Additive $N$-Step Markov Chains as Prototype Model of Symbolic Stochastic Dynamical Systems with Long-Range Correlations

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A theory of symbolic dynamic systems with long-range correlations based on the consideration of the binary $N$-step Markov chains developed earlier in Phys. Rev. Lett. 90, 110601 (2003) is generalized to the biased case (non equal numbers of zeros and unities in the chain). In the model, the conditional probability that the $i$-th symbol in the chain equals zero (or unity) is a linear function of the number of unities (zeros) among the preceding $N$ symbols. The correlation and distribution functions as well as the variance of number of symbols in the words of arbitrary length $L$ are obtained analytically and verified by numerical simulations. A self-similarity of the studied stochastic process is revealed and the similarity group transformation of the chain parameters is presented. The diffusion Fokker-Planck equation governing the distribution function of the $L$-words is explored. If the persistent correlations are not extremely strong, the distribution function is shown to be the Gaussian with the variance being nonlinearly dependent on $L$. An equation connecting the memory and correlation function of the additive Markov chain is presented. This equation allows reconstructing a memory function using a correlation function of the system. Effectiveness and robustness of the proposed method is demonstrated by simple model examples. Memory functions of concrete coarse-grained literary texts are found and their universal power-law behavior at long distances is revealed.

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INTRODUCTION

The problem of systems with long-range spatial and/or temporal correlations (LRCS) is one of the topics of intensive research in modern physics, as well as in the theory of dynamic systems and the theory of probability. The LRCS systems are usually characterized by a complex structure and contain a number of hierarchic objects as their subsystems. The LRCS are the subject of study in physics, biology, economics, linguistics, sociology, geography, psychology, etc. [1, 2, 3, 4].

One of the efficient methods to investigate the correlated systems is based on a decomposition of the space of states into a finite number of parts labeled by definite symbols. This procedure referred to as coarse graining can be accompanied by the loss of short-range memory between states of system but does not affect and does not damage its robust invariant statistical properties on the large scales. The most frequently used method of the decomposition is based on the introduction of two parts of the phase space. In other words, it consists in mapping the two parts of states onto two symbols, say 0 and 1. Thus, the problem is reduced to investigating the statistical properties of the symbolic binary sequences. This method is applicable for the examination of both discrete and continuous systems.

One of the ways to get a correct insight into the nature of correlations consists in an ability of constructing a mathematical object (for example, a correlated sequence of symbols) possessing the same statistical properties as the initial system. There are many algorithms to generate long-range correlated sequences: the inverse Fourier transform [5], the expansion-modification Li method [6], the Voss procedure of consequent random addition [7], the correlated Levy walks [8], etc. [9]. We believe that, among the above-mentioned methods, using the Markov chains is one of the most important. This was demonstrated in Ref. [9], where the Markov chains with the step-like memory function (MF) were studied. It was shown that there exist some dynamical systems (coarse-grained sequences of the Eukarya’s DNA and dictionaries) with correlation properties that can be properly described by this model.

The many-step Markov chain is the sequence of symbols of some alphabet constructed using a conditional probability function, which determines the probability of occurring some definite symbol of sequence depending on $N$ previous

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Formulation of the Problem

Conditional probability of the many-step additive Markov chain

Let us consider a stationary binary sequence of symbols $a_i$, $a_i = \{0, 1\}, i \in \mathbb{Z} = \ldots, -2, 0, 1, 2, \ldots$. To determine the $N$-step Markov chain we have to introduce the conditional probability $P(a_i \mid a_{i-N}, a_{i-N+1}, \ldots, a_{i-1})$ of occurring the definite symbol $a_i$ (for example, $a_i = 1$) after $N$-word $T_{N,i}$, where $T_{N,i}$ stands for the sequence of symbols $a_{i-N}, a_{i-N+1}, \ldots, a_{i-1}$. Thus, it is necessary to define $2^N$ values of the $P$-function corresponding to each possible configuration of the symbols $a_i$ in the $N$-word $T_{N,i}$.

We suppose that the conditional probability $P(a_i \mid T_{N,i})$ differs from zero and unity for any word $T_{N,i}$ that provides the metrical transitivity of the Markov chain (see Appendix). In turn, according the Markov theorem, this property leads to the ergodicity of the symbolic system under consideration.

Since we suppose to apply our theory to the sequences with long memory lengths of the order of $10^6$, some special restrictions to the class of $P$-functions should be imposed. We consider the memory function of the additive form,

$$P(a_i = 1 \mid T_{N,i}) = \frac{1}{N} \sum_{r=1}^{N} f(a_{i-r}, r). \tag{1}$$

Here the function $f(a_{i-k}, k)/N$ describes the additive contribution of the symbol $a_{i-r}$ to the conditional probability of occurring the symbol unity, $a_i = 1$, at the $i$th site. The homogeneity of the Markov chain is provided by independence of the conditional probability Eq. (1) of the index $i$. It is possible to consider Eq. (1) as the first term in expansion of conditional probability in the formal series, where each term corresponds to the additive (unary), binary, ternary, and so on functions up to the $N$-ary one.

It is reasonable to assume the function $f$ to be decreasing with an increase of the distance $r$ between the symbols $a_{i-r}$ and $a_i$ in the Markov chain. However, for the sake of simplicity we consider a step-like memory function $f(a_{i-r}, r)$ independent of the second argument $r$. As a result, the model is characterized by three parameters only, specifically by $f(0)$, $f(1)$, and $N$:

$$P(a_i = 1 \mid T_{N,i}) = \frac{1}{N} \sum_{r=1}^{N} f(a_{i-r}). \tag{2}$$

Note that the probability $P$ in Eq. (2) depends on the numbers of symbols 0 and 1 in the $N$-word but is independent of the arrangement of the elements $a_{i-k}$. Instead of two parameters $f(0)$ and $f(1)$ it is convenient to introduce new independent parameters $\nu$ and $\mu$ (see below Eq. (3)),

$$f(0) + f(1) = 1 + 2\nu, \quad |\nu| < 1/2. \tag{3}$$

Parameter $\nu$ provides the statistical inequality of the numbers of symbols zero and unity in the Markov chain under consideration. In other words, the chain is biased. Indeed, taking into account Eqs. (2) and (3) and the sequence of equations,

$$P(a_i = 1 \mid T_{N,i}) = \frac{1}{N} \sum_{r=1}^{N} f(\bar{a}_{i-r}) - 2\nu = P(a_i = 0 \mid \bar{T}_{N,i}) - 2\nu, \tag{4}$$
one can see the lack of symmetry with respect to interchange $\tilde{a}_i \leftrightarrow a_i$ in the Markov chain if $\nu \neq 0$. Here $\tilde{a}_i$ is the symbol "opposite" to $a_i$, $\tilde{a}_i = 1 - a_i$, and $\tilde{T}_{N,i}$ is the word "opposite" to $T_{N,i}$. Therefore, the probabilities of occurring the words $T_{N,i}$ and $\tilde{T}_{N,i}$ are not equal to each other for any word of the length $L$. At $L = 1$ this yields nonequal average probabilities that symbols 0 and 1 occur in the chain. Particularly, probability of occurring symbol 0 is greater by $2\nu$ than that of symbol 1. If $\nu = 0$ one has non-biased case.

Taking into account the symmetry of the conditional probability $P$ with respect to a permutation of symbols $a_i$ (see Eq. (2)), we can simplify the notations and introduce the conditional probability $p_k$ of occurring the symbol zero after the $N$-word containing $k$ unities, e.g., after the word $(11 \ldots 1 00 \ldots 0)$,

$$p_k = P(a_{N+1} = 0 \mid 11 \ldots 1 00 \ldots 0)$$

$$= \frac{1}{2} + \nu + \mu \left(1 - \frac{2k}{N}\right), \quad (5)$$

with the correlation parameter $\mu$ being defined by the relation

$$\mu = \frac{f(0) - f(1)}{2} = f(0) - \frac{1}{2} - \nu. \quad (6)$$

We focus mainly our attention on the region of $\mu$ determined by the persistence inequality $0 < \mu$. In this case, each of the symbols unity in the preceding $N$-word promotes the birth of new symbol unity. Nevertheless, the major part of our results is valid for the anti-persistent region $\mu < 0$ as well. Note that inequalities $|\nu| < 1/2$ and $|\mu + \nu| < 1/2$ follow from Eq. (5). Without loss of generality, we consider a case $\nu > 0$ only.

**Statistical characteristics of the chain**

In order to investigate the statistical properties of the Markov chain, we consider the distribution $W_L(k)$ of the words of definite length $L$ by the number $k$ of unities in them,

$$k_i(L) = \sum_{l=1}^{L} a_{i+l}, \quad (7)$$

and the variance of $k$,

$$D(L) = \overline{k^2} - \overline{k}^2, \quad (8)$$

where

$$\overline{g(k)} = \sum_{k=0}^{L} g(k)W_L(k). \quad (9)$$

If $\mu = 0$, one arrives at the known result for the non-correlated Brownian diffusion,

$$D(L) = L \left(\frac{1}{4} - \nu^2\right). \quad (10)$$

We will show that the distribution function $W_L(k)$ for the sequence determined by Eq. (5) (with nonzero but not extremely close to $1/2 - \nu$ parameter $\mu$) is the Gaussian with the variance $D(L)$ nonlinearly dependent on $L$. However, at $\mu \to 1/2 - \nu$ the distribution function can differ strongly from the Gaussian.

**Main equation**

For the stationary Markov chain, the probability $b(a_1 a_2 \ldots a_N)$ of occurring a certain word $(a_1, a_2, \ldots, a_N)$ satisfies the condition of compatibility for the Chapman-Kolmogorov equation (see, for example, Ref. [13]):

$$b(a_1 \ldots a_N) =$$
\[ b(a_1 \ldots a_{N-1})P(a_N \mid a, a_1, \ldots, a_{N-1}). \] (11)

Thus, we have \( 2^N \) homogeneous algebraic equations for the \( 2^N \) probabilities \( b \) of occurring the \( N \)-words and the normalization equation \( \sum b = 1 \). This set of equations is equivalent to that of Eq. (77). In the case under consideration, the set of Eqs. (11) can be substantially simplified owing to the following statement:

**Proposition 1:** The probability \( b(a_1 a_2 \ldots a_N) \) depends on the number \( k \) of unities in the \( N \)-word only, i.e., it is independent of the arrangement of symbols in the word \( (a_1, a_2, \ldots, a_N) \).

This statement illustrated by Fig. 1 is valid owing to the chosen simple model (2), (5) of the Markov chain. It can be easily verified directly by substituting the obtained below solution (15) into the set of Eqs. (11). Note that according to the Markov theorem, Eqs. (11) do not have other solutions [17].

Proposition 1 evidently leads to the very important property of isotropy: any word \( (a_1, a_2, \ldots, a_L) \) appears with the same probability as the inverted one, \( (a_L, a_{L-1}, \ldots, a_1) \).

Let us apply the set of Eqs. (11) to the word \( (11 \ldots 1 00 \ldots 0) \):

\[
\begin{align*}
    b(11 \ldots 1 00 \ldots 0) &= b(011 \ldots 1 00 \ldots 0)p_k + \\
    &+ b(11 \ldots 1 00 \ldots 0)p_{k+1}.
\end{align*}
\] (12)

This yields the recursion relation for \( b(k) = b(11 \ldots 1 00 \ldots 0) \),

\[
\begin{align*}
    b(k) &= \frac{1 - p_{k-1}}{p_k}b(k - 1) = \\
    &= \frac{N - 2\mu N - 2\mu(N - 2k + 2)}{N + 2\nu N + 2\mu(N - 2k)}b(k - 1).
\end{align*}
\] (13)

The probabilities \( b(k) \) for \( \mu > 0 \) satisfy the sequence of inequalities,

\[
\begin{align*}
    b \left( \frac{N}{2} \left( 1 + \frac{\nu}{\mu} \right) \right) &< b \left( \frac{N}{2} \left( 1 + \frac{\nu}{\mu} \right) - 1 \right) < \ldots < b(0), \\
    b \left( \frac{N}{2} \left( 1 + \frac{\nu}{\mu} \right) \right) &< b \left( \frac{N}{2} \left( 1 + \frac{\nu}{\mu} \right) + 1 \right) < \ldots < b(N),
\end{align*}
\] (14)
which is the reflection of persistent properties for the chain.

The solution of Eq. (11) is

\[
b(k) = A \cdot \Gamma(n_1 + k)\Gamma(n_2 + N - k)
\]

with the parameters \(n_1\) and \(n_2\) defined by

\[
n_1 = \frac{N(1 - 2(\mu + \nu))}{4\mu}, \quad n_2 = \frac{N(1 - 2(\mu - \nu))}{4\mu}.
\]

The constant \(A\) will be found below by normalizing the distribution function. Its value is,

\[
A = \frac{\Gamma(n_1 + n_2)}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_1 + n_2 + N)}.
\]

**DISTRIBUTION FUNCTION OF \(L\)-WORDS**

In this section we investigate into the statistical properties of the Markov chain, specifically, the distribution of the words of definite length \(L\) by the number \(k\) of unities. The length \(L\) can also be interpreted as the number of jumps of some particle over an integer-valued 1D lattice or as the time of the diffusion imposed by the Markov chain under consideration. The form of the distribution function \(W_L(k)\) depends, to a large extent, on the relation between the word length \(L\) and the memory length \(N\). Therefore, the first thing we will do is to examine the simplest case \(L = N\).

**Statistics of \(N\)-words**

The value \(b(k)\) is the probability that an \(N\)-word contains \(k\) unities with a definite order of symbols \(a_i\). Therefore, the probability \(W_N(k)\) that an \(N\)-word contains \(k\) unities with arbitrary order of symbols \(a_i\) is \(b(k)\) multiplied by the number \(C^k_N = N!/(k!(N-k)!\) of different permutations of \(k\) unities in the \(N\)-word,

\[
W_N(k) = C^k_N b(k).
\]

Combining Eqs. (15) and (18), we find the distribution function,

\[
W_N(k) = W_N(0)C^k_N \frac{\Gamma(n_1 + k)\Gamma(n_2 + N - k)}{\Gamma(n_1)\Gamma(n_2 + N)}.
\]

The normalization constant \(W_N(0)\) can be obtained from the equality \(\sum_{k=0}^{N} W_N(k) = 1\),

\[
W_N(0) = \frac{\Gamma(n_1 + n_2)\Gamma(n_2 + N)}{\Gamma(n_2)\Gamma(n_1 + n_2 + N)}.
\]

Comparing Eqs. (15), (18)-(20), one can get Eq. (17) for the constant \(A\) in Eq. (15).

**Limiting case of weak persistence, \(n_1, n_2 \gg 1\)**

In terms of the correlation parameter \(\mu\), this limiting case corresponds to the values of \(\mu\) not very close to 1/2,

\[
\frac{1 - 2(\mu + \nu)}{4\mu} \gg \frac{1}{N}.
\]

This inequality can be rewritten via the \(f\)-function (see Eqs. (2)—(6)),

\[
\frac{f(1)}{f(0) - f(1)} \gg \frac{1}{N}.
\]
In the absence of correlations, \( n_2 \to \infty \), Eq. (19) and the Stirling formula yield the Gaussian distribution at \( k, Nn_1/n_2, N - k \gg 1, k - k_0 \ll N \). Given the persistence is not too strong,

\[
n_2 \gg 1,
\]

one can also obtain the Gaussian form for the distribution function,

\[
W_N(k) = \frac{1}{\sqrt{2\pi D(N)}} \exp \left\{ -\frac{(k-k_0)^2}{2D(N)} \right\},
\]

with the \( \mu \)-dependent variance,

\[
D(N) = \frac{N(N+n_1+n_2)n_1n_2}{(n_1+n_2)^3} = \frac{N}{4(1-2\mu)} \left[ 1 - \frac{4\mu^2}{(1-2\mu)^2} \right],
\]

\[
k_0 = \frac{n_1}{n_1+n_2} N = \frac{N}{2(1-2\mu)} \left[ 1 - \frac{2\mu}{1-2\mu} \right].
\]

It is followed from Eq. (24) that \( N \)-words containing \( k_0 \) units are the most probable. It is interesting to note, that the persistence leads to a decrease of the variance \( D(N, \mu > 0) \) with respect to \( D(N, \mu = 0) = N (1/4 - \nu^2) \) if

\[
\nu > \frac{1-2\mu}{2\sqrt{3-6\mu+4\mu^2}}.
\]

In other case, for instance, at \( \nu = 0 \), the persistence results in an increase of the variance \( D(N, \mu) \). To put it differently, the persistence is conductive to the intensification of the diffusion under conditions opposite to inequality (27).

Inequality \( n_2 \gg 1 \) gives \( D(N) \ll N^2 \). Therefore, despite the increase of \( D(N) \), the fluctuations of \( (k-k_0) \) of the order of \( N \) are exponentially small.

**Intermediate case, \( n_2 \geq 1 \)**

If the parameters \( n_1 \) and \( n_2 \) are integers of the order of unity, the distribution function \( W_N(k) \) is a polynomial of degree \( n_1 + n_2 - 2 \). In particular, at \( n_1 = n_2 = 1 \), the function \( W_N(k) \) is constant,

\[
W_N(k) = \frac{1}{N+1}.
\]

At \( n_1 \neq 1 \), \( W_N(k) \) has a maximum within the interval \([0, N]\). At \( n_1 = 1 \) and \( n_2 > 1 \), \( W_N(k) \) decreases monotonously with an increase of \( k \).

**Limiting case of strong persistence**

If the parameter \( n_2 \) satisfies the inequality,

\[
n_2 \ll \ln^{-1} N,
\]

or

\[
1 - 2(\mu - \nu) \ll 1/N \ln(N), \quad f(1) \ll 1/N \ln(N),
\]

then one can neglect the parameters \( n_1 \) and \( n_2 \) in the arguments of the functions \( \Gamma(n_1+k) \), \( \Gamma(n_2+N) \), and \( \Gamma(n_2+N-k) \) in Eq. (19). In this case, the distribution function \( W_N(k) \) assumes its maximal values at \( k = 0 \) and \( k = N \),

\[
W_N(1) = W_N(0) \frac{n_1 N}{N-1} \ll W_N(0).
\]
Formula (31) describes the sharply decreasing $W_N(k)$ as $k$ varies from 0 to 1 (and from $N$ to $N - 1$). Then, at $1 < k < N/2$, the function $W_N(k)$ decreases more slowly with an increase in $k$,

$$W_N(k) = W_N(0)\frac{n_1N}{k(N-k)}.
(32)$$

At $k = N/2$, the probability $W_N(k)$ achieves its minimal value,

$$W_N\left(\frac{N}{2}\right) = W_N(0)\frac{4n_1}{N}.
(33)$$

It follows from normalization (20) that the values $W_N(0)$ and $W_N(N)$ are approximatively equal to $n_2/(n_1 + n_2)$ and $n_1/(n_1 + n_2)$ respectively. Neglecting the terms of the order of $n_2^2$, one gets

$$W_N(0) = \frac{n_2}{n_1 + n_2}(1 - n_1 \ln N),
(34)$$

$$W_N(N) = \frac{n_1}{n_1 + n_2}(1 - n_2 \ln N).
(35)$$

In the straightforward calculation using Eqs. (8) and (32) the variance $D$ is

$$D(N) = \frac{n_1n_2N^2}{(n_1 + n_2)^2} - \frac{n_1n_2N(N - 1)}{n_1 + n_2}.
(36)$$

Thus, the variance $D(N)$ is equal to $n_1n_2N^2/(n_1 + n_2)$ in the leading approximation in the parameter $n$. This fact has a simple explanation. The probability of occurrence the $N$-word containing $N$ units is approximatively equal to $n_2/(n_1 + n_2)$. So, the relations $k^2 \approx n_1N^2/(n_1 + n_2)$ and $\tilde{k} = n_1N^2/(n_1 + n_2)$ give (36). The case of strong persistence corresponds to the so-called ballistic regime of diffusion: if we chose randomly some symbol $a_i$ in the sequence, it will be surrounded by the same symbols with the probability close to unity.

The evolution of the distribution function $W_N(k)$ from the Gaussian form to the inverse one with a decrease of the parameters $n_1$ and $n_2$ is shown in Fig. 2. In the interval $\ln^{-1}N < n_2 < 1$ the curve $W_N(k)$ is concave and the maximum of function $W_N(k)$ inverts into minimum. At $N \gg 1$ and $\ln^{-1}N < n_2 < 1$, the curve remains a smooth function of its argument $k$ as shown by curve with $n = 0.5$ in Fig. 2. Below, we will not consider this relatively narrow region of the change in the parameter $n_2$.

Formulas (24), (25), (32) and (34) — (36) describe the statistical properties of $L$-words for the fixed "diffusion time" $L = N$. Below, we examine the distribution function $W_L(k)$ for more general situation, $L < N$.

![FIG. 2: The distribution function $W_N(k)$ for $N=20$ and different values of the parameters $n_1$ and $n_2$ shown near the curves.](image-url)
Statistics of \( L \)-words with \( L < N \)

Distribution function \( W_L(k) \)

The distribution function \( W_L(k) \) at \( L < N \) can be given as

\[
W_L(k) = \sum_{i=k}^{k+N-L} b(i) C_i^k C_{N-i-L}^{i-k}.
\]

(37)

This equation follows from the consideration of \( N \)-words consisting of two parts,

\[
\left\{ a_1, \ldots, a_{N-L}, a_{N-L+1}, \ldots, a_N \right\},
\]

(38)

The total number of unities in this word is \( i \). The right-hand part of the word (\( L \)-sub-word) contains \( k \) unities. The remaining \( (i-k) \) unities are situated within the left-hand part of the word (within \( (N-L) \)-sub-word). The multiplier \( C_i^k C_{N-i-L}^{i-k} \) in Eq. (37) takes into account all possible permutations of the symbols "1" within the \( N \)-word on condition that the \( L \)-sub-word always contains \( k \) unities. Then we perform the summation over all possible values of the number \( i \). Note that Eq. (37) is a direct consequence of the proposition 1 formulated in Subsec. C of the previous section.

The straightforward summation in Eq. (37) yields the following formula that is valid at any value of the parameters \( n_1 \) and \( n_2 \):

\[
W_L(k) = W_L(0) C_i^k \Gamma(n_1 + k) \Gamma(n_2 + L - k) / \Gamma(n_1) \Gamma(n_2 + L).
\]

(39)

where

\[
W_L(0) = \Gamma(n_1 + n_2) \Gamma(n_2 + L) / \Gamma(n_1 + n_2 + L).
\]

(40)

It is of interest to note that the parameters \( \mu, \nu \) and the memory length \( N \) are presented in Eqs. (39), (40) via the parameters \( n_1 \) and \( n_2 \) only. This means that the statistical properties of the \( L \)-words with \( L < N \) are defined by these "combined" parameters.

In the limiting case of weak persistence, \( n_2 \gg 1 \), at \( k, L n_1/n_2, L - k \gg 1 \), Eq. (39) along with the Stirling formula give the Gaussian distribution function,

\[
W_L(k) = 1 / \sqrt{2\pi D(L)} \exp \left\{ - (k-k_0)^2 / 2D(L) \right\}
\]

(41)

with the variance \( D(L) \)

\[
D(L) = n_1 n_2 L / (n_1 + n_2)^2 \left( 1 + L / (n_1 + n_2) \right) / \left( 1 + 2\mu L / N(1 - 2\mu) \right) \left[ 1 - 4\nu^2 / (1 - 2\mu)^2 \right]
\]

(42)

and

\[
k_0 = n_1 L / (n_1 + n_2) = L / 2 \left( 1 - 2\nu / (1 - 2\mu) \right).
\]

(43)

In the case of strong persistence \( \mu > 1 \), the asymptotic expression for the distribution function Eq. (39) can be written as

\[
W_L(k) = W_L(0) n_1 L / k(L-k), \quad k \neq 0, k \neq L,
\]

(44)

\[
W_L(0) = n_2 / (n_1 + n_2) (1 - n_1 \ln L), \quad W_L(L) = n_1 / (n_1 + n_2) (1 - n_2 \ln L).
\]

(45)

Both the distribution \( W_L(k) \) and the function \( W_N(k) \) have concave forms. The former assumes the maximal values \( \mu \) at the edges of the interval \( [0, L] \) and has a minimum at \( k = L/2 \).
Using the definition Eq. (8) and the distribution function Eq. (30), one can obtain a very simple formula for the variance $D(L)$,

$$D(L) = \frac{L n_1 n_2}{(n_1 + n_2)^2} \left[ 1 + \frac{(L - 1)}{1 + n_1 + n_2} \right] = \frac{L}{4} \left[ 1 + \frac{2 \mu (L - 1)}{N - 2 \mu (N - 1)} \right] \left[ 1 - \frac{4 \nu^2}{(1 - 2 \mu)^2} \right].$$

(46)

Eq. (46) shows that the variance $D(L)$ obeys the parabolic law independently of the correlation strength in the Markov chain.

In the case of weak persistence, at $n_2 \gg 1$, we obtain the asymptotics of Eq. (42). It allows one to analyze the behavior of the variance $D(L)$ with an increase in the “diffusion time” $L$. At small $\mu$, the dependence $D(L)$ follows the classical law of the Brownian diffusion, $D(L) \approx L(1/4 - \nu^2)$.

For the case of strong persistence, $n_2 \ll 1$, Eq. (46) gives the asymptotics,

$$D(L) = \frac{n_1 n_2 L^2}{(n_1 + n_2)^2} - \frac{n_1 n_2 L (L - 1)}{n_1 + n_2}.$$  

(47)

The ballistic regime of diffusion leads to the quadratic law of the $D(L)$ dependence in the zero approximation in the parameter $n_2 \ll 1$.

The unusual behavior of the variance $D(L)$ raises an issue as to what particular type of the diffusion equation corresponds to the nonlinear dependence $D(L)$ in Eq. (42). In the following subsection, when solving this problem, we will obtain the conditional probability $p(0)$ of occurring the symbol zero after a given $L$-word with $L < N$. The ability to find $p(0)$, with some reduced information about the preceding symbols being available, is very important for the study of the self-similarity of the Markov chain (see Subsubsection 4 of this Subsection).

**Generalized diffusion equation at** $L < N$, $n_2 \gg 1$

It is quite obvious that the distribution $W_L(k)$ satisfies the equation

$$W_{L+1}(k) = W_L(k)p(0)(k) + W_L(k-1)p(1)(k-1).$$

(48)

Here $p(0)(k)$ is the probability of occurring “0” after an average-statistical $L$-word containing $k$ unities and $p(1)(k-1)$ is the probability of occurring “1” after an $L$-word containing $(k-1)$ unities. At $L < N$, the probability $p(0)(k)$ can be written as

$$p(0)(k) = \frac{1}{W_L(k)} \sum_{i=k}^{k+N-L} p_i b(i) C^k_L C^{i-k}_{N-L}.$$  

(49)

The product $b(i) C^k_L C^{i-k}_{N-L}$ in this formula represents the conditional probability of occurring the $N$-word containing $i$ unities, the right-hand part of which, the $L$-sub-word, contains $k$ unities (compare with Eqs. (37), (38)).

The product $b(i) C^{i-k}_{N-L}$ in Eq. (49) is a sharp function of $i$ with a maximum at some point $i = i_0$ whereas $p_i$ obeys the linear law (3). This implies that $p_i$ can be factored out of the summation sign being taken at point $i = i_0$. The asymptotical calculation shows that point $i_0$ is described by the equation,

$$i_0 = \frac{N}{2} \left( 1 - \frac{2 \nu}{1 - 2 \mu} \right) - \frac{L/2}{1 - 2 \mu (1 - L/N)} \left( 1 - \frac{2 \nu}{L} - \frac{2 \nu}{1 - 2 \mu} \right).$$  

(50)

Expression (50) taken at point $i_0$ gives the desired formula for $p(0)$ because

$$\sum_{i=k}^{k+N-L} b(i) C^k_L C^{i-k}_{N-L}$$  

is obviously equal to $W_L(k)$. Thus, we have
\[ p^{(0)}(k) = \frac{1}{2} \left( 1 + \frac{2\nu}{1 - 2\mu} \right) + \frac{\mu L}{N - 2\mu(N - L)} \left( 1 - \frac{2k}{L} - \frac{2\nu}{1 - 2\mu} \right). \] (52)

Let us consider a very important point relating to Eq. (50). If the concentration of unities in the right-hand part of the word \((i_0 - k)/(N - L)\) is higher than \(1/2 - \nu/(1 - 2\mu)\), \(k/L > 1/2 - \nu/(1 - 2\mu)\), then the most probable concentration \((i_0 - k)/(N - L)\) of unities in the left-hand part of this word is likewise increased, \((i_0 - k)/(N - L) > 1/2 - \nu/(1 - 2\mu)\). At the same time, the concentration \((i_0 - k)/(N - L)\) is less than \(k/L\),

\[ \frac{1}{2} \left( 1 - \frac{2\nu}{1 - 2\mu} \right) < \frac{i_0 - k}{N - L} < \frac{k}{L}. \] (53)

This implies that the increased concentration of unities in the \(L\)-words is necessarily accompanied by the existence of a certain tail with an increased concentration of unities as well. Such a phenomenon is referred to us as the macro-persistence. An analysis performed in the following section will indicate that the correlation length \(l_c\) of this tail is \(\gamma N\) with \(\gamma \geq 1\) dependent on the parameters \(\mu\) and \(\nu\) only. It is evident from the above-mentioned property of the isotropy of the Markov chain that there are two correlation tails from both sides of the \(L\)-word.

Note that the distribution \(W_L(k)\) is a smooth function of arguments \(k\) and \(L\) near its maximum in the case of weak persistence and \(k, L - k, Ln_1/n_2 \gg 1\). By going over to the continuous limit in Eq. (48) and using Eq. (52) with the relation \(p^{(1)}(k - 1) = 1 - p^{(0)}(k - 1)\), we obtain the diffusion Fokker-Planck equation for the correlated Markov process,

\[ \frac{\partial W}{\partial L} = \frac{1}{8} \frac{\partial^2 W}{\partial \kappa^2} \left( 1 - \frac{4\nu^2}{(1 - 2\mu)^2} \right) - \frac{2\mu}{(1 - 2\mu)N + 2\mu L} \frac{\partial}{\partial \kappa} (\kappa W), \] (54)

where \(\kappa = k - L/2\). Equation (54) has a solution of the Gaussian form Eq. (51) with the variance \(D(L)\) satisfying the ordinary differential equation,

\[ \frac{dD}{dL} = \frac{1}{4} \left( 1 - \frac{4\nu^2}{(1 - 2\mu)^2} \right) + \frac{4\mu}{(1 - 2\mu)N + 2\mu L} D. \] (55)

Its solution, given the boundary condition \(D(0) = 0\), coincides with (52).

**Self-similarity of the persistent Brownian diffusion**

In this subsection, we point to one of the most interesting properties of the Markov chain being considered, namely, its self-similarity. Let us reduce the \(N\)-step Markov sequence by regularly (or randomly) removing some symbols and introduce the decimation parameter \(\lambda\),

\[ \lambda = N^*/N \leq 1. \] (56)

Here \(N^*\) is a renormalized memory length for the reduced \(N^*\)-step Markov chain. According to Eq. (52), the conditional probability \(p^*_k\) of occurring the symbol zero after \(k\) unities among the preceding \(N^*\) symbols is described by the formula,

\[ p^*_k = \frac{1}{2} + \nu^* + \mu^* \left( 1 - \frac{2k}{N^*} \right), \] (57)

with

\[ N^* = \lambda N, \quad \nu^* = \nu \frac{1}{1 - 2\mu(1 - \lambda)}, \quad \mu^* = \mu \frac{\lambda}{1 - 2\mu(1 - \lambda)}. \] (58)

The comparison between Eqs. (55) and (57) shows that the reduced chain possesses the same statistical properties as the initial one but it is characterized by the renormalized parameters \((N^*, \nu^*, \mu^*)\) instead of \((N, \nu, \mu)\). Thus,
Eqs. (56) and (58) determine the one-parametrical renormalization of the parameters of the stochastic process defined by Eq. 56.

The astonishing property of the reduced sequence consists in that the variance $D^*(L)$ is invariant with respect to the one-parametric decimation transformation (56), (58). In other words, it coincides with the function $D(L)$ for the initial Markov chain:

$$D^*(L) = \frac{Ln^*_1}{(n^*_1 + n^*_2)} \left[ 1 + \frac{(L-1)}{1 + n^*_1 + n^*_2} \right] = D(L), L < N^*. \quad (59)$$

Indeed, according to Eqs. (56), (58), the renormalized parameters $n^*_1 = N^*(1 - 2(\mu^* + \nu^*)/4\mu^*$ and $n^*_2 = N^*(1 - 2(\mu^* - \nu^*)/4\mu^*$ of the reduced sequence coincides exactly with the parameters $n_1$ and $n_2$ of the initial Markov chain. Since the shape of the function $W_L(k)$ Eq. 39 is defined by the invariant parameters $n_1 = n^*_1$ and $n_2 = n^*_2$, the distribution $W_L(k)$ is also invariant with respect to the decimation transformation.

The transformation $(N, \nu, \mu) \rightarrow (N^*, \nu^*, \mu^*) (56), (58)$ possesses the properties of semi-group, i.e., the composition of transformations $(N, \nu, \mu) \rightarrow (N^*, \nu^*, \mu^*)$ and $(N^*, \nu^*, \mu^*) \rightarrow (N^{**}, \nu^{**}, \mu^{**})$ with transformation parameters $\lambda_1$ and $\lambda_2$ is likewise transformation from the same semi-group, $(N, \nu, \mu) \rightarrow (N^{**}, \nu^{**}, \mu^{**})$, with parameter $\lambda = \lambda_1\lambda_2$.

The invariance of the function $D(L)$ at $L < N$ was referred to by us as the phenomenon of self-similarity. It is demonstrated in Fig. 3.

It is interesting to note that the property of self-similarity is valid for any strength of the persistency. Indeed, the result Eq. (52) can be obtained directly from Eqs. (15)-(17), and (49) not only for $n_2 \gg 1$ but also for the arbitrary value of $n_2$.

**FIG. 3**: The dependence of the variance $D$ on the tuple length $L$ for the generated sequence with $N = 100$, $\mu = 0.4$ and $\nu = 0.08$ (dotted line) and for the decimated sequences (the parameter of decimation $\lambda = 0.5$). Squares and circles correspond to the stochastic and deterministic reduction, respectively. The solid line describes the non-correlated Brownian diffusion, $D(L) = L(1/4 - \nu^2)$.

**MEMORY FUNCTION AND ITS CONNECTION WITH CORRELATION FUNCTION**

Typically, the correlation function and other moments are employed as the input characteristics for the description of the correlated random sequences. However, the correlation function describes not only the direct interconnection of the elements $a_i$ and $a_{i+r}$, but also takes into account their indirect interaction via all other intermediate elements. Our approach operates with the "origin" characteristics of the system, specifically, with the memory function. The correlation and memory functions are mutual-complementary characteristics of a random sequence in the following sense. The numerical analysis of a given random sequence enables one to directly determine the correlation function rather than the memory function. On the other hand, it is possible to construct a random sequence using the memory function, but not the correlation one. Therefore, we believe that the investigation of memory function of the correlated systems will permit one to disclose their intrinsic properties which provide the correlations between the elements.
The memory function used in Refs. \[9, 10\] was characterized by the step-like behavior and defined by two parameters only: the memory depth \(N\) and the strength of symbol’s correlations. Such a memory function describes only one type of correlations in a given system, the persistent or anti-persistent one, which results in the super- or sub-linear dependence \(D(L)\) \[22\]. Obviously, both types of correlations can be observed at different scales in the same system. Thus, one needs to use more complex memory functions for detailed description of the systems with both type of correlations. Besides, we have to find out a relation connecting the mutually-complementary characteristics of random sequence, the memory and correlation functions.

**Main equation**

Let us rewrite Eq. (60) in an equivalent form,

\[
P(a_i = 1 \mid T_{N,i}) = b + \sum_{r=1}^{N} F(r)(a_i - r - b),
\]

(60)

with

\[
b = \frac{\sum_{r=1}^{N} f(0, r)/N}{1 - \sum_{r=1}^{N} F(r)}, \quad F(r) = \frac{1}{N} [f(1, r) - f(0, r)].
\]

(61)

The constant \(b\) is the value of \(a_i\) averaged over the whole sequence, \(b = \bar{a}\):

\[
\bar{a} = \lim_{M \to \infty} \frac{1}{2M + 1} \sum_{i=-M}^{M} a_i.
\]

(62)

Indeed, according to the ergodicity of the Markov chain, \(\bar{a}\) coincides with the value of \(a_i\) averaged over the ensemble of realizations of the Markov chain. So, we can write

\[
\bar{a} = Pr(a_i = 1) = \sum_{T_{N,i}} P(a_i = 1 \mid T_{N,i}) Pr(T_{N,i}).
\]

(63)

Here \(Pr(a_i = 1)\) is the probability of occurring the symbol \(a_i\) equal to unity and \(Pr(T_{N,i})\) is the probability of occurring the definite word \(T_{N,i}\) in the considering ensemble of sequences. Substituting \(P(a_i = 1 \mid T_{N,i})\) from Eq. (60) into Eq. (63) and taking into account the obvious relation \(\sum_{T_{N,i}} Pr(T_{N,i}) = 1\), one gets,

\[
\bar{a} = b - b \sum_{r=1}^{N} F(r) + \sum_{r=1}^{N} F(r) \sum_{T_{N,i}} Pr(T_{N,i}) a_{i-r}.
\]

(64)

The sum \(\sum_{T_{N,i}} Pr(T_{N,i}) a_{i-r}\) does not depend on the subscript \(r\) and obviously coincides with \(\bar{a}\). So, we have \(\bar{a} = b + (\bar{a} - b) \sum_{r} F(r)\). From this equation we conclude that \(b = \bar{a}\). Thus, we can rewrite Eq. (60) as

\[
P(a_i = 1 \mid T_{N,i}) = \bar{a} + \sum_{r=1}^{N} F(r)(a_i - r - \bar{a}).
\]

(65)

We refer to \(F(r)\) as the **memory function** (MF). It describes the strength of influence of previous symbol \(a_{i-r}\) upon a generated one, \(a_i\). To the best of our knowledge, the concept of memory function for many-step Markov chains was introduced in Ref. \[9\]. The function \(P(\cdot \mid \cdot)\) contains the complete information about correlation properties of the Markov chain.

We suggest below two methods for finding the memory function \(F(r)\) of a random binary sequence with a known correlation function. The first one is based on the minimization of a ”distance” \(Dist\) between the Markov chain
Equation (70) can also be derived by straightforward calculation of the average yields the relationship between the correlation and memory function $s$,

$$P(r) = \frac{1}{2M} \sum_{i=-M}^{M} (a_i - P(a_i = 1 | T_{N,i}))^2,$$

with the conditional probability $P$ defined by Eq. (66).

Let us express distance (66) in terms of the correlation function,

$$K(r) = \bar{a}_{i-t} - \bar{a}, \quad K(0) = \bar{a}(1 - \bar{a}), \quad K(-r) = K(r).$$

From Eqs. (65), (66), one obtains

$$\text{Dist} = \sum_{r,r'} (a_{i-r} - \bar{a})(a_{i-r'} - \bar{a})F(r)F(r') - 2 \sum_{r} (a_i - \bar{a})(a_{i-r} - \bar{a})F(r) + (a_i - \bar{a})^2$$

$$= \sum_{r,r'} K(r - r')F(r)F(r') - 2 \sum_{r} K(r)F(r) + K(0).$$

The minimization equation,

$$\frac{\delta \text{Dist}}{\delta F(r)} = 2 \sum_{r'} K(r - r')F(r') - 2K(r) = 0,$$

yields the relationship between the correlation and memory functions,

$$K(r) = \sum_{r' = 1}^{N} F(r')K(r - r'), \quad r \geq 1.$$

Equation (70) can also be derived by straightforward calculation of the average $\bar{a}_{i-t} \bar{a}$ in Eq. (67) using definition (65) of the memory function.

![Graph showing the initial memory function Eq. (72) (solid line) and the reconstructed one (dots) vs the distance $r$. In inset, the correlation function $K(r)$ obtained by a numerical analysis of the sequence constructed by means of the memory function Eq. (72).](image)

The second method resulting from the first one, establishes a relationship between the memory function $F(r)$ and the variance $D(L)$,

$$M(r, 0) = \sum_{r' = 1}^{N} F(r')M(r, r'),$$

$$r > 1.$$
M(r, r') = D(r - r') - (D(-r') + r[D(-r' + 1) - D(-r')]).

It is a set of linear equations for \( F(r) \) with coefficients \( M(r, r') \) determined by \( D(r) \). The relations, \( K(r) = [D(r - 1) - 2D(r) + D(r + 1)]/2 \) obtained in Ref. 23 and \( D(-r) = D(r) \) are used here.

Let us verify the robustness of our method by numerical simulations. We consider a model "triangle" memory function,

\[
F(r) = 0.008 \begin{cases} 
    r, & 1 \leq r < 10, \\
    20 - r, & 10 \leq r < 20, \\
    0, & r \geq 20,
\end{cases}
\]  

(72)

presented in Fig. 4 by solid line. Using Eq. (58), we construct a random non-biased, \( \bar{a} = 1/2 \), sequence of symbols \( \{0, 1\} \). Then, with the aid of the constructed binary sequence of the length \( 10^6 \), we calculate numerically the correlation function \( K(r) \). The result of these calculations is presented in inset Fig. 4. One can see that the correlation function \( K(r) \) mimics roughly the memory function \( F(r) \) over the region \( 1 \leq r \leq 20 \). In the region \( r > 20 \), the memory function is equal to zero but the correlation function does not vanish 24. Then, using the obtained correlation function \( K(r) \), we solve numerically Eq. (70). The result is shown in Fig. 4 by dots. One can see a good agreement of initial, Eq. (72), and reconstructed memory functions \( F(r) \).

**Numerical simulations**

The main and very nontrivial result of our paper consists in the ability to construct a binary sequence with an arbitrary prescribed correlation function by means of Eq. (70). As an example, let us consider the model correlation function,

\[
K(r) = 0.1 \frac{\sin(r)}{r},
\]  

(73)

presented by the solid line in Fig. 6. We solve Eq. (70) numerically to find the memory function \( F(r) \) using this correlation function. The result is presented in inset Fig. 5. Then we construct the binary Markov chain using the obtained memory function \( F(r) \). To check up a robustness of the method, we calculate the correlation function \( K(r) \) of the constructed chain (the dots in Fig. 5) and compare it with Eq. (73). One can see an excellent agreement between the initial and reconstructed correlation functions.

Let us demonstrate the effectiveness of our concept of the additive Markov chains when investigating the correlation properties of coarse grained literary texts. First, we use the coarse-graining procedure and map the letters of the text of Bible 24 onto the symbols zero and unity (here, \( (a - m) \mapsto 0, (n - z) \mapsto 1 \)). Then we examine the correlation properties of the constructed sequence and calculate numerically the variance \( D(L) \). The result of simulation of the normalized variance \( D_n(L) = D(L)/4\bar{a}(1-\bar{a}) \) is presented by the solid line in Fig. 7. The dominator \( 4\bar{a}(1-\bar{a}) \) in the
equation for the normalized variance $D_n(L)$ is inserted in order to take into account the inequality of the numbers of zeros and unities in the coarse-grained literary texts. The straight dotted line in this figure describes the variance $D_0(L) = L/4$, which corresponds to the non-biased non-correlated Brownian diffusion. The deviation of the solid line from the dotted one demonstrates the existence of correlations in the text. It is clearly seen that the diffusion is anti-persistent at small distances, $L \lesssim 300$, (see inset Fig. 6) whereas it is persistent at long distances.

The memory function $F(r)$ for the coarse-grained text of Bible at $r < 300$ obtained by numerical solution of Eq. (71) is shown in Fig. 7. At long distances, $r > 300$, the memory function can be nicely approximated by the power function $F(r) = 0.25r^{-1.1}$, which is presented by the dash-dotted line in inset Fig. 7.

Note that the region $r \lesssim 40$ of negative anti-persistent memory function provides much longer distances $L \sim 300$ of anti-persistent behavior of the variance $D(L)$.

Our study reveals the existence of two characteristic regions with different behavior of the memory function and, correspondingly, of persistent and anti-persistent portions in the $D(L)$ dependence. This appears to be a prominent feature of all texts written in any language. The positive persistent portions of the memory functions are given in inset Fig. 4 for the coarse-grained English- and Russian-worded texts of Bible (dash-dotted and dashed lines, Refs. 24 and 25, correspondingly). Besides, for comparison, the memory function of the coarse-grained text of "Pygmalion" by B. Shaw 24 is presented in the same inset (dots), the power-law fitting is shown by solid line.
It is interesting to note that the memory function of any text mimics the correlation function, as it was found for the model example Eq. (73). This fact is confirmed by Fig. 8 where the correlation function of the coarse-grained text of Bible is shown. One can see that its behavior at both short and long scales is similar to the memory function presented in Fig. 7. However, the exponents in the power-law approximations of \( K(r) \) and \( F(r) \) functions differ essentially.

![FIG. 8: The correlation function \( K(r) \) for the coarse-grained text of Bible at short distances. In inset, the power-law decreasing portions of the \( K(r) \) plot for the same text. The solid line corresponds to power-law fitting of this function.](image)

**CONCLUSION**

Thus, the simple, exactly solvable model of the uniform binary \( N \)-step Markov chain is presented. The memory length \( N \), the parameter \( \mu \) of the persistent correlations and the biased parameter \( \nu \) are three parameters in our theory. The correlation function \( K(r) \) is usually employed as the input characteristics for the description of the correlated random systems. Yet, the function \( K(r) \) describes not only the direct interconnection of the elements \( a_i \) and \( a_i+r \), but also takes into account their indirect interaction via other elements. Since our approach operates with the "original" parameters \( N, \mu \) and \( \nu \), we believe that it allows us to reveal the intrinsic properties of the system which provide the correlations between the elements.

We have demonstrated the efficiency of description of the symbolic sequences with long-range correlations in terms of the memory function. An equation connecting the memory and correlation functions of the system under study is obtained. This equation allows reconstructing a memory function using a correlation function of the system. Actually, the memory function appears to be a suitable informative "visiting card" of any symbolic stochastic process. The effectiveness and robustness of the proposed method is demonstrated by simple model examples. Memory functions for some concrete examples of the coarse-grained literary texts are constructed and their power-law behavior at long distances is revealed. Thus, we have shown the complexity of organization of the literary texts in contrast to a previously discussed simple power-law decrease of correlations [4].

If the memory length \( N \) of the system under consideration is of order of the very system length then the Markov chain, modeling the system, could be non-stationary. In this case the proposed method does not allow to describe the system precisely, as distinct from the method proposed in [27, 28].

**APPENDIX. MATRIX OF THE CONDITIONAL PROBABILITY**

In this Appendix, we prove the property of metrical transitivity of the \( N \)-step Markov chains.

It is possible to look at the Markov chain from the other point of view and consider it as a 1-step vector Markov chain. To this end, we introduce the \( N \)-component vector-function \( X_l \),

\[
X_l = (a_{l+1}, a_{l+2}, ..., a_{l+N}), \quad l = ..., -2, -1, 0, 1, 2, ...
\]
The number of different sets of symbols \((a_{i+1}, a_{i+2}, ..., a_{i+N})\) is equal to \(Q = 2^N\). We number the different states of the vector \(X_i\) by their binary representation,

\[
D(a_N, a_{N-1}, ..., a_1) = a_N 2^0 + a_{N-1} 2^1 + ... + a_1 2^{N-1}, \quad 0 \leq D \leq 2^N - 1.
\]  

(75)

The matrix elements \(M_{ik}\) of the probability matrix \(M\), i.e. the probabilities of transition of the vector \(X = (a_1, a_2, ..., a_N)\) into the vector \(Y = (a'_1, a'_2, ..., a'_{N})\) can be expressed via the function of conditional probability \(P(a_i \mid T_{N,i})\). The subscripts \(i\) and \(k\) of the matrix \(M_{ik}\) are determined by the binary representations of the sequences \((a_1, a_2, ..., a_N)\) and \((a'_1, a'_2, ..., a'_{N})\), correspondingly: \(i = 1 + D(a_N, a_{N-1}, ..., a_1)\) and \(k = 1 + D(a'_1, a'_{N-1}, ..., a'_1)\).

Every matrix row contains only two non-zero elements since the vector \(X_i\) can take up two values only, namely, \((a_2, a_3, ..., a_N, 0)\) and \((a_2, a_3, ..., a_N, 1)\). For \(k \leq Q/2\), let us denote the probability of occurring of \(a_{N+1} = 0\) as \(1 - P_k\), where the index \(k\) is equal to \(k = 1 + D(a_N, a_{N-1}, ..., a_1)\) in the binary representation.

For the index \(k\) being in the range \(Q/2 + 1\) to \(Q\), we denote the probability of occurring of symbol \(a_{N+1} = 0\) after the word \(a_1, a_2, ..., a_N\) as \(P_k\). Then, \(1 - P_k\) is the probability of occurring of the symbol unity. Taking into account that \(a_N = 0\) for \(k \leq Q/2\) and obvious relations,

\[
D(a_{N-1}, ..., a_1, 0) = 2D(a_N, a_{N-1}, ..., a_1),
\]

\[
D(a_{N-1}, ..., a_1, 1) = 2D(a_N, a_{N-1}, ..., a_1) + 1,
\]

we get the transition probabilities matrix \(M\):

\[
M = \begin{pmatrix}
1 - P_1 & P_1 & 0 & 0 & ... & ... & 0 & 0 \\
0 & 0 & 1 - P_2 & P_2 & 0 & ... & 0 & 0 \\
... & ... & ... & ... & ... & ... & ... & ... \\
1 - P_{Q/2+1} & P_{Q/2+1} & 0 & 0 & ... & ... & 0 & 0 \\
0 & 0 & 1 - P_{Q/2+2} & P_{Q/2+2} & 0 & ... & 0 & 0 \\
... & ... & ... & ... & ... & ... & ... & ... \\
0 & 0 & ... & ... & ... & 0 & 1 - P_Q & P_Q
\end{pmatrix}.
\]  

(76)

Thus, to determine the vectors \(b\) of probability distribution of \(N\)-words in the stationary Markov chain we need to solve the system of equations,

\[
b_i = \sum_{k=1}^{Q} b_k M_{ki}, \quad \sum_{k=1}^{Q} b_k = 1.
\]  

(77)

In other words, one needs to obtain the normalized eigenvector corresponding to the eigenvalue \(\lambda = 1\) of the matrix \(M^{(N)}\) of the order \(Q = 2^N\). It is clear that if the vector \(b\) satisfies to the condition \(bM = b\) then for every integer \(k\) the condition \(bM^k = b\) is also true, here \(M^k\) is the power \(k\) of the matrix \(M\). Let us consider the matrix \(M^{(N)}\) and show that all matrix elements are positive. In this case, following the Markov theorem we can conclude that the matrix \(M\) determines uniquely the probability of the words distribution.

Let us suggest that for any \(k < N\) the matrix \(M^k\) satisfies to the next conditions: in the first row the elements \(M_{1i}\) for \(i = 1, ..., 2^k\) are positive, in the second row the positive elements are \(M_{2i}\) with \(i = 2^k + 1, ..., 2 \times 2^k\), ... in the \(2^{N-k}\)-th row — \(i = (2^{N-k} - 1)2^k, ..., 2^N\). In the next rows this order is repeated. Let us demonstrate that if the matrix \(M^k\) obeys to these rules, then it is true for the matrix \(M^{k+1}\) also.

After multiplication of matrixes \(M^k\) and \(M\) the elements of obtained matrix are defined by the expression:

\[
M^{k+1}(i, j) = \sum_l M^k(l, l) M(l, j).
\]  

(78)

Let us consider the first row of the matrix \(M^{k+1} - i = 1\). In each column of the matrix \(M\) only two elements are non-zero. After multiplication of the first row of the matrix \(M^k\) to some column of the matrix \(M\) the result is non-zero (positive) for \(j \leq 2 + 2^k\) only, because positive elements of the matrix \(M\) corresponds to the positive zone \(i < 2^k\) of the first row of matrix \(M^k\) only for this \(j\). So the described rule remains for the first row of the matrix \(M^{K+1}\). Similarly this fact can be proved for other rows.

The matrix \(M^1\) obeys to this rule, consequently, by induction, it is true for all \(M^k\). In according to this rule, if power \(k = N\), then all elements of the matrix \(M^N\) are positive.
Therefore, from the Markov theorem, there is the unique solution of the system $bM^N = b$ (or $bM = b$). This solution can be obtained by the method of successive approximations,

$$b^{k+1}_i = b^k_j M(j,i), \quad k = 0, 1, 2, \ldots, \tag{79}$$

if we start from the arbitrary initial distribution $b^0_j$. In the limit $k \to \infty$ we get to the stationary distribution of the probability vector $b$.

Taking into account the explicit form of the matrix $M$, the equation (77) comes to the next equations:

$$b_1(1 - P_1) + b_{i+Q/2}P_{i+Q/2} = b_{2i-1},$$

$$b_iP_1 + b_{i+Q/2}(1 - P_{i+Q/2}) = b_{2i}.$$ \hspace{1cm} (80)

For $Q = 2$ we get the well known result [21]:

$$M = \begin{pmatrix} 1 - P_1 & P_1 \\ P_2 & 1 - P_2 \end{pmatrix},$$

$$b_1 = \frac{P_2}{P_1 + P_2}, \quad b_2 = \frac{P_1}{P_1 + P_2}.$$ \hspace{1cm}

And in the case $Q = 4$ we obtain the next result:

$$M = \begin{pmatrix} 1 - P_1 & P_1 & 0 & 0 \\ 0 & 0 & 1 - P_2 & P_2 \\ P_3 & 1 - P_3 & 0 & 0 \\ 0 & 0 & P_4 & 1 - P_4 \end{pmatrix},$$

$$b_1 = \frac{P_3P_4}{P_1P_2 + 2P_1P_4 + P_3P_4}, \quad b_2 = b_3 = \frac{P_1P_4}{P_1P_2 + 2P_1P_4 + P_3P_4}, \quad b_4 = \frac{P_1P_2}{P_1P_2 + P_1P_4 + P_3P_4}.$$ \hspace{1cm}

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[11] Note that we here discuss the dependence of the variance $D$ upon the length $L$ that describes the persistent (or antipersistent) correlations in the words of different lengths $L$. This length does not coincides with the memory length $N$, $L \neq N$. The dependence $D(L)$ is completely different from the dependence $D$ on the memory length $N$ discussed in Ref. [14, 15].
The existence of the “additional tail” in the correlation function is in agreement with Ref. [9] and corresponds to the well-known fact that the correlation length is always larger than the region of memory function action.

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