An Inexact Manifold Augmented Lagrangian Method for Adaptive Sparse Canonical Correlation Analysis with Trace Lasso Regularization

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Abstract: Canonical correlation analysis (CCA for short) describes the relationship between two sets of variables by finding some linear combinations of these variables that maximizing the correlation coefficient. However, in high-dimensional settings where the number of variables exceeds sample size, or in the case of that the variables are highly correlated, the traditional CCA is no longer appropriate. In this paper, an adaptive sparse version of CCA (ASCCA for short) is proposed by using the trace Lasso regularization. The proposed ASCCA reduces the instability of the estimator when the covariates are highly correlated, and thus improves its interpretation. The ASCCA is further reformulated to an optimization problem on Riemannian manifolds, and an manifold inexact augmented Lagrangian method is then proposed for the resulting optimization problem. The performance of the ASCCA is compared with the other sparse CCA techniques in different simulation settings, which illustrates that the ASCCA is feasible and efficient.

Keywords: Canonical correlation analysis; Sparsity; Trace Lasso regularization; Manifold constrained optimization; Inexact manifold augmented Lagrangian method.

Mathematics Subject Classification: 65F15, 65K05, 90C50

1 Introduction

Canonical correlation analysis (CCA for short) firstly proposed by Hotelling [11] aims to tackle the associations between two sets of variables. It has wide applications in many important

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*This research was supported by the Natural Science Foundation of China (NSFC 11571074, 61672005).
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fields such as biology [20, 17, 13], medicine [5], image analysis [7, 13], etc.

Suppose that there are two data sets: $X \in \mathbb{R}^{n \times p}$ containing $p$ variables and $Y \in \mathbb{R}^{n \times q}$ containing $q$ variables, both are obtained from $n$ observations. The CCA seeks two linear combinations of these variables from $X$ and $Y$ with maximal correlation coefficient. Specially, let

$$\Sigma_{xx} = \frac{1}{n}X^TX, \Sigma_{yy} = \frac{1}{n}Y^TY$$

be the sample covariance matrices of $X$ and $Y$ respectively, and $\Sigma_{xy} = \frac{1}{n}X^TY$ be the sample cross-covariance matrix, then the CCA finds a pair $(u, v)$ such that

$$\text{corr}(Xu, Yv) = \frac{u^T\Sigma_{xy}v}{\sqrt{u^T\Sigma_{xx}u} \sqrt{v^T\Sigma_{yy}v}}$$

is maximized. The new variables $u$ and $v$ are canonical variables, and the correlations between canonical variables are called canonical correlations. The canonical variables $u$ and $v$ can be respectively obtained by the eigenvectors of matrix

$$\Sigma_{xx}^{-1}\Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx}.$$

The canonical correlations are given by the positive square root of those eigenvalues. Since the CCA model (1.1) is in form of fractional, it is difficult to optimize. A equivalent formulation of CCA is given by

$$\begin{cases}
\max_{u,v} u^T\Sigma_{xy}v \\
\text{s.t. } u^T\Sigma_{xx}u = 1, v^T\Sigma_{yy}v = 1,
\end{cases}$$

which can be regarded as an optimization problem on the generalized Stiefel manifolds. However, a potential disadvantage of the CCA is that, the learned solution is a linear combination of all original variables, which brings down the interpretability. If the number of variables exceeds sample size, traditional CCA cannot be performed due to that $\Sigma_{xx}$ and $\Sigma_{yy}$ are singular. Hence, many researchers proposed various sparse CCA (SCCA) to handle the case that the number of variables exceeds observations, and to improve the interpretability of canonical variables by restricting the linear combinations to a subset of original variables.

In this paper, we propose an adaptive sparse CCA model by incorporating the trace Lasso regularization. The matrix version of trace Lasso regularization can be adopted to both highly correlated and uncorrelated data. Our major contributions are summarized in follows:

1) We present a matrix version of trace Lasso regularization, and show that the new regularization function enjoys the properties of original trace Lasso.
2) By introducing trace Lasso regularization into the CCA model, we obtain an adaptive sparse CCA model (ASCCA). To our knowledge, our ASCCA is the first to takes the data correlation into account in the CCA model. In addition, our model consider multiple variables simultaneously.

3) The new model is reformulated to an optimization problem on the generalized Stiefel manifold. An manifold inexact augmented Lagrangian method is proposed for the resulted optimization problem, and the convergence is established under some assumptions.

4) The experimental results demonstrate that, the proposed ASCCA is superior to some existing sparse CCA models.

The rest of the paper is organized as follows. Section 2 briefly gives some reviews on the related works. Section 3 proposes an adaptive sparse version of the CCA introduced by the new trace Lasso regularization. Section 4 provides an optimization reformulation and the manifold inexact augmented Lagrangian method for the new model, and gives the convergence analysis. In Section 5 a simulation study is provided to show the validity and efficiency of the proposed method. Section 6 concludes this paper with some final remarks.

2 Related works

It is well known that, if the sample size exceeds dimension, the traditional CCA does not perform. To overcome this difficulty, various methods were proposed via incorporate different regularization function. Vinod [20] proposed a canonical ridge, which is an adaptation of the ridge regression for the CCA framework proposed by Hoerl and Kennard [10], and introduced an efficient sparsity penalty strategy. After that, various approaches for sparse CCA (SCCA for short) were proposed in literature, which includes $\ell_1$ regularization [18, 25], elastic net [21], group sparse and structured sparse [14, 4], etc. There also exists some limitations. If there is a group of variables which the pairwise correlation is high, the Lasso tends to select only one variable from this group, which may lead some misunderstands to the truth. Group sparse regularization needs the prior knowledge of group, which is unrealistic in some real applications. The proposed adaptive sparse CCA model utilized the new trace Lasso regularization, which incorporates data matrix into regularization, to adaptively deal with the correlation of covariation matrix.

The original SCCA model is difficult to handle, so many researchers simplify it by assuming that $\Sigma_x$ and $\Sigma_y$ are diagonal matrices or identity matrices. Parkhomenko, et al [18] assume that, the covariance matrices $\Sigma_{xx}, \Sigma_{yy}$ are the identity matrices, and used a sparse singular value
decomposition to derive sparse singular vectors. Wilms and Croux [24] converted the SCCA model into a penalized regression framework. Suo [19] presented an approximated SCCA model as follows

\[
\begin{align*}
\min_{u,v} & - u^T \Sigma_{xy} v + \lambda_1 \|u\|_1 + \lambda_2 \|v\|_1 \\
\text{s.t.} & \quad u^T \Sigma_{xx} u \leq 1, v^T \Sigma_{xx} v \leq 1,
\end{align*}
\]

(2.1)

and problem (2.1) was solved by a linearized alternating direction method of multipliers (LADMM). Witten [25] further relaxed (2.1) to

\[
\begin{align*}
\min_{u,v} & - u^T \Sigma_{xy} v \\
\text{s.t.} & \quad \|u\|_2^2 \leq 1, \|v\|_2^2 \leq 1, P_1(u) \leq c_1, P_2(v) \leq c_2,
\end{align*}
\]

(2.2)

where \(P_1(u)\) and \(P_2(v)\) are some regularizations for sparsity, and then developed a penalized matrix decomposition algorithm to solve model (2.2). Focusing on a sparse version of the original CCA model [12], Gao [8] proposed a two-stage method by a convex relaxation of CCA model. For the matrix case, many researchers adopted the residual model to obtain the high-order canonical variables [18, 25, 19]. In this paper, we get multiple variables simultaneously in our new model. In addition, all results on the matrix case mentioned above have not given convergence analysis for their algorithms, we proposed an efficient method to solve our new model, and provided the convergence analysis.

The original trace Lasso was proposed by Grave [9]. Trace Lasso regularization was successful applied to various scenarios including subspace clustering [23], sparse representation classification [22] and subspace segment [16], and so on. However, they only considered the trace Lasso regularization in vector case in literature. In this paper, we generalize the original trace Lasso regularization to matrix case, and adopt it as a new regularization for the SCCA, and get an adaptive SCCA model.

Notations: We use capital and lowercase symbols to represent matrix and vector, respectively. Let \(1_d \in \mathbb{R}^d\) denote the vector of all 1’s, \(e_i\) be a vector whose \(i\)-th entry is 1 and 0 for others, \(\text{Diag}(w)\) is a diagonal matrix where the \(i\)-th diagonal entry is \(w_i\), and \(\text{diag}(W)\) be a vector where the \(i\)-th entry is \(W_{ii}\). Let \(W_i\) and \(W_{ij}\) denote the \(i\)-th row and \(j\)-th column of \(W\), \(\text{tr}(W)\) be the trace of \(W\). For a vector \(w\), let \(\|w\|_1, \|w\|_2\) be the \(\ell_1\) and \(\ell_2\) norm. For a matrix \(W \in \mathbb{R}^{n \times p}\), let \(\|W\|_{2,1} = \sum_{i=1}^n \|W_i\|_2\) be the \(\ell_{2,1}\) norm, \(\|W\|_F\) and \(\|W\|_*\) denote the Frobenius norm and nuclear norm respectively, \(\|W\|_{op}\) denote the operator norm.
3 Adaptive sparse CCA using trace Lasso regularization

3.1 Trace Lasso in vector case

Consider the following linear estimator:
\[
\min_w \frac{1}{2}\|Xw - y\| + \lambda \Omega(w)
\] (3.1)
where \(X \in \mathbb{R}^{n \times p}\) is a data matrix. The trace Lasso is a correlation based penalized norm proposed by Grave et al. [9] for balancing the \(\ell_1\) and \(\ell_2\) norm. It is defined as follows
\[
\Omega(w) = \|X \text{diag}(w)\|_*
\]
where \(\| \cdot \|_*\) is nuclear norm. A main advantage of trace Lasso being superior to other norm is that, the trace Lasso involves the data matrix \(X\), makes it adaptive to the correlation of data. As shown in [9], if each column of \(X\) is normalized to 1, the trace Lasso interpolates between the \(\ell_1\) norm and \(\ell_2\) norm in the sense of
\[
\|w\|_1 \leq \|X \text{diag}(w)\|_* \leq \|w\|_2.
\] (3.2)
The inequality are tight. To see this, if the data are uncorrelated (\(X^TX = I_p\)), trace Lasso reduce to \(\|w\|_1\), and if the data are highly correlated (\(X = X_11^T\)), trace Lasso equals to \(\|w\|_2\).

3.2 Trace Lasso in matrix case

Let \(W \in \mathbb{R}^{p \times r}\), define a linear operator \(A_X \colon \mathbb{R}^{p \times r} \to \mathbb{R}^{n \times pr}\) as
\[
A_X(W) = (X \text{Diag}(W_1), \cdots, X \text{Diag}(W_r))
\]
and its adjoint operator \(A_X^* \colon \mathbb{R}^{n \times pr} \to \mathbb{R}^{p \times r}\) as
\[
A_X^*(M) = (\text{diag}(X^TM_1), \cdots, \text{diag}(X^TM_r))
\]
where \(M_i = M(:, (i-1)p+1 : ip)\) denotes \(i\)-th block matrix of \(M\). Then, the trace Lasso in matrix case is defined as follows
\[
\Omega(W) = \|A_X(W)\|_*
\] (3.3)
It is easy to show that, the trace Lasso regularizer in matrix case [3.3] has similar properties to that in vector case. If each column of \(X\) is normalized, then the linear operator \(A_X\) can be rewritten to
\[
A_X(W) = \sum_{i=1}^{r} \sum_{j=1}^{p} X_{ij} \bar{e}_{ij}^T \bar{e}_{ij} = \sum_{j=1}^{p} X_{ij} \cdot \left( \sum_{i=1}^{r} w_{ij} \bar{e}_{ij}^T \right)
\] (3.4)
where \(\bar{e}_{ij} \in \mathbb{R}^{pr \times 1}\) is an unit vector in which the \(((i-1)p + j)\)-th component is 1 and the others are 0. There are two special case:
1) If the data (i.e., column vectors of $X$) are uncorrelated, i.e., $X^T X = I_p$. Then (3.4) gives a singular value decomposition of $A_X(W)$. In the case, trace Lasso (3.3) reduces to the $\ell_{2,1}$ norm

$$\|A_X(W)\|_* = \sum_{j=1}^p \|X_j\|_2 \|W_j\|_2 = \|W\|_{2,1}.$$ 

2) If the data are highly correlated, especially if all columns of $X$ are identical and have unit size, we have

$$A_X(W) = X_1 \cdot \sum_{j=1}^p \sum_{i=1}^r W_{ij} \epsilon_{ij}^T = X_1 \cdot (\text{vec}(W))^T,$$

where $\text{vec}(W) = [W_1; \cdots; W_r]$. Then trace Lasso (3.3) reduces to Frobenius norm

$$\|A_X(W)\|_* = \|X_1 \cdot (\text{vec}(W))^T\|_* = \|X_1\|_2 \cdot \|\text{vec}(W)\|_2 = \|W\|_F.$$

The following proposition show that the trace Lasso (3.3) in matrix case is adaptive to the correlation of data, which is similar to the original trace Lasso.

**Proposition 3.1.** Let $X \in \mathbb{R}^{n \times p}$, and each column of $X$ is normalized, $W \in \mathbb{R}^{p \times r}$. Then

$$\|W\|_F \leq \|A_X(W)\|_* \leq \sqrt{r} \|W\|_{2,1}.$$  \hspace{1cm} (3.5)

**Proof.** We first show that $\|A_X(W)\|_F = \|W\|_F$. Specifically, we have

$$\|A_X(W)\|_F^2 = \sum_{i=1}^r \|X \text{diag}(W,i)\|_F^2 = \sum_{i=1}^r \sum_{j=1}^p W_{ij}^2 \|X_j\|_2^2 = \sum_{i=1}^r \sum_{j=1}^p W_{ij}^2 = \|W\|_F^2.$$

Then, for the first inequality of (3.5) we have

$$\|W\|_F = \|A_X(W)\|_F \leq \|A_X(W)\|_*.$$

Denote the $j$-th column of the $i$-th submatrix in $M$ by $M^i_{:,j}$, and let $\tilde{M}_j = [M^1_{:,j}, M^2_{:,j}, \cdots, M^r_{:,j}] \in \mathbb{R}^{n \times r}$. Then
\[ \|A_X(W)\|_* = \max_{\|M\|_{op} \leq 1} \langle M, A_X(W) \rangle \]
\[ = \max_{\|M\|_{2} \leq 1} \langle A_X^*(M), W \rangle \]
\[ = \max_{\|M\|_{2} \leq 1} \sum_{i=1}^{r} W_i^T \text{diag}(X^T M_i) \]
\[ \leq \max_{\|M\|_{2} \leq 1} \sum_{j=1}^{p} \|X_j\|_2 \cdot \left\| \sum_{i=1}^{r} W_{ij} M_i \right\|_2 \]
\[ \leq \max_{\|M\|_{2} \leq 1} \sum_{j=1}^{p} \|X_j\|_2 \cdot \|\hat{M}_j\|_F \|W_j\|_2 \]
\[ \leq \sqrt{r} \sum_{j=1}^{p} \|W_j\|_2 = \sqrt{r} \|W\|_{2,1}. \]

The first equality used the fact that the dual norm of the trace norm is the operator norm. The last inequality used that \(\|X_j\|_2 = 1(\forall j)\), and \(\|M_{i,:}\|_2 \leq 1\) which deduces \(\|\hat{M}_j\|_F \leq \sqrt{r}\).

**Remark 3.1.** If \(r = 1\), then Proposition 3.1 is indeed Proposition 3 in [9].

### 3.3 Regression framework of the adaptive SCCA

Given two data matrices \(X \in \mathbb{R}^{n \times p}\) and \(Y \in \mathbb{R}^{n \times q}\) on the same set of observations, where \(n\) is the sample size, \(p\) and \(q\) are the feature numbers. Without loss of generality, we assume that data matrices \(X\) and \(Y\) are mean centered. By \(\Sigma_{xx} = \frac{1}{n} X^T X, \Sigma_{yy} = \frac{1}{n} Y^T Y\) and \(\Sigma_{xy} = \frac{1}{n} X^T Y\), the CCA problem can be rewritten as

\[
\begin{align*}
(u^*, v^*) &= \arg \max_{u,v} u^T X^T Y v, \\
\text{s.t.} & \quad u^T X^T X u = 1, v^T Y^T Y v = 1.
\end{align*}
\]

For multiple canonical vectors, let \(U = [u_1, u_2, \cdots, u_r]\) and \(V = [v_1, v_2, \cdots, v_r]\) where \((u_i, v_i)\) denote the \(i\)-th pair of the canonical vectors, the multiple CCA problem is

\[
\begin{align*}
(U^*, V^*) &= \arg \max_{U,V} \text{tr}(U^T X^T Y V), \\
\text{s.t.} & \quad U^T X^T X U = I_r, V^T Y^T Y V = I_r
\end{align*}
\]

The CCA problem (3.7) can be reformulated to a constrained bilinear regression problem
of the form
\[
\begin{aligned}
(U^*, V^*) &= \arg \min_{U, V} \frac{1}{2} \|XU - YV\|^2_F, \\
\text{s.t. } U^T X^T X U &= I_r, V^T Y^T Y V &= I_r.
\end{aligned}
\] (3.8)

To adapt to the dependence of data, we consider an adaptive sparse CCA (SCCA) model with trace Lasso regularization. Specifically, we have
\[
\begin{aligned}
(U^*, V^*) &= \arg \min_{U, V} \frac{1}{2} \|XU - YV\|^2_F + \lambda_u \|A_X(U)\|_* + \lambda_v \|A_Y(V)\|_*, \\
\text{s.t. } U^T (X^T X) U &= I_r, V^T (Y^T Y) V &= I_r,
\end{aligned}
\] (3.9)

where \( U \in \mathbb{R}^{p \times r} \), \( V \in \mathbb{R}^{q \times r} \), and \( \lambda_u, \lambda_v \) are the penalty parameters, \( A_X : \mathbb{R}^{p \times r} \rightarrow \mathbb{R}^{n \times pr} \) and \( A_Y : \mathbb{R}^{q \times r} \rightarrow \mathbb{R}^{n \times qr} \) are linear operators.

4 Optimization method for SCCA (3.9)

The SCCA model (3.9) is a nonconvex and nonsmooth optimization problem, and it is difficult to be solved. Riemannian optimization methods are popular to solve a class constrained optimization problem with special structure. Hence, in this section we first reformulate problem (3.9) to a nonsmooth optimization problem on the generalized Stiefel manifolds, then adopt a manifold inexact augmented Lagrangian method in [6] to solve the resulting problem. Finally, we give a convergence analysis of the proposed method.

4.1 Augmented Lagrangian scheme

Let \( \mathcal{M}_1 = \{U \mid U^T X^T X U = I_r\} \), \( \mathcal{M}_2 = \{V \mid V^T Y^T Y V = I_r\} \), and \( g(\cdot) = \lambda_u \| \cdot \|_*, h(\cdot) = \lambda_v \| \cdot \|_* \), then problem (3.9) can be reformulated as
\[
\begin{aligned}
(U^*, V^*) &= \arg \min_{U, V} \frac{1}{2} \|XU - YV\|^2_F + g(P) + h(Q), \\
\text{s.t. } &U \in \mathcal{M}_1, V \in \mathcal{M}_2,
\end{aligned}
\] (4.1)

Here, we assume that \( X^T X \) and \( Y^T Y \) are positive definite.\(^\text{1}\) Then \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) can be regarded to generalized Stiefel manifolds, and problem (4.1) is an optimization problem on generalized Stiefel manifolds. We further reformulate (4.1) to
\[
\begin{aligned}
(U^*, V^*) &= \arg \min_{U, V} \frac{1}{2} \|XU - YV\|^2_F + g(P) + h(Q), \\
\text{s.t. } &A_X(U) = P, A_Y(V) = Q \\
&U \in \mathcal{M}_1, V \in \mathcal{M}_2,
\end{aligned}
\] (4.2)

\(^\text{1}\)If it is not positive definite, we can replace \( X^T X \) by \( (1 - \alpha)X^T X + \alpha I_p \).
The Lagrangian function associated with (4.2) is given by

\[ L(U, V, P, Q; \Lambda_1, \Lambda_2) = \frac{1}{2} \|XU - YV\|_F^2 + g(P) + h(Q) \]

\[ - \langle \Lambda_1, A_X(U) - P \rangle - \langle \Lambda_2, A_Y(V) - Q \rangle, \]

where \( \Lambda_1 \) and \( \Lambda_2 \) denote the Lagrangian multipliers. Let \( \rho \) be a penalty parameter. Then, the corresponding augmented Lagrangian function is given by

\[ L_\rho(U, V, P, Q; \Lambda_1, \Lambda_2) = L(U, V, P, Q; \Lambda_1, \Lambda_2) + \frac{\rho}{2} \|A_X(U) - P\|_F^2 + \frac{\rho}{2} \|A_Y(V) - Q\|_F^2 \] (4.4)

Then, the proposed manifold inexact augmented Lagrangian method for (4.2) is summarized in Algorithm 1.

**Algorithm 1** Manifold inexact augmented Lagrangian method for problem (4.2)

1. **Input:** Let \( \Lambda_{\text{min}} < \Lambda_{\text{max}}, X_0 \in \mathcal{M}, \) tolerance \( \epsilon_{\text{min}} \geq 0, \epsilon_0 > 0, \rho_0 > 1, \mu > 1, 0 < \tau < 1. \)
2. **for** \( k = 0, 1, \cdots \) **do**
3. Updating the primary variables by approximately solving

\[ (U^{k+1}, V^{k+1}, P^{k+1}, Q^{k+1}) = \arg \min_{U \in \mathcal{M}_1, V \in \mathcal{M}_2, P, Q} \{ \Psi_k = L_\rho(U, V, P, Q; \Lambda_1^k, \Lambda_2^k) \} \] (4.5)

such that a specified stopping criteria is hit.
4. Updating the dual variables via

\[ \Lambda_1^{k+1} = \Lambda_1^k - \rho_k (A_X(U^{k+1}) - P^{k+1}), \quad \Lambda_2^{k+1} = \Lambda_2^k - \rho_k (A_Y(V^{k+1}) - Q^{k+1}) \]

5. Updating \( \rho_{k+1} \) via

\[ \rho_{k+1} = \begin{cases} \rho_k & \text{if } \|R_i^k\|_\infty \leq \tau \|R_i^{k-1}\|_\infty, i = 1, 2, \\ \gamma \rho_k & \text{otherwise,} \end{cases} \]

where \( R_1^k = A_X(U^{k+1}) - P^{k+1}, R_2^k = A_Y(V^{k+1}) - Q^{k+1}. \)
6. **end for**

### 4.2 Convergence analysis

Let \( W = (U, V, P, Q) \in \mathbb{R}^{p \times r} \times \mathbb{R}^{r \times r} \times \mathbb{R}^{n \times pr} \times \mathbb{R}^{n \times qr} \) be a variable formed by concatenating \( U, V, P \) and \( Q. \) Let \( F(W) = f(U, V) + g(P) + g(Q) \) where \( f(U, V) = \frac{1}{2} \|XU - YV\|_F^2. \) Then, problem (4.2) can be rewritten as

\[ \min_W F(W) \quad \text{s.t.} \quad h(W) = 0, \ W \in \mathcal{N}, \]

(4.6)
where $\mathcal{N} := \mathcal{M}_1 \times \mathcal{M}_2 \times \mathbb{R}^{n \times pr} \times \mathbb{R}^{n \times qr}$, and $h(W)$ is given by

$$h(W) := \left( \mathcal{A}_X(U) - P, \mathcal{A}_Y(V) - Q \right).$$

The corresponding augmented Lagrangian can be rewritten as

$$\mathcal{L}_\rho(W; Z) = F(W) + \sum_{i,j} Z_{ij} [h(W)]_{ij} + \frac{\rho}{2} \sum_{i,j} [h(W)]^2_{ij} \quad (4.7)$$

The corresponding KKT condition is given by

$$0 \in \partial F(W^*) + \sum_{i=1}^m \sum_{j=1}^r Z_{ij} \text{grad}[h(W^*)]_{ij}, \quad h(W^*) = 0, \quad W^* \in \mathcal{N} \quad (4.8)$$

where $\partial F(W^*)$ is the Riemannian subdifferential of $F$ at $W^*$. To obtain an efficient implementation of Algorithm 1, we inexactly solve the iteration subproblem $\mathcal{L}_\rho(W; Z)$ in which the following stopping criteria is used:

$$\delta^k \in \partial \Psi_k(W^{k+1}) \quad \text{and} \quad \|\delta^k\| \leq \epsilon_k \quad (4.9)$$

where $\epsilon_k \to 0$ as $k \to \infty$.

Following Yang, Zhang and Song [27], we give the constraint qualifications of problem $\mathcal{L}_\rho(W; Z)$:

**Definition 4.1** (LICQ). Linear independence constraint qualifications (LICQ) are said to hold at $W \in \mathcal{N}$ for problem $(4.6)$ if

$$\{\text{grad}[h(W)]_{ij} | i = 1, \ldots, m; j = 1, \ldots, pr, \ldots, qr\} \quad (4.10)$$

**Theorem 4.1.** Suppose $\{W^k\}_{k \in \mathbb{N}}$ is a sequence generated by Algorithm 1 and the stopping criteria $(4.9)$ is hit at the $k$-th iteration. Then the limit point set of $\{W^k\}_{k \in \mathbb{N}}$ is nonempty. Let $W^*$ be a limit point of $\{W^k\}$, and the LICQ holds at $W^*$. Then $W^*$ is a KKT point of the problem $(4.6)$.

**Proof.** See [6].

**Lemma 4.1.** The LICQ always holds at $\forall W \in \mathcal{N}$ for problem $(4.6)$.

**Proof.** Let $h_1(W) := \mathcal{A}_X(U) - P, h_2(W) := \mathcal{A}_Y(V) - Q$, then

$$\nabla [h_1(W)]_{ij} = \nabla_U [h_1(W)]_{ij} \times \nabla_V [h_1(W)]_{ij} \times \nabla_P [h_1(W)]_{ij} \times \nabla_Q [h_1(W)]_{ij},$$

$$i = 1, \ldots, n; j = 1, \ldots, pr \quad (4.11)$$

$$\nabla [h_2(W)]_{ij} = \nabla_U [h_2(W)]_{ij} \times \nabla_V [h_2(W)]_{ij} \times \nabla_P [h_2(W)]_{ij} \times \nabla_Q [h_2(W)]_{ij},$$

$$i = 1, \ldots, n; j = 1, \ldots, qr.$$
For all $i = 1, \ldots, m, j = 1, \ldots, n$, let $E_{ij}^{m \times n}$ be a $m \times n$ matrix in which the entry at the $i$-th row and the $j$-column is 1, the others are 0. Then

$$
\begin{align*}
\nabla_P[h_1(W)]_{ij} &= E_{ij}^{m \times pr}, \quad \nabla_Q[h_1(W)]_{ij} = 0, \quad i = 1, \ldots, n; j = 1, \ldots, pr, \\
\nabla_P[h_1(W)]_{ij} &= 0, \quad \nabla_Q[h_1(W)]_{ij} = E_{ij}^{m \times qr}, \quad i = 1, \ldots, n; j = 1, \ldots, qr.
\end{align*}
$$

(4.12)

A basis of the normal cone of $\mathcal{M} : \mathcal{M}_1 \times \mathcal{M}_2$ at $(U, V)$, denoted by $N_U \mathcal{M}_1 \times N_V \mathcal{M}_2$, is given by

$$
\begin{align*}
\Sigma_{x=U}(e_i e_j^T + e_j e_i^T) : i = 1, \ldots, r, j = i, \ldots, r \} \times \{ \Sigma_{y=V}(e_i e_j^T + e_j e_i^T) : i = 1, \ldots, r, j = i, \ldots, r \}.
\end{align*}
$$

It is easy to show that, $\forall \ W \in \mathcal{N}$, if there exists $Z^1, Z^2$ such that

$$
\begin{align*}
\sum_{i=1}^n \sum_{j=1}^{pr} Z_{ij}^1 \nabla[h_1(W)]_{ij} + \sum_{i=1}^n \sum_{j=1}^{qr} Z_{ij}^2 \nabla[h_2(W)]_{ij} \in N_U \mathcal{N},
\end{align*}
$$

(4.13)

then $Z^1 = Z^2 = 0$. Since $\mathcal{N}$ is a submanifold of Euclidean space, it derives immediately from (4.13) that

$$
\begin{align*}
\sum_{i=1}^n \sum_{j=1}^{pr} Z_{ij}^1 \text{grad}[h_1(W)]_{ij} + \sum_{i=1}^n \sum_{j=1}^{qr} Z_{ij}^2 \text{grad}[h_2(W)]_{ij} = 0.
\end{align*}
$$

Which implies that LICQ holds at $W$ and completes the proof.

4.3 Riemannian gradient method for subproblem (4.5)

In section 4.1, we present an manifold inexact augmented Lagrangian method to solve problem (4.2). The main challenge in the proposed method (Algorithm 1) is to solve subproblem (4.5) efficiently. Problem (4.5) is a nonsmooth problem under manifold constrained. In this subsection, we first get an equivalence smooth optimization problem by using the Moreau envelop technique, then we present Riemannian gradient method to solve the equivalent problem.

The proximal mapping $\text{Prox}_p(\cdot)$ associated with $p$ is defined by

$$
\text{Prox}_p(U) = \arg \min_W \left\{ p(W) + \frac{1}{2}\| U - W \|_F^2 \right\},
$$

(4.14)

For fixed $\Lambda_1, \Lambda_2$ and $\rho$, we consider

$$
\min_{U,V,P,Q} \left\{ \psi(U, V, P, Q) := L_\rho(U, V, P, Q; \Lambda_1, \Lambda_2), \quad \text{s.t.} \quad U \in \mathcal{M}_1, V \in \mathcal{M}_2 \right\}
$$

Let

$$
\psi(U, V) = \inf_{P,Q} \psi(U, V, P, Q)
$$

$$
= \frac{1}{2}\| UX - Y V \|_F^2\| + g(\text{Prox}_{g/\rho}(A_X(U) - \Lambda_1/\rho)) + h(\text{Prox}_{h/\rho}(A_Y(V) - \Lambda_2/\rho))
+ \frac{\rho}{2}\| A_X(U) - \Lambda_1/\rho - \text{Prox}_{g/\rho}(A_X(U) - \Lambda_1/\rho)\|_F^2
+ \frac{\rho}{2}\| A_Y(V) - \text{Prox}_{h/\rho}(A_Y(V) - \Lambda_2/\rho)\|_F^2 - \frac{1}{2\rho}\| \Lambda_1 \|_F^2 - \frac{1}{2\rho}\| \Lambda_2 \|_F^2.
$$
Hence, if

\[(\tilde{U}, \tilde{V}, \tilde{P}, \tilde{Q}) = \arg\min_{U \in M_1, V \in M_2, P, Q} \Psi(U, V, P, Q), \]

then \((\tilde{U}, \tilde{V}, \tilde{P}, \tilde{Q})\) can be computed by

\[
\begin{cases}
(\tilde{U}, \tilde{V}) = \arg\min_{U \in M_1, V \in M_2} \psi(U, V), \\
\tilde{P} = \text{Prox}_{g/\rho}(A_X(\tilde{U}) - \Lambda_1/\rho), \\
\tilde{Q} = \text{Prox}_{h/\rho}(A_Y(\tilde{V}) - \Lambda_2/\rho).
\end{cases}
\] (4.15)

Notice that the subproblems for \(P\) and \(Q\) are proximal operators. Both \(g\) and \(h\) in (4.15) are nuclear norm functions, the proximal operator is indeed a singular value shrinkage operator, which is given by:

\[
D_\tau(Y) = \arg\min_X \left\{ \frac{1}{2} \|X - Y\|_F^2 + \tau \|X\|_* \right\} = UD_\tau(\Sigma)V',
\] (4.16)

where \(Y = U\Sigma V'\), \(\Sigma = \text{diag}\{\sigma_i\}_{1 \leq i \leq r}\) and \(D_\tau(\Sigma) = \text{diag}\{(\sigma_i - \tau)^+\}\).

Now we focus on the subproblem regarding jointed variable \((U, V)\) in (4.15). Recall that \((\tilde{U}, \tilde{V}) = \arg\min_{U, V} \{\psi(U, V), \text{ s.t. } U \in M_1, V \in M_2\}\). (4.17)

Let \(W := (U, V)\) and \(M := M_1 \otimes M_2\) be a product manifold. Then, problem (4.17) can be formulated

\[
\hat{W} = \arg\min_W \{\psi(W), \text{ s.t. } W \in M\}.
\] (4.18)

By Lemma B.1, \(\psi(W)\) is continuously differentiable in Euclidean space, and its Euclidean gradient is

\[
\nabla_\psi(W) = \begin{pmatrix}
\nabla_U \psi(W) \\
\nabla_V \psi(W)
\end{pmatrix} = \begin{pmatrix}
X^T(XU - YV) + \rho A_X(U) - \frac{1}{\rho} \Lambda_1 - \text{Prox}_{g/\rho}(A_X(U) - \frac{1}{\rho} \Lambda_1) \\
Y^T(YV - XU) + \rho A_Y(V) - \frac{1}{\rho} \Lambda_2 - \text{Prox}_{h/\rho}(A_Y(V) - \frac{1}{\rho} \Lambda_2)
\end{pmatrix}
\]

Since \(M\) is a Riemannian submanifold in Euclidean space, by lemma B.2 \(\psi(W)\) is retraction smooth, and its Riemannian gradient is

\[
\text{grad}_\psi(W) = \begin{pmatrix}
\text{grad}_U \psi(W) \\
\text{grad}_V \psi(W)
\end{pmatrix} = \begin{pmatrix}
P_{T_U M_1}(\nabla_U \psi(W)) \\
P_{T_V M_2}(\nabla_V \psi(W))
\end{pmatrix}.
\] (4.19)

It is shown that problem (4.18) is a smooth optimization problem on Riemannian manifold. In this paper, we adopt a Riemannian Barzilai-Borwein (RBB) gradient method [12] to solve problem (4.18), see Algorithm 2 for details.
Algorithm 2 Riemannian Barzilai-Borwein gradient method for subproblem (4.18), RBB

1: Given: \(W^0 \in \mathcal{M}\), tolerance \(\epsilon > 0\), initial step size \(\alpha_0^{BB}\). Let \(g^0 = \nabla \psi(W^0)\), the sufficient decrease parameter \(\gamma\) and the step length contraction factor \(\sigma \in (0, 1)\).

2: Initialize: \(k = 0\).

3: while \(\|g^k\| \geq \epsilon\) do

4: Find the smallest positive integer \(h\) such that

\[
\psi(R_{W^k}(-\sigma^h \alpha_k^{BB} g^k)) \leq \psi(W^k) - \gamma \sigma^h \alpha_k^{BB} \|g^k\|_W^2
\]  

(4.20)

and set \(\alpha_k := \sigma^h \alpha_k^{BB}\).

5: Compute \(W^{k+1} = R_{W^k}(-\alpha_k g^k)\), and \(g^{k+1} = \nabla \psi(W^{k+1})\).

6: Set

\[
\tau_{k+1} = \frac{\langle s^k, s^k \rangle_{W^k} \langle y^k, y^k \rangle_{W^{k+1}}}{\langle s^k, y^k \rangle_{W^k} \langle y^k, y^k \rangle_{W^{k+1}}},
\]

where \(s^k := g^{k+1} - T_{W^k \to W^{k+1}}(g^k)\) and \(y^k := T_{W^k \to W^{k+1}}(-\alpha_k g^k)\).

7: Compute the new step size \(\alpha_{k+1}^{BB}\)

\[
\alpha_{k+1}^{BB} = \left\{ \begin{array}{ll}
\min\{\alpha_{\text{max}}, \max\{\alpha_{\text{min}}, \tau_{k+1}\}\} & \text{if } \langle s^k, y^k \rangle_{W^{k+1}} > 0, \\
\alpha_{\text{max}} & \text{otherwise}.
\end{array} \right.
\]

8: Set \(k := k + 1\).

9: end while

5 Random Simulation

In the section, the performance of the SCCA model and the proposed method is verified by random simulation. The proposed adaptive trace Lasso regularization CCA in this paper is compared with the sparse CCA-\(\ell_1\) model (named as CoLaR) proposed by Gao [8]. The CoLaR is a computationally feasible two-stage method, which consists of a convex-programming-based initialization stage and a group-Lasso-based refinement stage. In the first stage, CoLaR solves the following convex minimization problem:

\[
\begin{align*}
\hat{F} &= \arg \min_{F} \left\{ \tilde{\Sigma}_{xy} F - \langle \tilde{\Sigma}_{xy}, F \rangle + \tau \|F\|_1 \right\} \\
\text{s.t.} \quad &\| (\hat{\Sigma}_x)^{1/2} F(\hat{\Sigma}_y)^{1/2} \|_* \leq r, \| (\hat{\Sigma}_x)^{1/2} F(\hat{\Sigma}_y)^{1/2} \|_2 \leq 1,
\end{align*}
\]  

(5.1)

where \(\hat{\Sigma}_{xy}, \hat{\Sigma}_x\) and \(\hat{\Sigma}_y\) are sample covariance matrices. Let \(U^0\) and \(V^0\) be the matrices whose column vectors are respectively the top \(r\) left and right singular vectors of \(\hat{F}\). Then, in the
second stage, CoLaR solve the following group-Lasso problem:

\[
\begin{align*}
U^1 &= \arg\min_U tr(U^T \hat{\Sigma}_x U) - 2tr(U^T \hat{\Sigma}_{xy} V^0) + \tau' \sum_{i=1}^p \|U_i\|_2, \\
V^1 &= \arg\min_V tr(V^T \hat{\Sigma}_y V) - 2tr((U^0)^T \hat{\Sigma}_{xy} V) + \tau' \sum_{i=1}^q \|V_i\|_2
\end{align*}
\] (5.2)

Finally, \(U^1\) and \(V^1\) are projected to generalized Stiefel manifolds \(U^T \hat{\Sigma}_x U = I_r\) and \(V^T \hat{\Sigma}_y V = I_r\).

The codes of CoLaR were downloaded from Ma’s homepage\(^2\). The first stage is terminated if it does not make much progress, or it reaches the maximum iteration number 10, the parameter in step (5.1) is set to \(\tau = 0.99\sqrt{(r + \log(q))/n}\). The obtained solution \((U^0, V^0)\) is used as the initial point for CoLaR. For our method, we also use \((U^0, V^0)\) obtained by (5.1) as the original point. The parameters of our Algorithm 1 are set to: \(\tau = 0.99, \sigma = 1.05, \rho_0 = \max\{\lambda_{\text{max}}(X^T X), \lambda_{\text{max}}(Y^T Y)\}, Z_{\text{min}} = -100 \cdot 1_{d \times r}, Z_{\text{max}} = 100 \cdot 1_{d \times r}, Z^0 = 0_{d \times r}\) and \(\epsilon_k = \max(10^{-3}, 0.9^k)\) where \(k \in \mathbb{N}\) is the iteration counter. We terminated Algorithm 1 if \(\max\{\|A_X(U^{k+1}) - P^{k+1}\|_F^2, \|A_Y(V^{k+1}) - Q^{k+1}\|_F^2\} \leq 10^{-8}\) or \(k \geq 500\). For Algorithm 2, the inner iteration of the Algorithm 1 it is terminated if \(\|\delta^k\| \leq \epsilon_k\), where \(\delta^k \in \partial \Psi_k(W^{k+1})\), or the inner iteration number exceeds 100.

**Parameters selection** We using the \(\kappa\)-fold cross-validation (CV) to select the optimal parameters for CoLaR and our method. For our method, we set \(\lambda_u = b \sqrt{(r + \log(p))/n}\) and \(\lambda_v = b \sqrt{(r + \log(q))/n}\). For CoLaR, the parameter \(\tau'\) is set to = \(\tau' = b \sqrt{(r + \log(\max(p, q)))/n}\). Then, we used \(\kappa\)-fold cross validation to select a common penalty parameter \(b\). In particular, for each choice of parameter \(b\), a \(\frac{1}{\kappa}\) proportion of the data (training sample) is used to obtain estimates \((\hat{U}, \hat{V})\). Then we evaluate the correlation between obtained canonical vectors in the remaining \(\frac{1}{\kappa}\) of the data set (testing sample), and compute average correlation over \(k\) CV steps. The optimal parameter \(b\) then corresponds to the highest average correlation. We set \(\kappa = 10\) in this paper.

### 5.1 Simulation data

The random simulation data is generated with respect to the following scheme: for each simulation design, there has \(p\) variables in data set \(X\) and \(q\) variables in data set \(Y\), and the sample size is \(n\). In all these settings, we set \(r = 2\) and \(\Lambda = \text{Diag}(\lambda_1, \lambda_2)\) where \(\lambda_1 = 0.9, \lambda_2 = 0.8\). The nonzero rows of both \(U\) and \(V\) are set to \(S = \{1, 6, 11, 16, 21\}\), the corresponding

\(^2\)http://www-stat.wharton.upenn.edu/ zongming/research.html
value at the nonzero coordinates are generated by normalizing random numbers drawn from the uniform distribution on finite set \{-2, -1, 0, 1, 2\}. We determinate the covariance matrix \( \Sigma_x \) and \( \Sigma_y \) via the following three procedure:

(1) Identity matrices: \( \Sigma_x = I_p, \Sigma_y = I_q \),

(2) Toeplitz matrices: \( \Sigma_x = 0.3^{\|i-j\|}, \text{for } i,j \in [p] \), and \( \Sigma_y = 0.3^{\|i-j\|}, \text{for } i,j \in [q] \),

(3) Correlation matrices:

\[
(\Sigma_x)_{ij} = \begin{cases} \sigma & (i,j) \in S \text{ and } i \neq j \\ 1 & i == j \\ 0 & \text{otherwise} \end{cases}, \quad (\Sigma_y)_{ij} = \begin{cases} \sigma & (i,j) \in S \text{ and } i \neq j \\ 1 & i == j \\ 0 & \text{otherwise} \end{cases},
\]

where \( \sigma \) is the correlation degree. In our experiments, we set \( \sigma = 0.3, 0.5, 0.8 \), respectively.

After determinate the covariance matrix, we generate the cross-covariance matrix \( \Sigma_{xy} \) by

\[
\Sigma_{xy} = \Sigma_x U \Lambda V \Sigma_y,
\]

and generate the data sets \( X \in \mathbb{R}^{n \times p} \) and \( Y \in \mathbb{R}^{n \times q} \) via

\[
\begin{pmatrix} x \\ y \end{pmatrix} \sim \mathcal{N} \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma_x & \Sigma_{xy} \\ \Sigma_{xy}^T & \Sigma_y \end{pmatrix} \right)
\]

Finally, we used the subspace distance between the estimation \((\hat{U}, \hat{V})\) and the ground truth \((U, V)\) as the prediction errors:

\[
\text{loss}_u = \|P_U - P_{\hat{U}}\|_F^2, \quad \text{loss}_v = \|P_V - P_{\hat{V}}\|_F^2.
\]

where \( P_U = U(U^TU)^{-1}U^T, P_V = V(V^TV)^{-1}V^T \) are projection matrices on the column space of \( U \) and \( V \), respectively.

Tables 5.1, 5.1, 5.1 display the simulation results: in each settings, the medians of the prediction error of CoLaR and our method over 100 repetitions for four different configurations of \((n; p; q)\) values are list for comparison.

In each table, the columns “lossu” and “lossv” report the smallest estimation errors of the medians out of the eleven trials on each simulated dataset, \((\rho_1, \rho_2)\) is the two canonical correlations. The columns “Init” report the results generated by the initialization step (5.1). The columns “CoLaR” and “Ours” report the results of the CoLaR and our method, respectively.

In case (1) and (2), i.e., \( \Sigma_x \) and \( \Sigma_y \) are the identity matrices and Toeplitz matrices, the results of CoLaR and our method are roughly the same. In case (3), our method consistently
Table 1: Performance for our method and CoLaR in case (1)

| (n, p, q)   | lossu | lossv | (ρ₁, ρ₂)   |
|------------|-------|-------|------------|
|            | Init  | CoLaR | Ours       |            | Init  | CoLaR | Ours       |            |
| (200,200,200) | 0.533 | 0.114 | **0.092** | 0.063      | (0.8325,0.6662) | (0.8852,0.7726) | **0.8891,0.7852** |
| (300,200,200) | 0.050 | 0.011 | **0.008** | **0.045** | (0.8832,0.6571) | (0.8987,0.7974) | **0.8988,0.7977** |
| (400,200,200) | 0.195 | 0.019 | **0.018** | **0.030** | (0.8527,0.7436) | (0.8948,0.7937) | **0.8949,0.7941** |
| (300,400,400) | 0.361 | 0.047 | **0.049** | 0.037      | (0.8442,0.6944) | (0.8962,0.7859) | (0.8972,0.7857) |

Table 2: Performance for our method and CoLaR in case (2)

| (n, p, q)   | lossu | lossv | (ρ₁, ρ₂)   |
|------------|-------|-------|------------|
|            | Init  | CoLaR | Ours       |            | Init  | CoLaR | Ours       |            |
| (200,200,200) | 0.249 | **0.071** | 0.079 | 0.096 | **0.081** | (0.8823,0.6879) | (0.8954,0.7767) | **0.8965,0.7773** |
| (300,200,200) | 0.275 | 0.032 | 0.032 | 0.187 | **0.022** | 0.027 | (0.8711,0.7475) | (0.8925,0.8101) | (0.8941,0.8100) |
| (400,200,200) | 0.211 | 0.027 | **0.022** | 0.237 | **0.022** | 0.030 | (0.8517,0.7470) | (0.8962,0.7807) | (0.8966,0.7791) |
| (300,400,400) | 0.250 | **0.040** | 0.043 | 0.394 | 0.039 | **0.038** | (0.8525,0.6726) | (0.8953,0.7804) | (0.8952,0.7796) |

outperform the CoLaR estimators. In particular, as the correlation parameter \( \sigma \) going to larger, the performance of our method is more significant inferiority than the CoLaR. This is in accordance to the theoretical results for trace Lasso regularization in Section 3.

6 Conclusions

In this paper, we presented a matrix variant of trace Lasso regularization, and proposed an adaptive sparse CCA model by incorporating the trace Lasso regularization into CCA problem. The proposed model can well cope with the situation in which the data set are correlate. The adaptive sparse CCA is further reformulated to an optimization problem on Riemannian manifolds, and a manifold inexact augmented Lagrangian method is then proposed for solving the resulting optimization problem. Note that the proposed manifold inexact augmented Lagrangian method can be used to solve the general manifold constrained optimization problem: \( \min_{U \in \mathcal{M}_1, V \in \mathcal{M}_2} \{ f(U, V) + g(U) + h(V) \} \), where \( g \) and \( h \) is may nonsmooth. Then, An Riemannian Barzilai-Borwein gradient method is adopted for the iteration subproblem of the proposed method, and the global convergence is established under some mild assumptions. We show that adaptive sparse CCA can significantly improve the performance compared with the
Table 3: Performance for our method and CoLaR in case (3)

| $\sigma$ | $(n,p,q)$ | lossu | lossv | $\rho_1,\rho_2$ |
|----------|-----------|-------|-------|-----------------|
|          | Init      | CoLaR | Ours  | Init | CoLaR | Ours | Init | CoLaR | Ours |
| 0.3      | (200,200,200) | 0.411 | 0.111 | **0.085** | 0.760 | 0.153 | **0.097** | (0.8555,0.6786) | (0.8846,0.7845) | 0.8880,0.7950 |
|          | (300,200,200) | 0.384 | 0.082 | **0.059** | 0.366 | 0.069 | **0.042** | (0.8394,0.5961) | (0.8878,0.7686) | **0.8941,0.7732** |
|          | (400,200,200) | 0.284 | 0.040 | **0.035** | 0.460 | 0.056 | **0.040** | (0.8034,0.7434) | (0.8890,0.7907) | (0.8918,0.7870) |
|          | (300,400,400) | 0.416 | 0.050 | 0.055 | 0.856 | 0.142 | **0.104** | (0.7650,0.6883) | (0.8667,0.7874) | 0.8737,0.7812 |
| 0.5      | (200,200,200) | 0.681 | 0.473 | **0.187** | 1.608 | 0.390 | **0.278** | (0.8250,0.5239) | (0.8880,0.6763) | (0.8849,0.7199) |
|          | (300,200,200) | 0.365 | 0.074 | **0.046** | 0.505 | 0.280 | **0.161** | (0.8584,0.6357) | (0.8820,0.7840) | **0.8930,0.7887** |
|          | (400,200,200) | 0.621 | 0.081 | **0.054** | 0.627 | 0.101 | **0.056** | (0.8196,0.7204) | (0.8790,0.7720) | **0.8852,0.7810** |
|          | (300,400,400) | 1.032 | 0.732 | **0.238** | 1.082 | 0.287 | **0.134** | (0.8480,0.5122) | (0.8512,0.7277) | **0.8762,0.7597** |
| 0.8      | (200,200,200) | 2.141 | 1.524 | **0.169** | 2.226 | 1.738 | **0.407** | (0.7891,0.2296) | (0.7850,0.2490) | **0.8519,0.7941** |
|          | (300,200,200) | 1.511 | 1.668 | **0.328** | 2.140 | 0.738 | **0.059** | (0.7275,0.4971) | (0.7841,0.6782) | **0.8877,0.7679** |
|          | (400,200,200) | 1.186 | 0.356 | **0.085** | 1.207 | 0.497 | **0.209** | (0.7793,0.6807) | (0.8113,0.7551) | **0.8926,0.7991** |
|          | (300,400,400) | 1.408 | 1.100 | **0.282** | 1.966 | 0.861 | **0.208** | (0.7105,0.2885) | (0.6924,0.3386) | **0.8101,0.7031** |

$\ell_1$-regularization based sparse CCA technique (CoLaR) in different simulation settings.

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A Riemannian submanifold

In this section, we introduce some necessary concepts and definitions of Riemannian optimization, we refer the reader to [1] for more details. A $d$-dimensional smooth manifold $\mathcal{M}$ is a Hausdorff and second-countable topological space, where each point has a neighborhood being locally homeomorphic to the $d$-dimensional Euclidean space via a family of charts, and the transition maps of intersecting charts are differentiable. At each point $x \in \mathcal{M}$, a tangent vector $\xi_x$ is defined as a mapping such that there exists a curve $\gamma$ on $\mathcal{M}$ with $\gamma(0) = x$, and satisfying

$$\xi_x f := \frac{d(f(\gamma(t)))}{dt} \bigg|_{t=0}$$

for all $f \in \mathcal{V}_x \mathcal{M}$. The tangent space $T_x \mathcal{M}$ to $\mathcal{M}$ at $x$ is defined as the set of all tangent vectors at $x$. A Riemannian manifold $(\mathcal{M}, \langle \rangle)$ is a real smooth manifold that quipped with a smoothly varying inner product $\langle \rangle_x$ on the tangent space $T_x \mathcal{M}$ of each point $x \in \mathcal{M}$. When $\mathcal{M}$ is a Riemannian submanifold of Euclidean space $\mathcal{E}$, the inner product is defined as Euclidean inner product: $\langle \eta_x, \xi_x \rangle_x = \langle \eta_x, \xi_x \rangle$, and the corresponding norm induced by the Riemannian metric is given by $\|\xi_x\|_x = \|\xi_x\|$. A retraction on manifold $\mathcal{M}$ is a smooth mapping $R : T \mathcal{M} \to \mathcal{M}$ with the following properties. Let $R_x : T_x \mathcal{M} \to \mathcal{M}$ be the restriction of $R$ to $T_x \mathcal{M}$:

- $R_x(0_x) = x$, where $0_x$ is zero element of $T_x \mathcal{M}$
- $dR_x(0_x) = id_{T_x \mathcal{M}}$, where $id_{T_x \mathcal{M}}$ is the identity mapping on $T_x \mathcal{M}$

To compare two tangent vector at different points, we need vector transport. The vector transport $T$ is a smooth mapping with $T \mathcal{M} \oplus T \mathcal{M} \to T \mathcal{M}$: $(\eta_x, \xi_x) \mapsto T_{\eta_x}(\xi_x) \in T \mathcal{M}$ for any $x \in \mathcal{M}$, where $T$ satisfies

- for any $\xi_x \in T_x \mathcal{M}$, $T_{0_x} \xi_x = \xi_x$,
- $T_{\eta_x}(a \xi_x + b \zeta_x) = a T_{\eta_x}(\xi_x) + b T_{\eta_x}(\zeta_x)$.

Vector transport preserves inner products, i.e. $\langle T_{\eta_x}(\xi_x), T_{\eta_x}(\zeta_x) \rangle_x = \langle \xi_x, \zeta_x \rangle_x$.

B Proximal operator and retraction-smooth

For a proper, convex and low semicontinuous function $g : \mathcal{E} \to \mathbb{R}$, the proximal operator with parameter $\mu \geq 0$, denoted by $\text{prox}_{\mu g}$, is defined by

$$\text{prox}_{\mu g}(v) := \arg \min_x \{g(x) + \frac{1}{2\mu} \|x - v\|^2\}. \quad (B.1)$$
The associated Moreau envelope is defined as a function $M_{\mu g} : \mathcal{E} \rightarrow \mathbb{R}$, given by

$$M_{\mu g}(v) := \min_x \left\{ g(x) + \frac{1}{2\mu} \|x - v\|^2 \right\} = g(\text{prox}_{\mu g}(v)) + \frac{1}{2\mu} \|\text{prox}_{\mu g}(v) - v\|^2.$$  \hspace{1cm} (B.2)

Note that the Moreau envelope is a continuously differentiable function, even when $g$ is not. The following lemma states this fact.

**Lemma B.1 (Theorem 6.60 in [2]).** Let $g : \mathcal{E} \rightarrow \mathbb{R}$ be a proper closed and convex function, $\mu \geq 0$. Then $M_{\mu g}$ is $\frac{1}{\mu}$-smooth over $\mathcal{E}$, and for any $v \in \mathcal{E}$ we have

$$\nabla M_{\mu g}(v) = \frac{1}{\mu} (v - \text{prox}_{\mu g}(v)).$$  \hspace{1cm} (B.3)

**Lemma (B.1)** states that the Moreau envelope is a continuously differentiable function over Euclidean space $\mathcal{E}$. The next results show the relationship between Retraction smoothness in a submanifold of Euclidean space and smoothness in Euclidean space.

**Definition B.1 (Retr-Smooth).** A function $f : \mathcal{M} \rightarrow \mathbb{R}$ is said to be retraction $L$-smooth if for any $x, y \in \mathcal{M}$,

$$f(y) \leq f(x) + \langle \text{grad} f(x), \xi \rangle_x + \frac{L}{2} \|\xi\|_x^2$$  \hspace{1cm} (B.4)

where $\xi \in T_x \mathcal{M}$ and $y = R_x(\xi)$.

Let $\mathcal{M}$ be a Riemannian submanifold of $\mathcal{E}$. The following lemma states that if $f : \mathbb{R}^n \rightarrow \mathbb{R}$ have Lipschitz continuous gradient, then $f$ is also retraction smooth.

**Lemma B.2 (Lemma 4 in [3]).** Let $\mathcal{E}$ be a Euclidean space (for example, $\mathcal{E} = \mathbb{R}^n$) and let $\mathcal{M}$ be a compact Riemannian submanifold of $\mathcal{E}$. Let $\text{Retr}$ be a retraction on $\mathcal{M}$. If $f : \mathcal{E} \rightarrow \mathbb{R}$ has Lipschitz continuous gradient in the convex hull of $\mathcal{M}$, then there exists a positive constant $L_g$ such that the pullbacks $f \circ \text{Retr}_x$ satisfies

$$f(\text{Retr}_x(\eta)) \leq f(x) + \langle \eta, \text{grad} f(x) \rangle + \frac{L_g}{2} \|\eta\|^2$$  \hspace{1cm} (B.5)

for all $\eta \in T_x \mathcal{M}$.