Rate-improved Inexact Augmented Lagrangian Method for Constrained Nonconvex Optimization

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Abstract First-order methods have been studied for nonlinear constrained optimization within the framework of the augmented Lagrangian method (ALM) or penalty method. We propose an improved inexact ALM (iALM) and conduct a unified analysis for nonconvex problems with either convex or nonconvex constraints. Under certain regularity conditions (that are also assumed by existing works), we show an $\tilde{O}(\varepsilon^{-\frac{5}{2}})$ complexity result for a problem with a nonconvex objective and convex constraints and an $\tilde{O}(\varepsilon^{-3})$ complexity result for a problem with a nonconvex objective and nonconvex constraints, where the complexity is measured by the number of first-order oracles to yield an $\varepsilon$-KKT solution. Both results are the best known. The same-order complexity results have been achieved by penalty methods. However, two different analysis techniques are used to obtain the results, and more importantly, the penalty methods generally perform significantly worse than iALM in practice. Our improved iALM and analysis close the gap between theory and practice. Numerical experiments are provided to demonstrate the effectiveness of our proposed method.

1 Introduction

First-order methods (FOMs) have been extensively used for solving large-scale optimization problems, partly due to its nice scalability. Compared to second-order or higher-order methods, FOMs generally have much lower per-iteration complexity and much lower requirement on machine memory. A majority of existing works on FOMs focus on problems without constraints or with simple constraints, e.g., [2,3,5,16,24]. Several recent works have made efforts on analyzing FOMs for problems with complicated functional constraints, e.g., [11,13,14,19,30,31,33].

In this paper, we consider nonconvex problems with nonlinear constraints, formulated as

$$f_0^* := \minimize_{x \in \mathbb{R}^n} \{ f_0(x) := g(x) + h(x), \text{ s.t. } c(x) = 0 \},$$

where $g$ is continuously differentiable but possibly nonconvex, $c = (c_1, \ldots, c_l) : \mathbb{R}^n \to \mathbb{R}^l$ is a vector function with continuously differentiable components, and $h$ is closed convex but possibly nonsmooth. Note that an
inequality constraint \( d(x) \leq 0 \) can be equivalently formulated as an equality constraint \( d(x) + s = 0 \) by enforcing the nonnegativity of \( s \). In addition, the stationary conditions of an inequality-constrained problem and its reformulation can be equivalent, as we will see later at the end of section 3. Hence, we do not lose generality by focusing on equality-constrained problems in the form of (1).

1.1 Related works

The augmented Lagrangian method (ALM) is one of the most popular approaches for solving nonlinear constrained problems. It first appeared in [7, 26]. Based on the augmented Lagrangian (AL) function, ALM alternatingly updates the primal variable by minimizing the AL function and the Lagrangian multiplier by dual gradient ascent. If the multiplier is fixed to zero, then ALM reduces to a standard penalty method. Early works often used second-order methods, such as the Newton’s method, to solve primal subproblems of ALM. With the rapid increase of problem size in modern applications and/or existence of non-differentiable terms, second-order methods become extremely expensive or even inapplicable. Recently, more efforts have been made on integrating first-order solvers into the ALM framework and analyzing the AL-based FOMs.

For convex affinely-constrained problems, [10] presents an AL-based FOM that can produce an \( \varepsilon \)-KKT point with \( O(\varepsilon^{-1}|\log \varepsilon|) \) gradient evaluations and matrix-vector multiplications. This result was extended to convex nonlinear constrained problems in [11, 12]. When an \( \varepsilon \)-optimal solution is desired, \( O(\varepsilon^{-1}) \) complexity results have been established for AL-based FOMs in several works, e.g., [11, 22, 25, 29, 31]. For strongly-convex problems, the complexity results can be respectively improved to \( O(\varepsilon^{-\frac{3}{2}}|\log \varepsilon|) \) for an \( \varepsilon \)-KKT point and \( O(\varepsilon^{-\frac{3}{2}}) \) for an \( \varepsilon \)-optimal solution; see [11, 12, 21, 22, 31] for example.

For nonconvex constrained problems, early works designed and analyzed FOMs in the framework of a penalty method. [4] first presents an FOM for minimizing composite functions and then applies it to nonlinear constrained nonconvex optimization within the framework of an exact-penalty method. To obtain an \( \varepsilon \)-KKT point, the FOM in [4] needs \( O(\varepsilon^{-5}) \) gradient evaluations. A follow-up paper gives a trust-region based FOM and shows an \( O(\varepsilon^{-2}) \) complexity result to produce an \( \varepsilon \)-Fritz-John point, which is weaker than an \( \varepsilon \)-KKT point. On solving affinely-constrained nonconvex problems, [9] gives a quadratic-penalty-based FOM and establishes an \( O(\varepsilon^{-3}) \) complexity result to obtain an \( \varepsilon \)-KKT point. When Slater’s condition holds, \( \tilde{O}(\varepsilon^{-\frac{3}{2}}) \) complexity results have been shown in [12, 13], which consider nonconvex problems with nonlinear convex constraints. While the FOMs in [12, 13] are penalty-based, the recent work [20] proposes a first-order proximal ALM for affinely-constrained nonconvex problems and obtains an \( \tilde{O}(\varepsilon^{-\frac{3}{2}}) \) result.

Besides AL and penalty-based FOMs, several other FOMs have been designed to solve nonlinear-constrained problems, such as the level-set FOM in [14] and the primal-dual method in [32] for convex problems. FOMs have also been proposed for minimax problems. For example, [6, 8] study FOMs for convex-concave minimax problems, and [15, 17, 18] analyzes FOMs for nonconvex-concave minimax problems. While a nonlinear-constrained optimization problem can be formulated as a minimax problem, its KKT conditions are stronger than the stationarity conditions of a nonconvex-concave minimax problem, because the latter with a compact dual domain cannot guarantee primal feasibility.

1.2 Contributions

Our contributions are three-fold. First, we propose a novel FOM in the framework of inexact ALM (iALM) for nonconvex optimization problems with nonlinear (possibly nonconvex) constraints. Due to nonlinearity
Rate-improved Inexact Augmented Lagrangian Method for Constrained Nonconvex Optimization

and large-scale, it is impossible to exactly solve primal subproblems of ALM, and the iALM instead solves each subproblem approximately to a certain desired accuracy. Different from existing works on iALMs, we use an inexact proximal point method (iPPM) to solve each ALM subproblem. The use of iPPM leads to more stable numerical performance and also better theoretical results. Second, we conduct complexity analysis to the proposed iALM. Under a regularity condition, we obtain an $\tilde{O}(\varepsilon^{-\frac{5}{2}})$ result if the constraints are convex and an $\tilde{O}(\varepsilon^{-3})$ result if the constraints are nonconvex. This yields a substantial improvement over the best known complexity results of AL-based FOMs, $\tilde{O}(\varepsilon^{-3})$ [12] and $\tilde{O}(\varepsilon^{-4})$ [27] respectively for the aforementioned convex and nonconvex constrained cases. While quadratic-penalty-based FOMs (under the same regularity condition as what we assume for nonconvex-constraint problems) [13] have achieved the same-order results as ours, their empirical performance is generally (much) worse. Hence, our results close the gap between theory and practice. Thirdly, our algorithm and analysis are unified for the convex-constrained and nonconvex-constrained cases. Existing works on penalty-based FOMs need different algorithmic designs and also different analysis techniques to obtain the $\tilde{O}(\varepsilon^{-\frac{5}{2}})$ and $\tilde{O}(\varepsilon^{-3})$ results, separately for the convex-constrained and nonconvex-constrained cases.

1.3 Complexity comparison on different methods

In Table 1, we summarize our complexity results and several existing ones of first order methods to produce an $\varepsilon$-KKT solution to (1). We consider several cases based on whether the objective and the constraints are convex. Here, constraints being convex means that the feasible set is convex, or in other words, equality constraint functions must be affine and inequality constraint functions must be convex. Our result matches the best-known existing results, which are achieved by penalty-type methods such as the iPPP in [13]. In practice, AL-type methods usually significantly outperform penalty-type methods. Hence, our method is competitive in theory and can be significantly better in practice, as we demonstrated in the numerical experiments.

1.4 Notations, definitions, and assumptions

We use $\| \cdot \|$ for the Euclidean norm of a vector and the spectral norm of a matrix. For a positive integer, $[n]$ denotes the set $\{1, \ldots, n\}$. The big-O notation is used with standard meaning, while $\tilde{O}$ suppresses all logarithmic terms of $\varepsilon$. Given $x \in \text{dom}(h)$, we denote $J_c(x)$ as the Jacobi matrix of $c$ at $x$. We denote the distance function between a vector $x$ and a set $X$ as $\text{dist}(x, X) = \min_{y \in X} \|x - y\|$. The augmented Lagrangian (AL) function of (1) is

$$L_{\beta}(x, y) = f_0(x) + y^T c(x) + \frac{\beta}{2} \|c(x)\|^2,$$

where $\beta > 0$ is a penalty parameter, and $y \in \mathbb{R}^l$ is the multiplier vector.

$^1$ An $\tilde{O}(\varepsilon^{-3})$ complexity is claimed in Corollary 4.2 in [27]. However, this complexity is based on an existing result that was not correctly referred to. The authors claimed that the complexity of solving each nonconvex composite subproblem is $O\left(\frac{\lambda_k^2 \varepsilon^2}{\varepsilon_{k+1}}\right)$, which should be $O\left(\frac{\lambda_k^2 \varepsilon^2}{\varepsilon_{k+1}^2}\right)$; see [27] for the definitions of $\lambda_k, \rho, \varepsilon_{k+1}$. Using the correctly referred result and following the same proof in [27], we get a total complexity of $\tilde{O}(\varepsilon^{-4})$. 

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Definition 1 (ε-KKT point) Given $\varepsilon \geq 0$, a point $x \in \mathbb{R}^n$ is called an $\varepsilon$-KKT point to (1) if there is a vector $y \in \mathbb{R}^l$ such that
\[
\|c(x)\| \leq \varepsilon, \quad \text{dist}(0, \partial f_0(x) + J^T_c(x) y) \leq \varepsilon.
\] (3)

Definition 2 ($L$-smooth) A differentiable function $f$ on $\mathbb{R}^n$ is $L$-smooth if $\|\nabla f(x_1) - \nabla f(x_2)\| \leq L\|x_1 - x_2\|$ for all $x_1, x_2 \in \mathbb{R}^n$.

Definition 3 ($\rho$-weakly convex) A function $g$ is $\rho$-weakly convex if $g + \frac{\rho}{2}\|\cdot\|^2$ is convex.

Remark 1 If $f$ is $L$-smooth, then it is also $L$-weakly convex. However, the weak-convexity constant of a differentiable function can be much smaller than its smoothness constant.

Throughout the paper, we make the following assumptions about (1).

Assumption 1 (smoothness and weak convexity) The function $g$ in the objective of (1) is $L_0$-smooth and $\rho_0$-weakly convex. For each $j \in [l]$, $c_j$ is $L_j$-smooth and $\rho_j$-weakly convex.

Assumption 2 (bounded domain) $h$ is a simple closed convex function with a compact domain, i.e.,
\[
D =: \max_{x,x' \in \text{dom}(h)} \|x - x'\| < \infty.
\] (4)

2 A novel AL-based FOM with improved convergence rate

In this section, we present a novel FOM (see Algorithm 3 below) for solving (1). It follows the ALM framework, similar to AL-based FOMs in [27,31]. Different from existing works, we use an inexact proximal point method (iPPM) to approximately solve each ALM subproblem. The complexity result of iPPM has the best dependence on the smoothness constant. This enables us to obtain order-reduced complexity results by geometrically increasing the penalty parameter in ALM, as compared to the AL-based FOMs in [12,27] for nonconvex constrained optimization.

Table 1 Comparison of the complexity results of several methods in the literature to our method to produce an $\varepsilon$-KKT solution to (1).

| Method      | type   | problem                                   | objective     | constraint | regularity | complexity                  |
|-------------|--------|-------------------------------------------|---------------|------------|------------|-----------------------------|
| iALM [12]   | AL     | $\min_x \{g(x) + h(x), \ s.t. \ Ax = b, f(x) \leq 0}\}$ | strongly convex | convex     | none       | $O(\varepsilon^{-\frac{3}{2}})$ |
| QP-AIPP [9] | penalty| $\min_x \{g(x) + h(x), \ s.t. \ Ax = b\}$ | nonconvex     | convex     | none       | $O(\varepsilon^{-3})$       |
| HiAPeM [12] | hybrid | $\min_x \{g(x) + h(x), \ s.t. \ Ax = b, f(x) \leq 0\}$ | nonconvex     | convex     | Slater’s condition          | $O(\varepsilon^{-\frac{5}{2}})$ |
| iPPP [13]   | penalty| $\min_x \{g(x) + h(x), \ s.t. \ c(x) = 0, f(x) \leq 0\}$ | nonconvex     | nonconvex  | Slater’s condition          | $O(\varepsilon^{-4})$       |
| iALM [27]   | AL     | $\min_x \{g(x) + h(x), \ s.t. \ c(x) = 0\}$ | nonconvex     | nonconvex  | Assumption 3              | $O(\varepsilon^{-\frac{7}{2}})$ |
| this paper  | AL     | $\min_x \{g(x) + h(x), \ s.t. \ c(x) = 0\}$ | nonconvex     | nonconvex  | Assumption 3              | $O(\varepsilon^{-4})$       |
Algorithm 1: Accelerated proximal gradient method: APG($G, H, \mu, L_G, \varepsilon$)

1. **Initialization:** choose $\bar{x}^{-1} \in \text{dom}(H)$ and set $\alpha = \sqrt{\frac{\mu}{L_G}}$.
2. Let $\bar{x}^0 = x^0 = \arg\min_x \langle \nabla G(\bar{x}^{-1}), x \rangle + \frac{L_G}{2} \| x - \bar{x}^{-1} \|^2 + H(x)$.
3. for $t = 0, 1, \ldots$ do
   4. Update the iterate by
      
      $$x^{t+1} = \arg\min_x \langle \nabla G(\bar{x}^t), x \rangle + \frac{L_G}{2} \| x - \bar{x}^t \|^2 + H(x),$$
      
      $\bar{x}^{t+1} = x^{t+1} + \frac{1 - \alpha}{1 + \alpha} (x^{t+1} - x^t).$
      
   if $\text{dist}( - \nabla G(x^{t+1}), \partial H(x^{t+1})) \leq \varepsilon$, then output $x^{t+1}$ and stop.

2.1 Accelerated proximal gradient (APG) method for convex composite problems

The kernel problems that we solve are a sequence of convex composite problems in the form of

$$\min_{x \in \mathbb{R}^n} F(x) := G(x) + H(x),$$

where $G$ is $\mu$-strongly convex and $L_G$-smooth, and $H$ is a closed convex function. Various optimal FOMs (e.g., [2, 23, 24]) have been designed to solve (7). We choose the FOM used in [12] for the purpose of obtaining near-stationary points. Its pseudocode is given in Algorithm 1.

The next lemma is from [12, Lemma 3]. It gives the complexity result of Algorithm 1.

**Lemma 1** Given $\varepsilon > 0$, within at most $T$ iterations, Algorithm 1 will output a solution $x^T$ that satisfies $\text{dist}(0, \partial F(x^T)) \leq \varepsilon$, where

$$T = \left\lceil \frac{64L_G^2}{\varepsilon^2} \log \left( \frac{L_G}{\mu} \| x^0 - x_* \|^2 + \mu \| x_* - x_0 \|^2 \right) + 1 \right\rceil.$$

2.2 Inexact proximal point method (iPPM) for nonconvex composite problems

Each primal subproblem of the ALM for (1) is a nonconvex composite problem in the form of

$$\Phi^* = \min_{x \in \mathbb{R}^d} \{ \Phi(x) := \phi(x) + \psi(x) \},$$

where $\phi$ is $L_\phi$-smooth and $\rho$-weakly convex, and $\psi$ is closed convex. We propose to use the iPPM to approximately solve the ALM subproblems. The iPPM framework has appeared in [9]. Different from [9], we propose to use APG in Algorithm 1 to solve each iPPM subproblem. The pseudocode of our iPPM is shown in Algorithm 2. It appears that our iPPM has more stable numerical performance.

The next theorem gives the complexity result. Its proof is given in the supplementary materials.

**Theorem 1** Suppose $\Phi^*$ is finite. Algorithm 2 must stop within $T$ iterations, where

$$T = \left\lceil \frac{64L^2}{\varepsilon^2} (\Phi(x^0) - \Phi^*) \right\rceil.$$

The output $x^S$ must be an $\varepsilon$-stationary point of (8), i.e., $\text{dist}(0, \partial \Phi(x^S)) \leq \varepsilon$. In addition, if $\text{dom}(\psi)$ is compact and has diameter $D_\psi < \infty$, then the total complexity is $O \left( \frac{\sqrt{D_\psi}}{\varepsilon^2} [\Phi(x^0) - \Phi^*] \log \frac{D_\psi}{\varepsilon} \right)$. 

Algorithm 2: Inexact proximal point method (iPPM) for (8): iPPM(φ, ψ, x0, ρ, Lφ, ε)

1. **Input:** x0 ∈ dom(ψ), smoothness Lφ, weak convexity ρ, stationarity tolerance ε
2. for k = 0, 1, ..., do
3.   Let G(·) = φ(·) + ρ∥· − xk∥2
4.   Call Algorithm 1 to obtain xk+1 ← APG(G, ψ, ρ, Lφ + 2ρ, ε/k)
5.   if 2ρ∥xk+1 − xk∥ ≤ ε, then return xk+1.

**Proof.** Let Φk(x) := Φ(x) + ρ∥x − xk∥2 and Φk = minx Φk(x) for each k ≥ 0. Note we have dist(0, ∂Φk(xk+1)) ≤ δ = ε/k, and also Φk is ρ-strongly convex. Hence Φk(xk+1) − Φ∗ ≤ δ2/2ρ, and Φ(xk+1) + ρ∥xk+1 − xk∥2 − Φ(xk) ≤ δ2/2ρ. Thus,

\[
\Phi(x^T) - \Phi(x^0) + \rho \sum_{k=0}^{T-1} \|x^{k+1} - x^k\|^2 \leq \frac{T \delta^2}{2\rho}
\]

\[
T \min_{0 \leq k \leq T-1} \|x^{k+1} - x^k\|^2 \leq \frac{1}{\rho} \left( \frac{T \delta^2}{2\rho} + [\Phi(x^0) - \Phi(x^T)] \right)
\]

\[
2\rho \min_{0 \leq k \leq T-1} \|x^{k+1} - x^k\|^2 \leq 2 \sqrt{\frac{\delta^2}{2} + \frac{\rho(\Phi(x^0) - \Phi^*)}{T}}. \tag{10}
\]

Since \(T \geq \frac{32\rho}{\varepsilon^2} \left[ \Phi(x^0) - \Phi^* \right] \) and \(\delta = \frac{\varepsilon}{k} \), we have

\[
\frac{\rho}{T} [\Phi(x^0) - \Phi^*] \leq \frac{\varepsilon^2}{32}, \tag{11}
\]

and thus (10) implies

\[
2\rho \min_{0 \leq k \leq T-1} \|x^{k+1} - x^k\|^2 \leq \frac{\varepsilon}{2}. \tag{12}
\]

Therefore, Algorithm 2 must stop within \(T \) iterations, from its stopping condition, and when it stops, the output \(x^S \) satisfies \(2\rho\|x^S - x^{S-1}\| \leq \frac{\varepsilon}{2} \). Now recall dist(0, ∂Φk(xk+1)) ≤ δ = ε/k, i.e.,

\[
\text{dist}(0, \partial\Phi(x_{k+1}) + 2\rho(x_{k+1} - x_k)) \leq \frac{\varepsilon}{2}, \quad \forall k \geq 0. \tag{13}
\]

The above inequality together with \(2\rho\|x^S - x^{S-1}\| \leq \frac{\varepsilon}{2} \) gives

\[
\text{dist}(0, \partial\Phi(x^S)) \leq \varepsilon,
\]

which implies that \(x^S \) is an \(\varepsilon\)-stationary point to (8).

Finally, we apply Lemma 1 to obtain the overall complexity and complete the proof.

\[\square\]

**Remark 2** A similar result has been shown in [9]. It has better dependence on \(Lφ\) than that in [5]. In addition, in the worst case, \(\Phi(x^0) - \Phi^*\) is in the same order of \(Lφ\). However, we will see that for our case, \(\Phi(x^0) - \Phi^*\) can be uniformly bounded when Algorithm 2 is applied to solve subproblems of ALM even if the penalty parameter (that is proportional to the smooth constant) in the AL function geometrically increases. This is the key for us to have order-reduced complexity results.
2.3 Inexact augmented Lagrangian method (iALM) for nonlinear constrained problems

Now we are ready to present an improved AL-based FOM for solving (1). Different from existing AL-based FOMs, our method uses iPPM, given in Algorithm 2, to approximately solve each subproblem, and also its dual step size is adaptive to the primal residual. The pseudocode is shown in Algorithm 3.

In the algorithm and the later analysis, we denote

\[
B_0 = \max_{x \in \text{dom}(h)} \max \{|f_0(x)|, \|\nabla g(x)\|\}, \quad B_c = \max_{x \in \text{dom}(h)} \|J_c(x)\|,
\]

\[
B_i = \max_{x \in \text{dom}(h)} \max \{|c_i(x)|, \|\nabla c_i(x)\|\}, \forall i \in [l],
\]

\[
\bar{B}_c = \sqrt{\sum_{i=1}^l B_i^2}, \quad \bar{L} = \sum_{i=1}^l L_i^2, \quad \rho_c = \sum_{i=1}^l B_i \rho_i, \quad L_c = \sum_{i=1}^l B_i L_i + B_i^2,
\]

where \(\{\rho_i\}\) and \(\{L_i\}\) are given in Assumption 1. Note that the above constants are all finite under Assumptions 1 and 2, and we do not need to evaluate them exactly but only need upper bounds.

Algorithm 3: Inexact augmented Lagrangian method (iALM) for (1)

1. **Initialization:** choose \(x^0 \in \text{dom}(f_0), y^0 = 0, z^0 = 0, \beta_0 > 0\) and \(\sigma > 1\)
2. for \(k = 0, 1, \ldots, \) do
3. Let \(\beta_k = \beta_0 \sigma^k, \phi(\cdot) = L_{\beta_k} (\cdot, y^k) - h(\cdot),\) and

\[
\hat{\rho}_k = \rho_0 + \bar{L}\|y^k\| + \beta_k \rho_c, \quad \hat{L}_k = \bar{L} + \beta_k L_c.
\]

4. Call Algorithm 2 to obtain \(x^{k+1} \leftarrow \text{iPPM}(\phi, h, x^k, \hat{\rho}_k, \hat{L}_k, \varepsilon)\)
5. Update \(y\) by

\[
y^{k+1} = y^k + w_k c(x^{k+1}),
\]

where

\[
w_k = w_0 \min \left\{1, \frac{\gamma_k}{\|c(x^{k+1})\|} \right\}.
\]

Algorithm 3 follows the standard framework of the ALM. The existing method that is the closest to ours is the iALM in [27]. The main difference is that we use the iPPM to solve ALM subproblems, while [27] applies the FOM in [5]. This change of subroutine, together with our new analysis, leads to order-reduced complexity results under the same assumptions.

3 Complexity results

In this section, we analyze the complexity result of Algorithm 3. In general, it is difficult to show convergence rates of AL-based FOMs on nonconvex constrained problems mainly due to two reasons. First, a stationary point of the AL function may not be (near) feasible, even a large penalty parameter is used. This is essentially different from penalty-based FOMs. Second, the Lagrangian multiplier cannot be bounded if the dual step size is not carefully set. We show that, with a regularity condition and a well-controlled dual step size, our AL-based FOM can circumvent both issues and achieve best-known convergence rates.
For simplicity, we let 
\[ \gamma_k = \frac{(\log 2)^2 \|c(x^1)\|}{(k + 1)(\log(k + 2))^2}, \]  
which has been adopted in [27]. This choice of \( \gamma_k \) will lead to a uniform bound on \( \{y^k\} \) and simplify our analysis. More complicated analysis with general \( \{\gamma_k\} \) is given in the supplementary materials.

Since it is impossible to find a (near) feasible solution of a general nonlinear system, a certain regularity condition must be made in order to guarantee near-feasibility. Following [27] and [13], we assume the regularity condition below on (1).

**Assumption 3 (regularity)** There is some \( v > 0 \) such that 
\[ v\|c(x_k)\| \leq \text{dist}(0, A^\top(Ax - b) + N_X(x)), \forall k \geq 1. \]  
\[ (19) \]

**Remark** [27] and [13] have given several nonconvex examples that satisfy the regularity condition. The LCQP problem in section 4.1 also has this property, as proven in subsection 3.1, where we also provide another convex example satisfying this property. Notice that we only require the existence of \( v \) but do not need to know its value in our algorithm.

### 3.1 Convex constraint examples with regularity condition

In this subsection, we prove that the regularity condition in Assumption 3 can hold for the LCQP problem (44) that we test. We will show it for all compact polyhedral set. Also, we prove the regularity condition for the constraint \( \{x \in \mathbb{R}^n : Ax = b, \|x\| \leq 1\} \).

#### 3.1.1 polyhedral constraint

Let \( X \subseteq \mathbb{R}^n \) be a compact polyhedral set and \( h(\cdot) = \iota_X(\cdot) \) be the indicator function on \( X \). Then for any \( \beta > 0 \) and \( x \in X \), \( \frac{\partial h(x)}{\partial \beta} = N_X(x) \), where \( N_X \) denotes the normal cone. We have the result in the claim below.

**Claim** If \( X \cap \{x \in \mathbb{R}^n : Ax = b\} \neq \emptyset \), then there is a constant \( v > 0 \) such that 
\[ v\|Ax - b\| \leq \text{dist}(0, A^\top(Ax - b) + N_X(x)), \forall x \in X, \]  
\[ (20) \]
which implies (19) with \( c(x) = Ax - b \) and \( h(x) = \iota_X(x) \).

By this claim, we let \( X = \{x \in \mathbb{R}^n : l_i \leq x_i \leq u_i, \forall i \in [n]\} \) and immediately have that the LCQP problem (44) satisfies the regularity condition in Assumption 3.

**Proof.**[Proof of Claim 3.1.1] Let \( X_* \) be the optimal solution set of 
\[ \min_{x \in X} f(x) := \frac{1}{2}\|Ax - b\|^2. \]  
\[ (21) \]
Then for any \( x \in X_* \), \( Ax - b = 0 \) by our assumption. From [28, Theorem 18], it follows that there is a constant \( \kappa > 0 \) such that 
\[ \|x - \text{Proj}_{X_*}(x)\| \leq \kappa \|x - \text{Proj}_X(x - \nabla f(x))\|, \forall x \in X, \]  
\[ (22) \]
where \( \text{Proj}_X \) denotes the Euclidean projection onto \( X \).

For any fixed \( x \in X \), denote \( u = \nabla f(x) \) and \( v = \text{Proj}_X(x - u) \). Then from the definition of the Euclidean projection, it follows that \( \langle v - x + u, v - x' \rangle \leq 0, \forall x' \in X \). Letting \( x' = x \), we have \( \|v - x\|^2 \leq \langle u, x - v \rangle \).

On the other hand, for any \( z \in N_X(x) \), we have from the definition of the normal cone that \( \langle z, x - x' \rangle \geq 0, \forall x' \in X \). Hence, letting \( x' = v \) gives \( \langle z, x - v \rangle \geq 0 \). Therefore, we have

\[
\|v - x\|^2 \leq \langle u, x - v \rangle + \langle z, x - v \rangle = \langle u + z, x - v \rangle \leq \|x - v\| \cdot \|u + z\|,
\]

which implies \( \|v - x\| \leq \|u + z\| \). By the definition of \( u \) and \( v \) and noticing that \( z \) is an arbitrary vector in \( N_X(x) \), we obtain

\[
\|x - \text{Proj}_X(x - \nabla f(x))\| \leq \text{dist} (0, \nabla f(x) + N_X(x)).
\]

The above inequality together with (22) gives

\[
\|x - \text{Proj}_X(x)\| \leq \kappa \cdot \text{dist} (0, \nabla f(x) + N_X(x)), \quad \forall x \in X.
\] (23)

Now by the fact \( A \text{Proj}_X(x) = b \), we have \( \|Ax - b\| \leq \|A\| \cdot \|x - \text{Proj}_X(x)\| \). Therefore, from (23) and also noting \( \nabla f(x) = A^\top (Ax - b) \), we obtain (20) with \( v = \frac{1}{\|A\|} \).

\[ \square \]

3.1.2 ball constraint

Let \( X = \{x \in \mathbb{R}^n : \|x\| \leq r\} \) be a ball of radius \( r > 0 \) and \( h \) be the indicator function on \( X \). Then we have the following result.

**Claim** Suppose \( A \) has full row-rank. In addition, there exists a \( \hat{x} \) in the interior of \( X \) such that \( A\hat{x} = b \). Then there is a constant \( v > 0 \) such that (20) holds.

**Proof.** [Proof of Claim 3.1.2] Without loss of generality, we assume \( r = 1 \) and \( AA^\top = I \), i.e., the row vectors of \( A \) are orthonormal. Notice that

\[
N_X(x) = \begin{cases} 
\{0\}, & \text{if } \|x\| < 1, \\
\{\lambda x : \lambda \geq 0\}, & \text{if } \|x\| = 1.
\end{cases}
\] (24)

Hence, if \( \|x\| < 1 \), (20) holds with \( v = 1 \) because \( AA^\top = I \). In the following, we focus on the case of \( \|x\| = 1 \).

When \( \|x\| = 1 \), we have from (24) that

\[
\text{dist} (0, A^\top (Ax - b) + N_X(x)) = \min_{\lambda \geq 0} \|A^\top (Ax - b) + \lambda x\|.
\] (25)

If the minimizer of the right hand side of (25) is achieved at \( \lambda = 0 \), then (20) holds with \( v = 1 \). Otherwise, the minimizer is \( \lambda = -x^\top A^\top (Ax - b) \geq 0 \). With this \( \lambda \), we have

\[
[\text{dist} (0, A^\top (Ax - b) + N_X(x))]^2 = \|A^\top (Ax - b) - x^\top A^\top (Ax - b)x\|^2 \\
= (Ax - b)^\top A(I - xx^\top)A^\top (Ax - b).
\]

Let

\[
v_* = \min_{x} \left\{ \lambda_{\min} (A(I - xx^\top)A^\top), \quad \text{s.t. } x^\top A^\top (Ax - b) \leq 0, \quad \|x\| = 1 \right\},
\] (26)
where \( \lambda_{\min}(\cdot) \) denotes the minimum eigenvalue of a matrix. Then \( \nu_\epsilon \) must be a finite nonnegative number. We show \( \nu_\epsilon > 0 \). Otherwise suppose \( \nu_\epsilon = 0 \), i.e., there is a \( x \) such that \( x^\top A^\top (Ax - b) \leq 0 \) and \( \|x\| = 1 \), and also \( A(I - xx^\top)A^\top \) is singular. Hence, there exists a \( y \neq 0 \) such that

\[
A(I - xx^\top)A^\top y = 0.
\]

By scaling, we can assume \( \|y\| = 1 \). Let \( z = A^\top y \). Then \( \|z\| = 1 \), and from (27), we have \( z^\top (I - xx^\top)z = 1 - (z^\top x)^2 = 0 \). This equation implies \( z = x \) or \( z = -x \), because both \( x \) and \( z \) are unit vectors. Without loss of generality, we can assume \( z = x \). Now recall \( b = Ax \) with \( \|x\| < 1 \) and notice

\[
x^\top A^\top (Ax - b) = z^\top A^\top (Az - b) = z^\top A^\top (y - Ax) = 1 - z^\top A^\top Ax > 0,
\]

where the inequality follows from \( \|A\| = 1 \), \( \|z\| = 1 \), and \( \|x\| < 1 \). Hence, we have a contradiction to \( x^\top A^\top (Ax - b) \leq 0 \). Therefore, \( \nu_\epsilon > 0 \).

Putting the above discussion together, we have that (20) holds with \( v = \min\{1, \nu_\epsilon\} \), where \( \nu_\epsilon \) is defined in (26). This completes the proof. \( \square \)

**Theorem 2 (total complexity of iALM)** Suppose that all conditions in Assumptions 1 through 3 hold. Given \( \epsilon > 0 \), then Algorithm 3 with \( \gamma_k \) given in (18) needs \( \tilde{O}(\epsilon^{-3}) \) APG iterations to produce an \( \epsilon \)-KKT solution of (1). In addition, if \( c(x) = Ax - b \) is affine, then \( \tilde{O}(\epsilon^{-2}) \) APG iterations are needed to produce an \( \epsilon \)-KKT solution of (1).

**Proof:** First, note that \( \mathcal{L}_{\beta_k} (\cdot, y^k) \) is \( \hat{L}_k \)-smooth and \( \hat{\rho}_k \)-weakly convex, with \( \hat{L}_k \) and \( \hat{\rho}_k \) defined in (15). Then by the \( x \) update in Algorithm 3, the stopping conditions of Algorithms 1 and 2, and following the same proof of \( \epsilon \) stationarity as in Theorem 1, we have

\[
\text{dist}(0, \partial_x \mathcal{L}_{\beta_k}(x^{k+1}, y^k)) \leq \epsilon, \forall k \geq 0.
\]

Next we give a uniform upper bound of the dual variable. By (16), (17), \( y^0 = 0 \), and also the setting of \( \gamma_k \), we have that \( \forall k \geq 0 \),

\[
\|y^k\| \leq \sum_{t=0}^{k-1} w_t\|c(x^{t+1})\| \leq \sum_{t=0}^{\infty} w_t\|c(x^{t+1})\| \leq \bar{c}w_0\|c(x^1)\|(\log 2)^2 = y_{\max},
\]

where we have defined \( \bar{c} = \sum_{t=0}^{\infty} \frac{1}{(t+1)^2 \log(t+2)^2} \) and \( y_{\max} = \bar{c}w_0\|c(x^1)\|(\log 2)^2 \).

Combining the above bound with the regularity assumption (19), we have the following feasibility bound: for all \( k \geq 1 \),

\[
\|c(x^k)\| \leq \frac{1}{\nu_\beta_{k-1}} \text{dist} (0, \partial h(x^k) + \beta_{k-1} J_c(x^k) \top c(x^k)) \\
= \frac{1}{\nu_\beta_{k-1}} \text{dist} (0, \partial_x \mathcal{L}_{\beta_k}(x^k, y^{k-1}) - \nabla g(x^k) - J_c(x^k) \top y^{k-1}) \\
\leq \frac{1}{\nu_\beta_{k-1}} \left( \text{dist} (0, \partial_x \mathcal{L}_{\beta_k}(x^k, y^{k-1})) + \|\nabla g(x^k)\| + \|J_c(x^k)\||(y^{k-1})\| \right) \\
\leq \frac{1}{\nu_\beta_{k-1}} (\epsilon + B_0 + B_cy_{\max}),
\]

\( \Box \)
where the third inequality follows from (28), (14a), (14c), and (29).

Now we define

\[ K = \lceil \log_2 C_e \rceil + 1, \text{ with } C_e = \frac{\varepsilon + B_0 + B_c y_{\text{max}}}{\mu \beta_0 \varepsilon}. \]  

(31)

Then by (30) and the setting of \( \beta_k \) in Algorithm 3, we have \( \| c(x^K) \| \leq \varepsilon \). Also recalling (28), we have

\[ \text{dist}(0, \partial f_0(x^{k+1}) + J_c(x^{k+1}) (y^k + \beta_k c(x^{k+1}))) \leq \varepsilon. \]

Therefore, \( x^K \) is an \( \varepsilon \)-KKT point of (1) with the corresponding multiplier \( y^{K-1} + \beta_{K-1} c(x^K) \), according to Definition 1.

In the rest of the proof, we bound the maximum number of iPPM iterations needed to stop Algorithm 2, and the number of APG iterations per iPPM iteration needed to stop Algorithm 1, for each iALM outer iteration.

Denote \( x^k_t \) as the \( t \)-th iPPM iterate within the \( k \)-th outer iteration of iALM. Then at \( x^k_t \), we use APG to minimize \( F^k_t(\cdot) := L_{\beta_k}(\cdot, y^k) + \hat{\rho}_k \| \cdot - x^k_t \|^2 \), which is \( \hat{L}_k := (\hat{L}_k + 2\hat{\rho}_k) \)-smooth and \( \hat{\rho}_k \)-strongly convex. Hence, by Lemma 1, at most \( T_{k}^{\text{APG}} \) (that is independent of \( t \)) APG iterations are required to find an \( \frac{\varepsilon}{4} \) stationary point of \( F^k_t(\cdot) \), where

\[ T_{k}^{\text{APG}} = \left\lceil \sqrt{\frac{L_k}{\hat{\rho}_k} \log \frac{1024L_k^2(\hat{L}_k + \hat{\rho}_k)D^2}{\varepsilon^2 \hat{\rho}_k}} \right\rceil + 1, \forall k \geq 0. \]  

(32)

In addition, recalling the definition of \( L_{\beta} \) in (2), observe that for all \( k \geq 1 \),

\[ L_{\beta_k}(x^k, y^k) \leq B_0 + \varepsilon + B_0 + B_c y_{\text{max}} \left( y_{\text{max}} + \frac{\sigma(\varepsilon + B_0 + B_c y_{\text{max}})}{2v} \right) \sigma^{1-k} \]  

(33)

and

\[ L_{\beta_0}(x^0, y^0) \leq B_0 + \frac{\beta_0}{2} \| c(x^0) \|^2, \]

where \( B_0 \) is given in (14a) and \( \hat{\beta}_0 := \frac{\varepsilon + B_0 + B_c y_{\text{max}}}{\mu \beta_0} \left( y_{\text{max}} + \frac{\sigma(\varepsilon + B_0 + B_c y_{\text{max}})}{2v} \right) \). Furthermore,

where \( \hat{B}_c \) is given in (14c).

Combining all three inequalities above with Theorem 1 and \( \hat{\rho}_k \)-weak convexity of \( L_{\beta_k}(\cdot, y^k) \), we conclude at most \( T_{k}^{\text{PPM}} \) iPPM iterations are needed to guarantee that \( x^{k+1} \) is an \( \varepsilon \) stationary point of \( L_{\beta_k}(\cdot, y^k, z^k) \), with

\[ T_{k}^{\text{PPM}} = \left\lceil \frac{32(\rho_0 + y_{\text{max}} L + \beta_k \rho_c)(2B_0 + y_{\text{max}} \hat{B}_c + \hat{\beta}_0)}{\varepsilon^2} \right\rceil, \forall k \geq 1 \]  

(35)

\[ T_{0}^{\text{PPM}} = \left\lceil \frac{32\rho_0}{\varepsilon^2} (2B_0 + y_{\text{max}} \hat{B}_c + \frac{\beta_0}{2} \| c(x^0) \|^2) \right\rceil. \]  

(36)
Consequently, we have shown that at most $T$ total APG iterations are needed to find an $\varepsilon$-KKT point of (1), where

$$T = \sum_{k=0}^{K-1} T_k^{PPM} T_k^{APG},$$

with $K$ given in (31), $T_k^{APG}$ given in (32), and $T_k^{PPM}$ given in (35).

The result in (37) immediately gives us the following complexity results.

By (31), we have $K = \tilde{O}(1)$ and $\beta_k = O(\varepsilon^{-1})$. Hence from (15), we have $\hat{\rho}_k = O(\beta_k), \hat{L}_k = O(\beta_k), \forall k \geq 0$. Then by (32), $T_k^{APG} = \tilde{O}(1), \forall k \geq 0$, and by (35), we have $T_k^{PPM} = O(\varepsilon^{-3}), \forall k \geq 0$. Therefore, in (37), $T = \tilde{O}(\varepsilon^{-3})$ for a general nonlinear $c(\cdot)$.

For the special case when $c(x) = Ax - b$, we have $\rho_c = 0$, and thus by (15), $\hat{\rho}_k = O(1), \forall k \geq 0$. Hence in (35), $T_k^{PPM} = O(\varepsilon^{-2}), \forall k \geq 0$, and in (32), $T_k^{APG} = O(\varepsilon^{-\frac{5}{2}}), \forall k \geq 0$. Therefore, by (37), $T = \tilde{O}(\varepsilon^{-\frac{5}{2}})$ for an affine $c(\cdot)$. This completes the proof.

In Theorem 2, we required the dual step size $w_k = w_0 \min \left\{ 1, \frac{\gamma_k}{\|c(x_k^+)\|} \right\}$, where as in (18),

$$\gamma_k = \frac{(\log 2)^2 \|c(x^1)\|}{(k+1)(\log(k+2))^2}.$$ 

Numerically, we observed better performance by slightly deviating from this setting. For example, we set $w_k = \frac{1}{\|c(x_k^+)\|}$ in all of our trials. This motivates us to give a more general version of Theorem 2. The following theorem considers $w_k = \frac{O(k^n)}{c(x_k^+)}$ and sacrifices an order of $(\log \varepsilon^{-1})^{n+1}$ in the total complexity compared to Theorem 2.

**Theorem 3 (complexity of iALM with general dual step sizes)** *In Algorithm 3, for some fixed $n \in \mathbb{Z}_+ \cup \{0\}$ and $M > 0$, let

$$w_k = \frac{M(k+1)^n}{\|c(x_k^+)\|} \forall k \geq 0.$$ 

Assume all other conditions of Theorem 2 hold. Then given $\varepsilon > 0$, Algorithm 3 with $w_k$ given in (38) needs $\tilde{O}(\varepsilon^{-3})$ APG iterations to produce an $\varepsilon$-KKT point of (1). In addition, if $c(x) = Ax - b$, then $\tilde{O}(\varepsilon^{-\frac{5}{2}})$ APG iterations are needed to produce an $\varepsilon$-KKT solution of (1).*

The proof of the above theorem is very similar to the proof of Theorem 2, except we have a nonuniform bound on the dual variable.

**Proof.** First, by (16), (38) and $y^0 = 0$, we have

$$\|y^k\| \leq \sum_{t=0}^{k-1} w_t \|c(x^{t+1})\| = \sum_{t=0}^{k-1} M(t+1)^n := y_k = O(k^{n+1}), \forall k \geq 0.$$ 

Following the first part of the proof of Theorem 2, we can easily show that at most $K = O(\log \varepsilon^{-1})$ outer iALM iterations are needed to guarantee $x^K$ to be an $\varepsilon$-KKT point of (1). Hence, $\beta_k = O(\varepsilon^{-1}), \forall 0 \leq k \leq K$.

Combining the above bound on $K$ with (39), we have

$$\|y^k\| \leq y_K := \sum_{t=0}^{K-1} M(K+1)^n = O(K^{n+1}) = O((\log \varepsilon^{-1})^{n+1}), \forall 1 \leq k \leq K.$$
Hence from (15), we have \( \hat{\rho}_k = O(\beta_k) = O(\varepsilon^{-1}) \), \( \hat{L}_k = O(\beta_k) = O(\varepsilon^{-1}) \), \( \forall 0 \leq k \leq K \).

Notice that (33) and (34) still hold with \( y_{\text{max}} \) replaced by \( y_k \). Hence,

\[
\mathcal{L}_{\hat{\beta}_k}(x^k, y^k) - \mathcal{L}_{\beta_k}(x, y^k) = O\left( y_k \left( 1 + \frac{y_k}{\hat{\beta}_k} \right) \right), \quad \forall k \leq K, \forall x \in \text{dom}(h).
\]

The above equation together with Theorem 1 gives that for any \( k \leq K \), at most \( T_{k}^{\text{PPM}} \) iPPM iterations are needed to terminate Algorithm 2 at the \( k \)-th outer iALM iteration, where

\[
T_{k}^{\text{PPM}} = \left\lceil \frac{32\hat{\rho}_k}{\varepsilon^2} \left( \mathcal{L}_{\hat{\beta}_k}(x^k, y^k) - \min_x \mathcal{L}_{\beta_k}(x, y^k) \right) \right\rceil = O\left( \frac{\hat{\rho}_ky_k}{\varepsilon^2} \left( 1 + \frac{y_k}{\hat{\beta}_k} \right) \right).
\]

Also, by Lemma 1, at most \( T_{k}^{\text{APG}} \) APG iterations are needed to terminate Algorithm 1, where

\[
T_{k}^{\text{APG}} = O\left( \frac{\sqrt{L_k}\rho_k \log \varepsilon^{-1}}{\varepsilon^2} \left( 1 + \frac{y_k}{\beta_k} \right) \right), \quad \forall k \geq 0.
\]

Therefore, for all \( k \leq K \),

\[
T_{k}^{\text{PPM}}T_{k}^{\text{APG}} = O\left( \frac{\sqrt{L_k}\rho_k \log \varepsilon^{-1}}{\varepsilon^2} \left( 1 + \frac{y_k}{\beta_k} \right) \right)
= O\left( \frac{y_k \log \varepsilon^{-1}}{\varepsilon^2} (\beta_k + y_k) \right)
= O\left( \frac{k^n \log \varepsilon^{-1}}{\varepsilon^2} (\sigma^k + k^n) \right)
= O\left( \frac{K^n \log \varepsilon^{-1}}{\varepsilon^2} (\sigma^K + K^n) \right)
= O\left( \frac{(\log \varepsilon^{-1})^n + 2}{\varepsilon^3} \left( \frac{1}{\varepsilon} + (\log \varepsilon^{-1})^n \right) \right)
= O\left( \frac{(\log \varepsilon^{-1})^n + 2}{\varepsilon^3} \right),
\]

where the second equation is from \( \hat{L}_k = O(\beta_k) \) and \( \hat{\rho}_k = O(\beta_k) \) for a general nonlinear \( c(\cdot) \), and the fifth one is obtained by \( K = O(\log \varepsilon^{-1}) \).

Consequently, for a general nonlinear \( c(\cdot) \), at most \( T \) APG iterations in total are needed to find the \( \varepsilon \)-KKT point \( x^K \), where

\[
T = \sum_{k=0}^{K-1} T_{k}^{\text{PPM}}T_{k}^{\text{APG}} = O \left( K\varepsilon^{-3}(\log \varepsilon^{-1})^n + 2 \right) = \tilde{O} \left( \varepsilon^{-3} \right).
\]
In the special case when \( c(x) = Ax - b \), we have \( \rho_c = 0 \), and thus by (15), \( \hat{\rho}_k = O(1), \forall k \geq 0 \). Then following the same arguments as above, we obtain that for any \( k \leq K \),

\[
T_k^{\text{PPM}}T_k^{\text{APG}} = O \left( \frac{\sqrt{L_k \hat{\rho}_k}}{\varepsilon^2} (\log \varepsilon^{-1})^{n+2} \right) = O \left( \varepsilon^{- \frac{5}{2}} (\log \varepsilon^{-1})^{n+2} \right).
\]

Therefore, at most \( T \) total APG iterations are needed to find the \( \varepsilon \)-KKT point \( x^K \), where

\[
T = \sum_{k=0}^{K-1} T_k^{\text{PPM}}T_k^{\text{APG}} = \tilde{O} \left( \varepsilon^{- \frac{5}{2}} \right),
\]

which completes the proof. \( \square \)

**Remark 4 (inequality constraints)** Although only equality constraints are considered in (1), our complexity result does not lose generality due to the boundedness of \( \{y^k\} \). Suppose we solve a problem with both equality and inequality constraints

\[
\min_x f_0(x), \quad \text{s.t.} \quad c(x) = 0, \quad d(x) \leq 0. \quad \text{(40)}
\]

Introducing a slack variable \( s \geq 0 \), we can have an equivalent formulation

\[
\min_{x, s \geq 0} f_0(x), \quad \text{s.t.} \quad c(x) = 0, \quad d(x) + s = 0. \quad \text{(41)}
\]

Suppose the conditions required by Theorem 2 hold. Then we can apply Algorithm 3 to (41) and obtain an \( \varepsilon \)-KKT point \((\bar{x}, \bar{s})\) with a corresponding multiplier \((\bar{y}, \bar{z})\), i.e.,

\[
\text{dist} \left( 0, \frac{\partial f_0(\bar{x})}{\mathcal{N}_+(\bar{s})} + \begin{bmatrix} J_c(\bar{x})^T \\ 0 \end{bmatrix} \bar{y} + \begin{bmatrix} J_d(\bar{x})^T \\ 1 \end{bmatrix} \bar{z} \right) \leq \varepsilon, \quad \text{(42a)}
\]

\[
||c(\bar{x})||^2 + ||d(\bar{x}) + \bar{s}||^2 \leq \varepsilon^2, \quad \bar{s} \geq 0, \quad \text{(42b)}
\]

where \( \mathcal{N}_+(\bar{s}) \) denotes the normal cone of the nonnegative orthant at \( \bar{s} \).

By (42a) and the definition of the normal cone, we have \( ||\bar{z}||_+ \leq \varepsilon \). Let \( \hat{z} = \bar{z} - [\bar{z}]_- \). Then \( \hat{z} \geq 0 \), and if \( ||J_d(\cdot)|| \) is uniformly bounded, then it follows from (42a) that

\[
\text{dist} \left( 0, \partial f_0(\bar{x})J_c(\bar{x})^T \bar{y} + J_d(\bar{x})^T \hat{z} \right) = O(\varepsilon). \quad \text{(43)}
\]

In addition, from (42b), it is straightforward to have \( ||c(\bar{x})||^2 + ||d(\bar{x})_+||^2 \leq \varepsilon^2 \). Furthermore, notice that if some \( \bar{s}_i = 0 \), then \( |d_i(\bar{x})| \leq \varepsilon \) from (42b), and if \( \bar{s}_i > 0 \), then \( |\bar{z}_i| \leq \varepsilon \) from (42a). Finally, use the boundedness of \( d \) and the fact that \( ||\hat{z}|| = O(1) \) is independent of \( \varepsilon \) from the proof of Theorem 2 to have \( ||\hat{z}^T d(\bar{x})|| = O(\varepsilon) \). Therefore, \( \bar{x} \) is an \( O(\varepsilon) \)-KKT point of the original problem (40), in terms of primal feasibility, dual feasibility, and the complementarity condition.
4 Numerical results

In this section, we conduct experiments to demonstrate the empirical performance of the proposed improved iALM. We consider the nonconvex linearly-constrained quadratic program (LCQP) and basis pursuit (BP).

More examples are given in the supplementary materials. We compare our method to the iALM in [27] and the HiAPeM method in [12]. All the tests were performed in MATLAB 2019b on a Macbook Pro with 4 cores and 16GB memory.

4.1 Experiments on nonconvex linearly-constrained quadratic programs (LCQP)

In this subsection, we test the proposed method on solving nonconvex LCQP:

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} x^T Q x + c^T x, \text{ s.t. } A x = b, \ x_i \in [l_i, u_i], \ \forall i \in [n],$$  \hspace{1cm} (44)

where $A \in \mathbb{R}^{m \times n}$, and $Q \in \mathbb{R}^{n \times n}$ is symmetric and indefinite (thus the objective is nonconvex). In the test, we generated all data randomly. The smallest eigenvalue of $Q$ is $-\rho < 0$, and thus the problem is $\rho$-weakly convex. For all tested instances, we set $l_i = -5$ and $u_i = 5$ for each $i \in [n]$.

We generated two groups of LCQP instances of different sizes. The first group had $m = 10$ and $n = 200$ and the second one $m = 100$ and $n = 1000$. In each group, we generated 10 instances of LCQP with $\rho = 1$. We compared the improved iALM in Algorithm 3 to the iALM in [27] and the HiAPeM method in [12]. HiAPeM adopted a hybrid setting ($N_0 = 10$, $N_1 = 1$) and a pure-penalty setting ($N_0 = 1$, $N_1 = 10^6$). The AL function $L_\beta(\cdot, y)$ of LCQP is $[Q + \beta A^T A]$-smooth and $\rho$-weakly convex. We set $\beta_k = \sigma^k \beta_0$ with $\sigma = 3$ and $\beta_0 = 0.01$ for both iALMs. For the subsolver of the iALM in [27], we set its step size to $\frac{1}{\sqrt{Q + \beta_k A^T A}}$ for the $k$-th outer iteration, as specified in [5]. The target error tolerance was set to $\varepsilon = 10^{-3}$ for all instances.

Besides the error tolerance, we set the maximum inner iteration to $10^6$ for all methods.

For each method, we report the primal residual, dual residual, running time (in seconds), and the number of gradient evaluations, shortened as pres, dres, time, and #Grad, respectively. The results for all trials are shown in Tables 2 and 3. From the results, we conclude that on average, to reach an $\varepsilon$-KKT point to the LCQP problem, the proposed improved iALM needs significantly fewer gradient evaluations and takes far less time than all other compared methods.

4.2 Experiments on basis pursuit

In this subsection, we test the proposed method on solving an equivalent but nonconvex reformulation of the basis pursuit (BP) problem:

$$\min_{x \in \mathbb{R}^d} ||x||^2, \text{ s.t. } B x^{\circ 2} - b = 0,$$  \hspace{1cm} (45)

where $B \in \mathbb{R}^{n \times d}$ and $b \in \mathbb{R}^n$, and $x^{\circ 2}$ denotes the component-wise square of $x$. The equivalence of (45) to a basis pursuit problem is shown in [27].

In this test, we set $B = (\hat{B}, -\hat{B}) \in \mathbb{R}^{n \times d}$ with the entries of $\hat{B}$ independently following from the standard Gaussian $\mathcal{N}(0,1)$. Also, we set $b = B z + \xi$, where $z \in \mathbb{R}^{d/2}$ has $k$ nonzero entries generated from $\mathcal{N}(0,1)$, and each entry of $\xi \in \mathbb{R}^n$ follows from $\mathcal{N}(0,10^{-6})$. With this setting, (45) is equivalent to

$$\min_{x \in \mathbb{R}^{d/2}} ||x||_1, \text{ s.t. } B z = b.$$  

Again, we generated two groups of instances of (45): one with the size $d = 100$, $n = 10$, $k = 20$ and the other $d = 1000$, $n = 100$, $k = 200$. Each group consisted of 10 instances. For (45), we were unable to
obtain an explicit formula of the smoothness constant $L_k$ and weak convexity constant $\rho_k$ of the AL function $L_{\beta_k}(\cdot, y^k)$ for any $k$. In the proposed iALM, we set $\rho_k = \beta_k$ as the input of iPPM and locally searched $L_k$ by backtracking. The iALM in [27] used the accelerated first-order method in [5]. We set its smoothness constant to $L_k = C\beta_k$, where $C$ was tuned to 100$\|B\|$. Numerically, we observed that this choice of $C$ enabled a relatively good trade-off between speed and stability. All other parameters of both iALMs were set in the same way as in the previous test.

For both iALMs, we report the primal residual, dual residual, running time (in seconds), and the number of gradient evaluation, shortened as pres, dres, time, and #Grad, respectively. For the proposed improved iALM, we also report the number of objective evaluation, shortened as #Obj, because we did a line search to obtain a local smoothness constant. The results for all instances are shown in Table 6. From the results, we conclude again that on average, to reach an $\varepsilon$-KKT point to (45), the proposed improved iALM method took significantly fewer gradient evaluations and much less time than the iALM in [27], for both small-sized and large-sized instances.
4.3 Experiments on generalized eigenvalue problem

In this subsection, we consider the generalized eigenvalue problem (EV) and compare our method to the iALM in [27].

The EV problem is

$$\min_{x \in \mathbb{R}^n} x^\top Q x, \quad \text{s.t. } x^\top B x - 1 = 0, \quad (46)$$

where $Q, B \in \mathbb{R}^{n \times n}$ are symmetric, and $B$ is positive definite. In the test, we set $Q = \frac{1}{2}(\hat{Q} + \hat{Q}^\top)$ with the entries of $\hat{Q}$ independently following from the standard Gaussian $N(0, 1)$. To ensure $B$ to be positive definite, we set $B = \overline{B} + (\|\overline{B}\| + 1)I_{n \times n}$, where $\overline{B}$ is generated in the same way as $Q$.

Again, we generated two groups of instances of (46), one with $n = 200$ and the other $n = 1000$. Each group consisted of 10 instances. For (46), we were unable to obtain an explicit formula of the smoothness constant $L_k$ and weak convexity constant $\rho_k$ of the AL function $L_{\beta_k}(\cdot, y^k)$ for any $k$. The iALM in [27] used the accelerated first-order method in [5]. We tuned its smoothness constant to $L_k = 2\|Q\| + 1000 + 100\beta_k$ when $n = 200$, and $L_k = 2\|Q\| + 100000 + 10000\beta_k$ when $n = 1000$. All other parameters of both iALMs were set in the same way as in the previous basis pursuit test in subsection 4.2.

The results for all instances are shown in Tables 4 and 5, where the meanings of pres, dres, time, #Grad, and #Obj are the same as in previous tests in section 4. From the results, we conclude that on average, to reach an $\varepsilon$-KKT point to (46), the proposed improved iALM took fewer gradient evaluations and less time than the iALM in [27], for all small-sized instances. The advantage of our method is even more significant for all large-sized instances.

| trial | pres | dres | time | #Obj | #Grad |
|-------|------|------|------|------|-------|
|       | pres | dres | time | #Obj | #Grad |
|       | proposed improved iALM | iALM in [27] | | | | |
| 1     | 1.39e-4 | 9.98e-4 | 1.09 | 46140 | 38245 | 1.39e-4 | 1.00e-3 | 2.84 | 233637 |
| 2     | 5.69e-4 | 9.87e-4 | 0.48 | 31456 | 25592 | 5.69e-4 | 1.00e-3 | 1.32 | 144750 |
| 3     | 2.57e-4 | 9.92e-4 | 0.60 | 32933 | 26112 | 2.57e-4 | 1.00e-3 | 2.21 | 150136 |
| 4     | 1.45e-4 | 9.98e-4 | 0.59 | 29408 | 25203 | 1.45e-4 | 1.00e-3 | 2.24 | 153485 |
| 5     | 1.52e-4 | 1.00e-3 | 0.93 | 37747 | 27434 | 1.51e-4 | 1.00e-3 | 1.63 | 153596 |
| 6     | 2.34e-4 | 9.71e-4 | 0.29 | 17765 | 14353 | 2.34e-4 | 1.00e-3 | 0.59 | 60643 |
| 7     | 9.06e-4 | 9.98e-4 | 0.42 | 26632 | 20886 | 9.06e-4 | 1.00e-3 | 1.05 | 109958 |
| 8     | 6.57e-4 | 9.97e-4 | 0.42 | 24184 | 19974 | 6.57e-4 | 1.00e-3 | 1.53 | 104508 |
| 9     | 2.44e-4 | 9.95e-4 | 0.45 | 27125 | 22390 | 2.44e-4 | 1.00e-3 | 1.20 | 126874 |
| 10    | 2.16e-4 | 9.98e-4 | 0.49 | 31238 | 26527 | 2.16e-4 | 1.00e-3 | 1.55 | 160941 |
| avg.  | 3.52e-4 | 9.03e-4 | 0.58 | 30376 | 24672 | 3.52e-4 | 1.00e-3 | 1.62 | 139823 |

4.4 Additional Plots

We provide additional plots of all three experiments above to demonstrate the empirical performance of the proposed iALM from another perspective.

Here we compare our method with the iALM in [27]. For clarity of presentation, we do not include the HiAPeM in [12] for comparison because HiAPeM has a different outer loop that generates completely
Table 5 Results by the proposed improved iALM and the iALM in [27] on solving a generalized eigenvalue problem (46) of size $n = 1000$.

| trial | pres | dres | time | #Obj | #Grad | pres | dres | time | #Grad |
|-------|------|------|------|------|------|------|------|------|------|
|       |      |      |      |      |      |      |      |      |      |
| proposed improved iALM | iALM in [27] | proposed improved iALM | iALM in [27] | proposed improved iALM | iALM in [27] | proposed improved iALM | iALM in [27] | proposed improved iALM | iALM in [27] |
| 1     | 6.87e-4 | 9.78e-4 | 60.77 | 56805 | 42620 | 6.86e-4 | 2.5e-3 | 5670.0 | 8585272 |
| 2     | 5.94e-4 | 9.92e-4 | 60.87 | 70884 | 49616 | 5.94e-4 | 1.00e-3 | 5070.0 | 8585272 |
| 3     | 4.20e-4 | 9.97e-4 | 51.08 | 74934 | 51707 | 4.20e-4 | 1.00e-3 | 6045.3 | 1000849 |
| 4     | 6.27e-4 | 9.99e-4 | 63.29 | 80454 | 60765 | 6.27e-4 | 1.60e-3 | 6733.4 | 10820619 |
| 5     | 2.92e-4 | 9.82e-4 | 36.16 | 41402 | 32164 | 2.90e-4 | 3.1e-3 | 3936.9 | 6588034 |
| 6     | 3.35e-4 | 9.95e-4 | 87.89 | 104069 | 74808 | 3.35e-4 | 2.1e-3 | 9183.8 | 15689148 |
| 7     | 4.47e-4 | 9.91e-4 | 51.12 | 60555 | 45578 | 4.46e-4 | 2.6e-3 | 5300.0 | 9039022 |
| 8     | 4.02e-4 | 9.91e-4 | 44.23 | 51399 | 39064 | 4.01e-4 | 2.6e-3 | 4771.7 | 8466906 |
| 9     | 9.32e-4 | 9.95e-4 | 79.42 | 98130 | 69322 | 9.32e-4 | 1.60e-3 | 8846.8 | 14688990 |
| 10    | 4.88e-4 | 9.91e-4 | 60.00 | 70996 | 51775 | 4.87e-4 | 2.24e-3 | 5975.1 | 10651152 |

Table 6 Results by the proposed improved iALM and the iALM in [27] on solving two groups of instances of (45) with size $(d, n, k) = (200, 10, 20)$ and $(d, n, k) = (1000, 100, 200)$.

| trial | pres | dres | time | #Obj | #Grad | pres | dres | time | #Grad | pres | dres | time | #Grad |
|-------|------|------|------|------|------|------|------|------|------|------|------|------|------|
|       |      |      |      |      |      |      |      |      |      |      |      |      |      |
| proposed improved iALM | iALM in [27] | proposed improved iALM | iALM in [27] | proposed improved iALM | iALM in [27] | proposed improved iALM | iALM in [27] | proposed improved iALM | iALM in [27] |
| 1     | 4.30e-4 | 9.29e-4 | 1.99 | 40335 | 34462 | 2.28e-9 | 7.06e-2 | 85.83 | 2210577 |
| 2     | 9.27e-4 | 9.58e-4 | 2.76 | 51771 | 44120 | 5.96e-4 | 9.94e-4 | 87.61 | 146243 | 123572 |
| 3     | 9.40e-4 | 8.46e-4 | 0.62 | 12596 | 10772 | 5.96e-4 | 1.80e-3 | 10451 | 14811722 |
| 4     | 9.82e-4 | 8.59e-4 | 0.52 | 10535 | 9111 | 5.67e-4 | 1.87e-3 | 167.88 | 261019 | 225558 |
| 5     | 4.83e-4 | 9.27e-4 | 1.86 | 10115 | 8365 | 6.21e-4 | 9.96e-4 | 148.24 | 261019 | 225558 |
| 6     | 8.89e-4 | 9.91e-4 | 1.23 | 20894 | 18322 | 5.45e-4 | 9.94e-4 | 75.02 | 243827 | 209174 |
| 7     | 8.89e-4 | 9.91e-4 | 1.23 | 20894 | 18322 | 5.45e-4 | 9.94e-4 | 75.02 | 243827 | 209174 |
| 8     | 8.57e-4 | 9.15e-4 | 0.56 | 12076 | 10376 | 5.11e-4 | 9.77e-4 | 234.71 | 8167564 | 6684666 |
| 9     | 4.36e-4 | 9.33e-4 | 2.44 | 51266 | 44120 | 4.02e-4 | 9.57e-4 | 142.02 | 278460 | 225558 |
| 10    | 7.07e-4 | 8.62e-4 | 0.54 | 11665 | 10027 | 1.73e-4 | 9.98e-4 | 617.13 | 1004482 | 851845 |
| avg.  | 7.98e-4 | 9.05e-4 | 1.29 | 24976 | 21508 | 5.36e-4 | 9.87e-4 | 194.22 | 314759 | 266326 |

different trajectories of primal and dual residuals. Specifically, rather than maintaining the dual residual below error tolerance as in our method, HiAPeM instead keeps the primal residual below error tolerance throughout all outer iterations. Since our method and the iALM in [27] both ensure the dual residual to be below a given error tolerance $\varepsilon$ at the end of each outer loop, it suffices to only compare their trajectories of the primal residual.

We conducted experiments on three problems (LCQP, BP, and EV), each with two different sized instances. For each of the six cases, we selected one representative instance and plot the primal residual versus the number of gradient evaluations. Notice that the number of gradient evaluations is a better metric than the number of inner iterations because the running time is roughly proportional to the former rather than the latter. Figure 1 shows the plots. From the plots, given the fact that the dual residual is below $\varepsilon$ at the end of each outer loop, we conclude again that in each problem case, our method reaches an $\varepsilon$-KKT point with far fewer gradient evaluations than the iALM in [27].
Fig. 1 Comparison of the proposed iALM and the existing iALM in [27] on solving the LCQP, BP, and EV problems. Each plot shows the primal residual. Dual residuals for both methods are similar, below a given tolerance $\varepsilon$. 

instance of LCQP (44) 
size $m = 20$ and $n = 100$

instance of LCQP (44) 
size $m = 100$ and $n = 1000$

instance of BP (45) 
size $(d, n, k) = (200, 10, 20)$

instance of BP (45) 
size $(d, n, k) = (1000, 100, 200)$

instance of EV (46) 
size $n = 200$

instance of EV (46) 
size $n = 1000$
5 Conclusions

We have presented an improved iALM for solving nonconvex constrained optimization. Different from existing iALMs, our iALM uses the iPPM to approximately solve each subproblem. Under a regularity condition, we explore the better convergence rate of iPPM and the boundedness of AL functions to establish improved complexity results. To reach an $\epsilon$-KKT solution, our method requires $\tilde{O}(\epsilon^{-2})$ proximal gradient steps for solving nonconvex optimization with convex constraints. The result is slightly worsened to $\tilde{O}(\epsilon^{-3})$ if the constraints are also nonconvex. Both complexity results are so far the best. Numerically, we demonstrated that the proposed improved iALM could significantly outperform one existing iALM and also one penalty-based FOM, though the latter can theoretically achieve a similar complexity result for convex-constrained nonconvex problems.

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