CONFORMAL ISOSYSTOLIC INEQUALITY OF BIEBERBACH
3-MANIFOLDS

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"A une femme unique, à celle qui m’inspire,
Florence Henault."

Abstract. The systole of a compact non simply connected Riemannian manifold is the smallest length of a non-contractible closed curve; the systolic ratio is the quotient (systole)/volume. Its supremum, over the set of all Riemannian metrics, is known to be finite for a large class of manifolds, including aspherical manifolds.

We study a singular metric $g_0$ which has a better systolic ratio than all flat metrics on 3-dimensional non-orientable Bieberbach manifolds (introduced in [El-La08]), and prove that it is extremal in its conformal class.

Key words and phrases. Systole; systolic ratio; singular Riemannian metric; Bieberbach manifold.

1. Introduction and main result

The systole of a compact non simply connected Riemannian manifold $(M^n, g)$ is the shortest length of a non contractible closed curve, we denote it by $\text{Sys}(g)$. To get an homogeneous Riemannian invariant, we introduce the systolic ratio $\frac{\text{Sys}(g)^n}{\text{Vol}(g)}$. It is important to note that this invariant is well defined even if $g$ is only continuous, i.e. a continuous section of the fiber bundle $S^2T^*M$ of symmetric forms.

An isosystolic inequality on a manifold $M$ is an inequality of the form

$$\frac{\text{Sys}(g)^n}{\text{Vol}(g)} \leq C$$

that holds for any Riemannian metric $g$ on $M$. The smallest such constant $C$ is called the systolic constant.

A systolic geodesic will be for us a closed curve, not homotopically trivial, whose length is equal to the systole.
In 1949, in an unpublished work, C. Loewner proved the following result. For any metric $h$ on the torus $T^2$ we have

$$\frac{\text{sys}^2(T^2, h)}{\text{Area}(T^2, h)} \leq 2/\sqrt{3}$$

Furthermore, the equality is achieved if and only if $(T^2, h)$ is isometric to a flat hexagonal torus.

In 1952, P.M. Pu proved an isosystolic inequality for the real projective plane (c.f. [Pu52]). The extremal metric has constant curvature, too. In the same paper, he proved a variant of the isosystolic inequality for the Möbius bands with boundary, valid for each conformal class of any metric.

There exists a third case, solved by C. Bavard in [Bav86], where the upper bound of the systolic ratio is known, and realized, this is the case of the Klein bottle. This time the extremal metric (for the isosystolic inequality) is singular, more precisely piecewise $C^\infty$ (see [Bav86] and [Bav93]). Furthermore, it has curvature equal to $+1$ where it is smooth.

In higher dimension, there exists non simply connected manifolds that do not satisfy any isosystolic inequality. The simplest example is $S^2 \times S^1$, or more generally the product of a simply connected manifolds by a non simply connected one. Making the volume of the simply connected factor tend to zero insures the explosion of the systolic ratio.

However a fundamental result of M. Gromov (cf. [Gro83]), insures that essential manifolds satisfy an isosystolic inequality. A compact manifold $M$ is essential if there exists a continuous map from $M$ in a $K(\pi, 1)$ ($\pi = \pi_1(M)$) which sends the fundamental class to a non trivial one. The essential manifolds include notably aspherical manifolds and the real projective spaces.

In dimensions $\geq 3$, hardly anything is known about metrics that realize the systolic constant (extremal metrics). It is not known for example, in the apparently simple cases of tori and real projective spaces, whether the metrics of constant curvature are extremal. A classical result in systolic geometry assures that if a metric is extremal, then the systolic geodesics cover the manifold.

In [El-La08] we proved with J. Lafontaine that flat metrics on Bieberbach non-orientable 3-manifolds are not extremal for the isosystolic inequality. A manifold is called Bieberbach if it carries a flat Riemannian metric. To prove this result, we showed that all flat Bieberbach non-orientable 3-manifolds are obtained by suspending a flat Klein bottle. Then we endowed the Klein bottle with the extremal metric discovered by Bavard in [Bav86] to construct singular metrics on each type of these manifolds (after adjusting the parameters). These metrics are better than all the flat ones. On each one of the four types of these manifolds (see [Wol74]), we denote this special metric by $g_0$.

In this paper we prove the following result:

**Theorem 1.** The metric $g_0$ on each non-orientable Bieberbach 3-manifolds is optimal in its conformal class.

Such a property is not verified by any flat metric on these manifolds. We also show that this metric has other "beautiful" properties of recovering the manifold that make it a good candidate to realize the systolic constant.
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2. Flat non-orientable 3-manifolds and the singular metric $g_0$

Compact flat manifolds are quotients $\mathbb{R}^n/\Gamma$, where $\Gamma$ is a discrete cocompact subgroup of affine isometries of $\mathbb{R}^n$ acting freely. By the theorem of Bieberbach, $\Gamma$ is an extension of a finite group $G$ by a lattice $\Lambda$ of $\mathbb{R}^n$. This lattice is the subgroup of the elements of $\Gamma$ that are translations, we obtain then the following exact sequence:

$$0 \rightarrow \Lambda \rightarrow \Gamma \rightarrow G \rightarrow 1$$

The classification of flat manifolds of dimension 3 results of a direct method of classification of discrete, cocompact subgroups of $\text{Isom}(\mathbb{R}^3)$ operating freely. This classification is due to W. Hantzsche and H. Wendt (1935), and exposed in the book [Wol74] of J.A. Wolf. There exist ten compact and flat manifolds of dimension 3 up to diffeomorphism, six are orientable and four are not. In [El-La08], we showed that each one of the non-orientable type is obtained by suspending a flat Klein bottle (that we denote by $K$) by an isometry. This can be done since the group $\text{Isom}(K)/\text{Isom}_0(K)$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

2.1. The Klein bottle. Flat Klein bottles are the quotient $\mathbb{R}^2/\Gamma$, where $\Gamma$ generated by the screw motion $(x,y) \mapsto (x + \frac{a}{2}, -y)$ and the translation $(x,y) \mapsto (x, y + b)$. Note that we can obtain a Klein bottle $K$ by gluing two Mobius bands along their boundary.

The connected component of the identity in $\text{Isom}(K)$ is formed of the horizontal translations $r_\alpha : (x,y) \mapsto (x + \alpha, y)$ where $\alpha$ is modulo $a$. The quotient $\text{Isom}(K)/\text{Isom}_0(K)$ is isomorphic to the Klein group, the three non trivial elements of this quotient can be represented by

1. a reflection with respect to a vertical geodesic, which is also a symmetry with respect to a point on a horizontal short geodesic. We denote by $S_1$ such a transformation.
2. a symmetry with respect to a point on the common boundary of the two Mobius bands composing the Klein bottle. We denote by $S_2$ such a symmetry.
3. the reflection with respect of the common boundary, it is the transformation (after going to the quotient) $(x,y) \mapsto (x,y + b/2)$. We denote by $T$ this last transformation.

2.2. Flat 3-manifolds. Up to a diffeomorphism, there are four types of flat non-orientable 3-manifolds (see [Wol74] and [El-La08])

1. Type $B_1$: it is the quotient of $K \times \mathbb{R}$ by the subgroup of isometries generated by $(p,t) \mapsto (r_\alpha(p), t + d)$, (for $p \in K$ and $t \in \mathbb{R}$)

   where $\alpha \in \mathbb{Z}/\pi \mathbb{Z}$, and $d \in \mathbb{R}^+$. 

2. Type $B_2$: it is the quotient of $K \times \mathbb{R}$ by the subgroup of isometries generated by $(p,t) \mapsto \left(r_\alpha(T(p)), t + d\right)$.
(3) Type $B_3$: it is the quotient of $K \times \mathbb{R}$ by the subgroup of isometries generated by 
\[(p, t) \mapsto (S_1(p), t + d).\]

(4) Type $B_4$: it is the quotient of $K \times \mathbb{R}$ by the subgroup of isometries generated by 
\[(p, t) \mapsto (S_2(p), t + d).\]

2.3. **The singular metric** $g_0$. We begin by defining the singular extremal Klein bottle. We introduce on $\mathbb{R}^2$ the Riemannian singular metric:

\[dv^2 + \psi^2(v)du^2,\]

where $\psi$ is a $\pi/2$-periodic function which coincides with $\cos v$ on the interval $[-\pi/2, \pi/2]$ (check [Bav86], [Bav93] and [El-La08]).

Finally, to obtain the singular extremal Klein bottle, we take the quotient of $\mathbb{R}^2$ endowed with the metric $[\mathbb{I}]$ by the subgroup generated by 
\[(u, v) \mapsto (u + \pi, -v) \quad \text{and} \quad (u, v) \mapsto (u, v + \pi).\]

We denote it by $(K, b)$.

**Remark 1.** The singular Klein bottle $(K, b)$ has the same group of isometries as a flat Klein bottle.

**Remark 2.** It is useful to note that the singular Klein bottle $(K, b)$ satisfies the following properties:

- The systolic geodesic of any systolic class do cover $(K, b)$. A systolic class is an element of the fundamental group that contains at least one systolic geodesic.
- Except on a set of null measure, there exists an infinite number of systolic geodesics going through each point of $(K, b)$.

Now we will define the singular metric $g_0$ on each of the four types $(B_i)_{1 \leq i \leq 4}$. This metric has "a lot" of systolic geodesics as we will see later.

1. $(B_1, g_0)$: we take the quotient of $(K, b) \times \mathbb{R}$ by the subgroup generated by $(p, t) \mapsto (r_\alpha(p), t + d)$ with $\alpha = \pi(2 - \sqrt{2})$ and $d = \pi(2\sqrt{2} - 2)^{1/2}$. For these values of $\alpha$ and $d$, we have, in addition to the family of geodesics in the surfaces $z = constant$, a $T^2$ family of systolic geodesics in the surfaces $\phi = 0$ and $\phi = \pm \pi/4$.

2. $(B_2, g_0)$: we take the quotient of $(K, b) \times \mathbb{R}$ by the subgroup generated by $(p, t) \mapsto (r_\alpha(T(p)), t + d)$ with $\alpha$ and $d$ verifying $\frac{\cos \alpha - 1}{2} = \cos \frac{\pi - \alpha}{\sqrt{2}}$ and $(d(r_\alpha(T(p)))^2 + d^2 = \pi^2$, where 
\[d(r_\alpha(T(p))) = \inf \{\frac{\pi - \alpha}{\sqrt{2}}, \arccos \frac{\cos \alpha - 1}{2}\}.\]

For these values of $\alpha$ and $d$, there is, in addition to the systolic geodesics in the surfaces $z = constant$, a systolic geodesic going through every point of $(B_2, g_0)$ which is homotopic to the transformation $(p, t) \mapsto (r_\alpha(T(p)), t + d)$. 
3. Proof of the theorem

We introduce on $\mathbb{R}^3$ the singular metric $g = \psi(v)du^2 + dv^2 + dz^2$, where $\psi$ is $\pi/2$-periodic and $\psi(v) = \cos(v)$ for $-\pi/4 \leq v \leq \pi/4$. The quotient of $(\mathbb{R}^3, g)$ by the group of translations generated by $u \rightarrow u + \pi$ and $v \rightarrow v + \pi/2$ is a singular 2-dimensional torus times $\mathbb{R}$. We denote this Riemannian manifold by $(T \times \mathbb{R}, g)$.

Now we take the quotient of $(T \times \mathbb{R}, g)$ by the transformation $(u, v, z) \rightarrow (u, -v, z)$. The manifold we obtain is the product $\mathbb{K}_b \times \mathbb{R}$, where $\mathbb{K}_b$ is the singular extremal Klein bottle.

In the following, we will prove the main result of this paper: "every metric constructed in [EL-La08] on each of the four Bieberbach non-orientable manifolds of dimension 3 is extremal in its conformal class". This result shows that these metrics are potentially extremal in the class of all Riemannian metrics. We will use the following result of C. Bavard [Bav92].

Let $(M, g)$ be a compact essential Riemannian manifold of dimension $n$ and let $\Gamma$ be the space of the systolic curves of $(M, g)$. For every Radon measure $\mu$ on $\Gamma$, we associate a measure $\mu^*$ on $M$ by setting for $\varphi \in C^0_0(M, \mathbb{R})$

$\langle \mu^*, \varphi \rangle = \langle \mu, \overline{\varphi} \rangle$

where $\overline{\varphi}(\gamma) = \int \varphi \circ \gamma(s) ds$, $ds$ is the arc length of $\gamma$ with respect to $g$. Then we have

**Theorem 2.** ([Bav88], [Bav92] and [Gro83])

The Riemannian manifold $(M, g)$ is minimal in its conformal class if and only if there exists a positive measure $\mu$, of mass 1, on $\Gamma$ such that

$\mu^* = \frac{\text{Sys}(g)}{\text{Vol}(g)} \cdot dg$

where $dg$ is the volume measure of $(M, g)$.

**Remark 3.** Actually we will only use the reciprocal of this theorem, this result was initiated by Jenkins in [Jenk57].

3.1. The manifold $\mathbb{K} \times S^1$. We endow the manifold $\mathbb{K} \times S^1$ with the metric $g = \psi(v)du^2 + dv^2 + dz^2$, where $\psi$ is defined as above. We now prove that this singular metric has the property to be optimal in its conformal class. We follow the strategy of C. Bavard in his paper ([Bav88]).
We consider in the band \{w = 0, -\pi/4 \leq v \leq \pi/4\} the geodesic \(\gamma_{(0,0)}\), \((-\pi/4 \leq a \leq \pi/4)\),
going through the points \((-\pi/2, 0, 0), (0, a, 0)\) and \((\pi/2, 0, 0)\). Then we consider its images \(\gamma_{a,\phi}\)by the isometries \((u, v, w) \rightarrow (u + \theta, v, w + \phi)\), and we put on the space \(\Gamma = \{\gamma_{a,\phi} : -\pi/4 \leq a \leq \pi/4, \theta \in \mathbb{R}/\pi\mathbb{Z} \text{ and } \phi \in \mathbb{R}/d\mathbb{Z}\}\) the following mesure:

\[
\mu = h(a)da \otimes d\theta \otimes d\phi
\]

We will calculate \(\ast \mu\). We write \(\gamma_{a,\phi}(t) = (t + \theta, v(t, a), \phi)\), where \(v\) is a continuous function.

Let \(\varphi\) be a continuous function of \((\mathbb{K} \times S^1, g)\):

\[
<\ast \mu, \varphi> = \int_{a=-\pi/4}^{\pi/4} \int_{\theta=-\pi/2}^{\pi/2} \int_{\phi=0}^{\pi/2} \varphi(t + \theta, v(t, a), \phi) \sqrt{\cos^2(v(t, a)) + \left(\frac{\partial v}{\partial t}(t, a)\right)^2} h(a)dt d\phi d\theta da
\]

Since \(t \rightarrow v(t, a)\) is an even function, we have

\[
<\ast \mu, \varphi> = 2 \int_{a=-\pi/4}^{\pi/4} \int_{\theta=-\pi/2}^{\pi/2} \int_{\phi=0}^{\pi/2} \varphi(t + \theta, v(t, a), \phi) \sqrt{\cos^2(v(t, a)) + \left(\frac{\partial v}{\partial t}(t, a)\right)^2} h(a)dt d\phi d\theta da
\]

The change of variable \(y = v(t, a)\) has a Jacobian equal to \(\frac{\partial v}{\partial t}(t, a)\) which is always positive when\(t \in [-\pi/2, 0]\). Let \(k(y, a) = t\) and \(f(y, a) = \frac{\partial v}{\partial t}(t, a)\), then we have

\[
<\ast \mu, \varphi> = 2 \int_{a=-\pi/4}^{\pi/4} \int_{\theta=-\pi/2}^{\pi/2} \int_{\phi=0}^{\pi/2} \varphi(k(y, a) + \theta, v, \phi) \sqrt{\cos^2(v) + f^2(y, a)\frac{h(a)}{|f(y, a)|}} dy d\phi d\theta da
\]

Now we do the change of variable \(x = k(y, a) + \theta\) and \(\phi = z\) whose Jacobian is equal to 1. We have

\[
<\ast \mu, \varphi> = 2 \int_{a=-\pi/4}^{\pi/4} \int_{x=-\pi/2}^{\pi/2} \int_{y=-\pi/2}^{\pi/2} \int_{z=0}^{\pi/4} \varphi(x, y, z) \sqrt{\left(\frac{\cos(y)}{f(y, a)}\right)^2} + h(a)dy dz dx da
\]

\[
= 2 \int_{x=-\pi/2}^{\pi/2} \int_{y=-\pi/2}^{\pi/2} \int_{z=0}^{\pi/4} \int_{a=|y|}^{\pi/4} \varphi(x, y, z) \sqrt{\left(\frac{\cos(y)}{f(y, a)}\right)^2} + h(a)da dy dz dx
\]

Finally

\[
\ast \mu(u, v, w) = 2 \frac{\chi(|v| \leq \pi/4)}{\cos(v)} \int_{a=|v|}^{\pi/4} \varphi(x, y, z) \sqrt{\left(\frac{\cos(y)}{f(y, a)}\right)^2} + h(a)da dy dz du
\]

To calculate the function \(f(y, a)\), we will use the fact that the curves \(\gamma_{a,\phi}\) are geodesics. Let \(\mathfrak{g}\) be the metric induced by \(g\) on the hypersurface \(w = \phi\). Then \(\mathfrak{g}\) can be written in the form \(g(y)(dx^2 + dy^2)\) and the geodesics \(\gamma_{a,\phi}\): \(-\pi/4 \leq a \leq \pi/4, \theta \in \mathbb{R}/\pi\mathbb{Z}\), will have to satisfy

\[
\frac{d}{dy} \left( x' \left( \frac{g(y)}{1 + x'^2} \right)^{1/2} \right) = 0
\]
where $x' = \frac{dx}{dy}$. In our case, we have $g(y) = \cos^2(v)$ and $x' = \frac{\cos(v)}{f(v,a)}$. Now the curves $\gamma_{\theta, \phi}$ satisfy

$$\frac{d}{dv} \left( \frac{\cos^2(v)}{\sqrt{f^2(v,a) + \cos^2(v)}} \right) = 0$$

We obtain a differential equation verified by $v \mapsto f^2(v,a)$, with $f^2(0,a) = 0$. The solution is $f^2(v,a) = \frac{\cos^2(v)}{\cos^2(a)} (\cos^2(v) - \cos^2(a))$. Finally we have

$$\ast \mu(u, v, w) = 2\chi(|v| \leq \pi/4) \int_{a = |y|}^{\pi/4} (\cos^2(v) - \cos^2(a))^{-\frac{1}{2}} h(a) da \ dg(u, v, w)$$

Now we are capable of calculating the function $h$. It should satisfy the equation

$$\ast \mu = dg$$

on the band $|v| \leq \pi/4$. Then we have

$$\int_{a = |y|}^{\pi/4} (\cos^2(v) - \cos^2(a))^{-\frac{1}{2}} h(a) da = 1/2.$$ 

We put $z = \cos^2(a) - 1/2$ and $t = \cos^2 v - 1/2$, then we get the equation

$$\int_0^t (t - z)^{-1/2} y(z) dz = 1/2$$

whose solution is

$$y(u) = \frac{f(0)}{\pi \sqrt{u}}.$$

Finally, we find

$$h(a) = \frac{\sin(2a)}{2\pi} \left( \cos^2(a) - 1/2 \right)^{-\frac{1}{2}} \text{ for } 0 \leq a \leq \pi/4$$

We define the same way $\mu$ on the family of systolic geodesics $s(\Gamma)$ where $s$ is the symmetry with respect to the surface $v = \pm \pi/4$.

Now, a simple calculation shows that $\mu = h(a) da \otimes d\theta \otimes d\phi$ has mass equal to $\frac{Vol(g)}{Sys(g)}$ when $Sys(g) = \pi \left(l(S^1) \geq \pi \right)$. The existence of this measure allows us to deduce that $g$ is extremal in its conformal class.
3.2. Extremality of $g$ for the manifolds of type $B_i$, $1 \leq i \leq 4$ \cite{El-La08}. The manifolds $(B_i, g_0)$, $1 \leq i \leq 4$ are the quotients of $(\mathbb{K} \times S^4, g)$ by a subgroup generated by $(p, v, z + d)$ where $\sigma$ is an element of $\text{Isom}(\mathbb{K})/\text{Isom}_0(\mathbb{K})$ (c.f. 2.1) and the parameters are fixed as in 2.3 we also suppose that $l(S^1) \geq \pi$.

For each one of the four types of manifolds $(B_i, g_0)$ we consider the space $\Gamma$ of the systolic curves $\gamma_{\theta, \phi}$. We endow this space with the measure

$$\mu = h(a) da \otimes d\theta \otimes d\phi$$

where $h$ is the function defined in the previous section. We can verify easily that it has mass equal to $\frac{\text{Vol}(g)}{\text{Sys}(g)}$ and that

$$\ast \mu = dg$$

**Corollary 1.** For each $1 \leq i \leq 4$, the manifold $(B_i, g_0)$ is extremal in its conformal class.

4. SOME USEFUL REMARKS

Among flat tori of dimension 3, the hexagonal one is the best for the isosystolic inequality. It is the quotient of $\mathbb{R}^3$ by the lattice that has a basis $(a_1, a_2, a_3)$ such that $(a_i, a_j) = \pi/3$ for $i \neq j$. We denote it by $T_{\text{hex}}$. It is known that $T_{\text{hex}}$ and the real projective space of dimension 3 endowed with its metric of constant curvature (we denote it by $(\mathbb{R}P^3, \text{sph})$), are very good candidates to realize the systolic constant of their topological manifold. Each one of these Riemannian manifolds satisfy the following properties (c.f. 2):

- The metric is optimal in its conformal class.
- The systolic geodesics of any systolic class of $T_{\text{hex}}$ (or $(\mathbb{R}P^3, \text{sph})$) cover the manifold (c.f. 2).

Our result shows that the singular metrics $(B_1, g_0)$, $(B_2, g_0)$, $(B_3, g_0)$ and $(B_4, g_0)$ satisfy the first property. Furthermore, the manifold $(B_2, g_0)$ satisfies the second one (see \cite{El-La08} 3. prop 2).

On the other hand $(\mathbb{R}P^3, \text{sph})$ and all the manifolds $(B_i, g_0)$ satisfy the following property: Except on a set of null measure, there exists an infinite number of systolic geodesics going through any point of the manifold. This property is not verified by the torus $T_{\text{hex}}$: there exist exactly 6 systolic geodesics going through any point of $T_{\text{hex}}$.

We think that as for $T_{\text{hex}}$ and $(\mathbb{R}P^3, \text{sph})$, the manifolds $(B_i, g_0)$ are very good candidates to realize the systolic constant because they have an abondance of systolic geodesics which satisfythe properties mentioned above. We also think that it would be fair if we give more importance to the manifolds $(\mathbb{R}P^3, \text{sph})$ and $(B_2, g_0)$ since they are the only ones to satify the remarkable property of being covered by the systolic geodesics of every systolic class. All this pushes us to make the following conjectures

**Conjecture 1.** (Classical) The manifold $(\mathbb{R}P^3, \text{sph})$ is extremal for the isosystolic inequality.

**Conjecture 2.** The manifold $(B_2, g_0)$ is extremal for the isosystolic inequality.
REFERENCES

[Bav86] Bavard, C.; Inégalité isosystolique pour la bouteille de Klein, Math. Ann. 274, 439–441 (1986).
[Bav88] Bavard, C.; Inégalités isosystoliques conformes pour la bouteille de Klein, Geom. Dedicata 27, 349–355 (1988).
[Bav92] Bavard, C.; Inégalités isosystoliques conformes, Comment. Math. Helv. 67 (1992), no. 1, 146–166 (1992).
[Bav93] Bavard, C.; Une remarque sur la géométrie systolique de la bouteille de Klein, Arch. Math. (Basel) 87 (2006), No 1, 72-74 (1993).
[Ber93] Berger, M.; Systoles et applications selon Gromov, Séminaire N. Bourbaki, exposé 771, Astérisque 216, 279–310 (1993).
[Bla61F] Blatter, C.; Über extremallängen auf geschlossenen Flächen, Comment. Math. Helvetici 35 (1961), 153–168.
[Bla61M] Blatter, C.; Zur Riemannschen Geometrie im Grossen auf dem Möbiusband, Compositio Math. 15 (1961), 88–107.
[El-La08] Elmir, C.; Lafontaine., J.; Sur la géométrie systolique des variétés de Bieberbach, Geom. Dedicata. 136, 95–110 (2008)
[Gro83] Gromov, M.; Filling Riemannian manifolds, J. Diff. Geom. 18, 1–147 (1983).
[Jenk57] Jenkins, J. A.; On the existence of certain general extremal metrics, Ann. of Math. 66, 440-453 (1957).
[Kat07] Katz, M.G, Systolic Geometry and Topology, Math. Surveys and Monographs 137, Amer. Math. Soc., Providence, R.I. (2007).
[Pu52] Pu, P. M.; Some inequalities in certain non-orientable riemannian manifolds. Pacific J.Math. 2, 55–71 (1952).
[Wol74] Wolf, J.A.; Spaces of constant curvature, Publish or Perish, Boston (1974).