On a constrained reaction-diffusion system related to multiphase problems

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Dedicated to MASAYASU MIMURA on the occasion of its 65th birthday

Abstract

We solve and characterize the Lagrange multipliers of a reaction-diffusion system in the Gibbs simplex of $\mathbb{R}^{N+1}$ by considering strong solutions of a system of parabolic variational inequalities in $\mathbb{R}^N$. Exploring properties of the two obstacles evolution problem, we obtain and approximate a $N$-system involving the characteristic functions of the saturated and/or degenerated phases in the nonlinear reaction terms. We also show continuous dependence results and we establish sufficient conditions of non-degeneracy for the stability of those phase subregions.

1 Introduction

This paper is motivated by the vector-valued reaction-diffusion equation

$$\partial_t U - \Delta U = F(x, t, U), \quad \text{in } Q,$$

for $U = U(x, t)$, defined from $Q = \Omega \times (0, T)$ into $\mathbb{R}^{N+1}$, with homogeneous Neumann condition on $\partial\Omega \times (0, T)$, where $\Omega$ is a bounded domain of $\mathbb{R}^n$ and $T > 0$ is arbitrary. We are interested in the case when every component $u_i = u_i(x, t)$ is nonnegative and the system is subject to the multiphase non-voids condition with $J = (1, \ldots, 1) \in \mathbb{R}^{N+1}$;

$$U \cdot J = \sum_{j=1}^{N+1} u_j = 1 \quad \text{in } Q.$$
From the equation (1) it is clear that the constraint (2) implies $F(x, t, U) \cdot J = 0$ in $Q$ and so the reaction vector $F$ should satisfy the necessary and very restrictive condition

$$F_{N+1}(x, t, V) = - \sum_{j=1}^{N} F_j(x, t, V) \quad \text{in} \quad Q, \quad \forall V = (v_1, \ldots, v_N, 1 - \sum_{j=1}^{N} v_j), \quad 0 \leq v_i \leq 1. \quad (3)$$

For instance, in replicator dynamics describing the evolution of certain frequencies in a population, one possible definition of the reaction term with this compatibility condition consists in choosing

$$F_i(x, t, V) = v_i[\phi_i(x, t, V) - \sum_{j=i}^{N+1} v_j\phi_j(x, t, V)] \quad \text{in} \quad Q, \quad i = 1, \ldots, N + 1, \quad (4)$$

where $v_i$ represents the $i$-frequency of the population and $\phi_i$ the respective fitness (see, for instance, [10] and [11]), the constraint (2) is essential to describe mixed strategies in evolutionary game theory in spatially homogeneous population dynamics (see [18] and its references) or to model the non-voids condition in biological tissue growing [15, 14]. In phase fields models, the condition (2) arises naturally in simulation of multiphase flows (ref 13) and multiphase systems with diffuse phase boundaries, as in solidification of alloys or in grain boundary motion (see [9] or [3]).

Of course, in the case (3), in particular, if $F = 0$, the problem becomes a simple one if the initial data $U(0) = U_0$ also satisfies the constraint (2). However the situation is entirely different in the general case of non trivial reactions, specially in multiphase problems where at least one phase “$i$” in a subregion of $Q$ is absent (i.e. $u_i = 0$), or fulfils another subregion (when $u_i = 1$).

Instead of solving the system (1) in the Gibbs (N+1)-simplex

$$\Psi = \{(v_1, \ldots, v_{N+1}) \in R^{N+1} : \sum_{j=1}^{N+1} v_j = 1 \text{ and } v_i \geq 0, \quad i = 1, \ldots, N + 1\},$$

we shall replace this problem by the study of a unilateral problem for the vector field of the first $N$ components $u = (u_1, \ldots, u_N)$ of $U$, with the $N + 1$ convex constraints

$$\sum_{i=j}^{N} u_j \leq 1 \quad \text{and} \quad u_i \geq 0 \quad \text{in} \quad Q, \quad i = 1, \ldots, N. \quad (5)$$

This corresponds to solve the system of parabolic variational inequalities, at each time
\[ t \in (0, T), \]
\[ u(t) \in \mathbb{K} : \quad \int_\Omega \partial_t u(t) \cdot (v - u(t)) + \int_\Omega \nabla u(t) \cdot \nabla (v - u(t)) \]
\[ \geq \int_\Omega f(u(t)) \cdot (v - u(t)), \quad \forall v \in \mathbb{K}, \quad (6) \]

under the initial condition
\[ u(0) = u_0 = (u_{01}, \ldots, u_{0N}) \in \mathbb{K}. \quad (7) \]

Here \( \mathbb{K} \) denotes the convex subset of the Sobolev space \( H^1(\Omega)^N \) defined by
\[ \mathbb{K} = \{ v \in H^1(\Omega)^N : \sum_{j=1}^{N} v_j \leq 1, \ v_i \geq 0, \ i = 1, \ldots, N, \text{ in } \Omega \}, \quad (8) \]
where \( v = (v_1, \ldots, v_N) \).

The reaction term may have a general form \( f_i(u) = f_i(x, t, U(x, t)), i = 1, \ldots, N, \) with \( (x, t) \in Q \) and \( U = (u_1, \ldots, u_N, 1 - \sum_{j=1}^{N} u_j) \). We denote \( \partial_t = \frac{\partial}{\partial t} \) and \( \nabla = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}) \).

The main part of this work is the analysis of the new unilateral problem (6)-(7) under general assumptions on \( f \): only continuity on \( u \) and integrability in \( (x, t) \in Q \). In particular, we prove that its solution \( u = u(x, t) \), which each component \( u_i \) satisfies a double obstacle problem
\[ 0 \leq u_i \leq 1 - \sum_{j \neq i} u_j \quad \text{in } Q, \quad i = 1, \ldots, N, \quad (9) \]
where \( \sum_{j \neq i} u_j \) denotes the sum of all \( N - 1 \) components but \( u_i \) is, in fact, also the solution of a reaction-diffusion system in the form
\[ \partial_t u_i - \Delta u_i = f_i(u) + f_i^-(u)\chi_{\{u_i=0\}} - \sum_{1 \leq i_1 < \ldots < i_k \leq N \atop i \in \{i_1, \ldots, i_k\}} \frac{1}{k} (f_{i_1}(u) + \cdots + f_{i_k}(u))^+ \chi_{i_1 \ldots i_k}, \quad \text{in } Q. \quad (10) \]

Here \( \sum_{1 \leq i_1 < \ldots < i_k \leq N \atop i \in \{i_1, \ldots, i_k\}} \) denotes the summation over all the subsets \( \{i_1, \ldots, i_k\} \) of \( \{1, \ldots, N\} \) to which \( i \) belongs, in particular, \( k \) varies from 1 to \( N \). We also denote \( g^+ = g \vee 0 \) and \( g^- = -(g \wedge 0) \) the positive and negative parts of a scalar function \( g = g^+ - g^- \), \( \chi_A \) the
characteristic function of the set $A$, (i.e., $\chi_A = 1$ in $A$ and $\chi_A = 0$ in $Q \setminus A$) and $\chi_{i_1\ldots i_k}$ the characteristic function of the set

$$I_{i_1\ldots i_k} = \{(x,t) \in Q : (u_{i_1} + \cdots + u_{i_k})(x,t) = 1, \ u_{i_j}(x,t) > 0, j = 1, \ldots, k\}, \ k \in \{1, \ldots, N\}.$$  

In particular $\{u_i = 1\} = \bigcap_{j \neq i}\{u_j = 0\}$, i.e., one component is fully saturated if and only if the others are absent. Hence from (10) we see that, in general, the respective reaction terms are coupled not only through the semilinear term $f(u)$ but also through the characteristic functions of the saturation sets of $I_{i_1\ldots i_k}$.

In this way, by setting for $i = 1, \ldots, N$,

$$F_i(U) = f_i(u) + f_i^-(u)\chi_{\{u_i = 0\}} - \sum_{1 \leq i_1 < \cdots < i_k \leq N, \ i \in \{1, \ldots, i_k\}} \frac{1}{k} (f_{i_1}(u) + \cdots + f_{i_k}(u))^+\chi_{i_1\ldots i_k},$$

with $U = (u, 1 - \sum_{j=1}^N u_j)$, we can solve the system (1) under the constraint (2) and identify the respective Lagrange multipliers $h_i \equiv F_i(U) - f_i(U)$ in a precise form.

To illustrate the meaning of the system (10), that contains $2^N - 1 + N$ characteristic functions, in general, we may consider the cases $N = 1, 2$ or 3. Denoting, for simplicity, $f_i = f_i(u)$, $\chi_i = \chi_{\{u_i = 1\}}$, we may write the Lagrange multipliers as

$$h_1 = f_1^-\chi_{\{u_1 = 0\}} - f_1^+\chi_1 - \frac{1}{2}(f_1 + f_2)^+\chi_{12} - \frac{1}{2}(f_1 + f_3)^+\chi_{13} - \frac{1}{3}(f_1 + f_2 + f_3)^+\chi_{123}$$

$$h_2 = f_2^-\chi_{\{u_2 = 0\}} - f_2^+\chi_2 - \frac{1}{2}(f_1 + f_2)^+\chi_{12} - \frac{1}{2}(f_2 + f_3)^+\chi_{23} - \frac{1}{3}(f_1 + f_2 + f_3)^+\chi_{123}$$

$$h_3 = f_3^-\chi_{\{u_3 = 0\}} - f_3^+\chi_3 - \frac{1}{2}(f_1 + f_3)^+\chi_{13} - \frac{1}{2}(f_2 + f_3)^+\chi_{23} - \frac{1}{3}(f_1 + f_2 + f_3)^+\chi_{123}$$

Ignoring the third equation and all the terms involving the third component, we may obtain the case $N = 2$. The first two terms of the right hand side of the first equation correspond, in the case $N = 1$, to the scalar two obstacles problem that has been proposed for phase separations in [4, 5].

The mathematical treatment of this unilateral system is done in the following three sections. In section 2, we consider the semilinear approximation of the unique solution of (6)-(7) in the case of the reaction $f$ is in $L^2(Q)^N$ and independent of the solution. Although there exits a large literature on parabolic variational inequalities (see, for instance, [16], [6], [12], [7] or [8]), the direct approach of the bounded penalization used for the two obstacles problem in [22] (see also [19]), extended here for the system (10), allows the use of monotone methods. This yields a direct way of obtaining Lewy-Stampacchia inequalities (26), obtained first by [7] for parabolic problems, implying the $W^{2,1}_p$ and Hölder regularity.
for the solution to (6). Similar results for the N-membranes stationary problem have been obtained in [1, 2]. We note in our case the simplification due to homogeneous Neumann condition.

In section 3, we extend the existence result to general nonlinear reaction \( f = f(u) \) taking values in \( L^1(Q)^N \). Here we explore the fact that the convex set (8) lies in the unit disc and we extend the direct technique of [20]. We show also a continuous dependence result and, in the case of \( \lambda I - f \) being monotone non-decreasing, in particular if \( f \) is Lipschitz continuous in \( u \), also the uniqueness of solution and their strong approximation by the penalized solutions.

Finally, in the last section, we characterize the solution of the variational inequality (6) as solutions of the reaction-diffusion system (10), by extending some remarks of [23] to the two obstacles parabolic problem. We also show that

\[
\{ u_i = 0 \} \subset \{ f_i(u) \leq 0 \} \quad \text{and} \quad I_{i_1...i_k} \subset \{ \sum_{j=1}^k f_{ij}(u) \geq 0 \}
\]

a.e. in \( Q \), for \( 1 \leq i_1 < \cdots < i_k \leq N, \forall k = 1, \ldots, N \) and we can modify the system (10) (see (77)) and show that the a.e. pointwise nondegeneracy assumptions

\[
\sum_{j=1}^k f_{ij}(u) \neq 0, \quad 1 \leq i_1 < \cdots < i_k \leq N, \quad k = 1, \ldots, N,
\]

are sufficient conditions for the local stability of the characteristic functions \( \chi_{\{u_i=0\}} \) and \( \chi_{i_1...i_k} \) with respect to the perturbation of the nonlinear reaction terms \( f \).

2 Approximation of strong solutions by semilinear problems

In this section we consider the case where \( f = (f_1, \ldots, f_N) \) depends only on \((x, t)\) and is given in \( L^2(Q)^N \).

To prove existence of solution of the variational inequality (6)-(7), we consider a family of approximating semilinear systems of equations. We define, for each \( \varepsilon > 0 \), \( \theta_\varepsilon : \mathbb{R} \rightarrow \mathbb{R} \) by

\[
\theta_\varepsilon(s) = \begin{cases} 
0 & \text{if } s \geq 0 \\
\frac{s}{\varepsilon} & \text{if } -\varepsilon < s < 0 \\
-1 & \text{if } s \leq -\varepsilon,
\end{cases}
\]

(11)
and we denote
\[ Pu = \partial_t u - \Delta u = (Pu_1, \ldots, Pu_N), \]
where \( \partial_t u = (\partial_t u_1, \ldots, \partial_t u_N) \) and \( \Delta u = (\Delta u_1, \ldots, \Delta u_N) \). We also denote \( Pu_i = \partial_t u_i - \Delta u_i, \quad i = 1, \ldots, N \). The approximating problems are given by the following weakly coupled parabolic system with Neumann condition
\[
Pu_i^\varepsilon + f_i^\varepsilon - \theta_\varepsilon(u_i^\varepsilon) - \sum_{1 \leq i_1 < \cdots < i_k \leq N} \frac{1}{k} (f_{i_1} + \cdots + f_{i_k})^\varepsilon \theta_\varepsilon(1 - u_{i_1\ldots i_k}) = f_i \quad \text{in } Q, \tag{12}
\]
\[
\frac{\partial u_i^\varepsilon}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0, T), \tag{13}
\]
\[
u_i(0) = u_{0i} \quad \text{in } \Omega, \quad (i = 1, \ldots, N) \tag{14}
\]
where \( \frac{\partial}{\partial n} \) is the outward normal derivative on \( \partial\Omega \times (0, T) \), the meaning of \( \sum_{1 \leq i_1 < \cdots < i_k \leq N} \) was explained in the introduction and
\[
\forall v = (v_1, \ldots, v_N) \quad \forall \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, N\} \quad v_{i_1\ldots i_k} = v_{i_1} + \cdots + v_{i_k}. \tag{15}
\]

Defining the penalization operator \( \Theta_\varepsilon \) by
\[
\Theta_\varepsilon u \cdot v = \sum_{i=1}^{N} \left[ f_i^\varepsilon \theta_\varepsilon(u_i) - \sum_{1 \leq i_1 < \cdots < i_k \leq N} \frac{1}{k} (f_{i_1} + \cdots + f_{i_k})^\varepsilon \theta_\varepsilon(1 - u_{i_1\ldots i_k}) \right] v_i \tag{16}
\]
\[
= \sum_{i=1}^{N} f_i^\varepsilon \theta_\varepsilon(u_i)v_i - \sum_{1 \leq i_1 < \cdots < i_k \leq N} \frac{1}{k} (f_{i_1} + \cdots + f_{i_k})^\varepsilon \theta_\varepsilon(1 - u_{i_1\ldots i_k}) v_{i_1\ldots i_k}, \tag{17}
\]
we formulate (12)-(13) in variational form for a.e. \( t \in (0, T) \),
\[
\int_\Omega \partial_t u^\varepsilon(t) \cdot v + \int_\Omega \nabla u^\varepsilon(t) \cdot \nabla v + \int_\Omega \Theta_\varepsilon(u^\varepsilon(t)) \cdot v = \int_\Omega f(t) \cdot v, \quad \forall v \in H^1(\Omega)^N, \tag{18}
\]
associated with the initial condition (14).

**Proposition 2.1.** Assuming that
\[
f = (f_1, \ldots, f_N) \in L^2(Q)^N \quad \text{and} \quad u_0 \in K, \tag{19}
\]
the problem (18)-(14) has a unique solution \( u^\varepsilon \in H^1(0, T; L^2(\Omega)^N) \cap L^\infty(0, T; H^1(\Omega)^N) \).
Proof. We begin by proving the monotonicity of the penalization operator $\Theta_\varepsilon$.

In fact, recalling that $\theta_\varepsilon$ is monotone nondecreasing and the definition (15) we have

$$(\Theta_\varepsilon u - \Theta_\varepsilon v) \cdot (u - v)$$

$$= \sum_{i=1}^{N} f_i^- (\theta_\varepsilon(u_i) - \theta_\varepsilon(v_i))(u_i - v_i)$$

$$- \sum_{1 \leq i_1 < \ldots < i_k \leq N} \frac{1}{k} (f_{i_1} + \ldots + f_{i_k})^+ (\theta_\varepsilon(1 - u_{i_1} \ldots i_k) - \theta_\varepsilon(1 - v_{1 \ldots i_k}))(u_{i_1 \ldots i_k} - v_{i_1 \ldots i_k}),$$

$$\geq 0,$$

since $f_j^-$ and $(f_{i_1} + \ldots + f_{i_k})^+$ are nonnegative functions.

The existence and uniqueness of solution $u_\varepsilon \in L^2(0, T; H^1(\Omega)^N)$ is immediate by applying the theory of monotone operators ([16], [25]).

Setting $v = (u_{1}^{\varepsilon}, \ldots, u_{N}^{\varepsilon})$ in the approximating problem (18) and integrating in time, letting

$$g_i^\varepsilon = Pu_i^\varepsilon = f_i - f_i^- \theta_\varepsilon(u_i^\varepsilon) + \sum_{1 \leq i_1 < \ldots < i_k \leq N} \frac{1}{k} (f_{i_1} + \ldots + f_{i_k})^+ \theta_\varepsilon(1 - u_{i_1}^{\varepsilon} \ldots i_k),$$

which is bounded in $L^2(Q)$ independently of $\varepsilon$, we obtain that, for every $0 < t < T$, with $Q_t = \Omega \times (0, t)$,

$$\frac{1}{2} \int_\Omega |u^\varepsilon(t)|^2 + \int_{Q_t} |\nabla u^\varepsilon|^2 \leq \frac{1}{2} \int_\Omega |u_0|^2 + \frac{1}{2} \int_{Q_t} |g|^2 + \frac{1}{2} \int_{Q_t} |u|^2.$$

The Gronwall inequality yields the uniform boundedness (in $\varepsilon$) of $u_\varepsilon$, first in $L^\infty(0, T; L^2(\Omega)^N)$ and afterwards also in $L^2(0, T; H^1(\Omega)^N)$.

Letting, formally, $v = \partial_t u_\varepsilon$ in (18) (in fact in the respective Faedo-Galerkin approximation) and integrating in time, we get

$$\int_{Q_t} |\partial_t u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon(t)|^2 \leq \int_{Q_t} |g|^2 + \int_\Omega |\nabla u_0|^2$$

and so $\partial_t u_\varepsilon$ is also bounded in $L^2(Q)^N$ and $\nabla u_\varepsilon$ in $L^\infty(0, T; L^2(\Omega)^N)$. Therefore

$$\{u^\varepsilon\}_{\varepsilon > 0} \text{ is bounded in } H^1(0, T; L^2(\Omega)^N) \cap L^\infty(0, T; H^1(\Omega)^N).$$

\(\square\)
Proposition 2.2. Assuming (19), the solution $u^\varepsilon$ of the problem (18)-(14) satisfies

$$u_i^\varepsilon \geq -\varepsilon, \quad i = 1, \ldots, N, \quad \sum_{i=1}^{N} u_i^\varepsilon \leq 1 + \varepsilon. \quad (21)$$

Proof. In fact, we are going to prove the following more general set of inequalities

$$u_i^\varepsilon \geq -\varepsilon, \quad i = 1, \ldots, N, \quad \text{and} \quad u_{i_1 \ldots i_r}^\varepsilon \leq 1 + \varepsilon, \quad \forall 1 \leq i_1 < \ldots < i_r \leq N$$

and the proof of the right-hand side inequalities will be done by induction on $r$.

Let us prove the case $r = 1$, i.e., $u_i^\varepsilon \leq 1 + \varepsilon$, for all $i \in \{1, \ldots, N\}$. Multiplying the $i$-th equation of the approximating system (12) by $(u_i^\varepsilon - (1 + \varepsilon))^+$ and integrating over $Q_t = \Omega \times (0, t)$, we have

$$\int_{Q_t} \theta_t u_i^\varepsilon (u_i^\varepsilon - (1 + \varepsilon))^+ + \int_{Q_t} \nabla u_i^\varepsilon \cdot \nabla (u_i^\varepsilon - (1 + \varepsilon))^+$$

$$= \int_{Q_t} \left[ f_i - \varepsilon^- \theta_t (u_i^\varepsilon) + \sum_{1 \leq i < \ldots < u \leq N} \frac{1}{k} (f_{i_1} + \cdots + f_{i_k})^+ \theta_t (1 - u_{i_1 \ldots i_k}^\varepsilon) \right] (u_i^\varepsilon - (1 + \varepsilon))^+$$

Recalling that $-1 \leq \theta_t \leq 0$ and that, in the set $\{u_i^\varepsilon > 1 + \varepsilon\}$, we have $\theta_t (u_i^\varepsilon) = 0$ and $\theta_t (1 - u_i^\varepsilon) = -1$ we get

$$\frac{1}{2} \int_{\Omega} |(u_i^\varepsilon - (1 + \varepsilon))^+(t)|^2 + \int_{Q} |\nabla (u_i^\varepsilon - (1 + \varepsilon))^+|^2 \leq \int_{Q} (f_i - f_i^+)(u_i^\varepsilon - (1 + \varepsilon))^+ \leq 0, \quad (22)$$

so $(u_i^\varepsilon - (1 + \varepsilon))^+ \equiv 0$, i.e. $u_i^\varepsilon \leq 1 + \varepsilon$.

Assuming we have proved that $u_{i_1 \ldots i_r}^\varepsilon \leq 1 + \varepsilon$, we are going to show that $u_{i_1 \ldots i_{r+1}}^\varepsilon \leq 1 + \varepsilon$.

We multiply the equations $i_1 j, j = 1, \ldots, r + 1$, by $(u_{i_1 \ldots i_{r+1}}^\varepsilon - (1 + \varepsilon))^+$, sum from 1 to $r + 1$ and integrate over $Q_t$. We obtain

$$\int_{Q_t} P u_{i_1 \ldots i_{r+1}}^\varepsilon (u_{i_1 \ldots i_{r+1}}^\varepsilon - (1 + \varepsilon))^+ = \int_{Q_t} \left[ \sum_{j=1}^{r+1} f_{i_j} - \sum_{j=1}^{r+1} f_{i_j}^- \theta_t (u_{i_j}) \right]$$

$$+ \sum_{j=1}^{r+1} \sum_{1 \leq i_1 < \ldots < i_k \leq N} \frac{1}{k} (f_{i_1} + \cdots + f_{i_k})^+ \theta_t (1 - u_{i_1 \ldots i_k}^\varepsilon) (u_{i_1 \ldots i_{r+1}}^\varepsilon - (1 + \varepsilon))^+.$$

Observe that, in the set $\{u_{i_1 \ldots i_{r+1}}^\varepsilon > 1 + \varepsilon\}$ we have $u_{i_j}^\varepsilon \geq 0$, for $j = 1, \ldots, r + 1$, since, by induction, $u_{i_1 \ldots i_r}^\varepsilon = u_{i_1}^\varepsilon + \cdots + u_{i_{r+1}}^\varepsilon - u_{i_j}^\varepsilon \leq 1 + \varepsilon$. So, in that set $\theta_t (u_{i_j}) = 0$ and, on
Proof. and obtained in (20), we know there exists \( u \) such that
\[
\text{Assuming Theorem 2.3.}
\]
that implies (6)-(7) has a unique solution \( u \) such that \( u \in H^1(0, T; L^2(\Omega)^N) \cap L^\infty(0, T; H^1(\Omega)^N) \) (23) and
\[
Pu \in L^2(Q)^N.
\] (24)

Proof. Let \( u^\varepsilon \) be the solution of the problem (18). Using the uniform estimates (in \( \varepsilon \)) obtained in (20), we know there exists \( u \) such that
\[
\begin{align*}
&u^\varepsilon \xrightarrow{\varepsilon} u \quad \text{in } L^2(Q)^N \text{ strong}, \\
&u^\varepsilon \xrightarrow{\varepsilon} u \quad \text{in } L^\infty(0, T; H^1(\Omega)^N) \text{ weak-*}, \\
&\partial_t u^\varepsilon \xrightarrow{\varepsilon} \partial_t u \quad \text{and} \quad Pu^\varepsilon \xrightarrow{\varepsilon} Pu \quad \text{in } L^2(Q)^N \text{ weak}.
\end{align*}
\]
We have \( u(t) \in K \), for a.e. \( t \in [0, T] \), because \( u^\varepsilon \) satisfies the inequalities (21).
Given \( v \in L^2(0, T; \mathbb{K}) \), set \( v(t) - u^\varepsilon(t) \) in (18) and integrate in time. Then

\[
\int_Q \partial_t u^\varepsilon \cdot (v - u) + \int_Q \nabla u^\varepsilon \cdot \nabla (v - u) \geq \int_Q f^\varepsilon \cdot (v - u^\varepsilon),
\]

since \( \int_Q (\Theta_\varepsilon(u^\varepsilon) - \Theta_\varepsilon(v)) \cdot (v - u^\varepsilon) \leq 0 \) and \( \Theta_\varepsilon(v(t)) = 0 \) if \( v(t) \in \mathbb{K} \). Passing to the limit when \( \varepsilon \to 0 \) and noting that

\[
\liminf_{\varepsilon \to 0} \int_Q (\partial_t u^\varepsilon \cdot u^\varepsilon + \nabla u^\varepsilon \cdot \nabla u^\varepsilon) \geq \int_Q (\partial_t u \cdot u + \nabla u \cdot \nabla u),
\]

we find that \( u \) satisfies (7) and

\[
\int_Q \partial_t u \cdot (v - u) + \int_Q \nabla u \cdot \nabla (v - u) \geq \int_Q f \cdot (v - u), \quad \forall v \in L^2(0, T; \mathbb{K}),
\]

which is easily seen to be equivalent to (6). The uniqueness is immediate. \( \square \)

We remark that no regularity of the boundary \( \partial \Omega \) has been required in (18) and, in fact, the Neumann boundary condition (13) is only formal. In the proof of Theorem 2.3 we have used the compactness of the sequence \( (u^\varepsilon)_\varepsilon \) in \( L^2(Q)^N \). This holds, for instance, for domains with Lipschitz boundaries, but also, since the sequence \( (u^\varepsilon)_\varepsilon \) is uniformly bounded in \( L^\infty(Q)^N \), for a larger class of bounded open subsets of \( \mathbb{R}^{N+1} \). However, the approximation by semilinear parabolic equations yields immediately an additional regularity of these strong solutions.

Indeed, from the definitions of \( \theta_\varepsilon \) and \( \Theta_\varepsilon \), from (18) with arbitrary \( \varphi \in \mathcal{D}(Q), \varphi \geq 0 \), we find

\[
f_i - \sum_{1 \leq i_1 < \ldots < i_k \leq N} \frac{1}{k} (f_{i_1} + \ldots + f_{i_k})^+ \leq Pu_i^\varepsilon = f_i - \Theta_\varepsilon(u^\varepsilon) \leq f_i + f_i^+ = f_i^+ \quad \text{a.e. in } Q.
\]

By the conclusion of Theorem 2.3 we also obtain, for each \( i = 1, \ldots, N \),

\[
f_i - \sum_{1 \leq i_1 < \ldots < i_k \leq N} \frac{1}{k} (f_{i_1} + \ldots + f_{i_k})^+ \leq Pu_i \leq f_i^+ \quad \text{a.e. in } Q
\]

and we can apply directly the second order linear parabolic theory (see [17]) in the Sobolev spaces

\[
W^{2,1}_p(Q) = W^{1,p}(0, T; L^p(\Omega)) \cap L^p(0, T; W^{2,p}(\Omega)), \quad 1 < p < \infty.
\]

These spaces satisfy the Sobolev imbeddings, for \( p > (n + 2)/(2 - k) \), with \( k = 0, 1 \),

\[
W^{2,1}_p(Q) \subset C^{k,0}_\alpha(Q), \quad 0 \leq \alpha < 2 - k - (n + 2)/p.
\]
where $C^{k,0}_\alpha(Q)$ denotes the spaces of Hölder continuous functions $v$ in $Q$, with exponent $\alpha$ in the $x$-variables and $\alpha/2$ in the $t$-variable and, in the case $k = 1$, with $\nabla v$ satisfying the same property (see [17], p. 80). Therefore, as a consequence of (27), we conclude.

\[ \square \]

Theorem 2.4. Assume that $\partial \Omega$ is smooth, say of class $C^2$ and

$$ f \in L^p(Q)^N \quad \text{and} \quad u_0 \in K \cap W^{2-2/p,p}(\Omega)^N, \quad 1 < p < \infty, \quad (28) $$

with each component $u_{0i}$ satisfying the compatibility condition $\frac{\partial u_{0i}}{\partial n} = 0$ on $\partial \Omega$ if $p > 3$.

Then the unique solution $u$ of the variational inequality (6)-(7) is such that

$$ u \in W^{2,1}_p(Q)^N \cap L^\infty(0,T;K), \quad (29) $$

and, in particular, is Hölder continuous in $\overline{Q}$ if $p > (n + 2)/2$ and has $\nabla u$ also Hölder continuous if $p > n+2$.

\[ \square \]

We observe that, when $p < 2$, the inclusion $W^{2,1}_p(Q) \subset L^2(0,T;H^1(\Omega))$ only takes place if $p \geq (2n+4)/(n+4)$ but, as we shall see in the next section and since $K$ is bounded, (6)-(7) is solvable for any $f \in L^1(Q)^N$.

### 3 Existence and uniqueness of variational solutions

In this section, requiring the compactness of the inclusion of $H^1(\Omega)$ into $L^2(\Omega)$ by assuming a Lipschitz boundary $\partial \Omega$, we show how we can still solve the variational inequality (25) for a more general initial condition

$$ u_0 \in \tilde{K} = \{v \in L^2(\Omega)^N : \sum_{j=1}^N v_j \leq 1, \ v_i \geq 0, \ i = 1, \ldots, N, \ \text{in} \ \Omega\} \quad (30) $$

and for general nonlinear $f = f(u)$ defining a continuous operator from $L^2(0,T;\tilde{K})$ in $L^1(Q)^N$. We shall assume that $f = f(x,t,v) : Q \times [0,1]^N \rightarrow \mathbb{R}^N$ satisfies

$$ f = f(x,t,v) \text{ is continuous in } v \text{ for a.e. } (x,t) \in Q, \quad (31) $$

$$ \exists \varphi_1 \in L^1(Q) : \quad |f(x,t,v)| \leq \varphi_1(x) \quad \forall v \in [0,1]^N, \text{ for a.e. } (x,t) \in Q. \quad (32) $$
However, now the solution has less regularity, namely

\[ u \in C([0,T]; L^2(\Omega) \cap \tilde{\mathbb{K}) } \cap L^2(0,T; H^1(\Omega)^N) \tag{33} \]

and its derivative may not be a function, since we only have

\[ \partial_t u \in L^1(Q)^N + L^2(0,T; (H^1(\Omega)^N)') \tag{34} \]

Hence the first term in the variational inequality (25) should be interpreted in the duality sense between \( L^1(Q)^N + L^2(0,T; (H^1(\Omega)^N)') \) and \( L^\infty(Q)^N \cap L^2(0,T; H^1(\Omega)^N) \), namely through the formula

\[ (\partial_t u, v)_t = \int_{Q_t} Pu \cdot v - \int_{Q_t} \nabla u \cdot \nabla v, \quad \forall v \in L^\infty(Q)^N \cap L^2(0,T; H^1(\Omega)^N), \tag{35} \]

for arbitrary \( t \in (0,T] \) since, as we shall see, (27) yields \( Pu \in L^1(Q)^N \).

**Theorem 3.1.** Under the assumptions (30), (31) and (32), the variational inequality (25) has a solution \( u \) satisfying (33), (34), (27) and \( u(0) = u_0 \) and we can write

\[ \int_Q (Pu - f(u)) \cdot (v - u) \geq 0, \quad \forall v \in L^2(0,T; \tilde{\mathbb{K}}), \tag{36} \]

**Proof.** We consider the closed convex subset of \( L^2(Q)^N \)

\[ K = L^2(0,T; \tilde{\mathbb{K}}) = \{ v \in L^2(Q)^N : u_i \geq 0, i = 1, \ldots, N, \sum_{i=1}^N u_i \leq 1 \text{ in } Q \} \]

and we define \( \Phi : K \to K \) as the nonlinear operator that associates to each \( w \in K \) the solution \( u_w = \Phi(w) \) of the variational inequality (25) with \( f \) replaced by \( g = f(x,t,w) \) and fixed initial data \( u_0 \in \tilde{\mathbb{K}} \).

By showing that \( \Phi \) is a continuous and compact operator, a fixed point \( u = \Phi(u) \), given by Schauder Theorem, will provide a solution with the required properties.

Indeed, first we observe that if we consider any sequence \( K \ni w_\nu \longrightarrow w \in K \) in \( L^2(Q)^N \), by (31) and (32), the Lebesgue Theorem implies

\[ g_\nu = f(w_\nu) \longrightarrow f(w) = g \quad \text{in } L^1(Q)^N. \]

Next, for any \( g \in L^1(Q)^N \) and any \( u_0 \in \tilde{\mathbb{K}} \) we consider sequences \( g_\nu \in L^2(Q)^N \) and \( u_{0\nu} \in \mathbb{K} \) such that

\[ g_\nu \longrightarrow g \quad \text{in } L^1(Q)^N \quad \text{and} \quad u_{0\nu} \longrightarrow u_0 \quad \text{in } L^2(\Omega)^N \]
and we denote by \( u_\nu \equiv S(u_{0\nu}, g_\nu) \) the unique solution of (25)-(7) given by Theorem 2.3, for each \( g_\nu \) and \( u_{0\nu} \). We observe that each component of \( Pu_\nu \) satisfies the inequality (27) with \( f_i \) replaced by \( (g_\nu)_i \). From (25) for \( u_\mu \) and \( u_\nu \), we easily find, for a.e. \( t \in (0,T) \)

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |u_\mu - u_\nu|^2 + \int_\Omega |\nabla (u_\mu - u_\nu)|^2 \leq \int_\Omega (g_\mu - g_\nu) \cdot (u_\mu - u_\nu)
\]

and, integrating in time, we obtain

\[
\sup_{0<t<T} \int_\Omega |u_\mu(t) - u_\nu(t)|^2 + \int_Q |\nabla (u_\mu - u_\nu)|^2 \leq \int_\Omega |u_{0\mu} - u_{0\nu}|^2 + 4 \int_Q |g_\mu - g_\nu|.
\]

This estimate shows that \( \{u_\nu\}_\nu \) is a Cauchy sequence in the Banach space

\[
W = C([0,T]; L^2(\Omega)^N) \cap L^2(0,T; H^1(\Omega)^N)
\]

with respect to the norm

\[
\|v\| = \left( \sup_{0<t<T} \int_\Omega |v(t)|^2 + \int_Q |\nabla v|^2 \right)^{1/2}
\]

and, hence, there exists a function \( u_\nu \rightarrow u_g \) in \( W \).

In addition, \( u_g \in L^2(0,T; \mathbb{H}) \cap C([0,T]; \mathbb{H}) \) and \( Pu_g \in L^1(Q)^N \), which implies, by (35), that \( \partial_t u_g \) satisfies (34). Hence, using (35), we may pass to the limit in \( \nu \) in

\[
\langle Pu_\nu - g_\nu, v - u_\nu \rangle = \int_Q (Pu_\nu - g_\nu) \cdot (v - u_\nu) \geq 0
\]

for an arbitrary \( v \in L^2(0,T; \mathbb{H}) \subseteq L^\infty(Q)^N \), and using the formula

\[
2\langle \partial_t u_g, u_g \rangle_t = \int_\Omega |u_g(t)|^2 - \int_\Omega |u_0|^2, \quad \forall t \in (0,T],
\]

we conclude that \( u_g = S(u_0, g) \) is the (unique) solution of the variational inequality (25) (or equivalently (36)) with data \( g \in L^1(Q)^N \) and \( u_0 \in \mathbb{H} \). In particular, from (37), we also obtain that, for fixed \( u_0 \in \mathbb{H} \), the operator \( \Sigma : g \mapsto u_g = S(u_0, g) \) is Hölder continuous of order 1/2, from \( L^1(Q)^N \) into \( W \).

Since \( \partial_t u_g \) satisfies the property (34), it is in fact in \( L^1(0,T; H^s(\Omega)^N) \), for \( s \) sufficiently large and, by a well known compactness embedding (see [24] or Theorem 3.11 of [25]), the compactness of \( H^1(\Omega) \subset L^2(\Omega) \) implies that, in fact, \( \Sigma \) regarded as an operator from \( L^1(Q)^N \) into \( K \subset L^2(Q)^N \) is, therefore, completely continuous. Hence, \( \Phi = \Sigma \circ f \) fulfills the requirements of the Schauder fixed point theorem and the proof is complete. \( \square \)
Remark 3.2. It is clear that if \( u_0 \in K \) and, in (32), \( \varphi_1 \in L^2(Q) \), we obtain in Theorem 3.1 the existence of a strong solution satisfying (23) and (24). Of course, if we have the regularity assumptions of Theorem 2.4, i.e., \( \varphi_1 \in L^p(Q) \), implying by the inequalities (27) that \( Pu \in L^p(Q)^N \), we also obtain solutions in \( W^{2,1}_p(Q)^N \), in particular Hölder continuous solutions if \( p > (n + 2)/2 \).

In general (36) may have more than one solution, but if we assume, in addition, that for some \( \lambda > 0 \), \( \lambda I - f \) is monotone non-decreasing in \([0, 1]^N\), i.e.

\[
\exists \lambda > 0 : \lambda |v - w|^2 - (f(x, t, v) - f(x, t, w)) \cdot (v - w) \geq 0, \quad (x, t) \in Q, \forall v, w \in [0, 1]^N,
\]

(40)
in particular, if \( f \) is Lipschitz continuous in \( v \), then there exists at most one solution \( u \) of the variational inequality (25) in the class (33) and initial condition \( u_0 \in K \).

In order to prove the uniqueness of solution, we suppose that \( u_1 \) and \( u_2 \) are two solutions of the variational inequality (25) with initial condition \( u_0 \in K \) and \( f = f(u_1), f = f(u_2) \) respectively. Then, choosing \( u_2 \) and \( u_1 \) as test functions, respectively, using (40) we find

\[
\frac{1}{2} \int_{\Omega} |u_2(t) - u_1(t)|^2 + \int_Q |\nabla(u_2 - u_1)|^2 \leq \int_Q (f(u_2) - f(u_1)) \cdot (u_2 - u_1) \leq \lambda \int_Q |u_2 - u_1|^2
\]

and so, by Grownval inequality \( u_1 = u_2 \) a.e. in \( Q \), since \( u_1(0) = u_2(0) = u_0 \).

We redefine the variational formulation of the approximating problem (18) in the framework of this section with \( \Theta_\varepsilon \) defined in (16) and with initial condition only in \( L^2(\Omega)^N \),

\[
\int_Q \partial_t u^\varepsilon \cdot v + \int_Q \nabla u^\varepsilon \cdot \nabla v + \int_Q \Theta_\varepsilon(u^\varepsilon) \cdot v = \int_Q f(u^\varepsilon) \cdot v, \quad \forall v \in L^2(0, T; H^1(\Omega)^N) \cap L^\infty(Q)^N.
\]

(41)

Arguing as in Theorem 3.1 we may prove the existence of a solution of the approximating problem (12), with initial condition \( u_0 \in K \) as long as \( f \) satisfies (31) and (32). We also have uniqueness if we assume (40).

Theorem 3.3. Suppose that \( f \) satisfies (31), (32) and (40) and \( u_0 \in K \).

Let \( u^\varepsilon \) and \( u \) be, respectively, the unique solution of the approximating problem (12) and of the variational inequality (25), both with initial condition \( u_0 \). Then there exists a positive constant \( c = c(\varphi_1, T) \) such that the following estimate in the norm (39) of \( W = C([0, T]; L^2(\Omega)^N) \cap L^2(0, T; H^1(\Omega)^N) \) holds,

\[
\|u^\varepsilon - u\| \leq c\sqrt{\varepsilon}.
\]

(42)
Proof. We choose in (41) \( v = u^\varepsilon - u \) as test function. Since \( u \in K \), then

\[
\int_Q \Theta_\varepsilon(u^\varepsilon) \cdot (u^\varepsilon - u) \geq 0
\]

and so

\[
\int_{Q_t} \partial_t u^\varepsilon \cdot (u^\varepsilon - u) + \int_{Q_t} \nabla u^\varepsilon \cdot (u^\varepsilon - u) \leq \int_{Q_t} f(u^\varepsilon) \cdot (u^\varepsilon - u). \tag{43}
\]

Choosing, as test function in (25) \( v^\varepsilon = ((u_1^\varepsilon - \varepsilon_N^+) \ldots , (u_N^\varepsilon - \varepsilon_1^+) \) we get

\[
\int_{Q_t} \partial_t u \cdot (u^\varepsilon - u) + \int_{Q_t} \nabla u \cdot \nabla (u^\varepsilon - u)
\geq \int_{Q_t} f(u) \cdot (u^\varepsilon - u) + \int_{Q_t} [Pu - f(u)] \cdot (u^\varepsilon - v^\varepsilon) \tag{44}
\]

and subtracting (44) from (43) we get

\[
\frac{1}{2} \int_\Omega |u^\varepsilon(t) - u(t)|^2 + \int_{Q_t} |\nabla (u^\varepsilon - u)|^2
\leq \int_{Q_t} (f(u^\varepsilon) - f(u)) \cdot (u^\varepsilon - u) + \int_{Q_t} [Pu - f(u)] \cdot (v^\varepsilon - u^\varepsilon) \tag{45}
\]

since \( \|v^\varepsilon - u^\varepsilon\|_{L^\infty(Q)^N} \leq \varepsilon \). Letting \( C = C(\varphi_1, T) = \|Pu - f(u)\|_{L^1(Q)^N} \) and dropping the nonnegative term \( \int_{Q_t} |\nabla (u^\varepsilon - u)|^2 \) in (45) we obtain, by application of the Gronwall inequality,

\[
\int_\Omega |u^\varepsilon(t) - u(t)|^2 \leq 2\varepsilon Ce^{2M}
\]

and using again (45), also

\[
\|u^\varepsilon - u\| \leq c\sqrt{\varepsilon}.
\]

□

With similar arguments we may give a continuous dependence result for solutions of the variational inequality (36).

Suppose we have a sequence \( f^\nu \overset{\nu}{\longrightarrow} f \) in the following sense

\[
\begin{align*}
  f^\nu &= f^\nu(x, t, v) \text{ are continuous in } v \in [0,1]^N, \text{ for a.e. } (x, t) \in Q \\
  f^\nu(\cdot, \cdot, v) &\overset{\nu}{\longrightarrow} f(\cdot, \cdot, v) \text{ in } L^1(Q)^N \text{ for all fixed } v \in [0,1]^N.
\end{align*} \tag{46}
\]
In addition, the assumption (32) is satisfied for all \( f \) uniformly in \( \nu \), i.e., there is a common \( \varphi_1 \) such that (32) holds for all \( \nu \), and the initial data are such that

\[
\tilde{K} \supset u_0^\nu \xrightarrow{\nu} u_0 \quad \text{in } L^2(\Omega)^N. 
\]

(47)

Hence, by Theorem 3.1, it is clear that there are solutions \( \{u^\nu\}_{\nu \in \mathbb{N}} \) to the corresponding problems associated with \( f^\nu \) and \( u_0^\nu \) and, moreover, they satisfy (33) and (34) uniformly in \( \nu \), i.e., their norms in those spaces are bounded by a constant independent of \( \nu \). Therefore, we have a function \( u \) in the same class (33) and (34), and a subsequence, still denoted by \( \nu \), such that

\[
\begin{align*}
&u^\nu \xrightarrow{\nu} u \quad \text{in } L^2(0,T;H^1(\Omega)^N) \quad \text{weak and in } L^\infty(0,T;\tilde{K}) \quad \text{weak-}* \\
&u^\nu \xrightarrow{\nu} u \quad \text{a.e. in } Q \quad \text{and in } L^p(Q)^N, \quad \forall 1 \leq p < \infty.
\end{align*}
\]

(48)

(49)

By assumption (46) and Lebesgue Theorem, we conclude first that \( f^\nu(u^\nu) \xrightarrow{\nu} f(u) \) a.e. in \( Q \) and in \( L^1(Q)^N \), as well as

\[
\begin{align*}
&\int_Q f^\nu(u^\nu) \cdot u^\nu \xrightarrow{\nu} \int_Q f(u) \cdot u, \\
&\int_Q (f^\nu(u^\nu) - f(u)) \cdot (u^\nu - u) \xrightarrow{\nu} 0,
\end{align*}
\]

(50)

(51)

since, in particular, \( |u^\nu| \leq 1 \) and \( |u| \leq 1 \) a.e. in \( Q \).

Recalling (27) for each \( \nu \), we may take the limit in

\[
\int_Q (Pu^\nu - f^\nu(u^\nu)) \cdot (v - u^\nu) \geq 0
\]

(52)

for a fixed \( v \in L^2(0,T;\tilde{K}) \). Using (50) and (48), that in particular imply

\[
Pu \in L^1(Q)^N \quad \text{and} \quad \liminf_{\nu} \int_{Q_t} Pu^\nu \cdot u^\nu \geq \int_{Q_t} Pu \cdot u, \quad \forall t \in (0,T),
\]

we conclude that \( u \) is a solution of (36) with initial condition \( u_0 \).

Using \( v = u\chi_{(0,t)} + u^\nu\chi_{(t,T)} \) in (52) and \( v = u^\nu\chi_{(0,t)} + u\chi_{(t,T)} \) in (36) we find, for a.e. \( t \in (0,T) \),

\[
\frac{1}{2} \int_\Omega |u^\nu(t) - u(t)|^2 + \int_{Q_t} |\nabla(u^\nu - u)|^2 \leq \int_{Q_t} [f^\nu(u^\nu) - f(u)] \cdot (u^\nu - u) + \frac{1}{2} \int_\Omega |u_0^\nu - u_0|^2
\]

and, by (51), we conclude that \( u^\nu \xrightarrow{\nu} u \) strongly in \( W \). Therefore, we have proved the following result
Theorem 3.4. If $u^{\nu}$ denotes the solution to the variational inequality (36) with $f^{\nu}$ satisfying the assumptions (46) and (32) uniformly in $\nu$ and initial condition satisfying (47), then there exists a subsequence $\{u^{\nu}\}_{\nu \in \mathbb{N}}$ such that

$$u^{\nu} \rightharpoonup u \quad \text{in} \quad C([0,T]; L^2(\Omega)^N \cap \tilde{\mathcal{K}}) \cap L^2(0,T; H^1(\Omega)^N) \cap L^p(Q)^N, \quad \forall 1 \leq p < \infty,$$

where $u$ is a solution to (36) corresponding to the limit $f$ and the limit initial condition $u_0$. In addition, if $f$ satisfies (40), by uniqueness of $u$, the whole sequence $\{u^{\nu}\}_{\nu \in \mathbb{N}}$ converges. □

4 The multiphases system and its characterization

In this section we consider a variational solution $u$ of (25) obtained in Theorem 3.1, i.e., satisfying (33) and (34). Setting

$$w_i(u) = 1 - \sum_{j \neq i} u_i, \quad i = 1, \ldots, N,$$

(53)

each component $u_i$ satisfies a double obstacle problem

$$0 \leq u_i(x,t) \leq w_i(x,t) \quad \text{a.e.} \quad (x,t) \in Q, \quad i = 1, \ldots, N.$$

(54)

For an arbitrary nonnegative and bounded function $\psi = \psi(x,t)$ defined for $(x,t) \in Q$, such that

$$\mathcal{K}_0^\psi = \{v \in L^2(0,T; H^1(\Omega)) : 0 \leq \psi \quad \text{in} \quad Q \} \neq \emptyset,$$

(55)

and for a given $g \in L^1(Q)$, we may introduce the parabolic double obstacle scalar problem

$$u \in \mathcal{K}_0^\psi : \quad \int_Q \partial_t u(v-u) + \int_Q \nabla u \cdot \nabla (v-u) \geq \int_Q g(v-u) \quad \forall v \in \mathcal{K}_0^\psi,$$

(56)

subject to a given compatible initial condition

$$u(0) = u_0 \quad \text{in} \quad \Omega.$$

(57)

For each $i = 1, \ldots, N$, we have $u_i \in \mathcal{K}_0^{w_i}$ and, by choosing in (25) $v \in L^2(0,T; \mathcal{K})$, such that $v_j = u_j$ for $j \neq i$ and $v_i = v \in \mathcal{K}_0^{w_i}$ arbitrarily, it is clear that $u_i$ is a solution of the scalar double obstacle problem (56) with $\psi = w_i$ and $g = f_i(u)$. Hence we can obtain further properties of our solution by applying the general theory of the obstacle problem. For the sake of completeness we prove here the result below.
Let
\[ \varphi \in L^2(0, T; H^1(\Omega)) \cap L^\infty(Q) \quad \text{with } \varphi \geq 0 \text{ a.e. in } Q, \]  
(58)
\[ \partial_t \varphi \in L^2(0, T; (H^1(\Omega))' ) \quad \text{with } P \varphi \in L^1(Q), \quad \frac{\partial \varphi}{\partial \mathbf{n}} = 0 \text{ on } \partial \Omega \times (0, T), \]  
(59)
and
\[ g \in L^1(Q), \quad u_0 \in L^2(\Omega), \quad 0 \leq u_0 \leq \varphi(0) \text{ in } \Omega. \]  
(60)

We observe that (59) means that \( \varphi \) satisfies the formula
\[ \langle \partial_t \varphi, v \rangle_t = \int_Q v P \varphi - \int_Q \nabla \varphi \cdot \nabla v, \quad \forall v \in L^2(0, T; H^1(\Omega)) \cap L^\infty(Q). \]

Proposition 4.1. Under the assumptions (58)-(60) the unique solution \( u \in K_0^\varphi \) to the scalar problem (56)-(57) is such that
\[ u \in C([0, T]; L^2(\Omega)) \cap L^\infty(Q), \quad \partial_t u \in L^1(Q) + L^2(0, T; (H^1(\Omega))'), \]  
(61)
and it satisfies the parabolic semilinear equation
\[ P u = g + g - \chi_{\{u = 0\}} - (P \varphi - g) \chi_{\{u = \varphi\}} \quad \text{a.e. in } Q. \]  
(62)

Proof. Using the function \( \theta_\varepsilon \) given by (11) and defining
\[ \theta_\varepsilon(v) = g - \theta_\varepsilon(v) - (P \varphi - g) \chi_{\{u = \varphi\}} \quad \text{a.e. in } Q, \]  
(63)
we can consider the approximating problem, for \( \varepsilon > 0 \),
\[ \int_Q (P u^\varepsilon + \theta_\varepsilon(u^\varepsilon)) v = \int_Q g v, \quad \forall v \in L^2(0, T; H^1(\Omega)) \cap L^\infty(Q), \]  
(64)
with the initial condition \( u^\varepsilon(0) = u_0 \in \Omega \). Since \( \theta_\varepsilon \) is monotone and \( \varphi \) is bounded, arguing as in Theorem 3.1, the problem (64) has a unique solution \( u^\varepsilon \) in the class (61). Moreover, it satisfies
\[ -\varepsilon \leq u^\varepsilon \leq \varphi + \varepsilon \quad \text{a.e. in } Q, \]  
(65)
as we can show by choosing, in (64), \( v = (u^\varepsilon - \varepsilon)^+ \) and \( v = (u^\varepsilon - \varphi - \varepsilon)^+ \), respectively. Indeed, in the first case we have
\[ \int_Q v P v = -\int_Q v P u^\varepsilon = \int_{\{v > 0\}} v (\theta(u^\varepsilon) - g) = \int_{\{u^\varepsilon \leq \varepsilon\}} (-g - g) \leq 0, \]
since \( \vartheta_\varepsilon(u^\varepsilon) = -1 \) and \( \vartheta_\varepsilon(\varphi - u_\varepsilon) = 0 \), because \( u^\varepsilon < -\varepsilon \) and \( \varphi - u_\varepsilon > \varepsilon \), and, in the second case,

\[
\int_Q vPv = \int_Q vP(u^\varepsilon - \varphi) = \int_{\{v > 0\}} v (g - \vartheta(u^\varepsilon) - P\varphi) \\
= \int_{\{\varphi - u^\varepsilon > \varepsilon\}} ((P\varphi - g) - (P\varphi - g)^-) \leq 0,
\]

since \( \vartheta_\varepsilon(\varphi - u^\varepsilon) = -1 \) and \( \vartheta_\varepsilon(u^\varepsilon) = 0 \) if \( \varphi - u_\varepsilon < -\varepsilon \) and \( u^\varepsilon > \varphi + \varepsilon \).

Hence, using the monotonicity argument, we easily conclude that \( u = \lim_{\varepsilon \to 0} u_\varepsilon \in \mathcal{K}_0^\varphi \) is the unique solution of the variational inequality (56). Remarking that, from (63) we have

\[
-g^- \leq \vartheta_\varepsilon(u^\varepsilon) \leq (P\varphi - g)^- \quad \text{a.e. in } Q,
\]

from (64) we deduce in the limit the Lewy-Stampacchia inequalities

\[
(P\varphi - g)^- \leq Pu - g \leq g^- \quad \text{a.e. in } Q.
\]

In particular, this yields \( Pu \in L^1(Q) \) and (56) implies that \( u \) also solves

\[
\int_Q (Pu - g)(v - u) \geq 0, \quad \forall v \in \tilde{\mathcal{K}}_0^\varphi,
\]

where \( \tilde{\mathcal{K}}_0^\varphi = \{v \in L^2(Q) : 0 \leq v \leq \varphi \text{ in } Q\} \subset L^\infty(Q) \).

Let \( \mathcal{O} \subset Q \) be an arbitrary measurable set and set \( v = u \) in \( Q \setminus \mathcal{O} \) and \( v = \delta \varphi \) in \( \mathcal{O} \), with \( \delta \in [0, 1] \), in (66). Since \( \mathcal{O} \) is arbitrary, we conclude the pointwise inequality

\[
(Pu - g)(\phi - u) \geq 0 \quad \forall \phi \in [0, \varphi(x, t)] \quad \text{a.e. in } Q,
\]

which implies, up to null measure subsets of \( Q \),

\[
Pu - g \geq 0 \quad \text{in } \{u = 0\}, \quad Pu - g \leq 0 \quad \text{in } \{u = \varphi\},
\]

\[
Pu = g \quad \text{in } \Lambda = \{0 < u < \varphi\}.
\]

On the other hand, arguing as in Lemma 2 of [23] and noting that \( V = (u, -\nabla u) \in L^1(Q)^{n+1} \) and \( D \cdot V = Pu \in L^1(Q) \), with \( D = (\partial_t, \partial_{x_1}, \ldots, \partial_{x_n}) \), we have

\[
Pu = 0 \quad \text{a.e. in } \{u = 0\} \quad \text{and} \quad Pu = P\varphi \quad \text{a.e. in } \{u = \varphi\}.
\]

Hence, by (67), up to negligible sets, we have \( \{u = 0\} \subset \{g \leq 0\} \) and \( \{u = \varphi\} \subset \{P\varphi \leq g\} \), and using also (68), we finally conclude (62). \( \square \)
Theorem 4.2. Any solutions \( u \) of the variational inequality (25) (or (36)) under the conditions of Theorem 3.1 satisfy the semilinear parabolic system

\[
P u_i = f_i(u) + f_i^-(u) \chi_{\{u_i=0\}} - \sum_{1 \leq i_1 < \ldots < i_k \leq N \atop i \in \{i_1, \ldots, i_k\}} \frac{1}{k} (f_{i_1}(u) + \cdots + f_{i_k}(u))^+ \chi_{i_1 \ldots i_k} \quad \text{a.e. in } Q, \tag{70}
\]

where \( \chi_{i_1 \ldots i_k} = \chi_{I_{i_1 \ldots i_k}} \), for \( k = 1, \ldots, N \), denotes the characteristic function of

\[
I_{i_1 \ldots i_k} = \{(x, t) \in Q : u_{i_1 \ldots i_k}(x, t) = 1, u_{i_j}(x, t) > 0 \text{ for all } j = 1, \ldots, k\}. \tag{71}
\]

Proof. We notice that the regularity (58), (59), holds for \( w_i = 1 - \sum_{j \neq i} u_j \), so \( w_i \) can be chosen as the upper obstacle of each component \( u_i, i = 1, \ldots, N \), of \( u \), to which we can apply the conclusions of Proposition 4.1. Since \( \{u_i = 0\} \subset \{f_i(u) \leq 0\} \) a.e., for each \( i = 1, \ldots, N \), we have

\[
P u_i = f_i(u) + f_i^-(u) \chi_{\{u_i=0\}} - (Pw_i - f_i(u))^+ \chi_{\{u_i=w_i, u_i>0\}} \quad \text{in } Q, \tag{72}
\]

and the condition (70) will follow if we show that

\[
(Pw_i - f_i(u))^+ \chi_{\{u_i=w_i, u_i>0\}} = \sum_{1 \leq i_1 < \ldots < i_k \leq N \atop i \in \{i_1, \ldots, i_k\}} \frac{1}{k} (f_{i_1}(u) + \cdots + f_{i_k}(u))^+ \chi_{i_1 \ldots i_k} \quad \text{in } Q.
\]

(73)

Observe that

\[
\{ u_i = w_i, u_i > 0 \} = \bigcup_{1 \leq i_1 < \ldots < i_k \leq N \atop i \in \{i_1, \ldots, i_k\}} I_{i_1 \ldots i_k},
\]

and these sets are a.e. disjoint. Here the union is taken also over all the subsets \( \{i_1, \ldots, i_k\} \) of \( \{1, \ldots, N\} \) that include \( i \) and over all \( k = 1, \ldots, N \). We remark that \( Pw_i = Pu_i \) in that subset and

- in the sets \( I_i = \{u_i = 1\} \), \( Pw_i = 0 \) and \( (Pw_i - f_i(u))^+ = f_i(u)^+ \), for \( i = 1, \ldots, N \);

- in each set \( I_{i_1 \ldots i_k} \), for \( k \geq 2 \), as we shall see,

\[
(Pu_i - f_i(u))^+ = \frac{1}{k} (f_{i_1}(u) + \cdots + f_{i_k}(u))^+,
\]

and this fact concludes the proof.

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Let \((x_0, t_0) \in I_{i_1 \ldots i_k}\). Recall that \(\{i_1, \ldots, i_k\}\) is the set of indexes for which we have \(0 < u_{i_j}(x_0, t_0)\) (notice that \(i \in \{i_1, \ldots, i_k\}\)). Denoting \(\alpha = \min\{u_{i_j}(x_0, t_0) : j = 1, \ldots, k\}\), the set \(O = \bigcap_{j=1}^k \{u_j > \alpha/2\}\) is measurable and contains \((x_0, t_0)\). Given any measurable set \(\omega \subset O\), choose, in (36), as test function \(\mathbf{v} = (v_1, \ldots, v_N)\) defined by

\[
v_{i_1} = u_{i_1} \pm \delta \chi_\omega, \quad v_{i_j} = u_{i_j} \mp \delta \chi_\omega \text{ for a fixed } j \in \{2, \ldots, k\}, \quad v_l = u_l \forall l \neq i_1, i_j,
\]

observing that

\[
\sum_{j=1}^N v_j = \sum_{j=1}^N u_j \pm \delta \chi_\omega = \sum_{j=1}^N u_j \leq 1
\]

and

\[
v_j \geq 0, \quad j = 1, \ldots, N, \quad \text{as long as } 0 < \delta \leq \alpha/2.
\]

Returning to the inequality (36) and setting \(S_j = Pu_j - f_j(u)\), we get

\[
\pm \delta \int_Q S_{i_1} \chi_\omega \mp \delta \int_Q S_j \chi_\omega \geq 0.
\]

Since \(\omega \supset \{(x_0, t_0)\}\) was taken arbitrarily in \(O\) and \((x_0, t_0)\) is a generic point of \(I_{i_1 \ldots i_k}\), we conclude that

\[
S_{i_1} = S_j, \quad \text{a.e. in } I_{i_1 \ldots i_k}, \quad \text{for any } j \in \{2, \ldots, k\}. \tag{74}
\]

Recalling that \(\sum_{j=1}^N Pu_j = Pu_{i_1 \ldots i_k} = 0\), in the set \(I_{i_1 \ldots i_k}\) we get, using (74), that

\[
kS_{i_1} = S_{i_1} + \cdots + S_{i_k} = (Pu_{i_1} - f_{i_1}) + \cdots + (Pu_{i_k} - f_{i_k}) = Pu_1 + \cdots + Pu_N - (f_{i_1} + \cdots + f_{i_k}),
\]

where, for simplicity, we set \(f_j = f_j(u)\), and so

\[
S_i = S_{i_1} = -\frac{1}{k}(f_{i_1} + \cdots + f_{i_k}).
\]

But in \(I_{i_1 \ldots i_k}\) we have \(S_i \leq 0\) (recall that \(u_i = w_i\) and (68)) and so

\[
(Pu_i - f_i(u))^\pm = -(Pu_i - f_i(u)) = -S_i = \frac{1}{k}(f_{i_1} + \cdots + f_{i_k}) = \frac{1}{k}(f_{i_1} + \cdots + f_{i_k})^\pm.
\]

\[\square\]

**Corollary 4.3.** Let \(\mathbf{u}\) be the solution of the variational inequality (25) (or (36)) under the conditions of Theorem 3.1.
Then, denoting by $|A|$ the $(n + 1)$-Lebesgue measure of $A \subset Q$, we have

$$\left| \left\{ \sum_{j=1}^{k} f_{ij}(u) < 0 \right\} \cap \left\{ \sum_{j=1}^{k} u_{ij} = 1, u_{ij} > 0, j = 1, \ldots, k \right\} \right| = 0$$

(75)

for each partial coincidence subset $I_{i_1 \ldots i_k}$, as well as

$$\left| \left\{ f_i(u) > 0 \right\} \cap \left\{ u_i = 0 \right\} \right| = 0, \quad i = 1, \ldots, N.$$  

(76)

Proof. Being $I_{i_1 \ldots i_k}$ defined in (71), using the equation (70), we obtain, for each $i_j$ with $j = 1, \ldots, k$, denoting $f_{ij} = f_{ij}(u)$,

$$Pu_{ij} = f_{ij} - \frac{1}{k}(f_{i_1} + \cdots + f_{i_k})^+ \quad \text{a.e in } I_{i_1 \ldots i_k}.$$  

Summing these $k$ equations, we have

$$0 = \sum_{j=1}^{k} Pu_{ij} = f_{i_1} + \cdots + f_{i_k} - (f_{i_1} + \cdots + f_{i_k})^+ = (f_{i_1} + \cdots + f_{i_k})^- \quad \text{a.e in } I_{i_1 \ldots i_k}.$$  

So, in $I_{i_1 \ldots i_k} = \left\{ \sum_{j=1}^{k} u_{ij} = 1, u_{ij} > 0, j = 1, \ldots, k \right\}$ we have $\sum_{j=1}^{k} f_{ij} \geq 0$ a.e. and (75) follows.

The proof of (76) is similar (recall (68)).

□

As a consequence of this corollary the semilinear system (70) can, in fact, be written in the equivalent form for $i = 1, \ldots, N$,

$$Pu_i = f_i(u) - f_i(u)\chi_{\{u_i=0\}} - \sum_{1 \leq i_1 < \cdots < i_k \leq N \atop i \in \{i_1, \ldots, i_k\}} \frac{1}{k}(f_{i_1}(u) + \cdots + f_{i_k}(u))\chi_i \quad \text{a.e in } Q,$$  

(77)

since $\{u_i = 0\} \subset \{f_i(u) \leq 0\}$ and $I_{i_1 \ldots i_k} \subset \left\{ \sum_{j=1}^{k} f_{ij}(u) \geq 0 \right\}$ up to a neglectable subset of $Q$.

This remark combined with the continuous dependence of the variational solutions obtained in Theorem 3.4 yields an interesting criteria of local stability of the characteristic functions of the coincidence sets in the Lebesgue measure. Denote

$$\chi_{i_1 \ldots i_k} = \chi_{\{u_{i_1 \ldots i_k} = 1, u_{ij} > 0 \forall j = 1, \ldots, k\}}, \quad 1 \leq i_1 < \cdots < i_k \leq N, \quad i \in \{i_1, \ldots, i_k\}.$$  

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Theorem 4.4. Let the assumptions and notations of Theorem 3.4 hold. Suppose that in some subset of positive measure \( \omega \subseteq Q \) the following assumption on the limit problem holds

\[
\sum_{j=1}^{k} f_{ij}(u) \neq 0 \quad \text{a.e. in } \omega, \quad 1 \leq i_1 < \cdots < i_k \leq N, \quad k = 1, \ldots, N. \tag{78}
\]

Then the associated characteristic functions are such that

\[
\chi_{\{u_{\nu}^i=0\}} \xrightarrow{\nu} \chi_{\{u_i=0\}} \quad \text{in } L^p(\omega), \quad \forall \ i = 1, \ldots, N, \tag{79}
\]

\[
\chi_{i_1 \ldots i_k}^{\nu} \xrightarrow{\nu} \chi_{i_1 \ldots i_k} \quad \text{in } L^p(\omega), \quad \forall i_1, \ldots, i_k, \tag{80}
\]

for all \( p, \ 1 < p < \infty \).

Proof. We observe that each \( u^\nu \) solves the system

\[
P u^\nu_i = f_i^\nu - f_i^\nu \chi_{\{u_{\nu}^i=0\}} - \sum_{1 \leq i_1 < \cdots < i_k \leq N \atop i \in \{i_1, \ldots, i_k\}} \frac{1}{k}(f_i^\nu + \cdots + f_{ik}^\nu) \chi_{i_1 \ldots i_k} \quad \text{a.e. in } Q \tag{81}
\]

where, for simplicity, we set \( f_i^\nu = f_i^\nu(u^\nu) \). By the convergence \( u^\nu \longrightarrow u \), we have \( P u^\nu \rightharpoonup P u \) in the distributional sense. Since \( 0 \leq \chi_{i_1 \ldots i_k}^\nu \leq 1 \), there exists \( \chi_{i_1 \ldots i_k}^* \), with \( 0 \leq \chi_{i_1 \ldots i_k}^* \leq 1 \) in \( Q \), such that

\[
\chi_{i_1 \ldots i_k}^\nu \xrightarrow{\nu} \chi_{i_1 \ldots i_k}^* \quad \text{in } L^\infty(Q) \text{ weak-}*.
\]

Analogously, for some \( \chi_{i,0}^* \), with \( 0 \leq \chi_{i,0}^* \leq 1 \) in \( Q \), we have

\[
\chi_{\{u_{\nu}^i=0\}} \xrightarrow{\nu} \chi_{i,0}^* \quad \text{in } L^\infty(Q) \text{ weak-}*.
\]

We are going to prove that, in fact,

\[
\chi_{i,0}^* = \chi_{\{u_i=0\}} \quad \text{and} \quad \chi_{i_1 \ldots i_k}^* = \chi_{i_1 \ldots i_k} \quad \text{a.e. in } \omega,
\]

which concludes the proof, since the weak convergence to characteristic functions in \( L^p(\omega) \) is in fact strong, as it is well known.

Passing to the limit in (81), we obtain

\[
P u_i = f_i - f_i \chi_{i,0}^* - \sum_{1 \leq i_1 < \cdots < i_k \leq N \atop i \in \{i_1, \ldots, i_k\}} \frac{1}{k}(f_{i_1} + \cdots + f_{i_k}) \chi_{i_1 \ldots i_k}^* \quad \text{a.e. in } Q
\]

where, for simplicity, we have also set \( f_{ij} = f_{ij}(u) \).
But each $u_i$ also solves the equation (77), so, by subtraction, we obtain a.e. in $Q$,

$$- f_i(\chi_{\{u_i=0\}} - \chi_{i,0}^*) - \sum_{\substack{1 \leq i_1 < \cdots < i_k \leq N \\ i \in \{i_1, \ldots, i_k\}}} \frac{1}{k}(f_{i_1} + \cdots + f_{i_k})(\chi_{i_1 \ldots i_k} - \chi_{i_1 \ldots i_k}^*) = 0. \quad (82)$$

Noticing that $\chi_{\{u_{\nu}=0\}} u_{\nu} = 0$, passing to the limit, we get $\chi_{i,0}^* u_i = 0$, which means that $\chi_{i,0}^* = 0$ whenever $u_i > 0$. To conclude that $\chi_{i,0}^* = \chi_{\{u_i=0\}}$ we only need to prove that $\chi_{i,0}^* = 1$ if $u_i = 0$.

Recall that the sets $\{u_i=0\}$ and $I_{i_1 \ldots i_k}$, $1 \leq i_1 < \cdots < i_k \leq N$, $i \in \{i_1, \ldots, i_k\}$, $k = 1, \ldots, N$, are mutually disjoint. Hence in $\{u_i = 0\}$ we obtain

$$- f_i(1 - \chi_{i,0}^*) + \sum_{\substack{1 \leq i_1 < \cdots < i_k \leq N \\ i \in \{i_1, \ldots, i_k\}}} \frac{1}{k}(f_{i_1} + \cdots + f_{i_k})\chi_{i_1 \ldots i_k}^* = 0$$

and since the left hand side is nonnegative, by the assumption (78) we conclude that

$$\chi_{i,0}^* = 1 \quad \text{and} \quad \chi_{i_1 \ldots i_k}^* = 0 \quad \text{in} \quad \{u_i = 0\} \cap \omega.$$ 

Since $\chi_{i_1 \ldots i_k}^*(1 - u_{i_1 \ldots i_k}) = 0$ a.e. in $Q$, taking the limit in $\nu$, we also obtain $\chi_{i_1 \ldots i_k}^*(1 - u_{i_1 \ldots i_k}) = 0$ a.e in $Q$, i.e. $\chi_{i_1 \ldots i_k}^* = 0$ if $u_{i_1 \ldots i_k} < 1$. It remains to evaluate $\chi_{i_1 \ldots i_k}^*$ when $u_{i_1 \ldots i_k} = 1$ and $u_{i_j} > 0$, for all $j = 1, \ldots, k$ or when $u_{i_j} = 0$, for some $j = 1, \ldots, k$.

In this later case, where $u_{i_j} = 0$, for some $j = 1, \ldots, k$, we have $\chi_{i_1 \ldots i_k}^* = 0$ and, since we already know that $\chi_{\{u_j=0\}} = \chi_{i_j,0}^*$, from (82) for the index $i_j$, we get

$$\sum_{\substack{1 \leq i_1 < \cdots < i_k \leq N \\ i_j \in \{i_1, \ldots, i_k\}}} \frac{1}{k}(f_{i_1} + \cdots + f_{i_k})\chi_{i_1 \ldots i_k}^* = 0.$$

Then, by the assumption (78) we have $\chi_{i_1 \ldots i_k}^* = 0$ in $(Q \setminus I_{i_1 \ldots i_k}) \cap \omega$.

Finally, in $I_{i_1 \ldots i_k} \cap \omega$, again from (82), we obtain

$$\frac{1}{k}(f_{i_1} + \cdots + f_{i_k})(1 - \chi_{i_1 \ldots i_k}^*) = 0$$

and the assumption (78) yields that $\chi_{i_1 \ldots i_k}^* = 1$, completing the proof. \hfill \Box
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