Generalized Wavefunctions for Correlated Quantum Oscillators IV: Bosonic and Fermionic Gauge Fields.

S. Maxson

Department of Physics
University of Colorado Denver
Denver, Colorado 80217

Abstract

The Hamiltonian quantum dynamical structures in rigged Hilbert spaces used in the preceding three installments to represent correlated Hamiltonian dynamics on phase space are shown to possess a well-defined covering structure, which is demonstrated explicitly in terms of Clifford algebras. The unitary Clifford algebras are described here for the first time, and arise from the intersection of the orthogonal and common symplectic (Weyl) Clifford algebras of the complexification of the canonical phase space. The convergence of the exponential map is possible in available topologies in our constructions, but it does not converge without additional assumptions in general. Continuous dynamics exists only in semigroups. A well-defined spin geometry exists for the unitary Clifford algebras in the appropriate Witt basis, which also affords us both bosonic and fermionic representations through alternative topological completions of the same structure, and physically represent the stable states of the system. Unitary Clifford algebras can be used to define dynamical gauge bundles for arbitrary numbers of correlated (unified) fields. The generic dynamical gauge group for four pairs of canonical variables (four fields) is shown to be isomorphic to $U(4) \times U(4)$, with the spectrum effectively determined by $S[U(4) \times U(3) \times U(1)]$ due to the constraint of geodesic transport of the generators of the dynamical group. It is conjectured this is an unified version of $U(4)$ gauge gravity wherein particles correspond to islands of dynamical stability. An isomorphism is shown explicitly demonstrating the ability to associate these structures over four pairs of canonical variables with covariant structures in a non-trivial spacetime with $(+,−,−,−)$ local signature. The covariance of the identity of elementary particles follows, and is of dynamical origin, and by inference PCT is of dynamical origin also. An area of fundamental conflict is demonstrated between notions of noncompact Hamiltonian dynamics and general covariance, and a resolution is proposed for the present constructions. This includes prediction of the existence of chimeric bosons, whose quantum numbers are not covariant so they may appear to have a different identity to different observers.
1 Introduction

In this fourth and concluding installment concerning correlations of quantum oscillators, we relate what has gone in the preceding installments to an overall mathematical structure which is (reasonably) well defined mathematically. Additionally, we will show ways in which a correlated, or unified, hamiltonian quantum field theory over canonical variables can be associated to extended objects in a curved relativistic spacetime. When a field theoretic interpretation is adopted for our constructions, there are some interesting generic implications for quantum field theory and particle physics, and we will demonstrate a dynamical fiber bundle structure of physical interest existing within our hamiltonian dynamics formalism. We will use this as the basis for a gauge field theory, but will not, however, make any effort in this first description to present anything like a mature theory of particle physics, and will argue extensively from analogy to the Standard Model. Our methods are largely generic to any probabilistic representation of correlated hamiltonian dynamics, and it is interesting to see confirmation of the gauge group of the Standard Model in the generic gauge structure for three fields. Our generic methods predict the spectrum of the Standard Model exactly, although there are differences in interpretive detail (which are mathematically driven in the present case).

We likewise make generic predictions of the gauge group for four fields or any other number of fields, and, for four fields a fairly straightforward generalization of the Standard Model is indicated. In addition to the gauge related particle spectrum, from the mere existence of hyperbolic dynamics in nature, we make the prediction of additional chimeric bosons which are "fundamental" but not "elementary", and these may possibly offer an explanation for the mysteries of Dark Matter and Dark Energy. These are characterized as chimeric since the quantum numbers which characterize them lack covariant associations, unlike the quantum numbers associated with the gauge group we deduce for four fields. They may appear as different particles to different observers in consequence, and one mechanism for this is shown.

This is a general probabilistic correlated dynamics, and we could, for instance, use this same general formalism to describe the dynamics of a fluid with a given number of pairs of canonical variables, either as an ensemble or as a field, which

Email address: steven.maxson@ucdenver.edu (S. Maxson).
incorporates the presence of internal correlation such as might result in the
case of long range forces with self-consistent interactions. Whatever physical
problem is being represented in our formalism, the dynamical fiber bundle
structure gives us a representation of the islands of stability within a richer
and fuller description of dynamics which includes resonances, and which is
also significantly more complex mathematically. We further will demonstrate
the mathematical and physical importance of understanding that our con-
structions are in fact spin constructions, and show that it is extremely useful
to regard our constructions as associated with representation of new types of
Clifford algebra, the unitary Clifford algebras, existing only for spaces possess-
ing both symplectic and orthogonal structure, such as phase space. A Clifford
algebra provides the covering structure to make this work well behaved math-
ematically, and our generalized wave functions are a part of a representation
of this Clifford algebra of phase space and its complexification.

Spinors are associated with the Clifford algebras and their representations,
and there are many subtleties we shall gloss and take an optimistic view of in
the present forum. Spinor structures and spin geometry can be problematic,
but we have grounds for feeling secure with respect to the key elements (the
unitary Clifford algebras) we depend on. We would also suggest that the unit-
ary Clifford algebras provide a well defined nucleus which may be extended
to a full symplectic Clifford algebraic structures without arbitrary conditions
imposed to insure convergence of the exponential map (as is the case for
the symplectic formal series type of Clifford algebra [3]). There appear to be
symplectic Clifford algebras containing well defined semigroups of symplectic
(=dynamical) transformations, even though full groups do not seem available
without additional, probably arbitrary, assumptions. This clearly implies, as
the physical interpretation of the mathematics we were compelled to adopt,
that irreversibility is intrinsic to dynamics, a long held contention of the late
Professor Prigogine: it appears that, from the structure of phase space it-
self, one can deduce continuous dynamics only for the case when correlation
exists within the dynamical system, and that this continuous dynamics is ir-
reversible, in general being associated with semigroups only, unless one adds
essentially arbitrary mathematical assumptions in order to obtain a group
structure and invertibility. (The semigroups are obtained by topological com-
pletion of polynomials into the exponential map using the only topology which
is naturally present in our complexification of phase space itself, and this is
a complex hyperbolic topology, the hyperbolic Kobayashi semidistance topol-
yogy. I know of no inequivalent topology in which convergence of continuous
dynamical transformations is well defined without arbitrary assumptions—but
I may be insufficiently clever to have found such a topology, so will assert only
an implication and not a well proven result.)

As to this nucleus, we have a sufficiently well defined spin structure to possess
a well defined Yang-Mills (principal fiber or gauge bundle) structure associ-
ated to it. We will restrict ourselves to those dynamical aspects of the spin geometry issues of concern to field theorists, and leave broader dynamical concerns for another day. It is noteworthy, however, that formulating our theory in a form capable of reflecting a dynamical arrow of time (the semigroups of symplectic transformations provide a vehicle for expressing the boundary and initial conditions of an irreversible dynamical process) is mathematically sufficient to make our hamiltonian quantum dynamical theory well defined. Whether such a semigroup formulation is also necessary seems to touch on areas of great subtlety and complexity, and will addressed in some detail, but not entirely resolved.

As indicated in installment one [1], our approach admits a field theoretic interpretation. If “dynamics is the geometry of behavior”, we would add that the most suitable description of geometry seems to be in terms of geometric (Clifford) algebra. The present paper emphasizes Clifford algebra issues and field theoretic interpretation of Clifford algebra representations for various Clifford algebras associated with phase spaces. After dealing with some left over matters from the preceding installments, we will establish the Lie algebra valued connection constructively, demonstrate the existence of generalized Yang-Mills gauge structures, and show how the basic gauge group of the Electroweak Theory and Standard Model, and also the canonical gauge structure for four fields, emerge in a breathtaking and natural way, merely by looking at the canonical transformations of appropriate numbers of oscillators (identified in the usual manner with fields). The elementary particles are the islands of stability in this dynamical structure of the potentials of the gauge fields. The gauge groups are exact, however, and there is no “spontaneous symmetry breaking” associated with their definition [69]. The unitary gauge group emerging from this unitary Clifford algebra approach possesses both fermionic (even dimensional) and bosonic (odd dimensional) representations, corresponding to whether we view \( U(N) \) as a subgroup of an orthogonal group, lying in an orthogonal Clifford algebra (fermionic) or as part of a symplectic (semi-)group, lying in a symplectic (bosonic) Clifford algebra of some sort. It is therefore a mathematical error to mix bosons and fermions in our hamiltonian formalism. We suggest that there is some correspondence between at least some parts of the even dimensional (fermionic) and odd dimensional (bosonic) representations, since they do represent the same group. This identification is probably related to topological notions, but we will not attempt any detailed justification at present. We do wish, however, to clearly indicate that at the present such terms as “quark–gluon plasma” have a very muddy meaning mathematically. We will ignore all but the most superficial representation issues.

As to the unitary Clifford algebras, the choice between bosons and fermions is between alternative topological completions, and, seeing that we can never conduct a Cauchy sequence of measurements, the two alternative topological
completions should be physically indistinguishable—matrix elements should not change. Bosons are products of the symplectic geometry of phase space, while fermions arise when you think of phase space from the perspective of orthogonal geometry. Seemingly, this is where inescapable mathematical necessity has led us in our pursuit of a well defined unified (correlated) Hamiltonian quantum field theory, and is either physically relevant or it isn’t. In any event, the Hamiltonian and Lagrangian approaches to field theory may lead to significantly different end points [70].

There are a couple of key ingredients in our geometric structure that play essential roles in our construction. Spinors are usually frame dependent (this is the reason there is no spinor calculus analogous to the tensor calculus), and the unitary transformations will leave our real Witt frames invariant (much like the orthogonal transformations leave conventional frames invariant). The real Witt bases enable us to obtain a well defined (but frame dependent) differential geometry for our symplectic spinors from the well known spin geometry of orthogonal spinors (a special topic in Riemannian geometry [22].) There is a simple mathematical trick using standard theorems of topology for linear spaces which we invoke to obtain this result. Secondly, we have a weak symplectic form (see [2,3]), meaning that Darboux’s theorem does not apply, and our geometry can be other than locally Euclidean, permitting the existence of non-trivial local invariants such as curvature, which are prerequisite for a non-trivial gauge theory.

A brief exercise will demonstrate that analytic continuation of the traditional Hilbert space does not result in vectors possessing Bose-Fermi symmetries which are well defined. This follows because the energy spectrum for vectors belonging to that analytically continued space is not necessarily bounded from below. Let us consider energy eigenvectors $|a\rangle$ and $|b\rangle$ belonging to some space for which there is a well defined “vacuum” or minimum energy eigenvector, $|0\rangle$. Then there is some transformation $A$ such that $|a\rangle = A|0\rangle$ and some transformation $B$ such that $|b\rangle = B|0\rangle$. Without loss of generality we may regard $A$ and $B$ as esa, and it follows that

$$\langle a|b\rangle = \langle 0| \frac{AB + BA}{2} + \frac{AB - BA}{2} |0\rangle$$

The uniqueness of this decomposition into symmetric and antisymmetric parts depends on the existence of a unique fiducial vector, such as $|0\rangle$. When the energy spectrum is unbounded below, there is no such fiducial vector, and many similar decompositions can exist, with nothing to distinguish any particular one. This brief demonstration illustrates that the traditional form of the boson-fermion superselection rule does not apply to analytically continued systems, in which the energy spectrum is not bounded from below. However, for our multicomponent state vectors there is a somewhat more complicated situation than this naive calculation is relevant to, which we elaborate in de-
For the multicomponent spinor formulation developed in preceding install-
ments, especially installment two \[2\], and further specified below, any bilinear 
form on phase space must be either strictly symmetric or strictly antisymmet-
ric. (This is a characteristic of Clifford algebras in general.) This compels us 
to choose one or the other bilinear form for the construction of our Clifford 
 algebra, although there is a special basis for phase space compatible with both 
the ordinary orthogonal (symmetric) form and the (antisymmetric) symplectic 
form. In this special basis, the real Witt basis, we can simultaneously generate 
representations of either, enabling us to form the non-trivial intersection of the 
orthogonal and symplectic Clifford algebras of phase space. The unitary Clif-
ford algebra which results thus has a canonical basis in which one may alter-
atively consider physical aspects associated with the orthogonal perspective, 
such as fermionic representations of bulk matter by Dirac spinors in a space-
time with local signature \((+,−,−,−,−)\), or those aspects associated with the 
symplectic perspective, such as dynamics, forces, interactions, etc., associated 
with bosons, represented by symmetric spinors. We will refer to such choices of 
representation as a choice of perspective for our state vectors, and is in some 
sense dependent on the choice of dimension for the representation: even di-
 dimensional representations are fermionic and odd dimensional representations 
are bosonic. We can thus think of a system as a bunch of fermions (particles) 
or as a bunch of bosons( intermediaries of the forces–the dynamical entities), 
but must consider a particle as either fermionic “lumps of geometry” or as 
 bosonic “lumps of dynamical fields”. These perspectives are alternative ways 
of looking at one physical structure in alternative representations, according 
to our constructions of those representations of physical structures using a 
unitary Clifford algebra. In this view, the boson-fermion dichotomy is an arti-
fact of the representation chosen for the stable structures of the theory (such 
as particles or other stable structures, as in, e.g., stable circulation of fluids), 
and need not be an intrinsic property of the underlying structure being repre-
seated. See also Section \[3.1\] below, but note the discussion at the end of this 
section. The representation chosen will be associated, in turn, with a choice of 
topological completion of an algebraic set, and topology is not an experimen-
tal observable (you can never conduct a Cauchy sequence of measurements). 
The alternative perspectives, once adopted, may have different observables in 
the physical context consistent with the underlying principles of the perspective 
which has been adopted. Our use of the terms boson and fermion may not, in 
consequence, exactly correspond to the usual conventions of quantum theory, 
but they have mathematical precision.

The decomposition of equation \(1\) can be said to be unique in a unitary Clif-
ford algebra such as we construct in Section \[3\] in the sense that each of the two 
terms is non-trivial in one perspective only, each perspective being associated 
with cofactors over ideals based on one or the other of the alternative bilin-
ear forms which exist separately on the space. The two terms cannot mix to define a mixed bilinear form on our spin-vectors, which form a representation of phase space (as part of the representation of the Clifford algebra of phase space). In our constructions, bosons and fermions are associated with separate and distinct, unique bilinear forms, and each bilinear form defines a perspective, but the perspectives (e.g., representations) may not mix. Thus, you may speak of the fermionic properties of bulk matter or you may speak of bosonic forces and dynamical evolution, but you must change perspective between these two alternatives, and really cannot properly talk of both simultaneously without exceeding the bounds of mathematical propriety. To consider the electrodynamic interaction of two electrons, for instance, one must consider each electron as a “conglomeration of dynamical field stuff” in order to speak of the exchange of photons (other “conglomerations of dynamical field stuff”) between them. *Dynamics* is the exclusive jurisdiction of the perspective associated with the symplectic form and factorization of the tensor algebra of phase space over that bilinear form yields a Clifford algebra suitable only for the representation of bosons (and that Clifford algebra is properly represented exclusively by odd dimensional symmetric spinors). There is no superselection rule in our RHS spin formulation in the same sense as such a rule is applied to the conventional Hilbert space quantum theory. Rather, there is a selection between perspectives (Clifford algebras and their representations).

All of the stable or quasi-stable states arising through symplectic transforms in our hamiltonian dynamical formalism are extremely closely related formally to the coherent and squeezed states of the electromagnetic field so well known to quantum optics. Using photons and Foch space in the usual formalism of quantum optics, for instance, there is limited localizability—localizability is limited by the position-momentum uncertainty principle. The notion of position-momentum minimum uncertainty comes from considering the area of an ellipse in phase space using familiar notions of Euclidean geometry, i.e., stems from an orthogonal (symmetric) metric (bilinear form), and not from the symplectic (antisymmetric) bilinear form. Our stable and quasi-stable states are nothing more than squeezed (minimum uncertainty) states of numerous fields, just as photons may represent squeezed states of the single electromagnetic field. The transition in one’s thinking from the notion of bosonic squeezed state to notion of fermionic minimum uncertainty state illustrates the subtlety of the transition between perspectives in our constructions. The creation–annihilation operator formalism suggests that we are operating on the “particle” side of wave-particle duality, although in Section 3.1 we shall indicate that such matters as energy scale really govern, just as they do in quantum optics and atom optics (e.g., Compton vs. de Broglie wavelength, and so on).

In terms of equation (1), we would say that there are fermionic and bosonic representations of the operators $A$ and $B$, appropriate to the two alternative
perspectives, and one or the other of the two mixed operator terms will vanish in a given perspective. All is predicated on our choice of the Witt basis, yet another instance of the basis dependence of spinors. The scalar product in equation (1) will survive as either symmetric or skew depending on whether the orthogonal (symmetric) or symplectic (skew) form is chosen for factorization of our tensor algebra. The strictly alternative representations are also distinguishable from the thing being represented, since they are isomorphic to alternative topological completions of a set, and, in any event, the representation isomorphisms are not natural isomorphisms so there is some inequivalence aside from any topological issues. (See, e.g., Section 9 and reference [59] for examples of inequivalence.)

2 Necessity of Spinor Structures

In the following two subsections, we pursue the reasons for use of spinors in our representation of the correlated combinations of oscillators problem. The puzzling structure motivating this is the conjugacy of the (complex) symplectic transformations between $iY$ and $Z$ seen in installment two [2]. Of course, spinors figure in group representations, providing the “fundamental representations”, and there are some technical mathematical reasons that make them appealing (even mandatory), but there are strong physical reasons as well. They make our hamiltonian dynamical structure well defined.

It is natural to avail ourselves of the spinoplectic covering structure or perhaps even to use the representation of the full Clifford algebra itself. This has the further virtue of making our representation structure into Clifford modules, which are well known and well studied [80].

Although we may have uncertainties about the best way to interpret stability implications of the conjugacy of $iY$ and $Z$ seen in installment two [2], at least there is a covering structure in which the conjugacy is well defined, without regard to the appearance of an apparently undefined inverse semigroup transformation in it. This is because we are working with spinors: our spaces of states possess an orientation by virtue of their symplectic structure (as do all spaces representing quantized systems according to accepted notions of geometric quantization) and $exp$ is holomorphic for us, so the first two Stieffel-Whitney classes vanish, making our generalized spaces of quantum states spin spaces by construction. To insist that our group representations be UIR’s would be a grievous mathematical error. Our representation spaces $\Phi_{sp(4,\mathbb{R})^C\pm}$ (and their function space realizations) possess a complex symplectic structure (since their automorphism group is $Sp(4,\mathbb{R})^C$). Spinors are ideals of Clifford algebras, so we conclude that our representation space(s) is part of the representation of some sort of a symplectic Clifford algebra [4].
The spinors physicists are most familiar with arise in the representation of orthogonal Clifford algebras, such as the Dirac spinors. Typical spinors of physics are associated with a Clifford algebra for some space $V$ which possesses an orthogonal structure (symmetric bilinear form, elliptic scalar product, etc.), denoted $Cl_O(V)$. This universal Clifford algebra for $V$ contains a group, the (orthogonal) Clifford group $G_O(V)$. The (orthogonal) Clifford group contains a spin group, $Spin(V)$, which in turn provides a double cover for the group of orthogonal transformations on $V$, $O(V)$. If we represent an orthogonal transformation on $V$ belonging to $Cl_O(V)$ by an exponential, $(e^{iX\theta/2})_{O(V)} \in O(V)$, then the orthogonal rotation of a vector $A \in V \subset Cl_O(V)$ about the direction given by the vector $X$ is represented as the conjugation

$$(e^{iX\theta/2})_{O(V)} A (e^{-iX\theta/2})_{O(V)} .$$

(2)

This exact same rotation is represented in the $Spin(V) \subset Cl_O(V)$ covering structure as

$$(e^{iX\theta})_{Spin(V)} A ,$$

(3)

i.e., as an operation from the left to right, without conjugation. In other words, a semigroup orthogonal rotation which is performed by conjugation (and therefore has only a conditional local meaning, at best) determines a well defined spin transformation which acts from the left only, therefore defining a unique geometric structure having a global meaning on our space of (generalized) states. We will show infra that the unitary transformations form what is effectively a group substructure within both the orthogonal and symplectic semigroups of transformations. The group structure of these unitary transformations are shown below in many places and in many ways to be the mathematical key to the well defined mathematical structures in our constructions.

There is a similar hierarchy of groups in the case of the symplectic Clifford algebras [5], namely a symplectic Clifford group, $G_S(V)$, covering the spinoplectic (or toroplectic) group $Sp_2(V)$, which is a non-trivial double cover of the symplectic group $Sp(2n,V)$, where $V$ is assumed to be a real space and $dim(V) = 2n$. There is also a metaplectic group, and higher covering spinoplectic groups $Sp_q(V)$, $q > 2$. There are subtleties with the symplectic Clifford algebras which we will overlook for the moment, since the common symplectic Clifford algebra (Weyl algebra) is a polynomial algebra, so that some additional structure must be added in order that the exponential map be defined. There are a number of alternatives for this extra structure. See [5] and also Section 3 below.

We will describe the construction of a symplectic Clifford algebra in the following section which is distinguished from the symplectic Clifford algebras in [5] (although it may contain some of those algebras) which is better adapted to our purposes and in which the exponential map is well defined, without arbi-
trary assumptions, although our description will not be exhaustively complete. Thus, our conjugation is by an element of the semigroup of symplectic transformations, which for the unitary sub-semigroup have a covering structure of orbits of spinoplectic transformations acting from the left only, and whose semigroup meaning is not qualified or restricted in any way once identified (locally but uniquely) with the appropriate spinoplectic covering structure. As to the unitary symplectic transformations, our (locally defined) conjugation fixes a group covering structure which is global in some sense, analogous to the orthogonal case, but full invertibility in the sense of a full group of transformations of the symplectic family of groups need not extend from the unitary transformations to the full group of symplectic transformations. There may be group covering structures to the symplectic group (or there may not be – we do not inquire further into this issue here.) We still associate dynamics with the symplectic transformations, which we are able to define in general only in semigroup form, and not with the spinoplectic and other covering structures, which may or may not be full group structures. It is only for the unitary transformations, a proper subset of the symplectic transformations, that we will have assurances of a full and proper group structure (up to sets of measure zero). Topological obstructions exist because the geodesics generated by the full symplectic Lie algebra include the orbits of hyperbolic generators, for whom inverses are only locally and infinitesimally uniquely defined, which is why macroscopically there are semigroups and not full groups of symplectic transformations.

We have a very strong motivation for working with Clifford algebras and their representations: we know instantly that we have a covering structure for which our constructions are well defined and have some sense of global meaning. Below, we establish only that the unitary transformations form an invertible subgroup structure within the semigroup of symplectic transformations—we will view the unitary transformations as forming the nucleus of larger families of transformations, and in our immediate concerns it is only the macroscopic invertibility in fact of the unitary transformations which is necessary for our present constructions to be well defined. One of our implicit lessons is that the symplectic Clifford constructions do not generally result in analogues to equation (2) and equation (3), and that the invertability of the unitary transformations within the symplectic Clifford algebras may be topologically dependent on their invertability in the orthogonal Clifford algebra. There is an important result in the next section that the general symplectic transformations are topologically obstructed from invertability and are inherently semigroup in nature, with the physical consequence that any sufficiently general hamiltonian dynamics is not macroscopically invertable, i.e., for general hamiltonian dynamics, the transition from equation (2) to equation (3) must be regarded as strictly local (infinitesimal).
3 Spinors and Clifford algebras

This brings us to an interesting juncture, which we illustrate with the simple case of the phase space over a single pair of canonical variables, which we will call $p$ and $q$. Our phase space, which we denote $T^x \mathbb{R}$, has a basis $e_p, e_q$. If we perform an analytic continuation of functions on $T^x \mathbb{R}$, we will get a space of functions over an “analytically continued” phase space upon which we construct both an orthogonal and a complex symplectic structure (which we take in its real or symplectic form), and which we may label the complex extension of the underlying phase space $\tilde{T^x \mathbb{R}} \equiv T^x \mathbb{R} \oplus i \circ T^x \mathbb{R}$. It is no great assumption to regard both $T^x \mathbb{R}$ and $\tilde{T^x \mathbb{R}}$ as inner product spaces (with alternative bilinear forms defining alternative products.) Below, we define a basis for $\tilde{T^x \mathbb{R}}$ which is simultaneously orthogonal and symplectic (compatible with both orthogonal and symplectic forms in their standard form), just as $e_q$ and $e_p$ provide such a basis for $T^x \mathbb{R}$. That basis is closely related to the creation and destruction operators or $\pm$ the unit imaginary times a creation or destruction operator (borrowing directly from [5], page 247):

Define \( a = \frac{(e_q + ie_p)}{\sqrt{2}} \) and \( a^\dagger = \frac{(e_q - ie_p)}{\sqrt{2}} \) (4)

so that the basis we choose for $\tilde{T^x \mathbb{R}}$ is

\[
\epsilon_1 = i a \sqrt{2} \quad \epsilon_1^* = a^\dagger \sqrt{2} \\
\epsilon_2 = a \sqrt{2} \quad \epsilon_2^* = ia^\dagger \sqrt{2}.
\]

(5)

This is essentially just an orthonormal analogue of the creation and destruction operators, stated in phase space rather than on our function space representations. Because they can identified with particular values of canonical position and canonical momentum, $a$ and $a^\dagger$ obey the familiar commutation relations of the creation and destruction operators ($A$ and $A^\dagger$ of [2]). In phase space, we have the Poisson bracket for $a$ and $a^\dagger$ corresponding to the commutator of $A$ and $A^\dagger$ on our function spaces; this is analogous to the Dirac canonical quantization $\{p, q\} \rightarrow \frac{1}{i\hbar}[P, Q]$, except with analogues to the creation and destruction operators on phase space, there is no $\hbar$ associated with the canonical quantization, suggesting something of a more fundamental geometric nature at work. See also the Appendix.

This means we have broken $\tilde{T^x \mathbb{R}}$ down into transverse hyperbolic spaces with bases $\{\epsilon_\alpha\}$ and $\{\epsilon_{\alpha^*}\}, \alpha = 1, 2$, respectively. These satisfy the relation

\[
\begin{align*}
\epsilon_\alpha, \epsilon_\beta) &= (\epsilon_{\alpha^*}, \epsilon_{\beta^*}) = 0 \\
\omega(\epsilon_\alpha, \epsilon_\beta) &= \omega(\epsilon_{\alpha^*}, \epsilon_{\beta^*}) = 0 \\
\omega(\epsilon_\alpha, \epsilon_{\beta^*}) &= \delta_{\alpha\beta}
\end{align*}
\]

(6)
where $(\cdot, \cdot)$ is the symmetric form (e.g., associated with the $\{\cdot, \cdot\}_+$ symmetric bracket on phase space and anticommutator on our representation function spaces, and with the symmetric scalar product), and where $\omega$ is the skew symmetric form (e.g., associated with the familiar $\{\cdot, \cdot\}_-$ Poisson bracket on phase space and the commutator on our representation function spaces, and with the skew symmetric scalar product). Geometrically, $\epsilon_1 \perp \epsilon_2$ and $\epsilon_\alpha \perp \epsilon_{\alpha*}$, because of the orthogonality of real and pure imaginary components. The other relations are straightforward. The $\epsilon_\alpha$ and $\epsilon_{\alpha*}$ thus form a real Witt basis for the metric of $\widetilde{T^*\mathbb{R}}$, the complexification of the phase space $T^*\mathbb{R}$. Generalization to higher dimensions is straightforward.

Note that to implement the constructions of [2], it is necessary to use the commutative real algebra $\mathbb{C}(1,i)$ for the ring of scalars of our Clifford algebras, rather than the field $\mathbb{C}$, for the reasons given in [2]. We will also adopt the notion of involution given there for our adjoint transformations. This is what [13] calls a $L^\alpha$ Clifford algebra, $L$ being a commutative algebra and $\alpha$ being an involution.

Of related importance to us is the notion of “correlation”. The scalar product $\langle \psi_{\text{out}} | \phi_{\text{in}} \rangle$ establishes the correlation between the prepared state $\phi_{\text{in}}$ and the observed effect $\psi_{\text{out}}$. The matrix elements of quantum theory are representations of correlations of this sort. With our multicomponent spin vectors there are more exotic correlations which it is possible to calculate. If on our space of states there is a symmetric (orthogonal) form

$$Q = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

where $I$ is the appropriate unit operator, and if there is a symplectic form

$$F = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

in addition to the scalar product $\langle \psi | \phi \rangle$ we can form the symmetric (bosonic) correlation $\langle Q\psi | \phi \rangle$ and the skew (fermionic) correlation $\langle F\psi | \phi \rangle$ [1]. The unit imaginary “$i$” is associated with a skew symmetric form as well, providing a further source of skew correlation. The significance of the unit imaginary “$i$” is that it is associated in our constructions with a complex hyperbolic (e.g., Lobachevsky) geometry, rather than some real hyperbolic structure—both are associated with a symplectic form [2]. The unit imaginary is a “correlation map” [13].

The point of this preliminary bit of algebra is that in our construction in part two of this series [2], we used the creation and destruction operators to form
polynomials acting as generators of (geodesic) transformations. Their role as vectors there is fully equivalent to their use within this paper hereinabove: this $a$, $ia$, $a^\dagger$ and $ia^\dagger$ are in nearly all regards identifiable with that $A$ and $A^\dagger$, etc., being an orthonormal (real Witt) basis associated with the later by taking special values of $p$ and $q$ in the phase space. In particular, they allow us to define an orthonormal basis according to the hyperbolic scalar product defined by the symplectic form. Here they provide a basis for the base space of a symplectic Clifford algebra and at the same time they can provide a basis for the base space of an orthogonal Clifford algebra for the same space (the complexification of phase space). Both orthogonal and symplectic Clifford algebras of a single phase space exist simultaneously in this basis! In [2], the creation and destruction operators were used to build the generators of infinitesimal translations (vectors) on the space used for the representation of hamiltonian dynamics. The roles of the $a$ and $a^\dagger$ and the $A$ and $A^\dagger$ are comparable on their respective spaces. In particular, $A$ and $A^\dagger$ may be substituted for $a$ and $a^\dagger$ in equation 5 and the relation equation 6 still holds: creation and annihilation operators provide a representation of the real Witt basis of the complexification of phase space.

The orthogonal Clifford algebras can be formally defined using the tensor algebra of a space. Thus, given a (real) space $E$ space with a symmetric bilinear (quadratic) form $Q$ defined on it, the orthogonal Clifford algebra $Cl_O(E)$ is defined as the quotient of the tensor algebra $\times(E)$ by the two sided ideal $\mathcal{N}(Q)$ generated by elements of the form [5], p. 37:

$$x \times x - Q(x), \quad x \in E \subset \times(E). \quad (7)$$

Due to invocation of the tensor algebra, existence is a fairly trivial issue. The orthogonal Clifford algebras contain the orthogonal Clifford group, $\mathcal{G}_O$, which contains the familiar pin, spin and orthogonal groups as subgroups. The orthogonal Clifford algebras are associated with the representations of fermions. (The familiar Dirac algebra is the even subalgebra of the orthogonal Clifford algebra for Minkowski space with signature $(+, -, -, -)$, [15], [14], pp 67, 75.)

Similarly, for $E$ an $n$-dimensional real vector space, and $F$ an antisymmetric bilinear form, we define the common symplectic Clifford algebra $Cl_S(E)$ by the quotient of the tensor algebra $\times(E)$ by the two-sided ideal $\mathcal{N}(F)$ generated by the elements [5], p. 233,

$$x \times y - y \times x - F(x, y), \quad x, y \in E. \quad (8)$$

The common symplectic Clifford algebras (Weyl algebras) are essentially polynomial algebras, for which the exponential map is not closed, and comprise subsets of a number of other symplectic Clifford algebras. Of particular interest to us are the formal symplectic Clifford algebras over $K((\hbar))$ containing
a symplectic Clifford group, $G_S$, which contains the torolectic (metaplectic), spinoplectic and familiar symplectic groups all as subgroups. We identify the ring $K$ as the commutative real algebra $\mathbb{C}(1, i)$, and $\hbar$ is Planck’s constant. (See [5] for details.) The symplectic Clifford algebras are associated with the representation of bosons.

If $Cl_O(E) = \times(E)/\mathcal{N}(Q)$ and $Cl_S(E) = \times(E)/\mathcal{N}(F)$, are both defined for a space $E$, there are obvious generalizations, such as, in particular,

$$Cl_U(E) = \times(E)/\{\mathcal{N}(Q) \cup \mathcal{N}(F)\} \quad .$$

(9)

It is straightforward that

$$Cl_U(E) = Cl_O(E) \cap Cl_S(E) \neq \emptyset \quad .$$

(10)

$Cl_U(E)$ contains the space $E$ and $I_E$, so is non-trivial. We will call it the unitary Clifford algebra, and note that its existence depends on the space $E$ having a real Witt basis (such as phase space or its complexification) and both symmetric and antisymmetric forms (such as phase space or its complexification). At this point, we must regard it as a set in the tensor algebra having algebraic properties, although not necessarily a fully endowed “algebra”, and in particular the exponential map is not closed on it (since this is the case for the Weyl algebras).

The orthogonal Clifford algebras have an exponential map which is complete as a by product of their ring of scalars being the reals, effectively using the same norm topology as $\mathbb{R}^n$. See [5]. Choosing this same separating norm topology for the completion of sequences formed of elements from $Cl_U(E)$, we can form what we will temporarily call the orthogonal completion (o-completion) of the unitary Clifford algebra, which we denote $Cl_{U-O}(E)$. It follows that $Cl_{U-O}(E) \subset Cl_O(E)$.

The common symplectic Clifford algebras (Weyl algebras) do not contain Lie groups since they are basically polynomial algebras and the exponential map is not complete in them. Some form of completion may be imposed on them in order to obtain an augmented symplectic Clifford algebra in which the exponential map converges. A linear topological space is an algebra plus a scalar product, so it is no great additional assumption if we treat $Cl_{U-O}(E)$ as a topological vector space complete in the o-topology as indicated above. We may freely regard our base spaces–phase space and its complexification using the commutative ring $\mathbb{C}(1, i)$–as scalar product spaces. Since the space $Cl_{U-O}(E) = Cl_{U-O}(\hat{T} \times \mathbb{R}^n)$ is now a linear topological space, it has a neighborhood of 0 of sets complete complete in $Cl_{U-O}(\hat{T} \times \mathbb{R}^n)$ in any finer topology than the o-topology previously chosen for the completion of $Cl_S(\hat{T} \times \mathbb{R}^n) \cap Cl_U(\hat{T} \times \mathbb{R}^n)$, and we obtain thereby a complete linear topological space in the finer topology.
Thereby, the Kobayashi semidistance on the complex hyperbolic space $\tilde{T} \times \mathbb{R}^n$, $n \geq 2$, provides a (effectively seminorm or weak) topology for $Cl_U(T^x \mathbb{R}^n)$ [20]. The notion of geodesic is well defined in this topology [20], which we will call the s-topology. Convergence of symplectic (=dynamical) transformations in semigroups is thus a sufficient condition for us to talk about bosons, e.g., a dynamical arrow of time is a sufficient condition for us to talk about bosonic fields continuously evolving in this construction.

By these standard theorems, $Cl_{U-O}(\tilde{T} \times \mathbb{R}^n)$ will also be complete in the finer seminorm s-topology [21]. Local convexity, and so on, easily follow from this. In the sequel, we will work with the algebraic set $Cl_U(T^x \mathbb{R}^n)$ as a completed linear topological space $Cl_{U-O}(\tilde{T} \times \mathbb{R}^n)$, with alternative normed o-topology and seminorm s-topology completions[74]. We will distinguish the individual completions, as necessary, by indicating the form of completion thus: $Cl_{U-O}(T^x \mathbb{R}^n)$ and $Cl_{U-S}(T^x \mathbb{R}^n)$.

Putting matters slightly differently, we can use the real Witt basis defined above as the basis for a unitary Clifford algebra, the intersection of the orthogonal and symplectic Clifford algebras of the complexification of phase space:

$$Cl_U(\mathbb{T}^x \mathbb{R}^n) = Cl_O(\mathbb{T}^x \mathbb{R}^n) \cap Cl_S(\mathbb{T}^x \mathbb{R}^n), \quad n \geq 2 \quad . \quad (11)$$

From the perspective of o-topology associated with the orthogonal form, we may identify $Cl_{U-O}(T^x \mathbb{R}^n)$ with representations of fermionic particles. From the perspective of s-topology associated with the symplectic form, we may identify this same $Cl_{U-S}(T^x \mathbb{R}^n)$ with the representation of correlated hamiltonian dynamics of bosonic fields. The complexification of phase space, $\tilde{T} \times \mathbb{R}^n$, $n \geq 2$, is used for the representation of correlated dynamics over $T^x \mathbb{R}^n$. We are dealing with the algebraic treatment of correlated dynamics from alternative perspectives by using the vehicle of Clifford algebras and their representations.

According to this prescription, all the spaces in our Gel’fand triplets of spaces in the Gadella diagrams are built by using alternative topological completions of Clifford algebra representations. The Kobayashi semidistance adapted to provide a seminorm above does not produce a countable family of seminorms, such as involved in the construction of our representation spaces in [1], although the topological completions obtained through its use are locally convex spaces with a nuclear part of our unitary Clifford algebras. Note that if phase space is taken as a scalar product space, it is then straightforward to define a rigged Hilbert space over the locally convex nuclear space $Cl_{U-S}(T^x \mathbb{R}^n)$ as follows,:

$$\overline{Cl}_{U-S}(T^x \mathbb{R}^n) \subset Cl_{U-O}(T^x \mathbb{R}^n) \cong Cl_{U-O}^x(\tilde{T} \times \mathbb{R}^n) \subset Cl_{U-S}^x(\tilde{T} \times \mathbb{R}^n) \quad , \quad (12)$$

where the over bar relates to the nuclear locally convex parts only.
3.1 A tentative physical view

In the preceding section, we indicated how the difference between fermions and bosons in one of choice of topological completion, with the bosons being associated with a finer topology, i.e., a topology which separates more points than does the topology used to construct the fermions. There is a natural way of translating this into physics. When the Compton and de Broglie wavelengths are comparable, the physical phenomena are intrinsically quantum and the wave nature is in evidence. It seems natural to associate the Compton wavelength with our finer topology and bosons of $CL_{U-S}(\mathbb{T} \times \mathbb{R}^n)$. We would infer that in the coarser topology of $CL_{U-O}(\mathbb{T} \mathbb{R}^n)$, the Compton wavelength may be smaller than the de Broglie wavelength and the particle nature is in evidence. This is of course reasoning by analogy, and not to be taken as law, but can explain such phenomena as why one never observes a free quark, for instance: there is no such thing (yet) as a non-relativistic free quark, just as there is no such thing as a non-relativistic photon. Thus, it should not be taken as canon law that topology has no observable consequences, in the sense that a choice of topology may reflect an assignment of relative mass and energy scales, etc., to the phenomena undergoing mathematical description. One has to adapt the mode of description to the phenomena being studied, and this may mean choosing a particular topology (perspective) for the representation. (Possible distinctions between bosonic and fermionic representations are given in [59], and we would conjecture this has observational consequences.)

3.2 Why focus on the unitary Clifford algebra?

Our ultimate goal is to define a set of structures in which every space in the associated Gadella diagrams is a spin space, since we have already seen in installment two [2] the presence of multicomponent vectors (which in fact satisfy technical requirements for being spinors). The unitary Clifford algebras are significant because they contain the relevant unitary group (or semigroup) as transformations groups, but they play a special role for us because they contain the relevant special unitary group (or semigroup). The unitary groups preserve the Witt bases which are the foundation of our construction. Because we have complex simple Lie groups, and not merely semisimple Lie groups, we are assured of unique spin structures [22].

We will focus on the special unitary groups, rather than the full unitary groups, because $SU(N)$ is simply connected, and spin manifolds are manifolds with simply connected structure groups, relevant to having well defined spinor bundles associated to the $SU(N)$ generated dynamical flow structure which we will use to set up a gauge theory in the following section. Also, $SU(N)$ has a bi-
invariant Riemannian (symmetric) metric, and this metric is identifiable with
harmonic forms, meaning that the group (and associated flows enerated by
it) will have a well defined harmonic structure, with kernels and propagators,
etc., well defined (initially in the o-topology). Likewise, one parameter sub-
groups are geodesic. $Cl_{U,-O}(\tilde{T} \times \mathbb{R}^n)$ (and its function space representations) is
thereby a very well behaved linear space, and associated to it are well defined
flows and harmonic structure on $\tilde{T} \times \mathbb{R}^n$. We thus have great confidence that
our mathematics is well defined when working with special unitary groups,
although it may be possible that such well-behavedness could be extended to
the full unitary groups.

The spin geometry for $Cl_O(E)$ is a specialized branch of Riemannian geom-
etry [22]. For $Cl_U(\tilde{T} \times \mathbb{R}^n)$, when the unitary Clifford algebra is completed in
the category of topological linear spaces using a seminorm topology, the real
Witt basis gives us a vehicle to obtain a well defined spin geometry for the
simply connected (locally convex, nuclear) completion $\overline{Cl}_{U-S}(\tilde{T} \times \mathbb{R}^n)$ as fol-
lows. The well defined spin structure on $Cl_{U-O}(\tilde{T} \times \mathbb{R}^n)$ derives from the spin
structure on $Cl_O(\tilde{T} \times \mathbb{R}^n)$. The first and second Stieffel-Whitney classes are
trivial on $Cl_{U-O}(\tilde{T} \times \mathbb{R}^n)$, and are homotopy invariants (characteristic classes).
(For proper homotopy theory, we must use groups and not semigroups. See
Section 4.3 below as to the semigroup $SU(N)\pm$ having no obstructions on
any set of positive measure to extrapolation to a full group structure.) It fol-
lows that this spin structure survives the change to a finer topology so that
$\overline{Cl}_{U-S}(\tilde{T} \times \mathbb{R}^n)$ also has a well defined spin structure (and harmonic structure,
etc.).

Because $exp$ maps dense sets to dense sets (topological notions!), even for
non-compact (e.g., hyperbolic) generators, the $\overline{Cl}_{U-S}(\tilde{T} \times \mathbb{R}^n)$ can serve as a
nucleus for a covering space of the common symplectic Clifford algebra or
Weyl algebra $Cl_S(\tilde{T} \times \mathbb{R}^n)$. By this device, we obtain a complete nuclear (pos-
sibly locally convex) linear topological space, the nuclear symplectic Clifford
algebra $\overline{Cl}_S(\tilde{T} \times \mathbb{R}^n)$ for which the exponential map is complete. This suggests
that there is in some qualified sense a well defined spin geometry on all or at
least parts of our nuclear sympletic Clifford algebra, $\overline{Cl}_S(\tilde{T} \times \mathbb{R}^n)$, and that the
exponential map of the Lie algebra of the symplectic semigroups $Sp(2n,\mathbb{R})_\pm$
is holomorphic with respect to our seminorm completion, and as extended in
this seminorm topology $\overline{Cl}_S(\tilde{T} \times \mathbb{R}^n)$ is also a Clifford algebra containing the
common symplectic Clifford algebra (Weyl algebra) as a sub-algabra. (Spin
geometry as currently understood is dependent on a simple connected struc-
ture group–the qualified sense of a well defined spin geometry as referred to
preceding may require an extension of spin geometry as currently accepted.
The topological obstructions that leave us with semigroups rather than groups
do not obstruct the exponential map. The nuclear symplectic Clifford algebras are locally path connected, but may not be simply connected. We are not concerned with these and related issues herein, but there are obvious issues that should be explored.)

As a noteworthy aside, since our Clifford algebras above are completed in the category of linear topological spaces, they also possess their own Clifford algebras. Thus, we have the possibility of constructing towers of algebras, and these have properties of interest also. In fact, as to the connected part, since we have a nuclear locally convex topology, these Clifford algebras may serve as the base space of a Gel’fand triplet, e.g., the $\Phi$ of the RHS $\Phi \subset \mathcal{H} \subset \Phi^\times$; for the abstract base space $\Phi_{sp(2N,|R)}^c$ of the Gel’fand triplets of [11] and [2], we could take $\overline{\text{Cl}}_{U-S}(\widetilde{T^xR^n})$, (or possibly $\overline{\text{Cl}}_S(T^xR^n)$). There are thus towers of Gel’fand triplets over the orthogonal and symplectic unitary Clifford algebras. The unitary Clifford algebras (both o-completion and s-completion) provide finite dimensional representation spaces of the (e.g., compact) unitary group and also provide a tower of (finite dimensional) Hilbert spaces. These infinite tower structures bring to mind the scale of Hilbert spaces, $\cdots \mathcal{H}_n \subset \mathcal{H}_{n+1} \subset \cdots \subset \mathcal{H}_\infty$, of [30]. This scale may be relaxed somewhat, and the mathematics is still sufficient to do interesting post-Hilbert space physics [31], and indeed the Schwartz space $\mathcal{S}$ may be obtained from taking the intersection of the spaces $\mathcal{H}_n = D(R^n)$. Our constructions seem to operate in the convergence of a lot of well defined mathematics with post-Hilbert space physics.

As to the unitary Clifford algebras, it is reasonably straightforward to obtain fiber bundles with unitary structure groups, much in the manner typical for frame bundles obtained from the base space and orthogonal transformations of an orthogonal Clifford algebra. In similarly straightforward and well known manner, one may obtain bosonic and fermionic principal fiber bundles with special unitary groups as the structure group. (We will discuss these and their physical relevance below.) There is also a suggestion of a type of “dynamical principal fiber bundle” with semigroups of symplectic transforms as structure (semi-)group [78]. We will provide a description of the spinor bundle structures of immediate relevance in Section 4.

Note that in general the Hamiltonian does not commute with the full symplectic Lie algebra, so that energy is not a constant of all possible dynamical evolutions (i.e., it is possible to represent open systems), and the energy eigenstates do not provide an irrep of the group–typical for spinor representations of groups. In order to include complex spectra in a mathematically well defined formalism, we have been led by mathematical necessity to representations which are neither unitary (they are “dynamical”, or more general) nor irreducible (they are spin)!
There are also other possible implications of defining the unitary and extended symplectic Clifford algebras as we have. The orthogonal Clifford algebras are associated with commutative geometry [60,62]. Because the symplectic Clifford algebras associated with the skew symmetric symplectic form rather than the symmetric orthogonal form, it is possible (and we will so conjecture) that our s-topology completion of the unitary Clifford algebra and the extended Clifford algebra obtained from it are associated with some type of noncommutative geometry. The Clifford algebras (all types) are \( \mathbb{Z}_2 \) graded and thus are superalgebras; when topologically completed as spaces they are superspaces as well. Although beyond the scope of these present inquiries, we will conjecture that most of the machinery of noncommutative geometry, superalgebras and superspaces, Hopf algebras, etc., (but not SUSY) is fairly close to hand even though not presently revealed. If these speculations are true, the unitary Clifford algebras possess both commutative and noncommutative geometric structures, depending on the choice of perspective (choice of bilinear form and topological completion). Another instance of the unitary Clifford algebras seeming to be a regime in which a lot of mathematical machinery is exceptionally well behaved, connecting a lot of disparate methodologies by having them defined over the same sets.

The spinor discussions in Section 2 refer to invertibility of what are nominally semigroups in the context of a covering structure which is spin. In the context of the orthogonal transformations, the well known spin groups provide the simply connected covering structure for obtaining equation (3) from equation (2), even in the case of semigroups of orthogonal transformations—simple connectedness is the key. The spinoplectic groups provide the analogous simply connected covering structure for the semigroups of symplectic transformations [5]. Thus, even though our use of a seminorm topology for the complex symplectic Clifford algebras formally results in semigroups of transformations, there are no obstructions on sets of positive measure to our extrapolating simply connected sub-semigroups of the unitary semigroup into full group structures—we can convert special unitary sub-semigroups such as \( SU(4)_\pm \) semigroups into full groups. These special unitary (effective) groups may be thought of as subgroups of symplectic and spinoplectic groups (in yet another topology!) We therefore conclude that the unitary sub-semigroups of our extended symplectic Clifford algebras effectively provide a spin representation of the special unitary group (and possibly the unitary group) within both s-topology and o-topology completions of the algebraic set \( Cl_U(T^*R^n) \).

3.3 Physical consequences

We will ultimately adopt a gauge field interpretation for these constructions, and this approach has some interesting physical consequences. Thus, dynam-
ics should be mediated by bosons, as represented by the spinors which in turn belong to a representation of some form of a symplectic Clifford algebra. On the other hand, bulk matter (fermions) should be represented by spinors representing an orthogonal Clifford algebra. Because there are more generators for $Sp(2n, \mathbb{R})$ than for $U(n)$, there is the formal possibility there could be bosons (intermediaries for dynamical forces) with no direct coupling to bulk matter properties. Likewise, we infer there are aspects of bulk matter not immediately associated with dynamics—i.e., apparently the gravitational force does not depend on the kind of bulk matter, but on the quantity of mass only. The true quantum geometrodynamics is contained only in the intersection of geometry and dynamics, the unitary Clifford algebra. We are working in a formal system in which there is only a limited overlap in which we can concurrently talk about all of the issues which are important to us. We are constrained to two separate perspectives, dynamics or geometry, which do not completely overlap. We must choose one or the other perspective exclusively when we choose to speak carefully, since there is no mathematically respectable way of speaking from both perspectives at once. There is, however, a domain of strong correspondences in which a single structure (an abstract spinor) may have alternative fermionic (even dimensional skew symmetric matrix) and bosonic (odd dimensional symmetric matrix) representations. A reminder that the representative is not necessarily the thing itself, and that isomorphism may not mean equivalence in all senses.

Our abstract unitary Clifford algebra possesses both symmetric and skew symmetric representations, corresponding to the bosonic and fermionic perspectives. In our carefully constructed mathematical structures, the notions of boson and fermion correspond to field (e.g., wave) and particle perspectives, respectively, but they no longer retain all of their traditional meaning (which we would argue arises from working in a mathematical formalism incompatible with resonances). The presence of composite bosons made up of fermions and recent developments in atom optics give examples of why the topological wave–particle assignments should probably not be taken in any absolute sense. There is nothing which prohibits bosonic and fermionic representations of the same thing, at least on occasion.

4 Spin Bundle Structures

4.1 Non-trivial dynamics

At this juncture, let us recapitulate the road to the mathematically well-defined covering structure for our hamiltonian quantum field theory, incorporating resonances and other coherence structures, and which is treated as a
form of dynamical system. We based this on a probability rather than point particle localization description of a dynamical system in phase space (e.g., over canonical coordinates). We tacitly assume that there is a lot of freedom in the dynamical system and also that we have some uncertainty in our specification of the initial state. A probabilistic field theory is the result. We added the notion of correlation maps (injective embeddings in the dual—which are related mathematically to conjugation and associated notions of coadjoint orbits, and related dynamically to momentum maps). For real symplectic (=dynamical) correlations of a simple two component system, we found that the system was either stationary or exponentially decays to some equilibrium configuration. There are more elaborate correlations—complex symplectic (=dynamical) correlations—which make more complex behavior possible, with complex spectra and possible oscillatory time evolution or damped oscillations occuring in the dynamical time evolution of the probability amplitudes. (E.g., such familiar phenomena as diffraction patterns, etc., are evidence of complex dynamical correlations.)

If we conjecture well behaved algebraic and topological properties for the constructions of the preceding installments, with proper algebras and topological linear spaces, we are led to multicomponent representations, our function spaces representing the probability amplitudes are \( L^2 \) spaces, and this coupled with the complex spectra forces us into a variant of the rigged Hilbert space formalism such as was outlined in [1] and [2]. The lesson of this fourth installment is that these multicomponent vectors are indeed spinors, and we identified the special roles played by the unitary Clifford algebras in providing a very well defined mathematical structure which forms a nucleus which may be enlarged to provide covering structures so that all of the relevant dynamics and geometry may be at least reasonably well defined mathematically. We understand that our well behaved spinor structures are well defined only in a special choice of basis, a real Witt basis.

The classical function space realizations of our abstract RHS, \( \Phi \subset \mathcal{H} \subset \Phi^\times \), was shown by Gadella to belong to the intersection of the Schwartz space (\( \mathcal{S} \)) and the spaces of Hardy class functions from above and below (\( \mathcal{H}^2 \_\pm \)) [7]. There are Clifford analogues of \( \mathcal{S} \) and \( \mathcal{H}^2 \_\pm \) [8][10][9], and so the function space realizations will be well defined if the abstract spaces are also well defined. (Recall the Gadella diagrams of installment one [1], and references therein.)

Gadella’s use of van Winter’s theorem [7] still provides the necessary and sufficient conditions for “analytic continuation”. Just as in the work of Bohm [11][12], there will be contours at infinity in integrals. We have performed our construction in such a way as to define and then preserve under dynamical transformation the complex hyperbolic (Lobachevsky) geometry of the tangent space to our space(s) of states. The functions spaces used are classic Schwartz and Hardy spaces as to their components, and so the necessity proof of Gadella-van
Winter suffices even in the spinorial RHS paradigm.

We identified in [3] many possible linkages with the classical treatments of dynamical systems, and in particular possible relationships with various notions of complex systems, statistical mechanics (and related thermodynamic ideas), fractals, etc., which seem compatible with the formalism, but which emerged as a by-product of largely mathematical considerations in our incorporation of correlation and associated resonances into a classical probability description of hamiltonian dynamics on phase space. The question arises then whether this is another instance in the long string of unreasonable effectiveness of mathematics in physics described by Wigner many years ago, or if we have wandered off the path somehow. In the following subsection, we will adapt this structure to exhibit principal bundle structures associated to our constructions, and interpret this structure as a gauge theory, setting the stage for calculations which make predictions which will ultimately tell us if this is a toy theory or has some relevance to the real world.

4.2 Special unitary spinor bundles

The unitary groups are compact and locally path connected, while the special unitary groups are simply connected, with geodesic subgroups. It might be supposed that because our sought after spin structure is obtained from a seminorm topology, there is no invertibility, notwithstanding that $SU(N)$ is compact and simply connected. The inverses used in conjugation are in one sense basically pullbacks along a single fiber to the nucleus (our base space), and so are well defined individually, but in general may be well defined only locally. The crucial issue then is the issue of whether or not there is a spin structure which will make global identifications possible. We will examine the existence of spin structures further below, but all these structures (and, for present inquiries, especially the spin structure) depend on both the existence of and choice of a special basis, the real Witt basis, and we must suppose that there would not be invertibility of any sort in a general basis. Given the extreme dependence on the choice of basis, which is a typical feature of any spin construction, there is an interesting interaction between a weak (seminorm/semidistance) topology (and associated semigroups), spin conjugation and momentum maps worthy of much further inquiry than will be undertaken in this first description. Our spin conjugation is not an “inner automorphism”, but is a momentum map, involving the dual, reinforcing our choice of completion of the unitary Clifford algebraic set as a linear topological space, with scalar product and dual. The existence of any principal bundle structure depends on the triviality of the structures, or, equivalently, on the existence of sections. Recall that with our spinors in the orthogonal case a locally defined conjugation was equivalent to a spin transformation that acted from the left.
only and which was part of a globally defined structure. This globally de-
nition should survive change to a finer topology, as outlined in Section 3.2.
This suggests the probability of extending bosonic transformations outside of
the unitary core discussed earlier and also below, but we will not tackle that
issue directly in this forum, and in the next subsection we indicate grounds to
believe such an effort should fail.

With respect to the special unitary semigroup orbits in \( \Phi_{su(N)}^\pm \subset \Phi_{sp(2N,\mathbb{R})}^\pm \),
notwithstanding the seminorm topology, there is no obstruction to invertibility
on any set of positive measure: we may regard the entirety of \( \Phi_{su(N)}^\pm \) as simply
connected fiber liftings of a simply connected base space. Conventionally, if
given a space \( E \) with base space \( B \) and and whose fiber \( F \) is isomorphic to
group \( G \), a principal fiber bundle structure associated to \( E \) has a global section
(making both \( P(E) \) and \( E \) trivial). This means there is a continuous mapping

\[
s : B \longrightarrow E
\]

which is invertible, i.e., there also exists a projection \( \pi \) such that

\[
\pi s(x) = x, \forall x \in B.
\]

This means, in effect, that \( s = \pi^{-1} \). See, e.g., [16]. The lifting \( s = \pi^{-1} \) is
in fact all we really have globally for the full symplectic (semi-)group in the
present case. The projection \( (\pi) \) is not defined as a continuous transform on
\( \Phi_{sp(2N,\mathbb{R})}^\pm \), due to topological obstruction associated with our semidistance
(weak or seminorm) topology.

Maximal compact subgroups are homotopy equivalent to the Lie groups that
contain them, e.g., \( U(N) \) is homotopy equivalent to \( Sp(2N,\mathbb{R}) \). However, hom-
otopy is based on groups, and semigroups won’t do! Thus, there are finite
dimensional UIR’s of \( U(N) \), but none of the noncompact \( Sp(2N,\mathbb{R}) \), recalling
Wigner’s definition of noncompact groups. This suggests, once again, that we
should think naturally of semigroups of symplectic transformations, and not
of groups—else, from this homotopy equivalence, one would expect there to be
finite dimensional representations of \( Sp(2N,\mathbb{R}) \) which are merely lifts of of
finite dimensional \( U(N) \) UIRs. (Similar statements could be made for other
noncompact groups containing compact subgroups.)

However, there is no obstruction to invertibility as to the special unitary sub-
group of the symplectic group itself. We have simply connected fibers and a
simply connected base space: the lifting of the base space are 1 : 1 and onto
the “sections”, and so are isomorphisms and invertible [32]. Thus, identify-
ing (both abstractly and as to the related very well behaved function space
realizations) for instance,
\begin{equation}
\begin{aligned}
B & \iff \{ |n_A\rangle \} \oplus \{ |n_B\rangle \} \\
G & \iff \exp \mathfrak{su}(N)^{\mathbb{C}} = SU(N)^{\mathbb{C}} \\
F & \iff \text{span} \left\{ \left( \theta \circ SU(N)^{\mathbb{C}} \right) \circ b \right\}, \quad b \in B \\
E & \iff \Phi_{\mathfrak{su}(N)^{\mathbb{C}}}
\end{aligned}
\end{equation}

where \( \theta \) is the representation mapping \( \theta : SU(N)^{\mathbb{C}} \to \text{Aut}(E) \), and, in the first instance, the representation must be framed in terms of the creation and destruction operators, i.e., the real Witt basis. We here have chosen to represent \( B \) by using the simple harmonic oscillator number states as a basis, e.g., the energy representation. \( E \) locally has the structure \( B \times F \) by construction, and we can readily invert the representation homeomorphism to identify an element of \( SU(N)^{\mathbb{C}} \), so that we have \([I \times \theta^{-1}] : B \times F \to B \times SU(N)^{\mathbb{C}},\)

\([I \times \theta^{-1}] \circ (x, g(x)) \mapsto (x, g), \ x \in B, \ g \in SU(N)^{\mathbb{C}}.\)

If in our candidate for \( P(E) \) we consider \( s(x) \in SU(N)^{\mathbb{C}} \) and \( g \in SU(N)^{\mathbb{C}} \), then \( gs(x) \) belongs to the fiber over \( x \in B \). \( P(E) \) thus has the global structure of a product between the base space \( B \) and a fiber \( SU(N)^{\mathbb{C}} \).

Because as to the \( SU(N)^{\mathbb{C}}_\pm \) sub-semigroup structure we have no obstructions (on sets of positive measure) to extrapolating the semigroup structure due to the weak Kobayashi semidistance topology into a full group structure, we have a candidate for a dynamical homotopy group for the base space. Reiterating, there is no such thing as a homotopy based on semigroups (possible distributional measures, such as Dirac measures confound the notions of continuity, analyticity, etc.), and the property of being spin is determined by characteristic classes, which are homotopy invariants. Our base is spin, our fibers are spin, and so we may properly talk of spinor bundles, and principal spinor bundles in particular, only as to the special unitary orbits within the overall dynamical structure, and all our discussion must be based on a real Witt basis and its representations. As indicated earlier, these may be represented by either bosonic or fermionic spinors when we do take a representation, depending on choice of bilinear form and associated topology (“perspective”), and whether the representation is even or odd dimensional.

### 4.3 The full dynamical spinor “bundles”

If there were full groups generally available for the full symplectic semigroups in this construction, there would be no problem thinking of the symplectic transformations of, for instance, the function space of energy eigenfunctions of two free quantum harmonic oscillators, which we will call \( \mathcal{F}_{b\pm} \), which produces a family of function spaces we will call \( \mathcal{F}_{\mathfrak{sp}(4,\mathbb{R})}^{\mathbb{C}}_\pm \). This space may be associated in some very loose (and as yet unspecified) sense to a principal G-bundle, e.g., an \( \mathfrak{sp}(4,\mathbb{R})^{\mathbb{C}} \)-bundle with \( \mathcal{F}_{b\pm} \) as base space. However, the only initial suggestion of invertibility is with the \( U(1) \) sub-semigroups which have
local actions on $\Phi_{\text{sp}(4,\mathbb{R})^C}$ (and the related very well behaved spin-function spaces). Invertibility generally does not extend to sub-semigroups larger than $SU(N)^C$, unless we can successfully implement globally defined spin transformations, such as in the manner alluded to in the preceding subsection. If we were to extend the $SU(N)$ representation nucleus into a representation of the $Sp(2N,\mathbb{R})^C$ algebra, it seems as if there should exist something like a spinor bundle (loosely, a vector bundle of spin type), but the topological obstruction which keeps our semigroups from being extrapolated into full dynamical groups probably also prevents formation of a full spinor bundle structure, notwithstanding a proper spinor bundle is contained (as a nucleus) somewhere within this extended structure. We do not have homotopy equivalence, and the characteristic classes which define the property of being “spin” are not preserved in arbitrary mathematical operations. Properly, we have special unitary spin bundles, with what we might call “improper homotopies” extending this to a larger multicomponent “symplectic bundle” which is not spin in the strictest sense, (just as the extended structure has no proper principal bundle structure in any accepted sense). We can have a lifting or a projection, but we cannot have both simultaneously, due to the semigroup rather than group nature of the “fibers”.

The dynamical liftings taken as a whole are not based upon isomorphisms, as in the conventional treatment of principal fiber bundles, since generally there is no invertibility to the lifting. It is precisely this lack of isomorphic liftings of the “paths along flows in phase space”—lifting of vector fields composed of state vectors—which enables us to convert two free oscillators into a pair of coupled oscillators or vice versa. If we identify this construction with particle-fields, pair production or destruction it is not a 1 : 1 mapping, so the typical quantum resonance processes of pair production and annihilation are not what we would think of as an isomorphism either. Dynamical pair production or destruction is not 1 : 1 in before : after, so in order to incorporate such processes into our overall dynamical structure we have lost the use of invertibility and thereby dynamical evolution is not an isomorphism in general. The existence of pair production in nature confirms for us that our non-invertible dynamical semigroup notions are sound physics. The compact generators of $SU(N)$ take us from “island of stability” to “island of stability”, while the noncompact generators of the full symplectic semigroup, which represents the full gamut of dynamics, take these “islands of stability” and make resonances out of them, which will evolve dynamically towards another “island of stability”. Bifurcations are possible, in much the same sense as that term is used in classical dynamical systems, except that probability may flow along both paths of the bifurcation, e.g., there may be pair production.

This lack of isomorphism in the “fiber” liftings raises questions as to the extent to which we may reasonably think of the more general constructions over the full symplectic semigroups as principal fiber bundles, which merely involve
semigroups rather than groups for the fiber of the “principal bundle”. The lack, in general, of invertibility in our structural semigroups, e.g., $Sp(4, \mathbb{R})_\pm^C$, is reflected in the flow structure of the representation space and provides a novel meaning to the term connection. As in the standard principal G-bundle construction, our Lie algebra provides the “connection”. The path of $\exp g$ connects “sections” in both cases, e.g., $e^{-iHt}$ for $t \geq 0$ is the semigroup which transports you from section (time slice) to section (time slice) in generalized state space. The Schrödinger equation is the equation for geodesic transport (parallel transport in this case), giving us a constant of the motion: energy is conserved. There seems to be a clear sense of meaning here, and clear mathematical analogies. Thus, that $e^{-iHt}$ is geodesic, with a conserved quantity (energy) does not prevent time evolution from being hyperbolic in appropriate cases, and yet time evolution is ergodic with an equilibrium end-point (installment three \[3\], and recall the presence of fractals), though the functions representing the system should be of bounded mean oscillation (bounded analytic functions) \[63\]. We would therefore infer that in the present construction the wave function for the universe as a whole does not permit unbounded continuous creation, that the energy of the universe has always been pretty much what it is now and will remain pretty much the same in the future, although the universe may continue to expand hyperbolically, to eventually become conformally flat, etc. (In Section \[5\] we offer a speculative interpretation of our bosonic spinors which offers an explanation of how the Big Bang could conserve energy.) But, can we reasonably treat this dynamical structure as some sort of fiber bundle, or at least what part of it may be so thought of?

These are physically appealing notions which which follow directly from the mathematics, although of course they need substantial elaboration to really make them respectable. There are also substantial physical interpretation issues to resolve, especially those contrasting the gauge transformations in the o-topology versus gauge transformations in the s-topology. In equation (1), we pointed out in the introduction that one or the other term vanishes because we will be expressing the operators in terms of creation and destruction operators, and that one of the two terms will vanish for either bose or fermi creation and destruction operators due to the properties of those operators in that perspective. Yet, mathematically \[64\] and physically we should be amazed if a “mere choice of topology” has any profound effect on the scalar product used to calculate the expectation of any physical observables—it is not possible physically to conduct a Cauchy sequence of experimental measurements, so whether we use a bose or fermi realization of the $SU(N)$ Lie algebra and Lie group should not effect the outcome of the computation of this scalar product. This means that, i.e., that the probability of observation should be a topological invariant for aspects of quantum dynamics associated with stability (like the energy levels of atoms or the mass spectrum of the elementary particles) and independent of bose–fermi notions, which are topologically dependent ideas.
Of course, there is more than topology separating the larger symplectic semigroup and orthogonal semigroup outside of their coincidence—or overlap—on the unitary group. There is nothing like any topological invariant there! We have shown that in installment two\cite{2} that the notion of resonance is associated with dynamics and symplectic geometry and foreign to orthogonal geometric notions, and hence we find differing energy eigenvalues depending on topological choice when working outside of the overlap of the orthogonal and symplectic semigroups of transformations—since on this domain orthogonal geometry and its associated topology has nothing whatsoever to do with dynamics! This suggests a clear divergence from frame bundle notions at the very least. We will discuss homotopy notions (which are the underlying notions in conventional treatments of gauge theory) more fully in the sequel, but clearly algebraic topology is different insignificant ways when done in strong and weak topologies.

Note also that, since there can only be a symmetric or skew-symmetric form on a space, the Grassmann algebras (upon which the notions of supersymmetry are based), cannot have any topological notions related to forms defined on them. Since the notion of convergence of the exponential map is topologically dependent, there can be no such thing as a unitary gauge group associated with any Grassmann algebra, which further suggests that notions of supersymmetry and notions of gauge group are mathematically incompatible to the extent that supersymmetry notions are identified with Grassmannian notions.

In addition, orthogonal gauge transformations are frequently regarded as “passive”, e.g., as simple changes of frame. This, for instance, is a common interpretation of a $U(1)$ gauge transformation in electromagnetism, and the associated effect on the vector potential. In installment two \cite{2}, care was used to make possible topologically transitive symplectic (=dynamical) transformations, and as to the unitary transformations in the s-topology we are not talking about “passive” frame transformations. This is “active” dynamics being represented in the s-topology on the dynamical gauge bundle, notwithstanding we are talking of transforming one “island of stability” into another. This should be born in mind when reading other parts of this Section 4 and Section 6.

4.4 Gauge bundles

We will consider four fields with the fullest correlation structure envisioned in our conservative constructions, based on the special unitary groups. (We will suggest possible larger constructions with possible enhanced physical interest in Section 3) The full dynamical structure thus has structure semigroup isomorphic to $Sp(8,\mathbb{R})^C$, and we may think of our base as the phase space of four pairs of oscillators complexified, and the function space representa-
tive of the base space being of the form $B \oplus i \circ B$, where $B$ may be, e.g., $B = \{|n_a\} \oplus \{|n_b\} \oplus \{|n_c\} \oplus \{|n_d\}$, a very well behaved spin space representing the number states of the four oscillators, taken as purely real. (We may think of $B$ as initially a Fock space, but the components will be mixed by dynamical correlations as the symplectic semigroup runs. Also, we need not choose energy eigenstates for a basis.)

Note that the gauge bundles we are discussing in this section involve active, topologically transitive transformations according to everything we have done thus far. Hence, they may change observables (!), unlike the usual passive gauge transformations (such as one meets in electromagnetism). We will ultimately understand them as transitioning from one “island of stability” (e.g., island of stability within the potentials of the multiple fields which we will eventually identify as a stable particle) to another.

There is direct mathematical analogy to this structure in the Whitney sum construction, in which if space $E$ has gauge group $G$, then $E \oplus E$ has gauge group isomorphic to $G \times G$. This analogy follows because the orbits of $e^{\alpha A}$ and $e^{i\beta B}$ are isomorphic, $\alpha, \beta \in \mathbb{R}_+$, $A, B \in \mathfrak{g}$, the Lie algebra of $G$, and we note that $\mathfrak{sp}(2n, \mathbb{R})$ and $i \circ \mathfrak{sp}(2n, \mathbb{R})$ are isomorphic.

Identifying only “stable” gauge transformations, which will, e.g., take stable states to stable states, one associates to each “block diagonal” subspace not the semigroups $Sp(8, \mathbb{R})_\pm$, but the largest compact sub-semigroup of transformations in $Sp(8, \mathbb{R})_\pm$, or $U(4)$. We arrive at $S[U(4)^C]$ from $U(4)^C$ by any of a number of routes: by insisting only on unimodular (unit Jacobean) transformations so that one avoids (for now) imputing any physical content to scale changes or inversions of coordinate orientations, or in order to preserve the normalized probability measure, or to obtain a simply connected sub-semigroup (which is thereby really a group), one identifies a representation of the maximal compact subgroups $S[U(4) \times U(4)]$ as (isomorphic to, according to the Whitney sum construction) the gauge transformations for the four correlated oscillator system whose structure (semi-)group was $Sp(8, \mathbb{R})^C_\pm$.

The general case of the correct compact gauge group is deduced from the relationship $U(N) \equiv Sp(2N, \mathbb{R}) \cap SO(2N)$. The largest subgroup of unimodular (unit Jacobean), simply connected group of gauge transformations is $S[U(2) \times U(2)]$ for the two oscillator system. A similar construction involving three oscillators will result in an algebra representation with gauge group $S[U(3) \times U(3)]$, and above we showed that four oscillators will yield $S[U(4) \times U(4)]$. The pattern is obvious.

The full range of these gauge groups is not available for any given transformation, but they provide the overall framework for such transformations. This is because the transformations are hamiltonian and operate by geodesic trans-
port (see again the constructions in [2], and also the generators of the special unitary groups act geodesically), and so along any particular evolution trajectory in the generalized state space not all quantum numbers can change. For instance, there must be some non-zero component along some eigenvector in the spectral resolution (of the generator of the generalized gauge transformation) which is non-vanishing under the gauge transformation, and so there must be some quantum number which is conserved. (This is analogous to saying that $e^{-iHt}\psi$ is not identically zero unless $\psi$ is orthogonal to all energy eigenvectors, which requires that $\psi \equiv 0$ if the energy eigenvectors form a complete set in our base space.)

Conservatively then, the maximum allowable group of spectrum generating gauge transformations which are transitive and dynamical in the case of four fields is $S[U(4) \times U(3)]$, for three fields, $S[U(3) \times U(2)]$, and for two fields $S[U(2) \times U(1)]$, all the result of the constraint that the dynamical evolution be geodesic. See Section 8 for more details.

In the hamiltonian treatment of electromagnetism, it generally turns out that the canonical variables do not involve the fields directly, but the electromagnetic potentials. (E.g., the transverse component of the electromagnetic vector potential is gauge invariant and may give rise to the Aharonov-bohm effect in a region where there are no fields.) By analogy, we conjecture that our dynamical gauge field theory describes the dynamics of the fields in terms of potentials rather than in terms of the fields themselves.

In electromagnetism, describing the electromagnetic field in terms of potentials introduces degrees of freedom which are not independent—e.g., the electromagnetic fields may remain unchanged by gauge transformations of the potentials. The particular choice of a gauge introduces constraint relations which are used to eliminate the redundant degrees of freedom. We thus infer that our theory also has constraints implicit in it somewhere, but not Dirac type constraints with arbitrary multipliers. In particular, our selection of the islands of stability within the potentials of the field probably figures largely in the constraint picture.

Some other reminders of the nature of our theory which are distinguished from classical electromagnetism:

- Our theory is phrased in terms of topologically transitive dynamics rather than a topologically intransitive form, such as the $U(1)$ electromagnetic gauge theory.
- The canonical variables of phase space do not provide a real Witt basis, and so the entire spin structure would be dubious if framed in terms of them. See also the Appendix in this regard. We require a special type of basis.
- The hamiltonian gauge field theory is framed in terms of occupation num-
bers because it is framed in terms of coherent and squeezed states and creation and destruction operators.

• The lack of manifest covariance is typical of gauge theories—recall the problems in this regard with $U(1)$ electromagnetic gauge theory. In the present context we have established covariant associations via a chain of isomorphisms—see the following two Sections.

• Recall that there is no configuration space wavefunction for the photon. It is hoped that by using phase space (spinorial) representation for the field bosons, those configuration space problems have been avoided. Recall that both position and momentum wave functions can be well defined simultaneously in the rigged Hilbert space formalism \[6\].

The implications of this section should be obvious to anyone familiar with the Electroweak and Standard Models. Whether or nor there is any deep lesson here for field theory remains to be seen \[73\]. It is possible to construct some representations of the Poincaré group in a rigged Hilbert space \[33\]. There is also a construction for relativistic Gamow vectors \[34,35\]. Given the highly generic nature of our methods, we seem to have confirmation that the Standard Model has the most general gauge structure one would expect from three fields in the absence of some new and special non-generic physics, although these gauge groups represent exact symmetries. They thus differ, at least in some details, from the Electroweak Theory and Standard Model. We will address the gauge structure further in Section \[81\].

5 Canonical Variables to Spacetime

Starting with a phase space for four pairs of conjugate variables, we can construct real probability amplitudes (distribution densities) over it, as sketched in the preceding Section \[4,3\]. We have thus constructed a very well behaved spin representation of the maximal compact subgroup $(U(4)^C \equiv U(4) \oplus i \cdot U(4))$ of the semigroup of dynamical (=symplectic) transformations on the complexification of phase space for four pairs of canonical variables $(Sp(8, \mathbb R)^C)$, whose action on the representation space is symplectic (=dynamical) as well.

We have two structures to relate to spacetime. We have the four canonical position coordinates to relate to spacetime coordinates, and we have a non-trivial dynamical structure containing resonances over the complex extensions (analytic continuation) of our phase space to relate to a similar evolution structure over spacetime. The most interesting subset of this dynamical structure over phase space is a special unitary group orbit.

The orthogonal Clifford algebra of the space spanned by the four canonical position coordinates can be associated with spacetime structures via the isomorphism $Cl_{O(4,0)} \cong Cl_{O(1,3)} \[76\]. For the dynamical structure, we require a
the largest compact subgroup of $Sp(8, \mathbb{R})^\mathbb{C}$ is isomorphic to $U(4) \oplus U(4)$, and we can identify this with a gauge structure $[77]$. The Whitney sum rule $[67]$ and the restriction to unimodular transformations gives us a gauge group isomorphic to $SU(4) \times SU(4)$, which may be identified with $SU(4, 4) \cap U(4)$ up to isomorphism $[18]$. We can consider the restriction of $SU(4, 4)$ to $SO(4, 4)$, and note we can represent $SO(4, 4)$ in $Mat(2, \mathbb{H})$, treating the quaternions as $4 \times 4$ real matrices. But, $Mat(2, \mathbb{H}) \cong ClO(1, 3)$, and so we see that $Cl_{U-O}(T^\times \mathbb{R}^4) \cap Cl_{O(1,3)} \supset Cl_{U-S}(T^\times \mathbb{R}^4) \cap Cl_{O(1,3)} \supset S[U(4) \times U(4)] \cap Cl_{O(1,3)} \neq \emptyset$. We have a subset of correlated Hamiltonian dynamics over four pairs of canonical variables identified via isomorphism with a subset of the universal Clifford algebra (geometric algebra) of a spacetime with local $(+, -, -, -)$ signature. (More on this point in Section 6 below.)

The $R^{1, 3}$ which is the foundation for the $Cl_{O(1,3)}$ above can be a general riemannian spacetime, and not merely a Minkowski spacetime $[36]$, suggesting our formalism is compatible with general relativity. (In Section 6 below we discuss a much stronger link than “compatibility”.) By considering the relatively straightforward chain of textbook isomorphisms above, one can identify at least a subset of our quantum canonical field theory’s dynamics with “geometrodynamics” of a riemannian spacetime. What is lacking in the above is any account of hyperbolic dynamics on either level, or even the demonstration of the existence of non-trivial connections on this spacetime associated to our hyperbolic dynamics. The linkage here needs much more careful exploration than we will attempt in the present forum.

The nature of the unanswered questions in this formulation is illustrated by considering the problems posed in describing “falling quantum rocks”. Quantum dynamical time evolution appears to be geodesic in the space of states as a consequence of Schrödinger’s equation. (But, recall our caution in attributing group properties to the noncompact generator orbits, because our Lie algebra may in fact generate something less than a riemannian, or pseudoriemannian, connection.) However, the dynamical evolution of a falling rock in general relativity is along an orthogonal to a geodesic. Thus, our isomorphism between the full scope of quantum dynamics and spacetime geometrodynamics must certainly map dense sets to dense sets, but apparently need not necessarily preserve all geodesics, but perhaps only those associated with stability in the quantum dynamics. Our chain of textbook isomorphisms above covered only the islands of stability, and not the resonances, and is thus seriously deficient as the source of covariant dynamics. It indicates, however, that we should at least be able to associate our “particles” which are coherences of four correlated fields to extended structures in a non-trivial relativistic spacetime. Naively, we will identify the generators of the compact transformations identifiable with unitary subgroups of dynamics with stability and geodesic behavior in spacetime, and the non-compact generators within the full semigroup of dynamical transformations “transverse” to the compact generators.
will be identified with resonances and trajectories orthogonal (transverse) to geodesics in spacetime.

There are also conceptual problems with reductionism in which a subpart of a large system is approximated as an isolated system, and we should prefer that there be some sort of analogy, at least, between the process of simplification and reductionism in our quantum canonical variable dynamics and the related description in spacetime geometrodynamics. We have also shown a possible linkage between energy-centric Hamiltonian dynamics and mass-centric general relativity.

6 “SU(4) canonical gauge gravity”

We must acknowledge a bit of ambiguity at the outset. Conventionally, one follows Wigner and thinks of a particle as being described by \( \mathcal{P} \otimes \mathcal{G} \), where \( \mathcal{P} \) is a UIR of the inhomogeneous Lorentz group, and \( \mathcal{G} \) is a UIR of the gauge group. Issues of fundamental (spin) representations versus UIR aside, in the preceding section, we have identified a possible association of our gauge group to the inhomogeneous Lorentz group directly, suggesting the possibility that a dynamical representation of \( \mathcal{G} \) could provide a representation of a particle directly in a manner which is equivalent to the traditional Wigner approach. Whether such a course will prove physically interesting is beyond our present scope, and we shall henceforth merely identify our candidate for \( \mathcal{G} \) as “the gauge group”, adopting a fairly traditionalist view which respects the traditional definition of a particle. Given the possible identification of our present work with a unified version of \( U(4) \) gauge gravity (indicated below), it seems likely there is a fairly strong and broad identification \( \mathcal{P} \Longleftrightarrow \mathcal{G} \), which requires further investigation. (Recall that any irrep of the compact unitary group would be equivalent to a UIR, according to well known theorems. Thereby, a dynamical representation of \( \mathcal{G} \) is equivalent to an unitary representation of \( \mathcal{G} \).)

The intersection of \( Sp(8, \mathbb{R}) \) and \( O(8) \) is \( U(4) = SU(4) \otimes U(1)/\mathbb{Z}_4 \) as a group and \( U(4) = SU(4) \times U(1)/\mathbb{Z}_4 \) as a manifold, so that \( U(4) \) has four sheets associated to it, just as the inhomogeneous Lorentz group does. As indicated previously, there maybe means of extending our well behaved \( SU(4) \) structure to larger structures (in both orthogonal and symplectic topologies), and it is interesting to speculate that such an extension may provide a dynamical analogue to PCT, especially given the associations with spacetime shown in the previous section. We identify the Lie transformation groups of primary interest as belonging to the even sub-algebras of the relevant Clifford algebras, and thereby identify a spin representation of our unitary group with a spin representation of some proper subgroup of the Lorentz group—whether we are
speaking of the connected part only or the full structure depends on whether we are working with the unitary or special unitary group.

As to the Lorentz group, we will conjecture that the Equivalence Principle allows us to interpret the direct sum of inequivalent irreps in our dynamical (spinor) representation as a sum of particles with correlation. The natural interpretation of our spin representations therefore appears to be the representation that it provides is for pairs of correlated particles, and is in some sense consistent with Wigner’s definition of a particle as a UIR of the inhomogeneous Lorentz group. We could extend Wigner’s notion of what is a particle to the notion of a dynamical representation, a superset of the unitary representations, but it seems natural to respect his definition in terms of UIR’s because thereby in our spinorial constructions we find we have incorporated the Equivalence Principle as a by product of the analytic continuation. This seems more interesting than regarding a single particle as a correlation between a pair of field potential structures, since, e.g., a photon in gauge electromagnetism is a single field potential structure. We have not made any justification as to use of the inhomogeneous transformations, however, and so we will merely be optimistic as to that issue for the present.

As indicated in Section 3.2 we shall choose the path of mathematical simplicity for the present and concentrate on simply connected special unitary groups only, and identify our basic gauge structure (e.g., possibly modulo dynamical PCT-type transformations) as $SU(4)$, which we may further simplify to consideration of $SU(4)$ or $SU(3)$ only. $SU(4)$ is one of the first quantum symmetries investigated, being used in nuclear physics and also figured as a “spontaneously broken” symmetry in an early competitor to the Standard Model (and also a GUT) [53,54,55,56,57,58]. It has been explored as a spectrum generating algebra and relevance to string orbifold theories [59]. Our primary concern is *dynamically based* gauge symmetries, the above symmetries are exact, and we abjure anything “spontaneous” [83]. (We conjecture that dynamical transformations which are not elements of the unitary or special unitary group may break our gauge symmetries, but we really need greater mathematical justification to make this assertion in a mathematically proper way. We are altogether devoted to a self-consistent dynamics.)

Our candidate for the full gauge group is of course obtained from the largest compact subgroup of $Sp(8,\mathbb{R})^\mathbb{C}$, or $U(4)^\mathbb{C}$, which may be thought of as $U(4) \times U(4)$ (Whitney sum construction), attracting immediate comparison to Poincaré gravity theory (PGT) and the resulting general $U(4)$ theory of gauge gravity, obtained from a simple form of the gravitational Hamiltonian, representing a generalization of the canonical Arnowitt–Deser–Misner (ADM) hamiltonian form of general relativity. See, e.g., [61] and references therein, especially chapter 5. Our basic dynamical structure reduced to be without correlation would
be identified with the gauge group $U(4)$, so our $U(4) \times U(4)$ theory represents a correlated, i.e., unified possibly two particle, analogue of the PGT and associated $U(4)$ theory of gauge gravity. It is possible to invert the chain of associations and go from $U(4)$ back to the ADM form of general relativity, and by implication to similarly go from $U(4) \times U(4)$ back to general relativity as well (with the added suggestion that we may possibly have explicitly incorporated a version of the equivalence principle into our correlation structure, as discussed above.) Uniqueness in the road back to general relativity is an interesting and open question here both physically—in the context of general covariance—and mathematically. We desire, of course, that all physical content be uniquely determined, within the scope of physical equivalence incorporated in the notions of general covariance, without appeal to arbitrary assumptions. Not only is our gauge theory compatible with general relativity, but there is a fairly well known and well developed parallel in the $U(4)$ theory of gravity, there is a suggestion of a dynamical basis for PCT, etc.

There is extensive discussion of the $SU(4)$ symmetry in the charmed baryons article in the Particle Data Book \cite{24,23} and in the quark model section also \cite{25}. In any event, the basic 16-plet and 20-plet structures, etc., we have pointed to in the Particle Data Book references should have counterparts in both bosonic and fermionic representations of $SU(4)$.

All of the above has been largely generic. If we were to start with a relativistic phase space, then a similar line of reasoning to the preceding could possibly lead to $U(1,3)$ as the principal symmetry \cite{71}, and in this case there is more than one way to look at the dynamical gauge group: should it be $S[U(1,3) \times U(3)]$ or should it be $S[U(1,3) \times U(1,2)]$? Perhaps both are relevant. In any event, this is meant to show quite clearly that there might possibly be different gauge groups for 4 generic fields than there will be if the fields are specifically identified with spacetime itself. The traditional identification of the gauge theory as a theory of the field potentials rather than of the fields themselves seems to be in tension with a spacetime based gauge theory in any event. Such alternatives to our generic approach could possibly conflict with the traditional interpretation of gauge theories, while our generic approach need not. The lack of manifest covariance in our dynamics is covered by the multiple ways in which we have illustrated covariant associations, and also by the already well known isomorphism between canonical transformations and Lorentz boosts \cite{68}. The dynamics of four generic field potentials will originate from the gauge group $S[U(4) \times U(4)]$ and the dynamics of spacetime itself might (speculatively) be viewed as originating (somehow) from the gauge group $S[U(1,3) \times U(1,3)]$, with the spectrum constrained as indicated above. Whether or not we choose to use a Planck length as a fundamental physical determinant in our dynamics could have consequences for the fundamental particles nature is made up of \cite{72}! We shall consider only the generic approach involving a gauge theory of the field potentials in the sequel, which
is a fairly naive adaptation of the Standard Model to our own notions of correlated dynamical fields as laid out herein and in the preceding three articles \[123\]. It is intended that this article be thought of as an outline for a more careful program of elaboration and development.

7 Bosons

We can conjecture what the resulting unified gauge field theory for four generic correlated fields will be like, by fairly straightforward extrapolation from the Standard Model, with allowances for the requirements for being mathematically well defined we have been noting ever since installment two \[2\] where we saw that the incorporation of complex spectra into a quantum dynamics forced us to depart from the use of unitary transformations for a more general class of dynamical (=symplectic) transformations, etc. Our exact $SU(4)$ gauge theory should have a lot in common with the exact $SU(3)$ symmetry of QCD of the Standard Model, and we will conjecture that $SU(4)$ is associated with a color chromodynamics of its own. In place of the “three color separation” of the RGB of QCD, we have a “four color separation” we may label CMYK (cyan, magenta, yellow, carmine), based on analogy the the color separations of the printing industry. The labels are, of course, arbitrary. Being a special unitary group, the even dimensional irreducible representations are fermionic and the odd dimensional irreducible representations are bosonic. We will discuss salient features of the bosonic representations first:

- Associated with the 15 generators of the $SU(4)$ gauge symmetry, we conjecture there are 15 color carrying gauge bosons–gluons–with zero rest mass and spin 1.
- Being spin representations of $SU(4)$, our bosonic spinors are a direct sum of inequivalent bosonic (i.e., odd dimensional) irreps of $SU(4)$. These irreps need not be UIR’s, since they are dynamical, a superset of the unitary irreps. We propose that rather than CUR’s we should be thinking in terms of CDR’s–complex dynamical representations, although we will ignore reality/complexity issues for the representations herein. Recall that our finite dimensional dynamical representationa are equivalent to UIR’s.
- There are four possible $SU(3)$’s contained in $SU(4)$, so it is very likely possible to identify the $SU(3)$ gluons of the strong interaction with $SU(4)$ gluons more or less directly, e.g., $R \leftrightarrow M, G \leftrightarrow Y$ and $B \leftrightarrow C$, may be taken as typical of the four alternative $SU(4)$ to $SU(3)$ decompositions. We envisage two, three and four color combination particles may be possible, if we have massles spin one gauge bosons. The issue of mass is discussed below, and for now we will discuss a four color QCD.
- Gluonium, glueballs, etc, exist for four colors (and four anti-colors) just as they exist for the three colors of standard QCD. We expect like colors to
repel and color-anticolor to attract, in analogy to the electromagnetic charge case.

- The massive $W^\pm$ and $Z^0$ bosons of the Standard Model raise numerous interesting issues of considerable physical moment. We would suggest the $SU(2)$ spontaneous symmetry breaking is dynamical and not “spontaneous”, and related to the proposed $SU(4)$ covering symmetry in some way. (Massive gauge bosons are discussed below.)

- The massless spin 2 graviton, if it exists, may be some sort of glueball, or perhaps stem from some special feature of the bosonic representation of $SU(4)$. Note there are possible repulsive color interactions, so there may be repulsive field boson interactions attributable to the fourth (gravitational) field as well as the attractive interaction intermediated by the graviton.

- In QCD, a significant percentage of mass of nucleons is associated with quark-gluon plasma. Although the mixed notions of fermionic quark and bosonic gluon violate our notion of separation of perspectives, an identification of a bosonic analogue of the quark is indicated below. We would conjecture that all mass is the result of color interactions, principally intermediated by gluons.

- Correlated bosonic field states in the spin representations we have adopted will be identifiable (by isomorphism) with the form (boson) $\oplus$ (boson), meaning the resulting representation will be equivalent with an even dimensional representation, but this even dimensional representation will not have fermionic exchange properties since it is entirely associated with a different bilinear form (and associated scalar product). We might call this a pseudo-fermionic representation of the basic gauge structure, i.e., correlated bosons may appear to be single fermions if exchange properties are not carefully dealt with.

- In our dynamical treatment, we will find only bosons. There is no proper place for quarks in this dynamical perspective, although there may be bosonic structures related to them (see below). The bosonic and fermionic $SU(4)$ representations need to be studied for their structure, and there should be similarities and correspondences between many, if not all, of the structures in the respective bosonic and fermionic perspectives.

As indicated in the last item above, the need for rigorous separation of perspectives means that our chromodynamics will differ significantly from the QCD associated with the Standard Model, and not just in the addition of one more color or with the attendant possibility of four color combinations in addition to two and three color combinations already familiar in QCD. For instance, the notion of a “quark-gluon plasma” seems an oxymoron, given the enforced separation of bosonic and fermionic perspectives (based, in part, on dimensionality of the irreducible representations our spinors are built up of). This seems to indicate the broad outline of our conjectured extrapolation from the Standard Model as to our generic hamiltonian four field gauge theory.
8 Symmetry Breaking and Bosonic Mass

Mass production in the Standard Model is due to what is there called spontaneous symmetry breaking and which we have preferred to call dynamical symmetry reduction. There are several implications to our construction which deserve at least speculative comment. Recall that the $U(N) \times U(N)$ structure arises (via isomorphism) from dynamical correlation being introduced in a specific way, and the spectrum comes from the Lie algebra of the generators of $S[U(N) \times U(N - 1)]$. One kind of correlation is binding, and binding is associated with negative energy, and after analytic continuation our energy spectrum is unbounded from below, at least formally. One possible implication of this construction is that the Big Bang may have been energetically neutral, with positive mass and energy balancing the negative correlation energy. The $U(N - 1)$ in the $S[U(N) \times U(N - 1)]$ might therefore be regarded as the positive mass part, being associated with a $U(N)$ of massless bosons in correlation, e.g., the $W^+$ particle is paired with various massless field intermediary particles (photons, gluons, gravitons, etc.) such that the total energy of each part is equal to that of the other part and the correlation effectively means they are equal and opposite to each other. This is of course only a speculative explanation of a possible basis for mass production, the equivalence of mass and energy, etc.

In the remainder of this subsection, we will explore other other possible implications in a pretty naive manner in the interest of brevity—we are attempting a fairly straightforward extrapolation from the Standard Model’s results to our constructions, and we make many speculations along the way to indicate issues in need of resolution. The point emphasized here is that the correlation need not be between matter and other matter, or between matter and antimatter, for instance, but might even serve as a source of particles in and of itself: the correlated potentials have the symmetry of the gauge group, and the spectrum arises from this symmetry which thereby provides the source of particles and antiparticles. One example of correlation of the type we refer to is bonding involving an attractive potential, which can result in the release of other forms of energy due to overall energy conservation. (Typically this is a kinetic energy release interpreted as a temperature rise, such as in chemical bond formation.) We will evade any efforts attempting to force us to specify particle formation mechanisms at this point. There are deep issues here best left for later.

The first hurdle to be faced in identifying the spectrum is to identify the fundamental symmetry of interest. As indicated previously, the basis symmetry for these islands of stability within our hamiltonian dynamical field-potential constructions is $U(4) \times U(4)$, but we are working with spin manifolds, and spin manifolds have simply connected fundamental groups—indicating that in this
case mathematical conservatism should restrict us to special unitary groups. Since we make no present attempt to extend the notion of a proper spin manifold, we shall consider \( SU(U(4) \times U(4)) \) as our simply connected fundamental group, and serves as our gauge group underlying these constructions. Now \( U(4) \equiv U(1) \otimes SU(4)/\mathbb{Z}_4 \), which we can think of in several ways, including \( U(4) \to \{U(1)/\mathbb{Z}_4\} \times SU(4) \) or \( U(4) \to U(1) \times \{SU(4)/\mathbb{Z}_4\} \), substituting a direct product for the tensor product. Proceeding naively, we can think of \( SU(U(4) \times U(4)) \) as

\[
S[\{U(1)/\mathbb{Z}_4\}_{PCT} \times SU(4)_{CMYK}] \times \{U(1) \times \{SU(4)/\mathbb{Z}_4\}\}.
\]

Because of the previously demonstrated injective embedding map \( \varphi : U(4) \to \mathcal{P} \), in the above decomposition we have conjecturally identified the \( (U(1)/\mathbb{Z}_4)_{PCT} \) term with dynamically based PCT transformations, and will henceforth ignore this factor. Because of this mapping (demonstrated in Section 5), \( SU(4)_{CMYK} \) 4-colored particles have covariant associations which we will conjecture is what makes them real and observable. This reality/observability should extend to subsymmetries of our color-\( SU(4) \), but not to the so called “fundamental particles” argued for in Section 10 below—these, we would argue, are real in some sense but lacking in covariant associations which make them a “particle” at this level of development [72].

Beginning with the spinorially well defined part of \( U(4) \times U(4) \), our fundamental symmetry has how been simplified to

\[
S[U(4) \times U(4)] \Rightarrow S[SU(4)_{CMYK} \times U(1) \times \{SU(4)/\mathbb{Z}_4\}] \quad .
\]  

(16)

The spectrum actually observed comes not from this, but, as indicated earlier, from the reduction of this by one generator to allow for the conservation effects of parallel transport, i.e., taking out the \( U(1) \) factor, for instance. Our basic gauge group is now \( SU(4)_{CMYK} \times \{SU(4)/\mathbb{Z}_4\} \). There are 4 copies of \( SU(3) \) in \( SU(4) \), so we can naively identify the effective dynamical gauge group as

\[
SU(4)_{CMYK} \times SU(3)_{Flavor}
\]

One can also speculate about a symmetry reduction \( SU(4)/\mathbb{Z}_4 \to SU(3)/\mathbb{Z}_4 \) to conjecture the possibility of there being 4 generations among the 8 generators of \( SU(3) \), an issue primarily of interest to fermions, see Table 1, e.g., thinking in terms of 4 pairs, or a total of 8, quarks being associated with the \( SU(3) \) symmetry.

It is recognized geometrically the process of “spontaneous symmetry breaking” is in fact reduction of a principal bundle. Because of our preoccupation with dynamical structures, and because our principal bundle has as its group the symmetry transformation relating to compact dynamical transformations, we much prefer the term “dynamical symmetry reduction” as being both physically more precise and also mathematically more precise.
We thus expect 15 (probably massless) gauge bosons associated with $SU(4)$ and 8 possibly massive gauge bosons associated with $SU(3)$. Note that there may not be any analog to the Goldstone theorem restricting the masses of the $SU(4)$ gauge bosons, and that if there are any analogues of the Higg’s boson they would be associated with the $SU(3)$ symmetry and could be vastly more massive than the Higg’s associated with the $SU(2)$ symmetry. Renormalizability is mathematically sufficient for spontaneous symmetry breaking, and our very well behaved (see installment one [1]) wavefunctions should be renormalizable, if not indeed already “renormalized”, so mathematical processes equivalent to spontaneous symmetry breaking should follow in our formalism, with the attendant massive gauge bosons resulting.

8.1 Spectrum generating algebras

It would appear to the author—who could probably be said to only understand sufficient of the matter to be dangerous—that from the perspective of spectrum generating algebras it is possible to recover the Standard Model from our constructions, at least in large measure. We note further that a representation of electromagnetic current and electromagnetic charge generators is known for $SU(4)$ [45]. Perhaps a representation for the graviton can be found as well. Also of interest that there are different fermionic and bosonic discrete subgroups to $SU(4)$ [59], suggesting that there may be observable choices to the choice of topology in addition to those alluded to back in Section 3.1.

We would conjecture that if one consistently follows a perspective, then any possible observables of that perspective should be observed in context. The context relevant to a perspective might be shaped by such considerations as energy scales (e.g., Compton vs. de Broglie wavelength), etc.

The spectrum generating algebra approach evolved from the “spectrum generating group” approach, also referred to as the dynamical group and as the non-invariance group. The present series of papers has adopted the approach well known from classical nonlinear dynamics that the dynamical group is the relevant group of symplectic transformations (also known as the group of area preserving maps in the case of simple systems). We have thus extended this notion from classical hamiltonian mechanics to the arena of hamiltonian quantum fields.

9 Fermions

In the fermionic perspective, we adopt quark flavor as the smallest fermionic analogue to the bosonic color charge, e.g., the fermionic counterpart of the two
Defining fermionic particles from the gauge group generators for $SU(3)/\mathbb{Z}_4$. It is assumed that a process equivalent to spontaneous symmetry breaking has led to mass generation. There is an additional pair of quarks ("inner" and "outer"), a fourth neutrino, and a fourth lepton (which we call the "zeta"), all associated with the massive $SU(3)$ symmetry. The $b'$ fourth generation search information in the Particle Data Book are relevant to the $i$ and $o$ quarks conjectured above. The $\zeta$ and $\nu_\zeta$ can only be expected to arise directly in significant numbers during the course of very high energy phenomena in nature, e.g., supernovae and the like, so the lack of observation of any fourth neutrino such as the $\nu_\zeta$ is not yet a criticism of the above predictions, but also indicates these may be very hard to confirm experimentally. If the mass of the $\nu_\zeta$ is assumed comparable to that of the three known neutrinos, there may be some effect on neutrino oscillations, which is probably the easiest place to look.

The fermionic effects of the dynamical apparent symmetry reduction can be predicted for $SU(3)$ and its eight generators, which we associate with quarks in the usual way. This is best visualized by the following Table 1 which extends a familiar table to a fourth column, reflecting the addition of a fourth generation.

Recall that underlying the choice of alternative bosonic or fermionic representations lies merely the choice of topology in which to complete Cauchy sequences. It is doubtful whether we can experimentally determine the actual topology of our spaces of states, and mathematically we do not expect the choice of topology to affect scalar products, so we very strongly expect that there should be some strong sense of equivalence, or at least an identification, between the observables of the two alternative perspectives. (In
Section 3.1, we indicated how the choice of topology may be relevant to mass and energy scales, etc., and the issue of wave–particle duality.) It is the matrix elements that matter, and the perspective chosen should not matter as to these–topology should not change the integrals, etc. At a minimum, a construction as outlined in the preceding section does seem to set the basic observables in each perspective, and establish a sort of invariance as to those basic observables using the mass-energy equivalence well known from relativity. If we take this model literally, we would expect to find some expression in nature of both the bosonic and the fermionic perspectives, meaning when we adopt a particular perspective as relevant to a particular experiment, we expect to find events in nature which may be interpreted consistently in that perspective. The unresolved mystery at this point of development is whether or not there is any strong identification, perhaps even equivalence, between any of the structures we may find in both perspectives. More issues to resolve in the future.

This suggests an identification between energy centric gauge bosons associated with hamiltonian dynamics, and mass centric gauge fermions associated with orthogonal (Riemannian or pseudo-Riemannian) geometry. Both types of spinors are associated in our construction with representations of unitary groups only, and any identification could not be extended to all of dynamics, for instance. This identification is between fermionic and bosonic stable structures (topological invariants?) , and does not extend to resonances. Whether we regard an elementary particle as a stable bit of geometry or as a stable coherence of fields depends on our choice of topology for the representation of that particle (and the consequent even or odd dimension of the representation). We have been taught to expect that the choice of topology in our space of states should have no observable consequences, absent a Cauchy (infinite) sequence of physical measurements. We would assert this principle applies to the mathematical representation of the system rather than acting as a constraint on the system itself: one must choose an appropriate vocabulary. If one adopts a vocabulary to make a prediction, then one’s results must be analyzed in a consistent vocabulary, but the model is not the thing itself.

The “islands of stability” in either perspective, i.e., the “elementary particles”, should in some sense be invariants, characteristic structures in our space(s) of states. A general gauge field theory is associated with the potentials of the gauge field, just as in the U(1) electromagnetism gauge theory, and for us the islands of stability represent topological features of the multiple field potentials which are associated with stability and it should not matter whether we think of them as a glueball or as a concatenation of quarks. The fermionic or bosonic perspectives represent a choice of topology and that choice should not be relevant for features resolvable in both topologies–however, there are different numbers of discrete fermionic and bosonic representations of $U(4)$, for instance. This means correspondences between “particulate lumps of matter”
and “particulate intermediaries of dynamics” will not be 1 : 1 correspondences in general.

Resonances are another matter!!! Although we should probably regard resonances as “particles”, we have shown in installment two [2] their intimate association with dynamics making representation of them by fermionic spinors physically and mathematically problematic— they are the essence of non-trivial dynamics. The preceding chapters have shown it is possible to associate bosonic and fermionic representations of the unitary group with fermionic representations of the Lorentz group, but whether or not there is any such identification extending to $Sp(8, \mathbb{R}) \backslash U(4)$ is unknown.

There are a lot of intriguing hints of how things should work out when an exact four color QCD is fully developed with strict separation of perspectives. Elaboration of these are, however, major undertakings which we will postpone completion of for another date and another forum. We have gone on long enough, so will conclude with a couple of sections making some related additional predictions and also in formally delimiting our work in various ways. We have identified our dynamical structures with a quantum field theory identifiable with certain features of particle physics because the particle formalism seems (in the author’s view) to offer the most immediate prospects for physical confirmation. The mathematical structures, and to some extent their physical interpretation, seem to be largely generic, however, and we thus anticipate that the preceding separation of unitary and hyperbolic dynamics should extend to any fluid with long range correlations, plasmas for instance.

10 Whence Hyperbolic Quantum Dynamics?

According to Wigner, particles are UIRs of the inhomogeneous Lorentz group. Whether or not we should think of resonances as particles depends on whether we identify a particle as $\mathcal{P} \otimes \mathcal{G}$ (where $\mathcal{G}$ is the gauge group) or allow dynamical semigroups $\mathcal{D}_\pm$ for which there exists a map $\phi$ of the type indicated in the preceding section such that $\phi \cdot \mathcal{D}_\pm \cap \mathcal{P} \neq \emptyset$, as well as the particular choice of dynamical semigroup $\mathcal{D}_\pm$ itself. There are very well behaved semigroup representations of the inhomogeneous Lorentz group [33], and so to identify a resonance as a particle, what is needed is to associate the dynamical semigroup with a representation of the inhomogeneous Lorentz semigroup (in the sense of an extension of the above $\phi \cdot \mathcal{D}_\pm \cap \mathcal{P} \neq \emptyset$). As a third alternative, one might extend Wigner’s definition of a particle from the elementary particles $\mathcal{P} \otimes \mathcal{G}$ to $\mathcal{P} \otimes \mathcal{D}_\pm$, where $\mathcal{D}_\pm$ is the dynamical semigroup containing the compact gauge group $\mathcal{G}$ as the maximal compact subgroup. This last course seems perfectly reasonable also, and in the present context concerning four fields we can identify $\mathcal{D}_\pm = Sp(8, \mathbb{R})^\mathcal{G}_\pm$ and $\mathcal{G} = S[U(4) \times U(4)]$, thereby ex-
pressly extending the notion of a particle to include resonances, but the first alternative is also interesting and will be explored below.

There is some degree of concordance between spacetime geometry and the non-trivial dynamics of quantum fields as previously indicated when we showed the links between the unitary Clifford algebras and the Clifford algebra of spacetime. That association concerns equilibrium and invertible dynamics. There are also links relative to that sub-aspect of all possible dynamics which is identifiable with semigroups and hyperbolic dynamical evolution, implying that hyperbolic dynamical evolution (including non-time-reversible dynamics) is possible in spacetime just as it is in the hyperbolic hamiltonian dynamics of our canonical variables. A complete quantum dynamics must be able to account for the falling quantum rocks previously alluded to - the resonances which are transverse to the stable states. Note also that there may be some possible aspects of even non-trivial dynamics which are not associated with influencing the geometry of spacetime (e.g., wholly internal irreversible transitions of a resonance, at least in toy models and gedanken experiments), and we must allow for possible aspects of spacetime geometry which do not have any non-trivial dynamical significance (e.g., stable or strictly periodic phenomena). This requires, for instance, that our cosmological thinking ought to include consideration of hyperbolic dynamics such as arise in non-conservative and/or open cosmologies, should we consider the wavefunction of the universe. Possible conservative, closed or cyclic cosmologies seem excluded by the experimental observations of accelerating expansion [39,40], and seems also in tension with the observation of (near) flatness on cosmological scales of distance.

The basic mass spectrum for observed particles seems to be deducible from the preceding section’s development of a generic hamiltonian four field gauge field theory, and we shall adopt an optimistic attitude concerning the outcome of those as yet unmade calculations and as yet unmade experimental tests. This seems reasonable because there seem to be relatively clear principles for extrapolating from the Standard Model to a hamiltonian quantum field theory along the lines we advocate for any number of fields, and in particular to four fields, thereby adding gravity to a modified Standard Model. (The principal differences from the Standard Model are conceptual and seem to stem from the separation of bosonic and fermionic perspectives, based on mathematical necessity.) There are further indicated tests for determining whether the additional fourth field should be thought of as “generic” or if one should think in terms of a quantized spacetime.

Our driving principle throughout has been to obtain a probabilistic description of hamiltonian correlated dynamics which is as well defined as the author knows how to make it. We have a core dynamical construction which is well defined over compact dynamical transformations—we can represent compact
dynamical transformations in a well defined spin geometry, associated to well
defined spin bundle structures, etc. Representing the generators of the com-
pact infinitesimal dynamical transformations are bosonic spinors, and asso-
ciated to them are representations of the same group structure by fermionic
spinors. Those fermionic spinors are associated with particle representations
(either in the sense of $\mathcal{P} \otimes \mathcal{G}$ or $\theta \cdot \mathcal{G} \cap \mathcal{P} \neq \emptyset$)–they are creatures of Rieman-
nian geometry (symmetric metric), so that we can think of their velocity and
position as sufficiently localized and well defined (probabilistically) in a way
that says “particle”. There is no comparable sense of “this is a particle” in
quite the same sense in the symplectic geometry of hamiltonian dynamics (the
uncertainty principle notwithstanding). Conversely, there is no way you can
speak with mathematical precision of the hyperbolic dynamics of fermions.
Recall how in installment two \[2\) we used conjugation by a semigroup element
of a generator to obtain a Breit-Wigner resonance–that conjugation was gen-
erated by a hyperbolic generator. It is natural to think of a stable particle as
a lift of another stable particle wave function–in our finite dimensional repre-
sentation of our compact gauge group, we can associate a family of particles
with a single representation in this way. In the sense of installment two \[2\)
then, interparticle interactions appear as multiplication (or conjugation) by
a group element, using the sense of symplectic action and duality indicated
in installment two. In the case in installment two \[2\) where we were look-
ing at the expectation of the Hamiltonian, $H$, we saw how the conjugation
changed the Hamiltonian, i.e., changed the interaction, etc., in addition to the
changed wavefunctions. The question emerges, why should we not consider the
infinitesimal generators of $\mathfrak{sp}(2n, \mathbb{R})_{\pm}^C \setminus \mathfrak{u}(n)$ as providing labels for particles as
well? Although this may possibly require acceptance of resonances as particles,
we know that resonances exist and that whatever produced them cannot have
anything to do with $\mathfrak{u}(n)$ and gauge bosons, e.g., there are actually existing in
nature “fundamental” particles which are not elementary particles in any ac-
cepted sense of what an elementary particle is. These “fundamental” particles
are the resonance makers–they transform stable particles into resonances.

You cannot extrapolate compact dynamics into non-compact dynamics using
only compact dynamics as a tool. The compact dynamical transformations
form a closed and compact group–there can be no noncompact dynamics, and
thereby no resonances, in any $U(n)$ elementary particle theory or which may
be obtained from such a theory in any finite limit. There is not any finite limit
(macrosopic or otherwise) whereby you can go from gauge bosons to hyper-
bolic dynamics of any kind. Complexity won’t do the job–complex dynamics
obtained from compact dynamics is merely complicated compact dynamics,
and can not rise to parabolic or hyperbolic dynamics. Repulsive color inter-
actions which are a part of compact dynamics cannot yield non-compact
dynamics in any finite limit, such as in any finite universe. You might adopt a
holistic approach to nature, but finite causal horizons say there can only be a
finite number of interacting particles–if there are finite causal horizons like the
Table 2

Defining particles from the dynamical semigroup generators. The “fundamental particles” are conjectured bosons associated with noncompact dynamics, and their interaction with an elementary particle would be the formation of a resonance. They may be possible sources of both Dark Matter and Dark Energy—they may be massive just as gauge bosons may be massive. Their conjectured existence makes large mathematical assumptions if they are to be accepted as well defined mathematically. We will call them chimeric bosons.

Big Bang, you cannot get to hyperbolic (or parabolic) dynamics by the necessary transfinite means through invoking them. Even speed of light exceeding accelerating expansion, such as may have occurred in the early universe, if it is of finite speed and duration will still produce finite causal horizons. Tachyonic matter, if it exists, is causally unrelated to ordinary matter. There seems to be no mechanism for producing hyperbolic dynamics, and yet resonances exist both microscopically and in form of the expanding universe as a whole—so that gauge field theories must provide an incomplete description of the dynamics of elementary particles, and therefore an incomplete description of nature. It would seem that the group description itself is too restrictive, and you must include dynamical semigroups in order to have a sufficiently rich description of nature to accommodate the existence of resonances. We shall conjecture the existence of particles associated with the dynamical semigroup but outside of the maximal compact subgroup, initially calling them “fundamental particles”, noting clearly they are not “elementary particles” in any accepted sense of that familiar term, and also because their description is constrained to symplectic geometry, meaning they are representable as bosons only, but not as gauge bosons. (We will adopt the name chimeric for these bosons in the next section, for the reasons given there.)

This observation that there must be fundamental particles which are not elementary particles (gauge bosons) has a number of important possible consequences. Most noticeable is the larger number of particles which emerges overall, as summarized in Table 2.

Thus, in the Standard Model $SU(3) \times SU(2) \times U(1)$ adopted to our view, there could be seven additional (probably massive) fundamental bosons in addition to the $W^\pm$ and $Z^0$ associated with the $SU(2)$ gauge symmetry, and another thirteen additional fundamental particles associated with the $SU(3)$ symmetry. (These thirteen would probably not be massive in this context.) Our four field generic gauge gravity would be based (modulo PCT-type transitions) on
\[ SU(4) \times SU(3) \times U(1), \] with eight massive gauge bosons (possibly regarded as including the \( W^\pm \) and \( Z^0 \)) and another thirteen possibly massive fundamental bosons, all associated with the \( SU(3) \) symmetry, plus fifteen elementary and an additional twenty-one fundamental (and presumably massless) bosons associated with the \( SU(4) \) symmetry. These emerge from our constructions when the notion of gauge symmetry reduction is extended to the full dynamical semigroup, and provide obvious candidates for Dark Matter and Dark Energy. It is hoped that the reader will at least receive this as a principled speculation worthy of further exploration. Such fundamental particles are mathematically well defined only if we can extend our spinor structures to the full dynamical semigroup, and can be massive (according to analogy to present understandings of mass mechanisms for gauge particles) only if we can speak coherently of something analogous to principal bundle reduction for “semigroup bundles”.

11 Covariant Identification of Particles

The relationship between the symplectic transformations and Lorentz transformations for photons is explored in chapter 7 of [68]. A Lorentz boost is equivalent to a symplectic transformation, and vice versa, in the case of photons, and a similar equivalence should hold for elementary particles associated with \( U(4) \) type gauge structures. As indicated before, we have adopted the position that all particles are minimum uncertainty eigenstates of various fields, and that the transformations between eigenstates correspond to squeezing transformations in the language of quantum optics, and to transitive canonical transformations in the language of hamiltonian dynamics. In the context of chimeric bosons, a general symplectic transformation need not be identifiable with a Lorentz boost, meaning that in general the difference between one particle and another may not be covariant: different observers may assign different quantum number identifications to the same particle, different participants to the same process, etc., according to our constructions (and this seems to be a generic problem of hamiltonian mechanics!).

If the quantum numbers which characterize a particle are derivative of the \( U(4) \) gauge symmetry, then they have covariant associations as indicated preceding. Thus, all observers should be able to agree that a given particle is an electron or a \( W^+ \), or whatever. The same cannot be said of the quantum numbers which we associate with the chimeric particles described in Section 10 which are “fundamental” but not gauge—these particles are chimeras in the sense that different observers might identify different quantum numbers for them, and so our preferred name for them is chimeric, derived from analogy to the chimeras of ancient mythology.

We therefore suggest the name chimeric bosons because of this lack of covari-
ant association. Like all of the states we have dealt with in this series of papers, they are minimum uncertainty eigenstates, but I am not prepared to discuss them much more than by simple analogy to the squeezed and coherent states of quantum optics, also obtained by the application of symplectic transforms to Foch states. See, e.g., [68], especially chapter seven, where the relationship between symplectic transforms and Lorentz transformations is explored in the case of photons—a Lorentz transformation is equivalent to a symplectic transformation (squeezing transformation in the language of quantum optics) in the case of a photon. In the present context, that symplectic transformation might work an apparent transformation of a chimeric boson into another type of particle, since in the case of chimeric particles there are no covariant associations permitting universal identification of its fundamental quantum numbers. Those non-equilibrium squeezed photon states have sub-Poisson statistics, as we have shown our own resonant states have. Our analogy would be to suggest that one observer might say p-squeezed and another say q-squeezed, to use the jargon of squeezed states in quantum optics. From our development of these notions from foundational notions of dynamics and the observation that the islands of stability in four-field dynamics have covariant associations, and the observation that stable bosons are associated with the generators of transformations between the islands of dynamical stability, we are led to conclude that there must be bosons associated with the carrying of our four fields of force which are responsible for the accelerating expansion of the universe (among other effects) which lack unique covariant identification. However highly we esteem it, general covariance is at odds with the hyperbolic dynamics implied in the accelerating expansion of the universe, or there are more than four forces in the universe, or we have some fundamental misunderstanding of hamiltonian dynamics, etc.

The author hopes that at the very least he has demonstrated the presence of a foundational crossroads and has endeavored to offer a resolution.

12 Possible Quaternion Structure?

In this section, we tie up some odds and ends.

We began with a pair of spaces in direct sum, each element of which is a sum of states associated with four correlated oscillators. If we were to allow full expression of the correlations possible between four oscillators, each of the states for four correlated oscillators would have a quaternionic structure. We have also made use of an isomorphism involving $\text{Mat}(2, \mathbb{H}) \cong \text{Cl}_{O(1,3)}$, further suggesting a possible quaternionic structure. We will not make further investigation of quaternionic structures, and dynamics with quaternionic correlation structures, in this forum, but will note in passing that quaternionic quantum
mechanics is a fairly mature field, with many points of interest. See, e.g., [37]. Our reason for avoiding quaternions is that quaternionic notions would raise tensions, if not outright conflict, within our fundamental structures.

The equivalence principle of Einstein is related to Mach’s principle, and all that is required for the equivalence principle (“your can’t have mass A without also having a mass B”, to paraphrase Wheeler’s well known version) is that \(|A\rangle \oplus |B\rangle\) lie in a space wherein there is a correlation between the \(|A\rangle\) and \(|B\rangle\) components. A Fock space structure alone will not suffice for the desired correlation. In the present context, we envision a complex symplectic structure between \(|A\rangle\) and \(|B\rangle\) components, reflecting the real form of our correlated dynamical semigroups, \(Sp(2n, \mathbb{R})^C\). (A real symplectic structure might possibly suffice, however, or, indeed, even an orthogonal correlation). In light of the quaternion issues with respect to \(|A\rangle\) and \(|B\rangle\) suggested above, if \(|A\rangle\) and \(|B\rangle\) were quaternionic then the existence of a complex symplectic structure (complex correlation) between them would raise the possible specter of non-associative octonionic structures, with consequent loss of functoriality of mappings, categoriality within the meaning of the mathematical theory of categories, and ultimately even the boolean structure underlying our probability interpretation [38] vanishes. We emphatically do not dismiss the quaternion and biquaternion approaches outright, but indicate there are possible problems if they are not used carefully and leave such issues for another day.

13 Concluding Remarks

The formalism we outline in this series of articles makes no assumptions about the nature of particles, their numbers or their conservation, only that there were some structures on phase space which could be localized in some sense (so that our measures are finite) and which possessed non-trivial dynamics [79]. Since we are working with distributions, dynamical evolution is by semigroups, and there is an intrinsic dynamic arrow of time to the formalism (subsequent to the analytic continuation, which physically introduces correlation into our considerations, and thereafter mathematical necessity places us in the distributions). Conversely, this arrow of time requires that our quantum descriptions make use of distributions and the rigged Hilbert space structure [41,42], such as we use here. Since we see no evidence for stationary or strictly cyclic cosmological dynamical behavior (in fact, quite the contrary), and since we do see expansion, we must expect that the rate of expansion increases exponentially (hyperbolically)—as apparently it actually does [39,40]. We have also illustrated how these hyperbolic dynamical system results may be obtained from a quantum description of the universe.

Because our results have touched on numerous foundational issues, including
the universal validity of general covariance (!!), it is perhaps appropriate at this point to engage in sort of an executive summary of the steps making up the developments we have undertaken, and differing substantially in its outlook from the summary in Section [4.1]. While the author has not been as fastidious with his mathematics throughout as a mathematician would have been, he has tried to exercise much greater mathematical care than is typical in physics, and we have repeatedly seen physically interesting notions emerge from greater mathematical care—this is lesson number one.

In the first installment [1], we made a recapitulation of the rigged Hilbert space formalism and

- Represented classical hamiltonian dynamics on phase space with probability amplitudes rather than point localizations. Because both position and momentum are simultaneously defined variables in the rigged Hilbert space formalism, there is no need to resort to Wigner functions or quasi-probability distributions, or the like, and one may work represent quantum dynamics on phase space directly.
- Showed how the analytic continuation of this hamiltonian probabilistic dynamical description is indistinguishable from quantum theory in the rigged Hilbert space generalization of the Schrödinger theory, with distributions and an arrow of time. Our probability amplitudes are generalized gaussian wave packets, of mathematical necessity.
- Showed how physically, the analytic continuation represents the introduction of a type of mathematical correlation. Recall that with analytic continuation, the energy spectrum may be unbounded from below—e.g., the correlation introduced may in some sense be identified with binding.
- Explored the multiple types of scalar product which can be defined on phase space, anticipating the bosonic and fermionic types of scalar products which emerged in our later constructions.
- Showed that the mathematics was consistent with a field theory interpretation, i.e., the formalism really does not distinguish whether a particle is a “marble” or a localized excitation of fields.

The second installment [2] demonstrated that the incorporation of resonances with their complex energy eigenvalues required us to meet a lot of mathematical and physical needs by using multicomponent vectors (spinors) and a novel definition of the adjoint:

- We required esa operators in order to represent the canonical Lie algebra inclusion $\mathfrak{g} \subset \mathfrak{g}^\times$, so that, e.g., the generators of the two algebras are identified with the same tangent vector to a geodesic. A consequence is that the spectral theorem applies, and we have well defined spectra. This suggested that the adjoint should be an algebra involution for the Lie algebra of dynamical observables—e.g., for the Lie algebra of the group of dynamical
transformations, and the largest possible and most general dynamical group is generally accepted to be the group of symplectic transformations.

- In order to incorporate complex spectra, we were required to adopt the time reversed scalar product if the Hamiltonian was to be the generator of dynamical transformations.
- For the adjoint operation to be a well defined algebra involution, we were required to adopt for our ring of scalars the commutative and associative real algebra (with two units) $\mathbb{C}(1, i)$ rather than the field $\mathbb{C}$ as has been the custom heretofore.
- We were careful to respect the Lie group and Lie algebra structures at all levels of construction and representation, in order to qualify our use of the Hamiltonian as the generator of dynamical time translations, and to preserve the underlying geodesic nature of the group of symplectic (= dynamical) transformations.
- We made use of the geodesic structure in the identifications generator=tangent vector to a geodesic=derivation and the geometric notions of covariant derivative. Thereby, dynamical evolution on phase space is represented by geodesic transport in the space of probability amplitudes, etc.

In the third installment [3], we demonstrated the existence of things we expect to find in hamiltonian dynamics were properly included in our probabilistic representation of hamiltonian dynamics:

- We demonstrated the connection between chaos and analytic continuation, e.g., how Devaney chaos should be an expected result emerging from our correlated dynamics, and in its representation by probability amplitudes which are both analytically continued.
- Because the symplectic group is geodesic and has hyperbolic generators, we expect the existence of fractals in both our dynamics on phase space and in our dynamically evolving probability amplitudes. Fractals are typically generated by hyperbolic affine transformations. Our resonances are associated with a Julia set, and are generated by the hyperbolic generators of a geodesic semigroup.
- In non-linear dynamics language, the equilibrium state which a resonance evolves towards would be called a strange attractor.
- We demonstrated explicitly the role of resonances in a weak, or local, version of the Second Law of Thermodynamics, generic to all non-compact dynamics. We argued it may be possible to arrive at a global, or strong, form of this law as well, also predicated on the presence of resonances.

In this fourth installment, we have:

- Looked for a well defined covering structure for what has gone before.
- For the islands of stability, we have found this in the unitary Clifford algebras.
• For the generators of resonances, there are orbits of transformations (and their representations) for which we have offered no mathematically secure construction, although there is grounds for optimism. There should be a sort of symplectic Clifford “semialgebra” for the representation of all of dynamics, reflecting a semigroup of symplectic transformations, etc., rather than a full group structure, etc.

• Demonstrated that there appears to be something akin to a rigged Hilbert space structure in the nuclear part of the unitary Clifford algebras for phase space, making the rigged Hilbert space structure of our representation spaces appropriate. (A similar structure exists for the nuclear symplectic Clifford algebras.) We also demonstrated an enormous diversity of seemingly well behaved mathematical structure associated with our unitary Clifford algebras.

• We showed, as a mathematical truth, that irreversibility is intrinsic to dynamics, and existence of irreversibility depends only on the presence of correlation in the continuous dynamics of a dynamical system. Without correlation, there is no continuous (geodesic) hamiltonian dynamics on phase space, unless it be arbitrarily postulated. We have seen that, in general, reversible dynamics does not exist if one relies upon the geometry of phase space only. One must postulate away topological obstructions otherwise possible if one wants continuous invertible dynamics in the general case. This is a mathematical confirmation of long held physical intuitions of the late Professor Prigogine when viewed in conjunction with the results of installment three [3], where we showed the presence of fractals and that resonances were evolving toward towards a strange attractor - the new equilibrium that a resonance evolves towards is an attractor so that there is an arrow to dynamical evolution.

• Showed how bosonic and fermionic representations of dynamics emerge from alternative topological completions of the same set. We argued that bosons are associated with the finer topology, and so seem more intimately associated with waves, while fermions are associated more intimately with the particle nature of the structures under study. The wave–particle duality was explored.

• Used the geodesic structure of the symplectic transformations to demonstrate the existence of Lie algebra valued connections, and used this in turn to define a gauge theory as to the islands of dynamical stability, showing a dynamical origin to the unitary gauge group. Is was argued that this mathematically well defined unitary core may be extended to the orbit of the semigroups of symplectic transformations spanning the remainder of the structure of possible dynamics. (Mathematical optimism at work.)

• In our construction, it would seem that the gauge freedom previously dealt with by contraints, multipliers, etc., has been tacitly dealt with by stability considerations. (Mathematical optimism at work.)

• We adopted a field theoretic interpretation for the constructions and explored the generic span of compact hamiltonian dynamics for various num-

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bers of fields. As a gauge field theory, we were dealing with the potentials of the fields rather than the fields directly. Lack of manifest covariance is dealt with by covariant associations due to isomorphism. Particles are associated with the islands of stability of the potentials of the fields.

- Particles are minimum uncertainty eigenstates, and the transformations between particles are analogous to the dynamical squeezings of the electromagnetic field in quantum optics.

- In the case of three fields, we recover the Standard Model spectrum exactly, although with some differences in interpretation. There are modifications to the mathematical understanding of bosons and fermions, and terms such as "quark gluon plasma" are mathematically found to be lacking in precision, although there may be some underlying associations between the fermionic and bosonic representations which make this and similar phrases physically meaningful in a qualified sense.

- In the case of four fields, we found a strong connection to general covariance and Hamiltonian General Relativity, but only so far as the islands of stability were concerned.

- From the four field associations of our stable Hamiltonian dynamics with the Lorentz group, we have deduced the existence of covariant associations for the characterizing quantum numbers for all elementary particles (e.g., whose quantum numbers are determined from a unitary gauge group) in the case of four fields. All observers should agree as to the identity of an elementary particle if there are four or fewer fields in nature.

- We have shown that the mere existence in nature of non-compact dynamics demands the inclusion (somehow) of particles into our Hamiltonian field dynamics which are not "elementary"—not derived from a unitary gauge group—and which are chimeric in that they may be characterized differently by different observers: whose characterization is, in short, non-covariant. This is demanded by the mathematics if we suppose that non-compact dynamics exists anywhere in nature, and if the universe is finite.

Hopefully, this will help focus in the mind the largely generic dynamical–mathematical approach which has led us to this seeming dichotomy between dynamics and general covariance. There are specific predictions which may be anticipated as stemming from the spectrum generating algebra approach to our four field theory, and so we have an initial touchstone to test our dynamical description. The predictions as to the non-equilibrium aspects of dynamics are largely inferential, given the lack of general covariance of the relevant quantum numbers, and so specific inferences must be looked for if they are to be accepted.
A Basis Normalization and Squeezing

In Section 3, we saw how the creation and destruction operators may be regarded as the canonical quantization of the real Witt basis, just as the familiar position and momentum operators of quantum theory may be thought of as the canonical (Dirac) quantization of the classical position and momentum. There are other operators related to the creation and destruction operators and the position and momentum operators (differing primarily in normalization) which are especially useful for some of the more elementary considerations of the squeezed and coherent states of quantum optics.

Taking the $A$ and $A^\dagger$ as in installment two [2], define

\begin{align*}
\hat{a}_1 &= \frac{A + A^\dagger}{2} \\
\hat{a}_2 &= \frac{A - A^\dagger}{2i}
\end{align*}

(A.1)

and then define the new normalization of the position and momentum operators as:

\begin{align*}
\hat{x} &= \sqrt{2} \hat{a}_1 \\
\hat{p} &= \sqrt{2} \hat{a}_2
\end{align*}

(A.2)

Then $[\hat{x}, \hat{p}] = i\mathbb{I}$, eliminating the $\hbar$ from the familiar commutation relation of position and momentum–we could obviously change the commutator to the identity by further alteration of the normalization. One of the simplest squeezing operators is [66]:

\begin{equation}
Z = -\frac{1}{2}(\hat{x}\hat{p} + \hat{p}\hat{x}) ,
\end{equation}

(A.3)

The full gamut of squeezing operators belong to the relevant symplectic Lie algebra, here the “single photon” lie algebra $\mathfrak{sp}(2, \mathbb{R}) \cong \mathfrak{su}(1, 1) \cong \mathfrak{o}(2, 1)$. Just as there are one and two photon squeezed states in quantum optics, there should be analogous one and two field squeezing operators, etc., in Hamiltonian field dynamics.

The traditional treatment of squeezed states has been in terms of real and imaginary parts of a wave function and not of $p - q$ quadratures. Note there is a symplectic structure in either situation, and the present paper follows the more recent trend of thinking in terms of $p - q$ quadratures and a rotating $(p, q)$ space ellipse being squeezed, etc.
The gist of this is that there should be multifield analogues to all the squeezing phenomena of quantum optics if the present mathematical development is to be believed.

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[67] If space $E$ has gauge group $G$, then $E \oplus E$ has gauge group isomorphic to $G \times G$.

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[69] The full symplectic semigroup—the semigroup of dynamical transformations—is of course larger than the unitary group which is our gauge group. Thus, the full scope of our dynamics is larger than the span of the gauge group structure. It is not unreasonable, then, to speak of dynamical symmetry reduction. But, the term “spontaneous symmetry breaking” is essentially meaningless, and deprecated in our formalism. What actually occurs is merely reduction of the principal fiber bundle whose group is a proper subgroup of the full dynamical semigroup. It is a representation of a dynamical process.

[70] If so, our hamiltonian formalism is largely generic, while the Lagrangian approach to field theory has been fine tuned over decades. If the two disagree, it seems probable that this would be the result of some non-generic element figuring significantly in the Lagrangian approach. Nature may or may not allow faithful description by generic means.

[71] Rather that using for the symplectic form the matrix $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, one might use $J' = \begin{pmatrix} 0 & g \\ -g & 0 \end{pmatrix}$, for instance. This would allow exploration of the actual quantization of spacetime itself by respecting the spacetime metric as applied to the canonical position and momentum variables.
There are associations between an affine frame bundle over a relativistic world manifold and $\mathbb{R}^4$, since its structure group is reducible to the linear group $GL_4$. See, e.g., section 4.1 of [65]. On this basis, there is grounds for speculation that our “generic” results have in fact some sort of covariant associations, but to extract anything well defined from this tenuous link is not trivial. If this speculation follows, it would seem that our generic results should apply regardless of whether or not there is any Planck length acting as a fundamental length to spacetime. By this, we therefore suggest speculatively that our generic approach is “strongly generic” and independent even of the precise nature of spacetime.

According to O’Raifeartaigh [29], the standard Model is more properly $S(U(3) \times U(2))$, with unbroken $U(3) = SU(3) \times U(1)/\mathbb{Z}_3$. $SU(4)$ figured in the first GUT proposal [53, 54]. See also [32]. $SU(4) \cong Spin(6)$ can be thought of as the compact version of $SU(2,2) \cong Spin(4)$, the twistor group or the conformal group which describes gravity. There may well be some connection of a more concrete nature with the twistor formalism of Penrose, given our (momentum spinor) $\times$ (position spinor) structure in the original representation of phase space.

It is interesting to speculate whether the use of a seminorm topology and resultant semigroups is also necessary mathematically, and this may be taken up by the author at a later date. The importance of initial conditions has been emphasized by R. Peierls and his school [26, 27]. It is known mathematically that the complete specification of initial and boundary conditions results in semigroups of evolution [28]. At the very least, our use of seminorm and resultant semigroups here provides us with a vehicle to receive the complete specification of dynamical initial and boundary conditions. Here, the specification would be probabilistic and not for the point localizations on phase space of classical mechanics.

Note that we are not in Hilbert space, and $a^\dagger$ is not necessarily the dual or adjoint operator of $a$. They are independent continuous operators (corresponding to the operator representation of a real Witt basis), still obeying the familiar commutation relations. Similarly, given that the Hamiltonian is our generator of time translations according to the Schrödinger equation, $H$ and $iH$ generate transverse orbits, reflecting to the direct sum structure of our dynamical group, $Sp(8, \mathbb{R})^C$.

This isomorphism is unique up to an equivalence. If there are isomorphisms from some aspect of our dynamical structure into two separate $Cl_{O(1,3)}$ algebras, then their substructures are isomorphic to each other not only because of the isomorphism of algebras, but also by way of their isomorphism to the geometric algebra of the canonical coordinate space, and so any alternative structures can differ by only the choice of frame used to describe them.

We can treat the space as if it were obtained by lifting direct sums of independent states, e.g., number states of a special sort, for which both creation and destruction operators are continuous. The existence of such states is known. See chapter 2 of [6].
One of the key ingredients in making extended dynamical structures well defined, e.g., beyond the island of stability associated with the unitary transformations and their orbit, is the fact that the symplectic (semi-)group is geodesic as to each locally compact sheet, meaning that its Lie algebra may be associated with connections. Thus, in electromagnetism the relation \( B = \nabla \times A \) is undefined at the singularity of a point charge if functional analysis considerations are all that prevail, but when viewed geometrically, with \( B \) as the curvature of the connection \( A \), this relation is well defined. This is well understood in the context of non-trivial bundles, and is the reason that gauge theoretic approaches have found much currency in modern mathematics, but it is the geodesic geometry and not the group structure (and well defined classical homotopy) which seems to be the governing factor here: a full blown gauge theory seems to assume more structure than is needed for a well defined affine (i.e., geodesic) geometry. There are deep mathematical issues here which we will merely express an optimistic view of at this time, since they are peripheral to our necessary concerns.

Since we have a non-trivial spacetime around, presumably there is a representation of the inhomogeneous Lorentz group available somewhere which meets Wigner’s definition of a particle.

By choosing the Clifford module structure, the physically noteworthy “time-reversed” involution automorphisms of installment two [2] become canonical. See [19].

As an interesting aside, note the complexification of the real representation on \( \Phi \) effectively results in a doubling of Hilbert spaces. If the real Lie algebra \( \mathfrak{g} \) generating \( G_0 \) is represented on \( \Phi_0 \subset \mathcal{H}_0 \subset \Phi^\times_0 \), then the analytic continuation \( \mathfrak{g} \rightarrow \mathfrak{g}^\mathbb{C} \) generating \( G^\mathbb{C} \) represented on \( \Phi \) of the Gel’fand triplet \( \Phi \subset \mathcal{H} \subset \Phi^\times \) can be seen by the preceding argument as leading to the representation of a group isomorphic to \( G_0 \times G_0 \) on \( \mathcal{H} \), e.g., looking like “\( G_0 \oplus G_0 \)” represented on “\( \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0 \)”. This looks reminiscent of the Liouville space formalism, and there is perhaps some connection.

There are two alternative spinorial covering structures for \( SU(N) \), e.g., alternatively associated with the orthogonal and symplectic forms. Both the spin and spinoplectic groups will suffice to make our conjugation well defined, in context of theperspective chosen. However, both exceed the limits of what we would understand as dynamics, which, tradition holds, should extend to the symplectic semigroup and no further. Hence, we conservatively consider only the \( SU(N) \) simply connected structure, with the understanding that there well may be larger well defined structures, such as the full \( U(N) \) group or semigroup, and even the full symplectic group or semigroups. I.e., we take it that dynamics is restricted to the orbit of the symplectic (semi-)groups, although we appeal to larger mathematical structures in order to find a formalism in which our constructions are well defined. The choice of covering structure has possible implications for the invertibility or non-invertibility of dynamics, which we choose to ignore.

Spontaneous symmetry breaking is
mathematically sufficient for renormalization. However, we seem to have no need for renormalization, given the very well behaved wavefunctions central to our formalism. Hence “spontaneous” symmetry breaking is not necessary in our hamiltonian formalism, as it was in the Lagrangian formalism. Whether one speaks of “spontaneous symmetry breaking” or of “dynamical apparent symmetry reduction”, you probably mean the same thing in both instances, although the second choice of phrase has the advantage of precision of meaning. There are also no constraints, *per se*, in the Dirac constraint sense, so there are no arbitrary multipliers in our formalism.