New approach to deriving gas dynamics equations

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Abstract

We derive the gas dynamics equations considering changes of velocity distribution function on the scale of a molecule free path. We define the molecule velocity distribution function in a specific form so that only molecule velocities after intermolecular collisions in a chosen fixed volume are taken into account. The obtained equations differ from the well-known Navier-Stokes equations and contain the new terms. The continuity equation includes the second derivatives of dynamical viscosity and the equation of energy conservation describes the transformation of gas mass velocity into the gas thermal energy. One can expect that these new equations of motion allow to construct a description of macroscopic physical phenomena in more complicated situations than the standard gas dynamics equations do.

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1 Introduction

The equations describing gas dynamics are used for studying various fundamental and applied problems in many areas of macrophysics. In particular, understanding a number of concrete macroscopic phenomena is based on solutions to Euler or Navier-Stokes equations determining the behavior of the gas velocity field. Standard macroscopic deriving of the equations of gas dynamics and discussion of their different applications are given, for example, in [1].

In contrast with the fundamental Newton dynamic equations, Euler and Navier-Stokes equations are approximate and provide a description of a macroscopic gas system in terms of a reduced number of degrees of freedom. As well known, it is impossible to find an exact solution to a system of Newton equations of motion for all interacting gas molecules. Therefore, the equations of gas motion are derived by methods of statistical mechanics, taking into account some approximations allowing to develop the description of gas in terms of macroscopic variables such as gas densities or mass velocities. One of the most known approaches leading to equations of gas dynamics in framework of statistical mechanics is the use of Boltzmann equation. Euler and Navier-Stokes equations are derived from Boltzmann equation under certain assumptions (viscosity being taken into account or neglected, respectively). See for details the ref. [2].

In its turn, the formulation of Boltzmann equation is also based on a number of physical hypotheses, which are discussed in detail, for example, in [2]. In particular, one of such hypotheses assumes that for a system, which is close to a locally equilibrium state, the molecules flying into an arbitrary phase volume have the same velocity distribution function as the molecules inside this phase volume. Within the physical meaning, this assumption implies that the molecule velocity distribution function does not change essentially on a free-path scale. Therefore, the physical phenomena associated with the variability of the velocity distribution function on a free-path scale can be neglected. As a result, deriving of the equations of a gas motion based on the Boltzmann equation takes into account only those physical phenomena, where the corresponding velocity distribution function varies on scale larger than a molecular free path in a gas.

The purpose of this paper is the deriving of the equations of gas motion for the physical situations, where the variations of velocity distribution function on a free-path scale is essential and can not be neglected. We consider the process of discrete molecule velocity changes, resulting from intermolecular collisions, as a mechanism responsible for the variation of the distribution function. The conventional definition of the velocity distribution function does not take into account molecule collisions on a free-path scale. Therefore, it will be the same for both a dense gas of colliding molecules and a highly rarefied gas without collisions. In this paper, we define a velocity distribution function of molecules in specific form in such a way that only the molecule velocities after intermolecular collisions in a chosen fixed volume contribute to this distribution function. The velocities of molecules, which pass through the volume without collisions, are not taken into account. The approach developed in this work, unlike the one based on Boltzmann equation, does not allow the determination of an explicit form of the molecule velocity distribution function. Nevertheless, this method enables deriving the equations of gas motion under the only assumption that the molecule velocity distribution function is spherically symmetric.

The paper is organized as follows. Section 2 is devoted to discussing the standard approach of deriving the gas dynamics equations. In Section 3 the new definition of
molecule velocity distribution function taking into account only the molecule velocities after intermolecular collisions is considered. Section 4 is devoted to the deriving the mass, momentum and energy fluxes for homogeneous gas on the base of new definition of the velocity distribution function. In Section 5 we discuss a general method to obtain the conservation equations for mass, momentum and energy in inhomogeneous gas. Section 6 is devoted to deriving the concrete form of mass conservation equation. In Section 7 we derive the momentum conservation equation. Section 8 is devoted to deriving the energy conservation equation. In Summary we briefly formulate the basic results of the paper.

2 Classical principles of deriving the equations of gas dynamics

Deriving the equations of gas motion by statistical mechanics methods is built upon several basic principles.

a. The gas fills some domain of space.

b. The gas is in an inhomogeneous state, i.e., density, pressure, and mass velocity of gas at different points in space are different.

c. It is possible to single out a small enough, but macroscopic domain of space, which contains a macroscopically large number of molecules. The changes in density, pressure, and mass velocity of the gas within the domain are small compared to their changes in the overall enclosing space.

d. Molecular distribution inside the domain is absolutely homogeneous. Molecules cross the boundaries of the domain on both sides thus changing the density, pressure, and mass velocity of the gas. The molecules flying into the domain increase gas density and pressure inside it, while the molecules flying out of the domain decrease them. The mass velocity of the gas in the chosen domain also changes.

e. The rate of change of the parameters depends on the number of molecules, which fly into or out of the domain, and also on the velocity of these molecules. It can be calculated from the conservation laws of mass, momentum and energy.

f. Mass, momentum and energy of the gas molecules are additive physical quantities, i.e., if one set of molecules is, on the whole, characterized by value $C_1$ for mass, momentum or energy and the other by $C_2$, then both sets of these molecules are characterized by value $C_s = C_1 + C_2$. Such physical quantities satisfy the local conservation laws for mass, momentum and energy, which are expressed as follows:

$$\frac{\partial C}{\partial t} + \text{div} \vec{J}_C = 0$$

(1)

Here $C$ is the density of the physical value under consideration (e.g., density of mass or density of energy), $\vec{J}_C$ is the density of the corresponding flux. If the physical value is a vector (e.g., momentum), the following equations must be fulfilled for the each of its $i$-th component.

$$\frac{\partial C_i}{\partial t} + \frac{\partial}{\partial x_j} \vec{J}_{ij} = 0$$

(2)

Thus, the density of a flux for a scalar quantity (mass, energy) will be a vector, and the one for a vector quantity (momentum) will be a second-rank tensor.
Equations (1) and (2) are valid for an arbitrary domain and if no sources of mass, energy or momentum are inside the domain, these equations allow us to construct a description of physical effects on the base of those assumptions, which are incorporated in the conservation law equations for density of mass, momentum and energy fluxes.

In their turn, the density of mass, momentum and energy fluxes in a gas can be calculated from the velocity distribution function $f(\vec{V}, \vec{r}, t)$. The equations for the density of mass, momentum and energy fluxes have the following form:

the $i$-th component of the density of flux $J^i_C$ for the physical value $C$ (mass, momentum, energy) is equal to

$$J^i_C = \rho \int_{-\infty}^{\infty} dV_x \int_{-\infty}^{\infty} dV_y \int_{-\infty}^{\infty} dV_z f(\vec{V}, \vec{r}, t) CV_i = \rho \int f(\vec{V}, \vec{r}, t) CV_i d\vec{V} \quad (3)$$

where $\rho$ is the density of gas at point $\vec{r}$ at time $t$, $V_i$ is the $i$-th component of the total velocity of molecules, $f(\vec{V}, \vec{r}, t)$ is the normalized function of molecule velocity distribution

$$\int_{-\infty}^{\infty} dV_x \int_{-\infty}^{\infty} dV_y \int_{-\infty}^{\infty} dV_z f(\vec{V}, \vec{r}, t) = 1 \quad (4)$$

Hence, if the molecular velocity distribution function is given, the calculation of the physical quantities is reduced to formal procedures. However, it seems that it is impossible to find the function $f(\vec{V}, \vec{r}, t)$ only from experimental data. Therefore to define this function, we have to use a theoretical approach. The generally accepted definition of the distribution function is formulated as follows.

**DEFINITION 1 (SNAPSHOT)**

The function $f(\vec{V}, \vec{r}, t)$ is given by a velocity histogram of the molecules in volume $dV$. Hereafter, the method, which is generally used to find this function will be referred to as "instant snapshot". The virtual camera located inside volume $dV$ centered around point $\vec{r}$ records at time $t$ all the molecules inside this volume. It is assumed that this camera also measures and records the velocities of all the molecules inside this volume at a given time $t$. Then one counts the number of molecules with a certain velocity vector (velocity modulus and direction) and then calculates the velocity histogram of the molecules.

After that the following formal transformations are fulfilled. The mass velocity $\vec{u}$ for all molecules in the domain under consideration is determined in the form

$$\vec{u} = \int f(\vec{V}, \vec{r}, t) \vec{V} d\vec{V} \quad (5)$$

A difference between the total velocity of a molecule $\vec{V}$ and the mass velocity $\vec{u}$ is called proper velocity of a molecule in the moving with gas reference frame and denoted $\vec{v}$. The proper velocity is called usually a thermal velocity. For proper velocity, the velocity distribution function $\tilde{f}(\vec{v}, \vec{r}, t)$ is also introduced, it is evident that this function is functionally connected to distribution function velocities $f(\vec{V}, \vec{r}, t)$ on total velocities.

Now, for a final derivation of the macroscopic equations of motion we have to define a form of distribution functions of molecules for the proper velocities. For this purpose the following assumption is usually introduced: the gas in considered domain has a locally equilibrium state. In this case the proper velocity distribution function is well-known Maxwell distribution function. After that, a formal consideration based on eqs. (1), (2) and (4) leads to the of Euler equation ( see the details in [2]).
To derive the Navier-Stokes equation, we have to represent the distribution function as the sum of the locally equilibrium function and some small unknown function (small perturbation). Then, taking into account the certain assumption about the small perturbation we can find the corrections to the Euler equation describing the effects associated with viscosity and thermal conductivity (Navier-Stokes equation; see the details in [2]).

3 Definition of proper velocity distribution function taking into account the molecular collisions

The DEFINITION 1 does not take into account the collisions of molecules and will be the same both for gas without collisions, and for gas with collisions. If our aim is to study the physical values in a gas for the situations where molecular collisions are essential, we have to find a distribution function containing the effects of molecular collisions in its expression from the very beginning. Therefore, we change the above DEFINITION 1 as follows.

Let's consider a domain filled with gas of colliding molecules and let the free path $\lambda$ be much smaller then the characteristic scale of the domain and much smaller then the characteristic scale of the problems under consideration. These assumptions are fulfilled for the most practically important problems. For example, free path of molecules in air at atmospheric pressure and room temperature is about $10^{-7}$ cm whereas the characteristic scale of any macroscopic problem is much larger.

To find the distribution function taking into account the molecular collisions we accept the following definition.

DEFINITION 2 (CINEMA FILM)

As before, we single out a domain $dV$ which size is small comparing to the free path of molecules in a gas. When we have considered the velocity histogram in DEFINITION 1, the velocities of molecules have been shot by the camera. Now we use a movie camera. It shoots on a film a series of consecutive pictures of molecules in the domain $dV$ during the time interval $dt$, duration of which will be defined later.

Let us divide the molecules, whose images are shot on the picture area of a film into two groups. The first group contains the molecules, which have taken part in the collisions inside the considered domain. Now we assume that the velocities for distribution function are taken into account only for molecules right after the collision. As a graphic illustration for such a situation we propose that colliding molecules in the volume $dV$ under consideration are visible and can be seen by the observer, while the molecules flying without collisions are invisible. The observer sees only painted molecules and, therefore, only their velocities define the distribution function for him. Second group of molecules, which pass through the above volume $dV$ without collisions, does not contribute to distribution function. We will define the normalized molecular velocity distribution function $f_s(\vec{V}, \vec{r}, t)$ on the base of set of velocities obtained in such a way. Namely this function will be further considered as the molecular velocity distribution function $f_s(\vec{V}, \vec{r}, t)$ at the given point of space, at the given time $t$.

In order to define this function statistically, in the same way as the function $f(\vec{V}, \vec{r}, t)$ according to the DEFINITION 1 it is necessary to take into account that a number of molecules, which is used to evaluate of the molecule velocity distribution function, would be equal to a total number of molecules in volume $dV$ at the time $t$. The number of
collisions is proportional to time. If the time interval $dt$ is approximately $\lambda/\bar{v}$, where $\bar{v}$ is the average modulus of proper velocity, a number of colliding molecules will be equal to a number of molecules in chosen volume $dV$. For gas under normal conditions this time is about $10^{-11}\text{s}$ and is much less than characteristic time of change of macroscopic parameters of gas. Since the distribution function is macroscopic characteristic of gas, the times $t$ and $t+dt$ are not distinguished from one another. Therefore, we can treat the function determined according to the above rules as molecule velocity distribution function at the time $t$. We will call this statement the DEFINITION 2.

One has to note that the above definition does not allow to obtain an explicit form of the distribution function in the same sense as in the DEFINITION 1. Nevertheless, it is possible a priori to draw a conclusion about the important features of such a function.

Let us introduce the definition of the mass velocity $\vec{u}$ and proper velocity $\vec{v} = \vec{V} - \vec{u}$ as above. According to the DEFINITION 2, the proper velocity of a molecule is the velocity of a molecule after collision in a reference frame moving with gas. The vectors of proper velocities of molecules after collisions have equiprobable directions in this reference frame. Therefore, the proper velocity distribution function $\tilde{f}_s(\vec{v}, \vec{r}, t)$ will exhibit a spherical symmetry. This property of spherical symmetry means that all directions of proper velocity vector of molecules $\vec{v}$ are equivalent.

The property of spherical symmetry is very important to find the explicit expression for macroscopic quantities according to the equation (3). Taking into account the spherical symmetry one define the dimensional distribution function function depending only on modulus of proper velocity $|\vec{v}|$, $\tilde{f}_s = \tilde{f}_s(|\vec{v}|)$. Then, the fluxes of mass, momentum and energy can be easily calculated.

To simplify the expressions we will write the distribution function of the modules of proper velocity as $f_s(v)$. The normalization condition for this function $f_s(v)$ looks like

$$\int_0^\infty \tilde{f}_s(|\vec{v}|) d|\vec{v}| = \int_0^\infty f_s(v) dv = 1$$

Namely the function $\tilde{f}_s$ will be used further to derive the macroscopic equations of motion.

4 Equations for flux of mass and momentum in homogeneous rest gas.

Volume filled with gas can be treated as continuous up to spatial scale of free path of molecules. It means, that only the collisions cause a change of velocity vector of a molecule. After collision, the velocity of the molecule does not vary, until it collides with another molecule. Between the collisions, the molecule passes an average distance approximately equal to free pass $\lambda$. The molecule, which had a collision, keeps a constant velocity during the free path (of course, if there is no external field).

Hence, the gas of colliding molecules can be considered as the set of points, each of them is spherical ”source” of molecules with velocity distribution function belonging to the given concrete point. Intensity of a molecular flux from each such a point decreases proportionally to an inverse square distance from the given point. Let us consider a sphere of the radius $\lambda$ surrounding the radiating point. It is clear that beyond this sphere, the intensity of a ”source” becomes equal to zero, since there should be a next collisions of molecules here. After the collisions, the velocities of molecules belong to other
distributions, which appropriate to other "radiating" points located on a surface of the same sphere.

The intensity of a considered point "source" is a product of velocity of molecules and their density. The above definition of velocity distribution function does not allow to find a density of colliding molecules in the volume under consideration. This density (we denote it $\rho'$) can be obtained using the following arguments. Let us introduce a surface into a gas and an observer behind this surface. This observer counts a number of molecules flying through the surface with different velocities. It is found that $N$ molecules have a flux under the certain angle $\theta, \varphi$ with the certain velocity $V$. It is not possible to say, from which of radiating points one or other molecule has started its flight. The observer is able to determine only a sum of intensities from all sources and this sum is equal to a product of the true density of the gas $\rho$ near the surface and the velocity of molecules. On the other hand, this sum is equal to the integral of the intensity of all radiating points at distances from 0 up to $\lambda$ from a point of observation. Therefore, one can write

$$\rho V(\theta, \varphi) = \int_0^\lambda \rho' V(\theta, \varphi) dl$$

Hence

$$\rho' = \frac{1}{\lambda} \rho$$

Let’s place in homogeneous non-moving gas of colliding molecules a surface $dS$ located at a point with the coordinates $x, y, z$, and small comparing to $\lambda^2$. Normal vector to this surface is directed along the axis X. Only the molecules, which had collisions inside the sphere of radius $\lambda$, cross such a surface. The molecules which had the collisions in hemisphere located to the left from $dS$ fly along the positive direction of X axis and the molecules which had the collisions in hemisphere located to the right from $dS$ fly along the negative direction of X axes. The total flux through $dS$ will be equal to the difference of two above fluxes.

The flux is considered at some time $t$. At this time point, only the molecules which had the collisions near the surface $dS$ and the molecules which had the collisions on the distance $l$ from the surface $dS$ fly through the given surface. Therefore, the distribution functions belong to the different time points. However, maximal difference in time is equal to $\lambda/V$, that for gas at normal conditions is around $10^{-11}$ s. Hence, one can neglect a difference in time between collisions and to consider, that all molecules had collisions at the same $t$ (simultaneously).

Let’s calculate a flux along positive directions of the axis. We set a polar coordinate system in the hemisphere, where a distance from $dS$ to a point inside a hemisphere is equal to $r$, and the angle between normal to $dS$ and direction of $r$ is equal to $\theta$. Let’s divide the hemisphere into layers parallel to $dS$ and located at the various distances $l$ from $dS$. A thickness of a layer is $dl$. A distance from $dS$ to a point at layer at the distance $l$ is equal to $r = l/cos\theta$. We associate a small volume $dV$ with each point of hemisphere attached to $dS$ and consider this volume $dV$ as a source of molecules with distribution on modules of proper velocity $f_s(v)$. The intensity of source for molecules with proper velocity from $v$ to $v+dv$ is equal to $\rho f_s(v)vdvdV/\lambda$. Density of a flux of molecules $dI$ decreases proportionally to inverse square of distance $r^2 = l^2/cos^2\theta$. The flux $dJ_+\rightarrow$ through $dS$ is equal to the product density $dI$ and a projection $dS$, which is
equal to \( dS \cos \theta \). Thus, flux \( dJ_{\rightarrow} \) from a single radiating point can be written as follows

\[
dJ_{\rightarrow} = \frac{\rho f_s(v)vdV dS \cos^3 \theta}{4\pi l^2 \lambda}
\]  

(9)

Let’s divide a hemisphere of radius \( \lambda \) into layers of thickness \( dl \), located at the distance \( l \) from a surface \( dS \). Each layer, in its turn, is divided into the rings with constant angle \( \theta \), where radius of the ring \( R = l \tan \theta \). Volume of the ring is \( RdRdl \). Also, \( dR = l \theta / \cos^2 \theta \).

The flux \( dJ_{\rightarrow} \) through a surface \( dS \) from such a ring is equal to

\[
dJ_{\rightarrow} = \frac{\rho f_s(v)vdvdS \cos \theta \tan \theta d\theta dl}{2\lambda}
\]  

(10)

For all rings belonging to a layer at the distance \( l \), the angle \( \theta \) varies from \( 0 \) to \( \arccos(l/\lambda) \). Therefore, the flux from the whole layer \( dJ_{\rightarrow} \) is equal to integral from \( dJ_{\rightarrow} \) over \( \theta \) from \( 0 \) to \( \arccos(l/\lambda) \). As a result we get

\[
dJ_{\rightarrow} = \int_0^{\arccos(l/\lambda)} dJ_{\rightarrow} \, d\theta = \frac{\rho f_s(v)vdS (\lambda - l)}{2\lambda^2} \frac{dl}{2}
\]  

(11)

The flux through \( dS \) is equal to the sum of fluxes from all layers, which are located away from \( dS \) at the distance from \( 0 \) to \( \lambda \).

\[
J_{m\rightarrow} = \int_0^\lambda dJ_{\rightarrow}dl = \frac{1}{4} \rho f_s(v)vdv dS
\]  

(12)

If we divide the expression obtained for any surface \( dS \) by \( dS \) we will have the expression for a mass flux density along axis for the molecules, whose velocity is equal to \( v \). The integration over \( v \) gives mass flux density along a positive direction of X-axis as a result of the proper velocity

\[
J_{mx\rightarrow} = \int_0^\infty dvJ_{m\rightarrow} = \frac{1}{4} \rho \bar{v}
\]  

(13)

Together with the flux of mass along a positive direction of an X-axis there exists a flux of mass coming along a negative direction of axes and having the same value. The resulting flux of mass is equal to a difference between two above fluxes, i.e. the total flux is equal to zero.

X-component of the momentum of the each individual molecule is equal to \( mv \cos \theta \). Multiplying (9) with \( v \cos \theta \) we obtain an equation for density of a flux for X-component of momentum from each radiating point. The similar calculations lead to the following result for the density of the flux X-component of the momentum along a positive direction of X-axis

\[
J_{px\rightarrow} = \frac{1}{6} \rho \bar{v}^2
\]  

(14)

Apart, a flux coming along a negative direction of an X-axis also exists. It has the same absolute value but opposite direction. The resulting density of a flux of X-component of the momentum is equal to a difference of above two fluxes. However the momentum, in contrast with a mass, is a vector quantity. Therefore, a difference of the fluxes of momentum along positive and along negative directions of X-axis is equal to the doubled value of (14). As a result, we obtain the known formula for hydrostatic pressure

\[
P = \frac{1}{3} \rho \bar{v}^2
\]  

(15)
For density of a flux of energy along a positive direction of an X-axis we have

\[ J_{Ex} = \frac{1}{8} \rho \bar{v}^3 \quad (16) \]

The energy is scalar quantity, therefore the resulting density of a flux is equal to zero, like for density of a flux of mass.

5 General approach of deriving the equations of motion for inhomogeneous gas.

Let’s consider, as in the previous section, a domain filled with gas, where a free path is much smaller than the size of the domain and much smaller than the characteristic scale of a problem. The gas is in a inhomogeneous state, i.e., a density and velocity distribution function at different points of the volume differ from each other.

Let’s place any surface in gas. This surface will be crossed by molecules which come from the both sides of the surface. Since the molecules do not change the velocity on the scale \( \lambda \), the effective sources of fluxes of molecules is located from each other at the distance determined by free path \( \lambda \). It means, the fluxes of molecules on the different sides of the surface will have the different parameters: density, temperature, mass velocity and the velocity distribution function. The difference of physical parameters of molecular fluxes leads to appearance of terms with spatial derivatives in expressions for fluxes for mass, momentum and energy.

The derivation of the equations is carried out under the following restrictions:

1. The linear approximation is considered. Accordingly to this, only the first spatial derivatives of density, mass and proper velocities are taken into account.

2. The approximation of small velocities is considered. It means, a module of the mass velocity \( |\vec{u}| \) is much smaller than average module of proper velocity \( |\bar{v}| \).

6 The mass conservation equation.

To derive the mass conservation equation, taking into account the collisions of molecules, it is necessary to find an expression for mass flux, suitable for these conditions. Let’s associate with a point \( x, y, z \) a small surface \( dS \), directed perpendicularly to X-axis in non-moving laboratory reference frame.

At the given point \( x, y, z \) gas is characterized by the density \( \rho \), mass velocity \( \bar{u} \) and distribution function on modules of proper velocity \( f_s(v) \).

The calculations for inhomogeneous gas are basically similar to the calculations for homogeneous gas. The sphere attached to \( dS \) is divided into layers and etc. Equation (6) is written as follows

\[ dJ_\rightarrow = \frac{\rho f_s(v)(v \cos \theta + u_x)dv dS \cos^2 \theta}{4\pi \lambda^2} \quad (17) \]

where \( u_x \) is X-th component of the mass velocity. The flux from a ring is written in appropriate way

\[ dJ_{c\rightarrow} = \frac{\rho f_s(v)(v \cos \theta + u_x)dS \tan \theta d\theta}{2} \quad (18) \]
For all rings belonging to the layer at the distance \( l \), angle \( \theta \) varies from 0 to \( \arccos(l/\lambda) \). Therefore, the total flux from all layer \( dJ_{l \to} \) is equal to integral from \( dJ_{c \to} \) over \( \theta \) from 0 to \( \arccos(l/\lambda) \).

\[
dJ_{l \to} = \int_{0}^{\arccos(l/\lambda)} dJ_{c \to} d\theta = \rho f_s(v)dv dS \left( v(\lambda - l) - u_x \ln \left( \frac{\lambda}{l} \right) \right) / 2\lambda
\] (19)

The flux through \( dS \) is equal to the sum of the fluxes from all layers, which are located away from \( dS \) at the distances from 0 to \( \lambda \). Since the inhomogeneous gas is considered, the density \( \rho \), proper velocity \( v \) and \( X \)-th component of the mass velocity \( u_x \) are the function of the distance from \( dS \) to the layer at the distance \( l \). In first approximation we restrict ourselves to the linear terms in spatial derivatives in expansion of all quantities.

\[
\rho(l) = \rho - \frac{\partial \rho}{\partial x} l
\] (20)

\[
v(l) = v - \frac{\partial v}{\partial x} l
\] (21)

\[
u_x(l) = u_x - \frac{\partial u_x}{\partial x} l
\] (22)

Partial derivative on \( y \) and \( z \) do not contribute to the expressions for flux on axis, since they are averaged under integration over ring.

The signs at partial derivatives in eqs. (17), (18) and (19) are defined by a direction of a flux. It means, the sign for a flux on left side from \( dS \) will be opposite to sign on right.

The flux of mass through \( dS \) is equal to the difference of fluxes on the left and on the right sides. Since the case \(|\vec{u}| << |v|\) is considered, the terms containing the powers of \( \lambda \) higher than first and the products of the quantities \( u_x \lambda \), which are small comparing to \( v\lambda \), are omitted. Expression for \( X \)-th component of a resulting vector of a mass flux resulted from the molecules with velocity \( v \) can be written in the following form

\[
J_{mxv} = \rho f_s(v)dv \left( \rho u_x - \frac{1}{6} \rho \lambda \frac{\partial v}{\partial x} - \frac{1}{6} v \lambda \frac{\partial \rho}{\partial x} \right)
\] (23)

For obtaining the final result, it is necessity to integrate eq.(20) over velocity. It is reasonable to assume, that the form of distribution function \( f_s(v) \) does not change on scale \( \lambda \), i.e. relative changes of individual velocities are proportional to changes of average velocity. Then we can write the relation

\[
\frac{\partial v}{\partial x} = \frac{v \frac{\partial \bar{v}}{\partial x}}{\bar{v}}
\] (24)

The equation (20) after integration over velocities looks like

\[
J_{mx} = \rho f_s(v)dv (\rho u_x - \frac{1}{6} \lambda \frac{\partial (\rho \bar{v})}{\partial x})
\] (25)

By definition, a product of the density, free path and the average velocity is proportional to the viscosity \( \mu \)

\[
\mu = \frac{1}{3} \rho \bar{v} \lambda
\] (26)
Then the relation (22) can be written as follows

\[ J_{mx} = \rho u_x - \frac{1}{2} \frac{\partial \mu}{\partial x} \]  

(27)

The expressions for Y-th and Z-th components of a vector of a mass flux density are written in the similar way

\[ J_{my} = \rho u_y - \frac{1}{2} \frac{\partial \mu}{\partial y} \]  

(28)

\[ J_{mz} = \rho u_z - \frac{1}{2} \frac{\partial \mu}{\partial z} \]  

(29)

Expression for the vector of mass flux density looks like

\[ \mathbf{J}_m = \rho \mathbf{u} - \frac{1}{2} \text{grad} \mu \]  

(30)

Using eqs. (1) we get the mass conservation equation in the form

\[ \frac{\partial \rho}{\partial t} + \text{div} (\rho \mathbf{u}) - \frac{1}{2} \text{div} (\text{grad} \mu) = 0 \]  

(31)

The expression (28) differs from the known continuity equation in the derivatives of viscosity. Factor 1/2 at the corresponding term is explained by hemispheres attached to each other which are the integral sources of molecules.

7 The momentum conservation equation

The momentum conservation equation is derived in the similar way to mass conservation equation with the help of substituting the appropriate quantities for momentum components by the untegrand of eq.(2). Density of a flux of X-th component of momentum along the X-axis \( dJ_{px} \rightarrow \) from the individual radiating point is equal to

\[ dJ_{px} \rightarrow = \frac{\rho f_s(v)(v \cos \theta + u_x)^2dv dS \cos^2 \theta}{4 \pi l^2} \]  

(32)

Density of the flux of X-th components of a momentum from a layer at the distance \( l \) \( dJ_{lpx} \rightarrow \) has the form

\[ dJ_{lpx} \rightarrow = \frac{\rho f_s(v)dv dS \left[v^2l^2 + 4 vu \lambda + 2 u^2 \ln \left(\frac{1}{\lambda}\right) \lambda^2 - v^2 \lambda^2 - 4 u \lambda^2\right]}{\lambda^2} \]  

(33)

Flux from whole left hemisphere is equal to integral from \( dJ_{lpx} \rightarrow \) over \( l \) from 0 to \( \lambda \). The flux from the right hemisphere \( dJ_{lpx} \leftarrow \) will be calculated by the same manner. The resulting flux through \( dS \) is equal to the sum of these fluxes and for fixed velocity \( v \) can be written as follows

\[ J_{Pxxv} = f_s(v) dv \left[ \frac{1}{3} \rho v^2 + \rho u_x^2 - \frac{1}{3} \lambda \left( \rho u_x \frac{\partial v}{\partial x} + v u_x \frac{\partial \rho}{\partial x} + \rho v \frac{\partial u_x}{\partial x} \right) \right] \]  

(34)

After integration over \( v \) we get the expression for density of a flux of X-th component of a momentum along the X-axis

\[ J_{Pxx} = \frac{1}{3} \rho \tilde{v}^2 + \rho u_x^2 - \frac{\partial \mu u_x}{\partial x} \]  

(35)
Density of a flux of Y-th component of a momentum along the X-axis $dj_{py} \to$ from a single radiating point is equal to

$$dj_{py} \to = \frac{\rho f_s(v)(v \cos \theta + u_x)u_y d\nu dS \cos^2 \theta}{4\pi l^2}$$  (36)

The calculations for density of a flux of Y-th component along the X-axis lead to the following expression

$$J_{pxy} = \rho u_x u_y - \frac{1}{2} \frac{\partial \mu u_y}{\partial x}$$  (37)

Expression for density of a flux of Z-th component of a momentum along the X-axis is equal to

$$J_{pxz} = \rho u_x u_z - \frac{1}{2} \frac{\partial \mu u_z}{\partial x}$$  (38)

Other components of a vector of a flux of a momentum are calculated in the same way. After substituting the expressions obtained into eq. (2) and subtracting from the result the conservation equation for mass (24), multiplied with $\vec{u}$, we get the following form of the momentum conservation equation

$$\rho \left( \frac{\partial}{\partial t} + u \nabla \right) u + \text{grad} \Pi - \text{grad}(\mu) \nabla u - D = 0$$  (39)

where the vector of pressure $\Pi$ is sum of hydrostatic pressure and a vector of hydrodynamic pressure. This pressure $\Pi$ is expressed in the following form

$$\Pi = \begin{cases} P - \mu \frac{\partial u_y}{\partial x} \\ P - \mu \frac{\partial u_y}{\partial y} \\ P - \mu \frac{\partial u_z}{\partial z} \end{cases}$$  (40)

The parameter $D$ in eq (39) is written as

$$D = 1/2 \mu \left[ \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2} \right]$$  (41)

Appearance of the hydrodynamical pressure in the expression for pressure $\Pi$ is explained by the following way. The pressure is defined as a flux of a momentum of molecules colliding with some surface. Quantitatively, this flux of momentum is equal to a product of momentum $mV$ of individual molecule with flux of number of molecules $nV$ colliding with the surface. Then, we get for hydrostatic pressure $P \sim nV mV < mV nV > = \rho < V^2 >$. The mass velocity leads to the change of both values of the momentum and the number of molecules, colliding with the surface. However, this effect works only at the distance of free path. Therefore, the correction to the hydrostatic pressure is equal to a product $mV$ with $nV \lambda \frac{\partial u_i}{\partial x_i}$, that is equal to $\mu \frac{\partial u_i}{\partial x_i}$. Thus, expression (37) defines the actual measurable pressure on the surface, oriented along axes X, Y and Z, respectively.
8 The energy conservation equation.

Density of an energy flux from the radiating point is equal to
\[
J_{Ex} = \frac{\rho f_j(v)(v \cos \theta + u_x)(v^2 + 2v u_x \cos \theta + \bar{w}^2)dv dS \cos^2 \theta}{8\pi l^2}
\] (42)

After calculation, we get the following expression for the resulting density of the energy flux along the X-axis
\[
J_{Ex} = \frac{1}{2}, u_x \rho \left( \frac{5\bar{w}^2}{3} + u^2 \right) - \frac{1}{12} \frac{d(\rho \bar{V}^3)}{dx} - \frac{3}{4} \frac{d(\mu u_x)}{dx} - \frac{1}{4} \frac{d(\mu u_y)}{dx} - \frac{1}{4} \frac{d(\mu u_z)}{dx}
\] (43)

Second term in the right hand side of the obtained expression represents the partial derivative of $\frac{1}{12} \rho \bar{V}^3$. Let’s transform it as follows. For an any velocity distribution function one can write
\[
\frac{1}{12} \rho \bar{V}^3 = \frac{1}{12} \rho \bar{V} \bar{V}^2 k(\theta) = \frac{1}{12} k(\theta) \mu \bar{V}^2 = K \bar{V}^2
\] (44)

Here the parameter $K$ can be interpreted as gas thermal conductivity (we point out that the $K$ depends on the temperature as well as the coefficient of viscosity $\mu$). In accordance to that, the expression for the vector of the energy flux takes the form
\[
J_E = \begin{pmatrix}
\frac{1}{2}, u_x \rho \left( \frac{5\bar{w}^2}{3} + u^2 \right) - \frac{d(K \bar{V}^2)}{dx} - \frac{3}{4} \frac{d(\mu u_x)}{dx} - \frac{1}{4} \frac{d(\mu u_y)}{dx} - \frac{1}{4} \frac{d(\mu u_z)}{dx} \\
\frac{1}{2}, u_y \rho \left( \frac{5\bar{w}^2}{3} + u^2 \right) - \frac{d(K \bar{V}^2)}{dy} - \frac{3}{4} \frac{d(\mu u_x)}{dy} - \frac{1}{4} \frac{d(\mu u_y)}{dy} - \frac{1}{4} \frac{d(\mu u_z)}{dy} \\
\frac{1}{2}, u_z \rho \left( \frac{5\bar{w}^2}{3} + u^2 \right) - \frac{d(K \bar{V}^2)}{dz} - \frac{3}{4} \frac{d(\mu u_x)}{dz} - \frac{1}{4} \frac{d(\mu u_y)}{dz} - \frac{1}{4} \frac{d(\mu u_z)}{dz}
\end{pmatrix}
\] (45)

The conservation equation for energy is obtained after substituting eq. (42) into the eq. (1) and the subsequent subtraction from the result of the mass conversation equation (28), multiplied with $u^2/2$, and momentum conservation equation (36), multiplied with a vector of mass velocity $\bar{u}$.
\[
\left( \frac{\partial}{\partial t} + \bar{u} \nabla \right) P + \frac{5}{3} \rho \varepsilon_d \bar{u} - \nabla^2 (K \bar{V}^2) - \mu/3 \sum_i \sum_j \left( \frac{\partial u_i}{\partial x_j} \right)^2 - B = 0
\] (46)

Where $B$ is equal
\[
B = 2\mu/3 \sum_i \left( \frac{\partial u_i}{\partial x_i} \right)^2 + 2/3 \sum_i u_i \left( \frac{\partial u_i}{\partial x_i} \frac{\partial \mu}{\partial x_i} \right) + \mu/3 \sum_i \left( u_i \frac{\partial^2 u_i}{\partial x_i^2} \right)
\] (47)

First three terms in the equation (46) are similar to the known energy conservation equation taking into account the process of heat transfer. Two last terms in the equation (46) describe a process of energy dissipation of the ordering motion into the thermal energy of gas due to the work of forces of viscosity. Energy dissipation can be understood as follows. There are the collisions of molecules at each space point. The molecules flying to the given point have the different velocity distribution, in particular they have the different mass velocities. After the collision, the molecules belong to the same velocity distribution. As a result, a part of energy of the ordering motion approximately equal to $\mu \left( \frac{\partial u_i}{\partial x_i} \right)^2$ will be always used for heating of gas. Hence, all processes in a gas, resulted from the change of the mass velocity, are irreversible processes. Thus, the equation (46) leads to the possibility to describe correctly the energy dissipation at the gas motion.
9 Summary

We have introduced the new definition of the molecule velocity distribution function describing the molecular motion on a free-path scale. Using this definition we derived the equations of gas dynamics containing the new terms in comparison with well known equations ref. [2]. The equation of mass conservation (31) includes the new contribution with second derivative of dynamic viscosity. The equations of momentum conservation (39 - 41) do not contain the volume viscosity. The equations of energy conservation (46 -47) describe the transformation of energy of gas mass velocity into the thermal energy of gas.

The new equations of gas dynamics derived here are approximate like the Euler and Navier-Stokes equations. We have restricted ourselves by the approximation of small velocities. Nevertheless, we hope that these new equations will be useful for description of the nonequilibrium phenomena in various areas of macroscopic physics. For example, these equations can be applicable to computation of atmospheric processes like turbulence and solution of the aerodynamic problems like modeling a subsonic fluxes. We plan to discuss such problems on the base of our new equations in forthcoming works.

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