INVARIANTS, KRONECKER PRODUCTS, AND
COMBINATORICS OF SOME REMARKABLE DIOPHANTINE SYSTEMS
(EXTENDED VERSION)

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ABSTRACT. This work lies across three areas (in the title) of investigation that are by themselves of independent interest. A problem that arose in quantum computing led us to a link that tied these areas together. This link consists of a single formal power series with a multifaceted interpretation. The deeper exploration of this link yielded results as well as methods for solving some numerical problems in each of these separate areas.

Key words: Invariant, Kronecker product, Diophantine system, Hilbert series
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1. INTRODUCTION

Since our work may be of interest to audiences of varied background we will try to keep our notation as elementary as possible and entirely self contained.

The problem in invariant theory that was the point of departure in our investigation is best stated in its simplest and most elementary version. Given two matrices 
\[ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \]
of determinants 1, or equivalently in \( SL[2] := SL(2, \mathbb{C}) \), we recall that their tensor product may be written in the block form
\[ A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix}. \] (1)

We also recall that the action of a matrix \( M = [m_{ij}]_{i,j=1}^{n} \) on a polynomial \( P(x) \) in \( \mathbb{R}[x_1, x_2, \ldots, x_n] \) may be defined by setting
\[ T_M P(x) = P(xM), \] (2)
where the symbol \( xM \) is to be interpreted as multiplication of a row \( n \)-vector by an \( n \times n \) matrix. This given, we denote by \( R_4^{SL[2]\otimes SL[2]} \) the ring of polynomials in \( \mathbb{R}_4 \) that are invariant under the action of \( A \otimes B \) for all pairs \( A, B \in SL[2] \). In symbols
\[ R_4^{SL[2]\otimes SL[2]} = \{ P \in \mathbb{R}_4 : T_A \otimes B P(x) = P(x) \}. \] (3)
Since the action in (2) preserves degree and homogeneity, \( R_4^{SL[2]\otimes SL[2]} \) is graded, and as a vector space it decomposes into the direct sum
\[ R_4^{SL[2]\otimes SL[2]} = \bigoplus_{m \geq 0} \mathcal{H}_m \left( R_4^{SL[2]\otimes SL[2]} \right), \]
where the \( m \)th direct summand here denotes the subspace consisting of the \( SL[2] \otimes SL[2] \)-invariants that are homogenous of degree \( m \). The natural problem then arises to determine the Hilbert series
\[ W_2(q) = \sum_{m \geq 0} q^m \dim \mathcal{H}_m \left( R_4^{SL[2]\otimes SL[2]} \right). \]
Now note that using (1) iteratively we can define the $k$-fold tensor product $A_1 \otimes A_2 \otimes \cdots \otimes A_k$, and thus extend (3) to its general form
\[
R_{2^k}^{SL(2) \otimes SL(2) \otimes \cdots \otimes SL(2)} = \{ P \in R_{2^k} : T_{A_1 \otimes A_2 \otimes \cdots \otimes A_k} P(x) = P(x) \}
\]
and set
\[
W_k(q) = \sum_{m \geq 0} q^m \dim \mathcal{H}_m(R_{2^k}^{SL(2) \otimes SL(2) \otimes \cdots \otimes SL(2)}).
\]

Remarkably, to this date only the series $W_2(q), W_3(q), W_4(q), W_5(q)$ are known explicitly. Moreover, although the three series $W_2(q), W_3(q), W_4(q)$ may be hand computed, so far $W_5(q)$ has only been obtained by computer.

The third named author, using branching tables calculated in [8], was able to predict the explicit form of $W_5(q)$ by computing a sufficient number of its coefficients. The computation of these tables took approximately 50 hours using an array of 9 computers.

The series $W_4(q), W_5(q)$ first appeared in print in works of Luque-Thibon [5, 6] which were motivated by the same problem of quantum computing. We understand that their computation of $W_5(q)$ was carried out by a brute force use of the partial fraction algorithm of the fourth named author, and it required several hours with the computers of that time.

The present work was carried out whilst unaware of the work of Luque-Thibon. Our main goal is to acquire a theoretical understanding of the combinatorics underlying such Hilbert series and give a more direct construction of $W_5(q)$ and perhaps bring $W_6(q)$ within reach of present computers.

Fortunately, as is often the case with a difficult problem, the methods that are developed to solve it may be more significant than the problem itself. This is no exception as we shall see.

Let us recall that the pointwise product of two characters $\chi^{(1)}$ and $\chi^{(2)}$ of the symmetric group $S_n$ is also a character of $S_n$, and we shall denote it here by $\chi^{(1)} \odot \chi^{(2)}$. This is usually called the Kronecker product of $\chi^{(1)}$ and $\chi^{(2)}$. An outstanding yet unsolved problem is to obtain a combinatorial rule for the computation of the integer
\[
c_{\lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(k)}}^\lambda
\]
and each $\lambda^{(i)}$ are irreducible Young characters of $S_n$. Using the Frobenius map $F$ that sends the irreducible character $\chi^\lambda$ onto the Schur function $s_\lambda$, we can define the Kronecker product of two homogeneous symmetric functions of the same degree $f$ and $g$ by setting
\[
f \odot g = F((F^{-1}f) \odot (F^{-1}g)).
\]
With this notation the coefficient in (4) may also be written in the form
\[
c_{\lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(k)}}^\lambda = \langle s_{\lambda^{(1)}} \otimes s_{\lambda^{(2)}} \otimes \cdots \otimes s_{\lambda^{(k)}} , s_\lambda \rangle,
\]
where $\langle , \rangle$ denotes the customary Hall scalar product of symmetric polynomials. The relevancy of all this to the previous problem is a consequence of the following identity.

**Theorem 1.1.**
\[
W_k(q) = \sum_{d \geq 0} q^{2d} \langle s_{d,d} \otimes s_{d,d} \otimes \cdots \otimes s_{d,d} , s_{2d} \rangle
\]
where, in each term, the Kronecker product has $k$ factors.

For this reason, we will often refer to the task of constructing $W_k(q)$ as the Sdd Problem. Using this connection and some auxiliary results on the Kronecker product of symmetric functions we derived in [3] that
\[
W_2(q) = \frac{1}{1 - q^2}, \quad W_3(q) = \frac{1}{1 - q^4}, \quad W_4(q) = \frac{1}{(1 - q^2)(1 - q^4)(1 - q^6)}.
\]
Although this approach is worth pursuing (see [3]), the present investigation led us to another
surprising facet of this problem.

Let us start with a special case. We are asked to place integer weights on the vertices of the unit
square so that all the sides have equal weights. Denoting by $P_{00}$, $P_{01}$, $P_{10}$, $P_{11}$ the vertices (see figure)
and by $p_{00}$, $p_{01}$, $p_{10}$, $p_{11}$ their corresponding weights, we are led to the following Diophantine system.

$$S_2 : \begin{align*}
    p_{00} + p_{01} - p_{10} - p_{11} &= 0 \\
    p_{00} - p_{01} + p_{10} - p_{11} &= 0
\end{align*}$$

The general solution to this problem may be expressed as the formal series

$$F_2(y_{00}, y_{01}, y_{10}, y_{11}) = \sum_{p \in S_2} y_{00}^{p_{00}} y_{01}^{p_{01}} y_{10}^{p_{10}} y_{11}^{p_{11}} = \frac{1}{(1 - y_{00}y_{11})(1 - y_{01}y_{10})}.$$ 

In particular, making the substitution $y_{00} = y_{01} = y_{10} = y_{11} = q$ we derive that the enumerator of
solutions by total weight is given by the generating function

$$G_2(q) = \sum_{d \geq 0} m_d(2)q^{2d} = \frac{1}{(1 - q^2)^2},$$

with $m_d(2)$ giving the number of solutions of total weight $2d$.

This problem generalizes to arbitrary dimensions. That is we seek to enumerate the distinct ways
of placing weights on the vertices of the unit $k$-dimensional hypercube so that all hyperfaces have the
same weight. Denoting by $p_{\epsilon_1\epsilon_2\cdots\epsilon_k}$ the weight we place on the vertex of coordinates $(\epsilon_1, \epsilon_2, \ldots, \epsilon_k)$ we obtain a Diophantine system $S_k$ of $k$ equations in the $2^k$ variables \{\(p_{\epsilon_1\epsilon_2\cdots\epsilon_k}\)\}_{\epsilon_i=0,1}.

For instance, using this notation, for the 3-dimensional cube we obtain the system

$$S_3 : \begin{align*}
    p_{00} + p_{01} + p_{10} + p_{010} - p_{100} - p_{101} - p_{110} - p_{111} &= 0 \\
    p_{00} + p_{01} - p_{10} - p_{11} + p_{100} + p_{101} - p_{110} - p_{111} &= 0 \\
    p_{00} - p_{01} + p_{10} - p_{11} + p_{100} - p_{101} + p_{110} - p_{111} &= 0
\end{align*}$$

In this case the enumerator of solutions by total weight is

$$G_3(q) = \sum_{d \geq 0} m_d(3)q^{2d} = \frac{1 - q^8}{(1 - q^2)^4(1 - q^4)^2}.$$ 

The relevance of all this to the previous problem is a consequence of the following identity.

**Theorem 1.2.** Denoting by $m_d(k)$ the number of solutions of the system $S_k$ of total weight $2d$ and setting

$$G_k(q) = \sum_{d \geq 0} m_d(k)q^{2d}, \tag{7}$$

we have

$$G_k(q) = \sum_{d \geq 0} q^{2d}(h_{d,d} \odot h_{d,d} \odot \cdots \odot h_{d,d} , S_{2d}),$$

where, $h_{d,d}$ denotes the homogenous basis element indexed by the two part partition $(d,d)$, and in each
term, the Kronecker product has $k$ factors.

For this reason, we will refer to the task of constructing the series $G_k(q)$ as the $Hdd$ Problem.

Theorem 1.2 shows that the algorithmic machinery of Diophantine analysis may be used in the
construction of generating functions of Kronecker coefficients as well as Hilbert series of ring of invariants.
More precisely we are referring here to the constant term methods of MacMahon partition
analysis which have been recently translated into computer software by Andrews et al. [11] and Xin [10].
To see what this leads to, we start by noting that using MacMahon’s approach the solutions of $S_2$ may be obtained by the following identity

$$F_2(y_{00}, y_{01}, y_{10}, y_{11}) = \sum_{p_{00} \geq 0} \sum_{p_{01} \geq 0} \sum_{p_{10} \geq 0} \sum_{p_{11} \geq 0} \frac{y_{00} p_{00} y_{01} p_{01} y_{10} p_{10} y_{11} p_{11}}{a_2 p_{00} - p_{01} + p_{10} - p_{11} a_2 p_{00} - p_{01} + p_{10} - p_{11}},$$

where the symbol $\left\lfloor \frac{q_0 a_2}{a_1 a_2} \right\rfloor$ denotes the operator of taking the constant term in $a_1, a_2$. This identity may also be written in the form

$$F_2(y_{00}, y_{01}, y_{10}, y_{11}) = \frac{1}{(1 - y_{00} a_1 a_2)(1 - y_{01} a_1 / a_2)(1 - y_{10} a_2 / a_1)(1 - y_{11} a_1 a_2)} a_1^q a_2^q.$$

In particular the enumerator of the solutions of $S_2$ by total weight may be computed from the identity

$$G_2(q) = \frac{1}{(1 - q a_1 a_2)(1 - q a_1 / a_2)(1 - q a_2 / a_1)(1 - q / a_1 a_2)} a_1^q a_2^q.$$

More generally we have

$$G_k(q) = \prod_{S \subseteq [1,k]} \frac{1}{1 - q \prod_{i \in S} a_i / \prod_{j \not\in S} a_j} a_1^q a_2^q \cdots a_k^q,$$

where we use (and will often use) $[m, n]$ to denote the set $\{m, m+1, \ldots, n\}$. Now, standard methods of Invariant Theory yield that we also have

$$W_k(q) = \prod_{S \subseteq [1,k]} \frac{\prod_{i=1}^k (1 - a_i^2)}{1 - q \prod_{i \in S} a_i / \prod_{j \not\in S} a_j} a_1^q a_2^q \cdots a_k^q.$$

A comparison of (5) and (6) strongly suggests that a close study of the combinatorics of Diophantine systems such as $S_k$ should yield a more revealing path to the construction of such Hilbert series. This idea turned out to be fruitful, as we shall see, in that it permitted the solution of a variety of similar problems (see [3], [2]). In particular, we were eventually able to obtain that

$$G_5(\sqrt{q}) = \frac{N_5}{(1 - q)^3(1 - q^2)^5(1 - q^3)^6(1 - q^4)^3(1 - q^5)},$$

with

$$N_5 = q^{44} + 7q^{43} + 220q^{42} + 2606q^{41} + 24229q^{40} + 169840q^{39} + 951944q^{38} + 4391259q^{37} + 17128360q^{36} + 57582491q^{35} + 169556652q^{34} + 442817680q^{33} + 1036416952q^{32} + 2192191607q^{31} + 4219669696q^{30} + 7433573145q^{29} + 12041305271q^{28} + 18003453305q^{27} + 24921751416q^{26} + 32017113319q^{25} + 38243274851q^{24} + 42524815013q^{23} + 44052440432q^{22} + 42524815013q^{21} + 38243274851q^{20} + 32017113319q^{19} + 24921751416q^{18} + 18003453305q^{17} + 12041305271q^{16} + 7433573145q^{15} + 4219669696q^{14} + 2192191607q^{13} + 1036416952q^{12} + 442817680q^{11} + 169556652q^{10} + 57582491q^9 + 17128360q^8 + 4391259q^7 + 951944q^6 + 169840q^5 + 24229q^4 + 2606q^3 + 220q^2 + 7q + 1.$$

Surprisingly, the presence of the numerator factor in (6) absent in (5) does not increase the complexity of the result, as we see by comparing (10) with the Luque-Thibon result

$$W_5(\sqrt{q}) = \frac{P_5}{(1 - q^2)^4(1 - q^3)(1 - q^4)^6(1 - q^5)^3(1 - q^6)^5},$$
with
\[ P_5 = q^54 + q^{52} + 16q^{50} + 9q^{49} + 98q^{48} + 154q^{47} + 465q^{46} + 915q^{45} + 2042q^{44} + 3794q^{43} + 7263q^{42} 
+ 12688q^{41} + 21198q^{40} + 34323q^{39} + 52205q^{38} + 77068q^{37} + 108458q^{36} + 147423q^{35} + 191794q^{34} 
+ 241863q^{33} + 292689q^{32} + 342207q^{31} + 386980q^{30} + 421057q^{29} + 443990q^{28} + 451398q^{27} 
+ 443990q^{26} + 421057q^{25} + 386980q^{24} + 342207q^{23} + 292689q^{22} + 241863q^{21} + 191794q^{20} 
+ 147423q^{19} + 108458q^{18} + 77068q^{17} + 52205q^{16} + 34323q^{15} + 21198q^{14} + 12688q^{13} 
+ 7263q^{12} + 3794q^{11} + 2042q^{10} + 915q^{9} + 465q^{8} + 154q^{7} + 98q^{6} + 9q^{5} + 16q^{4} + q^{2} + 1. \]

It should be apparent from the size of the numerators of \( W_5(q) \) and \( G_5(q) \) that the problem of computing these rational functions explodes beyond \( k = 4 \). In fact it develops that all available computer packages (including Omega and Latte) fail to directly compute the constant terms in \( \mathbb{Q} \) for \( k = 5 \). This notwithstanding, we were eventually able to get the partial fraction algorithm of Xin \( [10] \) to deliver us \( G_5(q) \).

This paper covers the variety of techniques we developed in our efforts to compute these remarkable rational functions. Our efforts in obtaining \( W_k(q) \) and \( G_k(q) \) are still in progress, so far they only resulted in reducing the computer time required to obtain \( W_5(q) \) and \( G_5(q) \). Using combinatorial ideas, group actions, in conjunction with the partial fraction algorithm of Xin, we developed three essentially distinct algorithms for computing these rational functions as well as other closely related families. Our most successful algorithm reduces the computation time for \( W_5(q) \) down to about five minutes. The crucial feature of this algorithm is an inductive process for successively computing the series \( G_k(q) \) and \( W_k(q) \), based on a surprising role of divided differences.

This paper is the extended version of [2]. We organize the contents in 5 sections. Section 1 is this introduction. In Section [2] we relate these Hilbert series to constant terms and derive a collection of identities to be used in later sections. In Section 3 we develop the combinatorial model that reduces the computation of our Kronecker products to solutions of Diophantine systems. In Section 4 we develop the divided difference algorithm for the computation of the complete generating functions yielding \( W_k(q) \) and \( G_k(q) \). In Section 5 after an illustration of what can be done with bare hands we expand the combinatorial ideas acquired from this experimentation into our three algorithms that yielded \( G_5(q) \) and our fastest computation of \( W_5(q) \).

The readers are referred to the papers of Luque-Thibon [3,4] and Wallach [5] for an understanding of how these Hilbert series are related to problem arising in the study of quantum computing.

### 2. Hilbert series of invariants as constant terms

Let us recall that given two matrices \( A = [a_{ij}]_{i,j=1}^n \) and \( B = [b_{ij}]_{i,j=1}^n \) we use the notation \( A \otimes B \) to denote the \( nm \times nm \) block matrix \( A \otimes B = [a_{ij}B_{ij}]_{i,j=1}^n \). For instance, if \( m = n = 2 \), then

\[
A \otimes B = \begin{bmatrix}
  a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\
  a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\
  a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\
  a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22}
\end{bmatrix}.
\]

Here and in the following, we define \( T_A P(x) \) to be the action of an \( m \times m \) matrix \( A = [a_{ij}]_{i,j=1}^n \) on a polynomial \( P(x) = P(x_1, x_2, \ldots, x_m) \) in \( \mathbb{R}_m := \mathbb{C}[x_1, x_2, \ldots, x_m] \) by

\[
T_A P(x_1, x_2, \ldots, x_m) = P\left( \sum_{i=1}^{m} x_ia_{i1}, \sum_{i=1}^{m} x_ia_{i2}, \ldots, \sum_{i=1}^{m} x_ia_{im} \right). \quad (11)
\]

In matrix notation (viewing \( x = (x_1, x_2, \ldots, x_m) \) as a row vector) we may simply rewrite this as

\[
T_A P(x) = P(xA).
\]
Recall that if \( G \) is a group of \( m \times m \) matrices we say that \( P \) is \( G \)-invariant if and only if
\[
T_A P(x) = P(x) \quad \forall \quad A \in G.
\]
The subspace of \( R_m^G \) of \( G \)-invariant polynomials is usually denoted \( R_m^G \). Clearly, the action in \( R_m^G \) preserves homogeneity and degree. Thus we have the direct sum decomposition
\[
R_m^G = \mathcal{H}_0(R_m^G) \oplus \mathcal{H}_1(R_m^G) \oplus \mathcal{H}_2(R_m^G) \oplus \cdots \oplus \mathcal{H}_d(R_m^G) \oplus \cdots
\]
where \( \mathcal{H}_d(R_m^G) \) denotes the subspace of \( G \)-invariants that are homogeneous of degree \( d \). The Hilbert series of \( R_m^G \) is simply given by the formal power series
\[
F_G(q) = \sum_{d \geq 0} q^d \dim \left( \mathcal{H}_d(R_m^G) \right).
\]
This is a well defined formal power series since \( \dim \mathcal{H}_d(R_m^G) \leq \dim \left( \mathcal{H}_d(R_m) \right) = \left( i + m - 1 \right) \).

When \( G \) is a finite group the Hilbert series \( F_G(q) \) is immediately obtained from Molien’s formula
\[
F_G(q) = \frac{1}{|G|} \sum_{A \in G} \frac{1}{\det (I - qA)}.
\]
For an infinite group \( G \) which possess a unit invariant measure \( \omega \) this identity becomes
\[
F_G(q) = \int_{A \in G} \frac{1}{\det (I - qA)} \, d\omega.
\]
For the present developments we need to specialize all this to the case \( G = SL[2]^{\otimes k} \), that is the group of \( 2^k \times 2^k \) matrices obtained by tensoring a \( k \)-tuple of elements of \( SL[2] \). More precisely
\[
SL[2]^{\otimes k} = \{ A_1 \otimes A_2 \otimes \cdots \otimes A_k : A_i \in SL[2] \quad \forall \quad i = 1, 2, \ldots, k \}.
\]

Our first task in this section is to derive the identity in (9). That is

**Theorem 2.1.** Setting for \( k \geq 1 \)
\[
W_k(q) = F_{SL[2]^{\otimes k}}(q) = \sum_{d \geq 0} q^d \dim \left( \mathcal{H}_d(SL[2]^{\otimes k}_m) \right),
\]
we have
\[
W_k(q) = \prod_{i=1}^{k} \frac{(1 - a_i^2)}{\prod_{S \subseteq [1,k]} \left( 1 - q \prod_{i \in S} a_i / \prod_{j \not\in S} a_j \right) a_i a_i a_i \cdots a_k}.
\]

We need the following result.

**Proposition 2.2.** If \( Q(a_1, a_2, \ldots, a_k) \) is a Laurent polynomial in \( \mathbb{C}[a_1, a_2, \ldots, a_k; 1/a_1, 1/a_2, \ldots, 1/a_k] \) then
\[
\left( \frac{1}{2\pi} \right)^k \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} Q(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_k}) \, d\theta_1 \, d\theta_2 \cdots d\theta_k = Q(a_1, a_2, \ldots, a_k) \bigg|_{a_1 a_1 \cdots a_k}.
\]

**Proof.** By multilinearity, it suffices to consider \( Q(a_1, a_2, \ldots, a_k) = a_1^{r_1} a_2^{r_2} \cdots a_k^{r_k} \), in which case (16) obviously holds.

**Proof of Theorem 2.1.** To keep our exposition within reasonable limits we will need to assume here some well known facts (see [8] for proofs). Since \( SL[2] \) has no finite measure the first step is to note that a polynomial \( P(x) \in \mathbb{C}[x_1, x_2, \ldots, x_{2k}] \) is \( SL[2]^{\otimes k} \)-invariant if and only if it is \( SU[2]^{\otimes k} \)-invariant, where \( SU[2] := SU(2, \mathbb{C}) \) and as in (13)
\[
SU[2]^{\otimes k} = \{ A_1 \otimes A_2 \otimes \cdots \otimes A_k : A_i \in SU[2] \quad \forall \quad i = 1, 2, \ldots, k \}.
\]
In particular we derive that \(F_{SL[2] \otimes k}(q) = F_{SU[2] \otimes k}(q)\). This fact allows us to compute \(F_{SL[2] \otimes k}(q)\) using Molien’s identity \(\ref{eq:12}\). Note however that if
\[
A = A_1 \otimes A_2 \otimes \cdots \otimes A_k
\]
and \(A_i\) has eigenvalues \(t_i, 1/t_i\) then (using plethystic notation) we have
\[
\frac{1}{\det(I - qA)} = \sum_{m \geq 0} q^m h_m [(t_1 + 1/t_1)(t_2 + 1/t_2) \cdots (t_k + 1/t_k)].
\]
Denoting by \(d\omega_i\) the invariant measure of the \(i^{th}\) copy of \(SU[2]\) we see that \(\ref{eq:12}\) reduces to
\[
F_{SU[2] \otimes k}(q) = \sum_{m \geq 0} q^m \int_{SU[2]} \cdots \int_{SU[2]} h_m [(t_1 + 1/t_1) \cdots (t_k + 1/t_k)] d\omega_1 \cdots d\omega_k.
\]
Now it is well known that if an integrand \(f(A)\) of \(SU[2]\) is invariant under conjugation then
\[
\int_{SU[2]} f(A) d\omega = \frac{1}{\pi} \int_{-\pi}^{\pi} f \left( \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \right) \sin^2 \theta d\theta.
\]
This identity converts the right-hand side of \(\ref{eq:17}\) to
\[
\sum_{m \geq 0} q^m \frac{1}{\pi^k} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} h_m [(e^{i\theta_1} + e^{-i\theta_1}) \cdots (e^{i\theta_k} + e^{-i\theta_k})] \sin^2 \theta_1 \cdots \sin^2 \theta_k d\theta_1 \cdots d\theta_k.
\]
The substitution
\[
\sin^2 \theta_j = \frac{1 - e^{2i\theta_j} + e^{-2i\theta_j}}{2}
\]
reduces the coefficient of \(q^m\) to
\[
\frac{1}{(2\pi)^k} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} h_m [(e^{i\theta_1} + e^{-i\theta_1}) \cdots (e^{i\theta_k} + e^{-i\theta_k})] \prod_{i=1}^{k} \left( 1 - \frac{e^{2i\theta_j} + e^{-2i\theta_j}}{2} \right) d\theta_1 \cdots d\theta_k.
\]
However the factor \(h_m [(e^{i\theta_1} + e^{-i\theta_1}) \cdots (e^{i\theta_k} + e^{-i\theta_k})]\) is invariant under any of the interchanges \(e^{i\theta_j} \leftrightarrow e^{-i\theta_j}\). Thus the integral in \(\ref{eq:19}\) may be simplified to
\[
\frac{1}{(2\pi)^k} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} h_m [(e^{i\theta_1} + e^{-i\theta_1}) \cdots (e^{i\theta_k} + e^{-i\theta_k})] \prod_{i=1}^{k} \left( 1 - e^{2i\theta_j} \right) d\theta_1 \cdots d\theta_k.
\]
Proposition \(\ref{prop:2.2}\) then yields that this integral may be computed as the constant term
\[
h_m [(a_1 + 1/a_1)(a_2 + 1/a_2) \cdots (a_k + 1/a_k)] \prod_{i=1}^{k} \left( 1 - a_i^2 \right) \bigg|_{a_1^0 a_2^0 \cdots a_k^0}.
\]
Using this in \(\ref{eq:18}\) we derive that
\[
F_{SU[2] \otimes k}(q) = \sum_{m \geq 0} q^m \left( \sum_{S \subseteq [1,k]} \prod_{j \in S} a_j \right) \prod_{i=1}^{k} \left( 1 - a_i^2 \right) \bigg|_{a_1^0 a_2^0 \cdots a_k^0}.
\]
This completes the proof of Theorem \(2.1\). \(\square\)
Note that if we restrict our action of $SU(2)^{\otimes k}$ to the subgroup of matrices
\[ T_2^{\otimes k} = \left\{ \begin{bmatrix} t_1 & 0 \\ 0 & T_1 \end{bmatrix} \otimes \begin{bmatrix} t_2 & 0 \\ 0 & T_2 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} t_k & 0 \\ 0 & T_k \end{bmatrix} : t_r = e^{i\theta} \right\} \]

then a similar use of Molien’s theorem yields the following result.

**Theorem 2.3.** The Hilbert series of the ring of invariants $R_2^{T_2^{\otimes k}}$ is given by the constant term
\[ F_{T_2^{\otimes k}}(q) \equiv \prod_{S \subseteq [1, k]} \left( 1 - q \prod_{i \in S} a_i / \prod_{j \notin S} a_j \right) \bigg|_{a_1 a_2 \cdots a_k}. \quad (20) \]

**Proof.** The integrand $1 / \det(1-qA)$ is the same as in the previous proof and only the Haar measure changes. In this case we must take $dw = d\theta_1 d\theta_2 \cdots d\theta_k / (2\pi)^k$ in $[12]$, and Molien’s theorem gives
\[ F_{T_2^{\otimes k}}(q) = \frac{1}{(2\pi)^k} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \prod_{S \subseteq [1, k]} \left( 1 - q \prod_{i \in S} t_i / \prod_{j \notin S} t_j \right) d\theta_1 d\theta_2 \cdots d\theta_k. \]

Thus (20) follows from Proposition 2.2. \qed

**Remark 2.4.** There is another path leading to the same result that is worth mentioning here since it gives a direct way of connecting Invariants to Diophantine systems. For notational simplicity we will deal with the case $k = 3$. Note that the element
\[ \begin{bmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{bmatrix} \in T_2^{\otimes 3} \]
is none other than the $8 \times 8$ diagonal matrix

\[
A(t_1, t_2, t_3) =
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

This gives that for any monomial $x^p = x_1^{p_1} x_2^{p_2} \cdots x_8^{p_8}$ we have
\[
A(t_1, t_2, t_3) x^p = t_1^{p_1 + p_2 + p_3 + p_4 - p_5 - p_6 - p_7 - p_8} t_2^{p_1 + p_2 - p_3 - p_4 + p_5 + p_6 - p_7 - p_8} t_3^{p_1 - p_2 + p_3 - p_4 + p_5 - p_6 + p_7 - p_8} x^p.
\]

Thus all the monomials are eigenvectors and a polynomial $P(x_1, x_2, \ldots, x_8)$ will be invariant if and only if all its monomials are eigenvectors of eigenvalue 1. It then follows that the Hilbert series $F_{T_2^{\otimes 3}}(q)$ of $\mathbb{C}[x_1, x_2, \ldots, x_8]^{T_2^{\otimes 3}}$ is obtained by $q$-counting these monomials by total degree. That is $q$-counting by the statistic $p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8$ the solutions of the Diophantine system
\[
S_3 = \begin{pmatrix}
p_1 + p_2 + p_3 + p_4 - p_5 - p_6 - p_7 - p_8 = 0 \\
p_1 + p_2 - p_3 + p_4 + p_5 + p_6 - p_7 - p_8 = 0 \\
p_1 - p_2 + p_3 - p_4 + p_5 - p_6 + p_7 - p_8 = 0
\end{pmatrix} \quad (21)
\]
and MacMahon partition analysis gives
\[
F_{T_2^{\otimes 3}}(q) = \prod_{\text{all } a_1 a_2 a_3} \left( 1 - q a_1 a_2 a_3 \right).
\]

This gives another proof of the case $k = 3$ of (20). It is also clear that the same argument can be used for all $k > 3$ as well.
Remark 2.5. Full information about the solutions of our systems is given by the complete generating function

\[ F_k(x_1, x_2, \ldots, x_{2k}) = \sum_{p \in S_k} x_1^{p_1} x_2^{p_2} \cdots x_{2k}^{p_{2k}}. \]  

(22)

Using the notation adopted for \( S_3 \) in (21), our system \( S_k \) may be written in vector form

\[ p_1 V_1 + p_2 V_2 + \cdots + p_{2k} V_{2k} = 0, \]

where \( V_1, V_2, \ldots, V_{2k} \) are the \( k \)-vectors \((\pm 1, \pm 1, \ldots, \pm 1)\) yielding the vertices of the hypercube of semiside 1 centered at the origin. In this notation, MacMahon partition analysis gives that the rational function in (22) is obtained by taking the constant term

\[ F_k(x_1, x_2, \ldots, x_{2k}) = \prod_{i=1}^{2k} \frac{1}{1 - x_i A_i} \mid_{a_1^0 a_2^0 \cdots a_k^0}, \]

with the \( A_i \) Laurent monomials in \( a_1, a_2, \ldots, a_k \) which may be written in the form

\[ A_i = \prod_{i=1}^{k} a_i^{1-2\epsilon_i}, \]

where \( \epsilon_1 \epsilon_2 \cdots \epsilon_k \) are the binary digits of \( i - 1 \).

In the same vein the companion rational function \( W_k(x_1, x_2, \ldots, x_{2k}) \) associated to the Sdd problem is obtained by taking the constant term

\[ W_k(x_1, x_2, \ldots, x_{2k}) = \prod_{j=1}^{2k} \frac{1}{1 - q \prod_{i \in S_j} a_i / \prod_{j \not\in S_j} a_j} \mid_{a_1^0 a_2^0 \cdots a_k^0}. \]  

(23)

Of course we have

\[ G_k(q) = F_k(x_1, x_2, \ldots, x_{2k}) \big|_{x_i = q} \quad \text{and} \quad W_k(q) = W_k(x_1, x_2, \ldots, x_{2k}) \big|_{x_i = q}. \]

In Section 4 we will show that, at least in principle, these rational functions could be constructed by a succession of elementary steps interspersed by single constant term extractions.

3. Diophantine systems, constant terms and Kronecker products

We have seen, by MacMahon partition analysis, that the generating function \( G_k(q) \) defined in (7), which counts solutions of the Diophantine system \( S_k \), is given by the constant term identity in (8):

\[ G_k(q) = \prod_{S \subseteq [1,k]} \frac{1}{1 - q \prod_{i \in S} a_i / \prod_{j \not\in S} a_j} \mid_{a_1^0 a_2^0 \cdots a_k^0}. \]  

(24)

In the last section we proved (in Theorem 2.1) that the Hilbert series \( W_k(q) \) of invariants in (14) is given by the constant term

\[ W_k(q) = \prod_{S \subseteq [1,k]} \frac{\prod_{i=1}^{2k} (1 - a_i^0)}{1 - q \prod_{i \in S} a_i / \prod_{j \not\in S} a_j} \mid_{a_1^0 a_2^0 \cdots a_k^0}. \]  

(25)

A comparison of (24) and (23) clearly suggests that these two results must be connected. This connection has a beautiful combinatorial underpinning which leads to another interpretation of the these remarkable constant terms. The idea is best explained in the simplest case \( k = 2 \). Then (25) reduces to

\[ W_2(q) = \frac{1 - a_1^2 - a_2^2 + a_1^2 a_2^2}{(1 - qa_1 a_2)(1 - qa_1/a_2)(1 - qa_2/a_1)(1 - q/a_1 a_2)} \mid_{a_1^0 a_2^0}. \]
Expanding the inner rational function as product of four formal power series in \( q \) we get

\[
W_2(q) = \sum_{p_00 \geq 0} \sum_{p_01 \geq 0} \sum_{p_{010} \geq 0} \sum_{p_{011} \geq 0} q^{p_00 + p_{01} + p_{010} + p_{011}} a_{1}^{p_{00} + p_{01} + p_{011}^{2}} a_{2}^{p_{00} + p_{01} + p_{011}^{2}} - \sum_{p_00 \geq 0} \sum_{p_01 \geq 0} \sum_{p_{010} \geq 0} \sum_{p_{011} \geq 0} q^{p_00 + p_{01} + p_{010} + p_{011}} a_{1}^{p_00 + p_{01} + p_{011}^{2}} a_{2}^{p_00 + p_{01} + p_{011}^{2}} \cdot \cdot \cdot (26)
\]

Now by MacMahon partition analysis, the the \( i^{th} \) term counts solutions of the Diophantine system

\[
S^i = \left\{ \begin{array}{c} p_00 + p_{01} - p_{10} - p_{11} = c_i \\ p_00 - p_{01} + p_{10} - p_{11} = d_i \end{array} \right\} \quad (27)
\]

where \((c_i, d_i)\) equals \((0, 0), (-2, 0), (0, -2), (-2, -2)\) for \( i = 1, 2, 3, 4 \), respectively. Note that the first term of [26] is none other than [24] for \( k = 2 \).

Applying the same decomposition in the general case we see that the series \( W_k(q) \) may be viewed as the end product of an inclusion exclusion process applied to a family of Diophantine systems.

To derive some further consequences of this fact, it is more convenient to use another combinatorial model for these systems. In this alternate model our family of objects consists of the collection \( F_d \) of \( d \)-subsets of the \( 2d \)-element set

\[
\Omega_{2d} = \{1, 2, 3, \ldots, 2d\}.
\]

For a given \( A = \{1 \leq i_1 < i_2 < \cdots < i_d \leq 2d\} \in F_d \) and \( \sigma \) in the symmetric group \( S_{2d} \) we set

\[
\sigma A = \{\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_d}\}.
\]

This clearly defines an action of \( S_{2d} \) on \( F_d \) as well as on the \( k \)-fold cartesian product

\[
F^k_d = F_d \times F_d \times F_d \times \cdots \times F_d.
\]

**Theorem 3.1.** The number \( m_d(k) \) of solutions of the Diophantine system \( S_k \) is equal to the number of orbits in the action of \( S_{2d} \) on \( F^k_d \).

**Proof.** It will be sufficient to see this for \( k = 2 \). Then leaving \( d \) generic we can visualize an element of \( F_d \times F_d \) by the Ven diagram of Figure 11. There we have depicted the pair \((A_1, A_2)\) as it lies in \( \Omega_{2d} \). Using these two sets we can decompose \( \Omega_{2d} \) into 4 parts labeled by \( A_{00}, A_{01}, A_{10}, A_{11} \). More precisely “\( A_{00} \)” labels the set \( A_1 \cap A_2 \), “\( A_{01} \)” labels the set \( A_1 \cap \bar{A}_2 \), “\( A_{10} \)” labels the set \( \bar{A}_1 \cap A_2 \) and “\( A_{11} \)” labels the set \( \bar{A}_1 \cap \bar{A}_2 \). Here we use “\( \bar{A}_i \)” to denote the complement of \( A_i \) in \( \Omega_{2d} \). This given, if we let \( p_{00}, p_{01}, p_{10}, p_{11} \) denote the respective cardinals of these sets, the condition that the pair \((A_1, A_2)\) belongs to \( F_d \times F_d \) yields that we must have

\[
\begin{align*}
p_{00} + p_{01} + p_{10} + p_{11} &= 2d \\
p_{00} + p_{01} &= |A_1| = d \\
p_{00} + p_{10} &= |A_2| = d
\end{align*}
\]

Note that this system of equations is equivalent to the system

\[
\begin{align*}
p_{00} + p_{01} + p_{10} + p_{11} &= 2d \\
p_{00} + p_{01} &= p_{10} - p_{11} = 0 \\
p_{00} - p_{01} + p_{10} - p_{11} &= 0
\end{align*}
\]

It is easily seen that for any solution \((p_{00}, p_{01}, p_{10}, p_{11})\) of this system, we can immediately construct a pair of subsets \((A_1, A_2) \in F_d \times F_d \) by simply filling the sets \( A_{00}, A_{01}, A_{10}, A_{11} \) in the diagram of Figure 11 with \( p_{00}, p_{01}, p_{10}, p_{11} \) respective elements from the set \( \Omega_{2d} \). Moreover, any two such fillings can be seen to be images of each other under suitable permutations of \( S_{2d} \). In other words by this construction we obtain a bijection between the orbits of \( F_d \times F_d \) under \( S_{2d} \) and the solutions of the
system $S_2$ we have previously encountered. This proves the theorem for $k = 2$. The general case follows by an entirely analogous argument.

Now we are ready to prove Theorem 1.2 and then Theorem 1.1.

Proof of Theorem 1.2. We are to show that

$$m_k(d) = \langle h_{d,d} \odot h_{d,d} \odot \cdots \odot h_{d,d}, s_{2d} \rangle. \tag{28}$$

It is well known that a transitive action of a group $G$ on a set $\Omega$ is equivalent to the action of $G$ on the left $G$-cosets of the stabilizer of any element of $\Omega$. In our case, pick the subset $[1, d]$ of $\Omega^2_d$. Then the stabilizer is the Young subgroup $S_{[1,d]} \times S_{[d+1,2d]}$ of $S_{2d}$ and thus the Frobenius characteristic of this action is the homogeneous basis element $h_{d,d} = h_d h_d$. It follows then that the Frobenius characteristic of the action of $S_{2d}$ on the $k$-tuples $(A_1, A_2, \ldots, A_k)$ of $d$-subsets of $\Omega_{2d}$ is given by the $k$-fold Kronecker product $h_{d,d} \odot h_{d,d} \odot \cdots \odot h_{d,d}$. Therefore the scalar product

$$\langle h_{d,d} \odot h_{d,d} \odot \cdots \odot h_{d,d}, s_{2d} \rangle$$

yields the multiplicity of the trivial under this action. But it is well known, and easy to see that this multiplicity is also equal to the number of orbits under this action. Thus (28) follows by Theorem 3.1.

Proof of Theorem 1.1. Again we will only need to do it for $k = 2$. To this end note that by Theorem 1.2 the number of solutions of the system $S_{2}^1$ in (27) is given by the scalar product

$$\langle h_{d,d} \odot h_{d,d}, s_{2d} \rangle. \tag{29}$$

In the same vein we see that the number of solutions to the system $S_{2}^1$ in (27) may be viewed as the number of orbits in the action of $S_{2d}$ on the pairs of subsets $(A_1, A_2)$ of $\Omega_{2d}$ where $|A_2| = |A_1| + 2$. We have seen that the Frobenius characteristic of the action of $S_{2d}$ on subsets of cardinality $d$ is $h_{d,d}$. On the other hand the action of $S_{2d}$ on sets of cardinality $d + 1$ is equivalent to the action of $S_{2d}$ on left cosets of $S_{[1,d+1]} \times S_{[d+2,2d]}$ yielding that the Frobenius characteristic for this action is $h_{d+1} h_{d-1}$. Thus the Frobenius characteristic of the action of $S_{2d}$ on such pairs must be the Kronecker product

$$h_{d+1} h_{d-1} \odot h_d h_d.$$

It then follows that the number of solutions of the system $S_{2}^1$ is given by the scalar product

$$\langle h_{d+1} h_{d-1} \odot h_d h_d, s_{2d} \rangle. \tag{30}$$

The same reasoning gives that the number of solutions of the systems $S_{2}^2$ and $S_{2}^3$ in (27) are given by the scalar products

$$\langle h_d h_d \odot h_{d+1} h_{d-1}, s_{2d} \rangle \quad \text{and} \quad \langle h_{d+1} h_{d-1} \odot h_{d+1} h_{d-1}, s_{2d} \rangle. \tag{31}$$
It follows then that the coefficient of \( q^{2d} \) in the alternating sum of formal power series in \([23]\) is none other than the following alternating sum of the scalar products in \([24] \), \([31]\) and \([31]\).

\[
W_2(q) = \langle h_d h_d, s_{2d} \rangle - \langle h_{d+1} h_{d-1}, s_{2d} \rangle
- \langle h_d h_d - h_{d+1} h_{d-1}, s_{2d} \rangle
= \langle h_d h_d - h_{d+1} h_{d-1} \rangle \langle h_d h_d - h_{d+1} h_{d-1} \rangle, s_{2d} \rangle = \langle s_{d, d} \rangle s_{d, d}, s_{2d} \rangle.
\]

Summing over \( d \) gives

\[
W_2(q) = \sum_{d \geq 0} q^{2d} \langle s_{d, d} \rangle s_{d, d}, s_{2d} \rangle.
\]

An entirely analogous argument proves the general identity in \([23]\). \( \square \)

4. Enter divided difference operators

There is a truly remarkable approach to the solutions of a variety of constant term problems which exhibit the same types of symmetries of the Hdd and Sdd problems. We will introduce the approach in some simple cases first. We define the \( \text{double} \) of the Diophantine system

\[
S_2 = \left\{ \begin{array}{l}
p_1 + p_2 - p_3 - p_4 = 0 \\
p_1 - p_2 + p_3 - p_4 = 0
\end{array} \right.
\]

to be the system

\[
SS_2 = \left\{ \begin{array}{l}
p_1 + p_2 - p_3 - p_4 + p_5 + p_6 - p_7 - p_8 = 0 \\
p_1 - p_2 + p_3 - p_4 + p_5 - p_6 + p_7 - p_8 = 0
\end{array} \right.
\]

As we can easily see we have simply repeated twice each linear form and appropriately increased the indices of the variables. Now suppose that we are in possession of the complete generating function of \( S_2 \), that is

\[
F_{S_2}(x_1, x_2, x_3, x_4) = \sum_{p \in S_2} x_1^{p_1} x_2^{p_2} x_3^{p_3} x_4^{p_4}.
\]

We claim that the complete generating function of \( SS_2 \) is simply given by

\[
F_{SS_2}(x_1, x_2, \ldots, x_8) = \delta_{1,5} \delta_{2,6} \delta_{3,7} \delta_{4,8} F_{S_2}(x_1, x_2, x_3, x_4),
\]

where for any pair of indices \( (i, j) \) we let \( \delta_{i,j} \) denote the divided difference operator defined for any function \( f(x) \) by

\[
\delta_{i,j} f(x) = \left. \frac{f(x) - f(x)}{x_i - x_j} \right|_{x_i = x_j = x}.
\]

**Proof of \([32]\).** By MacMahon partition analysis we have

\[
F_{S_2}(x_1, x_2, x_3, x_4) = \frac{1}{(1 - x_1 a_1 a_2)} \frac{1}{(1 - x_2 a_1 a_2)} \frac{1}{(1 - x_3 a_2 a_1)} \frac{1}{(1 - x_4 a_2 a_1)} \left|_{a_1^2 a_2^2} \right.
\]

Now note that since

\[
\delta_{1,5} \frac{1}{(1 - x_1 a_1 a_2)} = \left( \frac{1}{(1 - x_1 a_1 a_2)^2} - \frac{1}{(1 - x_5 a_2 a_1)^2} \right) \frac{1}{x_1 - x_5} = \frac{a_1 a_2}{(1 - x_1 a_2)(1 - x_5 a_2 a_1)},
\]

we obtain similarly

\[
\delta_{2,6} \frac{1}{(1 - x_2 a_1 a_2)} = \frac{a_1 a_2}{(1 - x_2 a_1 a_2)(1 - x_6 a_1 a_2)},
\]

\[
\delta_{3,7} \frac{1}{(1 - x_3 a_2 a_1)} = \frac{a_2 a_1}{(1 - x_3 a_2 a_1)(1 - x_7 a_2 a_1)},
\]

\[
\delta_{4,8} \frac{1}{(1 - x_4 a_1 a_2)} = \frac{a_1 a_2}{(1 - x_4 a_1 a_2)(1 - x_8 a_1 a_2)}.
\]
Theorem 4.1. If \(\delta_1,5\delta_2,6\delta_3,7\delta_4,8\) to both sides of (33) gives

\[
\delta_1,5\delta_2,6\delta_3,7\delta_4,8F_{S_2}(x_1, x_2, x_3, x_4) = \frac{1}{(1 - x_1a_1a_2)(1 - x_2a_1/a_2)(1 - x_3a_2/a_1)(1 - x_4/a_1a_2)}
\]

\[
(1 - x_5a_1a_2)(1 - x_6a_1/a_2)(1 - x_7a_2/a_1)(1 - x_8/a_1a_2)
\]

Now we can easily recognize that (34) is precisely the constant term that MacMahon partition analysis would yield for the system \(SS_2\). This proves (32).

Note that to obtain the equality in (34) we have used the simple fact that the divided difference operator and the constant term operator do commute. This is the fundamental property which is at the root of the present algorithm. This example should make it evident to have the following more general result (with double modified).

**Theorem 4.1.** If \(F_S(x_1, x_2, \ldots, x_n)\) is the complete generating function of the Diophantine system

\[
S = \begin{bmatrix}
  b_{11} & b_{12} & \cdots & b_{1n} \\
  \vdots & \vdots & \ddots & \vdots \\
  b_{r_1} & b_{r_2} & \cdots & b_{rn}
\end{bmatrix}
\]

\[
p_1 = \begin{bmatrix}
  c_1 \\
  \vdots \\
  c_r
\end{bmatrix}
\]

then the complete generating function of the doubling of \(S\) defined by

\[
SS = \begin{bmatrix}
  b_{11} & b_{12} & \cdots & b_{1n} & b_{11} & b_{12} & \cdots & b_{1n} \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  b_{r_1} & b_{r_2} & \cdots & b_{rn} & b_{r_1} & b_{r_2} & \cdots & b_{rn}
\end{bmatrix}
\]

\[
p_2 = \begin{bmatrix}
  c_1 - b_{11} - b_{12} - \cdots - b_{1n} \\
  \vdots \\
  c_r - b_{r_1} - b_{r_2} - \cdots - b_{rn}
\end{bmatrix}
\]

is given by the rational function

\[
F_{SS}(x_1, x_2, \ldots, x_{2n}) = \delta_{1,n+1}\delta_{2,n+2} \cdots \delta_{n,2n} F_S(x_1, x_2, \ldots, x_n).
\]

This result combined with the next simple observation yields a powerful algorithm for computing a variety of complete generating functions.

**Theorem 4.2.** Let \(F_S(x_1, x_2, \ldots, x_n)\) be the complete generating function of a Diophantine system \(S\) then the complete generating function \(F_{SE}(x_1, x_2, \ldots, x_n)\) of the system \(SE\) obtained by adding the equation

\[
E = \begin{bmatrix}
  r_1p_1 + r_2p_2 + \cdots + r_np_n = s
\end{bmatrix}
\]

to \(S\) is obtained by taking the constant term

\[
F_{SE}(x_1, x_2, \ldots, x_n) = a^{-s}F_S(a^{r_1}x_1, a^{r_2}x_2, \ldots, a^{r_n}x_n)|_{a^0}.
\]

**Proof.** By assumption

\[
F_S(x_1, x_2, \ldots, x_n) = \sum_{p \in S} x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}.
\]

Now we have

\[
a^{-s}F_S(a^{r_1}x_1, a^{r_2}x_2, \ldots, a^{r_n}x_n)|_{a^0} = \sum_{p \in S} x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n} a^{r_1p_1+r_2p_2+\cdots+r_np_n-s}|_{a^0}
\]

\[
= \sum_{p \in SE} x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}
\]

\[
= F_{SE}(x_1, x_2, \ldots, x_n).
\]

These two results provide us with algorithms for (at least in principle) computing all the Hdd series \(G_k(q)\) as well as the Sdd series \(W_k(q)\).
Algorithm 4.3 (Hdd Case).

a_1) Initially compute the complete generating function for the Hdd problem for k = 1. That is, compute the constant term
\[ F_1(x_1, x_2) = \frac{1}{(1 - x_1 a)(1 - x_2/a)} \bigg|_{a^0}. \]

a_k) With \( F_{k-1}(x_1, \ldots, x_{2^{k-1}}) \) from step b_{k-1}, compute by divided difference
\[ FF_{k-1}(x_1, \ldots, x_k) = \delta_{1,1+2^{k-1}} \cdots \delta_{2^{k-1},2^k} F_{k-1}(x_1, \ldots, x_{2^{k-1}}). \]

b_k) With \( FF_{k-1}(x_1, \ldots, x_{2^{k-1}}) \) from step a_k, compute the complete generating function for the Sdd problem for k by the following constant term:
\[ F_k(x_1, x_2, \ldots, x_k) = FF_{k-1}(ax_1, ax_2, \ldots, ax_{2^{k-1}} - 1, x_{2^{k-1}+1}/a, \ldots, x_k/a) \bigg|_{a^0}. \]

This sequence of steps in Algorithm 4.3 can be terminated by replacing step b_k) by

b'_k) The q-generating function \( G_k(q) \) is given by the constant term
\[ G_k(q) = F_{SS_{k-1}}(aq, aq, \ldots, aq, q/a, \ldots, q/a) \bigg|_{a^0}. \]

The steps up to b_3) can be carried out by hand. For further steps we need a computer, and to carry out step b_5) by computer we have to introduce one more tool as we shall see. Unfortunately Step b_6) appears beyond reach at the moment.

It will be instructive to see what the first several steps give.

a_1)
\[ F_{S_1}(x_1, x_2) = \frac{1}{1 - x_1 x_2}. \]

a_2)
\[ F_{SS_1}(x_1, x_2, x_3, x_4) = \frac{(1 - x_1 x_2 x_3 x_4)}{(1 - x_1 x_2)(1 - x_2 x_3)(1 - x_1 x_4)(1 - x_3 x_4)}. \]

b_2)
\[ F_{S_2}(x_1, x_2, x_3, x_4) = \frac{(1 - x_1 x_2 x_3 x_4)}{(1 - a^2 x_1 x_2)(1 - x_2 x_3)(1 - x_1 x_4)(1 - x_3 x_4/a^2)} \bigg|_{a^0} = \frac{1}{(1 - x_2 x_3)(1 - x_1 x_4)}. \]

a_3)
\[ F_{SS_2}(x) = \frac{(1 - x_1 x_4 x_5 x_6 x_7)(1 - x_2 x_3 x_6 x_7)}{(1 - x_1 x_8)(1 - x_2 x_7)(1 - x_3 x_6)(1 - x_4 x_5)(1 - x_2 x_3)(1 - x_6 x_7)(1 - x_5 x_8)}. \]

b_3)
\[ F_{S_3}(x) = \frac{(1 - x_1 x_4 x_5 x_6 x_7)(1 - x_2 x_3 x_6 x_7)}{(1 - x_1 x_8)(1 - x_2 x_7)(1 - x_3 x_6)(1 - x_4 x_5)} \times \frac{1}{(1 - a^2 x_1 x_4)(1 - a^2 x_2 x_3)(1 - x_6 x_7/a^2)(1 - x_5 x_8/a^2)} \bigg|_{a^0}. \] (35)

We can compute this constant term in many ways. In particular we could use one of the MacMahon identities given by Andrews in [1]. But it is interesting to point out that our divided difference algorithm has already provided us (in step a_2) a formula we can use in step b_3). In fact, the output of step a_2)
\[ F_{SS_1}(x_1, x_2, x_3, x_4) = \frac{(1 - x_1 x_2 x_3 x_4)}{(1 - x_1 x_2)(1 - x_2 x_3)(1 - x_1 x_4)(1 - x_3 x_4)} \]
is the complete generating function of the system \( p_1 - p_2 + p_3 - p_4 = 0 \), so by MacMahon partition analysis we should also have
\[ F_{SS_1}(x_1, x_2, x_3, x_4) = \frac{1}{(1 - ax_1)(1 - x_2/a)(1 - ax_3)(1 - x_4/a)} \bigg|_{a^0}. \]
This implies that
\[
\left. \frac{1}{(1-a^2x_1x_4)(1-a^2x_2x_3)(1-x_6x_7/a^2)(1-x_5x_8/a^2)} \right|_{a^0}
= \frac{(1-x_1x_2x_3x_4)}{(1-x_1x_2)(1-x_2x_3)(1-x_1x_4)(1-x_3x_4)}
\]
\[
\times \frac{x_1x_2x_3x_4}{x_3x_2x_3x_4x_5x_6x_7x_8}
= \frac{(1-x_1x_2x_3x_4x_5x_6x_7x_8)}{(1-x_1x_4x_5x_7)(1-x_6x_7x_2x_3)(1-x_1x_4x_5x_8)(1-x_2x_3x_5x_8)}
\]

Using this in (21) gives
\[
F_{S_3}(x_1, \ldots, x_8) = \frac{(1-x_1x_4x_5x_8)(1-x_2x_3x_6x_7)}{(1-x_1x_8)(1-x_2x_7)(1-x_3x_6)(1-x_4x_5)}
\]
\[
\times \frac{(1-x_1x_2x_3x_4x_5x_6x_7x_8)}{(1-x_1x_4x_6x_7)(1-x_6x_7x_2x_3)(1-x_1x_4x_5x_8)(1-x_2x_3x_5x_8)}
\]

Replacing all the \(x_i\) by the single variable \(q\), we thus obtain that
\[
G_1(q) = \frac{1}{1-q^2}, \quad G_2(q) = \frac{1}{(1-q^2)^2}, \quad G_3(q) = \frac{1-q^8}{(1-q^2)(1-q^4)^2} = \frac{1+q^4}{(1-q^2)(1-q^4)}
\]

Using the computer to carry out step \(b_4^k\) gives
\[
G_4(q) = \frac{1+q^2 + 21q^4 + 36q^6 + 74q^8 + 86q^{10} + 74q^{14} + 36q^{16} + 21q^{18} + q^{20}}{(1-q^2)^4(1-q^4)^4(1-q^6)}
\]

We shall see later what else has to be done to obtain \(G_5(q)\).

Our divided difference algorithm can also be adapted to compute the first 4 Sdd series as well. In fact, again due to the fact that divided difference operators commute with the constant term operators, we can also show that all the complete Sdd series can (in principle) be obtained by the following algorithm.

**Algorithm 4.4** (Sdd Case).  
\(a_1\) Initially compute the complete generating function for the Sdd problem for \(k = 1\). That is, compute the constant term
\[
W_1(x_1, x_2) = \left. \frac{1-a^2}{(1-x_1a)(1-x_2/a)} \right|_{a^0}.
\]

\(a_k\) With \(W_{k-1}(x_1, \ldots, x_{2^k-1})\) from step \(b_{k-1}\), compute by divided difference
\[
WW_{k-1}(x_1, \ldots, x_{2^k}) = \delta_{1,1+2^{k-1}} \cdots \delta_{2^{k-1}-1,2^k} W_{k-1}(x_1, \ldots, x_{2^k-1}).
\]

\(b_k\) With \(WW_{k-1}(x_1, \ldots, x_{2^k-1})\) from step \(a_k\), compute the complete generating function for the Sdd problem for \(k\) by the following constant term:
\[
W_k(x_1, x_2, \ldots, x_{2^k}) = \left. WW_{k-1}(ax_1, ax_2, \ldots, ax_{2^{k-1}} + 1/a, \ldots, x_{2^k}/a)(1-a^2) \right|_{a^0}.
\]

Note that similarly as for the Hdd-case, the sequence of steps in Algorithm 4.4 can be terminated by replacing step \(b_k\) by

\(b_k\) To obtain the generating function \(W_k(q)\) compute the constant term
\[
W_k(q) = \left. WW_{k-1}(aq, aq, \ldots, aq, q/a, \ldots, q/a)(1-a^2) \right|_{a^0}.
\]
Only steps \( a_1 \) and \( a_2 \) can be carried out by hand. Though steps 3 and 4 are routine they are too messy to do by hand. But step 5 again needs further tricks to be carried out by computer. Step 6 appears beyond reach at the moment.

It will be instructive to see what some of these steps give.

\( a_1 \)
\[
W_1(x_1, x_2) = \frac{1 - x_2^2}{1 - x_1 x_2}.
\]

\( a_2 \)
\[
WW_1(x_1, \ldots, x_4) = \frac{1 - x_2^2 - x_2 x_4 - x_3^2 + x_1 x_2^2 x_4 + x_1 x_2 x_3 x_4 + x_1 x_2 x_3^2 x_4 + x_2 x_3 x_4^2 - x_1 x_2^2 x_3 x_4}{(1 - x_1 x_2)(1 - x_3 x_4)(1 - x_1 x_4)(1 - x_3 x_4)}.
\]

\( a_3 \)
\[
W_2(x_1, x_2, x_3, x_4) = \frac{1 - x_2 x_4 - x_3 x_4 + x_4^2}{(1 - x_1 x_4)(1 - x_2 x_3)}.
\]

This gives
\[
W_2(q) = \frac{1}{1 - q^2}.
\]

\( b_2 \)
\[
W_2(x_1, \ldots, x_8) = \frac{(\text{large } \text{numerator})}{(1 - x_1 x_4)(1 - x_1 x_8)(1 - x_2 x_3)(1 - x_2 x_7)(1 - x_3 x_6)(1 - x_4 x_5)(1 - x_5 x_8)(1 - x_6 x_7)}.
\]

\( b_3 \)
\[
W_3(x_1, \ldots, x_8) = \frac{(\text{large } \text{numerator})}{(1 - x_1 x_8)(1 - x_2 x_7)(1 - x_3 x_6)(1 - x_4 x_5)(1 - x_1 x_4 x_6 x_7)(1 - x_2 x_3 x_5 x_8)}.
\]

\( b_3 \)
Notwithstanding the complexity of the previous results it turns out that to obtain \( W_3(q) \) we need only compute the constant term
\[
W_3(q) = \frac{1}{(1 - q^2)} \times \left. \frac{1 - a^2}{(1 - q^2 a^2)(1 - q^2/a^2)} \right|_{a^2 = q}.
\]

To this end we start by determining the coefficients \( A \) and \( B \) in the partial fraction decomposition
\[
\frac{(1 - a^2)a^2}{(1 - q^2 a^2)(a^2 - q^2)} = \frac{1}{q^2} + \frac{A}{1 - q^2 a^2} + \frac{B}{a^2 - q^2}
\]

obtaining
\[
A = \left. \frac{(1 - a^2)a^2}{(a^2 - q^2)} \right|_{a^2 = 1/q^2} = \left. \frac{(1 - 1/q^2)/q^2}{(1/q^2 - q^2)} \right|_{a^2 = 1/q^2} = -\frac{1}{q^2(1 + q^2)},
\]
\[
B = \left. \frac{(1 - a^2)a^2}{(1 - q^2 a^2)} \right|_{a^2 = q^2} = \left. \frac{(1 - q^2)q^2}{(1 - q^4)} \right|_{a^2 = q^2} = \frac{q^2}{1 + q^2},
\]

(the exact value of \( B \) is not needed) and we can write
\[
\frac{1 - a^2}{(1 - q^2 a^2)(1 - q^2/a^2)} = \frac{1}{q^2} - \frac{1}{q^2(1 + q^2)} \times \frac{1}{1 - a^2 q^2} + \frac{1}{1 + q^2} \times \frac{q^2/a^2}{1 - q^2/a^2}.
\]

Thus taking constant terms gives
\[
\left. \frac{1 - a^2}{(1 - q^2 a^2)(1 - q^2/a^2)} \right|_{a^2 = 0} = \frac{1}{q^2} - \frac{1}{q^2(1 + q^2)} + 0 = \frac{1}{1 + q^2}.
\]

Using this in \( b_3 \) we finally obtain
\[
W_3(q) = \frac{1}{1 - q^4}.
\]

\( a_4 \)
\[
WW_4(x_1, x_2, \ldots, x_{16}) = (\text{too large for typesetting})
\]
b') Notwithstanding the complexity of the previous result it turns out that to obtain \( W_4(q) \) we need only compute the constant term
\[
W_4(q) = \frac{(1 + q^4)(1 + q^6)}{(1 - q^2)(1 - q^4)^2} \times \frac{1 - a^2}{(1 - a^2 q^4)(1 - q^4/a^2)(1 - a^4 q^4)(1 - q^4/a^4)} a_0.
\]

To illustrate the power and flexibility of the partial fraction algorithm we will carry this out by hand. The reader is referred to [3] for a brief tutorial on the use of this algorithm. In the next few lines we will strictly adhere to the notation and terminology given in [3].

To begin we note that we need only calculate the constant term
\[
C(x) = \frac{1 - a}{(1 - ax)(1 - x/a)(1 - a^2 x)(1 - x/a^2)} a_0,
\]

since we can write
\[
W_4(q) = \frac{(1 + q^4)(1 + q^6)}{(1 - q^2)(1 - q^4)^2} \times C(q^4).
\]

Now we have
\[
\frac{1}{(1 - a^2 x)(1 - x/a^2)} = \frac{a^2}{(1 - a^2 x)(a^2 - x)} = \frac{1}{1 - x^2 - a^2 x} + \frac{1}{1 - x^2 1 - x/a^2}.
\]

Thus (37) may be rewritten in the form
\[
C(x) = \frac{1}{1 - x^2} \left( \frac{1 - a}{(1 - ax)(1 - x/a)(1 - a^2 x)} a_0 \right) + \frac{1 - a x/a^2}{(1 - x/a)(1 - x/a^2)} a_0.
\]

Note that in the first constant term we have only one dually contributing term and on the second we have only one contributing term. This gives
\[
\frac{(1 - a)}{(1 - ax)(1 - x/a)} \left. a_0 \right|_{a_0} = \frac{(1 - a x/a^2)}{(1 - x/a)(1 - x/a^2)} a_0.
\]

Using (40) and (41) in (39) we get
\[
C(x) = \frac{(1 - x)}{(1 - x^2)} \left( \frac{1}{1 - x^2} - \frac{1 - x}{1 - x^3} \right) = \frac{1 - x}{(1 - x^2)(1 - x^3)}.
\]

Together with (38), we get
\[
W_4(q) = \frac{(1 + q^4)(1 + q^6)}{(1 - q^2)(1 - q^4)^2} \times \frac{1 - q^4}{(1 - q^8)(1 - q^{12})} = \frac{1}{(1 - q^2)(1 - q^4)^2(1 - q^6)}.
\]

We will see in section 4 what needs to be done to carry out step b') on the computer.

The identities for \( W_2(q), W_3(q), W_4(q) \) in (b) have also been derived in [3] by symmetric function methods from the relation (5). In fact, all three results in (b) are immediate consequences of the following deeper symmetric function identity. (For a proof see [3], Section 2.)

**Theorem 4.5.**

\[
s_{d,d} \odot s_{d,d} = \sum_{\lambda \vdash 2d} s_{\lambda} \chi(\lambda \in EO_4)
\]

where EO_4 denotes the set of partitions of length 4 whose parts are \( \geq 0 \) and all even or all odd.

Note that the Kronecker product identity
\[
\langle s_{d,d} \odot s_{d,d} \odot s_{d,d} \odot s_{d,d} \odot s_{d,d} \odot s_{d,d}, s_{d,d} \rangle = \langle s_{d,d} \odot s_{d,d} \odot s_{d,d} \odot s_{d,d} \odot s_{d,d} \rangle.
\]

suggests obtaining \( W_5(q) \) by means of a combinatorial interpretation of the coefficients of the Schur function expansion of the Kronecker product \( s_{d,d} \odot s_{d,d} \odot s_{d,d} \). However, to this date no formula has been given for these coefficients, combinatorial or otherwise.
This section is divided into four parts. In the first subsection we start with our computer findings and end by giving a combinatorial decomposition that works nicely to obtain \( F_3(x) \). In the second subsection, this decomposition is described algebraically and, together with group actions, turned into manipulatory gyrations that will be used to extract \( G_5(q) \) and \( W_5(q) \) out of our computers. In the third subsection, by combining the idea of decomposition and the method of divided difference in Section 4, we give our best way that reduce the computation time for \( G_5(q) \) and \( W_5(q) \) down to a few minutes. In the final subsection, we give our first algorithm to obtain \( G_5(q) \) and \( W_5(q) \).

### 5.1. A combinatorial decomposition for \( F_5(x) \)

Our initial efforts at solving the Hdd and Sdd problems were entirely carried out by computer experimentation. After obtaining quite easily the series \( G_2(q) \), \( G_3(q) \), \( G_4(q) \) and \( W_2(q) \), \( W_3(q) \), \( W_4(q) \), all the computer packages available to us failed to directly deliver \( G_5(q) \) and \( W_5(q) \).

The computer data obtained for the Hdd problem for \( k = 2, 3 \) were combinatorially so revealing that we have been left with a strong impression that this problem should have a very beautiful combinatorial general solution. Only time will tell if this will ever be the case. To stimulate further research we will begin by reviewing our initial computer and manual combinatorial findings.

Recall that we denoted by \( \mathcal{F}_d \) the collection of all \( d \)-subsets of the \( 2d \) element set \( \Omega_{2d} \). We also showed (in Theorem 3.4) that the coefficient \( m_d(k) \) in the series \( G_k(q) = \sum_{d \leq 0} q^{2d} m_d(k) \) counts the number of orbits under the action of the symmetric group \( S_{2d} \) on the \( k \)-fold cartesian product \( \mathcal{F}_d \times \mathcal{F}_d \times \cdots \times \mathcal{F}_d \). Denoting by \((A_1, A_2, \ldots, A_k)\) a generic element of this cartesian product, then each orbit is uniquely determined by the \( 2^k \) cardinalities

\[
p_{\epsilon_1, \epsilon_2, \ldots, \epsilon_k} = |A_1^{\epsilon_1} \cap A_2^{\epsilon_2} \cap \cdots \cap A_k^{\epsilon_k}|
\]

where for each \( 1 \leq i \leq k \) we set

\[
A_i^{\epsilon_i} = \begin{cases} A_i & \text{if } \epsilon_i = 0, \\ A_i^{-1} & \text{if } \epsilon_i = 1. \end{cases}
\]

It is also convenient to set \( A_{\epsilon_1, \epsilon_2, \ldots, \epsilon_k} = A_1^{\epsilon_1} \cap A_2^{\epsilon_2} \cap \cdots \cap A_k^{\epsilon_k} \). This given we have seen that the condition \((A_1, A_2, \ldots, A_k) \in \mathcal{F}_d^2 \) is equivalent to the Diophantine system

\[
S_k = \left| \sum_{\epsilon_1=0}^{1} \sum_{\epsilon_2=0}^{1} \cdots \sum_{\epsilon_k=0}^{1} (1 - 2\epsilon_1)p_{\epsilon_1, \epsilon_2, \ldots, \epsilon_k} = 0, \right|
\]

\[
\sum_{\epsilon_1=0}^{1} \sum_{\epsilon_2=0}^{1} \cdots \sum_{\epsilon_k=0}^{1} (1 - 2\epsilon_2)p_{\epsilon_1, \epsilon_2, \ldots, \epsilon_k} = 0,
\]

\[
\vdots
\]

\[
\sum_{\epsilon_1=0}^{1} \sum_{\epsilon_2=0}^{1} \cdots \sum_{\epsilon_k=0}^{1} (1 - 2\epsilon_k)p_{\epsilon_1, \epsilon_2, \ldots, \epsilon_k} = 0,
\]

together with the condition \(|\Omega_{2d}| = 2d\), that is \( \sum_{\epsilon_1=0}^{1} \sum_{\epsilon_2=0}^{1} \cdots \sum_{\epsilon_k=0}^{1} p_{\epsilon_1, \epsilon_2, \ldots, \epsilon_k} = 2d \).

There are several algorithms available to solve such a system. See for instance [7, Chapter 4.6]. The algorithm we used for our computer experimentations is the MacMahon algorithm which has been recently implemented in MATHEMATICCA by Andrews, Paule and Riese and in MAPLE by Xin using the partial fraction method of computing constant terms.

The former can be downloaded from the web site

[http://www.risc.uni-linz.ac.at/research/combinat/software/Omega/](http://www.risc.uni-linz.ac.at/research/combinat/software/Omega/)

and the latter from the web site

[http://www.combinatorics.net.cn/homepage/xin/maple/ell2.rar](http://www.combinatorics.net.cn/homepage/xin/maple/ell2.rar)

For computer implementation we found it more convenient to use the alternate notation adopted in Remark 2.5. That is

\[
S_k = \| p_1 V_1 + p_2 V_2 + \cdots + p_{2k} V_{2k} = 0. \tag{42}
\]

These algorithms may yield quite a bit more than the number of solutions of such a system. For instance, in our case letting \( C_k \) denote the collection of solutions of the system \( S_k \), the “Omega
package” of Andrews, Paule and Riese should, in principle, yield the formal power series
\[ F_k(x_1, x_2, \ldots, x_{2k}) = \sum_{(p_1, p_2, \ldots, p_{2k}) \in C_k} x_1^{p_1} x_2^{p_2} \cdots x_{2k}^{p_{2k}}. \]

It follows from the general theory of Diophantine systems that \( F_k(x_1, x_2, \ldots, x_{2k}) \) is always the Taylor series of a rational function.

Now for \( S_2 \) and \( S_3 \) the Omega package gives
\[
\begin{align*}
F_2(x_1, x_2, x_3, x_4) &= \frac{1}{(1 - x_1 x_4)(1 - x_2 x_3)}, \\
F_3(x_1, x_2, \ldots, x_8) &= \frac{1 - x_2 x_3 x_5 x_6 x_7 x_8 x_4 x_5 x_7}{(1 - x_1 x_8)(1 - x_2 x_7)(1 - x_3 x_6)(1 - x_4 x_5)(1 - x_2 x_3 x_5 x_8)(1 - x_1 x_4 x_6 x_7)}.
\end{align*}
\]

But this is as far as this package went in our computers. However we could go further by giving up full information about the solutions and only ask for the series
\[ G_k(q) = F_k(x_1, x_2, \ldots, x_{2k}) \big|_{x_i = q}, \]
which can be computed from its constant term representation in \([5]\). For example, the program \texttt{Latte} by De Loera, Hemmecke, Tauscher, Yoshida, which is available at [http://www.math.ucdavis.edu/~latte/](http://www.math.ucdavis.edu/~latte/) computed the \( G_4(q) \) series in approximately 30 seconds. However, this is as far as \texttt{Latte} went on our machines. We should also mention that all the series \( G_k(q) \) and \( W_k(q) \) for \( k \leq 4 \) can be obtained in only a few seconds, from the software of Xin by computing the corresponding constant terms in \([8]\) and \([9]\).

To get our computers to deliver \( G_5(q) \) and \( W_5(q) \) in a matter of minutes a divide and conquer strategy had to be adopted. More precisely, these rational functions were obtained by decomposing the constant terms \([8]\) and \([9]\) as sums of constant terms. This decomposition had its origin from an effort to find a human proof of the identities in \([13]\) and \([14]\). More importantly, the surprising simplicity of \([13]\) and \([14]\) required a combinatorial explanation. Our findings there provided the combinatorial tools that were used in our early computations of \( G_5(q) \) and \( W_5(q) \). This given, before describing our work on these series, we will show how to obtain \([13]\) and \([14]\) entirely by hand.

Let us start by sketching the idea for \( k = 2 \). Beginning with
\[ S_2 = \{ p_1 + p_2 - p_3 - p_4 = 0, p_1 - p_2 + p_3 - p_4 = 0 \} \]
we immediately notice that \((1, 0, 0, 1)\) and \((0, 1, 1, 0)\) are solutions. Set
\[ a = \min(p_1, p_4) \quad \text{and} \quad b = \min(p_2, p_3). \]

It is clear that the following difference must also be a solution.
\[ (q_1, q_2, q_3, q_4) = (p_1, p_2, p_3, p_4) - (a, b, b, a) = (p_1 - a, p_2 - b, p_3 - b, p_4 - a). \]

Now \( q_1q_4 = 0 \) and \( q_2q_3 = 0 \). This gives us four possibilities for \((q_1, q_2, q_3, q_4)\):
\[
(0, 0, x, y), \quad (0, x, 0, y), \quad (x, 0, y, 0), \quad (x, y, 0, 0),
\]
for some nonnegative integers \( x, y \). Testing the first equation of \( S_2 \) immediately forces the first and last in \((45)\) to identically vanish. Similarly, the second equation of \( S_2 \) yields that the second and third in \((45)\) must also identically vanish. This proves that the general solution of \( S_2 \) is of the form \((a, b, b, a)\). We thus reobtain the full generating function \((13)\) of solutions of \( S_2 \):
\[
F_2(x_1, x_2, x_3, x_4) = \frac{1}{(1 - x_1 x_4)(1 - x_2 x_3)}. \]

It turns out that we can deal with \( S_3 \) in a similar manner. Again we begin by noticing the four symmetric solutions
\[
(1, 0, 0, 0, 0, 0, 0, 1), \quad (0, 1, 0, 0, 0, 1, 0), \quad (0, 0, 1, 0, 0, 1, 0), \quad (0, 0, 0, 1, 1, 0, 0). \]
Next we set

\[ a = \min(p_1, p_8), \quad b = \min(p_2, p_7), \quad c = \min(p_3, p_6), \quad d = \min(p_4, p_5), \]

and by subtraction we get a solution

\[ (q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8) = (p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) - (a, b, c, d, c, b, a) \] (46)

with the property \(q_i q_{9-i} = 0\) for \(1 \leq i \leq 4\). It will be good here and after to call the set

\[ \{i \in [1, n] : p_i \geq 1\} \]

the support of the composition \((p_1, p_2, \ldots, p_n)\). This given, we derive that the resulting composition in (46) will necessarily have its support contained in at least one of the following 16 patterns.

\[
\begin{align*}
(0, 0, 0, 0, *, *, *), & \quad (0, 0, 0, *, 0, *, *) , \quad (0, 0, *, 0, 0, *, ), \quad (0, 0, *, 0, 0, *, ), \\
(0, *, 0, 0, *, 0, *) , & \quad (0, *, 0, 0, 0, 0, *) , \quad (0, *, *, 0, *, 0, *) , \quad (0, *, *, 0, *, 0, *) , \\
(*, 0, 0, 0, *, 0, *) , & \quad (*, 0, 0, *, 0, 0, *) , \quad (*, 0, *, 0, 0, *, 0) , \quad (*, 0, *, 0, 0, *, 0) , \\
(*, *, 0, 0, *, 0, 0) , & \quad (*, *, 0, *, 0, 0, 0) , \quad (*, *, *, 0, 0, 0, 0), \quad (*, *, *, 0, 0, 0, 0).
\end{align*}
\] (47)

Unlike the case \(k = 2\) not all of these patterns force a trivial solution. To find out which it is helpful to resort to a Venn diagram imagery. To this end recall that a solution of \(S_3\) gives the cardinalities of the 8 regions of the Venn diagram of three \(d\)-subsets \(A_1, A_2, A_3\) of \(\Omega_{2d}\) (see Figure 2).

![Figure 2. The Venn diagram for \(S_3\).](image)

In Figure 3 each pattern is represented by a Venn diagram where in each region \(A_1^a \cap A_2^b \cap A_3^c\) that corresponds to a * in the pattern we placed a black dot. That means that only the regions with a dot may have \(\geq 0\) cardinality. The miracle is that all but the two patterns \((0, *, *, 0, *, 0, *, 0)\) and \((*, 0, 0, *, 0, 0, *, 0)\) can be quickly excluded by a reasoning that only uses the positions of the dots in the Venn diagram. In fact, in each of the excluded cases, we show that it is impossible to replace the dots by \(\geq 0\) integers in such a manner that the three sets \(A_1, A_2, A_3\) and their complements \(^cA_1, ^cA_2, ^cA_3\) end up having the same cardinality (except for all empty sets).

![Figure 3. The 16 support patterns for \(S_3\).](image)

The reasoning is so cute that we are compelled to present it here in full. In what follows the \(j^{th}\) diagram in the \(i^{th}\) row will be referred to as \(D_{ij}\):

1. \(D_{11}, D_{14}, D_{16}, D_{23}, D_{25},\) and \(D_{28}\) can be immediately excluded because one of \(A_1, A_2, A_3, A_1^c, A_2^c\) or \(A_3^c\) would be empty.
(2) In $D_{15}$ the dot next to 8 should give the cardinality of $A_{2}^2$ (say $d$) and then the dot next to the 2 should also give $d$. But that forces the dots next to 5 and 6 to be 0, leaving $A_3$ empty, a contradiction. The same reasoning applies to $D_{12}, D_{13}, D_{18}, D_{21}, D_{24}, D_{26},$ and $D_{27}.$

That leaves only the two diagrams $D_{17}$ and $D_{22}$ which clearly correspond to the two mentioned patterns. Now we see that for $D_{22}$ we must have the equalities $p_1 + p_4 = p_1 + p_6 = p_1 + p_7 = p_6 + p_7$. This forces $p_1 = p_4 = p_6 = p_7$. In summary this pattern can only support the composition $(u, 0, 0, u, 0, u, 0)$. The same reasoning yields that the diagram $D_{17}$ can only support the composition $(0, v, 0, v, 0, 0, v)$. It follows that the general solution of $S_3$ must be of the form $(a, b, c, d, d, c, b, a) + (u, v, u, v, u, v, u, v)$.

Now recall that after the subtraction of a symmetric solution we are left with an asymmetric solution. Thus to avoid over counting we must impose the condition $u = v = 0$. This leaves only three possibilities $u = v = 0$, $u > 0$, $v = 0$ or $u = 0$, $v > 0$. Thus

$$F_3(x) = \sum_{a \geq 0} \sum_{b \geq 0} \sum_{c \geq 0} \sum_{d \geq 0} (x_1x_8)^a(x_2x_7)^b(x_3x_6)^c(x_4x_5)^d \left( 1 + \sum_{u \geq 1} (x_1x_4x_6x_7)^u + \sum_{v \geq 1} (x_2x_3x_5x_8)^v \right)$$

$$= \frac{1}{(1 - x_1x_3)(1 - x_2x_7)(1 - x_3x_6)(1 - x_4x_5)} \left( 1 + \frac{x_1x_4x_6x_7}{1 - x_1x_4x_6x_7} + \frac{x_2x_3x_5x_8}{1 - x_2x_3x_5x_8} \right),$$

which is only another way of writing (44).

5.2. Algebraic decompositions and group actions. It is easy to see that the decomposition of a solution into a sum of a symmetric plus an asymmetric solution can be carried out for general $k$. In fact, note that if $0 \leq i \leq 2^k - 1$ has binary digits $\epsilon_1\epsilon_2\cdots\epsilon_k$ then the binary digits of $2^k - 1 - i$ are $\bar{\epsilon}_1\bar{\epsilon}_2\cdots\bar{\epsilon}_k$ (with $\bar{\epsilon} = 1 - \epsilon$). Thus we see from (12) that in each equation $p_i$ and $p_{2^k+1-i}$ appear with opposite signs. This shows that for each $k \geq 2$ the system $S_k$ has $2^{k-1}$ symmetric solutions, which may be symbolically represented by the monomials $x_i x_{i'}$ for $i = 1, \ldots, 2^{k-1}$, where we use (and will often use) $i'$ to denote $2^k + 1 - i$ when $k$ is fixed.

Proceeding as we did for $S_2$ and $S_3$ we arrive at a unique decomposition of each solution of $S_k$ into

$$(p_1, p_2, \ldots, p_{2^k}) = (u_1, u_2, \ldots, u_2, u_1) + (q_1, q_2, \ldots, q_{2^k})$$

with the first summand symmetric and the second asymmetric, that is $u_i = u_{i'}$ and $q_iq_{i'} = 0$ for $1 \leq i \leq 2^{k-1}$, and thereby obtain a factorization of $F_k(x)$ in the form

$$F_k(x) = \left( \prod_{i=1}^{2^{k-1}} \frac{1}{1 - x_i x_{i'}} \right) F_k^A(x)$$

(48)

with $F_k^A(x)$ denoting the complete generating function of the asymmetric solutions.

This given it is tempting to try to apply, in the general case, the same process we used for $k = 3$ and obtain the rational function $F_k^A(x)$ by selecting the patterns that do contain the support of an asymmetric solution. Note that the total number of asymmetric patterns to be examined is $2^{2^{k-1}}$ which is already 256 for $k = 4$. For $k = 5$ the number grows to 65,536 and doing this by hand is out of the question. Moreover, it is easy to see, by going through a few cases, that even for $k = 4$ the geometry of the Venn Diagrams is so intricate that the only way that we can find out if a given pattern contains the support of a solution is to solve the corresponding reduced system.

Nevertheless, using some inherent symmetries of the problem, the complexity of the task can be substantially reduced to permit the construction of $G_5^*(q)$ by computer. To describe how this was done we need some notation. We will start with the complete generating function of the system $S_k$ as given in Remark 2.5 that is

$$F_k(x_1, x_2, \ldots, x_{2^k}) = \prod_{i=1}^{2^k} \frac{1}{1 - x_i A_i} \bigg|_{a_0^0 a_2^0 \cdots a_{2^k}^0},$$
where $A_i = \prod_{i=1}^{k} a_i^{1-2\epsilon_i}$, with $\epsilon_1 \epsilon_2 \cdots \epsilon_k$ being the binary digits of $i - 1$. Note that since (as we previously observed) the binary digits of $2^k - 1 - i$ are $\overline{7_1 \overline{7_2 \cdots \overline{7_k}}}$, we have $A_i = 1/A_i$. It then follows that

$$\frac{1 - x_ix_{i'}}{(1 - x_iA_i)(1 - x_{i'}A_{i'})} = \left(\frac{1}{1 - x_iA_i} + \frac{x_{i'}/A_i}{1 - x_{i'}/A_i}\right).$$

Thus combining the factors containing $A_i$ and $A_{i'}$ we may rewrite (48) in the form

$$F_k(x_1, x_2, \ldots, x_2^k) = \prod_{i=1}^{2^k-1} \frac{1}{1 - x_i x_{i'}} \prod_{i=1}^{2^k-1} \left(\frac{1}{1 - x_i A_i} + \frac{x_{i'}/A_i}{1 - x_{i'}/A_i}\right) |_{a_1^{a_2^2 \cdots a_6^i}}. \quad (49)$$

Comparing with (48) we derive that the complete generating function of the asymmetric solutions is given by the following sum.

$$F_k^A(x) = \sum_{S \subseteq [1, 2^k-1]} F_S(x), \quad (50)$$

where

$$F_S(x) = \left(\prod_{i \in S} \frac{1}{1 - x_i A_i}\right) \times \left(\prod_{i \in S} \frac{x_{i'}/A_i}{1 - x_{i'}/A_i}\right) |_{a_1^{a_2^2 \cdots a_6^i}}. \quad (51)$$

In this way we have described our decomposition algebraically. Using notation as of (42), we can see that $F_S(x)$ is none other than the complete generating function of the reduced system

$$\sum_{i \in S} p_i V_i + \sum_{i \in S} p_{i'} V_{i'} = 0$$

with the added condition that $p_{i'} \geq 1$ for all $i \in S$.

Note that for $k = 3$ the summands in (50) correspond precisely to the 16 patterns in (44) with the added condition that the “*” in position $i \geq 5$ should represent $p_i \geq 1$ in the corresponding solution vector. This extra condition is precisely what is needed to eliminate overcounting.

Perhaps all this is best understood with an example. For instance for $k = 3$ the patterns

$$(*, 0, 0, *, 0, *, *) \quad \text{and} \quad (0, *, *, 0, 0, 0, *)$$

were the only ones that supported an asymmetric solution represent the two reduced systems

$$S_{14} = \begin{cases} p_1 + p_4 - p_6 - p_7 = 0, \\ p_1 - p_4 + p_6 - p_7 = 0, \\ p_1 - p_4 - p_6 + p_7 = 0 \end{cases} \quad S_{23} = \begin{cases} p_2 + p_3 - p_5 - p_8 = 0, \\ p_2 - p_3 + p_5 - p_8 = 0, \\ -p_2 + p_3 + p_5 - p_8 = 0 \end{cases}$$

and correspond to the following two summands of (50) for $k = 3$

$$F_{1,4}^k(x) = \frac{1}{1 - x_1x_2x_3/1 - x_1 a_1 a_2 a_3} \frac{1}{1 - x_1 x_2 a_1 a_2 /1 - x_4 a_1 a_2 a_3} \frac{x_6 a_2 / a_1 a_3}{x_7 a_3 / a_1 a_2} |_{a_1 a_2 a_3} = \frac{x_1 x_4 x_6 x_7}{1 - x_1 x_4 x_6 x_7} \quad (52)$$

$$F_{2,3}^k(x) = \frac{1}{1 - x_2 a_1 a_2 /1 - x_2 a_1 a_2 a_3} \frac{1}{1 - x_2 x_3 a_1 a_2 /1 - x_3 a_1 a_2 a_3} \frac{x_5 a_2 a_3 / a_1}{x_8 / a_1 a_2 a_3} |_{a_1 a_2 a_3} = \frac{x_2 x_3 x_5 x_8}{1 - x_2 x_3 x_5 x_8}. \quad (53)$$

A close look at these two expressions should reveal the key ingredient that needs to be added to our algorithms that will permit reaching $k = 5$ in the Hdd and Sdd problems. Indeed we see that $F_{1,4}^k(x)$ goes onto $F_{2,3}^k(x)$ if we act on the vector $(x_1, x_2, \cdots, x_8)$ by the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 1 & 2 & 7 & 8 & 5 & 6 \end{pmatrix} \quad (54)$$

and on the triple $(a_1, a_2, a_3)$ by the operation $a_2 \rightarrow a_2^{-1}$. In fact, $\sigma$ is none other than an image of the map $(\epsilon_1, \epsilon_2, \epsilon_3) \rightarrow (\epsilon_1, \epsilon_2, \epsilon_3)$ on the binary digits of 0, 1, $\ldots$, 7, as we can easily see when we replace each $i$ in (51) by the binary digits of $i - 1$

$$\sigma = \begin{pmatrix} 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \\ 010 & 011 & 000 & 001 & 110 & 111 & 100 & 101 \end{pmatrix}.$$
What goes on is quite simple. Recall that solutions $p$ of our system $S_k$ can also be viewed as assignments of weights to the vertices of the $k$-hypercube giving all hyperfaces equal weight. Then clearly any rotation or reflection of the hypercube will carry this assignment onto an assignment with the same property. Thus the Hyperoctahedral group $B_k$ will act on all the constructs we used to solve $S_k$.

To make precise the action of $B_k$ on $[1, 2^k]$ we need some conventions.

1. We will view the elements of $B_k$ as pairs $(\alpha, \eta)$ with a permutation $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in S_k$ and a binary vector $\eta = (\eta_1, \eta_2, \ldots, \eta_k)$.
2. Next, for any binary vector $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_k)$ let us set
   $$(\alpha, \eta)(\epsilon) = (\epsilon_{\alpha_1} + \eta_1, \epsilon_{\alpha_2} + \eta_2, \ldots, \epsilon_{\alpha_k} + \eta_k)$$
   with “$\text{mod } 2$” addition.
3. This given, to each element $g = (\alpha, \eta) \in B_k$ there corresponds a permutation $\sigma(g)$ by setting
   $$\sigma(g) = \begin{pmatrix} 1 & 2 & \cdots & 2^k \\ \sigma_1 & \sigma_2 & \cdots & \sigma_{2^k} \end{pmatrix},$$
   where $\sigma_i = j$ if and only if the $k$-vector $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_k)$ giving the binary digits of $i - 1$ is sent by $g$ onto the $k$-vector giving the binary digits of $j - 1$. In particular we will set
   $$g(x_1, x_2, \ldots, x_{2^k}) = (x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_{2^k}}).$$
4. In the same vein we will make $B_k$ act on the $k$-tuple $(a_1, a_2, \ldots, a_k)$ by setting, again for $g = (\alpha, \eta)$
   $$g(a_1, a_2, \ldots, a_k) = (a_{\alpha_1}^{1-\eta_1}, a_{\alpha_2}^{1-\eta_2}, \ldots, a_{\alpha_k}^{1-\eta_k}).$$

With these conventions we can easily derive that $gx_i A_i = x_{\sigma_i} A_{\sigma_i}$. Thus
   $$g \prod_{i=1}^{2^k} \frac{1}{1-x_i A_i} \big|_{a_i^0 a_{i'}^0 \ldots a_k^0} = \prod_{i=1}^{2^k} \frac{1}{1-x_{\sigma_i} A_{\sigma_i}} \big|_{a_i^0 a_{i'}^0 \ldots a_k^0} = \prod_{i=1}^{2^k} \frac{1}{1-x_i A_i} \big|_{a_i^0 a_{i'}^0 \ldots a_k^0},$$
from which we again derive the $B_k$ invariance of the complete generating function $F_k(x_1, x_2, \ldots, x_{2^k})$.

If we let $B_{k-1}$ not only act on the indices $1, 2, \ldots, 2^k - 1$, but also on $1', 2', \ldots, 2^{k-1}'$ by $\sigma_{i'} = \sigma_i'$. Then $B_{k-1}$ permutes the summands in (50) as well as the factors in the product
   $$\prod_{i=1}^{2^{k-1}} \frac{1}{1-x_{i'} A_{i'}}.$$
The orbit representative that corresponds to 1 is simply the case \( S = \emptyset \) in (49) and that corresponds to \( x_1x_4 \) is given in (52).

Thus from (52), (59) and (49) we derive (again) that

\[
G_3(q) = \frac{1}{(1 - q^2)^4} \left( 1 + \frac{q^4}{1 - q^4} \right) = \frac{1 + q^4}{(1 - q^2)^4(1 - q^4)}.
\]

For \( k = 4 \) we have \( 2^8 = 256 \) summands in (50) with 22 orbits but only 11 of these orbits do contribute to \( F_5^A \). The number of denominator factors for each term is 8 which is still a reasonable number for the partial fraction algorithm. The formula for \( F_4(x) \) obtained this way can be typed within a page, but we would like to introduce a nicer \( F_4(x) \) using the full group \( B_k \) instead of \( B_{k-1} \), as we will do in the next paragraph. For \( k = 5 \) we have 216 summands in (50) with 402 orbits but only 341 orbits do contribute to \( F_5^A \). The number of denominator factors for each term is 16 which is out of reach for the partial fraction algorithm to obtain \( F_5^A(x) \). Nevertheless, in this manner we can still produce \( G_5(q) \) in about 15 minutes.

The decomposition in (53) is only \( B_{k-1} \) invariant, and it is natural from the geometry of the hypercube labelings, to ask of a \( B_k \) invariant decomposition. To obtain such a decomposition of \( F_k(x) \) we will pair off the factors containing \( A_i \) and \( A_{i'} \) by means of the more symmetric identity

\[
\frac{1 - x_i x_{2k+1-i}}{(1 - x_k A_i)(1 - x_{2k+1-i} A_{2k+1-i})} = \left( 1 + \frac{x_i A_i}{1 - x_k A_i} + \frac{x_i A_{i'}}{1 - x_{i'} A_{i'}} \right)
\]

and derive that

\[
F_k(x) = \sum_{S \cup T \subseteq [1, 2^k-1]} F_{S,T}(x),
\]

where \( S \) and \( T \) are disjoint and

\[
F_{S,T}(x) = \left( \prod_{i \in S} \frac{x_i A_i}{1 - x_i A_i} \right) \left( \prod_{i' \in T} \frac{x_i'/A_{i'}}{1 - x_i'/A_{i'}} \right).
\]

Note that every pair \( (S, T) \) should be identified with the set \( S \cup \{i' : i \in T\} \subseteq [1, 2^k] \) when applying the action of \( B_k \).

**Example 5.1.** For \( k = 3 \) we have \( 3^4 = 81 \) summands with 9 orbits but only 2 orbits do contribute to \( F_3^A \). The two orbits corresponds to the monomials \( 1 \) and \( x_1x_4x_6x_7 \) with respective orbit sizes 1 and 2. The orbit representative that corresponds to 1 is simply the case \( F_{\emptyset,\emptyset} = 1|_{a_0,0}^{a_2,0}a_2^{a_0} = 1 \) and that corresponds to \( x_1x_4x_6x_7 \) is

\[
F_{\{1,4,\{2,3\}\}}(x) = \frac{x_1 A_1 x_4 A_4 x_6 A_6 x_7 A_7}{1 - x_1 A_1 - x_4 A_4 - x_6 A_6 - x_7 A_7|_{a_0,0}^{a_2,0}a_2^{a_0}} = \frac{x_1 x_4 x_6 x_7}{1 - x_1 x_4 x_6 x_7}.
\]

Therefore, we can reobtain \( G_3(q) \) by using (59) as follows.

\[
G_3(q) = \frac{1}{(1 - q^2)^4(1 - q^4)} = \frac{1 + q^4}{(1 - q^2)^4(1 - q^4)}.
\]

**Example 5.2.** For \( k = 4 \) we have \( 3^8 = 6561 \) summands with 62 orbits but only 10 orbits do contribute to \( F_4^A \). We obtain the following complete generating functions for the 10 orbit representatives:

\[
\begin{align*}
(1) & \quad 1 \\
(24) & \quad \frac{x_{15} x_{15} x_{4} x_{14}}{1 - x_{15} x_{15} x_{4} x_{14}} \\
(16) & \quad \frac{x_{16} x_{7} (x_9)^2 x_{6} x_{4}}{1 - x_{16} x_{7} x_{9} x_{6} x_{4}} \\
(96) & \quad \frac{x_{15} x_{3} x_{7} (x_9)^2 (x_3)^3}{(1 - x_{12} x_{7} x_{9} x_6)(1 - x_{15} x_{3} x_{12} x_{9} x_6^2)}
\end{align*}
\]
Here the regions without numbers are empty. The number 1 indicates that the region has only one element. For \( k = 4 \) we found that there are only three orbits, containing 24, 8 and 16 elements respectively, the corresponding diagrams are depicted below.
Note, for $k = 4$ each Venn diagram is depicted as a pair of Venn diagrams of $k = 3$. The first member of the pair renders the Venn diagram of $A_1 \cap A_2, A_1 \cap A_3, A_1 \cap A_4$ and the second member renders the Venn diagram of $\complement A_1 \cap A_2, \complement A_1 \cap A_3$, $\complement A_1 \cap A_4$.

For $k = 5$ we found that there are 2712 extreme rays which break up into 9 orbits. We give in Figure 4 a set of representatives depicted as assignments of weights to the vertices of the 5 dimensional hypercube. We imagine that the vertices of this hypercube are indexed by the binary digits of $0, 1, 2, \ldots, 31$ with 00000 the vertex at the origin and 11111 giving the coordinates of the opposite vertex. In Figure 4 each hypercube is represented by two rows of two cubes. The cubes in the first row, from left to right, have the vertices labeled with the binary digits of 1 to 16 (minus 1) and the cubes in the second row have the vertices labeled with the binary digits of 17 to 32 (minus 1). The vertices here have possible weights 0, 1, 2, 3 and, correspondingly, are surrounded by 0, 1, 2, 3 concentric circles. The integer on the top of each diagram gives the size of the corresponding orbit.

Figure 4. Representatives of extreme rays for $k = 5$.

Each of the corresponding solutions of our system $S_5$ is minimal, that is, it cannot be decomposed into a non-trivial sum of solutions. But we found that there are also 480 minimal solutions that do not come from extreme rays. The latter break up into two orbits, with representatives depicted in Figure 5.

5.3. Our fastest way for $G_5(q)$ and $W_5(q)$. With the notations in the previous subsection and Section 4 handy, we can describe our best way to obtain $G_5(q)$ and $W_5(q)$. 
Let us explain the idea for \( k = 5 \). In Example 5.2 we have obtained for \( F^4_1(x) \) 10 orbit representatives with corresponding orbit sizes. Denote them by \( R_i(x) \) the representatives and \( m_i \) the orbit sizes for \( i = 1, \ldots, 10 \). From this we can give explicit formula of \( F^4_4(x) \) and hence of \( F_4(x) \) with the help of \( B_4 \) action as follows:

\[
F_4(x) = \frac{F^4_4(x)}{\prod_{i=1}^{10} (1-x_i x_{17-i})} = \sum_{i=1}^{10} m_i \sum_{g \in B_4} g \prod_{i=1}^{10} (1-x_i x_{17-i}) \tag{60}
\]

Applying Algorithm 4.3 to (60), we can obtain \( F_4(x) \) by multilinearity.

\[
F_5(x) = \sum_{i=1}^{10} m_i \sum_{g \in B_4} \left( \delta_{1,17} \cdots \delta_{16,32} \frac{R_i}{\prod_{i=1}^{10} (1-x_i x_{17-i})} \right) \left[ j=1,2,\ldots,16 \atop x_j = x_j a, x_{16+j} = x_{16+j}/a \right] \tag{61}
\]

where we have used the straightforwardly checked fact: for any rational function \( R(x_1,\ldots,x_{16}) \) and \( g \in B_k \), it holds that

\[
\delta_{1,17} \cdots \delta_{16,32} g R(x_1,\ldots,x_{16}) = g \delta_{1,17} \cdots \delta_{16,32} R(x_1,\ldots,x_{16}),
\]

where \( g \) is extended to permute also indices \( 16+j \) by \( g(16+j) = 16 + g(j) \) for \( j = 1,\ldots,16 \).

Substituting \( x_j = q \) for all \( j \) into (61) gives

\[
G_5(q) = \sum_{i=1}^{10} m_i \left( \delta_{1,17} \cdots \delta_{16,32} \frac{R_i}{\prod_{i=1}^{10} (1-x_i x_{17-i})} \right) \left[ j=1,2,\ldots,16 \atop x_j = x_j a, x_{16+j} = x_{16+j}/a \right]. \tag{62}
\]

That is to say, we only need representatives of \( F_{k-1}(x) \) together with orbit sizes to compute \( F_k(x) \), and this clearly extends for general \( k \). Using (62), we can persuade Maple to deliver \( G_5(q) \) as in (10) in about 12 minutes.

The orbit reduction idea for \( G_5(q) \) works in a similar way for \( W_5(q) \). In fact, we can carry out almost verbatim the same steps that yielded the orbit decomposition of the complete generating function \( F_k(x_1, x_2,\ldots, x_{2k}) \) to obtain the complete generating function \( W_k(x_1, x_2,\ldots, x_{2k}) \) as we shall define. Recall that the \( W_k(x) \) was originally defined in (23) as the constant term

\[
W_k(x_1, x_2,\ldots, x_{2k}) = \prod_{j=1}^{k} (1-a_j^2) \prod_{i=1}^{2k} \frac{1}{1-x_i A_i} \bigg|_{a_1^0 a_2^0 \cdots a_k^0}. \tag{63}
\]

To carry out its decomposition we need only observe that if we let

\[
\tilde{W}_k(x_1, x_2,\ldots, x_{2k}) = \frac{k}{2^k} \prod_{j=1}^{k} (1-a_j^2)(1-a_j^{-2}) \prod_{i=1}^{2k} \frac{1}{1-x_i A_i} \bigg|_{a_1^0 a_2^0 \cdots a_k^0}, \tag{64}
\]

then

\[
W_k(q) = \tilde{W}_k(q).
\]

The reason for this is that when all the \( x_i \) are replaced by \( q \), we can easily show that the constant term in (63) is not affected if we replace any \( a_i \) by \( a_i^{-1} \). Thus if we average out the right hand side of...
over all these interchanges the result will be simply the right hand side of (64) due to the simple relation
\[ 1 - \frac{a_i^2 + a_i^{-2}}{2} = \frac{1}{2} (1 - a_i^2(1 - a_i^{-2}). \]

Now (64) brings to evidence that \( \tilde{W}_k(x) \) is \( B_k \) invariant while \( W_k(x) \) is not. Symmetrizing \( W_k(x) \) gives \( \tilde{W}_k(x) \). We can obtain either a \( B_{k-1} \) invariant decomposition or a \( B_k \) invariant decomposition of \( \tilde{W}_k(x) \) just as for \( F_k(x) \).

The orbit reduction can also be used to considerably speed up steps \( a_k \) and \( b'_k \) in Algorithm 4.4 of the divided difference. The idea is similar as for the computation of \( G_5(q) \), but is much harder to be carried out.

To be clearer, we note that in step \( b'_k \) we do not need the complete generating function \( W_{k-1}(x) \). One way is to replace it by the more symmetric \( \tilde{W}_{k-1}(x) \). We have
\[
\tilde{W}_{k-1}(x) = \frac{1}{\prod_{i=1}^{2^{k-2}} (1 - x_i x_{2^{k-1}+1-i})} \sum_{S,T \subseteq [1,2^{k-2}]} \tilde{W}_{S,T}(x),
\]
where \( S \) and \( T \) are disjoint as before. We only need to find orbit representatives
\[ \tilde{W}_{S_1,T_1}(x), \tilde{W}_{S_2,T_2}(x), \ldots, \tilde{W}_{S_N,T_N}(x) \]
with respective multiplicities \( m_1, m_2, \ldots, m_N \), since from them we can rebuilt \( \tilde{W}_{k-1}(x) \), just as in (60). Then in step \( a_k \) we can replace \( \tilde{W}_{k-1}(x) \) by the sum
\[
\tilde{W}_{k-1}'(x) = \frac{1}{\prod_{i=1}^{2^{k-2}} (1 - x_i x_{2^{k-1}+1-i})} \sum_{i=1}^{N} m_i \tilde{W}_{S_i,T_i}(x)
\]
and, with a similar reasoning as for \( G_k(q) \), obtain
\[
W_k(q) = \sum_{i=1}^{N} m_i \left( \delta_{1,1+2^{k-1}} \cdots \delta_{2^{k-1},2^{k}} \tilde{W}_{S_i,T_i}(x) \right)^j_{x_j = qa, x_{j+2^{k-1}-1} = q/a} (1 - a^2)_{a \in A}.
\]

When working with \( W_5(q) \), we need an analogue of the collection of orbit representatives together with orbit sizes as in Example 5.2. Although Maple gives such a collection, we find it too complicated to be handled by Maple when using (65).

We find a way to avoid this problem. The idea is that in a formula like (60), the \( R_i \) need not be chosen to have combinatorial meanings. This is best illustrated by the \( k = 3 \) case. We can clearly see the advantage of orbit reduction in producing a compressed version of \( \tilde{W}_k(x) \). For \( k = 3 \), the \( B_3 \) decomposition will give 9 orbits with only 7 of them contributing to \( \tilde{W}_3(x) \). We thus get
\[
\tilde{W}_3^A(x) = \frac{1}{|B_3|} \sum_{g \in B_3} g \left( 9 \text{ monomials} + \frac{27 \text{ monomials}}{1 - x_1 x_4 x_6 x_7} \right).
\]
The actual formula is a little complicated and its combinatorial meaning is not significant, but it is good enough for us to use the divided difference algorithm to compute \( W_4(q) \). From this, by symmetrizing and re-choosing representatives, we obtain a simpler representative. Namely we end up obtaining that
\[
\tilde{W}_3^A(x) = \frac{1}{|B_3|} \sum_{g \in B_3} g \left( -1 + 3 x_2 x_6 - x_1 x_2 x_6 x_4 + \frac{2 - 6 x_1 x_7 - x_1^2 + 6 x_1 x_4^2 x_7 - x_1^2 x_4^2 x_7^2}{(1 - x_1 x_6 x_4 x_7)} \right),
\]
which can also be used in our divided difference algorithm. Originally we hoped that this formula would enable us to compute \( W_4(q) \) entirely by hand, but we were unable to do so.

For \( k = 4 \), directly using the \( B_4 \) decomposition gives us 62 orbits with 27 of them contributing to \( \tilde{W}_4(x) \). The representatives obtained this way are too complex for further computation since several of them have thousands of monomials in their numerators. The similar idea of symmetrizing and
re-choosing applies to give us 10 reasonably simple representatives for \( \tilde{W}_4(x) \), but typesetting them will take several pages. Nevertheless we are able to use them in the divided difference algorithm.

Having noticed that for \( k = 2, 3, 4 \) the divided difference algorithm reduced the computation of \( W_k(q) \) to a rather simple constant term evaluation, we tried to see what it gave for \( k = 5 \). Adding the contributions of these 10 representatives, before taking the constant term, yielded a rational function of the form

\[
\frac{1}{(1 - q^2)(1 - q^4)^4 (1 - q^6)(1 - a^4q^6)(1 - a^2q^2)^3 (1 - a^2q^4)^3} \\
\times \frac{357 \text{ monomials}}{(1 - q^4)^2 (1 - a^4q^4)^2 (1 - q^6)(1 - a^2q^6)(1 - a^4q^6)(1 - a^6q^6)}.
\]

It turns out that this is actually a rational function in \( q^2 \) and \( a^2 \). Replacing \( q \) by \( q^{1/2} \) and \( a \) by \( a^{1/2} \) and then taking constant term in \( a \), we can obtain \( W_5(q^{1/2}) \). Using this approach Maple can deliver \( W_5(q) \) in only about 5 minutes in total which is the shortest time we have been able to compute this series.

5.4. **Our first algorithm to obtain \( G_5(q) \) and \( W_5(q) \).** Before closing it will be worthwhile to include a description of the first algorithm that was used to obtain \( G_5(q) \) and \( W_5(q) \) since it contains another trick that clearly shows the flexibility afforded by the partial fraction algorithm in the computation of constant terms.

In this approach we begin by replacing our system \( S_k \) by a system \( S'_k \) which has the same cone of solutions. To describe the new system we will use the \( k \)-tuple of sets model. The idea is that originally we got \( S_k \) by equating the cardinality of each set to the cardinality of its complement obtaining

\[
S_k = \left\{ |A_1| = |^cA_1|, |A_2| = |^cA_2|, \ldots, |A_k| = |^cA_k| \right\}
\]

Now it is quite clear that this is equivalent to set

\[
S'_k = \left\{ |A_1| = d, |A_2| = d, \ldots, |A_k| = d, |^cA_1| = d \right\}
\]

For instance, using the binary digit indexing of the variables, for \( k = 3 \) this results in the following system of 4 equations in 9 unknowns

\[
\begin{align*}
p_{000} + p_{001} + p_{010} + p_{011} & = -d = 0 \\
p_{000} + p_{001} + p_{100} + p_{101} & = -d = 0 \\
p_{000} + p_{010} + p_{100} + p_{110} & = -d = 0 \\
p_{000} + p_{101} + p_{110} + p_{111} & = -d = 0
\end{align*}
\]

This given, our rational function \( G_3(q) = G_3(q,1) \) may be also obtained by taking the following constant term

\[
G_3(q,t) = \frac{1}{1 - qa_1a_2a_3} \frac{1}{1 - qa_1a_2} \frac{1}{1 - qa_1a_3} \frac{1}{1 - qa_1} \\
\frac{1}{1 - qa_2a_3a_4} \frac{1}{1 - qa_2a_4} \frac{1}{1 - qa_3a_4} \frac{1}{1 - qa_4} \frac{1}{1 - t/a_1a_2a_3a_4} \left| a_0^q a_2 a_3 a_4^t \right|.
\]

(66)

Here we choose the order \( q < t < a_1 < a_2 < \cdots \) and we can not set \( t = 1 \) as this moment yet.
Now it turns out to be expedient to start by eliminating $a_4$. This can simply be done by omitting the factor $1/(1 - t/a_1a_2a_3a_4)$ and making the substitution $a_4 \rightarrow t/a_1a_2a_3$, obtaining

$$G_3(q, t) = \frac{1}{1 - qa_1a_2a_3} \frac{1}{1 - qa_1a_2 - 1 - qa_1a_3} \frac{1}{1 - qa_1}$$

$$\frac{1}{1 - qt/a_1} \frac{1}{1 - qt/a_1a_3} \frac{1}{1 - qt/a_1a_2} \frac{1}{1 - qt/a_1a_2a_3} \bigg|_{a_1^0a_2a_3^0}.$$  

Setting $t = 1$ is valid here. Grouping terms containing the same subset of the variables $a_1, a_2, a_3$ gives

$$G_3(q) = \frac{1}{1 - qa_1} \frac{1}{1 - q/a_1}$$

$$\frac{1}{1 - qa_1a_2} \frac{1}{1 - q/a_1a_2}$$

$$\frac{1}{1 - qa_1a_3} \frac{1}{1 - q/a_1a_3}$$

$$\frac{1}{1 - qa_1a_2a_3} \frac{1}{1 - q/a_1a_2a_3} \bigg|_{a_1^0a_2a_3^0}.$$  

Likewise, we can easily see that the general form of (66) is

$$G_k(q, t) = \left( \prod_{S \subseteq [2, k]} \frac{1}{1 - qa_1A(S)} \right) \left( \prod_{S \subseteq [2, k]} \frac{1}{1 - qA(S)ak+1} \right) \frac{1}{1 - t/a_1a_2 \cdots ak+1} \bigg|_{a_1^0a_2^0 \cdots a_k^0}.$$  

Removing the last factor and setting $a_{k+1} = t/a_1a_2 \cdots a_k$ gives

$$G_k(q, t) = \left( \prod_{S \subseteq [2, k]} \frac{1}{1 - qa_1A(S)} \right) \left( \prod_{S \subseteq [2, k]} \frac{1}{1 - qtA(S)/a_1a_2 \cdots ak} \right) \bigg|_{a_1^0a_2^0 \cdots a_k^0}.$$  

and by setting $t = 1$ this can be rewritten as

$$G_k(q) = \left( \prod_{S \subseteq [2, k]} \frac{1}{1 - qa_1A(S)} \frac{1}{1 - q/a_1A(S)} \right) \bigg|_{a_1^0a_2^0 \cdots a_k^0}.$$  

Now comes the next trick: grouping terms according as $A(S)$ contains $a_2$ or not. This gives

$$G_k(q) = \left[ \prod_{S \subseteq [3, k]} \frac{1}{1 - qa_1A(S)} \right] \left[ \prod_{S \subseteq [3, k]} \frac{1}{1 - qa_1a_2A(S)} \right] \bigg|_{a_1^0a_2^0 \cdots a_k^0}.$$  

To appreciate the significance of this step let us see what this gives for $k = 3$. Grouping terms in (67) as was done in (68) gives

$$G_3(q) = \frac{1}{1 - qa_1} \frac{1}{1 - q/a_1} \frac{1}{1 - qa_1a_3} \frac{1}{1 - q/a_1a_3}$$

$$\frac{1}{1 - qa_1a_2} \frac{1}{1 - q/a_1a_2} \frac{1}{1 - qa_1a_2a_3} \frac{1}{1 - q/a_1a_2a_3} \bigg|_{a_2^0a_3^0}.$$  

Let us now see what the partial fraction algorithm gives if we first eliminate $a_2$. This entails computing the constant term

$$Q = \frac{1}{1 - qa_1a_2} \frac{1}{1 - q/a_1a_2} \frac{1}{1 - qa_1a_2a_3} \frac{1}{1 - q/a_1a_2a_3} \bigg|_{a_2^0}.$$
Using the terminology of [3], we note that the first and third factors are contributing and the other two are dually contributing. Thus,

\[ Q = \frac{A_1}{1 - qa_1a_2} + \frac{A_3}{1 - qa_1a_2a_3} \bigg|_{a_2^0} = A_1 + A_3 \]  

(70)

with

\[ A_1 = \frac{a_1^2a_3}{(a_1a_2 - q)(1 - qa_1a_2a_3)(a_1a_2a_3 - q)} \bigg|_{a_2 = 1/qa_1} = \frac{1}{(1 - q^2)(1 - a_3)(1 - q^2/a_3)} \\
A_3 = \frac{a_1^2a_3}{(1 - qa_1a_2)(a_1a_2 - q)(a_1a_2a_3 - q)} \bigg|_{a_2 = 1/qa_1a_3} = \frac{a_3}{(a_3 - 1)(1 - q^2a_3)(1 - q^2)} \]

Using (71) in (69) gives

\[ G_3(q) = \frac{1}{1 - qa_1} \frac{1}{1 - q/a_1} \frac{1}{1 - q/a_1} \frac{1}{1 - q/a_1} \bigg( A_1 + A_3 \bigg) \bigg|_{a_1^0a_3^0} \]

\[ = \frac{1}{1 - qa_1} \frac{1}{1 - q/a_1} \frac{1}{1 - q/a_1} \frac{1}{1 - q/a_1} \bigg( A_1 + A_3 \bigg) \bigg|_{a_1^0a_3^0} \cdot \]  

(71)

The last equality is due to the fact that \( A_1 \) and \( A_3 \) do not contain \( a_1 \). Next we will compute the constant term

\[ Q' = \frac{B_1}{1 - qa_1} + \frac{B_3}{1 - qa_1a_3} \bigg|_{a_2^0} = B_1 + B_3 \]

with

\[ B_1 = \frac{a_1^2a_3}{(a_1 - q)(1 - qa_1a_3)(a_1a_3 - q)} \bigg|_{a_1 = 1/q} = \frac{A_3}{(1 - q^2)(1 - a_3)(1 - q^2/a_3)} = A_1 \]

\[ B_3 = \frac{a_1^2a_3}{(1 - qa_1)(a_1 - q)(a_1a_2a_3 - q)} \bigg|_{a_2 = 1/qa_1a_3} = \frac{a_3}{(a_3 - 1)(1 - q^2a_3)(1 - q^2)} = A_3. \]

Thus (74) becomes

\[ G_3(q) = (A_1 + A_3)^2 \bigg|_{a_3^0} = A_1^2 \bigg|_{a_3^0} + A_3^2 \bigg|_{a_3^0} + 2A_1A_3 \bigg|_{a_3^0}. \]

It is easy to see that the same collapse of terms occurs in the general case. Indeed we can rewrite (69) in the form

\[ G_k(q) = \left[ \prod_{S \subseteq [3, k]} \frac{1}{1 - qa_1a_2A(S)} \frac{1}{1 - q/a_1a_2A(S)} \bigg|_{a_2^0} \right] \left[ \prod_{S \subseteq [3, \ldots, k]} \frac{1}{1 - qa_1A(S)} \frac{1}{1 - q/a_1A(S)} \bigg|_{a_1^0} \right]. \]

We can see that, in both constant terms with respect to \( a_1 \) and \( a_2 \), the first member of each pair of factors contributes and the second dually contributes, and the partial fraction algorithm yields

\[ \prod_{S \subseteq [3, \ldots, k]} \frac{1}{1 - qa_1a_2A(S)} \frac{1}{1 - q/a_1a_2A(S)} \bigg|_{a_2^0} = \sum_{T \subseteq [3, \ldots, k]} \frac{C_T}{1 - qa_1a_2A(T)} \bigg|_{a_2^0} = \sum_{T \subseteq [3, \ldots, k]} C_T \]
with

\[
C_T = \left(1 - qa_1a_2A(T)\right) \prod_{S \subseteq \{3, \ldots, k\}} \frac{1}{1 - qa_1a_2A(S)} \frac{1}{1 - q/a_1a_2A(S)} \bigg|_{a_2 = 1/qa_1A(T)}
\]

\[
= \frac{1}{(1 - q/a_1a_2A(T))} \prod_{S \subseteq \{3, \ldots, k\}} \frac{1}{1 - qa_1a_2A(S)} \frac{1}{1 - q/a_1a_2A(S)} \bigg|_{a_2 = 1/qa_1A(T)}
\]

\[
= \frac{1}{(1 - q^2)} \prod_{S \subseteq \{3, \ldots, k\}} \frac{1}{1 - A(S)/A(T)} \frac{1}{1 - q^2A(T)/A(S)}
\]

and we see that, as in the case \(k = 3\), all of these coefficients are independent of \(a_1\). Moreover we can also easily see that

\[
\left(1 - qa_1A(T)\right) \prod_{S \subseteq \{3, \ldots, k\}} \frac{1}{1 - qa_1A(S)} \frac{1}{1 - q/a_1A(S)} \bigg|_{a_1 = 1/qA(T)} = C_T.
\]

This reduces the computation of \(G_k(q)\) to the sum of \(2^{k-2} + \binom{2^{k-2}}{2}\) constant terms of the form

\[
G_k(q) = \sum_{i=1}^{2^{k-2}} A_i^2 \left| \frac{1}{a_3^{a_3} \cdots a_k^{a_k}} + 2 \sum_{1 \leq i < j \leq 2^{k-2}} A_iA_j \left| \frac{1}{a_3^{a_3} \cdots a_k^{a_k}} \right. \right.
\]

Note that for \(k = 5\) we are reduced to the calculation of \(2^3 + \binom{2^3}{2} = 36\) constant terms. Most importantly in each of these constant terms the denominators have at most 14 factors. The latest version of the partial fraction algorithm (whose update is motivated by the computation of \(G_5(q)\)) posted in the web site

[http://www.combinatorics.net.cn/homepage/xin/maple/ell2.rar](http://www.combinatorics.net.cn/homepage/xin/maple/ell2.rar)

computed these 36 constant terms on a Pentium 4 Windows system computer with a 3G Hz processor in about 22 minutes which is a considerable time reduction from the 2 hours and 15 minutes that took previous versions of the algorithm to compute these constant terms.

The same approach can be used to calculate \(W_5(q)\), but in a much simpler way. The constant terms have to be appropriately modified. Again we will start with the case \(k = 3\).

The \(k\)-tuple of sets interpretation of the constant term in \(26\) given in Section \(3\) yields that to obtain \(W_k(q)\) we must compute the constant terms corresponding to the \(2^k\) systems obtained by requiring each \(A_i\) to have 2 or 0 more elements than its complement in all possible ways and then carry out an inclusion exclusion type alternating sum of the results.

A moment reflection should reveal that to get \(W_3(q) = W_3(q, 1)\) we need only modify \(66\) to

\[
W_3(q, t) = \left((1 - a_4/a_1)(1 - 1/a_2)(1 - 1/a_3)\right) \times \frac{1}{1 - qa_1a_2a_3} \frac{1}{1 - q/a_1a_2a_3} \frac{1}{1 - q/a_1a_3} \frac{1}{1 - q/a_1} \times \frac{1}{1 - qa_2a_3a_4} \frac{1}{1 - qa_2a_4} \frac{1}{1 - qa_3a_4} \frac{1}{1 - qa_4} \frac{1}{1 - t/a_1a_2a_3} \left| \frac{1}{a_1^{a_1}a_2^{a_2}a_3^{a_3}} \right.
\]

\[
= \frac{1}{1 - 1/a_2 - 1/a_3 - a_4/a_1 + a_4/a_1a_2 + a_4/a_1a_3 + 1/a_2a_3 - a_4/a_1a_2a_3}.
\]

In fact expanding the first factor gives the 8 terms

\[1 - 1/a_2 - 1/a_3 - a_4/a_1 + a_4/a_1a_2 + a_4/a_1a_3 + 1/a_2a_3 - a_4/a_1a_2a_3.\]
And we see that the 8 constant terms obtained by expanding this factor in (72) correspond in order to the following 8 modified versions of $S_3^a$:

\[
\begin{array}{cccc}
|A_1| &=& d & |A_1| &=& d+1 & |A_1| &=& d \\
|A_2| &=& d & |A_2| &=& d & |A_2| &=& d+1 \\
|A_3| &=& d' & |A_3| &=& d' & |A_3| &=& d+1 \\
\end{array}
\]

Now the elimination of $a_4$ in (72) and then setting $t=1$ (as for $G_3(q)$) gives

\[
W_3(q) = \left( (1 - 1/a_1^2a_2a_3)(1 - 1/a_2)(1 - 1/a_3) \right) \times
\]

\[
\frac{1}{1-qa_1a_2a_3} \frac{1}{1-qa_1a_2} \frac{1}{1-qa_1a_3} \frac{1}{1-qa_2a_3} \frac{1}{1-qa_1} \frac{1}{1-q/a_1} \frac{1}{1-q/a_2} \frac{1}{1-q/a_3} \frac{1}{1-q/a_2a_3} \frac{1}{a_1^2a_2a_3}
\]

For general $k$, we are left to compute the constant term

\[
W_k(q) = \left( 1 - 1/a_1^2a_2 \cdots a_k \right) \prod_{i=2}^{k} (1 - 1/a_i) \left( \prod_{S \subseteq \{2,k\}} \frac{1}{1-q/a_1A(S)} \frac{1}{1-q/a_1A(S)} \right) |a_1^2a_2a_3|^{k-3}.
\]

Using this formula, the updated package will directly deliver $W_5(q)$ in about 17 minutes. This is because the factors in the numerator nicely cancel some of the denominators of the intermediate rational functions.

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