String Holonomy and Extrinsic Geometry in
Four-dimensional Topological Gauge Theory

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Abstract

The most general gauge-invariant marginal deformation of four-dimensional abelian
$BF$-type topological field theory is studied. It is shown that the deformed quantum field
theory is topological and that its observables compute, in addition to the usual linking
numbers, smooth intersection indices of immersed surfaces which are related to the Euler
and Chern characteristic classes of their normal bundles in the underlying spacetime
manifold. Canonical quantization of the theory coupled to non-dynamical particle and
string sources is carried out in the Hamiltonian formalism and explicit solutions of the
Schrödinger equation are obtained. The wavefunctions carry a one-dimensional unitary
representation of the particle-string exchange holonomies and of non-topological string-
string intersection holonomies given by adiabatic limits of the worldsheet Euler numbers.
They also carry a multi-dimensional projective representation of the deRham complex
of the underlying spatial manifold and define a generalization of the presentation of its
motion group from Euclidean space to an arbitrary 3-manifold. Some potential physical
applications of the topological field theory as a dual model for effective vortex strings
are discussed.

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1 Introduction

Topological quantum field theories [1, 2] have many applications in both physics and mathematics (see [3] for a review). They are characterized by the fact that their partition function and observables are independent of the metric of the manifold on which they are defined and therefore yield topological invariants of the underlying spacetime. In some cases they constitute the effective quantum field theory of physical models where topological phenomena, for example generalized Aharonov-Bohm effects which arise from adiabatic transports of objects around one another, play a significant role. They also provide interesting connections between various seemingly disconnected branches of physics and mathematics, the classic example being the interrelation between knot theory, integrable models and conformal field theory [4].

In this paper we will study a class of Schwarz-type topological gauge theories [1, 3]. One of the most widely studied examples of such theories is given by the abelian Chern-Simons action [1, 3, 4]

\[ S_{CS}[A] = \int_{\mathcal{M}_3} A \wedge F_A \] (1.1)

where \( A \) is a \( U(1) \) gauge connection of a complex line bundle over a 3-manifold \( \mathcal{M}_3 \) and \( F_A \) is its curvature. The quantum field theory defined by (1.1) is strictly renormalizable and has appeared in a variety of physical applications ranging from string theory [5] to condensed matter physics [6, 7]. It provides a phenomenological realization of anyons (see [7, 8] for reviews), i.e. particles in \((2 + 1)\)-dimensions with fractional exchange statistics. Its observables yield linking numbers of embedded curves in \( \mathcal{M}_3 \) [3, 9] while those of its nonabelian generalizations are related to polynomial invariants of knots and links embedded in 3-manifolds [4].

A four-dimensional generalization of the topological field theory (1.1) is defined by the abelian \( BF \) action [1, 3, 10]

\[ S_{BF}[B, A] = \int_{\mathcal{M}_4} B \wedge F_A \] (1.2)

where \( B \) is a 2-form field defined on a 4-manifold \( \mathcal{M}_4 \) and \( F_A \) is defined as in (1.1). The corresponding partition function is related to the Ray-Singer analytic torsion [1, 3] which is a topological invariant of \( \mathcal{M}_4 \), and its observables compute the linking numbers of embedded curves and surfaces in 4-manifolds [3, 10]. It has been discussed in connection with a wide variety of physical systems which involve vortex-like configurations such as superconductors [11], cosmic strings [12], and axionic black holes [13]. Recent interest in the nonabelian generalizations of this model has come from its role as a dual model for quantum chromodynamics (QCD) whereby the exchange holonomies are relevant to the quark confinement problem [14].

The construction of explicit physical states of the quantum field theory (1.2) which exhibit particle-string “fractional statistics” was carried out in [15]. The physical relevance of these
holonomies in string theory is discussed in [16], and for Nielsen-Olesen strings in abelian Higgs models in [17]–[19]. In addition to their applications in condensed matter physics, these latter models have also been relevant to properties of the confining QCD string [20] and to the problem of baryogenesis in electroweak theory [21].

Unlike Chern-Simons theory, the quantum field theory defined by the BF action (1.2) is not stable under renormalization. Although it is well established that BF quantum field theories are ultraviolet finite in certain gauges [22], it is still natural to examine the physical and mathematical properties of the theory obtained by perturbing the action (1.2) by gauge-invariant marginal (or irrelevant) operators. Such deformations can be thought of as perturbing the quantum field theory (1.2) to an isomorphic one in the associated moduli space. In the following we will study the modification of this theory which is obtained by renormalizing it via the addition of all truly marginal local operators to the model (1.2). The renormalized model is a deformation of (1.2) by a non-topological, explicitly metric-dependent counterterm, but, as we demonstrate, the resulting action still defines a topological field theory. This is similar to the usual situation in a topological gauge theory, where the gauge-fixing couples the quantum action to the spacetime metric. Nonetheless, the theory is topological since the energy-momentum tensor is BRST-exact and therefore has vanishing matrix elements in physical states [3], i.e. there are no classical propagating degrees of freedom. The renormalized field theory will therefore work to describe the holonomy effects which occur in adiabatic transport in a theory of point charges and strings just as well as Chern-Simons theory describes anyons.

The crucial effect of the renormalized BF field theory is that its observables yield, in addition to the usual topological linking numbers of embedded curves and surfaces in \( \mathcal{M}_4 \), an effectively computable representation of a novel intersection number of surfaces immersed in the manifold. This quantity is only a smooth invariant of the surfaces and is related to the geometry of their normal bundles in \( \mathcal{M}_4 \). It can be expressed in terms of the extrinsic geometry of the surfaces and also the Euler and Chern characteristic classes of the normal bundles. We shall study this invariant in detail in both an effective field theory formalism and in the context of canonical quantization. In the former approach we will see that the effective action contains the action for vortex strings with rigid extrinsic curvature term and Polyakov \( \theta \)-term. It therefore serves as a dual model for the effective field theory of the QCD string (and other vortex-like configurations), but in a much different manner than the usual non-topological dual models of QCD do [14]. Some properties of these strings then become quite transparent when viewed in this dual formalism. A similar sort of relationship was established in [23], where the coupling of dynamical point particles to nonabelian BF gauge fields in two-dimensions was considered and related to two-dimensional extended Poincaré gravity.
In the canonical formalism we will find an adiabatic limit of these intersection numbers which in turn defines an adiabatic representation for the Euler numbers of the associated normal bundles. The wavefunctions then carry, in addition to the usual particle-string holonomies, a one-dimensional unitary representation of a sort of non-topological string-string holonomy. These holonomies give interesting representations of the extrinsic geometry of the string worldsheets and could have applications in the aforementioned models. By an explicit construction of the physical state wavefunctions, we show directly that the physical Hilbert space is finite-dimensional and recover all of the properties of the states of ordinary BF theory \([15]\), but in a more symmetric representation with respect to particle and string degrees of freedom which also naturally provides a representation of the secondary gauge constraints of the theory \([12,13]\). These properties include a multi-dimensional projective representation of the deRham complex of the underlying spatial 3-manifold and of its associated motion group (the generalization of the braid group in Chern-Simons field theory). The latter feature was described briefly in \([15]\) and here we shall expand somewhat on the properties of this representation. In particular, we find that the BF theory naturally defines the extension of the motion group presentation from \(\mathbb{R}^3\) to an arbitrary 3-manifold.

The organisation of this paper is as follows. In section 2 we introduce the perturbed BF field theory and establish its topological properties. In section 3 we examine the invariants represented by the effective field theory when the gauge fields are coupled to non-dynamical particle and string sources, and describe the potential physical applications of this effective model to theories of vortex strings. In section 4 we describe the canonical structure and reduced phase space of the theory, taking into careful account the reducible gauge symmetries that BF field theories possess. In section 5 we construct the physical state wavefunctions which solve the gauge constraints and the Schrödinger equation and develop the adiabatic representations of the topological linking numbers, the extrinsic intersection indices and the Euler numbers. In section 6 we describe in detail the transformation properties of the physical states under gauge transformations and adiabatic transports. We also explicitly construct a multi-dimensional representation of the motion group which is valid for arbitrary 3-manifolds.

2 Deformed BF Field Theory

Consider a real-valued 1-form field \(A\) and a real-valued 2-form field \(B\) defined on a closed orientable four-dimensional spacetime manifold \(M_4\) with metric \(g\) of Minkowski signature. These forms can take values in some flat vector bundle over \(M_4\). They have the abelian gauge transforms

\[
A \rightarrow A + \chi, \quad B \rightarrow B + \xi
\]

(2.1)
where $\chi$ is a closed 1-form and $\xi$ is a closed 2-form, $d\chi = d\xi = 0$. The $A$ field minimally couples to the particle current

$$j^\mu(x) = \sum_a q_a \int_{L_a} dl^\mu(r_a) \delta^{(4)}(x, r_a(\tau))$$

where

$$dl^\mu(r_a) = d\tau \frac{dr_\mu^a(\tau)}{d\tau}$$

is the differential particle worldline element and $r_\mu^a(\tau)$ is the imbedding of the worldline $L_a$ of particle $a$ with charge $q_a$ in $M_4$. It has dimension 3 and satisfies the continuity equation $\partial_\mu j^\mu = 0$ when the worldlines are closed. The $B$ field couples minimally to the antisymmetric string current

$$\Sigma^{\mu\nu}(x) = \sum_b \phi_b \int_{\Sigma_b} d\sigma^{\mu\nu}(X_b) \delta^{(4)}(x, X_b(\sigma))$$

where

$$d\sigma^{\mu\nu}(X_b) = d^2\sigma \epsilon^{\alpha\beta} \frac{\partial X^\mu_b(\sigma)}{\partial \sigma^\alpha} \frac{\partial X^\nu_b(\sigma)}{\partial \sigma^\beta}$$

is the differential string worldsheet area element and $X^\mu_b(\sigma)$ is the imbedding of the worldsheet $\Sigma_b$ of string $b$ with electromagnetic flux $\phi_b$ in $M_4$. It has dimension 2 and obeys the conservation law $\partial_\mu \Sigma^{\mu\nu} = 0$ when the worldsheets are closed. We do not distinguish between the worldlines and worldsheets and their imbeddings in $M_4$, which we assume to be disjoint, $L_a \cap \Sigma_b = \emptyset$.

A strictly renormalizable local field theory constructed from these fields must include all gauge-invariant operators of dimension 4 or less. The action is then

$$S[B, A; \Sigma, j] = \int_{M_4} d^4x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \kappa \frac{1}{4\pi} \epsilon^{\mu\nu\lambda\rho} B_{\mu\nu} \partial_\lambda A_\rho + \frac{\theta}{4\pi} \epsilon^{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho} + A_\mu j^\mu + \frac{1}{2} B_{\mu\nu} \Sigma^{\mu\nu} \right)$$

where $F = dA$ is the field strength of $A$ and we use the convention $\epsilon^{0123} = +1$. The first term in (2.6) is the usual Maxwell term for the gauge field $A$ while the second term is the action (1.2) of topological $BF$ theory. The third term is the usual topological action of four-dimensional Yang-Mills theory and is a total derivative. It is non-trivial only on spacetimes with non-contractible loops and it does not appear in the classical equations of motion.

The field equations which follow from the action (2.6) are

$$\frac{k}{4\pi} \epsilon^{\mu\nu\lambda\rho} F_{\lambda\rho} + \Sigma^{\mu\nu} = 0, \quad -\partial_\nu F^{\mu\nu} + \frac{k}{4\pi} \epsilon^{\mu\nu\lambda\rho} \partial_\nu B_{\lambda\rho} + j^\mu = 0$$

\footnote{Here and in the following we will not write explicit metric factors required to make all terms generally covariant.}
The first field equation in (2.7) confines electromagnetic flux to the string worldsheets and gives the analog of the Meissner effect in a BCS superconductor. The solutions of these field equations in a covariant gauge
\[ \partial^\mu A_\mu = \partial^\mu B_{\mu\nu} = 0 \]
are
\[ A_\mu = -\frac{2\pi}{k} \epsilon_{\mu\nu\lambda\rho} \partial^\nu \Sigma^{\lambda\rho}, \quad B_{\mu\nu} = \frac{16\pi^2}{k^2} \Sigma_{\mu\nu} - \frac{4\pi}{k} \epsilon_{\mu\nu\lambda\rho} \partial^\lambda j^\rho \]
where in this section and the next we shall ignore harmonic zero modes on \( \mathcal{M}_4 \). Substituting (2.8) into (2.6), we see that the effective classical action is
\[ \Gamma[\Sigma, j] = \int_{\mathcal{M}_4} d^4x \left( \frac{4\pi^2}{k^2} \Sigma_{\mu\nu} \Sigma^{\mu\nu} - \frac{4\pi\theta}{k^2} \epsilon_{\mu\nu\lambda\rho} \Sigma^{\mu\nu} \Sigma^{\lambda\rho} - \frac{2\pi}{k} j^\mu \epsilon_{\mu\nu\lambda\rho} \partial^\nu \Sigma^{\lambda\rho} \right) \] (2.9)

The effective action (2.9) shows that there are no propagating degrees of freedom in this theory and therefore the action (2.6) essentially defines a topological field theory. Unlike other topological field theories, however, the action (2.6) explicitly couples to the spacetime metric.

Nonetheless, from (2.7) and (2.1) it follows that, in the source-free case, the space of classical solutions of the field theory (2.6) is the finite-dimensional vector space \( H_1(\mathcal{M}_4) \oplus H_2(\mathcal{M}_4) \) which coincides with that of the \( BF \) field theory (1.2) (\( H^k(\mathcal{M}_4) \) is the deRham cohomology group of \( k \)-forms on \( \mathcal{M}_4 \) with values in a flat vector bundle). The trace-less gauge-invariant symmetric energy-momentum tensor is
\[ T_{\mu\nu} \equiv \frac{\delta S}{\delta g^{\mu\nu}} \bigg|_{j=\Sigma=0} = \frac{1}{4} g_{\mu\nu} F_{\lambda\rho} F^{\lambda\rho} - F_{\mu\lambda} F^{\nu\lambda} \]
and it vanishes when restricted to flat gauge connections (i.e. classical field configurations). It is also possible to prove that the quantum field theory defined by (2.6) is topological. For this, we regard the \( F^2 \) terms in the source-free action (2.6) as a deformation \( \Delta \) of the topological \( BF \) theory (1.2),
\[ S[B, A; 0, 0] = (k/4\pi) S_{BF}[B, A] + \Delta[A] \] (2.11)

Consider the local gauge symmetries (2.1) which are parametrized by smooth functions \( \chi' \) with \( \chi = d\chi' \) and 1-forms \( \xi' \) with \( \xi = d\xi' \). The gauge transformations are then
\[ \delta_{\chi'} A = d\chi', \quad \delta_{\chi'} B = 0 \]
\[ \delta_{\xi'} A = 0, \quad \delta_{\xi'} B = d\xi' \] (2.12)

For the \( BF \) field theory (1.2), this symmetry is off-shell reducible, and, therefore, in addition to the usual ghost fields required for gauge-fixing, there are ghost-for-ghost fields due to the secondary gauge invariance
\[ \delta_{\chi''} \xi' = d\chi'' \]
(2.13)

To gauge-fix the invariances (2.12), we introduce as usual a Faddeev-Popov 0-form ghost field \( c \) of ghost number 1 for \( A \), and a 1-form ghost field \( c_1 \) of ghost number 1 for \( B \). The
gauge-fixing of (2.13) is then achieved by introducing an additional 0-form ghost $c_0$ with grading 2. One can now introduce the usual Stuckelberg fields and Faddeev-Popov anti-ghost fields to write down the gauge-fixed quantum action corresponding to (1.2) [3, 10]. Here we shall work instead in the Batalin-Vilkovisky antifield-antibracket formalism [3, 24, 25]. For each set of fields $(A,c)$ and $(B,c_1,c_0)$ we introduce the corresponding set of antifields $(A^*,c^*)$ and $(B^*,c_3^*,c_4^*)$ of dual form degrees $(3,4)$ and $(2,3,4)$ and ghost numbers $(-1,-2)$ and $(-1,-2,-3)$, respectively. The fields and antifields together generate the BRST symmetry of the gauge-fixed topological field theory. The classical theory is recovered by setting all antifields to 0. The off-shell nilpotent BRST algebra is represented as [24, 25]

\[
\{Q, c_4^*\}_+ = -dc_3^* \quad \{Q, c^*\}_+ = -dA^* \\
\{Q, c_3^*\}_+ = -dB^* \quad \{Q, A^*\}_+ = -dB \\
\{Q, B^*\}_+ = -dA \quad \{Q, B\}_+ = -dc_1 \\
\{Q, A\}_+ = -dc \quad \{Q, c_1\}_+ = -dc_0 \\
\{Q, c\}_+ = 0 \quad \{Q, c_0\}_+ = 0
\]

(2.14)

where

\[
Q = \int_{\mathcal{M}_4} \left( B \wedge dA + A^* \wedge dc + B^* \wedge dc_1 + c_3^* \wedge dc_0 \right)
\]

is the metric-independent BRST supercharge in the field-antifield representation. The bracket in (2.14) denotes the graded antibracket acting in the $\mathbb{Z}$-graded algebra of functionals of the fields and antifields.

From (2.14) it follows that the deformation in (2.11) can be written as

\[
\Delta[A] = \int_{\mathcal{M}_4} \{Q, \Psi\}_+
\]

(2.16)

where

\[
\Psi = B^* \wedge \left( -\frac{1}{4} \star F + \frac{g}{4\pi} F \right)
\]

(2.17)

is a gauge fermion field. Here $\star$ denotes the Hodge duality operator which is constructed from the metric of $\mathcal{M}_4$ and acts on the exterior algebra $\Lambda \mathcal{M}_4$. The BRST-exact representation of the deformation immediately implies that (2.6) defines a topological quantum field theory. To see this, we consider the partition function $Z$ which, ignoring the irrelevant gauge-fixing terms for the present discussion\(^2\), is given symbolically by the path integral

\[
Z = \int DA \ DB \ e^{iS[B,A,0,0]}
\]

(2.18)

\(^2\)The gauge-fixing terms for the renormalized field theory (2.6) will be the same as those of the pure $BF$ theory (1.2). Consequently, the full gauge-fixed quantum field theory will also be topological in the same way that ordinary $BF$ field theory is. Furthermore, the associated Faddeev-Popov determinants and Stuckelberg Jacobians will cancel each other in the usual way [3].
Since all the metric dependence of the action (2.6) lies in the gauge fermion field (2.17), the variation of (2.18) with respect to the metric $g$ of $\mathcal{M}_4$ is given by

$$-i \frac{\delta Z}{\delta g^{\mu\nu}} = \langle T_{\mu\nu} \rangle = \left\langle \left\{ Q, \frac{\delta \Psi}{\delta g^{\mu\nu}} \right\} \right\rangle$$

where the averages denote vacuum expectation values in the source-free theory (2.6). By gauge invariance, the BRST charge $Q$ annihilates the vacuum of the quantum field theory and (2.19) vanishes. The partition function (2.18) is thus formally independent of the metric of $\mathcal{M}_4$ and there are no local degrees of freedom in this model. The same property can be derived for the path integral representing the vacuum expectation value of any metric-independent gauge-invariant operator $O[B,A]$ (so that $\{ Q, O \}_+ = 0$) of the source-free field theory (2.6).

The topological gauge theory (2.6) will be henceforth referred to as “deformed BF theory”. From (2.16) and (2.17) we see that deformed BF theory is a perturbation of ordinary BF theory in which the deformation can be considered as a natural metric-dependent BF term involving the Batalin-Vilkovisky antifield of the field $B$. The gauge fermion field $\Psi$ thus depends on fields of the non-minimal sector of this formalism, in that $B^*$ is not a gauge-fixing field but rather an auxiliary field required to close the BRST algebra (2.14). This is a significant difference between the counterterms in (2.6) and the usual metric-dependent gauge-fixing counterterms that are added to topological field theory actions. These properties also distinguish the model (2.6) from the deformations of BF theory which are dual models for Yang-Mills theory [25, 26]. These models perturb the action (1.2) by (gauge non-invariant) $B^2$ terms, and integrating over the $B$ field in the deformed quantum field theory yields exactly the Yang-Mills model.

In the physical sector of the quantum field theory (localized about the classical flat gauge field configurations), by definition $B^* = 0$, and thus the partition function of the model (2.6) will represent the same topological invariant (the Ray-Singer analytic torsion) of $\mathcal{M}_4$ as that of the unperturbed BF field theory (1.2) [1, 3, 10]. This will not be true, however, of its observables. In the following we shall be interested in the type of holonomy effects that the topological field theory (2.6) describes, i.e. the holonomy which occurs in adiabatic transport in a theory of point charges and strings where all other degrees of freedom are heavy. This has been extensively studied for the conventional $B \wedge F$ action [13] to which we shall compare the properties of the modified theory.
3 Holonomy Operators and Effective Field Theory

In this section we shall examine the topological and geometrical quantities represented by the effective action

$$\Gamma[\Sigma, j] = -i \log \int DA \, DB \, e^{iS[B,A;\Sigma, j]} = -i \log \left\langle \prod_{a,b} W[L_a] W[\Sigma_b] \right\rangle$$

(3.1)

which corresponds to the expectation values of the Wilson line and surface operators $W[L_a] = \exp i q_a \int L_a A$ and $W[\Sigma_b] = \exp i \phi_b \int_{\Sigma_b} B$ for the particles and strings in the pure gauge part of (2.6). It therefore represents the generic gauge- and topologically-invariant observables of the quantum field theory. Although in this paper we shall be primarily interested in the canonical structure of the quantum field theory (2.6), it is instructive to first examine what sort of invariants the theory will represent in a covariant framework. From (2.9) we see that the effective action consists of three separate terms, the first two being string-string interactions and the last one particle-string interactions. It represents the holonomy phase factors of the covariant particle-string composite states. We shall see that these phase factors can all be written in terms of the extrinsic geometry of the strings, and topological and geometrical intersection indices.

3.1 Topological Linking Numbers

The particle-string interaction term in (2.9) is the effective action of ordinary BF field theory and is a topological linking number of the string and particle trajectories in $\mathcal{M}_4$. To see this, we use the explicit representations (2.2) and (2.4) for the sources, the continuity equations and Stokes’ theorem to write it as

$$S_L[\Sigma, j] = -\frac{2\pi}{k} \int_{\mathcal{M}_4} d^4x \, j_{\mu}(x) \epsilon_{\mu\nu\lambda\rho} \frac{\partial^\nu}{\Box} \Sigma^\lambda\rho(x)$$

$$= -\frac{2\pi}{k} \int_{\mathcal{M}_4} d^4x \, d^4y \, \epsilon_{\mu\nu\lambda\rho} j^\mu(x) (x | \partial^\nu / \Box | y) \Sigma^\lambda\rho(y)$$

(3.2)

$$= -\frac{2\pi}{k} \sum_{a,b} q_a \phi_b I(L_a, \Sigma_b)$$

where

$$I(L_a, \Sigma_b) = \int_{B_a(\Sigma_b)} \int_{L_a} \delta^{(1,3)}(r_a(\tau), x) = -\int_{D_a(L_a)} \int_{\Sigma_b} \delta^{(2,2)}(X_b(\sigma), x)$$

(3.3)

is the covariant linking number of the worldline $L_a$ with the worldsheet $\Sigma_b$ in $\mathcal{M}_4$. Here $B(\Sigma_b)$ is a volume bounded by the surface $\Sigma_b$ and $D(L_a)$ is a disk whose boundary is the contour $L_a$ in $\mathcal{M}_4$. $\delta^{(1,3)}(x, y)$ is the Dirac delta-function in the exterior algebra $\Lambda^1(\mathcal{M}_4(x)) \otimes \Lambda^3(\mathcal{M}_4(y))$ with the property that $\int_{\mathcal{M}_4(y)} \delta^{(1,3)}(x, y) \wedge \alpha(y) = \alpha(x)$ for any 1-form $\alpha(x) \in \Lambda^1(\mathcal{M}_4(x))$. Its
contour integral over $L_a$ defines the deRham current 3-form $\triangle L_a$ which is the Poincaré dual of $L_a$ in $\mathcal{M}_4$, i.e. $\int_{\mathcal{M}_4} \triangle L_a \wedge \alpha = \int_{L_a} \alpha$. Likewise $\delta^{(2,2)}(x, y)$ is the Dirac delta-function in the space $\Lambda^2(\mathcal{M}_4(x)) \otimes \Lambda^2(\mathcal{M}_4(y))$ so that $\int_{\mathcal{M}_4(y)} \delta^{(2,2)}(x, y) \wedge \beta(y) = \beta(x)$ for any 2-form $\beta(x) \in \Lambda^2(\mathcal{M}_4(x))$. Its surface integral over $\Sigma_b$ gives the deRham current 2-form $\triangle \Sigma_b$ which is the Poincaré dual of $\Sigma_b$ in $\mathcal{M}_4$, i.e. $\int_{\mathcal{M}_4} \triangle \Sigma_b \wedge \beta = \int_{\Sigma_b} \beta$. The covariant linking number (3.3) can therefore be alternatively written in terms of cohomology classes dual to the worldline and worldsheet homology classes as

$$I(L_a, \Sigma_b) = \int_{B(\Sigma_b)} \triangle L_a = - \int_{D(L_a)} \triangle \Sigma_b$$

(3.4)

When $\mathcal{M}_4 = \mathbb{R}^1 \times \mathbb{R}^3$, (3.3) becomes the standard Gauss linking integral in four-dimensions [13].

The quantity (3.3) counts the number of times particle $a$ and string $b$ link themselves in $\mathcal{M}_4$ and is a topological invariant of the particle and string trajectories. This linking number is the signed intersection number of the line $L_a$ with the volume $B(\Sigma_b)$, or equivalently of the surface $\Sigma_b$ with the disk $D(L_a)$, in $\mathcal{M}_4$.

$$I(L_a, \Sigma_b) = \sum_{p \in L_a \cap B(\Sigma_b)} \text{sgn}(p) = - \sum_{p \in \Sigma_b \cap D(L_a)} \text{sgn}(p)$$

(3.5)

where $\text{sgn}(p) = \pm 1$ according to whether or not the orientation at the intersection point $p$ coincides with that of $\mathcal{M}_4$. In the path integral quantization of the field theory, it is the linking term (3.2) that endows the effective particle-string composite states with fractional exchange statistics. Here the statistics parameter is $\frac{k}{2\pi}$.

### 3.2 Extrinsic Geometry of Strings

The two local string-string interaction terms in the effective action (2.9) can be written in terms of local intersection indices and the extrinsic geometry of the string worldsheets in $\mathcal{M}_4$. Substituting into the first term of (2.9) the explicit form (2.4) of the string current and integrating over $x$, we see that it can be written as

$$S_E[\Sigma] \equiv \frac{4\pi^2}{k^2} \int_{\mathcal{M}_4} d^4x \, \Sigma^{\mu \nu}(x) \Sigma_{\mu \nu}(x) = \sum_{b, b'} S^{(bb')}_{E}$$

(3.6)

where

$$S^{(bb')}_{E} \equiv \frac{4\pi^2}{k^2} \phi_b \phi_{b'} \int_{\Sigma_b} d^2\sigma \sqrt{\eta_b(\sigma)} \int_{\Sigma_{b'}} d^2\sigma' \sqrt{\eta_{b'}(\sigma')} t_{b, \mu \nu}(\sigma) t^{\mu \nu}_{b'}(\sigma') \delta^{(4)}(X_{b'}(\sigma'), X_b(\sigma))$$

(3.7)

---

3Generically, in $d$ dimensions, a $p$-surface and a $(d - p)$-surface intersect transversally at distinct isolated points.
Here
\[ t_{b}^{\mu\nu}(\sigma) = \frac{1}{\sqrt{\eta_b(\sigma)}} \epsilon^{\alpha\beta} \frac{\partial X_b^\mu(\sigma)}{\partial \sigma^\alpha} \frac{\partial X_b^\nu(\sigma)}{\partial \sigma^\beta} \]  \hspace{1cm} (3.8)

is the antisymmetric local area element of the surface \( \Sigma_b \) obeying the identities
\[ t_{b,\mu\nu}(\sigma)t_{b}^{\mu\nu}(\sigma) = 2 \quad , \quad \epsilon^{\mu\nu\lambda\rho} t_{b,\mu\nu}(\sigma)t_{b,\lambda\rho}(\sigma) = 0 \]  \hspace{1cm} (3.9)

and
\[ \eta_{b,\alpha\beta}(\sigma) = \frac{\partial X_b^\mu(\sigma)}{\partial \sigma^\alpha} \frac{\partial X_{b,\mu}(\sigma)}{\partial \sigma^\beta} \]  \hspace{1cm} (3.10)

is the induced metric on the string worldsheet formed by the tangent vectors \( \frac{\partial X_b^\mu(\sigma)}{\partial \sigma^\alpha} \) to \( \Sigma_b \subset \mathcal{M}_4 \). It has matrix inverse \( \eta^{-1}_{b,\alpha\beta}(\sigma) \) and determinant \( \eta_b(\sigma) \).

The integrals in (3.7) for \( b = b' \) are localized onto the subset \( \mathcal{C}_b \) of the manifold \( \Sigma_b(\sigma) \otimes \Sigma_b(\sigma') \) of points \( \{\sigma, \sigma'\} \) for which \( X_b^\mu(\sigma) = X_{b'}^\mu(\sigma') \). It can be decomposed into two disjoint subsets, \( \mathcal{C}_b = \mathcal{E}_b \otimes \mathcal{N}_b \), where \( \mathcal{E}_b = \{\sigma, \sigma'\} | X_b(\sigma) = X_b(\sigma') \iff \sigma = \sigma' \) and \( \mathcal{N}_b = \{\sigma, \sigma'\} | X_b(\sigma) = X_b(\sigma'), \sigma \neq \sigma' \). The integrals (3.7) for \( b = b' \) are then the sum of the contributions \( S_E^{(bb)} = S_E(\mathcal{E}_b) + S_E(\mathcal{N}_b) \) from these two disjoint subsets. In this subsection we shall examine the contribution from the subset \( \mathcal{E}_b \) on which the string functions \( X_b \) are embeddings. The contribution from \( \mathcal{N}_b \), which corresponds to twists or self-intersections of the immersed surface \( \Sigma_b \), along with the \( b \neq b' \) terms in (3.7) will be described in the next subsection.

To describe these terms, let us briefly recall a few facts concerning the extrinsic geometry of embedded surfaces. The embedding of \( \Sigma_b \) in \( \mathcal{M}_4 \) can be used to define the tangent bundle \( T \Sigma_b \) over \( \Sigma_b \). The fibre over a point \( \sigma \in \Sigma_b \) is the space of tangent vectors at the point \( X_b^\mu(\sigma) \in \mathcal{M}_4 \) (i.e. linear combinations of \( \frac{\partial X_b^\mu(\sigma)}{\partial \sigma^\alpha} \)). The associated normal bundle \( N \Sigma_b \) can be similarly defined, with fibre over the point \( \sigma \in \Sigma_b \) the space of vectors orthogonal to the tangent vectors at \( X_b^\mu(\sigma) \). The normal fibres are spanned by normal vectors \( n_{b,\ell}^\mu, \ell = 1,2 \), which satisfy
\[ n_{b,\ell}^\mu n_{b,\ell';\mu} = \delta_{\ell\ell'} \quad , \quad n_{b,\ell;\mu} \frac{\partial X_b^\mu}{\partial \sigma^\alpha} = 0 \]  \hspace{1cm} (3.11)

The extrinsic curvature \( K_{b,\ell}^{\alpha\beta} \) of the normal bundle is defined by decomposing the Hessian of the embedding \( X_b \) as an endomorphism over the vector space \( T \Sigma_b \otimes N \Sigma_b \cong T \mathcal{M}_4 \),
\[ \frac{\partial^2 X_b^\mu(\sigma)}{\partial \sigma^\alpha \partial \sigma^\beta} = \Gamma_{b,\alpha\beta}^\gamma \frac{\partial X_b^\mu(\sigma)}{\partial \sigma^\gamma} + K_{b,\ell;\alpha\beta} n_{b,\ell}^\mu \]  \hspace{1cm} (3.12)

where \( \Gamma_{b,\alpha\beta}^\gamma \) is the Christoffel connection on \( T \Sigma_b \). Note that the intrinsic Gaussian curvature \( R_b \) of the worldsheet \( \Sigma_b \) is related to the curvature of the induced metric (3.10) by
\[ R_b = \left( K_{b,\ell;\alpha}^\beta \right)^2 - K_{b,\ell;\beta} K_{b,\ell;\alpha} \]  \hspace{1cm} (3.13)
To evaluate the delta-function in (3.14) over \( \Sigma_b \), we note that it is determined by the topology of the normal bundle \( N\Sigma_b \), because it can defined as the limit of non-singular forms with shrinking supports in the neighbourhood of \( \Sigma_b \), which in turn can be approximated by the zero section of \( N\Sigma_b \). Thus we write

\[
\delta^{(4)}(x, y) = \lim_{\lambda \to \infty} \psi_\lambda(x, y) \tag{3.14}
\]

where \( \{ \psi_\lambda(x, y) \}_{\lambda \in \mathbb{R}^+} \) is a one-parameter family of smoothly supported functions near \( x = y \) with \( \int_{\mathcal{M}_4} \psi_\lambda = 1 \). Working in Gaussian normal coordinates in the transverse space of \( \Sigma_b \) (i.e. in Cartesian coordinates in the fibre of \( N\Sigma_b \)), we can choose \( \psi_\lambda(x, y) = C_\lambda e^{-\lambda^2 \|x-y\|^2} \) where \( \|x\| \) denotes the geodesic length of \( x \) in \( \mathcal{M}_4 \). If \( \mathcal{M}_4 \) is an open infinite manifold, then the normalization constant \( C_\lambda \) diverges as \( \Lambda^4 \) for \( \lambda \to \infty \). In general we are therefore concerned with the evaluation of generic integrals of the form

\[
\mathcal{Z}_b(\sigma) = \lim_{\lambda \to \infty} C_\lambda \int_{\Sigma_b} d^2\sigma' \sqrt{\eta_b(\sigma')} \ K(\sigma, \sigma') \ e^{-\lambda^2 \|x_b(\sigma')-x_b(\sigma)\|^2} \tag{3.15}
\]

where \( K(\sigma, \sigma') \) is a local integration kernel on \( \Sigma_b(\sigma) \otimes \Sigma_b(\sigma') \). Since for \( \{\sigma, \sigma'\} \in \mathcal{E}_b \), \( X_b^\mu(\sigma) = X_b^\mu(\sigma') \) is equivalent to \( \sigma = \sigma' \), the geodesic function appearing in the argument of the exponential in (3.15) is a Morse function of \( \sigma' \in \Sigma_b \) with global minimum 0 at \( \sigma' = \sigma \). We can therefore apply the stationary-phase approximation to evaluate the integral (3.15) which yields the standard expansion [28]

\[
\mathcal{Z}_b(\sigma) = \lim_{\lambda \to \infty} C_\lambda \left\{ \left( -\frac{2\pi}{\Lambda^2} \right) \det^{-1/2} \mathcal{H}_b(\sigma) \right. \times \sum_{\ell=0,1} \left( -\frac{1}{2\Lambda^2} \right)^\ell \left[ \mathcal{H}_b^{-1}(\sigma)^{\alpha\beta} \frac{\partial^2}{\partial \sigma^\alpha \partial \sigma^\beta} \right]^\ell K(\sigma, \sigma') \Big|_{\sigma' = \sigma} + O(\Lambda^{-6}) \right\} \tag{3.16}
\]

where \( \mathcal{H}_b(\sigma) \) is the Hessian of the exponential argument in (3.15) at \( \sigma' = \sigma \). One easily finds \( \mathcal{H}_{b,\alpha\beta} = \eta_{b,\alpha\beta} \) and thus

\[
\mathcal{Z}_b(\sigma) = 2\pi \lim_{\lambda \to \infty} \left( \frac{C_\lambda K(\sigma, \sigma)}{\Lambda^2 \sqrt{\eta_b(\sigma)}} - \frac{C_\lambda \eta_{b,\alpha\beta}^\beta(\sigma) \frac{\partial^2 K(\sigma, \sigma')}{\partial \sigma^\alpha \partial \sigma^\beta}}{2\Lambda^4 \sqrt{\eta_b(\sigma)}} \Bigg|_{\sigma' = \sigma} \right) \tag{3.17}
\]

Substituting \( K(\sigma, \sigma') = t_{b,\mu\nu}(\sigma)t_{b,\mu\nu}(\sigma') \), integrating (3.17) by parts over \( \sigma \in \Sigma_b \) and using the identities (3.3) we arrive at the final result for the first \( b = b' \) contributions to (3.17),

\[
S_E(\mathcal{E}_b) = \mu_b \int_{\Sigma_b} d^2\sigma \sqrt{\eta_b(\sigma)} + \frac{1}{\alpha_b} \int_{\Sigma_b} d^2\sigma \sqrt{\eta_b(\sigma)} \eta_{b,\alpha\beta}^\beta(\sigma) \frac{\partial t_{b,\mu\nu}(\sigma)}{\partial \sigma^\alpha} \frac{\partial t_{b,\mu\nu}(\sigma)}{\partial \sigma^\beta} \tag{3.18}
\]

where

\[
\mu_b = \frac{16\pi^3}{k^2} \phi_b^2 \lim_{\lambda \to \infty} \frac{C_\lambda}{\Lambda^2}, \quad \frac{1}{\alpha_b} = -\frac{4\pi^3}{k^2} \phi_b^2 \lim_{\lambda \to \infty} \frac{C_\lambda}{\Lambda^4} \tag{3.19}
\]
The first surface integral in (3.18) is the area $A(\Sigma_b)$ of $\Sigma_b$ in $\mathcal{M}_4$, while the second integral can be integrated by parts and written in terms of the extrinsic curvature of $\Sigma_b$ using (3.12) as
\[
\int_{\Sigma_b} d^2\sigma \sqrt{\eta_b(\sigma)} \, \eta^\alpha_b(\sigma) \frac{\partial t_{b,\mu\nu}(\sigma)}{\partial \sigma^\alpha} \frac{\partial t^{\mu\nu}(\sigma)}{\partial \sigma^\beta} = \int_{\Sigma_b} d^2\sigma \sqrt{\eta_b(\sigma)} \, K_{b,\ell;\beta}(\sigma) K_{b,\ell;\alpha}(\sigma) \equiv 4\pi \chi^{(1)}(N\Sigma_b)
\]
(3.20)

Thus the contributions $S_E(\mathcal{E}_b)$ to the action term (3.8) describe the extrinsic geometry of the embedded surfaces in $\mathcal{M}_4$. Note that the curvature term (3.20), which is the Euler number $\chi^{(1)}(N\Sigma_b)$, can be written in the form
\[
\chi^{(1)}(N\Sigma_b) = \frac{1}{4\pi} \int_{\Sigma_b} d^2\sigma \sqrt{\eta_b(\sigma)} \, \eta^\alpha_b(\sigma) D_{b,\alpha} n^\mu_{b,\ell} D_{b,\beta} n_{b,\ell;\mu}
\]
(3.21)

where
\[
D_{b,\alpha} n^\mu_{b,\ell} = \frac{\partial n^\mu_{b,\ell}}{\partial \sigma^\alpha} + A_{b,\alpha;\ell'} n^\mu_{b,\ell'} = -K_{b,\ell;\alpha} \frac{\partial X^\mu_{b}}{\partial \sigma^\beta}
\]
(3.22)

and $A_{b,\alpha;\ell'} = n^\mu_{b,\ell} \frac{\partial n^\nu_{b,\ell'}}{\partial \sigma^\alpha}$ is an $SO(2)$ connection of the normal bundle $N\Sigma_b$.

### 3.3 Extrinsic Intersection Numbers

The remaining contribution $S_E(N_b)$ to the self-interaction terms in (3.7) come from the points $\{\sigma, \sigma'\} \in \Sigma_b(\sigma) \otimes \Sigma_b(\sigma')$ for which $X^\mu_b(\sigma) = X^\mu_{b'}(\sigma')$ but $\sigma \neq \sigma'$. The same structure occurs for the $b \neq b'$ terms in (3.7), for which only $N$-type points contribute. We can therefore write the delta-function appearing in (3.7) in terms of a delta-function on the space $\Sigma_b(\sigma) \otimes \Sigma_{b'}(\sigma')$ to get
\[
S_E(N_{b'}) = \frac{4\pi^2}{k^2} \phi_b \phi_{b'} \sum_{\{\sigma, \sigma'\}} \int_{\Sigma_b} d^2\sigma \sqrt{\eta_b(\sigma)} \int_{\Sigma_{b'}} d^2\sigma' \sqrt{\eta_{b'}(\sigma')}
\]
\[
\times t_{b,\mu\nu}(\sigma) t^{\mu\nu}(\sigma') \frac{\delta(2)(\sigma, \sigma') \delta(2)(\sigma', \sigma'')}{|J_i(\sigma, \sigma')|}
\]
(3.23)

where $(X_b \otimes X_{b'})(N_{b'}) = \Sigma_b \cap \Sigma_{b'}$ and the string worldsheets intersect transversally in $\mathcal{M}_4$ at finitely many isolated points $X^\mu_b(\sigma_i) = X^\mu_{b'}(\sigma_i')$. Here $J_i(\sigma, \sigma')$ is the Jacobian for the four-dimensional coordinate transformation $\{\sigma^\alpha, \sigma'^\alpha\} \rightarrow X^\mu_b(\sigma) - X^\mu_{b'}(\sigma')$ on $\Sigma_b(\sigma) \otimes \Sigma_{b'}(\sigma') \rightarrow \mathcal{M}_4$. After a Taylor expansion about the points $\sigma_i$ and $\sigma_i'$, we can work out this Jacobian at the points $\sigma = \sigma_i$ and $\sigma' = \sigma_i'$ and find
\[
J_i(\sigma_i, \sigma_i') = \frac{1}{4} \epsilon_{\mu\nu\lambda\rho} \epsilon^{\alpha\beta\gamma\delta} \frac{\partial X^\mu_b(\sigma_i)}{\partial \sigma^\alpha} \frac{\partial X^\nu_{b'}(\sigma_i)}{\partial \sigma'^\beta} \frac{\partial X^\lambda_{b'}(\sigma_i')}{\partial \sigma^{\beta'}} \frac{\partial X^\rho_{b'}(\sigma_i')}{\partial \sigma'^{\gamma}}
\]
(3.24)

Substituting (3.24) into the action (3.23) and integrating over $\sigma$ and $\sigma'$ we have
\[
S_E(N_{b'}) = \frac{16\pi^2}{k^2} \phi_b \phi_{b'} \nu_G(\Sigma_b, \Sigma_{b'})
\]
(3.25)
where

\[
\nu_G(\Sigma_b, \Sigma_{b'}) = \sum_{\{b, \sigma, \sigma'\} \in \mathcal{N}_{bb'}} \frac{t_{b,\mu\nu}(\sigma)t_{b',\mu'}(\sigma')}{\varepsilon_{\mu
u\lambda\rho}t_{b,\mu\nu}(\sigma)t_{b',\lambda\rho}(\sigma')} \text{sgn} \left( \varepsilon_{\mu\nu\lambda\rho} t_{b,\mu\nu}(\sigma)t_{b',\lambda\rho}(\sigma') \right)
\]  

(3.26)

The quantity (3.26) is a geometrical intersection number of the surfaces \( \Sigma_b \) and \( \Sigma_{b'} \) in \( \mathcal{M}_4 \) (for \( b = b' \) it is a self-intersection number of \( \Sigma_b \)). The sign function in (3.26) is the local intersection index of the intersection point \( p^\mu = X_b^\mu(\sigma) = X_{b'}^\mu(\sigma') \), and, since \( t_{b,\mu\nu} \) is the extrinsic area element, it takes the values \( \pm 1 \) depending on whether or not the orientation at \( p \) coincides with that of \( \mathcal{M}_4 \). The factor multiplying each local intersection index is a geometrical quantity which measures the transversality of the intersection. It vanishes as the normal vectors of \( \Sigma_b \) at the intersection point \( p \) become parallel. Thus the contributions to (3.26) from the subsets \( \mathcal{N}_{bb'} \) of points yields a signed geometric transversal intersection index of the string worldsheets.

The intersection number (3.26) can also be written in terms of cohomology classes using the fact \([24]\) that the deRham current of the surface \( \Sigma_b \) can be written locally as \( (\Delta_{\Sigma_b})_{\mu\nu}(x) = \int_{\Sigma_b} d\sigma_{\mu\nu}(X_b) \delta^{(4)}(x, X_b(\sigma)) \), so that

\[
\nu_G(\Sigma_b, \Sigma_{b'}) = \int_{\mathcal{M}_4} \Delta_{\Sigma_b} \wedge \star \Delta_{\Sigma_{b'}} = \int_{\Sigma_b} \int_{\Sigma_{b'}} \star (\delta^{(2,2)}(X_b(\sigma), X_{b'}(\sigma')))
\]  

(3.27)

This shows that the geometrical intersection number is not a topological invariant of the string worldsheets. When \( b = b' \) and the \( X_b \) are embeddings, then (3.27) gives the first and second fundamental forms of the embedded surface \( \Sigma_b \) in \( \mathcal{M}_4 \) as described in the previous subsection. This follows formally from the global property \([27]\)

\[
\Delta_{\Sigma_b} \wedge \Delta_{\Sigma_{b'}} = \Delta_{\Sigma_b \cap \Sigma_{b'}} \wedge \chi_E(N_{\Sigma_b} \cap N_{\Sigma_{b'}})
\]  

(3.28)

of the deRham current, where \( \chi_E \) denotes the Euler characteristic class. If the geometry of the normal bundle of \( \Sigma_b \) is such that its deRham current is self-dual, i.e. \( \star \Delta_{\Sigma_b} = \Delta_{\Sigma_b} \), then (3.27) coincides with the algebraic intersection number of \( \Sigma_b \) and \( \Sigma_{b'} \) (see the next subsection). There exist examples of Kähler 4-manifolds for which this is true \([29]\).

### 3.4 Topological Intersection Numbers

Finally, we come to the \( \theta \)-term in the effective action (2.9), which we can write as

\[
S_1[\Sigma] = -\frac{4\pi \theta}{k^2} \int_{\mathcal{M}_4} d^4x \epsilon_{\mu\nu\lambda\rho} \Sigma^{\mu\nu}(x) \Sigma^{\lambda\rho}(x) = \sum_{b, b'} S_1^{(bb')}
\]  

(3.29)

where

\[
S_1^{(bb')} = -\frac{4\pi \theta}{k^2} \phi_b \phi_{b'} \int_{\Sigma_b} d^2\sigma \sqrt{\eta_b(\sigma)} \int_{\Sigma_{b'}} d^2\sigma' \sqrt{\eta_{b'}(\sigma')}
\]

\[
\times \epsilon_{\mu\nu\lambda\rho} t_{b,\mu\nu}(\sigma)t_{b',\lambda\rho}(\sigma') \delta^{(4)}(X_{b'}(\sigma'), X_b(\sigma))
\]

(3.30)
The calculation proceeds as before. For the $b = b'$ terms in (3.30), the contribution from the points \{σ, σ'\} ∈ $E_b$ are calculated using (3.17) with $K(σ, σ') = \epsilon^{μνλρ}t_{b,μν}(σ)t_{b,λρ}(σ')$. Because of the identities (3.9), the first term in (3.17) is absent in this case, and integrating by parts over $Σ_b$ we find

$$S_I(E_b) = -\frac{16\pi^3θ}{k^2} φ_b^2 \left(\lim_{λ→∞} \frac{C_λ}{Λ^4}\right) ν_P(Σ_b) \equiv \theta_b ν_P(Σ_b) \quad (3.31)$$

where

$$ν_P(Σ_b) = \frac{1}{4π} \int_{Σ_b} d^2σ \sqrt{η_b(σ)} η_b^{αβ}(σ) \epsilon^{μνλρ} \frac{∂t_{b,μν}(σ)}{∂σ^α} \frac{∂t_{b,λρ}(σ)}{∂σ^β} \quad (3.32)$$

is the self-intersection index of the worldsheet $Σ_b$ [31–32]. It is the algebraic signed self-intersection number of the surface, and it can be related to the Chern number of the normal bundle of $Σ_b$ by noting that from (3.10), (3.12) and (3.9) we can write (3.32) as

$$ν_P(Σ_b) = \frac{1}{8π} \int_{Σ_b} d^2σ \ η_b^{αβ} e^{γδ} K_{b,γα}(σ) K_{b,δβ}(σ) \epsilon_{ℓℓ'} = \frac{1}{2} ch^{(1)}(NΣ_b) \quad (3.33)$$

where

$$ch^{(1)}(NΣ_b) = \frac{1}{2π} \int_{Σ_b} tr F_b = \frac{1}{8π} \int_{Σ_b} d^2σ \ ε_{ℓℓ'} \ ε^{αβ} F_{b,αβ}(σ) \quad (3.34)$$

is the Chern number of $NΣ_b$ and $F_{b,ℓℓ'} = dA_{b,ℓℓ'}$ is the curvature of the $SO(2)$ connection of the normal bundle defined in (3.22). Thus the self-intersection index also measures the nontriviality of the normal bundle of $Σ_b$ in $M_4$, and algebraically it is counted by the Chern number of the normals.

Next we evaluate the contribution from $S_I(N_{bℓ'})$. Following the steps which led to the expression (3.25), we find that it can be written as

$$S_I(N_{bℓ'}) = -\frac{16\piθ}{k^2} φ_bφ_{ℓ'} ν_T(Σ_b, Σ_{ℓ'}) \quad (3.35)$$

where

$$ν_T(Σ_b, Σ_{ℓ'}) = \sum_{\{σ, σ'\} ∈ N_{bℓ'}} sgn \left(\epsilon^{μνλρ}t_{b,μν}(σ)t_{b,λρ}(σ')\right) \quad (3.36)$$

is a topological intersection number of the string worldsheets $Σ_b$ and $Σ_{ℓ'}$. As in (3.27) it can be expressed in terms of cohomology classes as

$$ν_T(Σ_b, Σ_{ℓ'}) = \int_{M_4} Δ_Σ_b \wedge Δ_Σ_{ℓ'} = \int_{Σ_b} \int_{Σ_{ℓ'}} δ^{(2,2)}(X_b(σ), X_{ℓ'}(σ')) \quad (3.37)$$

showing that it is a topological invariant of the string worldsheets. The quantity (3.37) is the algebraic intersection number of the oriented surfaces $Σ_b$ and $Σ_{ℓ'}$ [28]. When $b = b'$ and the $X_b$ are embeddings, (3.37) coincides with (3.32) when the geometry of the normal bundle of $Σ_b$ is such that the Chern class of $NΣ_b$ coincides with the Poincaré class of $Σ_b$. In the effective field theory (2.9), the instanton number of the complex line bundle of the gauge theory, given by the topological Yang-Mills term in (1.6), is then the sum of the monopole numbers of the normal bundles of the string worldsheets $Σ_b$. 

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Collecting all of the contributions above, we find that the total effective action \( (2.9) \) of the deformed BF field theory \( (2.6) \) can be written in terms of geometrical and topological quantities as

\[
\Gamma[\Sigma, j] = -\frac{2\pi}{k} \sum_{a,b} q_a \phi_b I(L_a, \Sigma_b) + \sum_b \left( \mu_b A(\Sigma_b) + \frac{4\pi}{\alpha_b} \chi^{(1)}(N \Sigma_b) + \frac{\theta_b}{2} \text{ch}^{(1)}(N \Sigma_b) \right) \\
+ \frac{16\pi}{k^2} \sum_{b, b'} \phi_b \phi_{b'} \left( \pi \nu_G(\Sigma_b, \Sigma_{b'}) - \theta \nu_T(\Sigma_b, \Sigma_{b'}) \right)
\] (3.38)

Each contribution to (3.38) is a diffeomorphism invariant of the embedded trajectories in \( \mathcal{M}_4 \), as anticipated from the topological nature of the field theory defined by (2.6). The first sum in (3.38) shows that the composite particle-string states in the spectrum of this quantum field theory have fractional exchange statistics. The third sum yields analogous fractional geometrical and topological intersection phases for the strings.

The second sum in (3.38) has appeared in the context of the extrinsic geometry of the QCD string \([20, 31, 32]\), and more generally in the effective theory of Nielsen-Olesen vortex strings in abelian Higgs field theories \([18, 19]\). The first term is the Nambu-Goto area action while the second term is the rigid string action. The last term is the analog of the \( \theta \)-term of four-dimensional Yang-Mills theory. It is expected (from scale-invariance and loop equation arguments) that the correct string theory for QCD is one in which the Nambu-Goto term is absent or irrelevant, and the extrinsic curvature term controls the phase structure of the string theory. It is interesting to note that this property is reflected by the forms of the induced coefficients \((3.19)\). When \( \mathcal{M}_4 \) is an open infinite spacetime, the string tension \( \mu_b \) diverges, and to make the effective string theory well-defined it should be set to 0. This can be achieved in some limiting situation involving the parameters \( k \) and \( \phi_b \) of the deformed BF field theory. The only remnants of the action then are the two topological terms, with a negative rigidity factor \([18, 19]\). On the other hand, when \( \mathcal{M}_4 \) is a compact manifold, only the area form survives, consistent with the cohomological representations \((3.27)\) and \((3.37)\).

Notice that, for the special periodic value \( \theta = \pi \) of the vacuum angle, when the geometry of the string worldsheets is such that their deRham currents are self-dual 2-forms, the sum over intersection numbers in (3.38) vanishes and we are left with a pure effective string theory (plus an additional non-local particle-string Aharonov-Bohm interaction \([17, 19]\)). This is precisely the value of \( \theta \) that was found in \([32]\) to be induced by the dynamical cancellation of folded string configurations. The deformed BF field theory \((2.6)\) is thus a dual model for rigid vortex strings with \( \theta \)-vacua and additional non-local interaction terms \([19]\). This topological field theory approach might serve as a useful tool for investigating the physical properties of such systems. As we will see, the topological nature of the dual model allows a complete and
exact solution of the quantum theory.

4 Canonical Quantization of Deformed BF Theory

In the previous section we have uncovered a rich geometrical and topological structure for the renormalized theory (2.6) which has many potential physical applications. We can learn more about this topological field theory from canonical quantization. We will see that the quantization of it will yield some novel quantum representations of the geometrical and topological indices, just as the wavefunctions of ordinary BF theory do for the topological linking numbers and the cohomology of the underlying manifold. In the following we shall be interested in precisely how these objects are realized in the physical sector of the Hilbert space of this quantum field theory. In this section we will describe the canonical structure of the field theory (2.6), taking into careful account the first-stage reducibility of its gauge symmetries. The reduced phase space of similar antisymmetric tensor field theories has been studied in [33].

We choose the spacetime to be the product manifold $\mathcal{M}_4 = \mathbb{R} \times \mathcal{M}_3$, where $\mathbb{R}$ parametrizes the time and $\mathcal{M}_3$ is a 3-manifold without boundary. We may then work in an adiabatic limit of the field theory in which the temporal components of the particle and string source fields parametrize their worldlines and worldsheets, i.e. $r^0_a(\tau) = \tau$ and $X^0_b(\sigma^1, \sigma^2) = \sigma^1$. The temporal components $A_0$ and $B_{0i}$ are Lagrange multipliers which enforce the local gauge constraints

$$-\nabla_i F^{0i} + \frac{k}{2\pi} \nabla \cdot B + j^0 \approx 0, \quad \frac{k}{4\pi} \epsilon^{ijk} F_{jk} + \Sigma^{0i} \approx 0 \quad (4.1)$$

where $B^i(x) = \frac{1}{2} \epsilon^{ijk} B_{jk}(x)$ and $\epsilon^{ijk} \equiv \epsilon^{0ijk}$. From (4.1) it follows that, when $\mathcal{M}_3$ is compact, $A$ and $B$ are only globally defined differential forms on $\mathcal{M}_3$ when the total particle charge and total string flux vanish, $\sum_a q_a = \sum_b \phi_b = 0$. When they are non-zero, the fields are instead sections of a non-trivial complex line bundle over $\mathcal{M}_3$ and the action (2.6) must be appropriately modified [34]. From a physical point of view, the restriction to vanishing charge and flux sectors of the theory on a closed space is natural by flux conservation. We shall assume this constraint on the source currents in this paper. Some aspects of abelian BF theories on topologically non-trivial line bundles have been discussed recently in [35].

The canonical momenta in the temporal gauge $A_0 = B_{0i} = 0$ are

$$\pi^i \equiv \frac{\delta S}{\delta \dot{A}_i} \bigg|_{j=\Sigma=0} = \dot{A}^i - \frac{\theta}{\pi} \epsilon^{ijk} F_{jk}, \quad \pi^{ij} \equiv \frac{\delta S}{\delta \dot{B}_{ij}} \bigg|_{j=\Sigma=0} = \frac{k}{2\pi} \epsilon^{ijk} A_k \quad (4.2)$$

and in this gauge we may invoke the strong equalities $\pi^0 = \pi^{0i} = 0$ [33]. They yield the
non-vanishing canonical Poisson brackets

\[
\{ A_i(x), \pi^j(y) \}_P = \delta^j_i \delta^{(3)}(x,y) \quad , \quad \{ B_{ij}(x), \pi^{kl}(y) \}_P = \left( \delta^k_j \delta^l_i - \delta^l_j \delta^k_i \right) \delta^{(3)}(x,y) \tag{4.3} \]

and the canonical Hamiltonian

\[
H = \int_{M_3} d^3 x \left[ \frac{1}{2} \left( \pi^i + \frac{\theta}{\pi} \epsilon^{ijk} F_{jk} \right)^2 + \frac{1}{4} F_{ij} F^{ij} - A_i j^i - \frac{1}{2} B_{ij} \Sigma^{ij} \right] \tag{4.4} \]

We therefore have to quantize the constrained dynamical system with Hamiltonian (4.4), Poisson brackets (4.3) and primary constraint functions

\[
\lambda_{ij}^1(x) = \pi^{ij}(x) - \frac{k}{2\pi} \epsilon^{ijk} A_k(x) \\
\lambda_2(x) = \nabla \cdot \pi(x) - \frac{k}{2\pi} \nabla \cdot B(x) - j^0(x) \tag{4.5} \\
\lambda_3^i(x) = \frac{k}{4\pi} \epsilon^{ijk} F_{jk}(x) + \Sigma^{0i}(x)
\]

The constraints \( \lambda_a \approx 0 \) are first class constraints in the Dirac classification of constrained systems [36], since they generate an abelian Poisson-Lie algebra

\[
\{ \lambda_a, \lambda_b \}_P = 0 \quad , \quad a, b = 1, 2, 3 \tag{4.6} \]

Secondary constraints are generated by the compatibility conditions for the primary constraint functions (4.5) given by

\[
\frac{\partial}{\partial t} \lambda_a - \{ \lambda_a, H \}_P + c_{a}^{\ b} \lambda_b \approx 0 \tag{4.7} \]

where \( c_{a}^{\ b} \) is a 2-cocycle of the abelian Poisson-Lie group generated by the \( \lambda_a \)'s. This yields the additional constraint function

\[
\lambda_4^i(x) = \pi^i(x) - \frac{2\pi}{k} \epsilon^{ijk} \Sigma_{jk}(x) + \frac{\theta}{\pi} \epsilon^{ijk} F_{jk}(x) \tag{4.8} \]

It has vanishing Poisson bracket with itself and with \( \lambda_2 \), but non-vanishing Poisson brackets with \( \lambda_1 \) and \( \lambda_3 \) in (4.3),

\[
\{ \lambda_1^i(x), \lambda_4^j(y) \}_P = \frac{k}{2\pi} \epsilon^{ijk} \delta^{(3)}(x,y) \quad , \quad \{ \lambda_3^i(x), \lambda_4^j(y) \}_P = -\frac{k}{2\pi} \epsilon^{ijk} \nabla_k \delta^{(3)}(x,y) \tag{4.9} \]

There are no tertiary constraints, owing to the first-stage reducibility of the gauge theory, and (4.3), (4.8) constitute the complete set of constraint functions for the dynamical system.

We shall choose the pair \( \lambda_1, \lambda_4 \) as second class constraint functions and impose the strong equalities

\[
\pi^{ij} = \frac{k}{2\pi} \epsilon^{ijk} A_k \quad , \quad \pi^i = \frac{2\pi}{k} \epsilon^{ijk} \Sigma_{jk} - \frac{\theta}{\pi} \epsilon^{ijk} F_{jk} \tag{4.10} \]
Then the remaining phase space variables have the non-vanishing Dirac brackets

\[ \{A_i(x), B^j(y)\}_D \equiv \{A_i(x), B^j(y)\}_P \]

\[ - \int_{M_3} d^3x' \ d^3y' \ \{A_i(x), \lambda_a(x')\}_P (C^{-1})^{ab}(x', y') \ \{\lambda_b(y'), B^j(y)\}_P \]

\[ = \frac{2\pi}{k} \delta^i_j \delta^{(3)}(x, y) \]  \hspace{1cm} (4.11)

where \(C_{ab} = \{\lambda_a, \lambda_b\}_P\) is the Poisson bracket matrix of the constraint functions for \(a, b = 1, 4\).

The constrained Hamiltonian is

\[ H = \int_{M_3} d^3x \ \left( \frac{4\pi^2}{k^2} \Sigma_{ij} \Sigma^{ij} + \frac{1}{4} F_{ij} F^{ij} - A_i j^i - \frac{1}{2} B_{ij} \Sigma^{ij} \right) \]  \hspace{1cm} (4.12)

with the additional first class constraints determined by the pair \(\lambda_2, \lambda_3\),

\[ \frac{4\pi}{k} \nabla \cdot \Sigma - \frac{k}{2\pi} \nabla \cdot B - j^0 \approx 0 \quad , \quad \frac{k}{4\pi} \epsilon^{ijk} F_{jk} + \Sigma^{0i} \approx 0 \]  \hspace{1cm} (4.13)

where \(\Sigma_i = \frac{1}{2} \epsilon_{ijk} \Sigma^{jk}\).

From the identity

\[ \frac{\partial}{\partial t} \nabla \cdot \Sigma = - \frac{1}{2} \partial_\mu \left( \epsilon^{\mu\nu\lambda\rho} \partial_\nu \Sigma_{\lambda\rho} \right) = 0 \]  \hspace{1cm} (4.14)

and those for \(B\) and \(F\) we see that the constraint functions in (4.13) are time independent. These constraints will be treated as physical state conditions in the quantum field theory. From (4.11) it follows that the non-vanishing canonical quantum commutators of the field theory are

\[ [A_i(x), B^j(y)] = \frac{2\pi i}{k} \delta^i_j \delta^{(3)}(x, y) \]  \hspace{1cm} (4.15)

Notice that, in the absence of sources, the Hamiltonian (4.12) vanishes only on the physical subspace of the entire Hilbert space. This again reflects the "mild" topological nature of deformed BF theory, i.e. that the gauge fermion field introduced by the deformation is constructed from the conjugate momentum to the ghost field associated with the curvature constraint of the field theory, as we discussed in section 2. In this sector, the reduced classical phase space of the source-free field theory is the finite-dimensional vector space

\[ \mathcal{P} = H^1(M_3) \oplus H^2(M_3) \]  \hspace{1cm} (4.16)

The physical Hilbert space therefore contains topological information and yields quantum field theoretical representations of the deRham complex of the 3-manifold \(M_3\), as is the usual case in topological gauge theories. This space is studied in detail in the next section.
5 Construction of the Physical Hilbert Space

We now assume that $\mathcal{M}_3$ is a compact, path-connected, orientable 3-manifold without boundary, and let $p$ be the dimension of its first and second homology groups. From the induced Euclidean-signature metric of $\mathcal{M}_3$ we can construct the dual forms $\tilde{j}$ and $\tilde{\Sigma}$ of the vector fields (2.2) and (2.4). The field $A$ restricted to $\mathcal{M}_3$ can be decomposed into exact, co-exact and harmonic forms using the Hodge decomposition

$$A = d\vartheta + *dK' + a^{\ell} \alpha^{\ell}$$

where $\{\alpha^{\ell}\}^{p}_{\ell=1} \in H^1(\mathcal{M}_3)$ is an orthonormal basis of harmonic 1-forms and $*$ denotes the Hodge duality operator defined with respect to the metric of $\mathcal{M}_3$. The harmonic basis of 1-forms is chosen to be Poincaré dual to a canonical homology basis of 2-cycles of $\mathcal{M}_3$. Choosing an orthonormal basis $\{\beta^{\ell}\}^{p}_{\ell=1} \in H^2(\mathcal{M}_3)$ of harmonic 2-forms in an analogous way, these generators have the normalizations

$$\int_{\mathcal{M}_3} \alpha^{\ell} \wedge *\alpha_k = \int_{\mathcal{M}_3} \beta^{\ell} \wedge *\beta_k = \delta^{\ell k}, \quad \int_{\mathcal{M}_3} \alpha^{\ell} \wedge \beta_k = M^{\ell k}$$

where $M^{\ell k}$ is the inverse of the topologically-invariant, positive-definite integer-valued linking matrix $M^{k\ell}$ of the homology 1-cycles with the homology 2-cycles $[15, 27]$. The scalar field $\vartheta$, the 1-form field $K'$ and the harmonic coefficients $a^{\ell}$ are formally given by

$$\nabla^2 \vartheta = *d*A, \quad d*K' = F$$

$$a^{\ell}(t) = M^{k\ell} \int_{\mathcal{M}_3} A \wedge \beta_k$$

Similarly, the Hodge decompositions of the 1-form fields $*B$, $\tilde{j}$ and $\tilde{\Sigma}$ over $\mathcal{M}_3$ are

$$*B = d\vartheta' + *dK + b^{\ell} * \beta^{\ell}$$

$$\nabla^2 \vartheta' = *dB, \quad d*K = d*B$$

$$b^{\ell}(t) = M^{k\ell} \int_{\mathcal{M}_3} B \wedge \alpha_k$$

$$\tilde{j} = d\omega' + *d\Omega + j^{\ell} M^{k\ell} * \beta_k$$

$$\nabla^2 \omega' = -\frac{\partial j^0}{\partial t}, \quad d*\Omega = d\tilde{j}$$

$$j^{\ell}(t) = \sum_a q_a \frac{\partial}{\partial t} \left( \int_{r_a(t)} \alpha^{\ell} \right)$$

\footnote{The ensuing construction also applies to the case where $\mathcal{M}_3$ is flat Euclidean 3-space. There $p = 0$ and we assume that the fields vanish at infinity.}
\[ *\Sigma = d\varpi + d\Pi' + \Sigma^\mu M^{\mu k} \alpha_k \]
\[ \nabla^2 \varpi = *d\Sigma, \quad *d *d\Pi' = -\frac{\partial \Sigma_{\nu i}}{\partial t} \, dx^i \]  
(5.6)
\[ \Sigma_{\ell}(t) = \sum_b \phi_b \frac{\partial}{\partial t} \left( \int_{\Sigma_{\ell}(t)} \beta_\ell \right) \]
where \( r_0 \) is a fixed basepoint in \( M_3 \) and the surface \( \Sigma_b(t) \) represents the string worldsheet projected onto \( M_3 \) with boundary the string \( X_b(t, \sigma) \) at time \( t \). In (5.5) and (5.6) we have used the continuity equations \( \partial_{\mu} j^\mu = \partial_{\mu} \Sigma^{\mu \nu} = 0 \) and the explicit forms (2.2) and (2.4) of the sources.

It is convenient to introduce a holomorphic polarization for the harmonic components of the gauge fields \[15\]. Consider the \( 2p \)-dimensional phase space \((4.16)\) of harmonic forms which represents the topological degrees of freedom of the gauge fields that remain when there are no sources present. On this space we introduce a complex structure defined by a symmetric \( p \times p \) complex-valued matrix \( \rho \) such that \(-\rho\) is an element of the Siegel upper half-plane. Its imaginary part defines a metric
\[ G^{\ell k} = -2M^{\mu \ell} \, \text{Im} \rho_{pq} M^{pq} \]
(5.7)
on \( \mathcal{P} \) and the desired polarization is defined by the complex variables
\[ \gamma^\ell = a^\ell + M^{\mu \ell} \rho_{mk} b^k, \quad \bar{\gamma}^\ell = a^\ell + M^{\mu \ell} \bar{\rho}_{mk} b^k \]
(5.8)
In terms of the above decompositions, we find that the canonical quantum commutator \((4.15)\) can be represented by the derivative operators
\[ \vartheta^\ell(x) = \frac{2\pi i}{k} \frac{1}{\nabla^2 - \delta \vartheta(x)} \quad \text{,} \quad *F^i(x) = \frac{2\pi i}{k} P_{ij} \frac{\delta}{\delta K_j(x)} \]
(5.9)
\[ \gamma^\ell = \frac{2\pi}{k} G^{\ell k} \frac{\partial}{\partial \gamma^k} \]
where \( P_{ij} \) is the symmetric transverse projection operator on \( \Lambda^1(M_3) \) defined by
\[ \nabla^i P_{ij} = \nabla^j P_{ij} = 0 \quad \text{,} \quad P_{ij} A^j = A_i - \nabla_i \left( \frac{1}{\nabla^2} \nabla \cdot A \right) \]
(5.10)
and \( \nabla^2 \) denotes the scalar Laplacian with its zero modes removed. The projections onto the subspaces orthogonal to the zero modes can be achieved using time-independent gauge transformations and the vanishing condition on the total flux of the sources.

Substituting (5.1)–(5.6) and (5.9) into (4.12) and integrating by parts, we find that the quantum Hamiltonian can be decomposed into two commuting pieces as \( H = H_L + H_T \). The
The local Hamiltonian $H_L$ depends only on the local parts of the fields,

$$H_L = \int_{\mathcal{M}_3} d^3x \left[ \left( -\vartheta \frac{\partial j^0}{\partial t} + \frac{2\pi i}{k} \varpi(x) \delta \vartheta \right) + \left( -K_i \frac{\partial \Sigma^0_i}{\partial t} - \frac{2\pi i}{k} \Omega_i P^i_j \frac{\delta}{\delta K_j} \right) \right] + \frac{4\pi^2}{k^2} \left( \varpi(x) \nabla^2 \varpi(x) + \Pi'_i \left( \nabla^2 \Pi' \right)^i - \frac{1}{4} P_{ij} \frac{\delta}{\delta K_j} P^i_k \frac{\delta}{\delta K_k} \right)$$

(5.11)

where $\nabla^2_1 = *d*d$ is the Laplacian acting on co-exact 1-forms. The topological Hamiltonian $H_T$ depends only on the global harmonic parts of the fields,

$$H_T = \frac{4\pi^2}{k^2} M^{k\ell} M^{m\ell} \Sigma_k \Sigma_m + i(\Sigma_m - \bar{\rho}_{mn} M^{m\ell} j^\ell) M^{mp} G_{pk} \gamma^k - \frac{2\pi i}{k} \left( \Sigma_k - \rho_{km} M^{m\ell} j^\ell \right) M^{kn} \frac{\partial}{\partial \gamma^n}$$

(5.12)

where $G_{\ell k}$ is the matrix inverse of $G^{\ell k}$. In the Schrödinger picture, we can therefore separate the variables $\vartheta$, $K$ and $\gamma$ and solve for the physical state wavefunctions in the form

$$\Psi_{\text{phys}}[\vartheta, K, \gamma; t] = \Psi_L[\vartheta, K; t] \Psi_T(\gamma; t)$$

(5.13)

### 5.1 Local Gauge Symmetries and Adiabatic Linking Numbers

The local wavefunctionals $\Psi_L$ must solve the first class constraints (4.13), which using the representations (5.9) can be written as

$$\left( i \frac{\delta}{\delta \vartheta(x)} + j^0(x, t) - \frac{4\pi}{k} \nabla \cdot \Sigma(x, t) \right) \Psi_L[\vartheta, K; t] = 0$$

(5.14)

$$\left( i P^i_j \frac{\delta}{\delta K_j(x)} + \Sigma^0_i(x, t) \right) \Psi_L[\vartheta, K; t] = 0$$

They are solved by wavefunctionals of the form

$$\Psi_L[\vartheta, K; t] = \exp \left[ i \int_{\mathcal{M}_3} d^3x \left\{ \vartheta(x) \left( j^0(x, t) - \frac{4\pi}{k} \nabla \cdot \Sigma(x, t) \right) + K_i(x) \Sigma^0_i(x, t) \right\} \right] \tilde{\Psi}_L(t)$$

(5.15)

These wavefunctionals transform under the time-independent local gauge transformations in (2.1) which are exact,

$$A_i \to A_i + \nabla_i \chi' , \quad B_{ij} \to B_{ij} + \nabla_i \xi_j' - \nabla_j \xi_i'$$

(5.16)

This gauge symmetry is represented projectively in the states (5.15) as

$$\Psi_L[\vartheta + \chi', K + \xi'; t] = \exp \left[ i \int_{\mathcal{M}_3} d^3x \left\{ \chi'(x) \left( j^0(x, t) - \frac{4\pi}{k} \nabla \cdot \Sigma(x, t) \right) + \xi'_i(x) \Sigma^0_i(x, t) \right\} \right] \Psi_L[\vartheta, K; t]$$

(5.17)
in terms of a non-trivial local $U(1) \times U(1)$ 1-cocycle. The remaining piece $\tilde{\Psi}_L(t)$ is determined from the Schrödinger equation
\[ i \frac{\partial}{\partial t} \Psi_{\text{phys}}[\theta, K, \gamma; t] = H \Psi_{\text{phys}}[\theta, K, \gamma; t] \]  
(5.18)
which using (5.11) shows that it contains two contributions,
\[ \tilde{\Psi}_L(t) = e^{i \int_{-\infty}^{t} dt' \left( \mathcal{L}(t') + S(t') \right)} \]  
(5.19)

In this subsection we shall discuss the first contribution to the local wavefunctionals (5.19) which is the particle-string term
\[ \mathcal{L}(t) = \frac{2\pi}{k} \int_{\mathcal{M}_3} d^3 x \left( \varpi(x, t) j^0(x, t) - \Omega_i(x, t) \Sigma^{0i}(x, t) \right) \]  
(5.20)
This integral was evaluated in [15] using the relations (5.5), (5.6), (2.2), (2.4) and shown to give
\[ \mathcal{L}(t) = -\frac{1}{2k} \sum_{a,b} g_\alpha \phi_{\alpha b}(t) \frac{d\Phi_{ab}(t)}{dt} + \frac{2\pi}{k} j_i(t) M^{k\ell} \int_{-\infty}^{t} dt' \Sigma_k(t') \]  
(5.21)
where
\[ \Phi_{ab}(t) = 4\pi \int_{-\infty}^{t} \left( \int_{\Sigma_b(t')} \delta^{(1,2)}(r_a(t'), X_b(t', \sigma)) \right) dl^i(r_a(t')) + 4\pi \sum_{\lambda \neq 0} \frac{1}{\lambda^2} \psi_\lambda(r_a(t)) \left( \int_{\Sigma_b(t)} d\psi_\lambda \right) \]  
(5.22)
is the generalized solid angle function on $\mathcal{M}_3$. Here the Dirac delta-function is defined as described in subsection 3.1 except now over $\mathcal{M}_3$, and $\psi_\lambda(x)$ are the orthonormal eigenfunctions of the scalar Laplacian operator on $\mathcal{M}_3$ with eigenvalue $\lambda^2$,
\[ \nabla^2 \psi_\lambda(x) = *d * d\psi_\lambda(x) = \lambda^2 \psi_\lambda(x) \]  
,  
\[ \int_{\mathcal{M}_3} \psi_\lambda * \psi_{\lambda'} = \delta_{\lambda \lambda'} \]  
(5.23)
Note that the Dirac delta-function $\delta^{(3)}(x, y) \in \Lambda^0(\mathcal{M}_3)$ (or $\delta^{(0,3)}(x, y) \in \Lambda^0(\mathcal{M}_3(x)) \otimes \Lambda^3(\mathcal{M}_3(y))$) can then be represented in terms of the completeness relation
\[ \delta^{(3)}(x, y) = \sum_\lambda \psi_\lambda(x) \psi_\lambda(y) \quad \text{or} \quad \delta^{(0,3)}(x, y) d^3 y = \ast \delta^{(3)}(x, y) = \sum_\lambda \psi_\lambda(x) \otimes \ast \psi_\lambda(y) \]  
(5.24)
If we further introduce a basis of orthonormal co-exact 1-forms $\psi^{(c)}_\lambda$ which are the eigenstates of the Laplacian operator $\nabla_1^2$ with eigenvalue $\lambda^2$,
\[ \nabla_1^2 \psi^{(c)}_\lambda = *d * d\psi^{(c)}_\lambda = \lambda^2 \psi^{(c)}_\lambda \]  
,  
\[ \int_{\mathcal{M}_3} \psi^{(c)}_\lambda \wedge * \psi^{(c)}_{\lambda'} = \delta_{\lambda \lambda'} \]  
(5.25)
then the delta-function $\delta^{(1,2)}(x, y) \in \Lambda^1(\mathcal{M}_3(x)) \otimes \Lambda^2(\mathcal{M}_3(y))$ can be represented in terms of the completeness relation
\[ \delta^{(1,2)}(x, y) = -\sum_{\lambda \neq 0} \frac{d\psi_\lambda(x) \otimes * d\psi_\lambda(y)}{\lambda^2} + \sum_{\lambda \neq 0} \psi^{(c)}_\lambda(x) \otimes \psi^{(c)}_\lambda(y) + \alpha(t) \otimes M^{mt} \beta_m(y) \]  
(5.26)
The function (5.22) depends only on the topological classes of the particle and string trajectories in $\mathcal{M}_3$ and it represents the solid angle formed by a string along $X_b(t,\sigma)$ relative to a particle at $r_a(t)$ [13]. It has the property that it changes by $4\pi$ everytime that a particle is adiabatically transported around a fixed string (for which $\Sigma_b(t)$ is constant and only the first term in (5.22) contributes), or a string around a fixed particle in the opposite direction (for which $r_a(t)$ is constant and only the second term in (5.22) contributes), as long as these trajectories do not intersect. $\Phi_{ab}(t)$ is the multivalued angle function that one anticipates in a theory of adiabatic transports, and we see that the first term in (5.19) represents the non-trivial particle-string linkings. It is that part of the full wavefunction that represents the exotic exchange holonomies between particles and strings and is easily seen to be the adiabatic limit of the covariant linking number in (3.3) that arises in the effective field theory. When $\mathcal{M}_3 = \mathbb{R}^3$ the function (5.22) reduces to the usual form of a solid angle [13].

5.2 Adiabatic Intersection Indices and Euler Numbers

In this subsection we will evaluate the second contribution to (5.19) which is given by the string-string term

$$S(t) = -\frac{\pi^2}{k^2} \int_{\mathcal{M}_3} d^3x \left( -4\varpi(x,t) \nabla^2 \varpi(x,t) + 4\Pi'_i(x,t)(\nabla^2 \Pi')^i(x,t) + \Sigma_{0i}(x,t)\Sigma_{0i}(x,t) \right)$$

(5.27)

For this, we use the completeness relations (5.24) and (5.26) along with (5.6) and (2.4) to write

$$\Sigma_{0i}(x,t) dx^i = \sum_b \phi_b \sum_{\lambda \neq 0} \psi^{(c)}_{\lambda}(x) \left( \int \partial \Sigma_b(t) \psi^{(c)}_{\lambda} \right)$$

$$\varpi(x,t) = -\sum_b \phi_b \sum_{\lambda \neq 0} \psi^{(c)}_{\lambda}(x) \frac{\partial}{\partial t} \left( \int \Sigma_b(t) * d\psi^{(c)}_{\lambda} \right)$$

(5.28)

The 1-form $\Pi'$ is then written in terms of these eigenstates using (5.8). Substituting these decompositions into (5.27) and integrating by parts using (5.23) and (5.25) we then have

$$S(t) = \frac{4\pi^2}{k^2} \sum_{b,\nu} \phi_b \phi_{\nu} \left[ \sum_{\lambda \neq 0} \frac{1}{\lambda^2} \frac{\partial}{\partial t} \left( \int \Sigma_{b}(t) * d\psi^{(c)}_{\lambda} \right) \frac{\partial}{\partial t} \left( \int \Sigma_{\nu}(t) * d\psi^{(c)}_{\lambda} \right) \right.$$

$$\left. - \sum_{\lambda \neq 0} \frac{\partial}{\partial t} \left( \int \Sigma_{b}(t) * \psi^{(c)}_{\lambda} \right) \frac{\partial}{\partial t} \left( \int \Sigma_{\nu}(t) * \psi^{(c)}_{\lambda} \right) - \frac{1}{4} \lambda^2 \left( \int \Sigma_{b}(t) * \psi^{(c)}_{\lambda} \right) \left( \int \Sigma_{\nu}(t) * \psi^{(c)}_{\lambda} \right) \right]$$

(5.29)
Integrating the first two terms in (5.29) by parts over time gives
\[ S(t) = \frac{2\pi^2}{k^2} \sum_{b,b'} \phi_b \phi_{b'} \left[ \frac{\partial^2}{\partial t^2} \left\{ \sum_{\lambda \neq 0} \frac{1}{\lambda^2} \left( \int_{\Sigma_b(t)} *d\psi_\lambda \right) \left( \int_{\Sigma_{b'}(t)} *d\psi_\lambda \right) \right\} \right. 
- \sum_{\lambda \neq 0} \left( \int_{\Sigma_b(t)} *\psi_{\lambda}^{(c)} \right) \left( \int_{\Sigma_{b'}(t)} *\psi_{\lambda}^{(c)} \right) - \left( \int_{\Sigma_b(t)} *\alpha_\ell \right) \left( \int_{\Sigma_{b'}(t)} M^{m\ell} \beta_m \right) \right] 
+ \frac{\partial^2}{\partial t^2} \left\{ \left( \int_{\Sigma_b(t)} *\alpha_\ell \right) \left( \int_{\Sigma_{b'}(t)} M^{m\ell} \beta_m \right) \right\} 
- \sum_{\lambda \neq 0} \left( \int_{\Sigma_b(t)} *\psi_{\lambda}^{(c)} \right) \left( \int_{\Sigma_{b'}(t)} *\psi_{\lambda}^{(c)} \right) + \frac{\partial^2}{\partial t^2} \left( \int_{\Sigma_b(t)} *d\psi_\lambda \right) \left( \int_{\Sigma_{b'}(t)} *d\psi_\lambda \right) 
+ \sum_{\lambda \neq 0} \left( \int_{\Sigma_b(t)} *\psi_{\lambda}^{(c)} \right) \left( \int_{\Sigma_{b'}(t)} *\psi_{\lambda}^{(c)} \right) \left( \int_{\Sigma_{b'}(t)} *\psi_{\lambda}^{(c)} \right) 
\left. - \frac{1}{4} \sum_{\lambda \neq 0} \left( \int_{\partial \Sigma_b(t)} *d\psi_{\lambda}^{(c)} \right) \left( \int_{\Sigma_{b'}(t)} *\psi_{\lambda}^{(c)} \right) + \left( \int_{\Sigma_b(t)} *\psi_{\lambda}^{(c)} \right) \left( \int_{\partial \Sigma_{b'}(t)} *d\psi_{\lambda}^{(c)} \right) \right] \right\} 
(5.30)

The first three terms in (5.30) can be combined together to give the Dirac delta-function \(*\delta^{(1,2)}(x, x') \in \Lambda^2(\mathcal{M}_3(x)) \otimes \Lambda^2(\mathcal{M}_3(x'))\) with \(x(t) \in \Sigma_b(t)\) and \(x'(t) \in \Sigma_{b'}(t)\). The fourth term can be rewritten using the harmonic coefficients \(\Sigma_{\ell}(t)\) in (5.3) and Hodge duality to relate the harmonic 1-forms and 2-forms by \(*\alpha_\ell = M^{k\ell} \beta_k\). In this way we arrive finally at the expression
\[ S(t) = -\frac{1}{4k^2} \sum_{b,b'} \phi_b \phi_{b'} \frac{d\Upsilon_{b'}(t)}{dt} + \frac{4\pi^2}{k^2} M^{k\ell} M^{m\ell} \frac{d}{dt} \int_{-\infty}^t dt' \Sigma_k(t) \Sigma_m(t') \] 
(5.31)
where
\[ \Upsilon_{b'}(t) = 8\pi^2 \frac{d}{dt} \int_{\Sigma_b(t)} \int_{\Sigma_{b'}(t)} *\delta^{(1,2)}(X_b(t, \sigma), X_{b'}(t, \sigma')) + 8\pi^2 \Xi_E[N\Sigma_b(t) \cap N\Sigma_{b'}(t)] \] 
(5.32)
is the generalized intersection number of the projected surfaces \(\Sigma_b(t)\) and \(\Sigma_{b'}(t)\) onto \(\mathcal{M}_3\) with
\[ \Xi_E[N\Sigma_b(t) \cap N\Sigma_{b'}(t)] = \int_{-\infty}^t dt' \left[ \sum_{\lambda \neq 0} \frac{1}{\lambda^2} \left\{ \left( \int_{\Sigma_{b'}(t')} *d\psi_\lambda \right) \frac{d^2}{dt'^2} \left( \int_{\Sigma_{b'}(t')} *d\psi_\lambda \right) \right. 
+ \frac{d^2}{dt'^2} \left( \int_{\Sigma_{b'}(t')} *d\psi_\lambda \right) \left( \int_{\Sigma_{b'}(t')} *d\psi_\lambda \right) \right] 
- \sum_{\lambda \neq 0} \left( \int_{\Sigma_{b'}(t')} *\psi_{\lambda}^{(c)} \right) \frac{d^2}{dt'^2} \left( \int_{\Sigma_{b'}(t')} *\psi_{\lambda}^{(c)} \right) + \frac{d^2}{dt'^2} \left( \int_{\Sigma_{b'}(t')} *\psi_{\lambda}^{(c)} \right) \left( \int_{\Sigma_{b'}(t')} *\psi_{\lambda}^{(c)} \right) 
+ \frac{1}{4} \sum_{\lambda \neq 0} \left( \int_{\partial \Sigma_b(t')} *d\psi_{\lambda}^{(c)} \right) \left( \int_{\Sigma_{b'}(t')} *\psi_{\lambda}^{(c)} \right) + \left( \int_{\Sigma_b(t')} *\psi_{\lambda}^{(c)} \right) \left( \int_{\partial \Sigma_{b'}(t')} *d\psi_{\lambda}^{(c)} \right) \right\} \right] \] 
(5.33)
Unlike the solid angle function (5.22), this intersection function is not a homological invariant of the surfaces $\Sigma_b(t)$. Under a homologically trivial motion $\Sigma_b(t) \to \Sigma_b(t) + \partial B_b(t)$ of a given string for some volume $B_b(t) \subset M_3$, using Stokes’ theorem we find that the function (5.32) for $b \neq b'$ changes by

$$
\delta \Upsilon_{bb'}(t) = -16\pi^2 \int_{-\infty}^{t} dt' \sum_{\lambda \neq 0} \frac{d}{dt'} \left( \int_{B_b(t')} \psi_{\lambda}(x) \ d^3x \right) \frac{d}{dt'} \left( \int_{\Sigma_{b'}(t')} *d\psi_{\lambda} \right) \tag{5.34}
$$

which is non-vanishing in general. It is, however, a smooth invariant of the projected worldsheets. This is anticipated since (5.32) represents the adiabatic limit of the extrinsic intersection index in $M_4$ which we studied in section 3. If string $b'$ is held fixed and string $b \neq b'$ is adiabatically transported through space, then only the delta-function in (5.32) contributes and it counts extrinsic intersections of $\Sigma_b(t)$ and $\Sigma_{b'}(t)$. Each such intersection essentially contributes $8\pi^2$ to the function $\Upsilon_{bb'}(t)$. The time derivative acts to give an extrinsic variation in the normal direction to $M_3$ and it is the adiabatic analog of the transversality factor in (3.26). The function (5.33) in general acts to make the adiabatic intersection index well-defined for $b = b'$. When $b = b'$ and the surfaces do not self-intersect, then the delta-function term in (5.32) and the last sum in (5.33) represent the adiabatic area form of $\Sigma_b(t)$ while the remaining terms in (5.33) are the adiabatic limits of its extrinsic curvature form.

Thus the second term in (5.19) represents a non-topological string-string holonomy term which takes into account the intersections of the strings and also the extrinsic geometry of their worldsheets. The function $\Xi_E[N \Sigma_b(t) \cap N \Sigma_{b'}(t)]$ is the adiabatic limit of the Euler numbers, which provide global corrections to the local intersection indices in terms of the geometry and topology of the projected normal bundles of the string worldsheets in $\mathbb{R} \times M_3$ (see (3.21) and (3.28)), and also of the transversality factor in (3.26). This follows from the way that it is explicitly related to the spectrum of the Laplace-Beltrami operator of $M_3$ and that the one-forms $d\psi_{\lambda}, \psi^{(c)}_{\lambda}$ and $*d\psi^{(c)}_{\lambda}$ can be regarded as connections on $N \Sigma_b(t)$. $\Upsilon_{bb'}(t)$ yields another sort of multivalued “angle function” in the wavefunctions of the deformed $BF$ theory which is very different from the usual surface-surface linking terms.\footnote{Note that it is not possible to generically write down an adiabatic limit of the topological intersection numbers which we encountered in subsection 3.4, so that the canonical quantization procedure of section 4 has consistently removed such potential topological string terms from the Hamiltonian.}

For example, these intersection indices are different than those found in [37] where the coupling of dynamical point particles to $BF$ gauge fields was considered. The classical observables in this case are related to mod 2 intersection numbers of $p$-chains and $(d-p)$-chains, as well as the construction of global step functions, on $d$-dimensional manifolds.
5.3 Global Gauge Symmetries and Cohomology Representations

The topological wavefunctions in (5.13) represent the windings of the sources around non-contractible cycles in $\mathcal{M}_3$. From (5.12) and the Schrödinger equation (5.18) we see that the $y$ have the form

$$\Psi_T(\gamma; t) = \exp \left[ \int_{-\infty}^{t} dt' \left( \Sigma_m(t') - \bar{\rho}_{mn}(t')M^{\ell\ell'}j_{\ell'}(t') \right) M^{mp}G_{pk}\gamma^k \right. $$

$$- \frac{4i\pi^2}{k^2} M^{k\ell}M^{m\ell} \int_{-\infty}^{t} dt' \Sigma_k(t') \Sigma_m(t') - \frac{2\pi}{k} \int_{-\infty}^{t} dt' \left( \Sigma_m(t') - \rho_{mp}M^{p\ell}j_\ell(t') \right) $$

$$\times M^{mq}G_{qr}M^{kr} \int_{-\infty}^{t'} dt'' \left( \Sigma_k(t'') - \bar{\rho}_{ks}M^{sn}j_n(t'') \right) \left. \right] \Psi_0(\gamma; t)$$

(5.35)

where the function $\Psi_0$ is a solution of the equation

$$\frac{\partial \Psi_0(\gamma; t)}{\partial t} = -\frac{2\pi}{k} \left( \Sigma_k - \rho_{km}M^{m\ell}j_\ell \right) M^{kn} \frac{\partial \Psi_0(\gamma; t)}{\partial \gamma^n}$$

(5.36)

The equation (5.36) is solved by any function of the form

$$\Psi_0(\gamma; t) = \Psi_0 \left( \gamma^\ell - \frac{2\pi}{k} M^{k\ell} \int_{-\infty}^{t} dt' \left( \Sigma_k(t') - \rho_{km}M^{m\ell}j_m(t') \right) \right)$$

(5.37)

The function $\Psi_0$ is fixed by requiring that, in addition to the local gauge invariance (4.13), the theory also be invariant under large gauge transformations which are not connected to the identity in $\mathcal{P}$, i.e. those forms in (2.1) which are not exact and have non-trivial cohomological parts. For a consistent quantum theory, this global gauge symmetry must be restricted to those forms in (2.1) which have integer-valued cohomology [15], i.e. $\int_L \chi$ and $\int_\Sigma \xi$ are integer multiples of $2\pi$. When there are no sources present the wavefunctions $\Psi_0$ should coincide with the cohomological states which represent the invariance of the quantum field theory under these winding transformations.

In the source-free case the local gauge constraints (5.14) imply that the physical state wavefunctions are functions only of the global harmonic variables $\gamma$. Moreover, the Hamiltonian then vanishes when acting on these states and so they are also time independent. In terms of the holomorphic polarization (5.8) the restricted global gauge transformations are

$$\gamma^\ell \to \gamma^\ell + 2\pi(n^\ell + M^{m\ell}\rho_{mk}m^k), \quad \bar{\gamma}^\ell \to \bar{\gamma}^\ell + 2\pi(n^\ell + M^{m\ell}\bar{\rho}_{mk}m^k)$$

(5.38)

where $n^\ell$ and $m^\ell$ are integers. The invariance of the physical states under the winding transformations (5.38) has been studied in detail in [15] for the case when the coefficient $k$ of the $BF$ term in (2.6) is of the form $k = Mk_1/k_2$, where $M > 0$ is the integer-valued determinant of the linking matrix and $k_1$ and $k_2$ are positive integers with gcd($Mk_1, k_2$) = 1. We shall henceforth consider only these values of $k$. This invariance condition is then uniquely solved.
where \( q^\ell = 1, 2, \ldots, Mk_1 k_2 \) and \( \Theta \) are the doubly semi-periodic Jacobi theta-functions

\[
\Theta \left( \frac{c}{d} \bigg| \frac{z}{\Pi} \right) = \sum_{\{n^\ell\} \in \mathbb{Z}^p} \exp \left[ i\pi \left( \frac{n^\ell + c^\ell}{Mk_1 k_2} n^k + c^k \right) + 2\pi i \left( \frac{n^\ell + c^\ell}{Mk_1 k_2} (z^\ell + d^\ell) \right) \right]
\]

which are well-defined and holomorphic for \( c^\ell, d^\ell \in [0, 1] \), \( \{z^\ell\} \in \mathbb{C}^p \) and \( \Pi = [\Pi^k] \) in the Siegel upper half-plane.

The wavefunctions (5.39) are orthogonal in the inner product on \( \mathcal{P} \) determined by the canonical coherent state measure for the holomorphic polarization (5.8). The basis of states (5.39) thereby produces an orthonormal basis of the full physical Hilbert space. They are well-defined functions on the reduced topological phase space

\[
\mathcal{P}_{\text{red}} = H^1(\mathcal{M}_3) \oplus H^2(\mathcal{M}_3)/\Gamma_G
\]

where \( \Gamma_G \cong \mathbb{Z}^p \oplus \mathbb{Z}^p \) is the torsion-free part of the integer cohomology group \( H^1(\mathcal{M}_3; \mathbb{Z}) \oplus H^2(\mathcal{M}_3; \mathbb{Z}) \). The free parameters \( c \) and \( d \) appearing in (5.39) can then be fixed by choosing a spin structure on the complex \( p \)-torus (5.41). From the transformation properties of the Jacobi theta-functions under the modular group \( Sp(2p, \mathbb{Z}) \) (acting on the matrix \( \rho \)) it follows that the physical observables of the quantum field theory are independent of the choice of phase space complex structure, as expected from the topological properties of the theory.

If we denote the unitary quantum operators that generate the large gauge transformations (5.38) on the Hilbert space by \( U(n, m) \), then the wavefunctions (5.39) transform under them as

\[
U(n, m) \Psi_0^{(q)} \left( \frac{c}{d} \bigg| \frac{\gamma}{\ell_\Gamma} \right) = \sum_{q^{\ell}} [U(n, m)]_{qq^{\ell}} \Psi_0^{(q^{\ell})} \left( \frac{c}{d} \bigg| \frac{\gamma}{\ell_\Gamma} \right)
\]

where the unitary matrices

\[
[U(n, m)]_{qq^{\ell}} = \exp \left[ \frac{2\pi i}{k_2} \left( \frac{c^\ell M_{m^\ell} n^m + d^\ell M_{m^\ell} n^m}{Mk_1 k_2} - i\pi k n^m M_{m^\ell} m^\ell \right) \right] \delta_{q^{\ell} - k_1 M m^\ell, q^{k}}
\]

generate a \((k_2)^p\)-dimensional projective representation of the group \( \Gamma_G \) of large gauge transformations. The projective phases here are non-trivial global \( U(1)^p \times U(1)^p \) 1-cocycles and are cyclic with period \( k_2 \). The topological part of the full wavefunction therefore carries a non-trivial multi-dimensional representation of the discrete group \( \Gamma_G \) representing the windings around the non-trivial homology cycles of \( \mathcal{M}_3 \). The invariance of the physical state wavefunctions under these global gauge symmetries partitions the Hilbert space into superselection sectors labelled by the integer cohomology classes of \( \mathcal{M}_3 \), and thus the quantum
states of the deformed BF theory provide novel quantum representations of the cohomology ring $H^1(M_3) \oplus H^2(M_3)$. When combined with the explicit time dependence (5.13), we shall see in the next section that the full wavefunctional also carries a multi-dimensional projective representation of the algebra dual to the algebra of large gauge transformations.

6 Transformation Properties of the Physical States

We will now examine the properties of the basis of full physical wavefunctions (5.13), which is given by combining together all of the components (5.15), (5.19), (5.21), (5.31), (5.35), (5.37) and (5.39). Using (2.2) and (2.4), after some algebra we arrive at the total wavefunction

$$
\Psi^{(q)}_{phys}(c \otimes d) [\varphi, K, \gamma; t] = \exp \left[ i \sum_a q_a \varphi(r_a(t)) + i \sum_b \phi_b \int d\sigma \frac{\partial X^i_j(t, \sigma)}{\partial \sigma} \left( K_i(X_b(t, \sigma)) - \frac{4\pi}{k} \epsilon_{ijk} \frac{\partial X^j_k(t, \sigma)}{\partial t} \nabla^k \vartheta(X_b(t, \sigma)) \right) \right] 
$$

$$
\times \exp \left[ -\frac{i}{2k} \sum_{a,b} q_a \phi_b \left( \Phi_{ab}(t) - \Phi_{ab}(\infty) \right) - \frac{i}{4k^2} \sum_{b,b'} \phi_b \phi_{b'} \left( Y_{bb'}(t) - Y_{bb'}(\infty) \right) + \frac{k}{4\pi} \gamma^k G_{k\ell} \gamma^\ell + 2\pi \int_{-\infty}^{t} dt' j_i(t') M^{k\ell} \int_{-\infty}^{t'} dt'' \Sigma_k(t'') 
$$

$$
- \frac{i\gamma^k}{k} \int_{-\infty}^{t} dt' j_k(t') - \frac{i}{k} \int_{-\infty}^{t} dt' j_k(t') M^{pk} \rho_{pp} M^{pt} \int_{-\infty}^{t} dt'' j_l(t'') + \frac{4\pi^2}{k^2} \int_{-\infty}^{t} dt' \Sigma_k(t') M^{k\ell} M^{\ell m} \left( \Sigma_m(t) - \Sigma_m(t') \right) \right] 
$$

$$
\times \Theta \left( \frac{c+q}{Mk_1k_2} \right) \left( \frac{Mk_1}{2\pi} M_{k\ell} \gamma^k - k_2 \int_{-\infty}^{t} dt' \left( \Sigma_{\ell}(t') - \rho_{m} M^{mn} j_n(t') \right) \right) \right) 
$$

(6.1)

where $q^k = 1, 2, \ldots, Mk_1k_2$, $\ell = 1, \ldots, p$, $k = Mk_1/k_2$, and the topological components $j_{\ell}(t)$ and $\Sigma_{\ell}(t)$ of the sources are given in (5.3) and (5.6). The wavefunctions (6.1) span a finite-dimensional Hilbert space and reduce to the wavefunctions of pure BF theory in the large-$k$ limit (as does the effective action (2.3)), as anticipated since the coupling constant of the quantum field theory is $1/k$. They represent the (exact) one-loop renormalization of the wavefunctions of the canonical quantum field theory (1.2).

The first exponential in (6.1) represents the invariance of the physical states under local gauge transformations. It determines a one-dimensional projective representation of the local $U(1) \times U(1)$ gauge group with 1-cocycle

$$
\Delta[\chi', \xi'] = \frac{1}{2\pi} \sum_a q_a \chi'(r_a(t)) + \frac{1}{2\pi} \sum_b \phi_b \left[ \int_{\Sigma_k(t)} \ast \xi' - \frac{4\pi}{k} \frac{\partial}{\partial t} \left( \int_{\Sigma_k(t)} \ast d\chi' \right) \right] 
$$

(6.2)
where \((\chi', \xi') \in \Lambda^0(M_3) \oplus \Lambda^1(M_3)\). This cocycle mixes the local 1-form (particle) and 2-form (string) gauge degrees of freedom in a non-trivial way and thereby defines a twisted representation of the local gauge group. This projective representation therefore differs significantly from those of usual topological gauge theories. The mixing term in (6.2) can be absorbed into a secondary gauge transformation (2.13), so that the physical states also naturally carry a representation of the secondary gauge symmetry.

The next set of local contributions involving the solid angle functions \(\Phi_{bb}(t)\) represent the adiabatic topological linking numbers of the particle and string trajectories in \(M_3\) and it endows the particle-string wavefunctions with fractional exchange statistics, i.e. when a particle of charge \(q_a\) and a string of flux \(\phi_b\) are adiabatically rotated once around one another, the wavefunction acquires the phase

\[
\hat{\sigma} = e^{-\frac{2\pi i}{M} q_a \phi_b}
\]

The wavefunctionals (6.1) therefore carry, in addition to the local gauge symmetries, a one-dimensional unitary representation of the subgroup of the motion group of \(M_3\) [13, 18] (the fundamental homotopy group of the particle-string quantum configuration space) consisting of the particle-string exchange holonomies. If \(M_3\) is homologically trivial then this is the full motion group of the space. When \(M_3\) has non-trivial homology we will see that the full wavefunctions also carry a representation of the other part of the motion group associated with the windings of the particles and strings around the non-trivial homology cycles of \(M_3\). Likewise, the set of local terms involving the intersection functions \(\Upsilon_{bb'}(t)\) represent non-topological intersection numbers as well as the extrinsic geometry of the strings. They yield non-trivial phase factors in the wavefunctions arising from intersections and self-intersections of the worldsheets. They also provide a novel representation of the adiabatic limit of the Euler characteristic classes of the normal bundles of the string worldsheets.

When the particle and string sources are fixed, the topological components of the wavefunctions (6.1) carry a twisted projective representation of the global gauge group \(\Gamma_G\) with 1-cocycle

\[
\Delta \left( \frac{c^\ell}{d^\ell} \right) (n^\ell, m^\ell; k, M) = \frac{1}{k_2} \left( c^\ell M_{r\ell} n^r + d^\ell m^\ell + q^\ell M_{r\ell} n^r - \frac{Mk_1}{2} n^r M_{r\ell} m^\ell \right)
\]

where \((n, m) \in \Gamma_G\). The remaining components in (6.1) then yield a “topological duality” between representations of \(\Gamma_G\), when we consider homologically non-trivial motions of the particles and strings. Consider the source configuration whereby the strings are fixed and the particles wind \(t_k\) times, up to a time \(\tilde{t}\), around the \(k\)th homology 1-cycle of \(M_3\),

\[
\int_{-\infty}^{\tilde{t}} dt' \ j_k(t') = t_k \quad , \quad \int_{-\infty}^{\tilde{t}} dt' \ \Sigma_k(t') = 0
\]
and then afterwards the particles are fixed and the strings wind $s_k$ times, up to a time $t > \tilde{t}$, around the $k^{th}$ homology 2-cycle of $\mathcal{M}_3$,

$$\int_{\tilde{t}}^{t} dt' \ j_k(t') = 0 \quad , \quad \int_{\tilde{t}}^{t} dt' \ \Sigma_k(t') = s_k$$

(6.6)

The holonomies which arise from these configurations are taken into account by the functions $\Phi_{ab}(t)$ and $\Upsilon_{bb}(t)$. Modulo these holonomies, the wavefunctions (6.1) transform under these motions as

$$\Psi^{(q)}_{\text{phys}}[\vartheta, K, \gamma; t] \rightarrow \sum_{q'} \left[ \tilde{U}(s, t) \right]_{qq'} \Psi^{(q')}_{\text{phys}}[\vartheta, K, \gamma; -\infty]$$

(6.7)

where the unitary matrices

$$\left[ \tilde{U}(s, t) \right]_{qq'} = \exp \left[ \frac{2\pi i}{k_1 M} \left( d_k M \ell t_\ell - s_k c_k - s_k q^k \right) + \frac{2\pi i}{k} s_k M \ell t_\ell \right] \delta_{q' - k_2 M \ell t_\ell, q^k}$$

(6.8)

generate a $(k_1)^p$-dimensional projective representation of $\Gamma_G$. This representation is dual to the one in (5.42), in that the corresponding projective phase can be considered as the dual 1-cocycle to (6.4),

$$\tilde{\Delta} \begin{pmatrix} c^\ell \\ d_\ell \end{pmatrix} (s, t; k, M) = \Delta \begin{pmatrix} M \ell d_k \\ M \ell c_k \end{pmatrix} (\text{\small{-}}M_km^k, \text{\small{-}}M_kn^k, \frac{1}{k}, M^{-1})$$

(6.9)

A similar sort of topological duality has been exploited recently in [39] to provide a deformed topological field theory interpretation of the phenomenon of mirror symmetry in string theory.

This duality acts as both an $S$-duality $k \leftrightarrow \frac{1}{k}$, relating perturbative and non-perturbative regimes of the quantum field theory and interchanging electric charges with string fluxes in its spectrum, and also as a Poincaré-Hodge duality relating non-trivial integer cohomology classes and homology classes. The corresponding algebra of the unitary operators (6.8) consists of those generators of the motion group of $\mathcal{M}_3$ which are associated with windings of the particles and strings around the non-contractible cycles of $\mathcal{M}_3$. Combining them with the local generators represented by the phases (5.3) we obtain an intriguing representation of the full motion group of $\mathcal{M}_3$ which can be described as follows. Let $\hat{s}_{k}^{(\ell)} = \delta_{k\ell}$ and $\hat{t}_{k}^{(m)} = \delta_{km}$. Then the operators

$$\hat{\alpha}_\ell = \tilde{U} \left( \hat{s}_{k}^{(\ell)}, 0 \right) \quad , \quad \hat{\beta}_m = \tilde{U} \left( 0, \hat{t}_{k}^{(m)} \right)$$

(6.10)

are the generators of motions of the particles and strings associated with the $\ell^{th}$ homology 1-cycle and $m^{th}$ homology 2-cycle, respectively. The operators (6.10) together with (5.3) generate a $(k_1)^p$-dimensional representation of the group of motions of the particle worldlines and string worldsheets in the 3-manifold $\mathcal{M}_3$ with the relations

$$\left[ \hat{\alpha}_\ell, \hat{\alpha}_m \right] = \left[ \hat{\beta}_\ell, \hat{\beta}_m \right] = 0 = \left[ \hat{\alpha}_\ell, \hat{\sigma} \right] = \left[ \hat{\beta}_\ell, \hat{\sigma} \right]$$

$$\hat{\alpha}_\ell \hat{\beta}_m = e^{\frac{2\pi i}{k} M_{\ell m}} \hat{\beta}_m \hat{\alpha}_\ell$$

(6.11)

$$30$$
where we have used (6.8). This is an abelian representation of the motion group which generalizes the presentation given in [38] from $\mathbb{R}^3$ to an arbitrary compact closed 3-manifold $M_3$. Note that in this representation the linking matrix of $M_3$ plays a crucial role. $M^{k\ell}$ is the identity matrix typically only for 3-manifolds $M_3$ which are product spaces. Thus the $BF$ field theory also provides a natural way of defining the motion group of generic manifolds.

The wavefunctions (6.11) thus incorporate the topology of the underlying 3-manifold $M_3$ (via their dependence on the linking matrix $M^{k\ell}$) and of its motion group (via the representation (6.11)) in precisely the same way that the wavefunctions of ordinary $BF$ field theory do. The only overall effect of the topological perturbation to the $BF$ action is to incorporate a sort of non-topological holonomy factor for the intersections of the strings which is represented in the twisted local 1-cocycle (6.2) and the intersection function $\Upsilon_{bb'}(t)$. The wavefunctions (6.1) do, however, represent the particle and string degrees of freedom in more symmetric fashion and naturally incorporate the first-stage reducible gauge symmetries of the topological field theory.

The deformed $BF$ theory thus yields quantum field theoretical representations for new sorts of smooth invariants of 3-manifolds. The quantum holonomies induced by these string-string terms could be relevant to the physics of systems which involve vortex strings [11]–[13], [16]–[21]. In abelian Higgs field theories, where the charged particles are represented by dynamical scalar fields, the structures described in section 3 emerge as the leading orders of a large Higgs mass expansion. The present formalism, which involves non-dynamical point particles, naturally incorporates the particle-string Aharonov-Bohm phases [17, 19], extrinsic curvature terms [18, 19], and similar long-range string intersection interactions [19] that have been discussed extensively in Higgs models. The emergence of smooth surface invariants in this topological field theory is intriguing in light of recent work [40] on observables in non-abelian $BF$ theories which suggests that surface observables yield possibly new invariants of immersed surfaces in 4-manifolds. In the case of non-topological deformations of $BF$ theory, these observables may be relevant to the quark confinement problem [14]. It would be interesting to generalize the topological deformation we have considered in this paper to the case of higher-dimensional $BF$ theories and to see what sort of smooth invariants arise in these cases. This appears to be difficult to do within a general framework, as the given class of gauge-invariant marginal deformations depends crucially on the dimension of the underlying spacetime manifold.
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