Canonical Solutions to Nonconvex Minimization Problems over Lorentz Cone

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Abstract

This paper presents a canonical dual approach for solving nonconvex quadratic minimization problem. By using the canonical duality theory, nonconvex primal minimization problems over n-dimensional Lorentz cone can be transformed into certain canonical dual problems with only one dual variable, which can be solved by using standard convex minimization methods. Extremality conditions of these solutions are classified by the triality theory. Applications are illustrated.

Key Words: Conical optimization; nonlinear programming; constrained minimization; canonical duality; NP-hard problems; global optimization.

1 Primal Problem and It’s Canonical Dual

The primal problem (\(\mathcal{P}\)) proposed to solve is the so-called second order cone programming:

\[
\mathcal{P} : \min \left\{ P(x) = \frac{1}{2} \langle x, Qx \rangle - \langle x, c \rangle : x \in \mathcal{C} \right\},
\]

where \(Q \in \mathbb{R}^{n \times n}\) is a given symmetrical matrix; \(c \in \mathbb{R}^n\) is a given vector; \(\mathcal{C} \subset \mathbb{R}^n\) is the so-called Lorentz cone in \(\mathbb{R}^n\):

\[
\mathcal{C} = \{ (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} | \| x_2 \| \leq x_1, \ x_1 > 0 \},
\]

which is a special second order cone. The problem (\(\mathcal{P}\)) appears in many applications such as structural optimization, filter design, and grasping force optimization in robotics. Extensive research has been focused on this subject.

In this paper we present a canonical dual approach for solving the second order cone optimization problem (\(\mathcal{P}\)). By using the canonical duality theory developed in...
the canonical dual problem $(P^d)$ can be formulated as
\[
\max \left\{ P^d(\sigma) = -\frac{1}{2} \langle c, G^{-1}(\sigma)c \rangle : \sigma \in S_a \right\}, \tag{3}
\]
where \(G(\sigma) = Q + \sigma L_o\),
\[
L_o = \begin{pmatrix} -1 & 0 \\ 0 & I_{n-1} \end{pmatrix}
\]
is the Lorentz matrix, in which, \(I_{n-1}\) is an identical matrix in \(\mathbb{R}^{(n-1)\times(n-1)}\). The dual feasible space \(S_a \subset \mathbb{R}\) is defined by
\[
S_a = \{ \sigma \in \mathbb{R} | \sigma \geq 0, \det G(\sigma) \neq 0 \}. \tag{4}
\]

**Theorem 1** The problem \((P^d)\) is canonically dual to \((P)\) in the sense that the vector \(\bar{\sigma} \in S_a\) is a KKT point of \((P^d)\) if and only if the vector
\[
\bar{x} = G^{-1}(\bar{\sigma})c
\]
is a KKT point of \((P)\), and
\[
P(\bar{x}) = P^d(\bar{\sigma}). \tag{6}
\]

**Proof.** By the standard procedure of the canonical dual transformation, we rewrite the cone constraint \(\|x_2\| \leq x_1\) in the quadratic form \(\frac{1}{2} x^T L_o x \leq 0\) and introduce a nonlinear transformation (i.e. the so-called geometrical mapping) \(\epsilon = \Lambda(x) = \frac{1}{2} x^T L_o x : \mathbb{R}^n \to \mathbb{R}\). Thus, the cone constraint \(x \in C\) can be replaced identically by \(\epsilon(x) = \Lambda(x) \leq 0\). Let
\[
V(\epsilon) = \begin{cases} 0 & \text{if } \epsilon \leq 0, \\ +\infty & \text{otherwise}. \end{cases} \tag{7}
\]
The primal problem \((P)\) can be written in the following canonical form \[7\]:
\[
(P_c) : \min \left\{ P(x) = V(\Lambda(x)) + \frac{1}{2} x^T Q x - c^T x : x \in \mathbb{R}^n \right\}. \tag{8}
\]
According to the Fenchel transformation, the sup-conjugate \(V^\sharp\) of the function \(V(\epsilon)\) is defined by
\[
V^\sharp(\sigma) = \sup_{\epsilon \in \mathbb{R}} \{ \epsilon^T \sigma - V(\epsilon) \} = \begin{cases} 0 & \text{if } \sigma \geq 0, \\ +\infty & \text{otherwise}. \end{cases}
\]
Since \(V(\epsilon)\) is a proper closed convex function over \(\mathbb{R}_- := \{ \epsilon \in \mathbb{R} | \epsilon \leq 0 \}\), we know that
\[
\sigma \in \partial V(\epsilon) \iff \epsilon \in \partial V^\sharp(\sigma) \iff V(\epsilon) + V^\sharp(\sigma) = \epsilon \sigma. \tag{9}
\]
The pair of \((\epsilon, \sigma)\) is then called a generalized canonical dual pair on \(\mathbb{R}_- \times \mathbb{R}_+\) by the definition introduced in [2, 4]. Following the original idea of Gao and Strang [16], we replace \(V(\Lambda(x))\) in equation (5) by the Fenchel-Young equality \(V(\Lambda(x)) = \Lambda(x)^T \sigma - V^*(\sigma)\). Then the so-called total complementary function \(\Xi(x, \sigma) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}\) associated with the problem \((P_c)\) can be defined as below

\[
\Xi(x, \sigma) = \Lambda(x)^T \sigma - V^*(\sigma) + \frac{1}{2}x^T Qx - c^T x.
\]

(10)

By the definition of \(\Lambda(x)\) and \(V^*(\sigma)\), on \(\mathbb{R}^n \times \mathbb{R}_+\) we have

\[
\Xi(x, \sigma) = \frac{1}{2}x^T G(\sigma)x - x^T c.
\]

(11)

The criticality condition of \(\Xi(x, \sigma)\) leads to the equilibrium equation

\[
G(\sigma)x = c,
\]

(12)

and the KKT conditions

\[
\sigma \geq 0, \quad x^T L_0 x \leq 0, \quad \sigma x^T L_0 x = 0.
\]

(13)

Substituting (12) into the canonical dual transformation

\[
P^d(\sigma) = \text{sta}\{\Xi(x, \sigma) \mid x \in \mathcal{C}\},
\]

the canonical dual function \(P^d(\sigma)\) is then formulated.

It is easy to prove that if \(\bar{\sigma} \geq 0\) is a KKT point of \((P^d)\), then we have

\[
\bar{\sigma} \geq 0, \quad \nabla P^d(\bar{\lambda}) = \frac{1}{2}x(\bar{\sigma})^T L_0 x(\bar{\sigma}) \leq 0,
\]

(14)

\[
\bar{\sigma} \cdot \left(\frac{1}{2}x(\bar{\sigma})^T L_0 x(\bar{\sigma})\right) = 0,
\]

(15)

where \(x(\bar{\sigma}) = G^{-1}(\bar{\sigma})c\). This shows that \(x(\bar{\sigma})\) is also a KKT point of the primal problem \((P)\).

By the complementarity condition (15), and the fact of \(x = G^{-1}(\sigma)c\), we have

\[
P^d(\bar{\sigma}) = -\frac{1}{2}c^T G^{-1}(\bar{\sigma})c
\]

\[
= \frac{1}{2}c^T G^{-1}(\bar{\sigma})c - c^T G^{-1}(\bar{\sigma})c
\]

\[
= \frac{1}{2}(G^{-1}(\bar{\sigma})c)^T G(\bar{\sigma})G^{-1}(\bar{\sigma})c - c^T G^{-1}(\bar{\sigma})c
\]

\[
= \frac{1}{2}(G^{-1}(\bar{\sigma})c)^T (Q + \bar{\sigma} L_0) G^{-1}(\bar{\sigma})c - c^T G^{-1}(\bar{\sigma})c
\]

(16)
\[
\begin{align*}
&= \frac{1}{2} (G^{-1}(\bar{\sigma})c)^T QG^{-1}(\bar{\sigma})c - c^T G^{-1}(\bar{\sigma})c + \frac{\bar{\sigma}}{2} (G^{-1}(\bar{\sigma})c)^T L_0 G^{-1}(\bar{\sigma})c \\
&= \frac{1}{2} (G^{-1}(\bar{\sigma})c)^T QG^{-1}(\bar{\sigma})c - c^T G^{-1}(\bar{\sigma})c \\
&= \frac{1}{2} \bar{x}^T Q\bar{x} - c^T \bar{x} \\
&= P(\bar{x})
\end{align*}
\]

This proves the theorem. \qed

\section{Extremality Conditions}

Let

\[ S^+_a = \{ \sigma \in \mathbb{R} \mid \sigma \geq 0, \ G(\sigma) \succ 0 \}. \] (16)

**Theorem 2** Suppose that \( \bar{\sigma} \in S_a \) is a solution to (P) and

\[ \bar{x} = G^{-1}(\bar{\sigma})c. \]

If \( \bar{\sigma} \in S^+_a \), then \( \bar{x} \) is a global minimizer of \( P(x) \) on \( C \) and

\[ P(\bar{x}) = \min_{x \in C} P(x) = \max_{\sigma \in S^+_a} P^d(\sigma) = P^d(\bar{\sigma}). \] (17)

**Proof.** By Theorem 1 we know that the vector \( \bar{\sigma} \in S_a \) is a KKT point of the problem (\( P^d \)) if and only if \( \bar{x} = G^{-1}(\bar{\sigma})c \) is a KKT point of the problem (P), and

\[ P(\bar{x}) = \Xi(\bar{x}, \bar{\sigma}) = P^d(\bar{\sigma}). \] (18)

Particularly, if \( \bar{\sigma} \in S^+_a \), the canonical dual function \( P^d(\sigma) \) is concave. In this case, the total complementary function \( \Xi \) is a saddle function, i.e., it is convex in \( x \in \mathbb{R}^n \) and concave in \( \sigma \in S^+_a \). Thus, we have

\[
P^d(\bar{\sigma}) = \max_{\sigma \in S^+_a} P^d(\sigma) = \max_{\sigma \in S^+_a} \min_{x \in \mathbb{R}^n} \max_{\sigma \in S^+_a} \left\{ \left\{ \frac{1}{2} x^T Qx - x^T c + \max_{\sigma \in S^+_a} \left\{ \left( \frac{1}{2} x^T L_0 x \right) \sigma - V^+(\sigma) \right\} \right\} \right\} = \min_{x \in C} P(x)
\]
due to the fact that

\[
V(\Lambda(x)) = \max_{\sigma \in S^+_a} \left\{ \left( \frac{1}{2} x^T L_0 x \right) \sigma - V^+(\sigma) \right\} = \begin{cases} 0 & \text{if } x \in C, \\ \infty & \text{otherwise.} \end{cases}
\]
From Theorem 1 we have (17). □

In a special case when \( Q = \text{Diag}(q) \) is a diagonal matrix with \( q = \{q_i\} \in \mathbb{R}^n \) being its diagonal elements, we have

\[
G^{-1}(\sigma) = \left\{ \frac{1}{q_i + \sigma \delta_i} \right\},
\]

(19)

In this case,

\[
P^d(\sigma) = -\frac{1}{2} \sum_{i=1}^{n} \frac{c_i^2}{q_i + \sigma \delta_i},
\]

(20)

where

\[
\delta_i^- = \begin{cases} -1 & i = 1 \\ 1 & i \neq 1 \end{cases}, \quad \delta_i^+ = \begin{cases} 1 & i = 1 \\ 0 & i \neq 1 \end{cases}.
\]

(21)

For the given \( \{c_i\}, \) and \( \{q_i\} \) such that \(-q_n \leq q_{n-1} \leq \ldots \leq q_1\), the dual variable \( \sigma \) can be solved completely within each interval \(-q_1 \leq \sigma < q_1\) or \(-q_2 < \sigma < q_1\), such that \( q_i < q_{i+1} \) (\( i = 2, \ldots, n \)).

3 Applications

We now list a few examples to illustrate the applications of the theory presented in this paper.

3.1 Two-D nonconvex minimization

First of all, let us consider two dimensional concave minimization problem:

\[
(P) : \min \left\{ P(x) = \frac{1}{2}(q_1x_1^2 + q_2x_2^2) - c_1x_1 - c_2x_2 : \|x_2\| \leq x_1, (x_1, x_2) \in \mathbb{R}^2 \right\},
\]

(22)

On the dual feasible set

\[
S_a = \{ \sigma \in \mathbb{R} \mid \sigma \geq 0, \ (q_1 - \sigma)(q_2 + \sigma) \neq 0 \},
\]

(23)

the canonical dual function has the form of

\[
P^d(\sigma) = -\frac{1}{2} [c_1, c_2]^T \begin{bmatrix} \frac{1}{q_1-\sigma} \\ \frac{1}{q_2+\sigma} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.
\]

(24)

Assume \( q_1 = 0.1, q_2 = -0.3, c_1 = 0.5, c_2 = -0.3 \), so we have

\[
\sigma = 0.45 \in S^+_a = \{ \sigma \in \mathbb{R} \mid 0.3 < \sigma < 0.5 \}.
\]

(25)

By Theorem 1, we know that \( x = \{c_1/(q_1 - \sigma), c_2/(q_2 + \sigma)\} = \{2, -2\} \) is a global minimizer, it is easy to verify that \( P(x) = P^d(\sigma) = -0.4 \) (see Figure 1).
3.2 Two-D general nonconvex minimization

\[
(P) : \min \left\{ P(x) = \frac{1}{2} (q_1 x_1^2 + q_2 x_2^2 + 2q_3 x_1 x_2) - c_1 x_1 - c_2 x_2 : \|x_2\| \leq x_1, (x_1, x_2) \in \mathbb{R}^2 \right\},
\]

(26)

On the dual feasible set

\[
S_a = \{ \sigma \in \mathbb{R}^2 \mid \sigma \geq 0, \ (q_1 - \sigma)(q_2 + \sigma) - q_3^2 \neq 0 \},
\]

(27)

the canonical dual function has the form of

\[
P^d(\sigma) = -\frac{1}{2} [c_1, c_2]^T \begin{bmatrix} q_1 - \sigma & q_3 \\ a_3 & q_2 + \sigma \end{bmatrix}^{-1} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}
\]

(28)

If we choose \( q_1 = 1.8, q_2 = -0.6, q_3 = 0.4, c_1 = 0.5, c_2 = 0.6 \), then we have

\[
\sigma = 1.29 \in S_a^+ = \{ \sigma \in \mathbb{R} \mid (q_1 - \sigma)(q_2 - \sigma) - a_3^2 > 0 \}.
\]

(29)

By Theorem \( \Box \) we know that \( x = \begin{bmatrix} q_1 - \sigma & q_3 \\ q_3 & q_2 + \sigma \end{bmatrix}^{-1} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \{0.5500, 0.5499\} \) is a global minimizer, and \( P(x) = P^d(\sigma) = -0.3025 \) (see Figure 3-4).
3.3 Three-D general nonconvex minimization

\[
(P) : \min \left\{ P(x) = \frac{1}{2} \langle x, Qx \rangle - \langle x, c \rangle : \|x_2\| \leq x_1, (x_1, x_2) \in \mathbb{R}^3 \right\},
\]

where \(Q = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix}, c = \{c_1, c_2, c_3\}\). On the dual feasible set

\[
\mathcal{S}_p = \{\sigma \in \mathbb{R} | \sigma \geq 0, \det G(\sigma) \neq 0\},
\]

where

\[
G(\sigma) = \begin{bmatrix} q_{11} - \sigma & q_{12} & q_{13} \\ q_{12} & q_{22} + \sigma & q_{23} \\ q_{13} & q_{23} & q_{33} + \sigma \end{bmatrix},
\]

the canonical dual function has the form of

\[
P^d(\sigma) = -\frac{1}{2} [c_1, c_2, c_3]^T G^{-1}(\sigma) \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}
\]
Suppose \( q_{11} = 2, q_{22} = -2, q_{33} = 1, q_{12} = -1, q_{13} = 2, q_{23} = 0, c_1 = 1.5, c_2 = -0.5, c_3 = 1.5 \), we have

\[
\sigma = 0.4509 \in S^+_a = \{ \sigma \in \mathbb{R} | \sigma \geq 0, G(\sigma) \succ 0 \}.
\] (34)

By theorem 1, we know that \( x = G^{-1}(\sigma) \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \{0.4355, 0.0416, 0.4335\} \) is a global minimizer, and \( P(x) = P^d(\sigma) = -0.6413 \) (see Figure 5).

![Figure 5: Graph of \( P^d(\sigma) \) for general three dimensional problem.](image)

4 Conclusions

We have presented a concrete application of the canonical dual transformation and triality to conic optimization problems. Results show that by the use of this method, the nonconvex cone constrained problem \( P \) in \( \mathbb{R}^n \) can be reformulated as a perfect dual problem in \( \mathbb{R} \), also the \( KKT \) points and extremality conditions of the originally difficult problems are identified by Theorem 1 and 2. Physically speaking, each optimal point represents a stable equilibrium state of the system. Duality theory reveals the intrinsic pattern of duality relations of these critical points, and plays an important role in nonconvex analysis, detailed study and comprehensive applications of this theory were presented in monograph [2].

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