SOME REMARKS ON CONES OF PARTIALLY AMPLE DIVISORS

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ABSTRACT. We study the cones of $q$-ample divisors $q\text{Amp}$ on smooth complex varieties. In favourable cases, we identify a part where the closure $\overline{q\text{Amp}}$ and the nef cone have the same boundary. This is especially interesting for Fano (or almost Fano) varieties.

Totaro’s landmark paper [25] has given a new impetus to the study of partially ample divisors. Let $X$ be a smooth projective complex variety of dimension $n$, and $L$ on $X$ a line bundle. We recall that $L$ is called $q$-ample if for every coherent sheaf $\mathcal{F}$ there exists an integer $m_0$ such that

$$H^i(X, \mathcal{F} \otimes L^m) \text{ for all } i > q \text{ and } m > m_0.$$ 

From Serre’s criterion it follows that 0-amplicity coincides with ampleness. Totaro proves that the $q$-amplicity of $L$ only depends on the numerical equivalence class of $L$ [25, Theorem 8.3]. The definition can moreover be extended to $\mathbb{R}$-divisors [25, 8.2], in such a way that $q$-ample $\mathbb{R}$-divisors form an open cone $q\text{Amp}(X)$ in $N^1(X)$ (the space of $\mathbb{R}$-divisors modulo numerical equivalence). We thus get a series of cones

$$\text{Amp}(X) = 0\text{Amp}(X) \subset 1\text{Amp}(X) \subset \cdots \subset n\text{Amp}(X) = N^1(X).$$

While the ample cone $\text{Amp}(X)$ and the cone $(n - 1)\text{Amp}(X)$ are fairly well understood, the intermediate cones $q\text{Amp}(X)$ for $0 < q < n - 1$ are still quite elusive and mysterious (see for instance [25, section 11] for some fundamental open questions).

The modest goal of this paper is to identify a part of these cones $q\text{Amp}$. Indeed, it turns out that in favourable cases, part of the boundary of the closed cone $\overline{q\text{Amp}}$ coincides with the boundary of the nef cone. To start with, let’s restrict attention to the case that is easiest to state, that of the cone of 1-ample divisors $1\text{Amp}$. Let $\partial \text{Nef}(X)$ denote the boundary of the nef cone, and let $K_X \in N^1X$ denote the class of the canonical divisor. We define

$$\partial \text{Nef}(X)_\text{visible} \subset \partial \text{Nef}(X)$$

to be the part of the boundary that is visible from $K_X$; cf. Definition 17 for the precise definition. (We note that when $K_X$ is nef, we have $\partial \text{Nef}(X)_\text{visible} = \emptyset$ !)

This “$K_X$–visible part” of the boundary turns out to be closely related to the boundary of $1\text{Amp}(X)$. This is detailed in the following result, where $\text{Mob}(X)$ and $\text{Big}(X)$ denote the cone of mobile divisors resp. big divisors.

Theorem. (=Theorem 19) Let $X$ be a smooth projective complex variety.

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(i) \[ \partial \text{Nef}(X)_{\text{visible}} \cap \text{int}(\text{Mob}(X)) \]
is in the boundary of \( \text{1Amp}(X) \).

(ii) Suppose \( X \) is not the blow–up of a smooth projective variety along a smooth codimension 2 subvariety. Then
\[ \partial \text{Nef}(X)_{\text{visible}} \cap \text{Big}(X) \subset \partial \text{1Amp}(X) . \]

(iii) Suppose \( X \) is not a conic bundle over a smooth projective variety, nor a blow–up of a smooth projective variety along a smooth codimension 2 subvariety. Then
\[ \partial \text{Nef}(X)_{\text{visible}} \subset \partial \text{1Amp}(X) . \]

That is, with two exceptions (a blow–up and a conic bundle) the ample cone and the 1-ample cone look exactly the same when observed from \( K_X \), and hence the only places where 1Amp can grow larger than Amp are located in the “shadowy part” invisible from \( K_X \). This theorem is proven by exploiting the existence of an MMP for any adjoint divisor, as proven by Birkar–Cascini–Hacon–McKernan [5].

It follows from Theorem 19 that the cone 1Amp(X) is strictly convex for any \( X \) such that \( \partial \text{Nef}(X)_{\text{visible}} \cap \text{int}(\text{Mob}(X)) \neq \emptyset \) (Corollary 24). The following is also an immediate corollary:

**Corollary.** (=Corollary 23) Let \( X \) be a smooth projective variety, and suppose \( K_X \) is 1-ample. Then
\[ \partial \text{Nef}(X)_{\text{visible}} \subset \partial \text{Mob}(X) . \]

That is, if \( K_X \) is 1-ample the nef cone and the closed mobile cone look the same when observed from \( K_X \).

Of course, the above theorem is empty of content when \( K_X \) is nef (for then the \( K_X \)--visible part is empty), while the assertion grows stronger when \( K_X \) grows more negative (for then the \( K_X \)--visible part grows larger, which means that the 1-ample cone looks more and more like the ample cone). The limit case is when \( X \) is a Fano variety: then the whole boundary of \( \text{Nef}(X) \) is \( K_X \)--visible. In fact, we can prove more generally:

**Corollary.** (=Corollary 25) Let \( X \) be a smooth projective complex variety such that either (1) \( -K_X \) is ample, or (2) \( -K_X \) is \( \neq 0 \) and nef and \( \dim N^1 X \geq 3 \). Then:

(i) \[ \partial \text{Nef}(X) \cap \text{int}(\text{Mob}(X)) \]
is in the boundary of \( \text{1Amp}(X) \).

(ii) Suppose \( X \) is not the blow–up of a smooth projective variety \( Y \) along a smooth codimension 2 subvariety. Then
\[ \partial \text{Nef}(X) \cap \text{Big}(X) \subset \partial \text{1Amp}(X) . \]

(iii) Suppose \( X \) is not a conic bundle over a smooth projective variety \( Y \), nor a blow–up of a smooth projective variety along a smooth codimension 2 subvariety. Then
\[ \text{Amp}(X) = \text{1Amp}(X) . \]
(For Fano varieties, I proved this in [21]).

Here is an application of the above theorem: we can identify a part of the nef cone for which the weak Lefschetz principle holds. Let \( Y \subset X \) be a generic hyperplane section. If the dimension \( n \) of \( X \) is \( \geq 4 \), pull-back induces a natural isomorphism \( N^1X \cong N^1Y \). Thus it makes sense to ask whether the nef cones \( \text{Nef}(X) \) and \( \text{Nef}(Y) \) coincide. The answer is negative in general, as shown by Hassett–Lin–Wang [15]. On the other hand, the answer is positive for certain Fano varieties ([26], [15], [18], [1], [6], [24]). Using the above Theorem, it turns out that the \( K_X \)-visible part cuts out a part where weak Lefschetz holds for the nef cone:

**Corollary.** (=Corollary 27) Let \( X \) be a smooth projective complex variety of dimension \( n \geq 4 \), and let \( Y \subset X \) be any ample hypersurface. Then

\[
\partial \text{Nef}(X)_{\text{visible}} \cap \text{int}(\text{Mob}(X)) \subset \partial \text{Nef}(Y) \cap \partial \text{Nef}(X).
\]

This is proven using a result of Demailly–Peternell–Schneider [10] (cf. also [20]), which says that a divisor restricting to an ample divisor on \( Y \) is 1-ample on \( X \).

We prove a result similar to Theorem 19 by similar means, for the \( q \)-ample cone (where \( q \) may be > 1). This result is a bit more awkward to state. As a matter of notation, we introduce the cone \( Bq\text{Amp}(X) \); this is defined as the cone of those \( \mathbb{R} \)-divisors which have augmented base locus of dimension \( \leq q \).

**Theorem.** (=Theorem 31) Let \( X \) be a smooth projective variety of dimension \( n \). For any non-negative integer \( q \), we have

\[
\partial \text{Nef}(X)_{\text{visible}} \cap B(n - 1 - q)\text{Amp}(X) \subset \partial q\text{Amp}(X).
\]

Here is how this paper is organized. The first two sections are of a preliminary nature. The first concerns several cones of divisors related to the \( q \)-ample cones; the second contains some results about contractions that will be needed. Section 3 contains the proof of Theorem 19 and its corollaries. In section 4, we prove Theorem 31.

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**Convention.** In this paper, all varieties will be (quasi-)projective algebraic varieties defined over the complex numbers.

1. Cones

This section contains notation and basic results concerning several cones of divisors related to the \( q \)-ample cones. These cones have been introduced by Küronya [20] and de Fernex–Küronya–Lazarsfeld [13].

**Definition 1.** Let \( X \) be a projective variety. A line bundle \( L \) on \( X \) is called \( q \)-ample if for every coherent sheaf \( \mathcal{F} \) there exists an integer \( m_0 \) such that

\[
H^i(X, \mathcal{F} \otimes L^m) \text{ for all } i > q \text{ and } m > m_0.
\]

A \( \mathbb{Q} \)-Cartier divisor is called \( q \)-ample if some integral multiple is \( q \)-ample. An \( \mathbb{R} \)-Cartier divisor \( D \) is called \( q \)-ample if it can be written as a sum

\[
D = cL + A,
\]
where $c \in \mathbb{R}_{>0}$, $L$ is a q-ample line bundle and $A$ is an ample $\mathbb{R}$–Cartier divisor. We will denote
\[
q\text{Amp}(X) \subset N^1(X)
\]
the cone generated by q-ample divisors.

**Remark 2.** The consistency of the definition for $\mathbb{R}$–divisors with the one for $\mathbb{Q}$–divisors is proven by Totaro \([25, \text{Theorem 8.3}]\). The cones $q\text{Amp}(X)$ are open cones \([25, \text{Theorem 8.3}]\).

**Theorem 3.** \((25, \text{Theorem 9.1})\) Let $X$ be a projective variety of dimension $n$. The cone $(n-1)\text{Amp}(X)$ is the complement in $N^1 X$ of the negative of the pseudo–effective cone of $X$.

**Definition 4.** Let $X$ be a projective variety.

(i) An $\mathbb{R}$–divisor $L$ on $X$ is called $B_q$-ample if the augmented base locus $B_+(L)$ has dimension $\leq q$. We will denote
\[
Bq\text{Amp}(X) \subset N^1(X)
\]
the cone generated by $B_q$-ample divisors.

(ii) Let $H_1, \ldots, H_q$ be very ample divisors on $X$. An $\mathbb{R}$–divisor $L$ on $X$ is called $(H_1, \ldots, H_q)$-ample if the restriction
\[
L|_{h_1 \cap \cdots \cap h_q}
\]
is ample, for $h_i \in |H_i|$ generic. An $\mathbb{R}$–divisor is said to be $H$-q-ample if it is $(H_1, \ldots, H_q)$-ample, for certain very ample $H_1, \ldots, H_q$. We will denote
\[
Hq\text{Amp}(X) = \bigcup_{(H_1, \ldots, H_q) \text{very ample}} (H_1, \ldots, H_q)\text{Amp}(X) \subset N^1(X)
\]
the cone generated by $H_q$-ample divisors.

**Remark 5.** The augmented base locus $B_+(L) \subset X$ is the locus where $L$ fails to be ample; for the definition and properties, cf. \([11]\) and \([12]\).

**Remark 6.** It is easily seen that
\[
B0\text{Amp}(X) = H0\text{Amp}(X) = \text{Amp}(X),
\]
while $B(n-1)\text{Amp}(X) = \text{Big}(X)$. The cones $Bq\text{Amp}(X)$ are open \([8, \text{Theorem 4.5}]\), and $B(n-2)\text{Amp}(X)$ coincides with the interior of the cone of mobile divisors:
\[
B(n-2)\text{Amp}(X) = \text{Mob}(X) \setminus \partial\text{Mob}(X)
\]
\([9, \text{Lemma 3.1}]\).

**Remark 7.** The cones $Bq\text{Amp}$ (or rather, their closure) have been studied by Payne \([22]\) and Choi \([8]\). It is established by Choi \([8, \text{Theorem 4.5}]\) that the closure of $Bq\text{Amp}(X)$ can be described in terms of the diminished base locus:
\[
\overline{Bq\text{Amp}(X)} = \{ L \in N^1 X | \dim B_-(L) \leq q \}.
\]

**Proposition 8.** (Küronya \([20]\)) Let $X$ be a smooth projective variety. For any $0 \leq q \leq n-1$, there are inclusions of cones
\[
Bq\text{Amp}(X) \subset Hq\text{Amp}(X) \subset q\text{Amp}(X).
\]
Proof. For the first inclusion, it is easily seen that actually
\[ B_q \text{Amp}(X) \subset \bigcap_{(H_1, \ldots, H_q) \text{very ample}} (H_1, \ldots, H_q) \text{Amp}(X); \]
indeed, suppose \( L \) is such that \( \dim B_+(L) \leq q \). For any \( H_1, \ldots, H_q \) very ample and \( h_1 \in |H_i| \) generic, \( B_+(L) \cap h_1 \cap \cdots h_q \) has dimension \( \leq 0 \). But
\[ B_+(L|_{h_1 \cap \cdots \cap h_q}) \subset B_+(L) \cap h_1 \cap \cdots h_q \]
so \( L|_{h_1 \cap \cdots \cap h_q} \) is ample. The second inclusion is a vanishing theorem proven by K"uronya [20, Theorem 1.1]; this was also proven by Demailly–Peternell–Schneider [10, Theorem 3.4].

Remark 9. Both inclusions in Proposition 8 may be strict. For the second inclusion, K"uronya provides an example [20, Example 1.13] where
\[ H(n - 1) \text{Amp}(X) \neq (n - 1) \text{Amp}(X). \]
For the first inclusion, let \( X \) be a surface. Then any line bundle \( L \) which is not big and such that \(-L\) is not pseudo-effective is in
\[ H1 \text{Amp}(X) \setminus B1 \text{Amp}(X). \]
A more subtle example is [20, Example 1.7], which exhibits a big line bundle \( L \) on a threefold \( X \), satisfying
\[ L \in H1 \text{Amp}(X) \setminus B1 \text{Amp}(X). \]

2. MMP

In this section, we collect some results about minimal model theory and contractions.

Definition 10. ([12]) A divisor \( L \) is called stable if \( B_-(L) \) and \( B_+(L) \) coincide.

Proposition 11. ([12, Proposition 1.29]) The stable divisors form an open and dense subset in \( N^1 X \).

Lemma 12. Let \( X \) be a smooth projective variety, and \( L \) on \( X \) an \( \mathbb{R} \)-divisor which is big and stable. Let
\[ f : X \leftarrow X_{\min} \]
be an \( L \)-MMP, i.e. \( f_* L \) is nef. Let \( \text{Exc}(f) \subset X \) denote the complement of the maximal open subset over which \( f \) is an isomorphism. Then
\[ B_+(L) \supset \text{Exc}(f). \]

Proof. Let \( E \subset \text{Exc}(f) \) be an irreducible component. Then, there is some index \( 0 < i < r \), such that \(-(f_i)_* L \) is \( \psi_i \)-ample on the strict transform \( E_i \) of \( E \) in \( X_i \). This implies
\[ E_i \subset B_+(f_i)_* L \]
(indeed, \( E_i \) is covered by curves on which \((f_i)_* L \) is negative, and such curves lie in the stable base locus of \((f_i)_* L \)). But then, applying the following proposition to a resolution of indeterminacy of \( f_i \), we see that \( E \) must lie in \( B_+(L) \).
Proposition 13. (Boucksom–Broustet–Pacienza [7, Proposition 1.5]) Let \( \pi: \tilde{X} \to X \) be a birational morphism between normal projective varieties. Let \( F \) be an effective \( \pi \)-exceptional divisor. Then for any big \( \mathbb{R} \)-divisor \( L \) on \( X \), we have
\[
B_+ (\pi^* L + F) = \pi^{-1} (B_+ (L)) \cup \text{Exc}(\pi) .
\]

Remark 14. With some more work, one can in fact prove that equality holds in Lemma 12; we don’t need this in this paper.

Theorem 15. (Wiśniewski [26]) Let \( X \) be a smooth projective variety, and let
\[
\psi: X \to Z
\]
be the contraction of a \( K_X \)-negative extremal ray. Suppose all fibres of \( \psi \) are of dimension \( \leq 1 \). Then \( Z \) is smooth, and \( \psi \) is either the blow–up of \( Z \) along a smooth codimension 2 subvariety, or a conic bundle over \( Z \).

Proof. [26, Theorem 1.2] (cf. also [4, Theorem 4.1].)

Theorem 16. (Wiśniewski [26], Ionescu [17]) Let \( X \) be a smooth projective variety of dimension \( n \), and let \( R \) be a \( K_X \)-negative extremal ray of length
\[
\ell(R) := \min \{-K_X \cdot C | C \text{ rational curve}, C \in R\} .
\]
Let \( \psi \) be the contraction of \( R \), and let \( E \) be an irreducible component of the locus of \( R \). Let \( F \) be an irreducible component of a fiber of the restriction of \( \psi \) to \( E \). Then
\[
\dim E + \dim F \geq n + \ell(R) - 1 .
\]

Proof. [26, Theorem 1.1] or [17, Theorem 0.4].

3. 1-AMPLE

This section is about the cone of 1-ample divisors. Here we prove Theorem 19 stated in the introduction.

Definition 17. Let \( X \) be a projective variety. The \( K_X \)-visible part of \( \partial \text{Nef}(X) \) is defined as
\[
\partial \text{Nef}(X)_{\text{visible}} := \{ D \in \partial \text{Nef}(X) | \overline{K_XD \cap \text{Nef}(X)} = D \} .
\]
Here \( \overline{K_XD} \) denotes the line segment joining \( K_X \) to \( D \).

Remark 18. This notion is considered also in [19, Theorem 1]. The definition is interesting only when \( K_X \notin \text{Nef}(X) \); if \( K_X \) is nef, the line segment \( \overline{K_XD} \) contains more than one point and we have
\[
\partial \text{Nef}(X)_{\text{visible}} = \emptyset .
\]
The other extreme is when \( X \) is Fano; then we have
\[
\partial \text{Nef}(X)_{\text{visible}} = \partial \text{Nef}(X) .
\]

Theorem 19. Let \( X \) be a smooth projective variety.
(i) \[ \partial \text{Nef}(X)_{\text{visible}} \cap \text{int} \left( \text{Mob}(X) \right) \subset \partial \overline{\text{Amp}(X)}. \]

(ii) Suppose \( X \) is not the blow–up of a smooth projective variety \( Y \) along a smooth codimension 2 subvariety. Then
\[ \partial \text{Nef}(X)_{\text{visible}} \cap \text{Big}(X) \subset \partial \overline{\text{Amp}(X)}. \]

(iii) Suppose \( X \) is not a conic bundle over a smooth projective variety \( Y \), nor a blow–up of a smooth projective variety along a smooth codimension 2 subvariety. Then
\[ \partial \text{Nef}(X)_{\text{visible}} \subset \partial \overline{\text{Amp}(X)}. \]

Proof.
(i) We will prove the following:

**Proposition 20.** Let \( L = K_X + A \), where \( A \) is an ample \( \mathbb{R} \)-divisor. Suppose \( L \) is stable and
\[ L \in 1 \text{Amp}(X) \cap \text{int} \left( \text{Mob}(X) \right). \]
Then \( L \) is ample.

This suffices to prove Theorem 19(i). Indeed, suppose there is an element
\[ D \in \partial \text{Nef}(X)_{\text{visible}} \cap \text{int} \left( \text{Mob}(X) \right) \]
that is in the interior of \( 1 \text{Amp}(X) \) (i.e. \( D \) is 1-ample). Then we can also find
\[ D' \in \partial \text{Nef}(X)^{\circ}_{\text{visible}} \cap \text{int} \left( \text{Mob}(X) \right) \]
that is 1-ample. Here \( \partial \text{Nef}(X)^{\circ}_{\text{visible}} \) denotes the relative interior of \( \partial \text{Nef}(X)_{\text{visible}} \). By definition of the \( K_X \)-visible part, \( D' \) is of the form \( D = m(K_X + A) \), for some ample \( \mathbb{R} \)-divisor \( A \) and \( m \in \mathbb{R} \). Now, \( \frac{1}{m} D' = K_X + A \) is also in
\[ \partial \text{Nef}(X)_{\text{visible}} \cap \text{int} \left( \text{Mob}(X) \right) \cap 1 \text{Amp}(X). \]

What’s more,
\[ D'' = K_X + (1 - \epsilon) A \in \text{int} \left( \text{Mob}(X) \right) \cap 1 \text{Amp}(X) \]
for \( 0 < \epsilon \) small enough (since these are open cones). Since stable divisors are open and dense in \( N^1 X \), there exists \( \epsilon > 0 \) such that \( D'' \) is stable. Then Proposition 20 implies that \( D'' \) is ample, and hence \( D' \) is ample: contradiction.

So let’s prove Proposition 20.

Since \( A \) is ample, there exists an effective \( \mathbb{R} \)-divisor \( \Delta \) numerically equivalent to \( A \) and such that \( (X, \Delta) \) is klt. According to [5, Theorem 1.2], there is an \( L \)-MMP
\[ \phi: X = X_0 \to X_1 \to \cdots \to X_{\min}, \]
where \( \phi_* L \) on \( X_{\min} \) is nef. Each step \( \phi_i: X_i \to X_{i+1} \) in the program is the flip of a morphism
\[ \psi_i: X_i \to Z_i, \]
where \( \psi_i \) is the (birational) contraction of an \( L \)-negative extremal ray. Since \( L \) is stable, the exceptional locus of \( \phi \) is contained in \( B_+(L) \) (Lemma [12], hence it is of dimension \( \leq n - 2 \).
(where $n = \dim X$). That is, all the $\psi_i$ in the program must be small contractions. Consider now the first of these small contractions

$$\psi = \psi_0 : X \to Z_0.$$ 

Since $K_X < L$, $\psi$ is the contraction of a $K_X$–negative extremal ray. If all fibres of $\psi$ are of dimension $\leq 1$, the contraction $\psi$ cannot be small by Theorem 15 so there must exist a fibre with an irreducible component $F$ of dimension $f \geq 2$. Since $-L$ is $\psi$–ample, we have

$$-L|_F \in \Amp(F) \subset \Big(F\).$$

Using Theorem 3 this implies

$$L|_F \not\in (f - 1)\Amp(F).$$

But this leads to a contradiction: $L$ is 1-ample, so the restriction to any subvariety must be 1-ample as well.

We find that $\psi$ is the identity, so the MMP cannot get started and $X = X_{min}$. That is, $L$ must be nef. Since $L$ is stable, $B_+(L) = B_-(L) = \emptyset$ and $L$ is ample.

(ii) In analogous fashion to the proof of (i), it will suffice to prove:

**Proposition 21.** Let $X$ be as in Theorem 19(ii), and let $L = K_X + A$, where $A$ is an ample $\mathbb R$–divisor. Suppose $L$ is stable and

$$L \in 1\Amp(X) \cap \Big(X\).$$

Then $L$ is ample.

To prove the proposition, consider again an $L$–MMP (which exists thanks to [5]). Let

$$\psi : X \to Z$$

be the first contraction of the program. Since $L$ is big, the contraction $\psi$ is birational. Just as above, we find that $\psi$ cannot be small, so $\psi$ must be a divisorial contraction. If all fibres of $\psi$ have dimension $\leq 1$, $\psi$ is a blow–up of a smooth projective $Y$ with smooth center of codimension 2 (Theorem 15); this is excluded by hypothesis. So there must be a fibre with an irreducible component $F$ of dimension $\geq 2$, which again contradicts the fact that $L|_F$ is 1-ample.

(iii) It will suffice to prove the following statement:

**Proposition 22.** Let $X$ be as in Theorem 19(iii), and let $L = K_X + A$, where $A$ is an ample $\mathbb R$–divisor. Suppose $L$ is stable and 1-ample. Then $L$ is nef.

We first remark that in case $L$ is big, Proposition 22 follows from Proposition 21. In case $L$ is pseudo–effective, $L$ is a limit of big divisors which are stable and 1-ample, and it follows from Proposition 21 that $L$ is nef. Suppose $L$ is not pseudo–effective. According to [5, Corollary 1.3.2], there exists an $L$–MMP such that on $X_{min}$ there is a Mori fibre space structure, i.e. a morphism

$$g : X_{min} \to Y$$

such that $-\phi_\ast L$ is $g$–ample. Just as in case (ii), we find there can be no birational contraction in the program, so we have $X = X_{min}$. If the Mori fibre space has only fibres of dimension 1, it is
Corollary 23. Let $X$ be a smooth projective variety, and suppose $K_X$ is $1$-ample. Then
\[ \partial \text{Nef}(X)_{\text{visible}} \subset \partial \text{Mob}(X). \]

Proof. It suffices to prove that the relative interior $\partial \text{Nef}(X)_{\text{visible}}^\circ$ is in the boundary of the mobile cone. But if $K_X$ is $1$-ample, every $L$ on $\partial \text{Nef}(X)_{\text{visible}}^\circ$ is also $1$-ample (since $L$ is a sum of ample plus $1$-ample). But then Theorem [19](i) implies that $L$ cannot live in the interior of $\text{Mob}(X)$. □

Corollary 24. Let $X$ be a smooth projective variety, and suppose
\[ \partial \text{Nef}(X)_{\text{visible}} \cap \text{int}(\text{Mob}(X)) \neq \emptyset. \]
Then $1\text{Amp}(X)$ is a strictly convex cone.

Proof. The hypothesis implies that the dimension of $X$ is at least $3$. In case the Picard number of $X$ is $1$, the statement is clear from Theorem [3]. Suppose the Picard number is $2$. The cone $1\text{Amp}(X)$ has $2$ extremal rays, and by Theorem [19](i) one of them is also an extremal ray of $\text{Nef}(X)$. On the other hand, $1\text{Amp}(X)$ lies outside of $-\text{Amp}(X)$ (Theorem [3]), so $1\text{Amp}(X)$ must be convex.

The argument for Picard number $\geq 3$ is similar: in this case, we have
\[ \dim \partial \text{Nef}(X)_{\text{visible}} \geq 2, \]
which means that $\partial \text{Nef}(X)_{\text{visible}}$ contains infinitely many rays. Since the visible part is locally rationally polyhedral (this is the cone theorem, stated in this form in [19], Theorem 1), there exists a ray
\[ R \in \partial \text{Nef}(X)_{\text{visible}} \]
which lies in the relative interior of a face $F$ of $\text{Nef}(X)$. Let $h \subset N^1_X$ denote the unique hyperplane containing $F$: the claim is now that $1\text{Amp}(X)$ lies on one side of $h$. To see this, suppose (by contradiction) there exists a divisor $D \in 1\text{Amp}(X)$ which lies on the “non–ample” side of $h$. Let $h_2 \subset N^1_X$ denote the $2$–plane spanned by $R$ and $D$. We find that any divisor $L \in R$ can be written
\[ L = mD + A, \]
for some $m \in \mathbb{R}_{>0}$ and $A$ ample (this is most easily seen by restricting attention to the $2$–plane $h_2$: by construction, $h_2$ meets $\text{Amp}(X)$, and $D$ lies outside of $-\text{Amp}(X) \cap h_2$, again by Theorem [3]). But then $L$ is $1$–ample, contradicting Theorem [19](i). □

Corollary 25. ("almost Fano") Let $X$ be a smooth projective complex variety, and suppose that either (1) $-K_X$ is ample, or (2) $-K_X$ is $\neq 0$ and nef and $\dim N^1_X \geq 3$. Then:
(i) \( \partial \text{Nef}(X) \cap \text{int}(\text{Mob}(X)) \) is in the boundary of \( \overline{1\text{Amp}(X)} \).

(ii) Suppose \( X \) is not the blow–up of a smooth projective variety \( Y \) along a smooth codimension 2 subvariety. Then
\[
\partial \text{Nef}(X) \cap \text{Big}(X) \subset \partial \overline{1\text{Amp}(X)}.
\]

(iii) Suppose \( X \) is not a conic bundle over a smooth projective variety \( Y \), nor a blow–up of a smooth projective variety along a smooth codimension 2 subvariety. Then
\[
\text{Amp}(X) = 1\text{Amp}(X).
\]

Proof.
(i) If \( -K_X \) is ample, clearly \( \partial \text{Nef}(X)_{\text{visible}} = \partial \text{Nef}(X) \) and we are done. Suppose now
\[
-K_X \in \partial \text{Nef}(X) \setminus \{0\}.
\]
Then we have
\[
\partial \text{Nef}(X)_{\text{visible}} = \partial \text{Nef}(X) \setminus k,
\]
where \( k \) denotes the ray generated by \( -K_X \). Applying Theorem \[19\] i), we find an inclusion
\[
\left( \partial \text{Nef}(X) \setminus k \right) \cap \text{int}(\text{Mob}(X)) \subset \partial \overline{1\text{Amp}(X)}.
\]
Suppose (i) is not true, i.e.
\[
k \subset \text{int}(\text{Mob}(X)) \cap 1\text{Amp}(X).
\]
Then, since \( 1\text{Amp} \) is an open cone,
\[
D := -K_X - \epsilon A \in 1\text{Amp}(X)
\]
for any ample \( A \) and \( \epsilon \) sufficiently small. On the other hand, \( D \) lies outside the closed cone \( \text{Nef}(X) \). Let’s pick an ample \( \mathbb{R} \)-divisor \( A' \) close to \( A \), but outside the plane spanned by \( A \) and \( k \) (this is possible if the ample cone has dimension \( \geq 3 \)). Then the line segment connecting \( A' \) to \( D \) crosses
\[
\left( \partial \text{Nef}(X) \setminus k \right) \cap \text{int}(\text{Mob}(X));
\]
let’s call the point of intersection \( B \). The \( \mathbb{R} \)-divisor \( B \) is a sum of ample and 1-ample, hence \( B \) is 1-ample [25, Theorem 8.3]. On the other hand, \( B \) lies in the boundary of \( \overline{1\text{Amp}(X)} \) and the 1-ample cone is open, so \( B \) cannot be 1-ample: contradiction.

(ii) and (iii) Similar. \( \square \)

Remark 26. Suppose \( X \) is Fano, i.e. \( -K_X \) is ample. The pseudo–index of \( X \) is defined as
\[
\tau(X) = \min \{-K_X \cdot C \mid C \subset X \text{ rational curve}\}.
\]
If \( \tau(X) \) is \( \geq 2 \) (respectively \( \geq 3 \)), the hypothesis of Corollary \[25\] ii) (respectively (iii)) is satisfied (this follows from Theorem \[16\]). In this way, we recover [21, Proposition 29] as a special case of Corollary \[25\].
Corollary 27. ("weak Lefschetz") Let $X$ be a smooth projective complex variety of dimension $n \geq 3$, and let $Y \subset X$ be a generic hyperplane section.

(i) \[
\partial \Nef(X)_{\text{visible}} \cap \text{int} \left( \text{Mob}(X) \right) \subset \partial \Nef(Y) \cap \partial \Nef(X) \ .
\]

(ii) Suppose $X$ is not the blow–up of a smooth projective variety $Y$ along a codimension 2 smooth subvariety. Then \[
\partial \Nef(X)_{\text{visible}} \cap \text{Big}(X) \subset \partial \Nef(Y) \cap \partial \Nef(X) \ .
\]

(iii) Suppose $X$ is not a conic bundle over a smooth projective variety, nor a blow–up of a smooth projective variety along a smooth codimension 2 subvariety. Then \[
\partial \Nef(X)_{\text{visible}} \subset \partial \Nef(Y) \cap \partial \Nef(X) \ .
\]

The following is an alternative formulation of Corollary 27(i). The reformulation of points (ii) and (iii) is left to the diligent reader.

Corollary 28. ("ampleness criterion") Let $X$ be a smooth projective variety of dimension $n \geq 3$, and let $L$ on $X$ be a divisor of the form $L = K_X + A$, with $A$ an ample $\mathbb{R}$-divisor. Suppose $L \in \text{int} \left( \text{Mob}(X) \right)$. Then $L$ is ample if and only if $L | Y$ is ample for some generic hyperplane $Y \subset X$.

Combining Corollaries 25 and 27 we get in particular:

Corollary 29. ("weak Lefschetz for almost Fano") Let $X$ be a smooth projective complex variety of dimension $n \geq 3$. Suppose either (1) $-K_X$ is ample, or (2) $-K_X$ is nef and $\neq 0$ and $\dim N^1 X \geq 3$. Let $Y \subset X$ be a very ample divisor, generic in its linear system.

(i) \[
\partial \Nef(X) \cap \text{int} \left( \text{Mob}(X) \right) \subset \partial \Nef(Y) \cap \partial \Nef(X) \ .
\]

(ii) Suppose $X$ is not the blow–up of a smooth variety along a smooth codimension 2 subvariety. Then \[
\partial \Nef(X) \cap \text{Big}(X) \subset \partial \Nef(Y) \cap \partial \Nef(X) \ .
\]

(iii) Suppose $X$ is not a conic bundle over a smooth projective variety, nor a blow–up of a smooth projective variety along a smooth codimension 2 subvariety. Then \[
\partial \Nef(X) \subset \partial \Nef(Y) \ .
\]

(iv) Let $X$ be as in (iii) and $n \geq 4$. Then restriction induces an isomorphism \[
\Nef(X) \cong \Nef(Y) \ .
\]

Remark 30. The statement of Corollary 29(iv) for $X$ Fano was originally proven by Wiśniewski [26, p. 147 Corollary]. This provided the starting–block for much further work concerning weak Lefschetz for the ample cone ([15], [18], [1], [2], [6], [24]).
4. \(q\)-ample

This section is about the cone of \(q\)-ample divisors. We prove the result stated in the introduction:

**Theorem 31.** Let \(X\) be a smooth projective variety of dimension \(n\). For any non–negative integer \(q\), we have

\[
\partial \mathrm{Nef}(X) \cap B(n - 1 - q)\mathrm{Amp}(X) \subset \partial q\mathrm{Amp}(X).
\]

We actually prove a more general statement:

**Theorem 32.** Let \(X\) be a smooth projective variety of dimension \(n\), and define

\[
\tau = \min \{ \ell(R) \mid R \text{ is a } K_X\text{-negative extremal ray} \}.
\]

(i) For any non–negative integer \(q\) such that \(q \geq \tau - 2\), we have

\[
\partial \mathrm{Nef}(X) \cap B(n + \tau - q - 2)\mathrm{Amp}(X) \subset \partial q\mathrm{Amp}(X).
\]

(ii) Suppose \(X\) is not the blow–up of a smooth variety \(Y\) along a smooth subvariety of codimension \(\geq 2\). Then

\[
\partial \mathrm{Nef}(X) \cap \mathrm{Big}(X) \subset \partial(\tau)\mathrm{Amp}(X).
\]

**Proof.**
(i) As in the proof of Theorem 19, one can restrict attention to the relative interior \(\partial \mathrm{Nef}(X)\) and hence it suffices to prove the following:

**Proposition 33.** Let \(L\) be a divisor of the form \(L = K_X + A\), with \(A\) an ample \(\mathbb{R}\)-divisor. Suppose \(L\) is stable and \(L \in B(n + \tau - q - 2)\mathrm{Amp}(X) \cap q\mathrm{Amp}(X)\).

Then \(L\) is ample.

To prove the proposition, consider an \(L\)--MMP

\[
\phi: X = X_0 \to X_1 \to \cdots \to X_{\min},
\]

where either \(\phi_0L\) is semi–ample on \(X_{\min}\) (if \(L\) is big), or there exists a Mori fibre space structure on \(X_{\min}\) (if \(L\) is not pseudo–effective). This exists thanks to [5]. Let

\[
\psi: X \to Z
\]

denote the first contraction of the \(L\)--MMP, let \(V \subset X\) denote the exceptional locus of \(\psi\) and let \(F\) be a general fibre of \(\psi|_V\). Note that \(K_X < L\) so that \(\psi\) corresponds to the contraction of a \(K_X\)–negative extremal ray and Wiśniewski’s theorem (Theorem 16) applies. This gives

\[
\dim V + \dim F \geq n + \tau - 1.
\]

Since \(V \subset B_+(L)\) (Lemma 12), its dimension is \(\leq n + \tau - q - 2\). It follows that

\[
\dim F \geq q + 1.
\]
By construction, $-L$ is $\psi$–ample, hence
\[ L|_F \in -\Amp(F). \]
On the other hand the restriction of $L$ to any subvariety is $q$–ample, so in particular
\[ L|_F \in q\Amp(F). \]
But this is not possible if $\dim F \geq q + 1$:
\[ q\Amp(F) \subset (\dim F - 1)\Amp(F), \]
and the cone $(\dim F - 1)\Amp(F)$ is the complement of $-\Psef(F)$: contradiction.

Since the $L$–MMP cannot get started, it is trivial. That is, either $L$ is nef on $X$, or there exists a contraction of fibre type
\[ g: X \to Z \]
which is $L$–negative and $K_X$–negative. The second possibility can be excluded, again using Wiśniewski’s theorem: if $F$ is a general fibre of $f$, we have
\[ n + \dim F \geq n + \tau - 1, \]
i.e. there is a fibre $F$ of dimension $\geq \tau - 1$. But supposing there is a fibre type contraction, $L$ is not big which is only possible if $q = \tau - 2$. So $L$ and the restriction $L|_F$ are $(\tau - 2)$–ample, which contradicts the fact that
\[ L|_F \in -\Amp(F) \subset -\Big(F). \]

(ii) This follows once we have proven the following:

**Proposition 34.** Let $X$ be as in Theorem 32(ii), and let $L$ be a divisor of the form $L = K_X + A$, with $A$ an ample $\mathbb{R}$-divisor. Suppose $L$ is big and $\tau$-ample. Then $L$ is ample.

To prove the proposition, we apply [5, ] to get an $L$–MMP
\[ \phi: X = X_0 \to X_1 \to \cdots \to X_{\min}, \]
where $\phi_* L$ on $X_{\min}$ is nef. Consider the first contraction
\[ \psi: X \to Z \]
in this $L$–MMP. As above, let $V \subset X$ denote the exceptional locus of $\psi$ and let $F$ be a general fibre of $\psi|_V$. Note that $K_X < L$ so that $\psi$ corresponds to the contraction of a $K_X$–negative extremal ray and hence Wiśniewski’s theorem (Theorem [16] applies to $\psi$. If $\psi$ is a small contraction (i.e. $\dim V \leq n - 2$), Wiśniewski’s theorem gives
\[ \dim F \geq \tau + 1, \]
and we get a contradiction with the fact that $L|_F$ is $\tau$-ample. So $\psi$ must be a divisorial contraction, and all fibres of $\psi|_V$ must be of dimension equal to $\tau$ (by Wiśniewski’s theorem, each fibre has dimension $\geq \tau$, while the fact that $L$ is $\tau$-ample implies that each fibre has dimension $\leq \tau$). In this case, a result of Andreatta–Occhetta [3, Theorem 5.1] informs us that $\psi$ identifies $X$ with a blow–up of some smooth projective variety $Y$ along a smooth subvariety; this is excluded by hypothesis.
Altogether, we find there can be no contraction and hence $X = X_{\text{min}}$ and $L$ is already nef. It remains to prove ampleness of $L$. To this end, note that

$$L' = K_X + (1 - \epsilon)A$$

is still big and $(\tau - 1)$-ample for $\epsilon$ sufficiently small (since $\text{Big}(X)$ and $(\tau - 1)\text{Amp}(X)$ are open cones). Applying the above reasoning to $L'$, we find that $L'$ is nef. But then

$$L = L' + \epsilon A$$

is ample. □

**Corollary 35.** ("weak Lefschetz") Let $X$ and $\tau$ be as in Theorem 32.

(i) Let $Y \subset X$ be a generic complete intersection of codimension $q \leq n - 2$. Then

$$\partial \text{Nef}(X)_{\text{visible}} \cap B(n + \tau - q - 2)\text{Amp}(X) \subset \partial \text{Nef}(Y) \cap \partial \text{Nef}(X).$$

(ii) Suppose $X$ is not the blow–up of a smooth variety along a smooth subvariety of codimension $\geq 2$. Let $Y \subset X$ be a generic complete intersection of codimension $\tau$. Then

$$\partial \text{Nef}(X)_{\text{visible}} \cap \text{Big}(X) \subset \partial \text{Nef}(Y) \cap \partial \text{Nef}(X).$$

**Proof.** This is immediate from Theorem 31, once one knows that $H_q$-ample implies $q$-ample (Proposition 8).

□

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