ANALYZING THE STRUCTURE OF REPRESENTATIONS VIA APPROXIMATIONS

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Dedicated to the memory of Maurice Auslander.

Abstract. Primarily this paper presents an expository report on alternatives to the traditional methods of classifying representations of finite dimensional algebras. Some new results illustrating such alternatives for algebras with only finitely many isomorphism types of uniserial modules are included.

1. Introduction and notation

We discuss several avenues of approach to the structure of representations of a finite dimensional algebra $\Lambda$ by comparing the target objects with the objects from a more thoroughly understood reference class. Our starting point is the concept of a ‘right $\mathcal{A}$-approximation’ of a left $\Lambda$-module relative to a subcategory $\mathcal{A}$ of $\Lambda$-mod, as introduced by Auslander and Smalø in [3] (although originally not in this terminology), as well as the subsequent work on the subject by Auslander and Reiten [2].

Due to [2], the following holds for any resolving contravariantly finite subcategory $\mathcal{A}$ of $\Lambda$-mod: If there are $n$ simple left $\Lambda$-modules, up to isomorphism, and if $A_1, \ldots, A_n$ are their minimal right $\mathcal{A}$-approximations, then a left $\Lambda$-module belongs to $\mathcal{A}$ if and only if it is isomorphic to a direct summand of a module $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_l = 0$ with $M_i/M_{i+1} \in \{A_1, \ldots, A_n\}$. In other words, approximating the simple modules by objects in $\mathcal{A}$ yields basic structural information about arbitrary objects in $\mathcal{A}$. To give a flavor of the usefulness of this type of information, e.g., towards a homological understanding of $\mathcal{A}$, we point out that, in the described situation, we have $\sup\{\text{p dim } A \mid A \in \mathcal{A}\} = \sup\{\text{p dim } A_i \mid 1 \leq i \leq n\}$. However, the applicability of this structure theorem is bounded by the following facts: Numerous module categories of interest fail to be contravariantly finite; on top of it, deciding whether a given subcategory $\mathcal{A}$ of $\Lambda$-mod has this property is a difficult task in general. And even when the question of contravariant finiteness has been resolved in the positive, it may still be extremely challenging to pin down the minimal right $\mathcal{A}$-approximations of the simple objects.

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In response to these obstacles, we suggest three lines of investigation, all based on the concept of approximation or derivatives thereof. The present report is partly an overview over existing results in these directions, illustrated by examples, partly an extension of ideas in the literature (e.g., most of Section 8 is new), and partly a program to be pursued more fully in the future. Roughly, the three lines of approach are as follows.

(1) Find manageable positive and negative criteria for contravariant finiteness, and systematically broaden the classes of algebras and module categories for which the ‘classical’ approximation theory can be brought to bear. In particular, this calls for an enlarged arsenal of techniques to find and describe minimal $A$-approximations in the case of contravariant finiteness and for a study of ‘typical shapes’ occurring among the minimal approximations of the simple modules, in dependence of $\Lambda$ and $A$. This line is illustrated by Sections 3, 4, and 7 below.

(2) Given a class $\mathcal{C}$ of representations which cannot be mastered in the classical mode of direct sum decompositions and a complete listing of the indecomposable summands occurring, scan the objects of $\mathcal{C}$ by comparing them with the objects of a reference class which is tailored to measure for this purpose. More precisely, construct contravariantly finite subcategories $\mathfrak{A}$ of $\Lambda$-mod so that the following classification of $\mathcal{C}$ relative to $\mathfrak{A}$ yields useful information on $\mathcal{C}$: namely, modules $C$ and $D$ are considered ‘similar relative to $\mathfrak{A}$’ provided they have isomorphic minimal right $\mathfrak{A}$-approximations. This, usually coarser, alternative to the traditional classification in terms of precise structural data naturally yields information of varying degrees of precision, ranging from the very rough – if $\mathfrak{A}$ is small – to a complete picture – if $\mathcal{C} \subseteq \mathfrak{A}$.

Here we illustrate this idea for $\mathcal{C} = \mathcal{P}^\infty(\Lambda$-mod), the category of all modules of finite projective dimension in $\Lambda$-mod. The objects of the reference category we use are glued together from uniserial building blocks (Sections 5, 6, 8). Sections 5 and 6 provide the tools: The first addresses the classification of the uniserial left $\Lambda$-modules in an informal sketch of some of the author’s results in [Geom I,II,III]. The second proposes a pattern for pasting uniserial modules together, while keeping a tight grip on the structure of the new objects created; our choice of pattern is justified in Section 7. In Section 8, we deal with algebras of finite uniserial type or, more precisely, with algebras for which the reference categories constructed in Section 7 have finite representation type. We conclude with concrete instances of the outlined approximation strategy.

(3) In a variation of our theme, we describe infinite dimensional substitutes for minimal $\mathfrak{A}$-approximations as developed by the author and Happel in [10]. These encode the same type of information as finite dimensional approximations do when they exist. To outline the underlying idea we require a few concepts, the first of which is only a slight extension of the original definition of right $\mathfrak{A}$-approximations. If $\mathfrak{B} \subseteq \mathfrak{A}$ and $M$ is a left $\Lambda$-module, a map $f : A \to M$ is called a (right) $\mathfrak{B}$-approximation of $M$ inside $\mathfrak{A}$ if $A \in \mathfrak{A}$ and each map $g \in \text{Hom}_\Lambda(B, M)$ for $B \in \mathfrak{B}$ factors through $f$. Furthermore, a $\Lambda$-module $H$ is called an $\mathfrak{A}$-phantom of $M$ if there exists a nonempty finite subclass $\mathfrak{B}$ of $\mathfrak{A}$ such that each $\mathfrak{B}$-approximation of $M$ inside $\mathfrak{A}$ has $H$ as a subfactor; direct limits of such modules $H$ – not required to be finitely generated – are again named phantoms. We are particularly interested in phantoms $H_j$ with the property that all homomorphisms $B \to M$ with $B$ running through
a specified subclass $B_j$ of $B$ factor through $H_j$. Given a chain $B_1 \subseteq B_2 \subseteq B_3 \subseteq \ldots$ of subclasses of $B$, the corresponding sequence of phantoms $H_1, H_2, H_3, \ldots$ will not stabilize in general, and so a direct limit is the natural way to integrate the information represented by such a sequence. To visualize this type of information, suppose that $M = S$ is simple; then a phantom $H_j$ as above communicates – in the most compressed form possible within the category $A$ – the relations of those objects $B \in B_j$ which contain $S$ in their tops.

The key fact in this connection is that nontrivial phantoms always exist. In fact, a module $M \in \Lambda$-mod fails to have an $A$-approximation in the traditional sense if and only if $M$ has $A$-phantoms of countably infinite vector space dimension.

In Section 2 we review the basics of contravariantly finite subcategories, concentrating on the point of view to be pursued later. In particular, we will omit all results linking functorially finite categories to tilting theory or relative derived functors. Since the concept of covariant finiteness of a subcategory $A$ of $\Lambda$-mod is dual to that of contravariant finiteness, we restrict our attention to the latter; the lines of investigation we discuss can be dualized accordingly. The following convention will therefore not lead to ambiguities: namely, we will refer to right $A$-approximations simply as $A$-approximations.

Throughout, $\Lambda$ will be a finite dimensional algebra over a field $K$ with Jacobson radical $J$. We assume $\Lambda$ to be split, i.e., $\Lambda$ is of the form $KT/I$, where $\Gamma$ is a quiver and $I$ is an admissible ideal of the path algebra $KT$. Our convention for composing paths in $KT$ is as follows: if $p$ and $q$ are paths, then $pq$ stands for ‘$p$ after $q$’ if the endpoint of $q$ coincides with the starting point of $p$, and $pq = 0$ otherwise. A path $u$ is called a right subpath (respectively, a left subpath) of $p$ in case there exists a path $v$ such that $p = vu$ (respectively, $p = uv$); both left and right subpaths will also be referred to as subpaths of $p$.

Whenever we mention primitive idempotents of $\Lambda$, we mean those associated with the vertices of $\Gamma$; in fact, we identify these vertices with the corresponding idempotents. Given a left $\Lambda$-module $M$, a top element of $M$ is an element $x \in M \setminus JM$ such that $x = ex$ for some primitive idempotent $e$; in that case we also say that $x$ is a top element of type $e$ of $M$.

Like numerous other authors (see, e.g., [1] and [7]), we find it convenient to represent certain $\Lambda$-modules by graphs. A special breed of such graphs, based on sequences of top elements of the pertinent modules, will make a great deal of information available at a glance. We introduce these labeled and layered graphs informally by means of some illustrative examples.

**Example 1.1.** Let $\Lambda = KT/I$, where $\Gamma$ is the quiver

![Quiver Diagram](#)

and $I$ an ideal in $KT$. That a left $\Lambda$-module $M$ have (layered and labeled) graph
relative to top elements $x_1$ of type $e_1$ and $x_2$ of type $e_2$ means that:

$$M/JM \cong S_1 \oplus S_2; \quad JM/J^2M \cong S_2;$$
$$J^2M/J^3M \cong J^3M/J^4M \cong S_3; \quad J^4M \cong S_4;$$

$\alpha x_1$ and $\beta x_2$ each generate $JM$ modulo $J^2M$;
$
\tau x_1, \gamma x_2, \gamma \alpha x_1$ (and hence also $\gamma / \beta x_2$) each generate $J^2M$ modulo $J^3M$;

$\delta \gamma \alpha x_1$ generates $J^3M$ modulo $J^4M$;

$\epsilon \gamma \alpha x_1$ and $\epsilon \delta \gamma \alpha x_1$ each generate $J^4M$.

Finally, whenever $p$ is a path in $\Gamma$ starting in $e_i$ which is not recorded under $x_i$ in the graph of $M$, the product $px_i$ is zero.

On the other hand,

is not the graph of a left $\Lambda$-module in case $\epsilon \delta^3 = 0$ in $\Lambda$.

It is often convenient to communicate the ideal $I$ by giving the graphs of the indecomposable projective modules $\Lambda e_i$. The algebra $\Lambda$ with quiver $\Gamma$ as above and with the following indecomposable projectives will recur in Section 6.
We extend these conventions as follows: That $N \in \Lambda\text{-mod}$ has the following graph

is to signify that $J^2N = S_2^3$, and that there are top elements $x_1, x_2, x_3$ of the obvious types such that $\alpha x_1, \beta \alpha x_1, \beta x_2, \beta \sigma x_4$ generate $J^2N$, with every choice of three of these four elements $K$-linearly independent (while the set of four is $K$-linearly dependent). □

Finally, $\mathcal{P}^\infty(\Lambda\text{-mod})$ will stand for the full subcategory of $\Lambda\text{-mod}$ having as objects the finitely generated modules of finite projective dimension, and $\mathcal{P}^\infty(\Lambda\text{-Mod})$ will be the analogous subcategory of $\Lambda\text{-Mod}$. As usual, $\text{l\,findim}\Lambda$ denotes the supremum of the projective dimensions attained on $\mathcal{P}^\infty(\Lambda\text{-mod})$.

## 2. Contravariantly finite subcategories of $\Lambda\text{-mod}$

The notions of co- and contravariantly finite subcategories of $\Lambda\text{-mod}$ were first introduced by Auslander and Smalø in [3], in connection with the existence of internal almost split sequences of a subcategory $\mathfrak{A}$ and preinjective/preprojective partitions of $\mathfrak{A}$. The following definitions evolved in [3], [4] and [2].

Throughout, we let $\mathfrak{A} \subseteq \Lambda\text{-mod}$ be a full subcategory which is closed under isomorphisms, direct summands, and finite direct sums.

**Definitions 2.1.** (1) Given $M \in \Lambda\text{-mod}$, a homomorphism $f : A \to M$ with $A \in \mathfrak{A}$ is called a right $\mathfrak{A}$-approximation of $M$ if each $g \in \text{Hom}_\Lambda(B, M)$ with $B \in \mathfrak{A}$ factors through $f$. Left $\mathfrak{A}$-approximations are defined dually.
(2) \( \mathfrak{A} \) is called \textit{contravariantly finite} (resp., \textit{covariantly finite}) in \( \Lambda \)-mod if each module \( M \in \Lambda \)-mod has a right (resp., left) \( \mathfrak{A} \)-approximation.

In the following we will focus on contravariant finiteness and will briefly write ‘\( \mathfrak{A} \)-approximation’ for ‘right \( \mathfrak{A} \)-approximation’. In [2] it is shown that, whenever \( M \in \Lambda \)-mod has an \( \mathfrak{A} \)-approximation, any two \( \mathfrak{A} \)-approximations of minimal \( K \)-dimension are isomorphic. We therefore refer to the minimal \( \mathfrak{A} \)-approximation of \( M \) in that case.

The nomenclature in Definition 2.1(2) stems from the fact that \( \mathfrak{A} \) is contravariantly finite in \( \Lambda \)-mod if and only if the restricted contravariant Hom-functor \( \text{Hom}(\cdot, M)|_{\mathfrak{A}} : \mathfrak{A} \to \text{Ab} \) is finitely generated for each object \( M \in \Lambda \)-mod. The flavor of this finiteness condition of \( \mathfrak{A} \) relative to \( \Lambda \)-mod is caught in the following easy argument showing that each subcategory \( \mathfrak{A} \subseteq \Lambda \)-mod of finite representation type is contravariantly (as well as covariantly) finite [AS, Proposition 4.2]: Indeed, if \( A_1, \ldots, A_m \in \mathfrak{A} \) are such that each object in \( \mathfrak{A} \) is a finite direct sum of copies of the \( A_i \), then given \( M \in \Lambda \)-mod, the homomorphism \( \bigoplus_{i=1}^m \bigoplus_{j=1}^{d_i} f_{ij} : \bigoplus_{i=1}^m A_i^{d_i} \to M \) is an \( \mathfrak{A} \)-approximation of \( M \) provided that, for each \( i \leq m \), the family \((f_{ij})_{j \leq d_i}\) is a \( K \)-basis of Hom\(_{\Lambda}(A_i, M)\).

One of the first results to draw attention to these finiteness conditions for subcategories of \( \Lambda \)-mod was the following theorem of Auslander and Smalø.

**Theorem 2.2.** [4, Theorem 2.4] If \( \mathfrak{A} \) is covariantly and contravariantly finite in \( \Lambda \)-mod, then \( \mathfrak{A} \) has almost split sequences. \( \square \)

The following two results, due to Auslander and Reiten, are pivotal in our present investigation.

**Theorem 2.3.** [2, Proposition 3.7] If \( \mathfrak{A} \subseteq \Lambda \)-mod is a resolving subcategory (meaning that \( \mathfrak{A} \) contains the projectives in \( \Lambda \)-mod and is closed under extensions and kernels of epimorphisms), then \( \mathfrak{A} \) is contravariantly finite in \( \Lambda \)-mod if and only if each of the simple left \( \Lambda \)-modules has an \( \mathfrak{A} \)-approximation. \( \square \)

**Theorem 2.4.** [2, Proposition 3.8] Suppose that \( \mathfrak{A} \) is a resolving, contravariantly finite subcategory of \( \Lambda \)-mod. Moreover, suppose that \( \Lambda \)-mod contains \( n \) isomorphism classes of simple modules and that \( A_1, \ldots, A_n \) are the minimal \( \mathfrak{A} \)-approximations of representative simples. Then a left \( \Lambda \)-module \( X \) belongs to \( \mathfrak{A} \) precisely when \( X \) is a direct summand of a module having a filtration with consecutive factors in \( \{A_1, \ldots, A_n\} \). \( \square \)

In the next section, we give a first instalment of ‘positive’ examples, i.e., of instances of contravariantly finite subcategories. A powerful tool for extending the list of such subcategories is the following theorem of Sikko and Smalø, which subsumes earlier results by Grecht, Vossieck, de la Pena-Simson, Ringel, and Smalø.

**Theorem 2.5.** [22, Theorem 2.6] If \( \mathfrak{A} \) and \( \mathfrak{B} \) are contravariantly finite subcategories of \( \Lambda \)-mod, then the category of all direct summands of extensions of objects in \( \mathfrak{A} \) by objects in \( \mathfrak{B} \) has the same property. \( \square \)

3. First instalment of positive examples

Roughly speaking, a subcategory \( \mathfrak{A} \subseteq \Lambda \)-mod is likely to be contravariantly finite if it is either very small or very large. In the former situation one usually obtains very rough
approximations, in the latter one can approach the target objects very closely with objects from $\mathfrak{A}$. The extreme cases are that of a category of finite representation type on one end – we already know such a category to be contravariantly finite in $\Lambda$-$\text{mod}$ – and the case $\mathfrak{A} = \Lambda$-$\text{mod}$ on the other end. While the latter does not hold much interest, the former does. For example, in case the objects of $\mathfrak{A}$ are precisely the finitely generated projective left $\Lambda$-modules, the minimal $\mathfrak{A}$-approximations coincide with projective covers. So in particular, if $\text{fin} \dim \Lambda = 0$, the category $P^\infty(\Lambda$-$\text{mod})$ is contravariantly finite in $\Lambda$-$\text{mod}$. The most interesting cases of contravariant finiteness of $P^\infty(\Lambda$-$\text{mod})$, however, are those where $\text{fin} \dim \Lambda > 0$, while $\text{gl} \dim \Lambda = \infty$.

As was already observed by Auslander and Reiten in [2], $P^\infty(\Lambda$-$\text{mod})$ is contravariantly finite in $\Lambda$-$\text{mod}$ whenever $\Lambda$ is stably equivalent to a hereditary algebra. This includes the case where $J^2 = 0$, a situation in which it is easy to describe the minimal $P^\infty(\Lambda$-$\text{mod})$-approximations. According to [12], they look as follows: Given any $X \in \Lambda$-$\text{mod}$ with $JX = 0$, write $X = X_{\text{fin}} \oplus X_{\text{inf}}$, where $X_{\text{fin}}$ is the sum of those simple submodules of $X$ which have finite projective dimension, and $X_{\text{inf}}$ is the sum of those simples which have infinite projective dimension. Then, given any $M \in \Lambda$-$\text{mod}$ with projective cover $P \to M$ say, the induced epimorphism $P/\Omega^1(M)_{\text{fin}} \to M$ is the minimal $P^\infty(\Lambda$-$\text{mod})$-approximation of $M$. In particular, the minimal $P^\infty(\Lambda$-$\text{mod})$-approximation of a simple module $S = \Lambda e/Je$ has the form $\Lambda e/(Je)_{\text{fin}} \to S$. To be more specific, if $\Lambda = K\Gamma/\langle \text{all paths of length } 2 \rangle$, where $\Gamma$ is the quiver

![Quiver diagram](image)

the graph of the minimal $P^\infty(\Lambda$-$\text{mod})$-approximation of $S_1$ is the following ‘brush’:

![Brush diagram](image)

Moreover, Auslander and Reiten proved that for each Gorenstein algebra $\Lambda$, i.e., for each algebra $\Lambda$ which has finite injective dimension on both sides, the category $P^\infty(\Lambda$-$\text{mod})$ is contravariantly finite in $\Lambda$-$\text{mod}$ (see [2, p. 150]). This result applies to a particularly interesting situation to which the author was alerted by Kirkman and Kuzmanovich: Namely, let $\mathcal{O} \subseteq M_n(F)$ be a tiled classical order over a DVR $D$ with quotient field $F$; that $\mathcal{O}$ be tiled means that $\mathcal{O}$ contains a full set of $n$ primitive orthogonal idempotents. Moreover, let $\pi$ be a uniformizing parameter of $D$ and observe that then $\Lambda = \mathcal{O}/\pi\mathcal{O}$ is a finite dimensional algebra over the residue class field $K = D/(\pi)$ of $D$. If $\mathcal{O}$ has finite left and right injective dimension – a fortiori, if $\text{gl} \dim \mathcal{O} < \infty$ – one of the classical change
of rings theorems (see [20, Theorem 205]) yields that $\Lambda$ is a Gorenstein algebra. So in particular $\mathcal{P}^\infty(\Lambda\text{-mod})$ is contravariantly finite in $\Lambda\text{-mod}$ whenever $\text{gl dim } \mathcal{O} < \infty$; hence, $\ell \text{ fin dim } \Lambda = \sup\{p \text{ dim } A_i \mid 1 \leq i \leq n\}$, where the $A_i$ are the minimal $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximations of the simple left $\Lambda$-modules. On the other hand, $\ell \text{ fin dim } \Lambda = \ell \text{ fin dim } \mathcal{O} - 1 = \text{gl dim } \mathcal{O} - 1$ if $\text{gl dim } \mathcal{O} < \infty$, due to [9]. Via approximation theory, this equality may provide access to the possible values of $\text{gl dim } \mathcal{O}$, which have been the object of investigation for a long time (see [23], [18], [19], [21], [6] for more detail). A summary of the results to date pertaining to the global dimensions of tiled classical orders over DVRs can be found in [13].

**Problem 3.1.** Let $\Lambda = \mathcal{O}/\pi \mathcal{O}$ for a tiled classical order $\mathcal{O}$ over a DVR. Give an explicit description of the minimal $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximations of the simple left $\Lambda$-modules in terms of the valuated quiver of $\mathcal{O}$ (see [24]) or, equivalently, in terms of quiver and relations of $\Lambda$.

### 4. Negative examples and a criterion for failure of contravariant finiteness

The first example of a finite dimensional algebra $\Lambda$ for which $\mathcal{P}^\infty(\Lambda\text{-mod})$ fails to be contravariantly finite is due to Igusa, Smalø, Todorov [17]. It is a monomial relation algebra with vanishing radical cube which, in addition, is biserial. (In Section 7, we will see that, by contrast, $\mathcal{P}^\infty(\Lambda\text{-mod})$ is always contravariantly finite in $\Lambda\text{-mod}$ when $\Lambda$ is left serial.) However, the conclusion that contravariant finiteness of $\mathcal{P}^\infty(\Lambda\text{-mod})$ in $\Lambda\text{-mod}$ be a rare occurrence would be precipitous. In fact, the condition cuts diagonally across the standard groupings of finite dimensional algebras, being extremely sensitive to changes in quiver and relations.

**Criterion 4.1.** [10] Let $\mathfrak{A}$ be a full subcategory of $\Lambda\text{-mod}$. Moreover, suppose that $e_1, \ldots, e_m$ are pairwise orthogonal primitive idempotents of $\Lambda$, and $p_1, \ldots, p_m, q_1, \ldots, q_m$ elements of $J$, with $p_i = p_i e_i$ and $q_i = q_i e_i$, such that the following conditions are satisfied:

1. For each $n \in \mathbb{N}$, there is a module $M_n \in \mathfrak{A}$, together with a sequence $x_{n1}, \ldots, x_{nn}$ of top elements of $M_n$ which are $K$-linearly independent modulo $JM_n$ and have the property that $p_{r(i)} x_{ni} = q_{r(i+1)} x_{n,i+1}$ is nonzero for $1 \leq i \leq n$, where $r(i) \in \{1, \ldots, m\}$ is congruent to $i$ modulo $m$.

2. Whenever $A \in \mathfrak{A}$, the following hold:
   i. If $x \in A$ is a top element of type $e_1$, then $p_1 x \neq 0$, and
   ii. If $y, z \in A$ are such that $p_{r(i)} y = q_{r(i+1)} z \neq 0$, then $p_{r(i+1)} z \neq 0$.

It follows that $S_1 = \Lambda e_1 / Je_1$ does not have an $\mathfrak{A}$-approximation. □

**Example 4.2.** [17] Let $\mathfrak{A} = \mathcal{P}^\infty(\Lambda\text{-mod})$, where $\Lambda = K\Gamma / I$ is the monomial relation algebra with quiver

\[
\begin{array}{c}
1 \\
\downarrow \alpha \\
\beta \\
\downarrow \gamma \\
2
\end{array}
\]

where $\alpha, \beta,$ and $\gamma$ are arrows with no labels.
such that the indecomposable projective left $\Lambda$-modules have graphs

Set $m = 1$, $p_1 = \beta$, $q_1 = \alpha$, and $S_i = \Lambda e_i / Je_i$. Then the modules $M_n \in \mathcal{P}^\infty(\Lambda\text{-mod})$ with graphs

having $n$ linearly independent top elements of type $e_1$ modulo the radical clearly satisfy Condition (1) of Criterion 4.1.

Moreover, Condition (2i) of the criterion is trivially satisfied, in view of the fact that $\text{p dim } S_2 = \infty$; indeed, given any $A \in \Lambda\text{-mod}$ with top element $x = e_1 x$ and $p_1 x = \beta x = 0$, the simple module $S_2$ is a direct summand of $\Omega^1(A)$. As for (2ii), whenever $p_1 y = \beta y = q_1 z = \alpha z \neq 0$, the element $e_1 z$ is a top element of $A$ of type $e_1$. Thus, $S_1$ does not have a $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximation. □

To the following example we will refer back in Sections 8 and 9. It is an instance where Criterion 4.1 applies only for $m > 1$.

**Example 4.3.** This time, let $\Lambda = K\Gamma/I$, where $\Gamma$ is the quiver

and $I \subset K\Gamma$ is such that the indecomposable projective left $\Lambda$-modules have the following graphs:
Set $m = 2$, $p_1 = \beta$, $p_2 = \gamma$, $q_1 = \alpha$, $q_2 = \delta$, and for $n \in \mathbb{N}$, let $M_n \in \mathcal{P}^{\infty}(\Lambda\text{-mod})$ be the module with graph

having $n$ linearly independent top elements modulo the radical, of types alternating between 1 and 6. It is straightforward to see that Condition (2) of Criterion 4.1 is satisfied as well, so that $\mathcal{P}^{\infty}(\Lambda\text{-mod})$ again fails to be contravariantly finite in $\Lambda\text{-mod}$. □

Our two final examples illustrate the instability – under minor modifications of quiver and/or relations – of the condition that $\mathcal{P}^{\infty}(\Lambda\text{-mod})$ be contravariantly finite in $\Lambda\text{-mod}$. The first is a variant of the Igusa-Smalø-Todorov example (4.2); the second results from a further slight alteration of the relations.

**Example 4.4.** Let $\Lambda = K\Gamma/I$ where $\Gamma$ is

and $I$ is such that the indecomposable projective left $\Lambda$-modules have graphs

This algebra $\Lambda$ contains the algebra of Example 4.2, and the modules $M_n$ defined in 4.2 remain modules of finite projective dimension over the new algebra. However, this time $\mathcal{P}^{\infty}(\Lambda\text{-mod})$ is contravariantly finite in $\Lambda\text{-mod}$; see [10]. □
Example 4.5. The quiver of the algebra \( \Lambda \) is the same as that of Example 4.4, but we delete one of the relations in the previous example, to the effect that the indecomposable projective left \( \Lambda \)-modules take on the forms

\[
\begin{array}{ccc}
\alpha & 1 & \beta \\
2 & & 2 \\
\gamma & \downarrow & 1 \\
1 & & \\
\end{array}
\quad
\begin{array}{ccc}
2 & & 3 \\
\gamma & 1 & \delta \\
\alpha & \downarrow & \beta \\
2 & & 2 \\
\gamma & \downarrow & 1 \\
1 & & \\
\end{array}
\]

Again, Criterion 4.1 (with \( m = 1, p_1 = \beta \) and \( q_1 = \alpha \)) readily yields that \( S_1 \) fails to have a \( P^\infty(\Lambda\text{-mod}) \)-approximation. \( \square \)

5. First intermezzo: Uniserial representations

We give a rough sketch of results from \([14,15,16]\) which will provide the foundation for the construction of several useful contravariantly finite subcategories of \( \Lambda\text{-mod} \), the objects of which are well understood. The pivotal problems addressed are the following:

(I) Classify the uniserial left \( \Lambda \)-modules in terms of manageable isomorphism invariants.

(II) Characterize the split algebras of finite uniserial type, i.e., characterize those algebras \( \Lambda = K\Gamma/I \) for which there are only finitely many uniserial \( \Lambda \)-modules up to isomorphism.

As the author learned in the meantime, Auslander has proposed these problems since the mid-70’s. We include an excerpt of an email message of July 1993 from him to the author. “I have been raising the question of the classification or description of uniserial modules for artin algebras for many years now. I believe the first time I raised the question in public was at the special session on representation theory at the winter meeting of the AMS in Atlanta around 1975. The big shot group representation people, like \([\ldots]\) assured me that they would have an answer by the afternoon. I am still waiting. I have raised the question repeatedly since then \([\ldots]\) As to motivation. One reason I am interested in these modules, aside from the fact that they should in some sense be the simplest nonprojective modules, is the fact that they have bounded length for a given algebra. Therefore the second Brauer-Thrall is true for algebras with an infinite number of uniserial modules. Hence it would be interesting to know for which algebras there are only a finite number \([\ldots]\) Secondly, I am interested where the uniserial modules occur in AR-quivers and preprojective partitions since as you mentioned other modules can be in some sense approximated by uniserial modules. \([\ldots]\)”

The placement of uniserial modules in the Auslander-Reiten quiver will not be discussed in this intermezzo. We just mention that Axel Boldt is working on this subject in his dissertation, and has already settled the question in the hereditary case.
A preliminary subdivision of the class of uniserial left $\Lambda$-modules is in terms of their ‘masts’.

**Definition 5.1.** Given a uniserial left $\Lambda$-module $U$ of length $l + 1$, any path $p$ of length $l$ in $K\Gamma$ with $pU \neq 0$ is called a mast of $U$.

Observe that, if $U$ is uniserial with consecutive simple composition factors $J^iU/J^{i+1}U \cong \Lambda e(i)/Je(i)$ for $0 \leq i \leq l$, and with layered and labeled graph $G$, then the masts of $U$ correspond precisely to those edge paths in $G$ which pass exactly once through each of these vertices from top to bottom. E.g., a uniserial module with graph

```
  e(0)
  α_1  α_2
  (0)  (1)  γ
  α_1
  e(2)
```

has masts $\beta\alpha_1$ and $\beta\alpha_2$.

The starting point of our approach to uniserial representations is the following theorem which we state somewhat informally.

**Theorem 5.2.** [14,15] Let $p \in K\Gamma$ be a path.

1. There is an affine algebraic variety $V_p$, not necessarily irreducible, which parametrizes the isomorphism types of the uniserial left $\Lambda$-modules with mast $p$ in a natural fashion. Somewhat more precisely, there exists a canonical surjection

$$
\Phi_p : V_p \rightarrow \{\text{isomorphism types of uniserials in } \Lambda \text{-mod with mast } p\}.
$$

In particular, $V_p$ is nonempty if and only if there exists a uniserial left $\Lambda$-module with mast $p$. (Polynomials for this variety can be readily determined on the basis of a ‘coordinatization’ $\Lambda = K\Gamma/I$.)

2. If $p : e(0) \rightarrow e(1) \rightarrow \cdots \rightarrow e(l)$ does not have a right subpath of positive length from $e(0)$ to $e(0)$, the map $\Phi_p$ is bijective, and the points of $V_p$ serve as isomorphism invariants of the uniserial modules with mast $p$. More sharply, if $e(0)$ recurs $t$ times among the vertices $e(1),\ldots,e(l)$, then each fibre of $\Phi_p$ is contained in a closed subvariety of $V_p$ of dimension at most $t$.

3. If $V_p \subseteq \mathbb{A}^d$ and $t$ is as under (2), there exists a system of equations

$$
S_p(X,Y,Z) = S_p(X_1,\ldots,X_d,Y_1,\ldots,Y_d,Z_1,\ldots,Z_t),
$$

linear in $Z_1,\ldots,Z_t$, with the following property: Two points $k,k' \in V_p$ belong to the same fibre of $\Phi_p$ if and only if the linear system $S_p(k,k',Z)$ is consistent. (Just as $V_p$, the system $S_p(X,Y,Z)$ can be effectively computed from $\Gamma$ and $I$.) □
The traditional varieties of $\Lambda$-modules of a fixed $K$-dimension (or of the cyclics in $\Lambda$-$\text{mod}$) clearly contain the collection of uniserial modules as open subvarieties. However, these varieties are far too large and unwieldy to serve the purpose of an effective classification. By contrast, the varieties which we will consider here fit the collection of uniserials with a fixed composition series rather tightly. In fact, the bit of slack which may occur in the presence of certain oriented cycles is quite harmless; each such cycle just adds a copy of $A^1$ to the pertinent variety.

We will not give the equations defining the varieties $V_p$ here, but will instead give a very rough answer to the question of what data are recorded by the coordinates of their points. Suppose that $p = \alpha_1 \cdots \alpha_l$, where the $\alpha_i$ are arrows. Given any uniserial $U \in \Lambda$-$\text{mod}$ with $p$ and a top element $x$, the products $\alpha_m \cdots \alpha_1 x$, $0 \leq m \leq l$, clearly form a $K$-basis of $U$. Roughly speaking, the points of $V_p$ corresponding to $U$ are strings of coordinate vectors of elements $qx$, relative to this basis, where $q$ runs through certain paths in $K\Gamma$ (sufficiently many to pin down $U$ up to isomorphism). In particular, the graphs of uniserials are available at a glance from the corresponding points on the varieties $V_p$. For details, we refer to [14,15]. We include a fairly transparent example to illustrate the correspondence of Theorem 5.2, again suppressing all technical detail.

**Example 5.3.** Let $\Gamma$ be the quiver

![Quiver Diagram](image)

and $p$ the path $q_5q_4q_3q_2q_1$, where $q_i = \beta_i\alpha_i$ for $1 \leq i \leq 5$. Define $\Lambda = K\Gamma/I$, where $I$ is the ideal generated by the following relations:

$$
\gamma_5\gamma_4\gamma_3q_2q_1 - q_5q_4\gamma_3\gamma_2q_1 + q_5q_4q_3q_2\gamma_1, \quad q_5q_4q_3q_2\gamma_1 - q_5q_4q_3q_2q_1,
$$

$$
q_5q_4q_3q_2q_1 - q_5q_4\gamma_3q_2q_1, \quad q_5\gamma_4q_3q_2q_1 - q_5q_4q_3q_2q_1.
$$

Then $V_p = V(X_{21}X_{22} - X_{11}X_{12}X_{13} + X_{11}, \ X_{11} - X_{12}, \ X_{11} - X_{13}, \ X_{21} - X_{22})$. Observe that $V \cong V(X_2^2 - X_1(X_1^2 - 1))$ is the elliptic curve with $\mathbb{R}$-graph

![Elliptic Curve](image)

A host of questions supplementing Problems I and II pose themselves at this point. We will briefly address a few of the most immediate among these.
Additional Questions 5.4. (1) Suppose that $p,q \in K\Gamma$ are two paths of length $l$, both passing through the sequence of vertices $(e(0),\ldots,e(l))$. Then the uniserials with mast $p$ and those with mast $q$ have the same sequence of composition factors, namely $(\Lambda e(0)/Je(0),\ldots,\Lambda e(l)/Je(l))$. What can be said about the intersection $\Phi_p(V_p) \cap \Phi_q(V_q)$?

(2) The varieties $V_p$ are defined by polynomials over $K$ which depend on $\Gamma$ and $I$; the labeling as such is, in fact, tied to the coordinatization of $\Lambda$. Is there geometric information on the uniserial left $\Lambda$-modules with fixed sequence of composition factors which does not depend on the coordinatization of $\Lambda$?

(3) Which affine varieties arise as varieties of uniserial modules with a fixed sequence of composition factors?

Answers (sketch). (1) [14, Theorem D] Given any pair of irreducible components $U_p \subseteq V_p$ and $U_q \subseteq V_q$, the intersection $D = \Phi_p(U_p) \cap \Phi_q(U_q)$ is either empty or else $\Phi_p^{-1}(D)$ and $\Phi_q^{-1}(D)$ are dense open subsets of $U_p$ and $U_q$ respectively, and there exists an isomorphism of varieties $\Psi : \Phi_p^{-1}(D) \to \Phi_q^{-1}(D)$ which makes the following diagram commutative:

\[
\begin{array}{ccc}
\Phi_p^{-1}(D) & \xrightarrow{\Psi} & \Phi_q^{-1}(D) \\
\Phi_p & \downarrow & \Phi_q \\
D & & D
\end{array}
\]

In particular, $U_p$ and $U_q$ are birationally equivalent in the latter case.

(2) [16] Let $S = (S(0),\ldots,S(l))$ be a sequence of simple left $\Lambda$-modules. If $\Gamma$ has no double arrows, there is at most one path $p$ of length $l$ passing through a sequence of primitive idempotents of $\Lambda$ corresponding to the simples $S(i)$ and, in the positive case, the variety $V_p = V_S$ is determined up to isomorphism by the $K$-algebra isomorphism type of $\Lambda$.

In the general case, start with a coordinatization $\Lambda = K\Gamma/I$ and let $(e(0),\ldots,e(l))$ be the sequence of vertices of $\Gamma$ corresponding to the simple modules $S(i)$. Denote by $V_S$ the set of the birational equivalence classes of the irreducible components of the $V_p$’s, where $p$ runs through the paths of length $l$ passing through $(e(0),\ldots,e(l))$. Then $V_S$ is uniquely determined by the isomorphism type of $\Lambda$.

(3) Each affine variety occurs as a variety $V_p = V_S$, even under the additional requirements that the map $\Phi_p$ be bijective and that $V_p$ be determined up to isomorphism by the corresponding algebra $\Lambda$. More precisely: Given any affine algebraic variety $V$ over $K$, there exists an acyclic quiver $\Gamma$ without double arrows, together with a path $p$ in $K\Gamma$, such that $V \cong V_p$. □

To close in on the structure of the algebras of finite uniserial type, we require several additions to our conceptual framework (see [15]). Since, in the sequel, we will focus on algebras with the stronger property that all the varieties $V_p$ be finite, we will content ourselves with the far more straightforward characterization of these latter algebras. This characterization will be preceded by a strong necessary condition for finite uniserial type.
Theorem 5.5. [15] If $\Lambda$ has finite uniserial type, then the following condition (N) is satisfied: Whenever $\alpha : e \to e'$ is an arrow in $\Gamma$ and $p : e \to e'$ a mast of positive length, the path $p$ belongs to $K\Gamma \alpha \cup \alpha K\Gamma$, that is, $p$ is of the form

$$
\begin{array}{c}
e \xrightarrow{\alpha} p' \xrightarrow{c'} \bullet \xrightarrow{\alpha} e' \\
p \xrightarrow{\alpha} e' \xrightarrow{c} p
\end{array}
$$

where $c', c$ are oriented cycles which may be trivial.

A coordinate-free rendering of condition (N) is as follows: If there exists a uniserial left $\Lambda$-module $W$ of length 2 with top $S$ and socle $S'$, and if $U$ is any uniserial left $\Lambda$-module of length $l \geq 2$ with top $S$ and socle $S'$, then either $U/J^2U \cong W$ or else $J^{l-2}U \cong W$.

Condition (N), in turn, implies that:

(a) $\Gamma$ has no double arrows, meaning that each uniserial left $\Lambda$-module has a unique mast.

(b) Given any uniserial module $U$ with mast $p$, each graph of $U$ results from the superposition of graphs of the form

under identification of the edge path $p$; here $c$ is an oriented cycle of positive length and $\alpha$ an arrow such that $\alpha c$ is a subpath of $p$. □

The algebras we wish to describe turn out to be precisely those for which the second option in Condition (N) of Theorem 5.5 is excluded.

Theorem 5.6. [15] Given any algebra $\Lambda = K\Gamma/I$, the following statements are equivalent:

1. For each path $p \in K\Gamma$, the variety $V_p$ is finite.
2. For each path $p \in K\Gamma$, the variety $V_p$ is either empty or a singleton.
3. Whenever $\alpha : e \to e'$ is an arrow in $\Gamma$ and $p : e \to e'$ a mast of positive length, the path $p$ is equal to $c'\alpha$, where $c'$ is an oriented cycle which may be trivial, i.e., $p$ has the form

$$
\begin{array}{c}
e \xrightarrow{\alpha} p' \xrightarrow{c'} \bullet \xrightarrow{\alpha} e' \\
p \xrightarrow{\alpha} e'
\end{array}
$$
In coordinate-free terms: If there exists a uniserial left \( \Lambda \)-module \( W \) of length 2 with top \( S \) and socle \( S' \), and if \( U \) is any uniserial left \( \Lambda \)-module of length \( \geq 2 \) with top \( S \) and socle \( S' \), then \( U/J^2U \cong W \).

(4) There is a 1–1 correspondence between the isomorphism types and the graphs of the uniserial left \( \Lambda \)-modules.

(5) The only graphs of uniserial left \( \Lambda \)-modules are edge paths. \( \square \)

Clearly every left serial algebra satisfies condition (5) of Theorem 5.6. Natural instances of algebras satisfying the equivalent conditions of Theorem 5.6 are, moreover, the algebras \( \Lambda = O/\pi O \) where \( O \) is a tiled classical order over a DVR with uniformizing parameter \( \pi \) (see Section 3).

6. Second intermezzo: Saguaros

Let \( \Lambda = K\Gamma/I \) be a path algebra having a quiver \( \Gamma \) without double arrows, which means that each uniserial module has a unique mast in \( K\Gamma \).

Definition 6.1. Suppose that \( T_1, \ldots, T_m \) is a sequence of non-zero uniserial left \( \Lambda \)-modules, and let \( p_i \) be the mast of \( T_i \), respectively. A left \( \Lambda \)-module \( T \) is called a saguaro\(^1\) on \((T_1, \ldots, T_m)\) if

(i) \( T \cong (\bigoplus_{1 \leq i \leq m} T_i)/U \), where \( U \subseteq \bigoplus_{1 \leq i \leq m} JT_i \) is generated by a sequence of elements of the form \( q_it_i - q'_{i+1}t_{i+1} \), \( 1 \leq i \leq m - 1 \), where \( t_i \in T_i \) are suitable top elements and \( q_i, q'_i \) are right subpaths of the masts \( p_i \) such that \( q_it_i \neq 0 \), and \( q'_{i+1}t_{i+1} \neq 0 \); moreover, we require that

(ii) each \( T_j \) embeds canonically in \( T \) via

\[
T_j \xrightarrow{\text{can}} \left( \bigoplus_{1 \leq i \leq m} T_i \right)/U \cong T.
\]

The uniserial modules \( T_i \) are called the trunks of \( T \).

In the sequel, we will identify \( T \) with \( (\bigoplus_{1 \leq i \leq m} T_i)/U \). To avoid ambiguities, we will denote the canonical images of the trunks \( T_i \) by \( \hat{T}_i \) and the canonical images of the top elements \( t_i \) by \( \hat{t}_i \). Any such sequence \((\hat{t}_1, \ldots, \hat{t}_m)\) will be called a canonical sequence of top elements for \( T \).

Note that saguaros are particularly amenable to graphing, the shape of their graphs explaining their name. By a slight abuse of language, we will say that the graph of a saguaro \( T \) displays a canonical sequence of top elements if the simple summands of \( T/JT \) shown in the uppermost layer of the graph are generated by the terms of such a canonical sequence. It is clear that layered and labeled graphs displaying a canonical sequence of top elements always exist. To give an example, the following is a graph of a saguaro \( T \) over the algebra \( \Lambda \) of Example 1.1. Since \( \Gamma \) has no double arrows in this case, we can omit the labels on the edges without losing information.

\(^1\)The name is that of a cactus found in the Sonoran Desert.
Here $T = \left( \bigoplus_{i=1}^{5} T_{i} \right)/U$, where $U$ is generated by the elements $\alpha t_{1} - \beta t_{2}$, $\beta^{2} t_{2} - \delta \gamma t_{3}$, $\gamma t_{3} - \gamma t_{4}$, and $\delta^{2} \gamma t_{4} - \delta^{2} \gamma t_{5}$. The trunks of $T$ are the uniserials $T_{1}, \ldots, T_{5}$ in $\Lambda$-mod which, relative to suitable top elements $t_{i} \in T_{i}$, have graphs

respectively.

In Section 8, we will need slight upgrades of some of the observations proved in [5]; wherever additional care is required, we include the short proofs for the convenience of the reader.

**Observation 6.2.** (On the role of scalars.) If $T \cong \left( \bigoplus_{1 \leq i \leq m} T_{i} \right)/U$ is a saguaro, where $U$ is generated by the relations $q_{i} t_{i} - q_{i+1}' t_{i+1}$, $1 \leq i \leq m - 1$, as in Definition 6.1, and if $k_{1}, \ldots, k_{m-1}$ are non-zero scalars, then

$$T \cong \left( \bigoplus_{i=1}^{m} T_{i} \right)/ \left( \sum_{i \leq m-1} \Lambda(q_{i} t_{i} - k_{i} q_{i+1}' t_{i+1}) \right). \qed$$
Observe that for any two indices $i < j$, the subpaths $\hat{T}_i$ and $\hat{T}_j$ of $T$ are right subpaths of $T_i$ and $T_j$, respectively. Moreover, whenever $i < j$, we have $\hat{T}_i \cap \hat{T}_j = \hat{T}_i \cap \left( \sum_{l \geq j} \hat{T}_l \right) = \left( \sum_{l \leq i} \hat{T}_l \right) \cap \hat{T}_j$.

Proof. The first line of equalities is immediate from the definition. That $\Lambda \hat{q}_i \hat{t}_i \neq 0$ is a consequence of the facts that $q_i t_i$ is nonzero in $T_i$ by condition (i) of Definition 6.1, and that $T_i$ is isomorphic to its canonical image $\hat{T}_i$ in $T$ by condition (ii) of that definition. This clearly implies that $\text{Soc}(\hat{T}_i) = \text{Soc}(T_i)$ for all $i$.

To check that $\hat{T}_i \cap \hat{T}_j \supseteq \hat{T}_i \cap \left( \sum_{l \geq j} \hat{T}_l \right)$ for $i < j$, let $\lambda t_i = \sum_{l \geq j} \lambda_T t_i$ be suitable elements $\lambda$ and $\lambda_T$ in $\Lambda$. Note that, whenever $\mu \in \Lambda$ and $k < m$ are such that $\mu q_k t_k$ is equal to zero in $T_k$, then $\mu q_{k+1} t_{k+1}$ is zero in $T_{k+1}$ by condition (ii) of the definition. Using this fact and condition (i), we obtain $\lambda_T t_i - \sum_{l \geq j} \lambda_T t_i = \sum_{k \geq i} \mu_k (q_k t_k - q_{k+1} t_{k+1})$ in $\oplus_{l \leq m} T_l$ for certain $\mu_k \in \Lambda$. It follows that $\lambda_T t_i = \mu q_i t_i$, and hence that $\lambda_T t_i = \mu q_{i+1} t_{i+1} \in \hat{T}_{i+1} \cap \left( \sum_{l \geq j} \hat{T}_l \right)$, and an obvious induction on $i$ completes the proof. □

Note that the inclusion $\text{Soc}(\hat{T}_i) \subseteq \text{Soc}(T)$ is proper, in general.

Observation 6.4. (Additional information on intersections of trunks.) Let $T$ be a saguaro on $(T_1, \ldots, T_m)$ with canonical sequence of top elements $\hat{T}_1, \ldots, \hat{T}_m$, where $T_i$ has mast $p_i$. For any two indices $i < j$ in $\{1, \ldots, m\}$ there exist right subpaths $a$ of $p_i$ and $b$ of $p_j$ such that $a \hat{T}_i = b \hat{T}_j$ and $\hat{T}_i \cap \hat{T}_j = \Lambda a \hat{T}_i = \Lambda b \hat{T}_j$.

Proof. We proceed by induction on $j - i$. The case where $j - i = 1$ is covered by Observation 6.3, so we may assume $j - i \geq 2$. By the induction hypothesis, we can then find a right subpath $u$ of $p_{i+1}$ and a right subpath $v$ of $p_j$ with $u \hat{T}_{i+1} = v \hat{T}_j$ and such that $\hat{T}_{i+1} \cap \hat{T}_j = \Lambda u \hat{T}_{i+1}$. In view of Observation 6.3, we see moreover that $\hat{T}_i \cap \hat{T}_j = \hat{T}_i \cap \hat{T}_{i+1} \cap \hat{T}_j = (\hat{T}_i \cap \hat{T}_{i+1}) \cap (\hat{T}_{i+1} \cap \hat{T}_j) = \Lambda q_{i+1} t_{i+1} \cap \Lambda u \hat{T}_{i+1}$ where $q'_{i+1}$ is a right subpath of $p_{i+1}$ as in Definition 6.1.

If $\text{length}(q'_{i+1}) \leq \text{length}(u)$, we obtain $u = w q'_{i+1}$ for a suitable subpath $w$ of $p_{i+1}$, since both $u$ and $q_{i+1}$ are right subpaths of $p_{i+1}$. Observe that, in this case, $w q_i$ is a right subpath of $p_i$ and $w q_i \hat{T}_i = w q_{i+1} \hat{T}_{i+1} = u \hat{T}_{i+1} = v \hat{T}_j \neq 0$, which shows in particular that $\hat{T}_i \cap \hat{T}_j = \Lambda w q_i \hat{T}_i = \Lambda u \hat{T}_j$; thus our claim is satisfied with $a = w q_i$ and $b = v$.

If, on the other hand, $\text{length}(u) < \text{length}(q'_{i+1})$, there exists a subpath $w$ of $p_{i+1}$ with $w u = q'_{i+1}$, which implies $w \hat{T}_j = w \hat{T}_{i+1} = q'_{i+1} t_{i+1} = q_i \hat{T}_i$ and $\hat{T}_i \cap \hat{T}_j = \Lambda q_i \hat{T}_i = \Lambda w \hat{T}_j$; in other words, our claim is satisfied with $a = q_i$ and $b = w v$. □

It is an obvious consequence of the preceding observation that, given a saguaro $T = \sum_{i \in I} \hat{T}_i$ over a left serial algebra and $I_1 \subseteq I$, the sum $T' = \sum_{i \in I_1} \hat{T}_i$ is in turn a saguaro with trunks $\{T_i \mid i \in I_1\}$. 
Observation 6.5. (On the reordering of trunks.) Let $T$ be a saguaro on $(T_1, \ldots, T_m)$, and let $\pi \in S_m$ be any permutation. If for each $i \in \{1, \ldots, m-1\}$ we have $\hat{T}_\pi(i) \cap \hat{T}_\pi(i+1) \supseteq \hat{T}_\pi(i) \cap \hat{T}_\pi(j)$ for all $j > i$, then $T$ is also a saguaro on $(T_{\pi(1)}, \ldots, T_{\pi(m)})$.

Proof. The details of the proof can be derived from Observation 6.4 by induction on $j$. This yields: Given any index $i$, there exists a permutation $\pi \in S_m$ with $\pi(m) = j$ (or with $\pi(1) = j$) such that $T$ is a saguaro on $(T_{\pi(1)}, \ldots, T_{\pi(m)})$.

A less obvious consequence of Observation 6.5 is the following.

Observation 6.6. (Moving two trunks together.) Again let $T \in \Lambda\text{-mod}$ be a saguaro on $(T_1, \ldots, T_m)$, and let $(s, t)$ be a pair of distinct indices in $\{1, \ldots, m\}$. Then there exists a permutation $\pi \in S_m$ with the property that $T$ is a saguaro on $(T_{\pi(1)}, \ldots, T_{\pi(m)})$ and $(s, t) = (\pi(l), \pi(l+1))$ for some $l$.

Proof. Define $I(s), I(t) \subseteq \{1, \ldots, m\}$ as follows:

$$
I(s) = \{i \mid i \neq s, t \text{ and } \hat{T}_i \cap \hat{T}_s \supseteq \hat{T}_i \cap \hat{T}_t\}
$$

$$
I(t) = \{i \mid i \neq s, t \text{ and } \hat{T}_i \cap \hat{T}_s \subseteq \hat{T}_i \cap \hat{T}_t\}.
$$

If $|I(s)| = l - 1$ for $l \geq 1$, set $\pi(l) = s$, and define $\pi(l-1), \ldots, \pi(1)$ recursively: If $I(s) \neq \emptyset$, i.e., if $l \geq 2$, select $\pi(l-1) \in I(s)$ so that $\hat{T}_{\pi(l-1)} \cap \hat{T}_s$ is maximal among the intersections $\hat{T}_i \cap \hat{T}_s$ for $i \in I(s)$. If $I(s) \setminus \{\pi(l-1)\} = \emptyset$, choose an element $\pi(l-2) \in I(s) \setminus \{\pi(l-1)\}$ so that $\hat{T}_{\pi(l-2)} \cap \hat{T}_{\pi(l-1)}$ is maximal among the intersections $\hat{T}_i \cap \hat{T}_{\pi(l-1)}$, $i \in I(s) \setminus \{\pi(l-1)\}$, etc.

Next set $\pi(l+1) = t$, and if $I(t) \neq \emptyset$, select $\pi(l+2) \in I(t)$ so that $\hat{T}_{\pi(l+2)} \cap \hat{T}_t$ is maximal among the $\hat{T}_i \cap \hat{T}_t$, $i \in I(t)$. Continue as above: if $I(t) \setminus \{\pi(l+2)\} = \emptyset$, pick $\pi(l+3)$ in this set difference so that $\hat{T}_{\pi(l+3)} \cap \hat{T}_{\pi(l+2)}$ is maximal among the intersections $\hat{T}_i \cap \hat{T}_{\pi(l+2)}$, $i \in I(t) \setminus \{\pi(l+2)\}$, and so forth.

Going back to the definition of a saguaro, one verifies that $\pi$ satisfies the hypothesis of Observation 6.5. □

The category of finite direct sums of saguaros will serve as a key source of examples to illustrate the usefulness of approximations; see Sections 7 and 8.

7. Second instalment of positive examples: $\mathcal{P}^\infty(\Lambda\text{-mod})$ where $\Lambda$ is left serial

The importance of saguaros – or, more generally, modules built on similar patterns – came to our attention through their ‘natural’ occurrence as minimal $\mathcal{P}^\infty(\Lambda\text{-mod})$-approximations of the simple modules over a split left serial algebra $\Lambda$. 
**Theorem 7.1.** [5, Theorems 5.2, 5.3] Suppose that $\Lambda = K\Gamma/I$ is a left serial algebra. Then $P^\infty(\Lambda\text{-mod})$ is contravariantly finite, and the minimal $P^\infty(\Lambda\text{-mod})$-approximations of the simple left $\Lambda$-modules are saguaros with simple socles.

More precisely, the minimal $P^\infty(\Lambda\text{-mod})$-approximation of a simple left $\Lambda$-module $S = \Lambda e/Je$ can be described as follows: If $C \subseteq Je$ is maximal with respect to the property that $p \dim \Lambda e/C < \infty$, there is a unique saguaro $A(S)$ of maximal length in $P^\infty(\Lambda\text{-mod})$ such that $\Lambda e/C$ is a trunk of $A(S)$ and $\text{Soc} A(S)$ is simple. Then each canonical epimorphism $A(S) \to S$, which maps $\Lambda e/C$ onto $S$ and sends the other trunks of $A(S)$ to zero, is a minimal $P^\infty(\Lambda\text{-mod})$-approximation. $\square$

When $\Lambda$ is a left serial algebra, the minimal $P^\infty(\Lambda\text{-mod})$-approximations of the simple modules can actually be constructed algorithmically from quiver and relations of $\Lambda$. In view of Theorem 2.4, they form the basic structural components of arbitrary objects in $P^\infty(\Lambda\text{-mod})$.

**Corollary 7.2.** Let $\Lambda$ be a left serial algebra. Moreover, suppose that the saguaros $A(S_1), \ldots, A(S_n)$ are minimal right $P^\infty(\Lambda\text{-mod})$-approximations of the simple left $\Lambda$-modules $S_1, \ldots, S_n$. Then a finitely generated left $\Lambda$-module has finite projective dimension if and only if it is a direct summand of a submodule $M$ having a filtration $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_k = 0$, where each of the consecutive factors is isomorphic to some $A(S_i)$. $\square$

**Example 7.3.** Let $\Lambda$ be a left serial algebra whose quiver has 5 vertices and whose indecomposable projective modules $\Lambda e_i$ are given by the following graphs:

```
 1  2  3  4  5
|\alpha|\beta|\gamma|\delta|\epsilon|
5  3  5  3  5
|\epsilon|\gamma|\epsilon|\gamma|\epsilon|
5  5  5  5  5
|\epsilon|\epsilon|\epsilon|
5  5  5
|\epsilon|
5  5
```

Then the minimal $P^\infty(\Lambda\text{-mod})$-approximations of the the simple modules $S_i = \Lambda e_i/Je_i$ are as follows (given the fact that they are saguaros, they are in fact determined up to isomorphisms by their graphs):
The homological information to be gleaned from saguaros can be pushed further. For each \( d \geq 0 \), denote by \( \mathcal{P}(d) \) the full subcategory of \( \mathcal{P}^\infty(\Lambda \text{-mod}) \) whose objects are the modules of projective dimension at most \( d \).

**Theorem 7.4.** [5] Let \( \Lambda \) again be a left serial algebra. Then, for each \( d \geq 0 \), the category \( \mathcal{P}(d) \) is contravariantly finite in \( \Lambda \text{-mod} \), and the minimal \( \mathcal{P}(d) \)-approximations of the simple modules are saguaros.

A minimal \( \mathcal{P}(d) \)-approximation of the simple module \( S = \Lambda e/Je \) is as follows: Let \( T_d \) be the non-zero homomorphic image of \( \Lambda e \) within \( \mathcal{P}(d) \) which has smallest (positive) composition length. If \( A_d(S) \) is a suguaro in \( \mathcal{P}(d) \) having simple socle and trunk \( T_d \), which has maximal composition length with respect to these properties, then any homomorphism \( f : A_d(S) \to S \) which maps \( T_d \) onto \( S \) and all the other trunks of \( A_d(S) \) onto zero is a minimal \( \mathcal{P}(d) \)-approximation of \( S \). In particular, \( A_d(S) \) is unique up to isomorphism. \( \square \)

For each simple module \( S \), there is thus a sequence of minimal \( \mathcal{P}(d) \)-approximations \( A_1(S), A_2(S), A_3(S), \ldots \) which terminates in \( A_\delta(S) \), where \( \delta \) is the left finitistic dimension of \( \Lambda \) (note that this dimension is known to be finite for left serial algebras [11, Theorem 3]). Clearly \( A_\delta(S) \) coincides with the minimal \( \mathcal{P}^\infty(\Lambda \text{-mod}) \)-approximation of \( S \). As \( d \) increases, the trunks \( T_d \) may shrink from the bottom up, while the saguaros \( A_d(S) \) will increasingly ramify on what is left of these trunks. The following example illustrates this growth pattern; it encodes a great deal of homological information about \( \Lambda \) in a compact form.

**Example 7.5.** Let \( \Lambda \) be a left serial algebra whose indecomposable projective modules are represented by the following graphs.
The evolution of the $P^{(d)}$-approximations of the simple left $\Lambda$-module $S_1$ is graphically represented below. From left to right, we exhibit the minimal $P^{(1)}$, $P^{(2)}$, $P^{(3)}$-approximations of $S_1$; the last coincides with the minimal $P^\infty(\Lambda\text{-mod})$-approximation, since the left finitistic dimension of $\Lambda$ is 3. The $P^{(0)}$-approximation is simply $\Lambda e_1$ and is omitted from the list.

8. Approximations over algebras of finite uniserial type

Throughout this section we will assume that $\Lambda = K\Gamma/I$ has finite uniserial type. In view of Section 5, this implies in particular that $\Gamma$ has no double arrows. Let $S \subseteq \Lambda\text{-mod}$ and $S^\infty \subseteq \Lambda\text{-mod}$ denote the full subcategories having as objects all finite direct sums of saguaros in $\Lambda\text{-mod}$ in the first case, and all finite direct sums of saguaros of finite projective dimension in the second. If the categories $S$ and $S^\infty$ have finite representation type, they are of course contravariantly finite in $\Lambda\text{-mod}$ and can thus be used to group the objects of $\Lambda\text{-mod}$ according to their minimal $S$- or $S^\infty$-approximations. Whether this yields an
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effective classification depends on the algebra Λ and on the class of modules to be explored. The objects in S or S∞ are, in a way, ‘first approximations’ to ‘useful approximations’. In general, one is led to consider larger contravariantly finite subcategories of Λ-mod, such as CS ⊂ Λ-mod, the category of all finite direct sums of saguaros in Λ-mod and duals of saguaros in mod-Λ, or categories of objects glued together along uniserial submodules from certain cyclic building blocks. An effective method of creating new, more flexible – for approximation purposes – contravariantly finite subcategories from S is provided by the following corollary to Theorem 2.5: If A ⊆ Λ-mod is contravariantly finite, then so is the category E(A) having as objects all direct summands of extensions of modules in A by modules in A.

We believe that the conclusion of the following theorem remains true for arbitrary algebras of finite uniserial type. The somewhat more narrow situation which we address here is technically far less involved, however. As in Section 5, we denote by Vp the affine variety describing the uniserial left Λ-modules with mast p. For a characterization of the algebras satisfying the hypothesis of the next theorem, we refer back to Theorem 5.6.

**Theorem 8.1.** If Vp is finite for all paths p in KΓ, the subcategories S and S∞ of Λ-mod have finite representation type.

The line of the proof is akin to that of Theorem 5.1 of [BZ-H], but the present, far more general, situation calls for non-trivial supplements. We therefore include a fairly detailed argument.

In case the hypothesis of Theorem 8.1 is satisfied, the graphs of the saguaros in Λ-mod are essentially unique, as long as we insist that a canonical sequence of top elements be displayed (see Theorem 5.6(4),(5)). To state this uniqueness more concisely, we say that two layered and labeled graphs, G1 and G2, are equivalent if there exists an isomorphism of undirected graphs between them which preserves the layering, as well as the numbers attached to the vertices and the labels attached to the edges. Under this relation, the equivalence class of graphs of a saguaro T over an algebra Λ as above is uniquely determined, and it therefore makes sense to refer to the graph of T.

**Lemma 8.2.** If Vp is finite for all paths p in KΓ, then any two saguaros in Λ-mod having equivalent graphs (in the sense of Section 7) are isomorphic. More precisely, if T(1) and T(2) are saguaros with identical graphs and ĭ11, . . . , ĭ1s (respectively, ĭ21, . . . , ĭ2s) are canonical sequences of top elements corresponding to these graphs, then ĭ1i → ĭ2i induces an isomorphism T(1) → T(2).

**Proof.** Recall that Λ = KΓ/I satisfies ‘|Vp| < ∞ for all p’ precisely when each graph of a uniserial module is an edge path. In particular, this means that, given a uniserial module U ∈ Λ-mod with top elements x and x′, the left annihilator of x in Λ coincides with that of x′; in other words, the assignment x → x′ determines an automorphism U → U.

Now suppose that T(1) and T(2) in Λ-mod are saguaros with equivalent graphs; without loss of generality, we may assume these graphs to be identical. Then T(1) and T(2) have identical sequences of (isomorphism types) of trunks, (T1, . . . , Ts) say. Write T(j) = Σ1≤i≤s Tji for j = 1, 2, where the Tji are the canonical images of the Ti in T(j).
Moreover, if $p_i \in \mathcal{K}_T$ is the mast of $T_i$, the fact that the graphs of $T^{(1)}$ and $T^{(2)}$ are the same guarantees the existence of right subpaths $q_i$ of $p_i$, $q_i'$ of $p_i$ for $2 \leq i \leq s - 1$, $q'_s$ of $p_s$ such that $q_it_{ji} = q'_{i+1}t_{j,i+1}$, $j = 1, 2$, for canonical sequences of top elements $t_{ji}$ corresponding to the coinciding graphs of $T^{(1)}$ and $T^{(2)}$. By the first paragraph, $\hat{t}_{1i} \mapsto \hat{t}_{2i}$ then yields a well-defined isomorphism from $T^{(1)}$ to $T^{(2)}$. \hfill \Box

For the sake of the proof of Theorem 8.1 we require some additional concepts. The reader solely interested in an overview can safely skip them.

**Definition 8.3.** Let $T = \sum_{i=1}^{m} \hat{T}_i$ be a saguaro as in Definition 6.1, and set $I = \{1, \ldots, m\}$.

(a) A submodule $V$ of $T$ of the form $V = \sum_{i \in I_1} \hat{T}_i$, where $I_1 \subseteq I$, is called a subsaguaro of $T$ if, for all indices $j \in I \setminus I_1$,

$$\hat{T}_j \cap V \subseteq \bigcap_{i \in I_1} \hat{T}_i.$$

(b) Two subsaguaros $V^{(1)} = \sum_{i \in I_1} \hat{T}_i$ and $V^{(2)} = \sum_{i \in I_2} \hat{T}_i$ of $T$ are said to be isomorphic as subsaguaros of $T$ if the following is true: The index sets $I_1$ and $I_2$ have the same cardinality, $s$ say, and there exists an isomorphism $\phi : V^{(1)} \to V^{(2)}$, together with orderings $\hat{T}_{11}, \ldots, \hat{T}_{1s}$ and $\hat{T}_{21}, \ldots, \hat{T}_{2s}$ of the trunks indexed by $I_1$ and by $I_2$, respectively, such that $\phi$ induces an isomorphism $\hat{T}_{1j} \to \hat{T}_{2j}$ for $1 \leq j \leq s$ which restricts to the identity on the intersection $\bigcap_{i \in I_1 \cup I_2} \hat{T}_i$.

(c) The saguaro $T$ is called redundant if it contains nontrivial isomorphic subsaguaros $V^{(1)} = \sum_{i \in I_1} \hat{T}_i$ and $V^{(2)} = \sum_{i \in I_2} \hat{T}_i$ with $I_1 \cap I_2 = \emptyset$ such that, moreover, $\hat{T}_k \cap V^{(1)} = \hat{T}_k \cap V^{(2)}$ for all $k \in I \setminus (I_1 \cup I_2)$. Otherwise $T$ is called irredundant.

Observe that each subsaguaro of $T$ is a saguaro in its own right by the remark following Observation 6.4. Intuitively speaking, a subsum $V = \sum_{i \in I_1} \hat{T}_i$ of trunks of a saguaro $T = \sum_{i \in I} \hat{T}_i$ is a subsaguaro if there is no index $j \in I \setminus I_1$ such that the trunk $\hat{T}_j$ meets $V$ at a properly higher point in the graph of $T$ than it meets the intersection $\bigcap_{i \in I_1} \hat{T}_i$ of the trunks of $V$.

**Lemma 8.4.** Again suppose that all the uniserial varieties $V_p$ are finite, and let $T$ in $\Lambda \text{-mod}$ be a saguaro with graph $G$. Then $T$ is redundant if and only if $G$ has a non-trivial full subgraph of the form

![Diagram](image)

where $G_1$ and $G_2$ are equivalent trees sharing precisely one vertex $a$ of $G$ such that, moreover, each edge of $G$ which is contiguous with $G_1 \cup G_2$, without belonging to $G_1 \cup G_2$, is incident with $a$. 
Proof. Start by observing that, by Theorem 5.6, our hypothesis forces all graphs of saguaros in $\Lambda$-mod to be trees (all graphs of uniserial modules being edge paths). Hence redundancy of $T$ clearly implies the existence of a full subgraph of $G$ as postulated.

Conversely, suppose that $G$ has a subgraph as described in the claim. Without loss of generality, we may assume that $G_1$ and $G_2$ are identical trees. By Observations 6.5 and 6.6, we are, moreover, free to assume that the graph $G$ has a form as follows

\[
G_1 \quad G_2 \quad G_3 \quad G_4 \quad \ldots \quad G_m
\]

•

with a canonical sequence of top elements $\hat{t}_{11}, \ldots, \hat{t}_{1s}$ corresponding to the uppermost layer of vertices in $G_1$, with $\hat{t}_{21}, \ldots, \hat{t}_{2s}$ corresponding to the vertices in the uppermost layer of $G_2$, and $\hat{t}_3, \ldots, \hat{t}_l$ corresponding to the top vertices of $G_3, \ldots, G_m$. For $j = 1, 2$, define $\hat{T}_{(j,i)} = \Lambda \hat{t}_{ji}$ and $V^{(j)} = \sum_{i=1}^{s} \Lambda \hat{T}_{(j,i)}$. Then, clearly, $V^{(1)}$ and $V^{(2)}$ are subsaguaros of $T$, and the sets $I_j = \{(j,i) \mid 1 \leq i \leq s\}$, $j = 1, 2$, indexing their trunks are disjoint. We will show that $V^{(1)}$ and $V^{(2)}$ are isomorphic as subsaguaros of $T$. Indeed, from Lemma 8.2 we know that the assignment $\hat{t}_{1i} \mapsto \hat{t}_{2i}$ induces an isomorphism $\phi : V^{(1)} \cong V^{(2)}$. To see that $\phi$ induces the identity on $D = \bigcap_{i \leq s} \hat{T}_{(1,i)} \cap \bigcap_{i \leq s} \hat{T}_{(2,i)}$, it suffices to observe that there are right subpaths $a_i$ of $\text{mast}(\hat{T}_{(1,i)}) \cap \text{mast}(\hat{T}_{(2,i)})$, respectively, such that, for any choice of $i$ and $h$ in $\{1, \ldots, s\}$, we have $\hat{T}_{(1,i)} \cap \hat{T}_{(2,h)} = D = \Lambda a_i \hat{t}_{1i} = \Lambda a_h \hat{t}_{2h}$ and $a_i \hat{t}_{1i} = a_h \hat{t}_{2h}$; for details, see Observation 6.4. □

Proof of Theorem 8.1. In a first step we show that, up to isomorphism, there are only finitely many irredundant saguaros in $\Lambda$-mod. We proceed by induction on the Loewy length $L$ of $\Lambda$. The case $L = 1$ being clear, suppose that $L \geq 2$ and that there are $r$ isomorphism types of irredundant saguaros in $\Lambda/JL^{-1}$-mod; let $G_1, \ldots, G_r$ be the corresponding graphs, and recall that the $G_i$ are unique up to equivalence. The graphs of the additional irredundant saguaros in $\Lambda$-mod are all equivalent to graphs of the form...
where $t \geq 1$, $1 \leq i_j \leq r$, and the $\alpha_i$ are arrows such that $\alpha_j = \alpha_k$ implies $i_j \neq i_k$. The number of equivalence classes of graphs of this type is clearly bounded above by $2^{ra}$, where $a$ is the number of distinct arrows in $\Gamma$.

In a second step we prove that each saguaro in $\Lambda$-mod is a direct sum of irredundant ones. Suppose that $T$ is a saguaro on $(T_1, \ldots, T_m)$. This time, we proceed by induction on $m$. If $T$ is irredundant to begin with, there is nothing to prove; in particular, this is the case when $m = 1$. So suppose that $m \geq 2$ and that $T$ is redundant with non-zero isomorphic subsaguaros $V^{(1)} = \sum_{i \in I_1} \hat{T}_i$ and $V^{(2)} = \sum_{i \in I_2} \hat{T}_i$ such that $I_1 \cap I_2 = \emptyset$ and $\hat{T}_k \cap V^{(1)} = \hat{T}_k \cap V^{(2)} \subseteq V^{(1)} \cap V^{(2)}$ for all $k \not\in I_1 \cup I_2$ as in Definition 8.3. Moreover, let $\phi : V^{(1)} \to V^{(2)}$ be an isomorphism and $\hat{T}_{11}, \ldots, \hat{T}_{1s}$, resp. $\hat{T}_{21}, \ldots, \hat{T}_{2s}$, orderings of the trunks indexed by $I_1$, resp. by $I_2$, such that $\phi$ induces an isomorphism $\hat{T}_{1j} \to \hat{T}_{2j}$ for $1 \leq j \leq s$ and restricts to the identity on $D := \bigcap_{i \in I_1} \hat{T}_{1i} \cap \bigcap_{i \in I_2} \hat{T}_{2i}$. In particular, this implies that, given a top element $\hat{t}_{1j} \in \hat{T}_{1j}$, the image $\hat{t}_{2j} := \phi(\hat{t}_{1j})$ is a top element of $\hat{T}_{2j}$ for $1 \leq j \leq s$.

Observe that $\hat{T}_{1i} \cap \hat{T}_{2j} = D$ for all $i, j \in \{1, \ldots, s\}$, since $\hat{T}_{1i} \cap \hat{T}_{2j} \subseteq \bigcap_{1 \leq k \leq s} \hat{T}_{1k} \cap \bigcap_{1 \leq k \leq s} \hat{T}_{2k}$ due to the fact that $V^{(1)}$ and $V^{(2)}$ are subsaguaros of $T$ and $I_1 \cap I_2 = \emptyset$. It follows that the submodule $V := \sum_{1 \leq i \leq s} \Lambda(\hat{t}_{1i} - \hat{t}_{2i})$ of $T$ is isomorphic to $V^{(1)}/D$ and thus is a direct sum of saguaros, each of which has at most $s$ trunks. By the remark following Observation 6.4, $W := \sum_{i \in I \setminus I_1} \hat{T}_i$ is in turn a saguaro which clearly has fewer than $m$ trunks because $I_1 \neq \emptyset$. In view of the fact that we can reorder $T_1, \ldots, T_m$ into another legitimate sequence of trunks in such a way that the trunks indexed by $I_1$ precede those indexed by $I \setminus I_1$ (Observation 6.5), it is now routine to check that $V \cap W = 0$. We infer that $T = V \oplus W$ and apply the induction hypothesis to $W$ and to the saguaros occurring as direct summands of $V$ to complete the proof. $\Box$

In a first easy example – Example 4.3 revisited – we present a monomial relation algebra $\Lambda$ for which $\mathcal{P}^\infty(\Lambda$-mod) fails to be contravariantly finite in $\Lambda$-mod, while all the varieties $V_p$ describing the uniserial left $\Lambda$-modules are finite. By Theorem 8.1, this implies that $\mathcal{S}^\infty$ has finite representation type.

**Example 8.5.** If $\Lambda$ is the algebra of Example 4.3, then $\mathcal{P}^\infty(\Lambda$-mod) fails to be contravariantly finite as we saw earlier. However, $\Lambda$ satisfies the hypothesis of Theorem 8.1, and so $\mathcal{S}^\infty$ has finite type. A fortiori, $\mathcal{S}^\infty$ is contravariantly finite in $\Lambda$-mod. The minimal $\mathcal{S}^\infty$-approximations $A_i$ of the simple left $\Lambda$-modules $S_i$ are as follows:
Observe that \( \text{fin dim } \Lambda = 1 = \sup_{1 \leq i \leq 6} \text{p dim } A_i. \)

**Problem 8.6.** Characterize those algebras \( \Lambda \) of finite uniserial type for which the supremum of the minimal \( S^\infty \)-approximations of the simple left \( \Lambda \)-modules equals \( \text{fin dim } \Lambda \).

While there are algebras for which this equality fails, for instance among the algebras of type \( \Lambda = \mathcal{O}/\pi \mathcal{O} \) discussed in Section 4, the realm of validity of this equality even among non-monomial algebras appears to be fairly wide. We conclude with a not so straightforward binomial example. Here the computations leading to the precise shapes of the minimal approximations of the simple modules are a bit more involved; we will suppress them nonetheless.

**Example 8.7.** Let \( \Lambda = K\Gamma/I \), where \( \Gamma \) is the quiver

![Quiver Diagram]

and the indecomposable projective left \( \Lambda \)-modules are

![Module Diagram]
The only relations which cannot be gleaned from the graphs – the scalars occurring cannot be detected – we take to be $\alpha_9 \gamma_2 - \alpha_3 \alpha_2 \beta_2$ and $\alpha_3 \alpha_2 \beta_2 \alpha_1 - \alpha_8 \beta_1$.

Then $\Lambda$ has finite uniserial type, without satisfying the stronger hypothesis of Theorem 8.1. However, the categories $S$ and $S^\infty$ are still of finite representation type. One can compute the minimal $S^\infty$-approximation $A_1$ of $S_1$ to be given by the graph

```
1  1  4  5

8  ⊕  2  6

2  5

3
```

The minimal $S^\infty$-approximations $A_i$ of the remaining simples $S_i$ are:

```
A_2 : 2  9
2  4
3

A_3 : 7  9
7

A_7 : 7  9
A_8 : 7  9
```

and $A_i = S_i$ for $i = 4, 5, 6, 9$.

Moreover, one obtains $\text{sup}\{p\dim A_i \mid 1 \leq i \leq 9\} = 2$. We resolve $A_1$, to indicate how easy it is to find minimal projective resolutions of saguaros. Indeed, the first syzygy $\Omega^1(A_1)$ of $A_1$ has graph

```
2  9  8

9  ⊕  7
2

3

7
```

and $\Omega^2(A_1)$ has graph

```
3  7

7  ⊕  7
7
```
Thus $\Omega^2(A_1)$ is projective. With the ‘repetition method’ of [8], one can finally check that $\dim \Lambda = 2$, which yields the equality $\dim \Lambda = \sup \{p \dim A_i \mid 1 \leq i \leq 9\}$. □

9. Phantoms

When $\mathfrak{A} \subseteq \Lambda$-mod fails to be contravariantly finite, we abandon the requirement that the approximating objects, used to compare arbitrary finitely generated modules with the modules in $\mathfrak{A}$, be themselves finitely generated. In [10] it became apparent that this often yields information which is no less effective than that stored in classical (finite dimensional) $\mathfrak{A}$-approximations. Such generalized $\mathfrak{A}$-approximations, ‘phantoms’ as we will call them, of a simple module $\Lambda e/Je$ say, provide a synopsis of the relations present in those objects of $\mathfrak{A}$ which contain a top element of type $e$. We simply renounce the requirement that this picture should fit into a finitely generated module.

Definitions 9.1. [10] Let $\mathfrak{C} \subseteq \mathfrak{A}$ be full subcategories of $\Lambda$-mod and suppose that $\mathfrak{A}$ is closed under finite direct sums. Moreover, let $X$ be a finitely generated left $\Lambda$-module.

1. A $\mathfrak{C}$-approximation of $X$ inside $\mathfrak{A}$ is a homomorphism $f : A \to X$ with $A \in \mathfrak{A}$ such that each map $g \in \text{Hom}_\Lambda(C, X)$ with $C \in \mathfrak{C}$ factors through $f$.

2. An $\mathfrak{A}$-phantom of $X$ of the first kind is an object $H \in \Lambda$-mod (not necessarily in $\mathfrak{A}$) with the following property: There exists a finite nonempty subclass $\mathfrak{C}(H) \subseteq \mathfrak{A}$ such that for each $\mathfrak{C}(H)$-approximation $f : A \to X$ inside $\mathfrak{A}$, the module $H$ is a subfactor of $A$. Any direct limit of $\mathfrak{A}$-phantoms of $X$ of the first kind will be called an $\mathfrak{A}$-phantom of $X$ of the second kind.

We will refer to both kinds of phantoms as $\mathfrak{A}$-phantoms of $X$.

3. An $\mathfrak{A}$-phantom $H$ of $X$ is called $\mathfrak{C}$-effective if $H$ is a direct limit of objects in $\mathfrak{A}$ and there exists a homomorphism $f : H \to X$ with the property that each homomorphism $g \in \text{Hom}_\Lambda(C, X)$ with $C \in \mathfrak{C}$ factors through $f$.

The effective phantoms are in a sense the best possible substitutes for minimal $\mathfrak{A}$-approximations in the traditional sense. The crucial fact is that, given any module $X \in \Lambda$-mod, nontrivial phantoms exist. In case $X$ has an $\mathfrak{A}$-approximation in the sense of Auslander and Smalø, the minimal $\mathfrak{A}$-approximation $A(X)$ is the ‘best possible’ phantom; indeed, the $\mathfrak{A}$-phantoms of $X$ are precisely the subfactors of $A(X)$ in that situation. The case of interest is addressed by the following theorem, the proof of which indicates a construction pattern for phantoms of infinite $K$-dimension.

Theorem 9.2. [10] Suppose that $\mathfrak{A} \subseteq \Lambda$-mod is closed under finite direct sums and let $X \in \Lambda$-mod. Then the following conditions are equivalent:

1. $X$ fails to have an $\mathfrak{A}$-approximation.
2. $X$ has $\mathfrak{A}$-phantoms of countably infinite $K$-dimension.
3. There exists a countable subclass $\mathfrak{C} \subseteq \mathfrak{A}$ such that $X$ has $\mathfrak{C}$-effective $\mathfrak{A}$-phantoms of infinite $K$-dimension. □
Remarks 9.3. (1) From the proof of Criterion 4.1 it can be gleaned that, under the hypotheses of 4.1, with \(p_1, \ldots, p_m, q_1, \ldots, q_m\) being paths in \(K\Gamma\) of positive length, the simple module \(S_1\) has an \(\mathfrak{A}\)-phantom with graph

\[
\begin{array}{ccc}
e_1 & q_2 & e_2 \\
p_1 & & p_2 \\
& \cdots & \\
q_m & p_m & q_1 \\
& \cdots & \\
& \cdots & \\
e_2 & p_2 & \\
\end{array}
\]

The next remark gives a clue how to start building phantoms.

(2) Let \(\mathfrak{A} = \mathcal{P}^\infty(\Lambda\text{-mod})\) where \(\Lambda = K\Gamma/I\) is a monomial relation algebra, and suppose that the simple module \(S = \Lambda e/Je\) has infinite projective dimension. If \(\alpha_1, \ldots, \alpha_m\) are arrows \(\alpha_j : e \rightarrow e_j\) ending in distinct vertices \(e_1, \ldots, e_m\) such that \(\text{pdim } Je_j = \infty\) for \(1 \leq j \leq m\), then

\[
\begin{array}{ccc}
e & e_m \\
\alpha_1 & & \alpha_m \\
e_1 & e_2 & \cdots & e_m \\
\end{array}
\]

is the graph of a \(\mathcal{P}^\infty(\Lambda\text{-mod})\)-phantom of \(S\).

Moreover, if there exists a module \(M \in \mathcal{P}^\infty(\Lambda\text{-mod})\) with graph \(G\) having top elements \(m = em\) and \(m' = e'm'\) which correspond to the top vertices of a subgraph of \(G\) as follows

\[
\begin{array}{ccc}
e & e' \\
\alpha_m & & \beta \\
e_m & & e_m \\
\end{array}
\]

then

\[
\begin{array}{ccc}
e & e' \\
\alpha_1 & & \alpha_m \\
e_1 & e_2 & \cdots & e_m \\
\end{array}
\]

is the graph of a \(\mathcal{P}^\infty(\Lambda\text{-mod})\)-phantom of \(S\). □

Examples 9.4. We revisit the negative examples of Section 4. Throughout, \(\mathfrak{A}\) stands for the category \(\mathcal{P}^\infty(\Lambda\text{-mod})\) and \(S_1\) for the simple left \(\Lambda\)-module corresponding to the vertex ‘1’.

- Example 4.2. The module \(M = \lim M_n \in \mathcal{P}^\infty(\Lambda\text{-Mod})\) with graph

\[
\begin{array}{ccc}
1 & 1 & 1 & \cdots \\
\beta & \alpha & \beta & \beta \\
2 & 2 & 2 & \cdots \\
\end{array}
\]
is an $\mathfrak{A}$-phantom of $S_1$ which is $\mathfrak{C}_0$-effective, where $\mathfrak{C}_0 = \{ M_n \mid n \in \mathbb{N} \}$; here $M_n$ is defined as in 4.2.

On the other hand, the left $\Lambda$-module $N$ with graph

- while still an object of $\mathcal{P}^\infty(\Lambda\text{-Mod})$ with the property that each homomorphism $M_n \to S_1$, $n \in \mathbb{N}$, factors through it - is not an $\mathfrak{A}$-phantom of $S_1$.

Further non-finitely generated phantoms of $S_1$ are as follows. For each nonzero scalar $k \in K$ and $n \in \mathbb{N}$, let $x_1 = \cdots = x_n = e_1$, and consider the module

$$L_{nk} = \left( \bigoplus_{i=1}^n \Lambda x_i \right) / \left( \Lambda (\beta x_1 - k \alpha x_1) + \sum_{i=2}^n \Lambda (\beta x_i - k \alpha x_i - \alpha x_{i-1}) \right)$$

in $\mathcal{P}^\infty(\Lambda\text{-mod})$, which has the following graph relative to the top elements $\overline{x}_1, \ldots, \overline{x}_n$:

Clearly, $L_{nk}$ embeds canonically into $L_{mk}$ for $n < m$, and it is not difficult to see that the direct limit $L_k := \varinjlim L_{nk}$ with graph

is an $\mathfrak{A}$-phantom of $S_1$ which is $\mathfrak{C}_k$-effective, where $\mathfrak{C}_k = \{ L_{nk} \mid n \in \mathbb{N} \} \subseteq \mathfrak{A}$.

- Example 4.3. Here the direct limit $M = \varinjlim M_n$ with graph
is an $\mathfrak{A}$-phantom of $S_1$ which is $\{M_n \mid n \in \mathbb{N}\}$-effective. It is not difficult to see that the $\mathfrak{A}$-phantom $M$ is actually $\mathfrak{A}$-effective and hence encodes the full information stored in classical approximations when they exist.

- Example 4.5. Consider the subclasses $\mathcal{C} = \{M_n \mid n \in \mathbb{N}\}$ and $\mathcal{D} = \{N_n \mid n \in \mathbb{N}\}$ of $\mathfrak{A}$, where $M_n$ and $N_n$ are the left $\Lambda$-modules with graphs

\[
\begin{array}{c}
1 \\
\beta \\
\alpha \\
\beta \\
1
\end{array}
\quad
\begin{array}{c}
1 \\
\beta \\
\alpha \\
\beta \\
1
\end{array}
\quad \cdots 
\quad
\begin{array}{c}
1 \\
\beta \\
\alpha \\
\beta \\
1
\end{array}
\quad
\begin{array}{c}
1 \\
\beta \\
\alpha \\
\beta \\
1
\end{array}
\]

respectively, relative to $n$ top elements which are linearly independent modulo the radical in each case. Then both $M = \varinjlim M_n$ and $N = \varinjlim N_n$ are $\mathfrak{A}$-phantoms of $S_1$. The phantom $M$ is $\mathcal{C}$-effective, but not $\mathcal{D}$-effective, while $N$ is $(\mathcal{C} \cup \mathcal{D})$-effective; in other words, $N$ is the ‘better’ of the two $\mathfrak{A}$-phantoms of $S_1$, storing more information about $\mathfrak{A}$ than the phantom $M$. □

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