On the algebra of local unitary invariants of pure and mixed quantum states

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Abstract

We study the structure of the inverse limit of the graded algebras of local unitary invariant polynomials using its Hilbert series. For \( k \) subsystems, we show that the inverse limit is a free algebra and the number of algebraically independent generators with homogenous degree \( 2m \) equals the number of conjugacy classes of index \( m \) subgroups in a free group on \( k - 1 \) generators. Similarly, we show that the inverse limit in the case of \( k \)-partite mixed state invariants is free and the number of algebraically independent generators with homogenous degree \( m \) equals the number of conjugacy classes of index \( m \) subgroups in a free group on \( k \) generators. The two statements are shown to be equivalent. To illustrate the equivalence, using the representation theory of the unitary groups, we obtain all invariants in the \( m = 2 \) graded parts and express them in a simple form both in the case of mixed and pure states. The transformation between the two forms is also derived. Analogous invariants of higher degree are also introduced.

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1. Introduction

Entanglement of a quantum state in a composite system is captured by the values of entanglement measures, functions which are invariant under the action of some group modelling manipulations which are performed locally, without interaction between the subsystems. In most cases, this group is either the LU (local unitary) group which describes local transformations that can be applied with probability 1, or the stochastic local operations and classical communication (SLOCC) group corresponding to transformations that can be performed with nonzero probability [1]. For a review on quantum entanglement, see [2].

Previous works focus mostly on composite systems with distinguishable constituents [3–6], all of which has the same Hilbert space dimension. Consider that the LU group has the advantage that invariant polynomials can be written in a form that is independent of the Hilbert space dimension, at least when it is large enough.
The outline of the paper is as follows. In section 2, we summarize some facts about how the algebra of polynomials in the coefficients of a state and their conjugates decomposes to irreducible representations under the action of the local unitary group. In section 3, we present an exact formula for the dimension of the subspace of degree $2m$ LU-invariant polynomials for any number of subsystems provided the single-particle state spaces are all at least $m$ dimensional. We interpret these dimensions as that of the degree $m$ homogenous subspace of the inverse limit of the algebras of LU-invariant polynomials.

In section 4, we collect some facts about finite coverings of a graph. In particular, we describe a bijection between conjugacy classes of finite index subgroups of a free group, finite coverings of a certain graph and orbits of tuples of permutations under simultaneous conjugation following [7].

In section 5, we prove that the algebra of local unitary invariants is free by giving an algebraically independent generating set. Our proof makes use of the invariants introduced in [8].

In section 6, we refine the derivation of the stable dimension formula, and conclude that in the case when only one single-particle state space is large enough, a relatively simple formula can also be obtained, and has an interpretation in terms of mixed state polynomial invariants. Next we formulate and prove this correspondence rigorously, and we also repeat the inverse limit construction in the case of mixed state invariants.

In section 7, the space of the fourth-order LU-invariants of multipartite quantum systems with arbitrary dimensional single-particle states is described completely. It is shown, in particular, that the dimension of this space is $2^k - 1$ where $k$ denotes the number of subsystems. The invariants are expressed in terms of both pure and mixed states, and the transformation relating the two is also found. In section 8, analogous invariants with higher degree are introduced.

### 2. Polynomial invariants under the local unitary group

Let $k \in \mathbb{N}$ and $n = (n_1, \ldots, n_k) \in \mathbb{N}^k$, and consider the complex Hilbert space $\mathcal{H}_n = \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_k}$ describing the pure states of a composite system with $k$ distinguishable subsystems. The group of local unitary transformations, $LU_n = U(n_1, \mathbb{C}) \times \cdots \times U(n_k, \mathbb{C})$, acts on $\mathcal{H}_n$ in the obvious way, i.e. regarding $\mathbb{C}^{n_i}$ as the standard representation of $U(n_i, \mathbb{C})$.

The functions $\mathcal{H}_n \rightarrow \mathbb{C}$ which are polynomial in the coefficients and their conjugates with respect to an arbitrary fixed basis are in one-to-one correspondence with the vectors in $S(\mathcal{H}_n \oplus \mathcal{H}_n^*)$, the symmetric algebra on $\mathcal{H}_n \oplus \mathcal{H}_n^*$ on which an action of $LU_n$ is induced. We are looking for invariant functions, i.e. vectors in the symmetric algebra which are fixed under this action.

As each graded part is fixed by $LU_n$, it suffices to look at the homogenous subspaces. By the isomorphism

$$S^p(\mathcal{H}_n \oplus \mathcal{H}_n^*) \cong \bigoplus_{m=0}^{p} S^m(\mathcal{H}_n) \otimes S^{p-m}(\mathcal{H}_n^*), \quad (1)$$

these split further into LU-invariant subspaces. The action of $Z(U(n_1, \mathbb{C})) \times \{1\} \times \cdots \times \{1\} \cong U(1) \ni \lambda$ on $S^m(\mathcal{H}_n) \otimes S^{p-m}(\mathcal{H}_n^*)$ is a multiplication by $\lambda^{m-(p-m)} = \lambda^{2m-p}$, implying that the $m = \frac{p}{2}$ term is the only one we need to look at. In particular, the order of an invariant polynomial is always even. Note that this fact makes it convenient for us to use a grading on the algebra of invariants which is different from the usual one, and take homogenous degree $m$ elements to be polynomials with degree $m$ both in the components and their conjugates, which is actually a degree $2m$ (real) polynomial.
From now on, \( m \) will denote \( \frac{p^2}{2} \). Observe that 
\[
S^m(H_n^*) = S^m(H_n)^*.
\]
be the decomposition to the orthogonal sum of isotypic subspaces, where \( c_a \) are nonnegative integers and \( A \) is a set labelling the isomorphism-classes of irreducible representations of \( \text{LU}_n \). Then 
\[
S^m(H_n^*) \otimes S^m(H_n)^* \simeq \bigoplus_{a, a' \in A} c_a c_{a'} V_a \otimes V_{a'}^*.
\]
The multiplicity of the trivial representation in \( V_a \otimes V_{a'}^* \) is 1 if \( a = a' \) and 0 if \( a \neq a' \). We can conclude that for \( a \in A \) there are three possibilities:

1. \( c_a = 0 \): in this case, we do not obtain any invariants;
2. \( c_a = 1 \): in this case, there is a one-dimensional subspace of invariant polynomials in \( V_a \otimes V_{a'}^* \leq S^m(H_n^*) \otimes S^m(H_n)^* \), and we can choose a 'canonical' element spanning it;
3. \( c_a > 1 \): in this case, we obtain a \( c^2_a \)-dimensional space of invariants in which we cannot find a 'distinguished' basis in any obvious way.

In sections 7 and 8, we concentrate on those components for which the coefficient \( c_a \) is 1.

The above-mentioned distinguished element can be obtained as follows. Up to normalization, there exists a unique inner product on \( S^m(H_n) \) which is invariant under the induced action of the full unitary group acting on \( H_n \). This follows from the fact that \( S^m(H_n) \) carries an irreducible representation of this group. We choose the normalization so that for any \( \psi \in H_n \), the equation \( \|\psi^m\| = \|\psi\|^m \) holds, where \( \psi^m = \psi \otimes \cdots \otimes \psi \in S^m(H_n) \). Now let \( P_a \) denote the orthogonal projection onto the irreducible subrepresentation indexed by \( a \). The value of the distinguished invariant polynomial on \( \psi \in H_n \) is then \( \langle \psi^m, P_a \psi^m \rangle \).

3. Stabilized dimensions of the spaces of invariant polynomials

As was mentioned in section 2, the dimension of the space of degree \( 2m \) polynomial invariants on \( H_n \) is
\[
\sum_{a \in A} c_a^2,
\]
where \( A \) is an index set labelling the equivalence classes of irreducible representations of the LU group, and \( \{c_a\}_{a \in A} \) are the multiplicities of the irreducible representations in \( S^m(H_n) \).

Let \( \nu \) be a partition of \( m \) (this fact will be denoted by \( \nu \vdash m \)). Denoting the corresponding Schur functor by \( S_\nu \), we have the following isomorphism [9]:
\[
S_\nu H_n \simeq \bigoplus_{\lambda_1, \ldots, \lambda_k \vdash m} C_{\nu, \lambda_1, \ldots, \lambda_k} S_{\lambda_1} C^{\lambda_1} \otimes \cdots \otimes S_{\lambda_k} C^{\lambda_k},
\]
where \( C_{\nu, \lambda_1, \ldots, \lambda_k} \) is the multiplicity of \( V_\nu \) in \( V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_k} \) as a representation of the symmetric group \( S_m \), where \( V_\nu \) denotes the irreducible \( S_m \) representation indexed by the partition \( \nu \).

Denoting the character of \( V_\nu \) by \( \chi_\nu \), we can write
\[
C_{\nu, \lambda_1, \ldots, \lambda_k} = (\chi_\nu, \chi_{\lambda_1}, \chi_{\lambda_2}, \ldots, \chi_{\lambda_k})_{S_m},
\]
where we denoted by \( (\cdot, \cdot)_{S_m} \) the usual inner product on the space of class functions on \( S_m \).

Our convention is that this inner product is semilinear in the first and linear in the second argument, but this does not have much effect as all the appearing characters are real.

Note that \( S_\nu(H_n) \) is the zero vector space if and only if \( l(\lambda_i) > \dim H_i \). Therefore, when some of the Hilbert spaces are less than \( m \) dimensional, some of the terms in equation (5) may
vanish, decreasing the dimension of the space of invariants. On the other hand, if \( \forall i : n_i \geq m \), this dimension is independent of the exact values of the \( n_i \), meaning that for sufficiently large single-particle state spaces, the dimension stabilizes, and depends only on \( k \) and \( m \).

Let \( I_{k,n} \) denote the graded algebra of polynomials over \( \mathcal{H}_n \) which are invariant under the action of \( LU_n \). Suppose that \( n, n' \in \mathbb{N}^k \) such that \( n \leq n' \) with respect to the componentwise (product) order. Then we have the inclusion \( \iota_{n,n'} : \mathcal{H}_n \hookrightarrow \mathcal{H}_{n'} \) which is the tensor product of the usual inclusions \( \mathbb{C}^{n_i} \hookrightarrow \mathbb{C}^{n'_i} \), sending an \( n_i \) tuple to the first \( n_i \) components. We can similarly regard \( LU_n \) as a subgroup of \( LU_{n'} \) which stabilizes the image of \( \iota_{n,n'} \), and thus \( \iota_{n,n'} \) is an \( LU_{n'} \)-equivariant linear map. Therefore, it induces a morphism of graded algebras \( \varrho_{n,n'} : I_{k,n} \rightarrow I_{k,n} \) (note that the algebra of polynomials on a vector space is the symmetric algebra of its dual space, which is a contravariant construction).

Clearly, \( \iota_{n,n} \) is the identity and if \( n \leq n' \leq n'' \), then \( \iota_{n',n''} \circ \iota_{n,n'} = \iota_{n,n''} \), which implies that \( \varrho_{n,n} = \text{id}_{I_{k,n}} \) and \( \varrho_{n,n'} \circ \varrho_{n',n''} = \varrho_{n,n''} \). The central object which we study is the inverse limit of this system of graded algebras and their morphisms:

\[
I_k := \lim_{n \in \mathbb{N}^k} I_{k,n} = \left\{ \left( f_n \right)_{n \in \mathbb{N}^k} \in \prod_{n \in \mathbb{N}^k} I_{k,n} \mid \forall n \leq n' : f_n = \varrho_{n,n'} f_{n'} \right\}.
\]

Note that as the product is taken in the category of graded algebras, it consists of sequences with bounded degree. We will call \( I_k \) the algebra of \( LU \)-invariants.

**Lemma 1.** Suppose that \( n, n' \in \mathbb{N}^k \) and \( n \leq n' \). Let \( m \in \mathbb{N} \) such that for all \( i \) we have \( m \leq n_i \). Then the restriction of \( \varrho_{n,n'} \) is an isomorphism (of vector spaces) between the spaces of homogeneous degree \( m \) elements of \( I_{k,n'} \) and \( I_{k,n} \).

**Proof.** As was already noted, the dimension of the two homogenous parts is equal. We will show that \( \varrho_{n,n'} \) is injective on elements of degree at most \( m \).

Suppose first that for some \( 1 \leq i \leq k, n_i = n'_i - 1 \) and for \( j \neq i, n_j = n'_j \). Let \( \left\{ e_{1,1}, \ldots, e_{n_i,\ldots,n_i} \right\} \) be the basis of \( \mathcal{H}_{n_i} \) formed by tensor products of standard basis elements of the \( \mathbb{C}^{n_i} \). Then the algebra of real polynomials is generated by the coordinate functions \( \left\{ e^*_1, \ldots, e^*_{n_i,\ldots,n_i} \right\} \) and their conjugates \( \left\{ e'^*_1, \ldots, e'^*_{n_i,\ldots,n_i} \right\} \).

Let \( f \in I_{k,n'} \) be a degree \( m \) homogeneous polynomial such that \( \varrho_{n,n'} f = 0 \). This means that \( f \) vanishes on the image of \( \iota_{n,n'} \). Denoting by \( J_n \) the ideal generated by elements of the form \( \left\{ e_{j_1,\ldots,j_k} \right\} \) such that \( j_i = a \), we can reformulate this fact as \( f \in J_n \). Note that \( f \) is invariant under the action of \( LU_n \), and we have the subgroup \( S_{n_i} \leq LU_n \) which permutes the basis elements in the \( i \)th factor \( \mathbb{C}^{n_i} \); therefore, \( f \) is also contained in the ideals \( J_1, \ldots, J_{n_i} \).

But the intersection of the ideals \( J_1, \ldots, J_{n_i} \) is their product; therefore, \( f \) is in the ideal generated by \( n_i \)-fold products of the coordinate functions. As \( f \) is \( LU_n \)-invariant, its terms must contain the same number of conjugate coordinate functions, and therefore its homogenous parts of degree less than \( n_i \) vanish. \( m \leq n_i \) implies that \( f = 0 \).

For the general case, observe that if \( n \leq n' \), then \( \varrho_{n,n'} \) can be written as a composition of the maps considered above (or is the identity in the case of \( n = n' \)), and hence also injective. \( \square \)

This lemma means that every element of \( I_k \) is represented in some \( I_{k,n} \) (it suffices to take \( n \) to be \( (m, m, \ldots, m) \) with \( m \) the degree of the element), and that if \( n_{\min} = \min(n_i) \), then the factors of \( I_k \) and \( I_{k,n} \) by the ideals generated by homogenous elements of degree at least \( n_{\min} + 1 \) are isomorphic. Therefore, the algebras \( I_{k,n} \) and \( I_k \) are closely related, while the latter seems considerably simpler to study. Our next aim is to calculate the Hilbert series of \( I_k \).

Let \( d_{k,m} \) denote the stabilized dimension of degree \( m \) \( LU \)-invariants for a composite system of \( k \) subsystems for which a remarkable formula was found by Hero and Willenbring [10].
We present here a slightly different derivation for later convenience. The value of \( d_{k,m} \) can be expressed as follows:

\[
d_{k,m} = \sum_{\lambda_1,\ldots,\lambda_k \vdash m} C_{(m)\lambda_1,\ldots,\lambda_k}^2
\]

\[
= \sum_{\lambda_1,\ldots,\lambda_k \vdash m} (\chi(\lambda_1) \cdots \chi(\lambda_k))^2
\]

\[
= \sum_{\lambda_1,\ldots,\lambda_k \vdash m} (\chi(\lambda_1) \cdots \chi(\lambda_k))(\chi(\lambda_1) \cdots \chi(\lambda_k))
\]

\[
= \sum_{\lambda_1,\ldots,\lambda_k \vdash m} (\chi(\lambda_1) \cdots \chi(\lambda_k) \chi(\lambda_1) \cdots \chi(\lambda_k))
\]

\[
= \sum_{\lambda_1,\ldots,\lambda_k \vdash m} (\chi(\lambda_1) \chi(\lambda_1) \cdots \chi(\lambda_k) \chi(\lambda_1) \cdots \chi(\lambda_k - 1))
\]

\[
= \sum_{\lambda \vdash m} (\chi(\lambda_1) \cdots \chi(\lambda_k - 1))
\]

\[
= \left( \chi(\lambda), \sum_{\lambda_1,\ldots,\lambda_k \vdash m} \chi(\lambda_1) \cdots \chi(\lambda_k - 1) \right)
\]

\[
= \left( \chi(\lambda), \left( \sum_{\lambda \vdash m} \chi(\lambda) \right)^{k-1} \right).
\]

(8)

For a finite group \( G \), the sum of the squares of absolute values of the characters of inequivalent irreducible representations gives the character of the representation on the group algebra \( CG \) by conjugation. The value of this character on \( g \) is the number of the elements in \( G \) which commute with \( g \). Therefore, denoting by \( R \) a set of representatives of the conjugacy classes in \( S_m \), we have that

\[
\left( \chi(\lambda), \left( \sum_{\lambda \vdash m} \chi(\lambda) \right)^{k-1} \right) = \frac{1}{|G|} \sum_{g \in S_m} \chi_{\text{conj}.}(g)^{k-1}
\]

\[
= \frac{1}{|G|} \sum_{g \in R} |C_{\text{conjugacy}}(g)| \chi_{\text{conj}.}(g)^{k-1}
\]

\[
= \frac{1}{|G|} \sum_{g \in R} |S_m| |Z_{\text{conjugacy}}(g)|^{k-1}
\]

\[
= \sum_{g \in R} |Z_{\text{conjugacy}}(g)|^{k-2}.
\]

(9)

where \( Z_G(g) \) denotes the centralizer, and \( C_G(g) \) denotes the conjugacy class of \( g \).

Conjugacy classes in \( S_m \) may be conveniently labelled by cycle types, which are \( m \)-tuples of nonnegative integers, the \( j \)th integer being the number of \( j \)-cycles when an (arbitrary) element of the given conjugacy class is written as a product of disjoint cycles. Clearly, \( a = (a_1, \ldots, a_m) \) describes a cycle type if and only if it consists of nonnegative integers and

\[
\sum_{i=1}^{m} i a_i = m
\]

(10)

holds. We will denote this fact by \( a \vdash m \).

If \( g \) is an element with cycle-type \( a \), then the order of its centralizer is given by the formula

\[
|C_{\text{conjugacy}}(g)| = \prod_{i=1}^{m} i^{a_i} a_i !.
\]

(11)
Table 1. Stable dimensions for small values of $k$ and $m$.

| $m$ = | 1 | 2 | 3 | 4 | 5 |
|-------|---|---|---|---|---|
| 1     | 1 | 1 | 1 | 1 | 1 |
| 2     | 1 | 2 | 4 | 8 | 16|
| 3     | 1 | 3 | 11| 49| 251|
| 4     | 1 | 5 | 43| 681|14491|
| 5     | 1 | 7 | 161|14721|1730861|
| 6     | 1 | 11| 901|524137|⋯ |
| 7     | 1 | 15| 5579|25471105|⋯ |
| 8     | 1 | 22| 43206|⋯ |
| 9     | 1 | 30| 378360|⋯ |
| 10    | 1 | 42| 3742738|⋯ |

We conclude that

$$d_{k,m} = \sum_{a \vdash m} \left( \prod_{i=1}^{m} i^{a_i}! \right)^{k-2} t^m.$$  

(12)

Table 1 shows the values of $d_{k,m}$ for some small values of $k$ and $m$.

The Hilbert series of the algebra $I_k$ is the formal power series

$$\sum_{m \geq 0} d_{k,m} t^m.$$  

(13)

Using equation (12), this can be rewritten as

$$\sum_{m \geq 0} d_{k,m} t^m = \sum_{m \geq 0} \sum_{a \vdash m} \left( \prod_{i=1}^{m} i^{a_i}! \right)^{k-2} t^m$$

$$= \sum_{a_1, a_2, \ldots \geq 0} \prod_{i=1}^{m} (i^{a_i} t)^{k-2} t^{i^{a_i}}$$

$$= \prod_{i \geq 1} \left( \sum_{a \geq 0} (i^a t)^{k-2} t^a \right)$$

$$= \prod_{d \geq 1} (1 - t^d)^{-u_d(I_{k-1})}.$$  

(14)

where in the last row, $u_d(G)$ denotes the number of conjugacy classes of index $d$ subgroups of a group $G$ and $F_{k-1}$ is the free group on $k-1$ generators. This last equality can be found in [11]. Some values of $u_d(F_{k-1})$ are shown in table 2 [12].

We regard this equality as an indication that the algebra of LU-invariants $I_k$ of $k$-partite quantum systems might be free, and the number of degree 2 $d$ invariants in an algebraically independent generating set equals the number of conjugacy classes of index $d$ subgroups in the free group on $k-1$ generators. We will prove this in the next two sections.

4. Graph coverings

Let $G = (V, E)$ be a connected graph with directed edges (possibly multiple edges and/or loops). A graph $\tilde{G} = (\tilde{V}, \tilde{E})$ together with a projection $p : \tilde{G} \to G$ is said to be a covering
Table 2. Number of conjugacy classes of index \( d \) subgroups of the free group on \( k - 1 \) generators.

| \( d = 1 \) | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| \( k = 1 \) | 1 | 1 | 1 | 1 |
| 2 | 0 | 1 | 3 | 7 | 15 |
| 3 | 0 | 1 | 7 | 41 | 235 |
| 4 | 0 | 1 | 26 | 604 | 14120 |
| 5 | 0 | 1 | 97 | 13753 | 1712845 |
| 6 | 0 | 1 | 624 | 504243 | \( \ldots \) |
| 7 | 0 | 1 | 4163 | 24824785 | \( \ldots \) |
| 8 | 0 | 1 | 34470 | \( \ldots \) |
| 9 | 0 | 1 | 314493 | \( \ldots \) |
| 10 | 0 | 1 | 3202839 | \( \ldots \) |

of \( G \) if \( p_V : \tilde{V} \to V \) and \( p_E : \tilde{E} \to E \) are two surjections where the image of the head (tail) of an edge is the head (tail) of its image, such that the indegree and outdegree of every vertex \( \tilde{v} \in \tilde{V} \) are the same as those of \( p_V(\tilde{v}) \). A covering \( p : \tilde{G} \to G \) is said to be finite if \( |p_{V}^{-1}(v)| < \infty \) and \( m \)-fold if \( |p_{V}^{-1}(v)| = m \) for all \( v \in G \).

Two coverings \( p_1 : \tilde{G}_1 \to G \) and \( p_2 : \tilde{G}_2 \to G \) are said to be isomorphic if there exists an isomorphism \( \varphi : \tilde{G}_1 \to \tilde{G}_2 \) making the following diagram commute:

\[
\begin{array}{ccc}
\tilde{G}_1 & \xrightarrow{\varphi} & \tilde{G}_2 \\
p_1 \downarrow & & \downarrow p_2 \\
G & & G
\end{array}
\]

The set of isomorphism classes of finite coverings of a graph \( G \) will be denoted by \( \text{Iso}(G) \), the set of isomorphism classes of \( m \)-fold coverings by \( \text{Iso}(G, m) \), and the connected ones by \( \text{Isoc}(G) \) and \( \text{Isoc}(G, m) \), respectively. Our convention is that the empty graph \( (\emptyset, \emptyset) \) is a \( (0 \text{-fold}) \) covering of \( G \), but not connected.

Let \( G \) be the graph with a single vertex and \( k - 1 \) directed loops. It is well known that its fundamental group \( \pi_1(G) \) is the free group of rank \( k - 1 \), and a set of generators may be identified with the directed loops. There exists a bijection between \( \text{Isoc}(G, m) \) and conjugacy classes of subgroups of index \( m \) of \( \pi_1(G) \) (recall that a subgroup of \( \pi_1(G) \) corresponds to a basepointed covering, and changing the basepoint corresponds to conjugation). For simplicity, let us assume that the edges of \( G \) are coloured with \( 1, \ldots, k - 1 \), and for any covering \( \tilde{G} \) we pull back the colours with \( p \) to \( \tilde{G} \). A covering of \( G \) is then equivalently a directed coloured graph with one head and one tail of each colour at each vertex, and an isomorphism respecting the colours is precisely a covering isomorphism.

Let \( S_m \) denote the group of bijections from \( \{1, \ldots, m\} \) to itself. This group acts on \( S_m^{k-1} = S_m \times S_m \times \cdots \times S_m \) by simultaneous conjugation:

\[
\pi \cdot (\sigma_1, \ldots, \sigma_{k-1}) = (\pi \sigma_1 \pi^{-1}, \ldots, \pi \sigma_{k-1} \pi^{-1}).
\]

Let us denote the orbit of \( (\sigma_1, \ldots, \sigma_{k-1}) \in S_m^{k-1} \) by

\[
[\sigma_1, \ldots, \sigma_{k-1}] = \{ \pi \cdot (\sigma_1, \ldots, \sigma_{k-1}) | \pi \in S_m \}.
\]

With a (not necessarily connected) \( m \)-fold covering \( \tilde{G} \) of \( G \), we can associate an element in

\[
S_m^{k-1}/S_m = \{ [\sigma_1, \ldots, \sigma_{k-1}] | v_i : \sigma_i \in S_m \}
\]
as follows. Label the vertices of $\tilde{G}$ arbitrarily with the numbers $\{1, \ldots, m\}$ using each label exactly once. Let $\sigma_i$ be the permutation which sends $a$ to $b$ if there is a directed edge from $a$ to $b$ of colour $i$ in $G$. Note that this gives indeed a $(k-1)$-tuple of permutations as the indegree and outdegree of every vertex in $\tilde{G}$ are 1 in the subgraph determined by any colour. As relabelling corresponds to simultaneous conjugation, we have indeed a well-defined map $\Phi: \text{Iso}(G, m) \to S_m^{k-1}/S_m$.

Now let $\tilde{G}_1$ and $\tilde{G}_2$ be two coverings of $G$ where $\tilde{G}_1$ is $m_1$-fold and $\tilde{G}_2$ is $m_2$-fold. The disjoint union $\tilde{G} := \tilde{G}_1 \sqcup \tilde{G}_2$ is then an $(m_1+m_2)$-fold covering of $G$. We would like to relate the orbits of $(k-1)$-tuples of permutations of the three coverings. Let us choose the numbering of the vertices of $\tilde{G}$ so that $\tilde{G}_1$ is labelled with $\{1, \ldots, m_1\}$ and $\tilde{G}_2$ is labelled with $\{m_1+1, \ldots, m_1+m_2\}$.

Let $(\sigma_1^{(1)}, \ldots, \sigma_{k-1}^{(1)})$ be the representative of the orbit corresponding to $\tilde{G}_j$ and $(\sigma_1, \ldots, \sigma_{k-1})$ be that of $\tilde{G}$ which can be read off from the above-chosen labelling (after subtracting $m_1$ in the $j=2$ case). It is easy to see that for all $1 \leq i \leq k-1$ we have

$$\sigma_i(a) = \begin{cases} 
\sigma_i^{(1)}(a) & \text{if } a \leq m_1 \\
\sigma_i^{(2)}(a - m_1) + m_1 & \text{if } a > m_1
\end{cases} \tag{19}$$

given by the usual homomorphism $S_m \times S_{m_2} \to S_{m+m_2}$. This map clearly induces a map $\star: S_m^{k-1}/S_m \times S_m^{k-1}/S_{m_2} \to S_m^{k-1}/S_{m+m_2}$ on the orbits (we will use infix notation, i.e. the image of $(a, b)$ is $a \star b$). One can see immediately that $\star$ turns the set $\bigcup_{m=0}^{\infty} S_m^{k-1}/S_m$ into a commutative monoid. Also, $\text{Iso}(G)$ can be equipped with a monoid structure induced by disjoint union, and $\Phi$ is an isomorphism.

5. Algebraically independent generators of the algebra of LU-invariants

With an orbit in $S_m^{k-1}/S_m$ we can associate an element in $I_k$ as follows. We have seen that every element of $I_k$ is represented in some $I_{k,n}$. We will give a representative in $I_{k,n}$ where $n = (n_1, \ldots, n_k) \geq (m, \ldots, m)$ following [8]. A vector in $H_n$ is of the form

$$\psi = \sum_{i_1, \ldots, i_k} \psi_{i_1, \ldots, i_k} e_{i_1} \otimes \cdots \otimes e_{i_k}, \tag{20}$$

where in the sum $1 \leq i_j \leq n_j$ for all $1 \leq j \leq k$.

Let $(\sigma_1, \ldots, \sigma_{k-1}) \in S_m^{k-1}$ be a representative. The value of the associated polynomial on $\psi$ is

$$f_{[\sigma_1, \ldots, \sigma_{k-1}]}(\psi) = \sum_{i_1, \ldots, i_k} \psi_{i_1, \ldots, i_k} e_{i_1} \cdots e_{i_k} \cdot \psi_{\sigma_1^{(1)}(i_1), \ldots, \sigma_{k-1}^{(1)}(i_1)} \cdots \psi_{\sigma_1^{(n_k)}(i_k), \ldots, \sigma_{k-1}^{(n_k)}(i_k)}, \tag{21}$$

where the sum is over all $k \cdot m$-tuples of integers where $1 \leq i_j \leq n_j$ for all $1 \leq j \leq k$ and $1 \leq l \leq m$. Note that the expression defining $f_{[\sigma_1, \ldots, \sigma_{k-1}]}$ is independent of the choice of the representative, justifying the notation.

An important observation is the following.

Lemma 2. Let $[\sigma_1^{(1)}, \ldots, \sigma_{k-1}^{(1)}] \in S_m^{k-1}/S_m$ and $[\sigma_1^{(2)}, \ldots, \sigma_{k-1}^{(2)}] \in S_m^{k-1}/S_m$. Then

$$f_{[\sigma_1^{(n_1)}, \ldots, \sigma_{k-1}^{(n_1)}] \star [\sigma_1^{(n_k)}, \ldots, \sigma_{k-1}^{(n_k)}]} = f_{[\sigma_1^{(n_1)}, \ldots, \sigma_{k-1}^{(n_1)}]} f_{[\sigma_1^{(n_k)}, \ldots, \sigma_{k-1}^{(n_k)}]} \cdot$$

(22)
Proof. Let the representative of the orbit $[\sigma_1^{(1)}, \ldots, \sigma_k^{(1)}] \cdot [\sigma_1^{(2)}, \ldots, \sigma_k^{(2)}]$ given by equation (19) be $(\sigma_1, \ldots, \sigma_k) \in S_m^{k-1}$ and let us denote $m_1 + m_2$ by $m$. Then

$$
\sum_{i_1, \ldots, i_k} \psi_i^{(1)} \cdots \psi_i^{(k)} = \sum_{i_1, \ldots, i_k} \sum_{i_{k+1}, \ldots, i_m} \psi_i^{(1)} \cdots \psi_i^{(k)} \psi_i^{(k+1)} \cdots \psi_i^{(m)}
$$

$$
= \sum_{i_1, \ldots, i_k} \psi_i^{(1)} \cdots \psi_i^{(k)} \psi_i^{(k+1)} \cdots \psi_i^{(m)}
$$

$$
= \sum_{i_1, \ldots, i_k} \psi_i^{(1)} \cdots \psi_i^{(k)} \psi_i^{(k+1)} \cdots \psi_i^{(m)}
$$

$$
\cdot \sum_{i_1, \ldots, i_k} \psi_i^{(1)} \cdots \psi_i^{(k)}
$$

$$
(23)
$$

In other words, the map $\bigcup_{m=0}^{\infty} S_m^{k-1} / S_m \rightarrow I_k$ given by $s \mapsto f_s$ is a semigroup-homomorphism.

Now we are ready to prove our main theorem.

Theorem 3. $I_k$ is freely generated by the set

$$
F := \{ f_{\Phi(G)_i} | G \in \text{Iso}(G) \}. (24)
$$

Proof. In [8], it was shown that the set

$$
\{ f_s | s \in S_m^{k-1} / S_m \} = \{ f_{\Phi(G)} | G \in \text{Iso}(G, m) \}
$$

forms a basis of the degree $m$ homogenous subspace of $I_k,n$ (when represented as polynomials) if $n \geq m, \ldots, m)$. Therefore, it is also a basis of the degree $m$ homogenous subspace of $I_k$. As $I_k$ is the direct sum of its homogenous subspaces, we conclude that $\{ f_{\Phi(G)} | G \in \text{Iso}(G) \}$ is a basis of $I_k$. Note that this also implies that the map $G \mapsto f_{\Phi(G)}$ is injective.

An element of the form $f_s$ where $s \in S_m^{k-1} / S_m$ can be uniquely written as the product of some elements of $F$. Indeed, $\Phi^{-1}(s)$ is a covering of $G$, which can be uniquely written as a disjoint union of connected coverings $G_1, \ldots, G_d$ (up to isomorphism and ordering), and therefore

$$
f_s = f_{\Phi(G_1) \cdots \Phi(G_d)} = f_{\Phi(G_1)} \cdots f_{\Phi(G_d)}. (26)
$$

□

6. Mixed state entanglement

Observe that the derivation in equation (8) remains valid when we only assume that $\dim H_k \geq m$ with some modification. If $\mathcal{H} = H_1 \otimes \cdots \otimes H_k$ where $\dim H_i = n_i$ and $n_k \geq m$, then

$$
d_{n_1, \ldots, n_k+m} := \dim(S^{2m}(\mathcal{H} \oplus \mathcal{H}^*))^{LU}
$$

$$
= \chi(m), \prod_{i=1}^{k-1} \sum_{l(\lambda) \leq n_i} \chi_\lambda^2
$$

$$
(27)
$$
Moreover, in this case we do not need $n_k \to \infty$ to have stabilization for every $m$; the condition $n_k \geq n_1 \cdot n_2 \cdot \ldots \cdot n_{k-1}$ is sufficient.

An important special case is when $n_i = 2$ for $i \leq k$. In this case, the above formula reduces to

$$d_{(2,\ldots,2),m} = \dim(S^{2m}(\mathcal{H} \oplus \mathcal{H}^*))^{LU} = \left( \chi(m), \sum_{\lambda \vdash m} \chi^2_{\lambda} \right)_{k-1}.$$  (28)

One can also find a physical interpretation of this condition. Regarding $\mathcal{H}_S := \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{k-1}$ as the Hilbert space of an open quantum system interacting with its environment with state space $\mathcal{H}_{ENV} := \mathcal{H}_k$, the condition above ensures that every mixed state over $\mathcal{H}_S$ arises as the reduced state of a pure state of $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_{ENV}$.

Recall that in this case given a mixed state $\varrho \in \text{End}(\mathcal{H}_S)$, we can find a vector $\psi \in \mathcal{H}$ (called a purification of $\varrho$) such that $\varrho = \text{Tr}_{ENV} \psi \psi^*$ and that $\psi$ is unique up to transformations of the form $id_{\mathcal{H}_1} \otimes U$ where $U \in U(\mathcal{H}_{ENV})$. As $\text{Tr}_{ENV} : \text{End}(\mathcal{H}_S \otimes \mathcal{H}_{ENV}) \to \text{End}(\mathcal{H}_S)$ is $U(\mathcal{H}_S) \times U(\mathcal{H}_{ENV})$-equivariant, we have that purification gives a bijection between LU-equivalence classes of mixed states over $\mathcal{H}_S$ and pure states in $\mathcal{H}$ with the partial trace as inverse. Note that a special case of this phenomenon was already observed in [13].

Denoting the set of mixed states over a Hilbert space $\mathcal{H}$ by

$$D(\mathcal{H}) := \{ A \in \text{End}(\mathcal{H}) | A \geq 0, \text{Tr } A = 1 \}$$  (29)

and the set of unit vectors by

$$P(\mathcal{H}) := \{ \psi \in \mathcal{H} | \| \psi \|^2 = 1 \},$$  (30)

we can write the commutative diagram

$$ \begin{array}{ccc}
P(\mathcal{H}) & \xrightarrow{\text{Tr}_{ENV} \circ P} & D(\mathcal{H}_S) \\
\downarrow & & \downarrow \\
\frac{P(\mathcal{H})}{LU} & \xrightarrow{\sim} & \frac{D(\mathcal{H}_S)}{LU}
\end{array} $$ (31)

where $P : \mathcal{H} \to \text{End}(\mathcal{H})$ is defined by $\psi \mapsto \psi \psi^*$, $\text{LU} = U(\mathcal{H}_1) \times \cdots \times U(\mathcal{H}_{k-1}) \times U(\mathcal{H}_k)$ is the local unitary group acting in the obvious way and the vertical arrows are the factor maps. Note that when $n_k < n_1 \cdot n_2 \cdot \ldots \cdot n_{k-1}$, the lower horizontal map making this diagram commutative (as well as $\text{Tr}_{ENV} \circ P$) fails to be surjective.

We have seen that the LU-equivalence problem of mixed states can be reduced to the LU-equivalence problem of pure states. To deal with this latter problem, one usually seeks for (real) polynomials on the Hilbert space of the composite system which are invariant under the induced action of the LU-group. The reason for this is that invariant polynomials are relatively easy to handle, while still separate the orbits.

In the case of mixed states, polynomial invariants are not the ones which are physically most important for the quantification of entanglement, but they are equally well suited for determining whether two states are equivalent as in the case of pure states. Using the fact that $\text{Tr}_{ENV} \circ P : P(\mathcal{H}) \to D(\mathcal{H}_S)$ is an equivariant polynomial function (of degree 2), from a polynomial invariant $f : D(\mathcal{H}_S) \to \mathbb{C}$ on pure states we can always construct one on mixed states, namely $f \circ \text{Tr}_{ENV} \circ P$. Similarly, if we are given an invariant $g : P(\mathcal{H}) \to \mathbb{C}$, we can
pull it back via the isomorphism $D(\mathcal{H}_S)/LU \to P(\mathcal{H})/LU$ to obtain an invariant on mixed states:

$$
\begin{array}{c}
\overset{g}{\longrightarrow} \\
\overset{\text{Tr}_{\text{ENV}}}{\longrightarrow} \\
\overset{P(\mathcal{H})}{\longrightarrow} \\
\overset{D(\mathcal{H}_S)/LU}{\longrightarrow}
\end{array}
$$

The two constructions are clearly inverses of each other, but it is not clear that $f$ is polynomial whenever $f \circ \text{Tr}_{\text{ENV}} \circ P$ is polynomial.

To prove this, observe that the map $S^m(\text{End}(\mathcal{H}_S)) \to S^m(\mathcal{H} \oplus \mathcal{H}^*)$ defined by $f \mapsto f \circ \text{Tr}_{\text{ENV}} \circ P_1$ is an injective linear map where $P = P_2 \circ P_1$ and $P_1 : \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}^*$ is defined by $\psi \mapsto \psi \oplus \psi^*$ while $P_2 : \mathcal{H} \oplus \mathcal{H}^* \to \mathcal{H} \oplus \mathcal{H}^*$ is defined by $\psi \oplus \varphi^* \mapsto \varphi^*$. As the appearing vector spaces are by assumption finite dimensional, we need to show that these dimensions are equal.

Let $\mathcal{H}_S = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{k-1}$ be the state space of a composite quantum system, and $\mathcal{H}_{\text{ENV}}$ the state space of its environment as before, and let $\mathcal{H} = H_S \otimes \mathcal{H}_{\text{ENV}}$ denote the Hilbert space of the joint system composed of the two. Then

$$
S^m(\mathcal{H} \oplus \mathcal{H}^*)^{U(\mathcal{H}_{\text{ENV}})} \simeq S^m(\mathcal{H} \oplus S^m(\mathcal{H}^*))^{U(\mathcal{H}_{\text{ENV}})}
$$

$$
\simeq \left( \bigoplus_{\lambda \vdash m} S_\lambda \mathcal{H}_S \otimes S_\lambda \mathcal{H}_{\text{ENV}} \otimes S_n \mathcal{H}_S^* \otimes S_n \mathcal{H}_{\text{ENV}}^* \right)^{U(\mathcal{H}_{\text{ENV}})}
$$

$$
\simeq \left( \bigoplus_{\lambda \vdash m} S_\lambda \mathcal{H}_S \otimes S_\lambda \mathcal{H}_{\text{ENV}} \otimes S_n \mathcal{H}_S^* \otimes S_n \mathcal{H}_{\text{ENV}}^* \right)^{U(\mathcal{H}_{\text{ENV}})}
$$

$$
\simeq \left( \bigoplus_{\lambda \vdash m} S_\lambda \mathcal{H}_S \otimes S_\lambda \mathcal{H}_{\text{ENV}} \right)^{U(\mathcal{H}_{\text{ENV}})}
$$

as $U(\mathcal{H}_S) \times U(\mathcal{H}_{\text{ENV}})$-modules where we have used that

$$
\left( S_\lambda \mathcal{H}_S \otimes S_\lambda \mathcal{H}_{\text{ENV}} \right)^{U(\mathcal{H}_{\text{ENV}})} \simeq \begin{cases} 
\mathbb{C} & \text{if } \dim \mathcal{H}_{\text{ENV}} \geq l(\lambda) \\
0 & \text{if } \dim \mathcal{H}_{\text{ENV}} < l(\lambda)
\end{cases}
$$

with the trivial representation on $\mathbb{C}$ and that $S_\lambda \mathcal{H}_S \simeq 0$ iff $l(\lambda) > \dim \mathcal{H}_S \leq \dim \mathcal{H}_{\text{ENV}}$.

Similar to the case of pure state invariants, we may construct the inverse limit of all the algebras of invariant polynomials with a fixed number of subsystems.

For $k \in \mathbb{N}$ and $n \in \mathbb{N}^k$, let $I_k^{\text{mixed}}$ denote the algebra of $LU_n$-invariant (real) polynomials on $\text{End}(\mathcal{H}_n)$. The inclusions $\theta_{n,n'} : \mathcal{H}_n \to \mathcal{H}_{n'}$ induce also in this case the maps $\theta_{n,n'} : I_k^{\text{mixed}} \to I_{k,n'}^{\text{mixed}}$ with similar composition properties. Let us consider the inverse limit of this system:

$$
I_k^{\text{mixed}} := \lim_{n \to \infty} I_k^{\text{mixed}} = \left\{ (f_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} I_k^{\text{mixed}} \left| \forall n : f_n = \theta_{n,n'} f_{n'} \right. \right\}
$$

As in the case of pure state invariants, this inverse limit also has the property that every element is represented in some $I_k^{\text{mixed}}$, and that there is no difference between $I_k^{\text{mixed}}$ and $I_k^{\text{mixed}}$ when we consider only elements with degree at most the minimum of the dimensions $\{n_i\}_{1 \leq i \leq k}$.

The above-shown correspondence between mixed and pure state invariants is clearly reflected in the isomorphism $I_k^{\text{mixed}} \simeq I_{k+1}$ induced by the isomorphisms

$$
I_k^{\text{mixed}} \simeq I_{k+1}(a_1,t,\ldots,a_k)
$$
described above. Note that the grading of $I_k^{\text{mixed}}$ is the usual which is to be contrasted with the extra factor of 2 in the grading of $I_k$. With this convention, the map $f \mapsto f \circ T_{\text{ENV}} \circ P$ respects the degree.

We can also present a set of free generators as follows. Observe that each term on the right-hand side of equation (21) depends only on the reduced density matrix obtained when we trace over the last subsystem. Therefore, it is easy to translate the result to the case of mixed state local unitary invariants. Let $G$ be the graph with a single vertex and $k$ directed coloured edges. With a connected covering $\tilde{G} \in \text{Iso}(G, m)$, we associate the following invariant with $[\sigma_1, \ldots, \sigma_k] = \Phi(\tilde{G})$:

$$f_{[\sigma_1, \ldots, \sigma_k]}(\varrho) = \sum_{i_1, \ldots, i_k} e_{i_1} \otimes \cdots \otimes e_{i_k},$$

where $\varrho = \sum_{j_1, \ldots, j_k} e_{j_1} \otimes \cdots \otimes e_{j_k}$ is an arbitrary mixed state, and these invariants generate $I_k^{\text{mixed}}$ freely.

7. Degree 4 invariants

In this section, we investigate the $m = 2$ case. We will see that this case is special in that $S^2\mathcal{H}$ is the direct sum of irreducible representations with multiplicity at most 1. Indeed, we only need to observe that the two representations of $S_2$, the trivial and the alternating one, are both one dimensional; hence, the tensor product of their arbitrary powers is irreducible. To be specific, as $\chi(2)$ is constant 1 and $\chi(1,1)$ is 1 on the identity and $-1$ on the other element,

$$\left(\chi(2), \chi(2), \chi(1,1)\right)_{S_2} = \frac{1}{2}(1 + (-1)^b) = \begin{cases} 1 & b \text{ even} \\ 0 & b \text{ odd}. \end{cases}$$

Consequently, the irreducible components of $S^2(\mathcal{H})$ can be indexed by even-element subsets of $\{1, \ldots, k\}$, with a bijection sending the set $A$ to the component in which $\mathcal{H}_j’s$ alternating power appears iff $j \in A$. The subspace corresponding to $A$ will be denoted by $V_A$. It follows that the dimension of the space of fourth-order $G$-invariant polynomials is $2^{k-1}$. Following the strategy of [14], our next aim is to construct an orthonormal basis in $V_A$ for each possible subset $A$. We would like to express elements of $S^2(\mathcal{H})$ in terms of a computational basis in $\mathcal{H}$. Let $\{e_{j,i} \mid 1 \leq j \leq k, 1 \leq i \leq n_j\}$ be a set of vectors such that $\{e_{j,i} \mid 1 \leq j \leq n\}$ is an orthonormal basis in $\mathcal{H}_j$. Let us now introduce the following short notation: $e_{i_1, \ldots, i_k} := e_{1,i_1} \otimes e_{2,i_2} \otimes \cdots \otimes e_{k,i_k} \in \mathcal{H}$, where $1 \leq i_j \leq n_j$ (for all $1 \leq j \leq k$). The set of vectors of this form is an orthonormal basis in $\mathcal{H}$. Elements of the symmetric algebra $S(\mathcal{H})$ are polynomials in these vectors, in particular, a vector of $S^m(\mathcal{H})$ is a degree $m$ homogenous polynomial.

Let $i_0, 1, i_0, 2, \ldots, i_0, k, i_1, 1, i_1, 2, \ldots, i_1, k$ be fixed integers such that $1 \leq i_0, j \leq i_1, j \leq n_j$ for all $1 \leq j \leq k$, and $i_0, j \neq i_1, j$ whenever $j \in A$. Let us now consider the vector

$$v = \sum_{b_1, \ldots, b_k=0}^{1} (-1)^{\Lambda_{b_1, \ldots, b_k}} e_{i_0, i_1, 1, i_1, 2, \ldots, i_1, k} \otimes e_{i_1, i_1, 1, i_1, 2, \ldots, i_1, k},$$

where $B = \{ j \in [k] \mid b_j = 1 \}$. We claim that this is an element of $V_A$; moreover, vectors of this type form a basis of $V_A$ and are pairwise orthogonal.
Clearly, when we construct two vectors \( v \) and \( v' \) this way starting from different sets of indices, then not only \( v \) and \( v' \) are orthogonal, but any term appearing in the above expression of \( v \) is orthogonal to any term in \( v' \). It is also easy to see that the span of these vectors is \( G \)-invariant. The highest weights can be read off from the vector with smallest possible indices, namely for \( j \notin A \), the highest weight for the \( j \)th factor in \( G \) is \( 2 \), while for \( j \in A \) it is \((1,1)\). Therefore, vectors of this type span \( V_A \). Note that this is consistent with the fact that the number of admissible sets of indices is

\[
\prod_{j \in \{1, \ldots, k\} \setminus A} \left( \frac{n_j + 1}{2} \right) \prod_{j \in A} \left( \frac{n_j}{2} \right) = \dim V_A
\]

for a fixed subset \( A \).

We calculate next the norm squared of the elements of this basis. The sum has \( 2^k \) terms, but they are not necessarily distinct. More precisely, each term appears with the same multiplicity, which is easily seen to be \( 2^{k-1} \) if \( c := \{ j \in [k] | \bar{i}_0,j = i_1,j \} < k \) and \( 2^k \) if \( c = k \). The latter case can only be realized if \( A = \emptyset \). The norm of a single term is 1 if \( c = k \), and \( 2^{-k} \) otherwise.

To sum up, the norm squared of \( v \) is

\[
\|v\|^2 = \begin{cases} 
\frac{2^k}{2^{k+1}} (2^{k-1})^2 \frac{1}{2} = 2^{k+c} & \text{if } c < k \\
(2^k)^2 & \text{if } c = k.
\end{cases}
\]

A formula which gives back both cases is \( \|v\|^2 = 2^{k+c} \).

The invariant associated with the subrepresentation \( V_A \) is therefore given by

\[
I_A(\psi) = 2^{-k} \sum_{1 \leq |i| \leq |i'| \leq n_k} 2^{-c} \left| \frac{1}{|B|} \prod_{i \in B} \psi_{i_1, \ldots, i_{k-1}} \right|^2
\]

with \( B \) and \( c \) as above, and \( \psi = \sum \psi_{i_1, \ldots, i_{k-1}} e_{i_1i_2 \ldots i_{k-1}} \).

According to section 6, we can also find \( 2^{k-1} \) linearly independent polynomial invariants of degree 2 on the space of mixed states of a \((k-1)\)-partite quantum system. A convenient choice is the following one. Let \( A \subseteq [k-1] \) and let

\[
\text{Tr}_A : \text{End}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{k-1}) \rightarrow \text{End} \left( \bigotimes_{i \notin A} \mathcal{H}_i \right)
\]

be the partial trace over the subsystems whose index is in \( A \). Then for a mixed state \( \varrho \in \text{End}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{k-1}) \),

\[
\varrho \mapsto \text{Tr}((\text{Tr}_A \varrho)^2)
\]

is clearly a local unitary invariant, and as \( A \) runs through different subsets of \([k-1]\), we obtain this way a linearly independent set of local unitary invariants.

To relate the two bases, let us introduce an environment to the \((k-1)\)-particle system, and note that \( \text{Tr}((\text{Tr}_A \varrho)^2) = \text{Tr}((\text{Tr}_{A \cup \{k\}} \psi \psi^*)^2) = \text{Tr}((\text{Tr}_{[k-1] \setminus A} \psi \psi^*)^2) \) for some \( \psi \). We have the following proposition.

**Proposition 4.** Let \( I_A(\psi) \) be as above and \( J_A(\varrho) = \text{Tr}((\text{Tr}_A \varrho)^2) \). Then

\[
J_A(\psi^*) = \sum_{A \subseteq [k]} (-1)^{|A|} I_A(\psi)
\]
and

\[ I_A(\psi\psi^*) = 2^{-k} \sum_{B \subseteq [k]} (-1)^{|A \cap B|} J_B(\psi). \]  

(47)

**Proof.** Let \( \psi = \sum_{i_1, \ldots, i_k} \psi_{i_1 \ldots i_k} e_{i_1} \otimes \cdots \otimes e_{i_k} \). Then

\[ \sum_{A \subseteq [k]} (-1)^{|A \cap S|} I_A(\psi) = \sum_{i_1 \leq \cdots \leq i_l} 2^{-k+c} \sum_{A \subseteq [k]} (-1)^{|A \cap S|} \sum_{b_1, \ldots, b_k} (-1)^{|A \cap B|} (-1)^{|A \cap B'|}, \]

where \( B = \{ i | b_i = 1 \} \) and similarly for \( B' \) and \( S \), superscripts are understood modulo 2, and we have used the identity

\[ \sum_{A \subseteq [k]} (-1)^{|A \cap S|+|A \cap B|+|A \cap B'|} = \sum_{A \subseteq [k]} \prod_{a \in A} (-1)^{1_{\alpha S}+1_{\alpha \delta}+1_{\alpha \delta'}} = \prod_{a \in [k]} (1 + (-1)^{1_{\alpha S}+1_{\alpha \delta}+1_{\alpha \delta'}}) = \begin{cases} 2^k & \text{if } B' = S \Delta S' \\ 0 & \text{else} \end{cases}, \]  

(49)

where \( \Delta \) denotes the symmetric difference.
In the other direction,
\[
2^{-k} \sum_{B \subseteq [k]} (-1)^{|A \cap B|} J_B(\psi \psi^*) = 2^{-k} \sum_{B \subseteq [k]} (-1)^{|A \cap B|} \sum_{S \subseteq [k]} (-1)^{|B \cap S|} I_S(\psi) \\
= 2^{-k} \sum_{S \subseteq [k]} \sum_{B \subseteq [k]} (-1)^{|A \cap B|} I_S(\psi) \\
= I_A(\psi)
\]

since
\[
\sum_{B \subseteq [k]} (-1)^{|A \cap B|} = \sum_{B \subseteq [k]} \prod_{b \in B} (1 + (-1)^{1_{b \in A} + 1_{b \notin T}}) = \begin{cases} 
2^k & \text{if } A = B \\
0 & \text{else.}
\end{cases}
\] (51)

Note that one can easily form an entanglement monotone from each \(J_A\), as follows \([15]\):
\[
\eta_A = \frac{2^{|S|}}{2^{|S|} - 1} (1 - J_A)
\] (52)
and this quantity takes its values between 0 and 1.

From the Brennen \([16]\) form, it also follows that the Meyer–Wallach measure \([17]\) can be expressed as
\[
Q(\psi) = 2 - \frac{2}{k} \sum_{i=1}^{k} J_{[i]} = 2 - \frac{2}{k} \sum_{A \subseteq [k]} \sum_{i=1}^{k} (-1)^{|A \cap [i]|} I_A \\
= 2 - \frac{2}{k} \sum_{A \subseteq [k]} I_A \sum_{i=1}^{k} (-1)^{|A \cap [i]|} = 2 - \frac{2}{k} \sum_{A \subseteq [k]} (k - 2|A|) I_A \\
= 2 \sum_{A \subseteq [k]} I_A - \frac{2}{k} \sum_{A \subseteq [k]} (k - 2|A|) I_A = \sum_{A \subseteq [k]} \left( 2 - \frac{2}{k} (k - 2|A|) \right) I_A \\
= \sum_{A \subseteq [k]} \frac{4|A|}{k} I_A.
\] (53)

We would like to remark that the special case of mutliple qubits, that is, when \(n = (2, \ldots, 2)\) was investigated in \([6]\). There \(I_A\) (with a different normalization) was computed using covariants under the LU group which were obtained via transvections from a ground form.

8. Invariants of higher order

We turn to the general \(m \geq 2\) case, and construct invariants analogous to the ones introduced in section 7. We consider the two simplest irreducible representations of \(S_m\), the trivial and the alternating one, corresponding to symmetric and alternating powers in equation (5). It is easy to see that
\[
V(m) \otimes \cdots \otimes V(m) \otimes V(1,1,\ldots,1) \otimes \cdots \otimes V(1,1,\ldots,1) \simeq \begin{cases} 
V(1,1,\ldots,1) & \text{if } b \text{ is even} \\
V(m) & \text{if } b \text{ is odd.}
\end{cases}
\] (54)
Consequently, in the decomposition of $S^n(H)$ into irreducibles, we can find $S_{i_1} H_1 \otimes \ldots \otimes S_{i_k} H_k$ with multiplicity 1 (zero) when among the $\lambda$s only $(m)$ and $(1, 1, \ldots, 1)$ are present and the latter appears an even (odd) number of times (note that tensor product is commutative up to isomorphism).

This means that again we have a family of local unitary invariants, labelled by even-element subsets of $\{1, \ldots, k\}$, mapped bijectively to the set of irreducible subrepresentations of $S^n(H)$ built up from symmetric and alternating powers of the representations $H_j$ with an even number of alternating powers. Again, the subspace corresponding to the subset $A \subseteq \{1, \ldots, k\}$ will be denoted by $V_A$.

For a fixed subset $A$, let $(i_{j,l})_{1 \leq j \leq k, 1 \leq l \leq m}$ be integers such that if $j \notin A$, then $1 \leq i_{j,1} \leq i_{j,2} \leq \ldots \leq i_{j,m} \leq n_j$ and if $j \in A$, then $1 \leq i_{j,1} < i_{j,2} < \ldots < i_{j,m} \leq n_j$. From these indices, we can form the vector

$$\sum_{\pi_1, \ldots, \pi_m \in S_n} \prod_{j=1}^m \lambda_{j}(\pi_j)e_{i_{1,j}(1)/i_{2,j}(1)\ldots i_{m,j}(1)}e_{i_{1,j}(2)/i_{2,j}(2)\ldots i_{m,j}(2)}\cdots e_{i_{1,j}(m)/i_{2,j}(m)\ldots i_{m,j}(m)},$$

where $\lambda_j = (m)$ if $j \notin A$ and $\lambda_j = (1, 1, \ldots, 1)$ if $j \in A$. $\chi_{\lambda}$ is the character of the corresponding Specht module, i.e. constant 1 if $\lambda = (m)$ and the sign of the permutation if $\lambda = (1, 1, \ldots, 1)$. $v$ is then an element of $V_A$; vectors of this form span $V_A$ and they are pairwise orthogonal for different (multi)sets of indices. As there does not seem to exist a simple formula for the norm squared of these vectors, we cannot give the general form of the corresponding invariant.

It would be interesting to relate these invariants to those obtained from invariants on mixed states of a $(k-1)$-particle quantum system via the isomorphism $f \mapsto f \circ Tr_{ENV} \circ P$. It is interesting to note that formulas analogous to the $m = 2$ case stated in the proposition above but with the exponent $m$ in equation (45) instead of 2 do not hold.

9. Conclusion

We have shown that the inverse limit $I_k$ of the algebras of LU-invariant polynomials of pure states of $k$-partite quantum systems with finite-dimensional Hilbert spaces is free, and an algebraically independent generating set can be given in terms of finite connected coverings of a graph with a single vertex and $k$ loops. The number of homogenous degree $2d$ polynomials in the algebraically independent generating set equals the number of isomorphism classes of $d$-fold connected coverings, which in turn equals the number of conjugacy classes of index $d$ subgroups of a free group on $k-1$ generators.

Note that the statement is easily verified to be true in the simplest $k = 2$ case. The algebra of LU-invariants of a bipartite quantum system is known to be generated by the traces of the positive integer powers of the reduced density matrix, and therefore we have one generator in every even homogenous part, which is algebraically independent. On the other hand, the free group on one generator is isomorphic to $\mathbb{Z}$, which clearly has exactly one index $m$ subgroup for all $m \geq 1$.

The same program can be carried out in the case of mixed states, and the resulting algebras turn out to be the same as in the case of pure states with a shift in the number of subsystems. This phenomenon is already present in the case of a single $(k-1)$-partite mixed state over a Hilbert space with finite dimension and its purification, a $k$-partite pure state in the space obtained by tensoring with a sufficiently large Hilbert space representing the environment.

As an illustration we have given all fourth-order local unitary invariants of pure states of quantum systems with distinguishable constituents having arbitrary (and not necessarily equal) dimensional Hilbert spaces, as well as all degree 2 invariants of mixed states with one
less subsystems. The two vector spaces are given with two bases which are particularly simple from the two points of view, and the linear transformation relating the two bases is calculated. Analogous invariants of higher order have also been constructed.

Explicit descriptions of the algebras $I_{k,n}$ are known in only a limited number of cases, including $k = 2$, $n$ arbitrary, $k = 3$, $n = (2, 2, 2)$ [18] and $k = 4$, $n = (2, 2, 2, 2)$ [19]. It should be noted that for any $k \in \mathbb{N}$ and $n \in \mathbb{N}^k$, $I_{k,n}$ is a quotient of $I_k$. It would be interesting to determine the kernels of the quotient maps in each case.

We would like to emphasize that in spite of our lack of knowledge about the structure of every single $I_{k,n}$, fortunately the generators of $I_k$ can be directly interpreted as generators of the algebras of invariants of pure states of arbitrary $k$-partite quantum systems.

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