High-order Field Theory and Weak Euler-Lagrange-Barut Equation for Classical Relativistic Particle-Field Systems

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Abstract

It is widely accepted that conservation laws, especially energy-momentum conservation, have fundamental importance for both classical and quantum systems in physics. A widely used method to derive the conservation laws is based on Noether’s theorem. However, for classical relativistic particle-field systems, this process is still impeded. Different from the quantum situation, the obstruction emerged when we regard the particle’s field as a classical world line. The difficulties come from two aspects. One is the mass-shell constraint and the other comes from the heterogeneous-manifolds that particles and fields reside on. This study develops a general geometric (manifestly covariant) field theory for classical relativistic particle-field systems. In considering the mass-shell constraint, the Euler-Lagrange-Barut (ELB) equation as a geometric version of the Euler-Lagrange (EL) equation is applied to determine the world lines of the relativistic particles. As a differential equation in the standard field theory, the infinitesimal criterion of the symmetry condition is converted into an integro-differential equation. To overcome the second difficulty, we develop a weak ELB equation on the 4D space-time. The weak version of the ELB equation will play an essential role in establishing the connections between symmetries and local conservation laws. Using field theory together with the weak ELB equation developed here, the conservation laws can be systematically derived from the symmetries that the systems admit.

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I. INTRODUCTION

Classical relativistic particle-field systems, where many particles interact with the self-generated and high background fields, are often encountered in astrophysics [1–3], accelerator physics [4] and gyrokinetic systems in plasma physics [3, 5, 6]. For these systems, one of the significant topics is focused on the derivation of energy and momentum conservation laws [7–9].

As a widely accepted fundamental principle, conservation laws can be derive by the corresponding symmetry that the Lagrangian (or action) of the system admit. This is the so-called Noether’s theorem [10]. This method has been widely used in deriving the energy-momentum conservation in quantum field systems by the space-time translation symmetry [11]. However, for classical particle-field systems, the derivation process is still elusive. For example, for electromagnetic system coupled with relativistic particles, the energy-momentum conservation was first derived by Landau and Lifshitz [12], and just reformulated into a geometric form recently from the space-time translation symmetry [13].

Different from the quantum systems, the dynamics of the particles and the fields reside on heterogeneous manifolds. The fields, e.g., the electromagnetic fields, are defined on the 4D space-time domain, while the particles as world lines in Minkowski space are only defined on 1D parametric space (e.g., time-axis), which make the standard Euler-Lagrange (EL) breakdown [14]. This difficulty has been overcome recently by transformed the EL equation into a weak form [14–16], and the corresponding method has been applied in non-relativistic systems, such as the Vlasov-Possion (VP) system, Vlasov-Darwin (VD) system and gyrokinetic system in plasma physics.

However, for relativistic situation with the geometric setting, this theory was only reformulated for the Maxwell’s system [13]. In promoting the method to geometric formalism, the proper-time parameter rather than time parameter is often used, which yields the mass-shell constraint. In considering this constraint, the equation of motion, derived by the Hamilton’s principle, will be the Euler-Lagrange-Barut (ELB) equation instead of the standard EL equation [17, 18]. Besides, the using of proper-time parameter also makes the Lagrangian density a functional instead a function of particle’s world line. Due to the existing difficulties, for general classical relativistic particle-field systems, the weak form equation of particle’s motion and infinitesimal criterion of symmetry condition for is still unclear.
In this study, we will extend the theory constructed in Ref. [13] to general high-order field systems. High-order electromagnetic field theories always appear in the study of gyrokinetic systems for magnetized plasma [19] and the Podolsky system [20, 21] for the radiation reaction of classical charged particles. The weak ELB equation for Maxwell’s system is also extended to general situations. The infinitesimal criterion of symmetry condition is also reformulated in considering the geometric setting. Moreover, because the Lagrangian density is a functional, the criterion we derived here is then a integro-differential equation. This is quite different from the standard field theory, in which the Lagrangian density is a function, and the infinitesimal criterion is consequently a differential equation [22]. Using the general weak ELB equation and the infinitesimal criterion developed here, the conservation laws can be systematically derived from the underlying symmetries. As a simple but non-trivial high-order case, we will calculate the energy-momentum conservation laws for Podolsky system coupled with relativistic charged particles.

The rest of this paper is organized as follows. In Sec. II, we will introduce the action of a general classical relativistic particle-field systems and the weak ELB equation is established as well. In Sec. III, the geometric infinitesimal criterion of the symmetry is developed as required by the mass-shell constraint. Using the geometric weak Euler-Lagrange equation and geometric infinitesimal criterion, the conservation laws are obtained in Sec. IV. In the last section, we will derive the energy-momentum conservation laws for high-order electromagnetic systems coupled with charged particles.

II. CLASSICAL RELATIVISTIC PARTICLE-FIELD SYSTEMS AND GEOMETRIC WEAK EULER-LAGRANGE-BARUT EQUATION

We start from the geometric action of the particle-field system and extend the theory developed in Ref. [13, 15] to the general relativistic situation. The geometric action of a classical relativistic particle-field system is generally written as

\[ A = \sum_a \int_{\tau_{a1}}^{\tau_{a2}} L_a (\tau_a, \chi_a (\tau_a), \dot{\chi}_a (\tau_a), \text{pr}^{(n)} \psi (\chi_a (\tau_a))) \, d\tau_a + \int_{\Omega} L_F (\chi, \text{pr}^{(n)} \psi (\chi)) \, d^4 \chi, \]  

where the subscript \( a \) labels particles, \( \chi \) is the space-time position, \( \psi (\chi) \) is a vector (or any tensor) field defined on space-time. Here, \( \text{pr}^{(n)} \psi (\chi) \) is the prolongation of the field \( \psi (\chi) \),
which contains \( \psi(\chi) \) and its derivatives up to the \( n \)th order, i.e.,

\[
{\text{pr}}^{(n)} \psi(\chi) = (\psi(\chi), \partial_{\mu_1} \psi(\chi), \cdots, \partial_{\mu_1} \cdots \partial_{\mu_n} \psi(\chi)),
\]

(2)

where \( \partial_{\mu_i} (i = 1, 2, \cdots, n) \) represents derivative with respect to space time coordinates. In Eq. (1), \( \chi_a(\tau_a) \) is the world line of the \( a \)th particle and \( \tau_a \) is the proper time parameter. \( \chi_a(\tau_a) \equiv d\chi_a(\tau_a)/d\tau_a \) is the 4-velocity of the \( a \)th particle which satisfy the mass-shell constraint

\[
\dot{\chi}_a^\mu \dot{\chi}_{a\mu} = c^2,
\]

(3)

where the Lorentzian metric \( \eta_{\mu\nu} = \text{diag}\{1, -1, -1, -1\} \) is applied to define \( \dot{\chi}_{a\mu} \), i.e., \( \dot{\chi}_{a\mu} = \eta_{\mu\nu} \dot{\chi}^\nu_a \). Due to the subsidiary condition (3), the variation of the action \( A \) with respect to \( \delta \chi_a \) won’t yields the standard Euler-Lagrange equation but the equation as follows

\[
E_{\chi_a^\mu} (L_a) = 0,
\]

(4)

where the linear operator \( E_{\chi_a^\mu} \) is defined by

\[
E_{\chi_a^\mu} := \frac{\partial}{\partial \chi_a^\mu} - \frac{d}{d\tau_a} \left[ \frac{\partial}{\partial \dot{\chi}_a^\mu} + \frac{1}{c^2} \left( \text{id} - \dot{\chi}_a^\nu \frac{\partial}{\partial \dot{\chi}_a^\nu} \right) \right],
\]

(5)

which is different from the standard Euler operator by the existence of the last two terms. To be faithful to history, equation (4) was first obtained by Barut using the Lagrange multiplier method [17]. Therefore, we will refer to the linear operator \( E_{\chi_a^\mu} \) as the Euler-Lagrange-Barut (ELB) operator and call equation (4) the ELB equation.

The integral of the \( a \)th particle’s Lagrangian \( L_a \) in Eq. (1) is along an arbitrary time-like world line (denoted by \( l_a \)) which connected two fixed world points \( a_1 \) and \( a_2 \) at the space-time \( \mathbb{R}^4 \), while the integral of the field’s Lagrangian density \( \mathcal{L}_F \) is over the the space-time domain \( \Omega \). Hence, the integral of the Lagrangian density \( \mathcal{L}_F \) for the field \( \psi \) is over space-time, and the integral of Lagrangian \( L_a \) for the \( a \)th particle is over proper time only. As a consequence, the action in the form (1) is not easily applicable for Noether’s procedure of deriving local conservation laws.

To deal with this problem, we multiply the first part on the right-hand side of Eq. (1) by the following identity

\[
\int \delta_a d^4 \chi = 1,
\]

(6)
where $\delta_a \equiv \delta (\chi - \chi_a (\tau_a))$ is the Dirac’s delta function. The action $\mathcal{A}$ in Eq. (11) is then transformed into one integral over space-time,

$$\mathcal{A} = \int_{\Omega} \mathcal{L} (\chi, [\chi_a], \text{pr}^{(n)} \psi (\chi)) d^4 \chi,$$

(7)

where

$$\mathcal{L} = \sum_{a} \mathcal{L}_a + \mathcal{L}_F (\chi, \text{pr}^{(n)} \psi (\chi)),$$

(8)

$$\mathcal{L}_a = \int_{a_1}^{a_2} \ell_a d\tau_a, \quad \ell_a = \ell_a (\tau_a, \chi, \text{pr}^{(l)} \chi_a (\tau_a), \text{pr}^{(n)} \psi (\chi)) = L_a \delta_a.$$

(9)

Different from the non-geometric situation, the Lagrangian density $\mathcal{L}$ is not a function but a functional of the particle’s world line. To differ from the other local variables (such as $\chi$ and $\psi (\chi)$), we enclose $\chi_a$ by square brackets.

We now calculate how the action (7) varies in response to the variations of $\delta \chi_a$ and $\delta \psi$,

$$\delta \mathcal{A} = \sum_{a} \int_{a_1}^{a_2} \left[ \int_{\Omega} E_{\chi_a} (\ell_a) d^4 \chi \right] \delta \chi_a^\mu d\tau_a + \int_{\Omega} E_{\psi} (\mathcal{L}) \cdot \delta \psi d^4 \chi,$$

(10)

where

$$E_{\psi} = \frac{\partial}{\partial \psi} + \sum_{j=1}^{n} (-1)^j D_{\mu_1} \cdots D_{\mu_j} \frac{\partial}{\partial (\partial_{\mu_1} \cdots \partial_{\mu_j} \psi)}$$

(11)

is the Euler operator for the field $\psi$. Here, $\text{id}$ in Eq. (5) is the identity operator. Using Hamilton’s principle, we immediately obtain the equation of motion for particles and fields

$$\int_{\Omega} E_{\chi_a} (\ell_a) d^4 \chi = 0,$$

(12)

$$E_{\psi} (\mathcal{L}) = 0.$$

(13)

Similar to the submanifold EL equation given in Refs. [14, 15], equation (12) is called the submanifold ELB equation. Using the linear property of ELB operator, we can easily prove that the submanifold ELB equation (12) is equivalent to ELB equation (4).

To apply equation (12) to Noether’s method, we need the explicit expression of $E_{\chi_a} (\ell_a)$. For electromagnetic system, the expression was given in the previous work [13]. We now derive a general expression of $E_{\chi_a} (\ell_a)$. We first transform the first and the last three terms of Eq. (12) into following forms

$$\frac{\partial \ell_a}{\partial \chi_a^\mu} = \frac{\partial}{\partial \chi_a^\mu} (L_a \delta_a) = \frac{\partial \delta_a}{\partial \chi_a^\mu} L_a + \frac{\partial L_a}{\partial \chi_a^\mu} \delta_a = -\frac{\partial \delta_a}{\partial \chi_a^\mu} L_a + \frac{\partial L_a}{\partial \chi_a^\mu} \delta_a.$$
\[ - \frac{D}{D\chi^\mu} (L_a \delta_a) + \frac{\partial L_a}{\partial \chi^a} \delta_a = \frac{D}{D\chi^\nu} (-L_a \delta_a \eta^\nu) + \frac{\partial L_a}{\partial \chi^a} \delta_a. \tag{14} \]

\[ - \frac{d}{d\tau_a} \left\{ \frac{\partial L_a}{\partial \chi^a} + \frac{1}{c^2} \dot{\chi}_a \left( L_a - \dot{x}_a^\sigma \frac{\partial L_a}{\partial \chi^a} \right) \right\} \delta_a \]
\[ = - \frac{d \delta_a}{d\tau_a} \left[ \frac{\partial L_a}{\partial \chi^a} + \frac{1}{c^2} \dot{\chi}_a \left( L_a - \dot{x}_a^\sigma \frac{\partial L_a}{\partial \chi^a} \right) \right] - \frac{d}{d\tau_a} \left[ \frac{\partial L_a}{\partial \chi^a} + \frac{1}{c^2} \dot{\chi}_a \left( L_a - \dot{x}_a^\sigma \frac{\partial L_a}{\partial \chi^a} \right) \right] \delta_a. \tag{15} \]

The first term on the right-hand side of Eq. (15) can be rewritten as
\[ - \frac{d \delta_a}{d\tau_a} \left[ \frac{\partial L_a}{\partial \chi^a} + \frac{1}{c^2} \dot{\chi}_a \left( L_a - \dot{x}_a^\sigma \frac{\partial L_a}{\partial \chi^a} \right) \right] = - \dot{x}_a^\nu \frac{\partial \delta_a}{\partial \chi^\nu} \left[ \frac{\partial L_a}{\partial \chi^a} + \frac{1}{c^2} \dot{\chi}_a \left( L_a - \dot{x}_a^\sigma \frac{\partial L_a}{\partial \chi^a} \right) \right] \]
\[ = \frac{D}{D\chi^\nu} \left\{ \dot{x}_a^\nu \left[ \frac{\partial \ell_a}{\partial \chi^a} + \frac{1}{c^2} \dot{\chi}_a \left( \ell_a - \dot{x}_a^\sigma \frac{\partial \ell_a}{\partial \chi^a} \right) \right] \right\}. \tag{16} \]

Substituting Eqs. (14)–(16) into \( E_{\chi a\mu}(\ell_a) \), we have
\[ E_{\chi a\mu}(\ell_a) = \frac{D}{D\chi^\nu} \left\{ - \ell_a \eta^\nu + \dot{x}_a^\nu \left[ \frac{\partial \ell_a}{\partial \chi^a} + \frac{1}{c^2} \dot{\chi}_a \left( \ell_a - \dot{x}_a^\sigma \frac{\partial \ell_a}{\partial \chi^a} \right) \right] \right\}, \tag{17} \]
where we used the ELB equation (4). We will refer to Eq. (17) as the weak ELB equation, which as a differential equation is equivalent to the submanifold ELB equation (12). Just like the weak EL equation used in non-relativistic particle-field systems, the weak ELB equation is also elemental in deriving local conservation laws for relativistic situation.

**III. GEOMETRIC INFINITESIMAL CRITERION OF SYMMETRY CONDITION**

We now turn to discuss the symmetries of the relativistic particle-field systems. A symmetry of the action \( \mathcal{A} \) is a group of transformation

\[ g_\epsilon \cdot (\tau_a, \chi_a, \dot{\chi}_a (\tau_a), \psi (\chi)) \equiv (Z_{ac} (\tau_a, \chi, \chi_a, \psi), \Xi_\epsilon (\tau_a, \chi, \chi_a, \psi), \Theta_{ac} (\tau_a, \chi_a, \psi), \Phi_\epsilon (\tau_a, \chi_a, \psi)) \equiv \left( \tilde{\tau}_a, \tilde{\chi}, \tilde{\chi}_a, \tilde{\psi} (\tilde{\chi}) \right) \tag{18} \]

such that
\[ \int_\Omega \mathcal{L} (\chi, \chi_a) + \text{pr}(\mathcal{N}) \psi (\chi_a) \right) r^4 \chi = \int_\Omega \mathcal{L} (\tilde{\tau}_a, \tilde{\chi}, \tilde{\chi}_a, \tilde{\psi} (\tilde{\chi})) r^4 \tilde{\chi}, \tag{19} \]
and
\[ \tilde{x}_a^\mu \tilde{\chi}_a = \tilde{x}_a^\mu \tilde{\chi}_a = c^2, \tag{20} \]
where \( g_\epsilon = (Z_a, \Xi_a, \Theta_a, \Phi_\epsilon) \) constitutes a continuous group of transformations parameterized by \( \epsilon \). Equations (19) and (20) are called the symmetry conditions. Different from the standard field systems, the second condition (20) is needed now due to the mass-shell constraint.

To derive the local conservation laws, an infinitesimal version of the symmetry condition is required. For this purpose, we take the derivative of Eqs. (19) and (20) with respect to \( \epsilon \) at \( \epsilon = 0 \),

\[
\frac{d}{d\epsilon} \bigg|_{\epsilon=0} \int_\Omega \left[ \sum_a \int_{a1}^{a2} \ell_a \left( \tilde{T}_a, \tilde{X}_a, \text{pr}^{(1)} \tilde{X}_a(\tilde{T}_a), \text{pr}^{(n)} \tilde{\psi}(\tilde{X}) \right) d\tau_a + \mathcal{L}_F \left( \tilde{X}, \text{pr}^{(n)} \tilde{\psi}(\tilde{X}) \right) \right] d^4 \tilde{X} = 0.
\]

(21)

\[
\frac{d}{d\epsilon} \bigg|_{\epsilon=0} \left[ \tilde{\chi}_\mu \dot{\tilde{\chi}}_{\mu} \right] = 0
\]

(22)

Substituting Eqs. (8) and (9) into Eq. (21), we have

\[
\int_\Omega \frac{d}{d\epsilon} \bigg|_{\epsilon=0} \left\{ \left[ \sum_a \int_{a1}^{a2} \ell_a \left( \tilde{T}_a, \tilde{X}_a, \text{pr}^{(1)} \tilde{X}_a(\tilde{T}_a), \text{pr}^{(n)} \tilde{\psi}(\tilde{X}) \right) j_a(\epsilon) d\tau_a 
\right. 
+ \mathcal{L}_F \left( \tilde{X}, \text{pr}^{(n)} \tilde{\psi}(\tilde{X}) \right) \right\} \det J(\epsilon) \right\} d^4 \chi = 0,
\]

(23)

where

\[
J(\epsilon) \equiv \frac{\partial \tilde{X}}{\partial \chi},
\]

(24)

Because equation (23) survive for any small integral domains, the integrand must be zero, i.e.,

\[
\frac{d}{d\epsilon} \bigg|_{\epsilon=0} \left[ \sum_a \int_{a1}^{a2} \ell_a \left( \tilde{T}_a, \tilde{X}_a, \text{pr}^{(1)} \tilde{X}_a(\tilde{T}_a), \text{pr}^{(n)} \tilde{\psi}(\tilde{X}) \right) j_a(\epsilon) d\tau_a 
\right. 
+ \mathcal{L}_F \left( \tilde{X}, \text{pr}^{(n)} \tilde{\psi}(\tilde{X}) \right) \left. \right\} \det J(\epsilon) = 0.
\]

(25)

Equation (25) can be finally transformed into

\[
\sum_a \int_{a1}^{a2} \left[ \text{pr}^{(n)} \nu(\ell_a) + \ell_a \xi_a \right] d\tau_a \right) + \mathcal{L}_F \left( \tilde{X}, \text{pr}^{(n)} \tilde{\psi}(\tilde{X}) \right) + \mathcal{L} D_\mu \xi^\mu = 0,
\]

(26)

where

\[
\nu = \frac{d}{d\epsilon} \bigg|_{\epsilon=0} g_\epsilon \cdot (\tau_a, X_a, X_a, \psi) = \sum_a \xi_a \frac{\partial}{\partial \tau_a} + \xi \cdot \frac{\partial}{\partial \chi} + \sum_a \theta_a \cdot \frac{\partial}{\partial \chi_a} + \phi \cdot \frac{\partial}{\partial \psi}
\]

(27)
is the infinitesimal generator of the group transformation, and where

\[
\begin{align*}
\zeta_a &= \zeta_a(\tau_a, \chi, \chi_a, \psi) = \frac{d}{d\epsilon} \bigg|_0 Z_{ae}(\tau_a, \chi, \chi_a, \psi), \\
\xi &= \xi(\tau_a, \chi, \chi_a, \psi) = \frac{d}{d\epsilon} \bigg|_0 \Xi_{ae}(\tau_a, \chi, \chi_a, \psi), \\
\theta_a &= \theta_a(\tau_a, \chi, \chi_a, \psi) = \frac{d}{d\epsilon} \bigg|_0 \Theta_{ae}(\tau_a, \chi, \chi_a, \psi), \\
\phi &= \phi(\tau_a, \chi, \chi_a, \psi) = \frac{d}{d\epsilon} \bigg|_0 \Phi_{e}(\tau_a, \chi, \chi_a, \psi).
\end{align*}
\]

Here, \( \text{pr}^{(n)} v \) is the prolongation of \( v \) defined by

\[
\text{pr}^{(n)} v = \frac{d}{d\epsilon} \bigg|_0 \left( \tilde{\tau}_a, \tilde{\chi}_a, \text{pr}^{(1)} \tilde{\chi}_a, \text{pr}^{(n)} \tilde{\psi}(\tilde{\chi}) \right)
\]

\[
= v + \sum_a \theta^\mu_{a1} \frac{\partial}{\partial \tilde{\chi}_a^\mu} + \sum_{k=1}^n \phi^\alpha_{\mu_1 \cdots \mu_k} \frac{\partial}{\partial (D_{\mu_1} \cdots D_{\mu_k} \psi^\alpha)},
\]

where

\[
\begin{align*}
\theta^\mu_{a1} &= \zeta_a \tilde{\chi}_a^\mu + \dot{q}^\mu_a, \\
\phi^\alpha_{\mu_1 \cdots \mu_k} &= \xi^\mu D_{\mu_1} \cdots D_{\mu_k} (D_{\mu_1} \psi^\alpha) + D_{\mu_1} \cdots D_{\mu_k} Q^\alpha,
\end{align*}
\]

and

\[
\begin{align*}
q^\mu_a &\equiv \theta^\mu_a - \zeta_a \dot{\chi}_a, \\
Q^\alpha &\equiv \phi^\alpha - \xi^\alpha D_{\mu} \psi^\alpha,
\end{align*}
\]

are characteristics of the infinitesimal generator \( v \). Equation (26) is the infinitesimal criterion of the symmetry condition (19). Detailed derivation process of the prolongation formula (29) can be found in Ref. [22]. Different from the situation in the previous references, an integral along the particle’s world line appeared in the infinitesimal criterion (see Eq. (26)). This originally comes from the fact that the Lagrangian density \( L \) is a functional rather than a function of particle’s world lines. Owing to this integral, the infinitesimal criterion (31) is not a differential equation but an integro-differential equation.

Similarly, using Eq. (22), we can obtained another infinitesimal criterion as

\[
\dot{q}^\mu_a \dot{\chi}_{a\mu} = 0.
\]

In deriving Eq. (32), we used the following equations

\[
\begin{align*}
\frac{d}{d\epsilon} \bigg|_0 \dot{\chi}_a^\mu &= \theta^\mu_{a1} = \zeta_a \ddot{\chi}_a^\mu + \dot{q}^\mu_a, \\
\ddot{\chi}_a^\mu \dot{\chi}_{a\mu} &\equiv 0.
\end{align*}
\]
IV. CONSERVATION LAWS

Due to the particularities of the infinitesimal criterion shown in Sec. III, especially the existence of the additional infinitesimal criterion (32), the derivation process of local conservation law and the and the final results are quite different from the standard situation without constraints. We next show how the infinitesimal criterion (26) and (32) determine a conservation law.

We first transform Eq. (26) into another equivalent form. The first term in Eq. (26) can be rewritten by components as

\[ \text{pr}^{(n)}(\ell_a) + \ell_a \dot{\zeta}_a = \zeta_a \frac{\partial \ell_a}{\partial \tau_a} + \theta_a^\mu \frac{\partial \ell_a}{\partial \chi_a^\mu} + \theta_a^\alpha \frac{\partial \ell_a}{\partial \chi_a^\alpha} + \ell_a \dot{\zeta}_a + \xi^\mu \frac{\partial \ell_a}{\partial \chi_a^\mu} + \phi^\alpha \frac{\partial \ell_a}{\partial \psi^\alpha} + \sum_{k=1}^n \phi^\alpha_{\mu_1 \ldots \mu_k} \frac{\partial \ell_a}{\partial (D_{\mu_1} \ldots D_{\mu_k} \psi^\alpha)}. \tag{35} \]

From the first four terms of right-hand side of Eq. (35), we have

\[
\begin{align*}
\zeta_a \frac{\partial \ell_a}{\partial \tau_a} + \theta_a^\mu \frac{\partial \ell_a}{\partial \chi_a^\mu} + (\zeta_a \dot{\chi}_a^\mu + \dot{\theta}_a^\mu) \frac{\partial \ell_a}{\partial \chi_a^\mu} + \ell_a \dot{\zeta}_a \\
= \zeta_a \frac{d \ell_a}{d \tau_a} + \chi_a^\mu \frac{\partial \ell_a}{\partial \chi_a^\mu} + q_a^\mu \frac{\partial \ell_a}{\partial \chi_a^\mu} + \dot{q}_a^\mu \frac{\partial \ell_a}{\partial \chi_a^\mu} + \ell_a \dot{\zeta}_a \\
= \zeta_a \frac{d \ell_a}{d \tau_a} + \ell_a \dot{\zeta}_a + q_a^\mu \frac{\partial \ell_a}{\partial \chi_a^\mu} - \frac{d}{d \tau} \frac{\partial \ell_a}{\partial \chi_a^\mu} \\
= -\frac{d}{d \tau_a} (\ell_a \zeta_a) + \frac{d}{d \tau_a} \left[ q_a^\mu \frac{\partial \ell_a}{\partial \chi_a^\mu} - \frac{d}{d \tau} \frac{\partial \ell_a}{\partial \chi_a^\mu} \right] + q_a^\mu E_{\chi_a^\mu} (\ell_a) \\
= -\frac{d}{d \tau_a} \left[ \ell_a \zeta_a + q_a^\mu \frac{\partial \ell_a}{\partial \chi_a^\mu} - \frac{d}{d \tau} \frac{\partial \ell_a}{\partial \chi_a^\mu} \right] + q_a^\mu E_{\chi_a^\mu} (\ell_a) \tag{36}
\end{align*}
\]

where we used the second infinitesimal criterion (32) in the last step. The last three terms of Eq. (35) can be transformed into

\[
\begin{align*}
\xi^\mu \frac{\partial \ell_a}{\partial \chi_a^\mu} + \phi^\alpha \frac{\partial \ell_a}{\partial \psi^\alpha} + \sum_{k=1}^n \phi^\alpha_{\mu_1 \ldots \mu_k} \frac{\partial \ell_a}{\partial (D_{\mu_1} \ldots D_{\mu_k} \psi^\alpha)} \\
= \xi^\mu D_\mu \ell_a + D_\mu \psi^\mu_a + E_\psi (\ell_a) \cdot Q \tag{37}
\end{align*}
\]

by the standard derivation process (see Ref. [22]), where

\[
\psi^\mu_a = \sum_{i=1}^n \sum_{j=1}^i (-1)^{j+1} D_{\mu_{j+1}} \ldots D_{\mu_k} Q^\alpha \left[ D_{\mu_1} \ldots D_{\mu_{j-1}} \frac{\partial \ell_a}{\partial (D_{\mu_1} \ldots D_{\mu_{j-1}} D_{\mu_j} \ldots D_{\mu_k} \psi^\alpha)} \right]. \tag{38}
\]


Similarly, the last two terms in Eq. (41) can be written as

$$\Pr^{(n)}(L_F) + \mathcal{L}D_{\mu}\xi^{\mu} = \xi^{\mu}D_{\mu}L_F + \mathcal{L}D_{\mu}\xi^{\mu} + D_{\mu}\Pi_{F}^{\mu} + E_{\psi}(L_{F}) \cdot Q,$$

where

$$\Pi_{F}^{\mu} = \sum_{i=1}^{n} \sum_{j=1}^{i} (-1)^{j+1} D_{\mu_{j+1}} \cdots D_{\mu_{k}} Q^{\alpha} \left[ D_{\mu_{1}} \cdots D_{\mu_{j-1}} \frac{\partial L_{F}}{\partial (D_{\mu_{1}} \cdots D_{\mu_{j-1}} D_{\mu} D_{\mu_{j+1}} \cdots D_{\mu_{k}} \psi^{\alpha})} \right].$$

Combining Eqs. (35), (36), (37) and (39), equation (26) is then transformed into

$$\sum_{a} \int_{a_{1}}^{a_{2}} \left[ q_{a}^{\mu} E_{\chi_{a\mu}}(\ell_{a}) + \xi^{\mu} D_{\mu} \ell_{a} + D_{\mu} P_{a}^{\mu} + E_{\psi}(L_{F}) \cdot Q \right] d\tau_{a}$$

$$+ \xi^{\mu} D_{\mu} L_{F} + D_{\mu} \Pi_{F}^{\mu} + E_{\psi}(L_{F}) \cdot Q + \mathcal{L}D_{\mu}\xi^{\mu} = 0,$$

where we used

$$\sum_{a} \int_{a_{1}}^{a_{2}} \frac{d}{d\tau_{a}} \left[ \ell_{a} \zeta_{a} + q_{a}^{\mu} \frac{\partial \ell_{a}}{\partial \chi_{a}^{\mu}} + \frac{q_{a}^{\mu}}{c^{2}} \dot{\chi}_{a\mu} \left( \ell_{a} - \frac{\partial \ell_{a}}{\partial \chi_{a}^{\mu}} \dot{\chi}_{a}^{\mu} \right) \right] d\tau_{a}$$

$$= \left[ \ell_{a} \zeta_{a} + q_{a}^{\mu} \frac{\partial \ell_{a}}{\partial \chi_{a}^{\mu}} + \frac{q_{a}^{\mu}}{c^{2}} \dot{\chi}_{a\mu} \left( \ell_{a} - \frac{\partial \ell_{a}}{\partial \chi_{a}^{\mu}} \dot{\chi}_{a}^{\mu} \right) \right]_{a_{1}}^{a_{2}} = 0.$$  \hspace{1cm} (42)

Suppose $\xi$ and $Q$ are independent of $\tau_{a}$, equation (41) becomes

$$D_{\mu} [\mathcal{L} \xi^{\mu} + \Omega^{\mu}] + \sum_{a} \int_{a_{1}}^{a_{2}} \left[ q_{a}^{\mu} E_{\chi_{a\mu}}(\ell_{a}) \right] d\tau_{a} + E_{\psi}(L_{F}) \cdot Q = 0,$$

where

$$\Omega^{\mu} = \sum_{i=1}^{n} \sum_{j=1}^{i} (-1)^{j+1} D_{\mu_{j+1}} \cdots D_{\mu_{k}} Q^{\alpha} \left[ D_{\mu_{1}} \cdots D_{\mu_{j-1}} \frac{\partial L}{\partial (D_{\mu_{1}} \cdots D_{\mu_{j-1}} D_{\mu} D_{\mu_{j+1}} \cdots D_{\mu_{k}} \psi^{\alpha})} \right].$$

Using the EL equation (13) of the field $\psi$, the last term in the Eq. (43) vanishes. However, due to the the weak ELB equation (17), the second term (43) is not zero. If the characteristic $q_{a}$ are independent of $\chi$ and $\psi$, this term can be written as a divergence form, i.e.,

$$\sum_{a} \int_{a_{1}}^{a_{2}} \left[ q_{a}^{\mu} E_{\chi_{a\mu}}(\ell_{a}) \right] d\tau_{a} = D_{\mu} S^{\mu},$$

where

$$S^{\mu} = \sum_{a} \int_{a_{1}}^{a_{2}} d\tau_{a} \left\{ q_{a}^{\mu} \left[ -\ell_{a} \eta_{a}^{\mu} + \dot{\chi}_{a}^{\mu} \left[ \frac{\partial \ell_{a}}{\partial \chi_{a}^{\mu}} + \frac{1}{c^{2}} \dot{\chi}_{a\mu} \left( \ell_{a} - \dot{\chi}_{a}^{\sigma} \frac{\partial \ell_{a}}{\partial \chi_{a}^{\sigma}} \right) \right] \right] \right\}.$$  \hspace{1cm} (46)

Substituting Eq. (45) into Eq. (43), we finally arrive at the geometric conservation law

$$D_{\mu} [\mathcal{L} \xi^{\mu} + \Omega^{\mu} + S^{\mu}] = 0.$$  \hspace{1cm} (47)
V. GAUGE-SYMMETRIC ENERGY-MOMENTUM CONSERVATION LAWS FOR HIGH-ORDER ELECTROMAGNETIC SYSTEMS COUPLED WITH CHARGED PARTICLES

We now apply the general theory to high-order electromagnetic system coupled with charged particles. The Lagrangian $L_a$ for this system can be generally written as a gauge symmetric form as

\[ L_a = L_a \left( \chi^\mu_a, A_\mu, F_{\mu\nu}, \cdots, D^{(n)} F_{\mu\nu} \right), \]

(48)

\[ \mathcal{L}_F = \mathcal{L}_F \left( \chi^\mu, F_{\mu\nu}, D F_{\mu\nu}, \cdots, D^{(n)} F_{\mu\nu} \right). \]

(49)

where $D$ is the space-time derivative operator, i.e., $D = (1/c) \partial / \partial t, \nabla$), $A_\mu$ as the field $\psi$ is the 4-potential and $F_{\mu\nu}$ is the Faraday tensor defined by

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \]

(50)

Using Eq. (13), we can obtain the equation of motion for electromagnetic field,

\[ E_\mu^A (\mathcal{L}) = 0, \]

(51)

where $E_\mu^A$ is the Euler operator for $A_\mu$. To obtain gauge-symmetric energy-momentum conservation laws, we transform (51) into

\[ \partial_\mu D^{\mu\nu} = \frac{4\pi}{c} J^\nu_f \]

(52)

using the derivatives with respect to $F_{\mu\nu}$ (see Ref. [23]), where

\[ D^{\mu\nu} = -8\pi E_F^{[\mu\nu]} (\mathcal{L}) \]

(53)

is regarded as the electric displacement tensor. Here, $E_F^{\mu\nu}$ is the Euler operator for the Faraday tensor, i.e.,

\[ E_F^{\mu\nu} := \frac{\partial}{\partial F_{\mu\nu}} + \sum_{i=1}^n (-1)^i D_{\mu_1} \cdots D_{\mu_i} \frac{\partial}{\partial \partial_{\mu_1} \cdots \partial_{\mu_i} F_{\mu\nu}}, \]

(54)

$J^\nu_f$ is the free 4-current which is defined by

\[ J^\nu_f := -c \frac{\partial \mathcal{L}}{\partial A_\nu} = - \int_{a_1}^{a_2} \frac{\partial L_a}{\partial A_\nu} ds_a. \]

(55)
Here, $ds_a = c d\tau_a$ is the line element in the Minkowski space. The superscript $[\mu \nu]$ in Eq. (53) denotes the anti-symmetrization with respect of $\mu$ and $\nu$. Suppose Eq. (52) is gauge symmetric and the Lagrangian density obey the minimal coupling which can be written as

$$\mathcal{L} = \sum_a \int_{a_1}^{a_2} \left[ -\frac{q_a}{c} A_\mu \dot{\chi}_a^\mu \right] \delta_a d\tau_a + \text{GSP}(\mathcal{L}) = -\frac{1}{c} A_\mu J^\mu + \text{GSP}(\mathcal{L}),$$

where $q_a$ is the charge of the $a$th particle and “GSP (\mathcal{L})” denotes the gauge-symmetric parts of the Lagrangian density. To get the gauge-symmetric energy-momentum conservation laws, the prolongation formula is also need reconstructed by using the derivatives with respect to $F_{\mu\nu}$, that is

$$\text{pr}^{(n)}(v) = v + \sum_a \theta_0^\mu \frac{\partial}{\partial \chi_a^\mu} + [G_{\sigma\rho} + \xi^\sigma D_\sigma F_{\rho\sigma}] \frac{\partial}{\partial F_{\sigma\rho}}$$

$$+ \sum_i [D_{\mu_1} \cdots D_{\mu_i} G_{\sigma\rho} + \xi^\sigma D_\sigma D_{\mu_1} \cdots D_{\mu_i} F_{\sigma\rho}] \frac{\partial}{\partial (\partial_{\mu_1} \cdots \partial_{\mu_i} F_{\sigma\rho})},$$

where

$$G_{\sigma\rho} = \partial_\sigma Q_\rho - \partial_\rho Q_\sigma = 2\partial_\sigma Q_\rho.$$ 

Finally, the conservation law (47) now read

$$D_\mu \left\{ \mathcal{L} \xi^\mu - \frac{1}{4\pi} D^{\mu\sigma} Q_\sigma + \mathbb{P}^\mu + \mathbb{S}^\mu \right\} = 0,$$

where

$$\mathbb{P}^\mu = \sum_{i=1}^n \sum_{j=1}^i (-1)^{j+1} (D_{\mu_{j+1}} \cdots D_{\mu_i} G_{\sigma\rho}) \left[ D_{\mu_1} \cdots D_{\mu_{j-1}} \frac{\partial \mathcal{L}}{\partial (\partial_{\mu_1} \cdots \partial_{\mu_{j-1}} \partial_{\mu_j} \cdots \partial_{\mu_i} F_{\sigma\rho})} \right].$$

We now turn to discuss the space-time translation symmetry and energy-momentum conservation law. Suppose the action of the system is invariant under the space-time translation,

$$g_\epsilon \cdot (\tau_a, \chi^\mu, \chi_0^\mu, A^\mu) = (\tilde{\tau}_a, \tilde{\chi}^\mu, \tilde{\chi}_a^\mu, \tilde{A}^\mu) = (\tau_a, \chi^\mu + \epsilon \chi_0^\mu, \chi_0^\mu + \epsilon \chi_a^\mu, A^\mu).$$

The corresponding infinitesimal generator of the group transformation (61) is

$$v = \chi_0^\mu \frac{\partial}{\partial \chi^\mu} + \sum_a \chi_0^\mu \frac{\partial}{\partial \chi_a^\mu},$$

where $\zeta_a = \phi^a = 0$, $\xi_\mu = \theta_0^\mu = \chi_0^\mu$. The characteristics $Q_a$, $q_a$, and the term $G_{\sigma\rho}$ are calculated as

$$q_a^\mu \equiv \chi_0^\mu.$$
\[ Q^\alpha = -\chi^\nu_0 \partial_\nu A^\alpha, \]  
(64) 
\[ G_{\sigma\rho} = -\chi^\nu_0 \partial_\nu F_{\sigma\rho}. \]  
(65) 

It is clear that the second infinitesimal criterion (32) is satisfied by \( \dot{q}_a^\mu \equiv 0 \). Substituting Eqs. (63)-(65) into Eq. (59), we obtain the canonical energy-momentum conservation law,

\[ D_\mu T_{N}^{\mu\nu} = 0, \]  
(66) 
\[ T_{N}^{\mu\nu} = \mathcal{L} \eta^{\mu\nu} + \frac{1}{4\pi} D^{\mu\sigma} \partial_\sigma A_\nu - \Sigma^{\mu\nu} + \Pi^{\mu\nu}, \]  
(67)

where

\[ \Sigma^{\mu\nu} = \sum_{i=1}^{n} \sum_{j=1}^{i} (-1)^{j+1} (D_{\mu_{j+1}} \cdots D_{\mu_{i}} \partial_\nu F_{\sigma\rho}) \left[ D_{\mu_{i}} \cdots D_{\mu_{j-1}} \frac{\partial \mathcal{L}}{\partial (\partial_{\mu_{i}} \cdots \partial_{\mu_{j-1}} \partial_\nu F_{\sigma\rho})} \right]. \]  
(68)

\[ \Pi^{\mu\nu} = \sum_a \int_{a_1}^{a_2} d\tau_a \left\{ -\ell_a \eta^{\mu\nu} + \hat{\chi}_a^{\mu} \left[ \frac{\partial \ell_a}{\partial \hat{x}_{a\nu}} + \frac{1}{c^2} \hat{\chi}_{a\nu} \left( \ell_a - \hat{x}_{a\sigma} \frac{\partial \ell_a}{\partial \hat{x}_{a\sigma}} \right) \right] \right\}. \]  
(69)

Here, \( T_{N}^{\mu\nu} \) is the canonical energy-momentum tensor which is gauge dependent. We next “improve” \( T_{N}^{\mu\nu} \) to a gauge-symmetric form. We add the following identity

\[ D_\mu (D_\sigma F^{\sigma\mu}) = 0, \quad F^{\sigma\mu} \equiv \frac{1}{4\pi} D^{\sigma\mu} A^\nu \]  
(70)

to Eq. (66) to get the explicitly gauge invariant conservation law

\[ D_\mu T_{GS}^{\mu\nu} = 0, \]  
(71)

where

\[ T_{GS}^{\mu\nu} = \mathcal{L} \eta^{\mu\nu} + \frac{1}{c} J_\nu A^\nu + \Pi^{\mu\nu} + \frac{1}{4\pi} D^{\mu\sigma} F_\sigma^\nu - \Sigma^{\mu\nu} \]  
(72)

is the improved energy-momentum tensor, where we used Eq. (52). We next prove that \( T_{GS}^{\mu\nu} \) is gauge invariant. It is sufficient to show that the first three terms in the right-hand side of Eq. (72) is gauge invariant. Substituting Eqs. (56) and (69) into Eq. (72), these terms are

\[ \mathcal{L} \eta^{\mu\nu} + \frac{1}{c} J_\nu A^\nu + \Pi^{\mu\nu} \]

\[ = \mathcal{L} \eta^{\mu\nu} + \frac{1}{c} J_\nu A^\nu + \sum_a \int_{a_1}^{a_2} d\tau_a \left\{ -\ell_a \eta^{\mu\nu} + \hat{\chi}_a^{\mu} \left[ \frac{\partial \ell_a}{\partial \hat{x}_{a\nu}} + \frac{1}{c^2} \hat{\chi}_{a\nu} \left( \ell_a - \hat{x}_{a\sigma} \frac{\partial \ell_a}{\partial \hat{x}_{a\sigma}} \right) \right] \right\} \]

\[ = \mathcal{L} \eta^{\mu\nu} - \left( \sum_a \int_{a_1}^{a_2} d\ell_a \tau_a \right) \eta^{\mu\nu} + \frac{1}{c} J_\nu A^\nu \]
+ \sum_a \int_{a_1}^{a_2} d\tau_a \left\{ -\frac{q_a}{c} \dot{\chi}_a^\mu A^\nu + \dot{\chi}_a^\mu \text{GSP} \left( \frac{\partial \ell_a}{\partial \chi_{a\nu}} \right) + \frac{1}{c^2} \dot{\chi}_a^\mu \dot{\chi}_a^\nu \left[ \ell_a - \frac{q_a}{c} \chi_a^\sigma A_\sigma - \dot{\chi}_a^\sigma \text{GSP} \left( \frac{\partial \ell_a}{\partial \chi_{a\sigma}} \right) \right] \right\} \\
= L_{\cal F} \eta^{\mu\nu} + \frac{1}{c^2} \sum_a \int_{a_1}^{a_2} d\tau_a \left\{ \dot{\chi}_a^\mu \text{GSP} \left( \frac{\partial \ell_a}{\partial \chi_{a\mu}} \right) + \dot{\chi}_a^\mu \dot{\chi}_a^\nu \left[ \text{GSP} \left( \ell_a \right) - \dot{\chi}_a^\nu \text{GSP} \left( \frac{\partial \ell_a}{\partial \chi_{a\nu}} \right) \right] \right\}
+ \frac{1}{c^2} \sum_a \int_{a_1}^{a_2} d\tau_a \left\{ \dot{\chi}_a^\mu \left[ \text{GSP} \left( \ell_a \right) - \dot{\chi}_a^\nu \text{GSP} \left( \frac{\partial \ell_a}{\partial \chi_{a\nu}} \right) \right] \right\},
(73)

where Eq. (55) is used in the last step. Equation (73) confirms that $T_{\text{inv}}^{\mu\nu}$ is gauge invariant.

Lastly, we discuss the Podolsky system \[20\] coupled with charged particles, where the Lagrangian density of particles and fields are respectively written as

$$\ell_{Pa} = - \left( m_a c^2 + \frac{q_a}{c} A_\mu \dot{\chi}_a^\mu \right) \delta_a$$
\( (74) \)

$$L_{\text{PF}} = - \frac{1}{16\pi} F_{\sigma\rho} F^{\sigma\rho} - \frac{a^2}{8\pi} \partial_\sigma F^{\sigma\lambda} \partial_\rho F_{\rho\lambda}.$$ 
\( (75) \)

Substituting Eq. (74) into Eq. (69) to obtain $\Pi_P^{\mu\nu}$

$$\Pi_P^{\mu\nu} = \sum_a \int_{a_1}^{a_2} d\tau_a \left\{ \left( m_a c^2 + \frac{q_a}{c} A_\mu \dot{\chi}_a^\mu \right) \delta_a \eta^{\mu\nu} - \left( m_a \dot{\chi}_a^\mu \dot{\chi}_a^\nu + \frac{q_a}{c} \chi_a^\mu A^\nu \right) \delta_a \right\}.$$ 
\( (76) \)

$\Sigma^{\mu\nu}$ and $D^{\sigma\mu}$ in Eq. (72) has been shown in Ref. \[23\], we rewritten here as

$$\Sigma_P^{\mu\nu} = - \frac{a^2}{4\pi} \left( \partial_\nu F^{\mu}_\rho \right) \left( \partial_\sigma F^{\sigma\rho} \right),$$
\( (77) \)

$$D_P^{\sigma\mu} = - F^{\mu\sigma} + a^2 \left( \partial_\mu \partial_\lambda F^{\lambda\sigma} - \partial_\sigma \partial_\lambda F^{\lambda\mu} \right).$$
\( (78) \)

Using Eqs. (74)-(78), the gauge invariant energy-momentum tensor is now become

$$- T_{\text{inv}}^{\mu\nu} = \sum_a \int_{a_1}^{a_2} \left( m_a \dot{\chi}_a^\mu \dot{\chi}_a^\nu \delta_a \right) \eta^{\mu\nu} + \frac{a^2}{4\pi} \left( F^{\mu\sigma} F_{\nu\sigma} + \frac{1}{4} \left( F_{\sigma\rho} F^{\sigma\rho} \right) \right)$$
\[79\]

\( \left( \delta_\mu \partial_\rho F^{\mu\rho} \right) \eta^{\mu\nu} + \frac{a^2}{8\pi} \left( \partial_\mu \partial_\rho F^{\mu\rho} \right) \eta^{\mu\nu} + \frac{a^2}{4\pi} \left( \partial_\mu \partial_\rho F^{\mu\rho} \right) - \frac{a^2}{4\pi} \left( \partial_\mu F^{\mu\rho} \right) \left( \partial_\sigma F^{\sigma\rho} \right) \}

VI. CONCLUSION AND DISCUSSION

In this work, we developed a general geometric (or manifestly covariant) field theory for classical relativistic particle-field systems and established the connections between general
symmetries and local conservation laws for the systems. To achieve this goal, we overcame two difficulties.

The first difficulty associated with the mass-shell constraint (see Eq. (3)). As a result, the standard Euler-Lagrange (EL) equation is reconstructed by the Euler-Lagrange-Barut (ELB) equation (see Eq. (17)). Besides, the use of proper time parameter makes the Lagrangian density (8) a function of the field $\psi$ and also a functional of the particle’s world line, which directly lead to the standard infinitesimal criterion a integro-differential equation rather than a differential one (see Eq. (26)). Furthermore, to satisfy the mass-shell condition, an extra criterion (32) is derived.

The second difficulty comes from the heterogeneous-manifolds that the particles and fields reside on. The fields are defined on the 4D space-time, while each particle’s word line is defined only on the 1D parameter space. As a consequence, the standard Noether’s procedure for deriving local conservation laws from symmetries is not applicable without modification. To overcome this difficulty, we developed a weak version of the Euler-Lagrange-Barut (ELB) equation for particles on the 4D space-time, which is rewritten here as

$$E_{\chi^a \mu} (\ell_a) = D_\nu H_\mu^\nu,$$

(80)

where the definition of $H_\mu^\nu$ can be easily read from Eq. (17). This non-vanishing term $H_\mu^\nu$ is emerged in the transformed infinitesimal criterion (43) as

$$\sum_a \int_{a_1}^{a_2} [q_a^\mu D_\nu H_\mu^\nu] d\tau_a.$$

(81)

If the characteristic $q_a^\mu$ is independent of space-time position $\chi$ and the field $\psi$, the derivative operator $D_\nu$ can be moved out from the integral, i.e., $\sum_a \int_{a_1}^{a_2} [q_a^\mu H_\mu^\nu] d\tau_a = D_\nu J_\nu^\nu$, the new current $J_\nu^\nu \equiv \sum_a \int_{a_1}^{a_2} [q_a^\mu H_\mu^\nu] d\tau_a$ is then induced.

Combing the weak ELB equation and infinitesimal criterion of the symmetry conditions, the general geometric conservation laws is systematically derived. Using general theory constructed here, we obtain the energy-momentum conservation laws for high-order relativistic electromagnetic systems by the space-time translation symmetry.

Interestingly, when the characteristic $q_a^\mu$ of the transformation (18) is related with space-time position $\chi$ or the field $\psi$, the derivative operator $D_\nu$ cannot be moved out from the integral. As such, equation (81) cannot be transformed into a divergence form, and a conservation law cannot be given even the system admitting a continuous symmetry. We have
not find a suitable symmetry which won't lead to a conservation law. Here, we recommend it as an open question and it would be an exciting research project to shed more light on that question.

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