Inter-Class Orthogonal Main Effect Plans for Asymmetrical Experiments

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Abstract
In this paper we construct ‘inter-class orthogonal’ main effect plans (MEPs) for asymmetrical experiments. In such a plan, the factors are partitioned into classes so that any two factors from different classes are orthogonal. We have also defined the concept of “partial orthogonality” between a pair of factors. In many of our plans, partial orthogonality has been achieved when (total) orthogonality is not possible due to divisibility or any other restriction. We present a method of obtaining inter-class orthogonal MEPs. Using this method and also a method of ‘cut and paste’ we have obtained several series of inter-class orthogonal MEPs. One of them happens to be a series of orthogonal MEP (OMEPs) [see Theorem 3.6], which includes an OMEP for a $3^{30}$ experiment on 64 runs. We have also obtained a series of MEPs which are almost orthogonal in the sense that every contrast is non-orthogonal to at most one more. A member of this series is an MEP for a $3^{10}2^{10}$ experiment on 32 runs in which the only non-orthogonality is between the linear contrasts of pairs of three-level factors. Plans of small size ($\leq 15$ runs) are also constructed by ad-hoc methods. Among these plans there are MEPs for a $4^2.3^2.2$ and a $3^5.2$ experiment on 12 runs and a $5^2.3^2$ experiment on 15 runs.

AMS (2000) subject classification. 62k10.
Keywords and phrases. Main effect plans, Inter-class orthogonality

1 Introduction
In many industrial experiments like screening experiments, the interest lies only in the main effects of factors. The wide use of orthogonal main effect plans (OMEPs) for such experiments is due to their orthogonality property that ensures both uncorrelated (and the most precise) estimation of every main effect contrast of every factor, and a simple analysis.

However, orthogonality requires certain divisibility conditions and so an OMEP for an asymmetrical experiment often requires a large number of runs. [See Dey and Mukherjee (1999) and Hedayat et al. (1999) for details].
The proportional frequency (PF) plans of Addelman (1962), as we know, are OMEPs with possibly unequal replications for one or more factors, thus require weaker conditions for existence. However, very few unequally replicated PF plans are known, apart from Starks (1964) plan for a $3^7$ experiment on 16 runs. Thus, the problem of availability of an OMEP with economic run size remains. Therefore, the question arises whether one can find an alternative plan (with given run size) - something not as good as an OMEP but not too bad either.

Of late, departure from full orthogonality has been investigated in the context of main effect plans (MEPs). In the “nearly orthogonal” plans of Wang and Wu (1992) factors are allowed to be non-orthogonal to a few of the other factors. Subsequently, other nearly orthogonal MEPs having interesting combinatorial properties have been proposed and studied by others like Nguyen (1996), Huang et al. (2002), Ma et al. (2000), Xiao et al. (2012), & Jones and Nachtsheim (2013).

Why do we stop at near orthogonality? Why can’t we go to the far end and use a fully non-orthogonal plan? This would lead to tremendous flexibility. We could, for instance, make a $2^4$ experiment on 5 runs (instead of 8) [see plan $A_5(2)$ of Example 2.2] or a $3^5$ experiment on 12 runs (instead of 16) [see Plan $A_{12}(4)$ in Section 5]. One hurdle to the usability of such plans is the complexity in the data analysis. However, in the age of powerful computers, complexity of analysis is not a paramount consideration. The reduction in the precision is, of course, another problem.

In the present paper our main aim is to provide main effect plans (MEPs) for asymmetrical experiments with small run size, deviating “as little as possible” from the desirable properties like orthogonality and/or equal replications, so that the reduction in the precision is not too much and the analysis remains relatively simple. Specifically, we construct plans satisfying “inter-class orthogonality”, in which factors from different classes are orthogonal. In the process we have also obtained a series of orthogonal MEPs which includes an OMEP for a $3^{30}$ experiment on 64 runs (see Theorem 3.6).

We have also defined the concept of “partial orthogonality” between a pair of factors and derived sufficient condition for it. [See Definition 2.15, Theorem 2.21, Corollary 2.1.1 and the discussion thereafter.] In many of our plans, partial orthogonality has been achieved between one or more pair(s) of factors when (full) orthogonality is not possible due to divisibility or any other restriction.

We now present a small example of an inter-class orthogonal MEP with partial orthogonality between a pair of factors.
Notation 1.1. The factors are named as $A, B, \ldots$ in the natural order. Many of the plans presented in the paper will be accompanied by a graph describing its orthogonality structure. The vertices are the factors. Bold edges denote orthogonality and directed edges (arrows) partial orthogonality.

Example 1.1. An experiment is to be conducted to study the influence of the factors on the filtration rate of a chemical product. The factors are temperature ($A$), pressure ($B$), concentration of the reactants ($C$) and stirring rate ($D$). It is felt that effects of three different temperatures and three different pressures are needed to be seen while two concentrations and two stirring rates were adequate. Due to financial reasons, only 8 runs are possible. Thus, one needs a plan for a $3^2.2^2$ experiment on 8 runs. An OMEP for such an experiment on 8 runs can not exist. We suggest the following MEP $A_8(1)$. The data obtained from the experiment conducted based on $A_8(1)$ is shown in Table 1 below. The relation between the factors is shown in the following graph.

$A_8(1)$ has the following features.

(a) The set of factors can be divided into classes as follows. \( \{A, B\} \cup \{C\} \cup \{D\} \). Every factor is non-orthogonal to the other member of its own class and orthogonal to the members of the other class. Thus, it may be termed as inter-class orthogonal.

(b) The BLUE of the quadratic contrast of factor $A$ is orthogonal to the BLUEs of both the main effect contrasts of factor $B$, but the linear contrast of $A$ does not satisfy this. Similarly, the BLUE of only the quadratic contrast (and not the linear one) of $B$ is orthogonal to the BLUEs of both the main effect contrasts of factor $A$. Thus, we may say that factor $A$ (respectively $B$) is partially orthogonal to factor $B$ (respectively $B$).
Table 2: ANOVA for $A_8(1)$

| Source          | d.f | Adjusted S.S | Unadjusted S.S | M.S.S.  | F     |
|-----------------|-----|--------------|----------------|---------|-------|
| Class 1:        |     |              |                |         |       |
| $A$             | 2   | 1438.2       | 1249.0         | 624.5   | 41.63 |
| $B$             | 2   | 471.46       | 283.0          | 141.5   | 9.43  |
| Class total     | –   | 1721.2       | –              | –       | –     |
| Class 2:        |     |              |                |         |       |
| $C$             | 1   | 253.0        | 253.0          | 253.0   | 16.87 |
| Class 3:        |     |              |                |         |       |
| $D$             | 1   | 136.0        | 136.0          | 136.0   | 9.06  |
| Error           | 1   | 15.0         | –              | 15.0    | –     |
| Total           | 7   | 2125.0       | –              | –       | –     |

Conclusion: The data do not provide evidence of significant effect of any of the factors on the filtration rate, at 5% as well as 10% levels of significance.

The data analysis is shown in Table 2 below. One may note that Type I SS for factors $A$ and $B$ must not be used in the analysis, as they are not orthogonal to each other.

We present definition, notation and other details in Section 2. In Section 3 we construct a few series of “inter-class orthogonal” MEPs for asymmetrical experiments with factors having at most seven levels. In the process we also get a series of OMEPs for a symmetric experiment with three-level factors. In another series of MEPs for two- and three-level factors the only non-orthogonality is between the linear contrasts of pairs of three-level factors.

Using ad-hoc methods we have also constructed MEPs with factors nonorthogonal to one or two factors on at most 15 runs, which are in Section 4. These plans include saturated plans for the following experiments. $4^2.2$, $3^2.2^3$ and $4.3^2.2^2$ on 8 runs, $5^2.2$ on 10 runs, $4^2.3^2.2$ and $2.3^5$ on 12 runs and $5^2.3^2$ on 15 runs.

2 Inter-Class Orthogonal Main Effect Plans

Throughout this paper we shall be concerned with main effect plans, that is, plans aiming at gaining information only about the main effects, assuming interactions to be negligible. In all plans presented henceforth in this paper, rows represent factors, while columns represent runs.

Notation 2.1. (a) A plan on $n$ runs and having $m$ factors $A, B, \cdots$ will be referred to as an $m$-factor MEP.

(b) $\mathcal{F}$ will denote the set of all factors.
(c) The number of levels of factor $P$ will be denoted by $p$, $P = A, B, \cdots$.

(d) The plan will be represented by an $m \times n$ array $\rho(n, m; a, b, \cdots)$.

We assume an additive, fixed effects, main effects model with homoscedastic and uncorrelated errors having constant variance $\sigma^2$.

**Notation 2.2.** (a) $1_n$ will denote the $n \times 1$ vector of all-ones, while $J_{m \times n}$ will denote the $m \times n$ matrix of all-ones.

(b) $\alpha_j$ will denote the unknown effect of the $j$th level of the factor $A$. $\alpha$ will denote the column vector with entries $\alpha_1, \cdots \alpha_a$. Similar notation for other factors.

(c) $Y_u$ will denote the yield from the $u$th run, $u = 1, 2, \cdots n$. Assuming that in the $u$th run the factor $A$ is set at level $u_A$, $B$ at level $u_B$ and so on; denoting the general effect by $\mu$ and error by $\epsilon_u$, $Y_u$ is given by

\[ Y_u = \mu + \alpha_{uA} + \beta_{uB} + \cdots + \epsilon_u, u = 1, 2, \cdots n. \]

(d) Viewing the general effect as another factor, say $G$, the model is expressed in matrix form as

\[ Y = X \begin{bmatrix} \mu \\ \eta \end{bmatrix}, \text{ where } X = \begin{bmatrix} X_G & X_A & X_B & \cdots \end{bmatrix} \text{ and } \eta = \begin{bmatrix} \alpha & \beta & \cdots \end{bmatrix}'. \]

(2.1)

Here, $X_G = 1_n$. Further, $X_P$, the design matrix for $P$, is a $n \times p$ matrix with entries 0 and 1 - the $(u, t)$th entry of $X_P$ is 1 if in the $u$th run $P$ is set at level $t$ and 0 otherwise, $P \in \mathcal{F}$.

**Notation 2.3.** (a) $r_P$ will denote replication vector of the factor $P$. $R_P$ will denote a diagonal matrix with diagonal entries same as those of $r_P$ in the same order.

(b) $N_{PQ}$ will denote the factor $P$ versus factor $Q$ incidence matrix, $P, Q \in \mathcal{F}$. It may be noted that $N_{PQ} = X_P'X_Q$; thus when $Q = P, N_{PQ} = R_P$.

(c) $G$ and $T_P$ will denote grand total and the vector of raw totals of $P$, $P \in \mathcal{F}$, respectively.

**Definition 2.1.** A main effect contrast of a factor $P$ is said to be orthogonal to a main effect contrast of another factor $Q$ if the BLUEs of these contrasts are uncorrelated.

A pair of factors $P$ and $Q$ are said to be orthogonal if the BLUE of every main effect contrast of $P$ is uncorrelated with that of every main effect contrast of $Q$. 

Definition 2.2. Let us consider an \( m \)-factor MEP \( \rho \) on \( n \) runs. Suppose the set of factors \( \mathcal{F} \) can be partitioned into several classes, say as \( \mathcal{F} = \bigcup_{i=1}^{c} \mathcal{F}_i \) in such a way that \( P \in \mathcal{F}_i \) is orthogonal to \( Q \in \mathcal{F}_j, j \neq i \). Then \( \rho \) is called inter-class orthogonal. Note that a pair of factors from the same class may also happen to be orthogonal. A plan with at most \( t \) factors in every class may be referred to as an inter-class(t) orthogonal MEP.

An inter-class orthogonal main effect plan (IOMEP) on \( n \) runs with \( m \) factors and the factors in the \( i \)-th class having levels \( s_{i1}, s_{i2}, \ldots \) will be denoted by \( \rho(n,m;\{s_{11}, s_{12}, \ldots \}, \{s_{21}, s_{22}, \ldots \}, \ldots) \); except that, in case a class is singleton, we shall omit the curly bracket.

Remark 2.1. Any main-effect plan may be looked upon as an inter-class(t) orthogonal MEP, for some \( t \). For instance, an OMEP may be viewed as an inter-class(1) orthogonal MEP, while an MEP with \( p \) factors of which no one is orthogonal to any other is inter-class(p) orthogonal. The plan \( L'_{18}(3^4.2^8) \) (termed a “near orthogonal” array) of Wang and Wu (1992) is an inter-class(8) orthogonal MEP, according to the present terminology, as the 8 two-level factors are mutually non-orthogonal. We see that the term inter-class(t) orthogonal does not always convey the exact picture as there may be classes with size much smaller than \( t \), as in the case of \( L'_{18}(3^4.2^8) \). This term is informative when the class sizes are close to one another, which is the case for the plans constructed here.

Example 2.1. The plan \( A_8(1) \) in Table 1 is, according to Definition 2.5, denoted by \( \rho(8,4;\{3^2\}.2^2) \). Many more interclass orthogonal plans are presented later.

We now define another desirable property of an MEP: Orthogonality through another factor.

Definition 2.3. Consider three factors \( A, B \) and \( C \) of an MEP. We say that \( A \) is orthogonal to \( B \) “through” \( C \) if the incidence matrices \( N_{AB}, N_{BC} \) and \( N_{AC} \) satisfy the following condition.

\[
N_{AC}(R_C)^{-1}N_{CB} = N_{AB}. \tag{2.2}
\]

It may be noted that situations in which \( C \) is a nuisance factor have been handled earlier. Morgan and Uddin (1996) considered main effects plans on a nested row-column set up, where \( C \) was the column factor and Bagchi (2010) considered blocked main effects plans with \( C \) as the block factor.

Remark 2.2. Consider an MEP \( \rho_1 \) with three factors \( A, B \) and \( C \) such that \( B \) is orthogonal to \( C \) through \( A \). Let us compare with another MEP \( \rho_2 \) for
the same experiment in which there is no orthogonality among the factors. 
To estimate the main effect contrasts of $B$ from $\rho_1$ one has to eliminate 
only $A$, while if one uses $\rho_2$, both $A$ and $C$ are to be eliminated. Thus, the 
contrasts of $B$ are estimated with greater precision when $\rho_1$ is used. The 
same holds for $C$. Regarding $A$, however, the performances of both plans 
would be similar.

In the plan $A_5(1)$ in Example 2.2 below, $B$ and $C$ are mutually orthogonal 
through $A$. Further, in the plans $A_5(2)$ in Example 2.2 and $A_{10}$ and $A_{14}$ 
in Examples 2.3 and 2.4 respectively, every pair of factors among the last 
three are mutually orthogonal through the first one. Thus, there are three 
factors, the contrasts of which are estimated with greater precision than a 
non-orthogonal MEP. Moreover, in the plans $A_{10}$ and $A_{14}$ the linear contrast 
of every factor is orthogonal to every contrast of every factor [see Examples 
2 and 3 after Corollary 2.1.1].

These plans are used as ‘replacing arrays’ [ see Definition 3.1] in the 
construction of inter-class orthogonal plans. [See Theorems 3.3 and 3.4].

**Example 2.2.** Two MEPs on 5 runs with two- and three-level factors :

$$A_5(1) = \rho(5, 3; \{3 \times 2^2\}) = \begin{bmatrix} 0 & 0 & 1 & 1 & 2 \\
0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \end{bmatrix},$$

$$A_5(2) = \rho(5, 4; \{2^4\}) = \begin{bmatrix} 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

(2.3)

**Remark 2.3.** In the plan $A_5(2)$ the contrast of each of factors $B, C, D$ is 
estimated with the same precision as that of a factor in the plan obtained 
from an $OA(4, 3, 2, 2)$ [see Definition 3.2]. Thus $A_5(2)$ may be viewed as 
follows. By adding a run to $OA(4, 3, 2, 2)$ an additional two-level factor is 
accommodated without compromising the precisions in estimating the existing 
factors.

**Example 2.3.** An MEP on 10 runs with three-level factors :

$$A_{10}(1) = \rho(10, 4; \{3^4\}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 & 2 & 2 \\
1 & 1 & 2 & 0 & 2 & 0 & 1 & 1 & 2 & 0 \\
1 & 2 & 1 & 0 & 0 & 2 & 0 & 1 & 2 & 1 \\
1 & 2 & 0 & 1 & 2 & 0 & 0 & 1 & 1 & 2 \end{bmatrix}.$$ 

(2.4)
Example 2.4. An MEP on 14 runs with four-level factors:

\[ A_{14}(1) = \rho(14, 4; \{4^4\}) = \begin{bmatrix}
0 & 0 & 0 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
0 & 0 & 1 & 2 & 1 & 2 & 1 & 1 & 3 & 3 & 3 & 0 & 0 & 1 & 2 \\
0 & 1 & 0 & 2 & 2 & 1 & 1 & 3 & 1 & 3 & 2 & 0 & 1 & 0 \\
0 & 1 & 2 & 0 & 1 & 2 & 1 & 3 & 3 & 1 & 2 & 0 & 0 & 1 \\
\end{bmatrix} \quad (2.5) \]

2.1. Partial Orthogonality Between Factors

The relation between the factors A and B in \( A_8(1) \) motivates us to define the concept of partial orthogonality between two factors.

**Definition 2.4.** Consider a pair of factors A and B of an MEP.

We say that the factor A is partially orthogonal (PO) to another factor B if the following conditions are satisfied.

(a) No main effect contrast of A is confounded with any main effect contrast of B.

(b) The BLUE of at least one main effect contrast of A is orthogonal to the BLUE of every one of B.

Note that a two-level factor can not be partially orthogonal to another factor.

A statement like “A is PO to B” immediately raises the question “which contrast of A is orthogonal to B”? We shall now see how the incidence matrix \( N_{AB} \) helps us to find at least partial answer to this question.

We recall the proportional frequency condition of Addelman (1962).

**Definition 2.5.** [Addelman (1962)] Two factors A and B of main effect plan \( \rho \) on \( n \) runs are said to satisfy proportional frequency condition (PFC) if:

\[ n_{A,B}(i, j) = \frac{r_A(i).r_B(j)}{n}, \quad i = 1, 2, \ldots, a, \quad j = 1, 2, \ldots, b. \quad (2.6) \]

**Remark 2.4.** One may check that PFC is equivalent to the orthogonality between A and B “through” the general effect. [See Definition 2.3].

We now define PFC between one factor and one or more levels of another factor.

**Definition 2.6.** Consider two factors A and B, with a and b levels respectively, of a main effect plan \( \rho \) on \( n \) runs.
(a) If a pair of levels \( i \) and \( k \) of \( A \) satisfies
\[
n_{A,B}(i,j)/r_A(i) = n_{A,B}(k,j)/r_A(k), \quad j = 1, 2, \ldots, b, \tag{2.7}
\]
then the pair \( \{i, k\} \) of levels of \( A \) is said to satisfy PFC with factor \( B \).

(b) Suppose all the replication numbers of \( A \) except possibly the \( i \)th one are equal. If the level \( i \) of \( A \) satisfies
\[
n_{A,B}(i,j) = r_A(i).r_B(j)/n, \quad j = 1, 2, \ldots, b, \tag{2.8}
\]
then we say that the level \( i \) of \( A \) satisfies PFC with factor \( B \).

(c) Consider an \( a \times 1 \) vector \( l \). We say that the set of levels of \( A \), weighted by \( l \), satisfies PFC with \( B \), if
\[
l'(R_A)^{-1}N_{AB} = 0. \tag{2.9}
\]

We quote the well-known result of Addelman (1962).

**Lemma 2.1.** If the incidence matrix \( N_{AB} \) of two factors \( A \) and \( B \) satisfies PFC, then \( A \) and \( B \) are orthogonal to each other.

We extend this result to partial orthogonality. Towards that we present a result, which is immediate from the definition of inter-class orthogonality.

**Lemma 2.2.** Consider an IOMEP with \( c \) classes.

(a) After elimination of \( \hat{\mu} \) from the normal equations there are \( c \) equations, the \( t \)th one involving only the BLUEs of the effects of factors in \( \mathcal{F}_t \), \( t = 1, \ldots, c \).

(b) Fix \( t, 1 \leq t \leq c \). Suppose we denote the factors in \( \mathcal{F}_t \) by \( A, B \) and so on. Then the system of reduced normal equations for \( \alpha, \beta \) etc is given by
\[
\sum_{B \in \mathcal{F}_t} C_{AB}\hat{\beta} = Q_A, A \in \mathcal{F}_t, \text{ where} \tag{2.10}
\]
\[
C_{PQ} = N_{PQ} - (1/n)r р _{rQ}' \text{ and} \tag{2.11}
\]
\[
Q_P = T_P - (G/n)r_{P}, P, Q \in \mathcal{F}_t. \tag{2.12}
\]

**Theorem 2.1.** Consider an IOMEP \( \rho \). Fix a factor \( A \) in \( \mathcal{F}_t \) and let \( l \) be an \( a \times 1 \) vector satisfying \( l'1_a = 0 \). If the set of levels of \( A \), weighted by \( l \), satisfies PFC with every \( B \neq A, B \in \mathcal{F}_t \) then the BLUE of \( l'\alpha \) is orthogonal to the BLUE of every contrast of every factor.
Proof. In view of Lemma 2.2 it is enough to consider the system of reduced normal Eq. 2.10. Pre-multiplying it by \( l' (R_A)^{-1} \), using the next equation and Eq. 2.9 [which holds in view of the hypothesis] we get \( l' \hat{\alpha} = l'Q_A \). Since this equation does not involve the effect of any other factor, the result follows.

Using two specific choices for the vector \( l \) (given in (c) of Definition 2.6) in Theorem 2.1 we get the following result.

Corollary 2.1. Consider the main effect contrasts \( M_1 = \alpha_i - \alpha_k \) and \( M_2 = (a-1)\alpha_i - \sum_{t \neq i} \alpha_t \) of \( A \) in \( F_t \).

(a) If the pair of levels \( \{i, k\} \) of \( A \) satisfies PFC with every \( B \neq A, B \in F_t \), then the BLUE of \( M_1 \) is orthogonal to the BLUE of every main effect contrast of every factor.

(b) If the level \( j \) of \( A \) satisfy PFC with every \( B \neq A, B \in F_t \), then the BLUE of \( M_2 \) is orthogonal to the BLUE of every main effect contrast of every factor.

Proof. Fix \( B \in F_t, B \neq A \). Let \( l = (l_1, \ldots l_a)' \).

(a) Let \( l_i = 1, l_k = -1, l_u = 0, u \neq i, u \neq k \). By hypothesis, Eq. 2.9 is satisfied by \( l \). So, by Definition 2.6 the set of levels of \( A \), weighted by \( l \), satisfies PFC with \( B \). Now the result follows from Theorem 2.21.

(b) Let the replication numbers of \( A \) and the entries of the vector \( l \) be as follows.

\[
\begin{align*}
  r_A(i) &= r + s, \quad \text{where} \quad s = (n - ar)/a, \quad r_A(u) = r, \quad u \neq i, (2.13) \\
  l_i &= a - 1, \quad l_u = -1, \quad u \neq i, \quad 1 \leq u \leq a. (2.14)
\end{align*}
\]

One can verify that the hypothesis together with Eq. 2.13 and the next equation implies (2.9). Therefore, the result follows from Theorem 2.1.

Example 2.5. Consider Plan \( A_8(1) \) and the class \( F_1 = \{ A, B \} \). Level 1 of \( A \) satisfies PFC with factor \( B \), as shown in the Table 3 below. Thus, by Corollary 2.1.1 (b), the quadratic contrast of \( A \) is orthogonal to all contrasts of \( B \) and hence all the main effect contrasts. By the same argument the quadratic contrast of \( B \) is orthogonal to every main effect contrast (see Table 3).
Table 3: A x B incidence matrix

| A ↓ | 0 | 1 | 2 | $r_A$ ↓ | $r_A \cdot r_B' / n$ |
|-----|---|---|---|---------|---------------------|
| $N_{AB}$ |  |   |   |         |                     |
| 0   | 1 | 1 | 0 | 2       | 1/2                 |
| 1   | 1 | 2 | 1 | 4       | 1                   |
| 2   | 0 | 1 | 1 | 2       | 1/2                 |

Example 2.6. Consider plan $A_{10}(1)$, in which no factor satisfies PFC with is any other. However, the pair of levels 0 and 2 of $A$ satisfies PFC with $B, C$ and $D$. Hence, by Corollary 2.1.1 (a), the linear contrast of $A$ is orthogonal to all the main effect contrasts. By the same argument the linear contrasts of each of $B, C$ and $D$ is orthogonal to every main effect contrasts of every factor. Applying the same argument to plan $A_{14}(1)$ we see that linear contrasts of every factor is estimated with maximum possible precision, as is the case with $A_{10}(1)$.

Remark 2.5. In the MEP presented in (2) of Huang et al. (2002), the relation between every pair of three-level factors in that plan is just like the relation between $A$ and $B$ of $A_8(1)$. However, we are not sure whether Theorem 2.1 is applicable.

2.2. Discussion Partial orthogonality has been explored as a useful property for design settings where full orthogonality is not possible. It can also require a compromise. If $A$ and $B$ cannot be made mutually orthogonal, then partial orthogonality imparts the beneficial properties of orthogonality to some contrasts of the two factors. In some cases, achieving partial orthogonality may also mean sacrificing on some other desirable property, hence the compromise. Consider the following situations for two factors $A$ and $B$ with $a$ and $b$ levels respectively.

Case 1. $ab$ does not divide $n$ so that there does not exist any plan in which $A$ and $B$ are mutually orthogonal, each with equal frequency. If a proportional frequency plan exists then that is the best option. Suppose such a plan is not known, but there is a plan, say $\rho_1$, in which both $A$ and $B$ are orthogonal to others and $A$ is partially orthogonal to $B$. Then the experimenter is able to estimate at least a few among the main effect contrasts of $A$ with maximum precision. Yet the precision for the other $A$ contrasts should not be ignored. Suppose there is also a plan $\rho_2$, lacking partial orthogonality, for which all the contrasts of $A$
and \( B \) are estimated with precision higher than some contrasts under \( \rho_1 \) (and necessarily lower than some others). The choice between \( \rho_1 \) and \( \rho_2 \) depends on the importance attached to each individual contrast. [See Remark 3.3.]

**Case 2.** \( ab \) divides \( n \), so that orthogonality between \( A \) and \( B \) is possible. However, in the only available plan (say \( \rho_1 \)) in which \( A \) and \( B \) are mutually orthogonal, various other pairs of factors are mutually non-orthogonal. Suppose another plan \( \rho_2 \) is also available in which \( A \) is partially orthogonal to \( B \), and in which several pairs of factors non-orthogonal in \( \rho_1 \) are orthogonal in \( \rho_2 \). Which plan is preferred? Again the choice depends on the importance attached to different contrasts of different factors. [See Remark 4.1.]

Design questions become more complex when fully orthogonal plans are not available. By providing a wider range of options, nearly orthogonal, inter-class orthogonal, and other similar plans can improve experimental practice.

### 3 Construction of Inter-Class Orthogonal Plans

**Definition 3.1.** Consider an MEP \( \rho(n, m; a, b, \cdots) \). We find an MEP \( \rho_A(a, l; t_1, t_2, \cdots t_l) \), \((l \geq 2)\), such that the quantities \( l, t_1, \cdots t_l \) satisfies the following.

\[
\sum_{i=1}^{l} (t_i - 1) \leq a - 1.
\]  

We replace each occurrence of the level \( u \) of factor \( A \) by the \( u \)-th column (run) of \( \rho_A \), for each \( u \), \( 0 \leq u \leq a - 1 \). This yields an \((m + l - 1) \times n \) array which represents a new MEP \( \tilde{\rho} \). We say that the factor \( A \) is replaced by a class \( G_A \) of \( l \) factors related through \( \rho_A \) and \( \rho_A \) will be said to be the replacing array for \( A \). In the same way we can replace two or more factors of a given MEP, through two or more suitable replacing arrays.

**Remark 3.1.** Note that if the replacing array \( \rho_A \) do not satisfy condition (3.1) then it is supersaturated and therefore the resultant MEP obtained by replacing \( A \) by \( \rho_A \) is also supersaturated. Thus, condition (3.1) acts as a measure for avoiding supersaturated plans.

**Example 3.1.** Consider the replacing arrays

\[
\rho_A = \rho(3, 2; 2^2) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \rho_B = \rho(3, 2; 2^2) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.
\]
We now replace first two factors from an MEP $\rho$ using the replacing arrays $\rho_A$ and $\rho_B$ respectively to construct a MEP $\tilde{\rho}$ with a bigger set of factors.

$$\rho = \rho(9, 4; 3^4) = \begin{bmatrix}
0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\
0 & 1 & 2 & 1 & 2 & 0 & 2 & 0 & 1 \\
0 & 1 & 2 & 2 & 0 & 1 & 1 & 2 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2
\end{bmatrix}$$

is modified to $A_9(1) = \rho(9, 6; \{2^2\}^2.3^2) = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 2 & 2 & 0 & 1 & 1 & 2 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2
\end{bmatrix}$

Note that if we remove the second row of $\rho_A$, then the procedure of “replacement” is reduced to Addelman (1962) “collapsing” of (1st and 2nd) levels of the three-level factor to make it a two-level factor. Thus, the condition $l \geq 2$ in Definition 3.1 distinguishes the procedure of “replacement” from “collapsing”.

Now we see how properties of the resultant plan $\tilde{\rho}$ depends on the initial plan $\rho$ as well as the replacing arrays $\rho_A, \rho_B$ and so on.

**Theorem 3.1.** Consider a set of factors $\mathcal{R} = \{A, B, \cdots\}$ of an MEP $\rho$. Suppose $\tilde{\rho}$ is an MEP obtained from $\rho$ by replacing each factor of $\mathcal{R}$ by a class of factors. More precisely, the factor $A$ (respectively $B$) is replaced by the class of factors $G_A$ (respectively $G_B$) related through $\rho_A$ (respectively $\rho_B$), and so on. Then, the following hold.

(a) If $A$ and $B$ are mutually orthogonal (with equal or unequal frequency) in the original plan $\rho$ then every factor in the class $G_A$ is orthogonal to every factor in the class $G_B$ in the derived plan $\tilde{\rho}$, generally with unequal frequency.

(b) In $\tilde{\rho}$ two factors of $G_A$ will be partially (respectively totally) orthogonal if and only if the corresponding rows of $\rho_A$ are partially (respectively totally) orthogonal.

**Proof.** We shall prove (a). (b) will follow by similar argument.

Proof of (a): Let $\mathcal{A}$ (respectively $\mathcal{B}$) denote the set of levels of $A$ (respectively $B$). Then, we may (and do) index the set of columns of $\rho_A$ (respectively $\rho_B$) by $\mathcal{A}$ (respectively $\mathcal{B}$).
Fix a factor, say $K$ of $G_A$ and a factor $L$ of $G_B$. Let $\mathcal{K}_s$ (respectively $\mathcal{L}_t$) denote the set of columns of $\rho_A$ (respectively $\rho_B$) in which the level $s$ of $K$ (respectively $t$ of $L$) appear. Then, $\mathcal{K}_s \subset A$. Similarly, $\mathcal{L}_t \subset B$.

Let $\tilde{N}_{KL}, \tilde{r}_K, \tilde{r}_L$ denote the $K$ versus $L$ incidence matrix and the replication vectors of $K$ and $L$ respectively in $\tilde{\rho}$. Then, the $(s,t)$th entry of $\tilde{N}_{KL}$ is given by

$$\tilde{n}_{KL}(s,t) = \sum_{i \in \mathcal{K}_s} \sum_{j \in \mathcal{L}_t} n_{AB}(i,j).$$  \hspace{1cm} (3.2)

From this, we obtain that for a level $s$ of $K$,

$$\tilde{r}_K(s) = \sum_{i \in \mathcal{K}_s} \sum_{j} \sum_{t} n_{AB}(i,j) = \sum_{i \in \mathcal{K}_s} r_A(i).$$  \hspace{1cm} (3.3)

Similarly, $\tilde{r}_L(t) = \sum_{j \in \mathcal{L}_t} r_B(j)$. But $N_{AB}, r_A, r_B$ satisfy (2.6) by hypothesis.

Combining that with the relations above we see that $\tilde{N}_{KL}, \tilde{r}_K, \tilde{r}_L$ also satisfy (2.6).

We recall the well-known definition of an orthogonal array.

**Definition 3.2.** Let $m, n, t \geq 2$ be integers and $\vec{s} = (s_1, \ldots, s_m)$ be a vector of integers $\geq 2$. Then an orthogonal array of strength $t$ is an $m \times n$ array, with the entries of the $i$th row coming from a set of $s_i$ symbols satisfying the following. All $t$-tuples of symbols appear equally often as columns in every $t \times n$ subarray. Such an array is denoted by $OA(m, n, s_1 \times \cdots \times s_m, t)$. When $s_1 = s_2 = \cdots = s_m = s$, say, this array is denoted by $OA(n, m, s, t)$.

**Corollary 3.1.** Suppose there exists an orthogonal array $OA(n, m, s, 2)$. Let $\rho_i = \rho(s, l_i; t_{i1}, \ldots, t_{il_i}), 1 \leq i \leq k$ be any $k$ replacing arrays, $k \leq m$, satisfying $s - 1 \geq \sum_{j=1}^{l_i} (t_{ij} - 1), i = 1, 2, \cdots, k$.

Then, an IOMEP $\rho(n, l; \prod_{i=1}^{k} \{t_{i1} \times \cdots \times t_{il_i}\} \cdot s^{m-k})$ exists. Here $l = \sum_{i=1}^{k} l_i + m - k$.

**Proof.** Let $\rho_0$ be the orthogonal MEP represented by the given orthogonal array. We replace the $i$th factor by a group of factors related through $\rho_i$, $i = 1, \cdots, k$ to form a new MRP $\rho$. Clearly $\rho$ has $l = \sum_{i=1}^{k} l_i + m - k$ factors. That $\rho$ is interclass orthogonal with the given parameters follows from Theorem 3.1.

**Examples of replacing arrays with desirable properties** We have seen that to obtain an useful IOMEP, one needs replacing arrays with desirable properties. We now present a few such arrays. The set of $a$ levels
of a factor are taken to be the set of integers modulo $a$. Apart from a few exceptions, the plans in this paper are denoted by $A_n(t)$, where $A$ stands for array, $n$ denotes the number of runs and $t$ the serial number of the plan among all plans with $n$ runs presented here.

**Notation 3.1.**

$$R(p, q) = \rho(s, 2; \{p, q\}) = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 2 & \cdots & p-1 \\ 0 & 1 & \cdots & q-1 & 0 & 0 & \cdots & 0 \end{bmatrix}, \ s = p + q - 1.$$ 

**Remark 3.2.** In $R(p, q)$ the contrast $\alpha_i - \alpha_j$ of factor $A$ is orthogonal to the contrasts of $B$, for $i \neq j$, $i, j \geq 1$. Similarly, the contrast $\beta_i - \beta_j$ of $B$ is orthogonal to the contrasts of $A$, for $i \neq j$, $i, j \geq 1$.

**A List of Replacing Arrays of Small Size.** Arrays with 4 runs :

$$A_4(1) = \rho(4, 2; \{3, 2\}) = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 0 \end{bmatrix} \text{ and } A_4(2) = \rho(4, 2; \{3, 2\}) = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

**Remark 3.3.** It is rather surprising that the contrast of the two-level factor $B$ is estimated with the same precision in both the plans $A_4(1)$ and $A_4(2)$. Regarding the other factor $A$, however, the plans perform in a different manner.

In $A_4(1)$, Eq. 2.8 is satisfied by level 1 of factor $A$, so that the quadratic contrast is orthogonal to the BLUEs of the main effect contrasts of $B$. As a result, it is estimated with maximum possible precision (given the replication vector), while the linear contrast is estimated with much less precision. In $A_4(2)$, however, both contrasts of $A$ are estimated with the same precision. Thus, while replacing a four-level factor the experimenter may choose between $A_4(1)$ and $A_4(2)$ depending on whether equal importance is attached to both the contrasts of $A$ or not.

Arrays with 5 runs : Arrays $A_5(1)$ and $A_5(2)$ are given in Eq. 2.3.

$$A_5(3) = R(4, 2) \text{ and } A_5(4) = R(3, 3).$$

Arrays with 7 runs :

$$A_7(1) = R(6, 2), \ A_7(2) = R(4, 4), \ A_7(3) = \rho(7; \{3^3\}) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 2 & 2 \\ 0 & 1 & 2 & 0 & 1 & 0 & 2 \\ 0 & 1 & 2 & 1 & 2 & 0 & 0 \end{bmatrix}.$$

In the next section we use suitable arrays from the list above to replace one or more rows of existing orthogonal arrays and obtain IOMEPs.
3.1. Some Series of Inter-Class Orthogonal Main Effect Plans

Our starting point is an OA\((n,m,s,t)\) (see Definition 3.2).

**Theorem 3.2.** (a) Whenever an OA\((n,m,\prod_{i=1}^{m} s_i,2)\) exists, an IOMEP \(\rho(n,2m;\prod_{i=1}^{m}((s_i-t_i)(t_i+1))\) exists. Here \(t_i\) is an integer, \(1 \leq t_i \leq s_i-2\).

(b) These IOMEPs may be constructed so as to satisfy partial orthogonality property among the members of the same class similar to the description in Remark 3.2.

**Proof.**

(a) For every \(i, 1 \leq i \leq m\), one can choose a \(2 \times s_i\) array, say \(\rho_i\), with \(p = s_i - t_i\) symbols in the first and \(q = t_i + 1\) symbols in the second row. Now using \(\rho_i\) as a replacing array for the \(i\)th factor of the given OA, \(1 \leq i \leq m\), we get the required MEP.

(b) In particular, if \(\rho_i = R(p,q)\) of Notation 3.1 with \(p = s_i - t_i, q = t_i + 1\), then the members of the \(i\)th class will satisfy the stated partial orthogonality property.

**Theorem 3.3.** Suppose \(s = 3, 4, 5\) or \(7\). Whenever an OA\((n,m,s,2)\) exists, the following series of IOMEPs \(\tilde{\rho}\) exist. Here \(p, q, r, s, t\) are nonnegative integers.

Further, in the case \(s = 5\), there exists an IOMEP for which the factors within the classes of size \(\geq 3\) satisfy the property that all factors except the first one are pairwise orthogonal through the first factor.

\[
\tilde{\rho} = \begin{cases} 
\rho(n;3^p \times \{2^2\}^q), & p + q = m, \quad \text{if } s=3 \\
\rho(n;4^p.\{3.2\}^q.2^{3t}), & p + q + t = m, \quad \text{if } s=4 \\
\rho(n;5^p.\{4.2\}^q.\{3^2\}^r.\{3.2^2\}^t.\{2^4\}^u), & p + q + r + t + u = m, \quad \text{if } s=5 \\
\rho(n;7^p.\{6.2\}^q.\{4^2\}^r.\{3^3\}^t), & p + q + r + t = m, \quad \text{if } s=7 
\end{cases}
\]

(3.7)

**Proof.** Let \(O\) be an OA\((n,m,s,2)\). We keep \(p\) (out of \(m\)) of the factors of \(O\) as they are, replace every other factor by a class of factors related through an appropriate replacing array. This replacing array can be (i) an OA if it is available (which is the case when \(s = 4\)), (ii) one of the replacing arrays shown above or (iii) a replacing array of similar type. Corollary 3.1.1 implies that the MEP thus constructed is inter-class orthogonal with the required parameters.
The statement regarding Case $s = 5$ follows from the fact the replacing arrays can be chosen as $A_5(1), A_5(2), A_5(3)$ or $A_5(4)$ [see Eqs. 2.3 and 3.5.]

Similarly, in the case $s = 7$ the replacing arrays can be chosen as $A_7(1), A_7(2)$ or $A_7(3)$ presented in Eq. 3.6, implying the statement.

3.2. Discussion

1. Example 3.1 is obtained by putting $s = 3, n = 9, p = 4, q = 2$ in Theorem 3.3.

2. While applying Theorem 3.3 with $s = 4$, the experimenter has a choice between the replacing arrays $A_4(1)$ and $A_4(2)$. Remark 3.3 may be useful in making the choice.

3. Comparing an inter-class(2) orthogonal MEP, say $\rho_1$ constructed in Theorem 3.3 with $s = 4$ with existing plans for similar experiments with the same number of runs, we find the following.

(a) Asymmetric plans for two- and three-level factors are presented in Huang et al. (2002) & Xiao et al. (2012) and Jones and Nachtsheim (2013), the first one concentrates on main effects, while the second and third estimates two-factor interactions also. Regarding the relation between the factors we find the following. In all the three plans the two-level factors are orthogonal among themselves. However, they differ in estimating the contrasts of the three-level factors, as stated below.

(i) In the plans of Huang et al. (2002) the quadratic contrasts of three-level factors are orthogonal among themselves as well as orthogonal to every two-level factor. The linear contrasts are also orthogonal to the two-level factors. [See the last paragraph of section 3 (page 349) of Huang et al. (2002)].

(ii) In the plans constructed in Xiao et al. (2012) and in Jones and Nachtsheim (2013) the linear contrasts are orthogonal among themselves as well as orthogonal to every two-level factor. The quadratic contrasts are also orthogonal to the two-level factors. [See Table (b) of Figure 1 (page 126) of Jones and Nachtsheim (2013)].

(b) In $\hat{\rho}$ every three-level factor is orthogonal to both the contrasts of every other three-level factor and all but one two-level factors, (with which it is partially orthogonal in case $A_4(1)$ is used).
3.3. More Series of Inter-Class Orthogonal Main Effect Plans

In this section we present a two-stage construction. In the first stage, we start with an MEP, say $\rho$, fix a subset (say $\mathcal{R}$) of factors of $\rho$ such that one or more replacing arrays are available for each factor in $\mathcal{R}$. Then, we obtain a number of MEPs by replacing each factor in $\mathcal{R}$ by a class of factors, related through a replacing array. In the second stage we juxtapose these first stage MEPs in a suitable manner to form a bigger MEP, say $\rho^*$. 

Note that while generating different first stage MEPs we may use different replacing arrays for the same factor. In that case these replacing arrays must be compatible in order that the final outcome $\rho^*$ is a meaningful plan.

We now formally describe the two-stage procedure.

**Definition 3.3.** A two-stage $I \times J$ procedure : we start with an MEP $\rho$ and integers $I$ and $J$. We fix a subset $\mathcal{R}$ of $\mathcal{F}$. For $P \in \mathcal{R}$ with $p$ levels we choose $IJ$ replacing arrays $R_{ij}^P$, $1 \leq j \leq J$, $1 \leq i \leq I$.

(a) We say that these replacing arrays satisfy row-condition if for every $i$, $1 \leq i \leq I$,

$$R_{ij}^P = \rho(p,t_i;l_1 \cdots l_t_i), \text{ for each } j, 1 \leq j \leq J, \text{ for some } t_i, l_1 \cdots l_t_i, \text{ satisfying (3.1)}.$$

(b) $G_P(i)$ will denote a class of $t_i$ factors, the $u$th one having $l_u$ levels, $u = 1, \cdots t_i$, $1 \leq i \leq I$.

(c) $G_P = \bigcup_{i=1}^{I} G_P(i)$ denote a class of $\sum_{i=1}^{I} t_i$ factors.

(d) For every $P \in \mathcal{R}$, we replace $P$ in $\rho$ by the class of factors $G_P(i)$ using replacing array $R_{ij}^P$ and thus obtain $\rho_{ij}$, $j = 1, \cdots J$, $i = 1, \cdots I$. Let

$$\rho^* = \begin{bmatrix} \rho_{11} & \cdots & \rho_{1J} \\ \vdots & \ddots & \vdots \\ \rho_{I1} & \cdots & \rho_{IJ} \end{bmatrix}. \tag{3.8}$$

We say that $\rho^*$ is an MEP obtained from $\rho$ by a two-stage $I \times J$ procedure.

The following properties of $\rho^*$ can be verified easily.

**Lemma 3.1.** Consider an MEP $\rho^*$ constructed by a two-stage $I \times J$ procedure from $\rho$ as described in Definition 3.3.

Then $\rho^*$ satisfies the following.
(a) If \( P \) and \( Q \) are mutually orthogonal in \( \rho \), then every factor in the class \( G_P(i) \) is orthogonal to every factor in the class \( G_Q(i') \) in the derived plan \( \rho^* \), \( i, i' = 1, 2, \ldots I \).

(b) \( \rho^* \) can be viewed as an MEP directly obtained from \( \rho \) by replacing every \( P \) in \( \mathcal{R} \) with a class \( G_P \) of factors related through the following array.

\[
\begin{bmatrix}
  R_{11}^P & \cdots & R_{1J}^P \\
  \vdots & \ddots & \vdots \\
  R_{I1}^P & \cdots & R_{IJ}^P 
\end{bmatrix}, \quad \text{where } R_{ij}^P \text{'s are as in (a) of Definition 3.3.}
\]

(3.9)

The following properties of \( \rho^* \) are immediate consequence of (b) of Lemma 3.1.

**Corollary 3.2.** Consider the MEP \( \rho^* \) of Lemma 3.1. Each class \( G_P \) satisfies the following.

(a) The number of factors of \( G_P \) is the same as that of \( R_P \).

(b) The number of levels of a factor, say \( T \) of \( G_P \) is the same as that of the corresponding factor, say \( \tilde{T} \) of \( R_P \).

(c) Consider a pair of factors, say \( T \) and \( U \) of \( G_P \). \( T \) is orthogonal (respectively partially orthogonal) to \( U \) in \( \rho^* \) if and only if the corresponding factor, say \( \tilde{T} \) is so to \( \tilde{U} \) in \( R_P \).

**Application** It appears that in order to use this method we need to search for replacing arrays satisfying Definition 3.3. However, we can do better. We can incorporate the desirable properties of an available plan of appropriate run size into the classes of the constructed plan by a sort of cut-and-paste method, described below.

**A Cut and Paste Method** Suppose we want to replace a factor, say \( P \), with \( p \) levels of an existing MEP \( \rho \) and we know a plan \( R_P \) with desirable properties and run size a multiple of \( p \), say \( tp \), \( t \) is an integer > 1. In stage 1 we partition the rows and columns of \( R_P \), so that it is partitioned into subarrays, say \( R_{ij}, j = 1, \ldots J = t, i = 1, \ldots I \), each with \( p \) runs. Now we construct an MEP \( \rho^* \) by the two-stage procedure from \( \rho \) as described in Definition 3.3. More than one factor can be replaced in a similar manner, provided \( I \) and \( J \) are the same for every factor.

Here the following questions are pertinent.
(a) Is it always possible to partition the array $R_P$ so that the subarrays $R_{ij}$’s satisfy the row condition of Definition 3.3?

(b) In case it is not possible can we still proceed with the above method to construct $\rho^*$. What will be the performance of $\rho^*$ then?

Before finding answers to these questions in Lemma 3.2, we consider a situation without any such problem.

**Theorem 3.4.** The existence of an $OA(n, m, 4, 2)$, implies the existence of the IOMEP $\rho(2n, 4m; \{3^2\}^m.2^{2m})$. This MEP is, in fact, almost orthogonal since the only non-orthogonality is between the linear contrasts of the three-level factors belonging to the same class.

**Proof.** We first permute the second and third rows of the plan $A_8(1)$ [see Table 1], yielding the array $\tilde{A}_8$, say. Then we partition it into arrays $A_{11}, \cdots A_{22}$, each one is a $\rho(4, 2; \{3, 2\})$, as shown below.

$$
\tilde{A}_8(1) = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 1 & 2 & 1 & 0 & 1 & 2 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 2 & 2 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 & 1 & 0
\end{bmatrix}
$$

(3.10)

We note that each $A_{ij}$, $i, j = 1, 2$ satisfies the row-condition of Definition 3.3. We, therefore can (and do) proceed to replace every factor of the given OA using the cut and paste method yielding the array $\rho^*$.

By Lemma 3.1, it follows that $\rho^*$ may be viewed as the plan obtained by replacing every factor $P$ by the class $G_P$ of four factors related through the replacing array $\tilde{A}_8(1)$. Hence $\rho^*$ is an inter-class orthogonal (4) plan with the relation between the factors in every class being the same as the factors in $A_8(1)$. Now the rest follows from the property of $\tilde{A}_8(1)$ as noted in section 1.

We now consider situations where it is not possible to partition the available array $R$ so that the row condition of Definition 3.3 is satisfied by the subarrays. However, we can still proceed with the cut and paste method with a more flexible row condition, given below.

**Definition 3.4.** A flexible cut-and-paste method: Suppose the subarrays $R_{ij}$, $1 \leq i \leq I, 1 \leq j \leq J$, obtained by partitioning $R_P$ deviate from the conditions of Definition 3.1 in the following way.

(a) For a fixed $i, 1 \leq i \leq I$, the number of factors and the number of runs of $R_{ij}$ is the same, $1 \leq j \leq J$. But the number of levels of the corresponding factor of different $R_{ij}$’s may be different.
(b) $R_P$ satisfies the condition (3.1) of Definition 3.1, but one or more of $R_{ij}$'s don’t.

Then, by a flexible cut-and-paste procedure we mean the same procedure as described in Definition 3.3 with the subarrays $R_{ij}$, $1 \leq i \leq I, 1 \leq j \leq J$.

Let us now investigate the properties of the resultant array constructed by the method above.

**Lemma 3.2.** The resultant array $\rho^*$ constructed by the procedure of Definition 3.4 satisfies the properties Corollary 3.2. In particular $\rho^*$ is not supersaturated, even when one or more of $R_{ij}$’s are so.

**Proof.** Property (a) of Corollary 3.2 is trivial. So, we need to show the other two.

Suppose a factor $T$ of $R_P$ satisfies condition (a) of Definition 3.4. Specifically, suppose $T$ has $t$ levels, but $T_j$, the factor of $R_{ij}$ corresponding to $T$, has $t_j$ levels, $j = 1, \cdots, J$, where $t_j < t$ for one or more $j$’s for some $i$. Then we treat $T_j$ of $R_{ij}$ as one with $t$ levels of which $t - t_j$ are absent, $j = 1, \cdots, J$ and go ahead. By (b) of Lemma 3.3, the factor of $\rho^*$ corresponding to $T$ of $R_P$ has $t$ levels. Hence (b) of Corollary 3.2 is satisfied.

Now, suppose one or more of $R_{ij}$’s do not satisfy the condition (3.1), i.e. they are supersaturated. Then the corresponding $\rho_{ij}$’s would also be so. However, (b) of Lemma 3.3 says that in the process of generating $\rho^*$ from $\rho$, $R_P$ acts as the replacing array for $P$. Again, condition (b) of Definition 3.4 says that $R_P$ satisfies (3.1). Therefore, Theorem 3.1 is applicable. Now, (b) of the same theorem implies condition (c) of Corollary 3.2. Thus $\rho^*$ satisfies all the properties of Corollary 3.2. In particular $\rho^*$ is not supersaturated.

**Theorem 3.5.** (a) The existence of an $OA(n,m,s,2)$, $s = 5$ or $7$ implies the existence of the following IOMEPs.

$$
\rho_2 = \begin{cases} 
\rho(2n, 4m; \{3^4\}^m) & \text{if } s=5 \\
\rho(2n, 4m; \{4^4\}^m) & \text{if } s=7
\end{cases}
$$

(3.11)

(b) Further, the plans satisfy the following.

(i) Among the four members in the same class, any two among the last 3 are mutually orthogonal through the first one. [See Definition 2.3].

(ii) The linear contrast of every factor is orthogonal to every contrast of every factor.
Proof.

(a) We first prove the result for $s = 5$. Consider $A_{10}$ presented in Example 2.3. We get $\tilde{A}_{10}$ by permuting the columns of $A_{10}$ and then partition it as shown below.

$$
\tilde{A}_{10} = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & 1 & 2 & 0 & 1 & 2 & 2 & 2 \\
1 & 1 & 2 & 0 & 1 & 0 & 2 & 1 & 2 & 0 \\
1 & 2 & 1 & 2 & 0 & 0 & 0 & 1 & 2 & 1 \\
2 & 1 & 0 & 0 & 0 & 1 & 2 & 1 & 1 & 2
\end{bmatrix}
$$

(3.12)

We now take $OA(n, m, 5, 2)$ and proceed in a similar manner as in the proof of Theorem 3.4.

We may note that in $\tilde{A}_{10}$, the set of levels of the 2nd factor in $A_{21}$ is $\{0, 1, 2\}$ while that in $A_{22}$ it is $\{1, 2\}$. We take $G_P(2)$ to be a class of two three-level factors, for each $P$, with the level 0 is absent in the second factor of $A_{22}$.

By arguments similar to that in the proof of Theorem 3.4 and using Lemma 3.2 we find that $\rho^*$ is an inter-class orthogonal(4) plan with the relation between the factors in the same class being the same as the factors in $A_{10}$.

For case $s = 7$ we take $A_{14}$ presented in Example 2.4. The partitioned form of $\tilde{A}_{14}$ (obtained by permuting the columns of $A_{14}$) is presented in Eq. 3.13 below. Here we consider the 2nd factor in $A_{22}$ to be a four-level factor with the level 3 absent.

The rest follows from the same argument as in the case $s = 5$.

$$
\tilde{A}_{14} = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 1 & 2 & 2 & 3 & 3 & 0 & 0 & 1 & 2 & 2 & 3 & 3 \\
0 & 1 & 1 & 1 & 3 & 0 & 2 & 0 & 2 & 2 & 1 & 3 & 0 & 2 \\
1 & 0 & 2 & 3 & 1 & 0 & 0 & 0 & 2 & 1 & 1 & 3 & 2 & 0 \\
1 & 2 & 1 & 3 & 3 & 0 & 1 & 0 & 0 & 2 & 1 & 1 & 2 & 2
\end{bmatrix}
$$

(3.13)

(b) It follows from the properties of $A_{10}$ and $A_{14}$ as discussed before Examples 2.3 and 2.4.

3.4. A Series of Orthogonal Main Effect Plans Our next construction is based on the elegant plan of Starks (1964), which we present below, partitioned into two $7 \times 8$ arrays. [See Dey (1985), for instance, for more details about this plan.]
\[ R_{16} = \begin{bmatrix} R_8(1) & R_8(2) \\ 1 & 2 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 & 0 & 2 & 0 & 0 & 0 & 2 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 2 & 0 & 2 & 0 & 0 & 0 & 2 & 1 & 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 2 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 2 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 1 & 1 & 0 \\ 2 & 0 & 2 & 0 & 0 & 1 & 2 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 2 & 0 \end{bmatrix}. \]

Theorem 3.6. (a) The existence of an OA(n, m, 8, 2) implies the existence of an orthogonal MEP for 7m three-level factors on 2n runs.

(b) The existence of an OA(n, m, 4, 2) implies the existence of an orthogonal MEP for 6m three-level factors on 4n runs.

Proof. Let \( O \) denote the given OA.

(a) We first construct arrays \( \rho_j \) from \( O \) by using replacing array \( R_8(j) \) of Eq. 3.14 for every factor following the method of Corollary 3.1.1, \( j = 1, 2 \). Then we form the required plan \( \rho^* \) as \( \rho^* = [ \rho_1 \rho_2 ] \).

Note that \( R_8(j) \) is supersaturated and therefore so is \( \rho_j, j = 1, 2 \). But since \( R_{16} \) is an OMEP, it follows from Lemma 3.2 that \( \rho^* \) satisfies the required property. In particular, \( \rho^* \) is not supersaturated.

(b) Let \( \tilde{R}_{16} \) denote the \( 6 \times 16 \) array obtained by deleting the first row from \( R_{16} \). Now we partition \( \tilde{R}_{16} \) as follows.

\[ \tilde{R}_{16} = ((R_{ij}))_{1 \leq i,j \leq 4} = \begin{bmatrix} 0 & 1 & 2 & 2 & 0 & 2 & 0 & 0 & 0 & 2 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 2 & 0 & 2 & 0 & 0 & 0 & 2 & 1 & 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 2 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 2 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 1 & 1 & 0 \\ 2 & 0 & 2 & 0 & 0 & 1 & 2 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 2 & 0 \end{bmatrix}. \]

Then we proceed to replace every factor of the the given OA using the general cut-and-paste procedure described in Definition 3.4 yielding the array \( \rho^* \).

By (b) of Lemma 3.1 \( \rho^* \) is obtained by replacing every factor \( P \) (among the \( m \) ones) of \( O \) by the class \( G_P \) of six three-level factors, related through \( \tilde{R}_{16} \) and therefore it is an MEP for 6m three-level factors on 4n runs.

Note that for every \( j = 1, 2, 3, 4 \), \( \rho_{1j} \) and \( \rho_{1j} \) are supersaturated, but \( \rho_{3j} \) and \( \rho_{4j} \) are not. However, since \( \tilde{R}_{16} \) is an OMEP, Lemma 3.2 implies that \( \rho^* \) is also an OMEP.
4 Main Effect Plans Of Small Size

In this section, we present MEPs with fifteen or less runs obtained by ad-hoc methods. The factors have at most five levels, and the class size of each plan is at most three. The plans $A_{12}(1), A_{12}(3)$ and $A_{12}(4)$ have class-size three and the remaining have class-size two. Further, all plans except $A_{12}(2)$ and $A_{12}(3)$ are saturated. For the interpretation of the graph see Notation 1.1.

We begin with a general plan for two $p$-level and one two-level factors on $2p$ runs. If $p = 3$, the levels of $A$ form a balanced incomplete block design (BIBD) with those of $B$.

$$A_{2p}(1) = \rho(2p, 3; \{p^2\}; 2) = \begin{bmatrix}
0 & 1 & \cdots & p-1 & 0 & 1 & \cdots & p-1 \\
0 & 1 & \cdots & p-1 & 1 & 2 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1
\end{bmatrix}$$

(4.1)

Now a plan with 6 runs: $A_6(1) = \rho(6, 4; \{3.2\}; \{2^2\})$}

\[
\begin{bmatrix}
0 & 0 & 1 & 1 & 2 & 2 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}
\]

(4.2)

**Remark 4.1.** For the same experiment an equal-frequency plan is available - plan $L_6(3.2^3)$ of Wang and Wu (1992). The graphical representation of these two plans are shown below.

Regarding the performances, $A_6(1)$ estimates all but the quadratic contrast, say, $C_Q$ of $A$ with equal or more precision. One may check that the amount of computation is also less here. However, $C_Q$ may be more important for some experimenter, in which case the plan $L_6(3.2^3)$ would be preferable.

We now present plans on 8 runs. We first present a saturated plan by adding another two-level factor to $A_8(1)$ [see Table 1].
(a) The plan \( A_8(2) = \rho(8, 5; \{3^2\}, \{2^2\}, 2) = \)

\[
\begin{bmatrix}
1 & 0 & 1 & 2 & 1 & 0 & 1 & 2 \\
1 & 1 & 0 & 2 & 2 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 1
\end{bmatrix}
\]

(b) We take up the well-known \( OA(8, 4, 3^4, 2) \) and add one more two-level factor \((F)\) with unequal frequency such that it is orthogonal to all other factors except \( A \). However, \( A \) is PO to \( F \) as the linear contrast of \( A \) is orthogonal to \( F \).

(a) \( A_8(2) = \rho(8, 6; \{3 \times 2\}, 2^4) = \)

\[
\begin{bmatrix}
1 & 0 & 2 & 1 & 1 & 2 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0
\end{bmatrix}
\]

We now present two plans with 4-level factors on 8 runs. Note that on 8 runs, a four-level factor can be orthogonal to neither a four-level nor a three-level factor. Using non-orthogonality, we are able to accommodate two four-level factors in one plan and one four-level and one three-level factor in another plan on 8 runs.

(c) \( A_8(4)^* = \rho(8, 3; \{4^2\}, 2) = \) is obtained by putting \( p = 4 \) in \( A_{2p}(1) \) [see Eq. 4.1]. In this plan, the levels of the four-level factors \( A \) and \( B \) form a group divisible design \((m = n = 2, r = k = 2, \lambda_1 = 0, \lambda_2 = 1)\).

(d) In our next plan \( A_8(5) \) \( A \) is partially orthogonal to \( B \) – the linear contrast of \( A \) is orthogonal to the contrasts of \( B \). Further, \( B \) is also partially orthogonal to \( A \) as the quadratic contrast is orthogonal to all contrasts of \( A \).

\( A_8(5) = \rho(8, 4; \{4.3\}, 2^2) = \)

\[
\begin{bmatrix}
0 & 1 & 3 & 2 & 0 & 1 & 3 & 2 \\
1 & 0 & 2 & 1 & 2 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1
\end{bmatrix}
\]
A Plan on 10 Runs  \( A_{10}^*(2) = \rho(10, 3; \{5^2\}.2) = \) is obtained by putting \( p = 5 \) in \( A_{2p}(1) \). Here the levels of the five-level factors form a symmetric cyclic PBIBD with \( r = k = 2, \lambda_1 = 1, \lambda_2 = 0. \)

We shall now present plans on 12 runs: There is no plan in the literature accommodating one or more 4-level factors on 12 runs. So, we begin with such plans.

(a) \( A_{12}(1)^* = \rho(12, 5; \{2.4^2\}.\{3^2\}) = \)

\[
\begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 3 \\
1 & 2 & 3 & 2 & 3 & 0 & 3 & 0 & 1 & 0 & 1 & 2 \\
0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 2 & 0 & 1 \\
0 & 1 & 2 & 2 & 1 & 0 & 0 & 2 & 1 & 1 & 2 & 0
\end{bmatrix}
\]

In this equal frequency saturated plan, the levels of both the four-level factors B and C form a generalized group divisible design with the levels of the two-level factor A. Between themselves, they form a balanced incomplete block design (BIBD). D and E are partially orthogonal to each other as the linear contrast of D (respectively E) is orthogonal to both the contrasts of E (respectively D).

(b) \( A_{12}(2) = \rho(12, 4; \{3^2\}.\{3.4\}) = \)

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 \\
0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\
0 & 1 & 2 & 0 & 0 & 0 & 1 & 2 & 1 & 2 & 0 & 0 \\
0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3
\end{bmatrix}
\]

Here all the factors other except C has equal frequency. The levels of factors A and B form a balanced block design (BBD). D is partially orthogonal to C as the contrast \( \delta_1 - \delta_2 \) is orthogonal the contrasts of C. However, C is non-orthogonal to D.

Remark 4.2. In the plan \( A_{12}(2) \), the four-level factor D may be replaced by three mutually orthogonal two-level factors to obtain an almost orthogonal MEP for a \( 3^3.2^3 \) experiment.
Remark 4.3. The plan $A_{12}(3)$ is very similar to the plan $L'_{12}(3^4.2^3)$ of Wang and Wu (1992). In both the plans

(a) every three-level factor is non-orthogonal to every other one and its levels form a balanced incomplete block design (BIBD) with the levels of the other one.

(b) Every two-level factor is orthogonal to every other factor.

The difference is that $A_{12}(3)$ provides one more two-level factor and one less three-level factor and so has total d.f one less than $L'_{12}(3^4.2^3)$. On the other hand, since each three-level factor in $A_{12}(3)$ is non-orthogonal to fewer factors, (a) implies that its contrasts are estimated with greater precision.

(d) $A_{12}(4) = \rho(12, 6; \{3^3\}.\{3^2.2\}) =

\begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 \\
2 & 1 & 2 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 2 \\
0 & 1 & 2 & 1 & 0 & 2 & 1 & 2 & 1 & 2 & 0 & 0 \\
0 & 0 & 1 & 2 & 0 & 0 & 1 & 2 & 0 & 0 & 1 & 2 \\
2 & 1 & 0 & 0 & 0 & 1 & 0 & 2 & 2 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0
\end{bmatrix}

This inter-class (3) orthogonal MEP has accommodated five three-level factors together with a two-level factor. The levels of $A$ form a BBD with those of each of $B$ and $C$, while the levels of $B$ form a variance-balanced non-binary design with those of $C$.

A Plan on 15 Runs $A_{15} = \rho(15, 4; \{3^2\}.\{5^2\}) =

\begin{bmatrix}
0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\
0 & 1 & 2 & 2 & 0 & 1 & 1 & 2 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\
0 & 1 & 2 & 1 & 2 & 3 & 2 & 3 & 4 & 3 & 4 & 0
\end{bmatrix}
In this plan the levels of $A$ and $B$ form a BBD, while those of $C$ and $D$ form a BIBD. A list of Interclass orthogonal plans of small run size is provided in Table 4 below.

| Run size | Experiment | Name of plan | Reference |
|----------|------------|--------------|-----------|
| 4        | \{3.2\}   | $A_4(1)$ and $A_4(2)$ | (3.4) |
| 5        | \{3.2^2\} | $A_5(1)$ | Example 2.2 |
| 5        | \{2^4\}   | $A_5(1)$ | Example 2.2 |
| 5        | \{4.2\}   | $A_5(3)$ | (3.5) |
| 5        | \{3^2\}   | $A_5(4)$ | (3.5) |
| 6        | \{3.2\}.\{2^2\} | $A_6(1)$ | (4.2 ) |
| 7        | \{6.2\}   | $A_7(1) = R(6, 2)$ | (3.1) |
| 7        | \{3^3\}   | $A_7(2)$ | (3.6) |
| 8        | \{3.2\}.\{2^2\} | $A_8(1)$ | Table 1 |
| 8        | \{3^2\}.\{2^2\}.2 | $A_8(2)$ | Section 4 |
| 8        | \{3.2\}.2^4 | $A_8(3)$ | Section 4 |
| 8        | \{4^2\}.2 | $A_8(4)$ | Section 4 |
| 8        | \{4.3\}.2^2 | $A_8(5)$ | Section 4 |
| 9        | \{2^2\}.3^2 | $A_9(1)$ | Example 3.1 |
| 10       | \{3^4\}   | $A_{10}(1)$ | Example 2.3 |
| 10       | \{5^2\}.2 | $A_{10}(2)$ | Section 4 |
| 12       | \{2.4^2\}.\{3^2\} | $A_{12}(1)$ | Section 4 |
| 12       | \{3^2\}.\{3.4\} | $A_{12}(2)$ | Section 4 |
| 12       | 2^4.\{3^3\} | $A_{12}(3)$ | Section 4 |
| 12       | \{3^3\}.\{3^2\}.2 | $A_{12}(4)$ | Section 4 |
| 14       | \{4^4\}   | $A_{14}(1)$ | Example 2.4 |
| 15       | \{3^2\}.\{5^2\} | $A_{15}(1)$ | Section 4 |
| 16       | $4^p.\{3.2\}^q.2^3t$ | $p + q + t = 5$ | Theorem 3.7 |
| 25       | $5^p.\{4.2\}^q.\{3^2\}^r.\{3.2^2\}^t.\{2^4\}^u$ | $p + q + r + t + u = 6$ | Theorem 3.7 |
| 32       | \{3^2\}^5.2^{10} | $p + q + r + t + u = 6$ | Theorem 3.4 |
| 49       | $7^p.\{6.2\}^q.\{4^2\}^r.\{3^3\}^t$ | $p + q + r + t = 8$ | Theorem 3.4 |
| 50       | \{3^4\}^6 | $p + q + r + t + u = 6$ | Theorem 3.5 |
| 64       | $3^{10}$  | $p + q + r + t + u = 6$ | Theorem 3.6 |
5 Summary

This paper is devoted to inter-class orthogonal main effect plans (IOMEPs) (Definition 2.2). The concept of partial orthogonality between a pair of factors is also introduced here (Definition 2.4) and a sufficient condition for partial orthogonality has been derived (Theorem 2.1). Next we present methods of construction of IOMEPs. In the first method we replace one or more factors of an orthogonal array (OA) by a class of factors - specifically the runs of suitably chosen ‘replacing arrays’ replace the levels of the factors of the given OA (Theorems 3.2 and 3.3). In the next section we present a ‘two-stage construction’, in which a few IOMEPs constructed by the first method are juxtaposed in the second stage to yield a bigger IOMEPs. In the application, we have used a ‘cut-and-paste’ method - partitioned a ‘good’ MEP and used the parts as replacing arrays in the first stage, so as to make the final plan ‘as good as’ the ‘good’ MEP (Theorems 3.4 and 3.5). In another application we have used supersaturated replacing arrays to obtain two series of orthogonal main effect plans (Theorem 3.6).

Acknowledgments. The author is extremely grateful to the reviewers for all the thoughtful suggestions which have improved the presentation of the paper considerably. The author also express her gratitude to Professor J.P. Morgan of Virginia Tech for many fruitful discussions.

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Paper received: 18 December 2015.