On the relationship between sigma models and spin chains

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Abstract

We consider the two-dimensional O(3) non-linear sigma model with topological term using a lattice regularization introduced by Shankar and Read [Nucl.Phys. B336 (1990), 457], that is suitable for studying the strong coupling regime. When this lattice model is quantized, the coefficient $\theta$ of the topological term is quantized as $\theta = 2\pi s$, with $s$ integer or half-integer. We study in detail the relationship between the low energy behaviour of this theory and the one-dimensional spin-$s$ Heisenberg model. We generalize the analysis to sigma models with other symmetries.

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I. INTRODUCTION

The O(3) non–linear sigma model (NLSM) is a two–dimensional field theory of a 3-component, unit-vector field $n_\alpha$ ($\alpha = 1, 2, 3; n^2 = 1$), with the Euclidean action given by

$$S_\theta = \frac{1}{2g^2} \int d^2x (\partial_\mu n)^2 + i \theta T,$$

where $g$ and $\theta$ are dimensionless coupling constants and

$$T = \frac{1}{8\pi} \int d^2x \, e^{\mu \nu} n \cdot (\partial_\mu n \times \partial_\nu n)$$

is an integer-valued topological term which measures the instanton number of a configuration; it is well-defined provided that $n$ converges to the same limit at infinity in all directions in the plane. In this case the physical space, as well as the target space, is effectively a sphere, $S^2$, and the classical configurations are maps from $S^2$ to $S^2$; these are classified up to homotopy by the second homotopy group $\pi_2(S^2) \cong \mathbb{Z}$. The topological term measures the homotopy class of such a configuration and is computed as the total area of the plane wrapped around the sphere, divided by $4\pi$. Since $\theta$ enters the partition function as $e^{i\theta T}$ with $T$ integer, the model only depends on $\theta$ modulo $2\pi$. The theory is known to be asymptotically free, i.e. it behaves as a free theory at high energies but becomes strongly interacting at small energies \[2\]. For $\theta = 0$ the spectrum is known to be massive \[3\]. The topological term seems to have no effect on the perturbation theory \[4\], nevertheless for $\theta = \pi$ the strong coupling behaviour changes completely. In fact, at some value of $g$, the flow will reach a fixed point \[4, 5, 6, 7, 8, 9, 10, 11, 12\]. Thus the excitations of the model at $\theta = \pi$ are massless and it corresponds to the Renormalization Group (RG) flow between the $c = 2$ Conformal Field Theory and the SU(2)$_1$ Wess–Zumino–Witten (WZW) model at level 1, with central charge $c = 1$ \[3, 11, 14\]. The two values $\theta = 0, \pi$ are the only ones for which the action \[11\] is known to be integrable \[13\] and the two-particle S-matrix was suggested in \[3, 14\] for $\theta = 0$ and $\theta = \pi$ respectively. The spectrum for generic values of $\theta$ was discussed in \[15\].

There is general agreement that the NLSM \[11\] with

$$\theta = 2\pi s$$

(3)

describes the low energy behaviour of the one dimensional spin-$s$ quantum Heisenberg model.
with antiferromagnetic interaction \((J > 0)\)

\[
H_\text{II} = J \sum_{j=1}^{N} \mathbf{S}(j) \cdot \mathbf{S}(j + 1),
\]

(4)

where \(\mathbf{S}(j)\) are spin-\(s\) generators of SU(2). The relationship was first suggested by Haldane [6] who showed, using a coherent state path integral description of the Heisenberg model, that for large spin, \(s \gg 1\), in the continuum limit this model can be mapped onto the NLSM with \(\theta\) given by (3). Other alternative derivations can be obtained within the same approximation [7, 10, 17].

The identification (3), together with the periodicity of the sigma model in \(\theta\), leads to the famous prediction that the Heisenberg model is massive for integer spin and massless for half-integer spins. This prediction was confirmed by a recent study using non-Abelian bosonization [18], and is in agreement with the exact solution of [11] for \(s = \frac{1}{2}\) and numerical results on \(s = 1\) and \(\frac{3}{2}\) chains [20]. From the solution of the \(s = \frac{1}{2}\) chain one also finds that the IR behaviour is given by the SU(2) \(_1\) WZW model with a marginally irrelevant current-current interaction [21].

Shankar and Read [11] suggested an interesting alternative analysis. They constructed a lattice regularization of the NLSM with a topological term and showed that in a naive \(g \to \infty\) limit the \(\theta = \pi\) model can be mapped to the \(s = \frac{1}{2}\) Heisenberg model. Specifically, as in the \(\theta = 0\) case [24], this lattice Hamiltonian can be written as the sum of a kinetic and a potential term and, for \(g\) “large”, the latter can be treated as a perturbation. Shankar and Read were only interested in using this mapping to show that the NLSM with \(\theta = \pi\) is massless and did not consider in detail the relationship between the lattice sigma model and the Heisenberg model [11]. This is what we try to do in this paper. We use this analysis to gain some more insight into the validity of the Haldane mapping for general spin. As we will see, the perturbative approach can be easily generalized to any value of \(\theta = 2\pi s\) and it turns out to be related to the geometric quantization of \(S^2\). We carefully construct perturbation theory up to second order and identify its limit of validity. The first order result is the spin-\(s\) Heisenberg model, while second order introduces next-nearest neighbor interactions (and quadrupole couplings); these are strongly reduced relative to the leading term by a very small (\(s\) dependent) pre-factor. At least for the \(s = 1/2\) case it is possible to show that this correction is irrelevant. Then condition for the validity of the perturbative approach, (27), provides a sufficient condition for the validity of the Haldane mapping. In
other words this approach shows that the low energy behaviour of the spin-$s$ Heisenberg model is described by the O(3) NLSM with topological term $\theta$ given by (3) and $g$ that has to satisfy the condition \[ \theta \approx 0 \]. We also extend this approach to a more general class of sigma models. These models have been considered in Ref. [22] and, in the context of the Quantum Hall Effect and disordered systems, in Refs. [4, 8, 16].

The paper is organized as follows. In the next section we recall the Shankar-Read regularization procedure for the construction of a lattice regularized version of the NLSM and discuss its quantization in Section III. In Section IV we consider perturbation theory and study its implications for the Haldane mapping in Section V. The extension to more general sigma models is done in Sec. VI and the main results are discussed and summarized in the last section.

II. LATTICE REGULARIZATION

Of course the action (1) does not define a quantum theory unless one specifies how to regularize it. In fact, perturbation theory presents short distance divergences. In the asymptotically free regime ($g \ll 1$) standard renormalization techniques can be applied [23]. An alternative that is more suitable for studying the strong coupling regime, is to put the theory on a lattice [11, 24] and then quantize it. We replace the spatial coordinate $x$ by a lattice with spacing $a$ and $N$ sites, but leave the Euclidean time $\tau$ continuous. The lattice regularization of the first term of (1) is straightforward and is given by

\[
S_0 = \frac{1}{g^2} \int d\tau \sum_{j=1}^{N} \left\{ \frac{a^2}{2} (\partial_\tau \mathbf{n}(j))^2 - \frac{1}{a} \mathbf{n}(j) \cdot \mathbf{n}(j+1) \right\},
\]

plus an irrelevant constant. Clearly other regularizations, including e.g., next-nearest neighbor interactions, are possible. Nevertheless they would give an irrelevant contribution at large distances. This regularization correctly reproduces the spectrum of the NLSM with $\theta = 0$ in the strong coupling limit [24].

Now, following Shankar and Read [11], we must regularize the topological term, i.e., we have to construct a lattice term that approximates the area of the plane wrapped around the sphere. The lattice divides the plane into strips and we can compute the topological term from the sum of the areas covered by each of these. Because of the boundary condition, each lattice point traces out a closed curve on the sphere. The area associated to the strip
between $j$ and $j + 1$ is simply the difference $A(j + 1) - A(j)$ of the areas on the sphere enclosed by the curves traced out by $j$ and $j + 1$. Using Stokes’ theorem, the area $A(j)$ can be written as

$$A(j) = \int d\tau A(n(j)) \cdot \frac{dn(j)}{d\tau}$$

where $A(n)$ is chosen such that

$$\nabla \times A = n$$

along the unit sphere. It is not actually possible to choose such an $A$ over the entire sphere, but it is enough to define it locally and in the quantum theory $A$ plays the role of a background electromagnetic potential. Because of this, (6) may be incorrect by $\pm 4\pi$; however, this will eventually be irrelevant if $\theta$ is a multiple of $\pi$. We have to keep this in mind when discussing some of the results below that follow from the quantization of the theory.

The contribution to the topological term coming from the strip between two nearest sites is the difference between the associated areas: $A(j + 1) - A(j)$. This seems to be a very natural definition of the topological term. However, if we sum up these terms for all the strips, then everything cancels except a boundary term, thus not giving interesting physics. What Shankar and Read did was to carry out the summation only over every other strip and multiply by 2, to get

$$T \approx \frac{1}{2\pi} \int d\tau \sum_{j=1}^{N} (-)^{j} A(n(j)) \cdot \frac{dn(j)}{d\tau}.$$  

In effect, the integrand in $T$ was rewritten as a finite difference on the lattice. Integration was then approximated by summation, but in a way that is not inverse to the finite difference used in the first step. Nevertheless, one can see that for approximately continuous configurations, (8) indeed approximates the topological term. However, it is by no means the unique choice. Because the lattice has a different topology from the plane, it is not possible to systematically define a lattice-regularized topological term. At best, the “topological” term in a lattice model is merely analogous to the topological term in the continuum model. One justification for this is that, as we shall see below, it extends to a much more general analogy.

The lattice regularized form of the NLSM with topological term can then be written in the form

$$S_\theta = \int d\tau \sum_{j=1}^{N} \left\{ \frac{a}{2g^2} (\partial_\tau n(j))^2 + \frac{i\theta}{2\pi} (-)^{j} A(n(j)) \cdot \frac{dn(j)}{d\tau} - \frac{1}{ag^2} n(j) \cdot n(j + 1) \right\}$$

(9)
where \( a \) is the lattice spacing and the coupling constants in the rhs are all dimensionless. If we redefine \( n(j) \) as \((-)^j n(j)\), then we get rid of the alternating sign in the second term and change the sign of the last term\(^1\). Now we can perform a Wick rotation and construct the Hamiltonian. The result is the lattice regularized version of the NLSM suggested by Shankar and Read \[11\]

\[
H = \frac{1}{a} \sum_{j=1}^{N} \left\{ \frac{g^2}{2} L'(j)^2 + \frac{1}{g^2} n(j) \cdot n(j+1) \right\}
\]

where

\[
L' := n \times \left( \pi - \frac{\theta}{2\pi} A \right)
\]

and \( \pi(j) \) is the (formal) conjugate momentum to \( n(j) \). The first term in the sum is the kinetic energy of a charged particle moving on a unit sphere with a magnetic monopole at the center, the strength of the monopole being given by \( \theta/2\pi \) \[25\]. We will take this Hamiltonian as our starting point and use it to study the relationship between the (continuum) NLSM \[1\] and the spin-\( s \) Heisenberg model \[4\]. In particular we would like to verify the validity of the Haldane mapping for general values of \( s \). Clearly the Heisenberg model and the regularized sigma model \[10\] are in general not equivalent at all scales. The latter is a regularization of the NLSM that corresponds to the original theory \[1\] only at large distances (small energies). If we now carry out the comparison at the level of the two lattice models, we need to show that part of the low energy sector of the lattice theory \[10\] can be mapped to the low energy sector of the Heisenberg model \[4\].

### III. QUANTUM MODEL

In the lattice version of the NLSM \[10\], the degrees of freedom at a single lattice site are equivalent to a charged particle moving on a sphere around a magnetic monopole \[25\] with charge proportional to \( \theta \). The quantization of this model is now straightforward; it can be carried out systematically by using the formalism of geometric quantization \[26\], or one can use the known results for magnetic monopoles \[27\]. In the quantum model, the value of \( \theta \) is quantized and can only be a multiple of \( \pi \). This corresponds to the quantization of magnetic

\(^1\) \( A(j) \) can be defined independently at each site, this allows us to use the same form at every site in the final expression.
monopole charge. We can therefore write it as in (3) as \( \theta = 2\pi s \), where \( s \in \frac{1}{2}\mathbb{Z} \). This seems in contrast with other quantization procedures (10 and ref.’s therein) which work for other values of \( \theta \), but it is a consequence of the necessity of assuming this in the derivation of the regularized topological term.

The subtlety of quantizing the Hamiltonian (10) is entirely with the first term. Quantizing the operator \( \mathbf{L}' \) gives

\[
\mathbf{L}' := i\mathbf{n} \times \left( \nabla + i\frac{\theta}{2\pi} \mathbf{A} \right).
\]

Because \( \mathbf{A} \) satisfying Eq. (7) only exists over part of the sphere, this formula can only be applied over part of the sphere at one time. In order to define \( \mathbf{L}' \) over the entire sphere, we must interpret \( \frac{\theta}{2\pi} \mathbf{A} \) as a gauge potential, and \( \nabla + i\frac{\theta}{2\pi} \mathbf{A} \) as a covariant derivative. Patching together gauge-equivalent operators, we find that the Hilbert space \( \mathcal{H}_{\text{site}} \) for a single lattice site consists of sections of a topologically nontrivial line bundle over \( S^2 \). The curvature of the covariant derivative shows that the Chern character of the line bundle is proportional to \( \theta \), but the Chern character is always integral, thus \( \theta \) is quantized. If \( \theta \) was not \( 2\pi s \), then it would not be possible to patch together consistently to define \( \mathbf{L}' \).

Although it is not immediately apparent from Eq. (12), this lattice model preserves the SU(2) symmetry of the original model. The operator \( \mathbf{L}' \) is not quite the angular momentum, \( \mathbf{J} \), as it does not satisfy the correct commutation relations. Instead, \( \mathbf{L}' = \mathbf{J} - s\mathbf{n} \). Because \( \mathbf{L}'^2 = \mathbf{J}^2 - s^2 \), we can redefine the Hamiltonian by an irrelevant constant, and write it most conveniently in terms of \( \mathbf{J}^2 \).

Because the physics is the same for \( -\theta \) as for \( \theta \), we can assume without loss of generality that \( s \geq 0 \). As a representation of SU(2),

\[
\mathcal{H}_{\text{site}} = (s) \oplus (s + 1) \oplus (s + 2) \oplus \ldots
\]

is the infinite direct sum of the spin \( s \) representation, the spin \( s + 1 \) representation, and so on. In the case of \( s = 0 \), this is just the Hilbert space of functions on \( S^2 \) and it is the direct sum of all integer-spin representations. If we consider a finite lattice with \( N \) sites, then the Hilbert space \( \mathcal{H} \) for the model is the tensor product of \( N \) copies of \( \mathcal{H}_{\text{site}} \),

\[
\mathcal{H} = \bigotimes_{j=1}^{N} \mathcal{H}_{\text{site}}(j).
\]

It is clear at this point that changing \( \theta \) by \( 2\pi \) changes the model. This is in contrast to the formal partition function that we began with. We should expect that, if this is a valid
regularization, then all the models with integer (respectively, half-integer) $s$ will have the same long-wavelength behaviour.

It is convenient to rewrite the quantum Hamiltonian as

$$H = \frac{g^2}{a} K + \frac{1}{ag^2} V.$$  \hspace{1cm} (15)

The “kinetic” operator is

$$K = \frac{1}{2} \sum_j J^2(j),$$  \hspace{1cm} (16)

while the “potential” operator is

$$V = \sum_j n(j) \cdot n(j+1),$$  \hspace{1cm} (17)

where $n(j)$ is the position vector on the unit sphere for the lattice site $j$. Note that $\theta$ does not appear explicitly in the Hamiltonian. Instead it enters into the definitions of $\mathcal{H}_{\text{site}}$ and the angular momentum operators $J(j)$.

Let us consider a finite, circular lattice with $N$ sites and circumference $L = Na$. In this case, the operator $K$ is unbounded and has discrete spectrum, but the operator $V$ is bounded with continuous spectrum. In fact the angular momentum operator $J^2$ has spectrum $\{l(l+1) \mid l = s, s+1, s+2, \ldots\}$. On the other hand the inner product $n(j) \cdot n(j+1)$ takes values between $-1$ and $1$, and so has operator-norm 1. The operator $V$ is the sum of $N$ such terms, hence it has norm $N$ and its spectrum is the closed interval from $-N$ to $N$. The second term of the Hamiltonian (15) is thus bounded, with norm

$$\left\| \frac{1}{ag^2} V \right\| = \frac{N}{ag^2}.$$  \hspace{1cm} (18)

The lowest eigenvalue of $K$ is $Ns(s+1)/2$. This contributes an (irrelevant) constant to the vacuum energy of the model. The eigenspace $\mathcal{H}_0 \subset \mathcal{H}$, associated to this eigenvalue, is the tensor product of a copy of the spin-$s$ representation for each lattice site. This is precisely the Hilbert space of the spin-$s$ Heisenberg model. The second eigenspace is spanned by states with spin $s+1$ at one site, and $s$ at all other sites; it is separated from the ground state by a gap $\frac{(s+1)g^2}{a}$. If this gap is more than the band width of the potential term, i.e. if

$$(s+1)g^4 > 2N,$$  \hspace{1cm} (19)
then the gap must remain in the full theory \(15\). We notice that, at least on the bare level, inequality \(19\) is more likely satisfied for large spin\(^2\). In particular for a finite lattice and fixed \(g\), this condition is always satisfied as \(s \to \infty\).

IV. PERTURBATION THEORY

We shall analyze the Hamiltonian \(15\) using perturbation theory about the strong coupling, \(g \to \infty\) limit. Except for the overall factor of \(g^2/a\), this means taking \(K\) as the bare Hamiltonian, and treating \(g^{-4}V\) as a perturbation. The expansion parameter is thus \(g^{-4}\). Condition \(19\) is the sufficient condition for convergence of the perturbative series as well as for the validity of the perturbative approach. In fact, perturbation theory breaks down when distinct unperturbed energy levels intersect due to the perturbation. We expect that for \(g^4\) of the order \(N/(s+1)\), the highest energy level of the lower band will intersect with a higher level, and a perturbative approach will no longer be rigorous. In the next section we will discuss the validity and meaning of a weaker condition.

If the inequality \(19\) is satisfied, the relevant part of the energy spectrum will spring entirely from the highly degenerate lowest eigenvalue of \(K\). As \(g^{-1}\) increases from 0, this lowest eigenvalue of \(K\) splits into a discrete subspectrum of \(H\), and we can follow these eigenvalues continuously, as long as \(19\) is satisfied. The eigenspaces of these eigenvalues give a well-defined subspace \(\mathcal{H}_{\text{low}} \subset \mathcal{H}\), with \(\mathcal{H}_{\text{low}} = \mathcal{H}_0\) for \(g = \infty\). Using contour integrals, we can construct orthogonal projections \(\Pi\) and \(\Pi_0\) on \(\mathcal{H}_{\text{low}}\) and \(\mathcal{H}_0\) as,

\[
\Pi = \frac{1}{2\pi i} \oint \frac{dz}{z - K - g^{-4}V}
\]

where the contour is (e.g.) the circle of radius \((s+1)N\) centered at \(\frac{1}{2}s(s+1)N\). The remaining eigenstates have energy at least \(\frac{(s+1)g^2}{a} - \frac{N}{ag^2}\) above the ground state and for energies smaller then this gap, such states become physically irrelevant. As we are interested in the low energy behaviour of this system, we should concentrate on \(\mathcal{H}_{\text{low}}\).

Because the Hamiltonians (for all \(g\)) have the same symmetries by SU(2) and lattice translations, it is possible to equivariantly identify \(\mathcal{H}_{\text{low}}\) and \(\mathcal{H}_0\). If we choose some equivariant, unitary map \(u : \mathcal{H}_0 \to \mathcal{H}_{\text{low}}\), then we can describe this low energy sector with an\(^2\) One should be careful about the dependence on \(s\) of the behaviour of \(g\) under Renormalization Group and of the possible renormalization of \(s\) itself.
effective Hamiltonian,

\[ H_{\text{eff}} := u^* Hu = u^* \left( \frac{g^2}{a} K + \frac{1}{ag^2} V \right) u \]  

(20)
on \mathcal{H}_0.

If \( u \) depends analytically on \( g^{-4} \), then we can expand \( H_{\text{eff}} \) as a perturbative power series,

\[ H_{\text{eff}} = \frac{g^2}{a} h_0 + \frac{1}{ag^2} h_1 + \frac{1}{ag^6} h_2 + \ldots \]  

(21)

It is important to keep in mind that this depends on some choice of \( u \). The simplest choice of \( u \) is

\[ u = \prod \Pi_0 (\Pi_0 \Pi \Pi_0)^{-1/2}, \]

although the resulting perturbation series may diverge prematurely. A more robust choice is defined by the conditions \( u = \Pi_0 \) for \( g = \infty \) and \( u^* \frac{du}{dg} = 0 \). However, the effective Hamiltonians for these two choices agree to second order, which is as far as we shall compute here.

Now consider the terms of the perturbative expansion (21). At order 0, we have just the irrelevant constant \( \frac{g^2}{a} h_0 = Ns(s+1)g^2/2a \). The first order term is given by \( h_1 = \Pi_0 V \Pi_0 \). The projection \( \Pi_0 \) onto \( \mathcal{H}_0 \) is the product of the projection onto the spin-s representation \((s) \subset \mathcal{H}_{\text{site}}\) for each site. For a function \( f \) on the sphere, \( \Pi_0 f \Pi_0 \) is the Toeplitz operator of \( f \), used in the geometric quantization of the sphere [26]. In these terms, the operator \( h_1 \) is the sum of the products of Toeplitz operators of \( n \) at neighboring sites. The projection \( \Pi_0 \) is equivariant, so the Toeplitz operators \( \Pi_0 n(j) \Pi_0 \) must transform with spin 1; consequently, they must be proportional to \( S(j) \), the angular momentum operator on \((s) \). The proportionality constant can be determined straightforwardly by computing one matrix element explicitly. This gives,

\[ \Pi_0 n(j) \Pi_0 = \frac{1}{s+1} S(j) \]

From this, we see that the first order correction to the effective Hamiltonian is the Heisenberg Hamiltonian,

\[ \frac{1}{ag^2} h_1 = \frac{1}{ag^2} \Pi_0 V \Pi_0 = \frac{1}{ag^2(s+1)^2} \sum_j S(j) \cdot S(j+1). \]  

(22)

We can compute the second order correction explicitly. Let \( P \) be a parametrix (approximate inverse) to \( K - \frac{1}{2} Ns(s+1) \). Specifically, let \( P \) act as the inverse on vectors orthogonal to \( \mathcal{H}_0 \), and as 0 on \( \mathcal{H}_0 \). The second order correction to the effective Hamiltonian is given by

\[ h_2 = -\Pi_0 VP \Pi_0. \]
To write this explicitly, define the operator \( S^{\alpha\beta} \) as the traceless, symmetric part of \( S^\alpha S^\beta \); that is,
\[
S^{\alpha\beta} := \frac{1}{2} \left( S^\alpha S^\beta + S^\beta S^\alpha \right) - \frac{1}{3} s(s+1) \delta^{\alpha\beta}.
\] (23)

In terms of this,
\[
h_2 = - \sum_j \left\{ \frac{4s+1}{6(s+1)^3} S^\alpha(j) S^\alpha(j+1) + \frac{8s+5}{2(2s+1)^2(s+1)^2} S^{\alpha\beta}(j) S^{\alpha\beta}(j+1) \right. \\
+ \left. \frac{2}{3(s+1)^4} S^\alpha(j) S^\alpha(j+2) + \frac{2}{(2s+1)(s+1)^4} S^\alpha(j) S^{\alpha\beta}(j+1) S^{\beta}(j+2) \right\}. \] (24)

The computation is detailed in the appendix. We notice that the \( \frac{1}{ag^6} \) correction is further dumped by a coefficient that contains higher powers of \( \frac{1}{s+1} \) and then will not modify significantly the long distance behaviour of the theory even for finite values of \( g \). Because \( S^{\alpha\beta} = 0 \) for \( s = \frac{1}{2} \), (24) simplifies considerably
\[
h_2 = - \sum_j \left\{ \frac{4}{27} - \frac{16}{2\times13} S^\alpha(j) S^\alpha(j+1) + \frac{32}{2\times13} S^\alpha(j) S^\alpha(j+2) \right\}. \] (25)

We note that the sign of the next-nearest neighbor interaction is negative and thus it is irrelevant at large distances for any value of \( g \). This was not a prioriy obvious since a next-nearest neighbor interaction with positive coupling constant \( J' > 0.241 J \) produce a frustrating effect that completely modify the large distance behaviour of the theory \([28]\).

V. THE O(3) NLSM AND THE HEISENBERG MODEL

Inequality \([19]\) is a sufficient condition for the exact validity and convergence of the perturbative approach. The results of the previous section show that, if this is satisfied, it is possible to rigorously map the low energy sector of the regularized version of the O(3) NLSM onto a generalized Heisenberg model given by Eq. \([20]\). The full spectrum of the Heisenberg model lies below the lowest energy gap of the lattice NLSM and is produced by the restriction of this model to the Hilbert space \( \mathcal{H}_{\text{low}} \). Unfortunately, \([19]\) is extremely restrictive and will always be violated in the thermodynamic limit as \( N \to \infty \). This condition comes from requiring that the highest energy level of the lowest band does not intersect with a higher band. However, we are not really interested in these highest energy levels, but in the lowest ones. In fact, we have to remember that our original aim was to check whether, at low energies, the spin-\( s \) Heisenberg model is described by the (continuum) NLSM \([1]\) with
\( \theta = 2\pi s \). This is much less than what we have shown to be valid when \( (19) \) is satisfied. For our purposes the mapping needs to be established only between the lowest energy sectors of the two models. Thus if we focus only on the lowest energy levels of the lattice NLSM, it seems likely that these levels will not intersect levels coming from an upper band and will continue to depend analytically on \( g \), for much lower values of \( g \) than the one identified by the condition \( (19) \). Thus conclusions about the low energy states obtained from perturbation theory will continue to be valid even when \( (19) \) is not satisfied. In the rest of this section we will try to identify a condition for the validity of the perturbative analysis for the lowest energy levels of the lattice NLSM. Given the results of the analysis of the previous section, this translates to a condition for the validity of the mapping between the low energy sectors of the Heisenberg model and those of the NLSM.

Let us consider the lowest energy level of the second band (i.e., the part of the spectrum originating from the second eigenvalue of \( K \)). We expect the perturbative analysis to break down completely only when this level dives through the lower band and intersects with the ground state. The second band is not so different from the lowest band. It is dominated by states in which \( N - 1 \) sites have spin \( s \), and one site has spin \( s + 1 \). The difference between this energy and the ground state should be given essentially by the effect of changing one site. There is clearly a positive contribution of \( \frac{2}{a}(s + 1) \) from increasing the spin. In addition, modifying one site will decrease the potential term by no more than about \( \frac{2}{ag^2} \) from the coupling of this site to its nearest neighbors. We thus estimate that this state will be separated from the ground state by an energy

\[
\Delta_s(g) \gtrsim g^2(s + 1)/a - 2/ag^2. \tag{26}
\]

If the theory is massless (the spin \( s \) is half-integer), in the thermodynamic limit, there exist many states with energies arbitrarily close to the ground state. This suggests that as long as \( \Delta_s(g) > 0 \) the perturbative expansion for the lowest energy levels of the model remain valid and provides the much weaker condition:

\[
g^4(s + 1) \gtrsim 2. \tag{27}
\]

If the condition \( (27) \) is satisfied, one expects that the behaviour of the low energy sector of the lattice NLSM (which should be equivalent to that of its continuum limit) is correctly described by the perturbative analysis presented in the previous section and is then equivalent to the low energy behaviour of the Heisenberg model.
If \( s \) is integer the above reasoning has to be modified because the theory has a mass gap \( M \), that translates to a finite correlation length of the order of \( 1/M \), or \( 1/aM \) lattice sites. The most direct extension of the above results to take this into account is to require that \( \Delta_s(g) > M \). Now, we must keep in mind that the continuum approximation will break down if the correlation length is less than the lattice spacing \( a \); this means that we must assume \( aM < 1 \). Taking this into account when correcting (27), we find that the correction is small. The weak condition (27) should only be trusted as an order of magnitude estimate, and as such it is unchanged in the massive case. The same results can be obtained also in an alternative way. In fact the presence of a finite correlation length implies that physics is insensitive to the total size of the lattice, as long as \( N \gg 1/aM \). We can then use a relatively small lattice to test the relationship between the sigma and Heisenberg models and replace the lattice length \( N \) in (19) with the correlation length. This implies that conclusions based on perturbation theory will be valid if \((s + 1)ag^4M > 1\).

What do we learn from this analysis about the validity of the Haldane map for general spin \( s \)? We started from the assumption that the lattice model (10) provides a valid lattice regularization for the O(3) NLSM (1). There is no guarantee for this but it is a reasonable hypothesis previously used by other authors. This means that at small energies, the continuum and lattice models have the same behaviour. We showed that, if the requirement (27) is satisfied, the low energy behaviour of the lattice NLSM can be described within a perturbative framework, carefully developed in the previous section up to second order. The perturbative results show that when this approach is valid the low energy sector of the lattice NLSM can be mapped onto the low energy sector of the spin-\( s \) Heisenberg model, where \( s \) is related to the coefficient of the topological term by (3). Looking at things from the opposite direction we can say that the low energy sector of the spin-\( s \) Heisenberg model is correctly described by the O(3) NLSM with \( \theta \) given by (3) and \( g \) that satisfies the condition (27). Since the sigma model is known to renormalize toward strong coupling it is not necessary that the inequality (27) is satisfied on the bare level. Instead it can be satisfied at least at some intermediate scale. In principle, it is then possible that the mapping will be valid independent of the value of the bare coupling constant. Unfortunately the renormalization group behaviour of \( g \) is not known exactly, but, from what is known, it looks likely that condition (27) will be satisfied at least if the bare value of \( g \) is not too small.
VI. MORE GENERAL SIGMA MODELS

Let us consider more general sigma models with a topological term. Space-time will still be 2-dimensional, but instead of the sphere $S^2$, we allow a more general target space $\Sigma$. In order to preserve the symmetric character of the O(3) model, we assume that $\Sigma$ is a homogeneous space for some compact Lie group $G$ (i.e., it can be expressed as a coset space $G/H$). If we did not make this restriction, then the theory would have infinitely many coupling constants. We can also assume that $G$ is simple; if it were semisimple then we would essentially have a sum of two simpler theories. The case of $G$ Abelian is qualitatively different, so we don’t consider that here either.

A classical configuration of the model is a map $\phi : \mathbb{R}^2 \rightarrow \Sigma$ which converges to the same limit at $\infty$ in all directions. Because of this boundary condition, $\phi$ is effectively a map from the sphere $S^2$ to $\Sigma$. Such maps are classified topologically by the second homotopy group $\pi_2(\Sigma)$.

In cases where $\pi_2(\Sigma)$ is a torsion group, such as $\mathbb{Z}_2$, it is possible to define a topological term for the partition function, but this cannot be written as a well-defined term in the action.

We are interested in adding to the action a topological term which can be written as a local integral. This is given by a 2-form $\omega$ on $\Sigma$ which we integrate over the image of $S^2$. “Topological” means that this must be closed, $d\omega = 0$. Without loss of generality, we can assume that $\omega$ is $G$-invariant. This (with the assumption that $G$ is simple) implies that $\Sigma$ is a coadjoint orbit of $G$ (at least up to coverings).

The coadjoint space $g^*$ is just the dual vector space to the Lie algebra $g$ of $G$. (Because $G$ is simple, $g^*$ and $g$ are really the same for all intents and purposes.) There is a natural representation of $G$ on this space, and a coadjoint orbit is (by definition) the orbit of some point of $g^*$ under the $G$ action. The form $\omega$ can be constructed canonically from the Lie algebra structure of $g$.

The action for the more general sigma model is,

$$ S_\theta = \frac{1}{2g^2} \int d^2x \left( \partial_{\mu}\phi \right)^2 + i \frac{\theta}{2\pi} \int \phi^* \omega $$ \hspace{1cm} (28)

where $\phi(x) \in \Sigma \subset g^*$ is a vector-valued function restricted to the given coadjoint orbit and $\phi^* \omega$ is the pull-back of $\omega$ to the space-time plane.
If \( \Sigma \) is chosen to be an integral coadjoint orbit (see below) then the last integral in (28) is always a multiple of \( 2\pi \), and so changing \( \theta \) by \( 2\pi \) does not change the partition function. In the case of \( S^2 \), we take \( \omega \) as \( 1/2 \) the volume form so that \( \int_{S^2} \omega = 2\pi \).

Aside from the simple case of \( S^2 \), this class of models includes the \( U(2N)/U(N) \times U(N) \) sigma models (with topological term) \([4, 16]\) that have been proposed for modeling the transition between integer quantum Hall plateaus \([8]\).

Because \( g^* \) is a vector space, the first part of the action can be regularized on a lattice exactly as in Eq. (5),

\[
S_0 = \frac{1}{g^2} \int d\tau \sum_{j=1}^{N} \left\{ \frac{a}{2} (\partial_\tau \phi(j))^2 - \frac{1}{a} \phi(j) \cdot \phi(j+1) \right\}.
\]

The derivation of the lattice version of the topological term also proceeds very much as above. The lattice divides the plane into strips. The topological integral can be written as a sum over the strips. The contribution from a strip is the difference \( A(j + 1) - A(j) \) of terms from the two lattice points bounding the strip. Again, if we include all the strips, then there is a massive cancellation and we are left with nothing interesting. So, we again approximate the integral by summing over half of the strips and multiplying by 2.

This gives an alternating sum over the lattice sites. Because of the boundary condition, \( \phi(j) \) draws a closed curve on \( \Sigma \); \( A(j) \) is defined (actually only defined modulo \( 2\pi \)) by integrating \( \omega \) over any disc which spans this curve. We can rewrite this as a line integral,

\[
A(j) = \int \frac{d\phi(j)}{d\tau} \cdot A \, d\tau
\]

using some 1-form \( A \) such that \( dA = \omega \) (in a suitable neighborhood).

As in the derivation for \( S^2 \), we redefine the sign of every other \( \phi(j) \), Wick rotate, construct the Hamiltonian, and quantize. The contribution of the topological term becomes a \( U(1) \) potential \( \frac{\theta}{\pi} A \) on \( \Sigma \). The Hilbert space for a site is thus the space of square-integrable sections of a line bundle with curvature \( \frac{\theta}{\pi} \omega \). This line bundle and connection are \( G \)-equivariant. The Hamiltonian is,

\[
H = \sum_j \left\{ \frac{g^2}{2a} J^2(j) + \frac{1}{2ag^2} \phi(j) \cdot \phi(j+1) \right\}
\]

where \( J^2 \) is the quadratic Casimir operator for the \( G \)-action.

There is a correspondence between irreducible representations of \( G \) and coadjoint orbits. This is given by the “orbit method” of constructing representations \([29]\). Irreducible representations are classified by positive integral weights. The space of weights can be identified
with a subspace of $g^*$. Any coadjoint orbit is the orbit of a unique positive weight $\Lambda \in g^*$; it is called integral if $\Lambda$ is an integral weight (and thus corresponds to a representation).

To construct the representation, we first construct a line bundle with curvature $\omega$ over the coadjoint orbit. The Hilbert space $(\Lambda)$ is the space of holomorphic sections of this line bundle. This can also be characterized as the subspace of sections with the lowest eigenvalue of the Laplacian — or equivalently, the quadratic Casimir operator.

The Hilbert space $\mathcal{H}_{\text{site}}$ for one site in this model is the space of square integrable sections of a line bundle with curvature $\frac{\theta}{\pi}\omega$. This is precisely the line bundle used in constructing the representation $(\frac{\theta}{\pi}\Lambda)$ by the orbit method. This only exists if $\frac{\theta}{\pi}\Lambda$ is integral. If $\Lambda$ is chosen carefully, then this means that $\theta$ is a multiple of $\pi$.

We can decompose $\mathcal{H}_{\text{site}}$ into irreducible representations, including $(\frac{\theta}{\pi}\Lambda)$ which is the lowest eigenspace of $J^2$. The logic is the same as for the O(3) model. The lowest eigenspace $\mathcal{H}_0$ of the kinetic operator $K$ is a tensor product with a factor of $(\frac{\theta}{\pi}\Lambda)$ for each site. The gap between the two lowest eigenvalues depends linearly on $\theta$.

If $g$ is sufficiently large, then we can identify a subspace $\mathcal{H}_{\text{low}} \subset \mathcal{H}$ of low-energy eigenstates which is deformed from $\mathcal{H}_0$. Perturbation theory gives an effective Hamiltonian which is at first order the obvious generalization of the Heisenberg Hamiltonian $H$. The SU(2)-representation $(s)$ is simply replaced by the $G$-representation $(\frac{\theta}{\pi}\Lambda)$, and the angular momentum operators are replaced with the $g$-generators.

**VII. DISCUSSION**

In this paper we considered in some detail a lattice regularization of the O(3) non-linear sigma model with topological term $i\theta T$ in order to get some more insight on the validity of the Haldane map for general values of the spin. The lattice NLSM can be written as a sum of a kinetic and potential term (cf. Eq. (10)) and the latter can be treated as a perturbation for $g$ sufficiently large. When the model is quantized, $\theta$ is restricted to values $\theta = 2\pi s$, with $s$ integer or half-integer. The Hilbert space associated to the highly degenerate ground state of the unperturbed theory is a tensor product of spin-$s$ representations of SU(2) at each lattice site, i.e. exactly the Hilbert space of the Heisenberg model. We carefully constructed perturbation theory up to second order and identified its limits of validity. It turns out that, for the lowest energy levels, the condition for the applicability of the perturbative
approach is given by (27). If this condition is satisfied, one can show that the low energy sector of the lattice sigma model can be mapped onto the low energy sector of the spin-s Heisenberg model with effective coupling constant $J \approx 1/g^2(s + 1)^2a$. Thus the condition of applicability of perturbation theory also provides a sufficient condition for the validity of the Haldane mapping for general $s$. Due to renormalization this inequality only needs to be satisfied at some intermediate scale. The condition (27) depends on $s$ and shows explicitly that the mapping becomes more accurate (and valid on a wider range of energies) as $s$ increases.

We also extended the analysis to sigma models with a topological term and a target space that is a homogeneous space of some compact Lie group $G$. Again in this case, under some conditions, it is possible to introduce a suitable lattice regularization, and a mapping can be established between the low energy sector of these more general lattice sigma models and the low energy sector of generalized “Heisenberg” models where SU(2) representations are replaced by $G$-representations.

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APPENDIX A: PERTURBATION DETAILS

We now present the details of the computation of the expression (24) for the second order correction $h_2$ to the effective Hamiltonian.

The second order term is $h_2 = -\Pi_0V\Pi_0$. We can decompose the Hilbert space $\mathcal{H}$ into pieces based on the total spin at each site. $\Pi_0$ projects to $\mathcal{H}_0$, where every site has spin $s$. Multiplying a spin-$s$ state by the coordinate operators $n_\alpha(j)$ gives superpositions of spin $s$ and spin $s + 1$ states at $j$. Applying $n(j) \cdot n(j + 1)$ to a vector in $\mathcal{H}_0$ gives a superposition of a vector in $\mathcal{H}_0$, vectors with spin $s + 1$ just at $j$ or $j + 1$, and a vector with spin $s + 1$ at
and $j + 1$. The parametrix $P$ vanishes on the first component, has eigenvalue $\frac{1}{s+1}$ on the second two, and eigenvalue $\frac{1}{2(s+1)}$ on the last.

* A priori, 

$$h_2 = -\sum_{i,j=1}^{N} \Pi_0 \mathbf{n}(i) \cdot \mathbf{n}(i+1) P \mathbf{n}(j) \cdot \mathbf{n}(j+1) \Pi_0,$$

but the summands vanish unless $\{i, i+1\}$ overlaps with $\{j, j+1\}$. This naturally decomposes as

$$h_2 = \sum_{j} \left( h_{j}^{\text{triple}} + h_{j}^{\text{pair}} \right)$$

where $h_{j}^{\text{triple}}$ is the terms involving all three sites $j - 1$, $j$, and $j + 1$, and $h_{j}^{\text{pair}}$ is the terms involving the two sites $j$ and $j + 1$.

In $h_{j}^{\text{triple}}$, $P$ acts as 0 for spin-$s$ at $j$, and as $\frac{1}{s+1}$ for spin $s+1$ at $j$. We can therefore replace $P$ with $\frac{1}{s+1}(1 - \Pi_0)$ in this term,

$$h_{j}^{\text{triple}} = -\Pi_0 [n^\alpha(j-1) n_\alpha(j) P n^\beta(j+1) n_\beta(j+1) P n^\alpha(j-1) n_\alpha(j)] \Pi_0$$

$$= -\frac{1}{(s+1)^3} S^\alpha \otimes \Pi_0 (n_\alpha[1-\Pi_0]n_\beta + n_\beta[1-\Pi_0]n_\alpha) \Pi_0 \otimes S^\beta. \tag{A1}$$

Some explicit computation gives the useful identity, $\Pi_0 n_\alpha n_\beta \Pi_0 = \frac{2}{(2s+1)(s+1)} S_{\alpha\beta} + \frac{1}{3} \delta_{\alpha\beta}$. Using this, the middle factor of (A1) becomes,

$$2\Pi_0 n_\alpha n_\beta \Pi_0 - \frac{1}{(s+1)^2} (S_\alpha S_\beta + S_\beta S_\alpha) = \frac{4}{(2s+1)(s+1)} S_{\alpha\beta} + \frac{2}{3} \delta_{\alpha\beta} - \frac{2}{(s+1)^2} S_{\alpha\beta} - \frac{2s}{3(s+1)} \delta_{\alpha\beta}$$

$$= \frac{2}{(2s+1)(s+1)} S_{\alpha\beta} + \frac{2}{3(s+1)} \delta_{\alpha\beta}.$$ 

This gives,

$$h_{j}^{\text{triple}} = \frac{-2}{(2s+1)(s+1)^3} S^\alpha \otimes S_{\alpha\beta} \otimes S^\beta - \frac{2}{3(s+1)^4} S^\alpha \otimes 1 \otimes S_\alpha,$$

which are the last two terms of Eq. (24).

The other part,

$$h_{j}^{\text{pair}} = \Pi_0 n^\alpha(j) n_\alpha(j+1) P n^\beta(j) n_\beta(j+1) \Pi_0$$

is more complicated. There are only 4 combinations of spins that can occur in the middle of this, therefore we can substitute

$$(s + 1) P \rightarrow \frac{1}{2}(1 \otimes 1) + \frac{1}{2}(\Pi_0 \otimes 1 + 1 \otimes \Pi_0) - \frac{3}{2} \Pi_0 \otimes \Pi_0$$
where \( \Pi_0 \) now denotes the projection onto \((s)\) at one site. 

\[
\left( \frac{2}{(2s+1)(s+1)} S^{\alpha \beta} + \frac{1}{3} \delta^{\alpha \beta} \right) \otimes \left( \frac{2}{(2s+1)(s+1)} S_{\alpha \beta} + \frac{1}{3} \delta_{\alpha \beta} \right) = \frac{4}{(2s+1)^2(s+1)^2} S^{\alpha \beta} \otimes S_{\alpha \beta} + \frac{1}{3}.
\]

\( \Pi_0 \otimes 1 \) gives

\[
\frac{1}{(s+1)^2} S^{\alpha \beta} \otimes \left( \frac{2}{(2s+1)(s+1)} S_{\alpha \beta} + \frac{1}{3} \delta_{\alpha \beta} \right) = \left( \frac{1}{(s+1)^2} S^{\alpha \beta} + \frac{s}{3(s+1)} \delta^{\alpha \beta} \right) \otimes \left( \frac{2}{(2s+1)(s+1)} S_{\alpha \beta} + \frac{1}{3} \delta_{\alpha \beta} \right)
= \frac{2}{(2s+1)(s+1)^2} S^{\alpha \beta} \otimes S_{\alpha \beta} + \frac{s}{3(s+1)}.
\]

\( 1 \otimes \Pi_0 \) gives the same thing. \( \Pi_0 \otimes \Pi_0 \) gives

\[
\frac{1}{(s+1)^2} S^{\alpha \beta} \otimes S_{\alpha \beta} = \frac{1}{(s+1)^2} \left( S^{\alpha \beta} + \frac{s}{3(s+1)} \delta^{\alpha \beta} \right) \otimes \left( S_{\alpha \beta} + \frac{s}{3(s+1)} \delta_{\alpha \beta} \right)
= \frac{1}{(s+1)^2} S^{\alpha \beta} \otimes S_{\alpha \beta} - \frac{1}{2(s+1)^2} S^{\alpha} \otimes S_{\alpha} + \frac{s^2}{3(s+1)^2}.
\]

This adds up to

\[
h_{\text{pair}} = -\frac{8s + 5}{2(2s+1)^2(s+1)^2} S^{\alpha \beta} \otimes S_{\alpha \beta} + \frac{1}{2(s+1)^5} S^{\alpha} \otimes S_{\alpha} - \frac{4s + 1}{6(s+1)^3}
\]

which are the first 3 terms of (24).

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