Turbulence as Gibbs Statistics of Vortex Sheets

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We study the vortex sheet solutions of the Euler equation, which correspond to the tangent discontinuity of the velocity field. We observe that the stationary flows correspond to the Hamiltonian’s minimization by the tangent discontinuity density $\Gamma$. This observation means that the stationary flow represents the low-temperature limit of Gibbs distribution of the vortex sheet dynamics. An infinite number of Euler conservation laws leads to a degenerate vacuum of this system, which degeneracy explains the complexity of the turbulent statistics and provides the relevant degrees of freedom (random surfaces). We find an exact analytic solution for a spherical surface. This solution provides an example of the instanton advocated in our recent work, which is supposed to be responsible for the dissipation of the Navier-Stokes equation in the turbulent limit of vanishing viscosity at fixed energy flow. We further conclude that one can obtain the turbulent statistics from the Gibbs statistics of vortex sheets by adding Lagrange multipliers for the conserved volume inside closed surfaces, energy pumping, and energy dissipation via viscosity anomaly in the enstrophy. The effective temperature in our Gibbs distribution goes to zero as $Re^{-\frac{1}{2}}$, with Reynolds number $Re \sim \nu^{-\frac{5}{6}}$ in the turbulent limit, which opens the way for the quantitative theory of turbulence as low-temperature expansion in this Gibbs ensemble around minimal surfaces.

I. INTRODUCTION

Tangent velocity discontinuity has been around for ages. Make the water above and below a planar surface move fast in the opposite direction, and this discontinuity surface arises and then becomes unstable.

This phenomenon is the famous Kelvin-Helmholtz instability.\textsuperscript{[2]} The time evolution of this instability leads to the surface rolling up in one direction, creating a vorticity layer.

There were many simulations of this process, which is an onset of turbulence. Initially, it was analyzed as a 1-dimensional Birkhoff equation, neglecting one of the coordinates of the planar interface but later it was realized that this is a nonlinear 2D problem of a vorticity sheet moving in a self-generated velocity field.\textsuperscript{[5]} Later, there were several such studies\textsuperscript{[7]} with more references inside. The verdict was the same – this motion is unstable. It leads to a vorticity layer, ending with turbulence, which remains unsolved after centuries of good tries.

However, to our knowledge, nobody noticed that this surface discontinuity in Euler dynamics could be stationary up to re-parametrization under certain conditions. This stationary Euler flow is the primary subject of the present paper.

We advocated in\textsuperscript{[8]} stationary vortex sheets subject to a constant external force as the primary source of dissipation and multi-fractal scaling laws in a turbulent flow.

Let us define here the necessary equations.

Navier-Stokes equations

\begin{align}
\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla})\vec{v} + \vec{\nabla}p &= \nu \vec{\nabla}^2 \vec{v} ; \quad (1a) \\
\vec{\nabla} \cdot \vec{v} &= 0; \quad (1b)
\end{align}

can be rewritten as the equation for vorticity

\begin{align}
\vec{\omega} &= \vec{\nabla} \times \vec{v} ; \quad (2a) \\
\partial_t \vec{\omega} + (\vec{v} \cdot \vec{\nabla})\vec{\omega} - (\vec{\omega} \cdot \vec{\nabla})\vec{v} &= \nu \vec{\nabla}^2 \vec{\omega} ; \quad (2b)
\end{align}

As for the velocity, it is given by a Biot-Savart integral

\begin{equation}
\vec{v}(r) = -\vec{\nabla} \times \int \frac{d^3 r' \vec{\omega}(r')}{4\pi |r-r'|} \quad (3)
\end{equation}

which is a linear functional of the instant value of vorticity.

The Euler equation corresponds to setting $\nu = 0$ in $1a$. This limit is known not to be smooth, leading to a statistical distribution of vortex structures, which is the whole turbulence problem.

We addressed this problem in\textsuperscript{[8]}. Our initial goal here was not so ambitious: to study the vorticity sheet dynamics in the Euler-Lagrange equations and the stationary vortex sheets that seem impossible.

As an unexpected by-product of this study, we find the general relation between Gibbs statistics and the turbulent statistics for this vorticity sheet system, which moves us closer to our big goal: the Statistical Theory of turbulence.

II. STATIONARY DISCONTINUITY SURFACE IN THE LAGRANGE DYNAMICS

As it is well known, the Euler equation is equivalent to the motion of every point in the fluid with local velocity:

\begin{equation}
\partial_t \vec{r} = \vec{v}(\vec{r}) . \quad (4)
\end{equation}
This also applies to every point \( \tilde{r} = \tilde{X}(\xi_1, \xi_2) \) on the discontinuity surface.

The normal component \( v_n \) of velocity describes the surface’s genuine change—its global motion or the change of its shape. In a stationary flow, this normal component must vanish at every point of the surface.

The remaining two tangent components \( \tilde{v}_t \) of velocity, on the other hand, just move the points along the surface without changing its position and shape. To see that, we rewrite the tangent part \( \tilde{v}_t \) of velocity at the surface as time-dependent re-parametrization \( \xi \Rightarrow \xi(t) \):

\[
\tilde{v}_t = \partial_t \tilde{X}(\xi(t)) = \partial_a \tilde{X} \partial_t \xi^a;
\]

\[
\partial_t \xi^a = g^{ab} \partial_a \tilde{X} \cdot \tilde{v}_t;
\]

\[
g_{ab} = \partial_a \tilde{X} \cdot \partial_b \tilde{X};
\]

Here \( g_{ab} \) is an induced metric, and \( g^{ab} \) is its inverse.

Note that there are two different re-parametrizations of each side of the surface due to the tangent velocity discontinuity. In other words, these two sides are the same surface parameterized by different internal coordinates. There is no contradiction with this double parametrization as long as there is a shared boundary value (\( \Delta \tilde{v}(\tilde{r} \in C) = 0 \) in our case).

Note that this 2D transformation does not preserve the area, as the surface divergence \( \nabla \cdot \tilde{v}_t = -\partial_z v_z \neq 0 \) for a closed surface, these tangent motions will never leave the surface. For the surface with the fixed edge \( C \) there is a boundary condition that velocity normal to the edge vanishes \( \tilde{v}_t \times d\tilde{r} = 0; \forall \tilde{r} \in C \). In this case, the fluid will slide along \( C \) leading to its re-parametrization, but it never leaves the surface.

We will restrict ourselves to the parametric invariant functionals, which do not depend on this tangent flow, and therefore, they will stay stationary.

We introduced the Lagrange action of vortex sheets in the old paper. In that paper, we also conjectured the relation of turbulence to Random Surfaces.

In the next paper, this Lagrange vortex dynamics was simulated using a triangulated surface. We calculated the contributions to the velocity field from each triangle in terms of elliptic integrals. The positions of the triangle vertices served as dynamical degrees of freedom.

There were conserved variables related to the velocity gap as a function of a point at the surface. These points were passively moving together with the surface by the mean value of velocity on the surface’s two sides.

Later, our equations were recognized and reiterated in traditional terms of fluid dynamic and simulated with a larger number of triangles with similar results regarding Kelvin-Helmholtz instability against the roll-up of the vortex sheet. There were dozens of publications using various versions of the discretization of the surface and various simulation methods.

Let us reproduce this theory here for the reader’s convenience before advancing it further.

The following Ansatz describes the vortex sheet vorticity:

\[
\tilde{\omega}(\tilde{r}) = \int_\Sigma d\tilde{\Omega} \delta^3 \left( \tilde{X} - \tilde{r} \right)
\]

where the 2-form

\[
d\tilde{\Omega} = d\Gamma \wedge d\tilde{X} = d\xi_1 d\xi_2 c_{ab} \frac{\partial \Gamma}{\partial \xi_a} \frac{\partial \tilde{X}}{\partial \xi_b}
\]

The conservation of vorticity

\[
\tilde{\nabla} \cdot \tilde{\omega} = 0;
\]

is built into this Ansatz for arbitrary \( \Gamma(\xi) \), as can be verified by direct calculation

\[
\tilde{\nabla} \cdot \tilde{\omega} = \int_\Sigma d\tilde{\Omega} \cdot \frac{\partial}{\partial \xi} \delta^3 \left( \tilde{X} - \tilde{r} \right) = -\int_\Sigma d\tilde{\Omega} \cdot \frac{\partial}{\partial \tilde{X}} \delta^3 \left( \tilde{X} - \tilde{r} \right) = -\int_{\partial \Sigma} \Gamma d\delta^3 \left( \tilde{X} - \tilde{r} \right);
\]

In case there is a boundary of the surface, this \( \Gamma(\xi) \) must vanish (or, in general, be a constant) at the boundary for the conservation law to hold.

This Ansatz is equivalent to the transverse vorticity of our instanton with the discontinuity \( 2\pi n \) of Clebsch field \( \phi_2 \) and \( \Gamma = 2\pi n \phi_1 \).

Neither nor any subsequent papers noticed this relation between Clebsch field phase discontinuity and vortex sheets.

In this paper, we are not interested in this relation, as we study the vortex sheets by themselves as a generalized Hamiltonian system.

As we already noted in, the function \( \Gamma(\xi_1, \xi_2) \) is defined modulo diffeomorphisms \( \xi \Rightarrow \eta(\xi); \det \partial_i \eta_j > 0 \) and is conserved in Lagrange dynamics:

\[
\partial_t \Gamma = 0;
\]

This function is related to 1-form of velocity discontinuity

\[
d\Gamma = \Delta \tilde{\omega} \cdot d\tilde{X}
\]

where the velocity gap

\[
\Delta \tilde{\omega}(\xi) = \tilde{\omega}(X_+(\xi)) - \tilde{\omega}(X_-(\xi))
\]

The surface is driven by the self-generated velocity field (mean of velocity above and below the surface), as in . The Biot-Savart integral for the velocity field in this case yields

\[
\tilde{\omega}(\tilde{r}) = \frac{1}{4\pi} \int d\tilde{\Omega} \times \tilde{\nabla} \frac{1}{|\tilde{r} - \tilde{X}|}
\]
The Lagrange equations of motion for the surface
\[ \delta t \vec{X}(\xi) = \vec{v}(\vec{X}(\xi)) \] (16)
were shown in \[2\] to follow from the action
\[ S = \int \Gamma dV - \int Hdt; \] (17)
\[ dV = d\xi_1 d\xi_2 dt \frac{\partial \vec{X}}{\partial \xi_1} \times \frac{\partial \vec{X}}{\partial \xi_2} \cdot \delta t \vec{X}; \] (18)
\[ H = \frac{1}{2} \int d^3 r \vec{v}^2 = \frac{1}{2} \int \int \frac{d\bar{\Omega} \cdot d\bar{\Omega}^\prime}{4\pi |\vec{X} - \vec{X}'|}; \] (19)
This \( dV \) is the 3-volume swept by the surface area element in its movement for the time \( dt \).

In the case of the handle \( H \) on a surface, \( \Gamma \) acquires extra term \( \Delta \Gamma = \int_\gamma \Delta \vec{v} \cdot d\vec{r} \) when the point goes around one of the cycles \( \gamma = \{a, b\} \) of the handle.

This \( \Delta \Gamma \) does not depend on the path shape because there is no normal vorticity at the surface, and thus there is no flux through the surface. This topologically invariant \( \Delta \Gamma \) represents the flux through the handle cross-section.

This ambiguity in \( \Gamma \) makes our action multivalued as well.

Let us check the equations of motion emerging from the variation of the surface at fixed \( \Gamma \):
\[ \delta \int Hdt = \int d\bar{\Omega} \times \delta \vec{X} \cdot \vec{v}(\vec{X}) dt; \] (20)
\[ \delta \int \Gamma dV = \int d\bar{\Omega} \times \delta \vec{X} \cdot \delta t \vec{X} dt \] (21)
As we already discussed above, the tangent components of velocity at the surface create tangent motion, resulting in the surface’s re-parametrization.

One of the two tangent components of the velocity (along the line of constant \( \Gamma(\xi) \)) does not contribute to variation of the action, so that the correct Lagrange equation of motion following from our action reads
\[ \partial_t \vec{X}(\xi) = \vec{v}(\vec{X}(\xi)) \mod e^{i/\delta \Gamma} \partial \vec{X} \] (22)
We noticed this gauge invariance before \[2\], but now we see that both tangent components of the velocity could be absorbed into the re-parametrization of a surface and therefore do not represent an observable change.

However, the normal component of the velocity must vanish in a stationary solution, and this provides a linear integral equation for the conserved function \( \Gamma(\xi) \).

In the general case, when there is an ensemble of such surfaces \( S_n, n = 1 \ldots N \) each has its discontinuity function \( \Gamma_n(\xi) \). At each point on each surface \( \bar{r} \in S_n \), the net normal velocity adding up from all surfaces, including this one in the Biot-Savart integral, must vanish:
\[ \vec{\Sigma}_n(\xi) \cdot \vec{v}(\vec{X}_n(\xi)) = 0; \] (23)
\[ \vec{\Sigma}_n(\xi) = e^{i/\delta \Gamma} \vec{X}_n \times \delta_j \vec{X}_n; \] (24)
\[ \vec{v}(\bar{r}) = \frac{1}{4\pi} \sum_m \int d\bar{\Omega}_m \times \vec{\nu} \frac{1}{|\bar{r} - \vec{X}_m|} \] (25)
This requirement provides a linear set of \( N \) linear integral equations (called Master Equation in \[2\]) relating \( N \) independent surface functions \( \Gamma_1 \ldots \Gamma_n \). With this set of equations satisfied, the collection of surfaces \( S_1 \ldots S_N \) will remain stationary up to re-parametrization on each side of each surface independently.

Here is an essential new observation we are reporting in this paper.

The Master Equation is equivalent to the minimization of our Hamiltonian by \( \Gamma_n, n = 1 \ldots N \)
\[ H[\Gamma, \vec{X}] = \frac{1}{2} \sum_n \int_{S_n} \int \frac{d\bar{\Omega}_n \cdot d\bar{\Omega}_m}{4\pi |\vec{X}_n - \vec{X}_m|}; \] (26)
\[ \frac{\delta H[\Gamma, \vec{X}]}{\delta \Gamma_n(\xi)} = \vec{\Sigma}_n(\xi) \cdot \vec{v}(\vec{X}_n(\xi)); \] (27)
As for the tangential components of velocity, those are included in the parametric invariance, as noted above. They are equivalent to variations of the Hamiltonian by the parametrization of \( \Gamma, \vec{X} \) and are therefore satisfied identically.

Therefore, the normal velocity of the surface in the general case is equal to the Hamiltonian variation by \( \Gamma \), as if \( \Gamma \) is the conjugate momentum corresponding to the surface’s normal displacement. To be more precise, \( \Gamma \) in our action \[17\] is a conjugate momentum to the volume, which is locally equivalent – the variation of volume equals the area element times the normal displacement.

In other words, we can consider an extended dynamical system with the same Hamiltonian \[26\] but a wider phase space \( \Gamma, \vec{X} \mod Diff \). We can introduce an extended Hamiltonian dynamics with our action \[17\].

This system is degenerate in the sense that for an arbitrary evolution of \( \vec{X} \) providing an extremum of the action, the evolution for \( \Gamma \) is absent, i.e., \( \Gamma \) is constant. It is a conserved momentum in our Hamiltonian dynamics with volume as coordinate.

This conservation of \( \Gamma \) is a consequence of Kelvin’s theorem. To see this relation \[4\], we rewrite this \( \Gamma \) as a circulation over the loop \( C \) puncturing the surface in two points \( A, B \) and going along some curve \( \gamma_{AB} \) on one side, then back on the same curve \( \gamma_{BA} \) on another side. The circulation does not depend upon the shape of \( \gamma_{AB} \) because there is no normal vorticity at the surface.

Another way to arrive at the conservation of \( \Gamma \) is to notice that it is related to the Clebsch field on the discontinuity surface, as we mentioned in the introduction.

The stationary solution for \( \vec{X} \mod Diff \) corresponds to the Hamiltonian minimum as a (quadratic) functional of \( \Gamma \).

III. DOES STATIONARY SURFACE MEAN STATIONARY FLOW?

There is a subtle difference between the stationary discontinuity surface and stationary flow. After all, the
flow around a stationary object does not have to be stationary? There could be time-dependent motions in the bulk of the flow, while the normal component of the flow vanishes at the solid surface (as it always does).

This logic applies to the generic flow around stationary solid objects, but it does not apply here. The big difference is that by our assumption, there is no vorticity outside these discontinuity surfaces.

The Biot-Savart integral for the velocity field (15) is manifestly parametric invariant, if we transform both $\Gamma, \tilde{X}$

$$\Gamma(\xi) \Rightarrow \Gamma(\eta(t, \xi)); \quad \tilde{X}(\xi) \Rightarrow \tilde{X}(\eta(t, \xi)); \quad \partial_t \eta = \phi_a(\eta);$$ (28) (29) (30)

This transformation describes the flux of coordinates $\eta$ in parametric space with the velocity field $\phi_a(\eta)$. The tangent flow around the surface is equivalent to such a transformation of $\tilde{X}$, as we demonstrated in (16).

However, in the Lagrange dynamics of vortex sheets, the function $\Gamma^a(\xi)$ remains constant, not the constant up to re-parametrization, but an absolute constant so that $\partial_t \Gamma^a = 0$ where time derivative goes at fixed $\xi$.

Therefore, in general, the velocity field $\bar{v}(\vec{r})$ does change when the surface gets re-parametrized, but $\Gamma$ does not. Naturally, one could not get the stationary solution without solving some equations first :). However, in our stationary manifold, $\Gamma^a(\xi)$ is related to the surface by the master equation. Let us write it down once again

$$0 = \sum_a \sum_m \int S_m \Gamma^a_m \wedge \tilde{X}_m \times \nabla \tilde{X}_m \frac{1}{|X_n - X_m|}; \quad \Sigma_n = \epsilon^{ab} \partial_a \tilde{X}_n \times \tilde{X}_n \quad (31) \quad (32)$$

This equation is invariant by the parametric transformation of both variables $\Gamma^a, \tilde{X}$. This means that the solution of this equation for $\Gamma^a$ would come out as a parametric invariant functional of $\tilde{X}$, in addition invariant by translations of $\tilde{X}$.

As we have seen, this master equation leads to a vanishing normal velocity $\Sigma_n \cdot \partial_i \tilde{X}_n = 0$. The remaining tangent velocity leaves $\tilde{X}$ stationary up to re-parametrization. Therefore, in virtue of this master equation, the velocity field will also be stationary.

We introduced a family of stationary solutions of Euler equations, parametrized by an arbitrary set of discontinuity surfaces $\tilde{X}_n(\xi)$ with discontinuities $\Gamma^a_n(\xi)$ determined by the minimization of the Hamiltonian.

IV. CONSERVATION LAWS AND MINIMIZATION PROBLEM

As it is well known, the Euler dynamics has infinitely many integrals of motion. In the dynamics of vortex sheets, these integrals are generated by two-dimensional conserved function $\Gamma_n(\xi)$ on each discontinuity surface $S_n$.

One can also write down explicit integrals of motion, involving both $\Gamma$ and $X$ variables. In addition to the Hamiltonian (29), there is a helicity $\mathcal{H}$, momentum $\vec{P}$ and angular momentum $\vec{M}$

$$\mathcal{H} = \int d^3 r \bar{v} \cdot \bar{v} = \sum_{n, m} \int S_m \int S_n d\tilde{X}_n \times \nabla \tilde{X}_n \frac{1}{4 \pi |X_n - X_m|}; \quad (33)$$

$$\vec{P} = \int d^3 \vec{r} \bar{v} = \frac{1}{3} \sum_n \int S_n d\tilde{X}_n \times \tilde{X}_n; \quad (34)$$

$$\vec{M} = \int d^3 \vec{r} \times \bar{r} = \frac{1}{2} \sum_n \int S_n d\tilde{X}_n \tilde{X}_n^2. \quad (35)$$

For the closed surfaces there is also a conserved volume inside each of them

$$\nu_n = \int_{\partial B = S_n} d^3 r = \frac{1}{3} \int_{S_n} d^2 \bar{r} \times \bar{X} \cdot \epsilon_{ij} \partial_i \tilde{X} \times \partial_j \tilde{X} \quad (36)$$

This volume only depends on the surface, but not on the vorticity density $\Gamma$.

The viscosity anomaly coming from resolving $0 \times \infty$ in the enstrophy integral breaks the Hamiltonian conservation

$$-\partial_t \mathcal{H} = \mathcal{E} = \nu \int d^3 r \bar{\omega}^2 \rightarrow E[\Gamma, \tilde{X}]; \quad (37)$$

$$E[\Gamma, \tilde{X}] = \Lambda \sum_n \int S_n \sqrt{g} \epsilon^{ij} \partial_i \Gamma_n \partial_j \Gamma_n; \quad (38)$$

The parameter $\Lambda$ was computed in (29) by taking limit $\nu \rightarrow 0$ in the Navier-Stokes equation. We found Gaussian profile of vorticity with viscous width $\hbar$ in direction $z$ normal to the surface.

$$\delta_0(z) = \frac{1}{\hbar \sqrt{2\pi}} \exp \left(-\frac{z^2}{2\hbar^2} \right) \quad (39)$$

At $\nu \rightarrow 0$ this Gaussian profile is becomes $\delta(z)$, making it equivalent to our representation of vorticity. At finite but small viscosity, the resulting integral over the viscous layer provides an extra factor $\Lambda/\nu$ with

$$\Lambda = \frac{\nu}{2 \hbar \sqrt{\pi}} \quad (40)$$

We sketch this calculation below for the reader’s convenience.

In the steady state of turbulence, all dissipated energy is compensated by the work made by external random forces, which we introduce as a constant Gaussian random vector imposed at infinity as the boundary condition for pressure $p \rightarrow -\vec{f} \cdot \vec{r}$.

$$\mathcal{E} = \vec{f} \cdot \vec{P} \quad (41)$$
We have to minimize the free energy
\[ F[\Gamma, \tilde{X}, \tilde{\lambda}] = H[\Gamma, \tilde{X}] - \tilde{\lambda} \cdot (\tilde{P}[\Gamma, \tilde{X}] - \tilde{Q}(\tilde{f})); \]  \tag{42}
with Lagrange multiplier \( \tilde{\lambda} \) to be determined from the fixed momentum
\[ \tilde{P}[\Gamma, \tilde{X}] = \tilde{Q}(\tilde{f}); \]  \tag{43}

We argued in the previous section that the momentum could be expanded in terms of the external random force \( \tilde{Q}(\tilde{f}) = \tilde{Q} \cdot \dot{\tilde{f}} \) with some constant 3 \times 3 symmetric matrix \( \tilde{Q} \) depending on the distribution of vorticity. In this paper, we compute this momentum analytically and prove its linear dependence on \( \tilde{f} \).

The best way to solve this minimization problem is to go back to 3D space and use the fact that there is no vorticity in the space except for the discontinuity surfaces.

Let us assume that inside each surface, the velocity field is just a constant vector.
\[ \tilde{v}(\tilde{r} \text{ inside } S_n) = \tilde{c}_n; \]  \tag{44}
This constant velocity is a trivial solution of the Euler equation.

The surfaces nested in the other ones drop from the equations. Consider the inner surface \( S_{in} \) nested inside the outer surface \( S_{out} \). Constant velocity fields inside \( S_{in} \) and inside \( S_{out} \) have to match at every point of \( S_{in} \), after being projected on its normal vector \( \Sigma_{in} \). This matching is possible only if these constant velocities are equal, in which case there will be no discontinuity, and thus the inner surface drops from the equations.

Therefore, we always assume that the surfaces are not nested.

Outside all surfaces, the flow is purely potential. The potential \( \Phi \) is a harmonic function with Neumann boundary conditions on each sphere, following from the continuity of normal velocity
\[
\nabla \cdot \tilde{v} = \tilde{\nabla} \Phi; \\
\tilde{\nabla} \cdot \tilde{v} = \tilde{\nabla}^2 \Phi = 0; \\
\Sigma_n \cdot (\tilde{\nabla} \Phi - \tilde{c}_n)_{S_n} = 0; \\
\Gamma_n(\tilde{r}) = \Phi(\tilde{r}) - \tilde{r} \cdot \tilde{c}_n; \tilde{r} \in S_n; \]  \tag{47}
This boundary condition also applies to the open surface bounded by some contour. In that case, the condition involves a constant vector velocity \( \tilde{c}_n \), the same one on both sides of the surface. The inner region is absent in this case, or, better to say, it shrinks to a zero thickness layer between the two sides of the surface.

The net momentum equals to
\[ \tilde{p} = \frac{1}{3} \sum_n \int_{S_n} \Gamma_n d\tilde{s}; \]  \tag{49}
where \( d\tilde{s} = d\xi_1 d\xi_2 d\eta_1 \tilde{X} \times d_2 \tilde{X} \) is vector area element.

Let us assume that we solved the Neumann Laplace problem. Then we can express \( \Phi(\tilde{r}) \) as linear combination of all the vectors \( \tilde{c}_m \) which define the boundary conditions for the external potential.
\[
\Phi(\tilde{r}) = \sum_m \tilde{c}_m \cdot \tilde{\Psi}_m(\tilde{r}); \\
\Gamma_n(\tilde{r}) = \sum_m \tilde{c}_m \cdot \tilde{\Psi}_{mn}(\tilde{r}); \\
\tilde{\Psi}_{mn}(\tilde{r}) = \tilde{\Psi}_m(\tilde{r}) - \tilde{r} \tilde{\delta}_{mn}; \tilde{r} \in S_n; \]  \tag{52}
This harmonic vector field \( \tilde{\Psi}_m(\tilde{r}) \) satisfies the Neumann boundary conditions
\[
\left( \Sigma_n \cdot \nabla \tilde{\Psi}_m^\mu \right)_{S_n} = \Sigma^\mu \delta_{mn} \]  \tag{53}
This field is universal, given the geometry of discontinuity surfaces, which gives us the opportunity to minimize the free energy as a quadratic form made of \( \tilde{c}_n \).

Substituting this into the momentum equation yields the linear equation
\[
Q_\mu = M^{\mu \nu} c_\nu; \]  \tag{54}
\[
M^{\mu \nu} = \frac{1}{3} \sum_n \int_{S_n} d^\alpha \tilde{\Psi}_m^\mu \]  \tag{55}
The Hamiltonian can also be expressed in these functions
\[
H = \frac{1}{2} \sum_{mn} C^\mu_{mn} H^{\mu \nu}_{mn} C_\nu^{\nu}; \]  \tag{56}
\[
H^{\mu \nu}_{mn} = \sum_{kl} \int_{S_k} \int_{S_l} \frac{d \Omega^{\mu \nu}_{mk}(\tilde{r}) d \Omega^{\nu \alpha}_{nl}(\tilde{r})}{4\pi |\tilde{r} - \tilde{r}|}; \]  \tag{57}
\[
d \Omega^{\mu \nu}_{mk}(\tilde{r}) = d \Psi^{\mu \nu}_{mk}(\tilde{r}) \wedge d^\alpha \tilde{r}; \]  \tag{58}
We have to minimize the free energy
\[
F = \frac{1}{2} \sum_{mn} C^\mu_{mn} H^{\mu \nu}_{mn} C_\nu^{\nu} - \lambda^{\mu} \sum_m M^{\mu \nu} c_\nu + \lambda^\mu Q^\mu \]  \tag{59}
after which the vector parameters \( \tilde{c}_1 \ldots \tilde{c}_n, \tilde{\lambda} \) will be linearly related to \( \tilde{Q} \) and thus \( \Gamma_n \) will have a form
\[
\Gamma_n(\tilde{r}) = 3 \Sigma_n(\tilde{r}) \cdot \tilde{Q}; \]  \tag{60}
We are left with the last problem: to find the momentum \( \tilde{Q} \) from the energy balance equation \( E[\Gamma, \tilde{X}] = \tilde{Q} \cdot \dot{\tilde{f}} \). We find the following equation:
\[
Q_\alpha M^{\alpha \beta} Q_\beta = Q_\alpha f_\alpha; \]  \tag{61}
\[
M^{\alpha \beta} = \Lambda \sum_n \int_{S_n} \partial_\beta \Sigma_n \left( \delta_{\mu \nu} - \Sigma_n \Sigma_\nu \right) \partial_\nu \Sigma_n^\beta; \]  \tag{62}
where as before, \( \Sigma_n \) is the unit normal vector to the surface.

This is an equation for the momentum \( \tilde{Q} \) as a function of the force vector \( \tilde{f} \). The relevant nonzero solution is just a 3 \times 3 matrix inversion:
\[
\tilde{Q} = M^{-1} \cdot \tilde{f}; \]  \tag{63}
The energy flow after averaging over Gaussian vector \( \vec{f} \) with variance \( \sigma \)

\[
\langle \mathcal{E} \rangle = \left< \vec{f} \cdot \vec{Q} \right> = \sigma \text{tr} \hat{M}^{-1};
\]  

(64)

This minimization can be readily done for a sphere\(^{13}\)

\[
\Phi(\vec{r}) = -\frac{3\vec{r} \cdot \vec{Q}}{2\pi R^3} \quad \text{for } |\vec{r}| < R
\]

(65a)

\[
\frac{3\vec{r} \cdot \vec{Q}}{4\pi |\vec{r}|^3} \quad \text{for } |\vec{r}| > R;
\]

(65b)

\[
\Gamma(\vec{r}) = \frac{9\vec{r} \cdot \vec{Q}}{4\pi R^3} \quad \text{for } |\vec{r}| = R;
\]

(65c)

\[
\vec{Q} = \frac{2\pi R^4 \vec{f}}{27\Lambda};
\]

(65d)

\[
\langle \mathcal{E} \rangle = \frac{2\pi R^4 \sigma}{9\Lambda};
\]

(65e)

This solution is manifestly gauge invariant, as we expressed \( \Gamma \) as a function of a point \( \vec{r} \) in 3D space, projected on a surface, without specifying the parametrization of the surface.

It is a good problem for a grad student to minimize the free energy in its original form, as a functional of \( \Gamma(\vec{x}) \). With the spherical Ansatz \( \Gamma = \gamma \vec{r} \cdot \vec{Q} \), the free energy becomes the quadratic function of \( \gamma \). The whole problem is to compute the coefficients and verify the above solution using the 3D Laplace equation.

If they solve this problem, let them try two concentric spheres and verify that the smaller one drops from the solution (has zero \( \gamma \)). Everything looks different in 2D, but it is guaranteed to come out the same as in the 3D Laplace equation because this is a linear minimization problem with a positive non-degenerate quadratic form.

V. TOPOLOGICAL INVARIANTS

Let us now compare this system’s topology with the one discussed in our recent review paper\(^7\).

The simplest case is the one where all surfaces \( S_n \) are closed.

In that case, we can introduce Clebsch field \( \phi_1(\vec{r}), \phi_2(\vec{r}) \) as usual

\[
\vec{\omega} = \nabla \phi_1 \times \nabla \phi_2
\]

(66)

Then, the second field \( \phi_2 \) taking constant values inside each closed surface and zero outside would lead to our Ansatz with \( \Gamma = \phi_1 \Delta \phi_2 \). The values of the Clebsch field \( \phi_1 \) outside the surface drops from the equation. One can take any smooth interpolating field \( \phi_1 \) between these surfaces, and the vorticity field will stay zero outside and inside the surfaces.

Furthermore, the velocity circulation around any contractible loop at each surface vanishes because there is no normal vorticity on these surfaces.

The Clebsch topology plays no role in this case, unlike the case with open surfaces with edges \( C_n = \partial S_n \) because we considered \( \vec{r} \) in \( \Gamma(\vec{r}) \). There, the normal component of vorticity at the surface was present (and finite).

Still, our collection of closed vortex sheets in the general case has some nontrivial helicity\(^{13}\). This helicity is pseudoscalar, but it preserves the time-reversal symmetry.

The parity transformation \( P \) changes the sign of velocity, keeping vorticity invariant, whereas the time-reversal \( T \) changes the signs of both velocity and vorticity.

The helicity integral is \( T \)-even and \( PT \)-odd. It measures the knotting of vortex lines between these surfaces. (see Fig.\(^7\).) As it was noted in\(^1\) the surfaces avoid each other and themselves, so these are not just some random surfaces. This property was studied in their time evolution and recently in their statistics\(^7\).

The circulation around each contractible loop on the surface will still be zero, but the loop winding around a handle would produce a topologically invariant circulation \( n \Delta \Gamma \) for any loop winding \( n \) times.

This \( \Delta \Gamma \) is the period of \( \Gamma(\vec{z}) \) when the point \( \vec{z} \) goes around this handle\(^{13}\). This period depends upon the surface’s size and shape, but it does not change when the path varies along the surface as long as it winds around the handle.

There is a flux through the handle related to tangential vorticity inside the skin of the surface. This flux through any surface intersecting the handle is topologically invariant, and it equals to \( \Delta \Gamma \).

If we consider the circulation around some fixed loop in space, it will reduce to an algebraic sum of the circulations around all closed surfaces’ handles encircled by this loop. It will be topologically invariant when the loop moves in space without crossing any of the surfaces.

In particular, the closed vortex tube (topological torus) encircled by a fixed loop \( C \) in space would produce the
VI. VISCOSITY ANOMALY AND SCALING LAWS

Balancing the terms in the energy flow equation in the turbulent limit \( \nu \to 0, \mathcal{E} = \text{const} \) led us in [70] to new scaling laws, different from Kolmogorov scaling law.

Let us repeat these arguments now, with our new understanding of the stationary vortex sheets. We also reproduce in our solution the Gaussian profile[7] of vorticity in the viscous layer around Euler discontinuity surface.

Let us balance the powers in our equations, assuming that the width \( h \) of this layer goes to zero as some power of \( \nu \).

We could have three scaling laws

\[
\begin{align*}
\frac{d}{dh} \sim \nu^a; \\
\Phi, \vec{v}, \Gamma \sim \nu^b; \\
\int \sim \nu^c;
\end{align*}
\]

We shall find the unknown powers from the balance of energy and the equations of motion. All surfaces are finite so that the coordinates \( \vec{X} \) are not supposed to scale with viscosity.

The power balance in Navier-Stokes equation for the velocity field yields (assuming \( \nu z \to \nu' z \) where \( z \to 0 \) is the local normal coordinate to the surface and \( v_i \) is the tangent velocity near the surface, which is rapidly changing in the viscous layer)

\[
\begin{align*}
\nu \nabla^2 \vec{v}_i &\sim \nu \nabla^2 \vec{v}_i \sim \nu^{1-2\alpha+\beta}; \\
\vec{v} \cdot \nabla \vec{v}_i &\sim \nu' z \partial_z \vec{v}_i \sim \nu^{2\beta}
\end{align*}
\]

Note that the tangent derivative term \( \vec{v}_i \cdot \nabla \vec{v}_i \) has the same power counting \( \nu^{2\beta} \) as the normal derivative term we kept. We interpret these nonsingular terms as the surface’s re-parametrization, so we have to balance only the normal derivative terms.

This yield equation (with \( i = 1,2 \) corresponding to the local tangent plane, and \( z \to 0 \) to the normal coordinate)

\[
\begin{align*}
\nu \partial^2_z v_i = \nu' z \partial_z v_i; \\
\lim_{z \to \pm \infty} v_i(x,y,z) = \pm \frac{1}{2} \partial_z \Gamma(x,y)
\end{align*}
\]

with the solution found in [7]. In our terms

\[
\begin{align*}
v_i(x,y,z) &\to \frac{1}{2} \partial_z \Gamma(x,y) \text{erf} \left( \frac{z}{h \sqrt{2}} \right) + \ldots; \\
v_z(x,y,z) &\to -\frac{\nu}{h^2} z + \ldots
\end{align*}
\]

where dots stand for the smooth part at \( z \to 0 \).

The nonzero value of \( v_z(x,y,0) \) would result in the exponentially growing solution at \( \nu v_z(x,y,0) \to \infty \). Therefore, the existence of a finite solution demands the zero value of \( v_z(x,y,0) \) and finite negative value of \( \partial_z v_z(x,y,0) \), independent of the point \( x,y \) on the surface[72].

In the turbulent limit, \( h \to 0 \) error function becomes the sign function, and we recover our tangent discontinuity. In this limit, the relevant normal coordinate \( z \sim h \)
goes to zero, so this solution applies to the curved surface with \(x, y\) being the coordinates in a local tangent plane.

Comparing the powers in (73) we find the first relation
\[
\beta + 2\alpha = 1
\]  
(74)

The second relation follows from the forced energy pumping
\[
\mathcal{E} = \tilde{f} \cdot \tilde{p} \sim \nu^{\gamma + \beta};
\]  
(75)

Demanding finite \(\mathcal{E}\) we find the second relation
\[
\beta + \gamma = 0
\]  
(76)

Finally, the energy dissipation
\[
\mathcal{E} = \nu \int d^3r \tilde{\omega}^2 \sim \nu \int_{-\infty}^{\infty} dz \int_S d^2\xi \sqrt{\lambda} (\partial_x \bar{u}_i)^2
\]  
(77)

Here the normal derivatives of the velocity produce a Gaussian function of the local normal coordinate \(z\) which after integration yields
\[
\nu \int_{-\infty}^{\infty} dz \int_S d^2\xi \sqrt{\lambda} (\partial_x \bar{u}_i)^2 \propto \frac{\nu}{H^2} \int d^2\xi \sqrt{\lambda} \delta_{ij} \partial_i \Gamma \partial_j \Gamma \propto \nu^{1-\alpha + 2\beta}
\]  
(78)

Demanding a finite \(\mathcal{E}\) once again, we find the third relation
\[
1 - \alpha + 2\beta = 0
\]  
(79)

Solving these three equations we find (just as in [72]):
\[
\alpha = \frac{3}{5}; \quad \beta = -\frac{1}{5}; \quad \gamma = \frac{1}{5}.
\]  
(80a, 80b, 80c)

Note that the powers match in our exact spherical solution [65], as they should:
\[
\Lambda \sim \frac{\nu}{h} \sim \nu^{1-\alpha}; \quad \bar{u} \sim \bar{Q} \sim \frac{\tilde{f}}{\Lambda} \sim \nu^{\beta}; \quad \tilde{f} \sim \sqrt{\sigma} \sim \nu^\gamma.
\]  
(81, 82, 83)

Positive \(\alpha\) justifies our assumption of the viscous layer \(h\) shrinking to zero in the turbulent limit and the error function becoming the sign function.

Positive \(\gamma\) means that the external force goes to zero in the turbulent limit, but a large value of velocity field compensates that in energy pumping. In enstrophy, the large factor \(1/\nu\) comes from the large square of vorticity times the small width of the vorticity layer. The resulting large factor compensates the factor of \(\nu\) in front of the enstrophy leading to finite energy dissipation.

Therefore, just like in the critical phenomena, the infinitesimal external field is enhanced by a large susceptibility. The susceptibility is large due to the singular vorticity coming from large gradients of velocity in the viscous layer surrounding the Euler discontinuity surface.

This enhancement makes these vorticity sheets the dominant configuration in the turbulent limit, responsible for the energy dissipation.

The reader must have a natural question: what about the K41 scaling, which dominated the turbulence theory for half a century? It became even more complicated in the last 30 years, with multi-fractal scaling laws – nothing like simple rational indexes.

The real answer is that our scaling laws say nothing about the energy spectrum or spatial dependence of velocity/vorticity correlation functions. We addressed these issues in [72] and we found the qualitative and sometimes quantitative agreement with DNS, including the multi-fractal laws.

Moreover, these new scaling laws correspond to extreme turbulence, which is just beginning to reveal itself in DNS. We are planning large-scale DNS in collaboration with K.Iyer, to verify the predictions of instanton theory.

VII. TURBULENT STATISTICS

According to the approach to turbulent statistics that we presented in [72], the Hopf equation’s fixed point corresponds to a stationary flow with random initial data and Gaussian random force \(\tilde{f}\).

What is the random distribution of parameters in our stationary solution? After expressing the velocity discontinuities \(\Gamma\) at each surface in terms of the shapes \(\bar{X} (\xi)\) of these surfaces by minimizing the Hamiltonian, we are left with this \(\bar{X} (\xi)\) as the initial data to randomize.

On a second thought, we rather keep the quadratic Hamiltonian [26] with some large coefficient \(\beta\) (effective inverse temperature) and study this statistical system with both \(\Gamma (\xi), \bar{X} (\xi)\) fields present.

In that case, we have the Hamiltonian residual value at the stationary manifold \(\Gamma = \Gamma'' [\bar{X}]\) as an effective Hamiltonian for the surface degrees of freedom.

This approach seems like a natural choice for the effective Hamiltonian for these surface degrees of freedom, which was left arbitrary at this point. All we know so far is that there is a degenerate stationary solution of Euler dynamics involving arbitrary surfaces \(\bar{X}_n (\xi)\) and corresponding \(\Gamma_n\) minimizing our Hamiltonian.

We expect this solution to be unstable, but this is the whole point of the Gibbs distribution. For arbitrary ini-
tial values of these surfaces, assuming stationary values of $\Gamma^*$, we would have the surfaces evolve while preserving their topology and avoid each other and themselves. Eventually, they cover some manifold corresponding to a fixed point of the Hopf equation.

On the other hand, we know that Gibbs distribution $\exp\left(-\beta H[\Gamma, \bar{X}]\right)$ represents a stable fixed point of the Hopf equation.

We do not know whether this $\beta$ should tend to infinity. The Gibbs distribution solves the Hopf equation for arbitrary $\beta$.

The thermodynamics of the laminar fluid (large viscosity limit) involves different $\beta$, and in that case, there are no vortex sheets, and the description in terms of Wylde functional integral with weakly fluctuating velocity field would be an appropriate approach.

It is just the turbulent (strong coupling) phase with vortex sheets where the Gibbs distribution with energy conservation in our dynamics. The time derivative $\partial_t H[\Gamma, \bar{X}]$ fixes energy pumping and energy dissipation in the face is self-avoiding $V$

Another new aspect is the requirement that this surface $\Gamma$ be inside the closed surfaces, conserved in Euler dynamics. This gives a term $cV[\bar{X}]$ in the exponential, where

$$V[\bar{X}] = \frac{1}{3} \sum_n \int_{S_n} d^2\xi \bar{X} \cdot \epsilon_{ij} \partial_i \bar{X} \times \partial_j \bar{X};$$

We arrive at a slightly modified Gibbs canonical ensemble

$$Z(\beta, a, b, c, \bar{f}) = \int D\Gamma D\bar{X} \exp\left(-\frac{\bar{f}^2}{2\sigma} - \beta H^{eff}[\Gamma, \bar{X}]\right);$$

$$H^{eff}[\Gamma, \bar{X}] = H[\Gamma, \bar{X}] + a \bar{P}[\Gamma, \bar{X}] \cdot \bar{f} + b E[\Gamma, \bar{X}] + c V[\bar{X}];$$

Note that this distribution is Gaussian in terms of $\Gamma$ with quadratic part being essentially a 2D free massless field kinetic energy $E[\Gamma, \bar{X}]$. The new soft field appearing from the viscosity anomaly and dominating the turbulent statistics represents the main result of this work.

The nonlocal interaction is provided by (26), (35).

One can remove this nonlocal interaction by introducing a vector Gaussian field

$$\exp\left(-\beta H[\Gamma, \bar{X}]\right) \propto \int D\bar{\Psi} \exp\left(-\beta H_1[\Gamma, \bar{\Psi}, \bar{X}]\right)$$

$$H_1[\Gamma, \bar{\Psi}, \bar{X}] = \frac{1}{2} \int d^3r (\bar{\nabla} \bar{\Psi})^2 + \frac{1}{8} \sum_n \int_{S_n} d\bar{\Sigma}^a \bar{\Psi}^a;$$

Now we have a local theory of a free 2D field $\Gamma$ interacting with free vector 3D field $\bar{\Psi}$. The only nonlinear part is the interaction with the surface field $\bar{X}$.

This functional integral is well defined as the effective Hamiltonian $H^{eff}[\Gamma, \bar{X}]$ grows at large surfaces as well as large $\Gamma$. The Hamiltonian $H[\Gamma, \bar{X}]$ as well as momentum...
\( \hat{P}[\Gamma, \vec{X}] \) are manifestly stationary in our dynamics with action (17), and so is the Liouville measure \( D\Gamma DV[\vec{X}] \).

The energy dissipation \( E[\Gamma, \vec{X}] \) seems to present a problem as it is not stationary in our dynamics. However, it is stationary on a stationary manifold, corresponding to the minimum by \( \Gamma \). This fact follows from its parametric invariance. As we have seen, the surfaces \( \vec{X}_n \) are stationary up to the re-parametrization of coordinates on this manifold \( \Gamma = \Gamma^*[\vec{X}] \), therefore the parametric invariant functional \( E^*[\vec{X}] = E[\Gamma^*[\vec{X}], \vec{X}] \) is also stationary.

The Gaussian functional integral \( \int D\Gamma \) involved in the partition function reduces to

\[
Z(\beta, a, b, c) \propto \int DV[\vec{X}] \exp \left( -\beta H_{eff}[\Gamma^*[\vec{X}], \vec{X}] \right) \sqrt{\det Q} , \quad (97)
\]

\[
\dot{Q}_{nm}(\xi, \eta) = \frac{\delta^2 H_{eff}}{\delta \Gamma_n(\xi) \delta \Gamma_m(\eta)} ; \quad (98)
\]

Here everything is already stationary, including the functional determinant of \( Q \), which is independent of \( \Gamma \) and is parametric invariant functional of \( \vec{X} \).

Now, here is an important detail, which we did not mention so far. As we estimated, the turbulent limit of small viscosity at a fixed energy flow \( \dot{E} \) corresponds to large values of \( \Gamma^* \propto L^{-\frac{1}{2}} \propto \nu^{-\frac{1}{3}} \to \infty \).

On the other hand, the deviations \( \delta \Gamma = \Gamma - \Gamma^* \) are controlled by our effective temperature \( \delta \Gamma \sim \beta^{-\frac{1}{2}} \ll \Gamma^* \).

We have a WKB situation, where the fluctuations are small compared to the background variable, minimizing the Hamiltonian.

In the zeroth approximation we can neglect these fluctuations and we arrive at the effective \( \beta_{eff} \sim \beta \nu^{-\frac{3}{2}} \to \infty \).

This happens because the leading term in the Hamiltonian \( H[\Gamma^*, \vec{X}] \sim \nu^{-\frac{3}{2}} \). Therefore, our effective temperature in the Gibbs distribution goes to zero as \( T_{eff} = 1/\beta_{eff} \sim \nu^{\frac{3}{2}} \to 0 \).

The temperature will become small only in the limit of ultrahigh Reynolds number \( Re \sim \Gamma/v \sim \nu^{-\frac{3}{2}} \). We have to wait until \( \beta_{eff} \sim Re^{\frac{1}{3}} \) becomes large.

This relation between the effective temperature of the turbulent vortex sheet and Reynolds number perfectly matches the empirical fit of the data, suggesting an analogy between 3D turbulence and spin waves in 2D.

We interpret these observations as confirmation of our low-temperature Gibbs statistics, with \( \delta \Gamma \) playing the role of spin waves of the 2D XY model.

At low temperatures, the saddle point in \( \vec{X} \) (ground manifold) serves as the zeroth approximation. The optimal configuration for the closed surface would be a sphere, and for the open ones bounded by fixed loops, it would be a minimal surface\(#20\).

In the low-temperature expansion, the fluctuations around these minimal surfaces \( \delta \vec{X} \sim \sqrt{T_{eff}} \sim \nu^{\frac{1}{2}} \) are small, so the conditions of self-avoiding are not used.

With Gaussian approximation, we shall have a standard perturbation expansion around the ground manifold.

This expansion would involve some technical work beyond our present scope. However, let us stress that this perturbation expansion is opposite to the one in the Wylde functional integral. In our dual theory of random surfaces, we are working in the perturbative phase, corresponding to the non-perturbative phase of the original theory of fluctuating velocity field.

VIII. DISCUSSION

Mikhail Bulgakov wrote these words about exposing black magic in his immortal novel “The Master and Margarita”\(#21\): “An explanation is essential, otherwise your brilliant act will leave a painful impression. The audience demands an explanation . . .’

In the novel, the black magic was real, but the explanation was a mockery of a stupid Soviet bureaucrat demanding Satan to expose his magic.

Let us try to expose as many tricks as possible without killing the real magic.

We have already made an unsuccessful attempt in the 90-ties to represent turbulence as Gibbs statistics of vortex cells. We used in this paper many of the ideas of that approach. What is the difference?

The short answer is the viscosity anomaly of vortex sheets. The 3D vortex cells do not explain the persistent energy dissipation in the limit of vanishing viscosity in the Navier-Stokes equation.

The vortex sheets represent a particular case of vortex cells, but the singular vorticity of these sheets is the only explanation of the viscosity anomaly – with vorticity spread over the 3D cell, this would not happen.

With vortex cells, we could not derive the effective Hamiltonian of the Gibbs distribution from the first principles such as the Navier-Stokes equation, so we had to assume some generic terms in the effective Hamiltonian from the symmetry principle in a low gradient limit.

In this paper, we do not invent the Hamiltonian but simply use the existing one after adding constraints for an energy flow/dissipation.

Another essential ingredient is the manifold of the Hamiltonian’s stationary solutions, which we did not know at that time. The vortex sheet dynamics is highly degenerate, and this is the origin of statistical distribution in a deterministic mechanical system.

There is a stationary manifold \( \mathcal{G} : \Gamma = \Gamma^*[\vec{X}] \), parametrized by arbitrary set of surfaces \( \vec{X} \). As we discussed above, the stationary manifold satisfies the linear integral equation corresponding to the Hamiltonian’s minimization in terms of \( \Gamma_n \) with two more conditions: fixed energy pumping and the same energy dissipation.

There is also a ground manifold corresponding to the Hamiltonian’s degenerate minimum by \( \Gamma, \vec{X} \) variables.
This ground manifold is parametrized by scalar functions \( f(\xi) \) on each separate surface.

The Hamiltonian value at the ground manifold does not depend upon these functions \( f(\xi) \). They only affect the tangent flow \( \vec{v}(\vec{r}) \) at the surface. The normal velocity vanishes at the surface, and the tangent velocity goes along the line of constant \( \Gamma \), with coefficient proportional to \( f(\xi) \).

These \( f(\xi) \) represent part of surface’s re-parametrization, and reflect gauge invariance of the underlying vortex dynamics. One can fix this gauge to, say \( f(\xi) = 1 \), as the surface re-parametrization is unobservable anyway. Any correlation function like

\[
\left\langle (\vec{v}(\vec{r}) - \vec{v}(\vec{0}))^n \right\rangle
\]

or the loop average:

\[
W_C(\gamma) = \exp \left( i\gamma \oint_{\gamma} \vec{v} d\vec{r} \right)
\]

will come out parametric invariant.

Moreover, this circulation is topologically invariant, as we discussed earlier (see Fig. 2, Fig. 3). It reduces to the sum of periods of the multivalued function \( \Gamma \) on the handles \( \mathcal{H}(C) \) linked with the loop \( \mathcal{C} \).

\[
\oint_{\mathcal{C}} \vec{v} d\vec{r} = \sum_{h \in \mathcal{H}(C)} \Delta_h \Gamma
\]

The computation of the Gaussian integral over \( \Gamma \) for the loop average yields

\[
W_C(\gamma) = \exp \left( i\gamma \sum_{h} \Delta_h \Gamma^* - \frac{\gamma^2}{2\beta} \sum_{h,h'} \Delta_h \Delta_{h'} \hat{Q}^{-1}(h,h') \right) \]

Here the last averaging goes over the surface \( \hat{X} \) with weight \( \hat{W} \), and \( \Delta_h F \) stands for the function’s period on \( \mathcal{C} \) linked with \( h \).

At fixed surfaces \( \hat{X} \) this would be the Gaussian function of \( \gamma \), which would result in the Gaussian distribution of circulation (Fourier transform of \( W_C(\gamma) \)).

This solution is different from the instantons considered in [7], where there was some normal flux through the minimal surface, resulting in the exponential PDF for circulation.

There is no physical reason to choose \( \beta = \infty \), so it must remain one of the theory’s parameters. We go one step forward from the program of [7] where we suggested to use stationary flow with randomized parameters as turbulent statistics.

This new view of the turbulent statistics as a particular case of the Gibbs statistics saves us the necessity of studying the Lyapunov stability. Gibbs distribution status as a stable fixed point of the Hopf equation is reasonably well justified.

The ambiguity in the definition of the measure also goes away. According to Liouville’s theorem, the Gibbs distribution’s unique conserved measure is the symplectic measure in the phase space \( dt dV \). After Gaussian integration over \( \Gamma \), we have an additional factor \( (\det Q)^{-\frac{1}{2}} \)

in the distribution over the surfaces. This factor leads to nonlocal interactions between surfaces, in addition to the stationary value of the effective Hamiltonian.

The Kelvin-Helmholtz instability does not contradict the Gibbs distribution; it instead explains this distribution.

Without any stable configuration of surfaces in the stationary manifold, these surfaces’ dynamics would uniformly cover this manifold with the Gibbs measure. Otherwise, the system could get stuck in such a stable configuration.

There is no question that Gibbs statistics exist; the question always was – does it apply to turbulence? As we have just demonstrated, in the case of vortex sheets, it can describe the energy pumping by adding a constant Gaussian force interacting with the fluid momentum and energy dissipation via viscosity anomaly.

It required us to take a more general view of Gibbs distribution. The Gibbs distribution’s general philosophy is that all conserved quantities become distributed due to exchanging them with the thermostat.

In our case, the conserved quantities are functions \( \Gamma(\xi) \) for each surface. These conservation laws are broken by viscosity in the full Navier-Stokes dynamics.

These viscous effects smear velocity discontinuities and lead to a redistribution of \( \Gamma \), reconnection of surfaces, surface splitting, and other topological violations. However, the nature of these small interactions with the thermostat does not affect the resulting Gibbs distribution, which is still the fixed point of the Hopf equation for the ideal fluid.

For example, the ideal gas has all particle momenta conserved but ends up with Boltzmann distribution, thanks to the small interactions working behind the scenes to exchange momenta between particles and distribute them according to the Boltzmann law.

The beauty of the Gibbs statistics is that it does not depend on the actual dynamics and, as a consequence, the interactions leading to the equilibrium do not enter in the final Gibbs distribution for the ideal system. These interactions are all hidden in the temperature or, in general, in Lagrange multipliers for various external fields.

In the same way here, we expect the conserved discontinuity density \( \Gamma \) (our analog of the conserved momentum of free particles) to become distributed by the Gibbs distribution for an ideal fluid. Viscosity only displays itself in the energy dissipation, which is enhanced by vorticity singularities to persist in the turbulent limit.

We see no contradiction in keeping this turbulent temperature \( \beta \) finite in light of these arguments. As is well known, the Gibbs distribution is valid for any mechanical system, including an ideal fluid.

There is a folklore belief that the standard 3D Gibbs
distribution of the velocity field with Hamiltonian $\int \frac{1}{2} \nabla^2 \vec{v}$ does not describe turbulence – it has short-range velocity correlations and no sign of large vorticity structures.

We see here that for the subset of Euler dynamics – that of infinitely thin vortex sheets – the Gibbs distribution is very rich even at zero temperature, and it has the features needed for turbulent statistics, once we understand how to pump in the energy and how it dissipates.

In terms of modern QFT lingo, this is the duality – a description of a fluctuating vector field as a fluctuating geometry, with the strong coupling phase of one theory corresponding to another’s weak coupling phase.

We suggested this duality already in the old work but at that time, we did not dare to suggest the Gibbs distribution for turbulence. The viscosity anomaly was the missing clue.

We noted back then that these surfaces were self-avoiding and, as such, they may exist in three dimensions, where the free random surfaces cannot.

We explored this idea in a recent work where we conjectured the relation of vorticity correlation functions to the averages of the product of vertex operators in the Liouville theory with unknown parameters $a, Q$.

The Liouville field $\varphi(\xi)$ arises as an internal metric $g_{ij} = \delta_{ij} e^{a \varphi}$ on the surface. This internal degree of freedom reflects the random tangent motion of fluid along the surface without preserving the area, which is related to $\int d^2 \xi \sqrt{\det g} \varphi \propto \int d^2 \xi e^{a \varphi}$.

We are not trying to explain these heuristic connections here; one could find more details in that paper, including the multi-fractal scaling law derived from Liouville conjecture, in reasonable agreement with DNS.

Finally, there are exciting possible applications of our method to quantum turbulence in superfluid helium. The action principle opens the way to quantization. As we already mentioned in the old work, the action is multivalued, which leads to the quantization $\Delta \overline{\Gamma} \Delta V = 2\pi i \hbar$ in units of Planck’s constant.

It came to our attention that the vortex sheets were extensively studied in superfluid helium under the name of domain walls. The instantons of our paper correspond to the so-called KLS domain wall edged by the Alice string first suggested in early Universe cosmology (see for a recent review) and then applied to superfluid helium, where they were observed in real experiments.

In a recent review their topological stability was extensively studied. This review contains complete references to previous theoretical and experimental work. As a consequence of topological stability, we conclude that the winding number $n$ in our instanton must be half-integer rather than integer as we assumed, so that the $XY$ component of the $S_2$ vector $\vec{S}$ in Clebsch variables changes the sign when crossing the vortex sheet.

Nobody suspected, although, that such exotic animals can live in a water faucet!

**IX. CONCLUSION**

This work’s initial goal was to study the stationary discontinuity surface in Euler dynamics with viscosity anomaly and exactly solve these equations in some symmetric cases, which we did.

In doing so, we stumbled upon a very general observation that one can obtain turbulent statistics from the Gibbs distribution corresponding to the vortex sheet dynamics. One should add three extra global constraints: energy pumping, energy dissipation, and the volume inside closed sheets.

An infinite number of Euler conservation laws make this Gibbs distribution nontrivial. Instead of a stationary state, we have a stationary manifold (minimum of the effective Hamiltonian by $\Gamma$ at fixed $\vec{X}$), characterized by a collection of self-avoiding surfaces.

The ground manifold (absolute minimum of the effective Hamiltonian for both $\Gamma, \vec{X}$) corresponds to minimal surfaces, with low-temperature expansion corresponding to small fluctuations around these surfaces as well as fluctuations of $\Gamma$ around the stationary solution.

For the closed surface with fixed volume, the minimal shape is a sphere, and for the ones bounded by a stationary loop $C$ these are the well-known soap films.

In a turbulent limit of vanishing viscosity at fixed energy flow, we confirm the scaling laws recently obtained in the context of stationary flow.

As a result, we arrive at a particular distribution of discontinuity surfaces in the turbulent statistics. Effective temperature goes to zero in the turbulent limit as $\nu^{7/6}$, so that the low-temperature expansion around the ground manifold would be appropriate.

This theory leaves aside more complex vortex sheets with some normal vorticity present on the discontinuity surface. We are planning to reconcile these solutions soon.

Much technical work is left to elaborate this low-temperature expansion to the level usable in engineering applications. We leave this pleasant work for future young researchers, and it will take some string theory expertise plus some knowledge of statistical physics.

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**DATA AVAILABILITY**

Data sharing does not apply to this article, as no new data were created or analyzed in this study.
The constant symmetric traceless tensors in front of the components' evolution in three dimensions," Journal of Computational Physics 35, 2030018 (2020) arXiv:2007.12489v7 [hep-th].

By means of re-parametrization, we can reduce the internal metric on a surface to a conformal metric $g_{ij} = \hat{g}_{ij} \hat{e}^\varphi$ with $\hat{g}_{ij}$ being standard metric for this genus and $\varphi$ is some local field on a surface. This is so-called Liouville field, which as we argue is responsible for the multi-fractal scaling laws in turbulence.

I could not find anywhere online, but it exists in university libraries such as Princeton University Library and U.C. Berkeley Library.

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This is not the most general potential solution, of course. One can add higher polynomials, such as $B_{\mu \nu} t_\rho$ with some constant symmetric traceless tensor $B_{\mu \nu}$. We restricted ourselves to a constant velocity to give an example of the stationary Euler flow with energy pumping and dissipation. In the general case, the inside velocity would be a polynomial of $\vec{r}$, restricted by the incompressibility conditions. The constant symmetric traceless tensors in front of the components' product would become the minimization parameters.

We carefully verified this fact for $N$ concentric spheres with spherical Ansatz $\Gamma_n = \gamma_n Q \cdot \vec{r}$. The minimum of free energy requires all $\gamma_n = 0$ except for the largest sphere.

In this case, due to the symmetry, the constant velocity $\vec{v}(\vec{r}) = a \vec{Q}$ inside the sphere is the most general solution, linear in $\vec{Q}$.

By the way, let us notice that $\partial_\nu v_\nu = -\gamma \vec{Q}$ is constant on the surface, the same one at both sides in spite of the tangent discontinuity. Therefore, the surface velocity divergence $\vec{V}_1 \cdot \vec{v}_1 = +\frac{\gamma}{\rho_0} \vec{Q}$ is the same constant on both sides of the surface. This means that the internal area (amount of fluid on the surface) is the same on both sides, although it is not conserved. We believe that these area fluctuations are described by an internal metric on the surface, represented by Liouville field in a conformal gauge. This is a pure speculation at this point.

Let us stress that we are working inside the viscous layer, at its core $z = 0$. Therefore, this value of the slope $\partial_z v_\nu(x,y,0)$ does not apply to the Euler velocity. The slopes of normal velocities at two sides of the discontinuity surface in the Euler equation can be different and maybe positive, but they only have to match and be negative in the Navier-Stokes equation deep inside the vorticity layer.

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