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Well-Posedness and Porosity for Symmetric Optimization Problems

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Abstract: In the present work, we investigate a collection of symmetric minimization problems, which is identified with a complete metric space of lower semi-continuous and bounded from below functions. In our recent paper, we showed that for a generic objective function, the corresponding symmetric optimization problem possesses two solutions. In this paper, we strengthen this result using a porosity notion. We investigate the collection of all functions such that the corresponding optimization problem is well-posed and prove that its complement is a $\sigma$-porous set.

Keywords: complete metric space; generic element; lower semi-continuous function; porous set

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1. Introduction

In this paper, we study a class of symmetric minimization problems, which was studied recently in our paper [1]. The results of [1] and of the present paper have prototypes in [2,3], where some minimization problems arising in crystallography were considered. It was shown in [2,3] that a typical symmetric minimization problem possesses exactly two minimizers, and every minimizing sequence converges to them in some natural sense. In [1], we extend the results of [2,3] for a sufficiently large class of symmetric minimization problems by showing that for a generic objective function, the corresponding symmetric optimization problem possesses two solutions. In this paper, we strengthen this result using a porosity notion. We investigate the collection of all functions such that the corresponding optimization problem is well-posed and prove that its complement is a $\sigma$-porous set.

More precisely, we study an optimization problem

$$g(\xi) \to \min, \; \xi \in X,$$

where $X$ is a complete metric space and $g$ is a lower semi-continuous and bounded from below function.

It is well-known that the above problem possesses a minimizer when the space $X$ is compact or when the objective function $f$ possesses a growth property and all bounded subsets of the space $X$ satisfy certain compactness assumptions. Without such assumptions, the existence problem becomes more difficult. This difficulty is overcome by applying the Baire category approach, which was used for many mathematical problems [4–9].

Namely, it is known that the minimization problem stated above can be solved for a generic objective function [8–10]. More precisely, there is a collection $F$ in a complete metric space of objective functions, which is a countable intersection of open and everywhere dense sets such that for every objective function $f \in F$, the corresponding minimization problem has a unique solution, which is a limit of every minimizing sequence. See [9], which contains this result and its several extensions and modifications. Note that the generic approach in nonlinear analysis is used in [11–15], generic solvability of best approximation problems are discussed in [4,11,13], while generic existence of fixed points for nonlinear operators is established in [7,12,13].
In our recent paper [1] the goal was to establish a generic solvability of optimization problems with symmetry. These results have applications in crystallography [2,3]. In this paper, we strengthen this result using a porosity notion. We investigate the set of all functions for which the corresponding minimization problem is well-posed and show that its complement is a \( \sigma \)-porous set.

2. The Main Result

We begin this section recalling the following notion of porosity [3,4,7,9,12,13]. Suppose that \((Y, d)\) is a complete metric space and define

\[
B_d(y, r) = \{ \xi \in X : d(y, \xi) \leq r \}.
\]

We say that a set \( E \subseteq Y \) is porous with respect to \( d \) (or just porous if the metric is understood) if there are a real number \( \alpha \in (0, 1] \) and a positive number \( r_0 \) such that for every positive number \( r \leq r_0 \) and every point \( y \in Y \) there is a point \( z \in Y \) such that

\[
B_d(z, \alpha r) \subseteq B_d(y, r) \setminus E.
\]

We say that a set in the complete metric space \( Y \) is \( \sigma \)-porous with respect to \( d \) (or just \( \sigma \)-porous if the metric is understood) if this set is a countable union of porous (with respect to \( d \)) subsets of \( Y \).

For every function \( h : Y \to (-\infty, 0] \), where the set \( Y \) is nonempty, put

\[
\inf(h) = \inf\{ h(\xi) : \xi \in Y \}
\]

and

\[
\text{dom}(h) = \{ y \in Y : h(y) < \infty \}.
\]

Suppose that \((X, \rho)\) is a complete metric space. For every \( z \in X \) and every positive \( \Delta \) put

\[
B(z, \Delta) = \{ \xi \in X : \rho(z, \xi) \leq \Delta \}.
\]

For every \( z \in X \) and every subset \( D \neq \emptyset \) of the space \( X \), define

\[
\rho(z, C) = \inf\{ \rho(z, \xi) : \xi \in C \}.
\]

Denote by \( M_1 \) the collection of all functions \( f : X \to \mathbb{R}^1 \cup \{ \infty \} \), which are bounded from below, lower semi-continuous, and which are not identical infinity. For each \( h_1, h_2 \in M_1 \), define

\[
\tilde{d}(h_1, h_2) = \sup\{ |h_1(z) - h_2(z)| : z \in X \},
\]

\[
d(h_1, h_2) = \tilde{d}(h_1, h_2)(1 + \tilde{d}(h_1, h_2))^{-1}.
\]

Note that by convention, \( d(h_1, h_2) = 1 \) when \( \tilde{d}(h_1, h_2) = \infty \).

It is clear that \( d : M_1 \times M_1 \to [0, \infty) \) is a complete metric. We denote by \( M_c \) the collection of all continuous finite-valued functions \( f : X \to \mathbb{R}^1 \) which are bounded from below. Clearly, \( M_c \) is a closed set in the complete metric space \((M_1, d)\). We endow the space \( M_c \) with the metric \( d \) too.

Suppose that \( T : X \to X \) is a continuous operator such that

\[
T^2(z) = z \text{ for every } z \in X.
\]

We denote by \( M_{1,T} \) the collection of all functions \( f \in M_1 \) for which

\[
f(T(x)) = f(x) \text{ for every point } x \in X
\]

and define

\[
M_{c,T} = \{ f \in M_c : f \circ T = f \}.
\]
Evidently, $\mathcal{M}_{l,T}$ and $\mathcal{M}_{c,T}$ are closed subsets of the complete metric space $\mathcal{M}_l$. We endow them with the same metric $d$ too.

We investigate the optimization problem

$$f(x) \to \min, \ x \in X,$$

where the objective function $f \in \mathcal{M}_{l,T}$.

Given $f \in \mathcal{M}_{l,T}$, we say that the problem of minimization for $f$ on $X$ is well-posed with respect to $(\mathcal{M}_l, d)$ if the following properties are true:

There exists $x_f \in X$, which satisfies

$$\{x \in X : f(x) = \inf(f)\} = \{x_f, T(x_f)\}$$

and for every $\epsilon > 0$ there are an open neighborhood $\mathcal{U}$ of $f$ in $\mathcal{M}_l$ and a positive number $\delta$ such that if a function $g \in \mathcal{U}$ and if a point $z \in X$ satisfies $g(z) \leq \inf(g) + \delta$, then

$$|g(z) - f(x_f)| \leq \epsilon$$

and

$$\min\{\rho(z, \{x_f, T(x_f)\}), \rho(T(z), \{x_f, T(x_f)\})\} \leq \epsilon.$$

This notion has an analog in the optimization theory [9], where the set of minimizers is a singleton. Here, since the problem is symmetric, the set of minimizers contains two points in general.

The next theorem is our sole main result.

**Theorem 1.** Suppose that $A$ is either $\mathcal{M}_{l,T}$ or $\mathcal{M}_{c,T}$. Then, there is a set $B \subset A$ such that its complement $A \setminus B$ is $\sigma$-porous in the metric space $(A, d)$ and that for every function $f \in B$ the minimization problem for $f$ on the space $X$ is well-posed with respect to $(\mathcal{M}_l, d)$.

### 3. Auxiliary Results

**Lemma 1.** For every positive number $r \leq 1$, each $f, g \in \mathcal{M}_l$, which satisfy $d(f, g) \leq 4^{-1}r$ and each $x \in X$,

$$|g(x) - f(x)| \leq r.$$

**Proof.** Let $r \in (0,1], f, g \in \mathcal{M}_l$ satisfy

$$d(f, g) \leq 4^{-1}r$$

and let $x \in X$ be given. By (2) and (3),

$$d(f, g) \leq 4^{-1},$$

$$\bar{d}(f, g) = d(f, g)(1 - d(f, g))^{-1} \leq 2d(f, g) \leq 2^{-1}r.$$

In view of (1) and the equation above,

$$|g(x) - f(x)| \leq 2^{-1}r.$$

$\square$

**Lemma 2.** Suppose that $f \in \mathcal{M}_{l,T}, \epsilon \in (0,1), r \in (0,1]$. Then there are $\tilde{f} \in \mathcal{M}_{l,T}$ and $x \in X$ such that $\tilde{f} \in \mathcal{M}_{c,T}$ if $f \in \mathcal{M}_{c,T}$,

$$f(x) \leq \tilde{f}(x) \leq f(x) + r/2, \ x \in X$$

(4)
and that for each \( y \in X \), which satisfies
\[
\bar{f}(y) \leq \inf(f) + \epsilon r/4
\]
the equation
\[
\min\{\rho(y, \bar{x}), \rho(T(y), \bar{x})\} \leq \epsilon
\]
is true.

**Proof.** There exists \( \bar{x} \in X \) satisfying
\[
f(\bar{x}) \leq \inf(f) + \epsilon r/4.
\]
Define a function \( \bar{f} \in M_{I} \) as follows:
\[
\bar{f}(x) = f(x) + 2^{-1}r \min\{\rho(x, \bar{x}), \rho(T(x), \bar{x})\}, \ x \in X.
\]
Clearly, \( \bar{f} \in M_{I,T} \) and \( \bar{f} \in M_{c,T} \) if \( f \in M_{c,T} \) and (4) is true. Let \( y \in X \) and (5) hold.
By (5) and (6),
\[
f(y) + 2^{-1}r \min\{\rho(y, \bar{x}), \rho(T(y), \bar{x})\} = \bar{f}(y) \leq \inf(\bar{f}) + \epsilon r/4
\leq \bar{f}(\bar{x}) + \epsilon r/4 = f(\bar{x}) + \epsilon r/4 \leq f(y) + \epsilon r/2.
\]
Therefore,
\[
\min\{\rho(y, \bar{x}), \rho(T(y), \bar{x})\} \leq \epsilon.
\]

\( \square \)

**4. Proof of Theorem 1**

For every integer \( n \geq 1 \) let \( A_n \) be the collection of all functions \( f \in A \) such that:
(i) there are a point \( \bar{x} \in X \) and \( \delta > 0 \) such that if \( z \in X \) and \( f(z) \leq \inf(f) + \delta \), then the inequality \( \rho(\bar{x}, \{z, T(z)\}) \leq 1/n \) is valid.
Let a natural number \( n \) be given. We claim that the set \( A \setminus A_n \) is porous.
By Lemma 1, for every positive number \( r \leq 1 \), each \( f, g \in M_{I} \) satisfying \( d(f, g) \leq 4^{-1}r \) and each \( x \in X \),
\[
\|g(x) - f(x)\| \leq r.
\]
By Lemma 2 applied with \( \epsilon = (2n)^{-1} \), the following property is valid:
(ii) for each function \( f \in A \) and every positive number \( r \leq 1 \), there exist \( \bar{f} \in A \) and \( \bar{x} \in X \) such that
\[
\bar{d}(f, \bar{f}) \leq r/4
\]
and that for each \( y \in X \) satisfying
\[
\bar{f}(y) \leq \inf(\bar{f}) + 16^{-1}rn^{-1}
\]
the equation
\[
\min\{\rho(y, \bar{x}), \rho(T(y), \bar{x})\} \leq (2n)^{-1}.
\]
is valid.

Fix
\[
r = 4^{-1}, \ a = 80^{-1}n^{-1}.
\]
Let \( f \in A \) and a positive number \( r \leq r \) be given. By property (ii), there exist \( \bar{f} \in A \) and \( \bar{x} \in X \) such that
\[
\bar{d}(f, \bar{f}) \leq r/4
\]
and that the next property is true:
(iii) for every point \( y \in X \) satisfying (9), Equation (10) is true.
Let a function $g \in A$ satisfy
\[ d(g, f) \leq ar. \]  (13)

By (2) and (11)–(13),
\[ d(g, f) \leq ar + r/4 \leq r/2. \]  (14)

By (2), (11) and (13),
\[ d(g, f) \leq d(g, f)(1 - d(g, f))^{-1} \]
\[ \leq ar(1 - ar)^{-1} \leq 2ar \]  (15)

and
\[ |\inf(f) - \inf(g)| \leq 2ar. \]  (16)

Let a point $z \in X$ satisfy the inequality
\[ g(z) \leq \inf(g) + ar. \]  (17)

By (15),
\[ |g(z) - \bar{f}(z)| \leq 2ar. \]  (18)

By (11) and (16)–(18),
\[ \bar{f}(z) \leq g(z) + 2ar \leq \inf(g) + 3ar \leq \inf(f) + 5ar \]
\[ \leq \inf(f) + 16^{-1}rn^{-1}. \]  (19)

Property (iii), (9), (10) and (19) imply that
\[ \min\{\rho(z, \bar{x}), \rho(T(z), \bar{x})\} \leq (2n)^{-1}. \]

Thus
\[ g \in A_n \]
by definition. Together with (14), this implies that
\[ \{g \in A : d(g, f) \leq ar\} \subset \{g \in A : d(g, f) \leq r\} \cap A_n. \]

Thus, the set $A \setminus A_n$ is $\sigma$-porous. Then the set
\[ A \setminus \bigcap_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} (A \setminus A_n) \]
is $\sigma$-porous.

Let
\[ f \in \bigcap_{n=1}^{\infty} A_n. \]  (20)

By (20), for every integer $n \geq 1$, there are $x_n \in X$ and $\delta_n > 0$ such that the following property is valid:
(iv) if a point $z \in X$ satisfies the inequality $f(z) \leq \inf(f) + \delta_n$, then the equation
\[ \rho(x_n, \{z, T(z)\}) \leq 1/n \]
holds.

Suppose that a sequence $\{z_i\}_{i=1}^{\infty} \subset X$ satisfies
\[ \lim_{i \to \infty} f(z_i) = \inf(f). \]  (21)

Let a natural number $n$ be given. By (21) and property (iv), for every large enough positive integer $i$,
\[ \{\rho(x_n, \{z_i, T(z_i)\}) \leq n^{-1}. \]
Since \( n \) is an arbitrary positive integer, there is a sub-sequence \( \{z_{i_p}\}_{p=1}^{\infty} \) such that at least one of the sequences \( \{z_{i_p}\}_{p=1}^{\infty} \) and \( \{T(z_{i_p})\}_{p=1}^{\infty} \) converges. Since \( T \) is continuous and \( T^2 \) is the identity operator, they both converge and

\[
T\left(\lim_{p \to \infty} z_{i_p}\right) = \lim_{p \to \infty} T(z_{i_p}).
\]  
(22)

Set

\[
x_f = \lim_{p \to \infty} z_{i_p}.
\]  
(23)

By (21), (23) and the lower semi-continuity of \( f \),

\[
f(x_f) = f(T(x_f)) = \inf(f).
\]  
(24)

Applying property (iv) with \( z_i = x_f \) for every natural number \( i \), we obtain that

\[
\rho(x_n, \{x_f, T(x_f)\}) \leq n^{-1} \text{ for every natural number } n \geq 1.
\]  
(25)

Let \( \xi \in X \) be such that

\[
f(\xi) = \inf(f).
\]  
(26)

By (26) and property (iv) applied with \( z_i = \xi \) for every integer \( i \geq 1 \) we obtain that

\[
\rho(x_n, \{\xi, T(\xi)\}) \leq n^{-1} \text{ for every natural number } n.
\]  
(27)

Equations (25) and (27) imply that

\[
\min\{\rho(\xi, x_f), \rho(T(\xi), x_f), \rho(\xi, T(x_f)), \rho(T(\xi), T(x_f))\} \leq 2n^{-1}.
\]  
(28)

Since \( n \) is an arbitrary positive integer, we conclude that and at least one of the following equalities is true:

\[
\xi = x_f, \quad \xi = T(x_f).
\]  
(29)

Thus

\[
\{x \in X : f(x) = \inf(f)\} = \{x_f, T(x_f)\}.
\]  
(30)

Let \( \epsilon > 0 \). Fix a natural number \( n \) such that

\[
4n^{-1} < \epsilon.
\]  
(31)

Property (iv) and (25) imply that for every \( z \in X \) which satisfies the inequality

\[
f(z) \leq \inf(f) + \delta_n,
\]

we have

\[
\rho(x_n, \{z, T(z)\}) \leq 1/n
\]  
(32)

\[
\min\{\rho(z, x_f), \rho(T(z), x_f), \rho(z, T(x_f)), \rho(T(z), T(x_f))\} \leq 2n^{-1}.
\]  
(33)

Fix a positive number

\[
\delta < \min\{3^{-1}\delta_n, 8^{-1}\epsilon\}.
\]  
(34)

Let a function \( g \in \mathcal{M}_1 \) satisfy

\[
\tilde{d}(g, f) \leq \delta
\]  
(35)

and let a point \( z \in X \) be such that

\[
g(z) \leq \inf(g) + \delta.
\]  
(36)
By Equations (32)–(34), we have
\[ f(z) \leq g(z) + \delta \leq \inf(g) + 2\delta \leq \inf(f) + 3\delta \leq \inf(f) + \delta_n. \] (35)

It follows from (29), (31) and (35) that
\[ \min\{\rho(z, x_f), \rho(T(z), x_f), \rho(z, T(x_f)), \rho(T(z), T(x_f))\} \leq 2n^{-1} < \epsilon. \] (36)

Thus, (28) holds and for each function \( g \in M \) which satisfies (33) and every point \( z \in X \) satisfying (34) Equation (36) holds. By Equations (33) and (34), we have
\[ |g(z) - \inf(f)| \leq 2\delta < \epsilon. \]

Thus, the minimization problem for \( f \) on \( X \) is well-posed with respect to \((M, d)\) for all \( f \in \cap_{n=1}^{\infty} A_n \). Theorem 1 is proved.

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