Fuzzy Stochastic Linear Fractional Programming based on Fuzzy Mathematical Programming

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ABSTRACT
In this paper, we consider a Fuzzy Stochastic Linear Fractional Programming problem (FSLFP). In this problem, the coefficients and scalars in the objective function are the triangular fuzzy number and technological coefficients and the quantities on the right side of the constraints are fuzzy random variables with the specific distribution. Here we change an FSLFP problem to an equivalent deterministic Multi-objective Linear Fractional Programming (MOLFP) problem. Then by using Fuzzy Mathematical programming approach transformed MOLFP problem is reduced single objective Linear programming (LP) problem. A numerical example is presented to demonstrate the effectiveness of the proposed method.

KEYWORDS
Trapezoidal fuzzy number; linear fractional programming problem; fuzzy Stochastic linear fractional programming problem; chance-constrained programming; fuzzy mathematical programming; multi-objective linear fractional programming problem

1. Introduction
In real-world problems, often face problems to take decisions that optimise the department/equity ratio, profit/cost, inventory/sales, actual cost/standard cost, output/employee, student/cost, nurse/patient ratio, as financial and corporate planning, production planning, marketing and media selection, university planning and student admissions, health care and hospital planning, etc. such problems can be solved through Linear Fractional Programming (LFP) problems. In practice, due to the errors of measurement or vary with market conditions, etc., some or all of the coefficients are not exact. These situations can be modelled through Fuzzy Linear Fractional Programming (FLFP). Most of the FLFP problems can be modelled and solved by fuzzy goal programming approach [1–8], but very few authors considered FLFP problem where fuzzy coefficients are fuzzy numbers. Pop and Stancu Minasian [9], analysed a method to solve the fully fuzzified LFP problem, where all the variables and parameters are represented by triangular fuzzy numbers and Veeramani and Sumathi [10], proposed a method to solve FLFP problem where the cost of the objective function, the resources and the technological coefficients are triangular fuzzy numbers.

In some significant real-world problems, one has to base decisions on information which is both fuzzily imprecise and probabilistically uncertain [11–18]. As an example, consider a production problem that is cast into a linear program format. Assume that the second member components of constraints are demands which are random variables while coefficients
of the technological matrix are given by experts who prefer to express them as fuzzy numbers, in a way to couple their vague perceptions with hard statistical data. Fuzzy Stochastic Linear Programming aims at grappling with such hybrid situations.

On the other hand, the case of fuzzy stochastic multiobjective linear programming problems has been considered by utilising the possibility programming approach as well as the chance-constrained approach [1,6]. In this paper, we consider the FSLFP problem with fuzzy objective coefficient and technological coefficient and resources as fuzzy random variables, where the fuzzy coefficient is triangular fuzzy numbers. First the given FSLFP problem is transformed into a deterministic MOLFP problem. This transformation is obtained by using Zadeh extension principle. By using Fuzzy Mathematical programming approach transformed MOLFP problem is reduced single objective Linear Programming (LP).

Different sections of the paper are organised as follows: In Section 2, we review some concepts of fuzzy numbers. In Section 3, the method of converting LFP problem into an LP problem is discussed. The procedure for transforming MOLFP into MOLP problem and Fuzzy Mathematical programming technique is presented in Section 4. In Section 5, the method for solving FSLFP problem using Fuzzy Mathematical Programming approach is developed. The proposed procedure illustrated through a numerical example in Section 6.

2. Preliminaries

In this section, we recall some basic definitions involving fuzzy sets, fuzzy numbers and operations on fuzzy numbers are outlined.

Definition 2.1: Let X denote a universal set. Then a fuzzy subset $\tilde{A}$ of X is defined by its membership function

$$\mu_{\tilde{A}} : X \rightarrow [0, 1],$$

which assigns a real number $\mu_{\tilde{A}}(x)$ in the interval [0, 1] to each element $x \in X$, where $\mu_{\tilde{A}}(x)$ represents the grade of membership of $x$ in $\tilde{A}$. Thus, the nearer the value of $\mu_{\tilde{A}}(x)$ is unity, the higher the grade of membership of $x$ in $\tilde{A}$.

A fuzzy $\tilde{A}$ subset can be characterised as a set of ordered pairs of element $x$ and its grade $\mu_{\tilde{A}}(x)$ and is often written as

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X\}. \quad (2)$$

Definition 2.2: A fuzzy subset $\tilde{A}$ of the real line with membership function $\mu_{\tilde{A}} : \mathbb{R} \rightarrow [0, 1]$ is called fuzzy number if

1. $\tilde{A}$ is normal and convex fuzzy set.
2. Support of $\tilde{A}$ must be bounded.

The membership function $\mu_{\tilde{A}}$ of $\tilde{A}$ can be expressed as

$$\mu_{\tilde{A}}(x) = \begin{cases} L_{\tilde{A}}(x), & a \leq x \leq b, \\ R_{\tilde{A}}(x), & c \leq x \leq d, \\ 0, & \text{otherwise}, \end{cases} \quad (3)$$
where \( a \leq x \leq b \leq c \leq x \leq d, L_A(x) \) is increasing and left continuous function on \([a, b]\) and \( R_A(x) \) is decreasing and right continuous function on \([c, d]\).

### 2.1. Trapezoidal Fuzzy Number

In this section, the membership function of Triangular Fuzzy Number (TFN) is presented.

**Definition 2.3:** \( \tilde{a} \) is a TFN if \( \tilde{a} = \{a, a_0, \bar{a}\} \), where \( a \) is the least possible value, \( a_0 \) is the main value, and \( \bar{a} \) is the highest possible value. The triangular shaped membership function is \( \mu_{\tilde{A}}(a; \theta), \theta \in (0, 1] \), where \( \theta \) is the maximum value of the membership function, i.e. when \( a = a_0 \). Then

\[
\mu_{\tilde{A}}(a; \theta) = \begin{cases} 
0, & \text{if } a \langle a \text{ or } a \rangle \bar{a} \\
(a - a)a, & \text{if } a \leq a \leq a_0 \\
(a_0 - a)(\bar{a} - a), & \text{if } a_0 \leq a \leq \bar{a}.
\end{cases}
\]

Therefore,

\[
\theta \geq \mu_{\tilde{A}}(a; \theta) \geq 0.
\]

### 2.2. Random Variables

This section covers univariate random variables.

**Definition 2.4:** Let \((\Omega, \mathcal{F}, P)\) be a probability space. If \( X : \Omega \rightarrow \mathbb{R} \) is a real-valued function have as its domain elements of \( \Omega \), then \( X \) is called a random variable.

**Definition 2.5:** A random variable is called discrete if its range consists of a countable (possibly infinite) number of elements.

Discrete random variables are characterised by a Probability Mass Function (PMF) which gives the probability of observing a particular value of the random variable.

**Definition 2.6:** The probability mass function for a discrete random variable \( X \) is defined as \( f(x) = P(X) \), for all \( x \in R(X) \) and \( f(x) = 0 \), for all \( x \notin R(X) \), where \( R(X) \) is the range of \( X \) (i.e. the values for which \( X \) is defined).

**Definition 2.7:** A random variable is called continuous if its range is uncountably infinite and there exists a non-negative-valued function \( f(x) \) defined or all \( x \in (-\infty, \infty) \) such that for any event \( B \subset R(X), P(X) = \int_{x \in B} f(x) dx \) and \( f(x) = 0 \), for all \( x \in R(X) \), where \( R(X) \) is the range of \( X \) (i.e. the values for which \( X \) is defined).

The PMF of a discrete random variable is replaced with the probability density function (pdf) for continuous random variables.

**Definition 2.8:** For a continuous random variable, the function \( f \) is called the Probability Density Function (PDF). A function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is a member of the class of continuous density functions if and only if \( f(x) \geq 0 \), for all \( x \in (-\infty, \infty) \) and \( \int_{-\infty}^{\infty} f(x) = 1 \).
**Definition 2.9:** The Cumulative Distribution Function (CDF) for a random variable $X$ is defined as $F(c) = P(x \leq c)$, for all $c \in (-\infty, \infty)$.

The cumulative distribution function is used for both discrete and continuous random variables. When $X$ is a discrete random variable, the CDF is

$$F(x) = \sum_{s \leq x} f(s),$$

for $c \in (-\infty, \infty)$ and when $X$ is a continuous random variable, the CDF is

$$F(x) = \int_{-\infty}^{x} f(s) \, ds,$$

for $x \in (-\infty, \infty)$.

### 3. Linear Fractional Programming Problem

The linear fractional programming problem can be written as

$$\begin{cases}
\text{Max} & \sum_{j=1}^{n} c_{j} x_{j} + r \cdot \sum_{j=1}^{n} d_{j} x_{j} + s = P(x) / Q(x), \\
\text{s.t.} & x \in D = \{ x \in \mathbb{R}^{n} : Ax \leq b, x \geq 0 \},
\end{cases}$$

where $j = 1, \ldots, n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^{m}$, $c_{j}, d_{j} \in \mathbb{R}^{n}$ and $r, s \in \mathbb{R}$. For some $x$, $Q(x)$ may be equal to zero. To avoid such cases, one requires that either

$$\{ x \geq 0, x \leq b \Rightarrow Q(x) > 0 \} \cup \{ x \geq 0, Ax \leq b \Rightarrow Q(x) < 0 \}.$$

For convenience, assume that LFP satisfies the condition that

$$x \geq 0, Ax \leq b \Rightarrow Q(x) > 0.$$

**Theorem 3.1:** Assume that no point $(z;0)$ with $z \geq 0$ is feasible for the following linear programming problem

$$\begin{cases}
\text{Max} & c^{T} y + rt \\
\text{s.t.} & d^{T} y + st = 1 \\
& Ay - bt = 0 \\
& t \geq 0, y \geq 0, y \in \mathbb{R}^{n}, t \in \mathbb{R}
\end{cases}$$

Now assume that the condition (9), then the LFP (8) is equivalent to linear programming problem (10).
Consider the following two related problems

\[
\begin{align*}
\text{Max } & \quad tP\left(\frac{Y}{t}\right) \\
\text{s.t. } & \quad A\left(\frac{Y}{t}\right) - b \leq 0 \\
& \quad tQ\left(\frac{Y}{t}\right) = 1 \\
& \quad t > 0, y \geq 0
\end{align*}
\]

\[ (11) \]

and

\[
\begin{align*}
\text{Max } & \quad tP\left(\frac{Y}{t}\right) \\
\text{s.t. } & \quad A\left(\frac{Y}{t}\right) - b \leq 0 \\
& \quad tQ\left(\frac{Y}{t}\right) \leq 1 \\
& \quad t > 0, y \geq 0
\end{align*}
\]

\[ (12) \]

where (11) is obtained from (8) by the transformation \( t = \frac{1}{Q(x)} \), \( y = tx \) and (12) differs from (11) by replacing the equality constraint \( Q\left(\frac{Y}{t}\right) = 1 \) by an inequality constraint \( tQ\left(\frac{Y}{t}\right) \leq 1 \).

The problem (8) is said to be standard concave–convex programming problem, if \( P(x) \) is concave on \( D \) with \( P(\gamma) \geq 0 \) for some \( \gamma \in D \) and \( Q(x) \) is convex and positive on \( D \).

**Theorem 3.2:** Let for some \( \gamma \in D, P(\gamma) \geq 0 \), if (8) achieve to a (global) maximum at \( x = x^* \), then (12) achieve to a (global) maximum at \((t, y) = (t^*, y^*)\), where \( \frac{y^*}{t^*} = x^* \) and The objective functions are equal at this point.

**Theorem 3.3:** If (8) is a standard concave–convex programming problem which reaches a (global) maximum at a point \( x^* \), then the corresponding transformed problem (12) attains the same maximum value at a point \((t^*, y^*)\) Where \( \frac{y^*}{t^*} = x^* \). Moreover (12) has a concave objective function and a convex feasible set.

Let that, in (8) \( P(x) \) is concave, \( Q(x) \) is concave and positive on \( D \) and \( P(x) \) is negative for each \( x \in D \), then

\[
\max_{x \in D} \frac{P(x)}{Q(x)} \iff \min_{x \in D} \frac{-P(x)}{Q(x)} \iff \max_{x \in D} \frac{Q(x)}{-P(x)}
\]

\[ (13) \]

where \( -P(x) \) is convex and positive. Now, with the applications of the Theorem (3.1) and under the present hypotheses the fractional programming (8) transformed to the following linear programming problem

\[
\begin{align*}
\text{Max } & \quad tQ\left(\frac{Y}{t}\right) \\
\text{s.t. } & \quad A\left(\frac{Y}{t}\right) - b \leq 0 \\
& \quad -tP\left(\frac{Y}{t}\right) \leq 1 \\
& \quad t > 0, y \geq 0
\end{align*}
\]

\[ (14) \]
4. Multi-Objective Linear Fractional Programming problem

The MOLFP problem can be written as follows:

\[
\begin{align*}
\text{Max} & \quad Z_i = \sum_{j=1}^{n} \frac{c_j x_j + r}{d_j x_j + s} = \frac{P_i(x)}{Q_i(x)}, \\
\text{s.t.} & \quad x \in D = \{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\},
\end{align*}
\]

where \(i = 1, 2, \ldots, k\), \(j = 1, 2, \ldots, n\), \(A \in \mathbb{R}^{n \times m}\), \(b \in \mathbb{R}^m\), \(c_j, d_j \in \mathbb{R}^n\) and \(r, s \in \mathbb{R}\).

Let \(I\) be the index set such that 

\[I = \{i : P_i(x) \geq 0 \text{ for some } x \in D\},\]

and 

\[I^c = \{i : P_i(x) < 0 \text{ for some } x \in D\}.
\]

By using the transformation \(y = tx(t > 0)\), Theorem 3.2 and 3.3, MOLFP problem (15) may be written as follows

\[
\begin{align*}
\text{Max} & \quad g_i(y, t) = \begin{cases} tP_i \left( \frac{Y}{t} \right), & \text{for } i \in I; \\ tQ_i \left( \frac{Y}{t} \right), & \text{for } i \in I^c \end{cases} \\
\text{s.t.} & \quad tQ_i \left( \frac{Y}{t} \right) \leq 1, \text{ for } i \in I \\
& \quad -tP_i \left( \frac{Y}{t} \right) \leq 1, \text{ for } i \in I^c \\
& \quad A \left( \frac{Y}{t} \right) - b \leq 0, \\
& \quad t \geq 0, z \geq 0
\end{align*}
\]

In classical linear programming with objective functions represented by fuzzy sets, the complete solution set \((y, t)\) from theoretically well-defined membership function expression 

\[
\mu_Q(y, t) = \bigcap_{k=1}^{c} \mu_k(y, t),
\]

Zimmermann [3] proved that, if \(\mu_Q(y, t)\) had a unique maximum value \(\mu_Q(y^*, t^*) = \max \mu_Q(y, t)\), then \((y^*, t^*)\) which is an element of a complete solution set \((y, t)\) can be derived by solving a classical linear programming with one variable \(\lambda\). The complete solution set is composed of all those solution vectors which results \(\mu_Q(y, t) > 0\). If no solution vector \((y, t)\) can result \(\mu_Q(y, t) > 0\), we say that the complete solution set does not exist. If a complete solution set contains all solutions vectors with \(\mu_Q(y, t) > 0\) and \((y^*, t^*)\) with the unique \(\mu_Q(y^*, t^*) = \max \mu_Q(y, t)\) exists, it must be included in the complete solution set. If \(i \in I\), then membership function of each objective function can be written as

\[
\mu_i \left( tP_i \left( \frac{Y}{t} \right) \right) = \begin{cases} 0, & tP_i \left( \frac{Y}{t} \right) \leq 0, \\ \frac{tP_i \left( \frac{Y}{t} \right)}{Z_i}, & 0 \leq tP_i \left( \frac{Y}{t} \right) \leq Z_i, \\ 1, & tP_i \left( \frac{Y}{t} \right) \geq Z_i, \end{cases}
\]

(17)
If \( i \in \mathcal{I} \), then membership function of each objective function can be written as

\[
\mu_i \left( tQ_i \left( \frac{y}{t} \right) \right) = \begin{cases} 
0, & tQ_i \left( \frac{y}{t} \right) \leq 0, \\
\frac{tQ_i \left( \frac{y}{t} \right)}{Z_i}, & 0 \leq tQ_i \left( \frac{y}{t} \right) \leq Z_i, \\
1, & tQ_i \left( \frac{y}{t} \right) \geq Z_i.
\end{cases}
\] (18)

Using Zimmermann’s min operator the model (16) transformed to the crisp model as

\[
\begin{align*}
\text{Max} & \quad \lambda \\
\text{s.t.} & \quad \mu_i \left( tP_i \left( \frac{y}{t} \right) \right) \geq \lambda, \quad \text{for } i \in \mathcal{I} \\
& \quad \mu_i \left( tQ_i \left( \frac{y}{t} \right) \right) \geq \lambda, \quad \text{for } i \in \mathcal{I}^c \\
& \quad tQ_i \left( \frac{y}{t} \right) \leq 1, \quad \text{for } i \in \mathcal{I}^c \\
& \quad -tP_i \left( \frac{y}{t} \right) \leq 1, \quad \text{for } i \in \mathcal{I}^c \\
& \quad A \left( \frac{y}{t} \right) - b \leq 0, \\
& \quad t \geq 0, z \geq 0 \\
\end{align*}
\] (19)

5. Fuzzy Stochastic Linear Fractional Programming problem

A Fuzzy Stochastic Linear Fractional Programming (FSLFP) model involving fuzzy random variables can be expressed in its general form as

\[
\begin{align*}
\text{Max} & \quad \frac{\sum_{j=1}^{n} \tilde{c}_j x_j + \tilde{r}}{\sum_{j=1}^{n} \tilde{d}_j x_j + \tilde{s}} \\
\text{s.t.} & \quad \sum_{j=1}^{n} \tilde{a}_{ij}^x x_j \leq \tilde{b}_i^x, \quad i = 1, \ldots, m, \\
& \quad x_j \geq 0, \quad j = 1, \ldots, n \\
\end{align*}
\] (20)

where \( x_j, j = 1, \ldots, n \) denotes the vector of decision variables, \( \tilde{c}_j \) and \( \tilde{d}_j \) are fuzzy coefficients while \( \tilde{r} \) and \( \tilde{s} \) are two fuzzy scalars. \( \tilde{a}_{ij}^x \) and \( \tilde{b}_i^x \) are normally distributed fuzzy random variables. Thus, by incorporating predetermined tolerance measures \( \beta_i, i = 1, \ldots, m \), and by use the chance-constrained approach, the set of fuzzy stochastic constraint problem (20) can be transformed to their deterministic fuzzy equivalents as follows [7,19]:

\[
P \left( \sum_{j=1}^{n} \tilde{a}_{ij}^x x_j \leq \tilde{b}_i^x \right) \geq \beta_i, \quad i = 1, \ldots, m. \] (21)
Now let \( E(.) \) and \( \text{Var}(.) \) is represents the mean and the variance of random variables respectively, then the relationship (21) can be rewritten as follows:

\[
\sum_{j=1}^{n} E(\tilde{a}_{ij}^2)x_j - \Phi^{-1}(1 - \beta_i) \sqrt{\text{Var}(\tilde{b}_i^j)} + \sum_{j=1}^{n} \text{Var}(\tilde{a}_{ij}^2)x_j^2 \leq E(\tilde{b}_i^j),
\]

for \( i = 1, \ldots, m \). So problem (20) becomes as follows:

\[
\begin{aligned}
\text{Max} & \quad \frac{\sum_{j=1}^{n} \tilde{c}_j x_j + \tilde{r}}{\sum_{j=1}^{n} \tilde{d}_j x_j + \tilde{s}} \\
\text{s.t.} & \quad \sum_{j=1}^{n} E(\tilde{a}_{ij}^2)x_j - \Phi^{-1}(1 - \beta_i) \sqrt{\text{Var}(\tilde{b}_i^j)} + \sum_{j=1}^{n} \text{Var}(\tilde{a}_{ij}^2)x_j^2 \leq E(\tilde{b}_i^j),
\end{aligned}
\]

Since \( \tilde{a}_{ij}^i, i = 1, \ldots, m; j = 1, \ldots, n \) are normally distributed fuzzy random variables and the decision variables \( x_j \geq 0, j = 1, \ldots, n \) are unknowns, some fuzzy random variables, \( \tilde{h}_i, i = 1, \ldots, m \), can be introduced as \( \tilde{h}_i = \sum_{j=1}^{n} \tilde{a}_{ij}^i x_j, i = 1, \ldots, m \). Then \( \tilde{h}_i \) are normally distributed fuzzy random variables with the respective mean and variance given by \( m_{\tilde{h}_i} = \sum_{j=1}^{n} E(\tilde{a}_{ij}^i)x_j \) and \( \sigma_{\tilde{h}_i}^2 = \sum_{j=1}^{n} \text{Var}(\tilde{a}_{ij}^i)x_j, i = 1, \ldots, m \). Also \( m_{\tilde{b}_i^j} = E(\tilde{b}_i^j) \) and \( \sigma_{\tilde{b}_i^j}^2 = \text{Var}(\tilde{b}_i^j) \) now applying mean and variance of the fuzzy random variables, the above respective expression in (22) is converted into the following form:

\[
m_{\tilde{h}_i} - \Phi^{-1}(1 - \beta_i) \sqrt{\sigma_{\tilde{b}_i^j}^2 + \sigma_{\tilde{a}_i^j}^2} \leq m_{\tilde{b}_i^j}, i = 1, \ldots, m.
\]

In a fuzzy decision making situation, it is to be assumed that the mean and variance associated with the fuzzy random variables \( \tilde{h}_i \) are triangular fuzzy numbers, which are considered as follows:

\[
m_{\tilde{h}_i} = (m_{\tilde{h}_i}, m_{\tilde{h}_i}, \tilde{m}_{\tilde{h}_i}), \sigma_{\tilde{h}_i}^2 = (\sigma_{\tilde{a}_i^j}^2, \sigma_{\tilde{b}_i^j}^2, \sigma_{\tilde{a}_i^j}^2), f_i = 1, \ldots, m.
\]

Also the expected values \( E(\tilde{b}_i^j), i = 1, \ldots, m \), is shown as follows:

\[
m_{\tilde{b}_i^j} = (m_{\tilde{b}_i^j}, m_{\tilde{b}_i^j}, \tilde{m}_{\tilde{b}_i^j}), \sigma_{\tilde{b}_i^j}^2 = (\sigma_{\tilde{a}_i^j}^2, \sigma_{\tilde{b}_i^j}^2, \sigma_{\tilde{a}_i^j}^2), l = 1, \ldots, m.
\]

For fuzzy coefficients \( \tilde{c}_j \) and \( \tilde{d}_j \), we have

\[
\tilde{c}_j = (c_j, c_j, \tilde{c}_j), \Gamma \Gamma \tilde{d}_j = (d_j, d_j, \tilde{d}_j).
\]

Therefore, the problem (23) can be written as

\[
\begin{aligned}
\text{Max} & \quad \sum_{j=1}^{n} (c_j, c_j, \tilde{c}_j) x_j + (r, r, \tilde{r}) \\
\text{s.t.} & \quad (m_{\tilde{h}_i}, m_{\tilde{h}_i}, \tilde{m}_{\tilde{h}_i}) - \Phi^{-1}(1 - \beta_i) \sqrt{(\sigma_{\tilde{b}_i^j}^2, \sigma_{\tilde{b}_i^j}^2, \sigma_{\tilde{a}_i^j}^2) + (\sigma_{\tilde{a}_i^j}^2, \sigma_{\tilde{a}_i^j}^2, \sigma_{\tilde{a}_i^j}^2)} \leq (m_{\tilde{b}_i^j}, m_{\tilde{b}_i^j}, \tilde{m}_{\tilde{b}_i^j}), \\
x_j \geq 0, j = 1, \ldots, n
\end{aligned}
\]

(24)
By using Zadeh’s extension principle of fuzzy numbers, the problem (24) convert to an equivalent MOLP problem as follows:

\[
\begin{align*}
\text{Max } Z_1 &= \sum_{j=1}^{n} c_j x_j + \frac{r}{\sum_{j=1}^{n} d_j x_j + \bar{s}} \\
\text{Max } Z_2 &= \sum_{j=1}^{n} c_j x_j + r \\
\text{Max } Z_3 &= \sum_{j=1}^{n} c_j x_j + \tilde{r} \\
\text{s.t. } \sum_{j=1}^{n} \tilde{d}_j y_j + \bar{s} t &\leq 1, \\
\sum_{j=1}^{n} d_j y_j + s &\leq 1, \\
\sum_{j=1}^{n} d_j y_j + t &\leq 1, \\
\tilde{m}_h - \Phi^{-1}(1 - \beta_i) \sqrt{\frac{\sigma^2_{\tilde{b}_i}}{\bar{b}_i} + \frac{\sigma^2_{\tilde{a}_i}}{\bar{a}_i}} &\leq \tilde{m}_{\tilde{b}_i}, \\
\tilde{m}_h - \Phi^{-1}(1 - \beta_i) \sqrt{\frac{\sigma^2_{\tilde{b}_i}}{\bar{b}_i} + \frac{\sigma^2_{\tilde{a}_i}}{\bar{a}_i}} &\leq \tilde{m}_{\tilde{a}_i}, \\
\tilde{m}_h - \Phi^{-1}(1 - \beta_i) \sqrt{\frac{\sigma^2_{\tilde{b}_i}}{\bar{b}_i} + \frac{\sigma^2_{\tilde{a}_i}}{\bar{a}_i}} &\leq \tilde{m}_{\tilde{a}_i}, \\
x_j &\geq 0, \quad j = 1, \ldots, n.
\end{align*}
\]  

(25)

where is a MOLFP problem.

Let us assume that \(Z_1(x), Z_2(x), Z_3(x) \geq 0\) for the feasible region. By using Charnes-Cooper’s variable transformation [20], the MOLFP problem can be converted into the following MOLP problem:

\[
\begin{align*}
\text{Max } Z_1(y, t) &= \sum_{j=1}^{n} c_j y_j + r t \\
\text{Max } Z_2(y, t) &= \sum_{j=1}^{n} c_j y_j + r t \\
\text{Max } Z_3(y, t) &= \sum_{j=1}^{n} c_j y_j + \tilde{r} t \\
\text{s.t. } \sum_{j=1}^{n} \tilde{d}_j y_j + \bar{s} t &\leq 1, \\
\sum_{j=1}^{n} d_j y_j + s &\leq 1, \\
\sum_{j=1}^{n} d_j y_j + t &\leq 1, \\
\tilde{m}_h - \Phi^{-1}(1 - \beta_i) \sqrt{\frac{\sigma^2_{\tilde{b}_i}}{\bar{b}_i} + \frac{\sigma^2_{\tilde{a}_i}}{\bar{a}_i}} &- t m_{\tilde{b}_i} \leq 0, \\
m_h - \Phi^{-1}(1 - \beta_i) \sqrt{\frac{\sigma^2_{b_i}}{b_i} + \frac{\sigma^2_{a_i}}{a_i}} &- t m_{b_i} \leq 0, \\
\tilde{m}_h - \Phi^{-1}(1 - \beta_i) \sqrt{\frac{\sigma^2_{\tilde{b}_i}}{\bar{b}_i} + \frac{\sigma^2_{\tilde{a}_i}}{\bar{a}_i}} &- t \tilde{m}_{\tilde{b}_i} \leq 0, \\
x_j &\geq 0, \quad j = 1, \ldots, n.
\end{align*}
\]

(26)

Solving the problem (27) for each objective function, we achieve \(Z_1^*, Z_2^*\) and \(Z_3^*\). Suppose, \(Z_1(y, t) \geq Z_1^*, Z_2(y, t) \geq Z_2^*\) and \(Z_3(y, t) \geq Z_3^*\), by using the membership function defined in
(17) and (18) the above model reduces to the crisp model as follows:

\[
\begin{align*}
\text{Max} & \quad \lambda \\
\text{s.t.} & \quad \sum_{j=1}^{n} c_j y_j + rt - Z_1^n \lambda \geq 0, \\
& \quad \sum_{j=1}^{n} c_j y_j + rt - Z_2^n \lambda \geq 0, \\
& \quad \sum_{j=1}^{n} \bar{c}_j y_j + \bar{r} t - Z_3^n \lambda \geq 0, \\
& \quad \sum_{j=1}^{n} \bar{d}_j y_j + \bar{s} t \leq 1, \\
& \quad \sum_{j=1}^{n} d_j y_j + s t \leq 1, \\
& \quad \sum_{j=1}^{n} d_j y_j + s t \leq 1, \\
& \quad m_{h_i} - \Phi^{-1}(1 - \beta_i) \sqrt{\sigma_{\bar{a}_i}^2 + \sigma_{\bar{s}}^2} - t m_{\bar{b}_i} \leq 0, \\
& \quad m_{h_i} - \Phi^{-1}(1 - \beta_i) \sqrt{\sigma_{\bar{a}_i}^2 + \sigma_{\bar{s}}^2} - t m_{\bar{b}_i} \leq 0, \\
& \quad \bar{m}_{h_i} - \Phi^{-1}(1 - \beta_i) \sqrt{\sigma_{\bar{a}_i}^2 + \sigma_{\bar{s}}^2} - t \bar{m}_{\bar{b}_i} \leq 0, \\
& \quad x_j \geq 0, \Gamma j = 1, \ldots, n.
\end{align*}
\]

(27)

6. Numerical Example

Consider the following FSLFP:

\[
\begin{align*}
\text{Max} & \quad \frac{(\tilde{c}_1 x_1 + \tilde{c}_2 x_2)}{\tilde{d}_1 x_1 + \tilde{d}_2 x_2 + \tilde{s}} \\
\text{s.t.} & \quad \tilde{a}_{11}^i x_1 + \tilde{a}_{12}^i x_2 \leq \tilde{b}_1^i, \\
& \quad \tilde{a}_{21}^i x_1 + \tilde{a}_{22}^i x_2 \leq \tilde{b}_2^i, \\
& \quad x_1, x_2 \geq 0.
\end{align*}
\]

(28)

where \(\tilde{c}_i\) and \(\tilde{d}_i\), \(i = 1, 2\) are fuzzy variables, \(\tilde{s}\) is a scalar fuzzy where represented by following triangular fuzzy number:

\[
\tilde{c}_1 = (3, 5, 7), \tilde{c}_2 = (2, 3, 4) \\
\tilde{d}_1 = (4, 5, 6), \tilde{d}_2 = (1, 2, 3) \\
\tilde{s} = (0, 1, 2)
\]

\(\tilde{a}_{ij}\) and \(\tilde{b}_j\), \(i = 1, 2, j = 1, 2\) are independent fuzzy random normal variables. The mean and variance of the independent random variables, is given in Table 1.
Table 1. The mean and variance of the $\tilde{a}_{ij}^*$ and $\tilde{b}_j^*$.  

|       | $\tilde{a}_{11}^*$ | $\tilde{a}_{12}^*$ | $\tilde{a}_{21}^*$ | $\tilde{a}_{22}^*$ | $\tilde{b}_1^*$ | $\tilde{b}_2^*$ |
|-------|---------------------|---------------------|---------------------|---------------------|-----------------|-----------------|
| $E(.)$ | $\tilde{3}$         | $\tilde{5}$         | $\tilde{5}$         | $\tilde{2}$         | $\tilde{15}$    | $\tilde{10}$    |
| $\text{Var(.)}$ | $\tilde{3}$     | $\tilde{4}$         | $\tilde{9}$         | $\tilde{9}$         | $\tilde{5}$    | $\tilde{10}$    |

Then the problem (29) can be rewritten as

$$\begin{align*}
\max & \quad (3, 5, 7)x_1 + (2, 3, 4)x_2 \\
\text{s.t.} & \quad (4, 5, 6)x_1 + (1, 2, 3)x_2 + (0, 1, 2) \\
& \quad 3x_1 + 5x_2 - \Phi^{-1}(1 - \beta_1)\sqrt{3x_1^2 + 4x_2^2 + 5} \leq \tilde{15}, \\
& \quad 5x_1 + 2x_2 - \Phi^{-1}(1 - \beta_2)\sqrt{9x_1^2 + 9x_2^2 + 10} \leq \tilde{10}, \\
& \quad x_1, x_2 \geq 0.
\end{align*}$$

(30)

All fuzzy numbers associated with the system constraints are taken as

- $E(\tilde{\alpha}_{i1}^*) = \tilde{3} = (2, 3, 4)$, $\text{Var}(\tilde{\alpha}_{i1}^*) = \tilde{3} = (2, 3, 4)$,
- $E(\tilde{\alpha}_{i2}^*) = \tilde{5} = (3, 5, 7)$, $\text{Var}(\tilde{\alpha}_{i2}^*) = \tilde{4} = (3, 4, 5)$,
- $E(\tilde{\alpha}_{21}^*) = \tilde{5} = (4, 5, 6)$, $\text{Var}(\tilde{\alpha}_{21}^*) = \tilde{9} = (8, 9, 10)$,
- $E(\tilde{\alpha}_{22}^*) = \tilde{2} = (1, 2, 3)$, $\text{Var}(\tilde{\alpha}_{22}^*) = \tilde{9} = (7, 9, 11)$,
- $E(\tilde{\beta}_1^*) = \tilde{15} = (11, 15, 19)$, $\text{Var}(\tilde{\beta}_1^*) = \tilde{5} = (4, 5, 6)$,
- $E(\tilde{\beta}_2^*) = \tilde{10} = (8, 10, 12)$, $\text{Var}(\tilde{\beta}_2^*) = \tilde{10} = (9, 10, 11)$

Now the problem (30) using the above defined fuzzy numbers takes the form as:

$$\begin{align*}
\max & \quad (3, 5, 7)x_1 + (2, 3, 4)x_2 \\
\text{s.t.} & \quad (2, 3, 4)x_1 + (3, 5, 7)x_2 - \Phi^{-1}(1 - \beta_1)\sqrt{2, 3, 4)x_1^2 + (3, 4, 5)x_2^2 + (4, 5, 6)} \leq (11, 15, 19), \\
& \quad (4, 5, 6)x_1 + (1, 2, 3)x_2 - \Phi^{-1}(1 - \beta_2)\sqrt{(8, 9, 10)x_1^2 + (7, 9, 11)x_2^2 + (9, 10, 11)} \leq (8, 10, 12), \\
& \quad x_1, x_2 \geq 0
\end{align*}$$

(31)
The above problem is equivalent to the following MOLFP problem:

\[
\begin{align*}
\text{Max } Z_1 &= \frac{3x_1 + 2x_2}{4x_1 + x_2 + 2}, \\
\text{Max } Z_2 &= \frac{5x_1 + 3x_2}{5x_1 + 2x_2 + 1}, \\
\text{Max } Z_3 &= \frac{7x_1 + 4x_2}{4x_1 + x_2}, \\
\text{s.t. } &2x_1 + 3x_2 - \Phi^{-1}(1 - \beta_1)\sqrt{2x_1^2 + 3x_2^2 + 4} \leq 11, \\
&3x_1 + 5x_2 - \Phi^{-1}(1 - \beta_1)\sqrt{3x_1^2 + 4x_2^2 + 5} \leq 15, \\
&4x_1 + 7x_2 - \Phi^{-1}(1 - \beta_1)\sqrt{4x_1^2 + 5x_2^2} \leq 19, \\
&4x_1 + x_2 - \Phi^{-1}(1 - \beta_2)\sqrt{8x_1^2 + 7x_2^2 + 9} \leq 8, \\
&5x_1 + 2x_2 - \Phi^{-1}(1 - \beta_2)\sqrt{9x_1^2 + 9x_2^2 + 10} \leq 10, \\
&6x_1 + 3x_2 - \Phi^{-1}(1 - \beta_2)\sqrt{10x_1^2 + 11x_2^2 + 11} \leq 12, \\
&x_1, x_2 \geq 0
\end{align*}
\]

where $Z_1(x), Z_2(x)$ and $Z_3(x)$ for the feasible region. The above MOLFP problem is equivalent to the following MOLP problem:

\[
\begin{align*}
\text{Max } Z_1 &= 3y_1 + 2y_2, \\
\text{Max } Z_2 &= 5y_1 + 3y_2, \\
\text{Max } Z_3 &= 7y_1 + 4y_2, \\
\text{s.t. } &2y_1 + 3y_2 - \Phi^{-1}(1 - \beta_1)\sqrt{2y_1^2 + 3y_2^2 + 4t^2} \leq 11t, \\
&3y_1 + 5y_2 - \Phi^{-1}(1 - \beta_1)\sqrt{3y_1^2 + 4y_2^2 + 5t^2} \leq 15t, \\
&4y_1 + 7y_2 - \Phi^{-1}(1 - \beta_1)\sqrt{4y_1^2 + 5y_2^2} \leq 19t, \\
&4y_1 + y_2 - \Phi^{-1}(1 - \beta_2)\sqrt{8y_1^2 + 7y_2^2 + 9t^2} \leq 8t, \\
&5y_1 + 2y_2 - \Phi^{-1}(1 - \beta_2)\sqrt{9y_1^2 + 9y_2^2 + 10t^2} \leq 10t, \\
&6y_1 + 3y_2 - \Phi^{-1}(1 - \beta_2)\sqrt{10y_1^2 + 11y_2^2 + 11t^2} \leq 12t, \\
&4y_1 + y_2 + 2t \leq 1, \\
&5y_1 + 2y_2 + t \leq 1, \\
&4y_1 + y_2 \leq 1, \\
&y_1, y_2 \geq 0, t \geq 0.
\end{align*}
\]
If the MOLFP problem is solved for each of the objective function one by one. Let $Z_1(y, t) \geq \frac{24}{50}$, $Z_2(y, t) \geq \frac{37}{50}$ and $Z_3(y, t) \geq \frac{1}{2}$. Using the membership function defined in (17) and (18) the above model reduces to the linear programming problem as follows:

$$\begin{align*}
\text{Max} & \quad Z = \lambda \\
\text{s.t} & \quad 150y_1 + 100y_2 - 24\lambda \geq 0, \\
& \quad 250y_1 + 150y_2 - 37\lambda \geq 0, \\
& \quad 350y_1 + 200y_2 - 49\lambda \geq 0, \\
& \quad 2y_1 + 3y_2 - 11t \leq 0, \\
& \quad 3y_1 + 5y_2 - 15t \leq 0, \\
& \quad 4y_1 + 7y_2 - 19t \leq 0, \\
& \quad 4y_1 + y_2 - 8t \leq 0, \\
& \quad 5y_1 + 2y_2 - 10t \leq 0, \\
& \quad 6y_1 + 3y_2 - 12t \leq 0, \\
& \quad 4y_1 + y_2 + 2t \leq 1, \\
& \quad 5y_1 + 2y_2 + t \leq 1, \\
& \quad 4y_1 + y_2 \leq 1, \\
& \quad y_1, y_2 \geq 0, t \geq 0.
\end{align*}$$

The problem is solved and the solution of the above problem is $y_1 = 0, y_2 = 0.24, t = 0.1$ and $\lambda = 0.99$. Hence the solution of the original problem is $x_1 = 0, x_2 = 2.7$.

7. Conclusion

In the paper, a method of solving the FSLFP problems, where the cost of the objective function is triangular fuzzy numbers and the resources and technological coefficients are fuzzy random variables, is proposed. In the proposed method, FSLFP problem is transformed into a Multi Objective Linear Fractional programming (MOLFP) problem and the resultant problem is converted to a LP problem, using Fuzzy Mathematical programming method. The proposed approach can be extended for solving linear fractional programming problems, where the cost of the objective function, the resources and the technological coefficients are trapezoidal fuzzy numbers or non-linear membership functions and solving Fuzzy Multi-objective linear fractional programming problems.

Disclosure statement

No potential conflict of interest was reported by the authors.

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