High-speed Contraction of Transverse Rotations to Gauge Transformations*

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Abstract

The İnönü-Wigner contraction is applied to special relativity and the little groups of the Lorentz group. If the \( O(3) \) symmetry group for massive particle is boosted to an infinite-momentum frame, it becomes contracted to a combination of the cylindrical group and the two-dimensional Euclidean group. The Euclidean component becomes the Lorentz condition applicable to the electromagnetic four-potential, and the cylindrical component leads to the helicity and gauge degrees of freedom. The rotation around the cylindrical axis corresponds to the helicity, while the translation parallel to the axis on the cylindrical surface leads to a gauge transformation.

I. INTRODUCTION

In his 1939 paper on representations of the Poincaré group, Wigner formulated the internal space-time symmetries of relativistic particles in terms of the little groups whose transformations leave the four-momentum of a given free particle invariant [1]. He showed that the little groups for massive and massless particles are isomorphic respectively to \( O(3) \) (three-dimensional rotation group) and \( E(2) \) (two-dimensional Euclidean group).

In 1953, İnönü and Wigner introduced to physics the concept and techniques of group contractions [2], and they studied the contraction of \( O(3) \) to \( E(2) \) in detail. They also pointed out that the Lorentz group can be contracted to the Galilean group in the low-speed limit. In 1968 [3], Bacry and Chang suggested the idea that the high-speed limit of the Poincaré group could simplify quantum field theory in the infinite-momentum frame [4]. They then boosted all ten generators of the Poincaré group and observed that the contracted group has only seven generators. Since Wigner’s little groups are subgroups of the Poincaré group, the Bacry-Chang procedure contains the contractions of those little groups.

In this report, we work out the high-speed contraction of the \( O(3) \)-like little group in terms of ellipsoidal deformations of a sphere in the three-dimensional space, namely

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(American) football-like and pancake-like deformations. We shall see that both deformations are needed for the description of the high-speed limiting process. It will be shown that the football-like and pancake-like deformations lead to a cylindrical surface and a flat plane respectively. The flat-plane deformation leads to the contraction to the two-dimensional Euclidean group, while the football-like deformation leads to a cylindrical surface.

It is shown further that the flat-surface collapse corresponds to the Lorentz condition applicable to four-potentials for massless particles. Rotations around the cylindrical axis correspond to the helicity. It is shown also that the translation along the axis on the cylindrical surface corresponds to a gauge transformation.

In Sec. II, we explain what group contraction is in terms of the north-pole approximation of a spherical surface, and how it can it can be applied to contractions of the Lorentz group in the low-speed limit. In Sec. II, we review what Wigner did in his 1939 paper and what have been done since then. This paper has a stormy history because it did not explain where Maxwell’s covariant theory stands within his representation scheme. We now have a clear resolution of this question. In Sec. IV, we study the three-dimensional rotation group and its contractions to the cylindrical and two-dimensional Euclidean groups. It is shown that both of these contractions can be combined into a single four-by-four representation. In this section, the four-dimensional group theoretical language of the little groups is translated into a three-dimensional geometrical language based on spheres, ellipsoids, cylinders, and flat surfaces.

In Sec. V, the generators of the little groups are given the light-cone coordinate system. It is shown that these generators are identical with the combined geometry of the cylindrical group and the Euclidean group discussed in Sec. IV. The geometry of Sec. IV therefore gives a comprehensive description of the little groups for massive and massless particles. In Sec. VI, we study how the four-vectors become contracted in the high-speed limit and study also how the cylindrical group describes the helicity and gauge degrees of freedom for massless particles. In Sec. VII, the Lorentz boost is applied directly to the transformation matrices. In this way, we present a direct proof that transverse rotations become contracted to gauge transformations.

II. WHAT IS THE GROUP CONTRACTION?

If there is a curved line on a plane, we can define a straight line tangent to the curve at a given point. This led to the branch of mathematics known today as differential calculus. Likewise, if there is a curved surface in a three-dimensional space, we can define a flat plane tangent to the surface. We live and work on a sphere with a large radius called the earth, but the diameter of our city is much smaller than the radius of the earth. For this reason, the city appears to be on a flat plane, which is tangent to the spherical surface of the earth.

We are accustomed to associate the spherical surface with the three-dimensional rotation group or $O(3)$, but are not accustomed to think the flat plane in terms of the two-dimensional Euclidean group. On the other hand, it is not difficult to construct a group theory of transformations on this plane. It is commonly known as the two-dimensional Euclidean group or $E(2)$. With this point in mind, Inönü and Wigner in 1953 introduced to physics the concept and technique of group contractions [2]. They started with the set of generators and the closed Lie algebra for a given group. They then considered the case when some
of the generators change the form with a scale change. For instance, the generators of the three-dimensional rotation group are

\[ L_1 = -i \left( y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right), \quad L_2 = -i \left( z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right), \quad L_3 = -i \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right). \] (2.1)

They of course satisfy the commutation relations

\[ [L_i, L_j] = i \epsilon_{ijk} J_k. \] (2.2)

Let us next consider the sphere with a large radius. If we consider a small region on the north pole, the \( z \) variable can be replaced by \( R \), which is the radius of the sphere and is much larger than \( x \) and \( y \) variables. Then, \( L_3 \) remains unchanged, but \( L_1 \) and \( L_2 \) become

\[ L_1 = iR \frac{\partial}{\partial y} = -RP_2, \quad L_2 = -iR \frac{\partial}{\partial x} = RP_1, \] (2.3)

where \( P_1 \) and \( P_2 \) are generators of translations along the \( x \) and \( y \) directions respectively. They can also be written as

\[ P_1 = -i \frac{\partial}{\partial x} = \frac{1}{R} L_2, \quad P_2 = -i \frac{\partial}{\partial y} = -\frac{1}{R} L_1. \] (2.4)

This leads to the commutation relations

\[ [P_1, L_3] = -iP_2, \quad [P_2, L_3] = iP_1, \] (2.5)

and

\[ [P_1, P_2] = i \left( \frac{1}{R} \right)^2 J_3. \] (2.6)

The right hand side of this expression becomes zero when \( R \) becomes large. We then end up with a closed set of three commutation relations:

\[ [P_1, P_2] = 0, \quad [P_1, L_3] = -iP_2, \quad [P_2, L_3] = iP_1. \] (2.7)

These three operators generate two translations and rotations around the origin on the \( xy \) plane. They generate the \( E(2) \) group.

In order to see the contraction of the polar variables \((R, \theta, \phi)\), let us write \((x, y, z)\) as

\[ x = R \sin \theta \cos \phi, \quad y = R \sin \theta \sin \phi, \quad z = R \cos \theta. \] (2.8)

If \( R \) becomes infinite while \( x \) and \( y \) remain finite, the angle \( \theta \) has to be small, and we can write \((z, y, z)\) as

\[ x = R \theta \cos \phi, \quad y = R \theta \sin \phi, \quad z = R, \] (2.9)

with a finite value of \( R\theta \). This is how the angle variables \( \theta \) and \( \phi \) are transformed into \( x \) and \( y \).
We have illustrated above how $O(3)$ is contracted to $E(2)$ through a flat surface tangent to a sphere. A curved surface does not have to be a spherical. Instead of the spherical surface, we can consider a plane tangent to a hyperboloidal surface at its top or bottom. This is the contraction of $O(2, 1)$ to $E(2)$. The tangential surface does not have to be a plane. For the case of sphere, we can construct a cylinder which is tangential to the equatorial belt of the sphere. If we can construct a group theory on the surface of the cylinder, it is possible to contract $O(3)$ to the cylindrical group.

In their 1953 paper, Inönü and Wigner were primarily interested in contractions of representations within the framework of unitary representations. On the other hand, they also gave an example based on finite-dimensional matrix representations. They considered the contraction of the Lorentz group into the Galilean group for infinite value of the speed of light. Let us review their reasoning. If the Lorentz boost is made in the $x$ direction, the transformation matrix is

$$
\begin{pmatrix}
  x' \\
  ct'
\end{pmatrix} = \begin{pmatrix}
  \cosh \lambda & \sinh \lambda \\
  \sinh \lambda & \cosh \lambda
\end{pmatrix}
\begin{pmatrix}
  x \\
  ct
\end{pmatrix},
$$

(2.10)

where $c$ is the velocity of light. Now the space-time column vector can be written as

$$
\begin{pmatrix}
  x \\
  ct
\end{pmatrix} = \begin{pmatrix}
  1 & 0 \\
  0 & c
\end{pmatrix}
\begin{pmatrix}
  x \\
  t
\end{pmatrix}.
$$

(2.11)

Thus,

$$
\begin{pmatrix}
  x' \\
  t'
\end{pmatrix} = \begin{pmatrix}
  \cosh \lambda & c(\sinh \lambda) \\
  (\sinh \lambda)/c & \cosh \lambda
\end{pmatrix}
\begin{pmatrix}
  x \\
  t
\end{pmatrix}.
$$

(2.12)

Let us introduce a variable $v$, where

$$
v/c = \sinh \lambda.
$$

(2.13)

If $c$ becomes infinite while $v$ remains finite, $\lambda$ has to be vanishingly small, and the transformation matrix becomes

$$
\begin{pmatrix}
  x' \\
  t'
\end{pmatrix} = \begin{pmatrix}
  1 & v \\
  0 & 1
\end{pmatrix}
\begin{pmatrix}
  x \\
  t
\end{pmatrix},
$$

(2.14)

resulting in

$$
x' = x + vt, \quad t' = t.
$$

(2.15)

This is how the Lorentz group becomes contracted to the Galilean group.

In Sec. IV of this paper, we shall use the three-by-three matrices to contract $O(3)$ to $E(2)$ and to the cylindrical group. We then combine these two contractions to construct the high-speed contraction of the $O(3)$-like little group to the $E(2)$-like little group. Before doing this let us review the history of Wigner’s little groups.
III. HISTORICAL REVIEW OF WIGNER’S LITTLE GROUPS

In 1939, Wigner observed that internal space-time symmetries of relativistic particles are dictated by their respective little groups [1]. The little group is the maximal subgroup of the Lorentz group which leaves the four-momentum of the particle invariant. The Lorentz group is generated by three rotation generators $J_i$ and three boost generators $K_i$. If a massive particle is at rest, its momentum is invariant under three-dimensional rotations. Thus, its little group is generated by $J_1, J_2$, and $J_3$, and its spin orientation is changed under the little group transformation.

For a massless particle, it is not possible to find a Lorentz frame in which the particle is at rest. We can however assume that its momentum is in the $z$ direction. Then the momentum is invariant under the subgroup of the Lorentz group generated by

$$J_3, \quad N_1 = K_1 - J_2, \quad N_2 = K_2 + J_1. \quad (3.1)$$

Wigner noted in his 1939 paper that these generators satisfy the same set of commutation relations as those for the two-dimensional Euclidean group consisting of one rotation and translations in two different directions.

The 1939 paper indeed gives a covariant picture of massive particles with spins, and connects the helicity of massless particles with the rotational degree of freedom in the group $E(2)$. This paper also gives many homework problems for us to solve.

- First, like the three-dimensional rotation group, $E(2)$ is a three-parameter group. It contains two translational degrees of freedom in addition to the rotation. What physics is associated with the translational-like degrees of freedom for the case of the $E(2)$-like little group?

- Second, as is shown by Inönü and Wigner [2], the rotation group $O(3)$ can be contracted to $E(2)$. Does this mean that the $O(3)$-like little group can become the $E(2)$-like little group in a certain limit?

- Third, it is possible to interpret the Dirac equation in terms of Wigner’s representation theory [3]. Then, why is it not possible to find a place for Maxwell’s equations in the same theory?

- Fourth, the proton was found to have a finite space-time extension in 1955 [8], and the quark model has been established in 1964 [9]. The concept of relativistic extended particles has now been firmly established. Is it then possible to construct a representation of the Poincaré group for particles with space-time extensions?

The list could be endless, but let us concentrate on the above four questions. As for the first question, it has been shown by various authors that the translation-like degrees of freedom in the $E(2)$-like little group is the gauge degree of freedom for massless particles [10,11]. As for the second question, it is not difficult to guess that the $O(3)$-like little group becomes the $E(2)$-like little group in the limit of large momentum/mass [3,12]. However, the non-trivial result is that the transverse rotational degrees of freedom become contracted to the gauge degree of freedom [13].
Then there comes the third question. Indeed, in 1964 [14], Weinberg found a place for the electromagnetic tensor in Wigner’s representation theory. He accomplished this by constructing from the $SL(2, c)$ spinors all the representations of massless fields which are invariant under the translation-like transformations of the $E(2)$-like little group. Since the translation-like transformations are gauge transformations, and since the electromagnetic tensor is gauge-invariant, Weinberg’s construction should contain the electric and magnetic fields, and it indeed does [15].

Next question is whether it is possible to construct electromagnetic four-potentials. After identifying the translation-like degrees of freedom as gauge degrees of freedom, this becomes a tractable problem. It is indeed possible to construct gauge-dependent four-potentials from the $SL(2, c)$ spinors [15,16]. Yes, both the field tensor and four-potential now have their proper places in Wigner’s representation theory. The Maxwell theory and the Poincaré group are perfectly consistent with each other.

The fourth question is about whether Wigner’s little groups are applicable to high-energy particle physics where accelerators produce Lorentz-boosted extended hadrons such as high-energy protons. The question is whether it is possible to construct a representation of the Poincaré group for hadrons which are believed to be bound states of quarks [17,18]. This representation should describe Lorentz-boosted hadrons. Next question then is whether those boosted hadrons give a description of Feynman’s parton picture [19] in the limit of large momentum/mass. These issues have also been discussed in the literature [17,20].

The author of this report was fortunate enough to have a close collaboration with Professor Wigner in his late years. Sections IV and V of the present report are based on the 1990 joint paper by Wigner and the author [21], where the little groups were translated into a geometrical language.

**IV. THREE-DIMENSIONAL GEOMETRY OF THE LITTLE GROUPS**

The little groups for massive and massless particles are isomorphic to $O(3)$ and $E(2)$ respectively. It is not difficult to construct the $O(3)$-like geometry of the little group for a massive particle at rest [4]. In the three-by-three matrix representation, the generators applicable to the column vector $(x, y, z)$ are

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (4.1)$$

These matrices satisfy the three commutation relations given in Eq.(2.2). The Euclidean group $E(2)$ is generated by $L_3, P_1$ and $P_2$, with

$$P_1 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{pmatrix}, \quad (4.2)$$

and they satisfy the commutation relations given in Eq.(2.7) for the $E(2)$ group. The generator $L_3$ is given in Eq.(4.1). When applied to the vector space $(x, y, 1)$, $P_1$ and $P_2$ generate translations on in the $xy$ plane. The geometry of $E(2)$ transformations is quite familiar to our daily life.
Let us transpose the above algebra. Then $P_1$ and $P_2$ become $Q_1$ and $Q_2$ respectively, where

\[
Q_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & i & 0 \end{pmatrix},
\]

(4.3)

respectively. Together with $L_3$, these generators satisfy the same set of commutation relations as that for $L_3, P_1,$ and $P_2$ given in Eq.(2.7):

\[
[Q_1, Q_2] = 0, \quad [L_3, Q_1] = iQ_2, \quad [L_3, Q_2] = -iQ_1.
\]

(4.4)

These matrices generate transformations of a point on a circular cylinder. Rotations around the cylindrical axis are generated by $L_3$. The $Q_1$ and $Q_2$ matrices generate the transformation:

\[
\exp(-i\xi Q_1 - i\eta Q_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \xi & \eta & 1 \end{pmatrix}.
\]

(4.5)

When applied to the space $(x, y, z)$, this matrix changes the value of $z$ while leaving the $x$ and $y$ variables invariant. This corresponds to a translation along the cylindrical axis. We shall call the group generated by $L_3, Q_1$ and $Q_2$ the cylindrical group.

We can achieve the contractions to the Euclidean and to the cylindrical groups by taking the large-radius limits of

\[
P_1 = \frac{1}{R} B^{-1} L_2 B, \quad P_2 = -\frac{1}{R} B^{-1} L_1 B,
\]

(4.6)

and

\[
Q_1 = -\frac{1}{R} B L_2 B^{-1}, \quad Q_2 = \frac{1}{R} B L_1 B^{-1},
\]

(4.7)

where

\[
B(R) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{pmatrix}.
\]

(4.8)

The vector spaces to which the above generators are applicable are $(x, y, z/R)$ and $(x, y, Rz)$ for the Euclidean and cylindrical groups respectively. They can be regarded as the north-pole and equatorial-belt approximations of the spherical surface respectively.

Next, let us consider linear combinations:

\[
F_1 = P_1 + Q_1, \quad F_2 = P_2 + Q_2.
\]

(4.9)

Since $P_1(P_2)$ commutes with $Q_2(Q_1)$, these operators satisfy commutation relations:

\[
[F_1, F_2] = 0, \quad [L_3, F_1] = iF_2, \quad [L_3, F_2] = -iF_1.
\]

(4.10)
Indeed, this is another set of the $E(2)$-like commutation relations. However, we cannot make this addition using the three-by-three matrices for $P_i$ and $Q_i$ to construct three-by-three matrices for $F_1$ and $F_2$, because the vector spaces are different for the $P_i$ and $Q_i$ representations. We can accommodate this difference by creating two different $z$ coordinates, one with a contracted $z$ and the other with an expanded $z$, namely $(x, y, Rz, z/R)$. Then the generators become four-by-four matrices, and $F_1$ and $F_2$ take the form

$$F_1 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (4.11)$$

The rotation generator $L_3$ is also a four-by-four matrix:

$$L_3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (4.12)$$

These four-by-four matrices satisfy the $E(2)$-like commutation relations of Eq.(4.10). The $B(R)$ matrix of Eq.(4.8) can now be combined with its inverse to become a four-by-four matrix:

$$B(R) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & R & 0 \\ 0 & 0 & 0 & 1/R \end{pmatrix}. \quad (4.13)$$

Next, let us consider the transformation matrix generated by the above matrices. It is easy to visualize the transformations generated by $P_i$ and $Q_i$. It would be easy to visualize the transformation generated by $F_1$ and $F_2$, if $P_i$ commuted with $Q_i$. However, $P_i$ and $Q_i$ do not commute with each other, and the transformation matrix takes a somewhat complicated form:

$$\exp (-i\xi F_1 - i\eta F_2) = \begin{pmatrix} 1 & 0 & 0 & \xi \\ 0 & 1 & 0 & \eta \\ \xi & \eta & 1 & (\xi^2 + \eta^2)/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (4.14)$$

This matrix performs both the Euclidean and cylindrical transformations. We shall see what role this matrix plays in special relativity and the Lorentz group.

**V. LITTLE GROUPS IN THE LIGHT-CONES COORDINATE SYSTEM**

Let us now study the group of Lorentz transformations using the light-cone coordinate system. If the space-time coordinate is specified by $(x, y, z, t)$, then the light-cone coordinate variables are $(x, y, u, v)$ for a particle moving in the $z$ direction, where

$$u = (z + t)/\sqrt{2}, \quad v = (t - z)/\sqrt{2}. \quad (5.1)$$

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The transformation from the conventional space-time coordinate to the above system is achieved through the coordinate transformation

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}. \tag{5.2}$$

In the light-cone coordinate system, the generators of Lorentz transformations are

$$J_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & i \\ 0 & i & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad K_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & i & i \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$

$$J_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & i & -i \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & i \\ 0 & i & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix},$$

$$J_3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}. \tag{5.3}$$

where $J_1$, $J_2$, and $J_3$ are the rotation generators, and $K_1$, $K_2$, and $K_3$ are the generators of boosts along the three orthogonal directions.

If a massive particle is at rest, its little group is generated by $J_1$, $J_2$ and $J_3$. For a massless particle moving in the z direction, the little group is generated by $N_1$, $N_2$ and $J_3$, where

$$N_1 = K_1 - J_2, \quad N_2 = K_2 + J_1, \tag{5.4}$$

which can be written in the matrix form as

$$N_1 = \sqrt{2} \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad N_2 = \sqrt{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.5}$$

These matrices satisfy the commutation relations:

$$[J_3, N_1] = iN_2, \quad [J_3, N_2] = -iN_1, \quad [N_1, N_2] = 0. \tag{5.6}$$

Let us go back to $F_1$ and $F_2$ of Eq. (4.11). Indeed, they are proportional to $N_1$ and $N_2$ respectively:

$$N_1 = \sqrt{2}F_1, \quad N_2 = \sqrt{2}F_2. \tag{5.7}$$

Since $F_1$ and $F_2$ are somewhat simpler than $N_1$ and $N_2$, and since the commutation relations of Eq. (5.6) are invariant under multiplication of $N_1$ and $N_2$ by constant factors, we shall hereafter use $F_1$ and $F_2$ for $N_1$ and $N_2$. 

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In the light-cone coordinate system, the boost matrix takes the form
\[
B(R) = \exp\left( -i \rho K_3 \right) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & R & 0 \\
0 & 0 & 0 & 1/R
\end{pmatrix},
\] (5.8)
with \( \rho = \ln(R) \), and
\[
R = \left( \frac{1 + \beta}{1 - \beta} \right)^{1/2}
\] (5.9)
where \( \beta \) is the velocity parameter of the particle. The boost is along the \( z \) direction. This boost matrix takes the same form as the four-by-four contraction matrix given in Eq.(4.13). Under this transformation, \( x \) and \( y \) coordinates are invariant, and the light-cone variables \( u \) and \( v \) are transformed as
\[
u' = Ru, \quad v' = v/R.
\] (5.10)
If we boost \( J_2 \) and \( J_1 \) and divide them by \( \sqrt{2}R \), as
\[
W_1(R) = -\frac{1}{\sqrt{2}R} BJ_2 B^{-1} = \begin{pmatrix}
0 & 0 & -i/R^2 & i \\
0 & 0 & 0 & 0 \\
i & 0 & 0 & 0 \\
i/R^2 & 0 & 0 & 0
\end{pmatrix},
\]
\[
W_2(R) = \frac{1}{\sqrt{2}R} BJ_1 B^{-1} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & -i/R^2 & i \\
i & 0 & 0 & 0 \\
i/R^2 & 0 & 0 & 0
\end{pmatrix},
\] (5.11)
then \( W_1(R) \) and \( W_2(R) \) become \( F_1 \) and \( F_2 \) of Eq.(4.11) respectively in the large-\( R \) limit.

The algebra given in this section is identical with that of Sec. IV based on the three-dimensional geometry of a sphere going through a contraction/expansion of the \( z \) axis. Therefore, it is possible to give a concrete geometrical picture to the little groups of the Poincaré group governing the internal space-time symmetries of relativistic particles.

The most general form of the transformation matrix is
\[
D(\xi, \eta, \alpha) = D(\xi, \eta, 0) D(0, 0, \alpha),
\] (5.12)
with
\[
D(\xi, \eta, 0) = \exp(-i \xi F_1 - i \eta F_2), \quad D(0, 0, \alpha) = \exp(-i \alpha J_3).
\] (5.13)
The matrix \( D(0, 0, \alpha) \) represents rotations around the \( z \) axis and takes the form
\[
D(0, 0, \alpha) = \begin{pmatrix}
\cos \alpha & -\sin \alpha & 0 & 0 \\
\sin \alpha & \cos \alpha & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\] (5.14)
In the light-cone coordinate system, $D(\xi, \eta, 0)$ takes the form of Eq.(4.14):

$$D(\xi, \eta, 0) = \begin{pmatrix}
1 & 0 & 0 & \xi \\
0 & 1 & 0 & \eta \\
\xi & \eta & 1 & (\xi^2 + \eta^2)/2 \\
0 & 0 & 0 & 1
\end{pmatrix}. \quad (5.15)$$

This form is identical to the four-by-four matrix given in Eq.(4.14), and therefore performs both the Euclidean and cylindrical transformations.

\section*{VI. FOUR-VECTORS AND GAUGE TRANSFORMATIONS}

Let us consider a particle represented by a four-vector:

$$A^\mu(x) = A^\mu e^{i(kz-\omega t)}, \quad (6.1)$$

where $A^\mu = (A_1, A_2, A_3, A_0)$. This is not a massless particle. In the light-cone coordinate system,

$$A^\mu = (A_1, A_2, A_u, A_v), \quad (6.2)$$

where $A_u = (A_3 + A_0)/\sqrt{2}$, and $A_v = (A_0 - A_3)/\sqrt{2}$. If it is boosted by the matrix of Eq.(5.8), then

$$A'^\mu = (A_1, A_2, RA_u, A_v/R). \quad (6.3)$$

Thus the fourth component will vanish in the large-$R$ limit, while the third component becomes large.

The momentum-energy four-vector in the light-cone coordinate system is

$$P^\mu = \left(0, 0, (k + \omega)/\sqrt{2}, (\omega - k)/\sqrt{2}\right), \quad (6.4)$$

which in the rest frame becomes

$$P^\mu = \left(0, 0, m/\sqrt{2}, m/\sqrt{2}\right), \quad (6.5)$$

where $m$ is the mass. If we boost this four-momentum using the matrix of Eq.(5.8), then

$$P'^\mu = \left(0, 0, Rm/\sqrt{2}, m/\sqrt{2}R\right). \quad (6.6)$$

Here again, the fourth component becomes vanishingly small for large values of $R$, while the third component becomes large. We can transform the above four-momentum to that of a massless particle by ignoring the fourth component and renormalizing the third component, as we did for the matrices given in Eq.(5.11), the above four vector can be written as

$$P'^\mu = \left(0, 0, \sqrt{2}\omega, 0\right), \quad (6.7)$$

with $R = \sqrt{2}\omega/m$. We first took the limit of large $R$ to eliminate the fourth component. We then brought $R$ to a finite value to make the third component finite.
TABLE I. Applications of group contraction to special relativity. Massive and massless particles have different energy-momentum relations. Einstein’s special relativity gives one relation for both. Wigner’s little group unifies the internal space-time symmetries for massive and massless particles which are locally isomorphic to $O(3)$ and $E(2)$ respectively.

| Massive, Slow | COVARIANCE | Massless, Fast |
|---------------|------------|---------------|
| Energy-Momentum | $E = p^2/2m$ | Einstein’s | $E = \sqrt{p^2 + m^2}$ | $E = p$ |
| Internal space-time symmetry | $S_3$ | Wigner’s | $S_3$ | Little Group | Gauge Transformations |

We can follow the same procedure to obtain the four-potential without the fourth component:

$$A^\mu = (A_1, A_2, A_u, 0), \quad (6.8)$$

This process is the same as imposing the Lorentz condition

$$\partial^\mu A_\mu(x) = 0 \quad (6.9)$$
on the four-potential.

Let us apply the transformations of Eq.(5.13) and Eq.(5.14) to the above four-vectors. By definition of the little group, they do not change the four-momentum. As for the four-potential, the rotation matrix does not change the third and fourth component. If we apply $D(\xi, \eta, 0)$ to the four-potential satisfying the Lorentz condition,

$$\begin{pmatrix} 1 & 0 & 0 & \xi \\ 0 & 1 & 0 & \eta \\ \xi & \eta & 1 & (\xi^2 + \eta^2)/2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_u \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \xi & \eta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_u \\ 0 \end{pmatrix}. \quad (6.10)$$

This means that transformations of the little group can be simplified to a three-by-three matrix formalism, with the three-component column vector $(A_1, A_2, A_u)$. If $D(0,0,\alpha)$ is applied to this vector,

$$\begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_u \end{pmatrix} = \begin{pmatrix} A_1 \cos \alpha - A_2 \sin \alpha \\ A_1 \sin \alpha + A_2 \cos \alpha \\ A_u \end{pmatrix}, \quad (6.11)$$
which performs a rotation in the transverse plane. If \( D(\xi, \eta, 0) \) is applied to the three-component vector,

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
\xi & \eta & 1
\end{pmatrix}
\begin{pmatrix}
A_1 \\
A_2 \\
A_u
\end{pmatrix} =
\begin{pmatrix}
A_1 \\
A_2 \\
A_u + \xi A_1 + \eta A_2
\end{pmatrix}.
\] (6.12)

The above three-by-three matrix takes the same form as the cylindrical transformation matrix given in Eq.(4.5). This transformation does not change the transverse components, but changes the value of the third component. Indeed, the matrices \( D(0, 0, \alpha) \) and \( D(\xi, \eta, 0) \) perform cylindrical transformations.

Let us go back to Eq.(6.12). This transformation does not change the transverse component of the four-potential. It changes only the third component which is parallel to the momentum. For this reason, it performs a gauge transformation. Therefore, we come to the conclusion that transverse rotations become contracted to gauge transformations. The result is summarized in Table I.

VII. LORENTZ-BOOSTED ROTATION MATRICES

As we noted in Sec. 11, Inönü and Wigner considered the contraction of the Lorentz group into the Galilean group. There they started from the transformation matrix, instead of generators, and took the large-\( c \) limit. In this way, we can trace the transformation parameters during the limiting process. Let us follow this line of reasoning and take the high-speed limit of the rotation matrices.

For a particle at rest, we can perform the rotation around the \( x \) axis using the matrix

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \theta & -\sin \theta & 0 \\
0 & \sin \theta & \cos \theta & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\] (7.1)

applicable to the coordinate \((x, y, z, t)\). If we boost this rotation matrix,

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \cosh \lambda & \sinh \lambda \\
0 & 0 & \sinh \lambda & \cosh \lambda
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \theta & -\sin \theta & 0 \\
0 & \sin \theta & \cos \theta & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \cosh \lambda & -\sinh \lambda \\
0 & 0 & -\sinh \lambda & \cosh \lambda
\end{pmatrix},
\]

the result is

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \theta & -\sin \theta \cosh \lambda & \sin \theta \sinh \lambda \\
0 & \sin \theta \cosh \lambda & \cos \theta - (1 - \cos \theta) \sinh^2 \lambda & (1 - \cos \theta) \cosh \lambda \sinh \lambda \\
0 & \sin \theta \sinh \lambda & -(1 - \cos \theta) \cosh \lambda \sinh \lambda & \cos \theta + (1 - \cos \theta) \cosh^2 \lambda
\end{pmatrix}.
\] (7.2)

If we set \( \sin \theta \sinh \lambda = \sqrt{2} \eta \) with a finite value of \( \eta \), the angle \( \theta \) becomes very small in the large-\( \lambda \) limit:
\[ \theta = 2\sqrt{2}\eta \exp(-\lambda), \]  

(7.3)

and the above matrix becomes

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & -\sqrt{2}\eta & \sqrt{2}\eta \\
0 & \sqrt{2}\eta & 1 - \eta^2 & \eta^2 \\
0 & \sqrt{2}\eta & -\eta^2 & 1 + \eta^2
\end{pmatrix}.
\]

(7.4)

Likewise, we start from the rotation matrix around the y axis:

\[
\begin{pmatrix}
\cos \phi & 0 & -\sin \phi & 0 \\
0 & 1 & 0 & 0 \\
\sin \phi & 0 & \cos \phi & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

(7.5)

and set \( \sin \phi \sinh \lambda = -\sqrt{2}\xi \), the result is

\[
\begin{pmatrix}
1 & 0 & -\sqrt{2}\xi & \sqrt{2}\xi \\
0 & 1 & 0 & 0 \\
\sqrt{2}\xi & 0 & 1 - \xi^2 & \xi^2 \\
\sqrt{2}\xi & 0 & -\xi^2 & 1 + \xi^2
\end{pmatrix}.
\]

(7.6)

Both of the matrices in Eq.(7.4) and Eq.(7.6) appear complicated, but they commute with each other. The multiplication of these two leads to

\[
\begin{pmatrix}
1 & 0 & -\sqrt{2}\xi & \sqrt{2}\xi \\
0 & 1 & 0 & 0 \\
\sqrt{2}\xi & \sqrt{2}\eta & 1 - (\xi^2 + \eta^2) & (\xi^2 + \eta^2) \\
\sqrt{2}\xi & \sqrt{2}\eta & -(\xi^2 + \eta^2) & 1 + (\xi^2 + \eta^2)
\end{pmatrix}.
\]

(7.7)

This is the most general form of the contracted transverse rotation matrix. This form as a component of the \( E(2) \)-like little group is given in Wigner’s original paper \[1\], and in many later papers \[14\]. However, its complicated expression scared away many physicists in the past. In this report, we studied the physics and mathematics of this impossible form. In particular, we studied the cylindrical symmetry contained in the above expression.

If we transform the above expression into the light-cone coordinate system using the matrix given in Eq.(5.2), it becomes the four-by-four \( D(\xi, \eta, 0, 0) \) matrix given in Eq.(5.15). What is new in this section is the expression for the angle given in Eq.(7.3). This gives a physical interpretation for the large-R renormalization procedure used in Eq.(5.11). The physical content is that for a finite value of gauge parameter, the rotation angle becomes vanishingly small in the infinite-speed limit. Since the rotation angle is vanishingly small, gauge transformations are not observable.

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REFERENCES

[1] E. P. Wigner, Ann. Math. 40, 149 (1939).
[2] E. Inönü, and Wigner, E. P., Proc. Natl. Acad. Scie. (U.S.A.) 39, 510 (1953).
[3] H. Bacry and N. P. Chang, Ann. Phys. 47, 407 (1968).
[4] S. Weinberg, Phys. Rev. 150, 1313 (1966).
[5] R. Gilmore, Lie Groups, Lie Algebras, and Some of Their Applications (Wiley, New York, 1974).
[6] Y. S. Kim and E. P. Wigner, J. Math. Phys. 28, 1175 (1987).
[7] V. Bargmann and E. P. Wigner, Proc. Natl. Acad. Scie (U.S.A.) 34, 211 (1948).
[8] R. Hofstadter and R. W. McAllister, Phys. Rev. 98, 217 (1955).
[9] M. Gell-Mann, Phys. Lett. 13, 598 (1964).
[10] A. Janner and T. Jenssen, Physica 53, 1 (1971); ibid. 60, 292 (1972).
[11] J. Kuperzstych, Nuovo Cimento 31B, 1 (1976); D. Han and Y. S. Kim, Am. J. Phys. 49, 348 (1981); J. J. van der Bij, H. van Dam, and Y. J. Ng, Physica 116A, 307 (1982); D. Han, Y. S. Kim, and D. Son, Phys. Rev. D 26, 3717 (1982).
[12] S. P. Misra and J. Maharana, Phys. Rev. D 14, 133 (1976); S. Ferrara and C. Savoy, in Supergravity 1981, S. Ferrara and J. G. Taylor, eds. (Cambridge Univ. Press, Cambridge, 1982), p.151; D. Han, Y. S. Kim, and D. Son, J. Math. Phys. 27, 2228 (1986); P. Kwon and M. Villasante, J. Math. Phys. 29, 560 (1988); ibid. 30, 201 (1989).
[13] D. Han, Y. S. Kim, and D. Son, Phys. Lett. 131B, 327 (1983).
[14] S. Weinberg, Phys. Rev. 134, B882 (1964); ibid. 135, B1049 (1964).
[15] For the contraction of the Maxwell-like tensor, see S. Baskal and Y. S. Kim, Europhys. Lett. 40, 375 (1997).
[16] D. Han, Y. S. Kim, and D. Son, Am. J. Phys. 54, 818 (1986).
[17] Y. S. Kim and M. E. Noz, Theory and Applications of the Poincaré Group (Reidel, Dordrecht, 1986).
[18] R. P. Feynman, M. Kislinger, and F. Ravndal, Phys. Rev. D 3, 2706 (1971); Y. S. Kim, M. E. Noz, and S. H. Oh, J. Math. Phys. 20, 1341 (1979).
[19] R. P. Feynman, in High Energy Collisions, Proceedings of the Third International Conference, Stony Brook, New York, C. N. Yang et al., eds. (Gordon and Breach, New York, 1969); J. D. Bjorken and E. A. Paschos, Phys. Rev. 185, 1975 (1969).
[20] Y. S. Kim and M. E. Noz, Phys. Rev. D 15, 335 (1977); Y. S. Kim, Phys. Rev. Lett. 63, 348 (1989).
[21] Y. S. Kim and E. P. Wigner, J. Math. Phys. 31, 55 (1990).