Dynamics of the scalar field in 5-dimensional Kaluza-Klein theory

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Abstract

Using the language of differential forms, the Kaluza-Klein theory in 4+1 dimensions is derived. This theory unifies electromagnetic and gravitational interactions in four dimensions when the extra space dimension is compactified. Without any ad-hoc assumptions about the five-dimensional metric, the theory also contains a scalar field coupled to the other fields. By a conformal transformation the theory is transformed from the Jordan frame to the Einstein frame where the physical content is more manifest. Including a cosmological constant in the five-dimensional formulation, it is seen to result in an exponential potential for the scalar field in four dimensions. A similar potential is also found from the Casimir energy in the compact dimension. The resulting scalar field dynamics mimics realistic models recently proposed for cosmological quintessence.

1 Introduction

A unified formulation of Einstein’s theory of gravitation and Maxwell’s theory of electromagnetism in four-dimensional spacetime was first proposed by Kaluza[1] by assuming a pure gravitational theory in a five-dimensional spacetime. The metric components were to be independent of the fifth coordinate[2]. This so-called ‘cylinder condition’ was a few years later explained by Klein by invoking arguments from the newly established quantum mechanics when the extra dimension was compactified on a circle $S^1$ with a microscopic radius[3]. In the following years it was studied by a large number of authors and also extended to higher dimensions in order to incorporate non-Abelian gauge theories[4]. In the last couple of years it has seen a rebirth as the mathematical implementation of the exciting possibility that our Universe can have extra dimensions which in principle can be of macroscopic size[5][6][7][8][9].

The field content of the original Kaluza-Klein theory is given by the five-dimensional metric $\bar{g}_{MN}$ where the indices $M,N = 0,1,2,3,4$. Denoting the indices of our four-dimensional spacetime by Greek indices, its metric is given by the components $\bar{g}_{\mu\nu}$, while the electromagnetic vector potential is given by the components $\bar{g}_{\mu4} = \bar{g}_{4\mu}$. The remaining spatial component $\bar{g}_{44}$ was for no good reason set equal to a constant by Kaluza[11] and kept that way by the subsequent authors. This assumption was apparently first abandoned by Jordan[10] and then later by Thiry[11] who showed that $\bar{g}_{44}$ corresponds to a scalar field in our spacetime. At that time there was no need or place for such a field, but today we realize that it is a generic feature of almost any extension of Einstein’s theory of gravity. From the experimental point of view it can be related to the physics behind
the dark energy \[12\] in the Universe in the form of a cosmological constant \[13\] or variable quintessence \[14\].

The main motivation behind the present contribution is to investigate some of the physical properties and manifestations of such a scalar field. In order to make the presentation more accessible, we present the reduction of the five-dimensional theory down to our four-dimensional spacetime in more detail than is generally available in the literature. We will take advantage of the simplifications which follow from using differential forms as was first done by Thiry \[11\].

After this reduction, we find that the scalar field couples directly to the gravitational field. To get this part on the standard form, we perform a conformal transformation so that the gravitational part is described by the usual Einstein-Hilbert action. In this frame the scalar field couples only to the Maxwell field. It is shown that this form of the full action can can be obtained by performing a conformal transformation on the five-dimensional metric. The classical equations of motion for the different fields are then derived and discussed.

If there is a cosmological constant in the five-dimensional spacetime, we show that it gives rise to an exponential potential for the scalar field. Such potentials are phenomenologically very interesting in cosmology where they can drive the observed acceleration of the Universe \[14\]. A potential of the same form also follows from the Casimir energy in the compactified, extra dimension. Including both of these effects, we have an effective potential which is the sum of two exponential terms. Adjusting the parameters in this potential, one can then obtain a realistic cosmology of an accelerating Universe \[15\] \[16\].

## 2 Reduction to four dimensions

In the five-dimensional spacetime\(^1\) we have the line element \(d\bar{s}^2 = \bar{g}_{MN}dx^Mdx^N\) where the coordinates are \(x^M = (x^\mu, x^4 \equiv y)\). It is convenient to define \(\bar{g}_{44} = h\) and \(\bar{g}_{4\mu} = hA_\mu\) where the fields \(h(x)\) and \(A_\mu(x)\) at this stage depend on all five coordinates. Our four-dimensional spacetime is orthogonal to the basis vector \(\bar{e}_4\) in the extra direction. It is therefore spanned by the four basis vectors

\[
\bar{e}_\mu \perp = \bar{e}_\mu - \bar{e}_\mu \parallel = \bar{e}_\mu - \frac{\bar{e}_\mu \cdot \bar{e}_4}{\bar{e}_4 \cdot \bar{e}_4} \bar{e}_4 = \bar{e}_\mu - \frac{\bar{g}_{4\mu}}{\bar{g}_{44}} \bar{e}_4
\]

and is endowed with the metric

\[
g_{\mu\nu} = \bar{e}_\mu \perp \cdot \bar{e}_\nu \perp = \bar{g}_{\mu\nu} - \frac{\bar{g}_{4\mu} \bar{g}_{4\nu}}{\bar{g}_{44}} = \bar{g}_{\mu\nu} - hA_\mu A_\nu
\]

\(^1\)The flat metric in this spacetime has the diagonal components \(\eta_{MN} = (-1, 1, 1, 1, 1)\).
In this way we then have the following splitting of the five-dimensional metric
\[ \bar{g}_{MN} = \left[ \begin{array}{c|c} g_{\mu\nu} + h A_{\mu} A_{\nu} \hline h A_{\mu} \end{array} \right] \]
and correspondingly for the contravariant components
\[ \bar{g}^{MN} = \left[ \begin{array}{c} g^{\mu\nu} \hline -A_{\mu} \end{array} \right] - A_{\mu} h^{-1} \]
which satisfy \( \bar{g}_{MN} \bar{g}^{NP} = \delta^P_M \). The line element in this spacetime thus takes the form
\[ ds^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu + 2\bar{g}_{\mu 4} dx^\mu dx^4 + \bar{g}_{44} dx^4 dx^4 = g_{\mu\nu} dx^\mu dx^\nu + h (dy + A_{\mu} dx^\mu)^2 \]
The appearance of the gauge potential 1-form in the last term is characteristic for Kaluza-Klein theories also with additional extra dimensions\cite{4}. At this stage we impose the cylinder condition \( \bar{g}_{MN,4} = 0 \) which means that the fields \( g_{\mu\nu}(x), A_{\mu}(x) \) and \( h(x) \) are independent of the fifth coordinate \( y \).

### 2.1 Vierbeins and connection forms

In the four-dimensional spacetime we introduce the vierbeins \( V^\mu_{\nu} \) and their inverse \( V^\nu_{\mu} \) satisfying \( V^\mu_{\lambda} V^\lambda_{\nu} = \delta^\mu_{\nu} \) and \( V^\mu_{\nu} V^\nu_{\lambda} = \delta^\mu_{\lambda} \). The metric is thus \( g_{\mu\nu} = V^\alpha_{\mu} V^\beta_{\nu} \eta^\alpha_{\beta} \) where \( \eta^\alpha_{\beta} \) is the four-dimensional Minkowski metric in this spacetime. Using these vierbeins we find the orthonormal basis 1-forms as
\[ \omega^{\hat{\mu}} = V^\mu_{\nu} dx^\nu \]
In the fifth direction we see from the line element (5) that the basis 1-form is
\[ \omega^{\hat{4}} = \sqrt{h} (dy + A_{\mu} dx^\mu) \]
Now using standard methods\cite{17}, we can calculate the connection 1-forms and the curvature 2-forms needed to find the Riemann and Ricci tensors which enter the Einstein-Hilbert action. As an example, we find
\[ d\omega^{\hat{4}} = \frac{1}{2} h^{-1/2} h_{,\mu} dx^\mu \wedge (dy + A_{\mu} dx^\mu) + h^{1/2} A_{\mu,\rho} dx^\rho \wedge dx^\mu \]
since \( d^2 y = 0 \). The result can be written in the form \( d\omega^{\hat{4}} = -\Xi^{\hat{4}}_{\hat{\mu}} \wedge \omega^{\hat{\mu}} \) where \( \Xi^{\hat{4}}_{\hat{\mu}} \) is the corresponding five-dimensional connection 1-form. By using the antisymmetry of \( dx^\rho \wedge dx^\mu \) to introduce the field strength \( F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \), we find
\[ \Xi^{\hat{4}}_{\hat{\mu}} = \frac{1}{2} \sqrt{h} F_{\mu\nu} \omega^{\hat{\nu}} + \frac{1}{2h} h_{,\hat{\mu}} \omega^{\hat{4}} \]
where \( h_{ij} = h_{ij}^\nu V^\nu_{\mu} \). Using the antisymmetry in the two orthonormal indices, we then also have the value for \( \bar{\Omega}^{\hat{\mu}}_{\hat{4}} \) which we will need in the following.

The exterior derivative of the remaining basis forms is similarly given in five dimensions as

\[
d\omega^{\hat{\mu}} = -\bar{\Omega}^{\hat{\mu}}_{\hat{\rho}} \wedge \omega^{\hat{\rho}} - \bar{\Omega}^{\hat{\mu}}_{\hat{\lambda}} \wedge \omega^{\hat{\lambda}}
\]

while taken in the four-dimensional spacetime it would just be \( d\omega^{\hat{\mu}} = -\Omega^{\hat{\mu}}_{\hat{\rho}} \wedge \omega^{\hat{\rho}} \) where \( \Omega^{\hat{\mu}}_{\hat{\rho}} \) now is the the connection 1-form in four dimensions. We thus have the relationship

\[
\bar{\Omega}^{\hat{\mu}}_{\hat{\rho}} = \Omega^{\hat{\mu}}_{\hat{\rho}} + \frac{1}{2} \sqrt{\frac{h}{h}} F^{\hat{\mu}}_{\hat{\rho}} \omega^{\hat{\lambda}}
\]

between the connections in these two spacetimes.

### 2.2 Curvature forms and tensors

The curvature 2-forms are defined by the structure equation

\[
R^{\hat{\mu}}_{\hat{\nu}} = d\Omega^{\hat{\mu}}_{\hat{\nu}} + \Omega^{\hat{\mu}}_{\hat{\lambda}} \wedge \Omega^{\hat{\lambda}}_{\hat{\nu}} \equiv \frac{1}{2} R^{\hat{\mu}}_{\hat{\nu} \hat{\rho} \hat{\sigma}} \omega^{\hat{\rho}} \wedge \omega^{\hat{\sigma}}
\]

in the four-dimensional spacetime and correspondingly in five dimensions. Their components \( R^{\hat{\mu}}_{\hat{\nu} \hat{\rho} \hat{\sigma}} \) form the Riemann curvature tensor in the orthonormal frame we are working in. In the same frame the corresponding expansion of the connection forms is

\[
\Omega^{\hat{\mu}}_{\hat{\nu}} = \Omega^{\hat{\mu}}_{\hat{\nu} \sigma} dx^\sigma
\]

where \( \Omega^{\hat{\mu}}_{\hat{\nu} \sigma} \) are the connection coefficients. These will enter the calculation together with partial derivatives of the field strengths \( F^{\hat{\mu}}_{\hat{\nu}} \) to give the covariant derivative

\[
F^{\hat{\mu}}_{\hat{\nu} ; \sigma} = F^{\hat{\mu}}_{\hat{\nu} \sigma} + \Omega^{\hat{\mu}}_{\hat{\rho} \sigma} F^{\hat{\rho}}_{\hat{\nu}} - \Omega^{\hat{\rho}}_{\hat{\nu} \sigma} F^{\hat{\mu}}_{\hat{\rho}}
\]

and similarly for the other components.

We calculate first the curvature form \( \bar{R}^{\hat{\mu}}_{\hat{4}} \) from

\[
\bar{R}^{\hat{\mu}}_{\hat{4}} = d\bar{\Omega}^{\hat{\mu}}_{\hat{4}} + \bar{\Omega}^{\hat{\mu}}_{\hat{\nu}} \wedge \bar{\Omega}^{\hat{\nu}}_{\hat{4}}
\]

Along the direction \( \omega^{\hat{\rho}} \wedge \omega^{\hat{\sigma}} \) we then find the curvature tensor components

\[
\bar{R}^{\hat{\mu}}_{\hat{4} \hat{\rho} \hat{\sigma}} = \frac{1}{2} \sqrt{\frac{h}{h}} F^{\hat{\mu}}_{\hat{\rho} \sigma} \frac{\hat{\sigma}}{\hat{\mu}} + \frac{1}{4} \sqrt{\frac{h}{h}} \left( 2 h^{\hat{\mu}}_{\hat{\rho} \hat{\sigma}} F^{\hat{\rho}}_{\sigma} + h^{\hat{\mu}}_{\hat{\rho} \sigma} F^{\hat{\rho}}_{\sigma} - h^{\hat{\mu}}_{\hat{\sigma} \rho} F^{\hat{\rho}}_{\hat{\sigma}} \right)
\]

and similarly

\[
\bar{R}^{\hat{\mu}}_{\hat{4} \hat{\rho} \hat{\lambda}} = \frac{1}{4} F^{\hat{\mu} \hat{\sigma}} F^{\hat{\sigma}}_{\hat{\rho} \hat{\lambda}} - \frac{1}{2} h^{i \hat{\mu}}_{\hat{\rho} \hat{\sigma}} i_{\hat{\rho}} \hat{\sigma} + \frac{1}{4} h^{i \hat{\mu}}_{\hat{\rho} \hat{\sigma}} h^{i \hat{\rho} \hat{\sigma}}
\]
in the direction of $\omega^\phi \wedge \omega^\hat{\lambda}$. These are all the components of the Riemann tensor involving
the fifth index, since the Riemann tensor is antisymmetric in the first two and in the last
two indices. We have here introduced the notation $F_{\tilde{\sigma}\tilde{\rho}\tilde{\mu}\tilde{\nu}} = F_{\tilde{\sigma}\tilde{\rho}\tilde{\mu}} V^\nu_{\tilde{\tilde{\nu}}}$ for the covariant
derivative of the field tensor. We have also simplified the result using the Bianchi identity
$F_{\tilde{\sigma}\tilde{\rho}\tilde{\mu}} + F_{\tilde{\rho}\tilde{\mu}\tilde{\sigma}} + F_{\tilde{\mu}\tilde{\sigma}\tilde{\rho}} = 0$.

The remaining components of the curvature tensor follow now from the 2-form

\[ \bar{\Omega}^\mu_{\tilde{\nu}\tilde{\sigma}} = d\bar{\Omega}^\mu_{\tilde{\nu}\tilde{\sigma}} + \bar{\Omega}^\mu_{\tilde{\nu}\tilde{\rho}} \wedge \bar{\Omega}^\rho_{\tilde{\tilde{\sigma}}} \]

which can be evaluated along the same lines. It gives

\[ \bar{\Omega}^\mu_{\tilde{\nu}\tilde{\sigma}} = \bar{R}^\mu_{\tilde{\nu}\tilde{\sigma}} + \frac{1}{4} h \left( 2 F^\mu_{\tilde{\nu}} F_{\tilde{\rho}\tilde{\sigma}} - F^\mu_{\tilde{\rho}} F_{\tilde{\sigma}\tilde{\nu}} - F^\mu_{\tilde{\sigma}} F_{\tilde{\nu}\tilde{\rho}} \right) \]

which has the correct antisymmetry in the first and last two indices. We also notice that
the symmetry $\bar{R}^\mu_{\tilde{\nu}\tilde{\rho}\tilde{\sigma}} = \bar{R}^\mu_{\tilde{\rho}\tilde{\sigma}\tilde{\nu}}$ is satisfied, as well as $\bar{R}^\mu_{\tilde{\nu}\tilde{\rho}\tilde{\sigma}} = 0$.

The Ricci curvature tensor is defined in four dimensions to be $R^\nu_{\tilde{\nu}\tilde{\sigma}} = \bar{R}^\nu_{\tilde{\nu}\tilde{\rho}}$ and similarly
in five dimensions. It is symmetric in its two indices. We find its components to be

\[ R^\nu_{\tilde{\nu}\tilde{\sigma}} = R^\nu_{\tilde{\nu}\tilde{\sigma}} - \frac{1}{2} h F^\nu_{\tilde{\nu}} F^\mu_{\tilde{\sigma}} - \frac{1}{4} h \nabla^2 h + \frac{1}{2} h h_{\tilde{\nu}\tilde{\nu}} \]

For the scalar curvature $\bar{R} = \bar{R}^\mu_{\tilde{\nu}}$ expressed in a coordinate basis we then have

\[ \bar{R} = R - \frac{1}{4} h F^\mu\nu F_{\mu\nu} - \frac{1}{2} h \nabla^2 h + \frac{1}{2 h} (\nabla h)^2 \]

in agreement with Thiry [11]. Here we have introduced the $\nabla$-operator for the covariant
derivative and $\nabla^2 = \nabla^\mu \nabla_{\mu}$ is the four-dimensional d’Alembertian operator. Needless to
say, it is the appearance of the Maxwell Lagrangian in this higher-dimensional curvature
first derived by Kaluza and Klein, that we still don’t understand the full significance of.

2.3 Einstein-Hilbert action and equations of motion

These geometrical considerations become physical when we postulate that gravitation in
the five-dimensional space is governed by the corresponding Einstein-Hilbert action

\[ S = \frac{1}{2} M^3 \int d^5 x \sqrt{-g} \bar{R} \]
where $\tilde{M}$ is the Planck mass in this space and $\tilde{R}$ is the Ricci curvature scalar \[^{21}\]. In principle there can also be an additional term here corresponding to a cosmological constant. Its implications will be considered in section 4.1. From the 4 + 1 split of the metric in \(^{22}\) we see that its determinant is simply $\tilde{g} = hg$ where $g$ is the determinant of the four-dimensional metric $g_{\mu\nu}$. Using this in the action \(^{22}\) and then integrating out the fifth coordinate, we find the action
\[
S = \frac{1}{2} M^2 \int d^4x \sqrt{-\tilde{g}\sqrt{h}} \left[ R - \frac{1}{4} h F^{\mu\nu} F_{\mu\nu} - \frac{1}{h} \nabla^2 h + \frac{1}{2h^2} (\nabla_\mu h)^2 \right] \tag{23}
\]
when the extra dimension is a microscopic circle of radius $a$ so that $M^2 = 2\pi a \tilde{M}^3$ becomes the ordinary, four-dimensional Planck constant. The two last terms form a total derivative,
\[
\sqrt{h} \left[ \frac{1}{h} \nabla^2 h - \frac{1}{2h^2} (\nabla_\mu h)^2 \right] = 2 \nabla^2 \sqrt{h} \tag{24}
\]
Assuming that $h$ disappears far away these terms can be neglected from the action. We are therefore left with the final result
\[
S = \frac{1}{2} M^2 \int d^4x \sqrt{-\tilde{g}\sqrt{h}} \left[ R - \frac{1}{4} h F^{\mu\nu} F_{\mu\nu} \right] \tag{25}
\]
which is the Kaluza-Klein action.

In the general case where the scalar field $h(x)$ varies with position, the effective gravitational constant given by the coefficient of the Ricci scalar in \(^{25}\), is no longer a constant, but varies with time and position in the four-dimensional spacetime. Electromagnetic interactions described by the Maxwell part, will similarly have a variable coupling strength. For this reason the theory seems to be in disagreement with present-day observations although there have been recent indications that the fine-structure constant may vary over cosmological time scales\[^{18}\]. Kaluza-Klein theory thus belongs to a wider class of fundamental theories characterized by the extension of Einstein’s tensor theory of gravity to include also the effect of scalar interactions. Such scalar-tensor theories of gravitation were constructed by Jordan\[^{19}\] and later shown by Brans and Dicke\[^{20}\] to be compatible with gravitational experiments and cosmological tests.

The classical equations of motion for the three fields can be derived from the action \(^{25}\). But it is simpler to use the five-dimensional action \(^{22}\) which gives rise to the equation of motion $\tilde{R}_{\mu\nu} = 0$.

We have already the components of the Ricci tensor in orthonormal basis \(^{18}\)^{20}. We now transform these to coordinate basis using the vierbeins $V^\mu_\hat{\mu}$, along with the rest of the fünfbein components $V^\mu_4 = 0$, $V^\lambda_\mu = \sqrt{h} A_\mu$, and $V^4_4 = \sqrt{h}$, which we read out from \(^{6}\).
and (7). We then find
\[ \bar{R}_{44} = \frac{1}{4} h^2 F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} \nabla^2 h + \frac{1}{4 h} (\nabla_\mu h)^2 \] (26)
\[ \bar{R}_{\mu\nu} = \frac{1}{2} h \nabla^\nu F_{\mu\nu} + \frac{3}{4} F_{\mu\nu} \nabla^\nu h + A_\mu \left[ \frac{1}{4} h^2 F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} \nabla^2 h + \frac{1}{4 h} (\nabla_\mu h)^2 \right] \] (27)
\[ \bar{R}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} h F^\sigma_\mu F_{\sigma\nu} - \frac{1}{2 h} \nabla_\mu \nabla_\nu h + \frac{1}{4 h^2} (\nabla_\mu h)(\nabla_\nu h) + A_\mu A_\nu \left[ \frac{1}{4} h^2 F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} \nabla^2 h + \frac{1}{4 h} (\nabla_\mu h)^2 \right] \] (28)
\[ + A_\nu \left[ \frac{1}{2} h \nabla^\sigma F_{\mu\sigma} + \frac{3}{4} F_{\mu\sigma} \nabla^\sigma h \right] + A_\mu \left[ \frac{1}{2} h \nabla^\sigma F_{\nu\sigma} + \frac{3}{4} F_{\nu\sigma} \nabla^\sigma h \right] \]
By equating these components of the Ricci scalar to zero we find the corresponding four-dimensional equations of motion. With the help of the identity (24), we find from (26) for the scalar field
\[ \nabla^2 \sqrt{h} = \frac{1}{4} h^{3/2} F^{\mu\nu} F_{\mu\nu} \] (29)
while (27) combined with (29) gives the equation of motion for the Maxwell field,
\[ \nabla^\mu F_{\mu\nu} = - \frac{3}{2 h} F_{\mu\nu} \nabla^\mu h \] (30)
Finally, for the gravitational field we find from (28) when using (29) and (30)
\[ R_{\mu\nu} = \frac{1}{2} h F^\sigma_\mu F_{\sigma\nu} + \frac{1}{\sqrt{h}} \nabla_\mu \nabla_\nu \sqrt{h} \] (31)
These equations can also be found in Wesson [21]. Introducing the Einstein curvature tensor
\[ E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}, \]
we see that the last equation can be written as
\[ E_{\mu\nu} = \frac{1}{2} \left( F^\sigma_\mu F_{\sigma\nu} - \frac{1}{4} g_{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} \right) + \frac{1}{\sqrt{h}} \left( \nabla_\mu \nabla_\nu \sqrt{h} - g_{\mu\nu} \nabla^2 \sqrt{h} \right) \] (32)
when we express the Ricci scalar \( R \) in terms of the Maxwell and scalar fields using (21) with \( \bar{R} = 0 \). In the first term we recognize the energy-momentum tensor of the electromagnetic field, while the last term must be the corresponding entity for the scalar field in this representation.

From the equation of motion (29) for the scalar field we see that it can take a constant value, which can be chosen to be \( h = 1 \), provided the accompanying gauge field satisfies the special condition \( F^{\mu\nu} F_{\mu\nu} = 0 \). The magnitudes of the electric and magnetic fields must therefore be the same everywhere. This rather strong and unnatural condition was imposed for many years in investigations of the Kaluza-Klein theory [4] since there didn’t seem to be a real physical need for a scalar field on the same footing as the electromagnetic and gravitational fields. Today, however, the situation is different. In fact, scalar fields are at the core of the Higgs mechanism in particle physics, cosmological inflation in the early universe and dark energy in the late universe. In the following we will therefore keep the scalar field non-constant and study some of its physical implications.
3 Conformal transformations to the Einstein frame

A basic assumption in Einstein’s general theory of relativity is that all observers are equipped with standard measuring rods and clocks. The properties of these rods and clocks are coded into the components of the metric tensor $g_{\mu\nu}$ and has the consequence that the gravitational action is just given by the volume integral of the Ricci curvature scalar. This is obviously the case for the underlying, five-dimensional theory described by (22). But in the resulting, four-dimensional theory (25) we see that this term is multiplied by the scalar field $\sqrt{h}$ which is generally not constant. This corresponds to using non-standard measuring rods and clocks. We can now adjust these by changing the metric at every point by a Weyl transformation

$$g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu}$$

and choosing the scale factor $\Omega(x)$ appropriately. In a $D$-dimensional spacetime this results in the corresponding change

$$R_{\mu\nu} \rightarrow R_{\mu\nu} - \Omega^{-1} \nabla^2 \Omega \cdot g_{\mu\nu} - (D - 2)\Omega^{-1} \nabla_\mu \nabla_\nu \Omega - (D - 3)\Omega^{-2} (\nabla_\rho \Omega)^2 g_{\mu\nu} + 2(D - 2)\Omega^{-2} \nabla_\mu \Omega \nabla_\nu \Omega$$

(34)

of the Ricci tensor. This gives a change in the Ricci scalar

$$R \rightarrow \Omega^{-2} R - 2(D - 1)\Omega^{-3} \nabla^2 \Omega - (D - 1)(D - 4)\Omega^{-4} (\nabla_\mu \Omega)^2$$

(35)

as showed in [22].

3.1 Weyl transformations

When we are in $D = 4$ dimensions $\sqrt{-g} \rightarrow \Omega^4 \sqrt{-g}$ while the Ricci scalar changes according to (35). The first term in (25) thus changes as

$$\int d^4x \sqrt{-g}\sqrt{h}R \rightarrow \int d^4x \sqrt{-g}\Omega^4 \sqrt{h} \Omega^{-2} R + \ldots$$

(36)

For the coefficient of $R$ to take the canonical value we must therefore choose $\Omega = h^{-1/4}$. Including the Maxwell term in (25) and the second term of the Weyl transformation (35) we thus find the transformed Kaluza-Klein action

$$S = \frac{1}{2} M^2 \int d^4 x \sqrt{-g} \left[ R - \frac{1}{4} h^2 F_{\mu\nu} F^{\mu\nu} + \frac{3}{4} \frac{1}{h} \nabla^2 h - \frac{15}{8} \frac{1}{h^2} (\nabla_\mu h)^2 \right]$$

(37)

It can be simplified by combining the last two terms by a partial integration which results in

$$S = \frac{1}{2} M^2 \int d^4 x \sqrt{-g} \left[ R - \frac{1}{4} h^2 F_{\mu\nu} F^{\mu\nu} - \frac{3}{8} \frac{1}{h^2} (\nabla_\mu h)^2 \right]$$

(38)
After this Weyl transformation we are now in the Einstein frame. By construction the gravitational part in the first term of the action has now the canonical form. The last term describes a massless scalar field which is coupled to the electromagnetic field in the second term. This is the physical content of the Einstein frame theory. For similar transformations between the Jordan frame and the Einstein frame for dilatonic brane-worlds see [23].

We could have achieved the same result by performing the Weyl transformation (33) directly on the five-dimensional metric appearing in (22). Since we now have $\sqrt{-\bar{g}} \rightarrow \Omega^5 \sqrt{-g}$, we find that the transformation needed is $\Omega = h^{-1/6}$. Again using (35) where now also the last term contributes and the gradients act in five dimensions, we find

$$\int d^5x \sqrt{-g} R \rightarrow \int d^4x \sqrt{-g} \left[ \bar{R} + \frac{1}{3} h' \nabla^2 h' - \frac{1}{h'^2} (\nabla_{\mu} h')^2 \right]$$

(39)

For later convenience we have here denoted the scalar field $h'$ instead of $h$. By using (21) for the five-dimensional scalar curvature, we find the transformed action integral

$$S = \frac{1}{2} M^2 \int d^4x \sqrt{-g} \left[ R - \frac{1}{4} h' F^{\mu\nu} F_{\mu\nu} + \frac{1}{3} \frac{1}{h'} \nabla^2 h' - \frac{1}{2} \frac{1}{h'^2} (\nabla_{\mu} h')^2 \right]$$

(40)

Again we can use a partial integration to combine the two last terms, as in (38). The final result for the Kaluza-Klein action after a five-dimensional Weyl transformation is then

$$S = \frac{1}{2} M^2 \int d^4x \sqrt{-g} \left[ R - \frac{1}{4} h' F^{\mu\nu} F_{\mu\nu} - \frac{1}{6} \frac{1}{h'^2} (\nabla_{\mu} h')^2 \right]$$

(41)

We see that there is full agreement between the four-dimensional Weyl transformation $g_{\mu\nu} \rightarrow h^{-1/2} g_{\mu\nu}$ and the five-dimensional counterpart $\bar{g}_{MN} \rightarrow h^{-1/3} \bar{g}_{MN}$. If we put $h' = h^{3/2}$ in (41) this equation is transformed into (38).

This is also easily understood directly from the five-dimensional metrical structure (3). After the four-dimensional Weyl transformation the five-dimensional metric becomes

\[
\begin{bmatrix}
\frac{h^{-1/2} g_{\mu\nu} + h A_{\mu} A_{\nu}}{h A_{\mu}} & h A_{\mu} \\
\frac{h A_{\mu}}{h}
\end{bmatrix} = h^{-1/2}
\begin{bmatrix}
\frac{g_{\mu\nu} + h^2 A_{\mu} A_{\nu}}{h^2 A_{\mu}} & h^2 A_{\mu} \\
\frac{h^2 A_{\mu}}{h^2}
\end{bmatrix}
\]

Introducing here $h' = h^{3/2}$ we then have the Weyl transformation of the five-dimensional metric used above.

### 3.2 Canonical fields

Although the gravitational part of the action now has the standard form, the kinetic energies of scalar and electromagnetic fields do not have their canonical forms. However, this is simple to achieve. In the action (41) we introduce

$$h' = e^{\sqrt{6} \phi/M}$$

(42)
where now $\phi(x)$ is a scalar field with canonical normalization. Similarly, we redefine the electromagnetic field by

$$A_\mu \rightarrow \sqrt{2}A_\mu / M$$

and the Kaluza-Klein action takes its final form

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} M^2 R - \frac{1}{4} e^{\sqrt{6}\phi/M} F_{\mu\nu} F_{\mu\nu} - \frac{1}{2} (\nabla_\mu \phi)^2 \right]$$

When the scalar field $\phi$ has values much less than the Planck mass $M$, both the electromagnetic field and the scalar field are seen to be free.

### 3.3 Equations of motion in Einstein frame

Now that we have performed the Weyl transformation $\bar{g}_{MN} \rightarrow h^{-\frac{1}{2}} \bar{g}_{MN}$ the equations of motion are no longer the same as the equations \((29), (30)\) and \((32)\) in the Jordan frame. We can find the new equations of motion from the transformed four-dimensional action \((41)\). Varying this action with respect to $h$ gives

$$\nabla^2 h = \frac{3}{4} h^2 F_{\mu\nu} F^{\mu\nu} + \frac{1}{h} (\nabla_\mu h)^2$$

Similarly the equation for $A_\mu$ is found to be

$$\nabla^\nu F_{\mu\nu} = -\frac{1}{h} F_{\mu\nu} \nabla^\nu h$$

while varying with respect to $g_{\mu\nu}$ gives us

$$E_{\mu\nu} = \frac{1}{2} \left[ h T^{\text{el.mag.}}_{\mu\nu} + \frac{1}{3h^2} T^{\text{scalar}}_{\mu\nu} \right]$$

Here $T^{\text{el.mag.}}_{\mu\nu} = F_{\rho\lambda} F^{\rho\lambda} - \frac{1}{4} F_{\rho\sigma} F^{\rho\sigma} g_{\mu\nu}$ is the ordinary electromagnetic energy-momentum tensor, while $T^{\text{scalar}}_{\mu\nu} = \nabla_\mu h \nabla_\nu h - \frac{1}{2} (\nabla_\sigma h)^2 g_{\mu\nu}$ is the energy-momentum tensor for a free scalar field. By performing the canonical transformations \((42)\) and \((43)\) we then obtain the equations of motion in canonical normalization:

$$\nabla^2 \phi = \sqrt{\frac{3}{8M^2}} e^{\sqrt{6}\phi/M} F_{\mu\nu} F^{\mu\nu}$$

$$\nabla^\nu F_{\mu\nu} = -\frac{\sqrt{6}}{M} F_{\mu\nu} \nabla^\nu \phi$$

$$E_{\mu\nu} = \frac{1}{M^2} \left[ e^{\sqrt{6}\phi/M} T^{\text{el.mag.}}_{\mu\nu} + T^{\text{scalar}}_{\mu\nu} \right]$$

We can also get the equations \((45-47)\) by varying the five-dimensional action after the Weyl transformation. This gives $\bar{R}'_{MN} = 0$, where $\bar{R}'_{MN}$ is the transformed Ricci tensor.
which we find from (34). In our case \( \Omega^2 = h^{-\frac{1}{3}} \) and we get the following expression for the transformed Ricci tensor:

\[
\bar{R}'_{MN} = \bar{R}_{MN} - \frac{1}{4h^2} (\bar{\nabla}_{P} h)^2 \bar{g}_{MN} - \frac{5}{12h^2} \bar{\nabla}_M h \bar{\nabla}_N h + \frac{1}{2h} \bar{\nabla}_M \bar{\nabla}_N h + \frac{1}{6h} \bar{\nabla}^2 h \bar{g}_{MN}
\]  

(51)

Here we have barred the covariant derivative to remind the reader that it is the covariant derivative with respect to the five-dimensional metric. The relationship between the d’Alembertian operator in four and five dimensions is \( \bar{\nabla}^2 h = \nabla^2 h + \frac{1}{2k} (\nabla h)^2 \). By computing a couple of Christoffel symbols we find

\[
\bar{\nabla}_A \bar{\nabla}_A h = \frac{1}{2} (\nabla \mu h)^2
\]

(52)

\[
\bar{\nabla}_A \bar{\nabla}_\nu h = \frac{1}{2} (\nabla \mu h)^2 A_\nu - \frac{1}{2} h \nabla \mu h F_{\nu}^\mu
\]

(53)

\[
\bar{\nabla}_\mu \bar{\nabla}_\nu h = \nabla_\mu \nabla_\nu h + \frac{1}{2} (\nabla \sigma h)^2 A_\mu A_\nu - \frac{1}{2} h \nabla \rho h (A_\mu F_{\nu \rho} + A_\nu F_{\mu \rho})
\]

(54)

When we now equate the various components of \( \bar{R}'_{MN} \) to zero and use the expressions (26-28) for the untransformed Ricci tensor, we end up with equations (45), (46) and (47).

To get the last equation we must also use the fact that the transformed Ricci scalar \( \bar{R}' = 0 \).

### 4 Potential energy for the scalar field

So far the scalar field is massless and will therefore modify the gravitational interactions over cosmological distances. This is surely unwanted and is easily avoided by slightly enlarging the theory. By including a cosmological constant in five dimensions we will see that the scalar field develops a potential and thus also a non-zero mass. Alternatively, we will see that the Casimir vacuum energy induced by the presence of the compact, fifth dimension also generates a similar potential.

#### 4.1 Cosmological constant in five dimensions

With a cosmological constant \( \bar{\Lambda} \) in the original, five-dimensional theory, the fundamental action (22) is replaced by

\[
S = \frac{1}{2} M^3 \int d^5 x \sqrt{-\bar{g}} \left( \bar{R} - 2\bar{\Lambda} \right)
\]

(55)

Going through the same compactification as before, followed by the Weyl transformation in five dimensions, it immediately follows that

\[
S = \frac{1}{2} M^2 \int d^4 x \sqrt{-g} \left[ R - \frac{1}{4} h' F^{\mu \nu} F_{\mu \nu} - \frac{1}{6h'^{2}} (\nabla \mu h')^2 - 2\bar{\Lambda} h'^{-\frac{1}{3}} \right]
\]

(56)
Introducing here the canonically normalized fields, it takes the more informative form

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} M^2 R - \frac{1}{4} e^{\sqrt{3} \phi/M} F_{\mu\nu} F_{\mu\nu} - \frac{1}{2} (\nabla \phi)^2 - M^2 \bar{\Lambda} e^{\sqrt{3} \phi/M} \right]$$  \hspace{1cm} (57)

The five-dimensional cosmological constant is thus seen to correspond to an exponential potential in four dimensions. Its absolute sign is directly set by the sign of $\Lambda$.

Such an exponential potential for a scalar field addition to Einstein’s tensor theory was first considered by Wetterich [24]. It has been much studied since then in connection with models for quintessence [14]. Its cosmological evolution is completely characterized by the coefficient of $\phi$ in the exponent.

### 4.2 Casimir energy from the compact dimension

The cosmological constant is a contribution to the vacuum energy from physics on short scales, i.e. scales shorter than the size

$$L = \int_0^{2\pi a} dy \sqrt{h} = 2\pi a e^{\sqrt{3} \phi/M}$$  \hspace{1cm} (58)

of the compact dimension. But including quantum effects, the Casimir energy will contribute to the vacuum energy at the scale $L$ due to the confinement of the massless field quanta in a space with a compact, fifth dimension. The corresponding momentum is therefore quantized with the values $p = (2\pi/L)n$ where $n = 0, \pm 1, \pm 2, \ldots, \pm \infty$. In (58) we have chosen to calculate $L$ after the four-dimensional Weyl transformation, which means that the correspondence between $\phi(x)$ and $h(x)$ is given by $h = (e^{\sqrt{3} \phi/M})^{2/3} = e^{2\sqrt{2/3} \phi/M}$. The calculation of the Casimir energy is done in lowest order perturbation theory where the field quanta represent small oscillation around the ground state $\bar{g}_{MN} = \eta_{MN}$ which follows from the classical equations of motion.

Considering first the contribution from one such massless mode, it gives rise to the Casimir energy

$$E_0 = \frac{V}{2} \int \frac{d^3k}{(2\pi)^3} \sum_{n=-\infty}^{\infty} \sqrt{k^2 + (2\pi n/L)^2}$$  \hspace{1cm} (59)

where $V$ is a finite 3-volume. Using dimensional regularization, we do the momentum integral in $d$ dimensions where we have the general formula

$$\int \frac{1}{(2\pi)^3} \frac{1}{(k^2 + m^2)^N} = (4\pi)^{-d/2} \frac{\Gamma(N - d/2)}{\Gamma(N)} \frac{1}{(m^2)^{d/2 - N}}$$  \hspace{1cm} (60)

In our case $N = -1/2$ and we find

$$E_0 = \frac{V}{2} \sum_{n=-\infty}^{\infty} (4\pi)^{-d/2} \frac{\Gamma\left(-\frac{d+1}{2}\right)}{\Gamma\left(-\frac{1}{2}\right)} \left(\frac{2\pi n}{L}\right)^{d+1}$$
The sum over the compact quantum number $n$ is still divergent. It is made finite with zeta-function regularization which gives

$$\sum_{n=-\infty}^{\infty} n^{d+1} = 2\zeta(-d - 1)$$ (61)

Taking now $d \to 3$, we have

$$E_0 = -\frac{V}{L^4}\pi^2\Gamma(-2)\zeta(-4)$$ (62)

Although $\Gamma(-2)$ is infinite and $\zeta(-4)$ is zero, their product is finite as follows from the reflection formula

$$\Gamma\left(\frac{z}{2}\right)\pi^{-z/2}\zeta(z) = \Gamma\left(\frac{1-z}{2}\right)\pi^{-(1-z)/2}\zeta(1-z)$$ (63)

for zeta-functions. We put $z = -4$ in this formula and use it to rewrite $E_0$. The corresponding vacuum energy density $\mathcal{E}_0 = E_0/VL$ in five dimensions is then

$$\mathcal{E}_0 = -\frac{3\zeta(5)}{4\pi^2L^5}$$ (64)

This is in agreement with the result of Applequist and Chodos\cite{25} obtained by a more indirect approach. It also follows from the calculations of Ambjørn and Wolfram\cite{26} who studied Casimir energies in different geometries. More recently the corresponding Casimir energy has been calculated by Albrecht, Burgess, Ravndal and Skordis\cite{27} in a similar Kaluza-Klein theory in six dimensions of which two are compact in order to derive a corresponding scalar potential. Elizalde et. al. have calculated the corresponding Casimir energy for an anti-de Sitter background\cite{28}.

In five dimensions the graviton has five physical degrees of freedom. The total Casimir energy is thus the above calculated result (64) times five. With the length $L$ expressed by the scalar field as in (58), we put this energy into the five-dimensional action integral and obtain the potential

$$V(\phi) = 5\mathcal{E}_0 (2\pi a) \sqrt{h} \left(h^{-\frac{1}{2}}\right)^4 = -\frac{15\zeta(5)}{(2\pi)^6 a^4} e^{-6\sqrt{\frac{\phi}{M}}}$$ (65)

in the Einstein frame. This potential term enters the Kaluza-Klein action instead of the last term in (57). It has the same exponential form as the contribution from the cosmological constant, but the exponent is six times as large.

In general the potential will get contributions both from the small-scale cosmological constant and from the Casimir energy. It will thus have the form

$$V(\phi) = A e^{-\alpha\phi} + B e^{-\beta\phi}$$ (66)

where the exponents are fixed but the coefficients are unknown. We only know that at least one of the prefactors $A$ or $B$ must be negative in this particular theory.
Including also higher order quantum corrections to the Casimir energy, the simple exponential result will be modified. In the six-dimensional theory of Albrecht, Burgess, Ravn-dal and Skordis\cite{27} the radiative corrections add up to a polynomial in the field $\phi$. This modification can be important when such potentials are used in models for cosmological quintessence\cite{29}.

5 Conclusions

The original, five-dimensional theory of Kaluza and Klein is the simplest example of the more elaborate theories used today to describe physics with extra dimensions. In addition to unifying the electromagnetic and gravitational fields, it also contains a scalar field which codes the size of the single extra dimension here. Similar theories with additional compact dimensions will contain a corresponding scalar field which is usually called the radion\cite{6}. This represents an extension of Einstein’s tensor theory of gravity and can have important, cosmological consequences. In this connection the radion field appears as quintessence which can give rise to acceleration of the Universe at late times. This depends on the field dynamics which is governed by its effective potential which appears in the Einstein frame.

Here it is pointed out that this effective potential will in the lowest order approximation used here be a sum of two exponential terms. One is resulting from the small-scale cosmological constant in the higher-dimensional spacetime while the other is induced as a Casimir energy due to one or more compact dimensions. It has been shown by others that potentials of such a form allow for a consistent description of the evolution of the Universe since radiation domination until today when the dark energy dominates and gives acceleration\cite{15}\cite{16}. It would be even more satisfactory if also the inflationary mechanism could be explained by similar physics from extra dimensions.

References

[1] T. Kaluza, *Sitzungsber. Preuss. Akad. Wiss. Phys. Mat. Klasse*, 966 (1921).

[2] G. Nordström, *Phys. Zeit.* **15**, 504 (1914), had earlier used the same idea to formulate an in four dimensions unified theory of electromagnetism and his scalar theory of gravitation, based on Maxwell’s theory in a five-dimensional spacetime.

[3] O. Klein, *Zeit. f. Phys.* **37**, 895 (1926), *Nature* **118**, 516 (1926).

[4] T. Appelquist, A. Chodos and P.G.O. Freund, *Modern Kaluza-Klein Theories*, Addison-Wesley Publishing Company, Menlo Park, California (1987).

[5] I. Antoniadis, *Phys. Lett.* **B246**, 377 (1990);
[6] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, \textit{Phys. Lett.} \textbf{B429}, 263 (1998); \textit{Phys. Rev.} \textbf{D59}, 086004 (1999).

[7] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, \textit{Phys. Lett.} \textbf{B436}, 257 (1998);

[8] T. Han, J. D. Lykken and R. Zhang, \textit{Phys. Rev.} \textbf{D59}, 105006 (1999).

[9] G.F. Giudice, R. Rattazzi and J.D. Wells, \textit{Nucl. Phys.} \textbf{B544}, 3 (1999).

[10] P. Jordan, \textit{Ann. d. Physik} \textbf{1}, 219 (1947).

[11] Y. Thiry, \textit{Comptes Rendus} \textbf{226}, 216 (1948).

[12] P.J.E. Peebles and B. Ratra, \textit{astro-ph/0207347}.

[13] S. Weinberg, \textit{astro-ph/0005265}.

[14] For a recent summary, see M. Doran and C. Wetterich, \textit{astro-ph/0205267}.

[15] T. Barreiro, E.J. Copeland and N.J. Nunes, \textit{Phys. Rev.} \textbf{D61}, 127301 (1999).

[16] A.S. Majumdar, \textit{Phys. Rev.} \textbf{D64}, 083503 (2001).

[17] C.W. Misner, K.S. Thorne and J.A. Wheeler, \textit{Gravitation}, W.H. Freeman and Company, New York (1973).

[18] J.K. Webb et al., \textit{Phys. Rev. Lett.} \textbf{87}, 091301 (2001).

[19] P. Jordan, Nature \textbf{164}, 637 (1949); \textit{Schwerkraft und Weltall}, Friedr. Vieweg & Sohn, Braunschweig (1952).

[20] C. Brans and R.H. Dicke, \textit{Phys. Rev.} \textbf{124}, 925 (1961).

[21] P.S. Wesson, \textit{Space, time, matter: Modern Kaluza-Klein theory}, World Scientific, Singapore (1999).

[22] S.W. Hawking and G.F.R. Ellis, \textit{The large scale structure of space-time}, Cambridge University Press, Cambridge (1973).

[23] S. Nojiri, O. Obregon, S.D. Odintsov and V.I. Tkach \textit{Phys. Rev.} \textbf{D64}, 043505 (2001).

[24] C. Wetterich, \textit{Nucl. Phys.} \textbf{B302}, 668 (1988).

[25] T. Appelquist and A. Chodos, \textit{Phys. Rev.} \textbf{D28}, 772 (1983).

[26] J. Ambjørn and S. Wolfram, \textit{Ann. Phys. (NY)} \textbf{147}, 33 (1983).

[27] A. Albrecht, C.P. Burgess, F. Ravndal and C. Skordis, \textit{Phys. Rev.} \textbf{D65}, 123506, 123507 (2002).

[28] E. Elizalde, S. Nojiri, S.D. Odintsov and S. Ogushi, \textit{hep-th/0209242}.

[29] A. Albrecht and C. Skordis, \textit{Phys. Rev. Lett.} \textbf{84}, 2076 (2000).