SOME PROPERTIES AND APPLICATIONS OF BRIESKORN LATTICES

by

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Abstract. After reviewing the main properties of the Brieskorn lattice in the framework of tame regular functions on smooth affine complex varieties, we prove a conjecture of Katzarkov-Kontsevich-Pantev in the toric case.

Contents

1. Introduction ..................................................... 1
2. The Brieskorn lattice of a tame function ......................... 3
3. On a conjecture of Katzarkov-Kontsevich-Pantev ............ 6
References .................................................................. 10

1. Introduction

The Brieskorn lattice, introduced by Brieskorn in [Bri70] in order to provide an algebraic computation of the Milnor monodromy of a germ of complex hypersurface with an isolated singularity, has also proved central in the Hodge theory for vanishing cycles of such a singularity, as emphasized by Pham [Pha80, Pha83]. Hodge theory for vanishing cycles, as developed by Steenbrink [Ste76, Ste77, SS85] and Varchenko [Var82], makes it an analogue of the Hodge filtration in this context, and fundamental results have been obtained by M. Saito [Sai89] in order to characterize it among other lattices in the Gauss-Manin system of an isolated singularity of complex hypersurface. As such, it leads to the definition of a period mapping, as introduced and studied with much detail by K. Saito for some singularities [Sai83]. It is also a basic constituent of the period mapping restricted to the $\mu$-constant stratum [Sai91], where a natural Torelli problem occurs (see [Sai91], [Her99]).

2010 Mathematics Subject Classification. 14F40, 32S35, 32S40.
Key words and phrases. Brieskorn lattice, irregular Hodge filtration, irregular Hodge numbers, tame function.
For a holomorphic germ $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ with an isolated singularity, denoting by $t$ the coordinate on the target space $\mathbb{C}$, the space

$$\Omega^{n+1}_C / df \wedge d\Omega^{-1}_C$$

is naturally endowed with a $C\{t\}$-module structure (where $t$ acts as the multiplication by $f$), and the Brieskorn lattice is the $C\{t\}$-module (see [Br70, p. 125])

$$\mathcal{H}^n_{f,0} = \left( \Omega^{n+1}_C / df \wedge d\Omega^{-1}_C \right) / C\{t\}$$

Brieskorn shows that (1.2) is free of finite rank equal to the Milnor number $\mu(f, 0)$, and Sebastiani [Seb70] shows the torsion freeness of (1.1), which can thus also serve as an expression for $\mathcal{H}^n_{f,0}$. It is also endowed with a meromorphic connection $\nabla$ having a pole of order at most two at $t = 0$, and the $C\{t\}$-vector space with connection generated by $\mathcal{H}^n_{f,0}$ is isomorphic to the Gauss-Manin connection, which has a regular singularity there. $\mathcal{H}^n_{f,0}$ is thus a $C\{t\}$-lattice of this $C\{t\}$-vector space. While the action of $\nabla_{\partial_t}$, simply written as $\partial_t$, introduces a pole, there is a well-defined action of its inverse $\partial_t^{-1}$ that makes $\mathcal{H}^n_{f,0}$ a module over the ring of $C\{\partial_t^{-1}\}$ of 1-Gevrey series (i.e., formal power series $\sum_{n \geq 0} a_n \partial_t^{-n}$ such that the series $\sum_{n} a_n u^n / n!$ converges). It happens to be also free of rank $\mu$ over this ring ([Ma74, Ma75]). The relation between the rings $C\{t\}$ and $C\{\partial_t^{-1}\}$ is called microlocalization. In the global case below, we will use instead the Laplace transformation. The mathematical richness of this object leads to various generalizations.

For non-isolated hypersurface singularities, the objects with definition as in (1.2) (but in various degrees) have been introduced by Hamm in his Habilitationsschrift (see [Ham75]), who proved that they are $C\{t\}$-free of finite rank, but do not coincide with (1.1) in general. A natural $C\{\partial_t^{-1}\}$-structure still exists on (1.1), and Barlet and Saito [BS07] have shown that the $C\{t\}$-torsion and the $C\{\partial_t^{-1}\}$-torsion coincide, so that $\mathcal{H}^n_{f,0}$ remains $C\{\partial_t^{-1}\}$-free of finite rank.

The Brieskorn lattice has also a global variant. On the one hand, the Brieskorn lattice for tame regular functions on smooth affine complex varieties (see Section 2) is a direct analogue of the case of an isolated singularity, but the double pole of the action of $t$ with respect to the variable $\partial_t^{-1}$ cannot in general be reduced to a simple one by a meromorphic (even formal) gauge transformation i.e., the Gauss-Manin system with respect to the variable $\partial_t^{-1}$ has in general an irregular singularity. The properties of the Brieskorn module for regular functions on affine manifolds which are not tame have been considered by Dimca and M. Saito [DS01].

On the other hand, given a projective morphism $f : X \to \mathbb{A}^1$ on a smooth quasi-projective variety $X$, the Brieskorn modules, defined as the hypercohomology $C[\partial_t^{-1}]$-modules of the twisted de Rham complex $\Omega^*_X[\partial_t^{-1}]/d - \partial_t^{-1}df$, have been shown to be $C[\partial_t^{-1}]$-free (Barannikov-Kontsevich, see [Sab99]), and a similar result holds when one replaces $\Omega^*_X$ with $\Omega^*_X(\log D)$ for some divisor with normal crossings. More generally, one can adapt the definition of the Brieskorn modules for the twisted de Rham complex attached to a mixed Hodge module, and the $C[\partial_t^{-1}]$-freeness still holds, so that they can be called Brieskorn lattices (see loc. cit.). This enables one to use the push-forward operation by the map $f$ and reduce the study to that of Brieskorn lattices attached to mixed Hodge modules on the affine line, as for example
the mixed Hodge modules that the Gauss-Manin systems of \( f \) underlie. In such a way, the Brieskorn lattice has a purely Hodge-theoretic definition, which does not refer to the underlying geometry, and can thus be attached, for example, to any polarizable variation of Hodge structure on a punctured affine line (see [Sab08, §1.d]).

The Brieskorn lattice of tame functions is of particular interest and has been considered in [Sab06] for example. The Brieskorn lattice for families of such functions, considered in [DS03], has been investigated with much care for families of Laurent polynomials in relation with mirror symmetry by Reichelt and Reichelt-Sevenheck [RS15, Rei14, Rei15, RS17].

Lastly, in the global setting as above, the pole of order two of the action of \( t \) with respect to the variable \( \partial^{-1}t \) produces in general a truly irregular singularity, and the Brieskorn lattice is an essential tool to produce the irregular Hodge filtration attached to such a singularity (see [SY15, Sab17]).

The contents of this article is as follows. In Section 2, we review known results on the Brieskorn lattice for a tame function. We show in Section 3 how these results enables one to obtain a simple proof of a conjecture of Katzarkov-Kontsevich-Pantev in the toric case.

Acknowledgements. I thank the referee for his/her careful reading of the manuscript and interesting suggestions and Claus Hertling for pointing out Lemma 2.4.

2. The Brieskorn lattice of a tame function

In this section, we review the main properties of the Brieskorn lattice attached to a tame function on an affine manifold, following [Sab99a, Sab06, DS03].

Let \( U \) be a smooth complex affine variety of dimension \( n \) and let \( f \in \mathcal{O}(U) \) be a regular function on \( U \). There are various notions of tameness for such a function, which are not known to be equivalent, but for what follows they have the same consequences. One of the definitions, given by Katz in [Kat90, Th.14.13.3], is that the cone of \( f!\mathcal{C}_U \to Rf_*\mathcal{C}_U \) should have constant cohomology on \( \mathbb{A}^1 \). We will use the notion of a weakly tame function, as defined in [NS99], that is, either cohomologically tame or M-tame.

We assume that \( f \) is weakly tame. Let \( \theta \) be a new variable. The Brieskorn lattice attached to \( f \) is the \( \mathbb{C}[\theta] \)-module

\[
G_0 := \Omega^n(U)[\theta]/(\theta d - df)\Omega^{n-1}(U)[\theta].
\]

An expression like (1.1) also exists if \( U \) is the affine space \( \mathbb{A}^{n+1} \), but the above one is valid for any smooth affine variety \( U \). The variable \( \theta \) is for \( \partial^{-1}t \). We already notice that

\[
G_0/\theta G_0 \simeq \Omega^n(U)/df \wedge \Omega^{n-1}(U)
\]

has dimension equal to the sum \( \mu = \mu(f) \) of the Milnor numbers of \( f \) at all its critical points in \( U \). The following properties are known in this setting.

(1) The algebraic Gauss-Manin systems \( \mathcal{H}^k f_*\mathcal{O}_U \) are isomorphic to powers of the \( \mathbb{C}[t](\partial_t) \)-module \( (\mathbb{C}[t], \partial_t) \), except for \( k = 0 \), so their localized Laplace transforms vanish except that for \( k = 0 \). If we regard the Laplace transform of \( \mathcal{H}^0 f_*\mathcal{O}_U \) as a
$\mathbb{C}[\tau][\partial_\tau]$-module, we know that it has finite type as such, and its localized Laplace transform $G$, that is, the $\mathbb{C}[\tau, \tau^{-1}]$-module obtained by localization, is free of rank $\mu$. We have 

$$G = \Omega^n(U)[\tau, \tau^{-1}] / (d - \tau df) \Omega^{n-1}(U)[\tau, \tau^{-1}].$$

(2) Setting $\theta = \tau^{-1}$, we write 

$$G = \Omega^n(U)[\theta, \theta^{-1}] / (\theta d - df) \Omega^{n-1}(U)[\theta, \theta^{-1}],$$

and there is therefore a natural morphism $G_0 \to G$. This morphism is injective, so that $G_0$ is a free $\mathbb{C}[\theta]$-module of rank $\mu$ such that $\mathbb{C}[\theta, \theta^{-1}] \otimes_{\mathbb{C}[\theta]} G_0 = G$, i.e., $G_0$ is a $\mathbb{C}[\theta]$-lattice of $G$, on which the restriction of the Gauss-Manin connection has a pole of order at most two. Moreover, the action of $\theta^2 \partial_\theta$ on the class $[\omega]$ of $\omega \in \Omega^n(U)$ in $G_0$ is given by 

$$\theta^2 \partial_\theta [\omega] = [f\omega],$$

and the action of $\theta^2 \partial_\theta$ on a polynomial $\sum_{k \geq 0} [\omega_k \theta^k]$ is obtained by the usual formulas.

(3) Let $V_* G$ be the (increasing) $V$-filtration of $G$ with respect to the function $\tau$ (recall that $G$ has a regular singularity at $\tau = 0$, while that at infinity is usually irregular). It is a filtration by free $\mathbb{C}[\tau]$-modules of rank $\mu$ indexed by $\mathbb{Q}$. The jumping indices of the induced filtration $V_*(G_0/\theta G_0)$, together with their multiplicities (the dimension of $\text{gr}\_1 G_0/\theta G_0))$ form the spectrum of $f$ at $\infty$. The jumping indices are contained in the interval $[0, n] \cap \mathbb{Q}$ and the spectrum is symmetric with respect to $n/2$.

(4) On the other hand, for $\alpha \in [0, 1] \cap \mathbb{Q}$, the vector space $\text{gr}\_\alpha V_\alpha G$ is endowed with the nilpotent endomorphism $N$ induced by the action of $-(\tau \partial_\tau + \alpha)$ and with the increasing filtration $G_\alpha \text{gr}\_\alpha V_\alpha G$ naturally induced by the filtration $G_p = \theta^{-p} G_0$, i.e.,

$$G_p \text{gr}\_\alpha V_\alpha G = (G_p \cap V_\alpha G) / (G_p \cap V_{\alpha p} G),$$

where the intersections are taken in $G$. As a consequence, we have isomorphisms ($p \in \mathbb{Z}, \alpha \in [0, 1]$)

$$\text{gr}\_p G \text{gr}\_\alpha V_\alpha G \xrightarrow{\theta^p} \text{gr}\_p G_0 \text{gr}\_\alpha V_\alpha G_0.$$

(5) The $\mathbb{C}$-vector space $H_{\alpha \in (0, 1) \cap \mathbb{Q}} \text{gr}\_\alpha V_\alpha G$, resp. $H_1 := \text{gr}\_1 V_1 G$, endowed with

- the filtration

$$F^p H_{\alpha \in (0, 1) \cap \mathbb{Q}} := \bigoplus_{\alpha \in (0, 1) \cap \mathbb{Q}} G_{n-1-p} \text{gr}\_\alpha V_\alpha G$$

- and the weight filtration $W_* = \text{M}(\mathbb{N})[n-1]$ (resp. $\text{M}(\mathbb{N})[n]$), i.e., the monodromy filtration of $N$ centered at $n-1$ (resp. $n$),

is part of a mixed Hodge structure. In particular, $N$ strictly shifts by one the filtration $G_\alpha \text{gr}\_\alpha V_\alpha G$ and acts on the graded space $\text{gr}\_G \text{gr}_{\alpha+1} V_\alpha G$ as the degree-one morphism induced by $-\tau \partial_\tau$. We therefore have a commutative diagram, for any $\alpha \in [0, 1)$ and $p \in \mathbb{Z}$, (see [Var81] and [SSS85, §7] in the singularity case):

$$\begin{array}{ccc}
\text{gr}\_p G \text{gr}\_\alpha V_\alpha G & \xrightarrow{\theta^p} & \text{gr}\_p \text{gr}\_\alpha V_\alpha (\Omega^n(U) / df \wedge \Omega^{n-1}(U)) \\
\downarrow{[N]} & & \downarrow{[f]} \\
\text{gr}\_p G \text{gr}_{\alpha+1} V_\alpha G & \xrightarrow{\theta^{p+1}} & \text{gr}_p \text{gr}_{\alpha+1} V_\alpha (\Omega^n(U) / df \wedge \Omega^{n-1}(U))
\end{array}$$

(2.2) by using the relation $-\tau \partial_\tau = \theta \partial_\theta$. 
To see this, write the commutative diagram

\[
\begin{array}{c}
\text{gr}^G_p \text{gr}^V_α G \\ \text{gr}^G_{p+1} \text{gr}^V_α G
\end{array} \xrightarrow{\theta \theta_0 - \alpha} \begin{array}{c}
\text{gr}^V_{α+p} \text{gr}^G_0 G \\ \text{gr}^V_{α+p} \text{gr}^G_0 G
\end{array}
\]

and use that in the vertical morphisms, the constant part $α$ or $α + p$ induces the morphism 0.

(6) Recall that a mixed Hodge structure $(H_Q, F^*H_C, W, H_Q)$ is said to be of Hodge-Tate type if

(a) the filtration $W_*$ has only even jumping indices

(b) and $W_2, H_C$ is opposite to $F^*H_C$.

The description of the mixed Hodge structure given in (5) implies the following criterion. We will set $\nu = n - 1$ when considering $H_{\neq 1}$ and $\nu = n$ when considering $H_1$. We will then denote by $H$ either $H_{\neq 1}$, or $H_1$.

**Corollary 2.3.** The mixed Hodge structure that the triple $(H, F^*H, W, H)$ underlies is of Hodge-Tate type if and only if, for any integer $k$ such that $0 \leq k \leq [\nu/2]$, the $(\nu - 2k)$th power of $N$ induces an isomorphism

\[
[N]^{\nu - 2k} : \text{gr}^G_k H \xrightarrow{\sim} \text{gr}^G_{\nu - k} H.
\]

**Proof.** We define the filtration $W'_H$ indexed by $2\mathbb{Z}$ by the formula $W'_{2k}H = G_{\nu - k}H$, so that $G_kH = W'_{2(\nu - k)}H$. If we set $\ell = \nu - 2k$ for $0 \leq k \leq \nu/2$, we have $0 \leq \ell \leq \nu$ and the isomorphism in the corollary is written

\[
[N]^{\ell} : \text{gr}^W_{\nu + \ell} H \xrightarrow{\sim} \text{gr}^W_{\nu - \ell} H.
\]

We can conclude that $W'_H = W_* H$ if we know that $N^{\nu + 1} = 0$, that is, $\text{gr}^G_{\nu + 1} H = 0$. This is a consequence of the positivity of the spectrum [Sab06, Cor. 13.2], which says that, if $α \in [0, 1)$, we have $\text{gr}^G_{\nu} \text{gr}^V_α G = 0$ for $k \notin [0, \nu] \cap \mathbb{N}$. \qed

The following lemma was pointed out to me by Claus Hertling.

**Lemma 2.4.** A mixed Hodge structure $(H_Q, F^*H_C, W, H_Q)$ is Hodge-Tate if and only if we have, for all $p \in \frac{1}{2}\mathbb{Z}$,

\[
\dim \text{gr}^P_p H_C = \dim \text{gr}^W_{2p} H_Q.
\]

**Proof.** Indeed, one direction is clear. Conversely, if the equality of dimensions holds, then (6a) holds since $F^*H$ has only integral jumps; moreover, up to a Tate twist, one can assume that $W_{<0} H = 0$, so $\text{gr}^P_p H = 0$ for $k < 0$. It is enough to prove that $\text{gr}^P_p \text{gr}^W_{2i} H = 0$ for all $p \neq \ell$. We prove this by induction on $\ell$. If $\ell = 0$, the result follows from the property that $F^p H = 0$ for $p < 0$ and Hodge symmetry. Assume the result up to $\ell$. For $j < \ell$ we thus have $\dim \text{gr}^P_p \text{gr}^W_{2j} H = \dim \text{gr}^W_{2j} H = \dim \text{gr}^P_p H$ (the latter equality by the assumption), and therefore $\text{gr}^W_{2i} \text{gr}^P_p H = 0$ for $i \neq j$. In particular, taking $i = \ell + 1$, we have $\text{gr}^P_p \text{gr}^W_{2(\ell + 1)} H = 0$ for all $p \leq \ell$. By Hodge symmetry, we obtain $\text{gr}^P_p \text{gr}^W_{2(\ell + 1)} H = 0$ for all $p \neq \ell + 1$, as wanted. \qed
(7) We now consider the case where \( U = (\mathbb{C}^*)^n \), endowed with coordinates \( x = (x_1, \ldots, x_n) \). Let \( f \in \mathbb{C}[x, x^{-1}] \) be a Laurent polynomial in \( n \) variables, with Newton polyhedron \( \Delta(f) \). We assume that \( f \) is nondegenerate with respect to its Newton polyhedron and convenient (see [Kou76]). In particular, 0 belongs to the interior of its Newton polyhedron. It is known that such a function is M-tame.

For any face \( \sigma \) of dimension \( n - 1 \) of the boundary \( \partial \Delta(f) \), we denote by \( L_\sigma \) the linear form with coefficients in \( \mathbb{Q} \) such that \( L_\sigma \equiv 1 \) on \( \sigma \). For \( g \in \mathbb{C}[x, x^{-1}] \), we set \( \deg_\sigma(g) = \max_m L_\sigma(m) \), where the max is taken on the exponents of monomials \( x^m \) appearing in \( g \), and \( \deg_\sigma(g) = \max_\sigma \deg_\sigma(g) \). We denote the volume form \( dx_1/x_1 \wedge \cdots \wedge dx_n/x_n \) by \( \omega \), giving rise to an identification \( \mathbb{C}[x, x^{-1}] \to \Omega^m(U) \) and \( \mathbb{C}[x, x^{-1}]/(\partial f) \to G_0/\theta G_0 \) (see (2.1)).

The Newton increasing filtration \( N_\beta \Omega^n(U) \) indexed by \( \mathbb{Q} \) is defined by
\[
N_\beta \Omega^n(U) := \{ g \omega \in \Omega^n(U) \mid \deg_\beta(g) \leq \beta \}.
\]
We have \( N_0 \Omega^n(U) = 0 \) for \( \beta < 0 \) and \( N_0 \Omega^n(U) = \mathbb{C} \cdot \omega \). We can extend this filtration to \( \Omega^n(U)[\theta] \) by setting
\[
N_\beta \Omega^n(U)[\theta] := N_\beta \Omega^n(U) + \theta N_{\beta - 1} \Omega^n(U) + \cdots + \theta^k N_{\beta - k} \Omega^n(U) + \cdots
\]
and then naturally induce this filtration on \( G_0 \), to obtain a filtration \( N_\beta G_0 \) and then on \( G_0/\theta G_0 \). We have
\[
N_\beta G_0 = V_\beta G \cap G_0 \quad \text{and} \quad N_\beta(G_0/\theta G_0) = V_\beta(G_0/\theta G_0).
\]

Corollary 2.3 now reads, according to (2.2) and by using the above identification through multiplication by \( \omega \):

**Corollary 2.6.** The mixed Hodge structure that the triple \( (H, F^*H, W_\bullet H) \) underlies is of Hodge-Tate type if and only if, for any integer \( k \) such that \( 0 \leq k \leq \lfloor \nu/2 \rfloor \) (\( \nu = n - 1 \), resp. \( n \)), we have isomorphisms
\[
\text{gr}^{N}_{\alpha+k}(\mathbb{C}[x, x^{-1}]/(\partial f)) \xrightarrow{\sim} \text{gr}^{N}_{\alpha+n-1-k}(\mathbb{C}[x, x^{-1}]/(\partial f)) \quad \forall \alpha \in (0, 1),
\]
resp.
\[
\text{gr}^{N}_{k}(\mathbb{C}[x, x^{-1}]/(\partial f)) \xrightarrow{\sim} \text{gr}^{N}_{n-k}(\mathbb{C}[x, x^{-1}]/(\partial f)).
\]

### 3. On a conjecture of Katzarkov-Kontsevich-Pantev

In this section we use the algebraic Brieskorn lattice of a convenient nondegenerate Laurent polynomial to solve the toric case of the part “\( f^{p,q} = h^{p,q} \)” of Conjecture 3.6 in [KKP17] (the other equality “\( h^{p,q} = i^{p,q} \)” is obviously not true by simply considering the case of the standard Laurent polynomial mirror to the projective space \( \mathbb{P}^n \), see also another counter-example in [LP18]). We refer to [LP18, Har17, Sha17] for further discussion and positive results on this conjecture.

#### 3.a. The Brieskorn lattice and the conjecture of Katzarkov-Kontsevich-Pantev

Given a smooth quasi-projective variety \( U \) and a morphism \( f : U \to \mathbb{A}^1 \), every twisted de Rham cohomology \( H^k_{D^R}(U, d + df) \), i.e., the \( k \)th hypercohomology of
the twisted de Rham complex \((\Omega^*_U, d + df)\), is endowed with a decreasing filtration \(F^*_{\text{Yu}}H^k_{\text{DR}}(U, d + df)\) indexed by \(\mathbb{Q}\) (see [Yu14]). For \(\alpha \in [0, 1)\), the filtration indexed by \(\mathbb{Z}\) defined by \(F^*_{\text{Yu}, \alpha} = F^*_{\text{Yu}}\alpha\) can also be computed in terms of the Kontsevich complex \(\Omega^*_f(\alpha)\) together with its stupid filtration (see [ESY17, Cor. 1.4.5]). The irregular Hodge numbers \(h^p_{\alpha}(f)\) are defined as

\[
(\text{3.1}) \quad h^p_{\alpha}(f) := \dim \text{gr}^{p-\alpha}_{F_{\text{Yu}}} H^p_{\text{DR}}(U, d + df).
\]

It is well-known that \(\dim H^k_{\text{DR}}(U, d + df) = \dim H^k(U, f^{-1}(t))\) for \(|t| \gg 0\). This space is endowed with a monodromy operator (around \(t = \infty\)), and we will consider the case where this monodromy operator is unipotent. In such a case, the filtration \(F^*_{\text{Yu}}H^p_{\text{DR}}(U, d + df)\) is known to jump at integers only, and in (3.1) only \(\alpha = 0\) occurs. We then simply denote this number by \(h^p_{\alpha}(f)\), so that, in such a case,

\[
(\text{3.2}) \quad h^p_{\alpha}(f) := \dim \text{gr}^p_{F_{\text{Yu}}} H^p_{\text{DR}}(U, d + df).
\]

Let \(W_\alpha\) be the monodromy filtration on \(H^k(U, f^{-1}(t))\) centered at \(k\). The conjecture of [KKP17] that we consider is the possible equality (see [LP18, Har17, Sha17])

\[
\text{(3.2)} \quad h^p_{\alpha}(f) = \dim \text{gr}^W_{2p} H^{p+q}(U, f^{-1}(t)).
\]

If moreover \(U\) is affine and \(f\) is weakly tame, so that \(H^{p+q}_{\text{DR}}(U, d + df) = 0\) unless \(p + q = n\), [SY15, Cor. 8.19] gives, using the notation of Section 2:(1)

\[
h^p_{\alpha}(f) = \begin{cases} 
\dim \text{gr}^{V}_{n-p}(G_0(f)/\theta G_0(f)) = \dim \text{gr}^p_{\alpha} \text{gr}_0^V G & \text{if } p + q = n, \\
0 & \text{if } p + q \neq n, 
\end{cases}
\]

and this is the number denoted by \(f^{p,q}\) in [KKP17]. In such a case, we have \(H = H_1\) in the notation of Section 2(5).

The following criterion has been obtained, with a different approach of the irregular Hodge filtration, by Y. Shamoto.

**Proposition 3.3 ([Sha17]).** Assume \(U\) affine and \(f\) weakly tame with unipotent monodromy operator at infinity. Then (3.2) holds true if and only if the mixed Hodge structure of Section 2(5) on \(H = H_1\) is of Hodge-Tate type.

**Proof.** According to Lemma 2.4, proving the result amounts to identifying the space \(\text{gr}^V G\) endowed with its nilpotent operator \(N\) with the space \(H^k(U, f^{-1}(t))\) endowed with the nilpotent part of the (unipotent) monodromy (up to a nonzero constant). Choosing an extension \(F : X \to \mathbb{P}^1\) of \(f\) as a projective morphism on a smooth variety \(X\) such that \(X \smallsetminus U\) is a divisor, and setting \(\mathcal{F} = R_{j*} \mathcal{C}_U\) \((j : U \hookrightarrow X)\), we identify the dimension of \(H^k(U, f^{-1}(t))\) with that of the \(k\)th-hypercohomology on \(X\) of the Beilinson extension \(\Xi_{\mathcal{F}}\mathcal{F}\). Then the desired identification is given by [Sab97, Cor. 1.13].

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(1) The definition of \(\delta_e\) in [SY15] should read \(\dim \text{gr}^V G_0(f) / u G_0(f)\).
3.b. The toric case of the conjecture of Katzarkov-Kontsevich-Pantev, first part

As usual in toric geometry, we denote by $M$ the lattice $\mathbb{Z}^n$ in $\mathbb{C}^n$ and by $N$ its dual lattice. We fix a reflexive simplicial polyhedron $\Delta \subset \mathbb{R} \otimes M$ with vertices in $M$ and having 0 in its interior (it is then the unique integral point in its interior), see [Bat94, §4.1]. We denote by $\Delta^*$ the dual polyhedron with vertices in $N$, which is also simplicial reflexive and has 0 in its only interior point, and by $\Sigma \subset N$ the fan dual to $\Delta$, which is also the cone on $\Delta^*$ with apex 0. We assume that $\Sigma$ is the fan of nonsingular toric variety $X$ of dimension $n$, that is, each set of vertices of the same $(n-1)$-dimensional face of $\partial \Delta^*$ is a $\mathbb{Z}$-basis of $N$. We know that

- $X$ is Fano ([Bat94, Th. 4.1.9]),
- the Chow ring $A^*(X) \simeq H^{2*}(X, \mathbb{Z})$ is generated by the divisor classes $D_v$ corresponding to vertices $v \in V(\Delta^*)$ of $\Delta^*$, i.e., primitive elements on the rays of $\Sigma$ (see [Ful93, p. 101]),
- we have $c_1(TX) = c_1(K_X^*) = \sum_{v \in V(\Delta^*)} D_v$ in $H^{2*}(X, \mathbb{Z})$ (see [Ful93, p. 109]), which satisfies Hard Lefschetz on $H^{2*}(X, \mathbb{Q})$, by ampleness of $K_X^*$.

Let us fix coordinates $x = (x_1, \ldots, x_n)$ such that $\mathbb{Q}[N] = \mathbb{Q}[x, x^{-1}]$. We use the notation of Section 2(7). Due to the reflexivity of $\Delta^*$, $\La$ has coefficients in $\mathbb{Z}$ (it corresponds to a vertex of $\Delta$). For $g \in \mathbb{C}[x, x^{-1}]$, the $\sigma$-degree $\deg_\sigma(g) = \max_m L_\sigma(m)$ and the $\Delta^*$-degree $\deg_\Delta^*(g) = \max_m \deg_\sigma(g)$ are thus nonnegative integers.

**Proposition 3.4.** The case $\cdot f^p = h^{p,q}$ of [KKP17, Conj. 3.6] holds true if $f$ is the Laurent polynomial

$$f(x) = \sum_{v \in V(\Delta^*)} x^v \in \mathbb{Q}[x, x^{-1}].$$

The idea of the proof is to notice that the property for the second morphism in Corollary 2.6 to be an isomorphism is exactly the property that $c_1(TX)$ satisfies the Hard Lefschetz property, and thus to identify its source and target as the cohomology of $X$ in suitable degree.

**Lemma 3.5.** For $\Delta$ as above, any Laurent polynomial

$$f_\alpha(x) = \sum_{v \in V(\Delta^*)} a_v x^v \in \mathbb{C}[x, x^{-1}], \quad \alpha = (a_v \in V) \in (\mathbb{C}^*)^V(\Delta^*),$$

is convenient and non-degenerate in the sense of Kouchnirenko.

**Proof.** The Newton polyhedron of $f_\alpha$ is equal to $\Delta^*$, and 0 belongs to its interior. In order to prove the non-degeneracy, we note that the vertices of any $(n-1)$-dimensional face $\sigma$ of $\partial \Delta^*$ form a $\mathbb{Z}$-basis. It follows that, in suitable toric coordinates $y_1, \ldots, y_n$, the restriction $f_\alpha|\sigma$ can be written as $y_1 + \cdots + y_n$, and the non-degeneracy is then obvious. \(\square\)

**Proof of Proposition 3.4.** Note that $\deg_\Delta^*(f) = 1$, as well as $\deg_\Delta^*(x_i \partial f / \partial x_i) = 1$. The Jacobian ring $\mathbb{Q}[x, x^{-1}]/(\partial f)$ is endowed with the Newton filtration $\mathcal{N}_\bullet$ induced by the $\Delta^*$-degree $\deg_\Delta^*$, and corresponds to $\mathcal{N}_\bullet(G_0/\theta G_0)$ by multiplication by $\omega$. In
the present setting, [BCS05, Th.1.1] identifies the graded ring $A^*(X)_{\mathbb{Q}}$ with the graded ring
\[ \text{gr}^N_*(\mathbb{Q}[x,x^{-1}]/(\partial f)) \]
By applying Hard Lefschetz to $c_1(TX)$, we deduce that, for every $k \in \mathbb{N}$ such that $0 \leq k \leq [n/2]$, multiplication by the $(n-2k)$th power of the $N$-class $[f]$ of $f$ induces an isomorphism
\[ [f]^{n-2k} : \text{gr}^N_k(\mathbb{Q}[x,x^{-1}]/(\partial f)) \xrightarrow{\sim} \text{gr}^N_{n-k}(\mathbb{Q}[x,x^{-1}]/(\partial f)) \]
By Corollary 2.6 for $H = H_1$, we deduce the assertion of the proposition from Proposition 3.3.

3.c. The toric case of the conjecture of Katzarkov-Kontsevich-Pantev, second part

We now prove the main result of this note.

**Theorem 3.6.** The case “$f^{p,q} = h^{p,q}$” of [KKP17, Conj. 3.6] holds true for any Laurent polynomial
\[ f_\alpha(x) = \sum_{v \in V(\Delta^*)} a_v x^v \in \mathbb{C}[x,x^{-1}], \quad \alpha = (a_v \forall v) \in (\mathbb{C}^*)^{V(\Delta^*)}. \]

**Remark 3.7.** The case where $n = 3$ was already proved differently by Y. Shamoto [Sha17, §4.2].

**Proof.** Let us set $H(f_\alpha) = H_1(f_\alpha) = \text{gr}^V_0 G(f_\alpha)$, where $G(f_\alpha)$ is the localized Laplace transform of the Gauss-Manin system for $f_\alpha$ as in Section 2(2). By Lemma 3.5, we can apply the results of Section 2 to $f_\alpha$ for any $\alpha \in (\mathbb{C}^*)^{V(\Delta^*)}$. We will prove that, for fixed $p$, both terms $\dim \text{gr}^G_{n-p} H(f_\alpha)$ and $\dim \text{gr}^W_{n-p} H(f_\alpha)$ in Lemma 2.4 are independent of $\alpha$. Since they are equal if $\alpha = (1, \ldots, 1)$, after Proposition 3.4, they are equal for any $\alpha \in (\mathbb{C}^*)^{V(\Delta^*)}$, as wanted.

(1) For the first term, we will use [NS99]. We have denoted there $\dim \text{gr}^G_p H(f_\alpha)$ by $\nu_p(f_\alpha)$ and, since $\text{gr}^V_0 G = 0$ for $\alpha \notin \mathbb{Z}$, it is also equal to the number denoted there by $\Sigma_{p-1}^V(f_\alpha)$. By the theorem in [NS99] and Lemma 3.5, $\Sigma_{p-1}(f_\alpha)$ depends semi-continuously on $\alpha$. On the other hand, according to [Kou76], $\dim H(f_\alpha)$ is independent of $\alpha$ and is computed only in terms of $\Delta^*$. Since $\dim H(f_\alpha) = \sum_p \Sigma_{p-1}(f_\alpha)$, each term in this sum is also constant with respect to $\alpha$.

(2) We will prove the local constancy of $\dim \text{gr}^W_p H(f_\alpha)$ near any $\alpha_0 \in (\mathbb{C}^*)^{V(\Delta^*)}$. As noticed in [DS03, §4], we can apply the results of Section 2 of loc. cit. to $f_{\alpha_0}$. We fix a Stein open set $B^0$ adapted to $f_{\alpha_0}$ as in [DS03, §2a], and fix a neighbourhood $X$ of $\alpha_0$ so that it is also adapted to any $f_\alpha$ for $\alpha$ in this neighbourhood. By construction, all the critical points of $f_{\alpha_0}$ are contained in the interior of $B^0$ if $X$ is chosen small enough, and since $\mu(f_\alpha)$ is constant, the same property holds for $\alpha \in X$. By using successively Theorem 2.9, Remark 2.11 and Proposition 1.20(1) in [DS03], we deduce that, when $\alpha$ varies in $X$, the localized partial Laplace transformed Gauss-Manin systems $G(f_\alpha)$ form an $\mathcal{O}_X[\tau, \tau^{-1}]$-free module with integrable connection and regular singularity along $\tau = 0$, which is compatible with base change with respect to $X$. 
As a consequence, the monodromy of each $G(f_a)$ around $\tau = 0$ is constant, and the assertion follows.

Remark 3.8 (suggested by the referee). If we relax the condition in Section 3.b that the toric Fano variety $X$ is nonsingular, then we have to consider the orbifold Chow ring of $X$ as in [BCS05], or the Chen-Ruan orbifold cohomology of $X$. For the cohomology of the untwisted sector (i.e., the usual cohomology), the Hard Lefschetz theorem is still valid (see [Ste77]) and Proposition 3.4 still holds, i.e., (3.2) holds for $f$. Moreover, Part (2) of the proof of Theorem 3.6 also extends to this setting. However, the semicontinuity result of [NS99] used in Part (1) of the proof is not enough to imply the constancy (with respect to $a$) of $\nu_p(f_a)$.

On the other hand, one can also consider the various $h^{p,q}_\alpha(f)$ for $\alpha \in (0,1) \cap \mathbb{Q}$ and, correspondingly, the twisted sectors of the orbifold $X$. In such a case, Hard Lefschetz for $f$ may already give trouble (see [Fer06]).

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