Extending the Bruhat order and the length function from the Weyl group to the Weyl monoid

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Abstract
For a symmetrizable Kac-Moody algebra the category of admissible representations is an analogue of the category of finite dimensional representations of a semisimple Lie algebra. The monoid associated to this category and the category of restricted duals by a generalized Tannaka-Krein reconstruction contains the Kac-Moody group as open dense unit group and has similar properties as a reductive algebraic monoid. In particular there are Bruhat and Birkhoff decompositions, the Weyl group replaced by the Weyl monoid, [M1].

We determine the closure relations of the Bruhat and Birkhoff cells, which give extensions of the Bruhat order from the Weyl group to the Weyl monoid. We show that the Bruhat and Birkhoff cells are irreducible and principal open in their closures. We give product decompositions of the Bruhat and Birkhoff cells. We define extended length functions, which are compatible with the extended Bruhat orders. We show a generalization of some of the Tits axioms for twinned BN-pairs.

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Introduction

The Kac-Moody group $G$ constructed in [K,P1] by V. Kac and D. Peterson is a group analogue of a semisimple, simply connected algebraic group. In particular there are Bruhat and Birkhoff decompositions

\[ G = \bigcup_{w \in W} B^\epsilon wB^\delta \quad (\epsilon, \delta) \in \{ (+, +), (-, -), (-, +), (+, -) \} , \]
and also certain multiplicative decompositions of the Bruhat and Birkhoff cells like

\[ BwB = U_w \cdot wT \cdot U \quad \text{and} \quad B^- wB = (U^w)^- \cdot wT \cdot U. \]

(Here \( T \) denotes a maximal torus, \( W \) the Weyl group, and \( B = B^+, B^- \) denote opposite Borel subgroups containing \( T \).) V. Kac and D. Peterson equipped in [K,P 2] a symmetrizable Kac-Moody group with a coordinate ring, the algebra of strongly regular functions \( \mathbb{C}[G] \). This coordinate ring has many properties in common with the coordinate ring of a semisimple, simply connected algebraic group. It is an integrally closed domain, even a unique factorization domain. It admits a Peter and Weyl theorem, i.e.,

\[ \mathbb{C}[G] \cong \bigoplus_{\Lambda \in P^+} L^*(\Lambda) \otimes L(\Lambda) \]

as \( G \times G \)-modules. (Here \( L(\Lambda) \) denotes an irreducible highest weight module with highest weight \( \Lambda \), \( L^*(\Lambda) \) its restricted dual, and \( P^+ \) the set of dominant weights.) As a difference, in the non-classical case, the multiplication map and the inverse map of \( G \) do not induce comorphisms. V. Kac and D. Peterson showed that the Zariski closures of the following Bruhat and Birkhoff cells are obtained similarly to the classical case, using the Bruhat order of the Weyl group \( \hat{W} \):

\[ B^\epsilon wB^\epsilon = \bigcup_{w' \leq w} B^\epsilon w'B^\epsilon \quad \text{where} \quad \epsilon \in \{+,-\}, \quad B^- wB = \bigcup_{w' \geq w} B^- w'B. \]

In [M 1] we determined by a generalized Tannaka-Krein reconstruction the monoid with coordinate ring \( (\hat{G}, \mathbb{C}[\hat{G}]) \) associated to a natural category determined by the modules \( L(\Lambda), \Lambda \in P^+ \), and a category of duals determined by \( L^*(\Lambda), \Lambda \in P^+ \). For its history in connection with V. Kac, D. Peterson, and P. Slodowy please look at the introduction of [M 1]. The monoid \( \hat{G} \) contains the Kac-Moody group \( G \) as open dense unit group. In particular its coordinate ring \( \mathbb{C}[\hat{G}] \) is isomorphic to the algebra of strongly regular functions \( \mathbb{C}[G] \) by the restriction map.

This monoid is a purely infinite-dimensional phenomenon. In the classical case it reduces to a semisimple simply connected algebraic group. It is a proper analogue of such a group. For generalizing some results of classical invariant theory it should be more fundamental than the Kac-Moody group itself.

In [M 1] we also showed that the monoid \( \hat{G} \) has similar structural properties as a normal reductive algebraic monoid. In particular there are Bruhat and Birkhoff decompositions

\[ \hat{G} = \bigcup_{\hat{w} \in \hat{W}} B^\epsilon \hat{w}B^\delta \quad (\epsilon, \delta) \in \{ (+, +), (-, -), (-, +), (+, -) \}, \]

the Weyl group replaced by the Weyl monoid \( \hat{W} \), which is an analogue of a Renner monoid. It contains the Weyl group as unit group and its idempotents correspond bijectively to the faces of the Tits cone.
In $\mathbb{M}^2$ we determined and investigated the $\mathbb{C}$-valued points of $\mathbb{C}[\hat{G}]$. Identifying the elements of $\hat{G}$ with their evaluation morphisms, $\hat{G}$ embeds in the set of $\mathbb{C}$-valued points of $\mathbb{C}[\hat{G}]$. The Bruhat decompositions of $\hat{G}$ do not extend, but one of the Birkhoff decompositions of $\hat{G}$ extends to a decomposition of the $\mathbb{C}$-valued points.

In $\mathbb{M}^1$, $\mathbb{M}^2$ the Bruhat and Birkhoff cells have not been investigated further. In particular their closure relations, which determine extensions of the Bruhat order from the Weyl group to the Weyl monoid, have not been determined.

For reductive algebraic monoids these questions have already been investigated: L. E. Renner studied in $\mathbb{Re}^1$, $\mathbb{Re}^2$ the closure relation of the Bruhat cells of a reductive algebraic monoid. Transferred to the Renner monoid he called this order the Bruhat-Chevalley order. He showed that all maximal chains between two elements have the same length. He also showed a monoid version of one of the Tits axioms for $BN$-pairs. An algebraic description of the Bruhat-Chevalley order has been obtained by E. A. Pennel, M. S. Putcha, and R. E. Renner in $\mathbb{Pe, Pu, Re}$. An investigation of the lexicographic shellability has been started by M. S. Putcha in $\mathbb{Pu}$.

A length function for the Renner monoid of matrices over a finite field has first been introduced in $\mathbb{So}$ by L. Solomon. He also showed that this length function fits to the monoid generalization of one of the Tits axiom mentioned above. A length function for the Renner monoid of an arbitrary reductive algebraic monoid has been introduced and investigated by L. E. Renner in $\mathbb{Re}^2$ by a different approach. It has the property that the length of a maximal chain of the Bruhat-Chevalley order, which is contained in an orbit of the action of the product of the Weyl group on the Renner monoid, is given by the difference of the length of the maximal and the minimal element of the chain. He also showed that this length function fits to his monoid generalization of one of the Tits axioms. A subadditivity property has been shown by E. A. Pennel, M. S. Putcha, and R. E. Renner in $\mathbb{Pe, Pu, Re}$.

L. E. Renner gave in $\mathbb{Re}^3$ a product decomposition of a Bruhat cell of the wonderful compactification of a semisimple algebraic group.

Note also that many of these results just mentioned induce, by using the longest element of the Weyl group, corresponding results for the Birkhoff cells.

In this article we show that some of these results are also valid for the monoid $\hat{G}$ and the spectrum $\text{Specm} \mathbb{C}[\hat{G}]$: In Section 2 we obtain a similar algebraic description of the closure relations of the Bruhat and Birkhoff cells of $\hat{G}$, and of the Birkhoff cells of $\text{Specm} \mathbb{C}[\hat{G}]$. We show that these relations are actually order relations, which is not automatic in our infinite dimensional situation. Transferred to the Weyl monoid we call these orders the extended Bruhat and Birkhoff orders.

We show that the Bruhat and Birkhoff cells are irreducible and principal open in their closures. We equip the Bruhat and Birkhoff cells with their coordinate rings as principal open sets and give product decompositions of these cells.

In Section 3 we extend the length function on the Weyl group to functions on
the Weyl monoid, compatible with the extended Bruhat orders. We show a
generalization of some of the Tits axiom for groups with twinned $BN$-pairs.

To obtain these results is often more arduous as for reductive algebraic monoids,
and we have to use different methods for the following reasons: Most of the the-
orems of algebraic geometry, which are used to investigate algebraic groups and
monoids, break down for the infinite-dimensional varieties we have to use. In
the non-classical case the multiplication map of $\hat{G}$ is not a morphism, only the
left and right multiplications with elements of $\hat{G}$ are morphisms. The Weyl
group and the Weyl monoid have infinitely many elements. In particular there
is no longest element of the Weyl group, and there are infinitely many Bruhat
and Birkhoff cells.

For our algebraic geometric investigations we mainly use the following two aids:
The explicit description of the coordinate ring of $\hat{G}$ by the Peter and Weyl theo-
rem. This allows to use properties of the action of parts of $\hat{G}$ on the irreducible
highest weight representations $L(\Lambda), \Lambda \in P^+$, to obtain algebraic geometric
results. As an advantage our morphisms, closures, principal open sets are de-
scribed very explicitly by using matrix coefficients. The monoid $\hat{G}$ does not act
on the flag varieties, but it acts on the corresponding affine cones by morphisms.
This allows to make use of the combinatorial properties of these cones described
by V. Kac and D. Peterson in $[K,P1]$.

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1 Preliminaries

In this section we collect some basic facts about Kac-Moody algebras, minimal
and formal Kac-Moody groups, and the corresponding monoid completions,
which are used later. One aim is to introduce our notation. Another aim is
to put these things, which are scattered between many articles and books, on
equal footing appropriate for our goals.

All the material stated in this section about Kac-Moody algebras can be found
in the books $[K]$ (most results also valid for a field of characteristic zero with
the same proofs), $[Mo,Pi]$. The facts about the minimal Kac-Moody group can
be found in $[K,P1], [K,P3], [Mo,Pi]$, about the formal Kac-Moody group in
called the weight lattice $\Lambda$. The Weyl group $H$ is fixed.

We set $H^1 := \{ h \in H \mid h \in H \}$, about the spectrum of $F$-valued points of its coordinate ring in $M^2$.

We denote by $N = Z^+$, $Q^+$, and the sets of strictly positive numbers of $Z$, $Q$, and $R$, and the sets $N_0 = Z_0^+$, $Q_0^+$, $R^+_0$ contain, in addition, the zero. In the whole paper, $F$ is a field of characteristic 0 and $F^\times$ its group of units.

**Generalized Cartan matrices:** Starting point for the construction of a Kac-Moody algebra and its associated simply connected minimal and formal Kac-Moody groups is a generalized Cartan matrix, which is a matrix $A = (a_{ij}) \in M_n(Z)$ with $a_{ii} = 2$, $a_{ij} \leq 0$ for all $i \neq j$, and $a_{ij} = 0$ if and only if $a_{ji} = 0$. Denote by $l$ the rank of $A$, and set $I := \{1, 2, \ldots, n\}$.

For the properties of the generalized Cartan matrices, in particular their classification, we refer to the book [K]. In this paper we assume $A$ to be symmetrizable.

**Realizations:** A simply connected minimal free realization of $A$ consists of dual free $Z$-modules $H$, $P$ of rank $2n - l$, and linear independent sets $\Pi^i = \{h_1, \ldots, h_n\} \subseteq H_i$, $\Pi = \{\alpha_1, \ldots, \alpha_n\} \subseteq P$ such that $\alpha_i(h_j) = \delta_{ij}$, $i, j = 1, \ldots, n$. Furthermore there exist (non-uniquely determined) fundamental dominant weights $\Lambda_1, \ldots, \Lambda_n \in P$ such that $\Lambda_i(h_j) = \delta_{ij}$, $i, j = 1, \ldots, n$. $P$ is called the weight lattice, and $Q := Z$-span $\{ \alpha_i \mid i \in I \}$ the root lattice. Set $Q_0^+ := Z_0^+$-span $\{ \alpha_i \mid i \in I \}$, and $Q^+ := Q_0^+ \setminus \{0\}$.

We fix a system of fundamental dominant weights $\Lambda_1, \ldots, \Lambda_n$, and extend $h_1, \ldots, h_n \in H$, $\Lambda_1, \ldots, \Lambda_n \in P$ to a pair of dual bases $h_1, \ldots, h_{2n-l} \in H$, $\Lambda_1, \ldots, \Lambda_{2n-l} \in P$. We set $H_{rest} := Z$-span $\{ h_i \mid i = n + 1, \ldots, 2n-l \}$.

**The Weyl group:** Identify $H$ and $P$ with the corresponding sublattices of the following vector spaces over $F$:

$$h := h_F := H \otimes_Z F \quad , \quad h^* := h_F^* := P \otimes_Z F.$$ 

$h^*$ is interpreted as the dual of $h$. Order the elements of $h^*$ by $\lambda \leq \lambda'$ if and only if $\lambda' - \lambda \in Q_0^+$. Choose a symmetric matrix $B \in M_n(Q)$ and a diagonal matrix $D = \text{diag}(\epsilon_1, \ldots, \epsilon_n)$, $\epsilon_1, \ldots, \epsilon_n \in Q^+$, such that $A = DB$. Define a nondegenerate symmetric bilinear form on $h$ by:

$$(h_i \mid h) = (h \mid h_i) := \alpha_i(h) \epsilon_i \quad i \in I, \quad h \in h,$n

$$(h' \mid h'') := 0 \quad h', h'' \in h_{rest} := H_{rest} \otimes F.$$ 

Denote the induced nondegenerate symmetric form on $h^*$ also by $(\mid \mid)$.

The Weyl group $W = W(A)$ is the Coxeter group with generators $\sigma_i$, $i \in I$, and relations $\sigma_i^2 = 1$ $(i \in I)$, $(\sigma_i \sigma_j)^{m_{ij}} = 1$ $(i, j \in I, i \neq j)$. 

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The \( m_{ij} \) are given by: 
\[
\begin{array}{c|cccc}
   & 0 & 1 & 2 & 3 & 4 & 6 \\
\hline
2 & 0 & 1 & 2 & 3 & 4 & 6 \\
3 & 1 & 0 & 1 & 2 & 3 & 4 \\
4 & 2 & 1 & 0 & 1 & 2 & 3 \\
6 & 4 & 3 & 2 & 1 & 0 & 1 \\
\end{array}
\]
no relation between \( \sigma_i \) and \( \sigma_j \).

The Weyl group \( \mathcal{W} \) acts faithfully and contragrediently on \( \mathbf{h} \) and \( \mathbf{h}^* \) by

\[
\sigma_i h := h - \alpha_i(h) h_i , \quad \sigma_i \lambda := \lambda - \lambda(h_i) \alpha_i , \quad i \in I , \quad h \in \mathbf{h} , \quad \lambda \in \mathbf{h}^* ,
\]
leaving the lattices \( H, Q, P \), and the forms invariant. \( \Delta_{re} := \mathcal{W} \{ \alpha_i \mid i \in I \} \) is called the set of real roots, and \( \Delta^\vee_{re} := \mathcal{W} \{ h_i \mid i \in I \} \) the set of real coroots.

The map \( \alpha_i \mapsto h_i, \ i \in I \), can be extended to a \( \mathcal{W} \)-equivariant bijection \( \alpha \mapsto h_\alpha \).

**The Tits cone and its faces:** To illustrate the action of \( \mathcal{W} \) on \( \mathbf{h}_R^* \) geometrically, for \( J \subseteq I \) define

\[
\begin{align*}
F_J & := \{ \lambda \in \mathbf{h}_R^* \mid \lambda(h_i) = 0 \text{ for } i \in J, \ \lambda(h_i) > 0 \text{ for } i \in I \setminus J \}, \\
\overline{F}_J & := \{ \lambda \in \mathbf{h}_R^* \mid \lambda(h_i) = 0 \text{ for } i \in J, \ \lambda(h_i) \geq 0 \text{ for } i \in I \setminus J \}.
\end{align*}
\]

\( \overline{F}_J \) is a finitely generated convex cone with relative interior \( F_J \). The parabolic subgroup \( \mathcal{W}_J \) of \( \mathcal{W} \) is the stabilizer of every element \( \lambda \in F_J \). For \( \sigma \in \mathcal{W} \) call \( \sigma F_J \) a facet of type \( J \).

The fundamental chamber \( \overline{C} := \{ \lambda \in \mathbf{h}_R^* \mid \lambda(h_i) \geq 0 \text{ for } i \in I \} \) is a fundamental region for the action of \( \mathcal{W} \) on the convex cone \( X := \mathcal{W} \overline{C} \), which is called the Tits cone. The partition \( \overline{C} = \bigcup_{J \subseteq I} F_J \) induces a \( \mathcal{W} \)-invariant partition of \( X \) into facets.

We denote the set of faces of the Tits cone \( X \) by \( \mathcal{R}(X) \). These faces can be described as follows: A set \( \Theta \subseteq I \) is called special, if either \( \Theta = \emptyset \), or else all connected components of the generalized Cartan submatrix \( (a_{ij})_{i,j \in \Theta} \) are of non-finite type. Set \( \Theta^\perp := \{ i \in I \mid a_{ij} = 0 \text{ for all } j \in \Theta \} \). Every face of the Tits cone \( X \) is \( \mathcal{W} \)-conjugate to exactly one of the faces

\[
\begin{align*}
R(\Theta) & := X \cap \{ \lambda \in \mathbf{h}_R^* \mid \lambda(h_i) = 0 \text{ for all } i \in \Theta \} = \mathcal{W}_{\Theta^\perp} \overline{F}_\Theta , \quad \Theta \text{ special}.
\end{align*}
\]

The parabolic subgroup \( \mathcal{W}_\Theta \) is the pointwise stabilizer of \( R(\Theta) \), and the parabolic subgroup \( \mathcal{W}_{\Theta^\perp \cup \Theta^\perp} \) is the stabilizer of the set \( R(\Theta) \) as a whole.

The relative interior of \( R(\Theta) \) is given by the union of the facets \( \sigma F_{\Theta^\perp \cup \Theta^\perp} \), where \( \sigma \in \mathcal{W}_{\Theta^\perp} \), and \( \Theta_f \) is a subset of \( \Theta^\perp \), which is either empty, or else for which all connected components of \( (a_{ij})_{i,j \in \Theta_f} \) are of finite type.

**The Weyl monoid:** The Weyl group acts on the monoid \( (\mathcal{R}(X), \cap) \). The semidirect product \( \mathcal{R}(X) \rtimes \mathcal{W} \) consists of the set \( \mathcal{R}(X) \times \mathcal{W} \) with the structure of a monoid given by

\[
(R, \sigma) \cdot (S, \tau) := (R \cap \sigma S, \sigma \tau).
\]

For \( R \in \mathcal{R}(X) \) let \( Z_\mathcal{W}(R) := \{ \sigma \in \mathcal{W} \mid \sigma \lambda = \lambda \text{ for all } \lambda \in R \} \) be the pointwise stabilizer of \( R \). The Weyl monoid \( \overline{\mathcal{W}} \) is defined as the monoid \( \mathcal{R}(X) \rtimes \mathcal{W} \) factored by the congruence relation

\[
(R, \sigma) \sim (R', \sigma') \iff R = R' \text{ and } \sigma' \sigma^{-1} \in Z_\mathcal{W}(R).
\]
We denote the congruence class of \((R, \sigma)\) by \(\varepsilon(R)\sigma\).
Assigning to \(\sigma \in W\) the element \(\sigma := \varepsilon(X)\sigma \in \hat{W}\), the Weyl group \(W\) identifies with the unit group of \(\hat{W}\). It acts in the obvious way on \(\hat{W}\), the partition of \(\hat{W}\) into \(W \times W\)-orbits given by
\[
\hat{W} = \bigcup_{\Theta \text{ special}} W \varepsilon(R(\Theta)) W.
\]
Assigning to \(R \in \mathcal{R}(X)\) the element \(\varepsilon(R) := \varepsilon(R)1 \in \hat{W}\), the monoid \((\mathcal{R}(X), \cap)\) embeds into \(\hat{W}\). Its image are the idempotents of \(\hat{W}\). By this map, the action of \(W\) on \(\mathcal{R}(X)\) identifies with the restricted conjugation action of \(W\) on \(\hat{W}\), i.e.,
\[
we(R)w^{-1} = \varepsilon(wR), \quad w \in W, \quad R \in \mathcal{R}(X).
\]
Recall that a monoid \(M\) is called an inverse monoid, if for every element \(m \in M\) there exists a unique element \(m^{\text{inv}} \in M\), such that \(mm^{\text{inv}}m = m\) and \(m^{\text{inv}}mm^{\text{inv}} = m^{\text{inv}}\). The inverse map \(\varepsilon\) embeds into \(\hat{W}\) with inverse map \(\varepsilon\). Its image are the idempotents of \(\hat{W}\). By this map, the action of \(W\) on \(\mathcal{R}(X)\) identifies with the restricted conjugation action of \(W\) on \(\hat{W}\), i.e.,
\[
\varepsilon(R)w^{-1} = \varepsilon(wR), \quad w \in W, \quad R \in \mathcal{R}(X).
\]
The Kac-Moody algebra: The Kac-Moody algebra \(g = g(A)\) is the Lie algebra over \(\mathbb{F}\) generated by the abelian Lie algebra \(h\) and \(2n\) elements \(e_i, f_i, i \in I\), with the following relations, which hold for any \(i, j \in I\), \(h \in h\):
\[
[e_i, f_j] = \delta_{ij}h_i, \quad [h, e_i] = \alpha_i(h)e_i, \quad [h, f_i] = -\alpha_i(h)f_i, \\
(ad e_i)^{1-\alpha_{ij}}e_j = (ad f_i)^{1-\alpha_{ij}}f_j = 0 \quad (i \neq j).
\]
The Chevalley involution \(* : g \to g\) is the involutive anti-automorphism determined by \(e_i^* = f_i, f_i^* = e_i, h^* = h, (i \in I, h \in h)\).
The nondegenerate symmetric bilinear form \((\quad , \quad)\) on \(h\) extends uniquely to a nondegenerate symmetric invariant bilinear form \((\quad , \quad)\) on \(g\).
We have the root space decomposition
\[
g = \bigoplus_{\alpha \in h^*} g_\alpha \quad \text{where} \quad g_\alpha := \{ x \in g \mid [h, x] = \alpha(h)x \quad \text{for all} \quad h \in h \}.
\]
In particular \(g_0 = h, g_{\alpha_i} = \mathbb{F}e_i, \text{ and } g_{-\alpha_i} = \mathbb{F}f_i, i \in I\).
The set of roots \(\Delta := \{ \alpha \in h^* \setminus \{0\} \mid g_\alpha \neq \{0\} \}\) is invariant under the Weyl group, \(\Delta = -\Delta\), and \(\Delta\) spans the root lattice \(Q\). We have \(\Delta_{re} \subseteq \Delta\), and \(\Delta_{im} := \Delta \setminus \Delta_{re}\) is called the set of imaginary roots.
\(\Delta, \Delta_{re}, \text{ and } \Delta_{im}\) decompose into the disjoint union of the sets of positive and
negative roots $\Delta^\pm := \Delta \cap Q^\pm$, $\Delta_{re}^\pm := \Delta_{re} \cap Q^\pm$, $\Delta_{im}^\pm := \Delta_{im} \cap Q^\pm$.

There is the triangular decomposition $g = n^- \oplus h \oplus n^+$, where $n^\pm := \bigoplus_{\alpha \in \Delta^\pm} g_{\alpha}$.

Irreducible highest weight representations: For every $\Lambda \in h^*$ there exists, unique up to isomorphism, an irreducible representation $(L(\Lambda), \pi_\Lambda)$ of $g$ with highest weight $\Lambda$. It is $h$-diagonalizable, and we denote its set of weights by $P(\Lambda)$. Any such representation carries a nondegenerate symmetric bilinear form $\langle \cdot | \cdot \rangle : L(\Lambda) \times L(\Lambda) \rightarrow F$ which is contravariant, i.e., $\langle \langle v | xw \rangle \rangle = \langle \langle x^* v | w \rangle \rangle$ for all $v, w \in L(\Lambda), x \in g$. This form is unique up to a nonzero multiplicative scalar.

The category $O_{adm}$: The category $O$ is defined as follows: Its objects are the $g$-modules $V$, which have the properties:

(1) $V$ is $h$-diagonalizable with finite dimensional weight spaces.

(2) There exist finitely many elements $\lambda_1, \ldots, \lambda_m \in h^*$, such that the set of weights $P(V)$ of $V$ is contained in the union $\bigcup_{i=1}^m \{ \lambda \in h^* \mid \lambda \leq \lambda_i \}$.

The morphisms of $O$ are the morphisms of $g$-modules.

Call a $g$-module $V$ admissible, if $V$ is $h$-diagonalizable with set of weights $P(V) \subseteq P$ and the elements of $g_{\alpha}$ act locally nilpotent on $V$ for all $\alpha \in \Delta_{re}$. (If the generalized Cartan matrix is degenerate, then admissible is slightly stronger than integrable, which means $V$ is $h$-diagonalizable and the elements of $g_{\alpha}$ act locally nilpotent on $V$ for all $\alpha \in \Delta_{re}$.) Examples of admissible representations are the adjoint representation $(g, ad)$, and the irreducible highest weight representations $(L(\Lambda), \pi_\Lambda), \Lambda \in P^+ := P \cap C$.

We denote by $O_{adm}$, the full subcategory of the category $O$, whose objects are admissible modules. This category generalizes the category of finite dimensional representations of a semisimple Lie algebra, keeping the complete reducibility theorem. Every object of $O_{adm}$ is isomorphic to a direct sum of the admissible irreducible highest weight modules $L(\Lambda), \Lambda \in P^+$. The set of weights of a module of $O_{adm}$ is contained in $X \cap P$ because of $\bigcup_{\Lambda \in P^+} P(\Lambda) = X \cap P$.

The minimal and formal Kac-Moody groups $G$ and $G_f$, the monoids $\hat{G}$ and $\hat{G}_f$: The monoid $\hat{G}$, can be obtained by a Tannaka-Krein reconstruction from the category $O_{adm}$ and its corresponding category of restricted duals, compare [MT], Section 4. It can also be characterized as follows:

(a) The monoid $\hat{G}$ acts on every module of $O_{adm}$. Two elements $\hat{g}, \hat{g}' \in \hat{G}$ are equal if and only if for all modules $V$ of $O_{adm}$ and for all $v \in V$ we have $\hat{g}v = \hat{g}'v$.

(b) There are the following elements of $\hat{G}$ acting on the modules of $O_{adm}$ in a particular way:

(1) For every $h \in H, s \in F^\times$ there exists an element $t_h(s) \in \hat{G}$, such that for every module $V$ of $O_{adm}$ we have

$$t_h(s)v_{\lambda} = s^{\langle h | \lambda \rangle}v_{\lambda}, \quad v_{\lambda} \in V_{\lambda}, \lambda \in P(V).$$
(2) For every $x \in \mathfrak{g}_\alpha$, $\alpha \in \Delta_{re}$, there exists an element $\exp(x) \in \hat{G}$, such that for every module $V$ of $\mathcal{O}_{adm}$ we have
\[
\exp(x)v = \exp(\pi(x))v, \quad v \in V.
\]

(3) For every face $R$ of the Tits cone there exists an element $e(R) \in \hat{G}$, such that for every module $V$ of $\mathcal{O}_{adm}$ we have
\[
e(R)v_\lambda = \begin{cases}
v_\lambda & \lambda \in R \\
0 & \lambda \in X \setminus R
\end{cases}, \quad v_\lambda \in V_\lambda, \lambda \in P(V).
\]

$\hat{G}$ is generated by the elements of (1), (2), and (3).

The unit group $G$ of $\hat{G}$ is generated by the elements of (1) and (2). It is the minimal Kac-Moody group, which we call Kac-Moody group for short.

The Chevalley involution $* : \hat{G} \to \hat{G}$ is the involutive anti-isomorphism determined by $\exp(x_\alpha)^* := \exp(x_\alpha^*)$, $t^* := t$, $e(R)^* := e(R)$, where $x_\alpha \in \mathfrak{g}_\alpha$, $\alpha \in \Delta_{re}$, $t \in T$, and $R \in R(X)$. It is compatible with any nondegenerate symmetric contravariant form $\langle\langle \cdot | \cdot \rangle\rangle$ on any module $V$ of $\mathcal{O}_{adm}$, i.e., $\langle\langle xv | w \rangle\rangle = \langle\langle v | x^*w \rangle\rangle$, $v, w \in V$, $x \in \hat{G}$.

In this paper we are interested in the monoid $\hat{G}$ and the spectrum of $\mathbb{F}$-valued points of its coordinate ring, which will be defined soon. To describe the $\mathbb{F}$-valued points we need a second monoid $\hat{G}_f$, extending $\hat{G}$, which we define already now: Set $n_f := \prod_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ and $g_f := \mathfrak{n}^- \oplus \mathfrak{h} \oplus n_f$. The Lie bracket of $g$ extends in the obvious way to a Lie bracket of $g_f$. Every $\mathfrak{g}$-module of $\mathcal{O}_{adm}$ can be extended to a $g_f$-module. The monoid $\hat{G}_f$ can be characterized as follows:

(a) The monoid $\hat{G}_f$ acts on every module of $\mathcal{O}_{adm}$. Two elements $\hat{g}, \hat{g}' \in \hat{G}$ are equal if and only if for all modules $V$ of $\mathcal{O}_{adm}$, and for all $v \in V$, we have $\hat{g}v = \hat{g}'v$.

(b) $\hat{G}_f$ extends $\hat{G}$ and it contains the following elements:

(4) For every $x \in n_f$ there exists an element $\exp(x) \in \hat{G}_f$, such that for any admissible representation $(V, \pi)$ we have
\[
\exp(x)v = \exp(\pi(x))v, \quad v \in V.
\]

$\hat{G}_f$ is generated by $\hat{G}$ and the elements of (4).

The unit group $G_f$ of $\hat{G}_f$ is generated by $G$ and the elements of (4). It is the formal Kac-Moody group.

Note that the groups $G$, $G_f$ as well as the monoid $\hat{G}$, $\hat{G}_f$ act faithfully on the sum $\bigoplus_{\Lambda \in P^+} L(\Lambda)$.

The groups $G$ and $G_f$ have the following important structural properties:

(a) The elements of (1) induce an embedding of the torus $H \otimes \mathbb{Z}^\times \mathbb{F}^\times$ into $G \subseteq G_f$. Its image is denoted by $T$. For $\alpha \in \Delta_{re}$, the elements of (2) induce an embedding of $(\mathfrak{g}_\alpha, +)$ into $G \subseteq G_f$. Its image $U_\alpha$ is called the root group belonging to $\alpha$. 

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Let $\alpha \in \Delta_{re}^+$ and $x_\alpha \in g_\alpha$, $x_- \in g_-\alpha$ such that $[x_\alpha, x_-] = h_\alpha$. There exists an injective homomorphism of groups $\phi_\alpha : \text{SL}(2, F) \to G$ with

$$\phi_\alpha \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} = \exp(sx_\alpha), \quad \phi_\alpha \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \exp(sx_-), \quad (s \in \mathbb{F}^\times).$$

(b) Denote by $N$ the subgroup generated by $T$ and $n_\alpha := \phi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\alpha \in \Delta_{re}$. The Weyl group $W$ can be identified with the group $N/T$ by the isomorphism $\kappa : W \to N/T$ given by $\kappa(\sigma_\alpha) := n_\alpha T \alpha \in \Delta_{re}$. We denote an arbitrary element $n \in N$ with $n^{-1}(nT) = \sigma \in W$ by $n_\sigma$. The set of weights $P(V)$ of an admissible $g$-module $(V, \pi)$ is $W$-invariant, and $n_\alpha V_\lambda = V_{\sigma\lambda}$, $\lambda \in P(V)$.

(c) Let $U^\pm$ be the subgroups generated by $U_\alpha$, $\alpha \in \Delta_{re}^\pm$. Let $U_f := \exp(n_f) \alpha$. Then $U^\pm$ and $U_f$ are normalized by $T$. Set

$$B^\pm := T \rtimes U^\pm, \quad B_f := T \rtimes U_f.$$

The pairs $(B^\pm, N)$ are twinned BN-pairs of $G$ with the property $B^+ \cap B^- = B^\pm \cap N = T$. The pair $(B_f, N)$ is a BN-pair of $G_f$ with $B_f \cap N = T$. There are the Bruhat and Birkhoff decompositions

$$G = \bigcup_{\sigma \in W} B^\sigma R^\sigma, \quad G_f = \bigcup_{\sigma \in W} B^\sigma R_f, \quad \epsilon, \delta \in \{+, -, \}.$$  

(d) There are also Levi decompositions of the standard parabolic subgroups. In this article we only use the corresponding decompositions for the groups $U^\pm$ and $U_f$. Set $\Delta_\pm^J := \Delta^\pm \cap \sum_{j \in J} \mathbb{Z} \alpha_j$, and $(\Delta^J)^\pm := \Delta^\pm \setminus \sum_{j \in J} \mathbb{Z} \alpha_j$. Similarly define $(\Delta_\pm^J)_{re}$ and $(\Delta^J)^{\pm}_{re}$ by replacing $\Delta^\pm$ by $\Delta_{re}^\pm$. Set $(n_f)^\pm := \bigoplus_{\alpha \in \Delta^\pm} g_\alpha$, $(n_f)_J := \prod_{\alpha \in \Delta^J} g_\alpha$, and $(n_f)_J^J := \prod_{\alpha \in (\Delta^J)_+} g_\alpha$. We have

$$U^\pm = (U_f)_J \times (U_f)^J, \quad U_f = (U_f)_J \rtimes (U_f)^J.$$  

Here $U^\pm_\alpha$ is the group generated by $U_\alpha$, $\alpha \in (\Delta_\pm^J)_{re}$. $(U_f)^J$ is the smallest normal subgroup of $U^\pm$ containing $U_\alpha$, $\alpha \in (\Delta_f^J)_{re}$. This group equals $\bigcap_{\sigma \in W_f} \sigma U^\pm \sigma^{-1}$. Furthermore $(U_f)_J := \exp((n_f)_J)$ and $(U_f)_J^J := \exp((n_f)_J^J)$.

For $w \in W$ set $U_w := U \cap wU^-w^{-1}$, $U^w := U \cap wUw^{-1}$, and $(U_f)^w := U_f \cap wU_fw^{-1}$. Then $U_w = \prod_{\alpha \in \Phi_w} U_\alpha$ (arbitrary order of the product), where $\Phi_w = \Delta_{re}^+ \setminus \Delta_{re}^- = \Delta^+ \setminus \Delta^-$. The multiplications maps $U_w \times U^w \to U$ and $U_w \times (U_f)^w \to U_f$ are bijective. Set $U_w^* := (U_w)^*$ and $(U_w)^* := (U^w)^*.$

The derived minimal Kac-Moody group $G'$ is identical with the Kac-Moody group as defined in \[\textbf{K}, \textbf{P}, \textbf{J}\]. It is generated by the root groups $U_\alpha$, $\alpha \in \Delta_{re}$. We have $G = G' \rtimes T_{\text{rest}}$, where $T_{\text{rest}} := H_{\text{rest}} \otimes \mathbb{Z} \mathbb{F}$ is a subtorus of $T$. The group $G_f$ is identical with the Kac-Moody group of $[\text{SU}]$ for a simply connected minimal free realization.

The monoid $\hat{G}$ has the following important structural properties: The Kac-Moody group $G$ is the unit group of $\hat{G}$. Every idempotent is $G$-conjugate to
some idempotent $e(R(\Theta))$, $\Theta$ special. We have
\[
\hat{G} = \bigcup_{\Theta \text{ special}} Ge(R(\Theta))G.
\]

We get an abelian submonoid of $\hat{G}$ by $\hat{G} := \bigcup_{R \in R(X)} Ne(R)$. Define a congruence relation on $\hat{N}$ as follows:
\[
\hat{n} \sim \hat{n}' \iff \hat{n}T = \hat{n}'T \iff \hat{n} \in \hat{n}'T.
\]
The Weyl monoid $\hat{W}$ is isomorphic to the monoid $\hat{N}/T$, an isomorphism $\kappa : \hat{W} \to \hat{N}/T$ given by $\kappa(\sigma \epsilon(R)) = n_\sigma e(R)T$.

$\hat{G}$ has Bruhat and Birkhoff decompositions:
\[
\hat{G} = \bigcup_{\hat{n} \in \hat{N}} U^\epsilon \hat{n} U^\delta = \bigcup_{\sigma \in \hat{W}} B^\epsilon \hat{\sigma} B^\delta, \quad \epsilon, \delta \in \{+, -\}.
\]

For later reference we state the following formulas, which are useful for computations in $\hat{G}$:
(α) Let $R, S$ be faces of the Tits cone, and $n_\sigma \in N$. Then
\[
e(R)e(S) = e(R \cap S), \quad n_\sigma e(R)n_\sigma^{-1} = e(\sigma R).
\]
(β) An element $g$ of $T, N, U, U^-$, resp. $G$ satisfies
\[
e(R(\Theta))g = e(R(\Theta))
\]
if and only if it satisfies
\[
g^* e(R(\Theta)) = e(R(\Theta))
\]
if and only if it is contained in $T_\Theta, N_\Theta, U_\Theta, U^-_\Theta \prec (U^\Theta \cdots)^-, \text{ resp. } G_\Theta \prec (U^\Theta \cdots)^-$. Here $T_\Theta$ is the subtorus of $T$ generated by $th_j(s), j \in \Theta$, $s \in \mathbb{F}^\times$, $N_\Theta$ is the subgroup of $N$ generated by $T_\Theta$ and $n_\alpha j, j \in \Theta$, and $G_\Theta$ is the subgroup of $G$ generated by $U^\pm\alpha_j, j \in \Theta$.

(γ) An element $g$ of $T, N, U, U^-$, resp. $G$ satisfies
\[
ge(R(\Theta))g^{-1} = e(R(\Theta))
\]
if and only if it is contained in the groups $T, N_\Theta \cdots, U_\Theta, U^-_\Theta$, resp. $G_\Theta \cdots T$.

(δ) In particular we have
\[
U e(R(\Theta)) = U_\Theta e(R(\Theta)) = e(R(\Theta))U_\Theta^-,
\]
\[
e(R(\Theta))U^- = e(R(\Theta))U^-\Theta = U^-_\Theta e(R(\Theta)).
\]
Sets with coordinate rings: We call a point separating algebra of functions \( \mathbb{F}[A] \) on a set \( A \) a coordinate ring. The closed sets of the Zariski topology on \( A \) are given by the zero sets of the functions of \( \mathbb{F}[A] \). The set \( A \) is irreducible if and only if \( \mathbb{F}[A] \) is an integral domain.

A morphism of sets with coordinate rings \((A, \mathbb{F}[A])\) and \((B, \mathbb{F}[B])\) consists of a map \( \phi : A \to B \), whose comorphism \( \phi^* : \mathbb{F}[B] \to \mathbb{F}[A] \) exists. In particular a morphism is Zariski continuous.

If \((B, \mathbb{F}[B])\) is a set with coordinate ring, and \( A \) is a nonempty subset of \( B \), we get a coordinate ring on \( A \) by restricting the functions of \( \mathbb{F}[B] \) to \( A \).

If \((A, \mathbb{F}[A])\) is a set with coordinate ring and \( f \in \mathbb{F}[A] \setminus \{0\} \), the principal open set \( D_A(f) := \{ a \in A \mid f(a) \neq 0 \} \) is equipped with a coordinate ring by identifying the localization \( \mathbb{F}[A]_f \) in the obvious way with an algebra of functions on \( D_A(f) \). The principal open set \( D_A(f) \) is irreducible if and only if \( A \) is irreducible.

If \((A, \mathbb{F}[A])\) and \((B, \mathbb{F}[B])\) are sets with coordinate rings, then the product \( A \times B \) is equipped with a coordinate ring by identifying the tensor product \( \mathbb{F}[A] \otimes \mathbb{F}[B] \) in the obvious way with an algebra of functions on \( A \times B \). The product \( A \times B \) is irreducible if and only if \( A \) and \( B \) are irreducible.

The coordinate ring of \( \hat{G} \): By the Tannaka Krein reconstruction given in [M, 1], Section 4, the monoid \( \hat{G} \) is equipped with a natural coordinate ring. It can also be defined as follows: For a module \( V \) of \( \mathcal{O}_{adm} \), \( v, w \in V \), and \( \langle\langle \mid \rangle\rangle \) a nondegenerate symmetric contravariant bilinear form on \( V \), call the function \( f_{vw} : \hat{G} \to \mathbb{F} \) defined by \( f_{vw}(x) := \langle\langle v \mid xw \rangle\rangle, x \in \hat{G} \), a matrix coefficient of \( \hat{G} \).

The set of all such matrix coefficients \( \mathbb{F}[\hat{G}] \) is a coordinate ring on \( \hat{G} \), which is an integral domain. For a set \( M \subseteq \hat{G} \) we denote by \( \overline{M} \) its Zariski closure. The Chevalley involution \( \ast : \hat{G} \to \hat{G} \) is a morphism. We denote its comorphism also by \( \ast : \mathbb{F} [\hat{G}] \to \mathbb{F} [\hat{G}] \) and call it Chevalley involution. Right and left multiplications with elements of \( \hat{G} \) are morphisms. In particular an action of \( \hat{G} \times \hat{G} \) on \( \hat{G} \) from the right by morphisms is given by

\[
x (g, h) := g^* x h \quad x, g, h \in \hat{G}.
\]

The comorphisms induce an action of \( \hat{G} \times \hat{G} \) on \( \mathbb{F}[\hat{G}] \) from the left.

In this article we fix a nondegenerate symmetric contravariant bilinear form on \( L(\Lambda) \) for every \( \Lambda \in P^+ \). The coordinate ring \( \mathbb{F}[\hat{G}] \) admits a Peter-Weyl theorem: The map \( \bigoplus_{\Lambda \in P^+} L(\Lambda) \otimes L(\Lambda) \to \mathbb{F}[\hat{G}] \) induced by \( v \otimes w \mapsto f_{vw} \) is an isomorphism of \( \hat{G} \times \hat{G} \)-modules. It identifies the direct sum of the switch maps of the factors with the Chevalley involution.

The algebra of strongly regular functions \( \mathbb{F}[G] \) is obtained by restricting the functions of \( \mathbb{F}[\hat{G}] \) onto \( G \). The restriction map is an isomorphism from \( \mathbb{F}[\hat{G}] \) to \( \mathbb{F}[G] \). Restricting the functions of \( \mathbb{F}[G] \) onto \( G' \), resp. \( T_{rest} \) gives the algebras \( \mathbb{F}[G'] \), resp. \( \mathbb{F}[T_{rest}] \) the first identical with the algebra of strongly regular functions as defined in [K.P-2], the second the classical coordinate ring of the
torus $T_{\text{rest}}$. The multiplication map $G' \times T_{\text{rest}} \to G$ is an isomorphism.

The monoids $\hat{T}, \hat{N}, \hat{G}$ are the Zariski closures of $T, N, G$, and $G$ is the Zariski open dense unit group of $\hat{G}$.

Denote by $\leq$ the Bruhat order on $W$. Lemma 3.4 of [K,P 2] is also valid for the slightly enlarged group $G$, which we use here. It gives the relative closures of Bruhat and Birkhoff cells of $G$:

$$B^\epsilon_w B^\epsilon \cap G = \bigcup_{\substack{w' \in W \\ w' \leq w}} B^\epsilon_{w'} B^\epsilon$$

$$B^{-\epsilon} w B^{-\epsilon} \cap G = \bigcup_{\substack{w' \in W \\ w' \geq w}} B^{-\epsilon}_{w'} B^{-\epsilon}$$

where $\epsilon \in \{+, -\}$ and $w \in W$.

**The spectrum of $\mathbb{F}$-valued points of $\hat{G}$:** We denote the $\mathbb{F}$-valued points of $\mathbb{F}[\hat{G}]$, i.e., the homomorphisms of algebras from $\mathbb{F}[\hat{G}]$ to $\mathbb{F}$, by $\text{Specm} \mathbb{F}[\hat{G}]$.

A function $f \in \mathbb{F}[\hat{G}]$ induces a function on $\text{Specm} \mathbb{F}[\hat{G}]$, assigning to $\phi \in \text{Specm} \mathbb{F}[\hat{G}]$ the value $\phi(f)$. We denote this function also by $f$. In this way $\text{Specm} \mathbb{F}[\hat{G}]$ is equipped with a coordinate ring isomorphic to $\mathbb{F}[\hat{G}]$. Its Zariski topology coincides with the relative topology induced by the topology of the spectrum of $\mathbb{F}[\hat{G}]$. For a set $M \subseteq \text{Specm} \mathbb{F}[\hat{G}]$ we denote by $M^{\text{spm}}$ its Zariski closure.

As a set, the $\mathbb{F}$-valued points of $\mathbb{F}[\hat{G}]$ can be described as follows: There is a surjective map $\hat{\cdot} : \hat{G}_f \times \hat{G}_f \to \text{Specm} \mathbb{F}[\hat{G}]$ given by

$$(x \hat{\cdot} y)(f_{vw}) := \langle (xv \mid yw) \rangle, \quad x, y \in \hat{G}_f, \quad v, w \in L(\Lambda), \quad \Lambda \in P^+.$$ 

The set of fibres of this map coincides with the partition of $\hat{G}_f \times \hat{G}_f$ corresponding to the equivalence relation, which is generated by

$$(x, zy) \sim (z^* x, y), \quad x, y \in \hat{G}_f, \quad z \in \hat{G}.$$ 

In particular $x \hat{\cdot} zy = xz^* \hat{\cdot} y$, $x, y \in \hat{G}_f$, $z \in \hat{G}$.

The monoid $\hat{G}_f \times \hat{G}_f$ acts in a natural way on $\text{Specm} \mathbb{F}[\hat{G}]$ by morphisms from the right. The map $\hat{\cdot}$ is equivariant, i.e.,

$$(x \hat{\cdot} y)(\hat{x}, \hat{y}) = \hat{x} \hat{\cdot} y \hat{y}, \quad x, \hat{x}, y, \hat{y} \in \hat{G}_f.$$ 

The Chevalley involution of $\mathbb{F}[\hat{G}]$ induces an involutive morphism $\ast$ on $\text{Specm} \mathbb{F}[\hat{G}]$, also called Chevalley involution. Using the map $\hat{\cdot}$ it can be described by

$$(x \hat{\cdot} y)^\ast = \hat{y} \hat{\cdot} x, \quad x, y \in \hat{G}_f.$$ 

In [M 2] we investigated the $G_f \times G_f$-orbit decomposition (closure relation of the orbits, irreducibility of the orbits, coverings of the orbits by big cells, transversal stratified slices to the orbits). As an aid for these investigations, we showed the Birkhoff decomposition

$$\text{Specm} \mathbb{F}[\hat{G}] = \bigcup_{\hat{w} \in W} B_f \hat{\cdot} \hat{w} B_f$$.
Note that $\hat{G}_f$ embeds into $\text{Spec} \mathbb{F}[\hat{G}]$ by assigning $x \in \hat{G}_f$ the element $1 \circ x$. In this way this decomposition of $\text{Spec} \mathbb{F}[\hat{G}]$ extends one of the Birkhoff decompositions of $\hat{G}$, and a corresponding Birkhoff decomposition, which holds for $\hat{G}_f$.

**The Kostant cones:** Fix $\Lambda \in P^+$. Recall that we have fixed a nondegenerate symmetric contravariant bilinear form $\langle \langle \cdot | \cdot \rangle \rangle$ on $L(\Lambda)$. Equip $L(\Lambda)$ with the coordinate ring $\mathbb{F}[L(\Lambda)]$ generated by the matrix coefficients $f_v$, $v \in L(\Lambda)$, where $f_v(w) := \langle \langle v | w \rangle \rangle$ for all $w \in L(\Lambda)$. It is a symmetric algebra in the linear space given by these functions. For a set $M \subseteq L(\Lambda)$ denote by $\overline{M}$ its Zariski closure.

The set $V_\Lambda := G(L(\Lambda),\Lambda) \subseteq L(\Lambda)$ is Zariski closed. It is called *Kostant cone*.

Now assume that $\Lambda \in F_I$, $J \subseteq I$. Denote by $W^J$ the set of minimal coset representatives of $W/W_I$. Because the parabolic subgroup $W_I$ is the $W$-stabilizer of $\Lambda$, the evaluation map $W^J \to W^J \Lambda = W\Lambda$ is bijective. By this map the Bruhat order on $W^J$ induces an order on $W\Lambda$. The corresponding inverse order on $W\Lambda$ is denoted by $\preceq$. If $\lambda, \mu \in W\Lambda$ then $\lambda \preceq \mu$ implies $\lambda \leq \mu$.

For $v \in L(\Lambda)$ denote by $\text{supp}(v)$ the set of weights of the nonzero weight space components of $v$. Denote by $S(v)$ the convex hull of $\text{supp}(v)$ in $h^*_\mathbb{R}$.

For $v \in V_\Lambda \setminus \{0\}$ the vertices of $S(v)$ are given by $S(v) \cap W\Lambda$. The edges of $S(v)$ are parallel to real roots. The two vertices of an edge are comparable in $\preceq$. $S(v)$ has one maximal and one minimal vertex.

For $\lambda \in W\Lambda$ set

$$V^+(\lambda) := \{ v \in V_\Lambda \setminus \{0\} \mid \lambda \text{ is the minimal vertex of } S(v) \} ,$$

$$V^-(\lambda) := \{ v \in V_\Lambda \setminus \{0\} \mid \lambda \text{ is the maximal vertex of } S(v) \} .$$

Then $V_\Lambda(\lambda)^\epsilon = U^\epsilon(L(\Lambda)\setminus\{0\})$, and $V_\Lambda \setminus \{0\} = \bigcup_{\lambda \in W\Lambda} V_\Lambda(\lambda)^\epsilon$ where $\epsilon \in \{+,-\}$, and

$$\overline{V_\Lambda(\lambda)^+} \setminus \{0\} = \bigcup_{\mu \in W\Lambda, \mu \geq \lambda} V_\Lambda(\mu)^+ , \quad \overline{V_\Lambda(\lambda)^-} \setminus \{0\} = \bigcup_{\mu \in W\Lambda, \mu \leq \lambda} V_\Lambda(\mu)^- .$$

The Kostant cones are the affine cones of the flag varieties. For our investigations they are more important than the flag varieties, because the monoid $\hat{G}$ acts on the Kostant cones by morphisms, but it does not act on the flag varieties. In particular $V_\Lambda = \hat{G}(L(\Lambda),\Lambda)$.

### 2 Extensions of the Bruhat order

Since $\hat{G} \times \hat{G}$ acts on $\hat{G}$ by morphisms, the closures of the $B^{-\epsilon} \times B^\delta$-orbits, $\epsilon, \delta \in \{+,-\}$, are unions of $B^{-\epsilon} \times B^\delta$-orbits. Because of the Bruhat and Birkhoff decompositions

$$\hat{G} = \bigcup_{\hat{w} \in \hat{W}} B^\epsilon \hat{w} B^\delta$$

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the closure relations of the $B^{-\epsilon} \times B^4$-orbits determine relations on $\tilde{W}$:

**Definition 2.1** For $\tilde{w}, \tilde{w}' \in \tilde{W}$ and $\epsilon \in \{+, -\}$ define:

$$\tilde{w}' \leq_{\epsilon \epsilon} \tilde{w} : \iff B^\epsilon \tilde{w}' B^\epsilon \subseteq B^\epsilon \tilde{w} B^\epsilon,$$

$$\tilde{w} \leq_{-+} \tilde{w}' : \iff B^{-\epsilon} \tilde{w} B \subseteq B^{-\epsilon} \tilde{w}' B.$$

**Remarks:**

1. Due to this definition

$$B^\epsilon \tilde{w} B^\epsilon = \bigcup_{\tilde{w}' \in \tilde{W}, w' \leq_{\epsilon \epsilon} \tilde{w}} B^\epsilon \tilde{w}' B^\epsilon$$

and

$$B^{-\epsilon} \tilde{w} B = \bigcup_{\tilde{w}' \in \tilde{W}, \tilde{w} \leq_{-+} \tilde{w}'} B^{-\epsilon} \tilde{w}' B.$$

2. Due to equations (4) the relations $\leq_{++}$, $\leq_{--}$, and $\leq_{-+}$ extend the Bruhat order $\leq$ of the Weyl group $W$.

The relations $\leq_{++}$, $\leq_{--}$, and $\leq_{-+}$ are the closure relations of orbit decompositions. Therefore these relations are reflexive and transitive. But we can not conclude that they are antisymmetric, because for this we would need to know that the orbits are locally closed.

3. The Chevalley involution $\ast : \hat{G} \to \hat{G}$ is a morphism with $(B^\epsilon \tilde{w} B^\delta)^\ast = B^{-\delta} \tilde{w}^\text{inv} B^{-\epsilon}$, $\tilde{w} \in \tilde{W}$, $\epsilon, \delta \in \{+, -\}$. From the definition of the extended Bruhat orders follows immediately, that the inverse map $\text{inv} : W \to W$ is an isomorphism of the relations $(W, \leq_{\epsilon \epsilon})$ and $(W, \leq_{-\epsilon -\epsilon}), \epsilon = +, -$. It is an automorphism of $(W, \leq_{-+})$.

4. In this article we do not treat the Birkhoff cells $B \tilde{w} B^{-\epsilon}$, $\tilde{w} \in \tilde{W}$. These cells can not be treated in a similar way as the Birkhoff cells $B^{-\epsilon} \tilde{w} B$, $\tilde{w} \in \tilde{W}$, because the coordinate ring $\mathbb{F} \lbrack \hat{G} \rbrack$ does not contain the matrix coefficients of the admissible lowest weight representations. Even the relative closures of $B \tilde{w} B^{-\epsilon}$, $w \in W$, in the Kac-Moody group $G$ have not been determined.

Our first aim is to determine these relations explicitely. To this end we introduce three normal forms of the elements of $\tilde{W}$, which are similar to the standard form of an element of a Renner monoid introduced in [Pe, Pu, Re].

For $\tilde{J} \subseteq I$ denote by $W(\tilde{J})$ the minimal coset representatives of $W/W(\tilde{J})$, and denote by $JW$ the minimal coset representatives of $W(\tilde{J})W$.

**Proposition 2.2** Let $\tilde{w} \in \tilde{W}$.

1. There exists a uniquely determined special set $\Theta$, and uniquely determined elements $w_1 \in \Theta^\Theta, w_2 \in \Theta^\Theta W$, such that

$$\tilde{w} = w_1 \in (R(\Theta))w_2.$$

2. There exists a uniquely determined special set $\Theta$, and uniquely determined elements $w_1 \in \Theta^\Theta W$, such that

$$\tilde{w} = w_1 \in (R(\Theta))w_2.$$

3. There exists a uniquely determined special set $\Theta$, and uniquely determined elements $w_1 \in \Theta^\Theta W$, such that

$$\tilde{w} = w_1 w_2 \in (R(\Theta))w_3 = w_1 \in (R(\Theta))w_2 w_3.$$
Applying this map to an element \( \hat{w} \) to an element \( \hat{w}^{inv} \) in normal form I resp. II we obtain the element \( \hat{w}^{inv} \) in normal form II resp. I. By applying this map to an element \( \hat{w} \) in normal form III we obtain the element \( \hat{w}^{inv} \) in normal form III.

**Remarks:** (1) By applying the inverse map \( \hat{W} \rightarrow W \) of the same element of \( \hat{w} \in W \), we find

\[
\hat{W} \rightarrow W : w \rightarrow \hat{w}^{-1}
\]

Proof: 1) We first show the existence and uniqueness of normal form I: We have \( \hat{W} = \bigcup_{\Theta} W_\Theta \), and by using formulas (3) and (4) we find

\[
W \varepsilon (R(\Theta))W = W^{\Theta,\Theta'} W_{\Theta',\Theta} \varepsilon (R(\Theta))W = W^{\Theta,\Theta'} \varepsilon (R(\Theta))W_{\Theta',\Theta} W = W^{\Theta,\Theta'} \varepsilon (R(\Theta))W_{\Theta} \varepsilon (\Theta) = W^{\Theta,\Theta'} \varepsilon (R(\Theta))W^\Theta.
\]

To show the uniqueness let \( w_1 \varepsilon (R(\Theta))w_2 \) and \( w'_1 \varepsilon (R(\Theta'))w'_2 \) be normal forms of the same element of \( \hat{W} \). Then by using equation (2) we find

\[
\varepsilon (w_1 R(\Theta))w_1 w_2 = w_1 \varepsilon (R(\Theta))w_2 = w'_1 \varepsilon (R(\Theta'))w'_2 = \varepsilon (w'_1 R(\Theta'))w'_1 w'_2,
\]

which is equivalent to

\[
w_1 R(\Theta) = w'_1 R(\Theta') \quad \text{and} \quad w'_1 w_2 (w_1 w_2)^{-1} \in w'_1 W_{\Theta'} (w'_1)^{-1}.
\]

From the first equation follows \( \Theta = \Theta' \) and \( w_1 W_{\Theta',\Theta} = w'_1 W_{\Theta',\Theta} \). Since the minimal coset representatives \( w_1, w'_1 \) are uniquely determined we find \( w_1 = w'_1 \).

Inserting in the second expression we get \( w'_2 w_2^{-1} \in W_{\Theta} \), resp. \( w_{\Theta'} w'_2 = W_{\Theta} w_2 \). Because the minimal coset representatives \( w_2, w'_2 \) are uniquely determined this implies \( w_2 = w'_2 \).

2) The existence and uniqueness of normal form II follows from the existence and uniqueness of normal form I by using the inverse map \( \hat{W} \rightarrow W \).

3) If we show the bijectivity of the restricted multiplication maps \( \hat{W} \rightarrow W \), then the existence and uniqueness of normal form I and II together with formula (3) imply the existence and uniqueness of normal form III. We only have to show the bijectivity of the first map, the bijectivity of the second follows by applying the inverse map of \( W \).

We have \( W = W^{\Theta,\Theta'} W_{\Theta',\Theta} = W^{\Theta,\Theta'} W_{\Theta} W_{\Theta} \), and the corresponding multiplicative decomposition of the elements are unique. It remains to show that \( W^{\Theta,\Theta'} W_{\Theta} \subseteq W^\Theta \). If \( w_1 \in W^{\Theta,\Theta'} \) and \( w_2 \in W_{\Theta} \), then for all \( i \in \Theta \) we have

\[
w_1 w_2 \alpha_i = w_1 \alpha_i \in \Delta^+_r.
\]

Therefore \( w_1 w_2 \in W^\Theta \). $\square$
The following theorem describes the relations $\leq_{++}$, $\leq_{-+}$, and $\leq_{-}$ explicitly. The description of 1a) (iii) is similar to the result for reductive algebraic monoids obtained in Pe.Pu.Re.

For $J \subseteq I$ and $w \in W$ denote by $w^J$ the minimal coset representative of $wW_J$, and by $Jw$ the minimal coset representative of $W_Jw$.

**Theorem 2.3**

1a) Let $\hat{w} = w_1 \varepsilon (R(\Theta))w_2$, $\hat{w}' = w'_1 \varepsilon (R(\Theta'))w'_2$ be elements of $\hat{W}$ in normal form $I$. Then the following statements are equivalent:

(i) $\hat{w} \leq_{++} \hat{w}'$

(ii) $\Theta \supseteq \Theta'$ and there exists an element $w \in W_{\Theta'}$ such that $w_1 \leq (w'_1 w)^\Theta$ and $w_2 \geq (w'_1 w'_2)^\Theta$.

(iii) $\Theta \supseteq \Theta'$ and there exists an element $w \in W_{\Theta'} W_\Theta$ such that $w_1 \leq w'_1 w$ and $w_2 \geq w'_1 w'_2$.

b) Let $\hat{w} = w_1 \varepsilon (R(\Theta))w_2$, $\hat{w}' = w'_1 \varepsilon (R(\Theta'))w'_2$ be elements of $\hat{W}$ in normal form $II$. Then the following statements are equivalent:

(i) $\hat{w} \leq_{-+} \hat{w}'$

(ii) $\Theta \supseteq \Theta'$ and there exists an element $w \in W_{\Theta'}$ such that $w_1 \geq (w'_1 w)^\Theta$ and $w_2 \leq (w'_1 w'_2)^\Theta$.

(iii) $\Theta \supseteq \Theta'$ and there exists an element $w \in W_{\Theta'} W_\Theta$ such that $w_1 \geq w'_1 w$ and $w_2 \leq w'_1 w'_2$.

2) Let $\hat{w} = w_1 \varepsilon (R(\Theta))w_2$, $\hat{w}' = w'_1 \varepsilon (R(\Theta'))w'_2$ be elements of $\hat{W}$ both in normal form $I$, or both in normal form $II$. Then the following statements are equivalent:

(i) $\hat{w} \leq_{-} \hat{w}'$

(ii) $\Theta \supseteq \Theta'$ and there exists an element $w \in W_{\Theta'}$ such that $w_1 \geq (w'_1 w)^\Theta$ and $w_2 \geq (w'_1 w'_2)^\Theta$.

**Proof:** We only have to prove the statements of the theorem, which use normal form I. Then the statements which use normal form II follow by applying the inverse map $\text{inv} : \hat{W} \rightarrow \hat{W}$, compare Remark (3) following Definition 2.1 and Remark (1) following Proposition 2.

For $J \subseteq I$ and $w \in W$ denote by $w^J$ the minimal coset representative of $wW_J$, and by $Jw$ the minimal coset representative of $W_Jw$.

**Theorem 2.3**

1a) Let $\hat{w} = w_1 \varepsilon (R(\Theta))w_2$, $\hat{w}' = w'_1 \varepsilon (R(\Theta'))w'_2$ be elements of $\hat{W}$ in normal form $I$. Then the following statements are equivalent:

(i) $\hat{w} \leq_{++} \hat{w}'$

(ii) $\Theta \supseteq \Theta'$ and there exists an element $w \in W_{\Theta'}$ such that $w_1 \leq (w'_1 w)^\Theta$ and $w_2 \geq (w'_1 w'_2)^\Theta$.

(iii) $\Theta \supseteq \Theta'$ and there exists an element $w \in W_{\Theta'} W_\Theta$ such that $w_1 \leq w'_1 w$ and $w_2 \geq w'_1 w'_2$.

b) Let $\hat{w} = w_1 \varepsilon (R(\Theta))w_2$, $\hat{w}' = w'_1 \varepsilon (R(\Theta'))w'_2$ be elements of $\hat{W}$ in normal form $II$. Then the following statements are equivalent:

(i) $\hat{w} \leq_{-+} \hat{w}'$

(ii) $\Theta \supseteq \Theta'$ and there exists an element $w \in W_{\Theta'}$ such that $w_1 \geq (w'_1 w)^\Theta$ and $w_2 \leq (w'_1 w'_2)^\Theta$.

(iii) $\Theta \supseteq \Theta'$ and there exists an element $w \in W_{\Theta'} W_\Theta$ such that $w_1 \geq w'_1 w$ and $w_2 \leq w'_1 w'_2$.

**Proof:** We only have to prove the statements of the theorem, which use normal form I. Then the statements which use normal form II follow by applying the inverse map $\text{inv} : \hat{W} \rightarrow \hat{W}$, compare Remark (3) following Definition 2.1 and Remark (1) following Proposition 2.

(a) First we show the equivalence of (ii) and (iii) of 1a): Let (ii) be valid. Choose an element $\hat{w} \in W_\Theta$ such that $\Theta(w^{-1} w'_2) = \hat{w}^{-1} w_1 w'_2$. Then we have

$$w_1 \leq (w'_1 w)^\Theta = (w'_1 w\hat{w})^\Theta \leq w'_1 w\hat{w}^\Theta,$$

$$w_2 \geq (w^{-1} w'_2) = (w\hat{w})^{-1} w'_2.$$

Let (iii) be valid. Write the element $w$ in the form $w = \hat{w}w'$ with $\hat{w} \in W_{\Theta'}$ and $w' \in W_\Theta$. From $w_1 \leq w'_1 w$ follows $w_1 = \hat{w}^\Theta \leq (w'_1 w)^\Theta = (w'_1 \hat{w})^\Theta$. Similarly we get $w_2 \geq \Theta(w^{-1} w'_2)$.

The easy part in the following proof of the equivalence (i) and (ii) in 1), 2a) is the direction from (ii) to (i). The proof of this direction, which uses only the formulas for the relative closures of the Bruhat and Birkhoff cells of a Kac-Moody group, is for 1a) similar, and for 2) a modification of the corresponding proof in Pe.Pu.Re. The proof of the opposite direction is quite different.

(b) To prepare the proof of the direction from (i) to (ii) of 1a) and 2) first note:
Let $\Lambda \in P^+$, and let $v, v' \in V_\Lambda$ such that $\langle \langle v \mid v' \rangle \rangle \neq 0$. Since different weight spaces of $L(\Lambda)$ are $\langle \langle \mid \rangle \rangle$-orthogonal we have $\text{supp}(v) \cap \text{supp}(v') \neq \emptyset$. Let $\lambda$ be an element of this intersection. Then the biggest vertex of $S(v)$ is bigger than $\lambda$, which is bigger than the smallest vertex of $S(v')$ (with respect to $\preceq$).

(e) Now we can prove the direction from (i) to (ii) of 1a) and 2): Let $\epsilon \in \{+,-\}$ and $B^* \hat{w} B \subseteq B^* \hat{w}' B$.

Choose an element $\Lambda \in F_\Theta \cap P$. Choose elements $v_{w_1 \Lambda} \in L(\Lambda)_{w_1 \Lambda} \setminus \{0\}$ and $v_{w_2 \Lambda} \in L(\Lambda)_{w_2 \Lambda} \setminus \{0\}$. To cut short the notation set $g := f_{v_{w_1 \Lambda}^* v_{(w_2)\Lambda}^{-1}}$.

Since $\Lambda \in R(\Theta)$, for any element $\hat{n}_{\hat{w}} \in \hat{N}$ belonging to $\hat{w}$ we have

$$g(\hat{n}_{\hat{w}}) = \langle \langle v_{w_1 \Lambda} \mid \hat{n}_{w_1 \epsilon(R(\Theta))} v_{(w_2)\Lambda}^{-1} \rangle \rangle \neq 0.$$ 

Therefore $g$ does not vanish entirely on the closure $B^* \hat{w}' B$, which implies that it also can not vanish entirely on $B^* \hat{w}' B$. Therefore there exists an element $\hat{n}_{\hat{w}} \in \hat{N}$ such that $\hat{w} = \hat{w}$ and elements $u_\epsilon \in U^*$, $u \in U$, such that

$$g(u_\epsilon \hat{n}_{\hat{w}} u) = \langle \langle u_\epsilon^* v_{w_1 \Lambda} \mid \hat{n}_{w_1 \epsilon(R(\Theta))} v_{(w_2)\Lambda}^{-1} \rangle \rangle \neq 0.$$

The vertices of $S(u v_{(w_2)\Lambda}^{-1})$ are of the form

$$\hat{w} \Lambda \text{ with } \hat{w} \in W \text{ such that } \hat{w}^\Theta \leq (w_2)^{-1} \Theta = (w_2)^{-1}.$$ 

Therefore the vertices of $S(\hat{n}_{w_1 \epsilon(R(\Theta))} v_{w_2 \Lambda}^{-1})$ are of the form

$$w_1^\epsilon \hat{w} \Lambda \text{ with } \hat{w} \in W \text{ such that } \hat{w} \leq (w_2)^{-1} \text{ and } w_1^\epsilon \hat{w} \Lambda \in R(\Theta') .$$

Here the inequality $\hat{w} \leq (w_2)^{-1}$ is equivalent to $\Theta(\hat{w}^{-1}) \leq w_2$. By comparing the type of facets in the formula $w_1^\epsilon \hat{w} \Lambda \in R(\Theta')$ on the left and on the right we find

$$F_\Theta \subseteq F_{\Theta'} \text{ resp. } \Theta \geq \Theta' \text{ resp. } R(\Theta) \subseteq R(\Theta') .$$

We also get $w_2^\epsilon \hat{w} \in W_{\Theta', \Theta} W_{\Theta} = W_{\Theta'} W_{\Theta} = W_{\Theta'} W_{\Theta}$. Set $w := w_2^\epsilon \hat{w}$. We have shown that $\Theta \geq \Theta'$, and the vertices of $S(\hat{n}_{w_1 \epsilon(R(\Theta'))} v_{w_2 \Lambda}^{-1})$ are of the form

$$w_1^\epsilon \Lambda \text{ with } w \in W_{\Theta', \Theta} W_{\Theta} \text{ such that } \Theta(w^{-1} w_2') \leq w_2 .$$

$w_1 \Lambda$ is the biggest resp. smallest vertex of $(u_\epsilon^* v_{w_1 \Lambda})$ for $\epsilon = +$ resp. $\epsilon = -$. Using (b) we conclude that there exists an element $w \in W_{\Theta', \Theta} W_{\Theta}$ such that $\Theta(w^{-1} w_2') \leq w_2$ and

$$w_1 \Lambda \begin{cases} \geq & w_1^\epsilon \Lambda \text{ for } \epsilon = + \\ \leq & w_1^\epsilon \Lambda \text{ for } \epsilon = - \end{cases} .$$

These inequalities are equivalent to

$$w_1 = w_1^\Theta \begin{cases} \leq & (w_1^\epsilon w)^\Theta \text{ for } \epsilon = + \\ \geq & (w_1^\epsilon w)^\Theta \text{ for } \epsilon = - \end{cases} .$$
Obviously there also exists an element $w \in W_{\Theta^+}$, which satisfies the required inequalities.

**d** To prepare the proof of the direction from (ii) to (i) of 1a) and 2), we need the following formula: Let $w_1, w'_1 \in \mathcal{W}$ such that $w_1 \leq w'_1$ for $\epsilon = +$, and $w_1 \geq w'_1$ for $\epsilon = -$. Let $w_2 \in \Theta^+ \mathcal{W}$. By using equation (3) for the relative closures of the Bruhat and Birkhoff cells of $G$, and formula (d) stated in the part 'The minimal and formal Kac-Moody group $G$ and $G_f$, the monoids $\hat{G}$ and $\hat{G}_f$' of the section 'Preliminaries' we find

$$B^\epsilon w_1 \epsilon (R(\Theta)) w_2 B \subseteq B^\epsilon w_1 B \epsilon (R(\Theta)) w_2 B \subseteq \overline{B^\epsilon w_1 B \epsilon (R(\Theta)) w_2 B} \subseteq B^\epsilon w_1 \epsilon (R(\Theta)) w_2 B .$$

**e** Now we show the direction from (ii) to (i) of 1a) and 2): By using (d) and formulas 2, 3 we find

$$(w'_2)^{-1}w \epsilon (R(\Theta)) T w_2 = ((w'_2)^{-1}w) \Theta \epsilon (R(\Theta)) T w_2 \subseteq \overline{B (w'_2)^{-1} \epsilon (R(\Theta)) w_2 B}$$

$$= B \epsilon B \epsilon (w'_2)^{-1} (R(\Theta)) B .$$

The closure $\overline{B}$ is a monoid. It contains $B$. It contains the closure $\overline{T} = \hat{T}$, in particular it contains the elements $\epsilon(R), R \in R(X)$. Therefore the last set of the preceding formula is contained in $\overline{B}$. From this follows

$$w'_1 \epsilon (R(\Theta')) w \epsilon (R(\Theta)) w_2 T = w'_1 \epsilon (R(\Theta')) w_2 (w'_2)^{-1} w \epsilon (R(\Theta)) w_2 T \in w'_1 \epsilon (R(\Theta')) w_2 \overline{B} \subseteq \overline{B^\epsilon w_1 \epsilon (R(\Theta')) w_2 B} .$$

On the other hand we get by using formula 3:

$$w'_1 \epsilon (R(\Theta')) w \epsilon (R(\Theta)) w_2 T = w'_1 w \epsilon (R(\Theta')) \epsilon (R(\Theta)) w_2 T = w'_1 w \epsilon (R(\Theta)) w_2 T .$$

Therefore

$$B^\epsilon w'_1 \epsilon (R(\Theta)) w_2 B \subseteq \overline{B^\epsilon w_1 \epsilon (R(\Theta')) w_2 B} .$$

(7)

Due to the first inequalities of 1a) (ii) and 2) (ii) we get by using (d) and formula 4 once more:

$$B^\epsilon w_1 \epsilon (R(\Theta)) w_2 B \subseteq \overline{B^\epsilon (w'_2 w) \epsilon (R(\Theta')) w_2 B} = B^\epsilon w'_1 \epsilon (R(\Theta)) w_2 B .$$

(8)

From the inclusions (7) and (8) follows (i).

□

By using the explicit description of $\leq_{++}$, $\leq_{--}$, and $\leq_{-+}$ given in the last theorem, we now can show that these relations are order relations.

**Theorem 2.4** The relations $\leq_{++}$, $\leq_{--}$, and $\leq_{-+}$ are order relations on $\hat{\mathcal{W}}$, which extend the Bruhat order on $\mathcal{W}$.  

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Proof: It remains to show that these relations are antisymmetric. Let \( \hat{w} = w_1 e(R(\Theta)) w_2, \hat{w}' = w_1' e(R(\Theta')) w_2' \in \hat{W} \) be elements of \( \hat{W} \) in normal form I, and let \( \hat{w} \leq_{++} \hat{w}', \hat{w}' \leq_{++} \hat{w} \). Due to the last theorem we have \( \Theta = \Theta' \), and there exist elements \( w, \hat{w} \in W_{\Theta^+} \) such that

\[
\begin{align*}
    w_1 &\leq (w_1')^\Theta, \quad w_1' \leq (w_1 \hat{w})^\Theta, \\
    w_2 &\geq \Theta(w^{-1}w_2'), \quad w_2' \geq \Theta(\hat{w}^{-1}w_2).
\end{align*}
\]

(9) and (10)

Remark (2) following Proposition 2.2 implies \( W^\Theta W_{\Theta^+} = W_{\Theta \cup \Theta^+} W_{\Theta^+} = W^\Theta \).

Therefore the inequalities of (9) and (10) are equivalent to

\[
\begin{align*}
    w_1 &\leq w_1' w, \quad w_1' \leq w_1 \hat{w}, \\
    w_2 &\geq w^{-1}w_2', \quad w_2' \geq \hat{w}^{-1}w_2.
\end{align*}
\]

(11) and (12)

Since \( w_2, w_2' \in \Theta_{\cup \Theta^+} W \) and \( w^{-1}, \hat{w}^{-1} \in W_{\Theta^+} \subseteq W_{\Theta \cup \Theta^+} \) from (12)

\[
\begin{align*}
l(w_2) &\geq l(w^{-1}w_2') = l(w^{-1}) + l(w_2') \geq l(w_2') \geq l(\hat{w}^{-1}w_2) = l(\hat{w}^{-1}) + l(w_2) \geq l(w_2).
\end{align*}
\]

This implies \( l(w_2) = l(w_2') \), and inserting in this chain of inequalities we find \( l(w^{-1}) = l(\hat{w}^{-1}) = 0 \). Therefore \( w^{-1} = \hat{w}^{-1} = 1 \). Inserting in (11), (12), we get

\[
\begin{align*}
w_1 &\leq w_1', \quad w_1' \leq w_1, \\
w_2 &\geq w_2', \quad w_2' \geq w_2.
\end{align*}
\]

(13) and (14)

Due to the antisymmetry of the Bruhat order of \( W \) we get \( w_1 = w_1' \) and \( w_2 = w_2' \).

The antisymmetry of \( \leq_{++} \) can be proved similarly, only \( \leq' \) has to be exchanged by \( \geq' \) in (9), (11), (13).

Because the inverse map \( \text{inv} : \hat{W} \to \hat{W} \) is an isomorphism of the relations \( \leq_{++} \) and \( \leq_{--} \), also the relation \( \leq_{--} \) is antisymmetric. \( \square \)

\( \Delta \in P^+ \) and \( \lambda, \mu \in W \Lambda \) fix elements \( v_\lambda \in L(\Lambda, \lambda) \setminus \{0\}, v_\mu \in L(\Lambda, \mu) \setminus \{0\} \) and set

\[
g_{\lambda\mu} := f_{v_\lambda, v_\mu}
\]

(15) for short. By using the antisymmetry of the extended Bruhat orders just proved we show:

**Theorem 2.5** Let \( \hat{w} \in \hat{W} \). The Bruhat and Birkhoff cells \( B \hat{w} B, B^{-} \hat{w} B^{-} \), and \( B^{-} \hat{w} B \) are principal open in their closures, i.e., for

\[
(\epsilon, \delta) = \begin{cases} 
(+, +) & \text{let } \hat{w} = w_1 e(R(\Theta)) w_2 \text{ be } \begin{cases} \text{normal form I}, \\
\text{normal form II}, \\
\text{normal form I or II},
\end{cases} \\
(-, -) & \\
(-, +)
\end{cases}
\]

and let \( \Delta \in F_{\Theta} \cap P^+ \). Then

\[
B^{-} \hat{w} B^\Theta \cap D_{G}(g_{w_1 \Lambda w_2^{-1} \Lambda}) = B^\epsilon \hat{w} B^\delta.
\]

(16)
Proof: 1) Let \( \dot{w} = w_1 \varepsilon (e(R(\Theta))) w_2 \) be in normal form I. Let \( \epsilon \in \{+, -\} \) and \( \delta = +. \)

We first show the inclusion \( ' \subseteq ' \) of (16). Because of the Bruhat and Birkhoff decompositions of \( \tilde{G} \), any element of \( \tilde{G} \) can be written in the form \( u_v \tilde{n}_v u \) with \( u_v \in U^v, u \in U, \) and \( \tilde{n}_v \in \tilde{N} \) belonging to \( \tilde{w} \) and \( \tilde{w}' \) in \( \tilde{W} \). If \( u_v \tilde{n}_v u \in B^v \tilde{w} B \), then we also have \( B^v \tilde{w}' B \subseteq B^v \tilde{w} B \). By definition \( \tilde{w}' \leq +, \tilde{w} \) and \( \tilde{w}' \leq -, \tilde{w} \) for \( \epsilon = +, \) and \( \tilde{w}' \geq +, \tilde{w} \) for \( \epsilon = -, \)

In part (c) of the proof of Theorem 2.3 we have seen that

\[
g_{w_1 \Lambda w_2^{-1} \Lambda}(u_v \tilde{n}_v u) \neq 0
\]

implies \( \Theta \supseteq \Theta' \) and there exists an element \( w \in W_{\tilde{w}' \tilde{w}} \) such that \( \Theta(w^{-1}w_2) \leq w_2 \) and

\[
w_1 \left\{ \begin{array}{ll}
\leq & (w'_1 w)^\Theta \\
\geq & (w'_1 w)^\Theta
\end{array} \right. \text{ for } \epsilon = + \\
\leq & (w'_1 w)^\Theta \\
\geq & (w'_1 w)^\Theta
\end{array} \right. \text{ for } \epsilon = -.
\]

Due to Theorem 2.3 itself, from this follows \( \tilde{w}' \geq +, \tilde{w} \) for \( \epsilon = +, \) and \( \tilde{w}' \leq -, \tilde{w} \) for \( \epsilon = -, \)

Due to the last theorem the relations \( \leq +, \leq - \) are antisymmetric. Therefore we get \( \tilde{w}' = \tilde{w} \).

To show the inclusion \( ' \supseteq ' \) of (16) let \( u_v \in U^v, u \in U, \) and let \( \tilde{n}_v \in \tilde{N} \) belong to \( \tilde{w} \). Write \( \tilde{n}_v \) in the form \( \tilde{n}_v = n_{w_1} e(R(\Theta)) n_{w_2} \) with \( n_{w_1}, n_{w_2} \in N \) belonging to \( w_1, w_2 \in W \). We have

\[
g_{w_1 \Lambda w_2^{-1} \Lambda}(u_v \tilde{n}_v u) = \left\langle \left\langle n_{w_1}^* u^*_v w_1 \Lambda \mid e(R(\Theta)) n_{w_2} u v_{w_2^{-1} \Lambda} \right\rangle \right\rangle \tag{17}
\]

The vertices of \( S(n_{w_2} u v_{w_2^{-1} \Lambda}) \) are of the form \( w_2 \tilde{w} \Lambda \) where \( \tilde{w} \in W^\Theta \) and \( \tilde{w} \leq w_2^{-1} \). Furthermore \( \Lambda \) is a vertex.

Now we want to determine which vertices of this form are also vertices of \( S(e(R(\Theta)) n_{w_2} u v_{w_2^{-1} \Lambda}) \). Because of \( \Lambda \) is contained in the relative interior of \( R(\Theta) \) we find

\[
w_2 \tilde{w} \Lambda \in R(\Theta) \iff w_2 \tilde{w} R(\Theta) \subseteq R(\Theta) \iff w_2 \tilde{w} R(\Theta) = R(\Theta) \iff w_2 \tilde{w} \in W_{\Theta(\cup \Theta \cup \Theta)} \iff \tilde{w} \in w_2^{-1} W_{\Theta(\cup \Theta \cup \Theta)}
\]

In this case we get \( w_2^{-1} \leq \tilde{w} \) because \( w_2^{-1} \) is a minimal coset representative of \( W_{\Theta(\cup \Theta \cup \Theta)} \). Because of the antisymmetry of the Bruhat order of \( W \) this implies \( w_2^{-1} = \tilde{w} \). Therefore \( \Lambda \) is the only vertex of \( S(e(R(\Theta)) n_{w_2} u v_{w_2^{-1} \Lambda}) \) and we have

\[
e(R(\Theta)) n_{w_2} u v_{w_2^{-1} \Lambda} = n_{w_2} v_{w_2^{-1} \Lambda}
\]

Inserting in equation (17) we find

\[
g_{w_1 \Lambda w_2^{-1} \Lambda}(u_v \tilde{n}_v u) = \left\langle \left\langle n_{w_1}^* u^*_v w_1 \Lambda \mid n_{w_2} v_{w_2^{-1} \Lambda} \right\rangle \right\rangle
\]

\[
= \left\langle \left\langle n_{w_1}^* w_1 \Lambda \mid n_{w_2} v_{w_2^{-1} \Lambda} \right\rangle \right\rangle \neq 0
\]

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2) By applying the Chevalley involution \( * : \hat{G} \to \hat{G} \) on \( \Theta \) for \( \epsilon \in \{+,-\} \), and \( \delta = + \) we find the equation

\[
B^-((\hat{w})^{\text{inv}}B^{-\epsilon}) \cap D_{\hat{G}}(g^*) = B^-((\hat{w})^{\text{inv}}B^{-\epsilon}) ,
\]

from which follow the remaining statements of the theorem by Remark (1) following Proposition 2.2.

The next two theorems give product decompositions of \( B\hat{w}B \), \( B^-\hat{w}B^- \), and \( B^-\hat{w}B \), \( \hat{w} \in \hat{W} \), as principal open sets in their closures. These generalize the product decompositions of sets \( BwB = U_w \cdot wT \cdot U \), and \( B^-wB = (U^w)^- \cdot wT \cdot U \), \( w \in W \), given in [K.P] for Kac-Moody groups. The decomposition of \( B\hat{w}B \), \( \hat{w} \in \hat{W} \), is nearly similar to the corresponding decomposition of a Bruhat cell of the wonderful compactification of a semisimple algebraic group in [Re3].

**Theorem 2.6**

1a) Let \( \hat{w} = w_1 \in (R(\Theta))w_2 \) be an element of \( \hat{W} \) in normal form I, and let \( \hat{n} \) be a corresponding element of \( \hat{N} \). Then

\[
B \hat{w} B = U_{w_1} \hat{n}(w_2^{-1}T^{\Theta}w_2)(U \cap w_2^{-1}U^{\Theta}w_2) .
\]

b) Let \( \hat{w} = w_1 \in (R(\Theta))w_2 \) be an element of \( \hat{W} \) in normal form II, and let \( \hat{n} \) be a corresponding element of \( \hat{N} \). Then

\[
B^- \hat{w} B^- = (U^- \cap w_1(U^{\Theta})^-w_1^{-1})(w_1T^{\Theta}w_1^{-1}) \hat{n}U_{w_2}^{-1} .
\]

2) Let \( \hat{w} = w_1 \in (R(\Theta))w_2 \) be an element of \( \hat{W} \) in normal form I or in normal form II, and let \( \hat{n} \) be a corresponding element of \( \hat{N} \). Then

\[
B^- \hat{w} B = (U^- \cap w_1(U^{\Theta})^-w_1^{-1}) \hat{n}(w_2^{-1}T^{\Theta}w_2)(U \cap w_2^{-1}U^{\Theta}w_2) = (U^- \cap w_1(U^{\Theta})^-w_1^{-1})(w_1T^{\Theta}w_1^{-1}) \hat{n}(U \cap w_2^{-1}U^{\Theta}w_2) .
\]

**Remark:** It is easy to see that

\[
U_{w_1} = U \cap w_1U^-w_1^{-1} = U \cap w_1(U^{\Theta})^-w_1^{-1} ,
\]

\[
U_{w_2}^{-1} = U \cap w_2^{-1}U^-w_2 = U \cap w_2^{-1}(U^{\Theta})^-w_2 .
\]

In this sense the formulas of 1) are symmetric in the first and last factor.

**Proof:** We only have to show the statements of the theorem which involve normal form I. Then the statements which involve normal form II follow by applying the Chevalley involution \( * : \hat{G} \to \hat{G} \), together with the Remark (1) following Proposition 2.2.

Write \( \hat{n} \) in the form \( \hat{n} = n_1 \epsilon(R(\Theta))n_2 \) with \( n_1, n_2 \in N \) corresponding to \( w_1 \in W^{\Theta} \), \( w_2 \in \Theta^{\epsilon} \cdot \Theta^{-\epsilon} \cdot W \). For the following transformations we make use of the
formulas (β), (γ), and (δ) stated in the part "The minimal and formal Kac-Moody group G and \(G_f\), the monoids \(\widehat{G}\) and \(\widehat{G}_f\)" of the section "Preliminaries".

**1) By using the first equation of (δ) we find**

\[
B\hat{w}B = U_{w_1}U^{w_1}n_1 e(R(\Theta))Tn_2 U = U_{w_1}n_1 w_1^{-1}U^{w_1}w_1 e(R(\Theta))Tn_2 U \\
\subseteq U_{w_1}n_1 e(R(\Theta))U_{\Theta}Tn_2 U = U_{w_1}n_1 e(R(\Theta))Tn_2 w_2^{-1}U_{\Theta} w_2 U \\
= U_{w_1}n_1 e(R(\Theta))Tn_2 U = U_{w_1}n_1 e(R(\Theta))Tn_2 U w_2^{-1}U^{w_2^{-1}}w_2^{-1} \\
= U_{w_1}n_1 e(R(\Theta))w_2 U w_2^{-1}w_2^{-1}Tn_2 U^{w_2^{-1}}w_2^{-1}.
\]

\(w_2 U w_2^{-1} w_2^{-1}\) is generated by the root groups

\[U_\beta \text{ where } \beta \in \Delta_{re}^- , w_2^{-1} \beta \in \Delta_{re}^+ .\]

Since \(w_2^{-1} \in \mathcal{W}_{\Theta_u \Theta_i}\) these root groups coincide with the root groups

\[U_\beta \text{ where } \beta \in \Delta_{re}^- \setminus \mathcal{W}_{\Theta_u \Theta_i} \{ \alpha_i | i \in (\Theta \cup \Theta_i^-) \} , w_2^{-1} \beta \in \Delta_{re}^+ ,\]

which are contained in \((U^{\Theta_u \Theta_i})^-\). By using formula 2 we find

\[B\hat{w}B \subseteq U_{w_1}n_1 e(R(\Theta))Tn_2 U^{w_2^{-1}}w_2^{-1},\]

and the reverse inclusion is obvious. We have

\[U^{w_2^{-1}} \subseteq w_2^{-1}U w_2 = w_2^{-1}U_{\Theta}w_2 \times w_2^{-1}U^{\Theta}w_2 .\]

Because of \(w_2^{-1} \in \mathcal{W}_{\Theta_u \Theta_i}\), the group \(w_2^{-1}U_{\Theta}w_2\) is contained in \(U\). Clearly it is also contained in \(w_2^{-1}Uw_2\). Therefore it is contained in \(U^{w_2^{-1}}\).

An element \(x \in w_2^{-1}U w_2\) can be written in the form \(x = p_1(x)p_2(x)\) with \(p_1(x) \in w_2^{-1}U_{\Theta}w_2\) and \(p_2(x) \in w_2^{-1}U^{\Theta}w_2\). Obviously \(p_2(U^{w_2^{-1}}) \supseteq U^{w_2^{-1}} \cap w_2^{-1}U^{\Theta}w_2\).

The reverse inclusion follows because for \(x \in U^{w_2^{-1}}\) we have

\[w_2^{-1}U^{\Theta}w_2 \ni p_2(x) = p_1(x)^{-1}x \in U^{w_2^{-1}} .\]

We get

\[p_2(U^{w_2^{-1}}) = U^{w_2^{-1}} \cap w_2^{-1}U^{\Theta}w_2 = U \cap w_2^{-1}U w_2 \cap w_2^{-1}U^{\Theta}w_2 = U \cap w_2^{-1}U^{\Theta}w_2 .\]

By using this equation and two times formula (β) we find

\[
B\hat{w}B = U_{w_1}n_1 e(R(\Theta))Tn_2 U^{w_2^{-1}} = U_{w_1}n_1 e(R(\Theta)) w_2 p_1(U^{w_2^{-1}})w_2^{-1}Tn_2 p_2(U^{w_2^{-1}}) \\
\subseteq U_{\Theta} \\
= U_{w_1}n_1 e(R(\Theta))Tn_2(U \cap w_2^{-1}U^{\Theta}w_2) = U_{w_1}n_1 e(R(\Theta))T^{\Theta}n_2(U \cap w_2^{-1}U^{\Theta}w_2) \\
= U_{w_1}n_1 e(R(\Theta))n_2(w_2^{-1}T^{\Theta}w_2)(U \cap w_2^{-1}U^{\Theta}w_2) .
\]
2) By using the first equation of (δ) we get
\[
B^{-\hat{w}}B = (U^{\hat{w}_1})^{-1}U_{w_1}m_1e(R(\Theta))Tn_2U = (U^{\hat{w}_1})^{-1}w_{1}^{-1}U_{w_1}^{-1}w_{1}^{-1}e(R(\Theta))Tn_2U \\
\subseteq (U^{\hat{w}_1})^{-1}n_1e(R(\Theta))U_{\theta_1}Tn_2U = (U^{\hat{w}_1})^{-1}n_1e(R(\Theta))Tn_2w_1^{-1}U_{\theta_1}^{-1}w_{2}^{-1}U \\
= (U^{\hat{w}_1})^{-1}n_1e(R(\Theta))Tn_2U.
\]
The reverse inclusion is obvious. In the same way as before we get
\[
B^{-\hat{w}}B = (U^{\hat{w}_1})^{-1}n_1e(R(\Theta))Tn_2(U \cap w_2^{-1}U^{\theta_2}w_2) \\
= (U^{\hat{w}_1})^{-1}n_1e(R(\Theta))n_2(w_2^{-1}T^{\theta_2}w_2)(U \cap w_2^{-1}U^{\theta_2}w_2).
\]
Treating the first factor in a similar way we find
\[
B^{-\hat{w}}B = (U^{-1} \cap w_1(U^{\theta_2}^{-1}w_1^{-1})n_1e(R(\Theta))n_2(w_2^{-1}T^{\theta_2}w_2)(U \cap w_2^{-1}U^{\theta_2}w_2) \\
= (U^{-1} \cap w_1(U^{\theta_2}^{-1}w_1^{-1})(w_1T^{\theta_2}w_1^{-1})n_1e(R(\Theta))n_2(U \cap w_2^{-1}U^{\theta_2}w_2).
\]

We equip \(B^\epsilon \hat{w}B^\delta\) with its coordinate ring \(F[B^\epsilon \hat{w}B^\delta]\) as a principal open set in its closure, \((\epsilon, \delta) \in \{(+, +), (-), (-+), (-\epsilon)\}, \hat{w} \in \hat{W}\).

Recall that the torus \(T\) of the Kac-Moody group can be described by the following isomorphism of groups:
\[
H \otimes \mathbb{Z} F^\times \rightarrow \prod_{i=1}^{2n-l} h_i \otimes s_i \rightarrow T.
\]
The group algebra \(F[P]\) of the lattice \(P\) can be identified with the classical coordinate ring on \(T\), identifying \(\sum c_\lambda e_\lambda \in F[P]\) with the function on \(T\) defined by
\[
\left(\sum c_\lambda e_\lambda\right)\left(\prod_{i=1}^{2n-l} t_{h_i}(s_i)\right) := \sum c_\lambda \prod_{i=1}^{2n-l} (s_i)^{\lambda(h_i)}, \quad (s_i \in F^\times).
\]
Similarly, for \(J \subseteq I\) and \(w \in \hat{W}\), the classical coordinate ring of the torus \(wT^Jw^{-1}\), where
\[
T^J := \{ \prod_{i=1, i \in J}^{2n-l} t_{h_i}(s_i) \mid s_i \in F^\times \},
\]
is given by the group algebra \(F[wP^J]\), where
\[
P^J := \mathbb{Z}\text{-span} \{ \Lambda_i \mid i = 1, \ldots, 2n - l, i \notin J \}
\]
(In general the classical coordinate rings of these tori do not coincide with the restriction of the coordinate ring \(F[\hat{G}]\) onto these tori. It is possible to show that \(wT^Jw^{-1}\) is principal open in its closure, and the classical coordinate ring of \(wT^Jw^{-1}\) is the coordinate ring of this principal open set. But we do not need this for the following considerations.)
Theorem 2.7
1a) Let $\hat{w} = w_1 \varepsilon (R(\Theta))w_2$ be an element of $\hat{\mathcal{W}}$ in normal form I, and let $\hat{n}$ be a corresponding element of $\hat{N}$. The map
\[ m : U_{w_1} \times w_2^{-1}T^{\Theta}w_2 \times (U \cap w_2^{-1}U^{\Theta}w_2) \rightarrow B\hat{w}B, \]
defined by $m(u, t, \hat{u}) := u\hat{t}\hat{u}$, is an isomorphism.

b) Let $\hat{w} = w_1 \varepsilon (R(\Theta))w_2$ be an element of $\hat{\mathcal{W}}$ in normal form II, and let $\hat{n}$ be a corresponding element of $\hat{N}$. The map
\[ m : (U^{-} \cap w_1(U^{\Theta})^{-}w_1^{-1}) \times w_1T^{\Theta}w_1^{-1} \times U_{w_2^{-1}} \rightarrow B^{-}\hat{w}B^{-}, \]
defined by $m(u, t, \hat{u}) := ut\hat{n}u$, is an isomorphism.

2) Let $\hat{w} = w_1 \varepsilon (R(\Theta))w_2$ be an element of $\hat{\mathcal{W}}$ in normal form I or in normal form II, and let $\hat{n}$ be a corresponding element of $\hat{N}$. The map
\[ m : (U^{-} \cap w_1(U^{\Theta})^{-}w_1^{-1}) \times w_2^{-1}T^{\Theta}w_2 \times (U \cap w_2^{-1}U^{\Theta}w_2) \rightarrow B^{-}\hat{w}B, \]
defined by $m(u, t, \hat{u}) := ut\hat{n}u$, is an isomorphism.

Proof: We only have to show the statements of the theorem which involve normal form I. Then the statements which involve normal form II follow easily by using the Chevalley involution $*: \hat{G} \rightarrow \hat{G}$, together with Remark (3) following Definition 2.1 and Remark (1) following Proposition 2.2.

Due to the last theorem the maps $m$ of 1a) and 2) are surjective. We show that the corresponding comorphisms $m^*$ are well defined and surjective. This is sufficient, because the surjectivity of the maps $m$ imply the injectivity of the comorphisms $m^*$, and the surjectivity of the comorphisms $m^*$ imply the injectivity of the maps $m$.

To 1a): To show that the comorphism
\[ m^* : F[B\hat{w}B] \rightarrow F[U_{w_1}] \otimes F[w_2^{-1}P^{\Theta}] \otimes F[U \cap w_2^{-1}U^{\Theta}w_2] \]
is well defined, write $\hat{n}$ in the form $\hat{n} = n_1e(R(\Theta))n_2$ with $n_1, n_2 \in N$ corresponding to $w_1 \in W^{\Theta}, w_2 \in \Theta^{-}W$.

Fix an element $\Lambda \in F_{\Theta}$, and fix $v_1 \in L(\Lambda)_{w_1\Lambda} \setminus \{0\}, v_2 \in L(\Lambda)_{w_2^{-1}\Lambda} \setminus \{0\}$. For $u \in U_{w_1}, t \in w_2^{-1}T^{\Theta}w_2$, and $\hat{u} \in (U \cap w_2^{-1}U^{\Theta}w_2)$ we have
\[ f_{v_1v_2}(m(u, t, \hat{u})) = \langle \langle v_1 | un_1e(R(\Theta))n_2\hat{t}\hat{u}v_2 \rangle \rangle = \langle \langle u^*v_1 | n_1e(R(\Theta))n_2\hat{t}\hat{u}v_2 \rangle \rangle. \]

Because of $\hat{u} \in w_2^{-1}U^{\Theta}w_2$ we get $\hat{u}v_2 = v_2$. Because of
\[ u^* \in U_{w_1}^{\ast} = (U \cap w_1U^{-}w_1^{-1})^{\ast} = U^{-} \cap w_1Uw_1^{-1} \]
we get $uv_1 = v_1$. Therefore
\[ f_{v_1v_2}(m(u, t, \hat{u})) = \langle \langle v_1 | n_1e(R(\Theta))n_2tv_2 \rangle \rangle = \langle \langle v_1 | n_1e(R(\Theta))n_2v_2 \rangle \rangle e_{w_2^{-1}\Lambda}(t) \]
\[ = \{\langle v_1 | n_1n_2v_2 \rangle \} e_{w_2^{-1}\Lambda}(t). \]
Now let \( N \in P^+ \) and \( u, w \in L(N) \). Choose \( \langle \langle \ | \rangle \rangle \)-dual bases of \( L(N) \) by choosing \( \langle \langle \ | \rangle \rangle \)-dual bases

\[
(a_{\tau i})_{i=1, \ldots, m_r}, \quad (b_{\tau i})_{i=1, \ldots, m_r}
\]

of \( L(N) \) for every \( \tau \in P(N) \). For \( u \in U_w, t \in w_2^{-1}T^\Theta w_2 \), and \( \bar{u} \in (U \cap w_2^{-1}U^\Theta w_2) \) we have

\[
f_{vw}(m(u, t, \bar{u})) = \langle \langle v \ | \ un_1 e(R(\Theta)) n_2 t \bar{u} w \rangle \rangle = \sum_{\tau, i} \langle \langle v \ | \ un_1 e(R(\Theta)) n_2 t a_{\tau i} \rangle \rangle \langle \langle b_{\tau i} \ | \ \bar{u} w \rangle \rangle = \sum_{w_2 \tau \in \hat{\mathcal{B}}} \langle \langle v \ | \ un_1 n_2 a_{\tau i} \rangle \rangle \varepsilon_{\tau}(t) \langle \langle b_{\tau i} \ | \ \bar{u} w \rangle \rangle.
\]

A summand of this sum is nonzero at the most if \( w \neq 0 \), and if \( \tau \) is bigger than an element of \( \text{supp}(w) \), which is only possible for finitely many non-zero summands. Furthermore \( R(\Theta) \cap P = X \cap P^\Theta \). Taking into account (18) we therefore get for \( p \in \mathbb{N}_0 \):

\[
m^* \left( \frac{f_{vw}}{f_{v_1v}} \big|_B \right) = \sum_{\tau, i} \frac{1}{\langle \langle v_1 \ | \ n_1 n_2 v_2 \rangle \rangle^p} f_{v n_1 n_2 a_{\tau i}} \big|_{U w_1} \otimes \varepsilon_{\tau - p(w_2^{-1})} \big|_{U \cap w_2^{-1}U^\Theta w_2} \big|_{U w_2^{-1}U^\Theta w_2}.
\]

Because \( F[\hat{\mathcal{G}}] \) is spanned by the matrix coefficients \( f_{vw}, v, w \in L(N), N \in P^+ \), we have shown that the comorphism \( m^* \) is well defined.

To show the surjectivity of \( m^* \), it is sufficient to find elements of \( F[\hat{\mathcal{B}} \hat{\mathcal{B}} B] \), which are mapped onto a system of generators of \( F[\hat{\mathcal{G}}] \). Let \( v \in L(\Lambda) \). For \( u \in U_w, t \in w_2^{-1}T^\Theta w_2 \), and \( \bar{u} \in (U \cap w_2^{-1}U^\Theta w_2) \) we have

\[
\langle \langle v_1 \ | \ un_1 e(R(\Theta)) n_2 \bar{u} w \rangle \rangle = \langle \langle u^* v_1 \ | \ n_1 e(R(\Theta)) n_2 \bar{u} w \rangle \rangle = \langle \langle v_1 \ | \ n_1 e(R(\Theta)) n_2 \bar{u} \rangle \rangle = \langle \langle n_2 e(R(\Theta)) n_2^* v_1 \ | \ \bar{u} w \rangle \rangle = \langle \langle n_2^* v_1 \ | \ \bar{u} w \rangle \rangle e_{w_2}^\Lambda(t).
\]

Taking into account (18) we therefore get

\[
m^* \left( \frac{f_{vw}}{f_{v_1v}} \big|_B \right) = \frac{1}{\langle \langle v_1 \ | \ n_1 n_2 v_2 \rangle \rangle} \bigg|_{\neq 0} 1 \otimes 1 \otimes f_{n_2^* n_1^* v_1 \big|_{U \cap w_2^{-1}U^\Theta w_2}} \bigg|_{U \cap w_2^{-1}U^\Theta w_2}.
\]

We have \( n_1^* v_1 \in L(\Lambda) \setminus \{0\} \). Due to Theorem 5.6 of [11] the coordinate ring \( F[U^\Theta] \) is generated by the functions \( f_{n_1^* v_1 \big|_{U^\Theta}} \), \( v \in L(\Lambda) \). Therefore its restriction \( F[w_2 U w_2^{-1} \cap U^\Theta] \) is generated by

\[
f_{n_1^* v_1 \big|_{w_2 U w_2^{-1} \cap U^\Theta}} \quad , \quad v \in L(\Lambda).
\]
It is easy to check that we get an isomorphism
\[ \phi : U \cap w_2^{-1}U^{th}w_2 \to w_2Uw_2^{-1} \cap U^{th} \]
by \( \phi(u) := n_2un_2^{-1} \). For \( u \in U \cap w_2^{-1}U^{th}w_2 \) we have
\[ \phi^*(f_{n_1^*v_1n_2v}|_{w_2Uw_2^{-1}\cap(U^{th})})(u) = \langle\langle n_1^*v_1 | n_2un_2^{-1}n_2v \rangle\rangle = \langle\langle n_2^*n_1^*v_1 | uv \rangle\rangle. \]

Therefore the functions
\[ \phi^*(f_{n_1^*v_1n_2v}|_{w_2Uw_2^{-1}\cap(U^{th})}) = f_{n_2^*n_1^*v_1v}|_{U\cap w_2^{-1}U^{th}w_2}, \quad v \in L(\Lambda), \]
which appear in [20], generate the coordinate ring \( \mathbb{F}[U \cap w_2^{-1}U^{th}w_2] \).

b) Let \( N \in \mathbb{F}_0 \cap P \) and \( \tilde{v}_1 \in L(N)_{w_1N} \setminus \{0\}, \tilde{v}_2 \in L(N)_{w_2^{-1}N} \setminus \{0\} \). Let \( p \in \mathbb{N}_0 \). Similar to [18] we find
\[ m^* \left( \frac{f_{\tilde{v}_1\tilde{v}_2}}{(f_{v_1v_2})^p} |_{B \bar{w}B} \right) = \frac{\langle\langle \tilde{v}_1 | n_1n_2\tilde{v}_2 \rangle\rangle}{\langle\langle v_1 | n_1n_2v_2 \rangle\rangle} 1 \otimes e_{w_2^{-1}(N-p\Lambda)} \otimes 1. \quad (21) \]

It is easy to check that \( (\mathbb{F}_0 \cap P) - \mathbb{N}_0\Lambda = P^{th} \). Therefore the functions \( e_{w_2^{-1}(N-m\Lambda)}, N \in \mathbb{F}_0 \cap P, m \in \mathbb{N}_0, \) span the coordinate ring \( \mathbb{F}[w_2^{-1}P^{th}] \).

c) Let \( v \in L(\Lambda) \). For \( u \in U_{w_1}, t \in w_2^{-1}T^{th}w_2, \) and \( \tilde{u} \in (U \cap w_2^{-1}U^{th}w_2) \) we have
\[ \langle\langle v | un_1e(R(\Theta))n_2\tilde{u}v_2 \rangle\rangle = \langle\langle v | un_1e(R(\Theta))n_2tv_2 \rangle\rangle = \langle\langle v | un_1e(R(\Theta))n_2v_2 \rangle\rangle e_{w_2^{-1}\Lambda}(t) = \langle\langle v | un_1v_2 \rangle\rangle e_{w_2^{-1}\Lambda}(t). \]

Taking into account \[ [18] \] we therefore get
\[ m^* \left( \frac{f_{v_2}}{f_{v_1v_2}} |_{B \bar{w}B} \right) = \frac{1}{\langle\langle v_1 | n_1n_2v_2 \rangle\rangle} f_{v_1n_2v_2} |_{U_{w_1}} \otimes 1 \otimes 1. \quad (22) \]

\( U_{w_1} = U \cap w_1U^{-w_1^{-1}} \) is generated by the root groups \( U_\alpha, \alpha \in \Delta^{+}_{re} \cap w_1\Delta^{+}_{re} \).

Since \( w_1 \in W^{th} \) we have \( \Delta^{+}_{re} \cap w_1\Delta^{+}_{re} = \Delta^{+}_{re} \cap w_1(\Delta^{+}_{re} \setminus W_\Theta) \). This implies
\[ U_{w_1} = U \cap w_1U^{-w_1^{-1}} \subseteq U \cap w_1(U^{th})^{-w_1^{-1}}, \]
and the reverse inclusion is obvious. We have \( n_2v_2 \in L(\Lambda) \setminus \{0\} \). Due to Theorem 5.6 of [14] the coordinate ring \( \mathbb{F}[(U^{th})^{-}] \) is generated by the functions \( f_{v_1v_2} |_{(U^{th})^{-}}, v \in L(\Lambda) \). Therefore its restriction \( \mathbb{F}[w_1^{-1}U_{w_1} \cap (U^{th})^{-}] \) is generated by
\[ f_{n_1^*v_1n_2v_2} |_{w_1^{-1}U_{w_1} \cap (U^{th})^{-}}, \quad v \in L(\Lambda). \]

It is easy to check that we get an isomorphism
\[ \phi : U \cap w_1(U^{th})^{-w_1^{-1}} \to w_1^{-1}U_{w_1} \cap (U^{th})^{-}. \]
Next we want to show that the orbits are irreducible. For this we first state there is a similar theorem for due to Theorem 16 of \[M 2\] this closure coincides with the closure denoted by let \(D\) in \[M 2\]. Now Theorem 4 of \[M 2\] gives:

\[\begin{align*}
\phi(u) := n_1^{-1}u_n.
\end{align*}\]

For \(u \in U_{w_1} = U \cap w_1(U^\Theta)w_1^{-1}\) we have
\[\phi^*(f_{n_1^*v,n_2v_2}\mid_{w_1^{-1}U_{w_1\cap(U^\Theta)^{-1}}})(u) = \langle\langle n_1^*v \mid n_1^{-1}u_nv_2\rangle\rangle = \langle\langle v \mid u_nv_2\rangle\rangle\]

Therefore the functions
\[\phi^*(f_{n_1^*v,n_2v_2}\mid_{w_1^{-1}U_{w_1\cap(U^\Theta)^{-1}}}) = f_{v,n_1v_2}\mid_{U_{w_1}}, \quad v \in L_{\Lambda},\]

which appear in \[22\] generate the coordinate ring \(F[U_{w_1}]\).

To 2): We use the same notations as in the first part of this proof. For \(u \in U \cap w_1(U^\Theta)^{-1}\) we have \(u^* \in w_1U_{w_1}^{-1}\). Therefore also \(u^*v_1 = v_1\).

Completely parallel to the first part of this proof we get formula \[19\] with \(B\hat{w}B\) replaced by \(B^{-1}\hat{w}B\), and \(U_{w_1}\) replaced by \(U \cap w_1(U^\Theta)^{-1}\). In particular it follows that the comorphism
\[m^* : F[B^{-1}\hat{w}B] \rightarrow F[U \cap w_1(U^\Theta)^{-1}] \otimes F[w^{-1}_2P^\Theta] \otimes F[U \cap w^{-1}_2U_{w_1}w_2]\]

is well defined.

Also completely parallel to a) and b) of the first part of this proof we get the formulas \[20\] and \[21\] with \(B\hat{w}B\) replaced by \(B^{-1}\hat{w}B\). Therefore we have found elements of \(F[B^{-1}\hat{w}B]\), which are mapped onto a system of generators of \(1 \otimes 1 \otimes F[U \cap w^{-1}_2U_{w_1}w_2]\), and \(1 \otimes F[w^{-1}_2P^\Theta] \otimes 1\).

Completely parallel to the proof of formula \[22\], we get formula \[22\] with \(B\hat{w}B\) replaced by \(B^{-1}\hat{w}B\), and \(U_{w_1}\) replaced by \(U \cap w_1(U^\Theta)^{-1}\). An easy modification of the corresponding argument of a) of the first part of the proof shows that the functions \(f_{v,n_1v_2}\mid_{U \cap w_1(U^\Theta)^{-1}}, v \in L_{\Lambda}\), generate the coordinate ring \(F[U \cap w_1(U^\Theta)^{-1}]\).

Next we want to show that the orbits are irreducible. For this we first state a theorem of \[M 2\]. Equip \(\hat{G}_f\) with a coordinate ring by identifying with the coordinate ring of \(\hat{G}_f\). Denote the Zariski closure of \(M \subseteq \hat{G}_f\) by \(\overline{M}\). To use later note that due to Theorem 16 of \[M 2\] this closure coincides with the closure denoted by \(\overline{M}\) in \[M 2\]. Now Theorem 4 of \[M 2\] gives:

**Theorem 2.8** Let \(D_1, D_2\) be irreducible subgroups of \(G\), and \(x \in \hat{G}\). The \(D_1 \times D_2\)-orbit of the element \(1 \circ x \in SpecmF[\hat{G}]\) is irreducible.

There is a similar theorem for \(\hat{G}\):

**Theorem 2.9** Let \(D_1, D_2\) be irreducible subgroups of \(G\), and let \(x \in \hat{G}\). The \(D_1 \times D_2\)-orbit \(D_1^1x\) is irreducible.

The proof of this theorem can be extracted from a part of the proof of Theorem 2.8. For the convenience of the reader we sketch the proof.

**Proof:** Let \(x \in \hat{G}\) and \(\text{Or} := D_1^1x\) its \(D_1 \times D_2\)-orbit. Let \(A_1\) and \(A_2\) be

\[28\]
closed subsets of $\hat{G}$, such that $Or \subseteq A_1 \cup A_2$. We have to show $Or \subseteq A_1$ or $Or \subseteq A_2$.

Let $d_1 \in D_1$. The map $\gamma_{d_1} : D_2 \to \hat{G}$ defined by $\gamma_{d_1}(d_2) := d_1^* x d_2$, $d_2 \in D_2$, is a morphism.

Similarly, for $d_2 \in D_2$, the map $\delta_{d_2} : D_1 \to \hat{G}$ defined by $\delta_{d_2}(d_1) := d_2^* x d_1$, $d_1 \in D_1$, is a morphism.

Let $d_1 \in D_1$. Because of $\gamma_{d_1}(D_2) \subseteq Or$, we have $\gamma_{d_1}^{-1}(A_1) \cup \gamma_{d_1}^{-1}(A_2) = D_2$. Furthermore $\gamma_{d_1}^{-1}(A_1)$ and $\gamma_{d_1}^{-1}(A_2)$ are closed. Because of the irreducibility of $D_2$ we get $\gamma_{d_1}^{-1}(A_1) = D_2$ or $\gamma_{d_1}^{-1}(A_2) = D_2$.

Therefore the sets

$$B_1 := \{ d_1 \in D_1 \mid \gamma_{d_1}^{-1}(A_1) = D_2 \},$$

$$B_2 := \{ d_1 \in D_1 \mid \gamma_{d_1}^{-1}(A_2) = D_2 \}$$

satisfy $B_1 \cup B_2 = D_1$.

Note that for $d_1 \in D_1$ and $d_2 \in D_2$ we have $\gamma_{d_1}(d_2) = \delta_{d_2}(d_1)$. The set $B_1$ is closed, because of

$$B_1 = \{ d_1 \in D_1 \mid \gamma_{d_1}(d_2) \in A_1 \text{ for all } d_2 \in D_2 \} = \bigcap_{d_2 \in D_2} \delta_{d_2}^{-1}(A_1) \, \text{closed}.$$ 

Similarly, the set $B_2$ is closed. Because of the irreducibility of $D_1$ we get $B_1 = D_1$ or $B_2 = D_1$, which is equivalent to $Or \subseteq A_1$ or $Or \subseteq A_2$.

\[ \square \]

**Proposition 2.10** $B$ and $B^-$ are irreducible.

**Proof:** The Chevalley involution $* : \hat{G} \to \hat{G}$ is an isomorphism, which maps $B$ onto $B^-$. Therefore it is sufficient to show that $B$ is irreducible.

Due to Corollary 2.5 we have $D_B(gΛΛ|B) = B$. Due to Theorem 2.7 the multiplication map

$$m : U \times T \to D_B(gΛΛ|B)$$

is an isomorphism. Here $U$ is irreducible because its coordinate ring is a symmetric algebra, compare \[2.12\], Lemma 4.3. The torus $T$, which is equipped with its classical coordinate ring, is irreducible. Therefore also the principal open set $D_B(gΛΛ|B)$ is irreducible, which implies that $B$ is irreducible.

Combining the last proposition with Theorem 2.7 we get:

**Corollary 2.11** The Bruhat and Birkhoff cells $B\hat{w}B$, $B^-\hat{w}B^-$, and $B^-\hat{w}B$ are irreducible for every $\hat{w} \in W$. 

29
At last we consider the Birkhoff cells of \( \text{Specm } \mathbb{F}[\hat{G}] \). They have similar properties as the Birkhoff cells of \( \hat{G} \):

**Theorem 2.12** Let \( \hat{w} = w_1 \in (R(\Theta))w_2 \) be an element of \( \hat{\mathcal{W}} \) in normal form I or in normal form II, and let \( \hat{n} \) be a corresponding element of \( \hat{N} \).

The closure of the Birkhoff cell \( B_f \odot \hat{w}B_f \) is given by

\[
B_f \odot \hat{w}B_f^{\text{Spm}} = \bigcup_{\hat{w}' \in \hat{\mathcal{W}}} B_f \odot \hat{w}'B_f . 
\] (23)

The Birkhoff cell \( B_f \odot \hat{w}B_f \) is irreducible. It is principal open in its closure, i.e.,

\[
\overline{B_f \odot \hat{w}B_f} \cap D_{\text{Specm } \mathbb{F}[\hat{G}]}(g_{w_1, \lambda, w_2}) = B_f \odot \hat{w}B_f . 
\] (24)

 Equip \( B_f \odot \hat{w}B_f \) with its coordinate ring as a principal open set in its closure. Equip the torus \( w_2^{-1}T \Theta w_1 \) with its classical coordinate ring. Then the map

\[
m : (U_f \cap w_1 U_f w_1^{-1}) \times w_2^{-1}T \Theta w_2 \times (U_f \cap w_2^{-1}U_f w_2) \to B_f \odot \hat{w}B_f , 
\] (25)

defined by \( m(u, t, \tilde{u}) := u \odot \hat{w}t\tilde{u} \), is an isomorphism.

**Proof:** We first show the equation (23): It is easy to adapt step (c), \( \epsilon = -1 \), of the proof of Theorem 2.3 to show the inclusion ‘\( \subseteq \)’. To show the reverse inclusion, note that for \( M \subseteq \hat{G} \) we have

\[
1 \odot M \subseteq 1 \odot M^{\text{Spm}} .
\]

Let \( \hat{w}' \in \hat{\mathcal{W}} \) such that \( \hat{w}' \geq_{-+} \hat{w} \), then

\[
\begin{align*}
B \odot \hat{w}'B &= 1 \odot B^{-1}\hat{w}'B \subseteq 1 \odot B^{-1}\hat{w}B \subseteq 1 \odot B^{-1}\hat{w}B^{\text{Spm}} = B \odot \hat{w}'B^{\text{Spm}} \\
&\subseteq B_f \odot \hat{w}B_f^{\text{Spm}} .
\end{align*}
\]

Since \( B_f \times B_f \) acts on \( \text{Specm } \mathbb{F}[\hat{G}] \) by morphisms, we find by applying \( B_f \times B_f \) to this inclusion:

\[
B_f \odot \hat{w}'B_f \subseteq B_f \odot \hat{w}B_f^{\text{Spm}} .
\]

It is easy to adapt the proof of Theorem 2.8 to show equation (24).

Because of Theorem 2.8 to show the irreducibility of \( B_f \hat{w}B_f \), it is sufficient to show the irreducibility of \( B_f \subseteq \hat{G} \).

The relative topology on \( B \) induced by \( \hat{G} \) is the same as the relative topology on \( B \) induced by \( \hat{G} \). Due to Proposition 2.10 \( B \) is irreducible. Therefore it is sufficient to show

\[
\overline{B_f} = \overline{B} .
\] (26)

Due to [M 2], Theorem 9 (1) we have \( \overline{U} = U_f \). Due to Proposition 1 of [M 2] \( \overline{B} \) is a monoid. Because \( T \) and \( U \) are contained in \( B \) we find \( B_f = TU_f \subseteq B ,
\]
from which follows the inclusion \( \subseteq \) in (26). The reverse inclusion is obvious.

Due to Theorem 9 (2) of [M 2] we have \( \bar{U}^\Theta = U_f^\Theta \). Therefore the coordinate ring \( F[U^\Theta] \) is isomorphic to \( F[U_f^\Theta] \) by the restriction map. Due to part (a) of the proof of Theorem 21 in [M 2] we have \( e(R(\Theta))(U_f^\Theta) = e(R(\Theta)) \). Using these results it is not difficult to adapt the proof of Theorem 2.6 and Theorem 2.7 to show that the map \( m \) in (25) is an isomorphism.

\[ \square \]

3 Extensions of the length function

We first define three extensions of the length function \( l : \mathcal{W} \to \mathbb{N}_0 \) of the Weyl group to functions \( l_{++}, l_{--} : \hat{\mathcal{W}} \to \mathbb{Z} \) and \( l_{-+} : \hat{\mathcal{W}} \to \mathbb{N}_0 \) of the Weyl monoid by using the normal forms of the elements of \( \hat{\mathcal{W}} \). The length function \( l_{++} \) is similar to the length function of a Renner monoid given in [Re 2] up to additive constants on the orbits given by the action of the product of the Weyl group on the Renner monoid. The functions \( l_{++}, l_{--} \) are allowed to take positive and negative values. It is not possible to make \( l_{++}, l_{--} \) positive on the \( \mathcal{W} \times \mathcal{W} \)-orbits of \( \hat{\mathcal{W}} \) by adding constants, because in general these functions can take arbitrary positive and negative values on such orbits.

**Definition 3.1**

1a) Let \( \hat{w} = w_1 \varepsilon (R(\Theta))w_2 \) be an element of \( \hat{\mathcal{W}} \) in normal form I. Set

\[
l_{++}(\hat{w}) := l(w_1) - l(w_2) \in \mathbb{Z} .
\]

b) Let \( \hat{w} = w_1 \varepsilon (R(\Theta))w_2 \) be an element of \( \hat{\mathcal{W}} \) in normal form II. Set

\[
l_{--}(\hat{w}) := -l(w_1) + l(w_2) \in \mathbb{Z} .
\]

2) Let \( \hat{w} = w_1 \varepsilon (R(\Theta))w_2 \) be an element of \( \hat{\mathcal{W}} \) in normal form I or in normal form II. Set

\[
l_{-+}(\hat{w}) := l(w_1) + l(w_2) \in \mathbb{N}_0 .
\]

**Remarks:**

1. We have \( l_{++}(\hat{w}) = l_{--}(\hat{w}^{inv}) \), and \( l_{-+}(\hat{w}) = l_{--}(\hat{w}^{inv}) \).
2. If \( \hat{w} = w_1 w_2 \varepsilon (R(\Theta))w_3 = w_1 \varepsilon (R(\Theta))w_2 w_3 \) is an element of \( \hat{\mathcal{W}} \) in normal form III, then

\[
l_{++}(\hat{w}) = l(w_1) + l(w_2) - l(w_3) ,
\]

\[
l_{--}(\hat{w}) = -l(w_1) + l(w_2) + l(w_3) ,
\]

\[
l_{-+}(\hat{w}) = l(w_1) + l(w_2) + l(w_3) .
\]

The length function \( l : \mathcal{W} \to \mathbb{N}_0 \) is compatible with the Bruhat order on \( \mathcal{W} \) and the natural order on \( \mathbb{N}_0 \). Similar things hold for the the extensions of the length function restricted to the \( \mathcal{W} \times \mathcal{W} \)-orbits of \( \hat{\mathcal{W}} \).
Proposition 3.2 Let $\Theta$ be special, and $\hat{w}, \hat{w}' \in W \in (R(\Theta))W$. Equip $W \in (R(\Theta))W$ with the restriction of the extended Bruhat order $\leq_{\varepsilon, \delta}$, $(\varepsilon, \delta) \in \{(++), (-+), (-+)\}$.

a) Then $\hat{w} \leq_{\varepsilon, \delta} \hat{w}'$ implies $l_{\varepsilon, \delta}(\hat{w}) \leq l_{\varepsilon, \delta}(\hat{w}')$.

b) The length of every chain joining $\hat{w}$ and $\hat{w}'$ is finite, and does not extend $l_{\varepsilon, \delta}(\hat{w}') - l_{\varepsilon, \delta}(\hat{w})$. In particular there exists a maximal chain joining $\hat{w}$ and $\hat{w}'$.

**Proof:** b) follows immediately from a). We only have to prove the cases $(++)$, $(-+)$ of a). Then the case $(-+)$ follows from the case $(++)$ by using the inverse map $\text{inv} \tilde{W} \rightarrow \tilde{W}$, combining Remark (3) following Definition 3.1.

Recall that for elements $u, v \in W$ we have

$$|l(u) - l(v)| \leq l(uv) \leq l(u) + l(v),$$ (27)

and for $J \subseteq I$, $u, v \in W_J$, $J' u \in J' W$ we have

$$l(u J' u) = l(u) + l(J) u .$$ (28)

Let $\hat{w} = w_1 \varepsilon (R(\Theta))w_2$, $\hat{w}' = w_1' \varepsilon (R(\Theta))w_2'$ be in normal form $I$, i.e., $w_1, w_1' \in W^\Theta$ and $w_2, w_2' \in \Theta \subseteq W$.

a) If $\hat{w} \leq_{++} \hat{w}'$ then due to Theorem 2.3 there exist an element $w \in W_{\Theta^I}$, such that

$$w_1 \leq (w_1')^\Theta = w_1' w \quad \text{and} \quad w_2 \geq (w_2') = w_2' w_2' .$$

Furthermore we have $w_1 \neq w_1' w$ or $w_2 \neq w_2' w$ because otherwise we get by using formula (3):

$$\hat{w} = w_1 \varepsilon (R(\Theta))w_2 = w_1' \varepsilon (R(\Theta))w_2' = w_1 \varepsilon (R(\Theta))w_2' = \hat{w}' .$$

Now $l : (W, \leq) \rightarrow (\mathbb{N}_0, \leq)$ is an order morphism. By using equation (28) and inequality (27) we get

$$l_+(\hat{w}) = l(w_1) - l(w_2) \leq l(w_1 w) - l(w_1 w_2') = l(w_1' w) - l(w_1) - l(w_2') \leq l(w_1' w w_1) - l(w_2') = l_+(\hat{w}') .$$

b) If $\hat{w} \leq_{-+} \hat{w}'$ then due to Theorem 2.3 there exist an element $w \in W_{\Theta^I}$, such that

$$w_1' \geq (w_1 w)^\Theta = w_1 w \quad \text{and} \quad w_2' \geq (w_2 w_2') = w_2' w_2 .$$

Furthermore we have $w_1' \neq w_1 w$ or $w_2' \neq w_2$ because otherwise

$$\hat{w}' = w_1' \varepsilon (R(\Theta))w_2' = w_1 \varepsilon (R(\Theta))w_2 = w_1 \varepsilon (R(\Theta))w_2 = \hat{w} .$$

By using that $l : (W, \leq) \rightarrow (\mathbb{N}_0, \leq)$ is an order morphism, by using equation (28) and inequality (27) we get

$$l_+(\hat{w}') = l(w_1) + l(w_2') \geq l(w_1 w) + l(w_2) = l(w_1 w) + l(w_2) \geq l(w_1 w w_2') + l(w_2) = l_+(\hat{w}) .$$
For a reductive algebraic monoid the length of any maximal chain of the Bruhat-Chevalley order, which joins two elements in an orbit of the action of the product of the Weyl group on the Renner monoid, is given by the difference of the length functions of these elements. This follows from the definition of the length function in [Re 2], the algebraic description of the Bruhat Chevalley order in [Pe,Pu,Re], and the Theorem in Section 7 of [Re 1], which states that if an element covers another then the dimension difference of the corresponding Bruhat cells is one. The algebraic geometric proof of this theorem can not be generalized to the Kac-Moody setting.

A combinatorial proof working for a Renner monoid and the Weyl monoid is separate article this will be investigated further.

For an element \( w \in W \) and \( i \in I \) the length of \( \sigma_i w \) and \( w \), and the length of \( w\sigma_i \) and \( w \) are related in the following way:

\[
\begin{align*}
    l(\sigma_i w) &= \begin{cases} 
        l(w) + 1 & \text{for } w^{-1}\alpha_i \in \Delta^+_\Theta \\
        l(w) - 1 & \text{for } w^{-1}\alpha_i \in \Delta^-_\Theta 
    \end{cases} \\
    l(w\sigma_i) &= \begin{cases} 
        l(w) + 1 & \text{for } w\alpha_i \in \Delta^+_\Theta \\
        l(w) - 1 & \text{for } w\alpha_i \in \Delta^-_\Theta 
    \end{cases}
\end{align*}
\]

(29) \quad (30)

The next theorem gives the generalization for the extended length functions.

To cut short our notation we set \( W_J := W_J \{ \alpha_i \mid i \in J \}, J \subseteq I \).

**Theorem 3.3**

1) Let \( \hat{w} = w_1 \in (R(\Theta))w_2 \) be an element of \( \hat{W} \) in normal form I. Let \( i \in I \). We have:

\[
\begin{align*}
    l_{++}(\sigma_i \hat{w}) &= \begin{cases} 
        l_{++}(\hat{w}) + 1 & \text{for } w_1^{-1}\alpha_i \in \Delta^+_\Theta \setminus W_\Theta \\
        l_{++}(\hat{w}) & \text{for } w_1^{-1}\alpha_i \in W_\Theta \\
        l_{++}(\hat{w}) - 1 & \text{for } w_1^{-1}\alpha_i \in \Delta^-_\Theta \setminus W_\Theta 
    \end{cases} \\
    l_{+-}(\sigma_i \hat{w}) &= \begin{cases} 
        l_{--}(\hat{w}) + 1 & \text{for } w_1^{-1}\alpha_i \in (\Delta^-_\Theta \setminus W_{\Theta^{\perp}} \cup (\Delta^+_\Theta \cap W_{\Theta^{\perp}})) \cup (\Delta^-_\Theta \cap W_{\Theta^{\perp}}) \\
        l_{--}(\hat{w}) & \text{for } w_1^{-1}\alpha_i \in W_{\Theta^{\perp}} \\
        l_{--}(\hat{w}) - 1 & \text{for } w_1^{-1}\alpha_i \in (\Delta^-_\Theta \setminus W_{\Theta^{\perp}}) \cup (\Delta^-_\Theta \cap W_{\Theta^{\perp}}) 
    \end{cases} \\
    l_{-+}(\sigma_i \hat{w}) &= \begin{cases} 
        l_{--}(\hat{w}) + 1 & \text{for } w_1^{-1}\alpha_i \in \Delta^+_\Theta \setminus W_\Theta \\
        l_{--}(\hat{w}) & \text{for } w_1^{-1}\alpha_i \in W_\Theta \\
        l_{--}(\hat{w}) - 1 & \text{for } w_1^{-1}\alpha_i \in \Delta^-_\Theta \setminus W_\Theta 
    \end{cases}
\end{align*}
\]

In all three cases the extended lengths of \( \sigma_i \hat{w} \) and \( \hat{w} \) are equal if and only if the elements \( \sigma_i \hat{w} \) and \( \hat{w} \) are equal.

2) Let \( \hat{w} = w_1 \in (R(\Theta))w_2 \) be an element of \( \hat{W} \) in normal form II. Let \( i \in I \).

We have:

\[
\begin{align*}
    l_{++}(\hat{w}\sigma_i) &= \begin{cases} 
        l_{++}(\hat{w}) + 1 & \text{for } w_2\alpha_i \in (\Delta^-_\Theta \setminus W_{\Theta^{\perp}} \cup (\Delta^+_\Theta \cap W_{\Theta^{\perp}})) \cup (\Delta^-_\Theta \cap W_{\Theta^{\perp}}) \\
        l_{++}(\hat{w}) & \text{for } w_2\alpha_i \in W_{\Theta^{\perp}} \\
        l_{++}(\hat{w}) - 1 & \text{for } w_2\alpha_i \in \Delta^-_\Theta \setminus W_{\Theta^{\perp}} 
    \end{cases} \\
    l_{+-}(\hat{w}\sigma_i) &= \begin{cases} 
        l_{--}(\hat{w}) + 1 & \text{for } w_2\alpha_i \in \Delta^+_\Theta \setminus W_\Theta \\
        l_{--}(\hat{w}) & \text{for } w_2\alpha_i \in W_\Theta \\
        l_{--}(\hat{w}) - 1 & \text{for } w_2\alpha_i \in \Delta^-_\Theta \setminus W_\Theta 
    \end{cases} \\
    l_{-+}(\hat{w}\sigma_i) &= \begin{cases} 
        l_{--}(\hat{w}) + 1 & \text{for } w_2\alpha_i \in \Delta^-_\Theta \setminus W_{\Theta^{\perp}} \cup (\Delta^+_\Theta \cap W_{\Theta^{\perp}}) \cup (\Delta^-_\Theta \cap W_{\Theta^{\perp}}) \\
        l_{--}(\hat{w}) & \text{for } w_2\alpha_i \in W_{\Theta^{\perp}} \\
        l_{--}(\hat{w}) - 1 & \text{for } w_2\alpha_i \in \Delta^-_\Theta \setminus W_{\Theta^{\perp}} \cup (\Delta^+_\Theta \cap W_{\Theta^{\perp}}) \cup (\Delta^-_\Theta \cap W_{\Theta^{\perp}}) 
    \end{cases}
\end{align*}
\]

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In all three cases the extended lengths of \( \hat{w}_i \) and \( \hat{w} \) are equal if and only if the elements \( \hat{w}_i \) and \( \hat{w} \) are equal.

**Proof:** We only have to show the statements 2) of the theorem, which involve normal form II. Then the statements 1), which involve normal form I, follow easily by using Remark (1) following Definition 5.1 and Remark (1) following Proposition 2.2. We make use of the formulas (3) and (4) without further mention.

Let \( \hat{w} = w_1 \varepsilon(R(\Theta))w_2 \) be in normal form II, i.e., \( w_1 \in W_{\Theta,\Theta^\bot} \), \( w_2 \in \Theta W \). Let \( w_2 = bc \) with \( b \in W_{\Theta^\bot} \) and \( c \in \Theta \cup \Theta^\bot \).

a) We have:

\[
\hat{w}_i = \hat{w} \iff \varepsilon(R(\Theta))w_2 \sigma_i w_2^{-1} = \varepsilon(R(\Theta))
\]

\[
\iff \sigma_{w_2 \alpha_i} = w_2 \sigma_i w_2^{-1} \in W_{\Theta} \iff w_2 \alpha_i \in W_{\Theta} \Theta
\]

b) Let \( w_2 \alpha_i \in W_{\Theta^\bot} \Theta^\bot \), which is equivalent to \( c \alpha_i \in W_{\Theta^\bot} \Theta^\bot \). Then

\[
\hat{w}_i = w_1 b c (R(\Theta)) c \sigma = w_1 b c (R(\Theta)) \sigma_{c \alpha_i} c = w_1 b c \sigma_{c \alpha_i} \varepsilon(R(\Theta)) c
\]

Note that the expression on the right is in normal form III with \( b \sigma_{c \alpha_i} \) as middle part. Now \( c \alpha_i \) can be written in the form \( c \alpha_i = \sum_{j \in \Theta} n_j \alpha_j \) with either all \( n_j \in \mathbb{N}_0 \) or all \( n_j \in -\mathbb{N}_0 \). Applying \( c^{-1} \) we get

\[
\alpha_i = \sum_{j \in \Theta} n_j c^{-1} \alpha_j
\]

Because of \( c^{-1} \in W_{\Theta,\Theta^\bot} \) we have \( c^{-1} \alpha_j \in \Delta^+_{re} \subseteq \mathbb{N}_0 \cap \text{span}\{ \alpha_1, \ldots, \alpha_n \} \) for all \( j \in \Theta \). This implies that there exists an index \( j \in \Theta \) such that \( \alpha_i = c^{-1} \alpha_j \). In particular \( c \alpha_i \) is a simple root. By using Remark (2) following Definition 5.1 and equation (30) we find

\[
l_+ (\hat{w}_i) = l_+ (\hat{w}) \]

\[
l_- (\hat{w}_i) = l_- (\hat{w}) = l_+ (\hat{w}_i) = l_+ (\hat{w})
\]

\[
l (b \sigma_{c \alpha_i}) - l(b) = \begin{cases} 1 & \text{for } w_2 \alpha_i = b c \alpha_i \in \Delta^+_{re} \\ -1 & \text{for } w_2 \alpha_i = b c \alpha_i \in \Delta^+_{re} \end{cases}
\]

c) Let \( w_2 \alpha_i \in \Delta^+_{re} \backslash W_{\Theta^\bot \Theta^\bot} (\Theta \cup \Theta^\bot) \), which is equivalent to \( c \alpha_i \in \Delta^+_{re} \backslash W_{\Theta^\bot \Theta^\bot} (\Theta \cup \Theta^\bot) \).

We first show \( c \alpha_i \in \Theta \cup \Theta^\bot \): Because of \( c^{-1} \in W_{\Theta,\Theta^\bot} \) we have

\[
c^{-1} \alpha_j \in \Delta^+_{re} \quad \text{for all } j \in \Theta \cup \Theta^\bot
\]
Due to our assumption on $c\alpha_i$, from this follows
\[
c^{-1} \alpha_j \in \Delta_{re}^+ \setminus \{\alpha_i\} \subseteq \Delta_{re}^+ \setminus \{\alpha_i\} \cup \{-\alpha_i\} \quad \text{for all} \quad j \in \Theta \cup \Theta^\perp.
\]
Applying $\sigma_i$ we get
\[
\sigma_i c^{-1} \alpha_j \in \Delta_{re}^+ \quad \text{for all} \quad j \in \Theta \cup \Theta^\perp,
\]
which implies $\sigma_i c^{-1} \in W_{\Theta \cup \Theta^\perp}$, or equivalent $c\alpha_i \in \Theta \cup \Theta^\perp W$.

Now $\hat{w}\sigma_i = w_1 \varepsilon(R(\Theta))b\sigma_i$ is in normal form III with $c\sigma_i$ as last part. By using Remark (2) following Definition 3.1 and equation (30) we find
\[
l_{++}(\hat{w}\sigma_i) - l_{+}(\hat{w}) = \left\{
\begin{array}{ll}
+1 & \text{for} \quad c\alpha_i \in \Delta_{re}^- \setminus W_{\Theta \cup \Theta^+} \Theta \cup \Theta^+ \\
-1 & \text{for} \quad c\alpha_i \in \Delta_{re}^+ \setminus W_{\Theta \cup \Theta^+} \Theta \cup \Theta^+
\end{array}
\right.
\]
Here $c\alpha_i \in \Delta_{re}^\pm \setminus W_{\Theta \cup \Theta^+} \Theta \cup \Theta^+$ is equivalent to $w_2 \alpha_i = b\alpha_i \in \Delta_{re}^\pm \setminus W_{\Theta \cup \Theta^+} \Theta \cup \Theta^+$.

Putting together the cases a), b), and c) 2) of the theorem follows.

For the monoid of matrices over a finite field, and more general for a reductive algebraic monoid a generalization of one of the Tits axioms for BN-pairs has been given in [31], [32] by using the length function. The next theorem gives a generalization of some of the Tits axioms for groups with twinned BN-pairs, compare [14], for the monoid $G$:

**Theorem 3.4** Let $i \in I$, let $\hat{w} \in \hat{W}$. Let $\epsilon \in \{+,-\}$. We have:
\[
(B^\epsilon \sigma_i B^\epsilon) (B^\epsilon \hat{w} B^\epsilon) = \left\{
\begin{array}{ll}
B^\epsilon \hat{w} B^\epsilon & \text{if} \quad l_{ec}(\sigma_i \hat{w}) = l_{ec}(\hat{w}) \\
B^\epsilon \sigma_i \hat{w} B^\epsilon & \text{if} \quad l_{ec}(\sigma_i \hat{w}) = l_{ec}(\hat{w}) + 1 \\
B^\epsilon \sigma_i \hat{w} B^\epsilon \cup B^\epsilon \hat{w} B^\epsilon & \text{if} \quad l_{ec}(\sigma_i \hat{w}) = l_{ec}(\hat{w}) - 1
\end{array}
\right.
\]
\[
(B^\epsilon \hat{w} B^\epsilon) (B^\epsilon \sigma_i B^\epsilon) = \left\{
\begin{array}{ll}
B^\epsilon \hat{w} B^\epsilon & \text{if} \quad l_{ec}(\hat{w} \sigma_i) = l_{ec}(\hat{w}) \\
B^\epsilon \hat{w} \sigma_i B^\epsilon & \text{if} \quad l_{ec}(\hat{w} \sigma_i) = l_{ec}(\hat{w}) + 1 \\
B^\epsilon \hat{w} \sigma_i B^\epsilon \cup B^\epsilon \hat{w} B^\epsilon & \text{if} \quad l_{ec}(\hat{w} \sigma_i) = l_{ec}(\hat{w}) - 1
\end{array}
\right.
\]
We have:
\[
(B^{-\epsilon} \sigma_i B^{-\epsilon}) (B^{-\epsilon} \hat{w} B^{-\epsilon}) = \left\{
\begin{array}{ll}
B^{-\epsilon} \hat{w} B^{-\epsilon} & \text{if} \quad l_{ec}(\sigma_i \hat{w}) = l_{ec}(\hat{w}) \\
B^{-\epsilon} \sigma_i \hat{w} B^{-\epsilon} & \text{if} \quad l_{ec}(\sigma_i \hat{w}) = l_{ec}(\hat{w}) - 1 \\
B^{-\epsilon} \sigma_i \hat{w} B^{-\epsilon} \cup B^{-\epsilon} \hat{w} B^{-\epsilon} & \text{if} \quad l_{ec}(\sigma_i \hat{w}) = l_{ec}(\hat{w}) + 1
\end{array}
\right.
\]
\[
(B^\epsilon \hat{w} B^{-\epsilon}) (B^{-\epsilon} \sigma_i B^{-\epsilon}) = \left\{
\begin{array}{ll}
B^\epsilon \hat{w} B^{-\epsilon} & \text{if} \quad l_{ec}(\hat{w} \sigma_i) = l_{ec}(\hat{w}) \\
B^\epsilon \hat{w} \sigma_i B^{-\epsilon} & \text{if} \quad l_{ec}(\hat{w} \sigma_i) = l_{ec}(\hat{w}) - 1 \\
B^\epsilon \hat{w} \sigma_i B^{-\epsilon} \cup B^\epsilon \hat{w} B^{-\epsilon} & \text{if} \quad l_{ec}(\hat{w} \sigma_i) = l_{ec}(\hat{w}) + 1
\end{array}
\right.
\]
Proof: For the following transformations we use the formulas \((\beta)\) and \((\gamma)\) stated in the part “The minimal and formal Kac-Moody group \(G\) and \(G_f\), the monoids \(G\) and \(G_f\)” of the section “Preliminaries” several times.

1) Let \(\overline{w} = w_1 \epsilon (R(\Theta)) w_2\) be in normal form I, i.e., \(w_1 \in \mathcal{W}^\Theta\), \(w_2 \in \Theta \cup \Theta^\perp \mathcal{W}\).

a) We have

\[
\begin{align*}
\sigma_i B \overline{w} &= \sigma_i TU^i U_1 w_1 \epsilon (R(\Theta)) w_2 = TU^i \sigma_i U_1 w_1 \epsilon (R(\Theta)) w_2 \\
&= TU^i \sigma_i U_1 w_1 \epsilon (R(\Theta)) w_2 \\
&= \begin{cases}
TU^i \sigma_i U_1 w_1 \epsilon (R(\Theta)) w_2 & \text{for } w_1^{-1} \alpha_i \in \Delta^+_re \setminus W_{\Theta \cup \Theta^\perp} (\Theta \cup \Theta^\perp) \cup W_{\Theta^\perp} \\
TU^i \sigma_i U_1 w_1 \epsilon (R(\Theta)) U_{w_2^{-1} \alpha_i} w_2 & \text{for } w_1^{-1} \alpha_i \in W_{\Theta^\perp} \Theta^\perp 
\end{cases} \\
&= \begin{cases}
TU^i \sigma_i w & \text{for } w_1^{-1} \alpha_i \in \Delta^+_re \setminus W_{\Theta \cup \Theta^\perp} (\Theta \cup \Theta^\perp) \cup W_{\Theta^\perp} \\
TU^i \sigma_i U w_{w_2^{-1} \alpha_i} & \text{for } w_1^{-1} \alpha_i \in W_{\Theta^\perp} \Theta^\perp 
\end{cases} .
\end{align*}
\]

Since \(w_2^{-1} \in \mathcal{W}^\Theta\cup\Theta^\perp\) we have \(w_2^{-1} w_1^{-1} \alpha_i \in (\Delta^+_re)^\pm\) if \(w_1^{-1} \alpha_i \in W_{\Theta^\perp} \Theta^\perp \cap (\Delta^+_re)^\pm\). We conclude:

\[
\begin{align*}
B \sigma_i B \overline{w} B &= B \sigma_i w B & \text{for } w_1^{-1} \alpha_i \in (\Delta^+_re \setminus W_{\Theta^\perp} \Theta) \cup W_{\Theta^\perp} \Theta \\
B \sigma_i B \overline{w} B^{-} &= B \sigma_i w B^{-} & \text{for } w_1^{-1} \alpha_i \in (\Delta^+_re \setminus W_{\Theta^\perp} \Theta^\perp \cup (\Theta \cup \Theta^\perp)) \cup (W_{\Theta^\perp} \Theta \cup \Theta^\perp) \cup W_{\Theta^\perp} \Theta .
\end{align*}
\]

b) We have

\[
\begin{align*}
\sigma_i B \overline{w} &= \sigma_i TU^i U_1 w_1 \epsilon (R(\Theta)) w_2 = TU^i \sigma_i U_1 w_1 \epsilon (R(\Theta)) w_2 \\
&= \begin{cases}
TU^i (\sigma_i U_1 \sigma_i^{-1}) \sigma_i w_1 \epsilon (R(\Theta)) w_2 & \subseteq TU^i (U_1 \cup U_1 \sigma_i U_1) \sigma_i w_1 \epsilon (R(\Theta)) w_2 \\
B \sigma_i \overline{w} & \cup B \sigma_i \sigma_i U_1 w_{(w_1 \sigma_i)^{-1} \alpha_i} \epsilon (R(\Theta)) w_2 \\
&= \begin{cases}
B w_1 \epsilon (R(\Theta)) w_2 & \text{for } w_1^{-1} \alpha_i \in \Delta^+_re \setminus W_{\Theta \cup \Theta^\perp} (\Theta \cup \Theta^\perp) \\
B w_1 \epsilon (R(\Theta)) U_{w_2^{-1} \alpha_i} w_2 & \text{for } w_1^{-1} \alpha_i \in W_{\Theta^\perp} \Theta^\perp 
\end{cases} \\
&= \begin{cases}
B \overline{w} & \text{for } w_1^{-1} \alpha_i \in \Delta^+_re \setminus W_{\Theta \cup \Theta^\perp} (\Theta \cup \Theta^\perp) \\
B U_{w_2^{-1} \alpha_i} & \text{for } w_1^{-1} \alpha_i \in W_{\Theta^\perp} \Theta^\perp 
\end{cases} .
\end{align*}
\]

Here \(-w_2^{-1} w_1^{-1} \alpha_i \in (\Delta^+_re)^\perp\) if \(w_1^{-1} \alpha_i \in W_{\Theta^\perp} \Theta^\perp \cap (\Delta^+_re)^\pm\). Note also that \(\sigma_i U_1 \sigma_i^{-1}\) contains elements of \(U_1\) and \(U_1 \sigma_i U_1\). We conclude:

\[
\begin{align*}
B \sigma_i B \overline{w} B &= B \sigma_i w B \cup B \overline{w} B & \text{for } w_1^{-1} \alpha_i \in \Delta^+_re \setminus W_{\Theta^\perp} \Theta \\
B \sigma_i B \overline{w} B^{-} &= B \sigma_i w B^{-} \cup B \overline{w} B^{-} & \text{for } w_1^{-1} \alpha_i \in (\Delta^+_re \setminus W_{\Theta \cup \Theta^\perp}) \cup (W_{\Theta^\perp} \Theta \cup \Theta^\perp) \cup \Delta^+_re .
\end{align*}
\]

Due to Part 1) of the last theorem, we get from a) and b):

\[
\begin{align*}
B \sigma_i B \overline{w} B &= \begin{cases}
B \overline{w} B & \text{if } l_+ (\sigma_i \overline{w}) = l_+ (\overline{w}) \\
B \sigma_i \overline{w} B & \text{if } l_+ (\sigma_i \overline{w}) = l_+ (\overline{w}) + 1 \\
B \sigma_i \overline{w} B \cup B \overline{w} B & \text{if } l_+ (\sigma_i \overline{w}) = l_+ (\overline{w}) - 1 
\end{cases} \\
B \sigma_i B \overline{w} B^{-} &= \begin{cases}
B \overline{w} B^{-} & \text{if } l_- (\sigma_i \overline{w}) = l_- (\overline{w}) \\
B \sigma_i \overline{w} B^{-} & \text{if } l_- (\sigma_i \overline{w}) = l_- (\overline{w}) - 1 \\
B \sigma_i \overline{w} B^{-} \cup B \overline{w} B^{-} & \text{if } l_- (\sigma_i \overline{w}) = l_- (\overline{w}) + 1 
\end{cases} .
\end{align*}
\]
2) Let \( \hat{w} = w_1 \sigma (R(\Theta))w_2 \) be in normal form II, i.e., \( w_1 \in W_{\Theta \cup \Theta^\perp} \) and \( w_2 \in \Theta W \).

\( \text{a)} \) We have

\[
\begin{align*}
\hat{w}B\sigma_i & = w_1 \sigma (R(\Theta))w_2 U_i U^T \sigma_i = w_1 \sigma (R(\Theta))w_2 U_i \sigma_i U^T \\
& = w_1 \sigma (R(\Theta))U_{w_\alpha_1 \varepsilon} \sigma_i U^T \\
& = \begin{cases} 
\begin{align*}
& w_1 \sigma (R(\Theta))w_2 \sigma_i U^T \\
& w_1 U_{w_\alpha_1 \varepsilon}(R(\Theta))w_2 \sigma_i U^T
\end{align*}
& \text{for } w_2 \alpha_i \in \Delta_{\hat{w}} \setminus W_{\Theta \cup \Theta^\perp} (\Theta \cup \Theta^\perp) \cup W_{\Theta} \\
& \text{for } w_2 \alpha_i \in W_{\Theta^\perp}
\end{cases} \\
& = \begin{cases} 
\begin{align*}
& \hat{w} \sigma_i U^T \\
& U_{w_1 w_\alpha_1 \varepsilon} \hat{w} \sigma_i U^T
\end{align*}
& \text{for } w_2 \alpha_i \in \Delta_{\hat{w}} \setminus W_{\Theta \cup \Theta^\perp} (\Theta \cup \Theta^\perp) \cup W_{\Theta} \\
& \text{for } w_2 \alpha_i \in W_{\Theta^\perp}
\end{cases}.
\end{align*}
\]

Since \( w_1 \in W_{\Theta \cup \Theta^\perp} \) we have \( w_1 w_2 \alpha_i \in (\Delta_{\hat{w}})^\pm \) if \( w_2 \alpha_i \in W_{\Theta^\perp} \cup (\Delta_{\hat{w}})^\pm \). We conclude:

\[
\begin{align*}
B \hat{w}B\sigma_i B & = B \hat{w} \sigma_i B \quad \text{for } w_2 \alpha_i \in (\Delta_{\hat{w}} \setminus W_{\Theta \cup \Theta^\perp} (\Theta \cup \Theta^\perp)) \cup (W_{\Theta^\perp} \cup (\Delta_{\hat{w}})^\perp) \cup W_{\Theta} \\
B^{-1} \hat{w}B\sigma_i B & = B^{-1} \hat{w} \sigma_i B \quad \text{for } w_2 \alpha_i \in (\Delta_{\hat{w}} \setminus W_{\Theta \cup \Theta^\perp}) \cup W_{\Theta}.
\end{align*}
\]

\( \text{b)} \) We have

\[
\begin{align*}
\hat{w}B\sigma_i & = w_1 \sigma (R(\Theta))w_2 U_i U^T \sigma_i = w_1 \sigma (R(\Theta))w_2 U_i \sigma_i U^T \\
& = w_1 \sigma (R(\Theta))w_2 \sigma_i U^T \sigma_i \subseteq w_1 \sigma (R(\Theta))w_2 \sigma_i (U_i \cup U_i U_i) U^T \\
& = \hat{w} \sigma_i B \cup w_1 \sigma (R(\Theta))U_{w_\alpha_1 \varepsilon} \sigma_i w_2 B \\
& = \hat{w} \sigma_i B \cup \begin{cases} 
\begin{align*}
& w_1 \sigma (R(\Theta))w_2 B \\
& w_1 U_{w_\alpha_1 \varepsilon}(R(\Theta))w_2 B
\end{align*}
& \text{for } w_2 \alpha_i \in \Delta_{\hat{w}} \setminus W_{\Theta \cup \Theta^\perp} (\Theta \cup \Theta^\perp)
\end{cases} \\
& \cup \begin{cases} 
\begin{align*}
& \hat{w} B \\
& U_{w_1 w_\alpha_1 \varepsilon} \hat{w} B
\end{align*}
& \text{for } w_2 \alpha_i \in W_{\Theta^\perp}
\end{cases}.
\end{align*}
\]

Here \( -w_1 w_2 \alpha_i \in (\Delta_{\hat{w}})^\mp \) if \( w_2 \alpha_i \in W_{\Theta^\perp} \cup (\Delta_{\hat{w}})^\pm \). Note also that \( \sigma_i^{-1} U \cup U \sigma_i U \). We conclude:

\[
\begin{align*}
B \hat{w}B\sigma_i B & = B \hat{w} \sigma_i B \cup B \hat{w} B \quad \text{for } w_2 \alpha_i \in (\Delta_{\hat{w}} \setminus W_{\Theta \cup \Theta^\perp} (\Theta \cup \Theta^\perp)) \cup (W_{\Theta^\perp} \cup (\Delta_{\hat{w}}) \cup W_{\Theta}) \\
B^{-1} \hat{w}B\sigma_i B & = B^{-1} \hat{w} \sigma_i B \cup B^{-1} \hat{w} B \quad \text{for } w_2 \alpha_i \in \Delta_{\hat{w}} \setminus W_{\Theta}.
\end{align*}
\]

Due to Part 2) of the last theorem, from a) and b) follows:

\[
\begin{align*}
B \hat{w}B\sigma_i B & = \begin{cases} 
B \hat{w} B & \text{if } l_{++}(\hat{w} \sigma_i) = l_{++}(\hat{w}) \\
B \hat{w} \sigma_i B & \text{if } l_{++}(\hat{w} \sigma_i) = l_{++}(\hat{w}) + 1 \\
B \hat{w} \sigma_i B \cup B \hat{w} B & \text{if } l_{++}(\hat{w} \sigma_i) = l_{++}(\hat{w}) - 1
\end{cases} \\
B^{-1} \hat{w}B\sigma_i B & = \begin{cases} 
B^{-1} \hat{w} B & \text{if } l_{--}(\hat{w} \sigma_i) = l_{--}(\hat{w}) \\
B^{-1} \hat{w} \sigma_i B & \text{if } l_{--}(\hat{w} \sigma_i) = l_{--}(\hat{w}) - 1 \\
B^{-1} \hat{w} \sigma_i B \cup B^{-1} \hat{w} B & \text{if } l_{--}(\hat{w} \sigma_i) = l_{--}(\hat{w}) + 1
\end{cases}.
\end{align*}
\]

The remaining cases of the theorem follow by applying the Chevalley involution * \( \hat{G} \rightarrow \hat{G} \), together with Remark (3) following Definition 2.1 and Remark (1) following Definition 2.4.
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