MILNOR’S TRIPLE LINKING NUMBER AND GAUSS DIAGRAM FORMULAS OF 3-BOUQUET GRAPHS

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Abstract. In this paper, we introduce two functions such that the subtraction corresponds to the Milnor’s triple linking number; the addition obtains a new integer-valued link homotopy invariant of 3-component links. We also have found a series of integer-valued invariants derived from four terms whose sum equals the Milnor’s triple linking number. We apply this structure to give invariants of 3-bouquet graphs.

1. Introduction

Bouquet graphs are elementary topological objects which have been well studied. However, explicit Gauss diagram formulas [2] of 3-bouquet graphs have not been very few or may be unknown.

Let us consider flat vertex isotopy classes of 3-bouquet graphs. Since any flat vertex isotopy preserves the cyclic order of edges connecting to the flat vertex, we choose a cyclic order and fix it. It is graphically explained by the next paragraph.

We firstly take a small disk which center is the flat vertex and assign the fixed cyclic order to intersections between edges and the boundary of the disk. It means that we select a base point on the boundary of the disk $d$ (Fig. 1). From the base point, we read the endpoints on the boundary by the cyclic order and have the Gauss word: $p_1p_2p_3p_4p_5p_6$. Then for each pair $p_i, p_j$ ($i < j$) belonging to the same component, we orient the component by setting that $p_i$ is the starting point and $p_j$ is the end point. Since the boundary $\partial d = S^1$, the isomorphism $S^1 \setminus \{a \text{ base point}\} \rightarrow \mathbb{R}$ induces a mapping from a flat vertex isotopy classes of a 3-bouquet graph with a base point on $\partial d$ to a $(6, 0)$-tangle.

Throughout this paper, we call the flat vertex isotopy classes of 3-bouquet graphs equipped with a base point as above based flat vertex isotopy classes. In the following, every notation obeys Östlund [2] including Gauss diagram formulas and arrow diagrams.

Theorem 1. Let $G_k$ be a Gauss diagram of an ordered three-component link $k$, where $k_1, k_2, k_3$ are the components of $k$. Let $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}$; indices $i, j,$ and $k$ are assigned with three circles. Let

$$P_{\text{even}}(k) = \frac{1}{6} \sum_{\sigma: \text{even}} \langle 2\begin{tikzpicture} [baseline=(X.base)] 
    
    \node[draw, circle, minimum size=5mm] (X) at (0,0) {};
    \node[draw, circle, minimum size=5mm] (Y) at (1,0) {};
    \node[draw, circle, minimum size=5mm] (Z) at (2,0) {};
    \node[draw, circle, minimum size=5mm] (A) at (3,0) {};
    \node[draw, circle, minimum size=5mm] (B) at (4,0) {};
    \node[draw, circle, minimum size=5mm] (C) at (5,0) {};
    \draw (X) -- (Y); \draw (Y) -- (Z); \draw (Z) -- (A); \draw (A) -- (B); \draw (B) -- (C);
\end{tikzpicture}, G_k \rangle$$

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and
\[ P_{\text{odd}}(k) = \frac{1}{6} \sum_{\sigma \text{ odd}} (2 \circlearrowright + 2 \circlearrowright + \circlearrowright, G_k). \]

Then \( P_{\text{even}} + P_{\text{odd}} \) is an integer-valued base-point-free link homotopy invariant. In comparison, \( P_{\text{even}} - P_{\text{odd}} \mod \gcd(lk(k_2, k_3), lk(k_1, k_3), lk(k_1, k_2)) \) is the Milnor’s triple linking number that is torsion-valued base-point-free.

Remark 1. Note that \( P_{\text{even}} \pm P_{\text{odd}} \in \mathbb{Z} \). This is because the value of links with no crossings is obviously zero; the difference of values by applying a single crossing change is a multiple 6\(^1\).

Corollary 1. Let \( t \in \mathbb{Q} \) and \( \hat{\mu} = P_{\text{even}} + P_{\text{odd}} \). Then
\[ (1 - t)\mu + t\hat{\mu} \]
is link homotopy invariant for links with the fixed base points. If \( t = 1 \), it is the base-point-free invariant.

Theorem 2. Let
\[ P_1(k) = \sum_{\sigma \in S_3} \text{sign}(\sigma)(\circlearrowright + \circlearrowright + \circlearrowright, G_k), \]
and
\[ P_2(k) = \sum_{\sigma \in S_3} \text{sign}(\sigma)(\circlearrowright + \circlearrowright + \circlearrowright, G_k). \]
Figure 2. A third Reidemeister move with respect to 3-components and the corresponding Gauss diagrams. We use the move \( \Omega_{III}^{+3} \) included in [2, Table 1].

Then \( P_1, P_2, P_{even} + P_{odd}, \) and \( \mu_{123} = P_{even} - P_{odd} \) are invariants of based flat vertex isotopy classes of 3-bouquet graphs and satisfy \( \mu_{123}(k) = \frac{1}{6}(P_1(k) + P_2(k)) \). We also have that

\[
P_\ast \equiv \text{gcd}(2lk(k_2,k_3), 2lk(k_1,k_3), 2lk(k_1,k_2)) \quad (\ast = 1,2)
\]

is a base-point-free link homotopy invariant.

**Theorem 3.** Let \( G_k \) be a Gauss diagram of an ordered three-component link \( k \). Let 

\[
\sigma = \begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}.
\]

The 18 functions

\[
Q_1^\sigma(k) = \langle \begin{tikzpicture} \draw (0,0) .. controls (0.5,1) and (1,0) .. (1.5,1); \draw (2,0) .. controls (2.5,1) and (3,0) .. (3.5,1); \draw (4,0) .. controls (4.5,1) and (5,0) .. (5.5,1); \end{tikzpicture}, G_k \rangle,
\]

\[
Q_2^\sigma(k) = \langle \begin{tikzpicture} \draw (0,0) .. controls (0.5,1) and (1,0) .. (1.5,1); \draw (2,0) .. controls (2.5,1) and (3,0) .. (3.5,1); \draw (4,0) .. controls (4.5,1) and (5,0) .. (5.5,1); \end{tikzpicture}, G_k \rangle,
\]

and

\[
Q_3^\sigma(k) = \langle \begin{tikzpicture} \draw (0,0) .. controls (0.5,1) and (1,0) .. (1.5,1); \draw (2,0) .. controls (2.5,1) and (3,0) .. (3.5,1); \draw (4,0) .. controls (4.5,1) and (5,0) .. (5.5,1); \end{tikzpicture}, G_k \rangle
\]

are base-point-free link homotopy invariants, each of which is independent of the Milnor invariant \( \mu_{123} \) for \( k \). The 18 functions are also invariants of based flat vertex isotopy classes of 3-bouquet graphs.

If \( n \neq n' \),

\[
\left( Q_n^{id}, Q_n^{(123)}, Q_n^{(132)}, Q_n^{(123)}, Q_n^{(12)}, Q_n^{(13)} \right) \neq \left( Q_{n'}^{id}, Q_{n'}^{(123)}, Q_{n'}^{(132)}, Q_{n'}^{(123)}, Q_{n'}^{(12)}, Q_{n'}^{(13)} \right).
\]

2. Proofs of Theorems 1–3

In this section, every notation of Gauss diagram formulas \( \langle \cdot, \cdot \rangle \) obeys the paper [2] (Tables 1–3). We use the notion of diagram fragments in the paper [2, Section 4.4] of Östlund.

2.1. Invariance under changes of base points. Note that the four equalities hold:

\[
\langle \begin{tikzpicture} \draw (0,0) .. controls (0.5,1) and (1,0) .. (1.5,1); \draw (2,0) .. controls (2.5,1) and (3,0) .. (3.5,1); \end{tikzpicture}, \cdot \rangle = \langle \begin{tikzpicture} \draw (0,0) .. controls (0.5,1) and (1,0) .. (1.5,1); \draw (2,0) .. controls (2.5,1) and (3,0) .. (3.5,1); \end{tikzpicture}, \cdot \rangle,
\]

\[
\langle \begin{tikzpicture} \draw (0,0) .. controls (0.5,1) and (1,0) .. (1.5,1); \draw (2,0) .. controls (2.5,1) and (3,0) .. (3.5,1); \end{tikzpicture}, \cdot \rangle = \langle \begin{tikzpicture} \draw (0,0) .. controls (0.5,1) and (1,0) .. (1.5,1); \draw (2,0) .. controls (2.5,1) and (3,0) .. (3.5,1); \end{tikzpicture}, \cdot \rangle,
\]

\[
\langle \begin{tikzpicture} \draw (0,0) .. controls (0.5,1) and (1,0) .. (1.5,1); \draw (2,0) .. controls (2.5,1) and (3,0) .. (3.5,1); \end{tikzpicture}, \cdot \rangle = \langle \begin{tikzpicture} \draw (0,0) .. controls (0.5,1) and (1,0) .. (1.5,1); \draw (2,0) .. controls (2.5,1) and (3,0) .. (3.5,1); \end{tikzpicture}, \cdot \rangle,
\]

and

\[
\langle \begin{tikzpicture} \draw (0,0) .. controls (0.5,1) and (1,0) .. (1.5,1); \draw (2,0) .. controls (2.5,1) and (3,0) .. (3.5,1); \end{tikzpicture}, \cdot \rangle = \langle \begin{tikzpicture} \draw (0,0) .. controls (0.5,1) and (1,0) .. (1.5,1); \draw (2,0) .. controls (2.5,1) and (3,0) .. (3.5,1); \end{tikzpicture}, \cdot \rangle.
\]

Then it is sufficient to focus on two kinds of the base point moves

\[
\varepsilon ( \rightarrow ) \quad \varepsilon ( \rightarrow ) \quad \varepsilon ( \rightarrow ) \quad \varepsilon ( \rightarrow )
\]
since $-$ sign case is the same up to an overall sign. The differences before and after applying base point moves are as in Tables 1 and 2.

- For $P_\ast$ ($\ast = \text{even, odd}$) or $Q_\ast$, after each of these base point moves is applied, the value $\langle \cdot, G_k \rangle$ differs from the original one by 0.
- For $P_i$ ($i = 1, 2$), after each of these base point moves is applied, the value $\langle \cdot, G_k \rangle$ differs from the original one by $\pm 2l(k_1, k_2)$ or $\pm 2l(k_2, k_3)$.

Thus, Tables 1 and 2 imply the invariances of under the base point moves.

**Table 1.** Case I for permutations id and (13) (the other cases of permutations are easily recovered seeing them). Table indicates a base point move with + sign ($-$ sign case is the same up to an overall sign).

| Move | Difference (Right − Left) | $P_\ast$-type (upper) | $\hat{\mu}_\ast, Q_\ast$-type (lower) |
|------|--------------------------|------------------------|----------------------------------------|
| Counted fragment $1 \uparrow \uparrow \uparrow \downarrow \downarrow \downarrow \rightarrow 3 \uparrow \uparrow \uparrow \downarrow \downarrow \downarrow$ | $-2l(k_2, k_3)$ | 0 |
| Counted fragment $1 \uparrow \uparrow \uparrow \downarrow \downarrow \downarrow \rightarrow 3 \uparrow \uparrow \uparrow \downarrow \downarrow \downarrow$ | $-2l(k_2, k_3)$ | 0 |
| Counted fragment $3 \uparrow \uparrow \uparrow \downarrow \downarrow \downarrow \rightarrow 1 \uparrow \uparrow \uparrow \downarrow \downarrow \downarrow$ | $2l(k_1, k_2)$ | 0 |
| Counted fragment $3 \uparrow \uparrow \uparrow \downarrow \downarrow \downarrow \rightarrow 1 \uparrow \uparrow \uparrow \downarrow \downarrow \downarrow$ | $2l(k_1, k_2)$ | 0 |

**Table 2.** Case II for permutations id and (13) (the other cases of permutations are easily recovered seeing them). Table indicates a base point move with + sign ($-$ sign case is the same up to an overall sign).

| Move | Difference (Right − Left) | $P_\ast$-type (upper) | $\hat{\mu}_\ast, Q_\ast$-type (lower) |
|------|--------------------------|------------------------|----------------------------------------|
| Counted fragment $1 \uparrow \uparrow \uparrow \downarrow \downarrow \downarrow \rightarrow 3 \uparrow \uparrow \uparrow \downarrow \downarrow \downarrow$ | $-2l(k_2, k_3)$ | 0 |
| Counted fragment $1 \uparrow \uparrow \uparrow \downarrow \downarrow \downarrow \rightarrow 3 \uparrow \uparrow \uparrow \downarrow \downarrow \downarrow$ | $-2l(k_2, k_3)$ | 0 |
| Counted fragment $3 \uparrow \uparrow \uparrow \downarrow \downarrow \downarrow \rightarrow 1 \uparrow \uparrow \uparrow \downarrow \downarrow \downarrow$ | $2l(k_1, k_2)$ | 0 |
| Counted fragment $3 \uparrow \uparrow \uparrow \downarrow \downarrow \downarrow \rightarrow 1 \uparrow \uparrow \uparrow \downarrow \downarrow \downarrow$ | $2l(k_2, k_3)$ | 0 |
2.2. Invariance under Reidemeister moves.

2.2.1. Invariance under Reidemeister moves with respect to one/two component(s).

We will use the list of Reidemeister moves [2, Table 1] except for replacing \( \Omega^{3+}\) with \( \Omega^{3+} \) as in [1]. Noting that our formula consisting of four ordered Gauss diagrams \( \Gamma_1 \), \( \Gamma_2 \), \( \Gamma_3 \), and \( \Gamma_4 \), we have the invariance of Reidemeister moves with respect to one component \( (\Omega_{1+}, \Omega_{1-}, \Omega_{2+}, \Omega_{2-}, \Omega_{3+-}, \Omega_{3-+}) \) and with respect to two components: \( \Omega_{II+}, \Omega_{II-}, \Omega_{III+}, \Omega_{III-}, \Omega_{III+-}, \Omega_{III-+} \) [2, Table 1].

2.2.2. Invariance under \( \Omega_{III+} \). Recall that \( \Omega_{III+} \) is as in Fig. 2.

The complete table of the differences of counted fragments by a single Reidemeister move of type \( \Omega_{III+} \) as in Table 3. The contributions to \( P_{\text{even}} \pm P_{\text{odd}} \) or \( Q_i \) \((i = 1, 2, 3)\) do not change before and after applying \( \Omega_{III+} \) as in Table 3. We also have that for a given permutation \( \sigma \), the contributions to \( P_i(k) \) \((i = 1, 2)\) as in the left column of Table 3 are canceled out and the right also, respectively. Thus the invariance of each function holds.

### Table 3. Invariance under move \( \Omega_{III+} \).

| Move | Counted fragment | Counted fragment | Counted fragment |
|------|-----------------|-----------------|-----------------|
| ![Diagram](https://example.com/diagram.png) | \(1^+1^-2^+2^-1\) | \(6^-1^-1^+1^+\) | \(6^-1^-1^+1^+\) |
| ![Diagram](https://example.com/diagram.png) | \(6^-1^-1^+1^+\) | \(5^-2^-3^+3^-\) | \(5^-2^-3^+3^-\) |
| ![Diagram](https://example.com/diagram.png) | \(5^-2^-3^+3^-\) | \(5^-2^-3^+3^-\) | \(5^-2^-3^+3^-\) |

2.3. Identifying our invariants with Milnor’s triple linking number. Recalling [4, 2] gave Fact 1, which implies \( \mu_{123} = P_{\text{even}} - P_{\text{odd}} \).

**Fact 1.** Let \( \sigma = \begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix} \). \( \mu_{123}(k) \) equals

\[
\frac{1}{6} \sum_{\sigma \in S_3} \text{sign}(\sigma) \left( 2 \Gamma_1 + 2 \Gamma_2 + \Gamma_3 + \Gamma_4, G_k \right).
\]

Then \( \mu_{123}(k) \) mod gcd\((lk(k_2,k_3),lk(k_1,k_3),lk(k_1,k_2))\) is Milnor’s triple linking number.

2.4. \( \mu_{123} = P_{\text{even}} - P_{\text{odd}} \) and \( P_{\text{even}} + P_{\text{odd}} \) (or \( P_1, P_2 \)) are independent. Let \( k_1 \) be as in Fig. 3 and its Gauss diagram is as in Fig. 4. By definition, \( \mu_{123} = 0 \), whereas \( P_{\text{even}} + P_{\text{odd}} = m_1 \) for a given integer \( m_1 \). By defining similar links \( k_i \) \((i = 2, 3, 4, 5, \text{and } 6)\), let \( G_{k_i} \) be as in Fig. 4. If \( m_1 \) is odd, Table 4 implies the claim.

\[\text{If you chose the third Reidemeister move } \Omega_{3a} \text{ of [3, } \Omega_{3a}, \text{ the corresponding Gauss diagram here is not } \Omega_{3+} \text{ but } \Omega_{3+} \text{.}\]
2.5. Showing integer-valued bouquet graph invariants $P_1$ and $P_2$ are different and they are nontrivial. Note that a $(6, 0)$-tangle which fixes Gauss diagrams with three base points. Note also that integer-valued function $P_i$ ($i = 1, 2$) is invariant of Reidemeister moves preserving the base points. Therefore, we do not need the argument of Section 2.1 (i.e. invariance under base point moves are not requested). Table 5 implies the claim.

2.6. Independencies of $Q^n$. Let $G_{k_i}$ ($i = 1, 2, 3$) be as in Fig. 4.
**Table 6.** Values of $Q^r_n$ $(n = 1, 2, 3)$ of $k_r$ $(r = 1, 2, 3, 4, 5, \text{and } 6)$. 

| Milnor’s $\mu_{123}$ modulo linking numbers | $k_1$ | $k_2$ | $k_3$ | $k_4$ | $k_5$ | $k_6$ |
|---------------------------------------------|-------|-------|-------|-------|-------|-------|
| $Q^1_n(k)$                                  | $m_1$ | 0     | 0     | 0     | 0     | 0     |
| $Q^{12}_{n}(k)$                             | 0     | $m_2$ | 0     | $m_4$ | 0     | 0     |
| $Q^{13}_{n}(k)$                             | 0     | 0     | $m_3$ | 0     | $m_5$ | 0     |
| $Q^{23}_{n}(k)$                             | 0     | $m_2$ | 0     | $m_4$ | 0     | 0     |
| $Q^{12}_{n}(k)$                             | 0     | 0     | $m_3$ | 0     | $m_5$ | 0     |
| $Q^{13}_{n}(k)$                             | $m_3$ | 0     | 0     | 0     | 0     | $m_6$ |

**Table 7.** Values of $Q^1_{id}$, $Q^2_{id}$, and $Q^3_{id}$ of the based flat vertex isotopy of a bouquet graph $b$ of Fig. 1 and its mirror image $b_{mir}$.

|                  | $b$ | $b_{mir}$ |
|------------------|-----|-----------|
| $Q^1_{id}$       | −1  | −1        |
| $Q^2_{id}$       | −1  | 0         |
| $Q^3_{id}$       | 0   | −1        |

For these links, $\mu_{123} \equiv 0$ (Table 6), which implies the independence of $\mu_{123}$. Table 7 implies the difference $Q^id_n$ and $Q^id_{n'}$ ($n \neq n'$).

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