Domain Walls in $N = 1$ Supergravity

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ABSTRACT

We discuss a study of domain walls in $N = 1, d = 4$ supergravity. The walls saturate the Bogomol’nyi bound of wall energy per unit area thus proving stability of the classical solution. They interpolate between two vacua whose cosmological constant is non-positive and in general different. The matter configuration and induced geometry are static. We discuss the field theoretic realization of these walls and classify three canonical configurations with examples. The space-time induced by a wall interpolating between the Minkowski (topology $\mathbb{R}^4$) and anti-de Sitter (topology $S^1(time) \times \mathbb{R}^3(space)$) vacua is discussed.

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1. Introduction

Global and local topological defects are known to arise during symmetry breaking phase transitions if the vacuum manifold is not simply connected. Textures, monopoles, strings, domain walls and combinations thereof are examples. These objects may have important physical implications, especially in the cosmological context.

The inclusion of gravity in the study of topological defects is straightforward and usually leads to insignificant modifications to the otherwise stable topological defects. However, in superstring theories, for example, gravity and other moduli and matter fields are on an equal footing so the effects of gravity can yield qualitatively different features. With the advent of deeper understanding of semi-classical superstring theories in a topologically nontrivial sector, various stringy topological defects were discovered: stringy cosmic strings\textsuperscript{[1,2]}, axionic instantons\textsuperscript{[3,4]} as well as related heterotic five-branes and other solitons\textsuperscript{[5,6,7,8]}.

The above solutions were known to exist for free moduli fields, \textit{i.e.} vanishing superpotential. Additionally, there exist supersymmetric domain walls when a nontrivial superpotential for the moduli fields exists\textsuperscript{[9,10]}. These domain walls are interesting by themselves as well as in connection to the dynamical supersymmetry breaking mechanism in superstring theory\textsuperscript{[11,12]}. Additionally, they serve as a class of stringy topological defects in which a nonzero superpotential is essential to their existence.

The present discussion centers on the construction and properties of domain walls in $N = 1, d = 4$ supergravity. There are three major results of our analysis. The first is a proof of a positive energy density theorem for a topologically nontrivial extended object in which the matter part of the theory has a generic nonzero superpotential. It is known that the inclusion of gravity to reflection symmetric domain walls of infinite extent and infinitesimal thickness generically admits only \textit{time-dependent} solutions to Einstein's equations\textsuperscript{[13]}. We show that by allowing for a reflection asymmetric solution interpolating between either a Minkowski and anti-de Sitter space-time or anti-de Sitter and anti-de Sitter space-time, the metric and matter field can both be \textit{time-independent}.

The last result is that supersymmetric domain walls can interpolate between two vacua of different scalar potential energy: for example, between a supersymmetric vacuum with zero cosmological constant (Minkowski space-time) and an-
other with a negative cosmological constant (anti-de Sitter space-time). This result is at first counter to the notion of a domain wall interpolating between degenerate vacua. The point is that in defining degenerate energy solutions, one must include all the relevant energy in the theory; in this case both matter and gravity. It turns out that when the vacua of the theory preserve supersymmetry, their energy, which is defined in the appropriate way according to the ADM prescription \cite{14}, are the same regardless of the particular matter vacuum energy. This result is consistent with there being no semi-classical tunnelling bubble causing the decay of one supersymmetric vacua into another with a lower matter vacuum energy. In \cite{15} this result has been proven by showing the minimum energy bubble which one could conceivably form separating two supersymmetric vacua has an infinite radius and thus will never form. This result is complementary to positive energy theorems (see, for example \cite{16} and references therein) derived to show the stability of supergravity theories with a matter potential unbounded below. Indeed, without this result, one could never expect to find the domain wall solutions we wish to describe.

This paper is organized as follows. We start in chapter 2 with a discussion of the formal aspects of the realization of these walls in the supergravity theory. Chapter 3 gives a classification of the walls in terms of the various combination of vacua that they interpolate between. Chapter 4 presents examples of the three canonical wall configurations and chapter 5 gives the geodesic structure for the space-time induced by these walls. We finish with some further remarks on the wall interpolating between Minkowski and anti-de Sitter space-times.

Most of the work presented here is developed in the following references. The field theoretic results can be found in reference \cite{18}. Additional work addressing the problem of the stability of supersymmetric vacua can be found in reference \cite{15}. Discussion of the classification of the types of walls and their geodesic structure can be found in reference \cite{19}. Finally, the causal structure of the Minkowski-AdS wall as well as some phenomena related to quantum fields on this background is work in progress \cite{20}. 
2. Supergravity realization of the walls

Consider an $N = 1$ locally supersymmetric theory with one chiral matter superfield $T$. The bosonic part of the $N = 1$ supergravity Lagrangian is

$$e^{-1}L = -\frac{1}{2\kappa} R + K_{TT} g^{\mu\nu} \partial_{\mu} T \partial_{\nu} T - e^{\kappa K} (K^{T\bar{T}} |D_T W|^2 - 3\kappa |W|^2)$$  \hspace{1cm} (2.1)

where $e = |\text{det} g_{\mu\nu}|^{\frac{1}{2}}$, $K(T, \bar{T}) = \text{Kähler potential}$ and $D_T W \equiv e^{-\kappa K} (\partial_T e^{\kappa K} W)$.\[\dagger\]

In order to have stable domain wall solutions, topological arguments imply that the degenerate vacua be disconnected. Thus one must have isolated vacua of the matter potential. However, the inclusion of gravity will turn out to play an important role in removing the constraint that the isolated minima of the matter potential have to be degenerate. We shall see that with the inclusion of gravitational energy, the notion of degenerate vacua will be defined as supersymmetry preserving vacua just as in globally supersymmetric theories. Indeed, formal arguments for the stability follow from the existence of local supersymmetry charges which satisfy an algebra which is a generalization of the global algebra. Thus, the inclusion of gravity, when the dust settles, merely adds to the technology necessary to formulate the existence and stability criteria of these extended objects. Therefore, in a formal sense, the arguments are analogous to those in the global case\[9,10,18].

Supersymmetry preserving minimum of the potential in (2.1) satisfy $D_T W = 0$. This in turn implies that the supersymmetry preserving vacua have either zero cosmological constant (Minkowski space-time) when $W = 0$, or negative cosmological constant $-3e^{\kappa K} |\kappa W|^2$ (anti-de Sitter space-time) when $W \neq 0$. Note that we define the cosmological constant as follows. The energy momentum tensor when $T$ is at its vacuum value ($D_T W = 0$) is $T_{\mu\nu} = -3\kappa |W e^{\frac{K}{2}}|^2 g_{\mu\nu}$. Therefore, Einstein’s equation $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}$ can be written $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \Lambda g_{\mu\nu}$ with $\Lambda = -3|W e^{\frac{K}{2}}|^2$.

\[\dagger\] We use the conventions: $\gamma^{\mu} = e^{\mu}_{a} \gamma^{a}$ where $\gamma^{a}$ are the flat spacetime Dirac matrices satisfying $\{\gamma^{a}, \gamma^{b}\} = 2\eta^{ab}$; $e_{a} e_{b} = \delta_{a}^{b}$; $a = 0, \ldots, 3$; $\mu = t, x, y, z$; $\psi = \psi^{\dagger} \gamma^{t}$; $(+, -, -, -)$ space-time signature; and write $\kappa = 8\pi G_N$.\[\star\] We do not choose the commonly used Kähler gauge which introduces the potential function $G(T, \bar{T}) = K(T, \bar{T}) + ln|W(T)|^2$, since it is not adequate for situations in which the superpotential is allowed to vanish.
2.1. ADM Mass Density

In the following we obtain a lower bound on the mass density of domain walls living in this theory. In that regard, we employ the results of \[22\] and \[23\] who addressed the positivity of the ADM mass in general relativity, as well as certain generalizations to anti-de Sitter backgrounds \[16\]. We note that the ADM mass for spatially infinite objects is not well-defined \[24\]. However, as a weaker requirement, we will assume that the ADM procedure is valid for the mass per unit area rather than the mass of the domain wall. Indeed, this is the energy which is of interest since the total mass is, by definition, infinite.

Consider the supersymmetry charge density

\[
Q[\epsilon'] = \int_{\partial \Sigma} \bar{\epsilon}' \gamma^{\mu \rho} \psi_{\rho} d\Sigma_{\mu \nu}
\]

(2.2)

where \(\epsilon'\) is a commuting Majorana spinor, \(\psi_{\rho}\) is the spin 3/2 gravitino field, and \(\Sigma\) is a spacelike hypersurface. Taking a supersymmetry variation of \(Q[\epsilon']\) with respect to another commuting Majorana spinor \(\epsilon'\) yields

\[
\delta_{\epsilon} Q[\epsilon'] \equiv \{Q[\epsilon'], \bar{Q}[\epsilon]\}
= \int_{\partial \Sigma} N^{\mu \nu} d\Sigma_{\mu \nu} = 2 \int_{\Sigma} \nabla_{\nu} N^{\mu \nu} d\Sigma_{\mu}
\]

(2.3)

where \(N^{\mu \nu} = \epsilon' \gamma^{\mu \rho} \hat{\nabla}_{\rho} \epsilon\) is a generalized Nester’s form \[23\]. Here \(\hat{\nabla}_{\rho} \epsilon \equiv \delta_{\epsilon} \psi_{\rho} = [2\nabla_{\rho} + ie^{2K}(WP_{R} + WP_{L})\gamma_{\rho} - Im(K_{T} \sigma_{\rho} T)\gamma_{5}] \epsilon\) and \(\nabla_{\rho} \epsilon = (\partial_{\rho} + \frac{1}{2} \omega^{ab}_{\rho} \sigma_{ab}) \epsilon\). In (2.3) the last equality follows from Stoke’s law.

We consider an Ansatz for the space-time metric \(ds^2 = A(z, t)(dt^2 - dz^2) + B(z, t)(-dx^2 - dy^2)\) characteristic of space-times with a domain wall where \(z\) is the coordinate transverse to the wall. However, we do not assume \textit{a priori} that the metric is symmetric about the plane \(z = 0\). Nor do we assume a particular behavior of \(A\) and \(B\) at \(|z| \rightarrow \infty\) except that the asymptotic metric satisfies the vacuum Einstein equations with a zero or negative cosmological constant.

We are concerned with supercharge density and thus insist upon only \(SO(1, 1)\) covariance in the \(z\) and \(t\) directions. This in turn implies that the space-like hypersurface \(\Sigma\) in eq.(2.3) is the \(z\)-axis with measure \(d\Sigma_{\mu} = (d\Sigma_{t}, 0, 0, 0) = |g_{zz}|^{\frac{1}{2}} dz\). The boundary \(\partial \Sigma\) are then the two asymptotic points \(z \rightarrow \pm \infty\). Technical details in obtaining the explicit form of eq.(2.3) are given in reference \[18\] and will be omitted here.
After some algebra, the volume integral yields:

\[ 2 \int_{\Sigma} \nabla_{\mu} N_{\mu\nu} d\Sigma_{\mu} = \int_{-\infty}^{\infty} \left[ -\delta_{\epsilon'} \psi^i \delta_{\epsilon} \psi_j + K_{TT} \delta_{\epsilon'} \chi^i \delta_{\epsilon} \chi \right] dz \]  

(2.4)

where \( \delta_{\epsilon} \psi_i \) and \( \delta_{\epsilon} \chi \) are the supersymmetry variations of the fermionic fields in the bosonic backgrounds. Upon setting \( \epsilon' = \epsilon \) the expression (2.4) is a positive definite quantity which in turn (through eq.(2.3)) yields the bound \( \delta_{\epsilon} Q[\epsilon] \geq 0 \).

Analysis of the surface integral in (2.3) yields two terms: (1) The ADM mass density of the configuration, denoted \( \sigma \) and (2) The topological charge density, denoted \( C \). Positivity of the volume integral translates into the bound

\[ \sigma \geq |C| \]  

(2.5)

which is saturated iff \( \delta_{\epsilon} Q[\epsilon] = 0 \). In this case the bosonic backgrounds are supersymmetric; i.e. they satisfy \( \delta_{\epsilon} \psi = 0 \) and \( \delta_{\epsilon} \chi = 0 \) (see eq.(2.4)). Such configurations saturate the previous bound thus establishing their stability.

2.2. Self-Dual Equations

We now concentrate on solving for the space-time metric and matter field configuration in the supersymmetric case. This calculation involves an analysis of the first order equations \( \delta_{\epsilon} \psi_{\mu} = 0 \) and \( \delta_{\epsilon} \chi = 0 \) which is discussed in \(^{18}\). The self-dual equation for the matter field \( T(z) \) follows from \( \delta_{\epsilon} \chi = 0 \):

$$
\partial_{z} T(z) = i e^{i\theta} \sqrt{Ae^{K}} K^{T\bar{T}} D_{T\bar{T}} W 
$$  

(2.6)

with a constraint on the \( \epsilon \)-spinor:

$$
\epsilon_1 = e^{i\theta} \epsilon^*_2.
$$  

(2.7)

The undetermined phase \( e^{i\theta} \) is in general a space-time coordinate-dependent function.

Since we wish to define the ADM mass per unit area of the domain wall unambiguously, we look for a \textit{time-independent} metric solution. For the walls studied in \(^{13}\), the resulting reflection symmetric metric is time-dependent even though the

\footnote{\textit{\textsuperscript{*} We call these first order differential equations the Bogomol’\'nyi \cite{25} or self-dual equations. Their square gives the classical equations of motion.}}
energy-momentum tensor of the domain wall is time-independent (unless one takes a special value of mass to tension ratio that is not realized by generic field theory examples). With no assumed reflection symmetry of the space-time metric, a priori one cannot say if there exist nontrivial time-independent domain wall solutions. However, in order for our assumption of the time independence of the $T$-field to be consistent with the Bogomol’nyi equation (2.6), the metric component $A$ must be time-independent.

The self-dual equations for the metric components, following from $\delta \psi_z = \delta \psi_{x} = 0$, are
\[
\partial_z A^{-1/2} = \partial_z B^{-1/2} = -\kappa (ie^{-i\theta}) W e^{\frac{K}{2}}. \tag{2.8}
\]
Since the metric functions $A$ and $B$ are real, the phase $e^{i\theta}$ is required to meet a local constraint
\[
W = -i\zeta e^{i\theta} |W| \tag{2.9}
\]
where $\zeta = \pm$. Assuming continuity, $\zeta = \pm$ can change only at points where $W$ vanishes. This connection between the metric and matter superpotential restricts the possible $W$ admitting walls in the local theory. This result is in contrast to the global case in which all $W$ with degenerate vacua admit wall solutions. We comment on this result later.

$\delta \psi_z = 0$ yields the differential equation for the $z$ dependent phase $\theta$:
\[
\partial_z \theta = -Im(K_T \partial_z T). \tag{2.10}
\]
Consistency of (2.6), (2.8) and (2.10) with (2.9) leads to the following sufficient conditions for the existence of a static supersymmetric domain wall:
\[
Im(\partial_z T \frac{D_TW}{W}) = 0, \tag{2.11}
\]
\[
\partial_z T(z) = -\zeta \sqrt{A} |W| e^{\frac{K}{2}} K^{TT} \frac{D_TW}{W}, \tag{2.12}
\]
\[
\partial_z A^{-1/2} = \partial_z B^{-1/2} = \kappa \zeta \sqrt{A} |W| e^{\frac{K}{2}}, \tag{2.13}
\]
as well as the explicit expression for the ADM mass density (energy per area or surface tension) of the supersymmetric domain wall configuration
\[
\sigma = |C| \equiv 2 |(\zeta |W e^{\frac{K}{2}}|)_{z=+\infty} - (\zeta |W e^{\frac{K}{2}}|)_{z=-\infty}| \equiv \frac{2}{\sqrt{3}} \kappa^{-1} |\Delta (\zeta |A|^{1/2})| \tag{2.14}
\]
where $\Lambda \equiv -3 |\kappa We^{\frac{K}{2}}|^2$ is the cosmological constant for the supersymmetric vacuum.
Figure 1: The path in superpotential space traversed as the scalar field interpolates between degenerate vacua. The wall is realized in both the global and local theories for path (A) and just for the global theory in path (B).

We now comment on these equations.

It follows from (2.14) that there are no static domain walls saturating the Bogomol’nyi bound that interpolate between two supersymmetric vacua with zero cosmological constant. In this case $W(+\infty) = W(-\infty) = 0$ and thus there is no energy associated with such a domain wall since $|C| \equiv 0$. This result is in agreement with the results of reference [13], where for infinitesimally thin domain walls with asymptotically Minkowski space-times only time-dependent metric solutions were obtained. The result from (2.14) implies that static supersymmetric domain wall solutions exist only if at least one of the vacua is AdS.

Eq. (2.11) is a consistency constraint which specifies the geodesic path between two supersymmetric vacua in the supergravity potential space $e^{\frac{i \kappa}{2}} W \in \mathbb{C}$ when mapped from the $z$-axis $(-\infty, +\infty)$. This geodesic equation has qualitatively new features in comparison with the geodesic equation in the global supersymmetric case [9,10,18]. While in the global case geodesics are arbitrary straight lines in the $W-$plane, the local geodesic equation in the limit $\kappa \to 0$ (global limit of the local supersymmetric theory) leads to the geodesic equation $\text{Im}(s_i W) \equiv \partial_z \vartheta = 0$ where $W$ has been written as $W(z) = |W| e^{i\vartheta}$. This in turn implies that as $\kappa \to 0$ the local geodesic equation reduces to the constraint that $W$ has to be a straight line passing through the origin; i.e. the phase of $W$ has to be constant mod $\pi$. Figure 1 illustrates these points. This observation in turn implies that the introduction of gravity imposes a strong constraint on the type of domain wall solutions realized. In particular, domain wall solutions in the global case interpolating between vacua in the $e^{\frac{i \kappa}{2}} W$ plane that do not lie along a straight line passing through the origin do not have an analogous solution in the local case. This result is a manifestation of the singular nature of a perturbation in Newton’s constant as seen in (2.13).

Another way to understand the inability of all global walls to be realized in the local theory is that the space-time metric introduces an extra field degree of freedom to the local theory which allows for an extra direction to connect previously disconnected vacua.

Eq. (2.12) for the $T$ field (the “square root” of the equation of motion for $T$) and eq. (2.13) for the metric (the “square root” of the Einstein’s equation) are invariant under $z$ translation as well as under rescalings of $(A, B) \to \lambda^2 (A, B)$ and $z \to \lambda^{-1} z$. Additionally, eq. (2.12) implies that $\partial_z T(z) \to 0$ as one approaches the supersymmetric minima which are points where $D_T W = 0$, thus indicating
Figure 2: The projection of a typical scalar potential on a particular complex $T$ direction indicating the supersymmetric minima ($D_W T = 0$) between which the domain wall interpolates. These minima are in general not degenerate. However, since they are supersymmetric, when gravitational energy is included they become energetically the same thus allowing for domain wall solutions.

a solution smoothly interpolating between supersymmetric vacua. In general, the field $T$ reaches the supersymmetric minimum exponentially fast as a function of $z$.

3. Classification of the Walls

We now concentrate the equation (2.13) for the metric. Our aim is to classify all the qualitatively different metric configurations. First, we set $A(z) = B(z)$ without loss of generality which implies that the metric is conformally flat. Also, we emphasize in (2.13) the singular limit when gravity is turned off ($\kappa \to 0$). As noted earlier, the same singular limit ($\kappa \to 0$) is also responsible for the restrictive geodesics in the $W$-plane compared to a global theory which contains no gravitational information ($\kappa = 0$). For $\kappa = 0$, the conformal factor factor $A$ is constant in the whole space; i.e. we have flat space-time everywhere. However, the moment $\kappa > 0$, $A$ varies with $z$. Thus, our aim is to study the nature of the conformal factor $A(z)$. We classify three types of static domain wall configurations which depend on the nature of the potential of the matter field. For illustrative purposes to indicate the nature of the minima between which a wall interpolates, we sketch a typical scalar potential in Figure 2. The non-degeneracy, as emphasized throughout the paper, is deceptive since degeneracy is based on both gravity and matter energy; the scalar potential only involves the matter part.

(I) A wall interpolating between a supersymmetric AdS vacuum ($|W_{+\infty}| \neq 0$) and a Minkowski supersymmetric vacuum ($|W_{-\infty}| = 0$). From (2.13) one sees that on the Minkowski side the conformal factor approaches a constant which can be normalized to unity; i.e.

$$A(z) \to 1, \quad z \to -\infty.$$  \hspace{1cm} (3.1)

On the AdS side $A(z)$ falls off as $z^{-2}$ with the strength of the fall-off determined by the strength of the cosmological constant; i.e.

$$A(z) \to \frac{3}{|\Lambda_{+\infty}| z^2}, \quad z \to +\infty.$$  \hspace{1cm} (3.2)

The surface energy of this configuration as determined from (2.14) is

$$\sigma_I = \frac{2}{\sqrt{3}} \kappa^{-1}|\Lambda_{+\infty}|^{1/2}.$$  \hspace{1cm} (3.3)
Here, the cosmological constant of the supersymmetric AdS vacuum is $\Lambda_{+\infty} = -3|\kappa We^{\frac{dW}{d\phi}}|_{+\infty}^2$.

\textit{(II)} A wall interpolating between two supersymmetric AdS vacua and where the superpotential passes through zero in between. The cosmological constant need not be the same in both vacua. The point where $W = 0$ can be chosen at $z = 0$ without loss of generality due to the translational symmetry of the system. At this point $\zeta$ changes sign and thus $\zeta_{+\infty} = -\zeta_{-\infty} = 1$. The conformal factor has the same asymptotic behaviour on both sides of the domain wall:

$$A(z) \rightarrow \frac{3}{|\Lambda_{\pm\infty}|z^2}, \ z \rightarrow \pm\infty$$

while at $z = 0$, i.e. when $W = 0$, the conformal factor levels out, i.e. $\partial_z A(z)_{z=0} = 0$. In other words $A(z)$ has a characteristic (in general asymmetric) bell-like shape.

The surface energy of this configuration is

$$\sigma_{II} = \frac{2}{\sqrt{3}}\kappa^{-1}(|\Lambda|_{-\infty}^{1/2} + |\Lambda|_{+\infty}^{1/2})$$

\textit{(III)} A wall interpolating between two AdS vacua, while the superpotential does not pass through zero. Again, the cosmological constant need not be the same in both vacua. In this case, since $|W|$ is never zero, $\zeta$ has the same sign in the whole region, say, $+1$. Eq.(2.13) in turn implies that the conformal factor necessarily blows up at some coordinate $z^*$. In general, the matter field $T$ has long since interpolated between the two vacua by the time the metric reaches $z^*$. Thus, the domain wall, defined as the region over which $T$ moves from one vacuum to another, lies entirely within the coordinate region $z^* < z < +\infty$. The conformal factor has the asymptotic behaviour:

$$A(z) \rightarrow \frac{3}{|\Lambda_{+\infty}|z^2}, \ z \rightarrow +\infty$$

$$A(z) \rightarrow \frac{3}{|\Lambda_{z^*}|(z - z^*)^2}, \ z \rightarrow z^*.$$  

The surface energy of this configuration is

$$\sigma_{III} = \frac{2}{\sqrt{3}}\kappa^{-1}|\Lambda|_{z^*}^{1/2} - |\Lambda|_{+\infty}^{1/2}|.$$  

Note that the point $z^*$ is an infinite proper spatial distance away from any other point $z > z^*$ since $\int dz A^{1/2} \rightarrow \ln|z - z^*|$. 

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In order to understand this singularity as well as the distinctive $z^{-2}$ behaviour of the conformal factor on the AdS side of a wall, it is appropriate at this point to study AdS space-time in a coordinate system which singles out the $z$ direction. For this purpose, we consider the metric

$$ds^2 = (\alpha z)^{-2}(dt^2 - dx^2 - dy^2 - dz^2)$$

with $z > 0$. As noted above, this is the form of the metric on the AdS side of the domain wall when the $T$ field has reached its supersymmetric vacuum. In this context, $\alpha$ is related to the cosmological constant by $\Lambda = -3\alpha^2$.

Eq. (3.8) is the form for the metric describing AdS space-time where the translational invariance is broken in the $z$ direction. The curvature tensor, by definition of a maximally symmetric space-time, satisfies $R_{\mu\nu\sigma\rho} = \alpha^2(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma})$. One can represent four dimensional AdS space-time as the hyperboloid $\eta_{AB}Y^A Y^B = \alpha^{-2}$ embedded in the five dimensional space with flat metric $\eta^{AB} = \text{diag}(+-\cdots+)$. We found that the following choice of coordinates

$$Y^0 = te^{\alpha \tilde{z}}, \quad Y^1 = xe^{\alpha \tilde{z}}, \quad Y^2 = ye^{\alpha \tilde{z}}$$

$$Y^3 = (\alpha)^{-1} \sinh(\alpha \tilde{z}) - \frac{1}{2}\alpha e^{\alpha \tilde{z}}(x^2 + y^2 - t^2)$$

$$Y^4 = (\alpha)^{-1} \cosh(\alpha \tilde{z}) + \frac{1}{2}\alpha e^{\alpha \tilde{z}}(x^2 + y^2 - t^2)$$

yield the metric intrinsic to the surface

$$ds^2 = e^{2\alpha \tilde{z}}(dt^2 - dx^2 - dy^2) - d\tilde{z}^2.$$ 

(3.10)

This choice of intrinsic coordinates is motivated from the cosmological form for the metric in de Sitter space (see, for example [26]). By choosing $z = \alpha^{-1}e^{-\alpha \tilde{z}}$ we recover the form of the metric in (3.8).

These coordinates cover one-half of the AdS manifold since $Y^3 + Y^4 > 0$. By choosing $(Y^3, Y^4, z) \to (-Y^3, -Y^4, -z)$, we cover the $Y^3 + Y^4 < 0$ region and have the metric (3.8) for $z < 0$. This choice should be contrasted with the standard set of coordinates respecting spherical symmetry about an origin which completely covers AdS space-time [27]. In this case the metric has the form

$$ds^2 = (\alpha \cos \rho)^{-2}(dt_c^2 - d\rho^2 - \sin^2 \rho (d\theta^2 + \sin^2 \theta d\phi^2))$$

(3.11)

with $0 \leq \rho < \pi/2$, $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$, and $-\pi \leq t_c \leq \pi$. 

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The time-like coordinate \( t_c \) in (3.11) is a periodic coordinate. However, the coordinates (3.9), in which time ranges over \(-\infty < t < \infty\), exhibit no periodic structure. What we have effectively done in choosing the planar coordinates (3.9) is to sacrifice a complete covering of AdS for a non-periodic time-like variable. The coordinates (3.8) are extendible whereas those of (3.11) are not.

The previous discussion of the metric (3.8) now allows for a straightforward interpretation of the singular wall (type III) configuration. What we have is a domain wall separating two distinct regions of a generalized AdS space-time possessing a \( z \) dependent cosmological parameter which never passes through zero. The singular point \( z^* \) corresponds to the origin \( z = 0 \) in the metric (3.8). On the “other side” of \( z^* \) lives an AdS space-time symmetric to the \( z > z^* \) side. Together these two sides completely cover the whole of the generalized AdS space-time just as the regions \( z > 0 \) and \( z < 0 \) in the planar coordinates leading to (3.8) cover all of AdS.

4. Examples

The above discussion of the three types of domain wallsœfoot In Ref. œref-markœ the stringy examples based on the \( SL(2, \mathbb{C}) \) duality symmetry of the string theory is also discussed. is illustrated by a simple polynomial form for the superpotential, a flat Kähler manifold: \( K = T\bar{T} \), and a real \( T \). We choose the superpotential

\[
W = \gamma T\left[\frac{1}{5}T^4 - \frac{1}{3}T^2(a^2 + b^2) + a^2b^2\right].
\]

(4.1)

where \( \gamma \) is a mass dimension \(-2\) parameter which we set to unity and \( a^2 \) and \( b^2 \) are positive dimension 2 parameters. Depending on the value of the parameters \( a \) and \( b \), the superpotential (4.1) provides us with a set of theories which accommodate the above three classes of the domain walls.

Note that the geodesic constraint \( Im(\partial_z T \frac{D_T W}{W}) = 0 \) is always satisfied for \( T = \bar{T} \). The supersymmetric vacuum satisfies \( D_T W \equiv W_T + \kappa K_T W = 0 \), where \( W_T = (T^2 - a^2)(T^2 - b^2) \). Thus, for \( a, b << 1/\sqrt{\kappa} \), the supersymmetric vacua take place for real values of \( T \) near \( \pm a, \pm b \). Figures 3, 4 and 5 display the conformal factor \( A \) for the these three classes of the domain walls. Each example corresponds to a different choice of the parameters \( a \) and \( b \), which we took for simplicity to be in the range \( << 1/\sqrt{\kappa} \).
Figure 3: Type (I) conformal factor $A(z)$ for a space-time with $\Lambda_{-\infty} = 0$ (Minkowski: $z < 0$) separated by a domain wall from a space-time with $\Lambda_{+\infty} < 0$ (AdS: $z > 0$). The wall, i.e. the region over which the matter field $T$ changes is centered at $z = 0$ and has thickness $\approx 200$ in $\sqrt{\kappa}$ units. The superpotential (4.1) has parameters $a^2 = 0, b^2 = 0.1$ and $T$ interpolates between $T_{-\infty} = 0 = a$ and $T_{\infty} = .318 \approx b$.

Figure 4: Type (II) Conformal factor $A(z)$ for a space-time with negative cosmological constant separated by a domain wall from its mirror image (i.e. a $Z_2$ configuration). The wall is centered at $z = 0$ and has thickness $\approx 200$ in $\sqrt{\kappa}$ units. The superpotential (4.1) has parameters $a^2 = .025, b^2 = 0.1$. and $T$ interpolates between $T_{\mp\infty} = \pm 1.1598 \approx \pm a$.

Figure 5: Type (III) conformal factor $A(z)$ for a space with negative cosmological constant separated by a domain wall from a space with a different negative cosmological constant. The superpotential $W$ never passes through a zero as $T$ interpolates from one vacuum to another. The domain wall is centered at $z = 0$ and has thickness $\approx 200$ where $z$ is measured in $\sqrt{\kappa}$ units. The singularity is at $z^* \approx -5600$. The superpotential (4.1) has parameters $a^2 = .025, b^2 = 0.1$ and $T$ interpolates between $T_{-\infty} = .315 \approx b$ and $T_{\infty} = .160 \approx a$.

5. Geodesic Structure

We now turn to the study of the geodesic structure for the induced space-time. To do so, we analyze the motion of test particles in the background of a supersymmetric domain wall.

The motion of massless particles is trivial since the metric is conformally flat; they simply define the usual 45° null rays in a space-time diagram. Particles moving in constant $z$ planes will feel no force since the conformal factor is only a function of the transverse coordinate $z$. In other words, the metric is invariant under $x, y$ boosts and thus without loss we can move to an inertial frame in which there is no motion in these directions. Therefore, the only interesting geodesics will come from the $1+1$ metric $ds^2 = A(z)(dt^2 - dz^2)$. For massive particles, which live on time-like geodesics, we can parametrize the motion with the proper-time element $ds^2 = d\tau^2 > 0$. Rearranging the metric and introducing the conserved energy per mass $\epsilon \equiv A\frac{dt}{d\tau}$ of the particle yields the equation for the world-line

$$\left(\frac{dz}{dt}\right)^2 + \frac{A}{\epsilon^2} = 1.$$ (5.1)

On a time-like geodesic, $0 \leq (dz/dt)^2 < 1$, and so the turning point, i.e. $v \equiv dz/dt = 0$, of the motion is where $A/\epsilon^2 = 1$. 

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A convenient way to understand massive particle motion is to consider a particle with a given initial coordinate velocity \( v_0 \) at some coordinate \( z_0 \); from (5.1) \( \epsilon \) for such a particle is

\[
\epsilon^2 = A(z_0)(1-v_0^2)^{-1}.
\]

Equation (5.1) can be thought of as the conservation of energy with an effective potential \( V(z) \equiv (1 - v^2) = A(z)/(1 - v_0^2) \). Again, points where \( V(z) = 1 \) are turning points.

For particles incident upon the type I wall from the Minkowski side, passage through to the AdS side is always allowed. However, the reverse motion requires the initial velocity to satisfy \( v^2 > 1 - A(z_0) \); otherwise there is a turning point and the particle returns to the AdS side. Motion in the other domain wall backgrounds is analogous: a sketch of the effective potential \( V(z) = A(z)(1 - v_0^2)/A(z_0) \) makes the motion clear.

One can understand the repulsive nature of these space-times on the AdS side by calculating the force on a test particle which has a fixed position \( z \) (also known as a fiducial observer). This force can be obtained through the geodesic equation \( p_\alpha p_\beta = m f^\beta \) with \( p^\alpha = m \frac{dx^\alpha}{d\tau} \). The gravitational force acting on the fiducial observer is

\[
f^\beta = (0, 0, 0, -\frac{m}{2} A^{-2} \partial_z A).
\]

(5.2)

For a metric which falls off as \( \Lambda_\pm \) on the AdS side of a wall, this force is directed towards the AdS vacuum (e.g. \( z = +\infty \) in the type I wall depicted in figure 3). The magnitude of the acceleration is given by

\[
|a|^2 \equiv |f^a f_\alpha|/m^2 = \left(\frac{1}{2} \frac{\partial_z m A}{A^{1/2}}\right)^2 = (\kappa \Lambda_\pm /3)^2.
\]

(5.3)

For fiducial observers in the region where \( T \) is essentially at its vacuum value; i.e. far away from the wall, the proper acceleration has the constant magnitude \( |a|^2 = A_{\pm \infty}/3 |. \) In this region, integration of (5.1) yields the hyperbolic world line for freely falling test particles \( z^2 - t^2 = (A_{\pm \infty}/3)^{-1} \). Therefore, a fiducial observer situated far away from a type I or II wall in a \( A_{\pm \infty} \neq 0 \) region will feel a constant acceleration \( |A_{\pm \infty}/3|^{1/2} \) directed away from the wall as well as see freely falling test particles moving away from the wall with the a hyperbolic world line. Such a world line is also exhibited by a particle with a constant proper acceleration moving in Minkowski space-time. These particles, known as Rindler particles\(^{[17]} \), are not freely falling in the Minkowski background; their acceleration is provided by an external non-gravitational force. However, we see that AdS provides precisely this force due to the non-trivial curvature of the space-time. We add that on the Minkowski side of the walls, free test particles experience no gravitational force even though there is an infinite object nearby.
One can understand the no-force result for the particles living on the Minkowski side of the walls through the formalism of singular hypersurfaces\textsuperscript{[28]}. A straightforward calculation\textsuperscript{*} yields a negative effective gravitational mass/area due to the wall whereas AdS has exactly the opposite positive gravitational mass. Thus the observer on the Minkowski side of the wall does not feel any gravitational force.

This above result should be contrasted with the observation in Ref.\textsuperscript{[13]}, where infinitesimally thin reflection symmetric domain walls with asymptotically Minkowski space-times always repel the fiducial observer with a constant acceleration $\kappa\sigma/4$. Here, $\sigma$ is the energy per unit area of the domain wall.\textsuperscript{†} Recall that these domain walls always produce a time dependent metric. In our case everything is static. In particular, for the type (I) domain walls interpolating between AdS and Minkowski space-times, the asymptotic acceleration on the AdS side can be written as $a = \kappa\sigma_I/2$, where $\sigma_I$ is the energy per unit area of the domain wall (I) defined in (3.3). On the Minkowski side $a = 0$. For the type II domain wall when the potential has $\mathbb{Z}_2$ symmetry, the energy per unit area (3.5) is $\sigma_{II} = 4\kappa^{-1}|\Lambda_{\pm\infty}/3|^{1/2}$ and the fiducial observer is repelled on both sides of the domain wall with the same acceleration $a_{\pm\infty} \rightarrow \kappa\sigma_{II}/4$ which resembles remarkably the form for the acceleration for the domain walls discussed in Ref.\textsuperscript{[13]}. In our case the domain wall also respects the $\mathbb{Z}_2$ symmetry, however, it is completely static and its repulsive nature is due to the AdS nature of the asymptotic space-time.

6. Anti-de Sitter–Minkowski walls

In many ways, the walls separating flat Minkowski space-time from AdS are the most interesting. For example, it is known that AdS has the topology $S^1\times\mathbb{R}^3$ and thus has closed time-like curves. The common remedy for such loss of causality is to unwrap the time direction and work on the covering space CAdS. Nevertheless, this does not allow AdS to be globally hyperbolic; i.e. it has no Cauchy hypersurface and thus boundary conditions must be imposed at spatial infinity in order to properly specify the Cauchy problem\textsuperscript{[31,32]}. In the juxtaposition of AdS and Minkowski space-times, one must consider a proper formulation of the Cauchy problem in order to quantize a field living on this manifold. The problems of AdS in some ways are softened by the presence of the Minkowski side, yet one unfortunately cannot erase the problems associated with a lack of a Cauchy surface.

\textsuperscript{*} See\textsuperscript{[29]} for a nice example of this formalism applied to a planar geometry. In addition,\textsuperscript{[13]} employed these ideas in solving for the space-time around their domain walls.

\textsuperscript{†} Domain walls which separate two Minkowski vacua yet satisfy the nonstandard relation $\sigma = 2\tau$, where $\tau$ is the surface tension of the wall, produce no gravitational force on test particles. Walls of isotropically and uniformly distributed cosmic strings produce such an equation of state\textsuperscript{[30]}. 

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Figure 6: Penrose diagram for the covering space of the extended Minkowski-AdS domain wall system. The regions $M$ and $A$ are the Minkowski and AdS sides of the wall. The vertical line is the time-like line of the domain wall. The nulls separating AdS patches are sights of instabilities. The point $\Omega$ is on one such null. At this point, the observer experiences the complete history of his/her preceeding Minkowski region.

As an indication of the interesting causal structure obtained through the juxtaposition of AdS and Minkowski space-times, consider the $1+1$ Minkowski-AdS$_2$ wall and place an observer on the Minkowski side. Now allow the observer to send a moving mirror through the wall on a geodesic. The mirror will travel on a hyperbolic trajectory as it falls into the AdS space-time. If the Minkowski observer sends massless radiation at the mirror, s/he will receive more reflected radiation out than sent in due to the coupling of the mirror to the curved space-time and the resulting particle creation$^{[26]}$. In this way the Minkowski observer could deduce the structure of the space-time on the other side of the wall. In addition, it is known that the energy radiated from the mirror on a hyperbolic trajectory is zero$^{[33]}$ and thus the stability of the wall is not compromised.

Note that the moving mirror will reach the end of the coordinates $z,t$ within a finite proper time (i.e. these coordinates must be extended in order to cover AdS). However, the Minkowski observer considers $t$ his/her proper time. Such behaviour is true in the full $3+1$ case as well. Therefore, to construct the causal structure of the domain wall system, we must extend the coordinates on the AdS side, which means an extension onto a new half of AdS is necessary. However, by allowing for more AdS, we have added more effective gravitational mass to the system which must be cancelled by another identical wall. On the other side of the wall there is another Minkowski space. Moving to the covering space to avoid the closed time-like curves inherited from AdS gives us an infinite tower of 2-wall systems. The Penrose diagram for this configuration is shown in figure 6. Of particular interest is the null separating the AdS patches. It turns out that this surface corresponds to the Cauchy horizon$^{[20]}$ with the metric closely related to the one at the extremal Reissner Nordstöm (RN) black-hole horizon. Note the similarity of the Penrose diagram for our domain wall and the one of the extremal (RN) black hole; the only major difference is that in our case the singularity of the RN black-hole inside the Cauchy horizon is replaced by an identical wall. Space-time induced by such domain walls thus serves as an example of asymptotical Minkowski space-time with Cauchy horizon but without any singularity. Further study of the phenomena associated with the Minkowski-AdS wall is underway$^{[26]}$. 

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7. Summary

We studied the field theoretic realization of a new type of domain wall. These walls separate two maximally symmetric space-times of non-positive cosmological constant where one or both is AdS and the other can be Minkowski. These walls are found in $N = 1, d = 4$ supergravity and saturate a positive energy/area theorem thus providing stability to the classical configuration. We classified three canonical systems differing by the path in superpotential space traced out as the scalar field interpolated from one vacuum to the other. Equivalently, the form of the conformal factor on the conformally flat metric characterizes the three walls. Examples were given illustrating the three walls as realized by a particular superpotential. And the motion of test particles living in the background space-time induced by the walls was discussed. Finally, we pointed out some interesting behaviour in regard to the space-times induced from the Minkowski-AdS walls.

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Figure Captions

Figure 1: The path in superpotential space traversed as the scalar field interpolates between degenerate vacua. The wall is realized in both the global and local theories for path (A) and just for the global theory in path (B).

Figure 2: The projection of a typical scalar potential on a particular complex $T$ direction indicating the supersymmetric minima ($D_W T = 0$) between which the domain wall interpolates. These minima are in general not degenerate. However, since they are supersymmetric, when gravitational energy is included they become energetically the same thus allowing for domain wall solutions.

Figure 3: Type (I) conformal factor $A(z)$ for a space-time with $\Lambda_{-\infty} = 0$ (Minkowski: $z < 0$) separated by a domain wall from a space-time with $\Lambda_{+\infty} < 0$ (AdS: $z > 0$). The wall, i.e. the region over which the matter field $T$ changes is centered at $z = 0$ and has thickness $\approx 200$ in $\sqrt{\kappa}$ units. The superpotential (4.1) has parameters $a^2 = 0, b^2 = 0.1$ and $T$ interpolates between $T_{-\infty} = 0 = a$ and $T_{\infty} = .318 \approx b$.

Figure 4: Type (II) Conformal factor $A(z)$ for a space-time with negative cosmological constant separated by a domain wall from its mirror image (i.e. a $Z_2$ configuration). The wall is centered at $z = 0$ and has thickness $\approx 200$ in $\sqrt{\kappa}$ units. The superpotential (4.1) has parameters $a^2 = .025, b^2 = 0.1$. and $T$ interpolates between $T_{\pm\infty} = \pm .1598 \approx \pm a$.

Figure 5: Type (III) conformal factor $A(z)$ for a space with negative cosmological constant separated by a domain wall from a space with a different negative cosmological constant. The superpotential $W$ never passes through a zero as $T$ interpolates from one vacuum to another. The domain wall is centered at $z = 0$ and has thickness $\approx 200$ where $z$ is measured in $\sqrt{\kappa}$ units. The singularity is at $z^* \approx -5600$. The superpotential (4.1) has parameters $a^2 = .025, b^2 = 0.1$ and $T$ interpolates between $T_{-\infty} = .315 \approx b$ and $T_{\infty} = .160 \approx a$.

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