Searching closed form analytic solutions to some nonlinear fractional wave equations

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ABSTRACT
Numerous tangible incidents in physics, chemistry, applied mathematics and engineering are described successfully by means of models making use of the theory of derivatives of fractional order and research in this area has grown significantly. In this article, we establish exact solutions to some nonlinear fractional differential equations. The recently established rational (G'/G)-expansion method with the help of fractional complex transform is used to examine abundant further general and new closed form wave solutions to the nonlinear space-time fractional mBBM equation, the space-time fractional Burger’s equation and the space-time fractional ZKBBM in the sense of the Jumarie modified Riemann-Liouville derivative. The fractional complex transform reduces the nonlinear fractional differential equations into nonlinear ordinary differential equations and then the theories of ordinary differential equations are implemented effectively. It is observed that the performance of this method is reliable, useful and gives new and broad-ranging closed form solutions than the existing methods.

ARTICLE HISTORY
Received 27 May 2019
Revised 23 January 2021
Accepted 3 February 2021

KEYWORDS
Exact solutions; modified Riemann-Liouville derivative; nonlinear fractional complex transformation; nonlinear space-time fractional PDEs

1. Introduction
Fractional differential equation is the generalization of classical differential equation of integer order. In the recent decades, many researchers paid attention to analyze and examine closed form wave solutions to the fractional order differential equations, such as nonlinear partial differential equations (NPDEs) of fractional order. Nonlinear fractional differential equations have recently proved to be valuable tools to the modeling of many physical phenomena and have gained the focus of many studies due to their frequent appearance in various applications, such as fluid flow, diffusion, the signal processing, control theory, systems identification, solid state physics, condensed matter physics, plasma physics, optical fibers, chemical kinematics, electrical circuits, biogenetics and other areas (Hilfer, 2000; Kilbas, Srivastava, & Trujillo, 2006; Miller & Ross, 1993; Oldham & Spanier, 1974; Podlubny, 1999; Samko, Kilbas, & Marichev, 1993; West, Bologna, & Grigolini, 2003). The closed form wave solutions of these models (Baleanu, Diethelm, Scalas, & Trujillo, 2012; Kiryakova, 1994; Mainardi, 2010; Sabatier, Agrawal, & Machado, 2007; Yang, 2011; 2012) are greatly helpful to understand the structure of the phenomena as well as their further application in practical life. There are some attractive powerful approaches take into account in the recent research area related to fractional derivative associated problems (He, 2019, 2020a, 2020c; He et al., 2012; He & Ji, 2019; He & Jin, 2020). A large amount of the literature has been provided to construct the exact solutions of nonlinear fractional differential equations of physical interest. Consequently, the researchers established a number of useful and efficient methods for investigating fractional order NPDEs, as for instance the exponential function method (Ji et al., 2020), the two-scale dimension (Ain & He, 2019), the two-scale transform (He & Ain, 2020), the He-Laplace method (Li & Nadeem, 2019; Nadeem & Li, 2019; Suleman et al., 2019), the fractional sub-equation method (Alzaidy, 2013; Guo et al., 2012; Zhang & Feng, 2013; Zhang & Zhang, 2011), the Adomian decomposition method (El-Sayed et al., 2010; El-Sayed & Gaber, 2006; Hu, Luo, & Lu, 2008; Odbat & Momani, 2008), the variational iteration method (Inc, 2008; Liu et al., 2014; Odbat & Momani, 2009; Wu & Lee, 2010), the (G'/G)-expansion method and its various
modification (Feng et al., 2011; Gepreel & Omran, 2012; Zheng, 2012), the homotopy perturbation method (Gepreel, 2011), the improved homotopy perturbation method (Yang & Wang, 2019), the differential transformation method (Deng, 2009; Momani et al., 2007), the finite element method (Gao et al., 2012), the finite difference method (Li, Chen, & Ye, 2011), the semi-inverse method (He, 2020b), the Taylor series method (He, Shen, Ji, & He, 2020), and others (Halil & Seadawy, 2009; Seadawy, 2014, 2015, 2016a, 2016b, 2017; Seadawy & Rashidy, 2013).

Our object in this study is to investigate further general and advanced closed form wave solutions to some fractional NPDEs arise in mathematical physics and engineering, namely the space-time fractional mBBM equation, the space-time fractional Burger’s equation and the space-time fractional ZKBBM equation in the sense of the Jumarie’s modified Riemann-Liouville derivative (Jumarie, 2006; Li & He, 2010). The Jumarie’s modified Riemann-Liouville derivative of order \( x \) is defined as:

\[
D^\alpha_x f(t) = \left\{ \begin{array}{ll}
\frac{1}{\Gamma(x-\alpha)} \frac{d}{dt} \int_0^t \frac{f(\xi)-f(0)}{t^{\alpha}} d\xi, & 0 < \alpha < 1 \\
\frac{d}{dt} f(t), & n \leq \alpha \leq n + 1, n \geq 1
\end{array} \right.
\]

Some of the useful properties of this derivative are

(i) \( D^\alpha_x t^r = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)} t^{r-\alpha} \)

(ii) \( D^\alpha_x (f(t)g(t)) = g(t)D^\alpha_x f(t) + f(t)D^\alpha_x g(t) \)

(iii) \( D^\alpha_x f(g(t)) = \sigma g^\alpha f'(g(t)) + D^\alpha_x f(g(t))(g'(t)) \)

where \( \sigma \) is fractal index.

The Caputo fractional derivative (Oldham & Spanier, 1974; Podlubny, 1999; Samko et al., 1993) of order \( x \) is defined as:

\[
D^\alpha_x f(x) = I^{m-\alpha}D^m f(x)
\]

\[
= \frac{1}{\Gamma(m-\alpha)} \int_x^\infty (y-x)^{m-\alpha-1} \left( \frac{d}{dy} \right)^m f(y) dy,
\]

where \( \alpha > 0 \) for \( m \in \mathbb{N} \), \( m-1 < \alpha < m \), \( D^\alpha_x (.) \) and \( I^\alpha_x (.) \) denote the Caputo fractional derivative and Caputo fractional integral operator, respectively. This definition is used to examine numerical solutions.

Recently, Khalili et al. (2014) proposed a definition of derivative named conformable fractional derivative defined as:

\[
T_\varepsilon(f)(y) = \lim_{\varepsilon \to 0} \frac{f(y+\varepsilon y^{1-s})-f(y)}{\varepsilon}.
\]

Our assessment is highly satisfactory to obtain new and more general exact traveling wave solutions to the equations mentioned above by the suggested method.

2. Methodology

In this section, the rational \((G'/G)\)-expansion method is discussed to study nonlinear partial differential equations of fractional order.

Consider the following fractional partial differential equation in the independent variables, \( x_1, x_2, ..., x_n \):

\[
F\left( u_1, ..., u_k, \frac{\partial u_1}{\partial t}, ..., \frac{\partial u_k}{\partial t}, \frac{\partial u_1}{\partial x_1}, ..., \frac{\partial u_k}{\partial x_n} \right) = 0,
\]

(1)

where \( u_i = u_i(t,x_1,x_2, ..., x_n), i = 1, ..., k \) are unknown functions, \( F \) is a polynomial in \( u_i \) and it is various partial derivatives including fractional derivatives.

**Step 1:** We utilize the nonlinear fractional composite transformation

\[
\xi = \xi(t,x_1,x_2, ..., x_n), u_i = u_i(t,x_1,x_2, ..., x_n) = U_i(\xi)
\]

(2)

Making use of Kilbas et al. (2006), Eq. (1) turns to the following ordinary differential equation of integer order with respect to the variable \( \xi \):

\[
Q(U_1, ..., U_k, U'_1, ..., U'_k, U''_1, ..., U''_k, ..., U^{(n)}_k) = 0,
\]

(3)

where the primes in \( U_i's \) denote the order of derivatives of \( U \) with respect to \( \xi \).

**Step 2:** For convenience, integrate Eq. (3) one or more times as possibility and integral constant can be set to zero as soliton solutions are sought.

**Step 3:** In conformity with the rational \((G'/G)\)-expansion method the solution of (Miller & Ross, 1993) can be revealed in terms of \((G'/G)\) as follows (Islam, Akbar, & Azad, 2015):

\[
u(\xi) = \frac{a_0 + a_1(G'/G) + a_2(G'/G)^2 + \cdots + a_n(G'/G)^n}{b_0 + b_1(G'/G) + b_2(G'/G)^2 + \cdots + b_n(G'/G)^n},
\]

(4)

where \( a_i \) and \( b_i \) are non-zero real constants to be determined later and \( G = G(\xi) \) satisfies the second order linear ODE:

\[
G''(\xi) + \lambda G(\xi) + \mu G(\xi) = 0,
\]

(5)

where \( \lambda \) and \( \mu \) are real constants. Eq. (5) can be rewritten as:

\[
\frac{d}{d\xi} \left( \frac{G'}{G} \right) = -\frac{(G'/G)^2 - \lambda (G'/G) - \mu}{(G'/G)^2}.
\]

(6)

The general solutions of (6) are referred to below:
where \( A \) and \( B \) are constants of integration.

**Step 4:** To determine the positive integer \( n \), we substitute (4) along with (5) into (3) and balance between the highest order derivative and the highest order nonlinear terms appearing in (3). Furthermore, if the degree of \( u(\xi) \) is defined as \( \text{deg}(u(\xi)) = n \), the degree of the other terms are as follows:

\[
\text{deg}
\left[
\frac{d^m u(\xi)}{d\xi^m}
\right] = n + m, \text{deg}
\left[
\frac{d^m u(\xi)}{d\xi^m}
\right]^p
\]

\[= mn + p(n + l).
\]

**Step 5:** Substituting (4) together with (5) into (3), we will obtain a polynomial equation with indeterminate \((G'/G)\). Setting each coefficient of \((G'/G)\) to zero gives a system of algebraic equations. This system of equations can be unraveled for \(a_0\), \(b_0\), \(\lambda\), and \(\mu\) by means of the symbolic computation software Maple.

**Step 6:** We use the values of \(a_0\), \(b_0\), \(\lambda\), and \(\mu\) together with (7) into (4) to obtain the closed form traveling wave solutions of the nonlinear fractional partial differential equation (1).

### 3. Formulation of the solutions

The rational \((G'/G)\)-expansion approach is exploited in this section to search for general closed-form traveling wave solutions to the nonlinear space-time fractional mBBM equation, the space-time fractional Burger’s equation and the space-time fractional ZKBBM equation.

#### 3.1. The space-time fractional mBBM equation

Let us consider the space-time fractional mBBM equation

\[
D_t^p u + D_x^q u - vu^2 D_x u + D_x^{3q} u = 0,
\]

where \( v \) is a nonzero positive constant. This equation was first derived to describe an approximation for surface long waves in nonlinear dispersive media. It can also characterize the hydro-magnetic waves in cold plasma, acoustic waves in anharmonic crystals and acoustic gravity waves in compressible fluids.

Now we make use of the transformation of the fractional compound

\[
u(x, t) = u(\xi), \xi = \frac{bt^a}{(a + 1)} + \frac{mt^a}{(a + 1)} + \xi_0,
\]

Along with the chain rule (He et al., 2012), Eq. (8) is converted into the following ODE,

\[
(m + l)u - \frac{1}{3} v u^3 + l \sigma^2 u'' = 0,
\]

where \( \sigma \) is fractal index to be determined.

**Equation (10)** is integrable and hence integrating, yields

\[
(m + l)u - \frac{1}{3} v u^3 + l \sigma^2 u'' = 0.
\]

where the integral constant is supposed to be zero.

The homogeneous balance between nonlinear term \(u^3\) and the linear term \(u''\) appearing in (11) gives \( n = 1 \). Thus, the solution Eq. (4) has the form

\[
u(\xi) = \frac{a_0 + a_1 (G'/G)}{b_0 + b_1 (G'/G)}, \quad a_1, b_1 \neq 0.
\]

Substituting (12) into (11), the left-hand side becomes a polynomial in \((G'/G)\). We obtain an over-determined set of algebraic equations by zeroing each coefficient of this polynomial (for simplicity, we will omit to display them) for \(a_0, b_0, a_1, b_1, m, l\). Solving this set of equations by using the symbolic computation software Maple, provide the following results:

**Set 1:** \(a_0 = \pm \frac{\sqrt{6} \alpha \lambda b_1}{2 \sqrt{2}}, \quad a_1 = \pm \frac{\sqrt{6} \alpha b_0}{\sqrt{2}}, \quad b_1 = 0, m = -\frac{l}{2} (2 + 4 l \sigma^2 \mu - l \sigma^2 \sigma^2),\) (13)

where \(b_0, b_1, l, \lambda, \sigma \) and \( \mu \) are free parameters.

**Set 2:** \(a_0 = \pm \frac{\sqrt{6} \alpha b_1}{4 \sqrt{2}} (4 \mu - \lambda), a_1 = 0, b_0 = \frac{b_1 l}{2} m = -\frac{l}{2} (2 + 4 l \sigma^2 \mu - l \sigma^2 \sigma^2),\)

where \(b_1, l, \lambda, \sigma \) and \( \mu \) are free parameters. (14)
where \( b_0, l, \lambda, \sigma \) and \( \mu \) are free parameters.

Inserting the values provided in (13)–(15) into solution (12), three general solutions can be found, but for brevity, we have used only the solution Set 1. Thus, the subsequent solution is found:

\[
u_1(\xi) = \pm \frac{\sqrt{6} \sigma}{2 \sqrt{\nu}} \lambda \left( 2 + 4 \nu \mu \sigma^2 - l^2 \lambda^2 \sigma^2 \right), \tag{16}\]

where \( \xi = \frac{l}{1 + \nu} \left[ x^2 + (2 + 4 \nu \mu \sigma^2 - l^2 \lambda^2 \sigma^2) \right]. \)

The use of the general solutions presented in (7) into the solutions (16) results in the closed-form wave solutions to the space-time fractional mBBM equation via rational functions, hyperbolic function, and trigonometric. For simplicity, we record here only the obtained results taking into account the solution Set 1.

When \( \lambda^2 - 4 \mu > 0 \), letting the arbitrary constants \( A \neq 0 \) and \( B = 0 \), the hyperbolic function solution is found as

\[
u_1^{1.2}(\xi) = \pm \frac{\sqrt{6} \sigma}{2 \sqrt{\nu}} \lambda \left( \sqrt{\lambda^2 - 4 \mu}/2 \right), \tag{17}\]

where \( \xi = \frac{l}{1 + \nu} \left[ x^2 + (2 + 4 \nu \mu \sigma^2 - l^2 \lambda^2 \sigma^2) \right]. \)

Using \( \lambda = 3, \mu = 2, \sigma = 1, l = \sqrt{6}, \nu = 9 \), into solution (17) and simplifying, we attain

\[
u_1^{1.2}(\xi) = \pm \frac{\sqrt{6} \sigma}{2 \Gamma(x + 1) + 2 \sqrt{6} \nu} \tan \left( \frac{\sqrt{6} \sigma}{2 \Gamma(x + 1) + 2 \sqrt{6} \nu} \right), \tag{18}\]

When \( \lambda^2 - 4 \mu < 0 \), treating the arbitrary constants as \( A \neq 0 \) and \( B = 0 \), the trigonometric function solution is constructed as:

\[
u_1^{3.4}(\xi) = \pm \frac{\sqrt{6} \sigma}{2 \sqrt{\nu}} \sqrt{4 \mu - \lambda^2} \tan \left( \frac{\sqrt{4 \mu - \lambda^2}/2}{\xi} \right), \tag{19}\]

where \( \xi = \frac{l}{1 + \nu} \left[ x^2 + (2 + 4 \nu \mu \sigma^2 - l^2 \lambda^2 \sigma^2) \right]. \)

The substitution of \( \lambda = 2, \mu = 2, \sigma = 1, l = \sqrt{6}, \nu = 9, b_0 = 1 \), into solution (19) gives

\[
u_1^{3.4}(u, t) = \pm \frac{2 \sqrt{6} \sigma}{\Gamma(x + 1)} \frac{\sqrt{6} \sigma}{\Gamma(x + 1)} \tan \left( \frac{26 \sqrt{6} \sigma}{\Gamma(x + 1)} \right), \tag{20}\]

When \( \lambda^2 - 4 \mu = 0 \), the rational solution is derived as:

\[
u_1^{5.6}(\xi) = \pm \frac{\sqrt{6} \sigma}{2 \sqrt{\nu}} \times \frac{2 \beta}{\lambda + B} \tan \left( \frac{2 \beta}{\lambda + B} \right), \tag{21}\]

where \( \xi = \frac{l}{1 + \nu} \left[ x^2 + (2 + 4 \nu \mu \sigma^2 - l^2 \lambda^2 \sigma^2) \right]. \)

Choosing \( \lambda = 2, \mu = 1, \sigma = 1, l = \sqrt{6}, \nu = 9, b_0 = 1, A = 0 \), solution (21) turns into

\[
u_1^{5.6}(u, t) = \pm \frac{2}{\sqrt{6} \sigma} \frac{\sqrt{6} \sigma}{\Gamma(x + 1)} \frac{\sqrt{6} \sigma}{\Gamma(x + 1)} . \tag{22}\]

For terseness, we have estimated solutions for the values of Set 1 presented in (13). Thus, solution (12) provides the broad-spectrum solution (16) of the mBBM equation. For the different values of the free parameters, the solution (16) yields hyperbolic function solution (18), trigonometric solution (20) and rational function solution (22). Many other solutions can be sought for other choices of the parameters, but the remaining solutions are not written for minimalism and conciseness. It is worth noting that the hyperbolic solution represents the soliton of the kink shape soliton that descends from left to rising right and is constant towards infinity, and the periodic wave is represented by the triangular function solution. The rational function solution depicts a periodic type, breather type and rogue (unusually large, unexpected and suddenly appearing surface waves that can be extremely dangerous, even to large ships) type wave for different values of the free parameter.

### 3.2. The space-time fractional burger’s equation

We now determine the traveling wave solutions of the nonlinear space-time fractional Burgers equation through the introduced method. Consider the space-time fractional Burger’s equation in the underneath

\[
D_t^\alpha u + auD_x^\beta u + bD_x^\beta u = 0, \quad t > 0, x > 0, 0 < \alpha, \beta < 1, \tag{23}\]

where \( a \) and \( b \) are nonzero arbitrary constants. This equation has been found to apply in diverse fields, like, gas dynamics, heat conduction, elasticity, continuous stochastic processes, etc.

The fractional complex transformation

\[
\xi = \frac{k \nu^\beta}{\Gamma(\beta + 1)} + \frac{l \nu^\beta}{\Gamma(\beta + 1)}, \tag{24}\]

where \( k \) and \( l \) are constants, with the help of the chain rule (He, Elagan, & Li, 2012) permits us to reduce the Eq. (23) into the following ODE

\[
(\nu + ak\nu')^\sigma + bk^2 \nu'' = 0, \tag{25}\]

where \( \sigma \) is fractal index to be determined.

After integrating, Eq. (25) becomes

\[
u + \frac{a}{2} k^2 \nu'' = 0, \tag{26}\]

where the integral constant is supposed to be zero. The balance between \( \nu'' \) and \( \nu' \) in Eq. (26) yields \( n = 1 \). Then, the solution Eq. (4) is turned into the form

\[
u(\xi) = \frac{a_0 + a_1 (G'/G)}{b_0 + b_1 (G'/G)}, \tag{27}\]
Now, Eq. (26) with the help of Eq. (27) becomes a polynomial equation in \((G'/G)\). Equating the coefficients of like powers of \((G'/G)\) to zero derives a system of equations (for straightforwardness, the equations are omitted to display) for \(a_0, b_0, a_1, b_1, k, l\). This system of equations can be solved by the symbolic computation software Maple, and attain the subsequent solutions

\[
a_0 = \frac{\left(\pm \sqrt{\lambda^2 - 4 \mu}\right)a_1}{2}, \quad b_0 = \frac{aa_1 + bb_1k\sigma \left(\pm \sqrt{\lambda^2 - 4 \mu}\right)}{2bk},
\]

\[
l = \pm bk^2 \sigma \sqrt{\lambda^2 - 4 \mu},
\]

(28)

where \(a_1, b_1, k, \lambda, \sigma\) and \(\mu\) are arbitrary constants.

Making use of the results given in Eq. (28), the solution Eq. (27) becomes

\[
u(\zeta) = \frac{a_1 b k \sigma \left(\pm \sqrt{\lambda^2 - 4 \mu} + 2 (G'/G)\right)}{a a_1 \pm b b_1 k \sigma \left(\pm \sqrt{\lambda^2 - 4 \mu} + 2 (G'/G)\right)}.
\]

(29)

Substituting the general solutions provided in (7) into (29), yields the subsequent traveling wave solutions:

When \(\lambda^2 - 4 \mu > 0\), assigning the arbitrary constants as \(A \neq 0\) and \(B = 0\), the hyperbolic function solution is derived as:

\[
u_{1,2}(\zeta) = \frac{a_1 b k \sigma \left(\pm \tanh \sqrt{\lambda^2 - 4 \mu} / 2\right) \zeta}{aa_1 + bb_1 k \sigma \left(\pm \tanh \sqrt{\lambda^2 - 4 \mu} / 2\right) \zeta}.
\]

(30)

Setting \(a = \sqrt{2}, \quad a_1 = 1, \quad b = 2, \quad b_1 = 1, \quad k = 1, \quad \lambda = 2, \quad \mu = 1/2, \quad \sigma = 1\) and simplify, solution (30) transmuted to

\[
u_{1,2}(x, t) = \frac{1 \pm \tanh \left(\frac{x^2}{2} \left(1 + 1/7\right) - \frac{\pm \tanh \sqrt{2} x}{\pm \tanh \sqrt{2} x}\right) \zeta}{\left(1 \pm 2\right) \pm 2 \tanh \left(\frac{x^2}{2} \left(1 + 1/7\right) - \frac{\pm \tanh \sqrt{2} x}{\pm \tanh \sqrt{2} x}\right) \zeta}.
\]

(31)

When \(\lambda^2 - 4 \mu < 0\), selecting the integral constants as \(A \neq 0\) and \(B = 0\), the trigonometric function solution is obtained as:

\[
u_{3,4}(\zeta) = \frac{a_1 b k \sigma \sqrt{\lambda^2 - 4 \mu} \left(\pm 1 - \tan \left(\frac{4 \mu - \lambda^2 / 2}{2}\right) \zeta\right)}{aa_1 + bb_1 k \sigma \sqrt{\lambda^2 - 4 \mu} \left(\pm 1 - \tan \left(\frac{4 \mu - \lambda^2 / 2}{2}\right) \zeta\right)}.
\]

(32)

For the values \(a = \sqrt{2}, \quad a_1 = 1, \quad b = 2, \quad b_1 = 1, \quad k = 1, \quad \lambda = 2, \quad \mu = 1/2, \quad \sigma = 1\) and simplify solution (32) becomes

\[
u_{3,4}(x, t) = \frac{1 \pm \tanh \left(\frac{x^2}{2} \left(1 + 1/7\right) - \frac{\pm \tanh \sqrt{2} x}{\pm \tanh \sqrt{2} x}\right) \zeta}{\left(1 \pm 2\right) \pm 2 \tanh \left(\frac{x^2}{2} \left(1 + 1/7\right) - \frac{\pm \tanh \sqrt{2} x}{\pm \tanh \sqrt{2} x}\right) \zeta}.
\]

(33)

When \(\lambda^2 - 4 \mu = 0\), we obtain rational solution as:

\[
u_{5,6}(\zeta) = \frac{a_1 b k \sigma \sqrt{\lambda^2 - 4 \mu} \left(2B \mp (A + B^2)\right)}{2bk \sigma (aa_1 \pm \sqrt{\lambda^2 - 4 \mu}) (A + B^2)}.
\]

(34)

Setting the values \(a = \sqrt{2}, \quad a_1 = 1, \quad b = \sqrt{2}, \quad b_1 = 1, \quad k, \lambda = 2, \mu = 1/2, \quad A = 0, \quad \sigma = 1\) and simplify solution (34) becomes

\[
u_{5,6}(x, t) = \frac{2 \pm \sqrt{2} \left(\frac{x^2}{2} + \frac{x^2}{2}\right)}{2 \pm (1 \pm 1) \left(\frac{x^2}{2} + \frac{x^2}{2}\right)}.
\]

(35)

Solution (29) offers hyperbolic, trigonometric, and rational function solutions based on the parameters \(\lambda, \mu\) and \(\sigma\), shown in \(u_{1,2}(x, t), \quad u_{3,4}(x, t)\) and \(u_{5,6}(x, t)\), respectively. The solutions described in this study are wide-ranging and typical than existing solutions, investigated in earlier research. If we sort distinct values of the comprising parameters, further closed-form solutions to the space-time fractional Burger’s equation can be extracted, but for simplicity and conciseness the residual solutions have not been marked out. The solutions \(u_{1,2}(x, t), \quad u_{3,4}(x, t)\) characterize topological wave, smooth kink, ideal kink waves and \(u_{5,6}(x, t)\) typify the breather type soliton for making use the various values of the free parameters.

### 3.3. The space-time fractional ZKBBBM equation

In this subsection, we study the following nonlinear space-time fractional ZKBBBM equation:

\[
D_t^\alpha u + D_x^\beta u = -2auu_x - bD_x^\gamma D_t^\delta u = 0.
\]

(36)

where \(a\) and \(b\) are nonzero arbitrary constants. This equation arises as a description of gravity water waves in the long-wave regime.

Applying the fractional compound transformation

\[
u(x, t) = \nu(\zeta), \quad \zeta = \frac{kx}{\Gamma(x+1)} + \frac{m^2}{\Gamma(x+1)},
\]

(37)

together with the chain rule (He et al., 2012) reduces Eq. (36) into the following ODE

\[(mu' + lu - 2auu') - bm^2a^2u'' = 0,\]

(38)

where \(\sigma\) is fractal index to be determined.

Equation (38) is integrable and hence integrating, we get

\[(l + m)u - alu^2 - bm^2a^2u^m = 0,\]

(39)

where integral constant is supposed to be zero. Using the homogeneous balance theory it is found \(n = 2\), the solution Eq. (4) can be written as

\[
u(\zeta) = \frac{a_0 + a_1 (G'/G) + a_2 (G'/G)^2}{b_0 + b_1 (G'/G) + b_2 (G'/G)^2}, \quad a_2, b_2 \neq 0.
\]

(40)

Equation (39) with the aid of (40) reduces to a polynomial equation \((G'/G)\). Associating the
coefficients of the same power of \((G'/G)\) to zero, we get a set of algebraic equations (for simplicity, the equations are omitted to display) for \(a_0, b_0, a_1, b_1, a_2, b_2, l, m\). This set of equations is solved by the symbolic computation software Maple, which provides the following results

Set 1

\[
\begin{align*}
    a_0 &= 0, a_1 = \frac{b_0 b l^2 \sigma^2 (\lambda^2 - 4 \mu)^{3/2}}{\sqrt{3} a \mu (4 b l^2 \sigma^2 - 1 - b l^2 \sigma^2)}, \\
    a_2 &= \frac{b_0 b l^2 \sigma^2 \left(16 \mu \lambda - 2 \lambda^3 + \sqrt{3} (\lambda^2 - 4 \mu)^{3/2}\right)}{6 a l^2 (4 b l^2 \sigma^2 - 1 - b l^2 \sigma^2)}, \\
    b_1 &= \frac{3 \lambda - \sqrt{3} (\lambda^2 - 4 \mu)}{3 \mu}, \\
    b_2 &= \frac{2 \lambda^2 - 2 \mu - \lambda \sqrt{3} (\lambda^2 - 4 \mu)}{6 l^2}, \\
    m &= \frac{l}{4 b l^2 \sigma^2 - 1 - b l^2 \lambda^2 \sigma^2}.
\end{align*}
\]

where \(b_0, l, \lambda, \sigma, \mu\) are arbitrary constants.

Set 2

\[
\begin{align*}
    a_0 &= 0, a_1 = \frac{-b_0 b l^2 \sigma^2 (\lambda^2 - 4 \mu)^{3/2}}{\sqrt{3} a \mu (4 b l^2 \sigma^2 - 1 - b l^2 \sigma^2)}, \\
    a_2 &= \frac{b_0 b l^2 \sigma^2 \left(16 \mu \lambda - 2 \lambda^3 - \sqrt{3} (\lambda^2 - 4 \mu)^{3/2}\right)}{6 a l^2 (4 b l^2 \sigma^2 - 1 - b l^2 \sigma^2)}, \\
    b_1 &= \frac{3 \lambda + \sqrt{3} (\lambda^2 - 4 \mu)}{3 \mu}, \\
    b_2 &= \frac{2 \lambda^2 - 2 \mu + \lambda \sqrt{3} (\lambda^2 - 4 \mu)}{6 l^2}, \\
    m &= \frac{l}{4 b l^2 \sigma^2 - 1 - b l^2 \lambda^2 \sigma^2}.
\end{align*}
\]

where \(b_0, l, \lambda, \sigma, \mu\) are all arbitrary constants.

Using the values of the unknown constants given in (7), the solution (43) becomes

\[
\begin{align*}
    u_1(x, t) &= \frac{b l^2 \sigma^2 \left[-2 \sqrt{3} \mu (\lambda^2 - 4 \mu)^{3/2} + \lambda \left(16 \mu \lambda - 2 \lambda^3 - \sqrt{3} (\lambda^2 - 4 \mu)^{3/2}\right) \left(\frac{\sigma}{\sigma}\right) \left(\frac{\sigma}{\sigma}\right)\right]}{a (4 b l^2 \sigma^2 - b l^2 \lambda^2 \sigma^2 - 1) \left[6 l^2 + 2 \mu \left(3 \lambda + \sqrt{3} (\lambda^2 - 4 \mu)\right) \left(\frac{\sigma}{\sigma}\right) + \left(2 \lambda^2 - 2 \mu + \lambda \sqrt{3} (\lambda^2 - 4 \mu)\right) \left(\frac{\sigma}{\sigma}\right)\right]},
\end{align*}
\]

where \(\xi = \frac{b l^2 \sigma^2}{1 + \frac{\sigma}{\sigma}} + \frac{m t}{1 + \frac{\sigma}{\sigma}}\).

Now using the solutions derived in (7) instead of \((G'/G)\) into (4) and simplifying it gives the following traveling wave solutions:

For set 1:

When \(\lambda^2 - 4 \mu > 0\), using particular values \(a = b = l = 1, \lambda = 2, \mu = 1/2, \sigma = 1\) of the unknown parameters, we extract the realistic and significant hyperbolic function solution as follows:

\[
\begin{align*}
    u_1^1(x, t) &= -8.3 + 11.7 \tan\left(\frac{1}{\sqrt{2}} \left(\frac{\sigma}{\sigma} - \frac{1}{\sqrt{3}} \frac{\sigma}{\sigma}\right)\right) - 4 \tan^2\left(\frac{1}{\sqrt{2}} \left(\frac{\sigma}{\sigma} - \frac{1}{\sqrt{3}} \frac{\sigma}{\sigma}\right)\right), \\
    & \quad 2.3 + 9.8 \tan\left(\frac{1}{\sqrt{2}} \left(\frac{\sigma}{\sigma} - \frac{1}{\sqrt{3}} \frac{\sigma}{\sigma}\right)\right) - 3 \tan^2\left(\frac{1}{\sqrt{2}} \left(\frac{\sigma}{\sigma} - \frac{1}{\sqrt{3}} \frac{\sigma}{\sigma}\right)\right).
\end{align*}
\]

When \(\lambda^2 - 4 \mu < 0\), for the particular values \(a = b = l = 1, \lambda = 2, \mu = 1/2, \sigma = 1\) of the parameters, we obtain the following trigonometric function solution:

\[
\begin{align*}
    u_1^2(x, t) &= -8.3 - 11.7 \tan\left(\frac{1}{\sqrt{2}} \left(\frac{\sigma}{\sigma} - \frac{1}{\sqrt{3}} \frac{\sigma}{\sigma}\right)\right) + 4 \tan^2\left(\frac{1}{\sqrt{2}} \left(\frac{\sigma}{\sigma} - \frac{1}{\sqrt{3}} \frac{\sigma}{\sigma}\right)\right), \\
    & \quad 2.3 - 9.8 \tan\left(\frac{1}{\sqrt{2}} \left(\frac{\sigma}{\sigma} - \frac{1}{\sqrt{3}} \frac{\sigma}{\sigma}\right)\right) - 3 \tan^2\left(\frac{1}{\sqrt{2}} \left(\frac{\sigma}{\sigma} - \frac{1}{\sqrt{3}} \frac{\sigma}{\sigma}\right)\right).
\end{align*}
\]

When \(\lambda^2 - 4 \mu = 0\), for \(A = 0\) \(a = b = l = 1, \lambda = 2, \mu = 1/2, \sigma = 1\), we construct the rational solution as:

\[
\begin{align*}
    u_1^3(x, t) &= \frac{-1.6 \left(\frac{3}{1 + \frac{\sigma}{\sigma}} - \frac{1}{1 + \frac{\sigma}{\sigma}}\right)^2 - 11.9 \left(\frac{3}{1 + \frac{\sigma}{\sigma}} - \frac{1}{1 + \frac{\sigma}{\sigma}}\right)^2}{1.3 \left(\frac{3}{1 + \frac{\sigma}{\sigma}} - \frac{1}{1 + \frac{\sigma}{\sigma}}\right)^2 - 9.1 \left(\frac{3}{1 + \frac{\sigma}{\sigma}} - \frac{1}{1 + \frac{\sigma}{\sigma}}\right)^2}.
\end{align*}
\]

We have assessed the solutions for the values in set 1 for conciseness, provided in (41). The solution (43) thus provides wide-spectrum solutions namely, \(u_1^1(x, t)\), \(u_1^2(x, t)\) and \(u_1^3(x, t)\). In addition, for set 2, it might be achieved three types of exact traveling wave solutions, namely the hyperbolic, trigonometric and rational function solutions which are not documented here to avoid repetition. The characteristic of solution \(u_1^1(x, t)\) and \(u_1^2(x, t)\) represent kink shape wave and periodic wave, respectively, but gives rogue wave with several soliton for different values of parameters. The solution \(u_1^3(x, t)\) depicts irregular spike type soliton for different values of related constants.
4. Conclusions

In this article, the exact traveling wave solutions to NPDEs of fractional order, namely the space-time fractional mBBM equation, the space-time fractional Burger’s equation and the space-time fractional ZKBBM equation are established. To do this, we employ the recently established rational $(G'/G)$-expansion method which gives new and further general traveling wave solutions. Three types of closed form analytical solutions including the generalized hyperbolic function solutions, the generalized trigonometric function solutions and rational solutions for each of the above NPDEs are obtained successfully. These solutions might be further useful and effective for understanding the mechanisms of the intricate nonlinear physical phenomena occur in science and engineering. So far we know, the results gained in this article have not been reported in the literature.

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**References**

Ain, Q. T., & He, J. H. (2019). On two-scale dimension and its applications. *Thermal Science*, 23(3 Part B), 1707–1712. doi:10.2298/TSCI190408138A

Alzaidy, J. F. (2013). The fractional sub-equation method and exact analytical solutions for some fractional PDEs. *American Journal of Mathematical Analysis*, 1, 14–19.

Baleanu, D., Diethelm, K., Scalas, E., & Trujillo, J. J. (2012). *Fractional calculus: Models and numerical methods*, Vol. 3 of series on complexity, nonlinearity and chaos. Boston, MA: World Scientific Publishing.

Deng, W. (2009). Finite element method for the space and time fractional Fokker-Planck equation. *SIAM Journal on Numerical Analysis*, 47(1), 204–226. doi:10.1137/080714130

El-Sayed, A. M. A., Behiry, S. H., & Raslan, W. E. (2010). Adomian’s decomposition method for solving an intermediate fractional advection-dispersion equation. *Computers & Mathematics with Applications*, 59(5), 1759–1765. doi:10.1016/j.camwa.2009.08.065

El-Sayed, A. M. A., & Gaber, M. (2006). The Adomian decomposition method for solving partial differential equations of fractal order in finite domains. *Physics Letters A*, 359(3), 175–182. doi:10.1016/j.physleta.2006.06.024

Feng, J., Li, W., & Wan, Q. (2011). Using $(G'/G)$-expansion method to seek traveling wave solution of Kadomtsev-Petviashvili-Piskunov equation. *Journal of Applied Mathematics and Computing*, 217, 5860–5865.

Gao, G. H., Sun, Z. Z., & Zhang, Y. N. (2012). A finite difference scheme for fractional sub-diffusion equations on an unbounded domain using artificial boundary conditions. *Journal of Computational Physics*, 231(7), 2865–2879. doi:10.1016/j.jcp.2011.12.028

Gepreel, K. A. (2011). The homotopy perturbation method applied to nonlinear fractional Kadomtsev-Petviashvili-Piskunov equations. *Applied Mathematics Letters*, 24, 1458–1434.

Gepreel, K. A., & Omran, S. (2012). Exact solutions for nonlinear partial fractional differential equations. *Chinese Physics B*, 21(11), 110204–110207. doi:10.1088/1674-1056/21/11/110204

Guo, S., Mei, L., Li, Y., & Sun, Y. (2012). The improved fractional sub-equation method and its applications to the space-time fractional differential equations in fluid mechanics. *Physics Letters A*, 376(4), 407–411. doi:10.1016/j.physleta.2011.10.056

He, J. H., & Ain, Q. T. (2020). New promises and future challenges of fractal calculus: From two-scale thermodynamics to fractal variational principle. *Thermal Science*, 24, 659–681. doi:10.2298/TSCI200127065H

He, J. H., & Ji, F. Y. (2019). Two-scale mathematics and fractional calculus for thermodynamics. *Thermal Science*, 23(4), 2131–2133. doi:10.2298/TSCI1904131H

He, J. H., & Jin, X. (2020). A short review on analytical methods for the capillary oscillator in a nanoscale deformable tube. *Mathematical Methods in the Applied Science*, 1–8. doi:10.1002/mma.6321

He, J. H. (2019). A simple approach to one-dimensional convection-diffusion equation and its fractional modification for E reaction arising in rotating disk electrodes. *Journal of Electroanalytical Chemistry*, 854, 113565. doi:10.1016/j.jelechem.2019.113565

He, J. H. (2020a). A short review on analytical methods for a fully fourth-order nonlinear integral boundary value problem with fractal derivatives. *International Journal for Numerical Methods for Heat and Fluid Flow*, doi:10.1108/HFF-01-2020-0060

He, J. H. (2020b). Variational principle and periodic solution of the Kundu-Mukherjee-Naskar equation. *Results in Physics*, 17, 103031. doi:10.1016/j.rinp.2020.103031

He, J. H. (2020c). A fractal variational theory for one-dimensional compressible flow in a microgravity space. *Fractals*, 28, 2050024. doi:10.1142/S0218348X20500243

He, J. H., Elagan, S. K., & Li, Z. B. (2012). Geometrical explanation of the fractional complex transform and derivative chain rule for fractional calculus. *Physics Letters A*, 376(4), 257–259. doi:10.1016/j.physleta.2011.11.030

Helal, M. A., & Seadawy, A. R. (2009). Variational method for the derivative nonlinear Schroedinger equation with computational applications. *Physica Scripta*, 80(3), 035004–035360. doi:10.1088/0031-8949/80/03/035004

He, C. H., Shen, Y., Ji, F. Y., & He, J. H. (2020). Taylor series solution for fractal Bratu-type equation arising in electrospinning process. *Fractals*, 28(01), 2050011. doi:10.1142/S0218348X20500115

Hilfer, R. (2000). *Applications of fractional calculus in physics*. River Edge, NJ: World Scientific Publishing.

Hu, Y., Luo, Y., & Lu, Z. (2008). Analytical solution of the linear fractional differential equation by Adomian decomposition method. *The Journal of Computational and Applied Mathematics*, 215(1), 220–229. doi:10.1016/j.cam.2007.04.005

Inc, M. (2008). The approximate and exact solutions of the space- and time-fractional Burgers equations with initial conditions by variational iteration method. *Journal of Mathematical Analysis and Applications*, 345(1), 476–484. doi:10.1016/j.jmaa.2008.04.007
Islam, M. T., Akbar, M. A., & Azad, A. K. (2015). A rational \((G'/G)\)-expansion method and its application to the modified KdV-Burgers equation and the \((2+1)\)-dimensional Boussinesq equation. *Nonlinear Studies*, 6, 1–11.

Ji, F. Y., He, C. H., Zhang, J. J., & He, J. H. (2020). A fractal Boussinesq equation for nonlinear transverse vibration of a nano-fiber-reinforced concrete pillar. *Applied Mathematical Modelling*, 82, 437–448. doi:10.1016/j.apm.2020.01.027

Jumarie, G. (2006). Modified Riemann-Liouville derivative and fractional Taylor series of non-differentiable functions. Further results. *Computers & Mathematics with Applications*, 51(9–10), 1367–1376. doi:10.1016/j.camwa.2006.02.001

Khalil, R., Horani, M. A., Yousef, A., & Sababheh, M. (2014). *A new definition of fractional derivative. The Journal of Computational and Applied Mathematics.*, 264, 65–70. doi:10.1016/j.cam.2014.01.002

Kim, A. G., & Kim, S. W. (2007). Generalized fractional calculus and fractional differential equations. In *Computational and Applied Mathematics*. The Journal of Computational and Applied Mathematics, 51(9–10), 1367–1376. doi:10.1016/j.camwa.2006.02.001

Kiryakova, V. (1994). *Generalized fractional calculus and applications*, Vol. 301 of Pitman Research Notes in Mathematics Series. Harlow, UK: Longman Scientific and Technical.

Li, Z. B., & He, J. H. (2010). Fractional complex transform for fractional differential equations. *Mathematical and Computational Applications*, 15(5), 970–973. doi:10.3390/mca15050970

Li, F., & Nadeem, M. (2019). He-Laplace method for nonlinear vibration in shallow water waves. *Journal of Low Frequency Noise, Vibration and Active Control*, 38(3–4), 1305–1313. doi:10.1016/j.jlfn.2018.11.001

Li, C., Chen, A., & Ye, J. (2011). Numerical approaches to fractional calculus and fractional ordinary differential equation. *Journal of Computational Physics*, 230(9), 3352–3368. doi:10.1016/j.jcp.2011.01.030

Liu, H. Y., He, J. H., & Li, Z. B. (2014). Fractal calculus for nanoscale flow and heat transfer. *International Journal of Numerical Methods for Heat & Fluid Flow*, 24(6), 1227–1250. doi:10.1108/HFF-07-2013-0240

Mainardi, F. (2010). *Fractional calculus and waves in linear viscoelasticity: An introduction to mathematical models*. London, UK: Imperial College Press.

Miller, K. S., & Ross, B. (1993). *An introduction to the fractional calculus and fractional differential equations*. New York: John Wiley & Sons.

Momani, S., Odibat, Z., & Erturk, V. S. (2007). Generalized differential transform method for solving a space- and time-fractional diffusion-wave equation. *Physics Letters A.*, 370(5–6), 379–387. doi:10.1016/j.physleta.2007.05.083

Nadeem, M., & Li, F. (2019). He-Laplace method for nonlinear vibration systems and nonlinear wave equations. *Journal of Low Frequency Noise, Vibration and Active Control*, 38(3–4), 1060–1074. doi:10.1016/j.jlfn.2018.11.001

Odibat, Z., & Maman, S. (2008). A generalized differential transform method for linear partial differential equations of fractional order. *Applied Mathematics Letters*, 21(2), 194–199. doi:10.1016/j.aml.2007.02.022

Odibat, Z., & Maman, S. (2009). The variational iteration method: An efficient scheme for handling fractional partial differential equations in fluid mechanics. *Computers & Mathematics with Applications*, 58(11–12), 2199–2208. doi:10.1016/j.camwa.2009.03.009

Oldham, K. B., & Spanier, J. (1974). *The fractional calculus*. New York, NY: Academic Press.

Podlubny, I. (1999). *Fractional differential equations*, vol. 198 of *Mathematics in Science and Engineering*. San Diego, CA: Academic Press.

Sabatier, J., Agarwal, O. P., & Machado, J. A. T. (2007). *Advances in fractional calculus: Theoretical developments and applications in physics and engineering*. New York: Springer.

Samko, S. G., Kilbas, A. A., & Marichev, O. I. (1993). *Fractional Integrals and derivatives*. Yverdon, Switzerland: Gordon and Breach Science.

Seadawy, A. R., & Rashidy, K. E. (2013). Traveling wave solutions for some coupled nonlinear evolution equations. *Mathematical and Computer Modelling*, 57(5–6), 1371–1379. doi:10.1016/j.mcm.2012.11.026

Seadawy, A. R. (2014). Stability analysis for Zakharov-Kuznetsov equation of weakly nonlinear ion-acoustic waves in a plasma. *Computers & Mathematics with Applications*, 67(1), 172–180. doi:10.1016/j.camwa.2013.11.001

Seadawy, A. R. (2015). Fractional solitary wave solutions of the nonlinear higher-order extended KdV equation in a stratified shear flow: Part I. *Computers & Mathematics with Applications*, 70(4), 345–352. doi:10.1016/j.camwa.2015.04.015

Seadawy, A. R. (2016a). Stability analysis solutions for nonlinear three-dimensional modified Korteweg-de Vries-Zakharov-Kuznetsov equation in a magnetized electron-positron plasma. *Physica A: Statistical Mechanics and its Applications*, 455, 44–51. doi:10.1016/j.physa.2016.02.061

Seadawy, A. R. (2016b). Three-dimensional nonlinear modified Zakharov-Kuznetsov equation of ion-acoustic waves in a magnetized plasma. *Computers & Mathematics with Applications*, 71(1), 201–212. doi:10.1016/j.camwa.2015.11.006

Seadawy, A. R. (2017). Travelling-wave solutions of a weakly nonlinear two-dimensional higher-order Kadomtsev-Petviashvili dynamical equation for dispersive shallow-water waves. *The European Physical Journal Plus*, 132(11), 29. doi:10.1140/epjp/i2017-11313-4

Suleman, M., Lu, D., Yue, C., Rahman, J. U., & Anjum, N. (2019). He-Laplace method for general nonlinear periodic solitary solution of vibration equations. *Journal of Low Frequency Noise, Vibration and Active Control*, 38(3–4), 1297–1304. doi:10.1016/j.jlfn.2018.11.006

West, B. J., Bologna, M., & Grigolini, P. (2003). *Physics of fractal operators*. New York: Springer.

Wu, G. C., & Lee, E. W. M. (2010). Fractional variational iteration method and its application. *Physics Letters A*, 374(25), 2506–2509. doi:10.1016/j.physleta.2010.04.034

Yang, Y. J., & Wang, S. Q. (2019). An improved homotopy iteration method and its application. *Physics Letters A*, 375(7), 1069–1073. doi:10.1016/j.physleta.2011.01.029
Zhang, Y., & Feng, Q. (2013). Fractional Riccati equation rational expansion method for fractional differential equations. *Applied Mathematics & Information Sciences, 7*(4), 1575–1584. doi:10.12785/amis/070443

Zheng, B. (2012). \((G'/G)\)-expansion method for solving fractional partial differential equations in the theory of mathematical physics. *Communications in Theoretical Physics, 58*, 623–630.