AdS/CFT Correspondence and string/gauge duality

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Abstract

The AdS/CFT correspondence is an exact duality between string theory in anti-de Sitter space and conformal field theories on its boundary. Inspired in this correspondence some relations between strings and non conformal field theories have been found. Exact dualities in the non conformal case are intricate but approximations can reproduce important physical results. A simple approximation consists in taking just a slice of the AdS space with a size related to an energy scale. Here we will discuss how this approach can be used to reproduce the scaling of high energy QCD scattering amplitudes. Also we show that very simple estimates for glueball mass ratios can emerge from such an approximation.

1 Introduction

The AdS/CFT correspondence is a duality between the large $N$ limit of $SU(N)$ superconformal field theories and string theory in a higher dimensional anti-de Sitter spacetime. In this correspondence the AdS space shows up both as a near horizon geometry of a set of coincident D-branes or as a solution of supergravity. Precise prescriptions for the realization of the AdS/CFT correspondence were formulated by considering Poincare patches of AdS space. The Poincare coordinate system allows a simple definition for the boundary where the conformal field theory is defined. In this correspondence, bulk fields act as classical sources for boundary correlation functions. One of the striking features of this correspondence is that it is a realization of the holographic principle: “The degrees of freedom of a quantum theory with gravity can be mapped on the corresponding boundary.”

String theory, presently the main candidate to describe the fundamental interactions in a unified way, was originally proposed as a model for strong interactions with the successful results of Veneziano amplitude in the Regge regime. However this model was unable to reproduce the experimental hard and deep inelastic scatterings. These behaviours were latter explained by QCD which is today
the theory accepted to describe strong interactions. Despite of its success in high energies, QCD is non perturbative in the low energy regime, where one needs involved lattice calculations. In fact QCD and strings may hopefully be viewed as complementary theories. A strong indication found by ’t Hooft is the relation between large $N SU(N)$ gauge theories and strings. A recent remarkable result in the direction of understanding this duality was the AdS/CFT correspondence found by Maldacena.

One of the long standing puzzles for the string description of strong interactions is the high energy scattering at fixed angles. For string theory in flat space such a process is soft (the amplitudes decay exponentially with energy) while both experimental data and QCD theoretical predictions indicate a hard behavior (amplitudes decaying with a power of energy). A solution for this puzzle was proposed recently by Polchinski and Strassler, inspired in the AdS/CFT correspondence, considering strings not in flat space but rather in a 10 dimensional space which assymptotically tends to an AdS$_5$ times a compact space. This space is considered as an approximation for the space dual to a confining gauge theory which can be associated with QCD. An energy scale was introduced and identified with the lightest glueball mass. Then they found the glueball high energy scattering amplitude at fixed angles (with the correct QCD scaling) integrating the string amplitude over the warped AdS extra dimension weighted by the dilaton wave function. Other approaches to this problem have also been discussed recently.

Important related discussions on deep inelastic scattering of hadrons and on exclusive processes in QCD, both inspired in the AdS/CFT correspondence appeared recently.

In this paper we are going to review recent results regarding an approximate relation between strings and QCD. In particular we find the scaling of high energy glueball scattering at fixed angles and their masses. This paper is organized in the following way. In section 2 we will review the relation between AdS space and D-brane spaces coming from string theory. In section 3 we first present a mapping between AdS bulk and boundary states for scalar fields. Then we show in section 4 how this mapping can be used to find the same high energy scaling as that of glueball scattering, from the bulk theory. In section 5 we use this mapping to obtain an estimate for the ratio of scalar glueball masses.

2 Branes and AdS space

An $n+1$ dimensional AdS space can be defined as the hyperboloid ($R =$ constant)

$$X_0^2 + X_{n+1}^2 - \sum_{i=1}^{n} X_i^2 = R^2$$

in a flat $n+2$ dimensional space with measure

$$ds^2 = -dX_0^2 - dX_{n+1}^2 + \sum_{i=1}^{n} dX_i^2.$$
Usually the AdS space is represented by the so called global coordinates $\rho, \tau, \Omega_i$ given by \[4, 5\]

$$
X_0 = R \sec \rho \cos \tau \\
X_i = R \tan \rho \Omega_i ; \quad (\sum_{i=1}^{n} \Omega_i^2 = 1) \\
X_{n+1} = R \sec \rho \sin \tau.
$$

These coordinates are defined in the ranges $0 \leq \rho < \pi/2$ and $0 \leq \tau < 2\pi$ and the line element reads

$$
ds^2 = \frac{R^2}{\cos^2(\rho)} \left( -d\tau^2 + d\rho^2 + \sin^2(\rho)d\Omega^2 \right).$$

Note that the timelike coordinate $\tau$ in the above metric has a finite range. So in order to identify it with the usual time it is necessary to unwrap it. This is done taking copies of the original AdS space that together represent the AdS covering space\[28\]. From now on we take this covering space and call it simply as AdS space.

A consistent quantum field theory in AdS space\[29, 30\] requires the addition of a boundary at spatial infinity: $\rho = \pi/2$. This compactification of the space makes it possible to impose appropriate conditions and find a well defined Cauchy problem. Otherwise massless particles could go to or come from spatial infinity in finite times. Possible energy definitions in AdS spaces, relevant for the AdS/CFT correspondence have also been discussed\[31\].

Poincaré coordinates $z, \vec{x}, t$ are more useful for the study of the AdS/CFT correspondence. These coordinates are defined by

$$
X_0 = \frac{1}{2z} \left( z^2 + R^2 + \vec{x}^2 - t^2 \right) \\
X_j = \frac{R \vec{x}_j}{z} ; \quad (j = 1, ..., n-1) \\
X_n = -\frac{1}{2z} \left( z^2 - R^2 + \vec{x}^2 - t^2 \right) \\
X_{n+1} = \frac{Rt}{z},
$$

where $\vec{x}$ has $n-1$ components and $0 \leq z < \infty$. In this case the $AdS_{n+1}$ metric reads

$$
ds^2 = \frac{R^2}{(z)^2} \left( dz^2 + (d\vec{x})^2 - dt^2 \right).$$

The AdS boundary $\rho = \pi/2$ in global coordinates corresponds in Poincaré coordinates to the surface $z = 0$ plus a “point” at infinity ($z \to \infty$). The surface $z = 0$ defines a Minkowski space with coordinates $\vec{x}, t$. The connectedness of the AdS boundary was discussed by Witten and Yau\[32\].
The point at infinity can not be accommodated in the original Poincaré chart\(^{33,34}\) so that we have to introduce a second coordinate system to represent it properly. The consistency of the addition of a second chart requires ending the first chart at a finite \(z = z_{\text{max}}\). At this surface one needs to impose boundary conditions to guarantee the analyticity of fields. The imposition of boundary conditions will be essential to the discussion of the next section.

Finally it is interesting to see how AdS space shows up when we study D-brane systems in string theory. The ten dimensional metric generated by \(N\) coincident D3-branes can be written as\(^{35,2}\)

\[
\ds^2 = \left(1 + \frac{R^4}{r^4}\right)^{-1/2}(-\dt^2 + d\vec{x}^2) + \left(1 + \frac{R^4}{r^4}\right)^{1/2}(dr^2 + r^2 d\Omega_5^2) \tag{7}
\]

where \(R^4 = N/2\pi^2 T_3\) and \(T_3\) is the tension of a single D3-brane. Note that there is a horizon at \(r = 0\).

Now looking at the near horizon region \(r \ll R\) the metric (7) can be approximated as:

\[
\ds^2 = \frac{r^2}{R^2}(-\dt^2 + d\vec{x}^2) + \frac{R^2}{r^2}dr^2 + R^2 d\Omega_5^2. \tag{8}
\]

Changing the axial coordinate according to: \(z = R^2/r\) this metric takes the form

\[
\ds^2 = \frac{R^2}{z^2}(dz^2 + (d\vec{x})^2 - dt^2) + R^2 d\Omega_5^2. \tag{9}
\]

corresponding to a five dimensional AdS space (in Poincaré coordinates) times a five sphere: AdS\(_5\) \(\times\) S\(_5\).

### 3 Bulk boundary mapping

In the AdS/CFT framework there is a correspondence between (on shell) string theory in the AdS bulk and (off shell) conformal field theory on the boundary. This represents a realisation of the holographic principle which asserts that the degrees of freedom of a quantum theory with gravity in some space can be represented on the corresponding boundary. Based on this principle one can speculate on the possibility of finding a map between quantum states of theories defined in the bulk and on the boundary of a given space. Here we will find a map between AdS bulk and boundary states considering the simple situation of free scalar fields\(^{36}\). We will later identify the bulk scalar field with the dilaton which is a massless string excitation.

A free scalar field, like the dilaton, in a Poincaré AdS\(_5\) chart \(0 \leq z \leq z_{\text{max}}\) has the form\(^{33}\)

\[
\Phi(z,\vec{x},t) = \sum_{p=1}^{\infty} \int \frac{d^3k}{(2\pi)^3} \frac{z^2 J_2(u_p z)}{z_{\text{max}} u_p (k) J_5(u_p z_{\text{max}})} \times \{ a_p(\vec{k}) e^{-i\omega_p(\vec{k}) t + i\vec{k} \cdot \vec{x}} + \text{h.c.} \}, \tag{10}
\]
with \( w_p(\vec{k}) = \sqrt{u_p^2 + \vec{k}^2} \) and \( u_p \) defined by

\[
u_p z_{\text{max}} = \chi_{2,p},
\]

where \( \chi_{2,p} \) are the zeros of the Bessel function \( J_2(y) \).

The operators \( a_p, \ a_p^\dagger \) satisfy the commutation relations

\[
[a_p(\vec{k}), a_p^\dagger(\vec{k}')] = 2 \left(2\pi\right)^3 w_p(\vec{k}) \delta_{p,p'} \delta^3(\vec{k} - \vec{k}').
\] (12)

As the coordinate range \( z = z_{\text{max}} \) is arbitrary we can take it as large as we want so that most of the AdS space is described by one Poincaré chart. This way we will not need to use the second Poincaré chart explicitly. The size \( z = z_{\text{max}} \) corresponds to an energy scale as we will discuss below. From now on we call the Poincaré chart together with boundary conditions at \( z = z_{\text{max}} \) as the AdS slice.

On the four dimensional boundary (\( z = 0 \)) of the AdS slice we consider massive scalar fields \( \Theta(\vec{x},t) \) whose corresponding creation-annihilation operators are assumed to satisfy the algebra

\[
[b(\vec{K}), b^\dagger(\vec{K}')] = 2 \left(2\pi\right)^3 w(\vec{K}) \delta^3(\vec{K} - \vec{K}'),
\] (13)

where \( w(\vec{K}) = \sqrt{\vec{K}^2 + \mu^2} \) and \( \mu \) is the mass of the field \( \Theta \).

Note that if the above bulk and boundary field theories had continuous momenta it would be impossible to find a one to one mapping between the corresponding quantum states since they are defined in different dimensions. However, as we consider just a slice of AdS, the spectrum of the momentum \( u_p \) associated with the axial direction \( z \) is discrete. Then naturally the continuous part of bulk and boundary momenta \( \vec{k} \) and \( \vec{K} \) have the same dimensionality. So this discretization makes it possible to establish a one to one mapping between bulk and boundary momenta \( (\vec{k}, u_p) \) and \( \vec{K} \). Further we assume a trivial mapping between their angular parts and then look for a relation between their moduli \( K \equiv |\vec{K}|, \ k \equiv |\vec{k}| \). It is important to mention that the complete one to one bulk boundary mapping is only possible in the conformal limit \( \mu \to 0 \) of the \( \Theta \) field. In the discussion below we will be interested in an approximate map between the dilaton and massive fields so we will let \( \mu \neq 0 \). Naturally the massless limit is closest to the original AdS/CFT correspondence.

We may introduce a sequence of energy scales \( \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \ldots \) in order to map each interval of the boundary momentum modulus \( \mathcal{E}_{p-1} < K \leq \mathcal{E}_p \) with \( p = 1, 2, \ldots \) into the entire range of the transverse bulk momentum modulus \( k \), corresponding to some fixed axial momentum \( u_p \). This mapping between bulk and boundary momenta allow us to map the corresponding creation-annihilation operators \( \{12\}^\{13\} \).

For the first energy interval, corresponding to \( u_1 \), defined as \( 0 \leq K \leq \mathcal{E}_1 \), we can write

\[
k a_1(\vec{k}) = K b(\vec{K})
\]

\[
k a_1^\dagger(\vec{k}) = K b^\dagger(\vec{K}).
\] (14)
This mapping must preserve the physical consistency of both theories. In particular, Poincare invariance should not be broken neither for the boundary theory nor for the bulk theory at a fixed $z$. This is guaranteed imposing that the canonical commutation relations (12,13) are preserved by the mapping (14). Then substituting eq. (14) into relation (12) and using eq. (13) we find an equation in terms of the moduli of bulk and boundary momenta which solution can be written as:

$$k = \frac{u_1}{2} \left[ \frac{\mathcal{E}_1 + \sqrt{\mathcal{E}_1^2 + \mu^2}}{K + \sqrt{K^2 + \mu^2}} \right].$$

(15)

Similar relations can be obtained for the other intervals $\mathcal{E}_{p-1} < K \leq \mathcal{E}_p$ with $p = 2, 3, ...$. This way we found a one to one map between bulk and boundary creation-annihilation operators. This allows us to construct a similar map for quantum states. This will be used in the next section in order to relate bulk and boundary scattering amplitudes.

One might wonder if the trivial mapping $\vec{k} = \vec{K}$ would also be a solution. However this would not provide a one to one mapping between the entire bulk momenta $(\vec{k}, u_p)$ and the boundary momenta $\vec{K}$ for all different values of $p$ as long as there is only one boundary field with mass $\mu$.

The momentum operators in the bulk and boundary theories are respectively:

$$\langle \vec{P}, u \rangle = \sum_p \int \frac{d^3k}{2(2\pi)^3} \frac{a_p^\dagger(\vec{k})a_p(\vec{k})}{\sqrt{k^2 + u_p^2}}$$

$$\vec{\Pi} = \int \frac{d^3K}{2(2\pi)^3} \frac{b_p^\dagger(\vec{K})b_p(\vec{K})}{\sqrt{K^2 + \mu^2}}$$

(16)

(17)

Note that Poincare invariance in the $\vec{x}$ directions holds both in the boundary and bulk theories since the canonical commutation relations (12) and (13) are preserved by the mapping.

4 High energy scalling

In this section we are going to study the scattering of scalar particles at high energy using the map found in the previous section. This is inspired in the gauge /string duality sugested by the AdS/CFT correspondence, where the scaling of the high energy scattering of glueballs was obtained from string theory using the duality between glueballs and dilatons. Glueballs in QCD correspond to composite operators. So we do not expect the free boundary scalar field of the previous section to give a complete description of their dynamics. However, we will see that for high energies the map leads to the same scaling as that of QCD glueballs. This may be an indication that for this regime the asymptotic states of the glueballs may be approximated by free scalar fields.

So we will consider the mapping of the previous section in the following way: on the boundary we will approximate the asymptotic behaviour of the glueball operators by free massive scalar fields in four dimensions. In the bulk we take the
massless scalar fields in the AdS slice as representing the dilaton. A high energy scattering on the four dimensional AdS boundary will be mapped into a scattering process of dilaton states in the bulk.\footnote{23}

The equation (15) is understood as the relation between bulk and boundary momenta for the particles involved in the scatterings. Identifying $\mu$ with the mass of the lightest glueball and choosing the AdS size as

$$z_{\text{max}} \sim \frac{1}{\mu}$$

we find that $u_1 \sim \mu$, once $z_{\text{max}} \sim 1/u_1$ according to eq. (14). Note that the size $z_{\text{max}}$ can then be interpreted as an infrared cutoff for the boundary theory.

Further, we can take $E_1$ large enough so that the momenta associated with the high energy glueball scattering can fit into the region $\mu \ll K \ll E_1$. Then we can approximate relation (15) as

$$k \approx \frac{E_1 \mu^2}{K},$$

(19)

where we defined $E_1 \equiv E$ and disregarded the other energy intervals associated with higher axial momenta $u_p, p \geq 2$. Note that this mapping together with the conditions $\mu \ll K \ll E$ imply that $\mu \ll k \ll E$.

Choosing the string scale to be of the same order of the high energy cut off of the boundary theory, i.e., $E \sim 1/\sqrt{\alpha'}$, we find that the momenta $k$ associated with string theory correspond to a low energy regime well described by supergravity approximation.

The equations (14) and (19) represent a one to one holographic mapping between bulk dilaton and boundary glueball states. Now we are going to use these equations to relate the corresponding scattering amplitudes.

Let us consider, in the bulk string theory, a scattering of 2 particles in the initial state and $m$ particles in the final state, with all particles having axial momentum $u_1$. The $S$ matrix reads

$$S_{\text{Bulk}} = \langle \vec{k}_3; u_1; \ldots; \vec{k}_{m+2}, u_1; \text{out}| \vec{k}_1, u_1; \vec{k}_2, u_1; \text{in} \rangle = \langle 0| a_{\text{out}}(\vec{k}_3) \ldots a_{\text{out}}(\vec{k}_{m+2}) b_{\text{in}}^+(\vec{k}_1) b_{\text{in}}^+(\vec{k}_2)| 0 \rangle,$$

(20)

where $a \equiv a_1$ and the $\text{in}$ and $\text{out}$ states are defined as $|\vec{k}, u_1\rangle = a^+(\vec{k})(0)$.

Now using the mapping between creation-annihilation operators (14) one can rewrite the above $S$ matrix in terms of boundary operators. Considering fixed angle scattering, we take the bulk momenta to be of the form $k_i = \gamma_i K$ and the boundary momenta $K_i = \Gamma_i K$, where $\gamma_i$ and $\Gamma_i$ are constants with $i = 1, 2, \ldots, m + 2$. Then

$$S_{\text{Bulk}} \sim \langle 0| b_{\text{out}}(\vec{k}_3) \ldots b_{\text{out}}(\vec{k}_{m+2}) b_{\text{in}}^+(\vec{k}_1) b_{\text{in}}^+(\vec{k}_2)| 0 \rangle \left( \frac{K}{k} \right)^{m+2}$$

$$\sim \langle \vec{k}_3, \ldots, \vec{k}_{m+2}, \text{out}| \vec{k}_1, \vec{k}_2, \text{in} \rangle \left( \frac{K}{k} \right)^{m+2} K^{(m+2)(d-1)},$$

(21)

where the composite operators on the boundary have scaling dimension $d$ and then their $\text{in}$ and $\text{out}$ states are $|\vec{K}\rangle \equiv K^{1-d}b^+(\vec{K})(0)$, within the regime $K \gg \mu$. 

7
Using the relation (19) between bulk and boundary momenta we get
\[ S_{\text{Bulk}} \sim S_{\text{Bound.}} \left( \frac{\sqrt{\alpha'}}{\mu} \right)^{m+2} K^{(m+2)(1+d)} \] (22)

As the scattering amplitudes \( M \) are related to the corresponding \( S \) matrices (for non equal in and out states) by
\[
S_{\text{Bulk}} = M_{\text{Bulk}} \delta^4(k_1^\rho + k_2^\rho - k_3^\rho - \ldots - k_{m+2}^\rho) \\
S_{\text{Bound.}} = M_{\text{Bound.}} \delta^4(K_1^\rho + K_2^\rho - K_3^\rho - \ldots - K_{m+2}^\rho)
\] (23)

we find a relation between bulk and boundary scattering amplitudes \[23\]
\[
M_{\text{Bound.}} \sim M_{\text{Bulk}} S_{\text{Bound.}} \left( S_{\text{Bulk}} \right)^{-1} \left( \frac{K}{k} \right)^4 \\
\sim M_{\text{Bulk}} K^{8-(m+2)(d+1)} \left( \frac{\sqrt{\alpha'}}{\mu} \right)^{2-m}.
\] (24)

Now we must evaluate the bulk amplitude from the string low energy effective action. At energies much lower than the string scale \( 1/\sqrt{\alpha'} \) type IIB string theory can be described by the supergravity action \[2, 38\]
\[
S = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{G} e^{-2\Phi} \left[ R + G^{MN} \partial_M \Phi \partial_N \Phi + \ldots \right]
\] (25)

where \( G^{MN} \) is the ten dimensional metric, \( R \) is the Ricci scalar curvature, \( \Phi \) is the dilaton field and \( \kappa \sim g(\alpha')^2 \). We identify this ten dimensional space with an \( \text{AdS}_5 \times S^5 \) with radius \( R \) and measure \( \text{[4]} \). Furthermore we are take an AdS slice with "size" \( z_{\text{max}} \) representing an infrared cut off associated with the mass of the lightest glueball. Note that we are also considering the dilaton to be in the \( s-wave \) state, so we will not take into account variations with respect to \( S^5 \) coordinates. Thus, the action becomes
\[
S = \frac{\pi^3 R^8}{4\kappa^2} \int d^4x \int_0^{z_{\text{max}}} dz \frac{dz}{z^3} e^{-2\Phi} \\
\times \left[ R + (\partial_z \Phi)^2 + \eta^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + \ldots \right]
\] (26)

where \( \eta^{\mu\nu} \) is the four dimensional Minkowski metric.

Then the momentum dependence of the bulk amplitude can be determined using dimensional arguments. Note that the global constant \( R^8/\kappa^2 \) associated with this action is dimensionless and the only dimensionfull parameters are \( z_{\text{max}} \sim 1/\mu \) and the Ricci scalar \( R \sim 1/R^2 \). As \( \mu \ll k \) the relevant contribution to the bulk amplitude will not involve \( z_{\text{max}} \). Further, choosing the condition \( 1/R \ll k \) we can disregard the term involving the Ricci scalar. This condition does not fix completely the AdS radius \( R \) and we additionally impose that \( \mu \ll 1/R \). This implies that \( z_{\text{max}} \gg R \). Then, if one regularizes \[2\] the divergence \( (z = 0) \) of the bulk action
by cutting the axial coordinate \( z \) at \( R \), one still has a large portion of the original AdS space: \( R \leq z \leq z_{\text{max}} \). This guarantees that we keep the interesting AdS region which is an approximation for the near horizon geometry of \( N \) coincident \( D3 \)-branes, as in the Maldacena duality.

Taking into account the normalization of the states \( |k, u_1\rangle \) one sees that \( \mathcal{M}_{\text{Bulk}} \) is dimensionally \([\text{Energy}]^{4-n}\), where \( n \) is the total number of scattered particles. As \( k \) is the only dimensionfull quantity that is relevant at leading order for the bulk scattering in the regime considered we find:

\[
\mathcal{M}_{\text{Bulk}} \sim k^{2-m}.
\]  

(27)

Using again the relation between bulk and boundary momenta (19) and inserting this result in the boundary amplitude (24) we get

\[
\mathcal{M}_{\text{Boundary}} \sim K^{4-\Delta},
\]  

(28)

where \( \Delta = (m + 2)d \) is the total scaling dimension of the scattering particles associated with glueballs on the four dimensional boundary. Considering \( K \sim \sqrt{s} \) we find the expected QCD scaling behavior [19, 20]

\[
\mathcal{M}_{\text{Boundary}} \sim s^{2-\Delta/2}.
\]  

(29)

This shows that the bulk/boundary one to one mapping (14), (19) can be used to obtain the hard scattering behavior of high energy glueballs at fixed angles, from a low energy approximation of string theory.

It is interesting to relate the different energy scales used in the above derivation of the scattering amplitudes and check their consistency. The scales we discussed are

\[
\mu \ll \frac{1}{R} \ll \frac{1}{\sqrt{\alpha'}}.
\]  

(30)

Note that the relation between the AdS radius \( R \), the number of coincident branes \( N \), the string coupling constant \( g \) and scale \( \alpha' \) is \( R^4 \sim gN(\alpha')^2 \). Then the above condition between \( R \) and \( \alpha' \) corresponds to the ‘t Hooft limit [16]. Assuming that the dimensionless quantity \( \mu R \) is the parameter that relates the energy scales we find \( \sqrt{\alpha'} = \mu R^2 \) so that the lightest glueball mass is

\[
\mu^2 = \frac{1}{gN\alpha'}.
\]  

(31)

This result is in agreement with Maldacena and Nunez [39]. Further, the above relation between the energy scales together with the condition that \( k \gg 1/R \) and the mapping between bulk and boundary momenta (19) imply that

\[
\mu \ll K \ll \frac{1}{R} \ll k \ll \frac{1}{\sqrt{\alpha'}},
\]  

(32)

so that the absolute values of \( k \) are greater than those of \( K \), although the boundary scattering is a high energy process (with respect to \( \mu \)) while the bulk scattering is a low energy process (with respect to \( 1/\sqrt{\alpha'} \)). Furthermore \( K \) and \( k \) in this regime are inversely proportional showing a kind of infrared-ultraviolet duality as expected from holography [10].
5 Glueball masses

We can use the mapping between bulk and boundary scalar fields of the previous section to estimate the masses of scalar glueballs\cite{10}. In contrast to the previous case now we will not consider just one kind of glueball field. Rather we consider, on the boundary of the AdS slice, a sequence of different massive fields $\Theta_i(x, t)$ representing the states of scalar glueballs with masses $\mu_i$. Correspondingly we have a sequence of creation-annihilation operators $b_i b_i^\dagger$ satisfying equation (13) with $w_i(\vec{K}) = \sqrt{\vec{K}^2 + \mu_i^2}$.

Introducing momentum operators $\tilde{\Theta}_i(K)$ with momentum $K_j$ in the interval $E_{j-1} \leq K \leq E_j$ they would not be mapped in a one to one relation with bulk operators $\tilde{\Phi}(k, u_p)$ unless $j$ is limited, since $i$ and $p$ are unlimited. Then such a mapping is possible if we introduce a restriction on the index $j$. The simplest choice is to take just one value for $j$. This is obtained taking $E_1 \equiv \mathcal{E}$ large enough, which means that now $j = 1$. This recovers the one to one mapping of the previous section and in this case it reads

$$\tilde{\Theta}_i(K) \leftrightarrow \tilde{\Phi}(k, u_i),$$

where we have dropped the index of $K_1$ since this is the only relevant boundary momentum.

This mapping can be written explicitly in terms of bulk and boundary creation-annihilation operators. We will impose the same relation (14) for each pair $a_i$, $b_i$.

Requiring again that equations (14) preserve the canonical commutation relations (12,13) one finds that the moduli of the momenta are related by (15) where $0 \leq K \leq \mathcal{E}$ as before.

Now we associate the size $z_{\text{max}}$ of the AdS space with the mass of the lightest glueball which we choose to be $\mu_1$

$$z_{\text{max}} = \frac{\chi_{2,1}}{\mu_1},$$

so that from equation (11) we have

$$u_i = \frac{\chi_{2,i}}{\chi_{2,1}} \mu_1.$$ (34)

An approximate expression for the mapping (15) can be obtained choosing appropriate energy scales. We take $\mathcal{E}$ to be the string scale $1/\sqrt{\alpha'}$ assuming that $K \ll \mathcal{E}$. On the other side we restrict the momenta $K$ associated with glueballs to be much larger than their masses $\mu_i$. Then we have

$$\mu_i \ll K \ll \frac{1}{\sqrt{\alpha'}}.$$ (35)

In this regime the mapping (15) reduces to

$$k \approx \frac{u_i}{2 \sqrt{\alpha'} K}.$$ (36)
This approximate mapping gives a high energy scaling similar to QCD\cite{36}. Using the conditions \eqref{35} together with the above mapping we see that the bulk momenta satisfy

\[ u_i \ll k \ll \left( \frac{u_i}{\mu_i} \right)^{1/\sqrt{\alpha'}} \]  

(37)

Note that the supergravity approximation holds for \( k \ll 1/\sqrt{\alpha'} \). So in order to keep this approximation valid for all glueball operators \( \Theta_i \) the factor \( u_i/\mu_i \) should be nearly constant. We then impose that

\[ \frac{u_i}{\mu_i} = \text{constant} \].

(38)

So the glueball masses are related to the zeros of the Bessel functions by

\[ \frac{\mu_i}{\mu_1} = \frac{\chi_{2,i}}{\chi_{2,1}} \].

(39)

Using the values of these zeros one finds the ratio of the glueball masses for the state \( 0^{++} \) and its excitations. We are using the conventional notation for these states with spin zero and positive parity and charge conjugation. In order to compare our results from bulk/boundary holographic mapping with those coming from lattice we adopt the mass of the first state as an input. Our results\cite{40} are in good agreement with lattice\cite{41,42} and AdS-Schwarzschild black hole supergravity\cite{43,49} calculations.

It is interesting to mention that an approach to estimate glueball masses in Yang Mills* from a deformed AdS space was discussed very recently\cite{50}.

We can generalize the above results to AdS\(_{n+1}\). In this case massless bulk fields are expanded in terms of the Bessel functions \( J_{n/2} \) and the mass ratios for the \( n \) dimensional "glueballs" are given in terms of their zeros. In particular for AdS\(_4\), where one expects to recover results from QCD\(_3\), we find

\[ \frac{\mu_i}{\mu_1} = \frac{\chi_{3/2,i}}{\chi_{3/2,1}} \].

(40)

Using this relation we obtain\cite{40} the ratio of masses again in good agreement with lattice\cite{41,42} and AdS-Schwarzschild black hole supergravity calculations\cite{43,49}.

It is interesting to see if the AdS slice considered here can be related to the AdS-Schwarzschild black hole metric proposed by Witten\cite{51}. For the case of QCD\(_3\) the metric reads

\[
\begin{align*}
\text{d}s^2 &= R^2 \left( \rho^2 - \frac{b^4}{\rho^2} \right)^{-1} \text{d}\rho^2 + R^2 \left( \rho^2 - \frac{b^4}{\rho^2} \right) \text{d}\tau^2 \\
&\quad + R^2 \rho^2 (\text{d}\vec{x})^2 + R^2 d\Omega_5^2 ,
\end{align*}
\]

(41)

where \( \rho \geq b \), \( R^2 = l_s^2 \sqrt{4\pi g_s N} \) and \( b \) is inversely proportional to the compactification radius of \( S_1 \) where the \( \tau \) variable is defined.

If we qualitatively neglect the \( \tau \) contribution to the metric in the limit of very little compactification radius and then take the limit \( \rho \gg b \) this metric is approximated by
\[ ds^2 = \frac{R^2}{\rho^2} d\rho^2 + R^2 \rho^2 d\tau^2 + R^2 \rho^2 (d\vec{x})^2 + R^2 d\Omega_5^2. \]  

(42)

That is an AdS\(_4 \times S^5\) space that takes a form similar to eq. (6) if we change the axial coordinate to \( z = 1/\rho \). In Witten’s framework one must impose regularity conditions at \( \rho = b \) because of the presence of the horizon at this position. In our approximation in order to retain this physical condition we impose boundary conditions there and associate it to the cut of our slice \( (b = 1/z_{\text{max}}) \). This AdS\(_4\) slice is the one used to estimate the glueball mass ratios related to the three dimensional gauge theory (40). So we can think of our AdS\(_4\) slice as a naive approximation to Witten’s proposal.

An analogous situation could also be considered for the Witten’s proposal to QCD\(_4\). In that case the situation is more involved because of the form of the metric coming from the compactification of AdS\(_7 \times S^4\).

In conclusion we have seen that the bulk/boundary holographic mapping which reproduces the high energy scaling of QCD like theories can also be applied to estimate glueball mass ratios. We hope that this mapping can be used to describe other particle states that may be related to some properties of QCD.

It is important to remark that one can obtain a similar result for the ratio of the glueball masses considering other mappings between bulk and boundary creation-annihilation operators instead of eq. (14). For example one could take \( a_i = b_i \). This would contain the solution \( k = K \) implying that the masses of the glueballs are identically equal to the values of axial bulk momenta \( u_i \). However such a trivial mapping does not seem to reproduce the high energy QCD scaling.

Let us mention that we used a solution for the dilaton field corresponding to Dirichlet boundary conditions at \( z = 0 \) and \( z = z_{\text{max}} \). This allows the existence of Bessel functions but not the divergent Neumann solutions. Other boundary conditions can also be considered in the same context.

We have also obtained these mass ratios for scalar glueballs starting with the same AdS slice as discussed here without using the holographic mapping of ref [36] but assuming the stronger condition of relating directly the dilaton modes with the glueball masses [52]. The consistency between these results seems to indicate that the holographic mapping found before may indeed be valid within the approximations and the energy region considered.

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