QCD Sum-Rule Consistency of Lowest-Lying $q\bar{q}$ Scalar Resonances

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Abstract

We investigate lowest-lying scalar meson properties predicted from QCD Laplace sum rules based upon isovector and isoscalar non-strange $\bar{q}q$ currents. The hadronic content of these sum rules incorporates deviations from the narrow resonance approximation anticipated from physical resonance widths. The field theoretical content of these sum rules incorporates purely-perturbative QCD contributions to three-loop order, the direct single-instanton contribution in the instanton liquid model, and leading contributions from QCD-vacuum condensates. In the isovector channel, the results we obtain are compatible with $a_0(1450)$ being the lowest-lying $qq$ resonance, and are indicative of a non-$q\bar{q}$ interpretation for $a_0(980)$. In the isoscalar channel, the results we obtain are compatible with the lowest lying $q\bar{q}$ resonance being $f_0(980)$ or a state somewhat lighter than $f_0(980)$ whose width is less than half of its mass. The dilaton scenario for such a narrower $\sigma$-resonance is discussed in detail, and is found compatible with sum rule predictions for the resonance coupling only if the anomalous gluon-field portion of $\Theta^\mu_\mu$ dominates the matrix element $\langle \sigma | \Theta^\mu_\mu | 0 \rangle$. A linear sigma-model interpretation of the lowest-lying resonance’s coupling, when compared to the coupling predicted by sum rules, is indicative of a renormalization-group invariant light-quark mass between 4 and 6 MeV.

1 Introduction: Status of the Lowest-Lying Scalar Resonances

At present, there is a great deal of confusion concerning both the identity and interpretation of the lowest lying $I = 0$ and $I = 1$ scalar resonances, specifically the four states denoted in the Particle Data Guide by $f_0(400-1200)$, $f_0(980)$, $a_0(980)$, and $a_0(1450)$. The nearness of the $f_0(980)$ and $a_0(980)$ to a $K\bar{K}$ threshold has led to a widely held interpretation of these states as isopartner $K\bar{K}$-molecules [1,2,3], as opposed to light $qq$-resonances (linear combinations of $u\bar{u}$ and $d\bar{d}$ states). However, the assumption that these states are isopartners and/or $K\bar{K}$-molecules have both been subject to recent scrutiny. In particular, Morgan and Pennington [4] have disputed the $K\bar{K}$ interpretation of $f_0(980)$. An analysis using the Jülich model for $\pi\pi$ scattering [5] is compatible with a $K\bar{K}$ interpretation of $f_0(980)$, but sees
\(a_0(980)\) as a dynamical threshold effect, as opposed to a true resonance state. An even more recent analysis of OPAL data [6] supports the consistency of a \(q\bar{q}\) interpretation of \(f_0(980)\).

Theory is similarly ambivalent regarding the \(f_0(980)\) and \(a_0(980)\) scalar resonance states. A QCD sum rule analysis [7] based upon correlation-function currents chosen to project out \(K\bar{K}\)-molecule states concludes that \(f_0(1500)\) and \(f_0(1710)\) are better candidates for such multiquark states than either \(f_0(980)\) or \(a_0(980)\). Moreover, a very recent coupled channel analysis of \(\pi\pi\) scattering [8] suggests that the \(f_0(980)\) state may really be two distinct S-matrix poles (see also [4]), one corresponding to a \(K\bar{K}\) molecule and the other perhaps corresponding to a light \(q\bar{q}\) state.

Indeed, the determination of the lowest lying \(q\bar{q}\) \(I = 0\) and \(I = 1\) states is of genuine value as a test of our present understanding of QCD, particularly its nonperturbative content. Such lowest lying states, when first compared with QCD via sum rule methods [9], were necessarily found to be degenerate, as purely-perturbative and QCD-vacuum condensate contributions to scalar-current correlation functions cannot distinguish between \(I = 0\) and \(I = 1\) channels. Of course, this result was first seen to account for the degeneracy of \(f_0(980)\) and \(a_0(980)\) as lowest-lying \(q\bar{q}\) isopartners [9,10]. As has been emphasized repeatedly over the last twenty years, however, both scalar and pseudoscalar channels exhibit significant sensitivity not only to nonperturbative field-theoretical effects with infinite correlation length (QCD-vacuum condensates), but also to instanton effects, the nonperturbative content of the QCD vacuum characterized by finite correlation lengths [11,12,13,14,15,16]. The instanton component of the QCD vacuum is known to distinguish between \(I = 0\) and \(I = 1\) scalar (and pseudoscalar) states [12,13]. Such a distinction is, of course, quite evident in the pseudoscalar channel’s large \(\pi - \eta\) mass difference, though an understanding of the pseudoscalar \(I = 1\) channel necessarily must take into account the first pion-excitation state because of the near-masslessness (and concomitantly reduced sum-rule contribution) of the pion [17,18,19]. Similarly, the existence of instanton solutions in QCD necessarily imposes the theoretical expectation that a similar split occur between \(I = 0\) and \(I = 1\) \(q\bar{q}\) scalar resonance states, with the \(I = 0\) state substantially lighter than its \(I = 1\) isopartner [12,13]. In this regard, scalar meson spectroscopy is a genuine test of QCD.

Recent activity [20,21,22,23] in re-analyzing old \(\pi\pi\) and \(\pi N\) scattering data has led to the reinstatement of a lowest-lying \(I = 0\) scalar resonance that
is distinct from the $f_0(980)$, conservatively labelled by the 1996 Particle Data Group [1] as $f_0(400-1200)$. Perhaps of equal importance, assuming $a_0(980)$ and $f_0(980)$ really are either $K\bar{K}$ isopartners or other non-$q\bar{q}$ exotica, is the Crystal Barrel Collaboration’s recent confirmation of an $a_0(1450)$ $I = 1$ scalar resonance state [24]. If this isovector state is identified as the lowest-lying $I = 1 q\bar{q}$-scalar resonance (lattice simulations have shown glueballs below 1600 MeV to be unlikely), it is important to determine whether the identification of its $I = 0$ isopartner with any portion of the $f_0(400-1200)$ mass range (particularly the high end [4,25]) is compatible with the instanton-generated mass difference anticipated from QCD.

Moreover, the $f_0(400-1200)$ has been widely interpreted to be the $\sigma$-particle signature of chiral symmetry breaking anticipated from NambuJona-Lasinio (NJL) dynamics [26], the linear sigma-model (L$\sigma$M) spectrum [27], as well as in models for $q\bar{q}$ scattering in an instanton background [12,13] and in one boson exchange models of the nucleon-nucleon potential [28]. Such a $\sigma$-particle, however, does not characterize the nonlinear sigma model (NL$\sigma$M); indeed, the empirical absence of a credible $\sigma$ prior to 1996 has provided impetus for the development of NL$\sigma$M ideas into a chiral perturbation theory framework. Clearly a clarification of the properties, or even the existence [29], of a light $\sigma$-resonance is required to distinguish between L$\sigma$M and NL$\sigma$M alternatives for effective theories of low-energy hadron physics.

We choose here to distinguish, somewhat arbitrarily, four different alternatives for the $f_0(400-1200)$ resonance that each have some empirical support:

1. The resonance is both very light ($m_\sigma \lesssim 500$ MeV) and very broad ($\Gamma_\sigma \gtrsim 500$ MeV), as suggested by DM2 data [30] and by Törnqvist and Roos’s analysis of $\pi\pi$ scattering [23]. It must be recognized, however, that such a resonance may be generated dynamically [8] and is not necessarily a $q\bar{q}$ state.

2. The resonance is $\sigma$-like in mass (500-700 MeV) but substantially narrower in width ($\Gamma_\sigma \lesssim m_\sigma/2$), consistent with parameter ranges extracted by Svec [20], S. Ishida et al [21], and Harada, Sannino, and Schechter [22].

3. The lowest $I = 0 q\bar{q}$ scalar is to be identified with a $q\bar{q}$ pole at (or perhaps masked by) the $f_0(980)$ resonance [8], with a narrow ($\Gamma_\sigma \lesssim 150$ MeV) width.
4. The lowest \( I = 0 \) \( q\bar{q} \) scalar resonance is a very broad structure in the \( \pi\pi \) scattering amplitude characterized by a mass comparable to [4] or substantially above [25] that of the \( f_0 (980) \).

There is, of course, some blurring of the boundaries between these alternatives, particularly Alternatives 2 and 3. Svec [31] has recently reported a single-pole fit of \( \pi-N \)(polarized) scattering data leading to a mass \((775 \pm 17 \text{ MeV}) \) somewhat larger than NJL-L\( \sigma \)M expectations, and a width \((147 \pm 33 \text{ MeV}) \) substantially narrower than those already quoted in support of Alternative 2. Moreover, Harada, Sannino, and Schechter have demonstrated [32] how a very light, very broad state [Alternative 1] can transmute into a heavier, narrower state [Alternative 2] when \( \rho \) exchanges are taken into account. From a theoretical point of view, Alternatives 1 and 2 support the existence of a \( \sigma \)-particle, though straightforward \( \text{L}\sigma\text{M} \) expectations would favour the mass range of Alternative 2 and the broad width of Alternative 1 [33,34]. Theoretical arguments for a narrower \( \sigma \)-particle more fully consistent with Alternative 2 have been advanced through identifying the \( \sigma \) with the near-Goldstone particle of dilatation symmetry in the strong coupling limit [35,36]. Such a dilaton would be expected to have a width similar to that of the \( \rho \)-meson, corresponding to cancellation of an enhancement factor of \( 9/2 \) in a naive calculation of the width [33] against an anticipated suppression factor \( 1/d_{\sigma}^2 \approx 1/4 \), where \( d_{\sigma} \) is the anomalous mass dimension in the strong coupling limit. \footnote{Note that \( f_{\pi} \) appearing in eq. (13) of [35] should be understood to be 131 MeV, not 93 MeV, so that the \( d_{\sigma} = 1 \) prediction of \( \Gamma_{\sigma} \) coincides with the \( \text{L}\sigma\text{M} \)-equivalent prediction in [34]. This point has been verified through personal communication with V. A. Miransky.}

In the present manuscript, we employ QCD Laplace sum-rules as a technique particularly well-suited to relate the field-theoretical content of QCD to lowest-lying resonance properties [14]. We use sum-rule methodology specifically to address the following questions:

*Which, if any, of the four alternatives discussed above for the lightest \( I = 0 \) \( q\bar{q} \) scalar are supported by QCD sum rules?* In particular, do QCD sum rules rule out either of the alternatives that are consistent with an NJL/L\( \sigma \)M \( \sigma \)-like object? If the existence of a \( \sigma \) is consistent with QCD sum rules, is such a \( \sigma \) a broad \( \text{L}\sigma\text{M} \) object, or a narrower strong-coupling dilaton?
What is the mass range for the lightest $I = 1 q\bar{q}$ scalar resonance? In particular, can we rule out all but exotic interpretations of the $a_0(980)$, and does there exist sum-rule support for the recently confirmed $a_0(1450)$ being the lowest-lying $q\bar{q}$ object in this channel?

In the section that follows, we present the sum rule methodology necessary to address these questions. Specifically, we show how nonzero resonance widths can be incorporated into the hadronic contribution to QCD Laplace sum rules, which are argued to be particularly appropriate for studying lowest-lying resonance properties. We also demonstrate explicitly how a lowest-lying resonance’s nonzero width elevates a sum rule determination of that resonance’s mass.

In Section 3 we present the field-theoretical content of appropriate scalar current sum rules. We discuss the sum rule contribution arising from the 3-loop order purely-perturbative QCD contributions to the scalar current correlation function. We also present nonperturbative sum rule contributions arising from QCD-vacuum condensates and direct single-instanton contributions to the $I = 0$ and $I = 1$ scalar current correlation functions.

In Section 4, we utilize the results of the preceding two sections to obtain a sum-rule determination of the masses of lowest-lying scalar resonances. Stability curves are generated leading to estimates of such masses for a given choice of width and the continuum threshold above which perturbative QCD and hadronic physics are assumed to coincide. Detailed comparison is made with earlier sum-rule generated stability curves [9], showing how the separate incorporation of renormalization-group improvement, 3-loop perturbative effects, nonzero widths, and the contribution of instantons individually affect such curves.

In Section 5, we examine the isoscalar channel in further detail by obtaining values of the mass, width, continuum threshold, and coupling of the lowest-lying $qq$ resonance via a weighted least-squares fit to the overall Borel-parameter dependence of the first Laplace sum rule. A relationship between the anticipated resonance coupling and a phenomenologically estimable matrix element is developed in Appendix A. In Section 5 this relationship is utilized to obtain an estimate of the light quark mass. This relationship is also used to assess the sum-rule consistency of a dilaton interpretation for the lowest-lying resonance.

Finally in Section 6 we present our conclusions concerning the questions
we have raised above. We assess the compatibility of sum rule predictions for the lowest-lying non-exotic I=1 resonance with $a_0(980)$ and $a_0(1450)$. We examine in detail the four alternatives presented above for interpreting the $I = 0$ scalar resonance spectrum and argue that Alternatives 1 and 4 appear to be unsupported by a sum-rule based analysis of the lowest-lying $q\bar{q}$ state. We also discuss how our conclusions are affected by possible sum rule contamination from higher resonances.
2 Sum-Rule Methodology and Lowest-Lying
\(q\bar{q}\)-Scalar Resonances of Nonzero Width

In the narrow resonance approximation, subcontinuum resonance contributions to the light-quark scalar-current correlation function,\(^2\)

\[
\Pi(p^2) = i \int d^4 x e^{ip \cdot x} < 0|T j(x) j(0)|0>, \tag{1}
\]

\[j(x) \equiv [\bar{u}(x) u(x) \pm \bar{d}(x) d(x)]/2, \tag{2}\]

are proportional to a sum of delta functions:

\[
(Im \Pi(s))_{\text{res.}} = Im \sum_r \left[ \frac{-g_r}{(s - m_r^2) + im_r \Gamma_r} \right] = \sum_r \left[ \frac{g_r m_r \Gamma_r}{(s - m_r^2)^2 + m_r^2 \Gamma_r^2} \right] \to \sum_r \pi g_r \delta(s - m_r^2). \tag{3}\]

The coupling coefficient \(g_r\) is proportional to \(m_r^2\). However, the constant of proportionality is expected to be much larger for \(q\bar{q}\) resonances [i.e. resonances that couple directly to the field-theoretical operators in the scalar current (2)] than for exotic resonances [37]. It is for precisely this reason that sum-rule searches for non-\(q\bar{q}\) scalar resonance states, such as \(K\bar{K}\) molecules [7] or glueballs [37,38], utilize correlation functions based on appropriate \(K\bar{K}\)-or gluonic-currents that couple directly to such hadronic exotica.

Laplace sum rules \(R_k(\tau)\) for the scalar current correlation function are particularly sensitive to the lowest-lying \(q\bar{q}\) resonance of a given isostructure:

\[
R_k(\tau) \equiv \frac{1}{\pi} \int_0^\infty s^k Im[\Pi(s)] e^{-\tau s} ds
\]

\(^2\)We have normalized \(I=0\) (+) and \(I = 1\) (-) scalar currents in (2) so as to facilitate comparison with ref. [9].
\[
\sum_{r} g_r m_r^{2k} e^{-m_r^{2}\tau} \Theta(s_0 - m_r^{2}) \\
+ \frac{1}{\pi} \int_{s_0}^{\infty} s^k \text{Im}[\Pi(s)_{\text{pert.}}] e^{-s\tau} ds.
\]

The summation over resonances in (4) clearly follows from the final line of (3). The remaining integral in (4) reflects the anticipated duality [39] between purely-perturbative QCD and hadronic physics above some appropriately chosen continuum threshold \( s > s_0 \). As is evident from (4), higher-mass resonances are either absorbed into the continuum \( (m_r^2 > s_0) \), or if subcontinuum \( (m_r^2 < s_0) \), are exponentially suppressed relative to low-mass resonances. Consequently, Laplace sum rules are well-suited for determining properties of the lowest-lying resonance in a given channel. The subcontinuum resonance contribution \( R_k(\tau, s_0) \) to the \( k^{th} \) Laplace sum rule, defined as

\[
R_k(\tau, s_0) \equiv R_k(\tau) - \frac{1}{\pi} \int_{s_0}^{\infty} s^k \text{Im}[\Pi(s)_{\text{pert.}}] e^{-s\tau} ds,
\]

is clearly seen from (4) to satisfy the inequality

\[
\frac{R_{k+1}(\tau, s_0)}{R_k(\tau, s_0)} \geq m_\ell^2,
\]

where \( m_\ell \) is the mass of the lowest-lying resonance in a given channel. QCD sum-rule methodology for a given resonance channel generally involves obtaining an estimate of the mass \( m_\ell \) via minimization of the field-theoretical content of the left-hand side of (6) with respect to the Borel parameter \( \tau \) \([14,40]\). In practice, the ratio utilized is \( R_1/R_0 \), so as to avoid methodologically inconvenient enhancement of continuum and higher-mass resonance contributions, as well as heightened dependence on (unknown) higher dimensional condensates, through the factors \( m_r^{2k} \) appearing in (4).

The field theoretical content of the first two Laplace sum rules \( R_{0,1}(\tau, s_0) \) will be discussed in the section that follows. However, it is important to recognize that (6) requires significant modification if the lowest-lying resonance is broad. If one does not invoke the narrow resonance approximation in the final line of (3), but instead assumes a Breit-Wigner (or modified Breit-Wigner [1]) shape, one finds that
\[ R_k(\tau, s_0) = \frac{1}{\pi} \sum_r \frac{g_r m_r \Gamma_r}{r^2 + m_r^2 \Gamma_r^2} e^{-s \tau} ds \]

\[ = \sum_r g_r W_k(m_r, \Gamma_r, \tau) m_r^{2k} e^{-m_r^2 \tau} \Theta(s_0 - m_r^2), \] (7)

with the weighting functions \( W_k \) demonstrably related to the narrow resonance limit (4) via

\[ \lim_{\Gamma_r \to 0} W_k(m_r, \Gamma_r, \tau) = 1. \] (8)

The net effect of such weighting functions on the lowest-lying resonance contribution to (6) is to replace \( m_\ell^2 \) with \( m_\ell^2 W_{k+1}/W_k \). If the lowest-lying resonance is the dominant subcontinuum resonance in a given channel, then \( m_\ell \) can be extracted from the lowest-lying (\( \ell \)) resonance contribution to \( R_1/R_0 \) as follows [19,41]:

\[ \frac{W_0(m_\ell, \Gamma_\ell, \tau)}{W_1(m_\ell, \Gamma_\ell, \tau)} \left( \frac{R_1(\tau, s_0)}{R_0(\tau, s_0)} \right)_\ell = m_\ell^2. \] (9)

For a given choice of \( \Gamma_\ell \) and \( s_0 \), one can use field-theoretical expressions for \( R_{0,1}(\tau, s_0) \), including both perturbative QCD and nonperturbative QCD contributions of infinite and finite correlation lengths [Section 3], to obtain from (9) a self-consistent minimizing value of \( m_\ell \). This procedure constitutes the methodological foundation for the results we obtain in Section 4.

The weighting functions \( W_{0,1} \) can be derived from a Breit-Wigner resonance shape by expressing that shape as a Riemann sum of unit-area pulses \( P_{m_r} \) centred at \( s = m_r^2 \) [19]:

\[ P_M[s, \Gamma] \equiv \frac{1}{2M \Gamma} [\Theta(s - M^2 + M\Gamma) - \Theta(s - M^2 - M\Gamma)], \] (10)

\[ \frac{M \Gamma}{(s - M^2)^2 + M^2 \Gamma^2} = \lim_{n \to \infty} \frac{2}{n} \sum_{j=1}^{n} \sqrt{\frac{n-j+f}{j-f}} P_M \left[ s, \sqrt{\frac{n-j+f}{j-f}} \Gamma \right], \] (11)

where \( f \) is any arbitrarily chosen constant between 0 and 1. If one approximates the resonance shape via (11) by truncating \( n \) to some finite number of
pulses, then the approximation (unlike the $n \to \infty$ limit) becomes sensitive to the choice of $f$. The value for $f$ may be chosen to ensure that the area under the truncated sum is equal to the area under the “true” resonance shape

$$\int_{-\infty}^{\infty} \frac{M\Gamma}{(s - M^2)^2 + M^2\Gamma^2} ds = \pi. \quad (12)$$

In an $n = 4$ approximation, for example, one obtains an area of $\pi$ by choosing $f = 0.70$. One finds for this four-pulse approximation [Fig. 1] that [19]

$$W_k[M, \Gamma, \tau] \approx 0.5589\Delta_k(m, 3.5119 \Gamma, \tau) + 0.2294\Delta_k(M, 1.4412 \Gamma, \tau) + 0.1368\Delta_k(M, 0.8597 \Gamma, \tau) + 0.0733\Delta_k(M, 0.4606 \Gamma, \tau), \quad (13)$$

where

$$\Delta_k(M, \Gamma, \tau) = \int_{-\infty}^{\infty} P_M(s, \Gamma)s^k e^{-s\tau} ds. \quad (14)$$

In particular,

$$\Delta_0(M, \Gamma, \tau) = \frac{sinh(M\Gamma\tau)}{M\Gamma\tau}, \quad (15)$$

and

$$\Delta_1(M, \Gamma, \tau) = \frac{sinh(M\Gamma\tau)}{M\Gamma\tau} \left[1 + \frac{1}{M^2\tau}\right] - \frac{cosh(M\Gamma\tau)}{M^2\tau}. \quad (16)$$

As $\Gamma \to 0$, $\Delta_k \to 1$. Nevertheless, it is easy to show for small values of $\Gamma$ that [41]

$$\frac{\Delta_0(M, \Gamma, \tau)}{\Delta_1(M, \Gamma, \tau)} = 1 + \Gamma^2\tau/3 + O(\Gamma^4) \quad (17)$$

in which case we find from (13) that $W_0/W_1 > 1$.

It is evident from a comparison of (6) and (9) that the introduction of a small nonzero width necessarily increases a sum rule driven estimate of $m_\ell$ [41]. Specifically, if we regard (9) as a constraint implicitly defining a function $m_\ell(\tau, \Gamma_\ell, s_0)$, the result $W_0/W_1 > 1$ for nonzero $\Gamma$ implies that

$$m_\ell(\tau, \Gamma_\ell, s_0) > m_\ell(\tau, 0, s_0). \quad (18)$$

Such behaviour is evident from the analysis carried out in Section 4.
3 Field Theoretical Contributions to Scalar-Current Sum Rules

In this section, we seek to identify purely-perturbative and nonperturbative QCD contributions to \( R_0(\tau) \) and \( R_1(\tau) \), as defined by (4). The three-loop expression for the scalar-current [as defined by (2)] correlation function (1) in perturbative QCD \((n_f = 3; \; s = p^2 = -Q^2)\) is given by [42,43]

\[
\Pi_{\text{pert}}(s) = \frac{3}{16\pi^2} (-s) \ln \left( \frac{-s}{\mu^2} \right) \left\{ 1 + \left( \frac{\alpha_s}{\pi} \right) \left[ \frac{17}{3} - \elln \left( \frac{-s}{\mu^2} \right) \right] \right. \\
+ \left( \frac{\alpha_s}{\pi} \right)^2 \left[ 45.846 - \frac{95}{6} \elln \left( \frac{-s}{\mu^2} \right) + \frac{17}{12} \elln^2 \left( \frac{-s}{\mu^2} \right) \right] \right\}
\]

The imaginary part of (19), needed for the integrand of (4), is defined consistent with dispersion-relation conventions:

\[
2iIm\Pi(s) = \Pi(s + i\epsilon) - \Pi(s - i\epsilon),
\]

[e.g. \( Im(\elln(-s/\mu^2)) = -\pi \)], in which case one finds from (19) that

\[
Im(\Pi_{\text{pert}}(s)) = \frac{3s}{16\pi} \left\{ 1 + \left( \frac{\alpha_s}{\pi} \right) \left[ \frac{17}{3} - 2\elln \left( \frac{s}{\mu^2} \right) \right] \right. \\
+ \left( \frac{\alpha_s}{\pi} \right)^2 \left[ 31.864 - \frac{95}{3} \elln \left( \frac{s}{\mu^2} \right) + \frac{17}{4} \elln^2 \left( \frac{s}{\mu^2} \right) \right] \right\}
\]

We choose to set the renormalization scale via the Borel parameter by setting \( \mu^2 = 1/\tau \). Upon substituting (21) into (4), we obtain from (5) the following purely-perturbative contribution to \( R_0(\tau, s_0) \):

\[
[R_0(\tau, s_0)]_{\text{pert}} = \frac{3}{16\pi^2\tau^2} \left\{ 1 - (1 + s_0 \tau)e^{-s_0 \tau} \right\} \left[ 1 + \left( \frac{\alpha_s}{\pi} \right) \frac{17}{3} + \left( \frac{\alpha_s}{\pi} \right)^2 31.864 \right] \\
- \left( \frac{\alpha_s}{\pi} \right) \left[ 2 + \frac{95}{3} \left( \frac{\alpha_s}{\pi} \right) \right] \int_0^{s_0 \tau} w \elln(w)e^{-w}dw \\
+ \left( \frac{\alpha_s}{\pi} \right)^2 \frac{17}{4} \int_0^{s_0 \tau} w [\elln(w)]^2 e^{-w}dw
\]

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As is evident from (4) and (5), the subsequent sum rule $R_1(\tau, s_0)$ can be obtained from (22) by explicit differentiation with respect to the Borel parameter $\tau$.

QCD-vacuum condensate contributions to the scalar-current correlation function given by (1) and (2) are found via operator product methods to be [9,14,44]:

\[
\left( \Pi(s = -Q^2) \right)_{\text{cond}} = \left( \frac{3}{2Q^2} - \frac{m_q^2}{Q^4} \right) < m_q \bar{q}q > + \left( \frac{1}{16\pi Q^2} + \frac{7m_q^2}{24\pi Q^4} - \frac{m_q^2}{4\pi Q^4} \ln \left( \frac{Q^2}{\mu^2} \right) \right) < \alpha_s G^2 > + \left( \frac{m_q}{2Q^4} - \frac{3m_q^3}{2Q^6} \right) < \bar{q}G \cdot \sigma q > + \left( \frac{27m_q^2}{48\pi Q^6} - \frac{m_q^2}{4\pi Q^6} \ln \left( \frac{Q^2}{\mu^2} \right) \right) < \alpha_s G^3 > + O \left( m_q^4 \right) - \left( \frac{88\pi}{27Q^4} + O \left( m_q^2 \right) \right) < \alpha_s (\bar{q}q)^2 >, \quad (23)
\]

where $m_q = m_u = m_d$ in the SU(2)-flavour symmetry limit. The final term in (23) is obtained via the assumed vacuum saturation of several contributing dimension-6 operators [9].

Standard dispersion-relationship arguments link the expression (4) for $R_0(\tau)$ with $d\Pi/dQ^2$:

\[
- \frac{d\Pi}{dQ^2} = \frac{1}{\pi} \int_0^\infty \frac{\Im \Pi(s)}{(s + Q^2)^2} ds \quad (24)
\]

Equation (24) may be expressed as a Laplace transform (with respect to the variable $Q^2$) of the function $R_0(\tau)$. Since

\[
\frac{1}{(s + Q^2)^2} = \int_0^\infty (\tau e^{-\tau s}) e^{-Q^2 \tau} d\tau \equiv \mathcal{L}(\tau e^{-\tau s}), \quad (25)
\]

we see from (24) and (4) that

\[
\mathcal{L}^{-1} \left( -\frac{d\Pi}{dQ^2} \right) = \frac{1}{\pi} \int_0^\infty \tau e^{-\tau s} \Im \Pi(s) ds = \tau R_0(\tau). \quad (26)
\]
We note from the Laplace transform definition in (25) that $L[\tau] = 1/Q^4$, $L[1] = 1/Q^2$, and we find through substitution of (23) into the left-hand side of (26) that the leading-order condensate contribution to $R_0$ is given by [9]

$$[R_0(\tau)]_{\text{cond}} = \frac{3}{2} < m_q \bar{q}q > + \frac{1}{16\pi} < \alpha_s G^2 >$$

$$- \frac{88\pi}{27} < \alpha_s (\bar{q}q)^2 > \tau + \mathcal{O}(m_q).$$

(27)

We do not include $< m_q \bar{q}q >$ in the order $m_q$ terms, as this condensate’s magnitude ($= -f_{\pi}^2 m_{\pi}^2 / 4$) is independent of the quark mass [14,45].

As discussed in Section 1, both scalar and pseudoscalar correlation functions are also sensitive to nonperturbative contributions of finite correlation length, corresponding to the instanton structure of the QCD vacuum. The direct single-instanton contribution to the $I = 1$ pseudoscalar sum rule $R_0^\pi$ [16,46],

$$[R_0(\tau)]_{\text{inst}}^\pi = \frac{3\rho^2}{16\pi^2 \tau^3} e^{-\rho^2/2\tau} \left[ K_0 \left( \frac{\rho^2}{2\tau} \right) + K_1 \left( \frac{\rho^2}{2\tau} \right) \right],$$

(28)

is constructed via (4) from a correlation function based on the pseudoscalar current

$$j^p(x) \equiv i \left[ \bar{u}(x) \gamma_5 u(x) - \bar{d}(x) \gamma_5 d(x) \right] / 2.$$

(29)

This direct single-instanton correlator

$$\left( \Pi(p^2) \right)_{\text{inst}}^p \equiv i \int d^4 x e^{ip \cdot x} < 0 | T \left[ j^p(x) j^p(0) \right]_{\text{inst}} | 0 >$$

(30)

can be related to the direct single-instanton contribution to the $I = 1$ pseudoscalar channel of the quark-antiquark scattering amplitude [12] simply by tying the external fermion lines together [Fig. 2] to form the vacuum-polarization loop corresponding to (30). Consequently, useful properties of the single-instanton contribution to $q\bar{q}$ scattering are also applicable to (30). Specifically, the $I = 1$ pseudoscalar $q\bar{q}$ scattering amplitude in a single-instanton background is equivalent to the $I = 0$ scalar $q\bar{q}$ amplitude [12,13], the result of compensating sign changes occurring within the amplitude when
$i\gamma_5 \to 1$, and when $I = 1 \to I = 0$ [12]. These features allow the expression (28) to be identified with the instanton contribution to the $I = 0$ scalar-channel sum rule, as well as with the negative of the corresponding sum rule for the $I = 1$ channel:

$$[R_0(\tau)]_{\text{inst}}^I = [R_0(\tau)]_{\text{inst}}^{I=0} = - [R_0(\tau)]_{\text{inst}}^{I=1}.$$  \hspace{1cm} (31)

To summarize, the aggregate field-theoretical contribution from QCD to the leading Laplace sum-rule (5) is

$$[R_0(\tau, s_0)]_{I=0,1} = [R_0(\tau, s_0)]_{\text{pert}} + [R_0(\tau)]_{\text{cond}} + [R_0(\tau)]_{\text{inst}}^{I=0,1},$$ \hspace{1cm} (32)

where the terms on the right-hand side of (32) are respectively given by (22), (27), and (31) [via (28)]. Nonperturbative order parameters, specifically the condensates appearing in (27) and the instanton-size $\rho$ in (28), are known from other theoretical and phenomenological studies - the values we quote as standard in Section 4 are consistent with those of refs. 9, 14, and 16.

As noted earlier on, the next-to-leading sum-rule $R_1(\tau, s_0)$ can be extracted via differentiation with respect to the explicit $\tau$ dependence of (32):

$$R_1(\tau, s_0) = - \frac{\partial}{\partial \tau} R_0(\tau, s_0).$$ \hspace{1cm} (33)

However, the expressions $R_0(\tau, s_0)$ and $R_1(\tau, s_0)$ obtained from (32) and (33) require renormalization-group (RG) improvement to be useful - otherwise results obtained via (9) are unnaturally dependent on the specific choice for $\alpha_s$. Such is the case in ref. 9, where $\alpha_s$ is chosen to be 0.6.

QCD Laplace sum rules have been shown to satisfy RG equations with respect to the Borel scale parameter $\tau$ [47]. Consequently, the sum-rules $R_{0,1}$, once obtained from (32) and (33), can safely be RG-improved by replacing $\alpha_s$ with the running coupling-constant $\alpha_s(\tau^{-1/2})$. In the Section that follows, factors of $\alpha_s$ in $R_{0,1}$ will be understood to correspond to the PDG [1] 3-flavour expression for $\alpha_s(\tau^{-1/2})$ to 3-loop order.

\footnote{Such is the case in ref. 9, where $\alpha_s$ is chosen to be 0.6.}
4 Sum Rule Analysis of $q\bar{q}$ Scalar Resonances

For a given choice of $\Gamma_\ell$, $s_0$, and $\tau$, one can obtain a self-consistent value of the lowest-lying resonance mass $m_\ell$ by solving (9), as noted in Section 2, thereby defining a function $m_\ell(\tau, \Gamma_\ell, s_0)$ implicitly. Such a procedure is meaningful provided only the lowest-lying $q\bar{q}$ resonance dominates the sum-rule. This assumption is a reasonable one provided other subcontinuum resonances in the same channel are either much heavier, and thus exponentially suppressed (4), or are non-$q\bar{q}$ exotica, as has already been noted. A clear signal of this assumption’s validity is the absence of $a_0(980)$, if exotic, from the sum rule generated from $I = 1$ $\bar{q}q$-currents (2), even though $a_0(980)$ is the lowest lying $I = 1$ scalar resonance. One cannot explain such an absence if $a_0(980)$ is $q\bar{q}$; however its observed absence (see below) is consistent with it being both exotic and sum-rule decoupled.

For a given choice of $\Gamma_\ell$ and $s_0$, we shall examine the $\tau$-dependence of $m_\ell(\tau, \Gamma_\ell, s_0)$ obtained in a given channel from (9) via the field-theoretical expressions for $R_{0,1}(\tau, s_0)$ developed in Section 3. For such externally imposed values of $\Gamma_\ell$ and $s_0$, we shall identify the true value of $m_\ell$ with the minimum value of $m_\ell(\tau)$ obtained over an appropriate range of the Borel parameter $\tau \equiv 1/M^2$. Since $M = \tau^{-1/2}$ is itself the renormalization-scale ($\mu$) for the field-theoretical content of $R_{0,1}(\tau, s_0)$, a sum-rule calculation cannot be meaningful unless we restrict this mass scale to be not only well-above $\Lambda_{QCD}$, but also to be bounded from above by the continuum threshold: $M^2 \leq s_0$. This criterion is necessary to ensure that the continuum contribution to (22), which increases as $s_0 \tau = s_0/M^2$ becomes small, is not overly large compared to the remainder of the purely perturbative contribution, i.e. the $s_0 \to \infty$ limit of (22). A more realistic criterion, in which continuum effects are less than 30% of the purely perturbative contribution [14], would lead to an even tighter upper bound on $M$. Moreover, the true value of $m_\ell$ should not just be the global minimum of $m_\ell(1/M^2, \Gamma_\ell, s_0)$ with respect to $M$ chosen in this subcontinuum range; it should also exhibit insensitivity to small changes in the Borel parameter $M$ - a local minimum. This criterion of flatness, as well as the requirement for a sensible range of the Borel/renormalization scale $M$, is quite standard in sum rule applications [40]. We also employ

\footnote{i.e., a $KK$-molecule [2,3], a dynamical-threshold effect [5], or possibly an $s\bar{s}$ state with Zweig-Okubo-suppressed coupling to the nonstrange current (2).}
standard values [9,14,16] for QCD’s nonperturbative order parameters in our analysis: 
\[ \langle m_{q\bar{q}} \rangle = -\frac{f_{\pi}^2 m_{\pi}^2}{4} (f_{\pi} = 131 \text{MeV}), \quad \langle \alpha_s G^2 \rangle = 0.045 \text{GeV}^2, \quad \langle \alpha_s (q\bar{q}) \rangle^2 = 0.00018 \text{GeV}^6, \quad \rho = (600 \text{ MeV})^{-1}. \]
Factors of \( \alpha_s \) that are not absorbed in (approximately-) RG-invariant QCD-vacuum condensates are replaced with 3-loop \( n_f = 3 \) running (\( \Lambda_{QCD} = 150 \text{ MeV} \)) coupling constants \( \alpha_s (M) [M \equiv \tau^{-1/2}] \), as discussed at the end of Section 3.

### 4.1 \( I = 0 \) Scalar Channel: Narrow Resonance Limit

In Figure 3, the Borel-scale dependence of the lowest-lying \( I = 0 \) scalar resonance mass (denoted as \( M_\sigma \)) is displayed for a number of choices of \( s_0 \), assuming zero resonance width. The curves displayed are restricted to values of \( M_\sigma^2 \leq s_0 \). As in ref. [9], we see that the curves \( M_\sigma (M) \) increase with the choice for \( s_0 \). We also see that a subcontinuum local minimum does not develop until we consider values of \( s_0 \) larger than 1.6 GeV$^2$. Figure 4 displays explicitly the \( s_0 \) dependence of the values of \( M_\sigma \) that are local minima for each choice of \( s_0 \) in Fig. 3. As is evident from Fig. 4, the minimum value of \( M_\sigma \) for a given choice of \( s_0 \) also increases with \( s_0 \). Our analysis finds the onset of a local minimum occurring when \( s_0 = 1.61 \text{ GeV}^2 \), corresponding to \( M_\sigma = 680 \text{ MeV} \); the Figure 4 curve begins with this point.

It is useful to compare these results to the seminal 1982 sum-rule analysis of ref. [9], which included the condensate contributions (27), but which did not include the then-unknown instanton contribution (28,31) or correct higher-order perturbative contributions. Figure 5 demonstrates the crucial role instanton contributions play in lowering the lowest-lying \( I = 0 \) \( q\bar{q} \) resonance mass, though at the price of diminishing the broad range of Borel-parameter stability observed in [9] at comparable values of \( s_0 \). The use of RG-improvement also has a significant lowering effect on the lowest-lying \( I = 0 \) \( q\bar{q} \) resonance mass. In Figure 6, we compare the stability curves obtained from the full field-theoretical content of \( R_{0,1}(\tau, s_0) \) at \( s_0 = 1.55 \text{GeV}^2 \) with and without RG-improvement of \( [R_{0,1}(\tau, s_0)]_{pert} \). The upper curve is obtained with \( \alpha_s = 0.6 \), as in [9], while the lower curve utilizes \( \alpha_s (M) \). Finally, we note that instanton and RG-improvement effects are offset somewhat by higher-order perturbative contributions. Figure 7 demonstrates how going from two

\footnote{The analysis considered only \( O(\alpha_s/\pi) \) 2-loop contributions to (21), with 17/3 erroneously given as 13/3. The analysis also did not incorporate RG-improvement of \( \alpha_s \).}
to three loops can increase the estimate of the lowest-lying $I = 0$ resonance mass by as much as 150 MeV, suggesting higher-order effects as a clear source of theoretical uncertainty in scalar-channel sum-rule methodology. Since the leading 4-loop contribution to $\text{Im}[\Pi_{\text{pert}}(s)]$ is both large and the same sign [43] as the leading 3-loop $[O(\alpha_s^3)]$ contribution to (21), we can anticipate that sum-rule determinations of $M$ based upon (21) will likely underestimate $M_{\sigma}$, a point that will prove important in assessing the viability of the broad, very light $\sigma$ (Alternative 1) discussed in Section 1.

4.2 $I = 0$ Scalar Channel: Nonzero Widths

Figures 8-12 present stability curves analogous to Fig. 3 that are obtained from (9) with input values of $\Gamma_\ell$ between 100 and 500 MeV. The weighting functions $W_{0,1}$ are obtained via the four-pulse approximation leading to (13) and (14). For each value of $\Gamma_\ell$ considered, it is possible to construct a (Fig. 4 analog) plot of the minimizing value of $M_{\sigma}$ for each choice of $s_0$. These plots are presented in Fig. 13, and they clearly demonstrate how sum-rule determinations of the lowest-lying mass increase with increasing resonance width.

In comparing Figs. 8-12, we note that the onset of a local minimum below the continuum threshold is itself width-dependent. Specifically, the $\Gamma = 500$ MeV stability curves of Fig. 12 do not develop a locally flat minimum below $s_0$ until $s_0$ is larger than 1.8 GeV$^2$, corresponding to a minimizing value of $M_{\sigma}$ at onset that is substantially above 1 GeV. Local minima for larger values of $s_0$ are seen to lead to even larger values of $M_{\sigma}$, as is evident from the topmost curve of Fig. 13. The values of $s_0$ and $M_{\sigma}$ corresponding to the onset of local minima are tabulated in Table I for $0 \leq \Gamma \leq 500$ MeV, corresponding to the initial left-hand points of the six curves displayed in Fig. 13. The values of $M_{\sigma}$ in Table I correspond to the lowest sum-rule determinations of $M_{\sigma}$ possible for each choice of $\Gamma$ that are consistent with the methodological constraints delineated at the beginning of this Section. The Table entries clearly indicate that a light lowest-lying $I = 0 q\bar{q}$ scalar resonance cannot have a width larger than half of its mass.

\[\text{Our somewhat low value for } \Lambda_{\text{QCD}} \text{ is similarly motivated to bring down the size of perturbative contributions, although in practice our results are virtually unaffected by a 100 MeV increase in this parameter.}\]

\[\text{A qualitatively similar conclusion is stated in ref. [38].}\]
This conclusion is softened only slightly if we relax the flatness requirement and insist only on identifying $M_\sigma$ with the \textit{global} minimum over the Borel-parameter range $(\Lambda_{QCD})^2 < M^2 \leq s_0$. As $\Gamma$ increases from 100 to 500 MeV, Figs. 8-12 still show such a global minimum to be increasing with $\Gamma$. When $\Gamma = 100$ MeV, such a minimum occurs at about 530 MeV [see the $s_0 = 1.2 \text{ GeV}^2$ curve of Fig. 8], but as $\Gamma$ increases to 500 MeV, this global absolute minimum increases to 840 MeV [see the $s_0 = 1.4 \text{ GeV}^2$ curve of Fig. 12].

These qualitative conclusions concerning the non-viability of a very light, very broad $q\bar{q}$ resonance are not expected to be altered by further refinements in the treatment of the lowest-lying resonance shape. Once can obtain “improved” weighting-factors $W_{0,1}$, as opposed to those (13) based upon a four-pulse approximation of the Breit-Wigner resonance shape by increasing the value of $n$ used to truncate the Riemann sum (11). Figure 14 demonstrates that such an increase in the number of pulses used to approximate the Breit-Wigner shape does not appreciably alter the local-minimum value $M_\sigma$ obtained for a given choice of $s_0$ and $\Gamma$. It should be noted, however, that $M_\sigma$ increases slightly as $n$ increases, suggesting that a more precise modelling of the Breit-Wigner resonance shape would only serve to increase the $M_\sigma$ values of Table I.

A more fundamental issue is whether the Breit-Wigner shape is appropriate at all for the modelling of broad resonances, as the Breit-Wigner tail will extend significantly into Euclidean ($s < 0$) and continuum ($s > s_0$) regions if $\Gamma$ is sufficiently large. We have relied upon an admittedly crude $n=4$ (4-pulse) approximation in order to minimize such unphysical contributions. Larger choices of $n$ increase the sensitivity of $W_{0,1}$ to unphysical regions in $s$ by including square pulses of greater width than the largest pulse in Fig. 2.

It has already been argued that higher-order perturbative contributions will, if anything, increase the $M_\sigma$ values of Table I. Consequently, the only means compatible with QCD sum-rule methodology that is available for obtaining lower sum-rule estimates for $M_\sigma$ is to increase the size of the instanton contributions [c.f., Fig. 5], possibly via a decrease in the value for $\rho$ in (28). To do so, however, has ramifications for the $I = 1$ scalar resonance channel, as discussed below.
4.3 \( I = 1 \) Scalar Channel

We have seen in Section 3 that the sign of the instanton contribution to \( R_{0,1} \) in the \( I = 1 \) scalar channel is reversed from that of the \( I = 0 \) scalar channel. Consequently, instanton effects are now seen to increase the scale of sum-rule determinations of the lowest-lying contributing \( I = 1 \) resonance. The stability curves presented in Figs. 15 and 16 clearly indicate a much higher range of values for masses of such lowest-lying resonances, as well as a need to go to much higher values of \( s_0 \) in order to attain subcontinuum stability with respect to the Borel-parameter \( M \).

Table II lists values for the lowest-lying contributing \( I = 1 \) resonance mass \((M_\delta)\) associated with the onset of a subcontinuum local minimum. Since local-minima \( M_\delta \) increase with the choice of \( s_0 \), the \( M_\delta \) values in Table II, like the \( M_\sigma \) values in Table I, represent the minimum such values obtainable via sum-rule methodology outlined in the beginning of this Section. Table II clearly indicates a lowest-lying contributing \( I = 1 \) scalar resonance in excess of 1.49 GeV.

The much larger \( I = 1 \) values for the continuum threshold in Table II are comparable to values for \( s_0 \) obtained via sum-rule analysis of the \( I = 1 \) pseudoscalar channel [18,19]. In both \( I = 1 \) channels, sum rule methodology would suggest that Borel stability not occur for values of \( s_0^{1/2} \) that are less than the masses of contributing subcontinuum resonances. The large value of \( M_\delta \) evident from Table II necessarily requires values of \( s_0 \) larger than \( M_\delta^2 \), consistent with the values of \( s_0 \) actually listed.

As noted at the very beginning of this Section, the results of Table II clearly rule out a contribution from \( a_0(980) \), the lowest-lying scalar resonance in the \( I = 1 \) channel. Consequently, the coupling of this resonance to the \( \bar{q}q \)-current (2) [with a minus sign chosen for \( I = 1 \)] must be suppressed, as would be anticipated form a non-\( q\bar{q} \) interpretation of this state. The results of Table II, however, appear compatible with \( a_0(1450) \) being identified with the lowest-lying \( I = 1 \) \( q\bar{q} \)-scalar resonance, particularly if the width of this resonance is less than or of order 200 MeV.

As a final comment, we note that any attempt to obtain sum-rule support for a broad, very light \( q\bar{q} \) state by enhancing the magnitude of the instanton contribution (e.g. by decreasing the instanton size \( \rho \)) will necessarily drive up the mass of the lowest-lying contributing \( I = 1 \) state, \( i.e.\), the isovector \( q\bar{q} \) state. Thus the price of tuning instanton effects to accommodate a light
broad isoscalar $q\bar{q}$ resonance is likely to be an exotic interpretation for both $a_0(980)$ and $a_0(1450)$, the two lowest-lying I=1 scalar resonances.
5 Global-Fit of Lowest-Lying $I = 0$ Scalar Resonance Properties

In Section 4, we have utilized a specific choice of the Borel parameter corresponding to minimization of the sum-rule-derived mass. However, Leinweber [48] has stressed the value in an overall fit of the Borel-parameter dependence of a sum rule’s field-theoretical content to the dependence anticipated from resonance properties. For example, properties of the first pion-excitation state $\Lambda'$ have been obtained by fitting the QCD Borel-parameter dependence for the lowest $\Lambda'$-sensitive Laplace sum rule to its corresponding hadronic content [18,19]. In this section, we will apply this same procedure to the scalar $I = 0$ channel. Specifically, we will obtain a $\chi^2$-minimizing weighted least-squares fit of $[R_0(\tau,s_0)]^{I=0}$ to the $\tau$-dependence anticipated from the corresponding resonance contribution (7), assuming the contribution from the lowest-lying resonance dominates the $I=0$ scalar channel.

We begin first by modifying the scalar current (2) to include the non-strange quark mass in the $SU(2)_f$-invariant limit:

$$j_s(x) = m_q(\bar{u}(x)u(x) + \bar{d}(x)d(x))/2$$  \hspace{1cm} (34)

where $m_q \equiv (m_u + m_d)/2$. Although the additional factors of the quark mass cancel out in an $R_1/R_0$ determination of resonance properties, such as that in Section 4, these factors have nontrivial consequences when the overall $\tau$-dependence of $R_0$ itself is being analyzed. The current (34) now corresponds to an RG-invariant operator, and upon RG-improvement, the sum rules devolving from that current (as in Section 3) will now include the additional $\tau$-dependence associated with the running quark mass:

$$[R_0(\tau,s_0)]^{I=0}$$

$$= m_q^2(\tau) \left[ \frac{3}{16\pi^2 \tau^2} \left\{ \left[ 1 - (1 + s_0 \tau) e^{-s_0\tau} \right] \left[ 1 + 17\alpha_s(\tau)/3\pi \right] \right. \\
- \left. 2(\alpha_s(\tau)/\pi) \int_0^{s_0\tau} w \ln(w)e^{-w}dw \right\} \right.$$  \hspace{1cm} (35)

$$+ m_q^2(\tau) \left[ [R_0(\tau)]_{cond} + [R_0(\tau)]_{inst} \right];$$
\[ \alpha_s(\tau) = \frac{2\pi}{9L(\tau)} \left[ 1 - \frac{32}{81L(\tau)} \ln[L(\tau)] \right] \]  
\[ m_q(\tau) = \frac{\hat{m}}{L(\tau)^{4/9}} \left[ 1 - \frac{0.1989 - 0.1756 \ln[L(\tau)]}{L(\tau)} \right] \]  
\[ L(\tau) \equiv -\frac{1}{2} \ln \left( \tau \Lambda_{QCD}^2 \right). \]  

The condensate and instanton contributions to (35) are given respectively by (27) and (28). The running coupling (36) and mass (37), as well as the purely perturbative contribution to (35), are given only to two-loop order to accommodate the numerical limitations of our fitting procedure, which involves generating resonance parameters via a Monte-Carlo simulation of uncertainties (see below). The parameter \( \hat{m} \) in (37) is the RG-invariant nonstrange quark mass.

Our fit is generated by obtaining values for \( m_\sigma, \Gamma_\sigma, s_0, \) and the resonance coupling \( g_\sigma \) that minimize a least-squares fit of (35) to the lowest-lying resonance contribution anticipated via (7):

\[ [R_0(\tau, s_0)]^{I=0} = g_\sigma W_0(m_\sigma, \Gamma_\sigma, \tau) e^{-m_\sigma^2 \tau}. \]  

In (39), the weighting factor \( W_0 \) is given by the four-pulse approximation expression (13), with \( \Delta_0 \) given by (15). Our weighted least-squares fit is over the Borel-parameter range \( 0.4 \text{ GeV}^{-2} \leq \tau \leq 2.2 \text{ GeV}^{-2} [0.67 \text{ GeV} \leq M \leq 1.6 \text{ GeV}] \). The region in \( \tau \) is obtained by requiring an uncertainty of less than 20% on the theoretical contribution (35) to \( R_0 \), based upon a 30% continuum and a 50% power law uncertainty. Parameter values for perturbative and nonperturbative scale parameters are as given in Section 4. As mentioned above, uncertainties in the fitted parameters \( \{m_\sigma, \Gamma_\sigma, s_0, g_\sigma\} \) are obtained only for the two loop case via a Monte-Carlo simulation of the power law and continuum uncertainties described above, as well as a 15% variation in the size of \( \rho \), and a factor of 2 "vacuum-saturation uncertainty" in the value of \( <\alpha_s(\bar{q}q)^2>_s \). The parameter values for \( \{m_\sigma, \Gamma_\sigma, s_0, g_\sigma\} \) we obtain correspond to the fit of (35) to (39) that minimizes a \( \chi^2 \) weighted for the previously mentioned continuum and power law uncertainties.

\[ \text{Treatment of this quantity as well as the specific form of Monte-Carlo modelled uncertainties is discussed at length in ref. [18].} \]
Figure 17 demonstrates the success of these fitted parameters in matching theoretical (35) and hadronic (39) expressions for $R_0$. The results of this fit are as follows:

$$m_\sigma = 0.93 \pm 0.12 \text{GeV},$$  \hspace{1cm} (40a) \\
$$0 \leq \Gamma_\sigma \leq 0.26 \text{GeV},$$  \hspace{1cm} (40b) \\
$$s_0 = 3.20 \pm 1.20 \text{GeV}^2,$$  \hspace{1cm} (40c) \\
$$g_\sigma = \hat{m}^2 \cdot (0.039 \pm 0.014 \text{GeV}^4),$$  \hspace{1cm} (40d)

where the quoted uncertainties reflect 90% confidence levels. The results (40) are clearly indicative of a lowest-lying $q\bar{q}$-resonance that is neither very broad nor very light. These results are most consistent with the Alternatives 2 and 3 delineated in Section 1.

Pertinent to Alternative 3, the results (40) are certainly consistent with identifying the lowest-lying $I = 0$ scalar nonstrange $q\bar{q}$ state with $f_0(980)$, as has been suggested by a very recent OPAL analysis of $Z^0$ decays [6]. Indeed, the very broad region of Borel-parameter stability characterizing the $s_0 = 2.4$ GeV$^2$ curve of Fig. 3 [which yields an $f_0(980)$ lowest-lying resonance mass] corroborates parameter values for $m_\sigma, \Gamma_\sigma, \text{and } s_0$ within the fitted range (40).

However, the results (40) do not exclude an Alternative 2 $q\bar{q}$ state somewhat below the $f_0(980)$ in mass, consistent with a $K\bar{K}$ interpretation of $f_0(980)$. Such an $f_0(980)$ would be expected to be essentially decoupled from sum rules based upon the current (34), as has been discussed in Section 4. This Alternative 2 $q\bar{q}$ interpretation becomes particularly interesting upon examination of the fitted parameter $g_\sigma$, which cancels entirely from the $R_1/R_0$-based analysis of Section 4. Arguments within a $L\sigma M$ context (buttressed by more general PCAC arguments) have been advanced for obtaining the matrix element by which a scalar current couples a physical $\sigma$ to the vacuum [49]:

$$<\sigma|m_\bar{q} \left( \bar{u}(0)u(0) + \bar{d}(0)d(0) \right)|0 > = f_\pi m_\pi^2.$$  \hspace{1cm} (41)

In the above expression, $f_\pi$ is 93 MeV. In Appendix A we show that

$$g_\sigma = | <\sigma|j_s(0)|0 > |^2.$$  \hspace{1cm} (42)
Making use of the $I = 0$ scalar current $j_s$ defined via (34), we find from (41) that

$$g_\sigma = f^2_\pi m^4_\pi / 4. \tag{43}$$

Comparison of (40d) and (43) suggests that

$$\hat{m} = 4.6^{+1.2}_{-0.6} \text{MeV}, \tag{44}$$

a value somewhat on the low side of the expected range [1] for the nonstrange current quark mass. It should be noted, however, that a reasonable light-quark mass cannot be obtained in a dilaton scenario [35,36] in which

$$< \sigma | \Theta^\mu_\mu(0) | 0 > = f_\pi m^2_\sigma \geq f_\pi m^2_\pi,$$ \tag{45}

unless the contribution to the matrix element from the scalar-current component of $\Theta^\mu_\mu$ is negligible compared to the anomalous gluonic-field contributions. If one assumes that $\Theta^\mu_\mu \approx m_\eta (\bar{u}u + \bar{d}d)$, the quark mass one would obtain by replacing (41) with (45) [via comparison of (40d) with (42)] will be larger than (44) by a factor of $m^2_\sigma / m^2_\pi$.

As a final comment, it should be noted that the value for $s_0$ in (40c) is sufficiently high to exclude $f_0(1370)$ and $f_0(1500)$ from the continuum. Although such resonances (if subcontinuum) are expected either to be exponentially suppressed [via (7)], or if non-$q\bar{q}$ exotica, to be essentially decoupled from a sum rule based on $q\bar{q}$ currents, there is nevertheless reason to be concerned about the validity of assuming (as we have done in the section) that only one resonance contributes to $R_0$. To address the possibility of contributions from more than one resonance, we have examined whether the field theoretical content of $[R_0(\tau, s_0)]^{I=0}$ might be better fitted by contributions from two subcontinuum resonances by replacing (39) with

$$R_0(\tau, s_0)]^{I=0} = g_1 W_0(m_1, \Gamma_1, \tau) \exp(-m^2_1 \tau) + g_2 \exp(-m^2_2 \tau). \tag{46}$$

The second subcontinuum resonance [presumably $f_0(980)$] is assumed to be narrow: $W_0(m_2, 0, \tau) = 1$. The remaining resonance parameters are determined via a weighted least-squares minimization of $\chi^2$. We have found such minimization to be accompanied by equilibration of $m_1$ and $m_2$ to the previously fitted value (40a), indicating that the $\tau$-dependence of the field-theoretical content of $R_0$ [Fig. 17] is best fit by a single exponential (39), rather than by the sum of two distinct exponential contributions (46).
This result indicates that the leading I=0 scalar-current sum rule is dominated by a single resonance, as has been assumed in the body of this section. Such behaviour is to be contrasted with that of the leading I=1 pseudoscalar-current sum rule, for which a fit including two contributing resonances [π and π′] leads to a χ² an order of magnitude lower than that of a single resonance fit [18].
6 Conclusions

In this paper we have sought to determine which of several empirically justifiable interpretations of the scalar mesons are compatible with theoretical constraints based upon QCD sum-rule methodology.

For the isovector channel, we have found in Section 4 that $a_0(980)$ is decoupled entirely from the isovector sum rule based upon the scalar current correlation function given by (1) and (2), a result requiring an exotic (non-$q\bar{q}$) interpretation for this resonance. The Table II values for the mass of the lowest-lying resonance in this channel that does couple to the sum rule appear compatible with the lowest lying $q\bar{q}$ resonance being $a_0(1450)$.

For the isoscalar channel, we have delineated in the introductory section of this paper four Alternatives for the lowest-lying $q\bar{q}$ scalar resonance. We found in Section 4 that the lowest-lying I = 0 $q\bar{q}$ scalar resonance cannot be both very light and very broad [Alternative 1], provided this resonance is the sole contributing resonance to this channel’s leading QCD sum rules. Thus, the only way the results of Section 4 could accommodate such a resonance would be if this lowest-lying resonance’s contribution is itself masked by higher-resonance contamination of (7), the hadronic side of the sum rule. The results at the end of Section 5 show this to be unlikely. Unless we require the Alternative 1 scenario to specify $f_0(980)$ to be an additional light $q\bar{q}$ excitation, as opposed to a $K\bar{K}$ state, and unless the lowest-lying $q\bar{q}$ state is unnaturally decoupled, such contamination is itself possible only for values of $s_0$ large enough to include more massive resonances in the subcontinuum region, such as $f_0(1370)$ and $f_0(1500)$. The values of $s_0$ appearing in Table I are clearly below this threshold.

The results of Section 4 show demonstrable support for Alternative 2, a narrower and more massive lowest-lying resonance that is still lighter than the (presumably exotic) $f_0(980)$, particularly if the lowest lying resonance is the sole contributing resonance to the sum rule. Moreover, the results listed in Table I correlate masses for such a resonance with values of $s_0$ that preclude any contamination of the hadronic side of the sum rule by $f_0(1370)$ and $f_0(1500)$. These results clearly favour masses above 700 MeV and widths below 300 MeV, and are suggestive of Svec’s most recent single-resonance fits to $\pi N$ data [31].

It should be emphasized that the results of Section 4 do not exclude the possibility that $f_0(980)$ is itself the lightest I = 0 $q\bar{q}$ scalar resonance. In-
deed, this Alternative 3 scenario is strongly supported by the fit performed in Section 5, which gives results (40) consistent with $f_0(980)$ but problematical for a dilaton-explanation of Alternative 2. Specifically, the coupling $g_\sigma$ obtained in (40d) is much smaller than that anticipated for a dilaton, unless the anomalous gluonic component of $\Theta^\mu_\mu$ dominates the matrix element (45). Moreover, sum-rule contamination from higher $I = 0$ scalar resonances, though certainly possible for the fitted range (40c) of $s_0$, would be expected to *increase* rather than diminish the apparent value of $g_\sigma$ in a single resonance fit, leading to even a smaller true value for $g_\sigma$ and an even larger discrepancy from the scale (45) anticipated from a dilaton interpretation.

We therefore conclude that the support QCD sum rules provide to Alternative 2 [a lowest lying resonance below $f_0(980)$ whose width is less than half its mass] does *not* extend to a dilaton interpretation of this state, unless the matrix element (45) is driven almost entirely by $\Theta^\mu_\mu$'s anomalous gluon piece. As a final comment, the Alternative 4 scenario, in which the lowest-lying $q\bar{q}$ isoscalar resonance is broad and at least as massive as $f_0(980)$, clearly contradicts the QCD sum rule results of Section 5. Moreover, even if higher-mass resonance contamination of (7) were to occur, such additional hadronic contributions would be expected to drive the apparent mass of the lowest-lying state *above* its true value, as has already been noted. Thus the fitted result (40a) for the lowest-lying resonance mass in the absence of such contamination cannot be reconciled to a true value much in excess of 1 GeV.

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9Provided the $g_r$ in (7) are positive-definite: *e.g.*, (42).
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Appendix A: The $\sigma$ Contribution to the $I = 0$ Scalar Correlator

To obtain the explicit $\sigma$ contribution to the $I = 0$ scalar-current ($j_s$) correlation function (1), we begin by noting that insertion of a complete set of intermediate single-particle $\sigma$ states implies that

$$
<0| T j_s(x) j_s(0)|0 > = \Theta(x_0) <0| j_s(x) \left(\int \frac{d^3q}{(2\pi)^3} \frac{|\sigma(q)\rangle <\sigma(\bar{q})|}{2\sqrt{q^2 + m_\sigma^2}} j_s(0)|0 >
$$

$$
= \Theta(-x_0) <0| j_s(0) \left(\int \frac{d^3q}{(2\pi)^3} \frac{|\sigma(q)\rangle <\sigma(\bar{q})|}{2\sqrt{q^2 + m_\sigma^2}} j_s(0)|0 >
$$

$$
+ \text{(higher-resonance and multiple particle terms)} \quad (A.1)
$$

We will define the matrix element connecting a physical (on-shell) $\sigma$ to the vacuum via a scalar current to be

$$
<0| j_s(x)|\sigma(\bar{q}) > \equiv Me^{-i\sqrt{\bar{q}^2 + m_\sigma^2}x_0 - \bar{q} \cdot \bar{x}} \quad , \quad (A.2)
$$

where the Heaviside step function $\Theta(x_0)$ can be expressed as the following integral:

$$
\Theta(x_0) = \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{ix_0\tau}}{\tau - i|\epsilon|} d\tau. \quad (A.3)
$$

When $x_0 > 0$, the contour along the real $\tau$ axis can be closed by an infinite arc in the upper half plane, enclosing the residue at $\tau = i|\epsilon| : \Theta(x_0) = 1$. When $x_0 < 0$, the contour must be closed in the lower half plane, and the residue is not enclosed: $\Theta(x_0) = 0$. If we substitute (A.2) and (A.3) directly into (A.1), we obtain

$$
<0| T j_s(x) j_s(0)|0 >_\sigma = \lim_{\epsilon \to 0} \frac{-i|M|^2}{(2\pi)^4} \int_{-\infty}^{\infty} d^3q \int_{-\infty}^{\infty} d\tau \left[ e^{-i(\sqrt{\bar{q}^2 + m_\sigma^2} - \tau)x_0} e^{i\bar{q} \cdot \bar{x}} \right. \frac{e^{-i\sqrt{\bar{q}^2 + m_\sigma^2} \tau} e^{i\bar{q} \cdot \bar{x}}}{2(\tau - i|\epsilon|)\sqrt{\bar{q}^2 + m_\sigma^2}} \right]
$$
\[ \frac{e^{i(\sqrt{q^2 + m^2} - \tau)x_0} e^{-iq \cdot x}}{2(\tau - i|\epsilon|)\sqrt{q^2 + m^2}} \]  

(A.4)

where the \( \sigma \) subscript on the left-hand side of (A.4) represents the explicit contribution of single-\( \sigma \) intermediate states to the matrix element. To evaluate (A.4) further, we make the following change of variable from \( \tau \) to \( q_0 \):

\[ q_0 \equiv \sqrt{q^2 + m^2} - \tau, \quad dq_0 = -d\tau. \]  

(A.5)

We then obtain \( (d^4q \equiv d^3\vec{q} dq_0 ; q \cdot x \equiv q_0 x_0 - \vec{q} \cdot \vec{x} ) \):

\[ < 0 | T j_s(x) j_s(0) | 0 >_{\sigma} = \lim_{\epsilon \to 0} \frac{i|M|^2}{(2\pi)^4} \int_{-\infty}^{\infty} d^4q \left[ \frac{e^{-iq \cdot x}}{2(\sqrt{q^2 + m^2} - q_0 - i|\epsilon|)\sqrt{q^2 + m^2}} \right. \]
\[ + \left. \frac{e^{iq \cdot x}}{2(\sqrt{q^2 + m^2} - q_0 - i|\epsilon|)\sqrt{q^2 + m^2}} \right]  

= \lim_{\epsilon \to 0} \frac{i|M|^2}{(2\pi)^4} \int_{-\infty}^{\infty} d^4q \frac{e^{-iq \cdot x}}{2\sqrt{q^2 + m^2}} \left[ \frac{1}{\sqrt{q^2 + m^2} - q_0 - i|\epsilon|} \right. \]
\[ + \left. \frac{1}{\sqrt{q^2 + m^2} + q_0 - i|\epsilon|} \right] \]  

(A.6)

The final line of (A.6) is obtained from the intermediate line by letting \( q_\mu \to -q_\mu \) in the second portion of the integral. We find trivially from (A.6) that \( (q^2 \equiv q_0^2 - \vec{q}^2 ; \epsilon' = 2\sqrt{q^2 + m^2} \epsilon) \):

\[ < 0 | T j_s(x) j_s(0) | 0 >_{\sigma} = \lim_{\epsilon' \to 0} \frac{i|M|^2}{(2\pi)^4} \int_{-\infty}^{\infty} d^4q \frac{e^{-iq \cdot x}}{q^2 - m^2 + i|\epsilon'|}. \]  

(A.7)

Upon substitution of (A.7) into (1), we find that

\[ (\Pi(p^2))_{\sigma} = -\frac{|< \sigma | j_s(0) | 0 >|^2}{p^2 - m^2 + i|\epsilon'|} \]  

(A.8)
where we have utilized the explicit definition of the constant $M$ in (A.2). The propagator denominator in (A.8) can be modified in the standard way \[1\] for incorporating relativistically a Breit-Wigner width $\Gamma_\sigma$:

$$\left(\Pi(p^2)\right)_\sigma = \frac{|\langle \sigma | j_\sigma(0)|0 \rangle |^2}{p^2 - m_\sigma^2 + im_\sigma \Gamma_\sigma}.$$  \hspace{1cm} (A.9)

Consequently, we see from comparison to (3) that the coupling coefficient $g_r$ is given by

$$g_r = |\langle \sigma | j_\sigma(0)|0 \rangle |^2$$  \hspace{1cm} (A.10)
| $\Gamma$(MeV): | 0  | 100  | 200  | 300  | 400  | 500  |
|--------------|----|------|------|------|------|------|
| $s_0$(GeV):  | 1.61 | 1.61 | 1.62 | 1.65 | 1.70 | 1.82 |
| $m_\sigma$(MeV): | 680  | 687  | 716  | 778  | 884  | 1087 |

Table I: Sigma masses associated with the onset of a subcontinuum local minimum in the isoscalar channel.

| $\Gamma$(MeV): | 0  | 100  | 200  | 300  |
|--------------|----|------|------|------|
| $s_0$(GeV$^2$): | 3.24 | 3.26 | 3.38 | 3.62 |
| $M_\delta$(GeV): | 1.49 | 1.50 | 1.55 | 1.63 |

Table II: Lowest-lying masses ($M_\delta$) associated with the onset of a subcontinuum local minimum in the isovector channel.
Figure Captions:

**Figure 1:** An example of the 4 square-pulse approximation to the Breit-Wigner resonance shape obtained by truncating eq. (11) to $n=4$, and by choosing $f = 0.701$ to ensure that the area under the four pulses is equivalent to the total area under the Breit-Wigner curve. This particular example is for a mass $M = 680$ MeV and width $\Gamma = 100$ MeV.

**Figure 2:**

(a) The single-instanton contribution to the quark-antiquark scattering amplitude (c.f. fig. 5 of ref. [12]).

(b) The single-instanton contribution to the corresponding current-current correlation function.

**Figure 3:** Stability curves for determining via (9) the lowest-lying $I = 0$ resonance mass. The corresponding width of this resonance is assumed to be zero. The stability curves are truncated to exclude the region in which the Borel mass-scale $M$ exceeds $s_0^{1/2}$.

**Figure 4:** The $s_0$-dependence of the masses $M_\sigma$ corresponding to (local) minima of the stability curves presented in Fig. 3.

**Figure 5:** How instanton contributions affect $M_\sigma$: a comparison of two stability curves for the lowest-lying $I = 0$ resonance mass obtained by including or not including the direct single-instanton contributions to the field-theoretical expressions for $R_{0,1}(\tau, s_0)$. Both curves are obtained assuming zero resonance width and a continuum threshold $s_0 = 1.55$ GeV$^2$.

**Figure 6:** How renormalization-group improvement affects $M_\sigma$: a comparison of two stability curves for the lowest-lying $I = 0$ resonance are obtained by including or not including the $\tau$ dependence of the strong coupling. The constant coupling curve is obtained by choosing $\alpha_s = 0.6$, as in ref. [9]. The (3-loop) running coupling curve is obtained assuming $\Lambda_{QCD} = 150$ MeV. Both curves are obtained assuming zero resonance width and a continuum threshold $s_0 = 1.55$ GeV$^2$, and both curves incorporate direct single-instanton contributions to the field-theoretical expressions for $R_{0,1}(\tau, s_0)$. 

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**Figure 7:** How three-loop perturbative contributions affect $M_\sigma$: a comparison of two stability curves for the lowest-lying $I = 0$ resonance mass obtained by including or not including three-loop order perturbative contributions to the field-theoretical expressions for $R_{0,1}(\tau, s_0)$. As in Figs. 5 and 6, $s_0$ is assumed here to be $1.55$ GeV$^2$, and the resonance width $\Gamma$ is assumed to be zero.

**Figure 8:** Stability curves for determining via (9) the lowest-lying $I = 0$ resonance mass, assuming a resonance width of 100 MeV. As in Fig. 3, the stability curves are truncated to exclude the region in which the Borel mass scale $M$ exceeds $s_0^{1/2}$.

**Figure 9:** Stability curves for determining the lowest-lying $I = 0$ resonance mass, assuming a resonance width of 200 MeV.

**Figure 10:** Stability curves for determining the lowest-lying $I = 0$ resonance mass, assuming a resonance width of 300 MeV.

**Figure 11:** Stability curves for determining the lowest-lying $I = 0$ resonance mass, assuming a resonance width of 400 MeV.

**Figure 12:** Stability curves for determining the lowest-lying $I = 0$ resonance mass, assuming a resonance width of 500 MeV.

**Figure 13:** The $s_0$-dependence of the masses $M_\sigma$ corresponding to (local) minima of the stability curves presented in Figs. 8-12.

**Figure 14:** How improving the 4 square-pulse approximation affects $M_\sigma$: a comparison of three stability curves for the lowest-lying $I = 0$ resonance mass that are obtained via (9) from weighting functions $W_{0,1}(M_\sigma, \Gamma, \tau)$ based upon truncation of the Riemann sum (11) at $n = 4$, 8, and 14 pulses. A larger number of pulses corresponds to a more precise approximation of the Breit-Wigner shape than that presented in Fig. 1. In all three curves, $s_0$ is assumed to be $1.63$ GeV$^2$, and the resonance width $\Gamma$ is assumed to be 100 MeV.

**Figure 15:** Stability curves for determining via (9) the lowest-lying $I = 1$ resonance mass, denoted here as $M_\delta$. The corresponding width of this resonance is assumed to be zero. As before, the stability curves are truncated to exclude the region in which the Borel mass scale $M$ exceeds $s_0^{1/2}$.

**Figure 16:** Stability curves for determining the lowest-lying $I = 1$ resonance mass, assuming a resonance width of 200 MeV.
Figure 17: Weighted least-squares fit of field theoretical content (circles) of $R_0(\tau, s_0)/\hat{m}^2$ to the corresponding hadronic contribution (triangles) anticipated from the lowest-lying $I = 0$ scalar resonance. Parameter values leading to this fit are $m_\sigma = 0.95$ GeV, $\Gamma_\sigma = 0$, $s_0 = 3.3$ GeV$^2$, and $g_\sigma/\hat{m}^2 = 0.040$ GeV$^4$. 
Resonance Shape (GeV$^{-2}$)
\begin{align*}
\Gamma = 0 \\
S_0 = 3.0 \text{ GeV}^2 &\quad \square \\
S_0 = 2.8 \text{ GeV}^2 &\quad \bullet \\
S_0 = 2.6 \text{ GeV}^2 &\quad \square \\
S_0 = 2.4 \text{ GeV}^2 &\quad \square \\
S_0 = 2.2 \text{ GeV}^2 &\quad \diamond \\
S_0 = 2.0 \text{ GeV}^2 &\quad \bullet \\
S_0 = 1.8 \text{ GeV}^2 &\quad \triangle \\
S_0 = 1.6 \text{ GeV}^2 &\quad \blacktriangle \\
S_0 = 1.4 \text{ GeV}^2 &\quad \triangle \\
S_0 = 1.2 \text{ GeV}^2 &\quad \blacktriangledown \\
S_0 = 1.0 \text{ GeV}^2 &\quad \blacktriangleleft
\end{align*}
Figure 4
The diagram shows the relationship between $M^2$ (GeV$^2$) and $M_\sigma$ (GeV) for two coupling scenarios: Constant coupling and Running coupling. The graph illustrates how $M_\sigma$ decreases as $M^2$ increases for both types of coupling.
Three Loop

Two Loop
$\Gamma = 300$ MeV

$S_0 = 3.0 \text{ GeV}^2$

$S_0 = 2.8 \text{ GeV}^2$

$S_0 = 2.6 \text{ GeV}^2$

$S_0 = 2.4 \text{ GeV}^2$

$S_0 = 2.2 \text{ GeV}^2$

$S_0 = 2.0 \text{ GeV}^2$

$S_0 = 1.8 \text{ GeV}^2$

$S_0 = 1.6 \text{ GeV}^2$

$S_0 = 1.4 \text{ GeV}^2$

$S_0 = 1.2 \text{ GeV}^2$

$S_0 = 1.0 \text{ GeV}^2$
$\Gamma = 500 \text{ MeV}$

$M^2(\text{GeV}^2)$ vs. $M(\text{GeV})$ for various values of $S_0$: $S_0 = 3.0 \text{ GeV}^2$ (circles), $S_0 = 2.8 \text{ GeV}^2$ (black filled circles), $S_0 = 2.6 \text{ GeV}^2$ (squares), $S_0 = 2.4 \text{ GeV}^2$ (up triangles), $S_0 = 2.2 \text{ GeV}^2$ (down triangles), $S_0 = 2.0 \text{ GeV}^2$ (red filled triangles), $S_0 = 1.8 \text{ GeV}^2$ (gray filled circles), $S_0 = 1.6 \text{ GeV}^2$ (black filled squares), $S_0 = 1.4 \text{ GeV}^2$ (blue filled up triangles), $S_0 = 1.2 \text{ GeV}^2$ (gray filled squares), $S_0 = 1.0 \text{ GeV}^2$ (red filled down triangles).
\( \Gamma = 0 \)

\[ M^2 (\text{GeV}^2) \]

\[ \delta \]
$\Gamma = 200$ MeV

$\delta M^2 (\text{GeV}^2)$

- $S_0 = 3.6$ GeV$^2$
- $S_0 = 3.4$ GeV$^2$
- $S_0 = 3.2$ GeV$^2$
- $S_0 = 3.0$ GeV$^2$

$S_0 = 3.6$ GeV$^2$

$S_0 = 3.4$ GeV$^2$

$S_0 = 3.2$ GeV$^2$

$S_0 = 3.0$ GeV$^2$
Figure 17

$R_0/m^2 (\text{GeV}^{-4})$ vs $\tau (\text{GeV}^{-2})$