Existence and Blowup Behavior of Global Strong Solutions to the Two-Dimensional Baratropic Compressible Navier-Stokes System with Vacuum and Large Initial Data

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Abstract

For periodic initial data with initial density allowed to vanish, we establish the global existence of strong and weak solutions for the two-dimensional compressible Navier-Stokes equations with no restrictions on the size of initial data provided the shear viscosity is a positive constant and the bulk one is $\lambda = \rho^\beta$ with $\beta > 4/3$. These results generalize and improve the previous ones due to Vaigant-Kazhikhov ([Sib. Math. J. (1995), 36(6), 1283-1316]) which requires $\beta > 3$. Moreover, both the time-independent upper bound of the density and the large-time behavior of the strong and weak solutions are also obtained.

Keywords: compressible Navier-Stokes equations; global strong solutions; large initial data; vacuum states.

1 Introduction and main results

We study the two-dimensional barotropic compressible Navier-Stokes equations which read as follows:

\[
\begin{cases}
\rho_t + \text{div}(\rho u) = 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P = \mu \triangle u + \nabla((\mu + \lambda) \text{div} u),
\end{cases}
\]

(1.1)

where $\rho = \rho(x, t)$ and $u = (u_1(x, t), u_2(x, t))$ represent the density and velocity respectively, and the pressure $P$ is given by

\[ P(\rho) = a\rho^\gamma, \quad \gamma > 1. \]

(1.2)

We also have the following hypothesis on the shear viscosity $\mu$ and the bulk one $\lambda$:

\[ \mu = \text{const}, \quad \lambda(\rho) = b \rho^\beta, \quad b > 0, \quad \beta > 0. \]

(1.3)

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In the sequel, we set $a = b = 1$ without loosing any generality.

We consider the Cauchy problem with the given initial data $\rho_0$ and $m_0$, which are periodic with period 1 in each space direction $x_i, i = 1, 2$, i.e., functions defined on $T^2 = \mathbb{R}^2 / \mathbb{Z}^2$. We require that

$$\rho(x, 0) = \rho_0(x), \quad \rho u(x, 0) = m_0(x), \quad x \in T^2. \quad (1.4)$$

There is a huge literature concerning the theory of strong and weak solutions for the system of the multidimensional compressible Navier-Stokes equations with constant viscosity coefficients. The local existence and uniqueness of classical solutions are known in [25, 28] in the absence of vacuum and recently, for strong solutions also, in [3, 4, 27] for the case that the initial density need not be positive and may vanish in open sets. The global classical solutions were first obtained by Matsumura-Nishida [24] for initial data close to a non-vacuum equilibrium in some Sobolev space $H^s$. Later, Hoff [13] studied the problem for discontinuous initial data. For the existence of solutions for large data, the major breakthrough is due to Lions [23] (see also Feireisl [10, 11]), where he obtained global existence of weak solutions, defined as solutions with finite energy, when the exponent $\gamma$ is suitably large. The main restriction on initial data is that the initial energy is finite, so that the density is allowed to vanish initially. Recently, Huang-Li-Xin [17] established the global existence and uniqueness of classical solutions to the Cauchy problem for the isentropic compressible Navier-Stokes equations in three-dimensional space with smooth initial data which are of small energy but possibly large oscillations; in particular, the initial density is allowed to vanish, even has compact support. The compatibility conditions on the initial data of [17] are further relaxed by [14, 21].

However, there are few results regarding global strong solvability for equations of multi-dimensional motions of viscous gas with no restrictions on the size of initial data. One of the first ever ones is due to Vaigant-Kazhikhov [30] who obtained a remarkable result which can be stated that the two-dimensional system (1.1)-(1.4) admits a unique global strong solution for large initial data away from vacuum provided $\beta > 3$. Recently, Perepelitsa [26] proved the global existence of a weak solution with uniform lower and upper bounds on the density, as well as the decay of the solution to an equilibrium state in a special case that

$$\beta > 3, \quad \gamma = \beta, \quad (1.5)$$

when the initial density is away from vacuum. Very recently, under some additional compatibility conditions on the initial data, Jiu-Wang-Xin [18] considered classical solutions and removed the condition that the initial density should be away from vacuum in Vaigant-Kazhikhov [30] but still under the same condition $\beta > 3$ as that in [30].

Before stating the main results, we explain the notations and conventions used throughout this paper. We denote

$$\int f dx = \int_{T^2} f dx, \quad \bar{f} = \frac{1}{|T^2|} \int f dx. \quad (1.6)$$

For $1 \leq r \leq \infty$, we also denote the standard Lebesgue and Sobolev spaces as follows:

$$L^r = L^r(T^2), \quad W^{s, r} = W^{s, r}(T^2), \quad H^s = W^{s, 2}.$$

Then, we give the definition of weak and strong solutions to (1.1).
**Definition 1.1** If \((\rho, u)\) satisfies (1.1) in the sense of distribution, then \((\rho, u)\) is called a weak solution to (1.1).

If, for a weak solution, all derivatives involved in (1.1) are regular distributions and equations (1.1) hold almost everywhere in \(\mathbb{T}^2 \times (0, T)\), then the solution is called strong.

Thus, the first main result concerning the global existence and large-time behavior of strong solutions can be stated as follows:

**Theorem 1.1** Assume that
\[
\beta > \frac{4}{3}, \quad \gamma > 1, \quad (1.7)
\]
and that the initial data \((\rho_0, m_0)\) satisfy that for some \(q > 2\),
\[
0 \leq \rho_0 \in W^{1, q}, \quad u_0 \in H^1, \quad m_0 = \rho_0 u_0. \quad (1.8)
\]
Then the problem (1.1)-(1.4) has a unique global strong solution \((\rho, u)\) satisfying
\[
\begin{align*}
\rho & \in C([0, T]; W^{1, q}), \quad \rho_t \in L^\infty(0, T; L^2), \\
u & \in L^\infty(0, T; H^1) \cap L^{(q+1)/q}(0, T; W^{2, q}), \\
\frac{t^{1/2}}{2} u & \in L^2(0, T; W^{2, q}), \quad \frac{t^{1/2}}{2} u_t \in L^2(0, T; H^1), \\
\rho u & \in C([0, T]; L^2), \quad \sqrt{\rho} u_t \in L^2(\mathbb{T}^2 \times (0, T)),
\end{align*}
\]
for any \(0 < T < \infty\). Moreover, if
\[
\beta > \frac{3}{2}, \quad 1 < \gamma < 3(\beta - 1), \quad (1.10)
\]
there exists a constant \(C\) independent of \(T\) such that
\[
\sup_{0 \leq t \leq T} \|\rho(\cdot, t)\|_{L^\infty} \leq C, \quad (1.11)
\]
and the following large-time behavior holds:
\[
\lim_{t \to \infty} \left( \|\rho - \bar{\rho}_0\|_{L^p} + \|\nabla u\|_{L^p} \right) = 0, \quad (1.12)
\]
for any \(p \in [1, \infty)\).

The second result gives the global existence and large-time behavior of weak solutions.

**Theorem 1.2** Assume that (1.7) holds and that the initial data \((\rho_0, m_0)\) satisfy that
\[
0 \leq \rho_0 \in L^\infty, \quad u_0 \in H^1, \quad m_0 = \rho_0 u_0. \quad (1.13)
\]
Then the problem (1.1)-(1.4) has at least one weak solution \((\rho, u)\) in \(\mathbb{T}^2 \times (0, T)\) for any \(T \in (0, \infty)\). Moreover, if \(\beta\) and \(\gamma\) satisfy (1.10), there exists a constant \(C\) independent of \(T\) such that both (1.11) and (1.12) hold true.

Finally, similar to Li-Xin [20], we can obtain from (1.12) the following large-time behavior of the gradient of the density for the strong solution obtained in Theorem 1.1 when vacuum states appear initially.
Theorem 1.3 Let $\beta, \gamma$ satisfy (1.10). In addition to (1.8), assume further that there exists some point $x_0 \in \mathbb{T}^2$ such that $\rho_0(x_0) = 0$. Then the unique global strong solution $(\rho, u)$ to the Cauchy problem (1.1)-(1.4) obtained in Theorem 1.1 has to blow up as $t \to \infty$, in the sense that for any $2 < r \leq q$ with $q$ as in Theorem 1.1

$$\lim_{t \to \infty} \|\nabla \rho(\cdot, t)\|_{L^r} = \infty.$$ 

A few remarks are in order:

Remark 1.1 Theorems 1.1 and 1.2 generalize and improve the earlier results due to Vaigant-Kazhikhov [30] where they required that $\beta > 3$ and that the initial density is away from vacuum.

Remark 1.2 It should be mentioned here that it seems that $\beta > 1$ is the extremal case for the system (1.1)-(1.3) (see [30] or Lemma 3.7). Therefore, it would be interesting to study the problem (1.1)-(1.4) when $1 < \beta \leq 4/3$. This is left for the future.

Remark 1.3 In Theorem 1.1, the density is allowed to vanish initially just under the natural compatibility condition $m_0 = \rho_0 u_0$, and no more compatibility ones are required. In fact, our methods can be applied to obtain the local well-posedness of strong solutions to the three-dimensional system (1.1) just under the natural compatibility condition $m_0 = \rho_0 u_0$. This will be reported in a forthcoming paper [15].

Remark 1.4 With Theorem 1.1 at hand, one can easily check that similar to [14, 21], if $(\rho_0, m_0)$ satisfies for some $q > 2$,

$$0 \leq \rho_0 \in W^{2,q}, \quad u_0 \in H^2, \quad m_0 = \rho_0 u_0,$$

and the following additional compatibility condition:

$$-\mu \Delta u_0 - \nabla ((\mu + \lambda(\rho_0)) \text{div} u_0) + \nabla P(\rho_0) = \rho_0^{1/2} g,$$

with some $g \in L^2$, the strong solution obtained in Theorem 1.1 becomes a classical one for positive time. See [14, 18, 21] for details.

Remark 1.5 When the initial density is strictly away from vacuum, Perepelitsa [26] also obtained (1.11) and

$$\lim_{t \to \infty} (\|\rho - \rho_0\|_{L^\infty} + \|\nabla u\|_{L^2}) = 0,$$

under the stringent condition (1.5). Note that (1.5) is a particular case of (1.10) due to the fact that $3(\beta - 1) > \beta$ since $\beta > 3/2$. Thus, Theorems 1.1 and 1.2 improve the results of Perepelitsa [26].
on the blow-up criteria of strong solutions to (1.1), it turns out that the key issue in this paper is to derive the upper bound for the density which is independent of the lower bound of the initial density just under the condition \( \beta > \frac{4}{3} \). To do so, first, similar to [22,26], we rewrite (1.1) as (3.50) in terms of a sum of commutators of Riesz transforms and the operators of multiplication by \( u_i \) (see (3.34)). Then, by energy type estimates and the compensated compactness analysis [7, Theorem II.1], we show that \( \log(1 + \|\nabla u\|_{L^2}) \) does not exceed a polynomial function of \( \|\rho\|_{L^\infty} \) (see (3.30) and (3.65)). Next, using the \( W^{1,p} \)-estimate of the commutator due to Coifman-Meyer [6] (see (2.8)) and the Brezis-Wainger’s inequality (see (2.5)), we obtain an estimate on the \( L^\infty \) norm of the commutators in terms of \( \|\rho\|_{L^\infty} \) and \( \|\nabla u\|_{L^2} \).

Both estimates lead to the key a priori estimate on \( \|\rho\|_{L^\infty} \) which is independent of the lower bound of the initial density provided \( \beta > \frac{4}{3} \). See Proposition 3.6 and its proof.

The next main step is to bound the gradients of the density just under the natural compatibility condition \( m_0 = \rho_0 u_0 \). We first obtain the spatial weighted mean estimates on the material derivatives of the velocity which is achieved by modifying the basic estimates on the material derivatives of the velocity due to Hoff [13]. Then, following [16], the \( L^p \)-bound of the gradient of the density can be obtained by solving a logarithm Gronwall inequality based on a Beale-Kato-Majda type inequality (see Lemma 2.4), the a priori estimates we have just derived and some careful initial layer analysis; and moreover, such a derivation yields simultaneously also the bound for \( L^1((0,T) \times \mathbb{T}^2) \)-norm of the gradient of the velocity; see Proposition 4.3 and its proof.

The rest of the paper is organized as follows: In Section 2, we collect some elementary facts and inequalities which will be needed in later analysis. Section 3 is devoted to the derivation of time-independent and time-dependent upper bounds on the density which are independent of the lower of the initial density and needed to extend the local solution to all time. Based on the previous estimates, higher-order ones are established in Section 4. Then finally, the main results, Theorems 1.1–1.3, are proved in Section 5.

2 Preliminaries

The following well-known local existence theory, where the initial density is strictly away from vacuum, can be found in [27,29].

**Lemma 2.1** Assume that \((\rho_0, m_0)\) satisfies

\[
\rho_0 \in H^2, \quad u_0 \in H^2, \quad \inf_{x \in \mathbb{T}^2} \rho_0(x) > 0, \quad m_0 = \rho_0 u_0.
\]

Then there are a small time \( T > 0 \) and a constant \( C_0 > 0 \) both depending only on \( \|\rho_0\|_{H^2}, \|u_0\|_{H^2} \), and \( \inf_{x \in \mathbb{T}^2} \rho_0(x) \) such that there exists a unique strong solution \((\rho, u)\) to the problem (1.1)-(1.4) in \( \mathbb{T}^2 \times (0, T) \) satisfying

\[
\begin{cases}
\rho \in C([0,T];H^2), & \rho_t \in C([0,T];H^1), \\
u \in L^2(0,T;H^3), & u_t \in L^2(0,T;H^1), \\
u_t \in L^2(0,T;H^2), & u_{tt} \in L^2((0,T) \times \mathbb{T}^2),
\end{cases}
\]

and

\[
\inf_{(x,t) \in \mathbb{T}^2 \times (0,T)} \rho(x,t) \geq C_0 > 0.
\]
Remark 2.1 It should be mentioned that [27,29] dealt with the case that $\lambda = \text{const}$. However, after some slight modifications, their methods can also be applied to the problem (1.1)-(1.4).

Remark 2.2 In [27,29], instead of (2.2), it was shown that

$$\rho \in L^\infty(0, T; H^2), \quad \rho_t \in L^\infty(0, T; H^1).$$

However, one can use [22, Lemma 2.3] to derive (2.2) by standard arguments (see [3] for details). Moreover, one can also obtain Lemma 2.3 by standard arguments due to (2.2), 2.2\text{, and } 2.3.

The following Poincaré-Sobolev and Brezis-Wainger inequalities will be used frequently.

Lemma 2.2 ([2,8,19]) There exists a positive constant $C$ depending only on $\mathbb{T}^2$ such that every function $u \in H^1(\mathbb{T}^2)$ satisfies for $2 < p < \infty$,

$$\|u - \bar{u}\|_{L^p} \leq C p^{1/2} \|u - \bar{u}\|_{L^2}^{2/p} \|\nabla u\|_{L^2}^{1-2/p}, \quad \|u\|_{L^p} \leq C p^{1/2} \|u\|_{L^2}^{2/p} \|u\|_{H^1}^{1-2/p}. \quad (2.4)$$

Moreover, for $q > 2$, there exists some positive constant $C$ depending only on $q$ and $\mathbb{T}^2$ such that every function $v \in W^{1,q}(\mathbb{T}^2)$ satisfies

$$\|v\|_{L^\infty} \leq C \|
abla v\|_{L^2} \ln^{1/2}(e + \|\nabla v\|_{L^q}) + C \|v\|_{L^2} + C. \quad (2.5)$$

The following Poincaré type inequality can be found in [10, Lemma 3.2].

Lemma 2.3 Let $v \in H^1(\mathbb{T}^2)$, and let $\rho$ be a non-negative function such that

$$0 < M_1 \leq \int_{\mathbb{T}^2} \rho dx, \quad \int_{\mathbb{T}^2} \rho^\gamma dx \leq M_2,$$

with $\gamma > 1$. Then there is a constant $C$ depending solely on $M_1, M_2$ such that

$$\|v\|_{L^2(\mathbb{T}^2)}^2 \leq C \int_{\mathbb{T}^2} \rho^2 dx + C \|
abla v\|_{L^2(\mathbb{T}^2)}^2. \quad (2.6)$$

Then, we state the following Beale-Kato-Majda type inequality which was proved in [1] when $\text{div} u \equiv 0$ and will be used later to estimate $\|
abla u\|_{L^\infty}$ and $\|
abla \rho\|_{L^p}$.

Lemma 2.4 ([1,16]) For $2 < q < \infty$, there is a constant $C(q)$ such that the following estimate holds for all $\nabla u \in W^{1,q}(\mathbb{T}^2)$,

$$\|
abla u\|_{L^\infty} \leq C (\|\text{div} u\|_{L^\infty} + \|\text{rot} u\|_{L^\infty}) \log(e + \|\nabla^2 u\|_{L^q}) + C \|
abla u\|_{L^2} + C.$$

Next, let $\triangle^{-1}$ denote the Laplacian inverse with zero mean on $\mathbb{T}^2$ and $R_i$ be the usual Riesz transform on $\mathbb{T}^2 : R_i = (-\triangle)^{-1/2} \partial_i$. Let $H^1(\mathbb{T}^2)$ and $BMO(\mathbb{T}^2)$ stand for the usual HARDY and BMO space:

$$H^1 = \{ f \in L^1(\mathbb{T}^2) : \|f\|_{H^1} = \|f\|_{L^1} + \|R_1 f\|_{L^1} + \|R_2 f\|_{L^1} < \infty, \quad \bar{f} = 0 \}$$

$$BMO = \{ f \in L^1_{loc}(\mathbb{T}^2) : \|f\|_{BMO} < \infty \}$$
with
\[
\|f\|_{BMO} = \sup_{x \in \mathbb{T}^2, r \in (0, d)} \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} \left| f(y) - \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} f(z) \, dz \right| \, dy,
\]
where \( d \) is the diameter of \( \mathbb{T}^2 \), \( \Omega_r(x) = \mathbb{T}^2 \cap B_r(x) \), and \( B_r(x) \) is a ball with center \( x \) and radius \( r \). Consider the composition of two Riesz transforms, \( R_i \circ R_j(i, j = 1, 2) \). There is a representation of this operator as a singular integral
\[
R_i \circ R_j(f)(x) = p.v. \int K_{ij}(x - y) f(y) \, dy,
\]
where the kernel \( K_{ij}(x)(i, j = 1, 2) \) has a singularity of the second order at 0 and
\[
|K_{ij}(x)| \leq C|x|^{-2}, \ x \in \mathbb{T}^2.
\]
Given a function \( b \), define the linear operator
\[
[b, R_i R_j](f)(x) \triangleq b R_i \circ R_j(f) - R_i \circ R_j(b f), \ i, j = 1, 2.
\]
This operator can be written as a convolution with the singular kernel \( K_{ij} \),
\[
[b, R_i R_j](f)(x) \triangleq p.v. \int K_{ij}(x - y)(b(x) - b(y)) f(y) \, dy, \ i, j = 1, 2.
\]

The following properties of the commutator \([b, R_i R_j](f)\) will be useful for our analysis.

**Lemma 2.5** Let \( b, f \in C^\infty(\mathbb{T}^2) \). Then for \( p \in (1, \infty) \), there is \( C(p) \) such that
\[
\|[b, R_i R_j](f)\|_{L^p} \leq C(p)\|b\|_{BMO}\|f\|_{L^p}.
\] (2.7)
Moreover, for \( q_i \in (1, \infty)(i = 1, 2, 3) \) with \( q_1^{-1} = q_2^{-1} + q_3^{-1} \), there is \( C \) depending only on \( q_i(i = 1, 2, 3) \) such that
\[
\|\nabla[b, R_i R_j](f)\|_{L^{q_1}} \leq C\|\nabla b\|_{L^{q_1}}\|f\|_{L^{q_1}}.
\] (2.8)

**Remark 2.3** Properties (2.7) and (2.8) are due to Coifman-Rochberg-Weiss [5] and Coifman-Meyer [6] respectively.

Finally, the following Zlotnik inequality will be used to get the uniform (in time) upper bound of the density \( \rho \).

**Lemma 2.6** (31) Let the function \( y \) satisfy
\[
y'(t) = g(y) + h(t) \text{ on } [0, T], \ y(0) = y_0,
\]
with \( g \in C(R) \) and \( y, h \in W^{1,1}(0, T) \). If \( g(\infty) = -\infty \) and
\[
h(t_2) - h(t_1) \leq N_0 + N_1(t_2 - t_1)
\] (2.9)
for all \( 0 \leq t_1 < t_2 \leq T \) with some \( N_0 \geq 0 \) and \( N_1 \geq 0 \), then
\[
y(t) \leq \max \left\{ y_0, \tilde{\zeta} \right\} + N_0 < \infty \text{ on } [0, T],
\]
where \( \tilde{\zeta} \) is a constant such that
\[
g(\zeta) \leq -N_1 \text{ for } \zeta \geq \tilde{\zeta}.
\]
3 A priori estimates (I): upper bound of the density

In this section and the next, we will always assume that \((\rho_0, m_0)\) satisfies (2.1) and \((\rho, u)\) is the strong solution to (1.1)-(1.4) on \(\mathbb{T}^2 \times (0, T]\) obtained by Lemma 2.1.

3.1 Time-independent upper bound of the density

In this subsection, we will establish the following time-independent upper bound of the density provided (1.10) holds. Throughout this subsection, we use the convention that \(C\) denotes a generic positive constant independent of both the time \(T\) and the lower bound of the initial density, and we write \(C(\alpha)\) to emphasize that \(C\) depends on \(\alpha\).

**Proposition 3.1** If (1.10) holds, there is a positive constant \(C\) depending only on \(\mu, \beta, \gamma, \|\rho_0\|_{L^\infty}, \text{ and } \|u_0\|_{H^1}\) such that
\[
\sup_{0 \leq t \leq T} \|\rho\|_{L^\infty} \leq C. \tag{3.1}
\]

Before proving Proposition 3.1, we establish a series of a priori estimates, Lemmas 3.2-3.4. To proceed, we denote by

\[
\nabla^\perp = (\partial_2, -\partial_1), \quad \frac{D}{Dt} f = \dot{f} = f_t + u \cdot \nabla f,
\]

where \(\frac{D}{Dt} f\) is the material derivative of \(f\). Let \(G\) and \(\omega\) be the effective viscous flux and the vorticity respectively as follows:

\[
G \triangleq (2\mu + \lambda(\rho)) \text{div} u - (P - \bar{P}), \quad \omega \triangleq \nabla^\perp \cdot u = \partial_2 u_1 - \partial_1 u_2.
\]

Then, we define

\[
(A_1(t))^2 \triangleq \int_{\mathbb{T}^2} \left( (\omega(t))^2 + \frac{(G(t))^2}{2\mu + \lambda(\rho(t))} \right) dx, \tag{3.2}
\]

\[
(A_2(t))^2 \triangleq \int_{\mathbb{T}^2} \rho(t)|\dot{u}(t)|^2 dx, \tag{3.3}
\]

\[
(A_3(t))^2 \triangleq \int_{\mathbb{T}^2} ( (2\mu + \lambda(\rho(t)))(\text{div} u(t))^2 + \mu(\omega(t))^2) dx, \tag{3.4}
\]

and

\[
R_T \triangleq \sup_{0 \leq t \leq T} \|\rho\|_{L^\infty}. \tag{3.5}
\]

Without loss of generality, we assume that
\[
\int \rho_0 dx = 1,
\]

which together with (1.1) gives
\[
R_T \geq \|\rho(\cdot, t)\|_{L^\infty} \geq \int \rho(x, t) dx = \int \rho_0 dx = 1. \tag{3.6}
\]

Then, we have the following lemma.
Lemma 3.2 For any $\alpha \in (0, 1)$, there is a positive constant $C(\alpha)$ depending only on $\alpha, \mu, \beta, \gamma, \|\rho_0\|_{L^\infty}$, and $\|u_0\|_{H^1}$ such that

$$\frac{d}{dt} A_1^2(t) + A_2^2(t) \leq C(\alpha) \left( R_T \varphi^2 + \|\rho\|_{L^3}^{3/2} \right) A_3^2,$$  \hspace{1cm} (3.7)

where $\varphi$ is defined by

$$\varphi(t) \equiv 1 + A_1 R_T^{\alpha \beta/2} + \left\| \frac{P}{(2\mu + \lambda)^{3/2}} \right\|_{L^2} + \left\| \frac{P}{(2\mu + \lambda)^{1/2}} \right\|_{L^2}. \hspace{1cm} (3.8)$$

**Proof.** First, the standard energy inequality reads:

$$\sup_{0 \leq t \leq T} \int (\rho|u|^2 + \rho \gamma) \, dx + \int_0^T A_3^2(t) \, dt \leq C,$$  \hspace{1cm} (3.9)

which together with (3.10) gives that for $t \in [0, T]$,

$$CA_3^2(t) - C - CR_T^{-\beta} \leq A_1^2(t) \leq CA_3^2(t) + C + CR_T^{-\beta},$$  \hspace{1cm} (3.10)

$$C^{-1} \|
abla u\|_{L^2}^2 \leq A_3^2(t) \leq CR_T \|\nabla u\|_{L^2}^2,$$  \hspace{1cm} (3.11)

and

$$\|u\|_{H^1} \leq C + C \|\nabla u\|_{L^2} \leq C + CA_3.$$  \hspace{1cm} (3.12)

Next, direct calculations show that

$$\nabla \cdot \dot{u} = \frac{D}{Dt} \omega - (\partial_1 u \cdot \nabla)u_2 + (\partial_2 u \cdot \nabla)u_1 = \frac{D}{Dt} \omega + \omega \nabla u,$$  \hspace{1cm} (3.13)

and that

$$\text{div} \dot{u} = \frac{D}{Dt} \text{div} u + (\partial_1 u \cdot \nabla)u_1 + (\partial_2 u \cdot \nabla)u_2$$

$$= \frac{D}{Dt} \text{div} u - 2 \nabla u_1 \cdot \nabla \perp u_2 + (\text{div} u)^2$$  \hspace{1cm} (3.14)

$$= \frac{D}{Dt} \left( \frac{G}{2\mu + \lambda} \right) + \frac{D}{Dt} \left( \frac{P - \bar{P}}{2\mu + \lambda} \right) - 2 \nabla u_1 \cdot \nabla \perp u_2 + (\text{div} u)^2.$$

Then, we rewrite the momentum equations as

$$\rho \dot{u} = \nabla G + \mu \nabla \perp \omega.$$  \hspace{1cm} (3.15)

Multiplying (3.15) by $2\dot{u}$ and integrating the resulting equality over $\mathbb{T}^2$, we obtain after using (3.13) and (3.14) that

$$\frac{d}{dt} \int \left( \omega^2 + \frac{G^2}{2\mu + \lambda} \right) \, dx + 2A_2^2$$

$$= - \int \omega^2 \text{div} u \, dx + 4 \int G \nabla u_1 \cdot \nabla \perp u_2 \, dx - 2 \int G(\text{div} u)^2 \, dx$$

$$- \int \frac{(\beta - 1)\lambda - 2\mu}{(2\mu + \lambda)^2} G \text{div} u \, dx + 2\beta \int \frac{\lambda(P - \bar{P})}{(2\mu + \lambda)^2} G \text{div} u \, dx$$

$$- 2\gamma \int \frac{P}{2\mu + \lambda} G \text{div} u \, dx + 2(\gamma - 1) \int P \text{div} u \int \frac{G}{2\mu + \lambda} \, dx$$

$$= \sum_{i=1}^{7} I_i.$$
Each $I_i$ can be estimated as follows:

First, it follows from (3.15) that

$$\triangle G = \text{div}(\rho \dot{u}), \quad \mu \triangle \omega = \nabla^\perp \cdot (\rho \dot{u}).$$

(3.17)

which together with the standard $L^p$-estimate of elliptic equations implies that for $p \in (1, \infty)$,

$$\|\nabla G\|_{L^p} + \|\nabla \omega\|_{L^p} \leq C(p, \mu)\|\rho \dot{u}\|_{L^p}.$$  

(3.18)

In particular, we have

$$\|\nabla G\|_{L^2} + \|\nabla \omega\|_{L^2} \leq C(\mu)R_T^{1/2}A_2.$$  

(3.19)

This combining with (2.31) gives

$$\|\omega\|_{L^4} \leq C\left\|\omega\right\|_{L^2}^{1/2} \left\|\nabla \omega\right\|_{L^2}^{1/2} \leq CR_T^{1/4}A_1^{1/2}A_2^{1/2},$$

(3.20)

which leads to

$$|I_1| \leq C\left\|\omega\right\|_{L^4}^2 \|\text{div} u\|_{L^2} \leq \varepsilon A_2^2 + C(\varepsilon)R_T A_3^2 \varphi^2,$$

(3.21)

for $\varphi$ as in (3.8).

Next, we will use an idea due to Perepelitsa [26] to estimate $I_2$. Noticing that

$$\text{rot} \nabla u_1 = 0, \quad \text{div} \nabla^\perp u_2 = 0,$$

one thus derives from [7], Theorem II.1] that

$$\|\nabla u_1 \cdot \nabla^\perp u_2\|_{H^1} \leq C\|\nabla u\|_{L^2}^2.$$  

This combining with the fact that $\mathcal{BMO}$ is the dual space of $H^1$ (see [9]) gives

$$|I_2| \leq C\|G\|_{\mathcal{BMO}} \|\nabla u_1 \cdot \nabla^\perp u_2\|_{H^1} \leq C\|\nabla G\|_{L^2} \|\nabla u\|_{L^2}^2 \leq C\|G\|_{H^1 A_3} \|\nabla u\|_{L^2}^2 \leq C\|G\|_{H^1 A_3} \varphi,$$

(3.22)

where in the last inequality, we have used the following simple fact:

$$\|\nabla u\|_{L^2} \leq C\|\omega\|_{L^2} + C\|\text{div} u\|_{L^2} \leq C\|\omega\|_{L^2} + C \left\| \frac{G}{2\mu + \lambda} \right\|_{L^2} + C \left\| \frac{P - \bar{P}}{2\mu + \lambda} \right\|_{L^2} \leq C \varphi.$$  

Next, Holder’s inequality yields that

$$\sum_{i=3}^{6} |I_i| \leq C \int |G| \left\| \frac{G + P - \bar{P}}{2\mu + \lambda} \right\|_{L^2} \|\text{div} u\|_{L^2} dx + C \int \left\| \frac{G}{2\mu + \lambda} \right\|_{L^2} |\text{div} u| dx$$

$$+ C \int \left\| \frac{P|G|}{2\mu + \lambda} \right\|_{L^2} |\text{div} u| dx + C \int \left\| \frac{|G|\|\text{div} u\|}{2\mu + \lambda} \right\|_{L^2} dx \leq C \int \left\| \frac{G^2}{2\mu + \lambda} \right\|_{L^2} dx + C \int \left\| \frac{P|G|}{2\mu + \lambda} \right\|_{L^2} |\text{div} u| dx + C \int \left\| \frac{|G|\|\text{div} u\|}{2\mu + \lambda} \right\|_{L^2} dx$$  

$$\leq CA_3 \left\| \frac{G^2}{2\mu + \lambda} \right\|_{L^2} + CA_3 \|G\|_{L^4} \left\| \frac{P}{(2\mu + \lambda)^{3/2}} \right\|_{L^4} + C\|G\|_{L^2} A_3.$$  

(3.23)
It follows from (3.2) that
\[ \|G\|_{L^2} \leq CR_t^{\beta/2} A_1, \] (3.24)
which together with the Hölder inequality and (2.4) yields that for \(0 < \alpha < 1\),
\[
\left\| \frac{G^2}{\sqrt{2\mu + \lambda}} \right\|_{L^2} \leq C \left\| \frac{G}{\sqrt{2\mu + \lambda}} \right\|_{L^2}^{1-\alpha} \|G\|_{L^{2(1+\alpha)/\alpha}}^{1+\alpha} \\
\leq C(\alpha) A_1^{1-\alpha} \|G\|_{L^2}^{\alpha} \|G\|_{H^1} \\
\leq C(\alpha) A_1 R_t^{\beta/2} \|G\|_{H^1} \\
\leq C(\alpha) \|G\|_{H^1} \varphi. \tag{3.25}
\]
This combining with (3.23) and (3.24) gives
\[
\sum_{i=3}^{\infty} |I_i| \leq C(\alpha) \|G\|_{H^1} A_3 \varphi. \tag{3.26}
\]
Next, Hölder’s inequality leads to
\[
|I_7| \leq C \left( \frac{P}{(2\mu + \lambda)^{1/2}} \right) \|G\|_{L^2} A_3 \varphi. \tag{3.27}
\]
Finally, noticing that \(\tilde{G}\) satisfies
\[
|\tilde{G}| \leq C \|\rho\|_{L^6}^{\beta/2} A_3, \tag{3.28}
\]
we deduce from the Poincaré-Sobolev inequality and (3.19) that
\[
\|\omega\|_{H^1} + \|G\|_{H^1} \leq C \|\nabla\omega\|_{L^2} + C \|G - \tilde{G}\|_{L^2} + C \|\nabla G\|_{L^2} \\
\leq C \|\nabla\omega\|_{L^2} + C \|\nabla G\|_{L^2} + C \|\rho\|_{L^6}^{\beta/2} A_3 \\
\leq CR_t^{1/2} A_2 + C \|\rho\|_{L^6}^{\beta/2} A_3. \tag{3.29}
\]
Substituting (3.21), (3.22), (3.26), (3.27), and (3.29) into (3.16), we obtain that for any \(\varepsilon > 0\),
\[
\frac{d}{dt} A_1^2(t) + 2A_2^2(t) \leq \varepsilon A_2^2 + C(\varepsilon) R_t A_3^2 \varphi^2 + C(\alpha) \|G\|_{H^1} A_3 \varphi \\
\leq 2\varepsilon A_2^2 + C(\varepsilon, \alpha) R_t A_3^2 \varphi^2 + C(\alpha) \|\rho\|_{L^6}^{\beta/2} A_3^2 \varphi,
\]
which directly gives (3.7) after choosing \(\varepsilon\) suitably small. The proof of Lemma 3.2 is completed.

Lemma 3.3 directly yields that

**Lemma 3.3** For any \(\alpha \in (0,1)\), there is a constant \(C(\alpha)\) depending only on \(\alpha, \mu, \beta, \gamma, \|\rho_0\|_{L^\infty}, \text{ and } \|u_0\|_{H^1}\), such that
\[
\sup_{0 \leq t \leq T} \log(e + A_1^2(t) + A_3^2(t)) + \int_0^T \frac{A_2^2(t)}{e + A_1^2(t)} dt \leq C(\alpha) R_t^{1+\kappa+\alpha\beta}, \tag{3.30}
\]
with
\[
\kappa = \max\{0, (3\gamma - 6\beta)/2, \gamma - \beta, \beta - \gamma - 2\}. \tag{3.31}
\]
Proof. It follows from (3.8) and (3.9) that
\[ \varphi(t) \leq C + CA_1 R_{T}^{\alpha/2} + CR_{T}^{(3\gamma-6\beta)/4} + CR_{T}^{(\gamma-\beta)/2}, \]  
which together with (3.7) and (3.9) gives
\[ \frac{d}{dt} A_2^2(t) + A_3^2(t) \leq C(\alpha) R_T A_3^2 \varphi^2 + C(\alpha) \|\rho\|_{L^\infty}^\beta R_T^{-1} A_3^2 \]  
\[ \leq C(\alpha) R_T \left( R_T^{\beta/2} A_3^2 + R_T^{(3\gamma-6\beta)/2} + R_T^{\gamma-\beta} + R_T^{\gamma-2} + 1 \right) A_3^2. \]  
Dividing (3.33) by \( e + A_1^2(t) \) and integrating the resulting inequality over \((0, T)\), we obtain (3.30) after using (3.6), (3.9), and (3.10). We thus finish the proof of Lemma 3.3.

**Remark 3.1** Under the stringent condition (1.5), Perepelitsa [26] also obtained (3.30) with \( \kappa = 0 \).

Next, we denote the commutator \( F \) by
\[ F \triangleq \sum_{i,j=1}^2 [u_i, R_t R_j] (\rho u_j). \]  
The following lemma gives an estimate of \( F \) which will play an important role in obtaining the uniform upper bound of the density.

**Lemma 3.4** For any \( \varepsilon > 0 \), there is a positive constant \( C(\varepsilon) \) depending only on \( \varepsilon, \mu, \beta, \gamma, \|\rho_0\|_{L^\infty}, \) and \( \|u_0\|_{H^1} \) such that
\[ \|F\|_{L^\infty} \leq \frac{C(\varepsilon) R_T^{1-\kappa} A_2^2}{e + A_1^2} + C(\varepsilon) A_3^2 R_T^{(3+\kappa)/2+\varepsilon} + C(\varepsilon) R_T^{1+\varepsilon}, \]  
with \( \kappa \) as in (3.31).

Proof. First, it follows from (3.9) that
\[ \|\rho u\|_{L^{2\gamma/(\gamma+1)}} \leq \|\rho\|_{L^\infty}^{1/2} \|\rho^{1/2} u\|_{L^2} \leq C, \]  
which together with (3.12) gives
\[ |\bar{F}| \leq C\|u\|_{L^{2\gamma/(\gamma+1)}} \|\rho u\|_{L^{2\gamma/(\gamma+1)}} \leq C + CA_3. \]  
Then, we deduce from (2.7) that
\[ \|F\|_{L^q} \leq C(q) \|u\|_{BMO} \|\rho u\|_{L^q} \leq C(q) \|\nabla u\|_{L^2} \|\rho u\|_{L^q}, \]  
which together with the Gagliardo-Nirenberg inequality and (2.8) thus gives that for \( q \in (8, \infty) \),
\[ \|F - \bar{F}\|_{L^\infty} \leq C(q) \|F - \bar{F}\|_{L^q}^{(q-4)/q} \|\nabla F\|_{L^{4q/(q+4)}}^{4/q} \]  
\[ \leq C(q) \left( \|\nabla u\|_{L^2} \|\rho u\|_{L^q} + |\bar{F}| \right)^{(q-4)/q} \left( \|\nabla u\|_{L^2} \|\rho u\|_{L^q} \right)^{4/q} \]  
\[ \leq C(q) A_3^{(q-4)/q} \|\nabla u\|_{L^q}^{4/q} \|\rho u\|_{L^q} + C(q) R_T^{4/q} (A_3 + 1) \|u\|_{L^q}^{4/q}, \]  
\[ \|F - \bar{F}\|_{L^\infty} \leq C(q) \|F - \bar{F}\|_{L^q}^{(q-4)/q} \|\nabla F\|_{L^{4q/(q+4)}}^{4/q} \]  
\[ \leq C(q) \left( \|\nabla u\|_{L^2} \|\rho u\|_{L^q} + |\bar{F}| \right)^{(q-4)/q} \left( \|\nabla u\|_{L^2} \|\rho u\|_{L^q} \right)^{4/q} \]  
\[ \leq C(q) A_3^{(q-4)/q} \|\nabla u\|_{L^q}^{4/q} \|\rho u\|_{L^q} + C(q) R_T^{4/q} (A_3 + 1) \|u\|_{L^q}^{4/q}, \]
where in the last inequality, we have used (3.37) and the following simple fact:

\[ \| \rho u \|_{L^q} \leq CR_T \| u \|_{L^q} \leq C(q) R_T (1 + A_3) \]

due to (3.12).

Next, noticing that (3.10) gives

\[ e + A_1 \leq C + CA_3 + CR_T^{(\gamma - \beta)/2} \leq CR_T^{\max \{0,(\gamma - \beta)/2\}} (e + A_3), \]

we obtain from (3.20), (3.25), (3.29), and (3.32) that

\[
\begin{align*}
\| \nabla u \|_{L^4} & \leq C(\| \text{div} u \|_4 + \| \omega \|_4) \\
& \leq C \left( \left\| \frac{G + P - \bar{P}}{2 \mu + \lambda} \right\|_{L^4} + CR_T^{1/4} A_1^{1/2} A_2^{1/2} \right) \\
& \leq C \left( \left\| \frac{G^2}{\sqrt{2 \mu + \lambda}} \right\|_{L^2}^{1/2} + C \left\| \frac{P - \bar{P}}{2 \mu + \lambda} \right\|_{L^4} + CR_T^{1/4} A_1^{1/2} A_2^{1/2} \right) \\
& \leq C \left( R_T^{1/4} A_2^{1/2} + C \| \rho \|_{L^4}^{\beta/4} A_3^{1/2} \right) \varphi^{1/2} + CR_T^{(3\gamma - 4\beta)/4} \\
& \quad + CR_T^{(5 + \kappa)/4} (e + A_1) \left( \frac{R_T^{-4 - \kappa} A_2^2}{e + A_1^2} \right)^{1/4} \\
& \quad \leq CR_T^C (e + A_3) \left( \frac{R_T^{-4 - \kappa} A_2^2}{e + A_1^2} \right)^{1/4} + CR_T^C (e + A_3),
\end{align*}
\]

with some constant \( \tilde{C} > 1 \) depending only on \( \beta \) and \( \gamma \). This combining with (3.30) implies that for \( \alpha \in (0, 1) \),

\[
\begin{align*}
\log (e + \| \nabla u \|_{L^4}) & \leq C(\alpha) \log (e + R_T) + C \log (e + A_3) + C \log (e + \frac{R_T^{-4 - \kappa} A_2^2}{(e + A_1^2)(e + A_3)^6}) \\
& \leq C(\alpha) R_T^{1 + \kappa + \alpha \beta} + \frac{CR_T^{-4 - \kappa} A_2^2}{(e + A_1^2)(e + A_3)^6},
\end{align*}
\]

which together with (2.5) and (3.12) gives that for \( \alpha \in (0, 1) \),

\[
\begin{align*}
\| u \|_{L^\infty} & \leq C \| \nabla u \|_{L^2} \log^{1/2} (e + \| \nabla u \|_{L^4}) + C \| \nabla u \|_{L^2} + C \\
& \leq C(\alpha) A_3 R_T^{(1 + \kappa + \alpha \beta)/2} + C \left( \frac{R_T^{-4 - \kappa} A_2^2}{(e + A_1^2)(e + A_3)^4} \right)^{1/2} + C.
\end{align*}
\]

It thus follows from Holder’s inequality, (3.40), and (3.9) that for \( \alpha \in (0, 1) \) and \( q \in (8, \infty) \),

\[
\begin{align*}
\| \rho u \|_{L^q} & \leq C R_T^{1-1/q} \| \rho^{1/2} u \|_{L^2}^{2/q} \| u \|_{L^\infty}^{1-2/q} \\
& \leq C(\alpha) A_3^{1-2/q} R_T^{3 + \kappa + \alpha \beta/2} + C \left( \frac{R_T^{-1 - \kappa} A_2^2}{(e + A_1^2)(e + A_3)^4} \right)^{1/2 - 1/q} + CR_T.
\end{align*}
\]
Similarly, we have

\[ \|F - \overline{F}\|_{L^{\infty}} \leq C(q, \alpha) \|\nabla u\|_{L^4}^{4/q} A_3^{(2q-6)/q} R_T^{(3 + \kappa + \alpha \beta)/2} \]

\[ + C(q, \alpha) A_3^{(q-4)/q} \|\nabla u\|_{L^4}^{4/q} \left( \frac{R_T^{-1-\kappa} A_2^2}{e + A_1^2} \right)^{1/2} \]

\[ + C(q)R_A^3 A_3^{(q-4)/q} \|\nabla u\|_{L^4}^{4/q} + C(q) R_T^{4/q} (A_3 + 1) \|\nabla u\|_{L^4}^{4/q} \]

\[ \leq C(q, \alpha) A_3^{(2q-6)/q} (e + A_3)^{4/q} \left( \frac{R_T^{-1-\kappa} A_2^2}{e + A_1^2} \right)^{1/q} R_T^{(3 + \kappa + \alpha \beta)/2 + 4\tilde{C}/q} \]

\[ + C(q, \alpha) A_3^{(2q-6)/q} (e + A_3)^{4/q} R_T^{(3 + \kappa + \alpha \beta)/2 + 4\tilde{C}/q} \]

\[ + C(q, \alpha) \|\nabla u\|_{L^4}^{4/q} \left( \frac{R_T^{-1-\kappa} A_2^2}{e + A_1^2} \right)^{1/2} + C(q) R_T (A_3 + 1) \|\nabla u\|_{L^4}^{4/q} \]

\[ \triangleq \sum_{i=1}^{4} J_i. \]

Holder’s inequality implies that

\[ |J_1| \leq C(q, \alpha) \left( R_T^{(3 + \kappa + \alpha \beta)/2 + 4\tilde{C}/q} A_3^{(2q-6)/q} \right)^{q/(q-3)} + (e + A_3)^{4/q} \frac{R_T^{-1-\kappa} A_2^2}{e + A_1^2} + \frac{R_T^{-1-\kappa} A_2^2}{e + A_1^2} \]

\[ \leq C(q, \alpha) + C(q, \alpha) A_3^{2} R_T^{\tilde{\kappa}(\alpha, q)} + \frac{R_T^{-1-\kappa} A_2^2}{e + A_1^2}, \]

with

\[ \tilde{\kappa}(\alpha, q) \triangleq \left( \frac{3}{2} + \frac{\kappa}{2} + \frac{\alpha \beta}{2} + \frac{4\tilde{C}}{q} \right) \frac{q}{q - 3}. \]

Similarly, we have

\[ |J_2| \leq C(q, \alpha) + C(q, \alpha) A_3^{2} R_T^{\tilde{\kappa}(\alpha, q)}. \]

One thus deduces from (3.39) that for \( \eta \in (0, 1) \),

\[ \|\nabla u\|_{L^4}^q \leq C(\eta) R_T^{4\tilde{C}/(4-\eta)} (e + A_3)^{4/(4-\eta)} + \frac{R_T^{-1-\kappa} A_2^2}{e + A_1^2} \]

\[ + C(\alpha, \eta) R_T^{\tilde{\kappa}(\alpha, \eta)} (A_3 + 1) \]

\[ \leq C(\eta) R_T^{2\tilde{C}/(3+1) + 1} + \frac{R_T^{-1-\kappa} A_2^2}{e + A_1^2}, \]

which together with Holder’s inequality gives

\[ |J_3| \leq \|\nabla u\|_{L^4}^{8/q} + R_T^{-1-\kappa} A_2^2 \]

\[ + C(q, \alpha) \]

\[ \leq C(\alpha, q) R_T^{16\tilde{C}/q} + C(\alpha, q) R_T^{16\tilde{C}/q} A_3^2 + \frac{2R_T^{-1-\kappa} A_2^2}{e + A_1^2}. \]
It follows from (3.12) and (3.46) that
\begin{equation}
|J_4| \leq R_T \|\nabla u\|_{L^4}^{8/q} + C(q)R_T + C(q)R_T A_T^2
\leq C(q)R_T^{1+16\tilde{C}/q} + C(q)R_T^{1+16\tilde{C}/q}A_T^2 + \frac{R_T^{-1+\kappa}A_T^2}{e + A_T^2}.
\end{equation}

Substituting (3.43), (3.45), (3.47), and (3.48) into (3.42), we obtain (3.35) after using (3.37) and choosing \( q \) suitably large and then \( \alpha \) suitably small. The proof of Lemma 3.4 is completed.

Now, we are in a position to prove Proposition 3.1.

**Proof of Proposition 3.1.** It follows from (3.15) that \( G \) solves
\[ \Delta G = \text{div}(\rho \dot{u}) = \partial_t(\text{div}(\rho u)) + \text{divdiv}(\rho u \otimes u), \]
which implies
\[ G - \bar{G} + \frac{D}{Dt} ((-\Delta)^{-1} \text{div}(\rho u)) = F, \]
with \( F \) as in (3.34). The mass equation (1.11) leads to
\[ -\text{div}u = \frac{1}{\rho} D_t \rho, \]
which combining with (3.49) gives that
\[ \frac{D}{Dt} \theta(\rho) + P = \frac{D}{Dt} \psi + \bar{P} - \bar{G} + F, \]
with
\[ \theta(\rho) \triangleq 2\mu \log \rho + \beta^{-1} \rho^\beta, \quad \psi \triangleq (-\Delta)^{-1} \text{div}(\rho u). \]
Since the function \( y = \theta(\rho) \) is increasing for \( \rho \in (0, \infty) \), the inverse function
\[ \rho = \theta^{-1}(y) \]
exists for \( y \in (-\infty, \infty) \). We rewrite (3.50) as
\[ \frac{D}{Dt} y = g(y) + \frac{D}{Dt} h, \]
with
\[ y = \theta(\rho), \quad g(y) = -P(\theta^{-1}(y)), \quad h = \psi + \int_0^t (\bar{P} - \bar{G} + F) \, dt. \]
To apply Lemma 2.6 noticing that
\[ \lim_{y \to \infty} g(y) = -\infty, \]
we need to estimate \( h \). First, it follows from (2.5), (3.36), and (3.12) that
\begin{align}
\|\psi\|_{L^\infty} &\leq C\|\nabla\psi\|_{L^2} \log^{1/2}(e + \|\nabla\psi\|_{L^3}) + C\|\psi\|_{L^2} + C \\
&\leq C\|\rho u\|_{L^2} \log^{1/2}(e + \|\rho u\|_{L^3}) + C\|\rho u\|_{L^{2/(\gamma+1)}} + C \\
&\leq CR_T^{1/2} \log^{1/2}(e + R_T(1 + \|\nabla u\|_{L^2})) + C \\
&\leq CR_T^{1/2} \log^{1/2}(e + A_T^2) + CR_T,
\end{align}
(3.54)
which together with (3.30) gives that for \( \kappa \) as in (3.31)

\[
\| \psi \|_{L^\infty} \leq C \sup_{0 \leq t \leq T} R_T^{(3+\kappa)/2}.
\] (3.55)

Next, on one hand, (3.9) and (3.28) lead to

\[
|\bar{P} - \bar{G}| \leq C + C\|\rho\|_{L^\beta}^{\beta/2} A_3(t)
\leq C + CA_3^2(t) + CA_3^2(t) R_T^{\beta-\gamma}.
\] (3.56)

On the other hand, one deduces from (3.35), (3.30), and (3.9) that for any \( \epsilon > 0 \) and all \( 0 \leq t_1 \leq t_2 \leq T \)

\[
\int_{t_1}^{t_2} \|F\|_{L^\infty} ds \leq C(\epsilon) R_T^{\max\{(3+\kappa)/2 + \epsilon, \beta-\gamma\}} + C(\epsilon) R_T^{1+\epsilon}(t_2 - t_1).
\] (3.57)

This combining with (3.55) and (3.56) implies that for all \( 0 \leq t_1 \leq t_2 \leq T \) and any \( \epsilon > 0 \),

\[
|h(t_2) - h(t_1)| \leq C(\epsilon) R_T^{\max\{(3+\kappa)/2 + \epsilon, \beta-\gamma\}} + C(\epsilon) R_T^{1+\epsilon}(t_2 - t_1).
\]

Therefore, one can choose \( N_0 \) and \( N_1 \) in (2.9) as:

\[
N_0 = C(\epsilon) R_T^{\max\{(3+\kappa)/2 + \epsilon, \beta-\gamma\}}, \quad N_1 = C(\epsilon) R_T^{1+\epsilon}.
\]

For \( g(y) \) as in (3.53) with \( \rho = \theta^{-1}(y) \) as in (3.52) being the inverse function of \( y = \theta(\rho) \), we have

\[
g(\zeta) = -(\theta^{-1}(\zeta))^{\gamma} \leq -N_1 = -C(\epsilon) R_T^{1+\epsilon},
\]

for all \( \zeta \geq \tilde{\zeta} \triangleq C(\epsilon) R_T^{\beta(1+\epsilon)/\gamma} \). Lemma 2.6 thus yields that

\[
R_T^{\beta} \leq C(\epsilon) R_T^{\max\{(3+\kappa)/2 + \epsilon, \beta-\gamma, \beta(1+\epsilon)/\gamma\}},
\]

which together with (1.10) gives (3.1). We finish the proof of Proposition 3.1.

The following Proposition 3.5, which will play an important role in obtaining the large-time behavior of \((\rho,u)\), is a direct consequence of (3.33), (3.1), (3.9), (3.12), (3.29), and Gronwall’s inequality.

**Proposition 3.5** If (1.10) holds, there is a positive constant \( C \) depending only on \( \mu, \beta, \gamma, \|\rho_0\|_{L^\infty}, \) and \( \|u_0\|_{H^1} \) such that

\[
\sup_{0 \leq t \leq T} (\|\rho\|_{L^\infty} + \|u\|_{H^1}) + \int_0^T (\|\omega\|^2_{H^1} + \|G\|^2_{H^1} + A_2^2(t) + A_3^2(t)) dt \leq C.
\] (3.58)

### 3.2 Time-dependent upper bound of the density

The following Proposition 3.6 will give a time-dependent upper bound of the density which is the key to obtain higher order estimates provided (1.7) holds. Throughout this subsection, \( C \) denotes a generic positive constant independent of the lower bound of the initial density.
**Proposition 3.6** Assume that (1.7) holds. Then there is a positive constant $C(T)$ depending only on $T, \mu, \beta, \gamma, \|\rho_0\|_{L^\infty}$, and $\|u_0\|_{H^1}$ such that

$$
\sup_{0 \leq t \leq T} (\|\rho\|_{L^\infty} + \|u\|_{H^1}) + \int_0^T (\|\omega\|_{H^1} + \|G\|_{H^1} + A_2^2(t)) \, dt \leq C(T). \tag{3.59}
$$

Before proving Proposition 3.6, we establish some a priori estimates, Lemmas 3.8 and 3.9.

We first state the $L^p$-estimate of the density due to Vaigant-Kazhikhov (30).

**Lemma 3.7** (30) Let $\beta > 1$. For any $1 \leq p < \infty$, there is a positive constant $C(T)$ depending only on $T, \mu, \beta, \gamma, \|\rho_0\|_{L^\infty}$, and $\|u_0\|_{H^1}$ such that

$$
\sup_{0 \leq t \leq T} \|\rho(\cdot, t)\|_{L^p} \leq C(T) p^{\frac{2}{p-1}}. \tag{3.60}
$$

The following $L^p$-estimate of the momentum which plays an important role in the estimate of the upper bound of the density is a direct consequence of Lemma 3.7.

**Lemma 3.8** Let $\beta > 1$. For any $q > 4$, there is a positive constant $C(q, T)$ depending only on $T, q, \mu, \beta, \gamma, \|\rho_0\|_{L^\infty}$, and $\|u_0\|_{H^1}$ such that

$$
\|\rho u\|_{L^q} \leq C(q, T) R_T^{1+\beta(q-2)/(4q)} (1 + A_3)^{1-2/q}. \tag{3.61}
$$

**Proof.** First, we claim that there is a positive constant $\nu_0 \leq 1/2$ depending only on $\mu$ such that

$$
\sup_{0 \leq t \leq T} \int \rho |u|^{2+\nu} \, dx \leq C(T), \tag{3.62}
$$

with

$$
\nu = R_T^{-\beta/2} \nu_0 \in (0, 1/2].
$$

Then, let $r \triangleq (q - 2)(2 + \nu)/\nu > 2$ due to $q > 4$. It follows from Hölder’s inequality, (3.62), (2.3), and (3.12) that

$$
\|\rho u\|_{L^2} \leq C \|\rho u\|^{2/q}_{L^{2+r}} \|\rho u\|^{1-2/q}_{L^r} \leq C(T) R_T^{1-2/(2+\nu)q} \|\rho u\|^{1-2/q}_{L^r} \leq C(T) R_T \left( r^{1/2} \|u\|_{L^r}^2 \right)^{1-2/q} \leq C(q, T) R_T^{1+\beta(q-2)/(4q)} (1 + \|\nabla u\|_{L^2})^{1-2/q},
$$

which together with (3.11) shows (3.61).

Finally, it remains to prove (3.62). Multiplying (1.1) by $(2 + \nu)|u|^\nu u$, we get after integrating the resulting equation over $\mathbb{T}^2$ that

$$
\frac{d}{dt} \int \rho |u|^{2+\nu} \, dx + (2 + \nu) \int |u|^\nu \left( \mu |\nabla u|^2 + (\mu + \rho^2)(\text{div} u)^2 \right) \, dx \leq (2 + \nu) \nu \int (\mu + \rho^2)|\text{div} u||u|^\nu |\nabla u| \, dx + C \int \rho^\gamma |u|^\nu |\nabla u| \, dx
$$

$$
\leq \frac{2 + \nu}{2} \int (\mu + \rho^2)|\text{div} u|^2 |u|^\nu \, dx + \frac{2 + \nu}{2} \nu^2 (\mu + 1) \int |u|^\nu |\nabla u|^2 \, dx
$$

$$
+ \mu \int |u|^\nu |\nabla u|^2 \, dx + C \int \rho |u|^{2+\nu} \, dx + C \int \rho^{(2+\nu)\gamma-\nu/2} \, dx,
$$

and

$$
\frac{d}{dt} \int \rho |u|^{2+\nu} \, dx + (2 + \nu) \int |u|^\nu \left( \mu |\nabla u|^2 + (\mu + \rho^2)(\text{div} u)^2 \right) \, dx \leq (2 + \nu) \nu \int (\mu + \rho^2)|\text{div} u||u|^\nu |\nabla u| \, dx + C \int \rho^\gamma |u|^\nu |\nabla u| \, dx
$$

$$
\leq \frac{2 + \nu}{2} \int (\mu + \rho^2)|\text{div} u|^2 |u|^\nu \, dx + \frac{2 + \nu}{2} \nu^2 (\mu + 1) \int |u|^\nu |\nabla u|^2 \, dx
$$

$$
+ \mu \int |u|^\nu |\nabla u|^2 \, dx + C \int \rho |u|^{2+\nu} \, dx + C \int \rho^{(2+\nu)\gamma-\nu/2} \, dx,
$$

and
which, after choosing \(\nu_0(\mu)\) suitably small, together with Gronwall’s inequality and (3.60) thus gives (3.62). The proof of Lemma 3.8 is completed.

The next lemma will deal with the time-dependent estimate on the spatial \(L^\infty\)-norm of the commutator operator \(F\) defined by (3.34).

**Lemma 3.9** Let \(\beta > 1\). For any \(\varepsilon > 0\), there is a positive constant \(C(\varepsilon, T)\) depending only on \(\varepsilon, T, \mu, \beta, \gamma, \|\rho_0\|_{L^\infty}\), and \(\|u_0\|_{H^1}\) such that

\[
\|F\|_{L^\infty} \leq \frac{C(\varepsilon, T)A_2^2}{e + A_1^2} + C(\varepsilon, T)(1 + A_3^2)R_T^{1 + \beta/4 + \varepsilon}. \tag{3.63}
\]

**Proof.** First, it follows from (3.39) and (3.61) that for \(q > 8\),

\[
\begin{align*}
A_3^{(q-4)/4} &\|\nabla u\|_{L^q}^4 \|\rho u\|_{L^q} \\
&\leq C(q, T)R_T^{1 + \beta(q-2)/(4q)} \left(A_3^{2-6/q} + 1\right) \|\nabla u\|_{L^q}^4 \\
&\leq C(q, T)R_T^{1 + \beta(q-2)/(4q) + 4\tilde{C}/q} \left(A_3^{2-6/q} + 1\right) \left(1 + A_3^2\right)^{2/q} \left(\frac{A_2^2}{e + A_1^2}\right)^{1/q} \\
&\quad + C(q, T)R_T^{1 + \beta(q-2)/(4q) + 4\tilde{C}/q} \left(A_3^{2-6/q} + 1\right) \left(1 + A_3^2\right)^{2/q} \\
&\leq C(q, T)R_T^{1 + \beta(q-2)/(4q) + 4\tilde{C}/q} \left(A_3^{2-2/q} + 1\right) \left(\frac{A_2^2}{e + A_1^2}\right)^{1/q} \\
&\quad + C(q, T)R_T^{1 + \beta(q-2)/(4q) + 4\tilde{C}/q} \left(A_3^2 + 1\right) \\
&\leq C(q, T)R_T^{1 + \beta/4 + 4\tilde{C}/(q-1)} \left(1 + A_3^2\right) + \frac{A_2^2}{e + A_1^2}.
\end{align*}
\]

This combining with (3.38) and (3.38) yields that

\[
\|F - \bar{F}\|_{L^\infty} \leq C(q, T)R_T^{1 + \beta/4 + 4\tilde{C}/(q-1)} A_3^2 + C(q, T) \frac{A_2^2}{e + A_1^2}
\]

\[
+ C(q)R_T^{16\tilde{C}/q} + C(q)R_T^{16\tilde{C}/q} A_3^2,
\]

which together with (3.37) directly gives (3.63) after choosing \(q\) suitably large and then \(\alpha\) suitably small. The proof of Lemma 3.9 is completed.

**Proof of Proposition 3.6.** We deduce from (3.8) and (3.60) that for any \(\alpha \in (0, 1)\),

\[
\varphi(t) \leq C(T, \alpha) + C(T, \alpha)A_1 R_T^{\alpha\beta/2},
\]

which together with Lemmas 3.2 and 3.7 gives

\[
\sup_{0 \leq t \leq T} \log(e + A_1^2(t)) + \int_0^T \frac{A_2^2(t)}{e + A_1^2(t)} dt \leq C(T, \alpha)R_T^{1 + \alpha\beta}. \tag{3.65}
\]

Then, for \(\psi\) as in (3.51), it follows from (3.54) and (3.65) that

\[
\|\psi\|_{L^\infty} \leq C(T)R_T^{\alpha/\beta},
\]

which together with (3.50), (3.56), (3.60), (3.63), and (3.64) yields that for \(\varepsilon \in (0, 1)\),

\[
R_T^\beta \leq C(\varepsilon, T)R_T^{\max\{1 + \beta/4 + \varepsilon, \beta - \gamma, 4/3\}}.
\]
Due to (1.7), after choosing $\varepsilon$ suitably small, this directly gives
\[ \sup_{0 \leq t \leq T} \|\rho\|_{L^\infty} \leq C(T), \]
which together with (3.33), (3.9), (3.12), (3.29), and Gronwall’s inequality yields (3.59). We complete the proof of Proposition 3.6.

4 A priori estimates (II): higher order estimates

**Lemma 4.1** Assume that
\[ \sup_{0 \leq t \leq T} \|\rho\|_{L^\infty} \leq M, \] (4.1)
for some positive constant $M$. Then there is a positive constant $C(M)$ depending only on $M, \mu, \beta, \gamma, \|\rho_0\|_{L^\infty}$, and $\|u_0\|_{H^1}$ such that
\[ \sup_{0 \leq t \leq T} \int_\Omega \rho|\dot{u}|^2 dx + \int_0^T \sigma \|\nabla \dot{u}\|^2_{L^2} dt \leq C(M), \] (4.2)
with $\sigma(t) \equiv \min\{1, t\}$. Moreover, if (1.10) holds, for any $p \in [1, \infty)$, there is a positive constant $C(p)$ depending only on $p, \mu, \beta, \gamma, \|\rho_0\|_{L^\infty}$, and $\|u_0\|_{H^1}$ such that for any $T \in (1, \infty)$,
\[ \sup_{1 \leq t \leq T} \|\nabla u\|_{L^p} \leq C(p). \] (4.3)

**Proof.** Operating $\dot{u}^j [\partial_j \dot{t} + \text{div}(u^j)]$ to (1.1)\(\beta\), summing with respect to $j$, and integrating the resulting equation over $\Omega$, one obtains after integration by parts that
\[ \left( \frac{1}{2} \int \rho|\dot{u}|^2 dx \right)_t = - \int \dot{u}_j [\partial_j \dot{P}_t + \text{div}(\partial_j P u)] dx + \mu \int \dot{u}_j [\partial_t \Delta u_j + \text{div}(u \Delta u_j)] dx \]
\[ + \int \dot{u}_j [\partial_j ((\mu + \lambda)\text{div}u) + \text{div}(u\partial_j ((\mu + \lambda)\text{div}u))] dx \]
\[ \triangleq \sum_{i=1}^3 N_i. \]

First, using the equation (1.1)\(\alpha\), we obtain after integration by parts that
\[ N_1 = - \int \dot{u}_j [\partial_j \dot{P}_t + \text{div}(\partial_j P u)] dx \]
\[ = \int [-P' \rho \text{div}u \partial_j \dot{u}_j + \partial_k (\partial_j \dot{u}_j u_k) P - P \partial_j (\partial_k \dot{u}_j u_k)] dx \]
\[ \leq C(M) \|\nabla u\|_{L^2} \|\nabla \dot{u}\|_{L^2} \]
\[ \leq \frac{\mu}{8} \|\nabla \dot{u}\|^2_{L^2} + C(M) \|\nabla u\|^2_{L^2}. \] (4.5)

Then, integration by parts leads to
\[ N_2 = \mu \int \dot{u}_j [\partial_t \Delta u_j + \text{div}(u \Delta u_j)] dx \]
\[ = -\mu \int \left( |\nabla \dot{u}|^2 + \partial_i \dot{u}_j \partial_k u_k \partial_i u_j - \partial_i \dot{u}_j \partial_i u_k \partial_k u_j - \partial_i u_j \partial_i u_k \partial_k \dot{u}_j \right) dx \]
\[ \leq - \frac{3\mu}{4} \int |\nabla \dot{u}|^2 dx + C(M) \int |\nabla u|^4 dx. \] (4.6)
Similarly,

\[ N_3 = \int \dot{u}_j [\partial_{jt} \((\mu + \lambda)\text{div} u\) + \text{div}(u \partial_j \((\mu + \lambda)\text{div} u\))] \, dx \]
\[ = - \int \partial_j \dot{u}_j \[(\mu + \lambda)\text{div} u \] + \text{div}(u(\mu + \lambda)\text{div} u) \, dx \]
\[ - \int \dot{u}_j \text{div}(\partial_j u(\mu + \lambda)\text{div} u) \, dx \]
\[ \leq - \int \partial_j \dot{u}_j \[(\mu + \lambda)\text{div} u + \lambda_i \text{div} u + (u \cdot \nabla \lambda)\text{div} u + (\mu + \lambda)(u \cdot \nabla)\text{div} u \] \, dx \]
\[ + \frac{\mu}{8} \int |\nabla \dot{u}|^2 \, dx + C(M) \int |\nabla u|^4 \, dx \]
\[ = - \int \left( \frac{D}{Dt} \text{div} u + \partial_j u_i \partial_i u_j \right) [(\mu + \lambda) \frac{D}{Dt} \text{div} u - p \lambda'(\rho) \text{div} u] \, dx \]
\[ + \frac{\mu}{8} \int |\nabla \dot{u}|^2 \, dx + C(M) \int |\nabla u|^4 \, dx \]
\[ \leq - \frac{\mu}{2} \int \frac{D}{Dt} \text{div} u|^2 \, dx + \frac{\mu}{8} \int |\nabla \dot{u}|^2 \, dx + C(M)\|\nabla u\|^4_{L^4} + C(M)\|\nabla u\|^2_{L^2}. \quad (4.7) \]

Finally, substituting (4.3)–(4.7) into (4.4) shows that

\[ 2 \left( \int \rho |\dot{u}|^2 \, dx \right)_t + \mu \int |\nabla \dot{u}|^2 \, dx + \mu \int \frac{D}{Dt} \text{div} u|^2 \, dx \]
\[ \leq C(M)\|\nabla u\|^4_{L^4} + C(M)\|\nabla u\|^2_{L^2} \]
\[ \leq C(M)(\|G\|^4_{L^4} + \|\omega\|^4_{L^4} + \|P - \overline{\mathbf{P}}\|^4_{L^4} + \|\nabla u\|^2_{L^2}) \]
\[ \leq C(M)(\|G\|^2_{H^1} + \|\omega\|^2_{L^2} + \|\nabla \omega\|^2_{L^2} + \|P - \overline{\mathbf{P}}\|^2_{L^2} + \|\nabla u\|^2_{L^2}) \]
\[ \leq C(M)(\|G\|^2_{H^1} + \|\nabla \omega\|^2_{L^2} + \|\nabla u\|^2_{L^2}) \]
\[ \leq C(M)\|\rho^{1/2} \dot{u}\|^2_{L^2} + C(M)\|\nabla u\|^2_{L^2}, \quad (4.8) \]

where in the last inequality we have used (3.29) and (3.11). Multiplying (4.8) by \( \sigma \), integrating the resulting equation over \((0, T)\), we obtain (4.2) after using (3.59).

It remains to prove (4.3). Because of (1.10), we deduce from (1.10) and (3.1) that

\[ \|\nabla u\|_{L^p} \leq C(p)\|\text{div} u\|_{L^p} + C(p)\|\omega\|_{L^p} \]
\[ \leq C(p)\|G\|_{L^p} + C(p)\|P - \overline{\mathbf{P}}\|_{L^p} + C(p)\|\omega\|_{L^p} \]
\[ \leq C(p)\|G\|_{H^1} + C(p)\|\omega\|_{H^1} + C(p) \]
\[ \leq C(p)\|\rho^{1/2} \dot{u}\|_{L^2} + C(p), \]

where in the last inequality we have used (3.29), (3.11), and (3.58). This combining with (4.2) gives (4.4). We finish the proof of Lemma 4.1.

**Lemma 4.2** Assume that (1.7) holds. Then for any \( p > 2 \), there is a positive constant \( C(p, T) \) depending only on \( p, T, \mu, \beta, \gamma, \|\rho_0\|_{L^\infty}, \) and \( \|u_0\|_{H^1} \) such that

\[ \int_0^T (\|G\|_{L^\infty} + \|\nabla G\|_{L^p} + \|\nabla \omega\|_{L^p} + \|\rho \dot{u}\|_{L^p})^{1+1/p} \, dt \]
\[ + \int_0^T t \left( \|\nabla G\|^2_{L^p} + \|\nabla \omega\|^2_{L^p} + \|\dot{u}\|^2_{H^1} \right) \, dt \leq C(p, T). \quad (4.9) \]
For $T$ is a constant

Assume that Proposition 4.3 we directly derive (4.9) from (4.10) and (3.18). We finish the proof of Lemma 4.2.

Noticing that the Gargiardo-Nirenberg inequality and (3.59) yield that

$$
\int_0^T \left( \|\rho \dot{u}\|_{L^p}^{1+1/p} + t \|\dot{u}\|_{H^1}^2 \right) dt 
\leq C(p, T) \int_0^T \left( \|\rho \dot{u}\|_{L^p}^{1/2} \right) \leq C(p, T).
$$

(4.10)

Next, we deduce from standard $L^p$-estimate for elliptic system, (4.11), and (3.18) that

$$
\|\nabla^2 u\|_{L^q} \leq C(1 + \|\nabla u\|_{L\infty}) \|\nabla \rho\|_{L^q} + C \|[\nabla G]\|_{L^q} 
\leq C(1 + \|\nabla u\|_{L\infty}) \|\nabla \rho\|_{L^q} + C \|\rho \dot{u}\|_{L^q}.
$$

(4.14)

Proof. If follows from (2.6), (2.4), and (3.59) that

$$
\|\rho \dot{u}\|_{L^p} \leq C \|\rho \dot{u}\|_{L^2}^{2(p-1)/(p^2-2)} \|\dot{u}\|_{L^2}^{p(p-2)/(p^2-2)} 
\leq C \|\rho \dot{u}\|_{L^2}^{2(p-1)/(p^2-2)} \|\dot{u}\|_{H^1}^{p(p-2)/(p^2-2)} 
\leq C \|\rho \dot{u}\|_{L^2} + C \|\rho \dot{u}\|_{L^2}^{2(p-1)/(p^2-2)} \|\nabla \dot{u}\|_{L^2}^{p(p-2)/(p^2-2)},
$$

which together with (3.59), (4.2), and (2.6) implies that

$$
\int_0^T \left( \|\rho \dot{u}\|_{L^p}^{1+1/p} + t \|\dot{u}\|_{H^1}^2 \right) dt 
\leq C(p, T) \int_0^T \left( \|\rho \dot{u}\|_{L^p}^{1/2} \right) \leq C(p, T).
$$

(4.10)

Following [16], we will prove (4.12). First, denoting by $\Phi \triangleq (\Phi^1, \Phi^2)$ with $\Phi^i \triangleq (2 \mu + \lambda(\rho))\partial_i \rho$ ($i = 1, 2$), one deduces from (1.11) that $\Phi^i$ satisfies

$$
\Phi^i + (u \cdot \nabla) \Phi^i + (2 \mu + \lambda(\rho)) \partial_i u^j \partial_j \rho + \rho \partial_i G + \rho \partial_i P + \Phi^i \div u = 0.
$$

(4.13)

For $q > 2$, multiplying (4.13) by $|\Phi|^{q-2} \Phi^i$ and integrating the resulting equation over $\mathbb{T}^2$, we obtain after integration by parts and using (3.18) that

$$
\frac{d}{dt} \|\Phi\|_{L^q} \leq C(1 + \|\nabla u\|_{L\infty}) \|\nabla \rho\|_{L^q} + C \|[\nabla G]\|_{L^q} 
\leq C(1 + \|\nabla u\|_{L\infty}) \|\nabla \rho\|_{L^q} + C \|\rho \dot{u}\|_{L^q}.
$$

(4.14)

Next, we deduce from standard $L^p$-estimate for elliptic system, (4.11), and (3.18) that

$$
\|\nabla^2 u\|_{L^q} \leq C\|\nabla \div u\|_{L^q} + C \|[\nabla G]\|_{L^q} 
\leq C(\|\div [(2 \mu + \lambda)\div u]\|_{L^2} + C \|[\nabla G]\|_{L^2} + C \|\nabla \omega\|_{L^2} 
\leq C(\|\div [u]\|_{L^\infty} + 1) \|\nabla \rho\|_{L^q} + C \|[\nabla G]\|_{L^q} + C \|\nabla \omega\|_{L^q} 
\leq C(\|\rho \dot{u}\|_{L^q}^{q/(2(q-1))} + 1) \|\nabla \rho\|_{L^q} + C \|\rho \dot{u}\|_{L^q} 
\leq C \|\nabla \rho\|_{L^q}^{(2q-2)/(q-2)} + C \|\rho \dot{u}\|_{L^q} + C.
$$

(4.15)
Then, it follows from Lemma 2.4, (4.11), and (4.15) that
\[
\| \nabla u \|_{L^\infty} \leq C \left( \| \text{div} u \|_{L^\infty} + \| \omega \|_{L^\infty} \right) \log(e + \| \nabla^2 u \|_{L^q}) + C \| \nabla u \|_{L^2} + C
\]
\[
\leq C \left( 1 + \| \rho \dot{u} \|_{L^q}^{q/(2(q-1))} \right) \log(e + \| \rho \dot{u} \|_{L^q} + \| \nabla \rho \|_{L^q}) + C
\]
\[
\leq C \left( 1 + \| \rho \dot{u} \|_{L^q} \right) \log(e + \| \nabla \rho \|_{L^q}).
\]
(4.16)

Substituting (4.16) into (4.14), we deduce from Gronwall’s inequality and (4.9) that
\[
\sup_{0 \leq t \leq T} \| \nabla \rho \|_{L^q} \leq C,
\]
which combining with (4.15) and (4.9) shows
\[
\int_0^T \left( \| \nabla^2 u \|_{L^q}^{(q+1)/q} + t \| \nabla^2 u \|_{L^q}^2 \right) dt \leq C.
\]
(4.17)

Finally, it follows from (2.6), (3.59), (4.2), and (4.18) that
\[
\int_0^T t \| u_t \|_{H^1}^2 dt
\]
\[
\leq C \int_0^T t \left( \| \rho \|_{L^2}^2 + \| \nabla u \|_{L^2}^2 \right) dt
\]
\[
\leq C \int_0^T t \left( \| \rho \|_{L^2}^2 + \| u \cdot \nabla u \|_{L^2}^2 + \| \nabla u \|_{L^2}^2 + \| \nabla (u \cdot \nabla u) \|_{L^2}^2 \right) dt
\]
\[
\leq C + C \int_0^T \| \nabla u \|_{L^2}^2 dt + C \int_0^T t \| u \|_{H^1}^2 \| \nabla^2 u \|_{L^q}^2 dt
\]
\[
\leq C + C \int_0^T \| \nabla u \|_{L^2}^2 \| \nabla^2 u \|_{L^2} dt + C \int_0^T t \| \nabla^2 u \|_{L^q}^2 dt
\]
\[
\leq C.
\]
(4.19)

We obtain from (3.59), (3.18), and (4.17) that
\[
\| \nabla^2 u \|_{L^2} \leq C \| \nabla \omega \|_{L^2} + C \| \nabla \text{div} u \|_{L^2}
\]
\[
\leq C \| \nabla \omega \|_{L^2} + C \| \nabla ((2\mu + \lambda)\text{div} u) \|_{L^2} + C \| \text{div} u \|_{L^{2q/(q-2)}} \| \nabla \rho \|_{L^q}
\]
\[
\leq C \| \nabla \omega \|_{L^2} + C \| \nabla G \|_{L^2} + C + C \| \nabla u \|_{L^2}^{(q-2)/q} \| \nabla^2 u \|_{L^2}^{2/q}
\]
\[
\leq C + \frac{1}{2} \| \nabla^2 u \|_{L^2} + C \| \rho \dot{u} \|_{L^2},
\]
(4.20)

which together with (4.2) gives
\[
\sup_{0 \leq t \leq T} t \| \nabla^2 u \|_{L^2} \leq C.
\]

This combining with (4.17)–(4.19) and (3.59) yields (4.12). The proof of Proposition 4.3 is completed.

5 Proofs of Theorems 1.1–1.3

With all the a priori estimates in Sections 3 and 4 at hand, we are ready to prove the main results of this paper in this section. We first state the global existence of strong solution \((\rho, u)\) provided that (1.7) holds and that \((\rho_0, m_0)\) satisfies (2.1).
Proposition 5.1 Assume that (1.7) holds and that \((\rho_0, m_0)\) satisfies (2.1). Then there exists a unique strong solution \((\rho, u)\) to (1.1)-(1.4) in \(T^2 \times (0, \infty)\) satisfying (2.2) for any \(T \in (0, \infty)\). In addition, \((\rho, u)\) satisfies (4.12) with some positive constant \(C\) depending only on \(T, \mu, \beta, \gamma, \|\rho_0\|_{L^\infty},\) and \(\|u_0\|_{H^1}\) such that. Moreover, if (1.10) holds, there exists some positive constant \(C\) depending only on \(\mu, \beta, \gamma, \|\rho_0\|_{L^\infty},\) and \(\|u_0\|_{H^1}\) such that both (3.58) and (4.3) hold.

Proof. First, standard local existence result, Lemma 2.1, applies to show that the problem (1.1)-(1.4) with initial data \((\rho_0, m_0)\) has a unique local solution \((\rho, u)\), defined up to a positive time \(T_0\) which may depend on \(\inf_{x \in T^2} \rho_0(x)\), and satisfying (2.2) and (2.3).

We set
\[
T^* = \sup \left\{ T \mid \sup_{0 \leq t \leq T} \|(\rho, u)\|_{H^2} < \infty \right\}.
\]

Clearly, \(T^* \geq T_0\). We claim that
\[
T^* = \infty.
\]

Otherwise, \(T^* < \infty\). Then, we claim that there exists a positive constant \(\hat{C}\) which may depend on \(T^*\) and \(\inf_{x \in T^2} \rho_0(x)\) such that, for all \(0 < T < T^*\),
\[
\sup_{0 \leq t \leq T} \|\rho\|_{H^2} \leq \hat{C},
\]
where and what follows, \(\hat{C}\) denotes some generic positive constant depending on \(T^*\) but independent of \(T\). This together with (4.12) contradicts (5.1). The estimates (4.12), (4.3), and (3.58) directly follow from (2.2), Lemma 4.1 and Propositions 4.3 and 3.5.

It remains to prove (5.3). First, standard calculations together with (4.12) yield that for any \(T \in (0, T^*)\),
\[
\inf_{(x, t) \in T^2 \times (0, T)} \rho(x, t) \geq \inf_{x \in T^2} \rho_0(x) \exp \left\{ - \int_0^T \|\text{div} u\|_{L^\infty} \, dt \right\} \geq \hat{C}^{-1}.
\]

We define
\[
\sqrt{p_\rho}(x, t = 0) = \rho_0^{-1/2} (\mu \triangle u_0 + \nabla((\mu + \lambda(\rho_0))\text{div} u_0) - \nabla P(\rho_0)).
\]

Integrating (4.8) with respect to \(t\) over \((0, T)\) together with (2.1), (3.59), and (5.5) yields
\[
\sup_{0 \leq t \leq T} \int \rho |\dot{u}|^2 \, dx + \int_0^T \|\nabla \dot{u}\|_{L^2}^2 \, dt \leq \hat{C}.
\]

This combining with (3.17), (4.20), and (4.12) leads to
\[
\sup_{0 \leq t \leq T} \left( \|\nabla^2 u\|_{L^2} + \|\nabla G\|_{L^2} + \|\nabla \omega\|_{L^2} \right) + \int_0^T \left( \|\nabla^2 G\|_{L^2}^2 + \|\nabla^2 \omega\|_{L^2}^2 \right) \, dt
\]
\[
\leq \hat{C} \sup_{0 \leq t \leq T} \|\rho \dot{u}\|_{L^2} + \hat{C} \int_0^T \|\nabla (\rho \dot{u})\|_{L^2}^2 \, dt
\]
\[
\leq \hat{C} + \hat{C} \int_0^T \left( \|\nabla \rho\|_{L^4}^2 \|\dot{u}\|_{L^{2q/(q-2)}}^2 + \|\nabla \dot{u}\|_{L^2}^2 \right) \, dt
\]
\[
\leq \hat{C} + \hat{C} \int_0^T \|\dot{u}\|_{H^1}^2 \, dt \leq \hat{C},
\]
where in the last inequality, we have used (5.6).

Next, operating $\nabla$ to (4.13) and multiplying the resulting equality by $\nabla \Phi$, we obtain after integration by parts and using (5.4) and (5.7) that

$$
\frac{d}{dt} \| \nabla \Phi \|_{L^2} \leq \hat{C}(1 + \| \nabla u \|_{L^\infty}) (1 + \| \nabla \Phi \|_{L^2} + \| \nabla \rho \|_{L^4}^2 + \| \nabla^2 \rho \|_{L^2})
$$

$$
+ \hat{C} \| \nabla \rho \| \| \nabla^2 u \|_{L^2} + \hat{C} \| \nabla \rho \| \| \nabla G \|_{L^2} + \hat{C} \| \nabla^2 G \|_{L^2}.
$$

(5.8)

Note that (2.4) and (1.12) lead to

$$
\| \nabla^2 \rho \|_{L^2} + \| \nabla \rho \|_{L^4}^2 \leq \hat{C} \| \nabla \Phi \|_{L^2} + \hat{C} \| \nabla \rho \|_{L^4}^2
$$

$$
\leq \hat{C} \| \nabla \Phi \|_{L^2} + \hat{C} \| \nabla \rho \|_{L^4}^{\min(4,q)/2} \| \nabla^2 \rho \|_{L^2}^{(4-\min(4,q))/2}
$$

$$
\leq \hat{C} \| \nabla \Phi \|_{L^2} + \frac{1}{2} \| \nabla^2 \rho \|_{L^2} + \hat{C},
$$

which together with (5.4) yields that

$$
\| \nabla^2 P(\rho) \|_{L^2} + \| \nabla^2 \rho \|_{L^2} + \| \nabla \rho \|_{L^4}^2 \leq \hat{C} \| \nabla^2 \rho \|_{L^2} + \hat{C} \| \nabla \rho \|_{L^4}^2
$$

$$
\leq \hat{C} \| \nabla \Phi \|_{L^2} + \hat{C}.
$$

(5.9)

Then, on one hand, it follows from Holder’s inequality, (2.4), and (1.12) that

$$
\| \| \nabla \rho \| \| \nabla^2 u \|_{L^2} + \| \nabla \rho \| \| \nabla G \|_{L^2}
$$

$$
\leq \hat{C} \| \nabla \rho \|_{L^4} \| \nabla^2 u \|_{L^2}^{1-2/q} \| \nabla^3 u \|_{L^2}^{2/q} + \hat{C} \| \nabla \rho \|_{L^4} \| \nabla G \|_{L^2}^{1-2/q} \| \nabla^2 G \|_{L^2}^{2/q}
$$

$$
\leq \hat{C} \| \nabla^3 u \|_{L^2} + \hat{C} \| \nabla^2 G \|_{L^2}.
$$

(5.10)

On the other hand, the $L^2$-estimate of elliptic system leads to

$$
\| \nabla^3 u \|_{L^2} = \| \nabla \triangle u \|_{L^2}
$$

$$
\leq \hat{C} \| \nabla^2 \text{div} u \|_{L^2} + \hat{C} \| \nabla^2 \omega \|_{L^2}
$$

$$
\leq \hat{C} \| \nabla^2 ((2 \mu + \lambda(\rho)) \text{div} u) \|_{L^2} + \hat{C} \| \nabla \rho \| \| \nabla^2 u \|_{L^2}
$$

$$
+ \hat{C} \| \nabla^2 \rho \| \| \nabla \omega \|_{L^2} + \hat{C} \| \nabla \rho \| \| \nabla \omega \|_{L^2}
$$

$$
\leq \hat{C} \| \nabla^2 G \|_{L^2} + \hat{C} \| \nabla^2 \rho \|_{L^2} + \hat{C} \| \nabla \rho \|_{L^4}^2 + \hat{C} \| \nabla^2 \omega \|_{L^2}
$$

$$
+ \hat{C} \| \nabla \rho \| \| \nabla^2 u \|_{L^2} + \hat{C} \| \nabla \rho \|_{L^4}^2 \| \nabla u \|_{L^\infty}.
$$

(5.11)

Substituting (5.11) and (5.9) into (5.10) leads to

$$
\| \| \nabla \rho \| \| \nabla^2 u \|_{L^2} + \| \nabla \rho \| \| \nabla G \|_{L^2}
$$

$$
\leq \hat{C} \| \nabla^2 G \|_{L^2} + \hat{C} \| \nabla^2 \omega \|_{L^2} + \hat{C}(1 + \| \nabla \Phi \|_{L^2}) (1 + \| \nabla u \|_{L^\infty}),
$$

which together with (5.8) and (5.9) gives

$$
\frac{d}{dt} \| \nabla \Phi \|_{L^2} \leq \hat{C}(1 + \| \nabla u \|_{L^\infty}) (\| \nabla \Phi \|_{L^2} + 1) + \hat{C} \| \nabla^2 G \|_{L^2} + \hat{C} \| \nabla^2 \omega \|_{L^2}.
$$

This combining with (5.7) and Gronwall’s inequality yields

$$
\sup_{0 \leq t \leq T} \| \nabla \Phi \|_{L^2} \leq \hat{C},
$$
which together with (5.9) implies (5.3). The proof of Proposition 5.1 is finished.

Proof of Theorem 1.1. Let \((\rho_0, m_0)\) satisfying (1.8) be the initial data as described in Theorem 1.1. For constant \(\delta \in (0, 1)\), we define

\[
\rho_0^\delta \triangleq j_\delta \ast \rho_0 + \delta \geq \delta > 0, \quad u_0^\delta \triangleq j_\delta \ast u_0, \quad m_0^\delta = \rho_0^\delta u_0^\delta, \tag{5.12}
\]

where \(j_\delta\) is the standard mollifying kernel of width \(\delta\). Hence, we have \(\rho_0^\delta, u_0^\delta \in H^\infty\), and

\[
\lim_{\delta \to 0} \left( \|\rho_0^\delta - \rho_0\|_{W^{1,q}} + \|u_0^\delta - u_0\|_{H^1} \right) = 0.
\]

Proposition 5.1 thus yields that the problem \((1.1)-(1.4)\) with \((\rho_0, m_0)\) being replaced by \((\rho_0^\delta, m_0^\delta)\) has a unique global strong solution \((\rho^\delta, u^\delta)\) satisfying (4.12) for any \(T > 0\) and for some \(C\) independent of \(\delta\). Moreover, if (1.10) holds, there exists some positive constant \(C\) depending only on \(\mu, \beta, \gamma, \|\rho_0\|_{L^\infty}\), and \(\|u_0\|_{H^1}\) such that \((\rho^\delta, u^\delta)\) satisfies (3.58) and (4.3). Letting \(\delta \to 0\), standard arguments (see [14, 21, 26, 30]) thus show that the problem \((1.1)-(1.4)\) has a global strong solution \((\rho, u)\) satisfying the properties listed in Theorem 1.1 except (1.12) and the uniqueness of \((\rho, u)\) satisfying (1.9). Moreover, \((\rho, u)\) satisfies (3.58) and (4.3) for some positive constant \(C\) depending only on \(\mu, \beta, \gamma, \|\rho_0\|_{L^\infty}\), and \(\|u_0\|_{H^1}\) provided (1.10) holds.

Since the uniqueness of \((\rho, u)\) satisfying (1.9) is similar to that of Germain [12] and (1.12) will be proved in Theorem 1.2 we finish the proof of Theorem 1.1.

Proof of Theorem 1.2. Let \((\rho_0, m_0)\) satisfying (1.13) be the initial data as described in Theorem 1.2. For constant \(\delta \in (0, 1)\), let \((\rho_0^\delta, u_0^\delta)\) be as in (5.12). Hence, we have \(\rho_0^\delta, u_0^\delta \in H^\infty\), and for any \(p > 1\),

\[
\lim_{\delta \to 0} \left( \|\rho_0^\delta - \rho_0\|_{L^p} + \|u_0^\delta - u_0\|_{H^1} \right) = 0.
\]

Moreover,

\[
\rho_0^\delta \rightharpoonup \rho_0 \text{ weakly }^* \text{ in } L^\infty, \text{ as } \delta \to 0.
\]

Proposition 5.1 thus yields that the problem \((1.1)-(1.4)\) with \((\rho_0, m_0)\) being replaced by \((\rho_0^\delta, m_0^\delta)\) has a unique global strong solution \((\rho^\delta, u^\delta)\) satisfying (3.59), (4.2), and (4.9), for any \(T > 0\) and for some \(C\) independent of \(\delta\). Moreover, if (1.10) holds, there exists some positive constant \(C\) depending only on \(\mu, \beta, \gamma, \|\rho_0\|_{L^\infty}\), and \(\|u_0\|_{H^1}\) such that \((\rho^\delta, u^\delta)\) satisfies (3.58), (3.33), and (4.3).

We modify the compactness arguments in [26, 30] to obtain the compactness results of \((\rho^\delta, u^\delta)\).

First, it follows from (3.59) and (4.9) that

\[
\sup_{0 \leq t \leq T} \|u_t^\delta\|_{H^1} + \int_0^T t\|u_t^\delta\|_{L^2}^2 dt \leq C,
\]

which together with the Aubin-Lions lemma gives that, up to a subsequence,

\[
\begin{cases}
\quad u^\delta \rightharpoonup u \text{ weakly }^* \text{ in } L^\infty(0, T; H^1), \\
\quad u^\delta \rightarrow u \text{ strongly in } C([\tau, T]; L^p),
\end{cases}
\]

for any \(\tau \in (0, T)\) and \(p \in [1, \infty)\).
Next, let $A^\delta \triangleq (2\mu + \lambda(\rho^\delta))\text{div}\rho^\delta - P(\rho^\delta)$. One thus deduces from (3.13), (3.14), (3.59), and (4.9) that

$$\int_0^T \left( \|A^\delta\|_{L^2}^{4/3} + \|\omega^\delta\|_{H^1}^2 + \|A^\delta\|_{H^1}^2 + t\|\omega^\delta_t\|_{L^2}^2 + t\|A^\delta_t\|_{L^2}^2 \right) dt \leq C, \tag{5.13}$$

which implies that, up to a subsequence,

$$\begin{align*}
A^\delta &\rightharpoonup A \text{ weakly * in } L^{4/3}(0,T;L^\infty), \\
\omega^\delta &\rightharpoonup \omega = \text{curl}\text{u}, \ A^\delta \rightarrow A \text{ strongly in } L^2(\tau,T;L^p),
\end{align*}$$

for any $\tau \in (0,T)$ and $p \in [1,\infty)$.

Next, to obtain the strong limit of $\rho^\delta$, we deduce from (3.59) that, up to a subsequence,

$$\rho^\delta \rightharpoonup \rho \text{ weakly * in } L^\infty(0,T;L^\infty).$$

Let $f(s)$ be an arbitrary continuous function on $[0,C]$ with $C$ as in (3.59). Then, we have that, up to a subsequence, $f(\rho^\delta)$ converges weakly * in $L^\infty(0,T;L^\infty)$. Denote the weak-* limit by $\overline{f(\rho)}$:

$$f(\rho^\delta) \rightharpoonup \overline{f(\rho)} \text{ weakly * in } L^\infty(0,T;L^\infty).$$

Noticing that,

$$\text{div}\rho^\delta = \phi(\rho^\delta)A^\delta + \phi(\rho^\delta)P(\rho^\delta),$$

with $\phi(s) \triangleq 1/(2\mu + \lambda(s))$, we have

$$\text{div}\text{u} = \overline{\phi(\rho)A} + \overline{\phi(\rho)P(\rho)}, \quad \text{a.e. in } T^2 \times (0,T).$$

From (1.11), we obtain

$$\overline{(\rho^2)}_t + \text{div}(\overline{\rho^2u}) + \overline{A\rho^2\phi(\rho)} + \overline{\rho^2\phi(\rho)P(\rho)} = 0, \quad \text{in } D'(T^2 \times (0,\infty)).$$

Using [22] Lemma 2.3, we get by standard arguments that

$$(\rho^2)_t + \text{div}(\rho^2\text{u}) + A\rho^2\phi(\rho) + \rho^2\phi(\rho)P(\rho) = 0, \quad \text{in } D'(T^2 \times (0,\infty)).$$

Thus, for $\Phi \triangleq \overline{\rho^2} - \rho^2 \geq 0$, we have

$$\begin{align*}
\Phi_t + \text{div}(\Phi\text{u}) + A(\rho^2\overline{\phi(\rho)} - \rho^2\phi(\rho)) + A\rho^2(\phi(\rho) - \overline{\phi(\rho)}) + \rho^2\phi(\rho)P(\rho) \\
- \rho^2\phi(\rho)P(\rho) + \rho^2 \left( \phi(\rho)P(\rho) - \overline{\phi(\rho)P(\rho)} \right) &= 0, \quad \text{in } D'(T^2 \times (0,\infty)), \\
\Phi(x,t = 0) &= 0, \quad \text{a.e. } x \in T^2.
\end{align*}$$

$$\overline{(\rho^2)}_t + \text{div}(\overline{\rho^2u}) + \overline{A\rho^2\phi(\rho)} + \overline{\rho^2\phi(\rho)P(\rho)} = 0, \quad \text{in } D'(T^2 \times (0,\infty)).$$

By writing $(\rho^\delta)^2 - \rho^2 = 2\rho(\rho^\delta - \rho) + (\rho^\delta - \rho)^2$, we see that, up to a subsequence,

$$\lim_{\delta \to 0} \|\rho^\delta - \rho\|_{L^2}^2(t) \leq \int \Phi(x,t)dx, \quad \text{for } t > 0.$$

Also, for any $f(s) \in C^2([0,C])$ and $h(x) \in L^\infty(T^2)$, noticing that

$$f(\rho^\delta) - f(\rho) = f'(\rho)(\rho^\delta - \rho) + \int_0^1 \theta \int_0^1 f''(\rho + \theta\alpha)(\rho^\delta - \rho))d\alpha d\theta(\rho^\delta - \rho)^2,$$
we deduce from (3.59) that
\[
\left| \int h(x) \overline{(\rho f'(\rho) - f(\rho))} \, dx \right| \leq M \|h\|_{L^\infty} \int \Phi \, dx,
\]
for some constant $M > 0$. In particular, noticing that
\[
f_1(s) \triangleq s^2 \phi(s) \in C^2([0, C]), \quad f_2(s) \triangleq s^2 \phi(s) P(s) \in C^2([0, C]),
\]
we have
\[
\left| \int A \left( \overline{\rho^2 \phi(\rho)} - \rho^2 \phi(\rho) \right) \, dx \right| \leq M \|A\|_{L^\infty} \int \Phi \, dx,
\]
and
\[
\left| \int (\overline{\rho^2 \phi(\rho) P(\rho)} - \rho^2 \phi(\rho) P(\rho)) \, dx \right| \leq M \int \Phi \, dx.
\]
Let $g(s) \in C^1([0, C]) \cap C^2((0, C])$ be such that for any $\rho, s \in [0, C]$,
\[
\left| \rho^2 \int_0^1 \theta \int_0^1 g''(\rho + \theta \alpha(s - \rho)) d\alpha d\theta \right| \leq M.
\]
Note that
\[
\rho^2 (g(\rho^\delta) - g(\rho)) - \rho^2 g'(\rho)(\rho^\delta - \rho)
\]
\[
= \rho^2 \int_0^1 \theta \int_0^1 g''(\rho + \theta \alpha(\rho^\delta - \rho)) d\alpha d\theta (\rho^\delta - \rho)^2,
\]
which together with (5.18) yields that for any $h(x) \in L^\infty(T^2)$,
\[
\left| \int h(x) \rho^2 (\overline{g(\rho)} - g(\rho)) \, dx \right| \leq M \|h\|_{L^\infty} \int \Phi \, dx.
\]
Let $g_1(s) \triangleq \phi(s)$ and $g_2(s) \triangleq \phi(s) P(s)$. Since $g_i \in C^1([0, C]) \cap C^2((0, C]) (i = 1, 2)$ satisfy (5.18), from (5.19) we obtain that
\[
\left| \int A \rho^2 (\overline{\phi(\rho)} - \phi(\rho)) \, dx \right| \leq M \|A\|_{L^\infty} \int \Phi \, dx,
\]
and that
\[
\left| \int \rho^2 (\overline{\phi(\rho) P(\rho)} - \phi(\rho) P(\rho)) \, dx \right| \leq M \int \Phi \, dx.
\]
Substituting (5.16), (5.17), (5.20), and (5.21) into (5.15), after using Gronwall's inequality and (5.13), we arrive at
\[
\Phi = 0 \text{ a.e. in } T^2 \times (0, T),
\]
which gives that, up to a subsequence,
\[
\rho^\delta \to \rho \text{ strongly in } L^p(T^2 \times (0, T)),
\]
for any $p \in [1, \infty)$. This combining with (5.14) implies that, up to a subsequence,
\[
G^\delta \to G = (2\mu + \lambda(\rho)) \text{div} u + P(\rho) - \bar{P}, \text{ strongly in } L^2(T^2 \times (\tau, T)),
\]
for any \( \tau \in (0, T) \), where and what follows, \( \overline{\mathcal{T}} \) denotes the mean value of \( f \) over \( \mathbb{T}^2 \) as in (1.6).

Standard arguments thus show that the limit \((\rho, u)\) is a global weak solution of (1.1) - (1.2).

To finish the proof of Theorem 1.2, it only remains to prove (1.12).

First, it follows from (3.58) that
\[
\int_1^\infty \|P(\rho^\delta) - \overline{P(\rho^\delta)}\|_{L^2}^2 dt \leq C \int_1^\infty \left( \|G^\delta\|_{L^2}^2 + (A^\delta_0(t))^2 \right) dt \leq C,
\]
which combining with (5.22), (5.14), (5.23), and (3.58) gives
\[
\int_1^\infty \left( \|P(\rho) - \overline{P}\|_{L^2}^2 + \|G\|_{L^2}^2 + \|\omega\|_{L^2}^2 \right) dt \leq C. \tag{5.24}
\]

Simple calculations lead to
\[
\frac{d}{dt}(\|P(\rho^\delta) - \overline{P(\rho^\delta)}\|_{L^2}^2) = 2 \int (P(\rho^\delta) - \overline{P(\rho^\delta)})(P(\rho^\delta) - \overline{P(\rho^\delta)}) dx
\]
\[
= -2 \int (P(\rho^\delta) - \overline{P(\rho^\delta)})(u^\delta \cdot \nabla P(\rho^\delta) - \nabla \overline{P(\rho^\delta)}) + \rho^\delta P'(\rho^\delta) \text{div} u^\delta dx
\]
\[
+ 2 \int (P(\rho^\delta) - \overline{P(\rho^\delta)}) dx \int (\rho^\delta P'(\rho^\delta) - \overline{P(\rho^\delta)}) \text{div} u^\delta dx
\]
\[
\leq C\|P(\rho^\delta) - \overline{P(\rho^\delta)}\|_{L^2}^2 + C\|\nabla u^\delta\|_{L^2}^2
\]
\[
\leq C\|P(\rho^\delta) - \overline{P(\rho^\delta)}\|_{L^2}^2 + C\|G^\delta\|_{L^2}^2 + C\|\omega^\delta\|_{L^2}^2,
\]
which gives that, for any \( s, t \in [N, N + 1] \),
\[
\|P(\rho^\delta) - \overline{P(\rho^\delta)}\|_{L^2}^2(t) - \|P(\rho^\delta) - \overline{P(\rho^\delta)}\|_{L^2}^2(s)
\]
\[
\leq C \int_N^{N+1} \left( \|P(\rho^\delta) - \overline{P(\rho^\delta)}\|_{L^2}^2 + \|G^\delta\|_{L^2}^2 + \|\omega^\delta\|_{L^2}^2 \right) dt. \tag{5.25}
\]

Integrating (5.25) with respect to \( s \) over \((N, N + 1)\) yields that
\[
\sup_{N \leq t \leq N+1} \|P(\rho^\delta) - \overline{P(\rho^\delta)}\|_{L^2}^2(t)
\]
\[
\leq C \int_N^{N+1} \left( \|P(\rho^\delta) - \overline{P(\rho^\delta)}\|_{L^2}^2 + \|G^\delta\|_{L^2}^2 + \|\omega^\delta\|_{L^2}^2 \right) dt.
\]

From (5.22), (5.23), and (5.14), we have
\[
\sup_{N \leq t \leq N+1} \|P(\rho) - \overline{P}\|_{L^2}^2(t) \leq C \int_N^{N+1} \left( \|P(\rho) - \overline{P}\|_{L^2}^2 + \|G\|_{L^2}^2 + \|\omega\|_{L^2}^2 \right) dt.
\]

Letting \( N \to \infty \), this combining with (5.24) yields that
\[
\lim_{t \to 0} \|P(\rho) - \overline{P}\|_{L^2}^2(t) = 0. \tag{5.26}
\]

Next, standard arguments together with [22, Lemma 2.3] and (1.1) yield that \( P(\rho) \) satisfies
\[
(P(\rho))_t + \text{div}(P(\rho)u) + (\gamma - 1)P(\rho)\text{div} u = 0, \quad \text{in } \mathcal{D}'(\mathbb{T}^2 \times (0, \infty)),
\]
which gives that
\[
\int_1^\infty \left| \frac{d}{dt} P \right| dt \leq C \int_1^\infty \left| \int (P - \bar{P}) \text{div} u dx \right| dt \\
\leq C \int_1^\infty (\|P - \bar{P}\|_{L^2}^2 + \|\nabla u\|_{L^2}^2) dt \leq C,
\]
due to (3.58). Hence, there exists some positive constant \( \rho_s \) such that
\[
\lim_{t \to \infty} \bar{P}(t) = \rho_s^2,
\]
due to \( 0 < \bar{\rho}_0^2 \leq \bar{P} \leq C \). This combining with (5.26) and (3.58) yields that
\[
\lim_{t \to \infty} \|\rho - \rho_s\|_{L^p}(t) = 0,
\]
for any \( p \in [1, \infty) \). Thus, (3.6) gives
\[
\lim_{t \to \infty} \|\rho - \bar{\rho}_0\|_{L^p}(t) = 0, \quad (5.27)
\]
for any \( p \in [1, \infty) \).

Finally, similar to (5.25), from (3.33) and (3.58), we have
\[
(A_1^\delta(t))^2 \leq (A_1^\delta(s))^2 + C \int_N^{N+1} (A_3^\delta(t))^2 dt,
\]
for any \( s, t \in [N, N+1] \). This gives
\[
\sup_{N \leq t \leq N+1} (A_1^\delta(t))^2 \leq C \int_N^{N+1} \left( (A_1^\delta(t))^2 + (A_3^\delta(t))^2 \right) dt \\
\leq C \int_N^{N+1} \left( \|P(\rho^\delta) - \bar{P}(\rho^\delta)\|_{L^2}^2 + \|G^\delta\|_{L^2}^2 + \|\omega^\delta\|_{L^2}^2 \right) dt,
\]
which together with (5.14), (5.23), (5.24), and the fact that
\[
\|G^\delta\|_{L^2}^2 + \|\omega^\delta\|_{L^2}^2 \leq C(A_1^\delta(t))^2,
\]
leads to
\[
\lim_{t \to 0} (\|G\|_{L^2}^2 + \|\omega\|_{L^2}^2)(t) = 0.
\]
Because of (5.26), this shows
\[
\lim_{t \to \infty} \|\nabla u\|_{L^2} \leq C \lim_{t \to \infty} (\|G\|_{L^2} + \|\omega\|_{L^2} + \|P - \bar{P}\|_{L^2})(t) = 0,
\]
which combining with (4.3) and (5.27) directly yields (1.12). The proof of Theorem 1.2 is completed.

Proof of Theorem 1.3. Since the proof of Theorem 1.3 is similar to that of [20, Theorem 1.2], we omit it here.
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