Holonomy for Quantum Channels

David Kult\textsuperscript{1*}, Johan Åberg\textsuperscript{2†}, and Erik Sjöqvist\textsuperscript{1‡}
\textsuperscript{1}Department of Quantum Chemistry, Uppsala University, Box 518, S-751 20 Uppsala, Sweden
\textsuperscript{2}Centre for Quantum Computation, Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, United Kingdom

(Dated: February 2, 2008)

Abstract

A quantum holonomy reflects the curvature of some underlying structure of quantum mechanical systems, such as that associated with quantum states. Here, we extend the notion of holonomy to families of quantum channels, i.e., trace-preserving completely positive maps. By the use of the Jamiołkowski isomorphism, we show that the proposed channel holonomy is related to the Uhlmann holonomy. The general theory is illustrated for specific examples. We put forward a physical realization of the channel holonomy in terms of interferometry. This enables us to identify a gauge-invariant physical object that directly relates to the channel holonomy. Parallel transport condition and concomitant gauge structure are delineated in the case of smoothly parametrized families of channels. Finally, we point out that interferometer tests that have been carried out in the past to confirm the $4\pi$ rotation symmetry of the neutron spin, can be viewed as early experimental realizations of the channel holonomy.

PACS numbers: 03.65.Vf, 03.65.Yz, 03.75.Dg

I. INTRODUCTION

Quantum geometric phases and quantum holonomies have, since their initial discovery, proven to be a versatile structure that appears in many different contexts in quantum mechanics. Berry showed in his seminal paper that a state vector initially in an eigenspace of a non-degenerate Hamiltonian acquires a geometric phase factor in addition to the familiar dynamical phase factor after being adiabatically transported along a closed curve in the parameter space of the Hamiltonian. Wilczek and Zee [2] soon thereafter extended Berry’s work by showing that the geometric phase factor generalizes to a unitary state change, called a non-Abelian holonomy (or just holonomy for short), in the case of degenerate Hamiltonians. Another extension of Berry’s work was provided by Aharonov and Anandan [3] who removed the requirement of adiabaticity by showing that a geometric phase factor is defined for any cyclic evolution of a pure quantum state. This result was further generalized by Samuel and Bhandari [4] to include noncyclic evolution as well. A holonomy for curves of density operators was first introduced by Uhlmann [5].

The aforementioned geometric phases and holonomies may be classified in the following way. Firstly, we have holonomies for subspaces, such as eigenspaces of Hamiltonians [1, 2], subspaces selected by projective measurements [4], cyclic subspaces [2], and decoherence free subspaces [2, 5, 6]. Holonomic quantum computation [10, 11] is related to this class of holonomies. Secondly, we have geometric phases and holonomies for quantum states, both pure [3, 4] and mixed [5]. The geometric structures of these two classes are given by the fiber bundles associated with the mappings, “basis of subspace” $\rightarrow$ “subspace” and “purification” $\rightarrow$ “state”, respectively.

In this paper, we focus on the geometry related to a third major concept in quantum theory, namely quantum maps. More precisely, we are interested in the holonomy for sequences of quantum channels, i.e., trace-preserving completely positive maps. This concept, which we shall call “channel holonomy”, is associated with the geometry given by a fiber bundle structure related to the mapping “Kraus representation” $\rightarrow$ “completely positive map”. The aim with this analysis is to delineate this structure and to examine its physical relevance.

The outline of this paper is as follows. In the next section, we introduce the channel holonomy and examine its behavior under gauge transformations. The relation between the Uhlmann holonomy [5] for a sequence of density operators, constructed from the Jamiołkowski isomorphism [12], and the channel holonomy is analyzed in Sec. III. The case of smoothly parametrized families of quantum channels is discussed in Sec. IV. We derive the parallel transport condition and introduce a gauge potential associated with such families. In Sec. V the channel holonomy is calculated for specific types of channel sequences. A physical realization of the channel holonomy based on ancillary constructions in two-beam interferometry is demonstrated in Sec. VI. In particular, we demonstrate that the channel holonomy is related to the “gluing matrix” [13, 14] that arises when two channels are combined in an interferometer. The interferometer setup also provides means to identify a physically meaningful gauge-invariant object associated with the channel holonomy. In Sec. VII we examine the case of smoothly parametrized families of quantum channels using the ancillary construction. It should be noted that the analysis in Secs. VI and VII parallel to a large extent that of Secs. II and IV the main difference being that while the latter
utilizes directly the Kraus operators, the former utilizes ancillary systems. In Sec. VIII we examine the interferometer tests in Refs. 15, 16 of the 4π symmetry of the neutron spin in terms of the channel holonomy. The paper ends with the conclusions.

II. CHANNEL HOLOMONY

Consider a trace-preserving completely positive map (channel for short) $\mathcal{F}$ acting on a $D$-dimensional state space of a quantum system. The action of the channel on a state $\rho$ can be expressed as

$$\mathcal{F}(\rho) = \sum_{k=1}^{K} F_k \rho F_k^\dagger.$$  

(1)

The operators $F_k$ constitute a Kraus representation of $\mathcal{F}$ [17]. We assume that $K$ is the number of linearly independent Kraus operators needed to represent $\mathcal{F}$, i.e., $K$ is the Kraus number $K(\mathcal{F})$ of the channel [13]. From trace preservation it follows that $\sum_{k} F_k^\dagger F_k = 1$, where $I$ is the identity operator. The Kraus representation of a channel is not unique. If $\{F_k\}_k$ is a valid representation of $\mathcal{F}$ then so is $\{\tilde{F}_k\}_k$, where

$$\tilde{F}_k = \sum_{l=1}^{K} F_l U_{lk},$$  

(2)

$U$ being a unitary matrix [18].

Let $\{E_k\}_k$ be a linearly independent Kraus representation of a channel $\mathcal{E}$. Given another channel $\mathcal{F}$ with $K(\mathcal{F}) = K(\mathcal{E}) = K$, we wish to find a linearly independent Kraus representation $\{F_k\}_k$ of $\mathcal{F}$ that in some sense is parallel with $\{E_k\}_k$. A convenient choice would be to find the Kraus representation $\{F_k\}_k$ that minimizes [19]

$$\sum_{k} \|E_k - F_k\|^2 = 2D - 2\text{Re} \text{Tr} \mathcal{T},$$  

(3)

where $\| \cdot \|$ denotes the Hilbert-Schmidt norm and $\mathcal{T}$ is a matrix with elements $T_{kl} = \text{Tr}(F_k^\dagger E_l)$. Under a change of Kraus representation $\{F_k\}_k \rightarrow \{\tilde{F}_k\}_k$, as given by Eq. (2), the matrix $\mathcal{T}$ transforms as $\mathcal{T} \rightarrow \tilde{\mathcal{T}} = U^\dagger \mathcal{T} U$. Hence,

$$\sum_{k} \|E_k - \tilde{F}_k\|^2 = 2D - 2\text{Re} \text{Tr}(U^\dagger \mathcal{T} U).$$  

(4)

Assuming that $\mathcal{T}$ is of rank $K$ (see the appendix for an elaboration on the rank of $\mathcal{T}$) the minimum is obtained when $\tilde{\mathcal{T}} > 0$ which corresponds to the choice $U = \Phi(\mathcal{T}) = \sqrt{\mathcal{T}^{-1}}$. Thus, we say that two Kraus representations $\{E_k\}_k$ and $\{F_k\}_k$ are parallel if their corresponding $\mathcal{T}$ matrix is positive definite.

We are now in a position where we can define a holonomy corresponding to a sequence $\mathcal{E}_1, \ldots, \mathcal{E}_N$ of quantum channels with $K(\mathcal{E}_1) = \ldots = K(\mathcal{E}_N) = K$. This is done by choosing a Kraus representation $\{E^k\}_k$ for each $\mathcal{E}_n$ in the sequence and encoding the Kraus freedom in a family of unitary matrices $U_n$, $n = 1, \ldots, N$. In this way, we may express the parallelity conditions as

$$U_{n+1}^\dagger T_{n+1,n} U_n > 0, \quad n = 1, \ldots, N - 1,$$  

(5)

where the $K \times K$ matrices $T_{n+1,n}$ with elements

$$[T_{n+1,n}]_{kl} = \text{Tr}(E_n^{k+1} E_l^n)$$  

(6)

are all assumed to be of rank $K$. The conditions in Eq. (5) are satisfied by

$$U_{n+1} = \Phi(T_{n+1,n}) U_n.$$  

(7)

We obtain after iteration

$$U_N = \Phi(T_{N,N-1}) \cdots \Phi(T_{2,1}) U_1.$$  

(8)

Define

$$U_{\text{ch}}(\mathcal{E}_1, \ldots, \mathcal{E}_N) = \Phi(T_{1,N}) U_N U_1^\dagger = \Phi(T_{1,N}) \Phi(T_{N,N-1}) \cdots \Phi(T_{2,1})$$  

(9)

to be the channel holonomy for the sequence $\mathcal{E}_1, \ldots, \mathcal{E}_N$.

If we consider the set of linearly independent Kraus representations as a fiber bundle with the set of channels with a fixed Kraus number as base manifold, the change of Kraus representations, as in Eq. (9), can be interpreted as a gauge transformation. As seen from the definition in Eq. (9) the factor $U_N U_1^\dagger$ is multiplied from the left by $\Phi(T_{1,N})$. This construction guarantees that the matrix $U_{\text{ch}}(\mathcal{E}_1, \ldots, \mathcal{E}_N)$ transforms gauge covariantly. To see this we make the gauge transformation

$$E^k_n \rightarrow \sum_{l} E^l l \left[ V_n \right]_{lk}$$  

(10)

of the Kraus representation of each channel in the sequence. We obtain

$$\Phi(T_{n+1,n}) \rightarrow V_{n+1}^\dagger \Phi(T_{n+1,n}) V_n,$$  

(11)

which implies

$$U_{\text{ch}}(\mathcal{E}_1, \ldots, \mathcal{E}_N) \rightarrow V_{n+1}^\dagger U_{\text{ch}}(\mathcal{E}_1, \ldots, \mathcal{E}_N) V_1.$$  

(12)

Hence, the channel holonomy transforms gauge covariantly as required.

Concerning the gauge covariance, let us point out that the channel holonomy is described as a unitary matrix. The gauge covariance of this matrix is necessary since, in some sense, is a matrix representation of a gauge-invariant object, and the gauge covariance reflects the freedom of the choice of “basis” in this matrix representation. It is to be noted [20] that this is the case also for other type of holonomies, such as those of Refs. [2, 21]. In Sec. VII we elucidate what gauge-invariant object the channel holonomy represents, and in what sense the channel holonomy is a matrix representation of this object.
An alternative way to obtain the channel holonomy defined above is to make a gauge transformation yielding
\[ T_{n+1,n} \rightarrow \tilde{T}_{n+1,n} > 0, \] for all \( n = 1, \ldots, N - 1, \) i.e., \( \Phi(\tilde{T}_{n+1,n}) = I, I \) being the \( K \times K \) identity matrix. This choice amounts to parallel transport along the sequence \( \mathcal{E}_1, \ldots, \mathcal{E}_N. \) From Eq. (9) it follows that
\[ U_{\text{ch}}(\mathcal{E}_1, \ldots, \mathcal{E}_N) = \Phi(\tilde{T}_{1,N}), \] (13)
which is the expression for the channel holonomy in the parallel transport gauge.

### III. RELATION TO THE UHLJANN HOLONOMY

For any given channel \( \mathcal{E} \) acting on elements of \( \mathcal{L}(\mathcal{H}_q), \) i.e., the set of linear operators on the \( D \)-dimensional Hilbert space \( \mathcal{H}_q, \) one can find a corresponding density operator \( \rho \in \mathcal{L}(\mathcal{H}_q \otimes \mathcal{H}_q). \) This can be done via the Jamiołkowski isomorphism \[ 12, \] i.e., \( \mathcal{E} \mapsto \rho = \mathcal{E} \otimes I(\langle \psi | \psi \rangle), \) where \( I \) is the identity channel and \( | \psi \rangle = \frac{1}{\sqrt{D}} \sum_{k=1}^{D} | k \rangle \otimes | k \rangle \) with \( \{| k \rangle \}_k \) an orthonormal basis of \( \mathcal{H}_q. \) We show that the holonomy associated with a sequence of channels \( \mathcal{E}_1, \ldots, \mathcal{E}_N \) is related to the Uhlmann holonomy \[ 5 \] for the extended sequence of density operators \( \rho_1, \ldots, \rho_N, \rho_{N+1} = \rho_1 \) [22], where
\[ \rho_n = \mathcal{E}_n \otimes I(\langle \psi | \psi \rangle). \] (14)
To avoid technical complications, we assume that all channels have maximal Kraus number \( K(\mathcal{E}_n) = D^2 \). This guarantees that the corresponding density operators \( \rho_n \) are faithful [5], i.e., they are full rank.

There are several ways to calculate the Uhlmann holonomy associated with a sequence \( \rho_1, \ldots, \rho_{N+1} \) of faithful density operators. For each density operator \( \rho_n \) there corresponds Uhlmann amplitudes \( \tilde{W}_n = \sqrt{\rho_n} V_n, \) where
\[ V_n = \Phi(\tilde{W}_n) = \sqrt{\tilde{W}_n \tilde{W}_n^\dagger} \tilde{W}_n \] is a unitary operator. A sequence of such amplitudes is parallel if \( \tilde{W}_n^\dagger \tilde{W}_{n+1} > 0, \) which allows us to define the Uhlmann holonomy \( U_{\text{Uhl}} = V_{N+1} V_1^\dagger. \) If we instead consider another sequence of amplitudes \( W_n \) that is not parallel transported (but corresponds to the same sequence of density operators), we can make it into a parallel transported sequence \( W_n U_n \) by a choice of unitary operators \( U_n \) such that
\[ U_{n+1}^\dagger W_{n+1}^\dagger W_n U_n > 0, \] (15)
which implies that \( U_{n+1} = \Phi(W_{n+1}^\dagger W_n) U_n \) and we find \( U_{\text{Uhl}} = \Phi(W_{N+1}^\dagger U_{N+1}^\dagger) U_1 \Phi(W_1^\dagger). \) If we furthermore assume that the sequence of amplitudes is cyclic, i.e., \( W_{N+1} = W_1 \) (which implies that the underlying sequence of density operators is cyclic, i.e., \( \rho_{N+1} = \rho_1 \)), we obtain
\[ U_{\text{Uhl}} = \Phi(W_1) U_{N+1}^\dagger U_1^\dagger \Phi(W_1^\dagger). \] (16)
We also note that
\[ U_{N+1} = \Phi(W_1^\dagger W_N) \Phi(W_N^\dagger W_{N-1}) \cdots \Phi(W_1^\dagger W_1) U_1. \] (17)
Let us now return to the cyclic sequence of density operators \( \rho_1, \ldots, \rho_N, \rho_{N+1} = \rho_1 \) as defined by Eq. (14).

We fix an orthonormal basis \( \{| f_k \rangle \}_k \) of \( \mathcal{H}_q \otimes \mathcal{H}_q, \) and define the amplitudes
\[ W_n = \sum_k (E_k^n \otimes I) | \psi \rangle \langle f_k |, \] (18)
where \( \{ E_k^n \}_k \) is a linearly independent Kraus representation of \( \mathcal{E}_n. \) We furthermore find that
\[ W_{n+1}^\dagger W_n = \frac{1}{D} \sum_{kl} | f_k \rangle \langle f_l | \text{Tr}(E_{k+n+1}^l E_{k}^l) \]
\[ = \frac{1}{D} \sum_{kl} | f_k \rangle \langle f_l | [\Phi(T_{n+1,n})]_{kl}. \] (19)
It follows that the two amplitudes \( W_n \) and \( W_{n+1} \) defined in Eq. (18) are parallel if and only if \( T_{n+1,n} > 0. \) Hence, the parallelity condition for channels is, via this construction, closely related to the Uhlmann parallel transport of amplitudes. If we combine Eq. (17) with the fact that \( \Phi(W_{n+1}^\dagger W_n) = \sum_{kl} | f_k \rangle \langle f_l | \Phi(T_{n+1,n})_{kl} \) we can conclude that
\[ U_{N+1} = \sum_{kl} | f_k \rangle \langle f_l | [U_{\text{ch}}]_{kl} U_1, \] (20)
which implies
\[ [U_{\text{ch}}]_{kl} = \langle f_k | \Phi(W_1^\dagger) U_{\text{Uhl}} \Phi(W_1) | f_l \rangle. \] (21)
The right-hand side of the above equation transforms as in Eq. (12) under the gauge transformation in Eq. (10), as it should. We note that it is \( \Phi(W_1^\dagger) \) and \( \Phi(W_1) \) that are “responsible” for the gauge covariance, as \( U_{\text{Uhl}} \) remains invariant. Note also that the construction in Eq. (13) leading to Eq. (21) contains an arbitrary choice of basis \( \{| f_k \rangle \}_k, \) as well as an arbitrary maximally entangled state \( | \psi \rangle \). The channel holonomy \( U_{\text{ch}} \) should not depend on any of these arbitrary choices. (This invariance of \( U_{\text{ch}} \) with respect to the choice of \( \{| f_k \rangle \}_k \) and \( | \psi \rangle \) should not be confused with the covariance of \( U_{\text{ch}} \) under the gauge transformations in Eq. (10).) The Uhlmann holonomy \( U_{\text{Uhl}} \) is invariant under change of \( \{| f_k \rangle \}_k, \) and although \( \Phi(W_1) \) depends on \( \{| f_k \rangle \}_k, \) the combination \( \Phi(W_1^\dagger) | f_k \rangle \) does not, which leaves \( U_{\text{ch}} \) invariant under the choice of \( \{| f_k \rangle \}_k. \) Given a maximally entangled state \( | \psi \rangle, \) all other maximally entangled states can be obtained as \( [1 \otimes U]| \psi \rangle, \) where \( U \) is unitary. Both \( U_{\text{Uhl}} \) and \( \Phi(W_1) \) depend on the choice of \( | \psi \rangle. \) However, the combination \( \Phi(W_1^\dagger) U_{\text{Uhl}} \Phi(W_1) \) is independent of the choice of maximally entangled state \( | \psi \rangle. \) We can thus conclude that \( U_{\text{ch}} \) behaves as required.
IV. SMOOTHLY PARAMETRIZED FAMILIES OF CHANNELS

A. The parallel transport condition

So far we have considered the holonomies associated with sequences of channels. Here we consider the transition to families of channels \( \mathcal{E}_s \) smoothly parameterized by a real variable \( s \in [0,1] \). As before we assume that the Kraus number is constant within each family, i.e., \( K(\mathcal{E}_s) = K \) for all \( s \). Consider a smoothly parametrized family of linearly independent Kraus representations \( \{ E_k(s) \}_{k=1}^K \) of \( \mathcal{E}_s \). For sequences of Kraus representations, the parallel transport condition is \( T_{n+1,n} > 0 \). A generalization of this condition to smooth curves is obtained by requiring that the matrix with elements \( \text{Tr}[E_k(s + \delta s)E_l(s)] \) is positive definite to first order in the limit of small \( \delta s \). We find

\[
\text{Tr}[E_k(s + \delta s)E_l(s)] = Q_{kl} + \delta s R_{kl},
\]

where

\[
Q_{kl} = \text{Tr}[E_k(s)E_l(s)], \quad R_{kl} = \text{Tr}[\dot{E}_k(s)E_l(s)].
\]

(23)

Since \( \{ E_k(s) \} \) is a linearly independent set it follows that \( Q \) is positive definite. Therefore, a necessary condition for \( Q + \delta s R \) to be positive definite is that \( R \) is Hermitian. We shall now see that this is also a sufficient condition for small \( \delta s \). Since \( Q \) is positive definite it follows that

\[
Q + \delta s R = \sqrt{Q}(I + \delta s \sqrt{Q}^{-1} R \sqrt{Q}^{-1}) \sqrt{Q}.
\]

(24)

The assumption \( R^\dagger = R \) implies that \( \sqrt{Q}^{-1} R \sqrt{Q}^{-1} \) is a finite Hermitian matrix and consequently its eigenvalues are real and \( Q + \delta s R \) is positive definite for sufficiently small \( \delta s \). We can conclude that a smooth family of Kraus representations \( \{ E_k(s) \}_{k=1}^K \) of channels with constant Kraus number \( K \) is parallel transported if and only if

\[
\text{Tr}[\dot{E}_k(s)E_l(s)] = \text{Tr}[E_k(s)\dot{E}_l(s)], \quad k,l = 1,\ldots,K.
\]

(25)

This can be compared with the Uhlmann parallel transport condition for smoothly parameterized families of amplitudes \( W(s) \), which is \( W^\dagger(s)W(s) = W^\dagger W(s) \) [3].

B. The gauge potential

We now demonstrate how to introduce a gauge potential along the path \( C : [0,1] \ni s \mapsto \mathcal{E}_s \) of smoothly parametrized channels. Suppose we choose a family of Kraus representations \( \{ E_k(s) \}_{k=1}^K \) over \( C \) that is not parallel transported. We can make a gauge transformation of the form

\[
E_k(s) \rightarrow \bar{E}_k(s) = \sum_{l=1}^K E_l(s)U_{lk}(s)
\]

(26)

such that \( \{ \bar{E}_k(s) \}_{k=1}^K \) is parallel transported. By inserting Eq. (26) into Eq. (24) we obtain

\[
R - R^\dagger = UU^\dagger Q + QUU^\dagger,
\]

(27)

where \( Q \) and \( R \) are as in Eq. (23). Let \( A(s) \) be an anti-Hermitian matrix generating \( U(s) \) via \( \dot{U} = AU \), and substitute into Eq. (27) yielding (cf. Ref. [23])

\[
R - R^\dagger = QA + AQ.
\]

(28)

If we assume \( Q(s) > 0 \) for all \( s \), then, according to Theorem VII.2.3 in Ref. [24], we find that Eq. (28) has a unique solution \( A(s) \) for each \( s \), namely

\[
A(s) = \int_0^\infty e^{-rQ(s)}[R(s) - R^\dagger(s)]e^{-rQ(s)}dr.
\]

(29)

One can see from the right-hand side of this equation that \( A(s) \) is indeed an anti-Hermitian matrix for all \( s \in [0,1] \).

To see that \( A \) transforms as a proper gauge potential [25] we consider an arbitrary gauge transformation \( E_k(s) \rightarrow \sum_l E_l(s)V_{lk}(s) \), where \( V(s) \) is a smooth family of unitary operators. We find that \( Q \rightarrow V^\dagger QV \) and \( R \rightarrow V^\dagger RV \). From these transformation properties we obtain

\[
A \rightarrow V^\dagger AV + V^\dagger V,
\]

(30)

i.e., \( A \) indeed transforms as a gauge potential.

C. An example

The gauge potential \( A(s) \) along \( C \) can be found by solving Eq. (28). Although this is difficult in the general case, it can be done for Kraus number \( K = 2 \). Let \( \sigma = (\sigma_x, \sigma_y, \sigma_z) \) be the standard 2 \( \times \) 2 Pauli matrices. Since \( Q > 0 \) and \( TrQ = TrI \), we may write

\[
Q = I + x \cdot \sigma,
\]

(31)

where \( |x| \neq 1 \) (the eigenvalues of \( Q > 0 \) are \( 1 \pm |x| \)). Furthermore, from the definition of the matrices \( Q \) and \( R \), it follows that \( TrR + TrR^\dagger = TrQ = 0 \), which implies \( ReTrR = 0 \). Thus, we may put

\[
R = i\alpha I + (y + ix) \cdot \sigma,
\]

(32)

where \( 2y = x \). Finally, \( A \) is anti-Hermitian, which suggests that we can write

\[
A = iu_0 I + iu \cdot \sigma.
\]

(33)

We assume that \( x, z_0, y, z, u_0, u \) are all smooth real-valued functions of \( s \). By inserting Eqs. (31), (32), and (28) into Eq. (28), we obtain

\[
A = i \left( \frac{z_0 - x \cdot z}{1 - |x|^2} \right) I + i \left( \frac{z - z_0 x + x \times (x \times z)}{1 - |x|^2} \right) \cdot \sigma.
\]

(34)
along $C$, which is well-defined since $|x| \neq 1$. Note, in particular, that $A$ vanishes when $z_0$ and $z$ vanish, which from Eq. (32) can be seen to correspond to the parallel transport condition $\mathbf{R}^i = \mathbf{R}$. Thus, the form of the $K = 2$ gauge potential is consistent with the notion of parallel transport developed in Sec. [IV.A]. Note also that a result analogous to Eq. (31) for the Uhlmann holonomy in the case of smooth families of faithful qubit density operators has been obtained in Ref. [26].

D. Writing the holonomy in terms of the gauge potential

It remains to demonstrate that the holonomy of the smooth path $C : [0,1] \ni s \mapsto \mathbf{E}_s$ can be expressed in terms of $A$. Consider the polar decomposition $T_{s + \delta s,s} = [T_{s + \delta s,s}] \Phi(T_{s + \delta s,s}).$ To the first order in $\delta s$ we may write $T_{s + \delta s,s} = Q + \delta s H$ and $\Phi(T_{s + \delta s,s}) = I + \delta s J,$ where $H^\dagger = H$ and $J^\dagger = -J$. Thus, $T_{s + \delta s,s} = Q + \delta s(QJ + H) + O(\delta s^2)$ and we obtain

$$T_{s + \delta s,s} - T_{s + \delta s,s}^\dagger = \delta s(QJ + JQ) + O(\delta s^2).$$ (35)

Furthermore, from Eq. (28) and the expression $T_{s + \delta s,s} = Q + \delta s R + O(\delta s^2)$, we find

$$T_{s + \delta s,s} - T_{s + \delta s,s}^\dagger = \delta s(QA + QA) + O(\delta s^2).$$ (36)

Since $Q > 0$ it follows that $J = A$ and the channel holonomy can be written as

$$U_{ch}(C) = \Phi(T_{0,1}) \mathbf{P}_e \mathbf{I}_0(A \delta s),$$ (37)

where $\mathbf{P}$ denotes path ordering and $A$ is a solution of Eq. (25).

V. EXAMPLES

In this section, we consider two examples where the channel holonomy can be explicitly calculated. The first example concerns unitary channels, i.e., operations on closed quantum systems. It turns out that the channel holonomy for such channels is intimately connected to certain cases of the mixed state phase proposed in Ref. [27]. Our second example concerns channels for which there exist Kraus representations that are built up by subspace holonomies associated with smooth paths that, e.g., can be approximately generated by sequential projective measurements [21]. We call these “holonomic channels”. As we demonstrate below, the eigenvalues of the channel holonomy for smooth families of holonomic channels are directly related to the trace of the subspace holonomies, which reduces to Wilson loops in the sense of Ref. [28] for closed paths.

A. Unitary channels

Here, we provide a detailed analysis of the holonomy for sequences of unitary channels, each of which acts on a $D$-dimensional state space. Unitary channels are in a sense the simplest type of channels, since their Kraus representations consist of a single unitary operator ($K = 1$). Consequently, the Kraus freedom is encoded in a phase factor, and it follows that the holonomy in Eq. (9) is also a phase factor, which we denote $\gamma_{ch}$. For a sequence $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_N$ of unitary channels represented by the unitary operators $U_1, U_2, \ldots, U_N$, the holonomy $\gamma_{ch}$ can be written

$$\gamma_{ch} (\mathbf{E}_1, \ldots, \mathbf{E}_N) = \Phi(\text{Tr}(U_1^\dagger U_N)) \Phi(\text{Tr}(U_1^\dagger U_NU_{N-1})) \ldots \times \Phi(\text{Tr}(U_1^\dagger U_1)),$$ (38)

where $\Phi(z) = z/|z|$ for any nonzero complex number $z$. Note that $\gamma_{ch} (\mathbf{E}_1^\dagger, \ldots, \mathbf{E}_N^\dagger)$ is gauge invariant in the sense that it is unchanged under the gauge transformation $U_n \rightarrow e^{i\alpha_n} U_n$ for arbitrary real numbers $\alpha_n$, $n = 1, \ldots, N$. In particular, a gauge transformation $U_n \rightarrow \tilde{U}_n$ such that

$$\Phi(\text{Tr}(\tilde{U}_{n+1}^\dagger \tilde{U}_n)) > 0$$ (39)

yields

$$\gamma_{ch} (\mathbf{E}_1^\dagger, \ldots, \mathbf{E}_N^\dagger) = \Phi(\text{Tr}(\tilde{U}_N^\dagger \tilde{U}_N)),$$ (40)

which is the channel holonomy in the parallel gauge (cf. Eq. (38)).

Consider now a smoothly parametrized family of unitary channels $\mathbf{E}_s^\dagger (\rho) = U(s) \rho U_1^\dagger (s)$, $s \in [0,1]$, and consider the equation of motion $i \dot{U}(s) = H(s) U(s)$, $H(s)$ being the “Hamiltonian” of the system ($\hbar = 1$). This family defines the path $C : [0,1] \ni s \mapsto \mathbf{E}_s^\dagger$. To the first order in $\delta s$ we have $\text{Tr}[U^\dagger(s + \delta s) U(s)] = D + i \delta s \text{Tr}H(s)$, and $\Phi(\text{Tr}[U^\dagger(s + \delta s) U(s)]) = e^{\mathbf{I}_0(i \delta s \text{Tr}H(s))}$. In the $\delta s \rightarrow 0$ limit, Eq. (38) thus becomes

$$\gamma_{ch} (C) = \Phi(\text{Tr}[U^\dagger(0) U(1)]) \times \exp \left[ \frac{i}{\hbar} \int_0^1 ds \text{Tr}H(s) \right],$$ (41)

which resembles the Aharonov-Anandan geometric phase $\mathbf{B}$ with the “dynamical phase” $-\frac{i}{\hbar} \int_0^1 ds \text{Tr}H(s)$ removed from the “total phase” $\arg \Phi(\text{Tr}[U^\dagger(0) U(1)])$. Note also that this dynamical phase coincides with the one in Eq. (14) of Ref. [27] for the maximally mixed internal state $\mathbf{I}_0$ in an interferometer.

The channel holonomy related to the family of unitary channels can alternatively be calculated using the parallel transport gauge, as developed in Sec. [IV]. The parallel transport condition in Eq. (25) reduces to the requirement that $\text{Tr}[\tilde{U}(s) \tilde{U}(s)]$ should be a real number for all $s$. However, from the unitarity of $\tilde{U}(s)$ it follows that
this number can only be purely imaginary. We thus find the parallel transport condition

\[ \text{Tr}[\hat{U}(s)\hat{U}(s)] = 0. \] (42)

This condition is satisfied by the solution of the equation of motion \( i\hat{U}(s) = \{H(s) - 1/2\text{Tr}[H(s)]\}\hat{U}(s) \). The unitaries \( U(s) \) and \( \hat{U}(s) \) are related by the gauge transformation

\[ U(s) \rightarrow \hat{U}(s) = U(s)\exp\left[\frac{i}{D}\int_0^s ds'\text{Tr}H(s')\right]. \] (43)

The holonomy in the parallel transport gauge takes the form

\[ \gamma_{\text{ch}}(C) = \Phi(\text{Tr}[\hat{U}(0)\hat{U}(1)]), \] (44)

which can be seen by inserting Eq. (43) into Eq. (41).

Let us finally consider unitary channels for a single qubit (i.e., \( D = 2 \)). We claim that one-qubit holonomies only can take the values ±1. To see this, we first note that \( U(2) = \text{SU}(2) \times U(1) \). The U(1) parts of the sequence cannot contribute to the holonomy due to their cyclic appearance on the right-hand side of Eq. (38). The claim then follows from \( \Phi(\text{Tr}[\text{SU}(2)\text{SU}(2)]) = \Phi(\text{Tr}[\text{SU}(2)]) = ±1 \), since \( \text{Tr}[\text{SU}(2)] \) is a real number [24].

\section{B. Holonomic channels}

Consider a smoothly parametrized decomposition \( \mathcal{H}_q = \bigoplus_{k=1}^K \mathcal{H}_k(s') \) of a D-dimensional Hilbert space \( \mathcal{H}_q \), for \( s' \in [0, s] \). Assume \( \dim \mathcal{H}_k(s') = D_k \) is constant for all \( k \) on the interval \( [0, s] \) and let \( P_k(s') \) be the projection operator onto \( \mathcal{H}_k(s') \). The quantities

\[ \Gamma_k(s) = \lim_{\delta s \rightarrow 0} P_k(s)P_k(s - \delta s) \ldots P_k(0) \] (45)

can be expressed as [21]

\[ \Gamma_k(s) = \sum_{ij} [\mathcal{P} e_{s'}^{k} A_k(s')ds ]_{ij} [a_i^{(k)}(s)][a_j^{(k)}(0)], \] (46)

where \( \mathcal{P} \) denotes path ordering, the anti-Hermitian matrix \( A_k(s') \) has elements \( [A_k(s')]_{ij} = \langle a_i^{(k)}(s')|a_j^{(k)}(s') \rangle \), and \( \{a_i^{(k)}(s')\} \) is an orthonormal basis of \( \mathcal{H}_k(s') \). A holonomic channel is defined as

\[ \mathcal{E}^{\text{hol}}(\rho) = \sum_k \Gamma_k(s)\rho \Gamma_k(s). \] (47)

The condition for trace preservation \( \sum_k \Gamma_k(s)\Gamma_k(s) = 1 \) can be shown to be satisfied by using that \( \mathcal{P} e_{s'}^{k} A_k(s')ds' \) in Eq. (46) is a unitary matrix.

Let us now examine the holonomy corresponding to a curve \( C : [0, 1] \ni s \rightarrow E^{\text{hol}}_s \) of holonomic channels. The parallel transport condition in Eq. (25) is satisfied if the matrix with elements \( \text{Tr}[\hat{I}_s^{\text{hol}}(s)\Gamma_i(s)] \) is Hermitian. By a direct use of Eq. (49) we find that \( \text{Tr}[\hat{I}_s^{\text{hol}}(s)\Gamma_i(s)] = 0 \) for all \( k, l \) and \( s \). We thus conclude that \( \{\Gamma_k(s)\}_{k} \) is parallel transported. Consequently, the holonomy is given by

\[ U_{\text{ch}}(C) = \Phi(T_{0,1}). \] (48)

Here, \( T_{0,1} \) is a matrix with elements

\[ [T_{0,1}]_{kl} = \text{Tr}[P_k(0)\Gamma_l(1)] = \delta_{kl} \sum_{ij} [\mathcal{P} e_{s'}^{k} A_k(s')ds ]_{ij} |a_i^{(k)}(0)\rangle \langle a_i^{(k)}(1)| \]

where \( U_g(C_k) \) is the holonomy of the path \( C_k \) in the Grassmann manifold \( G(D; D_k) \), i.e., the space of \( D_k \)-dimensional subspaces of an \( D \)-dimensional Hilbert space. Thus, the holonomy takes the form

\[ U_{\text{ch}}(C) = \text{diag}\{\Phi(\text{Tr}[U_g(C_1)]), \Phi(\text{Tr}[U_g(C_2)]), \ldots, \Phi(\text{Tr}[U_g(C_k)]\}. \] (50)

Note that if \( C_k \) is a closed path, \( T_{0,1} \) equals the Wilson loop \( \text{Tr}[\mathcal{P} e_{s'}^{k} A_k(s')ds ] \) [23].

A possible way to implement the holonomic channels is to approximate them by sequences of projective measurements. We discretize the interval \( [0, s] \) with step size \( \delta s \). We first perform the measurement \( \{P_k(0)\}_{k_0} \) followed by \( \{P_k(\delta s)\}_{k_1} \), up to \( \{P_k(N\delta s)\}_{k_N} \) where \( N = N/\delta s \). Discarding the outcomes of these measurements the resulting operation on the input density operator \( \rho \) reads

\[ \rho \rightarrow \sum_{k_0, \ldots, k_N} P_{k_N}(N\delta s) \ldots P_{k_0}(0) \rho P_{k_0}(0) \ldots P_{k_N}(N\delta s) \]

\[ + \mathcal{R}(\rho) = M(\rho), \] (51)

which is a channel. Here,

\[ \mathcal{R}(\rho) = \sum_{k_0, \ldots, k_N \in \mathcal{K}} P_{k_N}(N\delta s) \ldots P_{k_0}(0) \rho P_{k_0}(0) \ldots P_{k_N}(N\delta s) \]

\times P_{k_0}(0) \ldots P_{k_N}(N\delta s), \] (52)

where \( \mathcal{K} \) is the set of values of \( k_0, \ldots, k_N \) excluding those where \( k_0 = \ldots = k_N. \) Note that both \( \sum_k P_k(N\delta s) \ldots P_k(0) \rho P_k(0) \ldots P_k(N\delta s) \) and \( \mathcal{R}(\rho) \) correspond to completely positive maps. In the \( \delta s \rightarrow 0 \) limit we have

\[ \lim_{\delta s \rightarrow 0} M(\rho) = E^{\text{hol}}(\rho) = \lim_{\delta s \rightarrow 0} \mathcal{R}(\rho). \] (53)

Now, \( M \) and \( E^{\text{hol}} \) are trace preserving from which it follows that \( \mathcal{R} \) must vanish in the \( \delta s \rightarrow 0 \) limit. Thus, we have

\[ \lim_{\delta s \rightarrow 0} M(\rho) = E^{\text{hol}}(\rho), \] (54)

which concludes our demonstration that the holonomic channels can be approximated by sequences of projective measurements.
VI. PHYSICAL REALIZATION

Given a physical system and a sequence of operations \( \mathcal{E}_1, \ldots, \mathcal{E}_N \) acting on this system, one might ask: what is the physical significance of the channel holonomy, and how should it be measured? Strictly speaking, these two questions have no well defined answer, given how we have constructed the channel holonomy. We have defined it as a change in Kraus representation, but the Kraus representation as such has no direct physical significance. Hence, there is no immediate way to attach measurable quantities to the channel holonomy. In other words, up to this point the channel holonomy has been a mathematical construction, rather than corresponding directly to a physical object or operation. However, we shall here obtain such a correspondence within the context of interferometry. Another related question is, what is the object that the channel holonomy \( U_{\text{ch}} \) represents? As \( U_{\text{ch}} \) is a gauge-covariant unitary matrix, it seems intuitively reasonable that it should be a matrix representation of a gauge-invariant object. Our interferometric construction will provide precisely such an object.

A. Yet another parallel transport

Every channel \( \mathcal{E} \) on a Hilbert space \( \mathcal{H}_q \) with dimension \( D \) can be obtained via a joint unitary evolution on the system and an ancillary system with Hilbert space \( \mathcal{H}_a \), according to

\[
\mathcal{E}(\rho) = \text{Tr}_a[\mathcal{U}(\rho \otimes |a\rangle\langle a|)\mathcal{U}^\dagger],
\]

where \( \mathcal{U} \) is a unitary operator on \( \mathcal{H}_q \otimes \mathcal{H}_a \) and \( |a\rangle \in \mathcal{H}_a \) is normalized. If \( \mathcal{U} \) is an arbitrary unitary operator on \( \mathcal{H}_a \), then both \( \mathcal{U} \) and \( (\mathcal{U} \otimes \mathcal{U})\mathcal{U} \) give rise to the same channel \( \mathcal{E} \). If we regard the set of unitary operators \( \mathcal{U} \) representing \( \mathcal{E} \) as a fiber, the transformation \( \mathcal{U} \rightarrow (\mathcal{U} \otimes \mathcal{U})\mathcal{U} \) can be seen as a gauge transformation resulting in a motion along the fiber.

Consider now the sequence of channels \( \mathcal{E}_1, \ldots, \mathcal{E}_N \), all with Kraus number \( K \). We assume an ancillary space \( \mathcal{H}_a \) of dimension \( K \), and consider a sequence of unitary operators \( \mathcal{U}_1, \ldots, \mathcal{U}_N \) on \( \mathcal{H}_q \otimes \mathcal{H}_a \), where \( \mathcal{U}_n \) represents \( \mathcal{E}_n \) via Eq. (55). We regard the sequence of unitaries as parallel transported if

\[
\text{Tr}_q[\mathcal{U}_n(\mathcal{U}_1 \otimes |a\rangle\langle a|)\mathcal{U}^\dagger_{n+1}] > 0.
\]

Given an arbitrary sequence of unitaries \( \mathcal{U}_1, \ldots, \mathcal{U}_N \) we can make it into a parallel transported sequence \( \mathcal{U}_n = (\mathcal{U} \otimes \mathcal{U}_n)\mathcal{U}_n \) by choosing unitary operators \( \mathcal{U}_1, \ldots, \mathcal{U}_N \) such that

\[
\text{Tr}_q[\mathcal{U}_n(\mathcal{U}_1 \otimes |a\rangle\langle a|)\mathcal{U}^\dagger_{n+1}] = U_n \text{Tr}_q[\mathcal{U}_n(\mathcal{U}_1 \otimes |a\rangle\langle a|)\mathcal{U}^\dagger_{n+1}] U_n^\dagger > 0.
\]

This requires that \( \text{Tr}_q[\mathcal{U}_n(\mathcal{U}_1 \otimes |a\rangle\langle a|)\mathcal{U}^\dagger_{n+1}] \) is of rank \( K \). It then follows that

\[
U_N = U_1 \Phi(\text{Tr}_q[\mathcal{U}_1(\mathcal{U}_1 \otimes |a\rangle\langle a|)\mathcal{U}^\dagger_1]) \cdots \Phi(\text{Tr}_q[\mathcal{U}_N(\mathcal{U}_1 \otimes |a\rangle\langle a|)\mathcal{U}^\dagger_N])
\]

and \( \mathcal{U}_N = (\mathcal{U} \otimes U_N)\mathcal{U}_N \).

As may be seen, the above construction strongly resembles the channel holonomy in Sec. III, as well as the Uhlmann holonomy in Sec. III. We shall see that this is not a mere coincidence, but that the above construction enables us to obtain the channel holonomy within the context of interferometry. With this purpose in mind we review some of the concepts and tools that are useful for the analysis of quantum operations in single-particle interferometry. A more thorough account of these theoretical tools can be found in Refs. 13, 14. See also Refs. 31, 52, 33 for related material.

B. Operations in interferometry

Suppose that we have a single particle with some internal degrees of freedom (e.g., spin or polarization) with Hilbert space \( \mathcal{H}_q \), and that the particle can propagate along two separated paths (e.g., the two paths in a Mach-Zehnder interferometer). These two paths correspond to two orthonormal vectors \( |0\rangle \) and \( |1\rangle \) spanning the “spatial” Hilbert space \( \mathcal{H}_p = \mathbb{S}p\{ |0\rangle, |1\rangle \} \). Thus, the total Hilbert space of the particle is \( \mathcal{H}_p \otimes \mathcal{H}_q \).

Now, suppose that we have two operations \( \Lambda_0 \) and \( \Lambda_1 \) acting on the internal degrees of freedom of the particle. Let \( \Lambda_0 \) operate on the particle if it passes path 0, and let \( \Lambda_1 \) operate on the particle if it passes path 1. The question is, what channels \( \Lambda \) acting on elements of \( \mathcal{L}(\mathcal{H}_p \otimes \mathcal{H}_q) \) would be compatible with the channels \( \Lambda_0 \) and \( \Lambda_1 \) acting in each path? (To be more precise, we require \( \Lambda \) to be a channel such that \( \Lambda(|0\rangle\langle 0|) = |0\rangle\langle 0| \otimes \Lambda_0(|0\rangle\langle 0|) \) and \( \Lambda(|1\rangle\langle 1|) = |1\rangle\langle 1| \otimes \Lambda_1(|1\rangle\langle 1|) \), for all density operators \( \rho \) on \( \mathcal{H}_q \).) The answer is as follows. Let \( \{V_n\}_n \) and \( \{W_m\}_m \) be linearly independent Kraus representations of the channels \( \Lambda_0 \) and \( \Lambda_1 \), respectively. Then

\[
\Lambda(\sigma) = |0\rangle\langle 0| \otimes \Lambda_0(|0\rangle\langle 0|) + |1\rangle\langle 1| \otimes \Lambda_1(|1\rangle\langle 1|)
\]

\[
+ |0\rangle\langle 0| \sum_{nm} C_{n,m} V_n(0\rangle\langle 0|) W_m^\dagger
\]

\[
+ |1\rangle\langle 1| \sum_{nm} C^*_{n,m} W_m(1\rangle\langle 1|) V_n^\dagger
\]

where the matrix \( C \) satisfies \( CC^\dagger \leq I \) and \( \sigma \in \mathcal{L}(\mathcal{H}_p \otimes \mathcal{H}_q) \). We refer to \( \Lambda \) as a “gluing” of the two channels \( \Lambda_0 \) and \( \Lambda_1 \), and to the matrix \( C \) as the “gluing matrix”.

All gluings can be obtained using a shared ancillary system between the two paths 13, 14. Let \( \mathcal{V}^{(0)} \) and \( \mathcal{V}^{(1)} \) be unitary operators representing the channels \( \Lambda_0 \) and \( \Lambda_1 \) via Eq. (55), where we assume the same ancilla. On the combined system of the two paths, the system, and
the ancilla, we can construct the unitary operator

$$U_{\text{tot}} = |0\rangle\langle 0| \otimes \mathcal{V}^{(0)} + |1\rangle\langle 1| \otimes \mathcal{V}^{(1)}.$$  \hspace{1cm} (60)

It turns out that

$$\Lambda(\sigma) = \text{Tr}_a(U_{\text{tot}} \sigma \otimes |a\rangle\langle a| U_{\text{tot}}^\dagger)$$ \hspace{1cm} (61)

is a gluing of the channels $\Lambda_0$ and $\Lambda_1$, and moreover, that every gluing can be obtained by the appropriate choices of $\mathcal{V}^{(0)}$ and $\mathcal{V}^{(1)}$. Hence, although various choices of $\mathcal{V}^{(0)}$ and $\mathcal{V}^{(1)}$ represent the same channels $\Lambda_0$ and $\Lambda_1$, respectively, these choices may nevertheless result in different gluings of $\Lambda_0$ and $\Lambda_1$.

C. Physical realization of the channel holonomy

1. Interferometric parallel transport procedure

We now consider the Mach-Zehnder setup. Let the particle start in path 0 and internal state $\rho \otimes |a\rangle\langle a|$, after which we apply a 50-50 beam splitter on the spatial degrees of freedom, acting as a Hadamard gate $\mathbb{H}$ on the spatial degrees of freedom regarded as a qubit. Thereafter, we apply $U_{\text{tot}}$ in Eq. (60) on the total system followed by a variable unitary operator $V^{(0)}$ on the ancillary Hilbert space in path 0 and a variable unitary operator $V^{(1)}$ in path 1, i.e., the unitary operator

$$F = |0\rangle\langle 0| \otimes \hat{1} \otimes \mathcal{V}^{(0)} + |1\rangle\langle 1| \otimes \hat{1} \otimes \mathcal{V}^{(1)}.$$ \hspace{1cm} (62)

Finally, we apply a second beam splitter and calculate the expectation value of the projector $|0\rangle\langle 0| \otimes \hat{1} \otimes 1$, i.e., the probability $p$ to find the particle in path 0, yielding

$$p = \text{Tr}[\rho \otimes \mathbb{H}F U_{\text{tot}} \times \mathbb{H}(|0\rangle\langle 0| \otimes \rho \otimes |a\rangle\langle a|) \mathbb{H}U_{\text{tot}}^\dagger F^\dagger \mathbb{H}^\dagger] = \frac{1}{2} + \frac{1}{2} \text{Re} \text{Tr}(V^{(0)} \mathcal{V}^{(0)}(\rho \otimes \langle a|\langle a|)) \mathcal{V}^{(1)}(\mathcal{V}^{(1)}))].$$ \hspace{1cm} (63)

If we regard the unitary operator $V^{(0)}$ as fixed, then we find that the maximum probability $p$ is obtained when $V^{(1)} = V^{(0)} \Phi(\mathcal{V}^{(0)}(\rho \otimes \langle a|\langle a|)) \mathcal{V}^{(0)}(\mathcal{V}^{(1)}))$. In the special case where $\rho = \frac{1}{2} \hat{1}$, i.e., the initial internal state is maximally mixed, we thus find that

$$V^{(1)} = V^{(0)} \Phi(\mathcal{V}^{(0)}(\hat{1} \otimes \langle a|\langle a|)) \mathcal{V}^{(1)}))$$ \hspace{1cm} (64)

maximizes the detection probability $p$.

Comparing with Eq. (57), we find that the parallel transport procedure can be implemented using this interferometric approach. We have a sequence of unitaries $U_1, \ldots, U_N$. We let $V^{(0)} = U_1$ and $V^{(1)} = U_2$, and choose the initial unitary operator $V^{(0)} = U_1$ arbitrarily. Next, we let the input internal state to the interferometer be the maximally mixed state $\rho = \frac{1}{2} \hat{1}$ and vary $V^{(1)}$ until maximal detection probability is obtained. The maximum is achieved when $V^{(1)} = U_2$. In the next step we repeat the process but let $V^{(0)} = U_2$ and $V^{(1)} = U_3$ and we let $V^{(0)} = U_2$, i.e., the unitary operator we obtained in the first step. Then we vary $V^{(1)}$ until we obtain maximal detection probability. We repeat this procedure up to $U_N$.

One may note that the parallel transport procedure just described is purely operational, in the sense that it is achieved as a result of optimizing detection probabilities. This is analogous to the approach to the Uhlenbeck holonomy as developed in Ref. 53.

2. Realizing the channel holonomy $U_{ch}$ as a gluing between the end point channels

After the $N$th step of the parallel transport procedure, we can construct the channel holonomy as a measurable object in the interferometer. We let $V^{(0)} = U_N$, $V^{(0)} = U_N$, $V^{(1)} = U_1$, and $V^{(1)} = U_1$. If we discard the ancillary system the resulting channel on $\mathcal{L}(\mathcal{H}_p \otimes \mathcal{H}_q)$ is

$$\Lambda_{\text{final}}(\sigma) = |0\rangle\langle 0| \otimes \mathcal{E}_N(|0\rangle\langle 0|) + |1\rangle\langle 1| \otimes \mathcal{E}_1(|1\rangle\langle 1|) + |0\rangle\langle 1| \otimes \mathcal{G}(\sigma) + |1\rangle\langle 0| \otimes \mathcal{G}(\sigma)^\dagger,$$ \hspace{1cm} (65)

where

$$\mathcal{G}(\sigma) = \text{Tr}_a[\hat{1} \otimes U_N(\mathcal{U}_N(|0\rangle\langle 1|) \otimes |a\rangle\langle a|) \mathcal{U}_1^\dagger],$$ \hspace{1cm} (66)

We shall now see that the channel $\Lambda_{\text{final}}$ is directly related to the channel holonomy.

Consider a unitary operator $U_n$ representing $\mathcal{E}_n$ via Eq. (57). Choose an arbitrary but fixed orthonormal basis $\{\ket{k}\}_{k=1}^N$ of $\mathcal{H}_n$ for the evaluation of $U_n$. For such a choice, the operators $E_k = \bra{k}\mathcal{U}_n\ket{a}$ form a linearly independent Kraus representation of the channel $\mathcal{E}_n$. We find

$$\mathcal{G}(\sigma) = \sum_{k,k'} \bra{k}\mathcal{U}_n^\dagger \mathcal{U}_n \mathcal{U}_k^\dagger \mathcal{U}_k^\dagger \langle 0|\sigma|1\rangle E_k^\dagger.$$ \hspace{1cm} (67)

One can furthermore show

$$\bra{ak}\mathcal{U}_n^\dagger \mathcal{U}_n \mathcal{U}_k^\dagger \mathcal{U}_k^\dagger |a_k\rangle = [\mathcal{T}_{n+1,n}]_{k'k}.$$ \hspace{1cm} (68)

(Note the reordering of $k$ and $k'$ between the left- and right-hand side.) As can be seen from the above equation, $\mathcal{U}_n^\dagger \mathcal{U}_n \mathcal{U}_k^\dagger \mathcal{U}_k^\dagger$ is of rank $K$ if and only if $\mathcal{T}_{n+1,n}$ is of rank $K$. Equation (68) implies

$$\bra{ak}\mathcal{U}_n \mathcal{U}_k^\dagger |a_k'\rangle = [\Phi(\mathcal{T}_{n+1,n})]_{k'k},$$ \hspace{1cm} (69)

which can be combined with Eq. (68) to give

$$\bra{ak}\mathcal{U}_n^\dagger \mathcal{U}_n |a_k'\rangle = [\Phi(\mathcal{T}_{n,N-1}) \ldots \Phi(\mathcal{T}_{2,1})]_{k'k},$$ \hspace{1cm} (70)
It follows that

$$\Lambda_{\text{final}}(\sigma) = |0\rangle\langle 0| \otimes E_N((0|\sigma|0)) + |1\rangle\langle 1| \otimes E_1((1|\sigma|1))$$

$$+ |0\rangle\langle 1| \otimes \sum_{k,k'=1}^{K} Z_{kk'} E_k^{N} (0|\sigma|1) E_k^{1\dagger}$$

$$+ |1\rangle\langle 0| \otimes \sum_{k,k'=1}^{K} E_{k'}^{1\dagger} (1|\sigma|0) E_{k'}^{N\dagger} Z_{kk'}^{\dagger},$$

(71)

where the two Kraus representations \( \{E_k^{1}\}_k \) and \( \{E_k^{N}\}_k \) are free and independent of each other. We may therefore choose a Kraus representation \( \{\mathcal{E}_k^{N}\}_k \) of the channel \( \mathcal{E}_N \), such that the overlap matrix \( \mathbf{T}(E_1^{1}, E_1^{1}) \) is positive definite, i.e., when the Kraus representation of \( \mathcal{E}_N \) is parallel with the initial Kraus representation, which we know is possible if the overlap matrix \( \mathbf{T}_{1N}[|k]\mathbf{T}(E_1^{1}, E_1^{1}) \) is invertible. We thus substitute \( E_k^{N\dagger} = k_{kk'} \left( \Phi_{k}(T_{1N}) \right)_{kk'} \) into Eq. (71), which yields

$$\Lambda_{\text{final}}(\sigma) = |0\rangle\langle 0| \otimes E_N((0|\sigma|0)) + |1\rangle\langle 1| \otimes E_1((1|\sigma|1))$$

$$+ |0\rangle\langle 1| \otimes \sum_{k,k'=1}^{K} [U_{ch}|kk'] E_{k}^{N} (0|\sigma|1) E_{k}^{1\dagger}$$

$$+ |1\rangle\langle 0| \otimes \sum_{k,k'=1}^{K} E_{k'}^{1\dagger} (1|\sigma|0) E_{k'}^{N\dagger} [U_{ch}|kk'].$$

(72)

Thus, the channel holonomy \( U_{ch} \) is nothing but the gluing matrix describing the gluing of the two end point channels \( \mathcal{E}_1 \) and \( \mathcal{E}_N \), with respect to the choice of parallel Kraus representations of these two channels. It is to be noted that a gauge transformation \( E_k^{1} \rightarrow V_{kk} E_k^{1} \) implies the same transformation \( E_k^{N\dagger} \rightarrow \sum_{k'} V_{kk'} E_{k'}^{N\dagger} \) due to the assumption of parallelity between the end points. One may also note that the gauge covariance of \( U_{ch} \), as described in Eq. (12) is necessary for \( \Lambda_{\text{final}} \) to be gauge invariant. Another way to put this is to say that \( U_{ch} \) in some sense is the matrix representation of the gluing \( \Lambda_{\text{final}} \), and as a matrix representation of a gauge-invariant object we expect \( U_{ch} \) to be gauge covariant. We also point out that in the special case where the two end point channels coincide, i.e., \( \mathcal{E}_N = \mathcal{E}_1 \), then \( \mathbf{E}_k^{N\dagger} = \mathbf{E}_k^{1\dagger} \).

The above construction of the channel holonomy as a matrix representation with respect to two parallel Kraus representations is analogous to the construction of the open-path holonomy in Ref. [21]. This is perhaps best seen in Eq. (12) in Ref. [21], where the open-path holonomy \( U_g \) can be expressed as \( \Gamma = \sum_{ak} U_{g}|ak||\mathcal{E}_k(1)\rangle\langle a_0|0\rangle \). Here \( \{\mathcal{E}_k(1)\}_k \) is a frame which is parallel to the frame \( \{|ak(0)\}_k\} \) in a sense that resembles the parallelity of Kraus representations. (Compare, e.g., Eq. (3) of the present paper with Eq. (8) in Ref. [21].)

We finally note that the gluing matrix is in principle possible to measure using interferometric setups \([14]\).

Hence, the channel holonomy resulting from the parallel transport procedure in Sec. [VII] is a measurable object.

### VII. THE CASE OF SMOOTH PARAMETRIZATION

In Sec. [V] we considered the smooth version of the iterative parallel transport in Sec. [IV]. Here we perform the analogous transition for the ancillary construction in Sec. [VI-A], finding conditions for parallel transport.

Consider a smooth family of unitary operators \( \tilde{U}(s) \) acting on the combined system and ancilla \( \mathcal{H}_q \otimes \mathcal{H}_a \), thus generating a smooth family of channels

$$\mathcal{E}_s(\rho) = \text{Tr}_q[\tilde{U}(s)\rho \otimes |a\rangle\langle a|\tilde{U}(s)^\dagger],$$

(73)

When can we say that the family \( \tilde{U}(s) \) is parallel transported? By recalling Eq. (56) it appears reasonable to require that

$$\text{Tr}_q[\tilde{U}(s)(\tilde{I} \otimes |a\rangle\langle a|)\tilde{U}(s + \delta s)] > 0$$

(74)

to the first order in \( \delta s \) in the limit of small \( \delta s \). We find

$$\text{Tr}_q[\tilde{U}(s)(\tilde{I} \otimes |a\rangle\langle a|)\tilde{U}(s + \delta s)] = Q(s) + \delta s R(s),$$

(75)

where

$$Q(s) = \text{Tr}_q[\tilde{U}(s)(\tilde{I} \otimes |a\rangle\langle a|)\tilde{U}(s)],$$

(76)

$$R(s) = \text{Tr}_q[\tilde{U}(s)(\tilde{I} \otimes |a\rangle\langle a|)\tilde{U}(s)^\dagger].$$

(77)

Note that \( Q(s) \geq 0 \). If we further assume that

$$Q(s) > 0,$$

(78)

then we find that the parallel transport condition is satisfied when \( R(s) \) is Hermitian, i.e.,

$$\text{Tr}_q[\tilde{U}(s)(\tilde{I} \otimes |a\rangle\langle a|)\tilde{U}(s)] = \text{Tr}_q[\tilde{U}(s)(\tilde{I} \otimes |a\rangle\langle a|)\tilde{U}(s)^\dagger].$$

(79)

Suppose we have a smooth family of unitaries \( U(s) \) that is not parallel transported. The question is, under what conditions we can make it parallel transported by multiplying with a unitary on the ancillary space

$$\tilde{U}(s) = [\tilde{I} \otimes U(s)]\tilde{U}(s),$$

(80)

i.e., a gauge transformation that leaves the path of channels \( \mathcal{E}_s \) in Eq. (76) unchanged? Here we elucidate when it is possible to find a time-dependent Hamiltonian \( H(s) \) on \( \mathcal{H}_a \) that generates \( U(s) \) via a Schrödinger equation. If we substitute Eq. (80) into Eq. (79) we can rewrite the result as

$$R(s) - R^\dagger(s) = Q(s)U^\dagger(s)\tilde{U}(s) + U^\dagger(s)\tilde{U}(s)Q(s).$$

(81)
If we now consider an anti-Hermitian operator $\mathcal{A}(s)$ generating $U(s)$ via $\dot{U}(s) = U(s)\mathcal{A}(s)$, and substitute into Eq. (81), we find $R - R^\dagger = AQ + QA$. Similarly as in Sec. V[4], we can use Theorem VII.2.3 in [24] to conclude that there exists a unique anti-Hermitian operator $\mathcal{A}(s)$ solving this equation. Hence, we can conclude that under the assumption $Q(s) > 0$, there exists a unitary family $U(s)$ creating a parallel transported family $\tilde{U}(s)$ via Eq. (80).

It is to be noted that we have described a kind of two-step procedure. First, the system and ancilla evolve jointly according to $U(s)$, which is not parallel transported. In the second step, we modify the state by unitarily evolve the ancilla according to $U(s)$. Hence, we have so far not obtained a joint Hamiltonian on the system and ancilla that generates the parallel transported family $U(s)$. However, the latter can in principle be obtained since the Hamiltonian $\tilde{H}(s) = i\tilde{u}\tilde{A}$ generates $\tilde{U}$ via the Schrödinger equation. Hence, the parallel transported evolution can in principle be tailored through a combined evolution on the system and ancilla. Finally, we note that the parallel transport is not obtained operationally in this smooth case, in the sense of the discrete case in Sec. VII C. Whether such a “interferometric parallel transport procedure” is possible in the smooth case we leave as an open question.

VIII. THE 4$\pi$ EXPERIMENTS

Bernstein [34] and Aharonov and Susskind [35] pointed out the possibility to observe the 4$\pi$ spinor symmetry of a spin-$\frac{1}{2}$ particle. The essence of their argument was that this symmetry may show up in the relative phase, say between two paths in an interferometer, one in which the spinor is rotated and one in which it is kept fixed. Subsequent neutron interferometer experiments [15, 16] were carried out to confirm this prediction.

These experiments used unpolarized neutrons that were sent through a two-beam interferometer, in which one beam was exposed to a time-independent magnetic field $B$ in the $z$-direction, i.e., $B = B\mathbb{1}$. In the weak-field limit, one can show that this corresponds to the family of unitary operations $U(s) = e^{-i\frac{\varphi}{2}s}$, $s \in \{0, \varphi]\$, where $\varphi \propto B$. A 4$\pi$ periodic $\varphi$ oscillation in the intensity in one of the output beams was observed by varying $B$.

Let us consider the standard interpretation of these experiments. The spinor part of unpolarized neutrons is described by the density operator $\rho = |\psi\rangle\langle\psi|$, which can be decomposed into an equal-weight mixture of any pair of orthogonal pure spin-$\frac{1}{2}$ states. Let the orthogonal vectors $|\psi\rangle$ and $|\psi^\perp\rangle$ represent such a choice of states. These vectors evolve into $U(\varphi)|\psi\rangle$ and $U(\varphi)|\psi^\perp\rangle$, respectively, under the action of the magnetic field. It follows that for $\varphi = n2\pi$, we obtain $|\psi\rangle \rightarrow (-1)^n|\psi\rangle$ and $|\psi^\perp\rangle \rightarrow (-1)^{n+1}|\psi^\perp\rangle$. Thus, irrespective how we choose $|\psi\rangle$ and $|\psi^\perp\rangle$, they have the desired 4$\pi$ periodicity needed to explain the experiments.

Here, we put forward another interpretation of these experiments, based on the channel holonomy. We demonstrate that the 4$\pi$ periodic oscillations seen in the experiments can be interpreted as a 4$\pi$ periodicity of the channel holonomy in this case. We also show that the channel holonomy itself can be measured, by a slight modification of the setup in Refs. [15, 16].

Let us first compute the holonomy of the path $C$ associated with the family of unitary channels $E^s_{\varphi}$ represented by the unitary operators $U(s) = e^{-i\frac{\varphi}{2}s}$, $s \in \{0, \varphi\}$. We obtain

$$\text{Tr}[U(s)U(\bar{s})] = -\frac{i}{2}\text{Tr}[\sigma_3] = 0,$$

(82)
i.e., $U(s)$ satisfies the parallel transport condition in Eq. (73). Thus, we may use Eq. (13) to deduce that

$$\gamma_{ch}[C] = \Phi(\text{Tr}[U(\varphi)]),$$

(83)
since $U(0) = \mathbb{1}$. Explicitly, $\Phi(\text{Tr}[U(\varphi)]) = \Phi(\cos \frac{\pi}{2})$, which is 4$\pi$ periodic in $\varphi$.

Next, we analyze the experimental setup in Refs. [15, 16] from the channel holonomy perspective. Let $|0\rangle, |1\rangle$ represent the two beam directions. These vectors constitute a basis for the spatial Hilbert space $H_p$. The experiment uses the standard interferometric sequence with an initial beam splitter, followed by an operation in the two paths, and finally a second beam splitter and a measurement. For the moment we focus only on the operation occurring in between the two beam splitters, but return below to the interferometer as a whole. Let $\sigma$ be some arbitrary total state on $H_p \otimes H_q$. Apply to the 0 beam the parallel transporting unitary family $U(s)$, $s \in \{0, \varphi\}$, that represents the family of unitary channels along $C$. This results in the map

$$A^u(\sigma) = |0\rangle\langle0| \otimes E^s_{\varphi}(|0\rangle\langle0|) + |1\rangle\langle1| \otimes E^s_{\varphi}(|1\rangle\langle1|)$$

$$+ |0\rangle\langle1| \otimes \gamma_{ch}[C]|\mathcal{U}(\varphi)|0\rangle\langle1|$$

$$+ |1\rangle\langle0| \otimes (1|\sigma\rangle\langle0|)\mathcal{U}(\varphi)|\gamma_{ch}[C]^*|,$$

(84)
where we have used Eq. (83), and where $\mathcal{U}(\varphi) = \Phi(\text{Tr}[U(\varphi)]) U(\varphi)$ is parallel to the initial unitary $U(0) = \mathbb{1}$, i.e., $\text{Tr}[\mathcal{U}(\varphi)] \geq 0$. We can identify this with Eq. (73). Hence, $A^u$ is the gauge-invariant gluing between the two unitary end point channels $E^u_0 = I$ and $E^u_{\varphi}$. Furthermore, the phase factor $\gamma_{ch}[C]$ is the corresponding gluing matrix with respect to the parallel Kraus representations $U(0) = \mathbb{1}$ and $\mathcal{U}(\varphi)$ of the end point channels.

In the experiment in Refs. [15, 16] the input state to the interferometer was $|0\rangle\langle0| \otimes \mathbb{1}^\perp$ (unpolarized neutrons). If we assume that the beam splitters can be represented by Hadamard gates, the passage through the first beam splitter results in the state $\sigma = \mathbb{1}^\perp(|0\rangle\langle0| + |1\rangle\langle1|) \otimes \mathbb{1}^\perp$. After the application of the channel $A^u$ on this state the particle passes through a second beam splitter, and the
probability to find the neutron in the 0 beam, say, is
\[ p(\varphi) = \frac{1}{2} \left( 1 + |\text{Tr}[U(\varphi)]| \cos[\arg \gamma_{ch}(C)] \right), \tag{85} \]
where we have used that \( \text{Tr}[U(\varphi)] = |\text{Tr}[U(\varphi)]| \). We see that \( p(\varphi) \) has period \( 4\pi \) in \( \varphi \). Since the visibility factor \( |\text{Tr}[U(\varphi)]| = |\cos \frac{\varphi}{2}| \) is \( 2\pi \) periodic in \( \varphi \), the observed periodicity of the interference oscillations must originate solely from the \( 4\pi \) periodicity of the channel holonomy \( \gamma_{ch}(C) \). This concludes our demonstration that Refs. \[15, 16\] can be viewed as experimental realizations of a channel holonomy and its \( 4\pi \) periodicity.

Finally, we wish to point out that the holonomy for this family of unitary channels could also be measured. It requires the following modification of the setup in Refs. \[15, 16\]: add a U(1) shift \( e^{i\chi} \) to the 1 beam and maximize the output detection probability \( p \) by varying \( \chi \). A direct calculation yields
\[ p(\chi) = \frac{1}{2} \left( 1 + |\text{Tr}[U(\varphi)]| \cos[\chi - \arg \gamma_{ch}(C)] \right), \tag{86} \]
which is maximal when \( \chi = \arg \gamma_{ch}(C) \).

\section{IX. CONCLUSIONS}

Investigations into quantum holonomy have yielded a unifying understanding of the geometry of some basic structures of quantum systems. These efforts have concerned not only the geometry of quantum states themselves, but also how the twisting of subspaces, realized, e.g., as eigenspaces of some adiabatically varying Hamiltonian, can be used to manipulate quantum states in a robust manner.

In this paper, we have extended the notion of holonomy to sequences of quantum channels. The proposed quantity transforms as a holonomy under change of Kraus representations of the channels. We have shown that the channel holonomy is related to the Uhlmann holonomy \[2\] for sequences of density operators constructed from the Jamiołkowski isomorphism \[12\]. We have delineated parallel transport and concomitant gauge potential for smooth families of channels.

In addition to the relation to the Uhlmann holonomy, we have found some results that connect to other known quantum holonomies. For smooth sequences of unitary channels, the channel holonomy reduces to the phase shift in Ref. \[27\] for unpolarized particles in an interferometer. Furthermore, we have analyzed a class of channels that have a direct relation to the subspace holonomies in Ref. \[21\]. For smooth families of such “holonomic” channels, the channel holonomy is completely determined by the trace of the holonomies associated with paths in the space of subspaces (i.e., the Wilson loops if these paths are closed).

We have demonstrated a physical realization of the channel holonomy in an interferometric setting, based on the idea of “gluings” of channels \[13, 14\]. The realization consists of a gauge invariant object related to the channel holonomy and a prescription for how this object can be used to extract the channel holonomy experimentally. Using this idea, we have been able to demonstrate that the neutron interferometer tests in Refs. \[13, 14\] of the \( 4\pi \) spinor symmetry can alternatively be interpreted in terms of the holonomy for unitary channels. Thus, one may be tempted to say that a particular realization of the channel holonomy already exists.

To avoid technical complications, we have consistently made simplifying assumptions about the nature of the channels. We have restricted the analysis to sequences of channels with fixed number of linearly independent Kraus operators and we have assumed that all relevant matrices and operators have a well-defined inverse. If these assumptions are relaxed, one could consider analogues to the admissible sequences of density operator as considered by Uhlmann \[5\], or to the partial holonomies described in Refs. \[21, 30\].

Let us end by a remark concerning the potential relevance of this work to holonomic quantum computation \[10, 11\]. The key point with this type of quantum computation is that it is believed to be resilient to certain errors, such as those induced by open-system effects. Thus, in order to examine the resilience of holonomic quantum computation, it becomes important to have a useful notion of geometric phase or holonomy for open quantum systems. This has been addressed from different perspectives in several recent papers \[37, 38, 39, 40, 41, 42, 43\].

Since open-system evolution may be described by quantum channels, it seems reasonable that the proposed channel holonomy, or some generalization of it allowing for variable Kraus number, might, in some way or another, play a role in the analysis of the resilience of holonomic quantum computation to open-system effects.

\section{ACKNOWLEDGMENTS}

J.A. wishes to thank the Swedish Research Council for financial support and the Centre for Quantum Computation at DAMTP, Cambridge, for hospitality. E.S. acknowledges financial support from the Swedish Research Council. The work by J.A. was supported by the European Union through the Integrated Project QAP (IST-3-015848), SCALA (CT-015714), SECOQC and the QIP IRC (GR/S821176/01).

\section{APPENDIX}

Consider channels \( \mathcal{E} \) and \( \mathcal{F} \) both with Kraus number \( K \). Any choice of Kraus operators for this pair of channels span \( K \)-dimensional subspaces \( \mathcal{L}_\mathcal{E} \) and \( \mathcal{L}_\mathcal{F} \), respectively, of the \( D^2 \)-dimensional space of linear operators acting on the \( D \)-dimensional state space. We show that a necessary and sufficient criterion for any \( T \) correspond-
Consider the set of channels with Kraus number \( L_{\mathcal{E}} \) and let \( L_{\mathcal{F}} \) be partially overlapping. Then, one can choose a Kraus representation \( \{ F_k \}_k \) for \( \mathcal{F} \) so that there exists a \( F_{k'} \) in this set lying in the orthogonal complement to \( L_{\mathcal{E}} \). It follows that \( T_{k' \ell} = \text{Tr}(F_{k'} E_{\ell}) = 0 \), \( \forall \ell \), and thus the rank of \( T \) is less than \( K \).

Conversely, if the rank of \( T \) is less than \( K \), then \( T \) has at least one singular value that is zero. Consider the singular value decomposition \( T = UDV \) (\( D \) diagonal and \( U, V \) unitary). Assume \( D_{kk'} = 0 \) and consider the transformation \( F_k \rightarrow \tilde{F}_k = \sum_l F_l U_{lk} \). This results in \( T \rightarrow \tilde{T} = DV \), which implies \( \tilde{T}_{k' \ell} = \text{Tr}(F_{k'}' E_{\ell}) = 0 \), \( \forall \ell \). Thus, \( \tilde{F}_{k'} \) lies in the orthogonal complement to \( L_{\mathcal{E}} \).

As an illustration, consider the following representations:

\[
\begin{align*}
\{ E_0, E_1 \} &= \left\{ \sqrt{1-p_e}, \sqrt{p_e} \sigma_z \right\}, \\
\{ F_0, F_1 \} &= \left\{ \sqrt{1-p_f}, \sqrt{p_f} \sigma_x \right\}, \\
(G_0, G_1) &= \left\{ \frac{1}{2} (1 + \sqrt{1-p_g}) \mathbb{1}, \frac{1}{2} (1 - \sqrt{1-p_g}) \sigma_z, \frac{1}{2} \sqrt{p_g} \sigma_+ \right\}.
\end{align*}
\]

(87)

of the phase flip (\( \mathcal{E} \)), bit flip (\( \mathcal{F} \)), and amplitude damping (\( \mathcal{G} \)) channels for a qubit (\( D = 2 \)). Here, the \( \sigma \)'s are the standard Pauli operators (with \( \sigma_+ = \sigma_x + i \sigma_y \) and \( p_e, p_f, p_g \in [0,1] \). Clearly, \( K(\mathcal{E}) = K(\mathcal{F}) = K(\mathcal{G}) = K = 2 \). By inspection, we see that \( F_1 \) and \( G_1 \) both lie in the orthogonal complement to \( L_{\mathcal{E}} \). Thus, both \( L_{\mathcal{F}} \) and \( L_{\mathcal{G}} \) overlap partially with \( L_{\mathcal{E}} \) and the corresponding \( T \) matrices have rank less than \( K = 2 \). Explicitly, one obtains

\[
T_{\mathcal{F},\mathcal{E}} = \begin{pmatrix} 2\sqrt{(1-p_e)(1-p_f)} & 0 \\ 0 & 0 \end{pmatrix},
\]

\[
T_{\mathcal{G},\mathcal{E}} = \begin{pmatrix} \sqrt{1-p_e} & 0 \\ 0 & \sqrt{1-p_g} \end{pmatrix}.
\]

Clearly, the rank of \( T_{\mathcal{F},\mathcal{E}} \) is 1 if \( p_e, p_f \neq 1 \) and 0 (\( L_{\mathcal{E}} \) and \( L_{\mathcal{F}} \) orthogonal) if \( p_e = 1 \) or \( p_f = 1 \). The rank of \( T_{\mathcal{G},\mathcal{E}} \) is 0 if \( p_e = 1 \) and \( p_g = 0 \), and 1 otherwise. On the other hand,

\[
T_{\mathcal{G},\mathcal{F}} = \begin{pmatrix} \sqrt{1-p_f} & 0 \\ 0 & 0 \end{pmatrix}.
\]

which has rank \( K = 2 \) if \( p_f \neq 1 \) and \( p_g \neq 0 \).

References:

[1] M. V. Berry, Proc. R. Soc. London A 392, 45 (1984).
[2] F. Wilczek and A. Zee, Phys. Rev. Lett. 52, 2111 (1984).
[3] Y. Aharonov and J. Anandan, Phys. Rev. Lett. 58, 1593 (1987).
[4] J. Samuel and R. Bhandari, Phys. Rev. Lett. 60, 2339 (1988).
[5] A. Uhlmann, Rep. Math. Phys. 24, 229 (1986).
[6] J. Anandan and A. Pines, Phys. Lett. A 141, 335 (1989).
[7] J. Anandan, Phys. Lett. A 133, 171 (1988).
[8] L.-A. Wu, P. Zanardi, and D. A. Lidar, Phys. Rev. Lett. 95, 130501 (2005).
[9] A. Carollo, M. França Santos, and V. Vedral, Phys. Rev. Lett. 96, 020403 (2006).
[10] P. Zanardi and M. Rasetti, Phys. Lett. A 264, 94 (1999).
[11] J. Pachos, P. Zanardi, and M. Rasetti, Phys. Rev. A 61, 010305(R) (2000).
[12] A. Jamiołkowski, Rep. Math. Phys. 3, 275 (1972).
[13] J. Åberg, Ann. Phys. (N.Y.) 313, 326 (2004).
[14] J. Åberg, Phys. Rev. A 70, 012103 (2004).
[15] H. Rauch, A. Zeilinger, G. Badurek, A. Wilking, W. Bauspiess, and U. Bonsie, Phys. Lett. A 54, 425 (1975).
[16] A. Werner, R. Colella, A. W. Overhauser, and C. F. Eagen, Phys. Rev. Lett. 35, 1053 (1975).
[17] K. Kraus, States, Effects, and Operations (Springer-Verlag, Berlin, 1983).
[18] More generally, the summation range on the right-hand side of Eq. 2 can be larger than the Kraus number \( K(\mathcal{F}) \), if we demand that \( U \) is a partial isometry. In such cases, the resulting operators \( \{ F_k \}_k \) become linearly dependent.
[19] Consider the set of channels with Kraus number \( K \), and two elements \( \mathcal{E} \) and \( \mathcal{F} \) from this set. Consider moreover
representations in Eq. (2) relates to the gauge transformation \( \mathcal{U} \rightarrow (1 \otimes \mathcal{U})\mathcal{U} \) according to \( \langle a_k | U | a_l \rangle = [U^{T}]_{kl} \), where \( T \) denotes transpose with respect to some orthonormal basis \( \{|a_k\rangle\}_{k=1}^{K} \) of \( \mathcal{H}_a \).

[31] D. K. L. Oi, Phys. Rev. Lett. 91, 067902 (2003).
[32] D. K. L. Oi and J. Åberg, Phys. Rev. Lett. 97, 220404 (2006).
[33] J. Åberg, D. Kult, E. Sjöqvist, and D. K. L. Oi, Phys. Rev. A, 75, 032106 (2007).
[34] H. J. Bernstein, Phys. Rev. Lett. 18, 1102 (1967).
[35] Y. Aharonov and L. Susskind, Phys. Rev. 158, 1237 (1967).
[36] E. Sjöqvist, D. Kult, and J. Åberg, Phys. Rev. A 74, 062101 (2006).
[37] M. Ericsson, E. Sjöqvist, J. Brännlund, D. K. L. Oi, and A. K. Pati, Phys. Rev. A 67, 020101(R) (2003).
[38] J. G. Peixoto de Faria, A. F. R. de Toledo Piza, and M. C. Nemes, Europhys. Lett. 62, 782 (2003).
[39] A. Carollo, I. Fuentes-Guridi, M. França Santos, and V. Vedral, Phys. Rev. Lett. 90 160402 (2003).
[40] K.-P. Marzlin, S. Ghose, and B. C. Sanders, Phys. Rev. Lett. 93, 260402 (2004).
[41] M. S. Sarandy and D. A. Lidar, Phys. Rev. A 73, 062101 (2006).
[42] A. Bassi and E. Ippoliti, Phys. Rev. A 73, 062104 (2006).
[43] H. Goto and K. Ichimura, Phys. Rev. A 76, 012120 (2007).