FOURIER TRANSFORMS ON $SL_2(\mathbb{Z}/p^n\mathbb{Z})$ AND RELATED NUMERICAL EXPERIMENTS

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Abstract. We detail an explicit construction of ordinary irreducible representations for the family of finite groups $SL_2(\mathbb{Z}/p^n\mathbb{Z})$ for odd primes $p$ and $n \geq 2$. For $n = 2$, the construction is a complete set of irreducible complex representations, while for $n > 2$, all but a handful are obtained. We also produce an algorithm for the computation of a Fourier transform for a function on $SL_2(\mathbb{Z}/p^2\mathbb{Z})$. With this in hand we explore the spectrum of a collection of Cayley graphs on these groups, extending analogous computations for Cayley graphs on $SL_2(\mathbb{Z}/p\mathbb{Z})$ and suggesting conjectures for the expansion properties of such graphs.

1. Introduction

Ever since the foundational work of Lubotzky, Phillips, and Sarnak [6] the expansion properties of the groups $SL_2(\mathbb{Z}/p\mathbb{Z})$ have been of significant interest. Work of Lafferty and Rockmore [3] [4] conjectured that random sets of generators for these groups (suitably defined) would generically give rise to expander graphs, a conjecture ultimately settled by Bourgain and Gamburd [1]. Lafferty and Rockmore studied these Cayley graphs by computing the Fourier transform of the characteristic function of the generators at a complete set of irreducible representations for these groups [5]. Since the dimension of the individual Fourier transforms is much smaller than the order of the group (in this case, roughly $p$ versus $p^3$) more extensive calculations could be accomplished. Additionally, this “microanalysis” suggested more fine scale conjectures about the spectra, that not only the expansion properties of random generators were generic, but even more, that the spectra of individual Fourier transforms was generic. This conjecture was only recently settled by Rivin and Sardari [10].

Analogous questions for the related family of groups $SL_2(\mathbb{Z}/p^n\mathbb{Z})$ are natural generalizations and have yet to be investigated. The goal of performing the necessary numerical experiments, à la [3] [4] [5] are the motivation for this paper. These experiments can only be accomplished if we have at hand a way to efficiently and explicitly construct a complete set of irreducible complex matrix representations for these groups. Achieving this first goal has the added attraction of producing yet another infinite family of important finite groups whose irreducible representations are then explicitly accessible. This in turn produces the basic material for considering the problem of efficiently computing a Fourier transform on these groups, thereby relating this work to another broad field of current research (see e.g., [8] [9]).
2. Construction of the irreducible representations of $SL_2(\mathbb{Z}/p^n\mathbb{Z})$ for $n \geq 2$

In this section we give an explicit construction of irreducible representations for $SL_2(\mathbb{Z}/p^n\mathbb{Z})$ for $n \geq 2$. Our work derives from that of Tanaka [12], filling in and correcting details as necessary as we work toward the goal of computing Fourier transforms on these groups.

The representations sort immediately into those representations that either don’t or do come from representations of $SL_2(\mathbb{Z}/p^{n-1}\mathbb{Z})$ via the composition

$$SL_2(\mathbb{Z}/p^n\mathbb{Z}) \xrightarrow{\pi} SL_2(\mathbb{Z}/p^{n-1}\mathbb{Z}) \xrightarrow{\rho} GL_d(\mathbb{C})$$

where $\pi$ is the natural projection map (via reduction of the matrix entries mod $p^{n-1}$) from $SL_2(\mathbb{Z}/p^n\mathbb{Z})$ to $SL_2(\mathbb{Z}/p^{n-1}\mathbb{Z})$ and $\rho$ is any representation of $SL_2(\mathbb{Z}/p^{n-1}\mathbb{Z})$. We will call these (irreducible) quotient representations. This gives a recursive roadmap for the construction: To compute the irreducible representations of $SL_2(\mathbb{Z}/p^n\mathbb{Z})$, first compute all the irreducible representations of $SL_2(\mathbb{Z}/p^{n-1}\mathbb{Z})$ and then compute any representations that don’t arise in this way. Continuing down this chain of projections we arise at a “base case” of $n = 1$ and the consideration of the irreducible representations of $SL_2(\mathbb{Z}/p\mathbb{Z})$. For that we draw on the construction in [3].

Tanaka [12] outlines the construction of those representations that don’t arise from projections to the quotients. Some readers may note that the mathematical structure underlying this constructions is derived from considering quadratic extensions of $p$-adic fields and some of the language used reflects this fact.

2.1. Outline of Construction. Following [12] we produce several large (and reducible) representations of $SL_2(\mathbb{Z}/p^n\mathbb{Z})$ that depend on three parameters: $k$, $\Delta$, $\sigma$. By varying these parameters, we produce representations that contain all non-quotient irreducible representations of $SL_2(\mathbb{Z}/p^n\mathbb{Z})$. We then extract the individual irreducible representations by computing the actions on carefully chosen subspaces of the reducible space.

The construction begins (for fixed $k$) with the group $G = \mathbb{Z}/p^n\mathbb{Z} \times \mathbb{Z}/p^{n-k}\mathbb{Z}$ and the associated free complex vector space, $\mathbb{C}[G]$, with canonical basis given by the group elements.

An action of $SL_2(\mathbb{Z}/p^n\mathbb{Z})$ of $\mathbb{C}[G]$ is then defined that requires a choice of the parameter $\sigma$ as well as a ring structure on $G$ that depends on the parameter $\Delta$. (For ease of reference, we keep most of Tanaka’s original notation). This ring we denote as $G(k, \Delta)$. The resulting representation given by turning this ring into a complex vector space indexed by the group (ring) elements is denoted $R_k(\Delta, \sigma)$.

The representation $R_k(\Delta, \sigma)$ is reducible and the most difficult part of the construction comes from finding the irreducible representations inside $R_k(\Delta, \sigma)$. This is accomplished by finding a specified (multiplicative) abelian group $C \subset G(k, \Delta)$. A specific set of so-called

\footnote{This is also the notation used for the group algebra generated by the additive group $G$, but we will not need this additional structure.}
principal characters $\chi \in \hat{C}$ then determine a basis for an irreducible sub-representation inside $R_k(\Delta, \sigma)$. We denote these irreducible sub-representations by $R_k(\Delta, \sigma, \chi)$.

To explicitly specify the representation, we only need to work out the action of generators of $SL_2(\mathbb{Z}/p^n\mathbb{Z})$. These are given by the factors in the Bruhat decomposition. For this, let $D$ denote the diagonal subgroup of $SL_2(\mathbb{Z}/p^n\mathbb{Z})$ and $U$, the unipotent subgroup. If we use the notation

$$d_a = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad u_b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

where $a \in (\mathbb{Z}/p^n\mathbb{Z})^\times$ and $b \in \mathbb{Z}/p^n\mathbb{Z}$, then given

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z}/p^n\mathbb{Z})$$

its Bruhat decomposition is given by

$$A = \begin{cases} (u_{\alpha\gamma^{-1}})(w)(d_{\gamma})(u_{\delta\gamma^{-1}}) & \text{if } \gamma \not\equiv 0 \pmod{p} \\
(w)(u_{-\gamma\alpha^{-1}})(w)(d_{-\gamma})(u_{\beta\alpha^{-1}}) & \text{if } \gamma \equiv 0 \pmod{p}. \end{cases} \quad (2)$$

Thus, to compute the matrix representation of any group element it suffices to specify (compute) the actions of $d_a$, $u_b$, and $w$ on a given representation space. The details now follow.

2.2. The Ring $G(k, \Delta)$ (case of $k < n$). For $0 \leq k \leq n - 1$, let $G$ denote the abelian (additive) group (the dependence on $k$ is understood) with point set

$$G = \mathbb{Z}/p^n\mathbb{Z} \times \mathbb{Z}/p^{n-k}\mathbb{Z}$$

As usual, let $\mathbb{C}[G]$ denote the free $\mathbb{C}$ vector space with basis $G$

$$\{\sum_{g \in G} \alpha_g g : \alpha_g \in \mathbb{C}\}$$

which will be the representation space for each of our representations.

Let $\Delta'$ be a square-free positive integer, such that $(\Delta', p) = 1$ and define $\Delta = p^k \Delta'$. We put a ring structure on $G$, by defining a multiplication determined by the parameter $\Delta'$ and denote the subsequent object $G(k, \Delta)$. To define the multiplication structure we make use of the embedding $\mathbb{Z}/p^{n-k}\mathbb{Z} \hookrightarrow \mathbb{Z}/p^n\mathbb{Z}$ given by $a \mapsto p^k \cdot a$. This gives a map from elements of $\mathbb{Z}/p^{n-k}\mathbb{Z}$ into $\mathbb{Z}/p^n\mathbb{Z}$, for example if $g_2 \in \mathbb{Z}/p^{n-k}\mathbb{Z}$ then $g_2\Delta = p^k \cdot g_2\Delta' \in \mathbb{Z}/p^n\mathbb{Z}$. For any $g = (g_1, g_2) \in G$, let $\tilde{g} = g_1 + g_2\sqrt{-\Delta}$ and define a multiplication $\ast$ on $G$ by
\[ g \ast h := \tilde{u} \cdot \tilde{v} \]
\[ = \left( g_1 + g_2 \sqrt{-\Delta} \right) \cdot \left( h_1 + h_2 \sqrt{-\Delta} \right) \]
\[ \equiv y_1 + y_2 \sqrt{-\Delta} \]
\[ = \tilde{y} \]

where \( y_1 = g_1 h_1 - (g_2 h_2) \Delta \) (mod \( p^n \)) and \( y_2 = g_1 h_2 + g_2 h_1 \) (mod \( p^{n-k} \)).

Having endowed \((G(\Delta, k), *)\) with a multiplicative structure, we further define conjugation \((\bar{\cdot})\), norm \((Nm)\) and trace \((Tr)\) maps:

\[ \bar{\cdot} : G \rightarrow G \]
\[ (g_1, g_2) \mapsto (g_1, -g_2) \]
\[ Nm: G \rightarrow \mathbb{Z}/p^n\mathbb{Z} \]
\[ Nm(g_1, g_2) \mapsto g_1^2 + \Delta g_2^2 \] (Multiplicative)
\[ Tr: G \rightarrow \mathbb{Z}/p^n\mathbb{Z} \]
\[ Tr(g_1, g_2) \mapsto 2g_1 \] (Additive)

Note that \( Nm \) and \( Tr \) are multiplicative and additive with respect to group multiplication on \( G(k, \Delta) \) and the standard componentwise additive structure on \( G \): \( Tr(g + h) = Tr(g) + Tr(h) \) and \( Nm(g \ast h) = Nm(g)Nm(h) \). We use the same embedding of \( \mathbb{Z}/p^{n-k}\mathbb{Z} \hookrightarrow \mathbb{Z}/p^n\mathbb{Z} \) to compute the norm.

2.2.1. The representation \( R_k(\Delta, \sigma) \). We define a representation space \( R_k(\Delta, \sigma) \) with map \( T_\sigma \) by specifying the actions of the Bruhat elements \( d_a, u_b \) and \( w \) on \( \mathbb{C}[G] \). We first let \( \sigma \in \mathbb{Z} \) determine a \( p^n \)-th-root of unity \( \zeta_\sigma = e^{2\pi i \sigma / p^n} \) for the characters. Now define the representation \( R_k(\Delta, \sigma) \) via the actions

\[ T_\sigma(d_a) [g] = \left( \frac{a}{p} \right)^k g \cdot a^{-1} \] (3)
\[ T_\sigma(u_b) [g] = \zeta_\sigma^{b Nm(g)} g \]
\[ T_\sigma(w) [g] = c \sum_{h \in G} \zeta_\sigma^{-Tr(g \ast \bar{h})} h \] (5)

where if \( g = (g_1, g_2) \) then \( g \cdot a = (ag_1, ag_2) \) and \( c \) is a constant given by

\[ c = p^{-n+(k/2)} \left( \frac{\Delta'}{p} \right)^{-n-k} \left( \frac{\sigma}{p} \right)^k e \]

where

\[ e = \begin{cases} 
1 & k \text{ odd} \\
-i & k \text{ odd} \\
-1^n & k \text{ even} 
\end{cases} \]

\[ \left( \frac{-1}{p} \right) = 1 \]
\[ \left( \frac{-1}{p} \right) = -1 \]
Tanaka [11] defines these actions on the dual space of functions. In order to effectively implement these methods, we select the standard basis to make use of the non-canonical isomorphism of \( \mathbb{C}[G]^* \) and \( \mathbb{C}[G] \).

2.2.2. Irreducible sub-representations \( R_k(\chi, \Delta, \sigma) \). Having built the large representation \( R_k(\Delta, \sigma) \) our next goal is to carve out irreducible subspaces of this large representation. To do this we consider the monoid \((G, *)\) under multiplication and define

\[
C = \{ u \in G : \text{Nm}(u) \equiv 1 \pmod{p^n} \}
\]

which is an abelian group within the monoid. Additionally, we define

\[
C_{n-1} = \{ c \in C : u_1 \equiv 1 \pmod{p^{n-1}} \text{ and } u_2 \equiv 0 \pmod{p^{n-1-k}} \}
\]

We say a character \( \chi \) on \( C \), is principal if the restriction of \( \chi \) to \( C_{n-1} \) is nontrivial. Otherwise we say it is decomposable. For a principal character \( \chi \), we consider the induced vector space \( \text{Ind}(V_\chi) \):

\[
\text{Ind}(V_\chi) := \{ \bar{v} \in \mathbb{C}[G] : \alpha_{c \ast g} = \chi(c) \alpha_g \quad c \in C, g \in G \}.
\]

The restriction of the representation on \( R_k(\Delta, \sigma) \) given in Section 2.2.1 to the subspace \( \text{Ind}(V_\chi) \) is irreducible (cf. [11] Section 4, Theorem i)). We denote this irreducible representation as \( R_k(\chi, \Delta, \sigma) \). Note that in the case \( n = 2 \) the group \( C \) is always cyclic [2]. This has numerous computational advantages, both for determining the principal characters and evaluating equations (3), (4), and (5). In fact, choices of the characters in this case can be computed directly for all parameter choices. We record these characters in the following theorem, noting that \( \chi_j \) and \( \chi_\ell \) give equivalent representations if and only if \( \chi_j = \chi_\ell \) or \( \chi_j = -\chi_\ell \) [12].

**Theorem 2.1.** In the case \( n = 2 \) a complete set of principal characters is given by

- When \( k = 0 \) and \( \left( -\frac{\Delta}{p} \right) = 1 \)
  \[
  \{ 0 < \ell < p(p-1) : (\ell, p) = 1 \}.
  \]
- When \( k = 0 \) and \( \left( -\frac{\Delta}{p} \right) = -1 \)
  \[
  \{ 0 < \ell < p(p+1) : (\ell, p) = 1 \}.
  \]
- When \( k = 1 \)
  \[
  \{ 0 < \ell < 2p : (\ell, p) = 1 \}.
  \]

2.3. Final representations – the case of \( (k = n) \). In the case of \( k = n \) the above construction has to be altered slightly in order to find irreducible representations and depends on [2]. Following the construction as in the case of \( k < n \) we see that

\[
G \cong \mathbb{Z}/p^n\mathbb{Z} \times \mathbb{Z}/p^{n-k}\mathbb{Z} \cong \mathbb{Z}/p^n\mathbb{Z} \times 0 \cong \mathbb{Z}/p^n\mathbb{Z}.
\]

Thus, the imposed multiplication structure no longer depends on \( \Delta \) and is just the usual multiplication on \( \mathbb{Z}/p^n\mathbb{Z} \). Note that this also implies that the representation only depends on \( \sigma \), so we will denote it as \( R_n(\sigma) \).
Within $R_n(\sigma)$ there are two important sub-representations $R_n(\sigma,\chi_i)$ for $i = \pm 1$ that we need to identify. To find them, observe first that in this case, the subgroup $C$ reduces to the two element group:

$$C = \{(1,0), (p^n-1,0)\}.$$  

The corresponding character group $\hat{C}$ has two elements: $\chi_1$ the trivial character and $\chi_{-1}$ the non-trivial character. Then according to [2] (pp. 371-372) we have the following:

**Theorem 2.2.** The representation $R_n(\sigma)$ has a decomposition into

$$R_n(\sigma) = R_n(\chi_1,\sigma) \oplus R_n(\chi_{-1},\sigma)$$

with

$$\dim_{\mathbb{C}} R_n(\chi_1,\sigma) = (p^n + 1)/2$$

and

$$\dim_{\mathbb{C}} R_n(\chi_{-1},\sigma) = (p^n - 1)/2.$$  

2.3.1. The case $k = n, n = 2$. In the case of $k = n = 2$ the representation $\dim_{\mathbb{C}} R_2(\chi_{-1},\sigma)$ is irreducible of dimension $(p^2 - 1)/2$. The representation $R_2(\chi_1,\sigma)$ is reducible and contains a copy of the trivial representation with the remaining representation of dimension $(p^2 - 1)/2$ irreducible. Among our new results is an explicit basis for computing these representations:

**Theorem 2.3.** The representation $R_2(\Delta,\sigma)$ decomposes into three irreducible sub-representations

$$R_2(\sigma) \cong R_2(1,\sigma) \oplus R_2(-1,\sigma) = \mathbb{C}[\mathbb{Z}/p^2\mathbb{Z} \times 0].$$

The elements

$$\left\{ \sum_{i=0}^{p^{-1}} (ip,0) \right\} \cup \left\{ (j,0) - (p^2 - j,0) : 1 \leq j \leq \frac{p^2 - 1}{2} \right\}$$

are a basis for $R_2(1,\sigma)$ with the first element corresponding to the trivial representation. The basis elements for $R_2(-1,\sigma)$ are of two kinds:

1. \(\left\{ (j,0) + (p^2 - j,0) : (j,p) = 1 \text{ and } 1 \leq j \leq \frac{p^2 - 1}{2} \right\}\)

2. \(\left\{ 2(0,0) - (i,0) - (p^2 - i,0) : (i,p) = 0 \text{ and } 1 \leq i \leq \frac{p^2 - 1}{2} \right\}\).

We are throughout identifying elements of $\mathbb{Z}/p^2\mathbb{Z} \times 0$ with their associated basis elements in $\mathbb{C}[\mathbb{Z}/p^2\mathbb{Z} \times 0]$.

With this, the case of $n = 2$ is now complete and we can state the following theorem:

**Theorem 2.4.** Let $p$ be an odd prime. The irreducible representations for $SL_2(\mathbb{Z}/p^2\mathbb{Z})$ may first be distinguished as those that arise as the projections of irreducible representations of $SL_2(\mathbb{Z}/p\mathbb{Z})$ (see Eq. (1)) and those that do not. Of the latter kind, select $\Delta_1$ and $\Delta_2$ so that

$$\left( \frac{-\Delta_1}{p} \right) = 1 \text{ and } \left( \frac{-\Delta_2}{p} \right) = -1$$

and $\sigma_1$ and $\sigma_2$ so that

$$\left( \frac{\sigma_1}{p} \right) = 1 \text{ and } \left( \frac{\sigma_2}{p} \right) = -1.$$  

Then Table 2.1, with notation as above, lists all the non-quotient irreducible representations.
Table 2.1. The non-quotient irreducible representations of $SL_2(\mathbb{Z}/p^2\mathbb{Z})$.

2.4. The case $k = n$, for $n \geq 3$. When $n \geq 3$, the representations $R_n(\chi_i, \sigma)$ for $i = \pm 1$ will not be irreducible. Let $\pi$ denote the natural projection $\mathbb{Z}/p^n\mathbb{Z} \rightarrow \mathbb{Z}/p^{n-1}\mathbb{Z}$ and $i$ the inclusion $\mathbb{Z}/p^n\mathbb{Z} \rightarrow p\mathbb{Z}/p^{n+1}\mathbb{Z}$. The composition of $i$ and the inverse image $\pi^{-1}$ yields a map

$$
\begin{array}{ccc}
\mathbb{Z}/p^n\mathbb{Z} & \xrightarrow{\pi^{-1}i} & \mathbb{Z}/p^{n-1}\mathbb{Z} \\
\mathbb{Z}/p^{n-1}\mathbb{Z} & \xleftarrow{\pi} & \mathbb{Z}/p^n\mathbb{Z}
\end{array}
$$

expressed as $\pi^{-1} \circ i : \mathbb{Z}/p^{n-2}\mathbb{Z} \rightarrow \mathbb{Z}/p^n\mathbb{Z}$ which induces a map on the group algebras.

$$
\mathbb{C}[\mathbb{Z}/p^{n-2}/\mathbb{Z}] \xrightarrow{\pi^{-1}i} \mathbb{C}[\mathbb{Z}/p^n\mathbb{Z}]
$$

Hence we have an embedding of the representations $R_{n-2}(\sigma) \hookrightarrow R_n(\sigma)$ and this map actually restricts to the subspaces carrying $R_{n-2}(\chi_0, \sigma) \hookrightarrow R_n(\chi_0, \sigma)$ and $R_{n-2}(\chi_1, \sigma) \hookrightarrow R_n(\chi_1, \sigma)$. Thus the representation $R_n(\sigma)$ decomposes as

$$
R_n(\sigma) \cong R_n(\chi_0, \sigma) \oplus R_n(\chi_1, \sigma)
$$
\[ \cong [R_n(\chi_0, \sigma) - R_{n-2}(\chi_0, \sigma)] \oplus [R_n(\chi_0, \sigma) - R_{n-2}(\chi_0, \sigma)] \oplus R_{n-2}(\sigma) \]

and the space does not simply decompose into a sum of new, irreducible representations.

2.5. Size and number of representations. Although the construction described above is fairly complex, the representations can be computed by simply varying a collection of parameters.

- \( k \): Determines the space for the representation. \( (0 \leq k \leq n) \).
- \( \Delta \): Determines arithmetic, depends on \( k \).
  \[
  \Delta = \Delta' p^k \in \mathbb{Z} \quad \text{with} \quad (\Delta', p) = 1 \quad \text{and squarefree.}
  \]
- \( \chi \): Non-decomposable character on an abelian group \( C \). Depends on \( \Delta \).
- \( \sigma \): Used to construct characters. \( \sigma \in \mathbb{Z} ; \quad (\sigma, p) = 1 \).

The results of Tanaka [12] specify which choices of parameters lead to inequivalent representations and describe the number and dimension of each space. We record these results in the following theorem.

**Theorem 2.5** (Tanaka). The equivalence of representations \( R_k(\chi, \Delta, \sigma) \) depends on the value \( k \) as follows:

- \( k = 0 \): When \( k = 0 \) two representations \( R_0(\chi_1, \Delta_1, \sigma_1) \) and \( R_0(\chi_2, \Delta_2, \sigma_2) \) are equivalent if and only if \( \left( \frac{\Delta_1}{p} \right) = \left( \frac{\Delta_2}{p} \right) \) and \( \chi_1 = \chi_2 \) or \( \chi_1 = \chi_2^{-1} \).
- \( 1 \leq k \leq n - 1 \): When \( 1 \leq k \leq n - 1 \) two representations \( R_0(\chi_1, \Delta_1, \sigma_1) \) and \( R_k(\chi_2, \Delta_2, \sigma_2) \) are equivalent if and only if \( \left( \frac{\Delta_1}{p} \right) = \left( \frac{\Delta_2}{p} \right) \), \( \left( \frac{\sigma_1}{p} \right) = \left( \frac{\sigma_2}{p} \right) \) and \( \chi_1 = \chi_2 \) or \( \chi_1 = \chi_2^{-1} \).
- \( k = n \): When \( k = n \) two representations \( R_0(\chi_1, \Delta_1, \sigma_1) \) and \( R_k(\chi_2, \Delta_2, \sigma_2) \) are equivalent if and only if \( \left( \frac{\sigma_1}{p} \right) = \left( \frac{\sigma_2}{p} \right) \).

The table below summarizes the properties of the objects constructed above:

| \( R_k(\chi, \Delta, \sigma) \) | \( k = 0 \) | \( 1 \leq k \leq n - 1 \) | \( k = n \) |
|----------------|---------|----------------|---------|
| Number | \( (p^{n-1})(p-1) \) | \( 4(p^{n-k} - p^{n-k-1}) \) | 4 |
| Dimension | \( p^n + \left( \frac{-\Delta}{p} \right) p^{n-1} \) | \( p^{n-2}(p^2 - 1)/2 \) | \( p^{n-2}(p^2 - 1)/2 \) |
| \( |C| \) | \( p^{n-1} \left( p - \left( \frac{-\Delta}{p} \right) \right) \) | \( 2p^{n-k} \) | 2 |
| \( |\chi| \) | \( p^{n-2}(p-1) \left( p - \left( \frac{-\Delta}{p} \right) \right) \) | \( 2(p^{n-k} - p^{n-k-1}) \) | 2 |

Table 2.2. Classification of non-quotient irreducible representations of \( SL_n(\mathbb{Z}/p^2\mathbb{Z}) \).
3. Example: Representations and character table for $SL_2(\mathbb{Z}/9\mathbb{Z})$

The representations are those that arise as representations of $SL_2(\mathbb{Z}/3\mathbb{Z})$ and those that don’t. For the former, we refer to [7] where these are worked out in some detail as an example of a different fast Fourier transform algorithm for the groups $SL_2(\mathbb{Z}/p\mathbb{Z})$. For the latter, we use the process detailed above and start with the characters of the “new” representations of $SL_2(\mathbb{Z}/9\mathbb{Z})$.

The group $SL_2(\mathbb{Z}/9\mathbb{Z})$ has 25 conjugacy classes with representatives:

$$\left\{ \begin{array}{ccccccccccc}
1 & 0 & 8 & 0 & 4 & 3 & 5 & 6 & 0 & 1 & 8 & 8 & 1 & 1 & 4 & 0 \\
0 & 1 & 0 & 8 & 3 & 7 & 6 & 2 & 8 & 0 & 8 & 7 & 1 & 2 & 6 & 7 \\
5 & 0 & 1 & 0 & 8 & 0 & 4 & 3 & 5 & 6 & 7 & 6 & 2 & 3 & 1 & 0 & 8 & 0 \\
3 & 2 & 1 & 1 & 8 & 8 & 7 & 1 & 2 & 8 & 4 & 1 & 5 & 8 & 2 & 1 & 7 & 8 \\
4 & 3 & 5 & 6 & 7 & 6 & 2 & 3 & 1 & 3 & 8 & 6 & 4 & 6 & 5 & 3 \\
2 & 4 & 7 & 5 & 2 & 7 & 7 & 2 & 0 & 1 & 0 & 8 & 3 & 7 & 6 & 2
\end{array} \right\}$$  \( (6) \)

Since $p = 3$ we can choose $\Delta' = \sigma_1 = 1$ and $\Delta'' = \sigma_2 = 2$. Following the procedure outlined in Section 2 we separate the representations into cases by $k$.

$(k = 0)$: As seen in Table 2.1 when $k = 0$ the principal characters depend only on $\left( \frac{-\Delta}{p} \right)$. When $\Delta = 1$ there are eight principal characters, indexed by $\{1, 11\}, \{2, 10\}, \{4, 8\}, \{5, 7\}$ (and grouped according to equivalence class). Selecting one character from each inequivalent class $\{1, 2, 4, 5\}$ gives four representations of dimension 6. When $\Delta = 2$ there are four principal characters, indexed by $\{1, 5\}, \{2, 4\}$. Class representatives are given by indices $\{1, 2\}$ producing two representations of dimension 12.

$(k = 1)$: In this case the principal characters are independent of $\Delta$ and $\sigma$. There are four characters $\{1, 5\}, \{2, 4\}$ (grouped according to equivalence class) so that $\{1, 2\}$ give class representatives. Independently choosing representative character, $\Delta$, and $\sigma$ determines eight four-dimensional representations.

$(k = 2)$: In this case we follow the procedure described in Section 2.3.1. This gives a a basis

$$\{(1, 0) - (8, 0), (2, 0) - (7, 0), (3, 0) - (6, 0), (4, 0) - (5, 0)\}$$

for the non-trivial part of $R_2(\sigma_j, \chi_1)$ and a basis

$$\{(1, 0) + (8, 0), (2, 0) + (7, 0), 2(0, 0) - (3, 0) - (6, 0), (4, 0) + (5, 0)\}$$

for $R_2(\sigma_j, \chi_{-1})$ to obtain the final four representations of dimensions four.

In total there are 18 non-quotient representations which along with the seven irreducible representations of $SL_2(\mathbb{Z}/3\mathbb{Z})$ gives the 25 irreducible representations of $SL_2(\mathbb{Z}/3^2\mathbb{Z})$ (recall that the number of irreducible representations is equal to the number of conjugacy classes). These representations are summarized in Table 3 below. The character table for the new representations is also constructed below with the rows indexed by conjugacy classes and the columns indexed by representations.
\[
\begin{array}{cccccccccccccccccccc}
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 6 & 6 & 6 & 6 & 12 & 12 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & -4 & 4 & -4 & 4 & -4 & 4 & -4 & -6 & 6 & 6 & -6 & -12 & 12 & 4 & -4 & 4 & -4 & 4 & -4 & 4 & -4 & 4 \\
-2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & 3 & 3 & 3 & 3 & 0 & 0 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 \\
-2 & 2 & -2 & 2 & -2 & 2 & -2 & 2 & -3 & 3 & 3 & -3 & 0 & 0 & -2 & 2 & -2 & 2 & -2 & 2 & -2 & 2 & -2 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -G_3 + G_2 & -1 & 1 & -G_3 + G_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -G_3 + G_1 & -1 & 1 & G_3 - G_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & -3 & -3 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 \\
\end{array}
\]

Table 3.1. Character Table for the non-quotient irreducible representations of \( SL_2(\mathbb{Z}/9\mathbb{Z}) \) The rows are indexed according to the conjugacy classes as listed in Eq. (3) and the columns correspond to the characters as listed previously according to \( k = 0, 1, 2. \)
Table 3.2. The irreducible representations of $SL_2(\mathbb{Z}/3^2\mathbb{Z})$. The “Type” for the quotient representations are either principal series (PS) or discrete series (DS). The notation for quotient representations as in [7], with details for the construction in [3]. The notation for the non-quotient representations is as in Section 2.
4. Numerical Experiments

The ability to construct representations of $SL_2(\mathbb{Z}/p^2\mathbb{Z})$ enables some experimentation on the spectra of Cayley graphs on these groups. Recall that for any group $G$ and generating set $S = \{s_1, \ldots, s_k\}$ such that $S = S^{-1}$, the Cayley graph $X(G, S)$ is the network with node set $G$ such that any $g \in G$ is connected to the each of the elements $gs_i$. The spectrum of $X(G, S)$ is the set of eigenvalues of the related adjacency matrix $A(G, S)$, the $|G| \times |G|$ matrix with $a_{g,gs_i} = 1$ for $i = 1, \ldots, k$ and 0 otherwise.

Note that

$$A(G, S) = \sum_{s \in S} \rho(s)$$

where $\rho$ is the right multiplication representation of $G$. By Wedderburn’s Theorem,

$$\text{spectrum } [A(G, S)] = \bigcup_{\eta \in \hat{G}} \text{spectrum } [\hat{\delta}_S(\eta)]$$

where the sum is over a complete set of irreducible representations of $G$ and $\hat{\delta}_S(\eta)$ denotes the Fourier transform of $\delta_S$, the characteristic function for the set $S$, at the irreducible representation $\eta$ of $G$, defined as

$$\hat{\delta}_S(\eta) = \sum_{s \in S} \eta(s).$$

Note that the normalized form would compute $\frac{1}{|S|} \hat{\delta}_S(\eta)$.

Since any irreducible representation can have dimension at most $|G|^{1/2}$, if $|G|$ is large, this observation can enable eigenvalue calculations for $X(G, S)$ that might have been intractable if attempted directly using $A(G, S)$ (see e.g., [3, 4]). In particular, for $G = SL_2(\mathbb{Z}/p^2\mathbb{Z})$ this is the difference between finding the eigenvalues of a matrix of order $p^6$ versus roughly $p^2$ matrices of order $p^2$.

Similarly, suppose $G$ generated by $S$ is acting on a set $\Gamma$. The Cayley graph on $\Gamma$ has point set $\Gamma$ and connects $\gamma$ to $s \gamma$ for any $s \in S$. If $\Gamma = G/B$ for a subgroup $B$ then the spectrum of the Cayley graph is equal to the union of the spectra over any representations that appear in the induction of the trivial representation from $B$ to $G$.

4.1. Spectra of some Cayley graphs on $SL_2(\mathbb{Z}/p^2\mathbb{Z})$. Here we consider Cayley graphs for three families of generators, defined below, which have been previously used for similar experiments, as well as randomly chosen generating sets. The first two were originally introduced in [3] and the “Selberg family” in [5, 10]. We expect based on the previous results that $G_1$ and $G_2$ should be associated to non–expanders while $G_3$ should give second eigenvalues much closer to the Ramanujan bound.

$$G_1 = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$$

$$G_2 = \left\{ \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$$

$$G_3 = \left\{ \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \right\}$$
4.2. Spectrum of the graph on the “projective line” $G/B$. Motivated by [10] we considered the “projective” Schreier graph formed by the action of the generators on the cosets of $G/B$, where $B$ is the upper triangular Borel subgroup. We first computed the second largest eigenvalue for the generating sets described in the previous section for primes less than 50 Figure 1 (a). As observed in [3], the eigenvalues appear to converge to a limit quickly. Note that these provide a lower bound for the second largest eigenvalue of the full Cayley graph with the same generators and hence $G_1$ and $G_2$ seem to share the behavior shown in Figures 4, 5, and 6 of [3]. The case of $G_3$ for $SL_2(Z/pZ)$ was considered in Figure 2 of [10] and our result appears similar, although over a much smaller range of primes. For $G_1$ the leading eigenvalue of the Schreier graph was equal to the leading eigenvalue of the full Cayley graph.

We next generated 100 pairs of generating elements $\{s, t\}$ for $SL_2(Z/p^2Z)$ for $3 \leq p \leq 19$ and computed the second largest eigenvalues for the Cayley graphs generated by $\{s, s^{-1}, t, t^{-1}\}$. These are displayed in Figure 1(b). This corresponds to the experiments considered in Figure 4 of [10] and Figure 8 of [3]. The mean of the values appears to be approaching the (normalized) Ramanujian bound of $2\sqrt{3}/4$ on average as $p$ increases which would correspond to the results in [10].

Figure 1. Eigenvalues associated to the Schreier graph $SL_2(Z/p^2Z)/B$. Figure (a) shows the behavior of the second largest eigenvalues for the three generating sets given in Section 4.1 as $p$ varies, normalized to lie in $[-1, 1]$. The red lines represent the largest normalized eigenvalue 1 and the corresponding Ramanujan bound $\sqrt{3}/2$. Of particular interest is the behavior of $G_3$ (pink dots) which also generates nearly Ramanujan graphs in the $SL_2(Z/pZ)/B$ case [10]. Figure (b) shows plots of the second largest eigenvalues for 100 random generating pairs, also normalized to lie in $[-1, 1]$. The red lines represent the largest eigenvalue 4 and the Ramanujan bound. For $p = 19$ the mean of the (un–normalized) values is approximately 3.41 and it is natural to conjecture that as $p$ increases the mean converges to $2\sqrt{3}$. 
4.3. **Spectra of some Cayley graphs on** $SL_2(\mathbb{Z}/25\mathbb{Z})$. We now restrict our attention to the particular case of $SL_2(\mathbb{Z}/25\mathbb{Z})$ to provide a more detailed analysis of the spectra. This group has 15,000 elements and 40 non-quotient irreducible representations. We began by comparing the eigenvalues of the associated Cayley graphs of $SL_2(\mathbb{Z}/25\mathbb{Z})$ and $SL_2(\mathbb{Z}/5\mathbb{Z})$. These values are displayed in Figure 2 with the $SL_2(\mathbb{Z}/5\mathbb{Z})$ eigenvalues marked in red.

There are many interesting questions relating the eigenvalues of the projection to the eigenvalues of the larger graph as well as relating the structure of the irreducible representations that occur on $SL_2(\mathbb{Z}/p\mathbb{Z})$ to the new representations for $SL_2(\mathbb{Z}/p^2\mathbb{Z})$. It would be interesting to compare the distributions of eigenvalues for the irreducibles, perhaps using a divergence measure.

![Figure 2. Comparison of the Cayley spectrum for $SL_2(\mathbb{Z}/25\mathbb{Z})$ and $SL_2(\mathbb{Z}/5\mathbb{Z})$. The eigenvalues corresponding to $SL_2(\mathbb{Z}/5\mathbb{Z})$ are marked in red, embedded in the larger set of eigenvalues.](image-url)
Next we present the full spectrum for each of the non–quotient irreducible representations for the generating sets $G_1$, $G_2$, and $G_4$. These plots begin to give a sense of the chromatic structure of the eigenvalues and suggest that for these representations it may be possible to describe the genericity in terms of $k$. The structure suggested by the eigenvalues that appear in all or most of the representations such as 1 for generating set $G_2$ presents an avenue for future study.

![Plots of the non–quotient eigenvalues of the full Cayley graphs for $SL_2(Z/25Z)$ for the generating sets described above, separated by irreducible representation. The values at height 1 represent the full spectrum for the non–quotient representations while the horizontal lines above each correspond to a single irreducible. The representations are ordered starting with $k = 0$, $\Delta = 1$, $\sigma = 1$, and $\chi = 1$ at height 2 and then varying $\chi$, $\sigma$, $\Delta$, and $k$ in that order ascending along the y axis. The red lines separate representations for different values of $k$ while the vertical black lines mark the values of $-1$ and 1.](image)

**Figure 3.** Plots of the non–quotient eigenvalues of the full Cayley graphs for $SL_2(Z/25Z)$ for the generating sets described above, separated by irreducible representation. The values at height 1 represent the full spectrum for the non–quotient representations while the horizontal lines above each correspond to a single irreducible. The representations are ordered starting with $k = 0$, $\Delta = 1$, $\sigma = 1$, and $\chi = 1$ at height 2 and then varying $\chi$, $\sigma$, $\Delta$, and $k$ in that order ascending along the $y$ axis. The red lines separate representations for different values of $k$ while the vertical black lines mark the values of $-1$ and 1.
Finally, for each of the irreducibles considered in the previous plot we extract the leading eigenvalues to get a sense of the behavior in these values over the parameter choices. The clearest behavioral pattern is determined by the parity of the associated character. Note that the adjacent elements (from bottom to top) that appear to have the same magnitude are due to skipping multiples of 5 in the choice of characters for the \( k = 0 \) case. In many of the examples for small primes that we computed, including ones not displayed here, the largest eigenvalue across the irreducibles occurred in the \( k = 2 \) representation where \( \left( \frac{2}{p} \right) = 1 \) and \( \chi = 1 \).

![Plots of the second largest eigenvalues of the non-quotient irreducible representations for \( SL_2(Z/25Z) \) ordered as in the previous figure. The red lines separate representations for different values of \( k \) while the vertical black line marks the normalized, leading eigenvalue 1. Note that the alternating behavior is determined by the parity of the character \( \chi_\ell \).](image_url)

**Figure 4.** Plots of the second largest eigenvalues of the non-quotient irreducible representations for \( SL_2(Z/25Z) \) ordered as in the previous figure. The red lines separate representations for different values of \( k \) while the vertical black line marks the normalized, leading eigenvalue 1. Note that the alternating behavior is determined by the parity of the character \( \chi_\ell \).
5. Conclusion and Directions for Future Work

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