On the Conjecture by Demyanov–Ryabova in Converting Finite Exhausters

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Abstract The Demyanov–Ryabova conjecture is a geometric problem originating from duality relations between nonconvex objects. Given a finite collection of polytopes, one obtains its dual collection as convex hulls of the maximal facet of sets in the original collection, for each direction in the space (thus constructing upper convex representations of positively homogeneous functions from lower ones and, vice versa, via Minkowski duality). It is conjectured that an iterative application of this conversion procedure to finite families of polytopes results in a cycle of length at most two. We prove a special case of the conjecture assuming an affine independence condition on the vertices of polytopes in the collection. We also obtain a purely combinatorial reformulation of the conjecture.

Keywords Demyanov–Ryabova · Exhausters · Polytope · Convex hull · Affinely independent · Combinatorial reformulation

Mathematics Subject Classification 90C27 · 52B11

1 Introduction

Given a finite collection of polytopes, we can obtain its dual by taking convex hulls of the support faces for every nonzero direction. Then, if we continue this process,
it will inevitably reach a cycle due to the finiteness of the problem. The Demyanov–Ryabova conjecture states that such a cycle will have length at most two. Essentially, we want to establish the uniqueness of a dual characterisation of a function by establishing a steady 2-cycle in the relevant dynamical system defined by the conversion operator.

The conjecture was first published in 2011 (see [1]) and is well known in the constructive nonsmooth analysis community. However, little progress has been made on this problem since publication. To our best knowledge, the only work done in this direction is [2], in which a special case of the problem was resolved. It is shown that when the collection of sets includes all subsets of the minimal cardinality, the conjecture is true. This condition is different from the affine independence assumption that we consider here in this paper. Note also that a more general problem of the existence of cycles for bounded families of compact convex sets is also open.

Exhausters are multiset objects that generalise the subdifferential of a convex function. An upper or lower exhauster is the collection of sub- or superdifferentials that correspond to the relevant upper or lower representations of the directional derivative. Introduced by Demyanov [3], exhausters attracted a noticeable following in the optimisation community [4–11]. Such constructions are popular in applied optimisation as they allow for exact calculus rules and easy conversion from ‘upper’ to ‘lower’ characterisations of the directional derivative. The Demyanov–Ryabova conjecture gives an elegant interpretation in terms of lower convex and upper concave representations of positively homogeneous functions. One of the main challenges in the calculus of exhausters is their lack of uniqueness, and whilst some works are dedicated to finding minimal objects [12], it is shown that a minimal exhauster may not exist (there exist different representations that cannot be reduced further; see [13]). The resolution of the Demyanov–Ryabova conjecture is key to identifying ‘stable’ upper and lower exhauster representations.

In this paper, we will focus on a geometric formulation of this conjecture that does not rely on nonsmooth analysis background. The contribution of this work is twofold: firstly, we prove that the conjecture is true in the special affinely independent case, when all vertices of the polytopes in the collection form a simplex (Theorem 3.1). So, we will restrict the conjecture to the case with \( n + 1 \) affinely independent vertices in an \( n \) dimensional space and prove it is true. Secondly, we will reformulate this geometric problem and obtain a combinatorial formulation by considering the orderings on the vertex set and forming a simplified map (Theorem 5.1). Then, we will show the combinatorial formulation and the geometric problem are equivalent. Apart from these, we have also done some numerical experiments on some random polytopes in 2D, and we did not find any counterexamples.

This paper is organised as follows. We begin with the precise statement of the conjecture and several technical observations in Sect. 2. We prove that the conjecture is true under the condition of affine independence in Sect. 3. Section 4 gives the formulation of the simplified map. Last Sect. 5 is dedicated to the algebraic reformulation of the conjecture. Throughout the paper, we use the standard Euclidean setting.
2 Preliminaries

For the convenience of our readers, I provide the following list of notations we are going to use throughout the paper.

- Standard scalar product: \( \langle x, y \rangle = x^T y \).
- The Euclidean norm: \( \|x\| = \|x\|_2 = \sqrt{\langle x, x \rangle} \).
- The unit sphere: \( S^{m-1} = \{ x \in \mathbb{R}^m : \|x\| = 1 \} \).
- The convex hull of a set \( S \subset \mathbb{R}^m \) is denoted by \( \text{conv} \ S \).
- The set of extreme points of a closed convex set \( C \subset \mathbb{R}^m \) is denoted by \( \text{ext} \ C \).
- A polytope is a convex hull of finitely many points in \( \mathbb{R}^m \), and it is denoted as \( P \).
- \( \Omega \) is the set of polytopes.
- \( \Omega_i \) is the set of polytopes after \( i \)th iteration.
- \( C \) is the convex hull of polytopes in the set.
- \( V \) is the set of vertices of polytopes in the set.
- \( F \) is the map from the original Demyanov–Ryabova conjecture.
- \( F' \) is the map from the new algebraic reformulation of the original conjecture.

**Definition 2.1** Let \( d \in S^{m-1} \) and let \( P \subset \mathbb{R}^m \) be a polytope. We define the maximal face or supporting face of \( P \) for the direction \( d \) as \( P_d := \text{Arg max}_{x \in P} \langle x, d \rangle \).

Geometrically, the maximal face \( P_d \) in the direction \( d \) is the subset of \( P \) of all points that project ‘the furthest’ along this direction (see the illustration in Fig. 1).

We note here that for any polytope \( P \), its face \( P_d \) is the convex hull of a subset of vertices of \( P \). We refer the reader to [14, Chaps. 1 and 2] for the basic results related to the facial structure of polytopes.

Consider a finite collection \( \Omega = \{ P_1, P_2, \ldots, P_k \} \), where \( P_i \subset \mathbb{R}^m \) is a polytope for each \( i \in \{ 1, \ldots, k \} \), and \( k \in \mathbb{N} \). For a direction \( d \in S^{m-1} \), let \( \Omega(d) := \text{conv} \{ P_d : P \in \Omega \} \).

Here we fix a direction \( d \) and take the convex hull of the maximal faces in this direction for all polytopes in the family. The new collection of sets generated in this fashion from all directions \( d \in S^{m-1} \) is the output of the Demyanov convertor,

\[
F(\Omega) := \{ \Omega(d) : d \in S^{m-1} \}. \tag{1}
\]

**Fig. 1** Maximal face of a polytope \( P \) for a given direction
It is not difficult to observe that for every \(d \in S^{m-1}\) the set \(\Omega(d)\) is a polytope, since each maximal face \(P_d\) is a convex hull of finitely many vertices of the polytope \(P\), and there are finitely many such polytopes. Moreover, since the total number of vertices of all polytopes in the collection is finite, there are finitely many subsets of these vertices that give finitely many possibilities to form different convex hulls. Hence, \(F(\Omega)\) is also a collection of finitely many polytopes. We can now define a sequence \((\Omega_i)\) recursively as

\[
\Omega_{i+1} = F(\Omega_i), \quad i \in \{0, 1, \ldots\}, \quad \Omega_0 := \Omega.
\]  

Conjecture 2.1 (Demyanov–Ryabova) Given a finite collection of polytopes \(\Omega_0\), the sequence \(\{\Omega_i\}_{i \in \mathbb{N}}\) defined via the recursive application of Demyanov convertor \(F\) to \(\Omega_0\) eventually reaches a cycle of length 2, i.e. there exists \(N \in \mathbb{N}\) such that \(\Omega_{n+2} = \Omega_n\) for all \(n > N\).

In the sequel, we will use an equivalent reformulation of Conjecture 2.1.

Definition 2.2 Let \(\Omega\) be a set of polytopes. We define \(V\) to be the set of all vertices of all polytopes in the collection \(\Omega\), i.e. \(V = \bigcup_{P \in \Omega} \text{ext } P\).

The following lemma is quite straightforward, but we will provide a rigorous proof for completeness.

Lemma 2.1 Let \(\Omega_0\) be a finite family of polytopes in \(\mathbb{R}^m\). Then, the following statements are equivalent.

(i) There exists an \(N \in \mathbb{N}\) such that \(\Omega_n = \Omega_{n+2}\) for all \(n > N\).

(ii) For any polytope \(P \subset \mathbb{R}^m\), there exists an \(N_P \in \mathbb{N}\) such that \(\forall n > N_P : P \in \Omega_n \Rightarrow P \in \Omega_{n+2}\).

Proof It is evident that (i) yields (ii) by letting \(N_P = N\) for all \(P\). Now, we want to show the reverse implication.

By the construction of our conversion process, the only polytopes that feature in any of the collections in \((\Omega_i)\) are the convex hulls of subsets of \(V\). There are finitely many such subsets. Therefore, for all but finitely many polytopes we can safely let \(N_P = 0\).

Consider the remaining finite set of polytopes \(\mathcal{P}\) that appear at least once in some of the collections in our conversion sequence. If \(P \in \mathcal{P}\) appears in the sequence finitely many times, then \(N_P\) has to be larger than the index \(n\) of the last collection \(\Omega_n\) that contains \(P\). If \(P\) features in infinitely many \(\Omega_i\’s\), then it has to be present in each \(\Omega_{N_P+2k}\) for \(i \geq N_P\) and \(k \in \mathbb{N}\), where \(N_P\) is the index that satisfies (ii).

It remains to assign the maximal number \(N_P\) over all polytopes to \(N\) and observe that (i) holds for this \(N\). \(\square\)

Note that whilst a large part of the discussion in this section can be repeated verbatim for the case when \(\Omega\) is a bounded family of compact convex sets, the reformulation of the conjecture given in Lemma 2.1 (ii) is not necessarily true for this case.
3 Affinely Independent Case

We first demonstrate the result that the convex hull $\text{conv } \Omega_i$ is a constant.

**Proposition 3.1** Let $\Omega$ be a finite collection of polyhedral sets in $\mathbb{R}^m$, then, $\text{conv } \Omega = \text{conv } \text{F}(\Omega)$, where $\text{conv } \Omega = \text{conv}\{P : P \in \Omega\}$ is the convex hull of all polytopes in $\Omega$, and $\text{F}(\Omega)$ is the output of the Demyanov convertor (1).

**Proof** It is evident that $\text{conv } \text{F}(\Omega) \subset \text{conv } \Omega$; we only need to show that no points of the convex hull are lost in the conversion. For this, it is sufficient to prove that for every vertex $v$ of $C = \text{conv } \Omega$ we have $v \in \text{conv } \text{F}(\Omega)$.

Since $C$ is a polytope, every vertex $v$ of $C$ is exposed. Thus, there exists $d \in S^{m-1}$ such that

$$C_d = \text{Arg max}_{x \in C} \langle x, d \rangle = \{v\}.$$ 

Since $C = \text{conv}\{P : P \in \Omega\}$, there exists a polytope $P \in \Omega$, such that $v \in P$ (since $v$ is an extreme point, it can not be represented as a convex combination of any other points in $C$).

On the other hand, since $P \subset C$, we have

$$\langle v, d \rangle \leq \max_{x \in P} \langle x, d \rangle \leq \max_{y \in C} \langle y, d \rangle = \langle v, d \rangle,$$

hence, $v \in P_d \subset \Omega(d)$, and therefore $v \in \text{conv } \text{F}(\Omega)$. \hfill $\square$

**Remark 3.1** According to the Proposition 3.1, the sequence $(\text{conv } \Omega_i)_{i \in \mathbb{R}}$ is constant. Thus, we can define $C$ to be the convex hull of all polytopes in the collection, i.e.

$$C = \text{conv}(\Omega_0) = \text{conv}(\Omega_1) = \cdots = \text{conv}(\Omega_i) \forall i$$

see Fig. 2.

The set $C$ is a polytope by the finiteness argument. Our goal is to prove the Demyanov–Ryabova conjecture for the special case when $C$ is a convex hull of an

![Fig. 2 Convex hull C of 5 sets](image)
affinely independent set, and all vertices of the polytopes in the collection belong to this set.

We denote $C_d$ as the maximal face of $C$ in direction $d \in S^{m-1}$.

Recall that a finite set of points $V = \{v_0, \ldots, v_k\} \subset \mathbb{R}^m$ is affinely independent if the vectors

$$p_i = v_i - v_0, \quad i \in \{1, 2, \ldots, k\}$$

span a $k$-dimensional linear subspace of $\mathbb{R}^m$, or equivalently if the convex hull of $V$ has dimension $k$. The following definition of a simplex will be useful for us in the sequel.

**Definition 3.1 (Simplex)** Let $k + 1$ points $v_0, v_1, \ldots, v_k \in \mathbb{R}^m$ be affinely independent. The simplex determined by this set of points is their convex hull:

$$C = \left\{ \lambda_0 v_0 + \cdots + \lambda_k v_k : \lambda_i \geq 0, \sum_{i=0}^{k} \lambda_i = 1 \right\}.$$

Thus, a $k$-simplex is a $k$-dimensional polytope that is the convex hull of its $k + 1$ vertices. Observe that every face of a simplex, called sub-simplex, is also a simplex of a lower dimension. We also mention here that every sub-simplex of a $k$-simplex is an exposed face of this simplex, i.e. it is a maximal face for some direction $d \in S^{m-1}$ (see [14]).

Our next goal is to prove the following special case of Conjecture 2.1.

**Theorem 3.1** Let $\Omega_0$ be a finite collection of polytopes in $\mathbb{R}^m$. Assume that there exists an affinely independent set $V = \{v_0, v_1, \ldots, v_k\} \subset \mathbb{R}^m$, such that $C = \text{conv}\{v_0, v_1, \ldots, v_k\}$, that is, $C$ is a $k$-simplex on the set of vertices $V$. Assume that every polytope $P \in \Omega_0$ is a sub-simplex of $C$, i.e. there exists $V_P \subset V$ such that $P = \text{conv} V_P$. Then, the conversion process (2) reaches a cycle of length 2, that is, there exists a sufficiently large $N \in \mathbb{N}$ such that we have $\Omega_{n+2} = \Omega_n$, for all $n > N$.

The proof of Theorem 3.1 is based on a technical claim that we prove in Lemma 3.1.

**Lemma 3.1** Let $\Omega$ be a finite collection of polytopes, let the sequence $(\Omega_i)_i$ be defined by (2) with $\Omega_0 = \Omega$ and let $C = \text{conv}(\Omega)$. If for some $d \in S^{m-1}$ and $n \in \mathbb{N}$ we have $C_d \in \Omega_n$, then $C_d \in \Omega_{n+2}$.

**Proof** Let $C_d \in \Omega_n$, where $d \in S^{m-1}$. Observe that $(C_d)_d = C_d$, hence,

$$C_d = (C_d)_d \subseteq \Omega_n(d) = \text{conv}\{P_d : P \in \Omega_n\},$$

therefore,

$$C_d = (C_d)_d \subseteq \Omega_n(d)_d. \quad (3)$$

On the other hand, since $\Omega_n(d) \subseteq C$, we have

$$\Omega_n(d)_d \subseteq C_d. \quad (4)$$
Putting (3) and (4) together, we obtain $C_d = \Omega_n(d)_d$. Since $\Omega_n(d) \in \Omega_{n+1}$, we have one inclusion $\Omega_{n+1}(d) \supseteq \Omega_n(d)_d = C_d$.

To finish the proof it remains to show the reverse inclusion $\Omega_{n+1}(d) \subseteq C_d$ (then $C_d = \Omega_{n+1}(d) \in \Omega_{n+2}$ and we are done). This is equivalent to show that if $P \in \Omega_{n+1}$, then $P_d \subseteq C_d$.

For any $P \in \Omega_{n+1}$ there exists $d' \in S^{m-1}$ such that $P = \Omega_n(d')$, therefore,

$$P = \Omega_n(d') = \text{conv}(P_{d'} : P \in \Omega_n) \supseteq (C_d)_{d'}.$$ 

Since $(C_d)_{d'}$ is nonempty, this yields $P \cap C_d \neq \emptyset$, and hence, $P_d = (P \cap C_d) \subseteq C_d$.

\[ \square \]

**Proof** (of Theorem 3.1) We will show the equivalent claim (see Lemma 2.1 (ii)) that for every polytope $P$ there exists $N_P \in \mathbb{N}$ such that

$$\forall n > N_P : P \in \Omega_n \Rightarrow P \in \Omega_{n+2}. \quad (5)$$

First consider the case when $P \neq C = \text{conv}(\Omega)$. There must be a direction $d \in S^{m-1}$, such that $P$ is a maximal face of $C$, i.e. $P = C_d$. Then (5) follows from Lemma 3.1.

It remains to show that (5) is true for $P = C$, i.e. there exists $N \in \mathbb{N}$, such that for all $n > N$, $C \in \Omega_n \Rightarrow C \in \Omega_{n+2}$.

Assume the contrary, then $C \in \Omega_N$ and $C \notin \Omega_{N+2}$ for some $N$. By the construction of our sequence $\{\Omega_i\}$, we know that $C = \Omega_{N-1}(d)$ for some $d \in S^{m-1}$. However, $\Omega_{N+1}(d) \neq C$ since $C \notin \Omega_{N+2}$ by the assumption.

There exists a vertex $a \in V$ of $C$ such that $a \notin \Omega_{N+1}(d)$. Since $a \in C \subset \Omega_{N-1}(d)$, this implies $a \in P_d$ for some $P \in \Omega_{N-1}$, hence, $P \notin \Omega_{N+1}$. Indeed, assuming the contrary, we would have $a \in P_d \subset \Omega_{N+1}(d)$, which contradicts our choice of the vertex $a$.

Since $P \in \Omega_{N-1}$ and $P \notin \Omega_{N+1}$, we have $P = C$ (otherwise we obtain a contradiction to Lemma 2.1). Hence, $C \in \Omega_{N-1}$ and $C \notin \Omega_{N+1}$. Now we can repeat the same argument with $N' = N - 1$ and deduce that $C \in \Omega_{N-2}$, $C \notin \Omega_N$, but the latter is a contradiction to our assumption that $C \in \Omega_N$. \[ \square \]

### 4 The Simplified Demyanov Convertor

In this section, we prove that Conjecture 2.1 has an equivalent formulation, which will be the combinatorial problem we are going to explain in details in Sect. 5. The first step in this reformulation is the observation that it is enough to consider a certain dense subset $S^{m-1}$ to obtain the conversion sequence $(\Omega_i)_i$ from the initial collection of polytopes $\Omega = \Omega_0$. The second step consists of further simplifications by mapping these directions to a finite subset of the symmetric group $S_k$, where $k$ is the cardinality of the set $V$ of all vertices, and realising the convertor as a transformation of collections of subsets of integers in $\{1, 2, \ldots, k\}$.

Our first step is to introduce a reduced transformation $F'$, that ignores all directions for which some of the support faces in the collection of polytopes are not singletons.
Given a finite collection of polytopes $\Omega$, and let $V$ be the set of all vertices of all polytopes in this collection, $V(\Omega) = V = \bigcup_{P \in \Omega} \text{ext } P$. We throw away all directions that can potentially result in a nonsingleton maximal face at some step of the conversion, and let

$$\hat{S}^{m-1} := \left\{ d \in S^{m-1} : \langle v, d \rangle \neq \langle w, d \rangle, \forall v \neq w \in V(\Omega) \right\}.$$  

We will use the fact that $\hat{S}^{m-1}$ is a dense subset of $S^{m-1}$ in later proofs. This is easy to see by noting that if $v_1, \ldots, v_k$ are the vertices of all polytopes in $\Omega$, then each of the sets

$$V_{ij} = \{ x : \langle v_i - v_j, x \rangle = 0 \}, \quad i \neq j, \quad i, j \in \{ 1, 2, \ldots, k \}$$

is a hyperplane in $\mathbb{R}^m$, so the cone $V = \mathbb{R}^m \setminus \bigcup_{i \neq j} V_{ij}$ is dense in $\mathbb{R}^m$, and hence $\hat{S}^{m-1} = S^{m-1} \cap V$ is dense in $S^{m-1}$.

We define a modified transformation $F'$ by ignoring the directions in $S^{m-1} \setminus \hat{S}^{m-1}$,

$$F'(\Omega) := \left\{ \Omega(d) : d \in \hat{S}^{m-1} \right\},$$  

and build the modified sequence $\Omega'_1, \Omega'_2, \ldots$ obtained by the recursive application of $F'$ to $\Omega_0 = \Omega$,

$$\Omega_{i+1}' = F(\Omega_i'), \quad i \in \{ 0, 1, \ldots \}, \quad \Omega_0 := \Omega.$$  

(7)

It is natural to ask whether $F'(\Omega_i) = F'(\Omega_i')$ and $F(\Omega_i) = F(\Omega_i')$ for all $i \in \mathbb{N}$. We answer this in the affirmative later on (see Theorem 4.1).

We will use two well-known results (see [14]), which we prove here for completeness.

**Lemma 4.1** Let $P \subset \mathbb{R}^m$ be a polytope, and let $d \in S^{m-1}$. There exists a neighbourhood $N_d$ of $d$, such that $\forall d' \in N_d : P_{d'} \subset P_d$.

*Proof* Assume the contrary, without lost of generality, there is a sequence $(d_k), d_k \to d$ and a vertex $v$ of $P$, such that $v \in P_{d_k}$, but $v \notin P_d$. This is impossible because

$$\max_{x \in P} \langle x, d \rangle = \lim_{k \to \infty} \max_{x \in P} \langle x, d_k \rangle = \lim_{k \to \infty} \langle v, d_k \rangle = \langle v, d \rangle.$$  

Therefore, $v \in P_d$. $\square$

**Lemma 4.2** Let $P$ be a polytope, and let $v$ be a vertex of $P_d$ for some $d \in S^{m-1}$. Then, in any neighbourhood $N_d$ of $d$ there exists $d' \in N_d \cap S^{m-1}$ such that $P_{d'} = \{ v \}$.

*Proof* Observe that if $P_d = \{ v \}$, there is nothing to prove. We hence assume that the dimension of $P_d$ is at least 2. Let $d'' \in S^{m-1}$ be some direction, that exposes the vertex $v$ within the affine hull of $P_d$, i.e. $d'' \in \text{aff } P_d - v$ and $(P_d)_{d''} = \{ v \}$. Consider

\[ \text{(7)} \]
the parametric family \( d_t := d + t(d'' - d) \), \( t \in [0, \infty) \). Since \( d_t \to d \) as \( t \to 0 \), by Lemma 4.1, there exists a sufficiently small \( t_0 \) such that

\[
\forall t \in [0, t_0) : \quad P_{d_t} \subset P_d.
\] (8)

On the other hand, observe that for any \( x \in P_d \), we have

\[
\langle x, d_t \rangle = (1 - t)\langle x, d \rangle + t \langle x, d'' \rangle = (1 - t)\langle v, d \rangle + t \langle x, d'' \rangle,
\]

hence,

\[
\text{Arg max}_{x \in P_d} \langle x, d_t \rangle = \text{Arg max}_{x \in P_d} \langle x, d'' \rangle = \{ v \},
\]

and together with (8), we have \( \forall t \in [0, t_0) : \quad P_{d_t} = \{ v \} \).

Observe that this relation is also true for the normalised vectors \( d_t / \|d_t\| \), which converge to \( d \) as well:

\[
\lim_{t \downarrow 0} \frac{d_t}{\|d_t\|} = d,
\]

and hence we are able to choose \( d'' \in S^{m-1} \) arbitrarily close to \( d \) so that \( P_{d''} = \{ v \} \). \( \Box \)

**Proposition 4.1** Let \( \Omega \) be a finite collection of polytopes in \( \mathbb{R}^m \), and let \( d \in S^{m-1} \).

For any \( P \in \Omega \) and any vertex \( v \) of \( P_d \), there exists another direction \( d' \in S^{m-1} \) and a neighbourhood \( N_{d'} \) of \( d' \), such that

\[
\forall d'' \in N_{d'}, \forall P' \in \Omega : \quad P_{d''} = \{ v \} \text{ and } P_{d''} \subset P'.'
\] (9)

**Proof** Let \( \Omega, P, v \) and \( d \) as in the statement of the proposition. By Lemma 4.1 for every polytope \( P' \in \Omega \), there exists a sufficiently small neighbourhood \( N_{d'}^{P'} \) of \( d' \), such that \( \forall d'' \in N_{d'}^{P'} : \quad P_{d''} \subset P'\).

We let \( N_d := \bigcap_{P' \in \Omega} N_{d'}^{P'} \), then, \( \forall d' \in N_d, \forall P' \in \Omega : \quad P_{d''} \subset P'\).

Since \( v \) is a vertex of \( P_d \), by Lemma 4.2, we can find another direction \( d' \in N_d \), such that \( P_{d'} = \{ v \} \).

Applying Lemma 4.1 to the direction \( d' \), and to our vertex \( v \) of \( P_{d'} \), we deduce that there is a neighbourhood \( N_{d''} \), such that for all \( d'' \in N_{d''}, P_{d''} \subset P_{d''} = \{ v \} \). Hence, \( \forall d'' \in N_{d''} \), \( P_{d''} = \{ v \} \).

To finish the proof we observe that the neighbourhood \( N_{d'} := N_{d'} \cap N_d \) satisfies (9). \( \Box \)

**Proposition 4.2** Let \( \Omega \) be a finite collection of polytopes in \( \mathbb{R}^m \), and let \( \tilde{S} \) be a dense subset of \( S^{m-1} \). Let \( v \) be a vertex of \( \Omega(d) \) for some \( d \in S^{m-1} \). Then, there exists \( \tilde{d} \in \tilde{S} \), such that \( \Omega(\tilde{d}) \subseteq \Omega(d) \) and \( v \in \Omega(\tilde{d}) \).
Proof If \( v \) is a vertex of \( \Omega(d) \), then there exists a polytope \( P \in \Omega \), such that \( v \) is a vertex of \( P_d \). By Proposition 4.1, there is another direction \( d' \in S^{m-1} \), such that

\[
\forall d'' \in N_{d'}, \forall P' \in \Omega : \quad P_{d''} = \{v\} \text{and} \quad P_{d''} \subset P' \quad (10)
\]

Since the set \( \tilde{S} \) is dense in \( S^{m-1} \), there exists \( \tilde{d} \in \tilde{S} \cap N_{d'} \neq \emptyset \) that satisfies (10). We have

\[
\Omega(\tilde{d}) = \text{conv}\{P_{\tilde{d}}, P \in \Omega\} \subseteq \text{conv}\{P_d, P \in \Omega\},
\]

and also \( v \in \{v\} = P_{\tilde{d}} \subset \Omega(\tilde{d}) \).

Proposition 4.3 For any finite collection of polytopes \( \Omega \), and any dense subset \( \tilde{S} \) of \( S^{m-1} \), we have

\[
F(F(\Omega)) = F(F'(\Omega)) \quad \text{and} \quad F'(F'(\Omega)) = F'(F(\Omega)), \quad (11)
\]

where \( F' \) is the reduced mapping associated with \( \tilde{S} \), that is, \( F'(\Omega) = \{\Omega(d) : d \in \tilde{S}\} \).

Proof Observe that by construction \( F'(\Omega) \subset F(\Omega) \). Therefore, for any direction \( d \in S^{m-1} \), we have

\[
(F'(\Omega))(d) = \text{conv}\{P_d : P \in F'(\Omega)\} \subseteq \text{conv}\{P_d : P \in F(\Omega)\} = (F(\Omega))(d).
\]

We will show that in fact \( (F'(\Omega))(d) = (F(\Omega))(d) \) for any \( d \in S^{m-1} \). Notice that this proves both relations in (11). It only remains to demonstrate the inclusion

\[
(F'(\Omega))(d) \supseteq (F(\Omega))(d). 
\]

For a fixed \( d \) choose any other direction \( d' \in S^{m-1} \), and let \( v \) be a vertex of \( (\Omega(d'))_d \). By Proposition 4.2, there exists a direction \( d'' \in \tilde{S} \), such that \( \Omega(d'') \subset \Omega(d') \) and \( v \in \Omega(d'') \).

This means that \( v \in (F'(\Omega))(d) \); hence, by the arbitrariness of \( v \), we have (12).

Our next goal is to prove that different ‘paths’ of transformations starting with \( \Omega_0 \) yield equivalent outcomes, so it does not matter if at some intermediate steps we use \( F \) or \( F' \), the \( i \)th application of \( F \) will lead to \( \Omega_i \).

Theorem 4.1 The following diagram commutes.

\[
\begin{array}{c}
\Omega_0 \xrightarrow{F} \Omega_1 \xrightarrow{F'} \Omega'_1 \\
\searrow \quad \quad \quad \searrow \\
\quad \Omega_1 \xrightarrow{F} \Omega_2 \xrightarrow{F} \Omega_3 \xrightarrow{F} \Omega_4 \cdots \\
\end{array}
\]

In other words, for any \( i \in \mathbb{N} \) we have \( F'(\Omega_i) = \Omega'_{i+1} \) and \( F(\Omega'_i) = \Omega_{i+1} \).
Before we proceed with the proof, we consider an example to clarify the meaning of the commutative diagram in Theorem 4.1.

**Example 4.1** Let $\Omega = \Omega_0$ be a collection of two dimensional polytopes, that consists of two opposite edges of a square, for instance,

$$\Omega_0 = \{((1, -1), (1, 1)), ((-1, -1), (-1, 1))\}.$$  

It is not difficult to verify that our commutative diagram reduces to the following chain of transformations (Fig. 3).

**Proof** (of Theorem 4.1) The set $\hat{S}^{m-1}$ is dense in $S^{m-1}$; hence, Proposition 4.3 yields

$$\Omega_2 = F(\Omega_1) = F(F(\Omega_0)) = F(F'(\Omega_0)) = F(\Omega'_1)$$

and

$$\Omega'_2 = F'(\Omega'_1) = F'(F'(\Omega_0)) = F'(F(\Omega_0)) = F'(\Omega_1),$$

which gives us the induction base. Assuming that for some $i \geq 2$

$$\Omega_i = F(\Omega_{i-1}) \quad \text{and} \quad \Omega'_i = F'(\Omega_{i-1}),$$

the relations (11) and (13) together yield

$$\Omega_{i+1} = F(F(\Omega_{i-1})) = F(F'(\Omega_{i-1})) = F(\Omega'_i)$$

and

$$\Omega'_{i+1} = F'(F'(\Omega'_{i-1})) = F'(F(\Omega'_{i-1})) = F'(\Omega_i).$$

The desired relations follow by induction on $i$.  \hfill \Box

![Fig. 3 An example on comparison between maps $F$ and $F'$](image)
5 Algebraic Reformulation and the Main Result

We have introduced some necessary components of our combinatorial formulation in the Sect. 4. We will state the main result of this section in Theorem 5.1 and present the proof.

We label all vertices in $V$ and encode each direction $d \in \hat{S}^{m-1}(\Omega)$ according to the order of the projections of the vertices on this direction. In other words, given an ordered list of vertices $V = \{v_1, \ldots, v_k\}$ to every direction $d \in \hat{S}^{m-1}$, we assign an element $\tau(d) \in S_k$, where $S_k$ is the set of orderings on the sequence $(1, 2, \ldots, k)$, that corresponds to the order of the projections of the vertices in the direction $d$,

$$\tau(d) = (i_1, i_2, \ldots, i_k) \in S_k \text{ such that } \langle v_{i_1}, d \rangle > \langle v_{i_2}, d \rangle > \cdots > \langle v_{i_k}, d \rangle. \quad (14)$$

Observe that this is well defined as we have discarded all directions for which we may encounter vertices, that project onto the same point. We can also encode each polytope $P \in \Omega_i$ as a subset of the vertex indices, $p \subset \{1, 2, \ldots, k\}$. Now that we have encoded our data in the discrete format, we are ready to explain the combinatorial equivalent of the conversion procedure. We first illustrate the ideas by a simple example.

Example 5.1 Consider a collection $\Omega$ that contains a line segment and a disjoint singleton in $\mathbb{R}^2$. We label the relevant vertices as $A$, $B$, and $C$ as shown in Fig. 4 (one can think of the labelling 1, 2, 3 instead).

Choose a direction $d$, and encode it using the order of the projections of the vertices along this direction (see Fig. 5).

Fig. 4 Two sets with three vertices in $\mathbb{R}^2$

Fig. 5 Encode the direction $d$ \[ \Rightarrow \text{ direction } d \rightarrow ACB \]
It is not difficult to observe that for our example, we obtain 6 different encodings of the reduced set of directions,

\begin{align*}
ACB, \ ABC, \ BCA, \ BAC, \ CAB \text{ and } \ CBA.
\end{align*}

(15)

We also encode the initial collection of polytopes as \(\omega_0 = \{AB, C\}\).

Now, suppose we want to construct \(\Omega_1'(d)\) for some direction \(d\) that is encoded as \(ACB\). To construct the ‘maximal faces’ for each encoded polytope in \(\omega_0\), we find the vertex that appears the earliest in the sequence that encodes our direction. For the first polytope \(AB\) this is \(A\), and for the second one we have the only possibility \(C\), hence \(\Omega_1'(d)\) corresponds to the encoded polytope \(AC\). It is evident from Fig. 5, that this produces the same polytope as the geometric construction.

If we apply this algorithm to every direction in (15), we end up with 6 polytopes

\[\{AC, \ AC, \ BC, \ BC, \ AC, \ BC\}\.\]

Removing the repetitions, we obtain \(\omega_1 = \{AC, \ BC\}\). Observe that the geometric conversion with the reduced set of directions yields exactly the same result.

We can continue this procedure using the same 6 directions with the set \(\omega_1\) to obtain another 6 polytopes, which are

\[\{AC, \ AB, \ CB, \ AB, \ C, \ C\}\,\]

and we have \(\omega_2 = \{AC, \ AB, \ CB, \ AB, \ C\}\). If we keep applying the same procedure, we get \(\omega_3 = \{AC, \ ABC, \ CB\}\) and \(\omega_4 = \{AC, \ AB, \ CB, \ AB, \ C\} = \{AC, \ AB, \ CB, \ C\}\). We have reached a cycle of length 2.

Our encoding of the set \(V\) for finite family \(\Omega\) of polytopes in \(\mathbb{R}^m\) results in the set of directions \(T = \{\tau_1, \tau_2, \ldots, \tau_r\} \subset S_k\), that correspond to the orderings of the projections of the vertices onto the directions in \(\hat{S}^{m-1}\). We also encode our family \(\Omega\) of polytopes \(P \in \mathbb{R}^m\) as subsets \(p\) of the set \(\{1, 2, \ldots, k\}\) of the vertex labels. We introduce the following discrete conversion procedure.

Given a collection \(w\) of nonempty subsets \(p\) of \(\{1, 2, \ldots, k\}\), for each ordering \(\tau \in S_k\). Let \(w(\tau) := \bigcup_{p \in w} \{\max_\tau(p)\}\), where \(\max_\tau(p)\) is the maximal element in \(p\) with respect to the ordering \(\tau\). In other words, it is the element of \(p\) that has the earliest place in the sequence \(\tau\).

The discrete conversion operator \(f\) is defined as

\[f(w) = \{w(\tau) : \tau \in T\}\.\]

(16)

Note that this operator maps elements of the space \(X\) of sets of nonempty subsets of \(\{1, 2, \ldots, k\}\) onto the elements of the same space.

We next explicitly identify the constructions, that we have just defined for the example considered previously.

**Example 5.2** We have the following structure based on our abstract algebraic formulation for the data given in the previous Example 5.1:
Conjecture 5.1  If a pair \( T \) and \( w \) are.

Definition 5.1  We say that the pair \( \{\Omega_i\}_{i=1}^{\infty} \) achieves a cycle of length 2, i.e. for some \( N \), \( \{\Omega_i\}_{i=1}^{N} \) reaches a cycle of length 2 if and only if \( \{w_i\}_{i=1}^{N} \) is.

The Demyanov–Ryabova conjecture (Conjecture 2.1) is true if and only if Conjecture 5.1 is true.

We first make the following obvious remark:

Proposition 5.1  Let \( \Omega \) be a finite collection of polytopes in \( \mathbb{R}^n \). Let \( (\Omega'_i)_i \) be the modified process starting from \( \Omega \), and \( (\omega_i)_i \) be the associated discrete process. Then, the sequence \( (\Omega'_i)_i \) reaches a cycle of length 2 if and only if the sequence \( (\omega_i)_i \) reaches a cycle of length 2.

The main challenge is to show that the modified process \( (\Omega'_i)_i \) reaches a cycle of length 2 if and only if \( (\Omega_i)_i \) does. The key to this is Theorem 4.1, which we have proved in the last section of the paper.

Theorem 5.1  The Demyanov–Ryabova conjecture (Conjecture 2.1) is true if and only if Conjecture 5.1 is true.

We first note that the equivalence of the modified process \( (\Omega'_i)_i \) to the discrete process \( (\omega_i)_i \) is evident: at each iteration, \( \Omega'_i \) can be reconstructed from \( \omega_i \), and vice versa. We hence have the following trivial claim.

Proof (of Theorem 5.1)  It follows immediately from Theorem 4.1 that the process \( (\Omega_i)_i \) reaches a cycle of length 2 if and only if \( (\Omega'_i)_i \) does. Since the process \( (\Omega'_i)_i \) is equivalent to the discrete process \( (\omega_i)_i \) by Proposition 5.1, we are done.
We end this section by showing the following proposition which gives a bound in dimension 2.

**Proposition 5.2** Let \( n \) be the number of vertices in \( \mathbb{R}^2 \); then, we have no more than \( n(n - 1) \) different directions in \( T \).

**Proof** Let \( d \) be an arbitrary direction. Then, we can rotate \( d \) clockwise to obtain all directions. We can encode \( d \) by writing the vertex set in order of furthest along \( d \) to closest along \( d \). As we rotate the direction \( d \) clockwise, each pair of letters swaps exactly twice. This implies that there are \( 2 \times \binom{n}{2} = n(n - 1) \) swaps in total, however, some swaps can occur simultaneously, so we only have an upper bound. \( \square \)

Note that when no 3 vertices are collinear and there are no more pairs of collinear vertices parallel to each other; then, the number of directions is precisely \( n(n - 1) \).

### 6 Conclusions

We have proved that the Demyanov–Ryabova conjecture is true assuming an affine independence condition; that is, when we restrict the number of vertices of polytopes in the collection to \( n + 1 \) affinely independent points for an \( n \) dimensional space.

We have also obtained a combinatorial reformulation of the conjecture by ordering vertices in the collection. The combinatorial formulation allows us to work on the conjecture using algebraic approaches. After we obtain the set of orderings on the vertex set that correspond to the set of restricted directions, we are able to forget about the geometry of the sets and proceed with the equivalent algebraic version of the conjecture. This means that we may can try to apply powerful algebraic and combinatorial tools to this problem. This should advance insight for the future work on solving the general conjecture using a purely algebraic approach.

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