On the number of distinct exponents in the prime factorization of an integer

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Abstract. Let \( f(n) \) be the number of distinct exponents in the prime factorization of the natural number \( n \). We prove some results about the distribution of \( f(n) \). In particular, for any positive integer \( k \), we obtain that

\[
\# \{ n \leq x : f(n) = k \} \sim A_k x
\]

and

\[
\# \{ n \leq x : f(n) = \omega(n) - k \} \sim \frac{B x (\log \log x)^k}{k! \log x},
\]

as \( x \to +\infty \), where \( \omega(n) \) is the number of prime factors of \( n \) and \( A_k, B > 0 \) are some explicit constants. The latter asymptotic extends a result of Aktaş and Ram Murty (Proc. Indian Acad. Sci. (Math. Sci.) 127(3) (2017) 423–430) about numbers having mutually distinct exponents in their prime factorization.

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1. Introduction

Let \( n = p_1^{a_1} \cdots p_s^{a_s} \) be the factorization of the natural number \( n > 1 \), where \( p_1 < \cdots < p_s \) are prime numbers and \( a_1, \ldots, a_s \) are positive integers. Several functions of the exponents \( a_1, \ldots, a_s \) have been studied, including their product [17], their arithmetic mean [2,4,5,7], and their maximum and minimum [11,13,15,18]. See also [3,8] for a more general function.

Let \( f \) be the arithmetic function defined by \( f(1) := 0 \) and \( f(n) := \{a_1, \ldots, a_s\} \) for all natural numbers \( n > 1 \). In other words, \( f(n) \) is the number of distinct exponents in the prime factorization of \( n \). The first values of \( f(n) \) are listed in sequence A071625 of OEIS [16].

Our first contribution is a quite precise result about the distribution of \( f(n) \).
Theorem 1.1. There exists a sequence of positive real numbers \((A_k)_{k \geq 1}\) such that, given any arithmetic function \(\phi\) satisfying \(|\phi(k)| < a^k\) for some fixed \(a > 1\), we have that the series

\[
M_\phi := \sum_{k=1}^{\infty} A_k \phi(k)
\]

(1)

converges and

\[
\sum_{n \leq x} \phi(f(n)) = M_\phi x + O_{a,\varepsilon}(x^{1/2+\varepsilon}),
\]

for all \(x \geq 1\) and \(\varepsilon > 0\).

From Theorem 1.1, it follows immediately that all the moments of \(f\) are finite and that \(f\) has a limiting distribution. In particular, we highlight the following corollary.

**COROLLARY 1.1**

For each positive integer \(k\), we have

\[
\#\{n \leq x : f(n) = k\} = A_k x + O_{k}(x^{1/2+\varepsilon}),
\]

for all \(x \geq 1\) and \(\varepsilon > 0\).

We also provide a formula for \(A_k\). Before stating it, we need to introduce some notations. Let \(\psi\) be the Dedekind function defined by

\[
\psi(n) := n \prod_{p \mid n} \left(1 + \frac{1}{p}\right)
\]

for each positive integer \(n\), and let \((\rho_k)_{k \geq 1}\) be the family of arithmetic functions supported on squarefree numbers and satisfying

\[
\rho_1(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \quad \rho_{k+1}(n) = \frac{1}{n-1} \sum_{d \mid n, d < n} \rho_k(d) \quad \text{if } n > 1,
\]

for all squarefree numbers \(n\) and positive integers \(k\).

**Theorem 1.2.** We have

\[
A_k = \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{\rho_k(n)}{\psi(n)}
\]

for each positive integer \(k\).

Clearly, \(f(n) \leq \omega(n)\) for all positive integers \(n\), where \(\omega(n)\) denotes the number of prime factors of \(n\). Motivated by a question of Recamán Santos [14], Aktaş and Ram Murty
[1] studied the natural numbers \( n \) such that all the exponents in their prime factorization are distinct, that is, \( f(n) = \omega(n) \). They called such numbers \textit{special numbers} (sequence A130091 of OEIS [16]) and they proved the following.

**Theorem 1.3.** The number of special numbers not exceeding \( x \) is

\[
\frac{B x}{\log x} + O\left( \frac{x}{(\log x)^2} \right),
\]

for all \( x \geq 2 \), where

\[
B := \sum \frac{1}{\ell}
\]

and the sum of over natural numbers \( \ell \) that are powerful and special.

Let \( g \) be the arithmetic function defined by \( g(n) := \omega(n) - f(n) \) for all positive integers \( n \). Hence, by the previous observation, \( g \) is a nonnegative function and \( g(n) = 0 \) if and only if \( n \) is a special number. We prove the following result about \( g \), which extends Theorem 1.3 and it is somehow dual to Corollary 1.1.

**Theorem 1.4.** For each nonnegative integer \( k \), we have

\[
\# \{ n \leq x : g(n) = k \} = \frac{B x (\log \log x)^k}{k! \log x} \left( 1 + O_k \left( \frac{1}{\log \log x} \right) \right),
\]

for all \( x \geq 3 \).

**Notation.** We employ the Landau–Bachmann “Big Oh” notation \( O \), as well as the associated Vinogradov symbol \( \ll \), with their usual meaning. Any dependence of the implied constants is explicitly stated. We let \( \epsilon \) denote an arbitrary small positive real number, not necessarily the same at each occurrence. We reserve the letter \( p \) for prime numbers.

**2. Preliminaries**

Recall that a natural number \( n \) is called \textit{powerful} if \( p \mid n \) implies \( p^2 \mid n \), for all primes \( p \). For all \( x \geq 1 \), let \( \mathcal{P}(x) \) be the set of powerful numbers not exceeding \( x \).

**Lemma 2.1.** We have \( \# \mathcal{P}(x) \ll x^{1/2} \) for every \( x \geq 1 \).

**Proof.** See [9]. \( \square \)

**Lemma 2.2.** We have

\[
\sum_{\ell \in \mathcal{P} \atop \ell > y} \frac{1}{\ell} \ll \frac{1}{y^{1/2}}, \quad \sum_{\ell \in \mathcal{P}(y)} \frac{1}{\ell^{1/2}} \ll \log y,
\]

for all \( y \geq 2 \).
Proof. By Lemma 2.1 and by partial summation, we have
\[
\sum_{\ell \in \mathcal{P}} \frac{1}{\ell} = \left. \frac{\#\mathcal{P}(t)}{t} \right|_{t=y}^{+\infty} + \int_{y}^{+\infty} \frac{\#\mathcal{P}(t)}{t^2} \, dt \ll \int_{y}^{+\infty} \frac{dt}{t^{1+1/2}} \ll \frac{1}{y^{1/2}}.
\]

The proof of the second claim is similar. \(\square\)

We need the following upper bound for the number of prime factors of a natural number.

**Lemma 2.3.** We have
\[
\omega(n) \ll \frac{\log n}{\log \log n}
\]
for all integers \(n \geq 3\).

**Proof.** See, for example, [6, Proposition 7.10]. \(\square\)

For every \(x \geq 1\) and every positive integer \(h\), let \(Q(x; h)\) denote the number of squarefree numbers not exceeding \(x\) and relatively prime with \(h\).

**Lemma 2.4.** We have
\[
Q(x; h) = \frac{6}{\pi^2} \frac{h}{\psi(h)} x + O(4^{\omega(h)}(x^{1/2} + 1))
\]
for all \(x \geq 1\) and all positive integers \(h\).

**Proof.** It follows easily from [10, Eq. 8]. \(\square\)

For every \(x \geq 1\) and every positive integers \(s, h\), let \(Q_s(x; h)\) denote the number of squarefree numbers not exceeding \(x\), having exactly \(s\) prime factors, and relatively prime with \(h\).

**Lemma 2.5.** We have
\[
Q_s(x; h) = \frac{x(\log \log x)^{s-1}}{(s-1)!\log x} \left(1 + O_{\delta,s} \left(\frac{\log \log \log (h + 15)}{\log \log x}\right)\right)
\]
for all \(x \geq 3\), \(0 < \delta < 1\), and for all integers \(1 \leq h \leq x^\delta\) and \(s \geq 1\).

**Proof.** For \(s = 1\), the claim follows from the Prime Number theorem, while for \(h = 1\), the claim is a classic result of Landau [12]. Hence, suppose \(s, h > 1\). Also, we can assume that \(x \geq 3^{1/(1-\delta)}\). If \(n \leq x\) is a squarefree number having exactly \(s\) prime factors such that \((n, h) > 1\), then \(n = pn'\), where \(p\) is a prime number dividing \(h\) and \(n' \leq x/p\) is a squarefree number having exactly \(s - 1\) prime factor. Therefore,
\[ 0 \leq Q_s(x; 1) - Q_s(x; h) \leq \sum_{p \mid h} Q_{s-1} \left( \frac{x}{p}, 1 \right) \ll_s \sum_{p \mid h} \frac{x \log \log (x/p)^{s-2}}{\log (x/p)}, \]

where we used the fact that \( p \leq x^\delta \) and the upper bound

\[ \sum_{p \mid h} \frac{1}{p} \ll \log \log (\omega(h) + 2) \ll \log \log (h + 15), \]

which in turn follows from Mertens’ second theorem [6, Theorem 4.5] and the simple bound \( \omega(h) \ll \log h \). Consequently,

\[ Q_s(x; h) = Q_s(x; 1) + O_{\delta, s} \left( \frac{x \log \log x^{s-1} \log \log (h + 15)}{\log x} \right) \log \log x, \]

as claimed. \( \square \)

Finally, we need a lemma about certain sums of powers.

**Lemma 2.6.** Let \( a_0 \) be an integer. For all \( x_1, \ldots, x_k > 1 \), we have

\[ \sum_{a_0 < a_1 < \cdots < a_k} \frac{1}{x_1^{a_1} \cdots x_k^{a_k}} = \frac{1}{(x_1 \cdots x_k)^{a_0}} \prod_{j=1}^{k} \frac{1}{x_j \cdots x_k - 1}, \]

where the sum is over all integers \( a_1, \ldots, a_k \) satisfying \( a_0 < a_1 < \cdots < a_k \).

**Proof.** We proceed by induction on \( k \). For \( k = 1 \), we have

\[ \sum_{a_0 < a_1} \frac{1}{x_1^{a_1}} = \frac{1}{x_1^{a_0+1}} \sum_{d=0}^{\infty} \frac{1}{x_1^{a_0} x_1^d} = \frac{1}{x_1^{a_0} x_1 - 1}, \]

as claimed. Suppose that the claim is true for \( k \), we shall prove it for \( k + 1 \). We have

\[ \sum_{a_0 < \cdots < a_{k+1}} \frac{1}{x_1^{a_1} \cdots x_{k+1}^{a_{k+1}}} = \sum_{a_0 < \cdots < a_k} \frac{1}{x_1^{a_1} \cdots x_k^{a_k}} \sum_{a_k < a_{k+1}} \frac{1}{x_k^{a_k+1}} \]

\[ = \sum_{a_0 < \cdots < a_{k+1}} \frac{1}{x_1^{a_1} \cdots x_{k-1}^{a_{k-1}} (x_k x_{k+1})^{a_k}} x_{k+1} - 1. \]
where we used (2), with $a_0$ and $x_1$ replaced respectively by $a_k$ and $x_{k+1}$, and the induction hypothesis. □

3. Proof of Theorem 1.1

We begin by proving that for each positive integer $k$, there exists $A_k > 0$ such that

$$N_k(x) := \#\{n \leq x : f(n) = k\} = A_k x + O(x^{1/2+\varepsilon}),$$

(3)

for all $x \geq 1$ and $\varepsilon > 0$. Clearly, every natural number $n$ can be written in a unique way as $n = m\ell$, where $m$ is a squarefree number, $\ell$ is a powerful number, and $(m, \ell) = 1$. If $m = 1$, then $n = \ell$ is powerful and, by Lemma 2.1, belongs to a set of cardinality $O(x^{1/2})$. If $m > 1$, then $f(n) = k$ is equivalent to $f(\ell) = k - 1$. Also, for each $\ell$, there are exactly $Q(x/\ell; \ell) - 1$ choices for $m > 1$. Therefore, we have

$$N_k(x) = \sum_{\ell \in \mathcal{P}(x)} \left( Q\left(\frac{x}{\ell}; \ell\right) - 1\right) + O(x^{1/2}),$$

(4)

for all $x \geq 1$. For each positive integer $\ell \leq x$, Lemma 2.3 gives $4^{o(\ell)} \ll \varepsilon x^\ell$. Consequently, by Lemma 2.4, we obtain

$$Q\left(\frac{x}{\ell}; \ell\right) = \frac{6}{\pi^2} \frac{x}{\psi(\ell)} + O\left(\frac{x^{1/2+\varepsilon}}{\ell^{1/2}}\right),$$

(5)

for all positive integers $\ell \leq x$. By Lemma 2.2, we have

$$\sum_{\ell \in \mathcal{P}} \frac{1}{\ell} \ll \sum_{\ell \in \mathcal{P}} \frac{1}{\ell^{1/2}},$$

(6)

for all $x \geq 1$. In particular, the series

$$A_k := \frac{6}{\pi^2} \sum_{\ell \in \mathcal{P}} \frac{1}{\psi(\ell)}$$

(7)

converges. Also, again by Lemma 2.2, we have

$$\sum_{\ell \in \mathcal{P}(x)} \frac{1}{\ell^{1/2}} \ll \log x \ll \varepsilon x^\varepsilon.$$

(8)

At this point, putting together (4) and (5), and using (6) and (8), we obtain
\[ N_k(x) = \sum_{\ell \in \mathcal{P}(x) \atop f(\ell) = k-1} \left( \frac{6}{\pi^2} \frac{x}{\psi(\ell)} + O_{\varepsilon} \left( \frac{x^{1/2+\varepsilon}}{\ell^{1/2}} \right) \right) + O(x^{1/2}) \]

\[ = A_k x + O \left( \sum_{\ell \in \mathcal{P}(x) \atop f(\ell) = k-1} \frac{x}{\psi(\ell)} \right) + O_{\varepsilon} \left( \sum_{\ell \in \mathcal{P}(x)} \frac{x^{1/2+\varepsilon}}{\ell^{1/2}} \right) + O(x^{1/2}) \]

\[ = A_k x + O_{\varepsilon}(x^{1/2+\varepsilon}), \]

as desired. Thus (3) is proved.

Now we shall show that

\[ A_k \leq \frac{6}{\pi^2} \frac{1}{(k-1)!} \quad \text{(9)} \]

for all positive integers \( k \). For \( k = 1 \), the claim is obvious since \( A_1 = 6/\pi^2 \). Hence, assume \( k \geq 2 \). If \( \ell \) is a powerful number such that \( f(\ell) = k - 1 \), then \( \ell = m_1^{a_1} \cdots m_{k-1}^{a_{k-1}} \) for some integers \( m_1, \ldots, m_{k-1} \geq 2 \) and \( 2 \leq a_1 < \cdots < a_{k-1} \). Consequently,

\[ \frac{\pi^2}{6} A_k = \sum_{\ell \in \mathcal{P} \atop f(\ell) = k-1} \frac{1}{\psi(\ell)} < \sum_{\ell \in \mathcal{P} \atop f(\ell) = k-1} \frac{1}{\ell} < \prod_{j=1}^{k-1} \sum_{m=j}^{\infty} \sum_{a_{j+1}}^{\infty} \frac{1}{m^a} \]

\[ = \prod_{j=1}^{k-1} \sum_{m=j}^{\infty} \frac{1}{m_j(m-1)} \leq \prod_{j=1}^{k-1} \frac{1}{j} = \frac{1}{(k-1)!}, \]

where we used the facts that

\[ \sum_{m=2}^{\infty} \frac{1}{m(m-1)} = \sum_{m=2}^{\infty} \left( \frac{1}{m-1} - \frac{1}{m} \right) = 1 \]

and

\[ \sum_{m=2}^{\infty} \frac{1}{m_j(m-1)} \leq \frac{1}{2j} + \frac{1}{3j \cdot 2} + \sum_{n=3}^{\infty} \frac{1}{n^{j+1}} \]

\[ < \frac{1}{2j} + \frac{1}{3j \cdot 2} + \int_{2}^{+\infty} \frac{dt}{t^{j+1}} = \frac{1}{2j} + \frac{1}{3j \cdot 2} + \frac{1}{j 2^j} \leq \frac{1}{j}, \]

for all integers \( j \geq 2 \). Thus (9) is proved.

Now let \( \phi \) be an arithmetic function satisfying \(|\phi(k)| < a^k\) for all positive integers \( k \), where \( a > 1 \) is some constant. From (9) it follows that series (1) converges. Define

\[ y := 2a + [C \log x / \log \log(x + 2)], \]

where \( C > 0 \) is some absolute constant. Since \( f(n) \leq \omega(n) \) for all positive integers \( n \), by Lemma 2.3, we can choose \( C \) sufficiently large so that \( f(n) \leq y \) for all natural numbers \( n \leq x \). Moreover, from (9) and \( y \geq 2a \), we get that

\[ \sum_{k > y} A_k \phi(k) \ll \sum_{k > y} \frac{a^k}{(k-1)!} < \frac{a^{y+1}}{y!} \sum_{j=0}^{\infty} \left( \frac{a}{y} \right)^j \ll a \frac{a^y}{y!} \ll a \frac{1}{x^{1/2}} \quad \text{(10)} \]
and
\[ a^y y \ll_{a, \varepsilon} x^\varepsilon, \tag{11} \]
for all \( x \geq 1 \). Therefore, putting together (3), (10) and (11), we have
\[
\sum_{n \leq x} \phi(f(n)) = \sum_{k \leq y} N_k(x)\phi(k) = \sum_{k \leq y} (A_k\phi(k)x + O_\varepsilon(\phi(k)x^{1/2+\varepsilon}))
\]
\[
= M_{\phi}x + O\left(\sum_{k > y} A_k\phi(k)x\right) + O_\varepsilon(a^y yx^{1/2+\varepsilon})
\]
\[
= M_{\phi}x + O_{a, \varepsilon}(x^{1/2+\varepsilon}),
\]
for all \( x \geq 1 \) and \( \varepsilon > 0 \). The proof is complete.

4. Proof of Theorem 1.2

Recall that \( A_k \) is defined by (7). For \( k = 1 \), the claim is obvious, since \( f(\ell) = 0 \) if and only if \( \ell = 1 \). Hence, assume \( k \geq 2 \). If \( \ell \) is a powerful number such that \( f(\ell) = k - 1 \), then \( \ell \) can be written in a unique way as \( \ell = m_1 \cdots m_{k-1} \), where \( 1 < a_1 < \cdots < a_{k-1} \) are integers and \( m_1, \ldots, m_{k-1} > 1 \) are pairwise coprime squarefree numbers. Therefore, from (7) and Lemma 2.6, we obtain
\[
\frac{\pi^2}{6} A_k = \sum_{m_1, \ldots, m_{k-1}} \sum_{1 < a_1 < \cdots < a_{k-1}} \frac{1}{\psi(m_1 \cdots m_{k-1})}
\]
\[
= \sum_{m_1, \ldots, m_{k-1}} \frac{1}{\psi(m_1 \cdots m_{k-1})} \sum_{1 < a_1 < \cdots < a_{k-1}} \sum_{m_1 \cdots m_{k-1}} \frac{1}{m_1 \cdots m_{k-1}}
\]
\[
= \sum_{m_1, \ldots, m_{k-1}} \frac{1}{\psi(m_1 \cdots m_{k-1})} \prod_{j=1}^{k-1} \frac{1}{m_j \cdots m_{k-1} - 1},
\]
where, here and in the rest of the proof, in summation subscripts \( m_1, \ldots, m_{k-1} \) are meant to be pairwise coprime, squarefree and greater than 1. At this point, it is enough to prove that
\[
\sum_{n = m_1 \cdots m_{k-1}} \prod_{j=1}^{k-1} \frac{1}{m_j \cdots m_{k-1} - 1} = \rho_k(n)
\]
for all squarefree numbers \( n > 1 \). We proceed by induction on \( k \). For \( k = 2 \), the claim is true since
\[
\frac{1}{n-1} = \rho_1(1) = \frac{1}{n-1} \sum_{d \mid n} \rho_1(d) = \rho_2(n),
\]
for all squarefree numbers \( n > 1 \). Assuming that the claim is true for \( k \), we shall prove it for \( k + 1 \). We have
\[
\sum_{n = m_1 \cdots m_{k+1}} \prod_{j=1}^{k+1} \frac{1}{m_j \cdots m_{k+1} - 1} = \rho_{k+1}(n)
\]
for all squarefree numbers \( n > 1 \).
\[
\sum_{n=m_1 \cdots m_k} \prod_{j=1}^k \frac{1}{m_j \cdots m_k - 1} = \frac{1}{n-1} \sum_{m_1 | n} \sum_{m_2 \cdots m_k = m_1} \prod_{j=2}^k \frac{1}{m_j \cdots m_k - 1}
= \frac{1}{n-1} \sum_{m_1 | n} \rho_k(n/m_1)
= \frac{1}{n-1} \sum_{d | n, d < n} \rho_k(d) = \rho_{k+1}(n),
\]
for all squarefree numbers \(n > 1\), as desired. The proof is complete.

5. Proof of Theorem 1.4

We have to count the number of positive integers \(n \leq x\) such that \(g(n) = k\). As in the proof of Theorem 1.1, every \(n\) can be written in a unique way as \(n = m \ell\), where \(m\) is a squarefree number, \(\ell\) is a powerful number, and \((m, \ell) = 1\). If \(m = 1\), then \(n = \ell\) is powerful and by Lemma 2.1, belongs to a set of cardinality \(O(x^{1/2})\). If \(m > 1\), then
\[
\omega(m) = \omega(n) - \omega(\ell) = g(n) + f(n) - f(\ell) - g(\ell) = k + 1 - g(\ell).
\]
In particular, \(1 \leq \omega(m) \leq k + 1\). Assume \(x\) sufficiently large, and put \(y := (\log x)^2\). Then, by Lemma 2.2, the number of \(n \leq x\) such that \(\ell > y\) is at most
\[
\sum_{\ell \in \mathcal{P}, \ell > y} \frac{x}{\ell} \ll \frac{x}{y^{1/2}} = \frac{x}{\log x}.
\]
Therefore,
\[
M_k(x) := \#\{n \leq x : g(n) = k\} = \sum_{s=1}^{k+1} \sum_{\ell \in \mathcal{P}(y)} \sum_{g(\ell) = k+1-s} Q_s \left( \frac{x}{\ell}; \ell \right) + O \left( \frac{x}{\log x} \right).
\]

(12)

For each nonnegative integer \(r\), put
\[
B_r := \sum_{\ell \in \mathcal{P}, g(\ell) = r} \frac{1}{\ell}.
\]
Note that, in light of Lemma 2.2, the series defining \(B_r\) converges and, more precisely,
\[
\sum_{\ell \in \mathcal{P}(y), g(\ell) = r} \frac{1}{\ell} = B_r + O \left( \frac{1}{y^{1/2}} \right) = B_r + O \left( \frac{1}{\log x} \right).
\]

(13)

Clearly, we can assume \(x\) sufficiently large so that \(x/y \geq 3\) and \(y \leq x^{\delta/(1+\delta)}\), for some fixed \(0 < \delta < 1\). Hence, applying Lemma 2.5, we obtain
\[
Q_s \left( \frac{x}{\ell}; \ell \right) = \frac{x (\log \log (x/\ell))^{s-1}}{\ell (s-1)! \log (x/\ell)} \left( 1 + O_k \left( \frac{\log \log \log (\ell + 15)}{\log \log (x/\ell)} \right) \right)
\]
\[
\frac{x (\log \log x)^{s-1}}{\ell (s-1)! \log x} \left( 1 + O_k \left( \frac{\log \ell}{\log x} \right) \right) \left( 1 + O_k \left( \frac{\log \log (\ell + 15)}{\log \log x} \right) \right)
\]

for all positive integers \( s \leq k + 1 \) and \( \ell \leq y \). Consequently,

\[
\sum_{\ell \in P(y)} \sum_{g(\ell) = k+1-s} Q_s \left( \frac{x}{\ell}; \ell \right)
= \frac{x (\log \log x)^{s-1}}{(s-1)! \log x} \sum_{\ell \in P(y)} \frac{1}{\ell} \left( 1 + O_k \left( \frac{\log (\ell + 1)}{\log \log x} \right) \right)
= \frac{x (\log \log x)^{s-1}}{(s-1)! \log x} \left( B_{k+1-s} + O \left( \frac{1}{\log x} \right) + O_k \left( \frac{1}{\log \log x} \right) \right)

= \frac{x (\log \log x)^{s-1}}{(s-1)! \log x} \left( B_{k+1-s} + O_k \left( \frac{1}{\log \log x} \right) \right),
\]

where we used (13) and the fact that the series

\[
\sum_{\ell \in P} \frac{\log (\ell + 1)}{\ell}
\]

converges. Thus, putting together (12) and (14), and noting that \( B_0 = B \), we obtain

\[
M_k(x) = \frac{B x (\log \log x)^k}{k! \log x} \left( 1 + O_k \left( \frac{1}{\log \log x} \right) \right),
\]

as desired. The proof is complete.

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