Dissipative and generative fractional electric elements in modeling \textit{RC} and \textit{RL} circuits

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Abstract Two types of constitutive equations consisting of instantaneous and power type hereditary contributions are proposed in order to model generalized capacitor (inductor). The first one, that expresses charge (magnetic flux) in terms of voltage (current) memory, proved to describe a dissipative electric element, while the second one, that expresses voltage (current) in terms of charge (magnetic flux) memory, proved to describe a generative electric element. These constitutive models are used in transient and steady-state regime analysis of the series \textit{RC} and \textit{RL} circuits subject to electromotive force, as well as in the study of circuits' frequency characteristics including their asymptotic behavior.

Keywords Dissipative and generative electric elements · Memory type fractional constitutive models · Fractional \textit{RC} and \textit{RL} circuits · Transient and steady-state regimes · Frequency and asymptotic analysis

1 Introduction

Constitutive equations corresponding to capacitor and inductor, as basic elements of electric circuits, are generalized in order to include memory effects, so that the physical quantities characteristic for the element are connected through the combination of instantaneous and hereditary terms, either by expressing the total charge \( q \) (total magnetic flux \( \phi \)) in terms of history of generalized capacitor’s voltage \( u_C \) (inductor’s current \( i_L \)) as

\[
q(t) = C u_C(t) + C_\alpha 0^{1-\alpha} u_C(t),
\]
\[
\phi(t) = L i_L(t) + L_\beta 0^{1-\beta} i_L(t),
\]

where the hereditariness is assumed as the power function decreasing in time, \( t > 0 \), and thus modeled by the Riemann–Liouville fractional integral of order \( 1 - \alpha, 1 - \beta \in [0, 1] \), defined by

\[
0^\xi \mathbf{f}(t) = \frac{t^{\xi-1}}{\Gamma(\xi)} \ast f(t) = \frac{1}{\Gamma(\xi)} \int_0^t \frac{f(t')}{(t-t')^{1-\xi}} \mathrm{d}t',
\]

with \( \ast \) denoting the convolution, or by expressing the voltage (current) on generalized capacitor (inductor) in terms of charge (flux) history as
where \( \mu, \nu \in [0, 1] \) and \( C[F], C_\alpha[F^{1-\alpha}], \) and \( C_\mu[F^{\mu}] \) are classical and fractional capacitances, while \( L[H], L_\beta[F^{1-\beta}], \) and \( L_\nu[H s^\nu] \) are classical and fractional inductances. Therefore, as the physical phenomenon is due to the different effects present in the material, as customary, the constitutive models are assumed as a superposition of terms accounting for these effects. If the generalized electric element displays instantaneous and memory effects, they may be constitutively modeled by the sum of classical constitutive model, accounting for the instantaneous effects, and a term containing the integral of physical quantity, accounting for the memory effects. The classical approach in modeling hereditary phenomena would include short-tail memory through the exponential function as the memory kernel, while in the approach used in postulating constitutive models (1) - (4), the long-tail memory is modeled through the power type hereditary kernel.

Although charge–voltage and voltage–charge constitutive relations (1) and (3), so as flux–current and current–flux constitutive relations (2) and (4), have exactly the same mathematical form, they describe different type of electric elements: the constitutive equations (1) and (2) describe the passive element, i.e., the element that dissipates electric energy, while the constitutive equations (3) and (4) describe the active element, i.e., the element that generates electric energy, as proved in Section 2.

The history dependence between the physical quantities described by the charge–voltage and flux–current constitutive equations (1) and (2) is emphasized by solving them with respect to the voltage and current, respectively, as

\[
\begin{align*}
  u_C(t) &= \frac{1}{C} q(t) + \frac{1}{C_\mu} \alpha_1^{\mu} q(t), \\
  i_L(t) &= \frac{1}{L} \phi(t) + \frac{1}{L_v} \alpha_1^{v} \phi(t),
\end{align*}
\]  

(3)\hspace{1cm}(4)

where \( \alpha_1 \) being the time derivative of one-parameter Mittag-Leffler function, defined as

\[
e_{\xi,\lambda}(t) = E_{\xi}(\lambda t^\xi),
\]

thus expressing them in the form analogous to the constitutive relations (3) and (4), having the memory kernel changed from the power to the Mittag-Leffler type. Similarly, by solving the voltage–charge and current–flux constitutive relations (3) and (4) with respect to charge and flux, respectively, as

\[
\begin{align*}
  q(t) &= C u_C(t) + C \dot{e}_{\mu, \lambda}(t) * u_C(t), \\
  \phi(t) &= L i_L(t) + L \dot{e}_{\nu, \lambda}(t) * i_L(t),
\end{align*}
\]  

(7)\hspace{1cm}(8)

one expresses the charge (flux) in terms of the voltage (current) history, as in constitutive relations (1) and (2) having again the memory kernel changed.

Moreover, the constitutive equations (1) and (7), as the equivalent form of (3), can be topologically viewed as the parallel connections of the classical and generalized capacitor, while the constitutive equations (3) and (5), as the equivalent form of (1), can be topologically viewed as the series connections of the classical and generalized inductor and similarly the series connections of the classical and generalized capacitor and similarly the series connections of the classical and generalized inductor are described by the relations (2) and (8), as the equivalent form of (4), while the parallel connections of the classical and generalized inductor are described by the relations (4) and (6), as the equivalent form of (2). Therefore, different topological generalizations of electric elements can still describe the same phenomenology of physical processes by choosing appropriate memory kernels.

Further, classical \( RC \) and \( RL \) circuits subject to the electromotive force are generalized by considering the capacitor and inductor displaying hereditary effects modeled by the previously mentioned memory type constitutive equations of fractional order rather than by the classical ones that are local in time. Such generalized circuits are analyzed by the means of Laplace transform method in the transient regime for a given electromotive force and especially for the electromotive force assumed as the Heaviside step function and as a harmonic function. The assumption of electromotive force in the form of the harmonic function enabled the comparison of the transient and steady-state regimes as well as the analysis of frequency characteristics. The charge–voltage and flux–current constitutive models
(1) and (2), describing the passive capacitor and inductor, are also used in the transient and steady-state regime analysis of the forced series fractional RLC circuit in [14].

Constitutive equations describing behavior of electric elements using fractional calculus found their application in modeling supercapacitors, ultracapacitors, and electrochemical double-layer capacitors (EDLC), that are used as energy storing elements in various construction devices. Models of supercapacitor and ultracapacitor range from the linear constitutive equations obtained by combining resistors and fractional capacitors as in [7, 26, 27] to nonlinear models like the one proposed in [9]. Moreover, fractional models of capacitor having the differentiation orders exceeding the first order are considered in [19], along with their behavior as circuit elements. Fractional order elements find their application in the study of complex electric networks as well, see [34, 43]. The review of supercapacitor’s fractional order models involving their applications can be found in [3]. Supercapacitors are also investigated analytically and experimentally at high frequencies in [2], while in [20] different models of fractional capacitors are presented and tested experimentally. Electric circuits containing fractional order elements are used to model the electrolyte processes in electrochemical double-layer capacitors, as demonstrated through the frequency analysis in [18], as well as by the analysis in the time domain in [23, 28]. Moreover, the presence of memory effects in electric double-layer capacitors is proved experimentally in [4]. The fractional order elements can be produced with desired characteristics, as demonstrated in [21, 24]. Not only that the capacitor is generalized to include memory effects, but also different phenomena in inductor can be modeled using fractional calculus, as done in [22, 35]. Fractional order capacitance and inductance are also considered in [40]. Review of fractional order element’s characteristics, along with their different realizations and their application in modeling various phenomena is given in [36]. Constitutive models of electric elements may even be derived from modeling the interaction of the electromagnetic field with material, see [25, 39].

The fractional order equations governing transient regime of electric circuits displaying memory effects are obtained in [11, 13] by replacing the ordinary time derivatives with the fractional ones in the equations corresponding to the classical RLC and RC circuits. Using analytical tools and by considering the series connection of resistor and fractional capacitor as a generalized capacitor, the transient regime analysis of the series RC\(_\alpha\) circuit is performed in [15], while by considering the series connection of resistor and fractional inductor as a generalized inductor, the transient regime analysis of the parallel RL\(_\beta\)C\(_\alpha\) is performed in [16, 17]. Simple RC and an example of a more complex circuit containing fractional capacitor and inductor is considered in [5] for the transient regime using analytical approach, while in [6, 37, 38] numerical tools are used to solve governing equations of fractionally generalized electric circuits in the time domain.

Frequency characteristics of the fractional RC, RL, and LC circuits including parameter optimization of RL\(_\beta\)C\(_\alpha\) circuit are investigated in [31, 32] and in [30]. Wien bridge oscillators and resonance phenomena in fractional order circuits are considered in [8, 33] and in [29, 42].

After introductory remarks and after formulating two types of hereditary constitutive models for capacitor and inductor, thermodynamic considerations leading to the model classification are performed in Section 2, while derivation and solution of the equations governing the transient regime of forced series fractional circuits are presented in Section 3, along with the numerical examples illustrating the transient responses in cases of capacitor models expressing either charge in terms of voltage memory or voltage in terms of charge memory. Transient response of the forced series fractional RL circuit is proved to be governed by equations of the same form as in the case of RC circuit. Section 4 contains the steady-state regime analysis along with the comparison of solutions for transient and steady-state regimes, while the frequency characteristics of the transfer function moduli and arguments, along with their asymptotics, are studied in Section 5. Finally, concluding remarks are given in Section 6.

The main contributions can be summarized as follows. By the thermodynamic considerations in the steady-state regime, passive capacitor (inductor), i.e., element that dissipates energy, is proved to be modeled by expressing charge (magnetic flux) in terms of voltage (current) memory through the fractional integral, while the active capacitor (inductor), i.e., element that generates energy, is proved to be modeled by expressing voltage (current) in terms of charge (magnetic flux) memory, again using the fractional integral. Proposed constitutive models are used in formulating equations governing the behavior of fractional RC and RL cir-
circuits subject to electromotive force, that for the transient regime yield the impulse response either as a pos-
tive monotonically decreasing convex function in the case of passive capacitor, or as a damped oscillatory
function in the case of active capacitor. It is shown that the fractional RC circuit having the active capac-
itor can either dissipate or generate energy depending on frequency and model parameters. The steady-state
regime of the fractional RC circuit proved to originate from transient regime assuming electromotive force as
a harmonic function. The asymptotics of transfer function modulus for low frequencies proved to be a linear
function of log \( \omega \) having slope proportional to the fractional integration order and intercept proportional
to the fractional time constant, while the transfer function argument asymptotics for low frequencies proved to be
dependent on the fractional integration order as well.

2 Thermodynamic considerations regarding constitutive equations

In order to analyze dissipativity properties of generalized electric elements: passive capacitor and inductor,
modeled by the charge–voltage (1) and flux–current (2) constitutive equations, as well as active capacitor
and inductor, modeled by the voltage–charge (3) and current–flux (4) constitutive equations, the element in
steady-state regime is considered by assuming its voltage as the harmonic function

\[
u(t) = u_0 e^{j\omega t}
\]

of amplitude \( u_0 \) and angular frequency \( \omega \), implying that its current is a harmonic function of the same frequency
as voltage (9), but shifted by phase angle \( \phi_1 \) and of amplitude \( i_0 \), taking the form:

\[
i(t) = i_0 e^{j(\omega t+\phi_1)}.
\]

due to the linearity of constitutive equations.

Since the quantities having physical meaning in (9) and (10) are

\[ u = \text{Re} \, u \quad \text{and} \quad i = \text{Re} \, i, \]

respectively, the energy on generalized electric element during harmonic function’s period \( T \)

\[
W = \int_{nT}^{(n+1)T} u(t) i(t) \, dt
= u_0 i_0 \int_{nT}^{(n+1)T} \cos(\omega t) \cos(\omega t + \phi_1) \, dt
= \frac{1}{2} u_0 i_0 T \cos \phi_1
\]
is dissipated if \( \cos \phi_1 > 0 \) and generated if \( \cos \phi_1 < 0 \).

Therefore, the element dissipates energy, i.e., it is considered to be passive, if its constitutive equation
in the steady-state regime, with voltage and current assumed in the forms given by (9) and (10), yields
\( \cos \phi_1 > 0 \) for all frequencies \( \omega \), while the element generates energy, i.e., it is considered to be active, if its
constitutive equation in the steady-state regime, with voltage and current assumed in the forms given by (9)
and (10), yields \( \cos \phi_1 < 0 \) for all frequencies \( \omega \).

In the case of passive capacitor, since electric current is \( i(t) = \frac{d}{dt} q(t) \), the charge–voltage constitutive
equation (1) differentiated with respect to time yields

\[
i(t) = C \frac{d}{dt} u_C(t) + C_\alpha \, 0D^\beta_t u_C(t),
\]

where \( 0D^\beta_t \), \( \beta \in (0, 1) \), denotes the operator of Riemann–Liouville fractional differentiation, defined as

\[
0D^\beta_t f(t) = \frac{d}{dt} 0I_t^{1-\beta} f(t) = \frac{d}{dt} \left( \frac{t^{-\beta}}{\Gamma(1-\beta)} \ast f(t) \right) \text{ with } \xi \in (0, 1),
\]

so that, by plugging capacitor’s voltage and current, assumed as (9) and (10), into (11) and by using

\[
0D^\beta_t e^{j(\omega t+\phi)} = (j\omega)^\xi e^{j(\omega t+\phi)} = \omega^{\xi} e^{j(\omega t+\phi + \frac{\xi \pi}{2})} \text{ as } t \to \infty,
\]

see [10], one finds that

\[
\sin \phi_1 = \frac{u_0}{i_0} \left( C \omega + C_\alpha \omega^\alpha \sin \frac{\alpha \pi}{2} \right) > 0 \quad \text{and}
\cos \phi_1 = \frac{u_0}{i_0} C_\alpha \omega^\alpha \cos \frac{\alpha \pi}{2} > 0,
\]

while for the passive inductor the flux–current model (2) differentiated with respect to time yields

\[
u_L(t) = L \frac{d}{dt} i(t) + L_\beta \, 0D^\beta_t i(t),
\]

since \( u_L(t) = \frac{d}{dt} \phi(t) \), so that, by plugging inductor’s voltage and current, assumed as (9) and (10), into (13),
once obtains
\[
\sin \phi_i = -\frac{i_0}{u_0} \left( L \omega + L_\beta \omega^\beta \sin \frac{\beta \pi}{2} \right) < 0 \quad \text{and} \\
\cos \phi_i = \frac{i_0}{u_0} L_\beta \omega^\beta \cos \frac{\beta \pi}{2} > 0 .
\]

On the other hand, the voltage–charge constitutive equation (3) describing the active capacitor, since electric charge is \( q(t) = \int_0^t i(t') \, dt' = 0 l_t^1 i(t) \) provided that \( q(0) = 0 \), becomes
\[
u C(t) = \frac{1}{C} 0 l_t^1 i(t) + \frac{1}{C_\mu} 0 l_t^{1+\mu} i(t) , \tag{14}
\]
where the semi-group property for fractional integrals, i.e., \( 0 l_t^\xi 0 l_t^\xi = 0 l_t^{\xi+\xi} \), is used, so that, by plugging capacitor’s voltage and current, assumed as (9) and (10), into (14) and by employing
\[
0 l_t^\xi e^{j(\omega t+\phi)} = \frac{1}{(j\omega)^\xi} e^{j(\omega t+\phi)} = \frac{1}{\omega^\xi} e^{j(\omega t+\phi-\frac{\xi \pi}{2})} \quad \text{as} \quad t \to \infty , \tag{15}
\]
see [10], one finds that
\[
\sin \phi_i = \frac{i_0}{u_0} \left( \frac{1}{C \omega} + \frac{1}{C_\mu \omega^{1+\mu} \sin \frac{\mu \pi}{2}} \right) > 0 \quad \text{and} \\
\cos \phi_i = -\frac{i_0}{u_0} C_\mu \omega^{1+\mu} \sin \frac{\mu \pi}{2} < 0 ,
\]
while for the active inductor the current–flux model (4), rewritten as
\[
i(t) = \frac{1}{L} 0 l_t^1 u_L(t) + \frac{1}{L_\nu} 0 l_t^{1+\nu} u_L(t) , \tag{16}
\]
using the magnetic flux taken as \( \phi(t) = \int_0^t u_L(t') \, dt' = 0 l_t^1 u_L(t) \) provided that \( \phi(0) = 0 \), yields
\[
\sin \phi_i = -\frac{u_0}{i_0} \left( \frac{1}{L \omega} + \frac{1}{L_\nu \omega^{1+\nu} \cos \frac{\nu \pi}{2}} \right) < 0 \quad \text{and} \\
\cos \phi_i = -\frac{u_0}{i_0} \frac{1}{L_\nu \omega^{1+\nu} \sin \frac{\nu \pi}{2}} < 0 ,
\]
that is obtained by plugging inductor’s voltage and current, assumed as (9) and (10), into (16).

Therefore, both capacitor and inductor, modeled by (1) and (2), dissipate energy and thus they are considered as passive elements, since \( \cos \phi_i > 0 \), while both capacitor and inductor, modeled by (3) and (4), generate energy and thus they are considered as active elements, since \( \cos \phi_i < 0 \). On the other hand, the sign of \( \sin \phi_i \) indicates whether the element has capacitive or inductive character, since in the former case \( \sin \phi_i > 0 \) implies that current leads the voltage, while in the latter case \( \sin \phi_i < 0 \) implies that current lags the voltage.

Passive capacitor’s constitutive model (11) in the limiting cases of fractional differentiation order reduces to the classical models, so that if \( \alpha = 0 \) one obtains the model of classical capacitor connected in parallel with resistor of conductivity \( G \equiv C_\alpha [S] \) as
\[
i(t) = C \frac{d}{dt} u_C(t) + G u_C(t) , \tag{17}
\]
describing dissipative element as well, while if \( \alpha = 1 \) one has
\[
i(t) = C_\cl \frac{d}{dt} u_C(t) , \tag{18}
\]
describing the classical capacitor of capacitance \( C_\cl = C + C_\alpha [F] \), that neither dissipates nor generates energy. On the other hand, active capacitor’s constitutive model (3) in the limiting case of the fractional integration order \( \mu = 0 \) reduces to
\[
u C(t) = \frac{1}{C_\cl} q(t) , \tag{19}
\]
describing the classical capacitor of capacitance \( C_\cl = \left( \frac{1}{C} + \frac{1}{C_\mu} \right)^{-1} [F] \), while in the case \( \mu = 1 \) constitutive model (3) reduces to a hereditary model of integer order
\[
u C(t) = \frac{1}{C} q(t) + \frac{1}{C_\alpha} \int_0^t q(t') \, dt' , \tag{20}
\]
corresponding to the series connection of classical capacitor and hereditary type element. Moreover, the constitutive model (20), with charge rewritten as \( q(t) = \int_0^t \nu i(t') \, dt' \) if \( q(0) = 0 \), describes the capacitive type element that generates energy, since it yields
\[
\sin \phi_i = -\frac{i_0}{u_0} \frac{1}{C \omega} > 0 \quad \text{and} \\
\cos \phi_i = -\frac{i_0}{u_0} \frac{1}{C_\mu \omega^2} < 0
\]
in the steady-state regime, i.e., when voltage and current are assumed as harmonic functions (9) and (10).

Similarly, the voltage-current constitutive relation (13), modeling the passive inductor, in the limiting cases of fractional differentiation order \( \beta = 0 \) and \( \beta = 1 \) becomes, respectively,
\[
u L(t) = L \frac{d}{dt} i(t) + R i(t) \quad \text{and} \quad u_L(t) = L_\cl \frac{d}{dt} i(t) ,
\]
corresponding to the series connection of the classical inductor and resistor of resistance \( R \equiv L_\beta [\Omega] \) and to the classical inductor of inductance \( L_\cl = L + L_\beta [H] \),

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while the current–flux model (4) for the active inductor reduces to
\[ i_L(t) = \frac{1}{L_\text{cl}} \phi(t) \] and
\[ i_L(t) = \frac{1}{L} \phi(t) + \frac{1}{L_v} \int_0^t \phi(t') \, dt' \] (21)

in the limiting cases of fractional integration orders \( v = 0 \) and \( v = 1 \), respectively, modeling the classical inductor of inductance \( L_\text{cl} = \left( \frac{1}{L} + \frac{1}{L_v} \right)^{-1} \) [H] and inductive type energy generating element, that may be considered as the parallel connection of classical inductor and hereditary type element, since when voltage and current are assumed as harmonic functions (9) and (10), model (21) yields
\[
\sin \phi_i = -\frac{u_0}{i_0} \frac{1}{L \omega} < 0 \quad \text{and} \quad 
\cos \phi_i = -\frac{u_0}{i_0} \frac{1}{L_v \omega^2} < 0,
\]
where \( \phi(t) = \int_0^t u_L(t') \, dt' \) with \( \phi(0) = 0 \) is used.

3 Transient response of fractional RC and RL circuits

Constitutive equations for generalized capacitor and inductor are used to model corresponding elements in RC and RL circuits, so that ordinary fractional equations governing transient processes are obtained by employing the second Kirchhoff’s law in addition to the constitutive relations. Further, governing equations are solved and used in order to produce illustrative numerical examples.

3.1 Derivation of governing equations and their solutions

Transient response of RC circuit, subject to the electromotive force \( \mathcal{E} \) and consisting of resistor of resistance \( R \) connected in series with generalized capacitor, is governed by the second Kirchhoff’s law
\[ \mathcal{E}(t) = R i(t) + u_C(t), \] (22)
coupled either with the current-voltage model (11), corresponding to the passive capacitor described by the charge–voltage relation (1), or with the voltage-current relation (14), describing the active capacitor modeled by the voltage–charge relation (3). Expressing the system of equations (11) and (22) in terms of capacitor voltage \( u_C \) vs. electromotive force \( \mathcal{E} \), as well as system (14) and (22) in terms of current \( i \) vs. \( \mathcal{E} \), the equations governing RC circuits’ responses are
\[ \tau_C \frac{d}{dt} u_C(t) + \frac{1}{\tau_\alpha} D^\alpha_0 u_C(t) + u_C(t) = \mathcal{E}(t) \] and
\[ i(t) + \frac{1}{\tau_\alpha} D^\alpha_0 i(t) + \frac{1}{\tau_\mu} D^{1+\mu}_0 i(t) = \frac{1}{R} \mathcal{E}(t), \] (23)

with \( \tau_C = RC \) [s], \( \tau_\alpha = RC_\alpha \) [s\(^\alpha\)], and \( \tau_\mu = RC_{\mu} \) [s\(^{1+\mu}\)] being classical and fractional time-constants. The governing equation (23) found its application in modeling charge and discharge processes of supercapacitor, studied in [41].

The transfer function connecting capacitor voltage with electromotive force and corresponding to RC circuit containing the passive capacitor is obtained from the governing equation (23) in the form
\[ \hat{g}_C^{(1)}(s) = \frac{\hat{u}_C(s)}{\mathcal{E}(s)} = \frac{1}{\tau_C s + \tau_\alpha s^\alpha + 1}, \] (25)

while the transfer function relating current to electromotive force in the case of RC circuit containing the active capacitor is obtained from the governing equation (24) as
\[ \hat{g}_C^{(2)}(s) = \frac{i(s)}{\mathcal{E}(s)} = \frac{1}{R} \frac{s^{1+\mu}}{s^{1+\mu} + \frac{1}{\tau_C} + \frac{1}{\tau_\mu}} \]
\[ = \frac{1}{R} \left( 1 - \hat{g}_C^{(2)}(s) \right), \quad \text{with} \]
\[ \hat{g}_C^{(2)}(s) = \frac{\tau_\alpha s^\alpha + 1}{\tau_\mu s^{1+\mu} + \frac{\tau_\alpha}{\tau_C} s^{1+\mu} + 1}, \] (26)

by applying the Laplace transform, defined as
\[ \hat{f}(s) = \mathcal{L}[f(t)](s) = \int_0^\infty f(t) e^{-st} \, dt, \quad \text{for} \quad \text{Re} s > 0, \]

and using the Laplace transforms of Riemann–Liouville fractional derivative and integral
\[ \mathcal{L} \left[ D^\xi_0 f(t) \right](s) = s^\xi \hat{f}(s) - \left[ \alpha_{1-\xi}^\xi f(t) \right]_{t=0} = s^\xi \hat{f}(s) \] and
\[ \mathcal{L} \left[ D^\xi_0 f(t) \right](s) = \frac{1}{s^\xi} \hat{f}(s), \]
holding for $f$ bounded at zero, with the remaining

\[ \hat{g}_i^{(1)}(s) = \frac{\hat{g}(s)}{\hat{C}(s)} = \frac{1}{R} \left( 1 - \hat{g}_C^{(1)}(s) \right) \]

and

\[ \hat{g}_C^{(2)}(s) = \frac{\hat{\mu}_C(s)}{\hat{C}(s)} \] (28)

being obtained from the Laplace transform of the second Kirchhoff’s law (22).

The impulse responses $g_C^{(1)}$ and $g_C^{(2)}$, i.e., the capacitor voltages as consequences of electromotive force assumed as Dirac delta function, are obtained by inverting the Laplace transform in (25) and (27), see also (28)2, yielding

\[ g_C^{(1)}(t) = \frac{1}{\pi} \int_0^\infty \frac{\tau_\mu \rho^\mu \sin(\alpha \pi)}{|1 - \tau_\rho \rho^\mu + \tau_\rho \rho^\mu e^{\pi i}|^2} e^{-\rho t} d\rho \]

and

\[ g_C^{(2)}(t) = -\frac{1}{\pi} \int_0^\infty \frac{\tau_\mu \rho^{1+\mu} \sin(\mu \pi)}{|1 - \tau_\rho \rho^\mu (\rho - \frac{1}{\tau_\rho}) e^{\pi i}|^2} e^{-\rho t} d\rho + 2 \text{Re} \left( \frac{s_0^{1-\mu} \left( \frac{s_0^{1+\mu}}{\tau_\rho} + 1 \right)}{(1 + \mu) \tau_\mu s_0 + \mu \frac{s_0^{1+\mu}}{\tau_\rho}} \right) e^{-|\text{Re} s_0| t}. \] (29)

(30)

with $s_0$ being the pole of $g_C^{(2)}$, given by (27), lying in the upper left complex quarter plane and obtained as a solution of the equation

\[ \tau_\mu s^{1+\mu} + \frac{\tau_\mu}{\tau_C} s^\mu + 1 = 0, \]

as proved in Appendix A.1. The impulse responses $g_C^{(1)}$ and $g_C^{(2)}$, given by (29) and (30), are calculated by the definition of inverse Laplace transform. More precisely, the impulse response $g_C^{(1)}$, given by (29), is not calculated, since it is a solution of the ordinary fractional differential equation (23), that is well known as the composite fractional relaxation equation, see Eq. (4.1) in [12], where it is solved and analyzed for asymptotic behavior, while the impulse response $g_C^{(2)}$, given by (30), representing the solution kernel of the fractional integral equation (24), is calculated in Appendix A.2. Once the impulse responses $g_C^{(1)}$ and $g_C^{(2)}$ are known, the responses $g_i^{(1)}$ and $g_i^{(2)}$ are easily calculated by the inverse Laplace transform of (28) and (26), respectively, yielding

\[ g_i^{(1,2)}(t) = \frac{1}{R} \left( \delta(t) - g_C^{(1,2)}(t) \right), \] (31)

where $\delta$ is the Dirac delta function.

Although originating from the constitutive equations that connect physical quantities by taking into account the instantaneous contribution and power type hereditariness of the physical quantity, namely charge–voltage relation (1) and voltage–charge relation (3), the impulse responses $g_C^{(1)}$ and $g_C^{(2)}$, given by (29) and (30), have utterly different qualitative properties, that may be the consequence of the fact that the former constitutive equation describes the dissipative element contrary to the latter one describing the generative element. The impulse response $g_C^{(1)}$ is completely monotonic, i.e., positive, decreasing, convex function, due to positivity of the integrand multiplying the exponential function in (29), while the impulse response $g_C^{(2)}$ is an oscillatory function having an exponentially decreasing amplitude, due to the second term in (30), with the first term being negative, increasing, concave function, due to the complete monotonicity of the integral.

Voltage $u_C$ on the generalized capacitor and current $i$ running through $RC$ circuit in the transient regime are expressed thorough the convolution of impulse response and electromotive force as

\[ u_C(t) = g_C^{(1,2)}(t) * \hat{E}(t) \] and \[ i(t) = g_i^{(1,2)}(t) * \hat{E}(t), \] (32)

by inverting the Laplace transforms in (25) and (28)1 in the case of passive capacitor and by the Laplace transform inversion in (28)2 and (26) in the case of active capacitor, with corresponding impulse responses given by (29) and (30) for voltages and by (31) for currents.

Considering $RL$ circuit, that consists of resistor connected in series either with passive inductor, constitutively modeled by flux–current relation (2), or with the active inductor, described by the current–flux relation (4), the transient response to electromotive force is obtained as a solution of the second Kirchhoff’s law

\[ \hat{E}(t) = R i(t) + u_L(t). \] (33)

coupled either with the voltage-current model (13) describing the passive inductor, or with the current-voltage relation (16) modeling the active inductor.

Equations governing the transient regime in $RL$ circuit

\[ \tau_L \frac{d}{dt} i(t) + \tau_\beta \delta D_\beta t i(t) + i(t) = \frac{1}{R} \hat{E}(t) \] and (34)
\[ u_L(t) + \frac{1}{\tau_L} 0^1_t u_L(t) + \frac{1}{\tau_v} 0^{1+\nu}_t u_L(t) = E(t), \quad (35) \]

with \( \tau_L = \frac{L}{R} \) [s], \( \tau_\beta = \frac{L_\beta}{R} \) [s\(^{\beta}\)], and \( \tau_v = \frac{L_v}{R} \) [s\(^{1+\nu}\)] being classical and fractional time-constants, are, respectively, obtained by reducing the system of equations (13), (33) to a single equation expressed in terms of current \( i \) in the case of passive inductor and by solving the system of equations (16), (33) with respect to voltage \( u_L \) in the case of active inductor.

If \( RL \) circuit contains passive inductor, then, due to the analogous forms of governing equations (34) for \( RL \) to (23) for \( RC \) circuit containing passive capacitor, the impulse response of current corresponding to \( \alpha \) has the same form as \( g_{c}^{(1)} \), given by (29), while if \( RL \) circuit contains active inductor, then, due to the analogous forms of governing equations (35) for \( RL \) to (24) for \( RC \) circuit containing active capacitor, the impulse response of inductor voltage corresponding to \( \alpha \) has the same form as \( g_{i}^{(2)} \), given by (31). Also, current running through \( RL \) circuit and inductor voltage are expressed thorough convolutions of the form analogous to the convolutions for capacitor voltage \( u_C \) and current \( i \) running through \( RC \) circuit, given by (32), respectively.

### 3.2 Numerical examples

Time evolution of current in \( RC \) circuit containing the passive capacitor, modeled by the charge–voltage constitutive relation (1), as a response to the electromotive force taken as the Heaviside step function, i.e., as \( E(t) = E_0 \cdot H(t) \), with \( E_0 \) being the constant intensity of the electromotive force, is calculated by (32) and presented in Figure 1, along with the responses of the classical \( RC \) circuits, see (17) and (18), that are calculated by:

\[
\begin{align*}
i(t) &= \frac{E_0}{R} e^{-\frac{t}{R}} \quad \text{for } \mu = 0 \quad \text{and} \\
i(t) &= \frac{E_0}{R} \begin{cases} e^{-\frac{t}{\tau_0}} \left( \cosh \left( \frac{\lambda t}{2\tau_c} \right) - \frac{\sinh \left( \frac{\lambda t}{2\tau_c} \right)}{\lambda} \right), & \lambda = \sqrt{\left( \frac{1}{2\tau_c} \right)^2 - \frac{1}{\tau_0}}, \\
(1 - \sigma t) e^{-\sigma t}, & \sigma = \frac{1}{2\tau_c} + \sqrt{\left( \frac{1}{2\tau_c} \right)^2 - \frac{1}{\tau_\mu}}, \\
\end{cases} \quad \text{for } \mu = 1. \end{align*}
\]

The time profiles of current for fractional order \( RC \) circuit, that are monotonically decreasing functions of time, for both small and large time lie between the corresponding time profiles for \( RC \) circuits containing classical elements, as obvious from Figure 1. The decrease rate of responses for small time is the greatest in the case of classical \( RC \) circuit with \( \alpha = 0 \), while responses’ decrease rate for fractional \( RC \) circuits decreases as the fractional differentiation order \( \alpha \) increases and finally the decrease rate is smallest in the case of classical \( RC \) circuit with \( \alpha = 1 \), as obvious from Figure 1a. The situation is reversed for large time, as depicted in Figure 1b, since the integer order response for \( \alpha = 0 \) is almost constant, fractional order responses decrease as the power type function according to the asymptotics

\[
i(t) \sim \frac{E_0}{R} \begin{cases} \tau_0 \left( \frac{t}{\tau_0} \right)^{\frac{\lambda}{\lambda_0}}, & \text{if } \alpha \in (0, \frac{1}{2}), \\
\tau_\mu \left( \frac{t}{\tau_\mu} \right)^{\lambda}, & \text{if } \alpha \in \left[ \frac{1}{2}, 1 \right], \end{cases} \quad \text{as } t \rightarrow \infty, \]

see [12], while integer order response for \( \alpha = 1 \) decreases exponentially. Note that all responses corresponding to \( \alpha \in (0, 1) \) tend to zero for large time, except for the response corresponding to \( \alpha = 0 \) that tends to a constant: \( \lim_{t \rightarrow \infty} i(t) = \frac{E_0}{R} \tau_\mu + 1 \).

Current in \( RC \) circuit containing the active capacitor, modeled by voltage–charge constitutive relation (3), as a response to the electromotive force taken as the Heaviside step function is calculated by (32) and corresponding time profiles are shown in Figure 2, along with the responses of the classical and integer order hereditary \( RC \) circuits, see (19) and (20), that are calculated by:

\[
i(t) = \frac{E_0}{R} \frac{\tau_0}{\tau_\mu + 1} \quad \text{for } \alpha = 0 \quad \text{and} \\
i(t) = \frac{E_0}{R} \frac{\tau_\mu}{\tau_0 + 1} \quad \text{for } \alpha = 1. \]
Dissipative and generative fractional electric elements in modeling RC and RL circuits

Similarly as in the previous case, as depicted in Figure 2, the time profiles of current corresponding to the fractional order voltage–charge model (3) lie between the profiles corresponding to the classical and integer order hereditary RC circuits, obtained according to (36) and (37), respectively. Classical RC circuit, according to (36), has a monotonic exponentially decreasing response for any values of the model parameters, while in the case of integer order hereditary RC circuit model parameters determine whether the response is aperiodic or oscillatory, so that the response from Figure 2a, obtained by (37)2, is aperiodic but also non-monotonic, while the response from Figure 2b, obtained by (37)3, has damped oscillatory character. The response of fractional RC circuit displays the damped oscillatory behavior, see curves for \( \mu = 0.5 \) and \( \mu = 0.75 \) from Figure 2b, that can be attenuated to such an extent, so that only one minimum remains, see all curves from Figure 2a and curve for \( \mu = 0.25 \) from Figure 2b.

As evident from Figures 1 and 2, there is a perfect agreement between curves obtained through analytical expressions and those calculated by the fixed Talbot numerical Laplace inversion Mathematica function, developed by J. Abate and P. P. Valkó according to [1] and available at: http://library.wolfram.com/INFOCENTER/MathSource/4738/.

4 Steady-state response of fractional RC and RL circuits

The steady-state regime of forced series fractional RC circuit containing either passive capacitor modeled by charge–voltage relation (1) or active capacitor modeled by voltage–charge constitutive equation (3) is considered by assuming the harmonic electromotive force

\[ \mathcal{E}(t) = \mathcal{E}_0 e^{j\omega t} \]  

(38)

of amplitude \( \mathcal{E}_0 \) and angular frequency \( \omega \), yielding the capacitor voltage and current in the form of harmonic functions

\[ u_C(t) = u_{C0} e^{j(\omega t + \phi_C)} \quad \text{and} \quad i(t) = i_0 e^{j(\omega t + \phi_i)}, \]  

(39)

as well, having amplitudes \( u_{C0} \) and \( i_0 \) and phase angles \( \phi_C \) and \( \phi_i \), due to linearity of fractional RC circuits’ governing equations (23) and (24). Note, the quantities having physical meaning in (38) and (39) are \( \text{Re} \mathcal{E}, \text{Re} u_C \), and \( \text{Re} i \).

If time is large enough, the fractional RC circuits enter the steady-state regime, due to the prevalence of harmonic forcing over the impulse responses \( g_C^{(1)} \) and \( g_i^{(2)} \), given by (29) and (31), since they decay to zero according to the final value Tauberian theorem:

\[
\lim_{t \to \infty} g^{(1)}_C(t) = \lim_{s \to 0} \left( s_0^2 g^{(1)}_C(s) \right) = 0 \quad \text{and} \\
\lim_{t \to \infty} g^{(2)}_i(t) = \lim_{s \to 0} \left( s_0^2 g^{(2)}_i(s) \right) = 0,
\]

The response of fractional RC circuit displays the damped oscillatory behavior, see curves for \( \mu = 0.5 \) and \( \mu = 0.75 \) from Figure 2b, that can be attenuated to such an extent, so that only one minimum remains, see all curves from Figure 2a and curve for \( \mu = 0.25 \) from Figure 2b.
Fig. 2 Case of active capacitor—time profiles of current in RC circuit as a response to electromotive force assumed as Heaviside's step function of intensity $E_0 = 1$, obtained analytically (lines) and numerically (dots) according to the formula for fractional derivative of harmonic function (12), so that passive capacitor’s voltage amplitude and phase angle are:

\[ u^{(1)}_C = \frac{E_0}{\sqrt{\tau_C^2 \omega^2 + 2\tau_C \tau_\alpha \omega^{2+\alpha} \sin \frac{\alpha \pi}{2} + \tau_\alpha^2 \omega^{2+\alpha} + 2\tau_\alpha \omega^\mu \cos \frac{\alpha \pi}{2} + 1}}, \]  \hspace{1cm} (40)  

\[ \phi^{(1)}_C = -\arctan \frac{\tau_C \omega + \tau_\alpha \omega^\mu \sin \frac{\alpha \pi}{2}}{\tau_\alpha \omega^\mu \cos \frac{\alpha \pi}{2} + 1}, \]  \hspace{1cm} (41)  

while the equation (24) governing the response of RC circuit containing active capacitor, with the electromotive force and current assumed as (38) and (39)$_2$, yields

\[ \sin \phi^{(2)}_i = \frac{R i_0}{E_0} \left( \frac{1}{\tau_C \omega} + \frac{1}{\tau_\mu \omega^{1+\mu}} \cos \frac{\mu \pi}{2} \right), \]  

\[ \cos \phi^{(2)}_i = \frac{R i_0}{E_0} \left( 1 - \frac{1}{\tau_\mu \omega^{1+\mu}} \sin \frac{\mu \pi}{2} \right), \]  

that solved with respect to the current amplitude and phase angle gives

\[ i^{(2)}_0 = \frac{E_0}{R} \sqrt{1 - \frac{1}{\tau_C \omega^{2+\mu}} \sin \frac{\mu \pi}{2} + \frac{1}{\tau_C^2 \omega^2 + 2\tau_C \tau_\mu \omega^{2+\mu} \cos \frac{\mu \pi}{2} + 1}} \]  \hspace{1cm} (42)  

\[ \phi^{(2)}_i = \arctan \frac{\frac{1}{\tau_C \omega} + \frac{1}{\tau_\mu \omega^{1+\mu}} \cos \frac{\mu \pi}{2}}{1 - \frac{1}{\tau_\mu \omega^{1+\mu}} \sin \frac{\mu \pi}{2}}, \]  \hspace{1cm} (43)  

4.1 Derivation of amplitudes and phase angles of steady-state responses

Plugging the electromotive force (38) and capacitor voltage (39)$_1$ into equation (23) governing the response of RC circuit containing passive capacitor, one finds

\[ \sin \phi^{(1)}_C = -\frac{u^{(1)}_C}{E_0} \left( \tau_C \omega + \tau_\alpha \omega^\mu \sin \frac{\alpha \pi}{2} \right), \]  

\[ \cos \phi^{(1)}_C = \frac{u^{(1)}_C}{E_0} \left( \tau_\alpha \omega^\mu \cos \frac{\alpha \pi}{2} + 1 \right), \]  

with the harmonic forcing (38) according to (32), for large time become harmonic as well.
where the formula for fractional integral of harmonic function (15) is used.

In order to obtain current in the steady-state regime of $RC$ circuit containing passive capacitor in the form given by (39)1, one rewrites equation (11) as

$$i(t) = \frac{1}{R} \left( \tau_C \frac{d}{dt} u_C(t) + \tau_a \alpha D^\alpha u_C(t) \right),$$

that along with passive capacitor’s voltage (39)1 yields

$$\sin(\phi^{(1)}_i - \phi^{(1)}_C) = \frac{u^{(1)}_{C0}}{R t^{(1)}_0} \left( \tau_C \omega + \tau_a \omega^\alpha \sin \frac{\alpha \pi}{2} \right),$$

$$\cos(\phi^{(1)}_i - \phi^{(1)}_C) = \frac{u^{(1)}_{C0}}{R t^{(1)}_0} \tau_a \omega^\alpha \cos \frac{\alpha \pi}{2},$$

where (12) is used, implying the current amplitude and phase angle in the following forms

$$i^{(1)}_0 = \frac{u^{(1)}_{C0}}{R \sqrt{\tau_C^2 \omega^2 + 2 \tau_C \tau_a \omega^{1+\alpha} \sin \frac{\alpha \pi}{2} + \tau_a^2 \omega^{2\alpha}},$$

$$\phi^{(1)}_i = \phi^{(1)}_C + \arctan \left( \frac{\tau_C \omega^{1-\alpha}}{\tau_a \omega + \tan \frac{\alpha \pi}{2}} \right),$$

with passive capacitor’s voltage amplitude and phase angle given by (40) and (41). On the other hand, active capacitor’s voltage in the steady-state regime of $RC$ circuit in the form given by (39)1 is obtained from its constitutive equation (14), rewritten as

$$u_C(t) = R \left( \frac{1}{\tau_C} \delta^{(1)}_1 i(t) + \frac{1}{\tau_\mu} \delta^{1+\mu}_t i(t) \right),$$

that along with the current (39)2 yields

$$\sin(\phi^{(2)}_C - \phi^{(2)}_i) = -\frac{R^{(2)}_0}{u^{(2)}_{C0}} \left( \frac{1}{\tau_C \omega} + \frac{1}{\tau_\mu \omega^{1+\mu}} \cos \frac{\mu \pi}{2} \right),$$

$$\cos(\phi^{(2)}_C - \phi^{(2)}_i) = -\frac{R^{(2)}_0}{u^{(2)}_{C0}} \frac{1}{\tau_\mu \omega \omega^{1+\mu}} \sin \frac{\mu \pi}{2},$$

where (12) is used, implying active capacitor’s voltage amplitude and phase angle in the following forms

$$u^{(2)}_{C0} = R t^{(2)}_0 \sqrt{\frac{1}{\tau_C^2 \omega^2} + \frac{2}{\tau_C \tau_\mu \omega^{2+\mu} \cos \frac{\mu \pi}{2}} + \frac{1}{\tau_\mu^2 \omega^{2+2\mu}},$$

$$\phi^{(2)}_C = \phi^{(2)}_i + \arctan \left( \frac{\frac{1}{\tau_\mu \omega} + \frac{1}{\tau_C \omega} \sin \frac{\mu \pi}{2}}{\cot \frac{\mu \pi}{2}} \right),$$

with current amplitude and phase angle given by (42) and (43).

If the passive inductor in the series fractional $RL$ circuit forced by the harmonic electromotive force (38) is modeled by the flux–current constitutive relation (2), then the current is given by (39)2, with amplitude and phase angle having the same form as the passive capacitor voltage amplitude $u^{(2)}_{C0}$ and phase angle $\phi^{(2)}_C$, given by (40) and (41), due to the same forms of governing equations (34) for $RL$ circuit and (23) for $RC$ circuit, while, by the analogy of constitutive equations (13) and (11), passive inductor’s voltage is a harmonic function of amplitude and phase angle having the same forms as the amplitude $i^{(2)}_0$ and phase angle $\phi^{(2)}_i$, given by (44) and (45). Also, in the case of active inductor modeled by the current–flux constitutive relation (4), due to the analogy of governing equations (35) for $RL$ and (24) for $RC$ circuit, active inductor’s voltage is a harmonic function having amplitude and phase angle of the same form as the current amplitude $i^{(2)}_0$ and phase angle $\phi^{(2)}_i$, given by (42) and (43), while, by the analogy of constitutive equations (16) and (14), the current is a harmonic function with amplitude and phase angle of the same form as for the active capacitor voltage amplitude $u^{(2)}_{C0}$ and phase angle $\phi^{(2)}_C$, given by (46) and (47).

### 4.2 Numerical examples

Assuming the harmonic electromotive forcing as

$$E(t) = E_0 \cos(\omega t),$$

the transition from transient to steady-state regime is illustrated in Figure 3 by showing the time profiles of current in the fractional $RC$ circuit containing passive capacitor, modeled by charge–voltage constitutive equation (1), as well as in Figure 4 in the case of active capacitor modeled by voltage–charge constitutive equation (3). The response in transient regime is calculated according to (32) as

$$i(t) = E_0 \left( g^{(1,2)}_C (t) \ast \cos(\omega t) \right) = \frac{E_0}{R} \cos(\omega t)$$

$$-\frac{E_0}{R} \int_0^t g^{(1,2)}_C (t - t') \cos(\omega t') dt',$$  \hspace{0.5cm} (48)
Since the impulse responses $g^{(1,2)_i}$ take the form (31), where $g^{(1,2)}_C$ are given by (29) and (30), while the steady-state response

$$i(t) = i^{(1,2)}_0 \cos(\omega t + \phi^{(1,2)}_i),$$

with the current amplitudes $i^{(1,2)}_0$ given by (44) and (42) and phase angles $\phi^{(1,2)}_i$ given by (45) and (43) is obtained as the real part of (39)$_2$.

As obvious from Figure 3, fractional RC circuit containing passive capacitor enters the steady-state regime quite rapidly regardless of the value of fractional differentiation order $\alpha$, since curves corresponding to the transient and steady-state regime overlap even for small time, presumably due to the complete monotonicity of impulse response $g^{(1)}_C$. The good agreement between the curves obtained through analytical expression for transient regime and by numerical Laplace transform inversion procedure is evident as well.

Having the angular frequency fixed, the time profiles of current in RC circuit containing the active capacitor, as well as the current amplitude and phase angle are depicted in Figure 4 taking different values of the fractional integration order $\mu$. Similarly as in the case of RC circuit containing the passive capacitor, the time profiles corresponding to transient and steady-state responses shown in Figure 4a start overlapping even for small time in case of low values of the parameter $\mu$, while with its increase the time needed to reach the steady-state increases as well. Again one notices the good agreement between the

$$|g^{(1)}_{R}(\omega)|_{\text{dB}} = 20 \log \left| \hat{g}^{(1)}_{R}(\omega) \right|$$

$$= 10 \log \left( 1 - \frac{2\tau_{\mu} \omega^{\alpha} \cos \frac{\alpha \pi}{2} + 1}{\tau_{C} \omega^2 + 2\tau_{C} \tau_{\mu} \omega^{1+\alpha} \sin \frac{\alpha \pi}{2} + \tau_{\mu}^{2} \omega^{2\alpha} + 2\tau_{\mu} \omega^{\alpha} \cos \frac{\alpha \pi}{2} + 1} \right)$$

$$\arg \hat{g}^{(1)}_{R}(\omega) = -\arctan \left( \frac{\tau_{C} \omega + \tau_{\mu} \omega^{\alpha} \sin \frac{\alpha \pi}{2}}{\tau_{\mu} \omega^{\alpha} \cos \frac{\alpha \pi}{2} + 1} \right) + \arctan \left( \frac{\tau_{C} \omega^{1-\alpha} \cos \frac{\alpha \pi}{2}}{\cos \frac{\alpha \pi}{2} \tan \frac{\alpha \pi}{2}} \right)$$

$$= \arctan \left( \frac{\tau_{C} \omega + \tau_{\mu} \omega^{\alpha} \sin \frac{\alpha \pi}{2}}{\tau_{C} \omega^2 + 2\tau_{C} \tau_{\mu} \omega^{1+\alpha} \sin \frac{\alpha \pi}{2} + \tau_{\mu}^{2} \omega^{2\alpha} + \tau_{\mu} \omega^{\alpha} \cos \frac{\alpha \pi}{2}} \right) + \arctan \left( \frac{\tau_{C} \omega \cos \frac{\alpha \pi}{2}}{\cos \frac{\alpha \pi}{2} \tan \frac{\alpha \pi}{2}} \right).$$

Curves obtained by analytical and numerical approach. The current amplitude, depicted in Figure 4b, is found to increase monotonically with the increase of parameter $\mu$, while, as can be seen from Figure 4c, the phase angle increases up to the maximum, attained at $(\mu, \phi_i) = (0.75286, 0.579321\pi)$, lying in the interval $[0.299579, 0.979149]$ of parameter $\mu$ with the values of phase angle $\phi_i$ greater than $\frac{\pi}{2}$, meaning that the resistor and active capacitor considered as a single element in this interval of parameter $\mu$ behave as a generative element, since $\cos \phi_i < 0$.

5 Frequency characteristics of fractional RC and RL circuits

In order to analyze the frequency characteristics of RC circuit containing either passive or active capacitor, one considers transfer functions

$$\hat{g}^{(1,2)}_{R} = R \hat{g}^{(1,2)}_{i},$$

where $\hat{g}^{(1,2)}_{i}$ are given by (26) and (28), with $\hat{g}^{(1,2)}_{i}$ corresponding to the current running through the RC circuit and $\hat{g}^{(1,2)}_{R}$ corresponding to the voltage on resistor $u_R$, so that the modulus and argument of transfer functions (50) are

$$\left| \hat{g}^{(1,2)}_{R}(\omega) \right| = \frac{u^{(1,2)}_{R}(\omega)}{E_0} = \frac{R i^{(1,2)}_0(\omega)}{E_0}$$

and

$$\arg \hat{g}^{(1,2)}_{R}(\omega) = \arg \hat{g}^{(1,2)}_{i}(\omega) = \phi^{(1,2)}_i,$$

where the current amplitude $i^{(1)}_0$ and phase angle $\phi^{(1)}_i$ are obtained as (44) and (45) in the steady-state regime of RC circuit containing the passive capacitor, so that (51) reads

$$|\hat{g}^{(2)}_{R}(\omega)|_{\text{dB}} = 20 \log |\hat{g}^{(2)}_{R}(\omega)|$$

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Fig. 3 Time profiles of current in RC circuit containing passive capacitor as a response to electromotive force assumed as a cosine function of amplitude \( E_0 = 1 \) and angular frequency \( \omega = 0.3 \), obtained by analytical expression (48) (line) and numerically (dots) in the transient regime and by analytical expression (49) (thin line) in the steady-state regime for model parameters: \( R = 1 \), \( \tau_C = 0.2 \), and \( \tau_\alpha = 0.5 \).

\[
\begin{align*}
\text{arg} \hat{g}_L^{(2)}(\omega) &= \arctan \left( \frac{\tau_\alpha \omega^{1+\mu}}{\tau_C \omega^{1+\mu} - \tau_\alpha \omega^{1+\mu}} \frac{\cos \frac{\mu \pi}{2}}{\sin \frac{\mu \pi}{2}} \right) \quad \text{and} \\
&= \arctan \left( \frac{\tau_\alpha \omega^{1+\mu}}{\tau_C \omega^{1+\mu} - \tau_\alpha \omega^{1+\mu}} \frac{\cos \frac{\mu \pi}{2}}{\sin \frac{\mu \pi}{2}} \right) \quad (54)
\end{align*}
\]

The expressions (52) - (55) for transfer function moduli and arguments are also obtained by substituting \( s = j\omega \) into transfer functions \( \hat{g}_L^{(1,2)}(s) \), given by (26) and (28), and subsequently finding their modulus and argument.

5.1 Asymptotic analysis

Asymptotic expansions of the transfer function modulus, obtained from (52) in the case of RC circuit containing the passive capacitor, for low and high frequencies reads

\[
\begin{align*}
\left| \hat{g}_R^{(1)}(\omega) \right|_{\text{dB}} &\sim 10 \log \begin{cases} \tau_\alpha^2 \omega^{2\alpha} - 2\tau_\alpha^3 \omega^{3\alpha} \cos \frac{\alpha \pi}{2}, & \text{if } \alpha \in (0, \frac{1}{2}) , \\ \tau_\alpha^2 \omega - \sqrt{2}\tau_\alpha \omega^{3/2} (\tau_\alpha^2 - \tau_C) , & \text{if } \alpha = \frac{1}{2} , \\ \tau_\alpha^2 \omega^{2\alpha} + 2\tau_\alpha \tau_C \omega^{1+\alpha} \sin \frac{\alpha \pi}{2} , & \text{if } \alpha \in (\frac{1}{2}, 1) , \end{cases} \\
\end{align*}
\]

Considering the frequency characteristics of RL circuit, due to the analogies between physical quantities of RL and RC circuits discussed in Section 4.1, one has that the frequency characteristics of the inductor voltage transfer function \( \hat{g}_L^{(1,2)} \) have the same form as the frequency characteristics of the transfer function \( \hat{g}_R^{(1,2)} \), given by (50), and also that the frequency characteristics of the transfer function corresponding to current in RL circuit have the same form as the frequency characteristics of the transfer function corresponding to the capacitor voltage in RC circuit.
Fig. 4 Current in RC circuit containing active capacitor as a response to electromotive force assumed as a cosine function of amplitude $E_0 = 1$ and angular frequency $\omega = 3.2$, obtained for model parameters: $R = 1$, $\tau_C = 1$, and $\tau_\mu = 0.1$

\[
\left| \hat{g}_R^{(1)}(\omega) \right|_{dB} \sim 10 \log \begin{cases} 
1 - \frac{1}{\tau_C^2 \omega^2} (2 \tau_\alpha \omega^{\alpha} \cos \frac{\alpha \pi}{2} + 1), & \text{if } \alpha \in \left(0, \frac{1}{2}\right), \\
1 - \frac{1}{\tau_C^2 \omega^2} (2 \tau_\alpha \omega^{\alpha} \cos \frac{\alpha \pi}{2} - 2 \tau_\alpha^2 \omega^{2\alpha-1} \sin(\alpha \pi) + 1), & \text{if } \alpha \in \left[\frac{1}{2}, \frac{3}{2}\right), \\
1 - \frac{1}{\tau_C^2 \omega^2} (2 \tau_\alpha \omega^{\alpha} \cos \frac{\alpha \pi}{2} - 2 \tau_\alpha^2 \omega^{2\alpha-1} \sin(\alpha \pi) + 2 \tau_\alpha^3 \omega^{3\alpha-2} \cos \frac{\alpha \pi}{2} (4 \sin^2 \frac{\alpha \pi}{2} - 1) - 1), & \text{if } \alpha \in \left[\frac{3}{2}, 1\right), \\
\end{cases}
\text{as } \omega \to \infty, \quad (57)
\]
respectively, while the asymptotic expansions of transfer function argument, obtained from (53), is

\[
\arg \hat{g}_R^{(1)} (\omega) \sim \arctan \left\{ \tan \frac{\alpha \pi}{2} \left( 1 - \frac{1}{\tau_C} \frac{1}{\sin \frac{\alpha \pi}{2}} \right) \right\} \quad \text{as } \omega \to 0, \\
\arg \hat{g}_R^{(1)} (\omega) \sim \arctan \left\{ \tan \frac{\alpha \pi}{2} \left( 1 - \frac{1}{\tau_C} \frac{1}{\sin \frac{\alpha \pi}{2}} \right) \right\} \quad \text{as } \omega \to \infty.
\]

Clearly, by retaining only the leading terms in the previous asymptotic expansions, one has

\[
|\hat{g}_R^{(1)} (\omega)|_{\text{dB}} \sim 20 \alpha \log \omega + 20 \log \tau_\alpha \quad \text{and}
\]

\[
\arg \hat{g}_R^{(1)} (\omega) \sim \frac{\alpha \pi}{2} \quad \text{as } \omega \to 0,
\]

\[
|\hat{g}_R^{(1)} (\omega)|_{\text{dB}} \sim 0 \quad \text{and}
\]

\[
\arg \hat{g}_R^{(1)} (\omega) \sim \arctan \left\{ \frac{1}{\tau_C} \frac{1}{\omega} \frac{1}{\sin \frac{\alpha \pi}{2}} \right\} \quad \text{as } \omega \to \infty,
\]

since \( \arctan \) \( x \sim x \) for \( x \ll 1 \), implying that for low frequencies the transfer function modulus is a linear function of \( \log \omega \) having slope proportional to the fractional differentiation order \( \alpha \) and intercept proportional to the fractional time constant \( \tau_\alpha \), while the transfer function argument is proportional to parameter \( \alpha \), and in the case of high frequencies, the transfer function argument tends to zero as a hyperbolic function with the coefficient inversely proportional to the time constant \( \tau_C \).

Asymptotic behavior of the transfer function modulus and argument, obtained from (54) and (55) in the case of RC circuit containing the active capacitor, is described by

\[
|\hat{g}_R^{(2)} (\omega)|_{\text{dB}} \sim -10 \log \left\{ \frac{1}{\tau_\mu} \frac{1}{\omega^{1+\mu}} + 2 \frac{1}{\tau_\mu} \omega^{1+\mu} \cos \frac{\mu \pi}{2}, \quad \text{as } \omega \to 0, \\
1 - 2 \frac{1}{\tau_\mu} \omega^{1+\mu} \sin \frac{\mu \pi}{2}, \quad \text{as } \omega \to \infty,
\]

\[
\arg \hat{g}_R^{(2)} (\omega) \sim \arctan \left\{ \tan \frac{(1+\mu) \pi}{2} \left( 1 + \frac{1}{\tau_C} \omega^{1+\mu} \cos \frac{\mu \pi}{2} \right) \right\} \quad \text{as } \omega \to 0,
\]

\[
|\hat{g}_R^{(2)} (\omega)|_{\text{dB}} \sim 20 (1 + \mu) \log \omega + 20 \log \tau_\mu \quad \text{and}
\]

\[
\arg \hat{g}_R^{(2)} (\omega) \sim \arctan \left\{ \frac{1}{\tau_C} \frac{1}{\omega} \frac{1}{\sin \frac{\mu \pi}{2}} \right\} \quad \text{as } \omega \to \infty,
\]

since \( \arctan \) \( x \sim x \) for \( x \ll 1 \), again implying that for low frequencies the transfer function modulus is a linear function of \( \log \omega \) having slope proportional to the fractional integration order \( 1 + \mu \) and intercept proportional to the fractional time constant \( \tau_\mu \), while the transfer function argument is proportional to parameter \( \mu \), and in the case of high frequencies, the transfer function argument tends to zero as a hyperbolic function with the coefficient inversely proportional to the time constant \( \tau_C \).

Therefore, regardless of the fact weather RC circuit contains passive or active capacitor, model parameters can easily be estimated from the asymptotic expressions of transfer function moduli and arguments. Derivation of asymptotic formulae (56)–(59), corresponding to transfer function \( \hat{g}_R^{(1)} \), and (61) and (62),
Fig. 5 Case of passive capacitor—frequency characteristics of transfer function $\hat{g}_R^{(1)}$ obtained for model parameters: $\tau_C = 0.2$ and $\tau_\alpha = 0.5$

corresponding to transfer function $\hat{g}_R^{(2)}$, is performed in Appendix B.

5.2 Numerical examples

Frequency characteristics corresponding to $RC$ circuit containing the passive capacitor, i.e., Bode plots, are presented in Figure 5, together with their asymptotics. As expected from the form of the transfer function $\hat{g}_R^{(1)}$, its modulus has a zero of non-integer order at the origin, since it decreases linearly to negative infinity as the frequency tends to zero, as predicted by the asymptotic expansion (60) and as observed from Figure 5a. Further, the transfer function modulus monotonically increases and, in accordance with its asymptotics (57), tends to zero for high frequencies regardless of the value of fractional differentiation order $\alpha$, as obvious from Figure 5b. Although it is not the case, such behavior of characteristics for high frequencies might be interpreted as if the transfer function has a real pole of the same order as its zero at the origin. On the other hand, the frequency characteristics of the transfer function argument change from non-monotonic function, which attains a maximum, to a monotonically decreasing function as the parameter $\alpha$ increases, see Figure 5c. The low frequency asymptotics of transfer function argument (60) shows that the frequency characteristics have a constant value depending on the parameter $\alpha$ confirming the conclusion derived from the asymptotics of transfer function modulus. Figure 5d shows the transfer function argument, along with its asymptotics, tending to zero for high frequencies regardless of the parameter $\alpha$, again misleading to the conclusion about the poles of transfer function.

In the case of $RC$ circuit containing the active capacitor, the frequency characteristics, including their asymptotics, are shown in Figure 6. Contrary to the
Dissipative and generative fractional electric elements in modeling $RC$ and $RL$ circuits

Fig. 6 Case of active capacitor—frequency characteristics of transfer function $\hat{g}_R^{(2)}$ obtained for model parameters: $\tau_C = 5$ and $\tau_{\mu} = 0.5$

According to asymptotic formula $(64)_2$, see Figures 6c and 6d.

It is clear from the frequency characteristics of the transfer function moduli that $RC$ circuit regardless whether it contains passive or active capacitor behaves as the high pass filter, see Figures 5a and 6a.

6 Conclusion

Classical constitutive equations, describing behavior of capacitor and inductor as basic elements of electric circuits, are generalized and proposed in the form that include element’s instantaneous and hereditary response, with the hereditariness modeled by the long memory kernel of power type, i.e., by the fractional integral, yielding two types of constitutive equations depending on the physical quantities whose memory is considered. Thermodynamical considerations imply
that charge–voltage (1) and flux–current (2) constitutive relations describe passive capacitor and inductor, while voltage–charge (3) and current–flux (4) models describe active capacitor and inductor. Also, equivalent models of generalized electric element can be obtained by the simultaneous change of its memory kernel and topology.

Charge–voltage and voltage–charge constitutive models (1) and (3) are further used in deriving the equations (23) and (24), governing transient regime in the series RC circuit subject to electromotive force, that yielded different qualitative behavior of the generalized capacitors’ impulse responses, obtained as (29) and (30) by the Laplace transform method. Namely, the impulse response corresponding to the passive capacitor, modeled by the charge–voltage relation (1), is a positive monotonic decreasing convex function, while the impulse response corresponding to the active capacitor, modeled by voltage–charge relation (3), is a damped oscillatory function, with the possibility of such an extensive damping that there are no visible oscillations. Numerical examples for the Heaviside function type electromotive force illustrated mentioned differences in the qualitative behavior of responses.

Considering the fractional RC circuit subject to harmonic electromotive forcing, numerical examples presenting the comparison of current obtained according to (48) for the transient regime in case of large time and current obtained according to (49) for the steady-state regime showed perfect agreement between curves corresponding to these two types of solutions. Moreover, depending on the fractional integration order in the constitutive relation of the active capacitor, it is proved that the fractional RC circuit in the steady-state regime can either consume or produce electric energy.

Frequency characteristics analysis of RC circuits’ transfer functions supported the conclusions about the order and nature of their poles and zeros drawn from the form of transfer functions and from the responses in transient regime and also proved that RC circuit behaves as the high pass filter regardless of the type of generalized capacitor. The asymptotics of transfer function modulus and argument in numerical examples showed good agreement with the characteristics providing the possibility to estimate model parameters of RC circuit.

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**Declarations**

**Conflict of interest** The authors declare that they have no conflict of interest.

**A Calculation of impulse response** \( g_C^{(2)} \)

Starting from the transfer function \( \hat{g}_C^{(2)} \), given by (27), the impulse response \( g_C^{(2)} \) is obtained in the form (30) by applying the Laplace inversion formula

\[
g_C^{(2)}(t) = \mathcal{L}^{-1}[\hat{g}_C^{(2)}(s)](t) = \frac{1}{2\pi i} \int_{\Gamma Br} \hat{g}_C^{(2)}(s) e^{st} ds
\]

and using the Cauchy residues theorem, claiming that

\[
\oint_{\Gamma} f(z)dz = 2\pi i \sum_k \text{Res}(f(z), z_k).
\]

if function \( f \) has poles \( z_k \) in the domain encircled by the contour \( \Gamma \), which is chosen to contain the Bromwich path \( \Gamma_{Br} \).

A.1 Nature of poles of transfer function \( \hat{g}_C^{(2)} \)

The existence of poles of the transfer function \( \hat{g}_C^{(2)} \) in the first Riemann sheet is determined by the occurrence of zeros of the denominator of \( \hat{g}_C^{(2)} \), rewritten as

\[
\psi(s) = as^{1+\mu} + bs^\mu + 1,
\]

with \( a = \tau_\mu \) and \( b = \frac{\tau_\mu}{\tau_C} \).

In order to find zeros of the function \( \psi \), given by (67), its real and imaginary parts are separated as

\[
\text{Re}\psi(\rho, \varphi) = \rho^{1+\mu} \cos((1+\mu)\varphi) + b\rho^\mu \cos(\mu\varphi) + 1,
\]

\[
\text{Im}\psi(\rho, \varphi) = \rho^{1+\mu} \sin((1+\mu)\varphi) + b\rho^\mu \sin(\mu\varphi),
\]

by substituting \( s = \rho e^{i\varphi} \) into (67). Properties of the real and imaginary parts of function \( \psi \)

\[
\text{Re}\psi(\rho, -\varphi) = \text{Re}\psi(\rho, \varphi)
\]

and
\[
\text{Im}\psi(\rho, -\varphi) = -\text{Im}\psi(\rho, \varphi)
\]

imply that it is symmetric with respect to the real axis, so that if \( \psi \) has a zero \( s_0 \) in the upper complex half-plane, then it also has its complex conjugate \( \overline{s_0} \) as a zero, thus it is sufficient to seek for zeros in the upper complex half-plane only. Moreover, function \( \psi \) does not have zeros in the upper right complex quarter-plane (and therefore in the lower right complex quarter-plane as well), since for \( \varphi \in (0, \frac{\pi}{2}) \), by (69), one has \( \text{Im}\psi(\rho, \varphi) > 0 \), while if \( \varphi = 0 \), then \( \text{Im}\psi(\rho, \varphi) = 0 \), but one has \( \text{Re}\psi(\rho, \varphi) > 0 \), by (68). Thus, if \( \psi \) has zeros, they lie in the upper left complex quarter-plane and their complex conjugates in the lower left complex quarter-plane.

The equation \( \text{Im}\psi(\rho, \varphi) = 0 \) solved with respect to \( \rho > 0 \) yields
\[
\rho = \frac{b}{a} \frac{\sin(\mu \varphi)}{\sin((1 + \mu) \varphi)} = \frac{b}{a} \frac{\sin(\mu \varphi)}{|\sin((1 + \mu) \varphi)|} \quad \text{for} \quad \varphi \in \left(\frac{\pi}{1 + \mu}, \pi\right),
\]
and when such obtained \( \rho \) is substituted into equation \( \text{Re}\psi(\rho, \varphi) = 0 \) one obtains
\[
\left(\frac{a}{\sin(\mu \varphi)}\right)^\mu \sin \varphi \left(\frac{b}{\sin((1 + \mu) \varphi)}\right)^{1+\mu}. \quad (70)
\]

It is unclear whether or not the equation (70) has a solution \( \varphi \in \left(\frac{\pi}{1+\mu}, \pi\right) \) and therefore the existence of zeros of function \( \psi \), given by (67), in the upper left complex quarter-plane is investigated by using the argument principle with the contour \( \gamma \) as in Figure 7, since \( s = 0 \) is its only branching point. Recall, the argument principle is stating that if the independent variable \( z \) changes along the closed contour \( \gamma \) in the complex plane, then the number of zeros \( N \) of function \( f(z) \) in the domain bounded by contour \( \gamma \) is determined by the change of argument: \( \Delta \arg f(z) = 2\pi N \), assuming that function \( f \) does not have poles in the mentioned domain.

Along the contour \( \gamma_1 \), parameterized by \( s = \rho e^{i\pi/2} \), with \( \rho \in (0, \infty) \), the imaginary part of function \( \psi \), by (69), is
\[
\text{Im}\psi\left(\rho, \frac{\pi}{2}\right) = a \rho^{1+\mu} \cos \frac{\mu \pi}{2} + b \rho^\mu \sin \frac{\mu \pi}{2} > 0,
\]
since \( \mu \in (0, 1) \), while in the limiting cases, one has
\[
\text{Re}\psi\left(\rho, \frac{\pi}{2}\right) \sim 1 \quad \text{and} \quad \text{Im}\psi\left(\rho, \frac{\pi}{2}\right) \sim b \rho^\mu \sin \frac{\mu \pi}{2} \rightarrow 0 \quad \text{as} \quad \rho \rightarrow 0,
\]
\[
\text{Re}\psi\left(\rho, \frac{\pi}{2}\right) \sim -a \rho^{1+\mu} \sin \frac{\mu \pi}{2} \rightarrow -\infty \quad \text{and}
\]
\[
\text{Im}\psi\left(\rho, \frac{\pi}{2}\right) \sim -a \rho^\mu \sin \frac{\mu \pi}{2} \rightarrow -\infty.
\]

Along the contour \( \gamma_2 \), parameterized by \( s = R e^{\mu \pi/2} \), with \( \rho \in (0, \infty) \), the real and imaginary parts of function \( \psi \), according to (68) and (69), read
\[
\text{Im}\psi\left(\rho, \frac{\pi}{2}\right) \sim a \rho^{1+\mu} \cos \frac{\mu \pi}{2} \rightarrow \infty \quad \text{as} \quad \rho \rightarrow \infty.
\]

Along the contour \( \gamma_2 \), parameterized by \( s = Re^\mu \varphi \), with \( \varphi \in \left[\frac{\pi}{2}, \pi\right] \) as \( R \rightarrow \infty \), according to (68) and (69), one has
\[
\text{Re}\psi(R, \varphi) \sim a R^{1+\mu} \cos((1 + \mu) \varphi) \quad \text{and}
\]
\[
\text{Im}\psi(R, \varphi) \sim a R^{1+\mu} \sin((1 + \mu) \varphi)
\]

implying
\[
|\psi| \sim a R^{1+\mu} \rightarrow \infty,
\]
while for \( \varphi = \frac{\pi}{2} \) and \( \varphi = \pi \) it holds
\[
\text{Re}\psi\left(R, \frac{\pi}{2}\right) \sim -a R^{1+\mu} \sin \frac{\mu \pi}{2} \quad \rightarrow -\infty \quad \text{and} \quad \text{Im}\psi\left(R, \frac{\pi}{2}\right) \sim a R^{1+\mu} \cos \frac{\mu \pi}{2} \rightarrow \infty,
\]
\[
\text{Re}\psi\left(R, \pi\right) \sim -a R^{1+\mu} \cos(\mu \pi) \quad \rightarrow \begin{cases} -\infty, & \text{if } \mu < \frac{1}{2} \\ \infty, & \text{if } \mu > \frac{1}{2} \end{cases} \quad \text{and}
\]
\[
\text{Im}\psi\left(R, \pi\right) \sim -a R^{1+\mu} \sin(\mu \pi) \rightarrow -\infty.
\]

Along the contour \( \gamma_3 \), parameterized by \( s = \rho e^{i\pi} \), with \( \rho \in (0, \infty) \), the real and imaginary parts of function \( \psi \), according to (68) and (69), read
Re\(\psi (\rho, \varphi) = -a\rho^{1+\mu} \cos(\mu\pi) + b\rho^\mu \cos(\mu\pi) + 1\)
\[= \rho^\mu \cos(\mu\pi) s(b - a\rho) + 1, \quad (71)\]
Im\(\psi (\rho, \varphi) = -a\rho^{1+\mu} \sin(\mu\pi) + b\rho^\mu \sin(\mu\pi)\)
\[= \rho^\mu \sin(\mu\pi) (b - a\rho). \quad (72)\]

The asymptotic behavior

Re\(\psi (\rho, \pi) \sim 1 \) and
Im\(\psi (\rho, \pi) \sim b\rho^\mu \sin(\mu\pi) \to 0^+ \) as \(\rho \to 0\),

Re\(\psi (\rho, \pi) \sim -a\rho^{1+\mu} \cos(\mu\pi) \to \begin{cases} -\infty, & \text{if } \mu < \frac{1}{2} \\ \infty, & \text{if } \mu > \frac{1}{2} \end{cases} \)

of \(\text{Re}\psi\) and \(\text{Im}\psi\), given by (71) and (72), implies that imaginary part of function \(\psi\) along \(\gamma_3\) changes its sign from negative to positive and therefore

\[\text{Im}\psi (\rho^*, \pi) = 0 \implies \rho^* = \frac{b}{a} \text{ and } \text{Re}\psi (\rho^*, \pi) = 1.\]

Along the contour \(\gamma_4\), parameterized by \(s = re^{i\varphi}\), with \(\varphi \in \left[\frac{\pi}{4}, \pi\right]\) as \(r \to 0\), according to (68) and (69), one has

\[\text{Re}\psi (r, \varphi) \sim 1 \text{ and } \text{Im}\psi (r, \varphi) \sim br^\mu \sin(\mu\pi) \to 0^+.\]

In conclusion, one finds that \(\Delta \arg \psi(s) = 2\pi\), so by the argument principle function \(\psi\) has a single zero in the upper left complex quarter-plane and therefore has a pair of complex conjugated zeros with negative real part in the first Riemann sheet.

A.2 Laplace transform inversion of transfer function \(\hat{g}_C^{(2)}\)

The impulse response \(\hat{g}_C^{(2)}\) is obtained from the transfer function \(\hat{g}_C^{(2)}\), given by (27), by the Laplace inversion formula (65) using the Cauchy residues theorem (66), i.e.,

\[
\int_{\Gamma} \hat{g}_C^{(2)}(s) e^{st} ds = 2\pi j \left( \text{Res} \left( \hat{g}_C^{(2)}(s) e^{st}, s_0 \right) + \text{Res} \left( \hat{g}_C^{(2)}(s) e^{st}, \tilde{s}_0 \right) \right), \quad (73)
\]

where the integration path \(\Gamma\) is chosen as in Figure 8,

![Figure 8](image.png)

Integration along contours \(\Gamma_3\) and \(\Gamma_5\), parameterized by \(s = \rho e^{i\pi}\) and \(s = \rho e^{-i\pi}\), with \(\rho \in (0, \infty)\), respectively, yields

\[
I_{\Gamma_3} = \lim_{R \to \infty} \int_{\Gamma_3} \hat{g}_C^{(2)}(s) e^{st} ds
\]
\[= \int_0^\infty \frac{\tau}{\tau C} \rho^\mu e^{i\mu\pi} + 1 \psi (\rho e^{i\pi}) e^{\rho t} d\rho = \int_0^\infty \frac{\tau}{\tau C} \rho^\mu e^{i\mu\pi} + 1 \psi (\rho e^{i\pi}) e^{\rho t} d\rho, \quad (74)\]

\[
I_{\Gamma_5} = \lim_{R \to \infty} \int_{\Gamma_5} \hat{g}_C^{(2)}(s) e^{st} ds
\]
\[= \int_0^\infty \frac{\tau}{\tau C} \rho^\mu e^{-i\mu\pi} + 1 \psi (\rho e^{-i\pi}) e^{-\rho t} d\rho = -\int_0^\infty \frac{\tau}{\tau C} \rho^\mu e^{-i\mu\pi} + 1 \psi (\rho e^{i\pi}) e^{-\rho t} d\rho, \quad (75)\]
since \( \psi(p e^{-j\pi}) = \bar{\psi}(p e^{j\pi}) \), where bar denotes complex conjugation, so that

\[
\frac{1}{2\pi j} \left( I_{\Gamma_3} + I_{\Gamma_5} \right) = \frac{1}{\pi} \int_0^\infty \frac{\tau \mu \rho^{1+i\mu} \sin(\mu \pi)}{|\psi(\rho e^{j\pi})|^2} e^{-\rho t} d\rho,
\]

(76)

according to (74) and (75).

Since integral along \( \Gamma_0 \) is given by (65) and since integrals along \( \Gamma_3 \) and \( \Gamma_5 \) are given by (76), while integrals along all other contours on Figure 8 tend to zero as \( R \to \infty \) and \( r \to 0 \), the Cauchy residues theorem (73) takes the form

\[
g_C^{(2)}(t) = \frac{1}{\pi} \int_0^\infty \frac{\tau \mu \rho^{1+i\mu} \sin(\mu \pi)}{|\psi(\rho e^{j\pi})|^2} e^{-\rho t} d\rho
\]

\[
= \text{Res} \left( \hat{g}_C^{(2)}(s) e^{st}, s_0 \right) + \text{Res} \left( \hat{g}_C^{(2)}(s) e^{st}, \bar{s}_0 \right),
\]

(77)

where the residues are calculated as

\[
\text{Res} \left( \hat{g}_C^{(2)}(s) e^{st}, s_0 \right) = \left. \frac{d}{ds} \psi(s) \right|_{s=s_0} e^{s_0 t} + \frac{\tau \mu}{\tau C} s_0^{\mu} \left. \psi(s) \right|_{s=s_0} e^{s_0 t}
\]

\[
\text{Res} \left( \hat{g}_C^{(2)}(s) e^{st}, \bar{s}_0 \right) = \left. \frac{d}{ds} \psi(s) \right|_{s=\bar{s}_0} e^{s_0 t} + \frac{\tau \mu}{\tau C} \bar{s}_0^{\mu} \left. \psi(s) \right|_{s=\bar{s}_0} e^{s_0 t}
\]

so that

\[
\text{Res} \left( \hat{g}_C^{(2)}(s) e^{st}, s_0 \right) + \text{Res} \left( \hat{g}_C^{(2)}(s) e^{st}, \bar{s}_0 \right) = 2 \text{Re} \left( \frac{\tau \mu}{\tau C} s_0^{\mu} + 1 \right) e^{s_0 t} e^{-|\text{Res}_0| t}
\]

since \( \frac{d}{ds} \psi(s) \big|_{s=s_0} = \frac{d}{ds} \psi(s) \big|_{s=\bar{s}_0} \), implying by (77)

\[
g_C^{(2)}(t) = -\frac{1}{\pi} \int_0^\infty \frac{\tau \mu \rho^{1+i\mu} \sin(\mu \pi)}{|\psi(\rho e^{j\pi})|^2} e^{-\rho t} d\rho
\]

\[
+ 2 \text{Re} \left( \frac{\tau \mu}{\tau C} s_0^{\mu} + 1 \right) e^{s_0 t} e^{-|\text{Res}_0| t}.
\]

It is left to prove that integrals along contours: \( \Gamma_1 \), \( \Gamma_2 \), \( \Gamma_4 \), \( \Gamma_6 \), and \( \Gamma_7 \) tend to zero as \( R \to \infty \) and \( r \to 0 \).

The integral along contour \( \Gamma_1 \), parameterized by \( s = p + jR \), with \( p \in (0, p_0) \) as \( R \to \infty \), yield

\[
I_{\Gamma_1} = \lim_{R \to \infty} \int_{\Gamma_1} \frac{\tau_\mu}{\tau C} (p + jR)^{\mu} + 1} \psi(p + jR) e^{(p+jR)t} dR,
\]

so that

\[
|I_{\Gamma_1}| \leq \lim_{R \to \infty} \int_0^{p_0} \frac{\tau_\mu}{\tau C} (p + jR)^{\mu} + 1} \psi(p + jR) e^{(p+jR)t} dR
\]

\[
\leq \lim_{R \to \infty} \int_0^{p_0} \frac{1}{\tau C R} e^{pR} dR \to 0 \quad \text{as} \quad R \to \infty,
\]

since for \( |s| = \sqrt{p^2 + R^2} \sim R \) and \( \arg s = \arctan \frac{R}{p} \sim \frac{R}{2} \), one has

\[
\left| \frac{\tau_\mu}{\tau C} R^{e^{j\pi}} \right| \sim \tau_\mu R^{(1+\mu)} \quad \text{as} \quad R \to \infty,
\]

with \( \psi \) given by (67), and therefore \( I_{\Gamma_1} \to 0 \) as \( R \to \infty \). Similar argumentation gives \( I_{\Gamma_1} \to 0 \) as \( R \to \infty \).

The integral along contour \( \Gamma_2 \), parameterized by \( s = Re^{j\phi} \), with \( \phi \in (\frac{\pi}{2}, \pi) \) as \( R \to \infty \), yield

\[
I_{\Gamma_2} = \lim_{R \to \infty} \int_{\Gamma_2} \frac{\tau_\mu}{\tau C} (s) e^{st} ds
\]

\[
= \lim_{R \to \infty} \int_\frac{\pi}{2}^\pi \frac{\tau_\mu}{\tau C} R e^{j\mu} e^{R e^{j\phi}} j Re^{j\phi} d\phi,
\]

so that

\[
|I_{\Gamma_2}| \leq \lim_{R \to \infty} \int_\frac{\pi}{2}^\pi \frac{\tau_\mu}{\tau C} R e^{j\mu} e^{R e^{j\phi}} d\phi
\]

\[
\leq \lim_{R \to \infty} \int_\frac{\pi}{2}^\pi \frac{1}{\tau C} e^{R e^{j\phi}} \cos \phi d\phi \to 0 \quad \text{as} \quad R \to \infty,
\]

since \( \cos \phi < 0 \) for \( \phi \in (\frac{\pi}{2}, \pi) \), while

\[
\left| \frac{\tau_\mu}{\tau C} R e^{j\mu} + 1 \right| \sim \tau_\mu R^{(1+\mu)} \quad \text{and}
\]

\[
\left| \psi(Re^{j\phi}) \right| \sim \tau_\mu R^{(1+\mu)} \quad \text{as} \quad R \to \infty,
\]

with \( \psi \) given by (67), so that \( I_{\Gamma_2} \to 0 \) as \( R \to \infty \). Similar argumentation gives \( I_{\Gamma_6} \to 0 \) as \( R \to \infty \).

The integral along contour \( \Gamma_4 \), parameterized by \( s = re^{j\phi} \), with \( \phi \in (\frac{\pi}{2}, \pi) \) as \( r \to 0 \), yield
\[ I_{\Gamma_4} = \lim_{r \to 0} \int_{\Gamma_4} g^{(2)}(s)e^{st} ds \]
\[ = \lim_{r \to 0} \left[ \int_{-\pi}^{\pi} \frac{\tau_{1c} r \mu e^{i\psi} + 1}{\psi (re^{i\psi})} e^{\tau_{1c} \omega \phi} jre^{i\phi} d\phi, \right. \]
so that
\[ |I_{\Gamma_4}| \leq \lim_{r \to 0} \int_{-\pi}^{\pi} \frac{\tau_{1c} r \mu e^{i\psi} + 1}{\psi (re^{i\psi})} e^{\tau_{1c} \omega \phi} \phi (re^{i\psi}) \tau_{1c} r \mu d\phi \]
\[ \leq \lim_{r \to 0} \int_{-\pi}^{\pi} r d\phi \to 0 \text{ as } r \to 0, \]

since by (67) one has \( |\psi (re^{i\psi})| \sim 1 \) as \( r \to 0 \), so that \( I_{\Gamma_4} \to 0 \) as \( r \to 0 \).

\[ 10 \log \left| \frac{\hat{g}^{(1)}(\omega)}{\tau_{1c} \omega^2} \right|_{\text{dB}} = 1 - \frac{1}{\tau_{1c} \omega^2} \left( 1 + 2 \tau_{1c} \omega \cos \frac{\alpha \pi}{2} \right) \]
\[ \times \left( 1 - 2 \tau_{1c} \omega \frac{1}{\alpha \omega - 1} \sin \frac{\alpha \pi}{2} - \frac{\tau_{1c}^2}{\alpha \omega - 1} \omega^2 \cos \frac{\alpha \pi}{2} - \tau_{1c}^2 \omega^2 \cos \frac{\alpha \pi}{2} + \frac{4 \tau_{1c}^3}{\alpha \omega - 1} \sin^2 \frac{\alpha \pi}{2} + \ldots \right) \]
\[ \sim 1 - \frac{1}{\tau_{1c} \omega^2} \left( 2 \tau_{1c} \omega \frac{1}{\alpha \omega - 1} \sin \frac{\alpha \pi}{2} - \tau_{1c}^2 \omega^2 \alpha \omega - 1 \sin \alpha \pi \right) \]
\[ + 2 \tau_{1c}^3 \omega^2 \cos \frac{\alpha \pi}{2} \left( 4 \sin^2 \frac{\alpha \pi}{2} - 1 \right) + \ldots \]

using the series expansion (79) up to the quadratic terms and by neglecting the higher-order terms, so that \( |\hat{g}_{R}^{(1)}(\omega)|_{\text{dB}} \) yields the asymptotic formula (56). In the case of high frequencies, the transfer function modulus \( |\hat{g}_{R}^{(1)}(\omega)|_{\text{dB}} \), given by (52), takes the form

\[ \tan \hat{g}_{R}^{(1)}(\omega) = \tan \frac{\alpha \pi}{2} \left( \frac{1 + \frac{\tau_{1c}}{\tau_{1c}} \omega^{1-\alpha} \frac{1}{\cos \frac{\alpha \pi}{2}}}{1 + \tau_{1c} \omega \tan \frac{\alpha \pi}{2} + \frac{\tau_{1c}^2}{\tau_{1c}} \omega^{2-\alpha} \frac{1}{\cos \frac{\alpha \pi}{2}}} \right) \]
\[ \times \left( 1 - \tau_{1c} \omega \frac{1}{\alpha \omega - 1} \cos \frac{\alpha \pi}{2} - 2 \tau_{1c} \omega \tan \frac{\alpha \pi}{2} \right) \]
\[ \times \left( 1 - \tau_{1c} \omega \frac{1}{\alpha \omega - 1} \sin \frac{\alpha \pi}{2} - 2 \tau_{1c} \omega \tan \frac{\alpha \pi}{2} \right) \]
\[ \times \left( 1 - \tau_{1c} \omega \frac{1}{\alpha \omega - 1} \sin \frac{\alpha \pi}{2} \cos \frac{\alpha \pi}{2} - 2 \tau_{1c} \omega \tan \frac{\alpha \pi}{2} \right) \]

### B Derivation of asymptotic formulae

In order to describe the asymptotic behavior of the transfer function modulus \( |\hat{g}_{R}^{(1)}| \) for low frequencies, one rewrites (52) in the form

\[ 10 \log \left| \frac{\hat{g}_{R}^{(1)}(\omega)}{\tau_{1c} \omega^2} \right|_{\text{dB}} = 1 - \left( 1 + 2 \tau_{1c} \omega \frac{1}{\alpha \omega - 1} \sin \frac{\alpha \pi}{2} - \tau_{1c}^2 \omega^2 \cos \frac{\alpha \pi}{2} - 4 \tau_{1c}^3 \omega^2 \cos \frac{\alpha \pi}{2} + \ldots \right) \]

using the series expansion (79) up to the quadratic terms and by neglecting the higher-order terms, so that the asymptotic formula (57) follows from (80) by neglecting the terms in bracket having negative powers.

The asymptotic expansion (58) for the transfer function argument \( \arg \hat{g}_{R}^{(1)}(\omega) \) in the case of low frequencies follows from (53), rewritten in the form
\[ \tan \frac{2\pi}{C} \left( \frac{1}{1 - \tau_a \omega^2} \frac{1}{\cos \frac{\alpha\pi}{2}} + \frac{1}{\cos^2 \frac{\alpha\pi}{2}} + \frac{1}{\sin \frac{\alpha\pi}{2}} \right) \approx \frac{\alpha\pi}{2} \left( 1 - \frac{\tau_a \omega^2}{1 - \tau_a \omega^2} \frac{1}{\cos \frac{\alpha\pi}{2}} + \frac{1}{\cos^2 \frac{\alpha\pi}{2}} + \frac{1}{\sin \frac{\alpha\pi}{2}} \right) \] as \( \omega \to 0, \)

using the series expansion (79) up to the quadratic terms, by neglecting the higher-order terms, and by selecting terms having the dominant contribution, while the transfer function argument \( g_R^{(1)} \), given by (53), for high frequencies transforms into

\[ \tan \arg g_R^{(1)}(\omega) \approx \frac{1}{\tau_C \omega} \left( 1 + \frac{\tau_a}{\tau_C \omega^1 - \sin \frac{\alpha\pi}{2}} \right) \frac{1}{\tau_C \omega^2 - 1 + \frac{\tau_a}{\tau_C \omega^1 - \sin \frac{\alpha\pi}{2}} \cos \frac{\alpha\pi}{2}} \] as \( \omega \to \infty, \)

using the series expansion (79) up to the linear terms and by neglecting the higher-order terms, yielding the asymptotic expansion (59).

The asymptotics (61) of the transfer function modulus \( |g_R^{(2)}| \) is easily obtained from (54) by retaining two terms containing the largest powers for low frequencies and smallest powers for high frequencies, while the transfer function argument \( g_R^{(2)} \), given by (55), transforms into the asymptotic expansions (62) by neglecting the appropriate terms in the denominator of \( \tan \arg g_R^{(2)} : \tau_C \tau_a \omega^{1+\mu} \) in the case of low and \( \tau_C \sin \frac{\alpha\pi}{2} \) in the case of high frequencies.

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