1. Introduction

In geometric analysis, one often studies the weak compactness of the moduli space of certain critical structures with various assumption on integral of curvatures. In absence of compactness of set of critical structures (for instance, a sequence of harmonic maps, or Yang Mills connections, or a sequence of Einstein metrics with certain integral bound on curvature etc), one often obtains strong convergence away from a singular set with real codimension $\geq 2$: It is then a critical problem in geometry analysis to understand when these singularities are removable. There are extensive research into this problem, to list a few, the removable singularities of K Uhlenbeck in 4 dimensional Yang Mills connections [10], Scheon-Uhlenbeck on Harmonic maps [12], Minimal surfaces [11] and Kähler Einstein metrics [6]. Perhaps, the most well known example is the one about harmonic function in puncture disc in $\mathbb{C}$: if the harmonic function is uniformly bounded, then it can be extended smoothly across the entire disc.

The existence of constant scalar curvature Kähler metric is a core problem in Kähler geometry. To derive its existence, we may expect that one can only obtain convergence away from a subvariety. Thus it is highly interested to understand the regularity theorem for CscK metric outside a subvariety: when it can be extend smoothly across singularities. This problem has already been studied recently, for instance, in [6]. Chen-Donaldson-Sun proved a removable singularities theorem that any KE metric outside a divisor with uniform equivalent to a smooth background metric must be smooth across singularities. We expect that similar situation might happen for constant scalar curvature Kähler metric with singularities. Therefore, it’s crucial and fundamental to understand how to extend CscK metric across a subvariety. More recently, C. LeBrun also studied a removable singularities for Einstein metrics [9].

The present paper is inspired by the last section of Chen-He [1]. In that paper, they proved, that for a CscK metric with isolated singularities, if the metric $g'$ is uniformly bounded near the singular points, i.e. $C^{-1}g \leq g' \leq Cg$ in a neighborhood of the singular points for some $C > 0$, then it can be extended to be a smooth CscK metric on the whole manifold.
One may ask, if we can generalize their result to more general case where the singularities is a sub variety?

In this paper, we consider a compact Kähler manifold \((M, g)\) of \(\dim \mathbb{C}M = n\) and a Divisor \(D\) on \(M\). A Kähler metric \(g_{\varphi,i\bar{j}} = g_{i\bar{j}} + \varphi_{i\bar{j}}\) is defined away from \(D\), where \(\varphi \in C^2(M - D) \cap W^{3,2}_{loc}(M - D)\). We want to extend \(g_{\varphi}\) to be a smooth metric on \(M\). An inspiring example of removable singularity of CscK metric is CscK metric in the punctured disk in \(\mathbb{C}\). Say \(D = \{ |z| < 1 \} \subset \mathbb{C}\) and 0 is the singular point. And we consider a metric \(g_u = e^{2u}(dx^2 + dy^2)\) defined by \(u \in C(D - \{0\}) \cap W^{1,2}_{loc}(D - \{0\})\). It has constant scalar curvature \(K\) on \(D - \{0\}\) means
\[
\Delta u = -Ke^{2u}
\]
holds in \(D - \{0\}\). And for this equation it’s easy to see that if \(\lim_{z \to 0} \frac{u}{\log r} = 0\), then \(u\) is actually a smooth function on \(D\). One easy case of it would be \(K = 0\), when \(u\) is actually a harmonic function on a punctured disk. In particular, we notice that there’s no singular solution of equation(1.1) with growth slower than \(\log r\) near the origin. In other words, there’s a gap between the smooth CscK metric and singular ones. In higher dimension, say \(n \geq 2\), situations are usually not so nice as in dimension 1. However, we can still observe the similar situation as what happened in dimension 1.

In this paper we’ll prove the following main theorem. Throughout this paper, we assume potential \(\varphi\) is bounded.

**Theorem 1.1** (Main Theorem, see Theorem 2.25). Consider \((M^n, g)\) is a compact Kähler manifold. And \(D\) is a divisor on \(M^n\). Consider \(\varphi \in C^2(M^n - D) \cap W^{3,2}_{loc}(M^n - D)\), such that \(g_{\varphi,i\bar{j}} = g_{i\bar{j}} + \varphi_{i\bar{j}} > 0\) on \(M^n - D\). And \(\varphi\) satisfies the CscK equation weakly on \(M^n - D\), i.e. For any \(\psi \in C^\infty_0(M^n - D)\), we have
\[
(1.2) \quad \int_M g_\varphi^{ij} \frac{\partial \log \det g_{\varphi}}{\partial z_i} \frac{\partial \psi}{\partial \bar{z}_j} \det g_{\varphi} dx = \int_M R \det g_{\varphi} \psi dx
\]
If we assume the following,
\(1\) \(g_{\varphi} \leq Cg\), where \(C > 0\) is a constant.
\(2\) \(\lim_{z \to D} \frac{[\log \det g_{\varphi}]^2}{\log d_g(z, D)} = 0\), where \(d_g(z, D)\) is the distance from \(z\) to the divisor \(D\) under the metric \(g\).
\(3\) \(\frac{\det g}{\det g_{\varphi}} \in L^p(M)\) for some \(p > n\)
then \(\varphi\) extends to be a smooth CscK metric on \(M\).

**Remark 1.2.** In this theorem, we assume the upper bounds for metric \(g_{\varphi}\), but allow that the lower bound of \(g_{\varphi}\) goes to 0 in certain rate when it’s approaching the divisor. Eventually, we prove that this “nicely” degenerate CscK metric \(g_{\varphi}\) is actually a smooth one. In the case of removing singularity theorem for Kähler-Einstein metrics outside of divisor in [6], condition
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(1)(3) are strengthened to the Kähler-Einstein metric is quasi isometric to background metric uniformly. So our theorem can be viewed as a generalization of this removable singularity theorem for Kähler-Einstein metrics.

Two byproducts in proving the Main Theorem are the following theorems,

**Theorem 1.3** (see Theorem 2.24). $S$ is an algebraic variety or a subset of an algebraic variety with complex codimension $d \geq 2$. $\varphi \in C^2(\mathbb{C}^n - S) \cap W^{3,2}_{\text{loc}}(\mathbb{C}^n - S)$ defines a CscK metric on $\mathbb{C}^n - S$ as $g_{\varphi,i\bar{j}} = \delta_{i\bar{j}} + \varphi_{i\bar{j}}$, i.e. For any $\psi \in C_0^\infty(\mathbb{C}^n - S)$, we have (1.3)

$$\int_M g_{\varphi,i\bar{j}} \frac{\partial \log \det g_\varphi}{\partial z_i} \frac{\partial \psi}{\partial z_j} \det g_\varphi \, dx = \int_M R \det g_\varphi \psi \, dx$$

Denote $\lambda$ and $\Lambda$ the least and the largest eigenvalues of $g_{\varphi,i\bar{j}}$. If we assume that $\Lambda \in L^\infty(\mathbb{C}^n)$ and $\lambda_A^{-1} = \frac{\Lambda}{\det g_\varphi} \in L^p_{\text{loc}}(\mathbb{C}^n)$ for $p > n$, then $\varphi$ extends smoothly on $\mathbb{C}$.

**Theorem 1.4** (see Theorem 2.17). Assume $u$ is a plurisubharmonic function on $D^n$ and $u \in W^{2,p}(D^n)$ satisfies that

$$\det(u_{i\bar{j}}) = f$$

a.e. on $D^n$. If $0 < f \in C^\alpha(D^n)$ and $\Delta u < \infty$, then $u \in C^{2,\beta}(D^n)$ for any $0 < \beta < \alpha$

**Remark 1.5.** In Y. Wang [2], he proved the above theorem for a $C^2$ solution. Here we generalize his result to a $W^{2,p}$ solution to the complex Monge-Ampère equation.

The strategy to prove the Main Theorem is as follows. First we consider the case at smooth points of the Divisor, which locally look like complex sub-manifold. This case is similar to the punctured disk. By choosing appropriate cut off functions, we can prove that the CscK equation holds in the distributional sense. Then doing a little work, we show that $\log \det g_\varphi$ is a weak solution to the degenerate elliptic equation $\Delta u = f$, where $f = -R \det g_\varphi$. Thus, next we do Moser Iteration for the weak solution. And eventually we prove that the weak solution is bounded. Therefore, $g_\varphi$ is bounded near divisor and it goes back to the case of an uniformly elliptic equation. Then by regularities of uniformly elliptic equation and Monge-Ampère equation, we can bootstrap to conclude that $g_\varphi$ is a smooth CscK metric on nonsingular point of the divisor. For nonsingular points of the divisor, since it’s a higher codimension subset and have some room to do analysis, we can use the same strategy to attack this problem. Actually we can prove that for singularities of real codimension $\geq 3$, we can extend the CscK metric smoothly by controlling the $L^p$ norm of $\frac{\det u}{\det g_\varphi}$ near the divisor.

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2. Proof of the main Theorem

Throughout this paper, we assume the potential $\varphi$ is bounded.

**Theorem 2.1.** $D^n := \{|z| < 1\} \subset \mathbb{C}^n$ and $\varphi \in C^2(D^n - \{z_1 = 0\}) \cap W^{3,2}_{\text{loc}}(D^n - \{z_1 = 0\})$ defines a kähler metric in $D^n - \{z_1 = 0\}$ as $g_{\varphi,ij} := \delta_{ij} + \varphi_{ij} > 0$. And $\varphi$ satisfies the CSCK equation weakly on $D^n - \{z_1 = 0\}$, i.e. For any $\psi \in C^\infty_0(D^n - \{z_1 = 0\})$, we have

\[
\int_{D^n} g_{\varphi,ij} \frac{\partial \log \det g_{\varphi}}{\partial z_i} \frac{\partial \psi}{\partial \bar{z}_j} \det g_{\varphi} \, dx = \int_{D^n} R \det g_{\varphi} \psi \, dx
\]

Denote $\lambda$ and $\Lambda$ the least and the largest eigenvalues of $g_{\varphi,ij}$. If we assume $\Lambda \in L^\infty(D^n)$ and $|\frac{\log \det g_{\varphi}}{\log |z_1|}| < C$, then $\varphi$ satisfies the CscK equation in the distributional sense on $D^n$.

**Remark 2.2.** Using results in [2], we can prove that $\varphi$ is a smooth function on $D^n - \{z_1 = 0\}$.

**Lemma 2.3.** $D = \{|z| < 1\} \subset \mathbb{C}$, for any $\epsilon > 0$, there exists a function $\eta_\epsilon \in C^\infty_0(D)$ such that

- $0 \leq \eta_\epsilon \leq 1$ and $\text{supp} \eta_\epsilon \subset D_{\epsilon^{1/2}}$ and $\eta_\epsilon \equiv 1$ in $D_{\frac{\epsilon}{2}}$
- $\int_D |\nabla \eta_\epsilon|^2 \, dx \leq -\frac{C}{\epsilon \ln \epsilon}$

**Proof.** For $\epsilon > 0$ and $\lambda > 0$

\[
\tilde{\eta}_{\epsilon,\lambda}(z) = \begin{cases} 
\frac{\ln \lambda}{-\ln |z| + \ln \lambda \epsilon} & \epsilon \leq |z| \leq \lambda \epsilon \\
0 & \text{otherwise}
\end{cases}
\]

Immediately we know that $\tilde{\eta}_{\epsilon,\lambda} \in W^{1,2}_0(C)$ and

\[
\int_D |\nabla \tilde{\eta}_{\epsilon,\lambda}|^2 \, dx = 2\pi \int_{\epsilon}^{\lambda \epsilon} \frac{1}{r^2} \, dr = 2\pi \ln \lambda
\]

Let $\lambda = e^{-\frac{1}{2}}$. Consider $\bar{\eta}_\epsilon = \frac{1}{2\pi \ln \lambda} \tilde{\eta}_{\epsilon,\lambda}$, we get

- $0 \leq \bar{\eta}_\epsilon \leq 1$ and $\text{supp} \bar{\eta}_\epsilon \subset D_{\epsilon^{1/2}}$ and $\bar{\eta}_\epsilon \equiv 1$ in $D_{\epsilon}$
- $\int_D |\nabla \bar{\eta}_\epsilon|^2 \, dx = -\frac{1}{\pi \ln \epsilon} = -\frac{1}{\pi \ln \epsilon}$

By approximating $\bar{\eta}_\epsilon$ by $C^\infty_0(D)$ functions $\bar{\eta}_{\epsilon,\delta} = \bar{\eta}_\epsilon * \theta_\delta$ in $W^{1,2}(D)$, we can get $\eta_\epsilon$ satisfying the conditions in the lemma. □
Lemma 2.4. Assumptions are the same as in Theorem 2.1, then

\begin{equation}
\int_{\{\|z\|<\frac{1}{2}\}} |\nabla \log \det g_\varphi|^2 \varphi \det g_\varphi dx < +\infty
\end{equation}

Proof. \(\tilde{\eta}_\epsilon\) as in Lemma 2.3 and view it as a function on \(D^n\) as \(\tilde{\eta}_\epsilon(z_1, \ldots, z_n) = \tilde{\eta}_\epsilon(z_1)\).

\[\psi \in C^\infty_0(D^n) \text{ with } \psi = 1 \text{ on } \{\|z\| < \frac{1}{2}\}, \text{ define } \sigma = (1 - \tilde{\eta}_\epsilon)\psi \]

\[|\nabla \varphi(\sigma \log det g_\varphi)|^2_\varphi = \langle \nabla \varphi(\sigma^2 \log det g_\varphi), \nabla \varphi \log det g_\varphi \rangle_\varphi + (\log det g_\varphi)^2 |\nabla \varphi \sigma|^2_\varphi \]

Therefore,

\[\int_{D^n} |\nabla \varphi \sigma \log det(g_\varphi)|^2_\varphi \det(g_\varphi) dx \]

\[= \int_{D^n} \langle \nabla \varphi \sigma^2 \log det(g_\varphi), \nabla \varphi \log det(g_\varphi) \rangle_\varphi \det(g_\varphi) dx \]

\[+ \int_{D^n} |\log det(g_\varphi)|^2_\varphi |\nabla \varphi \sigma|^2_\varphi \det(g_\varphi) dx \]

Using that \(\Lambda \in L^\infty(D^n)\) and \(|\frac{\log det g_\varphi}{\log r}| < C\), we get

\[I = -\int_{D^n} \sigma^2 \log det(g_\varphi) \Delta \varphi \log det(g_\varphi) \det(g_\varphi) dx \]

\[= \int_{D^n} \sigma^2 \log det g_\varphi R \det g_\varphi dx \]

\[\leq C_1(||\Lambda||_{L^\infty(D^n)}, |R|) \]
\[ II = \int_{D^n} [\log \det(g_\varphi)]^2 |\nabla \varphi|_\varphi^2 \det(g_\varphi) dx \]
\[ = \int_{D^n} [\log \det g_\varphi]^2 g_\varphi \frac{\partial \sigma}{\partial z_i} \frac{\partial \sigma}{\partial \bar{z}_j} \det(g_\varphi) dx \]
\[ \leq \int_{D^n} \frac{1}{\Lambda} [\log \det g_\varphi]^2 |\nabla \sigma|_\sigma^2 \det(g_\varphi) dx \]
\[ \leq \int_{D^n} [\log \det g_\varphi]^2 |\nabla \sigma|_\sigma^2 \Lambda^{n-1} dx \]
\[ \leq ||A||_{L_\infty^-(D^n)} \int_{z_2, \ldots, z_n} \int_0^{2\pi} \int_{\epsilon}^{\frac{1}{\pi L \epsilon}} \log det g_\varphi \frac{1}{r^2} \log |z_1| \quad d\theta \quad \frac{1}{r^2} \quad d\sigma \quad \frac{1}{r^2} \quad d\bar{z}_2 \quad \ldots \quad idz_n \quad d\bar{z}_n \]
\[ + \int_{|z_1| > \frac{1}{2}} \log det g_\varphi |\nabla \sigma|_\sigma^2 dx \]
\[ \leq ||A||_{L_\infty^-(D^n)} \int_{z_2, \ldots, z_n} \frac{1}{\pi (L \epsilon)^2} \int_{\epsilon}^{\frac{1}{\pi \epsilon}} \log \epsilon \quad d\theta \quad \frac{1}{(L \epsilon)^2} \quad d\sigma \quad \frac{1}{(L \epsilon)^2} \quad d\bar{z}_2 \quad \ldots \quad idz_n \quad d\bar{z}_n \quad C_2 \]
\[ \leq C_3(||A||_{L_\infty^-(D^n)}, n, C) \]

Where C is the const s.t. \( \frac{[\log \det g_\varphi]^2}{\log |z_1|} < C \)

So,
\[ \int_{|z_1| > \epsilon} \log det(g_\varphi) |\nabla \varphi|_\varphi^2 \det(g_\varphi) dx \leq \int_{D^n} |\nabla \varphi \log det(g_\varphi)|_\varphi^2 \det(g_\varphi) dx \]
\[ \leq C_1 + C_3 < \infty \]

Letting \( \epsilon \to 0 \), we proved the lemma.

\[ \square \]

Given two lemmas above, now we can prove Theorem 2.1

\textit{Proof.} Proof of Theorem 2.1

For any \( \Psi \in C_0^\infty(D^n) \),
\[ | \int_{D^n} g_\varphi \log \det(g_\varphi) \frac{\partial}{\partial z_i} \frac{\partial}{\partial z_j} dx - \int_{D^n} R \det(g_\varphi) \psi dx | \leq | \int_{D^n} g_\varphi \log \det(g_\varphi) \frac{\partial}{\partial z_i} \frac{\partial}{\partial z_j} dx | + | \int_{D^n} R \det(g_\varphi) \psi \eta dx | \]

\[ \leq (\int_{\{|z|<\epsilon^{1/2}\} \cap \text{supp } \psi} |\nabla \varphi \log \det(g_\varphi)|^2 dx)^{1/2} (\int_{\{|z|<\epsilon^{1/2}\} \cap \text{supp } \psi} |\nabla \varphi \eta|^2 dx)^{1/2} + C_5 \epsilon^2 \]

\[ \leq o(\epsilon)(\int_{\{|z|<\epsilon^{1/2}\} \cap \text{supp } \psi} |\nabla \varphi \eta|^2 dx)^{1/2} + C_5 \epsilon^2 \tag{2.3} \]

It’s easy to see that \( \int_{\{|z|<\epsilon\} \cap \text{supp } \psi} |\nabla \varphi \eta|^2 dx \) is uniformly bounded as \( \epsilon \to 0 \)

Let \( \epsilon \to 0 \) in (2.3) we proved the theorem. \( \square \)

Theorem 2.3 proves that \( \varphi \) satisfies CscK equation on \( D^n \) in the distributional sense. Moreover, \( \log \det g_\varphi \) can be viewed as a weak solution to \( \Delta \varphi u = -R \). Though it’s a degenerate elliptic equation, we can prove \( |\log \det g_\varphi| \) is bounded near divisor. Thus, we get \( g_\varphi \) is bounded near the divisor.

**Definition 2.5.** Let \( A = (a^{ij}(z)) > 0 \). For \( u, v \in C_0^\infty(D^n) \), define Hermitian inner product as

\[ \langle u, v \rangle = \int_{D^n} a^{ij} \frac{\partial u}{\partial z_i} \frac{\partial}{\partial z_j} dx \tag{2.4} \]

The completion of the \( C_0^\infty(D^n) \) under the Hermitian norm is called \( H_0(A, D^n) \). Denote \( \lambda_A \) and \( \Lambda_A \) the least and the largest eigenvalues of \( a^{ij} \).

**Remark 2.6.** If \( \lambda_A^{-1} \in L^1(D^n) \), then \( H_0(A, D^n) \) is a closed subspace of \( W_0^{1,1}(D^n) \).

**Definition 2.7.** a weak solution to equation \( -\frac{\partial}{\partial z_j}(a^{ij} \frac{\partial u}{\partial z_i}) = f \) on \( D^n \) is a function \( u \in W_0^{1,1}(D^n) \) such that \( \eta u \in H_0(A, D^n) \) for any \( \eta \in C_0^\infty(D^n) \) and for any \( \psi \in C_0^\infty(D^n) \), we have

\[ \int_{D^n} a^{ij} \frac{\partial u}{\partial z_i} \frac{\partial}{\partial z_j} dx = \int_{D^n} f \psi dx \tag{2.5} \]

**Remark 2.8.** Different from strongly uniformly elliptic equation, here \( a^{ij} \) is elliptic almost everywhere. In particular, it can degenerate on a divisor.

Denote \( A = (a^{ij}) = (g_\varphi \log \det g_\varphi) \). And in the rest of the paper, we always assume this.

**Lemma 2.9.** Assumptions are as in Theorem 2.4. Moreover, we assume \( \lim_{|z_1| \to 0} \frac{\log \det g_\varphi}{\log |z_1|} = 0 \) uniformly in \( z_2, \ldots, z_n \). If \( \gamma \in C_0^\infty(D^n) \), then \( u = \gamma \log \det g_\varphi \in H_0(A, D^n) \).
\textbf{Proof.} \( \tilde{\eta}_{\epsilon, \delta} \) as in Lemma \[2.3\]

Consider \( u_\epsilon = (1 - \tilde{\eta}_{\epsilon, \delta(\epsilon)})u \in C^\infty_0(D^n) \). \( \delta(\epsilon) \) will be determined later.

\[
\langle u - u_\epsilon, u - u_\epsilon \rangle = \int_{D^n} |\nabla \varphi \tilde{\eta}_{\epsilon, \delta} u|^2 \det g_\varphi dx
\]
\[
\leq \int_{|z_1| < \epsilon^2} |\nabla \varphi u|^2 \det g_\varphi dx + \int_{D^n} u^2 |\nabla \varphi \tilde{\eta}_{\epsilon, \delta}|^2 \det g_\varphi dx
\]
\[
\leq o(\epsilon) + ||\Lambda||^{n-1}_{L^\infty} \int_{D^n} u^2 |\nabla \tilde{\eta}_{\epsilon, \delta}|^2 dx
\]

For fixed \( \epsilon > 0 \), \( u^2 \) has upper bound away from divisor since \( u^2 \) is smooth away from divisor. We can choose \( \delta(\epsilon) \) small enough to make \( \int_{D^n} u^2 |\nabla \tilde{\eta}_{\epsilon, \delta}|^2 dx \leq 2 \int_{D^n} u^2 |\nabla \tilde{\eta}_\epsilon|^2 dx \), where \( \tilde{\eta}_\epsilon \) is in Lemma \[2.3\].

Therefore, we get

\[
\langle u - u_\epsilon, u - u_\epsilon \rangle \leq 2||\Lambda||^{n-1}_{L^\infty} \int_{z_2, \ldots, z_n} \frac{1}{\pi(Ln\epsilon)^2} \int_{\epsilon^2}^{2\epsilon} u^2 d\log ri dz_2 \wedge dz_2 \wedge \ldots \wedge idz_n \wedge dz_n + o(\epsilon)
\]
\[
\leq 2||\Lambda||^{n-1}_{L^\infty} \int_{z_2, \ldots, z_n} \frac{1}{\pi(Ln\epsilon)^2} \int_{\epsilon^2}^{2\epsilon} o(\epsilon) \log rd dz_2 \wedge dz_2 \wedge \ldots \wedge idz_n \wedge dz_n + o(\epsilon)
\]
\[
= o(\epsilon)
\]

So \( u \in H_0(A, D^n) \). This ends the proof. \( \square \)

\textbf{Theorem 2.10.} \( D^n := \{|z_1| < 1\} \subset C^\infty \) and \( \varphi \in C^2(D^n - \{z_1 = 0\}) \cap W^{2,2}_{loc}(D^n - \{z_1 = 0\}) \) defines a kähler metric in \( D^n - \{z_1 = 0\} \) as \( g_{\varphi, ij} := \delta_{ij} + \varphi_{ij} > 0 \). And \( \varphi \) satisfies the \( \text{CscK} \) equation weakly on \( D^n - \{z_1 = 0\} \), i.e. For any \( \psi \in C^\infty_0(D^n - \{z_1 = 0\}) \), we have

\[
\int_{D^n} g_{\varphi}^{ij} \frac{\partial \log \det g_{\varphi}}{\partial z_i} \frac{\partial \psi}{\partial z_j} \det g_{\varphi} dx = \int_{D^n} \text{R} \det g_{\varphi} \psi dx
\]

Denote \( \lambda \) and \( \Lambda \) the least and the largest eigenvalues of \( g_{\varphi, ij} \). If we assume \( \Lambda \in L^\infty(D^n) \) and

\[
\lim_{|z_1| \to 0} \frac{[\log \det g_{\varphi}]^2}{\log |z_1|} = 0 \text{ uniformly in } z_2, \ldots, z_n, \text{ then } \log \det g_{\varphi} \text{ is in } C^\alpha \text{ for some } 0 < \alpha < 1 \text{ across the divisor } \{z_1 = 0\}.
\]

\textbf{Proof.} According to Theorem \[2.1\], Lemma \[2.4\], and Lemma \[2.9\], \( \log \det g_{\varphi} \) is a weak solution to the degenerate elliptic equation \( -\frac{\partial}{\partial z_j}(a^{ij} \frac{\partial g_{\varphi}}{\partial z_i}) = R \det g_{\varphi} \) on \( D^n \). Using Moser Iteration, we can prove that the weak solution is bounded.

Since it’s a linear equation, without loss of generality, we can assume \( v = -\log \det g_{\varphi} > 1 \). Otherwise consider \( v + n \log[||\Lambda||_{L^\infty(D^n)}] + 1 \geq 1 \).
Lemma 2.11 ([1], Lemma 1.3). Let \( u \in W^{1,1}_{\text{loc}}(\Omega) \) and \( g \) a uniformly Lipschitz function on \( \mathbb{R} \). Then the composite function \( g(u) \in W^{1,1}_{\text{loc}}(\Omega) \) and
\[
Dg(u) = g'(u)Du.
\]
Furthermore if \( g(0) = 0 \) and \( u \) belongs to \( H_0(A, \Omega) \), then \( g(u) \) again belongs to \( H_0(A, \Omega) \).

Fix \( \eta \in C_0^\infty(D^n) \). For fixed \( N > 0, \beta \geq 1 \),
\[
F(t) = \begin{cases} 
|t|^{\beta} & |t| < N \\
\beta N^{\beta - 1}(|t| - N) + N^{\beta} & |t| \geq N
\end{cases}
\]
Consider \( \psi = \eta^2 F(v) \). Since \( \psi = \eta^2 F(v) = \eta^2 F(\gamma v) \), where \( \text{supp} \eta \subset \text{supp} \gamma \), according to Lemma 2.11, \( \psi \in H_0(A, D^n) \). Therefore, \( \psi \) is an eligible test function.

\[
\int_{D^n} \alpha^i \overline{\alpha^j} v_i (\eta^2 F(v)) \, dx = \int_{D^n} f \eta^2 F(v) \, dx
\]
\[
\int_{D^n} \alpha^i \eta^2 F'(v) v_i \overline{v_j} \, dx \leq C \left[ \int_{D^n} f \eta^2 F(v) \, dx + \int_{D^n} \alpha^i \eta \overline{\alpha^j} F^2(v) \right]
\]

Where \( f = -R \det g_\varphi \). Actually \( F \) is a convex function and \( F'(t) \geq 0 \) when \( t \geq 0 \). Denote
\[
\delta(w) = \int_0^w [F'(t)]^{\frac{1}{t}} dt
\]
It’s not hard to see that \( F^2 \leq F' \delta^2 \). So
\[
\int_{D^n} \lambda_A |\nabla (\eta \delta(v))|^2 \, dx \leq C \left[ \int_{D^n} f \eta^2 F(v) \, dx + \int_{D^n} \lambda_A |\nabla \eta|^2 \delta^2(v) \right]
\]

Lemma 2.12. If \( \lambda_A^{-1} \in L^t(D^n) \), then for any \( u \in H_0(A, D^n) \)
\[
\|u\|_{L^{t^*}} \leq C \|\lambda_A^{-1}\|_{L^t} \|u\|_{H_0(A, D^n)}
\]
where \( \frac{1}{t^*} = \frac{1}{t} \left( 1 + \frac{1}{t} \right) - \frac{1}{2n} \)

Proof. Assume \( u \in C_0^\infty(D^n) \)
\[
\|Du\|_{L^q}^2 \leq \|\lambda_A^{-1}\|_{L^t} \int \lambda_A |Du|^2 \, dx \leq \|\lambda_A^{-1}\|_{L^t} \|u\|_{H_0(A, D^n)}^2
\]
where \( \frac{1}{q} = (1 + \frac{1}{t}) \). Therefore,
\[
\|u\|_{L^{t^*}}^2 \leq \|Du\|_{L^q}^2 \leq \|\lambda_A^{-1}\|_{L^t} \|u\|_{H_0(A, D^n)}^2
\]
where \( \frac{1}{t^*} = \frac{1}{q} - \frac{1}{2n} \)
Claim 2.13. $\lambda_A^{-1} \in L^t_{loc}(D^n)$ for any $t > 1$

Proof. $\lambda_A^{-1} = \Lambda[\det g_\varphi]^{-1} \leq C[\det g_\varphi]^{-1}$. Since $\lim_{|z_1| \to 0} \frac{\log \det g_\varphi}{|z_1|} = 0$ uniformly in $z_2, \ldots, z_n$, we have

$$[\det g_\varphi]^{-1} \leq e^{\log \det g_\varphi} \leq e^0(|z_1|) \leq \left[ \frac{1}{|z_1|} \right]^{d(z)}$$

Where $d(z) = [-\log |z_1|]^{-\frac{1}{2}} o(|z_1|) \to 0$ uniformly as $|z_1| \to 0$. For $t > 1$, choose $\epsilon > 0$ sufficiently small s.t. $d(z) < \frac{1}{t}$ for any $z$ with $|z_1| < \epsilon$

$$\lambda_A^{-1} = \Lambda[\det g_\varphi]^{-1} \leq C\left[ \frac{1}{|z_1|} \right]^{d(z)}$$

Therefore for any $K^{\text{cpt}} \subset D^n$,

$$\int_K [\lambda_A^{-1}]^t dx \leq C \int_{\{|z_1| < \epsilon\} \cap K} \left[ \frac{1}{|z_1|} \right]^{td(z)} dx + \int_{\{|z_1| \geq \epsilon\} \cap K} [\lambda_A^{-1}]^t dx \leq \infty$$

So $\lambda_A^{-1} \in L^t_{loc}(D^n)$. In particular, $\lambda_A^{-1} \in L^t_{loc}(D^n)$ for $t > n$. 

Go back to the definition of $\delta$,

$$\delta(v) = \begin{cases} 
\frac{2 \delta^{\frac{1}{\gamma}}}{1 + \beta} v^{\frac{\beta + 1}{\gamma}} & v < N \\
\beta^{\frac{1}{\gamma}} N^{\frac{\beta - 1}{\gamma}} (v - N) + \frac{2 \delta^{\frac{1}{\gamma}}}{1 + \beta} N^{\frac{\beta + 1}{\gamma}} & v \geq N
\end{cases}$$

Let $\gamma = 1 + \beta \geq 2$

$$\left( \int_{\{|\eta| = 1\} \cap \{|v| < N\}} \nu^{\frac{\gamma}{\gamma - 2}} dx \right)^{\frac{\gamma - 2}{\gamma}} \leq C\gamma [\lambda_A^{-1}]\int_{\text{supp}(\eta)} (\eta + |D\eta|)^2 v^{\gamma} dx$$

Let $N \to \infty$

$$\|v\|_{L^{\frac{\gamma}{\gamma - 2}}(\{|\eta| = 1\})} \leq [C\gamma]^{\frac{1}{\gamma}} \|(\eta + |D\eta|)^{\frac{2}{\gamma}} v\|_{L^\gamma(\text{supp}(\eta))}$$
Where $t^* = 1 + \frac{1}{t} - \frac{1}{n} < 1$. Denote $\chi = \frac{t^*}{2} > 1$. So

$$(2.10) \quad \|v\|_{L^\chi(\{\eta=1\})} \leq [C\gamma]^\frac{1}{\chi} \|(\eta + |D\eta|)\|_{L^\gamma(supp(\eta))}^\frac{1}{\gamma} \|v\|_{L^\gamma(supp(\eta))}$$

Denote

$$(2.11) \quad \Psi(p, r) = (\int_{B_r} v^p)\frac{1}{p}$$

And obviously $\lim_{p \to \infty} \Psi(p, r) = \|v\|_{L^\infty(B_r)}$

Choose $\eta \in C_0^\infty(D^n)$. $\eta = 1$ in $B_{r_1}$ and $\eta = 0$ in $D^n - B_{r_2}$. Therefore 2.10 becomes

$$\Psi(\gamma_\gamma, r_1) \leq [C\gamma]^\frac{1}{\gamma} (\frac{C}{r_2 - r_1})^\frac{1}{\gamma} \Psi(\gamma, r_2)$$

Set $\gamma_m = 2\chi^m$ and $r_m = r(1 + 2^{-m})$. so

$$\Psi(2\chi^m, r) \leq (\frac{2C}{r^2})^\frac{1}{2} \chi^{-m} e^{\frac{1}{2}(\ln \chi + \ln 2)} \sum m\chi^{-m} \Psi(2, 2r) \leq C(r)\Psi(2, 2r)$$

Let $m \to \infty$, we get

$$(2.12) \quad \|v\|_{L^\infty(B_r)} \leq C(r)\|v\|_{L^2(B_{2r})} \leq C$$

Therefore, $\lambda_A = \frac{\det g_\varphi}{\lambda} \geq C$. Then $-\Delta \varphi u = f$ is actually a uniformly elliptic equation. Follow the standard weak solution theory of uniformly elliptic equation. We get $u = \log \det g_\varphi$ is a locally a $C^\alpha$ function for some $\alpha > 0$. This ends the proof.

\[\Box\]

**Theorem 2.14 (Y. Wang, [2])**. Let $u \in C^2(B_1)$ be a plurisubharmonic function that solves the equation

$$(2.13) \quad \det(u_{i\bar{j}}) = f$$

in $B_1$. Suppose that $0 < \alpha < 1$, $f^{1/n} \in C^\alpha(B_1)$, $inf_{B_1} f^{1/n} \geq \lambda > 0$ and $sup_{B_1} \Delta u \leq \Lambda$. Then, for any $0 < \beta < \alpha$, there exists a constant $C$ depending on $\beta, n, \lambda, \Lambda, \|f^{1/n}\|_{C^\alpha(B_1)}, \|u\|_{L^\infty(B_1)}$ such that $u \in C^{2,\beta}(\overline{B_{1/2}})$ and $\|u\|_{C^{2,\beta}(\overline{B_{1/2}})} \leq C$

**Remark 2.15**. In Theorem 2.10 we just proved that $\det(g_\varphi) \in C^\alpha$. However, we can’t directly apply Theorem 2.14 to conclude that $\varphi \in C^{2,\beta}$, since $\varphi$ is just a smooth function defined away from divisor but not a $C^2$ function. To get that $\varphi$ is $C^{2,\alpha}$, we have to go back to the proof of Theorem 2.14 and to modify it to serve our purpose.
Theorem 2.16. Suppose \( u \in C^2(D^n - \{z_1 = 0\}) \cap PSH(D^n - \{z_1 = 0\}) \) satisfies the following

1. \( \|u\|_{L^\infty(D^n)} < \infty \)
2. \( \Delta u \in L^\infty(D^n) \)

Then \( u \in W^{2,p}(D^n) \cap PSH(D^n) \), for any \( 1 < p < \infty \).

Proof. Let \( \Gamma(x) = \frac{1}{4n(1-n)\omega_{2n}}|x|^{2-2n} \) for \( n > 1 \). Then define \( N: C^\infty(\Omega) \rightarrow C^\infty(\Omega) \) as for \( g \in C^\infty(\Omega) \)

\[
N_g(x) = \int_\Omega \Gamma(x-y)g(y)dy
\]

\( N_g(x) \) is called the Newton Potential of \( g \). \( N \) can be extended to be a linear operator maps \( L^p \) function to a \( W^{2,p} \) function. In our case, \( Nf \in W^{2,p}(D^n) \) for any \( 1 < p < \infty \) and \( \Delta Nf = f \) a.e. on \( D^n \).

Next, consider \( v = u - Nf \). It’s not hard to see that \( v \) is harmonic in \( W^{1,2} \) weak sense, that is, for any \( \psi \in C^\infty_0(D^n) \) we have

\[
\int_{D^n} \langle \nabla v, \nabla \psi \rangle dx = 0
\]

To see this, one just follow the steps in proving Theorem 2.1 to prove that \( \nabla v \in L^2(D^n) \). Therefore, \( v \in C^\infty(D^n) \) and \( u \in W^{2,p}(D^n) \) for any \( 1 < p < \infty \). Since \( u \) is continues in \( D^n \), \( u \in PSH(D^n) \).

Theorem 2.17. Assume \( u \) is a plurisubharmonic function on \( D^n \) and \( u \in W^{2,p}(D^n) \) satisfies that

\( \det(u_{ij}) = f \)

a.e. on \( D^n \). If \( 0 < f \in C^\alpha(D^n) \) and \( \Delta u < \infty \), then \( u \in C^{2,\beta}(D^n) \) for any \( 0 < \beta < \alpha \).

Proof. In Y.Wang’s paper [2], one can view the complex Monge-Ampere equation to be a fully nonlinear uniformly elliptic equation on \( \text{Sym}(2n) \). The construction is as follows.

Let \( \text{Sym}(2n) \) be the space of \( 2n \times 2n \) real symmetric matrices and \( \text{Herm}(n) \) be the space of \( n \times n \) complex Hermitian matrices. Fix the following canonical complex structure

\[
J = \begin{pmatrix}
0 & -I \\
I & 0
\end{pmatrix},
\]

\( I \) is the \( n \times n \) identity matrix

on \( \mathbb{R}^{2n} \). Then \( \text{Herm}(n) \) can be identified to the subspace

\[
\{ M : [M, J] = MJ - JM = 0 \} \subset \text{Sym}(2n)
\]

by the map

\[
\iota : H = A + iB \mapsto \begin{pmatrix}
A & -B \\
B & A
\end{pmatrix}
\]
We view $\text{Herm}(n)$ as a subspace of $\text{Sym}(2n)$ according to the above identification. The complex structure $J$ gives rise the canonical projection $p: \text{Sym}(2n) \to \text{Herm}(n)$

$$p: M \mapsto \frac{M + J^tMJ}{2}$$

Just by easy calculation

**Claim 2.18.** $\det \frac{1}{\sqrt{2}}[p(M)] = \det C(H)$, if $\iota(H) = p(M)$, $M \in \text{Sym}(2n)$ and $H \in \text{Herm}(n)$.

Denote $F(M) = \log \det [p(M)]$

And $F$ is a concave function on the set $\{M \in \text{Sym}(2n) : p(M) > 0\}$.

**Definition 2.19.** Given $\theta > 0$, let $E_\theta \subset \text{Sym}(2n)$ consists of matrices $N$ such that $\theta I \leq p(N) \leq \theta^{-1}I$.

Define for all $M \in \text{Sym}(2n)$,

$$\tilde{F}(M) = \inf \{ \text{tr}[p(N)M] + c : (N, c) \in E_\theta \times \mathbb{R}, \text{s.t} \text{ tr}[p(N)X] + c \geq F(X) \}$$

**Remark 2.20.** By carefully checking definition of $\tilde{F}$, we get an equivalent definition

$$\tilde{F}(M) = \min_{N \in E_\theta} \text{tr}[p(N)^{-1}p(M - N)] + F(N)$$

**Lemma 2.21** ([2], Lemma 2.4). $\tilde{F}$ is concave and uniformly elliptic in $\text{Sym}(2n)$, i.e. there exists $\bar{\theta} > 0$ only depends on $\theta$ such that $\bar{\theta}\|P\| \leq \tilde{F}(M + P) - \tilde{F}(M) \leq \bar{\theta}^{-1}\|P\|, \forall M \in \text{Sym}(2n), P \geq 0$

Moreover $\tilde{F}(M) = F(M)$ for all $M \in E_\theta$.

Next, we want to prove that $u$ is a viscosity solution to

$$\tilde{F}(D^2u) = f(x)$$

Let’s first define the viscosity solution

**Definition 2.22** ([4], Definition 2.3). A continuous function $u$ in $\Omega$ is a viscosity subsolution [resp. viscosity supersolution] of $\tilde{F}(D^2u) = f(x)$ in $\Omega$, when the following condition holds: if $x_0 \in \Omega$, $\varphi \in C^2(\Omega)$ and $u - \varphi$ has a local maximum at $x_0$ then

$$\tilde{F}(D^2u(x_0)) \geq f(x_0)$$

[resp. if $u - \varphi$ has a local minimum at $x_0$ then $\tilde{F}(D^2u(x_0)) \leq f(x_0)$]

We say that $u$ is a viscosity solution of $\tilde{F}(D^2u) = f(x)$ when it is subsolution and supersolution.

To prove that $u$ is a viscosity subsolution of $\tilde{F}(D^2u)$ on $D^n$, say $\varphi \in C^2(D^n)$ such that

- $\varphi(x_0) = u(x_0)$. 
• \( \varphi(x) \geq u(x) \) in a small neighborhood of \( x_0 \).

Since \( D^2 u \in E_\theta \) a.e. on \( D^n \), it’s easy to prove \( \frac{\partial^2 \varphi}{\partial z_i \partial z_j} \geq \theta \delta_{ij} \).

\[
(2.17) \quad F(D^2 u(x)) - F(D^2 \varphi(x)) = 2a_{ij}(x) \frac{\partial^2}{\partial z_i \partial z_j}(u - \varphi)
\]

\[
(2.18) \quad = tr(\psi D^2(u - \varphi))
\]

where \( (a_{ij}(x)) = \int_0^1 ((1 - t) \varphi \tilde{\varphi}(x) + tu_i(x)) \) \( - dt \), it’s easy to see \( \psi(a_{ij}) \) is uniformly elliptic for fixed test function \( \varphi \). According to Theorem 9.6 in [3], \( u - \varphi \) can’t achieve a maximum at \( x_0 \) unless

\[
f(x_0) - F(D^2 \varphi(x_0)) \leq 0.
\]

Since \( F \) is concave on \( P = \{ M \in Sym(2n) : p(M) > 0 \} \). So \( F(M) \leq F(M) \) for any \( M \in P \). Therefore,

\[
\tilde{F}(D^2 \varphi(x_0)) \geq F(D^2 \varphi(x_0)) \geq f(x_0)
\]

we conclude that \( u \) is a viscosity subsolution of \( (2.15) \).

To prove that \( u \) is a viscosity supersolution of \( (2.15) \) on \( D^n \), say \( \varphi \in C^2(D^n) \) such that

\[
\begin{align*}
&\bullet \quad \varphi(x_0) = u(x_0), \\
&\bullet \quad \varphi(x) \leq u(x) \text{ in a small neighborhood of } x_0.
\end{align*}
\]

Denote operator \( L \psi := tr[p(D^2u)^{-1}p(D^2\psi)] := a^{ij}(x)D_{ij}\psi \). It’s easy to verify that \( \theta \delta_{ij} \leq a^{ij} \leq \theta^{-1} \delta_{ij} \). According to Theorem 9.6 in [3], \( m[\{ L(u - \varphi) < 0 \} \cap B_\epsilon(x_0) ] > 0 \) for any \( \epsilon > 0 \). So we can choose a sequence of points \( \{ x_i \} \) going to \( x_0 \) such that \( L(u - \varphi)(x_i) \leq 0 \) and \( F(D^2 u(x_i)) = f(x_i) \)

For each \( x_i \), we have

\[
(2.19) \quad \tilde{F}(D^2 \varphi(x_i)) \leq L(u - \varphi)(x_i) + F(D^2 u(x_i)) \leq F(D^2 u(x_i)) = f(x_i)
\]

Let \( i \to \infty \) in \( (2.19) \) then we get

\[
(2.20) \quad \tilde{F}(D^2 \varphi(x_0)) \leq f(x_0)
\]

Thus, \( u \) is a viscosity supersolution of \( (2.15) \) on \( D^n \).

Since \( u \) is a viscosity solution to \( (2.15) \), by the standard nonlinear elliptic theory (Theorem 6.6 and Theorem 8.1 in [7]), \( u \) is \( C^{2, \beta} \) in a small neighborhood of \( x_0 \), for \( 0 < \beta < \alpha \). \( \Box \)

**Corollary 2.23.** \( D^n := \{ |z_i| < 1 \} \subset \mathbb{C}^n \) and \( \varphi \in C^2(D^n - \{ z_1 = 0 \}) \) defines a kahler metric in \( D^n - \{ z_1 = 0 \} \) as \( g_{\varphi, ij} := \delta_{ij} + \varphi \tilde{\varphi} > 0 \). And \( \varphi \) satisfies the CscK equation weakly on \( D^n - \{ z_1 = 0 \} \), i.e. For any \( \psi \in C_0^\infty(D^n - \{ z_1 = 0 \}) \), we have

\[
(2.21) \quad \int_{D^n} g_{\varphi}^{ij} \frac{\partial \log \det g_{\varphi}}{\partial z_i} \frac{\partial \psi}{\partial z_j} \det g_{\varphi} dx = \int_{D^n} R \det g_{\varphi} \psi dx
\]
Denote $\lambda$ and $\Lambda$ the least and the largest eigenvalues of $g_{\varphi,ij}$. If we assume $\Lambda \in L^\infty(D^n)$ and $\lim_{|z_1| \to 0} \frac{|\log \det g_{\varphi}|^2}{\log |z_1|} = 0$ uniformly in $z_2,\ldots,z_n$, then $\varphi$ extends smoothly to be a CscK metric across the divisor.

Proof. According to Theorem 2.10 det $g_{\varphi} \in C^\alpha(D^n)$ for some $0 \leq \alpha < 1$. Using Theorem 2.10 and 2.17 we can get $\varphi \in C^{2,\beta}(D^n)$. Using Schauder Estimate for equation, $\Delta \varphi \log \det g_{\varphi} = -\bar{R} \det g_{\varphi}$, we get $\log \det g_{\varphi} \in C^{2,\beta}(D^n)$. Using results in [3], Lemma 17.16, if $u$ is strictly psh and $C^2$, then $\det(u_{ij}) \in C^{k,\beta}$ implies $u \in C^{k+2,\beta}$, where $k \geq 1$ and $0 < \beta < 1$. Therefore, we get that $\varphi \in C^{4,\beta}(D^n)$. By repeating arguments above, we get actually $\varphi$ is a smooth function.

Next, we want to extend the CscK metric across the divisor $D$ on a compact Kähler manifold $M$. Corollary 2.23 tells us that we can extend CscK metric across the nonsingular set of a divisor under some restrictions. And the singular set of a divisor is a higher codimension closed set of $M$.

**Theorem 2.24.** $S$ is an algebraic variety or a subset of an algebraic variety with complex codimension $d \geq 2$. $\varphi \in C^2(C^n - S) \cap W^{1,2}_{loc}(C^n - S)$ defines a CscK metric on $C^n - S$ as $g_{\varphi,ij} = \delta_{ij} + \varphi_{ij}$, i.e. For any $\psi \in C^\infty_0(C^n - S)$, we have

\[
(2.22) \quad \int_M g_{\varphi}^{ij} \frac{\partial \log \det g_{\varphi}}{\partial z_i} \frac{\partial \psi}{\partial z_j} \det g_{\varphi} dx = \int_M \bar{R} \det g_{\varphi} \psi dx
\]

Denote $\lambda$ and $\Lambda$ the least and the largest eigenvalues of $g_{\varphi,ij}$. If we assume that $\Lambda \in L^\infty_{loc}(C^n)$ and $\lambda^{-1}_A = \frac{\Lambda}{\det g_{\varphi}} \in L^p_{loc}(C^n)$ for $p > n$, then $\varphi$ extends smoothly on $C$.

Proof. The proof is essentially the same as what we did for nonsingular set of the divisor $D$. First we choose appropriate cut off functions $\eta_\varepsilon$ for $S$. Denote $S_\varepsilon$ as the $\varepsilon$-neighborhood of $S$, and consider the characteristic function of $S_\varepsilon$, denote by $\chi_\varepsilon$.

\[
\eta_\varepsilon = \chi_\varepsilon \ast \varphi_\delta = \int_{C^n} \chi_\varepsilon \frac{1}{\delta^{2n+1}} \varphi(\frac{x-y}{\delta}) dy
\]

where $\varphi \in C^\infty_0(C^n)$ with $\int \varphi dx = 1$, supp $\varphi \subset B_1(0)$ and $|D\varphi| \leq C$. $\delta$ will be determined later. Consider

\[
|D\eta_\varepsilon| \leq \int_{C^n} \chi_\varepsilon(y) \frac{1}{\delta^{2n+1}} |D\varphi(\frac{x-y}{\delta})| dy \leq C \int_{S_\varepsilon \cap B_\delta} \frac{1}{\delta^{2n+1}} dy \leq \frac{C}{\delta}
\]

Choose $\delta = \frac{\varepsilon}{2}$, so that $|D\eta_\varepsilon| \leq \frac{C}{\varepsilon}$ and $\eta_\varepsilon = 1$ on $S_{\frac{\varepsilon}{2}}$.

Applying the same argument as in the proof of Lemma 2.24 we can get for $\sigma = \psi(1 - \eta_\varepsilon)$, where $\psi \in C^\infty_0(D^n)$. 

\[ \int_{\psi=1}^{\{\psi=1\}} |\nabla_{\psi} \log \det (g_{\varphi})|_{\varphi}^2 \det (g_{\varphi}) dx \leq \int_{D^n} |\nabla_{\varphi} \log \det (g_{\varphi})|_{\varphi}^2 \det (g_{\varphi}) dx \]
\[ \leq C + \int_{S_{2\kappa}} [\log \det g_{\varphi}]^2 |\nabla g|_{\varphi}^2 \Lambda_{n-1}^{n-1} dx \]
\[ \leq C + C^2 \int_{S_{2\kappa}} [\log \det g_{\varphi}]^2 dx \]
\[ \leq C + C \left\| \log \det g_{\varphi} \right\|_{L^q(S_{2\kappa})}^2 \]

where \( \frac{2}{q} + \frac{4}{d} = 1 \). Condition \( \lambda_{A}^{-1} = \frac{A}{\det g_{\varphi}} \in L^p(D^n) \) will imply that \( \| \log \det g_{\varphi} \|_{L^q(D^n)} \) for any \( q > 1 \). Therefore, letting \( \epsilon \to 0 \), we get \( \int_{D^n} |\nabla_{\varphi} \log \det (g_{\varphi})|_{\varphi}^2 \det (g_{\varphi}) dx < \infty \). This is one crucial step in proving CscK equation holds in the distributional sense. By the same argument, we can get in this case, with singularity in codimension \( d \), CscK equation holds in distributional sense. Moreover, \( \log \det g_{\varphi} \) is a weak solution to the degenerate elliptic equation
\[-\frac{\partial}{\partial z_j} (a_{ij} \frac{\partial u}{\partial z_i}) = R \det g_{\varphi} \]. If \( \lambda_{A}^{-1} = \frac{A}{\det g_{\varphi}} \in L^p(D^n) \) for \( p > n \). Then by Moser iteration, we can get that \( \log \det g_{\varphi} \) is bounded. The rest of the proof just follows what we just did for the nonsingular set.

Combining results in Theorem 2.24 and Corollary 2.23, we get the following statement.

**Theorem 2.25** (Main Theorem). Consider \((M^n, g)\) is a compact Kähler manifold. And \( D \) is a divisor on \( M^n \). Consider \( \varphi \in C^2(M^n - D) \cap W^{3,2}(M^n - D) \), such that \( g_{\varphi,ij} = g_{ij} + \varphi_{ij} > 0 \) on \( M^n - D \). And \( \varphi \) satisfies the CscK equation weakly on \( M - D \), i.e. For any \( \psi \in C^0_0(M - D) \), we have
\[ \int_M g_{\varphi}^{ij} \frac{\partial \log \det g_{\varphi}}{\partial z_i} \frac{\partial \psi}{\partial z_j} \det g_{\varphi} dx = \int_M R \det g_{\varphi} \psi dx \]
If we assume the following,

1. \( g_{\varphi} \leq C g \), where \( C > 0 \) is a constant.

2. \( \lim_{z \to D} \frac{\log \frac{\det g_{\varphi}}{\det g}}{\log d_g(z, D)} = 0 \), where \( d_g(z, D) \) is the distance from \( z \) to the divisor \( D \) under the metric \( g \).

3. \( \frac{\det g}{\det g_{\varphi}} \in L^p(M) \) for some \( p > n \)

then \( \varphi \) extends to be a smooth CscK metric on \( M \).

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