An approximation for zero-balanced Appell function $F_1$ near $(1, 1)$

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Abstract. We suggest an approximation for the zero-balanced Appell hyper-
geometric function $F_1$ near the singular point $(1, 1)$. Our approximation can
be viewed as a generalization of Ramanujan’s approximation for zero-balanced
$2F_1$ and is expressed in terms of $3F_2$. We find an error bound and prove some
basic properties of the suggested approximation which reproduce the similar
properties of the Appell function. Our approximation reduces to the approx-
imation of Carlson-Gustafson when the Appell function reduces to the first
incomplete elliptic integral.

1. Introduction. The generalized hypergeometric function is defined by \[10\] formula 4.1(1)]

$$pF_q\left(\begin{array}{c}a_1, \ldots, a_p \\ b_1, \ldots, b_q\end{array} \mid z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{z^k}{k!},$$

where $(a)_0 = 1$, $(a)_k = a(a+1) \cdots (a+k-1)$, $k = 1, 2, \ldots$, is shifted factorial. This function is
called zero-balanced if $p = q + 1$ and $\sum_{i=1}^{p} a_i = \sum_{i=1}^{q} b_i$.

Ramanujan (see \[3\], \[4\], \[5\]) suggested the following approximations for zero-balanced $2F_1$ and
$3F_2$:

$$B(a, b)_{2F_1}(a, b; a + b; x) = -\ln(1 - x) + \gamma(a, b) + O((1 - x) \ln(1 - x)), \quad x \to 1-, \quad (2)$$

where

$$\gamma(a, b) = 2\psi(1) - \psi(a) - \psi(b), \quad \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}, \quad (3)$$

and

$$\frac{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}{\Gamma(b_1)\Gamma(b_2)}_{3F_2}\left(\begin{array}{c}a_1, a_2, a_3 \\ b_1, b_2\end{array} \mid x\right) = -\ln(1 - x) + L + O((1 - x) \ln(1 - x)), \quad x \to 1-, \quad (4)$$

where $\Re(a_3) > 0$ and

$$L = 2\psi(1) - \psi(a_1) - \psi(a_2) + \sum_{k=1}^{\infty} \frac{(b_2 - a_3)(b_1 - a_3)k}{k(a_1)(a_2)k^2}.$$

These formulas have been generalized to $q+1F_q$ by Nørlund \[17\], Saigo and Srivastava in \[18\],
Marichev and Kalla in \[15\] and Bühring in \[7\], see details in the survey paper by Bühring and
Srivastava \[8\].

The Appell function $F_1$ generalizes $2F_1$ to two variables and is defined by \[10\] :

$$F_1(\alpha; \beta_1, \beta_2; \gamma; z_1, z_2) = \sum_{k,n=0}^{\infty} \frac{(\alpha)_{k+n}(\beta_1)_k(\beta_2)_n}{(\gamma)_{k+n}k!n!} z_1^k z_2^n, \quad (4)$$

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for $|z_1| < 1$, $|z_2| < 1$ and by analytic continuation for other values of $z_1$, $z_2$. An asymptotic expansion for $F_1$ has been studied by Ferreira and López in [11] in the neighborhood of infinity. This approximation can be converted into an approximation around $(1,1)$ using the formula

$$F_1(a; b, c; d; 1 - xz, 1 - yz) = x^{-b}y^{-c}F_1\left(d - a; b, c; d; 1 - \frac{1}{xz}, 1 - \frac{1}{yz}\right).$$

It has been noticed by B.C. Carlson in [6] that the incomplete elliptic integral of the first kind is a particular case of $F_1$:

$$F(\lambda, k) = \lambda F_1(1/2; 1/2, 1/2; 3/2; \lambda^2, k^2\lambda^2).$$

(5)

Carlson and Gustafson studied the asymptotic approximation for $F(\lambda, k)$ in [9]. Their expansion can be shown to be a particular case of the expansion for $F_1$ given later in [11]. We will show below that both expansions (but not the error bounds!) can be obtained by simple rearrangement of (4) and use of known transformation formulas for $F_1$. More precise approximations for $F(\lambda, k)$ which cannot be reduced to expansions from [11] have been given recently by S.M. Sitnik and the author in [13].

The purpose of this paper is to give an analogue of (2) for the ”zero-balanced” Appell function $F_1$ with $\gamma = \alpha + \beta_1 + \beta_2$. Important properties of $F_1$ are permutation symmetry

$$F_1(\alpha; \beta_1, \beta_2; \gamma; z_1, z_2) = F_1(\alpha; \beta_2, \beta_1; \gamma; z_2, z_1),$$

(6)

reduction formulas

$$F_1(\alpha; \beta_1, \beta_2; \gamma; z, 1) = 2F_1(\alpha; \beta_2; \gamma; 1)F_1(\alpha; \beta_1; \gamma - \beta_2; z),$$

(7)

$$F_1(\alpha; \beta_1, \beta_2; \gamma; z, z) = 2F_1(\alpha; \beta_1 + \beta_2; \gamma; z),$$

(8)

and reduction formula (5). Our approximation reproduces the permutation symmetry (6), reduces to Ramanujan approximation given in (2) in cases given by (7) and (8) and reproduces Carlson-Gustafson approximation for the values of parameters given in (5).

Some new reduction formulas for $F_1$ have been discovered in [12].

2. Main results. To save space let us introduce the notation

$$f_{a,b_1,b_2}(x, y) = B(a, b_1 + b_2)F_1(a; b_1, b_2; a + b_1 + b_2; x, y).$$

(9)

Our main approximation is given by

$$g_{a,b_1,b_2}(x, y) = \ln\frac{1}{1 - x} + \gamma(a, b_1 + b_2) + \frac{b_2(y - x)}{(b_1 + b_2)(1 - x)} F_2\left(1, 1, b_2 + 1; 2, b_1 + b_2 + 1; \frac{y - x}{1 - x}\right),$$

(10)

where $\gamma(a, b_1 + b_2)$ is defined in [4]. The following theorem confirms that $g_{a,b_1,b_2}$ is indeed a correct analogue of the righthand side of (2).

**Theorem 1** For $0 \leq x < 1$, $0 \leq y < 1$, $a, b_1, b_2 > 0$:

$$f_{a,b_1,b_2}(x, y) = g_{a,b_1,b_2}(x, y) + R_{a,b_1,b_2}(x, y),$$

(11)

with

$$0 < R_{a,b_1,b_2}(x, y) < r (1 + a - a \ln(r)) = O(r \ln(r)).$$

(12)

where in the last formula $x, y \to 1$, $r = (1 - x)b_1 + (1 - y)b_2 \to 0$ is the ”rhombic” distance to $x = y = 1$ which is asymptotically equivalent to Euclidian distance.

**Corollary 1.1** Formulas (11) and (12) imply in particular the inequality

$$f_{a,b_1,b_2}(x, y) > g_{a,b_1,b_2}(x, y)$$

(13)

for all $x, y \in (0, 1)$. 

2
Applying the transformation \( \times \) 

It gives \( \frac{\Gamma(\eta)\Gamma(\beta)}{\Gamma(\eta + \beta)} F_1(\eta, \beta; \eta + \beta; z) = \sum_{n=0}^{\infty} \frac{(\eta)n(\beta)n}{(n!)^2} [-\log(1-z) + 2\psi(n+1) - \psi(\eta + n) - \psi(\beta + n)](1-z)^n. \) 

(15) 

It gives 

\[
\begin{align*}
\frac{\Gamma(\alpha)\Gamma(\beta_2)}{\Gamma(\alpha + \beta_2)} F_1(\alpha; \beta_1, \beta_2; \alpha + \beta_2; 1, 2) &= \sum_{n,k=0}^{\infty} \frac{(\alpha + k)n(\beta)n(\beta_1)k}{(n!)^2k!} [-\log(1-z_2) + 2\psi(1+n) - \psi(\beta_2 + n) - \psi(\alpha + k + n)](1-z_2)^n. \\
&= \sum_{n,k=0}^{\infty} \frac{(\alpha + k)n(\beta)n(\beta_1)k}{(n!)^2k!} [-\log(1-z_2) + 2\psi(1+n) - \psi(\beta_2 + n) - \psi(\alpha + k + n)](1-z_2)^n. 
\end{align*}
\] 

(16) 

Taking account of 
\( (\alpha)_{k+n} = (\alpha)_k(\alpha + k)_n = (\alpha)_n(\alpha + n)_k, \) 
the expression for Euler beta function
\[
B(\alpha, \beta_2) = \frac{\Gamma(\alpha)\Gamma(\beta_2)}{\Gamma(\alpha + \beta_2)}
\]
and the derivative formula
\[
2F_1'(a, b; c; x) = \frac{\partial}{\partial a} 2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{\psi(a + k)(a)b_k x^k}{(c)_k k!} - \psi(a)_2F_1(a, b; c; x),
\]
(17) 

identity (16) can be rewritten as:

\[
B(\alpha, \beta_2) F_1(\alpha; \beta_1, \beta_2; \alpha + \beta_2; 1, 2) = \sum_{n=0}^{\infty} \frac{(\alpha)n(\beta_2)n}{(n!)^2} (1-z_2)^n \times 
\left\{ [-\log(1-z_2) + 2\psi(1+n) - \psi(\alpha + n) - \psi(\beta_2 + n)]_2F_1(\alpha + n, \beta_1; \alpha; 1, 2) - 2F_1'(\alpha + n, \beta_1; \alpha; 1, 2) \right\}. 
\] 

(18) 

Applying the transformation 
\[
F_1(a; b_1, b_2; a + b_1 + b_2; x, y) = \left( \frac{1 - y}{1 - x} \right)^{b_1} F_1 \left( b_1 + b_2; b_1, a; a + b_1 + b_2; \frac{y - x}{1 - x}, y \right)
\]
to (16) and (18) in view of (19) gives 

\[
f_{a,b_1,b_2}(x, y) = \left( \frac{1 - y}{1 - x} \right)^{b_1} \times 
\left\{ \sum_{k,n=0}^{\infty} \frac{(b_1)_k\cdot(b_1 + b_2 + k)_n}{k!(n!)^2} (1 - y)^{-n} \right\} 
\left\{ \sum_{n=0}^{\infty} \frac{(a)n(b_1 + b_2)_n}{(n!)^2} \ln \frac{1}{1 - y} + 2\psi(1 + n) - \psi(a + n) - \psi(b_1 + b_2 + k + n) \right\} 
\left( \frac{y - x}{1 - x} \right)^k 
\times 2F_1 \left( b_1 + b_2 + n, b_1 + b_2; \frac{y - x}{1 - x} \right) (1 - y)^n 
\right\} 
\left\{ \sum_{n=0}^{\infty} \frac{(a)n(b_1 + b_2)_n}{(n!)^2} \times 2F_1' \left( b_1 + b_2 + n, b_1 + b_2; \frac{y - x}{1 - x} \right) (1 - y)^n \right\}. \) 

(19)
Taking $n = 0$ in the above formula and applying
\[ _2F_1 \left( b_1 + b_2, b_1 + b_2; \frac{y - x}{1 - x} \right) = \left( \frac{1 - y}{1 - x} \right)^{-b_1} \]
we get
\[ f_{a,b_1,b_2}(x, y) = \ln \frac{1}{1 - y} + 2\psi(1) - \psi(a) - \psi(b_1 + b_2) - \left[ \frac{1 - y}{1 - x} \right]^{b_1} \_2F_1' \left[ b_1 + b_2, b_1 \right| \frac{y - x}{1 - x} \right] + R, \quad (20) \]
where it is clear from (19) that
\[ R = O((1 - y) \ln(1 - y)), \]
which is equivalent to the second formula in (12). Formula (20) can be easily put into a different form by differentiating the identity
\[ _2F_1(a, b; c; x) = (1 - x)^{c-a-b} _2F_1(c - a, c - b; c; x) \]
with respect to $a$:
\[ _2F_1'(a, b; c; x) = -\ln(1 - x)(1 - x)^{c-a-b} _2F_1(c - a, c - b; c; x) - (1 - x)^{c-a-b} _2F_1'(c - a, c - b; c; x). \]
Hence:
\[ _2F_1' \left[ b_1 + b_2, b_1 \right| \frac{y - x}{1 - x} \right] = \left( \frac{1 - x}{1 - y} \right)^{b_1} \ln \frac{1 - x}{1 - y} - \left( \frac{1 - x}{1 - y} \right)^{b_1} _2F_1' \left[ 0, b_2 \right| \frac{y - x}{1 - x} \right]. \]
Since
\[ F'(a, b; c; z)|_{a=0} = \sum_{k=1}^{\infty} \frac{d}{da} (a) \frac{(b)_{k} z^{k}}{c_{k} k!} \bigg|_{a=0} = \sum_{k=1}^{\infty} \frac{(b)(k - 1)!}{(c)k!} z^{k} = \frac{b z}{c} \_3F_2 \left( \frac{1, 1, b + 1}{2, c + 1} \right| z \right), \quad (21) \]
we will have
\[ _2F_1' \left[ 0, b_2 \right| \frac{y - x}{1 - x} \right] = \frac{b_2(y - x)}{(b_1 + b_2)(1 - x)} \_3F_2 \left( \frac{1, 1, b_2 + 1}{2, b_1 + b_2 + 1} \right| \frac{y - x}{1 - x} \right). \]
In view of definition (10) of $g_{a,b_1,b_2}(x, y)$ formula (20) transforms into (11).

To estimate the remainder term we will use the ideas from (14). An application of the integral representation (10) formula 5.8(5)] and a change of variable give (u = 1 - x, v = 1 - y):
\[ F_1 \left( a; b_1, b_2; a + b_1 + b_2; 1 - u, 1 - v \right) = u^{-b_1} v^{-b_2} F_1 \left( b_1 + b_2; b_1, b_2; a + b_1 + b_2; 1 - \frac{1}{u}, 1 - \frac{1}{v} \right) \]
\[ = \frac{\Gamma(a + b_1 + b_2)}{\Gamma(a) \Gamma(b_1 + b_2)} \frac{t^{a-1}(1 + t)^{-a}}{\int_{0}^{\infty} (1 + ut)^{b_1}(1 + vt)^{b_2}} = \frac{\Gamma(a + b_1 + b_2)}{\Gamma(a) \Gamma(b_1 + b_2)} \int_{0}^{\infty} f_a(t) h_{b_1,b_2}(u, v; t) dt. \quad (22) \]
where
\[ f_a(t) = t^{a-1}(1 + t)^{-a} = \sum_{k=0}^{n} (-1)^{k} \frac{(a)_{k}}{k! k^{k+1}} + f_{a,n}(t), \quad t \to \infty, \quad (23) \]
\[ f_a(t) = O(t^{a-1}), \quad t \to 0 \Rightarrow f \in \mathcal{F}_{1,1-a}, \quad (24) \]
and
\[ h_{b_1,b_2}(u, v; t) = \frac{1}{(1 + tu)^{b_1}(1 + tv)^{b_2}} = \sum_{k=0}^{n} (-1)^{k} t^{k} \sum_{m=0}^{k} \frac{(b_1)_m(b_2)_m}{m!(k-m)!} u^{m} v^{k-m} + h_n(t), \quad t \to 0; \quad (25) \]
\[ h_{b_1, b_2}(u, v; t) = O(t^{-b_1 - b_2}), \quad t \to \infty \Rightarrow h \in \mathcal{H}_{a_1+b_2}. \]  

Spaces \( \mathcal{F} \) and \( \mathcal{H} \) are defined in \[14\]. Basically, they mean nothing other than the asymptotic formulas satisfied by \( f \) and \( h \), presented above. If \( a, b_1, b_2 \) are all positive conditions I and II from \[14\] are satisfied.

Representation \[22\] is not precisely a Mellin convolution. However, if we approach the point \( u = v = 0 \) (i.e. \( x = y = 1 \)) along straight lines we can put \( u = \gamma_1 \varepsilon, \quad v = \gamma_2 \varepsilon \), where \( \gamma_1 \) and \( \gamma_2 \) are positive constants and \( \varepsilon \to 0 \). It this case

\[ h_{b_1, b_2}(u, v; t) = h_{b_1, b_2, \gamma_1, \gamma_2}(\varepsilon t) \]

and \[22\] takes the form of Mellin convolution. Since every point \( u, v \) lies on some straight line with endpoint \((1, 1)\) and all our further speculations assume sufficiently small but fixed \( u, v \) there are always \( \gamma_1, \gamma_2 \) and \( \varepsilon \) (of course non-unique) which are implied. Hence the theory from \[14\] can be applied.

From

\[
 f_n(t) = \sum_{k=n}^{\infty} (-1)^k \frac{(a)_k}{k!} t^k = \frac{(-1)^n(a)_n 2F_1(a+n, 1; 1+n; -1/t)}{t^{n+1}n!} = \frac{(-1)^n(a)_n}{t^{n+1}(n-1)!} \int_0^1 \frac{(1-s)^{n-1}}{(1+s/t)^{n+a}} \, ds
\]

it is obvious that \( \text{sign}(f_n) = (-1)^n \). Similarly, from

\[
 h_{b_1, b_2, n}(u, v; t) = \sum_{k=n}^{\infty} (-1)^k \frac{b_1 b_2}{m!(k-m)!} t^m v^{k-m} = \sum_{k=n}^{\infty} (-1)^k \frac{(b_1 b_2)_k}{k!} \frac{k! k^k}{k!} 2F_1(b_1, -k; 1-b_2-k; u/v).
\]

it can be seen that \( \text{sign}(h_n) = (-1)^n \). This shows that the remainder is always positive which implies in particular inequality \[13\].

Now take \( n = 1 \) and apply \[14\] Theorem 4.3 which shows that the remainder has the form (since \( a = 0, b = 1 \) in terms of \[14\])

\[
 R_{a, b_1, b_2}(u, v) = \int_0^\infty f_{a, 1}(t) h_{b_1, b_2, 1}(u, v; t) \, dt = \int_0^\infty \left[ \frac{t^{a-1}}{(1+t)^a} - \frac{1}{t} \right] \left[ \frac{1}{(1+ut)^{b_1}(1+vt)^{b_2}} - 1 \right] \, dt. \tag{27}
\]

The bound for \( R_{a, b_1, b_2}(u, v) \) is based on the following lemma whose proof we postpone until the end of the proof of the theorem.

**Lemma 1** For all \( t \in (0, \infty) \) the inequalities

\[
 -a/t^2 < f_{a, 1}(t) < 0, \tag{28}
\]

\[
 -1/t < f_{a, 1}(t) < 0, \tag{29}
\]

\[
 -1 < h_{b_1, b_2, 1}(u, v; t) < 0, \quad -t(ub_1 + vb_2) < h_{b_1, b_2, 1}(u, v; t) < 0 \tag{31}
\]

hold true.

The integral in \(27\) may be decomposed as follows

\[
 R_1 = \int_0^1 f_{a, 1}(t) h_{b_1, b_2, 1}(u, v; t) \, dt + \int_1^{1/r} f_{a, 1}(t) h_{b_1, b_2, 1}(u, v; t) \, dt + \int_{1/r}^\infty f_{a, 1}(t) h_{b_1, b_2, 1}(u, v; t) \, dt,
\]
where \( r \) can be any positive number (it is not needed that \( r < 1! \)). Set \( r = ub_1 + vb_2 \) and use estimates (29) and (31) in the first integral, (28) and (31) in the second and (28) and (30) in the third. This gives the estimate (12). \( \Box \)

**Remark 1.** We could use Proposition 3.1 from [14] to give an estimate for the error term. However, in our specific situation we are able to derive a much better bound based on Lemma 1 using the method of proof of this proposition but not its statement.

**Proof of Lemma 1**

(a) Inequality (28). Write \( f_{a,1}(t) = g_a(t)/t^2 \), where

\[
g_a(t) = \frac{t^{a+1}}{(1+t)^a} - t.
\]

Then (28) is equivalent to \(-a < g_a(t) < 0\). Clearly, \( g_a(0) = 0 \). It is an easy exercise to check that \( g_a(\infty) = -a \). If we prove that \( g_a'(t) < 0 \) we are done. Differentiating and multiplying both sides by \((1 + t)^{a+1}\) we see that the required inequality takes the form

\[
(1 + a)(1 + t)t^a < (1 + t)^{a+1} + at^{a+1} \iff \frac{(1 + t)^{a+1}}{t^a(1 + a + t)} > 1 \iff (1 + x)^{a+1} > 1 + (1 + a)x,
\]

where \( x = 1/t \) and the last inequality is the classical Bernoulli inequality valid for \( a > 0 \) and \( x > -1 \) [16 formula III(1.2)].

(b) Inequality (29) is proved similarly but simpler.

(c) Inequality (30) is obvious from the definition (25) of \( h_{b_1, b_2}(u, v; t) \).

(d) To prove (31) we again apply Bernoulli’s inequality [16 formula III(1.2)] in the form \((b_1, b_2 > 0)\):

\[
(1 + tu)^{-b_1} > 1 - b_1 tu, \quad (1 + tu)^{-b_2} > 1 - b_2 tu.
\]

Multiplying these two inequalities we get the estimate

\[
1 - \frac{1}{(1 + tu)^{b_1}(1 + tv)^{b_2}} < t(ub_1 + vb_2) - t^2 uv b_1 b_2
\]

which is even stronger than (31). \( \Box \)

**Remark 2.** Application of (32) instead of (31) in the proof of theorem 1 leads to an estimate of the remainder term \( R \) which is better than (12). However, numerically it is only a very minor improvement, so we decided to keep the simpler estimate (12) in the theorem.

**Remark 3.** Representation (10) also leads to the following observation: for general values of parameters there exists no approximation for \( f_{a,b_1,b_2} \) in the neighbourhood of \((1, 1)\) in terms of elementary functions. Indeed, let

\[
f_{a,b_1,b_2}(x, y) = h(x, y) + o(1)
\]
as \( x, y \to 1 \) with an elementary \( h(x, y) \). Then from (11):

\[
g_{a,b_1,b_2}(x, y) - h(x, y) = o(1) \Rightarrow
\]

\[
h(x, y) + \ln(1-x) = 2\psi(1) - \psi(a) - \psi(b_1 + b_2) + \frac{b_2(y-x)}{(b_1 + b_2)(1-x)} \sum F \left( \begin{array}{c} 1,1,b_2+1 \\ 2,b_1+b_2+1 \end{array} \right) \frac{y-x}{1-x} + \varepsilon(x, y),
\]

and \( \varepsilon(x, y) \to 0 \) as \( x, y \to 1 \). Let \( x, y \to 1 \) along a straight line going through \((1, 1)\), so that

\[
(1 - y)/(1 - x) = \gamma = \text{const.}
\]

Then, due to

\[
(y - x)/(1 - x) = 1 - \gamma,
\]

we have for the elementary \( h_1(x, y) = h(x, y) + \ln(1 - x) \):

\[
h_1(x, y) = d(\gamma) + \varepsilon(x, y),
\]

which is valid for...
where
\[ d(\gamma) = 2\psi(1) - \psi(a) - \psi(b_1 + b_2) + \frac{b_2(1 - \gamma)}{(b_1 + b_2)} \cdot \text{exp} \left( \frac{\gamma_1}{z}, 1 - \frac{\gamma_2}{z} \right). \]

Since \( y = 1 - (1 - x)\gamma \), we can write the above as
\[ \tilde{h}_1(x, \gamma) = d(\gamma) + \epsilon(x, \gamma), \]
where \( \gamma \in (0, \infty) \) is arbitrary, but fixed. For \( x = 1 \) this gives \( \tilde{h}_1(1, \gamma) = d(\gamma) \) for all \( \gamma \in (0, \infty) \). Hence, a restriction of an elementary function \( \tilde{h}_1 \) gives \( \text{exp} \) for all values of its argument in the range \(( -\infty, 1)\), which is impossible, and so \( h(x, y) \) cannot be an elementary function.

**Remark 4.** Expansion [11] formula (33) can be cast into the form
\[ \frac{\Gamma(b_1 + b_2)\Gamma(a)}{\Gamma(a + b_1 + b_2)} F_1 \left( a; b_1, b_2; a + b_1 + b_2; 1 - \frac{\gamma_1}{z}, 1 - \frac{\gamma_2}{z} \right) = \sum_{k=0}^{n-1} \left[ \frac{D_k(a, b_1, b_2; \gamma_1, \gamma_2)}{\frac{a}{z^k}} + \log(z) \frac{E_k(a, b_1, b_2; x, y)}{\frac{a}{z^k}} \right] + R_n(a, b_1, b_2, \gamma_1, \gamma_2; z), \quad (33) \]

Substituting \( x = 1 - \frac{\gamma_1}{z}, y = 1 - \frac{\gamma_2}{z} \) into (19) we see that both (33) and (19) are asymptotic sequences \( z^{-k}, z^{-k}\log(z) \) and so their coefficients are the same. Hence, (19) can be viewed as a simpler form of [11] formula (33). The appearance of the coefficients \( D_k \) and \( E_k \) is very different from that of the coefficients of (19) and direct reduction is non-trivial. For instance, the first term of [11] formula (33) reads (after some simple manipulations) \( (F = 2F_1, M = (1 - y)/(1 - x)) \):
\[ B(a, b_1 + b_2)F_1(a; b_1, b_2; a + b_1 + b_2; x, y) = \psi(1) - \psi(a) + \frac{-\ln(1 - v) + \ln(M) + \psi(1)}{b_1 + b_2} \left( M_{b_2}F_1 \left[ 1, b_2 + 1; b_1 + b_2 + 1; 1 - M \right] + b_1F_1 \left[ 1, b_2 + 1; b_1 + b_2 + 1; 1 - M \right] \right) + \frac{1}{b_1 + b_2} \left( M_{b_2}F_1 \left[ 1, b_2 + 1; b_1 + b_2 + 1; 1 - M \right] + b_1F_1 \left[ 1, b_2 + 1; b_1 + b_2 + 1; 1 - M \right] \right) + R_1. \quad (34) \]

Now using the relation [10] formula 2.8(36)
\[ (c - a - b)F(a, b; c; z) - (c - a)F(a - 1, b; c; z) + b(1 - z)F(a, b + 1; c; z) = 0 \quad (35) \]
we immediately get
\[ M_{b_2}F_1 \left[ 1, b_2 + 1; b_1 + b_2 + 1; 1 - M \right] + b_1F_1 \left[ 1, b_2 + 1; b_1 + b_2 + 1; 1 - M \right] = b_1 + b_2. \]

Differentiating (35) with respect to \( a \) and putting \( a = 0 \) we get:
\[ (c - b - 1)F'(1, b; c; z) + b(1 - z)F'(1, b + 1; c; z) = F(1, b; c; z) + (c - 1)F'(0, b; c; z) - 1. \]

Using (21) we see
\[ M_{b_2}F_1 \left[ 1, b_2 + 1; b_1 + b_2 + 1; 1 - M \right] + b_1F_1 \left[ 1, b_2 + 1; b_1 + b_2 + 1; 1 - M \right] = F \left[ 1, b_2 + 1; b_1 + b_2 + 1; 1 - M \right] + \frac{b_2(b_1 + b_2)(1 - M)}{(b_1 + b_2 + 1)} \cdot \text{exp} \left( 1, 1, b_2 + 1; b_1 + b_2 + 2; 1 - M \right) - 1 \]
and
\[ B(a, b_1 + b_2)F_1(a; b_1, b_2; a + b_1 + b_2; x, y) = \frac{1}{1 - x} + 2\psi(1) - \psi(a) - \psi(b_1 + b_2) + \frac{1}{b_1 + b_2} \cdot \text{exp} \left( 1, 1, b_2 + 1; b_1 + b_2 + 2; 1 - M \right) - \frac{1}{b_1 + b_2} + R_1. \quad (36) \]
Finally, (35) is reduced to (10) with the help of the following formula found at http://functions.wolfram.com/07.27.03.0120.01:

\[ 3F_2(a, b, c; a + 1, e; z) = \frac{1}{a - e + 1} \left[ a_2F_1(b, c; e; z) - (e - 1)3F_2(a, b, c; a + 1, e - 1; z) \right]. \]

Recalling that \( M = (1 - y)/(1 - x) \) we get (10). The direct reduction for further terms is even more complicated.

**Theorem 2** The following properties are true:

1. The function \( g \) is permutation symmetric:

\[ g_{a,b_1,b_2}(x,y) = g_{a,b_2,b_1}(y,x). \] (37)

2. For \( y = 1 \) (and \( x = 1 \)) due to (37) the function \( g_{a,b_1,b_2}(x,y) \) reduces to the Ramanujan’s approximation:

\[ g_{a,b_1,b_2}(x,1) = \ln \frac{1}{1 - x} + 2\psi(1) - \psi(a) - \psi(b_1), \] (38a)

\[ g_{a,b_1,b_2}(1,y) = \ln \frac{1}{1 - y} + 2\psi(1) - \psi(a) - \psi(b_2). \] (38b)

3. For \( x = y \) the function \( g_{a,b_1,b_2}(x,y) \) becomes the Ramanujan’s approximation:

\[ g_{a,b_1,b_2}(x,x) = \ln \frac{1}{1 - x} + 2\psi(1) - \psi(a) - \psi(b_1 + b_2). \] (39)

4. For the values of parameters \( a = b_1 = b_2 = 1/2 \) we have

\[ g_{1/2,1/2,1/2} = \ln \frac{4}{\sqrt{1 - \lambda^2} + \sqrt{1 - k^2 \lambda^2}}, \] (40)

which is the approximation of Carlson-Gustafson.

**Proof.** To prove the first statement we need the following elementary lemma:

**Lemma 2** For \( b \neq 1 \) the following relation holds true:

\[ 3F_2 \left( \begin{array}{c} 1, b, c \\ 2, e \end{array} \middle| \frac{z}{z - 1} \right) = \frac{(1 - z)^b(c - e)}{c - 1} 3F_2 \left( \begin{array}{c} 1, b, e - c + 1 \\ 2, e \end{array} \middle| z \right) + \frac{(e - 1)(1 - z)(1 - (1 - z)^{b-1})}{(c - 1)(b - 1)z}. \] (41)

For \( b = 1 \) it reduces to

\[ 3F_2 \left( \begin{array}{c} 1, 1, c \\ 2, e \end{array} \middle| \frac{z}{z - 1} \right) = \frac{(z - 1)(e - c)}{c - 1} 3F_2 \left( \begin{array}{c} 1, 1, e - c + 1 \\ 2, e \end{array} \middle| z \right) + \frac{(e - 1)(1 - z)}{(c - 1)z} \ln \frac{1}{1 - z}. \] (42)

**Proof.** The proof is based on the following easily verifiable relation (which can be also found at http://functions.wolfram.com/07.27.03.0120.01):

\[ 3F_2 \left( \begin{array}{c} 1, b, c \\ 2, e \end{array} \middle| z \right) = \frac{e - 1}{(b - 1)(c - 1)z} \left[ 2F_1 \left( \begin{array}{c} b - 1, c - 1 \\ e - 1 \end{array} \middle| z \right) - 1 \right]. \] (43)

To prove (41) write this relation for \( z/(z - 1) \) in place of \( z \), apply

\[ 2F_1 \left( \begin{array}{c} b - 1, c - 1 \\ e - 1 \end{array} \middle| \frac{z}{z - 1} \right) = (1 - z)^{b-1} 2F_1 \left( \begin{array}{c} b - 1, e - c \\ e - 1 \end{array} \middle| z \right) \]

and substitute \( 2F_1 \) from the right-hand side by \( 2F_1 \) expressed from (43). To prove (42) let \( b \) tend to 1 and apply the L’Hopital rule. □
Combining (12) with the definition (10) of $g_{a,b_1,b_2}(x,y)$ we immediately obtain (37).

Next we check the behavior of the function $g_{a,b_1,b_2}(x,y)$ on the sides of the square $|x| < 1$, $|y| < 1$. Writing (13) for $z = 1$ and using the Gauss formula for $2F_1(1)$ we get

$$3F_2 \left( \frac{1}{2}, c \left| \frac{1}{1} \right. \right) = \frac{e - 1}{(b - 1)(c - 1)} \left[ 2F_1 \left( \frac{b - 1, c - 1}{e - 1} \right) \right] - 1 \right]$$

Now let $b \to 1$ and use the L’Hospital rule:

$$3F_2 \left( \frac{1, 1, e \left| \frac{1}{2} \right. \right) = \frac{(e - 1)\Gamma(e - 1) d \Gamma(e - b - c + 1)}{(c - 1)\Gamma(e - c) db \Gamma(e - b)} \bigg|_{b=1}$$

$$= \frac{\Gamma(e)}{(c - 1)\Gamma(e - c)} - \Gamma(e - c)\psi(e - c)\Gamma(e - 1) + \Gamma(e - 1)\psi(e - 1)\Gamma(e - c)$$

Substituting $e = b_1 + b_2 + 1$, $c = b_2 + 1$ gives (38).

Identity (39) is obvious from the definition (11) of $g_{a,b_1,b_2}(x,y)$.

Finally, formula (40) follows from the reduction formula

$$3F_2 \left( \frac{1, 1, 3/2 \left| \frac{1}{2} \right. \right) = - \frac{4}{z} \ln \left( \frac{1}{2} + \frac{\sqrt{1 - z}}{2} \right)$$

This completes the proof of the theorem. $\Box$

**Corollary 2.1** For $x, y \to 1$

$$f_{a,b_1,b_2}(x,y) = \ln \frac{1}{1 - xy} + O(1). \quad (44)$$

**Proof.** Assume first that $x$ and $y$ approach $(1, 1)$ in a way such $(1 - y)/(1 - x)$ stays bounded. We have

$$\ln \frac{1}{1 - xy} = \ln \frac{1}{1 - x + x - xy} = \ln \frac{1}{(1 - x)(\frac{x - y}{1 - x})} = \ln \frac{1}{1 - x} + \ln \frac{1}{1 + \frac{1 - y}{1 - x}}.$$

Hence,

$$\ln \frac{1}{1 - xy} - g_{a,b_1,b_2}(x,y)$$

$$= \ln \frac{1}{1 + \frac{1 - y}{1 - x}} - \gamma(a, b_1 + b_2) - \frac{b_2(y - x)}{(b_1 + b_2)(1 - x)}3F_2 \left( \frac{1, 1, b_2 + 1}{2, b_1 + b_2 + 1} \frac{y - x}{1 - x} \right) = O(1).$$

If $(1 - y)/(1 - x)$ is unbounded, than exchange the roles of $x$ and $y$ and use (37). $\Box$

Finally, we remark that the authors of [1] 2 consider monotonicity and ranges of the functions

$$\frac{1 - 2F_1(a, b; a + b; x)}{\ln(1 - x)}$$

and

$$B(a, b)_2F_1(a, b; a + b; x) + \ln(1 - x)$$

for $x \in (0, 1)$. Our Corollary 2.1 shows that similar problems can be considered for the combinations

$$\frac{1 - F_1(\alpha; \beta_1, \beta_2; \alpha + \beta_1 + \beta_2; x, y)}{\ln(1 - xy)}$$

and

$$f_{\alpha, \beta_1, \beta_2}(x, y) - \ln \frac{1}{1 - xy}$$

for $x, y \in (0, 1)$.
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