STABILITY OF STOCHASTIC HEROIN MODEL WITH TWO DISTRIBUTED DELAYS

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(Communicated by María J. Garrido-Atienza)

ABSTRACT. In this paper a stability of stochastic heroin model with two distributed delays is studied. Precisely, the deterministic model for dynamics of heroin users is extended by random perturbation that briefly describe how environmental fluctuations lead an individual to become a heroin user. By using a suitable Lyapunov function stability conditions for heroin use free equilibrium are obtained. Furthermore, asymptotic behavior around the heroin spread equilibrium of the deterministic model is investigated by using appropriate Lyapunov functional. Theoretical studies, based on real data, are applied on modeling of number of heroin users in the USA from 01.01.2014.

1. Introduction. Heroin is an illegal, highly addictive drug processed from morphine, a naturally occurring substance extracted from the seed pod of certain varieties of poppy plants [18]. Although the use of heroin in the general population is rather low, the number of people beginning to use heroin has been rising steadily since 2007. It is estimated that about 23 percent of individuals who use heroin become addicted to it [25].

The best way to keep people from becoming addicted to heroin is prevention. Staying away from heroin is the only way to positively prevent drug addiction. In order to do this scientists have developed many programs that are used as a protective means against drug abuse in families, school and other society groups. Different research-based programs [19] successfully bring down early use of heroin.

Like many other chronic diseases, addiction can be treated. A range of treatments including behavioral therapies and medications are effective in helping patients to stop using heroin and return to stable and productive lives [20]. Although behavioral and pharmacological treatments can be particularly useful when utilized alone, researches show that for some people, integrating both types of treatments is the most effective approach [21].

There are many mathematical models that have been established during the time, in order to investigate heroin epidemic. First, Macktintosh and Stewart [14] have presented an exponential model illustrating how the use of heroin spreads in epidemic fashion. Later, White and Comiskey [31] considered an ODE to analyze

2010 Mathematics Subject Classification. Primary: 60H10; Secondary: 92D30, 93E15.
Key words and phrases. Distributed delay, heroin free equilibrium, heroin spread equilibrium, Lyapunov functional, mean square stability.

The first author is supported by Grant No 174007 of MNTRS.
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the heroin model. This ODE model was updated by Mulone and Straughan [17]. They proved that the positive equilibrium of the deterministic model introduced in [31] is stable. However, the effects of the time delays are not taken into account in these papers. Considering a relapse distribution and assuming that the relapse time is not a constant, authors of papers [13] and [6] incorporated distributed delay into the ODE heroin epidemic model. Parallel with deterministic models, stochastic epidemic models with distributed time delay have been studied in [29] and [2], for instance. It is interesting to investigate the effects of two distributed delays in the modified White-Comiskey mathematical model for the dynamics of heroin users, which is similar to the popular SIR model.

In Section 2 a quick review on the main results of the paper [3] is made, where the appropriate deterministic model was introduced, and its stochastic version is presented. In Section 3 the existence and uniqueness of the global positive solution of considered system is proved and the boundedness of the solution is discussed. In Section 4, we turn our attention to establish stability properties of the considered model. In fact, by using a suitable constructed Lyapunov functionals, some sufficient stability conditions of the heroin use free equilibrium are obtained. Asymptotic behavior around the heroin spread equilibrium of the deterministic model is studied in Section 5. In Section 6, the numerical simulation of considered deterministic and stochastic system is presented. It is shown, that the stochastic model of dynamic of heroin users in the USA, with coefficients that are reliable data, is compatible with the mathematical results obtained through the paper. In order to make the theory more understandable, some comments and remarks about these results are also given.

2. The model. Throughout this section, the deterministic model for the dynamics of heroin users is described. Some detailed studies about the model may be found in White and Comiskey [31]. They developed a mathematical model of dynamics relating to heroin users, that is based on a model that explains the spread of infectious diseases. The reason for this is the fact that the spread of addiction among population is similar to spreading infectious diseases. The total number of high-risk human population at time $t$, denoted by $N(t)$, is divided into three distinct classes: susceptible individuals $S(t)$, heroin users that are not in treatment $U_1(t)$ and heroin users in treatment $U_2(t)$.

White and Comiskey model [31] is based on the ODE. In papers [13] and [6] modified White and Comiskey heroin model with one distributed time delay is formulated and time delay is used to describe the time needed for drug users to return to untreated users. To make a model more realistic, Fang et al. [3] took into account the additional time delay that is related to the time needed for a susceptible individual to become an infectious heroin user. It is assumed that both delays are finite.

The deterministic model for dynamics of heroin users, which is considered in [3], has the following form

$$\frac{dS(t)}{dt} = \Lambda - \beta S(t) \int_0^{h_1} f(\tau) e^{-(\mu + \delta_1 + p)\tau} U_1(t - \tau) d\tau - \mu S(t),$$

$$\frac{dU_1(t)}{dt} = \beta S(t) \int_0^{h_1} f(\tau) e^{-(\mu + \delta_1 + p)\tau} U_1(t - \tau) d\tau - (\mu + \delta_1 + p) U_1(t)$$

(1)
users in treatment may have relapse into the class of users not in treatment.
The last term in the model (1) is appeared as all the possible times at which heroin
∫
\[ g(t) e^{-(\mu + \delta_2)\tau} U_1(t-\tau) d\tau, \]
with the initial condition:
\[ S(0) = S^0, \quad U_1(\theta) = \varphi(\theta), \quad U_2(0) = U_2^0, \quad \theta \in [-h, 0], \quad h = \max\{h_1, h_2\}. \]

The parameters in model (1) have the following meaning:
\( \Lambda \) - the number of individuals in the general population entering the susceptible
population,
\( \beta \) - the rate of becoming a heroin user,
\( p \) - the rate of heroin users who enter treatment,
\( \delta_1 \) - a removal rate that includes heroin-related deaths of users not in treatment and
a spontaneous recovery rate: individuals not in treatment who stop using heroin
but are no longer susceptible,
\( \delta_2 \) - a removal rate that includes heroin-related deaths of users in treatment and
and a rate of successful “cure” that corresponds to recovery to a heroin free life and
immunity to heroin addiction for the duration of modeling time period,
\( \mu \) - the natural death rate of the susceptible population,
where \( \beta > 0, \delta > 0, p > 0, \mu > 0 \) and \( \delta_i > 0 \) for \( i = 1, 2 \).

In [3] authors assumed that the first time delay is the time necessary to be-
come an infectious heroin user and it is a distributed parameter over the finite
interval \([0, h_1]\), \( h_1 > 0 \). Therefore, the related incidence term, which is refer to
the bilinear or mass-action incidence, represents the force of infection and becomes
\[ \beta S(t) \int_0^{h_1} f(\tau) e^{-(\mu + \delta_1 + p)\tau} U_1(t-\tau) d\tau. \]
The function \( f \) represents the distribution of the infectivity of heroin users in susceptible individuals. Function \( f \) is non-negative, continuous function and satisfies \( \int_0^{h_1} f(\tau) d\tau = 1 \). The term \( e^{-(\mu + \delta_1 + p)\tau} \) expresses the probability that a susceptible individual will survive all the stages of progression to
become an infectious heroin user. The second time delay, which describes the time
needed for a heroin user to become an untreated heroin user once again, is consid-
ered to be the distributed parameter over the finite interval \([0, h_2]\). It is assumed
that \( g \) is the distribution function, which is non-negative, continuous and satisfies
\( \int_0^{h_2} g(\tau) d\tau = 1 \). The probability that the individuals will survive the moment they are treated until the moment they start using heroin again is \( e^{-(\mu + \delta_1)\tau} \). Therefore,
the last term in the model (1) is appeared as all the possible times at which heroin
users in treatment may have relapse into the class of users not in treatment.

The system (1) has a unique, non-negative solution \((S(t), U_1(t), U_2(t))\) for all
\( t \geq 0 \). The solutions of the system are also ultimately uniformly bounded in the
positive cone \( \Gamma \), that has a following form
\[ \Gamma = \left\{ (S, U_1, U_2) \in X \mid S(t) + U_1(t) + U_2(t) \leq \frac{\Lambda}{\mu} \right\}, \]
where \( X = R_+ \times C([-h, 0], R_+) \times R_+ \) and \( C([-h, 0], R_+) \) is the Banach space of
continuous functions mapping the interval \([-h, 0]\) into \( R_+ \) with the supremum norm
\( ||\varphi|| = \sup_{-h \leq \theta \leq 0} |\varphi(\theta)| \).

To make the calculation easier, the authors in [3] introduced the notation
\[ F(\tau) = f(\tau) e^{-(\mu + \delta_1 + p)\tau}, \quad G(\tau) = g(\tau) e^{-(\mu + \delta_2)\tau}, \]
and made some assumptions:
1. $F(\tau)$ and $G(\tau)$ are continuous on $[0, h]$.
2. $F(\tau) \geq 0$, $G(\tau) \geq 0$ for all $0 \leq \tau \leq h$ and

$$
\int_0^{h_1} F(\tau)d\tau = a, \quad \int_0^{h_2} G(\tau)d\tau = b.
$$

(2)

It is clear that $0 < a, b < 1$. The reproduction number $R_0$ of the heroin epidemic model (1) is a parameter that denotes the secondary number of heroin users produced by one heroin user, and has the following form

$$
R_0 = \frac{\beta a \Lambda}{\mu + \delta_1 + p(1 - b)}.
$$

The heroin use free equilibrium $E_0 = (A, 0, 0)$ represents the state in which the heroin users are absent and it belongs in $\Gamma$. The heroin spread equilibrium $E^* = (S^*, U_1^*, U_2^*)$, given by

$$
S^* = \frac{A}{\mu R_0}, \quad U_1^* = \frac{\mu}{\beta a}(R_0 - 1), \quad U_2^* = \frac{p(1 - b)\mu(R_0 - 1)}{\beta a(\mu + \delta_2)},
$$

is locally asymptotically stable if $R_0 > 1$. For $R_0 < 1$ the heroin use free equilibrium $E_0$ is globally asymptotically stable.

The causes of heroin addiction among individuals are different. However, researches have shown that a combination of many causes (genetic, physical, environmental) leads to explanation of how an individual becomes a heroin user and it is naturally assumed that the rate of becoming a heroin user $\beta$ is random. To describe this kind of randomness, we use stochastic model. Such a model distinctly includes all the environmental factors that could lead an individual to become a heroin user. Thus, we can construct a randomized model based on system (1) in order to find out how the aforementioned causes are affected in the given heroin model. More precisely, we establish the following stochastic system

$$
dS(t) = \left[ A - \beta S(t) \int_0^{h_1} F(\tau)U_1(t-\tau)d\tau - \mu S(t) \right] dt - \sigma S(t)U_1(t)dw_t,
$$

$$
dU_1(t) = \left[ \beta S(t) \int_0^{h_1} F(\tau)U_1(t-\tau)d\tau - (\mu + \delta_1 + p)U_1(t) + p \int_0^{h_2} G(\tau)U_1(t-\tau)d\tau \right] dt 
+ \sigma S(t)U_1(t)dw_t,
$$

$$
dU_2(t) = \left[ pU_1(t) - (\mu + \delta_2)U_2(t) - p \int_0^{h_2} G(\tau)U_1(t-\tau)d\tau \right] dt,
$$

with the initial condition:

$$
S(0) = S^0, \quad U_1(\theta) = \varphi(\theta), \quad U_2(0) = U_2^0, \quad \theta \in [-h, 0], \quad h = \max\{h_1, h_2\},
$$

(4)

where $\sigma > 0$ represents the intensity of noise, $w_t$ is a standard Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ with the filtration $\{\mathcal{F}_t\}_{t \geq 0}$, satisfying the usual conditions (it is right continuous and increasing, while $\mathcal{F}_0$ contains all $\mathcal{P}$-null sets) and $\mathcal{D}$ is the space of $\mathcal{F}_0$-adapted functions $\varphi \in \Gamma$.

3. **Existence, uniqueness and boundedness of the positive solution.** To investigate the dynamical behavior of model (3), we have to examine whether the solution has global existence. Moreover, for a population dynamics model, the positivity of solution is needed. Hence, in this section we prove that the solution of model (3) is positive and global. For a stochastic differential equation that has
a unique global (i.e. no explosion in finite time) solution for any given initial value it is needed that the coefficients of the equation satisfy the linear growth condition and local Lipschitz condition [15]. However, the coefficients of system (3) do not satisfy the linear growth condition, though they are locally Lipschitz continuous, so the solution of system (3) may explode at a finite time. We show global character for the solution of system (3).

**Theorem 3.1.** For any initial condition (4) in \( D \), system (3) has unique global solution \((S(t), U_1(t), U_2(t))\), for \( t \geq 0 \). Moreover, this solution remains in \( \Gamma \) with probability 1.

**Proof.** Since the coefficients of system (3) are locally Lipschitz continuous for any given initial condition (4) in \( D \), then there exists a unique local solution \((S(t), U_1(t), U_2(t))\) of system (3) for \( t \in [0, \tau_\varepsilon) \), where \( \tau_\varepsilon \) represents the explosion time. To show that this solution is global, we need to prove that \( \tau_\varepsilon = \infty \) a.s.

Let \( k_0 > 0 \) be sufficiently large so that \( S^0, \varphi(\theta), \theta \in [-h, 0] \) and \( U^0_2 \) lie within the interval \( \left[ \frac{1}{k_0}, \frac{1}{\mu} - \frac{1}{k_0} \right] \). For each integer \( k \geq k_0 \), define the stopping time

\[
\tau_k = \inf \left\{ t \in [0, \tau_\varepsilon) : S(t) \notin \left( \frac{1}{k}, \frac{\Lambda}{\mu} - \frac{1}{k} \right) \text{ or } U_1(t) \notin \left( \frac{1}{k}, \frac{\Lambda}{\mu} - \frac{1}{k} \right) \text{ or } U_2(t) \notin \left( \frac{1}{k}, \frac{\Lambda}{\mu} - \frac{1}{k} \right) \right\},
\]

where throughout this paper we set \( \inf \emptyset = \infty \) (\( \emptyset \) denotes the empty set).

Obviously, \( \tau_k \) is increasing as \( k \to \infty \). Set \( \tau_\infty = \lim_{k \to \infty} \tau_k \). Therefore \( \tau_\infty \leq \tau_\varepsilon \) a.s. If \( \tau_\infty = \infty \) a.s. is true, then \( \tau_\varepsilon = \infty \) a.s. and \((S(t), U_1(t), U_2(t)) \in \Gamma \) a.s for \( t \geq 0 \). Hence, it is required to show that \( \tau_\infty = \infty \) a.s. If this statement is false, then there exist a pair of constants \( T > 0 \) and \( \varepsilon \in (0, 1) \) such that \( P\{\tau_\infty \leq T\} > \varepsilon \). Consequently, for an integer \( k_1 \geq k_0 \), holds that

\[
P\{\tau_\infty \leq T\} \geq \varepsilon \quad \text{for all} \quad k \geq k_1. \tag{5}
\]

Let

\[
N(t) = S(t) + U_1(t) + U_2(t)
\]

denotes the total number of high-risk human population at time \( t \). We add all equations of system (3) and get

\[
\frac{dN(t)}{dt} = \Lambda - \mu N(t) - \delta_1 U_1(t) - \delta_2 U_2(t).
\]

For \( t \in [0, \tau_k) \) we conclude that

\[
\frac{dN(t)}{dt} < \Lambda - \mu N(t).
\]

By comparison theorem for differential equations, yields

\[
N(t) < \frac{\Lambda}{\mu} + \left( S^0 + \varphi(0) + U^0_2 - \frac{\Lambda}{\mu} \right) e^{-\mu t}, \quad t \in [0, \tau_k).
\]

Since \((S^0, \varphi(\theta), U^0_2) \in D\), we conclude that \( N(t) \leq \frac{\Lambda}{\mu} \) for \( t \in [0, \tau_k) \) almost surely.

Define a \( C^2 \)-function \( V : R^3_+ \to R_+ \) by

\[
V(S, U_1, U_2) = \frac{1}{S} + \frac{1}{\frac{\Lambda}{\mu} - S} + \frac{1}{U_1} + \frac{1}{\frac{\Lambda}{\mu} - U_1} + \ln \left( 1 + U^2_2 \right) - \ln U_2 + \frac{1}{\frac{\Lambda}{\mu} - U_2}.
\]
For simplicity denote by $I_1(t) = \int_0^{h_1} F(\tau) U_1(t-\tau) d\tau$ and $I_2(t) = \int_0^{h_2} G(\tau) U_1(t-\tau) d\tau$. Applying the Itô’s formula we obtain

$$dV(S,U_1,U_2) = LV(S,U_1,U_2)dt + \sigma SU_1 \left( \frac{1}{S^2} - \frac{1}{(\frac{\Lambda}{\mu}-S)^2} - \frac{1}{U_1^2} + \frac{1}{(\frac{\Lambda}{\mu}-U_1)^2} \right) dw_t,$$

where $LV : R_+^3 \rightarrow R_+$ has form

$$LV(S,U_1,U_2) = -\frac{\Lambda}{S^2} + \frac{\beta I_1 + \mu}{S} + \frac{\Lambda - \mu S}{(\frac{\Lambda}{\mu} - S)^2} - \frac{\beta S I_1}{S} + \frac{\sigma^2 U_1^2}{(\frac{\Lambda}{\mu} - S)^3} \frac{\beta S I_1}{U_1^2} + \frac{\mu + \delta_1 + p}{U_1} - \frac{p I_2}{U_1^2} + \frac{\beta S I_1}{U_1} - \frac{(\mu + \delta_1 + p) U_1}{(\frac{\Lambda}{\mu} - U_1)^2} + \frac{p I_2}{(\frac{\Lambda}{\mu} - U_1)^2} + \frac{\sigma^2 U_1^2}{S} + \frac{2p U_1 U_2}{1 + U_2^2} - \frac{2(\mu + \delta_2) U_2^2}{1 + U_2^2} - \frac{2p U_2 I_2 - p U_1}{U_2} + \mu + \delta_2 + \frac{p I_2}{U_2} + \frac{p U_1}{(\frac{\Lambda}{\mu} - U_2)^2} - \frac{p I_2}{(\frac{\Lambda}{\mu} - U_2)^2}.$$

Removing some nonpositive terms we have

$$LV(S,U_1,U_2) \leq -\frac{\beta I_1 + \mu}{S} + \frac{\Lambda - \mu S}{(\frac{\Lambda}{\mu} - S)^2} - \frac{\beta S I_1}{S} + \frac{\sigma^2 U_1^2}{(\frac{\Lambda}{\mu} - S)^3} \frac{\beta S I_1}{U_1^2} + \frac{\mu + \delta_1 + p}{U_1} - \frac{p I_2}{(\frac{\Lambda}{\mu} - U_1)^2} + \frac{2p U_1 U_2}{1 + U_2^2} + \mu + \delta_2 + \frac{p I_2}{U_2} + \frac{\sigma^2 U_1^2}{S} + \frac{2p U_1 U_2}{1 + U_2^2} - \frac{2(\mu + \delta_2) U_2^2}{1 + U_2^2} + \frac{p U_1}{(\frac{\Lambda}{\mu} - U_2)^2} - \frac{p I_2}{(\frac{\Lambda}{\mu} - U_2)^2}.$$

In view of $S + U_1 \leq \frac{\Lambda}{\mu}$ on $[0, \tau_k]$, we conclude that $S \leq \frac{\Lambda}{\mu} - U_1$ and $U_1 \leq \frac{\Lambda}{\mu} - S$. Applying elementary inequality $\frac{x + x^2}{1 + x^2} \leq \frac{1}{2}$ we get

$$LV(S,U_1,U_2) \leq 2\beta \frac{I_1 + \mu + 2\sigma^2 U_1^2}{S} + \frac{\mu + \delta_1 + p + 2\sigma^2 S^2}{U_1} + \frac{\mu}{\frac{\Lambda}{\mu} - S} + \frac{p I_2 - (\mu + \delta_1 + p) U_1}{(\frac{\Lambda}{\mu} - U_1)^2} + \frac{p U_1 + \mu + \delta_2}{U_2} + \frac{p}{\frac{\Lambda}{\mu} - U_2}.$$

Recalling that $S \leq \frac{\Lambda}{\mu}$ and $U_1 \leq \frac{\Lambda}{\mu}$ on $[0, \tau_k]$, yields that

$$2\beta I_1 = 2\beta \int_0^{h_1} F(\tau) U_1(t-\tau) d\tau \leq 2\beta \frac{\Lambda}{\mu} \int_0^{h_1} F(\tau) d\tau = \frac{2\beta \Lambda a}{\mu},$$

$$p I_2 = p \int_0^{h_2} G(\tau) U_1(t-\tau) d\tau \leq p \frac{\Lambda}{\mu} \int_0^{h_2} G(\tau) d\tau = p \frac{\Lambda b}{\mu},$$

and then

$$LV(S,U_1,U_2) \leq \frac{2\beta \Lambda a}{\mu} + \mu + 2\sigma^2 \left( \frac{\Lambda}{\mu} \right)^2 + \frac{\mu + \delta_1 + p + 2\sigma^2 \left( \frac{\Lambda}{\mu} \right)^2}{U_1} + \frac{\mu}{\frac{\Lambda}{\mu} - S} + \frac{p b \frac{\Lambda}{\mu} - (\mu + \delta_1 + p) U_1}{(\frac{\Lambda}{\mu} - U_1)^2} + \frac{\Lambda}{\mu} + \mu + \delta_2 + \frac{p b \frac{\Lambda}{\mu}}{U_2} + \frac{p}{\frac{\Lambda}{\mu} - U_2}.$$
By using elementary inequality \( \ln x \leq \frac{x^2-1}{2x}, \ x > 0 \) we obtain \( \frac{1}{x} \leq x + 2 \ln \frac{1+x^2}{x}. \) Since \( pb \leq \mu + \delta_1 + p \) and \( U_2 \leq \frac{\Lambda}{\mu} \) on \( [0, \tau_k] \), we conclude

\[
LV(S, U_1, U_2) \leq \frac{2\beta\Lambda}{\mu} + \mu + 2\sigma^2\left(\frac{\Lambda}{\mu}\right)^2 + \frac{\mu + \delta_1 + p + 2\sigma^2\left(\frac{\Lambda}{\mu}\right)^2}{U_1} + \frac{\mu}{\frac{\Lambda}{\mu} - S} + \frac{\mu + \delta_1 + p}{\frac{\Lambda}{\mu} - U_1}
\]

\[
+ \frac{p\Lambda}{\mu} + \mu + \delta_2 + \frac{pb\Lambda^2}{\mu^2} + 2pb\Lambda \ln \frac{1 + U_2^2}{U_2} + \frac{p}{\frac{\Lambda}{\mu} - U_2}.
\]

Let

\[
K_1 = \max\left\{\frac{2\beta\Lambda}{\mu} + \mu + 2\sigma^2\left(\frac{\Lambda}{\mu}\right)^2, \mu + \delta_1 + p + 2\sigma^2\left(\frac{\Lambda}{\mu}\right)^2, 2pb\Lambda, \frac{2p\Lambda}{\mu}\right\}, K_2 = \frac{p\Lambda}{\mu} + \mu + \delta_2 + \frac{pb\Lambda^2}{\mu^2}.
\]

Then,

\[
dV(S, U_1, U_2) \leq (K_1 V(S, U_1, U_2) + K_2) dt
\]

\[
+ \sigma SU_1 \left(\frac{1}{S^2} - \frac{1}{(\frac{\Lambda}{\mu} - S)^2} - \frac{1}{U_1^2} + \frac{1}{(\frac{\Lambda}{\mu} - U_1)^2}\right) dw_1. \tag{6}
\]

Integrating (6) from 0 to \( \tau_k \wedge T \), yields

\[
V(S(\tau_k \wedge T), U_1(\tau_k \wedge T), U_2(\tau_k \wedge T))
\]

\[
\leq V(S^0, \varphi(0), U_2^0) + \int_0^{\tau_k \wedge T} (K_1 V(S(z), U_1(z), U_2(z)) + K_2) dz
\]

\[
+ \sigma \int_0^{\tau_k \wedge T} S(z) U_1(z) \left(\frac{1}{S^2(z)} - \frac{1}{(\frac{\Lambda}{\mu} - S(z))^2} - \frac{1}{U_1^2(z)} + \frac{1}{(\frac{\Lambda}{\mu} - U_1(z))^2}\right) dw_z.
\]

By taking the expectation of both sides leads to

\[
EV(S(\tau_k \wedge T), U_1(\tau_k \wedge T), U_2(\tau_k \wedge T))
\]

\[
\leq V(S^0, \varphi(0), U_2^0) + \int_0^T (K_1 EV(S(\tau_k \wedge z), U_1(\tau_k \wedge z), U_2(\tau_k \wedge z)) + K_2) dz.
\]

The inequality of Gronwall-Bellman type [1] implies

\[
EV(S(\tau_k \wedge T), U_1(\tau_k \wedge T), U_2(\tau_k \wedge T)) \leq [V(S^0, \varphi(0), U_2^0) + K_2 T] e^{K_1 T}. \tag{7}
\]

Set \( \Omega_k = \{\tau_k \leq T\} \) for \( k \geq k_1 \). Then in view of (5), we obtain \( P(\Omega_k) \geq \varepsilon \). Note that for every \( \omega \in \Omega_k \), there is at least one of \( S(\tau_k, \omega), U_1(\tau_k, \omega) \) or \( U_2(\tau_k, \omega) \) equals either \( \frac{1}{k} \) or \( \frac{A}{\mu} - \frac{1}{k} \). Hence

\[
V(S(\tau_k), U_1(\tau_k), U_2(\tau_k))
\]

\[
\geq \left(k + \frac{1}{\mu} - \frac{1}{k}\right) \vee \left(\ln \frac{k^2 + 1}{k} + \frac{1}{\mu} - \frac{1}{k}\right) \vee \left(\ln \frac{1 + \left(\frac{A}{\mu} - \frac{1}{k}\right)^2}{k}\right).
\]

From (5) and (7) it follows that

\[
\infty > [V(S^0, \varphi(0), U_2^0) + K_2 T] e^{K_1 T} \geq E[I_{\Omega_k}(\omega) V(S(\tau_k), U_1(\tau_k), U_2(\tau_k))]
\]

\[
\geq \varepsilon \left[k + \frac{1}{\mu} - \frac{1}{k}\right] \vee \left(\ln \frac{k^2 + 1}{k} + \frac{1}{\mu} - \frac{1}{k}\right) \vee \left(\ln \frac{1 + \left(\frac{A}{\mu} - \frac{1}{k}\right)^2}{k}\right).
\]
where \( I_{\Omega_k} \) denotes the indicator function of \( \Omega_k \). Letting \( k \to \infty \), we get

\[
\infty > [V(S^0, \varphi(0), U^0) + K T] e^{K_1 T} \geq \infty,
\]

which leads to contradiction and we therefore have \( \tau_\infty = \infty \) a.s. Then it follows that \( (S(t), U_1(t), U_2(t)) \in \Gamma \) for \( t \geq 0 \) and \( \limsup_{t \to \infty} N(t) = \frac{A}{\mu} \). This completes the proof.

By the third equation of the system (3) we have

\[
U_2(t) = p \int_0^{\tau_2} g(\tau) \int_{t-\tau}^t e^{-(\mu+\delta_2)(t-s)} U_1(s) ds \, d\tau > 0, \quad t \geq 0
\]

and

\[
U_2(0) = p \int_0^{\tau_2} g(\tau) \int_{-\tau}^0 e^{(\mu+\delta_2)\tau} U_1(s) ds \, d\tau.
\]

Based on \( U_2(t) \) being completely determined by \( U_1(t) \), for \( t \geq -h \), it is sufficient to analyze system (3) without the third equation. Then system (3) becomes

\[
\begin{align*}
\frac{dS(t)}{dt} & = \left[ A - \beta S(t) \int_0^{h_1} F(\tau) U_1(t-\tau) d\tau - \mu S(t) \right] dt - \sigma S(t) U_1(t) dw_t, \\
\frac{dU_1(t)}{dt} & = \left[ \beta S(t) \int_0^{h_1} F(\tau) U_1(t-\tau) d\tau - (\mu + \delta_1 + p) U_1(t) + p \int_0^{h_2} G(\tau) U_1(t-\tau) d\tau \right] dt \\
& \quad + \sigma S(t) U_1(t) dw_t,
\end{align*}
\]

with initial condition

\[
S(0) = S^0, \quad U_1(\theta) = \varphi(\theta), \quad \theta \in [-h, 0], \quad h = \max\{h_1, h_2\}. \quad (9)
\]

4. **Stability of the heroin use free equilibrium.** The main interest in the sequel is to investigate conditions for coefficients of system (8) under which the heroin use free equilibrium \( E_0 \) is stochastically stable as well as under which the solution of system (8) converges in the mean to the heroin use free equilibrium. Under the transformation

\[
x = S - \frac{A}{\mu}, \quad y = U_1,
\]

system (8) becomes

\[
\begin{align*}
\frac{dx(t)}{dt} & = -\beta \left( x(t) + \frac{A}{\mu} \right) \int_0^{h_1} F(\tau) y(t-\tau) d\tau - \mu x(t) \right] dt - \sigma \left( x(t) + \frac{A}{\mu} \right) y(t) dw_t, \\
\frac{dy(t)}{dt} & = \left[ \beta \left( x(t) + \frac{A}{\mu} \right) \int_0^{h_1} F(\tau) y(t-\tau) d\tau - (\mu + \delta_1 + p) y(t) + p \int_0^{h_2} G(\tau) y(t-\tau) d\tau \right] dt \\
& \quad + \sigma \left( x(t) + \frac{A}{\mu} \right) y(t) dw_t,
\end{align*}
\]

with initial condition \( x(0) = S^0 - \frac{A}{\mu}, \ y(\theta) = \varphi(\theta), \ \theta \in [-h, 0] \).

In order to examine the stability of the heroin use free equilibrium of system (8), we study the stability of the trivial solution of system (10). To show the stochastic
asymptotic stability of trivial solution of system (10), we consider the linearized system
\[
\begin{align*}
    dX(t) &= \left[ -\beta \frac{\Lambda}{\mu} \int_{0}^{t} F(\tau)Y(t-\tau)\,d\tau - \mu X(t) \right] \,dt - \sigma \frac{\Lambda}{\mu} Y(t)\,dw_t, \\
    dY(t) &= \left[ \beta \frac{\Lambda}{\mu} \int_{0}^{t} F(\tau)Y(t-\tau)\,d\tau - (\mu + \delta_1 + \rho)Y(t) + p \int_{0}^{t} G(\tau)Y(t-\tau)\,d\tau \right] \,dt \\
    &\quad + \sigma \frac{\Lambda}{\mu} Y(t)\,dw_t.
\end{align*}
\]

Before we prove the main result, we establish some definitions and statements for the stochastic functional differential equations (see [4],[11],[24], for instance).

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) be a given complete probability space with the filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) satisfying the usual conditions, and let \(w(t)\) be an \(m\)-dimensional Brownian motion defined on this space. Denote that \(\mathcal{C} = C([-\tau, 0]; \mathbb{R}^d)\) is the family of continuous functions \(\varphi : [-\tau, 0] \to \mathbb{R}^d\) with the norm \(||\varphi|| = \sup_{-\tau \leq \theta \leq 0} |\varphi(\theta)|\) and \(\mathcal{D}\) is the space of \(\mathcal{F}_0\)-adapted function \(\varphi \in \mathcal{C}\).

Consider the \(d\)-dimensional stochastic functional differential equation
\[
\begin{align*}
    dy(t) &= f(t, y_t)\,dt + g(t, y_t)\,dw(t), \quad t \geq 0, \\
    y_0 &= \varphi = \{\varphi(\theta) : -\tau \leq \theta \leq 0\},
\end{align*}
\]
where \(y_t = \{y(t + \theta) : -\tau \leq \theta \leq 0\}\) is a \(C\)-valuated stochastic process and \(y_0 \in \mathcal{D}\), such that \(E||\varphi||^2 < \infty\), while \(f(t, \varphi)\) is \(d\)-dimensional vector and \(g(t, \varphi)\) is \(d \times m\)-dimensional matrix, both defined for \(t \geq 0\). We assume that Eq. (12) has a unique global solution \(y(t; \varphi)\), as well as that \(f(t, 0) = g(t, 0) \equiv 0\). So, Eq. (12) has the trivial solution \(y(t) \equiv 0\) corresponding to the initial condition \(y_0 = 0\).

**Definition 4.1.** The trivial solution of Eq. (12) is said to be stochastically stable if for every \(\varepsilon \in (0, 1)\) and \(r > 0\), there exists constant \(\delta = \delta(\varepsilon, r) > 0\) such that
\[
P\{||y(t; \varphi)|| > r, t \geq 0\} \leq \varepsilon,
\]
for any initial condition \(\varphi \in \mathcal{D}\) satisfying \(P\{||\varphi|| \leq \delta\} = 1\).

**Definition 4.2.** The trivial solution of Eq. (12) is said to be mean square stable if for every \(\varepsilon > 0\), there exists constant \(\delta > 0\) such that \(E||y(t; \varphi)||^2 < \varepsilon\) for any \(t \geq 0\) provided that \(\sup_{-\tau \leq \theta \leq 0} E||\varphi(\theta)||^2 \leq \delta\).

**Definition 4.3.** The trivial solution of Eq. (12) is said to be asymptotically mean square stable if it is mean square stable and \(\lim_{t \to \infty} E||y(t; \varphi)||^2 = 0\).

The differential operator associated to Eq. (12) is defined by the formula
\[
LV(t, \varphi) = \lim_{\Delta \to 0} \sup_{\Delta} \mathbb{E}_t \varphi V(t + \Delta, y_t + \Delta) - V(t, \varphi),
\]
where \(y(s), s \geq t\) is the solution of Eq. (12) satisfying the initial condition \(y_t = \varphi\), and \(V(t, \varphi)\) is a functional defined for \(t \geq 0\) and for functions \(\varphi \in \mathcal{D}\).

Following [24] let us reduce a class of functionals \(V(t, \varphi)\) so that the operator \(L\) can be calculated. First, for \(t \geq 0\) and function \(\varphi \in \mathcal{D}\), let \(V(t, \varphi) = V(t, \varphi(0), \varphi(\theta)))\), \(-\tau \leq \theta \leq 0\). Then, we define the function
\[
V_\varphi(t, y) = V(t, \varphi) = V(t, y_t) = V(t, y(y(t + \theta))), \quad -\tau \leq \theta \leq 0,
\]
where \(\varphi = y_t\), \(y = \varphi(0) = y(t)\).
Let us denote that $C_{1,2}$ is a class of functionals $V(t, \varphi)$ so that, for almost all $t \geq 0$, the first and second derivatives with respect to $y$ of $V_y(t, y)$ are continuous, and the first derivative with respect to $t$ is continuous and bounded. Then, the application of the generating operator $L$ of Eq. (12) yields

$$LV(t, y_t) = \frac{\partial V_y(t, y_t)}{\partial t} + f^T(t, y_t) \frac{\partial V_y(t, y_t)}{\partial y} + \frac{1}{2} \text{trace} \left[ g^T(t, y_t) \frac{\partial^2 V_y(t, y_t)}{\partial y^2} g(t, y_t) \right].$$

**Theorem 4.4.** Let there exist a functional $V(t, \varphi) \in C_{1,2}$ such that

$$c_1 E|y(t)|^2 - ELV(t, y_t) \leq c_2 \sup_{-\tau \leq \theta \leq 0} E|y(t + \theta)|^2,$$

for $c_i > 0$, $i = 1, 2, 3$. Then, the trivial solution of Eq. (12) is asymptotically mean square stable.

**Theorem 4.5.** Let there exist a functional $V(t, \varphi) \in C_{1,2}$ such that

$$c_1 |y(t)|^2 - V(t, y_t) \leq c_2 \sup_{-\tau \leq \theta \leq 0} |y(t + \theta)|^2 \quad \text{and} \quad LV(t, y_t) \leq 0,$$

for $c_i > 0$, $i = 1, 2$ and for any $\varphi \in D$ such that $P\{|\varphi| \leq \delta\} = 1$, where $\delta > 0$ is sufficiently small. Then, the trivial solution of Eq. (12) is stochastically stable.

Bearing in mind the previous facts, we give conditions ensuring asymptotically mean square stability of the trivial solution to the system (11).

**Theorem 4.6.** Assume that the parameters of system (11) satisfy condition $R_0 < 1$ and

$$\sigma^2 < \frac{2\mu^2}{\Lambda^2} \left( \mu + \delta_1 + p(1 - b))(1 - R_0) \right).$$

Then the trivial solution of system (11) is asymptotically mean square stable.

**Proof.** In order to show the asymptotic mean square stability of the trivial solution for system (11) it is necessary to construct the appropriate Lyapunov functional for the system. Consider the Lyapunov function $V_1(X, Y)$ of the form

$$V_1(X, Y) = \frac{1}{2} (X^2 + AY^2),$$

where $A$ is non-negative constant which will be chosen in the sequel. Let $L$ be the differential operator associated to system (11). Then we have

$$LV_1 = -\mu X^2 \left[ A \left( \mu + \delta_1 + p - \frac{\sigma^2 \Lambda^2}{2\mu^2} \right) - \frac{\sigma^2 \Lambda^2}{2\mu^2} \right] Y^2 - \frac{\beta \Lambda}{\mu} X \int_0^{h_1} F(\tau) Y(t - \tau) d\tau + A \frac{\beta \Lambda}{\mu} Y \int_0^{h_1} F(\tau) Y(t - \tau) d\tau + Ap Y \int_0^{h_2} G(\tau) Y(t - \tau) d\tau.$$

Obviously, $X(t) \leq 0$ for $t \geq 0$ and $Y(t) \geq 0$ for $t \geq -h$. By using elementary inequality $\pm 2xy \leq \varepsilon x^2 + \frac{y^2}{\varepsilon}$, $\varepsilon > 0$, and (2) we obtain

$$LV_1 \leq \left( \mu - \varepsilon \frac{\beta \Lambda}{2\mu} \right) X^2 \left[ A \left( \mu + \delta_1 + p - \frac{\beta \Lambda}{2\mu} - \frac{\beta \Lambda}{2\mu} \right) - \frac{\sigma^2 \Lambda^2}{2\mu^2} \right] Y^2 + \left( A + \frac{1}{\varepsilon} \right) \frac{\beta \Lambda}{2\mu} \int_0^{h_1} F(\tau) Y^2(t - \tau) d\tau + \frac{1}{2} \int_0^{h_2} G(s) Y^2(t - \tau) d\tau,$$
where $\varepsilon$ is a positive constant to be specified later. We choose functional $V_2$ to eliminate the terms with delay

$$V_2(t) = \left(A + \frac{1}{\varepsilon}\right)\frac{\beta \Lambda}{2 \mu} \int_0^{h_1} F(\tau) \int_t^{t+\tau} Y^2(s)dsd\tau + \frac{1}{2} A p \int_0^{h_2} G(\tau) \int_1^{t-\tau} Y^2(s)dsd\tau.$$

Finally,

$$LV = LV_1 + LV_2$$

$$\leq - \left(\mu - \varepsilon \frac{\beta \Lambda a}{2 \mu}\right) X^2$$

$$- \left[ A \left( \mu + \delta_1 + p (1-b) - \frac{\beta \Lambda a}{\mu} - \frac{\sigma^2 \Lambda^2}{2 \mu^2} \right) - \frac{1}{2} \left( \frac{\beta \Lambda a}{\varepsilon \mu} + \frac{\sigma^2 \Lambda^2}{\mu^2} \right) \right] Y^2.$$

We choose $0 < \varepsilon < \frac{2 \mu^2}{\beta \Lambda a}$. This implies that the quantity in the bracket multiplying $X^2$ is positive. Similarly, the positivity of expression in bracket multiplying $Y^2$ is guaranteed under condition (13) if the constant $A$ satisfies condition

$$A > \frac{\frac{\beta \Lambda a}{\varepsilon \mu} + \frac{\sigma^2 \Lambda^2}{\mu^2}}{2 \left( \mu + \delta_1 + p (1-b) - \frac{\beta \Lambda a}{\mu} - \frac{\sigma^2 \Lambda^2}{2 \mu^2} \right)}.$$

Hence, by virtue of Theorem 4.4, it follows that the trivial solution of the system (11) is asymptotically mean square stable.

**Theorem 4.7.** Assume that the parameters of system (8) satisfy condition (13). Then the heroin use free equilibrium of system (8) is stochastically stable.

The proof is omitted since the system (8), which is equivalent to the system (10), has the order of nonlinearity more than one. From [24] it follows that if the order of nonlinearity of the system under consideration is more than one, then the conditions, that are sufficient for asymptotic mean-square stability of the trivial solution of the linearized system, are sufficient for stochastic stability of the trivial solution of the initial system. Thus, if condition (13) holds, then the heroin use free equilibrium of system (8) is stochastically stable.

Next result describes another type of convergence of solution of system (8) to heroin use free equilibrium. When $R_0$ is close enough to 1, this theorem gives better bound for $\sigma^2$ in regard to (13).

**Theorem 4.8.** Let $(S(t), U_1(t))$ be the solution of system (8) with initial condition (9) and conditions $R_0 < 1$ and

$$\sigma^2 < \frac{2 \mu^3}{\Lambda^2} \left(1 + \frac{\mu + \delta_1 + p (1-b)}{2 \mu + \delta_1 + p} (1-R_0)\right).$$

hold. Then

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t S(r)dr = \frac{\Lambda}{\mu} \quad \text{and} \quad \lim_{t \to \infty} \frac{1}{t} \int_0^t U_1(r)dr = 0 \quad \text{a.s.}$$

**Proof.** Define Lyapunov functional $V = V_1 + AV_2 + V_3$, where

$$V_1(S, U_1) = S - \frac{\Lambda}{\mu} - \frac{\Lambda}{\mu} \ln \frac{S}{\mu} + U_1 \quad \text{and} \quad V_2(S, U_1) = \frac{1}{2} \left( \frac{S}{\mu} + U_1 \right)^2.$$
As a result, we obtain,

\[ dV(t) = LV(t)dt + \sigma \frac{\Lambda}{\mu} U_1(t)dw_t. \]

Applying operator \( L \) on \( V_1 \) and \( V_2 \), respectively, we get

\[
LV_1 = -\mu \frac{(S - \frac{\Lambda}{\mu})^2}{S} - (\mu + \delta_1 + p)U_1 + \beta \frac{\Lambda}{\mu} \int_0^{h_1} F(\tau)U_1(t-\tau)d\tau \\
+ p \int_0^{h_2} G(\tau)U_1(t-\tau)d\tau + \frac{\sigma^2}{2} \frac{\Lambda}{\mu} U_1^2 \\
\leq -\mu \left(S - \frac{\Lambda}{\mu}\right)^2 - (\mu + \delta_1 + p)U_1 + \beta \frac{\Lambda}{\mu} \int_0^{h_1} F(\tau)U_1(t-\tau)d\tau \\
+ p \int_0^{h_2} G(\tau)U_1(t-\tau)d\tau + \frac{\sigma^2}{2} \left(S - \frac{\Lambda}{\mu}\right)^2,
\]

\[
LV_2 = -\mu \left(S - \frac{\Lambda}{\mu}\right)^2 - (\mu + \delta_1 + p)U_1^2 + (2\mu + \delta_1 + p) \left(\frac{\Lambda}{\mu} - S\right)U_1 \\
+ p U_1 \int_0^{h_2} G(\tau)U_1(t-\tau)d\tau - p \left(\frac{\Lambda}{\mu} - S\right) \int_0^{h_2} G(\tau)U_1(t-\tau)d\tau \\
\leq -\mu \left(S - \frac{\Lambda}{\mu}\right)^2 - (\mu + \delta_1 + p)U_1^2 + (2\mu + \delta_1 + p) \left(\frac{\Lambda}{\mu} - S\right)U_1 \\
+ \frac{p b U_1^2}{2} + \frac{p}{2} \int_0^{h_2} G(\tau)U_1^2(t-\tau)d\tau.
\]

To eliminate terms with delay, we choose functional

\[
V_3(t) = \beta \frac{\Lambda}{\mu} \int_0^{h_1} F(\tau) \int_{t-\tau}^t U_1(s)ds d\tau \\
+ \frac{p}{2} \int_0^{h_2} G(\tau) \int_{t-\tau}^t U_1^2(s)ds d\tau.
\]

As a result, we obtain,

\[
LV \leq - \left(\frac{\mu^2}{\Lambda} - \frac{\Lambda \sigma^2}{2} + A\mu\right) \left(S - \frac{\Lambda}{\mu}\right)^2 - \left(\mu + \delta_1 + p (1-b) - \frac{\beta \Lambda a}{\mu} - A \frac{\Lambda}{\mu} (2\mu + \delta_1 + p)\right)U_1 \\
- A(\mu + \delta_1 + p (1-b))U_1^2.
\]

We choose \( A = \frac{\mu}{\Lambda} (1 - R_0) \frac{t^2 \mu + \delta (1-b)}{2\mu + \delta + \sigma} \) to eliminate the term multiplying \( U_1 \). Finally,\n
\[
dV(t) \leq - \left(K_1 \left(S(t) - \frac{\Lambda}{\mu}\right)^2 + K_2 U_1^2(t)\right) dt + \sigma \frac{\Lambda}{\mu} U_1(t)dw_t, \quad (15)
\]

where

\[
K_1 = \left(\frac{\mu^2}{\Lambda} - \frac{\Lambda \sigma^2}{2\mu} + A\mu\right), \quad K_2 = A(\mu + \delta_1 + p (1-b)).
\]

Positivity of constant \( K_1 \) is ensured from condition (14). Integrating (15) from 0 to \( t \) and dividing both sides by \( t \) we have\n
\[
\frac{V(t) - V(0)}{t} \leq - \frac{K_1}{t} \int_0^t \left(S(r) - \frac{\Lambda}{\mu}\right)^2 dr - \frac{K_2}{t} \int_0^t U_1^2(r)dr + \frac{M(t)}{t}, \quad (16)
\]
where $M(t) = \int_0^t \sigma \Lambda \mu U_1(r) \, dw_r$ is a local continuous martingale with $M(0) = 0$ and

$$\limsup_{t \to \infty} \langle M, M \rangle_t \leq \sigma^2 \left( \frac{\Lambda}{\mu} \right)^4 < \infty \text{ a.s.}$$

By using Strong law of large number for martingales we obtain that $\lim_{t \to \infty} M(t)/t = 0$ a.s.

Applying Hölder inequality on (16) and letting $t \to \infty$, we get

$$\lim_{t \to \infty} \left[ K_1 \left( \frac{1}{t} \int_0^t \left( S(r) - \frac{\Lambda}{\mu} \right) dr \right)^2 + K_2 \left( \frac{1}{t} \int_0^t U_1(r) dr \right)^2 \right] \leq 0 \text{ a.s.}$$

which completes the proof.

**Remark 1.** If $1 - \mu \left( 2\mu + \delta_1 + p \right) \left( \mu + \delta_1 + p \right)^{-1} < R_0 < 1$ holds, then Theorem 4.8 gives wider interval for $\sigma^2$, for which solution of system (8) converges to heroin use free equilibrium.

5. **Asymptotic behavior around the heroin spread equilibrium of the deterministic model.** In studying the spread of heroin addiction it is important when the heroin use persists in the population. In the deterministic model [3] the authors showed that the heroin spread equilibrium is globally stable, by using a suitable Lyapunov functionals. But there is none of heroin spread equilibrium in the stochastic system (8) which is the perturbed system of the system (1). Then, it is reasonable to consider that the heroin addiction will prevail if the solution of system (8) is going around the heroin spread equilibrium of the deterministic model (1) most of the time.

**Theorem 5.1.** Let $(S(t), U_1(t))$ be the solution of system (8) with initial condition (9) and conditions $R_0 > 1$ and

$$\sigma^2 < \frac{2}{\mu^2} U_1^*$$

hold. Then

$$\limsup_{t \to \infty} \frac{1}{t} E \int_0^t \left( S(r) - \frac{S^*}{1 - \mu \sigma^2} \right)^2 dr \leq \frac{\left( \frac{U_1^*}{2\mu^2} \frac{S^*}{\sigma^2} + \left( \frac{\Lambda}{\mu} \right)^2 \right) S^*}{2\mu^2 \left( 1 - \frac{\Lambda}{2\mu^2} U_1^* \sigma^2 \right)} - \sigma^2.$$  \hspace{1cm} (18)

Assume also

$$b < \frac{\delta_1 + p}{3p}.$$  \hspace{1cm} (19)

Then

$$\limsup_{t \to \infty} \frac{1}{t} E \int_0^t \left( U_1(r) - C_1 U_1^* \right)^2 dr \leq \left[ C_2 (S^* + U_1^*)^2 \frac{A}{2\mu^2} + C_1 (1 - A) U_1^2 \right] \left( \frac{\Lambda}{\mu} \right)^2 \sigma^2,$$  \hspace{1cm} (20)

if:

(I)

$$\mu > \delta_1 + p (1 - b)$$  \hspace{1cm} (21)

and

(i)

$$R_0 > \frac{2\mu}{\mu - \delta_1 - p (1 - b)};$$  \hspace{1cm} (22)

$$\frac{\mu + \delta_1 + p (1 - b) + \left( \frac{\Lambda}{\mu} \right)^2 \sigma^2}{\mu + 2\delta_1 + 2p - 4pb + \left( \frac{\Lambda}{\mu} \right)^2 \sigma^2} < A < 1,$$  \hspace{1cm} (23)

where $C_2$ and $C_1$ are positive constants.
Choosing \( - \)\( \end{equation}\)

where
\[
C_1 = \frac{A(\mu + 2\delta_1 + 2p - 4p b) - (\mu + \delta_1 + p(1-b))}{P},
\]
\[
C_2 = \frac{A(\mu + 2p b) - \mu + \delta_1 + p(1-b)}{P},
\]
\[
P = A\left( \mu + 2\delta_1 + 2p - 4p b + \left( \frac{\Lambda}{\mu} \right)^2 \right) - \left( \mu + \delta_1 + p(1-b) + \left( \frac{\Lambda}{\mu} \right)^2 \right);
\]

\[(ii)\]
\[
\frac{3\mu - 2p b}{2(\mu - \delta_1 - p(1-b))} < R_0 < \frac{2\mu}{\mu - \delta_1 - p(1-b)}
\]
\[
\max \left\{ \frac{\mu + \delta_1 + p(1-b) + \left( \frac{\Lambda}{\mu} \right)^2 \sigma^2}{\mu + 2\delta_1 + 2p - 4p b + \left( \frac{\Lambda}{\mu} \right)^2 \sigma^2}, \frac{2R_0(\mu - \delta_1 - p(1-b)) - 3\mu - \delta_1 - p(1-b)}{2R_0(\mu - \delta_1 - p(1-b)) - 3\mu - 2\mu b} \right\} < A < 1,
\]

where \( C_1 \) is defined as in \((24)\) and
\[
C_2 = \frac{A \left[ 2R_0(\mu - \delta_1 - p(1-b)) - 3\mu + 2p b \right] - \left[ 2R_0(\mu - \delta_1 - p(1-b)) - 3\mu - \delta_1 - p(1-b) \right]}{P};
\]

\[(iii)\]
\[
1 < R_0 < \frac{3\mu - 2p b}{2(\mu - \delta_1 - p(1-b))}
\]

and the constant \( A \) satisfies condition \((23)\), where \( C_1 \) is defined as in \((24)\) and \( C_2 \) as in \((28)\);

\[(II)\]
\[
\mu \leq \delta_1 + p(1-b)
\]

and condition \((23)\) holds, where \( C_1 \) and \( C_2 \) are given by \((24)\) and \((28)\), respectively.

**Proof.** Define a \( C^2 \)-function \( V_1 : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+ \) by
\[
V_1(S, U_1) = S - S^* - S^* \ln \frac{S}{S^*} + U_1 - U_1^* - U_1^* \ln \frac{U_1}{U_1^*}.
\]

The nonnegativity of this function follows from the property of function \( \Psi(u) = u - 1 - \ln u \geq 0 \), for any \( u > 0 \). By using \( \Lambda = \beta S^* U_1^* a + \mu S^* \) and \( (\mu + \delta_1 + p) U_1^* = \beta S^* U_1^* a + p b U_1^* \), we compute

\[
LV_1 = -\mu \frac{(S - S^*)^2}{S} + \beta S^* U_1^* \int_0^{h_1} F(\tau) \left[ \frac{S^*}{S} + \frac{U_1(t-\tau)}{U_1^*} - \frac{S}{S^*} \cdot \frac{U_1(t-\tau)}{U_1} \right] d\tau
\]
\[
+\mu U_1^* \int_0^{h_2} G(\tau) \left[ 1 - \frac{U_1(t-\tau)}{U_1} + \frac{U_1(t-\tau)}{U_1^*} \right] d\tau - (\mu + \delta_1 + p) U_1 + \frac{\sigma^2}{2} (S^* U_1^2 + U_1^* S^2).
\]

Choosing \( V_2 \) in the form
\[
V_2(t) = \beta S^* U_1^* \int_0^{h_1} F(\tau) \left[ \frac{U_1(s)}{U_1^*} - 1 - \ln \frac{U_1(s)}{U_1^*} \right] ds d\tau
\]
\[
+\mu U_1^* \int_0^{h_2} G(\tau) \left[ \frac{U_1(s)}{U_1^*} - 1 - \ln \frac{U_1(s)}{U_1^*} \right] ds d\tau,
\]
for functional $V = V_1 + V_2$ holds

$$LV = -\mu \frac{(S - S^*)^2}{S}$$

$$+ \beta S^* U_1^* \int_0^{b_1} F(\tau) \left[ 1 - \frac{S^*}{S} - \frac{S}{S^*} \cdot \frac{U_1(t - \tau)}{U_1} + 1 - \ln \frac{U_1}{U_1^*} + \ln \frac{U_1(t - \tau)}{U_1^*} \right] d\tau$$

$$+ p U_1^* \int_0^{b_2} G(\tau) \left[ 1 - \frac{U_1(t - \tau)}{U_1} - \ln \frac{U_1}{U_1^*} + \ln \frac{U_1(t - \tau)}{U_1^*} \right] d\tau + \frac{\sigma^2}{2} (S^* U_1^* + U_1^* S^2),$$

where we have used $[\beta S^* a - (\mu + \delta_1 + p (1 - b))] U_1^* = 0$. Since $-\Psi(u) \leq 0$ and $U_1(t) \leq \frac{\Lambda}{\mu}$, $t \geq 0$, then

$$LV \leq -\mu^2 \frac{(S - S^*)^2}{S} + \frac{\sigma^2}{2} U_1^* S^2 + \frac{\sigma^2}{2} \Lambda^2 \left( \frac{\Lambda}{\mu} \right)^2$$

$$+ \beta S^* U_1^* \int_0^{b_1} F(\tau) \left[ 1 - \frac{S^*}{S} + \ln \frac{S^*}{S} - \frac{S}{S^*} \cdot \frac{U_1(t - \tau)}{U_1} + 1 - \ln \frac{U_1}{U_1^*} \right] d\tau$$

$$+ p U_1^* \int_0^{b_2} G(\tau) \left[ 1 - \frac{U_1(t - \tau)}{U_1} + \ln \frac{U_1}{U_1^*} \right] d\tau \quad (31)$$

$$\leq -\left( \frac{\mu^2}{\Lambda} - \frac{\sigma^2}{2} U_1^* \right) \left( S - \frac{S^*}{1 - \frac{\Lambda}{2\mu^2} U_1^* \sigma^2} \right)^2 + \left( \frac{U_1^* S^*}{1 - \frac{\Lambda}{2\mu^2} U_1^* \sigma^2} \right)^2 \Lambda^2 \sigma^2 \frac{S^*}{2}.$$

Consequently,

$$dV(t) \leq \left( -\left( \frac{\mu^2}{\Lambda} - \frac{\sigma^2}{2} U_1^* \right) \left( S(t) - \frac{S^*}{1 - \frac{\Lambda}{2\mu^2} U_1^* \sigma^2} \right)^2 + M_\sigma \right) dt$$

$$+ \sigma \left( S^* U_1(t) - U_1^* S(t) \right) dw_t, \quad (32)$$

where $M_\sigma = \left( \frac{U_1^* S^*}{1 - \frac{\Lambda}{2\mu^2} U_1^* \sigma^2} \right) \Lambda^2 \sigma^2$. Integrating (32) from 0 to $t$ and taking the expectation, we obtain

$$0 \leq EV(t) \leq V(0) - \left( \frac{\mu^2}{\Lambda} - \frac{\sigma^2}{2} U_1^* \right) E \int_0^t \left( S(r) - \frac{S^*}{1 - \frac{\Lambda}{2\mu^2} U_1^* \sigma^2} \right)^2 dr + M_\sigma t.$$
Substituting $x = S - S^*$ and $y = U_1 - U_1^*$ into system (8) and by using $\Lambda = \beta S^* U_1^* a + \mu S^*$, we obtain system
\begin{align*}
\dot{x}(t) &= -[(\mu + \beta U_1^* a) x(t) - \beta x(t) I_1(t) - \beta S^* I_1(t)]dt \\
&\quad - \sigma(x(t) + S^*)(y(t) + U_1^*) d\omega_1, \\
\dot{y}(t) &= \beta U_1^* ax(t) - \beta \delta U_1^* (y(t) + \beta x(t) I_1(t) + \beta S^* I_1(t) + p I_2(t))]dt \\
&\quad + \sigma(x(t) + S^*)(y(t) + U_1^*) d\omega_1,
\end{align*}
with initial condition $x(0) = S^0 - S^*$, $y(\theta) = \varphi(\theta) - U_1^*$, $\theta \in [-h, 0]$, where $I_1(t) = \int_0^{h_1} F(\tau)y(t-\tau)d\tau$, $I_2(t) = \int_0^{h_2} G(\tau)y(t-\tau)d\tau$.

Define the Lyapunov $C^2$-function $V_3 : \mathbb{R}^2 \to \mathbb{R}_+$
\[ V_3(x, y) = x^2 + Ay^2 + Bxy, \]
where $A$ and $B$ are positive constants to be chosen later. Function $V_3$ is positive definite if these constants satisfy condition $A > \frac{B^2}{4}$. Then,
\[ LV_3 = -[2\mu + (2 - B)\beta U_1^* a] x^2 - 2A(\mu + \delta + p) y^2 - (B - 2)\beta S^* x I_1 \\
+ (2A - B)\beta S^* y I_1 + B p x I_2 + 2Ap y I_2 + (2A - B)\beta xy I_1 \\
+ [(2A - B)\beta U_1^* a - B(2\mu + \delta + p)] xy + (1 - B)\sigma^2 (x + S^*)^2 (y + U_1^*)^2. \]

Let us choose $2A = B$ in order to eliminate some terms in the last equality. From $A > \frac{B^2}{4}$ and $2A = B$ we deduce that $A \in (0, 1)$. Hence,
\[ LV_3 = -2[\mu + (1 - A)\beta U_1^* a] x^2 - 2A(\mu + \delta + p) y^2 - 2(1 - A)\beta S^* x I_1 - 2(1 - A)\beta S^* y I_1 \\
+ 2Ap x I_2 + 2Ap y I_2 - 2A(2\mu + \delta + p) xy + (1 - A)\sigma^2 (x + S^*)^2 (y + U_1^*)^2. \]

Since $-S^* < x = S - S^* \leq \frac{\Lambda}{\mu} - S^*$ and $-U_1^* < y = U_1 - U_1^* \leq \frac{\Lambda}{\mu} - U_1^*$, we conclude
\[ |x| \leq \max \left\{ S^*, \frac{\Lambda}{\mu} - S^* \right\} \quad \text{and} \quad |y| \leq \max \left\{ U_1^*, \frac{\Lambda}{\mu} - U_1^* \right\}. \]

From (35) and elementary inequality $\pm 2xy \leq x^2 + y^2$ we find that
\[ LV_3 \leq -2\mu - 2(1 - A)\beta U_1^* a + 2\beta a(1 - A) \max \left\{ U_1^*, \frac{\Lambda}{\mu} - U_1^* \right\} + (1 - A)\beta S^* a \\
+ ApB + A(2\mu + \delta + p) x^2 - (\delta + p (1-b)) y^2 + (1 - A)\beta S^* \int_0^{h_1} F(\tau)y^2(t-\tau)d\tau \\
+ 2Ap \int_0^{h_2} G(\tau)y^2(t-\tau)d\tau + (1 - A)\sigma^2 (x + S^*)^2 (y + U_1^*)^2. \]

In order to eliminate terms with delay, we consider the functional
\[ V_4(t) = (1 - A)\beta S^* \int_0^{h_1} F(\tau) \int_{t-\tau}^{t} y^2(s)ds d\tau + 2Ap \int_0^{h_2} G(\tau) \int_{t-\tau}^{t} y^2(s)ds d\tau. \]

For the functional $W = V_3 + V_4$, we obtain
\begin{align*}
LW &\leq \left[ A \left( 2\beta U_1^* a - 2\beta a \max \left\{ U_1^*, \frac{\Lambda}{\mu} - U_1^* \right\} - \beta S^* a + p b + 2\mu + \delta + p \right) \right] x^2 \\
&\quad - 2\mu - 2\beta U_1^* a + \beta S^* a + 2\beta a \max \left\{ U_1^*, \frac{\Lambda}{\mu} - U_1^* \right\} x^2 \\
&\quad - \left[ A(\delta + p (1-b) + \beta S^* a - 2p b) - \beta S^* a \right] y^2 \\
&\quad + (1 - A)\sigma^2 (x + S^*)^2 (y + U_1^*)^2. \]
\]
(I) Let condition (21) holds. 

(i) Condition (22) implies that max\(\{U^*_1, \frac{A}{\mu} - U^*_1\}\) = \(U^*_1\) and (36) becomes

\[
LW \leq [A(\mu+2p b) - \mu + \delta_1 + p(1-b)] x^2 - [A(\mu+2\delta_1 + 2p - 4p b) - (\mu + \delta_1 + p(1-b))] y^2 + (1-A)\sigma^2 (x + S^*)^2 (y + U^*_1)^2 .
\]

By Itô’s formula, we compute

\[
dW(t) = LW(t) dt - 2 \sigma(1-A)x(t)(x(t) + S^*)(y(t) + U^*_1) dw_t .
\]  

(37)

Integrating (37) from 0 to \(t\), taking expectations and applying the last inequality we get

\[
0 \leq EW(t)
\]

\[
\leq W(0) + [A(\mu+2p b) - \mu + \delta_1 + p(1-b)] E \int_0^t (S(r) - S^*)^2 dr
\]

\[
+ (1-A)\sigma^2 \left( \frac{\Lambda}{\mu} \right) E U^2_1(r) dr
\]

\[
- [A(\mu+2\delta_1 + 2p - 4p b) - (\mu + \delta_1 + p(1-b))] E \int_0^t (U_1(r) - U^*_1)^2 dr
\]

\[
=W(0) + [A(\mu+2p b) - \mu + \delta_1 + p(1-b)] E \int_0^t (S(r) - S^*)^2 dr
\]

\[
+ C_1(1-A)U^2_1 \left( \frac{\Lambda}{\mu} \right)^2 \sigma^2 t
\]

\[
- \left[ A \left( \mu + 2\delta_1 + 2p - 4p b + \left( \frac{\Lambda}{\mu} \right)^2 \right) + (\mu + \delta_1 + p(1-b) + \left( \frac{\Lambda}{\mu} \right)^2) \right] \times E \int_0^t (U_1(r) - C_1U^*_1)^2 dr ,
\]

where \(C_1\) is given in (24).

It is easy to proof

\[
\frac{\mu + \delta_1 + p(1-b) + \left( \frac{\Lambda}{\mu} \right)^2}{\mu + 2\delta_1 + 2p - 4p b + \left( \frac{\Lambda}{\mu} \right)^2} > \frac{\mu + \delta_1 + p(1-b)}{\mu + 2\delta_1 + 2p - 4p b} > \frac{\mu + \delta_1 + p(1-b)}{\mu + 2p b} .
\]

Then condition (23) ensures that quantities in the brackets multiplying \((S - S^*)^2\) and \((U_1 - U^*_1)^2\) are positive. Existence of constant \(A\), given by (23), yields from (19). Dividing both sides by \(t\) in (38), letting \(t \to \infty\) and using (33), we get (20), where constant \(C_2\) is given by (25).

(ii) Condition \(R_0 < \frac{2\mu}{\mu - \delta_1 - p(1-b)}\) guarantees that max\(\{U^*_1, \frac{A}{\mu} - U^*_1\}\) = \(\frac{A}{\mu} - U^*_1\). By using \(x + S^* \leq \frac{A}{\mu}\), (36) becomes

\[
LW(t)
\]

\[
\leq [A(2R_0(\mu - \delta_1 - p(1-b) - 3\mu + 2p b) - 2R_0(\mu - \delta_1 - p(1-b)) + 3\mu + \delta_1 + p(1-b))] x^2
\]

\[
- [A(\mu + 2\delta_1 + 2p - 4p b) - (\mu + \delta_1 + p(1-b))] y^2 + (1-A)\sigma^2 \left( \frac{\Lambda}{\mu} \right)^2 (y + U^*_1)^2 .
\]

Then, integrating (37) from 0 to \(t\), taking expectations and applying the last inequality, yields

\[
0 \leq EW(t) \leq W(0)
\]

\[
+ [A(2R_0(\mu - \delta_1 - p(1-b) - 3\mu + 2p b) - (2R_0(\mu - \delta_1 - p(1-b)) - 3\mu - \delta_1 - p(1-b))]\]

\[
\times E \int_0^t (S(r) - S^*)^2 dr + C_1(1 - A)U_1^*2 \left( \frac{\Lambda}{\mu} \right)^2 \sigma^2 t
\]

\[
- \left[ A \left( \mu + 2\delta_1 + 2p - 4p b \left( \frac{\Lambda}{\mu} \right)^2 \sigma^2 \right) - \left( \mu + \delta_1 + p(1 - b) \right) \left( \frac{\Lambda}{\mu} \right)^2 \sigma^2 \right]
\]

\[
\times E \int_0^t (U_1(r) - C_1 U_1^*2 )^2 dr,
\]

where the constant \( C_1 \) is given in (24). Condition (27) ensures that quantity in the bracket multiplying \((U_1 - U_1^*2)\) is positive and existence of constant \( A \in (0,1) \) is implied by condition (19). Inequalities (26) and (27) ensure that quantity in the bracket multiplying \((S - S^*)^2\) is positive. The existence \( R_0 \) in (26) is ensured since conditions (19) and (21) imply that \( \mu > 2p b \). Dividing both sides by \( t \), letting \( t \to \infty \) and using (33) we get (20), where the constant \( C_2 \) is given by (28).

(iii) If conditions (21) and (29) are satisfied then \( \max\{U_1^*, \frac{\Lambda}{\mu} - U_1^*\} = \frac{\Lambda}{\mu} - U_1^* \).

For constant \( A \), given by (23), estimation (20) holds with constants \( C_1 \) and \( C_2 \) defined in (24) and (28), respectively.

(II) Let condition (30) is satisfied. Then \( \max\{U_1^*, \frac{\Lambda}{\mu} - U_1^*\} = \frac{\Lambda}{\mu} - U_1^* \). Calculating operator \( LW \) and using \( x + S^* \leq \frac{\Lambda}{\mu} \) from (36), analogously as in the previous cases we get

\[
0 \leq EW(t) \leq W(0)
\]

\[
+ \left[ A(-2R_0(\delta_1 + p(1 - b) - \mu) - 3\mu + 2pb) + 2R_0(\delta_1 + p(1 - b) - \mu) + 3\mu + \delta_1 + p(1 - b) \right]
\]

\[
\times E \int_0^t (S(r) - S^*)^2 dr + C_1(1 - A)U_1^*2 \left( \frac{\Lambda}{\mu} \right)^2 \sigma^2 t
\]

\[
- \left[ A \left( \mu + 2\delta_1 + 2p - 4p b \left( \frac{\Lambda}{\mu} \right)^2 \sigma^2 \right) - \left( \mu + \delta_1 + p(1 - b) \right) \left( \frac{\Lambda}{\mu} \right)^2 \sigma^2 \right]
\]

\[
\times E \int_0^t (U_1(r) - C_1 U_1^*2 )^2 dr.
\]

Condition (23) ensures that quantity in the bracket multiplying \((U_1(r) - C_1 U_1^*2)\) is positive. In the time for any \( A < 1 \) quantity in the bracket multiplying \((S(r) - S^*)^2\) is positive. Similarly, as in the previous cases we conclude that (20) holds with constants \( C_1 \) and \( C_2 \) given by (24) and (28), respectively.

Authors in [3] highlight the fact that since \( b \) is the probability of leaving the treatment class and entering the untreated class, the long time treatment is beneficial to control the spread of habitual heroin use. Since practice has shown that this probability is large, therefore the condition (19) is limited. In order to weaken this condition, the system (34) is considered as a neutral system

\[
dx(t) = [- (\mu + \beta U_1^* a)x(t) - \beta x(t) I_1(t) - \beta S^* I_1(t)] \ dt - \sigma (x(t) + S^*)(y(t) + U_1^*) \ dw_t,
\]

\[
d \left( y(t) + p \int_0^{h_2} G(\tau) \int_0^t y(s) ds \ d\tau \right)
\]

\[
= [\beta U_1^* ax(t) - (\mu + \delta_1 + p(1 - b))y(t) + \beta x(t) I_1(t) + \beta S^* I_1(t)] \ dt
\]

\[
+ \sigma (x(t) + S^*)(y(t) + U_1^*) dw_t,
\]

with initial condition \( x(0) = S^0 - S^* \), \( y(\theta) = \varphi(\theta) - U_1^* \), \( \theta \in [-h, 0] \), where \( I_1(t) = \int_0^{h_1} F(\tau)y(t - \tau) d\tau \).
Theorem 5.2. Let \( (S(t), U_1(t)) \) be the solution of system (8) with initial condition (9) and let conditions \( R_0 > 1 \) and (17) hold. Assume also
\[
h_2 < \frac{\delta_1 + p (1 - b)}{pb(3\mu + 2\delta_1 + 2p(1 - b))}.
\]
Then (20) holds if:

(I) \( \mu > \delta_1 + p (1 - b) \)

and

(i) \( R_0 > \frac{2\mu}{\mu - \delta_1 - p(1-b)} \),
\( \frac{\mu + \delta_1 + p (1 - b) + (\frac{A}{\mu})^2 \sigma^2}{\mu + 2\delta_1 + 2p(1 - b) + (\frac{A}{\mu})^2 \sigma^2 - (3\mu + 2\delta_1 + 2p(1 - b))pbh_2} < A < 1, \)

where
\[
C_1 = \frac{A[\mu + 2\delta_1 + 2p(1-b)-(3\mu + 2\delta_1 + 2p(1-b))pbh_2] - (\mu + \delta_1 + p(1-b))}{D},
\]
\[
C_2 = \frac{A(\mu + pbh_2) - (\mu - \delta_1 - p(1-b))}{D},
\]
\[
D = A\left[\mu + 2\delta_1 + 2p(1-b) + (\frac{A}{\mu})^2 \sigma^2 - (3\mu + 2\delta_1 + 2p(1-b))pbh_2\right] - \left[\mu + \delta_1 + p(1-b) + (\frac{A}{\mu})^2 \sigma^2\right];
\]

(ii) \( \frac{3\mu + \delta_1 + p (1 - b)}{2(\mu - \delta_1 - p (1 - b))} < R_0 < \frac{2\mu}{\mu - \delta_1 - p (1 - b)}, \)
\( \max\{M, N\} < A < 1, \)

where
\[
M = \frac{\mu + \delta_1 + p (1 - b) + (\frac{A}{\mu})^2 \sigma^2}{\mu + 2\delta_1 + 2p(1 - b) + (\frac{A}{\mu})^2 \sigma^2 - (3\mu + 2\delta_1 + 2p(1 - b))pbh_2},
\]
\[
N = \frac{2R_0(\mu - \delta_1 - p (1 - b)) - (\mu + \delta_1 + p(1-b))}{2R_0(\mu - \delta_1 - p (1 - b)) - \mu(3 - pbh_2)},
\]
\( C_1 \) is defined as in (43) and
\[
C_2 = \frac{F}{D},
\]
\( F = A\left[2R_0(\mu - \delta_1 - p (1 - b)) - \mu(3 - pbh_2)\right] - [2R_0(\mu - \delta_1 - p(1-b)) - (3\mu + \delta_1 + p(1-b))]\),

(iii) \( 1 < R_0 < \frac{\mu(3 - pbh_2)}{2(\mu - \delta_1 - p (1 - b))} \)

and the constant \( A \) satisfies condition (42), where \( C_1 \) is defined as in (43) and \( C_2 \) as in (46);
(iv) 
\[
\frac{\mu (3 - \mu bh_2)}{2 (\mu - \delta_1 - p (1 - b))} < R_0 < \frac{3 \mu + \delta_1 + p (1 - b)}{2 (\mu - \delta_1 - p (1 - b))}
\]  
(48)
and the constant \( A \) satisfies condition (42), where \( C_1 \) is defined as in (43) and \( C_2 \) as in (46);

(II) 
\[
\mu \leq \delta_1 + p (1 - b),
\]  
(49)
then condition (42) holds, and \( C_1 \) and \( C_2 \) are given by (43) and (46) respectively.

Proof. Define a \( C^2 \)-functional \( W: R^2_+ \to R_+ \) by
\[
W(x, y) = x^2 + A \left( y + p \int_0^{h_2} G(\tau) \left( y(s)ds d\tau \right)^2 + Bx \left( y + p \int_0^{h_2} G(\tau) \int_{t-\tau}^{t} y(s)ds d\tau \right) \right),
\]
where \( A \) and \( B \) are positive constants to be chosen later. Functional \( W \) is positive definite if constants \( A \) and \( B \) satisfy condition \( A > \frac{B^2}{2} \). Application of the generating operator \( L \) on the functional \( W \) yields
\[
LW = -(2\mu + (2 - B)\beta U_1^* a)x^2 - 2A (\mu + \delta_1 + p (1 - b))y^2 + (B - 2)\beta x^2 I_1 + (B - 2)\beta S^* x_1 + (2A - B)\beta xy I_1 + (2A - B)\beta S^* y I_1 + [2A\beta U_1^* a - B(\mu + \delta_1 + p (1 - b) + \beta U_1^* a)]xy
\]
\[
+ [(2A - B)\beta (x + S^*) I_1 - 2Ap(\mu + \delta_1 + p (1 - b)) y] \int_0^{h_2} G(\tau) \int_{t-\tau}^{t} y(s)ds d\tau
\]
\[
+ [2Ap \beta U_1^* a - B(\mu + \beta U_1^* a)p] x \int_0^{h_2} G(\tau) \int_{t-\tau}^{t} y(s)ds d\tau
\]
\[
+ (1 + A - B)\sigma^2 (x + S^*)^2 (y + U_1^*)^2.
\]

Let us choose \( 2A = B \) in order to eliminate some terms in the last equality. From \( A > \frac{B^2}{2} \) and \( 2A = B \) we see that \( A \in (0, 1) \). Then we have,
\[
LW = -(2\mu + (2 - A)\beta U_1^* a)x^2 - 2A (\mu + \delta_1 + p (1 - b))y^2 - 2(1 - A)\beta x^2 I_1
\]
\[
- 2(1 - A)\beta S^* x_1 - 2A (2\mu + \delta_1 + p (1 - b) xy + (1 - A)\sigma^2 (x + S^*)^2 (y + U_1^*)^2
\]
\[
- 2Ap(\mu + \delta_1 + p (1 - b)) y + mx \int_0^{h_2} G(\tau) \int_{t-\tau}^{t} y(s)ds d\tau.
\]

By using the inequality (35), elementary inequality \( \pm 2xy \leq x^2 + y^2 \) and fact that \( \tau \leq h_2 \), we find that
\[
LW \leq \left[ 2(\mu + (1 - A)\beta U_1^* a) - (1 - A)\beta a \max \left\{ \left\{ \frac{A}{\mu} - U_1^* \right\} - (1 - A)\beta S^* a - A\beta bh_2 \right\}
\]
\[
- A (2\mu + \delta_1 + p (1 - b)) x^2 + Ap (2\mu + \delta_1 + p (1 - b)) \int_0^{h_2} G(\tau) \int_{t-\tau}^{t} y^2(s)ds d\tau
\]
\[
- [A (\delta_1 + p (1 - b)) - (1 - A)\beta S^* a - A (\mu + \delta_1 + p (1 - b)) p bh_2] y^2
\]
\[
+ (1 - A)\sigma^2 (x + S^*)^2 (y + U_1^*)^2.
\]
In order to eliminate term with delay we consider the functional
\[
V_2(t) = Ap (2\mu + \delta_1 + p (1 - b)) \int_0^{h_2} G(\tau) \int_{t-\tau}^{t} (s - t + \tau) y^2(s)ds d\tau.
\]
Then, for the functional $Q = W + V_2$ we obtain

$$LQ \leq \left[ -A \left( 2\beta U_1^* a - 2\beta a \max \left\{ U_1^*, \frac{\Lambda}{\mu} - U_1^* \right\} - \beta S^* a + 2\mu + \delta_1 + p(1-b) + \mu p b h_2 \right) 
+ 2\mu + 2\beta U_1^* a - \beta S^* a - 2\beta a \max \left\{ U_1^*, \frac{\Lambda}{\mu} - U_1^* \right\} \right] x^2 + (1-A)\sigma^2 (x + S^*)^2 (y + U_1^*)^2 
- \left[ A (\mu + 2\delta_1 + 2p(1-b) - (3\mu + 2\delta_1 + 2p(1-b))p b h_2) - (\mu + \delta_1 + p(1-b)) \right] y^2. $$

The discussion is quite similar to the one used in the proof of Theorem 5.1 and the details have been omitted here.

6. **Real world example.** The teenage years are typically a period of experimentation, regardless of parenting skills and influence. Parents typically worry about their child becoming dependent on drugs, such as cocaine, ecstasy, methamphetamines and heroin [8]. In accordance with this, the National Survey on Drug Use and Health (NSDUH) picks up information of illicit drug use for Americans aged 12 or older. They represent general population. The mean number of years from first heroin use is 8.5 years for heroin abuse and 9.7 years for dependence [32]. In agreement with this facts, if 01.01.2014. is chosen as initial date, in a group of susceptible individuals are all Americans aged 12 and older who used heroin at least once in their life from 2004 to 2012 but did not become dependent. The number of susceptible individuals $S^0$ is calculated as a sum of the number of heroin initiates who did not became dependent for each year from 2004 to 2012. For example, in 2004, there were 118 000 past year heroin initiates [12]. Then, if we suppose that 14% susceptible individuals became addicted, the number of susceptible individuals was $S^0 = 1 032 860$. The number of individuals in general population that join susceptible population for 2013 (past year heroin initiates) was $\Lambda = 169 000$ [12].

System (1) has two delays. The first time delay is used to describe the time needed for a susceptible individual to become an infectious heroin user, where is supposed that infectious heroin users are all people that abuse heroin or are dependent heroin users. Having in mind that the mean number of years from first heroin use to remission from the most recent episode was 8.5 years for heroin abuse and 9.7 years for dependence [32], we choose $h_1 = 10$ years. The second time delay $h_2$ is used to describe the time needed for a heroin user after treatment (in a period of treatment heroin users have abstinence) returns to untreated heroin user. The rate of relapse among heroin addicts is extremely high. Studies have shown that relapse rate among addicts during the first 3 months of abstinence is as high as 90% [7], so $h_2 = 0.25$ year. Then $h = \max\{h_1, h_2\} = 10$ years. The initial condition $U_1(\theta) = \varphi(\theta)$, where $\theta \in [-10, 0]$, represents the time function of the number of heroin users not in treatment, i.e. past heroin dependence or abuse among individuals aged 12 or older during the period from 2003 to 2013 and is obtained by interpolation of values that are taken over from Figure 6 [12]. For a given set of points we construct appropriate curve (see Figure 1).

Function $f(\tau)$, $\tau \in [0, 10]$ represents the distribution of the time needed for heroin dependence and abuse among individuals. Using Figure 3 from [32], where proportions of onset of heroin abuse or dependence by year after first heroin use among individuals with the corresponding heroin use disorder is presented, cumulative distribution function (CDF) is obtained. On the Figure 2 (left) the data from Figure 3 [32] and the graph of appropriate CDF are shown. This graph corresponds
to the CDF of truncated Weibull distribution over interval \([0, 10]\) with parameters \((0.25, 1.3)\). Finally,
\[
f(\tau) = \begin{cases} 
\frac{0.36688 - 0.912263 \tau^{0.35}}{\tau^{0.85}}, & 0 < \tau \leq 10, \\
0, & \text{else,}
\end{cases}
\]
and \(a = 0.768237\).

Function \(g(\tau), \tau \in [0, 0.25]\) represents the distribution of the time needed for a heroin user to return to heroin use after treatment. Study \([7]\) reports that relapse rate among addicts is as high as 90\% during the first three months of abstinence and that at least 59\% \([16]\) of those who use heroin will became heroin users within the first week. 80\% would relapse within a month after discharging from a detox program \([9]\). Those data are compared with CDF of double truncated Cauchy distribution over interval \([0, 0.25]\) with parameters \((-0.011, 0.005)\) and are shown on Figure 2 (right). Then,
\[
g(\tau) = \begin{cases} 
\frac{490.83}{1 + 4900(0.011 + \tau)^2}, & 0 < \tau \leq 0.25, \\
0, & \text{else,}
\end{cases}
\]
and \(b = 0.997409\).
If the population is growing, it seems that the mass-action incidence term must have an extra factor of $N$ hidden within the definition of $\beta$. Hethcote [5] introduced the incidence term by defining $\tilde{\beta}$ to be the average number of adequate contacts per person per unit time, and then for a given infective individual the chance that such a contact with susceptible individual is $\frac{S}{N}$. Of the people that try heroin, nearly one out of every four, approximately 24%, will become addicted [22], so $\tilde{\beta} = 0.24$. Multiply by the number of heroin users not in treatment to get the standard incidence $\tilde{\beta} = \int_0^{h_1} F(\tau) U_1(t - \tau) d\tau$ and then, comparing to the bilinear incidence, we see that we must have $\beta = \tilde{\beta} = \frac{0.24}{1032500 + 417000 + 526000}$, where 517 000 is the number of heroin users not in treatment, and 526 000 heroin users in treatment in 2013 [12].

Heroin addiction treatment includes many different changes in an addict’s life. Apart from many different medications, therapies and support groups, it requires a complete change of lifestyle of that heroin user, too. Researchers have speculated that there may be something akin to spontaneous remission among addicts, but until recently it was thought that the percentages of such recoveries were very small (5-15%) and insignificant [30], for example 5%. On the other side, some experts place the rate of relapse for heroin addicts as high as 90%, which means that the recovery rate may be as low as 10 percent [7], for example 7%. Since untreated heroin addiction carries a mortality rate of 2 to 3 percent per year [28], for example 2.5%, we conclude that mortality rate of heroin users in treatment is 1% (it is natural to assume that mortality rate of heroin users in treatment is less than mortality rate of untreated heroin users). Then, $\delta_1 = 0.05 + 0.025 = 0.075$ and $\delta_2 = 0.07 + 0.01 = 0.08$.

The Surgeon General’s report on Alcohol, Drugs and Health [26] points out that only about one in 10 people with a substance use disorder receive any of specialty treatment, and we suppose that the value for parameter $p$ is 0.08.

Finally, the natural death rate $\mu$ of the susceptible population is 7.389 per 1000, since death rate in the USA for 2013 was 8.21 per 1000 [27] and rate of non-natural deaths was less than 10% of total all-cause mortality [23].

If we change the constants from system (8) with their values, that are determined in this section, the endemic equilibrium is $E^* = (839722, 1971480)$, reproduction number is $R_0 = 27.2374$ and $\frac{\lambda}{\mu} = 2.28718 \cdot 10^7$. In order to the solution of system (3) goes around heroin spread equilibrium $E^*$ of deterministic system (1), from condition (17) the white noise has intensity $\sigma^2 < 3.27734 \cdot 10^{-16}$. In this case, since $\frac{\delta_1 + p}{3p} = 0.645833 < b$, condition (19) of Theorem 5.1 is violated. On the other side, it holds that $h_2 = 0.25 < 5.45973$, and conditions (30) and (II) of Theorem 5.2 are satisfied. This means that solution of system (8) fluctuates around heroin spread equilibrium (figures 3, 4 and 5 (left)). According to data from this section, $U_1(4)$ takes value 592 604, from deterministic system (1), and value 594 591 from stochastic system (8) on 1.1.2018. (Figure 5 (right)). From [10], 618000 people suffer from heroin use disorder for the same year. Considered model well approximates real data and show us some alarming futures data. The number of people who use heroin is large. Unless some more intensive measures to suppress the addiction are taken, prevention being the first one, the number of people using heroin will continue to grow.

Acknowledgments. The authors have expressed their appreciations to the anonymous referees for their very helpful suggestions which greatly improved the paper.
Figure 3. The graph of the deterministic model (1) and the stochastic trajectory of the number of susceptible individuals in USA from 1.1.2014.

Figure 4. The graph of the deterministic model (1) and the stochastic trajectory of the number of heroin users not in treatment in USA from 1.1.2014.

Figure 5. Stochastic trajectories of the number of susceptible individuals and heroin users not in treatment in USA from 1.1.2014. (left); stochastic trajectory of the number of heroin users not in treatment and real data (right)

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Received May 2019; revised July 2019.

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