Huygens synchronization of two pendulum clocks

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Abstract

The synchronization of two pendulum clocks hanging from a wall was first observed by Huygens during the XVII century. This type of synchronization is observed in other areas, and is fundamentally different from the problem of two clocks hanging from a moveable base. We present a model explaining the phase opposition synchronization of two pendulum clocks in those conditions. The predicted behavior is observed experimentally, validating the model.

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Introduction. – The synchronization between two periodic systems connected through some form of coupling is a recurrent, and still pertinent, problem in Nature, and in particular in Physics. During the 17th century Huygens, the inventor of the pendulum clock, observed phase or phase opposition coupling between two heavy pendulum clocks hanging either from a house beam and later from a board sitting on two chairs [1]. These two systems are inherently different in terms of the coupling process, and in consequence of the underlying model. The later case has been thoroughly studied [2, 3] by considering momentum conservation in the clocks-beam system. The first case has been approached in a few theoretical works [10–12]. We present a mathematical model where the coupling is assumed to be attained through the exchange of impacts between the oscillators (clocks). This model presents the additional advantage of being independent of the physical nature of the oscillators, and thus can be used in other oscillator systems where synchronization and phase locking has been observed [13]. The model presented starts from the Andronov [14] model of the phase-space limit cycle of isolated pendulum clocks and assumes the exchange of single impacts (sound solitons, for this system) between the two clocks at a specific point of the limit cycle. Two coupling states are obtained, near phase and near phase opposition, the latter being stable. Our experimental data, obtained using a pair of similar pendulum clocks hanging from an aluminum rail fixed to a masonry wall, match the theoretical predictions and simulations.

Andronov model. – The model for the isolated pendulum clock (Andronov clock) has been studied using models with viscous friction by physicists [2, 3, 5, 9]. However, Russian mathematicians lead by Andronov published a work [14] where the stability of the model with dry friction is established. The authors prove the existence and stability of the limit cycle.

We adopt as basis for our work the aforementioned model, assuming that dry friction predominates. Using the normalized angular coordinate $q$, the differential equation governing the pendulum clock is

$$\dot{q} + \mu \text{sign} q + \omega^2 q = 0,$$

where $\mu > 0$ is the dry friction coefficient, $\omega$ is the natural angular frequency of the pendulum and $\text{sign}(x)$ a function giving $-1$ for $x < 0$ and $+1$ otherwise. In [14] was considered that, in each cycle, a fixed amount of normalized kinetic energy $\frac{\Delta E}{\omega^2}$ is given by the escape mechanism to the pendulum to compensate the loss of kinetic energy due to dry friction in each complete cycle. We call to the transfer of kinetic energy a kick. We set the origin such that the kick is given when $q = -\frac{\mu}{\omega}$, which is very close to 0. The phase portrait is shown in Fig. 1.

We consider initial conditions $q(t = 0) = -\frac{\mu}{\omega}$ and $\dot{q}(t = 0) = v_0$. We draw a Poincaré section (II, page 268) as the half line $q = -\frac{\mu}{\omega}^+$ and $\dot{q} > 0$ [15]. The symbol + refers to the fact that we are considering that the section is taken immediately after the kick. Solving the differential equations sectionally we notice that there is a loss of velocity $-\frac{2\mu}{\omega}$ due to friction during a complete cycle. Considering $v_n = \dot{q} \left(\frac{2n\pi}{\omega}\right)$ the velocity at the Poincaré section in each cycle one obtains the discrete dynamical system

$$v_{n+1} = \sqrt{(v_n - \frac{4\mu}{\omega})^2 + h^2},$$

which has the asymptotically stable fixed point [15]

$$v_f = \frac{h^2}{8\mu} + \frac{2\mu}{\omega}.$$

The fixed point (2) attracts initial conditions $v_0$ in the interval $\left[\frac{2\mu}{\omega}, +\infty\right)$.

Model for two pendulum clocks. – We consider two pendulum clocks suspended at the same wall. When one clock receives the kick, the impact propagates in the wall and the second clock is slightly perturbed by a traveling wave. The perturbation is assumed instantaneous since
the time of travel of sound in the wall between the clocks is assumed very small compared to the period. The interaction was studied geometrically and qualitatively by Abraham [10, 11] without the heavy computations of analytical treatments. However, that approach does not give estimates on the speed of convergence.

In Vassalo-Pereira [12], the theoretical problem of the phase locking is tackled. The author makes the assumptions:

1. dry friction,
2. the pendulums have the same exact natural frequency $\omega$,
3. the perturbation in the momentum is always in the same vertical direction in the phase space, see also [10, 11].
4. The perturbation imposes a discontinuity in the momentum but not a discontinuity in the dynamic variable. The interaction between clocks takes the form of a Fourier series [13].

Vassalo-Pereira deduced that the two clocks synchronize with zero phase difference. This is the exact opposite of Huygens first remarks [1] and our experimental observations, where phase opposition was observed. Therefore, we propose a modified model accounting for a difference in frequency between the two clocks.

Consider two oscillators indexed by $i = 1, 2$, $\phi_i (t)$ is the time difference of clock $i$ relative to the other. Each oscillator satisfies the differential equation

$$\ddot{q}_i + \mu_i \text{sign} \dot{q}_i + \omega_i^2 q_i = -\alpha_i \delta (t - \phi_i (t)),$$

for $i = 1, 2$, (3)

when $q_i = -\frac{\mu_i}{\alpha_i}$, the kinetic energy of each oscillator is increased by the fixed amount $h_i$ as in the Andronov model.

The coupling term is the normalized force $\alpha_i \delta (t - \phi_i (t))$, where $\delta$ is the Dirac delta distribution and $\alpha_i$ a constant with acceleration dimensions.

The sectional solutions of the differential equation are obtainable when the clocks do not suffer kicks. To treat the effect of the kicks we construct a discrete dynamical system for the phase difference. The idea is similar to the construction of a Poincaré section. If there exists an attracting fixed point for that dynamical system, the phase locking occurs.

Our assumptions are

1. Dry friction.
2. The pendulums have natural angular frequencies $\omega_1$ and $\omega_2$ near each other with $\omega_1 = \omega + \varepsilon$ and $\omega_2 = \omega - \varepsilon$, where $\varepsilon \geq 0$ is a small parameter, typically $\varepsilon < 10^{-3}$.
3. The perturbation in the momentum is always in the same vertical direction in the phase space [10, 11].
4. Since the clocks have the same construction, the energy dissipated at each cycle of the two clocks is the same, $h_1 = h_2 = h < 2 \times 10^{-2}$. The friction coefficient is the same for both clocks, $\mu_1 = \mu_2 = \mu < 4 \times 10^{-4}$.
5. The kick is instantaneous. This is a reasonable assumption, since in general the perturbation propagation time between the two clocks is several orders of magnitude lower than the periods.
6. The interaction is symmetric, the coupling has the same constant $\alpha$ when the clock 1 acts on clock 2 and conversely. In our model we assume that $\alpha$ is very small.

All values throughout the paper are in SI units.

In this paper the function arctan when noted with two variables is the generalization of the usual arctan to incorporate the information about the quadrant. Thus, we define

$$\arctan (y, x) = \begin{cases} \arctan \frac{y}{x} & \text{if } x \geq 0, y \geq 0, \\
\pi + \arctan \frac{y}{x} & \text{if } x < 0, \\
2\pi + \arctan \frac{y}{x} & \text{if } x \geq 0, y < 0. \end{cases}$$

The phase of each clock at the limit cycle is defined as

$$\Phi_i = \begin{cases} \arctan \left( \frac{\omega_i q_i (t) + \frac{\mu_i}{\omega_i}, \dot{q}_i \right) & \text{if } \dot{q}_i \geq 0, \\
\arctan \left( \frac{\omega_i q_i (t) - \frac{\mu_i}{\omega_i}, \dot{q}_i \right) & \text{if } \dot{q}_i < 0. \end{cases}$$

If there is no interaction between the two clocks, the phase of each clock is

$$\Phi_i = \arctan (\sin \omega_i t, \cos \omega_i t) = \omega_i t, i = 1, 2.$$

For the purpose of analyzing the asymptotic properties of the interacting system, and without loss of generality,
we consider that at initial time oscillator 1 and 2 are isolated, both at each limit cycle. The phase difference is 
\( \phi(t) = \Phi_2(t) - \Phi_1(t) \), between the two Andronov clocks. When the two clocks are isolated from each other, and considering the period of the fastest \( T = \frac{2\pi}{\omega} \) the phase difference can be seen as a map from the circle \( S^1 \) in itself with solution

\[
\phi_n \equiv 4\pi n \frac{-\varepsilon}{\omega + \varepsilon} + \phi_0 \pmod{2\pi},
\]

which is a rigid rotation of the circle where the discrete variable \( n \) denotes the number of cycles of the fastest clock with natural frequency \( \omega_1 \).

When the coupling is established we have to modify \( \Phi \) to obtain the phase difference of the coupled system \( \phi_n \). The asymptotic properties of the discrete function \( \phi_n \) determine the limit properties of the phase difference of the coupled system.

The notation is simplified if we consider the function \( \gamma(\varphi) \) such that

\[
\gamma(x, y) = \begin{cases} 
\frac{h^2 y + 2\mu y}{8\mu}, & \text{if } 0 \leq x < \frac{\pi}{2}, \\
\frac{h^2 y}{8\mu}, & \text{if } \frac{\pi}{2} \leq x < \frac{3\pi}{2}, \\
\frac{h^2 y - 2\mu y}{8\mu}, & \text{if } \frac{3\pi}{2} \leq x < 2\pi,
\end{cases}
\]

We assume that the natural frequencies are close. A relatively large difference of 28 seconds per day in the movement of the clocks with natural periods in the order of 1.42s, implies that \( \varepsilon \sim 10^{-3} \).

This means that each clock will receive a perturbative kick from the other per cycle of the fastest one. This assumption would fail once in \( n = 10^4 \) cycles of the fastest clock. Suppose that the clocks are brought to interaction at \( t_0 = 0 \). The fastest clock (number 1) is at initial position

\[
\theta_1(0^-) = -\frac{\mu}{\omega + \varepsilon}, \quad \theta_1(0^-) = \frac{h^2 \omega + \varepsilon}{8\mu} - \frac{2\mu}{\omega + \varepsilon}.
\]

Solving sectionally the differential equations with the two small interactions, we can construct a discrete dynamical system taking into account the two interactions per cycle seen in figures 2 and 3. After that, we compute the phase difference when clock 1 returns to the initial position.

Expanding the solutions of the differential equations and computing the phase difference we get in first order of \( \alpha \) and \( \varepsilon \) the iterative scheme

\[
\phi_{n+1} = \Xi(\phi_n) = \phi_n + 4\pi \frac{-\varepsilon}{\omega + \varepsilon} + \frac{2\alpha \sin \phi_0}{\gamma(\phi_0, \omega)} + h.o.t.,
\]

The iteration of the map \( \Xi(x) \), defines the asymptotic properties of the phase difference. There are two fixed points of \( \Xi \) in the interval \([0, 2\pi]\). We have the first order approximation for the fixed point \( \phi_f \)

\[
\sin \phi_f = 2\pi \frac{h^2 \varepsilon}{4\alpha \mu}.
\]

FIG. 2: Interaction of clock 1 on clock 2 at \( t = 0^+ \). We see the original limit cycle, before interaction, and the new one in solid and the original limit cycle in dashed. Note that the value of \( \alpha \) and of \( h \) are greatly exaggerated to provide a clear view. The effect of the perturbation is secular and cumulative.

FIG. 3: Second interaction. Interaction of clock 2 on clock 1 when clock 2 reaches its impact position. All the features are similar to the Fig. 2.

with two solutions, respectively stable and unstable

\[
\phi_f = \pi - \arcsin \frac{\pi h^2 \varepsilon}{4\alpha \mu}, \quad \phi_u = \arcsin \frac{\pi h^2 \varepsilon}{4\alpha \mu}.
\]

The derivative of \( \Xi \) at the fixed point must be \( |\Xi'(x_f)| < 1 \) to have stability. The stability condition at \( \phi_f \in \left[ \frac{\pi}{2}, \pi \right] \) and the condition about the argument of the function arcsin gives

\[
\frac{\pi h^2 \varepsilon}{4\mu} < \alpha < \frac{h^2 \omega}{8\mu}, \text{ i.e., } \frac{h^2}{8\mu} \pi \Delta \omega < \alpha < \frac{h^2}{8\mu} \omega.
\]

The limit of the phase difference is, in first order, \( \pi - \arcsin \frac{\pi h^2 \varepsilon}{4\alpha \mu} \). When the system reaches this limit the corrections of phase are null for both clocks.

If the initial conditions are very near, with clock 1 slightly behind clock 2, one can have one overtaking but, after that one there are no more and the phase difference of the clocks tends to the same limit \( \pi - \arcsin \frac{\pi h^2 \varepsilon}{4\alpha \mu} \). The asymptotic behavior of the phenomenon is exactly the same.
Simulation.– To study the Huygens synchronization we used numerical simulations. We applied the map $\Xi(x)$ without performing the Taylor expansion. We used the environment of Wolfram Mathematica 9.0 to produce the computations. The values of $\mu$, $h$, $\omega$, $t_0$ were taken realistically from the experimental setup and kept fixed throughout the simulations. The coupling constant $\alpha$ and the half-difference between the clocks frequencies $\varepsilon$ are adjusted in simulations.

Additionally, we introduced noise in the model with normal distribution acting directly on the phase. The effect of noise is to mimic the small perturbations that occur in the lab, e.g., vibrations in the wall and the stochastic changes of the level of the interaction, cycle after cycle. The strength of this stochastic effect is given by the parameter $\rho$. When the noise function is not used, i.e., $\rho = 0$, if the parameters are in the convergence region given by conditions we have a fixed convergence point of $\Omega$.

Experimental.– Experiments were setup using a standard optical rail (Eurofysica) rigidly attached to a wall to which two similar clocks were fixed through an optical rail modified rail carriers. The sound propagation speed in Al is $6420 \text{ m s}^{-1}$, leading to a propagation time of the order of $3.0 \times 10^{-5} \text{s}$, which is negligible in front of the $1.4 \text{s}$ period of the oscillators. Other materials we tried (MDF and fiberglass) but no coupling could be observed. The mass-driven pendulum clocks used were Acctim 26268 Hatahaway. The chime mechanism was inhibited in order to reduce mechanical noise. Time was measured using one U-shaped LED emitter-receptor TCST 1103, connected to a Velleman K8055 USB data acquisition board operated with custom-developed software running in a standard personal computer (PC). The acquisition board operated with custom-developed software running in a standard personal computer (PC). The sound propagation speed for the PC was overcome by performing running averaging of the period data, up to 1000. The data files were then processed offline using Mathematica. In order to obtain the appropriate parameters for the simulation, the pendulums were filmed and the movement quantified using free software Tracker4.84 from OSP (https://www.cabrillo.edu/~dbrown/tracker/).

Results and discussion.– Observing the movement of the pendulum alone (as a damped oscillator; initially at the limit cycle), we noticed a decrease of the maximum velocity of the pendulum according to a linear fit $v_{\text{max}} = 0.2228 + 0.0023n$, where $n$ is the number of cycles, with correlation coefficient 0.994. The decrease of velocity per cycle predicted by the Andronov model is $\frac{4\varepsilon}{\omega}$. With the value of the period $T = 1.4$ and $\omega = \frac{2\pi}{T} = 4.48799$, we can estimate the value of $\mu = \frac{2\mu}{T} \approx 2.54 \times 10^{-3}$.

The value of $h$ can also be easily established by studying the movement at the limit cycle. We then have the maximum velocity $v_f = \frac{h^2 \omega}{8\mu} + \frac{2h}{\omega}$. We found consistently that the maximum velocity at the limit cycle is $v_f = 0.223$. Therefore, $h \approx 0.032$.

We used different possible values of natural frequency difference. For the average natural angular frequency we take the same value of $T = 1.4$, hence $\omega = 4.48799 = 2\pi/T$. The fastest clock has a natural frequency of $\omega + \varepsilon$ and the slowest $\omega - \varepsilon$. The angular frequency difference is in first order $\varepsilon = \frac{2\varepsilon}{T} \Delta t$, where $\Delta t$ is half of the period difference. Notice that when $\Delta t = 2 \times 10^{-4}$, with a delay between the clocks of $24.6 \text{s per day}$ for the non coupled pair of clocks, the value of $\varepsilon$ is $\varepsilon = 6.4 \times 10^{-4}$. We used values of $\varepsilon$ in the range $10^{-4} \rightarrow 10^{-3}$ as a realistic estimate for the performance of our setup.

The fixed parameters used for the simulations are then $\mu = 2.54 \times 10^{-3}$, $h = 0.032$, $\omega = 4.48799$, $t_0 = 0.8\pi$, and the phenomenological noise coefficient of $\rho = 0.093$, which fits the ripple observed at the experimental data. When we choose $\varepsilon = 3 \times 10^{-3}$, corresponding to a natural delay of 116 s per day, the clocks in the isolated state, a value of $\alpha = 7 \times 10^{-4}$ yielded results matching the experimental data. We expect more frequent escapes from stable states than if we choose $\varepsilon = 1.5 \times 10^{-4}$, corresponding to a natural delay of 2.9 s per day. These values correspond to the values that could realistically be obtained in our experimental setup. The plots can be seen in Fig. 4.

![FIG. 4: Simulations of delay between the two clocks in period units for two frequency differences. Upper curve: $\varepsilon = 1.5 \times 10^{-4} \text{s}^{-1}$. Lower curve: $\varepsilon = 3.0 \times 10^{-3} \text{s}^{-1}$.](image-url)
since the stability is much easier to reach and maintain.

This is strikingly similar to the behavior observed in Fig. 5 for the actual clocks (right axis). The number of synchronization plateaus is of the same order and can be fine-tuned using the stochastic parameter in the simulation. It was observed that the system could be unsettled by a number of external noise sources, e.g., from doors closing nearby in the building, people entering or leaving the room, or even the elevator stopping, than proceeding to the next synchronization plateau.

The periods of both clocks from the same experiment can be seen on the same figure (left axis). The periods vary together within an interval of about 1 ms around 1.427 s when the clocks are coupled with correlation coefficient above 0.97 in the coupled state. Notice the almost perfect coincidence of the two curves except when the system leaves coupled states. Notice also the instability of the coupled period, varying over an interval of almost 1 ms. When coupling is lost the period of one clock decreases sharply (up to 2 ms or more) and the period of the other clock increases by a (smaller) amount. These perturbations in the periods are coincident with the loss of phase opposition coupling. Although one may expect changes in the frequency even when the clocks are not coupled, due to the interaction between them, the difference in period can be estimated at around 2 ms, corresponding to a difference in frequency of the order of $6 \times 10^{-3}$ s$^{-1}$. This asymmetry of the coupled period relative to the original periods is predicted by the model. Both periods return to the previous baseline value when the coupling is restored.

Fig. 5 shows data for another experiment. In this case the free clock frequencies were closer. Both periods are remarkably coincident, but vary in an interval in excess of 10 ms, when the clocks are coupled. Since the frequencies are closer, the synchronization should be easier to maintain, hence the low number of plateaus, but also should be slower to attain, hence the longer transitions between plateaus. If the perturbation is large enough, especially if the frequencies are very close, it is possible to attain plateaus both above and below. Between $t = 100000T$ and $t = 112000T$ approximately the synchronization is lost, and the periods become separated by more than 100 µs (corresponding to frequency difference around $3 \times 10^{-4}$ s$^{-1}$), but become stable (within 1 µs). At instant $t = 148000T$ approximately clock #2 is stopped. From that moment on the period of the remaining working clock becomes stable within about 10 µs, an interval one order of magnitude below. This confirms that the clocks strongly disturb one another, but also that both periods are kept at the same value in order to keep the synchronization at the expense of some frequency instability.

**Conclusions.**– We have developed a model explaining the Huygens problem of synchronization between two clocks hanging from a wall. In this model each clock transmits once per cycle a sound pulse that is translated in a pendulum speed change. An equilibrium situation is obtained for almost half-cycle phase difference. These predictions match remarkably the experimental data obtained for two similar clocks hanging from a wall.

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1. C. Huygens, Letters to de Sluse. (letters; no. 1333 of 24 February 1665, no. 1335 of 26 February 1665, no. 1345 of 6 March 1665) (Societe Hollandaise Des Sciences, Martinus Nijho, La Haye, 1895).

2. M. Bennett, M. Schatz, H. Rockwood, and K. Wiesenfeld, Proceedings of the Royal Society of London: Mathematics, Physical and Engineering Sciences 458, 563 (2002).

3. K. Czolczynski, P. Perlikowski, A. Stefanski, and T. Kapitaniak, International Journal of Bifurcation and Chaos 07, 2047 (2011).

4. M. Kapitaniak, K. Czolczynski, P. Perlikowski, A. Stefanski, and T. Kapitaniak, Physics Reports 1, 1 (2012).

5. A. Fradkov and B. Andrievsky, International Journal of Non-Linear Mechanics 6, 895 (2007).

6. V. Jovanovic and S. Koshkin, Journal of Sound and Vibration 12, 2887 (2012).

7. E. A. Martens, S. Thutupalli, A. Fourriere, and O. Hallatschek, Proceedings of the National Academy of Sciences 26, 10563 (2013).

8. W. T. Oud, H. Nijmeijer, and A. Y. Pogromsky, A study of Huygens’ synchronization: experimental results (Springer, Berlin Heidelberg, 2006), vol. 336 of Lecture Notes in Control and Information Science, pp. 191–203.

9. M. Senator, Journal of sound and vibration 3–5, 566 (2006).

10. R. Abraham and A. Garfinkel (2003), http://www.ralph-abraham.org/articles/Blurbs/blurb111.shtml.

11. R. Abraham, Phase regulation of coupled oscillators and chaos (World Scientific, Singapore, 1991), pp. 49–78.

12. A. M. Nunes and J. Vassalo-Pereira, Physics Letters A 8, 362 (1985).

13. J. Vassalo-Pereira, in Dynamical Systems and Microphysics: Geometry and Mechanics, edited by A. Avez, A. Blaquiere, and A. Marzollo (Academic Press, New York, London, 1982), pp. 343–352.

14. A. Pikovsky, M. Rosenblum, and J. Kurths, Synchronization: A Universal Concept in Nonlinear Sciences, vol. 12 of Cambridge Nonlinear Science Series (Cambridge University Press, Cambridge, 2003), 1st ed., ISBN 978-0521533522.

15. A. A. Andronov, A. A. Vitt, and S. E. Khaikin, Theory of Oscillators (Pergamon Press, Oxford, New York, 1959/1963/1966).

16. G. D. Birkhoff, Collected Mathematical Papers (American Mathematical Society, Providence, Rhode Island, 1950).