EXPONENTIAL DECAY OF CORRELATIONS FOR SURFACE SEMI-FLOWS WITHOUT FINITE MARKOV PARTITIONS

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Abstract. We extend Dolgopyat’s bounds on iterated transfer operators to suspensions of interval maps with infinitely many intervals of monotonicity.

1. Statement of results

Let $0 < c_1 < \ldots < c_m < c_{m+1} < \ldots < 1$ be a finite or countable partition of $I = [0,1]$ into subintervals, and let $T : I \to I$ be so that $T|_{(c_m,c_{m+1})}$ is $C^2$ and extends to a homeomorphism from $[c_m,c_{m+1}]$ to $I$. We assume that $T$ is piecewise uniformly expanding: there are $C \geq 1$ and $\hat{p} < 1$ so that $|h(x) - h(y)| \leq C\hat{p}^n|x-y|$ for every inverse branch $h$ of $T^n$ and all $n$. Let $\mathcal{H}$ be the set of inverse branches $h : I \to [c_m,c_{m+1}]$ of $T$. We suppose (Renyi’s condition) that there is $K > 0$ so that every $h \in \mathcal{H}$ satisfies $|h'| \leq K|h'|$. Let $r : I \to \mathbb{R}^+$ be so that $r|_{(c_m,c_{m+1})}$ is $C^1$, and inf $r > 0$. Assume that there is $\sigma_0 > 0$ so that $\sum_{h \in \mathcal{H}} \sup \exp(-\sigma (r \circ h)|h'|) < \infty$ for all $\sigma > \sigma_0$, and that $|r' \circ h| \cdot |h'| \leq K$ for all $h \in \mathcal{H}$. For $n \geq 1$, write $r^{(n)}(x) = \sum_{k=0}^{n-1} r(T^k)(x)$.

We study the transfer operators, indexed by $s = \sigma + it$,

$$L_s f(x) = \sum_{T(y)=x} e^{-sr(y)} \frac{f(y)}{|T'(y)|} = \sum_{h \in \mathcal{H}} e^{-sr(h(x))} |h'(x)| \cdot (f \circ h)(x),$$

acting on $C^1(I)$, with norm $\|f\| = \sup |f| + \sup |f'|$. Note that $L_s = L_{s+htop}$, where $L_s$ is the transfer operator associated to the suspension semi-flow on the branched surface $\{(x,s) \in I \times \mathbb{R}^+ \mid s \leq r(x)\}$, with $(x,r(x)) \sim (T(x),0)$, defined by $\phi^t(x,s) = (x,s+t)$, and $htop$ is the topological entropy of $\phi^t$. See e.g. [5].

Finally, the following assumption is a translation of Dolgopyat’s ”uniform non-integrability of foliations” condition (see [11] for formulations closer to ours): we say that the pair $(T,r)$ satisfies UNI if there exist $D > 0$ and $n_0 \geq 1$ such that, for every integer $n \geq n_0 \geq 1$, there are two elements $h$, $k$ of the set $\mathcal{H}_n$ of inverse branches of $T^n$ so that the function on $I$ defined by $\psi_{h,k}(x) := r^{(n)}(h(x)) - r^{(n)}(k(x))$ satisfies $\inf |\psi_{h,k}| \geq D$. (See also Remark 2.5.4)

To state our main result, we use the equivalent norms $\|f\|_{1,t} = \sup |f| + \sup |f'|$ for $|t| \geq \epsilon_0 > 0$, on $C^1(I)$:

Theorem 1.1. Let $T$ and $r$ satisfy the assumptions above (in particular UNI for $D$). Then there is $A \geq n_0$ and $\gamma < 1$ so that for all $\sigma$ close enough to 0, all $|t| \geq \max(2\pi/D,4)$, and all $n \geq A \log |t|$, we have $\|L^n_s\|_{1,t} \leq \gamma^n$.
Theorem 1.1 was proved by Dolgopyat [3] when \( \mathcal{H} \) is finite. In [2], we considered the special case when \( T(x) = \{1/x\} \) (or analogues of the Gauss map) and \( r = \log |T'| \), working with a different version of UNI, adapted to “algebraic” situations. Note that the present UNI assumption also holds in the setting of [2]: if \( h \in \mathcal{H}_n \) is a linear fraction \((ax + b)/(cx + d)\) then \( h''(x)/h'(x) = -2c/(cx + d)\) so that \(|\psi_{h,h}(x)| = |2c/(cx + d) - \hat{c}/(\hat{c}x + \hat{d})|\). Write \( \mathcal{F}_n \) for the nth Fibonacci number and \( \hat{\mathcal{F}}_n \) for the sequence 0, 1, \( 2 \mathcal{F}_{n-1} + \hat{\mathcal{F}}_{n-2} \). For \( h \) and \( h \) in \( \mathcal{H}_n \) associated to the sequence of digits 1, 1, \ldots, 1, and 2, 2, \ldots, 2, we get \( a = \mathcal{F}_{n-2}, b = c = \mathcal{F}_{n-1}, \) and \( d = \mathcal{F}_n \), while \( \hat{a} = \hat{\mathcal{F}}_{n-2}, \hat{b} = \hat{c} = \hat{\mathcal{F}}_{n-1}, \) and \( \hat{d} = \hat{\mathcal{F}}_n \). We conclude by using \( \lim_{n \to \infty} \mathcal{F}_n / \hat{\mathcal{F}}_{n-1} = (1 + \sqrt{5})/2 \) and \( \lim_{n \to \infty} \hat{\mathcal{F}}_n / \mathcal{F}_{n-1} = (1 + \sqrt{8})/2 \).

From Theorem 1.1 one easily gets (see e.g. [2]):

**Corollary 1.2.** For every \( 0 < \alpha < 1 \) there is \( t_0 \) so that for all \(|t| > t_0\) and \( \sigma \) close to 0, we have \( \| (I - L_\sigma)^{-1} \|_{1,t} \leq |t|^{\alpha} \).

Theorem 1.1 implies [3] section 4] exponential decay of correlations for \( C^1 \) observables and the absolutely continuous invariant probability (SRB) measure of the semi-flow \( \phi^t \). We hope this will be a useful step towards proving exponential decay of correlations for (continuous-time) planar Sinai billiards, using [8]. (For the moment, only open continuous-time billiards have been considered [2]. They admit finite Markov sections.) See Remark 2.1 for extensions to other Gibbs states.

2. **Proof of Theorem 1.1**

We basically follow Dolgopyat’s proof, as detailed in [3]. A key point is the Federer property of any absolutely continuous measure \( \nu \) with continuous density bounded from above and from below: There are \( C, C'>0 \) so that if \( I, J \) are adjacent intervals with \(|I| \geq |J|/C \) then \( \nu(I) \geq \nu(J)/C' \). To exploit this information when considering \( L_\sigma \) for \( \sigma \neq 0 \), the arguments in [3] (e.g. last lines of p. 367) and [11] (e.g. first lines of p. 43) use that there is \( \alpha_\sigma \to 1 \) when \( \sigma \to 0 \) so that \( \tilde{L}_\sigma f(x) \leq \alpha_{\sigma} \tilde{L}_0 f(x) \), for the normalised operators in \( \tilde{L}_\sigma \) and positive \( f \). The above inequality uses that there are finitely many branches and is for example not true for the Gauss map. To bypass this problem, we exploit carefully the Cauchy-Schwartz decomposition in Lemma 2.1 below (see also [2], Lemma 2).

**Remark 2.1.** Beware that even when there are finitely many branches, the Federer property is not true for arbitrary Gibbs measures \( \nu \), in particular the measures \( \nu_{\sigma} \) introduced below for \( \sigma \neq 0 \), contrary to what is stated in [3] Proposition 7; [5] Lemma 1; and [6] Lemma 4. (Proposition 7 of [3] true e.g. if \( T \) is a \( C^2 \) circle map, and if \( r \) is \( C^1 \) on the circle, and not only piecewise \( C^1 \). For a counterexample, take \( T(x) = 2x \) modulo 1 with \( \exp(r) \equiv 3 \) on \([0,1/2]\) and \( \exp(r) \equiv 3/2 \) on \((1/2,1]\), and consider the intervals of size \( 1/2^n \) to the right and to the left of \( 1/2 \). By adding \( \epsilon \sin(2\pi x) \) to \( r \), this example can probably be made to satisfy the UNI condition [5] p. 537.) When there are finitely many branches, the Federer property does hold [1] for Gibbs measures and “most” adjacent intervals from the partitions in [3], [6], [1]. This is enough e.g. to recover the results in [3], in particular Theorem 1. When \( \mathcal{H} \) is infinite, the situation is more complicated but we expect that Theorem 1.1 will also hold for some general transfer operators \( L_{s,g} f(x) = \sum_{T(y)=x} e^{-sT(y)} g(y) f(y) \) associated to suitable positive \( g \).
Preliminary steps

Fix from now on $\hat{\rho} < \rho < 1$. The inverse branches of $T^n$ satisfy $|h''| \leq K|h'|$ for all $n$ and the distortion constant $K = K/(1 - \rho)$. Similarly for $(r^{(n)})' \circ h$. As a consequence it is easy to prove that, for every $n \geq 1$, and each pair $h, k$ in $H_n$, the function $\psi_{h,k}(x) = r^{(n)} \circ h - r^{(n)} \circ k$ satisfies $\sup |\psi_{h,k}'| \leq 2K$. We next recall spectral properties of the $L_s$ (see e.g. [2] and references therein). Let $\sigma > \sigma_0$ be real. The essential spectral radius $\lambda^s_\sigma$ of $L_s$ is strictly smaller than its spectral radius $\lambda_\sigma$ (in fact $\lambda^s_\sigma \leq \rho \lambda_\sigma$). Since $T$ is topologically mixing, the operator $L_s$ has a unique (simple) eigenvalue $\lambda_\sigma$ of maximal modulus, for a strictly positive $C^1$ eigenfunction $f_\sigma$, the rest of the spectrum is in the disc of radius $\rho \lambda_\sigma$ for some $\rho < 1$. The eigenvector $\mu_\sigma$ of $L^*_\sigma$ for $\lambda_\sigma$ is Lebesgue measure for $\sigma = 0$, and for all $\sigma > \sigma_0$ a positive Radon measure $\mu_\sigma$. We may assume $\mu_\sigma(1) = 1$ and $\mu_\sigma(f_\sigma) = 1$ so that $\nu_\sigma = f_\sigma \mu_\sigma$ is a probability measure. Note that $L_s : C^1(I) \to C^1(I)$ depends continuously on $\sigma$, so that $\lambda^{s,1}_\sigma, \tau_\sigma, f^s_\sigma, f^s_\sigma$ depend continuously on $\sigma$ (and therefore satisfy uniform bounds in any compact subset $\Sigma \subset (\sigma_0, \infty)$). Also, $\sigma \mapsto \lambda_\sigma$ is a nonincreasing function. Finally, the spectral radius of $L_{\sigma + it}$ is not larger than $\lambda_\sigma$ and its essential spectral radius is not larger than $\rho \lambda_\sigma$ for all $t \in \mathbb{R}$.

It will be convenient to work with the normalised operators

$$
(2.1) \quad \overline{L}_s(f) = \frac{L_s(f_\sigma \cdot f)}{\lambda_\sigma f_\sigma}, \quad s = \sigma + it.
$$

If $s = \sigma > \sigma_0$, the operator $\overline{L}_s$ acting on $C^1(I)$ has spectral radius 1, essential spectral radius $\leq \rho$, and fixes the constant function $\equiv 1$. Clearly $\overline{L}^*_s$ preserves the probability measure $\nu_\sigma = f_\sigma \cdot \mu_\sigma$. Our starting point is a Lasota-Yorke inequality:

**Lemma 2.2. (Lasota-Yorke)** For every compact $\Sigma \subset (\sigma_0, \infty)$, there is a constant $C = C(\Sigma, K) > 0$, so that for all $n \geq 1$, all $s \in \Sigma$ and all $f \in C^1(I)$:

$$
(2.2) \quad |(\overline{L}_s^n f')(x)| \leq C(\Sigma, K) |s| \cdot \overline{L}_s^n (|f|)(x) + \rho^n \cdot \overline{L}_s^n (|f'|)(x).
$$

**Proof.** The Leibniz sum for the derivative of each term $\exp(-s(r^{(n)} \circ h))|h'| \cdot \frac{1}{\lambda_\sigma f_\sigma} \cdot (f_\sigma f) \circ h$ forming $(\overline{L}_s^n f)'$ $(h \in H_n)$ contains four terms. We can bound the first for all $s$ using our “distortion” assumption on $r$ since $|s||(r^{(n)})' \circ h||h'|e^{-s(r^{(n)}h)} \leq |s|K'e^{-s(r^{(n)}h)}$. The second one is controlled by the Renyi assumption on $T$. Compactness of $\Sigma$ and continuity of $\sigma \mapsto \lambda_\sigma$ and $\sigma \mapsto f_\sigma$ imply $\sup_{\sigma \in \Sigma} |f_\sigma'| < \infty$ and $\inf_{\sigma \in \Sigma} f_\sigma > 0$, so that the third term may be controlled by $\frac{|f_\sigma'|}{\lambda_\sigma f_\sigma} \leq C_\Sigma \frac{1}{\lambda_\sigma f_\sigma}$ for some $C_\Sigma > 0$. Finally the last term can be estimated using

$$
|(f_\sigma \cdot f)' \circ h||h'| \leq \rho^n |\overline{L}_s^n (f_\sigma \cdot f)' \circ h + (f_\sigma \cdot |f'|)' \circ h|.
$$

We can ensure $K|s| + 2C_\Sigma + 2\rho C_\Sigma \leq C(\Sigma, K)|s|$ (for fixed $\Sigma$, if $|s|$ is large, i.e., if $|t|$ is large enough, then $C(\Sigma, K)$ is close to $K$).
We next state and prove an elementary lemma about complex numbers with almost opposite phases. Note that $2/3 < (\sqrt{7} - 1)/2 < 1$.

**Lemma 2.3. (Calculus lemma)** For each $\eta \in [(\sqrt{7} - 1)/2, 1)$ and every pair of complex numbers, $z_1 = r_1 \exp(i\theta_1)$ and $z_2 = r_2 \exp(i\theta_2)$,

\[
\cos(\theta_1 - \theta_2) \leq 1/2 \Rightarrow |z_1 + z_2| \leq \max(\eta r_1 + r_2, r_1 + \eta r_2).
\]

**Proof.** Up to exchanging $z_1$ and $z_2$, we can suppose that $r_1 \leq r_2$ so that $\eta r_1 + r_2 \geq r_1 + \eta r_2$. Our assumption on $\cos(\theta_1 - \theta_2)$ implies

\[
|z_1 + z_2|^2 = r_1^2 + r_2^2 + 2r_1r_2 \cos(\theta_1 - \theta_2) \leq r_1^2 + r_2^2 + r_2 r_1.
\]

Since $(\eta r_1 + r_2)^2 = \eta^2 r_1^2 + r_2^2 + 2\eta r_1 r_2$, we must show $r_1^2(1 - \eta^2) + 2r_1 r_2(1/2 - \eta) \leq 0$.

Since $\eta - 1/2 \geq 1 - \eta^2 \geq 0$ (use $3 \geq \sqrt{7} \geq 2$), we get

\[
r_2^2(1 - \eta^2) + 2r_1 r_2(1/2 - \eta) \leq r_2^2(\eta - 1/2) + r_1 r_2(1/2 - \eta) \leq r_1(\eta - 1/2)(r_1 - r_2) \leq 0.
\]

\[\square\]

**Preparatory lemmas in view of $L^2$ contraction**

In the next lemma, we combine $\text{UNI}$ and Lemma 2.3 to obtain cancellation-type estimates on terms appearing when applying iterates of $L_{\sigma}$ to a suitable pair $(u, v)$ of initial test functions in $C^1(I)$. We first introduce the “cone” condition that $(u, v)$ must satisfy: there are $C > 0$ and $t \in \mathbb{R}$ so that

\[
u > 0, \ 0 \leq |v| \leq u, \ \max(|u'(x)|, |v'(x)|) \leq 2C |t| u(x).
\]

**Lemma 2.4. (Exhibiting cancellations)** Assume that $\text{UNI}$ holds for $D$ and $n_0$. Then, for all $C > 0$, there are $n_1 \geq n_0$, $\delta > 0$ and $\Delta > 0$, so that for any $|t| > 2\pi/D$, and all $u, v \in C^1(I)$ satisfying (2.4) for $C$ and $t$, we have the following:

Fix $n \geq n_1$, and let $h, k \in \mathcal{H}_n$ be the branches from $\text{UNI}$. For every $x_0 \in I$, there is $x_1 \in I$ with $|x_0 - x_1| < \Delta/|t|$, so that the function

\[
F(x) := e^{-\sigma t} \mathcal{F}^{(n)}(h(x)) |h'(x)| ((u \cdot f_\sigma) \circ h)(x) + e^{-\sigma t} \mathcal{F}^{(n)}(k(x)) |k'(x)| ((u \cdot f_\sigma) \circ k)(x)
\]

satisfies for all $x$ s.t. $|x - x_1| < \delta/|t|$, all $\sigma > \sigma_0$, and all $\eta > (\sqrt{7} - 1)/2$

\[
|F(x)| \leq \max\left[\eta e^{-\sigma t} |h'(x)| (u \cdot f_\sigma) \circ h)(x) + e^{-\sigma t} |k'(x)| ((u \cdot f_\sigma) \circ k)(x),
\]

\[
e^{-\sigma t} |h'(x)| (u \cdot f_\sigma) \circ k)(x) + \eta e^{-\sigma t} |k'(x)| ((u \cdot f_\sigma) \circ k)(x)\right].
\]

When the maximum in (2.5) is attained by the expression where the $\eta$ factor is attached to branch $h$ we say we are “in case $h$,” and otherwise “in case $k$.”

It follows from the proof that $n_1 \geq n_0$ so that $3 \times 16 C\rho^{n_1} < 1/24$ works. In the application of Lemma 2.3 in Lemma 2.4 we require $C \geq C(\Sigma, K)$ from Lemma 2.3.

**Proof.** Let us fix $x_0 \in I$. Assume first (this case does not require $\text{UNI}$) that

\[
|e(h(x_0))| \leq \max(u(h(x_0))/2, u(k(x_0))/2).
\]

Let us suppose the maximum is realised for $u \circ h$ (the other case is symmetric). Then it is easy to see that for any $\epsilon > 0$, if $|x - x_0| < \delta_1/|t|$, with $\delta_1(2C\rho^{n\sigma}) = \epsilon$, we have $\exp(-\epsilon) \leq \frac{u(h(x_0))}{u(h(x_0))} \leq \exp(\epsilon)$. (Use $\exp(\log u(h(x)) - \log u(h(x_0))) dx \leq \exp h(x_0) |(\log u(y))'| dy$, the assumed bound on $|u'|/u$ from 2.4, and $n \geq n_0.$)
To prove (2.6), it is then enough to check that \(|x - x_0| < \delta_1/|t|\) implies \(|v(h(x)))| < \eta' u(h(x_0))\) for some \(\eta' > 2/3\) with \(\eta' \exp(\epsilon) \leq \eta\); indeed, we would then have \(|v(h(x)))| \leq \eta' \exp(\epsilon) u(h(x)) \leq \eta u(h(x))\) whenever \(|x - x_0| < \delta_1/|t|\), so that (2.6) would hold. Assume for a contradiction that no such \(\eta'\) exists, i.e. for each \(2/3 < \eta' \leq \eta \exp(-\epsilon)\) there is \(x_1\) with \(|x_1 - x_0| \leq \delta_1/|t|\) and \(|v(h(x_1)))| \geq \eta'(u(h(x_0)))\), so that (use (2.6)) \(|v(h(x_1)))| - v(h(x_1))| \geq (\eta' - 1/2)u(h(x_0)).\) On the other hand, (2.4) and the choice of \(\epsilon\) imply that there is \(y\) with \(|y - x_0| \leq \delta_1/|t|\) so that

\[|v(h(y))) - v(h(x_1))| \leq u(h(y)))2C|t|\rho^{n_0}\delta_1/|t| \leq u(h(x_0)))e^{\epsilon}2C\rho^{n_0}\delta_1 = u(h(x_0)))e^{\epsilon},\]

a contradiction if \(\epsilon \exp(\epsilon) < 1/6\). This ends the easy case, where we can take \(x_1 = x_0\) (i.e. \(\Delta_1 = 0\)) and \(\delta_1 = \epsilon/(2C\rho^{n_0})\) for small (independently of \(u, v, C, \text{etc.}\)) \(\epsilon > 0\). (The dependence of \(\delta_1\) on \(C\) can be removed by taking large enough \(n\).)

Let us now move to the more interesting situation when

\[(2.7)\]

\[|v(h(x)))| > \max(u(h(x)))2, u(k(x)))/2.\]

We shall use UNI to show that we are in a position to apply Lemma 2.3 to the sum forming \(F(x)\), for \(x\) in an \(\delta_2/|t|\)-interval around a point \(x_1\) which is \(\Delta_2/|t|\) close to \(x_0\). Since \(f_\sigma\) is real and positive, the difference \(\theta(x)\) between the argument of the two terms of \(F(x)\) can be decomposed as \(\theta(x) = tv_\theta h_\theta(x) + \arg(v(h(x)) - \arg(v(k(x)))).\)

Let us first show the claim by assuming that we found \(\delta_2, \Delta_2\) so that \(\cos \theta(x) \leq 1/2\), for all \(x\) with \(|x - x_1| \leq \delta_2/|t|\), some \(x_1\) with \(|x_1 - x_0| < \Delta_2/|t|\), leaving the (nontrivial) proof of this fact for the end. We have \(r_1(x) = e^{-\sigma \tau^{(n)}(h(x))}h'(x)((v \cdot f_\sigma)(h(x))\text{ and }r_2(x) = e^{-\sigma \tau^{(n)}(k(x))}k'(x)((v \cdot f_\sigma)(k(x))\). Fix \(x\) with \(|x - x_1| \leq \delta_2/|t|\), and assume (the other case is analogous) that \(r_1(x) \leq r_2(x).\) Lemma 2.3 then yields the claim:

\[|F(x)| \leq \eta e^{-\sigma \tau^{(n)}(h(x))}h'(x)((v \cdot f_\sigma)(h(x)) + e^{-\sigma \tau^{(n)}(k(x))}k'(x)((v \cdot f_\sigma)(k(x))\]

\[\leq \eta e^{-\sigma \tau^{(n)}(h(x))}h'(x)((u \cdot f_\sigma)(h(x)) + e^{-\sigma \tau^{(n)}(k(x))}k'(x)((u \cdot f_\sigma)(k(x))\].

It remains to prove that \(\cos \theta(x) \leq 1/2\) for \(x\) as above and some \(\delta_2, \Delta_2\). For this, the following consequence of (2.4) and (2.6) will be helpful: for all \(y, z\) with \(|z - x_0| \leq |y - x_0| \leq \xi/|t|\)

\[|v(h(y)))| \geq |v(h(x)))| - |v(h(x))) - v(h(y)))|\]

\[(2.8)\]

\[\geq u(h(x)))/2 - \rho^{n_0}\xi/2C|t|u(h(z))\]

\[\geq u(h(x)))/(1 - \exp(\epsilon)\rho^{n_0}\xi/2C) \geq u(h(x)))/4.\]

Next observe that, because of (2.4), \(V(x) = \arg(v(h(x))) - \arg(v(k(x)))\) does not vary too much around \(x_0\). More precisely:

\[(2.9)\]

\[|V(x) - V(x_0))| = |\log|v(h(x)))|/v(k(x)))| - \log|v(h(x)))|/v(k(x)))|\]

\[\leq |\log|v(h(x)))|/v(h(x)))| + |\log|v(k(x)))|/v(k(x)))|,\]

and, if \(|x - x_0| \leq \xi/|t|\),

\[|\log|v(h(x)))|/v(h(x)))| \leq h(x) - h(x_0)|v(h(y)))|/v(h(y)))| \leq \rho^n\xi/2C|t|e^\epsilon u(h(x_0)))e^\epsilon u(h(x))) \leq 8C\rho^n e^\epsilon.\]
(We used $|y-x_0| \leq |x-x_0|$ and (2.8). We may control $|\log(\frac{\|x\|}{\|x_0\|})|$, mutatis
mutandis, and we have for $|x-x_0| < \xi/|t|$;

\begin{equation}
|V(x) - V(x_0)| \leq \xi 16 C \exp(\epsilon) \rho^n.
\end{equation}

Recall that we have to show $\cos \theta(x) \leq 1/2$ in a suitable interval. We first find
$x_1$ with $|x_1 - x_0| < \Delta_2/|t|$ such that $|\theta(x_1) - \pi| \leq \pi/24$. For this, we use UNI which
ensures that, since $t(\psi(z) - \psi(x_0)) = t(z-x_0)\psi'(y)$ for $y \in I$, if $\Delta_2 = 2\pi/D$, then
$\{|t(\psi(z) - \psi(x_0))| \mod 2\pi : |z-x_0| \leq \Delta_2/|t|\} = [0, 2\pi)$. (We use here $|t| > 2\pi/D$.)
In particular there is $z = x_1$ so that $t(\psi(x_1) - \psi(x_0)) = \pi - \theta(x_0)$ (mod $2\pi$).

Applying (2.10) to $x = x_1$, $\xi = \Delta_2$, we find

\begin{equation}
|\theta(x_1) - \pi| = |\theta(x_0) + t(\psi(x_1) - \psi(x_0)) + (V(x_1) - V(x_0)) - \pi|
\leq |V(x_1) - V(x_0)| < \Delta_2 16 C \exp(\epsilon) \rho^n < \pi/24,
\end{equation}

if $n$ is large enough (depending on $C$ and, via $\Delta_2$, on $D$).

To conclude, we apply (2.10) and the “distorsion” upper bound, using $|x-x_0| < |x-x_1| + |x_1-x_0| < (\delta_2 + \Delta_2)/|t|$ and $|x-x_1| < \Delta_2/|t|$ to get, if $n$ is large enough
(depending on $C$ and $D$) and $0 < \delta_2 \leq \Delta_2$ is small enough (depending on $K$):

\begin{equation}
|\theta(x) - \pi| \leq \pi/24 + |\theta(x) - \theta(x_1)|
\leq \pi/24 + |t||\psi(x) - \psi(x_1)| + |V(x) - V(x_0)| + |V(x_1) - V(x_0)|
\leq \pi/24 + 2\delta_2 + 16 C \exp(\epsilon) \rho^n D|t|\frac{\delta_2 + 2\Delta_2}{|t|} < \pi/12.
\end{equation}

Taking $\delta = \min(\delta_1, \delta_2)$ and $\Delta = \Delta_2$, we have proved the lemma. \hfill \square

Remark 2.5. If we replace UNI by the assumption that there exist $D > 0$, $n_0$, and
two inverse branches $h$ and $k$ of $T^{n_0}$ so that $\inf |\psi_{h,k}^{\prime}| \geq D$, then for every $n \geq n_0$
there are $\hat{h}, \hat{k} \in \mathcal{H}_n$ so that $\inf |\psi_{h,k}^{\prime}| \geq \rho^{n-n_0} D$. (Take $\hat{h} = h \circ \ell$, $\hat{k} = k \circ \ell$, for
$\ell \in \mathcal{H}_{n-n_0}$ and observe that $\psi_{\hat{h},\hat{k}}(x) = \psi_{h,k}(\ell(x))$.) However, this is not enough.

In (2.11) we would get (in view of the definition of $\Delta_2$)
\begin{equation}
\frac{2\pi}{\rho^n} 8 C \exp(\epsilon) \rho^n = \frac{8 \pi}{\rho^n} \exp(\epsilon) \rho^n,
\end{equation}
which is independent of $n$ and not necessarily smaller than $\pi/24$.

(The constant 16 can be reduced, but not below 1.) Unfortunately, the strategy presented on p. 545 of [6]
seems to suffer from the same problem.

The following consequence of Lemma 2.4 will be instrumental towards Lemma 2.8.

Corollary 2.6. Let $T$ satisfy UNI for $D$. Let $C > 0$ and let $n_1 = n_1(C)$, $\delta = \delta(C)$,
$\Delta = \Delta(C)$ be given by Lemma 2.4. Fix $n \geq n_1$, let $h, k \in \mathcal{H}_n$ come from UNI, and
let $\rho_n, C = \min(\min |h^{\prime}|, \min |k^{\prime}|) (we have $0 < \rho_n, C \leq \rho^n$).

Then for every $|t| > 2\pi/D$, every $u$, $v \in C^1(I)$ satisfying (2.4) for $C$ and $|t|$, and each $\eta > (\sqrt{7} - 1)/2$ (recall Lemma 2.3), there are:

- a finite set of (disjoint) intervals $[a_j, b_j + 1] = I_j \subset I$, $j = 0, \ldots, N - 1$, with
  $|I_j| \geq \delta/|t|$, $a_0 \leq \Delta/|t|$, and $b_N \geq 1 - \Delta/|t|$; also, setting $J_j = [b_j, a_j]$, we have
  $0 < |J_j| \leq 2\Delta/|t|$; to each $I_j$ is associated type$(I_j) \in \{h, k\}$; we write $\hat{I}_j$ for the
  middle third interval of $I_j$;
Lemma 2.7. (Invariance of “cone condition”) Let $T$ satisfy UNI for $D$ and fix $\Sigma$ a compact subset of $(\sigma_0, \infty)$. Let $C(\Sigma, K)$ be from Lemma 2.2 and fix $C > 1$ so that: $C \geq C(\Sigma, K) \cdot \max(1, \max_{\sigma \in \Sigma} |\sigma|D/(2\pi))$.

Then, there is $n_2 \geq n_1$ ($n_1$ from Lemma 2.4) so that for every large enough $|t| > 2\pi/D$, each $u, v$, satisfying [2.4] for $C$ and $t$, and all $n \geq n_2$, taking $\eta = \eta(n) < 1$ close enough to 1, and $\chi = \chi(u, v, \eta)$ from Corollary 2.6 the pair $\hat{\nu} = \tilde{L}_n^\sigma(\chi u)$, $\tilde{\nu} = \tilde{L}_n^\sigma(v)$, satisfies [2.4], for the same $|t|$ and $C$, and for all $s = \sigma + it$ with $\sigma > \sigma_0$, we have $|\tilde{L}_n^\sigma(v)(x)| \leq \tilde{L}_n^\sigma(\chi u)(x)$, $\forall x \in I$.

Proof. Corollary 2.6 says that $|\hat{\nu}(x)| = |\tilde{L}_n^\sigma(v)(x)| \leq \tilde{L}_n^\sigma(\chi u)(x) = \hat{\nu}(x)$ for all $x \in I$. We also have inf $\hat{\nu} > 0$ since inf$(\chi u) > 0$ and $\tilde{L}_\sigma$ preserves the cone of strictly positive functions. To check the condition on $\max(|\hat{\nu}|, |\tilde{\nu}|)$ we shall (finally!) invoke the Lasota-Yorke inequality from Lemma 2.2 (recalling also that $\tilde{L}_\sigma$ is normalised so that sup $L_\sigma f \leq \sup |f|$). We first consider $\hat{\nu}$ and get, using $|u'| \leq 2C|t|u$, $\chi \geq \eta$ and $|\chi'| \leq 1$ ($\eta = \eta(C, n)$ is close to 1):

$$\left| \frac{d}{dx} L_n^\sigma(\chi u)(x) \right| \leq C(\Sigma, K)\sigma L_n^\sigma(\chi u)(x) + \rho^n L_n^\sigma(|\chi'| u + \chi u')(x),$$

if $n \geq n_2 \geq n_1$ and $C \geq C(\Sigma, K)$.

The computation for $|\tilde{\nu}|$ is similar:

$$\left| \frac{d}{dx} L_n^\sigma(v)(x) \right| \leq C(\Sigma, K)|s|L_n^\sigma(|v|)(x) + \rho^n L_n^\sigma(|v'|)(x)$$

$$\leq \frac{C(\Sigma, K)|s| + 2C|t|\rho^n L_n^\sigma(\chi u)(x)}{\eta} \leq 2C|t|\tilde{L}_n^\sigma(\chi u)(x),$$

if $n \geq n_2 \geq n_1$ and $C(\Sigma, K)|s| \leq C|t|$. \hfill $\Box$

Proof of the $\mathcal{L}^2$ contraction and proof of Theorem 1.1

We shall see below that the case $\sup|f'| > 2C|t|\sup |f|$ is easy. We next prove the key “$\mathcal{L}^2$ contraction lemma” (see [3] Lemma 4) to handle the other case:

Lemma 2.8. ($C^2(\nu)$ contraction) Assume UNI. Let $\Sigma$, $C$, $n \geq n_2$, $|t| > 2\pi/D$, be as in Lemma 2.7. There is $\beta < 1$ so that for all $\sigma$ close enough to 0, and for all $0 \neq f \in C^1$ with $|f'| \leq 2C|t|\sup |f|$, $\forall m \geq 1$,

$$\int |\tilde{L}_{\sigma+it}^m f|^2 \, dv_0 < \beta^m \sup |f|^2, \forall m \geq 1.$$
Lemma 2.7 implies that all \((u_m, v_m)\) satisfy (2.4) for \(C, t,\) and all \(m.\) (Note also that \(u_m \leq 1\) for all \(m.\)) In particular, \(|L_m^n(f/\sup |f|)| = |v_m| \leq u_m,\) and to prove the lemma, it is enough to show that there is \(\beta_1 < 1,\) so that \(\int u_{m+1}^2 \, dv_0 \leq \beta_1 \int u_m^2 \, dv_0\) for all \(m\) (note that \(\int u_0^2 \, dv_0 = 1).\)

The definition of \(u_{m+1}\) and the Cauchy-Schwartz inequality imply

\[
\lambda_0^{2n} f_0^2(x) u_{m+1}^2(x) = \left( \sum_{\ell \in H_n} e^{-2\sigma \ell(n) \ell(x)} |\ell'(x)| (\chi_\sigma \cdot f_{\sigma} \cdot u_m)(\ell(x)) \right)^2 \\
\leq \max_{l} \frac{f_0}{f_0} \sum_{\ell \in H_n} |\ell'(x)| (f_0 \cdot u_m^2)(\ell(x)) \\
\cdot \max_{l} \frac{f_\sigma}{f_\sigma} \sum_{\ell \in H_n} e^{-2\sigma \ell(n) \ell(x)} |\ell'(x)| (\chi_\sigma \cdot f_{\sigma})(\ell(x)).
\]

Now, if \(x \in \hat{J}_j\) for \(\chi_m,\) of type \(h,\) say (type \(k\) is similar), we get

\[
\frac{1}{\lambda_0^{2n} f_0^2(x)} \sum_{\ell \in H_n} e^{-2\sigma \ell(n) \ell(x)} |\ell'(x)| (\chi_\sigma^2 \cdot f_{\sigma})(\ell(x)) \\
\leq 1 - (1 - \eta^2) e^{-2\sigma \ell(n) h(x)} |h'(x)| f_{\sigma}(h(x)) / (\lambda_0^{2n} f_{\sigma}(x)) \leq 1 - \epsilon (1 - \eta^2) = \eta' < 1
\]

(we used \(e^{-2\sigma \ell(n) h(x)} |h'(x)| f_{\sigma}(h(x)) / (\lambda_0^{2n} f_{\sigma}(x)) \geq \epsilon > 0\) if \(n\) and \(h\) are fixed; obviously, \(\eta'\) depends on \(n).\) Denote

\[
\xi(\sigma, n) = \frac{\lambda_0^{2n} f_0(x) f_{\sigma}(x)}{\lambda_0^{2n} f_0^2(x)} \cdot \max_{l} \frac{f_0}{f_0} \cdot \max_{l} \frac{f_\sigma}{f_\sigma}.
\]

We showed that for some \(\eta' < 1\) and all \(x \in \hat{J}_j\) (recall \(\lambda_0 = 1)\)

\[
u_{m+1}^2(x) \leq \eta' \xi(\sigma, n) \bar{L}_0^n(u_m^2)(x).
\]

If \(x \notin \cup_j \hat{J}_j,\) the Cauchy-Schwartz inequality just gives, since \(\chi_m \leq 1,\)

\[
u_{m+1}^2(x) \leq \xi(\sigma, n) \bar{L}_0^n(u_m^2)(x).
\]

We claim that there is \(\hat{\delta},\) independent of \(m, n,\) and \(t,\) so that if \(\hat{J}_j\) is the union of the rightmost third of \(I_j, J_j,\) and the leftmost third of \(\hat{J}_{j+1},\) then

\[
(2.15) \quad \int_{\hat{J}_j} \bar{L}_0^n(u_m^2) \, dv_0 \geq \hat{\delta} \cdot \int_{J_j} \bar{L}_0^n(u_m^2) \, dv_0.
\]
We finish the proof assuming (2.13); if \( \tilde{\delta}(\beta_2 - \eta') \geq (1 - \beta_2) \) (e.g., \( \beta_2 = \frac{1+\eta'\delta}{1+\delta} < 1 \)),
\[
\int_I u_{m+1}^2 \, d\nu_0 \leq \xi(\sigma, n) \sum_j \left( \eta' \int_{I_j} \tilde{L}_0^n(u_m^2) \, d\nu_0 + \int_{I_j} \tilde{L}_0^n(u_m^2) \, d\nu_0 \right)
\leq \xi(\sigma, n)\beta_2 \left( \sum_j \int_{I_j} \tilde{L}_0^n(u_m^2) \, d\nu_0 + \int_{I_j} \tilde{L}_0^n(u_m^2) \, d\nu_0 \right)
= \xi(\sigma, n)\beta_2 \int_I \tilde{L}_0^n(u_m^2) \, d\nu_0 = \xi(\sigma, n)\beta_2 \int_I u_m^2 \, d\nu_0.
\]
(In the last line we used that the dual of \( \tilde{L}_0^n \) leaves \( \nu_0 \) fixed.) By taking \( \sigma \) sufficiently close to 0 (depending on \( n \), which is fixed) we can assume that \( \xi(\sigma, n)\beta_2 < 1 \).

It remains to show (2.15). It suffices to prove that \( \int_I u^2 \, d\nu_0 \geq \tilde{\delta} \int_I u^2 \, d\nu_0 \) for all \( C^1 \) functions \( u \) with \( |u'| \leq 2C|t|u(z) \) (recall Lemma 2.7 and use Lemma 2.2 and \( \bar{L}_0^n \equiv 1 \)). Note that such \( u \) satisfy, for all \( x \in I_j, y \in \bar{J}_j \):
\[
w^2(y) = exp2(log w(x) - log w(y)) = exp2 \int_x^y (w'/w)(z) \, dz \leq exp(4C(2\Delta + \delta)).
\]
Applying the above inequality, and making use of the Federer property (for intervals with length-ratio \( 3\Delta \)), of \( \nu_0 \) which has density \( f_0 \) (bounded from above and from below) with respect to Lebesgue measure, we find
\[
\int_{I_j} u^2 \, d\nu_0 \geq \nu_0(I_j) \min_{I_j} u^2 \geq \tilde{\delta} e^{-4C(2\Delta + \delta)} \nu_0(I_j) \max_{I_j} u^2 \geq \tilde{\delta} \int_{I_j} u^2 \, d\nu_0.
\]
\[\square\]

We are finally ready to prove the theorem:

**Proof.** Since there is \( B \) so that \( (\lambda_\sigma \text{ is semisimple}) \|L_\sigma^n\|_{1,t} \leq B\lambda_\sigma^n \|\bar{L}_\sigma^n\|_{1,t} \) for all \( n \geq 1 \), and since \( \lambda_0 = 1 \) and \( \sigma \) is a neighbourhood of 0, it is enough to show that there is \( \sigma \) and \( \hat{\gamma} < 1 \) so that \( \|\tilde{L}_0^n\|_{1,t} \leq \gamma^n \), for \( n \geq A \log |t| \). Clearly, this will follow from the existence of \( n_4 \) and \( \hat{\gamma} \) so that \( \|\tilde{L}_0^n\|_{1,t} \leq \gamma^n \) for all \( m \geq A \log |t| \) (write \( n = qn_4 + r \), with \( q, r \in \mathbb{Z}^+ \) and \( 0 \leq r < n_4 \), and use \( \|\tilde{L}_0^n\|_{1,t} \leq \bar{B} \)).

Let (see Lemma 2.1) \( C = \max(3/2, C(\Sigma, \bar{K}), \max(1, D/(2\pi))) \), and let \( n_2 \) be given by Lemma 2.1. Let \( n_3 \geq n_2 \) be so that \( \rho^{n_3} < 1/4 \).

Let us first deal with the easy case \( |f'| \geq 2C|t| \sup |f| \). Setting \( \gamma_1 = \max((2C|t|)^{-1}, \rho^{n_3} + 3/4) < 1 \), we have \( \sup |\tilde{L}_0^n\| \leq \sup |f| \leq \frac{1}{2C|t|} \sup |f'| \leq \gamma_1 \|f\|_{1,t} \), and, by Lemma 2.2,
\[
\left| \frac{(\tilde{L}_0^n f')'}{|t|} \right| \leq C(\Sigma, \bar{K}) \frac{|s|}{|t|} \sup |f| + \rho^{n_3} \sup |f'|
\leq \left( \sqrt{\max(|\sigma|^2 + |t|^2)} + \rho^{n_3} \right) \frac{\sup |f'|}{|t|} \leq \gamma_1 \|f\|_{1,t}.
\]

If \( \sup |g'| < 2C|t| \sup |g| \), then the function \( g^2 \) satisfies (2.14) for \( 2C \max(1, \sup |g|) \) for which Lemmas 2.7, 2.8 hold. Note also that a slight modification of the Cauchy-Schwartz argument in the beginning of the proof of Lemma 2.8 yields
\[
|\tilde{L}_0^{n_3}(g)(x)|^2 \leq K \frac{\lambda_\sigma^{n_3}}{\lambda_\sigma^{n_3}} \tilde{L}_0^{n_3}(g^2)(x),
\]
for some $K$ independent of $mn_3$ and $f$. Next, assume $\sup |f| < 2C|t|$ and assume $\|f\|_{1,t} = 1$. By the spectral properties of $\tilde{L}_0$ on the space of Lipschitz functions endowed with the norm $\sup |g| + \text{Lip}(g)$ (with $\text{Lip}(g)$ the smallest Lipschitz constant of $g$), there are $R_\sigma < \infty$, $\tau_\sigma^L < 1$ (independent of $f$ and $t$), with:

$$\sup |\tilde{L}_s^{2mn_3}(f)|^2 \leq \sup |\tilde{L}_s^{mn_3}(\tilde{L}_s^{mn_3}(f))|^2 \leq K\frac{\lambda_{2}\lambda_{mn_3}}{\lambda_{\sigma}^2}\sup \tilde{L}_0^{mn_3}(|\tilde{L}_s^{mn_3}(f)|^2)$$

$$\leq K\frac{\lambda_{2}\lambda_{mn_3}}{\lambda_{\sigma}^2}\left(\int |\tilde{L}_s^{mn_3}(f)|^2 \, dv_0 + R_\sigma(\tau_\sigma^L)^{mn_3}\sup \text{Lip}(|\tilde{L}_s^{2mn_3}(f)|^2)\right)$$

$$\leq K\frac{\lambda_{2}\lambda_{mn_3}}{\lambda_{\sigma}^2}\left(\sup |f|^2 \beta^m + R_\sigma(\tau_\sigma^L)^{mn_3}\sup \text{Lip}(|\tilde{L}_s^{mn_3}(f)|^2)\right)$$

using Lemma 2.8 for $n = n_3$ and Cauchy-Schwartz). Lemma 2.7 gives

\begin{equation}
\sup |\tilde{L}_s^{mn_3}(f)|^2 = \sup |f|^2|v_m|^2 \leq \sup |f|^2|u_m|^2 \leq \|f\|^2 \leq 1,
\end{equation}

and $\text{Lip}(\tilde{L}_s^{mn_3}(f))^2 \leq 2\sup |f|^2 \cdot \sup |f| \sup |v_m'| \leq 2\sup |f|^22C|t| \leq 4C|t|$, since $\text{Lip}(|v_m'|) \leq \text{Lip}(|v_m|) = \sup |v_m'|$.

In order to find $\max(\beta, \tau_\sigma^m) = \frac{\lambda_{2}\lambda_{mn_3}}{\lambda_{\sigma}^2} < \gamma_2^m < 1$ so that (for all $m$)

$$K\frac{\lambda_{2}\lambda_{mn_3}}{\lambda_{\sigma}^2}\cdot \left(\beta^m + R_\sigma(\tau_\sigma^L)^{mn_3}(1 + 4C|t|)\right) \leq \gamma_3^m,$$

it is enough to require $m \geq \tilde{A}\log |t|$ for some $\tilde{A} > 0$ (and $\sigma$ close enough to 0).

To control the derivative, invoke Lemma 2.2 exploiting the bounds just obtained:

$$\sup \left|\frac{(\tilde{L}_s^{2mn_3}(f))'}{|t|}\right| \leq \frac{C(\Sigma, \tilde{K})}{|t|}\sup (\tilde{L}_s^{2mn_3}|f|) + \frac{\rho^{mn_3}}{|t|}\sup (\tilde{L}_s^{2mn_3}|(f')|)$$

$$\leq \frac{C(\Sigma, \tilde{K})}{|t|}\gamma_2^m + 2C\rho^{mn_3}\sqrt{K}\frac{\lambda_{2}\lambda_{mn_3}}{\lambda_{\sigma}^2} \leq \gamma_3^m.$$

Take $n_4 = 2n_3$ and large enough $A \geq \tilde{A}$. 

□

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