The Even and the Odd Spectral Flows on the N=2 Superconformal Algebras

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ABSTRACT

There are two different spectral flows on the N=2 superconformal algebras (four in the case of the Topological algebra). The usual spectral flow, first considered by Schwimmer and Seiberg, is an even transformation, whereas the spectral flow previously considered by the author and Rosado is an odd transformation. We show that the even spectral flow is generated by the odd spectral flow, and therefore only the latter is fundamental. We also analyze thoroughly the four “topological” spectral flows, writing two of them here for the first time. Whereas the even and the odd spectral flows have quasi-mirrored properties acting on the Antiperiodic or the Periodic algebras, the topological even and odd spectral flows have drastically different properties acting on the Topological algebra. The other two topological spectral flows have mixed even and odd properties. We show that the even and the even-odd topological spectral flows are generated by the odd and the odd-even topological spectral flows, and therefore only the latter are fundamental.
1 Introduction

The N=2 Superconformal algebras provide the symmetries underlying the N=2 strings [1]. In addition, the topological version of the algebra is realized in the world-sheet of the bosonic string [2], as well as in the world-sheet of the superstrings [3]. The spectral flows, which are denoted as $U_\theta$ and $A_\theta$, transform the N=2 Superconformal algebras into isomorphic algebras for any value of $\theta$. In particular, for half-integer values of $\theta$ they interchange the Antiperiodic NS algebra and the Periodic R algebra. For specific values of this parameter they also transform primary states into primary states (and singular vectors into singular vectors, consequently). As we will show, the spectral flow $U_\theta$ (the usual spectral flow) is an even transformation while the spectral flow $A_\theta$ is odd.

The even spectral flow $U_\theta$ was written down in [1]. It has been considered in various papers, for example [4][5][6][7]. The topological twisting of this spectral flow was analyzed for the first time in [8], although only the applications to the chiral representations of the Topological algebra were taken into account. Some other applications of the “topological even spectral flow” can be seen in [9] and [10], although an appropriate analysis of the issue has never been done.

The odd spectral flow $A_\theta$ was written down in [8], where it was denoted as “the other” spectral flow. The topological twisting of this spectral flow has never been considered, although the “topological odd spectral flow” for $\theta = 1$, which is an automorphism of the Topological algebra denoted simply as $A$, has been applied in several occasions to map topological singular vectors into each other [9][10][11].

In this paper we intend to fill the existing gaps completing the whole picture. In section 2 we first review the main properties of the even and the odd spectral flows acting on the Antiperiodic NS and the Periodic R algebras, giving some new insights. Then we present the composition rules of the two spectral flows, showing that the even spectral flow $U_\theta$ is generated by the odd spectral flow $A_\theta$, which is therefore the only fundamental one. In section 3 we analyze thoroughly the topological twistings of the two spectral flows, giving rise to four different “topological” spectral flows, denoted as $U_\theta$, $A_\theta$, $\hat{U}_\theta$ and $\hat{A}_\theta$, two of which are written here for the first time. We point out the main properties of the four topological spectral flows and we write down their composition rules. We find that $U_\theta$ is even, $A_\theta$ is odd, $\hat{U}_\theta$ has even-odd properties, and $\hat{A}_\theta$ has odd-even properties. The odd and the odd-even spectral flows generate the even and the even-odd spectral flows, and therefore they are the only fundamental topological spectral flows. In section 4 we write the conclusions and final remarks.
2 Spectral Flows on the N=2 NS and R Algebras

The N=2 Superconformal algebra can be expressed as

\[
\begin{align*}
[L_m, L_n] &= (m-n)\delta_{m+n,0} + \frac{\zeta}{12}(m^3 - m), \\
[H_m, H_n] &= \frac{\zeta}{3}m\delta_{m+n,0}, \\
[L_m, G_r^\pm] &= \left(\frac{m}{2} - r\right)G_{m+r}^\pm, \\
[H_m, G_r^\pm] &= \pm G_{m+r}^\pm, \\
[L_m, H_n] &= -nH_{m+n}, \\
\{G_r^-, G_s^+\} &= 2L_{r+s} - (r-s)H_{r+s} + \frac{\zeta}{3}(r^2 - \frac{1}{4})\delta_{r+s,0},
\end{align*}
\]

(2.1)

where $L_m$ and $H_m$ are the spin-2 and spin-1 bosonic generators corresponding to the stress-energy momentum tensor and the U(1) current, respectively, and $G_r^+$ and $G_r^−$ are the spin-3/2 fermionic generators. These are half-integer moded for the case of the Antiperiodic NS algebra, and integer moded for the case of the Periodic R algebra.

In order to simplify the analysis that follows we will unify the notation for the $U(1)$ charge of the states of the Antiperiodic NS algebra and the states of the Periodic R algebra. Namely, the $U(1)$ charge of the Ramond states will be denoted by $h$, instead of $h \pm \frac{1}{2}$, like the $U(1)$ charge of the NS states. In addition, the relative charge $q$ of a secondary state will be defined as the difference between the $U(1)$ charge of the state and the $U(1)$ charge of the primary on which it is built. Therefore, the relative charges of the Ramond states are defined to be integer, like the relative charges of the NS states.

2.1 The Even and the Odd Spectral Flows

The spectral flows $U_\theta$ and $A_\theta$ are one-parameter families of transformations providing a continuum of isomorphic N=2 Superconformal algebras. The “usual” spectral flow $U_\theta$ is even, given by

\[
\begin{align*}
U_\theta L_m U_\theta^{-1} &= L_m + \theta H_m + \frac{\zeta}{6}\theta^2\delta_{m,0}, \\
U_\theta H_m U_\theta^{-1} &= H_m + \frac{\zeta}{3}\theta\delta_{m,0}, \\
U_\theta G_r^+ U_\theta^{-1} &= G_{r+\theta}^+, \\
U_\theta G_r^- U_\theta^{-1} &= G_{r-\theta}^-,
\end{align*}
\]

(2.2)

satisfying $U_\theta^{-1} = U_{(-\theta)}$. For $\theta = 0$ it is just the identity operator, i.e. $U_0 = 1$. It transforms the $(L_0, H_0)$ eigenvalues, i.e. the conformal weight and the $U(1)$ charge, $(\Delta, h)$ of a given state as $(\Delta - \theta h + \frac{\zeta}{6}\theta^2, h - \frac{\zeta}{3}\theta)$. From this one gets straightforwardly that the level $l$ of any secondary state changes to $l - \theta q$, while the relative charge $q$ remains equal.
The spectral flow $A_\theta$ is odd, given by

\begin{align*}
A_\theta L_m A_\theta^{-1} &= L_m + \theta H_m + \frac{\xi}{6} \theta^2 \delta_{m,0}, \\
A_\theta H_m A_\theta^{-1} &= -H_m - \frac{\xi}{3} \theta \delta_{m,0}, \\
A_\theta G^+_r A_\theta^{-1} &= G^-_{r-\theta}, \\
A_\theta G^-_r A_\theta^{-1} &= G^+_{r+\theta},
\end{align*}

(2.3)

satisfying $A_\theta^{-1} = A_\theta$. It is therefore an involution. $A_\theta$ is “quasi” mirror symmetric to $U_\theta$: under the exchange $H_m \rightarrow -H_m$, $G^+_r \leftrightarrow G^-_r$ and $\theta \rightarrow -\theta$. For $\theta = 0$ it is the mirror map, i.e. $A_0 = \mathcal{M}$. It transforms the $(L_0, H_0)$ eigenvalues of the states as $(\Delta + \theta h + \frac{\xi}{6} \theta^2, -h - \frac{\xi}{3} \theta)$. The level $l$ of the secondary states changes to $l + \theta q$, while the relative charge $q$ reverses its sign.

The table below summarizes the properties of $U_\theta$ and $A_\theta$ for general values of $\theta$. Observe that $U_\theta |\chi_i^{(q)}\rangle$ and $A_\theta |\chi_i^{(q)}\rangle$ are not mirror symmetric because of the sign of $\theta$.

|                   | conformal weight | $U(1)$ charge | level | relative charge |
|-------------------|-----------------|---------------|-------|-----------------|
| $U_\theta |\chi_i^{(q)}\rangle$ | $\Delta - \theta h + \frac{\xi}{6} \theta^2$ | $h - \frac{\xi}{3} \theta$ | $l - \theta q$ | $q$ |
| $A_\theta |\chi_i^{(q)}\rangle$ | $\Delta + \theta h + \frac{\xi}{6} \theta^2$ | $-h - \frac{\xi}{3} \theta$ | $l + \theta q$ | $-q$ |

### 2.2 Main Properties

Let us review the main properties of the spectral flows $U_\theta$ and $A_\theta$ for the most interesting values of $\theta$, those mapping primary states to primary states, adding some new insights. More details can be found in refs. [4], [6], [8] and [12].

For half-integer values of $\theta$ the two spectral flows interpolate between the Antiperiodic NS algebra and the Periodic R algebra. In particular, for $\theta = \pm 1/2$ the primary states of the NS algebra are transformed into primary states of the R algebra with helicities $(\mp)$ (i.e. annihilated by $G^+_0$ and $G^-_0$, respectively). As a result the NS singular vectors are transformed into R singular vectors with helicities $(\mp)$ built on R primaries with the same helicities. In addition, $U_{1/2}$ and $A_{-1/2}$ map the chiral NS primaries (annihilated by $G^+_{-1/2}$) to the set of R ground states (annihilated by both $G^+_0$ and $G^-_0$), whereas $U_{-1/2}$ and $A_{1/2}$ map the antichiral NS primaries (annihilated by $G^-_{-1/2}$) to the R ground states. As a consequence $U_{1/2}$ and $A_{-1/2}$ transform the NS singular vectors built on chiral primaries...
into helicity (±) R singular vectors built on R ground states, respectively, whereas $U_{-1/2}$ and $A_{1/2}$ transform the NS singular vectors built on antichiral primaries into helicity (±) R singular vectors built on R ground states, respectively.

For integer values of $\theta$ the NS algebra and the R algebra map back to themselves, although the states that were primary with respect to the initial algebra are in general not primary with respect to the final algebra.

In the case of the NS algebra only primary states that are chiral or antichiral can be mapped back into NS primary states, which turn out to be also chiral or antichiral. In addition, only $U_1$ and $A_0$ map chiral primaries into antichiral primaries, only $U_{-1}$ and $A_0$ map antichiral primaries into chiral primaries, only $A_{-1}$ and $U_0$ map chiral primaries into chiral primaries, and only $A_1$ and $U_0$ map antichiral primaries into antichiral primaries. An interesting consequence of this is that for no value of $\theta$, except $\theta = 0$, the spectral flows map singular vectors of the NS algebra back into NS singular vectors. In other words, the identity and the mirror map are the only spectral flows that transform NS singular vectors back into NS singular vectors. This is due to the fact that there are no chiral singular vectors built on chiral primaries neither antichiral singular vectors built on antichiral primaries. Observe that for no values of $\theta \neq 0$ does the odd spectral flow $A_0$ interpolate between the chiral ring and the antichiral ring, and inversely, for no values of $\theta \neq 0$ does the even spectral flow $U_0$ map the chiral and the antichiral rings back to themselves.

In the case of the R algebra only $U_{\pm 1}$ and $A_0$ transform primary states with helicity (±) into primary states with helicity (±), whereas only $A_{\pm 1}$ and $U_0$ transform primary states with helicity (±) back into helicity (±) primaries (observe that $A_{\pm 1}$ does not reverse the helicity as $U_{\pm 1}$ does). As a consequence, all the singular vectors of the R algebra with the same helicity as the primaries on which they are built (and only these) can be mapped back into R singular vectors using spectral flows with either $\theta = 1$ or $\theta = -1$, in addition to the identity $U_0$ and the mirror map $A_0$, which transform all kinds of R singular vectors into R singular vectors. Regarding the R ground states, for no value of $\theta \neq 0$ they are mapped back to R ground states. Under $U_{\pm 1}$ and $A_{\pm 1}$ they are transformed into helicity (±) primaries with the additional condition of being annihilated by $G_{\pm 1}^\pm$.

### 2.3 Composition Rules

The composition rules are the following. For the even spectral flow one has simply

$$U_{\theta_2} U_{\theta_1} = U_{(\theta_2 + \theta_1)}, \quad (2.4)$$
from which one obtains $U_0 = 1$ and $U_{\theta}^{-1} = U_{(-\theta)}$. For the odd spectral flow one finds

$$A_{\theta_2} A_{\theta_1} = U_{(\theta_2 - \theta_1)},$$

(2.5)

$$A_{\theta_2} U_{\theta_1} = A_{(\theta_2 - \theta_1)}, \quad U_{\theta_2} A_{\theta_1} = A_{(\theta_2 + \theta_1)},$$

(2.6)

from which one obtains $A_{\theta}^{-1} = A_{\theta}$, as well as the relations

$$U_{\theta} = A_{\theta} A_0 = A_0 A_{(-\theta)},$$

(2.7)

$$A_{\theta} = U_{\theta} A_0 = A_0 U_{(-\theta)}, \quad A_0 = A_{\theta} U_{\theta} = U_{\theta} A_{(-\theta)},$$

(2.8)

$A_0$ is the mirror map $H_m \leftrightarrow -H_m$, $G^+_{r} \leftrightarrow G^-_{r}$, as we pointed out before. We see that the odd spectral flow $A_{\theta}$ generates the even spectral flow $U_{\theta}$, and therefore it is the only fundamental spectral flow. Observe that $U_{\theta}$ and $A_{\theta}$ do not commute, and $U_{\theta}$ commutes with itself whereas $A_{\theta}$ does not. Notice also that $U_{\theta}^{-1} = U_{(-\theta)}$ while $A_{\theta}^{-1} = A_{\theta}$, i.e. the inverse of the even spectral flow with parameter $\theta$ is the one with parameter $(-\theta)$, while the odd spectral flow is its own inverse, i.e. an involution.

3 Spectral Flows on the N=2 Topological Algebra

The Topological N=2 Superconformal algebra reads [14]

$$[L_m, L_n] = (m - n)L_{m+n}, \quad [H_m, H_n] = \frac{\xi}{3}m\delta_{m+n,0},$$

$$[L_m, G_n] = (m - n)G_{m+n}, \quad [H_m, G_n] = G_{m+n},$$

$$[L_m, Q_n] = -nQ_{m+n}, \quad [H_m, Q_n] = -Q_{m+n}, \quad m, n \in \mathbb{Z}.$$  

(3.1)

where the fermionic generators $Q_m$ and $G_m$ correspond to the spin-1 BRST current and the spin-2 fermionic current, respectively. The Topological algebra (3.1) can be viewed as a rewriting of the algebra (2.1) using one of the two topological twists:

$$L_m^{(1)} = L_m + \frac{1}{2}(m + 1)H_m,$$

$$H_m^{(1)} = H_m,$$

$$G_m^{(1)} = G^+_{m+\frac{1}{2}}, \quad Q_m^{(1)} = G^-_{m-\frac{1}{2}},$$

(3.2)
and

\[ \begin{align*}
L_m^{(2)} &= L_m - \frac{1}{2}(m+1)H_m, \\
H_m^{(2)} &= -H_m, \\
G_m^{(2)} &= G_{m+\frac{1}{2}}, \\
Q_m^{(2)} &= G_{m-\frac{1}{2}},
\end{align*} \]

which we denote as \( T_{W1} \) and \( T_{W2} \), respectively. Observe that the two twists are mirrored. In particular \((G^{1/2}, G^{-1/2})\) results in \((G^{(1)}_0, Q^{(1)}_0)\), while \((G^{-1/2}, G^{+1/2})\) gives \((G^{(2)}_0, Q^{(2)}_0)\), so that the topological chiral primaries (annihilated by both \( Q_0 \) and \( G_0 \)) correspond to the antichiral and the chiral primaries of the NS algebra under the twists \( T_{W1} \) and \( T_{W2} \), respectively.

### 3.1 The Topological Spectral Flows

The “topological” spectral flows are obtained by twisting the spectral flows (2.2) and (2.3). There are two ways to proceed in each case: either using the same twist on the left and on the right-hand sides of expressions (2.2) and (2.3), or using different twists on the left and on the right. In the first case one obtains the even and the odd spectral flows

\[ \begin{align*}
U_\theta L_m^{(1)} U_{-\theta}^{-1} &= L_m^{(1)} + \theta H_m^{(1)} + \frac{\xi}{6}(\theta^2 + \theta)\delta_{m,0}, \\
U_\theta H_m^{(1)} U_{-\theta}^{-1} &= H_m^{(1)} + \frac{\xi}{3}\theta\delta_{m,0}, \\
U_\theta G_m^{(1)} U_{-\theta}^{-1} &= G_{m+\theta}, \\
U_\theta Q_m^{(1)} U_{-\theta}^{-1} &= Q_{m-\theta},
\end{align*} \]

with \( U_{-\theta}^{-1} = U_{(-\theta)} \), and

\[ \begin{align*}
A_\theta L_m^{(1)} A_{-\theta}^{-1} &= L_m^{(1)} + (\theta - m - 1)H_m^{(1)} + \frac{\xi}{6}(\theta^2 - \theta)\delta_{m,0}, \\
A_\theta H_m^{(1)} A_{-\theta}^{-1} &= -H_m^{(1)} - \frac{\xi}{3}\theta\delta_{m,0}, \\
A_\theta G_m^{(1)} A_{-\theta}^{-1} &= G_{m+1-\theta}, \\
A_\theta Q_m^{(1)} A_{-\theta}^{-1} &= Q_{m-1+\theta},
\end{align*} \]

with \( A_{-\theta}^{-1} = A_{-\theta} \). One obtains the same expressions for the generators with label \( (2) \) but with \( \theta \to -\theta \). These topological spectral flows satisfy the same composition rules as the even and the odd untwisted spectral flows (2.2) and (2.3). For this reason we denote them in the same way.
The topological even spectral flow $U_\theta$ looks (and behaves) almost identical as its untwisted partner \((2.2)\). It transforms the \((L_0, H_0)\) eigenvalues \((\Delta, h)\) of any given state as \((\Delta - \theta h + \frac{\xi}{6}(\theta^2 - \theta), h - \frac{\xi}{3}\theta)\). The level \(l\) of the secondary states changes to \(l - \theta q\), whereas the relative charge \(q\) remains unchanged.

The topological odd spectral flow $A_\theta$ looks and behaves quite differently from its untwisted partner \((2.3)\). It has never been considered in the literature before, although the specific transformation $A_1$ denoted simply as $A$, has been used in several papers \([3,10,12]\) to map singular vectors into each other. $A_\theta$ transforms the \((L_0, H_0)\) eigenvalues \((\Delta, h)\) of a given state as \((\Delta + ((\theta - 1)h + \frac{\xi}{2}(\theta^2 - \theta), -h - \frac{\xi}{3}\theta))\). The level \(l\) of the secondary states changes to \(l + (\theta - 1)q\), and the relative charge \(q\) reverses its sign.

By using different twists on the left and on the right-hand sides of the spectral flows \((2.2)\) and \((2.3)\) one obtains the expressions

\[
\begin{align*}
\hat{U}_\theta L_m^{(1)} & = L_m^{(2)} - (\theta + m + 1)H_m^{(2)} + \frac{\xi}{6}(\theta^2 + \theta)\delta_{m,0}, \\
\hat{U}_\theta H_m^{(1)} & = -H_m^{(2)} + \frac{\xi}{3}\theta\delta_{m,0}, \\
\hat{U}_\theta G_m^{(1)} & = G_m^{(2)}, \\
\hat{U}_\theta Q_m^{(1)} & = Q_m^{(2)},
\end{align*}
\]

and

\[
\begin{align*}
\hat{A}_\theta L_m^{(1)} & = L_m^{(2)} - \theta H_m^{(2)} + \frac{\xi}{6}(\theta^2 - \theta)\delta_{m,0}, \\
\hat{A}_\theta H_m^{(1)} & = H_m^{(2)} - \frac{\xi}{3}\theta\delta_{m,0}, \\
\hat{A}_\theta G_m^{(1)} & = G_m^{(2)}, \\
\hat{A}_\theta Q_m^{(1)} & = Q_m^{(2)}.
\end{align*}
\]

One exchanges the labels \((1) \leftrightarrow (2)\) in these expressions by setting \(\theta \rightarrow -\theta\) simultaneously. The spectral flows $\hat{U}_\theta$ and $\hat{A}_\theta$, which we denote as even-odd and odd-even respectively, connect the generators of the topological theory \((1)\) to the generators of the topological theory \((2)\), having mixed even and odd properties. Their inverses, which connect the generators of the topological theory \((2)\) to the generators of the topological theory \((1)\), are given by $\hat{U}_\theta^{-1} = \hat{U}_{(-\theta)}$ and $\hat{A}_\theta^{-1} = \hat{A}_\theta$, i.e. the same expressions as for the even and the odd spectral flows, respectively.

The topological even-odd spectral flow $\hat{U}_\theta$ looks and behaves very similarly as the topological odd spectral flow $A_\theta$ \((3.5)\), in spite of not being an involution. It was written down in ref. [9], although only the values $\theta = \pm 1$ were taken into account\[^\text{1}\]. It transforms

\[^{1}\text{In ref. [9] only topological chiral primaries were considered. They are mapped into each other by $\hat{U}_\theta$ only for $\theta = \pm 1$.}\]
the \((\mathcal{L}_0^{(1)}, \mathcal{H}_0^{(1)})\) eigenvalues \((\Delta^{(1)}, h^{(1)})\) of the states of the topological theory \(1\) into \((\mathcal{L}_0^{(2)}, \mathcal{H}_0^{(2)})\) eigenvalues of the states of the topological theory \(2\) as \((\Delta^{(2)}, h^{(2)}) = (\Delta^{(1)} - (\theta + 1)h^{(1)} + \frac{\xi}{6}(\theta^2 + \theta), -h^{(1)} + \frac{\xi}{3}\theta)\). It modifies the level \(l\) of the secondary states as \(l^{(2)} = l^{(1)} - (\theta + 1)q^{(1)}\), reversing the sign of the relative charge, i.e. \(q^{(2)} = -q^{(1)}\).

The topological odd-even spectral flow \(\hat{A}_\theta\) looks and behaves very similarly as the even spectral flows \(U_\theta\), in spite of being an involution. It has never been considered in the literature before. It transforms the \((\mathcal{L}_0^{(1)}, \mathcal{H}_0^{(1)})\) eigenvalues of the states of the topological theory \(1\) into \((\mathcal{L}_0^{(2)}, \mathcal{H}_0^{(2)})\) eigenvalues as \((\Delta^{(2)}, h^{(2)}) = (\Delta^{(1)} + \theta h^{(1)} + \frac{\xi}{6}(\theta^2 + \theta), h^{(1)} + \frac{\xi}{3}\theta)\). It modifies the level \(l\) of the secondary states as \(l^{(2)} = l^{(1)} + \theta q^{(1)}\), letting the relative charge invariant, i.e. \(q^{(2)} = q^{(1)}\).

The table below summarizes the properties of the topological spectral flows \(U_\theta, A_\theta, \hat{U}_\theta\) and \(\hat{A}_\theta\), for general values of \(\theta\), acting on the states of the topological theory \(1\). One finds the same table for the spectral flows acting on the states of the topological theory \(2\) but with \(\theta \rightarrow -\theta\).

| \(|\chi^{(1)}\rangle_{l}^{(q)}\) | conformal weight | U(1) charge | level | relative charge |
|---------------------------------|-----------------|-------------|-------|-----------------|
| \(U_\theta \mid \chi^{(1)}\rangle_{l}^{(q)}\) | \(\Delta - \theta h + \frac{\xi}{6}(\theta^2 - \theta)\) | \(h - \frac{\xi}{3}\theta\) | \(l - \theta q\) | \(q\) |
| \(A_\theta \mid \chi^{(1)}\rangle_{l}^{(q)}\) | \(\Delta + (\theta - 1)h + \frac{\xi}{6}(\theta^2 - \theta)\) | \(-h + \frac{\xi}{3}\theta\) | \(l + (\theta - 1)q\) | \(-q\) |
| \(\hat{U}_\theta \mid \chi^{(1)}\rangle_{l}^{(q)}\) | \(\Delta - (\theta + 1)h + \frac{\xi}{6}(\theta^2 + \theta)\) | \(-h + \frac{\xi}{3}\theta\) | \(l - (\theta + 1)q\) | \(-q\) |
| \(\hat{A}_\theta \mid \chi^{(1)}\rangle_{l}^{(q)}\) | \(\Delta + \theta h + \frac{\xi}{6}(\theta^2 + \theta)\) | \(h + \frac{\xi}{3}\theta\) | \(l + \theta q\) | \(q\) |

### 3.2 Main Properties

Now let us discuss the main properties of the topological spectral flows \(U_\theta, A_\theta, \hat{U}_\theta\) and \(\hat{A}_\theta\) for the most interesting values of \(\theta\), those mapping primary states to primary states.

*The Even Spectral Flow \(U_\theta\)*

The action of the topological even spectral flow \(U_\theta\) (3.4) on the states of the Topological algebra is very similar to the action of its untwisted partner \(U_\theta\) (2.2) on the states of the Periodic R algebra. Apart from the identity \(U_0\), only \(\mathcal{U}_{\pm 1}\) map primary states
into primary states, under restricted conditions though. Under $\mathcal{U}_1$ only $\mathcal{G}_0$-closed primaries (i.e. annihilated by $\mathcal{G}_0$) are transformed back into primaries. These turn out to be $\mathcal{Q}_0$-closed (i.e. annihilated by $\mathcal{Q}_0$ and therefore BRST-invariant). Inversely, under $\mathcal{U}_{-1}$ only $\mathcal{Q}_0$-closed primaries are transformed into primaries, which turn out to be $\mathcal{G}_0$-closed. As a consequence, under $\mathcal{U}_1$ only $\mathcal{G}_0$-closed singular vectors built on $\mathcal{G}_0$-closed primaries are transformed into singular vectors as well. These turn out to be $\mathcal{Q}_0$-closed built on $\mathcal{Q}_0$-closed primaries. The inverse situation occurs under $\mathcal{U}_{-1}$.

For no value of $\theta \neq 0$ do the topological chiral primaries (i.e. annihilated by $\mathcal{G}_0$ and $\mathcal{Q}_0$) transform back into topological chiral primaries. Under $\mathcal{U}_1$ they transform into $\mathcal{Q}_0$-closed primaries with the additional constraint of being annihilated by $\mathcal{Q}_{-1}$, whereas under $\mathcal{U}_{-1}$ they transform into $\mathcal{G}_0$-closed primaries with the additional constraint of being annihilated by $\mathcal{G}_{-1}$. Hence the topological even spectral flow $\mathcal{U}_\theta$ “destroys” topological chiral Verma modules. Chiral singular vectors in turn, built on $\mathcal{G}_0$-closed or $\mathcal{Q}_0$-closed primaries (they do not exist on chiral primaries) transform under $\mathcal{U}_1$ or $\mathcal{U}_{-1}$, respectively, into non-chiral singular vectors annihilated by $\mathcal{Q}_{-1}$ or $\mathcal{G}_{-1}$, as a result.

The Odd Spectral Flow $\mathcal{A}_\theta$

The action of the topological odd spectral flow $\mathcal{A}_\theta$ (3.5) on the states of the Topological algebra is drastically different from the action of the topological even spectral flow $\mathcal{U}_\theta$ and also drastically different from the action of the untwisted odd spectral flow $\mathcal{A}_\theta$ (2.3) on the states of the Periodic R algebra and the states of the Antiperiodic NS algebra. The main difference consists of the existence of a value of $\theta$, namely $\theta = 1$, for which the topological odd spectral flow becomes a “universal” transformation, in the sense that all kinds of primary states are mapped back into primary states. In addition, the Topological algebra automorphism $\mathcal{A}$, as $\mathcal{A}_1$ is denoted [9] [10] [12], transforms topological chiral primaries into topological chiral primaries. As a consequence, $\mathcal{A}$ transforms all kinds of singular vectors into singular vectors, and singular vectors in topological chiral Verma modules back to singular vectors in topological chiral Verma modules.

Let us say a few more words about the Topological algebra automorphism $\mathcal{A}$, because of its importance. It reads [9]

$$
\begin{align*}
\mathcal{A} \mathcal{L}_m \mathcal{A} &= \mathcal{L}_m - m \mathcal{H}_m , \\
\mathcal{A} \mathcal{H}_m \mathcal{A} &= - \mathcal{H}_m - \frac{5}{3} \delta_{m,0} , \\
\mathcal{A} \mathcal{G}_m \mathcal{A} &= \mathcal{Q}_m , \\
\mathcal{A} \mathcal{Q}_m \mathcal{A} &= \mathcal{G}_m .
\end{align*}
$$

†Curiously, this spectral flow has been considered in ref. [11] acting on topological chiral Verma modules.
It transforms the $(L_0, H_0)$ eigenvalues $(\Delta, h)$ of the states as $(\Delta, -h - \xi)$. Hence it does not modify the conformal weight. As a result the level of the secondary states remains unchanged whereas the relative charge is reversed $(q \rightarrow -q)$. In addition $A$ also reverses the BRST-invariance properties of the states: $G_0$-closed states are mapped to $Q_0$-closed states, and the other way around, and chiral states are mapped to chiral states, consequently.

There are two other values of $\theta$ for which $A_\theta$ maps primary states into primary states although with restrictions: $\theta = 0$ and $\theta = 2$. $A_0$ transforms only $G_0$-closed primary states back into primary states, which are $G_0$-closed as well, mapping chiral primary states to non-chiral $G_0$-closed primaries annihilated by $G_{-1}$. The complementary mapping is performed by $A_2$; that is, it transforms only $Q_0$-closed primary states into primary states, which are $Q_0$-closed as well, mapping chiral primaries to non-chiral $Q_0$-closed primaries annihilated by $Q_{-1}$. As a consequence only $G_0$-closed ($Q_0$-closed) singular vectors built on $G_0$-closed ($Q_0$-closed) primaries are transformed by $A_0$ ($A_2$) back into singular vectors. These turn out to be again $G_0$-closed ($Q_0$-closed) singular vectors built on $G_0$-closed ($Q_0$-closed) primaries. Chiral singular vectors are transformed in the same manner, with the additional constraint of being annihilated by $G_{-1}$ ($Q_{-1}$).

Observe the striking differences between the topological $A_0$ and the untwisted $A_0$, which is the mirror map transforming all the primaries and singular vectors of the NS algebra and the R algebra into mirrored primaries and mirrored singular vectors, and the chiral and the antichiral primaries into each other.

*The Even-Odd Spectral Flow $\hat{U}_\theta$*

The topological spectral flow $\hat{U}_\theta$ (3.6) is very similar to the odd spectral flow $A_\theta$ (3.3), although it is not an involution and satisfy the same composition rules as the even spectral flows. For this reason we denote it as “even-odd”. For $\theta = 0$ it is the “identity” which gives the exact relation between the generators of the topological theory (1) and the generators of the topological theory (2), as deduced from the twists (3.2) and (3.3). However, $\hat{U}_0$ transforms only the $G_0$-closed primary states of one theory into primary states of the other theory, which are also $G_0$-closed, in complete analogy with the action of $A_0$. The complementary transformations mapping $Q_0$-closed primary states into $Q_0$-closed primary states are given by $\hat{U}_{\pm 2}$ ($\theta = 2$ for the mapping from the states of theory (2) to the states of theory (1), and the other way around for $\theta = -2$). As a consequence, only $G_0$-closed singular vectors built on $G_0$-closed primaries are transformed into singular vectors under $\hat{U}_0$, which are also $G_0$-closed built on $G_0$-closed primaries. Similarly, only $\hat{U}_0$ has been considered in ref. [13] where it was denoted as the topological mirror map. However the authors treated $A_0$ as an isolated transformation, not as a particular case of a family of transformations, i.e. of a spectral flow.
$Q_0$-closed singular vectors built on $Q_0$-closed primaries are transformed under $\hat{U}_{\pm 2}$ into singular vectors, which are also $Q_0$-closed built on $Q_0$-closed primaries. Chiral primaries, and chiral singular vectors built on $G_0$-closed or $Q_0$-closed primaries, are transformed under $\hat{U}_0$ and $\hat{U}_{\pm 2}$ into non-chiral $G_0$-closed and $Q_0$-closed primaries, and singular vectors, annihilated by $G_{-1}$ and $Q_{-1}$, respectively.

In turn, $\hat{U}_{\pm 1}$ are the “universal” mappings which transform all kinds of primary states of one theory into primary states of the other theory, and chiral primary states into chiral primary states ($\theta = 1$ for the mapping from the states of theory (2) to the states of theory (1), and the other way around for $\theta = -1$). As a result all the singular vectors of each theory are mapped into singular vectors of the other theory. In particular singular vectors in chiral Verma modules of one theory are transformed into singular vectors in chiral Verma modules of the other theory. Let us write $\hat{U}_1$ explicitly, because of its importance:

$$
\begin{align*}
\hat{U}_1 L_m^{(2)} \hat{U}_1^{-1} &= L_m^{(1)} - m H_m^{(1)}, \\
\hat{U}_1 H_m^{(2)} \hat{U}_1^{-1} &= -H_m^{(1)} - \frac{c}{2} \delta_{m,0}, \\
\hat{U}_1 G_m^{(2)} \hat{U}_1^{-1} &= Q_m^{(1)}, \\
\hat{U}_1 Q_m^{(2)} \hat{U}_1^{-1} &= G_m^{(1)}. 
\end{align*}
$$

(3.9)

We see that $\hat{U}_1$, with the generators of the topological theory (2) on the left-hand side, looks exactly like the Topological algebra automorphism $\mathcal{A}$ (3.8).

The Odd-Even Spectral Flow $\hat{A}_\theta$

The topological spectral flow $\hat{A}_\theta$ (3.7) is very similar to the even spectral flow $U_\theta$ (3.4), although it is an involution and satisfy the same composition rules as the odd spectral flows. For this reason we denote it as “odd-even”. For $\theta = 0$ it just produces the interchange of labels (1) $\leftrightarrow$ (2); that is, the interchange between the two topological theories. Therefore $\hat{A}_0$ maps primary states and singular vectors between the two topological theories in a trivial way. For $\theta = \pm 1$ it transforms primaries of one theory into primaries of the other theory under restricted conditions: $\hat{A}_1$ maps $Q_0^{(1)}$-closed primary primary states of theory (1) into $G_0^{(2)}$-closed primary states of theory (2), and the other way around, whereas $\hat{A}_{-1}$ maps $G_0^{(1)}$-closed primary primary states of theory (1) into $Q_0^{(2)}$-closed primary states of theory (2), and the other way around. As a consequence, $Q_0$-closed singular vectors built on $Q_0$-closed primaries and $G_0$-closed singular vectors built on $G_0$-closed primaries are mapped into each other, between the two topological theories, using either $\hat{A}_1$ or $\hat{A}_{-1}$.

$^\dagger$As a matter of fact, the Topological algebra automorphism $\mathcal{A}$ was found in ref. [9] just by erasing the labels (1) and (2) in expression (3.3).
For no value of $\theta \neq 0$ the chiral primaries are transformed back into chiral primaries. Therefore $\hat{A}_\theta$ “destroys” chiral Verma modules, like the even spectral flow $U_\theta$ \[^{[3,4]}\]. Under $\hat{A}_1$ and $\hat{A}_{-1}$ the chiral primaries are mapped to either $G_0$-closed or $Q_0$-closed primaries annihilated by either $G_{-1}$ or $Q_{-1}$, respectively, depending on the specific transformation. Therefore chiral singular vectors built on $G_0$-closed or $Q_0$-closed primaries are transformed into singular vectors annihilated either by $G_{-1}$ or by $Q_{-1}$.

### 3.3 Composition Rules

The topological spectral flows $U_\theta$ and $\hat{U}_\theta$, on the one hand, and $A_\theta$ and $\hat{A}_\theta$ on the other hand, satisfy the same composition rules as their untwisted partners $U_\theta$ and $A_\theta$, taking into account a “hat number” conservation (mod 2), in spite of the striking differences between the untwisted and the topological spectral flows.

For the even and the even-odd spectral flows, separately, the rules are

$$U_{\theta_2} U_{\theta_1} = \hat{U}_{\theta_2} \hat{U}_{\theta_1} = U_{(\theta_2 + \theta_1)}, \quad \hat{U}_{\theta_2} U_{\theta_1} = U_{\theta_2} \hat{U}_{\theta_1} = \hat{U}_{(\theta_2 + \theta_1)},$$

\hspace{1cm} (3.10)

from which one obtains $U_0 = 1$, $U_{\theta}^{-1} = U_{(-\theta)}$, $\hat{U}_{\theta}^{-1} = \hat{U}_{(-\theta)}$, and the relations

$$U_\theta = \hat{U}_\theta \hat{U}_0 = \hat{U}_0 \hat{U}_\theta, \quad \hat{U}_\theta = U_\theta \hat{U}_0 = \hat{U}_0 U_\theta,$$

\hspace{1cm} (3.11)

$$\hat{U}_0 = \hat{U}_\theta U_{(-\theta)} = U_\theta \hat{U}_{(-\theta)}.$$\hspace{1cm} (3.12)

For the odd and the odd-even spectral flows the rules are

$$A_{\theta_2} A_{\theta_1} = \hat{A}_{\theta_2} \hat{A}_{\theta_1} = U_{(\theta_2 - \theta_1)}, \quad \hat{A}_{\theta_2} A_{\theta_1} = A_{\theta_2} \hat{A}_{\theta_1} = \hat{U}_{(\theta_2 - \theta_1)},$$

\hspace{1cm} (3.13)

$$A_{\theta_2} U_{\theta_1} = \hat{A}_{\theta_2} \hat{U}_{\theta_1} = A_{(\theta_2 - \theta_1)}, \quad \hat{A}_{\theta_2} U_{\theta_1} = A_{\theta_2} \hat{U}_{\theta_1} = \hat{A}_{(\theta_2 - \theta_1)},$$

\hspace{1cm} (3.14)

$$U_{\theta_2} A_{\theta_1} = \hat{U}_{\theta_2} \hat{A}_{\theta_1} = A_{(\theta_2 + \theta_1)}, \quad \hat{U}_{\theta_2} A_{\theta_1} = U_{\theta_2} \hat{A}_{\theta_1} = \hat{A}_{(\theta_2 + \theta_1)},$$

\hspace{1cm} (3.15)

from which one obtains $A_{\theta}^{-1} = A_{\theta}$, $\hat{A}_{\theta}^{-1} = \hat{A}_{\theta}$, as well as the relations

12
\[ U_\theta = \hat{A}_\theta \ A_0 = \hat{A}_\theta \ \hat{A}_0 = A_0 \ \mathcal{A}_{(-\theta)} = \hat{A}_0 \ \hat{A}_{(-\theta)}, \]  
(3.16)

\[ \hat{U}_\theta = \hat{A}_\theta \ A_0 = \hat{A}_\theta \ A_0 = \hat{A}_0 \ \mathcal{A}_{(-\theta)} = \hat{A}_0 \ \hat{A}_{(-\theta)}, \]  
(3.17)

\[ \hat{U}_0 = \hat{A}_\theta \ A_\theta = \mathcal{A}_\theta \ \hat{A}_\theta, \]  
(3.18)

\[ \mathcal{A}_\theta = U_\theta \ A_0 = \hat{U}_\theta \ \hat{A}_0 = \hat{A}_0 \ U_{(-\theta)} = \hat{A}_0 \ \hat{U}_{(-\theta)} = \hat{A}_\theta \ \hat{U}_0 = \hat{U}_0 \ \hat{A}_\theta, \]  
(3.19)

\[ \hat{A}_\theta = \hat{U}_\theta \ A_0 = \hat{U}_0 \ \hat{A}_0 = \hat{A}_0 \ U_{(-\theta)} = \hat{A}_0 \ \hat{U}_{(-\theta)} = \hat{A}_\theta \ \hat{U}_0 = \hat{U}_0 \ A_\theta, \]  
(3.20)

\[ \mathcal{A}_0 = \mathcal{A}_\theta \ U_\theta = \hat{A}_\theta \ \hat{U}_\theta = \hat{U}_\theta \ \hat{A}_\theta = \hat{U}_0 \ \hat{A}_{(-\theta)}, \]  
(3.21)

\[ \hat{A}_0 = \hat{A}_\theta \ U_\theta = \hat{A}_\theta \ \hat{U}_\theta = \hat{U}_\theta \ \hat{A}_\theta = \hat{U}_0 \ \hat{A}_{(-\theta)} \]  
(3.22)

We see that the odd and the odd-even spectral flows \( \mathcal{A}_\theta \) and \( \hat{A}_\theta \) generate the even and the even-odd spectral flows \( U_\theta \) and \( \hat{U}_\theta \), and therefore they are the only fundamental topological spectral flows. Observe that \( U_\theta \) and \( \hat{U}_\theta \) commute with each other, as well as with themselves, whereas \( \mathcal{A}_\theta \) and \( \hat{A}_\theta \) do not commute with each other, neither with \( U_\theta \) and \( \hat{U}_\theta \), nor with themselves.

### 4 Conclusions and Final Remarks

In this paper we have analyzed in much detail the spectral flows on the N=2 Superconformal algebras. For the Antiperiodic NS algebra and the Periodic R algebra there are two spectral flows: the “usual” \( U_\theta \), written down by Schwimmer and Seiberg [4], and the spectral flow \( \mathcal{A}_\theta \), written down by the author and Rosado [8], which is an involution quasi-mirror symmetric to \( U_\theta \). We have shown that the spectral flow \( \mathcal{A}_\theta \) is odd and generates the spectral flow \( U_\theta \), which is even. Therefore only \( \mathcal{A}_\theta \) is a fundamental spectral flow.

For the twisted Topological algebra we have found four different spectral flows. Two of them act inside the same topological theory: we denote them also as \( U_\theta \) and \( \mathcal{A}_\theta \).
because they satisfy the same composition rules as their untwisted partners. The other two spectral flows interpolate between the two topological theories corresponding to the two twistings of the Antiperiodic NS algebra: we denote them as $U_\theta$ and $A_\theta$. They satisfy the same composition rules as $U_\theta$ and $A_\theta$, up to a “hat number” conservation (mod 2), and both have mixed even and odd properties.

The even and the even-odd topological spectral flows $U_\theta$ and $\hat{U}_\theta$ have been considered in the literature before, but not analyzed properly. The odd and the odd-even topological spectral flows $A_\theta$ and $\hat{A}_\theta$, which are involutions, have never appeared in the literature before. They generate the even and the even-odd topological spectral flows, and therefore they are the only fundamental topological spectral flows.

We have analyzed the main properties of all the spectral flows. We have discussed the properties for general values of $\theta$, as well as the properties for the most interesting values of $\theta$, for which primary states are mapped to primary states (and singular vectors to singular vectors). Whereas the even and the odd spectral flows $U_\theta$ and $A_\theta$ have quasi-mirrored properties acting on the states of the NS algebra and the states of the R algebra, the even and the odd topological spectral flows $U_\theta$ and $A_\theta$ have drastically different properties acting on the states of the Topological algebra. The even-odd and the odd-even topological spectral flows $\hat{U}_\theta$ and $\hat{A}_\theta$ are very similar to the topological odd and even spectral flows $A_\theta$ and $U_\theta$, respectively.

Finally, we would like to emphasize that the topological spectral flow transformations $\hat{U}_{\pm 1}$ and $\hat{A}_1$, denoted simply as $A$, are “universal” in the sense that they transform all kinds of topological primary states and topological singular vectors back to primary states and singular vectors, mapping chiral primaries to chiral primaries. All other spectral flow transformations, except the trivial ones and the mirror map, either do not map primary states to primary states (the most general situation), or they do it under restricted conditions. For example, the topological spectral flows $U_\theta$ and $\hat{A}_\theta$ do not map topological chiral primaries back to chiral primaries for any values of $\theta \neq 0$ (they “destroy” topological chiral Verma modules). Observe that there are no universal transformations for the untwisted spectral flows, but only for the topological spectral flows $\hat{U}_\theta$ and $\hat{A}_\theta$. The case of the NS algebra is the most restricted one: only primary states that are chiral or antichiral can be mapped back to primary states (using different spectral flow transformations in each case). As a result no NS singular vectors can be mapped back to NS singular vectors, other than the mirrored ones, using the spectral flows, since there are no chiral NS singular vectors built on chiral primaries, neither antichiral NS singular vectors built on antichiral primaries.

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