices on the boundary of $\tilde{\Omega}$. Denote this $(d - 1)$-simplex by $\tilde{\Delta}_i$. Because a projection onto a convex set is a contraction, the diameter of $\tilde{\Delta}_i$ is no larger than $\sqrt{d}/n$, and the set of all vertices of these simplices forms a $\sqrt{d}/n$-net of the boundary of $\tilde{\Omega}$. Let $\tilde{P}$ be the convex hull of $\tilde{\Delta}_i, 1 \leq i \leq m_k$. Then $\tilde{P}$ is a simplicial sphere contained in $\tilde{\Omega}$ with $m_k \leq (4n)^{d-1}d!$ facets, each of which has a diameter no larger than $\sqrt{d}/n$. Furthermore, $\tilde{\Omega} \subset \tilde{P} + B_d(0, \sqrt{d}/n)$, which implies that $\Omega$ is contained in the interior of $\tilde{P}$ if $n > \sqrt{d}$. We show that for $n > \sqrt{d}$, we actually have $\tilde{\Omega} \subset \tilde{P} + B_d(0, d/n^2)$. Indeed, for any $x$ on the boundary of $\tilde{\Omega}$, by the definition of $\tilde{\Omega}$, there exists $y \in \tilde{\Omega}$, such that $\text{dist}(y, \partial \tilde{\Omega}) = \text{dist}(x, y) = 1$. Because $y$ is an interior point of $\tilde{P}$, the line segment $yx$ intersects the boundary of $\tilde{P}$ at some point, say $u$. Let $F$ be a facet of $\tilde{P}$ that contains $u$. Consider the hyperplane passing through $u$ and orthogonal to $xy$. Because all the vertices of $F$ are outside the interior of the ball $B_d(y, 1)$, if $\text{dist}(y, u) < \sqrt{1 - d/n^2}$, then the vertices and $y$ must lie on different sides of the hyperplane $H$. Since $u$ is a convex combination of these vertices, $u$ cannot lie on the hyperplane $H$, which is a contradiction.
Thus,

\[
dist(x, u) = dist(x, y) - dist(u, y) \leq 1 - \sqrt{1 - d/n^2} < d/n^2.
\]

Therefore, \( dist(x, \tilde{P}) < d/n^2 \). Consequently, \( \tilde{\Omega} \subset \tilde{P} + B_d(0, d/n^2) \). In particular, for any \( 0 < t < 1 \), we choose \( n \) as the smallest positive integer such that \( d/n^2 < t \). Then the simplicial sphere \( \tilde{P} \) has \( O(t^{-(d-1)/2}) \) facets, and the claim follows.

Now we construct a simplicial sphere \( P \) on \( \Omega \) with \( O(t^{-(d-1)/2}) \) facets such that \( \Omega \subset P + B(0, t) \). For each facet \( \tilde{D}_i \) of \( \tilde{P} \), we project its \( d \) vertices onto \( \Omega \). The projections of these vertices onto \( \Omega \) form a \((d - 1)\)-simplex. (If it is degenerate, we simply do not include it in our next step). Denote it by \( D_i \). Let \( P \) be the convex hull of these \( D_i \). Thus \( P \) is a simplicial sphere in \( \Omega \) with \( O(t^{-(d-1)/2}) \) facets. It remains to show that \( \Omega \subset P + B_d(0, t) \). Indeed, for any \( U \) on the boundary of \( \Omega \), let \( V \) be the projection of \( U \) onto \( P \). Suppose \( dist(U, V) > t \). The ray starting from \( V \) and containing \( U \) intersects the boundary of \( \tilde{\Omega} \) at some point \( W \). Since \( P \) is convex, \( dist(W, P) = dist(W, U) + dist(U, V) > 1 + t \). Thus \( W \notin P + B_d(0, 1 + t) \). However,

\[
P + B_d(0, 1 + t) = P + B_d(0, 1) + B_d(0, t) \supset \tilde{P} + B_d(0, t) \supset \tilde{\Omega} \supset W.
\]

This is a contradiction. Hence \( dist(U, V) \leq t \). This implies that for any \( U \in \partial \Omega \), \( dist(U, P) \leq t \). Therefore, \( \Omega \subset P + B_d(0, t) \).

From what we have proved so far, we can summarize that for any \( d/n^2 < 1 \), there exists a simplicial sphere \( P_n \) that contains no more than \( (4n)^{d-1}dd! \) facets of diameter at most \( \sqrt{d}/n \) such that \( \Omega \subset P + B_d(0, d/n^2) \). Furthermore, each facet of \( P_n \) is generated through a \((d - 1)\)-simplex that is contained in a \((d - 1)\)-cube of edge-length \( 1/n \) on the boundary of \([-2, 2]^d\).

Now, we construct a simplicial approximation sequence of \( \Omega \) as follows. Let \( k = \lceil \sqrt{d} \rceil + 1 \). Then \( d/k^2 < 1 \). Let \( O \) be a fixed interior point of \( P_k \). For each facet \( F_i \) of \( P_k \), let \( D_i \) be the convex hull of \( F_i \) and \( O \). Thus, \( D_i \) is a \( d \)-simplex, and \( P_k \) can be partitioned into \( d \)-simplices \( D_1, D_2, \ldots, D_{s_1} \), where \( s_1 \leq (4k)^{d-1}dd! \) is the number of facets in \( P_k \).

Consider the set \( P_{2k} \setminus P_k \). For each facet \( J_i \) of \( P_{2k} \), let \( Q_i \) be the convex hull of \( J_i \) and \( O \). Because \( J_i \) is generated by a \((d - 1)\)-simplex contained in a \((d - 1)\)-cube of edge-length \( 6/(2k) \) on the boundary of \([-2, 2]^d \), which only intersects with no more than \( d! \) \((d - 1)\)-simplices that generate the facets of \( P_k \), the \( d \)-simplex \( Q_i \) intersects with at most \( d! \) facets of \( P_k \). Thus each set \( Q_i \cap (P_{2k} \setminus P_k) \) can be triangulated into at most \( c(d) d \)-simplices, where \( c(d) \) is a constant depending only on \( d \). Consequently, \( P_{2k} \setminus P_k \) can be triangulated into no more than \( 8k^{d-1}dd! \cdot c(d) \). Denote these simplices by \( D_{s_1+1}, D_{s_1+2}, \ldots, D_{s_2} \).

We continue this process for \( P_{3k} \setminus P_{2k} \) and so on to obtain a simplicial approximation sequence \( D = \{D_1, D_2, \ldots\} \). Now we estimate \( S_D(t, \Omega) \). For any \( 0 < t < 1 \), we choose \( r \) to be the smallest integer such that \( \sigma_{d-1}d2^{-2(r-1)} < t \), where \( \sigma_{d-1} \) is the surface of \( d \)-dimensional unit surface. Thus, for \( n \geq r \), we have

\[
|\Omega \setminus \bigcup_{i=1}^n D_i| \leq |\Omega \setminus P_{2^r-1}k| \leq \sigma_{d-1}\cdot d2^{-2(r-1)} < t.
\]
Hence,
\[ S_D(t, \Omega) \leq s_r \leq (4k)^{d-1} dd! + (8k)^{d-1} dd! \cdot c(d) + \cdots + (2^{r-1} \cdot 2k)^{d-1} dd! \cdot c(d) \leq k_d t^{-(d-1)/2}, \]
where \( k_d \) is a constant depending only on \( d \).

A direct computation of the integrals in Theorem 2 gives the concrete upper bounds stated in Theorem 3. (When \( p < \frac{dr}{d+(d-1)r} \), the term \( \varepsilon^{-d/2} \) comes from the second integral.)

The proof for the statement of the bracketing entropy when \( r = \infty \) is similar, and the details are thus omitted.

### 2.9 General Lower Bound: Theorem 1 (i)

In this subsection, we prove the general lower bound stated in Theorem 1 (i). By Lemma 1 and Lemma 2, we only need to prove it for the case when \( \Omega \) is contained in \([0, 1]^d\) and has volume at least 1/d!\). Indeed, by Lemma 1, if \( \Omega \subset \mathbb{R}^d \) is closed and convex, \( \Omega \subset R \) for a box \( R \) with \(|R| \leq d!|\Omega|\). Let \( T \) be any affine transformation that maps \( R \) onto \([0, 1]^d\). Then by Lemma 2, and the fact that \( C_r(T(\Omega)) \supset C_\infty(T(\Omega)) \), we have

\[
N(\|\Omega\|^{\frac{1}{p}} / \varepsilon, C_r(\Omega), \| \cdot \|_{L^p(\Omega)}) = N((\|\Omega|/|R|)^{\frac{1}{p}} / \varepsilon, C_r(T(\Omega)), \| \cdot \|_{L^p(T(\Omega)))}) \\
\geq N(\varepsilon, C_\infty(T(\Omega)), \| \cdot \|_{L^p(T(\Omega)))}).
\]

Thus it suffices to establish a lower bound for the case when \( \Omega \) is contained in \([0, 1]^d\) and has volume at least 1/d!\).

We choose a function \( f \) so that \( f \) is supported on \([0, 1]^d\), with \( 0 \leq f \leq \frac{1}{20} \) and \( \|f\|_1 \geq \frac{1}{80d} \). Furthermore, the Hessian matrix of \( f \) at every \((x_1, x_2, \ldots, x_d) \in [0, 1]^d\) is a diagonal matrix with each entry bounded by \( 1 \). One such function is

\[
f(x_1, x_2, \ldots, x_d) = \begin{cases} 0 & \text{if } (x_1, x_2, \ldots, x_d) \in [0, 1]^d, \\ \frac{1}{20d} \sum_{i=1}^d \sin^3(\pi x_i) & \text{if } (x_1, x_2, \ldots, x_d) \notin [0, 1]^d. \end{cases}
\]

For each fixed \( 0 < \varepsilon < (10d!)^{-2} \), and each \( I = (i_1, i_2, \ldots, i_d) \in \mathbb{N}^d \), define

\[
f_I(x_1, x_2, \ldots, x_d) = \varepsilon^2 \cdot f \left( \frac{x_1 - i_1 \varepsilon}{\varepsilon}, \frac{x_2 - i_2 \varepsilon}{\varepsilon}, \ldots, \frac{x_d - i_d \varepsilon}{\varepsilon} \right).
\]

Then, \( f_I \) is supported on

\[ B_I := [i_1 \varepsilon, (i_1 + 1) \varepsilon] \times [i_2 \varepsilon, (i_2 + 1) \varepsilon] \times \cdots \times [i_d \varepsilon, (i_d + 1) \varepsilon], \]

with \( 0 \leq f_I \leq \frac{\varepsilon^2}{20} \cdot \|f_I\|_1 \geq \frac{\varepsilon^2}{80d} \cdot \varepsilon^d \), and furthermore, the Hessian matrix of \( f_I \) at every \((x_1, x_2, \ldots, x_d) \in B_I \) is a diagonal matrix with each entry bounded by \( 1 \).
We write

$$I = \{ I \mid I = (i_1, i_2, \ldots, i_d) \in \mathbb{N}^d, B_I \subset \Omega \}.$$ 

Let $\xi_I \in \{0, 1\}$, $I \in \mathcal{I}$, be i.i.d. random variables with $\mathbb{P}(\xi_I = 1) = \mathbb{P}(\xi_I = 0) = 1/2$, and define the random function

$$F(x; \xi) = \sum_{I \in \mathcal{I}} \xi_I f_I(x).$$

Then for each realization of $\xi = (\xi_I)_{I \in \mathcal{I}}$, we have $0 \leq F \leq \frac{\epsilon^2}{20}$, and the Hessian matrix of $F$ is diagonal with each entry bounded by 1. Therefore, for each realization of $\xi$, the function

$$G(x; \xi) = \frac{1}{d} \left( x_1^2 + x_2^2 + \cdots + x_d^2 - F(x; \xi) \right)$$

is convex and bounded by 1. Hence, $G(\cdot; \xi) \in C_\infty([0, 1]^d)$.

There are $2^{|\mathcal{I}|}$ realizations of $G(\cdot; \xi)$. Between two realizations, we define the Hamming distance

$$H(G(\cdot; \xi^{(1)}), G(\cdot; \xi^{(2)})) = \# \left\{ I \in \mathcal{I} \mid \xi_I^{(1)} \neq \xi_I^{(2)} \right\}.$$ 

For $r = \lfloor |\mathcal{I}|/10 \rfloor$, consider the set

$$U(G(\cdot; \xi), r) = \left\{ G(\cdot; \xi^{(2)}) : H(G(\cdot; \xi), G(\cdot; \xi^{(2)})) \leq r \right\}.$$ 

For each $G(\cdot; \xi)$, the set $U(G(\cdot; \xi), r)$ contains no more than

$$\sum_{k=0}^{r} \binom{|\mathcal{I}|}{k} \leq 2^{|\mathcal{I}|}/10$$

elements. Thus, by the pigeonhole principle, we can find $m \geq 2^{|\mathcal{I}|} \div 2^{|\mathcal{I}|}/10 = 2^{|\mathcal{I}|}/10$ realizations of $G(\cdot; \xi^{(k)})$, $1 \leq k \leq m$, such that for any $1 \leq i < j \leq m$, we have

$$H \left( G(\cdot; \xi^{(i)}), G(\cdot; \xi^{(j)}) \right) \geq |\mathcal{I}|/10.$$
Note that
\[ \int_{\Omega} \left| G(x; \xi^{(i)}) - G(x; \xi^{(j)}) \right| d\lambda(x) = \frac{1}{d^2} \int_{\Omega} \sum_{I \in \mathcal{I}} |\xi^{(i)}_I - \xi^{(j)}_I| |f_I(x)| d\lambda(x) \]
\[ \geq \frac{1}{d} \sum_{I \in \mathcal{I}} |\xi^{(i)}_I - \xi^{(j)}_I| \frac{\varepsilon^2}{80d} \cdot \varepsilon^d \]
\[ \geq \frac{1}{d} \cdot |\mathcal{I}| / 10 \cdot \frac{\varepsilon^2}{80d} \cdot \varepsilon^d. \quad (18) \]

We show that the cardinality $|\mathcal{I}|$ of $\mathcal{I}$ is at least $1 / 2d! \varepsilon - d$. Indeed, because $\Omega \subset [0, 1]^d$ has volume at least $1 / d!$, and $\Omega$ is convex, the set $[0, 1]^d \setminus \Omega_{\sqrt{d}\varepsilon}$ has volume at most $1 - 1 / d! + 2d \cdot \sqrt{d}\varepsilon$. Thus, $[0, 1]^d \setminus \Omega_{\sqrt{d}\varepsilon}$ contains no more than $e^{-d} \cdot [1 - 1 / d! + 2d \cdot \sqrt{d}\varepsilon]$ cubes $B_I$. Any cube $B_I \subset [0, 1]^d$ that is not contained in $[0, 1]^d \setminus \Omega_{\sqrt{d}\varepsilon}$ does not intersect with $[0, 1]^d \setminus \Omega$, thus must be contained in $\Omega$. Since $[0, 1]^d$ contains $[1 / \varepsilon]^d$ such cubes, we conclude that $\Omega$ contains at least
\[ \frac{1}{\varepsilon} - d \cdot \left[ 1 - 1 / d! + 2d \cdot \sqrt{d}\varepsilon \right] \geq \frac{1}{2d!} \varepsilon - d \]
cubes provided that $\varepsilon$ is small, say $\varepsilon < (10d!)^{-2}$.

Now plugging the inequality $|\mathcal{I}| \geq 1 / 2d! \varepsilon - d$ into (18), we obtain
\[ \int_{\Omega} \left| G(x; \xi^{(i)}) - G(x; \xi^{(j)}) \right| d\lambda(x) \geq c \varepsilon^2 \]
for some constant $c$ depending only on $d$. This implies that $C_\infty([0, 1]^d)$ contains
\[ m \geq 2^{|\mathcal{I}|/10} \geq e^{c' \varepsilon - d} \]
functions whose mutual $L^1(\Omega)$ distance is at least $c \varepsilon^2$.

This implies that
\[ \log N(\varepsilon, C_\infty([0, 1]^d), \| \cdot \|_{L^1(\Omega)}) \geq c'' \varepsilon^{-d/2} \]
for some $c'' > 0$ depending only on $d$.

Since $|\Omega| \geq 1 / d!$, for any $p \geq 1$, we have $\| \cdot \|_{L^p(\Omega)} \geq (d!)^{-1/p} \| \cdot \|_{L^1(\Omega)}$; this implies that
\[ \log N(\varepsilon, C_\infty([0, 1]^d), \| \cdot \|_{L^p(\Omega)}) \geq c \varepsilon^{-d/2} \]
for some constant $c$ depending on $p$ and $d$, provided that $|\Omega| \geq 1 / d!$.

Together with the discussion at the beginning of this subsection and the fact that bracketing entropy is bounded below by metric entropy, we conclude that the lower bound statements of Theorem 1 are true.
2.10 Lower Bound for the Ball: Theorem 4

The \((d - 1)\)-dimensional area of the unit sphere \(S^{d-1}\) in \(\mathbb{R}^d\) is \(2\pi^{d/2}/\Gamma(d/2)\), while the \((d - 1)\)-dimensional area of a cap with height \(h\) is \((\pi^{d/2}/\Gamma(d/2))I_{2h-h^2}((d-1)/2, 1/2)\) \(\sim c_d h^{(d-1)/2}\), where \(I_x(a, b)\) is the regularized incomplete beta function. Thus there exist \(s := \alpha_d h^{-(d-1)/2}\) disjoint spherical caps with height \(h\). The \(d\)-dimensional volume of each spherical cap is \(\beta_d h^{(d+1)/2}\). Let \(x_1, \ldots, x_s\) be the spherical center of the caps. For each \(1 \leq i \leq s\), we define a random function \(f_i\) on the closed unit ball \(\Omega = B_d(0, 1)\) such that, for \(y \in B_d(0, 1)\),

\[
f_i(y) = \begin{cases} 0, & \langle y, x_i \rangle \leq 1 - h, \\ \xi_i \frac{\langle y, x_i \rangle - (1-h)}{h}, & \langle y, x_i \rangle > 1 - h, \end{cases}
\]

where \(\xi_i\) is either 0 or 1. Now \(f_i\) is convex on the closed unit ball, and supported on the \(i\)-th cap \(C_i\); \(f_i(y) = 0\) if \(y/\|y\|_2 \notin C_i\). Furthermore, since the caps are disjoint, the sum \(f = \sum_{i=1}^s f_i\) is also convex and bounded by \(1\). There are \(2^s\) different possibilities for \(f\). By the same argument as we used in the proof of the lower bound of Theorem 1, we can find a set \(W\) of \(2^{s/2}\) functions in which any two functions \(f\) and \(g\) are different on at least \(s/10\) caps.

On each cap where the two functions are defined differently, \(|f - g| \geq 1/2\) the top half height of the cap which has a volume \(\gamma_d h^{(d+1)/2}\). Consequently the \(L^p\) distance between any two functions \(f, g \in W\) is at least

\[
\frac{1}{2} (s/10) \cdot \gamma_d h^{(d+1)/2} \leq \delta_d h^{1/p}.
\]

Letting \(\delta_d h^{1/p} = \varepsilon\), we have

\[
N(\varepsilon, C_{\infty}(B_d(0, 1)), \| \cdot \|_{L^p(B_d(0, 1))}) \geq \exp \left( C\varepsilon^{-(d-1)p/2} \right).
\]

When \((d - 1)p \leq d\), the lower bound above should be replaced by the universal lower bound \(\varepsilon^{-d/2}\) proved in the last section.

2.11 Optimality of Lower Bound for Polytopes: Remark 1

Clearly, it is enough to show (1) for the case \(r = \infty\), and \(p = 1\). Let \(\Omega\) be the regular \((m + 2)\)-gon inscribed in the unit circle, which can be triangulated into \(m\) triangles. If \(m < 4\), the statement simply follows from Theorem 1 (i) with \(c_2 = c_1/6\). If \(m \geq 4\), by connecting all other vertices, we can cut off \(n = \lfloor m/2 \rfloor\) isosceles triangles from \(\Omega\). Denote these isosceles triangles by \(\Delta_i, 1 \leq i \leq n\). Each \(\Delta_i\) has base-length \(2 \sin(\pi/m+2)\) and height \(1 - \cos(\pi/m+2)\). If \(\varepsilon < 1/4m^{-2}\), each \(\Delta_i\) contains \(cm^{-3}\varepsilon^{-2} \geq 2\) disjoint squares of side-length \(\varepsilon\). All together, these \(n\) isosceles triangles contain \(c'm^{-2}\varepsilon^{-2}\) disjoint squares of side-length \(\varepsilon\) for some constant \(c'\). We denote by \(\mathcal{F}\) the class of these small squares.
Note that the base line of each isosceles triangles separates the isosceles triangle from the rest of $\Omega$. If $f_0$ is a fixed function, and each $f_i$ is a function defined on $\Omega$ and supported on $\Delta_i$, such that $f_0 \pm f_i$ is convex, then for all choices of $\epsilon_i \in \{0, 1\}$, $1 \leq i \leq n$, the function $g = f_0 + \sum_{i=1}^{n} \epsilon_i f_i$ is also convex on $\Omega$. Therefore, replacing the class $I$ in Sect. 2.9 by the class $J$ defined above, the same argument in Sect. 2.9 gives

$$\log N(\epsilon, \Omega, \| \cdot \|_p) \geq c'' \sqrt{|J|} \geq c'' m \epsilon^{-d/2}$$

for some constants $c''$ and $c'''$. This finishes the proof of the statement in Remark 1.

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