The Algebraic and Geometric Classification of Nilpotent Bicommutative Algebras

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Abstract
We classify the complex 4-dimensional nilpotent bicommutative algebras from both algebraic and geometric approaches.

Keywords Bicommutative algebras · Nilpotent algebras · Algebraic classification · Central extension · Geometric classification · Degeneration

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1 Introduction

One of the classical problems in the theory of non-associative algebras is to classify (up to isomorphism) the algebras of dimension \(n\) from a certain variety defined by some family of polynomial identities. It is typical to focus on small dimensions, and there are two main directions for the classification: algebraic and geometric. Varieties as Jordan, Lie, Leibniz or Zinbiel algebras have been studied from these two approaches ([1, 11, 15–18, 26, 27, 31, 34, 45] and [3, 5, 8, 11, 13, 14, 25, 26, 28–30, 35–38, 41–48], respectively). In the present paper, we give the algebraic and geometric classification of 4-dimensional nilpotent bicommutative algebras.
The variety of bicommutative algebras is defined by the following identities of right- and left-commutativity:

\[(xy)z = (xz)y, \quad x(yz) = y(xz).\]

It contains the commutative associative algebras as a subvariety. One-sided commutative algebras first appeared in the paper by Cayley [9] in 1857. The structure of the free bicommutative algebra of countable rank and its main numerical invariants were described by Dzhumadildaev, Ismailov and Tulenbaev [23], see also the announcement [22]. Bicommutative algebras were also studied in [20, 21, 24], and in [6, 7] under the name of LR-algebras.

The key step in our method for algebraically classifying bicommutative nilpotent algebras is the calculation of central extensions of smaller algebras. In short, an algebra \(A\) is called an extension of another algebra \(B\) by \(K\) if there exists a short exact sequence \(0 \rightarrow K \rightarrow A \rightarrow B \rightarrow 0\). The simplest example is the direct sum \(B \oplus K\) with the inclusion and the projection. Imposing additional conditions, we find important special types of extensions such as the split extensions, the HNN-extensions and many others. In this paper, we will deal with central extensions, i.e. extensions in which the annihilator of \(A\) contains \(K\). Some important algebras can be constructed as central extensions; for example, the Virasoro algebra is the universal central extension of the Witt algebra, and the Heisenberg algebra is a central extension of a commutative Lie algebra.

The theory of central extensions also plays an important role in physics, especially in different areas of quantum theory as quantum mechanics (Wigner’s theorem) or quantum field theory (conformal field theory, string theory, \(M\)-theory). Roughly speaking, central extensions are needed in physics because the symmetry group of a quantized system is usually a central extension of the classical symmetry group, and in the same way the corresponding symmetry Lie algebra of the quantum system is, in general, a central extension of the classical symmetry algebra. In particular, affine Kac–Moody algebras (the universal central extensions of loop algebras) have been conjectured to be the symmetry Lie algebras of a unified superstring theory.

With this background, it comes as no surprise that the central extensions of Lie and non-Lie algebras have been exhaustively studied for years. It is interesting both to describe them and to use them to classify different varieties of algebras [2, 32, 33, 40, 49, 50]. Firstly, Skjelbred and Sund devised a method for classifying nilpotent Lie algebras employing central extensions [49]. Afterwards, using this method, all non-Lie central extensions of the 4-dimensional Malcev algebras were described [33], and also all non-associative central extensions of the 3-dimensional Jordan algebras [32], all anticommutative central extensions of the 3-dimensional anticommutative algebras [10], and all central extensions of the 2-dimensional algebras [12]. Moreover, the method is especially indicated for the classification of nilpotent algebras and it was used to describe all 4-dimensional nilpotent associative algebras [18], all 4-dimensional nilpotent Novikov algebras [39], all 5-dimensional nilpotent Jordan algebras [31], all 5-dimensional nilpotent restricted Lie algebras [16], all 6-dimensional nilpotent Lie algebras [15, 17], all 6-dimensional nilpotent Malcev algebras [34] and some others.

On the other hand, the geometric classification of the algebras of fixed dimension from a certain variety defined by some family of polynomials \(T\) consists of finding the irreducible components of the variety \(\mathbb{L}(T)\) formed by the algebra structures on \(V\) satisfying the identities of \(T\), which has a structure of affine variety with the Zariski topology. In particular, it is interesting to describe the so-called rigid algebras, since the closures of their orbits under
the action of the generalized linear group form irreducible components. For example, the rigid algebras in the varieties of all 2-dimensional algebras [45], all 3-dimensional nilpotent algebras [25], all 3- and 4-dimensional Lie algebras [8], all 5- and 6-dimensional nilpotent Lie algebras [29, 30, 48], all 4-dimensional Leibniz algebras [36], all 4-dimensional Malcev algebras [44], all 2- and 3-dimensional Novikov algebras [4, 5], all 4-dimensional nilpotent Novikov algebras [39], all 4-dimensional nilpotent associative algebras [35], all 6-dimensional nilpotent binary Lie algebras [1], all 2-dimensional terminal algebras [11], all \((n + 1)\)-dimensional Filippov or \(n\)-Lie algebras [47], and many others have been classified.

The paper is organized as follows. Section 1 is devoted to the algebraic classification of 4-dimensional nilpotent bicommutative algebras: after a description of the analogue of the Skjelbred-Sund method for bicommutative algebras, we compute the second cohomology space of 3-dimensional nilpotent bicommutative algebras, and then their 1-dimensional central extensions. This leads to the classification of all pure 4-dimensional nilpotent bicommutative algebras, which is explicitly stated in Table 1 (Appendix A). After that, in Section 2 we give a short introduction to the geometric classification of algebras, and we also explain the methods employed in this paper. Finally, Subsection 3.3 together with Table 2 (Appendix A) presents the main result in the paper: the geometric classification of 4-dimensional nilpotent bicommutative algebras.

2 The Algebraic Classification of Nilpotent Bicommutative Algebras

2.1 Method of Classification of Nilpotent Algebras

The objective of this section is to give an analogue of the Skjelbred-Sund method for classifying nilpotent bicommutative algebras. As other analogues of this method were carefully explained in, for example, [12, 32, 33], we will give only some important definitions, and refer the interested reader to the previous sources.

Let \((A, \cdot)\) be a bicommutative algebra of dimension \(n\) over \(\mathbb{C}\) and \(V\) a vector space of dimension \(s\) over \(\mathbb{C}\). We define the \(\mathbb{C}\)-linear space \(Z^2(A, V)\) as the set of all bilinear maps \(\theta: A \times A \rightarrow V\) such that \(\theta(xy, z) = \theta(xz, y), \theta(x, yz) = \theta(y, xz)\).

These maps will be called cocycles. Consider a linear map \(f\) from \(A\) to \(V\), and set \(\delta f: A \times A \rightarrow V\) with \(\delta f(x, y) = f(xy)\). Then, \(\delta f\) is a cocycle, and we define \(B^2(A, V) = \{\theta = \delta f : f \in \text{Hom}(A, V)\}\), which is a linear subspace of \(Z^2(A, V)\). Its elements are called coboundaries. The second cohomology space \(H^2(A, V)\) is defined to be the quotient space \(Z^2(A, V) / B^2(A, V)\).

Let \(\text{Aut}(A)\) be the automorphism group of the bicommutative algebra \(A\) and let \(\phi \in \text{Aut}(A)\). Every \(\theta \in Z^2(A, V)\) defines \(\phi \theta(x, y) = \theta(\phi(x), \phi(y))\), with \(\phi \theta \in Z^2(A, V)\). It is easily checked that \(\text{Aut}(A)\) acts on the right on \(Z^2(A, V)\), and that \(B^2(A, V)\) is invariant under the action of \(\text{Aut}(A)\). So, we have that \(\text{Aut}(A)\) acts on \(H^2(A, V)\).

Let \(\theta\) be a cocycle, and consider the direct sum \(A_\theta = A \oplus V\) with the bilinear product \("[-, -]_\theta\" defined by \([x + x', y + y']_\theta = xy + \theta(x, y)\) for all \(x, y \in A, x', y' \in V\). It is straightforward that \(A_\theta\) is a bicommutative algebra if and only if \(\theta \in Z^2(A, V)\); it is then a \(s\)-dimensional central extension of \(A\) by \(V\).
We also call the set \( \text{Ann}(\theta) = \{ x \in A : \theta(x, A) + \theta(A, x) = 0 \} \) the annihilator of \( \theta \). We recall that the annihilator of an algebra \( A \) is defined as the ideal \( \text{Ann}(A) = \{ x \in A : xA + Ax = 0 \} \). Observe that \( \text{Ann}(A_{\phi}) = (\text{Ann}(\theta) \cap \text{Ann}(A)) \oplus V \).

**Definition 1** Let \( A \) be an algebra and \( I \) be a subspace of \( \text{Ann}(A) \). If \( A = A_0 \oplus I \) then \( I \) is called an annihilator component of \( A \).

**Definition 2** A central extension of an algebra \( A \) without annihilator component is called a non-split central extension.

The following result is fundamental for the classification method.

**Lemma 3** Let \( A \) be an \( n \)-dimensional bicommutative algebra such that \( \dim(\text{Ann}(A)) = s \neq 0 \). Then there exists, up to isomorphism, a unique \((n - s)\)-dimensional bicommutative algebra \( A' \) and a bilinear map \( \theta \in Z^2(A, V) \) with \( \text{Ann}(A) \cap \text{Ann}(\theta) = 0 \), where \( V \) is a vector space of dimension \( s \), such that \( A \cong A'_\theta \) and \( A/\text{Ann}(A) \cong A' \).

For the proof, we refer the reader to [33, Lemma 5].

Then, in order to decide when two bicommutative algebras with nonzero annihilator are isomorphic, it suffices to find conditions in terms of the cocycles.

Let us fix a basis \( \{ e_1, \ldots, e_s \} \) of \( V \), and \( \theta \in Z^2(A, V) \). Then \( \theta \) can be uniquely written as \( \theta(x, y) = \sum_{i=1}^{s} \theta_i(x, y) e_i \), where \( \theta_i \in Z^2(A, \mathbb{C}) \). It holds that \( \theta \in B^2(A, V) \) if and only if all \( \theta_i \in B^2(A, \mathbb{C}) \), and it also holds that \( \text{Ann}(\theta) = \text{Ann}(\theta_1) \cap \ldots \cap \text{Ann}(\theta_s) \). Furthermore, if \( \text{Ann}(\theta) \cap \text{Ann}(A) = 0 \), then \( A_{\theta} \) has an annihilator component if and only if \([\theta_1], \ldots, [\theta_s]\) are linearly independent in \( \mathbb{H}^2(A, \mathbb{C}) \) (see [33, Lemma 13]).

Recall that, given a finite-dimensional vector space \( V \) over \( \mathbb{C} \), the Grassmannian \( G_k(V) \) is the set of all \( k \)-dimensional linear subspaces of \( V \). Let \( G_s(\mathbb{H}^2(A, \mathbb{C})) \) be the Grassmannian of subspaces of dimension \( s \) in \( \mathbb{H}^2(A, \mathbb{C}) \). For \( W = \langle [\theta_1], \ldots, [\theta_s] \rangle \in G_s(\mathbb{H}^2(A, \mathbb{C})) \) and \( \phi \in \text{Aut}(A) \), define \( \phi W = \langle [\phi \theta_1], \ldots, [\phi \theta_s] \rangle \). It holds that \( \phi W \in G_s(\mathbb{H}^2(A, \mathbb{C})) \), and this induces an action of \( \text{Aut}(A) \) on \( G_s(\mathbb{H}^2(A, \mathbb{C})) \). We denote the orbit of \( W \in G_s(\mathbb{H}^2(A, \mathbb{C})) \) under this action by \( \text{Orb}(W) \). Let

\[
W_1 = \langle [\theta_1], \ldots, [\theta_1] \rangle, \quad W_2 = \langle [\theta_1], \ldots, [\theta_s] \rangle \in G_s(\mathbb{H}^2(A, \mathbb{C})).
\]

Similarly to [33, Lemma 15], in case that \( W_1 = W_2 \), it holds that

\[
\bigcap_{i=1}^{s} \text{Ann}(\theta_i) \cap \text{Ann}(A) = \bigcap_{i=1}^{s} \text{Ann}(\theta_i) \cap \text{Ann}(A),
\]

and therefore the set

\[
T_s(A) = \left\{ W = \langle [\theta_1], \ldots, [\theta_s] \rangle \in G_s\left(\mathbb{H}^2(A, \mathbb{C})\right) : \bigcap_{i=1}^{s} \text{Ann}(\theta_i) \cap \text{Ann}(A) = 0 \right\}
\]

is well defined, and it is also stable under the action of \( \text{Aut}(A) \) (see [33, Lemma 16]).
Now, let $V$ be an $s$-dimensional linear space and let us denote by $E(A, V)$ the set of all non-split $s$-dimensional central extensions of $A$ by $V$. We can write

$$E(A, V) = \left\{ A_\theta : \theta(x, y) = \sum_{i=1}^{s} \theta_i(x, y) e_i \text{ and } \langle [\theta_1], \ldots, [\theta_s] \rangle \in T_s(A) \right\}.$$

Having established these results, we can determine whether two $s$-dimensional non-split central extensions $A_\theta, A_\vartheta$ are isomorphic or not. For the proof, see [33, Lemma 17].

**Lemma 4** Let $A_\theta, A_\vartheta \in E(A, V)$. Suppose that $\theta(x, y) = \sum_{i=1}^{s} \theta_i(x, y) e_i$ and $\vartheta(x, y) = \sum_{i=1}^{s} \vartheta_i(x, y) e_i$. Then the bicommutative algebras $A_\theta$ and $A_\vartheta$ are isomorphic if and only if

$$\text{Orb} \langle [\theta_1], \ldots, [\theta_s] \rangle = \text{Orb} \langle [\vartheta_1], \ldots, [\vartheta_s] \rangle.$$

Then, it exists a bijective correspondence between the set of $\text{Aut}(A)$-orbits on $T_s(A)$ and the set of isomorphism classes of $E(A, V)$. Consequently we have a procedure that allows us, given a bicommutative algebra $A'$ of dimension $n-s$, to construct all its non-split central extensions.

**Procedure**

Let $A'$ be a bicommutative algebra of dimension $n-s$.

1. Determine $H^2(A', \mathbb{C})$, $\text{Ann}(A')$ and $\text{Aut}(A')$.
2. Determine the set of $\text{Aut}(A')$-orbits on $T_s(A')$.
3. For each orbit, construct the bicommutative algebra associated with a representative of it.

It follows that, thanks to this procedure and to Lemma 3, we can classify all the nilpotent bicommutative algebras of dimension $n$, provided that the nilpotent bicommutative algebras of dimension $n-1$ are known.

**2.2 Notations**

Let $A$ be a bicommutative algebra and fix a basis $\{e_1, \ldots, e_n\}$. We define the bilinear form $\Delta_{ij} : A \times A \to \mathbb{C}$ by $\Delta_{ij}(e_i, e_m) = \delta_{i}^{j} \delta_{jm}$. Then the set $\{\Delta_{ij} : 1 \leq i, j \leq n\}$ is a basis for the linear space of the bilinear forms on $A$, and in particular, every $\theta \in Z^2(A, V)$ can be uniquely written as $\theta = \sum_{1 \leq i, j \leq n} c_{ij} \Delta_{ij}$, where $c_{ij} \in \mathbb{C}$. Let us fix the following notations:

- $\mathcal{B}_{ij}$ — $j$th $i$-dimensional nilpotent non-pure bicommutative algebra (with identity $xyz=0$);
- $\mathcal{B}_{ij}^*$ — $j$th $i$-dimensional nilpotent pure bicommutative algebra (without identity $xyz=0$);
- $\mathfrak{N}_i$ — $i$-dimensional algebra with zero product;
- $(A)_{i,j}$ — $j$th $i$-dimensional central extension of $A$.  

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2.3 The Algebraic Classification of 3-Dimensional Nilpotent Bicommutative Algebras

There are no non-trivial 1-dimensional nilpotent bicommutative algebras, and there is only one non-trivial 2-dimensional nilpotent bicommutative algebra, which is exactly the non-split central extension of the 1-dimensional algebra with zero product \( \mathfrak{M}_1 \):

\[
B^{2*}_{01} : (\mathfrak{M}_1)_{2,1} : e_1 e_1 = e_2.
\]

From this algebra, we construct the 3-dimensional nilpotent bicommutative algebra \( B^{3*}_{01} = B^{2*}_{01} \oplus \mathbb{C}e_3 \).

Also, [12] gives the description of all central extensions of \( B^{2*}_{01} \) and \( \mathfrak{M}_2 \). Choosing the bicommutative algebras between them, we have the classification of all non-split 3-dimensional nilpotent bicommutative algebras:

| Algebra | \( (\mathfrak{M}_2)_{3,1} \) | \( (\mathfrak{M}_2)_{3,2} \) | \( (\mathfrak{M}_2)_{3,3} \) |
|---------|----------------|----------------|----------------|
| \( B^{3*}_{02} \) | \( e_1 e_1 = e_3 \) | \( e_2 e_2 = e_3 \) | \( e_2 e_1 = -e_3 \) |
| \( B^{3*}_{03} \) | \( e_1 e_2 = e_3 \) | \( e_2 e_1 = -e_3 \) | \( e_2 e_2 = e_3 \) |
| \( B^{3*}_{04} (\alpha)_{\alpha \neq 0} \) | \( e_1 e_1 = \alpha e_3 \) | \( e_2 e_1 = e_3 \) | \( e_2 e_2 = e_3 \) |
| \( B^{3*}_{04} (0) \) | \( e_1 e_2 = e_3 \) | \( e_2 e_1 = e_3 \) | \( e_2 e_2 = e_3 \) |
| \( B^{3}_{01} \) | \( (B^{3*}_{01})_{3,1} : e_1 e_1 = e_2 \) | \( e_2 e_1 = e_3 \) | \( e_2 e_2 = e_3 \) |
| \( B^{3}_{02} (\alpha) \) | \( (B^{3*}_{01})_{3,2} : e_1 e_1 = e_2 \) | \( e_2 e_1 = e_3 \) | \( e_2 e_2 = e_3 \) |

Remark 5 The reasons for considering separately the cases \( B^{3*}_{04} (\alpha) \) with \( \alpha \neq 0 \) and \( B^{3*}_{04} (0) \) will become apparent in next subsection, as their cohomology spaces are rather different.

Note that these seven algebras provide seven bicommutative algebras of dimension 4 by embedding them into \( \mathbb{C}^4 \). They will be denoted by \( B^{4*}_{01}, B^{4*}_{02}, B^{4*}_{03}, B^{4*}_{04} (\alpha)_{\alpha \neq 0}, B^{4*}_{04} (0), B^{4}_{01} \) and \( B^{4}_{02} \), respectively.

2.4 1-Dimensional Central Extensions of 3-Dimensional Nilpotent Bicommutative Algebras

2.4.1 The Description of Second Cohomology Space of 3-Dimensional Nilpotent Bicommutative Algebras

In the following table we give the description of the second cohomology space of 3-dimensional nilpotent bicommutative algebras:

| Algebra | \( Z^2 (A) \) | \( B^2 (A) \) | \( H^2 (A) \) |
|---------|----------------|----------------|----------------|
| \( B^{3}_{01} \) | \( \{\Delta_{11}, \Lambda_{11}, \Lambda_{13}, \Lambda_{21}, \Lambda_{31}, \Lambda_{33}\} \) | \( \{\Delta_{12}, [\Lambda_{13}], [\Lambda_{21}], [\Lambda_{31}], [\Lambda_{33}]\} \) | \( \{[\Lambda_{12}], [\Lambda_{13}], [\Lambda_{21}], [\Lambda_{31}], [\Lambda_{33}]\} \) |
| \( B^{3}_{02} \) | \( \{\Lambda_{11} + \Lambda_{22}\} \) | \( \{\Delta_{12}, [\Lambda_{21}], [\Lambda_{22}]\} \) | \( \{[\Lambda_{12}], [\Lambda_{21}], [\Lambda_{22}]\} \) |
| \( B^{3}_{03} \) | \( \{\Delta_{12}, [\Lambda_{13}], [\Lambda_{21}], [\Lambda_{22}]\} \) | \( \{\Delta_{12}, [\Lambda_{13}], [\Lambda_{21}], [\Lambda_{22}]\} \) | \( \{[\Lambda_{12}], [\Lambda_{13}], [\Lambda_{21}], [\Lambda_{22}]\} \) |
| \( B^{3}_{04} (\alpha)_{\alpha \neq 0} \) | \( \{\alpha \Delta_{11} + \Delta_{21} + \Delta_{22}\} \) | \( \{\Delta_{12}, [\Lambda_{21}], [\Lambda_{22}]\} \) | \( \{[\Lambda_{12}], [\Lambda_{21}], [\Lambda_{22}]\} \) |
| \( B^{3}_{04} (0) \) | \( \{\Delta_{12}, [\Lambda_{13}], [\Lambda_{21}], [\Lambda_{22}], [\Lambda_{32}]\} \) | \( \{\Delta_{12}, [\Lambda_{13}], [\Lambda_{21}], [\Lambda_{22}], [\Lambda_{32}]\} \) | \( \{[\Lambda_{12}], [\Lambda_{13}], [\Lambda_{21}], [\Lambda_{22}], [\Lambda_{32}]\} \) |
| \( B^{3}_{01} \) | \( \{\Delta_{11}, [\Lambda_{13}], [\Lambda_{21}], [\Lambda_{31}]\} \) | \( \{\Delta_{12}, [\Lambda_{31}]\} \) | \( \{[\Lambda_{12}], [\Lambda_{31}]\} \) |
| \( B^{3}_{02} (\alpha) \) | \( \{\Delta_{11}, \Lambda_{13} + \alpha \Lambda_{22} + \Lambda_{31} + \alpha \Lambda_{31}\} \) | \( \{\Delta_{11}, \Lambda_{12} + \alpha \Lambda_{21}\} \) | \( \{[\Lambda_{12}], [\Lambda_{21}], [\Lambda_{31}]\} \) |

Remark 6 From the description of the cocycles of the algebras \( B^{3*}_{02}, B^{3*}_{03} \) and \( B^{3*}_{04} (\alpha)_{\alpha \neq 0} \), it follows that the 1-dimensional central extensions of these algebras are 2-dimensional central extensions of 2-dimensional nilpotent bicommutative algebras. Thanks to [12] we have
the description of all non-split 2-dimensional central extensions of 2-dimensional nilpotent bicommutative algebras:

\[ B_{03}^4 : (B_{01}^{2*})_{4,1} : e_1 e_1 = e_2, \ e_1 e_2 = e_4, \ e_2 e_1 = e_3. \]

Then, in the following subsections we study the central extensions of the other algebras.

2.4.2 Central Extensions of \( B_{01}^{3*} \)

Since the second cohomology spaces and automorphism groups of \( B_{01}^{3*} \) and \( N_{01}^{3*} \) coincide, these algebras have the same central extensions. Therefore, thanks to \([39]\) we have all the new 4-dimensional nilpotent bicommutative algebras constructed from \( B_{01}^{3*} \):

\[ B_{04}^4(\alpha), \ B_{05}^4, \ B_{06}^4(\alpha)_{\alpha \neq 0}, \ B_{07}^4, \ B_{08}^4, \ B_{09}^4. \]

The multiplication tables of these algebras can be found in Table 1 (Appendix).

2.4.3 Central Extensions of \( B_{04}^{3*}(0) \)

Let us use the following notations:

\[ \nabla_1 = [\Delta_{11}], \nabla_2 = [\Delta_{13}], \nabla_3 = [\Delta_{21}], \nabla_4 = [\Delta_{22}], \nabla_5 = [\Delta_{32}]. \]

The automorphism group of \( B_{04}^{3*}(0) \) consists of invertible matrices of the form

\[ \phi = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ z & t & xy \end{pmatrix}. \]

Since

\[ \phi^T \begin{pmatrix} \alpha_1 & 0 & \alpha_2 \\ \alpha_3 & \alpha_4 & 0 \\ 0 & \alpha_5 & 0 \end{pmatrix} \phi = \begin{pmatrix} x(x\alpha_1 + z\alpha_2) & \alpha^* & x^2 y\alpha_2 \\ xy\alpha_3 & y(y\alpha_4 + t\alpha_5) & 0 \\ 0 & xy^2\alpha_5 & 0 \end{pmatrix}, \]

we have that the orbit of the action of \( \text{Aut}(B_{04}^{3*}(0)) \) on the subspace \( \sum_{i=1}^{5} \alpha_i \nabla_i \) is given by

\[ \left\{ \sum_{i=1}^{5} \alpha_i^* \nabla_i \right\}, \] where

\[ \alpha_1^* = x(x\alpha_1 + z\alpha_2), \ \alpha_2^* = x^2 y\alpha_2, \ \alpha_3^* = x y\alpha_3, \]

\[ \alpha_4^* = y(y\alpha_4 + t\alpha_5), \ \alpha_5^* = xy^2\alpha_5. \]

It is easy to see that the elements \( \alpha_1 \nabla_1 + \alpha_3 \nabla_3 + \alpha_4 \nabla_4 \) give algebras which are central extensions of 2-dimensional algebras. We find the following new cases:

1. \( \alpha_2 \neq 0, \alpha_3 \neq 0, \alpha_5 \neq 0 \). Choosing \( x = \frac{\alpha_3}{\alpha_2}, \ y = \frac{\alpha_3}{\alpha_5}, \ z = -\frac{x\alpha_1}{\alpha_2} \) and \( t = -\frac{y\alpha_4}{\alpha_5} \), we have the representative \( \langle \nabla_2 + \nabla_3 + \nabla_5 \rangle \).

2. \( \alpha_2 \neq 0, \alpha_3 = 0, \alpha_5 \neq 0 \). Choosing \( y = \frac{\alpha_3}{\alpha_5}, \ z = -\frac{x\alpha_1}{\alpha_2} \) and \( t = -\frac{y\alpha_4}{\alpha_5} \), we have the representative \( \langle \nabla_2 + \nabla_5 \rangle \).

3. \( \alpha_2 = 0, \alpha_3 \neq 0, \alpha_5 \neq 0 \):

- (a) if \( \alpha_1 \neq 0 \), then choosing \( y = \frac{\alpha_3}{\alpha_5}, \ x = \frac{y\alpha_3}{\alpha_1} \) and \( t = -\frac{y\alpha_4}{\alpha_5} \), we have the representative \( \langle \nabla_1 + \nabla_3 + \nabla_5 \rangle \).

- (b) if \( \alpha_1 = 0 \), then choosing \( y = \frac{\alpha_3}{\alpha_5} \) and \( t = -\frac{y\alpha_4}{\alpha_5} \), we have the representative \( \langle \nabla_3 + \nabla_5 \rangle \).

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(4) \(\alpha_2 \neq 0, \alpha_3 \neq 0, \alpha_5 = 0:\)
   
   (a) if \(\alpha_4 \neq 0\), then choosing \(x = \frac{x_2}{\alpha_2}, \ y = \frac{x_3}{\alpha_4}\) and \(z = -\frac{x_1}{\alpha_2}\), we have the representative \(\langle \nabla_2 + \nabla_3 + \nabla_4 \rangle\). 
   
   (b) if \(\alpha_4 = 0\), then choosing \(x = \frac{\alpha_3}{\alpha_2}\) and \(z = -\frac{x_1}{\alpha_2}\), we have the representative \(\langle \nabla_2 + \nabla_3 \rangle\).

(5) \(\alpha_2 \neq 0, \alpha_3 = 0, \alpha_5 = 0:\)
   
   (a) if \(\alpha_4 \neq 0\), then choosing \(y = \frac{x^2_3}{\alpha_4}\) and \(z = -\frac{x_1}{\alpha_2}\), we have the representative \(\langle \nabla_2 + \nabla_4 \rangle\).
   
   (b) if \(\alpha_4 = 0\), then choosing \(z = -\frac{x_1}{\alpha_2}\), we have the representative \(\langle \nabla_2 \rangle\).

(6) \(\alpha_2 = 0, \alpha_3 = 0, \alpha_5 \neq 0:\)
   
   (a) if \(\alpha_1 \neq 0\), then choosing \(x = \frac{\alpha_1 x_3}{\alpha_2}\) and \(t = -\frac{\alpha_4}{\alpha_5}\), we have the representative \(\langle \nabla_1 + \nabla_5 \rangle\).
   
   (b) if \(\alpha_1 = 0\) then choosing \(t = -\frac{\alpha_4}{\alpha_5}\), we have the representative \(\langle \nabla_5 \rangle\).

Now we have all the new 4-dimensional nilpotent bicommutative algebras constructed from \(B_{04}^3(0)\):

\[
B_{10}^4, \ldots, B_{19}^4.
\]

The multiplication tables of these algebras can be found in Table 1 (Appendix).

2.4.4 Central Extensions of \(B_{01}^3\)

Let us use the following notations:

\[
\nabla_1 = [\Delta_{12}], \nabla_2 = [\Delta_{31}].
\]

The automorphism group of \(B_{01}^3\) consists of invertible matrices of the form

\[
\phi = \begin{pmatrix}
    x & 0 & 0 \\
    y & x^2 & 0 \\
    z & xy & x^3
\end{pmatrix}.
\]

Since

\[
\phi^T \begin{pmatrix}
    0 & \alpha_1 & 0 \\
    0 & 0 & 0 \\
    \alpha_2 & 0 & 0
\end{pmatrix} \phi = \begin{pmatrix}
    \alpha^* & x^3 \alpha_1 & 0 \\
    \alpha^* & 0 & 0 \\
    x^4 \alpha_2 & 0 & 0
\end{pmatrix},
\]

we have that the orbit of the action of \(\text{Aut}(B_{01}^3)\) on the subspace \(\left< \sum_{i=1}^{2} \alpha_i \nabla_i \right>\) is given by

\[
\left< \sum_{i=1}^{2} \alpha_i^* \nabla_i \right> , \text{ where}
\]

\[
\alpha_1^* = x^3 \alpha_1, \ \alpha_2^* = x^4 \alpha_2.
\]

It is straightforward that the elements \(\alpha_1 \nabla_1\) lead to central extensions of 2-dimensional algebras. The new cases are the following:

(1) \(\alpha_1 \neq 0, \alpha_2 \neq 0\). Choosing \(x = \frac{\alpha_1}{\alpha_2}\), we have the representative \(\langle \nabla_1 + \nabla_2 \rangle\).

(2) \(\alpha_1 = 0, \alpha_2 \neq 0\). Choosing \(x = \frac{1}{\sqrt{\alpha_2}}\), we have the representative \(\langle \nabla_2 \rangle\).
Now we have all the new 4-dimensional nilpotent bicommutative algebras constructed from $B^3_{01}$:

\[ B^3_{20}, B^4_{21}. \]

The multiplication tables of these algebras can be found in Table 1 (Appendix).

### 2.4.5 Central Extensions of $B^3_{02}(\alpha)$

Let us use the following notations:

\[ \nabla_1 = [\Delta_211], \nabla_2 = \alpha[\Delta_22] + [\Delta_13] + \alpha[\Delta_31]. \]

The automorphism group of $B^3_{02}(\alpha)$ consists of invertible matrices of the form

\[
\phi = \begin{pmatrix} x & 0 & 0 \\ y & x^2 & 0 \\ z & (\alpha + 1)xy & x^3 \end{pmatrix}.
\]

Since

\[
\phi^T \begin{pmatrix} 0 & 0 & \alpha_2 \\ \alpha_1 & \alpha \alpha_2 & 0 \\ \alpha \alpha_2 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha_1^* & \alpha \alpha_2^* & \alpha \alpha_2^* \\ \alpha_1^* + \alpha \alpha_2^* & \alpha \alpha_2^* & 0 \\ \alpha \alpha_2^* & 0 & 0 \end{pmatrix},
\]

we have that the orbit of the action of $\text{Aut}(B^3_{02}(\alpha))$ on the subspace $\langle \sum_{i=1}^{2} \alpha_i \nabla_i \rangle$ is given by

\[
\langle \sum_{i=1}^{2} \alpha_i^* \nabla_i \rangle,
\]

where

\[
\alpha_1^* = x^2(\alpha \alpha_1 + \alpha(1-\alpha)y \alpha_2), \quad \alpha_2^* = x^4 \alpha_2.
\]

The elements $\alpha_1 \nabla_1$ give central extensions of 2-dimensional algebras, so we will consider only cases with $\alpha_2 \neq 0$. We find the following new cases:

1. $\alpha = 0$ or $\alpha = 1$:
   
   (a) if $\alpha_1 \neq 0$, then choosing $x = \frac{\alpha_1}{\alpha_2}$, we have the representative $\langle \nabla_1 + \nabla_2 \rangle$.
   
   (b) if $\alpha_1 = 0$, then choosing $x = \frac{1}{\sqrt{\alpha_2}}$, we have the representative $\langle \nabla_2 \rangle$.

2. $\alpha \neq 0, 1$, then choosing $x = \frac{1}{\sqrt{\alpha_2}}$ and $y = \frac{-\alpha \alpha_1}{\alpha_2 \alpha(1-\alpha)}$, we have the representative $\langle \nabla_2 \rangle$.

Now we have all the new 4-dimensional nilpotent bicommutative algebras constructed from $B^3_{02}(\alpha)$:

\[ B^4_{22}, B^4_{23}, B^4_{24}(\alpha). \]

The multiplication tables of these algebras can be found in Table 1 (Appendix).

### 2.5 The Algebraic Classification of 4-Dimensional Nilpotent Bicommutative Algebras

We distinguish two main classes of bicommutative algebras: the **pure** and the **non-pure** or **trivial** ones. By the non-pure ones, we mean those satisfying the identities $(xy)z = 0$ and $x(yz) = 0$; the pure ones are the rest.

These trivial algebras can be considered in many varieties of algebras defined by polynomial identities of degree three (associative, Leibniz, Zinbiel...), and they can be expressed
as central extensions of suitable algebras with zero product. Those with dimension 4 are already classified: the list of the non-anticommutative ones can be found in [19], and there is only one nilpotent and anticommutative.

Regarding the pure 4-dimensional nilpotent bicommutative algebras, we have the following theorem, whose proof is based on the classification of 3-dimensional nilpotent bicommutative algebras and on the results of Section 2.4.

**Theorem 7** Let $B$ be a nonzero 4-dimensional nilpotent pure bicommutative algebra over $\mathbb{C}$. Then, $B$ is isomorphic to one of the algebras listed in Table 1 (Appendix).

### 3 The Geometric Classification of Nilpotent Bicommutative Algebras

#### 3.1 Definitions and Notation

Let $V$ be an $n$-dimensional vector space. The set $\text{Hom}(V \otimes V, V) \cong V^* \otimes V^* \otimes V$ is a vector space of dimension $n^3$, and it has the structure of the affine variety $\mathbb{C}^n$. Indeed, if we fix a basis $\{e_1, \ldots, e_n\}$ of $V$, then any $\mu \in \text{Hom}(V \otimes V, V)$ is determined by $n^3$ structure constants $c^i_{jk} \in \mathbb{C}$ such that $\mu(e_i \otimes e_j) = \sum_{k=1}^n c^i_{jk} e_k$. A subset of $\text{Hom}(V \otimes V, V)$ is called Zariski-closed if it can be defined by a set of polynomial equations in the variables $c^i_{jk}$ ($1 \leq i, j, k \leq n$).

Let $T$ be such a set of polynomial identities. It holds that every algebra structure on $V$ satisfying polynomial identities from $T$ forms a Zariski-closed subset of the variety $\text{Hom}(V \otimes V, V)$; it is denoted by $\mathbb{L}(T)$. There exists a natural action of the general linear group $GL(V)$ on $\mathbb{L}(T)$ defined by

$$(g \ast \mu)(x \otimes y) = g \mu(g^{-1}x \otimes g^{-1}y)$$

for $x, y \in V$, $\mu \in \mathbb{L}(T)$ and $g \in GL(V)$. Then, $\mathbb{L}(T)$ can be decomposed into $GL(V)$-orbits corresponding to the isomorphism classes of the algebras. We will denote by $O(\mu)$ the orbit of $\mu \in \mathbb{L}(T)$ under the action of $GL(V)$, and by $\overline{O(\mu)}$ the Zariski closure of $O(\mu)$.

Let $A$ and $B$ be two $n$-dimensional algebras satisfying the identities from $T$, and let $\mu, \lambda \in \mathbb{L}(T)$ represent $A$ and $B$, respectively. We say that $A$ degenerates to $B$, and write $A \to B$, if $\lambda \in \overline{O(\mu)}$. Note that, in particular, it holds that $\overline{O(\lambda)} \subset \overline{O(\mu)}$. Hence, the definition of a degeneration does not depend on the choice of $\mu$ and $\lambda$. If $A \not\cong B$, then the assertion $A \to B$ is called a proper degeneration. Also, we write $A \not\to B$ if $\lambda \notin \overline{O(\mu)}$.

Now consider $A(\alpha)_I := \{A(\alpha)\}_{\alpha \in I}$ and $B(\beta)_J := \{B(\beta)\}_{\beta \in J}$ two infinite families of algebras parameterized by $\alpha$ and $\beta$, respectively, and let $A(\alpha)$, for $\alpha \in I$, be represented by the structure $\mu(\alpha) \in \mathbb{L}(T)$, and $B(\beta)$, for $\beta \in J$, by the structure $\lambda(\beta) \in \mathbb{L}(T)$. Then $A(\alpha) \to B$ means $\lambda(\beta) \in \overline{O(\mu(\alpha))}$, and $A(\alpha) \not\to B$ means $\lambda(\beta) \notin \overline{O(\mu(\alpha))}$. On the other hand, $A \to B(\beta)$ means that $\lambda(\beta) \in \overline{O(\mu)}$ for all $\beta \in B$ except a finite number of instances, and $A \not\to B(\beta)$ means that $\lambda(\beta) \notin \overline{O(\mu)}$ for infinite $\beta \in B$.

Moreover, we call $A$ rigid in $\mathbb{L}(T)$ if $O(\mu)$ is an open subset of $\mathbb{L}(T)$. Recall that a subset of a variety is called irreducible if it cannot be represented as a union of two non-trivial closed subsets, and that a maximal irreducible closed subset of a variety is called an irreducible component. It is well known that any affine variety can be represented as a finite union of its irreducible components in a unique way. Then, we have the following characterization of rigidity: $A$ is rigid if $\mathbb{L}(T)$ if and only if $\overline{O(\mu)}$ is an irreducible component of $\mathbb{L}(T)$. 

\[ \square \]
Henceforth, given the vector spaces $U$ and $W$, we will write simply $U > W$ instead of $\dim U > \dim W$.

### 3.2 Method of the Description of Degenerations of Algebras

In the present work we use the methods applied to Lie algebras in [8, 29, 30, 48]. Let $\mathcal{D}er(A)$ denote the Lie algebra of derivations of $A$. Our first and useful consideration is that if $A \to B$ and $A \not\cong B$, then $\mathcal{D}er(A) < \mathcal{D}er(B)$ and $A^2 \geq B^2$. Then, we will compute the dimensions of algebras of derivations and will check the assertion $A \to B$ only for $A$ and $B$ such that $\mathcal{D}er(A) < \mathcal{D}er(B)$. Among them, we will calculate the dimension of the squares of the algebras and check $A \to B$ only for $A$ and $B$ such that $A^2 \geq B^2$.

Now, we explain our method for proving degenerations. Let $A$, $A(\ast)$, $B$ and $B(\ast)$ be as in Section 3.1. Fixed a basis $\{e_1, \ldots, e_n\}$ of $V$, let $c^k_{ij}$ ($1 \leq i, j, k \leq n$) be the structure constants of $\lambda$ in this basis, and $c^k_{ij}(\beta)$ ($1 \leq i, j, k \leq n$) be the structure constants of $\lambda(\beta)$. On the one hand, if there exist $a_{ij} : \mathbb{C}^* \to \mathbb{C}$ ($1 \leq i, j \leq n$), such that $E^t_i = \sum_{j=1}^n a_{ij}(t) e_j$, for $1 \leq i \leq n$, form a basis of $V$ for any $t \in \mathbb{C}^*$, and the structure constants of $\mu$ in the basis $\{E^t_1, \ldots, E^t_n\}$ are such polynomials $c^k_{ij}(t) \in \mathbb{C}[t]$ that $c^k_{ij}(0) = c^k_{ij}$, then $A \to B$. In this case $\{E^t_1, \ldots, E^t_n\}$ is called a parameterized basis for $A \to B$.

Also, if there exist $a_{ij} : J \times \mathbb{C}^* \to \mathbb{C}$ ($1 \leq i, j \leq n$), such that $E^t_i(\beta) = \sum_{j=1}^n a_{ij}(\beta, t) e_j$, for $1 \leq i \leq n$, form a basis of $V$ for any $t \in \mathbb{C}^*$ and all $\beta \in J$ except for a finite number, and the structure constants of $\mu$ in the basis $\{E^t_i(\beta), \ldots, E^t_n(\beta)\}$ are such polynomials $c^k_{ij}(\beta, t) \in \mathbb{C}[t]$ that $c^k_{ij}(\beta, 0) = c^k_{ij}(\beta)$, then $A \to B(\ast)$. The basis $\{E^t_1(\beta), \ldots, E^t_n(\beta)\}$ is called a parameterized basis for $A \to B(\ast)$.

On the other hand, if we construct $a_{ij} : \mathbb{C}^* \to \mathbb{C}$ ($1 \leq i, j \leq n$) and $f : \mathbb{C}^* \to I$ such that $E^t_i = \sum_{j=1}^n a_{ij}(t) e_j$, for $1 \leq i \leq n$, form a basis of $V$ for any $t \in \mathbb{C}^*$, and the structure constants of $\mu(f(t))$ in the basis $\{E^t_1, \ldots, E^t_n\}$ are such polynomials $c^k_{ij}(t) \in \mathbb{C}[t]$ that $c^k_{ij}(0) = c^k_{ij}$, then $A(\ast) \to B$. In this case $\{E^t_1, \ldots, E^t_n\}$ and $f(t)$ are called a parameterized basis and a parameterized index for $A(\ast) \to B$, respectively.

### 3.3 The Geometric Classification of 4-Dimensional Nilpotent Bicommutative Algebras

The main result of the present section is the following theorem.

**Theorem 8** The variety of 4-dimensional nilpotent bicommutative algebras has two irreducible components defined by the rigid algebra $B^4_{10}$ and the infinite family of algebras $B^4_{24}(\alpha)$.

**Proof** After having computed the dimensions of the spaces of derivations of all 4-dimensional nilpotent bicommutative algebras, which can be consulted in Table 1 (Appendix) and having checked that $\mathcal{D}er(B^4_{10}) < \mathcal{D}er(B)$ for all $B$ 4-dimensional nilpotent bicommutative algebra, $B \not\cong B^4_{10}$, it follows that there are not nilpotent bicommutative algebras degenerating to $B^4_{10}$. Also, the dimension of the square of $B^4_{10}$ is 2, and $B^4_{24}(\alpha)$ has 3-dimensional space, so it cannot degenerate from $B^4_{10}$. Therefore, if we prove that these...
two algebras degenerate to the rest of the nilpotent bicommutative algebras of dimension 4, the theorem is proved.

Recall that the full description of the degeneration system of 4-dimensional trivial bicommutative algebras was given in [43]. Using the cited result, we have that the variety of 4-dimensional trivial bicommutative algebras has two irreducible components given by the two following families of algebras:

\[ \mathfrak{N}_{02}(\alpha) : e_1 e_1 = e_3 \quad e_1 e_2 = e_4 \quad e_2 e_1 = -\alpha e_3 \quad e_2 e_2 = -e_4 \]
\[ \mathfrak{N}_{03}(\alpha) : e_1 e_1 = e_4 \quad e_1 e_2 = \alpha e_4 \quad e_2 e_1 = -\alpha e_4 \quad e_2 e_2 = e_4 \quad e_3 e_3 = e_4. \]

The algebra \( B^4_{10} \) degenerates to both \( \mathfrak{N}_{02}(\alpha) \) and \( \mathfrak{N}_{03}(\alpha) \). We will explain in detail the degeneration \( B^4_{10} \rightarrow \mathfrak{N}_{03}(\alpha) \alpha \neq 0, \pm i; \) as for \( B^4_{10} \rightarrow \mathfrak{N}_{02}(\alpha) \), it is similar, but easier. It can be found in Table 2 (Appendix).

Let us consider the following parametric basis of \( B^4_{10} \) : \( \{F_i = \sum_{j=1}^{4} a_{ij}(t)e_j\} \). The multiplication table in the new basis is given below:

\[ F_1 F'_1 = \frac{a_{11}a_{12}}{a_{33}} F'_3 + \frac{a_{11}a_{13} + a_{11}a_{12} + a_{12}a_{13} - a_{11}a_{12}a_{13} - a_{11}a_{12}a_{13}}{a_{44}} F'_4 \]
\[ F_1 F'_2 = \frac{a_{11}a_{22}}{a_{33}} F'_3 + \frac{a_{11}a_{23} + a_{11}a_{22} + a_{12}a_{13}}{a_{44}} F'_4 \]
\[ F_2 F'_1 = \frac{a_{11}a_{12}}{a_{44}} F'_4 \]
\[ F_1 F'_3 = \frac{a_{11}a_{13}}{a_{44}} F'_4 \]
\[ F'_3 F'_2 = \frac{a_{22}a_{33}}{a_{44}} F'_4. \]

To make the computations easier, we will consider a new basis \( f_1, f_2, f_3, f_4 \) in \( \mathfrak{N}_{03}(\alpha) \) such that

\[ f_2 f_3 = 0, \quad f_3 f_3 = 0, \quad f_4 \mathfrak{N}_{03}(\alpha) = \mathfrak{N}_{02}(\alpha)f_4 = 0. \]

Such a basis can be defined as

\[ f_1 = e_1, \quad f_2 = e_2, \quad f_3 = e_1 + \alpha e_2 + i\sqrt{\alpha^2 + 1} e_3, \quad f_4 = e_4. \]

The multiplication table of \( \mathfrak{N}_{03}(\alpha) \) with this new basis is

\[ f_1 f_1 = f_4 \quad f_2 f_2 = f_4 \quad f_3 f_3 = (1 + \alpha^2)f_4 \quad f_3 f_1 = (1 - \alpha^2)f_4 \quad f_3 f_2 = 2\alpha f_4. \]

Some routine calculations show that by taking

\[ \alpha_{11} = (1 + \alpha^2)t \quad \alpha_{12} = (1 - \alpha^2)t \quad \alpha_{13} = -2t \quad \alpha_{14} = 0 \]
\[ \alpha_{22} = 2\alpha t \quad \alpha_{23} = -2\alpha t \quad \alpha_{24} = 0 \]
\[ \alpha_{33} = -4\alpha^2t \quad \alpha_{34} = -4\alpha^2t \frac{a_{33} - a_{34}}{a_{44}} \]
\[ \alpha_{44} = -4\alpha^2t^2, \]
we obtain exactly

\begin{align*}
F_0^0 F_1^0 &= F_4^0 \\
F_0^1 F_2^0 &= \alpha F_4^0 \\
F_1^0 F_3^0 &= (1 + \alpha^2) F_4^0 \\
F_2^0 F_3^0 &= 2\alpha F_4^0.
\end{align*}

Then, it suffices to take

\begin{align*}
E_1^t &= F_1^t, \\
E_2^t &= F_2^t, \\
E_3^t &= \frac{i}{\sqrt{\alpha^2 + 1}} (F_1^t + F_2^t - F_3^t), \\
E_4^t &= F_4^t,
\end{align*}

so that we have the desired degeneration \( B_{10}^4 \to \mathcal{M}_{03}(\alpha) \) by the method described in the previous subsection. Namely,

\begin{align*}
E_1^t &= \tau ((1 + \alpha^2)e_1 + (1 - \alpha^2)e_2 - 2e_3) \\
E_2^t &= 2\alpha t (e_2 - e_3) \\
E_3^t &= \frac{i t}{\sqrt{1 + \alpha^2}} \left( (1 + \alpha^2)e_1 + (1 + \alpha^2)e_2 - 2(1 - \alpha^2)e_3 + \frac{4\alpha^2(\alpha^2 - 3)}{1 + \alpha^2} e_4 \right) \\
E_4^t &= -4\alpha^{-2} t^2 e_4.
\end{align*}

Regarding the pure nilpotent bicommutative algebras, similar computations show that \( B_{10}, B_{12}, B_{14}, B_{20} \) or \( B_{24}^4(\alpha) \) degenerate to them. The explicit degenerations can be seen in Table 2 (Appendix).

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## Appendix

### Table 1

The list of 4-dimensional nilpotent “pure” bicommutative algebras

| \( A \) | \( \text{Der} A \) | Multiplication table |
|--------|-----------------|---------------------|
| \( B_{01}^4 \) | 6                | \( e_1 e_1 = e_2 \) \( e_2 e_1 = e_3 \) |
| \( B_{02}^4(\alpha) \) | 6                | \( e_1 e_1 = e_2 \) \( e_1 e_2 = e_3 \) \( e_2 e_1 = \alpha e_3 \) |
| \( B_{03}^4 \) | 4                | \( e_1 e_1 = e_2 \) \( e_1 e_2 = e_4 \) \( e_2 e_1 = e_3 \) |
| \( B_{04}^4(\alpha) \) | 4                | \( e_1 e_1 = e_2 \) \( e_1 e_2 = e_4 \) \( e_2 e_1 = \alpha e_4 \) \( e_3 e_1 = e_4 \) |
| \( B_{05}^4 \) | 4                | \( e_1 e_1 = e_2 \) \( e_1 e_2 = e_4 \) \( e_1 e_3 = e_4 \) \( e_2 e_1 = e_4 \) \( e_3 e_2 = e_4 \) |
| \( B_{06}^4(\alpha \neq 0) \) | 5                | \( e_1 e_1 = e_2 \) \( e_1 e_2 = e_4 \) \( e_1 e_3 = e_4 \) \( e_2 e_1 = \alpha e_4 \) |
| \( B_{07}^4 \) | 4                | \( e_1 e_1 = e_2 \) \( e_1 e_2 = e_4 \) \( e_3 e_1 = e_4 \) |
| \( B_{08}^4 \) | 5                | \( e_1 e_1 = e_2 \) \( e_1 e_3 = e_4 \) \( e_2 e_1 = e_4 \) |
| \( B_{09}^4 \) | 5                | \( e_1 e_1 = e_2 \) \( e_1 e_2 = e_4 \) \( e_3 e_1 = e_4 \) |
| \( B_{10}^4 \) | 2                | \( e_1 e_2 = e_3 \) \( e_1 e_3 = e_4 \) \( e_2 e_1 = e_4 \) \( e_3 e_2 = e_4 \) |
| \( B_{11}^4 \) | 3                | \( e_1 e_2 = e_3 \) \( e_1 e_3 = e_4 \) \( e_3 e_2 = e_4 \) |
| \( B_{12}^4 \) | 3                | \( e_1 e_2 = e_3 \) \( e_1 e_1 = e_4 \) \( e_2 e_1 = e_4 \) \( e_3 e_2 = e_4 \) |
| \( B_{13}^4 \) | 4                | \( e_1 e_2 = e_3 \) \( e_2 e_1 = e_4 \) \( e_3 e_2 = e_4 \) |
Table 1  (continued)

| A       | Det A | Multiplication table                         |
|---------|-------|-----------------------------------------------|
| \(B_{14}^i\) | 3     | \(e_1e_2 = e_3\) \(e_1e_3 = e_4\) \(e_2e_1 = e_4\) \(e_2e_2 = e_4\) |
| \(B_{15}^i\) | 4     | \(e_1e_2 = e_3\) \(e_1e_3 = e_4\) \(e_2e_1 = e_4\) |
| \(B_{16}^i\) | 4     | \(e_1e_2 = e_3\) \(e_1e_3 = e_4\) \(e_2e_2 = e_4\) |
| \(B_{17}^i\) | 5     | \(e_1e_2 = e_3\) \(e_1e_3 = e_4\) |
| \(B_{18}^i\) | 4     | \(e_1e_2 = e_3\) \(e_1e_1 = e_4\) \(e_2e_2 = e_4\) |
| \(B_{19}^i\) | 5     | \(e_1e_2 = e_3\) \(e_2e_2 = e_4\) |
| \(B_{20}^i\) | 3     | \(e_1e_1 = e_2\) \(e_2e_1 = e_3\) \(e_2e_2 = e_4\) |
| \(B_{21}^i\) | 4     | \(e_1e_2 = e_3\) \(e_1e_3 = e_4\) \(e_2e_2 = e_4\) |
| \(B_{22}^i\) | 3     | \(e_1e_1 = e_2\) \(e_2e_2 = e_3\) \(e_2e_1 = e_4\) |
| \(B_{23}^i\) | 3     | \(e_1e_1 = e_2\) \(e_1e_2 = e_3\) \(e_1e_3 = e_4\) \(e_2e_2 = e_4\) |
| \(B_{24}^i(\alpha)\) | (\(\alpha \neq 0, 1\)) | \(e_1e_1 = e_2\) \(e_1e_2 = e_3\) \(e_1e_3 = e_4\) \(e_2e_2 = \alpha e_4\) \(e_3e_1 = \alpha e_4\) |

Table 2  Degenerations of 4-dimensional nilpotent bicommutative algebras

| \(B_{10}^i\) | \(\mathfrak{nil}_2^4(\alpha)\) | \(E_1^i = t(e_1 + e_3)\) | \(E_3^i = t^2 e_4\) |
| \(B_{10}^i\) | \(B_{01}^4\) | \(E_1^i = t e_1 + e_2\) | \(E_3^i = t e_4\) |
| \(B_{10}^i\) | \(B_{02}^4(\alpha)\) | \(E_1^i = \alpha e_2\) | \(E_3^i = \alpha e_4\) |
| \(B_{10}^i\) | \(B_{03}^4\) | \(E_1^i = t e_1\) | \(E_3^i = t^3 e_3\) |
| \(B_{10}^i\) | \(B_{04}^4(\alpha)\) | \(E_1^i = -t^2 e_1 - \alpha t^2 e_2 + ((\alpha + 1)t^2 + t^4) e_3\) | \(E_3^i = t^3 e_1 - \alpha t^3 e_3\) |
| \(B_{10}^i\) | \(B_{05}^4\) | \(E_1^i = \frac{1}{2} t^2 (e_1 + e_2 + (i t - 2) e_3)\) | \(E_3^i = \frac{1}{2} i t^2 (e_1 + e_3)\) |
| \(B_{10}^i\) | \(B_{06}^4(\alpha)\) | \(E_1^i = \frac{1}{2} e_2\) | \(E_3^i = \frac{1}{2} i e_4\) |
| \(B_{10}^i\) | \(B_{07}^4\) | \(E_1^i = \frac{1}{2} t^2 e_2\) | \(E_3^i = \frac{1}{2} t^2 e_4\) |
| \(B_{10}^i\) | \(B_{08}^4\) | \(E_1^i = t e_1 + e_2 - (1 + t^{-1}) e_3\) | \(E_3^i = t e_2 + t e_3\) |
| \(B_{20}^i\) | \(B_{09}^4\) | \(E_1^i = t e_1\) | \(E_3^i = t^3 e_3\) |
| \(B_{10}^i\) | \(B_{11}^4\) | \(E_1^i = t^{-1} e_1\) | \(E_3^i = t e_3 + e_4\) |
| \(B_{10}^i\) | \(B_{12}^4\) | \(E_1^i = e_2\) | \(E_3^i = t e_4\) |
| \(B_{10}^i\) | \(B_{13}^4\) | \(E_1^i = e_2\) | \(E_3^i = t e_4\) |
| \(B_{10}^i\) | \(B_{14}^4\) | \(E_1^i = e_1\) | \(E_3^i = t e_2 + e_3\) |

\(\mathfrak{nil}_2(\alpha)\) Springer
Table 2 (continued)

| $B^4_{10}$ | $B^4_{15}$ | $E^1_1 = e_1$ | $E^1_2 = te_2$ | $E^1_3 = i^2e_3$ | $E^1_4 = i^4e_4$ |
|------------|------------|---------------|-----------------|-------------------|-----------------|
| $B^4_{14}$ | $B^4_{16}$ | $E^1_1 = t^{-1}e_1$ | $E^1_2 = t^2e_2$ | $E^1_3 = t^{-3}e_3$ | $E^1_4 = t^{-4}e_4$ |
| $B^4_{10}$ | $B^4_{17}$ | $E^1_1 = t^{-1}e_1$ | $E^1_2 = e_2$ | $E^1_3 = t^{-3}e_3$ | $E^1_4 = t^{-4}e_4$ |
| $B^4_{12}$ | $B^4_{18}$ | $E^1_1 = t^{-2}e_1$ | $E^1_2 = e_2$ | $E^1_3 = t^{-3}e_3$ | $E^1_4 = t^{-4}e_4$ |
| $B^4_{10}$ | $B^4_{19}$ | $E^1_1 = e_1$ | $E^1_2 = t^{-1}e_2$ | $E^1_3 = t^{-3}e_3$ | $E^1_4 = t^{-4}e_4$ |
| $B^4_{24}(1/2)$ | $B^4_{20}$ | $E^1_1 = te_1 + t^{-1}e_2$ | $E^1_2 = t^2e_2 + (1 + 2t) t^{-1}e_3 + t^4 e_4$ | $E^1_3 = t^2e_3$ | $E^1_4 = t^4 e_4$ |
| $B^4_{20}$ | $B^4_{21}$ | $E^1_1 = t^{-1}e_1$ | $E^1_2 = t^{-2}e_2$ | $E^1_3 = t^{-3}e_3$ | $E^1_4 = t^{-4}e_4$ |
| $B^4_{24}(t)$ | $B^4_{22}$ | $E^1_1 = te_1 + te_2$ | $E^1_2 = t^2e_2 + (t^2 + t^3)e_3 + t^3 e_4$ | $E^1_3 = t^2e_3 + (3t^3 - 2t^4)e_4$ | $E^1_4 = t^4 e_4$ |
| $B^4_{24}(1 - t)$ | $B^4_{23}$ | $E^1_1 = te_1 + te_2$ | $E^1_2 = t^2e_2 + (2t^2 - t^3)e_3 + t^2(1 - t)e_4$ | $E^1_3 = t^2e_3 + (3t^3 - 2t^4)e_4$ | $E^1_4 = t^4 e_4$ |

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