A New Proof of Hilbert’s Theorem on Ternary Quartics

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Abstract
Hilbert proved that a non-negative real quartic form $f(x,y,z)$ is the sum of three squares of quadratic forms. We give a new proof which shows that if the plane curve $Q$ defined by $f$ is smooth, then $f$ has exactly 8 such representations, up to equivalence. They correspond to those real 2-torsion points of the Jacobian of $Q$ which are not represented by a conjugation-invariant divisor on $Q$.

1. Introduction
A ternary quartic is a homogeneous polynomial $f(x,y,z)$ of degree 4 in three variables. If $f$ has real coefficients, then $f$ is non-negative if $f(x,y,z) \geq 0$ for all real $x, y, z$. Hilbert [5] showed that every non-negative real ternary quartic form is a sum of three squares of quadratic forms. His proof (see [8], [9] for modern expositions) was non-constructive: The map
$$\pi: (p,q,r) \mapsto p^2 + q^2 + r^2$$
from triples of real quadratic forms to non-negative quartic forms is surjective, as it is both open and closed when restricted to the preimage of the (dense) connected set of non-negative quartic forms which define a smooth complex plane curve. An elementary and constructive approach to Hilbert’s theorem was recently begun by Pfister [6].

A quadratic representation of a complex ternary quartic form $f = f(x,y,z)$ is an expression
$$f = p^2 + q^2 + r^2$$
where $p, q, r$ are complex quadratic forms. A representation $f = (p')^2 + (q')^2 + (r')^2$ is equivalent to this if $p, q, r$ and $p', q', r'$ have the same linear span in the space of quadratic forms.
Powers and Reznick [7] investigated quadratic representations computationally, using the Gram matrix method of [1]. In several examples of non-negative ternary quartics, they always found 63 inequivalent representations as a sum of three squares of complex quadratic forms; 15 of these were sums or differences of squares of real forms. We explain these numbers, in particular the number 15, and show that precisely 8 of the 15 are sums of squares.

If the complex plane curve $Q$ defined by $f = 0$ is smooth, it has genus 3, and so the Jacobian $J$ of $Q$ has $2^6 - 1 = 63$ non-zero 2-torsion points. Coble [2] showed that these are in one-to-one correspondence with equivalence classes of quadratic representations of $f$. If $f$ is real, then $Q$ and $J$ are defined over $\mathbb{R}$. The non-zero 2-torsion points of $J(\mathbb{R})$ correspond to signed quadratic representations $f = \pm p_1 \pm p_2 \pm p_3$, where $p_i$ are real quadratic forms. If $f$ is also non-negative, the real Lie group $J(\mathbb{R})$ has two connected components, and hence has $2^4 - 1 = 15$ non-zero 2-torsion points. We use Galois cohomology to determine which 2-torsion points give rise to sum of squares representations over $\mathbb{R}$.

**Theorem 1** Suppose that $f(x, y, z)$ is a non-negative real quartic form which defines a smooth complex plane curve $Q$. Then the inequivalent representations of $f$ as a sum of three squares are in one-to-one correspondence with the eight 2-torsion points in the non-identity component of $J(\mathbb{R})$, where $J$ is the Jacobian of $Q$.

Wall [10] studies quadratic representations of (possibly singular) complex ternary quartic forms $f$. Again, in the irreducible case, the non-trivial 2-torsion points on the generalized Jacobian give equivalence classes of quadratic representations of $f$. These representations are special in that they have no basepoints.

Quadratic representations with a given base locus $B$ correspond to the 2-torsion points on the Jacobian of a curve $\tilde{Q}$, which is the image of $Q$ under the complete linear series of quadrics through $B$. Classifying all possibilities for $B$ gives the number of inequivalent quadratic representations of $f$. If $f$ is real and non-negative, this classification, together with arguments from Galois cohomology, gives all inequivalent representations of $f$ as a sum of squares. This complete analysis will appear in an unabridged version.

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2. **Basepoint-free quadratic representations**

Let $f(x, y, z)$ be an irreducible quartic form over $\mathbb{C}$, and let $Q$ be the complex plane curve $f = 0$. The Picard group Pic$(Q)$ of $Q$ is the group of Weil divisors on the regular part of $Q$, modulo divisors of rational functions which are invertible around the singular locus of $Q$. Let $J_Q$ be the generalized Jacobian of $Q$, so that $J_Q(\mathbb{C})$ is the identity component of Pic$(Q)$. Its structure may be determined from the Jacobian of the normalization $\tilde{Q}$ of $Q$ via the exact sequence [4, Ex. II.6.9]

$$0 \to \bigoplus_{p \in Q} \tilde{O}_p^*/O_p^* \to J_Q(\mathbb{C}) \to J_{\tilde{Q}}(\mathbb{C}) \to 0,$$

where $O_p$ is the local ring of $Q$ at $p$, $\tilde{O}_p$ is its normalization, and $^*$ indicates the group of units.

The base locus $B$ of a quadratic representation (1) of $f$ is the zero scheme of the homogeneous ideal generated by the forms $p, q, r$. The closed subscheme $B$ is supported on the singular locus of $Q$. We say that (1) is basepoint-free if $B$ is empty.

**Proposition 1** [Coble [2], Wall [10]] The non-trivial 2-torsion points of $J_Q$ are in one-to-one correspondence with the equivalence classes of basepoint-free quadratic representations of $f$.

**Proof.** Given a basepoint-free quadratic representation (1), consider the map

$$\varphi : \mathbb{P}^2 \to \mathbb{P}^2, \quad x \mapsto (p(x) : q(x) : r(x)).$$
The image of $Q$ under $\varphi$ is the conic $C$ defined by the equation $y_0^2 + y_1^2 + y_2^2 = 0$. Let $y$ be a point in $C$ whose preimages are regular points of $Q$. Then $\varphi^*(y)$ is an effective divisor of degree 4 that is not the divisor of a linear form. Indeed, after a linear change of coordinates we can assume $y = (0 : 1 : i)$. A linear form vanishing on $\varphi^*(y)$ would divide each conic $\alpha p + \beta(q + ir)$ through $\varphi^*(y)$, and thus would divide

$$f = p^2 + (q + ir)(q - ir),$$

contradicting the irreducibility of $f$.

Fix a linear form $\ell$ that does not vanish at any singular point of $Q$. Then $L := \text{div}(\ell)$ is an effective divisor of degree 4 on $Q$. Let $\zeta = [\varphi^*(y) - L]$. Since $2y$ is the divisor of a linear form (the tangent line to $C$ at $y$), $\varphi^*(2y)$ is the divisor on $Q$ of a quadratic form. Thus $2\zeta = 0$. Moreover, $\zeta \neq 0$ as $\varphi^*(y)$ is not the divisor of a linear form. The 2-torsion point $\zeta$ of $J_Q$ depends only upon the map $\varphi$.

Conversely, suppose that $\zeta \in J_Q(\mathbb{C})$ is a non-zero 2-torsion point. Let $D \neq D'$ be effective divisors which represent the class $\zeta + [L]$ in $\text{Pic}(Q)$. As $Q$ has arithmetic genus 3, the Riemann-Roch Theorem implies that there is a pencil of such divisors. Then $2D, 2D'$ and $D + D'$ are effective divisors of degree 8, and are all linearly equivalent to $2L$, the divisor of a conic. By the Riemann-Roch Theorem there are quadratic forms $g_0, g_1$ and $q_2$ such that

$$\text{div}(g_0) = 2D, \quad \text{div}(g_1) = 2D' \quad \text{and} \quad \text{div}(q_2) = D + D'.$$

Therefore, the rational function $g := g_0q_1/q_2^2$ on $Q$ is constant. Scaling $g_1$ and $q_2$ appropriately, we may assume that $g \equiv 1$ on $Q$ and also that $f = g_0q_1 - q_2^2$. Diagonalizing the quadratic form $g_0q_1 - q_2^2$ gives a quadratic representation for $f$. This defines the inverse of the previous map. □

3. Quadratic representations of real quartics

Suppose now that $f$ is a non-negative real quartic form defining a real plane curve $Q$ with complexification $Q_{\mathbb{C}} = Q \otimes_{\mathbb{R}} \mathbb{C}$. The elements of $\text{Pic}(Q)$ can be identified with those divisor classes in $\text{Pic}(Q_{\mathbb{C}})$ that are represented by a conjugation-invariant divisor. Let $J$ be the generalized Jacobian of $Q$.

If $\zeta \in J(\mathbb{C})$ is the 2-torsion point corresponding to a signed quadratic representation

$$f = \pm p^2 + q^2 \pm r^2$$

consisting of real polynomials $p, q, r$, then $\zeta = \bar{\zeta}$, i.e., $\zeta \in J(\mathbb{R})$.

Conversely, let $0 \neq \zeta \in J(\mathbb{R})$ with $2\zeta = 0$. Choose a real linear form $\ell$ not vanishing on the singular points of $Q$, and let $L := \text{div}(\ell)$. We can choose effective divisors $D \neq \overline{D}$ on $Q_{\mathbb{C}}$ representing the class $\zeta + [L]$. Then $2D, 2\overline{D}$ and $D + \overline{D}$ are each equivalent to $2L$. Let $r$ be a real quadratic form with divisor $D + \overline{D}$, and let $g$ be a (complex) quadratic form with divisor $2D$ (both divisors taken on $Q_{\mathbb{C}}$).

Since $D \sim \overline{D}$, there is a rational function $h$ on $Q_{\mathbb{C}}$, invertible around $Q_{\text{sing}}$, with $\text{div}(h) = D - \overline{D}$. Let $c = h(r)$, a nonzero real constant on $Q$. Since $\text{div}(r) = \text{div}(g) + \text{div}(h)$, there is a complex number $\alpha \neq 0$ with $\frac{\alpha}{\bar{\alpha}} = \alpha h$ on $Q$, which implies that

$$c|\alpha|^2 = \frac{r}{g} \overline{\frac{r}{g}} = \frac{r^2}{p^2 + q^2}$$

on $Q$, where $p, q$ are the real and imaginary parts of $g = p + iq$. So the quartic form

$$u := r^2 - c|\alpha|^2(p^2 + q^2)$$

vanishes identically on $Q$. Since $u \neq 0$, $f$ is a constant multiple of $u$. If $c > 0$, we get a signed quadratic representation of $f$, with both signs occurring. If $c < 0$, $f$ must be a positive multiple of $u$ since $f$ is non-negative, and we get a representation of $f$ as a sum of three squares of real forms.
We now calculate the sign of \( c \). For this we use the exact sequence

\[
0 \to \text{Pic}(Q) \to \text{Pic}(Q_C)^G \xrightarrow{\partial} \text{Br}(\mathbb{R}) \to H^2_{\text{ét}}(Q, \mathbb{G}_m) \tag{2}
\]

of étale cohomology groups. It arises from the Hochschild-Serre spectral sequence for the Galois covering \( Q_C \to Q \) and coefficients \( \mathbb{G}_m \). Here \( G = \text{Gal}(\mathbb{C}/\mathbb{R}) \) acts on \( \text{Pic}(Q_C) \) by conjugation, and \( \text{Pic}(Q_C)^G \) is the group of \( G \)-invariant divisor classes. Moreover, \( \text{Br}(\mathbb{R}) = H^2_{\text{ét}}(\text{Spec} \mathbb{R}, \mathbb{G}_m) \) is the Brauer group of \( \mathbb{R} \) (which is of order 2), and \( \text{Br}(\mathbb{R}) \to H^2_{\text{ét}}(Q, \mathbb{G}_m) \) is the restriction map.

It is easy to see that \( c < 0 \) if and only if \( \partial(\zeta) \) is the non-trivial class in \( \text{Br}(\mathbb{R}) \). If \( Q \) has an \( \mathbb{R} \)-point, then \( \text{Br}(\mathbb{R}) \to H^2_{\text{ét}}(Q, \mathbb{G}_m) \) has a splitting given by that point, and hence \( \partial \) vanishes identically.

If \( Q \) is smooth, then \( f \) non-negative forces \( Q(\mathbb{R}) = \emptyset \), and the map \( \text{Br}(\mathbb{R}) \to H^2_{\text{ét}}(Q, \mathbb{G}_m) \) is zero. In this case, \( \text{Pic}(Q_C)^G \) contains an odd degree divisor if and only if the genus of \( Q \) is even and \( J(\mathbb{R})^0 \), the identity connected component of the real Lie group \( J(\mathbb{R}) \), is the kernel of the restriction \( J(\mathbb{R}) \to \text{Br}(\mathbb{R}) \) of \( \partial \) [11,3]. Since in our case \( g(Q) = 3 \), this implies that the sequence

\[
0 \to J(\mathbb{R})^0 \to J(\mathbb{R}) \xrightarrow{\partial} \text{Br}(\mathbb{R}) \to 0
\]

is (split) exact. If \( Q \) is singular with \( Q(\mathbb{R}) = \emptyset \), one compares sequence (2) for \( Q \) to the same sequence for the normalization \( \tilde{Q} \) of \( Q \) and concludes that \( \partial : J(\mathbb{R}) \to \text{Br}(\mathbb{R}) \) is surjective as well.

We complete the proof of Theorem 1. Since \( f \) is non-negative and \( Q \) is smooth of genus 3, \( J(\mathbb{R})^0 \cong (S^1)^3 \) as a real Lie group. By the facts just mentioned, there exist \( 2^4 - 1 = 15 \) non-zero 2-torsion elements in \( J(\mathbb{R}) \). The 8 that do not lie in \( J(\mathbb{R})^0 \), or equivalently, which cannot be represented by a conjugation-invariant divisor on \( Q_C \), are precisely those that give rise to sums of squares representations of \( f \).

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