Abstract. For a connected semisimple group $G$ over the field of real numbers $\mathbb{R}$, using a method of Onishchik and Vinberg, we compute the first Galois cohomology set $H^1(\mathbb{R}, G)$ in terms of Kac labelings of the affine Dynkin diagram of $G$.

0. Introduction

0.1. Let $G$ be a linear algebraic group defined over the field of real numbers $\mathbb{R}$. For the definition of the first (nonabelian) Galois cohomology set $H^1(\mathbb{R}, G)$ see Serre’s book [23, Sect. I.5]; see also Section 1 below. Galois cohomology can be used to answer many natural questions (on classification of real forms of algebraic varieties with additional structure, on classification of real orbits in a homogeneous space, etc.); see Serre [23, Sect. III.1]. In this article we shall consider the case when $G$ is connected and semisimple. Note that it is very interesting to know the Galois cohomology of a semisimple or reductive group over a number field, but this actually reduces to a calculation of Galois cohomology over $\mathbb{R}$ (and a calculation of a certain abelian group); see Sansuc [21, Cor. 4.5] and Borovoi [4, Thm. 5.11].

0.2. In this article, by a semisimple or reductive algebraic group we always mean a connected semisimple or reductive algebraic group, unless otherwise specified. A real algebraic group $G$ is called $\mathbb{R}$-simple if it has no positive-dimensional normal algebraic subgroups defined over $\mathbb{R}$. A real algebraic group $G$ is called absolutely simple if it is both $\mathbb{R}$-simple and $\mathbb{C}$-simple. The set of absolutely simple algebraic groups forms a prime subfield of the field of algebraic groups (see [9, Sect. 5.3]).
simple if it remains simple after extension of scalars from $\mathbb{R}$ to $\mathbb{C}$.

The Galois cohomology sets $H^1(\mathbb{R}, G)$ of the classical groups are well known. Recently the sets $H^1(\mathbb{R}, G)$ were computed for "most" of the absolutely simple $\mathbb{R}$-groups by Adams and Taibi [1], in particular, for all simply connected absolutely simple $\mathbb{R}$-groups by Adams and Taibi [1] and by Borovoi and Evenor [6]. A simply connected $\mathbb{R}$-simple group $G$ that is not absolutely simple is isomorphic to a group obtained by the Weil restriction of scalars from a simply connected simple $\mathbb{C}$-group. It follows that $H^1(\mathbb{R}, G) = 1$ in this case. Now if $G$ is any simply connected semisimple $\mathbb{R}$-group, then $G$ is the direct product of its (simply connected) $\mathbb{R}$-simple normal subgroups:

$$G = \prod_i G_i,$$

and hence

$$H^1(\mathbb{R}, G) = \prod_i H^1(\mathbb{R}, G_i),$$

where we know $H^1(\mathbb{R}, G_i)$ for each $G_i$, see above. Thus $H^1(\mathbb{R}, G)$ is known for all simply connected semisimple $\mathbb{R}$-groups $G$.

Kac [18] used infinite-dimensional Lie algebras to classify the conjugacy classes of automorphisms of finite order of a simple Lie algebra over the field of complex numbers $\mathbb{C}$ in terms of what is now called Kac diagrams. For another method giving the same description of the conjugacy classes of automorphisms of finite order in terms of Kac diagrams, see Onishchik and Vinberg [20, Sect. 4.4], and also Gorbatevich, Onishchik, and Vinberg [14, Sect. 3.3]; the authors write that their approach goes back to Gantmacher [12]. In particular, these methods and results give a classification of involutions; see [20, Sect. 5.1.5] and [14, Sect. 4.1.4]. A slight modification of these methods and results gives $H^1(\mathbb{R}, G)$ for all absolutely simple, adjoint $\mathbb{R}$-groups $G$. Arguing as above, we see that $H^1(\mathbb{R}, G)$ is actually known for all adjoint semisimple $\mathbb{R}$-groups $G$.

In the present article we consider a general semisimple $\mathbb{R}$-group $G$, not necessarily simply connected or adjoint. We use the method of Onishchik and Vinberg to compute $H^1(\mathbb{R}, G)$ in terms of Kac diagrams, or, as we say, in terms of Kac labelings of the affine Dynkin diagram of $G$. Our main result is Theorem 13.3. It gives a simple uniform combinatorial description of the Galois cohomology set $H^1(\mathbb{R}, G)$ for any semisimple $\mathbb{R}$-group $G$. This permits one to describe easily additional structures on $H^1(\mathbb{R}, G)$, see below. Using this result, we shall compute the Galois cohomology of reductive $\mathbb{R}$-groups in a forthcoming article.

0.3. In general, the Galois cohomology set $H^1(\mathbb{R}, G)$ of a linear algebraic $\mathbb{R}$-group $G$ has no natural group structure. It has only a distinguished point, the neutral element; see Subsection 1.4. We see that $H^1(\mathbb{R}, G)$ is just a pointed set, and one is tempted to conclude that it has no other structure.

However, these pointed sets $H^1(\mathbb{R}, G)$ together have an important additional structure: functoriality. Namely, the correspondence $G \leadsto H^1(\mathbb{R}, G)$ is a functor. This means that for any homomorphism of algebraic $\mathbb{R}$-groups $\varphi: G \to G'$ we have the induced morphism of pointed sets

$$\varphi_*: H^1(\mathbb{R}, G) \to H^1(\mathbb{R}, G').$$

For applications it is important to know $\varphi_*$. 
Our description of $H^1(\mathbb{R}, G)$ for a semisimple $\mathbb{R}$-group $G$ in Main Theorem 13.3 permits one to compute easily the map $\varphi_*$ in the case when $\varphi$ is a normal homomorphism, that is, when $\varphi(G)$ is normal in $G'$. See Section 14, where the case of an isogeny $G \to G'$ is treated in detail.

Another additional structure on $H^1(\mathbb{R}, G)$ is twisting. Let $G$ be an algebraic $\mathbb{R}$-group and $a \in Z^1(\mathbb{R}, G)$ be a cocycle. Consider the inner twist $\vartheta_a: H^1(\mathbb{R}, G) \to H^1(\mathbb{R}, G)$ sending the neutral element of $H^1(\mathbb{R}, aG)$ to the class of $a$ in $H^1(\mathbb{R}, G)$. For a semisimple $\mathbb{R}$-group $G$, using our description of $H^1(\mathbb{R}, G)$ in Main Theorem 13.3, we can easily compute the map $\vartheta_a$; see Proposition 14.3.

For many applications of Galois cohomology (for instance, for classification of real orbits in real loci of complex homogeneous spaces) it is important to know explicit cocycles representing each cohomology class in $H^1(\mathbb{R}, G)$. Our Main Theorem 13.3 gives such representatives for semisimple $G$.

0.4. We describe the contents of the article.

Sections 1–3 contain preliminary material on real algebraic groups, Galois cohomology over $\mathbb{R}$, and structure, automorphisms, and real forms of reductive groups. In Section 4 we explain how to reduce the computation of Galois cohomology of a reductive $\mathbb{R}$-group $G$ to a calculation with a maximal compact subtorus $T_0$ of $G$. In Section 5 we identify $H^1(\mathbb{R}, G)$ with the orbit set for the action of a certain subgroup of the normalizer of $T_0$ on the set $(T_0)_2$ of elements in $T_0$ of order $\leq 2$ by twisted conjugation. We get rid of the twist in Section 6 by shifting $(T_0)_2$ to a disconnected group with identity component $G$.

Section 7 collects some standard facts about restricted roots of $T_0$ and the (twisted) affine Weyl group, to be used in the next section. In Section 8 we reduce our problem to a description of the orbit set for a certain discrete group of affine transformations of the Lie algebra of $T_0$. We obtain a description for this orbit set in Sections 9–11 for simple $\mathbb{R}$-groups and in Section 12 for an arbitrary semisimple $\mathbb{R}$-group.

In Section 13 we deduce our main result describing $H^1(\mathbb{R}, G)$ for semisimple $G$ in terms of Kac labelings. Functoriality of our description is discussed in Section 14. Sections 15–17 contain examples of explicit computation of Galois cohomology by our method.

0.5. Notation and conventions

- $\mathbb{Z}$ denotes the ring of integers.
- $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the fields of rational numbers, of real numbers, and of complex numbers, respectively.
- $i \in \mathbb{C}$ is such that $i^2 = -1$. (Our results do not depend on the choice of $i$.)
- $\Gamma = \text{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \gamma\}$, the Galois group of $\mathbb{C}$ over $\mathbb{R}$, where $\gamma$ is the complex conjugation.
- We denote real algebraic groups by boldface letters $G$, $T$, $\ldots$, their complexifications by respective Roman (non-bold) letters $G = G \times_\mathbb{R} \mathbb{C}$, $T = T \times_\mathbb{R} \mathbb{C}$, $\ldots$, and the corresponding complex Lie algebras by respective lowercase Gothic letters $g = \text{Lie } G$, $t = \text{Lie } T$, $\ldots$. 
For a homomorphism \( \varphi : G \to H \) of algebraic (or Lie) groups, the differential at the unity \( d\varphi : g \to h \) is a homomorphism of Lie algebras. By abuse of notation, we often write \( \varphi \) instead of \( d\varphi \).

- \( G^0 \) denotes the identity component of an algebraic (or Lie) group \( G \).
- \( \text{Inn} G \) denotes the group of inner automorphisms of a group \( G \).
- \( \text{inn}(g) : x \mapsto gxg^{-1} \) denotes the inner automorphism of a group \( G \) corresponding to an element \( g \) of \( G \).
- \( G \) is a connected reductive \( \mathbb{R} \)-group, unless otherwise specified.
- \( Z(G) \) denotes the center of \( G \).
- \( Z_G(\ ) \) denotes the centralizer in \( G \) of the set in the parentheses.
- \( N_G(\ ) \) denotes the normalizer in \( G \) of the group in the parentheses.
- \( G^\text{ad} := G/Z(G) \) denotes the corresponding adjoint group.
- \( G^\text{der} = [G, G] \), the derived group of \( G \).
- \( G^\text{sc} \) denotes the universal cover of the connected semisimple group \( G^\text{der} \).
- \( T \subseteq G \) is a maximal torus (defined over \( \mathbb{R} \)).
- \( T^\text{ad} := T/Z(G) \) denotes the image of \( T \) in \( G^\text{ad} \), which is a maximal torus in \( G^\text{ad} \).
- \( T^\text{sc} \) denotes the preimage of \( T \) in \( G^\text{sc} \), which is a maximal torus in \( G^\text{sc} \).
- \( \chi^*(T) = \text{Hom}(T, \mathbb{C}^\times) \), the character group of \( T \), where \( \text{Hom} \) denotes the group of homomorphisms of algebraic \( \mathbb{C} \)-groups. We regard \( \chi^*(T) \) as a lattice in the dual space \( t^* \) of \( t \), in view of the canonical embedding \( \chi^*(T) \hookrightarrow t^* \), \( \chi \mapsto d\chi \).
- \( \chi_*(T) = \text{Hom}(\mathbb{C}^\times, T) \), the cocharacter group of \( T \). We regard \( \chi_*(T) \) as a lattice in \( t \), in view of the canonical embedding \( \chi_*(T) \hookrightarrow t \), \( \nu \mapsto d\nu(1) \).
- \( A_2 \) denotes the set of elements of order dividing 2 in a subset \( A \) of some group.
- \( A_2^a \) denotes the set of elements of \( A \) with square \( a \).
- By an involution (of an algebraic group, based root datum etc.) we mean an automorphism with square identity. In particular, we regard the identity automorphism as the trivial involution.

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1. Real algebraic groups and Galois cohomology

1.1. Let \( G \) be a real linear algebraic group. In the coordinate language, the reader may regard \( G \) as a subgroup in the general linear group \( \text{GL}_n(\mathbb{C}) \) (for some integer \( n \)) defined by polynomial equations with real coefficients in the matrix entries; see Borel [2, Sect. 1.1]. More conceptually, the reader may assume that \( G \)
is an affine group scheme of finite type over \( \mathbb{R} \); see Milne [19, Def. 1.1]. With any of these two equivalent definitions, \( G \) defines a covariant functor

\[ A \mapsto G(A) \]

from the category of commutative unital \( \mathbb{R} \)-algebras to the category of groups. Applying this functor to the \( \mathbb{R} \)-algebra \( \mathbb{R} \), we obtain a real Lie group \( G(\mathbb{R}) \). Applying this functor to the \( \mathbb{R} \)-algebra \( \mathbb{C} \) and to the morphism of \( \mathbb{R} \)-algebras

\[ \gamma: \mathbb{C} \to \mathbb{C}, \quad z \mapsto \bar{z} \quad \text{for} \quad z \in \mathbb{C}, \]

we obtain a complex Lie group \( G(\mathbb{C}) \) together with an anti-holomorphic involution \( G(\mathbb{C}) \to G(\mathbb{C}) \), which will be denoted by \( \sigma_G \). The Galois group \( \Gamma \) naturally acts on \( G(\mathbb{C}) \); namely, the complex conjugation \( \gamma \) acts by \( \sigma_G \). We have \( G(\mathbb{R}) = G(\mathbb{C})^\Gamma \) (the subgroup of fixed points).

We shall consider the linear algebraic group \( G \times_{\mathbb{R}} \mathbb{C} \) obtained from \( G \) by extension of scalars from \( \mathbb{R} \) to \( \mathbb{C} \). In this article we shall denote \( G \times_{\mathbb{R}} \mathbb{C} \) by \( G \), the same Latin letter, but non-boldface (though the standard notation is \( G_{\mathbb{C}} \)). By abuse of notation we shall identify \( G \) with \( G(\mathbb{C}) \); in particular, we shall write \( g \in G \) meaning that \( g \in G(\mathbb{C}) \).

Since \( G \) is an affine group scheme over \( \mathbb{C} \), we have the ring of regular function \( \mathbb{C}[G] = \mathbb{R}[G] \otimes_{\mathbb{R}} \mathbb{C} \). Our anti-holomorphic involution \( \sigma_G \) of \( G(\mathbb{C}) \) is anti-regular in the following sense: when acting on the ring of holomorphic functions on \( G \), it preserves the subring \( \mathbb{C}[G] \). An anti-regular involution of \( G \) is called also a real structure on \( G \).

1.2. Remark. If \( G \) is a reductive algebraic group over \( \mathbb{C} \) (not necessarily connected), then any anti-holomorphic involution of \( G \) is anti-regular. The hypothesis that \( G \) is reductive is necessary: the abelian algebraic group \( \mathbb{C} \times \mathbb{C}^\times \) has the anti-holomorphic involution \( (z, w) \mapsto (\bar{z}, \exp(i\bar{z})\bar{w}) \) that is not anti-regular. See Adams and Taïbi [1, Lemma 3.1] and Cornulier [11].

1.3. A morphism of real linear algebraic groups \( G \to G' \) induces a morphism of pairs \( (G, \sigma_G) \to (G', \sigma_{G'}) \). In this way we obtain a functor \( G \mapsto (G, \sigma_G) \). By Galois descent, see Serre [22, V.4.20, Cor. 2 of Prop. 12] or Jahnel [16, Thm. 2.2], this functor is an equivalence of categories. In particular, any pair \( (G, \sigma) \), where \( G \) is a complex linear algebraic group and \( \sigma \) is a real structure on \( G \), is isomorphic to a pair coming from a real linear algebraic group \( G \), and any morphism of pairs \( (G, \sigma) \to (G', \sigma') \) comes from a unique morphism of the corresponding real algebraic groups.

From now on, when mentioning a real algebraic group \( G \), we shall actually work with a pair \( (G, \sigma) \), where \( G \) is a complex algebraic group and \( \sigma \) is a real structure on \( G \). We shall write \( G = (G, \sigma) \). We shall shorten “real linear algebraic group” to “\( \mathbb{R} \)-group”.

1.4. Let \( A \) be a \( \Gamma \)-group, that is, an abstract group (not necessarily abelian) endowed with an action of \( \Gamma = \text{Gal}(\mathbb{C}/\mathbb{R}) \). We denote this action by putting a superscript on the left, so that \( \gamma \) sends \( a \in A \) to \( \gamma a \). We consider the first
cohomology pointed set $H^1(\Gamma, A)$; see Serre [23, Sect. I.5]. Recall that the group $A$ acts on the set of 1-cocycles

$$Z^1(\Gamma, A) = \{ z \in A \mid z \cdot \gamma z = 1 \}$$

by “twisted conjugation”:

$$a : z \mapsto a \cdot z \cdot \gamma a^{-1} \quad \text{for } a \in A, \ z \in Z^1(\Gamma, A).$$

By definition, $H^1(\Gamma, A)$ is the set of orbits of $A$ in $Z^1(\Gamma, A)$. In general $H^1(\Gamma, A)$ has no natural group structure, but it has a neutral element, the class of the cocycle $1 \in Z^1(\Gamma, A) \subseteq A$.

If the group $A$ is abelian, we consider the subgroup of 1-coboundaries

$$B^1(\Gamma, A) = \{ a \cdot \gamma a^{-1} \mid a \in A \}$$

of the abelian group $Z^1(\Gamma, A)$. Then we have

$$H^1(\Gamma, A) = Z^1(\Gamma, A) / B^1(\Gamma, A).$$

Thus $H^1(\Gamma, A)$ is naturally an abelian group in this case.

Let $G = (G, \sigma)$ be a real algebraic group. The Galois group $\Gamma = \{1, \gamma\}$ acts on $G = G(\mathbb{C})$ (namely, $\gamma$ acts as $\sigma$), and the first Galois cohomology set of $G$ is defined as

$$H^1(\mathbb{R}, G) = H^1(\Gamma, G).$$

2. Complex reductive groups

2.1. Let $G$ be a (connected) reductive group over $\mathbb{C}$, and let $T \subset G$ be a maximal torus. We denote by

$$X = X^*(T) \quad \text{and} \quad X^\vee = X_*(T)$$

the character and cocharacter lattices of $T$, respectively. We have a canonical pairing

$$\langle \cdot, \cdot \rangle : X \times X^\vee \to \mathbb{Z}, \quad \langle \chi, \nu \rangle \mapsto \langle \chi, \nu \rangle \quad \text{for } \chi \in X, \ \nu \in X^\vee,$$

where $\chi \circ \nu = (z \mapsto z^{(\chi, \nu)})$ for $z \in \mathbb{C}^\times$. Under the canonical embeddings $X \hookrightarrow T^*$ and $X^\vee \hookrightarrow t$ of 0.5, this pairing is compatible with the pairing between $t^*$ and $t$, and makes the lattices $X$ and $X^\vee$ dual to each other.

2.2. We consider the root system $R = R(G, T)$. Then

$$g = t \oplus \bigoplus_{\alpha \in R} g_\alpha,$$

where $g_\alpha$ is the eigenspace corresponding to the root $\alpha$. Let $B \subset G$ be a Borel subgroup containing $T$; then

$$b = t \oplus \bigoplus_{\alpha \in R_+} g_\alpha,$$
be a maximal torus, and $S$ the set of coroots. We denote by $D = D(G) = D(G, T, B)$ the Dynkin diagram of $G$; then $S$ is identified with the set of vertices of $D(G, T, B)$. We do not assume that the Dynkin diagram $D(G)$ is connected.

The Dynkin diagram $D(G)$ does not determine $G$ uniquely up to an isomorphism. We recall the notion of the based root datum of $G$.

Let $G$ be a reductive group over $\mathbb{C}$, $T \subset G$ be a maximal torus, and $B \subset G$ be a Borel subgroup containing $T$. Let

$$BRD(G) = BRD(G, T, B) = (X, X^\vee, R, R^\vee, S, S^\vee)$$

denote the based root datum of $G$. Here $X = X^*(T)$ is the character group of $T$, $X^\vee = X^*_s(T)$ is the cocharacter group of $T$, $R = R(G, T) \subset X$ is the root system, $R^\vee \subset X^\vee$ is the coroot system, $S = S(G, T, B) \subset R$ is the basis of $R$ defined by $B$, $S^\vee \subset R^\vee$ is the corresponding basis of $R^\vee$. We have a canonical pairing $X \times X^\vee \rightarrow \mathbb{Z}$ and compatible canonical bijections $R \leftrightarrow R^\vee$, $S \leftrightarrow S^\vee$. See Springer [24, Sects. 1 and 2] for details.

2.3. Recall the notions of the lattices of roots, weights, coroots and coweights for an abstract (not necessarily reduced) root system $R$: the root lattice $Q = Q(R)$ is spanned by all roots in $R$, the coroot lattice $Q^\vee = Q(R^\vee)$ is spanned by all coroots in the dual root system $R^\vee$, and the weight lattice $P = P(R)$, resp. the coweight lattice $P^\vee = P(R^\vee)$, is the dual lattice of $Q^\vee$, resp. of $Q$; see Bourbaki [8, Sect. VI.1.9] for details.

With any basis $S$ of $R$ (that is, a set of simple roots) one canonically associates a basis $S^\vee$ of $R^\vee$, called the set of simple coroots relative to $S$; see [8, Sect. VI.1.5, Rem. 5]. The sets $S$ and $S^\vee$ are bases of the lattices $Q$ and $Q^\vee$, respectively. Note that for a non-reduced root system $R$ the set $S^\vee$ is not the set of coroots $\alpha^\vee$ over all $\alpha \in S$. The set of fundamental weights (resp. coweights) relative to $S$ is the basis of $P$ (resp. of $P^\vee$) dual to $S^\vee$, resp. to $S$; see [8, Sect. VI.1.10]. (Note that in loc. cit. the fundamental (co)weights are defined only for a reduced root system, though this restriction is unnecessary.)

2.4. We recall a construction of the canonical homomorphism

$$\text{Aut } G \rightarrow \text{Aut } BRD(G), \quad \varphi \mapsto \varphi_* \quad \text{for } \varphi \in \text{Aut } G$$

with kernel $\text{Inn } G$. Let $\varphi \in \text{Aut } G$. Consider the maximal torus $\varphi(T) \subset G$ and the Borel subgroup $\varphi(B) \subset G$ containing $\varphi(T)$. Then there exists $g \in G$ such that

$$(2.5) \quad g \cdot \varphi(T) \cdot g^{-1} = T \quad \text{and} \quad g \cdot \varphi(B) \cdot g^{-1} = B.$$ 

Moreover, if $g' \in G$ is another such element, then $g' = tg$ for some $t \in T$. The automorphism $\text{inn}(g) \circ \varphi$ of $G$ preserves $T$ and $B$, and thus induces an automorphism $\varphi_*$ of $BRD(G, T, B)$. Namely,

$$(\varphi_* \chi)(t) = \chi(\varphi^{-1}(g^{-1}tg)) \quad \text{for } \chi \in X, \ t \in T,$$
$$(\varphi_* \nu)(z) = g\varphi(\nu(z))g^{-1} \quad \text{for } \nu \in X^\vee, \ z \in \mathbb{C}^\times.$$
One checks immediately that $\varphi_*$ preserves the pairing $X \times X^\vee \to \mathbb{Z}$. It follows from (2.5) that $\varphi_*$ preserves the subsets $S \subset R \subset X$ and $S^\vee \subset R^\vee \subset X^\vee$. One can easily check that $\varphi_*$ does not depend on the choice of $g \in G$ and that the map $\varphi \mapsto \varphi_*$ is a homomorphism. See, for instance, [7, Sect. 3.2 and Prop. 3.1(a)] for details. Note that there is a canonical homomorphism

$$\text{Aut } \text{BRD}(G) \to \text{Aut } D(G),$$

which is injective when $G$ is semisimple.

2.6. Definition. A pinning of $(G, T, B)$ is a family $(e_\alpha)_{\alpha \in S}$, where $e_\alpha \in \mathfrak{g}_\alpha, e_\alpha \neq 0$ for all $\alpha \in S$.

2.7. Lemma (see Conrad [9, Prop. 1.5.5] or Milne [19, Prop. 23.44]).

Let $(e_\alpha)_{\alpha \in S}$ be a pinning of $(G, T, B)$. Then the natural homomorphism

$$\text{Aut}(G, T, B, (e_\alpha)) \to \text{Aut } \text{BRD}(G, T, B)$$

is an isomorphism.

Inverting the isomorphism of Lemma 2.7, we obtain a homomorphism

$$\text{Aut } \text{BRD}(G, T, B) \xrightarrow{\sim} \text{Aut}(G, T, B, (e_\alpha)) \hookrightarrow \text{Aut}(G),$$

which is a splitting of the homomorphism

$$\text{Aut } G \to \text{Aut } \text{BRD}(G), \quad \varphi \mapsto \varphi_*.$$

It follows that the latter homomorphism is surjective. By abuse of notation, we identify $\text{Aut } \text{BRD}(G)$ with its image in $\text{Aut } G$ under the embedding (2.8) and get an isomorphism

$$\text{Aut } G \xrightarrow{\sim} \text{Inn } G \rtimes \text{Aut } \text{BRD}(G)$$

depending on the chosen pinning $(e_\alpha)$.

3. Compact reductive groups

A (connected) reductive $\mathbb{R}$-group $G = (G, \sigma)$ is called compact if the real Lie group $G(\mathbb{R}) = G^\sigma$ is compact. In this case the real structure $\sigma$ on $G$ is called compact.

3.1. Proposition (Weyl, Chevalley). Any reductive $\mathbb{C}$-group $G$ admits a compact real structure. Any two compact real structures on $G$ are conjugate by an inner automorphism of $G$.

Proof. See, for instance, Onishchik and Vinberg [20, 5.2.3, Thms. 8 and 9].

3.2. Definition (Adams and Taibi [1, Def. 3.12]). Let $\sigma$ be a real structure on a reductive $\mathbb{C}$-group $G$. A Cartan involution for $\sigma$ is an involutive regular automorphism $\theta$ of $G$ commuting with $\sigma$ and such that the anti-regular automorphism $\sigma \circ \theta = \theta \circ \sigma$ is a compact real structure on $G$.

The Cartan involution $\theta$ as defined above is the complexification of the classical Cartan involution of the real Lie group $G^\sigma$. 
3.3. Proposition. Let $\sigma$ be a real structure on a reductive $\mathbb{C}$-group $G$. Then there exists a Cartan involution $\theta$ for $\sigma$. The correspondence $\sigma \sim \theta$ induces a bijection between the set of conjugacy classes of real structures on $G$ and the set of conjugacy classes of involutive regular automorphisms of $G$, where in both cases the conjugation action of inner automorphisms of $G$ is considered.

This result is well known for semisimple $G$; see for instance Onishchik and Vinberg [20, 5.1.4, Thms. 3 and 4]. For a proof for a general reductive group (even not necessarily connected) see Adams and Taïbi [1, Thm. 3.13(1)(a) and Cor. 3.17].

3.4. By Proposition 3.3 any reductive $\mathbb{R}$-group is obtained from a compact reductive $\mathbb{R}$-group $G = (G, \sigma_c)$ by twisting by an involutive real automorphism $\theta$ of $G$. In other words, any reductive $\mathbb{R}$-group is of the form

$$\rho G = (G, \sigma),$$

where $\sigma = \theta \circ \sigma_c$, $\sigma_c$ is a compact structure on $G$, $\theta \in \text{Aut} G$, $\theta^2 = \text{id}$. Our aim is to compute $H^1(\mathbb{R}, \rho G)$. To this end, we need to analyze the structure of $\theta$.

The fixed point subgroup $G^\theta$ is reductive (though possibly disconnected), and $(G^\theta)^0$ is compact. Choose a maximal torus $T_0 \subset (G^\theta)^0$.

3.5. Lemma. $T = Z_G(T_0)$ is a $\theta$-stable maximal torus in $G$.

This is a particular case of a more general statement about any semisimple automorphism $\theta$ of $G$; see for instance [14, 3.3.8, Thm. 3.13]. For convenience of the reader, we provide a simple proof adapted to our case.

Proof. The group $T$ is connected and reductive; see Humphreys [15, Thm. 22.3 and Cor. 26.2.A]. Since $T$ is $\theta$-stable, $t$ is a graded Lie subalgebra of $g$, where the grading $g = g^\theta \oplus g^{-\theta}$ modulo 2 is given by the decomposition into the sum of eigenspaces $g^{\pm \theta}$ for the differential of $\theta$ with eigenvalues $\pm 1$, respectively.

Since $t_0$ is a Cartan subalgebra in $g^\theta$, it coincides with its centralizer in $g^\theta$. Hence we have $t \cap g^\theta = t_0$. The reductive Lie algebra $t$ is the direct sum of its center $z(t)$ and the derived subalgebra $t^{\text{der}} = [t, t]$. Both summands are graded and, since $z(t) \supseteq t_0 = t \cap g^\theta$, we have $t^{\text{der}} \subseteq g^{-\theta}$. It follows that $[t^{\text{der}}, t^{\text{der}}] \subseteq g^\theta$. Since also $[t^{\text{der}}, t^{\text{der}}] \subseteq t^{\text{der}} \subseteq g^{-\theta}$, we see that $[t^{\text{der}}, t^{\text{der}}] = 0$. We conclude that $t$ is abelian, because otherwise $t^{\text{der}}$ would be nonzero and semisimple, hence nonabelian. Therefore $T$ is a torus, and clearly it is a maximal torus containing $T_0$. \hfill \Box

3.6. We have a decomposition $T = T_0 \cdot T_1$ into an almost direct product of algebraic tori, where $\theta$ acts on $T_0$ trivially and on $T_1$ as inversion. The respective Lie algebra decomposition is $t = t_0 \oplus t_1$, where $t_0 = t \cap g^\theta$ and $t_1 = t \cap g^{-\theta}$. Note that $\rho T_0$ is a maximal torus in $\rho G$, $\rho T_1$ is the maximal split subtorus in $\rho T$, and $\rho T_0 = T_0$ is the maximal compact subtorus in $\rho T$. Since $\rho T$ is the centralizer of $T_0$ in $\rho G$, we see that $T_0$ is a maximal compact torus in $\rho G$.

Consider the root system $R = R(G, T)$, on which $\theta$ acts naturally. The restrictions of all roots $\alpha \in R$ to $T_0$ are nonzero (in the additive terminology), because $Z_G(T_0) = T$. Choose a sufficiently general lattice vector $\nu \in (X^\vee)^\theta = X_*(T_0)$, so that $\langle \alpha, \nu \rangle \neq 0, \forall \alpha \in R$, and define the set of positive roots $R_+ \subset R$ by the
condition $\langle \alpha, \nu \rangle > 0$. Then $R_+$ and the corresponding set of simple roots $S \subset R_+$ are preserved under $\theta$. We denote by $B$ the Borel subgroup of $G$ containing $T$ which corresponds to $R_+$.

3.7. Proposition. Let $G = (G, \sigma_c)$ be a compact reductive $\mathbb{R}$-group, $\theta$ an involutive real automorphism of $G$, and $T = Z_G(T_0)$ be the centralizer of a maximal torus $T_0 \subset G^\theta$. Set $\tau = \theta_\ast \in \text{Aut BRD}(G, T, B)$. There exists a pinning $(e_\alpha)_{\alpha \in S}$ such that

$$\theta = \text{inn}(t_0) \circ \tau,$$

where $\tau$ is regarded as an involutive automorphism of $(G, T, B, (e_\alpha))$ via the isomorphism (2.8) and $t_0 \in T_0 = (T^\tau)^0$ is such that $t_0^2 \in Z(G^{\text{der}})$.

Again, this is a particular case, for which we provide a simple proof, of a more general statement about any semisimple automorphism $\theta$ of $G$; see for instance [14, 3.3.8, Thms. 3.12(4) and 3.13].

Proof. First, take an arbitrary pinning $(e_\alpha)$ of $(G, T, B)$ and consider the respective automorphism $\tau$ of $(G, T, B, (e_\alpha))$. Then $\theta$ and $\tau$ act similarly on $T$, whence $\theta = \text{inn}(t_0) \circ \tau$ for some $t \in Z_G(T) = T$.

We decompose: $t = t_0 t_1$, where $t_0 \in T_0$ and $t_1 \in T_1$. Choose $s \in T_1$ such that $s^2 = t_1$. Easy calculations show that the automorphism $\text{inn}(t_1) \circ \tau$ is still involutive and preserves the pinning $(\text{Ad}(s) e_\alpha)$. Replacing $\tau$ by $\text{inn}(t_1) \circ \tau$, we obtain $\theta = \text{inn}(t_0) \circ \tau$, as desired.

Finally, note that $T_0$ decomposes into a product of maximal compact tori in $\varrho G^{\text{der}}$ and in $Z(\varrho G)^0$. Decomposing $t_0$ respectively, we see that only the first factor contributes to $\text{inn}(t_0)$. Thus we may assume $t_0 \in T_0 \cap G^{\text{der}}$. Since $\tau$ commutes with $\text{inn}(t_0)$, and both $\tau$ and $\theta$ are involutive, $\text{inn}(t_0)$ is also involutive, whence $t_0^2 \in Z(G^{\text{der}})$. \hfill $\Box$

3.8. Remark. Since $t_0$ is an element of finite order, it belongs to $T_0(\mathbb{R})$. Hence $\text{inn}(t_0)$ and $\tau$ commute with $\sigma_c$, that is, belong to $\text{Aut } G$.

4. Reduction to a maximal compact torus

From now on we shall use the following notations:

4.1. Notation

- $G = (G, \sigma_c)$ is a compact connected reductive $\mathbb{R}$-group. We denote the action of $\sigma_c$ on $G$ by bar: $\bar{g} = \sigma_c(g)$ for $g \in G$.
- $T \subset G$ is a maximal torus.
- $B \subset G$ is a Borel subgroup containing $T$.
- $\tau \in \text{Aut BRD}(G, T, B)$, $\tau^2 = \text{id}$. We also regard $\tau$ as a real automorphism of $(G, T, B)$ (see Remark 3.8) preserving a given pinning $(e_\alpha)_{\alpha \in S}$.
- $T_0 = (T^\tau)^0$ and $T_1 = (T^{-\tau})^0$, where $T^{\pm \tau} = \{ t \in T \mid \tau(t) = t^{\pm 1} \}$. Then $T = T_0 \cdot T_1$. Note that both $T_0$ and $T_1$ are compact tori and $T(\mathbb{R}) = T_0(\mathbb{R}) \cdot T_1(\mathbb{R})$.
- $t_0 \in T_0$ is such that $t_0^2 \in Z(G^{\text{der}})$.
- $\theta = \text{inn}(t_0) \circ \tau$ is an involutive automorphism of $G$. It preserves $T$ and $B$, and acts on the Dynkin diagram $D(G)$ by the involution $\tau$. 
• \( \varrho G = (G, \sigma) \), where \( \sigma = \theta \circ \sigma_\varphi \).
• \( N = N_G(T) \) and \( N_0 = N_G(T_0) \). Note that \( N_0 \subseteq N \), because \( T = Z_G(T_0) \).
• \( W = N/T \) and \( W_0 = N_0/T \) are the Weyl groups of \( T \) and \( T_0 \) in \( G \), respectively.

4.2. We wish to compute \( H^1(\mathbb{R}, \varrho G) \). The set of 1-cocycles for \( \varrho G \) is by definition

\[
Z^1(\mathbb{R}, \varrho G) = \{ a \in G \mid a \cdot \theta(\bar{a}) = 1 \}.
\]

The group \( \varrho G(\mathbb{C}) = G \) acts on \( Z^1(\mathbb{R}, \varrho G) \) by the formula

\[
g : a \mapsto g \cdot a \cdot \theta(\bar{g})^{-1} \quad \text{for } g \in G, \ a \in Z^1(\mathbb{R}, \varrho G),
\]

and by definition \( H^1(\mathbb{R}, \varrho G) \) is the set of orbits for this action. See Subsection 1.4.

We shall show that the action of \( N_0 \subset G \) on \( T \) by formula (4.3) preserves \( Z^1(\mathbb{R}, \varrho T) \) and induces an action of \( W_0 \) on \( H^1(\mathbb{R}, \varrho T) \).

Let \( n \in N_0, s \in T \); then \( nsn^{-1} \in T \). We define actions \( *_{\theta} \) and \( \check{s}_{\theta} \) of \( N_0 \) on \( T \) by

\[
(4.4) \quad n *_{\theta} s := ns \theta(n)^{-1} = nsn^{-1} \cdot n \theta(n)^{-1},
\]

\[
(4.5) \quad n \check{s}_{\theta} s := ns \theta(\bar{n})^{-1} = nsn^{-1} \cdot n \theta(\bar{n})^{-1}.
\]

These actions are well defined by the following lemma.

4.6. Lemma. If \( n \in N_0, s \in T \), then \( n *_{\theta} s \in T \) and \( n \check{s}_{\theta} s \in T \).

Proof. Since \( \theta \) acts trivially on \( T_0 \) and since \( W_0 \) naturally embeds into \( \text{Aut}(T_0) \), we see that \( \theta \) acts trivially on \( W_0 \). It follows that \( n \cdot \theta(n)^{-1} \in T \), and hence

\[
n *_{\theta} s = nsn^{-1} \cdot n \theta(n)^{-1} \in T.
\]

Since \( T_0 \) is a compact torus, all its complex automorphisms are defined over \( \mathbb{R} \); hence the complex conjugation acts trivially on \( \text{Aut}(T_0) \) and on \( W_0 \), whence

\[
n \bar{n}^{-1} \in T \quad \text{and} \quad \theta(n) \cdot \theta(\bar{n})^{-1} = \theta(n \bar{n}^{-1}) \in T.
\]

Consequently,

\[
n \cdot \theta(\bar{n})^{-1} = n \theta(n)^{-1} \cdot \theta(n) \theta(\bar{n})^{-1} \in T,
\]

thus

\[
n \check{s}_{\theta} s = ns \theta(\bar{n})^{-1} = nsn^{-1} \cdot n \theta(\bar{n})^{-1} \in T. \quad \Box
\]

Let \( n \in N_0 \) and let \( s \in Z^1(\mathbb{R}, \varrho T) \subset T \), then by Lemma 4.6 we have \( n \check{s}_{\theta} s \in T \). Comparing (4.3) and (4.5), we see that \( n \check{s}_{\theta} s \) is cohomologous to \( s \) in \( Z^1(\mathbb{R}, \varrho G) \), and hence \( n \check{s}_{\theta} s \) is a cocycle, that is, \( n \check{s}_{\theta} s \in Z^1(\mathbb{R}, \varrho T) \). We obtain an action of \( N_0 \) on \( Z^1(\mathbb{R}, \varrho T) \), which we again denote by \( \check{s}_{\theta} \).

4.7. Lemma. The action \( \check{s}_{\theta} \) of \( N_0 \) on \( Z^1(\mathbb{R}, \varrho T) \) induces an action of \( N_0 \) on \( H^1(\mathbb{R}, \varrho T) \), which in turn induces an action of \( W_0 \) on \( H^1(\mathbb{R}, \varrho T) \).
Proof. Let $s, s' \in Z^1(\mathbb{R}, \rho T)$, and assume that $s' \sim s$, that is, $s' = ts\theta(t)^{-1}$ for some $t \in T$. Then

$$n \bar{s}_\theta s' = ns'\theta(n)^{-1} = nts\theta(t)^{-1}\theta(n)^{-1} = ntn^{-1} \cdot ns\theta(n)^{-1} \cdot \theta(n)\theta(t)^{-1}\theta(n)^{-1} = t' \cdot n \bar{s}_\theta s \cdot \theta(t')^{-1}$$

where $t' = ntn^{-1}$. Thus $n \bar{s}_\theta s' \sim n \bar{s}_\theta s$ and we see that indeed $N_0$ acts on $H^1(\mathbb{R}, \rho T)$. If $t \in T \subset N_0$, then

$$t \bar{s}_\theta s = t \cdot s \cdot \theta(t)^{-1} \sim s.$$

Thus $T$ acts on $H^1(\mathbb{R}, \rho T)$ trivially, and $\bar{s}_\theta$ indeed induces an action of $W_0$ on $H^1(\mathbb{R}, \rho T)$. \qed

We denote the action of $W_0$ on $H^1(\mathbb{R}, \rho T)$ of Lemma 4.7 again by $\bar{s}_\theta$. We denote the space of orbits of this action by $H^1(\mathbb{R}, \rho T)/W_0$.

4.8. Theorem (Borovoi [3, Thm. 1], [5, Thm. 9]). The embedding of the abelian group $Z^1(\mathbb{R}, \rho T)$ into $Z^1(\mathbb{R}, \rho G)$ induces a bijection

$$H^1(\mathbb{R}, \rho T)/W_0 \cong H^1(\mathbb{R}, \rho G).$$

4.9. Remark. Write $W = N/T$. It is easy to see that $\rho W(\mathbb{R}) = W_0$ and that the action $\bar{s}_\theta$ of $W_0$ on $H^1(\mathbb{R}, \rho T)$ is the action of $\rho W(\mathbb{R})$ related to the cohomology exact sequence

$$\cdots \rightarrow \rho W(\mathbb{R}) \rightarrow H^1(\mathbb{R}, \rho T) \rightarrow H^1(\mathbb{R}, \rho N) \rightarrow \cdots$$

coming from the short exact sequence

$$1 \rightarrow \rho T \rightarrow \rho N \rightarrow \rho W \rightarrow 1.$$

See Serre [23, I.5.5, before Prop. 39].

4.10. Theorem 4.8 reduces the computation of $H^1(\mathbb{R}, \rho G)$ to computing $H^1(\mathbb{R}, \rho T)$ and describing the orbit set for the above action of $W_0$ on $H^1(\mathbb{R}, \rho T)$. We compute $H^1(\mathbb{R}, \rho T)$ here and postpone describing the orbit set of $W_0$ to Section 5 (see Proposition 5.6).

Let $s \in (T_0)_2$; then clearly $s \in Z^1(\mathbb{R}, T_0) \subseteq Z^1(\mathbb{R}, \rho T)$. For any $s \in T_0 \cap T_1$, the equalities $\theta(s) = s$ and $\theta(s) = s^{-1}$ imply $s^2 = 1$; hence $T_0 \cap T_1 \subseteq (T_0)_2$. If $s \in T_0 \cap T_1$, we have $s \in Z^1(\mathbb{R}, \rho T_1)$, because $s \in (T_0)_2 \subset Z^1(\mathbb{R}, \rho T)$. Furthermore, then $s$ is a coboundary in $\rho T_1$ (because $\rho T_1$ is split and by Hilbert’s Theorem 90 $H^1(\mathbb{R}, \rho T_1) = 1$); hence $s$ is a coboundary in $\rho T$. We see that the embedding $(T_0)_2 \hookrightarrow Z^1(\mathbb{R}, \rho T)$ induces a homomorphism

$$(T_0)_2/(T_0 \cap T_1) \rightarrow H^1(\mathbb{R}, \rho T).$$

4.11. Lemma ([3, Lem. 1], [5, Lem. 3]). The canonical homomorphism of finite abelian groups

$$(T_0)_2/(T_0 \cap T_1) \rightarrow H^1(\mathbb{R}, \rho T)$$

is an isomorphism.
5. The twisted normalizer of a maximal compact torus

5.1. Construction. We define the twisted normalizer of $T_0$ to be the algebraic $\mathbb{R}$-subgroup $N_\tau$ of $N_0$ such that

$$N_\tau = \{ n \in N_0 \mid n \cdot T_0 \cdot \theta(n)^{-1} = T_0 \} = \{ n \in N_0 \mid n \cdot \theta(n)^{-1} \in T_0 \}.$$  

It depends only on $\tau \in \text{Aut}(G, T, B)$ and not on $t_\theta$. Indeed,

$$n \cdot \theta(n)^{-1} = n \cdot t_\theta \tau(n)^{-1} t_\theta^{-1} = n t_\theta n^{-1} \cdot n \tau(n)^{-1} \cdot t_\theta^{-1},$$

where the two extreme factors belong to $T_0$ whenever $n \in N_0$.

5.2. Lemma.  

(i) $n \cdot \theta(n)^{-1} \in (T_0)_2$ for all $n \in N_\tau$.

(ii) $T \cap N_\tau = \{ t_0 t_1 \mid t_0 \in T_0, t_1 \in T_1, t_1^2 \in T_0 \cap T_1 \}$.

(iii) The $*_\theta$-action of $N_\tau$ preserves the three sets $T_0 \supset T_0(\mathbb{R}) \supset (T_0)_2$.

Proof. (i) Put $t = n \cdot \theta(n)^{-1} \in T_0$. Since $t \in T_0$, we have $\theta(t) = t$. Thus

$$t = \theta(t) = \theta(n) \cdot n^{-1} = t^{-1},$$

hence $t^2 = 1$, that is, $t \in (T_0)_2$, as required.

(ii) If $t \in T$, $t = t_0 t_1$, where $t_0 \in T_0$, $t_1 \in T_1$, then $t \cdot \theta(t)^{-1} = t_1^2$. Hence $t \in T \cap N_\tau$ if and only if $t_1^2 \in T_0 \cap T_1$.

(iii) The conjugation action of any $n \in N_\tau$ preserves each of the three sets. The multiplication by $n \cdot \theta(n)^{-1}$ also preserves these sets by (i). Now it follows from (4.4) that the $*_\theta$-action preserves the three sets too. □

5.3. Lemma.

(i) $N_0 = N_\tau(\mathbb{R}) \cdot T$;

(ii) $N_\tau = N_\tau(\mathbb{R}) \cdot T_0$.

Proof. Since $N_\tau \subseteq N_0$ and $N_0/T = W_0$, for (i) it suffices to show that any element $w \in W_0$ can be represented by some element $m \in N_\tau(\mathbb{R})$. Let $w \in W_0$. Since $G$ is compact, $w$ can be represented by some $n \in N(\mathbb{R})$. Since $w \in W_0$, we have $n \in N_0 \cap N(\mathbb{R}) = N_0(\mathbb{R})$. Set $t = n \ast_\theta 1 = n \cdot \theta(n)^{-1} \in G(\mathbb{R})$. Then by Lemma 4.6 we have $t \in (T_0(\mathbb{R}) \cap T = T(\mathbb{R})$. Write $t = t_0 t_1$, where $t_i \in T_i(\mathbb{R})$, and let $s \in T_1(\mathbb{R})$ be such that $s^2 = t_1$. We set $m = s^{-1} n$, then $m \in N_0(\mathbb{R})$ and

$$m \cdot \theta(m)^{-1} = s^{-1} n \theta(n)^{-1} s^{-1} = t_1^{-1} t = t_0 \in T_0(\mathbb{R}) \subset T_0,$$

hence $m \in N_\tau(\mathbb{R})$. Clearly, $m$ represents $w$, which proves (i).

Now let $n \in N_\tau$. By (i), we may write $n = m t$, where $m \in N_\tau(\mathbb{R})$ and $t \in T$. Then $t \in N_\tau \cap T$. By Lemma 5.2(ii) we have $t = t_0 t_1$, where $t_0 \in T_0$, $t_1 \in T_1$, and $t_1^2 \in (T_0)_2$. Then $t_1 \in T(\mathbb{R})$ and hence $t_1 \in T(\mathbb{R}) \cap N_\tau \subset N_\tau(\mathbb{R})$. Thus $n = m t_1 \cdot t_0$, where $m t_1 \in N_\tau(\mathbb{R})$ and $t_0 \in T_0$, which proves (ii). □

By Lemma 5.2(iii), the group $N_\tau$ acts on $(T_0)_2$ via $*_\theta$, and we write $(T_0)_2/N_\tau$ for the set of orbits of this action. By Lemma 5.3(ii) we have $N_\tau = N_\tau(\mathbb{R}) \cdot T_0$, and $T_0$ acts on $(T_0)_2$ trivially. It follows that

$$(T_0)_2/N_\tau = (T_0)_2/N_\tau(\mathbb{R}).$$

Note that the action $*_\theta$ of $N_\tau(\mathbb{R})$ coincides with $*_\theta$.  

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5.4. Lemma. Each coset of $T_0 \cap T_1$ in $(T_0)_2$ is an orbit for the action $*_\theta$ of $T \cap N_\tau$ on $(T_0)_2$.

Proof. Any element $t \in T \cap N_\tau$ decomposes as $t = t_0 t_1$ with $t_0 \in T_0$, $t_1 \in T_1$, $t_1^2 \in T_0 \cap T_1$, by Lemma 5.2(ii). For any $s \in (T_0)_2$ we have

$$t * s = ts \theta(t)^{-1} = st_1^2,$$

hence the $(T \cap N_\tau)$-orbit of $s$ is contained in the coset $s \cdot (T_0 \cap T_1)$. Conversely, if $s' \in s \cdot (T_0 \cap T_1)$, $s' = st'$ with $t' \in T_0 \cap T_1$, then we may write $t' = t_1^2$ for some $t_1 \in T_1$. By Lemma 5.2(ii) we have $t_1 \in T \cap N_\tau$, and by (5.5) we have $t_1 * s = st' = s'$, which proves the lemma. \qed

5.6. Proposition. The inclusion $(T_0)_2 \hookrightarrow Z^1(\mathbb{R}, \theta \mathbf{T})$ induces a bijection

$$(T_0)_2/N_\tau \cong H^1(\mathbb{R}, \theta \mathbf{G}).$$

Proof. In view of Lemma 4.7 and Theorem 4.8, it suffices to show that the inclusion $(T_0)_2 \hookrightarrow Z^1(\mathbb{R}, \theta \mathbf{T})$ induces a bijection between the orbit sets for the actions $*_{\theta}$ of $N_\tau$ on $(T_0)_2$ and $*_{\theta}$ of $N_0$ on $Z^1(\mathbb{R}, \theta \mathbf{T})$.

We know from Lemma 4.11 that any orbit of $T \subset N_0$ in $Z^1(\mathbb{R}, \theta \mathbf{T})$ for the action $*_{\theta}$ intersects $(T_0)_2$ in a coset of $T_0 \cap T_1$. By Lemma 5.4, each coset is an orbit for the action $*_{\theta}$ of $T \cap N_\tau$ on $(T_0)_2$. Thus the inclusion $(T_0)_2 \hookrightarrow Z^1(\mathbb{R}, \theta \mathbf{T})$ induces a bijective map of orbit sets

$$Z^1(\mathbb{R}, \theta \mathbf{G})/T \cong (T_0)_2/(T \cap N_\tau)$$

which sends a $T$-orbit $O \subseteq Z^1(\mathbb{R}, \theta \mathbf{T})$ to the $(T \cap N_\tau)$-orbit $O' = O \cap (T_0)_2$.

The group $N_\tau(\mathbb{R})$ acts on $(T_0)_2$ by $*_{\theta}$ and on $Z^1(\mathbb{R}, \theta \mathbf{T})$ by $*_{\theta}$, or, which is the same, by $*_{\theta}$, so that $(T_0)_2$ is an $N_\tau(\mathbb{R})$-stable subset of $Z^1(\mathbb{R}, \theta \mathbf{T})$. Since $N_\tau(\mathbb{R})$ normalizes $T$ and $T \cap N_\tau$, we have induced actions of $N_\tau(\mathbb{R})$ on the orbit sets $Z^1(\mathbb{R}, \theta \mathbf{G})/T$ and $(T_0)_2/(T \cap N_\tau)$, so that the bijection (5.7) is $N_\tau(\mathbb{R})$-equivariant.

By Lemma 5.3 any orbit of $N_0$ in $Z^1(\mathbb{R}, \theta \mathbf{T})$ is of the form $Q = N_\tau(\mathbb{R}) *_{\theta} O$ for some $T$-orbit $O \subseteq Z^1(\mathbb{R}, \theta \mathbf{T})$ and, similarly, any orbit of $N_\tau$ in $(T_0)_2$ is of the form $Q' = N_\tau(\mathbb{R}) *_{\theta} O'$ for some $(T \cap N_\tau)$-orbit $O' \subseteq (T_0)_2$. Furthermore, if $O' = O \cap (T_0)_2$, then $Q' = Q \cap (T_0)_2$. We see that for any $N_0$-orbit $Q$ in $Z^1(\mathbb{R}, \theta \mathbf{T})$ under $*_{\theta}$, the intersection $Q \cap (T_0)_2$ is an orbit of $N_\tau$ under $*_{\theta}$. This yields the desired bijection between the orbit sets. \qed

5.8. Remark. Proposition 5.6 describes $H^1(\mathbb{R}, \theta \mathbf{G})$ in terms of the regular action $*_{\theta}$ of the complex algebraic group $N_\tau$.

6. Shifting the action of the twisted normalizer

6.1. In order to compute the Galois cohomology, we need an explicit description of the quotient set $(T_0)_2/N_\tau$.

Consider the semidirect product $\widehat{G} = G \rtimes \langle \hat{\tau} \rangle$, where $\langle \hat{\tau} \rangle$ is the group of order 1 or 2 whose generator $\hat{\tau}$ acts on $G$ as $\tau$. Then $\widehat{G}$ acts on its normal subgroup $G$ by conjugation. In particular, the element

$$\hat{\theta} := t_\theta \cdot \hat{\tau} \in G \cdot \hat{\tau} \subset \widehat{G}$$

acts on $G$ by $\text{inn}(t_\theta) \circ \tau = \theta$. 

6.2. Notation. Set
\[ z = \hat{\theta}^2 = (t_\theta \cdot \hat{\tau})^2 = t_\theta^2 \in Z(G^{\text{der}}) \cap T_0 \subset \tilde{G}. \]
Recall that \((T_0 \cdot \hat{\tau})_2^2\) denotes the subset of elements with square \(z\) in \(T_0 \cdot \hat{\tau}\).

6.3. Remark. If \(t \in T_0\), then \((t \cdot \hat{\tau})^2 = t^2\). Since \(z\) belongs to the finite group \(Z(G^{\text{der}})\), \(t\) is an element of finite order whenever \(t \cdot \hat{\tau} \in (T_0 \cdot \hat{\tau})_2^2\); in particular, \(t \in T_0(\mathbb{R})\) in this case. Thus
\[(T_0 \cdot \hat{\tau})_2^2 = (T_0(\mathbb{R}) \cdot \hat{\tau})_2^2 = \{t \cdot \hat{\tau} \mid t \in T_0(\mathbb{R}), t^2 = z\}.

6.4. Lemma.
(i) The bijective map
\[(T_0 \cdot \hat{\tau})_2^2 \rightarrow T_0 \cdot \hat{\tau}, \quad s \mapsto s \cdot \hat{\theta} = st_\theta \cdot \hat{\tau}, \]
is \(N_\tau\)-equivariant, where \(N_\tau\) acts on \(T_0\) by \(*_{t_\theta}\), see (4.4), and on \(T_0 \cdot \hat{\tau}\) by conjugation.

(ii) The map (6.5) restricts to an \(N_\tau\)-equivariant bijection between \((T_0)_2^2\) and \((T_0 \cdot \hat{\tau})_2^2\).

Proof. (i) For \(n \in N_\tau\) and \(s \in T_0\) we have
\[ n s \theta(n)^{-1} \cdot \hat{\theta} = n s \cdot \hat{\theta} n^{-1} \cdot \hat{\theta} = n (s \cdot \hat{\theta}) n^{-1}. \]
(ii) \((st_\theta)^2 = z\) if and only if \(s^2 = 1\).

6.6. Corollary (from Proposition 5.6 and Lemma 6.4(ii)). The map
\[(T_0 \cdot \hat{\tau})_2^2 \rightarrow (T_0)_2^2 \leftrightarrow Z^1(\mathbb{R}, \mathfrak{g}G), \quad t \cdot \hat{\tau} \mapsto tt_\theta^{-1} \]
induces a bijection
\[(T_0 \cdot \hat{\tau})_2^2 / N_\tau \simeq H^1(\mathbb{R}, \mathfrak{g}G), \]
where \(N_\tau\) acts on \((T_0 \cdot \hat{\tau})_2^2\) by conjugation.

7. Lattices, restricted roots, and affine Weyl groups

7.1. In Section 8 we shall reduce the computation of Galois cohomology to computations in the Lie algebra of \(T_0\). To this end, we need some preparations.

Recall that \(X = X^*(T)\) and \(X^\vee = X_*(T)\) are the character and cocharacter lattices of \(T\), respectively. We regard \(X\) as a lattice in \(t^*\) and \(X^\vee\) as a lattice in \(t\). A standard property of the exponential mapping implies the formula
\[(\nu(2\pi i z) = \exp(2\pi i dv(z)) = \exp(2\pi i z \cdot \nu) \quad \text{for any} \ \nu \in X^\vee, \ z \in \mathbb{C}, \]
where \(\nu\) is regarded as a homomorphism \(\mathbb{C}^\times \rightarrow T\) on the left-hand side, and as a vector in \(t\), identified with \(dv(1)\), on the right-hand side.

The scaled exponential mapping
\[ \mathcal{E}: t \rightarrow T, \quad x \mapsto \exp(2\pi x) \]
is a universal covering and a homomorphism from the additive group of \(t\) to \(T\) with kernel \(iX^\vee\). Note that \(iX^\vee \subset \mathfrak{t}(\mathbb{R}) := \text{Lie}\ T(\mathbb{R})\) and the restriction of \(\mathcal{E}\) to \(t(\mathbb{R})\) is the universal covering of the compact torus \(T(\mathbb{R})\).
Set \( X_0 = X^*(T_0) \). Then the restriction homomorphism \( X \to X_0 \) is surjective. We have a natural embedding \( X_0 \to t_0^* \). Then \( X_0 \subset t_0^* \) is the projection of \( X \subset t^* \) corresponding to the embedding \( t_0 \hookrightarrow t \).

We have \( t_0 = \text{Lie}(T_0) = t^\tau \). Set \( X_0^\tau = X_e(T_0) \). Then \( X_0^\tau = (X^\tau)^\tau \) is a lattice in \( t_0 \). The restriction
\[
\mathcal{E}_0 : t_0 \to T_0
\]
of the scaled exponential map \( \mathcal{E} \) to \( t_0 \) is a universal covering and a homomorphism with kernel \( iX_0^\tau \subset t_0(\mathbb{R}) = \text{Lie}T_0(\mathbb{R}) \). Its restriction to \( t_0(\mathbb{R}) \) is a universal covering of \( T_0(\mathbb{R}) \).

**7.4.** We have a canonical projection map
\[
t \to t_0, \quad x \mapsto \frac{1}{2}(x + \tau(x)).
\]
with kernel \( t_1 \). We denote by \( \tilde{X}_0^\tau \) the projection of \( X^\tau \) to \( t_0 \), that is,
\[
\tilde{X}_0^\tau = \{ \frac{1}{2}(\nu + \tau(\nu)) \mid \nu \in X^\tau \subset t \}.
\]
There are inclusions
\[
X_0^\tau \subseteq \tilde{X}_0^\tau \subseteq \frac{1}{2}X_0^\tau.
\]

**7.5. Lemma.** The exponential map \( \mathcal{E}_0 \) sends \( \frac{i}{2}X_0^\tau \) onto \( (T_0)_2 \) and \( i\tilde{X}_0^\tau \) onto \( T_0 \cap T_1 \), and thus induces an isomorphism
\[
\frac{1}{2}X_0^\tau / X_0^\tau \simeq i\tilde{X}_0^\tau / iX_0^\tau \thicksim (T_0)_2,
\]
which restricts to an isomorphism \( \tilde{X}_0^\tau / X_0^\tau \thicksim T_0 \cap T_1 \).

**Proof.** It suffices to show that \( \mathcal{E}(i\tilde{X}_0^\tau) = T_0 \cap T_1 \).

Let \( \nu \in \tilde{X}_0^\tau \). Then \( i\nu \in t_0 \); hence \( \mathcal{E}(i\nu) \in T_0 \). We have \( \nu = \frac{1}{2}(\nu' + \tau(\nu')) \) for some \( \nu' \in X^\tau \). Set \( y = \frac{1}{2}(\tau(\nu') - \nu') \in t_1 \); then \( i\nu = i\nu' + y \). We see that
\[
\mathcal{E}(i\nu) = \mathcal{E}(i\nu') \cdot \mathcal{E}(y) = 1 \cdot \mathcal{E}(y) \in T_1.
\]
Thus \( \mathcal{E}(i\nu) \in T_0 \cap T_1 \).

Conversely, let \( t \in T_0 \cap T_1 \). Write \( t = \mathcal{E}(x) \) with \( x \in t_0 \), and \( t = \mathcal{E}(y) \) with \( y \in t_1 \); then \( x = y + i\nu' \) for some \( \nu' \in X^\tau \). Since \( \tau(x) = x \) and \( \tau(y) = -y \), we get
\[
x = \frac{i}{2}(\nu' + \tau(\nu')) \in i\tilde{X}_0^\tau.
\]
Thus \( t = \mathcal{E}(x) \in \mathcal{E}(i\tilde{X}_0^\tau) \), as required. \( \square \)

**7.6.** In consideration of roots and Weyl groups, we may reduce most issues to the case of a semisimple group. For simplicity, we assume from now till the end of Section 7 that \( G \) is semisimple. Recall that \( G^{\text{ad}} \) denotes the adjoint group of \( G \) and \( G^{\text{sc}} \) is the universal cover of \( G \). Also, \( T^{\text{ad}} \) and \( T^{\text{sc}} \) are compatible with \( T \) maximal tori in \( G^{\text{ad}} \) and \( G^{\text{sc}} \), respectively. We identify the Lie algebras of \( T^{\text{ad}} \) and \( T^{\text{sc}} \) with \( t = \text{Lie}T \).
As usual, we write
\[ P = X^*(T^{\text{sc}}) \quad \text{and} \quad Q = X^*(T^{\text{ad}}); \]
these are the weight lattice and the root lattice. Then \( P \) and \( Q \) naturally embed into \( t^* \), and we obtain three lattices in \( t^* \):
\[ Q \subseteq X \subseteq P. \]

Also we write
\[ P^\vee = X_*\ (T^{\text{ad}}) \quad \text{and} \quad Q^\vee = X_*\ (T^{\text{sc}}); \]
these are the coweight lattice and the coroot lattice. Then \( P^\vee \) and \( Q^\vee \) naturally embed into \( t \), and we obtain three lattices in \( t \):
\[ P^\vee \supseteq X^\vee \supseteq Q^\vee. \]

Note that the lattices \( P^\vee \) and \( Q^\vee \) are dual to \( Q \) and \( P \), respectively.

We set
\[ C = P^\vee/Q^\vee, \quad F = X^\vee/Q^\vee. \]
Then \( F \subseteq C \) are finite abelian groups. The scaled exponential map \( E^{\text{sc}} : t \to T^{\text{sc}} \) with kernel \( iQ^\vee \) induces isomorphisms (depending on the choice of \( i \))
\begin{align*}
C & \xrightarrow{\sim} \ker[T^{\text{sc}} \to T^{\text{ad}}] = Z(G^{\text{sc}}) = \pi_1(G^{\text{ad}}), \\
F & \xrightarrow{\sim} \ker[T^{\text{sc}} \to T] = \pi_1(G). 
\end{align*}

Thus \( F \) is the fundamental group of \( G \).

**7.8.** Recall that \( W = N_G(T)/T \) is the Weyl group of \((G, T)\), of \((G^{\text{sc}}, T^{\text{sc}})\), and of \((G^{\text{ad}}, T^{\text{ad}})\). The scaled exponential maps from \( t \) to \( T \), to \( T^{\text{sc}} \), and to \( T^{\text{ad}} \) are \( W \)-equivariant. Since \( W \) acts trivially on \( Z(G^{\text{sc}}) \), we see from the \( W \)-equivariant isomorphism (7.7) that \( W \) acts trivially on \( C = P^\vee/Q^\vee \).

We set
\[ \tilde{W} = X^\vee \rtimes W, \quad \tilde{W}^{\text{ad}} = P^\vee \rtimes W, \quad \tilde{W}^{\text{sc}} = Q^\vee \rtimes W; \]
then \( \tilde{W}^{\text{sc}} \subseteq \tilde{W} \subseteq \tilde{W}^{\text{ad}} \). Since \( W \) acts on \( P^\vee/Q^\vee \) trivially, one checks that \( \tilde{W}^{\text{sc}} \) is a normal subgroup of \( \tilde{W}^{\text{ad}} \) and
\[ \tilde{W}^{\text{ad}}/\tilde{W}^{\text{sc}} \simeq C, \quad \tilde{W}/\tilde{W}^{\text{sc}} \simeq F. \]

The lattices \( Q^\vee, X^\vee, P^\vee \) act on \( t \) by translations:
\begin{equation}
\nu : x \mapsto x + i\nu \quad \text{for } \nu \in P^\vee, \ x \in t.
\end{equation}
Combined with the natural linear action of \( W \) on \( t \), the action (7.9) gives rise to actions of \( \tilde{W}^{\text{sc}}, \tilde{W}, \) and \( \tilde{W}^{\text{ad}} \) on \( t \) by affine transformations. Each of these actions preserves \( t(\mathbb{R}) \) (for which we added the multiplier \( i \) in (7.9)) and restricts to an action on \( t(\mathbb{R}) \) by affine isometries preserving the Euclidean structure induced by the Killing form on \( g \). The group \( \tilde{W}^{\text{sc}} \) regarded as a group of motions of the Euclidean space \( t(\mathbb{R}) \) is a crystallographic group generated by reflections, known as the affine (or extended) Weyl group of the root system \( R = R(G, T) \); see Bourbaki [8, Sect. VI.2.1], Gorbatsevich, Onishchik, and Vinberg [14, Sect. 3.3.6, Prop. 3.10(1)], and also Section 9 below for details.
7.10. For any character $\lambda \in X$, let $\bar{\lambda} \in X_0$ denote the restriction of $\lambda$ to $T_0$. In particular, we consider the restricted roots $\bar{\alpha}$ for all $\alpha \in R$.

Let $g^{\pm \tau}$ denote the $(\pm 1)$-eigenspaces for $\tau$ in $g$. Then $g^{\tau}$ is the Lie algebra of a reductive (in fact, semisimple) subgroup $G^\tau \subseteq G$ and $g^{-\tau}$ is preserved under the adjoint action of $G^\tau$.

If a root $\alpha \in R$ is $\tau$-invariant, then $\bar{\alpha}$ is either a root of $g^{\tau}$ or a weight of $g^{-\tau}$ with respect to $T_0$, depending on whether $g_{\alpha} \subseteq g^{\tau}$ or $g_{\alpha} \subseteq g^{-\tau}$, respectively. If $\alpha \in R$ is not $\tau$-invariant, then $\bar{\alpha} = \tau(\alpha)$ is both a root of $g^{\tau}$ and a weight of $g^{-\tau}$ with eigenvectors $e_{\alpha} \pm e_{\tau(\alpha)}$, respectively, where $e_{\alpha}$ is any generator of $g_{\alpha}$ and e_{\tau(\alpha)} = \tau(e_{\alpha}) \in g_{\tau(\alpha)}$.

7.11. Proposition.

(i) The set $\overline{R} = \{ \bar{\alpha} \mid \alpha \in R \}$ is a root system in $X_0$ (possibly non-reduced).

(ii) The set $\overline{S} = \{ \bar{\alpha} \mid \alpha \in S \}$ is a basis of $\overline{R}$.

(iii) The Weyl group $W(\overline{R})$ of $\overline{R}$ is naturally isomorphic to $W_0$.

Proof. For (i) and (ii) see Gorbatevich, Onishchik, and Vinberg [14, Sect. 3.3.9], in particular, Theorem 3.14(3) of loc. cit., or Timashev [25, Lem. 26.8]. A proof of (iii) is implicit in [14, Sect. 3.3.9] for the case of simple $G$; we provide a direct general argument.

Note that $W_0 \subseteq W$ acts on $T$ preserving $T_0$, and the induced action of $W_0$ on $X$ preserves $R$. Hence $W_0$ can be identified with a group of automorphisms of $X_0$ preserving $\overline{R}$. Each restricted simple root $\bar{\alpha}$ ($\alpha \in \overline{S}$) is a root of $g^{\tau}$ with respect to $T_0$, whose root subspace is spanned by $e_{\alpha} + e_{\tau(\alpha)}$, where, since $\tau$ respects the pinning $(e_{\beta})$, we have $e_{\alpha} + e_{\tau(\alpha)} = e_{\alpha} + \tau(e_{\alpha}) \subseteq g^{\tau}$. Hence each simple reflection in $W(\overline{R})$ and, furthermore, each element of $W(\overline{R})$, can be represented by some $n \in N_{G^\tau}(T_0) = N_0^\tau$. Therefore $W(\overline{R}) \subseteq W_0$.

On the other hand, each $w \in W_0$ induces an automorphism of $\overline{R}$; hence the set $w(\overline{S})$ is a basis of $\overline{R}$. Then $w(\overline{S}) = w'(\overline{S})$ for some $w' \in W(\overline{R})$, and $w'' = w^{-1}w'$ preserves $\overline{S}$. We claim that $w''$ preserves $\overline{S}$, too. Indeed, if $\alpha \in S$ and $\beta = w''(\alpha)$, then $\beta = w''(\bar{\alpha}) \in \overline{S}$; hence $\beta = \gamma$ for some $\gamma \in S$. Since the eigenspaces of $T_0$ in $g^{\pm \tau}$ are one-dimensional by [14, Sect. 3.3.9, Thm. 3.14(i)], it follows from the description of these eigenspaces given right before Proposition 7.11 that $\beta = \gamma$ or $\beta = \tau(\gamma) \in S$. This proves our claim. As $w'' \in W$ preserves $S$, we have $w'' = 1$, whence $w = w' \in W(\overline{R})$. Therefore $W_0 \subseteq W(\overline{R})$. \hfill \-box

7.12. Denote by $T_0^{\text{rad}}$ and $T_0^{\text{sc}}$ the subtori in $T^{\text{rad}}$ and in $T^{\text{sc}}$, respectively, with Lie algebra $t_0$. We set

$$Q_0 = X^*(T_0^{\text{rad}}) \quad \text{and} \quad P_0 = X^*(T_0^{\text{sc}}).$$

We obtain three lattices in $t_0^*$:

$$Q_0 \subseteq X_0 \subseteq P_0.$$

Note that $Q_0$ and $P_0$ are the projections to $t_0^*$ of $Q$ and $P$, respectively.

We also set

$$P_0^\vee = X_*(T_0^{\text{rad}}) = (P^\vee)^\tau \quad \text{and} \quad Q_0^\vee = X_*(T_0^{\text{sc}}) = (Q^\vee)^\tau.$$
These lattices in $t_0$ are dual to $Q_0$ and $P_0$, respectively.

Choose simple roots $\alpha_1, \ldots, \alpha_\ell \in S$ which are representatives of the $\tau$-orbits in $S$. Let $\alpha^\vee_i \in S^\vee$, $\omega_i \in P$, and $\omega^\vee_i \in P^\vee$ denote the simple coroot, fundamental weight and coweight corresponding to $\alpha_i$ ($i = 1, \ldots, \ell$), respectively.

7.13. Lemma.

(i) $Q_0$ and $P_0$ are the root lattice and the weight lattice of $\overline{R}$, respectively. The restricted simple roots $\overline{\alpha}_1, \ldots, \overline{\alpha}_\ell$ comprise the basis $\mathcal{S}$ of $Q_0$ and the restricted fundamental weights $\overline{\omega}_1, \ldots, \overline{\omega}_\ell$ comprise a basis of $P_0$.

(ii) $Q_0^\vee$ and $P_0^\vee$ are the coroot lattice and the coweight lattice of $\overline{R}$, respectively. The simple coroots of $\overline{R}$ with respect to $\mathcal{S}$, which comprise a basis of $Q_0^\vee$, are $\overline{\alpha}_j^\vee = \alpha_j^\vee$ and $\overline{\alpha}_k^\vee = \alpha_k^\vee + \tau(\alpha_k^\vee)$, where $j, k$ run over all integers $1, \ldots, \ell$ such that $\alpha_j$ is fixed and $\alpha_k$ is moved by $\tau$, respectively. The fundamental coweights of $\overline{R}$ with respect to $\mathcal{S}$, which comprise a basis of $P_0^\vee$, are $\overline{\omega}_j^\vee = \omega_j^\vee$ and $\overline{\omega}_k^\vee = \omega_k^\vee + \tau(\omega_k^\vee)$, with the same notation.

Proof. The claims about the bases of $Q_0$, $P_0$, $Q_0^\vee$, and $P_0^\vee$ are clear. It is also clear that $Q_0$ is the root lattice of $\overline{R}$ and $\overline{\alpha}_i$ are the simple roots. It follows that the dual lattice $P_0^\vee$ is the coweight lattice of $\overline{R}$ and the vectors $\omega_j^\vee$ and $\omega_k^\vee + \tau(\omega_k^\vee)$, which comprise the dual basis of $P_0^\vee$ for $\{\overline{\alpha}_1, \ldots, \overline{\alpha}_\ell\}$, are the fundamental coweights.

Under the identification of $t_0$ with $t_0^*$ via the Killing form, the simple coroots of $\overline{R}$ are proportional to the simple roots $\overline{\alpha}_i$ of $\overline{R}$, which are the projections of $\alpha_i \in t^*$ to $t_0$. Similarly, the simple coroots $\alpha_j^\vee$ of $R$ are proportional to $\alpha_i$. It follows that the vectors $\alpha^\vee_j$ and $\alpha^\vee_k + \tau(\alpha_k^\vee)$ in $Q_0^\vee$ are proportional to the simple coroots of $\overline{R}$.

To show that these are indeed the simple coroots, it suffices to check that the pairing of such a vector with the respective simple root $\overline{\alpha}_i$ ($i = j$ or $k$) is equal to 2 unless $2\overline{\alpha}_i \in \overline{R}$, in which case the pairing should be equal to 1.

We have $\langle \overline{\alpha}_j, \alpha_j^\vee \rangle = \langle \alpha_j, \alpha_j^\vee \rangle = 2$. Note that $2\overline{\alpha}_j \notin \overline{R}$. Indeed, if $2\overline{\alpha}_j = \overline{\alpha}$ for some $\alpha \in R$, then $\langle \alpha, \alpha_j^\vee \rangle = 4$, which is not possible.

Also we have

$$\langle \overline{\alpha}_k, \alpha_k^\vee + \tau(\alpha_k^\vee) \rangle = \langle \alpha_k, \alpha_k^\vee + \tau(\alpha_k^\vee) \rangle = 2$$

if $\alpha_k$ and $\tau(\alpha_k)$ are not linked in $D(G)$. In this case we have $2\overline{\alpha}_k \notin \overline{R}$. Indeed, if $2\overline{\alpha}_k = \beta$ for some $\beta \in R$, then $\overline{\alpha}_k = \tau(\alpha_k) = \beta/2$ and

$$\langle \beta, \alpha_k^\vee + \tau(\alpha_k^\vee) \rangle = \langle \alpha_k + \tau(\alpha_k), \alpha_k^\vee + \tau(\alpha_k^\vee) \rangle = 4,$$

which may happen only if $\alpha_k = \beta = \tau(\alpha_k)$, a contradiction.

Finally, if $\alpha_k$ and $\tau(\alpha_k)$ are linked in $D(G)$, then $\langle \alpha_k, \tau(\alpha_k^\vee) \rangle = -1$. In this case we have $2\overline{\alpha}_k = \beta$ for $\beta = \alpha_k + \tau(\alpha_k) \in R$ and

$$\langle \overline{\alpha}_k, \alpha_k^\vee + \tau(\alpha_k^\vee) \rangle = \langle \alpha_k, \alpha_k^\vee + \tau(\alpha_k^\vee) \rangle = 1.$$

Since the lattice $Q_0^\vee$ is generated by the simple coroots $\alpha_j^\vee$ and $\alpha_k^\vee + \tau(\alpha_k^\vee)$, it is the coroot lattice of $\overline{R}$. Hence the dual lattice $P_0$ is the weight lattice of $\overline{R}$. The vectors $\overline{\omega}_i$, which comprise the basis of $P_0$ dual to the basis of $Q_0^\vee$ consisting of the simple coroots $\alpha_j^\vee$, $\alpha_k^\vee + \tau(\alpha_k^\vee)$, are the fundamental weights. \[\square\]
Let $\tilde{P}^\vee_0$ and $\tilde{Q}^\vee_0$ denote the projections of $P^\vee$ and $Q^\vee$ to $t_0$, respectively. We have three lattices in $t_0$:

$$\tilde{Q}^\vee_0 \subseteq \tilde{X}^\vee_0 \subseteq \tilde{P}^\vee_0.$$ 

A basis of $\tilde{Q}^\vee_0$ consists of all vectors $\tilde{a}_j^\vee$ and $\frac{1}{2}\tilde{\alpha}_k^\vee$, and a basis of $\tilde{P}^\vee_0$ consists of all vectors $\tilde{\omega}_j^\vee$ and $\frac{1}{2}\tilde{\omega}_k^\vee$, with the notation of Lemma 7.13(ii).

We set

$$C_0 = \tilde{P}^\vee_0 / \tilde{Q}^\vee_0, \quad F_0 = \tilde{X}^\vee_0 / \tilde{Q}^\vee_0;$$

then $F_0 \subseteq C_0$. The canonical epimorphisms

$$P^\vee \to \tilde{P}^\vee_0, \quad X^\vee \to \tilde{X}^\vee_0, \quad Q^\vee \to \tilde{Q}^\vee_0$$

induce epimorphisms

$$F \to F_0 \quad \text{and} \quad C \to C_0.$$ 

The lattices $\tilde{X}^\vee_0$, $\tilde{P}^\vee_0$, and $\tilde{Q}^\vee_0$ are preserved by the action of $W_0$ on $t_0$ and, similarly to Subsection 7.8, we define

$$\tilde{W}_0 = \tilde{X}^\vee_0 \rtimes W_0, \quad \tilde{W}_0^{\text{ad}} = \tilde{P}^\vee_0 \rtimes W, \quad \tilde{W}_0^{\text{sc}} = \tilde{Q}^\vee_0 \rtimes W,$$

so that $\tilde{W}_0^{\text{sc}} \subseteq \tilde{W}_0 \subseteq \tilde{W}_0^{\text{ad}}$, where the subgroup $\tilde{W}_0^{\text{sc}}$ is normal in $\tilde{W}_0^{\text{ad}}$, and

$$\tilde{W}_0^{\text{ad}} / \tilde{W}^{\text{sc}} \simeq C_0, \quad \tilde{W}_0 / \tilde{W}_0^{\text{sc}} \simeq F_0.$$

The action (7.9) restricts to an action of $\tilde{X}^\vee_0$, $\tilde{P}^\vee_0$, and $\tilde{Q}^\vee_0$ on $t_0(\mathbb{R})$ by translations, which, in turn, gives rise to an action of $\tilde{W}_0$, $\tilde{W}_0^{\text{ad}}$, and $\tilde{W}_0^{\text{sc}}$ on $t_0(\mathbb{R})$ by affine isometries.

**7.15. Lemma** ([14, Sect. 3.3.10, Prop. 3.15(1)]). The group $\tilde{W}_0^{\text{sc}}$ regarded as a group of motions of the Euclidean space $t_0(\mathbb{R})$ is a crystallographic group generated by reflections along the hyperplanes $\{\tilde{\alpha}(x) = ik\}$ and $\{\tilde{\beta}(x) = ik/2\}$ ($k \in \mathbb{Z}$) over all roots $\tilde{\alpha}$ of $g^\tau$ and all weights $\tilde{\beta}$ of $g^{-\tau}$.

The group $\tilde{W}_0^{\text{sc}}$ is known as the affine Weyl group associated with $\tau$; see Onishchik and Vinberg [20, Sect. 4.4.5], Gorbachevich, Onishchik, and Vinberg [14, Sect. 3.3.10], and also Sections 10–11 below for details.

### 8. Logarithm of the action of the twisted normalizer

**8.1.** We wish to describe $(T_0 \cdot \hat{\cdot})^\sharp / N_\tau = (T_0(\mathbb{R}) \cdot \hat{\cdot})^\sharp / N_\tau$, where the group $N_\tau$ acts on the $\mathbb{C}$-variety $T_0 \cdot \hat{\cdot}$ by conjugation:

$$n: \quad t \cdot \hat{\cdot} \mapsto n(t \cdot \hat{\cdot})n^{-1} = nt\tau(n)^{-1} \cdot \hat{\cdot} = ntn^{-1} \cdot n\tau(n)^{-1} \cdot \hat{\cdot} \quad \text{for } n \in N_\tau, \ t \in T_0.$$ 

**8.3. Lemma.** The kernel of the action (8.2) is $T^\tau = T_0 \cdot (T_1)_2$. 


Proof. Let \( n \in N_\tau \subset N_0 \) be in the kernel of the action (8.2). Substituting \( t = 1 \), we obtain that \( \tau(n) = n \). It follows that for all \( t \in T_0 \) we have \( ntn^{-1} = t \), and hence \( n \in T \). Since \( \tau(n) = n \), we conclude that \( n \in T^\tau \).

For \( t = t_0t_1 \in T \) with \( t_0 \in T_0 \), \( t_1 \in T_1 \), the condition \( \tau(t) = t \) means that \( t_0t_1^{-1} = t_0t_1 \), that is, \( t_1^2 = 1 \). It follows that \( T^\tau = T_0 \cdot (T_1)_2 \). Thus the kernel of the action (8.2) is contained in \( T^\tau = T_0 \cdot (T_1)_2 \). The inverse inclusion is clear. \( \Box \)

8.4. Notation. We define \( \widehat{W}_0 = N_\tau / T^\tau \).

By Lemma 8.3, the action (8.2) induces an effective action of \( \widehat{W}_0 \) on \( T_0 \cdot \hat{\tau} \). The inclusion map \( N_\tau \hookrightarrow N_0 \) induces a homomorphism \( N_\tau \rightarrow W_0 \) (which is surjective by Lemma 5.3) with kernel \( N_\tau \cap T \). Since \( T^\tau \subseteq N_\tau \cap T \), we obtain an induced homomorphism

\[
\widehat{W}_0 \rightarrow W_0
\]

with kernel \((N_\tau \cap T)/T^\tau\). We have

\[
N_\tau \cap T = \{ t = t_0t_1 \in T \mid t_0 \in T_0, \ t_1^2 \in (T_0)_2 \},
\]

\[
T^\tau = \{ t = t_0t_1 \in T \mid t_0 \in T_0, \ t_1^2 = 1 \}.
\]

The homomorphism

\[
N_\tau \cap T \rightarrow T_1, \quad t = t_0t_1 \mapsto t \cdot \tau(t)^{-1} = t_1^2
\]

with kernel \( T^\tau \) and with image \( T_0 \cap T_1 \) yields an isomorphism

\[
(N_\tau \cap T)/T^\tau \cong T_0 \cap T_1.
\]

The inverse isomorphism is given as follows:

\[
(8.5) \quad T_0 \cap T_1 \cong (N_\tau \cap T)/T^\tau, \quad s \mapsto \sqrt{s} \cdot T^\tau \quad \text{for} \ s \in T_0 \cap T_1,
\]

where \( \sqrt{s} \) is any element \( t_1 \in T_1 \) such that \( t_1^2 = s \). Thus we obtain a short exact sequence

\[
(8.6) \quad 1 \rightarrow T_0 \cap T_1 \rightarrow \widehat{W}_0 \rightarrow W_0 \rightarrow 1,
\]

where the homomorphism \( T_0 \cap T_1 \rightarrow \widehat{W}_0 \) is given by (8.5).

We construct a canonical splitting of (8.6). Consider the subgroup

\[
N_0^\tau = \{ n \in N_0 \mid \tau(n) = n \} \subseteq N_\tau.
\]

The inclusion map \( N_0^\tau \hookrightarrow N_0 \) induces a homomorphism

\[
(8.7) \quad N_0^\tau \rightarrow W_0,
\]

with kernel \( N_0^\tau \cap T = T^\tau \).

8.8. Lemma. The homomorphism (8.7) is surjective.
Proof. By Proposition 7.11(iii), the group $W_0$ regarded as a group of automorphisms of $X_0$ is spanned by reflections along the restricted simple roots $\tilde{\alpha} \in \tilde{S}$, which also comprise a set of simple roots for $G^\tau$ with respect to $T_0$. Hence each element $w \in W_0$ can be represented by some $n \in N_{G^\tau}(T_0) = N_0^\tau$.

8.9. By Lemma 8.8, the injective homomorphism $N_0^\tau / T^\tau \to W_0$ is surjective, and hence is an isomorphism. We obtain a canonical splitting

$$W_0 \xrightarrow{\sim} N_0^\tau / T^\tau \hookrightarrow N_\tau / T^\tau = \hat{W}_0,$$

which defines a canonical isomorphism

$$(T_0 \cap T_1) \times W_0 \xrightarrow{\sim} \hat{W}_0.$$

We describe the action of $\hat{W}_0 = (T_0 \cap T_1) \times W_0$ on $T_0 \cdot \hat{\tau}$. An element $s \in T_0 \cap T_1$ acts by

$$s: \ t \cdot \hat{\tau} \mapsto \sqrt{s}(t \cdot \hat{\tau})(\sqrt{s})^{-1} = st \cdot \hat{\tau} \quad \text{for } t \in T_0.$$

The group $W_0$ acts by

$$w: \ t \cdot \hat{\tau} \mapsto n(t \cdot \hat{\tau})n^{-1} = ntn^{-1} \cdot \hat{\tau} = w(t) \cdot \hat{\tau} \quad \text{for } w \in W_0, \ t \in T_0,$$

where $n \in N_0^\tau$ is a representative of $w$.

8.10. By Lemma 7.5 the scaled exponential mapping $E$ restricts to a surjective homomorphism

(8.11) $E_0: \ i\tilde{X}_0^\vee \to T_0 \cap T_1$.

We construct a map

(8.12) $E_W: \ \hat{W}_0 \to \hat{W}_0, \ \nu \cdot w \mapsto E(i\nu) \cdot w \quad \text{for } \nu \in \tilde{X}_0^\vee, \ w \in W_0,$

Since the homomorphism (8.11) is $W_0$-equivariant, the map (8.12) is a homomorphism. It is clear that $E_W$ is surjective and its kernel is $X_0^\vee$.

8.13. Lemma. The map

(8.14) $\tilde{E}: \ t_0(\mathbb{R}) \to T_0(\mathbb{R}) \cdot \hat{\tau}, \ y \mapsto \exp(2\pi y) \cdot \hat{\tau} \quad \text{for } y \in t_0(\mathbb{R})$

is compatible with the homomorphism $E_W: \hat{W}_0 \to \hat{W}_0$ and thus induces a bijection of the orbit sets

(8.15) $E_{\text{orb}}: \ t_0(\mathbb{R}) / \hat{W}_0 \xrightarrow{\sim} (T_0(\mathbb{R}) \cdot \hat{\tau}) / \hat{W}_0$. 
Proof. It is routine to check that \( \hat{E} \) is \( \mathcal{E}_W \)-equivariant, that is,

\[
\hat{E}(\tilde{w} \cdot y) = \mathcal{E}_W(\tilde{w}) \cdot \hat{E}(y) \quad \text{for } \tilde{w} \in \tilde{W}_0, \ y \in t_0(\mathbb{R}).
\]

Hence the map on orbits \( \mathcal{E}_{\text{orb}} \) of (8.15) is well defined. Since the map \( \hat{E} \) is surjective, the map \( \mathcal{E}_{\text{orb}} \) is surjective as well.

We show that the map \( \mathcal{E}_{\text{orb}} \) is injective. Let \( \tilde{W}_0 \cdot y_1 \) and \( \tilde{W}_0 \cdot y_2 \) be two orbits in \( t_0(\mathbb{R}) \) with the same image in \( (T_0(\mathbb{R}) \cdot \hat{\tau})/\tilde{W}_0 \). This means that

\[
\hat{E}(y_2) = \hat{w} \cdot \hat{E}(y_1) \quad \text{for some } \hat{w} \in \tilde{W}_0.
\]

Since the homomorphism \( \mathcal{E}_W \) is surjective, there exists \( \tilde{w}' \in \tilde{W}_0 \) such that \( \hat{w} = \mathcal{E}_W(\tilde{w}') \). Set \( y_2' = \tilde{w}' \cdot y_1 \); then \( \hat{E}(y_2') = \hat{E}(y_2) \). It follows that

\[
y_2 = i\nu + y_2' \quad \text{for some } \nu \in X_0^\vee.
\]

Set \( \tilde{w} = \nu \cdot \tilde{w}' \in \tilde{W}_0 \). Then \( y_2 = \tilde{w} \cdot y_1 \). Thus \( \tilde{W}_0 \cdot y_1 = \tilde{W}_0 \cdot y_2 \). We have proved that the map \( \mathcal{E}_{\text{orb}} \) of (8.15) is injective, and hence bijective, as required. \( \square \)

8.16. Suppose now that \( G \) is semisimple. Recall that the group \( \tilde{W}_0 = X_0^\vee \rtimes W_0 \) contains a subgroup of finite index \( \tilde{W}_0^{sc} = \tilde{Q}_0^\vee \rtimes W_0 \). We have a finite group

\[
F_0 = X_0^\vee / \tilde{Q}_0^\vee \simeq \tilde{W}_0 / \tilde{W}_0^{sc}
\]

and a canonical bijection of the orbit sets

\[
(t_0(\mathbb{R})/\tilde{W}_0^{sc})/F_0 \sim t_0(\mathbb{R})/\tilde{W}_0,
\]

and thus a chain of bijections

\[
(t_0(\mathbb{R})/\tilde{W}_0^{sc})/F_0 \sim t_0(\mathbb{R})/\tilde{W}_0 \sim (T_0(\mathbb{R}) \cdot \hat{\tau})/\tilde{W}_0.
\]

In order to compute \( (T_0(\mathbb{R}) \cdot \hat{\tau})/\tilde{W}_0 \), we wish to compute \( (t_0(\mathbb{R})/\tilde{W}_0^{sc})/F_0 \). Note that the group

\[
C_0 = \tilde{P}_0^\vee / \tilde{Q}_0^\vee \simeq \tilde{W}_0^{ad} / \tilde{W}_0^{sc}
\]

naturally acts on the orbit set \( t_0(\mathbb{R})/\tilde{W}_0^{sc} \), and the group \( F_0 \) acts on \( t_0(\mathbb{R})/\tilde{W}_0^{sc} \) via the embedding \( F_0 \hookrightarrow C_0 \). So we wish to describe the set \( t_0(\mathbb{R})/\tilde{W}_0^{sc} \) together with the action of \( C_0 \), where we may and shall assume that \( G \) is simply connected.

8.17. A pair \((G, \theta)\) with simply connected \( G \) naturally decomposes into a direct product of indecomposable pairs. It suffices to describe the set \( t_0(\mathbb{R})/\tilde{W}_0^{sc} \) together with the action of \( C_0 \) for any indecomposable pair \((G, \theta)\). There are three cases:

(a) \( G \) is simple and \( \theta \) is an inner involution of \( G \);
(b) \( G \) is simple and \( \theta \) is an outer involution of \( G \);
(c) \( G = G' \times_{\mathbb{R}} G'' \), where \( \theta \) swaps the isomorphic simple factors \( G' \) and \( G'' \).

In the next three sections, for an indecomposable pair \((G, \theta)\) of each of the types (a), (b), and (c) above, we shall describe the orbit set \( t_0(\mathbb{R})/\tilde{W}_0^{sc} \) and the action of \( C_0 \) on it.
9. Case of an inner form of a simple compact group

9.1. In this section $G$ is a simply connected, simple, compact $\mathbb{R}$-group, and the involution $\theta$ of $G$ is inner, that is, $\tau = \text{id}$. Then $T_0 = T$, $t_0(\mathbb{R}) = t(\mathbb{R})$, $\tilde{W}^sc = W^sc$, $C_0 = C$, etc.; see Section 7 for notation.

Let $\tilde{D} = \hat{D}(G, T, B) = \hat{D}(R, S)$ denote the extended Dynkin diagram; the set of vertices of $\tilde{D}$ is $\tilde{S} = \{\alpha_0, \alpha_1, \ldots, \alpha_\ell\}$, where $\alpha_1, \ldots, \alpha_\ell$ are the simple roots enumerated as in [20, Table 1], and $\alpha_0$ is the lowest root. These roots $\alpha_0, \alpha_1, \ldots, \alpha_\ell$ are linearly dependent, namely,

$$m_0\alpha_0 + m_1\alpha_1 + \cdots + m_\ell\alpha_\ell = 0,$$

where the coefficients $m_i$ are positive integers for all $i = 0, 1, \ldots, \ell$, and $m_0 = 1$. These coefficients $m_i$ are tabulated in [20, Table 6] and in [14, Table 3]; see also Table 1. In the diagrams on the left, we write only the coefficients $m_i$ that are $\leq 2$. In the diagrams on the right, we write the coefficients $c_i$ (modulo $\mathbb{Z}$) of the decomposition into a linear combination of simple roots $\alpha_i$ for a representative $(\omega_1, \omega_\ell, \text{etc.})$ of a generator of the group $P/Q$. We write only the coefficients $c_i$ for the simple roots $\alpha_i$ for which $m_i \leq 2$. The vertex corresponding to $\alpha_0$ is painted in black.

9.3. Following [14, Sect. 3.3.6], we introduce the barycentric coordinates $x_0, x_1, \ldots, x_\ell$ of a point $x \in t(\mathbb{R})$ by setting

$$\alpha_i(x) = ix_i \quad \text{for } i = 1, \ldots, \ell, \quad \alpha_0(x) = i(x_0 - 1).$$

It follows from (9.2) that

$$m_0x_0 + m_1x_1 + \cdots + m_\ell x_\ell = 1.$$

By Bourbaki [8, Sect. VI.2.2, Proposition 5(i)], see also [14, Sect. 3.3.6, Proposition 3.10(2)], the closed simplex $\Delta \subset t(\mathbb{R})$ given by the inequalities

$$x_0 \geq 0, \quad x_1 \geq 0, \ldots, \quad x_\ell \geq 0$$

is a fundamental domain for the affine Weyl group $\tilde{W}^sc$ acting on $t(\mathbb{R})$.

9.5. The action of $C = P^\vee/Q^\vee \simeq \tilde{W}^{ad}/\tilde{W}^sc$ on $\Delta \simeq t(\mathbb{R})/\tilde{W}^sc$ is given by permutations of barycentric coordinates corresponding to a subgroup of the automorphism group of the extended Dynkin diagram acting simply transitively on the set of vertices $\alpha_i$ with $m_i = 1$. This action is described in [8, Sect. VI.2.3, Prop. 6]. Namely, the nonzero cosets of $C$ are represented by the fundamental coweights $\omega_i^\vee$ such that $i\omega_i^\vee \in \Delta$, that is, $m_i = 1$. Let $w_0$, resp. $w_i$, denote the longest element in $W$, resp. in the Weyl group $W_i$ of the root subsystem $R_i$ generated by $S \setminus \{\alpha_i\}$. Then the transformation $x \mapsto w_iw_0x + i\omega_i^\vee$ preserves $\Delta$ whenever $m_i = 1$ and gives the action of the respective coset $[\omega_i^\vee] \in C$ on $\Delta$.

We describe the action of $C$ on $\tilde{D}$ explicitly case by case, using [8, Plates I–IX, assertion (XII)]. If $G$ is of one of the types $E_8$, $F_4$, $G_2$, then $C = 0$. If $G$ is of one of the types $A_1$, $B_\ell$ ($\ell \geq 3$), $C_\ell$ ($\ell \geq 2$), $E_7$, then $C \simeq \mathbb{Z}/2\mathbb{Z}$, and the nontrivial element of $C$ acts on $\tilde{D}$ by the only nontrivial automorphism of $\tilde{D}$. 

\[ \text{Figure: Extended Dynkin diagram}\]
Table 1: Coefficients $m_i$ and $c_i$ on extended Dynkin diagrams

\[ A_1, A_{\ell} (\ell \geq 2) \]

\[ B_{\ell} (\ell \geq 3), C_{\ell} (\ell \geq 2) \]

\[ D_{\ell} (\ell \geq 4) \]
Table 1: (continued)

\[\begin{array}{c}
E_6 \\
E_7 \\
E_8 \\
F_4 \\
G_2 \\
\end{array}\]

\[\begin{array}{c}
\begin{aligned}
E_6 & : \\
\omega_1 & : \\
E_7 & : \\
\omega_7 & : \\
E_8 & : \\
F_4 & : \\
G_2 & : \\
\end{aligned}
\end{array}\]
It remains to consider the cases $A_\ell$ ($\ell \geq 2$), $D_\ell$, and $E_6$. The action of $C$ on $\widetilde{D}$ is uniquely determined by restriction to the set of vertices $\alpha_j$ of $\widetilde{D}$ with $m_j = 1$. These are the images of $\alpha_0$ under the automorphism group of $\widetilde{D}$.

Let $D$ be of type $A_\ell$, $\ell \geq 2$. The generator $[\omega^\vee_Y]$ of $C \simeq \mathbb{Z}/(\ell + 1)\mathbb{Z}$ acts on $\widetilde{D}$ as the cyclic permutation $0 \mapsto 1 \mapsto \cdots \mapsto \ell - 1 \mapsto \ell \mapsto 0$:

![Diagram of $A_\ell$ action](image)

Let $D$ be of type $D_\ell$, $\ell \geq 4$ is even. We have $C \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, and the classes $[\omega^\vee_Y]$ and $[\omega^\vee_{\ell-1}]$ are generators of $C$. These generators act on $\widetilde{D}$ as follows: $[\omega^\vee_Y]$ acts as $0 \leftrightarrow 1$, $\ell - 1 \leftrightarrow \ell$, and $[\omega^\vee_{\ell-1}]$ acts as $0 \leftrightarrow \ell - 1$, $1 \leftrightarrow \ell$:

![Diagram of $D_\ell$ action](image)

Let $D$ be of type $D_\ell$, $\ell \geq 5$ is odd. We have $C \simeq \mathbb{Z}/4\mathbb{Z}$, and the element $[\omega^\vee_{\ell-1}]$ is a generator of $C$. This generator acts on $\widetilde{D}$ as the 4-cycle $0 \mapsto \ell - 1 \mapsto 1 \mapsto \ell \mapsto 0$:

![Diagram of $D_\ell$ action](image)

Let $D$ be of type $E_6$. The generator $[\omega^\vee_Y]$ of $C \simeq \mathbb{Z}/3\mathbb{Z}$ acts as the 3-cycle $0 \leftrightarrow 1 \leftrightarrow 5 \mapsto 0$:

![Diagram of $E_6$ action](image)

We state results of this section as a proposition.

**9.6. Proposition.** With the assumptions and notation of this section, the inclusion map $\Delta \hookrightarrow \mathfrak{t}(\mathbb{R})$ induces a $C$-equivariant bijective correspondence between $\Delta$ and the set of orbits of $\tilde{W}^{\text{sc}}$ in $\mathfrak{t}(\mathbb{R})$, where $C$ acts on $\mathfrak{t}(\mathbb{R})/\tilde{W}^{\text{sc}}$ via the isomorphism $C \simeq \tilde{W}^{\text{ad}}/\tilde{W}^{\text{sc}}$, and on $\Delta$ by permutations of barycentric coordinates as described above.
10. Case of an outer form of a simple compact group

10.1. In this section, $G$ is a simply connected, simple, compact $\mathbb{R}$-group, and $\tau \in \text{Aut} \text{BRD}(G)$ is of order 2. Then $G$ is of one of the types $A_m$ ($m \geq 2$), $D_m$ ($m \geq 3$), or $E_6$. We describe the quotient set $t_0(\mathbb{R})/\widetilde{W}^\infty_0$, the group $C_0 = \mathcal{P}_0^\vee/\mathcal{Q}_0^\vee$ and the action of $C_0$ on $t_0(\mathbb{R})/\widetilde{W}^\infty_0$. We use the notation from Section 7.

The restricted root systems $\mathcal{R}$ are tabulated in Gorbatsevich, Onishchik and Vinberg [14, Sect. 3.3.9, p. 119]. If $\mathcal{R}$ is of type $A_{2l}$ ($l \geq 1$), then $\mathcal{R}$ is of type $BC_l$; if $\mathcal{R}$ is of type $A_{2l-1}$ ($l \geq 3$), then $\mathcal{R}$ is of type $C_l$; if $\mathcal{R}$ is of type $D_{l+1}$ ($l \geq 2$), then $\mathcal{R}$ is of type $B_l$; if $\mathcal{R}$ is of type $E_6$, then $\mathcal{R}$ is of type $F_4$.

Recall that $g$ denote the $(\pm 1)$-eigenspaces for $\tau$ in $g$.

10.2. Lemma ([14, Sect. 3.3.9]). $g^\tau$ is a simple Lie subalgebra of $g$ and the adjoint representation of $g^\tau$ in $g^{-\tau}$ is irreducible.

Proof. In [14] the statement is verified by explicit calculations in the four above cases. Here is a conceptual argument.

We compute the pairing between restricted simple roots and coroots, with the notation of Lemma 7.13:

$$\langle \alpha_i, \alpha_j^\vee \rangle = \langle \alpha_i, \alpha_j^\vee \rangle, \quad \langle \alpha_i, \alpha_k^\vee \rangle = \langle \alpha_i, \alpha_k^\vee + \tau(\alpha_k^\vee) \rangle = \langle \alpha_i, \alpha_k^\vee \rangle + \langle \alpha_i, \tau(\alpha_k^\vee) \rangle.$$ 

It follows from this computation that simple roots $\alpha_p, \alpha_q$ of $R$ which are linked in the Dynkin diagram $D$ of $R$, that is, have negative Cartan number, restrict to simple roots $\alpha_p, \alpha_q$ of $\mathcal{R}$ which are linked in the Dynkin diagram $\mathcal{D}$ of $\mathcal{R}$. Hence $\mathcal{D}$ is connected. Since $\mathcal{D}$ is also the Dynkin diagram of $g^\tau$ and the number of its vertices is $\ell = \dim t_0$, the latter Lie algebra is simple. Now the irreducibility of $g^{-\tau}$ follows by an argument in [14, Sect. 3.3.11], right above Lemma 3.17 of loc. cit. \hfill \Box

10.3. Let $\bar{\alpha}_0$ denote the lowest weight of $T_0$ in $g^{-\tau}$. Then $\mathcal{S} = \{\bar{\alpha}_0, \bar{\alpha}_1, \ldots, \bar{\alpha}_\ell\}$ is an admissible system of roots in the sense that the Cartan numbers of all pairs $\bar{\alpha}_p, \bar{\alpha}_q$ with $p \neq q$ are non-positive. It is encoded by a twisted affine Dynkin diagram $\widetilde{D} = \widetilde{D}(G, T, B, \tau) = \widetilde{D}(R, S, \tau)$ in the usual way; see [14, Sect. 3.1.7]. There is a unique linear dependence

$$m_0\bar{\alpha}_0 + m_1\bar{\alpha}_1 + \cdots + m_\ell\bar{\alpha}_\ell = 0,$$

where $m_i$ are positive even integers and $m_0 = 2$. These coefficients $m_i$ are tabulated in [14, Sect. 3.3.9]; see also Table 2. We write only the coefficients $m_i = 2$. For $^2A_{2\ell-1}$ and $^2D_{\ell+1}$, we write also the coefficients $c_i$ (modulo $\mathbb{Z}$) in the decomposition of a representative of the generator of $P_0/Q_0 \simeq \mathbb{Z}/2\mathbb{Z}$ into a linear combination of simple restricted roots. We write these coefficients $c_i$ only for simple roots with $m_i = 2$. The vertex corresponding to $\alpha_0$ is painted in black.
Table 2: Coefficients $m_i$ and $c_i$ on twisted affine Dynkin diagrams

| $\tilde{\mathcal{G}}_1$ | $\tilde{\mathcal{G}}_2$ |
|-------------------------|-------------------------|
| $2A_2$                  | $2A_{2\ell-1}$          |
| $2A_{2\ell}$ ($\ell \geq 2$) | $2A_{2\ell-1}$ ($\ell \geq 3$) |
| $2D_{\ell+1}$ ($\ell \geq 2$) | $2E_6$ |

**Diagram:**

- $\tilde{\mathcal{G}}_1$: Twisted affine Dynkin diagram for $2A_2$.
- $\tilde{\mathcal{G}}_2$: Twisted affine Dynkin diagram for $2A_{2\ell-1}$.$\ell \geq 3$).
Each \( x \in t_0(\mathbb{R}) \) has barycentric coordinates \( x_0, x_1, \ldots, x_\ell \) defined by

\[
\bar{\alpha}_i(x) = ix_i \quad (i = 1, \ldots, \ell), \quad \bar{\alpha}_0(x) = i(x_0 - \frac{1}{2}).
\]

They satisfy the identity coming from (10.4):

\[
m_0x_0 + m_1x_1 + \cdots + m_\ell x_\ell = 1.
\]

10.5. The simplex

\[
\Delta = \{ x \in t_0(\mathbb{R}) \mid x_0, x_1, \ldots, x_\ell \geq 0 \}
\]

is a fundamental domain for the \( \hat{W}_0^{sc} \)-action on \( t_0(\mathbb{R}) \); see [14, Sect. 3.3.10, Prop. 3.15(3)]. The group \( C_0 = \tilde{P}_0^\vee / \tilde{Q}_0^\vee \simeq \hat{W}_0^{ad} / \hat{W}_0^{sc} \) acts on \( \Delta \simeq t_0(\mathbb{R}) / \hat{W}_0^{sc} \) by permuting the barycentric coordinates in accordance with automorphisms of \( \hat{D} \). A general description of this action is very similar to the one given in Subsection 9.5 for the action of \( C \) on the fundamental simplex of \( \hat{W}^{sc} \). Namely, the nonzero cosets of \( C_0 \) are represented by the vectors \( \nu \in \tilde{P}_0^\vee \) such that \( i\nu \in \Delta \), which are exactly \( \nu = \tilde{\omega}_k^\vee / 2 \) such that \( \alpha_k \) is moved by \( \tau \) and \( m_k = 2 \). If \( w_0 \), resp. \( w_k \), denote the longest element in \( W_0 \), resp. in the Weyl group \( W_k \) of the root subsystem \( R_k \) generated by \( S \setminus \{ \tilde{\alpha}_k \} \), then the transformation \( x \mapsto w_k w_0 x + i\tilde{\omega}_k^\vee / 2 \) preserves \( \Delta \) and gives the action of the respective coset \( [\tilde{\omega}_k^\vee / 2] \in C_0 \) on \( \Delta \). The proof given in [8, Sect. VI.2.3, Prop. 6] for the non-twisted case works in the twisted case with obvious modifications.

Specifically, if \( G \) is of type \( A_{2l} (\ell \geq 1) \) or \( E_6 \), then \( C_0 = \{ 0 \} \). If \( G \) is of type \( A_{2l-1} (\ell \geq 3) \) or \( D_{l+1} (\ell \geq 2) \), then \( C_0 \simeq \mathbb{Z} / 2\mathbb{Z} \) is generated by \( [\tilde{\omega}_k^\vee / 2] \), where \( \tilde{\alpha}_k \) is the unique vertex of \( \hat{D} \) which is interchanged with \( \alpha_0 \) under the unique nontrivial (involutive) automorphism of \( \hat{D} \). The coset \( [\tilde{\omega}_k^\vee / 2] \) acts by this automorphism.

We state results of this section as a proposition.

10.6. Proposition. With the assumptions and notation of this section, the inclusion \( \Delta \hookrightarrow t_0(\mathbb{R}) \) induces a \( C_0 \)-equivariant bijective correspondence between \( \Delta \) and the orbit set of \( \hat{W}_0^{sc} \) in \( t_0(\mathbb{R}) \), where \( C_0 \) acts on \( t_0(\mathbb{R}) / \hat{W}_0^{sc} \) via the isomorphism \( C_0 \simeq \hat{W}_0^{ad} / \hat{W}_0^{sc} \), and on \( \Delta \) by permutations of coordinates as described above.

11. Case of an \( \mathbb{R} \)-simple non-absolutely simple group

11.1. Now suppose that \( G \) is simply connected and not simple, but \( q G \) is \( \mathbb{R} \)-simple. This means that \( G \simeq G' \times_{\mathbb{R}} G' \), where \( G' \) is a compact, simply connected, simple \( \mathbb{R} \)-group, and \( \theta \) swaps the two simple factors of \( G \), that is, \( \theta(g_1, g_2) = (g_2, g_1) \) for \( g_1, g_2 \in G' \).

We choose a maximal torus \( T = T' \times T' \) in \( G \), where \( T' \) is a maximal torus in \( G' \). We also choose a Borel subgroup \( B = B' \times B' \) in \( G \) containing \( T \), where \( B' \) is a Borel subgroup in \( G' \) containing \( T' \).

Then \( \theta \) preserves \( T \) and \( B \). Furthermore, a pinning of \( (G', T', B') \) gives rise to a “doubled” pinning of \( (G, T, B) \) preserved by \( \theta \). Hence \( \theta = \tau \).
The torus \( T_0 = T^\tau \) given by
\[
T_0 = \{(t,t) \mid t \in T'\} \subset T
\]
is a maximal torus in the diagonal subgroup \( G^\tau \subset G = G' \times_R G' \). The torus \( T_1 \) is given by
\[
T_1 = \{(t,t^{-1}) \mid t \in T'\} \subset T.
\]
We shall identify \( G^\tau \) with \( G' \) and \( T_0 \) with \( T' \), both embedded diagonally. The representation of \( G^\tau \) in \( g^{-\tau} = \{ (\xi,-\xi) \mid \xi \in g' \} \) is equivalent to the adjoint representation of \( g' \).

11.2. Let \( P', Q', P'^\vee, Q'^\vee \) denote the lattices of weights, roots, coweights, and coroots of \( G' \) with respect to \( T' \), correspondingly. With the notation of Section 7, we have decompositions:
\[
\begin{align*}
P &= P' \oplus P', & Q &= Q' \oplus Q', \\
P'^\vee &= P'^\vee \oplus P'^\vee, & Q'^\vee &= Q'^\vee \oplus Q'^\vee.
\end{align*}
\]

The restriction of characters from \( T \) to \( T_0 = T' \) is the map
\[
\chi = (\chi_1, \chi_2) \mapsto \bar{\chi} = \chi_1 + \chi_2.
\]
The restricted root system \( \bar{\Gamma} \) is just the root system of \( G' \) with respect to \( T' \). The restricted simple roots \( \bar{\alpha}_1, \ldots, \bar{\alpha}_\ell \) are the simple roots of \( G' \) and \( \bar{\alpha}_0 \) is the lowest root of \( G' \). The lattices \( \bar{P}_0 \) and \( \bar{Q}_0 \) are identified with \( P' \) and \( Q' \), respectively. The lattices \( \bar{P}'_0 = P'^\vee \) and \( \bar{Q}'_0 = Q'^\vee \) are diagonally embedded in \( P'^\vee \) and \( Q'^\vee \), respectively. The projection from \( t \) to \( t_0 \) is the map
\[
x = (x_1, x_2) \mapsto \bar{x} = \frac{1}{2}(x_1 + x_2).
\]
Under these identifications, we have
\[
\bar{P}'_0 = \frac{1}{2} P'^\vee \quad \text{and} \quad \bar{Q}'_0 = \frac{1}{2} Q'^\vee.
\]

11.3. The group \( W_0 \) is isomorphic to the Weyl group of \( (G', T') \) and \( \tilde{W}'^{sc} \) is isomorphic to the affine Weyl group of \( G' \). The simplex \( \Delta = \frac{1}{2} \Delta' \) is a fundamental domain in \( t_0(\mathbb{R}) = t'(\mathbb{R}) \) for \( \tilde{W}'^{sc} \), where \( \Delta' \) is the fundamental simplex of the affine Weyl group of \( G' \), as in Section 9.

The barycentric coordinates on \( t_0(\mathbb{R}) = t'(\mathbb{R}) \) are defined by
\[
\alpha_i(x) = ix_i \quad \text{for} \quad i = 1, \ldots, \ell, \quad \alpha_0(x) = i(x_0 - \frac{1}{2}),
\]
which coincides with definition (9.4) in Section 9 except for \( \frac{1}{2} \) instead of 1 in the last equality, so that \( x_0, x_1, \ldots, x_\ell \geq 0 \) are still the defining inequalities for \( \Delta \). We set \( m_i = 2m'_i \) (\( i = 0, 1, \ldots, \ell \)), where \( m'_i \) are the coefficients for \( G' \) as in (9.2). Then
\[
m_0x_0 + m_1x_1 + \cdots + m_\ell x_\ell = 1.
\]
The group \( C_0 \simeq P'^\vee/Q'^\vee \) acts on \( \Delta \) by permuting the barycentric coordinates in accordance with automorphisms of the extended Dynkin diagram \( \bar{D} = \bar{D}(G', T', B') \), as in Section 9.

We state results of this section as a proposition.
11.4. Proposition. With the assumptions and notation of this section, the inclusion \( \Delta \hookrightarrow t_0(\mathbb{R}) \simeq t'(\mathbb{R}) \) induces a \( C_0 \)-equivariant bijective correspondence between \( \Delta \) and the orbit set of \( \widetilde{W}^\text{sc}_0 \) in \( t_0(\mathbb{R}) \), where \( C_0 \simeq P^\vee/Q^\vee \) acts on \( t_0(\mathbb{R})/\widetilde{W}^\text{sc}_0 \) via the isomorphism \( C_0 \simeq \widetilde{W}^\text{rad}_0/\widetilde{W}^\text{sc}_0 \), and on \( \Delta = \frac{1}{2} \Delta' \) by permutations of coordinates as described in Section 9.

12. Square roots of a central element

12.1. We retain Notation 4.1 assuming additionally that \( G = (G, \sigma_c) \) is a compact semisimple \( \mathbb{R} \)-group, not necessarily simply connected. We write \( G^\text{sc} \) for the universal cover of \( G \). The involutions \( \sigma_c \) and \( \theta \) lift to commuting involutions of \( G^\text{sc} \), which we denote by the same letters. Consider the twisted group \( \theta G \) and its universal cover \( \theta G^\text{sc} \). We have a decomposition into \( \mathbb{R} \)-simple factors

\[ \theta G^\text{sc} = \theta G^{(1)} \times_\mathbb{R} \cdots \times_\mathbb{R} \theta G^{(r)}. \]

It corresponds to a decomposition

\[ G^\text{sc} = G^{(1)} \times_\mathbb{R} \cdots \times_\mathbb{R} G^{(r)}, \]

in which each pair \( (G^{(k)}, \theta^{(k)}) \) is indecomposable, where \( \theta^{(k)} \) is the restriction of \( \theta \) to \( G^{(k)} \).

We consider the spaces \( t_0, t_0^*, t_0(\mathbb{R}) \), and various lattices in them, see Section 7; each of these objects decomposes into a direct sum of the corresponding objects for \( G^{(k)} \), which we denote by adding the superscript \( (k) \). In particular,

\[ t_0(\mathbb{R}) = t_0^{(1)}(\mathbb{R}) \oplus \cdots \oplus t_0^{(r)}(\mathbb{R}), \]

Similarly, the groups \( W_0 \) and \( \widetilde{W}^\text{sc}_0 \) decompose into the direct products of the corresponding groups for \( G^{(k)} \). We obtain a decomposition of the orbit set

\[ t_0(\mathbb{R})/\widetilde{W}^\text{sc}_0 = t_0^{(1)}(\mathbb{R})/\widetilde{W}^{(1)}_0 \times \cdots \times t_0^{(r)}(\mathbb{R})/\widetilde{W}^{(r)}_0 \]

compatible with the decomposition of the group

\[ C_0 = C_0^{(1)} \oplus \cdots \oplus C_0^{(r)}, \]

where the group \( C_0 = \widetilde{P}^\vee_0/Q^\vee_0 \) acts on the orbit set \( t_0(\mathbb{R})/\widetilde{W}^\text{sc}_0 \), and each group \( C_0^{(k)} = \widetilde{P}_0^{(k)\vee}/\widetilde{Q}_0^{(k)\vee} \) acts on the respective orbit set \( t_0^{(k)}(\mathbb{R})/\widetilde{W}^{(k)\text{sc}}_0 \).

12.2. There are three cases:

(a) \( G^{(k)} \) is simple and \( \theta^{(k)} \) is an inner involution of \( G \);
(b) \( G^{(k)} \) is simple and \( \theta^{(k)} \) is an outer involution of \( G \);
(c) \( G^{(k)} = G^{(k)'} \times_\mathbb{R} G^{(k)''} \) and \( \theta^{(k)} \) swaps the isomorphic simple factors \( G^{(k)'} \) and \( G^{(k)''} \).

They were examined in detail in Sections 9, 10, and 11, respectively.
Let $\tilde{S} = \tilde{S}^{(1)} \cup \cdots \cup \tilde{S}^{(r)}$ denote the union of the extended sets of simple restricted roots of $(G^{(k)}, \tau^{(k)})$, where $\tau^{(k)} = \theta^{(k)}_* \in \text{Aut} \ BRD(G^{(k)})$; see Subsections 9.1, 10.3, and 11.2. We identify $\tilde{S}$ with the set of vertices of the diagram

$$\tilde{D} = \tilde{D}(G, T, B, \tau) = \tilde{D}^{(1)} \sqcup \cdots \sqcup \tilde{D}^{(r)},$$

where $\tilde{D}^{(k)} = \tilde{D}(G^{(k)}, T^{(k)}, B^{(k)}, \tau^{(k)})$ is an affine Dynkin diagram (twisted or not), as defined in Sections 9–11.

Each point $x = (x^{(1)}, \ldots, x^{(r)}) \in t_0(\mathbb{R})$ has barycentric coordinates $(x_\beta)_{\beta \in \tilde{S}}$, which are defined separately for each component $x^{(k)} \in t_0^{(k)}(\mathbb{R})$ (see Subsections 9.3, 10.3, and 11.3). They satisfy

$$(12.3) \quad \sum_{\beta \in \tilde{S}^{(k)}} m_\beta x_\beta = 1 \quad \text{for each } k = 1, \ldots, r,$$

where $m_\beta$ are the coefficients of the linear dependence of $\tilde{S}^{(k)}$ normalized as in Subsections 9.3, 10.3, and 11.3.

The product

$$\Delta = \Delta^{(1)} \times \cdots \times \Delta^{(r)}$$

is a fundamental domain for the group $\tilde{W}_0^{sc}$ acting on $t_0(\mathbb{R})$, where $\Delta^{(k)}$ are the fundamental simplices for the respective affine Weyl groups $\tilde{W}_0^{(k)}$ acting on $t_0^{(k)}(\mathbb{R})$; see Subsections 9.3, 10.5, and 11.3. The defining inequalities for $\Delta$ are $x_\beta \geq 0$ over all $\beta \in \tilde{S}$. The quotient group $F_0 = \tilde{W}_0/\tilde{W}_0^{sc} \simeq \tilde{X}_0^{\vee}/\tilde{Q}_0^{\vee}$ acts on $\Delta \simeq t_0(\mathbb{R})/\tilde{W}_0^{sc}$ as a subgroup of $C_0$, where $C_0$ acts by permuting the barycentric coordinates in accordance with automorphisms of $\tilde{D}$.

12.4. Proposition. The inclusion $\Delta \hookrightarrow t_0(\mathbb{R})$ and the map $\hat{E} : t_0(\mathbb{R}) \to T_0(\mathbb{R}) \cdot \hat{\tau}$ of (8.14) induce bijective correspondences between the orbit sets for the actions of $F_0$ in $\Delta$, of $\tilde{W}_0$ in $t_0(\mathbb{R})$, and of $\tilde{W}_0$ in $T_0(\mathbb{R}) \cdot \hat{\tau}$.

Proof. Propositions 9.6, 10.6, and 11.4 show that for each $k = 1, \ldots, r$, the $C_0^{(k)}$-equivariant map $\Delta^{(k)} \to t_0^{(k)}(\mathbb{R})/\tilde{W}_0^{(k)sc}$ is bijective. Now it follows from the product structure of $\Delta$, $t_0(\mathbb{R})$, $\tilde{W}_0^{sc}$, and $C_0$, that the $C_0$-equivariant map

$$\Delta \to t_0(\mathbb{R})/\tilde{W}_0^{sc}$$

is bijective. Since $F_0 \subseteq C_0$, this bijective map is $F_0$-equivariant and induces a bijection of the orbit sets

$$\Delta/F_0 \sim (t_0(\mathbb{R})/\tilde{W}_0^{sc})/F_0 = t_0(\mathbb{R})/\tilde{W}_0.$$

Finally, Lemma 8.13 gives a bijection $t_0(\mathbb{R})/\tilde{W}_0 \sim (T_0(\mathbb{R}) \cdot \hat{\tau})/\tilde{W}_0$. $\Box$

Our aim is to describe a smaller orbit set for the action of $\tilde{W}_0$ on $(T_0 \cdot \hat{\tau})_{\frac{1}{2}} \subset T_0(\mathbb{R}) \cdot \hat{\tau}$. We do it in terms of a combinatorial notion which we introduce now.
12.5. **Definition.** A *Kac labeling* of $\widetilde{D}$ is a family of nonnegative integer numerical labels $p = (p_\beta)_{\beta \in \widetilde{D}}$ at the vertices $\beta \in \widetilde{S}$ of $\widetilde{D}$ satisfying

$$
\sum_{\beta \in \widetilde{S}^{(k)}} m_\beta p_\beta = 2 \quad \text{for each } k = 1, \ldots, r.
$$

Labelings of this kind were used by Kac [18] in classification of automorphisms of finite order (specifically, involutions) of semisimple Lie algebras. We denote the set of Kac labelings of $\widetilde{D}$ by $K(\widetilde{D})$.

Note that a Kac labeling $p$ of $\widetilde{D} = \widetilde{D}^{(1)} \sqcup \cdots \sqcup \widetilde{D}^{(r)}$ is the same as a family $(p^{(1)}, \ldots, p^{(r)})$, where each $p^{(k)}$ is a Kac labeling of $\widetilde{D}^{(k)}$.

12.7. To any $x \in \mathfrak{t}_0(\mathbb{R})$, we assign a family $p = p(x) = (p_\beta)_{\beta \in \mathbb{R}S}$ of real numbers $p_\beta = 2x_\beta$, which satisfy (12.6) by (12.3). This correspondence identifies $\mathfrak{t}_0(\mathbb{R})$ with the subspace of $\mathbb{R}S$ defined by equations (12.6). We denote the inverse correspondence as

$$
p \mapsto x(p) = \frac{i}{2} \sum_{\beta \in S} p_\beta \tilde{\omega}_\beta^\vee,
$$

where $\tilde{\omega}_\beta^\vee = \tilde{\omega}_i^\vee$ is the fundamental coweight corresponding to a simple restricted root $\beta = \tilde{\alpha}_i$, with the notation of Lemma 7.13. We have $x \in \Delta$ if and only if $p_\beta \geq 0$ for all $\beta \in \tilde{S}$.

For any character $\lambda \in X_0$, $\lambda = \sum_{\beta \in S} c_\beta \beta$, $c_\beta \in \mathbb{Q}$, put

$$
\langle \lambda, p \rangle = \sum_{\beta \in S} c_\beta p_\beta = 2\lambda(x(p))/i.
$$

Note that $\langle \lambda, p \rangle \in \mathbb{Q}$ whenever $p \in K(\widetilde{D})$, and $\langle \lambda, p \rangle \in \mathbb{Z}$ if furthermore $\lambda \in Q_0$.

Recall that $\theta = \text{inn}(t_\theta) \circ \tau$, where $t_\theta \in T_0$ and $z = t_\theta^2 \in Z(G) \cap T_0$. We write

$$
z = \exp 2\pi i \zeta, \quad \text{where } \zeta \in \mathfrak{t}_0.
$$

For a character $\lambda \in X_0 \subset \mathfrak{t}_0^*$, we have

$$
\lambda(z) = \lambda(\exp 2\pi i \zeta) = \exp \lambda(2\pi i \zeta) = \exp 2\pi i \lambda(\zeta).
$$

Since $z$ is an element of finite order in $T_0$, we see that $\lambda(z)$ is a root of unity. Hence by (12.9) $\lambda(\zeta) \in \mathbb{Q}$ and, moreover, $\lambda(\zeta) \in \mathbb{Z}$ for $\lambda \in Q_0$ since $z \in Z(G)$ and therefore $\lambda(z) = 1$ in this case. It follows from (12.9) that the image of $\lambda(\zeta)$ in $\mathbb{Q}/\mathbb{Z}$ depends only on $z$ and not on the choice of $\zeta$, which is determined modulo $X_0^\vee$.

The following theorem gives a combinatorial description of the set $(T_0 \cdot \hat{\tau})^{\mathbb{Z}}/\hat{W}_0$ in terms of Kac labelings.

12.10. **Theorem.** Let $\zeta \in \mathfrak{t}_0$ be as in (12.8). There is a canonical bijection between the set $(T_0 \cdot \hat{\tau})^{\mathbb{Z}}/\hat{W}_0$ and the set of $F_0$-orbits in the set of Kac labelings $p \in K(\widetilde{D})$ satisfying

$$
\langle \lambda, p \rangle \equiv \lambda(\zeta) \pmod{\mathbb{Z}} \quad \text{for all } [\lambda] \in X_0/Q_0.
$$

The bijection sends the $F_0$-orbit of $p \in K(\widetilde{D})$ to the $\hat{W}_0$-orbit of $\hat{\mathcal{E}}(x(p)) \in T_0 \cdot \hat{\tau}$ with the notation of Section 8.
Proof. Any element of $T_0(\mathbb{R}) \cdot \hat{\tau}$ is of the form $t \cdot \hat{\tau} = \hat{\mathcal{E}}(x)$ for some $x \in t_0(\mathbb{R})$, so that $t = \mathcal{E}(x) \in T_0$ with the notation of Section 7. We have $t \cdot \hat{\tau} \in (T_0 \cdot \hat{\tau})_2^\circ$ if and only if $t^2 = z$. Since $t^2 = \mathcal{E}(2x)$ and $z = \mathcal{E}(i\zeta)$, the condition $t^2 = z$ is equivalent to

$$2x = i\zeta + iv$$

for some $v \in X_0^\vee$

$$\iff i(\lambda, p) = 2\lambda(x) = i\lambda(\zeta) + i\lambda(\nu) \equiv i\lambda(\zeta) \pmod{i\mathbb{Z}} \quad \text{for all } \lambda \in X_0$$

$$\iff \langle \lambda, p \rangle \equiv \lambda(\zeta) \pmod{\mathbb{Z}} \quad \text{for all } \lambda \in X_0.$$

In particular, taking $\lambda = \beta \in \mathcal{S} \subset Q_0$, we get $p_\beta \equiv \beta(\zeta) \in \mathbb{Z}$. Equality (12.6) implies $p_\beta \in \mathbb{Z}$ for all $\beta \in \mathcal{S}$, that is, $p \in \mathbb{Z}\mathcal{S}$.

The set of all $x \in t_0(\mathbb{R})$ such that $p = p(x)$ satisfies (12.11) is $\hat{\mathcal{W}}_0$-stable, being the preimage of a $\hat{\mathcal{W}}_0$-stable set $(T_0 \cdot \hat{\tau})_2^\circ$ under $\hat{\mathcal{E}}$. By Proposition 12.4, the $\hat{\mathcal{W}}_0$-orbits in this subset of $t_0(\mathbb{R})$ are the preimages of the $\hat{\mathcal{W}}_0$-orbits in $(T_0 \cdot \hat{\tau})_2^\circ$, and intersect $\Delta$ in $F_0$-orbits. This yields a bijection between $(T_0 \cdot \hat{\tau})_2^\circ / \hat{\mathcal{W}}_0$ and the set of $F_0$-orbits of all $x \in \Delta$ with $p = p(x)$ satisfying (12.11). The condition $x \in \Delta$ reads as $p_\beta \geq 0$ for all $\beta \in \mathcal{S}$. Thus all $p_\beta$ are integer and non-negative, that is, $p \in \mathcal{K}(\hat{D})$, which completes the proof. □

12.12. Theorem 12.10 implies, in particular, that the element $\hat{\theta} = t_\theta \cdot \hat{\tau} \in (T_0 \cdot \hat{\tau})_2^\circ$ is $\hat{\mathcal{W}}_0$-equivalent to an element of the form $\hat{\mathcal{E}}(x(q))$ for some Kac labeling $q \in \mathcal{K}(\hat{D})$. The action of $\hat{\mathcal{W}}_0$ on $(T_0 \cdot \hat{\tau})_2^\circ$ comes from the conjugation action (8.2) of $\mathbf{N}_\tau(\mathbb{R})$ (see Lemma 5.3(ii) and Lemma 8.3). If we replace $\theta$ by an $\mathbf{N}_\tau(\mathbb{R})$-conjugate, the involution $\theta$ will be replaced by a conjugate one $\text{inn}(g) \circ \theta \circ \text{inn}(g)^{-1}$ with $g \in \mathbf{N}_\tau(\mathbb{R})$. This does not change the $\mathbb{R}$-group $\mathcal{G}$ up to an isomorphism. Thus we may assume without loss of generality that $\hat{\theta} = \hat{\mathcal{E}}(x(q))$ and $t_\theta = \mathcal{E}(x(q))$.

The corresponding semisimple $\mathbb{R}$-group $\mathcal{G}$ is defined by the following data:

- a Dynkin diagram $D$, which determines the Lie algebra $\mathfrak{g}$ and the simply connected group $G^\text{sc}$;
- a lattice $X^\vee$ in between $P^\vee$ and $Q^\vee$ or, equivalently, a subgroup $F = X^\vee / Q^\vee \subset C = P^\vee / Q^\vee \simeq Z(G^\text{sc})$, which determines the complex algebraic group $G = G^\text{sc} / F$;
- a diagram automorphism $\tau \in \text{Aut} D$ preserving $F$, which determines the class of $\theta$ in $\text{Aut}(G) / \text{Inn}(G)$;
- a Kac labeling $q \in \mathcal{K}(\hat{D})$, which determines $t_\theta = \mathcal{E}(x(q))$ and, in turn, the involution $\theta = \text{inn}(t_\theta) \circ \tau$ and the real structure $\sigma = \theta \circ \sigma_c$ on $G$.

We write

$$\mathcal{G} = \mathbf{G}(D, F, \tau, q).$$

The above discussion shows that every semisimple $\mathbb{R}$-group is isomorphic to a group of the form $\mathbf{G}(D, F, \tau, q)$ for some $D$, $F$, $\tau$, and $q$.

13. Main theorem

13.1. We are now ready to state the main result of this article. We retain Notation 4.1 and the notation from Sections 7 and 12.
Consider a semisimple \( \mathbb{R} \)-group \( G(D, F, \tau, q) \), as defined in Subsection 12.12, where \( D \) is a Dynkin diagram, \( F \) is a subgroup of \( C = P^\vee/Q^\vee \), \( \tau \in \text{Aut} \, D \) is a diagram involution, and \( q \in \mathcal{K}(\tilde{D}) \) is a Kac labeling.

For any two Kac labelings \( p, q \in \mathcal{K}(\tilde{D}) \), set
\[
\nu_{p, q} = 2(x(p) - x(q))/i = \sum_{\beta \in \mathcal{S}} (p_\beta - q_\beta)\hat{\omega}_\beta^\vee \in P_0^\vee.
\]

Let \( \mathcal{K}(\tilde{D}, X_0, q) \) denote the set of Kac labelings \( p \in \mathcal{K}(\tilde{D}) \) satisfying the condition
\[
\langle \lambda, p \rangle \equiv \langle \lambda, q \rangle \pmod{\mathbb{Z}} \quad \text{for all } [\lambda] \in X_0/Q_0.
\]
Recall that the finite abelian group \( F_0 = \hat{X}_0^\vee/\hat{Q}_0^\vee \) acts by diagram automorphisms on \( \tilde{D} \), and hence on \( \mathcal{K}(\tilde{D}) \).

13.3. Main Theorem. For \( (D, F, \tau, q) \) as above, the group \( F_0 \), when acting on \( \mathcal{K}(\tilde{D}) \), preserves \( \mathcal{K}(\tilde{D}, X_0, q) \). We have \( \nu_{p, q} \in X_0^\vee = \text{Hom}(\mathbb{C}^\times, T_0) \) whenever \( p \in \mathcal{K}(\tilde{D}, X_0, q) \), and the map
\[
\mathcal{K}(\tilde{D}, X_0, q) \to (T_0)_2 \hookrightarrow Z^1(\mathbb{R}, G(D, F, \tau, q)), \quad p \mapsto \nu_{p, q}(-1) \in (T_0)_2
\]
induces a bijection between the set of \( F_0 \)-orbits in \( \mathcal{K}(\tilde{D}, X_0, q) \) and the first Galois cohomology set \( H^1(\mathbb{R}, G(D, F, \tau, q)) \).

**Proof.** Recall that we have \( t_\theta = \mathcal{E}(x(q)) \) and \( z = t_\theta^2 = (t_\theta \cdot \hat{\tau})^2 \). Condition (12.11) for a Kac labeling \( p \in \mathcal{K}(\tilde{D}) \) is equivalent to condition (13.2), since \( q \) itself satisfies (12.11) by Theorem 12.10. For \( p \in \mathcal{K}(\tilde{D}, X_0, q) \), it follows from (13.2) that
\[
\lambda(\nu_{p, q}) = 2\lambda(x(p) - x(q))/i = \langle \lambda, p \rangle - \langle \lambda, q \rangle \in \mathbb{Z} \quad \text{for all } \lambda \in X_0,
\]
that is, \( \nu_{p, q} \in X_0^\vee \). Again by Theorem 12.10, the map
\[
\mathcal{K}(\tilde{D}, X_0, q) \to (T_0 \cdot \hat{\tau})^2_2, \quad p \mapsto \mathcal{E}(x(p)) = \mathcal{E}(x(p)) \cdot \hat{\tau}
\]
duces a bijection \( \mathcal{K}(\tilde{D}, X_0, q)/F_0 \sim (T_0 \cdot \hat{\tau})^2_2/\hat{W}_0 \). By Corollary 6.6 the map
\[
(T_0 \cdot \hat{\tau})^2_2 \to (T_0)_2 \hookrightarrow Z^1(\mathbb{R}, G(D, F, \tau, q)), \quad t \cdot \hat{\tau} \mapsto tt_\theta^{-1} = t \cdot \mathcal{E}(-x(q))
\]
duces a bijection \( (T_0 \cdot \hat{\tau})^2_2/\hat{W}_0 \sim H^1(\mathbb{R}, G(D, F, \tau, q)) \). The composite bijection sends the \( F_0 \)-orbit of \( p \) to the cohomology class of
\[
\mathcal{E}(x(p)) \cdot \mathcal{E}(-x(q)) = \mathcal{E}(x(p) - x(q)) = \exp 2\pi i (x(p) - x(q)) = \exp \pi i \nu_{p, q} = \nu_{p, q}(-1),
\]
where the last equality follows from (7.2). This completes the proof. \( \square \)

Let us see what Theorem 13.3 gives us in the two particular cases of simply connected and adjoint groups. A semisimple \( \mathbb{R} \)-group \( G(D, F, \tau, q) \) is simply connected if and only if \( F = 0 \). In this case, we have \( F_0 = 0 \), \( X = P \), and \( X_0 = P_0 \).
13.5. Corollary. There is a canonical bijection between the sets $\mathcal{K}(\widetilde{D}, P_0, q)$ and $H^1(\mathbb{R}, G(D, 0, \tau, q))$ given by the map (13.4).

This result in the special case when $\tau = \text{id}$ was stated by E. B. Vinberg in a 2008 email message to the first-named author.

In the adjoint case, we have $X = Q$, $F = C$, and hence $X_0 = Q_0$, $F_0 = C_0$. It follows that condition (13.2) is empty and $\mathcal{K}(\widetilde{D}, Q_0, q) = \mathcal{K}(\widetilde{D})$.

13.6. Corollary. There is a canonical bijection between the sets $\mathcal{K}(\widetilde{D})/C_0$ and $H^1(\mathbb{R}, G(D, C, \tau, q))$ given by the map (13.4).

The adjoint $\mathbb{R}$-group $G(D, C, \tau, q) = \theta G$ is the identity component of the algebraic group $\text{Aut} \ G$ equipped with the real structure given by the conjugation action of $\sigma = \theta \circ \sigma_c$. It is well known that the cohomology classes in $H^1(\mathbb{R}, \theta G)$ are in a bijective correspondence with the $G$-conjugacy classes of real structures $\sigma' = \theta' \circ \sigma_c$ on $G$ such that $\theta' * = \tau$; see Serre [23, Sect. III.1]. Thus Corollary 13.6 gives a classification of inner forms of a semisimple group $\theta G$ in terms of Kac labelings. This result goes back to Kac [17], [18]; see also [20, Sects. 5.1.4–5.1.6] and [14, Sects. 4.1.3–4.1.5].

14. Twisting and functoriality

14.1. Recall that if $G = (G, \sigma)$ is a real algebraic group and $a \in Z^1(\mathbb{R}, G)$, then the $a$-twist of $G$ is $aG := (G, a \circ \sigma)$. Here $G$ is identified with the group $\text{Inn} G$ equipped with the real structure given by the conjugation action of $\sigma$. The same notation $aG$ is used for the twist of $G$ by a cocycle $a \in Z^1(\mathbb{R}, G)$ acting on $G$ by $\text{inn}(a)$. See Serre [23, Sect. I.5.3].

We use the notation of Subsection 12.12. Having fixed $D$, $F$, and $\tau$, we shall write $G_q = G(D, F, \tau, q)$ for brevity.

Let $E^\text{ad} : t \mapsto T^\text{ad}$ denote the scaled exponential map $x \mapsto \exp(2\pi i x)$ in the adjoint group. Identifying $G^\text{ad}$ with $\text{Inn} G$, we have $E^\text{ad}(x) = \text{inn}(E(x))$.

14.2. Proposition. Let $q' \in \mathcal{K}(\widetilde{D})$ and consider the 1-cocycle

$$a = E^\text{ad}(x(q') - x(q)) \in (T_0^\text{ad})_2 \subset Z^1(\mathbb{R}, G_q^\text{ad}).$$

Then there is a canonical isomorphism $aG_q \sim G_{q'}$. 

Proof. Let $\sigma_q$ denote the real structure on $G$ corresponding to the real form $G_q$; then $\sigma_q = E^\text{ad}(x(q)) \circ \tau \circ \sigma_c$. Similarly, $\sigma_{q'} = E^\text{ad}(x(q')) \circ \tau \circ \sigma_c$. We see that

$$\begin{align*}
a \circ \sigma_q &= E^\text{ad}(x(q') - x(q)) \circ E^\text{ad}(x(q)) \circ \tau \circ \sigma_c = E^\text{ad}(x(q')) \circ \tau \circ \sigma_c = \sigma_{q'},
\end{align*}$$

as required. $\square$

Now let $q' \in \mathcal{K}(\widetilde{D}, X_0, q)$, that is, $q' \in \mathcal{K}(\widetilde{D})$ and $\langle \lambda, q' \rangle - \langle \lambda, q \rangle \in \mathbb{Z}$ for all $[\lambda] \in X_0/Q_0$. With the notation of Theorem 13.3, consider the 1-cocycle

$$a = \nu_{q', q}(-1) = E(x(q') - x(q)) \in (T_0)_2 \subset Z^1(\mathbb{R}, G_q)$$

and the twisting bijection $T_a : H^1(\mathbb{R}, aG_q) \to H^1(\mathbb{R}, G_q)$ of Serre [23, I.5.3, Prop. 35 bis]. By Proposition 14.2 we may identify $aG_q$ with $G_{q'}$. Thus we obtain a map

$$T_a : H^1(\mathbb{R}, G_{q'}) \to H^1(\mathbb{R}, G_q)$$

sending the neutral cohomology class $[1] \in H^1(\mathbb{R}, G_{q'})$ to $[a] \in H^1(\mathbb{R}, G_q)$. 

14.3. **Proposition.** \( T_a[\nu_{p,q'}(-1)] = [\nu_{p,q}(-1)] \) for all elements \( p \in \mathcal{K}(\tilde{D}, X_0, q) = \mathcal{K}(\tilde{D}, X_0, q) \).

**Proof.** Note that \( \nu_{p,q} = \nu_{p,q'} + \nu_{q',q} \). The map \( T_a \) is induced by the map on cocycles

\[
Z^1(\mathbb{R}, G_{q'}) \to Z^1(\mathbb{R}, G_q), \quad a' \mapsto a'a,
\]

sending \( \nu_{p,q'}(-1) \) to

\[
\nu_{p,q'}(-1) \cdot a = \nu_{p,q'}(-1) \cdot \nu_{q',q}(-1) = \nu_{p,q}(-1).
\]

\( \square \)

14.4. Let

\[ \varphi : G \to G' \]

be a *normal* homomorphism of not necessarily compact semisimple \( \mathbb{R} \)-groups. Here “normal” means that \( \varphi(G) \) is normal in \( G' \). It is easy to describe the induced map

\[
(14.5) \quad \varphi_* : H^1(\mathbb{R}, G) \to H^1(\mathbb{R}, G')
\]

using the descriptions of \( H^1(\mathbb{R}, G) \) and \( H^1(\mathbb{R}, G') \) given in Main Theorem 13.3. In Proposition 14.7 below we state the corresponding assertion in the special case when \( \varphi \) is an isogeny (a surjective homomorphism with finite kernel). We leave the task of stating the general assertion to the interested reader.

14.6. Let \( \varphi : G \to G' \) be an isogeny of semisimple *not necessarily compact* \( \mathbb{R} \)-groups. We compute the induced map (14.5) in Galois cohomology.

Let \( T \subseteq G \) be a maximal torus containing a maximal compact torus \( T_0 \) of \( G \). Set \( T' = \varphi(T), T'_0 = \varphi(T_0) \subseteq T' \). Then \( T' \) is a maximal torus of \( G' \) and \( T'_0 \) is a maximal compact torus of \( G' \). Let \( X, X', X_0, X'_0 \) denote the character lattices of \( T, T', T_0, T'_0 \), respectively. Write

\[ G = G(D, F, \tau, q), \quad G' = G(D, F', \tau, q) \]

(with the same \( D, \tau, q \)). Here \( F = X/Q \) and \( F' = X'/Q \). For any \( p \in \mathcal{K}(\tilde{D}) \), the vector \( \nu_{p,q} \) may be regarded as a cocharacter of \( T_0 \) if \( p \in \mathcal{K}(\tilde{D}, X_0, q) \), and as a cocharacter of \( T'_0 \) if \( p \in \mathcal{K}(\tilde{D}, X'_0, q) \), in which case we denote it as \( \nu'_{p,q} \). Then by Main Theorem 13.3 we have

\[
H^1(\mathbb{R}, G) = \{ [\nu_{p,q}(-1)] \mid p \in \mathcal{K}(\tilde{D}, X_0, q) \},
\]

\[
H^1(\mathbb{R}, G') = \{ [\nu'_{p,q}(-1)] \mid p \in \mathcal{K}(\tilde{D}, X'_0, q) \}.
\]

Note that \( \mathcal{K}(\tilde{D}, X_0, q) \subseteq \mathcal{K}(\tilde{D}, X'_0, q) \), since \( X_0 \supseteq X'_0 \).

14.7. **Proposition.** For \( p \in \mathcal{K}(\tilde{D}, X_0, q) \) we have \( \varphi(\nu_{p,q}(-1)) = \nu'_{p,q}(-1) \), and hence

\[ \varphi_*(\nu_{p,q}(-1)) = [\nu'_{p,q}(-1)]. \]

**Proof.** Obvious. \( \square \)
15. **Example: real forms of $E_7$**

Let $G$ be the simply connected compact group $G$ of type $E_7$. In the figure below, on the left we give the extended Dynkin diagram $\tilde{D}$ of $G$ with the numbering of vertices of Onishchik and Vinberg [20, Table 7, Type I]. We paint the lowest root in black. On the right we give $\tilde{D}$ with the coefficients $m_i$ from [20, Table 6]; see (9.2).

We have $X = P$, and $P/Q \simeq \mathbb{Z}/2\mathbb{Z}$ is generated by $[\lambda]$ for

$$\lambda = \frac{1}{2}(\alpha_1 + \alpha_3 + \alpha_7),$$

see [20, Table 3]. In the diagram on the right we paint in black the roots appearing (with non-integer half-integer coefficients) in formula (15.1):

![Diagram](image_url)

The Kac labelings of $\tilde{D}$ are:

- $q^{(1)} = 000000 \quad q^{(2)} = 200000$
- $q^{(3)} = 1000000 \quad q^{(4)} = 0000010$
- $q^{(5)} = 0100000 \quad q^{(6)} = 0000000$

Since all automorphisms of $G$ are inner, the real forms of $E_7$ correspond to the elements of $H^1(\mathbb{R}, G^\text{ad})$, and by Corollary 13.6 they correspond to the orbits of $C = P^\vee/Q^\vee \simeq \mathbb{Z}/2\mathbb{Z}$ in the set $K(\tilde{D})$ of Kac labelings of $\tilde{D}$. These orbits are:

$$\{q^{(1)}, q^{(2)}\}, \quad \{q^{(3)}\}, \quad \{q^{(4)}, q^{(5)}\}, \quad \{q^{(6)}\},$$

hence $\#H^1(\mathbb{R}, G^\text{ad}) = 4$. We write $G_q$ for the real form of $G$ defined by the Kac labeling $q$. In particular, the compact form corresponds to $q^{(1)}$.

Concerning $H^1(\mathbb{R}, G_q)$, condition (13.2) defining $K^{(q)} := K(\tilde{D}, P, q)$ reads as

$$\frac{1}{2}(p_1 + p_7) \equiv \frac{1}{2}(q_1 + q_7) \pmod{\mathbb{Z}},$$

(note that $p_3 = 0$ and $q_3 = 0$), which is equivalent to

$$p_1 + p_7 \equiv q_1 + q_7 \pmod{2}.$$

We say that a labeling $p \in K$ is even (resp. odd) if the sum over the black vertices

$$p_1 + p_7$$
is even (resp. odd). Then \(\mathcal{K}^{(q)}\) is the set of the labelings \(p \in \mathcal{K}\) of the same parity as \(q\). By Corollary 13.5, the first Galois cohomology set \(H^1(\mathbb{R}, G_q)\) is in a bijection with the set \(\mathcal{K}^{(q)}\).

For \(G_q = E_7(-133)\) (the compact form) we take \(q = q^{(1)}\). For \(G_q = EVI = E_7(-5)\) we take \(q = q^{(5)}\); see [20, Table 7]. Both labelings \(q^{(1)}\) and \(q^{(5)}\) are even. We see that in both cases the set \(\mathcal{K}^{(q)}\) is the set
\[
\mathcal{K}^{\text{even}} = \{q^{(1)}, q^{(2)}, q^{(4)}, q^{(5)}\}
\]
of all even labelings of \(\tilde{D}\). The set \(H^1(\mathbb{R}, G_q)\) is in a bijection with the set \(\mathcal{K}^{\text{even}}\). In particular, \(\#H^1(\mathbb{R}, G_q) = 4\) in both the compact case and EVI.

For \(G_q = EV = E_7(7)\) (the split form) we take \(q = q^{(6)}\), and for \(G_q = EVII = E_7(25)\) (the Hermitian form) we take \(q = q^{(3)}\); see [20, Table 7]. Both labelings \(q^{(6)}\) and \(q^{(3)}\) are odd. In both cases the set \(\mathcal{K}^{(q)}\) is the set
\[
\mathcal{K}^{\text{odd}} = \{q^{(3)}, q^{(6)}\}
\]
of all odd labelings of \(\tilde{D}\). The set \(H^1(\mathbb{R}, G_q)\) is in a bijection with the set \(\mathcal{K}^{\text{odd}}\). In particular, \(\#H^1(\mathbb{R}, G_q) = 2\) in both cases EV and EVII.

In each case the element \(q \in \mathcal{K}^{(q)}\) corresponds to the neutral element of \(H^1(\mathbb{R}, G_q)\).

Note that \(H^1(\mathbb{R}, E_7(-5))\) was earlier computed by Garibaldi and Semenov [13, Example 5.1], and \(H^1(\mathbb{R}, E_7(7))\) was earlier computed by Conrad [10, Proof of Lemma 4.9]. The cardinalities of \(H^1(\mathbb{R}, G)\) for all simply connected absolutely simple groups \(G\), in particular, for all four real forms of the simply connected group of type \(E_7\), were computed by Adams and Taibi [1] and also by Borovoi and Evenor [6].

16. Example: half-spin groups

Let \(G\) be the compact group of type \(D_\ell\) with even \(\ell \geq 4\) with the cocharacter lattice
\[
X^\vee = \langle Q^\vee, \omega_{\ell-1}^\vee \rangle.
\]
This compact group is neither simply connected nor adjoint, and it is isomorphic to \(SO_{2\ell}\) only if \(\ell = 4\). It is called a half-spin group.

In the figure below, on the left we give the extended Dynkin diagram \(\tilde{D}\) of \(G\) with the numbering of vertices of Onishchik and Vinberg [20, Table 7, Type I]. We paint the lowest root in black. On the right we give \(\tilde{D}\) with the coefficients \(m_i\) from [20, Table 6]; see (9.2).

We show that the character lattice \(X\) is generated by \(Q\) and the weight

\[
\lambda := (\alpha_1 + \alpha_3 + \alpha_5 + \cdots + \alpha_{\ell-3} + \alpha_\ell)/2.
\]

Indeed, \(\langle \lambda, \omega_{\ell-1}^\vee \rangle = 0\) and \(\langle \lambda, \alpha^\vee \rangle = 0, 1, -1 \in \mathbb{Z}\) for any \(\alpha \in S\). Since \(\omega_{\ell-1}^\vee\) and the coroots \(\alpha^\vee\) generate \(X^\vee\), we see that \(\lambda \in X\). Since \(\lambda \notin Q\) and \([X : Q] = 2\), we conclude that \(X = \langle Q, \lambda \rangle\). In the diagram on the right we paint in black the roots that appear (with non-integer half-integer coefficients) in formula (16.1) for \(\lambda\).
Let $p$ be a Kac labeling of $\tilde{D}$. We say that $p$ is even (resp., odd) if the sum over the black vertices

$$p_1 + p_3 + p_5 + \cdots + p_{\ell-3} + p_{\ell}$$

is even (resp., odd). If $q \in \mathcal{K}(\tilde{D})$, then by (13.2) the set $\mathcal{K}(q) := \mathcal{K}(\tilde{D}, X, q)$ consists of all Kac labelings of the same parity as $q$.

The group $F = X^\vee / Q^\vee = \{0, [\omega_{\ell-1}^\vee]\}$ acts on $\tilde{D}$ and on $\mathcal{K}(\tilde{D})$. The nontrivial element $[\omega_{\ell-1}^\vee] \in F$ acts as the reflection with respect to the vertical axis of symmetry of $\tilde{D}$ (see Subsection 9.5) and clearly preserves the parity of labelings. We say that an $F$-orbit in $\mathcal{K}$ is even (resp., odd), if it consists of even (resp., odd) labelings.

Let $q$ be a Kac labeling of $\tilde{D}$. By Main Theorem 13.3, the cohomology set $H^1(\mathbb{R}, G_q)$ is in a bijection with the set $\mathcal{K}(q) = F$, that is, with the set of $F$-orbits in $\mathcal{K}$ of the same parity as $q$. Thus in order to compute $H^1(\mathbb{R}, G_q)$ for all labelings $q$ of $\tilde{D}$, it suffices to compute the sets of the even and odd $F$-orbits, respectively. We compute also the cardinalities $h_{\text{even}}(D_\ell)$ and $h_{\text{odd}}(D_\ell)$ of these sets.

For representatives of even $F$-orbits we take

$$\begin{align*}
1 & \cdots 1 \\
0 & 0 \cdots 0
\end{align*}, \begin{align*}
1 & 0 \cdots 0 \\
0 & 0 \cdots 1
\end{align*}, \begin{align*}
2 & 0 \cdots 0 \\
0 & 0 \cdots 0
\end{align*}, \begin{align*}
0 & 0 \cdots 0 \\
0 & 0 \cdots 1
\end{align*}, \begin{align*}
0 & 0 \cdots 1 \\
0 & 0 \cdots 0
\end{align*}$$

with $1$ at $2j$ for each integer $j$ with $1 < 2j \leq \ell/2$. Thus

$$h_{\text{even}}(D_\ell) = \lfloor \ell/4 \rfloor + 4.$$

For representatives of odd $F$-orbits we take

$$\begin{align*}
1 & 0 \cdots 0 \\
1 & 0 \cdots 1
\end{align*}, \begin{align*}
1 & 0 \cdots 0 \\
0 & 0 \cdots 1
\end{align*}, \begin{align*}
0 & 0 \cdots 1 \\
0 & 0 \cdots 0
\end{align*}$$

with $1$ at $2j + 1$ for each integer $j$ with $1 < 2j + 1 \leq \ell/2$. Thus

$$h_{\text{odd}}(D_\ell) = \lfloor \ell/4 \rfloor + 1.$$

We conclude that if $q$ is an even labeling, then $\#H^1(\mathbb{R}, G_q) = \lfloor \ell/4 \rfloor + 4$, while if $q$ is an odd labeling, then $\#H^1(\mathbb{R}, G_q) = \lfloor \ell/4 \rfloor + 1$.

Note that if $\ell > 4$, then our compact half-spin group $G$ has no outer automorphisms; hence all its real forms are inner forms, and we have computed the Galois cohomology for all real forms of $G$.

Note also that for the compact half-spin group $G$ we have

$$\#H^1(\mathbb{R}, G) = h_{\text{even}}(D_\ell) = \lfloor \ell/4 \rfloor + 4.$$

For comparison, $\#H^1(\mathbb{R}, \text{SO}_{2\ell}) = \ell + 1$. We have $\lfloor \ell/4 \rfloor + 4 = \ell + 1$ for an even natural number $\ell$ if and only if $\ell = 4$. (In this case, because of triality, our half-spin group $G$ is isomorphic to $\text{SO}_8$.)
17. Example: an almost direct product of $E_7$ and $SL_{m,\mathbb{H}}$.

Let $G^\text{sc} = E_7 \times SL_{m,\mathbb{H}}$, where by $E_7 = E_7(-133)$ we denote the compact simply connected $\mathbb{R}$-group of type $E_7$, and $SL_{m,\mathbb{H}}$ is the quaternionic real form of $SL_{2m}$. Let $\mu_{2m}$ denote the cyclic group generated by a primitive $2m$-th root of unity $\zeta_{2m}$. We identify $\mu_{2m}$ with the center of $SL_{2m}(\mathbb{C})$ consisting of scalar matrices, and embed $\mu_{2m}$ into $Z(G^{\text{sc}})$ by sending $\zeta_{2m}$ to $(-1, \zeta_{2m})$, where $-1$ denotes the nontrivial central element of $E_7$. We wish to compute the Galois cohomology for all inner forms of $G$ (which are outer forms of a compact form of $G$).

By Corollary 13.6, the inner forms of $G$ are classified by Kac labelings $q$ of the diagram $\tilde{D} = \tilde{D}_E \sqcup \tilde{D}_A$, where $\tilde{D}_E$ is as in Section 15 and $\tilde{D}_A$ is given at the figure below. We denote various objects (restricted roots, Kac labelings, ...) related to $\tilde{D}_E$ or $\tilde{D}_A$ by the subscripts (or superscripts) $E$ and $A$, respectively.

On the left-hand side of the figure below we give the twisted affine Dynkin diagram $\tilde{D}_A$ of $SL_{m,\mathbb{H}}$ with the numbering of vertices of [20, Table 7, Type III]. We paint the root $\alpha^E_0$ in black. On the right-hand side we give $\tilde{D}_A$ with the coefficients $m_i$ from the Table in [14, Sect. 3.3.9]; see (10.4).

The group $X_0/Q_0 \simeq \mathbb{Z}/2\mathbb{Z}$ is generated by $[\lambda]$ with

$$\lambda = \frac{1}{2}(\alpha^E_1 + \alpha^E_3 + \alpha^E_7 + \alpha^A_m).$$

In the diagram on the right we paint in black the root $\alpha^A_m$ appearing (with non-integer half-integer coefficient) in formula (17.1):

$$\lambda = \frac{1}{2}(\alpha^E_1 + \alpha^E_3 + \alpha^E_7 + \alpha^A_m).$$

The Kac labelings $q_E^{(1)}, \ldots, q_E^{(6)}$ of $\tilde{D}_E$ are written down in Section 15. The labelings $q_E^{(1)}, q_E^{(2)}, q_E^{(4)}$, and $q_E^{(5)}$ are even, while the labelings $q_E^{(3)}$ and $q_E^{(6)}$ are odd.

We write down the Kac labelings of $\tilde{D}_A$:

$$q_A^{(1)} = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}, \quad q_A^{(2)} = \begin{pmatrix} 0 & 1 & \cdots & 0 \end{pmatrix}, \quad q_A^{(3)} = \begin{pmatrix} 0 & 0 & \cdots & 1 \end{pmatrix}.$$

The labelings $q_A^{(1)}$ and $q_A^{(2)}$ are even (that is, the labels $q_{A,m}^{(1)}$ and $q_{A,m}^{(2)}$ are even), while the labeling $q_A^{(3)}$ is odd.

We have $\#K(\tilde{D}_E) = 6$, $\#K(\tilde{D}_A) = 3$; hence $\#K(\tilde{D}) = 6 \cdot 3 = 18$.

We say that a labeling $q = (q_E, q_A)$ is even (resp. odd) if the sum
$$q_1^E + q_3^E + q_7^E + q_m^A$$

is even (resp. odd). Clearly, $q = (q_E, q_A)$ is even if and only if either both labelings $q_E$ and $q_A$ are even or they both are odd. By (13.2) the set $K(\tilde{D}, X_0, q)$ consists of all Kac labelings of the same parity as $q$.

The group $F_0 = \tilde{X}_0^e / \tilde{Q}_0^e$ is of order 2 with the nontrivial element acting on the diagrams $\tilde{D}_E$ and $\tilde{D}_A$ by the only nontrivial automorphisms of these diagrams, respectively. This action clearly preserves the parity of $q = (q_E, q_A)$. We say that an orbit of $F_0$ in $K(\tilde{D})$ is even (resp. odd) if it consists of even (resp. odd) labelings.

We write down all even $F_0$-orbits in $K(\tilde{D})$:

$$(q_E^{(1)}, q_A^{(1)}) \leftrightarrow (q_E^{(2)}, q_A^{(2)}), \quad (q_E^{(1)}, q_A^{(2)}) \leftrightarrow (q_E^{(2)}, q_A^{(1)}),$$

$$(q_E^{(4)}, q_A^{(1)}) \leftrightarrow (q_E^{(5)}, q_A^{(2)}), \quad (q_E^{(4)}, q_A^{(2)}) \leftrightarrow (q_E^{(5)}, q_A^{(1)}),$$

$$(q_E^{(3)}, q_A^{(3)}).$$

We write down all odd $F_0$-orbits in $K(\tilde{D})$:

$$(q_E^{(1)}, q_A^{(3)}) \leftrightarrow (q_E^{(2)}, q_A^{(3)}), \quad (q_E^{(4)}, q_A^{(3)}) \leftrightarrow (q_E^{(5)}, q_A^{(3)}),$$

$$(q_E^{(3)}, q_A^{(1)}) \leftrightarrow (q_E^{(3)}, q_A^{(2)}), \quad (q_E^{(6)}, q_A^{(1)}) \leftrightarrow (q_E^{(6)}, q_A^{(2)}).$$

We see that there are six even $F_0$-orbits and four odd $F_0$-orbits in $K(\tilde{D})$.

Now let $G_q$ denote the twisted form of our group $G = (E_7 \times SL_{m, \mathbb{H}})/\mu_{2m}$ corresponding to the Kac labeling $q = (q_E, q_A)$. By Main Theorem 13.3, if $q$ is even, then $H^1(\mathbb{R}, G_q)$ is in a bijection with the set of even $F_0$-orbits in $K(\tilde{D})$ and hence $\#H^1(\mathbb{R}, G_q) = 6$. If $q$ is odd, then $H^1(\mathbb{R}, G_q)$ is in a bijection with the set of odd $F_0$-orbits in $K(\tilde{D})$ and hence $\#H^1(\mathbb{R}, G_q) = 4$.

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