Reynolds number dependency of small scale structures in steady anisotropic turbulence produced by implicit large-eddy simulation

Mayuka Oshibuchi¹, Hiroki Suzuki²,³ and Shinsuke Mochizuki¹

¹Graduate School of Sciences and Technology for Innovation, Yamaguchi University, 2-16-1 Tokiwadai, Ube-shi, Yamaguchi 755-8611, Japan
²Graduate School of Natural Science and Technology, Okayama University, 3-1-1 Tsushima-naka, Kita-ku, Okayama-shi, Okayama 700-8530, Japan
³Corresponding author’s e-mail: h.suzuki@okayama-u.ac.jp

Abstract. This study presents small-scale fluctuation characteristics of anisotropic steady turbulence reproduced by implicit large eddy simulation (LES). The Reynolds number dependence of this small-scale fluctuation characteristic is approached in this study. This study focuses on that, a small scale turbulence field is not needed to be isotropic if the Reynolds number is sufficiently high. The anisotropic steady turbulence is maintained steady by using the forcing terms in the governing equations. The results of the implicit LES are compared with those obtained by direct numerical simulation (DNS) and LES based on the Smagorinsky model. Spatial derivatives are discretized using a fourth-order central difference scheme that conserves the kinetic energy of the turbulence field. The governing equations are integrated for the temporal direction using the fourth-order Runge-Kutta method. The results of enstrophy and small-scale turbulence characteristics quantified by the isotropy parameter are found to be consistent between the implicit LES and DNS.

1. Introduction

Large-eddy simulation (LES) [1] technique has been widely used in the field of fluids engineering. For example, the LES technique is used to develop industrial equipments related to fluids engineering. In recent years, the performance of computers has improved remarkably, so the technique of LES can be applied more widely. Turbulent flow fields are generally expected to have small-scale fluctuations with universal properties when the Reynolds number is sufficiently high. In LES techniques, by focusing on this point and modeling the small-scale turbulence fluctuations, the required computational resources can be significantly reduced compared with that of direct numerical simulation.

In LES, a sub-grid scale (SGS) model needs to describe small scale turbulence fluctuations. The most widely used SGS model is the Smagorinsky model [1]. The Smagorinsky model is constructed based on the dimensional analysis of turbulent eddy viscosity. The Smagorinsky model has been widely used also in general-purpose solvers such as OpenFOAM® [2]. Also, implicit large-eddy simulation [3,4] has been used as an LES method without the use of the SGS model. In ILES, the truncation error due to the spatial discretization is equivalent quantitatively to an SGS model of a large-eddy simulation. The results of these LES methods have been validated by comparing with those of DNS (e.g., [5-7]). Since small-scale turbulence fluctuations are not modeled in the DNS technique, it is necessary to analyze
small-scale turbulent fields as well as large-scale fields but also with sufficient accuracy. Therefore, the computational cost of DNS is significantly larger than that of LES techniques.

Assumption that small-scale fluctuations are universal and isotropic used in the SGS model of the LES techniques can be held when the Reynolds number is sufficiently high. At conditions of low Reynolds numbers, small-scale turbulence fields may not be isotropic if large-scale fields are not isotropic [1]. For example, a turbulence field near the wall in wall turbulence is known to be anisotropic. The SGS eddy viscosity coefficient introduced in the Smagorinsky model is assumed to be isotropic. On the other hand, implicit LES may not assume that small-scale fields are isotropic because of the use of discretization error as an SGS model. This study considers that the small-scale turbulence field reproduced by implicit LES, an LES method without the use of the SGS model, should be compared with that by an LES based on the Smagorinsky model when analyzing low Reynolds number turbulence with LES.

This study aims to approach small-scale turbulence structures of steady turbulence obtained by a technique of implicit LES. In particular, this study aims to clarify the basic properties of small-scale structures predicted by using implicit large-eddy simulation in anisotropy turbulence, which is ideal turbulence. Most of the previous studies used complex flows, such as a turbulent channel flow. For this reason, this study considers that the characteristics of the small-scale structures obtained by ILES were not sufficiently clear. In this study, the accuracy of ILES was clarified by using steady anisotropic turbulence as idealized turbulence. Here, the Reynolds number dependency of anisotropic steady turbulence is analyzed. In anisotropic turbulence with sufficiently high Reynolds number, small-scale turbulence fields are maintained to be isotropic. As the Reynolds number decreases, small-scale turbulent flow fields may become anisotropic. In this study, the small-scale turbulent flow fields obtained by the present implicit LES is analyzed using the viewpoint of small-scale isotropy. Results of implicit LES are compared with those of direct numerical simulation and LES based on the standard Smagorinsky model. Two types of forcing terms, which are isotropic and anisotropic forcing terms, included in the governing equations are used to maintain steady turbulent flow to be anisotropic and isotropic.

2. Numerical methods

Figure 1 shows a schematic figure of the present simulation. The governing equations are the continuity and Navier-Stokes equations. Steady turbulent flows reproduced in a periodic box are analyzed. The forcing terms of the governing equation \( F, (F_x, F_y, F_z) \), are used to maintain this steady turbulence, where \( x, y \) and \( z \) are streamwise, transverse, and spanwise directions, respectively. In this study, isotropic and anisotropic steady turbulent flows are analyzed. The forcing terms for the isotropic and anisotropic turbulence fields are given, respectively, as follows:

\[
(F_x, F_y, F_z) = (-\cos(x) \sin(y), \sin(x) \cos(y), 0)
\]
and

\[
(F_x, F_y, F_z) = \left( \frac{2}{3^{1/2}}, \frac{1}{2} \right) \left( -\cos(x) \sin(y) + \cos(y) \sin(z) \right), \\
\quad \left( \frac{2}{3^{1/2}}, \frac{1}{2} \right) \left( -\cos(y) \sin(z) + \cos(z) \sin(x) \right), \\
\quad \left( \frac{2}{3^{1/2}}, \frac{1}{2} \right) \left( -\cos(z) \sin(x) + \cos(x) \sin(y) \right). 
\]

Here, these forcing terms satisfy the following conditions: \( \nabla \cdot F = 0 \) and \( \nabla^2 F = -F \) [8]. Therefore, the property of the forcing terms is similar between the isotropic and anisotropic turbulence fields.

This numerical analysis is based on previous studies [9-12]. Spatial derivatives in the governing equations are discretized using the fourth-order central difference scheme proposed by the previous study, (Morinishi et al. (1998)) [13]. This central difference scheme can conserve the kinetic energy explicitly. The governing equations are analyzed using the fractional step method based on the fourth-order Runge-Kutta method [14]. The Poisson’s equations in fractional steps are solved using the fast Fourier transform as the direct solver. The conservation error of mass described by the continuity equation is of the order of machine epsilon.

Three types of analysis conditions are used in this analysis. First, the implicit large-eddy simulation (ILES) does not require the SGS eddy viscosity model. In this research, the implicit LES is mainly analyzed to solve the present purpose, to approach small-scale turbulence structures obtained by implicit LES. The results of the implicit LES are validated by comparing with those of direct numerical simulation (DNS). The implicit LES results are further compared with those obtained by the LES based on the Smagorinsky model (SLES) [1]. Here, the value of the constant of the Smagorinsky model, for which the value should be set, is set to 0.1 as a commonly used value. Here, the Smagorinsky model is used in this study because this model is commonly used in the field of fluids engineering. The isotropic and anisotropic steady turbulence fields are reproduced in a periodic box in this analysis. The computational Reynolds number is changed in the range of 10 - 500 to investigate the Reynolds number dependency of the turbulent flow field. The number of grid points \( N^3 \) is set to \( N^3 = 16^3 \sim 128^3 \) for the DNS condition. In the two types of LES, \( N^3 \) is set as \( N^3 = 32^3 \). Turbulence statistics are calculated in the time range where the isotropic and anisotropic turbulence fields are maintained steady.

Figure 2. Validation results using kinetic energy conservation obtained by using periodic inviscid flow. Here the line denotes a function proportional to \( \Delta t^4 \) and the square plots indicate the present results.
3. Results and discussion
Before showing the present numerical results, a verification result of this numerical analysis is shown in this paragraph and Figure 2. Here, the conservation characteristics of kinetic energy are examined. A three-dimensional inviscid periodic fluctuation field is used to verify this kinetic energy conservation. The kinetic energy is analytically conserved in this flow field. The size of the computational region is \( L^3 = 2\pi^3 \). The number of grid points is \( N^3 = 16^3 \). The initial velocity field is the field of a Gaussian random number series [15-16] that satisfies the continuity equation [8]. Due to the influence of the kinetic energy conservation error, the kinetic energy slightly deviates from the initial value as time proceeds. The rate of temporal change of the kinetic energy at the initial time is shown in Figure 2 as a time increment function. As shown in the figure, the rate of change of the kinetic energy with time is proportional to the fourth-order of the time increment, where the second-order and third order Runge-Kutta schemes were also used to check our numerical code by using this flow (not shown). When the fourth-order Runge-Kutta method is used, the kinetic energy conservation error is proportional to the fourth-order of the increment. The agreement of the accuracy order to the time increment shows that this spatial discretization scheme that explicitly conserves the kinetic energy is correctly constructed in the present numerical code, which is in-house. When the time increment is small, the rate of change of the kinetic energy with time increment can be negligible. This result shows that the kinetic energy is sufficiently conserved in this numerical analysis.

Then, the present study examines the large-scale structures of the anisotropic turbulence field. Here this study introduces the Lumley triangle [1] to characterize large-scale structures. The anisotropic tensor is expressed as follows: \( b_{ij} = \langle u_i u_j \rangle / \langle u_k u_k \rangle - (1/3) \delta_{ij} \). By using this anisotropic tensor, two quantities \( \xi \) and \( \eta \) [1] are defined as follows: \( 6\xi^2 = b_{ij} b_{ij} \) and \( 6\eta^2 = b_{ij} b_{ji} \), where the two quantities are nondimensional in the present analysis. By using these two quantities, a large-scale structure is represented, as shown in Figure 3(a). If these two quantities are zero, the related structure can be considered to have the isotropic nature shown in Figure3 (a). When the two quantities are non-zero, types of the shape of Reynolds-stress ellipsoids for 1C and 2C are found to be line and ellipse, respectively. In this study, these two quantities for the isotropic turbulence field are certain zero. Therefore, the characteristics of the isotropic turbulent flow field are located at the point of “iso” in the triangle. On the other hand, the characteristics of the present anisotropic turbulence fields are located on the straight line connecting the point 2C to “iso”.

Figure 3(b) shows the model-constant dependency of the large-scale structure characterized by the Lumley triangle. Here, as shown in Figure 3(b), implicit LES results are slightly different from those of DNS. Still, these results are comparable between the two types of simulations. These comparable results are found for both quantities of the Lumley triangle. For the LES based on the Smagorinsky model, values of the two quantities that characterize the large-scale structure are in agreement with those of the implicit LES. Furthermore, when the Smagorinsky model is used, large-scale structures are insensitive to the value of the model constant. These results show that the properties of large-scale structures are similar between two types of LES when comparing the properties of small-scale structures between implicit LES and LES based on the Smagorinsky model. Also, as shown in the figure, the effects of the set value of the model constant required in the cases using the Smagorinski model on the results can be considered to be negligible.

The small-scale structure is calculated and examined. The present analysis approaches small-scale structures using a quantity characterizing local isotropy. Therefore, this study uses the isotropy parameter to quantify the nature of small-scale structures. The isotropy parameter of the small-scale structure is defined as follows:

\[
I_s = \frac{\langle \tilde{w} \tilde{w} \tilde{z} \rangle}{\langle \tilde{u} \tilde{w} \tilde{x} \rangle^2},
\]

where \( u \) and \( w \) are velocity components for \( x \) and \( z \) directions, respectively. Here, the value of the isotropy parameter is unity when the calculated field is isotropic. Therefore, if a parameter value is a unity, the local isotropy is considered to hold. Figure 4(a) shows a parameter value of the small-scale
velocity field when the large-scale field is isotropic. As shown in the figure, since the flow field of a large-scale is isotropic, the small-scale velocity field is also found to be isotropic. This result is found among DNS and two types of LES. This result verifies the use of the isotropy parameter in this numerical analysis.

Figure 4(b) shows a value of isotropy parameter of the small-scale velocity field when the large-scale flow field is anisotropic. As shown in the figure, the parameter value of DNS results is unity when the computational Reynolds number is sufficiently high. This value is also found in the results of the two types of LES. Therefore, when the computational Reynolds number is small, the small-scale velocity field is considered to be anisotropic. The parameter values are consistent between the results of implicit LES and LES using the Smagorinsky model.

The Reynolds number dependency of enstrophy is shown in Figure 5. Here, the enstrophy is defined as the intensity of vorticity as follows:

\[ I_\eta = \frac{\langle (\hat{\varepsilon}u/\hat{\varepsilon}y)^2 \rangle}{\langle (\hat{\varepsilon}u/\hat{\varepsilon}y)^2 \rangle}. \]  

Figure 4(c) shows the Reynolds number dependency of the isotropy parameter based on the lateral spatial derivatives. As shown in the figure, for the LES based on the Smagorinsky model, the value of this parameter deviates from unity at conditions of the low computational Reynolds number. The value of this parameter obtained using the implicit LES is in good agreement with that of the LES based on the Smagorinsky model. Also, under conditions of high computational Reynolds number, the value of this parameter is unity for the two types of LES. As shown in this result, in small-scale turbulent structures characterized by the lateral spatial differentiation, the implicit LES can give similar results as the LES based on the Smagorinsky model.

The Reynolds number dependency of enstrophy is shown in Figure 5. Here, the enstrophy is defined as the intensity of vorticity as follows:

\[ I_\eta = \frac{\langle (\hat{\varepsilon}u/\hat{\varepsilon}y)^2 \rangle}{\langle (\hat{\varepsilon}u/\hat{\varepsilon}y)^2 \rangle}. \]  

Figure 3. Effects of changing model coefficient value of SLES on the large-scale structure characterized by Lumley triangle.
\[ \Omega = \frac{1}{2} \left( \langle \omega_x^2 \rangle + \langle \omega_y^2 \rangle + \langle \omega_z^2 \rangle \right), \]

where \( \omega_x, \omega_y, \) and \( \omega \) are components of the vorticity for \( x, y, \) and \( z \) directions, respectively, and \( \langle \ldots \rangle \) denotes ensemble average. Since the enstrophy is calculated from the vorticity, this turbulence quantity can be considered to characterize the nature of small-scale fluctuations. As shown in the figure, the value of enstrophy in DNS increases as the computational Reynolds number increases. This result is found in both of the isotropic and anisotropic turbulence fields. The enstrophy value of the isotropic turbulence is larger than that of the anisotropic field in the whole range of the Reynolds number. One of the significant differences between isotropic and anisotropic turbulence fields is the presence or absence of the pressure-strain correlation terms of the kinetic energy equation. This larger enstrophy in the isotropic field may be related to the absence of the pressure-strain correlation terms of the kinetic energy equation. The enstrophy of implicit LES is in good agreement with that of DNS. This agreement is found in both the isotropic and anisotropic turbulence fields. In LES using the Smagorinsky model, the enstrophy value is reduced, especially in the range of higher computational Reynolds number. As the number of Reynolds number decreases, the magnitude of the small-scale turbulent components to be modeled by using SGS dissipation decreases. This reduced enstrophy is considered to be due to the SGS dissipation used in the LES model. This study has focused on showing the analysis results related to small-scale structures given by ILES. We consider that detailed discussion to clarify the reasons will be planned in the future using additional analyses.
4. Conclusions
The purpose of this study is to validate small-scale turbulence structures obtained by implicit large-eddy simulation. Here, anisotropic steady turbulence is used to address the present purpose. We focus on the Reynolds number dependency of the small-scale structure obtained by the present implicit LES. The results of this implicit LES are compared with those obtained by LES based on the Smagorinsky model and DNS. The isotropic and anisotropic steady turbulence flow fields used in this study are set using force terms of the governing equations. Here, the property of the force terms is set to be quite similar between the isotropic and anisotropic flow fields. Two types of steady turbulence are reproduced using the computational domain of a periodic cube. By changing the computational Reynolds number, the Reynolds number dependency of the steady turbulence is set. If the Reynolds number is sufficiently high, the small-scale turbulence field is held to be isotropic even if the large-scale turbulence field is anisotropic.

First, the kinetic energy conservation accuracy was verified to be sufficiently high in this numerical analysis using the inviscid periodic fluctuation field. The Lumley triangle is introduced to investigate turbulence structures of the large-scale structure. The large-scale turbulent structures obtained by the implicit LES are found to be similar to that of LES based on the Smagorinsky model using this Lumley triangle. Also, in the LES based on the Smagorinsky model, the set values of the model constant hardly affect the results of Lumley's triangle. The small-scale turbulence structures obtained by the implicit LES were investigated using two isotropy parameters. Here, the two isotropic parameters are defined by using the longitudinal and lateral spatial derivatives, respectively. The Reynolds number dependency of the isotropy parameter by the implicit LES agrees well with those of the LES based on the Smagorinsky model and DNS. The Reynolds number dependency of the enstrophy was shown. The result of enstrophy obtained by the implicit LES was in good agreement with that obtained by DNS.

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