DISCRETE FOURIER RESTRICTION THEOREMS IN TWO DIMENSIONS.

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ABSTRACT. Consider the group $\mathbb{R}^2$ with the discrete topology, and denote its Fourier algebra by $A(\mathbb{R}^2_d)$. We reformulate a theorem of V.A. Yudin as a statement about restrictions of functions in $A(\mathbb{R}^2_d)$ to the boundary of a strictly convex domain when those functions vanish outside that boundary. We give visual proofs of that statement and a complementary one.

1. Introduction

Yudin’s theorem [14] is about the Fourier coefficients, $\hat{f}(\vec{n})$ say, of an integrable function $f$ on the product $\mathbb{T} \times \mathbb{T}$ of two copies of the unit circle group $\mathbb{T}$. Those coefficients are defined on the product $\mathbb{Z} \times \mathbb{Z}$ of two copies of the integer group $\mathbb{Z}$. He used a dual method to estimate the $\ell^2$ norm of their restriction to the integer lattice points in the boundary of a strictly convex domain in $\mathbb{R}^2$ when $\hat{f}$ vanishes outside that boundary. We give direct proofs of that estimate and of the corresponding estimate when $\hat{f}$ vanishes inside the boundary.

As usual,

$$\hat{f}(n_1, n_2) = \left(\frac{1}{2\pi}\right)^2 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(t_1, t_2) e^{-n_1 t_1} e^{-n_2 t_2} dt_1 dt_2.$$ 

Use the same measure $(1/2\pi)^2 dt_1 dt_2$ in computing $L^p$ norms. Given a subset $D$ of $\mathbb{R}^2$, denote its interior by $\text{Int}(D)$, its complement by $D^c$ and its boundary by $\Gamma$.

Our main goal in this paper is to give visual proofs of both parts of an extension of the following statement.
Theorem 1.1. There is a constant $C$ so that if $D$ is a strictly convex set in $\mathbb{R}^2$ with boundary $\Gamma$, and if $f \in L^1(T^2)$, then the estimate

$$\left( \sum_{\vec{n} \in \Gamma \cap \mathbb{Z}^2} |\hat{f}(\vec{n})|^2 \right)^{1/2} \leq C \|f\|_1$$

follows from either of the following conditions:

1. $\hat{f}$ vanishes on $\text{Int}(D) \cap \mathbb{Z}^2$.
2. $\hat{f}$ vanishes on $\text{Int}(D^c) \cap \mathbb{Z}^2$.

Call these the interior and exterior cases. As in [14, p. 861], no uniform estimate of the form (1.1) is possible in either case for a family of sets $D$ whose boundaries contain arbitrarily long arithmetic progressions in the integer lattice $\mathbb{Z}^2$.

The validity of inequality (1.1) in the exterior case is Yudin’s theorem; we give a new proof of it in Section 4. The fact that the inequality also holds in the interior case seems to be new; we prove it in a direct way in Section 4 and outline a dual proof in Section 6. We explain in Section 2 how both cases have single-variable precedents in Yves Meyer’s paper [8] and related work. We describe the common part of our direct proofs of the two cases in Section 3, and discuss refinements of those methods in Section 5. In an appendix, we outline proofs of two known lemmas that we use throughout the paper.

The restriction theorem above applies to a subspace of $L^1(T^2)$ defined by requiring that some Fourier coefficients vanish. Related conclusions hold [2], [15, Theorem 1] without the latter requirement when $L^1(T^2)$ is replaced by $L^p(T^2)$, where $p \geq 4/3$. Unlike most Fourier restriction theorems, that result and ours give global $\ell^2$ estimates rather than local $L^2$ estimates.

In Section 7 we consider examples where our methods also yield $\ell^2$ estimates on suitable subsets of shifted copies of $\Gamma$. These sometimes lead to global $L^2$ estimates of the following kind.

Example 1.2. Let $D = \{(u,v) \in \mathbb{R}^2 : v > u^2\}$, let $f \in L^1(\mathbb{R}^2)$, and let $k \in \mathbb{R}$. If $\hat{f}$ vanishes on $\text{Int}(D)$ or $\text{Int}(D^c)$, then

$$\left( \int_{-\infty}^{\infty} |\hat{f}(u,u^2+k)|^2 du \right)^{1/2} \leq C'' |k|^{1/4} \|f\|_1.$$

2. Contagion of weakness of size in Fourier algebras

The standard notation for the set of Fourier coefficients of functions in $L^1(T^2)$ is $A(\mathbb{Z}^2)$. This set is a Banach algebra under pointwise operations because $L^1(T^2)$ is a Banach algebra under convolution. The
norm of \(\hat{f}\) in \(A(\mathbb{Z}^2)\) is defined to be \(\|f\|_1\). Denote the restriction of \(\hat{f}\) to a set \(S\) by \(\hat{f}|S\), and rewrite inequality (1.1) in the form

\[
\|\hat{f}|(\Gamma \cap \mathbb{Z}^2)\|_2 \leq C\|\hat{f}\|_{A(\mathbb{Z}^2)}.
\]

Also view \(A(\mathbb{Z}^2)\) as the set of sequences on \(\mathbb{Z}^2\) that factor as convolution products of sequences in \(\ell^2(\mathbb{Z}^2)\); this corresponds to the fact that \(L^1 = L^2 \cdot L^2\) pointwise. Moreover, \(\|\hat{f}\|_{A(\mathbb{Z}^2)}\) is the infimum of the products \(\|g\|_2\|h\|_2\) over all pairs \((g,h)\) of sequences on \(\mathbb{Z}^2\) for which \(g \ast h = \hat{f}\). Given such a convolution factorization of \(\hat{f}\), extend those factors the discrete group \(\mathbb{R}^d\) by letting them vanish off \(\mathbb{Z}^2\). The corresponding extension of \(\hat{f}\) belongs to \(A(\mathbb{R}^d)\), with a norm that is clearly no larger than the norm of \(\hat{f}\) in \(A(\mathbb{Z}^2)\).

Theorem 1.1 follows immediately from the next statement.

**Theorem 2.1.** There is a constant \(C\) so that if \(D\) is a strictly convex set in \(\mathbb{R}^2\) with boundary \(\Gamma\), and if \(w \in A(\mathbb{R}^d)\), then the estimate

\[
\|w|\Gamma\|_2 \leq C\|w\|_{A(\mathbb{R}^d)}
\]

follows from either of the following conditions:

1. \(w\) vanishes on \(\text{Int}(D)\).
2. \(w\) vanishes on \(\text{Int}(D^c)\).

Here we use the notion of “boundary” in the usual topology on \(\mathbb{R}^2\). This makes the corresponding statement for the space \(A(\mathbb{R}^2)\) true but trivial, because functions in \(A(\mathbb{R}^2)\) are continuous relative to the usual topology on \(\mathbb{R}^2\), and they vanish on \(\Gamma\) if they do so on \(\text{Int}(D)\) or \(\text{Int}(D^c)\).

Meyer’s result [8, pp. 532–533] on \(\mathbb{Z}\) extends to \(\mathbb{R}^d\) as follows.

**Theorem 2.2.** Let \((x_j)_{j=1}^J\) be a sequence of positive numbers satisfying the condition that \(x_{j+1} \geq (1 + \delta)x_j\) for some positive constant \(\delta\) and all \(j\). Let \(w \in A(\mathbb{R}^d)\). Then an estimate

\[
\left[ \sum_{j=1}^J |w(x_j)|^2 \right]^{1/2} \leq C(\delta)\|w\|_{A(\mathbb{R}^d)}.
\]

follows from either of the following conditions:

1. \(w\) vanishes on each of the intervals \((x_j/(1 + \delta), x_j)\).
2. \(w\) vanishes on each of the intervals \((x_j, (1 + \delta)x_j)\).

We will not prove this here, but we note that, as in [4], the first part, about coefficients after long-enough gaps, follows by the method that we use to prove the first part of Theorem 2.1. As in [5, page 214], the second part above follows from Remark 5.3 below.
Meyer used other methods to prove the version of Theorem 2.2 for \( A(\mathbb{Z}) \). He described the pattern in his theorem as a “contagion of weakness of size.” On any infinite discrete abelian group \( G_d \), use the \( \ell^2 \) norm to measure this weakness, noting that \( \|w\|_{A(G_d)} \leq \|w\|_2 \), and recalling that the most one generally say about the size of a function in \( A(G_d) \) is that it belongs to \( c_0(G_d) \), which strictly includes \( \ell^2(G_d) \).

Denote the indicator function of a set \( S \) by \( 1_S \). If \( w|\text{Int}(D) \) belongs to \( \ell^2(D) \), then applying the first part of Theorem 2.1 to \( w - w \cdot 1_{\text{Int}(D)} \) yields that

\[
\|w|\Gamma\|_2 \leq C\|w\|_{A(\mathbb{R}^2_d)} + C\|w|\text{Int}(D)\|_2.
\]

Similarly, if \( w|\text{Int}(D^c) \) belongs to \( \ell^2(D^c) \), then

\[
\|w|\Gamma\|_2 \leq C\|w\|_{A(\mathbb{R}^2_d)} + C\|w|\text{Int}(D^c)\|_2.
\]

In the setting of Theorem 2.2 replace \( \text{Int}(D) \) or \( \text{Int}(D^c) \) with the union of long-enough gaps ending or beginning at the numbers \( x_j \). In each case, weakness of a member of \( A(\mathbb{R}^2_d) \) or \( A(\mathbb{R}_d) \) on a suitable set propagates to the boundary of that set in \( \mathbb{R}^2 \) or \( \mathbb{R} \).

**Remark 2.3.** The methods for the second part of Theorem 2.1 can also be used \([6, 5, 13]\) to prove Paley’s theorem about coefficients of functions in the classical space \( H^1(\mathbb{T}) \). In that setting, weakness on any Hadamard set of positive integers follows from weakness on the set \( \mathbb{Z}_- \) of negative integers. It is less clear how Hadamard sets in \( \mathbb{Z}_+ \) can be regarded as parts of some boundary of \( \mathbb{Z}_- \). But they share with the strictly-convex examples the property that certain combinations of “boundary points” must belong to the set where weakness is assumed to occur. See Remark 5.3 for more on this.

**Remark 2.4.** Recall that \( \mathbb{R}_d \) is dual to the Bohr compactification \( b\mathbb{R} \) of the real line. As in \([8, \text{page 534}]\), applying standard duality arguments to Theorem 2.2 yields that if \( (v_j) \in \ell^2 \), then there exist functions \( G \) and \( H \) in \( L^\infty(b\mathbb{R}) \) with the following properties.

1. \( \|G\|_\infty \) and \( \|H\|_\infty \) are both no larger than \( C(\delta)\|v\|_2 \).
2. \( \hat{G}(x_j) = \hat{H}(x_j) = v_j \) for all \( j \).
3. \( \hat{G} \) vanishes outside the union of the intervals \( (x_j/(1 + \delta), x_j] \).
4. \( \hat{H} \) vanishes outside the union of the intervals \( [x_j, (1 + \delta)x_j) \).

If \( x_{j+1} \geq (1 + \delta)^2x_j \) for all \( j \), then the supports of \( \hat{G} \) and \( \hat{H} \) are disjoint except for the numbers \( x_j \). Work by Goes \([7, \S 4]\) exhibited similar patterns in a different context. As in \([5, \text{pp. 214–215}]\), they yield an easy proof of the Grothendieck inequality, which follows in the same way from the duals of Theorem 1.1 and 2.1 that we discuss in Section 6.
3. Two Lemmas

In our proofs of the nontrivial cases of Theorem 2.1, we write each value of $w$ as an inner product of one function in $\ell^2(\mathbb{R}^d_2)$ with a translate of another such function. Recall that for a function $v$ on an additive abelian group and a point $x$ in that group, the function $\tau_x v$ maps each point $y$ to $v(y - x)$, and the function $v^*$ maps each point $y$ to $v(-y)$. Rename the factor $h$ in $w = g * h$ as $h^*$, with no effect on norms. Since $(g * h^*)(x) = \sum_{y \in \mathbb{R}^2} g(y)h^*(x - y) = \sum_{y \in \mathbb{R}^2} g(y)h(y - x),$

\[ w(x) = (g, \tau_x h). \] (3.1)

Proving Theorem 2.1 therefore reduces to bounding $\sum_{j=1}^{J} |(g, \tau_x h)|^2$ for finite sequences $(x_j)_{j=1}^{J}$ of distinct points in $\Gamma$.

We apply the lemmas below with $H = \ell^2(\mathbb{R}^d_2)$ and $A_j = \tau_{x_j}$. The first lemma goes back to [3], and led to a rediscovery [4] of Meyer's result about coefficients after gaps. The second lemma is more recent [6], and was used there to reprove the extension [5, Theorem 2] of Paley's theorem that yields the part of Theorem 2.2 about coefficients before gaps.

In the next section, we specify subspaces with the properties required in the lemmas. We outline proofs of the lemmas in Appendix A.

**Lemma 3.1.** Let $H$ be a Hilbert space and $M_1 \subset M_2 \subset \cdots \subset M_J$ be closed subspaces of $H$. Let $A_1, A_2, \ldots, A_J$ be unitary operators on $H$ for which

\[ A_1 M_1 \subset A_2 M_2 \subset \cdots \subset A_J M_J. \]

Let $g$ and $h$ be members of $H$ satisfying the following conditions for all indices $j < J$.

1. $A_j h \in A_{j+1} M_{j+1}$.
2. The vector $g$ is orthogonal to the subspace $A_{j+1} M_j$.

Then

\[ \left[ \sum_{j=1}^{J} |(g, A_j h)|^2 \right]^{1/2} \leq 2(\|g\|_H)\|h\|_H. \] (3.2)

**Lemma 3.2.** Let $H$ be a Hilbert space and $L_1 \supset L_2 \supset \cdots \supset L_J$ be closed subspaces of $H$. Let $A_1, A_2, \ldots, A_J$ be unitary operators on $H$ for which

\[ A_2 L_1 \subset A_3 L_2 \subset \cdots \subset A_J L_{J-1}. \]

Let $g$ and $h$ be elements of $H$ satisfying the following conditions:

1. $A_j h \in A_{j+1} L_j$ for all $j < J$.
2. The vector $g$ is orthogonal to the subspace $A_j L_j$ for all $j > 1$. 




Then
\[
\left( \sum_{j=1}^{J} |(g, A_j h)|^2 \right)^{1/2} \leq 2(\|g\|_H)\|h\|_H.
\]

4. Visual proofs

Given the convolution factorization \( w = g \ast h^* \) and a subset \( E \) of \( \mathbb{R}_d^2 \), let \( V(E, h) \) denote the closure in \( H = \ell^2(\mathbb{R}_d^2) \) of the subspace spanned by the translates \( \tau_x h \) for which \( x \in E \). In the interior case of Theorem 2.1, we will apply Lemma 3.1 with \( M_j = V(E_j, h) \) for suitable sets \( E_j \). In the exterior case, we will apply Lemma 3.2 with \( L_j = V(D_j, h) \) for suitable sets \( D_j \).

The nesting and membership conditions in Lemma 3.1 hold if
\[
E_j \subset E_{j+1}, \quad x_j + E_j \subset x_{j+1} + E_{j+1}, \quad \text{and} \quad x_j \in x_{j+1} + E_{j+1}
\]
for all \( j < J \). The orthogonality condition holds if \( (g, \tau_y h) = 0 \) for all \( y \) in \( x_{j+1} + E_j \). Equation (3.1) makes this equivalent to having \( w(y) = 0 \) for all such \( y \).

Let \( F_j = x_j + E_j \) for all \( j \), and let \( \Delta x_j = x_{j+1} - x_j \) when \( j < J \). The last condition in the previous paragraph is equivalent to requiring that \( w \) vanish on all the sets \( F_j + \Delta x_j \) with \( j < J \). In the interior case, this happens if those sets are all included in \( \text{Int}(D) \). Translate the other conditions on the sets \( E_j \) to see that it suffices in that case to find sets \( F_j \) satisfying the following four conditions for all \( j < J \).

\[
F_j + \Delta x_j \subset \text{Int}(D), \quad F_j + \Delta x_j \subset F_{j+1},
\]
\[
F_j \subset F_{j+1}, \quad \text{and} \quad x_j \in F_{j+1}.
\]
That is,
\[
F_j + \Delta x_j \subset \text{Int}(D) \cap F_{j+1} \quad \text{and} \quad F_j \cup \{x_j\} \subset F_{j+1}.
\]
Call these the shifted inclusions and the unshifted inclusions.

Similarly, the subspaces \( L_j \) and their images \( A_{j+1} L_j \) nest as prescribed in Lemma 3.2 if
\[
D_1 \supset D_2 \supset \cdots \supset D_J,
\]
and \( x_2 + D_1 \subset x_3 + D_2 \subset \cdots \subset x_J + D_{J-1} \).

The membership condition in the lemma holds if \( x_j \in x_{j+1} + D_j \) for all \( j < J \), and the orthogonality condition holds in the exterior case if \( x_j + D_j \subset \text{Int}(D^c) \) for all \( j > 1 \).

Consider the sets \( G_{j+1} = x_{j+1} + D_j \), creating another point \( x_{J+1} \) to cover the case where \( j = J \). Translate the conditions on the sets \( D_j \) to
see that it suffices that

\[(4.5) \quad G_{j+1} - \Delta x_j \subset \text{Int}(D^c), \quad G_{j+1} - \Delta x_j \subset G_j, \]

\[(4.6) \quad G_j \subset G_{j+1}, \quad \text{and} \quad x_j \in G_{j+1}. \]

In this case, the shifted inclusions and unshifted inclusions state that

\[(4.7) \quad G_{j+1} - \Delta x_j \subset \text{Int}(D^c) \cap G_j \quad \text{and} \quad G_j \cup \{x_j\} \subset G_{j+1}. \]

If the boundary \( \Gamma \) of \( D \) is the graph of a strictly convex or strictly concave function defined on all of \( \mathbb{R} \), and the points \( x_j \) run from left to right along \( \Gamma \), then we can use sets \( F_j \) and \( G_j \) that are very similar.

For such a concave function \( \phi \), write \( x_j = (u_j, v_j) \), and

\[(4.8) \quad \text{let} \quad F_j = \{(u, v) \in \mathbb{R}^2 : u < u_j \text{ and } v \leq \phi(u)\}, \]

\[(4.9) \quad \text{while} \quad G_j = \{(u, v) \in \mathbb{R}^2 : u < u_j \text{ and } v \geq \phi(u)\}. \]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{similar_sets.png}
\caption{Similar Sets}
\end{figure}

The unshifted inclusions clearly hold for both \( F_j \) and \( G_j \). By strict concavity, any part of the boundary ending at \( x_j \) rises strictly more rapidly or falls strictly more slowly than any part of the same width to the right of it. Shifting such a part ending at \( x_j \) by \( \Delta x_j \) gives a curve that ends at \( x_{j+1} \) and lies strictly below \( \Gamma \) except at \( x_{j+1} \). This yields the shifted inclusions for the sets \( F_j + \Delta x_j \). The corresponding inclusions for the sets \( G_{j+1} - \Delta x_j \) follow in a similar way. Both cases of Theorem 2.1 therefore hold with \( C = 2 \) for such sets \( D \).

Every unbounded, strictly convex set can be rotated to have the form specified above, except that the domain of the function \( \phi \) may not be all of \( \mathbb{R} \). In that case, add the requirement that \( u \) belong to the domain of \( \phi \) in defining \( F_j \). If the domain of \( \phi \) is bounded on the left, also include all vertical lines to the left of \( D \) in defining \( G_j \).
When $D$ is bounded and strictly convex, follow [14] in recalling that there are vertical support lines at two boundary points, listed from left to right as $x_0$ and $x_\infty$ say. In the exterior case, let $\Gamma_0$ be the upper boundary with $x_\infty$ excluded.

Consider points $x_j$ running from left to right in $\Gamma_0$, starting with $x_0$. As above, let $G_j$ consist of all points in $\mathbb{R}^2$ that lie strictly to the left of $x_j$, and that do not lie directly below $\Gamma_0$. Then the inclusions (4.7) hold for all $j \geq 0$, so that $\|w|\Gamma_0\|_2 \leq 2\|w\|_{A(\mathbb{R}^2)}$. Rotate by 180° to get a similar estimate on the rest of $\Gamma$, and that

$$\|w|\Gamma\|_2 \leq 2\sqrt{2}\|w\|_{A(\mathbb{R}^2)}.$$  

In the interior case for the same set $D$, shear vertically and shift to place both of the points $x_0$ and $x_\infty$ on the $u$-axis; this does not affect $\|w\|_{A(\mathbb{R}^2)}$. Then the lower boundary lies below the $u$-axis. There will be one point, $x_J$ say, on the upper boundary with a horizontal support line. Place that point on the $v$-axis. Then the upper boundary in the second quadrant is the graph of an increasing function.

Consider points $\{x_j\}_{j=1}^{J-1}$ running from left to right in the interior of that graph. Find the midpoint of the line segment from $x_0$ to $x_j$; then rotate the part of boundary curve running from $x_0$ to $x_j$ by 180° about that midpoint to get a lower curve returning to $x_0$ from $x_j$. Form the convex hull of that lower curve and the upper boundary curve from $x_0$ to $x_j$, and delete the vertices $x_0$ and $x_j$ to get the set $F_j$. Form $F_J$ in the same way. We show this in Figure 2(a) below.

![Figure 2](imageurl)

**Figure 2.** Interior Inclusions

It is obvious that $\{x_j\} \subset F_{j+1}$. The sets $F_j$ and $F_{j+1}$ are mapped onto themselves by the 180° rotations, $\psi_j$ and $\psi_{j+1}$ say, about their centroids. Note that $\psi_{j+1}$ is equal to $\psi_j$ followed by the shift by $\Delta x_j$; so $\psi_{j+1}$ maps $F_j$ onto $F_j + \Delta x_j$. Since the upper boundary of $F_j$ is, by definition, an initial part of the upper boundary of $F_{j+1}$, the lower
boundary of $F_j + \Delta x_j = \psi_{j+1}(F_j)$ is a final part of the lower boundary of $F_{j+1}$.

As in Figure 1(b), the upper boundary of $F_j + \Delta x_j$ lies strictly below the upper boundary of $F_{j+1}$ except at the missing point $x_{j+1}$. Hence $F_j + \Delta x_j \subset F_{j+1}$; applying $\psi_{j+1}$ again then makes $F_j \subset F_{j+1}$.

The lower boundary of $F_j$ runs from $x_0$ to $x_J$, and is the graph of an increasing function. Hence $F_j$ lies strictly inside the second quadrant, as do its subsets $F_j + \Delta x_j$ with $j < J$. These sets therefore do not meet the lower boundary or right-hand boundary of $D$. Since the shifted sets $F_j + \Delta x_j$ lie strictly below the upper boundary of $D$ in the second quadrant, they are included in $\text{Int}(D)$, as required.

Let $\Gamma_2$ be the part of $\Gamma$ inside the second quadrant, together with $x_J$. Then $\|w|\Gamma_2\|_2 \leq 2\|w\|_{A(R^2)}$ in the interior case. Similar arguments on three other parts of $\Gamma$ yield that

$$\|w|\Gamma\|_2 \leq 4\|w\|_{A(R^2)}.$$  

5. **Weaker hypotheses**

Our methods work when $w$ vanishes on some sets that are smaller than the ones used in Section 4. In the next section, we discuss dual methods that also work with those weaker hypotheses.

Fix a finite sequence $(x_j)_{j=1}^J$. It will turn out to suffice that $w$ vanish on suitable subsets of the additive group generated by the points $x_j$. All points $x$ in that group have the form

$$x = \sum_{i=1}^J \varepsilon_i x_i,$$

where the coefficients $\varepsilon_i$ are integers.

The application of Lemma 3.1 to lacunary Fourier series was analysed in [4, Remark 3]. In the present context, the same reasoning shows that it suffices for $w$ to vanish on the set $\text{Alt}((x_j))$ of points $x$ with alternating sum representations

$$x = x_{j_1} - x_{j_2} + \cdots + x_{j_{2i-1}} - x_{j_{2i}} + x_{j_{2i+1}}$$

with at least 3 terms and a strictly-increasing index sequence $(j_\ell)$. Let $F_{j+1}$ be the set of points $x$ as above with $j_{2i+1} \leq j + 1$, but only impose the requirement that the sum (5.2) have at least 3 terms when $j_{2i+1} = j + 1$. These sums belong to the fatter sets $F_{j+1}$ shown in Figures 1(a) and 2(a). The inclusions (4.4) hold for the smaller sets $F_j$ and $F_{j+1}$, and Lemma 3.1 applies.
For Lemma 3.2, the analysis in [6, Section 5] yields the sets $G_{j+1}$ consisting of all points $x$ with a representation

$$x = x_i - \sum_{j' \geq i} n_{j'} \Delta x_{j'},$$

satisfying the following conditions:

1. $i \leq j + 1$.
2. The coefficients $n_{j'}$ are nonnegative integers.
3. If $i = j + 1$, then $n_{j'} \neq 0$ for some $j'$.

The points in this version of $G_{j+1}$ belong to the fatter set $G_{j+1}$ shown in Figure 1(a). The desired inclusions hold for the smaller sets $G_j$ and $G_{j+1}$.

The lemma applies provided that $w$ vanishes on the union $\text{Sch}((x_j))$ of the smaller sets $G_{j+1} - \Delta x_j$. The points $x$ in that union are those with a representation (5.3) satisfying condition (2) with $n_{j'} \neq 0$ for some $j'$. They are also given by the sums of the form (5.1) where the integer coefficients $\varepsilon_i$ have the following properties:

- The full sum $\sum_{i=1}^{j'} \varepsilon_i$ is equal to 1.
- All partial sums of the full sum are nonnegative.
- All partial sums after the first positive one are positive.
- Some partial sum is greater than 1.

These conditions also arose in [5] and [14].

**Remark 5.1.** In the setting of Remark 2.3, Paley’s theorem holds because the set $\text{Sch}((n_j))$ is included in $\mathbb{Z}$ when the sequence $(n_j)$ is sufficiently lacunary. This was used in a dual way in [5] and [13], and in a direct way in [6].

**Remark 5.2.** We made one choice of the sets $G_j$ in proving the exterior case of Theorem 2.1 and another just above. For both choices, the corresponding sets $D_j$ are additive semigroups. This can be used [6, Remark 5.7] to define suitable partial orders on $\mathbb{R}^2$, relating that case of Theorem 2.1 to Paley’s theorem.

**Remark 5.3.** It can happen that $|\varepsilon_i| > 1$ in the sums (5.1) representing points in $\text{Sch}((x_j))$. Let $S((x_j))$ consist of all points $x$ with representations (5.1) in which the coefficients $\varepsilon_i$ belong to the set $\{-1, 0, 1\}$ and satisfy the four conditions for membership of $x$ in $\text{Sch}((x_j))$. Arguments in [5] and [6] each combine with the application above of Lemma 3.2 to show that

$$\|w|X\|_2 \leq 4\|w\|_{A(\mathbb{R}^2)}$$

when $w$ vanishes on $S((x_j))$. 

\[5.4\]
Remark 5.4. For $\text{Alt}((x_j))$, rewrite the representation (5.2) in the form

\begin{equation}
x = x_{j2i+1} - \sum_{j'<j_{2i+1}-1} n_{j'} \Delta x_{j'}
\end{equation}

where the coefficients $n_{j'}$ take the values 0 and 1 only and the latter occurs at least once. For $F_{j+1}$, keep those conditions on $(n_{j'})$, put $j_{2i+1} = j + 1$, and require instead that $j' \leq j$ in the sum.

6. Dual Constructions

Denote the Bohr compactification of $\mathbb{R}^2$ by $b\mathbb{R}^2$. The duality arguments in [12] or [8, page 534] show that Theorem 2.1 is equivalent to the one below. Theorem 1.1 has a similar dual.

**Theorem 6.1.** Let $D$ be a strictly convex set in $\mathbb{R}^2$ with boundary $\Gamma$. Then for each function $v$ in $\ell^2(\Gamma)$, there exist functions $G$ and $H$ in $L^\infty(b\mathbb{R}^2)$ with the following properties:

1. $\|G\|_\infty$ and $\|H\|_\infty$ are both no larger than $C\|v\|_2$.
2. The restrictions of $\hat{G}$ and $\hat{H}$ to $\Gamma$ both coincide with $v$.
3. $\hat{G}$ vanishes on $\text{Int}(D^c)$.
4. $\hat{H}$ vanishes on $\text{Int}(D)$.

Theorem 2.1 can be proved by constructing suitable functions $G$ and $H$ when the support of $v$ is finite. Choose points $x_j$ as in Section 4. Let $v$ vanish off the set $X = \{x_j\}_{j=1}^J$, with $\|v\|_2 = 1$. The modification of the Rudin-Shapiro construction in [1] produces a trigonometric polynomial $G$ with the following properties.

- $\|G\|_\infty \leq C$.
- $\hat{G}|X = v$ if the sets $X$ and $\text{Alt}((x_j))$ are disjoint.
- $\hat{G}$ vanishes off the set $X \cup \text{Alt}((x_j))$.

This yields the first part of Theorem 2.1 since the strict convexity of the unbounded set $D$ makes $\text{Alt}((x_j))$ a subset of $\text{Int}(D)$ in the diagrams in Section 4.

It also follows that $\text{Sch}((x_j)) \subset \text{Int}(D^c)$ in those cases. For the second part of the theorem, it suffices to construct a function $H$ with the following properties.

- $\|H\|_\infty \leq 1$.
- $v(x_j)\hat{H}(x_j) \geq (1/C)|v(x_j)|^2$ for all $x_j$.
- $\hat{H}$ vanishes off the set $X \cup \text{Sch}((x_j))$.

Yudin refined a method of Pigno and Smith [10][13] for this, and noted that a construction in [3] would work too. In both of these methods, one can satisfy the middle condition above by making $\hat{H}|K = (1/C)v$. 

In Example 1.2, let $k > 0$ and $h > \sqrt{k/2}$. We will show that

$$
\left[ \sum_{j=-\infty}^{\infty} \left\{ \sup_{jh \leq u < (j+1)h} |\hat{f}(u, u^2 - k)| \right\} \right]^{1/2} \leq \sqrt{2C\|f\|_1},
$$
in the interior case, and that

$$
\left[ \sum_{j=-\infty}^{\infty} \left\{ \sup_{jh \leq u < (j+1)h} |\hat{f}(u, u^2 + k)| \right\} \right]^{1/2} \leq \sqrt{2C\|f\|_1}
$$
in the exterior case. Inequality (1.2) then follows because

$$
\int_{(j+1)h}^{(j+1)h} |g(u)|^2 \, du \leq h \left\{ \sup_{jh \leq u < (j+1)h} |g(u)| \right\}^2
$$

for all measurable functions $g$.

The “amalgam norm” estimates (7.1) and (7.2) follow from $\ell^2$ estimates on sets of suitably separated points, $x_j = (u_j, u_j^2)$ say, in $\Gamma$. Let $w \in A(\mathbb{R}^2)$ and require it to vanish that $w$ vanishes on the region where $v > u^2 + k$, or on the region where $v < u^2 - k$. Then

$$
\left[ \sum_j |w(x_j)|^2 \right]^{1/2} \leq C\|w\|_{A(\mathbb{R}^2)} \quad \text{if } \Delta u_j > \sqrt{k/2} \text{ for all } j.
$$

Apply this to shifted copies $\hat{w}$ of $\hat{f}$, and choose points $x_j$ in alternate intervals $[j'h, (j'+1)h)$ to get the estimates (7.1) and (7.2).

In proving inequality (7.3), we consider more general sets $D$ of the form $\{(u, v) : v \geq \phi(u)\}$, where $\phi''(u) \geq c > 0$. Our methods apply to $A(\mathbb{R}^2)$, and yield inequality (7.3) if the sets $\text{Alt}(x_j)$ and $\text{Sch}(x_j)$ are respectively included in the sets $\text{Int}(D) + (0, k)$ and $\text{Int}(D^c) - (0, k)$.

Given a point $x$ in $\text{Sch}(x_j)$ in the form (5.3), let $n = \sum_{j' = m}^{n_0} n_{j'}$ and say that $x$ is an $n$-th generation descendant of $x_i$. Subtracting another copy of $\Delta x_j$, where $j \geq i$, from $x$ gives an $(n + 1)$-st descendant, $x'$ say. All descendants $(u, v)$ of $x_i$ share the property that $u < u_i$. Visual arguments in the style of Section 4 show that if $\phi$ is stricly convex and $x \in \text{Int}(D^c - (0, k))$, then $x' \in \text{Int}(D^c - (0, k))$ too.

So it suffices to check that first-generation points in $\text{Sch}(x_j)$ belong to $\text{Int}(D^c - (0, k))$. They have the form $x_i - \Delta x_j$ where $j \geq i$. Rewriting this as $(u, v) = (u_i, v_i) - (\Delta u_j, \Delta v_j)$ reduces matters to showing...
that \( \phi(u) - v > k \). Now

\[
v_i = \phi(u_i) = \phi(u) + \int_{u}^{u_i} \phi'(r) \, dr, \quad \phi(u) = v_i - \int_{0}^{\Delta u_j} \phi'(u + s) \, ds, \]

\[
\Delta v_j = \int_{u_j}^{u_{j+1}} \phi'(r) \, dr, \quad \text{and} \quad v = v_i - \Delta v_j = v_i - \int_{0}^{\Delta u_j} \phi'(u_j + s) \, ds.
\]

Therefore,

\[
\phi(u) - v = \int_{0}^{\Delta u_j} \left[ \phi'(u_j + s) - \phi'(u + s) \right] \, ds
\]

\[
= \int_{0}^{\Delta u_j} \left[ \int_{u}^{u_j} \phi''(t + s) \, dt \right] \, ds \geq c(\Delta u_j)^2.
\]

Use the representation (5.5) to introduce a similar notion of generations of descendants in \( \text{Alt}(x_j) \), but add the requirement that the extra nonzero coefficient \( n_{j'} \) for the child \( x' = (u', v') \) occurs before all nonzero coefficients for the parent \( x \). Rename \( j_{2i+1} \) as \( j + 1 \); then \( u' \geq u_{j'} + \Delta u_j \). Argue visually to reduce matters to first-generation cases where

\[
x' = (u', v') = (u_{j+1}, v_{j+1}) - (\Delta u_{j'}, \Delta v_{j'}), \quad \text{and} \quad j' < j.
\]

As above,

\[
v' - \phi(u') = \int_{0}^{\Delta u_{j'}} \left[ \int_{u}^{u'} \phi''(t + s) \, dt \right] \, ds \geq c(\Delta u_{j'})\Delta u_j.
\]

The inclusions \( \text{Sch}(x_j) \subset \text{Int}(D^c - (0, k)) \) and \( \text{Alt}(x_j) \subset \text{Int}(D + (0, k)) \) follow if \( \Delta u_j > \sqrt{k/c} \) for all \( j \).

The outcome changes if the graph of \( \phi \) has an asymptote.

**Example 7.1.** Let \( D_\alpha = \{ (u, v) : u > 0, v > u^{-\alpha} \} \), where \( \alpha \) is a positive constant. Let \( f \in L^1(\mathbb{R}^2) \), and let \( k > 0 \). If \( \hat{f} \) vanishes on \( \text{Int}(D_\alpha^c) \), then

\[
\int_{0}^{\infty} \left| \hat{f} (u, u^{-\alpha} + k) \right|^2 \frac{du}{u} \frac{1}{2} \leq C\| f \|_1.
\]

There are cases where \( f \in L^1(\mathbb{R}^2) \) and \( \hat{f} \) vanishes on \( \text{Int}(D_\alpha) \) but

\[
\int_{0}^{\infty} \left| \hat{f} (u, u^{-\alpha} - k) \right|^2 \frac{du}{u} = \infty.
\]

The positive result here follows from the extension of Paley’s inequality to functions \( f \) in \( L^1(\mathbb{R}^2) \) for which \( \hat{f}(u, v) = 0 \) on the “negative”
semigroup, \(-P\) say, where \(u \leq 0\) and \(v < 0\) if \(u = 0\). That extension gives an \(\ell^2\) estimate for \((f(x_j))\) when the sequence \((x_j)\) satisfies the Hadamard condition that \(2x_j - x_{j+1} \in -P\) for all \(j\). So do the appropriate methods in Sections 5 or 6. These approaches all show that

\[
\left[ \int_{0}^{\infty} \sup_{v \in \mathbb{R}} \left| \hat{f}(u, v) \right| \frac{2}{u} \, du \right]^{1/2} \leq C \|f\|_1.
\]

To get the negative results, use the fact that for each parallelogram, \(B\) say, with positive area, there is a function in the unit ball of \(A(\mathbb{R}^2)\) that vanishes outside \(B\) and that exceeds \(1/4\) on \(1/4\) of the area of \(B\). One way to confirm this fact runs via the argument applied to arithmetic progressions in [14, p. 861].

Similar reasoning, going back to [12], shows that if a nonnegative measure \(\nu\) has the property that

\[
\int_{0}^{\infty} \left| \hat{f}(u, u^{-\alpha} + k) \right|^2 \, d\nu(u) < \infty
\]

whenever \(f \in L^1(\mathbb{R}^2)\) and \(\hat{f}\) vanishes on Int\((D^\alpha)\), then

\[
\nu((2^j, 2^{j+1})] \leq C' \quad \text{for all } j.
\]

**Remark 7.2.** Affine arclength measure is prominent in restriction theorems [9] for transforms of functions in \(L^p(\mathbb{R}^2)\) when \(p > 1\). The measure \(du\) on the graphs of \(v = u^2 \pm k\) is affine invariant, but the measure \(du/u\) on the graph of \(v = \phi_\alpha(u) + k\) is not, except when \(\alpha = 1\).

**Appendix A. Two orthogonality steps**

We prove both lemmas by splitting the sequence \((g, A_jh)_{j=1}^\ell\) as a sum of two sequences whose \(\ell^2\) norms are easy to bound.

In Lemma 3.2 let \(P_j\) and \(Q_j\) be the orthogonal projections onto the subspaces \(L_j\) and \(A_{j+1}L_j\) respectively, with \(j < J\) in the latter case. Also let \(Q_J = I\) and \(Q_0 = 0\). By the membership condition in the lemma,

\[(A.1) \quad (g, A_jh) = (g, Q_jA_jh) = (Q_jg, A_jh) = a_j + b_j,
\]

where \(a_j = ((Q_j - Q_{j-1})g, A_jh)\) and \(b_j = (Q_{j-1}g, A_jh)\) for all \(j\). Then \(b_1 = (Q_0g, A_jh) = 0\), and \(b_j = (g, A_j(P_{j-1} - P_j)h)\) when \(j > 1\), since \(A_jP_{j-1} = Q_{j-1}A_j\) and \((g, A_jP_jh) = 0\) in that case. The projections \(Q_j - Q_{j-1}\) have mutually orthogonal ranges, as do the projections \(P_{j-1} - P_j\). By Cauchy-Schwarz, \(\|(a_j)\|_2\) and \(\|(b_j)\|_2\) are both bounded above by \((\|g\|_H)\|h\|_H\), and inequality (3.3) follows.
In Lemma 3.1, consider the orthogonal projections $P_j$ and $Q_j$ onto the subspaces $M_j$ and $A_j M_j$. Also let $Q_{j+1} = I$ and $P_0 = 0$. This time, $(g, A_j h) = (Q_{j+1} g, A_j h)$, which splits as
\[(A.2) \quad ((Q_{j+1} - Q_j) g, A_j h) + (g, A_j (P_j - P_{j-1}) h),\]
since $(g, A_j P_{j-1} h) = 0$ and $(g, A_j P_j h) = (Q_j g, A_j h)$. Finish as above.

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