Quark Mass Matrices in the $A_4$ Model

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Abstract

If the standard model of quark interactions is supplemented by a discrete $A_4$ symmetry (which may be relevant for the lepton sector), the spontaneous breaking of the electroweak gauge symmetry allows arbitrary quark masses, but all mixing angles are predicted to be zero. A pattern of the explicit breaking of $A_4$ is proposed, which results in a realistic charged-current mixing matrix.
In the standard model of quark interactions, the mass matrices of the *up* and *down* sectors are diagonalized separately, with unitary matrices $V_u$ and $V_d$ for $(u, c, t)_L$ and $(d, s, b)_L$ respectively. The observed charged-current mixing matrix is then given by

$$V_{CKM} = V_u^\dagger V_d.$$  

(1)

There are two important issues to be understood here. (I) Why are quark masses hierarchical in each sector? and (II) why is $V_{CKM}$ almost the identity matrix? In this short note, there is no proposed answer to (I), but given (I), there is a possible explanation of (II) in the context of a recently proposed model [1] of nearly degenerate Majorana neutrino masses, using the discrete symmetry $A_4$.

There are 4 irreducible representations of $A_4$, i.e. $\mathbf{1}$, $\mathbf{1}'$, $\mathbf{1}''$, and $\mathbf{3}$, with the decomposition

$$
\mathbf{3} \times \mathbf{3} = \mathbf{1} + \mathbf{1}' + \mathbf{1}'' + \mathbf{3} + \overline{\mathbf{3}}.
$$

(2)

In particular,

$$
\mathbf{1} = a_1 a_2 + b_1 b_2 + c_1 c_2, \\
\mathbf{1}' = a_1 a_2 + \omega^2 b_1 b_2 + \omega c_1 c_2, \\
\mathbf{1}'' = a_1 a_2 + \omega b_1 b_2 + \omega^2 c_1 c_2,
$$

(3) \hspace{1cm} (4) \hspace{1cm} (5)

where the components of $\mathbf{3}$ are denoted by $(a, b, c)$ and the complex number $\omega$ is the cube root of unity, i.e. $e^{2\pi i/3}$. Hence $1 + \omega + \omega^2 = 0$.

Under $A_4$, the quarks are assumed to transform as follows.

$$
(u_i, d_i)_L \sim \mathbf{3}, \\
\begin{align*}
u_{1R}, \ d_{1R} & \sim \mathbf{1}, \\
u_{2R}, \ d_{2R} & \sim \mathbf{1}', \\
u_{3R}, \ d_{3R} & \sim \mathbf{1}'',
\end{align*}
$$

(6) \hspace{1cm} (7) \hspace{1cm} (8) \hspace{1cm} (9)
in exact analogy with the left-handed lepton doublets and the right-handed charged-lepton
singlets as proposed previously [1]. The same three Higgs scalar doublets

$$\Phi_i = (\phi_i^+, \phi_i^0) \sim \mathbf{3}$$

are also used. Consequently, the Lagrangian of this model contains the following invariant
Yukawa terms:

$$\mathcal{L}_Y = h^u_{ijk} (u_i, d_i)_L \tilde{\Phi}_k + h^d_{ijk} (u_i, d_i)_L d_R \Phi_k + H.c.,$$

where \(\tilde{\Phi}_k = (\phi_k^0, -\phi_k^-)\), and

$$h^u_{11k} = h^u_1 \delta_{ik},$$
$$h^u_{22k} = h^u_2 \delta_{ik} \omega^{-1},$$
$$h^u_{33k} = h^u_3 \delta_{ik} \omega^{1-i}.$$

As \(\phi_i^0\) acquire nonzero vacuum expectation values \(v_i\), the quark mass matrices are of the
form

$$\mathcal{M}_{u,d} = \begin{bmatrix}
h^u_1 v_1 & h^u_2 v_1 & h^u_3 v_1 \\
h^u_1 v_2 & h^u_2 v_2 \omega & h^u_3 v_2 \omega^2 \\
h^u_1 v_3 & h^u_2 v_3 \omega^2 & h^u_3 v_3 \omega
\end{bmatrix}.$$\hspace{1cm}(15)

If the Higgs potential is invariant under \(A_4\), it has been shown [1] that \(v_1 = v_2 = v_3 = v\)
is a possible solution. In that case, \(\mathcal{M}_{u,d}\) is easily diagonalized, i.e.

$$\frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix} \mathcal{M}_{u,d} = \begin{bmatrix} \sqrt{3} h^u_1 v & 0 & 0 \\ 0 & \sqrt{3} h^u_2 v & 0 \\ 0 & 0 & \sqrt{3} h^u_3 v \end{bmatrix}.\hspace{1cm}(16)$$

This allows quark (and charged-lepton) masses to be hierarchical, even though neutrino
masses are degenerate [1]. Since both \(\mathcal{M}_u\) and \(\mathcal{M}_d\) are diagonalized by the same unitary
matrix, this model predicts \(V_{CKM} = 1\) at this level, which is a good answer to (II).
To obtain a realistic $V_{CKM}$, the $A_4$ symmetry must be broken. Here the simple assumption is to add terms in $\mathcal{L}_Y$ of Eq. (11) which are not just $1$ under $A_4$, but also $1'$ and $1''$, such that

$$|h''_i| << |h'_i| << |h_i|$$

for each $i$. In that case, the right-hand side of Eq. (16) becomes proportional to

$$\begin{bmatrix}
  h_1 & h'_2 & h''_3 \\
  h''_1 & h_2 & h'_3 \\
  h'_1 & h''_2 & h_3
\end{bmatrix},$$

where the superscript $(u, d)$ has been dropped for simplicity. Note also that $|h_1| << |h_2| << |h_3|$ in each sector because they are proportional to $(m_u, m_c, m_t)$ or $(m_d, m_s, m_b)$. By rotating $q_i R$, the above matrix may be written as

$$\begin{bmatrix}
  h_1 & h'_2 & h''_3 \\
  0 & h_2 & h'_3 \\
  0 & 0 & h_3
\end{bmatrix},$$

(19)

to a very good approximation, because $|h''_1| << |h_2|$, or $|h'_3|$, and $|h'_1|, |h''_3| << |h_3|$. As a result, $V_{CKM}$ differs from the identity matrix by small amounts, i.e. [2]

$$V_{us} \simeq \frac{h'_2}{h_2}, \; V_{ub} \simeq \frac{h''_3}{h_3}, \; V_{cb} \simeq \frac{h'_1}{h_3}.$$  \hspace{1cm} (20)

This explains why each is small, as well as why $|V_{ub}| << |V_{cb}|$.

Consider now the 3 Higgs doublets of this model. Let

$$\begin{bmatrix}
  \Phi \\
  \Phi' \\
  \Phi''
\end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix}
  1 & 1 & 1 \\
  1 & \omega & \omega^2 \\
  1 & \omega^2 & \omega
\end{bmatrix} \begin{bmatrix}
  \Phi_1 \\
  \Phi_2 \\
  \Phi_3
\end{bmatrix},$$

(21)

then $\Phi$ has the properties of its standard-model counterpart, whereas $\Phi'$ and $\Phi''$ are degenerate in mass [1]. Since only $\Phi$ has a nonzero vacuum expectation value, flavor-changing neutral currents are absent at tree level as far as $\Phi$ is concerned. However, $\Phi'$ and $\Phi''$ have
the following predicted Yukawa interactions:

\[
L = \left( \frac{m_t}{v}(u, d)_{L}t_{R} + \frac{m_c}{v}(t, b)_{L}c_{R} + \frac{m_u}{v}(c, s)_{L}u_{R} \right) \Phi' + \left( \frac{m_t}{v}(c, s)_{L}t_{R} + \frac{m_c}{v}(u, d)_{L}c_{R} + \frac{m_u}{v}(t, b)_{L}u_{R} \right) \Phi'' + \left( \frac{m_b}{v}(u, d)_{L}b_{R} + \frac{m_s}{v}(t, b)_{L}s_{R} + \frac{m_d}{v}(c, s)_{L}d_{R} \right) \Phi' + \left( \frac{m_b}{v}(c, s)_{L}b_{R} + \frac{m_s}{v}(u, d)_{L}s_{R} + \frac{m_d}{v}(t, b)_{L}d_{R} \right) \Phi'' + H.c.
\]

This shows that flavor-changing neutral currents involving only the first 2 families are suppressed. In fact, the most severe constraint comes from $B^0 - \overline{B^0}$ mixing which occurs through $(\phi')^0$ exchange:

\[
\frac{\Delta m_{B^0}}{m_{B^0}} \approx \frac{G_F m_b^2}{4\sqrt{2}} B_B f_B^2 \left( \frac{1}{m_R^2} - \frac{1}{m_I^2} \right),
\]

where $G_F/\sqrt{2} = 1/12v^2$ and $m_{R,I}$ are the masses of the real and imaginary parts of $(\phi')^0$. Using $f_B = 170$ MeV, $B_B = 1.0$, $m_b = 4.2$ GeV, and the experimental value [3] of $5.9 \times 10^{-14}$ for the above fraction, the condition

\[
(m_R^{-2} - m_I^{-2})^{-1/2} >> 4.22 \text{ TeV}
\]

is obtained. This means that $m_R$ and $m_I$ should be almost equal, if each is of order a few hundred GeV.

In conclusion, it has been shown how a realistic $V_{CKM} \simeq 1$ may be obtained in the context of the $A_4$ model of nearly degenerate Majorana neutrino masses. It has specific verifiable predictions as given by Eq. (22). In particular, the model requires $(\phi')^\pm$, $(\phi'')^\pm$ to have the same mass, $\text{Re}(\phi')^0$, $\text{Re}(\phi'')^0$ to have the same mass, and $\text{Im}(\phi')^0$, $\text{Im}(\phi'')^0$ to have the same mass. Phenomenologically, the latter two pairs should also have almost the same mass, assuming that each is of order a few hundred GeV.

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