5-colorable visibility graphs have bounded size or 4 collinear points

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Abstract

We investigate the question of finding a bound for the size of a $\chi$-colorable finite visibility graph that have at most $\ell$ collinear points. This can be regarded as a relaxed version of the Big Line - Big Clique [5] conjecture. We prove that any finite point set that has at least 2311 points has either 4 collinear points or a visibility graph that cannot be 5-colored.

1 Introduction

Let $X$ be a finite set of points in the Euclidean plane. For a pair of points $u, v \in X$ the open line segment with endpoints $u$ and $v$ will be denoted by $(uv)$. The visibility graph $G_X$ is a simple graph with vertex set $X$, where the pair $u, v \in X$ is connected if and only if $(uv)$ does not contain any point from $X$.

The starting point of our investigation is the following conjecture.

Conjecture 1 (Big line, big clique [5]). For any fixed $\ell$ and $k$ there is a constant $c = c(k, \ell)$ such that every finite planar point set which has size at least $c$ has either $\ell$ collinear points, or its visibility graph has a clique of size $k$.

This conjecture is currently open for all $k \geq 6$ and $\ell \geq 4$. Note that the finiteness here is necessary, there is a counterexample if we allow infinite point sets [7].

Let $mc_\ell(k)$ be the maximum cardinality of a finite set $X$ that has at most $\ell$ collinear points and its visibility graph can be colored with at most $k$ colors. If there is no such maximum, then let $mc_\ell(k) = \infty$. Note that $mc_\ell(k) \leq c(k + 1, \ell + 1)$, because maximum clique size is at least the chromatic number. Based on this inequality the following weaker conjecture can be formulated:

Conjecture 2 ([8]). $mc_\ell(k) < \infty$ for all $k, \ell \geq 2$.

The values of $mc_\ell(\leq 3)$ have been established in a paper of Kára, Pór and Wood in [3]. The value of $mc_3(4)$ was found later by Aloupis et al. [1], but proven in a slightly different framework than ours in. They also showed some lower bounds for $mc_3(k)$. The best known bounds for $k = 4$ and general $\ell$ can be derived from the theorem of Barát et al. about empty pentagons [2].

We summarized the progress on finding upper bounds for $mc_\ell(k)$ in the below table. The new bound (our main result) is underlined.
2 Blocking lemmas

Let $X$ be any point set. We call $X$ properly colored if any pair of distinct points $x, y \in X$ are not visible to each other if they share the same color.

A subset $U$ of $X$ in which each point has color $c$ is called $c$-empty if every point in $X \cap \text{conv}(U) \setminus U$ has different color than $c$. $U$ is $k$-color-blocked by a colored set $B \subset \text{conv}(U) \setminus U$ if $U \cup B$ is a properly colored set and $B$ has at most $k$ colors.

A set of three non collinear points is called a triangle.

**Definition 1.** (Equivalence of two colored point sets) Let $X$ and $Y$ be two arbitrary colored point sets on the plane. We call $X$ and $Y$ equivalent if there exists a bijection $\phi : X \rightarrow Y$ satisfying the following conditions:

- For any $x_0 \in X$ and any finite subset $\{x_1, x_2, \ldots, x_k\}$ of $X$:

  $$x_0 \in \text{conv}(\{x_1, \ldots, x_k\}) \iff \phi(x_0) \in \text{conv}(\{\phi(x_1), \ldots, \phi(x_k)\})$$

- For any $x', x'' \in X$, $x'$ and $x''$ have the same color in $X$ if and only if $\phi(x')$ and $\phi(x'')$ have the same color in $Y$.

**Lemma 2.** A unicolored triangle cannot be 2-color-blocked.

*Proof.* Consider a minimal counterexample: a unicolored triangle $T$ blocked by a colored set $B$, such that $B$ is minimal among all such counterexamples. Assume that the color of $T$ is black and the colors of $B$ are blue and red.

There must be a red or blue point on each side of $T$. As a consequence of minimality, we have that $T$ is black-empty, since choosing the black point inside which is closest to one of the sides of $T$ and the two endpoints of this side would define a unicolored triangle that is 2-color-blocked by less points (it does not contain the blocking points on the two other sides of $T$).

If the three blocking points on the sides of $T$ have the same color then they form a unicolored triangle that is 2-color-blocked by fewer points than $T$, again contradicting minimality.
Hence we may assume (w.l.o.g.) that one of the points on the sides is red \((R_1)\) and two are blue \((B_1, B_2)\). Now \(B_1\) and \(B_2\) must be separated by a red point \((R_2)\), then \(R_1\) and \(R_2\) must be separated by a blue point \(B_3\). Finally \(B_1, B_2\) and \(B_3\) form a unicolored triangle that is 2-color-blocked by fewer points than \(T\), contradiction.

From this point onwards we fix \(\ell = 3\). It follows that in a properly coloured set, the points of any color class are in general position.

**Claim 3** (\([\mathbb{B}]\)). \(mc_3(3) = 6\).

**Proof.** On one hand, if there were more than 6 points in a 3-colored point set, at least three of them would have the same color.

Three unicolored points cannot be collinear (else at least two of them would be adjacent in the visibility graph, as \(\ell = 3\)), so they form a unicolored triangle. Then this triangle would be 2-color-blocked, that contradicts Lemma \([2]\).

On the other hand, the figure on the right shows a properly 3-colored set with 6 points.

**Claim 4.** Any properly 3-colored 6-point set is equivalent to the one shown on the figure \(mc_3(3)\).

**Proof.** There must be exactly two points in every color class. Let \(a\) and \(a'\) be two points from the first color class (red). These two must be blocked by a point \(b\) with different color (blue). \(b\) has a pair \(b'\), that has the same color. \(b\) and \(b'\) cannot be blocked by either \(a\) or \(a'\), as it would mean that \(a, a', b\) and \(b'\) would be on the same line. So \(b\) and \(b'\) are blocked by a new point \(c\) (from the third color class, green). \(c\) has a pair \(c'\) with the same color. These two cannot be blocked by \(b\) or \(b'\), that would cause 4 points on the same line. So they are blocked by \(a\) or \(a'\). Since \(a\) and \(a'\) played a symmetric role so far, we may assume w.l.o.g. that the blocker is \(a\). Now our notations give the desired bijection to Figure \(mc_3(3)\).

For any point set \(X\), we denote by \(\text{iconv}(X)\) the interior of \(\text{conv}(X)\).
Lemma 5. Suppose that $\ell = 3$ and consider a unicocolored, color-empty triangle $T$. If $T$ is 3-color-blocked by a set $B \subset \text{conv} T$, then $B \cup T$ is equivalent to one of the five instances below:

![Instances](image)

Proof. $B$ must have one point on each side of $T$, we denote the set of these three points by $B^s$, and $B^{in} = B \setminus B^s$. $B$ is a properly 3-colored set, so $|B| \leq 6$ by Claim 3. Hence $|B^{in}| \leq 3$.

Case 1: $|B^{in}| = 0$

The points of $B^s$ see each other, so they must have different colors. It means that $B \cup T$ is equivalent to Instance 1.

Case 2: $|B^{in}| = 1$

The only point $p$ of $B$ in $\text{iconv}(T)$ can block only one pair of $B^s$, so $B^s$ needs at least 2 colors. $p$ sees all the other points of $T \cup B$, so it must have a unique color. Then only two colors remain for $B^s$, the only way to color it with two colors is shown in Instance 2.

Case 3: $|B^{in}| = 2$

The two points of $B^{in}$ see each other, so they have different colors. They cannot block all three visibilities between $B^s$, so $B^{in}$ must have at least two colors, too. Hence there are points $i \in B^{in}$ and $s_1 \in B^s$ with the same color. $i$ and $s_1$ can only be blocked by the other point of $B^{in}$, call it $i'$. Now $i$ and $i'$ has different colors, and both can see both points of $B^s \setminus \{s_1\} = \{s_2, s_3\}$, hence $s_2$ and $s_3$ must have the same color. So the visibility between $s_2$ and $s_3$ has to be blocked. The blocking point can be either $i$ or $i'$, those correspond to the cases (3) and (4).

Case 4: $|B^{in}| = 3$

Now $|B| = 6$ and $B$ is properly 3-colored, so by Claim 4 $B$ is equivalent to the set shown on Figure $mc_3(3)$. It is easy to check that $B^s$ must be formed by $a'$, $b'$ and $c'$ (unless some of them would be outside $T$), then the equivalence is straightforward.

The following 2 lemmas and Theorem 10 are established in [11]. Despite the differences between the definitions, their proofs are directly applicable here. We included the proofs
Lemma 6 (II). Let \( Q \) be the set of the vertices of a convex quadrilateral. Suppose that \( Q \) is unic和平 and color-empty. Then any blocking set \( B \) of \( Q \) is equivalent to the one shown on the figure at the right.

Remark 7 (II). The above 9-point set is maximal, i.e. if a 4-colored visibility graph has 4 unicolored points in convex position, then it has exactly 9 points.

Lemma 8 (II). A unicolored concave set of 4 points can be blocked by 3 colors only the following way:

Remark 9 (II). The above 10-point set is maximal, i.e. if a 4-colored visibility graph has 4 unicolored points in concave position, then it has exactly 10 points.

Theorem 10. \( mc_3(4) = 12 \).

3 Main result

Theorem 11. A unicolored convex hexagon cannot be 4-color-blocked if \( \ell = 3 \).

Proof. In a unicolored convex hexagon the blocking set \( X \) has to block 15 segments defined by the 6 vertices and endpoints. We denote the vertex set of the hexagon by \( H \). The 6 edges of the hexagon need distinct blocking points. All other points may block at most two diagonals, except maybe one blocker: if the diagonals connecting opposite vertices are concurrent, then these can be blocked by one point. It follows that the number of blocking points needed is at least 10 (6 for the edges, and at least 4 for the 9 diagonals).

By Theorem 10 it follows that a blocking set cannot have more than 12 points. Thus it is enough to show that \( H \) cannot be 4-color-blocked by 10, 11 or 12 points.
We may assume that our blocking set $X$ lies in $\text{conv}(H)$. It is easy to observe that $X$ has at least 6 points in $\partial \text{conv}(X)$: the ones that block the edges of the hexagon.

To prove this theorem, we will use the following simple lemmas.

**Lemma 12.** The biggest color class of $X$ has size at most 3.

*Proof.* Suppose that a color class $C$ has at least 4 points. By Lemmas 6 and 8, we get that $X$ contains one of two configurations. Since both of these configurations are maximal, $X$ is also equivalent to one of these configurations. We observe that for both these configurations the number of points on the boundary of the convex hull is less than 6, thus the equivalent of these configurations cannot block the hexagon.

It follows that there are at least two unicolored triangles in the blocking set.

**Lemma 13.** A unicolored triangle $T$ of color $c$ cannot have all its vertices on the edges of the hexagon.

*Proof.* Take any unicolored triangle $T' \neq T$ of color $c'$. To block $T'$, there must be points of each of the 3 blocking colors that differ from $c'$. Thus $T'$ must contain a point of color $c$. The only way this is possible inside the convex hull of the hexagon is if two vertices of $T'$ lie on the same edge as a point of color $c$. But then there would be 5 points on this edge: we arrived at a contradiction.

We are now ready to prove the theorem. We will check the cases $|X| = 10, 11, 12$.

**Case 1:** $|X| = 10$. Take any unicolored triangle $T$. At least one of its vertices must be in $\text{iconv}(H)$ by Lemma 13. We need at least 3 points to block $T$, and all of these points lie in $\text{iconv}(H)$. It follows that all 4 points in $\text{iconv}(H)$ lie on the boundary of $T$, and they are in convex position. On the other hand, it can be verified that the inner points of a 10-point blocking set of a convex hexagon need to be in concave position (the intersection $p$ of the 3 diagonals that connect opposite vertices will be in the convex hull of the three other inner blocking points).
Case 2: \(|X| = 11\). It follows that there are at least two points in \(\text{iconv}(H)\) that have the same color (red), we denote them by \(p\) and \(q\). We distinguish two subcases based on the number of red points.

(2a) There are two red points: \(p\) and \(q\). It follows that all other classes have 3 points. Let \(v\) be the (blue) blocking point of \(p\) and \(q\). The blue triangle \(T = (vv'v'')\) can not have any red point on \(vv'\) or \(vv''\), because then there would be 4 collinear points. It also can not have any red point in \(\text{iconv}(T)\), because then there would be 5 points in \(\text{iconv}(H) \cap \text{conv}(T)\), and \(\text{iconv}(H)\) has one more point outside \(\text{conv}(T)\) since either \(p\) or \(q\) is there. It follows that there is a red point on \(v'v''\), suppose it is \(p\).

The green blocking point \(g\) on the segment \(vv''\) can see all points on the right side of \(- \rightarrow pq\), thus the remaining green points must lie on the left side. The only point that can block them is the black point \(b\). Consequently \(p\) and \(q\) lie on the sides of the green triangle. Similarly, \(p\) and \(q\) lie on the sides of the black triangle. It follows that there is a black and a green blocking point on the right side of \( \overline{v'v''} \), and these points are on the sides of \(H\). So \(v'\) and \(v''\) lie on the opposite sides of the hexagon. Since all inner points are on the left side of \( \overline{v'v''} \), we get that there is a diagonal \(d\) that is not blocked.

(2b) There are 3 red points. It follows that all inner points are on the convex hull of the red triangle (because the red triangle needs at least 3 more inner points to be blocked). Let \(v\) be the blue point that blocks the two inner red points \(p\) and \(q\). The other inner points need to be black (\(b\)) and green (\(g\)). Wlog. we can suppose \(v\) is on the left side of \( \overline{bg} \).

There cannot be 3 blue points, because only \(b\) and \(g\) can block points from \(v\), but then there would be two blue points on the right side of \( \overline{bg} \). Since all inner points are on the left side (or on line \(bg\) itself), these would see each other. Thus there are 2 blue points, 3 black points and 3 green points.

The remaining 2 black points could be blocked from \(b\) by \(v,p\), or \(g\). If a black point lies on the line \(bp\), then it will see every possible for the third black point on \(bg\) or \(bv\), thus no black point lies on \(bp\). Similarly, there is no further green point on \(gq\). It follows that
at least one of the dashed lines will have more than one of the remaining 5 points, which
would mean 4 points on one line.

![Diagram](image)

**Case 3:** \(|X|=12\). Each color class has 3 points. By earlier observations, no color class
can lie exclusively on the sides of \(H\). It follows that the size of color classes in \(\text{iconv}(H)\)
is either 3, 1, 1, 1 or 2, 2, 1, 1.

(3a) There is a unicolored triangle in \(\text{iconv}(H)\). This case is very similar to case (2b).
A blue point can not be blocked from \(v\) by \(r\), because such a point would see all other
possible third blue points on line \(vb\) and \(vq\), and the same can be said for the pair \(b,p\) and
\(q,g\). So the remaining 6 points would need to lie on the three dashed lines, resulting in at
least 4 points on one line.

![Diagram](image)

(3b) The size of color classes in \(\text{iconv}(H)\) is 2, 2, 1, 1. Suppose there are 2 red and 2
blue points. At least one of the inner red and blue point pairs is blocked by a green or
black point, wlog. we can suppose the two red points \(r_1\) and \(r_2\) are blocked by the green
point \(g\). We denote by \(G\) the vertices of the green triangle. There must be at least one
red point \((r_1)\) in \(\text{conv}(G) \setminus G\). Consequently \(r_2\) lies outside \(\text{conv}(G)\).

First we consider the case where \(r_1\) lies in \(\text{iconv}(G)\). There must be 4 points in
\(\text{conv}(G) \setminus G\). It follows that \(r_1\) is the blocker of the two inner blue points \(v_1\) and \(v_2\), and
the inner black point \(b\) lies on the third side of the triangle determined by \(G\).

If the beam \(v_1v_2\) lies on the neighbouring sides of \(g\), then the third blue point \(v\) can
not be blocked from both \(v_1\) and \(v_2\). To see this we can suppose wlog. that \(v\) is on the
same side of \(r_1r_2^2\) as \(v_1\). But then there are no points that could block \(v\) and \(v_1\).

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If the beam \( v_1v_2 \) is positioned otherwise, then the only point that can block the third red point \( r \) from \( r_1 \) is \( b \). It follows that \( r \) and \( r_2 \) can see each other.

Now we consider the case where \( r_1 \) lies on a side of \( GT \). The red triangle \( RT = (r_1r_2r) \) lies in a closed half plane determined by the line \( r_1r_2 \), and a side \( s \) of \( GT \) necessarily lies outside this half plane. The blocking point of \( s \) lies outside \( \text{conv}(RT) \), so \( RT \) must be blocked by 3 points of different colors: \( p, q \) and \( g \). Note that one of these three blocking points is also needed to block a side of \( GT \), suppose this point is \( p \).

\( RT \) will have a black point \( b \) and an inner blue point (suppose \( v_1 \)) on its sides. (So \( \{p,q\} = \{b,v_1\} \).) Let \( v_2 \) be the other point that is outside \( RT \) and blocks a side of \( GT \). Since there are 2 blue points in \( \text{iconv}(H) \), \( v_2 \) must be blue.

We can distinguish two cases depending on the role of \( p \): it can either block \( r_1r \) or \( r_2r \). It is easy to verify that in both cases \( v_2 \) can see both \( p \) and \( q \), but one of them is blue, which concludes the proof of this theorem.

\begin{definition}
Let \( h(s) \) be the smallest number such that a planar point set of \( s \) points in general position contains an empty \( s \)-gon.
\end{definition}

It is known that \( h(4) = 5 \), \( h(5) = 10 \) \[3\], and the best known upper bound for \( h(6) \) is 463 \[6\]. Horton \[4\] showed that \( h(s) = \infty \) for all \( s \geq 7 \).

\begin{theorem}
\( mc_3(5) \leq 5h(6) - 5 \leq 2310 \).
\end{theorem}
Proof. The proof will be by contradiction. Take a properly 5-colored point set $P$ that has at least $5h(6) - 4$ points. It follows that the largest color class $C$ has at least $h(6)$ points. Since $\ell = 3$, the points of $C$ are in general position, so they contain an empty convex hexagon $H$. It follows that $H$ is 4-color blocked, which contradicts Theorem 11.

4 Conclusions and remarks

We have shown that empty convex hexagons cannot be 4-color-blocked, and with this result we were able to derive the first upper bound for the value $mc_3(5)$. We believe that similar techniques could be used to investigate whether points in non-convex positions can be blocked by only a few colors, and such an investigation could lead to resolving Conjecture 2 or at least a lot of progress in bounding the values of $mc_\ell(k)$.

However, these proofs should be automated. We believe that it is possible to develop an algorithm that systematically checks all cases, using a search tree that is kept relatively small with proper pruning techniques.

Another interesting question would be the relationship of the maximum clique size and the chromatic number in visibility graphs. Is there a sequence $G_n$ of visibility graphs such that $\omega(G_n) = o(\chi(G_n))$? An answer to this question would illustrate the relationship between Conjecture 1 and Conjecture 2.

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Appendix

Proof of Lemma 6

Let \( Q = \{q_1, q_2, q_3, q_4\} \) denote the unicolored vertex set of the convex quadrilateral. Consider any blocking set \( B \) of three colors, suppose that the color of \( Q \) is black and the colors of \( B \) are red, green and blue. Let \( B' \) denote \( \text{conv}(Q) \cap B \).

On one hand, \( B' \) is a properly colored set with 3 colors and no more than 3 points on a line, so by claim 3, \( |B'| \leq 6 \). On the other hand, \( B' \) must have a point on any side of the quadrilateral (we denote them by \( y_{12}, y_{23}, y_{34}, y_{41} \)), and at least one point in \( \text{iconv}(Q) \) to block the visibilities along the diagonals of the quadrilateral. So there are two cases:

Case 1: \( |B'| = 5 \). In this case, the only point in \( \text{iconv}(Q) \) must block both diagonals, so it must be at the intersection of the diagonals, denote it by \( z \). \( z \) is visible by all other points of \( Q \cup B' \), so it must have a unique color (green). Then all the \( y_i \)-s will be red or blue. Visibility between \( y_i \)-s of neighbouring sides of the quadrilateral cannot be blocked by any point, hence neighbouring \( y_i \)-s must have different colors. So opposite pairs of \( y_i \)-s must have the same color, so they have to be blocked by \( z \). We got the figure above.

Case 2: \( |B'| = 6 \). One of the two points in the interior is visible by all the other points of \( Q \cup B' \), it must have a unique color, say green. Then the remaining five points must share two colors: red and blue. Hence there is a unicolored triangle among them. If \( B' \cup X \) is properly colored, this triangle is blocked by only two colors, that contradicts lemma 2.

What is left is to prove that there is no point of \( B \) outside \( \text{conv}(X) \). Suppose the contrary. We may assume (w.l.o.g.) that \( B \setminus B' \) has some points in the convex territory bordered by rays \( zq_1 \) and \( zq_2 \). Let \( p \) be a point among those with a minimal distance to the line \( q_1q_2 \). Now consider the following pairs: \( (p, x_1), (p, y_{12}), (p, z), (p, y_{23}), (p, y_{41}) \). Using the minimal distance property of \( p \) we have that the first three pairs are all visible (\( y_{12} \) cannot block \( (p, z) \) as that would mean \( p, y_{12}, z \) and \( y_{34} \) are all on the same line). One of the last two pairs may be blocked by \( y_{12} \) but then the other one will be a visible pair. \( x_1 \) is black, \( y_{12} \) is red, \( z \) is green and \( y_{23}, y_{41} \) are blue, so \( p \) cannot have any color, contradiction.

Proof of Lemma 8

Assume that the concave set \( X = (x_1, x_2, x_3, x_4) \) is red. (Here \( x_4 \in \text{iconv}(x_1, x_2, x_3) \).)

Suppose there are 4 unicolored (blue) points in the blocking set in \( \text{conv}(X) \setminus X \). It follows that there are 4 blue points \( B = \{b_1, b_2, b_3, b_4\} \) that form a blue-empty set. Suppose the points of \( B \) are in convex position. By Lemma 4 \( B \) can only be blocked one way, but that configuration does not contain a concave 4-point set, and it is also maximal. Thus the points of \( B \) are in concave position. But then there would be a triangle \( b_1, b_2, b_3 \in B \)
such that \((\text{conv}(b_1, b_2, b_3) \setminus \{b_1, b_2, b_3\}) \cap X = \emptyset\). (In particular only \(x_4\) can be in \(\text{conv}(B)\), but it cannot be both in \(\text{iconv}(x_1, x_2, x_4), \text{iconv}(x_2, x_3, x_4), \text{iconv}(x_3, x_1, x_4)\).)

Thus every color set in \(\text{conv}(X) \setminus X\) has at most 3 points. Since the segments \((x_i, x_j)\) need to be blocked, there are at most 3 points in \(\text{iconv}(T_1) \cup \text{iconv}(T_2) \cup \text{iconv}(T_3)\) (Here \(T_i\) \((i = 1, 2, 3)\) denote the red-empty triangles of \(X\).) We distinguish 7 cases based on the distribution of points in \(T_1, T_2, T_3\). The shorthand notation \((a_1, a_2, a_3)\) means that \(|T_i \cap S| = a_i\). From the discussion above it follows that \(a_1 + a_2 + a_3 \leq 3\). We may assume without loss of generality that \(a_1 \leq a_2 \leq a_3\).

We denote by \(s_{ij}\) the point lying on the segment \(x_i x_j\), and let \(S_k = \{s_{ij} | 1 \leq i \leq j \leq 4, i, j, k \text{ are distinct}\}\). Note that beams exist if and only if \(a_i = 1\) or \(a_i = 2\) by Lemma 2.

Case 1: \((0, 0, 0)\)
The three points in \(s_{i4}\) \((i = 1, 2, 3)\) can all see each other, thus their colors are distinct. Since each \(T_i\) contains at least one point of each color, there must be 2 points in each color class. It follows that the blockers of the opposing edges of the tetrahedron have the same color. Since \(\ell = 3\), \(s_{i4}\) and \(s_{jk}\) cannot be blocked by \(x_4\), so they are blocked by either \(s_{j4}\) or \(s_{k4}\). (Here \(\{i, j, k\} = \{1, 2, 3\}\).) It is easy to observe that we arrive at the configuration defined in the statement of the lemma (or its reflection).

Case 2: \((0, 0, 1)\)
Let \(p\) be the blocking point of the beam of \(T_3\). The blocking point cannot lay on the line \(x_3 x_4\) since there would be 4 collinear points on the line. Thus it lies on one side of line \(x_3 x_4\). We distinguish two subcases based on the position of the beam in \(T_3\).

(2a) If the beam is on \(s_{14}\) and \(s_{24}\), then we may assume that \(p\) lies on the left side of the ray \(\overrightarrow{x_3 x_4}\). Note that \(s_{24}\) cannot block any points from \(p\) again, because \(\ell = 3\). Thus \(p\) can see both \(s_{24}, s_{23}\) and \(s_{34}\), three points with different colors, which is a contradiction.

(2b) If the beam is on \(s_{14}\) and \(s_{12}\), and \(p\) lies on the right side of \(\overrightarrow{x_3 x_4}\), then a similar argument shows that \(p\) can see all points in \(S_2\). Otherwise since \(s_{24}\) and \(s_{34}\) cannot be blue (since \(T_3\) and \(T_2\) already has its blue points), \(s_{23}\) is blue. It follows that \(s_{34}\) has the same color as \(p\). Now the segments \(s_{23} s_{12}\) and \(p s_{34}\) can only be blocked by \(s_{24}\), but since the segments are disjoint, it cannot block both of them.
Note that the third possible beam position can be handled like this because it can be obtained by a reflection from this case.

Case 3: (0, 0, 2)
Again, we have 4 subcases based on the position of the beam in $T_3$, and the way $T_3$ is blocked. Let $p$ be the midpoint of the beam, and let $q$ be the other point in $T_3$. Note that $s_{14}, s_{24}$ and $q$ cannot block anything from $p$ because $\ell = 3$. If $p$ is on the right side of the ray $\overrightarrow{x_3 x_4}$, then it can see all points in $S_2$, one of which has the same color as $p$. If $p$ is on the left side, it will see all points in $S_1$.

Case 4: (0, 0, 3)
If $p$ is on the right side of $\overrightarrow{x_3 x_4}$, then it can see all points in $S_2$. Otherwise $p$ and $q$ are both on the left side, but then $q$ can see all points in $S_1$. 
Case 5: \((0, 1, 1)\)

We have 6 subcases based on beam positions.

(5a) Beams are \(s_{13}s_{14}\) and \(s_{14}s_{12}\). It follows that \(s_{13}, s_{14}\) and \(s_{12}\) have the same color (blue). But triangle \(T_1\) also needs a blue point, so we have 4 blue points, contradiction.

(5b) Beams are \(s_{24}s_{14}\) and \(s_{14}s_{34}\). But then \(s_{34}\) and \(s_{24}\) have the same color and they can see each other since \(\text{int}(T_1)\) has no points.

(5c) Beams are \(s_{13}s_{34}\) and \(s_{12}s_{24}\). The endpoints of the two beams must have distinct colors because \(s_{34}\) and \(s_{24}\) must be different to block \(T_1\). Assume the endpoints are blue and black. It is easy to observe that \(s_{14}\) has to be green, thus the point \(p\) in \(\text{int}(T_3)\) has to be blue. If \(p\) lies on the left side of \(\overrightarrow{x_3x_4}\), then it can see \(s_{34}\), otherwise it has to be blocked from \(s_{34}\) and \(s_{13}\), but the only point we can use for blocking is \(s_{14}\), so \(p\) will see at least one of \(s_{34}\) and \(s_{13}\).

(5d) Beams are \(s_{14}s_{34}\) and \(s_{12}s_{14}\). Let \(q\) be the blocker of beam \(s_{14}s_{34}\). Either \(s_{24}\) is black, in which case it can see \(q\), or there are 2 black points in \(T_1 \cup \text{conv}(x_2x_4s_{14}s_{12})\), one of which can see \(q\), because \(q\) can be blocked from them only by \(x_4\).

(5e) Beams are \(s_{14}s_{24}\) and \(s_{13}s_{34}\). Let \(p\) be the blocker of beam \(s_{14}s_{24}\). Now depending on which side of \(\overrightarrow{x_3x_4}\) point \(p\) is, \(p\) can see all points in \(T_1\) or all points in \(\text{conv}(x_4s_{34}s_{13}x_1)\).

(5f) Beams are \(s_{34}s_{13}\) and \(s_{14}s_{12}\). If \(p\) is on the right side of \(\overrightarrow{x_3x_4}\), then it can see all points in \(\text{conv}(x_4s_{34}s_{13}x_1)\). Otherwise \(s_{24}\) can only block one segment out of \(s_{34}p\) and \(s_{23}s_{12}\).

Note that the cases (5d), (5e), (5f) have reflections which can be handled the same way.
**Case 6:** (0, 1, 2)

We have 9 subcases based on beam positions. Let \( p \) be the midpoint of the beam in \( T_3 \). Notice that in the previous case, all subcases except (5c) and (5d) ended with the conclusion that \( p \) can see all points in \( T_1 \) or all points in \( T_2 \). All these arguments can be carried over here, since the new point \( q \) in this case cannot block any of these points from \( p \). It is also easy to see that the argument in (5d) and its reflection works here as well. So the only remaining case is corresponding to the beam position (5c).

(6c) Beams are \( s_{13}s_{34} \) and \( s_{12}s_{24} \). The endpoints of the two beams have different colors, suppose that \( s_{13}, s_{34} \) are blue and \( s_{12}, s_{24} \) are black. The colors of the rest of the points are determined as shown in the figures. If \( p \) lies on the left side of \( \overrightarrow{x_3x_4} \), then it can see all points in \( S_1 \) otherwise \( s_{13} \) and \( s_{34} \) can see the blue point in \( T_3 \).

![Case (6c)](image)

**Case 7:** (1, 1, 1)

(7a) The beams form a path of length 3. In this case the beam endpoints have the same color, and there is 4 of them. (With rotations and reflections, this case covers 9 beam positionings.)

(7b) The beams form a circle of length 3. The blue triangle \( s_{14}s_{24}s_{34} \) has one point inside, so it must be blocked as option 2 in Lemma 2. Without loss of generality we may suppose that \( q \) and \( r \) have the same color. They need to be blocked from \( s_{12} \), but the only point that can block either of them is \( p \), and it cannot block both \( s_{12}q \) and \( s_{12}r \).

(7c) The beams form a path of length two, and one of the endpoints of the path is not on the sides of \( x_1x_2x_3 \). We use the notations of the figure. Point \( p \) hast to be black or green. Depending on its position in relation to ray \( \overrightarrow{x_1x_4} \) it can see all green and black points in \( T_3 \) or \( T_2 \). (This case corresponds to 6 beam positions after taking rotations and reflections.)

(7d) The beams form a path of length two, and both endpoints of the path are on the sides of \( x_1x_2x_3 \). In this case the path is disjoint from triangle \( T_1 \), but \( T_1 \) must contain at least one more blue point, so we would have 4 blue points. (This case corresponds to 9 beam positions.)

(7e) The endpoints of the beams are disjoint. We use the notations of the figure. If
$q$ is on the right side of $x_3x_4$, then it can see $s_34$, otherwise $s_{12}$ is on the left side, and it can see $r$.

![Diagram](image1)

**Case (7b)**

![Diagram](image2)

**Case (7c)**

![Diagram](image3)

**Case (7d)**

![Diagram](image4)

**Case (7e)**

Proof of Theorem 10

Let $X$ be a properly 4-colored configuration. Assume $|X| \geq 13$. The largest color class $C$ contains at least 4 points, so there is a convex or concave $C$-empty set of 4 points. It is necessarily blocked by 3 colors, meaning that the blocking configuration is equivalent to the one described in Lemma 6 or Lemma 8. Both of these configurations are maximal, it follows that $|X| \leq 10$, a contradiction.

The two configurations below show properly 4-colored 12-point sets, and they prove $mc_3(4) = 12$.

![Diagram](image5)

![Diagram](image6)