REFLEXIVITY OF RINGS VIA NILPOTENT ELEMENTS

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Abstract. An ideal $I$ of a ring $R$ is called left $N$-reflexive if for any $a \in \text{nil}(R)$, $b \in R$, being $aRb \subseteq I$ implies $bRa \subseteq I$ where $\text{nil}(R)$ is the set of all nilpotent elements of $R$. The ring $R$ is called left $N$-reflexive if the zero ideal is left $N$-reflexive. We study the properties of left $N$-reflexive rings and related concepts. Since reflexive rings and reduced rings are left $N$-reflexive, we investigate the sufficient conditions for left $N$-reflexive rings to be reflexive and reduced. We first consider basic extensions of left $N$-reflexive rings. For an ideal-symmetric ideal $I$ of a ring $R$, $R/I$ is left $N$-reflexive. If an ideal $I$ of a ring $R$ is reduced as a ring without identity and $R/I$ is left $N$-reflexive, then $R$ is left $N$-reflexive. If $R$ is a quasi-Armendariz ring and the coefficients of any nilpotent polynomial in $R[x]$ are nilpotent in $R$, it is proved that $R$ is left $N$-reflexive if and only if $R[x]$ is left $N$-reflexive. We show that the concept of $N$-reflexivity is weaker than that of reflexivity and stronger than that of left $N$-right idempotent reflexivity and right idempotent reflexivity which are introduced in Section 5.

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1. Introduction

Throughout this paper, all rings are associative with identity. A ring is called reduced if it has no nonzero nilpotent elements. A weaker condition than “reduced” is defined by Lambek in [11]. A ring $R$ is said to be symmetric if for any $a, b, c \in R$, $abc = 0$ implies $acb = 0$. Equivalently, $abc = 0$ implies $bac = 0$. An equivalent condition on a ring to be symmetric is that whenever a product of any number of elements is zero, any permutation of the factors still yields product zero.

In [12], a right ideal $I$ of $R$ is said to be reflexive if $aRb \subseteq I$ implies $bRa \subseteq I$ for any $a, b \in R$. $R$ is called a reflexive ring if $aRb = 0$ implies $bRa = 0$ for any $a, b \in R$. And in [14], $R$ is said to be a weakly reflexive ring if $aRb = 0$ implies $bRa \subseteq \text{nil}(R)$ for any $a, b \in R$. In [8], a ring $R$ is said to be nil-reflexive if $aRb \subseteq \text{nil}(R)$ implies that $bRa \subseteq \text{nil}(R)$ for any $a, b \in R$. Let $R$ be a ring. In [11], $R$ is called a reflexivity with maximal ideal axis ring, in short, an RM ring if $aMb = 0$ for a maximal ideal.
$M$ and for any $a, b \in R$, then $bMa = 0$, similarly, $R$ has reflexivity with maximal ideal axis on idempotents, simply, $RMI$, if $eMf = 0$ for any idempotents $e, f$ and a maximal ideal of $M$, then $fMe = 0$. In [10], $R$ has reflexive-idempotents-property, simply, $RIP$, if $eRf = 0$ for any idempotents $e, f$, then $fRe = 0$, A left ideal $I$ is called idempotent reflexive if $eRa \subseteq I$ for any idempotents $e, a \in R$. A ring $R$ is called idempotent reflexive if $0$ is an idempotent reflexive ideal. And Kim and Baik [7], introduced the left and right idempotent reflexive rings. A two sided ideal $I$ of a ring $R$ is called right idempotent reflexive if $aRe \subseteq I$ implies $eRa \subseteq I$, for any $a, e \in R$. A ring $R$ is called right idempotent reflexive if $0$ is the right idempotent reflexive ideal. Left idempotent reflexive ideals and rings are defined similarly. If a ring $R$ is left and right idempotent reflexive then it is called an idempotent reflexive ring.

In this paper, motivated by these classes of types of reflexive rings, we introduce left $N$-reflexive rings and right $N$-reflexive rings. We prove that some results of reflexive rings can be extended to the left $N$-reflexive rings for this general setting. We investigate characterizations of left $N$-reflexive rings, and that many families of left $N$-reflexive rings are presented.

In what follows, $\mathbb{Z}$ denotes the ring of integers and for a positive integer $n$, $\mathbb{Z}_n$ is the ring of integers modulo $n$. We write $M_n(R)$ for the ring of all $n \times n$ matrices, $U(R)$, nil($R$) will denote the group of units and the set of all nilpotent elements of $R$, $U_n(R)$ is the ring of upper triangular matrices over $R$ for a positive integer $n \geq 2$, and $D_n(R)$ is the ring of all matrices in $U_n(R)$ having main diagonal entries equal.

2. N-reflexivity of rings

In this section, we introduce a class of rings, so-called left $N$-reflexive rings and right $N$-reflexive rings. These classes of rings generalize reflexive rings. We investigate which properties of reflexive rings hold for the left $N$-reflexive case. We supply an example to show that there are left $N$-reflexive rings that are neither right $N$-reflexive nor reflexive nor reversible. It is shown that the class of left $N$-reflexive rings is closed under finite direct sums. We have an example to show that homomorphic image of a left $N$-reflexive ring is not left $N$-reflexive. Then, we determine under what conditions a homomorphic image of a ring is left $N$-reflexive. We now give our main definition.

**Definition 2.1.** Let $R$ be a ring and $I$ an ideal of $R$. $I$ is called left $N$-reflexive if for any $a \in \text{nil}(R), b \in R$, being $aRb \subseteq I$ implies $bRa \subseteq I$. The ring $R$ is called left $N$-reflexive if the zero ideal is left $N$-reflexive. Similarly, $I$ is called right $N$-reflexive
if for any \( a \in \text{nil}(R), b \in R, \) being \( bRa \subseteq I \) implies \( aRb \subseteq I. \) The ring \( R \) is called right \( N\)-reflexive if the zero ideal is right \( N\)-reflexive. The ring \( R \) is called \( N\)-reflexive if it is left and right \( N\)-reflexive.

Clearly, every reflexive ring and every semiprime ring are \( N\)-reflexive. There are left \( N\)-reflexive rings which are not semiprime and there are left \( N\)-reflexive rings which are neither reduced nor reversible.

Let \( F \) be a field and \( R = F[x] \) be the polynomial ring over \( F \) with \( x \) an indeterminate and \( \alpha : R \rightarrow R \) be a homomorphism defined by \( \alpha(f(x)) = f(0) \) where \( f(0) \) is the constant term of \( f(x) \). Let \( D_2^0(R) \) denote skewtrivial extension of \( R \) by \( R \) and \( \alpha \). So \( D_2^0(R) = \left\{ \begin{pmatrix} f(x) & g(x) \\ 0 & f(x) \end{pmatrix} \mid f(x), g(x) \in R \right\} \) is the ring with componentwise addition of matrices and multiplication:
\[
\begin{pmatrix} f(x) & g(x) \\ 0 & f(x) \end{pmatrix} \begin{pmatrix} h(x) & t(x) \\ 0 & h(x) \end{pmatrix} = \begin{pmatrix} f(x)h(x) & \alpha(f(x))t(x) + g(x)h(x) \\ 0 & f(x)h(x) \end{pmatrix}.
\]
There are left \( N\)-reflexive rings which are neither reflexive nor semiprime. The \( N\)-reflexive property of rings is not left-right symmetric.

**Example 2.2.** Let \( D_2^0(R) \) denote skewtrivial extension of \( R \) by \( R \) and \( \alpha \) as mentioned above. Then by [14 Example 3.5], \( D_2^0(R) \) is not reflexive. We show that \( D_2^0(R) \) is left \( N\)-reflexive. Note that the set of all nilpotent elements of \( D_2^0(R) \) is the set \( \left\{ \begin{pmatrix} 0 & f(x) \\ 0 & 0 \end{pmatrix} \mid f(x) \in R \right\}. \) Let \( A = \begin{pmatrix} 0 & f(x) \\ 0 & 0 \end{pmatrix} \) be a nilpotent in \( D_2^0(R) \) and \( B = \begin{pmatrix} h(x) & g(x) \\ 0 & h(x) \end{pmatrix} \) any element in \( D_2^0(R) \). Assume that \( AD_2^0(R)B = 0 \). We may assume \( f(x) \neq 0 \). Then an easy calculation, \( AD_2^0(R)B = 0 \) reveals that \( h(x) = 0 \), and also \( BD_2^0(R)A = 0 \). Hence \( D_2^0(R) \) is left \( N\)-reflexive.

Next we show that \( D_2^0(R) \) is not right \( N\)-reflexive. Let \( A = \begin{pmatrix} 0 & f(x) \\ 0 & 0 \end{pmatrix} \) be a nilpotent and \( B = \begin{pmatrix} xh(x) & g(x) \\ 0 & xh(x) \end{pmatrix} \) any element in \( D_2^0(R) \) with both \( f(x) \) and \( h(x) \) nonzero. By definitions \( BD_2^0(R)A = 0 \). Since \( xh(x)f(x) \) is nonzero, \( AD_2^0(R)B = \begin{pmatrix} 0 & f(x)r(x)xh(x) \\ 0 & 0 \end{pmatrix} \) is nonzero for some nonzero \( r(x) \in R \). So \( D_2^0(R) \) is not right \( N\)-reflexive.

On the other hand, \( \text{nil}(D_2^0(R)) \) is an ideal of \( D_2^0(R) \) and \( (\text{nil}(D_2^0(R)))^2 = 0 \) but \( \text{nil}(D_2^0(R)) \neq 0 \). Therefore \( D_2^0(R) \) is not semiprime.

**Proposition 2.3.** Let \( R \) be a left \( N\)-reflexive ring. Then for any idempotent \( e \) of \( R, eRe \) is also left \( N\)-reflexive.
Proof. Let \( eae \in eRe \) be a nilpotent and \( ebe \in eRe \) arbitrary element with \( eaeRebe = 0 \). Then we have \( ebeReae = 0 \) since \( R \) is left N-reflexive. □

Examples 2.4. (1) Let \( F \) be a field and \( R = M_2(F) \). In fact, \( R \) is a simple ring, therefore prime. Let \( A, B \in R \) with \( ARB = 0 \). Since \( R \) is prime, \( A = 0 \) or \( B = 0 \). Hence \( BRA = 0 \). So \( R \) is reflexive. Therefore \( R \) is N-reflexive.

(2) Let \( F \) be a field and consider the ring \( R = D_3(F) \). Then \( R \) is neither left N-reflexive nor right N-reflexive. Let \( A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \) be nilpotent in \( D_3(F) \) and \( B = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in D_3(F) \). Then \( ARB = 0 \). For \( C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \), \( BCA = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0 \). Hence \( R \) is not left N-reflexive.

Next we show that \( R \) is not right N-reflexive either. Now assume that \( A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \text{nil}(R) \) and \( B = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in R \). It is clear that \( BRA = 0 \).

For \( C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \in R \), we have \( ACB \neq 0 \). Hence \( R \) is not right N-reflexive.

Lemma 2.5. N-reflexivity of rings is preserved under isomorphisms.

Theorem 2.6. Let \( R \) be a ring. Assume that \( M_n(R) \) is left N-reflexive. Then \( R \) is left N-reflexive.

Proof. Suppose that \( M_n(R) \) is a left N-reflexive ring. Let \( e_{ij} \) denote the matrix unit which \((i, j)\)-entry is 1 and the other entries are 0. Then \( R \cong Re_{11} = e_{11}M_n(R)e_{11} \) is N-reflexive by Proposition 2.3 and Lemma 2.5. □

Proposition 2.7. Every reversible ring is left and right N-reflexive.

Proof. Clear by the definitions. □

The converse statement of Proposition 2.7 may not be true in general as shown below.

Example 2.8. By Examples 2.4(1), \( M_2(F) \) is both left and right N-reflexive. But it is not reversible.
Theorem 2.9. Let $R$ be a ring. Then the following are equivalent.

1. $R$ is left N-reflexive.
2. $IRJ = 0$ implies $JRI = 0$ for any ideal $I$ generated by a nilpotent element and any nonempty subset $J$ of $R$.
3. $IJ = 0$ implies $JI = 0$ for any ideal $I$ generated by a nilpotent element and any ideal $J$ of $R$.

Proof. (1) $\Rightarrow$ (2) Assume that $R$ is left N-reflexive. Let $I = RaR$ with $a \in R$ nilpotent and $\emptyset \neq J \subseteq R$ such that $IRJ = 0$. Then for any $b \in J$, $aRb = 0$. This implies that $bRa = 0$, hence $bR(RaR) = bRI = 0$ for any $b \in J$. Thus $JRI = 0$.

(2) $\Rightarrow$ (3) Let $I = RaR$ with $a \in R$ nilpotent and $J$ be an ideal of $R$ such that $IJ = 0$. Then $J = RJ$, so $IRJ = 0$. By (2), $JRI = 0$, thus $JI = 0$.

(3) $\Rightarrow$ (1) Let $a \in R$ be nilpotent and $b \in R$ with $aRb = 0$. Then $(RaR)(RbR) = 0$. By (3), $(RbR)(RaR) = 0$. Hence $bRa = 0$. Therefore $R$ is left N-reflexive.

For any element $a \in R$, $r_R(a) = \{b \in R \mid ab = 0\}$ is called the right annihilator of $a$ in $R$. The left annihilator of $a$ in $R$ is defined similarly and denoted by $l_R(a)$.

Proposition 2.10. Let $R$ be a ring. Then $R$ is N-reflexive if and only if for any nilpotent element $a$ of $R$, $r_R(a) = l_R(Ra)$.

Proof. For the necessity, let $x \in r_R(aR)$ for any nilpotent element $a \in R$. We have $(aR)x = 0$. The ring $R$ being N-reflexive implies $xRa = 0$. So $x \in l_R(Ra)$. It can be similarly showed that $l_R(Ra) \subseteq r_R(aR)$.

For the sufficiency, let $a \in \text{nil}(R)$ and $b \in R$ with $aRb = 0$. Then $b \in r_R(aR)$. By hypothesis, $b \in l_R(Ra)$, and so $bRa = 0$. Thus $R$ is N-reflexive.

For a field $F$, $D_3(F)$ is neither left N-reflexive nor right N-reflexive. Subrings of left N-reflexive rings or right N-reflexive rings need not be left N-reflexive or right N-reflexive, respectively. But there are some subrings of $D_3(F)$ that are left N-reflexive or right N-reflexive as shown below.

Proposition 2.11. Let $R$ be a reduced ring (i.e., it has no nonzero nilpotent elements). Then the following hold.

1. Let $S = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \mid a, b, c \in R \right\}$ be a subring of $D_3(R)$. Then $S$ is N-reflexive.
2. Let $S = \left\{ \begin{pmatrix} a & 0 & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} \mid a, b, c \in R \right\}$ be a subring of $D_3(R)$. Then $S$ is N-reflexive.
Proof. (1) Let \( A = \begin{pmatrix} 0 & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in S \) be any nonzero nilpotent element and \( B = \begin{pmatrix} u & v & t \\ 0 & u & 0 \\ 0 & 0 & u \end{pmatrix} \in S \). Assume that \( ASB = 0 \). This implies \( AB = 0 \), and so \( bu = 0 \) and \( cu = 0 \). For any \( C = \begin{pmatrix} x & y & z \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix} \in S \), \( BCA = \begin{pmatrix} 0 & uxb & uxc \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \). Being \( bu = cu = 0 \) implies \((uxb)^2 = (uxc)^2 = 0\). Since \( R \) is reduced, \( uxb = uxc = 0 \). Then \( BCA = 0 \). Hence \( BSA = 0 \). Thus \( R \) is left \( N \)-reflexive. A similar proof implicates that \( S \) is right \( N \)-reflexive.

(2) By [14, Proposition 2.2]. □

The condition \( R \) being reduced in Proposition 2.11 is not superfluous as the following example shows.

Example 2.12. Let \( F \) be a field and \( R = F \langle a, b \rangle \) be the free algebra with noncommuting indeterminates \( a, b \) over \( F \). Let \( I \) be the ideal of \( R \) generated by \( aRa \) and \( a^2 \). Consider the ring \( \overline{R} = R/I \). Let \( \overline{a}, \overline{b} \in \overline{R} \). Then \( \overline{aRa} = 0 \). But \( \overline{bRa} \neq 0 \) since \( ba \notin I \). Note that \( \overline{R} \) is not reduced. Consider the ring \( S = \left\{ \begin{pmatrix} \pi & \eta & \tau \\ \overline{u} & \overline{u} & \overline{u} \\ \overline{u} & \overline{u} & \overline{u} \end{pmatrix} \mid \pi, \eta, \tau \in \overline{R} \right\} \). Let \( A = \begin{pmatrix} \pi & \overline{1} & \overline{1} \\ \overline{0} & \overline{0} & \overline{0} \\ \overline{0} & \overline{0} & \overline{0} \end{pmatrix} \in \text{nil}(S) \) and \( B = \begin{pmatrix} \overline{0} & \overline{0} & \overline{0} \\ \overline{0} & \overline{0} & \overline{0} \\ \overline{0} & \overline{0} & \overline{0} \end{pmatrix} \in S \). Then \( ASB = 0 \) since \( \overline{aRa} = 0 \). However \( BA \neq 0 \) since \( \overline{bRa} \neq 0 \). Hence \( S \) is not left \( N \)-reflexive.

Definition 2.13. Let \( R \) be a ring. We call that \( R \) is left \( N \)-reversible if for any nilpotent \( a \in R \) and \( b \in R \), \( ab = 0 \) implies \( ba = 0 \).

In [13], a ring \( R \) is called \textit{nil-semicommutative} if for every nilpotent \( a, b \in R \), \( ab = 0 \) implies \( aRa = 0 \).

Theorem 2.14. If a ring \( R \) is \( N \)-reversible, then \( R \) is \textit{nil-semicommutative} and \( N \)-reflexive.

Proof. Assume that \( R \) is \( N \)-reversible. Let \( a \in R \) be nilpotent and \( b \in R \) with \( aRa = 0 \). Then \( ab = 0 \). For any \( r \in R \), being \( abr = 0 \) implies \( bra = 0 \), and so \( bRa = 0 \). Hence \( R \) is left \( N \)-reflexive. By a similar discussion, \( R \) is right \( N \)-reflexive.
So \( R \) is \( N \)-reflexive. In order to see that \( R \) is nil-semicommutative, let \( a, b \in R \) be nilpotent with \( ab = 0 \). \( N \)-reversibility of \( R \) implies \( ba = 0 \), and so \( bar = 0 \) for any \( r \in R \). Again by the \( N \)-reversibility of \( R \), we have \( arb = 0 \). Thus \( aRb = 0 \). □

Note that in a subsequent paper, \( N \)-reversible rings will be studied in detail by the present authors.

Let \( R \) be a ring and \( I \) an ideal of \( R \). Recall by [4], \( I \) is called \( \text{ideal-symmetric} \) if \( ABC \subseteq I \) implies \( ACB \subseteq I \) for any ideals \( A, B, C \) of \( R \). In this vein, we mention the following result.

**Proposition 2.15.** Let \( R \) be a ring and \( I \) an ideal-symmetric ideal of \( R \). Then \( R/I \) is an \( N \)-reflexive ring.

**Proof.** Let \( \overline{a} \in R/I \) be nilpotent and \( \overline{b} \in R/I \) with \( \overline{a}(R/I)\overline{b} = 0 \). Then \( aRb \subseteq I \). \( N \)-reversibility of \( R \) implies \( bRa \subseteq I \). Therefore \( bRa \subseteq I \), and so \( \overline{b}(R/I)\overline{a} = 0 \). It means that \( R/I \) is left \( N \)-reflexive. Similarly, it can be shown that \( R/I \) is also right \( N \)-reflexive. □

Let \( R \) be a ring and \( I \) an ideal of \( R \). In the short exact sequence \( 0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0 \), \( I \) being \( N \)-reflexive (as a ring without identity) and \( R/I \) being \( N \)-reflexive need not imply that \( R \) is \( N \)-reflexive.

**Example 2.16.** Let \( F \) be a field and consider the ring \( R = D_3(F) \). Let \( I = \begin{pmatrix} 0 & F & F \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix} \). Then \( I \) is \( N \)-reflexive since \( I^3 = 0 \). Also, \( R/I \) is \( N \)-reflexive since \( R/I \) is isomorphic to \( F \). However, by Examples 2.4(2), \( R \) is not \( N \)-reflexive.

**Theorem 2.17.** Let \( R \) be a ring and \( I \) an ideal of \( R \). If \( I \) is reduced as a ring (without identity) and \( R/I \) is left \( N \)-reflexive, then \( R \) is left \( N \)-reflexive.

**Proof.** Let \( a \) be nilpotent in \( R \) and \( b \in R \) with \( aRb = 0 \). Then \( \overline{a}(R/I)\overline{b} = 0 \) and \( \overline{a} \) is nilpotent in \( R/I \). By hypothesis, \( \overline{b}(R/I)\overline{a} = 0 \). Hence \( bRa \subseteq I \). Since \( I \) is reduced and \( bRa \) is nil, \( bRa = 0 \). □

The reduced condition on the ideal \( I \) in Theorem 2.17 is not superfluous.

**Example 2.18.** Let \( F \) be a field and \( I = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in F \right\} \) denote the ideal of \( D_3(F) \). Then by Examples 2.4(2), \( D_3(F) \) is not left and right \( N \)-reflexive. The ring \( D_3(F)/I \) is isomorphic to \( F \) and so it is left and right \( N \)-reflexive. Note that \( I \) is not reduced since \( I^3 = 0 \).
The class of left (or right) N-reflexive rings are not closed under homomorphic images.

**Example 2.19.** Consider the rings $R$ and $\overline{R} = R/I$ in the Example 2.12 where $I$ is the ideal of $R$ generated by $aRb$ and $a^2$. Then $R$ is reduced, hence left (and right) N-reflexive. Let $\overline{a}, \overline{b} \in \overline{R}$. Then $\overline{aRb} = 0$. But $\overline{bRa} \neq 0$ since $ba \notin I$. Hence $R/I$ is not left N-reflexive.

Let $e$ be an idempotent in $R$. $e$ is called left semicentral if $re = ere$ for all $r \in R$. $S_l(R)$ is the set of all left semicentral elements. $e$ is called right semicentral if $er = ere$ for all $r \in R$. $S_r(R)$ is the set of all right semicentral elements of $R$. We use $B(R)$ for the set of central idempotents of $R$. In [3], a ring $R$ is called left(right) principally quasi-Baer (or simply, left(right) p.q.-Baer) ring if the left(right) annihilator of a principal right ideal of $R$ is generated by an idempotent.

**Theorem 2.20.** The following hold for a ring $R$.

1. If $R$ is right N-reflexive, then $S_l(R) = B(R)$.
2. If $R$ is left N-reflexive, then $S_r(R) = B(R)$.

**Proof.** (1) Let $e \in S_l(R)$ and $a \in R$. Then $(1 - e)Re = 0$. It follows that $(1 - e)Re(a - ae) = (1 - e)R(ea - eae) = 0$. Since $ea - eae$ is nilpotent and $R$ is right N-reflexive, $(ea - eae)R(1 - e) = 0$. Hence $(ea - eae)(1 - e) = 0$. This implies $ea - eae = 0$. On the other hand, $(1 - e)R(a - ea)e \subseteq (1 - e)Re = 0$. Thus $(1 - e)R(ae - eae) = 0$, and so $(1 - e)(ae - eae) = 0$. Then $ae - eae = 0$. So we have $ea = ae$, i.e., $e \in B(R)$. Therefore $S_l(R) \subseteq B(R)$. The reverse inclusion is obvious.

(2) Similar to the proof of (1). □

**Theorem 2.21.** Let $R$ be a right p.q-Baer ring. Then the following conditions are equivalent.

1. $R$ is a semiprime ring.
2. $S_l(R) = B(R)$.
3. $R$ is a reflexive ring.
4. $R$ is a right N-reflexive ring.

**Proof.** (1) ⇔ (2) By [3, Proposition 1.17(i)].
(1) ⇔ (3) By [3, Proposition 3.15].
(3) ⇒ (4) Clear by definitions.
(4) ⇒ (2) By Theorem 2.20(1). □

**Theorem 2.22.** Let $R$ be a left p.q-Baer ring. Then the following conditions are equivalent.
(1) $R$ is a semiprime ring.
(2) $S_r(R) = B(R)$.
(3) $R$ is a reflexive ring.
(4) $R$ is a left $N$-reflexive ring.

**Proof.** Similar to the proof of Theorem 2.21. \qed

**Proposition 2.23.** Let $R$ be a ring and $e^2 = e \in R$. Assume that $R$ is an $N$-reflexive. Then $aRe = 0$ implies $ea = 0$ for any nilpotent element $a$ of $R$.

**Proof.** Suppose that $aRe = 0$ for any $a \in \text{nil}(R)$. Since $R$ is $N$-reflexive, $eRa = 0$, and so $ea = 0$. \qed

**Question:** If a ring $R$ is $N$-reflexive, then is $R$ a 2-primal ring?

There is a 2-primal ring which is not $N$-reflexive.

**Example 2.24.** Consider the 2 by 2 upper triangular matrix ring $R = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$ over the field $\mathbb{Z}_2$ of integers modulo 2. For $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \text{nil}(R)$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in R$, we have $ARB = 0$ but $BRA \neq 0$. But $R$ is 2-primal by [2, Proposition 2.5].

**Proposition 2.25.** Let $\{R_i\}_{i \in I}$ be a class of rings. Then $R = \prod_{i \in I} R_i$ is left $N$-reflexive if and only if $R_i$ is left $N$-reflexive for each $i \in I$.

**Proof.** Assume that $R = \prod_{i \in I} R_i$ is left $N$-reflexive. By Proposition 2.23 for each $i \in I$, $R_i$ is left $N$-reflexive. Conversely, let $a = (a_i) \in R$ be nilpotent and $b = (b_i) \in R$ with $aRb = 0$. Then $a_iR_i b_i = 0$ for each $i \in I$. Since each $a_i$ is nilpoent in $R_i$ for each $i \in I$, by hypothesis, $b_i R_i a_i = 0$ for every $i \in I$. Hence $bRa = 0$. This completes the proof. \qed

### 3. Extensions of $N$-reflexive rings

In this section, we study some kinds of extensions of $N$-reflexive rings to start with, the Dorroh extension $D(R, \mathbb{Z}) = \{(r, n) \mid r \in R, n \in \mathbb{Z}\}$ of a ring $R$ is a ring with operations $(r_1, n_1) + (r_2, n_2) = (r_1 + r_2, n_1 + n_2)$ and $(r_1, n_1)(r_2, n_2) = (r_1 r_2 + n_1 r_2 + n_2 r_1, n_1 n_2)$, where $r_i \in R$ and $n_i \in \mathbb{Z}$ for $i = 1, 2$.

**Proposition 3.1.** A ring $R$ is left $N$-reflexive if and only if the Dorroh extension $D(R, \mathbb{Z})$ of $R$ is left $N$-reflexive.
Proof. Firstly, we note that nil($D(R, Z)$) = \{(r, 0) \mid r \in \text{nil}(R)\}. For the necessity, let $(a, b) \in D(R, Z)$ and $(r, 0) \in \text{nil}(D(R, Z))$ with $(r, 0)D(R, Z)(a, b) = 0$. Then $(r, 0)(s, 0)(a, b) = 0$ for every $s \in R$. Hence $rs(a + b1_R) = 0$ for all $s \in R$, and so $rR(a + b1_R) = 0$. Since $R$ is left $N$-reflexive, $(a + b1_R)Rr = 0$. Thus $(a, b)(x, y)(r, 0) = ((a + b1_R)(x + y1_R)r, 0) = 0$ for any $(x, y) \in D(R, Z)$. For the sufficiency, let $s \in R$ and $r \in \text{nil}(R)$ with $rRs = 0$. We have $(r, 0) \in \text{nil}(D(R, Z))$. This implies $(r, 0)D(R, Z)(s, 0) = 0$. By hypothesis, $(s, 0)D(R, Z)(r, 0) = 0$. In particular, $(s, 0)(x, 0)(r, 0) = 0$ for all $x \in R$. Therefore $sRr = 0$. So $R$ is left $N$-reflexive.

Let $R$ be a ring and $S$ be the subset of $R$ consisting of central regular elements. Set $S^{-1}R = \{s^{-1}r \mid s \in S, r \in R\}$. Then $S^{-1}R$ is a ring with an identity.

**Proposition 3.2.** For a ring $R$, $R[x]$ is left $N$-reflexive if and only if $(S^{-1}R)[x]$ is left $N$-reflexive.

**Proof.** For the necessity, let $f(x) = \sum_{i=0}^{m} s_i^{-1}a_ix^i$ be nilpotent and $g(x) = \sum_{i=0}^{n} t_i^{-1}b_ix^i \in (S^{-1}R)[x]$ satisfy $f(x)(S^{-1}R)[x]g(x) = 0$. Let $s = s_0s_1\ldots s_m$ and $t = t_0t_1t_2\ldots t_n$. Then $f_1(x) = sf(x)$ is nilpotent and $g_1(x) = tg(x) \in R[x]$ and $f_1(x)R[x]g_1(x) = 0$. By hypothesis, $g_1(x)R[x]f_1(x) = 0$. Then $g(x)(S^{-1}R)[x]f(x) = 0$. The sufficiency is clear. □

**Corollary 3.3.** For a ring $R$, $R[x]$ is left $N$-reflexive if and only if $R[x;x^{-1}]$ is left $N$-reflexive.

According to [5], a ring $R$ is said to be quasi-Armendariz if whenever $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)R[x]g(x) = 0$, then $a_iRb_j = 0$ for each $i, j$.

The left $N$-reflexivity or right $N$-reflexivity and the quasi-Armendariz property of rings do not imply each other.

**Examples 3.4.** (1) Let $F$ be a field and consider the ring $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Then $R$ is quasi-Armendariz by [5] Corollary 3.15. However, $R$ is not left $N$-reflexive. For $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \text{nil}(R)$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in R$, we have $ARB = 0$ but $BA \neq 0$.

(2) Consider the ring $R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z}_4 \right\}$. Since $R$ is commutative, $R$ is $N$-reflexive. For $f(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} x$ and $g(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} x$, we have $f(x)R[x]g(x) = 0$. Thus $R$ is quasi-Armendariz.
Let \( R \) be a quasi-Armendariz ring. Assume that coefficients of any nilpotent polynomial in \( R[x] \) are nilpotent in \( R \). Then \( R \) is left N-reflexive if and only if \( R[x] \) is left N-reflexive.

**Proof.** Suppose that \( R \) is left N-reflexive and \( f(x) = \sum_{i=0}^{m} a_i x^i \), \( g(x) = \sum_{j=0}^{n} b_j x^j \in R[x] \) with \( f(x)R[x]g(x) = 0 \) and \( f(x) \) nilpotent. The ring \( R \) being quasi-Armendariz implies \( a_i R b_j = 0 \) for all \( i \) and \( j \), and \( f(x) \) being nilpotent gives rise to all \( a_0, a_1, a_2, \ldots, a_m \) nilpotent. By supposition \( b_j R a_i = 0 \) for all \( i \) and \( j \). Therefore \( g(x)R[x]f(x) = 0 \), and so \( R[x] \) is left N-reflexive. Conversely, assume that \( R[x] \) is left N-reflexive. Let \( a \in R \) be nilpotent and \( b \in R \) any element with \( aRb = 0 \). Then \( aR[x]b = 0 \). Hence \( bRa = 0 \) and \( R \) is left N-reflexive. \( \square \)

Note that in commutative case, the coefficients of any nilpotent polynomial are nilpotent. However, this is not the case for noncommutative rings in general. Therefore in Proposition 3.5, the assumption “coefficients of any nilpotent polynomial in \( R[x] \) are nilpotent in \( R \)” is not superfluous as the following example shows.

**Example 3.6.** Let \( S = M_n(R) \) for a ring \( R \). Consider the polynomial \( f(x) = e_{21} + (e_{11} - e_{22})x - e_{12}x^2 \in S[x] \), where the \( e_{ij} \)'s are the matrix units. Then \( f(x)^2 = 0 \), but \( e_{11} - e_{22} \) is not nilpotent.

### 4. Applications

In this section, we study some subrings of full matrix rings whether or not they are left or right N-reflexive rings.

**The rings \( H_{(s,t)}(R) \):** Let \( R \) be a ring and \( s, t \) be in the center of \( R \). Let

\[
H_{(s,t)}(R) = \left\{ \begin{pmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{pmatrix} \in M_3(R) \mid a, c, d, e, f \in R, a - d = sc, d - f = te \right\}.
\]

Then \( H_{(s,t)}(R) \) is a subring of \( M_3(R) \). Note that any element \( A \) of \( H_{(s,t)}(R) \) has the form

\[
\begin{pmatrix} sc + te + f & 0 & 0 \\ c & te + f & e \\ 0 & 0 & f \end{pmatrix}.
\]

**Lemma 4.1.** Let \( R \) be a ring, and let \( s, t \) be in the center of \( R \). Then the set of all nilpotent elements of \( H_{(s,t)}(R) \) is

\[
\begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} x \in R[x], \text{ we have } f(x)Rg(x) = 0, \text{ and so by [5, Lemma 2.1] } f(x)R[x]g(x) = 0, \text{ but } \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0. \text{ Thus } R \text{ is not quasi-Armendariz.}
\[ \text{nil}(H(s,t)(R)) = \left\{ \begin{pmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{pmatrix} \in H(s,t)(R) \mid a, d, f \in \text{nil}(R), c, e \in \mathbb{R} \right\}. \]

**Proof.** Let \( A = \begin{pmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{pmatrix} \in \text{nil}(H(s,t)(R)) \) be nilpotent. There exists a positive integer \( n \) such that \( A^n = 0 \). Then \( a^n = d^n = f^n = 0 \). Conversely assume that \( a^n = 0, d^m = 0 \) and \( f^k = 0 \) for some positive integers \( n, m, k \). Let \( p = \max\{n, m, k\} \). Then \( A^{2p} = 0 \). \( \square \)

**Theorem 4.2.** The following hold for a ring \( R \).

1. If \( R \) is a reduced ring, then \( H(0,0)(R) \) is \( N \)-reflexive but not reduced.
2. If \( R \) is reduced, then \( H(1,0)(R) \) is \( N \)-reflexive but not reduced.
3. If \( R \) is reduced, then \( H(0,1)(R) \) is \( N \)-reflexive but not reduced.
4. \( R \) is reduced if and only if \( H(1,1)(R) \) is reduced.

**Proof.**

1. Let \( A = \begin{pmatrix} a & 0 & 0 \\ c & a & e \\ 0 & 0 & a \end{pmatrix} \in \text{nil}(H(0,0)(R)) \) be nilpotent. By Lemma 4.1, \( a \) is nilpotent. By assumption, \( a = 0 \). Let \( B = \begin{pmatrix} k & 0 & 0 \\ l & k & n \\ 0 & 0 & k \end{pmatrix} \in H(0,0)(R) \) with \( AB = 0 \). \( AB = 0 \) implies \( ck = 0 \) and \( ek = 0 \). For any \( X = \begin{pmatrix} x & 0 & 0 \\ y & x & u \\ 0 & 0 & x \end{pmatrix} \in H(0,0)(R) \), \( AXB = \begin{pmatrix} 0 & 0 & 0 \\ cxk & 0 & exk \\ 0 & 0 & 0 \end{pmatrix} = 0 \). Then \( cxk = 0 \) and \( exk = 0 \) for all \( x \in R \). The ring \( R \) being reduced implies \( kxc = 0 \) and \( kxe = 0 \) for all \( x \in R \). Then \( BXA = \begin{pmatrix} 0 & 0 & 0 \\ kxc & 0 & kxe \\ 0 & 0 & 0 \end{pmatrix} = 0 \) for all \( X \in H(0,0)(R) \). Hence \( H(0,0)(R) \) is left \( N \)-reflexive. A similar discussion reveals that \( H(0,0)(R) \) is also right \( N \)-reflexive. Note that being \( R \) reduced does not imply \( H(0,0)(R) \) is reduced because \( A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in H(0,0)(R) \) is a nonzero nilpotent element.

2. Let \( A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e \\ 0 & 0 & 0 \end{pmatrix} \in \text{nil}(H(1,0)(R)) \) and \( B = \begin{pmatrix} f + c & 0 & 0 \\ c & f & d \\ 0 & 0 & f \end{pmatrix} \in H(1,0)(R) \).
with $AH_{(1,0)}(R)B = 0$. For any $C = \begin{pmatrix} m + n & 0 & 0 \\ n & m & u \\ 0 & 0 & m \end{pmatrix} \in H_{(1,0)}(R)$, $ACB = 0$. Then $emf = 0$ and $fme = 0$. This implies $BCA = 0$. Therefore $H_{(1,0)}(R)$ is left $N$-reflexive. Similarly, $H_{(1,0)}(R)$ is also right $N$-reflexive.

(3) Let $A = \begin{pmatrix} 0 & 0 & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \text{nil}(H_{(0,1)}(R))$ and $B = \begin{pmatrix} e + f & 0 & 0 \\ a & e + f & e \\ 0 & 0 & f \end{pmatrix} \in H_{(0,1)}(R)$

with $AH_{(0,1)}(R)B = 0$. For any $C = \begin{pmatrix} m + n & 0 & 0 \\ k & m + n & m \\ 0 & 0 & n \end{pmatrix} \in H_{(0,1)}(R)$, $ACB = 0$. Then $c(m + n)(e + f) = 0$ and $(e + f)(m + n)c = 0$. This implies $BCA = 0$. Therefore $H_{(0,1)}(R)$ is left $N$-reflexive. Similarly, $H_{(0,1)}(R)$ is also right $N$-reflexive.

(4) Let $A = \begin{pmatrix} c + e + f & 0 & 0 \\ c & e + f & e \\ 0 & 0 & f \end{pmatrix} \in \text{nil}(H_{(1,1)}(R))$ be nilpotent. Then $f$ is nilpotent and so $f = 0$. In turn, it implies $e = c = 0$. Hence $A = 0$. Conversely, assume that $H_{(1,1)}(R)$ is reduced. Let $a \in R$ with $a^n = 0$. Let $A = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \in H_{(1,1)}(R)$. Then $A$ is nilpotent. By assumption $a = 0$. □

5. Generalizations, Examples and Applications

In this section, we introduce left $N$-right idempotent reflexive rings and right $N$-left idempotent reflexive rings generalize reflexive idempotent rings in Kwak and Lee [3], and Kim [6], Kim and Baik in [7]. An ideal $I$ of a ring $R$ is called idempotent reflexive if $aRe \subseteq I$ implies $eRa \subseteq I$ for any $a \in R$ and $e^2 = e \in R$. A ring $R$ is said to be idempotent reflexive if the ideal 0 is idempotent reflexive. In [10], a ring $R$ is called to have the reflexive-idempotents-property if $R$ satisfies the property that $eRf = 0$ implies $fRe = 0$ for any idempotents $e$ and $f$ of $R$. We introduce following some classes of rings to produce counter examples related to left $N$-reflexive rings. These classes of rings will be studied in detail in a subsequent paper by authors.

Definition 5.1. Let $I$ be an ideal of a ring $R$. Then $I$ is called left $N$-right idempotent reflexive if being $aRe \subseteq I$ implies $eRa \subseteq I$ for any nilpotent $a \in R$ and $e^2 = e \in R$. A ring $R$ is called left $N$-right idempotent reflexive if 0 is a left $N$-right idempotent reflexive ideal. Left $N$-right idempotent reflexive ideals and rings are defined similarly. If a ring $R$ is left $N$-right idempotent reflexive and right $N$-left idempotent reflexive, then it is called an $N$-idempotent reflexive ring.
Every left N-reflexive ring is a left N-right idempotent reflexive ring. But there are left N-right idempotent reflexive rings that are not left N-reflexive.

Examples 5.2. (1) Let $F$ be a field and $A = F<X,Y>$ denote the free algebra generated by noncommuting indeterminates $X$ and $Y$ over $F$. Let $I$ denote the ideal generated by $YX$. Let $R = A/I$ and $x = X + I$ and $y = Y + I \in R$. It is proved in [6, Example 5] that $R$ is abelian and so $R$ has reflexive-idempotents-property but not reflexive by showing that $xRy \neq 0$ and $yRx = 0$. Moreover, $xyRx \neq 0$ and $xRy \neq 0$. This also shows that $R$ is not left N-reflexive since $xy$ is nilpotent in $R$.

(2) Let $F$ be a field and $A = F<X,Y>$ denote the free algebra generated by noncommuting indeterminates $X$ and $Y$ over $F$. Let $I$ denote the ideal generated by $X^3$, $Y^3$, $XY$, $YX^2$, $Y^2X$ in $A$. Let $R = A/I$ and $x = X + I$ and $y = Y + I \in R$. Then in $R$, $x^3 = 0$, $y^3 = 0$, $xy = 0$, $yx^2 = 0$, $y^2x = 0$. In [1, Example 2.3], $xRy = 0$, $yRx \neq 0$ and idempotents in $R$ are 0 and 1. Hence for any $r \in \text{nil}(R)$ and $e^2 = e \in R$, $rRe = 0$ implies $eRe = 0$. Thus $R$ is left N-right idempotent reflexive. We show that $R$ is not a left N-reflexive ring. Since any $r \in R$ has the form $r = k_0 + k_1x + k_2x^2 + k_3y + k_4y^2 + k_5yx$ and $x$ is nilpotent, as noted above, $xRy = 0$. However, $yRx \neq 0$ since $yx \neq 0$. Thus $R$ is not left N-reflexive.

(3) Let $F$ be a field of characteristic zero and $A = F<X,Y,Z>$ denote the free algebra generated by noncommuting indeterminates $X$, $Y$ and $Z$ over $F$. Let $I$ denote the ideal generated by $XAY$ and $X^2 - X$. Let $R = A/I$ and $x = X + I$, $y = Y + I$ and $z = Z + I \in R$. Then in $R$, $xRy = 0$ and $x^2 = x$. $xy = 0$ and $yx$ is nilpotent and $x$ is idempotent and $xRyx = 0$. But $yxRx \neq 0$. Hence $R$ is not right N-left idempotent reflexive. In [9, Example 3.3], it is shown that $R$ is right idempotent reflexive.

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