Multidimensional Costas Arrays and Their Periodicity

Jaziel Torres and Ivelisse Rubio

Abstract—A novel higher-dimensional definition for Costas arrays is introduced. This definition works for arbitrary dimensions and avoids some limitations of previous definitions. Some non-existence results are presented for multidimensional Costas arrays preserving the Costas condition when the array is extended periodically throughout the whole space. In particular, it is shown that three-dimensional arrays with this property must have the least possible order; extending an analogous two-dimensional result by H. Taylor. Said result is conjectured to extend for Costas arrays of arbitrary dimensions.

Index Terms—Costas arrays, multidimensional arrays, periodicity, permutation.

I. INTRODUCTION

A COSTAS array is a permutation array, i.e., a square binary array with a single 1 per row and per column, with the property that the vectors joining pairs of 1’s are all distinct, this being called the Costas condition [1] or Costas property [2]. Costas arrays are useful in many applications, especially in radar/sonar detection and wireless communications [3], [4], [5], and their study preserves contemporary validity as, to this day, their useful appearance continues to find new applications [6], [7], [8], [9]. Costas arrays have also been an interesting object for mathematical research, with researchers looking at usual mathematical questions of existence, distribution, structure, constructions, and generalizations [10], [11]. For a comprehensive review on the history and basic theory of Costas arrays, see [12]. In this paper, we introduce a new multidimensional generalization of Costas arrays and study their periodicity, not only because it is an interesting mathematical inquiry, but also because multidimensional analogs of Costas arrays are also useful in radar and optical communications [13], [14], digital watermarking [15] and digital holography [16].

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To obtain a higher-dimensional analog of Costas arrays one has to generalize the two defining properties: being a permutation array and having no repeated difference vectors, i.e., the Costas condition. Some multidimensional analogs of Costas arrays have been proposed before [16], [17], [18], [19], [20], all satisfying the same multidimensional Costas condition, as it generalizes naturally; however, they differ in the generalization of a permutation array, as this can be done in different ways. Nonetheless, the generalization in [16, §2], which produces arrays of the type defined in [21, Definition 8], have an extremely low density of 1’s, thus these arrays “tend not to be very interesting” [21, p. 4]. The generalization in [17, Definition 6], further studied in [21], is problematic for odd dimensions. The arrays in [18, Definition 2] are only defined for three dimensions and their restriction to two dimensions do not produces a two-dimensional Costas array. Finally, [19, Definition 1] treats the arrays, and thus the vectors, over finite abelian groups, which is not consistent with the usual treatment of two-dimensional Costas arrays. We propose a new multidimensional definition of Costas arrays that works for arbitrary dimensions, is consistent with the definition of a two-dimensional Costas array when restricted to two dimensions, and produces arrays with density of 1’s equal to the square root of the number of entries. Moreover, [20, Definition 3.2] and [17, Definition 6] are special cases of our definition when restricted to permutations with one-dimensional domain to and arrays of even dimensions, respectively.

After introducing our definition, we study the existence of multidimensional arrays preserving the Costas condition when extended periodically to the whole space. In this paper we focus on studying the higher-dimensional extensibility of the following result.

**Theorem 1 (H. Taylor [22]):** For $n > 2$, let an $n \times n$ matrix of $n$ non-attacking rooks be extended doubly periodically over the whole plane. Then there must exist at least one $n \times n$ window in which some difference appears twice.

The non-attacking rooks configuration in Theorem 1 is equivalent to a permutation matrix. It is clear that when any...
permutation array of order \( n > 2 \) is extended periodically to the whole plane, every \( n \times n \) window contains a permutation array. Nonetheless, Theorem 1 is saying that in the periodic extension of a permutation array of order \( n \) there is at least one \( n \times n \) window that is not a Costas array, i.e., the Costas condition fails. We show that an analogous result holds for three-dimensional arrays and for higher-dimensional arrays with odd number of 1’s, and conjecture it holds for all higher-dimensional arrays.

Our motivation to study the existence of multidimensional arrays preserving the Costas condition and extending Theorem 1 to higher-dimensional arrays is based on the early work in Costas arrays by S. W. Golomb, O. Moreno, and H. Taylor. Firstly, Golomb and Taylor [1], by citing Theorem 1, stated that for \( n > 2 \), “there does not exist a doubly periodic pattern with a Costas array in every \( n \times n \) window” (p. 1154), and pointed at the Welch construction as the closest to such configuration. Then, Golomb and Moreno [23] introduced circular Costas sequences, which are equivalent to an \( n \times n \) permutation matrix with the addition of an empty row, in which all the vectors joining pairs of 1’s are distinct taken modulo the size of the array, i.e., modulo \( n \) in their horizontal component and modulo \( n + 1 \) in their vertical component. The addition of the empty row was necessary because there are no Costas arrays with all vectors being distinct after taking modulo \( n \) in both components. Although a fairly simple pigeonhole argument works to see the latter, the existence of such array (with vectors distinct modulo \( n \) in both components) would imply the existence of doubly periodic patterns with a Costas array in every \( n \times n \) window, which does not exist by Theorem 1. Golomb and Moreno conjectured that the only circular Costas arrays are those from the Welch construction [23, Conjecture 1], and this was proved by Muratović-Rubić et al. in [24, Theorem 3.4]. Our intention is to walk down and explore this chain of results on the periodicity of Costas arrays, but in the multidimensional context. This paper is our first step as we explore a multidimensional analog of Theorem 1.

The rest of the paper is structured as follows. In Section II preliminaries on multidimensional binary arrays are discussed, establishing all necessary definitions and notations. In Section III a novel higher-dimensional definition of Costas arrays is introduced. Lastly, Section IV contains several nonexistence results regarding the periodicity of Costas arrays.

II. PRELIMINARIES ON BINARY ARRAYS

Throughout the rest of this paper, \( m \) is a natural number greater than 1.

A binary array of dimension \( m \) is a function \( A : \Lambda \rightarrow \{0, 1\} \) where \( \Lambda \) is the hyper-rectangular subset of \( \mathbb{N}^m \) given by

\[
\Lambda = \{(a_1, a_2, \ldots, a_m) \in \mathbb{N}^m : a_k \leq n_k, \ k = 1, 2, \ldots, m\},
\]

for some natural numbers \( n_1, n_2, \ldots, n_m \). Equivalently, \( \Lambda = [n_1] \times [n_2] \times \cdots \times [n_m] \), where \([n] = \{1, 2, \ldots, n\}\). We say that \( \Lambda \) is the index set for the array \( A \), and that \( A \) has size \( n_1 \times n_2 \times \cdots \times n_m \). If in the index \( \Lambda \), \( n_i = 1 \) for some \( i \), the \( i \)-th dimension of \( A \) would be trivial, so we avoid those cases. Hence, whenever we consider an \( m \)-dimensional array, we implicitly assume that the size in each dimension is at least 2.

For \( \alpha = (a_1, \ldots, a_m) \in \mathbb{Z}^m \), we denote by \( \alpha_\Lambda \) the unique tuple \((a'_1, \ldots, a'_m) \in \Lambda \) satisfying \( a'_i \equiv a_i \pmod{n_i} \), \( \forall i \in [m] \). The periodic extension of \( A \), denoted by \( \Lambda \), is the \( m \)-dimensional infinite array defined by

\[
\Lambda(\alpha) = A(\alpha_\Lambda), \ \ \ \forall \alpha \in \mathbb{Z}^m.
\]

For two distinct dots \( \alpha = (a_1, \ldots, a_m) \) and \( \omega = (w_1, \ldots, w_m) \) in an \( m \)-dimensional binary array \( A \), the difference vector from \( \alpha \) to \( \omega \) is the vector

\[
\omega - \alpha = (w_1 - a_1, \ldots, w_m - a_m) \in \mathbb{Z}^m.
\]

The toroidal vector [25] from \( \alpha \) to \( \omega \) is the vector

\[
((w_1 - a_1) \mod n_1, \ldots, (w_m - a_m) \mod n_m) \in \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_m}.
\]

To evade degenerate cases, whenever we consider difference or toroidal vectors, we assume the dots \( \alpha \) and \( \omega \) to be distinct, i.e., \( \alpha \neq \omega \). Notice our convention: we use parenthesis \((\cdot)\) for dots, and angled brackets \((\cdot)\) for vectors.

For each pair of dots in a binary array, there are two distinct difference vectors joining them: from \( \alpha \) to \( \omega \) and vice versa. Hence, both the number of difference vectors and toroidal vectors in a binary array with \( n \) dots is \( n(n-1) \), counting repetitions.

Definition 1: For an \( m \)-dimensional binary array \( A \) of size \( n_1 \times \cdots \times n_m \), define the following multisets (a set allowing repetitions):

- \( T_A \) is the multiset of toroidal vectors occurring in \( A \).
- \( H_A \) is the multiset of toroidal vectors \((h_1, \ldots, h_m)\) occurring in \( A \) for which \( h_i = n_i/2 \), for some \( i \in [m] \).

We use the letter “H” because a toroidal vector belongs to \( H_A \) if it has a component that is half the length of the array in the corresponding direction. As discussed before, for a binary array \( A \) with \( n \) dots, \(|T_A| = n(n-1)|\).

Lemma 1: Let \( A \) be an \( m \)-dimensional binary array with index set \( \Lambda = [n_1] \times \cdots \times [n_m] \), and \( \Lambda \) its periodic extension to \( \mathbb{Z}^m \). If \( S \) is an \( n_1 \times \cdots \times n_m \) window of \( A \), then \( A \) and \( S \) have the same multiset of toroidal vectors. That is, \( T_S = T_A \).

Proof: Let \( S \) be an \( n_1 \times \cdots \times n_m \) window of \( A \), and let \( \psi \) be the function that maps every dot \( \alpha \in S \) to the unique dot \( \alpha_\Lambda \in A \). Since the window containing the array \( S \) has the same size as the original array \( A \), \( \psi \) is a bijection from the dots of \( S \) to the dots of \( A \). By the definition of \( \alpha_\Lambda \) and the definition of periodic extension, the toroidal vector from \( \alpha \) to \( \omega \) is equal to the toroidal vector from \( \alpha_\Lambda \) to \( \omega_\Lambda \). Hence, by the bijectivity of \( \psi \), \( T_S = T_A \).

Proposition 1: Let \( A \) be an \( m \)-dimensional binary array of size \( n_1 \times \cdots \times n_m \) and \( \Lambda \) its periodic extension to \( \mathbb{Z}^m \). If \( A \) has a repeated toroidal vector \((h_1, \ldots, h_m) \notin H_A \), then there is an \( n_1 \times \cdots \times n_m \) window of \( A \) having a repeated difference vector.

Proof: Let \( A \) be a binary array with index set \( \Lambda = [n_1] \times \cdots \times [n_m] \), and let \( \Lambda \) be its periodic extension to \( \mathbb{Z}^m \). Assume that \((h_1, \ldots, h_m) \) appears (at least) twice as a
toroidal vector occurring in $A$, with $0 \leq h_i \leq n_i - 1$ and $h_i \neq n_i/2$, for all $i \in [m]$. Then, there exist two pairs of dots of $A$,

$\alpha_1 = (a_{11}, \ldots, a_{1m}), \omega_1 = (w_{11}, \ldots, w_{1m})$ and $\alpha_2 = (a_{21}, \ldots, a_{2m}), \omega_2 = (w_{21}, \ldots, w_{2m})$, such that

$$w_{1i} - a_{1i} \equiv w_{2i} - a_{2i} \equiv h_i \pmod{n_i}, \quad \forall i \in [m].$$

We need to show that there are two pairs of dots of $A$,

$\alpha_1' = (a'_{11}, \ldots, a'_{1m}), \omega_1' = (w'_{11}, \ldots, w'_{1m})$ and $\alpha_2' = (a'_{21}, \ldots, a'_{2m}), \omega_2' = (w'_{21}, \ldots, w'_{2m})$

satisfying, for every $i$,

(i) $w'_{1i} - a'_{1i} = w'_{2i} - a'_{2i}$, and

(ii) $k_i \leq a'_{1i}, w'_{1i}, a'_{2i}, w'_{2i} \leq k_i + n_i - 1$, for some $k_i \in \mathbb{Z}$.

Since the components are independent, it is enough to prove (i) and (ii) for $i = 1$. To simplify notation, let $n := n_1, h := h_1$, and $k := k_1$. Notice that $\alpha_1, \omega_1 \in [n_1]$, hence $-(n_1 - 1) \leq \omega_1 - \alpha_1 \leq n_1 - 1$. Therefore, $h = (\omega_1 - \alpha_1) \mod n$ implies $\omega_1 - \alpha_1 = h$ or $\omega_1 - \alpha_1 = h - n$. Similarly, $\omega_2 - \alpha_2 = h$ or $\omega_2 - \alpha_2 = h - n$. If $\omega_1 - \alpha_1 = \omega_2 - \alpha_2$, choose $\alpha_1' = \alpha_1$, $\omega_1' = \omega_1$, $\alpha_2' = \alpha_2$, and $\omega_2' = \omega_2$. These four values satisfy (i) and (ii) with $k = 0$.

Otherwise, $\omega_1 - \alpha_1 \neq \omega_2 - \alpha_2$ so one difference is equal to $h$ and the other one is equal to $-h$, and also $h > 0$. Without loss of generality, assume $\omega_1 - \alpha_1 = h - n$ and $\omega_2 - \alpha_2 = h$.

Hence

$$\omega_1 - \alpha_1 < 0 \implies \omega_1 \leq \alpha_1 - 1, \quad \text{and} \quad (1)$$

$$\omega_2 - \alpha_2 > 0 \implies \omega_2 \leq \alpha_2 \leq \omega_2 - 2 \leq \omega_2 - n. \quad \text{(2)}$$

There are three cases: $\alpha_1 \leq \omega_2 - 1, \omega_2 \leq \alpha_1 - 1$ or $\alpha_1 = \omega_2$.

**Case 1.** If $\alpha_1 \leq \omega_2 - 1$, we set $\alpha_1' = \alpha_1, \omega_1' = \omega_1, \alpha_2' = \omega_2 - 1, \omega_2' = \omega_2$. Notice that $\omega_1' - \alpha_1' = \omega_2 - 1 - \alpha_2' = \omega_2 - n$, so (i) is satisfied. By the inequalities (1) and (2), and the assumption $\alpha_1 \leq \omega_2 - 1$, (ii) is satisfied with $h = \omega_2'$.

**Case 2.** If $\omega_2 \leq \alpha_1 - 1$, we set $\alpha_1' = \alpha_1 - n, \omega_1' = \omega_1, \alpha_2' = \omega_2, \omega_2' = \omega_2$. In this case, $\omega_1' - \alpha_1' = \omega_2 - \alpha_2' = h$.

Then, (i) is satisfied. By the inequalities (1) and (2), we have

$$\omega_1' - \alpha_1' = \omega_2' - \alpha_2' = h.$$

Therefore, $\omega_1' - \alpha_2' \geq 0$, hence $\alpha_2' \leq \omega_2'$. On the other hand, $\omega_1' + h < n$ so that $\omega_1' < \omega_2' + n$. Using (2) we conclude

$$\alpha_2' \leq \alpha_1', \omega_1' \geq \omega_2', \omega_2' \leq \omega_2' + n - 1,$$

and (ii) is satisfied with $k = \alpha_2'$.

**Case 3.** If $n - h < n/2$ we set $\alpha_1' = \alpha_1, \omega_1' = \omega_1, \alpha_2' = \omega_2 + n, \omega_2' = \omega_2$. Condition (i) is satisfied because $\omega_1' - \alpha_1' = \omega_2' - \alpha_2' = h - n$. As in Case 3a, we have $\omega_1' < \omega_2' < \alpha_2' + n$. Also $\omega_2' - \omega_1' = (\alpha_2' - \omega_2') + (\alpha_1' - \omega_1') = (n - h) + (n - h) < n$, implying $\omega_1' \leq \alpha_2'$ and $\omega_2' < \omega_2' + n$. Hence

$$\omega_1' \leq \alpha_1', \omega_1' \leq \omega_2' \leq \omega_1' + n - 1,$$

and (ii) is satisfied with $k = \omega_1'$.

These are all the cases, so the proof is finished.

In the proof of Proposition 1, the assumption that the repeated toroidal vector does not belong to $\mathcal{H}_A$ is only used in Case 3. Hence, if a binary array has a repeated toroidal vector $\tau$ not falling under Case 3, we can follow the proof of Proposition 1 to obtain the same result, even when $\tau \in \mathcal{H}_A$.

We state this result as a corollary, which will be used in Theorem 5 hereinafter.

**Corollary 1:** Let $A$ be an $m$-dimensional binary array of size $n_1 \times \cdots \times n_m$ and $A$ its periodic extension to $\mathbb{Z}^m$. Assume $A$ has two difference vectors $\omega_1 - \alpha_1$ and $\omega_2 - \alpha_2$ that are equal as toroidal vectors to $(h_1, \ldots, h_m)$, where $\alpha_1 = (a_{11}, \ldots, a_{1m}), \omega_1 = (w_{11}, \ldots, w_{1m}), \alpha_2 = (a_{21}, \ldots, a_{2m}), \omega_2 = (w_{21}, \ldots, w_{2m})$ are dots of $A$. If for $i = 1, \ldots, m$,

$$h_i = \frac{n_i}{2} \implies w_{1i} - a_{1i} = w_{2i} - a_{2i}, \quad \alpha_1 \neq w_{1i}, \quad \alpha_2 \neq w_{1i},$$

then there is an $n_1 \times \cdots \times n_m$ window of $A$ having a repeated difference vector.

Notice that Proposition 1 has the flavor of a higher-dimensional analog of Theorem 1, but there is a subtle difference. First and most obviously, Proposition 1 has the additional assumption that $(h_1, \ldots, h_m) \not\in \mathcal{H}_A$. However, it is more general than Theorem 1, in the sense that it is stated for arbitrary binary arrays, not permutation arrays (the non-attacking rooks configuration).

## III. MULTIDIMENSIONAL COSTAS ARRAYS

And now, the higher-dimensional generalization of Costas arrays we announced all along. As discussed in the Introduction, the novelty of our definition resides in our definition of an $m$-dimensional permutation array.

**Definition 2:** Let $A = [n_1] \times \cdots \times [n_m]$. A binary array $A : \Lambda \to \{0, 1\}$ is an $m$-dimensional permutation array if there is a bijection

$$\varphi : [n_1] \times \cdots \times [n_k] \to [n_{k+1}] \times \cdots \times [n_m],$$

for some $k$, $1 \leq k < m$, such that, for $\alpha = (a_{11}, \ldots, a_{k1}, a_{k+1}, \ldots, a_{km}) \in \Lambda, A(\alpha) = 1$ if and only if $\varphi(a_{11}, \ldots, a_{k1}) = (a_{k+1}, \ldots, a_{km})$. For a permutation array $A$, we define the order, denoted $n$, to be the number of dots in $A$, that is, $n = n_1 n_2 \cdots n_k = n_{k+1} n_{k+2} \cdots n_m$.

**Definition 3:** An $m$-dimensional Costas array is an $m$-dimensional permutation array having no repeated difference vectors.

**Example 1:** Let $A : [2] \times [2] \times [4] \to \{0, 1\}$ be the array with set of dots $\{1, 1, 1, 2, 2, 2, 3, 2, 1, 4\}$. This array can be seen in Figure 2. We can verify that this is a three-dimensional Costas array by computing all the difference vectors.
Notice that in Definition 2, if \( m = 2 \), then \( k = 1 \), \( \Lambda = [n_1] \times [n_2] \), and \( \varphi \) is a bijection \( \varphi : [n_1] \rightarrow [n_2] \), so \( n_1 = n_2 \). Therefore, when \( m = 2 \), the array \( A \) in Definition 3 is a permutation array with no repeated difference vectors, which is exactly the definition of a two-dimensional Costas array. If in Definition 2 we let \( m \) to be even, \( k = \frac{m}{2} \) and \( n_1 = \cdots = n_m \), a Costas array with this structure is precisely what is given in [17, Definition 6]. Furthermore, if in Definition 2 we let \( k = 1 \) so that \( \varphi : [n_1] \rightarrow [n_2] \times \cdots \times [n_m] \) is a bijection, this special configuration is equivalent to a Costas array of dimension \( m - 1 \) and type \( n_2, \ldots, n_m \), as defined in [20, Definition 3.2]. Definition 3 works for arbitrary dimensions, is consistent with the definition of two-dimensional Costas arrays when restricted to two dimensions, and produces arrays with square-root density: \( n \) entries with 1’s out of a total of \( n^2 \) entries. The multidimensional analogs of Costas arrays proposed in [16], [17], [18], and [19] lack at least one of the aforementioned features. A downside of our definition is that we do not know any systematic way of constructing multidimensional Costas arrays other than the reshaping technique described in [17, §4] for the special case of arrays with \( m \) even and \( n_1 = \cdots = n_m \).

To ease notation, from now on, let \( X = [n_1] \times \cdots \times [n_k] \) and \( Y = [n_{k+1}] \times \cdots \times [n_m] \), where the \( n_i \)’s are integers greater than 1, and \( |X| = |Y| \).

**Remark 1:** Having no repeated difference vectors in an \( m \)-dimensional Costas array defined by a bijection \( \varphi : X \rightarrow Y \) is equivalent to the so called **distinct difference property:** for any \( h \in \mathbb{Z}^k \), \( \varphi(i + h) - \varphi(i) = \varphi(j + h) - \varphi(j) \implies i = j \) or \( h = (0, \ldots, 0) \), for \( i, i + h, j, j + h \in X \).

If a bijection \( \varphi : X \rightarrow Y \) defines an \( m \)-dimensional Costas array, the inverse map \( \varphi^{-1} \) is also a bijection that defines an \( m \)-dimensional Costas array since the dots of the latter would be just a swap between the first \( k \) coordinates and the last \( m - k \) coordinates of the former, so all difference vectors are going to be distinct. We consider those arrays to be equivalent. Moreover, if a bijection \( \varphi : X \rightarrow Y \) defines an \( m \)-dimensional Costas array, any permutation of the coordinates in \( X \) or in \( Y \) will produce another Costas array, as this only permutes the components of the difference vectors, so they are going to be distinct. We consider those arrays to be equivalent.

**IV. PERIODICITY OF MULTIDIMENSIONAL COSTAS ARRAYS**

The non-existence of two-dimensional periodic patterns preserving the Costas condition was settled by Taylor [22] with Theorem 1: there are no two-dimensional Costas arrays of order \( n > 2 \), for which its periodic extension contains a Costas array in every \( n \times n \) window. Does the same happen for multidimensional Costas arrays? Exploring this question is appropriate and relevant in the higher-dimensional context as there is no apparent reason why an analogous result should hold.

Without making any assessment of whether these arrays could exist, it is intuitive to consider the following two types of Costas periodicity, i.e., multidimensional arrays for which the Costas condition is preserved when periodically extending a multidimensional array.

**Definition 4:** An \( m \)-dimensional Costas array of size \( n_1 \times \cdots \times n_m \) is **periodic Costas** if any \( n_1 \times \cdots \times n_m \) window of its periodic extension has no repeated difference vectors, i.e., every window is an \( m \)-dimensional Costas array.

**Remark 2:** Based on Definition 4, Theorem 1 can be rephrased as: if \( A \) is a two-dimensional periodic Costas array of order \( n \), then \( n = 2 \).

**Definition 5:** An \( m \)-dimensional Costas array of size \( n_1 \times \cdots \times n_m \) is **modular Costas** if any \( n_1 \times \cdots \times n_m \) window of its periodic extension has no repeated toroidal vectors.

**Remark 3:** In Definition 5 there is no need to consider the toroidal vectors in every window in the periodic extension because, by Lemma 1, an \( m \)-dimensional Costas array is modular Costas if and only if it has no repeated toroidal vectors.

**Definition 6:** Let \( \Lambda = [n_1] \times \cdots \times [n_m] \) be an index set with \( n_1 n_2 \cdots n_k = n_{k+1} n_{k+2} \cdots n_m \), for some \( k < m \). Define the following sets.
we don't consider a valid toroidal vector. Similarly, for \( \tau \)
but, if \( A \) is a valid toroidal vector in \( \tau \), it implies
a value set of toroidal vectors in a permutation array \( \Lambda \).

Proposition 2: Let \( A : \Lambda \rightarrow \{0, 1\} \) be an \( n \)-dimensional
permutation array. If \( \tau \) is a toroidal vector in \( A \), \( \tau \in \Lambda \).

Proof: Let \( \varphi : X \rightarrow Y \) be the bijection defining the dots
of \( A \), where \( X = [n_1] \times \cdots \times [n_k] \) and \( Y = [n_k+1] \times \cdots \times [n_m] \).
Then \( A \) is indexed by \( \Lambda = X \times Y \). Let \( Z_1 = Z_1 \times \cdots \times Z_{n_k} \)
and \( Z_2 = Z_{n_k+1} \times \cdots \times Z_{n_m} \). If \( \tau = (h_1, \ldots, h_m) \) is
a toroidal vector occurring in \( A \), it is clear that \( \tau \in Z_1 \times Z_2 \).
But, if \( \tau \) is a toroidal vector from \( \alpha = (a_1, \ldots, a_m) \in A 
\omega = (w_1, \ldots, w_m) \in A \) with \( h_1 = \cdots = h_k = 0 \),
it implies \( a_1 = w_1, \ldots, a_k = w_k \). Therefore \( \varphi(a_1, \ldots, a_k) = \varphi(w_1, \ldots, w_k) \Rightarrow (ak+1, \ldots, am) = (wk+1, \ldots, w_m) \),
so that \( \alpha = \omega \). In such case, \( \tau \) is the zero vector, which
we don’t consider a valid toroidal vector. Similarly, for \( \tau \) to
be a valid toroidal vector in \( A \), \( h_k+1 = \cdots = h_m = 0 \) cannot
happen. We conclude that \( \tau \in \Lambda A = (Z_1 \times Z_2) \setminus (Z_1 \times \{0\} \cup \{0\} \times Z_2) \).

As we can see from Proposition 2, the cardinality of the
value set of toroidal vectors in a permutation array \( \Lambda : \Lambda \rightarrow \{0, 1\} \) of order \( n \) is

\[
[T_\Lambda] = \left( \prod_{i=1}^{k} n_i - 1 \right) \left( \prod_{i=k+1}^{m} n_i - 1 \right) = (n-1)^2. \tag{4}
\]

Multidimensional permutation arrays of order \( n \) have \( n(n-1)\)
toroidal vectors out of \( (n-1)^2 \) possible vectors. Therefore, \( T_\Lambda \)
has at least \( n-1 \) repeated elements, counting multiplicities. We have proven the next result.

Theorem 2: Multidimensional modular Costas arrays do not
exist.

The result in Theorem 2 is the same as [28, Theorem 1],
which, for \( n > 2 \), could be seen as a consequence of
[26, Theorem 6]. The cited results are stated for finite
abelian groups, which is a more general context than ours.
However, our approach reveals a clear connection between
[28, Theorem 1], [26, Theorem 6] and multidimensional
Costas arrays.

We show next a nice picture to make sense of the multisets
\( T_\Lambda, H_\Lambda \) in Definition 1 and the sets \( T_\Lambda, H_\Lambda \) in Definition 6.
Consider the permutation array in Figure 1, which is also
Costas, and has toroidal vectors:

\[
T_\Lambda = \{\{(1,1), (2,3), (3,2), (3,3), (1,2), (2,1), (2,1), (3,2), (1,3), (1,2), (2,3), (3,1)\}\},
\]

where we use double curly braces \{\} to denote a multiset.
Construct a frequency array (two-dimensional in this case),
as in Figure 3, such that position \((x, y)\) contains the number of
times \((x, y)\) appears as a toroidal vector in the Costas array
in Figure 1. Of course, an analogous frequency array can be
constructed for any higher-dimensional permutation array. The
entries with coordinate \((0, 0)\) and \((\cdot, 0)\) are obscured because,
in a permutation array, toroidal vector with those coordinates
do not appear (those are the vectors in \( Z_1 \times \{0\} \cup \{0\} \times Z_2 \); see
Definition 6). In the case of a three-dimensional permutation
array of size \( n_1 \times n_2 \times n_1 n_2 \), the obscured entries (excluded
toroidal vectors) of the frequency array are those of the form
\((0,0,\cdot)\) and \((\cdot,\cdot,0)\); something like the shaded region in
Figure 4.

The set \( T_\Lambda \) is the set of all toroidal vectors corresponding
to the not excluded boxes (white boxes). For our example in Figure 1 and Figure 3, \( T_\Lambda = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\} \).
The set \( H_\Lambda \) represents the boxes in the frequency array corresponding
to toroidal vectors with some component half
the length of the matching side of the array. We highlight those
boxes in yellow. For the array \( A \) in Figure 1, whose order is 4,
these are the toroidal vectors with a component equal to 2,
so we highlight column 2 and row 2, shown in Figure 5.
Therefore,\( H_\Lambda = \{(1,2), (2,1), (2,2), (2,3), (3,2)\} \).

By construction, \( T_\Lambda \) is the multiset of all toroidal vectors
in the frequency array, each appearing repeatedly the number of
times given by the number in the corresponding box. For example,
the toroidal vector \((1,2)\) appears twice in \( T_\Lambda, (3,1)\) appears
once, and \((2,2)\) does not appear in \( T_\Lambda \). The multiset
\( H_\Lambda \) contains the elements of \( T_\Lambda \) corresponding to yellow
boxes in the frequency array.

Notice that any \( 4 \times 4 \) permutation array will have the same
boxes painted yellow as the ones in Figure 5. That is the reason
for our notation: we put \( \Lambda \) as a subscript in \( H_\Lambda \) to emphasize
that the set only depends on the shape of the permutation array,
not the entries, and the shape is determined by the index set \( \Lambda \).
On the other hand, the numbers on the array with yellow boxes
do depend on the permutation array \( A \); thus, our subscript
in \( H_\Lambda \).
After Theorem 2, if there is any hope for the existence of arrays preserving the Costas condition periodically, it must rely solely on periodic Costas arrays, not modular Costas. However, using the next lemma, we will show that multidimensional periodic Costas arrays do not exist for some classes of Costas arrays.

**Theorem 3:** Let \( A \) be an \( m \)-dimensional permutation array of size \( n_1 \times \cdots \times n_m \), index set \( \Lambda = [n_1] \times \cdots \times [n_m] \), and order \( n \). If \( |\mathcal{H}_A| - |\mathcal{H}_A| < n - 1 \), there is an \( n_1 \times \cdots \times n_m \) window in the periodic extension of \( A \) which has a repeated difference vector.

**Proof:** We have

\[
|\mathcal{H}_A| - |\mathcal{H}_A| < n - 1
\]

\[
\Rightarrow |\mathcal{H}_A| - |\mathcal{H}_A| < n(n - 1) - (n - 1)^2
\]

\[
\Rightarrow (n - 1)^2 - |\mathcal{H}_A| < n(n - 1) - |\mathcal{H}_A|
\]

\[
|T_A - \mathcal{H}_A| < |T_A - \mathcal{H}_A|
\]

Since the number of toroidal vectors of \( A \) not in \( \mathcal{H}_A \) is greater than the number of all possible values for toroidal vectors not in \( \mathcal{H}_A \), by the pigeonhole principle, \( A \) must have a repeated toroidal vector not in \( \mathcal{H}_A \). By Proposition 1, an \( n_1 \times \cdots \times n_m \) window of the periodic extension of \( A \) has a repeated difference vector. Right away, we obtain a non-existence result for multidimensional periodic Costas arrays of odd order.

**Corollary 2:** If \( A \) is an \( m \)-dimensional permutation array of odd size \( n_1 \times \cdots \times n_m \), there is an \( n_1 \times \cdots \times n_m \) window in the periodic extension of \( A \) which has a repeated difference vector. In particular, multidimensional Costas arrays of odd order are not periodic Costas.

**Proof:** Since the toroidal vectors in \( A \) have integer components, none can have the \( i \)-th component equal to \( n_i/2 \). Then, \( 0 = |\mathcal{H}_A| - |\mathcal{H}_A| < n - 1 \). By Theorem 3, \( A \) is not periodic Costas.

Proving the non-existence of multidimensional periodic Costas arrays is more complicated for even order, at least with our approach. The problem is that with an arbitrary permutation array of even order, we do not know how to obtain a sufficiently low upper bound for \( |\mathcal{H}_A| \). However, the following lemma will help us count some toroidal vectors in \( \mathcal{H}_A \) for an \( m \)-dimensional permutation array \( A \) of even order.

**Lemma 2:** Let \( A \) be an \( m \)-dimensional permutation array of even order defined by a bijection \( \varphi : X \to Y \) and \( E \subseteq [m] \), where \( n_i \) is even for all \( i \in E \). Denote \( \mathcal{H}_E \) the multiset of toroidal vectors \( \langle h_1, \ldots, h_m \rangle \) occurring in \( A \) for which \( h_i = n_i/2 \) for all \( i \in E \). If \( E \subseteq \{1, \ldots, k\} \) or \( E \subseteq \{k + 1, \ldots, m\} \), then

\[
|\mathcal{H}_E| \prod_{i \in E} n_i = n^2.
\]

**Proof:** Notice that \( E \neq \emptyset \) given that \( A \) has even order. Assume \( E \subseteq [k] \). After reordering indices we may assume \( E = [t] \), for some \( t \leq k \). For any \( \alpha = (a_1, \ldots, a_m) \in E \), set \( w_i \) to be the unique number in \( [n_i] \) that is equivalent to \( a_i + n_i/2 \) modulo \( n_i \), for all \( i \in [t] \). Choose \( w_{t+1}, \ldots, w_k \) freely and set \( w_{t+1}, \ldots, w_k = \varphi(w_{t+1}, \ldots, w_k) \). By construction, \( \omega = (w_1, \ldots, w_m) \in A \), and if \( \langle h_1, \ldots, h_m \rangle \) is the toroidal vector from \( \alpha \) to \( \omega \), \( h_i = n_i/2 \) for all \( i \in E \). Hence, for each \( \alpha \in A \), we can choose freely \( w_{t+1}, \ldots, w_k \) to obtain a toroidal vector in \( A \) with \( h_i = n_i/2 \) for all \( i \in E \). It is easy to see that this is the only way to obtain such toroidal vectors. By simple counting we have

\[
|\mathcal{H}_E| = n!_{t+1} \cdots n_k = n! \prod_{i \in E} n_i = n! \prod_{i \in E} n_i,
\]

and the result follows. The case \( E \subseteq \{k + 1, \ldots, m\} \) is analogous.

We illustrate Lemma 2 with an example. Consider the \( [2] \times [2] \times [4] \) array in Example 1, which is a 3-dimensional Costas array of order 4. Its multiset of toroidal vectors is given by

\[
\{(0, 1, 3), (1, 0, 1), (1, 1, 2), (0, 1, 1), (1, 1, 2), (1, 0, 1), (0, 1, 1), (1, 1, 2), (1, 0, 1), (0, 1, 3)\}
\]

Let \( E = \{3\} \), so that \( \mathcal{H}_E \) is the multiset of toroidal vectors with a 2 in the third coordinate, i.e.,

\[
\mathcal{H}_E = \{(1, 1, 2), (1, 1, 2), (1, 1, 2), (1, 1, 2)\}
\]

Hence \( |\mathcal{H}_E| = 4 \) and \( |\mathcal{H}_E| \prod_{i \in E} n_i = 4 \cdot 4 = 16 \), which is equal to the order squared, as guaranteed by Lemma 2.

The next lemma provides a tidy formula for a counting, done by inclusion-exclusion, that will be used in Theorem 4 below. It is proved by induction, and we omit the proof.

**Lemma 3:** Let \( K = \{n_1, \ldots, n_k\} \) be a non-empty multiset of \( k \) natural numbers. Then,

\[
\sum_{i \subseteq [k]} (-1)^{|i|+1} \prod_{i \in I} n_i - 1 = \prod_{i \in [k]} n_i - 1.
\]

We will tackle the existence of periodic Costas Arrays of even order only for arrays defined by bijections with one-dimensional image, which are equivalent, by taking the inverse, to bijections with a one-dimensional domain. We have two reasons for it. First and foremost, the counting argument gets very convoluted for higher-dimensional image sets. Secondly, every (non-equivalent) three-dimensional Costas array must have one-dimensional image, so the three-dimensional case will be covered.

**Theorem 4:** Let \( A \) be an \( m \)-dimensional permutation array of even order \( n \) defined by a bijection \( \varphi : [n_1] \times \cdots \times [n_m-1] \rightarrow [n_m] \). Denote \( \theta = 1 - \prod_{i \in E} \frac{n_{i-1}}{n_i} \), where \( E = \{i \in [m-1] : n_i \text{ is even}\} \). If \( \theta < \frac{n^2}{2} \), there is an \( n_1 \times \cdots \times n_m \) window in the periodic extension of \( A \) which has a repeated difference vector. In particular, if \( A \) is an \( m \)-dimensional Costas array with \( \theta < \frac{n^2}{2} \), it is not periodic Costas.

**Proof:** Let \( n \) be the order of \( A \). That is, \( n = n_1n_2 \cdots n_{m-1} = n_m \), which is even by assumption. By Theorem 3, it is enough to check that \( |\mathcal{H}_A| - |\mathcal{H}_A| < n - 1 \). First we count \( |\mathcal{H}_A| \). For it, define the following multisets:

- \( \mathcal{U} \) is the multiset of toroidal vectors \( \langle h_1, \ldots, h_m \rangle \in \mathcal{H}_A \) with \( h_i = n_i/2 \) for some \( i \in [m-1] \).
- \( \mathcal{V} \) is the multiset of toroidal vectors \( \langle h_1, \ldots, h_m \rangle \in \mathcal{H}_A \) with \( h_m = n_m/2 \).

It is clear that \( |\mathcal{H}_A| = |\mathcal{U}| + |\mathcal{V}| - |\mathcal{U} \cap \mathcal{V}| \), where the intersection is in the context of multisets, i.e., including repetitions. By the inclusion-exclusion principle \( |\mathcal{U}| \) is the sum of the number of
toroidal vectors with $h_i = n_i/2$ for a single $i \in [m-1]$, minus the toroidal vectors with $h_i = n_i/2$ for $i \in \{i_1, i_2\} \subseteq [m-1]$, plus the toroidal vectors with $i \in \{i_1, i_2, i_3\} \subseteq [m-1]$, and so on. By Lemma 2, for any $J \subseteq E$, the number of toroidal vectors with $h_i = n_i/2$, for all $i \in I$, is $\sum_{i \in I} n_i^2$. Therefore,

$$|\mathcal{U}| = \sum_{i \in I} (-1)^{|i|+1} \frac{n_i^2}{\prod_{i \in I} n_i}.$$ 

By Equation (5), $|\mathcal{U}| = n^2 \theta$. Also by Lemma 2,

$$|\mathcal{V}| = \frac{n^2}{\prod_{i \in (m)} n_i} = n.$$

Therefore, $|\mathcal{H}_A| \leq n(n \theta + 1)$.

To count $|\mathcal{H}_A|$, define the following subsets of $H_A$:

- $U = \{(h_1, \ldots, h_m) \in H_A : h_i = n_i/2$ for some $i \in [m-1]\}$, and
- $V = \{(h_1, \ldots, h_m) \in H_A : h_m = n_m/2\}$.

Then $|\mathcal{H}_A| = |U| + |V| - |U \cap V|$. Notice that, for fixed $i \in [m-1]$, there is a total of

$$(n-1) \prod_{j \in [m-1], j \neq i} n_j = (n-1) \frac{n}{n_i}$$

toroidal vectors $\langle h_1, \ldots, h_m \rangle \in H_A$ with $h_i = n_i/2$ (all entries are free to choose, except the $i$-th entry, which is fixed, and the $n-1$ factor is because $h_m$ could be any reminder modulo $n_m$, except zero). It follows from the inclusion-exclusion principle and Equation (5) that

$$|U| = (n-1) \sum_{i \in E, i \neq \emptyset} (-1)^{|i|+1} \frac{n}{\prod_{i \in I} n_i} = (n-1) n \theta.$$ 

On the other hand, $|V| = n-1$ because, for $\langle h_1, \ldots, h_m \rangle \in V$, $h_1, \ldots, h_{m-1}$ can be chosen freely except for zero. Finally, $|U \cap V| = \frac{|\mathcal{U}|}{n} = n \theta$. Then,

$$|\mathcal{H}_A| = (n-1) n \theta + (n-1) - n \theta = (n-1)(n \theta + 1) - n \theta.$$ 

We conclude that

$$|\mathcal{H}_A| - |\mathcal{H}_A| \leq n(n \theta + 1) - (n-1)(n \theta + 1) + n \theta = 2n \theta + 1.$$ 

By assumption, $2n \theta + 1 < n - 1$, so the result follows from Theorem 3.

$$\square$$

Notice that for a permutation array of even order defined by a bijection $\varphi : [n_1] \times \cdots \times [n_{m-1}] \to [n_m]$, if $n_i$ is very large for all $i \in [m]$, then $\theta$, as defined in Theorem 4, is close to zero, while $\frac{n}{2m}$ is close to $1/2$, so we will have $\theta < \frac{n_2}{2m}$. Therefore, sufficiently large Costas arrays defined by a bijection with one-dimensional image or one-dimensional domain are not periodic Costas.

The proof of Theorem 4 reveals another reason why we considered Costas arrays defined by a bijection with one-dimensional image. Notice that the multisets $\mathcal{U}$ and $\mathcal{V}$ can be defined for arbitrary bijections, not only those having one-dimensional image. That is, if $\varphi : [n_1] \times \cdots \times [n_{k-1}] \to [n_k] \times \cdots \times [n_m]$ defines a permutation array, define $\mathcal{U}$ as the multiset of toroidal vectors for which at least one of the first $k$ components is half the length of the array in the corresponding direction. Similarly, define $\mathcal{V}$ as the multiset of toroidal vectors with at least one of the last $m - k$ components being half the length of the array in the corresponding direction. As in the proof of Theorem 4, $|\mathcal{H}_A| = |\mathcal{U}| + |\mathcal{V}| - |\mathcal{U} \cap \mathcal{V}|$. The number $|\mathcal{H}_A|$ is easy to count using the inclusion-exclusion principle. Hence, to obtain $|\mathcal{H}_A| - |\mathcal{H}_A| < n - 1$, the hurdle is to get a sufficiently low upper bound for $|\mathcal{H}_A|$. This can be done by obtaining a large lower bound on $|\mathcal{U} \cap \mathcal{V}|$, but this appears to be a difficult task. Of course, $|\mathcal{U} \cap \mathcal{V}| \geq 0$ so zero is the worst possible lower bound. When the image of $\varphi$ is one-dimensional, even the worst possible lower bound is good enough; zero is good enough. When the image of $\varphi$ has a higher-dimensional image, zero is not in general a reasonable lower bound for $|\mathcal{U} \cap \mathcal{V}|$.

**Corollary 3.** Let $A$ be a three-dimensional permutation array of size $n_1 \times n_2 \times n_3 n_2$. Any of the following imply that there is an $n_1 \times n_2 \times n_3 n_2$ window in the periodic extension of $A$ having a repeated difference vector:

(i) $n_1$ and $n_2$ are odd.
(ii) $n_1$ and $n_2$ are even, and one of them is greater than 4.
(iii) $n_1$ is even greater than 2 and $n_2$ is odd, or vice versa.

**Proof:** Let $A$ be a permutation array defined by a bijection $\varphi : [n_1] \times [n_2] \rightarrow [n_1 n_2]$; hence, the order of $A$ is $n = n_1 n_2$. To avoid a degenerate three-dimensional array, we are implicitly assuming $n_1 > 1$ and $n_2 > 1$. If (i) holds, $A$ has odd order and the result follows by Corollary 2. Now assume $n_1$ and $n_2$ are even. Then $\theta = 1 - \frac{(n_1 - 1)(n_1 - 2)}{n_1 n_2}$ and

$$\theta < \frac{n_1 n_2 - 2}{2n_1 n_2} \iff n_1 n_2 - (n_1 - 2)(n_2 - 1) < \frac{n_1 n_2 - 2}{2} \iff n_1 + n_2 - 1 < \frac{n_1 + n_2 - 2}{2} \iff 2 < \frac{n_1 n_2}{n_1 + n_2}.$$ 

(6) Inequality (6) holds if $n_1 > 4$ or $n_2 > 4$. The result follows by Theorem 4.

Finally, assume $n_1$ is even, $n_2$ is odd, and $n_1 > 2$. In this case, $\theta = 1 - \frac{n_1 - 1}{n_1}$. Hence,

$$\theta < \frac{n_1 n_2 - 2}{2n_1 n_2} \iff n_1 n_2 - n_1 (n_1 - 1) < \frac{n_1 n_2 - 2}{2} \iff 2 n_2 < n_1 n_2 - 2 \iff 2 < n_1 (n_1 - 2).$$ 

(7) But $n_2$ odd, $n_2 > 1$, and $n_1 > 2$ ensures that inequality (7) holds. The result follows by Theorem 4. 

**With a bit more work, we can say even more than in Corollary 3.**

**Theorem 5:** If $A$ is a three-dimensional periodic Costas array, it has order 4.

**Proof:** Let $A$ be a three-dimensional periodic Costas array. Without loss of generality, we assume $A$ is defined by a bijection $\varphi : [n_1] \times [n_2] \rightarrow [n_1 n_2]$, where $n_1 n_2$ is the order of $A$ and $n_1 \leq n_2$. Since $A$ is periodic Costas, Corollary 3 leaves only four possibilities:

- Case 1: $n_1$ odd, $n_2$ even
- Case 2: $n_1$ even, $n_2$ odd
- Case 3: $n_1$ even, $n_2$ even
- Case 4: $n_1$ odd, $n_2$ odd
We must show (ii)–(iv) cannot happen. By exhaustive computation we checked that there are no periodic Costas arrays among all the 8! bijections \( \varphi : [2] \times [4] \rightarrow [8] \) and all 16! bijections \( \varphi : [4] \times [4] \rightarrow [16] \).

Now we focus on case (iv). Assume \( A \) is defined by a bijection \( \varphi : [2] \times [k] \rightarrow [2k] \), for \( k \) odd. Fix \( (x_0, y_0) \in \mathbb{Z}_2 \times \mathbb{Z}_{2k} \), with \( (x_0, y_0) \neq (0,0) \). Let \( \alpha = (a_1, a_2, a_3) \in A \). If \( \omega = (w_1, w_2, w_3) \in A \) is such that the toroidal vector from \( \alpha \) to \( \omega \) has the form \( (x_0, y_0, z) \), for some \( z \in \mathbb{Z}_{2k} \), then
\[
 w_1 - a_1 \equiv x_0 \pmod{2} \quad \text{and} \quad w_2 - a_2 \equiv y_0 \pmod{n}.
\]
Since \( w_1 \in [2] \) and \( w_2 \in [k] \), their values are unique. But \( A \) is defined by the bijection \( \varphi \), so we must have \( w_3 = \varphi(w_1, w_2) \).

Therefore, for each \( \alpha \in A \), we found a unique \( \omega \in A \) such that the toroidal vector from \( \alpha \) to \( \omega \) has the from \( (x_0, y_0, z) \), for some \( z \in \mathbb{Z}_{2k} \). We conclude that there are exactly \( 2k \) toroidal vectors \( \omega \) of such form. However, there are only \( 2k - 1 \) possible choices for \( z \) in a toroidal vector of the form \( (x_0, y_0, z) \). Thus, by the pigeonhole principle, for each pair \( (x_0, y_0) \in (\mathbb{Z}_2 \times \mathbb{Z}_{2k})^2 \), there must be some \( z_0 \in \mathbb{Z}_{2k} \) such that \( (x_0, y_0, z_0) \) is a repeated toroidal vector. In particular, let \( x_0 = 0 \), so there is a repeated toroidal vector with the form \( (0, y_0, z_0) \). Notice that \( y_0 \neq k/2 \) because \( k \) is odd. If \( z_0 \neq 2k/2 = k \), by Proposition 1, \( A \) is not periodic Costas, and the proof would be finished.

Let \( z_0 = k \). That is, assume \( A \) has a repeated toroidal vector of the form \( (0, y, k) \), for some \( y \in \mathbb{Z}_k \). We claim that this repeated toroidal vector will satisfy the conditions in Corollary 1, so \( A \) is not periodic Costas. For the sake of a contradiction, assume \( A \) has four dots
\[
\begin{align*}
\alpha_1 &= (a_{11}, a_{12}, a_{13}), \\
\omega_1 &= (w_{11}, w_{12}, w_{13}), \\
\alpha_2 &= (a_{21}, a_{22}, a_{23}), \\
\omega_2 &= (w_{21}, w_{22}, w_{23}),
\end{align*}
\]
not satisfying the conditions in Corollary 1 and for which \( \omega_1 - \alpha_1 \) and \( \omega_2 - \alpha_2 \) are equal as toroidal vectors to \( (0, y, k) \in \mathbb{Z}_2 \times \mathbb{Z}_k \times \mathbb{Z}_{2k} \). Then, since \( 0 \neq n_1/2 = 2/2 = 1 \) and also \( y \neq n_2/2 = k/2 \) because \( k \) is odd, if the four dots do not satisfy the conditions in Corollary 1, we must have \( w_{13} - a_{13} = k, \ w_{23} - a_{23} = -k, \) and \( a_{13} = w_{23} \). Then, \( w_{13} - a_{13} = -(w_{23} - a_{23}) = -(a_{13} - a_{23}) \), implying \( w_{13} = a_{23} \). \( A \) is defined by the bijection \( \varphi \), so all the dots of \( A \) can be expressed as \( (\varphi^{-1}(z), z) \), for some \( z \in [2k] \). Hence, \( \alpha_1 = (\varphi^{-1}(a_{13}), a_{13}) \) and \( \omega_2 = (\varphi^{-1}(w_{23}), (w_{23})), \) which implies \( \alpha_1 = \omega_2 \), given that \( \alpha_1 = a_{13} = w_{23} \). Similarly, \( \omega_1 = \alpha_2 \) because \( w_{13} = a_{23} \). Therefore, \( \omega_1 - \alpha_1 \) and \( \omega_2 - \alpha_2 = \alpha_1 - \omega_1 \) are both equal as toroidal vectors to \( (0, y, k) \).

Given that \( \omega_1 - \alpha_1 \) and \( \omega_2 - \alpha_2 = -(\omega_1 - \alpha_1) \) are both equal as toroidal vectors to \( (0, y, k), \ w_{12} - a_{12} \equiv y \pmod{k} \) and \( -(w_{12} - a_{12}) \equiv y \pmod{k} \). Then, \( 2(w_{12} - a_{12}) \equiv 0 \pmod{k}, \) but \( k \) is odd and \( a_{12}, w_{12} \in [k], \) so \( w_{12} = a_{12} \). Then we must have \( y = 0 \). This is a contradiction because \( y \in \mathbb{Z}_k \).

The statement of Theorem 5 raises a natural question. Are there periodic Costas arrays of order 4? As the reader should expect, the answer is yes. Periodic Costas arrays of size \( 2 \times 2 \times 4 \) do exist. There are \( 4! = 24 \) distinct bijections
\[
\varphi : \{(1,1), (1,2), (2,1), (2,2)\} \rightarrow \{1,2,3,4\}.
\]

By exhaustive computation, we found that, out of these 24 bijections, 16 define Costas arrays, and 8 are periodic Costas.

**Example 2:** Consider the three-dimensional Costas array described in Example 1 (see also Figure 2). Although quite a task to do by hand, by checking all the 16 possibly distinct windows of size \( 2 \times 2 \times 4 \) in its periodic extension, we can see that every window is a Costas array. Therefore, the array in Example 1 is a periodic Costas array.

Based on the above results and some exhaustive computations we performed, we finish this paper with a conjecture, which is a higher-dimensional analog of Theorem 1.

**Conjecture 1:** Let \( A \) be an \( m \)-dimensional Costas array of order \( n \) defined by a bijection \( \varphi : [n_1] \times \cdots \times [n_k] \rightarrow [n_{k+1}] \times \cdots \times [n_{m}], \) where \( 2k \geq m \). If \( A \) is periodic Costas, \( n_1 = n_2 = \cdots = n_k = 2 \).

Notice that, by Theorem 1 and Theorem 5, the above conjecture is true for two-dimensional and three-dimensional Costas arrays, respectively.

**V. Conclusion**

We proposed a new multidimensional generalization of Costas arrays. Our definition works for arbitrary dimensions, the restriction to two-dimensions is consistent with the well-known two-dimensional definition, produces arrays with density equal to the square root of the number of entries, and is more general than [17, Definition 6] and [20, Definition 3.2]. We studied arrays whose periodic extension contains a Costas array in every window of the same size of the original array. In the two-dimensional case, it was shown by H. Taylor [22] that those arrays must have order 2. We showed partial results on the higher-dimensional extensibility of Taylor’s theorem, and conjectured it holds for arbitrary dimensions.

With our definition for multidimensional Costas arrays there are as many research directions as there are for two-dimensional Costas arrays. In fact, any result that is known for two-dimensional Costas arrays becomes a question in the higher-dimensional context. We propose in [27] a higher-dimensional analog to circular Costas arrays such that their relationship to multidimensional Costas arrays as defined in Definition 3 is consistent with the two-dimensional case. The work includes results on the multidimensional extensibility of the Golomb-Moreno conjecture [23], shown in [24] to be true for two-dimensional Costas arrays, thus validating Definition 3 as a feasible higher-dimensional definition of a Costas array.

A few interesting questions in the theoretical side for further directions: Are there any systematic algebraic methods for constructing an \( m \)-dimensional Costas array (cf. [29])? Does the proportion of Costas arrays among permutations decays exponentially as the size and/or the dimension increases (cf. [30])? What can we say about the deficiency and ambiguity of multidimensional Costas arrays (cf. [25], [26])? In particular, is there any relation between deficiency/ambiguity
and the non-existence of periodic Costas arrays? Are there any structural constraints for multidimensional Costas arrays (cf. [31], [32])?

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