Behavior of the voltage produced by an electrode in a membrane by means of the finite element method

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Abstract. The spatial distribution of an electrical potential in a cell membrane subjected to an electric field was numerically obtained using an equivalent electrical circuit where the spatial variables that depend on the geometry are combined and an electrical circuit that relates the dynamics in the time of said excitation in four branches that represent the middle. It was observed that the potential decreases linearly in the geometry of the membrane due to the characteristics of the medium (sodium, potassium). On the other hand, the finite element method was developed for a two-dimensional domain that represents the geometry of a membrane, in such a way that it is possible to qualitatively analyze the behavior of the potential at any point of the membrane for an electrical pulse (electrode).

1. Introduction

By examining a cell membrane during its activation, we can according to [1], model the electric current flowing through it by means of parallel conductance, separating sodium, potassium and Leakage ions separately, generating four components in the current. To model mathematically the phenomenon produced when the membrane is excited by a variant source in time (electrode) it is necessary to construct by means of basic laws of electrical circuits a group of equations; in such a way that if the cell membrane is excited with a constant source, the dynamics produced by the source is zero. In this way the model of the propagation in the membrane due to the impulse will be the distribution of the potential through a membrane in a uniform medium with characteristics imposed by sodium, potassium and other chemical combinations characteristic of the cell membrane.

According to the classification of partial differential equations, the partial differential equation that results as a model of this problem is found within the parabolic differential equations. For problems such as this, computational techniques such as linear base interpolation and domain discretization are used, known as the finite element method, which allows an approximation of an ordinary or partial differential equation in one, two or three dimensions. To solve differential equations by means of the finite element method it is necessary to discretize the domain into uniformly spaced fragments depending on the domain, i.e. linear elements in one dimension, triangles or quadrilaterals in two dimensions or tetrahedral elements of 5 or 8 nodes depending on the desired precision in three dimensions. As a consequence of the partitioning of the domain into fragments, the need arises to find a set of linear equations that allow the construction, from the bases described above, of a global function that characterizes the solution of the differential equation and this implies a directly proportional relationship between the precision and the quantity of linear coefficients. On the other hand, the
advantage of this method over others is the ability to adapt the finite elements in the form of the domain of interest despite the existence of discontinuities [1,2].

The following manuscript shows in a few pages a beautiful application of the finite element method and allows us to conjecture that there seems to be no limit to the solution of an important variety of problems in physics, medicine, mathematics and engineering.

2. Finite element method in two dimensions

The finite element method approximates the solution by means of the division of known geometrical figures that complete the original domain including boundary conditions as integral by Galerkin residual theory, so the construction procedure is independent of the boundary conditions of each problem [3].

Consider the following partial differential equation, Equation (1).

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad (1)
\]

with \((x, y) \in \Omega\), and boundary conditions \(u(x_0, y_0) = u_0\) \(y(u, x) = g(x, y)\). For domains \(\Omega \subset \mathbb{R}^2\) by means of triangular elements as in Figure 1(a), we will build a mesh. The boundary conditions are evaluated on each element (Figure 1(b)), then, \(\Omega\) is fragmented in subdomains \(\Omega_1, \Omega_2, \Omega_3, \ldots, \Omega_q\) so that each element is consecutive with each element [4,5].

![Figure 1. (a) Triangular elements, and (b) domain divided into triangular elements.](image)

The values of the variable of interest in each node \((i \leftrightarrow 1, j \leftrightarrow 2, k \leftrightarrow 3)\) \(\phi_i, \phi_j, \phi_k\) can be determined in the following system describe by Equation (2).

\[
\begin{bmatrix}
\phi_i \\
\phi_j \\
\phi_k
\end{bmatrix} =
\begin{bmatrix}
1 & x_i & y_i \\
1 & x_j & y_j \\
1 & x_k & y_k
\end{bmatrix}
\begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3
\end{bmatrix}, \quad (2)
\]

From the system of equations proposed in Equation (2) we can obtain the values of \(\alpha_1, \alpha_2, \alpha_3\) function of the values of \(\phi_i, \phi_j, \phi_k\) in each triangular element and the area of each of them like Equation (3) and Equation (4).

\[
\begin{align*}
2A^q\alpha_1 &= [(x_jy_k - x_ky_j)\phi_i + (x_ky_i - x_iy_k)\phi_j + (x_iy_j - x_jy_i)\phi_k] \\
2A^q\alpha_2 &= [(y_j - y_k)\phi_i + (y_k - y_j)\phi_j + (y_i - y_j)\phi_k] \\
2A^q\alpha_3 &= [(x_k - x_j)\phi_i + (x_i - x_k)\phi_j + (x_j - x_i)\phi_k] \\
\end{align*}, \quad (3)
\]
parallel conductance, which allows to separate ions of sodium (Na), potassium (K) and leakage ions separately (branches) [7], in this way four components are generated.

\[ 2A^q = (x_i y_j - x_j y_i) + (x_k y_1 - x_1 y_k) + (x_j y_k - x_k y_j). \]  
\[ \text{(4)} \]

The numerical solution is written as a linear combination of degree 1 polynomial functions in a triangular shape as expressed by Equation (5) [6].

\[ \phi = N_i(x, y)\phi_1 + N_j(x, y)\phi_j + N_k(x, y)\phi_k, \]  
\[ \text{(5)} \]

where triangular functions are given by Equation (6), Equation (7), and Equation (8).

\[ 2A^q N_i^q(x, y) = [(x_j y_k - x_k y_j) + (y_j - y_k)x + (x_k - x_j)y]. \]  
\[ \text{(6)} \]

\[ 2A^q N_j^q(x, y) = [(x_k y_1 - x_1 y_k) + (y_k - y_1)x + (x_1 - x_k)y]. \]  
\[ \text{(7)} \]

\[ 2A^q N_k^q(x, y) = [(x_1 y_j - x_j y_1) + (y_j - y_1)x + (x_j - x_1)y]. \]  
\[ \text{(8)} \]

Taking \( x = x_1 \) and \( y = y_1 \) in Equation (6) we can obtain the value of \( N_1^q \) in node 1, and in the node 2 and node 3 and in the others that are not part of the element \( q \) takes values equal to zero. The gradients of the variable \( \phi \) are given by the Equation (9) and Equation (10).

\[ \frac{\partial \phi}{\partial x} = \frac{\partial N_i}{\partial x} \phi_1 + \frac{\partial N_j}{\partial x} \phi_j + \frac{\partial N_k}{\partial x} \phi_k, \]  
\[ \text{(9)} \]

\[ \frac{\partial \phi}{\partial y} = \frac{\partial N_i}{\partial y} \phi_1 + \frac{\partial N_j}{\partial y} \phi_j + \frac{\partial N_k}{\partial y} \phi_k. \]  
\[ \text{(10)} \]

By means of the discretization we found a functional to represent the numerical solution of the equation in each node and making use of the method of Galerkin it is possible to find the solution to the Equation (1). As proposed in [7] and [8], we write the solution as the Equation (13).

\[ V(x, y) = \sum_{i=1}^{N} \gamma_i \phi_i(x, y) \in \mathbb{K}, \]  
\[ \text{(13)} \]

with the approximation of the Galerkin method [9,10], a linear system of equations of the form \( [\alpha] \bar{v} = \bar{p} \) is obtained and the internal product between the functions of interpolation (functional) is defined as the integral represented in Equation (14).

\[ I(u(x, y)) = \int_{K} \left\{ \frac{1}{2} \left[ \left( \frac{\partial u(x, y)}{\partial x} \right)^2 + \left( \frac{\partial u(x, y)}{\partial y} \right)^2 \right] + f(x, y)u(x, y) \right\} \, dx \, dy. \]  
\[ \text{(14)} \]

3. Analysis and results

The electric current that circulates through a cell membrane during the activation of these, can be modelled by means of the circuit of Figure 2 as proposed in [11]. This circuit equivalent is known as parallel conductance, which allows to separate ions of sodium (Na), potassium (K) and leakage ions separately (branches) [7], in this way four components are generated.
• Current due to Na ions.
• Current due to K ions.
• Current due to other ions, ions of fugue (leakage ions).
• Displacement current due to the shape of the membrane (uniform).

**Figure 2.** Equivalent circuit for action on a membrane.

In Figure 2, each of the four currents mentioned above is assumed independent for each channel or branch. To mathematically model the phenomenon produced by exciting the membrane by means of a time-varying source (electrode), it is necessary to construct the Equation (15) to Equation (19) from the circuit of Figure 2 by means of basic laws of electrical circuits.

\[
im = i_c + i_{GNa} + i_{GK} + i_L, \tag{15}
\]

\[
i_{GNa} = G_{Na}(V_m - V_{Na}), \tag{16}
\]

\[
i_{GK} = G_{K}(V_m - V_{K}), \tag{17}
\]

\[
i_L = G_{L}(V_m - V_{L}), \tag{18}
\]

\[
i_c = C_m \frac{\partial V_m}{\partial t}. \tag{19}
\]

Replacing each one of the produced currents Equation (16) to Equation (19) by the Na, K and leakage ions in Equation (15), we have Equation (20).

\[
im = C_m \frac{\partial V_m}{\partial t} + G_{Na}(V_m - V_{Na}) + G_{K}(V_m - V_{K}) + G_{L}(V_m - V_{L}). \tag{20}
\]

If the cell membrane is excited with a constant source, the dynamics produced by the source is zero, that is, \(\frac{\partial V_m}{\partial t} = 0\), on the other hand, this dynamic is considered negligible, therefore due to various experiments it is necessary to feed said membrane with sources variables in time. The propagation in the membrane due to the impulse must be modeled as the distribution of the potential through a membrane in a uniform medium with characteristics imposed by Na, K and other chemical combinations typical of the cell membrane.

Starting from Figure 2 and using the fact that the impedance gathers all the resistances that can be accumulated in a single one through equivalents in series and parallel, we can obtain the equivalent circuit represented in Figure 3, in it we can see how the electric potential is represented as a source and the field current passes through the impedance that opposes a resistance to the electric fluid. This type of circuit is a simplified model that reflects and models ohm’s law and in which Kirchhoff’s equations can be used to perform circuit analysis. This model can be simplified into a single stream that stores the sodium and potassium currents, the displacement current, and those due to other ions. From the circuit shown in Figure 3 it is possible to obtain by means of Ohm’s law a relation between the current \(i_m\) and \(V_m\) in Laplacian terms as proposed in [12], Equation (21).
\[ \text{im} = rm \nabla^2 (Vm) = rm \left( \frac{\partial^2 V_m}{\partial x^2} + \frac{\partial^2 V_m}{\partial y^2} \right). \] (21)

In this way we obtain an equation in partial derivatives, Equation (22).

\[ rm \left( \frac{\partial^2 V_m}{\partial x^2} + \frac{\partial^2 V_m}{\partial y^2} \right) = C_m \frac{\partial V_m}{\partial t} + G_Na(V_m - V_Na) + G_K(V_m - V_K) + G_L(V_m - V_L). \] (22)

Figure 3. Simplified model dependent on the geometry of the membrane.

The Figure 4 show the time dynamics of the voltage on the membrane (Vm) due to a unitary pulse type excitation with duration of 0.5 ms and a surface load density amplitude of 2000 \( \mu A/cm^2 \). This impulse propagates through the whole membrane displacing loads and generating a potential in the whole geometry of the membrane. The intensity of this potential is presented in the color band as follows: red for higher intensity and blue for lower intensity due to the characteristics of the medium.

Figure 4. Behavior of the Vm in 12 instants of time, while a pulse last.

The pulse shown in the Figure 5, represents the behavior for any instant of time \( t \in [0 - 15 \text{ ms}] \) for the point \((0,0)\) of the membrane. The maximum and minimum values are 112 mV and \(-54.4 \text{ mV}\) respectively. This type of pulse allows to simulate an external excitation and the way in which the different layers would react, it is also desired through such a short pulse to generate a transient which in classical systems is not taken into account and is neglected on the contrary, facilitating the solution to the mathematical model; on the contrary, when considering it, it should not only be assumed that there is a local disturbance but also a limited variation.
4. Conclusion
Deep brain stimulation has become very precise due to the implementation of computational techniques that allow us to approximate the numerical solution of the partial differential equations that model the effects of the potential difference in the human brain. This technique allows surgeons to place electrodes in virtually any area of the brain, and turn it on or off, such as a radio or a thermostat, to correct defects. According to the results shown in Figure 4 and Figure 5 it is possible to determine which areas of the brain are involved by the interaction of potential sources; bioelectric sources originate electrochemical activity in brain cells that are inactive from so that they generate controlled movements in the human body. It is not only possible to study the interaction of the skull model with constant sources (electrodes with constant potential difference, (Figure 3) but with the interaction of time varying sources. In addition, as we studied in this document how was the potential behavior in different media for an isotropic and uniform model of the human head, we can also study the behavior of time-variant sources in applications such as electrocardiography to activate inactive regions of the heart, electromyography to improve the functioning of muscles in patients with conditions that are related to movement. The finite element method used in the development of this document is shown as a powerful tool for solving problems in physics, mathematics and in general in various applied fields where it is required to solve some complex type of partial differential equations.

References
[1] Hofmann G A, Dev S B, Dimmer S, Nanda G S 1999 IEEE Trans. Biomed. Eng. 46 752-759
[2] Plonsey R, Heppner D B 1967 Bull. Math. Biophys. 29 657–664
[3] Fedoryuk M 1999 Asymptotic Methods for Partial Differential Equations (Berlin: Springer-Verlag)
[4] Wolters C H, Kuhl M, Anwander A and Reitzinger S 2002 Comput. Visual. Sci. 5 165–177
[5] González J E, García D F 2017 Tecnologias 20 27-39
[6] Yang W, Wu Y H, Yin D, Koeffler H P, Sawcer D E, Vernier P T, Gundersen M A 2011 Technol. Cancer Res. Treat. 10 281-286
[7] Leontiadou H, Mark A E, Marrink S J 2004 Biophys J. 86(4) 2156-2164
[8] Weaver J C 2003 IEEE Trans. Dielectr. Electr. Insul. 10(5) 754-768
[9] Shamis Y, Croft R, Taube A, Crawford R J, Ivanova E P 2012 Appl. Microbiol. Biotechnol. 96(2) 319-325
[10] Sheetz M P, Dai J 1996 Trends. Cell. Biol. 6 85-89
[11] Chen M, Jiang C, Vernier PT, Wu Y H, Gundersen M A 2009 Two-dimensional nanosecond electric field mapping based on cell electropermeabilization PMC Biophysics 2 9:1-16
[12] Haueisen J, Touch D S, Ramon C, Schimpf P H, Wedeen V J, Georgee J S, Belliveau J W 2002 Neuro Image 15 159-166

Figure 5. $V_m(t)$ at point $(0,0)$ of the membrane geometry.