On Hamilton cycles in Erdős–Rényi subgraphs of large graphs

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Abstract

Given a graph $\Gamma = (V, E)$ on $n$ vertices and $m$ edges, we define the Erdős-Rényi graph process with host $\Gamma$ as follows. A permutation $e_1, \ldots, e_m$ of $E$ is chosen uniformly at random, and for $t \leq m$ we let $\Gamma_t = (V, \{e_1, \ldots, e_t\})$. Suppose the minimum degree of $\Gamma$ is $\delta(\Gamma) \geq (1/2 + \varepsilon)n$ for some constant $\varepsilon > 0$. Then with high probability$^1$, $\Gamma_t$ becomes Hamiltonian at the same moment that its minimum degree becomes at least two.

Given $0 \leq p \leq 1$ we let $\Gamma_p$ be the Erdős-Rényi subgraph of $\Gamma$, obtained by retaining each edge independently with probability $p$. When $\delta(\Gamma) \geq (1/2 + \varepsilon)n$, we provide a threshold function $p_0$ for Hamiltonicity, such that if $(p - p_0)n \to -\infty$ then $\Gamma_p$ is not Hamiltonian whp, and if $(p - p_0)n \to \infty$ then $\Gamma_p$ is Hamiltonian whp.

1 Introduction

Given a Hamiltonian graph $\Gamma$ on $n$ vertices and $m$ edges, pick a random ordering $e_1, \ldots, e_m$ of its edges, and let $\Gamma_t$ (or $\Gamma_{n,t}$) be the subgraph consisting of $e_1, \ldots, e_t$. Let $\delta(G)$ denote the minimum degree of a graph, and define

$$\tau_2 = \min\{t : \delta(\Gamma_t) \geq 2\},$$

$$\tau_H = \min\{t : \Gamma_t \text{ contains a Hamilton cycle}\}.$$  

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$^1$An event $\mathcal{E}_n$ holds with high probability (whp) if $\Pr\{\mathcal{E}_n\} \to 1$ as $n \to \infty$.  

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It is trivial to see that $\tau_2 \leq \tau_H$. A celebrated result, independently shown by Ajtai, Komlós and Szemerédi [1] and by Bollobás [4], is the fact that $\tau_2 = \tau_H$ with high probability in the case $\Gamma = K_n$. We generalize this result to a large class of graphs.

**Theorem 1.** Let $\beta > 1/2$ be constant and suppose $\delta(\Gamma) \geq \beta n$. Then $\tau_H = \tau_2$ whp.

Dirac’s theorem [8] states if $\Gamma$ has $n$ vertices and $\delta(\Gamma) \geq n/2$, then $\Gamma$ contains a Hamilton cycle. In many random graph models, it is enough that the random graph has constant minimum degree, see e.g. [3, 9, 25]. The most striking example of this phenomenon remains the Erdős-Rényi graph, and the connection between Hamiltonicity and minimum degree 2 has been well studied when $\Gamma = K_n$. Alon and Krivelevich [2] recently proved that the probability that $G_{n,p}$ contains no Hamilton cycle is $(1 + o(1))\Pr \{\delta(G) < 2\}$ for all values of $p$. It is known that at the moment the graph process reaches minimum degree $2k$, $G_{n,t}$ contains $k$ edge disjoint Hamilton cycles [5, 20, 15]. Briggs et al [6] showed that the edges upon insertion can be coloured in one of $k$ colours, with each colour class containing a Hamilton cycle at the moment the minimum degree reaches $2k$.

Given a graph $\Gamma = (V, E)$ and $0 \leq p \leq 1$, we also define the Erdős-Rényi subgraph $\Gamma_p$ (or $\Gamma_{n,p}$) as the random subgraph of $\Gamma$ obtained by independently retaining each edge with probability $p$. For $\Gamma = K_n$ (so $\Gamma_p = G_{n,p}$), we have [23, 17, 16] that if $p = (\log n + \log \log n + c_n)/n$, then

$$\lim_{n \to \infty} \Pr \{G_{n,p} \text{ is Hamiltonian} \} = \begin{cases} 0, & c_n \to -\infty, \\ e^{-e^{-c}}, & c_n \to c, \\ 1, & c_n \to \infty. \end{cases}$$

We show analogous results for graphs with $\delta(\Gamma) \geq (1/2 + \varepsilon)n$ below.

Traditionally, research on random graphs has mostly been concerned with random subgraphs of specific graphs such as $K_n$ or the complete bipartite graph $K_{n,n}$ (see e.g. [10]). Research on Erdős-Rényi subgraphs of graphs $\Gamma$ with large minimum degree was initiated by Krivelevich, Lee and Sudakov [18], who among other things showed that if $\delta(\Gamma) \geq n/2$ and $p = C \log n/n$ for some large constant $C$, then $\Gamma_p$ is Hamiltonian whp. This article determines the exact value of $C$ when $\delta(G) \geq (1/2 + \varepsilon)n$ for some constant $\varepsilon > 0$. The same authors showed [19] that if $\delta(G) \geq k$ for any $k$ tending to infinity with $n$, and $pk$ tends to infinity with $n$, then $\Gamma_p$ contains a cycle of length $(1 - o_k(1))k$ whp. Riordan [24] subsequently gave a short proof of this result, and Glebov, Naves and Sudakov [14] showed that if
\[ p \geq (\log k + \log \log k + \omega)/k \] for some \( \omega \) tending to infinity then \( \Gamma_p \) contains a cycle of length \( k + 1 \) whp.

A related topic is resiliency. A graph \( G \) is said to be \( \alpha \)-resilient with respect to a property held by \( G \) if the graph \( G \setminus H \) also has the property for any spanning subgraph \( H \subseteq G \) where \( d_H(v) \leq \alpha d_G(v) \) for all \( v \). Nenadov, Steger and Trujić [22] and independently Montgomery [21] showed among other things that \( G_{n,t} \) is \( (1/2 - o(1)) \)-resilient with respect to Hamiltonicity at the moment the minimum degree reaches 2. Very recently, Condon et al [7] showed a resiliency version of Pósa’s theorem in \( G_{n,p} \).

Many of these results have analogues for perfect matchings. Most pertinent to the result presented here is that of Glebov, Luria and Simkin [13], who showed that if \( \Gamma \) is a \( d \)-regular bipartite graph with \( d = \Omega(n) \), then whp \( \Gamma_t \) obtains a perfect matching at the same moment it loses its last isolated vertex.

The author and Frieze [11] considered graphs \( \Gamma \) with \( \delta(\Gamma) \geq (1/2 + \varepsilon)n \) and studied the Hamiltonicity of random \( k \)-out subgraphs of \( \Gamma \). In the context of \( k \)-out graphs with \( k = O(1) \), the positive constant \( \varepsilon \) is needed for the random subgraph to be connected whp. We note that this is not the case for Erdős-Rényi graphs, and that it is still unknown whether the hitting time result presented here can be extended to all \( \Gamma \) with \( \delta(\Gamma) \geq n/2 \).

Theorem 1 implies that finding a threshold for Hamiltonicity is simply a matter of finding the threshold for minimum degree 2.

**Theorem 2.** Let \( \varepsilon > 0 \) be constant and suppose \( \omega = o(\log \log n) \) tends to infinity arbitrarily slowly with \( n \). Suppose \( (\Gamma_n)_{n \in \mathbb{N}} \) is some graph sequence where \( \Gamma_n = (V_n, E_n) \) has \( n \) vertices and minimum degree \( \delta(\Gamma_n) \geq \varepsilon n \). For each \( n \) define \( p_0(n) \) as the unique solution in the interval \( 0 \leq p \leq 1 \) to the equation

\[
\sum_{v \in V_n} (1 - p)^{d(v)} \log n = 1.
\]

Then

\[
Pr\{\delta(\Gamma_{p_0 - \omega/n}) \geq 2\} = o(1),
\]

\[
Pr\{\delta(\Gamma_{p_0 + \omega/n}) \geq 2\} = 1 - o(1).
\]

If a large number of vertices of \( \Gamma_n \) have degree equal to the minimum degree \( \beta n \) for all large \( n \), we obtain

\[
p_0(n) = \frac{\log n + \log \log n + c(\Gamma_n)}{\beta n}
\]
for some bounded $c(\Gamma_n)$, which shows that the well-known threshold function for $\Gamma = K_n$ (where $\beta = 1$ and $c(K_n) \to 0$) is scaled by a factor of $\beta^{-1}$. The precise statement is the following.

**Corollary 3.** Let $\varepsilon > 0$. Suppose $\beta > 1/2$ is constant and that $(\Gamma_n)$ is a graph sequence with $\delta(\Gamma_n) = \beta n$, where at least $\varepsilon n$ vertices of $\Gamma_n$ have degree $\beta n$, and let $p = (\log n + \log \log n + c_n)/\beta n$. Then

$$\lim_{n \to \infty} \Pr\{\Gamma_{n,p} \text{ is Hamiltonian}\} = \begin{cases} 0, & c_n \to -\infty, \\ 1, & c_n \to \infty. \end{cases}$$

Furthermore, if $\Gamma_n$ is $\beta n$–regular and $c_n \to c$ for some constant $c$, then

$$\lim_{n \to \infty} \Pr\{\Gamma_{n,p} \text{ is Hamiltonian}\} = e^{-e^{-\beta c}}.$$

The corollary follows from calculating the probability that no vertex has degree less than 2. We omit some of the calculations here; see e.g. [12, Theorem 3.1] for a proof in the $\Gamma = K_n$ case.

The paper is laid out as follows. In Section 2 we define the random graph $\Gamma_{\tau_2}$ stopped at the moment the minimum degree reaches two, as well as an auxiliary subgraph $G_{\tau_2}$. Section 3 discusses Pósa’s rotation-extension technique, and Section 4 is devoted to proving Theorem 2. In Section 5 (and its many subsections) we prove Theorem 1.

We use the asymptotic notation $O$, $\Omega$, $o$, with the convention that $f(n) = \Omega(g(n))$ and $f(n) = O(g(n))$ both require $f(n)$ to be nonnegative. All logarithms are taken in the natural basis.

## 2 The random graph model

Let $\varepsilon > 0$ be constant and suppose $\Gamma = (V,E)$ is a graph with minimum degree $\delta(\Gamma) \geq (1/2 + \varepsilon)n$ and $m$ edges. Suppose $(e_1, \ldots, e_m)$ is some permutation of the edges, chosen uniformly at random. We define $\Gamma_t = (V, \{e_1, \ldots, e_t\})$ for all $1 \leq t \leq m$. Define $\tau_2$ as the smallest $t$ for which $\Gamma_t$ has minimum degree at least 2.

We let $q = \omega/\log n$ for some $\omega = o(\log \log n)$ tending to infinity arbitrarily slowly with $n$. Upon insertion, edges are independently coloured red with probability $q$ and blue with probability $1 - q$. Let $\Gamma_t^b$ denote the blue subgraph, i.e. the subgraph consisting of all blue edges.

Set $\sigma = 1/100$. We define SMALL as the set of vertices with degree less than $\sigma \log n$ in $\Gamma_{\tau_2}$, and LARGE $= V \setminus$ SMALL. We also define MEDIUM as the set of vertices with degree less than $\sigma \log n$ in $\Gamma_{\tau_2}$. Let $G_{\tau_2} \subseteq \Gamma_{\tau_2}$ be
the graph consisting of the blue edges, along with all red edges with at least one endpoint in MEDIUM. Note that SMALL ⊆ MEDIUM. By design, $G_{\tau_2}$ and $\Gamma_{\tau_2}$ agree on any property concerning only vertices of degree at most $\sigma \log n$ and their incident edges.

The following lemma will allow us to switch between $\Gamma_t$ and $\Gamma_p$. A property $P$ is increasing if for any $G \in P$ and any graph $H$, we have $G \cup H \in P$, and decreasing if $G \in P$ implies $G \setminus H \in P$ for any $H$. A property is monotone if it is either increasing or decreasing.

**Lemma 4.** Let $P$ be a graph property, and $\Gamma$ a graph with $m$ edges. If $p = t/m$ then

$$\Pr \{\Gamma_t \in P\} \leq 10t^{1/2}\Pr \{\Gamma_p \in P\}.\]

If $P$ is a monotone property and $t = o(m)$ tends to infinity with $n$, then

$$\Pr \{\Gamma_t \in P\} \leq 10\Pr \{\Gamma_p \in P\}.$$

This result is well known when $\Gamma = K_n$ (see e.g. [12, Lemmas 1.2, 1.3]), and the straightforward generalization to general dense $\Gamma$ is omitted here.

## 3 Rotation and extension

We define rotations of longest paths, as introduced by Pósa [23]. Suppose $G = (V, E)$ is a graph containing no Hamilton cycle, and let $P = (v_0, v_1, \ldots, v_\ell)$ be a path of maximal length in $G$. If $i \leq \ell - 2$ and $\{v_i, v_{i+1}\} \in E$, then the path $P' = (v_0, v_1, \ldots, v_{i-1}, v_i, v_{i+1}, v_{i+2}, \ldots, v_{\ell-1}, v_{\ell}, \ldots, v_{i+2}, v_{i+1})$ is also a path of maximal length. We say that $P'$ is obtained from $P$ by rotation with $v_0$ fixed. Let $EP(v_0)$ be the set of endpoints, other than $v_0$, that appear as a result of rotations with $v_0$ fixed.

We note that if $G$ is connected, then there is no edge between $v_0$ and $EP(v_0)$ for any $v_0$, as this forms a cycle which may be extended to form a longer path, contradicting the maximality of $P$. Given a graph $G = (V, E)$, we write

$$N_G(S) = \{v \in V \setminus S \mid \exists u \in S : \{u, v\} \in E\}.$$

We will use the following lemma of Pósa [23, Lemma 1].

**Lemma 5.** Suppose $G$ contains no Hamilton cycle and let $P = (v_0, \ldots, v_\ell)$ be a path of maximum length in $G$. Then

$$|N_G(EP(v_0))| < 2|EP(v_0)|.$$
4 The threshold

Suppose the underlying graph sequence is \((\Gamma_n)_{n \in \mathbb{N}}\), where \(\Gamma_n\) has \(n\) vertices \(V_n\). We define \(p_0 = p_0(n)\) as the unique solution to

\[
\sum_{v \in V_n} (1 - p)^{d_v(v)} \log n = 1.
\]

This exists as the left-hand side equals \(n \log n > 1\) for \(p = 0\), zero for \(p = 1\), and is strictly decreasing in \(p\). We note that

\[
1 = \sum_{v \in V_n} (1 - p_0)^{d_v(v)} \leq n \exp \{-p_0 \delta(\Gamma_n)\},
\]

so \(p_0 \leq \delta(\Gamma) - 1 \log n \leq 2 \log n\). We also have

\[
\sum_{v \in V_n} \left(1 - \frac{\log n}{n}\right)^{d_v(v)} \log n \geq n \left(1 - \frac{\log n}{n}\right)^n \log n \sim \log n,
\]

so \(p_0 \geq \frac{\log n}{n}\). The following bounds, both of which use \(\delta(\Gamma) \geq (1/2 + \varepsilon)n\), will be used frequently:

\[
\sum_{v \in V_n} (1 - p)^{d_v(v)} \geq \frac{e^{\omega/2}}{\log n}, \quad p = p_0 - \frac{\omega}{n}, \quad (1)
\]

\[
\sum_{v \in V_n} (1 - p)^{d_v(v)} \leq \frac{e^{-\omega/2}}{\log n}, \quad p = p_0 + \frac{\omega}{n}. \quad (2)
\]

Here (1) follows from summing the following inequality over \(v\):

\[
(1 - p)^{d_v(v)} = (1 - p_0)^{d_v(v)} \left(\frac{1 - p}{1 - p_0}\right)^{d_v(v)}
\]

\[
\geq (1 - p_0)^{d_v(v)} \left(1 + \frac{\omega}{n(1 - p_0)}\right)^{\delta(\Gamma)} \geq (1 - p_0)^{d_v(v)} e^{\omega/2},
\]

and (2) follows similarly.

The following lemma is easily generalized to smaller \(\beta\), and we insist that \(\beta > 1/2\) for notational convenience only.

**Lemma 6.** Let \(\beta > 1/2\) be constant. Suppose \(\omega = o(\log \log n)\) tends to infinity arbitrarily slowly with \(n\), and suppose \(\delta(\Gamma_n) \geq \beta n\) for all \(n\), and that \(\Gamma_n\) has \(m\) edges. Let

\[
T = \left(p_0 - \frac{\omega}{n}\right)m, \quad T' = \left(p_0 + \frac{\omega}{n}\right)m.
\]
Then

\[
Pr \{ \delta(G_T) \geq 2 \} = o(1), \quad Pr \{ \delta(G_T') \geq 2 \} = 1 - o(1).
\]

Proof. Let \( p = p_0 - \omega/n \). Then

\[
Pr \{ d_G(v) < 2 \} = (1 - p)^{d_G(v)} + d_G(v) p (1 - p)^{d_G(v) - 1}
\]

\[
= pd_G(v) (1 - p)^{d_G(v)} (1 + o(1)).
\]

Here we used the facts that \( d_G(v) = \Omega(n) \) and \( p = \Theta\left(\frac{\log n}{n}\right) \). Let \( I_v \) be the indicator variable for \( \{d_G(v) < 2\} \), and write \( X_n = \sum_v I_v \). We then have

\[
E[X_n] = (1 + o(1)) \sum_v pd_G(v) (1 - p)^{d_G(v)}.
\]

**Lower bound:** \( p = p_0 + \omega/n \). In this case, as \( pd_G(v) \leq 2 \log n \),

\[
E[X_n] \leq 3 \log n \sum_v (1 - p)^{d_G(v)} \leq 3 e^{-\omega/2}, \tag{3}
\]

by (2). Markov’s inequality implies that \( X_n = 0 \) whp, and Lemma 4 shows that \( \delta(G_{T'}) \geq 2 \) whp, as this is a monotone property.

**Upper bound:** \( p = p_0 - \omega/n \). We apply the second moment method. Using (1), similarly to (3) we have \( E[X_n] \geq (1 - o(1))e^{\omega/2} \), and in particular \( E[X_n] \) tends to infinity with \( n \). We also have

\[
E[X_n(X_n - 1)] = \sum_{u \neq v} E[I_u I_v].
\]

Firstly, if \( \{u, v\} \notin E(\Gamma) \) then \( I_u \) and \( I_v \) are independent and \( E[I_u I_v] = E[I_u] E[I_v] \). If \( \{u, v\} \in E(\Gamma) \) then

\[
E[I_u I_v] = p(1 - p)^{d_G(u) + d_G(v)} + (d_G(u) - 1)(d_G(v) - 1)p^2(1 - p)^{d_G(u) + d_G(v) - 3}
\]

\[
= d_G(u) d_G(v) p^2(1 - p)^{d_G(u) + d_G(v)} (1 + O(n^{-1}))
\]

\[
= E[I_u] E[I_v] (1 + O(n^{-1})).
\]

We then have

\[
E[X_n^2] = (1 + o(1)) \sum_{u \neq v} E[I_u] E[I_v] = (1 + o(1)) \left( \sum_v pd_G(v) (1 - p)^{d_G(v)} \right)^2.
\]

It follows that \( \text{Var}(X_n) = o(E[X_n^2]) \), and Chebyshev’s inequality implies that \( X_n > 0 \) whp, and so \( \delta(G_T) < 2 \) whp. This is a monotone event, so \( \delta(G_T) < 2 \) also holds whp by Lemma 4. \( \square \)
5 Proof of Theorem 1

We set up the main calculation. We fix the constants \( K = 10, \sigma = 1/100 \) and \( \alpha = e^{-2000} \), and also fix some \( \omega = o(\log \log n) \) tending to infinity arbitrarily slowly with \( n \). Define the following events concerning \( \Gamma_{\tau_2} \) (see Section 2 for definitions concerning our random graphs).

\[
\mathcal{H} = \{ \Gamma_{\tau_2} \text{ is Hamiltonian} \},
\]
\[
\mathcal{P}_\ell = \{ \Gamma_{\tau_2} \text{ is not Hamiltonian, and its longest path has } \ell \text{ vertices} \},
\]
\[
\mathcal{S} = \{|\text{SMALL}| \leq n^{0.1}\},
\]
\[
\mathcal{E}_1 = \left\{ \text{in } \Gamma_{\tau_2}, \text{ every } |S| \leq 6\alpha n \text{ has } e(S) \leq \frac{\sigma \log n}{K}|S| \right\},
\]
\[
\mathcal{E}_2 = \{ \Gamma_{\tau_2} \text{ contains no path of length } \leq 4 \text{ between vertices of SMALL, and no cycle of length } \leq 4 \text{ intersects SMALL} \},
\]
\[
\mathcal{E} = \mathcal{E}_1 \cap \mathcal{E}_2,
\]
\[
\mathcal{N} = \mathcal{E} \cap \mathcal{S}.
\]

We define \( \mathcal{H}', \mathcal{P}_\ell', \) etc., as the corresponding events with \( G_{\tau_2} \) replacing \( \Gamma_{\tau_2} \), noting that the set of vertices of degree less than \( \sigma \log n \) is unchanged. We further define the following events concerning \( G_{\tau_2} \).

\[
\mathcal{C}' = \{ G_{\tau_2} \text{ is connected} \},
\]
\[
\mathcal{M}' = \left\{ |\text{MEDIUM}| \leq \frac{\omega n}{\log \log n} \right\},
\]
\[
\mathcal{R}' = \left\{ e(\Gamma_{\tau_2}) - e(G_{\tau_2}) \geq \frac{\omega n}{2} \right\},
\]
\[
\mathcal{G}_\ell' = \mathcal{P}_\ell' \cap \mathcal{N}' \cap \mathcal{C}' \cap \mathcal{M}' \cap \mathcal{R}'.
\]

We note that \( \mathcal{N} \subseteq \mathcal{N}' \), as the properties involved are either decreasing or only concern vertices of degree less than \( \sigma \log n \). We finally define, with \( T = (p_0 - \omega/n)m \) as in Section 4,

\[
\mathcal{T} = \{ \tau_2 \geq T \},
\]
\[
\mathcal{A} = \bigcup_{\ell=1}^n (\mathcal{P}_\ell \cap \mathcal{P}_\ell').
\]

In words, \( \mathcal{A} \) is the event that the longest path in \( G_{\tau_2} \) has the same length (in terms of the number of vertices) as the longest path in \( \Gamma_{\tau_2} \). The following lemma is proved in Section 5.5.
Lemma 7. If $E$ occurs, then any set $S$ of at most $\alpha n$ vertices satisfies $|N(S)| \geq 2|S|$ in $\Gamma_{\tau_2}$ and in $G_{\tau_2}$.

We will show in the upcoming two sections that

\[ \Pr \{ A \mid P_{\ell} \cap N \cap T \} \geq \exp \left\{ -O \left( \frac{\omega n}{\log n} \right) \right\}, \quad (4) \]

and that

\[ \Pr \{ A \cap P_{\ell} \cap N \cap T \} \leq \exp \left\{ -\Omega \left( \frac{\omega n}{\log n} \right) \right\}, \quad (5) \]

from which we conclude that

\[ \Pr \{ P_{\ell} \cap N \cap T \} = \Pr \{ A \cap P_{\ell} \cap N \cap T \} \Pr \{ A \mid P_{\ell} \cap N \cap T \} \leq e^{-\Omega(\omega n/\log n)} e^{-O(\omega n/\log n)} = o(n^{-1}). \]

We then have (as $Pr \{ T \} = 1 - o(1)$ by Lemma 6),

\[ \Pr \{ \mathcal{H} \} \leq \Pr \{ \mathcal{N} \cup \mathcal{T} \} + \sum_{\ell} \Pr \{ P_{\ell} \cap N \cap T \} = \Pr \{ \mathcal{N} \} + o(1). \]

To finish the argument, we show in Section 5.4 that

\[ \Pr \{ \mathcal{N} \} = 1 - o(1). \quad (6) \]

5.1 Proof of (4)

Suppose $\Gamma_{\tau_2} = G$, where $G$ is a graph with some longest path (or Hamilton cycle) $P$, on edges $f_1, \ldots, f_\ell$ with $\ell \leq n$. If each edge of $P$ is coloured blue then $P$ must also appear in $G_{\tau_2}$. So, as $q = \omega/\log n$,

\[ \Pr \{ A \mid \Gamma_{\tau_2} = G \} \geq \left( 1 - \frac{\omega}{\log n} \right)^\ell = \exp \left\{ -O \left( \frac{\omega n}{\log n} \right) \right\}. \]

As the event $P_{\ell} \cap N$ is of the form $\{ \Gamma_{\tau_2} \in \mathcal{G} \}$ for some class of graphs $\mathcal{G}$, (4) follows.

5.2 Proof of (5)

By the discussion in Section 5, we have $\mathcal{N} \subseteq \mathcal{N}'$. Recall $\mathcal{G}'_{\ell} = P_{\ell}' \cap \mathcal{N}' \cap \mathcal{C}' \cap \mathcal{M}' \cap \mathcal{R}'$. Then as $\mathcal{N}' \subseteq \mathcal{E}'$,

\[ A \cap P_{\ell} \cap N \cap T \subseteq (A \cap \mathcal{G}'_{\ell}) \cup (\overline{\mathcal{C}' \cap \mathcal{E}'}) \cup (\overline{\mathcal{M}' \cap \mathcal{R}' \cap T}). \]
We also have $A \cap \mathcal{P}_\ell = A \cap \mathcal{P}_\ell'$, so

$$
\Pr \{A \cap \mathcal{P}_\ell \cap \mathcal{N} \cap \mathcal{T}\} \\
\leq \Pr \{A \cap \mathcal{P}'_\ell\} + \Pr \{\overline{\mathcal{C}} \cap \mathcal{E}'\} + \Pr \{\overline{\mathcal{M}} \cap \mathcal{M}' \cap \mathcal{T}\}
$$

$$
\leq \Pr \{\mathcal{P}_\ell \mid \mathcal{G}'\} + \Pr \{\overline{\mathcal{C}} \cap \mathcal{E}'\} + \Pr \{\overline{\mathcal{M}} \cap \mathcal{T}\} + \Pr \{\overline{\mathcal{R}} \cap \mathcal{M}' \cap \mathcal{T}\}. \ (7)
$$

In this section we show that the first term is at most $e^{-\Omega(\omega n)}$, while the other terms are postponed for Section 5.3.

So we condition on $\mathcal{G}'_\ell = \mathcal{P}'_\ell \cap \mathcal{N}' \cap \mathcal{C}' \cap \mathcal{M}' \cap \mathcal{R}'$. Then $\mathcal{G}_\ell = \mathcal{G}_\ell'$ is a connected graph such that $|N_G(S)| \geq 2|S|$ for any $|S| \leq \alpha n$ (see Lemma 7), and there exists a set $|L| = n - o(n)$, such that $\Gamma_\tau_2$ is obtained from $G$ by randomly adding $r \geq \omega n/2$ edges from $\Gamma$ with both endpoints in $L$. Let $R$ denote the set of red edges fully contained in $L$. The longest path of $G$ has length $\ell$, and we will show that it is very unlikely that adding the edges $R$ does not increase the length of the longest path.

Let $P$ be a longest path in $G$, and let $x, y$ be its two endpoints. Let $EP(x)$ be the set of opposite endpoints obtainable from $P$ by rotations with $x$ fixed. By Lemma 7 we have $|N_G(S)| \geq 2|S|$ whenever $|S| \leq \alpha n$, so Lemma 5 implies $|EP(x)| \geq \alpha n$. Let $EP$ be the set of all endpoints of longest paths in $G$. The total number of boosters in $G$ is

$$
b = \frac{1}{2} \sum_{x \in EP} \sum_{y \in EP(x)} |\{x, y\} \in E(\Gamma)|,
$$

where we write $[\mathcal{B}] = 1$ if the statement $\mathcal{B}$ is true, and 0 otherwise. We divide the set $R$ of random edges into two parts $R_1 \cup R_2$ of as equal size as possible. We use $R_1$ to build $\Omega(n^2)$ boosters, in case $G$ does not already have $\Omega(n^2)$ boosters.

**Lemma 8.** With probability at least $1 - e^{-\Omega(n)}$, $G \cup R_1$ either has a path of longer length than $\ell$ (or is Hamiltonian), or it has $\Omega(n^2)$ boosters.

**Proof.** Assign some arbitrary order $R_1 = \{f_1, \ldots, f_{r/2}\}$ to the edges of $R_1$. Let $L = \text{LARGE} \setminus \text{MEDIUM}$. We can treat the $f_i$ as independent uniform edges in $E(\Gamma) \cap L^2$, as doing so only introduces repetitions which decreases the probability of producing many boosters.

Let $P$ be a longest path on vertex set $U$, and let $EP$ be the set of endpoints of paths spanning $U$. Let $EP_L = EP \cap L$. Suppose first that there are at least $(\varepsilon n)^2$ edges in $\Gamma$ between $EP_L$ and $\overline{U} \cap L$. Adding any of these edges extends the path. Each $f_i$ has probability at least $\varepsilon$ of landing
in this set, so the probability that the path is not extended by such an edge is at most 
\((1 - \varepsilon)^{r/2} \leq e^{-\Omega(\omega_n)}\).

Suppose that there are less than \((\varepsilon n)^2\) edges between \(EP_L\) and \(\overline{U} \cap L\).
As \(|MEDIUM| = o(n)|\), there are \(o(n^2)|\) edges incident to \(MEDIUM\). So at most 
\(2\varepsilon n\) vertices of \(EP_L\) can have more than \(\varepsilon n/2\) edges to \(U\). Say that 
\(y \in EP\) is good if it is in \(L\) and has at least \((1 + \varepsilon)n/2\) edges to \(U\) in 
\(\Gamma\). For any \(x\), the set \(EP(x)\) of opposite endpoints must contain at least 
\((\alpha - 2\varepsilon)n - o(n) \geq \alpha n/2\) good vertices.

We aim to show that in \(G \cup R_1\), with probability \(1 - e^{-\Omega(\omega_n)}\), either there is a path of longer length than \(\ell\) (or is Hamiltonian), or all good endpoints are incident to at least \(\varepsilon n/8\) boosters. Say that \(x \in EP_L\) is settled if there are at least \(\varepsilon n/8\) vertices \(y \in EP_L(x)\) such that \(\{x, y\} \in E(\Gamma)\), and unsettled otherwise.

For now, we consider \(G_0 = G\), i.e. with no edges of \(R_1\) added. Suppose 
\(x\) is good and unsettled. Let \(y\) be a good vertex of \(EP(x)\), and let \(Q = (x = 
v_0, v_1, \ldots, v_t = y)\) be a longest path on \(U\) between \(x\) and \(y\). As both \(x\) and 
\(y\) are good, there are at least \(\varepsilon n/2\) indices \(i\) such that \(\{x, v_{i+1}\}, \{y, v_i\} \in 
E(\Gamma)\). As \(MEDIUM\) has size \(o(n)|\), there must be at least \(\varepsilon n/4\) such indices 
where \(v_i\) and \(v_{i+1}\) are both in \(L\). As \(x\) is unsettled, at most \(\varepsilon n/8\) such 
indices have \(v_{i+1} \in EP(x)\), and for each such index it follows that \(\{y, v_i\} \notin 
E(G)|\), as otherwise a rotation around \(\{y, v_i\}\) would contradict \(v_{i+1} \notin EP(x)\).
We conclude that there are at least \(\varepsilon n/8\) indices \(i\) such that \(\{x, v_{i+1}\} \in 
E(\Gamma), \{y, v_i\} \in E(\Gamma) \setminus E(G)\), and all four vertices are in \(L\). Let \(A_0(x,y)\) be the set of such semiboosters \(\{y, v_i\}\), and \(B_0(x,y)\) the set of such \(v_{i+1}\). Repeat this for all good \(y \in EP(x)\) (if there is more than one longest path 
between \(x\) and \(y\), pick one arbitrarily). Let \(A_0(x)\) be the union of \(A_0(x,y)\) over all 
good \(y \in EP(x)\). As there are at least \(\alpha n/2\) good \(y \in EP(x)\), and each edge in 
\(A_0(x,y)\) is incident to \(y\), we have

\[
|A_0(x)| = \sum_{\substack{y \in EP(x)\ \text{good}}} |A_0(x,y)| \geq \frac{\alpha n}{2} \times \frac{\varepsilon n}{8} = \frac{\alpha \varepsilon}{16} n^2.
\]

The sets \(B_0(x,y) \subseteq U \cap N_1(x)\) are not necessarily disjoint for different \(y\).
However, each \(z \in U \cap N_1(x)\) appears in \(B_0(x,y)\) for at most \(|EP(x)| < n\) 
different \(y\).

Each \(e \in A_0(x)\) is associated with some vertex \(z = z(e) \in U \cap N_1(x)\) 
such that adding \(e\) implies that \(\{x, z\}\) is a new booster (or extends the path 
if \(\{x, z\}\) is in \(G\)). Let \(G_1 = G_0 \cup \{f_1\}\). If \(f_1 \in A_0(x)\), then the booster 
\(\{x, z(f_1)\}\) has been added. Any \(e \in A_0(x) \setminus \{f_1\}\) with \(z(e) = z(f_1)\) may no 
longer be a semibooster, and we remove such \(e\) from our set of semiboosters.
In general, given \( A_j(x) \), we reveal \( f_{j+1} \). If \( f_{j+1} \in A_j(x) \) (a success) we set \( A_{j+1}(x) = A_j(x) \setminus z^{-1}(f_{j+1}) \). If \( f_{j+1} \notin A_j(x) \) (a failure), we let \( A_{j+1}(x) = A_j(x) \). If revealing \( f_1, \ldots, f_j \) results in \( s \) successes, then
\[
|A_j(x)| \geq |A_0(x)| - sn \geq \frac{\alpha \varepsilon}{16} n^2 - sn.
\]

As long as there are less than \( \alpha \varepsilon n/32 \) successes, there are at least \( \alpha \varepsilon n^2/32 \) semiboosters left in \( A_j(x) \). In this case, \( f_{j+1} \) has probability at least \( |A_j(x)|/\binom{n}{2} \geq \frac{\alpha \varepsilon}{16} \) of succeeding. So the probability that adding all of \( R_1 \) results in less than \( \alpha \varepsilon n/32 \) successes is bounded by
\[
\Pr \left\{ \text{Bin} \left( \frac{r}{2}, \frac{\alpha \varepsilon}{16} \right) < \frac{\alpha \varepsilon n}{32} \right\} = e^{-\Omega(\omega n)}.
\]

Suppose \( G \cup R_1 \) has \( \Omega(n^2) \) boosters. We expose \( R_2 \). The probability that none of the \( \Omega(n^2) \) boosters are in \( R_2 \) is \( e^{-\Omega(\omega n)} \). As we condition on \( G' \subseteq C' \), adding a booster extends the length of the longest path or forms a Hamilton cycle. We conclude that
\[
\Pr \{ P_\ell \mid G' \} \leq e^{-\Omega(\omega n)}.
\]

5.3 Properties of \( G_{\tau_2} \)

Recall from (7) that it remains to show that \( \Pr \{ \overline{C} \cap \mathcal{E}' \} \), \( \Pr \{ \overline{M'} \cap \mathcal{T} \} \) and \( \Pr \{ \overline{G'} \cap M' \cap T \} \) are all at most \( e^{-\Omega(\omega n/\log \log n)} \).

5.3.1 Connectivity

We bound \( \Pr \{ \overline{C} \cap \mathcal{E}' \} \). Suppose \( \mathcal{E}' \) holds. Lemma 7 implies that in \( G = G_{\tau_2} \), any \( S \) with \( |S| \leq \alpha n \) has \( |N_G(S)| \geq 2|S| \). In particular, any connected
component has size at least $3\alpha n$. Suppose $S$ is a set of size $3\alpha n \leq s \leq n/2$. Then $e_{\Gamma}(S, \overline{S}) \geq s(\beta n - s) \geq \varepsilon sn$, and for $p \geq \frac{\log n}{n}$,

$$\Pr \{ \exists 3\alpha n \leq |S| \leq n/2 : e_G(S, \overline{S}) = 0 \} \leq \sum_{s=3\alpha n}^{n/2} \binom{n}{s}(1-p)^{\varepsilon sn} \leq \sum_{s=3\alpha n}^{n/2} \left( \frac{ne}{s} \left( 1 - \frac{\log n}{n} \right) \right)^s \leq \sum_{s=3\alpha n}^{n/2} \left( \frac{e}{3\alpha n} \right)^s \leq e^{-\Omega(\omega n)}.$$

### 5.3.2 The set MEDIUM

Define $M'_t$ as the event that the blue subgraph $\Gamma_b^t$ has $|M| \leq \omega n/ \log \log n$, noting that $M'_s \subseteq M'_t$ whenever $s \leq t$, so

$$\overline{M'_t} \cap T \subseteq \overline{M'_s} \cap T \subseteq \overline{M'_s}.$$

By an argument similar to the one behind Lemma 4, we can couple $\Gamma_b^t$ to $\Gamma_p$ where $p = (p_0 - \omega/n)(1 - q) \geq \frac{\log n}{2n}$, and we move to bounding the probability that $\Gamma_p$ has more than $\omega n/ \log \log n$ vertices of degree less than $\sigma \log n$.

Let $S$ be a set of $s = \omega n/ \log \log n$ vertices. For any $v \in S$, the probability that $v$ has degree less than $\sigma \log n$ in $\Gamma_p$, is at most the probability that it has less than $\sigma \log n$ edges to $\overline{S}$. As $v$ has at least $\delta(\Gamma) - s$ potential edges to $\overline{S}$, we have

$$\Pr \{ e(v, \overline{S}) < \sigma \log n \} \leq \sum_{k=0}^{\sigma \log n} \binom{d_{\Gamma}(v)}{k} p^k (1-p)^{\delta(\Gamma)-s-k}.$$ 

Letting $b_k$ denote the summand, we have

$$\frac{b_{k+1}}{b_k} = \frac{p}{1-p} \frac{d_{\Gamma}(v) - k}{k + 1} \geq (1 - o(1)) \frac{\log n}{2n} \frac{n/2}{\sigma \log n + 1} > 2,$$

as $\sigma = 1/100$. If follows that $\sum b_k \leq 2b_\sigma \log n$, and as $\frac{\log n}{2n} \leq p \leq 2 \frac{\log n}{n}$ and $\delta(\Gamma) > (1 + \varepsilon)n/2 + 2\sigma \log n$,

$$\Pr \{ e(v, \overline{S}) < \sigma \log n \} \leq 2 \left( \frac{ne}{2\sigma \log n} \frac{2 \log n}{n} \right)^{\sigma \log n} \left( 1 - \frac{\log n}{2n} \right)^{(1+\varepsilon)n/2} \leq 2 \left( \frac{e}{\sigma} \frac{\sigma \log n}{n} \right)^{(1+\varepsilon)/4}.$$
With $\sigma = 1/100$, this is at most $n^{-1/8}$. As the $e(v, \overline{S})$ are independent random variables for all $v \in S$, the probability that $e(v, \overline{S}) < \sigma \log n$ for all $v \in S$ is at most $n^{-s/8}$. It follows that

$$\Pr \{|MEDIUM| > s\} \leq \left(\frac{n}{s}\right)^{n^{-s/8}} \leq \left(\frac{ne}{s \cdot n^{1/8}}\right)^s \leq \exp \left\{\frac{7s}{8} \log n - s \log s\right\} \leq \exp \left\{-\frac{\omega n}{16 \log \log n}\right\},$$

as $s = \omega n / \log \log n$.

### 5.3.3 There are enough red edges outside MEDIUM

Let $D_t$ be the event that at least $\omega n$ edges are coloured red in $\Gamma_t$. This is increasing in $t$, and $T = \Omega(n \log n)$, so

$$\Pr \{D_{\tau_2} \cap \mathcal{T}\} \leq \Pr \{D_T \cap \mathcal{T}\} \leq \Pr \{D_T\} = \Pr \{\text{Bin} \left(\frac{T}{\log n}, \frac{\omega}{\log n}\right) < \omega n\} \leq e^{-\Omega(\omega n)}.$$

Conditioning on $D_{\tau_2}$, the probability that there exists some set $M$ of $s = o(n)$ vertices such that more than $\omega n/2$ red edges have at least one endpoint incident to $M$ is at most $e^{-\Omega(\omega n)}$. As $\mathcal{M}'$ implies the existence of such a set, we have

$$\Pr \{\overline{\mathcal{M}} \cap \mathcal{M}' \cap \mathcal{T}\} \leq \Pr \{D_{\tau_2} \cap \mathcal{T}\} + \Pr \{\overline{\mathcal{M}} \cap \mathcal{M}' \mid D_{\tau_2}\} \leq e^{-\Omega(\omega n)}.$$

### 5.4 Properties of $\Gamma_{\tau_2}$

In (6) we claim that $\Pr \{\mathcal{N}\} = \Pr \{\mathcal{S} \cap \mathcal{E}_1 \cap \mathcal{E}_2\} = 1 - o(1)$, which we now prove.

#### 5.4.1 $\mathcal{S} - \text{SMALL is small}$

The set $\text{SMALL}$ is defined as the set of vertices of degree at most $\sigma \log n$ in $\Gamma_{\tau_2}$. We will show that $\text{SMALL}$ is small in $\Gamma_T$, which is enough as $\mathcal{S}$ is
an increasing property. With \( p = p_0 - \omega/n \), repeating the calculations in Section 5.3.2,

\[
\Pr \{ d_T(v) < \sigma \log n \} \leq 2 \left( \frac{2e}{\sigma} \right)^{\sigma \log n} (1 - p)^{d_T(v)}.
\]

With \( \sigma = 1/100 \) we have \( \sigma \log(2e/\sigma) < 0.08 \), and (1) shows that \( \sum_v (1 - p)^{d_T(v)} = o(n^{0.01}) \), so

\[
\mathbb{E}|\text{SMALL}| \leq 2^{n^{0.08}} \sum_v (1 - p)^{d_T(v)} \leq n^{0.09}.
\]

Markov’s inequality implies that \(|\text{SMALL}| \leq n^{0.1}\) whp.

### 5.4.2 \( \mathcal{E}_1 \) - Small sets are sparse

Let \( \Gamma_n \) have \( m \) edges and minimum degree \( \beta n \). Recall, with the threshold \( p_0 \) as defined in Theorem 2, that we define for some \( \omega \),

\[
T = \left( p_0 - \frac{\omega}{n} \right) m, \quad T' = \left( p_0 + \frac{\omega}{n} \right) m.
\]

By Lemma 6, the hitting time \( \tau_2 \) for having minimum degree 2 satisfies \( T \leq \tau_2 \leq T' \). In this section we show that \(|N_G(S)| \geq 2|S| \) for all \(|S| \leq \varepsilon n \) whp in \( G = \Gamma_{\tau_2} \).

**Lemma 9.** Suppose \(|p - p_0| \leq \omega/n \) and let \( G = \Gamma_p \). Whp, no \( S \subseteq V \) with \(|S| \leq 6\alpha n\) contains more than \( \frac{\sigma \log n}{K} |S| \) edges.

**Proof.** Recall from Section 4 that as \( \delta(\Gamma_n) \geq n/2 \), we have \( p_0 \leq 2 \frac{\log n}{n} \). The lemma follows from the first moment method: a set \( S \) of size \( s \) contains at most \( \binom{s}{2} \) edges of \( \Gamma \), so

\[
\Pr \left\{ \exists |S| \leq 6\alpha n : e_G(S) > \frac{\sigma \log n}{K} |S| \right\} \leq \sum_{s=\frac{\sigma \log n}{2K}}^{6\alpha n} \binom{n}{s} \left( \frac{\binom{s}{2}}{\frac{\sigma \log n}{K}} \right)^{\frac{\sigma \log n}{K} s} p^s \leq \sum_{s=\frac{\sigma \log n}{2K}}^{6\alpha n} \left( \frac{ne}{s} \left( \frac{Ks}{2n\sigma} \right)^{\frac{\sigma \log n}{K}} \right)^s \leq \sum_{s=\frac{\sigma \log n}{2K}}^{6\alpha n} \left( n \frac{Ke\alpha}{2\sigma} \right)^{\frac{\sigma \log n}{K}} s. \]
The constants $K = 10$, $\sigma = 1/100$, $\alpha = e^{-2000}$ were chosen so that $\frac{K}{K} \ln(Ke\alpha/2\sigma) < -1$, so the summand is $o(1)$, and the sum tends to zero.

5.4.3 $\mathcal{E}_2$ – There are no small structures

Recall from Section 4 that we define $T = (p_0 - \omega/n)m$ and $T' = (p_0 + \omega/n)m$, with $\omega$ as chosen in Section 5. The following lemma implies that $\Pr \{ \mathcal{E}_2 \} = o(1)$, as $T \leq t \leq T'$ whp by Lemma 6.

**Lemma 10.** Whp, the following holds in $\Gamma_{\tau_2}$. No two vertices $u, v \in \text{SMALL}$ are connected by a path of length at most 4, and no vertex in $\text{SMALL}$ is on a cycle of length at most 4.

**Proof.** We let $S$ be the set of vertices of degree less than $\sigma \log n$ in $\Gamma_T$, noting that $\text{SMALL} \subseteq S$, and bound the probability that $\Gamma_T$ contains a short path or cycle involving $S$ as described. As $T \leq \tau_2 \leq T'$ whp, the lemma will follow.

We show the proof for the path $P_4$ on three edges, and later explain how the other will follow. Write $\Gamma_t \leftarrow P_4$ for the event that $P_4$ is in $\Gamma_t$ with its two endpoints in $S$. We consider a path $P$ on vertex set $U = \{u_1, u_2, u_3, u_4\}$, where $u_1, u_4$ are the endpoints. Summing over injective maps $\phi : U \rightarrow V$, writing $v_i = \phi(u_i)$, we bound the probability that $v_1, v_4 \in S$ and that $G[\phi(U)]$ contains three edges as follows.

Let $p = (p_0 - \omega/n)m$ and $p_1 = (p_0 + \omega/n)m$. We pick four vertices $v_1, v_2, v_3, v_4$, and three edges forming a path on these vertices. These edges are included in $\Gamma_{p_1}$ with probability $p_3^3$. This does not significantly change the probability that $v_1$ and $v_4$ are small, and repeating the calculations in Section 5.4.1 we can bound

$$\Pr \left\{ \gamma_{p_1} \leftarrow P_4 \right\} \leq \sum_{v_1, v_2, v_3, v_4} p_3^3 \Pr \{ v_1, v_4 \in \text{SMALL} \}$$

$$\leq n^2 \left( \frac{2 \log n}{n} \right)^3 \left( \sum_v \Pr \{ d_T(v) \leq \sigma \log n \} \right)^2$$

$$\leq 8n^{0.2 - 1}$$

We apply Lemma 4, noting that $(T')^{1/2} = O(n^{1/2} \log n)$, and conclude that

$$\Pr \left\{ \gamma_{T'} \leftarrow P_4 \right\} = O((T')^{1/2}n^{-0.95}) = o(n^{-1/4}),$$

and $\Gamma_{T'} \leftarrow P_4$ whp implies that $\Gamma_t \leftarrow P_4$ for all $T \leq t \leq T'$. In general, suppose $H$ is a small graph on $u$ vertices with $f$ edges, and $s$ vertices required...
to be in $S$, such that $f + s - u \geq 1$ and $s \leq 2$. This is the case for all the graphs considered, and repeating the above calculations gives

$$\Pr \{ \exists T \leq t \leq T' : \Gamma_t \leftarrow H \} = O(T^{1/2} n^{0.05 - f - s + u}) = o(n^{-1/4}).$$

\[\square\]

### 5.5 Expansion: Proof of Lemma 7

What now remains is to prove Lemma 7, which states that if $E$ then $|N(S)| \geq 2|S|$ whenever $|S| \leq \alpha n$. As expansion is an increasing property and $G_{\tau_2} \subseteq \Gamma_{\tau_2}$, we only need to show that this holds for $G = G_{\tau_2}$.

**Lemma 11.** Suppose $E$ occurs and $S \subseteq \text{LARGE}$ has $|S| \leq \alpha n$. Then $|N_G(S)| \geq 5|S|$ in $G = G_{\tau_2}$.

**Proof.** If $|N_G(S)| < 5|S|$, then $|S \cup N_G(S)| \leq 6\alpha n$. As $E_1 \subseteq E'_1$, this implies that $e_G(S \cup N_G(S)) \leq \frac{\sigma \log n}{K} |S|$. We then have

$$|S| \sigma \log n \leq \sum_{v \in S} d(v) = 2e_G(S) + e_G(S, N_G(S))$$

$$\leq e_G(S) + e_G(S \cup N_G(S))$$

$$\leq \frac{\sigma \log n}{K} (|S| + |S \cup N_G(S)|)$$

$$\leq |S| \frac{2\sigma}{K} \log n + |N_G(S)| \frac{\sigma \log n}{K},$$

and we conclude that

$$\frac{|N_G(S)|}{|S|} \geq \frac{K}{\sigma \log n} \left( \sigma - \frac{2\sigma}{K} \right) \log n \geq K - 2.$$

As $K = 10$, this proves the lemma. \[\square\]

Now let $S$ be any set of at most $\alpha n$ vertices, and let $S_1 = S \cap \text{SMALL}$ and $S_2 = S \cap \text{LARGE}$. Then

$$|N_G(S)| = |N_G(S_1)| + |N_G(S_2)|$$

$$- |N_G(S_1) \cap S_2| - |N_G(S_2) \cap S_1| - |N_G(S_1) \cap N_G(S_2)|$$

$$\geq |N_G(S_1)| + |N_G(S_2)| - |S_2| - |N_G(S_2) \cap S_1| - |N_G(S_1) \cap N_G(S_2)|.$$
We have $|N_G(S_2)| \geq 5|S_2|$ by Lemma 11. As $E_2 \subseteq E_2'$, there are no paths of length 2 between vertices of $S_1$, which implies $|N_G(S_1)| \geq \sum_{v \in S_1} d(v) \geq 2|S_1|$ (as $\delta(G_{\tau_2}) \geq 2$), and $|N_G(S_2) \cap S_1| \leq |S_2|$. Finally, as there are no short cycles intersecting $S_1$ and no paths of length $\leq 4$ between vertices of $S_1$, again by $E_2$, we have $|N_G(S_1) \cap N_G(S_2)| \leq |S_2|$. This implies that

$$|N_G(S)| \geq 2|S_1| + 5|S_2| - 3|S_2| \geq 2|S|.$$

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