Discrete Quantum Field Theories
and the Intersection Form

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Abstract

It is shown that the standard mod-$p$ valued intersection form can be used to define Boltzmann weights of subdivision invariant lattice models with gauge group $\mathbb{Z}_p$. In particular, we discuss a four dimensional model which is based upon the assignment of field variables to the 2-simplices of the simplicial complex. The action is taken to be the intersection form defined on the second cohomology group of the complex, with coefficients in $\mathbb{Z}_p$. Subdivision invariance of the theory follows when the coupling constant is quantized and the field configurations are restricted to those satisfying a mod-$p$ flatness condition. We present an explicit computation of the partition function for the manifold $\pm \mathbb{C}P^2$, demonstrating non-triviality.

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1 Introduction

In [1, 2], a general strategy for the construction of subdivision invariant lattice models with $Z_p$ gauge group has been presented. One of the main ingredients in these models is the assignment of field variables (termed colours) to simplices of various dimensions. Subdivision invariance is secured by restricting to quantized values of the coupling parameter, and summing over field configurations which satisfy a mod-$p$ flatness condition, which are thus in correspondence with the cohomology classes of the simplicial complex $K$.

In three dimensions, it was shown [1] that the model of Dijkgraaf-Witten [3] is defined in terms of a 1-colour field $A$, with an action of the form $A \cup \delta A$, where the cup product and coboundary operator are defined in [1]. Of special interest is a model in four dimensions which relies on the introduction of a 2-colour $B$ and a 1-colour $A$, with an action of the form $B \cup \delta A$. Indeed, non-triviality of this model has been established in [2].

In each of these examples, one notes a formal resemblance of the action to the mod-$p$ intersection form. However, since these actions are “kinetic” in the sense that they contain the coboundary operator, this resemblance is indeed only formal.

On the contrary, the purpose of the present note is to observe that one can consider models defined with an action given in terms of the standard mod-$p$ intersection form. The basic model which we shall concentrate on here is defined in terms of a 2-colour $B$ with an action of the form $B \cup B$. In this case, subdivision invariance requires the coupling to be parametrized as a $p$th root of unity. Due to this fact, the action is identified with the mod-$p$ intersection form on the second cohomology group of $K$ with coefficients in $Z_p$, evaluated on the fundamental homology cycle of $K$.

2 Subdivision Invariant Models and the Intersection Form

For details of the general formalism we refer the reader to [1, 4]. Let us begin by recalling the four dimensional model. The Boltzmann weight of an
ordered 4-simplex \([0, 1, 2, 3, 4]\) is defined by:

\[
W[[0, 1, 2, 3, 4]] = \exp\{\beta < B \cup \delta A, [0, 1, 2, 3, 4]>\} = \exp\{\beta B_{012} (A_{23} + A_{34} - A_{24})\},
\]

(1)

where \(B\) and \(A\) are 2- and 1-colour fields. When the gauge group is taken to be \(Z_p\), a \(k\)-colour field is just an assignment of an integer in the set \(\{0, \ldots, p-1\}\) to each of the \(k\)-simplices in the simplicial complex. Here, \(\beta\) is the coupling parameter, and we shall also find it convenient to use the scale \(s = \exp[\beta]\).

One can now check the behaviour of this theory under subdivision of the underlying simplicial complex. It will suffice to recall here the behaviour when an additional vertex \(x\) is placed at the centre of the 4-simplex \([0, 1, 2, 3, 4]\), and linked to the other 5 vertices. The original 4-simplex is then replaced by an assembly of five 4-simplices, written symbolically as:

\[
[0, 1, 2, 3, 4] \rightarrow [x, 1, 2, 3, 4] - [x, 0, 2, 3, 4] + [x, 0, 1, 3, 4] - [x, 0, 1, 2, 4] + [x, 0, 1, 2, 3],
\]

(2)

where we declare the new vertex \(x\) to be the first in the total ordering of all vertices.

One easily verifies that the Boltzmann weight behaves in the following way under such a subdivision move:

\[
W[[0, 1, 2, 3, 4]]s^{-\delta B \cup \delta A, [x, 0, 1, 2, 3, 4]} = W[[x, 1, 2, 3, 4]]
\]

(3)

\[
W[[x, 0, 2, 3, 4]]^{-1} W[[x, 0, 1, 3, 4]] W[[x, 0, 1, 2, 4]]^{-1} W[[x, 0, 1, 2, 3]].
\]

It is clear that subdivision invariance can be secured once the insertion factor on the left hand side is trivialized. This is accomplished by imposing a quantization of the coupling, \(s^{\beta} = 1\), and restricting the colourings to those satisfying the “flatness” conditions

\[
[\delta B] = [\delta A] = 0.
\]

(4)

Thus, on the 2-simplex \([0, 1, 2]\), the 1-colour field is restricted by

\[
[\delta A]_{012} \equiv [A_{12} - A_{02} + A_{01}] = 0,
\]

(5)

while on the 3-simplex \([0, 1, 2, 3]\), the restriction on the 2-colour takes the form:

\[
[\delta B]_{012} \equiv [B_{123} - B_{023} + B_{013} - B_{012}] = 0.
\]

(6)
Here the bracket notation indicates that the quantity is to be taken modulo $p$ ($[x] = 0$ means $x = 0 \mod p$). With these restrictions, the product $\delta B \cup \delta A$ is a multiple of $p^2$ and hence the above insertion becomes unity.

It is also worth noting that the flatness conditions (4) are not the classical field equations which result from the action $B \cup \delta A$. One is genuinely summing over field configurations in the partition function which can have non-zero field strength ($\delta A$ or $\delta B$).

The subdivision invariant Boltzmann weight for the 4-simplex $[0, 1, 2, 3, 4]$ is then given by:

$$W[[0, 1, 2, 3, 4]] = \exp\left\{2\piik\frac{B_{012}}{p^2}(A_{23} + A_{34} - [A_{23} + A_{34}])\right\}, \quad (7)$$

with $k \in \{0, 1, \cdots, p - 1\}$.

One other important feature of the Boltzmann weight is that it possesses a local gauge invariance when the simplicial complex is closed. The gauge transformation of the $A$ field defined on the ordered 1-simplex $[0, 1]$ is defined by:

$$A'_{01} = [A - \delta \omega]_{01} = [A_{01} - \omega_1 + \omega_0], \quad (8)$$

where $\omega$ is a 0-colour field defined on the vertices of the complex. For the 2-colour field $B$ defined on the ordered 2-simplex $[0, 1, 2]$, we have a gauge transformation given by:

$$B'_{012} = [B - \delta \lambda]_{012} = [B_{012} - \lambda_{12} + \lambda_{02} - \lambda_{01}], \quad (9)$$

where $\lambda$ is a 1-colour defined on 1-simplices.

Invariance of the theory under the above transformations is not manifest, but requires both the quantization of the coupling parameter, together with the restriction on the allowed field configurations. One easily finds that under the transformation of $B$,

$$s^{B \cup \delta A} = s^{B \cup \delta A} s^{-\delta \lambda \cup \delta A} = s^{B \cup \delta A} s^{-\delta(\lambda \cup \delta A)}, \quad (10)$$

where the first equality uses the fact that $\delta A$ is proportional to $p$ due to the flatness constraint, and that $s$ is a $p^2$-root of unity. Hence the Boltzmann weight is invariant up to a total boundary term and the product of
all these cancels for a closed oriented complex. Invariance under the $A$ field transformation follows in the same way if one first notes the simple identity,

$$s^{B \delta A} = s^{-\delta B \cup A} s^{\delta (B \cup A)} . \quad (11)$$

Similarly, in three dimensions, a Boltzmann weight of the form $[1]$: 

$$W[[0, 1, 2, 3]] = \exp\{\beta < A \cup \delta A, [0, 1, 2, 3] >\} = \exp\{\beta A_{01} (A_{12} + A_{23} - A_{13})\} , \quad (12)$$

leads to a subdivision invariant model, which is known as the Dijkgraaf-Witten model $[3]$. 

In each of these models one notices that the action is "kinetic" in the sense that the coboundary operator is involved. As a result, these actions bear only a formal resemblance to the mod-$p$ valued intersection form.

Let us now deal with the main order of business which is to consider models which are indeed based upon the intersection form. In four dimensions, let us consider again the 2-colour field $B$, along with the Boltzmann weight

$$W[[0, 1, 2, 3, 4]] = \exp\{\beta < B \cup B, [0, 1, 2, 3, 4] >\} = \exp\{\beta B_{012} B_{234}\} . \quad (13)$$

The behaviour of this model under the subdivision move described above is given by:

$$W[[0, 1, 2, 3, 4]] s^{-<\delta B \cup B + B \cup \delta B >, [x, 0, 1, 2, 3, 4]} = W[[x, 1, 2, 3, 4]] \quad (14)$$

$$W[[x, 0, 2, 3, 4]]^{-1} W[[x, 0, 1, 3, 4]] W[[x, 0, 1, 2, 4]]^{-1} W[[x, 0, 1, 2, 3]] .$$

In this case, we see that if the coupling parameter is chosen to satisfy $s^p = 1$, and the field configurations restricted to satisfy the flatness conditions, we again have a subdivision invariant Boltzmann weight.

The fact that the subdivision invariant Boltzmann weight requires the scale parameter to a $p$th root of unity means it can equally well be written as:

$$W[[0, 1, 2, 3, 4]] = s^{[B_{012} B_{234}]} , \quad (15)$$

where the square brackets indicate that the product of $B$ fields is now taken modulo $p$. Thus, for a given closed simplical complex $K$, the form of the
action is specified as the mod-$p$ intersection form on the second cohomology group (with $Z_p$ coefficients) evaluated on the fundamental homology cycle of $K$.

The verification of gauge invariance, and independence of the choice of vertex ordering follows as described in [1, 2].

3 The Partition Function for $CP^2$

We continue in this section by evaluating the partition function of the $B \cup B$ theory (equation (13)) for complex projective space. An economical simplicial complex for the manifold $CP^2$ with a minimal number of 9 vertices has been given in [4], and we label its vertices by elements in the set \{1, ..., 9\}. The complex is fully determined by specifying the 4-simplices which are 36 in number and are given explicitly by,

\[ +[1, 2, 4, 5, 6] + [4, 5, 7, 8, 9] + [1, 2, 3, 7, 8] + [2, 3, 4, 5, 6] \\
+ [5, 6, 7, 8, 9] + [1, 2, 3, 8, 9] - [1, 3, 4, 5, 6] - [4, 6, 7, 8, 9] \\
- [1, 2, 3, 7, 9] - [1, 2, 4, 5, 9] - [3, 4, 5, 7, 8] - [1, 2, 6, 7, 8] \\
- [2, 3, 5, 6, 7] - [1, 5, 6, 8, 9] - [2, 3, 4, 8, 9] - [1, 3, 4, 6, 8] \\
- [2, 4, 6, 7, 9] - [1, 3, 5, 7, 9] + [2, 3, 4, 6, 9] + [3, 5, 6, 7, 9] \\
+ [1, 3, 6, 8, 9] + [1, 3, 4, 5, 7] + [1, 4, 6, 7, 8] + [1, 2, 4, 7, 9] \\
- [1, 2, 5, 6, 8] - [2, 4, 5, 8, 9] - [2, 3, 5, 7, 8] + [1, 3, 5, 6, 9] \\
+ [3, 4, 6, 8, 9] + [2, 3, 6, 7, 9] + [1, 2, 4, 6, 7] + [1, 4, 5, 7, 9] \\
+ [1, 3, 4, 7, 8] - [2, 3, 4, 5, 8] - [2, 5, 6, 7, 8] - [1, 2, 5, 8, 9] \], \quad (16)

where the signs denote the relative orientations of each simplex. The corresponding simplicial complex for the orientation reversed complex projective space, $-CP^2$, is obtained by reversing the signs of each 4-simplex in (16).

The number of simplices of each dimension contained in $K$ is:

\[
0 - \text{simplices} : 9 \\
1 - \text{simplices} : 36 \\
2 - \text{simplices} : 84
\]
\[ \text{3} - \text{simplices} : 90 \]
\[ \text{4} - \text{simplices} : 36 \]  
\[ (17) \]

We must first solve the flatness conditions \((6)\) subject to the gauge equivalence \((9)\). Here, we have 84 \(B\) fields subject to 90 constraints. As always when solving these constraints, it is convenient to make use of the gauge freedom, and one readily determines that a maximal number of 28 fields can be gauge fixed. The resulting solution is expressed in terms of a single independent field variable \(x \in \{0, 1, \cdots, p - 1\}\).

As described in \([2]\), a certain scaling factor is required in the definition of the subdivision invariant partition function. Since in the present case we have only a single 2-colour field, the partition function is defined by:

\[ Z = \frac{1}{|G|^{|N_1 - N_0|}} \sum_{flat} W[K] \]  
\[ (18) \]

For the gauge group \(Z_2\), we have two independent solutions and the partition function evaluated at the non-trivial root of unity, \(s = -1\) in equation \((15)\), takes the form:

\[ \begin{align*} 
Z[\pm CP^2, Z_2] &= \frac{1}{2^{27}} 2^{28} (1 - 1) = 0 , \\
\end{align*} \]

where the factor \(2^{28}\) accounts for the gauge equivalent copies of the solution.

With gauge group \(Z_3\), there are three gauge inequivalent solutions to the flatness constraints and the partition function evaluated at \(s = \exp[2\pi i/3]\) is given by:

\[ \begin{align*} 
Z[+CP^2, Z_3] &= \frac{1}{3^{27}} 3^{28} (1 + 2 \exp[-\frac{2\pi i}{3}]) = -3 \sqrt{3} i , \\
Z[-CP^2, Z_3] &= \frac{1}{3^{27}} 3^{28} (1 + 2 \exp[\frac{2\pi i}{3}]) = +3 \sqrt{3} i . 
\end{align*} \]

We see therefore that the partition function is sensitive to orientation. Of course, a reversal of orientation always takes \(Z\) into its complex conjugate.

For the sake of comparison, and in order to establish the normalization, it is useful to compare the above result with that of the 4-sphere \(S^4\). A
simplicial complex is given simply by the boundary of a single 5-simplex; here one has \( N_0 = 6, N_1 = 15, N_2 = 20, N_3 = 15, \) and \( N_4 = 6. \) The partition function is,

\[
Z[S^4, G] = \frac{1}{|G|^{15-6}} |G|^{10} 1 = |G|. \tag{21}
\]

4 Observables

In any quantum field theory, one typically defines a set of gauge invariant observable operators which yield additional interesting information beyond the partition function. To illustrate the general structure of these observables, let us consider the \( B \cup \delta A \) model in four dimensions. It is evident that one can construct a gauge invariant observable involving the \( A \) field as follows:

\[
W(\gamma_1) = \exp\{\beta' < A, \gamma_1 >\}, \tag{22}
\]

where \( \gamma_1 \) symbolically denotes a homology 1-cycle in the complex \( K. \) Here, \( \beta' \) is a coupling parameter, and we shall also use the notation \( s' = \exp[\beta']. \)

In order for this quantity to be gauge invariant, and dependent only on the homology class of \( \gamma_1, \) we are again forced to quantize this coupling. From (8), we see that when \( (s')^p = 1, \) the gauge transformed observable is given by

\[
W'(\gamma_1) = \exp\{\beta' < A - \delta \omega, \gamma_1 >\}
= W(\gamma_1)(s' - <\delta \omega, \gamma_1>)
= W(\gamma_1). \tag{23}
\]

This is a simple consequence of the fact that \( ([x+y] - x - y) \) takes the value 0 or \( p, \) and that \( \gamma_1 \) is a homology cycle.

A similar arugument shows that \( W(\gamma_1) \) depends only on the homology class of the cycle \( \gamma_1. \) Consequently, such an observable is trivial when \( \gamma_1 \) is homologically trivial.

Additionally, a gauge invariant observable involving the \( B \) field can be defined by

\[
W(\gamma_2) = \exp\{\beta' < B, \gamma_2 >\}, \tag{24}
\]
where one verifies that the observable depends only on the homology class of the 2-cycle $\gamma_2$. One can also contemplate the evaluation of an observable such as

$$W(\gamma_3) = \exp\{\beta' < B \cup A, \gamma_3 >\}, \quad (25)$$

where $\gamma_3$ is a homology 3-cycle.

Finally, we should mention that interesting structures may also emerge by considering the coupling of the intersection and kinetic models. In this respect, one may view the addition of the $B \cup B$ term to the kinetic $B \cup \delta A$ action as the evaluation of the observable $B \cup B$ in the kinetic theory.

## 5 Remarks

Our construction gives a topological quantum field theory (TQFT) flavor to the intersection form, a well known topological invariant. One can expect that all the axioms of a TQFT will be obeyed. Clearly there is nothing sacred about the particular example we considered in four dimensions, and the same construction can be undertaken in any dimension, possibly with a mix of fields of various colour types. Subdivision invariance is achieved through a quantization of the coupling parameter, together with a restriction of the field configurations in the theory. One might also view this $B \cup B$ theory as an additional coupling (or observable) to the “kinetic” term $B \cup \delta A$ introduced in [1, 2].

## References

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