Chern-Simons Theory with Complex Gauge Group on Seifert Fibred 3-Manifolds

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Abstract

We consider Chern-Simons theory with complex gauge group and present a complete non-perturbative evaluation of the path integral (the partition function and certain expectation values of Wilson loops) on Seifert fibred 3-Manifolds. We use the method of Abelianisation. In certain cases the path integral can be seen to factorize neatly into holomorphic and anti-holomorphic parts. We obtain closed formulae of this factorization for the expectation values of torus knots.
1 Introduction

Chern-Simons theory with a complex gauge group $G_C$ (viewed as a complexification of some compact gauge group $G$) has been under study since it was first considered in [31]. Formal aspects of gauge fixing for semi-simple gauge groups appeared in [2] and a Hamiltonian version of quantization appears in [11]. In the case of $G_C = SL(2,\mathbb{C})$ Chern-Simons theory appears as a gravitational theory in three dimensions [32, 17] in first order formalism. The ‘volume conjecture’ which relates certain (limits of) quantum knot invariants to the hyperbolic volume of the knot complement [20, 24] renewed interest in complex Chern-Simons theory. This is mainly for hyperbolic manifolds (or knots).

Rather more recently the 3d-3d correspondence [15] and [12, 13] has revived the study of Chern-Simons theory with complex gauge group and in particular requires one to know the partition function on manifolds which are not necessarily hyperbolic. Short of having an ab initio derivation, many of the results are in fact conjectural and are used to test, rather than to prove the veracity of, the proposed correspondence.

In the study of the quantum theory canonical quantization of the complexified theory has its difficulties. The phase space is the symplectic manifold of flat $G_C$ connections on a Riemann surface $\Sigma$ up to gauge transformations. That moduli space has the standard description

$$M_C(\Sigma) = \text{Hom}(\pi_1(\Sigma), G_C)/\text{Ad}G_C \quad (1.1)$$

As explained by Witten [31], rather than quantizing this non-compact space, one alternative is to consider instead the space of flat $G$ connections, up to gauge equivalence,

$$M(\Sigma) = \text{Hom}(\pi_1(\Sigma), G)/\text{Ad}G \quad (1.2)$$

and square integrable sections of a pre-quantum line bundle over this space. Dealing directly with the quantization of (1.1) has been pushed forward in [14] and in particular in [18], where an integral grading of the infinite dimensional Hilbert space into finite dimensional subspaces has been given.

One of our aims here is to instead perform the path integral directly (foregoing the canonical procedure completely). In a series of papers we have evaluated the Chern-Simons partition function with compact and simply connected gauge group on (and invariants for certain links in) Seifert fibered 3-manifolds [6, 7, 8] (by a suitable gauge fixing and pushing the problem down to a 2-dimensional Abelian gauge theory). While we have applied these ideas to three dimensional $BF$ theory [32, 5] in [7, 10] we have not dealt with the complex case.

The computations that we perform are for Seifert 3-manifolds and Seifert manifolds are never hyperbolic. There are, however, advantages to this choice of 3-manifold. For the most part the situation here is vastly simplified as (with suitable gauge fixing) almost all
of the path integrals that we encounter are Gaussian (with sources). The final answer for
the partition function with complex gauge group takes a very simple form (7.6) as an integral over the complex Cartan subalgebra and a summation whose range depends on the order $|d|$ of $H_1(M, \mathbb{Z})$,

$$Z_M[t, \bar{t}] = \frac{1}{|W|} \sum_{1 \leq n \leq d} \int_{\mathcal{C}} d^{2\text{rk}(G)} z \exp \left( iI^n(t, \bar{t}) \right) \frac{\tau_M^{1/2}(z; a_1, \ldots, a_N)}{V(t)} \tag{1.3}$$

The integrand involves the Ray-Singer torsion $\tau_M$ of the 3-manifold with connections in the complexified group and the exponent $I^n(t, \bar{t})$ is a polynomial of degree two in the integration variables. The complex parameter $t = k + is$ is the complex ‘level’ or coupling constant of the theory. The circle $V$-bundle of the line $V$-bundle $L$ over $S^2$ with $N$ orbifold points is the 3-manifold $M$ in question. Note that (1.3) is very similar to (a square of) the general formula given [22], except that we have an integral rather than a sum to perform over one of the factors. The continuous variable reflects the non-compact nature of the gauge group.

Another motivation for this work is to understand the holomorphic factorization of $G_C$ Chern-Simons theory [33]. The path integral under consideration may be written as

$$Z[M, t, \bar{t}] = \int D\mathcal{C} D\bar{\mathcal{C}} e^{(itI(\mathcal{C}) + i\bar{t}I(\bar{\mathcal{C}}))} \tag{1.4}$$

where the overline indicates complex conjugation. The natural question is how close are we to being able to make sense of a factorization of the path integral into holomorphic and anti-holomorphic parts

$$Z[M, t, \bar{t}] \approx Z[M, t] \cdot \overline{Z[M, t]} \tag{1.5}$$

where

$$Z[M, t] = \int D\mathcal{C} e^{(itI(\mathcal{C}))} \quad \text{and} \quad \overline{Z[M, t]} = \int D\bar{\mathcal{C}} e^{(i\bar{t}I(\bar{\mathcal{C}}))}$$

and if such partition functions are to make sense what ‘contour’ path integrals are being performed on the right hand side? It turns out that around a given flat connection (1.5) holds in perturbation theory. In [16], equation (1.6), and [33], equation (2.27), it is shown that the correct formula is

$$Z[M, t, \bar{t}] = \frac{1}{|W|} \sum_{\rho, \bar{\rho}} n_{(\rho, \bar{\rho})} Z^\rho[M, t] \cdot \overline{Z^\rho[M, t]} \tag{1.6}$$

for some numbers $n_{(\rho, \bar{\rho})}$, where $\rho$ labels the solution to the flatness equation, and for particular contours. We will discuss below, under which circumstances and in which sense our exact non-perturbative result (1.3) displays this holomorphic factorization.

In the next section we introduce Chern-Simons theory with a complex gauge group proper, taking it to be the complexification $G_C$ of a compact gauge group $G$. There
we point out that there exists a symmetry of the path integral under the exchange $t \leftrightarrow \hat{t}$ which in part motivates holomorphic factorization. Various formal limits of the coupling constants $k$ and $s$ are considered in section 3. These limits allow us to get an understanding of what may be gleaned from allowing for a complex gauge group. Sections 4 and 5 are devoted to our choice of gauge and choice of manifold. We note that the choice of gauge is unitary and so avoids the pitfalls outlined in [2] of the more usual Lorentz gauge.

In section 6 we integrate out the fibre dependence of all the fields leaving us with an effective two dimensional Abelian gauge theory (6.15). This theory can also be seen to formally factorize into holomorphic and anti-holomorphic components and we see that the Ray-Singer torsion enjoys this factorization. In section 7 we integrate all non-constant modes in the two dimensional theory to be left with the finite dimensional integral (1.3). This integral is such that the integrand factorises into holomorphic and anti-holomorphic parts itself and so we have been able to follow the factorization at each step of the evaluation of the path integral. In particular for (1.3) factorization, at least for certain $M$, is based upon the fact that the following integral

$$\int d^2z \exp \left( az^2 + b \bar{z}^2 + cz + d \bar{z} \right) = \frac{\pi}{\sqrt{-ab}} \exp \left( -\frac{c^2}{4a} - \frac{d^2}{4b} \right)$$

may be expressed as the product of two contour integrals (not closed contours)

$$e^{-i\pi/2} \int_{\Gamma} dz \exp (az^2 + cz) \cdot \int_{\Gamma'} d\bar{z} \exp (b\bar{z}^2 + d\bar{z})$$

for example with both $\Gamma$ and $\Gamma'$ being the real axis. These last formulae immediately imply that the ‘holomorphic’ factor of the $G_C$ theory is just the partition function of the $G$ theory (at least up to framing and a change of level) once we sum over $n$.

Given that we have established factorization the question of what we are summing over (or what does $n$ represent geometrically) arises. In section 8 by comparing with expectations of the $s \to \infty$ limit, we are also able to show that, at least in that limit, the sum is over particular flat Abelian connections. At least for the Lens spaces this is in keeping with the expectations of [16, 14, 33].

While one can perform the integral (1.3) exactly (we have done it for the Lens spaces) it is instructive to use the factorization formula (1.8) to evaluate the holomorphic and anti-holomorphic parts separately. As expected, up to framing dependent phases and a shift in the level, these calculations agree for $G_C = SL(2, \mathbb{C})$ with those of the partition function $SU(2)$. We perform the calculations on $M(n, 0, (a_1, b_1), (a_2, b_2))$ (Seifert manifolds with $N = 2$ orbifold points on the base $S^2$) giving a redundant (diffeomorphic) description of large numbers of Lens spaces in section 9. This description allows us to evaluate the expectation values of Torus knots (in $S^3$, though it is also just as easy to evaluate them in Lens spaces).

Somewhat less obvious is factorization for Seifert manifolds with $N \geq 3$ exceptional fibres. We do not claim to have shown this, though (1.3) holds. However, in the $s \to \infty$
limit holomorphic factorization holds (with the holomorphic theory being a kind of ‘square root’ of BF theory).

For the future, we note that the techniques used here can equally well be used to evaluate the partition functions and observables in Yang-Mills theory with complex gauge group and the $G_C/G_C$ model (at least on $S^2$) and for supersymmetric theories.

2 Complex Chern-Simons Theory

Let $G$ denote a simple and simply connected compact Lie group and let $G_C$ be its complexification. Consider a 3-manifold $M$ and the trivial $G_C$ bundle over it $P_C = M \times G_C$. Let $\mathcal{A}_C$ be the affine space of connections on $P_C$. The Chern-Simons action is,

$$I(t, \hat{t}) = \frac{t}{8\pi} \int_M \text{Tr} \left( \mathcal{C} \wedge d\mathcal{C} + \frac{2}{3} \mathcal{C} \wedge \mathcal{C} \wedge \mathcal{C} \right)$$

$$+ \frac{\hat{t}}{8\pi} \int_M \text{Tr} \left( \overline{\mathcal{C}} \wedge d\overline{\mathcal{C}} + \frac{2}{3} \overline{\mathcal{C}} \wedge \overline{\mathcal{C}} \wedge \overline{\mathcal{C}} \right),$$

(2.1)

for $\mathcal{C} \in \mathcal{A}_C$ and with $t, \hat{t} \in \mathbb{C}$ (but not necessarily complex conjugates). The overline indicates minus Hermitian conjugation (which is the same as complex conjugation within the trace). We may write

$$\mathcal{C} = \mathcal{A} + i\mathcal{B}, \quad \mathcal{A}, \mathcal{B} \in \Omega^1(M, \text{Lie } G)$$

(2.2)

where both $\mathcal{A}$ and $\mathcal{B}$ are anti-Hermitian. Under this split the action becomes

$$I(k, s) = \frac{k}{4\pi} \int_M \text{Tr} \left( \mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} - \mathcal{B} \wedge d\mathcal{A} \mathcal{B} \right)$$

$$- \frac{s}{2\pi} \int_M \text{Tr} \left( \mathcal{B} \wedge F_{\mathcal{A}} - \frac{1}{3} \mathcal{B} \wedge \mathcal{B} \wedge \mathcal{B} \right)$$

(2.3)

where $t = k + is, \hat{t} = k - is$. We let $I$ denote the Chern-Simons action for the compact structure group $G$,

$$I = \frac{1}{4\pi} \int_M \text{Tr} \left( \mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right)$$

(2.4)

As the exponential of the action should be invariant under $G \subset G_C$ transformations we must have that $k \in \mathbb{Z}$, however, this imposes no constraint on $s \in \mathbb{C}$. Unitarity of the theory is another issue. As explained in [31] unitarity for a Euclidean theory amounts to requiring that the argument of the path integral under a change orientation is the same as complex conjugation. Consequently, on putting the manifold dependence in $I(t, \hat{t})$ [2.1],

$$-I(M, t, \hat{t}) = I(-M, t, \hat{t})$$

(2.5)

There are two choices for $s$ which lead to a unitary theory [31]:

4
1. $s \in \mathbb{R}$ with $\mathcal{C} \rightarrow \overline{\mathcal{C}}$ under orientation reversal or

2. $s \in i\mathbb{R}$ and under a reversal of orientation $\mathcal{C} \rightarrow \overline{\mathcal{C}}$

The unitary theory that we consider here is the one where $s \in \mathbb{R}$ so that $\hat{t} = \overline{t}$.

2.1 A Symmetry $t \leftrightarrow \hat{t}$

We note that by simply sending $\mathcal{B} \rightarrow -\mathcal{B}$ in (2.2) we are exchanging $\mathcal{C} \leftrightarrow \overline{\mathcal{C}}$ so that the theory (providing one’s regularization preserves this transformation) is invariant under an interchange of $t \leftrightarrow \hat{t}$. This transformation is independent of the unitarity conditions just described. The interchange $t \leftrightarrow \hat{t}$ amounts to saying that the theory is invariant under $s \rightarrow -s$.

3 Various Limits

One may get a feel for what the invariants are by taking various limits of the parameter $s$ that appears in the theory. This provides some intuition for the formal structure of the theory. For the rest of this section the 3-manifold $M$ is taken to be a rational homology sphere ($\mathbb{Q}$HS), $H_1(M, \mathbb{Q}) = 0$, so that the moduli space of flat $G_C$ connections is a set of isolated points. $M$ being a $\mathbb{Q}$HS means that we do not have to worry about zero modes associated with flat directions in the path integral.

3.1 The $s \rightarrow 0$ Limit

A potential source of zero modes are solutions to the equations
\[
d_{\mathcal{A}} \mathcal{B} = 0
\]
however, by our choice of $M$, we do not have to confront solutions to this equation about a flat connection $\mathcal{A}$. This means that $I(k, 0)$ is a perfectly good action in this case. Performing the Gaussian Integration over $\mathcal{B}$ (including gauge fixing) gives, up to a phase,
\[
\sqrt{\tau_M(\mathcal{A})}
\]
where $\tau_M$ is the Ray-Singer torsion on $M$ of the given flat connection. The phase that one gets from this path integral exactly compensates that that one would get on a perturbative evaluation of the path integral over $\mathcal{A}$ [2].

While it is pleasing to see the Ray-Singer Torsion (3.2) arise, its presence in a non-perturbative treatment of the path integral complicates matters. The argument of the torsion in general is not a flat connection and the integral over the space of $G$ connections $\mathcal{A}$ now has its measure given by (3.2) which is in principle very complicated.
3.2 The $s \to \infty$ Limit

To ensure that $\exp (iI(k, s))$ does not oscillate wildly as $s \to \infty$ we also scale 

$$\mathcal{B} \to \mathcal{B}/s$$

Then the leading terms in the action become 

$$\frac{k}{4\pi} \int_M \text{Tr} \left( \mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right) - \frac{1}{2\pi} \int_M \text{Tr} (\mathcal{B} \wedge F_{\mathcal{A}})$$

which is a combination of $BF$ and Chern-Simons theories. Had we kept higher order terms in $1/s$ we would have been able to develop a perturbation theory in this limit. Formally, the measure of the theory in this limit together with the scaling (up to $\pi$ factors) goes as 

$$\prod_i (dC_i \sqrt{t}) (dC_i \sqrt{\hat{t}}) \rightarrow \prod_i (dA_i d\mathcal{B}_i \sqrt{k^2 + s^2})$$

for $i$ some complete basis indexing set of the forms that appear in the path integral. This is not quite right, as we have not taken the volume of the gauge group into account. A correct analysis will require us to multiply the measure by a factor so that the right hand limit in (3.4) is what one obtains. One can see the problem directly in (1.3) where the proposed limit always gives zero. We will come back to this in Section 8.

Notice that for consistency in this limit we need also to make the Wigner-Inönü group contraction 

$$G_C \to IG$$

(3.5)

The path integral is straightforward to perform and, at least formally, one obtains that the partition function is 

$$Z(k, \infty) = \sum_{\mathcal{A} \in M} \exp (ikI(\mathcal{A})) . \tau_M(\mathcal{A})$$

(3.6) 

where $M$ is the moduli space of flat $G$ connections on $M$. One can do better and also formally evaluate the expectation value of Wilson loops of the $G$ connection $\mathcal{A}$, 

$$\left\{ \prod_i \text{Tr}_{R_i} \left( P e^{\int_{\gamma_i} \mathcal{A}} \right) \right\} = \sum_{\mathcal{A} \in M} \exp (ikI(\mathcal{A})) . \prod_i \text{Tr}_{R_i} \left( P e^{\int_{\gamma_i} \mathcal{A}} \right) \tau_M(\mathcal{A})$$

(3.7)

In section 9 of [7] and in [10] we were able to evaluate the path integral explicitly for certain Seifert manifolds (for example the computation gives the explicit form of the Ray-Singer torsion for each flat connection). One may also obtain explicit expressions with the insertion of Wilson loops which wrap along the fibre of $M$ and the class of Wilson loops include those involving $\mathcal{B}$ with finite and infinite dimensional representations [10].

Before leaving this example we note that there is no extra phase arising from performing the Gaussian integrals in this limit. This is completely in line with the observations of Bar-Natan and Witten [2].
3.3 The $k \to 0$ Limit

The $k \to 0$ limit greatly simplifies the action of the finite dimensional theory that we arrive at, as one can see in [8.3].

This limit is interesting from the gravitational viewpoint as, if one considers $\mathcal{B}$ to be a (possibly singular) dreibein, the action is that of gravity with a cosmological constant [32]. Predominantly, the manifolds of current interest for applications to quantum gravity are non-compact and require considerations like boundary conditions, an issue we do not discuss here. However, there is also, via the 3d-3d correspondence at $k = 0$ [26], a relationship between complex Chern-Simons theory and the superconformal index of an associated 3d $N = 2$ theory. This comparison requires one to take $s$ purely imaginary and thus requires us to analytically continue our results to that regime.

4 Contact 3-Manifolds and Gauge Conditions

Technically we will need to make use of two facts having to do with gauge fixing. The first fact is that one can impose that the component $\phi$ of the connection $\mathcal{C}$ along the fibre direction of the Seifert manifold can be gauge fixed to be constant along the fibre. The second fact is that one may then, at a price, conjugate $\phi$ into a Cartan sub-algebra of the Lie algebra of the compact group [9]. Once the gauge fixing has been performed one finds that the evaluation can be pushed down to a problem in two dimensional Abelian gauge theory. One decomposes the connection in terms of Fourier modes (along the fibre) and then one integrates over all the massive Fourier modes of the connection leaving an effective theory on the (orbifold) base. Obviously the integration over the infinite number of Fourier components of the connection requires its own set of techniques.

To set notation we let $\mathfrak{g}$ be the Lie algebra of the compact group $G$, $\mathfrak{t}$ a Cartan subalgebra and $\mathfrak{k}$ the complement to $\mathfrak{t}$ in $\mathfrak{g}$.

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{k} \quad (4.1)$$

Likewise the complexification of $\mathfrak{g}$, $\mathfrak{g}_C$, is then the Lie algebra of $G_C$ and we let $\mathfrak{t}_C$ be the complexification of $\mathfrak{t}$ so that we have

$$\mathfrak{g}_C = \mathfrak{t}_C \oplus \mathfrak{k}_C \quad (4.2)$$

All 3-manifolds admit a contact structure, that is $M$ can be equipped with a globally defined one-form $\kappa$ such that $\kappa \wedge d\kappa \neq 0$. When $M$ is considered to be a contact manifold $(M, \kappa)$, (and $M$ now not necessarily a QHS), there is a natural decomposition of the tangent bundle and consequently of the cotangent bundle so that we may decompose connections as

$$\mathcal{A} = A + \kappa \phi, \quad \mathcal{B} = B + \kappa \lambda. \quad (4.3)$$
Given a contact structure one also has a Reeb vector field $K$ such that

$$\iota_K \kappa = 1, \quad \iota_K d\kappa = 0 \quad (4.4)$$

When $M$ is Seifert 3-manifold we take $\kappa$ to be the connection 1-form and $K$ the generator of $S^1$ on $M$.

Given this decomposition (4.3) the action becomes

$$I(k, s) = \frac{k}{4\pi} \int_M \kappa \land \text{Tr} \left[ -A \land L_\phi A + B \land L_\phi B - 2\lambda d_A B + 2\phi dA + d\kappa \left( \phi^2 - \lambda^2 \right) \right]$$

$$- \frac{s}{2\pi} \int_M \kappa \land \text{Tr} \left[ -B \land L_\phi A + B \land d\phi + \lambda \land dA + \frac{1}{2} B \land [\lambda, B] \right.$$

$$\left. - \frac{1}{2} A \land [\lambda, A] + d\kappa \lambda \phi \right] \quad (4.5)$$

where the twisted Lie derivative is

$$L_\phi \equiv \iota_K \circ d_\phi + d_\phi \circ \iota_K \quad (4.6)$$

and the covariant derivative that enters is twisted along the flow

$$d_\phi = d + \kappa \phi \quad (4.7)$$

Now we impose the condition that

$$\iota_K d \iota_K C = 0 \Rightarrow \iota_K d\phi = 0, \quad \iota_K d\lambda = 0 \quad (4.8)$$

Furthermore, we also impose the conditions that along the fibre the connection is in the Cartan subalgebra of $G_C$,

$$\iota_K C^k = 0 \Rightarrow \phi^k = 0, \quad \lambda^k = 0 \quad (4.9)$$

We will impose these conditions in a unitary manner. Let $\mathcal{D}$ be a $g_C$-valued 0-form and add to the action

$$\int_M \text{Tr} \left[ \mathcal{D} * \iota_K dt_K C + \overline{\mathcal{D}} * \iota_K dt_K \overline{C} \right] \quad (4.10)$$

as Lagrange multiplier fields imposing (4.8). Notice that this is a real choice and that the components of $\mathcal{D}$ that do not depend on the fibre direction of $M$ do not enter. We let $\mathcal{E}$ and $\mathcal{F}$ be $g_C$-valued 0-form ghost fields, so that the ghost action becomes

$$\int_M \text{Tr} \left[ \mathcal{E} * \iota_K dt_K d_\phi \mathcal{F} + \overline{\mathcal{E}} * \iota_K dt_K d_\phi \overline{\mathcal{F}} \right] \quad (4.11)$$

As long as we agree to not include $L_K$ zero modes we may simplify the ghost action to,

$$\int_M \text{Tr} \left[ \mathcal{E} * \iota_K d_\phi \mathcal{F} + \overline{\mathcal{E}} * \iota_K d_\phi \overline{\mathcal{F}} \right] \quad (4.12)$$
We also want to impose that the now constant fields lie in the Cartan subalgebra. We use the constant $k_C$ part of $D$ to do that,

$$\int_M \text{Tr} \left[ D^* C + D^* \overline{C} \right]$$

(4.13)

The ghost terms are then correctly incorporated in (4.12) where we understand that the only components of the ghosts which do not appear are those constant along the fibre and simultaneously take values in the Cartan subalgebra.

5 Seifert Rational Homology Spheres

The 3-manifolds of interest are circle $V$-bundles over 2 dimensional orbifolds $\Sigma$ of genus $g$ - shortly we will fix on the base being $\mathbb{P}^1$ with $N$ orbifold points. The Seifert manifold is written as $M[\text{deg } \mathcal{L}, g, (a_1, b_1), \ldots, (a_N, b_N)]$ where the $a_i$ are the isotropies of the orbifold points, the $b_i$ are the weights of the line $V$-bundle at the orbifold points and $\text{deg } \mathcal{L}$ is the degree of that line bundle. The local model at each orbifold point, for the associated line $V$-bundle, is

$$(z, w) \simeq (\zeta z, \zeta^b w), \quad \zeta^a = 1$$

(5.1)

The Seifert manifold is smooth if $\gcd(a_i, b_i) = 1$ for each $i$. It is an $\mathbb{Z}$HS iff the line bundle $\mathcal{L}_0$ that defines it satisfies

$$g = 0, \quad c_1(\mathcal{L}_0) = \pm \frac{1}{a_1 \ldots a_N}$$

(5.2)

one consequence of these conditions is that $\gcd(a_i, a_j) = 1$ for $i \neq j$. If one takes a tensor power of this line $V$-bundle, $\mathcal{L}_0^{\otimes d}$ then the Seifert manifold is a $\mathbb{Q}$HS with

$$g = 0, \quad c_1(\mathcal{L}_0^{\otimes d}) = \pm \frac{d}{a_1 \ldots a_N}$$

(5.3)

and

$$|d| = |H_1(M, \mathbb{Z})|$$

(5.4)

We want to make contact with the $L(p, q)$ notation used for Lens spaces. It is a theorem that all Seifert manifolds satisfying (5.3) with $N \leq 2$ are Lens spaces (see [25] page 99). Indeed all Seifert manifolds satisfying (5.2) with $N \leq 2$ are $S^3$. Fix $N = 2$ and recall that for a Lens space $L(p, q)$ the order of the first integral homology group is $|p|$ so that we may identify $p$ with $d$ in (5.3). This gives us a formula, namely

$$p = a_1 a_2 c_1(\mathcal{L}_0^{\otimes d}) = a_1 a_2 \left( n + \frac{b_1}{a_1} + \frac{b_2}{a_2} \right)$$

(5.5)

where $n = \text{deg } \mathcal{L}_0^{\otimes d}$. The $q$ of the Lens space is then determined in the following way. Let $r$ and $s$ be integers satisfying $ra_1 - s(na_1 + b_1) = 1$ then

$$q = ra_2 - sb_2$$

(5.6)
6 Calculation of the Determinants

The calculation of the ratio of the determinants in the $G_C$ theory is very similar to that for gauge group $G$ [8]. We will concentrate mostly on the differences with that exposition.

The ratio of determinants that we wish to evaluate is, see (5.5) of [8],

$$
\frac{\text{Det} \left( i \tilde{L}(\phi, \lambda) \right)_{\Omega^0(M, t) \otimes \Omega^0(M, t)}}{\sqrt{\text{Det} \left( * \kappa \wedge i \tilde{L}(\phi, \lambda) \right)_{\Omega^1_H(M, t) \otimes \Omega^1_H(M, t)}}}
$$

(6.1)

where the operator $* \kappa \wedge i \tilde{L}(\phi, \lambda)$ acts on horizontal $\mathfrak{k}$ valued forms

$$
* \kappa \wedge i \tilde{L}(\phi, \lambda) : \Omega^1_H(M, \mathfrak{k}) \otimes \Omega^1_H(M, \mathfrak{k}) \rightarrow \Omega^1_H(M, \mathfrak{k}) \otimes \Omega^1_H(M, \mathfrak{k})
$$

(6.2)

while the ghost operator is

$$
i \tilde{L}(\phi, \lambda) : \Omega^0(M, \mathfrak{k}) \otimes \Omega^0(M, \mathfrak{k}) \rightarrow \Omega^0(M, \mathfrak{k}) \otimes \Omega^0(M, \mathfrak{k})
$$

(6.3)

and neither of these are diagonal, though both are Hermitian. As usual we decompose the space of forms into modes along the circle direction as

$$
\Omega^1_H(M, \mathfrak{k}) = \bigoplus \Omega^1(\Sigma, \mathcal{L} \otimes n \otimes V_\alpha),
\Omega^0(M, \mathfrak{k}) = \bigoplus \Omega^0(\Sigma, \mathcal{L} \otimes n \otimes V_\alpha)
$$

(6.4)

and we also decompose the charge space into roots

$$
V_\mathfrak{k} = \bigoplus \alpha V_\alpha
$$

(6.5)

On expanding the fields $A$ and $B$ in terms of Fourier modes and charges (concentrating on those that take values in the complement of the Cartan subalgebra) we may write the operators in matrix form as

$$
* \kappa \wedge i \tilde{L}(\phi, \lambda) \big|_{\Omega^1(\Sigma, \mathcal{L} \otimes n \otimes V_\alpha) \otimes \Omega^1(\Sigma, \mathcal{L} \otimes n \otimes V_\alpha)} = \frac{1}{\sqrt{(k^2 + s^2)}} \begin{pmatrix}
k(n + i\alpha(\phi)) - si\alpha(\lambda) & -s(n + i\alpha(\phi)) - k\alpha(\lambda) \\
-s(n + i\alpha(\phi)) - k\alpha(\lambda) & k(n + i\alpha(\phi)) + si\alpha(\lambda)
\end{pmatrix}
$$

(6.6)

the factor of $(k^2 + s^2)^{-1/2}$ is there so that the measure for the fields is simply $\prod_n dA_n dB_n$, however, there is still a factor of $\sqrt{\mathfrak{t}}$ for each of the modes of the fields which we have not integrated. To put the ghost operator in a similar form we expand as before

$$
\mathcal{E} = E + i\mathcal{E}, \quad \mathcal{F} = F + i\mathcal{F}
$$

(6.7)

and the ghost operator is

$$
i \tilde{L}(\phi, \lambda) \big|_{\Omega^0(\Sigma, \mathcal{L} \otimes n \otimes V_\alpha) \otimes \Omega^0(\Sigma, \mathcal{L} \otimes n \otimes V_\alpha)} = \begin{pmatrix}
n + i\alpha(\phi) & i\alpha(\lambda) \\
i\alpha(\lambda) & -n - i\alpha(\phi)
\end{pmatrix}
$$

(6.8)
Given an operator $T$ one may define the absolute value of its determinant following [30] as the positive root $\sqrt{\det(T^T)}$. The operators in question are Hermitian so we want to take their square. The matrix square of (6.6) is

$$[(n + i\alpha(\phi))^2 + (i\alpha(\lambda))^2] \cdot I_{2\times2}$$

(6.9)

where $I_{2\times2}$ is the two by two identity matrix and the matrix square of (6.8) is

$$[(n + i\alpha(\phi))^2 + (i\alpha(\lambda))^2] \cdot I_{2\times2}$$

(6.10)

Though these last two expressions appear to be the same one must recall that they act on different spaces of forms. Nevertheless, they may be ‘holomorphically factorized’ as

$$[n + i\alpha(\phi) + \alpha(\lambda)] \cdot [n + i\alpha(\phi) - \alpha(\lambda)] \cdot I_{2\times2}$$

(6.11)

Following through the same reasoning as presented in section 5 of [8] we find that the absolute value of the ratio of determinants (6.1) is simply

$$\sqrt{\tau_M(\phi + i\lambda, a_1, \ldots, a_N)} \cdot \sqrt{\tau_M(\phi - i\lambda, a_1, \ldots, a_N)}$$

(6.12)

where

$$\tau_M(\Phi, a_1, \ldots, a_N) = \tau_{S^1}(\Phi)^{2-2g-N} \cdot \prod_{i=1}^{N} \tau_{S^1}(\Phi/a_i)$$

(6.13)

and the Ray-Singer Torsion of the circle being

$$\tau_{S^1}(\Phi) = \prod_{m,\alpha} (2\pi m + i\alpha(\Phi))$$

$$= \prod_{\alpha} 2 \sin(i\alpha(\Phi))$$

(6.14)

Incidentally, this calculation implies that the Ray-Singer torsion for non-unitary flat connections, $\tilde{\tau}_M$, factorises holomorphically, at least for these Seifert manifolds and for these particular Abelian connections.

We have not explicitly calculated the phase of the determinants but one way to see that there is no overall phase correction is to note that both matrices (6.6) and (6.8) are traceless. Being traceless means that the two eigenvalues are of the same absolute value but with the opposite sign and so cancel each others contribution in the evaluation of the $\eta$ invariant.

Having integrated out all the non-zero modes in the $S^1$ direction of $M$ and all those zero modes in the $\xi_C$ part of the Lie algebra, we are left with an Abelian theory on the orbifold base of the fibration,

$$Z_M[t, \vec{t}] = \sum_{n} \int D\Phi D\bar{\Phi} D\Lambda D\bar{\Lambda} \exp\left(iP_{\Sigma}(t, \vec{t})\right) \frac{\tau_{1/2}^M(\Phi; a_1, \ldots, a_N)}{\tilde{\tau}_M}$$

(6.15)
Here we have set $\Phi = \phi + i\lambda$, $A = A + iB$, 
\[
\hat{\tau}_M(\Phi; a_1, \ldots, a_N) = \tau_M(\Phi, a_1, \ldots, a_N) \cdot \tau_M(\Phi, a_1, \ldots, a_N)
\] (6.16)
and the action is 
\[
I^R_\Sigma(t, \overline{t}) = \frac{t}{4\pi} \int_{\Sigma} \text{Tr} (\Phi \wedge (F_A + i2\pi n \omega)) + \frac{\overline{t}}{4\pi} \int_{\Sigma} \text{Tr} (\overline{\Phi} \wedge (F_A - i2\pi n \omega)) + \frac{t}{8\pi} \int_{\Sigma} \text{Tr} (\Phi^2) \wedge \omega + \frac{\overline{t}}{8\pi} \int_{\Sigma} \text{Tr} (\overline{\Phi}^2) \wedge \omega
\] (6.17)
The sum over $\text{rank}(G)$ $U(1)$ bundles is there to take into account the non-triviality of the Abelian gauge fixing, so that $A$ is a connection on a product of trivial line bundles over $\Sigma$. We have not specified the range of the summation as that depends on the choice of 3-manifold. In the next section we will pick a certain class of 3-manifolds.

Notice that the form of (6.15) with action (6.17) still suggests a ‘holomorphic’ factorization of the path integral,
\[
Z_M[t, \overline{t}] \simeq \sum_n Z^R_n[\tau] \cdot \overline{Z^R_n[\overline{\tau}]}
\] (6.18)
with
\[
Z^R_n[\tau] = \int D\Phi \, D\lambda \, \exp \left( iI^R_\Sigma(t) \right) \, \tau_M^{1/2}(\Phi, a_1, \ldots, a_N),
\] (6.19)
\[
I^R_\Sigma(t) = \frac{t}{4\pi} \int_{\Sigma} \text{Tr} (\Phi \wedge (F_A + i2\pi n \omega)) + \frac{\overline{t}}{8\pi} \int_{\Sigma} \text{Tr} (\Phi^2) \wedge \omega
\] (6.20)
We could have included a phase factor that only depends on the gauge group and on the 3-manifold $M$ in the definition of the holomorphic partition function -providing it cancels against a similar phase from the anti-holomorphic partition function.

7 Finite Dimensional Integrals

At this point we must make an assumption about the 3-manifold $M$. The Abelian theory (6.15) has zero modes in that elements $B \in H^1(\Sigma, it)$ do not appear in the argument of the path integral. There are also $A$ zero modes but large gauge transformations would ensure that they are compact directions. To avoid this issue we assume that the genus of $\Sigma$ is zero.

With this assumption the path integral (6.15) can be simplified further on noting that integration over $\Lambda$ and $\overline{\Lambda}$ imply that $d\Phi = 0$.

\[
Z_M[t, \overline{t}] \simeq \sum_n \int_{i\mathbb{C}} \exp \left( iI^R(t, \overline{t}) \right) \, \hat{\tau}_M^{1/2}(\Phi; a_1, \ldots, a_N)
\] (7.1)
with
\[ I^n(t, \bar{t}) = \text{Tr} \left( \frac{t}{2} \Phi n - i \frac{\bar{r}}{2} \Phi n \right) - c_1(\mathcal{L}) \text{Tr} \left( \frac{t}{8\pi} \Phi^2 + \frac{\bar{r}}{8\pi} \Phi^2 \right) \] (7.2)

As \( c_1(\mathcal{L}) \in \mathbb{Q} \) we set \( c_1(\mathcal{L}) = d/P \). The finite dimensional ‘action’ (7.2) transforms as
\[ I^n(t, \bar{t}) \rightarrow I^n(t, \bar{t}) - 2\pi kP \text{Tr}(n r) - \pi kP d \text{Tr}(r r) \] (7.3)
under
\[ \Phi \rightarrow \Phi + 2\pi i r, \quad n \rightarrow n + d r \] (7.4)
and provided that \( \text{Tr}(r r) \in 2\mathbb{Z} \) the exponential in (7.1) is invariant under these transformations. On taking \( P = \prod a_i \) the Ray-Singer torsion \( \hat{\tau}_M \) is also invariant under (7.4). These transformations correspond to shifts by the integral lattice \( I \) of \( t \) (not the complexified integral lattice for which there is no such invariance).

One should also consider the action of the Weyl group which acts naturally on \( t \) as well as on \( t_\mathbb{C} \). This is a symmetry of finite dimensional theory as it is part of the (ungauged) gauge group. On quotienting out by these residual invariances we have either of two formulae. One by using the symmetry to reduce the integrals to \( t_\mathbb{C}/I \rtimes W \), the other to restrict the range of the summation. Consequently,
\[ Z_M[t, \bar{t}] = \sum_n \int_{t_\mathbb{C}/I \rtimes W} \exp \left( i I^n(t, \bar{t}) \right) \hat{\tau}_M^{1/2}(\Phi; a_1, \ldots, a_N) \] (7.5)
or
\[ Z_M[t, \bar{t}] = \frac{1}{|W|} \sum_{1 \leq n \leq d} \int_{t_\mathbb{C}} \exp \left( i I^n(t, \bar{t}) \right) \hat{\tau}_M^{1/2}(\Phi; a_1, \ldots, a_N) \] (7.6)
Clearly (7.1), (7.5) and (7.6) appear to have the holomorphic decomposition
\[ Z_M[t, \bar{t}] = \frac{\exp(-i\pi/2)}{|W|} \sum_n Z_{t_\mathbb{C}}^n[t] \cdot \overline{Z_{t_\mathbb{C}}^n[t]} \] (7.7)
with
\[ Z_{t_\mathbb{C}}^n[t] = \int_{t_\mathbb{C}} \exp \left( i I_{t_\mathbb{C}}^n[t] \right) \tau_M^{1/2}(\Phi; a_1, \ldots, a_N) \] (7.8)
where
\[ I_{t_\mathbb{C}}^n[t] = \text{Tr} \left( \frac{t}{2} \Phi n - c_1(\mathcal{L}) \frac{t}{8\pi} \Phi^2 \right) \] (7.9)
and there is either a restriction on the range of summation or on the integration. We could have also introduced a phase into the definition, but prefer not to. This has consequences for the interpretation of the factorization formula as we will see. The contour \( t_\mathbb{C} \) needs to be defined. In the case of Lens spaces one may take it to be \( t_\mathbb{C} = t \).
Once one has factorized it is no longer necessarily true that
\[ Z_{n+d\Gamma}^n[t] = Z_{n+d\Gamma}^n[t] \] (7.10)
as the symmetry (7.4) holds through a cancellation between the holomorphic and anti-holomorphic parts of the action, so that the best one can hope for is
\[ Z_{n+d\Gamma}^n[t] = \pm Z_{n+d\Gamma}^n[t] \] (7.11)

The Ray-Singer Torsion \( \tau_M(\Phi) \) (6.13) diverges at the zeros of the sine function (6.14) when the number of orbifold points is such that \( N + 2 \, g > 2 \). This means that the torsion \( \hat{\tau}_M \) will also have poles. The zeros of \( \sin(x + iy) \) are at \((x, y) = (m\pi, 0)\) for \( m \in \mathbb{Z} \).

If one first makes use of the symmetry (7.4) to restrict the range of integration of \( \Phi \), actually of \( \phi \), then the integrals to be performed over \( \phi \) range from over \(-\pi P \leq \phi \leq \pi P\) for each component in the Cartan subalgebra. The sum over the integers \( n \) then implies that \( k\phi - s\lambda = r\pi \) for \( r \in \mathbb{Z}^{G_k} \). The singularities occur at \((\phi, \lambda) = (m\pi, 0)\) whence the divergences are at \( k|r| \). For compact gauge group \( G \) the divergences are at \((k + c_g)|r|\) (but we do not see the shift in the level in \( G_c \)). We may give the charged gauge fields (those in the orthocomplement of \( t_c \)) a small mass while preserving the maximal torus symmetry of the action. As for Chern-Simons theory with compact gauge group \( G \) this means that at \( \lambda = 0 \) we have the ratio
\[ \sin^2 (i\alpha(\phi))/\sin^N (i\alpha(\phi) + \epsilon) \] (7.12)
which is non-singular on the walls, indeed vanishes there.

8 Summing over Flat Connections

A comparison of (7.8) with our formula for compact \( G \) (6.2) in [8] shows that we have
\[ Z_G[M, k] = \sum_r \exp \left( \frac{4\pi i\Phi(L)}{|W|} \right) \cdot Z_{t_c}^r[M, 2(k + c_g)] \] (8.1)
while the formula for the partition function of Chern-Simons theory with gauge group \( G_c \) is
\[ Z_{G_c}[M, k] = \exp \left( \frac{-i\pi/2}{|W|} \right) \sum_r Z_{t_c}^r[M, t] \cdot Z_{t_c}^r[M, \hat{t}] \] (8.2)
In passing to (8.1) one is losing some information as \( t \) is a Gauss integer, so it has integral real part, while \( 2(k + c_g) \) is even.

The summation in the factorization formulae is over an integer which, presently, appears to have no geometric significance from the point of view of the original theory. Here we will show that, quite miraculously, the summation is at least in a limit over certain
flat connections. To establish this relationship we will need to take a closer look at the large $s$ limit.

In components the action of the finite dimensional integral for any Seifert QHS $M$ is

$$I^n(t, \bar{t}) = \text{Tr}[(k\phi - s\lambda)i\mathbf{n}] + c_1(\mathcal{L}) \text{Tr} \left[ \frac{k}{4\pi} (\phi^2 - \lambda^2) - \frac{s}{2\pi} \phi \lambda \right]$$

(8.3)

In order to compare with the large $s$ limit of Section 3 we must scale $\lambda \rightarrow \lambda/s$ in which case the action goes over to

$$I^n(t, \bar{t}) \rightarrow \text{Tr}[(k\phi - \lambda)i\mathbf{n}] + c_1(\mathcal{L}) \text{Tr} \left[ \frac{k}{4\pi} \phi^2 - \frac{1}{2\pi} \phi \lambda \right]$$

(8.4)

Unfortunately, the scaling multiplies the integral with a prefactor of $|s|^{-\text{rk}(G)}$, which in the limit means that the path integral vanishes. In order to pass to the BF theory we must multiply the path integral by $|c_1(\mathcal{L}_0^d)|^{-\text{rk}(G)}$ and we presume that this has been done.

This scaling also turns the argument of the Ray-Singer Torsion from $\Phi$ to $\phi$,

$$\tau^{1/2}_M(\Phi) \rightarrow \tau_M(\phi)$$

(8.5)

As $\lambda$ now only appears in the action (8.4), the integral over $\lambda$ imposes the condition

$$c_1(\mathcal{L}_0^d) \phi = -2\pi i\mathbf{n}$$

that is

$$\phi = 2\pi i\frac{\mathbf{n}}{d}$$

(8.6)

as a delta function constraint. In this way we obtain for the partition function

$$Z_M[t, \bar{t}] \xrightarrow{s \rightarrow \infty} \sum \exp \left( i\pi k P \text{Tr} \frac{\mathbf{n}^2}{d} \right) \cdot \tau_M(2\pi i\frac{\mathbf{n} P}{d})$$

(8.7)

Some things about this result requires comment. Note that we do not attempt to sum over flat connections at all, yet the final sum may be interpreted as that over flat Abelian connections. We have introduced a sum over non-trivial $U(1)$ bundles in (6.17) which may be viewed as arising from background connections with values in the Cartan subalgebra of $G$ which are Yang-Mills connections (that is they satisfy the Yang-Mills equations) with the explicit connection being

$$A = 2\pi P\frac{\mathbf{n}}{d}\kappa, \quad F_A = 2\pi \mathbf{n}\omega \quad \text{and} \quad d_A \ast F_A = 0$$

(8.8)

As explained by Atiyah and Bott [1] section 6 and, in a version closer to our needs, in section 5 of Beasley and Witten [4] there is a close relationship between the space of flat connections on $M$ a circle bundle over $\Sigma$ and Yang-Mills connections on $\Sigma$. To explain
this we need the generators and relations defining the various fundamental groups. As
M is a QHS the generators of \( \pi_1(M) \), \( c_i, i = 1, \ldots, N \) and \( h \) satisfy

\[
[c_i, h] = 1, \quad c_j^a h^b = 1, \quad \text{and} \quad \prod_{j=1}^N c_j = h^n \quad (8.9)
\]
clearly \( h \) is central and when \( h = 1 \) these relations reproduce those of the fundamental
group of the orbifold \( \Sigma \). Yang-Mills connections on \( \Sigma \) correspond to having a central
extension of the fundamental group according to [1], while in [4] it is noted that as

\[
1 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(M) \longrightarrow \pi_1(\Sigma) \longrightarrow 1 \quad (8.10)
\]
that the extra generator \( h \) in \( \pi_1(M) \) then provides that extension. In turn the Yang-Mills
connection (8.8) provides us with a representation of the extension \( h \)

\[
\rho(h) = \exp \left( 2\pi \oint_{S^1} (P n/d) \kappa \right) = \exp (2\pi P n/d) \quad (8.11)
\]
Given an \( m \)-dimensional representation,

\[
\rho : \pi_1(M(n, 0, (a_1, b_1), \ldots (a_N, b_N))) \longrightarrow \text{GL}_m(F) \quad (8.12)
\]
the Reidemeister torsion for a Seifert manifold \( M \) is given by [21] (see Lemma 4.3 there
and note our definition of the torsion is the inverse of the one used there)

\[
\tau_M(\rho) = \det (\rho(h) - I)_{2N} \prod_{i=1}^N \det (\rho(h^{a_i} c_i^r) - I) \quad (8.13)
\]
where

\[
a_is_i - b_ir_i = 1 \quad (8.14)
\]
Our representation (8.11) is into the diagonal matrices so that by (8.14) and the homo-
topy relations (8.9) we have that

\[
\rho(h^{a_i} c_i^r) = \rho(h)^{a_i - r_i/b_i} = \rho(h)^{1/a_i} \quad (8.15)
\]
so that

\[
\tau_M(\rho) = \prod_\alpha 4 \sin (2\pi P\alpha(r)/d)_{2N} \prod_{i=1}^N \sin (2\pi P\alpha(r)/da_i) \quad (8.16)
\]
We are, therefore, able to understand the partition function (8.7) as being given by a
sum over flat connections as required by (3.6), however, not a sum over all possible
flat connections but rather over a class of Abelian flat connections. To complete the
comparison we also note that the exponent in (8.7) is just the evaluation of the Chern-
Simons action on the flat connection of interest.
In summary one interprets the summation as being over Abelian connections for all allowed \( M \) then (1.6) holds where the representations \( \rho \) arise from very special flat connections, namely those that are Abelian, are in the Cartan subalgebra of \( G \) and correspond to the generator \( h \). For such \( \rho \) we would then also have that \( n_{(\rho, \bar{\rho})} = 1 \), and that the factors \( Z_M^{\rho}[t] \) are, in some way, the complex Chern-Simons theory exactly evaluated about our preferred Abelian connection. In the following section we show in which sense that this is the case for Lens spaces.

9 Some Calculations on \( L(p, q) \) with \( G_C = SL(2, \mathbb{C}) \)

In this section we will derive some exact formulae for the partition function on \( L(p, q) \) with complex gauge group \( SL(2, \mathbb{C}) \). A bit further on we consider the expectation value of some knots too. These calculations allow one to see the explicit dependence on framing. We represent these Lens spaces as Seifert manifolds either with one or two exceptional fibres.

9.1 Holomorphic Factorization

According to (1.7) and (1.8) the partition function factorises with holomorphic part

\[
Z_{tr}^r[L(p, q), t] = \frac{1}{\sqrt{4\pi^2 a_1 a_2}} \int \exp \left( -itzr + i\frac{tp}{4\pi a_1 a_2} z^2 \right) \frac{4}{\sqrt{4}} \sin \left( \frac{2\pi}{t} \right) \sin \left( \frac{2\pi}{\hat{t}} \right)
\]  

(9.1)

On \( S^3 \) we have that \( p = 1 \) which means that \( r = 0 \) but we consider \( S^3 \) as the Seifert 3-manifold, \( M(n, 0, (a_1, b_1), (a_2, b_2)) \) with \( n \) chosen so that \( c_1 = \pm 1/a_1 a_2 \). The standard Hopf fibration corresponds to \( a_1 = a_2 = 1, \ b_1 = b_2 = 0 \) and \( n = \pm 1 \). As \( r = 0 \) we have a very simple integral to perform indeed. By writing the product of the sine functions as sums of exponentials we arrive at

\[
Z_{tr}[S^3, t] = \exp \left( \frac{-\pi i}{4} - i\frac{\pi a_1^2 + a_2^2}{a_1 a_2} \right) \frac{8}{\sqrt{4\pi t}} \sin \left( \frac{2\pi}{t} \right) \sin \left( \frac{2\pi}{\hat{t}} \right)
\]  

(9.2)

The phase is a consequence of our choice of framing. Combining with the anti-holomorphic part we obtain

\[
Z_{SL(2,\mathbb{C})}[S^3, t, \hat{t}] = \exp \left( -i\pi \frac{a_1^2 + a_2^2}{a_1 a_2} \left( \frac{1}{t} + \frac{1}{\hat{t}} \right) \right) \frac{8}{\sqrt{4\pi t \hat{t}}} \sin \left( \frac{2\pi}{t} \right) \sin \left( \frac{2\pi}{\hat{t}} \right)
\]  

(9.3)

The phase prefactor is neatly written as

\[
\exp (2\pi im(c_L - c_R)/24), \quad m = (a_1^2 + a_2^2)/a_1 a_2
\]  

(9.4)

We now view the general Lens spaces as the Seifert Manifolds \( L(p, q) \equiv M(n, 0, (a, b)) \) where \( q = a \) and \( p = na + b \) ([25] page 99). That is we let \( (a_1, b_1) = (1, 0) \) and
\((a_2, b_2) = (q, b)\) and use \((9.1)\) to calculate

\[
Z_{r_t}^r[L(p,q), t] = \frac{2i}{\sqrt{tp}} e^{\left(\frac{i\pi}{4} - \frac{i\pi}{tp}(q + 1/q)\right)} e^{\left(-\frac{i\pi}{tp}tqr^2\right)} \sum_{\epsilon=\pm 1} \epsilon \cos\left(\frac{2\pi r(q + \epsilon)}{p}\right) e^{-i\frac{2\pi i}{tp}}
\]

Notice that this expression is not necessarily invariant under \(r \rightarrow r + p\). Apart from a phase and an overall factor related to the order of the Weyl group this expression does, however, compare favourably with that of Jeffrey \([19]\) Theorem 3.4. on taking \(t = 2(k + 2)\).

As we have the explicit \(s\) dependence of the path integral in terms of known functions we may extend the formulae to the complex \(s\) plane.

### 9.2 Inclusion of Knots

There is great advantage in having different fibrations represent the same topological space. Given a Seifert fibred 3-manifold the fibre over a regular point of the base is a Torus knot \(K_{a_1, a_2}\). The type of Torus knot depends on the choice of fibration. This means that Wilson loops in the vertical direction can be invariants for different knots in the same manifold by changing the fibration. This approach was used by Beasley in \([3]\) in a study of Chern-Simons theory with knot invariants within the context of non-Abelian localisation. In a finite dimensional representation \(R\), the possible Wilson loops along the fibre are

\[
\text{Tr}_R \left( P \exp i \oint \Phi \right), \quad \text{and} \quad \text{Tr}_R \left( P \exp i \oint \Phi \right)
\]

Once we have diagonalised then path ordering is no longer required and, with the Cartan subalgebra being made up of diagonal matrices, such traces are just finite sums of exponentials of \(\Phi\) or \(\Phi\). For example if we fix on \(G_\mathbb{C} = SL(2, \mathbb{C})\) then in a, finite dimensional representation \(R\) of dimension \(n + 1\)

\[
\text{Tr}_R \left( P \exp i \oint \Phi \right) = \frac{\sin ((n + 1)i\Phi)}{\sin (i\Phi)} = \sum_{j=-n}^{n} \exp(j\Phi)
\]


\( j/2 \) is the spin. Being sums of exponentials we can use the factorization formulae to evaluate expectation values of products of Wilson loops, at least in the case of Lens spaces.

As before, for simplicity, we fix on \( G_C = SL(2, \mathbb{C}) \) and on \( S^3 \). We consider Torus knots of the form \( K_{a_1, a_2} \) and to do so we consider \( S^3 \) to be given by the Seifert fibration \( M(n, 0, (a_1, b_1), (a_2, b_2)) \) while ensuring that the first Chern class of the fibration satisfies \( c_1 = \pm 1/a_1 a_2 \). The \( K_{a_1, a_2} \) Torus knot is then a generic fibre of \( M(n, 0, (a_1, b_1), (a_2, b_2)) \).

The factorization (1.7, 1.8) extends to sums of exponentials,

\[
\int d^2 z \exp (a z^2 + b \overline{z}^2) f(z) g(\overline{z}) = \exp (-i\pi/2) \int \Gamma \exp (a z^2) f(z) \int_{\Gamma'} d\overline{z} \exp (b \overline{z}^2) g(\overline{z})
\]

where \( f(z) = \sum \epsilon_i \exp (c_i z) \) and \( g(\overline{z}) = \sum \eta_j \exp (d_j \overline{z}) \) and

\[
\int d^2 z \exp (a z^2 + b \overline{z}^2) f(z) g(\overline{z}) = \frac{\pi}{\sqrt{-ab}} \sum_{i,j} \epsilon_i \eta_j \exp \left( -\frac{\sum c_i^2}{4a} - \frac{\sum d_j^2}{4b} \right)
\]

We want to use holomorphic factorization once more, so we evaluate the non-normalised ‘holomorphic’ expectation value of a Torus knot \( K_{a_1, a_2} \),

\[
Z_{lc}[S^3, K_{a_1, a_2}, t] \equiv \sum_{j=-n}^{n} \frac{1}{\sqrt{4\pi^2 a_1 a_2}} \int \Gamma \exp \left( i j z + i \frac{t}{4\pi a_1 a_2} z^2 \right) 4 \sin \left( z/a_1 \right) \sin \left( z/a_2 \right)
\]

\[
= \frac{2i}{\sqrt{t}} \sum_{\epsilon = \pm 1} \sum_{j=-n}^{n} \epsilon \exp \left( \frac{\pi}{4} - \frac{i\pi}{a_1 a_2 t} (a_1 a_2 j + a_1 + \epsilon a_2)^2 \right)
\]

One can compare this with the formula for \( G = SU(2) \), [23] [29].

The perturbative holomorphic partition function with the inclusion of the figure eight knot, shown in Figure 2, for an infinite dimensional representation, was determined in [14], but our methods, unfortunately, do not extend to give a non-perturbative evaluation in this case.

Figure 2: The figure 8 knot is the simplest knot which is not a Torus knot. It is designated the 4_1 knot in Rolfsen’s list [28].

9.3 Beyond Lens Spaces

We begin this section with an observation on the partition function in the large \( s \) limit (8.7), namely that it can be written in factorized form since we can take the root of the
Ray-Singer Torsion. This suggests, that in this limit at least, the partition function for
general Seifert manifolds also factorises.
The large $s$ limit makes no sense in (8.1) as $s$ has been set to zero there. What real
time theory does the factorization correspond to then? We need a theory that gives us as
its ‘holomorphic’ part the square root of the Ray-Singer Torsion evaluated at a flat
connection. One possibility is an action which includes a Chern-Simons term and a $BF$
term to land on flat connections but such a theory leads to the Ray-Singer Torsion and
not its square root. This situation can be remedied with the addition of

$$\int_M \text{Tr} \left( \overline{\psi} d_A \psi + \rho d_A \rho \right)$$

(9.9)
to the action where $\psi$ and $\overline{\psi}$ are Grassmann odd Lie algebra valued 1-forms and $\rho$ is a
Grassmann even Lie algebra valued 1-form. This term is, by itself, not gauge invariant
under gauge transformations for the new fields, e.g. under $\rho \to \rho + d_A \sigma$ but can be
compensated for by transforming $B$ (by shifting it by a multiple of $[\rho, \sigma]$ in this case).

We should point out that the derivation of the path integral is also correct when the
3-manifold is $S^2 \times S^1$ even though this is not a QHS. The components $\phi$ and $\lambda$
are now those along the $S^1$ direction and one sets $p = 0$ in (7.1) and subsequent formulae. This
path integral is usually normalised to unity (as in the compact case one is counting
conformal blocks).
The restriction that the Seifert manifold be a QHS comes from the fact that we have
no control over Abelian $B$ modes that are simultaneously constant on the fibre and
harmonic on the base. These modes are not damped in the path integral and so lead to
a divergence. The corresponding $A$ modes are compact and so do not give rise to any
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