On some properties of weak solutions to elliptic equations with divergence-free drifts

Nikolay Filonov and Timofey Shilkin

Abstract. We discuss the local properties of weak solutions to the equation
\[-\Delta u + b \cdot \nabla u = 0.\]
The corresponding theory is well-known in the case \(b \in L^n\), where \(n\) is the dimension of the space. Our main interest is focused on the case \(b \in L^2\). In this case the structure assumption \(\text{div} \, b = 0\) turns out to be crucial.

1. Introduction and Notation

Assume \(n \geq 2\), \(\Omega \subset \mathbb{R}^n\) is a smooth bounded domain, \(b : \Omega \to \mathbb{R}^n\), \(f : \Omega \to \mathbb{R}\). In this paper we investigate the properties of weak solutions \(u : \Omega \to \mathbb{R}\) to the following scalar equation
\[\tag{1.1} -\Delta u + b \cdot \nabla u = f \quad \text{in} \quad \Omega.\]
This equation describes the diffusion in a stationary incompressible flow. If it is not stated otherwise, we always impose the following conditions
\[b \in L^2(\Omega), \quad f \in W^{-1}_2(\Omega)\]
(see the list of notation at the end of this section). We use the following

Definition 1.1. Assume \(b \in L^2(\Omega), \ f \in W^{-1}_2(\Omega)\). The function \(u \in W^{1}_2(\Omega)\) is called a weak solution to the equation (1.1) if the following integral identity holds:
\[\tag{1.2} \int_{\Omega} \nabla u \cdot (\nabla \eta + b\eta) \, dx = \langle f, \eta \rangle, \quad \forall \ \eta \in C^\infty_0(\Omega).\]

Together with the equation (1.1) one can consider the formally conjugate (up to the sign of the drift) equation
\[\tag{1.3} -\Delta u + \text{div}(bu) = f \quad \text{in} \quad \Omega.\]

1991 Mathematics Subject Classification. 35B65.

Key words and phrases. elliptic equations, weak solutions, regularity.

Both authors are supported by RFBR grant 17-01-00099-a.

The research of the second author leading to these results has received funding from the People Programme (Marie Curie Actions) of the European Union’s Seventh Framework Programme FP7/2007-2013/ under REA grant agreement n 319012 and from the Funds for International Co-operation under Polish Ministry of Science and Higher Education grant agreement n 2853/7.PR/2013/2. The author also thanks the Technische Universität of Darmstadt for its hospitality.
DEFINITION 1.2. Assume \( b \in L_2(\Omega) \), \( f \in W^{-1}_2(\Omega) \). The function \( u \in W^{1}_2(\Omega) \) is called a weak solution to the equation (1.3) if

\[
\int_{\Omega} (\nabla u - bu) \cdot \nabla \eta \, dx = \langle f, \eta \rangle, \quad \forall \ \eta \in C^\infty_0(\Omega).
\]

The advantage of the equation (1.3) is that it allows one to define weak solutions for a drift \( b \) belonging to a weaker class than \( L_2(\Omega) \). Namely, Definition 1.2 makes sense for \( u \in W^{1}_2(\Omega) \) if

\[
b \in L_s(\Omega) \quad \text{where} \quad s = \begin{cases} \frac{2n}{n+2}, & n \geq 3, \\ 1 + \varepsilon, & \varepsilon > 0, \quad n = 2. \end{cases}
\]

Nevertheless, it is clear that for a divergence-free drift \( b \in L_2(\Omega) \) the Definitions 1.1 and 1.2 coincide.

Together with the equation (1.1) we discuss boundary value problems with Dirichlet boundary conditions:

\[
\begin{cases}
-\Delta u + b \cdot \nabla u = f & \text{in } \Omega, \\
u|_{\partial \Omega} = \varphi.
\end{cases}
\]

For weak solutions the boundary condition is understood in the sense of traces. Assume \( f \) is “good enough” and \( b \in L_2(\Omega) \), \( \text{div} \, b = 0 \). Our main observation is that the regularity of solution \( u \) inside \( \Omega \) can depend on the behaviour of its boundary values. If the function \( \varphi \) is bounded, then the solution \( u \) is also bounded (see Theorem 3.4 below). If the function \( \varphi \) is unbounded on \( \partial \Omega \), then the solution \( u \) can become infinite in internal points of \( \Omega \) (see Example 3.6 below). So, we distinguish between two cases: the case of general boundary data \( \varphi \in W^{1/2}_2(\partial \Omega) \), and the case of bounded boundary data

\[
\varphi \in L_\infty(\partial \Omega) \cap W^{1/2}_2(\partial \Omega).
\]

Discussing the properties of weak solutions to the problem (1.6) we also distinguish between another two cases: in Section 2 we consider sufficiently regular drifts, namely, \( b \in L_n(\Omega) \), and in Section 3 we focus on the case of drifts \( b \) from \( L_2(\Omega) \) satisfying \( \text{div} \, b = 0 \). Section 4 is devoted to possible ways of relaxation of the condition \( b \in L_n(\Omega) \) in the framework of the regularity theory. In Appendix for reader’s convenience some proofs (most of which are either known or straightforward) are gathered.

Together with the elliptic equation (1.1) it is possible to consider its parabolic analogue

\[
\partial_t u - \Delta u + b \cdot \nabla u = f \quad \text{in } \Omega \times (0, T),
\]

but it should be a subject of a separate survey. We address the interested readers to the related papers [Z], [LZ], [NU], [Sem], [SSSZ], [SV], [SVZ] and references there.

In the paper we explore the following notation. For any \( a, b \in \mathbb{R}^n \) we denote by \( a \cdot b \) its scalar product in \( \mathbb{R}^n \). We denote by \( L_\mu(\Omega) \) and \( W^{k}_p(\Omega) \) the usual Lebesgue and Sobolev spaces. The space \( \dot{W}^{1}_p(\Omega) \) is the closure of \( C_0^\infty(\Omega) \) in \( W^{1}_p(\Omega) \) norm. The negative Sobolev space \( W^{-1}_p(\Omega) \), \( p \in (1, +\infty) \), is the set of all distributions which are bounded functionals on \( W^{1}_p(\Omega) \) with \( p' := \frac{p}{p-1} \). For any \( f \in W^{-1}_p(\Omega) \)
and \( w \in \overset{\circ}{W}_{p'}^1(\Omega) \) we denote by \( \langle f, w \rangle \) the value of the distribution \( f \) on the function \( w \). We use the notation \( W_{2}^{1/2}(\partial\Omega) \) for the Slobodetskii–Sobolev space. By \( C(\bar{\Omega}) \) and \( C^\alpha(\bar{\Omega}), \alpha \in (0, 1) \) we denote the spaces of continuous and Hölder continuous functions on \( \bar{\Omega} \). The space \( C^{1+\alpha}(\bar{\Omega}) \) consists of functions \( u \) whose gradient \( \nabla u \) is Hölder continuous. The index “loc” in notation of the functional spaces \( L_{\infty,\text{loc}}(\Omega), C^\alpha_{\text{loc}}(\Omega), C^{1+\alpha}_{\text{loc}}(\Omega) \) etc implies that the function belongs to the corresponding functional class over every compact set which is contained in \( \Omega \). The symbols \( \rightharpoonup \) and \( \rightarrow \) stand for the weak and strong convergence respectively. We denote by \( B_R(x_0) \) the ball in \( \mathbb{R}^n \) of radius \( R \) centered at \( x_0 \) and write \( B_R \) if \( x_0 = 0 \). We write also \( B \) instead of \( B_1 \).

2. Regular drifts

2.1. Local properties. For sufficiently regular drifts we have the local Hölder continuity of a solution.

**Theorem 2.1.** Assume

\[
(2.1) \quad b \in L_\alpha(\Omega) \quad \text{if} \quad n \geq 3, \quad \int_{\Omega} |b|^2 \ln(2 + |b|^2) \, dx < \infty \quad \text{if} \quad n = 2.
\]

Let \( u \in W^1_2(\Omega) \) be a weak solution to \( (1.1) \) with \( f \) satisfying

\[
f \in L_p(\Omega), \quad p > \frac{n}{2}.
\]

Then

\[
u \in C^\alpha_{\text{loc}}(\Omega) \quad \text{with} \quad \left\{ \begin{array}{ll}
\alpha = 2 - \frac{n}{p}, & p < n, \\
\forall \quad \alpha < 1, & p \geq n.
\end{array} \right.
\]

The local Hölder continuity of weak solutions in Theorem 2.1 with some \( \alpha \in (0, 1) \) is well-known, see [ST, Theorem 7.1] or [NU, Corollary 2.3] in the case \( f \equiv 0 \). The Hölder continuity with arbitrary \( \alpha \in (0, 1) \) was proved in the case \( f \equiv 0 \), for example, in [F]. The extension of this result for non-zero right hand side is routine.

If \( b \) possesses more integrability then the first gradient of a weak solution is locally Hölder continuous.

**Theorem 2.2.** Let \( b \in L_p(\Omega) \) with \( p > n \), and \( u \in W^1_2(\Omega) \) be a weak solution to \( (1.1) \) with \( f \in L_p(\Omega) \). Then \( u \in C^{1+\alpha}_{\text{loc}}(\Omega) \) with \( \alpha = 1 - \frac{n}{p} \).

For the proof see [LU, Chapter III, Theorem 15.1].

2.2. Boundary value problem. We consider the second term \( \int_{\Omega} \nabla u \cdot b \eta \, dx \) in the equation \( (1.2) \) as a bilinear form in \( \overset{\circ}{W}^1_2(\Omega) \). It defines a linear operator \( T : \overset{\circ}{W}^1_2(\Omega) \to \overset{\circ}{W}^1_2(\Omega) \) by the relation

\[
(2.2) \quad \int_{\Omega} \nabla(Tu) \cdot \nabla \eta \, dx = \int_{\Omega} \nabla u \cdot b \eta \, dx, \quad \forall \ u, \eta \in \overset{\circ}{W}^1_2(\Omega).
\]

The following result is well-known.

**Theorem 2.3.** Let \( b \) satisfy \( (2.1) \). Then the operator \( T : \overset{\circ}{W}^1_2(\Omega) \to \overset{\circ}{W}^1_2(\Omega) \) defined by \( (2.2) \) is compact.
Indeed, if $n \geq 3$ then the estimate

\[(2.3) \quad \int_{\Omega} \nabla u \cdot b \eta \, dx \leq C_b \|\nabla u\|_{L^2(\Omega)} \|\nabla \eta\|_{L^2(\Omega)} \quad \forall \, u, \eta \in \overset{\circ}{W}^1_2(\Omega)\]

follows by the imbedding theorem and the Hölder inequality. In the case $n = 2$ such estimate can be found for example in [F, Lemma 4.3]. Next, the operator $T$ can be approximated in the operator norm by compact linear operators $T_{\varepsilon}$ generated by the bilinear forms $\int_{\Omega} \nabla u \cdot b_{\varepsilon} \eta \, dx$ where $b_{\varepsilon} \in C^\infty(\bar{\Omega})$.

**Remark 2.4.** The condition $b \in L^2(\Omega)$ in the case $n = 2$ is not sufficient. For example, one can take $\Omega = B_{1/3}$, $b(x) = \frac{x}{|x|^2 \ln |x|^{3/4}}$, and $u(x) = \eta(x) = \ln |x|^{3/8} - (\ln 3)^{3/8}$.

Then $\int_{\Omega} \nabla u \cdot b \eta \, dx = \infty$, and therefore, the corresponding operator $T$ is unbounded.

**Remark 2.5.** The issue of boundedness and compactness of the operator $T$ in the case of the whole space, $\Omega = \mathbb{R}^n$, is investigated in full generality in [MV], see Theorem 4.1 below. In this section we restrict ourselves by considering assumptions on $b$ only in $L^p$-scale.

Now, the problem (1.6) with $\varphi \equiv 0$ reduces to the equation $u + Tu = h$ in $\overset{\circ}{W}^1_2(\Omega)$ with an appropriate right hand side $h$. The solvability of the last equation follows from the Fredholm theory. Roughly speaking, “the existence follows from the uniqueness”.

The uniqueness in the case $b \in L_n(\Omega)$, $n \geq 3$, and $\text{div} \, b = 0$ is especially simple. In this situation

$$\int_{\Omega} b \cdot \nabla u \, dx = 0 \quad \forall \, u \in \overset{\circ}{W}^1_2(\Omega),$$

and the uniqueness for the problem (1.6) follows. In the general case of drifts satisfying (2.1) without the condition $\text{div} \, b = 0$ the proof of the uniqueness is more sophisticated. It requires the maximum principle which can be found, for example, in [NU], see Corollary 2.2 and remarks at the end of Section 2 there.

**Theorem 2.6.** Let $b$ satisfy (2.1). Assume $u \in W^1_2(\Omega)$ is a weak solution to the problem (1.6) with $f \equiv 0$ and $\varphi \in L_\infty(\partial \Omega) \cap W^{1/2}_2(\partial \Omega)$. Then either $u \equiv \text{const}$ in $\Omega$ or the following estimate holds:

$$\text{essinf}_{\partial \Omega} \varphi < u(x) < \text{esssup}_{\partial \Omega} \varphi, \quad \forall \, x \in \Omega.$$  

**Corollary 2.7.** Let $b$ satisfy (2.1). Then a weak solution to the problem (1.6) is unique in the space $W^1_2(\Omega)$.

Now, the solvability of the problem (1.6) is straightforward.

**Theorem 2.8.** Let $b$ satisfy (2.1). Then for any $f \in W^{-1}_2(\Omega)$ and $\varphi \in W^{1/2}_2(\partial \Omega)$ the problem (1.6) has the unique weak solution $u \in W^1_2(\Omega)$, and

$$\|u\|_{W^1_2(\Omega)} \leq C \left( \|f\|_{W^{-1}_2(\Omega)} + \|\varphi\|_{W^{1/2}_2(\partial \Omega)} \right).$$
Proof. For \( \varphi \equiv 0 \) Theorem 2.8 follows from Fredholm’s theory. In the general case the problem (1.6) can be reduced to the corresponding problem with homogeneous boundary conditions for the function \( v := u - \tilde{\varphi} \), where \( \tilde{\varphi} \) is some extension of \( \varphi \) from \( \partial \Omega \) to \( \Omega \) with the control of the norm \( \| \tilde{\varphi} \|_{W^2_1(\Omega)} \leq c \| \varphi \|_{W^{1/2}_2(\partial \Omega)} \). The function \( v \) can be determined as a weak solution to the problem

\[
\begin{aligned}
-\Delta v + b \cdot \nabla v &= f + \Delta \tilde{\varphi} - b \cdot \nabla \tilde{\varphi} \quad \text{in} \quad \Omega, \\
v|_{\partial \Omega} &= 0
\end{aligned}
\]

Under assumption (2.1) the right hand side belongs to \( W^{-1}_n(\Omega) \) due to Theorem 2.3.

Note that for \( n \geq 3 \) the problems (1.6) and (2.4) are equivalent only in the case of “regular” drifts \( b \in L^n(\Omega) \). If \( b \in L^2(\Omega) \) and additionally \( \text{div} \, b = 0 \), then \( b \cdot \nabla \tilde{\varphi} \in W^{-1}_n(\Omega) \), \( n' = \frac{n}{n-1} \), and the straightforward reduction of the problem (1.6) to the problem with homogeneous boundary data is not possible.

Finally, to investigate in Section 3 the problem (1.6) with divergence-free drifts from \( L^2(\Omega) \) we need the following maximum estimate.

Theorem 2.9. Let \( b \) satisfy (2.1). Assume \( \varphi \) satisfies (1.7) and let \( u \in W^1_2(\Omega) \) be a weak solution to (1.6) with some \( f \in L^p(\Omega) \), \( p > n/2 \). Then

1) \( u \in L^\infty(\Omega) \) and

\[
\| u \|_{L^\infty(\Omega)} \leq \| \varphi \|_{L^\infty(\partial \Omega)} + C \| f \|_{L^p(\Omega)}.
\]

2) If \( \text{div} \, b = 0 \) then \( C = C(n, p, \Omega) \) does not depend on \( b \).

We believe Theorem 2.9 is known though it is difficult for us to identify the precise reference to the statement we need. So, we present its proof in Appendix.

Remark 2.10. For \( n \geq 3 \) consider the following example:

\[
\Omega = B, \quad u(x) = \ln |x|, \quad b(x) = (n - 2) \frac{x}{|x|^2}
\]

The statements of Theorem 2.1, Theorem 2.6 and Corollary 2.7 are violated for these functions. On the other hand, \( -\Delta u + b \cdot \nabla u = 0 \), \( u \in W^1_2(\Omega) \) and \( b \in L^p(\Omega) \) for any \( p < n \). It means that for non-divergence free drifts the condition \( b \in L^n(\Omega) \) in (2.1) is sharp.

Remark 2.11. For \( n = 2 \) the condition \( b \in L^2(\Omega) \) is not sufficient. The statements of Theorem 2.1, Theorem 2.6 and Corollary 2.7 are violated for the functions

\[
\begin{aligned}
u(x) &= \ln |\ln |x||, \\
b(x) &= \frac{x}{|x|^2 \ln |x|}
\end{aligned}
\]

in a ball \( \Omega = B_{1/e} \), nevertheless \( b \in L^2(\Omega) \).

Conversely, if in the case \( n = 2 \) we assume that \( b \in L^2(\Omega) \) and \( \text{div} \, b = 0 \), then the estimate (2.3) is fulfilled (see [MV] or [F]), and all statements of this section (Theorems 2.1, 2.3, 2.6, 2.8 and 2.9) hold true, see [F] or [NU]. So, this case can be considered as the regular one. See also Remark 4.3 below.
3. Non-regular divergence-free drifts

In this section we always assume that \( \text{div} \, b = 0 \). It turns out that this assumption plays the crucial role in local boundedness of weak solutions if one considers drifts \( b \in L_p(\Omega) \) with \( p < n \), \( n \geq 3 \). Recall that the case \( n = 2 \), \( b \in L_2(\Omega) \) and \( \text{div} \, b = 0 \) can be considered as a regular case, see Remark 2.11. Thus, below we restrict ourselves to the case \( n \geq 3 \).

3.1. Boundary value problem. We have the following approximation result.

**Theorem 3.1.** Assume \( b \in L_2(\Omega) \), \( \text{div} \, b = 0 \), and let \( u \in W_2^1(\Omega) \) be a weak solution to (1.1). Assume also \( b_k \in L_n(\Omega) \), \( \text{div} \, b_k = 0 \) is an arbitrary sequence satisfying
\[
\|b_k - b\|_{L_2(\Omega)} \to 0
\]
and let \( u_k \in W_2^1(\Omega) \) be the unique weak solution to the problem
\[
\begin{align*}
-\Delta u_k + b_k \cdot \nabla u_k &= f, \\
\|u_k\|_{\partial \Omega} &= \varphi,
\end{align*}
\]
where \( \varphi = u|_{\partial \Omega} \). Then
\[
u_k \to u \quad \text{in} \quad L_q(\Omega) \quad \text{for any} \quad q < \frac{n}{n-2}.
\]
Moreover, if \( \varphi \in L_\infty(\partial \Omega) \) then
\[
u_k \to u \quad \text{in} \quad W_2^1(\Omega).
\]
Finally, if \( \varphi \equiv 0 \) then the energy inequality holds:
\[
\int_{\Omega} |\nabla u|^2 \, dx \leq \langle f, u \rangle.
\]

The convergence (3.2) is proved (in its parabolic version) for \( q = 1 \) in [Z] Proposition 2.4. Note that the proof in [Z] uses the uniform Gaussian upper bound of the Green functions of the operators \( \partial_t u - \Delta u + b_k \cdot \nabla u \) (see [A]). In Appendix we present an elementary proof of Theorem 3.1 based on the maximum estimate in Theorem 2.9 and duality arguments.

Theorem 3.1 has several consequences. The first of them is the uniqueness of weak solutions, see [Z] and [Zhi]:

**Theorem 3.2.** Let \( b \in L_2(\Omega) \), \( \text{div} \, b = 0 \). Then a weak solution to the problem (1.0) is unique in the class \( W_2^1(\Omega) \).

Indeed, \( u \) is a \( L_q \)-limit of the approximating sequence \( u_k \), and such limit is unique. The alternative proof of the uniqueness (which is in a sense “direct”, i.e. it does not hang upon the approximation result of Theorem 3.1) for \( b \in L_2(\Omega) \), \( \text{div} \, b = 0 \), can be found in [Zhi] (see also some development in [Su]). Note that in [Zhi] it was also shown that the uniqueness can break for weak solutions to the equation (1.3) if \( b \) satisfy (1.5) (actually a little better than (1.5)) and \( \text{div} \, b = 0 \), but \( b \notin L_2(\Omega) \).

Another consequence of Theorem 3.1 is the existence of weak solution.

**Theorem 3.3.** Let \( b \in L_2(\Omega) \), \( \text{div} \, b = 0 \). Then for any \( f \in W_2^{-1}(\Omega) \) and any \( \varphi \) satisfying (1.7) there exists a weak solution to the problem (1.0).
Theorem 3.3 is proved in Appendix.

Finally, Theorem 3.1 allows one to establish the global boundedness of weak solutions whenever the boundary data are bounded.

**THEOREM 3.4.** Let \( b \in L_2(\Omega) \), \( \text{div} \, b = 0 \), \( f \in L_p(\Omega) \), \( p > n/2 \), and \( \varphi \) satisfies (1.4). Assume \( u \in W^1_2(\Omega) \) is a weak solution to (1.6). Then \( u \in L_\infty(\Omega) \) and
\[
\| u \|_{L_\infty(\Omega)} \leq \| \varphi \|_{L_\infty(\partial \Omega)} + C \| f \|_{L_p(\Omega)},
\]
where the constant \( C = C(n, p, \Omega) \) is independent on \( b \).

Theorem 3.3 is proved in Appendix.

### 3.2. Local properties

Note that any weak solution to (1.1) belonging to the class \( W^1_2(\Omega) \) can be viewed as a weak solution to the problem (1.6) with some \( \varphi \in W^1_2(\Omega) \).

**THEOREM 3.5.** Assume \( \text{div} \, b = 0 \) and
\[
b \in L_p(B) \quad \text{where} \quad p = 2 \quad \text{if} \quad n = 3 \quad \text{and} \quad p > \frac{n}{2} \quad \text{if} \quad n \geq 4.
\]
Let \( u \in W^1_2(B) \) be a weak solution to (1.1) in \( B \) with some \( f \in L_q(B), q > n/2 \). Then \( u \in L_\infty(B_{1/2}) \) and
\[
\| u \|_{L_\infty(B_{1/2})} \leq C \left( \| u \|_{W^1_2(B)} + \| f \|_{L_q(B)} \right)
\]
where the constant \( C \) depends only on \( n, p, q \) and \( \| b \|_{L_p(B)} \).

Theorem 3.5 was proved (in the parabolic version) in [Z]. For the reader’s convenience we present the proof of this theorem in Appendix.

Let us consider the following

**EXAMPLE 3.6.** Assume \( n \geq 4 \) and put
\[
u(x) = \ln r, \quad b = (n - 3) \left( \frac{1}{r} \, e_r - (n - 3) \frac{z}{r^2} \, e_z \right),
\]
where \( r^2 = x^2_1 + \ldots + x^2_{n-1} \), \( z = x_n \), and \( e_r, e_z \) are the basis vectors of the corresponding cylindrical coordinate system in \( \mathbb{R}^n \). Then \( u \in W^1_2(\Omega) \), and
\[-\Delta u + b \cdot \nabla u = 0.
\]
Next, \( \text{div} \, b = 0 \), \( b(x) = O(r^{-2}) \) near the axis of symmetry, and hence
\[
b \in L_p(B) \quad \text{for any} \quad p < \frac{n - 1}{2}.
\]

Clearly, the assumption \( b \in L_2(\Omega) \) leads to the restriction \( n \geq 6 \). So, for divergence-free drifts \( b \in L_2(\Omega) \) we have the following picture. Assume \( u \in W^1_2(\Omega) \) is a weak solution to (1.6) with \( f \in L_p(\Omega), p > n/2 \). Theorem 3.4 means that
\[
\varphi \in L_\infty(\partial \Omega) \cap W^{1/2}_2(\partial \Omega) \implies u \in L_\infty(\Omega) \quad \text{for any} \quad n \geq 2.
\]
The Example 3.6 shows that for general \( \varphi \) we have
\[
\varphi \in W^{1/2}_2(\partial \Omega) \implies \begin{cases} \text{if } n \leq 3 \text{ then } u \in L_{\infty,\text{loc}}(\Omega), \\ \text{if } n \geq 6 \text{ then it is possible } u \notin L_{\infty,\text{loc}}(\Omega), \\ \text{if } n = 4, 5 \text{ – open questions.} \end{cases}
\]

Theorem 3.3 and Example 3.6 together establish an interesting phenomena: for drifts \( b \in L_2(\Omega) \), \( \text{div} \, b = 0 \), the property of the elliptic operator in (1.1) to
improve the “regularity” of weak solutions (in the sense that every weak solution is locally bounded) depends on the behavior of a weak solution on the boundary of the domain. If the values of $\varphi := u|_{\partial \Omega}$ on the boundary are bounded then this weak solution must be bounded as Theorem 3.4 says. On the other hand, if the function $\varphi$ is unbounded on $\partial \Omega$ then the weak solution can be unbounded even near internal points of the domain $\Omega$ as Example 3.6 shows. To our opinion such a behavior of solutions to an elliptic equation is unexpected. Allowing some abuse of language we can say that non-regularity of the drift can destroy the hypoellipticity of the operator.

Theorem 3.4 impose some restrictions on the structure of the set of singular points of weak solutions. Namely, let us define a singular point of a weak solution as a point for which the weak solution is unbounded in any its neighborhood, and then define the singular set of a weak solution as the set of all its singular points. It is clear that the singular set is closed. Theorem 3.4 shows that if for some weak solution its singular set is non-empty then its 1-dimensional Hausdorff measure must be positive.

Theorem 3.7. Let $b \in L_2(\Omega)$, $\text{div } b = 0$, and let $u \in W^1_2(\Omega)$ be a weak solution to (1.1) with $f \in L_p(\Omega)$, $p > n/2$. Denote by $\Sigma \subset \Omega$ the singular set of $u$ and assume $\Sigma \cap \Omega \neq \emptyset$. Then any point of the set $\Sigma \cap \Omega$ never can be surrounded by any smooth closed $(n-1)$-dimensional surface $S \subset \bar{\Omega}$ such that $u|_S \in L_\infty(S)$. In particular, this means that

$$(3.6) \quad \mathcal{H}^1(\Sigma) > 0, \quad \Sigma \cap \partial \Omega \neq \emptyset,$$

where $\mathcal{H}^1$ is one-dimensional Hausdorff measure in $\mathbb{R}^n$.

Proof. The first assertion is clear. Let us prove (3.6). Assume $\Sigma \cap \Omega \neq \emptyset$ and $x_0 \in \Sigma \cap \Omega$. Denote $d := \text{dist}\{x_0, \partial \Omega\}$. Let $z_0 \in \partial \Omega$ be a point such that $|z_0 - x_0| = d$ and denote by $[x_0, z_0]$ the straight line segment connecting $x_0$ with $z_0$. Let us take arbitrary $\delta > 0$ and consider any countable covering of $\Sigma$ by open balls $\{B_{\rho_i}(y_i)\}$ such that $\rho_i \leq \delta$. For any $i$ denote $r_i := |x_0 - y_i|$. If $r_i \leq d$ then denote $z_i := [x_0, z_0] \cap \partial B_{r_i}(x_0)$. By Theorem 3.4 for any $r \leq d$ we have $\Sigma \cap \partial B_{r_i}(x_0) \neq \emptyset$. Therefore,

$$[x_0, z_0] \subset \bigcup_{r_i \leq d} B_{\rho_i}(z_i).$$

This inclusion means that

$$\mathcal{H}^1(\Sigma) \geq \mathcal{H}^1([x_0, z_0]) = d > 0.$$ 

□

Theorem 3.7 in particular implies that no isolated singularity is possible. This exactly what Example 3.6 demonstrates: the singular set in this case is the axis of symmetry.

Note that the divergence free condition brings significant improvements into the local boundedness results. Without the condition $\text{div } b = 0$ one can prove local boundedness of weak solutions to (1.1) only for $b \in L_n(\Omega)$ ($n \geq 3$), while if $\text{div } b = 0$ the local boundedness is valid for any $b \in L_p(\Omega)$ with $p > \frac{n}{2}$. Note also that for the moment of writing of this paper we can say nothing about analogues of neither Theorem 3.5 nor Example 3.6 if $p \in \left[\frac{n-1}{2}, \frac{n}{2}\right]$. We state this problem as an open question.
The final issue we need to discuss is the problem of further regularity of solutions to the equation (1.1). The example of a bounded weak solution which is not locally continuous was constructed originally in [SSSZ] for $n = 3$ and $b \in L_1(\Omega)$, div $b = 0$ (actually the method of [SSSZ] allowed to extend their example for $b \in L_p$, $p \in [1, 2]$). Later the first author in [F] generalized this example for all $n \geq 3$ and for all $p \in [1, n)$.

**Theorem 3.8.** Assume $n \geq 3$, $p < n$. Then there exist $b \in L_p(B)$ satisfying div $b = 0$ and a weak solution $u$ to (1.1) with $f \equiv 0$ such that $u \in W^1_2(B) \cap L_\infty(B)$ but $u \notin C(B_{1/2})$.

The latter result shows that if one is interested in the local continuity of weak solutions then the assumption $b \in L_n(\Omega)$ can not be weakened in the Lebesgue scale and the structure condition div $b = 0$ does not help in this situation.

It is not difficult to construct also a weak solution to (1.1) which is continuous but not Hölder continuous.

**Example 3.9.** Assume $n \geq 4$ and take

$$u(x) = \frac{1}{\ln r}, \quad b = \left(\frac{n - 3}{r} - \frac{2}{r \ln r}\right) e_r + \left(\frac{(n - 3)^2}{r^2} - \frac{2(n - 3)}{r^2 \ln r} - \frac{2}{r^2 \ln^2 r}\right) z e_z.$$  

Here $r^2 = x_1^2 + \ldots + x_{n-1}^2$, $z = x_n$, and $e_r$, $e_z$ are the basis vectors of the cylindrical coordinate system. Then $u \in W^1_2(B_{1/2}) \cap C(B_{1/2})$, $-\Delta u + b \cdot \nabla u = 0$, div $b = 0$ in $B_{1/2}$ and $b \in L_p(B_{1/2})$ for any $p < \frac{n-1}{2}$.

Thus, for weak solutions of (1.1) with $b \in L_2(\Omega)$, div $b = 0$, in large space dimensions (at least for $n \geq 6$) the following sequence of implications can break at any step:

$$u \in W^1_2(\Omega) \neq\Rightarrow u \in L_{\infty, loc}(\Omega) \neq\Rightarrow u \in C_{loc}(\Omega) \neq\Rightarrow u \in C^\alpha_{loc}(\Omega).$$

### 4. Beyond the $L_p$-scale

Theorem 3.8 shows that in order to obtain the local continuity of weak solutions to (1.1) for drifts weaker than $b \in L_n(\Omega)$ one needs to go beyond the Lebesgue scale.

We start with the question of the boundedness of the operator $T$ defined by the formula (2.2). The necessary and sufficient condition on $b$ is obtained in [MV] in the case $\Omega = \mathbb{R}^n$.

**Theorem 4.1.** The inequality (2.3) holds true if and only if the drift $b$ can be represented as a sum $b = b_0 + b_1$, where the function $b_0$ is such that

$$\int_{\mathbb{R}^n} |b_0|^2 |\nabla \eta|^2 \, dx \leq C \int_{\mathbb{R}^n} |\nabla \eta|^2 \, dx, \quad \forall \eta \in C^\infty_0(\mathbb{R}^n),$$

$b_1$ is divergence-free, div $b_1 = 0$, and $b_1 \in BMO^{-1}(\mathbb{R}^n)$. It means that $b_1(x) = \text{div } A(x)$, $A(x)$ is a skew-symmetric matrix, $A_{ij} = -A_{ji}$, and $A_{ij} \in BMO(\mathbb{R}^n)$.

Here $BMO(\Omega)$ is the space of functions $f$ with bounded mean oscillation, i.e.

$$\sup_{x \in \Omega} \frac{1}{r^n} \int_{B_r(x) \cap \Omega} |f(y) - (f)_{B_r(x) \cap \Omega}| \, dy < \infty,$$

where $(f)_\omega = \frac{1}{|\omega|} \int_{\omega} f(y) \, dy$.

Clearly, each divergence-free vector $b_1$ can be represented as $b_1 = \text{div } A$ with a skew-symmetric matrix $A(x)$. 

This Theorem mentions that the behaviour of the bilinear form \( \int_{\Omega} \nabla u \cdot b \eta \, dx \) already distinguish between general drifts and divergence-free drifts. First, let us discuss general drifts. If \( b \) satisfies (2.1) then it satisfies the estimate (4.1) too. But we can not use the condition (4.1) instead (2.1) for the regularity theory, as the example of Remark 2.10 shows. Indeed, for functions satisfying

\[
|b(x)| \leq \frac{C}{|x|}
\]

the estimate (4.1) is fulfilled by the Hardy inequality.

On the other hand, the case of the drift \( b \) having a one-point singularity (say, at the origin) with the asymptotics which includes homogeneous of degree \(-1\) functions like (4.2), is also interesting. There are several papers, see [LZ], [Sem], [SSSZ] and [NU], dealing with different classes of divergence-free drifts which cover (4.2). All these papers contain also the results for parabolic equation (1.8), but we discuss only (simplified) elliptic versions of them. We address the interested readers to the original papers.

The approach of [SSSZ] seems to be the most general one. Assume \( b \in BMO^{-1}(\Omega) \) and \( \text{div} b = 0 \). In this case we understand the equation \(-\Delta u + b \nabla u = 0\) in the sense of the integral identity

\[
\int_{\Omega} (\nabla u \cdot \nabla \eta + A \nabla u \cdot \nabla \eta) \, dx = 0 \quad \forall \, \eta \in C^\infty_0(\Omega),
\]

where the skew-symmetric matrix \( A \in BMO(\Omega) \) is defined via \( \text{div} A(x) = b(x) \).

**Theorem 4.2.** Let \( b \in BMO^{-1}(\Omega) \) and \( \text{div} b = 0 \). Then

1) The maximum principle holds. If \( u \in W^1_2(\Omega) \) satisfies (1.3) and \( \varphi := u|_{\partial\Omega} \) is bounded, then \( \|u\|_{L^\infty(\Omega)} \leq \|\varphi\|_{L^\infty(\partial\Omega)} \). In particular, the weak solution to (1.1) is unique.

2) Any weak solution \( u \) to (1.1) is Hölder continuous, \( u \in C^\alpha_{\text{loc}}(\Omega) \) for some \( \alpha > 0 \).

For the proof see [NU] or [SSSZ]. The regularity theory developed in Section 2 is slightly better as it guarantees that weak solutions are locally Hölder continuous with any exponent \( \alpha < 1 \). Nevertheless, Theorem 4.2 means that divergence-free drifts from \( BMO^{-1} \) can be also considered as regular ones.

**Remark 4.3.** Note that the case \( n = 2, b \in L_2(\Omega), \text{div} b = 0 \), is the particular case of this situation. Indeed, such drifts can be represented as a vector-function with components \( b_1 = \partial_2 h, b_2 = -\partial_1 h \), where \( h \) is a scalar function \( h \in W^1_2(\Omega) \). By the imbedding theorem \( W^1_2(\Omega) \subset BMO(\Omega) \) we have

\[
A(x) = \begin{pmatrix} 0 & -h(x) \\ h(x) & 0 \end{pmatrix} \in BMO(\Omega).
\]

5. Appendix

First we prove Theorem 2.9

**Proof.** We present the proof in the case \( n \geq 3 \) only. The case \( n = 2 \) differs from it by routine technical details.
1) The statement similar to our estimate (2.5) (for more general equations) can be found in [Su]. In particular, in [Su, Theorem 4.2] the following estimate for weak solutions to the problem

\[
\begin{cases}
-\Delta u + b \cdot \nabla u = f & \text{in } \Omega, \\
u|_{\partial \Omega} = 0,
\end{cases}
\]

was proved:

\[
\|u\|_{L_\infty(\Omega)} \leq C \left( \|f\|_{L_p(\Omega)} + \|u\|_{L_2(\Omega)} \right).
\]

On the other hand,

\[
\|u\|_{W^{1,2}_0(\Omega)} \leq C \|f\|_{W^{-1,2}_0(\Omega)}
\]
due to Theorem 2.8. Hence we can exclude the weak norm of \(u\) from the right hand side of (5.2) and obtain the estimate (2.5) in the case \(\varphi \equiv 0\). In general case we can split a weak solution \(u\) of the problem (1.6) as \(u = u_1 + u_2\), where \(u_1\) is a weak solution of (5.1) and \(u_2\) is a weak solution to the problem (1.6) with the boundary data \(\varphi\) and zero right hand side. For \(u_1\) we have (5.3) and for \(u_2\) we have

\[
\|u_2\|_{L_\infty(\Omega)} \leq \|\varphi\|_{L_\infty(\partial \Omega)} \quad \text{by Theorem 2.6.}
\]

2) As \(b \in L^n(\Omega)\) we can complete the integral identity (1.2) up to the test functions \(\eta \in W^{1,2}_0(\Omega)\). Denote \(k_0 := \|\varphi\|_{L_\infty(\partial \Omega)}\) and assume \(k \geq k_0\). Take in (1.2)

\[
\eta = (u - k)_+, \quad \text{where we denote } (u)_+ := \max\{u, 0\}.
\]
As \(k \geq k_0\) we have \(\eta \in W^{1,2}_0(\Omega)\) and \(\nabla \eta = \chi_{A_k} \nabla u\) where \(\chi_{A_k}\) is the characteristic function of the set

\[
A_k := \{ x \in \Omega : u(x) > k \}.
\]

We obtain the identity

\[
\int_{A_k} |\nabla u|^2 \, dx + \int_{A_k} b \cdot (u - k) \nabla u \, dx = \int_{A_k} f(u - k) \, dx.
\]

The second term vanishes

\[
\int_{A_k} b \cdot (u - k) \nabla u \, dx = \frac{1}{2} \int_{\Omega} b \cdot \nabla [(u - k)_+]^2 \, dx = 0,
\]
as \(\text{div } b = 0\), and hence

\[
\int_{A_k} |\nabla u|^2 \, dx = \int_{A_k} f(u - k) \, dx, \quad \forall k \geq k_0.
\]

The rest of the proof goes as in the usual elliptic theory. 
Applying the imbedding theorem we obtain

\[
\left( \int_{A_k} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \leq C(n) \left( \int_{A_k} [f]_{W^{0,2}}^{\frac{n+2}{n+2-2}} \, dx \right)^{\frac{n+2}{n+2-2}}
\]

and using the Hölder inequality we get

\[
\|f\|_{L_{\frac{n+2}{n+2-2}}^p(A_k)} \leq \|A_k\|^{\frac{n+2}{n+2-2} - \frac{1}{p}} \|f\|_{L_p(A_k)}.
\]
So we arrive at
\[
\int_{A_k} |\nabla u|^2 \, dx \leq C(n) \|f\|_{L_p(\Omega)}^2 |A_k|^{1-\frac{2}{pq}} , \quad \forall \, k \geq k_0,
\]
where \( \varepsilon := 2 \left( \frac{2}{n} - \frac{1}{p} \right) > 0 \). This inequality yields the following estimate, see \cite[Chapter II, Lemma 5.3]{LU},
\[
\text{esssup}_{\Omega} (u - k_0)_+ \leq C(n, p, \Omega) \|f\|_{L_p(\Omega)}.
\]
The estimate of \( \text{essinf}_{\Omega} u \) can be obtained in a similar way if we replace \( u \) by \( -u \).

In order to prove Theorem 5.1 we need some auxiliary results.

**Theorem 5.1.** Assume \( n \geq 3, \, b \in C^\infty(\bar{\Omega}), \, \text{div} \, b = 0 \) in \( \Omega, \, f \in L_1(\Omega), \) and assume \( u \in W^1_2(\Omega) \) is a weak solution of \( (1.6) \) with \( \varphi \equiv 0 \). Then for any \( q \in \left[ 1, \frac{n}{n-2} \right) \) the following estimate holds:
\[
(5.4) \quad \|u\|_{L_q(\Omega)} \leq C(n, q, \Omega) \|f\|_{L_1(\Omega)}.
\]

**Proof.** Assume \( q \in \left( 1, \frac{n}{n-2} \right) \). By duality we have
\[
\|u\|_{L_q(\Omega)} = \sup_{g \in L_{q'}(\Omega), \, \|g\|_{L_{q'}(\Omega)} \leq 1} \int_{\Omega} ug \, dx,
\]
where \( q' := \frac{n}{n-1}, \, q' > \frac{n}{2} \). For any \( g \in L_{q'}(\Omega) \) denote by \( w_g \in W^2_1(\Omega) \) a solution to the problem
\[
\begin{aligned}
-\Delta w_g - b \cdot \nabla w_g &= g \quad \text{in} \quad \Omega, \\
w_g|_{\partial \Omega} &= 0.
\end{aligned}
\]
From Theorem 2.3 we conclude that for \( w_g \) the following estimate holds:
\[
\|w_g\|_{L_{\infty}(\Omega)} \leq C(n, q, \Omega) \|g\|_{L_{q'}(\Omega)}.
\]
Integrating by parts we obtain
\[
\int_{\Omega} ug \, dx = \int_{\Omega} u (-\Delta w_g - b \cdot \nabla w_g) \, dx = \int_{\Omega} \nabla u \cdot (\nabla w_g + bw_g) \, dx = \int_{\Omega} f w_g \, dx.
\]
Then for any \( g \in L_{q'}(\Omega) \) such that \( \|g\|_{L_{q'}(\Omega)} \leq 1 \) we get
\[
\int_{\Omega} ug \, dx = \int_{\Omega} f w_g \, dx \leq \|f\|_{L_1(\Omega)} \|w_g\|_{L_{\infty}(\Omega)} \leq C(n, q, \Omega) \|f\|_{L_1(\Omega)}.
\]
Hence we obtain \( (5.4) \). \( \square \)

Another auxiliary result we need is the following extension theorem.

**Theorem 5.2.** Assume \( \Omega \subset \mathbb{R}^n \) is a bounded domain of class \( C^1 \). Then there exists a bounded linear extension operator \( T : L_\infty(\partial \Omega) \cap W^{1/2}_2(\partial \Omega) \to L_\infty(\Omega) \cap W^1_2(\Omega) \) such that
\[
T \varphi|_{\partial \Omega} = \varphi, \quad \forall \, \varphi \in L_\infty(\partial \Omega) \cap W^{1/2}_2(\partial \Omega),
\]
\[
\|T \varphi\|_{W^1_2(\Omega)} \leq C(\Omega) \|\varphi\|_{W^{1/2}_2(\partial \Omega)}, \quad \|T \varphi\|_{L_\infty(\Omega)} \leq C(\Omega) \|\varphi\|_{L_\infty(\partial \Omega)}.
\]
Proof. For the sake of completeness we briefly recall the proof of Theorem 5.2. After the localization and flattening of the boundary it is sufficient to construct the standard operator from $\mathbb{R}^{n-1}$ to $\mathbb{R}^n_+ := \mathbb{R}^{n-1} \times (0, +\infty)$. Then we can take the operator

$$(T\varphi)(x', x_n) = \eta(x_n) \int_{\mathbb{R}^{n-1}} \varphi(x' - x_n \xi') \psi(\xi') \, d\xi', \quad (x', x_n) \in \mathbb{R}^n_+,$$

where $x' := (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$, $\eta \in C_0^\infty(\mathbb{R})$, $\eta(0) = 1$, $\psi \in C_0^\infty(\mathbb{R}^{n-1})$, and $\int_{\mathbb{R}^{n-1}} \psi(\xi') \, d\xi' = 1$. This operator is bounded from $W^{1/2}_2(\mathbb{R}^{n-1})$ to $W^1_2(\mathbb{R}^n_+)$ and also from $L_\infty(\mathbb{R}^{n-1})$ to $L_\infty(\mathbb{R}^n_+)$. More details can be found in [31N].

Now we can give an elementary proof of Theorem 3.1.

Proof. The function $v_k := u_k - u \in \overset{\circ}{W}^1_2(\Omega)$ is a weak solution to the problem

$$\begin{cases}
-\Delta v_k + b_k \cdot \nabla v_k = f_k & \text{in } \Omega, \\
v_k|_{\partial \Omega} = 0,
\end{cases}$$

where $f_k := (b - b_k) \cdot \nabla u$, $f_k \in L_1(\Omega)$, $\|f_k\|_{L_1(\Omega)} \to 0$.

Assume $q \in [1, \frac{n}{n-2})$. By Theorem 5.1 we have

$$\|v_k\|_{L_q(\Omega)} \leq C(n, \Omega) \|f_k\|_{L_1(\Omega)} \to 0,$$

and hence (3.2) follows.

Now assume additionally $\varphi \in L_\infty(\partial \Omega)$. Denote $\tilde{\varphi} := T\varphi$ where $T$ is the extension operator from Theorem 5.2. Taking in the integral identity (1.4) for $u_k$ and $b_k$ the test function $\eta = u_k - \tilde{\varphi} \in \overset{\circ}{W}^1_2(\Omega)$ we obtain

$$\int_{\Omega} |\nabla u_k|^2 \, dx - \int_{\Omega} u_k b_k \cdot \nabla (u_k - \tilde{\varphi}) \, dx = \int_{\Omega} \nabla u_k \cdot \nabla \tilde{\varphi} \, dx + \langle f, u_k - \tilde{\varphi} \rangle.$$

Using the condition $\text{div} b_k = 0$ we get

$$\int_{\Omega} u_k b_k \cdot \nabla (u_k - \tilde{\varphi}) \, dx = \int_{\Omega} \tilde{\varphi} b_k \cdot \nabla (u_k - \tilde{\varphi}) \, dx.$$
Now we turn to the proof of Theorem 3.3.

**Proof.** We take a sequence \( b_k \in C^\infty(\overline{\Omega}) \), \( \text{div} b_k = 0 \), such that \( b_k \to b \) in \( L_2(\Omega) \). Let \( u_k \in W^1_2(\Omega) \) be a weak solution to the problem (3.1). Repeating the arguments in the proof of Theorem 3.1, we obtain the estimate (5.5) with a constant \( C \) independent on \( k \). Using this estimate we can extract a subsequence satisfying (3.3) for some \( u \in W^1_2(\Omega) \). The weak convergence (3.3) and the strong convergence \( b_k \to b \) in \( L_2(\Omega) \) allow us to pass to the limit in the integral identities (1.2) corresponding to \( u_k \) and \( b_k \). Therefore, \( u \) is a weak solution to (1.6). \( \square \)

Now we present the proof of Theorem 3.4.

**Proof.** Let \( b_k \) be smooth divergence-free vector fields such that \( b_k \to b \) in \( L_2(\Omega) \). Denote by \( u_k \) the weak solution to the problem (3.1). By Theorem 2.9 (5.6)
\[
\| u_k \|_{L_\infty(\Omega)} \leq \| \varphi \|_{L_\infty(\partial \Omega)} + C \| f \|_{L_p(\Omega)}
\]
with the constant \( C \) depending only on \( n, p \) and \( \Omega \). From Theorem 3.1 we have the convergence \( u_k \to u \) in \( L_1(\Omega) \) and hence we can extract a subsequence (for which we keep the same notation) such that \( u_k \to u \) a.e. in \( \Omega \). Passing to the limit in (5.6) we obtain (3.5). \( \square \)

Finally we give the proof of Theorem 3.5.

**Proof.** To simplify the presentation we give the proof only in the case \( f \equiv 0 \). The extension of the result for non-zero right hand side can be done by standard methods, see [HL, Theorem 4.1] or [FSh]. First we derive the estimate (5.7)
\[
\| u \|_{L_\infty(B_{1/2})} \leq C \left( 1 + \| b \|_{L_p(B)} \right)^\mu \| u \|_{L_{2p'}(B)}, \quad p' := \frac{p}{p-1}
\]
(with some positive constants \( C \) and \( \mu \) depending only on \( n \) and \( p \)) under additional assumption \( u \in C^\infty(B) \). We explore Moser’s iteration technique, see [Mo]. Assume \( \beta \geq 0 \) is arbitrary and let \( \zeta \in C_0^\infty(B) \) be a cut-off function. Take a test function \( \eta = \zeta^2 |u|^\beta \) in the identity (1.2). Denote \( w := |u|^{2+\beta} \). Then after integration by parts and some routine calculations we obtain the inequality (5.8)
\[
\int_B |\nabla (\zeta w)|^2 \, dx \leq C \int_B |w|^2 \left( |\nabla \zeta|^2 + |b| |\nabla \zeta| \right) \, dx
\]
Applying the imbedding theorem and the Hölder inequality and choosing the test function \( \zeta \) in an appropriate way we arrive at the inequality
\[
\| w \|_{L_\frac{2n}{n-2}(B_r)} \leq C \left( \frac{1}{R-r} + \| b \|_{L_p(B_R)} \right) \| w \|_{L_{2p'}(B_r)},
\]
which holds for any \( \frac{1}{2} \leq r < R \leq 1 \). Note that \( \frac{2n}{n-2} > 2p' \) as \( p > \frac{n}{2} \) if \( n \geq 4 \) and \( p = 2 \) if \( n = 3 \). The latter inequality gives us the estimate (5.9)
\[
\| u \|_{L_{\frac{2n}{n-2}}(B_r)} \leq C^{\frac{2}{\beta}} \left( \frac{1}{R-r} + \| b \|_{L_p(B_R)} \right) \| u \|_{L_{2p'}(B_R)}
\]
with an arbitrary \( \gamma \geq 2, \gamma := \beta + 2 \). Denote \( s_0 = 2p', s_m := \chi s_m - 1, \) where 
\( \chi := \frac{n(p-1)}{p(n-2)}, \) and denote also \( R_m = \frac{1}{2} + \frac{1}{2m+1} \). Taking in \( (5.9) \) \( r = R_m, R = R_{m-1}, \gamma = \frac{2m+1}{p} \) we obtain
\[
\|u\|_{L_{sm}(B_{R_m})} \leq \left( C \frac{2m+1}{2} + C\|b\|_{L_p(B)} \right) \frac{1}{\chi^{m+1}} \|u\|_{L_{s_m+1}(B_{R_{m-1}})}
\]
Iterating this inequality we arrive at \((5.7)\).

Now we need to get rid of the assumption \( u \in C^\infty(B) \). Assume \( u \in W^1_2(B) \) is an arbitrary weak solution to \((1.1)\). Let \( \zeta \in C^\infty_0(B) \) be a cut-off function such that \( \zeta \equiv 1 \) on \( B_{3/6} \) and denote \( v := \zeta u \). Then \( v \) is a weak solution to the boundary value problem
\[
\begin{cases}
-\Delta v + b \cdot \nabla v = g & \text{in } B \\
v_{|\partial B} = 0
\end{cases}
\]
where
\[
g := -u\Delta \zeta - 2\nabla u \cdot \nabla \zeta + bu \cdot \nabla \zeta.
\]
Note that \( g \equiv 0 \) and \( v \equiv u \) on \( B_{3/6} \). As \( b \in L_p(B) \) with \( p > \frac{2}{3} \) we have \( g \in W^{1-1}(B) \).

Now we take a sequence \( b_k \in C^\infty(B) \), \( \text{div } b_k = 0 \), such that \( b_k \rightarrow b \) in \( L_{p^*}(B) \) and let \( v_k \) be the weak solution to the problem
\[
\begin{cases}
-\Delta v_k + b_k \cdot \nabla v_k = g & \text{in } B \\
v_k_{|\partial B} = 0
\end{cases}
\]
From Theorem \(3.1\) we have \( v_k \rightharpoonup v \) in \( W^{1}_2(B) \) and as \( p > \frac{2}{3} \) we can extract a subsequence (for which we keep the same notation) such that \( v_k \rightarrow v \) a.e. in \( B \) and \( v_k \rightarrow v \) in \( L_{2p^*}(B) \). As \( g \equiv 0 \) on \( B_{3/6} \) from the usual elliptic theory (see \( [L,U] \)) we conclude that \( v_k \in C^\infty(B_{3/6}) \). Applying \( (5.7) \) (with the obvious modification in radius) we obtain the estimate
\[
\|v_k\|_{L_{u}(B_{1/2})} \leq C (1 + \|b_k\|_{L_{p}(B)}) \|v_k\|_{L_{2p^*}(B_{3/4})}.
\]
Hence \( v_k \) are equibounded on \( B_{1/2} \). Passing to the limit in the above inequality and taking into account that \( v = u \) on \( B_{3/6} \) we obtain
\[
\|u\|_{L_{u}(B_{1/2})} \leq C (1 + \|b\|_{L_{p}(B)}) \|u\|_{L_{2p^*}(B_{3/4})}.
\]
To conclude the proof we remark that for \( p > \frac{2}{3} \) from the imbedding theorem we have
\[
\|u\|_{L_{2p^*}(B)} \leq C(n,p) \|u\|_{W^1_2(B)}.
\]
\( \square \)

References

[A] D. G. Aronson, Non-negative solutions of linear parabolic equations, Ann. Scuola Norm. Sup. Pisa, 22 (1968), pp. 607-694.

[BIN] O. V. Besov, V. P. Il’in, S. M. Nikol’skii, Integral representations of functions and imbedding theorems, Moscow, 1975.

[F] N. Filonov, On the regularity of solutions to the equation \(-\Delta u + b \cdot \nabla u = 0\), Zap. Nauchn. Sem. of Steklov Inst. 410 (2013), 168-186; reprinted in J. Math. Sci. (N.Y.) 195 (2013), no. 1, 98-108.

[FSh] N. Filonov, T. Shilkin, On the local boundedness of weak solutions to elliptic equations with divergence-free drifts, Preprint 2714, Technische Universität Darmstadt, 2017.

[HL] Q. Han, F. H. Lin, Elliptic partial differential equations, Courant Lecture Notes in Mathematics, AMS, 1997.
[LU] O. A. Ladyzhenskaya, N. N. Uraltseva, *Linear and quasilinear equations of elliptic type*, Academic Press, 1968.

[LZ] V. Liskevich, Q. S. Zhang, *Extra regularity for parabolic equations with drift terms*, Manuscripta Math. **113** (2004), no. 2, 191-209.

[M] V. G. Mazja, *Sobolev Spaces*, Springer, 1985.

[MV] V. G. Mazja, I. E. Verbitskiy, *Form boundedness of the general second-order differential operator*, Comm. Pure Appl. Math. **59** (2006), 1286-1329.

[Mo] J. Moser, *A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations*, Comm. Pure and Appl. Math., **13** (1960), no. 3, pp. 457-468.

[NU] A. I. Nazarov, N. N. Uraltseva, *The Harnack inequality and related properties of solutions of elliptic and parabolic equations with divergence-free lower-order coefficients*, St. Petersburg Math. J. **23** (2012), no. 1, 93-115.

[Sem] Y. A. Semenov, *Regularity theorems for parabolic equations*, J. Funct. Anal. **231** (2006), no. 2, 375-417.

[SSSZ] G. Seregin, L. Silvestre, V. Sverak, A. Zlatos, *On divergence-free drifts*, J. Differential Equations **252** (2012), no. 1, 505-540.

[SV] L. Silvestre, V. Vicol, *Hölder continuity for a drift-diffusion equation with pressure*, Ann. Inst. H. Poincare Anal. Non Lineaire **29** (2012), no. 4, 637-652.

[SVZ] L. Silvestre, V. Vicol, A. Zlatos, *On the loss of continuity for super-critical drift-diffusion equations*, Arch. Ration. Mech. Anal. **207** (2013), no. 3, 845-877.

[St] G. Stampacchia, *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus. (French)* Ann. Inst. Fourier (Grenoble) **15** (1965) fasc. 1, 189-258.

[Su] M. D. Surnachev, *On the uniqueness of a solution to a stationary convection-diffusion equation with a generalized divergence-free drift*, [arXiv:1706.00389](https://arxiv.org/abs/1706.00389), 2017.

[Z] Q. S. Zhang, *A strong regularity result for parabolic equations*, Commun. Math. Phys., **244** (2004), no. 2, pp. 245-260.

[Zhi] V. V. Zhikov, *Remarks on the uniqueness of the solution of the Dirichlet problem for a second-order elliptic equation with lower order terms*, Funct. Anal. Appl. **38** (2004), no. 3, 173-183.

V.A. Steklov Mathematical Institute, St.-Petersburg, Fontanka 27, 191023, Russia 
E-mail address: filonov@pdmi.ras.ru

V.A. Steklov Mathematical Institute, St.-Petersburg, Fontanka 27, 191023, Russia 
E-mail address: shilkin@pdmi.ras.ru