CONFORMAL CHANGE OF SPECIAL FINSLER SPACES

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Abstract. The present paper is a continuation of a foregoing paper [Tensor, N. S., 69 (2008), 155-178]. The main aim is to establish an intrinsic investigation of the conformal change of the most important special Finsler spaces, namely, $C^h$-recurrent, $C''$-recurrent, $C^0$-recurrent, $C_2$-like, quasi-$C$-reducible, $C$-reducible, Berwald space, $S^v$-recurrent, $P^*$-Finsler manifold, $R_3$-like, $P$-symmetric, Finsler manifold of $p$-scalar curvature and Finsler manifold of $s$-$ps$-curvature. Necessary and sufficient conditions for such special Finsler manifolds to be invariant under a conformal change are obtained. Moreover, the conformal change of Chern and Hashiguchi connections, as well as their curvature tensors, are given.

Keywords. Conformal change, $C^h$-recurrent, $C''$-recurrent, $C^0$-recurrent, $C_2$-like, quasi-$C$-reducible, $C$-reducible, Berwald space, $S^v$-recurrent, $P^*$-Finsler manifold, $R_3$-like, $P$-symmetric, Chern connection, Hashiguchi connection.

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Introduction

Studying Finsler geometry one encounters substantial difficulties trying to seek analogues of classical global, or sometimes even local, results of Riemannian geometry. These difficulties arise mainly from the fact that in Finsler geometry all geometric objects depend not only on positional coordinates, as in Riemannian geometry, but also on directional arguments.

The infinitesimal transformations in Riemannian and Finsler geometries are important, not only in differential geometry, but also in application to other branches of science, especially in the process of geometrization of physical theories.

The theory of conformal changes in Riemannian geometry has been deeply studied (locally and intrinsically) by many authors. As regards to Finsler geometry, an almost complete local theory of conformal changes has been established ([1], [6], [7], [8], [9], [11], [12], · · · , etc.).

In [13], we investigated intrinsically conformal changes in Finsler geometry, where we got, among other results, a characterization of conformal changes. Also the conformal change of Barthel connection and its curvature tensor were studied. Moreover, the conformal changes of Cartan and Berwald connections as well as their curvature tensors, were obtained.

The present paper is a continuation of [14] where we present an intrinsic theory of conformal changes of special Finsler spaces. Moreover, we study the conformal change of Chern and Hashiguchi connections.

The paper consists of two parts preceded by an introductory section (§1), which provides a brief account of the basic definitions and concepts necessary for this work.

In the first part (§2), the conformal change of Chern and Hashiguchi connections, as well as their curvature tensors, are given.

In the second part (§3), we provide an intrinsic investigation of the conformal change of the most important special Finsler spaces, namely, $C^h$-recurrent, $C^s$-recurrent, $C^0$-recurrent, $S^v$-recurrent, $P^*$-manifold, $R_3$-like, Finsler manifold of $p$-scalar curvature and of $s$-$ps$-curvature, · · · , etc. Moreover, we obtain necessary and sufficient conditions for such special Finsler manifolds to be invariant under a conformal change.

Finally, it should be noted that the present work is formulated in a prospective modern coordinate-free form, without being trapped into the complications of indices. However, some important results of [6], [8], [9] and others (obtained in local coordinates) are immediately derived from the obtained global results (when localized).

1. Notation and Preliminaries

In this section, we give a brief account of the basic concepts of the pullback approach to global Finsler geometry necessary for this work. For more detail, we refer to [2], [3] and [13]. We assume that all geometric objects treated are of class $C^\infty$. The following notations are to be used throughout this paper:

- $M$: a real paracompact differentiable manifold of finite dimension $n$ and of class $C^\infty$,
- $\mathcal{F}(M)$: the $\mathbb{R}$-algebra of differentiable functions on $M$,
- $\mathcal{X}(M)$: the $\mathcal{F}(M)$-module of vector fields on $M$,
- $\pi_M : TM \longrightarrow M$: the tangent bundle of $M$. 


\[ \pi^*_M : T^*M \rightarrow M \]: the cotangent bundle of \( M \),
\[ \pi : TM \rightarrow M \]: the subbundle of nonzero vectors tangent to \( M \),
\[ V(TM) \]: the vertical subbundle of the bundle \( TTM \),
\[ P : \pi^{-1}(TM) \rightarrow TM \]: the pullback of the tangent bundle \( TM \) by \( \pi \),
\[ P^* : \pi^{-1}(T^*M) \rightarrow T^*M \]: the pullback of the cotangent bundle \( T^*M \) by \( \pi \),
\[ \mathcal{X}(\pi(M)) \]: the \( \mathcal{F}(TM) \)-module of differentiable sections of \( \pi^{-1}(TM) \),
\[ \mathcal{X}^*(\pi(M)) \]: the \( \mathcal{F}(TM) \)-module of differentiable sections of \( \pi^{-1}(T^*M) \),
\[ i_X \]: the interior product with respect to \( X \in \mathcal{X}(M) \),
\[ df \]: the exterior derivative of \( f \in \mathcal{F}(M) \),
\[ d_L := [i_L, d] \), \( i_L \) being the interior derivative with respect to a vector form \( L \).

Elements of \( \mathcal{X}(\pi(M)) \) will be called \( \pi \)-vector fields and will be denoted by barred letters \( \overline{X} \). Tensor fields on \( \pi^{-1}(TM) \) will be called \( \pi \)-tensor fields. The fundamental \( \pi \)-vector field is the \( \pi \)-vector field \( \overline{\eta} \) defined by \( \overline{\eta}(u) = (u, u) \) for all \( u \in TM \).

We have the following short exact sequence of vector bundles
\[ 0 \rightarrow \pi^{-1}(TM) \xrightarrow{\gamma} T(TM) \xrightarrow{\rho} \pi^{-1}(TM) \rightarrow 0, \]
where the bundle morphisms \( \rho \) and \( \gamma \) are defined respectively by \( \rho := (\pi_{TM}, d\pi) \) and \( \gamma(u, v) := j_u(v) \), \( j_u \) being the natural isomorphism \( j_u : T_{\pi_M(u)}M \rightarrow T_u(T_{\pi_M(u)}M) \). The vector 1-form \( J \) on \( TM \) defined by \( J := \gamma \circ \rho \) is called the natural almost tangent structure of \( TM \). The vertical vector field \( C \) on \( TM \) defined by \( C := \gamma \circ \overline{\eta} \) is called the canonical or Liouville vector field.

Let \( D \) be a linear connection (or simply a connection) on the pullback bundle \( \pi^{-1}(TM) \). The map
\[ K : TTM \rightarrow \pi^{-1}(TM) : X \mapsto D_X \overline{\eta} \]
is called the connection map or the deflection map associated with \( D \). A tangent vector \( X \in T_u(TM) \) is said to be horizontal if \( K(X) = 0 \). The vector space \( H_u(TM) \) of the horizontal vectors at \( u \in TM \) is called the horizontal space of \( M \) at \( u \). The connection \( D \) is said to be regular if
\[ T_u(TM) = V_u(TM) \oplus H_u(TM) \quad \forall u \in TM. \tag{1.1} \]

If \( M \) is endowed with a regular connection \( D \), then the maps
\[ \gamma : \pi^{-1}(TM) \rightarrow V(TM), \]
\[ \rho|_{H(TM)} : H(TM) \rightarrow \pi^{-1}(TM), \]
\[ K|_{V(TM)} : V(TM) \rightarrow \pi^{-1}(TM) \]
are vector bundle isomorphisms. Let \( \beta := (\rho|_{H(TM)})^{-1} \), called the horizontal map associated with \( D \), then
\[ \rho \circ \beta = \text{id}_{\pi^{-1}(TM)}, \quad \beta \circ \rho = \begin{cases} \text{id}_{H(TM)} & \text{on } H(TM) \\ 0 & \text{on } V(TM) \end{cases} \tag{1.2} \]

The (classical) torsion tensor \( T \) of the connection \( D \) is given by
\[ T(X,Y) = D_X \rho Y - D_Y \rho X - \rho[X,Y] \quad \forall X,Y \in \mathcal{X}(TM), \]
from which the horizontal or (h)h-torsion tensor \( Q \) and the mixed or (h)hv-torsion tensor \( T \) are defined respectively by
\( Q(\overline{X}, \overline{Y}) = T(\beta \overline{X} \beta \overline{Y}), \quad T(\overline{X}, \overline{Y}) = T(\gamma \overline{X}, \beta \overline{Y}) \quad \forall \overline{X}, \overline{Y} \in \pi(M). \)

The (classical) curvature tensor \( K \) of the connection \( D \) is given by

\[
K(X, Y)\rho Z = -D_X D_Y \rho Z + D_Y D_X \rho Z + D_{[X,Y]}\rho Z \quad \forall X, Y, Z \in \pi(TM),
\]

from which the horizontal (h-), mixed (hv-) and vertical (v-) curvature tensors, denoted by \( R, P \) and \( S \) respectively, are defined by

\[
R(\overline{X}, \overline{Y})Z = K(\beta \overline{X} \beta \overline{Y})Z, \quad P(\overline{X}, \overline{Y})Z = K(\beta \overline{X}, \gamma \overline{Y})Z, \quad S(\overline{X}, \overline{Y})Z = K(\gamma \overline{X}, \gamma \overline{Y})Z.
\]

The contracted curvature tensors \( \hat{R}, \hat{P} \) and \( \hat{S} \), also known as the (v)h-, (v)hv- and (v)v-torsion tensors, are defined by

\[
\hat{R}(\overline{X}, \overline{Y}) = R(\overline{X}, \overline{Y})\eta, \quad \hat{P}(\overline{X}, \overline{Y}) = P(\overline{X}, \overline{Y})\eta, \quad \hat{S}(\overline{X}, \overline{Y}) = S(\overline{X}, \overline{Y})\eta.
\]

If \( M \) is endowed with a metric \( g \) on \( \pi^{-1}(TM) \), we write

\[
R(\overline{X}, \overline{Y}, Z, W) := g(R(\overline{X}, \overline{Y})Z, W), \quad \cdots, \quad S(\overline{X}, \overline{Y}, Z, W) := g(S(\overline{X}, \overline{Y})Z, W). \tag{1.3}
\]

On a Finsler manifold \((M, L)\), there are canonically associated four linear connections on \( \pi^{-1}(TM) \): the Cartan connection \( \nabla \), the Chern (Rund) connection \( D^c \), the Hashiguchi connection \( D^h \) and the Berwald connection \( D^v \). Each of these connections is regular with (h)hv-torsion \( T \) satisfying \( T(\phi X, \phi \eta) = 0 \).

**Definition 1.1.** Let \((M, L)\) be a Finsler manifold and \( g \) the Finsler metric defined by \( L \). Let \( T \) be the (h)hv-torsion tensor and \( S, P, R \) are the v-, hv- and h-curvature tensors associated with the Cartan connection \( \nabla \). We define

\[
\ell(X) := L^{-1}g(X, \eta), \quad h := g - \ell \otimes \ell : the \ angular \ metric \ tensor,
\]

\[
T(\overline{X}, \overline{Y}, \overline{Z}) := g(T(\overline{X}, \overline{Y}), \overline{Z}) : the \ Cartan \ tensor,
\]

\[
C(\overline{X}) := Tr\{\overline{Y} \mapsto T(\overline{X}, \overline{Y})\} : the \ contracted \ torsion,
\]

\[
g(C, \overline{X}) := C(\overline{X}) : C \ is \ the \ \pi-vector \ field \ associated \ with \ the \ \pi-form \ C, \ by \ duality,
\]

\[
Ric^c(\overline{X}, \overline{Y}) := Tr\{\overline{Z} \mapsto S(\overline{X}, \overline{Z})\overline{Y}\} : the \ vertical \ Ricci \ tensor,
\]

\[
Ric^h(\overline{X}, \overline{Y}) := Tr\{\overline{Z} \mapsto R(\overline{X}, \overline{Z})\overline{Y}\} : the \ horizontal \ Ricci \ tensor,
\]

\[
g(Ric^c_0(\overline{X}), \overline{Y}) := Ric^v(\overline{X}, \overline{Y}) : the \ vertical \ Ricci \ map \ Ric^c_0,
\]

\[
g(Ric^h_0(\overline{X}), \overline{Y}) := Ric^h(\overline{X}, \overline{Y}) : the \ horizontal \ Ricci \ map \ Ric^h_0,
\]

\[
Sc^v := Tr\{\phi X \mapsto Ric^c_0(\phi X)\} : the \ vertical \ scalar \ curvature,
\]

\[
Sc^h := Tr\{\phi X \mapsto Ric^h_0(\phi X)\} : the \ horizontal \ scalar \ curvatures.
\]

The following two results \([17]\) give an explicit expression for each of the Berwald, Chern and Hashiguchi connections in terms of the Cartan connection \( \nabla \).

**Theorem 1.2.** The Chern connection \( D^c \) is given, in terms of Cartan connection, by

\[
D^c_X \overline{Y} = \nabla_X \overline{Y} - T(KX, \overline{Y}) = D^c_X \overline{Y} - \hat{P}(\rho X, \overline{Y}).
\]

In particular, we have

(a) \( D^c_{\gamma X} \overline{Y} = \nabla_{\gamma X} \overline{Y} - T(X, \overline{Y}) = D^c_{\gamma X} \overline{Y}. \)
Theorem 1.3. The Hashiguchi connection $D^*$ is given, in terms of Cartan connection, by

$$D^*_X Y = \nabla_X Y + \widehat{P}(\rho X, Y) = D^*_X Y + T(KX, Y).$$

In particular, we have

(a) $D^*_X Y = \nabla_X Y = D^*_X Y + T(X, Y)$.

(b) $D^*_X Y = \nabla_X Y + \widehat{P}(X, Y) = D^*_X Y$.

Now, we give some concepts and results concerning the Klein-Grifone approach to intrinsic Finsler geometry. For more details, we refer to [4], [5] and [10].

Proposition 1.4. Let $(M, L)$ be a Finsler manifold. The vector field $G$ on $TM$ defined by $i_G \Omega = -dE$ is a spray, where $E := \frac{1}{2}L^2$ is the energy function and $\Omega := dd^cE$. Such a spray is called the canonical spray.

A nonlinear connection on $M$ is a vector 1-form $\Gamma$ on $TM$, $C^\infty$ on $TM$, such that $J\Gamma = J$, $\Gamma J = -J$. The horizontal and vertical projectors associated with $\Gamma$ are defined by $h := \frac{1}{2}(I + \Gamma)$ and $v := \frac{1}{2}(I - \Gamma)$ respectively. The torsion and curvature of $\Gamma$ are defined by $t := \frac{1}{2}[J, \Gamma]$ and $R := \frac{1}{2}[h, v]$ respectively. A nonlinear connection $\Gamma$ is homogenous if $[\mathcal{C}, \Gamma] = 0$. It is conservative if $d_hE = 0$.

Theorem 1.5. On a Finsler manifold $(M, L)$, there exists a unique conservative homogeneous nonlinear connection with zero torsion. It is given by:

$$\Gamma = [J, G],$$

where $G$ is the canonical spray. Such a nonlinear connection is called the canonical connection, the Cartan nonlinear connection or the Barthel connection associated with $(M, L)$.

It can be proved [17] that the nonlinear connection associated with each of the four canonical linear connections coincide with the Barthel connection.

We terminate this section by the following fact. Under an arbitrary change $L \rightarrow \tilde{L}$ of Finsler structures on $M$, let the corresponding Cartan connections $\nabla$ and $\tilde{\nabla}$ be related by

$$\tilde{\nabla}_X Y = \nabla_X Y + \omega(X, Y).$$

If we denote

$$A(X, Y) := \omega(\gamma X, Y), \quad B(X, Y) := \omega(\beta X, Y), \quad N(X) := B(X, \eta), \quad N_o := N(\eta),$$

then we have

Proposition 1.6. [14] Under an arbitrary change $L \rightarrow \tilde{L}$ of Finsler structures on $M$, the corresponding Barthel connections $\Gamma$ and $\tilde{\Gamma}$ are related by

$$\tilde{\Gamma} = \Gamma - 2L, \text{ with } L := \gamma o N o \rho.$$

Moreover, we have $\tilde{h} = h - L$, $\tilde{v} = v + L$. 


2. Conformal change of the fundamental regular connections and their curvature tensors

In this section, we first review some concepts and results concerning the conformal changes of the Cartan and Berwald connections [14]. Then, using these results, the conformal changes of Chern and Hashiguchi connections, as well as their curvature tensors, are investigated.

Definition 2.1. Let \((M, L)\) and \((M, \tilde{L})\) be two Finsler manifolds. The two associated metrics \(g\) and \(\tilde{g}\) are said to be conformal if there exists a positive differentiable function \(\sigma(x)\) such that \(\tilde{g}(\tilde{X}, \tilde{Y}) = e^{2\sigma(x)}g(X, Y)\). Equivalently, \(g\) and \(\tilde{g}\) are conformal iff \(L^2 = e^{2\sigma(x)}L\). In this case, the transformation \(L \rightarrow \tilde{L}\) is said to be a conformal transformation and the two Finsler manifold \((M, L)\) and \((M, \tilde{L})\) are said to be conformal or conformally related.

Definition 2.2. Let \((M, L)\) and \((M, \tilde{L})\) be two conformal Finsler manifolds with \(\tilde{g} = e^{2\sigma(x)}g\).

(a) A geometric object \(W\) is said to be conformally invariant (resp. conformally \(\sigma\)-invariant) if \(\tilde{W} = W\) (resp. \(\tilde{W} = e^{2\sigma(x)}W\)).

(b) A property \(\xi\) is said to be a conformal invariant property if whenever it is possessed by \((M, L)\), it is also possessed by \((M, \tilde{L})\).

Definition 2.3. The vertical gradient of a function \(f \in \mathfrak{X}(TM)\), denoted \(\operatorname{grad}_v f\), is the vertical vector field \(J X\) defined by

\[
\operatorname{grad}_v f(Y) = \tilde{g}(J X, J Y), \quad \text{for all } Y \in \mathfrak{X}(TM),
\]

where \(\tilde{g}\) is the metric on \(V(TM)\) defined in [4].

\[
\tilde{g}(J Y, J Z) = \Omega(J Y, Z), \quad \text{for all } Y, Z \in \mathfrak{X}(TM).
\]

Lemma 2.4. Let \((M, L)\) and \((M, \tilde{L})\) be conformally related Finsler manifolds with \(\tilde{g} = e^{2\sigma(x)}g\). The associated Barthel connections \(\Gamma\) and \(\tilde{\Gamma}\) are related by

\[
\tilde{\Gamma} = \Gamma - 2L,
\]

where

\[
L := d\sigma \otimes C + \sigma J - dJ E \otimes \operatorname{grad}_v \sigma - EF = \gamma \circ \omega p,
\]

\[
\sigma_1 := dG \sigma \quad \text{and} \quad F := [J, \operatorname{grad}_v \sigma].
\]

Consequently, \(\tilde{h} = h - L, \quad \tilde{v} = v + L\) or, equivalently, \(\tilde{\beta} = \beta - L \circ \beta, \quad \tilde{K} = K + K \circ L\).

Concerning the conformal change of the Cartan and Berwald connections and their curvature tensors, we have the following two results [14].

Theorem 2.5. If \((M, L)\) and \((M, \tilde{L})\) are conformally related Finsler manifolds, then the associated Cartan connections \(\nabla\) and \(\tilde{\nabla}\) are related by:

\[
\tilde{\nabla}_X \tilde{Y} = \nabla_X \tilde{Y} + \omega(X, \tilde{Y}),
\]

where

\[
\omega(X, \tilde{Y}) := (h X \cdot \sigma(x)) \tilde{Y} + (\beta Y \cdot \sigma(x)) \rho X - g(\rho X, \tilde{Y}) \tilde{F} - T(N \tilde{Y}, \rho X) + T'(L X, \beta Y),
\]
\( \bar{P} \) being a \( \pi \)-vector field defined by
\[
g(\bar{P}, \rho Z) = hZ \cdot \sigma(x)
\]
and \( T' \) being a 2-form on \( TM \), with values in \( \pi^{-1}(TM) \), defined by
\[
g(T'(LX, hY), \rho Z) = g(T(N\rho Z, \rho Y), \rho X).
\]
In particular,
(a) \( \tilde{\nabla}_{\gamma X} \bar{Y} = \nabla_{\gamma X} \bar{Y} \),
(b) \( \tilde{\nabla}_{\beta X} \bar{Y} = \nabla_{\beta X} \bar{Y} - \nabla_{L_{\beta X}} \bar{Y} + B(\bar{X}, \bar{Y}) \).
The associated curvature tensors are related by:
(a) \( \tilde{\mathcal{S}}(X, Y)Z = \mathcal{S}(X, Y)Z \),
(b) \( \tilde{\mathcal{P}}(X, Y)Z = P(X, Y)Z + V(X, Y)Z \),
(c) \( \tilde{\mathcal{R}}(X, Y)Z = R(X, Y)Z + H(X, Y)Z \),
where \( H \) and \( V \) are the \( \pi \)-tensor fields defined by
\[
V(X, Y)Z = (\nabla_{\gamma Y}B)(X, Z) + B(T(Y, X), Z) - S(NX, \rho Y)Z,
\]
\[
H(X, Y)Z = S(NX, \rho Y)Z - \mathcal{L}_{\rho Y}(((\nabla_{\beta X}B)(Y, Z) - (\nabla_{L_{\beta X}}B)(Y, Z))
+ P(NX, \rho Y)Z - B(X, B(Y, Z)) - B(T(NX, \rho Y), Z));
\]
\( B \) being defined by \((1.4)\) and \( \mathcal{L} \) by \((2.1)\).

**Theorem 2.6.** If \((M, L)\) and \((M, \tilde{L})\) are conformally related Finsler manifolds, then the associated Berwald connections \( D^\circ \) and \( \tilde{D}^\circ \) are related by:
\[
\tilde{D}^\circ_X \bar{Y} = D^\circ_X \bar{Y} + \omega^\circ(X, Y),
\]
where \( \omega^\circ(X, Y) = K([\gamma Y, L]X) + D^\circ_{LX} \bar{Y} \).
In particular, we have
(a) \( \tilde{D}^\circ_{\gamma X} \bar{Y} = D^\circ_{\gamma X} \bar{Y} \)
(b) \( \tilde{D}^\circ_{\beta X} \bar{Y} = D^\circ_{\beta X} \bar{Y} - D^\circ_{NX\beta X} \rho Y + B^\circ(\rho X, \rho Y) \).
The associated curvature tensors are related by:
(a) \( \tilde{\mathcal{S}}^\circ(X, Y)Z = \mathcal{S}^\circ(X, Y)Z = 0 \).
(b) \( \tilde{\mathcal{P}}^\circ(X, Y)Z = P^\circ(X, Y)Z + (D^\circ_{\gamma Y}B^\circ)(X, Z) \).
(c) \( \tilde{\mathcal{R}}^\circ(X, Y)Z = R^\circ(X, Y)Z + \mathcal{L}_{\rho Y}(((D^\circ_{\gamma X}B^\circ)(Y, Z) - (D^\circ_{\beta X}B^\circ)(Y, Z)
+ P^\circ(Y, NX)Z - B^\circ(X, B^\circ(Y, Z))));
\]
where \( B^\circ(\rho X, \rho Y) := \omega^\circ(\beta X, \bar{Y}) \).

Now, we turn our attention to the Chern and Hashigauchi connections.
Theorem 2.7. Let \((M, L)\) and \((\tilde{M}, \tilde{L})\) be conformally related Finsler manifolds with \(\tilde{g} = e^{2\sigma(x)}g\). The associated Chern connections \(D^c\) and \(\tilde{D}^c\) are related by
\[
\tilde{D}^c_XY = D^c_XY + \omega^c(X, Y),
\]
where
\[
\omega^c(X, Y) := (hX \cdot \sigma(x))Y + (\beta Y \cdot \sigma(x))\rho X - g(\rho X, \tilde{Y})\tilde{P} - T(N\tilde{Y}, \rho X) + T'(LX, \beta Y) - T(N\rho X, \tilde{Y}),
\]
In particular, we have
(a) \(\tilde{D}^c_{\gamma X}Y = D^c_{\gamma X}Y\)
(b) \(\tilde{D}^c_{\beta X}Y = D^c_{\beta X}Y - D^c_{\gamma N\rho X}Y + B^c(X, Y)\),
where \(B^c(X, Y) := \omega^c(\beta X, Y)\).

Proof. Formula (2.2) follows from Theorem 1.2, Theorem 2.5 and Lemma 2.4, taking into account the fact that the (h)hv-torsion tensor \(T\) is conformally invariant [14].

In more details,
\[
\tilde{D}^c_XY = \tilde{\nabla}_X Y - \tilde{T}(\tilde{K}X, Y) = \nabla_X Y + \omega(X, Y) - T(KX, \tilde{Y}) - T(KLX, \tilde{Y}) = D^c_XY + \omega^c(X, Y).
\]
Relations (a) and (b) follow from (2.2) by setting \(X = \gamma X\) and \(X = \beta X\) respectively.

In view of the above theorem, we have

Theorem 2.8. Under a Finsler conformal change \(\tilde{g} = e^{2\sigma(x)}g\), we have
(a) \(\tilde{S}^c(X, Y)Z = S^c(X, Y)Z = 0\),
(b) \(\tilde{P}^c(X, Y)Z = P^c(X, Y)Z + (D^c_{\gamma X}B^c)(X, Y)\),
(c) \(\tilde{R}^c(X, Y)Z = R^c(X, Y)Z + \{D^c_{\gamma N\rho X}(Y, Z) - (D^c_{\beta X}B^c)(Y, Z) - (D^c_{\gamma N\rho X}B^c)(Y, Z)\} + P^c(Y, N\tilde{X})Z - B^c(X, B^c(Y, Z))\).

Theorem 2.9. Let \((M, L)\) and \((\tilde{M}, \tilde{L})\) be conformally related Finsler manifolds with \(\tilde{g} = e^{2\sigma(x)}g\). The associated Hashiguchi connections \(D^*\) and \(\tilde{D}^*\) are related by
\[
\tilde{D}^*_XY = D^*_X Y + \omega^*(X, Y),
\]
where \(\omega^*(X, Y) = (D^*_{\gamma Y} N)(\rho X) + NT(Y, \rho X)\).
In particular, we have
(a) \(\tilde{D}^*_{\gamma X}Y = D^*_{\gamma X}Y\)
(b) \(\tilde{D}^*_{\beta X}Y = D^*_{\beta X}Y - D^*_{\gamma N\rho X}Y + B^*(X, 0Y)\),
where \(B^*(X, Y) := \omega^*(\beta X, Y)\).
Proof. Formula (2.3) follows from Theorem 1.3 and Theorem 2.5(b). In more details,

\[ \tilde{D}^*_X \tilde{\nabla} = \tilde{\nabla}^* X + \tilde{P}(\rho X, \tilde{Y}) \eta \]

\[ = \nabla^*_X + B(\rho X, \tilde{Y}) + P(\rho X, \tilde{Y}) \eta + V(\rho X, \tilde{Y}) \eta \]

\[ = D^*_X \tilde{\nabla} + B(\rho X, \tilde{Y}) + \nabla_{\rho Y} B(\rho X, \eta Y) - B(\rho Y, \rho X, \eta Y) \]

\[ = D^*_X \tilde{\nabla} + (\nabla_{\rho Y} N)(\rho X) + NT(\rho Y, \rho X). \]

Relation (a) follows from (2.3) by setting \( X = \gamma \tilde{X} \) noting that \( \rho \circ \gamma = 0 \), whereas relation (b) follows from the same formula by setting \( X = \tilde{\beta} \tilde{X} \), noting that \( \tilde{\beta} = \beta - L \circ \beta \).

Theorem 2.10. Under a Finsler conformal change \( \tilde{g} = e^{2\sigma(x)} g \), we have

(a) \( \tilde{S}^*(X, \tilde{Y}) Z = S^*(X, Y) Z \),

(b) \( \tilde{P}^*(X, \tilde{Y}) Z = P^*(X, Y) Z - S^*(N X, Y) Z + (D^*_\gamma X B^*)(X, Z) + B^*(T(Y, X), Z), \)

(c) \( \tilde{R}^*(X, \tilde{Y}) Z = R^*(X, Y) Z + S^*(N X, N Y) Z - \mathfrak{L}_{X, Y} (P^*(X, N Y) Z + (D^*_\gamma X B^*)(Y, Z) - (D^*_\gamma N X B^*)(Y, Z) + B^*(X, B^*(Y, Z)) - B^*(T(N X, Y), Z)). \)

3. Conformal change of special Finsler spaces

In this section, we establish an intrinsic investigation of the conformal change of the most important special Finsler spaces. Moreover, we obtain necessary and sufficient conditions for such special Finsler spaces to be conformally invariant.

Throughout this section, \( g, \tilde{g}, \nabla \) and \( D^0 \) denote respectively the Finsler metric on \( \pi^{-1}(TM) \), the induced metric on \( \pi^{-1}(T^*M) \), the Cartan connection and the Berwald connection associated with a Finsler manifold \((M, L)\). Also, \( R, P \) and \( S \) denote respectively the h-, hv- and v-curvature tensors of Cartan connection, whereas \( R^0, P^0 \) and \( S^0 \) denote respectively the h-, hv- and v-curvature tensors of Berwald connection. Finally, \( T \) denotes the (h)hv-torsion tensor of Cartan connection.

We first set the intrinsic definitions of the special Finsler spaces that will be treated. These definitions are quoted from [15], where we have made a systematic intrinsic study of special Finsler spaces.

Definition 3.1. A Finsler manifold \((M, L)\) is:

(a) Riemannian if the metric tensor \( g(x, y) \) is independent of \( y \) or, equivalently, if \( T = 0 \).
(b) locally Minkowskian if the metric tensor \( g(x, y) \) is independent of \( x \) or, equivalently, if
\[
\nabla_{\beta X} T = 0 \quad \text{and} \quad R = 0.
\]

The above conditions are also equivalent to
\[
\hat{R} = 0 \quad \text{and} \quad P^o = 0.
\]

**Definition 3.2.** A Finsler manifold \((M, L)\) is said to be:

(a) Berwald if the torsion tensor \( T \) is horizontally parallel. That is,
\[
\nabla_{\beta X} T = 0.
\]

(b) \( C^h \)-recurrent if the torsion tensor \( T \) satisfies the condition
\[
\nabla_{\beta X} T = \lambda_o(\overline{X}) T,
\]
where \( \lambda_o \) is a \( \pi \)-form of order one.

(c) \( P^* \)-Finsler manifold if the \( \pi \)-tensor field \( \nabla_{\beta Y} T \) is expressed in the form
\[
\nabla_{\beta Y} T = \lambda(x, y) T,
\]
where \( \lambda(x, y) = \frac{\tilde{g}(\nabla_{\beta Y} C, C)}{C^2} \) and \( C^2 := \tilde{g}(C, C) = C(\overline{C}) \neq 0. \)

**Definition 3.3.** A Finsler manifold \((M, L)\) is said to be:

(a) \( C^o \)-recurrent if the torsion tensor \( T \) satisfies the condition
\[
\nabla_{\gamma X} T = \lambda_o(\overline{X}) T.
\]

(b) \( C^0 \)-recurrent if the torsion tensor \( T \) satisfies the condition
\[
D_{\gamma X}^o T = \lambda_o(\overline{X}) T.
\]

**Definition 3.4.** A Finsler manifold \((M, L)\) is said to be:

(a) semi-\( C \)-reducible if \( \dim M \geq 3 \) and the Cartan tensor \( T \) has the form
\[
T(X, Y, Z) = \frac{1}{n+1} \left\{ h(X, Y)C(Z) + h(Y, Z)C(X) + h(Z, X)C(Y) \right\}
+ \frac{\tau}{C^2} C(X)C(Y)C(Z),
\]
where \( \mu \) and \( \tau \) are scalar functions on \( TM \) satisfying \( \mu + \tau = 1. \)

(b) \( C \)-reducible if \( \dim M \geq 3 \) and the Cartan tensor \( T \) has the form
\[
T(X, Y, Z) = \frac{1}{n+1} \left\{ h(X, Y)C(Z) + h(Y, Z)C(X) + h(Z, X)C(Y) \right\}.
\]
(c) $C_2$-like if $\dim M \geq 2$ and the Cartan tensor $T$ has the form
\[ T(X, Y, Z) = \frac{1}{C^2} C(X)C(Y)C(Z). \]

**Definition 3.5.** A Finsler manifold $(M, L)$, where $\dim M \geq 3$, is said to be quasi-$C$-reducible if the Cartan tensor $T$ is written as:
\[ T(X, Y, Z) = A(X, Y)C(Z) + A(Y, Z)C(X) + A(Z, X)C(Y), \]
where $A$ is a symmetric indicatory $(2)$-form $(A(X, \eta) = 0$ for all $X$).

**Definition 3.6.** A Finsler manifold $(M, L)$ is said to be:

(a) $S_3$-like if $\dim M \geq 4$ and the vertical curvature tensor $S$ has the form:
\[ S(X, Y, Z, W) = \frac{Sc^v}{(n-1)(n-2)} \{ h(X, Z)h(Y, W) - h(X, W)h(Y, Z) \}. \]

(b) $S_4$-like if $\dim M \geq 5$ and the vertical curvature tensor $S$ has the form:
\[ S(X, Y, Z, W) = h(X, Z)F(Y, W) - h(Y, Z)F(X, W) \]
\[ + h(Y, W)F(X, Z) - h(X, W)F(Y, Z), \]
(3.1)

where $F = \frac{1}{n-3} \{ Ric^v - \frac{Sc^v}{2(n-2)} \}$.

**Definition 3.7.** A Finsler manifold $(M, L)$ is said to be $S^v$-recurrent if the $v$-curvature tensor $S$ satisfies the condition
\[ (\nabla_X S)(Y, Z, W) = \lambda(X)S(Y, Z)W, \]
where $\lambda$ is a $\pi$-form of order one.

**Definition 3.8.** A Finsler manifold $(M, L)$ is said to be:

(a) a Landsberg manifold if
\[ \hat{P} = 0, \text{ or equivalently } \nabla_{\beta\eta} T = 0. \]

(b) a general Landsberg manifold if
\[ Tr \{ Y \rightarrow \hat{P}(X, Y) \} = 0 \ \forall X \in \mathfrak{X}(\pi(M)), \text{ or equivalently } \nabla_{\beta\eta} C = 0. \]

**Definition 3.9.** A Finsler manifold $(M, L)$ is said to be $P$-symmetric if the mixed curvature tensor $P$ satisfies
\[ P(X, Y)Z = P(Y, X)Z, \ \forall X, Y, Z \in \mathfrak{X}(\pi(M)). \]
Definition 3.10. A Finsler manifold \((M, L)\), where \(\dim M \geq 3\), is said to be \(P_2\)-like if the mixed curvature tensor \(P\) has the form:

\[
P(X, Y, Z, W) = \alpha(Z)T(X, Y, W) - \alpha(W)T(X, Y, Z),
\]

where \(\alpha\) is a \((1)\pi\)-form, positively homogeneous of degree 0.

Definition 3.11. A Finsler manifold \((M, L)\), where \(\dim M \geq 3\), is said to be \(P\)-reducible if the \(\pi\)-tensor field \(P(X, Y, Z) := g(\hat{P}(X, Y), Z)\) is expressed in the form:

\[
P(X, Y, Z) = \delta(X)h(Y, Z) + \delta(Y)h(Z, X) + \delta(Z)h(X, Y),
\]

where \(\delta\) is the \(\pi\)-form defined by \(\delta = \frac{1}{n+1} \nabla_{\eta} C\).

Definition 3.12. A Finsler manifold \((M, L)\), where \(\dim M \geq 3\), is said to be \(h\)-isotropic if there exists a scalar \(k_o\) such that the horizontal curvature tensor \(R\) has the form

\[
R(X, Y)Z = k_o\{g(X, Z)Y - g(Y, Z)X\}.
\]

Definition 3.13. A Finsler manifold \((M, L)\), where \(\dim M \geq 3\), is said to be:

(a) of scalar curvature if there exists a scalar function \(k : TM \rightarrow \mathbb{R}\) such that the horizontal curvature tensor \(R\) satisfies the relation

\[
R(\eta, X, \eta, Y) = kL^2h(X, Y).
\]

(b) of constant curvature if the function \(k\) in (a) is constant.

Definition 3.14. A Finsler manifold \((M, L)\) is said to be \(R_3\)-like if \(\dim M \geq 4\) and the horizontal curvature tensor \(R\) is expressed in the form

\[
R(X, Y, Z, W) = g(X, Z)F(Y, W) - g(Y, Z)F(X, W) + g(Y, W)F(X, Z) - g(X, W)F(Y, Z),
\]

where \(F\) is the \((2)\pi\)-form defined by \(F = \frac{1}{n-2}\{Ric^h - \frac{Sc^h}{2(n-1)}\}\).

Definition 3.15. A Finsler manifold \((M, L)\) is called of perpendicular scalar (simply, \(p\)-scalar) curvature if the \(h\)-curvature tensor \(R\) satisfies the condition

\[
R(\phi(X), \phi(Y), \phi(oZ), \phi(oW)) = R_o\{h(\phi(X), \phi(oZ))h(\phi(oY), oW) - h(X, oW)h(oY, oZ)\},
\]

where \(R_o\) is a function on \(TM\), called perpendicular scalar curvature, and \(\phi\) is the \(\pi\)-tensor field defined by \(\phi(X) := X - L^{-1}l(\phi(X))\eta\).

Definition 3.16. A Finsler manifold \((M, L)\) is called of \(s-ps\) curvature if \((M, L)\) is both of scalar curvature and of \(p\)-scalar curvature.

Definition 3.17. A Finsler manifold \((M, L)\) is said to be symmetric if the \(h\)-curvature tensor \(R^o\) of the Berwald connection \(D^o\) is horizontally parallel: \(D^o_{\delta\xi}(R^o) = 0\).

Now, we focus our attention to the change of the above mentioned special Finsler manifolds under a conformal transformation \(g \rightarrow \tilde{g} = e^{2\sigma(x)}g\). In what follows we assume that the Finsler manifolds \((M, L)\) and \((\tilde{M}, \tilde{L})\) are conformally related.
Proposition 3.18.

(a) \((M, L)\) is a Riemaniann manifold if, and only if, \((M, \tilde{L})\) is a Riemaniann manifold.

(b) Assume that \(D^\gamma_X B^\circ = 0\) and \(H(X, \phi Y)\phi \eta = 0\). Then, \((M, L)\) is Locally Minkowskian if, and only if, \((M, \tilde{L})\) is Locally Minkowskian.

Proof.

(a) Follows from Definition 3.1 together with the fact that the \((h)hv\)-torsion tensor \(T\) is conformally invariant.

(b) By Theorem 2.5(c)' and Theorem 2.6(b)', we get

\[
\mathring{R}(X, \phi Y)\phi \eta = R(X, \phi Y)\phi \eta, \quad \text{and} \quad \tilde{P}^\circ(X, \phi Y)\phi Z = P^\circ(X, \phi Y)\phi Z.
\]

The result follows then from Definition 3.1.

Let us introduce the \(\pi\)-tensor field

\[
A(X, Y, Z) := T(U(\beta X, Y), Z) + T(U(\beta X, Z), Y) - U(\beta X, T(Y, Z)),
\]

where \(U(\beta X, Y) := B(X, Y) - \nabla_{L_\beta X} Y\).

One can show that the \(\pi\)-tensor field \(A\) has the property that \(A(X, Y, \eta) = 0\).

Proposition 3.19. Assume that the \(\pi\)-tensor field \(A\) vanishes. Then, \((M, L)\) is a Berwald (resp. \(C^h\)-recurrent) manifold if, and only if, \((M, \tilde{L})\) is a Berwald (resp. \(C^h\)-recurrent) manifold.

Proof. Using Theorem 2.5(b), taking into account the fact that \(T\) is conformally invariant, we get

\[
(\mathring{\nabla}_{\beta X} \tilde{T})(Y, Z) = \nabla_{\beta X} T(Y, Z) - T(\nabla_{\beta X} Y, Z) - T(Y, \nabla_{\beta X} Z) - \{T(U(\beta X, Y), Z) + T(Y, U(\beta X, Z)) - U(\beta X, T(Y, Z))\}.
\]

Consequently,

\[
(\mathring{\nabla}_{\beta X} \tilde{T})(Y, Z) = (\nabla_{\beta X} T)(Y, Z) - A(X, Y, Z).
\]

Hence, under the given assumption, we have

\[
\mathring{\nabla}_{\beta X} \tilde{T} = \nabla_{\beta X} T.
\]

Therefore, \((M, L)\) is Berwald iff \((M, \tilde{L})\) is Berwald.

On the other hand, if \((M, L)\) is \(C^h\)-recurrent, then the \((h)hv\)-torsion tensor \(T\) has the property that \(\nabla_{\beta X} T = \lambda_0(X) T\), where \(\lambda_0\) is a \(\pi\)-form.

Now, from (3.3), we obtain

\[
\mathring{\nabla}_{\beta X} \tilde{T} = \lambda_0(X) \tilde{T}.
\]

This implies that \((M, \tilde{L})\) is \(C^h\)-recurrent. The converse can be proved similarly.

Proposition 3.20. Assume that the \(\pi\)-tensor field \(A\) has the property that \(i_\eta A = 0\). Then, \((M, L)\) is a \(P^*\)-Finsler manifold if, and only if, \((M, \tilde{L})\) is a \(P^*\)-Finsler manifold.
Proof. From relation (3.4), we have
\[ \nabla_{\beta\sigma\eta} T = \tilde{\nabla}_{\hat{\beta}\hat{\sigma}\hat{\eta}} \tilde{T}. \]
Hence, the \(\pi\)-tensor field \(\nabla_{\beta\sigma\eta} C\) is conformally invariant. This, together with the fact that \(\tilde{C} = C\), imply that the scalar function \(\lambda(x, y)\) defined by \(\lambda(x, y) := \frac{\tilde{g}(\nabla_{\beta\sigma\eta} C, \tilde{C})}{\tilde{g}(C, C)}\) is also conformally invariant. Hence the result.

\[\square\]

**Proposition 3.21.** A Finsler manifold \((M, L)\) is \(C^n\)-recurrent (resp. \(C^0\)-recurrent) if, and only if, \((M, \tilde{L})\) is \(C^n\)-recurrent (resp. \(C^0\)-recurrent).

Proof. If \((M, L)\) is \(C^n\)-recurrent, then the \((h)\nu\)-torsion tensor \(T\) has the form \(\nabla_{\gamma\nu} T = \lambda_v(\bar{X})T\), where \(\lambda_v\) is a \(\pi\)-form. Since the map \(\nabla_{\gamma\nu} : \nu Y \longmapsto \nabla_{\gamma\nu} \nu Y\) and the torsion tensor \(T\) are conformally invariant, it follows that
\[ \tilde{\nabla}_{\gamma\nu} \tilde{T} = \lambda_v(\bar{X})\tilde{T}. \]
This implies that \((M, \tilde{L})\) is \(C^n\)-recurrent. The converse is proved similarly. The same argument can be applied to the \(C^0\)-recurrence property.

\[\square\]

**Proposition 3.22.** A Finsler manifold \((M, L)\) is semi-\(C\)-reducible if, and only if, \((M, \tilde{L})\) is semi-\(C\)-reducible. Consequently, \((M, L)\) is \(C\)-reducible (resp. \(C_2\)-like) if, and only if, \((M, \tilde{L})\) is \(C\)-reducible (resp. \(C_2\)-like).

Proof. The semi-\(C\)-reducibility property is expressed as
\[ T(\bar{X}, \nu Y, \nu Z) = \frac{\mu}{n + 1} \tilde{g}_{\mu \nu Y, \nu Z} \{ h(\bar{X}, \nu Y) C(\nu Z) \} + \frac{\tau}{C^2} C(\bar{X}) C(\nu Y) C(\nu Z), \]
where \(\mu\) and \(\tau\) are scalar functions satisfying \(\mu + \tau = 1\) and the symbol \(\tilde{g}_{\mu \nu Y, \nu Z}\) denotes cyclic sum over \(\nu Y, \nu Z\).

Since \(C^2 := \tilde{g}(\tilde{C}, \tilde{C}) = e^{-2\sigma} \tilde{g}(C, C) = e^{-2\sigma} C^2, \tilde{T}(\bar{X}, \nu Y, \nu Z) = e^{2\sigma} T(\bar{X}, \nu Y, \nu Z)\) and the angular metric tensor \(h\) is conformally \(\sigma\)-invariant, the above relation is equivalent to
\[ \tilde{T}(\bar{X}, \nu Y, \nu Z) = \frac{\mu}{n + 1} \{ \tilde{h}(\bar{X}, \nu Y) \tilde{C}(\nu Z) \} + \frac{\tau}{C^2} \tilde{C}(\bar{X}) \tilde{C}(\nu Y) \tilde{C}(\nu Z). \]
Hence, the semi-\(C\)-reducibility property is preserved.

Finally, the proof of the cases of \(C\)-reducibility and \(C_2\)-likeness is similar.

\[\square\]

**Proposition 3.23.** A Finsler manifold \((M, L)\) is quasi-\(C\)-reducible if, and only if, \((M, \tilde{L})\) is quasi-\(C\)-reducible.

**Theorem 3.24.** A necessary and sufficient condition for a Finsler manifold to be conformal to a Landsberg manifold is that
\[ \tilde{P} = i_{\nu A}. \]

Proof. We have [16]
\[ \tilde{P} = \nabla_{\beta\sigma\eta} T. \]
From which, together with (3.3), we obtain
\[ \tilde{P} - \hat{P} = \tilde{\nabla}_{\hat{\beta}\hat{\sigma}\hat{\eta}} \tilde{T} - \nabla_{\beta\sigma\eta} T = -i_{\nu A}. \]
Hence, the result follows.

\[\square\]
Let us define the \(\pi\)-tensor field
\[
\mc{A}_o(X) := \text{Tr}\{\phi Y \mapsto (i_\pi A)(X, Y)\},
\]  
(3.7)
where \(A\) is the \(\pi\)-tensor field defined by (3.3).

**Proposition 3.25.**
(a) Assume that \(i_\pi A = 0\). Then, \((M, L)\) is Landsberg if, and only if, \((M, \tilde{L})\) is Landsberg.
(b) Assume that \(A_o = 0\). Then, \((M, L)\) is general Landsberg if, and only if, \((M, \tilde{L})\) is general Landsberg.

**Proof.**
(a) Setting \(X = \phi \eta\) in (3.4), we get
\[
\tilde{\nabla}_{\beta \eta} \tilde{T} = \nabla_{\beta \eta} T - i_\pi A,
\]  
(3.8)
from which, under the given assumption, \(\nabla_{\beta \eta} T\) is conformally invariant. Hence the result.
(b) Taking the trace of (3.8), we obtain
\[
\tilde{\nabla}_{\beta \eta} \tilde{C} = \nabla_{\beta \eta} C - A_o.
\]
From which the result. \(\square\)

**Proposition 3.26.** Assume that \(i_\pi A = 0\). Then, \((M, L)\) is \(P\)-reducible if, and only if, \((M, \tilde{L})\) is \(P\)-reducible.

**Proof.** Under a conformal change, the angular metric tensor \(h\) is conformally \(\sigma\)-invariant. On the other hand, \(\hat{P}\) is conformally invariant by our assumption together with (3.6). Consequently, \(\nabla_{\beta \eta} T\) is conformally invariant, which implies that \(\nabla_{\beta \eta} C\) (or \(\delta\) of Definition 3.11) is also conformally invariant.

Now, since \(P(X, \phi Y, \phi Z) = g(\hat{P}(X, \phi Y), \phi Z)\) is conformally \(\sigma\)-invariant, then, the tensor field
\[
U_1(X, \phi Y, \phi Z) := g(\hat{P}(X, \phi Y), \phi Z) - \mathcal{G}_{X, \phi Y, \phi Z}\{\delta(X)h(Y, Z)\}
\]
is conformally \(\sigma\)-invariant. From which, the result follows (provided that \(\sigma \neq 0\). \(\square\)

**Proposition 3.27.** \((M, L)\) is \(S_3\)-like (resp. \(S_4\)-like) if, and only if, \((M, \tilde{L})\) is \(S_3\)-like (resp. \(S_4\)-like).

**Proof.** Let \(U\) be the \(\pi\)-tensor field defined by
\[
U(X, Y, Z, W) := S(X, Y, Z, W) - \frac{Sc}{(n-1)(n-2)}\{h(X, Z)h(Y, W) - h(X, W)h(Y, Z)\}.
\]
Under a conformal transformation, the \(\pi\)-tensor field \(Sc^\nu h\) is conformally invariant and \(S(X, Y, Z, W) = e^{2\sigma(x)}S(X, Y, Z, W)\).
Hence,
\[
\tilde{U}(X, Y, Z, W) = e^{2\sigma(x)}U(X, Y, Z, W).
\]  
(3.9)
This means that the \(\pi\)-tensor field \(U\) is conformally \(\sigma\)-invariant.
On the other hand, let $V$ be the $\pi$-tensor field defined by

$$V(X, Y, Z, W) := S(X, Y, Z, W) - \hbar(Z, X)F(W, Y) + h(Z, Y)F(W, X)
- h(W, Y)F(Z, X) - h(W, X)F(Z, Y).$$

Since both the angular metric tensor $\hbar$ and the $v$-curvature tensor $S$ are conformally $\sigma$-invariant and $F(X, Y) := \frac{1}{n-3}\left\{Ric^v(X, Y) - \frac{2e^{\sigma h(X, Y)}}{2(n-2)}\right\}$ is conformally invariant, we conclude that

$$\tilde{V}(X, Y, Z, W) = e^{2\sigma(x)}V(X, Y, Z, W), \quad (3.10)$$

which means that the $\pi$-tensor field $V$ is conformally $\sigma$-invariant. The result follows from (3.9) and (3.10).

**Proposition 3.28.** $(M, L)$ is $S^v$-recurrent if, and only if, $(M, \tilde{L})$ is $S^v$-recurrent.

**Proof.** Follows from the fact that both the map $\nabla_{\gamma X} : \phi Y \mapsto \nabla_{\gamma X} \phi Y$ and the $v$-curvature tensor $S$ are conformally invariant. \qed

**Proposition 3.29.** Assume that the $\pi$-tensor field $H$ defined in Theorem 2.5 has the property that $H(\eta, X)\eta = 0$ for all $X \in X(\pi(M))$. Then, $(M, L)$ is of scalar curvature if, and only if, $(M, \tilde{L})$ is of scalar curvature.

**Proof.** By Theorem 2.5(c)', we have

$$\tilde{R}(\eta, X, \eta, Y) = e^{2\sigma(x)}R(\eta, X, \eta, Y) + e^{2\sigma(x)}g(H(\eta, X)\eta, Y),$$

which implies, by hypothesis, that

$$\tilde{R}(\eta, X, \eta, Y) = e^{2\sigma(x)}R(\eta, X, \eta, Y). \quad (3.11)$$

Now, let $(M, L)$ be of scalar curvature, then the $h$-curvature tensor $R$ has the form

$$R(\eta, X, \eta, Y) = kL^2h(X, Y).$$

This, together with (3.11), imply that

$$\tilde{R}(\eta, X, \eta, Y) = e^{2\sigma(x)}kL^2h(X, Y) = e^{-2\sigma(x)}kL^2\tilde{h}(X, Y),$$

where we have used the fact that both $L^2$ and $h$ are conformally $\sigma$-invariant. Hence

$$\tilde{R}(\eta, X, \eta, Y) = k_oL^2\tilde{h}(X, Y),$$

where $k_o = e^{-2\sigma(x)}k$. \qed

**Proposition 3.30.** Assume that the given conformal change is homothetic. Then, we have

(a) $(M, L)$ is $P_2$-like if, and only if, $(M, \tilde{L})$ is $P_2$-like.

(b) $(M, L)$ is $h$-isotropic if, and only if, $(M, \tilde{L})$ is $h$-isotropic.

(c) $(M, L)$ is of constant curvature if, and only if, $(M, \tilde{L})$ is of constant curvature.

(d) $(M, L)$ is of $p$-scalar curvature if, and only if, $(M, \tilde{L})$ is of $p$-scalar curvature.
(e) $(M, L)$ is of $s$-$ps$-curvature if, and only if, $(M, \tilde{L})$ is of $s$-$ps$-curvature.

(f) $(M, L)$ is $R_3$-like if, and only if, $(M, \tilde{L})$ is $R_3$-like.

(g) $(M, L)$ is symmetric if, and only if, $(M, \tilde{L})$ is symmetric.

(h) $(M, L)$ is $P$-symmetric if, and only if, $(M, \tilde{L})$ is $P$-symmetric.

Proof. The proof follows from the fact that:
\[ \sigma(x) \text{ is constant } \iff \tilde{\nabla}_X Y = \nabla_X Y \quad \square \]

Summing up, the results of this section can be gathered in the following

**Theorem 3.31.** The following properties are conformally invariant:
- being Riemannian,
- being $C$-reducible,
- being quasi-$C$-reducible,
- being $C^0$-recurrent,
- being $S_3$-like,
- being $S_3$-like,
- being $S_3$-like,
- being $S_3$-like,
- being $S_3$-like,
- being $S_3$-like,
- being $S_3$-like.

The following properties are conformally invariant under certain conditions:
- being locally Minkowskian,
- being $C^h$-recurrent,
- being Landsberg,
- being $P$-symmetric,
- being $P$-reducible,
- being of scalar curvature,
- being $R_3$-like,
- being of $s$-$ps$ curvature,
- being semi-$C$-reducible,
- being $C_2$-like,
- being $C^v$-recurrent,
- being $S^v$-recurrent,
- being $S^v$-recurrent,
- being $S^v$-recurrent.

**Remark 3.32.** It should be noted that some important results of [6], [8], [9] (obtained in local coordinates) are retrieved from the above mentioned global results (when localized).

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