CALABI-YAU PROBLEM FOR LEGENDRIAN CURVES IN $\mathbb{C}^3$ AND APPLICATIONS

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Abstract. We construct a complete, bounded Legendrian immersion in $\mathbb{C}^3$. As direct applications of it, we show the first examples of a weakly complete bounded flat front in hyperbolic 3-space, a weakly complete bounded flat front in de Sitter 3-space, and a weakly complete bounded improper affine front in $\mathbb{R}^3$.

1. Introduction

In a series of previous papers, the authors have constructed the first examples of complete bounded null holomorphic immersion $\nu : \mathbb{D}_1 \rightarrow \mathbb{C}^3$ of the unit disc $\mathbb{D}_1 \subset \mathbb{C}$, where null means that $\nu_z \cdot \nu_z$ vanishes identically; here $\nu_z := d\nu/dz$ is the derivative of $\nu$ with respect to the complex coordinate $z$ of $\mathbb{D}_1$ and the dot denotes the canonical complex bilinear form. The existence of such an immersion has important consequences. Actually, as a short and direct application of the main result in [14], by using different kinds of transformations, the following objects were constructed:

1. complete bounded minimal surfaces in the Euclidean 3-space $\mathbb{R}^3$ ([14, Theorem A]),
2. complete bounded holomorphic curves in $\mathbb{C}^2$ ([14, Corollary B]).
3. weakly complete bounded maximal surfaces in the Lorentz-Minkowski 3-space $\mathbb{R}^3_1$ ([14, Corollary D]),
4. complete bounded constant mean curvature one surfaces in the hyperbolic 3-space $H^3$ ([14, Theorem C]).

Moreover, we constructed higher genus examples of the first three objects in [15].

Recently, Alarcón and López [1] have constructed a complete bounded null proper holomorphic immersion of a given Riemann surface of an arbitrary topology into a convex domain in $\mathbb{C}^3$ (see also [2]). Their method is different from ours.

It is known that null curves in $\mathbb{C}^3$ are closely related to Legendrian curves in $\mathbb{C}^3$ (cf. Bryant [4] and also Ejiri-Takahashi [5] for the corresponding SL(2, $\mathbb{C}$)-case). In this paper, we use the techniques develop by the authors in [14] to produce a

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complete bounded Legendrian holomorphic immersion

\[ F : \mathbb{D}_1 \rightarrow \mathbb{C}^3. \]

Recall that \( F \) is called Legendrian if the pull-back of the canonical contact form

\[ \Omega_{\mathbb{C}} := dx_3 + x_2dx_1 \]

by \( F \) vanishes, where \((x_1, x_2, x_3)\) is the canonical complex coordinate system of \( \mathbb{C}^3 \). The existence of such an \( F \) is non-trivial, since the correspondences between null curves and Legendrian curves given in [4] and [5] seem not to preserve neither boundedness nor completeness. Also, the authors do not know whether the method in [1] can be applied for Legendrian holomorphic immersions by using a suitable modification or not.

As applications, we are able to construct the following new examples:

1. a weakly complete bounded flat front in \( \mathbb{H}^3 \) (Theorem 4.1),
2. a weakly complete bounded flat front in the de Sitter 3-space \( S_3^1 \) (Theorem 4.2),
3. a weakly complete bounded improper affine front in \( \mathbb{R}^3 \) (Theorem 4.3).

It should be remarked that there are no compact flat fronts in \( \mathbb{H}^3 \) and \( S_3^1 \) (resp.
improper affine fronts in \( \mathbb{R}^3 \)). See Remark 4.4.

A holomorphic map \( E : \mathbb{D}_1 \rightarrow SL(2, \mathbb{C}) \) is called Legendrian if the pull-back
\[ E^*\Omega_{SL} \]
vanishes on \( \mathbb{D}_1 \), where \( \Omega_{SL} \) is the complex contact form on \( SL(2, \mathbb{C}) \) defined as

\[ \Omega_{SL} := x_{11}dx_{22} - x_{12}dx_{21}. \]

Here, elements in \( SL(2, \mathbb{C}) \) are represented by matrices \((x_{ij})_{i,j=1,2} \). A holomorphic immersion \( E : \mathbb{D}_1 \rightarrow SL(2, \mathbb{C}) \) is said to be complete if the pull-back metric \( E^*g_{SL} \) of the canonical Hermitian metric \( g_{SL} \) on \( SL(2, \mathbb{C}) \) is complete, see (2.2).

To construct the bounded holomorphic immersion \( F : \mathbb{D}_1 \rightarrow \mathbb{C}^3 \), we show the following

**Main Theorem.** There exists a complete holomorphic Legendrian immersion of the unit disk \( \mathbb{D}_1 \subset \mathbb{C} \) into \( SL(2, \mathbb{C}) \) such that its image is contained an arbitrary bounded domain in \( SL(2, \mathbb{C}) \).

By Darboux’s theorem, the contact structure of \( SL(2, \mathbb{C}) \) is locally Legendrian equivalent to that of \( \mathbb{C}^3 \). Moreover, the following explicit transformation

\[ F : \mathbb{C}^3 \ni (x, y, z) \mapsto \begin{pmatrix} e^{-z} & xe^{-z} \\ ye^{z} & e^{z}(1 + xy) \end{pmatrix} \in SL(2, \mathbb{C}) \]

maps holomorphic contact curves in \( \mathbb{C}^3 \) to those in \( SL(2, \mathbb{C}) \). Then if one take a complete Legendrian immersion of \( \mathbb{D}_1 \) into \( SL(2, \mathbb{C}) \) with sufficiently small image in \( SL(2, \mathbb{C}) \), a Legendrian immersion into \( \mathbb{C}^3 \) is obtained. Completeness follows from the same argument as [14, Lemma 3.1]. In fact, since the image is bounded, the metrics induced from \( SL(2, \mathbb{C}) \) and \( \mathbb{C}^3 \) are equivalent.

The paper is organized as follows: In Section 2, we establish our formulations and state the key-lemma to prove the main theorem, which is proved in Section 3. In Section 4, we give the applications as above. In the appendix, we prepare a Runge-type theorem for Legendrian curves in \( SL(2, \mathbb{C}) \) which is needed in Section 3.

Finally, we mention the corresponding real problem, that is, the existence of complete bounded Legendrian submanifolds immersed in \( \mathbb{R}^{2n+1} \). When \( n = 1 \), there exists a closed Legendrian curve immersed in an arbitrarily given open subset
in $\mathbb{R}^3$. In fact, in [9, Section 2], it is shown the existence of a Legendrian curve contained in an arbitrary given open ball of $P^3 = T_1S^2$ (i.e. the unit cotangent bundle of 2-sphere) as a lift of an eye-figure curve. Since any contact structure is locally rigid, it gives an existence of a closed Legendrian curve immersed in any ball of $\mathbb{R}^3$. Also, as an application of our construction, we can construct a complete bounded Legendrian immersion $L : D_1 \rightarrow B(\subset \mathbb{R}^5)$: There exists a canonical projection (cf. [11, Page 159])

$$\pi : SL(2, \mathbb{C}) \rightarrow T^*_1H^3,$$

where $T^*_1H^3$ is a unit cotangent bundle of the hyperbolic 3-space $H^3$. Then the projection of our complete bounded holomorphic Legendrian curve gives a complete bounded Legendrian submanifold immersed in an arbitrarily given open subset of $T^*_1H^3$. By Darboux’s rigidity theorem, this implies the existence of a complete Legendrian immersion $L : D_1 \rightarrow B$, where $B$ is an arbitrary ball in $\mathbb{R}^3$.

2. The Main Lemma

In this section, we state the main lemma, which is an analogue of [14, Main Lemma in page 121]. The main theorem in the introduction can be obtained as a direct conclusion of the main lemma in the same way as in [14].

2.1. Preliminaries. We denote $i = \sqrt{-1}$ and

$$D_r := \{ z \in \mathbb{C} ; |z| < r \}, \quad \overline{D}_r := \{ z \in \mathbb{C} ; |z| \leq r \}$$

for a positive number $r$. Throughout this paper, the prime ‘ means the derivative with respect to the complex coordinate $z$ on $\mathbb{C}$.

**Proposition 2.1.** A holomorphic immersion $X : \overline{D}_1 \rightarrow SL(2, \mathbb{C})$ is Legendrian if and only if $X^{-1}X'$ is anti-diagonal:

$$\psi_X dz := X^{-1}dX = \begin{pmatrix} 0 & \theta \\ \omega & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \varphi_1 + i\varphi_2 \\ \varphi_1 - i\varphi_2 & 0 \end{pmatrix} dz,$$

where $\varphi_1$ and $\varphi_2$ are holomorphic functions on $\overline{D}_1$. The metric induced by $X$ from the canonical Hermitian metric $g_{SL}$ of $SL(2, \mathbb{C})$ is represented as

$$ds^2_X := |\omega|^2 + |\theta|^2 = (|\varphi_1|^2 + |\varphi_2|^2)|dz|^2.$$

In particular, $\varphi_1$ and $\varphi_2$ have no common zeros on $\overline{D}_r$.

The holomorphic 1-forms $\omega$ and $\theta$ in (2.1) are called the **canonical one forms** for the flat front corresponding to $X$, see [12].

**Definition 2.2.** A pair of holomorphic functions $\varphi = (\varphi_1, \varphi_2)$ on $\overline{D}_1$ is called non-degenerate if $\varphi_1$ and $\varphi_2$ have no common zeros. The pair $(\varphi_1, \varphi_2)$ given by (2.1) is called the **holomorphic data** of $X$. The matrix valued function

$$M_\varphi := \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \varphi_1 + i\varphi_2 \\ \varphi_1 - i\varphi_2 & 0 \end{pmatrix}$$

is called the **matrix form** of the pair $\varphi$. 
2.2. The Main Lemma. To state the lemma, we define the matrix norm $|A|$ of a $2 \times 2$-matrix $A$ as

\begin{equation}
|A| := \sqrt{\text{trace}(AA^*)} = \sqrt{\sum_{i,j=1,2} |A_{ij}|^2} \quad (A = (A_{ij})_{i,j=1,2}).
\end{equation}

Note that if $A \in \text{SL}(2, \mathbb{C})$, then $|A| \geq \sqrt{2}$ holds. The equality holds if and only if $A$ is the identity matrix.

For a vector $v = (v_1, v_2) \in \mathbb{C}^2$, we set $|v| = \sqrt{|v_1|^2 + |v_2|^2}$.

Main Lemma. Let $X : \mathbb{D}_1 \to \text{SL}(2, \mathbb{C})$ be a holomorphic Legendrian immersion $X : \mathbb{D}_1 \to \text{SL}(2, \mathbb{C})$ satisfies the following properties:

1. $X(0) = \text{id}$, where id is the identity matrix.
2. $\mathbb{D}_1$ contains the geodesic disc of radius $\rho$ centered at the origin with respect to the induced metric $ds_X^2$.
3. There exists a number $\tau > \sqrt{2}$ such that $|X| \leq \tau$ holds on $\mathbb{D}_1$.

Then, for any positive numbers $\varepsilon$ and $s$, there exists a holomorphic Legendrian immersion $Y : \mathbb{D}_1 \to \text{SL}(2, \mathbb{C})$ such that

1. $Y(0) = \text{id}$,
2. $\mathbb{D}_1$ contains the geodesic disc of radius $\rho + s$ centered at the origin with respect to the induced metric $ds_Y^2$,
3. $|Y| \leq \tau \sqrt{1 + 32s^2 + \varepsilon}$ in $\mathbb{D}_1$,
4. $|Y - X| < \varepsilon$ and $|\varphi_Y - \varphi_X| < \varepsilon$ in $\mathbb{D}_{1-\varepsilon}$, where $\varphi_X$ and $\varphi_Y$ denote holomorphic data of $X$ and $Y$, respectively.

The main theorem in the introduction is obtained by the same argument as [14, Section 3.4]).

2.3. Key Lemma. Now we state the key lemma, as an analogue of [14, Key Lemma in page 129]. The main lemma in the previous subsection can be obtained directly from the key lemma.

We work on the Nadirashvili’s labyrinth [17]. Let us give a brief description of this labyrinth: Let $N$ be a (sufficiently large) positive number. For $k = 0, 1, 2, \ldots, 2N^2$, we set

\begin{equation}
r_k = 1 - \frac{k}{N^2} \quad \left( r_0 = 1, r_1 = 1 - \frac{1}{N^2}, \ldots, r_{2N^2} = 1 - \frac{2}{N} \right),
\end{equation}

and let

\begin{equation}
\mathbb{D}_r_k = \{ z \in \mathbb{C} ; |z| < r_k \} \quad \text{and} \quad S_r_k = \partial \mathbb{D}_r_k = \{ z \in \mathbb{C} ; |z| = r_k \}.
\end{equation}

We define an annular domain $A$ as

\begin{equation}
A := \mathbb{D}_1 \setminus \mathbb{D}_{r_{2N^2}} = \mathbb{D}_1 \setminus \mathbb{D}_{1 - \frac{2}{N}},
\end{equation}

and

\begin{align*}
A &= \bigcup_{k=0}^{N^2-1} \mathbb{D}_{r_{2k}} \setminus \mathbb{D}_{r_{2k+1}}, \quad \tilde{A} := \bigcup_{k=0}^{N^2-1} \mathbb{D}_{r_{2k+1}} \setminus \mathbb{D}_{r_{2k+2}},
L &= \bigcup_{k=0}^{N-1} l_{\frac{2k\pi}{N}}, \quad \tilde{L} = \bigcup_{k=0}^{N-1} l_{\frac{(2k+1)\pi}{N}},
\end{align*}
where \( l_\theta \) is the ray \( l_\theta = \{ re^{i\theta} : r \geq 0 \} \). Let \( \Sigma \) be a compact set defined as
\[
\Sigma := L \cup \overline{L} \cup S, \quad S = \bigcup_{j=0}^{2N^2} D_{r_j} = \bigcup_{j=0}^{2N^2} S_{r_j},
\]
and define a compact set \( \Omega \) by
\[
\Omega = A \setminus U_{1/(4N^3)}(\Sigma),
\]
where \( U_\varepsilon(\Sigma) \) denotes the \( \varepsilon \)-neighborhood (of the Euclidean plane \( \mathbb{R}^2 = \mathbb{C} \)) of \( \Sigma \). Each connected component of \( \Omega \) has width \( 1/(2N^3) \). For each number \( j = 1, \ldots, 2N \), we set
\[
\omega_j := (l_{\infty} \cap A) \cup \left( \text{connected components of } \Omega \text{ which intersect with } l_{\infty} \right),
\]
\[
\overline{\omega}_j := U_{1/(4N^3)}(\omega_j).
\]
Then \( \omega_j \)'s are compact sets.

**Key Lemma.** Assume that a holomorphic Legendrian immersion \( \mathcal{L} = \mathcal{L}_0 : \mathbb{D}_1 \rightarrow \text{SL}(2, \mathbb{C}) \) satisfies:

(A-1) \( \mathcal{L}(0) = \text{id} \),

(A-2) \( \mathbb{D}_1 \) contains the geodesic disc of radius \( \rho \) centered at the origin with respect to the metric \( ds_L^2 \).

Then for any positive number \( \varepsilon \) and positive number \( s \in (0, 1/3) \), there exists a sufficiently large integer \( N \) and a sequence of holomorphic Legendrian immersions \( \mathcal{L}_0 = \mathcal{L}, \mathcal{L}_1, \ldots, \mathcal{L}_{2N} \) of \( \mathbb{D}_1 \) such that

(C-1) \( \mathcal{L}_j(0) = \text{id} \ (j = 0, \ldots, 2N) \),

(C-2) for each \( j = 1, \ldots, 2N \), \( |\varphi_j - \varphi_{j-1}| < \varepsilon/(2N^2) \) holds on \( \mathbb{D}_1 \setminus \overline{\omega}_j \), where \( \varphi_j \) is the non-degenerate holomorphic data of \( \mathcal{L}_j \),

(C-3) for each \( j = 1, \ldots, 2N \),
\[
|\varphi_j| \geq \begin{cases} 
cN^{3.5} & \text{on } \omega_j \\
cN^{-0.5} & \text{on } \overline{\omega}_j
\end{cases}
\]
holds, where \( c \) is a positive constant depending only on \( \mathcal{L} = \mathcal{L}_0 \),

(C-4) \( \mathbb{D}_1 \) contains the geodesic disc of radius \( \rho + s \) centered at the origin with respect to the metric \( ds_{\mathcal{L}_{2N}}^2 \),

(C-5) on \( \overline{\mathbb{D}}_g \) as in (C-4), it holds that
\[
|\mathcal{L}_{2N}| \leq \left( \max_{\mathbb{S}_1} |\mathcal{L}_0| \right) \sqrt{1 + 32s^2 + (b/\sqrt{N})},
\]
where \( b \) is a positive constant depending only on \( \mathcal{L} = \mathcal{L}_0 \).

The proof is given in Section 3.

3. Proof of the Key Lemma

3.1. Flat fronts in hyperbolic 3-space. We denote by \( H^3 \) the hyperbolic 3-space, that is, the connected and simply connected 3-dimensional space form of constant sectional curvature \(-1\), which is represented as
\[
H^3 = \text{SL}(2, \mathbb{C})/\text{SU}(2) = \{ aa^* : a \in \text{SL}(2, \mathbb{C}) \}
\]
\[
= \{ X \in \text{Herm}(2) ; \text{det } X = 1, \text{trace } X > 0 \}, \quad (a^* = \bar{a}).
\]
where $\text{Herm}(2)$ is the set of $2 \times 2$ Hermitian matrices. Identifying $\text{Herm}(2)$ with the Lorentz-Minkowski 4-space $\mathbb{R}_1^4$ as

$$
(3.2) \quad \text{Herm}(2) \ni \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix} \quad \leftrightarrow \quad (x_0, x_1, x_2, x_3) \in \mathbb{R}_1^4,
$$

the hyperbolic space $H^3$ can be considered as the connected component of the two-sheeted hyperboloid

$$
\{ (x_0, x_1, x_2, x_3) \in \mathbb{R}_1^4 : -(x_0)^2 + (x_1)^2 + (x_2)^2 + (x_3)^2, x_0 > 0 \}. 
$$

A Legendrian immersion $\mathcal{L} : \mathbb{D}_1 \to \text{SL}(2, \mathbb{C})$ induces a flat front

$$
l = \mathcal{L}^* : \mathbb{D}_1 \to H^3.
$$

Here flat fronts in $H^3$ are flat surfaces with certain kind of singularities, see Section 4.1. The pull-back of the metric of $H^3$ by $l$ is computed as

$$
(3.3) \quad ds_l^2 := |\omega|^2 + |\theta|^2 + \omega \theta + \bar{\omega} \bar{\theta} = |\omega + \bar{\theta}|^2
$$

$$
\left( \mathcal{L}^{-1} d\mathcal{L} = \begin{pmatrix} 0 & \theta \\ \omega & 0 \end{pmatrix} \right),
$$

which is positive semi-definite and may degenerate. On the other hand, let $ds_{\mathcal{L}}^2$ be the pull-back of the canonical Hermitian metric of $\text{SL}(2, \mathbb{C})$ by $\mathcal{L}$. Then by (2.2), we have

$$
ds_{\mathcal{L}}^2 - \frac{1}{2} ds_l^2 = \frac{1}{2} (|\omega|^2 + |\theta|^2 - \omega \theta - \bar{\omega} \bar{\theta}) = \frac{1}{2} |\omega - \bar{\theta}|^2 \geq 0,
$$

and hence

$$
(3.4) \quad ds_l^2 \leq 2 ds_{\mathcal{L}}^2
$$

holds. For any path $\gamma$ in $\mathbb{D}_1$ joining $x$ and $y \in \mathbb{D}_1$, it holds that

$$
(3.5) \quad \text{Length}_{ds_l^2} \gamma := \int_{\gamma} ds_l \geq \text{dist}_{H^3} \left( l(x), l(y) \right),
$$

where $\text{dist}_{H^3}$ denotes the distance in the hyperbolic 3-space. On the other hand,

$$
(3.6) \quad 2 \cosh \text{dist}_{H^3} \left( o, l(x) \right) = |\mathcal{L}(x)|^2
$$

holds (see [14, Lemma A.2]), where we set

$$
o := \text{id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
$$

namely $o$ is the point on $H^3$ which corresponds to the origin of the Poincaré ball model.

3.2. Inductive construction of $\mathcal{L}_j$'s. In this section, we describe the recipe to construct a sequence $\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_{2N}$ in Key Lemma. Assume $\mathcal{L}_0, \ldots, \mathcal{L}_{j-1}$ are already obtained, and we shall now construct $\mathcal{L}_j$ as follows: Let

$$
(3.7) \quad \xi_j := \left( 1 - \frac{2}{N} - \frac{4}{N^3} \right) e^{i\pi j/N}
$$

be the base point of the compact set $\omega_j$ given in [14, Fig. 1]. We set

$$
E_0(z) := \mathcal{L}_{j-1}^{-1} \mathcal{L}_{j-1}(z), \quad f_0(z) := E_0(z) E_0^*(z).
$$

That is, $E_0$ is the Legendrian immersion with the same holomorphic data as $\mathcal{L}_{j-1}$ such that $E_0(\xi_j) = \text{id}$, and $f_0$ the corresponding flat front. Here, if we write

$$
f_0(0) = \begin{pmatrix} \xi_0 + \xi_3 & \xi_1 + i\xi_2 \\ \xi_1 - i\xi_2 & \xi_0 - \xi_3 \end{pmatrix} \in H^3,
$$
then there exist real numbers \( t \) and \( \hat{C}_1 \) such that

\[
(3.9) \quad a f_0(0)a^* = \left( \begin{array}{cc} \xi_0 + \xi_3 & \hat{\xi}_1 \\ \hat{\xi}_1 & \xi_0 - \xi_3 \end{array} \right) \in H^3, \quad \left( a = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \right) \in \text{SL}(2, \mathbb{C}).
\]

We now set

\[
(3.10) \quad E := aE_0a^*, \quad \psi := E^{-1}E'.
\]

Then it holds that

\[
\psi = a^*\psi_{j-1}a \quad (\psi_{j-1} := \mathcal{L}_{j-1}^{-1}\mathcal{L}_{j-1}).
\]

Let \( \varphi = (\varphi_1, \varphi_2) \) be the holomorphic data induced from \( E \) such that \( \psi = M_{\varphi} \) (see (2.3)). Applying Lemma A.1 in the appendix to this \( \varphi \), we get a new holomorphic data \( \hat{\varphi} \). Take \( \hat{E} \) such that

\[
(3.11) \quad \hat{E}^{-1}E' = \hat{\psi}, \quad \hat{E}(\zeta_j) = \text{id} \quad (\hat{\psi} = M_{\hat{\varphi}}).
\]

Finally, we set

\[
(3.12) \quad \mathcal{L}_j := a^*\{\hat{E}(0)\}^{-1}\hat{E}a,
\]

which gives the desired Legendrian immersion.

3.3. **Estimation of interior distance.** Applying the inductive construction, we get a sequence of Legendrian immersion \( \mathcal{L}_1, \ldots, \mathcal{L}_{2N} \). Then one can prove (C-1), (C-2), (C-3) and (C-4) by the exactly same argument as in [14, Page 129]. In fact, although the data \( \varphi \) (a pair of holomorphic functions) is different from that in [14] (a triple of holomorphic functions), the proof of the key lemma (as in [14, Page 129]) only needs the norm \( |\varphi| \) of the data \( \varphi \). In [14], we were working on the metric \( ds_{\mathcal{L}}^2 = |\psi_j|^2dz\,d\bar{z} \). In this paper, we now use the metric \( ds_{\mathcal{L}}^2 = |\psi_j|^2dz\,d\bar{z} \), where \( \psi_j \) is given in (3.10). If one replaces \( |\psi_j| \) in [14] by this new \( |\psi_j| \) as above, then the completely same argument as [14] works in our case.

3.4. **Extrinsic distance.** Thus, the only remaining assertion we should prove is (C-5) of Key Lemma. We shall now prove it: By (C-4), \( (\mathcal{D}, ds_{\mathcal{L}}^2) \) contains a geodesic disc \( \mathcal{D}_g \) centered at origin with radius \( \rho + s \). By the maximum principle, it is sufficient to show that for each \( p \in \partial\mathcal{D}_g \), it holds that

\[
(3.13) \quad |\mathcal{L}_{2N}(p)| \leq \left( \max_{\mathcal{F}_1} |\mathcal{L}_0| \right) \sqrt{1 + 32s^2 + (b/\sqrt{N})} \quad (p \in \partial\mathcal{D}_g),
\]

where \( b \) is a positive constant depending only on the initial immersion \( \mathcal{L}_0 \).

If \( p \in \partial\mathcal{D}_g \) is not in \( \mathcal{W}_1 \cup \cdots \cup \mathcal{W}_{2N} \), the same argument as [14, Page 129] implies the conclusion. So, it is sufficient to consider the case that there exists \( j \in \{1, \ldots, 2N\} \) such that

\[
(3.14) \quad p \in \partial\mathcal{D}_g \cap \mathcal{W}_j.
\]

We fix such a \( p \). From now on, the symbols \( c_k \) (\( k = 1, 2, \ldots \)) denote suitable positive constants, which depend only on the initial data \( \mathcal{L} = \mathcal{L}_0 \).

Like as in the proof of the inequality [14, (4.8)], we can apply [14, Corollary A.6] for \( X := \mathcal{L}_j \) and \( Y := \mathcal{L}_{j-1} \). Then we have

\[
(3.15) \quad \text{dist}_{H^3}(l_j(z), l_{j-1}(z)) \leq \frac{c_1\varepsilon}{2N^2} \quad \text{(on } \mathcal{D}_1 \setminus \mathcal{W}_j\text{)},
\]
where \( l_j := \mathcal{L}_j \mathcal{L}^*_j \) and \( l_{j-1} := \mathcal{L}_{j-1} \mathcal{L}^*_{j-1} \). Taking (3.15) into account, inequality (3.13) reduces to the following inequality

\[
|\mathcal{L}_j(p)| \leq \left( \max_{z \in \overline{\mathbb{B}_1}} |\mathcal{L}_0(z)| \right) \sqrt{1 + 32s^2 + \left( \frac{c_2}{\sqrt{N}} \right)} \quad (p \in \partial \mathcal{D}_g \cap \varpi_j).
\]

Thus, by (A-2), we have

\[
\text{dist}_{ds^2}(\hat{p}, p) \leq s + \frac{c_3}{\sqrt{N}}.
\]

In fact, let \( \gamma_1 \) (resp. \( \gamma_2 \)) be the subarc of \( \gamma_0 \) which joins \( 0 \) and \( \hat{p} \) (resp. \( \hat{p} \) and \( p \)). Since \( \gamma_0 \) is the geodesic and \( \mathcal{D}_g \) is the disc with radius \( \rho + s \), (C-2) and (C-3) yields that

\[
\rho + s = \int_{\gamma_0} |\varphi_2| |dz| \geq \int_{\gamma_1} |\varphi_0| |dz| - \int_{\gamma_1} |\varphi_2 - \varphi_0| |dz| + \int_{\gamma_2} |\varphi_2| |dz|
\]

\[
\geq \int_{\gamma_1 \cup \gamma_2} |\varphi_0| |dz| - \int_{\gamma_2} |\varphi_0| |dz| - \frac{\varepsilon}{N} + \text{dist}_{ds^2}(\hat{p}, p)
\]

which implies (3.17).

Moreover, since \( \zeta_j \) and \( \hat{p} \) can be joined by a path \( \gamma \) on \( \overline{\mathbb{D}_1 \setminus (\varpi_1 \cup \cdots \cup \varpi_{2N})} \) whose Euclidean length is not greater than \( c_8/N \) (see [14, Fig. 1]), we have

\[
\text{dist}_{ds^2}(\zeta_j, p) \leq s + \frac{c_8}{\sqrt{N}}
\]

(see [14, (4.9)]).

**Lemma 3.1.** In the above setting, the following inequalities hold:

\[
|\mathcal{L}_{j-1}(\zeta_j)|^2 \leq \left( \max_{z \in \overline{\mathbb{B}_1}} |\mathcal{L}_0(z)|^2 \right) \left( 1 + \frac{c_{10}}{N} \right),
\]

\[
\text{dist}_{H^3}(l_{j-1}(\zeta_j), l_j(\zeta_j)) \leq \frac{c_{11}}{N^2}.
\]

**Proof.** The first inequality holds because

\[
|\mathcal{L}_{j-1}(\zeta_j)|^2 = 2 \cosh\left( \text{dist}_{H^3}(0, l_{j-1}(\zeta_j)) \right) \leq 2 \cosh\left( \text{dist}_{H^3}(0, l_0(\zeta_j)) + \frac{c_{12}}{N} \right),
\]

see (3.6). The second inequality directly follows from (3.15). \( \square \)

Now, recall the procedure constructing \( \mathcal{L}_j \) from \( \mathcal{L}_{j-1} \): Two Legendrian immersions \( E, \hat{E} \) in (3.10) and (3.11) are congruent to \( \mathcal{L}_{j-1}, \mathcal{L}_j \), respectively, and satisfy \( E(\zeta_j) = \hat{E}(\zeta_j) = \text{id} \). Take flat fronts \( f := EE^* \) and \( \hat{f} = \hat{E}\hat{E}^* \) associated to \( E \) and
\[ \Pi := \left\{ \begin{pmatrix} x_0 + x_3 \\ x_1 \\ x_0 - x_3 \end{pmatrix} \in H^3; \ x_0, x_1, x_3 \in \mathbb{R} \right\}. \]

Let \( \Lambda \) be the geodesic of \( H^3 \) passing through the origin \( \text{id} \) and perpendicular to \( \Pi \) (i.e., the \( x_2 \)-axis), and let \( q \) be the foot of the perpendicular from \( \tilde{f}(p) \) to the line \( \Lambda \).

**Lemma 3.2.** In the above circumstances, one has:
\[
\text{dist}_{H^3}(o, q) \leq 2s + \frac{c_{13}}{\sqrt{N}}.
\]

**Proof.** The triangle \( \triangle oq\tilde{f}(p) \) is a right triangle such that the angle \( q \) is the right angle. Then by (3.20), (3.18) and (3.4), we have
\[
\text{dist}_{H^3}(o, \tilde{f}(p)) = \text{dist}_{H^3}(l_{j-1}(\zeta_j), l_j(p))
\]
\[
\leq \text{dist}_{H^3}(l_{j-1}(\zeta_j), l_j(\zeta_j)) + \text{dist}_{H^3}(l_j(\zeta_j), l_j(p))
\]
\[
\leq \frac{c_{11}}{N} + 2\text{dist}_{d\zeta_j}(\zeta_j, p)
\]
\[
\leq \frac{c_{15}}{N} + 2\left(s + \frac{c_8}{\sqrt{N}}\right) \leq 2s + \frac{c_{14}}{\sqrt{N}}.
\]

Thus we have the conclusion. \( \square \)

**Lemma 3.3.** Under the hypotheses above, one has:
\[
\text{dist}_{H^3}(q, \tilde{f}(p)) \leq 14s^2 + \frac{c_{10}}{\sqrt{N}}.
\]

**Proof.** Let \( \varphi = (\varphi_1, \varphi_2) \) and \( \tilde{\varphi} = (\tilde{\varphi}_1, \tilde{\varphi}_2) \) be the holomorphic data of the Legendrian immersions \( E \) and \( \tilde{E} \), respectively, that is,
\[
\psi := E^{-1}E' = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \varphi_1 + i\varphi_2 \\ \varphi_1 - i\varphi_2 & 0 \end{pmatrix} = M_\varphi,
\]
\[
\tilde{\psi} := \tilde{E}^{-1}\tilde{E} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \tilde{\varphi}_1 + i\tilde{\varphi}_2 \\ \tilde{\varphi}_1 - i\tilde{\varphi}_2 & 0 \end{pmatrix} = M_{\tilde{\varphi}}
\]
hold. Set
\[
F(z) := \int_{\zeta_j}^{\tilde{z}} \psi(z) \, dz, \quad \text{and} \quad \tilde{F}(z) := \int_{\zeta_j}^{\tilde{z}} \tilde{\psi}(z) \, dz.
\]

We define two values \( \Delta \) and \( \tilde{\Delta} \) by
\[
E(p) = \text{id} + F(p) + \Delta, \quad \tilde{E}(p) = \text{id} + \tilde{F}(p) + \tilde{\Delta}.
\]

Since \( E(\zeta_j) = \tilde{E}(\zeta_j) = \text{id} \), [14, Appendix A.4] yields that
\[
|\Delta| \leq \left[ \max_{\gamma} |E| \right] \int_{\gamma} |\psi| \, |dz| \right]^{\frac{2}{2}}, \quad |\tilde{\Delta}| \leq \left[ \max_{\gamma} |\tilde{E}| \right] \int_{\gamma} |\tilde{\psi}| \, |dz| \right]^{\frac{2}{2}}.
\]

Here \( \gamma \) is a path joining \( \zeta_j \) and \( p \) as in [14, Fig. 1]. This argument is completely parallel to that in [14, Page 132]. Since the Euclidean length of \( \gamma \) in \( \mathbb{R}^2 \) is bounded by \( c_{17}/N \), we have
\[
|F(p)| \leq \int_{\gamma} |\psi| \, |dz| \leq \frac{c_{18}}{N}.
\]
Then, reasoning as in [14, Page 133], we have
\[ |\tilde{F}(p)| \leq \int |\tilde{\psi}| \, dz = \int ds \leq s + \frac{c_{19}}{\sqrt{N}}, \]

On the other hand, since
\[ \max_{\gamma} |E|^2 = \max_{\gamma} \left\{ 2 \cosh \text{dist}_{H^3}(o, f(z)) \right\} \]
\[ = \max_{\gamma} \left\{ 2 \cosh \text{dist}_{H^3}(l_{j-1}(z_j), l_{j-1}(z)) \right\} \leq 2 \cosh \frac{c_{29}}{N} \leq 2 \left( 1 + \frac{c_{21}}{N} \right), \]
\[ \max_{\gamma} |\tilde{E}|^2 \leq 2 \cosh \text{dist}_{H^3}(o, \tilde{f}(p)) = 2 \cosh \text{dist}_{H^3}(l_{j-1}(z_j), l_j(p)) \]
\[ \leq 2 \cosh \{ \text{dist}_{H^3}(l_{j-1}(z_j), l_j(z_j)) + \text{dist}_{H^3}(l_{j-1}(z_j), l_j(p)) \} \]
\[ \leq 2 \cosh \left( 2s + \frac{c_{22}}{\sqrt{N}} \right) \leq 2 \left( 1 + 4s^2 + \frac{c_{23}}{\sqrt{N}} \right), \]
we have
\[ |\Delta| \leq \frac{c_{24}}{N}, \quad |\tilde{\Delta}| \leq 4s^2 + \frac{c_{25}}{\sqrt{N}}, \]
whenever \( s < 1/3 \). Now, we set
\[ \left( x_0 + x_3 \quad x_1 + i x_2 \right) \quad \left( x_1 - i x_2 \quad x_0 - x_3 \right) := \tilde{f}(p) = (id + \tilde{F} + \tilde{\Delta})(id + \tilde{F}^* + \tilde{\Delta}^*) \]
\[ = id + \tilde{F} + \tilde{F}^* - F - F^* + \delta. \]

Then, reasoning as in [14, Page 133], we have
\[ |\delta| = |F + F^* + \tilde{F}\tilde{F}^* + \Delta + \tilde{\Delta}\tilde{F}^* + \tilde{F}\tilde{F}^* + \Delta\Delta^*| < 14s^2 + \frac{c_{26}}{\sqrt{N}}. \]

Moreover, if we define
\[ h(z) := F(z) + F^*(z) = \sqrt{2} \int_{\zeta_j}^{z} \begin{pmatrix} 0 & \text{Re} \varphi_1 + i \text{Re} \varphi_2 \\ \text{Re} \varphi_1 - i \text{Re} \varphi_2 & 0 \end{pmatrix}, \]
\[ \tilde{h}(z) := \tilde{F}(z) + \tilde{F}^*(z) = \sqrt{2} \int_{\tilde{\zeta}_j}^{z} \begin{pmatrix} 0 & \text{Re} \tilde{\varphi}_1 + i \text{Re} \tilde{\varphi}_2 \\ \text{Re} \tilde{\varphi}_1 - i \text{Re} \tilde{\varphi}_2 & 0 \end{pmatrix}, \]
the \( x_3 \)-component of \( \tilde{h} - h \) vanishes, and the \( x_1 \)-component is
\[ \sqrt{2} \int \text{Re}(\tilde{\varphi}_1 - \varphi_1) = |\tilde{h}(z) - h(z)| \frac{u_2}{u_1} \leq \left( s + \frac{c_{27}}{\sqrt{N}} \right) \frac{u_2}{u_1} \leq \frac{c_{28}}{N}, \]
because of the property of \( u \) in Lemma A.1. Thus, we have
\[ |x_1| \leq 14s^2 + \frac{c_{29}}{\sqrt{N}}, \quad |x_3| \leq 14s^2 + \frac{c_{30}}{\sqrt{N}}. \]

Since \( \text{dist}_{H^3}(\tilde{f}(p), q) \) is the distance between \( \tilde{f}(p) \) and \( x_2 \)-axis, we have
\[ \text{dist}_{H^3}(\tilde{f}(p), q) = \sinh^{-1} \sqrt{x_1^2 + x_3^2} \leq 14s^2 + \frac{c_{16}}{\sqrt{N}}, \]
which proves the conclusion. \( \square \)
3.5. **Proof of Key Lemma.** Note that for any positive numbers $x$ and $y$, it holds that

\begin{equation}
\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y 
\leq \cosh x (\cosh y + \sinh y) = e^y \cosh x.
\end{equation}

By (3.15), we have

\begin{equation}
\text{dist}_{H^3}(o, l_{j-1}(\zeta_j)) \leq \text{dist}_{H^3}(o, l_0(\zeta_j)) + \frac{c_{31}}{N^2}.
\end{equation}

Under the situation here, $f(\zeta_j)$ and $o (= \text{id})$ lie on the $x_1x_3$-plane, and $q$ is on the $x_2$-axis, the geodesic triangle $\triangle f(\zeta_j) o q$ in $H^3$ is a right triangle. Then by the hyperbolic Pythagorean theorem, we have

\[
\cosh \text{dist}_{H^3}(f(0), q) = \cosh \text{dist}_{H^3}(f(0), o) \cosh \text{dist}_{H^3}(o, q) \\
= \cosh \text{dist}_{H^3}(f(0), f(\zeta_j)) \cosh \text{dist}_{H^3}(o, q) \\
= \cosh \text{dist}_{H^3}(l_{j-1}(0), l_{j-1}(\zeta_j)) \cosh \text{dist}_{H^3}(o, q) \\
= \cosh \left( \text{dist}_{H^3}(o, l_0(\zeta_j)) + \frac{c_{31}}{N^2} \right) \cosh \text{dist}_{H^3}(o, q) \\
\leq \exp \left( \frac{c_{31}}{N^2} \right) \cosh \text{dist}_{H^3}(o, l_0(\zeta_j)) \cosh \text{dist}_{H^3}(o, q) \\
= \frac{1}{2} |\mathcal{L}_0(\zeta_j)|^2 \exp \left( \frac{c_{31}}{N^2} \right) \cosh \text{dist}_{H^3}(o, q). \\
\leq \left( \frac{1}{2} \max_{\mathcal{L}_1} |\mathcal{L}_0|^2 \right) \cosh \left( 2s + \frac{c_{32}}{\sqrt{N}} \right) \left( 1 + \frac{c_{32}}{N^2} \right) \\
\leq \left( \frac{1}{2} \max_{\mathcal{L}_1} |\mathcal{L}_0|^2 \right) \left( 1 + \frac{c_{32}}{N^2} \right) \left( 1 + 4s^2 + \frac{c_{33}}{\sqrt{N}} \right) \\
\leq \left( \frac{1}{2} \max_{\mathcal{L}_1} |\mathcal{L}_0|^2 \right) \left( 1 + 4s^2 + \frac{c_{34}}{\sqrt{N}} \right).
\]

Thus, by using Lemmas 3.1, 3.2, and 3.3, we have

\[
\frac{1}{2} |\mathcal{L}_f(p)|^2 = \cosh \text{dist}_{H^3}(o, l_j(p)) = \cosh \text{dist}_{H^3}(\tilde{f}(0), \tilde{f}(p)) \\
\leq \cosh \left( \text{dist}_{H^3}(\tilde{f}(0), f(0)) + \text{dist}_{H^3}(f(0), q) + \text{dist}_{H^3}(q, \tilde{f}(p)) \right) \\
\leq \cosh \left( \text{dist}_{H^3}(l_j(\zeta_j), l_{j-1}(\zeta_j)) + \text{dist}_{H^3}(f(0), q) + \text{dist}_{H^3}(q, \tilde{f}(p)) \right) \\
\leq \cosh \left( \frac{c_{11}}{N^2} + \text{dist}_{H^3}(f(0), q) + \text{dist}_{H^3}(q, \tilde{f}(p)) \right) \\
\leq \exp \left( \frac{c_{11}}{N^2} \right) \cosh \left( \text{dist}_{H^3}(f(0), q) + \text{dist}_{H^3}(q, \tilde{f}(p)) \right) \\
\leq \left( 1 + \frac{c_{15}}{N^2} \right) \cosh \left( \text{dist}_{H^3}(q, \tilde{f}(p)) \right) \cosh \text{dist}_{H^3}(f(0), q) \\
\leq \left( 1 + \frac{c_{15}}{N^2} \right) \exp \left( 14s^2 + \frac{c_{16}}{\sqrt{N}} \right) \cosh \text{dist}_{H^3}(f(0), q) \\
\leq \left( 1 + \frac{c_{15}}{N^2} \right) \left( 1 + 2 \left( 14s^2 + \frac{c_{16}}{\sqrt{N}} \right) \right) \left( \frac{1}{2} \max_{\mathcal{L}_1} |\mathcal{L}_0|^2 \right) \left( 1 + 4s^2 + \frac{c_{36}}{\sqrt{N}} \right).
\]
\[
\leq \left( \frac{1}{2} \max_{b_1} |\mathcal{L}_0|^2 \right) \left( 1 + 32s^2 + \frac{c_2}{\sqrt{N}} \right)
\]
which proves the conclusion.

4. Applications

This section is devoted to prove some applications of the main theorem as we stated in the introduction.

A smooth map \( f : \mathbb{D}_1 \to M^3 \) into a 3-manifold \( M^3 \) is called a (wave) front if there exists a Legendrian immersion \( L_f : \mathbb{D}_1 \to P(T^* M^3) \) with respect to the canonical contact structure of the projective cotangent bundle \( \pi : P(T^* M^3) \to M^3 \) such that \( \pi \circ L_f = f \).

4.1. Flat fronts in hyperbolic 3-space. Recall that \( H^3 := \text{SL}(2, \mathbb{C})/\text{SU}(2) = \{ aa^* ; a \in \text{SL}(2, \mathbb{C}) \} \) \((a^* = \bar{a}^t)\).

For a Legendrian immersion \( \mathcal{L} : \mathbb{D}_1 \to \text{SL}(2, \mathbb{C}) \), the projection
\[
\begin{aligned}
\rho := \frac{\theta}{\omega} : \mathbb{D}_1 \to \mathbb{C} \cup \{ \infty \}.
\end{aligned}
\]

Then a point \( z \in \mathbb{D}_1 \) is a singular point if and only if \( |\rho(z)| = 1 \). We denote by \( \Sigma_f \) the singular set of \( f \):
\[
\Sigma_f := \{ z \in \mathbb{D}_1 ; |\rho(z)| = 1 \}.
\]

We have the following

**Theorem 4.1.** There exists a weakly complete flat front \( f : \mathbb{D}_1 \to H^3 \) whose image is bounded in \( H^3 \) such that \( \mathbb{D}_1 \setminus \Sigma_f \) is open dense in \( \mathbb{D}_1 \).

**Proof.** Let \( \mathcal{L} : \mathbb{D}_1 \to \text{SL}(2, \mathbb{C}) \) be a Legendrian immersion as in the main theorem, and set \( f := \mathcal{L} \mathcal{L}^* : \mathbb{D}_1 \to H^3 \), which gives a flat front. The boundedness of \( f \) follows from that of \( \mathcal{L} \), and the weak completeness of \( f \) follows from the completeness of \( \mathcal{L} \).

Finally, we shall prove that \( \mathbb{D}_1 \setminus \Sigma_f \) is open dense: If \( \Sigma_f \) has an interior point, then \( \rho \) in (4.2) satisfies \( |\rho| = 1 \) identically because of the analyticity of \( \rho \). However, if we take a initial immersion so that \( |\rho| \) is not constant, then the resulting bounded weakly complete flat front has the same property, since our iteration can be taken to be small enough near the origin of \( \mathbb{D}_1 \). \( \square \)

4.2. Flat fronts in de Sitter 3-space. The de Sitter 3-space \( S^3_1 \) is the connected and simply connected Lorentzian space form of constant curvature 1, which is represented as
\[
S^3_1 = \text{SL}(2, \mathbb{C})/\text{SU}(1, 1) = \{ ae_3 a^* ; a \in \text{SL}(2, \mathbb{C}) \} \quad (e_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}).
\]
Let $\mathcal{L}: \mathbb{D}_1 \to \text{SL}(2, \mathbb{C})$ be a Legendrian immersion. Then the projection

$$f: \mathcal{L}_2 \mathcal{L}^*: \mathbb{D}_1 \to S^3_1$$

gives a flat front. We remark that $f$ is not an immersion at $z$ if and only if $|\rho(z)| = 1$, where $\rho = \theta/\omega$ as in (4.2). The singular set $\Sigma_f$ of $f$ is characterized by $|\rho(z)| = 1$. Similar to the case of flat fronts in $H^3$, $f$ is said to be weakly complete if the metric induced from the canonical Hermitian metric of $\text{SL}(2, \mathbb{C})$ by $\mathcal{L}$ is complete. Then we have

**Theorem 4.2.** There exists a weakly complete flat front $f: \mathbb{D}_1 \to H^3$ whose image is bounded in $S^3_1$ such that $\mathbb{D}_1 \setminus \Sigma_f$ is open dense in $\mathbb{D}_1$, where $\Sigma_f$ is the set of singular points of $f$.

**Proof.** Let $\mathcal{L}: \mathbb{D}_1 \to \text{SL}(2, \mathbb{C})$ be a Legendrian immersion as in the main theorem, and set $f := \mathcal{L}_2 \mathcal{L}^*$. Then $f$ is a weakly complete flat front in $S^3_1$. Moreover, one can see that $\mathbb{D}_1 \setminus \Sigma_f$ is open dense by the same argument as in Theorem 4.1. \hfill $\square$

See [7], [10] and [3] for the relationships between flat surfaces and linear Weingarten surfaces in $H^3$ or $S^3_1$.

### 4.3. Improper Affine Front in affine 3-space.

A notion of IA-maps in the affine 3-space has been introduced by A. Martínez in [16]. IA-maps are improper affine spheres with a certain kind of singularities. Since all of IA-maps are wave fronts (see [18, 19]), we call them improper affine fronts (The terminology ‘improper affine fronts’ has been already used in Kawakami-Nakajo [8]). The precise definition of improper affine fronts is given in [19, Remark 4.3]. The set of singular points $\Sigma_f$ of an improper affine front $f: \mathbb{D}_1 \to \mathbb{R}^3$ is represented as $\Sigma_f = \{ z \in \mathbb{D}_1 \mid |\rho(z)| = 1 \}$, where $\rho(z) := dG/dF$. In [19], weak completeness of improper affine fronts is introduced. Then we have

**Theorem 4.3.** There exists a weakly complete improper affine front $f: \mathbb{D}_1 \to \mathbb{R}^3$ whose image is bounded such that $\mathbb{D}_1 \setminus \Sigma_f$ is open dense in $\mathbb{D}_1$, where $\Sigma_f$ is the set of the singular points of $f$.

**Proof.** Let $\mathcal{L} = (F,G,H)$ be a complete bounded Legendrian immersion into $\mathbb{C}^3$. Since $\mathcal{L}$ is Legendrian, (1.1) yields that $dH = -F \ dG$. Here, by completeness of $\mathcal{L}$, the induced metric

$$ds^2_L = |dF|^2 + |dG|^2 + |dH|^2 = |dF|^2 + |dG|^2 + |FdG|^2 = |dF|^2 + (|F|^2 + 1)|dG|^2$$

is complete. Moreover, since the image of $\mathcal{L}$ is bounded, we have

$$ds^2_L \leq C(|dF|^2 + |dG|^2) \quad (C > 0 \text{ is a constant}).$$

Thus, the metric

$$d\tau^2 := |dF|^2 + |dG|^2$$

is complete. Hence, we have an improper affine front $f$ using Martínez’ representation formula [16] with respect to $(F,G)$:

$$f = \left( G + \frac{1}{2} (|G|^2 - |F|^2) + \text{Re} \left( GF - 2 \int F \ dG \right) \right)$$

$$= \left( G + \frac{1}{2} (|G|^2 - |F|^2) + \text{Re} \left( GF + 2H \right) \right): \mathbb{D}_1 \to \mathbb{R}^3.$$
Since $d\tau^2$ in (4.3) is complete, $f$ is weakly complete, by definition of weak completeness given in [19]. Boundedness of $f$ follows from that of $L$.

We next prove that $D^1_1(\Sigma_f)$ is open dense: If we take an initial immersion so that $|dG/dF|$ is not constant, then the resulting bounded weakly complete improper affine front has the same property, since the singular set is characterized by $|dG/dF| = 1$.

\[\Box\]

Remark 4.4. As mentioned in the introduction, there exist no compact flat fronts in $H^3$ (resp. $S^3_1$). In fact, [11, Proposition 3.6] implies non-existence of compact flat fronts in $H^3$. On the other hand, suppose that there exists a flat front $f: \Sigma^2 \to S^3_1$ where $\Sigma^2$ is a compact 2-manifold. Then the unit normal vector $\nu$ of $f$ induces a flat front $\nu: \Sigma^2 \to H^3$, which makes a contradiction.

Next, we shall show the non-existence of compact improper affine fronts as mentioned in the introduction. An improper affine front $f: \Sigma^2 \to \mathbb{R}^3$ defined on a Riemann surface $\Sigma^2$ is represented as in (4.4), where $F$ and $G$ are holomorphic functions on $\Sigma^2$. If $\Sigma^2$ is compact, $G + \overline{F}$ is equal to a constant $c \in \mathbb{C}$ because it is harmonic. Then we have that $G = -\overline{F} + c$, where the left-hand side is holomorphic whereas the right-hand side is anti-holomorphic. Hence we can conclude that $F$ and $G$ are both constants. Hence $d\tau^2$ as in (4.3) vanishes identically, which contradicts the definition of improper affine fronts.

Appendix A. An application of Runge’s Theorem

To prove the Key Lemma, we prepare the following assertion, which is an analogue of [14, Lemma 4.1].

Lemma A.1. Let $\varphi = (\varphi_1, \varphi_2)$ be a non-degenerate pair of holomorphic functions on $D^1_1$ (see Definition 2.2), and let $\varepsilon > 0$ and $N$ be a positive number and a sufficiently large integer N, respectively. Then for each $j$ ($j = 1, \ldots, 2N$), there exists a non-degenerate pair $\tilde{\varphi} = (\tilde{\varphi}_1, \tilde{\varphi}_2)$ of holomorphic functions satisfying the following three conditions:

(a) $|\tilde{\varphi} - \varphi| < \frac{C N}{2N}$ on $D^1_1 \setminus \omega_j$.

(b) $|\tilde{\varphi}| \geq \begin{cases} CN^{3.5} & \text{on } \omega_j, \\ CN^{-0.5} & \text{on } \overline{\omega}_j, \end{cases}$

where $C$ is a constant depending only on $\varphi$.

(c) There exists a unit vector $u = (u_1, u_2) \in \mathbb{R}^2$ ($|u| = 1$) such that

$$u \cdot (\varphi - \tilde{\varphi}) = 0, \quad |u_1| > 1 - \frac{2}{N},$$

where we set $\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2$ for $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$.

As pointed out in Remark A.2 below, we can prove the lemma as a modification of [14, Lemma 4.1] for null holomorphic curves in $\mathbb{C}^3$. However, we believe that the theory of Legendrian curves should be established independently from the theory of null curves. So we give here a self-contained proof of the lemma, which might be convenient for the readers. In fact, our proof is easier than that of [14, Lemma 4.1].

Proof. In the proof, the symbols $c_k$ ($k = 1, 2, \ldots$) denote suitable positive constants, which depend only on the initial data $\varphi$. Since $\varphi$ has no zeroes, we can take $\nu$ and
where

\[(A.1) \quad 0 < \nu \leq |\varphi| \leq m \quad (\text{on } \overline{D}_1).\]

Set

\[\hat{\varphi} = (\hat{\varphi}_1, \hat{\varphi}_2) = ((\cos t) \varphi_1 + (\sin t) \varphi_2, -(\sin t) \varphi_1 + (\cos t) \varphi_2)\]

for \(t \in \mathbb{R}\). To prove the lemma, we need to prove that one can choose \(t \in [0, \frac{\pi}{2}]\) such that

\[(A.2) \quad \sin t \leq \sqrt{\frac{2}{N}} \quad |\hat{\varphi}_k| \geq \frac{\nu}{2\sqrt{N}} \quad (\text{on } \omega_j, k = 1, 2).\]

In fact, if \(|\varphi_k| \geq \nu/(2\sqrt{N}) (k = 1, 2)\) holds on \(\omega_j\), (A.2) holds obviously for \(t = 0\).

By exchanging the roles of \(\varphi_1\) and \(\varphi_2\), we may assume that there exists \(x \in \omega_j\) such that \(|\varphi_1(x)| < \nu/(2\sqrt{N})\) without loss of generality. Here, notice that the diameter (as a subset of \(\mathbb{C} = \mathbb{R}^2\)) of \(\omega_j\) satisfies \(\text{diam}_{\mathbb{C}}(\omega_j) \leq c_1/N\), where \(c_1\) is a positive constant. Since the derivative of \(\varphi: D_1 \rightarrow \mathbb{C}^2\) is bounded, it holds that, \(\text{diam}_{\mathbb{C}^2}(\varphi(\omega_j)) \leq c_2/N\) for \(c_2 > 0\). Then it follows that

\[|\varphi_1(y)| \leq |\varphi_1(x)| + |\varphi(y) - \varphi(x)| < \frac{\nu}{2\sqrt{N}} + c_2 N\]

for any \(y \in \omega_j\). Here, noticing \(|\varphi|^2 \geq \nu^2\), we have

\[|\varphi_2(y)| \geq \sqrt{\nu^2 - |\varphi_1(y)|^2} \geq \sqrt{\nu^2 - \left(\frac{\nu}{2\sqrt{N}} + c_2 N\right)^2} \geq \nu \left(1 - \frac{1}{4N}\right)\]

since \(N\) is sufficiently large. We choose \(t\) as \(\sin t = \sqrt{\frac{2}{N}}\). Then noticing

\[1 - \frac{2}{N} \leq \cos t \leq 1 - \frac{1}{N}\]

we have

\[(A.3) \quad |\hat{\varphi}_1(y)| = |(\cos t) \varphi_1(y) + (\sin t) \varphi_2(y)| \geq (\sin t)|\varphi_2(y)| - (\cos t)|\varphi_1(y)|
\]

\[\geq \frac{\sqrt{2}\nu}{\sqrt{N}} \left(1 - \frac{1}{4N}\right) - \left(1 - \frac{1}{N}\right) \left(\frac{\nu}{2\sqrt{N}} + c_2 N\right) \geq \frac{\nu}{2\sqrt{N}}.\]

\[|\hat{\varphi}_2(y)| \geq (\cos t)|\varphi_2(y)| - (\sin t)|\varphi_1(y)|
\]

\[\geq \left(1 - \frac{2}{N}\right) \left(1 - \frac{1}{4N}\right) \nu - \frac{2}{N} \left(\frac{\nu}{2\sqrt{N}} + c_2 N\right) \geq \frac{\nu}{2\sqrt{N}}.\]

Hence we have (A.2).

We now fix a real number \(t\) in (A.2) and will prove (a), (b) and (c): Since \(\omega_j\) and \(D_1 \setminus \omega_j\) are compact set such that \(\mathbb{C} \setminus (\omega_j \cup (D_1 \setminus \omega_j))\) is connected, Runge’s theorem implies that there exists a holomorphic function \(h\) (cf. [17, (4)]) such that \(h \neq 0\) on \(D_1\) and

\[(A.4) \quad \begin{cases} |h - 2N^4| < \frac{1}{2N^2} & \text{on } \omega_j, \\ |h - 1| < \frac{\varepsilon}{2N^2m} & \text{on } \overline{D}_1 \setminus \omega_j, \end{cases}\]

where \(m\) is as in (A.1). We set

\[\tilde{\varphi} = (\tilde{\varphi}_1, \tilde{\varphi}_2) := (\hat{\varphi}_1, h\hat{\varphi}_2).\]
Then
\[ \tilde{\varphi} := \left((\cos t)\varphi_1 - (\sin t)\varphi_2, (\sin t)\varphi_1 + (\cos t)\varphi_2\right) \]
satisfies the desired properties: In fact,
\[ |\tilde{\varphi} - \varphi| = |\tilde{\varphi} - \tilde{\varphi}| = |h - 1| |\varphi_2| < \frac{\varepsilon}{2N^2 m} |\varphi| \leq \frac{\varepsilon}{2N^2} \]
which implies (a). On the other hand, by (A.3),
\[ |\tilde{\varphi}| = |\varphi| \geq |\varphi_1| = |\tilde{\varphi}_1| \geq \frac{\nu}{2\sqrt{N}} \]
holds on \( \omega_j \), which proves the first inequality of (b).
It holds on \( \omega_j \) that
\[ |\tilde{\varphi}| = |\tilde{\varphi}| \geq |h| |\tilde{\varphi}_2| \geq |2N^4 - |h - 2N^4|| |\tilde{\varphi}_2| \geq \left(2N^4 - \frac{1}{2N^2}\right) \frac{\nu}{2\sqrt{N}} \geq \frac{\nu}{2N^{3.5}}. \]
Hence we have the second inequality of (b). Finally, we set \( u = (\cos t, \sin t) \). Then (c) holds.

**Remark A.2.** Lemma A.1 can be proved directly from the corresponding assertion for null curves in \( \mathbb{C}^3 \) given in [14, Lemma 4.1] as follows: Let \( \varphi = (\varphi_1, \varphi_2) \) be as in Lemma A.1. Since \( \varphi_1, \varphi_2 \) have no common zeros, there exists a holomorphic function \( \varphi_3 \) defined on \( \overline{D_1} \) such that \( (\varphi_1)^2 + (\varphi_2)^2 + (\varphi_3)^2 = 0 \). We apply [14, Lemma 4.1] for \( \Phi := (\varphi_1, \varphi_2, \varphi_3) \) and get a new Weierstrass data \( \tilde{\Phi} := (\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3) \). Then \( \tilde{\varphi} := (\tilde{\varphi}_1, \tilde{\varphi}_2) \) satisfies (a) which follows immediately from (a) of [14, Lemma 4.1]. Next, by the proof of [14, Corollary B] it holds that
\[ 4(|\tilde{\varphi}_1|^2 + |\tilde{\varphi}_2|^2) \geq |\tilde{\varphi}_1|^2 + |\tilde{\varphi}_2|^2 + |\tilde{\varphi}_3|^2, \]
which implies that (b) of our lemma follows from (b) of [14, Lemma 4.1]. (c) of our lemma does not follows from (c) of [14, Lemma 4.1] directly. However, we can choose \( u = (u_1, u_2, u_3) \) in the proof of [14, Lemma 4.1] in such a way that \( u \in \mathbb{R}^3 \) and \( u_3 = 0 \) without loss of generality. So this gives a alternative proof of the lemma.

**References**

[1] A. Alarcón and F. J. López, *Null curves in \( \mathbb{C}^3 \) and Calabi-Yau conjectures*, preprint, arXiv:0912.2847.
[2] A. Alarcón and F. J. López, *Compact complete null curves in Complex 3-space*, preprint, arXiv:1106.0684.
[3] J. Aledo and J. Espinar, *A conformal representation for linear Weingarten surfaces in the de Sitter space*, J. Geom. Phys. 57 (2007), 1669–1677.
[4] R. L. Bryant, *Surfaces in conformal geometry*, Proceedings of Symposia in Pure Mathematics, Volume 48 (1988), 227–240.
[5] N. Ejiri and M. Takahashi, *The Lie transform between null curves and contact curves in PSL(2, \( \mathbb{C} \)),* Proceedings of the 16th OCU International Academic Symposium 2008, OCAMI Studies, Volume 3 (2009), 265–277.
[6] J. A. Gálvez, A. Martínez and F. Milán, *Flat surfaces in hyperbolic 3-space*, Math. Ann., 316 (2000) 419–435.
[7] J. A. Gálvez, A. Martínez and F. Milán, *Complete linear Weingarten surfaces of Bryant type. A Plateau problem at infinity*, Trans. Amer. Math. Soc. 356 (2004), 3405–3428.
[8] Y. Kawakami and D. Nakajo, *Value distribution of the Gauss map of improper affine spheres*, preprint, arXiv:1004.1484.
[9] Y. Kitagawa and M. Umehara, *Extrinsic diameter of immersed flat tori in \( S^3 \),* to appear in Geom. Dedicata, DOI 10.1007/s10711-011-9580-5.
[10] M. Kokubu and M. Umehara, *Orientability of linear Weingarten surfaces, spacelike CMC-1 surfaces and maximal surfaces*, to appear in Math. Nachrichten, arXiv:0907.2284.
[11] M. Kokubu, M. Umehara and K. Yamada, *Flat fronts in hyperbolic 3-space*, Pacific J. Math., 216 (2004) 149–175.

[12] M. Kokubu, W. Rossman, M. Umehara and K. Yamada, *Flat fronts in hyperbolic 3-space and their caustics*, J. Math. Soc. Japan, 59 (2007) 265–299.

[13] M. Kokubu, W. Rossman, M. Umehara and K. Yamada, *Asymptotic behavior of flat surfaces in hyperbolic 3-space*, J. Math. Soc. Japan, 61 (2009) 799–852.

[14] F. Martín, M. Umehara and K. Yamada, *Complete bounded null curves immersed in $\mathbb{C}^3$ and $\text{SL}(2, \mathbb{C})$*, Calculus of Variations and PDE’s, 36 (2009), 119–139.

[15] F. Martín, M. Umehara and K. Yamada, *Complete bounded holomorphic curves immersed in $\mathbb{C}^2$ with arbitrary genus*, Proc. Amer. Math. Soc. 137 (2009), 3437–3450.

[16] A. Martínez, *Improper Affine maps*, Math. Z., 249 (2005) 755–766.

[17] N. Nadirashvili, *Hadamard’s and Calabi-Yau’s conjectures on negatively curved and minimal surfaces*, Invent. Math., 126 (1996), 457–465.

[18] D. Nakajo, *A representation formula for indefinite improper affine spheres*, Result. Math., 55 (2009), 139–159.

[19] M. Umehara and K. Yamada, *Applications of a completeness lemma in minimal surface theory to various classes of surfaces*, Bull. of the London Math. Soc., 43 (2011), 191–199.

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