First-Order Approximation to the Boltzmann-Curtiss Equation for Flows with Local Spin

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Abstract The first-order approximation to the solution of the Boltzmann-Curtiss transport equation is derived. The resulting distribution function treats the rotation or gyration of spherical particles as an independent classical variable, deviating from the quantum mechanical treatment of molecular rotation found in the Wang Chang-Uhlenbeck equation. The Boltzmann-Curtiss equation, therefore, does not treat different rotational motions as separate molecular species. The first-order distribution function yields momentum equations for the translational velocity and gyration that match the form of the governing equations of morphing continuum theory (MCT), a theory derived from the approach of rational continuum thermomechanics. The contribution of the local rotation to the Cauchy stress and the viscous diffusion are found to be proportional to an identical expression based off the relaxation time, number density, and equilibrium temperature of the fluid. When gyration is equated to the macroscopic angular velocity, the kinetic description reduces to the first-order approximation for a classical monatomic gas, and the governing equations match the form of the Navier-Stokes equations. The relaxation time used for this approximation is shown to be more complex due to the additional variable of local rotation. The approach of De Groot and Mazur is invoked to give an initial approximation for the relaxation of the gyration. The incorporation of this relaxation time, and other physical parameters, into the coefficients of the governing equations provides a more in-depth physical treatment of the new terms in the MCT equations, allowing experimenters to
test these expressions and get a better understanding of new coefficients in MCT.

1 Introduction

Researchers have been heavily relying on vorticity to characterize the flow physics of vortices, eddies and molecular rotations [43]. However, vorticity-based descriptions are often found inadequate or inconsistent when the vorticity field deviates from local rotations [34]. Those cases are considered as flows with strong local spins. It should be emphasized that spin is different from vorticity. Vorticity is defined by curl of velocity. In other words, it is a consequent motion of the translation. On the other hand, spin can be an independent motion. One could use the solar system as an example. Each planet spins on its own axis while moving on its own orbits. The translational velocities of any two planets are used to calculate their corresponding vorticity; however, such quantity does not represent the spinning of either planets.

Flows with strong local spin have been the focus of extensive theoretical, experimental, and numerical work for decades [24,35,37,38,56,64]. High-speed, turbulent, compressible, reacting, and polyatomic gas flows all involve complex interactions based on strong local spin. The Wang Chan-Uhlenbeck equation accounts for molecular spin through the lens of quantum mechanics, treating each different quantum state as a separate species of molecule [66]. This additional rigor adds more complexity to the distribution function and the dynamics of the collisional integral. For classical physics, however, local rotation may affect the dynamics of the entire flow. Turbulent flows, in particular, may produce additional angular momentum from the smallest eddies. The rotation of these smallest eddies affects the energy and momentum transfer at the inertial length scales, requiring researchers to develop methods that capture this additional small-scale angular momentum. The most effective of these analytical methods have revealed deeper physical or mathematical characteristics to previously well-tested theories of fluid dynamics [1,24,25,26,40,62].

Several different fields of research have adapted to flows with local spin, by either modifying classical theories or developing entirely new approaches. Meng et. al. constructed a thermal lattice Boltzmann model based on the ellipsoidal statistical Bhatnagar-Gross-Krook (ES-BGK) equation to capture dynamics of rarefied gas thermal flows [45]. When these flows approach higher Mach numbers, the higher nonequilibrium flows were more difficult to capture in the transition regime without driving up computational costs. For hypersonic flows, Munafo et. al. proposed a Boltzmann rovibrational collisional coarse-grained model, which grouped internal energies associated with vibration and rotation into separate energy bins [49]. These groups of internal energies were treated as separate species. The main flow equation was simplified to a one-dimensional inviscid flow, also to save on computational resources. For polyatomic gases, theoretical approaches often treat local spin
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as an internal degree of freedom, similar to molecular vibration. Arima et al. modified the approach of rational extended thermodynamics to treat the molecular and vibrational relaxation processes in polyatomic gases as separate processes, but included all effects of vibration and rotation in a separate variable denoting internal motions in the gas [4]. For rotation in nonequilibrium flows, Eu added to his generalized hydrodynamic relations [27,28] by introducing excess normal stress associated with a bulk viscosity [29]. Myong et al. developed computational models based on Eu’s relations to analyze high Knudsen number, rarefied diatomic gas flows [50,51,52].

The models previously discussed have typically been modifications to classical approaches, treating rotation in separate closure models. The history of theoretical work on flows with local spin, however, shows that much can be learned about these flows by challenging the assumptions behind the classical approach. From the perspective of statistical mechanics, Grad developed the generalized thermodynamic relations for nonequilibrium distributions of molecules [32]. These relations were then applied to systems of molecules where individual molecules possessed internal rotation. When molecular rotation was treated as an internal variable dependent on the coordinates local to a molecule, the angular momentum equation’s dependence on the linear momentum equation no longer held for nonequilibrium flows. Additionally, the pressure tensor become asymmetric due to the added internal rotation. De Groot considered the effects of an asymmetric pressure tensor on the production of entropy, linear momentum, and angular momentum [23]. The stresses produced by the difference between internal rotation and the vorticity in the flow were characterized by the “rotational viscosity” [23]. Since this new coefficient appeared in the linear and angular momentum equations, the new parameter played the role of a coupling coefficient. Furthermore, when the material parameters were assumed to be homogeneous in space, the Navier-Stokes equations were recovered, with the rotational viscosity included [23].

Snider later generalized this work to account for more complex rotational motions in anisotropic fluids, or fluids where the local equilibrium properties depended on the presence of this local spin [60]. At the small and large scales, fundamental forces and properties of the fluid are recharacterized when molecules possess strong internal spin. The pressure tensor becomes asymmetric, and the torque caused by local spin gives rise to a couple stress tensor. This couple stress is not to be confused with Stokes’ formulation of the couple stress, which emerges from the vorticity vector [61]. Evans added depth to Snider and De Groot’s work by calculating the transport coefficients in the linear constitutive equations that related stresses to deformations in the fluid [30]. Molecular dynamics simulations of dense polyatomic fluids between parallel plates produced preliminary data on the so-called “vortex viscosity,” which corresponded to De Groot’s coupling coefficient, and showed how the variable for internal rotation approached the angular velocity in the classical limit [30]. The characteristic time that the local rotation approached the macroscopic angular velocity was referred to as the relaxation time. De Groot and Evans both derived an expression for the relaxation constant for constant
vorticity and zero classical viscosity [23,30]. For this simple case, the relaxation time was shown to be inversely proportional to the coupling coefficient. This analysis was the first indication that the return to equilibrium for these flows required a reformulation of the relaxation time. This concept will be treated with greater depth when the kinetic theory approach to polyatomic gases is discussed.

The presence of the new transport properties in multiple approaches to the problem of flows with local spin [23,30], and their relevance to the departure from classical fluids suggests the need for a deeper treatment of their physical meaning. As has been shown for the Navier-Stokes equations, the physics behind derived material constants can only come from an approach that achieves a macroscopic description of the fluid from modeling the interactions of individual particles. Maxwell and Boltzmann showed that these classical fluid descriptions could arise from the collisions of several particles in a monatomic gas [8,42]. Given enough collisions, a probability distribution function could predict how many particles would occupy a given point in space and possess a certain translational velocity. Maxwell and Boltzmann showed that a zeroth-order approximation of this distribution function demonstrates symmetries in physical and velocity space [8,42]. Furthermore, the substitution of the Maxwell-Boltzmann distribution into the balance laws for mass, momentum, and energy yield governing equations that take the form of the Euler equations. The first-order approximation to the exact solution to the Boltzmann transport equation, in turn, yields equations that mirror the Navier-Stokes formulation.

This initial work for monatomic gases expanded to fluids that contained particles with structure. Curtiss extended the Boltzmann transport equation to account for variables associated with separate internal motions apart from the translational velocity [19][20][21]. Additional rotation from diatomic molecules [19] and molecules of arbitrary structure [20] all resulted in additional terms to the Boltzmann transport equation, and complicated the physics behind collisional integrals and transport coefficients. The resulting transport equation became known as the Boltzmann-Curtiss transport equation. From this bottom-up approach to flows with structure, Curtiss employed the Chapman-Enskog method [21] to obtain explicit expressions for the transport coefficients of a dilute gas mixture of rigid, non-spherical, symmetric-top molecules. Curtiss found that the resulting kinetic theory for this specific case matched the “loaded-sphere” formalism presented by Dahler and Sather [57]. She et. al. generalized this approach to molecules with arbitrary internal degrees of freedom with intermolecular central potential forces [58], deriving a solution to the Boltzmann-Curtiss equation for molecules with translational and rotational motion. She also applied the Chapman-Enskog approach to find transport coefficients for this limited case [58]. When well-known potentials were substituted in for the perturbations to the equilibrium distribution, the resulting coefficients compared with well-known models such as the “rough-sphere” approximation. Solutions to the Boltzmann-Curtiss equation that accounted for nonequilibrium entropy production consistent with the second law of ther-
modynamics were introduced by Eu [27,28] and investigated by Myong [50, 51, 52, 53] for cases of high thermal nonequilibrium, where the complexity in the constitutive equations required a more in-depth theoretical and numerical treatment. The effects of rotational motion in Eu’s solution were encapsulated by the rotational Hamiltonian, but the fluid was assumed to have no intrinsic angular momentum [28].

The extension of kinetic theory to polyatomic gases also required a detailed investigation into the process of the gas departing from and returning to an equilibrium state. As mentioned in the statistical mechanical approach, the relaxation time typically gives a characteristic time for the return to equilibrium. In the classical approach, this relaxation time could be approximated as the time between two collisions. For the Boltzmann distribution, the only parameters needed to obtain this time were the mean free path and the most probable velocity value [36]. For kinetic theories involving local rotation as an independent variable, the transfer of kinetic energy between translation and rotation needed to be considered. Parker et. al. focused on deriving the rotational relaxation time for homonuclear diatomic molecules [54]. This description relied on the simplification that the translational energies already reached equilibrium, and that the rotational energy was initially unexcited. Furthermore, Parker did not consider rotation as a fundamental degree of freedom with its own equilibrium temperature [51]. In the loaded-sphere approximation, Dahler et. al. established local equilibrium temperatures for both translation and rotation, and derived the rotational relaxation time from an approximation of the rate of transfer of energy between these motions [22]. This approximation came from assuming a pair distribution for translation and rotation as the product of each of the two local Boltzmann distributions. Monchick et. al. was later able to establish singular relaxation times for rotation and other internal degrees of freedom based off of the Chapman-Enskog process to the linearized Boltzmann equation [46]. Determination of the relaxation time associated with rotation or vibration of gas molecules was determined experimentally via absorption of ultrasound frequencies and by measurements of heat conductivity [9, 47]. Recently, molecular dynamics simulations have been performed to give better approximations of the rotational relaxation time [65] as well as other transport properties such as the shear viscosity. Still, these methods treat rotation as an internal degree of freedom, and evaluate how well the values match with classical treatments of the rotation in the flow. The effects of molecular rotation to the total relaxation time of a polyatomic gas still require a more detailed treatment.

When rotation is treated as an internal or quantum state, the theoretician is challenged with the task of isolating its contribution from all other internal or quantum states. Distribution functions corresponding to each quantum state, as solutions to the Wang Chan-Uhlenbeck equation are formulated [66], must be obtained and compared with available data. This challenge is eliminated if rotation is treated as an explicit, independent variable, allowing for a single distribution function to describe the small-scale rotation in the system. Recently, an additional theory derived from the perspective of rational contin-
num thermomechanics (RCT) has provided governing equations for the mass, momenta, and energy of a fluid composed of spherical particles \[15,16,24,25,26\]. These equations start from a description of the fluid that deviates from any classical mechanical fluid. In the framework of MCT, the fluid is now posed as a morphing continuum, composed of individual spheres that possess intrinsic rotation as a separate, independent motion. It should not be confused between the pioneering works by Eringen \[24\] and Stokes \[61\] and the current study. Both Erigen’s and Stokes’ formulations were deduced from rational continuum mechanics and thermodynamic irreversible processes. The current study was built upon the basis of kinetic theory and statistical mechanics.

Similar to Snider’s work \[60\], the governing equations of morphing continuum theory (MCT) present a series of new coefficients directly related to the contribution of local spin to various stresses in the fluid. Chen showed that the inviscid equations of MCT could be derived from a zeroth-order approximation to the solution of the Boltzmann-Curtiss transport equation \[14\]. The meaning of the additional coefficients in MCT, and their precise contribution to turbulent flows with local spin, has yet to be explained through kinetic theory. Peddieson et. al. derived dimensionless parameters \[55\] that produced a range of boundary layer profiles, including profiles that demonstrated aspects of turbulence. Still, the choice of the values for these parameters was arbitrary, with no expectation for which material constants were indispensable for the generation of turbulent fluctuations. When governing equations derived from kinetic theory descriptions of a fluid mirror the form of similar equations derived from first principles, further insight into the role of new material constants arises. Material properties are shown to have intrinsic dependencies on other properties of the fluid. Furthermore, differences in the two sets of equations may reveal hidden assumptions in the mathematical approach or the need for higher orders of accuracy in the kinetic theory description. The focus of this paper is to apply the Chapman-Enskog approach from kinetic theory to extend Chen’s analysis of fluids with spherical particles to the first-order approximation to the Boltzmann-Curtiss transport equation. The final form of this approximation is intended to give a deeper insight into the new coefficients introduced by MCT, and to present a kinetic description of a fluid possessing independent local rotation. The benefit of this approach of treating the local rotation as independent will be evident as the need for developing complex constitutive relations is avoided. These first-order equations should possess familiar terms from the Navier-Stokes equations, and introduce key terms that arise from local rotation. An analysis of these new equations will require a discussion of the relaxation time used to make this first-order approximation, due to the presence of local rotation.

Several recent studies have shown that MCT governing equations can predict the turbulence velocity profile in a pipe flow \[44\], plane Couette flow \[3\] and flat plate \[70\]. These equations have also been used for transonic and supersonic \[67\] turbulence \[12\]. All these results are obtained through a direct numerical simulation (DNS) with a fraction of the computational resources needed for NS-based DNS. MCT is considered as a computationally friendly
approach for direct numerical simulation (DNS). Therefore, it is crucial to see
the correlation between rational continuum thermomechanics and kinetic
time for morphing continua.

Section 2 specifies the assumptions for the fluid and outlines the
mathematical consequences of making these assumptions. The distribution func-
tion, conservation equations, and balance laws will obtain a certain form from
these assumptions. In section 3, the first-order approximation to the distrib-
ution function is derived from the zeroth order balance laws by following
the Chapman-Enskog approach. With the distribution function, the expres-
sions for key stresses in the original balance laws are derived and discussed
briefly. Then, section 4 derives the governing equations by substituting the
stress tensors back into the original balance laws. A brief comparison with the
Navier-Stokes and MCT linear momentum equations is done to highlight the
new terms brought about by the local rotation of the spherical particles. In
section 5, the physics underlying the new relaxation time is investigated. For
section 6, the equivalence between the gyration and the macroscopic angular
velocity results in a reduction of the governing equations to the Navier-Stokes
description. Finally, section 7 concludes by remarking on the next steps for
verifying and expanding the influence of this work to pressing problems for
turbulent flows and other flows involving strong local spin.

2 Background

For monatomic gases composed of infinitesimal particles, any kinetic theory
needs to track only the position and translational velocity of the particles.
These assumptions greatly simplify the probability distribution of particles,
as well as the transport equation used to describe the evolution of that dis-
tribution. When the particles are given a finite size and allowed to rotate,
additional motions bring additional degrees of freedom to the system. If the
angular motion of the particles is independent from the translational motion
and is dependent on its orientation, then the transport equation has the form
\[ \left( \frac{\partial}{\partial t} + \frac{p_i}{m} \frac{\partial}{\partial x_i} + \frac{M_i}{I} \frac{\partial}{\partial \Phi_i} \right) f = \left( \frac{\partial f}{\partial t} \right)_{coll} \]  
(1)

Here \( m \) denotes the mass of a particle, \( p_i \) represents the linear momentum,
\( M_i \) the angular momentum, \( I \) the moment of inertia of a particle, and \( \Phi_i \) the
Euler angle with respect to the center of mass of the particle.

The solution \( f(p_i, \Phi_i, x_i, t) \) gives the probability a particular particle will
possess the values of the given variables, and generalizes the motion of the sys-
tem by simplifying the interactions of individual particles. For instance, this
solution is absent of dependencies on vibrational energy or vibrational mo-
tion, as the dynamics of individual collisions are assumed to be independent
of these variables. The right-hand side of equation (1) accounts for the cumula-
tive effect of collisions on the distribution. For this description, the particles
are treated as spheres, so all axial orientations of the distribution are equivalent, i.e. independent of the Euler angle. Therefore, the Boltzmann-Curtiss transport equation becomes \([14,20]\):

\[
\left( \frac{\partial}{\partial t} + \frac{p_i}{m} \frac{\partial}{\partial x_i} \right) f = \left( \frac{\partial f}{\partial t} \right)_{\text{coll}} \tag{2}
\]

Equilibrium solutions to this equation should look similar to the Maxwell-Boltzmann distribution function, as the remaining terms are concerned with linear momentum. Still, the presence of an independent angular rotation, \(\omega_i\), changes the distribution of kinetic energy of the particles. From Boltzmann’s principle, the equilibrium solution to equation 2 can be approximated as \([14]\):

\[
f_0(x_i, v_i, \omega_i, t) = n \left( \frac{\sqrt{mI}}{2\pi\theta} \right)^{3/2} \exp\left(-\frac{m(v'_i v'_l) + I(\omega'_p \omega'_p)}{2\theta}\right) \tag{3}
\]

Here, the perturbed velocity, \(v'_i = v_i - U_i\), for mean velocity \(U_i\) and the perturbed gyration, \(\omega'_p = \omega_p - W_p\) for mean gyration \(W_p\), are introduced. The form of this distribution function differs from the classical Boltzmann distribution function \([41,63]\), which assigns a \(3/2\) power to the terms in front of the exponential. The increased exponential in equation 2 arises due to the additional contribution to the momentum by the gyration, \(\omega'_p\). The number density, \(n\), of the particles is found by integrating the distribution function \(f\) over all the perturbed variables, \(v'\) and \(\omega'\), which is now a six-dimensional integral:

\[
n = \int \int d^3v' d^3\omega' f^0 \tag{4}
\]

The superscript indicates that this function only serves as a zeroth-order approximation to the true solution. In equation 3, the mean thermal energy \(\theta\), mean velocity and mean gyration are assumed to vary slowly in time due to the rapid number of collisions, ensuring a rapid return to equilibrium. The thermal energy, \(\theta = kT\), contains the Boltzmann constant \(k\) and absolute temperature \(T\). Classical kinetic approaches by Huang \([36]\) and by Gupta et. al. for granular fluids \([33]\) often group the Boltzmann constant with the characteristic temperature to focus on the thermal energy of the system. The velocity and gyration perturbations represent the rapid fluctuations of the spheres, and provide the main source of any dynamics at equilibrium. Furthermore, the moment of inertia of a sphere can be expressed in terms of a parameter \(j\) \([17]\), known as the microinertia. This parameter comes from the averaging of spatial coordinates attached to the sphere, allowing one to show that \(j = \frac{5}{24}d^2\), where \(d\) is the diameter of the sphere \([17]\). Substituting \(I = mj\) into equation 3 yields:

\[
f^0(x_i, v_i, \omega_i, t) = n \left( \frac{m \sqrt{2j}}{2\pi\theta} \right)^{3/2} \exp\left(-\frac{m(v'_i v'_l) + j\omega'_p \omega'_p}{2\theta}\right) \tag{5}
\]

This equilibrium distribution function represents the starting point for the kinetic theory derivation, providing an abstract description of the system. To account for the evolution of the physical motion of a particle, the balance laws
must be derived. The average of a quantity $A$ is here defined by the following expression:

$$\langle A \rangle = \frac{1}{n} \int \int A f(x_i, v_i, \omega_i, t) d^3v' d^3\omega'$$  \hspace{1cm} (6)$$

where $n$ is the number density of the particles and is found by integrating the distribution function $f$ over all the perturbed variables, $v'$ and $\omega'$. The mean velocity and gyration are naturally obtained from $\langle v \rangle$ and $\langle \omega \rangle$. Therefore, any balance laws governing the mean velocity and mean gyration must come by averaging the transport equation (2) for some conserved quantity $\chi(x_i, p_i)$:

$$\frac{\partial}{\partial t} \langle n\chi \rangle + \frac{\partial}{\partial x_i} \langle n p_i m \chi \rangle - n \langle \frac{p_i}{m} \partial \chi / \partial x_i \rangle = 0$$  \hspace{1cm} (7)$$

Note that all potential time derivatives vanished as $\chi$ is a function of momentum and position alone. The collisional term emerging from the averaging of the right-hand side of equation (2) is also presumed to vanish, namely, $\langle \chi(x_i, p_i) \partial f / \partial t \rangle_{coll} = 0$. Huang proved this statement for any conserved quantity [36], and his proof will be discussed in section 3 when the effects of collisions are discussed in more detail.

The balance laws come by letting $\chi$ equal the conserved values of mass $m$, linear momentum $m(v_i + \epsilon_{ipl} r_{lp} \omega_p)$, angular momentum $mr_i r_p \omega_p$, and total energy $m(e + \frac{1}{2} v_i v_i + r_p r_q \omega_p \omega_q)$ where $e = \frac{1}{2} (v'_l v'_l + r_p r_q \omega'_p \omega'_q)$ is the internal energy density [14] and $r_i$ is the distance vector emerging from the center of mass of the particle. The new velocity associated with the linear momentum arises from the combined motion of the classical translational velocity, $v_i$, and the contribution of the gyration to the total velocity, $\epsilon_{ipl} r_{lp} \omega_p$. The angular momentum is the standard expression involving the cross product of the local angular velocity induced by the gyration, $r_p \omega_p$, and the distance vector emerging from the center of mass of the particle, $r_i$. The Levi-Civita tensor, $\epsilon_{ipq}$, is used for cross products of two vectors, and has the properties:

$$\epsilon_{ipq} = \begin{cases} +1, & \text{if } (i, p, q) = (x, y, z), (z, x, y) \text{ or } (y, z, x) \\ -1, & \text{if } (i, p, q) = (y, x, z), (z, y, x) \text{ or } (x, z, y) \\ 0, & \text{otherwise} \end{cases}$$  \hspace{1cm} (8)$$

Finally, the conserved quantity of energy contains the kinetic energy associated with the local angular velocity, $r_{lp} \omega_p$, and adds this to the traditional translational kinetic energy. Substituting the conserved quantities of mass, linear momentum, angular momentum, and energy for $\chi$ into the conservation
Continuity \( (\chi_1 = m) \)
\[
\frac{\partial}{\partial t} \langle mn \rangle + \frac{\partial}{\partial x_i} \langle mnv_i \rangle = 0 \tag{9}
\]

Linear Momentum \( (\chi_2 = m(v_i + \epsilon_{ipl}r_{pl}\omega_p)) \)
\[
\frac{\partial}{\partial t} \langle mnv_i \rangle + \frac{\partial}{\partial x_i} \langle mn\epsilon_{ipl}r_{pl}\omega_p \rangle + \frac{\partial}{\partial x_i} \langle mn\epsilon_{ipl}v_s r_{pl} r_{ql} \omega_p \rangle = 0 \tag{10}
\]

Angular Momentum \( (\chi_3 = mr_i r_p \omega_p) \)
\[
\frac{\partial}{\partial t} \langle mnr_i r_p \omega_p \rangle + \frac{\partial}{\partial x_i} \langle mnr_i s_{pl} W_p \rangle = 0 \tag{11}
\]

Energy \( (\chi_4 = m(e + \frac{1}{2}[v_i v_i + r_p r_q \omega_p \omega_q])) \)
\[
\frac{\partial}{\partial t} \langle mne \rangle + \frac{\partial}{\partial x_i} \langle mne v_i \rangle + \frac{\partial}{\partial x_i} \frac{1}{2} \langle mnv_i v_i' v_i' r_p r_q \omega_p \omega_q \rangle - \rho \langle v_i \frac{\partial e}{\partial x_i} \rangle = 0 \tag{12}
\]

Here, \( e \) is internal energy and is already itself a mean quantity of the system. Since the velocity, \( v \), and gyration, \( \omega \), are separate coordinates, any derivative of the positional coordinate with respect to these variables vanishes. Letting the averages of the variables equal their mean values and splitting total variables into mean and fluctuating components, the balance laws become:

Continuity
\[
\frac{\partial}{\partial t} \rho + \frac{\partial}{\partial x_i} (\rho U_i) = 0 \tag{13}
\]

Linear Momentum
\[
\frac{\partial}{\partial t} (\rho U_s) + \frac{\partial}{\partial x_i} (\rho U_s U_i) + \frac{\partial}{\partial x_i} (\rho (v_i' v_i' + \epsilon_{spq} v_i' r_i \omega_q')) = 0 \tag{14}
\]

Angular Momentum
\[
\frac{\partial}{\partial t} (\rho i_{sp} W_p) + \frac{\partial}{\partial x_i} (\rho i_{sp} W_p U_i) + \frac{\partial}{\partial x_i} (\rho (i_{sp} \omega_i' v_i')) = 0 \tag{15}
\]

Energy
\[
\frac{\partial}{\partial t} (\rho e) + \frac{\partial}{\partial x_i} (\rho e U_i) + \frac{\partial}{\partial x_i} \frac{1}{2} (\rho v_i' v_i' v_i' + i_{pq} \omega_p' \omega_q' v_i') - \rho \langle v_i \frac{\partial e}{\partial x_i} \rangle = 0 \tag{16}
\]

Here, the properties \( \langle v_i' \rangle = 0 \) and \( \langle \omega_i' \rangle = 0 \) are employed. Additionally, the term \( \langle \epsilon_{ipl} v_s r_{pl} W_p \rangle = 0 \) as this can be viewed as an integral of the fluctuating component of the total velocity \([5,14]\). Also, the term \( i_{pq} = r_p r_q \) is used to
represent the product of the coordinates, \( r_p \), emerging from the center of mass of the particle. These coordinates measure the relative deformation of a particle, tracking how the surface varies about the center of mass. The tensor \( i_{pq} \) is related to the earlier parameter \( j \), known as the microinertia. For spherical particles, \( i_{pq} \) is reduced to \( i_{pp} \delta_{pq} = i_{pp} \), which can be shown to equal \( \frac{3}{2} j^2 \) [17]. Applying this reduction to \( i_{pq} \), the balance laws become:

**Continuity**
\[
\frac{\partial}{\partial t} \rho + \frac{\partial}{\partial x_l} (\rho U_l) = 0 \tag{17}
\]

**Linear Momentum**
\[
\frac{\partial}{\partial t} (\rho U_s) + \frac{\partial}{\partial x_l} (\rho U_s U_l) + \frac{\partial}{\partial x_l} (\rho (v_s v'_l + \rho \epsilon_{spq} v'_l r_q \omega'_p)) = 0 \tag{18}
\]

**Angular Momentum**
\[
\frac{\partial}{\partial t} \left( \frac{3}{2} \rho j W_s \right) + \frac{\partial}{\partial x_l} \left( \frac{3}{2} \rho j W_s U_l \right) + \frac{\partial}{\partial x_l} \rho \left( \frac{3}{2} j \omega'_l v'_l \right) = 0 \tag{19}
\]

**Energy**
\[
\frac{\partial}{\partial t} (\rho e) + \frac{\partial}{\partial x_l} (\rho e U_l) + \frac{\partial}{\partial x_l} \left( \frac{1}{2} \rho v'_s v'_l + \frac{3}{2} j \omega'_p v'_p v'_l - \rho v_l \frac{\partial e}{\partial x_l} \right) = 0 \tag{20}
\]

These conservation equations feature material derivatives for the mean flow variables as well as gradients of products of perturbed variables. These perturbations are variables in the distribution function, and so can be treated separately. Defining these expressions in the following way:

\[
q_{\alpha} = \frac{1}{2} \rho v'_l v'_l + \frac{3}{2} j \omega'_p \omega'_l v'_l \tag{21}
\]

\[
i_{\alpha \beta}^{\text{Bol}} = -\rho (v'_s v'_s) \tag{22}
\]

\[
i_{\alpha \beta}^{\text{Cur}} = -\rho (v'_s \epsilon_{spq} r_q \omega'_p) \tag{23}
\]

\[
m_{\alpha \beta} = -\rho \left( \frac{3}{2} j \omega'_p v'_l \right) \tag{24}
\]

Here, \( q_{\alpha} \) denotes the heat flux, \( i_{\alpha \beta}^{\text{Bol}} \) gives the Boltzmann stress, \( i_{\alpha \beta}^{\text{Cur}} \) yields the Curtiss stress, and \( m_{\alpha \beta} \) introduces the moment stress. Plugging these
expressions into the balance laws gives:

**Continuity**
\[ \frac{\partial}{\partial t} \rho + \frac{\partial}{\partial x_l} (\rho U_l) = 0 \]  \hfill (25)

**Linear Momentum**
\[ \frac{\partial}{\partial t} (\rho U_s) + \frac{\partial}{\partial x_l} (\rho U_s U_l) - \frac{\partial}{\partial x_l} (t_{Is}^{Bol} + t_{Is}^{Cur}) = 0 \]  \hfill (26)

**Angular Momentum**
\[ \frac{\partial}{\partial t} (\rho j W_s) + \frac{\partial}{\partial x_l} (\rho j W_s U_l) - \frac{2}{3} \frac{\partial}{\partial x_l} (m_{Is}) = 0 \]  \hfill (27)

**Energy**
\[ \frac{\partial}{\partial t} (\rho e) + \frac{\partial}{\partial x_l} (\rho e U_l) + \frac{\partial q_l}{\partial x_l} - \rho (v_j \frac{\partial e}{\partial x_l}) = 0 \]  \hfill (28)

At the moment, they represent only source or sink terms for the momentum and energy of the flow. These terms can be determined from the definition of the average in equation 6 using the equilibrium distribution in equation 5, which would give a very rough approximation of how they contribute to the balance laws. A more thorough treatment of their contribution, however, requires the derivation of a distribution function that accounts for departures in the fluid from equilibrium. For this function, the Chapman-Enskog process is followed to derive a first order approximation to the solution of the Boltzmann transport equation 2.

### 3 First-Order Approximation

#### 3.1 Distribution Function

The right-hand side of the transport equation 2 tracks the gain or loss of particles due to collisions in some small time interval. For the equilibrium distribution function in equation 3 the assumption was made that a large number of binary collisions occurred over a short time interval, meaning that any deviation from equilibrium would result in a rapid return to equilibrium. These binary collisions affect the initial rotation and velocity of the particle instantaneously at the moment the particles collide. Huang studied these binary collisions considering molecules with only translational velocities [36]. The existence of spin within molecules was treated through the lens of quantum mechanics, denoting different spin states as separate species of molecules. In order to account for these different spins, then, one would need to solve the Wang Chang-Uhlenbeck equation [66] for the distribution function of each of these molecular species, with a collisional integral that accounts for the cross-section calculated from the quantum states of these species. Here, the Boltzmann-Curtiss distribution function described in equation 5 treats gyration as an
additional classical variable applicable to the same molecules throughout the domain, thus requiring only one solution to describe the distribution of rotation throughout the system. Additionally, the collisional integral is easier to calculate since the rotational motion is treated as a classical motion.

The collision rate on the right-hand side of the Boltzmann transport equation (2) is given by the following integral:

$$ \frac{\partial f}{\partial t}_{\text{coll}} = \int d^3p_2 \, d^3p'_1 \, d^3p'_2 \, \delta^4(P_f - P_i) |T_{f_i}|^2 (f'_2 f'_1 - f_2 f_1) $$ (29)

Here, \( P_f \) and \( P_i \) refer to the total final and initial momenta, \( p_1 \) and \( p_2 \) refer to the initial momenta of the colliding particles while their primed counterparts, \( p'_1 \) and \( p'_2 \), each refer to their respective final linear momentum. As mentioned in the previous section, these linear momenta contain an added term to the classical linear momentum, \( p_i = mv_i \), found in the Boltzmann transport equation. Here, the Boltzmann-Curtiss linear momentum, \( p_i = m(v_i + \epsilon_{ipl} \omega_p) \), includes an additional contribution from the component of the local rotation moving in the direction of the translational velocity. The transition matrix \( T_{fi} \) contains the elements of the operator \( T(E) \) that converts the particle from its initial to final state in the collision. Finally, the distribution functions \( f_1 \) and \( f_2 \) refer to the distributions of particles containing momenta \( p_1 \) and \( p_2 \) respectively while the primed distribution functions contain the final momenta values denoted by the primed counterparts \( p'_1 \) and \( p'_2 \). Any conserved quantity for a particle initiating a binary collision, \( \chi \), integrated with the collision integral (29) vanishes. Huang proved this result by interchanging the momenta variables before and after the collision and integrating over pre-collision and post-collision linear momenta [36]. When equation (29) is used on the right-hand side of the Boltzmann-Curtiss transport equation (2) the Wang Chan-Uhlenbeck equation is obtained [66]:

$$ \left( \frac{\partial}{\partial t} + \frac{p_i}{m} \frac{\partial}{\partial x_i} \right) f = \int d^3p_2 \, d^3p'_1 \, d^3p'_2 \, \delta^4(P_f - P_i) |T_{f_i}|^2 (f'_2 f'_1 - f_2 f_1) $$ (30)

This treatment will look at a simplified version of this equation.

In observing the effect of collisions on equation (30) it is important to recognize that \( \frac{\partial f^0}{\partial t}_{\text{coll}} = 0 \) for the equilibrium Boltzmann-Curtiss distribution function defined in equation (5). This result emerges from the fact that the coefficients in equation (5) do not depend on the velocity \( v_i \) [36]. To get a good approximation of the collision integral (29) higher order approximations of \( f \) are needed. If \( g \), the deviation from the equilibrium distribution, is defined by the expression:

$$ g(x_i, p_i, t) = f(x_i, p_i, t) - f^0(x_i, p_i, t) $$ (31)
then the collision integral, e.g. equation 29 can be approximated with the following expression:

$$
\frac{\partial f}{\partial t}_{\text{coll}} \approx \int d^3p_2 \ d^3p_1' \ d^3p_2' \ \delta^4(P_f - P_i) |T_{fi}|^2
$$

$$
(f_2' g_1' - f_1 g_1 + g_2' f_1' - g_2 f_1')
$$

(32)

where squared terms involving \( g \) have been neglected due to their presumed smaller magnitude in relation to \( f_0 \). Indices associated with different distribution functions again correspond to the initial and final distributions of the particles in the binary collisions. To assess the relative magnitude of the terms within equation 32, the second term on the right-hand side can be calculated by the expression:

$$
-g_1(x,p_1,t) \int d^3p_2 \ d^3p_1' \ d^3p_2' \ [\delta^4(P_f - P_i) |T_{fi}|^2 f_0^2 = - \frac{g_1}{\tau}
$$

(33)

Here, the relaxation time constant \( \tau \) incorporates all the physics associated with the transition from initial to final states, including the transfer of angular momentum through the new variable of gyration. A more in-depth treatment of the gyration and the characteristic time constants associated with its evolution will be given in the next section.

Given the order-of-magnitude estimate to the collision integral 32, the right-hand side of the Boltzmann transport equation can be given a simpler treatment with the expression:

$$
\frac{\partial f}{\partial t}_{\text{coll}} = - \frac{f - f_0}{\tau} = - \frac{g}{\tau}
$$

(34)

The deviation function, \( g \), measures the probability that large numbers of particles will exit their equilibrium state purely through collisions. The relaxation time, \( \tau \), now gives an approximation for the entire distribution departing from equilibrium through collisions. Therefore, this time constant should characterize the transition of all degrees of freedom to and from their equilibrium states. If the time-scale of the problem is reduced such that only one motion departs from equilibrium, as Parker considered for internal rotation [54], then this time constant can be scaled to focus on this relaxation process. If further approximations are needed to account for additional physics, the relaxation time can be expanded into a series of terms that take into account these additional interactions. Chen et. al. applied this approach [13] to generate an expression for the characteristic collisional time scale of turbulent eddy interactions. Such expansions have the benefit of incorporating multiple physical processes within one time constant, allowing for the interaction of rotation and translation to affect the relaxation of the distribution function simultaneously.

If equation 34 is substituted into the transport equation 2, an approximate form of the transport equation known as the Bhatnagar, Gross, and Krook
(BGK) equation is obtained \[7\]:

\[
g = -\tau \left( \frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x_i} \right) (f^0 + g) \tag{35}
\]

For this paper, it suffices to show what forces and properties are influencing the mean flow when slight deviations to equilibrium occur. Therefore, it can be assumed that \(g << f^0\), reducing equation 35 to the form:

\[
g = -\tau \left( \frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x_i} \right) f^0 \tag{36}
\]

This equation gives a formula for finding \(g\) entirely in terms of derivatives of \(f^0\). Still, the variables in the transport equation 36 are present in \(f^0\) only through its independent variables. Therefore, to get the spatial derivatives of \(f^0\), the following derivatives of its independent variables are calculated:

\[
\frac{\partial f^0}{\partial \rho} = \frac{f^0}{\rho} \tag{37}
\]

\[
\frac{\partial f^0}{\partial \theta} = -(3 - \frac{m(v'^2 + j\omega'^2)}{2\theta}) \frac{f^0}{\theta} \tag{38}
\]

\[
\frac{\partial f^0}{\partial U_i} = \frac{mv'_{i}}{\theta} f^0 \tag{39}
\]

\[
\frac{\partial f^0}{\partial W_{i}} = \frac{mj\omega'_{i}}{\theta} f^0 \tag{40}
\]

Using the chain rule, the expression for \(g\) in equation 36 can be written as:

\[
g = -\tau f^0 \left( \frac{1}{\rho} D(\rho) + \frac{1}{\theta} \left( 3 - \frac{m(v'^2 + j\omega'^2)}{2\theta} \right) \frac{f^0}{\theta} \right) D(\theta) + \left( \frac{mv'_{i}}{\theta} D(U_{i}) \right) + \left( \frac{mj\omega'_{i}}{\theta} D(W_{i}) \right) \tag{41}
\]

where \(D(X) = \left( \frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x_i} \right) X\). The material derivatives present in equation 41 can be derived from the zeroth order balance laws. To obtain the zeroth order approximations of the equations 25, 26, 27, 28, the terms related to the perturbation of the velocity and gyration are eliminated, yielding:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho U_i}{\partial x_i} = 0 \tag{42}
\]

\[
\frac{\partial}{\partial t}(\rho U_s) + \frac{\partial}{\partial x_i}(\rho U_i U_s) = -\frac{\partial}{\partial x_s}(n\theta) \tag{43}
\]

\[
\frac{\partial}{\partial t}(\rho j W_s) + \frac{\partial}{\partial x_i}(\rho j W_s U_i) = 0 \tag{44}
\]

\[
\frac{\partial}{\partial t}(n\theta) + \frac{\partial}{\partial x_i}(n\theta U_i) = -\frac{n\theta}{3} \frac{\partial U_q}{\partial x_q} \tag{45}
\]
It should be noticed that the angular momentum equation, e.g. equation [44], is decoupled with the linear momentum equation, e.g. equation [43], at the equilibrium state. At the same time, the linear momentum equation (equation [43]) is identical to the classical Euler equation.

From these approximations to the balance laws, the material derivatives found in equation [41] are obtained:

\[
D(\rho) = \frac{\partial}{\partial x_1} \rho - \frac{\rho U_q}{\partial x_1} \quad \text{(46)}
\]

\[
D(\theta) = \frac{\partial}{\partial x_1} \theta - \frac{1}{3} \frac{\partial U_2}{\partial x_1} \quad \text{(47)}
\]

\[
D(U_i) = \frac{\partial}{\partial x_1} U_i - \frac{1}{\rho} \frac{\partial (n\theta)}{\partial x_1} \quad \text{(48)}
\]

\[
D(W_i) = \frac{\partial}{\partial x_1} W_i \quad \text{(49)}
\]

With these final expressions substituted back into equation [41], the final form of \( g \) is given as:

\[
g = \tau f^{(0)} \left[ \frac{1}{\rho} \left( v_i \frac{\partial \rho}{\partial x_1} - \rho \frac{\partial U_i}{\partial x_1} \right) - \frac{3}{\theta} \left( \frac{m(v^2 + j\omega^2)}{2\theta^2} \right) \left( v_i \frac{\partial \theta}{\partial x_1} - \frac{\theta}{3} \frac{\partial U_2}{\partial x_1} \right) \right. \\
\left. + \left( \frac{mv_i'}{\theta} \right) \left( v_i \frac{\partial U_i}{\partial x_1} - \frac{1}{\rho} \frac{\partial (n\theta)}{\partial x_1} \right) \right.
\]

\[
+ \left( \frac{mj\omega_i'}{\theta} \right) \left( v_i \frac{\partial W_i}{\partial x_1} \right) \right]
\]

(50)

Here, the first order distribution is now expressed entirely in terms of the mean and perturbed flow properties. All that remains is to find the first order approximations to the equations [21], [22], [23], and [24] to obtain non-zero expressions for the missing terms in the first-order balance laws [17], [18], [19], and [20].

3.2 Stresses and Heat Flux

With the definition of the heat flux in equation [21] and the stresses in equations [22], [23], and [24] along with the definition of the average in equation [6], the zeroth and first-order approximations to the missing terms in the balance laws can be calculated. Beginning with the zeroth order approximations, the averaging
is carried out with $f^0$ [13]:

$$q^0_{\alpha} = \frac{m\rho}{2n} \int (v'_\alpha v'_\alpha + j\omega'_\alpha \omega'_\alpha)(\frac{m\sqrt{J}}{2\pi\theta})^3$$

$$exp(-\frac{m(v'_\alpha v'_\alpha + j\omega'_\alpha \omega'_\alpha)}{2\theta}) \, d^3v' d^3\omega'$$ \hspace{1cm} (51)

$$= 0$$

$$\iota_{\alpha\beta}^{\text{Bol}.0} = -\rho \int v'_\alpha v'_\beta (\frac{m\sqrt{J}}{2\pi\theta})^3$$

$$exp(-\frac{m(v'_\alpha v'_\beta + j\omega'_\alpha \omega'_\beta)}{2\theta}) \, d^3v' d^3\omega'$$ \hspace{1cm} (52)

$$= -n\theta \delta_{\alpha\beta}$$

$$\iota_{\alpha\beta}^{\text{Cur}.0} = -\rho \epsilon_{\beta pq} r'_p \int \omega'_q v'_\alpha (\frac{m\sqrt{J}}{2\pi\theta})^3$$

$$exp(-\frac{m(v'_\alpha v'_\beta + j\omega'_\alpha \omega'_\beta)}{2\theta}) \, d^3v' d^3\omega'$$ \hspace{1cm} (53)

$$= 0$$

$$m^0_{\alpha\beta} = -\frac{3\rho j}{2} \int \omega'_\beta v'_\alpha (\frac{m\sqrt{J}}{2\pi\theta})^3$$

$$exp(-\frac{m(v'_\alpha v'_\beta + j\omega'_\alpha \omega'_\beta)}{2\theta}) \, d^3v' d^3\omega'$$ \hspace{1cm} (54)

$$= 0$$

Due to the functional form of $f^0$, those integrals possessing odd powers of $v'_p$ or $\omega'_l$ vanish. A key note is that the Boltzmann stress in equation 52 yields the hydrostatic pressure $P$ from $n\theta$ due to the assumption that the fluid is an ideal gas. This result is expected as the zeroth-order Boltzmann stress should reflect the zeroth order pressure of an ideal gas at equilibrium.

For the first-order approximations to the above stresses, the definitions must now involve volume integrals of the first-order distribution function $g$:

$$q^1_{\alpha} = \frac{m\rho}{2n} \int \int (v'_\alpha v'_\alpha + \omega'_p \omega'_p v'_\alpha) gd^3v' d^3\omega'$$ \hspace{1cm} (55)

$$\iota_{\alpha\beta}^{\text{Bol}.1} = -\rho \int \int v'_\alpha v'_\beta gd^3v' d^3\omega'$$ \hspace{1cm} (56)

$$\iota_{\alpha\beta}^{\text{Cur}.1} = -\rho \int \int \epsilon_{\beta pq} r'_p \omega'_q v'_\alpha gd^3v' d^3\omega'$$ \hspace{1cm} (57)

$$m^1_{\alpha\beta} = -\frac{3\rho j}{2} \int \int \omega'_\beta v'_\alpha gd^3v' d^3\omega'$$ \hspace{1cm} (58)
These volume integrals are more easily evaluated if they can be converted into surface integrals. Since there is no angular dependence in these integrals, the spherical symmetry of the integrands implies:

$$\int \int G(v', \omega') d^3v' d^3\omega' = 16\pi^2 \int \int v'^2 \omega'^2 G(v', \omega') dv' d\omega'$$

(59)

Additionally, due to the functional form of \( f^0 \), terms involving the average of vector components of different indices, such as \( \langle v'_\alpha v'_\beta \rangle \), retain non-zero values for the integral only when indices match. Therefore, the identities \( \langle v'_\alpha v'_\beta \rangle = \delta_{\alpha\beta} \frac{\langle v'^2 \rangle}{4} \) and \( \langle v'_\alpha v'_\beta v'_i v'_l \rangle = \frac{\langle v'^4 \rangle}{15} (\delta_{\alpha\beta} \delta_{il} + \delta_{\alpha i} \delta_{\beta l} + \delta_{\alpha l} \delta_{\beta i}) \) are employed for both the velocity and gyration variables. Applying all these properties to our volume
integrals yields:

\[
q^1_\alpha = - \left[ \frac{8\pi^2 m \rho \tau}{3} \int \int dv' d\omega' (v'^6 \omega'^2 + j v'^4 \omega'^4) \right. \\
+ \left. \left\{ \frac{4}{\theta} + m(v'^2 + j \omega'^2) \right\} \left( \frac{m \sqrt{J}}{2\pi \theta} \right)^3 \exp\left( - \frac{m(v'^2 + j \omega'^2)}{2\theta} \right) \right] \frac{\partial \theta}{\partial x_\alpha}
\]

\[= -(4n\tau \theta) \frac{\partial \theta}{\partial x_\alpha}\]

\[
\lambda^{\text{Bol.1}}_{\alpha \beta} = \left[ \frac{16\pi^2 \tau \rho}{15\theta} \eta^6 \omega'^2 (\frac{m \sqrt{J}}{2\pi \theta})^\frac{3}{2} \right. \\
\exp\left( - \frac{m(v'^2 + j \omega'^2)}{2\theta} \right) dv' d\omega' \right] \\
\left( \frac{\partial U_\alpha}{\partial x_\beta} + \frac{\partial U_\beta}{\partial x_\alpha} + \delta_{\alpha \beta} \frac{\partial U_l}{\partial x_l} \right) \\
- \left[ \frac{16\pi^2 \tau \rho}{3\theta} \int v'^4 \omega'^2 \left( \frac{m \sqrt{J}}{2\pi \theta} \right)^\frac{3}{2} \right. \\
\exp\left( - \frac{m(v'^2 + j \omega'^2)}{2\theta} \right) dv' d\omega' \left. \right] \delta_{\alpha \beta} \frac{\partial U_l}{\partial x_l}
\]

\[= n\tau \theta \left( \frac{\partial U_\alpha}{\partial x_\beta} + \frac{\partial U_\beta}{\partial x_\alpha} \right) - \frac{n\tau \theta}{3} \left( \frac{\partial U_l}{\partial x_l} \delta_{\alpha \beta} \right)\]

\[
\lambda^{\text{Cur.1}}_{\alpha \beta} = \left[ \frac{16\pi^2 \tau \rho m_j}{9\theta} \left( \frac{m \sqrt{J}}{2\pi \theta} \right)^3 \int v'^4 \omega'^4 \right. \\
\exp\left( - \frac{m(v'^2 + j \omega'^2)}{2\theta} \right) dv' d\omega' \left. \right] \epsilon_{\beta pq} \frac{\partial W_p}{\partial x_\alpha}
\]

\[= (n\tau \theta) \epsilon_{\beta pq} \frac{\partial W_p}{\partial x_\alpha}\]

\[
m^1_{\alpha \beta} = \left[ \frac{48\pi^2 \tau \rho j^2 m}{2\theta} \int \omega'^2 \omega'^2 \omega' \omega' v' v' (\frac{m \sqrt{J}}{2\pi \theta})^3 \right. \\
\exp\left( - \frac{m(v'^2 + j \omega'^2)}{2\theta} \right) dv' d\omega' \left. \right] \frac{\partial W_l}{\partial x_p}
\]

\[= \left( \frac{3n\tau j \theta}{2} \right) \frac{\partial W_p}{\partial x_\alpha}\]

The reduced forms of these stresses appear to follow familiar patterns. The heat flux in equation 60 appears to demonstrate a direct proportionality rela-
relationship with the temperature gradient. The Boltzmann stress contains terms related to the familiar strain-rates and divergences of the velocity. Still, these stresses all have nonlinear dependence on the temperature, meaning that simplifications will have to be made before direct comparisons with classical fluids can occur.

4 Governing Equations

Equations 60, 61, 62 and 63 serve as the constitutive models for the first order approximation to the Boltzmann-Curtiss distribution and close the governing equations. A direct substitution of the stresses found in equations 60, 61, 62, and 63 into the first-order balance laws 25, 26, 27, and 28 yields:

Continuity
\[
\frac{\partial}{\partial t} \rho + \frac{\partial}{\partial x_i} (\rho U_i) = 0
\]  

Linear Momentum
\[
\frac{\partial}{\partial t} (\rho U_s) + \frac{\partial}{\partial x_l} (\rho U_l U_s) - \frac{\partial}{\partial x_l} \left[ -P \delta_{sl} + n \tau \theta \left( \frac{\partial U_l}{\partial x_s} + \frac{\partial U_s}{\partial x_l} \right) - \frac{n \tau \theta}{3} \frac{\partial U_q}{\partial x_q} \delta_{sl} \right] - \frac{\partial}{\partial x_l} (n \tau \theta \epsilon_{spq} r_q \frac{\partial W_p}{\partial x_l}) = 0
\]

Angular Momentum
\[
\frac{\partial}{\partial t} (\rho j W_s) + \frac{\partial}{\partial x_l} (\rho j W_s U_l) - \frac{\partial}{\partial x_l} \left[ (n \tau \theta) \frac{\partial W_s}{\partial x_l} \right] = 0
\]

Energy
\[
\frac{\partial}{\partial t} (\rho e) + \frac{\partial}{\partial x_l} (\rho e U_l) - \frac{\partial}{\partial x_l} \left( 4 n \tau \theta \frac{\partial \theta}{\partial x_l} \right) - \rho \left( \frac{\partial v_l}{\partial x_l} \frac{\partial e}{\partial x_l} \right) = 0
\]

These equations contain derivatives of nonlinear terms and products of spatially varying variables. For this first-order approximation to the balance laws, the products of gradients of terms are presumed to vanish. Furthermore, equation 66 contains a spatial derivative of the spatial coordinate \( r_p \) that has its origin at the center of mass of the spherical particle. Looking at Figure 1, the expression for this coordinate is easily derived in terms of the Eulerian coordinates: \( r_i = x'_i - x_i \). Therefore:

\[
\frac{\partial r_i}{\partial x_l} = -\delta_{il}
\]

Clearly the derivative is zero unless the components of \( x \) and \( r \) are the same. Taking this derivative into account, removing terms associated with products of gradients, and allowing for the existence of body forces, the governing equa-
tions become:

**Continuity**
\[
\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x_l}(\rho U_l) = 0
\]

(69)

**Linear Momentum**
\[
\frac{\partial}{\partial t}(\rho U_s) + \frac{\partial}{\partial x_l}(\rho U_s U_l) + \frac{\partial P}{\partial x_s} - n\tau\theta \left(\frac{\partial^2 U_s}{\partial x_l \partial x_l} + \frac{2}{3} \frac{\partial^2 U_l}{\partial x_l \partial x_l}\right) - n\tau\theta \epsilon_{spq} \frac{\partial W_q}{\partial x_p} - \rho F_s = 0
\]

(70)

**Angular Momentum**
\[
\frac{\partial}{\partial t}(\rho j W_s) + \frac{\partial}{\partial x_l}(\rho j W_s U_l) - n\tau j \theta \frac{\partial^2 W_s}{\partial x_l \partial x_l} - \rho L_s = 0
\]

(71)

**Energy**
\[
\frac{\partial}{\partial t}(\rho e) + \frac{\partial}{\partial x_l}(\rho e U_l) - (4n\tau\theta)\frac{\partial^2 \theta}{\partial x_l \partial x_l} - \rho \langle v_l \frac{\partial e}{\partial x_l} \rangle - \rho H = 0
\]

(72)

In the preceding equations the body forces \(\rho F_s\) and \(\rho L_s\) have been introduced to account for external phenomena unrelated to the stresses previously introduced. Body forces for the linear momentum are easily found from the classical approach and require no special treatment. In the independent angular momentum equation, however, the factors affecting \(\rho L_s\) are more subtle.

Fig. 1 illustrates a body force created by the presence of vorticity near an individual particle. The connection between the two particles is symbolized by the coefficient \(\nu_r\). The motion of the right-hand particle creates the classical rotational motion, or macroscopic angular velocity, which induces the local rotation of the left particle. The amount of influence the angular velocity has on the gyration is determined by the value of \(\nu_r\). The body force disappears once the
local rotation of the left particle equals the angular velocity, represented by half of the vorticity. De Groot and Mazur characterized this body force as an asymmetric pressure tensor \[23\], which had a linear relationship with the difference between the gyration and the angular velocity:

\[
\rho L_{\text{interior}} = \nu_r (\epsilon_{spq} \frac{\partial U_q}{\partial x_p} - 2W_s) \quad (73)
\]

Here, \(\nu_r\) is designated as the “rotational viscosity,” measuring the strength of induced gyration on a particle caused by the presence of a difference between its gyration and the local vorticity. This interior body force couples the local rotation with the translational velocity, ensuring that the linear momentum equation \[70\] and angular momentum equation \[71\] remain intertwined as long as the value of \(\rho L_{\text{interior}}\) remains non-zero. The total angular momentum body force, \(\rho L_s\), can be viewed as the sum of this induced interior force and any external body moment force, \(\rho L_s = \rho (L_{\text{interior}} + L_{\text{exterior}})\).

The continuity equation \[69\] is clearly the classical continuity equation for the mean velocity field. The deviation from classical kinetic theory becomes clear in the momenta equations. The compressible Navier-Stokes linear momentum equation, with the assumed satisfaction of Stokes’s hypothesis, has the form:

\[
\frac{\partial}{\partial t} (\rho U_s) + \frac{\partial}{\partial x_i} (\rho U_i U_s) + \frac{\partial P}{\partial x_s} - \mu \frac{\partial^2 U_s}{\partial x_i \partial x_i} + \frac{2\mu}{3} \frac{\partial^2 U_i}{\partial x_i \partial x_s} - \rho F_s = 0 \quad (74)
\]

Here, \(\mu\) is the dynamic viscosity of the classical fluid. Comparing equations \[74\] and \[70\] the formulations are very similar, with the molecular viscosity from the Navier-Stokes equations represented by the expression \(n r \theta\), as is expected from the first-order approximation to the Boltzmann transport equation \[36\]. The reduction of equations \[69\], \[70\] and \[71\] to the Navier-Stokes description will be discussed in more detail in the next section.

The new term introduced by the preceding kinetic description has the form, \(n r \theta s_{pq} \frac{\partial W_q}{\partial x_p}\). Here, the new variable of gyration, \(W_q\), produces an additional source of linear momentum due to its transverse gradient. A notable insight
is that the expression in front of the gyration gradient is also $n \tau \theta$, suggesting that the new term may contain a coefficient similar to the viscosity presented in classical fluids theory. To understand the meaning and importance of this new term in the linear momentum equation, the linear momentum equation from MCT is presented [70]:

$$\frac{\partial}{\partial t}(\rho U_s) + \frac{\partial}{\partial x_i}(\rho U_s U_l) + \frac{\partial P}{\partial x_s} - (\lambda + \mu) \frac{\partial^2 U_s}{\partial x_i \partial x_l} - (\mu + \kappa) \frac{\partial^2 U_s}{\partial x_l \partial x_l} - \kappa \epsilon_{spq} \frac{\partial W_q}{\partial x_p} - \rho F_s = 0 \quad (75)$$

Here, $\lambda$ represents the second coefficient of viscosity and a new coupling coefficient, $\kappa$, is added to the total viscosity of the MCT fluid. Additionally, this coupling coefficient corresponds to the coefficient described in [2] as it determines the strength of the force induced by relative rotation within the MCT fluid. This theory, derived from the approach of rational continuum thermomechanics (RCT) [16,24,25,26], starts with the same picture of the fluid and derives governing equations from kinematic and thermodynamic principles for a fluid with spherical particles. Comparing equations 70 and 75, the term associated with the transverse gradient in the kinetic equation now has a counterpart term associated with the coupling coefficient $\kappa$. Therefore, the first-order approximation to the Boltzmann-Curtiss transport equation produces a linear momentum equation consistent with the MCT formulation. Comparisons between the expressions for the coefficients in front of identical terms in these equations will shed light into the validity of these expressions for the new coefficients in MCT.

5 Physical Meaning of Relaxation Time

The simplification of the collisional term in equation [34] presumes that a singular relaxation time can be used to describe the transition from the real distribution function $f$ to the equilibrium distribution function $f^0$. Due to the extra degrees of freedom introduced by the local rotation of the molecules, this relaxation time cannot be equated directly to the case of classical fluids. Still, as Chen et. al. demonstrated [13], expressions for a singular relaxation time can incorporate multiple processes or models involving several degrees of freedom. These expressions typically start from a base time constant applied to the relaxation of the motions of the molecular motion. In the current treatment, this base relaxation time would apply to the gyration.

De Groot and Mazur investigated the case of viscous flow in an isotropic fluid, but allowed for asymmetry in the pressure tensor. This asymmetry required for the consideration of an independent conservation theorem for angular momentum. Furthermore, pressure asymmetry generated “internal angular momentum,” $S_p$, which arose from the local angular velocity, $\omega_p$, of groups of
particles at a point in the system. From conservation of angular momentum, De Groot and Mazur derived a balance equation for the internal angular momentum [23]:

\[ \rho \frac{dS_q}{dt} = -2 \Pi_q \]  

(76)

Here, \( \Pi_q \) is the asymmetrical component of the pressure tensor. Internal angular momentum could be easily related to the angular velocity through \( S_q = I \omega_q \), where \( I \) denotes the average moment of inertia of the constituent particles. The asymmetric pressure tensor, however, needed a more nuanced treatment. By deriving relations for the conservation of internal energy and entropy production, De Groot and Mazur found the thermodynamic force associated with the asymmetric pressure tensor [23]. This force emerged from a difference between the local and classical angular velocities, \( \omega_s - \frac{1}{2} \epsilon_{spq} v_{q,p} \).

Invoking Curie’s principle [18] regarding thermodynamic fluxes and forces, De Groot and Mazur derived the following relation [23]:

\[ \Pi_s = \nu_r (2 \omega_s - \epsilon_{spq} \frac{\partial v_q}{\partial x_p}) \]  

(77)

Clearly, the asymmetric pressure tensor mirrors the body force found in equation 73 indicating that the body force of the kinetic description can be obtained from a consideration of thermodynamic fluxes and forces. Given this closure relation, the conservation of internal angular momentum in equation 76 became:

\[ \frac{d\omega_s}{dt} = -2 \nu_r \rho I (2 \omega_s - \epsilon_{spq} \frac{\partial v_q}{\partial x_p}) \]  

(78)

This equation is equivalent to the kinetic angular momentum equation 71 with the diffusion terms eliminated. Therefore, the kinetic theory is shown to obtain a more general form of a conservation equation. For the case of initially zero local angular velocity and constant vorticity, the solution to equation 78 becomes:

\[ \omega_s = \frac{1}{2} \epsilon_{spq} \frac{\partial v_q}{\partial x_p} (1 - e^{-t/\tau_o}) \]  

(79)

where the decay of the local angular velocity is characterized by a relaxation time constant, \( \tau_o \), that has the form:

\[ \tau_o = \frac{\rho I}{4 \nu_r} \]  

(80)

Measurements of diatomic hydrogen and deuterium mixtures at \( p = 1 \, \text{atm} \) and \( T = 77K \) by Montero et. al. give a value of \( 2.20 \times 10^{-8} \, \text{s} \) for the rotational relaxation time [48]. Thus, the assumptions of zero initial local rotation, constant vorticity, and absence of external forces, leads to the derivation of a characteristic relaxation time that exclusively applied to internal angular momentum. These assumptions become relevant when the characteristic time
is sufficiently reduced such that macroscale phenomena, such as the vorticity, can be approximated as constant compared with the evolution of local rotation. In these short time scales, equilibrium is achieved for the local rotation once it approaches the constant vorticity. Equation 80 provides a suitable first approximation of the characteristic relaxation time, $\tau$, used in our kinetic theory description. De Groot’s characterization of local angular velocity as the mean angular velocity of groups of particles matches the physical picture of our kinetic theory description. The addition of body forces into the governing kinetic theory equations can also incorporate the thermodynamic forces found in De Groot and Mazur’s treatment. The rotational viscosity, $\nu_r$, has a counterpart through the coupling coefficient $\kappa$ in the MCT linear and angular momentum equations [15]. Therefore, numerical simulations of the kinetic and MCT descriptions should be able to determine the appropriate conditions for the use of equation 80 in this first order approximation.

6 Reduction to Navier-Stokes Equations

The introduction of local rotation, $\omega_s$, as an independent variable has resulted in a slightly different physical picture from the classical fluids description shown in the Navier-Stokes equations. The angular momentum equation 71 is not derived from the linear momentum equation 70, while the classical vorticity equation can only be derived from the classical linear momentum equation previously shown in equation 74. Still, the physical picture from which equations 69, 70, 71, and 72 are derived differs from Boltzmann’s classical picture of a monatomic gas only through the introduction of the variable of gyration. When the gyration of a particle is distinct from macroscopic rotation, as defined by the angular velocity, $\frac{1}{2} \epsilon_{sab} \frac{\partial U_b}{\partial x_a}$, the new form of the linear momentum equation 70 and the independent angular momentum equation 71 can provide an alternative description to the classical Navier-Stokes picture. The difference of vorticity and gyration forms an objective (frame-indifferent) description of rotational motion, absolute rotation, as

$$\Omega_{AR} = \epsilon_{sab} \frac{\partial U_b}{\partial x_a} - 2W_s$$  

The disappearance of absolute rotation indicates that vorticity is solely responsible for the local rotation and the dependence of gyration vanishes. Thus, such relation is called as angular equivalence. When zero absolute rotation occurs, i.e. angular equivalence, the gyration provides no new insight from the classical description. Therefore, the governing equations derived in previous sections should reduce to the Navier-Stokes equations. Setting $W_s = \frac{1}{2} \epsilon_{sab} \frac{\partial U_b}{\partial x_a}$ in the
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governing momentum equations 70 and 71 yields:

Reduced Linear Momentum
\[
\frac{\partial}{\partial t} (\rho U_s) + \frac{\partial}{\partial x_l} (\rho U_s U_l) + \frac{\partial P}{\partial x_s} - n \tau \theta \left( \frac{\partial^2 U_s}{\partial x_l \partial x_l} + \frac{2}{3} \frac{\partial^2 U_l}{\partial x_l \partial x_s} \right) - n \tau \theta \varepsilon_{s p q} \left( \frac{\partial U_b}{\partial x_a} \right) = 0
\] (82)

Reduced Angular Momentum
\[
\frac{\partial}{\partial t} (\rho \epsilon_{s a b} \frac{\partial U_b}{\partial x_a}) + \frac{\partial}{\partial x_l} (\rho \epsilon_{s a b} \frac{\partial U_b}{\partial x_a} U_l) - n \tau \theta \frac{\partial^2}{\partial x_l \partial x_l} (\epsilon_{s a b} \frac{\partial U_b}{\partial x_a}) - 2 \rho L_{\text{exterior}} = 0
\] (83)

The common terms of the microinertia \( j \) and \( \frac{1}{2} \) have been eliminated from equation 83. A key observation from equation 83 is the absence of the interior body force, \( \rho L_{\text{interior}} \), described in equation 73. The difference in rotational motions necessary for the inducement of gyration on a particle has vanished, thus making \( \rho L_{\text{interior}} = 0 \). Meanwhile, equation 83 matches the form of the vorticity equation, derived from the curl of the Navier-Stokes linear momentum equation 74:
\[
\frac{\partial}{\partial t} (\rho \epsilon_{s a b} \frac{\partial U_b}{\partial x_a}) + \frac{\partial}{\partial x_l} (\rho \epsilon_{s a b} \frac{\partial U_b}{\partial x_a} U_l) - \mu \frac{\partial^2}{\partial x_l \partial x_l} (\epsilon_{s a b} \frac{\partial U_b}{\partial x_a}) - \rho \epsilon_{s a b} \frac{\partial F_b}{\partial x_a} = 0
\] (84)

Looking at the reduced linear momentum equation 82, further manipulations will show how this equation matches the classical picture. Using the identity for the Levi-Civita tensor \( \epsilon_{s a b} \epsilon_{p q r} = \delta_{a p} \delta_{b q} - \delta_{a q} \delta_{b p} \) and contracting the appropriate indices, equation 82 becomes:
\[
\frac{\partial}{\partial t} (\rho U_s) + \frac{\partial}{\partial x_l} (\rho U_s U_l) + \frac{\partial P}{\partial x_s} - n \tau \theta \left( \frac{\partial^2 U_s}{\partial x_l \partial x_l} + \frac{2}{3} \frac{\partial^2 U_l}{\partial x_l \partial x_s} \right) - \frac{n \tau \theta}{2} \left( \frac{\partial^2 U_p}{\partial x_s \partial x_p} - \frac{\partial^2 U_s}{\partial x_q \partial x_q} \right) - \rho F_s = 0
\] (85)

Grouping together like terms yields the type II of the Navier-Stokes linear momentum equation:
\[
\frac{\partial}{\partial t} (\rho U_s) + \frac{\partial}{\partial x_l} (\rho U_s U_l) + \frac{\partial P}{\partial x_s} - n \tau \theta \frac{\partial^2 U_s}{\partial x_l \partial x_l} - \frac{7 n \tau \theta}{6} \frac{\partial^2 U_p}{\partial x_s \partial x_p} - \rho F_s = 0
\] (86)

The form of the classical momenta equations is achieved when local rotation is indistinguishable from macroscopic rotation. Still, the precise formulation found in equations 86 and 83 requires a more detailed treatment. Following
B-C Equations for Flows with Spin

Boltzmann-Curtiss Equation

\[ \frac{\partial}{\partial \phi_i} = 0 \quad \text{No angular dependence} \]

Boltzmann Equation

MCT

NS Type I

NS Type II

Euler

\[ \Delta S = 0 \]

In the type II of the Navier-Stokes equations, i.e. equation (86), the expression for the coefficient in front of the diffusion term is half that value in the Type I, i.e. equation (87), due to the contribution from the new term associated with the curl of the gyration. This term originally contained a coefficient that matched the form of the classical viscosity, but applied to the contribution of local rotation not found in the classical description. The temperature dependence of viscous rotational motion appears to have a slightly different limiting behavior as the particle rotation begins to resemble macroscopic motion.

Figure 3 shows the map between morphing continuum theory and Navier-Stokes theory of the classical kinetic theory formulation and Boltzmann distribution, it leads to the Type I of the Navier-Stokes equations [36]:

\[
\frac{\partial}{\partial t} (\rho U_s) + \frac{\partial}{\partial x_l} (\rho U_s U_l) + \frac{\partial P}{\partial x_s} - n\tau \theta \frac{\partial^2 U_s}{\partial x_l \partial x_l} - n\tau \theta \frac{\partial^2 U_p}{3 \partial x_s \partial x_p} - \rho F_s = 0 \quad (87)
\]

In the type II of the Navier-Stokes equations, i.e. equation (86) the expression for the coefficient in front of the diffusion term is half that value in the Type I, i.e. equation (87) due to the contribution from the new term associated with the curl of the gyration. This term originally contained a coefficient that matched the form of the classical viscosity, but applied to the contribution of local rotation not found in the classical description. The temperature dependence of viscous rotational motion appears to have a slightly different limiting behavior as the particle rotation begins to resemble macroscopic motion.

Figure 3 shows the map between morphing continuum theory and Navier-Stokes equations from both the perspective of kinetic theory and rational continuum mechanics. From a kinetic point of view, Boltzmann equation can be obtained by dropping the angular dependence in the Boltzmann-Curtiss equation. For Boltzmann equations, two different distribution functions can be used to further deriving the conservation equations. The first one is the classical Boltzmann distribution. When the system is at the Boltzmann distribution (zero-th order approximation), i.e. equilibrium, the Boltzmann equation leads to Euler’s equations. Furthermore, if the system is linearly deviated from the Boltzmann distribution (first order approximation), type I of the Navier-Stokes equation can be obtained. Similarly, if the Boltzmann-Curtiss distribution is
adopted, the zero-th order approximation also leads to the Euler’s equation. It is noticed that the first order approximation of the Boltzmann-Curtiss distribution is assumed for the system, the transport equation leads to morphing continuum theory as presented in this study. Interestingly, one of the correlations between morphing continuum theory and the type II of the Navier-Stokes equations is the angular motion equivalence or the vanishing absolute rotation. As discussed in the previous section, the type I and II of Navier-Stokes equation differ in the angular motion dependence of the distribution functions. This concludes the theoretical development and relations between classical and morphing continua.

7 Conclusion

With the North American X-15 flying Mach 6.72 in the hypersonic regime and the anticipation of American supersonic drones by the 2030s, understanding the nonequilibrium phenomena in aerothermodynamic flows becomes a priority. Before the internal rotation and vibrations modes are excited, the molecular rotations first impact the hypersonic flows. This phenomenon is considered one of the prominent examples for flows with local spin. Turbulence is another example requiring local spin. It is well-known that turbulence can be viewed as a tangle of vortex and eddy filaments [39]. These vortex and eddy filaments are extremely rotational and severely impact the mean flow. Traditionally, these rotational motions are associated with the translational velocities through vorticity. The vorticity, however, changes depending on the observer [34], thus limiting its applicability to turbulence physics. It has been known that the classical NS theory does not yield satisfactory results for flow physics at high Mach number since 1920s [9]. It is often believed that the nonequilibrium phenomena in high speed flows is not captured by the first order constitutive models from Boltzmann-Maxwell distribution in NS theory.

In the meantime, a high order continuum theory considering local spin, e.g. morphing continuum theory [15], can be derived under rational continuum thermomechanics (RCT) or constructed from a high order distribution function, e.g. Boltzmann-Curtiss distribution [14]. Furthermore, MCT has shown promise in analyzing turbulent flows [12,55,59,70] by considering local spin caused by eddies and vortices as an independent variable. The current study even show that even a first order deviation to the Boltzmann-Curtiss distribution provides satisfactory results equivalent to DSMC or Burnett equations or Super Burnett equations for hypersonic flows without any computational burdens or numerical deficiencies [2]. Therefore, it is crucial to provide the physical meaning of RCT-based continuum theories with the physics-based kinetic theory from the Boltzmann-Curtiss transport equation.

The first-order approximation to the Boltzmann-Curtiss transport equation was able to yield governing equations with terms corresponding to particular stresses in the Navier-Stokes equations and MCT. Furthermore, new material parameters, introduced by Chen in the zeroth-order approximation
to fluids with spherical particles [14], received expressions based on relaxation
time, number density, and equilibrium thermal energy. Resulting equations
showed that the contribution of local rotation to the Cauchy stress and vis-
cous diffusion were weighted equally in the kinetic description of the linear
momentum equation. When transverse gradients in gyration disappear, the
kinetic equation becomes the classical linear momentum equation, with the
expression for the total viscosity equivalent to the result in classical fluids. It
should be noticed that the current formulation is at the continuum level. The
rotation here refers to the spin of the whole molecule and should not be conf-
fused with the rotation inside a molecule. The internal rotation and vibration
modes are not within the scope and should be treated separately while these
effects are dominant.

Further work is needed to address the proposal that the separate variables
of gyration and translation each contain their own relaxation time, and that
the relaxation of the gyration can be approximated by equation 80. Exper-
imental efforts into flows where local rotation can be treated independently
from translation will help answer this question, and will also test the expres-
sions derived for the coefficients of the viscous diffusion and rotational Cauchy
stress terms. The predictions of the rotational relaxation time should be com-
pared with Parker [54] and Monchick [47] to determine whether the treatment
of gyration as a non-internal, independent variable is an accurate treatment
of the flow. Previous MCT simulations indicate [10,11,68,69,70] that treating
local rotation as an independent variable leads to successful modeling of turbu-
lement fluctuations. Therefore, the successful derivation of the MCT equations
by this kinetic description suggests that the independence of molecular rotation
should result in realistic predictions of flows with strong local rotation.

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