HITTING TIMES TO SPHERES OF BROWNIAN MOTIONS WITH AND WITHOUT DRIFTS

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Abstract. Explicit formulae for the densities of the first hitting times to the sphere of Brownian motions with drifts and the asymptotic behavior of the tail probabilities are shown. For this purpose we present an explicit formula for the Laplace transform of the joint distribution of the first hitting time to a sphere and the hitting position, which is different from the known formulae in the literature and is of independent interest.

1. Introduction

For $d \geq 2$ let $B = \{B_t\}_{t \geq 0}$ be a standard $d$-dimensional Brownian motion starting from a fixed point $x \in \mathbb{R}^d$ defined on a probability space $(\Omega, \mathcal{F}, P)$. Letting $v \in \mathbb{R}^d$ be a non-zero constant vector, we consider a Brownian motion $B^{(v)} = \{B_t^{(v)}\}_{t \geq 0}$ with constant drift $v$ given by $B_t^{(v)} = B_t + vt$. We are concerned with the first hitting time $\sigma^{(v)}$ of $B^{(v)}$ to the sphere $S_r^{d-1}$ with radius $r$ and centered at the origin.

The purpose of this article is two-fold: firstly to show a new explicit expression for the density $p^{(v)}(t; x)$, where $\nu = \frac{d-2}{2}$, of $\sigma^{(v)}$, and secondly to study the asymptotic behavior of the tail probability $P(t < \sigma^{(v)} < \infty)$ as $t \to \infty$ by using the first result. The results in this article will be applied in a study of the mean volume of the Wiener sausage associated with $B^{(v)}$ ([8]).

In the course of the study we show the joint Laplace transform of $\sigma^{(v)}$ and the hitting position $B_{\sigma^{(v)}}^{(v)}$. For the standard Brownian motion, Wendel [16] has given a Laplace-Gegenbauer transform and Aizenman-Simon [1] has developed an expression for the density. For the Brownian motion with constant drift, Yin [17] has extended the result by Wendel and Yin-Wang [18] has shown an expression for the density of the joint distribution. See the reference in [18] for other studies.

Our approach and results are quite different from these known results. We reduce the study on the Brownian motion with constant drift $B^{(v)}$ to that on the standard Brownian motion $B$ by the Cameron-Martin theorem and, for the study of $(\sigma, B\sigma)$, $\sigma$ being $\sigma^{(0)}$, we use the skew-product representation of $B$. By using stochastic analysis and solving some special differential equations, we will obtain the Laplace transform of $(\sigma, B\sigma)$. The density $p^{(v)}(t; x)$ will be expressed as an
infinite sum involving the modified Bessel functions, the Gegenbauer polynomials and the densities of the first hitting times of standard Brownian motions to the sphere.

We now give the main results. We denote by \( p_\nu(t; x) \) the density of \( \sigma \). While the density \( p_\nu(t; x) \) is a classical object (see [10][12] and the references therein), the detailed results on the asymptotics as \( t \to \infty \) are recently known and, in the course of the studies, new explicit expressions have been shown. See Byczkowski and Ryznar [3], Uchiyama [14] and [5–7].

**Theorem 1.1.** Assume that \( x \neq 0 \). When \( d = 2 \), one has
\[
p^{(v)}_0(t; x) = e^{-\langle v, x \rangle - \frac{1}{2} |v|^2 t} \left\{ I_0(\|v\| r) p_0(t; x) + \sum_{n=1}^{\infty} n C_n^0 \left( \frac{\langle v, x \rangle}{\|v\| |x|} \right) I_n(\|v\||r) \frac{|x|^n}{r^n} p_n(t; x) \right\}.
\]

When \( d \geq 3 \),
\[
p^{(v)}(t; x) = 2^\nu \Gamma(\nu) e^{-\langle v, x \rangle - \frac{1}{2} |v|^2 t} \times \sum_{n=0}^{\infty} (\nu + n) C_n^\nu \left( \frac{\langle v, x \rangle}{\|v\| |x|} \right) I_{\nu+n}(\|v\||r) \frac{|x|^n}{|v|^\nu r^{\nu+n}} p_{\nu+n}(t; x).
\]

Here \( I_\mu (\mu \geq 0) \) is the modified Bessel function of the first kind and \( C_n^\nu \) is the Gegenbauer polynomial.

We refer to Magnus-Oberhettinger-Soni [11] and Watson [15] for the special functions. We note again that explicit expressions and the asymptotic behavior of \( p_\nu(t; x) \) are known.

For the asymptotic behavior of the tail probabilities, we show the following. To mention the result, we recall ([3][5][14]) that
\[
p_0(t; x) = \frac{L(0)}{t(\log t)^2} \left( 1 + o(1) \right) \quad \text{and} \quad p_\nu(t; x) = \frac{L(\nu)}{t^{\nu+1}} \left( 1 + o(1) \right) \quad (\nu > 0)
\]
hold as \( t \to \infty \), where \( L(0) = 2 \log \frac{|x|}{r} \) and, for \( \nu > 0 \),
\[
L(\nu) = \frac{r^{2\nu}}{2^\nu \Gamma(\nu)} \left( 1 - \left( \frac{r}{|x|} \right)^{2\nu} \right).
\]

**Theorem 1.2.** Assume that \( |x| > r \). Then, one has
\[
P(t < \sigma^{(v)} < \infty) = \frac{2L(0)}{|v|^2} I_0(\|v\| \cdot r) e^{-\frac{1}{2} \|v\|^2 t} \frac{e^{-\frac{1}{2} |v|^2 t}}{t(\log t)^2} (1 + o(1))
\]
as \( t \to \infty \) when \( d = 2 \), and
\[
P(t < \sigma^{(v)} < \infty) = \frac{2^{\nu+1} L(\nu) \Gamma(\nu+1)}{|v|^2} I_\nu(|v||r|) \frac{e^{-\frac{1}{2} |v|^2 t}}{t^{\nu+1}} (1 + o(1))
\]
when \( d \geq 3 \).

This article is organized as follows. In the next section, we consider a Brownian motion \( B \) without drift and study the distribution of \( (\sigma, B_\sigma) \). In Section 3, by combining the results in Section 2 with the Cameron-Martin theorem, we prove Theorem 1.1. Section 4 is devoted to a proof of Theorem 1.2. Some precise estimates for the modified Bessel functions and the Gegenbauer polynomials, which are necessary to prove the main results, are shown in the Appendix.
2. Distribution of \((\sigma, B_\sigma)\)

We show an explicit expression for the joint Laplace transform of the first hitting time \(\sigma = \inf\{t > 0; |B_t| = r\}\) to the sphere \(S^{d-1}_\nu\) and the position \(B_\sigma\) of the Brownian motion.

To present the main results of this section, we define the function \(Z^{(v),\lambda}_\mu\) for \(\mu \geq 0, \lambda > 0\) and \(v \in \mathbb{R}^d\) by

\[
Z^{(v),\lambda}_\mu(\xi, \eta) = \frac{K_\mu(\xi \sqrt{2 \lambda + |v|^2})}{K_\mu(\eta \sqrt{2 \lambda + |v|^2})} \quad \text{if } \xi > \eta > 0
\]

and

\[
Z^{(v),\lambda}_\mu(\xi, \eta) = \frac{I_\mu(\xi \sqrt{2 \lambda + |v|^2})}{I_\mu(\eta \sqrt{2 \lambda + |v|^2})} \quad \text{if } \eta > \xi > 0,
\]

where \(K_\mu\) is the modified Bessel function of the second kind (the Macdonald function). Since \(K_\mu\) is decreasing and \(I_\mu\) is increasing on \((0, \infty)\), \(Z^{(v),\lambda}_\mu \leq 1\).

**Theorem 2.1.** Assume that \(x \neq 0\). Let \(v \in \mathbb{R}^d \setminus \{0\}\) and \(\lambda > 0\). When \(d = 2\), one has

\[
E[e^{-\lambda \tau + (v, B_\tau)} 1_{\{\tau < \infty\}}] = I_0(|v| r) Z^{(0),\lambda}_0(|x|, r) + \sum_{n=1}^\infty n C_n^\nu(\gamma_v) I_n(|v| r) Z^{(0),\lambda}_n(|x|, r),
\]

where \(\gamma_v = \frac{(v, x)}{|v| |x|}\). When \(d \geq 3\), one has

\[
E[e^{(v, B_\tau) - \lambda \tau} 1_{\{\tau < \infty\}}] = 2^\nu \Gamma(\nu) \sum_{n=0}^\infty (\nu + n) C_n^\nu(\gamma_v) I_{\nu+n}(|v| r) Z^{(0),\lambda}_{\nu+n}(|x|, r).
\]

For a proof of the theorem, we recall the skew-product representation of Brownian motions. Let \(R = \{R_t\}_{t \geq 0}\) be a \(d\)-dimensional Bessel process (with index \(\nu = \frac{d-2}{2}\)) and \(\theta = \{\theta_t\}_{t \geq 0}\) be a Brownian motion on the unit sphere \(S^{d-1} = S_1^{d-1}\) with \(\theta_0 = \frac{x}{|x|}\) and assume that \(R \) and \(\theta\) are independent. Recall also that, embedding \(S^{d-1}\) in \(\mathbb{R}^d\), we can realize \(\theta\) as a solution of a stochastic differential equation, which is the so-called Stroock’s representation of a spherical Brownian motion.

Set \(S_t = \int_0^t (R_u)^{-2} du\). Then, \(\{R_t \theta_{S_t}\}_{t \geq 0}\) is a \(d\)-dimensional Brownian motion (\([9]\) Chapter 7). Hence, we have

\[
E[e^{-\lambda \tau + (v, B_\tau)} 1_{\{\tau < \infty\}}] = \int_0^\infty \int_0^\infty e^{-\lambda \tau} E_{\theta_0}^{\theta_0} [e^{(v, \theta_t)}] P_{\nu,|x|} (\tau \in dt, S_t \in du),
\]

where \(E_{\theta_0}^{\theta_0}\) denotes the expectation with respect to the probability law of \(\theta\) starting from \(\theta_0\), \(P_{\nu,|x|}\) is the probability law of \(R\) with \(R_0 = |x|\) and \(\tau\) is its first hitting time to \(r\).

It is known (cf. \([2]\) p. 407) that, for \(\alpha, \beta > 0\),

\[
E_{\nu,|x|}[e^{-\alpha \tau - \frac{1}{2} \beta^2 S^2_t}] = \frac{r^\nu}{|x|^\nu} Z^{(0),\alpha}_{\nu + \beta^2}(|x|, r),
\]

where \(E_{\nu,|x|}\) is the expectation with respect to \(P_{\nu,|x|}\).

About the distribution of \(\theta_t\) for fixed \(t\), we show the following.
Proposition 2.2. Let $\xi > 0$. Then, when $d = 2$, one has
\[
E_{\theta_0}^\theta [e^{\xi (v, \theta_t)}] = I_0(|v|\xi) + \sum_{n=1}^\infty n C_n^0(\gamma_v)e^{-\frac{1}{2}n^2t} I_n(|v|\xi)
\]
and, when $d \geq 3$,
\[
E_{\theta_0}^\theta [e^{\xi (v, \theta_t)}] = 2^\nu \Gamma(\nu) \sum_{n=0}^\infty (\nu + n) C_n^\nu(\gamma_v)e^{-\frac{1}{2}n(n+2\nu)t} I_{\nu+n}(|v|\xi) (|v|\xi)\nu.
\]

It may be worthwhile to note here that the formula (cf. [11, p. 227]) for the Gegenbauer polynomial
\[
e^{\alpha \xi} = 2^\mu \Gamma(\mu) \sum_{n=0}^\infty (\mu + n) C_n^\mu(\alpha) \xi^{-\mu} I_{\nu+n}(\xi) \quad (\alpha \in \mathbb{R}, \, \xi > 0, \, \mu > 0)
\]
is recovered by letting $t \downarrow 0$ in (2.3).

Using the estimates for $I_\mu$ and $C_n^\mu$ given in the Appendix, we can show, when $d \geq 3$,
\[
\int_0^\infty \int_0^\infty e^{-\lambda t} (\nu + n) C_n^\nu(\gamma_v)e^{-\frac{1}{2}n(n+2\nu)u} I_{\nu+n}(|v|\xi) (|v|\xi)\nu P_{\nu,|x|}(\tau \in dt, S_x \in du)
\]
\[
\leq \rho_\nu \left( \frac{r}{2|x|} \right)^\nu e^{\xi (v, t)} (2|v|r)^n n!.
\]

We can show a similar estimate for the case where $d = 2$. Hence, we can change the order of integration and the infinite sum and obtain Theorem 2.1 from Proposition 2.2.

For a proof of the proposition, we first show that $f_\mu(t, \xi) = E_{\theta_0}^\theta [e^{\xi (v, \theta_t)}]$ satisfies
\[
\frac{\partial f_\mu}{\partial t} = -\frac{1}{2} \xi^2 \frac{\partial^2 f_\mu}{\partial \xi^2} - \frac{d - 1}{2} \xi \frac{\partial f_\mu}{\partial \xi} + \frac{1}{2} |v|^2 \xi^2 f_\mu, \quad t > 0, \, \xi > 0,
\]
together with the boundary conditions
\[
f_\mu(0, \xi) = e^{\xi (v, \theta_0)}, \quad f_\mu(t, 0) = 1, \quad \frac{\partial f_\mu}{\partial \xi}(t, 0) = \langle v, \theta_0 \rangle e^{-\frac{d-1}{2}t}.
\]

Note that the analytic solution for the equation (2.5) with the boundary condition (2.6) of the form $\sum \alpha_n(t) \xi^n$ is unique by the Cauchy-Kowalevski theorem (cf., e.g., [4]).

For this purpose, we recall Stroock’s representation of spherical Brownian motion (cf. [13]). $\theta$ may be realized as a solution of the stochastic differential equation based on a $d$-dimensional Brownian motion $\{w_s = (w^1_s, w^2_s, \ldots, w^d_s)\}_{s \geq 0}$ which is given by
\[
d\theta^i_s = \sum_{j=1}^d (\delta_{ij} - \theta^i_s \theta^j_s) \circ dw^j_s, \quad i = 1, 2, \ldots, d.
\]

Then, by Itô’s formula,
\[
d(e^{\xi (v, \theta_t)}) = \xi \sum_{i=1}^d v^i e^{\xi (v, \theta_t)} \sum_{j=1}^d (\delta_{ij} - \theta^i_s \theta^j_s) dw^j_t
\]
\[
- \frac{d - 1}{2} \xi \sum_{i=1}^d v^i e^{\xi (v, \theta_t)} \theta^i_t dt + \frac{\xi^2}{2} \sum_{i,j=1}^d v^i v^j e^{\xi (v, \theta_t)} (\delta_{ij} - \theta^i_t \theta^j_t) dt.
\]
From this, taking the expectation, we easily obtain (2.5) because \( \frac{\partial^n}{\partial t^n} E[e^{\xi(v, \theta t)}] = E[(v, \theta t)^n e^{\xi(v, \theta t)}], n = 1, 2. \) (2.6) immediately follows from the definition.

For simplicity, consider the function \( g_\nu \) given by
\[
g_\nu(t, \xi) = f_\nu \left( 2t, \frac{\xi}{|v|} \right).
\]

Then \( g_\nu \) is a smooth function which satisfies
\[
(2.7) \quad \frac{\partial g_\nu}{\partial t} = -\xi^2 \frac{\partial^2 g_\nu}{\partial \xi^2} - (d - 1) \xi \frac{\partial g_\nu}{\partial \xi} + \xi^2 g_\nu, \quad t > 0, \ \xi > 0,
\]
and
\[
(2.8) \quad g_\nu(0, \xi) = e^{-\nu \xi}, \quad g_\nu(t, 0) = 1, \quad \frac{\partial g_\nu}{\partial \xi}(t, 0) = \gamma_\nu e^{-(d-1)t}.
\]

If \( u(t, \xi) = e^{-\lambda t} \phi(\xi) \) satisfies (2.7), we should have
\[
\xi^2 \phi''(\xi) + (2\nu + 1) \xi \phi'(\xi) - (\xi^2 + \lambda) \phi(\xi) = 0.
\]

The system of the fundamental solutions of this second order differential equation is given by \( \xi^{-\nu} I_{\sqrt{\lambda + \nu^2}}(\xi) \) and \( \xi^{-\nu} K_{\sqrt{\lambda + \nu^2}}(\xi) \). For the function \( \phi \) to be smooth at \( \xi = 0 \), we should choose \( \xi^{-\nu} I_{\sqrt{\lambda + \nu^2}}(\xi) \). Moreover, \( n = \sqrt{\lambda + \nu^2} - \nu \) should be a non-negative integer and \( \lambda = n(n + 2\nu) \).

The following lemma is easily shown and we omit the proof.

**Lemma 2.3.** (1) The function \( \varphi_{\nu,n}(\xi) = \xi^{-\nu} I_{\nu+n}(\xi) \) satisfies
\[
\xi^2 \varphi''_{\nu,n}(\xi) + (2\nu + 1) \xi \varphi'_{\nu,n}(\xi) - \xi^2 \varphi_{\nu,n}(\xi) = n(n + 2\nu) \varphi_{\nu,n}(\xi), \quad \xi > 0.
\]

(2) One has
\[
\varphi'_{\nu,1}(0) = \frac{1}{2^{\nu+1}\Gamma(\nu + 2)} \quad \text{and} \quad \varphi'_{\nu,n}(0) = 0 \quad (n \neq 1).
\]

The following proposition immediately implies Proposition 2.2.

**Proposition 2.4.** When \( d = 2 \), one has
\[
(2.9) \quad g_0(t, \xi) = I_0(\xi) + \sum_{n=1}^{\infty} n C_n^0(\gamma_\nu) e^{-n^2t} I_n(\xi), \quad t \geq 0, \ \xi \geq 0,
\]
and, when \( d \geq 3 \),
\[
(2.10) \quad g_\nu(t, \xi) = 2^{\nu} \Gamma(\nu) \sum_{n=0}^{\infty} (\nu + n) C_n^{\nu}(\gamma_\nu) e^{-n(\nu+2\nu)t} \xi^{-\nu} I_{\nu+n}(\xi).
\]

**Proof.** First of all we note that the sums on the right hand sides of (2.9) and (2.10) are absolutely convergent at each \((t, \xi)\), which is seen from (A.1) and (A.2).

Letting \( \varphi_{\nu,n} \) be the function defined in Lemma 2.3 we set
\[
h_\nu(t; \xi) = \sum_{n=1}^{\infty} (\nu + n) C_n^{\nu}(\gamma_\nu) e^{-n(\nu+2\nu)t} \varphi_{\nu,n}(\xi).
\]

We have shown that the sum on the right hand side is absolutely convergent. Moreover, noting
\[
\varphi'_{\nu,n}(\xi) = \frac{n}{\xi} \varphi_{\nu,n}(\xi) + \varphi_{\nu,n+1}(\xi),
\]
we see, in a similar way to the right hand sides of (2.9) and (2.10), that
\[
\sum_{n=1}^{\infty} (\nu + n) C_n^\nu(\gamma_v) e^{-n(n+2\nu)t} \varphi_{\nu,n}(\xi) \quad \text{and} \quad \sum_{n=1}^{\infty} (\nu + n) C_n^\nu(\gamma_v) e^{-n(n+2\nu)t} \varphi_{\nu,n}''(\xi)
\]
converge uniformly on compact sets in \(\xi \in (0, \infty)\) and are equal to \(\frac{\partial}{\partial \xi} h_\nu(t, \xi)\) and \(\frac{\partial^2}{\partial \xi^2} h_\nu(t, \xi)\), respectively.

Next we look at \(h_\nu(t, \xi)\) as a function in \(t > 0\). By (A.1) and (A.2) we have
\[
\sum_{n=1}^{\infty} (\nu + n) C_n^\nu(\gamma_v) |n(n+2\nu)| e^{-n(n+2\nu)t} \varphi_{\nu,n}(\xi)
\]
\[
\leq \rho_\nu \sum_{n=1}^{\infty} (\nu + n) 4^n \Gamma(\nu + n) \frac{n(n+2\nu)}{n!} \frac{\xi^n}{2^{\nu+n} \Gamma(n+1) \Gamma(\nu + n + 1)} e^\xi
\]
\[
= \frac{\rho_\nu}{2\nu} \xi^2 \left\{ \sum_{n=1}^{\infty} \frac{(n-1)(2\xi)^n}{(n-1)!} + \sum_{n=1}^{\infty} \frac{(2\nu + 1)(2\xi)^n}{(n-1)!} \right\}
\]
\[
= \frac{\rho_\nu}{2\nu-2} \xi^2 e^{3\xi} + \frac{\rho_\nu(2\nu + 1)\xi}{2\nu-1} e^{3\xi},
\]
and we may differentiate term by term to obtain
\[
\frac{\partial}{\partial t} h_\nu(t, \xi) = -\sum_{n=1}^{\infty} (\nu + n) C_n^\nu(\gamma_v) e^{-n(n+2\nu)t} n(n+2\nu) \varphi_{\nu,n}(\xi).
\]
Combining the identities above, we see that the function \(g_\nu(t, \xi)\) given by (2.9) and (2.10) satisfies (2.7).

The boundary condition (2.8) in the case of \(d \geq 3\) may be checked by (2.4) and the fact \(C_n^\nu(\alpha) = 1\) and \(C_n^\nu(\alpha) = 2\nu\alpha\) (cf. [11, p. 218]).

For a check when \(d = 2\), we rewrite (2.4) as
\[
e^{\alpha \xi} = 2^{\mu} \Gamma(\mu + 1) \xi^{-\mu} I_{\mu}(\xi) + 2^{\mu} \sum_{n=1}^{\infty} \Gamma(\mu + n) C_n^\mu(\alpha) \xi^{-\mu} I_{\mu+n}(\xi)
\]
\[
= 2^{\mu} \Gamma(\mu + 1) \left\{ \xi^{-\mu} I_{\mu}(\xi) + \sum_{n=1}^{\infty} (\mu + n) C_n^\mu(\alpha) \mu \xi^{-\mu} I_{\mu+n}(\xi) \right\},
\]
which holds for any \(\mu > 0\). Note \(C_n^\mu(\alpha) \downarrow C_n^0(\alpha)\) as \(\mu \downarrow 0\). Then, by using (A.1) and (A.2), we can show that we may apply the dominated convergence theorem and obtain
\[
e^{\alpha \xi} = I_0(\xi) + \sum_{n=1}^{\infty} n C_n^0(\alpha) I_n(\xi),
\]
which is exactly the condition \(g_0(0, \xi) = e^{\gamma_v \xi}\).

Another boundary condition \(g_0(t, 0) = 1\) follows from \(I_0(0) = 1\) and the other condition \(\frac{\partial}{\partial \xi} g_0(t, 0) = \gamma_v e^{-t}\) follows from the formula \(C_1^0(\alpha) = \alpha\).

We have now completed the proof of Proposition 2.4. \(\square\)

Remark 2.5. The function \(z^{-\nu} I_{\nu+n}(z)\) may be regarded as a holomorphic function on \(C\). Hence, by using (A.1) and (A.2), we can show that the functions
\[
E[e^{z(\nu, \theta_t)/k}] \quad \text{and} \quad \sum_{n=1}^{\infty} (\nu + n) C_n^\nu(z) e^{-\frac{1}{2} n(n+2\nu)} z^{-\nu} I_{\nu+n}(z)
\]
are holomorphic in $z \in \mathbb{C}$. From this we obtain the Fourier-Laplace transform of the joint distribution of $(\sigma, B_\sigma)$. For example, when $d \geq 3$, we can show

$$E[e^{i\langle v, B_\sigma \rangle - \lambda \sigma}\mathbf{1}_{\{\sigma < \infty\}}] = 2^\nu \Gamma(\nu) \sum_{n=0}^{\infty} i^n (\nu + n) C_n^{\nu} \gamma_v^{\nu} J_{\nu+n}(\|v\| r) Z_{\nu+n}(\|x\|, r),$$

where $J_\mu$ is the usual Bessel function.

3. PROOF OF THEOREM 1.1

First we reduce the study on $(\sigma^{(v)}, B_{\sigma^{(v)}})$ to that on the distribution of $(\sigma, B_\sigma)$ developed in the previous section.

By the Cameron-Martin theorem and the strong Markov property of Brownian motion, we have

$$E[e^{-\lambda \sigma^{(v)} + \langle \mu, B_{\sigma^{(v)}} \rangle}\mathbf{1}_{\{\sigma^{(v)} \leq t\}}] = e^{-\langle v, x \rangle} E[e^{\langle v, B_t \rangle - \frac{1}{2} \|v\|^2} e^{-\lambda \sigma + \langle \mu, B_{\sigma} \rangle}\mathbf{1}_{\{\sigma \leq t\}}]$$

Hence, letting $t \to \infty$, we obtain

$$E[e^{-\lambda \sigma^{(v)} + \langle \mu, B_{\sigma^{(v)}} \rangle}\mathbf{1}_{\{\sigma^{(v)} < \infty\}}] = e^{-\langle v, x \rangle} E[e^{-\lambda \sigma + \langle \mu + \mu, B_{\sigma} \rangle}\mathbf{1}_{\{\sigma < \infty\}}].$$

By using Theorem 2.1, we obtain the joint Laplace transform of $(\sigma^{(v)}, B_{\sigma^{(v)}})$.

**Theorem 3.1.** Let $\mu \in \mathbb{R}^d$. Then, under the same condition of Theorem 1.1, one has

$$E[e^{-\lambda \sigma^{(v)} + \langle \mu, B_{\sigma^{(v)}} \rangle}\mathbf{1}_{\{\sigma^{(v)} < \infty\}}] = e^{-\langle v, x \rangle} \left\{ I_0(\|v + \mu\| r) Z_0^{(v),\lambda}(\|x\|, r) + \sum_{n=1}^{\infty} n C_n^{\nu} \gamma_v^{\nu} I_n(\|v + \mu\| r) Z_n^{(v),\lambda}(\|x\|, r) \right\}$$

when $d = 2$ and, when $d \geq 3$,

$$E[e^{-\lambda \sigma^{(v)} + \langle \mu, B_{\sigma^{(v)}} \rangle}\mathbf{1}_{\{\sigma^{(v)} < \infty\}}] = e^{-\langle v, x \rangle} 2^\nu \Gamma(\nu) \sum_{n=0}^{\infty} (\nu + n) C_n^{\nu} \gamma_v^{\nu} I_{\nu+n}(\|v + \mu\| r) Z_{\nu+n}(\|x\|, r).$$

By Theorem 3.1 we have, when $d = 2$,

$$E[e^{-\lambda \sigma^{(v)}}] = e^{-\langle v, x \rangle} \left\{ I_0(\|v\| r) Z_0^{(v),\lambda}(\|x\|, r) + \sum_{n=1}^{\infty} n C_n^{\nu} \gamma_v^{\nu} I_n(\|v\| r) Z_n^{(v),\lambda}(\|x\|, r) \right\}$$

and, when $d \geq 3$,

$$E[e^{-\lambda \sigma^{(v)}}] = e^{-\langle v, x \rangle} 2^\nu \Gamma(\nu) \sum_{n=0}^{\infty} (\nu + n) C_n^{\nu} \gamma_v^{\nu} I_{\nu+n}(\|v\| r) Z_{\nu+n}(\|x\|, r).$$

It is well known (cf. [10]) that the density $p_\mu(s; x)$ of $\sigma_\mu$, the first hitting time of a Bessel process with index $\mu$ starting from $|x|$, is characterized by

$$E[e^{-\lambda \sigma_\mu}] = \int_0^\infty e^{-\lambda s} p_\mu(s; x) ds = \frac{r^\mu}{\|x\|^\mu} Z_0^{(0),\lambda}(\|x\|, r).$$

We use the same notation for the density since our main concern is on the special case where the index $\mu$ is a half integer, and there is no fear of confusion.
To prove Theorem 1.1, we compute the Laplace transform of the right hand sides of (1.1) and (1.2) by changing the order of the integrations and the infinite sums. Using the estimates given in the Appendix and (2.1), we have, for \( \nu \geq 0 \),

\[
\sum_{n=1}^{\infty} (\nu + n)|C_n(\gamma_v)|I_{\nu+n}(|v|r) \frac{|x|^n}{|v|^\nu r + n} \int_0^\infty e^{-\left(\lambda + \frac{1}{2}|v|^2\right)t} p_{\nu+n}(t; x)dt
\]

(3.2)

\[
\leq \frac{\rho_\nu r^{\nu}}{2^\nu |x|^\nu} e^{|v|r} \sum_{n=1}^{\infty} \frac{(\nu + n)\Gamma(\nu + n)(2|v|)^n}{n!\Gamma(\nu + n + 1)} Z_{\nu+n}^{(v),\lambda}(|x|, r).
\]

Since \( Z_{\nu}^{(v),\lambda} \leq 1 \), the above is bounded by

\[
\frac{\rho_\nu r^{\nu}}{2^\nu |x|^\nu} e^{|v|r} \sum_{n=0}^{\infty} \frac{(2|v|)^n}{n!} = \frac{\rho_\nu r^{\nu}}{2^\nu |x|^\nu} e^{3|v|r}.
\]

Hence, we may apply Fubini’s theorem and see, from (3.1), that the Laplace transforms of the right hand sides of (1.1) and (1.2) are equal to those of \( p_\nu^{(v)}(t; x) \) in both cases.

We have now proved Theorem 1.1.

4. PROOF OF THEOREM 1.2

We set

\[
f^{(v)}(t; x) = P(t < \sigma^{(v)} < \infty) = \int_t^\infty p^{(v)}(s; x)ds.
\]

In order to apply Theorem 1.1, we need to change the order of the integration and the summation.

For this purpose, we note from (A.1) and (A.2)

\[
\sum_{n=1}^{\infty} (\nu + n)|C_n(\gamma_v)|I_{\nu+n}(|v|r) \frac{|x|^n}{|v|^\nu r + n} \int_t^\infty e^{-\left(\frac{1}{2}|v|^2\right)s} p_{\nu+n}(s; x)ds
\]

\[
\leq \sum_{n=1}^{\infty} (\nu + n)\rho_\nu \frac{4^n\Gamma(\nu + n)}{n!} \frac{(|v|r)^n e^{|v|r}}{2^\nu + n\Gamma(\nu + n + 1)} \frac{|x|^n}{r^n} e^{-\left(\frac{1}{2}|v|^2\right)t}
\]

\[
\leq \frac{\rho_\nu}{2^\nu} e^{|v|r} \sum_{n=1}^{\infty} \frac{(2|v|)|x|^n}{n!} \leq \frac{\rho_\nu}{2^\nu} e^{(r+2|x|)}.
\]

Then we can apply Fubini’s theorem and obtain from Theorem 1.1

\[
f^{(v)}(t; x) = e^{-\left<\nu, x\right>} I_0(|v|r) \int_t^\infty e^{-\left(\frac{1}{2}|v|^2\right)s} p_0(s; x)ds + f_0(t)
\]

when \( d = 2 \), and, when \( d \geq 3 \),

\[
f^{(v)}(t, x) = e^{-\left<\nu, x\right>} 2^\nu \Gamma(\nu + 1) \frac{I_\nu(|v|r)}{|v|^\nu r} \int_t^\infty e^{-\left(\frac{1}{2}|v|^2\right)s} p_\nu(s; x)ds + f_\nu(t),
\]

where the second terms on the right hand sides are given by

\[
f_\nu(t) = e^{-\left<\nu, x\right>} 2^\nu \Gamma(\nu + 1)
\]

\[
\times \sum_{n=1}^{\infty} (\nu + n)C_n(\gamma_v) \frac{|x|^n I_{\nu+n}(|v|r)}{|v|^\nu r + n} \int_t^\infty e^{-\left(\frac{1}{2}|v|^2\right)s} p_{\nu+n}(s; x)ds.
\]
First we prove the theorem when \( \nu > 0 \). We have
\[
p_\nu(t; x) = \frac{L(\nu)}{t^{\nu+1}(1 + o(1))}
\]
and, by L’Hospital’s rule,
\[
(4.1) \quad \int_t^\infty e^{-\frac{1}{2}|x|^2s}p_\nu(s; x)ds = \frac{2L(\nu)}{|x|^2}t^{-\nu-1}e^{-\frac{1}{2}|x|^2t}(1 + o(1)).
\]
Note that this identity holds for any \( \nu > 0 \).

In order to show that \( f_\nu(t) \) is negligible when \( \nu > 0 \), we use the argument in Section 2 of [7]. Then we obtain
\[
\int_t^\infty e^{-\frac{1}{2}|x|^2s}p_{\nu+n}(s; x)ds \leq e^{-\frac{1}{2}|x|^2t}P(t < \sigma^{(\nu+n)} < \infty)
\]
\[
\leq e^{-\frac{1}{2}|x|^2t}E_{\nu+n, |x|}[(R_t)^{-2(\nu+n)}],
\]
where \( E_{\nu+n, |x|} \) denotes the expectation with respect to the probability law of the Bessel process \( \{R_t\}_{t \geq 0} \) with index \( \nu + n \) and starting from \( |x| \).

Using the explicit expression of the transition density of the Bessel process, we obtain
\[
E_{\nu, |x|}[(R_t)^{-2(\nu+n)}] = \frac{1}{(2t)^{\nu+n}}e^{-\frac{|x|^2}{2t}}\sum_{m=0}^\infty \frac{|x|^{2m}}{\Gamma(\nu + n + m + 1)(2t)^m}
\]
\[
\leq \frac{1}{(2t)^{\nu+n}}e^{-\frac{|x|^2}{2t}}\sum_{m=0}^\infty \frac{|x|^{2m}}{\Gamma(\nu + n + 1)(m + 1)(2t)^m}
\]
\[
= \frac{1}{\Gamma(\nu + n + 1)(2t)^{\nu+n}}.
\]
Hence, by using (A.1) and (A.2) again, we obtain for \( n \geq 1 \) and \( t \geq 1 \)
\[
t^{\nu+1}e^{\frac{1}{2}|x|^2t(\nu + n)}J_n^\nu(\gamma_t)|x|^nI_{\nu+n}(|x|^r)\int_t^\infty e^{-\frac{1}{2}|x|^2s}p_{\nu+n}(s; x)ds
\]
\[
\leq \frac{\nu e^{|x|^r}}{4^\nu n!\Gamma(\nu + n + 1)}|x|^n.
\]

The quantity on the right hand side is independent of \( t \geq 1 \) and is summable in \( n \). Therefore, since (4.1) holds when we replace \( \nu \) by \( \nu + n \), we can apply the dominated convergence theorem and obtain
\[
\lim_{t \to \infty} t^{\nu+1}e^{\frac{1}{2}|x|^2t}f_\nu(t) = 0.
\]

When \( d = 2 \), we have
\[
p_0(t; x) = \frac{L(0)}{t(\log t)^2}(1 + o(1)), \quad L(0) = 2\log \frac{|x|}{r},
\]
and
\[
\int_t^\infty e^{-\frac{1}{2}|x|^2s}p_0(s; x)ds = \frac{L(0)}{t(\log t)^2}e^{-\frac{1}{2}|x|^2t(1 + o(1)).
\]

In the same way as in the case of \( \nu > 0 \), we can show by the dominated convergence theorem
\[
\lim_{t \to \infty} t(\log t)^2 e^{\frac{1}{2}|x|^2t}f_0(t) = 0,
\]
and we also obtain the assertion of Theorem 1.2 in this case.
Remark 4.1. The estimate for the tail probability $P(t < \sigma < \infty)$ was first given in Byczkowski and Ryznar [3]. We need an explicit upper bound here.

**Appendix A. Auxiliary estimates**

In this section we show some estimates for the modified Bessel function $I_\mu$ and for the Gegenbauer polynomial $C_n^\nu$.

First we show an estimate for $I_\mu$,

$$I_\mu(\xi) = \sum_{m=0}^{\infty} \frac{(\xi/2)^{\mu+2m}}{\Gamma(m+1)\Gamma(m+\mu+1)}.$$

**Lemma A.1.** For $\mu \geq 0$ and $n \geq 1$, one has

$$\xi^{-\mu}I_{\mu+n}(\xi) \leq \frac{\xi^n}{2^{\mu+n}\Gamma(\mu+n+1)} e^\xi, \quad \xi > 0.$$

**Proof.** Note that $\Gamma(p+q) \geq \Gamma(p+1)\Gamma(q)$ holds for $p \geq 0$ and $q \geq 1$, which can be seen from

$$\frac{\Gamma(p+1)\Gamma(q)}{\Gamma(p+q)} = pB(p,q) \leq p \int_0^1 x^{p-1} dx = 1, \quad p > 0.$$

Then we have

$$\xi^{-\mu}I_{\mu+n}(\xi) \leq \frac{\xi^n}{2^{\mu+n}\Gamma(\mu+n+1)} \sum_{m=0}^{\infty} \frac{(\xi/2)^{2m}}{\Gamma(m+1)^2\Gamma(\mu+n+1)}$$

$$\leq \frac{\xi^n}{2^{\mu+n}\Gamma(\mu+n+1)} \left( \sum_{m=0}^{\infty} \frac{(\xi/2)^m}{m!} \right)^2,$$

which shows (A.1). \hfill \Box

Next we give an estimate for the Gegenbauer polynomial $C_n^\nu$. When $\nu > 0$, it is given by

$$C_n^\nu(\xi) = \frac{1}{\Gamma(\nu)} \sum_{m=0}^{\lfloor n/2 \rfloor} (-1)^m \frac{\Gamma(\nu+n-m)}{m!(n-2m)!} (2\xi)^{n-2m},$$

which is characterized by the relation

$$(1 - 2t\xi + t^2)^{-\nu} = \sum_{n=0}^{\infty} C_n^\nu(\xi)t^n.$$

When $\nu = 0$, $C_0^0(\xi) = 1$ and, when $n \geq 1$, $C_n^0$ is given by

$$C_n^0(\xi) = \sum_{m=0}^{\lfloor n/2 \rfloor} (-1)^m \frac{\Gamma(n-m)}{\Gamma(m+1)\Gamma(n-2m+1)} (2\xi)^{n-2m}.$$

**Lemma A.2.** For $\alpha \in \mathbb{R}$ with $|\alpha| \leq 1$, $\nu \geq 0$ and $n \geq 1$, one has

$$|C_n^\nu(\alpha)| \leq \rho_\nu \frac{4^n\Gamma(\nu+n)}{n!},$$

where $\rho_0 = 1$ and $\rho_\nu = (\Gamma(\nu))^{-1}$ for $\nu > 0$. 


Proof. When $\nu = 0$, since $C_0^n(\cos \theta) = \frac{2}{n} \cos(n\theta)$, we have

$|C_n^0(\alpha)| \leq \frac{4^n}{n} = \frac{4^n\Gamma(n)}{n!}$.

When $\nu > 0$, we have

$|C_n^\nu(\alpha)| \leq \frac{1}{\Gamma(\nu)} \sum_{m=0}^{[n/2]} \frac{2^{n-2m}\Gamma(\nu + n - m)}{\Gamma(\nu + n)m!(n-2m)!} \cdot \frac{1}{\Gamma(\nu + n)(n-2m)!}$.

If $1 \leq m \leq [n/2]$, it holds that

$\frac{\Gamma(\nu + n - m)}{\Gamma(\nu + n)(n-2m)!} \leq \frac{1}{(n-1)(n-2)\cdots(n-m)(n-2m)!} = \frac{(n-m-1)\cdots(n-(2m-1))}{(n-1)!} \leq \frac{n^m}{n!}$.

Hence, we get

$|C_n^\nu(\alpha)| \leq \frac{1}{\Gamma(\nu)} \sum_{m=0}^{[n/2]} \frac{1}{m!} \frac{n^m}{n!} 2^{n-2m}$

$\leq \frac{2^n}{\Gamma(\nu)n!} \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{n}{4}\right)^m \leq \frac{2^n}{\Gamma(\nu)n!} e^n \leq \frac{4^n}{\Gamma(\nu)n!}$

because $e^{\frac{1}{4}} \leq 2$.

\[ \Box \]

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References

[1] M. Aizenman and B. Simon, Brownian motion and Harnack inequality for Schrödinger operators, Comm. Pure Appl. Math. 35 (1982), no. 2, 209–273, DOI 10.1002/cpa.3160350206. MR644024

[2] Andrei N. Borodin and Paavo Salminen, Handbook of Brownian motion—facts and formulae, 2nd ed., Probability and its Applications, Birkhäuser Verlag, Basel, 2002. MR1912205

[3] T. Byczkowski and M. Ryznar, Hitting distributions of geometric Brownian motion, Studia Math. 173 (2006), no. 1, 19–38, DOI 10.4064/sm173-1-2. MR2204460

[4] Gerald B. Folland, Introduction to partial differential equations, 2nd ed., Princeton University Press, Princeton, NJ, 1995. MR1357411

[5] Yuji Hamana and Hiroyuki Matsumoto, The probability densities of the first hitting times of Bessel processes, J. Math-for-Ind. 4B (2012), 91–95. MR3072521

[6] Yuji Hamana and Hiroyuki Matsumoto, The probability distributions of the first hitting times of Bessel processes, Trans. Amer. Math. Soc. 365 (2013), no. 10, 5237–5257, DOI 10.1090/S0002-9947-2013-05799-6. MR3074372

[7] Yuji Hamana and Hiroyuki Matsumoto, Asymptotics of the probability distributions of the first hitting times of Bessel processes, Electron. Commun. Probab. 19 (2014), no. 5, 5pp., DOI 10.1214/ECP.v19-3215. MR3164752

[8] Yuji Hamana and Hiroyuki Matsumoto, A formula for the expected volume of the Wiener sausage with constant drift, Forum Math. (to appear), available at arXiv:1512.03541[Math.PR].

[9] K. Itô and H.P. McKean, Jr., Diffusion Processes and Their Sample Paths, Springer–Verlag, 1965. MR0345224 (49:9963)
[10] John T. Kent, *Eigenvalue expansions for diffusion hitting times*, Z. Wahrsch. Verw. Gebiete 52 (1980), no. 3, 309–319, DOI 10.1007/BF00538895. MR576891

[11] Wilhelm Magnus, Fritz Oberhettinger, and Raj Pal Soni, *Formulas and theorems for the special functions of mathematical physics*, Third enlarged edition. Die Grundlehren der mathematischen Wissenschaften, Band 52, Springer-Verlag New York, Inc., New York, 1966. MR0232968

[12] Jim Pitman and Marc Yor, *Bessel processes and infinitely divisible laws*, Stochastic integrals (Proc. Sympos., Univ. Durham, Durham, 1980), Lecture Notes in Math., vol. 851, Springer, Berlin, 1981, pp. 285–370. MR620995

[13] Daniel W. Stroock, *On the growth of stochastic integrals*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 18 (1971), 340–344. MR0287622

[14] Kôhei Uchiyama, *Asymptotics of the densities of the first passage time distributions for Bessel diffusions*, Trans. Amer. Math. Soc. 367 (2015), no. 4, 2719–2742, DOI 10.1090/S0002-9947-2014-06155-2. MR3301879

[15] G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1995. Reprint of the second (1944) edition. MR1349110

[16] J. G. Wendel, *Hitting spheres with Brownian motion*, Ann. Probab. 8 (1980), no. 1, 164–169. MR556423

[17] Chuancun Yin, *The joint distribution of the hitting time and place to a sphere or spherical shell for Brownian motion with drift*, Statist. Probab. Lett. 42 (1999), no. 4, 367–373, DOI 10.1016/S0167-7152(98)00231-4. MR1707182

[18] Chuancun Yin and Chunwei Wang, *Hitting time and place of Brownian motion with drift*, Open Stat. Prob. J. 1 (2009), 38–42, DOI 10.2174/1876527000901010038. MR2520345

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