Dynamic First Order Wave Systems with Drift Term on Riemannian Manifolds.

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An abstract first order differential equation of hyperbolic type with drift term on a Riemannian manifold is considered. For proving its well-posedness, transmutator and commutator relations are needed, which are studied in a general functional analytic setting.

1 Introduction

In isotropic and homogeneous media linear acoustic waves are governed by a system combining Euler’s force equation (momentum balance)

\[ D_t v + \text{grad } p = 0 \]  

with the continuity equation (mass balance)

\[ D_t p + \text{div } v = f, \]  

where we have for simplicity reduced all parameters by rescaling (in effect having in particular 1 as the speed of sound). Here \( p \) denotes the pressure and \( v \) the velocity field associates with the acoustic wave, \( f \) is a given source term. In a rest frame situation \( D_t = \partial_t \), the partial derivative with respect to time. In moving media, however, we have – by a suitable rotation of coordinates in \( \mathbb{R}^3 \) – \( D_t = \partial_t + v_0 \partial_3 \), with \( v_0 \in \mathbb{R} \) denoting the velocity of the drift with direction \( e_3 = (0,0,1) \) of the underlying media, the so-called convective, substantial or material derivative. Assuming with much loss of generality an irrotational velocity field we can reduce this to a bi-isotropic, homogeneous, but otherwise standard acoustic wave system by introducing

\[
\begin{pmatrix} \tilde{p} \\ \tilde{v} \end{pmatrix} := \begin{pmatrix} 1 & v_0 e_3^\top \\ v_0 e_3 & 1 \end{pmatrix} \begin{pmatrix} p \\ v \end{pmatrix}
\]

as new unknowns yielding the block system

\[
\begin{pmatrix} \partial_0 M_0 + \begin{pmatrix} 0 & \text{div} \\ \text{grad} & 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \tilde{p} \\ \tilde{v} \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}
\]

with

\[
M_0 = \begin{pmatrix} \frac{1}{1-v_0^2} e_3 & -\frac{v_0}{1-v_0^2} e_3^\top \\ -\frac{v_0}{1-v_0^2} e_3 & \frac{1}{1-v_0^2} \end{pmatrix}.
\]

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‡This is mimicking the approach of constructing the Maxwell-Hertz-Cohn system of “pre-relativity” electrodynamics in moving media.

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Since for well-posedness of this system we should have that $M_0$ is strictly positive definite, we obtain

$$v_0^2 < 1,$$

\[ (3) \]

i.e. Mach number less than 1, as a reasonable constraint. This constraint also recurs in the perspective of a second order approach. Indeed, eliminating\(^2\) the velocity field $v$ from the system (1), (2) yields

$$\partial_0^2 p + 2v_0 \partial_0 \partial_3 p - (\partial_1^2 + \partial_2^2 + (1 - v_0^2) \partial_3^2) p = \partial_0 f$$

and requiring ellipticity of the second order spatial operator $(\partial_1^2 + \partial_2^2 + (1 - v_0^2) \partial_3^2)$ imposes again (3). In contrast, looking at the original system as a standard evolution equation in $L^2 (\mathbb{R}^3, \mathbb{R}^4)$ we see that

\[
\begin{pmatrix}
\partial_0 + \left( \begin{array}{cccc}
v_0 \partial_3 & \partial_1 & \partial_2 & \partial_3 \\
\partial_1 & v_0 \partial_3 & 0 & 0 \\
\partial_2 & 0 & v_0 \partial_3 & 0 \\
\partial_3 & 0 & 0 & v_0 \partial_3 \\
\end{array} \right) & \begin{pmatrix} p \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} f \\ 0 \\ 0 \\ 0 \end{pmatrix}
\end{pmatrix}
\]

(4)

results in a well-posed system for arbitrary $v_0 \in \mathbb{R}$, since the spatial operator is essentially skew-selfadjoint by the function calculus of the commuting skew-selfadjoint partial derivatives provided by the spatial Fourier transform. This exposes the constraint (3) as a mathematical artefact and strongly suggests a direct approach via (4) and for the general anisotropic, inhomogeneous case via Friedrichs type systems. This has been successfully done in [6] in the framework of [10] for the Euclidean case with sufficiently smooth boundary and material properties. The purpose of this paper is to explore the Friedrichs type approach from a more functional analytical perspective with the aim of including in particular non-smooth boundaries. Indeed, no boundary regularity is needed, which is a benefit of the carefully constructed functional analytical setting. In particular, the common-place use of boundary traces is avoided. To include a variety of geometries we found it also helpful to give the discussion a more differential geometric flavor by generalising the discussion to a differential form setting on Riemannian manifolds. This allows to conveniently handle coordinate transformations since the differential form calculus on manifolds provides a machinery for this. As a benefit we also cover with our approach wave propagation for example on surfaces (with or without boundary). Since there is no added mathematical difficulty we include as a by-product forms of arbitrary degree, thus addressing for example also Maxwell’s equations in a unified setting. More precisely, we consider equations of the form

$$\left( \partial_0 M_0 + \alpha \begin{pmatrix} \nabla X_0 & 0 \\ 0 & \nabla X_0' \end{pmatrix} \right) M_0 + M_1 + \begin{pmatrix} 0 & -d^* \\ d & 0 \end{pmatrix} U = F$$

(5)

on a non-empty open subset of a Riemannian manifold. Here, $d$ denotes the exterior derivative and $\nabla X_0$, the covariant derivative in direction of a suitable vector field $X_0$. The term $\begin{pmatrix} 0 & -d^* \\ d & 0 \end{pmatrix}$ generalises the operator $\begin{pmatrix} 0 & \text{div} \\ \text{grad} & 0 \end{pmatrix}$ in the euclidean situation and $\alpha \begin{pmatrix} \nabla X_0 & 0 \\ 0 & \nabla X_0' \end{pmatrix}$ is the generalisation of the concrete drift term in direction $e_3$ from above. The operators $M_0$ and $M_1$ are assumed to be bounded and incorporate material parameters such as mass density, conductivity, etc. in applications.

The main goal is to prove the well-posedness of problem (5) in a suitable sense. The main idea to tackle this problem is to replace the drift operator by another drift operator, which is given in terms of the Lie-derivative of $X_0$, to decompose this new drift operator in its skew-selfadjoint and

\[^{2}\text{Eliminating instead the pressure $p$ yields the Galbrun equation.} \]

\[^{3}\text{The latter is of course a mere historical comment, since Maxwell’s equations in moving media are properly considered in the frame work of relativity theory, which essentially removes the difficulty of having to deal with a separate drift term.} \]
selfadjoint part and to show that the sum of the skew-selfadjoint part and the spatial operator 
\begin{pmatrix} 0 & -d^* \\ d & 0 \end{pmatrix}
is essentially skew-selfadjoint. Thus, the problem can be rewritten as an equation of the form
\[
(\partial_0 M_0 + \tilde{M}_1 + A) U = F,
\]
where \(A\) is skew-selfadjoint, \(M_0\) and \(\tilde{M}_1\) are bounded, such that \(M_0\) is selfadjoint and strictly positive definite. Then it is easy to see, that the problem is well-posed by invoking the theory of 
\(C_0\)-semigroups (note that \(-\sqrt{M_0^{-1}(\tilde{M}_1 + A)}\sqrt{M_0^{-1}}\) generates a \(C_0\)-semigroup) or by the theory of evolutionary equations, which will be introduced in the next section. The crucial part in the proof is to show the essential skew-selfadjointness mentioned above. For doing so, commutator, or more generally, transmutator relations between different differentiation operators are needed. Hence, we will provide some abstract results on commutators and transmutators of operators in Hilbert spaces in Section 3, which may be useful also for other applications. Finally, in Section 4 we deal with problem (5). First we introduce all differential operators on Riemannian manifolds needed in the forthcoming subsections. Then in Subsection 4.2 we prove our main result Theorem 4.13 showing the well-posedness of (5) in suitable sense under certain restrictions on the vector field \(X_0\). Moreover, we show that in cylindrical domains \(\Sigma \times \mathbb{R}\) or \(\Sigma \times [1/2, 1/2]\) for \(X_0 = e_3\) the assumptions on the vector field are satisfied, so that our solution theory applies in these cases. Moreover, we comment on how the solution theory carries over to isometrically transformed manifolds, allowing to deal with deformed pipes, etc. In the last subsection we provide an abstract localisation technique, which allows to "glue together" different different open subsets \(\Omega_1, \Omega_2\) of a manifold, so that the solution theory on each part carries over to their union \(\Omega_1 \cup \Omega_2\).

2 Some Hilbert Space Solution Theory

We recall the basic Hilbert space setting for dealing with evolutionary equations; that is, differential equations of the form
\[
(\partial_0 M_0 + M_1 + A) U = F.
\]
The results are based on the observations made in [7] (see also [8, Chapter 6] and [11]). Throughout, let \(H\) be a real Hilbert space.

Definition. For \(\rho \geq 0\) we define the space
\[
L_{2,\rho}(\mathbb{R}; H) := \{ f : \mathbb{R} \to H : f \text{ measurable}, \int_\mathbb{R} \|f(t)\|^2 e^{-2\rho t} dt \}
\]
equipped with the obvious inner product. Moreover, we define the weighted Sobolev space
\[
H^1_{0,\rho}(\mathbb{R}; H) := \{ f \in L_{2,\rho}(\mathbb{R}; H) : f' \in L_{2,\rho}(\mathbb{R}; H) \},
\]
where \(f'\) is meant in the sense of distributions, and the operator
\[
\partial_0, \rho : H^1_{0,\rho}(\mathbb{R}; H) \subseteq L_{2,\rho}(\mathbb{R}; H) \to L_{2,\rho}(\mathbb{R}; H), \quad f \mapsto f'.
\]

Proposition 2.1. The operator \(\partial_0, \rho\) is normal with \(\text{sym} \partial_0, \rho = \frac{1}{2} (\partial_0, \rho + \partial_0, \rho) = \rho\). Moreover, \(\partial_0, \rho\) is invertible if and only if \(\rho > 0\) and in this case
\[
(\partial_0, \rho^{-1} f)(t) = \int_{-\infty}^t f(s) ds \quad (t \in \mathbb{R}, f \in L_{2,\rho}(\mathbb{R}; H)).
\]
If the choice of \(\rho\) is clear from the context, we drop the additional index and just write \(\partial_0\).
Theorem 2.2 ([7, Solution Theory]). Let \( M_0, M_1 \in L(H) \) with \( M_0 = M_0^* \) and \( A : \text{dom}(A) \subseteq H \to H \) a skew-selfadjoint operator. We extend all these operators to \( L_{2,\rho}(\mathbb{R}; H) \) in the canonical way. Moreover, we assume that there exists \( \rho_0 \geq 0 \) and \( c > 0 \) such that
\[
(\rho M_0 + \frac{1}{2}(M_1 + M_1^*))x, x \geq c\|x\|^2 \quad (x \in H)
\]
for all \( \rho \geq \rho_0 \). Then \( \partial_0 M_0 + M_1 + A \) is closable in \( L_{2,\rho}(\mathbb{R}; H) \) for each \( \rho \geq \rho_0 \) and the closure is continuously invertible with
\[
\| (\partial_0 M_0 + M_1 + A)^{-1} \|_{L(L_{2,\rho}(\mathbb{R}; H))} \leq \frac{1}{c}.
\]
Moreover, the operator \( S_\rho := (\partial_0 M_0 + M_1 + A)^{-1} \) is causal; i.e., for all \( f \in L_{2,\rho}(\mathbb{R}; H) \) such that \( \text{spt} f \subseteq [a, \infty) \) for some \( a \in \mathbb{R} \) it follows that \( \text{spt} S_\rho f \subseteq [a, \infty) \), and the operator \( S_\rho \) is independent of the choice of \( \rho \geq \rho_0 \) in the sense that \( S_\rho f = S_\mu f \) for each \( f \in L_{2,\rho}(\mathbb{R}; H) \cap L_{2,\mu}(\mathbb{R}; H) \) and \( \mu, \rho \geq \rho_0 \).

Remark 2.3.

(a) The latter theorem shows that the problem of finding \( u \in L_{2,\rho}(\mathbb{R}; H) \) such that
\[
(\partial_0 M_0 + M_1 + A)u = f
\]
for some \( f \in L_{2,\rho}(\mathbb{R}; H) \) is well-posed in the sense of Hadamard. Indeed, the bijectivity of \( (\partial_0 M_0 + M_1 + A) \) yields the existence and uniqueness of a solution for each \( f \in L_{2,\rho}(\mathbb{R}; H) \), while the continuity of the inverse shows the continuous dependence of the solution \( u \) on the data \( f \). Moreover, the causality shows that the equation models a physically reasonable process in time.

(b) The latter theorem is just a special case of [7, Solution Theory], where equations of the form
\[
(\partial_0 M(\partial_0^{-1}) + A)u = f
\]
for a suitable operator-valued function \( M \) of \( \partial_0^{-1} \) are considered. The special case in Theorem 2.2 corresponds to the choice \( M(\partial_0^{-1}) = M_0 + \partial_0^{-1} M_1 \). Moreover, several generalisations of Theorem 2.2 can be found in the literature, for instance allowing to treat non-autonomous problems ([9, 14]) or non-linear problems ([12, 13]).

As an immediate consequence of Theorem 2.2, we obtain the following perturbation result.

Corollary 2.4. Let \( M_0, M_1 \in L(H) \) with \( M_0 = M_0^* \) and \( A : \text{dom}(A) \subseteq H \to H \) a skew-selfadjoint operator. Moreover, assume that there exists \( c > 0 \) such that
\[
(M_0 x, x) \geq c\|x\|^2 \quad (x \in H).
\]
Then there exists \( \rho_0 \geq 0 \) such that for all \( \rho \geq \rho_0 \) the operator \( \partial_0 M_0 + M_1 + A \) is closable and continuously invertible in \( L_{2,\rho}(\mathbb{R}; H) \).

Proof. We choose \( \rho_0 \) such that \( \tilde{c} := \rho_0 c - \|M_1\| > 0 \). Then for \( \rho \geq \rho_0 \) we have that
\[
(\rho M_0 + \frac{1}{2}(M_1 + M_1^*))x, x \geq \tilde{c}\|x\|^2
\]
and hence, the claim follows from Theorem 2.2.  \( \square \)
3 Sums of Skew-Selfadjoint Operators: Weak=Strong

There is a well-developed general theory of sums of discontinuous operators in Banach spaces, see [2]. For sake of simplicity and transparency, however, we chose instead a more “pedestrian” approach fitted to the Hilbert space setting and in keeping with Friedrichs original approach to positive symmetric systems, [4], in which the classical question of the relation between weak and strong extensions are of significance, [3].

3.1 Transmutators and Commutators

We begin with defining the notions of transmutators and commutators for operators on Hilbert spaces. For doing so, let $H_0, H_1$ be Hilbert spaces.

**Definition.** Let $L : H_1 \to H_1$ and $R : H_0 \to H_0$ be continuous linear operators and $C : \text{dom}(C) \subseteq H_0 \to H_1$ a densely defined closed linear operator such that $R[\text{dom}(C)] \subseteq \text{dom}(C)$. We define

$$[L, C, R] := LC - CR : \text{dom}(C) \subseteq H_0 \to H_1$$

the transmutator of $L, R$ and $C$. Moreover, if $H_0 = H_1$ we set

$$[R, C] := [R, C, R]$$

the commutator of $R$ and $C$ and for convenience

$$[C, R] := -[R, C].$$

**Lemma 3.1.** Let $L : H_1 \to H_1$ and $R : H_0 \to H_0$ be continuous linear operators and $C : \text{dom}(C) \subseteq H_0 \to H_1$ a densely defined closed linear operator such that $R[\text{dom}(C)] \subseteq \text{dom}(C)$. Assume that $[L, C, R]$ is continuous.

(a) Then $LC$ is closable with

$$\overline{LC} \subseteq CR + [L, C, R].$$

(b) If, additionally, $\text{dom}(C)$ is a core for $CR$, then

$$LC = CR + [L, C, R]$$

and hence, in particular

$$\text{dom}(LC) = \text{dom}(CR).$$

**Proof.** We note that $[L, C, R] : H_0 \to H_1$ is continuous, since $[L, C, R]$ is densely defined and continuous by assumption. Moreover, since $C$ is closed, so is $CR$. Since clearly

$$LC \subseteq CR + [L, C, R]$$

we infer (a) holds. For showing (b), assume now that $\text{dom}(C)$ is a core for $CR$ and let $x \in \text{dom}(CR)$. Then we find a sequence $(x_n)_{n \in \mathbb{N}}$ in $\text{dom}(C)$ with $x_n \to x$ and $CRx_n \to CRx$ as $n \to \infty$. Thus, we have

$$LCx_n = CRx_n + [L, C, R]x_n \to CRx + [L, C, R]x \quad (n \to \infty)$$

due to the continuity of $[L, C, R]$. The latter proves $x \in \text{dom}(LC)$, which shows the claim. □

**Definition.** Let $C : \text{dom}(C) \subseteq H_0 \to H_1$ and $D : \text{dom}(D) \subseteq H_0 \to H_1$ be densely defined closed linear operators. We call $C$ and $D$ *essentially equal*, if $\text{dom}(C) = \text{dom}(D)$ and $C - D$ is continuous.

**Remark 3.2.**

(a) If $C$ and $D$ are essentially equal, we infer that $C = D + C - D$.

(b) In the situation of Lemma 3.1 (b) we have that $\overline{LC}$ and $CR$ are essentially equal.
3.2 Commutators with resolvents of $m$-accretive Operators and Convergence results.

We now focus on the case $H := H_0 = H_1$ and $L = R$. A class of operators for which we will apply Lemma 3.1 is the class of $m$-accretive operators. For doing so, we recall the definition of these operators.

**Definition.** Let $C : \text{dom}(C) \subseteq H \to H$ be a linear densely defined closed operator. Then $C$ is called accretive, if
\[ \forall x \in \text{dom}(C) : \langle x, Cx \rangle \geq 0. \]
Moreover, $C$ is called $m$-accretive, if $C$ is accretive and $1 + C$ is onto.

**Remark 3.3.**
(a) We remark that a linear operator $C : \text{dom}(C) \subseteq H \to H$ is $m$-accretive if and only if $(1 + \eta C)^{-1} : H \to H$ is continuous for all $\eta \geq 0$ and
\[ \| (1 + \eta C)^{-1} \| \leq 1. \]
In particular, the closedness and the dense domain of $C$ follow from this uniform bound of the resolvents.
(b) From (a) we see that $C^\ast$ is $m$-accretive if $C$ is $m$-accretive.

**Definition.** Let $C : \text{dom}(C) \subseteq H \to H$ be a linear densely defined closed operator. We call $C$ quasi-$m$-accretive, if there exists an $\eta_0 > 0$ such that $(1 + \eta C)^{-1} \in L(H)$ for each $0 \leq \eta \leq \eta_0$ and
\[ \sup_{0 \leq \eta \leq \eta_0} \| (1 + \eta C)^{-1} \| < \infty. \]

**Lemma 3.4.** If $C$ is quasi-$m$-accretive, then $(1 + \eta C)^{-1} \to 1$ strongly as $\eta \to 0+$.

**Proof.** Since the resolvents are uniformly bounded near zero, it suffices to prove the convergence for elements in a dense set, say $\text{dom}(C)$. For $x \in \text{dom}(C)$ we compute
\[ (1 + \eta C)^{-1} x - x = -\eta (1 + \eta C)^{-1} (Cx) \to 0 \quad (\eta \to 0+), \]
where we again have used the uniform boundedness of the resolvents near zero. \hfill $\Box$

**Lemma 3.5.** Let $C : \text{dom}(C) \subseteq H \to H$ be linear densely defined and closed. Moreover, let $B \in L(H)$ such that $C - B$ is $m$-accretive. Then $C$ is quasi-$m$-accretive.

**Proof.** We choose $\eta_0 > 0$ such that $\eta_0 \| B \| < 1$. Then we estimate for each $0 \leq \eta \leq \eta_0$
\[ \langle (1 + \eta C)x, x \rangle = \| x \|^2 + \eta \langle (C - B)x, x \rangle + \eta \langle Bx, x \rangle \]
\[ \geq \| x \|^2 - \eta \| B \| \| x \|^2 \]
\[ \geq (1 - \eta_0 \| B \|) \| x \|^2, \]
which shows that $(1 + \eta C)$ is injective. For showing that $1 + \eta C$ is onto, we take $y \in H$. By the contraction mapping theorem, there exists $x \in H$ such that
\[ x = (1 + \eta(C - B))^{-1} (y - \eta Bx) \]
and it is immediate, that this $x$ satisfies
\[ (1 + \eta C)x = y. \]

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4Equivalently, if $C$ and $C^\ast$ are accretive.
Finally, the estimate above shows that
\[
\sup_{0 \leq \eta \leq \eta_0} \| (1 + \eta C)^{-1} \| \leq \frac{1}{1 - \eta_0 \| \mathcal{B} \|}
\]
and hence, the assertion follows.

Lemma 3.6. Let \( C : \text{dom}(C) \subseteq H \to H \) be quasi-\( m \)-accretive and \( \alpha : H \to H \) continuous with \( \alpha[\text{dom}(C)] \subseteq \text{dom}(C) \). Moreover, we assume that \( [\alpha, C] \) is continuous. Then
\[
[(1 + \eta C)^{-1}, \alpha] = \eta (1 + \eta C)^{-1} [\alpha, C](1 + \eta C)^{-1}
\]
for sufficiently small \( \eta \geq 0 \). In particular
\[
[(1 + \eta C)^{-1}, \alpha] \to 0 \quad (\eta \to 0+)
\]
in operator norm.

Proof. Since
\[
\alpha C \subseteq C\alpha + [\alpha, C]
\]
we infer that
\[
\alpha(1 + \eta C) \subseteq (1 + \eta C)\alpha + \eta [\alpha, C] \quad (\eta \geq 0).
\]
Hence,
\[
(1 + \eta C)^{-1} \alpha \subseteq \alpha(1 + \eta C)^{-1} + \eta (1 + \eta C)^{-1}[\alpha, C](1 + \eta C)^{-1}
\]
for a \( \eta \geq 0 \) sufficiently small. Since both sides are continuous operators on \( H \), we derive that
\[
[(1 + \eta C)^{-1}, \alpha] = (1 + \eta C)^{-1} \alpha - \alpha(1 + \eta C)^{-1} = \eta (1 + \eta C)^{-1} [\alpha, C](1 + \eta C)^{-1}.
\]
Finally, since \( (1 + \eta C)^{-1} \) is uniformly bounded in \( \eta \), we infer the asserted convergence result.

We can prove an even stronger convergence result, than the one in the previous lemma.

Proposition 3.7. Let \( C : \text{dom}(C) \subseteq H \to H \) be quasi-\( m \)-accretive and \( \alpha : H \to H \) continuous with \( \alpha[\text{dom}(C)] \subseteq \text{dom}(C) \). Moreover, we assume that \( [\alpha, C] \) is continuous. Then \( C[(1 + \eta C)^{-1}, \alpha] : H \to H \) is continuous and
\[
C[(1 + \eta C)^{-1}, \alpha] \to 0
\]
strongly as \( \eta \to 0+ \).

Proof. By Lemma 3.6 we have that
\[
[(1 + \eta C)^{-1}, \alpha] = \eta (1 + \eta C)^{-1} [\alpha, C](1 + \eta C)^{-1}
\]
and thus
\[
C[(1 + \eta C)^{-1}, \alpha] = \eta C(1 + \eta C)^{-1} [\alpha, C](1 + \eta C)^{-1}
\]
\[
= [\alpha, C](1 + \eta C)^{-1} - (1 + \eta C)^{-1} [\alpha, C](1 + \eta C)^{-1}
\]
for each sufficiently small \( \eta \geq 0 \). The latter expression yields the claim as \( (1 + \eta C)^{-1} \to 1 \) strongly as \( \eta \to 0+ \) by Lemma 3.5.

With these preparations at hand, we can state our main theorem of this subsection.

Theorem 3.8. Let \( C : \text{dom}(C) \subseteq H \to H \) be quasi-\( m \)-accretive and \( \alpha : H \to H \) continuous with \( \alpha[\text{dom}(C)] \subseteq \text{dom}(C) \). Moreover, we assume that \( [\alpha, C] \) is continuous. Then
\[
\overline{\alpha C} = C\alpha + [\alpha, C].
\]
Proof. By Lemma 3.1 it suffices to prove that dom(C) is a core for Cα. For doing so, let \( x \in \text{dom}(C\alpha) \) and define
\[
x_n := \left(1 + \frac{1}{n}C\right)^{-1} x \quad (n \in \mathbb{N} \text{ sufficiently large}).
\]
Then \( x_n \in \text{dom}(C) \) for each \( n \in \mathbb{N} \) and \( x_n \to x \) as \( n \to \infty \) by Lemma 3.3. Moreover,
\[
C\alpha x_n = C\alpha \left(1 + \frac{1}{n}C\right)^{-1} x
= C \left(1 + \frac{1}{n}C\right)^{-1} \alpha x - C \left(1 + \frac{1}{n}C\right)^{-1} \frac{1}{n} \alpha x
= \left(1 + \frac{1}{n}C\right)^{-1} C\alpha x - C \left(1 + \frac{1}{n}C\right)^{-1} \frac{1}{n} \alpha x
\to C\alpha x \quad (n \to \infty)
\]
by Proposition 3.6, which yields the claim. \( \square \)

Remark 3.9. The latter theorem can be interpreted as a “weak=strong” result in the following sense:
\( \triangleright \) \((C^*)^* = C^{**} = \overline{C} \) for a closable densely defined linear operator \( C \) (“weak=strong” for operators). This yields in particular \( \overline{\alpha C} = (C^*\alpha^*)^* \).
\( \triangleright \) With our commutator results we get \( \overline{\alpha C} = C\alpha + [\alpha, C] = (\alpha^*C^*)^* + [\alpha, C] \) showing that \( \overline{\alpha C} \) and \( (\alpha^*C^*)^* \) are essentially equal (“weak=strong” for operator products).

We conclude this subsection by the following corollary.

Corollary 3.10. Let \( C : \text{dom}(C) \subseteq H \to H \) linear closed and densely defined, and \( \alpha : H \to H \) linear and continuous. Assume that \( \alpha[\text{dom}(C)] \subseteq \text{dom}(C) \) and \( [\alpha, C] \) is continuous. Then
\[
[\alpha, C]^* = [C^*, \alpha^*].
\]
If, additionally, \( C \) is quasi-m-accretive, we have that
\[
(\alpha C)^* = \alpha^*C^* + [C^*, \alpha^*].
\]
Proof. Since we have \( \alpha^*C^* \subseteq (C\alpha)^* \), it follows that
\[
\alpha^*C^* + [\alpha, C]^* \subseteq (C\alpha)^* + [\alpha, C]^* \subseteq (C\alpha + [\alpha, C])^* = (\alpha C)^* = C^*\alpha^*,
\]
which proves that
\[
[\alpha, C]^* x = [C^*, \alpha^*] x \quad (x \in \text{dom}(C^*)).
\]
Since \( \text{dom}(C^*) \) is dense and \( [\alpha, C]^* \) is bounded, we derive
\[
[\alpha, C]^* = [C^*, \alpha^*].
\]
For the second statement, we recall that
\[
\overline{\alpha C} = C\alpha + [\alpha, C]
\]
by Theorem 3.8. Taking adjoints on both sides and using the result above, we obtain
\[
(\alpha C)^* = (C\alpha)^* + [C^*, \alpha^*].
\]
Since
\[
\overline{\alpha^*C^*} = (\alpha^*C^*)^{**} = (C\alpha)^*,
\]
the assertion follows. \( \square \)
3.3 Commutators for Quasi-Skew-Selfadjoint Operators

Throughout, let $H$ be a Hilbert space, $\alpha \in L(H)$ and $C : \text{dom}(C) \subseteq H \rightarrow H$ linear closed and densely defined.

**Definition.** We call $C$ quasi-skew-selfadjoint, if $\text{dom}(C) = \text{dom}(C^*)$ and $\text{sym} C := \frac{1}{2}(C + C^*)$ is bounded.

**Remark 3.11.**
(a) If $C$ is skew-selfadjoint, then $C$ is quasi-skew-selfadjoint, since in this case $\text{sym} C = 0$.
(b) If $C$ is quasi-skew-selfadjoint, we have

$$C = -C^* + 2 \text{sym} C$$

and thus,

$$C - \text{sym} C = -(C^* - \text{sym} C)$$

$$= -(C - \text{sym} C)^*,$$

since $\text{sym} C$ is selfadjoint and bounded, and hence, $C - \text{sym} C$ is skew-selfadjoint and thus, $C$ is quasi-$m$-accretive by Lemma 3.5 (note that skew-selfadjoint operators are $m$-accretive).

**Theorem 3.12.** Let $C$ be quasi-skew-selfadjoint and $\alpha \in L(H)$ selfadjoint. Moreover, assume that $\alpha[\text{dom}(C)] \subseteq \text{dom}(C)$ and $[\alpha, C]$ is continuous. Then

$$\text{skew}(\alpha C) := \frac{1}{2} \alpha C - (\alpha C)^*$$

is skew-selfadjoint.

**Proof.** Using Lemma 3.10 (note that $C$ is quasi-$m$-accretive) we compute for $x \in \text{dom}(C) = \text{dom}(C^*)$

$$\frac{1}{2} (\alpha C - (\alpha C)^*) x = \frac{1}{2} (\alpha C x - \alpha C^* x - [C^*, \alpha] x)$$

$$= \alpha C x - \frac{1}{2} \alpha (C + C^*) x - \frac{1}{2} [C^*, \alpha] x$$

$$= \alpha C x - \alpha \text{sym}(C) x - \frac{1}{2} [C^*, \alpha] x,$$

which yields

$$\text{skew}(\alpha C) = \overline{\alpha C} - \alpha \text{sym}(C) - \frac{1}{2} [C^*, \alpha]. \quad (6)$$

Thus,

$$\text{skew}(\alpha C)^* = (\alpha C)^* - \text{sym}(C) \alpha - \frac{1}{2} [C^*, \alpha]^*$$

$$= (C^* - \text{sym}(C)) \alpha - \frac{1}{2} [\alpha, C]$$

again by Lemma 3.10. Since $C$ is quasi-skew-selfadjoint, we obtain

$$C^* - \text{sym}(C) = - (C - \text{sym}(C))$$

and hence,

$$\text{skew}(\alpha C)^* = - \left( (C - \text{sym}(C)) \alpha + \frac{1}{2} [\alpha, C] \right).$$
Hence, it suffices to prove that

\[(C - \text{sym}(C))\alpha + \frac{1}{2}[\alpha, C] = \text{skew}(\alpha C)\].

By Theorem 3.8 we get that

\[C\alpha + [\alpha, C] = \alpha C\].

Hence, using formula (6) we need to show that

\[-\text{sym}(C)\alpha - \frac{1}{2}[\alpha, C] = -\alpha \text{sym}(C) - \frac{1}{2}[C^*, \alpha].\]

Since the operators on both sides are all bounded, it suffices to check this equality on a dense set, say dom(C). However, on dom(C) we have

\[
\text{sym}(C)\alpha + \frac{1}{2}[\alpha, C] = \frac{1}{2}(C\alpha + C^*\alpha) + \frac{1}{2}(\alpha C - Ca) \\
= \frac{1}{2}C + \frac{1}{2}C^*\alpha \\
= \alpha \text{sym}(C) + \frac{1}{2}(C^*\alpha - \alpha C^*) \\
= \alpha \text{sym}(C) + \frac{1}{2}[C^*, \alpha],
\]

which shows the claim. \(\square\)

### 3.4 Transmutators and Sums of Operators

**Lemma 3.13.** Let \(C : \text{dom}(C) \subseteq H_0 \rightarrow H_1\) and \(D : \text{dom}(D) \subseteq H_0 \rightarrow H_1\) two densely defined closed linear operators such that \(\text{dom}(C) \cap \text{dom}(D)\) is dense in \(H_0\). Let \(R \in L(H_0), L \in L(H_1)\) with \(R[\text{dom}(C) \cap \text{dom}(D)] \subseteq \text{dom}(C) \cap \text{dom}(D)\) and \([L, C + D, R]\) is continuous. Then \(L^*[\text{dom}((C + D)^*)] \subseteq \text{dom}((C + D)^*)\) and

\[
[R^*, (C + D)^*, L^*] \subseteq -[L, C + D, R]^*.
\]

Moreover, if \(\text{dom}((C + D)^*)\) is dense, we obtain

\[
[R^*, (C + D)^*, L^*] = -[L, C + D, R]^*.
\]

**Proof.** Let \(x \in \text{dom}((C + D)^*)\) and \(y \in \text{dom}(C + D) = \text{dom}(C) \cap \text{dom}(D)\). Then we compute

\[
\langle (C + D)y, L^*x \rangle = \langle L(C + D)y, x \rangle \\
= \langle [L, C + D, R]y + (C + D)Ry, x \rangle \\
= \langle y, [L, C + D, R]^*x + R^*(C + D)^*x \rangle,
\]

which shows \(L^*x \in \text{dom}((C + D)^*)\) and

\[(C + D)^*L^*x = [L, C + D, R]^*x + R^*(C + D)^*x.\]

In other words

\[
[R^*, (C + D)^*, L^*] \subseteq -[L, C + D, R]^*.
\]

If additionally, \(\text{dom}((C + D)^*)\) is dense the continuity of the right-hand side yields

\[
[R^*, (C + D)^*, L^*] = -[L, C + D, R]^*.
\]

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Theorem 3.14. Let $C : \text{dom}(C) \subseteq H_0 \to H_1$ and $D : \text{dom}(D) \subseteq H_0 \to H_1$ two densely defined closed linear operators such that $\text{dom}(C) \cap \text{dom}(D)$ is dense in $H_0$. Moreover, let $L_n \in L(H_1), R_n \in L(H_0)$ such that $R_n[\text{dom}(C) \cap \text{dom}(D)] \subseteq \text{dom}(C) \cap \text{dom}(D)$ and

$$[L_n, C + D, R_n]$$

is continuous for each $n \in \mathbb{N}$. If $(C + D)^* L_n^* = (C^* + D^*) L_n^*$ for each $n \in \mathbb{N}$ and $R_n^* \overset{\mathcal{D}}{\to} 1$, $L_n^* \overset{\mathcal{D}}{\to} 1$ as well as

$$[L_n, C + D, R_n]^* \overset{\mathcal{D}}{\to} 0,$$

as $n \to \infty$, we obtain

$$C^* + D^* \subseteq (C + D)^*.$$

Proof. Since always $C^* + D^* \subseteq (C + D)^*$ it follows that $C^* + D^* \subseteq (C + D)^*$. To prove the remaining inclusion let $v \in \text{dom}((C + D)^*)$. Then $L_n^*v \to v$ as $n \to \infty$ and by Lemma 3.13 we obtain

$$(C^* + D^*) L_n^* v = (C + D)^* L_n^* v$$

$$= R_n^*(C + D)^* v - [R_n^*, (C + D)^*, L_n^*]v$$

$$= R_n^*(C + D)^* v + [L_n, (C + D), R_n]^* v$$

$$\to (C + D)^* v.$$

The latter gives $v \in \text{dom}(C^* + D^*)$ and thus, the assertion follows.

4 An Application to Maxwell’s and Acoustics Equations with Drift Term

4.1 Lie Derivative and Co-Variant Derivative

Throughout, let $(M, g)$ be a smooth Riemannian manifold of odd dimension $n$ and $\Omega \subseteq M$ open. We begin to define the space of tensor fields on $\Omega$.

**Definition.** Let $k, \ell \in \mathbb{N}$.

(a) Let $p \in \Omega$. By $T^k_\ell(T_p M)$ we denote the set of all tensors of type $(k, \ell)$ on the tangent space $T_p M$, i.e.

$$T^k_\ell(T_p M) = \bigotimes_{j=1}^{k} (T_p M) \otimes \bigotimes_{i=1}^{\ell} T_p M.$$

(b) A function $X : \Omega \to \bigcup_{p \in \Omega} T_p M$ with $X(p) \in T_p M$ for each $p \in \Omega$ is called a vectorfield on $\Omega$.

(c) A function $Y : \Omega \to \bigcup_{p \in \Omega} (T_p M)'$ with $Y(p) \in (T_p M)'$ for each $p \in \Omega$ is called a covectorfield on $\Omega$.

is smooth. We denote by $T^k_\ell(\Omega)$ the set of all vectorfields on $\Omega$.

$$\Omega \ni p \mapsto X(p)(f) \in \mathbb{R}$$

is smooth. We denote by $T^0_0(\Omega)$ the set of all covectorfields on $\Omega$.

$$\Omega \ni p \mapsto Y(p)(X(p)) \in \mathbb{R}$$
(d) A function $T : \Omega \to \bigcup_{p \in \Omega} T^k(T_p M)$ with $T(p) \in T^k(T_p M)$ for each $p \in \Omega$ is called a tensorfield of type $(k, \ell)$ on $\Omega$, if for all $X_1, \ldots, X_k \in T^0_0(\Omega)$ and all $Y_1, \ldots, Y_\ell \in T^0_0(\Omega)$ the mapping

$$\Omega \ni p \mapsto T(p)(X_1(p), \ldots, X_k(p), Y_1(p), \ldots, Y_\ell(p)) \in \mathbb{R}$$

is smooth. We denote by $T^k_\ell(\Omega)$ the set of all tensorfields of type $(k, \ell)$ on $\Omega$.

(e) A tensorfield $T$ of type $(k, 0)$ on $\Omega$ is called a $k$-form on $\Omega$, if for all $p \in \Omega$

$$T(p) \in \Lambda^k(T_p M),$$

where

$$\Lambda^k(T_p M) := \{ f \in T^k_0(T_p M) : f \text{ alternating} \}.$$

The set of $k$-forms on $\Omega$ is denoted by $\Lambda^k(\Omega)$.

Remark 4.1. Note that $\Lambda^k(\Omega) = \{ 0 \}$ for each $k > n$.

Since we want to employ the framework of evolutionary equations, see Section 2, we need to define a suitable Hilbert space structure. This is done as follows.

**Definition.** Let $k \in \mathbb{N}$. We define the set $L^k_2(\Omega)$ as the completion of

$$\{ T \in \Lambda^k(\Omega) : \int_\Omega \| T(p) \|^2_{\Lambda^k(T_p M)} dV(p) < \infty \}$$

with respect to the norm

$$T \mapsto \left( \int_\Omega \| T(p) \|^2_{\Lambda^k(T_p M)} dV(p) \right)^{1/2}.$$

Here $V$ denotes the volume element of the Riemannian manifold $M$.

Remark 4.2. For $k \in \mathbb{N}$ we have

$$L^k_2(\Omega) = \overline{\Lambda^k_c(\Omega)},$$

where

$$\Lambda^k_c(\Omega) := \{ T \in \Lambda^k(\Omega) : T \text{ compactly supported} \}.$$

We first inspect three different differentiation operators on $L^k_2(\Omega)$ and their relations, namely the Lie-derivative, the exterior derivative and the covariant derivative. First, we define the operators to be considered.

**Definition.** Let $X \in T^0_1(\Omega)$ a vector field, $k \in \mathbb{N}$.

(a) We define the operator $\mathcal{L}_{X,c}$ by

$$\mathcal{L}_{X,c} : \Lambda^k_c(\Omega) \subseteq L^k_2(\Omega) \to L^k_2(\Omega), \quad T \mapsto \mathcal{L}_X T,$$

where $\mathcal{L}_X$ denotes the Lie-derivative on $\Lambda^k(\Omega)$ in direction $X$.

(b) We define the operator $\nabla_{X,c}$ by

$$\nabla_{X,c} : \Lambda^k_c(\Omega) \subseteq L^k_2(\Omega) \to L^k_2(\Omega), \quad T \mapsto \nabla_X T,$$

where $\nabla_X$ denotes the covariant derivative on $\Lambda^k(\Omega)$ in direction $X$.

(c) We define the operator $d_c$ by

$$d_c : \Lambda^k_c(\Omega) \subseteq L^k_2(\Omega) \to L^{k+1}_2(\Omega), \quad T \mapsto dT,$$

where $d$ denotes the exterior derivative on $\Lambda^k(\Omega)$.
Proposition 4.3. Let $X \in T^0_1(\Omega)$. Then
\[- \ast L^*_X \subseteq L^*_X,\]
\[-\nabla_X \subseteq \nabla_X,\]
where $\ast : L^2_2(\Omega) \to L^{2-k}_2(\Omega)$ denotes the Hodge-star operator. Moreover
\[(-1)^{k+1} \ast d_c \subseteq d_c^c\]
as an operator from $L^{k+1}_2(\Omega) \to L^k_2(\Omega)$.

Proof. Let $S, T \in \Lambda^c(\Omega)$. Then we have, using the Hodge star operator $\ast : L^2_2(\Omega) \to L^{n-k}_2(\Omega),$
\[
\langle L^*_X S, T \rangle_{L^2_2(\Omega)} = \int_\Omega (L^*_X S) \wedge (\ast T) \, dV
= \int_\Omega L^*_X (S \wedge (\ast T)) - S \wedge (L^*_X \ast T) \, dV
= \langle S, -\ast L^*_X \ast T \rangle_{L^2_2(\Omega)},
\]
which proves
\[- \ast L^*_X \subseteq L^*_X.\]
Here we have used Cartan’s formula and Stokes’ Theorem to compute
\[
\int_\Omega L^*_X (S \wedge (\ast T)) \, dV = \int_\Omega (d\iota_X + \iota_X d)(S \wedge (\ast T)) \, dV
= \int_\Omega d\iota_X (S \wedge (\ast T)) \, dV
= 0,
\]
since $\iota_X (S \wedge (\ast T)) \in \Lambda^{n-1}_c(\Omega)$, where $\iota_X$ denotes the interior derivative in direction $X$. Similarly, we compute
\[
\langle \nabla_X S, T \rangle_{L^2_2(\Omega)} = \int_\Omega \nabla_X \langle S, T \rangle = \int_\Omega \langle S, \nabla_X T \rangle \, dV
= \int_\Omega \langle L^*_X S \wedge (\ast T) \rangle \, dV - \langle S, \nabla_X T \rangle_{L^2_2(\Omega)}
= \langle S, -\nabla_X T \rangle_{L^2_2(\Omega)},
\]
i.e.
\[\nabla_X \subseteq \nabla_X.\]
Finally, we compute for $S \in \Lambda^c(\Omega)$ and $T \in \Lambda^{k+1}_c(\Omega)$
\[
\langle d_c S, T \rangle_{L^{k+1}_2(\Omega)} = \int_\Omega (dS) \wedge (\ast T) \, dV
= \int_\Omega dS(\wedge (\ast T)) \, dV - (-1)^k \int_\Omega \langle S, d(\ast T) \rangle \, dV
= \langle S, (-1)^{k+1} \ast d \ast T \rangle_{L^2_2(\Omega)}.
\]

The latter proposition shows that each of the operators $L^*_X, \nabla_X$ and $d_c$ is closable and hence, we may define the following operators.
Definition. Let \( X \in T^0_1(M) \). We set
\[
\hat{\mathcal{L}}_X := \mathcal{L}_{X,c}, \quad \mathcal{L}_X := - \mathcal{L}_{X,c}^*, \\
\hat{\nabla}_X := \nabla_{X,c}, \quad \nabla_X := - \nabla_X^*
\]
and
\[
d := d_c, \quad d := (-1)^{n-k} \circ d_c^*
\]
as operators from \( L^k_2(\Omega) \) to \( L^{k+1}_2(\Omega) \) for \( k \in \mathbb{N} \cap \mathbb{N}^* \).

Remark 4.4. By Proposition 4.3 we have
\[
\hat{\mathcal{L}}_X \subseteq \mathcal{L}_X, \quad \hat{\nabla}_X \subseteq \nabla_X, \quad \hat{d} \subseteq d.
\]

Lemma 4.5. Let \( X \in T^0_1(M) \) and assume that
\[
C := \sup \{ \| \nabla_X^c \|_\infty ; Y \in T^0_1(\Omega), \| Y \|_\infty \leq 1 \} < \infty.
\]
Then \( \mathcal{L}_X - \nabla_X \) is continuous.

Proof. By density, it suffices to prove that \( \mathcal{L}_{X,c} - \nabla_{X,c} \) is continuous on \( \Lambda^k_c(\Omega) \). Moreover, by induction it suffices to show the assertion for \( k = 0 \) and \( k = 1 \). Since \( \mathcal{L}_{X,c} \) and \( \nabla_{X,c} \) agree on \( \Lambda^0_c(\Omega) = \mathcal{C}^\infty_c(\Omega) \), there is nothing to show for the case \( k = 0 \). So, let \( \alpha \in \Lambda^1_c(\Omega) \). We then have for all \( Y \in T^0_1(\Omega) \) with compact support
\[
((\mathcal{L}_{X,c} - \nabla_{X,c})(\alpha) - (\nabla_{X,c} \alpha))(Y) = \mathcal{L}_X(\alpha(Y)) - \alpha(\mathcal{L}_X Y) - \nabla_X(\alpha(Y)) + \alpha(\nabla_X Y)
\]
\[
= \alpha(\nabla_X Y - \mathcal{L}_X Y)
\]
where we have used that \( \nabla \) is torsion-free. Hence,
\[
\|(\mathcal{L}_{X,c} - \nabla_{X,c})(\alpha)\|_{L^2(\Omega)}^2 = \int_\Omega \|(\mathcal{L}_{X,c} - \nabla_{X,c})(\alpha)(p)\|_{L^2(T^c_p M)}^2 dV(p)
\]
\[
= \int_\Omega \sup_{Z \in T^c_p M, \| Z \| \leq 1} |((\mathcal{L}_{X,c} - \nabla_{X,c})(\alpha)(p), Z)|^2 dV(p)
\]
\[
= \sup_{Y \in T^0_1(\Omega), \| Y \|_\infty \leq 1} \int_\Omega |\alpha(\nabla_X Y)(p)|^2 dV(p)
\]
\[
\leq \sup_{Y \in T^0_1(\Omega), \| Y \|_\infty \leq 1} \int_\Omega \| \alpha(p) \|^2 \| \nabla_X Y \|_\infty^2 dV(p)
\]
\[
= C^2 \| \alpha \|_{L^4_2(\Omega)}^2,
\]
which shows the claim. \( \square \)

Lemma 4.6. Let \( X \in T^0_1(M) \). It is
\[
d \mathcal{L}_X = \mathcal{L}_X d
\]
on \( \Lambda^k(\Omega) \).

Proof. We apply Cartan’s magic formula stating that
\[
\mathcal{L}_X = d_i X + i X d
\]
on $\Lambda^k(\Omega)$. Thus,
\[
d\mathcal{L}_X = dd_X + x_X d \\
= x_X d \\
= (d_x + c_x d) d \\
= \mathcal{L}_X d
\]
on $\Lambda^k(\Omega)$.

**Proposition 4.7.** Let $\mathcal{L}_X$ be quasi-skew-selfadjoint. Moreover, let $\eta \in \mathbb{R}$ with $|\eta|$ small enough and assume that

\[
(1 + \eta \mathcal{L}_X) [\Lambda^k(\Omega) \cap \text{dom}(\mathcal{L}_X) \cap \text{dom}(d)] \cap \text{dom}(d) \text{ is dense in } \text{dom}(d).
\]

Then
\[
(1 + \eta \mathcal{L}_X)^{-1} d \subseteq \hat{d} (1 + \eta \mathcal{L}_X)^{-1}.
\]

**Proof.** Since $\mathcal{L}_X$ is quasi-skew-selfadjoint and hence quasi-m-accretive, $(1 + \varepsilon \mathcal{L}_X)^{-1}$ defines a bounded operator on $L^2(\Omega)$ for $\varepsilon > 0$ small enough.

Let now $\alpha \in \Lambda^k(\Omega) \cap \text{dom}(\mathcal{L}_X) \cap \text{dom}(d)$ such that $(1 + \eta \mathcal{L}_X) \alpha \in \text{dom}(d)$. By Lemma 4.6 we have that

\[
d(1 + \eta \mathcal{L}_X)^{-1}(1 + \eta \mathcal{L}_X) \alpha - (1 + \eta \mathcal{L}_X)^{-1}d(1 + \eta \mathcal{L}_X) \alpha \\
= d\alpha - \alpha = 0,
\]
i.e.

\[
d(1 + \eta \mathcal{L}_X)^{-1} - (1 + \eta \mathcal{L}_X)^{-1}d = 0
\]
on $(1 + \eta \mathcal{L}_X) [\Lambda^k(\Omega) \cap \text{dom}(\mathcal{L}_X) \cap \text{dom}(d)] \cap \text{dom}(d)$. Let now $\alpha \in \text{dom}(d)$ and $(\alpha_n)_n$ in $(1 + \eta \mathcal{L}_X) [\Lambda^k(\Omega) \cap \text{dom}(\mathcal{L}_X) \cap \text{dom}(d)] \cap \text{dom}(d)$ with $\alpha_n \to \alpha$ in $\text{dom}(d)$. Then

\[
d(1 + \eta \mathcal{L}_X)^{-1} \alpha_n = (1 + \eta \mathcal{L}_X)^{-1} \hat{d} \alpha_n \to (1 + \eta \mathcal{L}_X)^{-1} \hat{d} \alpha
\]
as $n \to \infty$. Since $(1 + \eta \mathcal{L}_X)^{-1} \alpha_n \to (1 + \eta \mathcal{L}_X)^{-1} \alpha$ as $n \to \infty$, we infer that $(1 + \eta \mathcal{L}_X)^{-1} \alpha \in \text{dom}(\hat{d})$ and

\[
d(1 + \eta \mathcal{L}_X)^{-1} \alpha = (1 + \eta \mathcal{L}_X)^{-1} \hat{d} \alpha,
\]
which shows the asserted operator inclusion.

**4.2 The Equations on smooth Riemannian manifolds**

We now come to the equation we want to study. Let $\Omega \subseteq M$ be open for a smooth Riemannian manifold $M$ of odd dimension $n$. Moreover, let $1 \leq k < n$ and set $H := \Lambda^k(\Omega) \times \Lambda^{k+1}(\Omega)$. We assume that $M_0, M_1 \in L(H)$ such that $M_0$ is selfadjoint and $M_0 \geq c > 0$. Moreover, let $X_0 \in T^1(M)$ and $\alpha \in L^\infty(\Omega; \mathbb{R})$. We consider the equation

\[
\begin{pmatrix}
\partial_0 M_0 + \alpha \\
0
\end{pmatrix}
\begin{pmatrix}
\nabla_{X_0} \\
0
\end{pmatrix}
M_0 + M_1 + \begin{pmatrix}
0 \\
0 \\
\hat{d} \alpha
\end{pmatrix}
U = F
\]

for given $F \in L_2(\rho; H)$ for some $\rho \in \mathbb{R}$ big enough, where we identify $\alpha$ with its induced multiplication operator in $H$. We impose the following conditions on the vector field $X_0$ and the operators $\alpha$ and $M_0$. 

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Hypotheses 1. We assume that
\[ \sup \{ \| \nabla Y X_0 \|_\infty ; Y \in T_{1,\text{c}}^0(\Omega), \| Y \|_\infty \leq 1 \} < \infty \]
as well as \( L_{X_0} \) is quasi-skew-selfadjoint. Moreover, we assume that \( M_0[\text{dom}(L_{X_0}) \times \text{dom}(L_{X_0})] \subseteq \text{dom}(L_{X_0}) \times \text{dom}(L_{X_0}) \), \( \alpha[\text{dom}(L_{X_0})] \subseteq \text{dom}(L_{X_0}) \) as well as
\[ \left( L_{X_0} \begin{pmatrix} 0 \\ \partial X_0 \end{pmatrix}, M_0 \right) \text{ and } \left[ \alpha, L_{X_0}^* \right] \]
are continuous. Finally, we assume that
\[ (1 \pm \varepsilon L_{X_0})[\Lambda^k(\Omega) \cap \text{dom}(L_{X_0}) \cap \text{dom}(d) \cap \text{dom}(\partial d)] \text{ is dense in } \text{dom}(\partial d) \]
for all \( \varepsilon > 0 \) small enough and that \( M_0 \alpha = \alpha M_0 \).

Under this hypotheses we observe the following operator relations.

Lemma 4.8. The operators
\[ \left( \begin{pmatrix} \nabla X_0 \\ 0 \end{pmatrix}, M_0 \right) - \left( \begin{pmatrix} L_{X_0} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -L_{X_0}^* \end{pmatrix} \right), \left[ \begin{pmatrix} L_{X_0} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -L_{X_0}^* \end{pmatrix} \right], M_0 \text{ and } [\alpha, L_{X_0}^*] \]
are continuous and densely defined.

Proof. We obtain
\[ \left( \begin{pmatrix} \nabla X_0 \\ 0 \end{pmatrix}, M_0 \right) - \left( \begin{pmatrix} L_{X_0} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -L_{X_0}^* \end{pmatrix} \right) = \left( \begin{pmatrix} \nabla X_0 - L_{X_0} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \nabla X_0 - L_{X_0}^* \end{pmatrix} \right) + \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, 0 \right) \]
which is continuous by Lemma 4.5 and the quasi-skew-selfadjointness of \( L_{X_0} \). For the second operator we compute
\[ \left[ \begin{pmatrix} L_{X_0} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -L_{X_0}^* \end{pmatrix} \right], M_0 \right] = \left[ \begin{pmatrix} L_{X_0} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ L_{X_0}^* \end{pmatrix} \right], M_0 \right] - \left[ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right], M_0 \right], \]
and both operators on the right-hand side are continuous, which yields the assertion. The continuity of \([\alpha, L_{X_0}^*] \) follows by arguing in the same way.

In order to study (7), we rewrite the second operator on the left-hand side in the following way

\[ \alpha \left( \begin{pmatrix} \nabla X_0 \\ 0 \end{pmatrix}, M_0 \right) = \alpha \left( \begin{pmatrix} L_{X_0} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -L_{X_0}^* \end{pmatrix} \right) M_0 + \alpha \left( \begin{pmatrix} \nabla X_0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -L_{X_0}^* \end{pmatrix} \right) M_0 \]

where we have used Lemma 4.8. Thus, ignoring the bounded operators, we may restrict ourselves to the study of the operator
\[ \partial_0 M_0 + \alpha M_0 \left( \begin{pmatrix} L_{X_0} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -L_{X_0}^* \end{pmatrix} \right) + \left( \begin{pmatrix} 0 \\ \partial \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right). \]
The main idea is now to decompose the second operator in the above sum in its symmetric and skew-symmetric part, which are studied in the next proposition.
Proposition 4.9. The operator

\[ C := \text{sym} \alpha M_0 \begin{pmatrix} \mathcal{L}_{X_0} & 0 \\ 0 & -\mathcal{L}_{X_0}^* \end{pmatrix} \]

is bounded and selfadjoint. Moreover

\[ D := \text{skew} \alpha M_0 \begin{pmatrix} \mathcal{L}_{X_0} & 0 \\ 0 & -\mathcal{L}_{X_0}^* \end{pmatrix} \]

is skew-selfadjoint.

Proof. We note that \( \begin{pmatrix} \mathcal{L}_{X_0} & 0 \\ 0 & -\mathcal{L}_{X_0}^* \end{pmatrix} \) is quasi-skew-selfadjoint in the sense of Subsection 3.3. Indeed, we have that

\[ \text{dom} \left( \begin{pmatrix} \mathcal{L}_{X_0} & 0 \\ 0 & -\mathcal{L}_{X_0}^* \end{pmatrix} \right) = \text{dom} \left( \mathcal{L}_{X_0} \right) \times \text{dom} \left( \mathcal{L}_{X_0}^* \right) = \text{dom} \left( \begin{pmatrix} \mathcal{L}_{X_0}^* & 0 \\ 0 & -\mathcal{L}_{X_0} \end{pmatrix} \right) \]

since \( \mathcal{L}_{X_0} \) is quasi-skew-selfadjoint and

\[ \left( \begin{pmatrix} \mathcal{L}_{X_0} & 0 \\ 0 & -\mathcal{L}_{X_0}^* \end{pmatrix} \right)^* + \left( \begin{pmatrix} \mathcal{L}_{X_0}^* & 0 \\ 0 & -\mathcal{L}_{X_0} \end{pmatrix} \right) = \left( \mathcal{L}_{X_0}^{\ast} + \mathcal{L}_{X_0} \right) \]

is bounded, again by the quasi-skew-selfadjointness of \( \mathcal{L}_{X_0} \).

Next we note that \( \alpha M_0 \) is selfadjoint and that

\[ \alpha M_0 \left[ \text{dom} \left( \mathcal{L}_{X_0} \right) \times \text{dom} \left( \mathcal{L}_{X_0}^* \right) \right] \subseteq \text{dom} \left( \mathcal{L}_{X_0} \right) \times \text{dom} \left( \mathcal{L}_{X_0}^* \right) \]

by assumption. Moreover,

\[ \left[ \alpha M_0, \begin{pmatrix} \mathcal{L}_{X_0} & 0 \\ 0 & -\mathcal{L}_{X_0}^* \end{pmatrix} \right] = \alpha \left[ M_0, \begin{pmatrix} \mathcal{L}_{X_0} & 0 \\ 0 & -\mathcal{L}_{X_0}^* \end{pmatrix} \right] + \alpha \left[ \begin{pmatrix} \mathcal{L}_{X_0}^* & 0 \\ 0 & -\mathcal{L}_{X_0} \end{pmatrix} \right] M_0 \]

is continuous by Lemma 4.8 and hence, \( D \) is skew-selfadjoint by Theorem 3.12. Moreover, \( C \) is densely defined, since

\[ \text{dom} \left( \mathcal{L}_{X_0} \right) \times \text{dom} \left( \mathcal{L}_{X_0}^* \right) \subseteq \text{dom} \left( \mathcal{L}\right) \]

and

\[ \alpha M_0 \left[ \mathcal{L}_{X_0} \right] \subseteq \text{dom} \left( \mathcal{L}\right) \]

shows that \( C \) is continuous and hence, selfadjoint. \( \square \)

It is now our goal to prove that the operator

\[ D + \begin{pmatrix} 0 & -d^* \\ d & 0 \end{pmatrix} \]

with \( D \) given as in Proposition 4.9 is essentially skew-selfadjoint; i.e., its closure is skew-selfadjoint. For doing so, we want to apply Theorem 3.14 with

\[ L_\varepsilon := \begin{pmatrix} (1 - \varepsilon \mathcal{L}_{X_0})^{-1} & 0 \\ 0 & (1 + \varepsilon \mathcal{L}_{X_0})^{-1} \end{pmatrix}, \quad R_\varepsilon := \begin{pmatrix} (1 + \varepsilon \mathcal{L}_{X_0})^{-1} & 0 \\ 0 & (1 - \varepsilon \mathcal{L}_{X_0})^{-1} \end{pmatrix} \quad (8) \]
for $\varepsilon > 0$ small enough. Note that these operators are well-defined and bounded, since by Proposition \[4.7\] the operator $\mathcal{L}_{X_0}$ is quasi-skew-selfadjoint and hence, $\pm \mathcal{L}_{X_0}$ and $\pm \mathcal{L}^*_{X_0}$ are quasi-m-accretive. Note further that this yields

$$L^*_\varepsilon = \begin{pmatrix} 1 - \varepsilon \mathcal{L}_{X_0} & 0 \\ 0 & 1 + \varepsilon \mathcal{L}^*_{X_0} \end{pmatrix}^{-1} \rightarrow 1_H$$

strongly as $\varepsilon \to 0$ by Lemma \[3.3\] and similarly $R^*_\varepsilon \to 1_H$ strongly as $\varepsilon \to 0$.

**Lemma 4.10.** Let $\varepsilon > 0$ small enough and $D$ as in Proposition \[4.9\]. Then the following statements hold:

(a) We have

$$\text{ran}(R_\varepsilon) \subseteq \text{dom} \left( \begin{pmatrix} \mathcal{L}_{X_0} & 0 \\ 0 & -\mathcal{L}_{X_0} \end{pmatrix} \right) \cap \text{dom} \left( \begin{pmatrix} \alpha M_0 & \mathcal{L}_{X_0} \\ 0 & -\mathcal{L}^*_{X_0} \end{pmatrix} \right) \subseteq \text{dom}(D)$$

and

$$R_\varepsilon \left[ \text{dom}(d) \times \text{dom}(d^*) \right] \subseteq \text{dom}(d) \times \text{dom}(d^*).$$

(b) We have

$$\left( D + \begin{pmatrix} 0 & -\hat{d}^* \\ \hat{d} & 0 \end{pmatrix} \right)^* L^*_\varepsilon = - \left( D + \begin{pmatrix} 0 & -\hat{d}^* \\ \hat{d} & 0 \end{pmatrix} \right) L^*_\varepsilon.$$

**Proof.** (a) Note that $\text{ran}(R_\varepsilon) \subseteq \text{dom}(\mathcal{L}_{X_0}) \times \text{dom}(\mathcal{L}^*_{X_0}) = \text{dom}(\mathcal{L}_{X_0}) \times \text{dom}(\mathcal{L}_{X_0})$. Moreover, we have that

$$\text{dom} \left( \begin{pmatrix} \mathcal{L}_{X_0} & 0 \\ 0 & -\mathcal{L}_{X_0} \end{pmatrix} \right) \cap \text{dom} \left( \begin{pmatrix} \alpha M_0 & \mathcal{L}_{X_0} \\ 0 & -\mathcal{L}^*_{X_0} \end{pmatrix} \right) \supseteq \text{dom}(\mathcal{L}_{X_0}) \times \text{dom}(\mathcal{L}_{X_0})$$

and hence,

$$\text{ran}(R_\varepsilon) \subseteq \text{dom}(\mathcal{L}_{X_0}) \times \text{dom}(\mathcal{L}_{X_0})$$

$$= \text{dom} \left( \begin{pmatrix} \mathcal{L}_{X_0} & 0 \\ 0 & -\mathcal{L}_{X_0} \end{pmatrix} \right) \cap \text{dom} \left( \begin{pmatrix} \alpha M_0 & \mathcal{L}_{X_0} \\ 0 & -\mathcal{L}^*_{X_0} \end{pmatrix} \right) \subseteq \text{dom}(D).$$

Moreover, we have $(1 \pm \varepsilon \mathcal{L}_{X})^{-1} \hat{d} \subseteq \hat{d}(1 \pm \varepsilon \mathcal{L}_{X})^{-1}$ by Proposition \[4.7\] which implies $(1 \pm \varepsilon \mathcal{L}_{X})^{-1} \hat{d}^* \subseteq \hat{d}^*(1 \pm \varepsilon \mathcal{L}_{X})^{-1}$. Hence

$$R_\varepsilon \left[ \text{dom}(d) \times \text{dom}(d^*) \right] \subseteq \text{dom}(d) \times \text{dom}(d^*),$$

which completes the proof for statement (a).

(b) It suffices to prove

$$\text{dom} \left( \begin{pmatrix} D + \begin{pmatrix} 0 & -\hat{d}^* \\ \hat{d} & 0 \end{pmatrix} \end{pmatrix} L^*_\varepsilon \right) \subseteq \text{dom} \left( \begin{pmatrix} D + \begin{pmatrix} 0 & -\hat{d}^* \\ \hat{d} & 0 \end{pmatrix} \end{pmatrix} L^*_\varepsilon \right).$$

So let $(x, y) \in \text{dom} \left( \begin{pmatrix} D + \begin{pmatrix} 0 & -\hat{d}^* \\ \hat{d} & 0 \end{pmatrix} \end{pmatrix} L^*_\varepsilon \right)$; i.e.,

$$L^*_\varepsilon (x, y) = \begin{pmatrix} (1 - \varepsilon \mathcal{L}_{X_0})^{-1} x \\ (1 + \varepsilon \mathcal{L}^*_{X_0})^{-1} y \end{pmatrix} \in \text{dom} \left( \begin{pmatrix} D + \begin{pmatrix} 0 & -\hat{d}^* \\ \hat{d} & 0 \end{pmatrix} \end{pmatrix} L^*_\varepsilon \right).$$
Since \( L^*_\varepsilon(x,y) \in \text{dom}(L_{X_0}) \times \text{dom}(L_{X_0}) \subseteq \text{dom}(D) = \text{dom}(D^*) \) by (9), the assertion would follow if \( \text{dom}(D) \) is a core for \( \begin{pmatrix} 0 & -d^* \\ \tilde{d} & 0 \end{pmatrix} \). Since \( (1 + \varepsilon L_X)^{-1} \tilde{d} \subseteq \tilde{d}(1 + \varepsilon L_X)^{-1} \) and \( (1 + \varepsilon L_X)^{-1} \rightarrow 1 \) strongly as \( \varepsilon \rightarrow 0 \), it follows that \( \text{dom}(L_X) \) is a core for \( \tilde{d} \). In the same way, it follows that \( \text{dom}(L_{X^*}) = \text{dom}(L_X) \) is a core for \( \tilde{d}^* \) and hence, since \( \text{dom}(L_X) \times \text{dom}(L_X) \subseteq \text{dom}(D) \) by (9) the claim follows. This proves statement (b).

**Lemma 4.11.** For \( \varepsilon_0 > 0 \) small enough there exists \( K \geq 0 \) such that
\[
\| [R^*_\varepsilon, \alpha M_0, L^*_\varepsilon] \| \leq K2\varepsilon\|C\| \quad (0 < \varepsilon \leq \varepsilon_0),
\]
where \( C \) is the operator given in Proposition 4.9.

**Proof.** We first observe that
\[
\varepsilon \left[ \begin{pmatrix} L_{X_0}^* & 0 \\ 0 & -L_{X_0} \end{pmatrix}, \alpha M_0, \begin{pmatrix} -L_{X_0} & 0 \\ 0 & L_{X_0}^* \end{pmatrix} \right] = \left[ \begin{pmatrix} 1 + \varepsilon L_{X_0}^* & 0 \\ 0 & 1 - \varepsilon L_{X_0} \end{pmatrix}, \alpha M_0 \right] - \alpha M_0 \left[ \begin{pmatrix} 1 - \varepsilon L_{X_0} & 0 \\ 0 & 1 + \varepsilon L_{X_0} \end{pmatrix}, \alpha M_0 \right]
\]
and thus,
\[
[R^*_\varepsilon, \alpha M_0, L^*_\varepsilon] = -\varepsilon R^*_\varepsilon \left[ \begin{pmatrix} L_{X_0}^* & 0 \\ 0 & -L_{X_0} \end{pmatrix}, \alpha M_0, \begin{pmatrix} -L_{X_0} & 0 \\ 0 & L_{X_0}^* \end{pmatrix} \right] L^*_\varepsilon
\]
\[
= -\varepsilon R^*_\varepsilon \left[ \begin{pmatrix} L_{X_0}^* & 0 \\ 0 & -L_{X_0} \end{pmatrix}, \alpha M_0 \left( -L_{X_0} \alpha M_0 + \alpha M_0 \left( -L_{X_0} \alpha M_0 \right) \right) \right] L^*_\varepsilon
\]
\[
= -\varepsilon R^*_\varepsilon \left[ \alpha M_0 \left( L_{X_0} \alpha M_0 \right) \right] + \alpha M_0 \left( \begin{pmatrix} L_{X_0}^* & 0 \\ 0 & -L_{X_0} \end{pmatrix} \right) \alpha M_0 \alpha
\]
\[
= -2\varepsilon R^*_\varepsilon CD L^*_\varepsilon.
\]
The assertion follows with \( K := \sup_{0 < \varepsilon \leq \varepsilon_0} \| R^*_\varepsilon \| \| L^*_\varepsilon \| \), which is finite, since \( L_X \) is quasi-skew-selfadjoint and thus, \( -L_X, L_X^* \) are quasi-m-accretive.

With these preparations at hand, we are able to prove the essentially skew-selfadjointness of \( D + \begin{pmatrix} 0 & -d^* \\ \tilde{d} & 0 \end{pmatrix} \).

**Proposition 4.12.** The operator \( D + \begin{pmatrix} 0 & -d^* \\ \tilde{d} & 0 \end{pmatrix} \) is essentially skew-selfadjoint, where \( D \) is given in Proposition 4.9.

**Proof.** We will apply Theorem 3.14 with the operators \( L_\varepsilon, R_\varepsilon \) given in (8). Thanks to Lemma 4.10 we only need to check that
\[
\left[ L_\varepsilon, D + \begin{pmatrix} 0 & -d^* \\ \tilde{d} & 0 \end{pmatrix}, R_\varepsilon \right]
\]
is continuous and that
\[
\left[ L_\varepsilon, D + \begin{pmatrix} 0 & -d^* \\ \tilde{d} & 0 \end{pmatrix}, R_\varepsilon \right]^* \rightarrow 0
\]

strongly as $\varepsilon \to 0$. We compute

$$\left[ L\varepsilon, D + \begin{pmatrix} 0 & -\dot{d}^* \\ d & 0 \end{pmatrix}, R\varepsilon \right] = [L\varepsilon, D, R\varepsilon] + \left[ L\varepsilon, \begin{pmatrix} 0 & -\dot{d}^* \\ d & 0 \end{pmatrix}, R\varepsilon \right]$$

$$= [L\varepsilon, D, R\varepsilon],$$

where we have used $(1 \pm \varepsilon \mathcal{L}_{X_0})^{-1} \dot{d} \subseteq \dot{d}(1 \pm \varepsilon \mathcal{L}_{X_0})^{-1}$ by Proposition 4.7, which implies $(1 \pm \varepsilon \mathcal{L}_{X_0})^{-1} \dot{d} \subseteq \dot{d}(1 \pm \varepsilon \mathcal{L}_{X_0})^{-1}$ and thus, the second transmutator vanishes. Now we have

$$\frac{1}{2} \left( \alpha M_0 \left( \mathcal{L}_{X_0} 0 \\ 0 -\mathcal{L}_{X_0} \right) - \left( \mathcal{L}_{X_0} 0 \\ 0 -\mathcal{L}_{X_0} \right) M_0 \alpha \right) \left( (1 \pm \varepsilon \mathcal{L}_{X_0})^{-1} 0 \\ 0 \right) (1 \mp \varepsilon \mathcal{L}_{X_0})^{-1}$$

$$= \frac{1}{2} \left( \left[ \alpha M_0 \left( \mathcal{L}_{X_0} 0 \\ 0 -\mathcal{L}_{X_0} \right) \right] + \alpha M_0 \left( \mathcal{L}_{X_0} -\mathcal{L}_{X_0} \right) \right) \left( (1 \pm \varepsilon \mathcal{L}_{X_0})^{-1} 0 \\ 0 \right) (1 \mp \varepsilon \mathcal{L}_{X_0})^{-1}$$

$$= \frac{1}{2} \left[ \alpha M_0 \left( \mathcal{L}_{X_0} 0 \\ 0 -\mathcal{L}_{X_0} \right) \right] \left( (1 \pm \varepsilon \mathcal{L}_{X_0})^{-1} 0 \\ 0 \right) (1 \mp \varepsilon \mathcal{L}_{X_0})^{-1} +$$

$$+ \alpha M_0 \left( (\mathcal{L}_{X_0} -\text{sym}(\mathcal{L}_{X_0})) (1 \pm \varepsilon \mathcal{L}_{X_0})^{-1} \right) \left( (1 \mp \varepsilon \mathcal{L}_{X_0})^{-1} \right),$$

which is continuous. Note that in one case this operators equals $D R\varepsilon$, and in the other case it equals $D L\varepsilon^*$, since $L\varepsilon D \subseteq (D^* L\varepsilon^*)^* = (-D L\varepsilon^*)^*$, we infer that also $L\varepsilon D$ is continuous and hence, so is $[L\varepsilon, D, R\varepsilon]$.

Now we come to the second claim. We compute

$$[L\varepsilon, D, R\varepsilon]^* = (L\varepsilon D - D R\varepsilon)^*$$

$$= (L\varepsilon D)^* - (D R\varepsilon)^*$$

$$= D^* L\varepsilon^* - (D R\varepsilon)^*$$

$$= -(D L\varepsilon^* + (D R\varepsilon)^*)$$

and so, we have to show that $D L\varepsilon^* + (D R\varepsilon)^* \to 0$ strongly as $\varepsilon \to 0$. By the computation above we have that

$$D L\varepsilon^* = \frac{1}{2} \left[ \alpha M_0 \left( \mathcal{L}_{X_0} 0 \\ 0 -\mathcal{L}_{X_0} \right) \right] L\varepsilon^* + \alpha M_0 \left( \mathcal{L}_{X_0} 0 \\ 0 -\mathcal{L}_{X_0} \right) L\varepsilon^* + \alpha M_0 \left( -\text{sym}(\mathcal{L}_{X_0}) 0 \\ 0 \text{sym}(\mathcal{L}_{X_0}) \right) L\varepsilon^*$$

and

$$(D R\varepsilon)^* = R\varepsilon^* \left[ \alpha M_0 \left( \mathcal{L}_{X_0} 0 \\ 0 -\mathcal{L}_{X_0} \right) \right]^* + R\varepsilon^* \left( \mathcal{L}_{X_0} 0 \\ 0 -\mathcal{L}_{X_0} \right) M_0 \alpha +$$

$$+ R\varepsilon^* \left( -\text{sym}(\mathcal{L}_{X_0}) 0 \\ 0 \text{sym}(\mathcal{L}_{X_0}) \right) M_0 \alpha,$$

where we have used Corollary 4.10 in the last equality. Now, on $\text{dom}(\mathcal{L}_{X_0}) \times \text{dom}(\mathcal{L}_{X_0})$ we obtain

$$D L\varepsilon + (D R\varepsilon)^* \to -\frac{1}{2} \left[ \alpha M_0 \left( \mathcal{L}_{X_0} 0 \\ 0 -\mathcal{L}_{X_0} \right) \right] + \alpha M_0 \left( \mathcal{L}_{X_0} 0 \\ 0 -\mathcal{L}_{X_0} \right) +$$

$$+ \alpha M_0 \left( -\text{sym}(\mathcal{L}_{X_0}) 0 \\ 0 \text{sym}(\mathcal{L}_{X_0}) \right) + \frac{1}{2} \left[ \left( \mathcal{L}_{X_0} 0 \\ 0 -\mathcal{L}_{X_0} \right) \right] M_0 \alpha$$

for $\varepsilon \to 0$. This proves the claim.
\[
\alpha M_0 \left( \frac{\partial X_0}{\partial \nu} - \frac{\partial X_0}{\partial \nu} \right) + \alpha M_0 \left( \frac{\partial X_0}{\partial \nu} - \frac{\partial X_0}{\partial \nu} \right) M_0 = \alpha M_0 \frac{1}{\varepsilon} (-1 + L_\varepsilon) + \frac{1}{\varepsilon} (1 - R_\varepsilon) M_0 \alpha \\
= \frac{1}{\varepsilon} (\alpha M_0 L_\varepsilon - R_\varepsilon M_0) \\
= -\frac{1}{\varepsilon} ([R_\varepsilon, \alpha M_0, L_\varepsilon]).
\]

This term, however, is uniformly bounded by Lemma 4.11. □

**Theorem 4.13.** Problem (7) is well-posed in the following sense: There exists \(\rho_0 > 0\) such that for each \(\rho \geq \rho_0\) the operator

\[
\partial_0 M_0 + \overline{M}_1 + \left( D + \begin{pmatrix} 0 & -d^* \\ d & 0 \end{pmatrix} \right)_{L^2,\rho}(\mathbb{R}; H)
\]

is continuously invertible. Here

\[
\overline{M}_1 := M_1 + \alpha \left( \begin{pmatrix} \nabla X_0 & 0 \\ 0 & \nabla X_0 \end{pmatrix} - \begin{pmatrix} \partial X_0 & 0 \\ 0 & \partial X_0 \end{pmatrix} - \begin{pmatrix} \partial X_0 & 0 \\ 0 & \partial X_0 \end{pmatrix} \right) M_0 + \alpha \left( \begin{pmatrix} \partial X_0 & 0 \\ 0 & \partial X_0 \end{pmatrix}, M_0 \right) + C
\]

and \(C\) and \(D\) are given as in Proposition 4.9. Moreover, denoting by \(S_{\rho}\) the inverse of (10), we have that \(S_{\rho}\) is causal and independent of the choice of \(\rho \geq \rho_0\) in the sense that for each \(F \in H_{\rho,0}(\mathbb{R}; H) \cap H_{\mu,0}(\mathbb{R}; H)\) with \(\rho, \mu \geq \rho_0\) we have that

\[
S_{\rho} F = S_{\mu} F.
\]

**Proof.** The claim follows from Corollary 2.4, where we use that \(\overline{M}_1\) is bounded by Lemma 4.11 and Proposition 4.9 and that \(\left( D + \begin{pmatrix} 0 & -d^* \\ d & 0 \end{pmatrix} \right)\) is skew-selfadjoint by Proposition 4.12. □

### 4.3 Cylindrical domains

We shall consider cylindrical domains as two reference cases:

1. The infinite straight tube: \(\Omega = \Sigma \times \mathbb{R}, \Sigma \subseteq \mathbb{R}^2\).
2. The finite straight tube: \(\Omega = \Sigma \times [-1/2, 1/2], \Sigma \subseteq \mathbb{R}^2\), (with top and bottom identified this is a torus).

We assume a metric tensor \(g\) for \(\Omega\). We discuss the Hypotheses [11] in both cases for the vector field \(X_0 := e_3\).

In both cases we recall that \(L_{e_3}\) is essentially \(\partial_3\) (up to lower order terms). In particular, \(L_{e_3}\) is quasi-skew-selfadjoint, since in both cases \(\partial_3\) is skew-selfadjoint (note that we impose periodic boundary conditions in the second case). Moreover, the assumption

\[
\sup \{ \|\nabla Y e_3\|_{\infty} : Y \in T^0_{1,e}(\Omega), \|Y\|_{\infty} \leq 1 \} < \infty
\]

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is a constraint on the Riemannian metric $g$. Indeed, we compute
\[
\nabla_Y e_3 = \nabla_Y \left( \sum_{i=1}^{n} g^i(e_3) g_i \right) = \sum_{i=1}^{n} (\nabla_Y g^i(e_3)) g_i + g^i(e_3) \nabla_Y g_i
\]
for each $Y \in T^1_0(\Omega)$ and we require that the metric $g$ is given such that
\[
\sup \left\{ \| \nabla_Y e_3 \|_\infty ; Y \in T^0_{1,c}(\Omega), \| Y \|_\infty \leq 1 \right\} < \infty
\]
is satisfied. Thus, we are left to discuss the assumption
\[
(1 \pm \varepsilon \mathcal{L}_X)[\Lambda^k(\Omega) \cap \text{dom}(\mathcal{L}_X) \cap \text{dom}(d)] \cap \text{dom}(d) \text{ is dense in } \text{dom}(d).
\]
We will do this in both cases separately.

\textbf{Case 1:} 
\textbf{Proposition 4.14.} For $|\eta|$ small enough we have
\[
(1 + \eta \mathcal{L}_{e_3}) \left[ \Lambda^k(\Omega) \cap \text{dom}(\mathcal{L}_{e_3}) \right] \cap \text{dom}(d) \text{ is dense in } \text{dom}(d).
\]
\textit{Proof.} The set $C_c^\infty(\Omega)$ is dense in $\text{dom}(\hat{d})$ and – as can be shown by a standard cut-off technique –
\[
\hat{Z}(\Omega) := \left\{ \phi \in C^\infty(\Omega) \cap \text{dom}(d) ; \text{supp}(\phi) \subseteq \tilde{\Sigma} \times \mathbb{R} \text{ for some } \tilde{\Sigma} \text{ relatively compact in } \Sigma \right\} \subseteq \text{dom}(\hat{d}).
\]
We show that $(1 + \eta \mathcal{L}_{e_3})^{-1} f \in \hat{Z}(\Omega)$ for $f \in C_c^\infty(\Omega)$. Indeed, setting $u := (1 + \eta \mathcal{L}_{e_3})^{-1} f$ and $b := \mathcal{L}_{e_3} - \partial_3$, we infer that
\[
(1 + \eta \partial_3) u + \eta bu = f.
\]
Since $b$ is a smooth multiplication operator, $u \in C^\infty(\Omega) \cap \text{dom}(d)$ and $\text{supp}(u) \subseteq \tilde{\Sigma} \times \mathbb{R}$ where $\tilde{\Sigma} \subseteq \Sigma$ is relatively compact such that $\text{supp} f \subseteq \tilde{\Sigma} \times \mathbb{R}$. Thus, we have in particular
\[
\hat{z}(\Omega) := (1 + \eta \mathcal{L}_{e_3})^{-1} [C_c^\infty(\Omega)] \subseteq \hat{Z}(\Omega) \subseteq \text{dom}(\hat{d})
\]
and so
\[
(1 + \eta \mathcal{L}_{e_3}) \left[ \hat{z}(\Omega) \right] = C_c^\infty(\Omega) \text{ dense in } \text{dom}(\hat{d}).
\]
Since
\[
\hat{z}(\Omega) \subseteq \Lambda^k(\Omega) \cap \text{dom}(\mathcal{L}_{e_3}),
\]
the desired density property follows. 

\textbf{Case 2:} 
\textbf{Proposition 4.15.} We have
\[
(1 + \eta \mathcal{L}_{e_3}) \left[ \Lambda^k(\Omega) \cap \text{dom}(\mathcal{L}_{e_3}) \right] \cap \text{dom}(d) \text{ is dense in } \text{dom}(\hat{d}).
\]
\textit{Proof.} As in Proposition 4.14 we see that
\[
\hat{Z}(\Omega) := \left\{ \phi \in C^\infty(\Omega) \cap \text{dom}(d) ; \phi(\cdot, \cdot, -1/2) = \phi(\cdot, \cdot, 1/2), \text{supp}(\phi) \subseteq \tilde{\Sigma} \times [-1/2, 1/2] \text{ for some } \tilde{\Sigma} \text{ relatively compact in } \Sigma \right\} \subseteq \text{dom}(\hat{d}).
\]
We show that $(1 + \eta \mathcal{L}_c)^{-1} f \in \tilde{Z}(\Omega)$ for $f \in C^\infty_c(\Omega)$. Indeed, setting $u := (1 + \eta \mathcal{L}_c)^{-1} f$ and $b := \mathcal{L}_c - \partial_3$, we infer that

$$(1 + \eta \partial_3) u + \eta bu = f.$$ 

Since $b$ is a periodic multiplication operator, we obtain $u \in \tilde{Z}(\Omega)$ and thus, we have in particular

$$\tilde{z} (\Omega) := (1 + \eta \mathcal{L}_c)^{-1} [C^\infty_c(\Omega)] \subseteq \tilde{Z} (\Omega) \subseteq \text{dom} \left( \tilde{d} \right).$$

and so

$$(1 + \eta \mathcal{L}_c) [\tilde{z} (\Omega)] \subset C^\infty_c(\Omega) \text{ dense in } \text{dom} \left( \tilde{d} \right).$$

Since

$$\tilde{z} (\Omega) \subseteq \Lambda^k(\Omega) \cap \text{dom}(\mathcal{L}_c),$$

the desired density property follows. \hfill \Box

**Remark 4.16.** We have provided two examples, where the solution theory developed in Subsection 4.2 can be applied. Using isometries between Riemannian manifolds, we can apply our solution theory to a broader class of examples. More precisely, let $M$ and $N$ be two Riemannian manifolds and $\Phi : M \to N$ be smooth and orientation preserving. We denote the associated pull-back of cotangential vectorfields (and tensors thereof) by $\Phi^*$ and the push-forward of tangential vectorfields (and tensors thereof) by $\Phi_*$. We assume that $\Phi$ is an isometry; that is,

$$\Phi^* g_N = g_M,$$

where $g_N$ and $g_M$ denote the Riemannian metrics on $N$ and $M$, respectively. An easy computation shows that $\Phi^*$ commutes with the Hodge-star operator in the sense that

$$\Phi^* \ast_N = \ast_M \Phi^*$$

and also with the exterior derivative. Thus, in particular

$$\Phi^* \begin{pmatrix} 0 & -d^*_N \\ d_N & 0 \end{pmatrix} = \begin{pmatrix} 0 & -d^*_M \\ d_M & 0 \end{pmatrix} \Phi^*.$$

Moreover, $\Phi^*$ interacts with the Lie-derivative in the following way

$$\Phi^* \mathcal{L}_X = \mathcal{L}_{\Phi^{-1} X} \Phi^*$$

for a vectorfield $X$ on $N$. Hence, we can transform the equation on $N$ via $\Phi^*$ into a corresponding equation on $M$. Moreover, due to the isometry of $\Phi$, the condition on $M_0$, defined on forms on $N$ (selfadjointness and positive definiteness) carries over to the transformed operator $\Phi^* \mathcal{L}_0 \Phi^*$. Thus, if we can apply the solution theory to the problem posed on the reference manifold $M$, we also derive the well-posedness on the manifold $N$. Hence, by transforming the two reference situations above, we can also treat the case of a infinite deformed pipe or a deformed torus.

**4.4 An abstract localisation technique**

In this section we inspect a localisation technique, which will allow us to glue together different open subsets of a Riemannian manifold. Before we come to the concrete application, we will present this technique in an abstract functional analytic setting. Throughout this section, we consider the following setting:

Let $H$ be a Hilbert space and $U \subseteq H$ a closed subspace. We denote the canonical embedding of $U$ into $H$ by $\iota_U : U \to H$ and remark that the adjoint $\iota_U^* : H \to U$ assigns each element in $H$ its best approximation in $U$. Note that then $\iota_U \iota_U^* : H \to H$ is the orthogonal projector onto $U$ and $\iota_U^* \iota_U : U \to U$ is the identity on $U$. Moreover, let $A : \text{dom}(A) \subseteq H \to H$ and $B : \text{dom}(B) \subseteq U \to U$ be two densely defined closed linear operators and we assume that $A^*$ leaves $U$ invariant. Finally, let $S : H \to H$ be a bounded linear operator such that $\text{ran}(S), \text{ran}(S^*) \subseteq U$, $S[\text{dom}(A)] \subseteq \text{dom}(A)$, and $[S, A]$ is continuous.
Example 4.17. A typical example for the situation above is as follows: Let $\Omega \subseteq M$ an open subset of a Riemannian manifold and $\tilde{\Omega} \subseteq \Omega$ open. We set $H := \Lambda^k(\Omega) \oplus \Lambda^{k+1}(\Omega)$ for some $k \in \mathbb{N}$ and $U := \Lambda^k(\tilde{\Omega}) \oplus \Lambda^{k+1}(\tilde{\Omega})$. Moreover, let

$$A := \begin{pmatrix} 0 & -d\Omega \\ -d\Omega & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 0 & -d\tilde{\Omega} \\ -d\tilde{\Omega} & 0 \end{pmatrix},$$

where $d\Omega$ and $d\tilde{\Omega}$ denote the exterior derivative with homogeneous Dirichlet boundary conditions on $\Omega$ and $\tilde{\Omega}$, respectively. Moreover, let $\phi : \Omega \to \mathbb{R}$ be smooth with $\text{supp} \phi \subseteq \tilde{\Omega}$ and denote by $S$ the multiplication operator with $\phi$.

We start with two useful observations.

Lemma 4.18. $S^*[\text{dom}(A^*)] \subseteq \text{dom}(A^*)$ and $[A^*, S^*] = [S, A^*]$.

Proof. We recall from Lemma 3.1 (a) $SA \subseteq AS + [S, A]$. Taking adjoints on both sides yields

$$A^*S^* = (SA)^* \supseteq (AS)^* + [S, A] \supseteq S^*A^* + [S, A].$$

The latter gives

$$S^*A^* \subseteq A^*S^* + [A, S]^*,$$

which shows the claim.

Lemma 4.19. Assume that there exists a mapping $E : \text{dom}(B) \to \text{dom}(A)$ such that $\iota_U^*Ex = x$ and $\iota_U^*AEx = Bx$ for each $x \in \text{dom}(B)$. Let $v \in \text{dom}(A^*)$. Then $\iota_U^*S^*v \in \text{dom}(B^*)$ with $B^*\iota_U^*S^*v = \iota_U^*A^*S^*v$.

Proof. For $u \in \text{dom}(B)$ we compute

$$\langle Bu, \iota_U^*S^*v \rangle = \langle SU^*Bu, v \rangle = \langle SU^*\iota_U^*AEx, v \rangle = \langle \iota_U^*\iota_U^*AExu, S^*v \rangle = \langle AExu, S^*v \rangle = \langle SAEu, v \rangle = \langle ASEu, v \rangle + ([S, A]Eu, v) = \langle Eu, (S^*A^* + [S, A]^*)v \rangle = \langle Eu, A^*S^*v \rangle.$$

To finish the proof, observe that $A^*S^*v \in U$ and hence

$$\langle Eu, A^*S^*v \rangle = \langle Eu, \iota_U^*A^*S^*v \rangle = \langle \iota_U^*Eu, \iota_U^*A^*S^*v \rangle = \langle u, \iota_U^*A^*S^*v \rangle.$$

This shows $\iota_U^*S^*v \in \text{dom}(B^*)$ and $B^*\iota_U^*S^*v = \iota_U^*A^*S^*v$.

We now come to the concrete application. Let $\Omega \subseteq M$ be open for a smooth Riemannian manifold $M$ of odd dimension $n$. Moreover, let $1 \leq k < n$ and set $H := \Lambda^k(\Omega) \times \Lambda^{k+1}(\Omega)$. We assume that
\[ M_0, M_1 \in L(H) \text{ such that } M_0 \text{ is selfadjoint and } M_0 \geq c > 0. \] Moreover, let \( X_0 \in T^1_0(M) \) and \( \alpha \in L_\infty(\Omega; \mathbb{R}) \). We again consider an equation of the form
\[
\left( \partial_0 M_0 + \alpha \begin{pmatrix} \nabla X_0 & 0 \\ 0 & \nabla X_0 \end{pmatrix} \right) M_0 + M_1 + \begin{pmatrix} 0 & -d^\ast \\ d & 0 \end{pmatrix} U = F
\]
but now we assume that the vectorfield \( X_0 \) and the function \( \alpha \) are supported in an open subset \( \tilde{\Omega} \subseteq \Omega \). Moreover, we assume that \( M_0 \) and \( M_1 \) are local operators (e.g. multiplication operators) and that the hypotheses are satisfied on two open subdomains \( \tilde{\Omega} \subseteq \Omega \) and \( \hat{\Omega} \subseteq \Omega \) with \( \Omega = \Omega \cup \hat{\Omega} \) and \( \alpha \) and \( X_0 \) vanish on \( \hat{\Omega} \). Then we can solve the problem separately on \( \Omega \) and \( \hat{\Omega} \). We now employ the localisation technique to illustrate how this yields a solution theory for the original problem on \( \Omega \). The crucial point for showing the well-posedness is the essential skew-selfadjointness of \( D + \begin{pmatrix} 0 & -d^\ast \\ d & 0 \end{pmatrix} \) with \( D := \text{skew} \alpha M_0 \begin{pmatrix} \mathcal{L}_{X_0} & 0 \\ 0 & -\mathcal{L}_{X_0}^\ast \end{pmatrix} \). In order to show this, we employ the result above in the following two situations: Let \( \phi : \Omega \to [0, 1] \) be smooth with \( \text{supp} \phi \subseteq \tilde{\Omega} \) and \( \text{supp} X_0 \subseteq [\phi = 1] \). Moreover, we set
\[
A := \text{skew} \alpha M_0 \begin{pmatrix} \mathcal{L}_{X_0} & 0 \\ 0 & -\mathcal{L}_{X_0}^\ast \end{pmatrix} + \begin{pmatrix} 0 & -d^\ast \\ d & 0 \end{pmatrix}
\]

\[ U_1 := \Lambda^k(\tilde{\Omega}) \times \Lambda^{k+1}(\tilde{\Omega}), \quad B_1 := \iota_{U_1}^* \text{skew} \alpha M_0 \begin{pmatrix} \mathcal{L}_{X_0} & 0 \\ 0 & -\mathcal{L}_{X_0}^\ast \end{pmatrix} \iota_{U_1} + \begin{pmatrix} 0 & -d^\ast \Omega \\ d & 0 \end{pmatrix} \] and \( S_1 := \phi(m) \) the multiplication operator with \( \phi \). In order to apply Lemma 4.19 we have to assume the existence of an extension operator \( E_1 : \text{dom}(B_1) \to \text{dom}(A) \), which is an implicit regularity assumption for the part of the boundary of \( \Omega \) in the interior of \( \Omega \).

\[ U_2 := \Lambda^k(\hat{\Omega}) \times \Lambda^{k+1}(\hat{\Omega}), \quad B_2 := \begin{pmatrix} 0 & -\partial^\ast \Omega \\ \partial & 0 \end{pmatrix} \] and \( S_2 := 1 - \phi(m) \) the multiplication operator with \( 1 - \phi \). Again, we have to assume the existence of an extension operator \( E_2 : \text{dom}(B_2) \to \text{dom}(A) \), which is an implicit regularity assumption for the part of the boundary of \( \Omega \) in the interior of \( \Omega \).

**Theorem 4.20.** Assume that \( B_1 \) and \( B_2 \) are skew-selfadjoint operators on \( U_1 \) and \( U_2 \), respectively. Then \( A \) is skew-selfadjoint on \( H \).

**Proof.** Since \( A \) is clearly skew-symmetric, it suffices to prove \( \text{dom}(A^*) \subseteq \text{dom}(A) \). For doing so, let \( v \in \text{dom}(A^*) \). By Lemma 4.19 we have that \( \iota_{U_1}^* S_1 v \in \text{dom}(B_1^*) = \text{dom}(B_1) \) as well as \( \iota_{U_2}^* S_2 v \in \text{dom}(B_2^*) = \text{dom}(B_2) \). Next, we observe that \( S_1 v, S_2 v \in \text{dom}(A) \). Indeed, since \( \iota_{U_1}^* S_1 v \in \text{dom}(B_1) \), we find a sequence \( (\psi_n)_n \) in \( \text{dom} \left( \iota_{U_1}^* \text{skew} \alpha M_0 \begin{pmatrix} \mathcal{L}_{X_0} & 0 \\ 0 & -\mathcal{L}_{X_0}^\ast \end{pmatrix} \iota_{U_1} + \begin{pmatrix} 0 & -d^\ast \Omega \\ d & 0 \end{pmatrix} \right) \) with
\[
\psi_n \to \iota_{U_1}^* S_1 v \quad \text{and} \quad \left( \iota_{U_1}^* \text{skew} \alpha M_0 \begin{pmatrix} \mathcal{L}_{X_0} & 0 \\ 0 & -\mathcal{L}_{X_0}^\ast \end{pmatrix} \iota_{U_1} + \begin{pmatrix} 0 & -d^\ast \Omega \\ d & 0 \end{pmatrix} \right) \psi_n \to B_1 \iota_{U_1}^* S_1 v \quad (n \to \infty)
\] in \( U_1 \). Take a smooth function \( \zeta : \Omega \to \mathbb{R} \) with \( \text{supp} \zeta \subseteq \tilde{\Omega} \) and \( \zeta = 1 \) on \( \text{supp} \phi \). Moreover, we denote by \( \tilde{\psi}_n \) the extension of \( \psi_n \) to \( \Omega \) by 0. Then
\[
\zeta \tilde{\psi}_n \to S_1 v
\]
in $H$ and

\[
\left( \text{skew } \alpha M_0 \left( \frac{\mathcal{L}X_0}{0} - \frac{\mathcal{L}X_0^*}{0} \right) \right) \zeta \widetilde{\psi}_n
\]

\[
= \iota U_i \left( \iota^* \iota U_i \text{skew } \alpha M_0 \left( \frac{\mathcal{L}X_0}{0} - \frac{\mathcal{L}X_0^*}{0} \right) \iota U_i + \left( \frac{0 - \tilde{d}_0}{\partial \Omega} \right) \right) \zeta \psi_n
\]

\[
= \iota U_i \zeta \left( \iota^* \iota U_i \text{skew } \alpha M_0 \left( \frac{\mathcal{L}X_0}{0} - \frac{\mathcal{L}X_0^*}{0} \right) \iota U_i + \left( \frac{0 - \tilde{d}_0}{\partial \Omega} \right) \right) \psi_n +
\]

\[
+ \iota U_i \left[ \iota^* \iota U_i \text{skew } \alpha M_0 \left( \frac{\mathcal{L}X_0}{0} - \frac{\mathcal{L}X_0^*}{0} \right) \iota U_i + \left( \frac{0 - \tilde{d}_0}{\partial \Omega} \right) \right] \psi_n.
\]

Since $\zeta$ is smooth, the commutator $\left[ \iota^* \iota U_i \text{skew } \alpha M_0 \left( \frac{\mathcal{L}X_0}{0} - \frac{\mathcal{L}X_0^*}{0} \right) \iota U_i + \left( \frac{0 - \tilde{d}_0}{\partial \Omega} \right) \right] \zeta$ is continuous and hence, the latter term converges in $H$ as $n \to \infty$. Thus, $S_1 v \in \text{dom}(A)$ by definition of $A$. In the same way one obtains $S_2 v \in \text{dom}(A)$ and thus, $v = S_1 v + S_2 v \in \text{dom}(A)$, which completes the proof.

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**References**

[1] A.-S. Bonnet-Ben Dhia, G. Legendre, and É. Lunéville. Analyse mathématique de l’équation de galbrun en écoulement uniforme. *Comptes Rendus de l’Académie des Sciences - Series IIB - Mechanics*, 329(8):601 – 606, 2001.

[2] G. da Prato and P. Grisvard. Sommes d’opérateurs linéaires et équations différentielles opérationnelles. *J. Math. Pures Appl. (9)*, 54:305–387, 1975.

[3] K. O. Friedrichs. The identity of weak and strong extensions of differential operators. *Trans. Am. Math. Soc.*, 55:132–151, 1944.

[4] K. O. Friedrichs. Symmetric positive linear differential equations. *Commun. Pure Appl. Math.*, 11:333–418, 1958.

[5] H. Galbrun. Propagation d’une onde sonore dans l’atmosphere et théorie des zones de silence. Paris: Gauthier-Villars & Cie., 1931.

[6] L. Hägg and M. Berggren. On the well-posedness of Galbrun’s equation, arXiv: 1912.04364, 2019.

[7] R. Picard. A structural observation for linear material laws in classical mathematical physics. *Math. Methods Appl. Sci.*, 32(14):1768–1803, 2009.

[8] R. Picard and D. McGhee. *Partial differential equations. A unified Hilbert space approach.*, volume 55. Berlin: de Gruyter, 2011.

[9] R. Picard, S. Trostorff, M. Waurick, and M. Wehowski. On non-autonomous evolutionary problems. *J. Evol. Equ.*, 13(4):751–776, 2013.
[10] J. Rauch. Symmetric positive systems with boundary characteristic of constant multiplicity. *Trans. Am. Math. Soc.*, 291:167–187, 1985.

[11] C. Seifert, S. Trostorff, and M. Waurick. Evolutionary equations, arXiv: 2003.12403, 2020.

[12] S. Trostorff. An alternative approach to well-posedness of a class of differential inclusions in Hilbert spaces. *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, 75(15):5851–5865, 2012.

[13] S. Trostorff. Well-posedness for a general class of differential inclusions. *J. Differ. Equations*, 268(11):6489–6516, 2020.

[14] M. Waurick. On non-autonomous integro-differential-algebraic evolutionary problems. *Math. Methods Appl. Sci.*, 38(4):665–676, 2015.