Topology of delocalization in the nonlinear Anderson model and anomalous diffusion on finite clusters

A.V. Milovanov\textsuperscript{1,2,4}, A. Iomin\textsuperscript{3,4}

\textsuperscript{1} ENEA National Laboratory, Centro Ricerche Frascati, I-00044 Frascati, Rome, Italy
\textsuperscript{2} Space Research Institute, Russian Academy of Sciences, 117997 Moscow, Russia
\textsuperscript{3} Department of Physics and Solid State Institute, Technion, Haifa 32000, Israel
\textsuperscript{4} Max-Planck-Institut für Physik komplexer Systeme, 01187 Dresden, Germany

This study is concerned with destruction of Anderson localization by a nonlinearity of the power-law type. We suggest using a nonlinear Schrödinger model with random potential on a lattice that quadratic nonlinearity plays a dynamically very distinguished role in that it is the only type of power nonlinearity permitting an abrupt localization-delocalization transition with unlimited spreading already at the delocalization border. For super-quadratic nonlinearity the borderline spreading corresponds to diffusion processes on finite clusters. We have proposed an analytical method to predict and explain such transport processes. Our method uses a topological approximation of the nonlinear Anderson model and, if the exponent of the power nonlinearity is either integer or half-integer, will yield the wanted value of the transport exponent via a triangulation procedure in an Euclidean mapping space. A kinetic picture of the transport arising from these investigations uses a fractional extension of the diffusion equation to fractional derivatives over the time, signifying non-Markovian dynamics with algebraically decaying time correlations.

PACS numbers: 05.40.-a

\textbf{I. INTRODUCTION}

We consider the problem of dynamical localization of waves in a nonlinear Schrödinger model with random potential on a lattice and arbitrary power nonlinearity,

\begin{equation}
\partial_t \psi_n = \hat{H}_L \psi_n + \beta |\psi_n|^{2s} \psi_n, \tag{1}
\end{equation}

where \( s \geq 1 \) is a real number;

\begin{equation}
\hat{H}_L \psi_n = \varepsilon_n \psi_n + V(\psi_{n+1} + \psi_{n-1}) \tag{2}
\end{equation}

is the Hamiltonian of a linear problem in the tight binding approximation; \( \beta (\beta > 0) \) characterizes the strength of nonlinearity; on-site energies \( \varepsilon_n \) are randomly distributed with zero mean across a finite energy range; \( V \) is hopping matrix element; and the total probability is normalized to \( \sum_n |\psi_n|^2 = 1 \). For \( \beta \to 0 \), the model in Eqs. (1) and (2) reduces to the original Anderson model in Ref. [1]. In the absence of randomness, the nonlinear Schrödinger equation (NLSE) in Eq. (1) is completely integrable.

Experimentally, Anderson localization has been reported for electron gases [2], acoustic waves [3], light waves [4,5], and matter waves in a controlled disorder [6]. It is generally agreed that the phenomena of Anderson localization are based on interference between multiple scattering paths, leading to localized wave functions with exponentially decaying profiles and dense eigenspectrum [1,7]. Theoretically, nonlinear Schrödinger models offer a mean-field approximation, where the nonlinear term containing \( |\psi_n|^{2s} \) absorbs the interactions between the components of the wave field.

It has been discussed by a few authors [8–10] that NLSE with quadratic nonlinearity (i.e., \( s = 1 \)) observes a localization-delocalization transition above a certain critical strength of nonlinear interaction. That means that the localized state is destroyed, and the nonlinear field can spread across the lattice despite the underlying disorder, provided just that the \( \beta \) value exceeds a maximal allowed value. Below the delocalization border, the field is dynamically localized similarly to the linear case.

A generalization of this result to super-quadratic nonlinearity, with \( s > 1 \), is far from trivial. In a recent investigation of NLSE with disorder, we have shown [11] that the critical strength destroying localization is only preserved through dynamics, if \( s = 1 \). For \( s > 1 \) (and similarly for \( 0 < s < 1 \), a regime not considered here), the critical strength is dynamic in that it involves a dependence on the number of already excited modes (the latter are the exponentially localized modes of the linear disordered lattice). If the field is spread across \( \Delta n \) states, then the conservation of the probability implies that \( |\psi_n|^2 \propto 1/\Delta n \). As the number of already excited modes is proportional to \( \Delta n \), the distance between the frequencies obeys \( \delta \omega \propto 1/\Delta n \); whereas the nonlinear frequency shift varies as \( \Delta \omega_{NL} \propto 1/(\Delta n)^{s} \). Hence \( \delta \omega/\Delta \omega_{NL} \propto (\Delta n)^{s-1} \) is only independent of \( \Delta n \), if the nonlinearity is quadratic, i.e., \( s = 1 \). The implication is that the effect of quadratic nonlinearity does not depend on the range of field distribution; but the effect of super-quadratic (as well as sub-quadratic, with \( 0 < s < 1 \)) power nonlinearity does. Hence, if initial behavior is chaotic, say, chaos remains while spreading only for \( s = 1 \). For \( s > 1 \), a transition to regularity occurs, which blocks spreading in vicinity of the criticality beyond a certain limiting number of excited modes \( \Delta n_{\text{max}} \) \( (\Delta n_{\text{max}} \gg 1) \).

One sees that quadratic nonlinearity, characterized by \( s = 1 \), plays a dynamically very distinguished role in that it is the only type of power nonlinearity permitting an abrupt localization-delocalization transition with
unlimited spreading of the wave field already at the delocalization border. This localization-delocalization transition bears signatures, enabling to associate it with a percolation transition on the infinite Cayley tree (Bethe lattice) \(\text{(Bethe lattice)}\) \([10, 11]\). The main idea here is that delocalization occurs through infinite clusters of chaotic states on a Bethe lattice, with occupancy probabilities decided by the strength of nonlinear interaction. Then the percolation transition threshold can be translated into a critical value of the nonlinearity control parameter, such that above this value the field spreads to infinity, and is dynamically localized in spite of these nonlinearities otherwise. This critical value when account is taken for hierarchical geometry of the Cayley tree is found to be \(\beta_c = 1/\ln 2 \approx 1.4427\) \([10, 11]\), a fancy number representing the topology of nonlinear interaction posed by the quadratic power term. It was argued based on a random walk approach that in vicinity of the criticality the spreading of the wave field is subdiffusive in the limit \(t \to +\infty\), and that the second moments grow with time as a power law

\[
\langle (\Delta n)^2(t) \rangle \propto t^\alpha, \quad t \to +\infty,
\]

with \(\alpha = 1/3\) exactly. This critical regime is modeled as a next-neighbor random walk at the onset of percolation on a Cayley tree. The phenomena of critical spreading find their significance in some connection with the general problem \([12]\) of transport along separatrices of dynamical systems with many degrees of freedom and are mathematically related to a description \([13–16]\) in terms of Hamiltonian pseudochaos (random non-chaotic dynamics with zero Lyapunov exponents) \([17, 18]\) and time-fractional diffusion equations.

For \(s > 1\), the phenomena of field spreading are limited to finite clusters at the onset of delocalization \([11]\). Mathematically, this regime of field spreading is complicated by the fact that finiteness of clusters on which the transport processes concentrate conflicts with the assumptions of threshold percolation, breaking the universal scaling laws \([16, 22]\) which pertain to the infinite clusters. It is not clear, therefore, how to predict and explain transport on finite clusters, avoiding as the conceptual key element the use of percolation, and what the ensuing transport laws would be. The goal of the present study is to present a general solution to this problem.

The approach, which we advocate, is based on topological methods and in a sketchy form comprises three basic steps explored in Sec. II:

- **Step 1:** Enabling an equivalent reduced dynamical model of field-spreading based on backbone map.
- **Step 2:** Projecting dynamical equations on a Cayley tree with appropriately large coordination number which accommodates the power nonlinearity \(s (s \geq 1)\).
- **Step 3:** Calculating the index of anomalous diffusion based on combinatorial arguments, using a triangulation procedure in the mapping space and the notion of one-bond-connected (OBC) polyhedron.

It is shown in Sec. III that the transport on finite clusters is subdiffusive with a power law memory kernel (for time scales for which the dynamics concentrate on a self-similar geometry) and pertains to a class of non-Markovian transport processes described by generalized diffusion equations with the fractional derivative in time. We summarize our findings in Sec. IV.

## II. THE THREE-STEP TOPOLOGICAL APPROACH

Expanding \(\psi_n\) over a basis of linearly localized modes, the eigenfunctions of the linear problem, \(\{\phi_{n,m}\}, m = 1, 2, \ldots\), we write, with time depending complex coefficients \(c_m(t)\),

\[
\psi_n = \sum_m c_m(t)\phi_{n,m}. \tag{4}
\]

We consider \(\psi_n, \psi_n \in \{\psi_n\}\), as a vector in functional space whose basis vectors \(\phi_{n,m}\) are the Anderson eigenstates. For strong disorder, dimensionality of this space is infinite (countable). It is convenient to think of each node \(n\) as comprising a countable number of “compactified” dimensions representing the components of the wave field. So these hidden dimensions when account is taken for Eq. (4) are “expanded” via a topological mapping procedure to form the functional space \(\{\psi_n\}\). We consider this space as providing the embedding space for dynamics. Further, given any two vectors \(\psi_n \in \{\psi_n\}\) and \(\phi_n \in \{\phi_n\}\), we define the inner product, \(\langle \psi_n \phi_n \rangle\),

\[
\langle \psi_n \phi_n \rangle = \sum_n \phi_n\phi_n. \tag{5}
\]

where star denotes complex conjugate. To this end, the functional space \(\{\psi_n\}\) becomes a Hilbert space, permitting the notions of length, angle, and orthogonality by standard methods \([23]\). With these implications in mind, we consider the functions \(\phi_n, m\) as “orthogonal” basis vectors obeying

\[
\sum_n \phi_{n,m}^*\phi_{n,k} = \delta_{m,k}, \tag{6}
\]

where \(\delta_{m,k}\) is Kronecker’s delta. Then the total probability being equal to 1 implies

\[
\langle \psi_n \psi_n \rangle = \sum_n \psi_n\psi_n = \sum_m c_m^*(t) c_m(t) = 1. \tag{7}
\]

### A. Step 1: The backbone map

We define the power \(2s (2s \geq 2)\) of the modulus of the wave field as the power \(s\) of the probability density, i.e., \(|\psi_n|^{2s} = |\psi_n\phi_n^*|^{2s}\). Then in the basis of linear localized
modes we can write, with the use of \( \psi_n = \sum_m \sigma_m \phi_{n,m} \),
\[
|\psi_n|^{2s} = \left[ \sum_{m_1,m_2} \sigma_{m_1} \sigma_{m_2}^* \phi_{n,m_1} \phi_{n,m_2}^* \right]^s.
\]

It is convenient to consider the expression on the right-hand side as a functional map
\[
\hat{F}_s : \{ \phi_{n,m} \} \rightarrow \left[ \sum_{m_1,m_2} \sigma_{m_1} \sigma_{m_2}^* \phi_{n,m_1} \phi_{n,m_2}^* \right]^s
\]
from the vector field \( \{ \phi_{n,m} \} \) into the scalar field \( |\psi_n|^{2s} \).

Consider expanding the power law on the right-hand side of Eq. (3). If \( s \) is a positive integer, then a regular expansion can be obtained as a sum over \( s \) pairs of indices \((m_{1,1}, m_{1,2}) \ldots (m_{s,1}, m_{s,2})\). The result is a homogeneous polynomial, an \( s \)-quadratic form \([24]\). In contrast, for fractional \( s \), a simple procedure does not exist. Even so, with the aid of Eq. (10), one might circumvent the algebraic structure of \( \hat{F}_s \) that the map in Eq. (9) is positive definite, and that it contains a self-similarity character in it, such that by stretching the basis vectors (by a stretch factor \( \lambda \)) the value of \( \hat{F}_s \) is just renormalized (multiplied by \( |\lambda|^{2s} \)).

We have, accordingly,
\[
\hat{F}_s \{ \phi_{n,m} \} = |\lambda|^{2s} \hat{F}_s \{ \phi_{n,m} \}.
\]

Consider expanding the power law on the right-hand side of Eq. (3). If \( s \) is a positive integer, then a regular expansion can be obtained as a sum over \( s \) pairs of indices \((m_{1,1}, m_{1,2}) \ldots (m_{s,1}, m_{s,2})\). The result is a homogeneous polynomial, an \( s \)-quadratic form \([24]\). In contrast, for fractional \( s \), a simple procedure does not exist. Even so, with the aid of Eq. (10), one might circumvent the algebraic structure of \( \hat{F}_s \) that the map in Eq. (9) is positive definite, and that it contains a self-similarity character in it, such that by stretching the basis vectors (by a stretch factor \( \lambda \)) the value of \( \hat{F}_s \) is just renormalized (multiplied by \( |\lambda|^{2s} \)).

We have, accordingly,
\[
\hat{F}_s \{ \phi_{n,m} \} = |\lambda|^{2s} \hat{F}_s \{ \phi_{n,m} \}.
\]

Consider expanding the power law on the right-hand side of Eq. (3). If \( s \) is a positive integer, then a regular expansion can be obtained as a sum over \( s \) pairs of indices \((m_{1,1}, m_{1,2}) \ldots (m_{s,1}, m_{s,2})\). The result is a homogeneous polynomial, an \( s \)-quadratic form \([24]\). In contrast, for fractional \( s \), a simple procedure does not exist. Even so, with the aid of Eq. (10), one might circumvent the algebraic structure of \( \hat{F}_s \) that the map in Eq. (9) is positive definite, and that it contains a self-similarity character in it, such that by stretching the basis vectors (by a stretch factor \( \lambda \)) the value of \( \hat{F}_s \) is just renormalized (multiplied by \( |\lambda|^{2s} \)).

We have, accordingly,
\[
\hat{F}_s \{ \phi_{n,m} \} = |\lambda|^{2s} \hat{F}_s \{ \phi_{n,m} \}.
\]

Consider expanding the power law on the right-hand side of Eq. (3). If \( s \) is a positive integer, then a regular expansion can be obtained as a sum over \( s \) pairs of indices \((m_{1,1}, m_{1,2}) \ldots (m_{s,1}, m_{s,2})\). The result is a homogeneous polynomial, an \( s \)-quadratic form \([24]\). In contrast, for fractional \( s \), a simple procedure does not exist. Even so, with the aid of Eq. (10), one might circumvent the algebraic structure of \( \hat{F}_s \) that the map in Eq. (9) is positive definite, and that it contains a self-similarity character in it, such that by stretching the basis vectors (by a stretch factor \( \lambda \)) the value of \( \hat{F}_s \) is just renormalized (multiplied by \( |\lambda|^{2s} \)).

We have, accordingly,
\[
\hat{F}_s \{ \phi_{n,m} \} = |\lambda|^{2s} \hat{F}_s \{ \phi_{n,m} \}.
\]

In what follows, we consider the backbone as representing the algebraic structure of \( \hat{F}_s \) in the sense of Eq. (10). So, for fractional \( s \), our analysis will be based on a reduced model which is obtained by replacing the original map \( \hat{F}_s \) by the backbone map \( \hat{F}'_s \). The claim is that the reduction \( \hat{F}_s \to \hat{F}'_s \) does not really alter the scaling exponents behind the wave-spreading, since the algebraic structure of the original map is there anyway. Note that \( \hat{F}_s \) and \( \hat{F}'_s \) both have the same degree \( 2s \), which is the sum of the exponents of the variables that appear in their terms. Note, also, that the original map coincides with its backbone in the limit \( s \to 1 \). This property illustrates the significance of the quadratic nonlinearity vs. arbitrary power nonlinearity. Turning to NLSE \([11]\), if we now substitute the original power nonlinearity with the backbone map, in the orthogonal basis of the Anderson eigenstates we find
\[
\dot{\sigma}_k - \omega_k \sigma_k = \beta \sum_{m_1,m_2,m_3} V_{k,m_1,m_2,m_3} \sigma_{m_1}^* \sigma_{m_2}^* \sigma_{m_3} \tag{12}
\]
where
\[
V_{k,m_1,m_2,m_3} = \sum_n \phi^*_{n,k} \phi^*_{n,m_1} \phi_{n,m_2} \phi_{n,m_3}
\]
are complex coefficients characterizing the overlap structure of the nonlinear field, and we have reintroduced the eigenvalues of the linear problem, \( \omega_k \), satisfying \( \hat{H}_L \phi_{n,k} = \omega_k \phi_{n,k} \). Although obvious, it should be emphasized that the use of the backbone map \( \hat{F}'_s \) in place of the original map \( \hat{F}_s \) preserves the Hamiltonian character of the dynamics, but with a different interaction Hamiltonian, \( \hat{H}_{\text{int}} \),
\[
\dot{\sigma}_k = \omega_k \sigma_k^* + \frac{\beta}{1+s} \sum_{k,m_1,m_2,m_3} V_{k,m_1,m_2,m_3} \sigma_k^* \sigma_{m_1}^* \sigma_{m_2}^* \sigma_{m_3} \tag{14}
\]
Note that \( \hat{H}_{\text{int}} \) includes self-interactions through the diagonal elements \( V_{k,k,k,k} \). Another important point worth noting is that the strength of the interaction vanishes in the limit \( s \to \infty \) (as \( \sim 1/s \)). Therefore, keeping the \( \beta \) parameter finite, and letting \( s \to \infty \), one generates a regime where the nonlinear field is asymptotically localized. One sees that high-power nonlinearities act as to reinstall the Anderson localization. We shall confirm this by the direct calculation of respective transport exponents. Equations (12) define a system of coupled nonlinear oscillators with a parametric dependence on \( s \). Similarly to the NLSE model with a quadratic power nonlinearity, each nonlinear oscillator with the Hamiltonian
\[
\dot{\sigma}_k = \omega_k \sigma_k^* + \frac{\beta}{1+s} \sum_{k,m_1,m_2,m_3} V_{k,m_1,m_2,m_3} \sigma_k^* \sigma_{m_1}^* \sigma_{m_2}^* \sigma_{m_3} \tag{15}
\]
and the equation of motion
\[
\dot{\sigma}_k - \omega_k \sigma_k = \beta V_{k,k,k,k} \sigma_k^* \sigma_k^* \sigma_k = 0 \tag{16}
\]represents one nonlinear eigenstate in the system — identified by its wave number \( k \), unperturbed frequency \( \omega_k \), and nonlinear frequency shift \( \Delta \omega_k = \beta V_{k,k,k,k} \sigma_k^* \sigma_k^* \). We reiterate that non-diagonal elements \( V_{k,m_1,m_2,m_3} \) characterize couplings between each four eigenstates with wave numbers \( k, m_1, m_2, \) and \( m_3 \). The comprehension of Hamiltonian character of the dynamics paves the way for a consistency analysis of the various transport scenarios behind the Anderson localization problem (with the topology of resonance overlap taken into account) \([10, 11]\). To this end, the transport problem for the wave function becomes essentially a topological problem in phase space.
The “edge” character of onset transport corresponds to infinite chains of next-neighbor interactions with a minimized number of links at every step. For the reasons of symmetry, when summing on the right-hand side of Eq. (12), the only combinations of terms to be taken into account, apart from the self-interaction term \( \sigma_k \sigma_k \sigma_k \), are, essentially, \( \sigma_{k-1} \sigma_k \sigma_{k+1} \) and \( \sigma_{k+1} \sigma_k \sigma_{k-1} \). These terms will come with respective interaction amplitudes \( V_{k,k,k,k} \), \( V_{k-1,k+1,k,k} \), and \( V_{k+1,k-1,k,k} \), which we shall denote simply by \( V_k \), \( V'_k \), and \( V''_k \). Then on the right-hand side (r.h.s.) of Eq. (12) we have

\[
r.h.s. = \beta V_k \sigma_k^* \sigma_k^s \sigma_k + \beta \sum_{k} V_k^\pm \sigma_k^* \sigma_{k\pm 1}^s \sigma_{k\mp 1}.
\] (17)

The interaction Hamiltonian in Eq. (14) becomes

\[
\hat{H}_{\text{int}} = \frac{\beta}{1+s} \sum_{k} \left[ V_k \sigma_k^* \sigma_k^s \sigma_k + \sum_{\pm} V_k^{\pm} \sigma_k^* \sigma_{k\pm 1}^s \sigma_{k\mp 1} \right].
\] (18)

representing the effective reduced \( \hat{H}_{\text{int}} \) for arbitrary real power \( s \geq 1 \). Assuming that the exponent \( s \) is confined between two integer numbers, i.e., \( j \leq s < j + 1 \), in the next-neighbor interaction term we can write

\[
\hat{H}'_{\text{int}} = \frac{\beta}{1+s} \sum_{k} \sum_{\pm} V_k^{\pm} \left[ \sigma_k^s \sigma_{k\pm 1}^s \sigma_{k\mp 1}^s \right] \sigma_{k\pm 1}^{s-1} \sigma_{k\mp 1}^{s-j},
\] (19)

where the prime symbol indicates that we have extracted the self-interactions. When drawn on a graph in wave-number space, the terms raised to the power \( s - j \) will correspond to disconnected bonds, thought as Cantor sets with the fractal dimensionality \( 0 < s - j < 1 \). Hence, they will not contribute to field-spreading. These terms, therefore, can be cut off from the interaction Hamiltonian, suggesting that only those terms raised to the integer power, \( j \), should be considered. We have, accordingly,

\[
\hat{H}'_{\text{int}} \to \frac{\beta}{1+s} \sum_{k} \sum_{\pm} V_k^{\pm} \left[ \sigma_k^s \sigma_{k\pm 1}^s \sigma_{k\mp 1}^s \right] \sigma_{k\pm 1}^{s-1} \sigma_{k\mp 1}^{s-j}.
\] (20)

This is the desired result. Equation (20) defines the effective reduced interaction Hamiltonian in the parameter range of onset spreading for \( j \leq s < j + 1 \).

Focusing on the transport problem for the wave field, because the interactions are next-neighbor-like, it is convenient to project the system of coupled dynamical equations (17) on a Cayley tree, such that each node with the coordinate \( k \) represents a nonlinear eigenstate, or nonlinear oscillator with the equation of motion (16); the outgoing bonds represent the complex amplitudes \( \sigma_{k\pm 1}^s \) and \( \sigma_{k\pm 1}^s \); and the ingoing bonds, which involve complex conjugation, represent the complex amplitudes \( \sigma_k^s \). To make it with the amplitudes \( \sigma_k^s \) when raised to the algebraic power \( s \) one needs for each node a fractional number \( s \) of the ingoing bonds. Confining the \( s \) value between two nearest integer numbers, \( j \leq s < j + 1 \), we carry on with \( j \) connected bonds, which we charge to receive the interactions, and one disconnected bond, which corresponds to a Cantor set with the fractal dimensionality \( s - j \), and which cannot transmit the waves. At this point we cut this bond off the tree. A similar procedure applied to the amplitudes \( \sigma_{k+1}^s \), coming up in the algebraic power \( s \), generates \( j \) outgoing bonds, leaving one disconnected bond behind. Lastly, the remaining amplitude \( \sigma_{k\mp 1}^s \), which does not involve a nonlinear power, contributes with one outgoing bond for each combination of the indexes. One sees that the mapping requires a Cayley tree with the coordination number \( z = 2j + 1 \).

C. Step 3: Obtaining the connectivity index

If the interactions are next-neighbor-like, and if the number of excited modes after \( t \) time steps is \( \Delta n(t) \), then self-similarity will imply that

\[
\langle (\Delta n)^2(t) \rangle \propto t^{2/(2+\theta)}, \quad t \to +\infty,
\] (21)

where \( \theta \) is the connectivity exponent of the structure on which the spreading processes occur. This exponent accounts for the deviation from the usual Fickian diffusion in a self-similar geometry [19, 21, 25] and observes remarkable invariance properties under homeomorphic maps of fractals [22, 26]. The scaling law in Eq. (21) has been discussed by Gefen et al. [25] for anomalous diffusion on percolation clusters. The crucial assumption behind this scaling, however, is the assumption of self-similarity (and not of percolation) extending the range of validity of Eq. (21) to any self-similar fractal. Here we apply the scaling law in Eq. (21) to Cayley trees by appropriately choosing the \( \theta \) value. We note in passing that self-similarity of the Cayley trees is not necessarily manifest in their folding in the embedding space, but is inherent in their connectedness and topology [27]. Indeed a Cayley tree is a graph without loops, where each node hosts the same number of branches (known as the coordination number). Therefore, one might expect from the outset that the value of \( \theta \) will be a function of the coordination number, given that the dynamics occur on a Cayley tree. The coordination number, in its turn, will depend on the power \( s \), thus paving the way to predict onset spreading in association with the topology of interaction between the components of the wave field.
More so, the power nonlinearity in Eq. (11) suggests that the connectivity value is a multiplicative function of $s$. Also one might expect this function to naturally reproduce the known value $\theta = 4$ [19, 20, 28] for mean-field percolation on Bethe lattices in the limit $s \to 1$. Then the obvious dependence satisfying these criteria is $\theta = 4s$, where $s \geq 1$. For the integer and half-integer $s$, this dependence can also be derived using the standard renormalization-group procedure [29] for self-similar clusters in Hilbert space. So restricting ourselves to the short times for which the dynamics concentrate on a self-similar geometry, we write, with $\Delta n_{\text{max}} \gg 1$,

\[ \langle (\Delta n(t)^2) \rangle \propto t^{1/(2s+1)}, \quad 1 \ll t \ll (\Delta n_{\text{max}})^{2(2s+1)}, \quad (22) \]

from which the scaling dependence in Eq. (3) can be deduced for quadratic nonlinearity, i.e., $\alpha = 1/3$. Numerically, the field-spreading on finite clusters has been already discussed [30, 31] based on computer simulation results, using one-dimensional disordered Klein-Gordon chains with tunable nonlinearity. It is noticed that the exponent of the powerlaw, $\alpha = 1/(2s + 1)$, vanishes in the limit $s \to \infty$, conformally with the previous considerations.

To illustrate the determination of $\theta$ and to address the origin of subdiffusion in the regime of next-neighbor communication rule, we look directly into the connectivity properties of finite clusters. For this, we need a simple procedure by which calculations can be done exactly. We formulate such a procedure for integer and half-integer values of $s$ using topological triangulation [32, 33] of the Cayley tree. For the purpose of formal analysis, it is essential to choose a node on a Cayley tree and a reference system of $z$ next-neighbor connecting bonds (for a Cayley tree with the coordination number $z = 2s + 1$). Next we dispose the selected node of the Cayley tree and immerse it into a $z$-dimensional Euclidean space $R^{2z+1}$. The latter space is built on $2s + 1$ orthonormal basis vectors. Note that there is a one-to-one correspondence between the basis vectors in $R^{2z+1}$ and the reference bonds on the Cayley tree. Connecting the ending points of the basis vectors generates a polyhedron in $R^{2z+1}$, which reflects the connectivity of the original Cayley tree and the hierarchical composition of this. For the standard Cayley tree with $z = 3$ the associated geometric construction is illustrated in Fig. 1.

More so, we apply the above procedure to all nodes of the original Cayley tree, such that the nodes which communicate via a next-neighbor rule on the tree go to the ending points of the corresponding basis vectors in $R^{2s+1}$. One sees that this procedure generates an infinite chain of mutually overlapping polyhedrons. The number of internal one-bond-connections (OBC’s) is obtained as the minimal number of bonds belonging to the same polyhedron and enabling an infinite connected mesh. We distinguish between “nodes” which compose a polyhedron, that is analyzed, and “node-vertexes” which are nodes pertaining to neighboring polyhedrons. In what follows, we identify the nodes with numbers, and node-vertexes with letters. For instance, the only node having a full family of nearest neighbors in Figs. 2 and 3 is the node marked as 1. The connectivity index $\theta$ is obtained as the number of paths (routes without self-crossings) connecting node 1 to node $2s + 2$ via any nodes of the same polyhedron.

To illustrate, consider quadratic nonlinearity first, with $s = 1$ (see Fig. 2). Clearly, there are just three OBC’s defined by a tetrahedron with nodes 1, 2, 3 and 4. So one identifies these OBC’s with the bonds $(1 - 2), (1 - 3)$, and $(1 - 4)$. It is noticed that the connection between nodes 3 and 4 occurs via the node-vertex $c$; the connection between nodes 2 and 3 occurs via the node-vertex $b$; and the connection between nodes 2 and 4, via the node-vertex $a$. The connectivity index $\theta$ is the number of paths connecting node 1 to node $2s + 2 = 4$. These paths are just four, namely, $(1 - 4), (1 - 2 - 4), (1 - 3 - 4)$, and $(1 - 2 - 3 - 4)$. Hence, $\theta = 4$. This result is to be expected, as it also characterizes mean-field transport on lattice animals [19, 20] and trees [28]. With the aid of Eqs. (21) and (22) one also obtains $\langle (\Delta n(t)^2) \rangle \propto t^{1/3}$ for $t \ll (\Delta n_{\text{max}})^{6}$ consistently with the result of Ref. [11].

Let us now calculate the connectivity index $\theta$ for half-integer $s = 3/2$. Here one constructs a pentahedron in
also get for quadratic power nonlinearity. The transport rates, with the fastest transport obtained
tivities \((s > 1/s)\) has a progressively weakening effect over the transport rates, with the fastest transport obtained for quadratic power nonlinearity.

\[ s = 5/2 \quad \text{and} \quad s = 7/2. \]

Fig. 3: Schematic representation of polyhedrons for the half-integer \(s = 5/2\) and \(s = 7/2\).

\(R^2\), with nodes marked 1, 2, 3, 4 and 5. The OBC’s are the bonds \((1 - 2), (1 - 3), (1 - 4),\) and \((1 - 5)\) (see Fig. 2). There are exactly six paths connecting node 1 to node 5, that is, \((1 - 5), (2 - 6), (4 - 5),\) \((1 - 3 - 2 - 5),\) \((1 - 3 - 4 - 5),\) and \((1 - 2 - 3 - 4 - 5).\) Thus, \(\theta = 6,\) leading to a subdiffusive scaling of second moments \(\langle (\Delta n)^2(t) \rangle \propto t^{1/4}\) for \(t \ll (\Delta n_{\text{max}})^8.\)

The same triangulation procedure applied to a hexahedron in \(R^3\) generates for \(s = 2\) the following eight paths (see Fig. 2): \((1 - 6), (1 - 2 - 6), (1 - 5 - 6), (1 - 3 - 2 - 6),\) \((1 - 4 - 5 - 6), (1 - 4 - 3 - 2 - 6), (1 - 3 - 4 - 5 - 6),\) and \((1 - 2 - 3 - 4 - 5 - 6),\) leading to \(\theta = 8.\) The scaling of second moments is given by \(\langle (\Delta n)^2(t) \rangle \propto t^{1/5}\) for \(t \ll (\Delta n_{\text{max}})^10.\)

In Fig. 3 we also present for reader’s convenience respective geometric constructions corresponding to half-integer \(s = 5/2\) and \(s = 7/2,\) facilitating the calculation of the paths and of respective connectivity values.

By mathematical induction the connectivity index for integer and half-integer \(s\) is given by \(\theta = 4s,\) yielding for the transport exponent \(\alpha = 2/(2 + \theta) = 1/(2s + 1)\) consistently with the subdiffusive scaling in Eq. (22). Eliminating \(s\) with the aid of coordination number, \(z = 2s + 1,\) we also get \(\alpha = 1/z.\) One sees that the transport is slowed down by complexity elements of clusters, contained in the \(z\) value. All in all, one sees that higher-order nonlinearities \((s > 1)\) have a progressively weakening effect over the transport rates, with the fastest transport obtained for quadratic power nonlinearity.

**III. NON-MARKOVIAN DIFFUSION EQUATION**

The next-neighbor communication rule which we associate with the phenomena of onset spreading must have implications for anomalous diffusion on the short times for which the dynamics concentrate on a self-similar geometry of finite clusters. This equation has been already discussed \([13, 16]\) and has been shown to be a non-Markovian variant of the diffusion equation with power-law memory kernel:

\[
\frac{\partial}{\partial t} f(t, \Delta n) = \frac{\partial}{\partial t} \int_0^t \frac{dt'}{(t - t')^{1-\alpha}} \frac{\partial^2}{\partial (\Delta n)^2} [W_\theta(f(t', \Delta n))],
\]

where \(1 - \alpha = \theta/(2 + \theta)\): \(\theta\) is the connectivity exponent; \(W_\theta\) absorbs the parameters of the transport model; and we have chosen \(t = 0\) as the beginning of the system’s time evolution. The integral term on the right-hand side has the analytical structure of fractional time the so-called Riemann-Liouville fractional derivative \([23]\). In a compact form,

\[
\frac{\partial}{\partial t} f(t, \Delta n) = \frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}} \frac{\partial^2}{\partial (\Delta n)^2} [W_\theta(f(t, \Delta n))].
\]

The fractional order of time differentiation in Eq. (24) is determined by the connectivity value through \(1 - \alpha = \theta/(2 + \theta)\) and is exactly zero for \(\theta = 0.\) Then the fractional derivative of the zero order is a unity operator, implying that no fractional properties come into play for homogeneous spaces. Also in writing Eq. (23) we have adopted results of Refs. [13, 15] to diffusion processes on a single cluster. Equations (23) and (24) when account is taken for the initial value problem can be rephrased \([13]\) in terms of the Caputo fractional derivative \([24]\), which shows a better behavior under transformations.

One sees that the dispersion law in Eq. (24) can be obtained as a second moment of the fractional diffusion equation \([22]\), with \(\alpha = 2/(2 + \theta).\) Using for the connectivity exponent \(\theta = 4s,\) one also finds the fractional order \(1 - \alpha = 2s/(2s + 1)\) in the entire parameter range \(s \geq 1,\) showing that ordinary differentiation is reinstated on the right-hand side of Eq. (24) in the limit \(s \to \infty.\) For \(s = 1,\) one gets \(1 - \alpha = 2/3,\) implying that the diffusion process is essentially non-Markovian with power-law correlations in the regime of quadratic nonlinearity. We associate this non-Markovian character of field-spreading with the effect of complexity elements of the Cayley tree, contained in the \(z = 2s + 1\) value.

It is noticed that the fractional diffusion equation in Eq. (24) is “born” within the exact mathematical framework of nonlinear Schrödinger equation with usual time differentiation. Indeed, no ad hoc introduction of fractional time differentiation in the dynamic Eqs. (1) has been assumed to obtain this subdiffusion. It is, in fact, the interplay between nonlinearity and randomness, which leads to a non-Markovian transport of the wave function at criticality, and to a time-fractional kinetic equation in the end. This observation also emphasizes the different physics implications behind the fractional kinetic vs. dynamical equations \([15, 33, 36]\). Equation (24) shows that the onset spreading is a matter of fractional, or “strange,” kinetics \([27, 32]\) consistently with the implication of critical behavior \([13, 16, 18].\)
The fundamental solution or Green’s function of the fractional Eq. (24) is evidenced in Table 1 of Ref. [40].

IV. CONCLUSIONS

This study is concerned with destruction of Anderson localization by a nonlinearity of the power-law type. It has been proposed using an NLSE with random potential on a lattice that quadratic nonlinearity plays a dynamically very distinguished role in that it is the only type of power nonlinearity permitting an abrupt localization-delocalization transition with unlimited spreading already at the delocalization border. For super-quadratic nonlinearity the borderline spreading corresponds to a diffusion process on finite clusters. We have suggested an analytical method to predict and explain such transport processes. Our method uses a topological approximation of the nonlinear Anderson model and, if the exponent of the power nonlinearity is either integer or half-integer, will yield the wanted value of the transport exponent via a triangulation procedure in an Euclidean mapping space. Also we predict that the transport of waves at the border of delocalization is subdiffusive, with the exponent \( \alpha \) which is inversely proportional with the power nonlinearity increased by one. For quadratic nonlinearity we have \( \langle (\Delta n)^2(t) \rangle \propto t^{1/3} \) for \( t \rightarrow +\infty \) consistently with the previous investigations [10, 11]. A kinetic picture of the transport arising from these investigations uses a fractional extension of the diffusion equation to fractional derivatives over the time, signifying non-Markovian dynamics with algebraically decaying time correlations.

Acknowledgments

A.V.M. and A.I. thank the Max-Planck-Institute for the Physics of Complex Systems for hospitality and financial support. This work was supported in part by the Israel Science Foundation (ISF) and by the ISSI project “Self-Organized Criticality and Turbulence” (Bern, Switzerland).

[1] Anderson P W 1958, Phys. Rev. 109 1492
[2] Akkermans E and Montambaux G 2006 Mesoscopic Physics of Electrons and Photons (Cambridge: Cambridge Univ. Press)
[3] Weaver R L 1990 Wave Motion 12 129
[4] Størzer M, Gross P, Aegerter C. M. and Maret G 2006 Phys. Rev. Lett. 96, 063904
[5] Schwartz T, Bartal G, Fishman S, and Segev M 2007 Nature (London) 446 52
[6] Billy J, Josse V, Zuo Z, Bernard A, Hambrecht B, Lugan P, Clément D, Sanchez-Palencia L, Bouyer P and Aspect A 2008 Nature (London) 453 891
[7] Abou-Chacra R, Anderson P W and Thouless D J 1973 J. Phys. C: Solid State Phys. 6 1734
[8] Shepelyansky D L 1993 Phys. Rev. Lett. 70 1787
[9] Pikovsky A S and Shepelyansky D L 2008 Phys. Rev. Lett. 100 094101
[10] Milovanov A V and Iomin A 2012 Europhys. Lett. 100 10006
[11] Milovanov A V and Iomin A 2014 Phys. Rev. E accepted.
[12] Chirikov B V and Vecheslavov V V 1997 Zh. Éksp. Teor. Fiz. 112 1132
[13] 2009 Phys. Rev. E 79, 046403 (2009).
[14] Milovanov A V 2010 Europhys. Lett. 89 60004
[15] Milovanov A V 2011 New J. Phys. 13 043034
[16] Milovanov A V 2013 Self-Organized Criticality Systems ed. Aschwanden M J (Berlin: Open Academic Press) (Percolation Models of Self-Organized Critical Phenomena, Chapter 4, pp. 103-182).
[17] Lyubomudrov O, Edelman M and Zaslavsky G M 2003 Int. J. Mod. Phys. B 17 4149
[18] Zaslavsky G M Phys. Rep. 371 461
[19] Havlin S and ben-Avraham D 2002 Adv. Phys. 51 187
[20] Havlin S and ben-Avraham D 2002 Diffusion and Reactions in Fractals and Disordered Systems (Cambridge: Cambridge University Press)
[21] Nakayama T, Yakubo K and Orbach R L 1994 Rev. Mod. Phys. 66 381
[22] Zelenyi L M and Milovanov A V 2004 Phys. Uspekhi 47 749
[23] Hirsch M W 1997 Differential Topology (New York: Springer)
[24] Cox D, Little J and O’Shea D 1998 Using Algebraic Geometry (New York: Springer-Verlag)
[25] Gefen Y, Aharony A and Alexander S 1983 Phys. Rev. Lett. 50 77
[26] Milovanov A V 1997 Phys. Rev. E 56 2437
[27] Schroeder M R 1991 Fractals, Chaos, Power Laws: Minutes from an Infinite Paradise (New York: Freeman)
[28] Coniglio A 1982 J. Phys. A 15 3829
[29] O’Shaughnessy B and Procaccia I 1985 Phys. Rev. Lett. 54 455
[30] Iomin A 2010 Phys. Rev. E 81 017601
[31] Skokos Ch and Flach S Phys. Rev. E 82 016208
[32] Nash C and Sen S 1987 Topology and Geometry for Physicists (London: Academic Press)
[33] Fomenko A T and Fuks D B 1989 A Course of Homotopic Topology (Moscow: Nauka)
[34] Podlubny I 1999 Fractional Differential Equations (San Diego: Academic Press)
[35] Iomin A 2009 Phys. Rev. E 80 022103
[36] Iomin A 2011 Chaos, Solitons & Fractals 44 348
[37] Metzler R and Klafter J 2000 Phys. Rep. 339 1
[38] Shlesinger M F, Zaslavsky G M and Klafter J 1993 Nature (London) 363 31
[39] Sokolov I M, Klafter J and Blumen A 2002 Phys. Today 55 48
[40] Metzler R and Klafter J 2004 J. Phys. A: Math. Gen. 37 R161