Fluctuations of the Self-Normalized Sum in the Curie-Weiss Model of SOC

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Abstract
We extend the main theorem of [2] about the fluctuations in the Curie-Weiss model of SOC. We present a short proof using the Hubbard-Stratonovich transformation with the self-normalized sum of the random variables.

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1 Introduction
In [2], Raphaël Cerf and Matthias Gorny designed a Curie-Weiss model of self-organized criticality. It is the model given by an infinite triangular array of real-valued random variables \((X^k_n)_{1 \leq k \leq n}\) such that for all \(n \geq 1\), \((X^1_n, \ldots, X^n_n)\) has the distribution

\[
\tilde{\mu}_n,\rho(x_1, \ldots, x_n) = \frac{1}{Z_n} \exp \left( \frac{1}{2} \left( \frac{x_1 + \cdots + x_n}{x_1^2 + \cdots + x_n^2} \right)^2 \right) \mathbb{1}_{\{x_1^2 + \cdots + x_n^2 > 0\}} \prod_{i=1}^n d\rho(x_i),
\]

where \(\rho\) is a probability measure on \(\mathbb{R}\) which is not the Dirac mass at 0, and where \(Z_n\) is the normalization constant. This model is a modification of the generalized Ising Curie-Weiss model by the implementation of an automatic control of the inverse temperature.

For any \(n \geq 1\), we denote

\[
S_n = X^1_n + \cdots + X^n_n, \quad T_n = (X^1_n)^2 + \cdots + (X^n_n)^2.
\]

By using Cramér’s theory and Laplace’s method, Cerf and Gorny proved in [2] that, if \(\rho\) satisfies

\[
\exists v_0 > 0 \quad \int_{\mathbb{R}} e^{v_0 z^2} d\rho(z) < +\infty \quad (\ast)
\]
and if $\rho$ has a bounded density, then

$$
\frac{S_n}{n^{3/4}} \xrightarrow{\rho} \left( \frac{4\mu_4}{3\sigma^8} \right)^{1/4} \Gamma \left( \frac{1}{4} \right) \exp \left( -\frac{\mu_4}{12\sigma^8} s^4 \right) ds.
$$

The case where $\rho$ is a centered Gaussian measure has been studied in [6]. This fluctuation result shows that this model is a self-organized model exhibiting critical behaviour. Indeed it has the same behaviour as the critical generalized Ising Curie-Weiss model (see [4]) and, by construction, it does not depend on any external parameter.

This result has been extended in [5] to the case where $\rho$ satisfies some Cramér condition, which is fulfilled in particular when $\rho$ has an absolutely continuous component. However the proof is very technical and it does not deal with the case where $\rho$ is discrete for example.

In this paper we prove that the convergence in distribution of $S_n/n^{3/4}$, under $\tilde{\mu}_{n,\rho}$, is true for any symmetric probability measure $\rho$ on $\mathbb{R}$ which satisfies (**). To this end, we study the fluctuations of the self-normalized sum $S_n/\sqrt{T_n}$. With this term, it is possible to use the so-called Hubbard-Stratonovich transformation as in lemma 3.3 of [4], which is the key ingredient for the proof of the fluctuations theorem in the generalized Ising Curie-Weiss model.

**Theorem 1.** Let $\rho$ be a symmetric probability measure on $\mathbb{R}$ which is not the Dirac mass at 0 and which has a finite fifth moment. We denote by $\sigma^2$ the variance of $\rho$ and by $\mu_4$ its fourth moment. Then, under $\tilde{\mu}_{n,\rho},$

$$
\frac{S_n}{n^{1/4}\sqrt{T_n}} \xrightarrow{\rho} \left( \frac{4\mu_4}{3\sigma^8} \right)^{1/4} \Gamma \left( \frac{1}{4} \right) \exp \left( -\frac{\mu_4}{12\sigma^8} s^4 \right) ds.
$$

Remark: the hypothesis that $\rho$ has a fifth moment may certainly be weakened by assuming instead that

$$
\exists \varepsilon > 0 \quad \int_{\mathbb{R}} |z|^{4+\varepsilon} d\rho(z) < +\infty.
$$

We prove theorem 1 in section 2. If we add the hypothesis that $\rho$ satisfies (**) then, under $\tilde{\mu}_{n,\rho}$, $T_n/n$ converges in probability to $\sigma^2$. This result is proved in section 3 of [5] using Cramér’s theorem, Varadhan’s lemma (see [3]) and a conditioning argument. Moreover

$$
\forall n \geq 1 \quad \frac{S_n}{n^{3/4}} = \sqrt{\frac{T_n}{n}} \times \frac{S_n}{n^{1/4}\sqrt{T_n}},
$$

and condition (**) implies that $\rho$ has finite moments of all orders. Therefore the following theorem is a consequence of theorem 1 and Slutsky lemma (theorem 3.9 of [1]).

**Theorem 2.** Let $\rho$ be a symmetric probability measure on $\mathbb{R}$ which is not the Dirac mass at 0 and such that

$$
\exists v_0 > 0 \quad \int_{\mathbb{R}} e^{v_0 z^2} d\rho(z) < +\infty.
$$

Then, under $\tilde{\mu}_{n,\rho},$

$$
\frac{S_n}{n^{3/4}} \xrightarrow{\rho} \left( \frac{4\mu_4}{3\sigma^8} \right)^{1/4} \Gamma \left( \frac{1}{4} \right) \exp \left( -\frac{\mu_4}{12\sigma^8} s^4 \right) ds.
$$
2 Proof of theorem 1

Let \((X_k^n)_{1 \leq k \leq n}\) be an infinite triangular array of random variables such that, for any \(n \geq 1\), \((X_1^n, \ldots, X_n^n)\) has the law \(\tilde{\mu}_{n, \rho}\). Let us recall that

\[
\forall n \geq 1, \quad S_n = X_1^n + \cdots + X_n^n \quad \text{and} \quad T_n = (X_1^n)^2 + \cdots + (X_n^n)^2,
\]

and that \(T_n > 0\) almost surely. We use the Hubbard-Stratonovich transformation: let \(W\) be a random variable with standard normal distribution and which is independent of \((X_k^n)_{1 \leq k \leq n}\). Let \(n \geq 1\) and let \(f\) be a bounded continuous function on \(\mathbb{R}\). We put

\[
E_n = E\left[ f \left( \frac{W}{n^{1/4}} + \frac{S_n}{n^{1/4} \sqrt{T_n}} \right) \right].
\]

We introduce \((Y_i)_{i \geq 1}\) a sequence of independent random variables with common distribution \(\rho\). We have

\[
E_n = \frac{1}{Z_n \sqrt{2\pi}} \mathbb{E} \left[ \int_{\mathbb{R}} f \left( \frac{w}{n^{1/4}} + \frac{Y_1 + \cdots + Y_n}{n^{1/4} \sqrt{Y_1^2 + \cdots + Y_n^2}} \right) \right. \\
\left. \times \exp \left( \frac{1}{2} \frac{(Y_1 + \cdots + Y_n)^2}{Y_1^2 + \cdots + Y_n^2} - \frac{w^2}{2} \right) \mathbb{1}_{\{Y_1^2 + \cdots + Y_n^2 > 0\}} \right] dw.
\]

We make the change of variable

\[
z = \frac{w}{n^{1/4}} + \frac{Y_1 + \cdots + Y_n}{n^{1/4} \sqrt{Y_1^2 + \cdots + Y_n^2}}
\]

in the integral and we get

\[
E_n = \frac{n^{1/4}}{Z_n \sqrt{2\pi}} \mathbb{E} \left[ \mathbb{1}_{\{Y_1^2 + \cdots + Y_n^2 > 0\}} \right. \\
\left. \times \int_{\mathbb{R}} f(z) \exp \left( -\sqrt{n} z^2 + zn^{1/4} \frac{Y_1 + \cdots + Y_n}{\sqrt{Y_1^2 + \cdots + Y_n^2}} \right) dz \right].
\]

Let \(U_1, \ldots, U_n, \varepsilon_1, \ldots, \varepsilon_n\) be independent random variables such that the distribution of \(U_i\) is \(\rho\) and the distribution of \(\varepsilon_i\) is \((\delta_{-1} + \delta_1)/2\), for any \(i \in \{1, \ldots, n\}\). Since \(\rho\) is symmetric, the random variables \(\varepsilon_1 U_1, \ldots, \varepsilon_n U_n\) are also independent with common distribution \(\rho\). As a consequence

\[
E_n = \frac{n^{1/4}}{Z_n \sqrt{2\pi}} \mathbb{E} \left[ \mathbb{1}_{\{U_1^2 + \cdots + U_n^2 > 0\}} \right. \\
\left. \times \int_{\mathbb{R}} f(z) \exp \left( -\sqrt{n} z^2 + \sum_{i=1}^{n} \frac{zn^{1/4} \varepsilon_i U_i}{\sqrt{U_1^2 + \cdots + U_n^2}} \right) dz \right].
\]

For any \(i \in \{1, \ldots, n\}\), we denote (in the case where \(U_1^2 + \cdots + U_n^2 > 0\))

\[
A_{i,n} = \frac{U_i}{\sqrt{U_1^2 + \cdots + U_n^2}}.
\]
By using Fubini’s theorem and the independence of $\varepsilon_i, U_i, i \in \{1, \ldots, n\}$, we obtain

$$E_n = \frac{n^{1/4}}{Z_n \sqrt{2\pi}} \mathbb{E} \left[ \mathbf{1}_{\{U_1^2 + \cdots + U_n^2 > 0\}} \int_{\mathbb{R}} f(z) \exp \left( -\frac{\sqrt{n}z^2}{2} \right) \right.$$ 

$$\times \mathbb{E} \left( \prod_{i=1}^{n} \exp \left( zn^{1/4} \varepsilon_i A_{i,n} \right) \bigg| (U_1, \ldots, U_n) \right) \right. dz] .$$

$$= \frac{n^{1/4}}{Z_n \sqrt{2\pi}} \mathbb{E} \left[ \mathbf{1}_{\{U_1^2 + \cdots + U_n^2 > 0\}} \int_{\mathbb{R}} f(z) \exp \left( -\frac{\sqrt{n}z^2}{2} \right) \right.$$ 

$$\times \exp \left( \sum_{i=1}^{n} \ln \cosh \left( zn^{1/4} A_{i,n} \right) \right) \right. dz] .$$

We define the function $g$ by

$$\forall y \in \mathbb{R} \quad g(y) = \ln \cosh y - \frac{y^2}{2}.$$ 

It is easy to see that $g(y) < 0$ for $y > 0$. We notice that $A_{1,n}^2 + \cdots + A_{n,n}^2 = 1$, so that

$$E_n = \frac{n^{1/4}}{Z_n \sqrt{2\pi}} \mathbb{E} \left[ \mathbf{1}_{\{U_1^2 + \cdots + U_n^2 > 0\}} \int_{\mathbb{R}} f(z) \exp \left( \sum_{i=1}^{n} g\left( zn^{1/4} A_{i,n} \right) \right) \right. \] .$$

Now we use Laplace’s method. Let us examine the convergence of the term in the exponential: for any $i \in \{1, \ldots, n\}$, the Taylor-Lagrange formula states that there exists a random variable $\xi_i$ such that

$$g\left( zn^{1/4} A_{i,n} \right) = -\frac{(zn^{1/4} A_{i,n})^4}{12} + \frac{(zn^{1/4} A_{i,n})^5}{5!} g^{(5)}(\xi_i).$$

By a simple computation, we see that the function $g^{(5)}$ is bounded over $\mathbb{R}$. As a consequence

$$\sum_{i=1}^{n} g\left( zn^{1/4} A_{i,n} \right) = -\frac{z^4}{12} \frac{(Y_1^4 + \cdots + Y_n^4)/n}{(Y_1^2 + \cdots + Y_n^2)/n}^2$$

$$+ z^5 \frac{(Y_1^5 + \cdots + Y_n^5)/n}{(Y_1^2 + \cdots + Y_n^2)/n}^{5/2} O \left( \frac{1}{n^{1/4}} \right).$$

By hypothesis, the distribution $\rho$ has a finite fifth moment. Hence the law of large numbers implies that

$$\forall z \in \mathbb{R} \quad \sum_{i=1}^{n} g\left( zn^{1/4} A_{i,n} \right) \xrightarrow{\text{n} \to +\infty} -\frac{\mu_4 z^4}{12 \sigma^4} \quad \text{a.s.}$$

**Lemma 3.** There exists $c > 0$ such that

$$\forall z \in \mathbb{R} \quad \forall n \geq 1 \quad \sum_{i=1}^{n} g\left( zn^{1/4} A_{i,n} \right) \leq -\frac{cz^4}{1 + z^2 / \sqrt{n}}.$$
Proof. We define $h$ by

$$
\forall y \in \mathbb{R}\setminus\{0\} \quad h(y) = \frac{1 + y^2}{y^4} g(y).
$$

It is a negative continuous function on $\mathbb{R}\setminus\{0\}$. Since $g(y) \sim -y^4/12$ in the neighbourhood of 0, the function $h$ can be extended to a function continuous on $\mathbb{R}$ by putting $h(0) = -1/12$. Next we have

$$
\forall y \in \mathbb{R}\setminus\{0\} \quad h(y) = \frac{1 + y^2}{y^2} \times \left( \frac{\ln \cosh y}{y^2} - \frac{1}{2} \right),
$$

so that $h(y)$ goes to $-1/2$ when $|y|$ goes to $+\infty$. Therefore $h$ is bounded by some constant $-c$ with $c > 0$. Next we easily check that $x \mapsto x^2/(1 + x)$ is convex on $[0, +\infty]$ so that, for any $z \in \mathbb{R}$ and $n \geq 1$,

$$
\sum_{i=1}^{n} g(zn^{1/4} A_{i,n}) \leq -nc \frac{1}{n} \sum_{i=1}^{n} \frac{(zn^{1/4} A_{i,n})^4}{1 + (zn^{1/4} A_{i,n})^2} \\
\leq -nc \frac{1}{n} \sum_{i=1}^{n} (zn^{1/4} A_{i,n})^2 = - \frac{cz^4}{1 + z^2/\sqrt{n}},
$$

since $A_{1,n}^2 + \cdots + A_{n,n}^2 = 1$.

If $|z| \leq n^{1/4}$ then $1 + z^2/\sqrt{n} \leq 2$ and thus, by the previous lemma,

$$
\left| \mathbf{1}_{\{t_1^2 + \cdots + t_2^2 > 0\}} \mathbf{1}_{|z| \leq n^{1/4}} \exp \left( \sum_{i=1}^{n} g(zn^{1/4} A_{i,n}) \right) \right| \leq \exp \left( -\frac{cz^4}{2} \right).
$$

Since

$$
\mathbb{E} \left[ \int_{\mathbb{R}} \|f\|_{\infty} \exp \left( -\frac{cz^4}{2} \right) \, dz \right] < +\infty,
$$

the dominated convergence theorem implies that

$$
\mathbb{E} \left[ \mathbf{1}_{\{t_1^2 + \cdots + t_2^2 > 0\}} \int_{\mathbb{R}} \mathbf{1}_{|z| \leq n^{1/4}} f(z) \exp \left( \sum_{i=1}^{n} g(zn^{1/4} A_{i,n}) \right) \, dz \right] \\
\longrightarrow_{n \to +\infty} \int_{\mathbb{R}} f(z) \exp \left( \frac{\mu_4 z^4}{12\sigma^4} \right) \, dz.
$$

If $|z| > n^{1/4}$ then $1 + z^2/\sqrt{n} \leq 2z^2/\sqrt{n}$ and thus, by the previous lemma,

$$
\left| \mathbf{1}_{\{t_1^2 + \cdots + t_2^2 > 0\}} \mathbf{1}_{|z| > n^{1/4}} \exp \left( \sum_{i=1}^{n} g(zn^{1/4} A_{i,n}) \right) \right| \leq \exp \left( -c\sqrt{nz^2} \right).
$$

Hence

$$
\mathbb{E} \left[ \mathbf{1}_{\{t_1^2 + \cdots + t_2^2 > 0\}} \int_{\mathbb{R}} \mathbf{1}_{|z| > n^{1/4}} f(z) \exp \left( \sum_{i=1}^{n} g(zn^{1/4} A_{i,n}) \right) \, dz \right] \leq \frac{\|f\|_{\infty} \sqrt{2\pi}}{n^{1/4} \sqrt{c}}.
$$
and thus

\[
E \left[ \mathbb{1}_{\{U_1^2 + \cdots + U_n^2 > 0\}} \int_{\mathbb{R}} f(z) \exp \left( \sum_{i=1}^{n} g(zn^{1/4}A_{i,n}) \right) \; dz \right] \xrightarrow{n \to +\infty} \int_{\mathbb{R}} f(z) \exp \left( -\frac{\mu_4 z^4}{12\sigma^4} \right) \; dz.
\]

If we take \( f = 1 \), we get

\[
\frac{Z_n\sqrt{2\pi}}{n^{1/4}} \xrightarrow{n \to +\infty} \int_{\mathbb{R}} \exp \left( -\frac{\mu_4 z^4}{12\sigma^4} \right) \; dz.
\]

We have proved that

\[
\frac{W}{n^{1/4}} + \frac{S_n}{n^{1/4}\sqrt{T_n}} \xrightarrow{n \to +\infty} \left( \int_{\mathbb{R}} \exp \left( -\frac{\mu_4 z^4}{12\sigma^4} \right) \; dz \right)^{-1} \exp \left( -\frac{\mu_4}{12\sigma^4} s^4 \right) \; ds.
\]

Since \( (n^{-1/4}W)_{n \geq 1} \) converges in distribution to 0, Slutsky lemma (theorem 3.9 of [1]) implies that

\[
\frac{S_n}{n^{1/4}\sqrt{T_n}} \xrightarrow{n \to +\infty} \left( \int_{\mathbb{R}} \exp \left( -\frac{\mu_4 z^4}{12\sigma^4} \right) \; dz \right)^{-1} \exp \left( -\frac{\mu_4}{12\sigma^4} s^4 \right) \; ds.
\]

By an ultimate change of variables we compute that

\[
\int_{\mathbb{R}} \exp \left( -\frac{\mu_4 z^4}{12\sigma^4} \right) \; dz = \left( \frac{3\sigma^4}{4\mu_4} \right)^{1/4} \Gamma \left( \frac{1}{4} \right).
\]

This ends the proof of theorem 1.

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