We consider a chain of Josephson-junction rhombi (proposed originally in [1]) in quantum regime. In a regular chain with no disorder in the maximally frustrated case when magnetic flux through each rhombi $\Phi_r$ is equal to one half of superconductive flux quantum $\Phi_0$, Josephson current is due to correlated transport of \textit{pairs of Cooper pairs}, i.e. charge is quantized in units of $4e$. Sufficiently strong deviation $\delta\Phi \equiv |\Phi_r - \Phi_0/2| > \delta\Phi_c$ from the maximally frustrated point brings the system back to usual $2e$-quantized supercurrent. For a regular chain $\delta\Phi_c$ was calculated in [5]. Here we present detailed analysis of the effect of quenched disorder (random stray charges and random fluxes piercing rhombi) on the pairing effect.
I. INTRODUCTION

Pairing of Cooper pairs in frustrated Josephson junction arrays was theoretically proposed recently \cite{1,2,3,4} in the search of topologically protected nontrivial quantum liquid states. The simplest system where such a phenomenon could be observed was proposed by Douçot and Vidal in \cite{1}. It consists of a chain of rhombi (each of them being small ring of 4 superconductive islands connected by 4 Josephson junctions) placed into transverse magnetic field. cf. Fig. 1.

It was shown in \cite{1} that in the fully frustrated case (i.e. magnetic flux through each rhombus $\Phi_r = \frac{1}{2} \Phi_0 = \frac{hc}{4e}$) usual tunnelling of Cooper pairs along the chain is blocked due to destructive interference of tunneling going through two paths within the same rhombus, while correlated 2-Cooper-pair transport survives. Sufficiently strong deviation $\delta \Phi = \Phi_r / 2 > \delta \Phi^c$ from the maximally frustrated point brings the system back to usual $2e$-quantized supercurrent.

In ref. \cite{5} rhombi chain was studied for the experimentally relevant situation when its Coulomb energy is determined by capacitance of junctions and not by capacitance of superconductive islands themselves. Expression for the critical deflection $\delta \Phi^c$ was derived. However, ref. \cite{5} dealt with regular chain with no disorder. In any real system two intrinsic sources of disorder are always present: a) some weak randomness of fluxes $\Phi^{\text{a}}_r$ penetrating different rhombi (due to unavoidable differences in their areas), and b) random stray charges $q_n$ which produce, due to Aharonov-Casher effect, some random phase factors to the phase slip tunnelling amplitudes leading to suppression of quantum fluctuations in the chain (compare analogous discussion in \cite{6}).

In this paper we adapt method used in ref. \cite{5} for the case of rhombi chain with quenched disorder and study influence of random stray charges on pairing effect, derive "phase diagram" of the chain with fixed realization of disorder, calculate probability $P_{4e}(E_J/E_C, N, \delta \Phi)$ to find the chain in the regime with dominating $4e$- supercurrent and finally estimate critical deflection from the maximally frustrated point $\delta \Phi^c$ destroying $4e$-supercurrent in a chain with stray charges; in Sec. IV we consider modulation of the supercurrent in a clean and disordered chain by external capacitively coupled gates; in Sec. V. we analyse influence of randomness of fluxes in rhombi on...
pairing of Cooper pairs. Finally, in Sec. VI. we present our conclusions and suggestions.

II. EFFECTIVE HAMILTONIAN IN PRESENCE OF DISORDER

We study a chain of $N$ rhombi shown in Fig. 1. Each rhombi consists of four superconductive islands connected by tunnel junctions with Josephson coupling energy $E_J = \hbar I_c^0 / 2e$; charging energy $E_C$ is determined by capacitance $C$ of junctions, $E_C = e^2 / 2C$ (we neglect self-capacitances of islands which are assumed to be much smaller than $C$). Below we consider Josephson current along the chain of $N \gg 1$ rhombi and assume that the chain is of the ring shape, with total magnetic flux $\Phi_c$ inside the ring. We also denote by $\Phi_n^r$ the flux through $n$-th rhombus and define phases $\gamma$ and $\varphi_n$: 

$$
\gamma = 2\pi \frac{\Phi_c}{\Phi_0}, \quad \varphi_n = 2\pi \frac{\Phi_n^r}{\Phi_0},
$$

A regular chain with no disorder (no random stray charges and all $\varphi_n \equiv \varphi$) is described by the imaginary-time action 

$$
S_E = \int dt \sum_{n=1}^{N} \sum_{m=1}^{4} \left\{ \frac{1}{16E_C} \left( \frac{d\varphi_n^{(m)}}{dt} \right)^2 - E_J \cos \varphi_n^{(m)} \right\}.
$$

and additional conditions 

$$
\sum_{m=1}^{4} \varphi_n^{(m)} = \varphi, \quad n = 1, 2, \ldots, N,
$$

Figure 1: The chain of rhombi with random stray charges. Also shown is an external capacitively coupled gate (c.f. Sec. IV).
\[
\sum_{n=1}^{N} \left( -\theta_n^{(3)} - \theta_n^{(4)} \right) = \gamma. \quad (4)
\]

Here the variable \( \theta_n^{(m)} \) is the phase difference across the \( m \)-th junction in the \( n \)-th rhombus (see Fig. 1).

This regular model was analyzed in details in ref. [5] within the limit \( E_J \gg E_C \). At \( E_C = 0 \) the phases \( \theta_n^{(m)} \) are classical variables which do not fluctuate. Emerging classical states of the chain \( |m, \{ \sigma_n^z \} \rangle \) can be characterized by a set of binary variables \( \{ \sigma_n^z \} \) (one for each rhombi) and an integer-valued variable \( m \). Note that each binary variable \( \sigma_n^z \) can be considered as \( z \)-projection of spin \( \frac{1}{2} \) ascribed to each rhombi. Energies of these classical states are

\[
E_{m, \sigma} \approx \frac{E_J \sqrt{\gamma}}{4N} (\tilde{\gamma} - \pi N/2 - \pi S^z - 2\pi m)^2 - \sqrt{2}\delta S^z E_J + \text{Const.} \quad (5)
\]

\[
s_n^z = \frac{1}{2} \sigma_n^z, \quad S^z = \frac{1}{2} \sum_{n=1}^{N} \sigma_n^z, \quad \tilde{\gamma} = \gamma + \frac{N\phi}{2}. \quad (6)
\]

Finite \( E_J/E_C \) gives rise to quantum phase slips (QPS) in the chain, which mix different classical states leading to formation of truly quantum ground state. Let us denote as \( \upsilon \) the amplitude of classical states in one contact. At large \( N \gg 1 \) this amplitude does not differ from the "spin flip" amplitude for a single rhombus at \( \Phi_r \approx \Phi_0/2 \). In this approximation we can use result from Ref. [2]:

\[
\upsilon \approx k \left( \frac{E_J^3 E_C}{E_C} \right)^{1/4} \exp \left( -1.61 \sqrt{\frac{E_J}{E_C}} \right). \quad (7)
\]

Here \( k \) is a numerical factor of order one which slightly depends on \( E_J/E_C \) (see ref. [5] for details).

In [5] it was shown that calculation of the persistent current in the large-\( N \) limit can be reduced to the solution of a Schrödinger equation for a particle having a large spin \( S = \frac{1}{2}N \) moving in a periodic cos-like potential, with appropriate boundary condition.

More precisely, the quantum ground-state energy \( E \) of the chain in the limit \( E_J \gg E_C \) can be obtained from solution of Schrödinger equation

\[
\frac{\partial^2 \psi}{\partial x^2} + \left( \tilde{E} - 2w \cos 2x \sum_{n=1}^{N} \tilde{S}_n^z + 2h \sum_{n=1}^{N} \tilde{S}_n^x \right) \psi = 0, \quad (8)
\]

where

\[
\tilde{E} = \frac{16NE}{\sqrt{2E_J\pi^2}}, \quad w = \frac{64N\upsilon}{\sqrt{2E_J\pi^2}}, \quad h = \frac{8N\delta}{\pi^2}. \quad (9)
\]
Here $\nu$ — amplitude for quantum phase slip in one rhombi and $\delta = \pi - \varphi$. Function $\psi \equiv \psi(x, \{\sigma_n\})$. $\hat{S}_x$ and $\hat{S}_z$ are standard operators of $x$ and $z$ projections of spin $\frac{1}{2}$ acting on $\sigma_n$. The magnetic flux inside the whole ring enters the problem through the twisted boundary condition

$$e^{i\pi \hat{S}_z + i\pi N/2} \psi(x + \pi/2, \sigma) = e^{i\tilde{\gamma}} \psi(x, \sigma), \quad \tilde{\gamma} = \gamma + \frac{N\Phi}{2}$$

(10)

In [5] on the basis of this formulation of the problem critical deflection $\delta\Phi^c$ was derived

$$(\delta\Phi^c)_{\text{reg}} \approx 0.2 \left( \frac{\nu}{E_J} \right)^{2/3} \Phi_0$$

(11)

The aim of this section is to derive effective Hamiltonian similar to (8) for the chain with random fluxes $\Phi^r_n$ through rhombi or random stray charges.

To account for random stray charges we present the electrostatic charging energy in the form

$$H_C = \sum_{n_1,n_2=1}^{N} \sum_{k_1,k_2=1}^{3} \frac{1}{2} \left[ C^{-1} \right]_{k_1n_1}^{k_2n_2} \left[ Q_n^{(k_1)} - q_n^{(k_1)} \right] \left( Q_n^{(k_2)} - q_n^{(k_2)} \right)$$

(12)

where $\left[ C^{-1} \right]_{k_1n_1}^{k_2n_2}$ is the matrix of inverse capacitances. Indices $n$ and $k$ numerates superconducting islands (3 rows, $N$ islands in a row). $Q_n^{(k)}$ — charge of the $n$-th island in $k$-th row. Parameters $q_n^{(k)}$ are determined by the random stray charges.

Starting from this charging energy we may derive an additional term in action (2) emerging from presence of random stray charges (compare with [6])

$$\delta S = -i \int dt \sum_{n=1}^{N} \sum_{m=1}^{4} p_n^m d\theta_n^m$$

(13)

Parameters $p_n^m$ can be expressed in terms of charges $q_n^{(k)}$. Corresponding expressions are a bit cumbersome and we do not present them here. Below we write down some special combinations of $p_n^m$ which we will need in our paper.

Additional term (13) has a form of total derivative. Hence it does not change neither the classical states of the chain nor the classical equations of motion and the real part of classical action on a single tunneling trajectory. The only effect of this term to give a tunneling amplitude along each path its on phase factor. Since there are several QPS trajectories between two classical states, all having the same real part of tunneling action, these additional phase factors may give rise to destructive interference of tunneling processes, leading to reduction of total matrix element, connecting two classical states.
Following ref. [5] and taking into account complications due to random stray charges we write tight-binding Hamiltonian

$$\hat{H}|m, \sigma > = E_{m\sigma}|m, \sigma > + \frac{\nu}{2} \sum_{n=1}^{N} \exp \left\{ \frac{i\pi}{2} \left( 3p_n^1 - p_n^2 - p_n^3 - p_n^4 \right) \right\} \hat{\sigma}_n^+ |m, \sigma > +$$

$$\frac{\nu}{2} \sum_{n=1}^{N} \exp \left\{ -\frac{i\pi}{2} \left( 3p_n^3 - p_n^1 - p_n^2 - p_n^4 \right) \right\} \hat{\sigma}_n^+ |m, \sigma > +$$

$$\frac{\nu}{2} \sum_{n=1}^{N} \exp \left\{ -\frac{i\pi}{2} \left( 3p_n^4 - p_n^1 - p_n^2 - p_n^3 \right) \right\} \hat{\sigma}_n^+ |m-1, \sigma > + h.c. \quad (14)$$

Performing Fourier transformation over variable $m$ according to

$$|x, \sigma > = \sum_{m} \exp \left\{ 2i \left( 2m - \frac{\gamma}{\pi} + S^z + \frac{N}{2} \right) x \right\} |m, \sigma > , \quad (15)$$

we obtain the effective Schrödinger equation

$$\frac{\partial^2 \psi}{\partial x^2} + \left( \tilde{E} - 2w \cos 2x \sum_{n=1}^{N} a_n \hat{S}_n^x - 2w \sin 2x \sum_{n=1}^{N} b_n \hat{S}_n^y + 2h\hat{S}_z \right) \psi = 0, \quad (16)$$

and twisted boundary condition

$$e^{i\pi \hat{S}_z + i\pi/2} \psi(x + \pi/2, \sigma) = e^{i\tilde{\gamma}} \psi(x, \sigma). \quad (17)$$

Here parameters $\tilde{E}, w$ and $h$ are described by equation (9); $a_n$ and $b_n$ are random coefficients which can be expressed in term of random charges $q_{n}^{(k)}$

$$a_n = \frac{\cos \pi Q_n^1 + \cos \pi Q_n^2}{2}, \quad b_n = \frac{\cos \pi Q_n^1 - \cos \pi Q_n^2}{2} \quad (18)$$

$$Q_n^1 \equiv p_n^1 - p_n^2 = q_{n}^{(2)} - \frac{1}{3N} \sum_{n=1}^{N} \sum_{k=1}^{3} q_{n}^{(k)} \quad Q_n^2 \equiv p_n^3 - p_n^4 = q_{n}^{(3)} - \frac{1}{3N} \sum_{n=1}^{N} \sum_{k=1}^{3} q_{n}^{(k)} \quad (19)$$

Here we measure charges $q_{n}^{(k)}$ in units $2e$.

We turn now to generalization of (8) for the case of random fluxes in rhombi. Imaginary-time action for this problem is given by equation (2) but additional conditions (3) should be changed to

$$\sum_{m=1}^{4} \theta_n^{(m)} = \phi_n, \quad n = 1, 2, \ldots, N, \quad (20)$$
Following the same steps as before we see that to account for disorder in flux piercing the rhombi one should just replace the last term in (8) by

\[ 2 \sum_{n=1}^{N} h_n \hat{S}_z^n \]  

(21)

Here \( h_n = 8N(\pi - \varphi_n)/\pi^2 \).

Now we have effective Hamiltonians of the chain in presence of disorder. In the rest of the paper we will analyse these Hamiltonians in order to find out influence of disorder on the crossover point between 4\(e\)- and 2\(e\)-regimes.

### III. INFLUENCE OF RANDOM CHARGES ON CROSSOVER POINT

In this section we study influence of random stray charges on the on pairing effect in rhombi chain. It is important to note that, generally speaking, even at maximally frustrated point \( h = 0 \) (in contrast to regular chain) symmetry properties of Schrödinger equation \((16)\) do not prohibit 2\(e\)-supercurrent. This is due to the fact that asymmetric realizations of random charges with \( q_n^{(2)} \neq q_n^{(3)} \) (and thus \( b_n \neq 0 \)) break symmetry between two tunneling trajectories of a Cooper pair within the same rhombus. Of course, random charges preserve classical states of the rhombi chain and in a chain with no QPS there would be no 2\(e\)-supercurrent at maximally frustrated point, but full quantum Hamiltonian of the chain does not possess corresponding symmetry. Nevertheless, we will see later that for typical realizations of random charges and at \( \Phi_r = \Phi_0/2 \) the 2\(e\)-supercurrent is small as compared to 4\(e\)-supercurrent. The reason for that is as follows: if at least in one rhombi \( b_n = 0 \) (or \( a_n = 0 \)) then 2\(e\)-supercurrent is prohibited. Here we start with analysis of general situation of rhombi chain with random charges and \( h \neq 0 \) and then make some conclusion on disordered rhombi chain in maximally frustrated point.

We investigate the grand partition function for the system described by equations \((16, 17)\)

\[ Z = \int \mathcal{D}x(\tau) \exp \left( -\int_0^\beta \frac{x^2}{2} \right) \prod_{n=1}^{N} \text{Tr} \left[ \hat{U}_n(\beta) \right] \]  

(22)

with \( \beta \) being the inverse temperature, \( \beta \to \infty \). Operators \( \hat{U}_n \) act in the space of spin \( \frac{1}{2} \) each and are functionals of \( x(\tau) \) defined as

\[ \frac{d\hat{U}_n}{d\tau} = -\left( w f_n(\tau) \hat{S}^x + w g_n(\tau) \hat{S}^y - h \hat{S}^z \right) \hat{U}_n \]  

(23)
\[ f_n(\tau) = a_n \cos 2x(\tau), \quad g_n(\tau) = b_n \sin 2x(\tau) \] (24)

As was shown in \[5\], in a regular chain with \( E_J \gg E_C \) the borderline between 4\( e\)- and 2\( e\)-supercurrents is at rather large \( \Phi_r - \Phi_0/2 \), in the sense that at the crossover point \( h \gg w \). So in this paper we will also consider this limit only.

Under such condition equation (23) can be solved for arbitrary functions \( f(\tau) \) and \( g(\tau) \). For \( \text{Tr} \hat{U}_n(\beta) \) we then find

\[
\text{Tr} \hat{U}_n(\beta) = \exp \left( \frac{\beta h}{2} + \frac{w^2}{4h^2} (f_n(0) - i g_n(0)) (f_n(\beta) + ig_n(\beta)) + \frac{w^2}{8} \int_0^\beta d\tau_1 d\tau_2 (f_n(\tau_1) f_n(\tau_2) + g_n(\tau_1) g_n(\tau_2)) e^{-h|\tau_1 - \tau_2|} + i\frac{w^2}{4} \int_0^\beta d\tau_1 d\tau_2 f_n(\tau_1) g_n(\tau_2) e^{-h|\tau_1 - \tau_2|} \text{sign}(\tau_1 - \tau_2) \right) \tag{25}
\]

From equation (25) we derive effective action for variable \( x \)

\[
Z = \int_0^\beta Dx(\tau) e^{-S[x(\tau)]} \quad S = S_{\text{bound}} + S_\tau \tag{26}
\]

\[
S_{\text{bound}} = -\frac{w^2}{4h^2} \sum_{n=1}^N \left( a_n \cos 2x(\beta) + ib_n \sin 2x(\beta) \right) \left( a_n \cos 2x(0) - ib_n \sin 2x(0) \right) \tag{27}
\]

\[
S_\tau = \int_0^\beta d\tau \frac{x^2(\tau)}{2} - A \frac{w^2}{8} \int_0^\beta d\tau_1 d\tau_2 \cos 2x(\tau_1) \cos 2x(\tau_2) \exp(|\tau_1 - \tau_2|) - B \frac{w^2}{8} \int_0^\beta d\tau_1 d\tau_2 \sin 2x(\tau_1) \sin 2x(\tau_2) \exp(|\tau_1 - \tau_2|) - iC \frac{w^2}{4h} \int_0^\beta d\tau_1 d\tau_2 \cos 2x(\tau_1) \sin 2x(\tau_2) \exp(|\tau_1 - \tau_2|) \text{sign}(\tau_1 - \tau_2) \tag{28}
\]

Where

\[
A = \sum_{n=1}^N a_n^2, \quad B = \sum_{n=1}^N b_n^2, \quad C = \sum_{n=1}^N a_n b_n \tag{29}
\]

Note that similar approximation was used previously in \[5\] for a regular chain. In a regular chain semiclassical analysis which do not rely on linearization with respect to \( w/h \) is also possible. It turns out that exact value of the critical deflection \( \delta \Phi^c \) differs only by 12% from the one obtained by linearization even for the case when at the crossover point \( w/h = 1 \).

Action (28) is nonlocal and looks a bit terrific, but since

\[
\left( \frac{\partial^2}{\partial \tau_1^2} - h^2 \right) e^{-h|\tau_1 - \tau_2|} = -2h \delta(\tau_1 - \tau_2) \tag{30}
\]
we can perform Hubbard-Stratonovich transformation and derive a representation for grand partition function with local action (after redefinition of time scale according to \( \tau \rightarrow t/h \)).

\[
Z = \int_0^\beta \mathcal{D}x(\tau) \mathcal{D}y(\tau) \mathcal{D}z(\tau) e^{-S[\mathcal{X}(\tau)]} 
\]  

(31)

\[
S = h \int_0^{\beta h} \mathcal{D}\tau \left( \frac{x^2 + y^2 + z^2}{2} + \frac{y^2 + x^2}{2} + \alpha_1 d\tau \cos 2x + i\beta_1 d\tau \sin 2x + \alpha_2 d\tau \sin 2x + i\beta_2 d\tau \cos 2x - \frac{d^2}{2}(\beta_1^2 \sin^2 2x + \beta_2^2 \cos^2 2x) \right) 
\]  

(32)

Here \( \alpha_1, \alpha_2, \beta_1 \) and \( \beta_2 \) should satisfy equations

\[
\alpha_1^2 + \beta_1^2 = \frac{A}{N}, \quad \beta_1^2 + \alpha_2^2 = \frac{B}{N}, \quad \alpha_1 \beta_1 - \alpha_2 \beta_2 = \frac{C}{N} 
\]  

(33)

and

\[
d = \sqrt{\frac{\omega^2 N}{2h^3}} = \frac{\sqrt{2\pi} \nu}{\delta^{3/2} E_J} 
\]  

(34)

As we see from equations (33) we have some freedom in definitions of these parameters. Namely, if we set

\[
\alpha_1 = \sqrt{\frac{A}{N}} \sin \kappa_1, \quad \beta_2 = \sqrt{\frac{A}{N}} \cos \kappa_1 
\]  

(35)

\[
\alpha_2 = \sqrt{\frac{B}{N}} \sin \kappa_2, \quad \beta_1 = \sqrt{\frac{B}{N}} \cos \kappa_2 
\]  

(36)

then (33) reduces to

\[
\sin(\kappa_1 - \kappa_2) = \frac{C}{\sqrt{AB}} 
\]  

(37)

and we have only one equation for two parameters \( \kappa_1 \) and \( \kappa_2 \). This freedom will not be important for us. In present paper for the reasons described below we will mostly consider action (32) under condition \( C = 0 \) and chose

\[
\alpha_1 = \sqrt{\frac{A}{N}}, \quad \alpha_1 = \sqrt{\frac{A}{N}}, \quad \beta_1 = 0, \beta_2 = 0 
\]  

(38)

The main feature of the result presented above is that we have reduced original problem including large number of random variables \( \sim N \) to a problem with only three random parameters \( A, B, \) and \( C \). More over, since \( A, B \) and \( C \) arise as sums of large number \( N \) of independent random variables (see equation (29)), it is natural to expect that they obey Gaussian statistics. Let us assume that parameters \( Q_n^1 \) and \( Q_n^2 \) in (19) are uniformly distributed on \([-1, 1]\). It corresponds to
strong fluctuations of stray charges from sample to sample. Under such an assumption one can easily find

\[ \langle A \rangle = \langle B \rangle = \frac{N}{4} \]

\[ \langle A^2 \rangle = \langle B^2 \rangle = \frac{N^2}{16} + \frac{5N}{64}, \quad \langle AB \rangle = \frac{N^2}{16} - \frac{3N}{64} \]

\[ \langle C \rangle = 0, \quad \langle C^2 \rangle = \frac{N}{64} \]

(39)

Our strategy in the rest part of this section will be to analyse properties of the system described by equations (32, 17) with fixed \( A, B \) and \( C \) and then make some statistical analysis. Under fixed \( A, B \) and \( C \) the main subject of our investigation will be whether the system is in regime of dominating 4\( e \)-supercurrent or not.

First of all let us analyze action (32) for a trajectory where \( x(\tau), y(\tau) \) and \( z(\tau) \) are constant. We have

\[ S_{st} = \beta h \left( \frac{y^2 + z^2}{2} + \alpha_1 dy \cos 2x + \alpha_2 dz \sin 2x - \frac{d^2}{2} \left( \beta_1^2 \sin^2 2x + \beta_2^2 \cos^2 2x \right) \right) \]

(40)

This static action has two groups of minima (we call them even and odd, suppose \( A > B \))

\[ x = \pi n, \quad y = -\alpha_1 d, \quad z = 0 \]

(41)

and

\[ x = \frac{\pi}{2} + \pi n, \quad y = \alpha_1 d, \quad z = 0 \]

(42)

where \( n \) is an arbitrary integer. All these minima correspond to the same value of \( S_{st} \). So we have to consider two types of tunnelling trajectories. Trajectories of the first type connect minima of the same group, i.e. "even-even" and "odd-odd", and corresponding variation of the variable \( x \) between minima is \( \pm \pi \), whereas \( y \) returns to its original value. Trajectories of the second type connect minima of opposite parity (i.e. opposite signs of \( y \)), and change \( x \) variable by \( \pm \frac{\pi}{2} \). It is not difficult to see from Eqs.(15,16,17), that increment \( \Delta x \) of the variable \( x \) along tunnelling trajectory is in one-to-one correspondence to the elementary charge transported along the rhombi chain: \( q_0 = \frac{4e}{\pi} \Delta x \). Therefore trajectories of the first type lead to 4\( e \) - supercurrent, whereas trajectories of the second type produces usual 2\( e \)-quantized supercurrent. In semiclassical approximation amplitudes of the supercurrent components are determined primarily by the classical actions on corresponding trajectories:

\[ I(\gamma) = I_{2e} \sin \tilde{\gamma} + I_{4e} \sin(2\tilde{\gamma}), \]

(43)
where

\[ I_{4e} = A_{4e} \exp(-S_{4e}^e) \]  
\[ I_{2e} = A_{2e} \exp(-S_{2e}^e) \]  
\[ (44) \]

To find out whether 4e-supercurrent dominates in the system, we should compare classical actions on the trajectories of two types described above. So we examine classical equations of motion for action \( (32) \). For simplicity here we put coefficient \( C \) to zero. It can be shown that relatively small \( C \) of order \( \sqrt{N} \) (compare to \( (39) \)) is not important for our future purpose.

\[ \ddot{x} + 2\alpha_1 dy \sin 2x - 2\alpha_2 dz \cos 2x = 0 \]  
\[ (45) \]

\[ \ddot{y} - y = \alpha_1 d \cos 2x \]  
\[ (46) \]

\[ \ddot{z} - z = \alpha_2 d \sin 2x \]  
\[ (47) \]

Here we will analyse equations \( (45, 46, 47) \) and find analytic expression for the borderline between 2e- and 4e-regimes under conditions

\[ Ad^2/N \gg 1, \quad A - B \ll A \]  
\[ (48) \]

We also present results of numerical computation of the borderline in general situation.

Let us start with 2e-trajectory. Note that for variables \( x \) and \( y \) characteristic frequency is 1. Let us suppose that on 2e-trajectory \( x \) varies slowly so that characteristic frequency \( \omega_x \ll 1 \). Then we can eliminate \( y \) and \( z \) from equations of motion in adiabatic approximation and obtain

\[ \ddot{x} - d_1^2 \sin 4x = 0 \]  
\[ (49) \]

\[ d_1^2 = \frac{(A - B)d^2}{N + 2d^2(A + B)} \]  
\[ (50) \]

We see that under conditions \( (48) \) \( \omega_x \sim d_1 \) is indeed small. Equation \( (49) \) has solution corresponding to 2e-trajectory.

\[ x(\tau) = \frac{1}{2} \arccos (-\tanh (2d_1 \tau)) \]  
\[ (51) \]

Classical action on this solution is

\[ S_{2e} = h \sqrt{\frac{d^2(A - B)}{N} \left( 1 + 2\frac{(A + B)d^2}{N} \right)} \]  
\[ (52) \]
Let us now turn to examination of $4e$-trajectories. Here we suppose that $x$ is a fast variable i.e. $\omega_x \gg 1$. We can neglect then variation of $y$ and $z$ on classical trajectory and put $y = -\alpha_1 d$, $z = 0$. This is consistent with boundary conditions for $4e$-trajectory. Equation for $x$ becomes

$$\ddot{x} - \frac{2A}{N} d^2 \sin 2x = 0$$

Again, due to (48), we see that $x$ indeed varies fast on $4e$-trajectory since $\omega_x \sim \sqrt{Ad^2/N} \gg 1$. So we find $4e$-trajectory and corresponding action

$$x(\tau) = \arccos \left( -\tanh \left( 2\sqrt{\frac{Ad^2}{N}} \tau \right) \right),$$

$$S_{4e} = 4\hbar \sqrt{\frac{Ad^2}{N}}$$

Comparing equations (52) and (55) we find the line of crossover between $4e$- and $2e$- regimes: the set of points $(A, B)$ such that $S_{4e} = S_{2e}$.

$$\frac{Bd^2}{N} = \sqrt{\left( \frac{Ad^2}{N} \right)^2 - 8 \frac{Ad^2}{N}}$$

The result presented above was derived under assumption $C = 0$ but proceeding in the same way with $C \lesssim \sqrt{N}$ one can show that at large $N$ nonzero $C$ does not affect the crossover line (56). Therefore in the rest part of this paper we will completely ignore coefficient $C$.

The borderline obtained from numerical solution of classical equations of motion (45, 46, 47) and its asymptotic form (56) is presented on figure 2. Instead of $A$ and $B$ we have used here more convenient variables

$$u = d^2 \frac{(A - B)}{N}, \quad v = d^2 \frac{(A + B)}{2N}$$

Note that by definition $v > |u|/2$. From (56) we see that in terms of $u$ and $v$ at large $v$ the borderline between $2e$- and $4e$-regimes is given by

$$u = f(v) \approx \frac{4v}{v - 2} \rightarrow 4, \quad v \rightarrow \infty$$

On figure 2b "phase diagram" of the disordered chain is presented in a more intuitive manner: we emphasize here that $4e$-supercurrent exist in a small vicinity of the maximally frustrated point.

Equipped with this result, for any given set of quenched random charges (characterized by coefficients $A$ and $B$) we can (in principle) determine whether $4e$-supercurrent dominates in the chain. Of course experimentally we have no access to such quantities as $A$ and $B$. Hence we need
Figure 2: "Phase diagram" of disordered chain. Solid lines on the both figures mark the crossover 2e- and 4e-regimes. Note that these lines does not correspond to any phase transition. The crossover however is sharp at large $N$ since actions $S_{4e}$ and $S_{2e}$ are proportional to the number of rhombi. Subplot a) presents crossover line in variables $u$ and $v$ defined by (57) which are useful for calculations. Subplot b) depicts the same in a bit more intuitive way. Here we introduce factors $\tilde{A} = \left(\frac{NA^2}{A^2 + B^2}\right)^{1/3}$ and $\tilde{B} = \left(\frac{NB^2}{A^2 + B^2}\right)^{1/3}$ describing realization of disorder; $(\delta\Phi^c)_{reg}$ — critical deviation for a clean chain with same $E_J/E_C$.

some statistical description of rhombi chain with random charges. Such a description is provided by the probability $P(E_J/E_C, N, \delta)$ to find dominating 4e-supercurrent in the system.

Assuming Gaussian statistics for $u$ and $v$ and taking into account (39) one can derive probability distribution for $u$ and $v$

$$P(u, v) = \frac{8N}{\pi d^4} \exp\left(\frac{-2Nu^2}{d^4}\right) \exp\left(\frac{-32N(v - v_0)^2}{d^4}\right)$$

$$v_0 = \frac{d^2}{4}$$

Required probability can be written as

$$\mathcal{P}_{4e}(E_J/E_C, N, \delta) = 1 - 2 \int_0^{+\infty} dv \int_0^{f(v)} du P(u, v)$$

Maximum of probability distribution $P$ lies at $u = 0, v = d^2/4$. This point for $d = 9$ is marked on figure 2 with a cross. At sufficiently large $d$ we can replace $f(v)$ in (61) by 4. We then get

$$\mathcal{P}_{4e}(E_J/E_C, N, \delta) = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{32N/d^4}} du \exp\left(-u^2\right) = 1 - \text{Erf}\left(\sqrt{\frac{32N}{d^4}}\right)$$
Figure 3: Probability $P_{4e}$ as function of deviation from maximally frustrated point. Subplots a), b), c), d), correspond to $N = 10, 20, 30, 70$.

Let us introduce parameter $\kappa$ and define critical deviation from the maximally frustrated point $\delta \Phi_c$ as deviation under which the probability (62) equals $\kappa$. Reasonable choice for $\kappa$ is 0.5 or 0.75 or something else. From (62) we then get central result of this section

$$
\frac{\delta \Phi_c}{\Phi_0} = \frac{(\text{Erf}^{-1}(1 - \kappa))^{1/6}}{2^{3/2} \pi^{1/3}} \frac{1}{N^{1/6}} \left( \frac{\nu}{E_J} \right)^{2/3}
$$

Comparing (63) and (11) we see that for any experimentally reasonable $N$ critical deviation from the maximally frustrated point does not differ from the one in a regular chain. However one should remember that theory presented above has several limitations. To clarify this question we present here several graphics for $P_{4e}(\delta \Phi, E_J/E_C, N)$ at different $E_J/E_C$ and $N$ together with the boundaries of the validity region of the approximation used (fig. 3). Subplots a), b), c), d) correspond to different number of rhombi $N = 10, 20, 30, 70$ respectively. Solid and dashed lines 1,2,3,4 correspond to different $E_J/E_C = 10, 12, 15, 20$. Dashed lines were obtained with the aid of equation (62) relying on the condition $d^2 > 1$ whereas solid lines were produced by exploiting equation (61) with $f(v)$ determined from numerical calculations. The validity region should be determined as follows. First of all, we have used assumption $h >> w$. Taking $h = w$ as a criterium for the edge of our validity region one finds the dash-doted curves on figure 3. Strictly speaking
only those parts of solids curves are valid, which lie under the corresponding dash-dotted curves. Now we have to ensure that rhombi chain is in quantum regime, i.e. currents are exponentially small. Simple estimates show that this is true above the dotted line on figure[3]. At large N this limitation is very soft. Finally, our approximations are not valid at probabilities \( P_{4e} \) very small or very close to unity since they rely on gaussian statistics for quantities \( A \) and \( B \) which certainly does not describe rare events.

Qualitatively, results presented above can be interpreted as follows: at large \( E_J/E_C \) phase fluctuations in a single rhombi are weak and this makes random charges inefficient, however in a long chain global supercurrent is still exponentially suppressed. From the analysis presented above one concludes that we can guarantee with high probability existence of dominating, suppressed by quantum fluctuations, 4\( e \)-supercurrent in long chain \((N \sim 20)\) with large \( E_J/E_C \sim 20 \) and for such a chain critical deflection \( \delta \Phi^c/\Phi_0 \sim 10^{-3} \). If we compare this result with critical deflection \( (\delta \Phi^c)_{reg}/\Phi_0 \approx 1.2 \times 10^{-3} \) for a regular chain with the same \( E_J/E_C \) we conclude that the influence of disorder on pairing effect is really small. On the other hand in a regular chain it was possible to obtain critical deflection of order \( 10^{-2} \) by choosing \( E_J/E_C \sim 8 \). For this set of parameters we cannot, use our theory quantitatively; we expect however that qualitatively the same behaviour as described above for larger values \( N \) and \( E_J/E_C \) should be observed here as well.

IV. MODULATION OF THE SUPERCURRENT BY CAPACITIVE GATE

In this section we will discuss briefly influence of regular gates on pairing effect in maximally frustrated rhombi chain. We suppose that we have a gate, which is capacitively connected to all superconducting islands in one row (fig. 1).

Let us consider first rhombi chain with no disorder. Our system still can be described by equations (16, 17), but now \( a_n \) and \( b_n \) are no more random. This coefficients can be expressed in terms of the gate voltage according to

\[
a_n = \frac{1 + \cos \pi C_g V_g}{2} \equiv a, \quad b_n = \frac{1 - \cos \pi C_g V_g}{2} \equiv b
\]

(64)

We will show here that the supercurrent in the chain is rather sensitive to the gate voltage. Since the gate voltage enters the problem in a particularly quantum way (via phases of virtual QPS), this dependence may provide an experimental test for the quantum nature of chain’s state near the maximally frustrated point.
The problem we are considering now is much simpler as compared to the problem of random stray charges discussed above. Now our Hamiltonian commutes with the total spin of the chain and we can apply semiclassical approximation directly (e.g. by introducing spin coherent state path integral, c.f. [5, 7]). It’s easy to see that the ground state corresponds to the maximal total spin $S = N/2$. After proper redefinition of the time scale we can write imaginary-time action in the form

$$S_E = \frac{N}{2} \int d\tau \left( -i \cos \theta \dot{\phi} + \frac{w x^2}{N} + (a \cos 2x \cos \phi + b \sin 2x \sin \phi) \sin \theta \right) = N\tilde{S}_E(E_J/E_C, C_g V_g)$$

(65)

Here angles $\theta$ and $\phi$ parameterize coherent states of the spin. Again one easily find two types of tunneling trajectories corresponding to $4e$- and $2e$-supercurrents respectively (e.g. $4e$-trajectory which connects $|x = 0, \theta = \pi/2, \phi = -\pi\rangle$ with $|x = \pi, \theta = \pi/2, \phi = -\pi\rangle$ and $2e$-trajectory connecting $|x = 0, \theta = \pi/2, \phi = -\pi\rangle$ with $|x = \pi/2, \theta = \pi/2, \phi = 0\rangle$). Action $\tilde{S}_E$ is a function of $E_J/E_C$ and $C_g V_g$ and does not depend on the number of rhombi.

From (65) we have classical equations of motion (it is convenient to change variables according to $\theta \rightarrow \pi/2 + i\theta$)

$$\frac{w}{N} \ddot{x} + (a \sin 2x \cos \phi - b \cos 2x \sin \phi) \cosh \theta = 0$$

(66)

$$\dot{\theta} - (a \cos 2x \sin \phi - b \sin 2x \cos \phi) = 0$$

(67)

$$\cosh \theta \dot{\phi} - (a \cos 2x \cos \phi + b \sin 2x \sin \phi) \sinh \theta = 0$$

(68)

Note that $w/N \sim \nu$ is a very good small parameter at $E_J \gg E_C$. It allow us to obtain actions $\tilde{S}_{4e}$ and $\tilde{S}_{2e}$ analytically. Characteristic frequency of variable $x$ is $\omega_x \sim \sqrt{N/w} \gg 1$, while for spins such a frequency is $\omega_s \sim 1$. This means that on $4e$-trajectory, which does not require any change in spin, we can assume spin to be constant (at least while coefficients $a$ and $b$ are not too close). This immediately leads to

$$S_{4e} \approx \frac{16N}{21/4\pi} \sqrt{\frac{\nu}{E_J}} \left( 1 + \cos \pi C_g V_g \right)$$

(69)

On the other hand, on $2e$-trajectory smallness of $w/N$ allows us to omit $w\ddot{x}/N$ in equation for $x$. Resulting equations can be integrated analytically and we obtain $2e$-action which, practically does not depend on $E_J/E_C$

$$S_{2e} \approx N \operatorname{arcsh} \frac{\sqrt{|a^2 - b^2|}}{b} = N \operatorname{arcsh} \frac{2\sqrt{|\cos \pi C_g V_g|}}{1 - \cos \pi C_g V_g}$$

(70)
Actions $\tilde{S}_{4e}$ and $\tilde{S}_{2e}$ can also be evaluated numerically. Results of numerical calculations are presented on figure [4]. Analytical results ([69], [70]) are in very good agreement with numerical data. Note that $2e$-supercurrent dominates over the current of pairs of Cooper pairs only in small vicinity of the point $C_g V_g/e = 1$.

Note that in a regular chain change in $4e$-action upon applying external gate voltage is negative and proportional to the number of rhombi, i.e. external gate leads to significant increase of otherwise suppressed by fluctuations $4e$-supercurrent.

Let us now discuss influence of a regular gate on a chain with random stray charges. We will treat this problem within the same limit as in the previous section: we consider chain which is rather far from the maximally frustrated point in the sense $h \gg w$. From the preceding analysis we know that $4e$-action is given by equation (55). Coefficient $A$ acquires now the following form

$$A = \frac{1}{4} \sum_{n=1}^{N} \left( \cos \pi Q_n + \cos \pi (Q_n^2 + C_g V_g) \right)^2 = A_0 + \Delta A$$

(71)

Here $A_0$ is the value of $A$ at zero external bias. Assuming that fluctuations of random charges are strong one easily finds

$$\langle \Delta A \rangle = 0 \quad \langle \Delta A^2 \rangle = \frac{N}{32} \left( \sin^2 \pi C_g V_g + 8 \sin^2 \frac{\pi C_g V_g}{2} \right)$$

(72)

So typical change in $4e$-action can be estimated as

$$\delta S_{4e} \sim \frac{4\sqrt{2} d \delta}{\pi^2} \sqrt{N \left( \sin^2 \pi C_g V_g + 8 \sin^2 \frac{\pi C_g V_g}{2} \right)}$$

(73)

This result differs from the analogous one for a regular (without offset charges) chain in two important aspects: i) we see that change in action is now proportional to $\sqrt{N}$ instead of $N$; ii) variation
of the tunnelling action can now be both negative or positive, i.e. applying external voltage we can decrease $4e$-supercurrent as well as increase it, depending on the realization of charge disorder. From the above analysis we conclude that in a disordered chain current modulation by external gates is random and much weaker than in a regular chain, but it is still present and can be used to demonstrate quantum coherence of the tunnelling processes which occur in the chain.

V. INFLUENCE OF "MAGNETIC DISORDER" UPON THE CROSSOVER POINT

We now consider the effect of randomness in the values of magnetic fluxes through different rombi. Just as in the previous section we start from the grand partition function given by (22) with

$$\frac{d\hat{U}_n}{d\tau} = -\left( w \cos 2x(\tau)S^z - h_n\hat{S}^z \right) \hat{U}_n $$

(74)

We presume that fluxes $\Phi_n^+$ are uniformly distributed on $(\Phi_r^+ - \Delta\Phi, \Phi_r^+ + \Delta\Phi)$, i.e. probability distribution for $h_n$ is

$$P(h) = \begin{cases} \frac{1}{2\pi}, & |h - h_0| < \sigma \\ 0, & |h - h_0| > \sigma \end{cases} $$

(75)

where

$$h_0 = \frac{16N(\Phi_r - \Phi_0/2)}{\pi\Phi_0}, \quad \sigma = \frac{16N\Delta\Phi}{\pi\Phi_0}. $$

(76)

Actually particular form of $P(h)$ is not important for us as we assume that $\sigma \ll h_0$ and use perturbation theory in $\sigma/h_0$. We are interested in the critical deviation from the maximally frustrated point $\Phi_r = \Phi_0/2$ destroying $4e$-supercurrent. Therefore we presume that $h_0 \gg w$. In such an approximation we can use equation (25) to calculate $\text{Tr} \hat{U}_n(\beta)$ and get effective action for variable $x$

$$S = \int_0^\beta d\tau \frac{x^2(\tau)}{2} - \frac{w^2 hN}{4} \int \cos 2x(\tau_1) D(\tau_1 - \tau_2) \cos 2x(\tau_2) $$

(77)

$$D(\tau_1 - \tau_2) = \frac{1}{2h_0N} \sum_{n=1}^N \exp(-h_n|\tau_1 - \tau_2|) $$

(78)

At large $N$ we can replace $D(\tau_1 - \tau_2)$ in (77) by its mean value

$$\overline{D}(\tau_1 - \tau_2) = \frac{1}{2h_0} \int dh \exp(-h|\tau_1 - \tau_2|)P(h) $$

(79)

Fourier transformation of $\overline{D}$ for small $\sigma/h_0$ reads

$$\overline{D}(\omega) = \frac{1}{h_0^2 + \omega^2} \left( 1 + \sigma^2 \frac{h_0^2 - 3\omega^2}{3(h_0^2 + \omega^2)^2} \right) $$

(80)
Again, using the fact that

$$D^{-1}(\omega) = \omega^2 + h_0^2 + \sigma^2 - \frac{4}{3} \sigma^2 h_0^2 \frac{1}{h_0^2 + \omega^2}$$

(81)

we can perform Hubbard-Stratonovich transformation and find representation for partition function with local action (after redefinition of time scale)

$$S = \hbar \int d\tau \left( \frac{x^2}{2} + \frac{y^2}{2} + \frac{\dot{x}^2}{2} + \left( 1 + \frac{\sigma^2}{h_0^2} \right) \frac{y^2}{2} + \frac{\dot{y}^2}{2} + dy \cos 2x + \sqrt{\frac{4}{3} \sigma} yz \right)$$

(82)

Note that our formulation of the problem is a bit specific: we fix relative diversity $\sigma/h_0$ and look for $h_0$ which brings the chain to the point with equal $4e$- and $2e$-supercurrents.

Discussion of the previous section concerning two types of tunneling trajectories and their connection to components of supercurrent can be literally applied to action (82). So we need to estimate classical action for $2e$- and $4e$-trajectories. Problem with $\sigma = 0$ was analysed in [5]. It was shown that $2e$-supercurrent dominates at $d \ll 1$ while $4e$-supercurrent — at $d \gg 1$. Crossover takes place at $d \approx 3.24$. From (82) one can easily see that corrections to classical actions due to nonzero $\sigma$ should be of order $(\sigma/h)^2$. We analyse action (82) numerically. Results are presented on figure 5.

It may come as a surprise, that with this formulation of the problem critical deviation $\delta \Phi^c$ (determined through mean flux in rhombi) grows with $\sigma/h_0$. However, this is quit reasonable
since zero deviation from the maximally frustrated point for one rhombi in a chain immediately prohibits $2e$-supercurrent. Allowing for diversity of fluxes in rhombi we allow some of the rhombi to be closer to the maximally frustrated point than a "mean" rhombi. This fact causes strong suppression of $2e$-supercurrent.

So we conclude that randomness in fluxes is not important for the pairing effect, at least if the standard scatter of these fluxes $\Delta \Phi$ is not too large in comparison with the critical value $\Phi^c$ found for a regular chain, cf. Eq. (11).

VI. CONCLUSIONS

In this paper we provide detailed calculations of superconductive current in a long frustrated rhombi chain with quenched disorder. We have considered two types of disorder: random stray charges and random fluxes in rhombi. We found that small (as compared to the critical deflection $(\Phi^c)_{reg}$ destroying $4e$-supercurrent in a clean chain) fluctuations of fluxes piercing rhombi are not that important for the pairing effect.

Main results of our paper concern effect of quenched random stray charges on pairing of Cooper pairs. For a chain which is relatively far from the maximally frustrated point in the sense $h \gg w$ we managed to calculate probability to find the system in the regime with dominating $4e$-supercurrent. Stray charges, in principal, may significantly affect properties of the rhombi chain. In particular, in such a chain $2e$-supercurrent exist even at the maximally frustrated point. However, as we have demonstrated in Sec. III, at large $E_J/E_C$ and $\Phi_r = \Phi_0/2$ probability to find large $2e$-supercurrent is small. This result itself is not a great surprise: in a perfectly classical chain stray charges have no effect at all. More important are two things: i) it is possible to combine low probability of finding significant $2e$-supercurrent at the maximally frustrated point with exponential suppression of the supercurrent by quantum fluctuations; ii) in a perfectly classical chain critical deflection $\Phi^c$ scales with number of rhombi as $1/N$ (this can be easily seen from eq. (5) describing energy spectrum of classical rhombi chain) while in a disordered quantum chain this dependence is much weaker $\Phi^c \sim 1/N^{1/6}$, c.f. (63).

In Sec. IV we have considered modulation of the supercurrent by capacitively coupled gate. In a regular chain, applying gate voltage one suppresses quantum fluctuations of rhombi and increases the supercurrent. Its dependence on the gate voltage is very strong: change of the logarithm of the supercurrent is linear in $N$. On the contrary, in a chain with strong random stray charges
applying external gate can both increase or decrease supercurrent. Dependence of the supercurrent on the gate voltage is now much weaker: typical change of the logarithm of the supercurrent is now proportional to $\sqrt{N}$. Still we see that dependence of the current on gate voltage should be measurable. Such a dependence is one of the possible ways to demonstrate coherence of quantum phase slips in the chain.

We are grateful to L. B. Ioffe and B. Pannetier for many useful and inspiring discussions. This research was supported by the Program “Quantum Macrophysics” of the Russian Academy of Sciences, by the Russian Ministry of Education and Science via the contract RI-112/001/417 and by Russian Foundation for Basic Research under the grant No. 04-02-16348. I.V.P. acknowledges financial support from the Dynasty Foundation and Landau-Juelich Scholarship.

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