QUANTUM $ax+b$ GROUP AS QUANTUM AUTOMORPHISM GROUP OF $k[x]$

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Abstract. By introducing a result that guarantees a given bialgebra to be a Hopf algebra under a natural condition, we show that the quantum automorphism group of the algebra $k[x]$ of polynomials over a field $k$ (of any characteristic) is the universal quantum $ax+b$ group $A$, generalizing the fact that the automorphism group of $k[x]$ is the $ax+b$ group. The $q$-deformation of the $ax+b$ group is then seen as one among a certain family $A_{q,n}$ ($q \in k$, $n$ an integer) of quantum subgroups of this universal quantum group.

1. Introduction

In [12], adapting the philosophy that quantum groups should be viewed as the mathematical objects encoding quantum symmetries of (noncommutative) spaces [3], instead of as deformations of ordinary groups, we described the quantum automorphism groups of matrix algebras over $\mathbb{C}$. Just as automorphism groups of matrix algebras over $\mathbb{C}$ are compact Lie groups, the quantum automorphism groups of such algebras are compact matrix quantum groups in the sense of Woronowicz (see [14, 9, 10, 11, 12]). These quantum automorphism groups are of a quite different nature from the $q$-deformations of Lie groups, although their representation ring is the same as the representation ring of $SO(3)$ for the generic situation, as cleverly shown by Banica [1]. These quantum automorphism groups contain the ordinary automorphism (Lie) groups as proper subgroups. Hence, quantum group symmetries are significantly richer than ordinary group symmetries.

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In this note, adapting the same philosophy as in \[5, 12\] and using “algebra of functions on quantum group approach”, we determine explicitly the quantum automorphism group of the polynomial algebra \(k[x]\) of one variable over a field \(k\) (of any characteristic). It is well known that the automorphism group of the algebra \(k[x]\) is the \(ax + b\) group over the field \(k\). Namely, every automorphism \(\alpha\) of the \(k\)-algebra \(k[x]\) is of the form

\[\alpha(x) = ax + b\]

for some \(a, b \in k\) with \(a \neq 0\). In exactly the same way, we find that the quantum automorphism group of \(k[x]\) is the universal quantum \(ax + b\) group, which has been studied earlier by Sweedler \[6\] in a different context. As a matrix quantum group, its algebra \(A\) of functions has generators \(a, a^{-1}, b\) (\(a\) and \(b\) are free with respect to each other) and is represented by

\[
T = \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}.
\]

We only describe this quantum group at the purely algebraic level in this note, its \(C^\ast\)-algebraic (topological) description (e.g. when \(k = \mathbb{R}\) or \(\mathbb{C}\)) would require a rather involved analysis of unbounded operators \[15\]. This problem seems to have not been solved yet, although there have been serious attempts at it. Note that in \[4\], Majid describes a very big quantum semigroup (bialgebra) for \(k[x]\). This quantum semigroup is not a quantum group in our sense (i.e. it has no antipode), and it not homogeneous in the sense that it does not preserve the degrees of the polynomials in \(k[x]\).

\section*{2. From Bialgebras to Hopf Algebras: A Sufficient Condition}

In practice, there are a profusion of bialgebras (i.e. quantum semigroups), cf. \[3, 12, 4\]. However, as Drinfeld pointed out in his paper \[3\], before the discovery of quantum groups there were very few non-trivial examples of noncommutative and noncocommutative Hopf algebras and it was very hard to find such. This is one of the reasons why the Drinfeld-Jimbo construction of such examples has attracted much attention in both physics and mathematics. In this section, we
give a natural condition that guarantees a given bialgebra to be a Hopf algebra. It is an analog of the result of Woronowicz in [16]. This result is very useful in constructing examples of Hopf algebras. For instance, the quantum groups in [3] can be viewed as such. See also [9, 7, 12].

For a given algebra $\mathcal{A}$ over a field $k$ (of any characteristic), denote by $M_n(\mathcal{A})$ the algebra of $n \times n$ matrices over $\mathcal{A}$.

**Theorem 1.** Let $(\mathcal{A}, \Delta, \epsilon)$ be a unital bialgebra over a field $k$ that is generated by a multiplicative matrix $u = (u_{ij})_{i,j=1}^n$ and relations $R$. That is, $\mathcal{A}$ is generated by $u_{ij}$ ($i, j = 1 \cdots n$) and relations $R$ such that

$$\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}, \quad \epsilon(u_{ij}) = \delta_{ij}.$$ 

Then $\mathcal{A}$ is a Hopf algebra if and only if $u$ is invertible in $M_n(\mathcal{A})$ and the entries of $u^{-1}$ satisfy the opposite relations $R^{op}$. When this condition is satisfied, the antipode $S$ is given on the generators $u_{ij}$ by

$$S(u) = u^{-1}.$$ 

**Proof.** (cf Manin [5].) The necessary condition follows from the antipodal property.

We prove that the condition is sufficient. Since the entries of $u^{-1}$ satisfy the opposite relations $R^{op}$ we can define a anti-homomorphism $S$ on $\mathcal{A}$ by the formula

$$(S(u_{ij}))_{i,j=1}^n = u^{-1}.$$ 

We show that $S$ satisfies the antipodal property

$$m(S \otimes id)\Delta(a) = \epsilon(a)1 = m(id \otimes S)\Delta(a)$$

for all $a \in \mathcal{A}$. From the definition of $S$, this is clearly true for $a = u_{ij}$. Since $\mathcal{A}$ is generated by the $u_{ij}$’s as an algebra, to prove the antipodal property, we show that if $a, b \in \mathcal{A}$ satisfy the antipodal property, then so does $ab$. Using Sweedler’s notation, let

$$\Delta(a) = \sum a_{(1)} \otimes a_{(2)}, \quad \Delta(b) = \sum b_{(1)} \otimes b_{(2)}.$$
Then
\[
m(S \otimes id)\Delta(ab) = m(S \otimes id)\Delta(a)\Delta(b)
\]
\[
= m(S \otimes id)(\sum a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)})
\]
\[
= m \sum S(b_{(1)})S(a_{(1)}) \otimes a_{(2)}b_{(2)}
\]
\[
= \sum S(b_{(1)})S(a_{(1)})a_{(2)}b_{(2)} = \sum S(b_{(1)})[S(a_{(1)})a_{(2)}]b_{(2)}
\]
\[
= \sum S(b_{(1)})\epsilon(a)b_{(2)} = \epsilon(a)\epsilon(b)1 = \epsilon(ab)1.
\]
Similarly, we have
\[
m(id \otimes S)\Delta(ab) = \epsilon(ab)1.
\]
Q.E.D.

Remark 1. Let \( u \) be a matrix corepresentation of a Hopf algebra (or a multiplicative matrix in the sense of Manin \[5\]). Let \( S \) be the antipode of \( H \). Then both \( u \) and \( u^t \) are invertible, and we have \( u^{-1} = S(u) \), but the inverse \((u^t)^{-1}\) is not equal to \((u^{-1})^t\) in general. That is, we have in general
\[
S(u^t) = S(u)^t = (u^{-1})^t \neq (u^t)^{-1}.
\]
The reader may convince himself of this by taking the Hopf algebra of \( SU_q(2) \) for \( q \in (0, 1) \), or the universal quantum group \( A_u(Q) \) for \( Q > 0 \) but \( Q \neq cI_n \) \[7\].

If we restrict to compact quantum groups in the sense of Woronowicz \[14\], then \( S(u^t) = (u^t)^{-1} \) if and only if the quantum group is of Kac type, i.e. if and only if \( S^2 = id \). To see this, it suffices to use the fact that \( S \) preserves the involution in this case.

3. The Universal Quantum \( ax + b \) Group as a Quantum Automorphism Group

In this section, we describe the quantum automorphism group of \( k[x] \). It turns out that its Hopf algebra is exactly the same as the one studied earlier by Sweedler in a different context (p89 of \[3\]).

Note we are taking the “functions-on-group” point of view. This means that a **quantum group** is a Hopf algebra (over a field \( k \) of any characteristic) of “noncommutative functions”. An **action** of a
quantum group on an algebra is defined to be a right coaction of the Hopf algebra on the algebra. A quantum group endowed with an action on an algebra is called a quantum transformation group over the algebra.

Let \((A_1, \alpha_1)\) and \((A_2, \alpha_2)\) be two quantum transformation groups over an algebra \(B\). A morphism \(\pi\) from \((A_1, \alpha_1)\) to \((A_2, \alpha_2)\) is defined to be a morphism \(\pi\) of Hopf algebras from \(A_2\) to \(A_1\) with the property

\[
\alpha_1 = (id \otimes \pi) \alpha_2.
\]

We refer the reader to \([12]\) for more discussions on categories of quantum transformation groups.

Let \(k[x]_n \subset k[x]\) be the subspace of polynomials of degrees less or equal to \(n\) \((n \geq 0)\). Note that the automorphism group of \(k[x]\) leaves the subspaces \(k[x]_n\) invariant. Therefore, it is natural to expect that its quantum automorphism group also has this property. On the other hand, it is clear that the class of quantum groups acting on \(k[x]\) that leave \(k[x]_n\) invariant is a category (compare with Definition 2.1 of \([12]\)). Let \(\mathcal{C}\) denote this category. We now show that the category \(\mathcal{C}\) has a universal object (then it is unique up to isomorphism by abstract nonsense). We explicitly describe it in terms of generators and relations. This universal object is the quantum automorphism group of \(k[x]\) (compare with \([12]\)).

Let \((A_0, \alpha_0)\) be an object in \(\mathcal{C}\), where \(\alpha_0\) denotes the action of the quantum group \(A_0\) on \(k[x]\). Since \(k[x]_1\) is invariant under \(\alpha_0\), there are elements \(a_0, b_0 \in A_0\) such that

\[
\alpha_0(x) = x \otimes a_0 + 1 \otimes b_0.
\]

Since (see Section 2 of \([5]\) and Definition 2.1 of \([12]\))

\[
(1 \otimes \Delta_0) \alpha_0 = (\alpha_0 \otimes 1) \alpha_0, \quad (id \otimes \epsilon_0) \alpha_0 = id,
\]

where \(\Delta_0\) and \(\epsilon_0\) are respectively the coproduct and counit on \(A_0\), we see that

\[
\begin{pmatrix}
  a_0 & 0 \\
  b_0 & 1 
\end{pmatrix}
\]
is a multiplicative matrix (see Section 2 of [3]). Namely, letting $T_0 = (t^0_{ij})$ be the above matrix, then

$$\Delta_0(t^0_{ij}) = \sum_{r=1}^{2} t^0_{ir} \otimes t^0_{rj}; \quad \epsilon_0(t^0_{ij}) = \delta_{ij}.$$  

This combined with the antipodal property for $\mathcal{A}_0$ gives

$$\sum_{r=1}^{2} S(t^0_{kr})t^0_{rl} = \delta_{kl}, \quad \sum_{r=1}^{2} t^0_{kr}S(t^0_{rl}) = \delta_{kl}, \quad (\ast)$$

where $S_0$ is the antipode of $\mathcal{A}_0$, $k, l = 1, 2$. Simplifying the above relations $(\ast)$ we obtain

$$S_0(a_0) = a_0^{-1}, \quad S_0(b_0) = -b_0a_0^{-1}.$$  

Now we can state the following

**Theorem 2.** Let $\mathcal{A}$ be the universal algebra over the field $k$ generated by $a, a^{-1}, b$, subjecting to the following two relations

$$aa^{-1} = 1 = a^{-1}a.$$  

Then

(1). $\mathcal{A}$ is a quantum group with Hopf algebra structure

$$\Delta(a) = a \otimes a, \quad \Delta(b) = b \otimes a + 1 \otimes b,$$

$$S\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} = \begin{pmatrix} a^{-1} & 0 \\ -ba^{-1} & 1 \end{pmatrix}, \quad \epsilon\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$  

(2). The quantum group $\mathcal{A}$ admits a natural action $\alpha$ on the algebra $k[x]$ given by

$$\alpha(x) = x \otimes a + 1 \otimes b.$$  

The pair $(\mathcal{A}, \alpha)$ is the quantum automorphism group of $k[x]$ (i.e. a universal object in the category $\mathcal{C}$), and it naturally contains (in the sense of [3]) the ordinary automorphism group $\text{Aut}(k[x])$ as a subgroup.

Note that part (1) of the above theorem appeared in Sweedler [4] pp89-90 in a completely different context, where the result was stated without proof. Although the proof of (1) can be done by straightforwardly
verifying the axioms of a Hopf algebra one by one, it is also illuminating to prove it by using Theorem 1 above, which we leave to the reader as an exercise. The proof of (2) follows easily from the calculations above. Note that the proof of this theorem is easier than the proofs of the corresponding results in [12], partly because this theorem is of a purely algebraic nature whereas the results in [12] concerns analytic aspect as well (i.e. Woronowicz Hopf $C^*$-algebras).

In view of this theorem, the quantum group $A$ can be called the universal quantum $ax+b$ group.

**Remark 2.** The quantum automorphism group $(A, \alpha)$ above is the universal object in the category of all Hopf algebras coacting from the right on the algebra $k[x]$. It is easy to see that the pair $(A^{op}, \alpha^{op})$ is the universal object in the category of all Hopf algebras that coact from the left on the algebra $k[x]$ and leave subspaces $k_n[x]$ invariant, where $A^{op}$ is the Hopf algebra that has the same elements as $A$ but with the opposite product $m^{op}$ and opposite coproduct $\Delta^{op}$ (see [3]), and $\alpha^{op}$ is defined by

$$\alpha^{op}(x) = a \otimes x + b \otimes 1.$$ 

The multiplicative matrix for $(A^{op}, \alpha^{op})$ is

$$
\begin{pmatrix}
a & b \\
0 & 1
\end{pmatrix}.
$$

Hence in a sense we obtain the same quantum group with left coaction.

**4. $q$-Deformation $A_q$ of the $ax+b$ Group and Other Quantum Subgroups $A_{q,n}$ of $A$**

Let $q \in k^*$ be a non-zero element. Let $I_q$ be the ideal of $A$ generated by $ab - qba$. Then one easily verifies that $I_q$ is a Hopf ideal of $A$. Let $A_q$ be the quotient $A/I_q$. Then $A_q$ is a deformation of the ordinary $ax+b$ group over $k$ in the sense that $A_1$ is the ordinary algebra of coordinate functions on this group.

More generally, with $q$ as above and $n \in \mathbb{Z}$ an integer, let $I_{q,n}$ be the ideal of $A$ by $a^n b - qba^n$. Then $I_{q,n}$ is a Hopf ideal of $A$. To see this,

$$\Delta(a^n b - qba^n) = (a^n b - qba^n) \otimes a^{n+1} + a^n \otimes (a^n b - qba^n),$$
hence, \( \Delta(\mathcal{I}_{q,n}) \subseteq \mathcal{I}_{q,n} \otimes \mathcal{A} + \mathcal{A} \otimes \mathcal{I}_{q,n} \);
\[
\epsilon(a^n b - qba^n) = 0, \quad \text{hence}, \quad \epsilon(\mathcal{I}_{q,n}) = 0;
\]
\[
S(a^n b - qba^n) = a^{-n}(-a^n b + qba^n)a^{n-1},
\]
\[
\text{hence,} \quad S(\mathcal{I}_{q,n}) \subseteq \mathcal{I}_{q,n}.
\]
Let \( \mathcal{A}_{q,n} \) be the quotient \( \mathcal{A}/\mathcal{I}_{q,n} \). Then \( \mathcal{A}_{q,n} \) is a quantum subgroup of the universal quantum group \( \mathcal{A} \), and of \( \mathcal{A}_{q,m,n} \) for integers \( m \neq 0 \) (because \( a^m b - q^m ba^m = 0 \) in \( \mathcal{A}_{q,n} \)). In particular, since \( \mathcal{A}_q = \mathcal{A}_{q,1} \), \( \mathcal{A}_q \) is a quantum subgroup of \( \mathcal{A}_{q,m,n} \) for all \( m \neq 0 \). The quantum groups \( \mathcal{A}_{q,n} \) can be viewed as a quantum deformation of the quantum groups \( \mathcal{A}_{1,n} \) (\( \mathcal{A}_{1,1} \) is just \( \mathcal{A}_1 \) treated in the previous paragraph.) Note that \( \mathcal{A}_q \) is a proper quantum subgroup of the universal quantum group \( \mathcal{A} \).

In the degenerate cases, we have
\[
\mathcal{A}_0 = \mathcal{A}_{0,n} = \mathcal{A}_{q,0} = k[a, a^{-1}], \quad \text{for all} \quad n \in \mathbb{Z},
\]
which is commutative and cocommutative. It is clear that this is a quantum subgroup of \( \mathcal{A}_{q,n} \) for all \( q, n \).

The underlying algebra of \( \mathcal{A}_q \) above is exactly the same as the dual algebra \( B \) of the function algebra \( \mathcal{A} \) of the quantum \( E(2) \) group in Van Daele and Woronowicz [8], but it has a different Hopf algebra structure from \( B \). The above description is purely algebraic. It would interesting to see whether one can adopt the \( C^* \)-version of \( B \) in [8] to obtain a \( C^* \)-version of \( \mathcal{A}_q \). For the same reason as in Proposition 3.4 of [7], we may not expect the universal quantum group \( \mathcal{A} \) to have a \( C^* \)-version.

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