Federated Clustering via Matrix Factorization Models: From Model Averaging to Gradient Sharing

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Abstract

Recently, federated learning (FL) has drawn significant attention due to its capability of training a model over the network without knowing the client’s private raw data. In this paper, we study the unsupervised clustering problem under the FL setting. By adopting a generalized matrix factorization model for clustering, we propose two novel (first-order) federated clustering (FedC) algorithms based on principles of model averaging and gradient sharing, respectively, and present their theoretical convergence conditions. We show that both algorithms have a $O(1/T)$ convergence rate, where $T$ is the total number of gradient evaluations per client, and the communication cost can be effectively reduced by controlling the local epoch length and allowing partial client participation within each communication round. Numerical experiments show that the FedC algorithm based on gradient sharing outperforms that based on model averaging, especially in scenarios with non-i.i.d. data, and can perform comparably as or exceed the centralized clustering algorithms.

Keywords— Federated learning, Clustering, Matrix factorization, Model averaging, Gradient Sharing

I. INTRODUCTION

As one of the most fundamental data mining tasks, unsupervised clustering has a vast range of applications [1]. In view of the increasing volume of real-life data, distributed clustering methods that can process large-scale datasets in parallel computing environments have gained significant interests in the last decade [2], [3], [4]. However, recent emphasis on user privacy has called for new distributed schemes that can perform clustering without directly accessing the users’ raw data. Specific examples include processing distributed patient medical records stored in multiple hospitals [5] and daily personal data of mobile device users [6].

As an emerging distributed learning paradigm, federated learning (FL) has been introduced by Google [7] recently to enable collaborative model learning among massive clients (e.g., mobile devices or institutions) under the orchestration of a central server without the need of knowing the clients’ raw

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private data. Compared with the traditional distributed setting, FL faces new challenges in addition to protecting user privacy and handling non-convex learning objectives, including massively distributed clients, unbalanced and non-i.i.d. data distribution, and limited network connection [7], [6]. To train a model under the challenging FL setting, several distributed learning algorithms have been proposed [8], [9], [10], [11], [12], [13], mostly based on the classical stochastic gradient descent (SGD) method. In particular, [8] proposed a model averaging algorithm, called federated averaging (FedAvg), where the server coordinates the training by iteratively averaging the local models learned by the clients via SGD. A salient feature of FedAvg over the classical gradient sharing approach is that allowing the clients to perform multiple epochs of local SGD before model averaging can effectively reduce the required number of communication rounds for achieving a desired learning accuracy [8], [12], [13]. We notice that, while many successful efforts have been made for supervised FL, little work has been done for the unsupervised clustering task.

In this paper, we are interested in studying the clustering problem under the FL setting. Since many of the clustering methods, such as the K-means [14] and its enhanced counterparts [15], [16], can be formulated as a matrix factorization (MF) model, we study the federated clustering (FedC) problem through a distributed MF model (see Eqn. (4)). Note that the existing FL algorithms are not always applicable since the MF model involves two blocks of variables.

A. Related Works

Here let us briefly present the literature on distributed clustering and distributed MF methods.

**Distributed clustering**: There are two main categories for distributed clustering. In the first category, the methods are simply parallel implementations of the centralized clustering algorithms, such as K-means [17], [18], [19], [3] and density based DBSCAN [20], [21]. They usually assume a parallel computing environment with cheap communication links and shared memory, which, however, is opposite to the FL setting.

Distributed clustering methods in the second category target at approximating the centralized clustering methods via constructing so-called coreset, which is a small-sized set of weighted samples whose cost approximates the cost of the original dataset. Thus, clustering over the coreset is approximately the same as clustering over the original dataset, which resolves the large-scale clustering issue. For example, in [22], [23] distributed clients generate local coresets based on local data, and their union constitutes a global coreset, while in [2], [24], a global coreset is directly constructed from locally clustering results. Impressively, in these methods the clients require to communicate with the sever for one or
two rounds only. Approximation ratios with respect to the referenced algorithms (such as K-means/K-median/K-centers) are also guaranteed [2], [23], [4]. However, these coreset methods can never exceed their referenced algorithms.

**Distributed MF:** A large body of the existing distributed MF methods are parallel implementations of the centralized sequential SGD or alternating least square (ALS) algorithms, either on MapReduce [25], [26], [27] or Parameter Sever [28]. Analogously, parallel implementations of multiplicative rule [29] and block coordinate descent [30], [31] on MapReduce are developed for non-negative MF (NMF) models. Again, these works usually assume that there is a shared memory that all nodes can access, and careful model/data partition is required for efficient parallelization.

Decentralized MF methods such as [32], [33], [34], [35], [36] assume the absence of the central server, and the network topology is more flexible than that in FL. The consensus-type methods are often employed where the distributed nodes exchange messages with their neighbors only. However, the key issues of FL such as communication overhead and unbalanced/non-i.i.d. data are not considered therein. Besides, convergence results are limited to smooth and unconstrained problems.

The recent works in [37] and [38] have considered the FL scenarios and presented distributed MF algorithms based on gradient sharing. However, unlike the proposed FedCGds, neither multiple local epochs nor partial client participation are considered in these works. There is no theoretical convergence result therein neither.

**B. Contributions**

By adopting the popular alternating minimization strategy [39], we proposed in this paper two novel (first-order) FedC algorithms based on model average (MA) and gradient sharing (GS), respectively. Our technical contributions include:

- We first adopt the MA approach and propose a first-order iterative FedC algorithm, termed FedCAvg, where in each round, the clients perform multiple epochs of projected gradient descent (PGD) with respect to the two blocks of variables sequentially, followed by averaging the local centroid model at the server. We present theoretical conditions for which FedCAvg has a $O(1/T)$ convergence rate, where $T$ is the total number of gradient evaluations per client, and that the communication overhead can be effectively reduced by controlling the local epoch length.

- In view of the fact that MA methods are likely vulnerable to non-i.i.d. data, we further propose a GS based FedC algorithm, termed FedCGds, where in each round, the clients compute the gradient information and send it to the sever for updating the global centroid model. Interestingly, thanks
to the linear structure of the gradient vector, the clients can simply send the server the differential gradient information. Moreover, like FedAvg [8], FedCGds allows partial client participation in each communication round. Theoretical analysis further shows that the communication overhead can also be reduced if the sever and clients performs multiple epochs of PGD within each round.

- The performance of the proposed FedC algorithms is examined by numerical experiments based on both synthetic dataset and the MNIST handwriting digit dataset. In addition to revealing useful insights, the experimental results corroborate with the theorems that controlling the local epoch length can effectively improve the convergence speed of both FedAvg and FedCGds. Moreover, FedCGds is resilient to non-i.i.d. data and can achieve clustering performance comparable as or better than the centralized clustering algorithms. To the best of our knowledge, the current work is the first for FedC algorithms.

**Synopsis:** Section II reviews the MF model for clustering and introduces the FedC problem. Section III presents the proposed MA based FedCAvg algorithm and its theoretical convergence properties, and Section IV presents the GS based FedCGds algorithm and its convergence analysis. Extensive experimental results are presented in Section V. Finally, concluding remarks and future directions are discussed in Section VI.

II. PROBLEM FORMULATION

A. Clustering via Matrix Factorization Models

Let \( X = [x_1, x_2, \ldots, x_N] \in \mathbb{R}^{M \times N} \) be a data matrix that contains \( N \) data samples where each data sample \( x_i \) has \( M \) features. The task of clustering is to partition the \( N \) data samples into \( K \) non-overlapping and meaningful clusters in which data samples belonging to one cluster are close to each other based on an appropriate distance metric. It is known that the popular K-means algorithm can be written as the following integer-constrained MF problem [15], [40]

\[
\begin{align}
\min_{W,H} & \quad \|X - WH\|_F^2 \\
\text{s.t.} & \quad 1^T h_j = 1, \quad [H]_{ij} \in \{0, 1\}, \quad \forall i \in \mathcal{K}, j \in \mathcal{N}.
\end{align}
\]

where \( \mathcal{K} \triangleq \{1, \ldots, K\} \) and \( \mathcal{N} \triangleq \{1, \ldots, N\} \); \( \| \cdot \|_F \) is the matrix Frobenius norm, \( [H]_{ij} \) is the \((i,j)\)th entry of \( H = [h_1, \ldots, h_N] \in \mathbb{R}^{K \times N} \), and \( 1 \) is the all-one vector. In (1), columns of \( W \in \mathbb{R}^{M \times K} \) represent centroids of the \( K \) clusters, while \( H \) is the cluster assignment matrix where \([H]_{ij} = 1\) indicates that the \( j \)th data sample is uniquely assigned to cluster \( i \). Then, the K-means algorithm is equivalent to solving the above MF problem (1) via alternating minimization [15], [16].
However, due to the non-convexity and integer constraints, the K-means algorithm is sensitive to the choice of initial points and is likely to yield undesirable clustering results. In view of this, various MF models either with relaxed constraints or with structured regularization \[41\], \[42\], \[43\], \[40\], \[16\] have been developed in order to achieve improved clustering performance over the K-means. Mathematically, one may formulate this line of methods as the following more general MF model

\[
\min_{\mathbf{W}, \mathbf{H}} \frac{1}{N} \| \mathbf{X} - \mathbf{WH} \|_F^2 + R_W(\mathbf{W}) + R_H(\mathbf{H}) \tag{2a}
\]

s.t. \( \mathbf{W} \in \mathcal{W}, \mathbf{H} \in \mathcal{H}, \tag{2b} \)

where \( R_W(\cdot) \) and \( R_H(\cdot) \) are the (smooth) regularization functions for \( \mathbf{W} \) and \( \mathbf{H} \), respectively, for promoting cluster-interpretable solutions, and \( \mathcal{W} \) and \( \mathcal{H} \) are some compact and convex constraints. For example, one may have \( \mathcal{H} = \{ \mathbf{H} \mid 1^\top \mathbf{h}_j = 1, [\mathbf{H}]_{ij} \in [0, 1], \forall i \in \mathcal{K}, j \in \mathcal{N} \} \) as the convex relaxation of (1b), and \( \mathcal{W} = \{ \mathbf{W} \mid \| [\mathbf{W}]_{ij} \| \leq \bar{W} \} \) for constraining the maximum values of centroids.

It is worth mentioning that alternating minimization is the most popular strategy to handle the MF-type problems. In particular, the (proximal) alternating linearized minimization (PALM) algorithm \[39\] has been recognized as a computationally efficient method. By applying PALM to \[2\], the algorithm iteratively and alternatively performs PGD steps with respect to \( \mathbf{H} \) and \( \mathbf{W} \): for \( t = 1, 2, \ldots, \)

\[
\mathbf{H}^{t+1} = \mathcal{P}_H\{\mathbf{H}^t - \frac{1}{c^t} \nabla_H F(\mathbf{W}^t, \mathbf{H}^t)\}, \tag{3a}
\]

\[
\mathbf{W}^{t+1} = \mathcal{P}_W\{\mathbf{W}^t - \frac{1}{d^t} \nabla_W F(\mathbf{W}^t, \mathbf{H}^{t+1})\}, \tag{3b}
\]

where \( c^t \) and \( d^t \) are two step size parameters, and \( \mathcal{P}_H \) and \( \mathcal{P}_W \) are the two projection operations onto the sets \( \mathcal{H} \) and \( \mathcal{W} \), respectively. As will be seen shortly, the PALM method is employed in the developed FedC algorithms.

B. Federated Clustering Problem

By considering the FL setting, we assume that the data samples are partitioned as \( \mathbf{X} = [\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_P] \) and respectively owned by \( P \) distributed clients. Specifically, each client \( p \) owns non-overlapping data \( \mathbf{X}_p \in \mathbb{R}^{M \times N_p} \), where \( N_p \) is the number of samples of client \( p \) and \( \sum_{p=1}^{P} N_p = N \). Besides, we assume that there is a server who coordinates the \( P \) clients to accomplish the unsupervised clustering task with all the distributed data \( \mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_P \) being considered. Note that, under the FL scenario, the number of clients \( P \) could be large, the data size \( N_p, p = 1, \ldots, P \), could be unbalanced, and the data samples \( \mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_P \) could be non-i.i.d. \[7\], \[6\].

\(^1\)For ease of algorithm development, we assume that the regularization terms are smooth functions.
Let $H = [H_1, \ldots, H_P]$ be partitioned in the same fashion as $X$, and let $\omega_p = N_p/N$, $p = 1, \ldots, P$. Moreover, assume that $R_H(H) = \sum_{p=1}^P R_H(H_p)$ and $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2 \cdots \times \mathcal{H}_P$ which are separable with respect to the partitioned assignment matrices $H_1, \ldots, H_P$. Then, one can write the MF problem (2) as

$$\min_{W, H_p, p=1, \ldots, P} F(W, H) = \sum_{p=1}^P \omega_p F_p(W, H_p)$$

(4a)

subject to $W \in \mathcal{W}, H_p \in \mathcal{H}_p$, $\forall p = 1, \ldots, P$, (4b)

where

$$F_p(W, H_p) = \frac{\|X_p - WH_p\|^2}{N_p} + R_H(H_p) + R_W(W)$$

(5)

is the local cost function of each client $p$. In this paper, we study the FedC problem based on the above distributed MF model (4). The FedC algorithm should enable the server to coordinate the distributed clients to jointly solve the MF problem (4) without the need of the clients revealing their private raw data. The MF formulation (4) resembles the finite-sum problem that is widely studied in the FL literature [8]. However, (4) involves two blocks of variables, and thereby the existing FL algorithms are not always applicable.

III. FEDERATED CLUSTERING BY MODEL AVERAGING

As the MA approach is extensively adopted in the FL literature [8]. It is natural to extend this idea to the FedC task. In this section, we present such an algorithm, termed FedCAvg, and establish its theoretical convergence properties.

A. The FedCAvg Algorithm

Directly applying the idea of the model averaging to the distributed MF problem (4) would lead to an iterative algorithm as follows. For round $s = 1, 2, \ldots$, each client $p$ obtains an approximate solution to the corresponding local subproblem of (4), i.e.,

$$(W_p^s, H_p^s) \approx \arg \min_{W, H_p} F_p(W, H_p)$$

(6a)

subject to $W \in \mathcal{W}, H_p \in \mathcal{H}_p$. (6b)

Since the centroid matrix $W$ in (4) is the common variable to all clients, the server takes certain average of $W_1^s, \ldots, W_P^s$, denoted by $\bar{W}^s$, and broadcasts the average $\bar{W}^s$ to the clients for the next round of
updates. One way to handling (6) is simply employing one step of the PALM method. Specifically, given \( W_{s-1} \) and \( H_{s-1} \) in the previous round, each client \( p \) performs

\[
H_p^s = \mathcal{P}_H\{H_{p}^{s-1} - \frac{1}{c_p} \nabla_{H_p} F_p(W_{s-1}, H_{p}^{s-1})\},
\]

(7)

\[
W_p^s = \mathcal{P}_W\{W_{s-1} - \frac{1}{d_p} \nabla_{W_p} F_p(W_{s-1}, H_{p}^s)\},
\]

(8)

and then sends the local model \( W_p^s \) to the sever for model averaging. However, this may demand a lot of communication resources since the PALM algorithm would require a large number of iterations to yield a satisfactory solution.

It has been analytically shown in [13], [12] that the FedAvg algorithm [8], where the clients perform multiple epochs of SGD in each communication round, is effective in reducing the communication overhead. Following the FedAvg algorithm, we propose to perform \( Q_1 \geq 1 \) consecutive steps of PGD with respect to \( H_p \) (see Eqn. (10)), followed by \( Q_2 \geq 1 \) steps of gradient descent with respect to \( W_p \) (see Eqn. (12)) in each communication round. After a total number of \( Q = Q_1 + Q_2 \) local model updates, each client \( p \) sends its local model of \( W_p^{s,Q} \) to the server. The server takes the weighted average and applies to it projection operation \( \mathcal{P}_W \) (see Eqn. (9)). The details of the proposed FedCAvg algorithm is shown in Algorithm 1.

**Remark 1** It is arguable that the proposed client update steps in (10)-(13) for approximating (6) are not unique. For example, one may instead apply \( Q/2 \) consecutive PALM steps (7)-(8) locally at each client \( p \). Intriguingly, our numerical experiments suggest that this may not be a good strategy (see Fig. 1(d) in Section V). To gain the insight, one can see that when \( Q \to \infty \) the updates in (10)-(13) merely correspond to applying a single step of (two-)block coordinate descent to the local problem (6), whereas applying \( Q/2 \) PALM steps (7)-(8) with \( Q \to \infty \) would reach a stationary point of (6) [39]. Given solely locally observable data at the clients, the latter strategy would be too greedy and may not always benefit the global algorithm convergence. Similar insights are also observed for the FedAvg algorithm [13], [12].

**B. Convergence Analysis of FedCAvg**

We first make some proper assumptions on problem (4).

**Assumption 1** All local cost functions \( F_p \) are lower bounded (i.e., \( F_p(W, H_p) > F, \forall W \in \mathcal{W}, H_p \in \mathcal{H}_p \)), and continuously differentiable.
Algorithm 1 FedCAvg

1: **Input:** initial values of $W_1^{0,Q} = \cdots = W_p^{0,Q}$ at the server side and initial values of $\{H_p^{0,Q}\}_{p=1}^P$ at the clients.

2: for round $s = 1$ to $S$ do

3: **Server side:** Compute

$$W^s = \mathcal{P}_W \left( \sum_{p=1}^P \omega_p W^{s-1,Q}_p \right). \tag{9}$$

and broadcast $W^s$ to all clients.

4: **Client side:** For client $p = 1$ to $P$ in parallel do

5: Set $H_p^{s,0} = H_p^{s-1,Q}$ and $W_p^{s,0} = W^s$.

6: for epoch $t = 1$ to $Q_1$ do

$$H_p^{s,t} = \mathcal{P}_H \{ H_p^{s,t-1} - \frac{\nabla H_p F_p(W_p^{s,t-1}, H_p^{s,t-1})}{c_p^s} \}, \tag{10}$$

$$W_p^{s,t} = W_p^{s,t-1}. \tag{11}$$

7: end for

8: for epoch $t = Q_1 + 1$ to $Q$ do

$$W_p^{s,t} = W_p^{s,t-1} - \frac{\nabla W F_p(W_p^{s,t-1}, H_p^{s,t-1})}{d_s^s}, \tag{12}$$

$$H_p^{s,t} = H_p^{s,t-1}. \tag{13}$$

9: end for

10: Upload $W_p^{s,Q}$ to the server.

11: end for

12: end for

**Assumption 2** $\nabla H_p F_p(W^s, \cdot)$ is Lipschitz continuous on $H_p$ with constant $L_{H_p}^s$, and $\nabla W F_p(\cdot, H_p^{s,Q})$ is Lipschitz continuous on $W$ with constant $L_{W}^s$.

**Assumption 3** Both $L_{W}^s$ and $L_{H_p}^s$ are bounded sequences.

**Assumption 4** There exists a constant $G > 0$ such that $\nabla W F_p(W, H_p) \leq G$ holds for any bounded $(W, H_p)$. 
Note that by Assumption 2, $\nabla_W F(\cdot, H)$ is Lipschitz continuous with constant $L_{W}^s = \sqrt{\sum_{p=1}^{P} \omega_p (L_{W}^p)^2}$.
Assumptions 3 and 4 would hold as long as the generated sequence $(W^{s,t}_p, H^{s,t}_p)$ are bounded. We also define the following sequence
\[ \tilde{W}^{s,t} = \mathcal{P}_W \left( \sum_{p=1}^{P} \omega_p W^{s,t}_p \right), \quad \tilde{W}^{s,0} = W^{s}. \] (14)
as the instantaneous weighted average of local models. We have the main theoretical results for FedCAvg as follows.

**Theorem 1** Let $c^s_p = \frac{\gamma}{2} L_{H}^s$, where $\gamma > 1$, $d^s = (s+1)L_{W}^s$, and let $L_{W}^s \geq L_{W} > 0$. Then, under Assumptions 1, 2 and 4, the sequence $\{ (\tilde{W}^{s,t}, H^{s,t}) \}$ generated by Algorithm 1 where $H^{s,t} = [H^{s,t}_1, \ldots, H^{s,t}_P]$, satisfies
\[
\frac{1}{T} \left[ \frac{\gamma - 1}{2} \sum_{s=1}^{S} \sum_{t=1}^{Q_s} \sum_{p=1}^{P} \omega_p L_{H}^s \|H^{s,t}_p - H^{s,t-1}_p\|^2_F \right. \\
\left. + \frac{1}{2} \sum_{s=1}^{S} L_{W}^s \sum_{t=Q_s+1}^{Q} \|\tilde{W}^{s,t} - \tilde{W}^{s,t-1}\|^2_F \right] \\
\leq \frac{F(\tilde{W}^{1,0}, H^{1,0}) - F}{T} + \frac{5}{6} \frac{Q_2^2 (Q_2 - 1)}{L_{W}} \frac{1}{T}, \] (15)
where $T = SQ$ is the total number of gradient evaluations per client.

The proof of Theorem 1 is presented in Section III-C. Equation (15) shows that Algorithm 1 has a $O(1/T)$ convergence rate. Moreover, since the number of communication rounds is $S = T/Q$, Eqn. (15) implies that Algorithm 1 with $Q_1 > 1$ and/or $Q_2 > 1$ can effectively reduce the communication overhead if $T$ is given and fixed. However, it is interesting to observe that a large value of $Q_2$ may deteriorate the convergence speed. This corroborates our discussion in Remark 1 that being too greedy in solving the local problem (6) may not always benefit global convergence.

Alternatively, one can have a diminishing $Q_2$, for example, by setting $Q_2^s = \lfloor \hat{Q}/s \rfloor + 1$ for a preset value $\hat{Q} > 0$. This allows to set $d^s = \gamma L_{W}^s$ instead of using the (diminishing) step size rule $d^s = (s+1)L_{W}^s$. as in Theorem 1. Then, by following a similar proof as for Theorem 1 one can show the following result.

**Theorem 2** Let $c^s_p = \frac{\gamma}{2} L_{H}^s$ and $d^s = \gamma L_{W}^s$, where $\gamma > 1$, and let $Q_2^s = \lfloor \hat{Q}/s \rfloor + 1$. Under Assumptions 1-4 any limit point of the sequence $\{ (W^s, H^{s,0}) \}$ generated by Algorithm 1 is a stationary solution of problem (4).

The proof of Theorem 2 is presented in Section III-E. It is found numerically that the alternative scheme with diminishing $Q_2$ has favorable convergence behavior than that with constant $Q_2$ as we will demonstrate in Sec. \(\nabla\).
Remark 2 Analogous to the existing MA algorithms, the FedCAvg algorithm presented above is inherently susceptible to non-i.i.d. data distribution in practice. As suggested by Theorem 1 and Theorem 2, careful algorithm design (either diminishing step size or diminishing $Q_2$) will be needed for FedCAvg to yield desirable performance. This motivates us to develop the second FedC scheme in Section IV based on GS which can be resilient to non-i.i.d. data.

The proofs of Theorem 1 and Theorem 2 are given in the next three subsections. Readers who are not interested in the detailed proofs may skip them and directly go to Section IV.

C. Proof of Theorem 1

By following (14), we define

$$
\tilde{W}^{s,t} = \mathcal{P}_W(W^{s,t}), \quad W^{s,t} = \sum_{p=1}^{P} \omega_p W_p^{s,t},
$$

for $t = 0, 1, \ldots, Q$. Then, by (11), we have

$$
\tilde{W}^{s,t} = \tilde{W}^{s,0} = W^s, \quad t = 0, 1, \ldots, Q_1, \quad p = 1, \ldots, P. \tag{17}
$$

Objective Descent w.r.t. $H$: Recall from (10) and (11) in Algorithm 1 that in each round $s$, the client $p$ updates

$$
H_p^{s,t} = \mathcal{P}_{H_p} \left\{ H_p^{s,t-1} - \frac{\nabla H_p F_p(W_p^{s,t-1}, H_p^{s,t-1})}{c_p^s} \right\}, \tag{18}
$$

$$
W_p^{s,t} = W_p^{s,t-1}, \tag{19}
$$

for $t = 1, \ldots, Q_1$. According to [39, Lemma 3.2], (18), (19) and (17) implies

$$
F_p(\tilde{W}^{s,t}, H_p^{s,t}) - F_p(\tilde{W}^{s,t-1}, H_p^{s,t-1}) \leq - \left( c_p^s - \frac{L_{H_p}}{2} \right) \|H_p^{s,t-1} - H_p^{s,t}\|_F^2
$$

$$
= - \frac{\gamma - 1}{2} L_{H_p} \|H_p^{s,t-1} - H_p^{s,t}\|_F^2, \tag{20}
$$

for $t = 1, \ldots, Q_1$, where (20) is due to $c_p^s = \frac{\gamma L_{H_p}}{2}$.

Summing up (20) from $t = 1$ to $Q_1$ yields

$$
F_p(\tilde{W}^{s,Q_1}, H_p^{s,Q_1}) - F_p(\tilde{W}^{s,0}, H_p^{s,0}) \leq - \frac{\gamma - 1}{2} \sum_{t=1}^{Q_1} L_{H_p} \|H_p^{s,t-1} - H_p^{s,t}\|_F^2. \tag{21}
$$

As a result, the objective function $F$ descends with local updates of $H$ as follows

$$
F(\tilde{W}^{s,Q_1}, H_p^{s,Q_1}) - F(\tilde{W}^{s,0}, H_p^{s,0})
$$

$$
= \sum_{p=1}^{P} \omega_p \left( F_p(\tilde{W}^{s,Q_1}, H_p^{s,Q_1}) - F_p(\tilde{W}^{s,0}, H_p^{s,0}) \right)
$$
Note by (13) that $H^s_{p,t} = H^s_{p,t-1}$ for $t = Q_1 + 1, \ldots, Q$. Since $\nabla_W F(\cdot, H^s, Q)$ is Lipschitz continuous under Assumption 2, by the descent lemma [39, Lemma 3.1], we have

$$F(\widetilde{W}^{s,t}, H^s) \leq F(\widetilde{W}^{s,t-1}, H^s)$$

$$+ \langle \nabla_W F(\widetilde{W}^{s,t-1}, H^s), \widetilde{W}^{s,t} - \widetilde{W}^{s,t-1} \rangle + \frac{L^s_W}{2} \| \widetilde{W}^{s,t} - \widetilde{W}^{s,t-1} \|^2_F. \tag{23}$$

Let us bound the term (a) as follows. Firstly, by the optimality of $W^{s,t}$, we have

$$0 = \sum_{p=1}^{P} \omega_p \left( \nabla_W F_p(W_p^{s,t-1}, H_p^{s,t-1}) + d^s(W_p^{s,t-1} - W_p^{s,t}) \right)$$

$$= \sum_{p=1}^{P} \omega_p \nabla_W F_p(W_p^{s,t-1}, H_p^{s,t-1}) + d^s(W^{s,t} - W^{s,t-1}). \tag{24}$$

Secondly, consider the following term

$$\langle \nabla_W F(\widetilde{W}^{s,t-1}, H^s), \widetilde{W}^{s,t-1} - \widetilde{W}^{s,t} \rangle$$

$$= \langle \nabla_W F(\widetilde{W}^{s,t-1}, H^s), \widetilde{W}^{s,t-1} - \widetilde{W}^{s,t} \rangle - \sum_{p=1}^{P} \omega_p \nabla_W F_p(\widetilde{W}_p^{s,t-1}, H_p^{s,t-1}), \widetilde{W}^{s,t-1} - \widetilde{W}^{s,t} \rangle$$

$$\geq (\nabla_W F(\widetilde{W}^{s,t-1}, H^s), - \sum_{p=1}^{P} \omega_p \nabla_W F_p(\widetilde{W}_p^{s,t-1}, H_p^{s,t-1}), \widetilde{W}^{s,t-1} - \widetilde{W}^{s,t} \rangle$$

$$\geq (\nabla_W F(\widetilde{W}^{s,t-1}, H^s), - \sum_{p=1}^{P} \omega_p \nabla_W F_p(\widetilde{W}_p^{s,t-1}, H_p^{s,t-1}), \widetilde{W}^{s,t-1} - \widetilde{W}^{s,t} \rangle, \tag{25}$$

where (25) holds due to (24), and (26) follows because

$$\langle \widetilde{W}^{s,t} - \widetilde{W}^{s,t}, \widetilde{W}^{s,t-1} - \widetilde{W}^{s,t} \rangle \geq 0, \quad \langle \widetilde{W}^{s,t-1} - \widetilde{W}^{s,t}, \widetilde{W}^{s,t} - \widetilde{W}^{s,t-1} \rangle \geq 0. \tag{27}$$

Inequalities in (27) are obtained by the fact that $\widetilde{W}^{s,t} = P_W\{W^{s,t}\}$ and $\widetilde{W}^{s,t-1} = P_W\{W^{s,t-1}\}$, and the application of the optimality condition $\langle x^* - z, x - x^* \rangle \geq 0, \forall x \in \mathcal{X}$ of the projection problem $x^* = \arg\min_{x \in \mathcal{X}} \frac{1}{2}\|x - z\|_2^2$, where $\mathcal{X}$ is a closed convex set [44, Proposition 3.1.1]. Rearranging the terms in (26) yields

$$\langle \nabla_W F(\widetilde{W}^{s,t-1}, H^s), \widetilde{W}^{s,t} - \widetilde{W}^{s,t-1} \rangle \leq -d^s(\widetilde{W}^{s,t}, \widetilde{W}^{s,t-1})$$

$$+ \langle \nabla_W F(\widetilde{W}^{s,t-1}, H^s), \sum_{p=1}^{P} \omega_p \nabla_W F_p(\widetilde{W}_p^{s,t-1}, H_p^{s,t-1}), \widetilde{W}^{s,t} - \widetilde{W}^{s,t-1} \rangle. \tag{28}$$
Thus, substituting (28) into (23) gives rise to
\[
F(\tilde{W}^{s,t}, H^{s,t}) \leq F(\tilde{W}^{s,t-1}, H^{s,t-1}) - (d^s - \frac{L^s}{2})\|\tilde{W}^{s,t} - \tilde{W}^{s,t-1}\|^2_F \\
+ \langle \nabla_W F(\tilde{W}^{s,t-1}, H^{s,t-1}) - \sum_{p=1}^P \omega_p \nabla_W F_p(W_p^{s,t-1}, H_p^{s,t-1}), \tilde{W}^{s,t} - \tilde{W}^{s,t-1} \rangle. 
\]
(29)
We bound the term (b) as follows
\[
(b) = \langle \nabla_W F(\tilde{W}^{s,t-1}, H^{s,t-1}) - \sum_{p=1}^P \omega_p \nabla_W F_p(W_p^{s,t-1}, H_p^{s,t-1}), \tilde{W}^{s,t} - \tilde{W}^{s,t-1} \rangle \\
= \sum_{p=1}^P \omega_p \langle \nabla_W F_p(\tilde{W}^{s,t-1}, H_p^{s,t-1}) - \nabla_W F_p(W_p^{s,t-1}, H_p^{s,t-1}), \tilde{W}^{s,t} - \tilde{W}^{s,t-1} \rangle \\
\leq \sum_{p=1}^P \omega_p \left( \frac{1}{2d^s} \|\nabla_W F_p(\tilde{W}^{s,t-1}, H_p^{s,t-1}) - \nabla_W F_p(W_p^{s,t-1}, H_p^{s,t-1})\|^2_F + \frac{d^s}{2} \|\tilde{W}^{s,t} - \tilde{W}^{s,t-1}\|^2_F \right) \\
\leq \sum_{p=1}^P \omega_p \left( \frac{(L^s)^2_{W_p}}{2d^s} \|\tilde{W}^{s,t-1} - W_p^{s,t-1}\|^2_F + \frac{d^s}{2} \|\tilde{W}^{s,t} - \tilde{W}^{s,t-1}\|^2_F \right) \\
= \frac{1}{2d^s} \sum_{p=1}^P \omega_p (L^s)^2 \|\tilde{W}^{s,t-1} - W_p^{s,t-1}\|^2_F + \frac{d^s}{2} \|\tilde{W}^{s,t} - \tilde{W}^{s,t-1}\|^2_F, 
\]
(30)
where (30) follows from the basic inequality \(a, b \leq \frac{1}{2c} \|a\|^2 + \frac{c}{2} \|b\|^2\), for any \(c > 0\), and (31) holds by the Lipschitz continuity of \(\nabla_W F_p(\cdot, H_p^{s,Q})\). To bound the \(\|\tilde{W}^{s,t-1} - W_p^{s,t-1}\|^2_F\), we need the following lemma (proved in Section III-D).

**Lemma 1** Under Assumption 4, we have for all \(t = Q_1, \ldots, Q - 1\),
\[
\|\tilde{W}^{s,t} - W^{s,t}\|^2_F \leq \frac{(t - Q_1)^2}{(d^s)^2} G^2, \quad \|W^{s,t} - W_p^{s,t}\|^2_F \leq \frac{4(t - Q_1)^2}{(d^s)^2} G^2. 
\]
(33)

Then, by applying Lemma 1 we have
\[
\|\tilde{W}^{s,t-1} - W_p^{s,t-1}\|^2_F = \|\tilde{W}^{s,t-1} - W^{s,t-1}_p + W^{s,t-1} - W_p^{s,t-1}\|^2_F \\
\leq 2\|\tilde{W}^{s,t-1} - W^{s,t-1}_p\|^2_F + 2\|W^{s,t-1} - W_p^{s,t-1}\|^2_F \\
\leq \frac{2(t - 1 - Q_1)^2}{(d^s)^2} G^2 + 8(t - 1 - Q_1)^2 \frac{G^2}{(d^s)^2} \\
= \frac{10(t - 1 - Q_1)^2}{(d^s)^2} G^2. 
\]
(34)
By substituting (35) into (32), we obtain

\[
(b) = \langle \nabla_W F(W^{s,t-1}, H^{s,t-1}) - \sum_{p=1}^{P} \omega_p \nabla_W F_p(W^{s,t-1}_p, H^{s,t-1}_p), W^{s,t} - \tilde{W}^{s,t} \rangle 
\]

\[
\leq \frac{5(t - 1 - Q_1)^2}{(d^s)^3} G^2 \sum_{p=1}^{P} \omega_p (L^s_{W_p})^2 + \frac{d^s}{2} \| \tilde{W}^{s,t} - \tilde{W}^{s,t-1} \|_F^2. 
\] (36)

Thus, by (36), (29) can be written as

\[
F(W^{s,t}, H^{s,t}) \leq F(W^{s,t-1}, H^{s,t-1}) - \frac{d^s - L^s_W}{2} \| \tilde{W}^{s,t} - \tilde{W}^{s,t-1} \|_F^2 + \frac{5(t - 1 - Q_1)^2 G^2}{(d^s)^3} \sum_{p=1}^{P} \omega_p (L^s_{W_p})^2. 
\] (37)

By summing up (37) from \( t = Q_1 + 1 \) to \( Q \), we have

\[
F(W^{s,Q}, H^{s,Q}) \leq F(W^{s,Q_1}, H^{s,Q_1}) - \frac{d^s - L^s_W}{2} \sum_{t=Q_1+1}^{Q} \| \tilde{W}^{s,t} - \tilde{W}^{s,t-1} \|_F^2 
\]

\[
+ \sum_{t=Q_1+1}^{Q} \frac{5(t - 1 - Q_1)^2 G^2}{(d^s)^3} \sum_{p=1}^{P} \omega_p (L^s_{W_p})^2 
\]

\[
\leq F(W^{s,Q_1}, H^{s,Q_1}) - \frac{d^s - L^s_W}{2} \sum_{t=Q_1+1}^{Q} \| \tilde{W}^{s,t} - \tilde{W}^{s,t-1} \|_F^2 
\]

\[
+ \frac{5Q_2(Q_2 - 1)(2Q_2 - 1)}{6(d^s)^3} G^2 \sum_{p=1}^{P} \omega_p (L^s_{W_p})^2. 
\] (38)

**Derivation of the Main Result:** By combining (20) and (38), we obtain

\[
\frac{\gamma - 1}{2} \sum_{t=1}^{Q_1} \sum_{p=1}^{P} \omega_p L^s_{H_p} \| H^{s,t}_p - H^{s,t-1}_p \|_F^2 + \frac{d^s - L^s_W}{2} \sum_{t=Q_1+1}^{Q} \| \tilde{W}^{s,t} - \tilde{W}^{s,t-1} \|_F^2 
\]

\[
\leq F(W^{s,0}, H^{s,0}) - F(W^{s,Q}, H^{s,Q}) + \frac{C}{(d^s)^3} \sum_{p=1}^{P} \omega_p (L^s_{W_p})^2. 
\] (39)

where \( C \triangleq \frac{5Q_2(Q_2 - 1)(2Q_2 - 1)G^2}{6} \). By further summing up (39) from \( s = 1 \) to \( S \), we have

\[
\frac{\gamma - 1}{2} \sum_{s=1}^{S} \sum_{t=1}^{Q_1} \sum_{p=1}^{P} \omega_p L^s_{H_p} \| H^{s,t}_p - H^{s,t-1}_p \|_F^2 
\]

\[
+ \sum_{s=1}^{S} \frac{d^s - L^s_W}{2} \sum_{t=Q_1+1}^{Q} \| \tilde{W}^{s,t} - \tilde{W}^{s,t-1} \|_F^2 
\]

\[
\leq F(W^{1,0}, H^{1,0}) - F(W^{S,Q}, H^{S,Q}) + C \sum_{s=1}^{S} \sum_{p=1}^{P} \omega_p (L^s_{W_p})^2 
\]

\[
\leq F(W^{1,0}, H^{1,0}) - F + C \sum_{s=1}^{S} \frac{(L^s_{W})^2}{(d^s)^3}, 
\] (40)
where (40) follows because of $F(\tilde{W}^{S,Q}, H^{S,Q}) > F$ under Assumption 1 and $L^s_W = \sqrt{\sum_{p=1}^P \omega_p (L^s_p)^2}$.

Now, by substituting the step size $d^s = (s + 1)L^s_W$ into (40), we can bound it as

$$
\frac{\gamma - 1}{2} \sum_{s=1}^S \sum_{t=1}^Q \sum_{p=1}^P \omega_p L^s_p \|H^s_{p,t} - H^s_{p,t-1}\|_F^2 + \frac{1}{2} \sum_{s=1}^S \sum_{t=Q+1}^1 \|\tilde{W}^s_{s,t} - \tilde{W}^s_{s,t-1}\|_F^2
$$

$$
\leq F(\tilde{W}^{1,0}, H^{1,0}) - F + C \sum_{s=1}^S \frac{1}{(s+1)^3 L^s_W},
$$

$$
\leq F(\tilde{W}^{1,0}, H^{1,0}) - F + \frac{C}{2} \frac{1}{L_W},
$$

$$
\leq F(\tilde{W}^{1,0}, H^{1,0}) - F + \frac{5}{6} Q^2 (Q_2 - 1),
$$

(41)

(42)

where (41) is due to $\sum_{s=1}^S \frac{1}{(s+1)^3} \leq \frac{1}{2}$ and (42) follows because $C = \frac{5}{2} Q^2 (Q_2 - 1)(2Q_2 - 1) \leq \frac{5}{6} Q^2 (Q_2 - 1)$. Lastly, dividing both sides of (42) by $T = SQ$ gives rise to

$$
\frac{1}{T} \left[ \frac{\gamma - 1}{2} \sum_{s=1}^S \sum_{t=1}^Q \sum_{p=1}^P \omega_p L^s_p \|H^s_{p,t} - H^s_{p,t-1}\|_F^2 + \frac{1}{2} \sum_{s=1}^S \sum_{t=Q+1}^1 \|\tilde{W}^s_{s,t} - \tilde{W}^s_{s,t-1}\|_F^2 \right]
$$

$$
\leq \frac{F(\tilde{W}^{1,0}, H^{1,0}) - F}{T} + \frac{5}{6} Q^2 (Q_2 - 1)/L_W,
$$

(43)

where we have used the fact that $s L^s_W \geq L_W$ since $s \geq 1$. This completes the proof.

D. Proof of Lemma 1

Note that

$$
\|\tilde{W}^s_{s,t} - \tilde{W}^s_{s,t}\|_F \leq \|W^s - \tilde{W}^s_{s,t}\|_F^2
$$

(44)

since $\tilde{W}^s_{s,t} = \mathcal{P}_W(W^s_{s,t})$. Besides, according to the definition of $W^s_{s,t}$ in (16), we have

$$
W^s_{s,t} = \sum_{p=1}^P \omega_p W^s_{p,t}
$$

$$
= \sum_{p=1}^P \omega_p \left( W^s - \frac{1}{d^s} \sum_{j=Q_1}^{t-1} \nabla_W F_p(W^s_{p,j}, H^s_{p,j}) \right)
$$

(45)

$$
= W^s - \frac{1}{d^s} \sum_{j=Q_1}^{t-1} \sum_{p=1}^P \omega_p \nabla_W F_p(W^s_{p,j}, H^s_{p,j}),
$$

(46)
where (45) is obtained by

\[
W_p^{s,t} = W^s - \frac{1}{d^s} \sum_{j=Q_1}^{t-1} \nabla_W F_p(W_p^{s,j}, H_p^{s,j})
\]  

(47)

from (12) and (17). As a result, by (44) and (46), we can bound

\[
\|\tilde{W}^{s,t} - W^{s,t}\|_F^2 \leq \|W^s - W^s + \frac{1}{d^s} \sum_{j=Q_1}^{t-1} \sum_{p=1}^P \omega_p \nabla_W F_p(W_p^{s,j}, H_p^{s,j})\|_F^2
\]

(48)

\[
= \frac{1}{(d^s)^2} \| \sum_{j=Q_1}^{t-1} \sum_{p=1}^P \omega_p \nabla_W F_p(W_p^{s,j}, H_p^{s,j})\|_F^2
\]

(49)

\[
\leq \frac{(t - Q_1)}{(d^s)^2} \| \sum_{j=Q_1}^{t-1} \sum_{p=1}^P \omega_p \nabla_W F_p(W_p^{s,j}, H_p^{s,j})\|_F^2
\]

(50)

\[
\leq \frac{(t - Q_1)}{(d^s)^2} \| \sum_{j=Q_1}^{t-1} \sum_{p=1}^P \omega_p \| \nabla_W F_p(W_p^{s,j}, H_p^{s,j})\|_F^2
\]

(51)

\[
\leq \frac{(t - Q_1)^2}{(d^s)^2} G^2,
\]

(52)

where (48) follows from (46), (50) is obtained by the basic inequality \( \| \sum_{i=1}^n a_i \|_2^2 \leq n \sum_{i=1}^n \| a_i \|_2^2 \), (51) is obtained by the Jensen’s inequality for the convex function \( \| \cdot \|_F^2 \), and the last inequality (52) is due to Assumption 4.

Similarly, using (46) and (47), we can bound

\[
\|\bar{W}^{s,t} - W_p^{s,t}\|_F^2 = \|W^s - \frac{1}{d^s} \sum_{j=Q_1}^{t-1} \sum_{p=1}^P \omega_p \nabla_W F_p(W_p^{s,j}, H_p^{s,j}) - W^s + \frac{1}{d^s} \sum_{j=Q_1}^{t-1} \nabla_W F_p(W_p^{s,j}, H_p^{s,j})\|_F^2
\]

(53)

\[
= \frac{1}{(d^s)^2} \| \sum_{j=Q_1}^{t-1} \sum_{p=1}^P \omega_p \nabla_W F_p(W_p^{s,j}, H_p^{s,j}) - \sum_{j=Q_1}^{t-1} \nabla_W F_p(W_p^{s,j}, H_p^{s,j})\|_F^2
\]

\[
\leq \frac{(t - Q_1)}{(d^s)^2} \| \sum_{j=Q_1}^{t-1} \sum_{p=1}^P \omega_p \nabla_W F_p(W_p^{s,j}, H_p^{s,j}) - \nabla_W F_p(W_p^{s,j}, H_p^{s,j})\|_F^2
\]

\[
\leq \frac{2(t - Q_1)}{(d^s)^2} \sum_{j=Q_1}^{t-1} \left( \| \sum_{p=1}^P \omega_p \nabla_W F_p(W_p^{s,j}, H_p^{s,j})\|_F + \| \nabla_W F_p(W_p^{s,j}, H_p^{s,j})\|_F \right)
\]

\[
\leq \frac{2(t - Q_1)}{(d^s)^2} \sum_{j=Q_1}^{t-1} \left( \| \sum_{p=1}^P \omega_p \nabla_W F_p(W_p^{s,j}, H_p^{s,j})\|_F^2 + G^2 \right)
\]

\[
\leq \frac{2(t - Q_1)}{(d^s)^2} \sum_{j=Q_1}^{t-1} \left( \| \omega_p \nabla_W F_p(W_p^{s,j}, H_p^{s,j})\|_F^2 + G^2 \right)
\]

\[
\leq \frac{4(t - Q_1)^2}{(d^s)^2} G^2.
\]

(54)

The proof is thus complete.
E. Proof of Theorem 2

Here we consider diminishing $Q_2^s$, i.e., $Q_2^s = \lfloor \frac{Q_2}{s} \rfloor + 1$, and step size $d^s = \gamma L_W^s$. We denote $Q^s = Q_1 + Q_2^s$. Then, following the same proof procedure as obtaining (40), one can obtain

$$\frac{\gamma - 1}{2} \sum_{s=1}^{S} \sum_{t=1}^{Q_1} \sum_{p=1}^{P} \omega_p L^s_{H_p} \| H^s_{p} - H^{s-1}_{p} \|^2_F$$

$$+ \frac{\gamma - 1}{2} \sum_{s=1}^{S} \sum_{t=Q_1+1}^{Q'} L^s_W \| \tilde{W}^s_{t} - \tilde{W}^{s-1}_{t} \|^2_F$$

$$\leq F(\tilde{W}^{1,0}, H^{1,0}) - F + \frac{1}{\gamma^3} \sum_{s=1}^{S} \frac{C^s}{L^s_W}, \quad (55)$$

where $C^s \triangleq \frac{5Q^s(Q^s-1)(2Q^s-1)}{6}G^2$. Since $C^s = 0$ if $Q^s = 1$, we have $C^s = 0$ whenever $s > \hat{Q}$. Therefore, it holds that

$$\lim_{S \to \infty} \sum_{s=1}^{S} C^s < \infty. \quad (56)$$

Meanwhile, due to the fact that $F(W, H)$ is lower-bounded by Assumption 1, and $L^s_W \geq L_W > 0$, we must have from (55) that

$$\lim_{s \to \infty} \sum_{t=1}^{Q_1} \sum_{p=1}^{P} \omega_p \| H^s_{p} - H^{s-1}_{p} \|^2_F = 0, \quad \lim_{s \to \infty} \sum_{t=Q_1+1}^{Q'} \| \tilde{W}^s_{t} - \tilde{W}^{s-1}_{t} \|^2_F = 0. \quad (57)$$

which further implies that

$$\lim_{s \to \infty} \| H^s_{p} - H^{s,1}_{p} \|^2_F = 0, \quad p = 1, \ldots, P, \quad (58)$$

$$\lim_{s \to \infty} \| H^s_{p} - H^{s,Q_1}_{p} \|^2_F = 0, \quad p = 1, \ldots, P, \quad (59)$$

$$\lim_{s \to \infty} \| \tilde{W}^s_{Q_1} - \tilde{W}^{s,Q_1+1} \|^2_F = 0. \quad (60)$$

We next show that any limit point of $(W^s, H^{s,0})$ is a stationary point to problem (4). Let $(W^\infty, H^\infty)$ be a limit point of the sequence $(W^s, H^{s,0})$ when $s \to \infty$, and let $c^\infty_p$ and $d^\infty$ be the two limit values of $c^s_p = \frac{\gamma}{2} L^s_{H_p}$ and $d^s = \gamma L^s_W$, respectively, under Assumption 3.

Firstly, (10) implies

$$H^{s,1}_{p} = \mathcal{P}_{H_p} \{ H^s_{p} - \frac{\nabla H_p F_p(W^s_{p}, H^{s,0}_{p})}{c^s_p} \}, \quad p = 1, \ldots, P. \quad (61)$$
Moreover, by (58), there exists a subsequence such that $H_{s,1} \to H_\infty$, $H_{s,0} \to H_\infty$, $W_{s,0} = W_s \to W_\infty$ and $c_{s,p} \to c_\infty$. By applying this to (61), we obtain

$$H_\infty = \mathcal{P}_{H_\infty} \{ \frac{\nabla H_{p} F_p(W_\infty, H_\infty)}{c_\infty} \}, \ p = 1, \ldots, P.$$  \hfill (62)

Secondly, note that

$$\tilde{W}^{s, Q_1+1} = \mathcal{P}_W \left( \sum_{p=1}^{P} \omega_p W^{s, Q_1+1} \right)$$

$$= \mathcal{P}_W \left( \sum_{p=1}^{P} \omega_p W^{s} - \frac{1}{d^s} \nabla W F_p(W^{s, Q_1}) \right)$$

$$= \mathcal{P}_W \left( W^{s} - \frac{1}{d^s} \sum_{p=1}^{P} \omega_p \nabla W F_p(W^{s}, H^{s, Q_1}_{p}) \right)$$

$$= \mathcal{P}_W \left( \tilde{W}^{s, Q_1} - \frac{1}{d^s} \nabla W F(\tilde{W}^{s, Q_1}, H^{s, Q_1}) \right),$$  \hfill (65)

where (64) is due to (12)-(13), and (65) is obtained by the fact that $\tilde{W}^{s, Q_1} = W$ from (17). Besides, by (59) and (60), there exists a subsequence such that $H^{s, Q_1} \to H_\infty$, $\tilde{W}^{s, Q_1+1} \to W_\infty$, $\tilde{W}^{s, Q_1} = W_s \to W_\infty$ and $d^s \to d_\infty$. By applying this to (65), we obtain

$$W_\infty = \mathcal{P}_W \left( W_\infty - \frac{1}{d_\infty} \nabla W F(W_\infty, H_\infty) \right).$$  \hfill (66)

Equations (62) and (66) imply that $(W_\infty, H_\infty)$ is a stationary point of problem (4).

IV. FEDERATED CLUSTERING BY GRADIENT SHARING

In this section, we present an improved FedC algorithm, termed FedCGds, and its convergence conditions. Comparison between FedCGds and FedCAvg is also discussed.

A. The FedCGds Algorithm

One key observation for the distributed MF problem (4) is that it is separable with respect to the assignment matrices $H_1, \ldots, H_P$. Therefore, direct application of the PALM algorithm in (3) to problem (4) naturally leads to a distributed algorithm as follows. The server is in charge of updating the centroid $W$

$$W^s = \mathcal{P}_W \{ W^{s-1} - \frac{1}{d^s} \nabla W F(W^{s-1}, H^s) \},$$  \hfill (67)

while each client $p$ updates $H_p$ locally

$$H_p^s = \mathcal{P}_H \{ H_p^{s-1} - \frac{1}{c^s} \nabla H_p F_p(W^{s-1}, H_p^{s-1}) \},$$  \hfill (68)
for \( p = 1, \ldots, P \) \([37], [38]\).

Like the FedAvg algorithm for reducing the communication overhead, we propose to let each client \( p \) perform \( Q_1 \) consecutive epochs of PGD with respect to \( H_p \) (see Eqn. (70)) and let the server perform \( Q_2 \) epochs of PGD with respect to \( W \) (see Eqn. (74)) in each communication round.

**Differential gradient sharing:** Under the FL setting, to enable the server to update the global centroid \( W \), the clients are required to compute and upload the gradient information \( \nabla W F \) for the server. The separable structure of \( \nabla W F(W, H) \) makes it computable in a distributed fashion among the clients. Specifically, by (5), we have

\[
\nabla W F(W, H) = 2\sum_{p=1}^P H_p H_p^\top \frac{N}{2} - 2\sum_{p=1}^P X_p H_p^\top \frac{N}{2} + \nabla R_W(W).
\]

Thus, it is sufficient for each client \( p \) to send \( H_p^{s, Q_1}(H_p^{s, Q_1})^\top \) and \( X_p(H_p^{s, Q_1})^\top \) to the server. Interestingly, due to the additive structure in (69), the client can alternatively send the server the differential information between two consecutive rounds, i.e., \( U_p^s \triangleq H_p^{s, Q_1}(H_p^{s, Q_1})^\top - H_p^{s-1, Q_1}(H_p^{s-1, Q_1})^\top \) and \( V_p^s \triangleq X_p(H_p^{s, Q_1})^\top - X_p(H_p^{s-1, Q_1})^\top \) (see (71)), while the server is still able to construct the gradient \( \nabla W F(\cdot, H^{s, Q_1}) \) out of \( (U_p^s, V_p^s) \) as shown in (72) and (73). Since asymptotically \( H_p^{s, Q_1} - H_p^{s-1, Q_1} \to 0 \) when the algorithm converges, the differential messages \( (U_p^s, V_p^s) \) would become sparse and contain zeros mostly. This would reduce the size of messages for uplink transmissions.

**Partial client participation:** In FL scenarios, the number of clients could be large. However, due to limited communication resources, it is desirable that only a small number of clients are active and access the network in each communication round. In the FedCGds algorithm, we let the server select only a small subset of clients to participate in the FedC task in each communication round. Specifically, the server selects a subset \( A_s^s \subset \{1, \ldots, P\} \) with size \( |A_s^s| = m \ll P \) in each round \( s \). Then, only clients in \( A_s^s \) perform local PGD in (70), and upload required messages to the server in each round \( s \). The details of the FedCGds algorithm are summarized in Algorithm 2.

**B. Convergence Analysis of FedCGds**

Here we assume that the client subset \( A_s^s \) is obtained by the server through uniform sampling of \( \{1, \ldots, P\} \) without replacement. Then, we can obtain the following convergence results for FedCGds.

**Theorem 3** Let \( c_p^s = \frac{\gamma}{2} L_{H_p}^s \) and \( d_p^s = \frac{\gamma}{2} L_{W}^s \), where \( \gamma > 1 \), and that \( A_s^s \) (with \( |A_s^s| = m \)) is obtained by uniform sampling without replacement.
Algorithm 2 FedCGds

1: **Input:** Initial values of $H_1^{0,Q_1}, \ldots, H_{p}^{0,Q_1}$ at the clients, and initial value of $W^{0,Q}$ and $G_1^{0} = \sum_{p=1}^{P} \frac{2}{N} H_{p}^{0,Q_1}(H_{p}^{0,Q_1})^\top$, $G_2^{0} = \sum_{p=1}^{P} \frac{2}{N} X_p(H_{p}^{0,Q_1})^\top$, at the server.

2: **for** round $s = 1$ to $S$ **do**

3: **Server side:** Select a subset of clients $A^s \subset \{1, \ldots, P\}$ (with size $|A^s| = m$), and broadcast $W^s = W^{s-1,Q}$ to the clients in $A^s$.

4: **Client side:**

5: **for** client $p = 1$ to $P$ in parallel **do**

6: if client $p \notin A^s$ then

7: Set $H_{p}^{s,t} = H_{p}^{s-1,Q_1}$, $t = 1, \ldots, Q_1$.

8: else if client $p \in A^s$ then

9: Set $H_{p}^{s,0} = H_{p}^{s-1,Q_1}$.

10: **for** epoch $t = 1$ to $Q$

11: $H_{p}^{s,t} = \mathcal{P}_{H_{p}} \{ H_{p}^{s,t-1} - \frac{\nabla H_{p} F_p(W^{s}, H_{p}^{s,t-1})}{\epsilon_{p}^{s}} \}$.

12: **end for**

13: Send the server

14: $U_{p}^{s} = H_{p}^{s,Q_1}(H_{p}^{s,Q_1})^\top - H_{p}^{s,0}(H_{p}^{s,0})^\top$, $V_{p}^{s} = X_p(H_{p}^{s,Q_1})^\top - X_p(H_{p}^{s,0})^\top$.

15: **end if**

16: **end for**

17: for epoch $t = Q_1 + 1$ to $Q$

18: $\nabla W F(W^{s,t-1}, H^{s,Q_1}) = W^{s,t-1} G_{1}^{s} - G_{2}^{s}$,

19: $W^{s,t} = \mathcal{P}_W \{ W^{s,t-1} - \frac{\nabla W F(W^{s,t-1}, H^{s,Q_1})}{d^s} \}$.

20: **end for**

21: **end for**
(i) Under Assumptions 1 and 2, the sequence \((W^{s,t}, H^{s,t})\) generated by Algorithm 2 satisfies
\[
\frac{1}{SQ} \left[ \sum_{s=1}^{m} \sum_{t=1}^{Q_1} \sum_{p=1}^{P} \omega_p L^s_{H_p} \|H^{s,t}_p - H^{s,t-1}_p\|_F^2 + \sum_{s=1}^{m} \sum_{t=Q_1+1}^{Q} L^s_{W} \|W^{s,t} - W^{s,t-1}\|_F^2 \right] \leq \frac{2(F(W^{1,0}, H^{1,0}) - F)}{SQ(\gamma - 1)}. \tag{75}
\]
(ii) Further assume that Assumption 3 holds and \(L^s_{H_p} \geq L^s_{H_p} > 0\). Then any limit point of \((W^{s,0}, H^{s,0})\) is a stationary point of problem (4) when \(s \to \infty\) almost surely.

The proof of Theorem 3 is presented in Section IV-C. Analogously, (75) shows that Algorithm 2 has a \(O(1/T)\) convergence rate, and that Algorithm 2 with \(Q > 1\) can reduce the communication overhead if \(T\) is given and fixed. Experimental results presented in Section V will further illustrate how the partial client participation \((m)\) can influence the practical local epoch length \(Q_1\) and \(Q_1\).

Remark 3 The upload message size of \((U^s_p, V^s_p)\) in FedCGds is \(MK + K^2\) per client and per communication round, while that of \(W_p\) in FedCAvg is \(MK\). Nevertheless, the FedCGds algorithm can exhibit a better convergence behavior. As one can observe from (75), the FedCGds algorithm can monotonically converge to a stationary solution with constant step size parameters. Since from a global point of view, the FedCGds algorithm is nothing but implementation of the two-block block gradient descent method over the FL network, the load balancing and data distribution would not have critical impact on the algorithm performance. This will be further verified in Section V.

C. Proof of Theorem 3
Here, we present the proof of Theorem 3. Readers who are not interested in the detailed proof may directly go to Section V for experimental results.

Proof of Theorem 3(i): From Algorithm 2, firstly note that
\[
W^{s,Q} = W^{s+1,0}, \quad W^{s,t} = W^{s,t-1}, \quad t = 1, \ldots, Q_1, \tag{76}
\]
\[
H^{s,Q} = H^{s+1,0}, \quad H^{s,t} = H^{s,t-1}, \quad t = Q_1 + 1, \ldots, Q. \tag{77}
\]
Secondly, under partial client participation, the local updates of \(H^{s,t}_p\) are
\[
H^{s,t}_p = \begin{cases} 
\mathcal{P}_{H_p} \left\{ H^{s,t-1}_p - \frac{\nabla H_p F_p(W^{s,t-1}_p, H^{s,t-1}_p)}{c^s_p} \right\}, & \text{if } p \in \mathcal{A}^s, \\
H^{s,t-1}_p, & \text{otherwise},
\end{cases} \tag{78}
\]
for \(t = 1, \ldots, Q_1\), where only clients in \(\mathcal{A}^s\) perform PGD. So we define the following virtual variables assuming that all clients perform PGD in each round \(s\):
\[
\tilde{H}^{s,0}_p = H^{s,0}_p, \quad \tilde{H}^{s,t}_p = \mathcal{P}_{H_p} \left\{ \tilde{H}^{s,t-1}_p - \frac{\nabla H_p F_p(W^{s,0}_p, \tilde{H}^{s,t-1}_p)}{c^s_p} \right\}, \quad t = 1, \ldots, Q_1. \tag{79}
\]
Let us consider the descent of the objective function with respect to the
\( \mathcal{E}^{s-1} = \{ A^1, \ldots, A^{s-1}, \{ H_1^{1,t} \}_{t=0}^1, \ldots, \{ H_1^{s-1,t} \}_{t=0}^{s-1} \} \)
\{ W_1^{1,t} \}_{t=0}^1, \ldots, \{ W_1^{s-1,t} \}_{t=0}^{s-1}, \{ c_p^1 \}_{p=1}^{P_1}, \ldots, \{ c_p^{s-1} \}_{p=1}^{P_1}, d_1^1, \ldots, d_1^{s-1} \} (80)
as the collection of historical events up to round \((s - 1)\), and denote \( \mathbb{I}(p \in A^s) \) as the indicator function which is one if the event \( p \in A^s \) is true and zero otherwise.

**Objective Descent w.r.t. \( H \):** Let us consider the descent of the objective function with respect to the update of \( H \) when \( \mathcal{E}^{s-1} \) is given. Specifically, we have the following chain

\[
\mathbb{E}[F(W^{s,0}, H^{s,0}) | \mathcal{E}^{s-1}] - F(W^{s,0}, H^{s,0}) \\
= \mathbb{E} \left[ \sum_{p \in A^s} \omega_p \left( F_p(W^{s,0}, \tilde{H}_p^{s,Q_1}) - F_p(W^{s,0}, H_p^{s,0}) \right) | \mathcal{E}^{s-1} \right] \\
= \mathbb{E} \left[ \sum_{p=1}^{P} \mathbb{I}(p \in A^s) \omega_p \left( F_p(W^{s,0}, \tilde{H}_p^{s,Q_1}) - F_p(W^{s,0}, H_p^{s,0}) \right) | \mathcal{E}^{s-1} \right] \\
= \sum_{p=1}^{P} \mathbb{E}[\mathbb{I}(p \in A^s) | \mathcal{E}^{s-1}] \omega_p \left( F_p(W^{s,0}, \tilde{H}_p^{s,Q_1}) - F_p(W^{s,0}, H_p^{s,0}) \right) \\
= \frac{m}{P} \sum_{p=1}^{P} \omega_p \left( F_p(W^{s,0}, \tilde{H}_p^{s,Q_1}) - F_p(W^{s,0}, H_p^{s,0}) \right) \\
= \frac{m}{P} \sum_{p=1}^{P} \omega_p \sum_{t=1}^{Q_1} \left( F_p(W^{s,0}, \tilde{H}_p^{s,t}) - F_p(W^{s,0}, \bar{H}_p^{s,t-1}) \right) \\
\leq - \frac{m(\gamma - 1)}{2P} \left( \sum_{t=1}^{Q_1} \sum_{p=1}^{P} \omega_p L_{H_p}^s \| \tilde{H}_p^{s,t-1} - \bar{H}_p^{s,t} \|_F^2 \right), \\
\leq - \frac{m(\gamma - 1)}{2P} \left( \sum_{t=1}^{Q_1} \sum_{p=1}^{P} \omega_p L_{H_p}^s \| H_p^{s,t-1} - \bar{H}_p^{s,t} \|_F^2 \right),
\]

where (81) is due to (70) and (79); (82) holds because \( F_p(W^{s,0}, \tilde{H}_p^{s,0}) - F_p(W^{s,0}, \bar{H}_p^{s,0}) \) is deterministic given \( \mathcal{E}^{s-1} \); (83) is true since \( \mathbb{E}[\mathbb{I}(p \in A^s) | \mathcal{E}^{s-1}] = m/P \) when uniform sampling without replacement is employed; (85) follows (20) according to [39, Lemma 3.2] and \( c_p^s = \frac{1}{2} L_{H_p}^s \); lastly, (86) is true because \( H_p^{s,t} = \bar{H}_p^{s,t-1} \) for \( p \notin A^s \) and \( H_p^{s,t} = \tilde{H}_p^{s,t} \) for \( p \in A^s \), for \( t = 1, \ldots, Q_1 \).

Further taking the expectation of (86) w.r.t. \( \mathcal{E}^{s-1} \), we obtain

\[
\mathbb{E}[F(W^{s,0}, H^{s,0})] - \mathbb{E}[F(W^{s,0}, H^{s,0})] \\
\leq - \frac{m(\gamma - 1)}{2P} \mathbb{E} \left[ \sum_{t=1}^{Q_1} \sum_{p=1}^{P} \omega_p L_{H_p}^s \| H_p^{s,t-1} - \bar{H}_p^{s,t} \|_F^2 \right],
\]

for \( p = 1, \ldots, P \). We also define

\[
\mathcal{E}^{s-1} = \{ A^1, \ldots, A^{s-1}, \{ H_1^{1,t} \}_{t=0}^1, \ldots, \{ H_1^{s-1,t} \}_{t=0}^{s-1} \} \]
\{ W_1^{1,t} \}_{t=0}^1, \ldots, \{ W_1^{s-1,t} \}_{t=0}^{s-1}, \{ c_p^1 \}_{p=1}^{P_1}, \ldots, \{ c_p^{s-1} \}_{p=1}^{P_1}, d_1^1, \ldots, d_1^{s-1} \} (80)
Objective Descent w.r.t. $W$: By applying \cite[Lemma 3.2]{39} to the update of $W$ in (74)

$$W^{s,t} = P_W\{W^{s,t-1} - \nabla_W F(W^{s,t-1}, H^{s,Q_1})\}, \quad t = Q_1 + 1, \ldots, Q,$$

(88)

with $d^s = \frac{\gamma}{2}L_W^s$, we immediately obtain

$$F(W^{s,Q_1}, H^{s,Q_1}) - F(W^{s,Q_1}, H^{s,Q_1}) = \sum_{t=Q_1+1}^Q \left( F(W^{s,t}, H^{s,Q_1}) - F(W^{s,t-1}, H^{s,Q_1}) \right) \leq -\frac{\gamma - 1}{2} \sum_{t=Q_1+1}^Q L_W^s \|W^{s,t} - W^{s,t-1}\|^2_F. \quad (89)$$

By noting from (76) and (77) that $W^{s,Q_1} = W^{s,0}$, $W^{s,Q} = W^{s+1,0}$ and $H^{s,Q_1} = H^{s+1,0}$, and by taking expectation of (89), we have

$$E[F(W^{s+1,0}, H^{s+1,0})] - E[F(W^{s,0}, H^{s,Q_1})] \leq -\frac{\gamma - 1}{2} \sum_{t=Q_1+1}^Q L_W^s \|W^{s,t} - W^{s,t-1}\|^2_F. \quad (90)$$

Derivation of the Main Result: After combing (87) and (90), we have

$$E[F(W^{s+1,0}, H^{s+1,0})] - E[F(W^{s,0}, H^{s,0})] \leq -\frac{m(\gamma - 1)}{2P} \sum_{s=1}^S \sum_{t=Q_1}^P \sum_{p=1}^P \omega_p L_{H_p}^s \|H_{p}^{s,t} - H_{p}^{s,t-1}\|^2_F$$

$$-\frac{\gamma - 1}{2} \sum_{t=Q_1+1}^Q L_W^s \|W^{s,t} - W^{s,t-1}\|^2_F. \quad (91)$$

Lastly, taking the telescope sum of (91) from $s = 1$ to $S$ and dividing both sides by $SQ$ yields

$$\frac{1}{SQ} \left[ \frac{m}{P} \sum_{s=1}^S \sum_{t=Q_1}^P \sum_{p=1}^P \omega_p L_{H_p}^s \|H_{p}^{s,t} - H_{p}^{s,t-1}\|^2_F$$

$$+ \sum_{s=1}^S \sum_{t=Q_1+1}^Q L_W^s \|W^{s,t} - W^{s,t-1}\|^2_F \right] \leq \frac{2(F(W^{1,0}, H^{1,0}) - E[F(W^{S+1,0}, H^{S+1,0})])}{SQ(\gamma - 1)}$$

$$\leq \frac{2(F(W^{1,0}, H^{1,0}) - F)}{SQ(\gamma - 1)}, \quad (92)$$

which is (75).

Proof of Theorem 3(ii): We will need the following the supermartingale convergence result of Robbins and Siegmund from \cite[Lemma 11]{45]
Lemma 2 Let \( \nu_k, \mu_k \) and \( \alpha_k \) be three sequences of non-negative random variables such that

\[
\mathbb{E}[\nu_{k+1}|\mathcal{F}_k] \leq (1 + \alpha_k)\nu_k - \mu_k, \forall k = 0, 1, \ldots,
\]

and \( \sum_{k=0}^{\infty} \alpha_k \leq \infty \), almost surely (a.s.), where \( \mathcal{F}_k \) denotes the collection of historical events \( \{\nu_0, \ldots, \nu_k, \mu_0, \ldots, \mu_k, \alpha_0, \ldots, \alpha_k\} \) up to \( k \). Then, we have

\[
\lim_{k \to \infty} \nu_k = \nu, \text{ a.s.,}
\]

for a random variable \( \nu \geq 0 \) and \( \sum_{k=0}^{\infty} \mu_k < \infty \), a.s.

By recalling (85) and combining it with (89), we obtain

\[
\mathbb{E}[F(W^{s+1,0}, H^{s+1,0})|\mathcal{E}^{s-1}] - F(W^{s,0}, H^{s,0}) 
\leq - \frac{m(\gamma - 1)}{2P} \left( \sum_{s=1}^{\infty} \sum_{p=1}^{P} \omega_p L_{H_p}^s \left\| \bar{H}_{p}^{s,t-1} - \bar{H}_{p}^{s,t} \right\|_F^2 \right) 
- \frac{\gamma - 1}{2} \sum_{t=Q_t+1}^{Q} L_{W}^s \left\| W^{s,t} - W^{s,t-1} \right\|_F^2 |\mathcal{E}^{s-1}|
\leq - \frac{m(\gamma - 1)}{2P} \sum_{p=1}^{P} \omega_p L_{H_p}^s \left\| H_{p}^{s,0} - \bar{H}_{p}^{s,1} \right\|_F^2.
\]

(93)

Then after applying Lemma 2 to (93), we have

\[
\lim_{s \to \infty} F(W^{s,0}, H^{s,0}) = \overline{F}, \text{ a.s.,}
\]

(94)

\[
\sum_{s=1}^{\infty} \sum_{p=1}^{P} \omega_p L_{H_p}^s \left\| H_{p}^{s,0} - \bar{H}_{p}^{s,1} \right\|_F^2 < \infty, \text{ a.s.,}
\]

(95)

for some random variable \( \overline{F} \). Under the assumption that \( L_{H_p}^s \geq L_{H_p} > 0 \), (95) implies

\[
\lim_{s \to \infty} \left\| H_{p}^{s,0} - \bar{H}_{p}^{s,1} \right\|_F^2 = 0, \text{ a.s.}
\]

(96)

Then, by following a similar argument as in (61) and (62), one can obtain

\[
H_{p}^{\infty} = p_{H_{p}} \left\{ H_{p}^{\infty} - \frac{\nabla_{H_p} F_p(W_{p}^{\infty}, H_{p}^{\infty})}{c_{p}^{\infty}} \right\}, \ p = 1 \ldots, P, \text{ a.s.,}
\]

(97)

where \( (W_{p}^{\infty}, H_{p}^{\infty}) \) be a limit point of the sequence \( (W_{p}^{s,0}, H_{p}^{s,0}) \) when \( s \to \infty \), and \( c_{p}^{\infty} \) is a limit value of \( c_{p}^{s} = \frac{\gamma}{2} L_{H_p}^s \).

On the other hand, note that according to the update in (70) and following (22) in the proof of Theorem 1, it always holds that

\[
F(W_{p}^{s,0}, H_{p}^{s,Q_t}) - F(W_{p}^{s,0}, H_{p}^{s,0}) 
\leq - \frac{\gamma - 1}{2} \sum_{s=1}^{Q_t} \sum_{p=1}^{P} \omega_p L_{H_p}^s \left\| H_{p}^{s,t-1} - H_{p}^{s,t} \right\|_F^2.
\]

(98)
for any realization of the random sequence. Combing (98) with (89) yields that
\[ F(W^{s+1,0}, H^{s+1,0}) - F(W^{s,0}, H^{s,0}) \]
\[ \leq - \frac{\gamma - 1}{2} \sum_{t=1}^{Q_s} \sum_{p=1}^{P} \omega_p L^s_{W_p} \| H^{s,t-1}_p - H^{s,t}_p \|^2_F \]
\[ - \frac{\gamma - 1}{2} \sum_{t=Q_s+1}^{Q} L^s_W \| W^{s,t} - W^{s,t-1} \|^2_F \]
\[ \leq - \frac{\gamma - 1}{2} L^s_W \| W^{s,Q_1} - W^{s,Q_1+1} \|^2_F \] (99)
holds for any realization of the random sequence. Since, according to (94) it holds that
\[ \lim_{s \to \infty} F(W^{s+1,0}, H^{s+1,0}) - F(W^{s,0}, H^{s,0}) = 0, \ a.s., \] (100)
(99) implies
\[ \lim_{s \to \infty} \| W^{s,Q_1} - W^{s,Q_1+1} \|^2_F = 0, \ a.s. \] (101)
Then, by following a similar argument as in (65) and (66), we can obtain
\[ W^\infty = \mathcal{P}_W \left( W^\infty - \frac{1}{d^\infty} \nabla_W F(W^\infty, H^\infty) \right), \ a.s., \] (102)
where \( d^\infty \) is a limit value of \( d^s = \frac{\gamma - 1}{2} L^s_W \). As a result, by (97) and (103), any limit point of \( (W^{s,0}, H^{s,0}) \) will be a stationary point to problem (4) almost surely.

V. EXPERIMENT RESULTS

In this section, we examine the practical convergence behavior and clustering performance of the proposed algorithms.

Model: We consider the orthogonal NMF based clustering model in [16] which corresponds to problem (4) with \( R_W(W) = 0 \),
\[ R_H(H_p) = \frac{P}{2} \sum_{j=1}^{N_p} \left( (1^T h_{p,j})^2 - \| h_{p,j} \|^2_2 \right) + \frac{\nu}{2} \| H_p \|^2_F, \] (104)
\[ \mathcal{W} = \{ W \in \mathbb{R}^{M \times K} | W \geq [W]_{ij} \geq \bar{W}, \forall i,j \}, \]
\[ \mathcal{H}_p = \{ H_p \in \mathbb{R}^{K \times N_p} | H_{p,ij} \geq 0, \forall i,j \}, \]
where \( \bar{W} \) (resp. \( W \)) is the maximum (resp. minimum) value of \( X \), and \( \rho, \nu > 0 \) are two penalty parameters.
If not mentioned specifically, we set \( \rho = 10^{-8} \) and \( \nu = 10^{-10} \).

Datasets: Both synthetic data and the MNIST database [46] are considered. Specifically, we follow the linear model \( X = WH + E \) as in [40] to generate a synthetic dataset with \( M = 2000, N = \)
10000 and $K = 20$, where $\mathbf{E} \in \mathbb{R}^{M \times N}$ denotes the measurement noise and the signal to noise ratio $= 10 \log_{10}(\|\mathbf{W}\|_{F}^2/\|\mathbf{E}\|_{F}^2)$ dB is set to $-3$ dB. We distribute the 10000 samples to $P = 100$ clients in two ways. The first one follows [2] which gives a balanced and i.i.d. dataset and is denoted by syn_unf. The second way follows the similarity-based partition where the K-means algorithm is applied to the dataset to cluster it into 100 clusters, and each of the cluster is assigned to one client. This leads to a highly unbalanced and non-i.i.d. dataset, which is denoted as syn_noniid. In addition, following [13], we generate two non-i.i.d. MNIST_unls and MNIST_bals datasets with $M = 784, N = 10000, K = 10$ and $P = 100$, where each of the client contains images of two digits only. The former one is highly unbalanced with the number of samples among clients following a power law while the latter one has the same number of samples among the clients.

Parameter setting: For FedCAvg, the step size $c_p^s$ and $d_s$ are set to $c_p^s = \frac{1}{2} \lambda_{\text{max}}((\mathbf{W}_p^{s,0})^\top \mathbf{W}_p^{s,0})$ and $d_s^s = \lambda_{\text{max}}(\mathbf{H}^{s,Q_1}(\mathbf{H}^{s,Q_1})^\top)$, respectively, where $\lambda_{\text{max}}$ denotes the maximum eigenvalue. For FedCGds,
Fig. 2: Convergence curve versus number of rounds of FedCGds for different local epoch lengths.

it is set to $e_p^s = \frac{1}{2} \lambda_{\text{max}}((W^{s,0})^T W^{s,0})$ and $d^s = \frac{1}{2} \lambda_{\text{max}}(H^{s,Q_1} (H^{s,Q_1})^T)$. For partial client participation, the set $\mathcal{A}^s$ is obtained by uniform sampling from the 100 clients without replacement. The stopping condition for both algorithms is that the normalized change of the objective value

$$
\varepsilon = \frac{|F(W^s, H^{s,Q_1}) - F(W^{s-1}, H^{s-1,Q_1})|}{F(W^{s-1}, H^{s-1,Q_1})}
$$

is smaller than a preset number or a maximum number of rounds is achieved. All algorithms under test are initialized with 10 common, randomly generated initial points, and the presented results are averaged over the 10 experiment trials.

**Communication cost:** Here, we consider only the uplink transmission since it is the primary communication bottleneck when the client number is large. We define the communication cost as the accumulated number of real values to be sent to the sever. By Remark 3 for the $s$th round, the communication cost of FedCAvg is $(MK)Ps$, while that of FedCGds is $(MK + K^2)ms$. 
Fig. 3: Convergence curve versus number of rounds of FedCGds for different local epoch lengths.

A. Convergence of FedCAvg

We first examine the FedCAvg algorithm with respect to different values of local epoch length $Q_1$ and $Q_2$. As shown in Fig. 1(a), with fixed $Q_2 = 1$, the algorithm with $Q_1 > 1$ converges faster than that with $Q_1 = 1$. However, it is observed that for $Q_1 > 10$ the convergence speedup is not as significant as for $Q_1 = 10$. From Figs. 1(b) and 1(c) one can further observe that increasing $Q_2$ can speedup the convergence when the data is uniform and i.i.d., whereas larger value of $Q_2$ can slowdown the convergence when the data is non-i.i.d. In the two figures, we also display the convergence of FedCAvg when $Q_2$ is diminishing as described in Theorem 2. Specifically, we set $\hat{Q} = 400$ and $\hat{Q} = 10$ for syn_unf and syn_noniid datasets, respectively. Interestingly, one can observe that this dynamic strategy can perform best in both uniform and non-i.i.d. scenarios. Lastly, in Fig. 1(a) the convergence curve of the naive strategy discussed in Remark 1 (termed FedCPALM) which employs $Q/2$ steps of PALM locally followed by centroid averaging is plotted. As seen, it does not perform as well as FedCAvg and can converge even poorly when $Q$ is larger as shown in Fig. 1(d).

B. Convergence of FedCGds

In Fig. 2 and Fig. 3 the convergence curves of the FedCGds algorithm on the syn_unf dataset and syn_noniid are displayed. One can see from Fig. 2(a) that FedCGds with $m = 100$ (full client participation) and $Q_1 = 10$ can have monotonically improved convergence speed when $Q_2$ increases. However, as shown in Fig. 2(b) and 2(c) when $m = 10$ (partial participation), increasing $Q_2$ can improve the convergence only if $Q_1$ is also large ($Q_1 = 100$). We remark that similar insights apply to the non-i.i.d datasets, which are shown in Fig 2(d) and 2(e).

On the other hand, Fig. 3(a) shows that increasing $Q_1$ can speed up the convergence as well, but the speedup with $Q_1 > 10$ is not as significant as that with $Q_1 = 10$. Intriguingly, as shown in Fig. 3(b)
Fig. 4: Convergence curve versus number of rounds/communication cost of FedCAvg and FedCGds on the synthetic datasets. For FedCGds, it is set that \( Q_2 = 100 \), and \( Q_1 = 10 \) for \( m \geq 50 \) and \( Q_1 = 100 \) for \( m < 50 \).

Increasing \( Q_1 \) can monotonically improve the convergence rate when \( m = 10 \), which is also exhibited in Fig 3(c) for non-i.i.d. datasets. In summary, one can conclude that the algorithm convergence can benefit from a large \( Q_2 \) and small \( Q_1 \) when \( m \) is large while from both large \( Q_1 \) and \( Q_2 \) when \( m \) is small.

C. Comparison between FedCAvg and FedCGds

Fig. 4(a) and Fig. 4(b) show the convergence of FedCAvg and FedCGds on the syn_unf dataset with respect to communication round number and communication cost, respectively. By comparing the curve of FedCAvg with that of FedCGds (\( m = 100 \)), one can observe that the two algorithms perform comparably. However, as shown in Fig. 4(c) and Fig. 4(d) where the syn_noniid dataset is considered, the convergence speed of FedCAvg deteriorates a lot whereas FedCGds can keep almost unchanged convergence performance. Similar observations can be made from their convergence behaviors on the
MNIST ubls dataset shown in Fig. 5(a) and Fig. 5(b), and on the MNIST bls dataset shown in Fig. 5(c) and Fig. 5(d).

One can also conclude from Figs. 4(b), 4(d), 5(b), 5(d) that partial client participation ($m = 10$ in Figs. 4(b) and 4(d) and $m = 50$ in Fig. 5(b) and 5(d)) can considerably improve the communication efficiency.

D. Clustering Performance

To evaluating the clustering performance of the proposed algorithms, we follow the successive non-convex penalty (SNCP) approach in [16] to gradually increase the penalty parameter $\rho$ in (104) whenever problem (4) is solved with sufficiently small $\varepsilon$. Specifically, the initial $\rho$ is set to $10^{-8}$ and is updated by $\rho = 1.5 \times \rho$ whenever $\varepsilon < 2 \times 10^{-5}$ if the synthetic dataset is used and $\varepsilon < 8 \times 10^{-5}$ otherwise. The stopping condition is set to $\varepsilon < 1 \times 10^{-8}$. For FedCGds, we set $Q_1 = 100$ and $Q_2 = 100$; for

![Convergence curve](image-url)

Fig. 5: Convergence curve versus number of rounds/communication cost of FedCAvg and FedCGds on the MNIST datasets. For FedCGds, it is set that $Q_2 = 100$, and $Q_1 = 10$ for $m \geq 50$ and $Q_1 = 100$ for $m < 50$. 
FedCAvg, we set $Q_1 = 10$ and $\hat{Q} = 10$. In addition, the centralized SNCP method [16 Algorithm 1 & 2] and the popular K-means++ [47] are also implemented as two benchmarks.

Fig. 6 presents the clustering accuracy versus accumulated round number and communication cost on

![Fig. 6: Clustering accuracy versus number of accumulated rounds/communication cost of FedCAvg and FedCGds for different datasets.](image-url)
different datasets. One can observe that FedCGds outperforms FedCAvg and achieves much higher clustering accuracy than K-means++. From Fig. 3(a)-(b) for the syn_noniid and Fig. 6(e)-(f) for the MNIST_bals dataset, one can see that FedCGds yields comparable clustering accuracy as the centralized SNCP. Surprisingly, one can see from Fig. 6(c)-(d) that FedCGds can even perform better than the centralized SNCP on the MNIST_bals dataset.

VI. CONCLUSION AND FUTURE WORK

In this paper, we have presented two novel FedCAvg and FedCGds algorithms for federated clustering. We have theoretically shown that the two algorithms can have a $O(1/T)$ convergence rate and controlling the local epoch length $Q_1$ and $Q_2$ can reduce the communication overhead. Experimental results have demonstrated consistent convergence behaviors of the proposed algorithms on both synthetic and real datasets, showing insights on the practical values of $Q_1$ and $Q_2$ that can improve the convergence speed. It has also been shown that FedCGds is more robust against the non-i.i.d data than FedCAvg, and partial client participation can significantly reduce the communication cost.

As the future works, it is worthwhile to devise FedC algorithms for general MF models that can handle outlier and noisy data [48]. Enhancing privacy and security [38] of FedC algorithms is also of great importance.

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