A New Algorithm for General Cyclic Heptadiagonal Linear Systems Using Sherman–Morrison–Woodbury formula

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ABSTRACT

In this paper, a new efficient computational algorithm is presented for solving cyclic heptadiagonal linear systems based on using of heptadiagonal linear solver and Sherman–Morrison–Woodbury formula. The implementation of the algorithm using computer algebra systems (CAS) such as MAPLE and MATLAB is straightforward. Numerical example is presented for the sake of illustration.

Key Words: Cyclic heptadiagonal matrices; LU factorization; Determinants; Inverse matrix; Sherman–Morrison–Woodbury formula; Linear systems; Computer Algebra System(CAS).

1. INTRODUCTION

The cyclic heptadiagonal linear systems often occur in several fields such as numerical solution of ordinary and partial differential equations, interpolation problems, boundary value problems, etc. [1,2]. In many of these areas, it is necessary to obtain the solution of the cyclic heptadiagonal linear systems.

In this paper, we consider general cyclic heptadiagonal linear systems of the form

\[ HX = R, \]  

where

\[
H = \begin{bmatrix}
    d_1 & a_1 & A_1 & C_1 & 0 & 0 & \cdots & B_1 & b_1 \\
    b_2 & d_2 & a_2 & A_2 & C_2 & \ddots & \ddots & \cdots & \vdots & B_2 \\
    B_3 & b_3 & d_3 & a_3 & A_3 & \ddots & \ddots & \ddots & \vdots & \vdots \\
    D_4 & B_4 & b_4 & d_4 & a_4 & \ddots & \ddots & \ddots & \ddots & \vdots \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
    a_{n-3} & A_{n-3} & C_{n-3} & 0 & 0 & \cdots & b_{n-1} & d_{n-1} & a_{n-1} \\
    0 & \cdots & \cdots & d_{n-2} & a_{n-2} & A_{n-2} & \ddots & \ddots & \ddots & \vdots \\
    A_{n-1} & 0 & \cdots & \cdots & b_{n-1} & d_{n-1} & a_{n-1} & \ddots & \ddots & \ddots & \vdots \\
    a_n & A_n & 0 & \cdots & \cdots & D_n & B_n & b_n & d_n & \ddots
\end{bmatrix}
\]  

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\[ X = (x_1, x_2, \ldots, x_n) \] , \[ R = (R_1, R_2, \ldots, R_n) \]^T , \[ n \geq 8 \].

In paper [3], Karawia gave an efficient symbolic algorithm to obtain the inverse of heptadiagonal matrix of the form (1.2) and then the solution of the systems (1.1) based on LU decomposition. There are many special cases of cyclic heptadiagonal linear systems (See [4-11]).

Recently in [12], a new efficient computational algorithm is presented for solving nearly penta-diagonal linear systems based on the use of any penta-diagonal linear solver. In this paper, we are going to compute the solution of a general cyclic heptadiagonal system of the form (1.1) without imposing any restrictive conditions on the elements of the matrix \( H \) in (1.2). Our approach is mainly based on getting the natural generalization of the algorithm presented in [12]. The development of a symbolic algorithm is considered in order to remove all cases where the numerical algorithm fails.

The paper is organized as follows. In Section 2, symbolic computational algorithm for the solution of heptadiagonal linear system, that will not break, is constructed. In Section 3, Sherman–Morrison–Woodbury formula is given. An illustrative example is given in section 4. In Section 5, Conclusions of the work are presented.

2. Heptadiagonal linear solver

In this section, we shall focus on the construction of new symbolic computational algorithms for computing the solution of general heptadiagonal linear system of the form:

\[ H_h X_h = R_h, \] (2.1)

where

\[
H_h = \begin{bmatrix}
    d_1 & a_1 & A_1 & C_1 & 0 & 0 & \cdots & 0 & 0 \\
    b_2 & d_2 & a_2 & A_2 & C_2 & \cdots & \cdots & \cdots & 0 \\
    B_3 & b_3 & d_3 & a_3 & A_3 & & & & \cdots \\
    D_4 & B_4 & d_4 & a_4 & & & & & \cdots \\
    0 & \cdots & \cdots & \cdots & a_{m-3} & A_{m-3} & C_{m-3} \\
    0 & \cdots & \cdots & \cdots & d_{m-2} & a_{m-2} & A_{m-2} \\
    0 & 0 & \cdots & \cdots & b_{m-1} & d_{m-1} & a_{m-1} \\
    0 & 0 & 0 & \cdots & \cdots & D_m & B_m & b_m & d_m
\end{bmatrix} \] (2.2)

\[ X_h = (x_{h_1}, x_{h_2}, \ldots, x_{h_m}) \] , \[ R = (R_{h_1}, R_{h_2}, \ldots, R_{h_m}) \]^T , \[ m \geq 4 \].

Firstly, we begin with computing the LU factorization of the matrix \( H_h \). It is as in the following:

\[ H_h = LU \] (2.3)

where
The elements in the matrices $L$ and $U$ in (2.4) and (2.5) satisfy:

\[
\alpha_i = \begin{cases} 
1 & \text{if } i = 1 \\
1 & \text{if } i = 2 \\
d_3 - e_3 e_{i-1} - d_3 g_i & \text{if } i = 3 \\
d_3 - D_{i-3} C_{i-3} - e_{i-2} z_{i-2} - d_3 g_i & \text{if } i = 4,5,\ldots,m,
\end{cases}
\]

(2.6)

\[
f_i = \begin{cases} 
\frac{b_i}{\alpha_1} & \text{if } i = 2 \\
\frac{b_i - e_i g_i}{\alpha_2} & \text{if } i = 3 \\
\frac{1}{\alpha_{i-1}} \left(b_i - \frac{D_{i-3}}{\alpha_{i-3}} e_{i-3} z_{i-2}\right) & \text{if } i = 4,5,\ldots,m,
\end{cases}
\]

(2.7)
\[ e_j = \begin{cases} B_3 & \text{if } i = 3 \\ \frac{1}{\alpha_j} \left( B_i - \frac{D_i}{\alpha_{i-2}} g_{i-3} \right) & \text{if } i = 4, 5, \ldots, m, \end{cases} \tag{2.8} \]

\[ g_i = \begin{cases} a_i & \text{if } i = 1 \\ a_i - f_i z_i & \text{if } i = 2 \\ a_i - f_i z_{i-3} - e_i C_{i-2} & \text{if } i = 3, 4, \ldots, m - 1, \end{cases} \tag{2.9} \]

\[ z_i = \begin{cases} A_3 & \text{if } i = 1 \\ A_i - f_i C_{i-1} & \text{if } i = 2, 3, \ldots, m - 2. \end{cases} \tag{2.10} \]

We also have:

\[ \det H_{h_i} = \prod_{i=1}^{n} \alpha_i. \tag{2.11} \]

Then, the solution is given by

\[ x_{h_i} = \begin{cases} \frac{Q_{h_0}}{\alpha_m} & \text{if } i = m \\ \frac{1}{\alpha_{m-1}} \left( Q_{h_{m-1}} - g_{m-1} x_{h_{m-1}} \right) & \text{if } i = m - 1 \\ \frac{1}{\alpha_{m-2}} \left( Q_{h_{m-2}} - g_{m-2} x_{h_{m-2}} - z_{m-2} x_{h_{m-2}} \right) & \text{if } i = m - 2 \\ \frac{1}{\alpha_i} \left( Q_{h_i} - g_i x_{h_{i+1}} - z_i x_{h_{i+1}} - C_i x_{h_{i+1}} \right) & \text{if } i = m - 3, m - 4, \ldots, 1, \end{cases} \tag{2.12} \]

where

\[ Q_{h_i} = \begin{cases} R_{h_1} & \text{if } i = 1 \\ R_{h_i} - f_i Q_{h_i} & \text{if } i = 2 \\ R_{h_i} - e_i Q_{h_{i-1}} - f_i Q_{h_{i-1}} & \text{if } i = 3 \\ R_{h_i} - \frac{D_i}{\alpha_{i-3}} Q_{h_{i-3}} - e_i Q_{h_{i-2}} - f_i Q_{h_{i-2}} & \text{if } i = 4, 5, \ldots. \end{cases} \tag{2.13} \]

At this point it is convenient to formulate our first result. It is a symbolic algorithm for computing the solution of a heptadiagonal linear system of the form (2.1).

**Algorithm 2.1.** To compute the solution of a heptadiagonal linear system of the form (2.1), we may proceed as follows:

**Step 1:** Set \( \alpha_j = d_j \). If \( \alpha_j = 0 \) then \( \alpha_j = t \) end if. Set \( g_{1} = a_{1}, z_{1} = a_{1}, k_{1} = a_{1}/\alpha_{1}, k_{1} = b_{1}, f_{1} = b_{1}/\alpha_{1}, e_{2} = b_{2}/\alpha_{2}, \alpha_{3} = d_{3} - f_{2} * g_{2} \). If \( \alpha_{2} = 0 \) then \( \alpha_{2} = t \) end if. Set \( g_{2} = a_{2} - f_{2} * z_{1}, f_{2} = (b_{2} - e_{2} * g_{2})/\alpha_{2}, \alpha_{3} = d_{3} - e_{2} * z_{1} * f_{1} * g_{2} \). If \( \alpha_{2} = 0 \) then \( \alpha_{2} = t \) end if.

**Step 2:** Compute and simplify:

For \( i \) from 4 to \( m \) do

- \( e_{i} = (B_{i} - D_{i} * g_{i} * a_{i})/a_{i} \)
- \( f_{i} = (b_{i} - D_{i} * z_{i}/a_{i} - e_{i} * g_{i} * a_{i})/a_{i} \)
- \( z_{i} = a_{i} * f_{i} * C_{i} \)
- \( g_{i} = a_{i} * f_{i} * z_{i} * e_{i} * C_{i} \)
\[ \alpha_i = (d_i - D_i \alpha_i - C_i - e_i z_i f_i g_{i,1}) \]

If \( \alpha_i = 0 \) then \( \alpha_i = t \) end if

**Step 3:** Compute \( \det H_i = \left( \prod_{t=1}^{n} \alpha_t \right) \).

**Step 4:** If \( \det H_i \neq 0 \) then do

**Step 5:** Set \( Q_{i=} = R_{ii} \), \( Q_{i2} = R_{i2} f_2 Q_{i1} \), \( Q_{i3} = R_{i3} e_3 Q_{i2} f_3 Q_{i2} \),

Compute and simplify:

For \( i \) from 4 to \( m \) do

\[ Q_{hi} = R_{hi} - D_i Q_{hi} - C_i z_i f_i g_i \]

End do

**Step 6:** Set \( x_{m} = Q_{mm} \), \( x_{m-1} = (Q_{m-2} - R_{m-1} x_{m-2}) / \alpha_{m-1} \), \( x_{m-2} = (Q_{m-3} - R_{m-2} x_{m-3}) / \alpha_{m-2} \),

Compute and simplify:

For \( i \) from \( m-3 \) by -1 to 1 do

\[ x_i = (Q_{i} - R_{i-1} x_{i+1} - C_i z_i x_{i+2} - g_i x_{i+3}) / \alpha_i \]

End do

**Step 8:** Compute and simplify the solution:

For \( i \) from 1 to \( m \) do

\[ x_i = (x_i)_{t=0} \]

End do

Else

OUTPUT("The matrix \( H_i \) is singular"); Stop.

End if

The new algorithm 2.1 is very useful to check the nonsingularity of the matrix \( H_i \).

### 3. Sherman–Morrison–Woodbury formula[13]

In this section, we are going to formulate a new computational algorithm for solving cyclic heptadiagonal linear systems of the form (1.1) based on the previous heptadiagonal linear solver. The heptadiagonal linear system of the form (1.1) can be written in the form:

\[
\begin{pmatrix}
M_1 & V \\
U^T & M_2
\end{pmatrix}
\begin{pmatrix}
x' \\
x''
\end{pmatrix} = \begin{pmatrix}
R' \\
R''
\end{pmatrix}
\]  

(3.1)

where

\[
M_1 = \begin{bmatrix}
d_1 & a_1 & A_1 & C_1 & 0 & 0 & \ldots & 0 & 0 \\
d_2 & a_2 & A_2 & C_2 & \ldots & \ldots & \ldots & 0 \\
B_3 & B_3 & A_3 & \ldots & \ldots & \ldots & \ldots & \ldots \\
D_4 & D_4 & A_4 & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & \ldots & \ldots & d_{n-4} & a_{n-4} & A_{n-4} \\
0 & 0 & \ldots & \ldots & \ldots & d_{n-3} & a_{n-3} & A_{n-3} \\
0 & 0 & \ldots & \ldots & \ldots & d_{n-2} & A_{n-2} & C_{n-2} \\
0 & 0 & \ldots & \ldots & \ldots & d_{n-1} & A_{n-1} & C_{n-1} \\
\end{bmatrix}
\]

\[1 \cdot M_2 = \begin{bmatrix}
d_{n-1} & a_{n-1} \\
b_{n} & d_{n}
\end{bmatrix}
\]
\[
V = \begin{bmatrix}
  v_1^T \\
v_2^T
\end{bmatrix}^T = \begin{bmatrix}
  B_1 & 0 & 0 & \cdots & \cdots & C_{n-4} & A_{n-3} & a_{n-2} \\
b_1 & B_2 & 0 & \cdots & \cdots & D_{n-1} & B_{n-1} & b_{n-1}
\end{bmatrix}^T,
\]
\[
U^T = \begin{bmatrix}
  A_{n-1} & 0 & 0 & \cdots & \cdots & 0 \\
a_n & A_n & 0 & \cdots & \cdots & 0
\end{bmatrix},
\]
\[
x' = (x_1, x_2, \ldots, x_{n-2})^T, x'' = (x_{n-1}, x_n)^T, R' = (R_1, R_2, \ldots, R_{n-2})^T, \text{ and } R'' = (R_{n-1}, R_n)^T.
\]

Thus (3.1) is equivalent to
\[
M_1 x' + V x'' = R'
\]
\[
U^T x' + M_2 x'' = R''
\]

Assume that \( M_2 \) is nonsingular. After elimination of \( x'' \) from (3.2), we get the linear systems
\[
M x' = \hat{R}
\]

where \( M = M_1 - VM_2^{-1}U^T \), \( \hat{R} = R' - VM_2^{-1}R'' \).

If we applying the Sherman–Morrison–Woodbury formula to \( M \), we obtain
\[
M^{-1} = M_1^{-1} + M_1^{-1}V (M_2 - U^T M_1^{-1}V)^{-1} U^T M_1^{-1} \text{ and } x' = M^{-1}\hat{R} = y + M_1^{-1}V (M_2 - U^T M_1^{-1}V)^{-1} U^T y
\]

where \( y \) is the solution of \( M_1 y = \hat{R} \). It is clear that the solution \( x'' \) can be found from the above formula by successive calculation of the expressions
\[
y = M_1^{-1}\hat{R}, \quad M_1^{-1}V, \quad U^T M_1^{-1}V, \quad (M_2 - U^T M_1^{-1}V)^{-1}, \text{ and } (M_2 - U^T M_1^{-1}V)^{-1} U^T y.
\]

The main part of above calculations is in finding the first two expressions, which is equivalent to solving three \((n-2)\)-by-\((n-2)\) heptadiagonal linear systems with the same coefficient matrix \( M_1 \) and different right-hand sides. After finding of \( x' \), we can get \( x'' \) from the second equation of (3.2) by formula
\[
x'' = M_2^{-1} (R'' - U^T x').
\]

At this point it is convenient to formulate our second result. It is a symbolic algorithm for computing the solution of a cyclic heptadiagonal linear system of the form (1.1) and can be considered as natural generalization of the symbolic algorithm 1 in [12].

**Algorithm 3.1.** To compute the solution of a cyclic heptadiagonal linear system of the form (1.1), we may proceed as follows:

**Step 1:** Find \( M_1, M_2, U^T, V, R', R'' \), and \( \hat{R} = R' - VM_2^{-1}R'' \).

**Step 2:** Solve \( M_1 y = \hat{R}, M_1 q_1 = v_1, \) and \( M_1 q_2 = v_2 \) by algorithm 2.1, then obtain \( y \) and \( M_1^{-1}v = (q_1, q_2) \).

**Step 3:** Compute \( x' = y + (q_1, q_2)(M_2 - U^T (q_1, q_2))^{-1}U^T y, x'' = M_2^{-1}(R'' - U^T x'). \)

**Step 4:** Compute the solution \( x = \left(\begin{array}{c}
x'
\end{array}\right)_{i=0} \).

Three systems \( M_1 y = \hat{R}, M_1 q_1 = v_1, \) and \( M_1 q_2 = v_2 \) in algorithm 3.1 can be solved in parallel.
4. An illustrative example

In this section, we give simple numerical experiment to illustrate the effectiveness of our algorithms.

Example 4.1. Consider the 10-by-10 cyclic heptadiagonal systems coming from [3]

\[
\begin{bmatrix}
1 & -1 & 1 & -2 & 0 & 0 & 0 & 0 & 2 & -1 \\
1 & 1 & 1 & 1 & -1 & 0 & 0 & 0 & 1 & \\
2 & 1 & -1 & 1 & 2 & 3 & 0 & 0 & 0 & 0 \\
2 & -2 & 3 & 1 & 5 & -6 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 0 & 0 \\
0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 \\
0 & 0 & 0 & 2 & 2 & 2 & 2 & 3 & 1 & -3 \\
0 & 0 & 0 & 0 & -2 & -2 & 1 & 1 & 3 & 5 \\
3 & 0 & 0 & 0 & 0 & 3 & 1 & 3 & 4 & -1 \\
2 & 4 & 0 & 0 & 0 & 0 & 2 & 3 & 4 & 1 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7 \\
x_8 \\
x_9 \\
x_{10} \\
\end{bmatrix} = 
\begin{bmatrix}
2 \\
15 \\
33 \\
33 \\
43 \\
-24 \\
47 \\
70 \\
78 \\
94 \\
\end{bmatrix}
\tag{4.1}
\]

Solution: The application of Algorithm 3.1 gives

Step1: 

\[
M_1 = 
\begin{bmatrix}
1 & -1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & -1 & 1 & 2 & 3 & 0 & 0 & 0 & 0 \\
2 & -2 & 3 & 1 & 5 & -6 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 0 & 0 \\
0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 \\
0 & 0 & 0 & 2 & 2 & 2 & 2 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 & -2 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
, M_2 = 
\begin{bmatrix}
4 & -1 \\
4 & 1 \\
\end{bmatrix}
, 
U^T = 
\begin{bmatrix}
3 & 0 & 0 & 0 & 0 & 3 & 1 & 3 \\
2 & 4 & 0 & 0 & 0 & 0 & 2 & 3 \\
\end{bmatrix}
, V = 
\begin{bmatrix}
v_1^T \\
v_2^T \\
\end{bmatrix} = 
\begin{bmatrix}
2 & 0 & 0 & 0 & 0 & 1 & 1 & 3 \\
-1 & 1 & 0 & 0 & 0 & 0 & -3 & 5 \\
\end{bmatrix}^T
, 
R' = \begin{bmatrix}
2 & 15 & 33 & 0 & 43 & -24 & 47 & 70 \\
\end{bmatrix}^T, R'' = \begin{bmatrix}
78 & 94 \\
\end{bmatrix}^T,
\]

\[
\tilde{R} = R' - VM_2^{-1} R'' = (-33, 7, 33, 0, 43, \frac{-91}{2}, 99, \frac{-69}{2})^T.
\]

Step2: 

\[
y = \begin{bmatrix}
-2814 & 3345 & 2208 & 1308 & 2654 & 4442 & 15739 & -7685 \\
199 & 199 & 199 & 199 & 199 & 199 & 199 & 199 \\
\end{bmatrix}^T,
\]

\[
M_1^T V = (q_1, q_2) = 
\begin{bmatrix}
242 & -98 & -150 & -104 & -110 & -212 & -184 & 297 \\
199 & 199 & 199 & 199 & 199 & 199 & 199 & 199 \\
6 & 861 & -132 & -394 & 142 & -895 & 4630 & -861 \\
199 & 199 & 199 & 199 & 199 & 199 & 199 & 199 \\
\end{bmatrix}^T
\]

Step3: 

\[
x' = y + (q_1, q_2)(M_2 - U^T(q_1, q_2))^{-1}U^T y = (1, 2, 3, 4, 5, 6, 7, 8),
\]
\[ x^* = M^{-1}_2 (R^* U^T x') = (9, 10). \]

Step 4: The solution of (4.1) is
\[ x = \begin{pmatrix} x' \\ x^* \end{pmatrix} = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10)^T. \]

5. Conclusions

In this paper, we derived a computational algorithm for solving the cyclic heptadiagonal linear systems. Since the algorithm uses the symbolic algorithm 2.1 method for the LU factorization of the heptadiagonal matrix, the factorization never suffers from breakdown, and this leads to the fast and reliable solution of cyclic heptadiagonal linear systems. The realization of the method needs \( O(n) \) operations. The algorithms are natural generalizations of some algorithms in current use.

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