Existence of global solutions for a Keller-Segel-fluid equations with nonlinear diffusion

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Abstract: We consider a coupled system consisting of the Navier-Stokes equations and a porous medium type of Keller-Segel system that model the motion of swimming bacteria living in fluid and consuming oxygen. We establish the global-in-time existence of weak solutions for the Cauchy problem of the system in dimension three. In addition, if the Stokes system, instead Navier-Stokes system, is considered for the fluid equation, we prove that bounded weak solutions exist globally in time.

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1 Introduction

We study a mathematical model describing the dynamics of oxygen, swimming bacteria, and viscous incompressible fluids in $\mathbb{R}^3$. More precisely, we consider the Cauchy problem for the coupled Keller-Segel-Navier-Stokes system in $\mathbb{R}^3 \times [0,T)$ with $0 < T$

\begin{equation}
\begin{cases}
\partial_t n + u \cdot \nabla n = \Delta n^{1+\alpha} - \nabla \cdot (\chi(c)n\nabla c), \\
\partial_t c + u \cdot \nabla c = \Delta c - \kappa(c)n, \\
\partial_t u + \tau(u \cdot \nabla)u + \nabla p = \Delta u - n\nabla \phi, \\
\text{div } u = 0,
\end{cases}
\end{equation}

where $n, c, u$ and $p$ are the cell density, oxygen concentration, velocity field and pressure of the fluid. Here $\alpha > 0$ is a positive constant such that porous medium type equation of $n$ is under our consideration.

It is known that the above system models the motion of swimming bacteria, so called Bacillus subtilis, which live in fluid and consume oxygen. The functions $\chi : \mathbb{R} \to \mathbb{R}$ and $\kappa : \mathbb{R} \to \mathbb{R}$ represent the chemotactic sensitivity and consumption rate of oxygen. The constant $\tau$ in the third equation is 0 or 1. When $\tau = 1$, $u$ becomes the velocity vector of fluid solving Navier-Stokes equation. If the fluid motion is so slow, one may assume $\tau = 0$ so that $u$ is the velocity vector satisfying Stokes system.

The above system \cite{11} was proposed by Tuval et al. in \cite{13} for the case $\alpha = 0$, which can be extended to the case $\alpha > 0$ when the diffusion of bacteria is viewed like movement in a porous medium (see e.g. \cite{3}, \cite{5}, \cite{8}, \cite{11}, \cite{12}). Throughout this paper, we call the above system a Keller-Segel-Navier-Stokes equations (KSNS) if $\tau = 1$ and Keller-Segel-Stokes system (KSS) in case that $\tau = 0$. Compared to the Keller-Segel model (KS) of porous medium type:

\begin{equation}
\begin{cases}
\partial_t n = \Delta n^{1+\alpha} - \nabla \cdot (\chi n\nabla c), \\
\tau \partial_t c = \Delta c - c + n,
\end{cases}
\end{equation}

where \( \chi \) is a positive constant and \( \tau = 0 \) or \( 1 \), we emphasize that the chemical substance (oxygen) in (1.1) is consumed, rather than produced by the bacteria. The existence of global-in-time bounded weak solution for Cauchy problem to (KS) with arbitrary large initial data is guaranteed when \( \alpha > 1/3 \) for three dimensional case. Otherwise, the solution may blow up in finite time unless sufficiently small initial condition is assumed (see the results [10] and [7]).

The main concern of this paper is to specify the values of \( \alpha \) so that the global-in-time weak solutions for (KSNS) and bounded weak weak solutions for (KSS) are established, under the some conditions of \( \chi \) and \( \kappa \). The notions of weak and bounded weak solutions mentioned above are defined as follows:

**Definition 1 (Weak solutions)** Let \( \alpha > 0 \) and \( 0 < T < \infty \). A triple \((n, c, u)\) is said to be a weak solution of the system (1.1) if the followings are satisfied:

(i) \( n \) and \( c \) are non-negative functions and \( u \) is a vector function defined in \( \mathbb{R}^3 \times (0, T) \) such that

\[
n(1 + |x| + |\log n|) \in L^\infty(0, T; L^1(\mathbb{R}^3)), \quad \nabla n^{\frac{1+\alpha}{2}} \in L^2(0, T; L^2(\mathbb{R}^3)), \\
n \in L^\infty(0, T; L^p(\mathbb{R}^3)), \quad \nabla n^{\frac{4}{p}} \in L^2(0, T; L^2(\mathbb{R}^3)), \quad 1 \leq p \leq 1 + \alpha \\
c \in L^\infty(0, T; H^1(\mathbb{R}^3)) \cap L^2(0, T; H^2(\mathbb{R}^3)), \quad c \in L^\infty(\mathbb{R}^3 \times [0, T)), \\
u \in L^\infty(0, T; L^2(\mathbb{R}^3)), \quad \nabla u \in L^2(0, T; L^2(\mathbb{R}^3)),
\]

(ii) \((n, c, u)\) satisfies the equation (1.1) in the sense of distributions, namely,

\[
\int_0^T \int_{\mathbb{R}^3} \left(-n_\psi t + \nabla n^{1+\alpha} \cdot \nabla \varphi - nu \cdot \nabla \varphi - n\chi(c) \nabla n \cdot \nabla \varphi\right) \, dx \, dt = \int_{\mathbb{R}^3} n_0 \varphi(\cdot, 0) \, dx,
\]

\[
\int_0^T \int_{\mathbb{R}^3} \left(-c_\psi t + \nabla c \cdot \nabla \varphi - cu \cdot \nabla \varphi + n\kappa(c) \varphi\right) \, dx \, dt = \int_{\mathbb{R}^3} c_0 \varphi(\cdot, 0) \, dx,
\]

\[
\int_0^T \int_{\mathbb{R}^3} \left(-u_\psi t + \nabla u \cdot \nabla \psi + (\tau(u \cdot \nabla) u) \cdot \psi + n \nabla \psi \cdot \psi\right) \, dx \, dt = \int_{\mathbb{R}^3} u_0 \cdot \psi(\cdot, 0) \, dx
\]

for all test functions \( \varphi \in C_0^\infty(\mathbb{R}^3 \times [0, T)) \) and \( \psi \in C_0^\infty(\mathbb{R}^3 \times [0, T), \mathbb{R}^3) \) with \( \nabla \cdot \psi = 0 \).

**Definition 2 (Bounded weak solutions)** Let \( \alpha > 0 \) and \( 0 < T < \infty \). A triple \((n, c, u)\) is said to be a bounded weak solution of the system (1.1) if \((n, c, u)\) is a weak solution in Definition 1 and furthermore satisfies the following: For any \( p \in [1, \infty) \) and \( q \in [2, \infty) \)

(i) \( n \in L^\infty((0, T) \times \mathbb{R}^3), \quad \nabla n^{\frac{4}{p}} \in L^2(0, T; L^2(\mathbb{R}^3)). \)

(ii) \( c \in L^q(0, T; W^{2,q}(\mathbb{R}^3)), \quad c_t \in L^q(0, T; L^q(\mathbb{R}^3)). \)

(iii) \( u \in L^q(0, T; W^{2,q}(\mathbb{R}^3)), \quad u_t \in L^q(0, T; L^q(\mathbb{R}^3)). \)

Before we state our main results, we recall some known results in case that \( \alpha > 0 \) (compare to e.g. [1], [2], [4], [9] and [15] for the case that \( \alpha = 0 \) and references therein). It was proved in [5] that weak solutions of (KSS) for bounded domains exist in case that \( \alpha \in (1/2, 1] \) in two dimensions or in case that \( \alpha \in \left(\frac{-5+\sqrt{217}}{12}, 1\right] \) in three dimensions. It was also shown that if domain is the whole space, i.e. \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \), then weak solutions exists globally when \( \alpha = 1 \).
In [8], the exponent $\alpha$ is reduced up to $1/3$ for the case that spatial domain is $\mathbb{R}^3$, under the following assumptions on $\chi$ and $\kappa$:

$$\chi(c), \kappa(c), \chi'(c), \kappa'(c) \geq 0, \quad \kappa(0) = 0, \quad \frac{\chi(c) \kappa(c)'}{\chi(c)} \geq 0, \quad \frac{\kappa(c)''}{\chi(c)} < 0. \quad (1.2)$$

For the case of bounded domains in dimension two, bounded weak solutions of (KSS) are constructed in [11] for any $\alpha > 0$. In case that fluid equation is the Navier-Stokes equations, it was proved recently in [3, Theorem 1.8] that bounded weak solutions of (KS-NS) exist for any $\alpha > 0$ in $\mathbb{R}^2$.

On the other hand, in three dimensional case, [12] considered a special case of $\chi(c) = 1$ and $\kappa(c) = c$ and showed that global weak solutions of (KSS) exist whenever $\alpha > 1/7$ for bounded domains. For the case of $\mathbb{R}^3$, [3] proved that global bounded weak solutions of (KSS) exist when $\alpha > 1/4$ under the hypothesis either $\chi'(-) \geq \chi_0 > 0$ or $\kappa'(-) \geq \kappa_0 > 0$, where $\chi_0$ and $\kappa_0$ are positive constants. It was shown, very recently, in [16] that in case that $\alpha > 1/6$, bounded weak solutions are constructed in a bounded convex domain $\Omega \subset \mathbb{R}^3$ with smooth boundary with more relaxed conditions on $\chi$ and $\kappa$ in general setting (see [16, Theorem 1.1] for the details).

The main purpose of this paper is to establish existence of global weak and global bounded weak solutions for the Cauchy problem of the system (KS-NS) in $\mathbb{R}^3$ under rather relaxed conditions of $\chi$ and $\kappa$ compared to (1.2), and smaller value of $\alpha$ ever known.

Before stating our results, we address some conditions for $\kappa$. To preserve the non-negativity of the density of bacteria $n(x,t)$ and the oxygen $c(x,t)$ for $0 < t < T$, we need the condition $\kappa(0) = 0$ (see the proof of Lemma 1). The condition $\kappa'(-) \geq 0$ is also essential since the bacteria is consuming the oxygen. Namely, we assume the following hypothesis:

$$\kappa(-) \geq 0 \quad \text{and} \quad \kappa(0) = 0. \quad (1.3)$$

We also present two different types of further assumptions on chemotactic sensitivity $\chi$ and consumption rate $\kappa$ together with the range of $\alpha$. The first one is related to weak solutions.

**Assumption 1** We suppose that $\chi, \kappa$ satisfy $\chi' \in L^\infty_{\text{loc}}$ and $\kappa \in L^\infty_{\text{loc}}$ with (1.3). We assume further that one of the following holds:

(i) $\alpha > 1/6$.

(ii) $\alpha > 0$ and $\chi'(-) \geq \chi_0$ for some constant $\chi_0 > 0$.

(iii) $\alpha > 0$ and $\kappa'(-) \geq \kappa_0$ for some constant $\kappa_0 > 0$.

Another hypothesis is prepared for bounded weak solutions.

**Assumption 2** We suppose that $\chi, \kappa$ satisfy $\chi' \in L^\infty_{\text{loc}}$ and $\kappa \in L^\infty_{\text{loc}}$ with (1.3). We assume further that one of the following holds:

(i) $\alpha > 1/6$.

(ii) $\alpha > 1/8$ and $\chi'(-) \geq \chi_0$ for some constant $\chi_0 > 0$.

(iii) $\alpha > 1/8$ and $\kappa'(-) \geq \kappa_0$ for some constant $\kappa_0 > 0$. 

3
We are now in a position to state the main results. The first main result of this paper is the existence of global-in-time weak solution of the system (KS-NS) with $\tau = 1$, which means that the fluid equations are the Navier-Stokes equations. More precisely, the first result reads as follows:

**Theorem 1** Let $\tau = 1$ and the Assumption $[\mathfrak{I}]$ hold. Suppose that initial data $(n_0, c_0, u_0)$ satisfies

$$
n_0(1 + |x| + |\log n_0|) \in L^1(\mathbb{R}^3), \quad n_0 \in L^{1+\alpha}(\mathbb{R}^3), \quad c_0 \in L^\infty(\mathbb{R}^3) \cap H^1(\mathbb{R}^3), \quad u_0 \in L^2(\mathbb{R}^3). \quad (1.4)
$$

Then, for each $T > 0$, there exists a weak solution $(n, c, u)$ for the system (1.1) such that it satisfies

$$
sup_{0 \leq t \leq T} \left( \int_{\mathbb{R}^3} n(t) (|\log n(t)| + 2|x|) \, dx + \|n(t)\|_{L^{1+\alpha}}^{\frac{1}{2}} + \|\nabla c(t)\|_{L^2}^2 + \|u(t)\|_{L^2}^2 \right)
+ \int_0^T \left( \left\| \nabla n^{\frac{1+\alpha}{2}}(t) \right\|_{L^2}^2 + \left\| \nabla n^{\frac{1+2\alpha}{2}}(t) \right\|_{L^2}^2 + \|\Delta c(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 \right) \, dt < C, \quad (1.5)
$$

where $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$ and $C = C(T, \|n_0\|_{L^{1+\alpha}} \cap L^\infty, \|n_0 \log n_0\|_{L^1}, \|n_0(x)\|_{L^1}, \|\nabla c_0\|_{L^2}, \|u_0\|_{L^2}$).

If the fluid equations are restricted to be the Stokes equations, i.e. $\tau = 0$, and if the range of $\alpha$ in the hypothesis in Theorem [II] is a bit more restrictive, we can construct bounded weak solutions for the Cauchy problem. More precisely, our second result reads as follows:

**Theorem 2** Let $\tau = 0$ and the Assumption $[\mathfrak{I}]$ hold. Suppose that initial data $(n_0, c_0, u_0)$ satisfy (1.4) and for any $q \in [2, \infty)$

$$
n_0 \in L^\infty(\mathbb{R}^3), \quad c_0 \in W^{1,q}(\mathbb{R}^3), \quad u_0 \in W^{1,q}(\mathbb{R}^3). \quad (1.6)
$$

Then, for each $T > 0$, there exists a bounded weak solution $(n, c, u)$ for the system (1.1) such that it satisfies

$$
\|n\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))} + \left\| \nabla n^{\frac{1+\alpha}{2}} \right\|_{L^2((0,T) \times \mathbb{R}^3)} + \|c\|_{L^q(0,T;W^{2,q}(\mathbb{R}^3))} + \|\partial_t c\|_{L^q(0,T;L^q(\mathbb{R}^3))} + \|u\|_{L^q(0,T;W^{2,q}(\mathbb{R}^3))} + \|\partial_t u\|_{L^q(0,T;L^q(\mathbb{R}^3))} < C, \quad (1.7)
$$

where $C = C(T, \|n_0\|_{L^\infty}, \|c_0\|_{W^{1,q}}, \|u_0\|_{W^{1,q}})$.

**Remark 1** The results in Theorem [II] can be rephrased as follows: In case that $\alpha > 1/6$, the bounded weak solutions can be constructed with the assumptions that $\chi, \kappa$ satisfy $\chi', \kappa \in L^\infty_{loc}$ and $\kappa \in L^\infty_{loc}$ with $\kappa(\cdot) \geq 0$ and $\kappa(0) = 0$ (compare to [I] for bounded domain case). If we assume further that either $\chi'(\cdot) \geq \chi_0 > 0$ or $\kappa'(\cdot) \geq \kappa_0 > 0$, then bounded weak solutions exist for $\alpha > 1/8$, weaker than $\alpha > 1/6$.

Our main concern is construction of weak and bounded weak solutions for the Cauchy problems of the system (1.1) in $\mathbb{R}^3$ but it can be extended without so much difficulty to bounded domains with Neumann boundary conditions for $n$ and $c$ and no-slip boundary conditions for $u$. To be more precise, let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary and we consider the system (1.1) in $\Omega \times [0, T]$ with boundary conditions

$$
\frac{\partial n}{\partial \nu} = \frac{\partial c}{\partial \nu} = 0, \quad u = 0 \quad \text{on} \quad \partial \Omega. \quad (1.8)
$$

We then obtain following results for the case of bounded domains by following almost same arguments as in Theorem [I] and Theorem [II].
Then, for each $T > 0$ such that it satisfies (1.8) any domain wider than $\Omega$ with smooth boundary. Suppose that initial data $(n_0, c_0, u_0)$ satisfies

$$n_0 \in L^1(\Omega) \cap L^{1+\alpha}(\Omega), \; c_0 \in L^\infty(\Omega) \cap H^1(\Omega), \; u_0 \in L^2(\Omega). \tag{1.9}$$

Then, for each $T > 0$, there exists a weak solution $(n, c, u)$ for the system (1.1) and (1.8), and it satisfies

$$\sup_{0 \leq t \leq T} \left( \int_{\mathbb{R}^3} n(t) |\log n(t)| \, dx + \|n(t)\|_{L^{1+\alpha}} + \|\nabla c(t)\|_{L^2}^2 + \|u(t)\|_{L^2}^2 \right)
$$

$$+ \int_0^T \left( \|\nabla n^{\frac{1+\alpha}{2}}\|_{L^2}^2 + \|\nabla n^{\frac{1+2\alpha}{2}}\|_{L^2}^2 + \|\Delta c(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 \right) \, dt < C,$$

where $C = C(T, \|n_0\|_{L^1 \cap L^{1+\alpha}}, \|n_0 \log n_0\|_{L^1}, \|\nabla c_0\|_{L^2}, \|u_0\|_{L^2}).$

**Theorem 4** Let $\tau = 0$ and the Assumption 2 hold with replacement of $\mathbb{R}^3$ by a bounded domain $\Omega$ with smooth boundary. Suppose that initial data $(n_0, c_0, u_0)$ satisfy (1.1) and (1.8) such that it satisfies

$$\|n\|_{L^\infty((0,T) \times \Omega)} + \|\nabla n^{\frac{p\alpha}{2}}\|_{L^2((0,T) \times \Omega)} + \|c\|_{L^q(0,T;W^{2,q}(\Omega))}
$$

$$+ \|\partial_t c\|_{L^q(0,T;L^q(\Omega))} + \|u\|_{L^q(0,T;W^{2,q}(\Omega))} + \|\partial_t u\|_{L^q(0,T;L^q(\Omega))} < C,$$

where $C = C(T, \|n_0\|_{L^\infty}, \|c_0\|_{W^{1,q}}, \|u_0\|_{W^{1,q}}).$

Since verifications of Theorem 3 and Theorem 4 are similar to those of Theorem 1 and Theorem 2, only differences are indicated in the proofs in section 4.

**Remark 2** Theorem 4 and Theorem 2 are improvements over the results in [3, Theorem 1.5 and Theorem 1.7]. We also remark that Theorem 4 covers the result of [12], since the case $\chi = 1$ and $\kappa(c) = c$ is the special case of (iii) in the Assumption 2 and besides, $\alpha > 1/8$ is wider than $\alpha > 1/7$ in [12].

This paper is organized as follows. In section 2 and section 3 a priori estimates of weak and bounded weak solutions are established. In section 4 we present the proofs of main results.

## 2 Weak solutions

Throughout this section, we study the solutions of the approximate problem of (1.1) given by

$$\begin{aligned}
\partial_t n_\varrho + u_\varrho \cdot \nabla n_\varrho &= \Delta(n_\varrho + \varrho)^{1+\alpha} - \nabla \cdot \left( \chi(c_\varrho) n_\varrho \nabla c_\varrho \right), \\
\partial_t c_\varrho + u_\varrho \cdot \nabla c_\varrho &= \Delta c_\varrho - \kappa(c_\varrho) n_\varrho, \\
\partial_t u_\varrho + \tau(u_\varrho \cdot \nabla) u_\varrho + \nabla p_\varrho &= \Delta u_\varrho - n_\varrho \nabla \varphi, \quad \text{div} u_\varrho = 0, 
\end{aligned} \tag{2.1}$$
in \( \mathbb{R}^3 \times (0, T) \) with smooth initial data \((n_0, c_0, u_0)\) given by
\[
\begin{align*}
n_0 &= \psi_\varepsilon * n_0, \\
c_0 &= \psi_\varepsilon * c_0, \\
u_0 &= \psi_\varepsilon * u_0
\end{align*}
\]
where \( \psi_\varepsilon \) denotes the usual mollifier with \( \varepsilon \in (0, 1) \).

It is known that, due to the standard theory of existence and regularity as done in \([5]\) and \([12]\), there exists a classical solution of the equation (2.1) locally in time for each \( \varrho \in (0, 1) \). The main objective of this section is to derive appropriate uniform estimates, independent of \( \varrho \), of the solutions. The estimates are crucially used in Section 3 to extend the above local solution to any given time interval \((0, T)\) and to construct the weak solutions and the bounded weak solutions of the equation (1.1).

We start with some notations. For \( 1 \leq q \leq \infty \), we denote by \( W^{k,q}(\mathbb{R}^3) \) the usual Sobolev spaces, namely \( W^{k,q}(\mathbb{R}^3) = \{ f \in L^q(\mathbb{R}^3) : D^\alpha f \in L^q(\mathbb{R}^3), 0 \leq |\alpha| \leq k \} \). The set of \( q \)-th power integrable functions on \( \mathbb{R}^3 \) is denoted by \( L^q(\mathbb{R}^3) \). In what follows, for simplicity, \( \| \cdot \|_p \) denotes \( \| \cdot \|_{L^p(\mathbb{R}^3)} \) for \( 1 \leq p \leq \infty \), unless there is any confusion to be expected. We also denote by \( W^{-k,q}(\mathbb{R}^3) \) dual space of \( W^{k,q}(\mathbb{R}^3) \), where \( q \) and \( q' \) are Hölder conjugates. The letters \( C = C(\ast, \ast, \ast, \ast) \) and \( C' = C'(\ast, \ast, \ast, \ast) \) are used to represent generic constants, depending on \( \ast, \ast, \ast, \ast \), which may change from line to line.

We first recall that maximal regularity estimate of the inhomogeneous heat equation, which we use later. Let \( 0 < T < \infty \) and \( 1 < p < \infty \), and we consider
\[
v_t - \Delta v = f \quad \text{in } \mathbb{R}^3 \times (0, T)
\]
with initial datum \( v(x, 0) = v_0(x) \) with \( v_0 \in W^{1,p}(\mathbb{R}^3) \). Then following \( L^p \) estimate is well-known:
\[
\|v_t\|_{L^p((0,T) \times \mathbb{R}^3)} + \|v\|_{L^p((0,T) \times \mathbb{R}^3)} \leq C \left( \|f\|_{L^p((0,T) \times \mathbb{R}^3)} + \|v_0\|_{W^{1,p}(\mathbb{R}^3)} \right). \tag{2.2}
\]

In the following lemma, we give an estimate of solutions of (2.1) under the Assumption [H]. The estimate is used in Section 3 to construct the weak solution of the equation (1.1). For the sake of simplicity, throughout Section 2, we denote \( n_\varrho, c_\varrho \) and \( u_\varrho \) by \( n, c \) and \( u \). Also, we define the functionals \( E_M(t) \) and \( D(t) \) as follows:
\[
E_M(t) := \int_{\mathbb{R}^3} n(t) \left( \|\log n(t)\| + 2\langle x \rangle \right) dx + \|n(t)\|_{L^{1+\alpha}}^{1+\alpha} + \|\nabla c(t)\|_2^2 + \frac{M + 2}{2} \|u(t)\|_2^2 \tag{2.3}
\]
and
\[
D(t) := \left\| \nabla n^{\frac{1+\alpha}{2}}(t) \right\|_2^2 + \left\| \nabla n^{\frac{1+2\alpha}{2}}(t) \right\|_2^2 + \|\Delta c(t)\|_2^2 + \|\nabla u(t)\|_2^2, \tag{2.4}
\]
where \( \langle x \rangle = (1 + |x|^2)^{\frac{1}{2}} \) and \( M \) is a positive constant, which will be specified later.

**Lemma 1** Let \( T > 0 \). Suppose that \((n, c, u)\) is a classical solution for the system (2.1) with the smooth initial datum \((n_0, c_0, u_0)\) satisfies the initial condition (1.1) independently of \( \varrho \). Assume further that \( \chi, \kappa \) and \( \alpha \) satisfy Assumption [H]. Then, there exists \( C > 0 \) and \( M > 0 \), which are independent of \( \varrho \), such that for any \( 0 < t \leq T \), \( E_M(t) \) and \( D(t) \) defined in (2.3)-(2.4) satisfy
\[
\sup_{0 \leq \tau \leq t} E_M(\tau) + \int_0^t D(\tau) d\tau < C, \tag{2.5}
\]
where \( C = C(T, \|n_0\|_{L^1}, \|n_0 \log n_0\|_{L^1}, \|n_0(x)\|_{L^1}, \|n_0\|_{L^{1+\alpha}}, \|\nabla c_0\|_{L^2}, \|u_0\|_{L^2}) \).
Proof. We observe, by integrating both sides of (2.1), that the total mass of \( n \) is preserved, i.e. \( \|n(t)\|_1 \equiv \|n_0\|_1 \). It is also obvious by applying maximal principle to (2.1) that \( \|c\|_{L^\infty(\mathbb{R}_T^3)} \leq \|c_0\|_\infty \), where \( \mathbb{R}_T^3 := \mathbb{R}^3 \times [0,T) \). We also note here that \( n \) and \( c \) preserves non-negativity of the initial data by the condition \( \kappa(0) = 0 \) and the parabolic comparison principle.

*Case (i) in the Assumption* \( \Box \) Here we consider the case that \( \alpha > 1/6 \). We separate the range of \( \alpha \) into three parts, that is, \( 1/6 < \alpha \leq 1/3 \), \( 1/3 < \alpha \leq 1 \) and \( \alpha > 1 \), and treat each case individually. We start with the case that \( 1/6 < \alpha \leq 1/3 \).

(Case \( 1/6 < \alpha \leq 1/3 \)): Multiplying (2.1) with \((1 + \log n)\) and integrating it parts,

\[
\frac{d}{dt} \int_{\mathbb{R}^3} n \log n \, dx + \int_{\mathbb{R}^3} \nabla \log n \cdot \nabla (n + \varrho)^{1+\alpha} \, dx = \int_{\mathbb{R}^3} \nabla n \cdot (\chi(c) \nabla c) \, dx.
\]

Since \( \nabla \log n \cdot \nabla n = 4 |\nabla n^{1/2}|^2 \), we have

\[
\int_{\mathbb{R}^3} \nabla \log n \cdot \nabla (n + \varrho)^{1+\alpha} \, dx = \int_{\mathbb{R}^3} \nabla \log n \cdot (1 + \alpha)(n + \varrho)^\alpha \nabla n \, dx
\geq \int_{\mathbb{R}^3} \nabla \log n \cdot (1 + \alpha)n^\alpha \nabla n \, dx = \frac{4}{1 + \alpha} \|\nabla n^{1+\alpha}\|^2_2.
\]

Taking into account \( 1/6 < \alpha \leq 1/3 \), which admits \((1 - \alpha)/2 > 0\), we obtain

\[
\frac{d}{dt} \int_{\mathbb{R}^3} n \log n \, dx + \frac{4}{1 + \alpha} \|\nabla n^{1+\alpha}\|^2_2 \leq \int_{\mathbb{R}^3} \nabla n \cdot (\chi(c) \nabla c) \, dx
\leq \frac{2\chi}{1 + \alpha} \int_{\mathbb{R}^3} |\nabla n^{1+\alpha}| \left| \frac{1}{2} \nabla n^\alpha \right| \, dx,
\]

where \( \chi := \sup_{\mathbb{R}^3} |\chi(c(\cdot))| \). Applying Young’s inequality, we observe

\[
\int_{\mathbb{R}^3} |\nabla n^{1+\alpha}| \left| \frac{1}{2} \nabla n^\alpha \right| \, dx \leq \epsilon \int_{\mathbb{R}^3} \|\nabla n^{1+\alpha}\|^2_2 \, dx + C(\epsilon) \int_{\mathbb{R}^3} n^{1-\alpha} |\nabla c|^2 \, dx.
\]

Combining above estimate and choosing sufficiently small \( \epsilon > 0 \), we have

\[
\frac{d}{dt} \int_{\mathbb{R}^3} n \log n \, dx + C' \|\nabla n^{1+\alpha}\|^2_2 \leq C \int_{\mathbb{R}^3} n^{1-\alpha} |\nabla c|^2 \, dx. \tag{2.6}
\]

Reminding that \( 1 - 3\alpha \geq 0 \) (since \( \alpha \leq 1/3 \)) and that \( c \in L^\infty(\mathbb{R}_T^3) \), the term \( \int_{\mathbb{R}^3} n^{1-\alpha} |\nabla c|^2 \, dx \) is estimated as

\[
\int_{\mathbb{R}^3} n^{1-\alpha} |\nabla c|^2 \, dx = \int_{\mathbb{R}^3} n^{1-\alpha} \nabla c \cdot \nabla c \, dx
\leq C \left( \int_{\mathbb{R}^3} |\nabla n^{1-\alpha}| |\nabla c| \, dx + \int_{\mathbb{R}^3} n^{1-\alpha} |\Delta c| \, dx \right)
\leq \int_{\mathbb{R}^3} (\epsilon |\nabla n^{1+\alpha}|^2 + C(\epsilon)n^{1-3\alpha} |\nabla c|^2) \, dx + C \int_{\mathbb{R}^3} n^{1-\alpha} |\Delta c| \, dx.
\]

Choosing small \( \epsilon > 0 \) in the above, we estimate \(2.6\) as follows:

\[
\frac{d}{dt} \int_{\mathbb{R}^3} n \log n \, dx + C'_1 \left| \frac{1}{2} \nabla n^{1+\alpha} \right| \leq C_1 \int_{\mathbb{R}^3} n^{1-3\alpha} |\nabla c|^2 + |n^{1-\alpha}| |\Delta c| \, dx. \tag{2.7}
\]
for some $C_1 > 0$ and $C'_1 > 0$. Next, Multiplying (2.1) with $n^\alpha$ and integrating it by part gives

$$
\frac{1}{1 + \alpha} \frac{d}{dt} \|n\|_{1+\alpha}^{1+\alpha} + \frac{4\alpha(1 + \alpha)}{(1 + 2\alpha)^2} \left\| \nabla n \right\|_2^{1+2\alpha} = \int_{\mathbb{R}^3} \nabla n^\alpha \cdot n \chi(c) \nabla c \, dx \\
\leq \frac{2\alpha \chi}{1 + \alpha} \int_{\mathbb{R}^3} \left| \nabla n \right|^{1+2\alpha} \left( n^\frac{1}{2} |\nabla c| \right) \, dx.
$$

Applying Young’s inequality to the integrand in the right-hand side of the above, we have

$$
\frac{d}{dt} \|n\|_{1+\alpha}^{1+\alpha} + C' \left\| \nabla n \right\|_2^{1+2\alpha} \leq C \int_{\mathbb{R}^3} n |\nabla c|^2 \, dx
$$

for some $C' > 0$. Since the integral term of the right-hand side in (2.8) is estimated as

$$
\int_{\mathbb{R}^3} n |\nabla c|^2 \, dx = \int_{\mathbb{R}^3} n \nabla c \cdot \nabla c \, dx \\
\leq C \left( \int_{\mathbb{R}^3} |\nabla n| |\nabla c| \, dx + \int_{\mathbb{R}^3} n |\Delta c| \, dx \right) \\
\leq C \left( \int_{\mathbb{R}^3} n^{1-2\alpha} |\nabla c|^2 + n |\Delta c| \, dx \right),
$$

it follows from Young’s inequality that

$$
\frac{d}{dt} \|n\|_{1+\alpha}^{1+\alpha} + C'_2 \left\| \nabla n \right\|_2^{1+2\alpha} \leq C_2 \left( \int_{\mathbb{R}^3} n^{1-2\alpha} |\nabla c|^2 + n |\Delta c| \, dx \right) (2.9)
$$

for some $C_2 > 0$ and $C'_2 > 0$. On the other hand, multiplying (2.1) with $-\Delta c$ and using integration by part, we get

$$
\frac{1}{2} \frac{d}{dt} \|\nabla c\|_2^2 + \|\Delta c\|_2^2 \leq \int_{\mathbb{R}^3} (u \cdot \nabla c) \Delta c \, dx + \mathcal{R} \int_{\mathbb{R}^3} n |\Delta c| \, dx
$$

where $\mathcal{R} := \max_{\mathbb{R}^3} |\kappa(c(\cdot))|$. Since the term $\int_{\mathbb{R}^3} (u \cdot \nabla c) \Delta c \, dx$ in the above is estimated as

$$
\int_{\mathbb{R}^3} (u \cdot \nabla c) \Delta c \, dx = \sum_{1 \leq i, j \leq 3} \int_{\mathbb{R}^3} u_i c_{x_j} c_{x_j, x_i} \, dx = - \sum_{1 \leq i, j \leq 3} \int_{\mathbb{R}^3} u_{i, x_j} c_{x_j} \, dx \\
= \sum_{1 \leq i, j \leq 3} \int_{\mathbb{R}^3} u_{i, x_j} c c_{x_j, x_i} \, dx \leq C \|\nabla u\|_2 \|\nabla^2 c\|_2
$$

where $u_i$ and $c_{x_i}$ denote the $i$-th component of $u$ and $\frac{\partial c}{\partial x_i}$ respectively, we have

$$
\frac{d}{dt} \|\nabla c\|_2^2 + \|\Delta c\|_2^2 \leq C_3 \left( \|\nabla u\|_2 \|\Delta c\|_2 + \int_{\mathbb{R}^3} n |\Delta c| \, dx \right) (2.10)
$$

for some $C_3 > 0$. For the fluid equation, energy estimate gives

$$
\frac{d}{dt} \|u\|_2^2 + 2 \|\nabla u\|_2^2 \leq C_4 \int_{\mathbb{R}^3} n \|u\| \, dx
$$

for some $C_4 > 0$. Finally, to bound $\int n |\log n|$ in (2.7), we recall that (see e.g. [5] and [2])

$$
\int_{\mathbb{R}^3} n |\log n| \, dx \leq \int_{\mathbb{R}^3} n \log n \, dx + 2 \int_{\mathbb{R}^3} \langle x \rangle n \, dx + C, \quad (2.12)
$$
where \( \langle x \rangle = (1 + |x|^2)^{1/2} \) and (see e.g. (18) of [3])

\[
\frac{d}{dt} \int_{\mathbb{R}^3} \langle x \rangle n dx = \int_{\mathbb{R}^3} nu \cdot \nabla \langle x \rangle + \int_{\mathbb{R}^3} (n + 2n \Delta \langle x \rangle) + \int_{\mathbb{R}^3} \nabla \langle x \rangle \cdot n \chi(c) \nabla c
\leq C \left(1 + \|u\|_2^2 + \|\nabla c\|^2_2\right) + \left(C(\epsilon) + \epsilon \|\nabla n^{\frac{1+\alpha}{2}}\|_2^2\right). \quad (2.13)
\]

Summing up (2.7) \(\sim\) (2.13) with sufficiently small \(\epsilon > 0\), we have

\[
\frac{d}{dt} \left(\int_{\mathbb{R}^3} n(\log n + 2\langle x \rangle) dx + \|n\|_{1+\alpha}^2 + \|\nabla c\|^2_2 + \|u\|_2^2\right)
+ C_5' \left(\|\nabla n^{\frac{1+\alpha}{2}}\|_2^2 + \|\nabla n^{\frac{1-3\alpha}{2}}\|_2^2 + \|\Delta c\|^2_2 + \|\nabla u\|^3_2\right)
\leq C_5 \left(\int_{\mathbb{R}^3} n^{1-3\alpha} |\nabla c|^2 dx + \int_{\mathbb{R}^3} n^{1-2\alpha} |\nabla c|^2 dx + \int_{\mathbb{R}^3} n^{1-\alpha} |\Delta c|^2 dx
+ \int_{\mathbb{R}^3} n |\Delta c|^2 dx + \|\nabla u\|^2_2 \|\Delta c\|^2_2 + \int_{\mathbb{R}^3} n |u|^2 dx\right).
:= C_5 (I + II + III + IV + V + VI).
\]

for some \(C_5' > 0\) and \(C_5 > 0\). Due to \(0 \leq 1 - 3\alpha < 2/3\) and \(0 < 1 - 2\alpha < 2/3\), we note that

\[
I \leq \left\{\begin{array}{ll}
\int_{\mathbb{R}^3} \left(C(\epsilon_1) + \epsilon_1 n^{\frac{2}{3}}\right) |\nabla c|^2 dx, & \text{if } 1/6 < \alpha < 1/3; \\
\|\nabla c\|^2_2, & \text{if } \alpha = 1/3,
\end{array}\right.
\]

and

\[
II \leq \int_{\mathbb{R}^3} \left(C(\epsilon_2) + \epsilon_2 n^{\frac{2}{3}}\right) |\nabla c|^2 dx,
\]

where the positive constants \(\epsilon_1\) and \(\epsilon_2\) will be determined later. Hence, it follows from Hölder inequality and Sobolev embedding that

\[
I \leq \left\{\begin{array}{ll}
C(\epsilon_1) \|\nabla c\|^2_2 + \epsilon_1 \|n\|_1^2 \|\Delta c\|^2_2, & \text{if } 1/6 < \alpha < 1/3; \\
\|\nabla c\|^2_2, & \text{if } \alpha = 1/3,
\end{array}\right.
\]

and

\[
II \leq C(\epsilon_2) \|\nabla c\|^2_2 + \epsilon_2 \|n\|_1^2 \|\Delta c\|^2_2. \quad (2.15)
\]

To estimate III, we apply Hölder and Young inequality to have \(III \leq C(\epsilon) \|n\|_{2-\alpha}^{2-\alpha} + \epsilon \|\Delta c\|^2_2\).

Noting that \(\frac{4}{3} \leq \frac{6-6\alpha}{2+3\alpha} < 2\) (which is obtained from the condition \(1/6 < \alpha \leq 1/3\)), we obtain, from Young's inequality and Gagliardo-Nirenberg inequality, the following:

\[
\|n\|_{2-\alpha}^{2-\alpha} \leq C \|n_0\|_{1+\alpha_0}^{\frac{2+3\alpha}{2+3\alpha}} \left\|\nabla n^{\frac{1+\alpha}{2}}\right\|_2^{\frac{6-6\alpha}{2+3\alpha}} \leq C(\epsilon') \|n_0\|_{1+\alpha_0}^{\frac{1+4\alpha}{2+3\alpha}} + \epsilon' \left\|\nabla n^{\frac{1+\alpha}{2}}\right\|_2^2.
\]

Hence by choosing sufficient \(\epsilon' > 0\), we have

\[
III \leq C(\epsilon_3) + \epsilon_3 \left\|\nabla n^{\frac{1+4\alpha}{2}}\right\|_2^2 + \epsilon_3 \|\Delta c\|^2_2. \quad (2.17)
\]
where \( \epsilon_3 > 0 \) will be specified later. We estimate \( IV \) via a calculation similar to the above as follows. Hölder and Young inequality give \( IV \leq C(\epsilon) \|n\|_2^2 + \epsilon \|\Delta c\|_2^2 \). Since \( \frac{3}{2} \leq \frac{6}{2 + 6\alpha} < 2 \) via \( 1/6 < \alpha \leq 1/3 \), it follows from Young and Gagliardo-Nierenberg inequality that

\[
\|n\|_2^2 \leq C \|n_0\|_1^{\frac{14+6\alpha}{2+6\alpha}} \left\| \nabla n^{\frac{1+2\alpha}{2}} \right\|_2^{\frac{6}{2+6\alpha}} \leq C(\epsilon') \|n_0\|_1^{\frac{14+6\alpha}{2+6\alpha}} + \epsilon' \|\nabla n^{\frac{1+2\alpha}{2}}\|_2^2.
\]

Therefore, we have

\[
IV \leq C(\epsilon_4) + \epsilon_4 \left\| \nabla n^{\frac{1+2\alpha}{2}} \right\|_2^2 + \epsilon_4 \|\Delta c\|_2^2. \tag{2.18}
\]

It is straightforward that \( V \) can be estimated as

\[
V \leq C(\epsilon_5) \|\nabla u\|_2^2 + \epsilon_5 \|\Delta c\|_2^2. \tag{2.19}
\]

The positive constants \( \epsilon_4 \) in \( (2.18) \) and \( \epsilon_5 \) in \( (2.19) \) will also be specified later. Finally, we estimate \( VI \) as follows. Via Hölder and Young inequality and Sobolev embedding, we obtain

\[
VI \leq C \|n\|_6^2 \|u\|_6^2 \leq C(\epsilon) \|n\|_6^2 + \epsilon \|\nabla u\|_2^2.
\]

Since \( 1 < \frac{6}{5} < 3 + 6\alpha \) and \( 0 < \frac{2}{2 + 6\alpha} < 2 \), \( \|n\|_6^2 \) is estimated, via Gagliardo-Nierenberg inequality and Young’s inequality, as

\[
\|n\|_6^2 \leq \|n_0\|_1^{\frac{5}{2+6\alpha}} \left\| \nabla n^{\frac{1+2\alpha}{2}} \right\|_2^{\frac{2}{2+6\alpha}} \leq C(\epsilon') + \epsilon' \left\| \nabla n^{\frac{1+2\alpha}{2}} \right\|_2^2.
\]

Thus, we conclude that

\[
VI \leq C(\epsilon_6) + \epsilon_6 \left\| \nabla n^{\frac{1+2\alpha}{2}} \right\|_2^2 + \epsilon_6 \|\nabla u\|_2^2. \tag{2.20}
\]

Summing up \( (2.15) \sim (2.20) \) with choosing sufficiently small \( \epsilon_1, ..., \epsilon_6 > 0 \), we finally have

\[
\frac{d}{dt} \left( \int_{\mathbb{R}^3} n(\log n + 2(x)) \ dx + \|n\|_{1+\alpha}^{1+\alpha} + \|\nabla c\|_2^2 + \|u\|_2^2 \right) + C_6' \left( \left\| \nabla n^{\frac{1+\alpha}{2}} \right\|_2^2 + \left\| \nabla n^{\frac{1+2\alpha}{2}} \right\|_2^2 + \|\Delta c\|_2^2 + \|\nabla u\|_2^2 \right) \leq C_6 \left( \|\nabla c\|_2^2 + \|\nabla u\|_2^2 + 1 \right)
\]

for some \( C_6' > 0 \) and \( C_6 > 0 \). To absorb the term \( C_6 \|\nabla u\|_2^2 \) in the right-hand side of the above, we test \( u \) to the \( (2.1)_3 \) and then multiply sufficiently large constant \( M > C_6 \) to have

\[
\frac{M}{2} \frac{d}{dt} \|u\|_2^2 + M \|\nabla u\|_2^2 \leq CM \int_{\mathbb{R}^3} n \ |u| \ dx. \tag{2.22}
\]

As \( (2.20) \), the integral in the right-hand side of the above is estimated as

\[
\int_{\mathbb{R}^3} n \ |u| \ dx \leq C(\epsilon) + \epsilon \left\| \nabla n^{\frac{1+2\alpha}{2}} \right\|_2^2 + \epsilon \|\nabla u\|_2^2.
\]
Hence substituting it to (2.22), choosing sufficiently small $\epsilon > 0$ and then adding it to (2.21), we have

$$
\frac{d}{dt} \left( \int_{\mathbb{R}^3} n \log n + 2 \langle x \rangle \, dx + \|n\|_{1+\alpha}^{1+\alpha} + \|\nabla c\|^2_2 + \frac{M + 2}{2} \|u\|^2_2 \right) 
+ C' \left( \|\nabla n\|^{1+\alpha}_{1+\alpha} + \|\nabla n\|^2_2 + \|\Delta c\|^2_2 + \|\nabla u\|^2_2 \right)
\leq C \left( \|\nabla c\|^2_2 + 1 \right).
$$

Integrating in time variables and combining estimate (2.12), we deduce (2.5).

(Case $1/3 < \alpha \leq 1$): Performing similar calculations as in the above case, we obtain

$$
\frac{d}{dt} \int_{\mathbb{R}^3} n \log n \, dx + C'_1 \left( \|\nabla n\|^{1+\alpha}_{1+\alpha} + \|\nabla c\|^2_2 \right) \leq C_1 \int_{\mathbb{R}^3} n^{1-\alpha} |\nabla c|^2 \, dx,
$$

$$
\frac{d}{dt} \int_{\mathbb{R}^3} n^{1+\alpha} + C'_2 \left( \|\nabla n\|^{1+2\alpha}_{1+\alpha} \right)^2 \leq C \int_{\mathbb{R}^3} n |\nabla c|^2 \, dx 
\leq C_{2,\epsilon} \left( \int_{\mathbb{R}^3} n^{1-\alpha} |\nabla c|^2 + n |\Delta c| \, dx \right) + \epsilon \left( \|\nabla n\|^{1+\alpha}_{1+\alpha} \right)^2,
$$

$$
\frac{d}{dt} \|\nabla c\|^2_2 + \|\Delta c\|^2_2 \leq C_3 \left( \|\nabla u\|_2 \|\Delta c\|_2 + \int_{\mathbb{R}^3} n |\Delta c| \, dx \right),
$$

$$
\frac{d}{dt} \|u\|^2_2 + \|\nabla u\|^2_2 \leq C_4 \int_{\mathbb{R}^3} n |u| \, dx
$$

and

$$
\frac{d}{dt} \int_{\mathbb{R}^3} \langle x \rangle n \, dx = \int_{\mathbb{R}^3} nu \cdot \nabla \langle x \rangle + \int_{\mathbb{R}^3} (n + g)^{1+\alpha} \Delta \langle x \rangle + \int_{\mathbb{R}^3} \nabla \langle x \rangle \cdot n \chi(c) \nabla c
\leq C_5 \left( 1 + \|u\|^2_2 + \|\nabla c\|^2_2 \right) + \left( C(\epsilon) + \epsilon \|\nabla n^{1+\alpha}_{1+\alpha} \right)^2.
$$

Summing up (2.24)~(2.28) with sufficiently small $\epsilon > 0$, we have

$$
\frac{d}{dt} \left( \int_{\mathbb{R}^3} n \log n + 2 \langle x \rangle \, dx + \|n\|_{1+\alpha}^{1+\alpha} + \|\nabla c\|^2_2 + \|u\|^2_2 \right)
+ C'_6 \left( \|\nabla n\|^{1+\alpha}_{1+\alpha} + \|\nabla n\|^2_2 + \|\Delta c\|^2_2 + \|\nabla u\|^2_2 \right)
\leq C_6 \left( \int_{\mathbb{R}^3} n^{1-\alpha} |\nabla c|^2 \, dx + \int_{\mathbb{R}^3} n |\Delta c| \, dx + \|\nabla u\|_2 \|\Delta c\|_2 + \int_{\mathbb{R}^3} n |u| \, dx \right)
=: C_6 (I + II + III + IV).
$$

for some $C'_6 > 0$ and $C_6 > 0$. Since $1/3 < \alpha \leq 1$, it is direct that $0 \leq 1 - \alpha < 2/3$ and $0 < \frac{\alpha}{2+6\alpha} < 2$, and thus, we obtain the same estimations as those in the previous case for I~IV. To be more precise, we have

$$
I \leq \begin{cases} 
C(\epsilon_1) \|\nabla c\|^2_2 + \epsilon_1 \|n_0\|^2_1 \|\Delta c\|^2_2, & \text{if } 1/3 < \alpha < 1; \\
\|\nabla c\|^2_2, & \text{if } \alpha = 1,
\end{cases}
$$

(3.20)
II \leq C(\epsilon_2) + \epsilon_2 \left\| \nabla n^{\frac{1+2\alpha}{2}} \right\|_2^2 + \epsilon_2 \left\| \Delta c \right\|_2^2,
(2.31)

III \leq C(\epsilon_3) \left\| \nabla u \right\|_2^2 + \epsilon_3 \left\| \Delta c \right\|_2^2
(2.32)

and

IV \leq C(\epsilon_4) + \epsilon_4 \left\| \nabla n^{\frac{1+2\alpha}{2}} \right\|_2^2 + \epsilon_4 \left\| \nabla u \right\|_2^2.
(2.33)

Following the same procedure as that in the previous case, we conclude that

\[
\frac{d}{dt} \left( \int_{\mathbb{R}^3} n(\log n + 2\langle x \rangle) \, dx + \|n\|_{1+\alpha}^2 + \|\nabla c\|_2^2 + \frac{M + 2}{2} \|u\|_2^2 \right) + C' \left( \left\| \nabla n^{\frac{1+2\alpha}{2}} \right\|_2^2 + \left\| \nabla n^{\frac{1+2\alpha}{2}} \right\|_2^2 + \|\Delta c\|_2^2 + \|\nabla u\|_2^2 \right)
\leq C \left( \left\| \nabla c \right\|_2^2 + 1 \right),
\]

for some \( M, C, C' > 0 \), which gives the result after time integration together with estimate (2.12).

(Case \( \alpha > 1 \)): As in the previous cases, we obtain

\[
\frac{d}{dt} \int_{\mathbb{R}^3} n \log n dx + C'_1 \left\| \nabla n^{\frac{1+2\alpha}{2}} \right\|_2^2 \leq \int_{\mathbb{R}^3} \nabla n \cdot (\chi(c)\nabla c) \, dx 
\leq C_1 \int_{\mathbb{R}^3} n|\nabla c|^2 + n|\Delta c| \, dx,
(2.35)
\]

\[
\frac{d}{dt} \|n\|_{1+\alpha}^2 + C'_2 \left\| \nabla n^{\frac{1+2\alpha}{2}} \right\|_2^2 \leq C_2 \int_{\mathbb{R}^3} n|\nabla c|^2 \, dx \quad \text{(see (2.39))},
(2.36)
\]

\[
\frac{d}{dt} \|\nabla c\|_2^2 + \|\Delta c\|_2^2 \leq C_3 \left( \|\nabla u\|_2 \|\Delta c\|_2 + \int_{\mathbb{R}^3} n|\Delta c| \, dx \right),
(2.37)
\]

\[
\frac{d}{dt} \|u\|_2^2 + 2\|\nabla u\|_2 \leq C_4 \int_{\mathbb{R}^3} n|u| \, dx
(2.38)
\]

and

\[
\frac{d}{dt} \int_{\mathbb{R}^3} \langle x \rangle n dx = \int_{\mathbb{R}^3} nu \cdot \nabla \langle x \rangle + \int_{\mathbb{R}^3} (n + \epsilon)\chi^{1+\alpha} \Delta \langle x \rangle + \int_{\mathbb{R}^3} \nabla \langle x \rangle \cdot n\chi(c)\nabla c
\leq C_5 (1 + \|u\|_2^2 + \|\nabla c\|_2^2) + \left( C(\epsilon) + \epsilon \|\nabla n^{\frac{1+2\alpha}{2}} \|_2^2 \right)
(2.39)
\]

for some constants \( C'_1, C'_2, C_1 \sim C_5 > 0 \). Summing up (2.35)~(2.39) with sufficiently small \( \epsilon > 0 \), we have

\[
\frac{d}{dt} \left( \int_{\mathbb{R}^3} n(\log n + 2\langle x \rangle) \, dx + \|n\|_{1+\alpha}^2 + \|\nabla c\|_2^2 + \|u\|_2^2 \right)
+ C'_5 \left( \left\| \nabla n^{\frac{1+2\alpha}{2}} \right\|_2^2 + \left\| \nabla n^{\frac{1+2\alpha}{2}} \right\|_2^2 + \|\Delta c\|_2^2 + \|\nabla u\|_2^2 \right)
\leq C_5 \left( \int_{\mathbb{R}^3} n|\nabla c|^2 \, dx + \int_{\mathbb{R}^3} n|\Delta c| \, dx + \|\nabla u\|_2 \|\Delta c\|_2 + \int_{\mathbb{R}^3} n|u| \, dx \right)
:= C_5 (I + II + III + IV)
\]
for some $C'_5 > 0$ and $C_5 > 0$. We now estimate $I$ as follows. Taking into account that $\frac{1+\alpha}{2} > 1$ via $\alpha > 1$, we have $n \leq C(\varepsilon) + \epsilon n^{\frac{1+\alpha}{2}}$ due to Young’s inequality and, therefore, we get

$$I \leq C(\varepsilon) \|\nabla c\|_2^2 + \epsilon \int_{\mathbb{R}^3} n^{\frac{1+\alpha}{2}} |\nabla c|^2 \, dx.$$

The second term in the right-hand side of the above is estimated by

$$\int_{\mathbb{R}^3} n^{\frac{1+\alpha}{2}} \nabla c \cdot \nabla c \, dx \leq C \left( \int_{\mathbb{R}^3} \nabla n^{\frac{1+\alpha}{2}} \cdot \nabla c + n^{\frac{1+\alpha}{2}} \Delta c \, dx \right) \leq C \left( \|\nabla n^{\frac{1+\alpha}{2}}\|_2^2 + \|\nabla c\|_2^2 + \|n\|_{1+\alpha} + \|\Delta c\|_2^2 \right).$$

We also note, due to Gagliardo-Nierenberg inequality and Young’s inequality that

$$\|n\|_{1+\alpha} \leq C \|n_0\|_{\frac{1+2\alpha}{1+3\alpha}} \|\nabla n^{\frac{1+\alpha}{2}}\|_{\frac{3\alpha}{1+3\alpha}} \leq C' + \|\nabla n^{\frac{1+\alpha}{2}}\|_2^2,$$

where we used that $\frac{3\alpha}{1+3\alpha} < 2$. Combining estimates, we conclude that

$$I \leq C(\varepsilon_1) \left( \|\nabla c\|_2^2 + 1 \right) + \epsilon_1 \left( \|\nabla n^{\frac{1+\alpha}{2}}\|_2^2 + \|\Delta c\|_2^2 \right). \quad (2.41)$$

We observe that the other terms $II \sim IV$ are estimated exactly the same as those in previous case. More precisely, we have

$$II \leq C(\varepsilon_2) + \epsilon_2 \left( \|\nabla n^{\frac{1+2\alpha}{2}}\|_2^2 + \epsilon_2 \|\Delta c\|_2^2 \right), \quad (2.42)$$

$$III \leq C(\varepsilon_3) \|\nabla u\|_2^2 + \epsilon_3 \|\Delta c\|_2^2, \quad (2.43)$$

$$IV \leq C(\varepsilon_4) + \epsilon_4 \left( \|\nabla n^{\frac{1+2\alpha}{2}}\|_2^2 + \|\nabla u\|_2^2 \right). \quad (2.44)$$

Following similar procedures as before, we conclude that

$$\frac{d}{dt} \left( \int_{\mathbb{R}^3} n \log n \, dx + \|n\|_{1+\alpha} + \|\nabla c\|_2^2 + \frac{M + 2}{2} \|u\|_2^2 \right) \leq C \left( \|\nabla c\|_2^2 + 1 \right) \quad (2.45)$$

for some $M$, $C$, $C' > 0$, which implies (2.5). Since its verifications are just repetition of previous computations, the details are omitted.

- **Case (ii) and (iii) in the Assumption** Here, we only deal with only the case (iii), since the case (ii) can be proved in almost the same manner. Due to the previous result, since the case $\alpha > 1/6$ is direct, it suffices to consider the case $0 < \alpha \leq 1/6$. Multiplying (2.12) with $-\Delta c$ and integrating it by part gives

$$\frac{1}{2} \frac{d}{dt} \|\nabla c\|_2^2 + \|\Delta c\|_2^2 \leq \int_{\mathbb{R}^3} (u \cdot \nabla c) \Delta c \, dx + \int_{\mathbb{R}^3} \kappa(c)n\Delta c \, dx.$$
Since the last integral of the above is estimated as
\[
\int_{\mathbb{R}^3} \kappa(c)n\Delta c \, dx = -\int_{\mathbb{R}^3} (\kappa'(c)n\nabla c + \kappa(c)\nabla n) \nabla c \, dx
\]
\[
\leq -\kappa_0 \int_{\mathbb{R}^3} n|\nabla c|^2 + \kappa \int_{\mathbb{R}^3} |\nabla n| |\nabla c| \, dx,
\]
we have
\[
\frac{d}{dt} \|\nabla c\|_2^2 + \|\Delta c\|_2^2 + 2\kappa_0 \int_{\mathbb{R}^3} n|\nabla c|^2 \, dx
\]
\[
\leq C \left( \|\nabla u\|_2 \|\Delta c\|_2 + \int_{\mathbb{R}^3} |\nabla n|^{1+\alpha} \left( n^{1+\alpha} |\nabla c| \right) \, dx \right). \tag{2.46}
\]
Multiplying (\ref{2.47}) with \((1 + \log n)\) and integrating it by parts, we get
\[
\frac{d}{dt} \int_{\mathbb{R}^3} n \log n \, dx + \frac{4}{1+\alpha} \left\| n^{1+\alpha} \right\|_2^2 \leq \int_{\mathbb{R}^3} \nabla n \cdot (\chi(c)\nabla c) \, dx
\]
\[
\left\| n^{1+\alpha} \right\|_2^2 \leq \frac{2\chi}{1+\alpha} \int_{\mathbb{R}^3} |\nabla n|^{1+\alpha} \left( n^{1+\alpha} |\nabla c| \right) \, dx. \tag{2.47}
\]
Since the integral in the last inequality of (\ref{2.46}) and (\ref{2.47}) is estimated, due to Young’s inequality, as
\[
\int_{\mathbb{R}^3} |\nabla n|^{1+\alpha} \left( n^{1+\alpha} |\nabla c| \right) \leq \epsilon \left\| n^{1+\alpha} \right\|_2^2 + C(\epsilon) \int_{\mathbb{R}^3} |n^{1+\alpha} \nabla c|^2,
\]
we obtain by choosing a sufficiently small \(\epsilon\)
\[
\frac{d}{dt} \left[ \int_{\mathbb{R}^3} n \log n \, dx + \|\nabla c\|_2^2 \right] + \left[ \left\| n^{1+\alpha} \right\|_2^2 + \|\Delta c\|_2^2 + 2\kappa_0 \int_{\mathbb{R}^3} n|\nabla c|^2 \, dx \right]
\]
\[
\leq C \left( \|\nabla u\|_2 \|\Delta c\|_2 + \int_{\mathbb{R}^3} n^{1-\alpha} |\nabla c|^2 \, dx \right). \tag{2.48}
\]
Multiplying (\ref{2.47}) with \(n^{\alpha}\) and integrating it by parts gives
\[
\frac{1}{1+\alpha} \|n\|_{1+\alpha}^{1+\alpha} + \frac{4\alpha(1+\alpha)}{(1+2\alpha)^2} \left\| n^{1+2\alpha} \right\|_2^2 = \int_{\mathbb{R}^3} \nabla n^{\alpha} \cdot n\chi(c)\nabla c \, dx
\]
\[
\leq \frac{2\alpha\chi}{1+\alpha} \int_{\mathbb{R}^3} |\nabla n|^{1+2\alpha} \left( n^{1/2} |\nabla c| \right) \, dx.
\]
Young’s inequality implies
\[
\frac{1}{1+\alpha} \frac{d}{dt} \|n\|_{1+\alpha}^{1+\alpha} + \left\| n^{1+2\alpha} \right\|_2^2 \leq C_0 \int_{\mathbb{R}^3} n|\nabla c|^2 \, dx
\]
for some \(C_0 > 0\). Now we choose sufficiently small \(\epsilon_0 > 0\) satisfying \(\epsilon_0 C_0 < 2\kappa_0\). Multiplying it by the both sides of the above gives
\[
\frac{\epsilon_0}{1+\alpha} \frac{d}{dt} \|n\|_{1+\alpha}^{1+\alpha} + \epsilon_0 \left\| n^{1+2\alpha} \right\|_2^2 \leq \epsilon_0 C_0 \int_{\mathbb{R}^3} n|\nabla c|^2 \, dx. \tag{2.49}
\]
Summing up (2.48) and (2.49), we have
\[ \frac{d}{dt} \left[ \int_{\mathbb{R}^3} n \log n + \| \nabla c \|_2^2 + \frac{\varepsilon_0}{1 + \alpha} \| n \|_{1+\alpha}^{1+\alpha} \right] \]
\[ + \left[ \| \nabla n \|_2^2 + \varepsilon_0 \| \nabla n \|_2^2 \right] + \| \nabla c \|_2^2 + \delta_0 \int_{\mathbb{R}^3} n |\nabla c|^2 \, dx \]
\[ \leq C \left( \| \nabla u \|_2 \| \Delta c \|_2 + \int_{\mathbb{R}^3} n^{1-\alpha} |\nabla c|^2 \, dx \right), \]
where \( \delta_0 := 2\kappa_0 - \varepsilon_0 C_0 > 0 \). Therefore, applying Young’s inequality \( n^{1-\alpha} \leq C(\epsilon) + \epsilon n \) with sufficiently small \( \epsilon > 0 \), we obtain
\[ \frac{d}{dt} \left[ \int_{\mathbb{R}^3} n \log n + \| \nabla c \|_2^2 + \frac{\varepsilon_0}{1 + \alpha} \| n \|_{1+\alpha}^{1+\alpha} \right] \]
\[ + \left[ \| \nabla n \|_2^2 + \varepsilon_0 \| \nabla n \|_2^2 \right] + \| \nabla c \|_2^2 + \frac{\delta_0}{2} \int_{\mathbb{R}^3} n |\nabla c|^2 \] \[ \leq C \left( \| \nabla c \|_2^2 + \| \nabla u \|_2^2 \right). \]
Finally, absorbing the term \( \| \nabla u \|_2^2 \) as we have done previously (see (2.22)) and combining estimate (2.13), we deduce (2.15). This completes the proof. \( \square \)

### 3 Bounded weak solutions

In this section, we provide some uniform estimates of solutions to the system (2.1) with \( \tau = 0 \), which is used in Section 4 to construct the bounded weak solution of the equation (1.1) and prove Theorem 2. To be more precise, our result is read as follows:

**Lemma 2** Let \( \tau = 0 \) and \( T > 0 \) be given. Suppose that \((n, c, u)\) is a classical solution for the system (2.1), with the smooth initial datum \((n_0, c_0, u_0)\) which satisfies the initial conditions (1.4) and (1.6). Assume further that \( \chi, \kappa \) and \( \alpha \) satisfies Assumption 3. Then, for any \( 0 < t \leq T \)
\[ n \in L^\infty(0, T; L^p(\mathbb{R}^3)), \quad 1 \leq p \leq \infty, \] (3.1)
\[ \nabla n^{\frac{p+\alpha}{2}} \in L^2(0, T; L^2(\mathbb{R}^3)), \quad 1 \leq p < \infty, \] (3.2)
\[ c, u \in L^\infty(0, T; W^{1,q}(\mathbb{R}^3)) \cap L^q(0, T; W^{2,q}(\mathbb{R}^3)), \quad 2 \leq q < \infty, \] (3.3)
\[ c_\ell, u_\ell \in L^q(0, T; L^q(\mathbb{R}^3)), \quad 2 \leq q < \infty. \] (3.4)

**Proof.** Since Assumption 2 is stronger than Assumption 1, it is automatic, due to Lemma 1, that
\[ n \in L^\infty(0, T; L^p(\mathbb{R}^3)) \quad \text{and} \quad \nabla n^{\frac{p+\alpha}{2}} \in L^2(0, T; L^2(\mathbb{R}^3)), \quad 1 \leq p \leq 1 + \alpha. \] (3.5)
In what follows, we complete the proof by showing that for each \( \alpha > 1/8 \), if \( n \) satisfies (3.5) then (3.1) \sim (3.4) hold. We remark that the first case (i) in the Assumption 2, i.e. \( \alpha > 1/6 \), cannot be relaxed as \( \alpha > 1/8 \), since (3.5) is not available in case only \( \alpha > 1/8 \) is assumed.

First, we consider \( L^p \) estimate of (2.1) \[ \frac{1}{p} \frac{d}{dt} \| n \|_p^p + \int_{\mathbb{R}^3} \nabla n^{p-1} \cdot \nabla (n + \varrho) \, dx = -\int_{\mathbb{R}^3} n^{p-1} \nabla \cdot (\chi(c)n \nabla c) \, dx. \]
Since $\nabla n^{p-1} \cdot \nabla n = (4(p - 1)/p^2) \left| \nabla n^{p/2} \right|^2 \geq 0$, we have
\[
\frac{1}{p} \frac{d}{dt} \left\| n \right\|_p^p + \frac{4(p-1)(1+\alpha)}{(p+\alpha)^2} \left\| \nabla n^{\frac{p+\alpha}{2}} \right\|_2^2 \leq \left\| \int_{\mathbb{R}^3} n^{p-1} \nabla \cdot (\chi(c)n \nabla c) \, dx \right\|.
\]
Since the right-hand side of the above is estimated as
\[
\left\| \int_{\mathbb{R}^3} n^{p-1} \nabla \cdot (\chi(c)n \nabla c) \, dx \right\| \leq C \int_{\mathbb{R}^3} \left\| \nabla n^{\frac{p+\alpha}{2}} \right\|^2 \left\| n^{\frac{p-\alpha}{2}} \right\| \left\| \nabla c \right\| \, dx,
\]
it follows from Young’s inequality that
\[
\frac{1}{p} \frac{d}{dt} \left\| n \right\|_p^p + \left\| \nabla n^{\frac{p+\alpha}{2}} \right\|^2 \leq C p \int_{\mathbb{R}^3} n^{p-\alpha} \left\| \nabla c \right\|^2 \, dx.
\]
Using Hölder inequality, Sobolev embedding and Young’s inequality, we estimate the integral in the right-hand side of the above as
\[
\int_{\mathbb{R}^3} n^{p-\alpha} \left\| \nabla c \right\|^2 \, dx \leq \left\| n^{p-\alpha} \right\|_{\frac{p}{p-\alpha}} \left\| \nabla c \right\|_{\frac{2}{p}}^2 \leq C p \left\| n \right\|_{p}^{p-\alpha} \left\| \Delta c \right\|_{\frac{2}{p+3\alpha}}^2 \leq C p \left( \frac{\alpha}{p} + \frac{p-\alpha}{p} \left\| n \right\|_p^p \right) \left\| \Delta c \right\|_{\frac{2}{p+3\alpha}}^2 \left\| \nabla c \right\|_{\frac{2}{p+3\alpha}}^2.
\]
Combining estimates, we get
\[
\frac{d}{dt} \left\| n \right\|_p^p \leq C p^3 \left( \frac{\alpha}{p} + \frac{p-\alpha}{p} \left\| n \right\|_p^p \right) \left\| \Delta c \right\|_{\frac{2}{p+3\alpha}}^2 \left\| \nabla c \right\|_{\frac{2}{p+3\alpha}}^2 \leq C p^3 \left\| \Delta c \right\|_{\frac{2}{p+3\alpha}}^2 \left\| n \right\|_p^p + C p^2 \left\| \Delta c \right\|_{\frac{2}{p+3\alpha}}^2 \left\| \nabla c \right\|_{\frac{2}{p+3\alpha}}^2.
\]
Thus it follows from Gronwall inequality that
\[
\left\| n \right\|_p^p \leq \exp \left( C p^3 \int_0^t \left\| \Delta c(s) \right\|_{\frac{2}{p+3\alpha}}^2 \, ds \right) \int_0^t \left\| \Delta c \right\|_{\frac{2}{p+3\alpha}}^2 \, ds + \left\| n_0 \right\|_p^p
\]
and hence (3.1) holds except for $\alpha > 8$ whenever the following holds:
\[
\int_0^T \left\| \Delta c(s) \right\|_{\frac{2}{p+3\alpha}}^2 \, ds < \infty, \quad 1+\alpha < p < \infty.
\]
We derive (3.7) by treating two cases, i.e. $\alpha > 1/3$ and $1/8 < \alpha \leq 1/3$ separately.

(Case $\alpha > 1/3$): Applying maximal regularity estimate (2.2) of heat equation to (2.1)_2,
\[
\int_0^T \left\| \Delta c(s) \right\|_{\frac{2}{p+3\alpha}}^2 \, ds \leq C \left( \left\| \nabla c_0 \right\|_{\frac{2}{p+3\alpha}}^2 + \int_0^T \left\| n(s) \right\|_{\frac{2}{p+3\alpha}}^2 \, ds + \int_0^T \left\| u \cdot \nabla c \right\|_{\frac{2}{p+3\alpha}}^2 \, ds \right)
\]
\[
:= C \left( \left\| \nabla c_0 \right\|_{\frac{2}{p+3\alpha}}^2 + I + \Pi \right).
\]
Now, for \( p > 1 + \alpha \), we obtain via interpolation
\[
I \leq C \int_0^T \left\| \nabla (s) \right\|_{1, \alpha}^{2} \left\| n(s) \right\|_{3, 6 \alpha}^{2} \, ds \\
\leq C \int_0^T \left\| \nabla (s) \right\|_{1, \alpha}^{2} \left\| n(s) \right\|_{3, 6 \alpha}^{2} \, ds = C \int_0^T \left\| \nabla (s) \right\|_{1, \alpha}^{2} \, ds,
\]
for some \( \delta_p > 0 \). We note that the last equality of the above holds because \( \alpha > 1/3 \). Before estimating II, we first observe that
\[
u \in L^\infty(0, T; L^6(\mathbb{R}^3)).
\]
Indeed, since \( \tau = 0 \), the vorticity, \( \omega = \nabla \times u \), satisfies \( \omega_t - \Delta \omega = -\nabla \times (n \nabla \phi) \), and its \( L^2 \) estimation becomes
\[
\frac{d}{dt} \left\| \omega \right\|^2_2 + \| \nabla \omega \|^2_2 \leq C \int_{\mathbb{R}^3} \nabla \omega \cdot ndx \leq \epsilon \| \nabla \omega \|^2_2 + C(\epsilon) \| n \|^2_2.
\]
Hence we have
\[
\frac{d}{dt} \left\| \omega \right\|^2_2 \leq C \| n \|^2_2 \leq C \| n \|^2_{1, \alpha} \left\| \nabla (s) \right\|_{2, 3 \alpha} \, ds.
\]
(3.9)
Since \( 0 < \frac{6}{2 + 3 \alpha} < 2 \) due to \( \alpha > 1/3 \), we obtain (3.8) by Sobolev embedding. Next we estimate II. Applying maximal regularity estimate (2.2) for heat equation and (3.8), we have for \( p > 1 + \alpha \),
\[
II \leq \int_0^T \left\| u(s) \right\|^2_6 \left\| \nabla c(s) \right\|_{6 \alpha}^{2 \delta_p} \, ds \leq C \int_0^T \left\| \Delta c(s) \right\|_{2 \alpha}^{2 \delta_p} \, ds \\
\leq C \left( \left\| \nabla c_0 \right\|_{2 \alpha}^{2 \delta_p} + \int_0^T \left( \left\| n(s) \right\|^2_{2 \alpha} + \| u \cdot \nabla c(s) \|^2_{2 \alpha} \right) ds \right) \\
\leq C + C \int_0^T \left( \left\| n(s) \right\|^2_{1, \alpha} \left\| n(s) \right\|_{3, 6 \alpha}^{2 \delta_p} \, ds \right) \\
\leq C + C \int_0^T \left( \left\| \nabla n \right\|_{2, 3 \alpha}^{2 \delta_p} \, ds + \left\| \nabla c(s) \right\|_{2, 3 \alpha}^{2 \delta_p} \, ds \right) \\
=: C + C \int_0^T \left( \left\| \nabla n \right\|_{2, 3 \alpha}^{2 \delta_p} \, ds + \left\| \nabla c(s) \right\|_{2, 3 \alpha}^{2 \delta_p} \, ds \right),
\]
where \( 0 < \delta_p := \frac{1}{2 + 3 \alpha} + \frac{3\alpha}{p(1 + 3 \alpha)} < 2 \). We note here that for each \( q \) with \( 2 \leq q \leq 6 \),
\[
\int_0^T \left\| \nabla c \right\|^2_6 \, ds < \infty \text{ due to the results in Lemma[1]}
\]
Therefore we conclude that \( n \in L^\infty(0, T; L^p(\mathbb{R}^3)) \) and \( \nabla n \in L^2(0, T; L^2(\mathbb{R}^3)) \) for \( 1 \leq p < \infty \), which can be easily extended to \( 1 + \alpha < p < \infty \) from (3.5). It remains to show that \( n \) is bounded. Indeed, since \( n \in L^\infty(0, T; L^p(\mathbb{R}^3)) \) for all \( 1 \leq p < \infty \), we can see that \( c_t, \nabla^2 c, u_t \) and \( \nabla^2 u \) belong to \( L^q((0, T) \times \mathbb{R}^3) \) for all \( q < \infty \) and therefore, we also note that \( \nabla c \in L^\infty((0, T) \times \mathbb{R}^3) \). Using the estimate (3.6) and \( \nabla c \in L^\infty((0, T) \times \mathbb{R}^3) \), we obtain
\[
\frac{d}{dt} \left\| n \right\|^p_p \leq C \left\| n \right\|^p_{p - \alpha} \leq C \left\| n \right\|^\alpha_{1, T} \left\| n \right\|^p_p \left\| n \right\|^p_{p - 1} \leq C \left\| n \right\|^p_{p(1 - \beta)}.
\]
where \( \beta = \alpha/(p - 1) \). Via Gronwall’s inequality, we observe that
\[
\left\| n(t) \right\|^p_p \leq \left( C \left\| n \right\|^p_{p(1 + \beta)} \right) t + \left\| n_0 \right\|^p_p, \quad t \leq T.
\]
Passing $p$ to the limit, we obtain for all $t < T$

$$\|n(t)\|_{L^\infty(\mathbb{R}^3)} \leq 1 + \|n_0\|_{L^\infty(\mathbb{R}^3)}. \quad (3.10)$$

This completes the case that $\alpha > 1/3$.

(Case $1/8 < \alpha \leq 1/3$): We first show that

$$n \in L^\infty(0, T; L^p(\mathbb{R}^3)) \text{ and } \nabla n \frac{p+4}{2} \in L^2(0, T; L^2(\mathbb{R}^3)), \quad 1 \leq p < 1 + 4\alpha \quad (3.11)$$

and we then derive \([3.7]\) via \([3.11]\). To show \([3.11]\), we compute $L^p$ estimate of $n$ as follows:

$$\|n(t)\|_p^p + \int_0^t \left\| \nabla n \frac{p+4}{2} (s) \right\|^2_2 \, ds \leq \int_0^t \int_{\mathbb{R}^3} n^{p-1} \nabla \cdot (\chi(c) n \nabla c) \, dx \, ds + \|n_0\|_p^p.$$ 

From Young's inequality, the first term in the righthand side of the above is estimated as

$$\int_0^t \int_{\mathbb{R}^3} n^{p-1} \nabla \cdot (\chi(c) n \nabla c) \, dx \, ds = - \int_0^t \int_{\mathbb{R}^3} \nabla n^{p-1} \cdot (\chi(c) n \nabla c) \, dx \, ds$$

$$\leq \epsilon_1 \int_0^t \left\| \nabla n \frac{p+4}{2} \right\|^2_2 \, ds + C(\epsilon_1) \int_0^t \int_{\mathbb{R}^3} n^{p-\alpha} |\nabla c|^2 \, dx \, ds.$$

We note that the last integral of the above is estimated as follows:

$$\int_0^t \int_{\mathbb{R}^3} n^{p-\alpha} |\nabla c|^2 \, dx \, ds = \int_0^t \int_{\mathbb{R}^3} n^{p-\alpha} \nabla c \cdot \nabla c \, dx \, ds$$

$$\leq C \left( \int_0^t \int_{\mathbb{R}^3} |\nabla n^{p-\alpha}| \, |\nabla c| \, dx \, ds + \int_0^t \int_{\mathbb{R}^3} |n^{p-\alpha}| \, |\Delta c| \, dx \, ds \right)$$

$$\leq \epsilon_2 \int_0^t \left\| \nabla n \frac{p+4}{2} \right\|^2_2 \, ds + C(\epsilon_2) \int_0^t \int_{\mathbb{R}^3} n^{p-3\alpha} |\nabla c|^2 \, dx \, ds + C \int_0^t \int_{\mathbb{R}^3} n^{p-\alpha} |\Delta c| \, dx \, ds.$$

Hence choosing sufficiently small $\epsilon_1$ and $\epsilon_2 > 0$, we have

$$\|n(t)\|_p^p + \int_0^t \left\| \nabla n \frac{p+4}{2} \right\|^2_2 \, ds$$

$$\leq C \left( \int_0^t \int_{\mathbb{R}^3} |n^{p-3\alpha}| \, |\nabla c|^2 \, dx \, ds + \int_0^t \int_{\mathbb{R}^3} |n^{p-\alpha}| \, |\Delta c| \, dx \, ds \right) + \|n_0\|_p^p$$

$$\leq C \left( \int_0^t \int_{\mathbb{R}^3} |n^{p-3\alpha}| \, |\nabla c|^2 \, dx \, ds + \int_0^t (\|n\|_{p-\alpha+1}^{p-\alpha+1} + \|\Delta c\|_{p-\alpha+1}^{p-\alpha+1}) \, ds \right) + \|n_0\|_p^p$$

$$:= C (I + II) + \|n_0\|_p^p.$$ 

For each $p$ with $1 + \alpha < p < 1 + 4\alpha$, we estimate $I$ via maximal regularity estimate \([2.2]\)

$$I \leq C \int_0^t \left\| n^{p-3\alpha} (s) \right\|_{\frac{1+\alpha}{p-3\alpha}} \left\| \nabla c(s) \right\|^2_{\frac{1+\alpha}{1+4n-\alpha}} \, ds$$

$$\leq C \int_0^t \|\Delta c(s)\|_{r_1}^2 \, ds \leq C \left( \|\nabla c_0\|_{r_1}^2 + \int_0^t (\|n(s)\|_{r_1}^2 + \|u \cdot \nabla c(s)\|_{r_1}^2) \, ds \right), \quad (3.12)$$

18
where \( r_1 := \frac{6 + 6\alpha}{4 + 14\alpha - 3p} \). We note that Hölder inequality is applicable to the first inequality of the above due to the conditions of \( \alpha \) and \( p \), that is, \( 1/8 < \alpha \leq 1/3 \) and \( 1 < p < 1 + 4\alpha \). Now let us estimate the term \( \int_0^t \| u \cdot \nabla c \|_{r_1}^2 \, ds \) in (3.12). Considering the following \( L^2 \)- estimation of the vorticity equation:

\[
\frac{d}{dt} \| \omega \|_2^2 + \| \nabla \omega \|_2^2 \leq C \int_{\mathbb{R}^3} \nabla \omega \cdot ndx \leq C \| n \|_2^2 + C(\epsilon) \| n \|_2^2,
\]

we have

\[
\frac{d}{dt} \| \omega \|_2^2 \leq C \| n \|_2^2 \leq C \left( \| \nabla c \|_{\frac{6 \alpha}{2 + 3\alpha}}^2 + \int_0^t \left( \| n \|_{\frac{6 \alpha}{2 + 3\alpha}}^2 + \| u \cdot \nabla c \|_{\frac{6 \alpha}{2 + 3\alpha}}^2 \right) \, ds \right),
\]

which enables us to apply the interpolation inequality to the above. Therefore, we have

\[
u \in L^\infty(0,T; L^6(\mathbb{R}^3)).
\]

Taking into account that \( 1 \leq r_1 < 3 \) and the above, we note, due to Hölder inequality, that

\[
\int_0^t \| u \cdot \nabla c \|_{r_1}^2 \, ds \leq \int_0^t \| n \|_2^2 \| \nabla c \|_{\frac{6 \alpha}{2 + 3\alpha}}^2 \, ds \leq C \int_0^t \| \Delta c \|_{\frac{6 \alpha}{2 + 3\alpha}}^2 \, ds \leq C \left( \| \nabla c \|_{\frac{6 \alpha}{2 + 3\alpha}}^2 + \int_0^t \left( \| n \|_{\frac{6 \alpha}{2 + 3\alpha}}^2 + \| u \cdot \nabla c \|_{\frac{6 \alpha}{2 + 3\alpha}}^2 \right) \, ds \right),
\]

Substituting it to (3.12) and applying interpolation inequality, we obtain

\[
I \leq C \left( \| \nabla c \|_{\frac{6 \alpha}{2 + 3\alpha}}^2 + \| \nabla c \|_{\frac{6 \alpha}{2 + 3\alpha}}^2 + \int_0^t \left( \| n \|_{\frac{6 \alpha}{2 + 3\alpha}}^2 + \| u \cdot \nabla c \|_{\frac{6 \alpha}{2 + 3\alpha}}^2 \right) \, ds \right)
\]

where \( \theta_1 = \frac{(p + \alpha)(3p - 14\alpha + 1)}{2(3p + 2\alpha - 1)} \) and \( \theta_2 = \frac{3(p + \alpha)(p - 3\alpha)}{2(3p + 2\alpha - 1)} \). We remark that it is not difficult to verify \( 1 + \alpha < r_1 < 3p + 3\alpha \) and \( 1 < \frac{6 \alpha}{2 + 3\alpha} < 3p + 3\alpha \) whenever \( 1/8 < \alpha \leq 1/3 \) and \( 1 + \alpha < p < 1 + 4\alpha \), which enables us to apply the interpolation inequality to the above. Therefore, we have

\[
I \leq C \left( \| \nabla c \|_{\frac{6 \alpha}{2 + 3\alpha}}^2 + \| \nabla c \|_{\frac{6 \alpha}{2 + 3\alpha}}^2 + \int_0^t \left( \| n \|_{\frac{6 \alpha}{2 + 3\alpha}}^2 + \| u \cdot \nabla c \|_{\frac{6 \alpha}{2 + 3\alpha}}^2 \right) \, ds \right)
\]

Similarly, for each \( p \) with \( 1 + \alpha < p < 1 + 4\alpha \), we estimate

\[
II \leq C \int_0^t \left( \| n \|_{r_2}^2 + \| u \|_6^2 \| \nabla c \|_{\frac{6 \alpha}{2 + 3\alpha}}^2 \right) \, ds + \| \nabla c \|_{r_2}^2
\]

where \( r_2 := p - \alpha + 1 \). We note that in the first inequality of the above, Hölder inequality is applied due to the conditions of \( \alpha \) and \( p \), that is, \( 1/8 < \alpha \leq 1/3 \) and \( 1 + \alpha < p < 1 + 4\alpha \). Since
1 + \alpha < r_2 < 3p + 3\alpha, 1 < \frac{6r_2}{6 + r_2} < 3p + 3\alpha and 2 < r_2 < 6, we obtain the following estimate via applying interpolation inequality:

\[
II \leq C \left( \int_0^t \left( \|n(t)\|_{\frac{p_2(1 - \theta_3)}{1 + \alpha}}^{r_2(1 - \theta_3)} + \|n(t)\|_{\frac{r_2 \theta_3}{3p + 3\alpha}}^{r_2(1 - \theta_3)} \right) \|n(t)\|_{\frac{r_2 \theta_4}{3p + 3\alpha}} \right) ds
\]

\[
+ \int_0^t \left( \|\nabla c\|_2^{r_2(1 - \theta_5)} \|\nabla c\|_{r_2^*}^{r_2(1 - \theta_5)} \right) ds + \|\nabla c_0\|_{\frac{r_2}{6 + r_2}}^{r_2(\theta_5)} + \|\nabla c_0\|_{r_2}^{r_2(\theta_5)},
\]

where

\[
\theta_3 := \frac{3(p + \alpha)(r_2 - 1 - \alpha)}{r_2(3p + 2\alpha - 1)}, \quad \theta_4 := \frac{(p + \alpha)(5r_2 - 6)}{r_2(6p + 6\alpha - 2)} \quad \text{and} \quad \theta_5 := \frac{3(p - \alpha - 1)}{2r_2}.
\]

Therefore, we have

\[
II \leq C \left( \int_0^t \left( \|\nabla n\|_{\frac{p_2 + \delta_1}{p + \alpha}}^{\frac{2r_2 \delta_1}{p + \alpha}} + \|\nabla n\|_{\frac{p_2 + \delta_1}{p + \alpha}}^{\frac{2r_2 \delta_1}{p + \alpha}} + \|\Delta c\|_2^{\delta_2(1 - \theta_5)} ds + \|\nabla c_0\|_{\frac{r_2}{6 + r_2}}^{\theta_5} + \|\nabla c_0\|_{r_2}^{\theta_5} \right) \]

\[
=: C \left( \int_0^t \left( \|\nabla n\|_{\frac{p_2 + \delta_1}{p + \alpha}}^{\delta_1} + \|\nabla n\|_{\frac{p_2 + \delta_1}{p + \alpha}}^{\delta_1} + \|\Delta c\|_2^{\delta_2(1 - \theta_5)} ds + \|\nabla c_0\|_{\frac{r_2}{6 + r_2}}^{\theta_5} + \|\nabla c_0\|_{r_2}^{\theta_5} \right) \right).
\]

Since one can see that 0 < \delta_i < 2, i = 1, 2, ..., 5 whenever 1/8 < \alpha \leq 1/3 and 1 + \alpha < p < 1 + 4\alpha, it follows from Young’s inequality that

\[
\|n(t)\|_p^p + C_1 \int_0^t \left( \|\nabla n\|_{\frac{p_2 + \delta_1}{p + \alpha}}^2 \right)^{\frac{\delta_1}{2}} ds \leq C_2 \left( \int_0^T \|\nabla c\|_2^{\theta_1} ds + 1 \right), \quad 0 < t < T
\]

for p with 1 + \alpha < p < 1 + 4\alpha. We know that, for each q with 2 \leq q \leq 6, \int_0^T \|\nabla c\|_q^q ds < \infty.

Since 2 \leq r_1 \leq 6 whenever \(\frac{2 + 11\alpha}{3} < p < 1 + 4\alpha\), we have \(n \in L^\infty(0, T; L^p(\mathbb{R}^3))\) and \(\nabla n \in L^2(0, T; L^2(\mathbb{R}^3))\), for \(\frac{2 + 11\alpha}{3} < p < 1 + 4\alpha\), which can be easily extended to \(1 + \alpha < p < 1 + 4\alpha\) from (3.5). Thus we conclude that (3.11) holds.

We are now ready to derive (3.7) for the case 1/8 < \alpha \leq 1/3. Let \(p_0 = \frac{3}{2} - \frac{3\alpha}{4}\). Since 1 \leq p_0 < 1 + 4\alpha whenever 1/8 < \alpha \leq 1/3, it is evident from the above conclusion that

\[
n \in L^\infty(0, T; L^{p_0}(\mathbb{R}^3)).
\]

From the maximal regularity estimate (2.22) for heat equation, we have

\[
\int_0^T \|\Delta c\|_2^{\frac{6p}{2p + 3\alpha}} \leq C \left( \int_0^T \|n\|_{\frac{2}{2p + 3\alpha}}^{6p} ds + \int_0^T \|u \cdot \nabla c\|_2^{\frac{6p}{2p + 3\alpha}} ds \right)
\]

\[
=: C \left( I + II \right).
\]

For p > 1 + \alpha, we have \(p_0 < \frac{6p}{2p + 3\alpha} < 3p_0 + 3\alpha\) and hence it follows from (3.14) that

\[
I \leq C \int_0^T \|n(s)\|_{\frac{p_0}{3p_0} \frac{p_0 \alpha (6p - 2p_0 - 3\alpha p_0)}{2(2p_0 + 3\alpha) \alpha (6p - 2p_0 - 3\alpha p_0)}} ds
\]

\[
\leq C \int_0^T \|\nabla n\|_{\frac{p_0 + \alpha}{p_0}} \|\nabla n\|_{\frac{p_0 + \alpha}{p_0}} \|\nabla n\|_{\frac{p_0 + \alpha}{p_0}} \|\nabla n\|_{\frac{p_0 + \alpha}{p_0}} ds
\]

\[
=: C \int_0^T \|\nabla n\|_{\frac{p_0 + \alpha}{p_0}} \|\nabla n\|_{\frac{p_0 + \alpha}{p_0}} \|\nabla n\|_{\frac{p_0 + \alpha}{p_0}} \|\nabla n\|_{\frac{p_0 + \alpha}{p_0}} ds
\]
for some $\delta_p > 0$. On the other hands, by the maximal regularity estimate (2.2) for the heat equation, we have for $p > \frac{\alpha(1+\alpha)}{1-\alpha}$ (in fact, $p > 1 + \alpha$ since $0 < \frac{\alpha(1+\alpha)}{1-\alpha} < 1 + \alpha$)

$$\Pi \leq \int_0^T \|u(s)\|_6^2 \|\nabla c(s)\|_{2p/3\alpha}^2 ds \leq C \int_0^T \|\Delta c(s)\|_{2p/3\alpha}^2 ds$$

$$\leq C \left( \|\nabla c_0\|_{2p/3\alpha}^2 + \int_0^T \left( \|n(s)\|_{2p/3\alpha}^2 + \|u \cdot \nabla c(s)\|_{2p/3\alpha}^2 \right) ds \right)$$

$$\leq C \left( \|\nabla c_0\|_{2p/3\alpha}^2 + \int_0^T \left( \left\|\nabla \left( n^{\frac{1+2\alpha}{2}} \right) (s) \right\|_{2p/3\alpha}^2 + \left\|\nabla c(s)\right\|_{2p/3\alpha}^2 \right) ds \right)$$

$$= C \left( \|\nabla c_0\|_{2p/3\alpha}^2 + \int_0^T \left( \left\|\nabla \left( n^{\frac{1+2\alpha}{2}} \right) (s) \right\|_{2p/3\alpha}^2 + \left\|\nabla c(s)\right\|_{2p/3\alpha}^2 \right) ds \right),$$

where $0 < \delta_p := \frac{2+16\alpha}{2(2\alpha + 6\alpha^2)} < 2$. We remark that $\int_0^T \|\nabla c\|^2 ds < \infty$ for each $q$ with $2 \leq q \leq 6$, due to the results in Lemma 1.

So far, we have shown (3.7) holds for each $\alpha > 1/8$. Therefore we conclude that $n \in L^\infty(0,T; L^p(\mathbb{R}^3))$ for $1 + \alpha < p < \infty$. Following similar procedures as in (3.10), we can show that $L^\infty$-norm of $n$ is bounded. Since arguments are on the same track, we omit the details. $\square$

4 Proofs of Theorems

To prove Theorem 1 and Theorem 2, using the uniform estimates established previously, we construct weak and bounded weak solutions. Using the uniform estimates established in the previous section. Since the argument is rather standard (compare to [2, 5, 11, 12, 14]), we omit the details and give the sketch of how our constructions are made instead.

Proofs of Theorem 1 and Theorem 2. We consider only the case of Theorem 2 since the proof of Theorem 1 is essentially same. First, we recall the regularized system (1.1) with the initial data $(n_0, c_0, u_0)$ which are chosen as smooth approximations of $(n_0, c_0, u_0)$:

$$n_{0\rho} = \psi_\rho * n_0, \quad c_{0\rho} = \psi_\rho * c_0 \quad \text{and} \quad u_{0\rho} = \psi_\rho * u_0,$$

where $\phi_\rho$ denotes the usual mollifier. The convergence of $(n_{0\rho}, c_{0\rho}, u_{0\rho})$ entails that the estimates obtained in Lemma 1 and Lemma 2 are uniform, independent of $\rho$, precisely, the constant $C$ and $M$ in (2.5) can be chosen independent of $\rho$. Likewise, there exists a constant $C$ such that for $q < \infty$

$$\|n_{0\rho}\|_{L^\infty((0,T) \times \mathbb{R}^3)} + \left\|\nabla n_{0\rho}\right\|_{L^2((0,T) \times \mathbb{R}^3)} < C, \quad (4.1)$$

$$\|c_{0\rho}\|_{L^\infty(0,T; W^{1,q}(\mathbb{R}^3))} + \|c_{0\rho}\|_{L^q(0,T; L^q(\mathbb{R}^3))} + \left\|\partial_t c_{0\rho}\right\|_{L^q(0,T; L^q(\mathbb{R}^3))} < C, \quad (4.2)$$

$$\|u_{0\rho}\|_{L^\infty(0,T; W^{1,q}(\mathbb{R}^3))} + \|u_{0\rho}\|_{L^q(0,T; L^q(\mathbb{R}^3))} + \left\|\partial_t u_{0\rho}\right\|_{L^q(0,T; L^q(\mathbb{R}^3))} < C. \quad (4.3)$$

According to the estimates we have derived, a bootstrap argument can extend the local solution to any given time interval $(0, T)$ (compare to [5, 12] and [10] for more detail). Let $k$ be
any number with \( k \geq 2 + \alpha \). We then show that \( \partial_t n, n_0 \) and \( \partial_t n_0^k \) are, independent of \( \varphi \), in \( L^1(0; W^{-2,2}(\mathbb{R}^3)) \), where \( W^{-2,2}(\mathbb{R}^3) \) is the dual space of \( W^{2,2}(\mathbb{R}^3) \) (compare to [12]). Then via Aubin-Lions Lemma, by passing to the limit, we have some weak limit \((n, c, u)\), which turns out to be a weak solution. Its verification is rather straightforward, and thus the details are skipped. It is also direct that \((n, c, u)\) is a bounded weak solution and satisfies the estimates (4.1)-(4.3). This completes the proof.

Next we present the proofs of Theorem 3 and Theorem 4. As mentioned in the Introduction, we indicate only difference compared to the case of \( \mathbb{R}^3 \).

**Proofs of Theorem 3 and Theorem 4** We note that unlike \( \mathbb{R}^3 \), \( L^1 \) estimate of \( n(x) \) is not necessary, because negative part of \( \int_{\Omega} n \log n \) is controlled by \( \|n\|_{L^1(\Omega)} \). We also observe that Gagliardo-Nirenberg inequality should be slightly modified, for example, we used in the case \( \mathbb{R}^3 \) (see the inequality right above (2.18))

\[
\|n\|^2_{L^2(\mathbb{R}^3)} \leq C \|n\|_{L^1(\mathbb{R}^3)}^{\frac{1+6\alpha}{2+6\alpha}} \|\nabla n\|_{L^2(\mathbb{R}^3)}^{\frac{6}{2+6\alpha}}.
\]

In the case of bounded domains, it is replaced by

\[
\|n\|^2_{L^2(\Omega)} \leq C \|n\|_{L^1(\Omega)}^{\frac{1+6\alpha}{2+6\alpha}} \|\nabla n\|_{L^2(\Omega)}^{\frac{6}{2+6\alpha}} + C \|n\|^2_{L^1(\Omega)}.
\]

Major modifications lie in the estimate of vorticity, \( \omega \), because the boundary condition of \( \omega \) is not prescribed. More precisely, in order to obtain (3.8), we used the equation of vorticity, which is not useful to the case of bounded domains. Here we show (3.8) differently not by using vorticity equations. Let \( Q = \Omega \times (0, T) \). Using \( L^p \)–type estimate of the Stokes system (see e.g. [9]), we note that

\[
\|u_t\|_{L^2(Q)} + \|u\|_{L^2((0,T);H^2(\Omega))} + \|\nabla p\|_{L^2(Q)} \leq C \left( \|n\|_{L^2(Q)} + \|u_0\|_{H^1(\Omega)} \right).
\]

Testing \(-\Delta u\) to the fluid equations,

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla u|^2 \, dx + \int_{\Omega} |\Delta u|^2 \, dx \leq \frac{1}{4} \int_{\Omega} |\Delta u|^2 \, dx + C \int_{\Omega} \nabla u \cdot \nabla p \, dx \\
\leq \frac{1}{2} \int_{\Omega} |\Delta u|^2 \, dx + C \int_{\Omega} \nabla u \cdot \nabla p \, dx.
\]

Therefore, after integrating the above in time over \((0, t)\) for any \( t < T \) and combining the estimate (4.4), we obtain

\[
\|\nabla u(t)\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(Q_t)} \leq \|\nabla u_0\|^2_{L^2(\Omega)} + \|\nabla p\|^2_{L^2(Q_t)} + C \left( \|n\|^2_{L^2(Q_t)} + \|u_0\|^2_{H^1(\Omega)} \right) \\
\leq C \left( \|u_0\|_{W^{1,2}(\Omega)} + \|n\|^2_{L^2(Q_t)} \right),
\]

where \( Q_t = (0, t) \times \Omega \) and we used (4.4). We note that \( L^2 \) norm of \( n \) can be estimated in the same ways as in (3.9) and (3.9), which implies that \( \nabla u \in L^\infty((0, T); L^2(\Omega)) \) and therefore, it is automatic that \( u \in L^\infty((0, T); L^6(\Omega)) \) via Sobolev embedding. The rest parts of proofs are essentially the same as the cases of whole space, and thus we omit the details.

□
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