Moduli spaces of invariant metrics of positive scalar curvature on quasitoric manifolds

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We show that the higher homotopy groups of the moduli space of torus-invariant positive scalar curvature metrics on certain quasitoric manifolds are non-trivial.

1 Introduction

In recent years it became fashionable to study the homotopy groups of the space of Riemannian metrics of positive scalar curvature on a given closed, connected manifold and its moduli space, see for example the papers [BHSW10], [HSS14], [BERW14], [Wal14], [Wal13], [Wal11], [Wra16] and the book [TW15]. As far as the moduli space is concerned these results are usually only for the so-called observer moduli space of positive scalar curvature metrics, not for the naive moduli space.

The definition of the naive and the observer moduli space are as follows. The diffeomorphism group of a manifold $M$ acts by pullback on the space of metrics of positive scalar curvature on $M$. The naive moduli space of metrics of positive scalar curvature on $M$ is the orbit space of this action.

The observer moduli space of metrics is the orbit space of the action of a certain subgroup of the diffeomorphism group, the so-called observer diffeomorphism group. It consists out of those diffeomorphisms $\varphi$, which fix some point $x_0 \in M$ and whose differential $D_{x_0} \varphi : T_{x_0} M \to T_{x_0} M$ at $x_0$ is the identity.

This group does not contain any compact Lie subgroup and therefore acts freely on the space of metrics on $M$. Hence, the observer moduli space can be treated from a homotopy theoretic viewpoint more easily than the naive moduli space.

In this paper we deal with the equivariant version of the above problem: We assume that there is a torus $T$ acting effectively on the manifold and that all our metrics are invariant under this torus action. To be more specific we study invariant metrics on so-called torus manifolds and quasitoric manifolds.

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A torus manifold is a $2n$-dimensional manifold with a smooth effective action of an $n$-dimensional torus such that there are torus fixed points in the manifold. Such a manifold is called locally standard if it is locally weakly equivariantly diffeomorphic to the standard representation of $T = (S^1)^n$ on $\mathbb{C}^n$. If $M$ is locally standard, then the orbit space of the $T$-action on $M$ is naturally a manifold with corners. $M$ is called quasitoric if it is locally standard and $M/T$ is diffeomorphic to a simple convex polytope.

In this paper we use the following notations: Let $M$ be a compact manifold. For a compact connected Lie subgroup $G$ of $\text{Diff}(M)$ we denote by

- $\mathcal{R}(M,G)$ the space of $G$-invariant metrics on $M$.
- $\mathcal{R}^+(M,G)$ the space of $G$-invariant metrics of positive scalar curvature on $M$.
- $D(M,G) = N_{\text{Diff}(M)}(G)/G$ the normalizer of $G$ in $\text{Diff}(M)$ modulo $G$.
- $\mathcal{M}(M,G) = \mathcal{R}(M,G)/D(M,G)$
- $\mathcal{M}^+(M,G) = \mathcal{R}^+(M,G)/D(M,G)$.

With this notation our main result is as follows:

**Theorem 1.1 (Theorem 4.2)** There are quasitoric manifolds $M$ of dimension $2n$ such that for $0 < k < \frac{n}{2} - 7$, $n$ odd and $k \equiv 0 \mod 4$, $\pi_k(\mathcal{M}^+) \otimes \mathbb{Q}$ is non-trivial, where $\mathcal{M}^+$ is some component of $\mathcal{M}^+(M; T^n)$.

We also show that if a simple combinatorial condition on the orbit polytope of a quasitoric manifold $M$ is satisfied, then the above theorem holds for $M$. We believe that this condition holds for “most” quasitoric manifolds.

Note that $\mathcal{M}^+(M; T^n)$ is the analogue of the naive moduli space of metrics of positive scalar curvature in the equivariant situation and not the analogue of the observer moduli space for which so far most results have been proven.

Moreover, we think that the above theorem is the first step to understand the topology of the full naive moduli space of metrics of positive scalar curvature on quasitoric manifolds. This is because this moduli space is stratified by the rank of the isometry groups of metrics on $M$. The above theorem is a non-triviality result for the homotopy type of a minimal stratum of $\mathcal{M}^+(M; \{\text{Id}\})$. If one also has non-triviality results for all higher strata, one might expect non-triviality results for the full moduli space.

The idea of proof for Theorem 1.1 is similar to the ideas used in [BHSW10]: We show that $\mathcal{M}(M, T)$ is a rational model for the classifying space $BG$ for a certain subgroup $G$ of $D(M, T)$. The classifying map of an $M$-bundle with structure group $G$, total space $E$ and base $B$ is then given by $b \mapsto [g|_{E_b}]$, where $g$ is any $T$-invariant Riemannian metric on $E$ and $g|_{E_b}$ denotes the restriction of $g$ to the fiber over $b \in B$. The proof of the theorem is then completed by constructing a non-trivial bundle as above with $B = S^k$, such that there is a metric on $E$ whose restriction to any fiber has positive scalar curvature.

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2 The action of $D(M,T)$ on $\mathcal{R}(M,T)$ for $M$ a torus manifold

In this section we describe the action of $D(M,T)$ on $\mathcal{R}(M,T)$ where $M$ is a torus manifold. We give sufficient criteria for the rational homotopy groups of $\mathcal{M}(M,T)$ to be isomorphic to the rational homotopy groups of the classifying space of $D(M,T)$.

**Lemma 2.1** Let $M$ be a closed manifold. If $T$ is a maximal torus in $\text{Diff}(M)$, then the isotropy groups of the natural $D(M,T)$-action on $\mathcal{R}(M,T)$ are finite.

**Proof.** The isotropy group of the $D(M,T)$-action of an element $g \in \mathcal{R}(M,T)$ is the normalizer $W$ of the torus $T$ in the isometry group $K$ of $g$ modulo $T$. Since $M$ is compact $K$ is a compact Lie group. Moreover, because $T$ is a maximal torus of $K$, $W$ is the Weyl group of $K$ which is a finite group. Therefore the statement follows. \qed

For each torus manifold $M$ there is a natural stratification of the orbit space $M/T$ by the identity components of the isotropy groups of the orbits. That is, the open strata of $M/T$ are given by the connected components of $$(M/T)_H = \{ Tx \in M/T; (T_x)^0 = H \}$$ for connected closed subgroups $H$ of $T$. We call the closure of an open stratum a closed stratum. The closed strata are naturally ordered by inclusion. We denote by $\mathcal{P}$ the poset of closed strata of $M/T$.

There is a natural map $$\lambda: \mathcal{P} \to \{ \text{closed connected subgroups of } T \}$$ such that $\lambda((M/T)_H) = H$. We call $(\mathcal{P}, \lambda)$ the characteristic pair of $M$.

An automorphism of $(\mathcal{P}, \lambda)$ is a pair $(f, g)$ such that $f$ is an automorphism of the poset $\mathcal{P}$ and $g$ is an automorphism of the torus $T$ so that $\lambda(f(x)) = g(\lambda(x))$ for all $x \in \mathcal{P}$. The automorphisms of $(\mathcal{P}, \lambda)$ naturally form a group $\text{Aut}(\mathcal{P}, \lambda)$.

There is a natural action of $D(M,T)$ on $M/T$ which preserves the above stratification. Therefore $D(M,T)$ acts by automorphisms on the characteristic pair $(\mathcal{P}, \lambda)$.

**Lemma 2.2** Let $M$ be a torus manifold. Then there is a finite index subgroup $G$ of $D(M,T)$ which acts freely on $\mathcal{R}(M,T)$. To be more precise, $G$ is the kernel of the natural homomorphism $D(M,T) \to \text{Aut}(\mathcal{P}, \lambda) \subset \text{Aut}(\mathcal{P}) \times \text{Aut}(T)$, where $(\mathcal{P}, \lambda)$ is the characteristic pair associated to $M$.

**Proof.** Denote by $(\mathcal{P}, \lambda)$ the labeled face poset of $M/T$.

At first we show that $\text{Aut}(\mathcal{P}, \lambda)$ is a finite group. To see this note that $\text{Aut}(\mathcal{P})$ is finite because $\mathcal{P}$ is finite. Moreover, the natural map $\text{Aut}(\mathcal{P}, \lambda) \to \text{Aut}(\mathcal{P})$ has finite kernel, because $T$ is generated by the image of $\lambda$ (see [Wie14 Section 2] for details).

Let $G$ be the kernel of the natural map $D(M,T) \to \text{Aut}(\mathcal{P}, \lambda)$. Then $G$ has finite index since $\text{Aut}(\mathcal{P}, \lambda)$ is a finite group.
Let $T \subset H \subset \tilde{G}$ be a compact Lie group which fixes some metric $g \in R(M)$, where $\tilde{G}$ is the preimage of $G$ in $N_{\text{Diff}(M)}(T)$. Then each element of $H$ fixes every $x \in M^T$ and commutes with $T$. Hence, the differential of the $H$-action on $T_xM$ gives an injective homomorphism $H \to O(2n)$. Since $T$ is identified with a maximal torus of $O(2n)$ under this map, it follows that the centralizer of $T$ in $O(2n)$ is $T$ itself. Hence it follows that $H = T$. Therefore $G = \tilde{G}/T$ acts freely on $R(M,T)$.

\[\square\]

**Definition 2.3** A $2n$-dimensional quasitoric manifold $M$ is a locally standard torus manifold, such that $M/T$ is diffeomorphic to a simple convex $n$-dimensional polytope $P$. We denote by $\pi : M \to P$ the orbit map.

Similarly to the automorphism group of the characteristic pair $(P, \lambda)$ we define the group $\text{Diff}(M/T, \lambda) \subset \text{Diff}(M/T) \times \text{Aut}(T)$ of those pairs $(f, g) \in \text{Diff}(M/T) \times \text{Aut}(T)$ with $\lambda(f(x)) = g(\lambda(x))$ for all $x \in M/T$. Then we have the following lemma.

**Lemma 2.4** For a quasitoric manifold, the group $D(M,T)$ is naturally isomorphic to $(\text{C}^\infty(M/T,T)/T) \times \text{Diff}(M/T, \lambda)$ as topological groups.

In particular the group $G$ of the previous lemma is homotopy equivalent to the group of all diffeomorphisms of $M/T$, which leave all faces of $M/T$ invariant.

**Proof.** First we show that the kernel of the natural map $\varphi : D(M,T) \to \text{Diff}(M/T, \lambda)$ is isomorphic to $C^\infty(M/T,T)/T$. Since $T$ is abelian, there is a natural map from $C^\infty(M/T,T)/T$ to the kernel of $\varphi$ which is induced by the map $C^\infty(M/T,T) \to N_{\text{Diff}(M)}(T)$ with $f \mapsto F$, where $F(x) = f(Tx)x$ for $x \in M$.

We show that this map is a homeomorphism. To do so, let $F \in N_{\text{Diff}(M)}(T)$, such that $FT \in \ker \varphi$. Then $F$ leaves all $T$-invariant subsets of $M$ invariant. Since $M$ is quasitoric, there is a covering of $M$ by open invariant subsets $U_1, \ldots, U_k$ which are weakly equivariantly diffeomorphic to $\mathbb{C}^n$ with the standard $T$-action.

The restriction of $F$ to $U_j \cong \mathbb{C}^n$ is of the form

\[F(z_1, \ldots, z_n) = (z_1 f_1(z_1, \ldots, z_n), \ldots, z_n f_n(z_1, \ldots, z_n)),\]

where $(z_1, \ldots, z_n) \in \mathbb{C}^n$ and $f_k(z_1, \ldots, z_n) \in S^1$ for $k = 1, \ldots, n$ depends only on $(|z_1|^2, \ldots, |z_n|^2)$.

We have to show that $f_k$ is smooth for all $k$.

Smoothness in points with $z_k \neq 0$ follows from the smoothness of $F$. We show that $f_k$ is also smooth in points with $z_k = 0$.

Since $F$ is smooth, we have for $(z_1, \ldots, z_n) \in \mathbb{C}^n$,

\[z_k f_k(z_1, \ldots, z_n) = F_k(z_1, \ldots, z_n) = \int_0^1 (D_{z_k} F_k(z_1, \ldots, z_{k-1}, 0, z_{k+1}, \ldots, z_n))(z_k) \, dt,\]

where

\[(D_{z_k} F_k(z_1, \ldots, z_n))(z) = \left(\frac{\partial F_k}{\partial x_k}(z_1, \ldots, z_n), \frac{\partial F_k}{\partial y_k}(z_1, \ldots, z_n)\right)(x, y)^t\]
with \( z_l = x_l + iy_l \) for \( l = 1, \ldots, n \) and \( z = x + iy, x_l, x, y_l, y \in \mathbb{R} \).

Since \( F \) is \( T \)-equivariant, it follows that

\[
\begin{align*}
  z_k f^k(z_1, \ldots, z_n) &= \int_0^1 (Dz_k F^k(z_1, \ldots, z_k-1, z_k t, z_{k+1}, \ldots, z_n)) (z_k) \, dt \\
  &= \int_0^1 z_k (Dz_k F^k(z_1, \ldots, z_k-1, |z_k| t, z_{k+1}, \ldots, z_n) (1) \, dt \\
  &= z_k \int_0^1 (Dz_k F^k(z_1, \ldots, z_k-1, |z_k| t, z_{k+1}, \ldots, z_n) (1) \, dt.
\end{align*}
\]

Since \( F \) is \( T \)-equivariant, it follows that \( t \mapsto (Dz_k F^k(z_1, \ldots, z_k-1, t, z_{k+1}, \ldots, z_n)) (1) \), \( t \in \mathbb{R} \), is an even function. Therefore the integrand in the last integral depends smoothly on \( (z_1, \ldots, z_n) \) and \( f_k \) is smooth everywhere. Because \( f_k \) is \( T \)-invariant, it induces a smooth map on the orbit space, whose derivatives depend continuously on the derivatives of \( F \).

Hence it is sufficient to show that there is a section to \( \varphi \).

There is a canonical model \( M = ((M/T) \times T)/\sim \), where \( (x, t) \sim (x', t') \) if and only if \( x = x' \) and \( t' t^{-1} \in \lambda(x) \). Therefore every \( (f, g) \in \text{Diff}(M/T, \lambda) \) of \( M/T \) lifts to a homeomorphism of \( M \) given by \( f \times g \). One can show (see [GP13, Lemma 2.3]), that this homeomorphism is actually a diffeomorphism. Therefore we have a section of \( \varphi \) and the first statement follows.

The second statement follows because \( H = C^\infty(M/T, T)/T \) is contractible because \( M/T \) is contractible.

\[
\square
\]

**Lemma 2.5** If in the situation of Lemma 2.2, \( M \) is quasitoric and the natural homeomorphism \( \text{Aut}(P, \lambda) \to \text{Aut}(P) \) is trivial, then \( \pi_k(M(M, T)) \otimes \mathbb{Q} \cong \pi_k(BD(M, T)) \otimes \mathbb{Q} \) for \( k > 1 \).

**Proof.** Since \( G \) acts freely and properly on \( R(M, T) \), it follows from Ebin’s slice theorem [Ebi70] (see also [Bou75]) that \( R(M, T) \to R(M, T)/G \) is a locally trivial fiber bundle. Because \( R(M, T) \) is contractible, \( R(M, T)/G \) is weakly homotopy equivalent to \( BG \).

Let \( H \) be as in the proof of the previous lemma. Then \( H \) is contractible. Hence it follows that \( R(M, T) \) and \( R(M, T)/H \) are weakly homotopy equivalent.

It follows from Ebin’s slice theorem that all \( H \)-orbits in \( R(M, T) \) are closed. Since there is a \( D(M, T) \)-invariant metric on \( R(M, T) \), it follows that \( R(M, T)/H \) is metrizable. Hence, \( R(M, T)/H \) is paracompact and completely regular.

The \( D(M, T) \)-invariant metric on \( R(M, T) \) can be constructed as follows. Ebin constructs in his paper a sequence of Hilbert manifolds \( R^s \), \( s \in \mathbb{N} \), such that \( R(M, \{\text{Id}\}) = \bigcap_{s \in \mathbb{N}} R^s \). On each \( R^s \) he constructs a \( \text{Diff}(M) \)-invariant Riemannian structure. This structure induces a \( \text{Diff}(M) \)-invariant metric \( d^s \) on \( R^s \). The restrictions of all these metrics \( d^s \) to \( R(M, \{\text{Id}\}) \) together induce the \( C^\infty \)-topology on \( R(M, \{\text{Id}\}) \). Therefore the metric

\[
d(x, y) = \sum_{s \in \mathbb{N}} \min\{d^s(x, y), 2^{-s}\}
\]
is \( \text{Diff}(M) \)-invariant and induces the \( C^\infty \)-topology on \( R(M, \{1\}) \).

Since \( \text{Aut}(P, \lambda) \to \text{Aut}(P) \) is trivial, there is a splitting \( \psi : \text{Aut}(P, \lambda) \to D(M, T) \).

Here an element \( \tau = (g, f) \in \text{Aut}(P, \lambda) \) acts on \( M = ((M/T) \times T)/\sim \) as identity on the first factor and by \( f \in \text{Aut}(T) \) on the second. To see that this is a diffeomorphism of \( M \), we note that there are invariant charts \( U \subset M \) which are weakly equivariantly diffeomorphic to \( \mathbb{C}^n \) such that \( U \cap ((M/T) \times (\mathbb{Z}_2)^n))/\sim \) is mapped to \( \mathbb{R}^n \subset \mathbb{C}^n \). For a construction of such charts see \([GP13, \text{Section 2}]\). The action of \( \tau \) in this chart is given by complex conjugation on some of the factors of \( \mathbb{C}^n \).

Note that \( H \) and \( G \) are normalized by \( \text{im} \psi \). Moreover, \( \text{im} \psi \) commutes with \( G/H \) in \( D(M, T)/H \).

Since \( H_1 = \langle T, \text{im} \psi \rangle \) is a compact Lie subgroup of \( \text{Diff}(M) \), there is an \( H_1 \)-invariant metric on \( M \).

Therefore it follows from \([Bre72, \text{Chapter II.6}]\) that \( R(M, T)/H/\text{im} \psi \) is simply connected. Moreover, by \([Bre72, \text{Theorem III.7.2}]\), one sees that \( R(M, T)/H/\text{im} \psi \) is rationally acyclic.

Hence, by the Whitehead theorem, all rational homotopy groups of \( R(M, T)/H/\text{im} \psi \) vanish.

By Lemma \([22] \) we know that the identity components of \( G \) and \( D(M, T) \) are the same. Therefore the higher homotopy groups of \( BG \) and \( BD(M, T) \) are naturally isomorphic. Therefore, by Lemma \([24] \) and Ebin’s slice theorem, it now suffices to show that \( G/H \) acts freely on \( R(M, T)/H/\text{im} \psi \).

Let \( g \in R(M, T), h_1 \in G, h_2 \in H \) such that \( h_1 g = \tau h_2 g \) with \( \tau \in \text{im} \psi \). Then we have

\[
\tau^{-1} h_1 g = \tau^{-1} \tau h_2 g = h_2 g.
\]

Since the isotropy group of \( g \) in \( D(M, T) \) is finite, it follows that \( \tau^{-1} h_1 \) has finite order in \( D(M, T)/H \).

In particular, \( h_1 \) induces a diffeomorphism of finite order \( m \) on \( M/T \) which leaves all faces of \( M/T \) invariant because \( \tau \) induces the identity on this space. Since the principal isotropy group of a \( \mathbb{Z}_m \)-action on a manifold with boundary is equal to the principal isotropy group of the restricted action on the boundary, it follows by induction on the dimension of the faces of \( M/T \) that the diffeomorphism induced by \( h_1 \) on \( M/T \) is trivial. This means that \( h_1 \) is contained in \( H \) and the lemma is proved.

The proof of the above lemma also shows that the bundle \( G/H \to R(M, T)/H/\text{im} \psi \to \mathcal{M}(M, T) \) is (rationally) a classifying bundle for principal \( G/H \)-bundles. We shall describe the classifying map for bundles \( M \to E \to B \) with structure group \( G/H \) and fiber \( M \) where \( E \) and \( B \) are manifolds. Since \( G/H \) is a subgroup of the group of \( T \)-equivariant diffeomorphisms of \( M \) the \( T \)-action on each fiber extends to an \( T \)-action on \( E \). Hence \( E \) is a \( T \)-manifold and we may choose a \( T \)-invariant Riemannian metric \( g \) on \( E \). If \( E = B \times M \) is trivial, we therefore have a map

\[
E = B \times M \to R(M, T)/H/\text{im} \psi \times M \quad (b, x) \mapsto ([g|_{E_b}], x),
\]

where \( g|_{E_b} \) denotes the restriction of \( g \) to the fiber of \( E \) over \( b \in B \).
If $E$ is only locally trivial, we still get a map

$$E \to (\mathcal{R}(M,T)/H/\text{im } \psi) \times_{G/H} M$$

where on the right-hand side we take the quotient of the diagonal $G/H$ action. This map makes the following diagram into a pull-back square,

$$
\begin{array}{ccc}
E & \to & (\mathcal{R}/H/\text{im } \psi) \times_{G/H} M \\
\downarrow & & \downarrow \\
B & \to & M(M,T)
\end{array}
$$

where the bottom map, given by $b \mapsto [g|_{E_b}]$, is the classifying map for the bundle $E$.

**Example 2.6** We give an example of quasitoric manifolds satisfying the assumptions of the previous lemma.

Let $M_0$ be the projectivization of a sum of $n-1$ complex line bundles $E_0, \ldots, E_{n-2}$ over $\mathbb{C}P^1$, such that $c_1(E_0) = 0$ and the first Chern classes of the other bundles are non-trivial, not equal to one and pairwise distinct. Then $M_0$ is a generalized Bott manifold and in particular a quasitoric manifold over $I \times \Delta^{n-2}$, where $I$ is the interval and $\Delta^{n-2}$ denotes an $n-2$-dimensional simplex.

Let $M_1 = \mathbb{C}P^1 \times M_0$ and $M_2$ the blow up of $M_1$ at a single point. The orbit space of $M_1$ is $I \times I \times \Delta^{n-2}$. The orbit space of $M_2$ is the orbit space of $M_1$ with a vertex cut off.

The combinatorial types of the facets of $M_2/T$ are given as in table 1 below. Since the combinatorial types of facets in the lines in this table are pairwise distinct, it follows that the lines in the table are invariant under the action of $\text{Aut}(\mathcal{P}, \lambda)$. Therefore the facets in the first two lines are fixed by the action of this group. The facets in lines 3 and 4 are fixed, because in each of these lines there appears one facet $F$ with $\lambda(F) = \{(z,1,\ldots,1) \in T^n; z \in S^1\}$ but the values of $\lambda$ on the other facets are distinct.

Finally the facets $F_1, \ldots, F_{n-2}$ in the last line are fixed, by all $(f,g) \in \text{Aut}(\mathcal{P}, \lambda)$ because $g$ must permute the subgroups $\lambda(F_1), \ldots, \lambda(F_{n-2})$, which are the coordinate subgroups in $\{(1,1)\} \times (S^1)^{n-2}$, and must also fix the subgroups $\lambda(F')$ with $F'$ from line 3.

Note that depending on the choices of the bundles $E_0, \ldots, E_{n-2}$, $M_2$ can be spin or non-spin.

### 3 The homotopy groups of $D(M, T)$ for $M$ a quasitoric manifold

In this section we show that, for quasitoric manifolds of dimension $2n$, $n$ odd, the rational homotopy groups of $D(M, T)$ are non-trivial in certain degrees.

Now let $M$ be a quasitoric manifold with orbit polytope $\mathcal{P}$. 

7
2 \times \Delta^{n-2} \times I \times \Delta^{n-3} \times I \times I \times \Delta^{n-3} \times I 
\begin{array}{c|c|c}
\text{combinatorial type} & (\alpha_1, \ldots, \alpha_n) \\
\hline
1 & \Delta^{n-1} & (1, \ldots, 1) \\
1 & I \times I \times \Delta^{n-3} & (0, 0, 1, \ldots, 1) \\
2 & I \times \Delta^{n-2} & (1, 0, 0, \ldots, 0) \\
& & (0, 1, k_1, \ldots, k_{n-2}) \\
& & \text{with } k_i \text{ pairwise distinct and non-zero} \\
2 & I \times \Delta^{n-2} \text{ with vertex cut off} & (0, 1, 0, \ldots, 0) \\
& & (0, 1, 0, \ldots, 0) \\
n - 2 & I \times I \times \Delta^{n-3} \text{ with vertex cut off} & (0, 0, 0, \ldots, 0, 1, 0, \ldots, 0) \\
\end{array}

Table 1: The combinatorial types of the facets of $M_2/T$. In the first column the numbers of facets of these type are given. In the last column the values of $\lambda(F) = \{(z_1, \ldots, z_n) \in T^n; z \in S^1\}$ are given.

Let $D^n \hookrightarrow P$ be an embedding into the interior of $P$ such that $K = P - D^n$ is a collar of $P$. Then we have a decomposition

$$M = (D^n \times T^n) \cup \pi^{-1}(K) = (D^n \times T^n) \cup N.$$ 

From this decomposition we get a homomorphism $\psi : \text{Diff}(D^n, \partial D^n) \to G/H \hookrightarrow \text{Diff}(M, T^n)$ by letting a diffeomorphism of $D^n$ act on $M$ in the natural way on $D^n$ and by the identity on $T^n$ and $N$.

Note that the natural map $\text{Diff}(D^n, \partial D^n) \to \text{Diff}(P)$ factors through $\psi$.

**Lemma 3.1** For $0 < k < \frac{n}{6} - 8$, $n$ odd and $k \equiv -1 \mod 4$. The natural map

$$\pi_k(\text{Diff}(D^n, \partial D^n)) \otimes \mathbb{Q} \to \pi_k(\text{Diff}(P)) \otimes \mathbb{Q}$$

is injective and non-trivial. In particular $\psi$ induces an injective non-trivial homomorphism on these homotopy groups.

**Proof.** We have exact sequences

$$1 \to \text{Diff}(D^n, \partial D^n) \to \widetilde{\text{Diff}}(P) \to \text{Diff}(K),$$

where $\widetilde{\text{Diff}}(P)$ is the group of diffeomorphisms of $P$ which preserve $K$, and

$$1 \to \text{Diff}(K, \partial D^n) \to \text{Diff}(K) \to \text{Diff}(\partial D^n).$$

Note that $\widetilde{\text{Diff}}(P)$ is weakly homotopy equivalent to $\text{Diff}(P)$, by the uniqueness of collars up to isotopy. Moreover, the images of the right-hand maps in the above sequences have finite index.

In the first sequence this is because the group of those diffeomorphisms of a sphere which extend to diffeomorphisms of the disc has finite index in all diffeomorphisms of the sphere.
For the second sequence one can argue as follows to see that $\text{Diff}(K) \to \text{Diff}(\partial D^n)$ is surjective. Since $K$ is homeomorphic to $\partial D^n \times I$ and every diffeomorphism $\varphi$ of $\partial D^n$ is isotopic to a diffeomorphism of $\partial D^n$ which is the identity on some big embedded disc $D^{n-1} \subset \partial D^n$, $\varphi$ can be extend to a homeomorphism $\phi$ of $K$ which satisfies:

- $\phi$ is a diffeomorphism on $\partial D^n \times [0, \frac{1}{2}] \cup \hat{F} \times [0, 1]$.
- $\phi$ is the identity on a neighborhood of $(\partial P - \hat{F}) \times [\frac{1}{2}, 1]$.

Hence $\phi$ is a diffeomorphism of $K$.

Therefore we get exact sequences of rational homotopy groups

$$\pi_{k+1}(\text{Diff}(P)) \otimes \mathbb{Q} \to \pi_{k+1}(\text{Diff}(K)) \otimes \mathbb{Q} \to \pi_k(\text{Diff}(D^n, \partial D^n)) \otimes \mathbb{Q} \to \pi_k(\text{Diff}(P)) \otimes \mathbb{Q}$$

and

$$\pi_{k+1}(\text{Diff}(K, \partial D^n)) \otimes \mathbb{Q} \to \pi_{k+1}(\text{Diff}(K)) \otimes \mathbb{Q} \to \pi_{k+1}(\text{Diff}(\partial D^n)) \otimes \mathbb{Q}.$$ 

By Farrell and Hsiang [FH78], we have $\pi_{k+1}(\text{Diff}(\partial D^n)) \otimes \mathbb{Q} = 0$.

Moreover every family of diffeomorphisms of $K$ which lies in the image of $\pi_{k+1}(K, \partial D^n)$ extends to a family of diffeomorphisms of $P$, by defining the extension to be the identity on $D^n$.

Therefore the map $\pi_{k+1}(\text{Diff}(K)) \otimes \mathbb{Q} \to \pi_k(\text{Diff}(D^n, \partial D^n)) \otimes \mathbb{Q}$ is the zero map and the claim follows from Farrell and Hsiang [FH78].

\[\square\]

4 $\pi_k(M^+)$ is non-trivial

In this section we show that $\pi_k(M^+(M, T))$ is non-trivial for manifolds as in Example 2.6.

To do so, we need the following theorem which is an equivariant version of Theorem 2.13 of [Wal11].

**Theorem 4.1** Let $G$ be a compact Lie group. Let $X$ be a smooth compact $G$-manifold of dimension $n$ and $B$ a compact space. Let $B = \{ g_b \in R^+(X, G) : b \in B \}$ be a continuous family of invariant metrics of positive scalar curvature. Moreover, let $\iota : G \times_H (S(V) \times D_1(W)) \to X$ be an equivariant embedding, with $H \subset G$ compact, $V, W$ orthogonal $H$-representations with $\dim G - \dim H + \dim V + \dim W = n + 1$ and $\dim W > 2$. Here $S(V)$ and $D_1(W)$ denote the unit sphere and the unit disc in $V$ and $W$, respectively.

Finally let $g_{G/H}$ be any $G$-invariant metric on $G/H$ and $g_V$ be any $H$-invariant metric on $S(V)$.

Then, for some $1 > \delta > 0$, there is a continuous map

$$B \to R^+(X, G)$$

$$b \to g^b_{\text{id}}$$

satisfying
1. Each metric $g_{std}^b$ makes the map $G \times_H (S(V) \times D_\delta(W)) \to (G/H, g_{G/H})$ into a Riemannian submersion. Each fiber of this map is isometric to $(S(V) \times D_\delta(W), g_V + g_{tor})$, where $g_{tor}$ denotes a torpedo metric on $D_\delta(W)$. Moreover $g_{std}^b$ is the original metric outside a slightly bigger neighborhood of $G \times_H (S(V) \times \{0\})$.

2. The original map $B \to \mathbb{R}^+(X, G)$ is homotopic to the new map.

The proof of this theorem is a direct generalization of the proof of Theorem 2.13 of [Wal11] using the methods of the proof of Theorem 2 in [Han08]. Therefore we leave it to the reader.

Let $E$ be the total space of a Hatcher disc bundle [Goe01] over $S^k$ and fiber $D^n$ with structure group $\text{Diff}(D^n, \partial D^n)$, that is a disc bundle over $S^k$ such that its classifying map $S^k \to B \text{Diff}(D^n, \partial D^n)$ represents a non-trivial element in $\pi_k(B \text{Diff}(D^n, \partial D^n))$.

Moreover, let

$$F = (E \times T^n) \cup (S^k \times N)$$

with $N$ as in the previous section. Let $M_1 \subset N$ be a characteristic submanifold and denote by $M_1$ a small equivariant tubular neighborhood of $M_1$. Then $F$ is a bundle over $S^k$ with fiber the quasitoric manifold $M$ and structure group $\text{Diff}(D^n, \partial D^n)$. Note that $F$ has a natural fiberwise $T^n$-action.

By Theorem 2.9 of [BHSW10], we have a metric on $E$ with fiberwise positive scalar curvature. Together with an invariant metric of non-negative scalar curvature on $T^n \times D^2$ we get from this metric an invariant metric $g$ on $\partial(E \times T^n \times D^2)$ with fiberwise positive scalar curvature, i.e. the restriction to any fiber of the bundle has positive scalar curvature.

On $(N - M_1) \times D^2$, there is an equivariant Morse function without critical orbits of coindex less than three. The global minimum of this Morse function is attained on $(\partial N) \times D^2$. Using this Morse function and Theorem 1.1 we get a fiberwise invariant metric of positive scalar curvature on $\partial((F - (S^k \times M_1)) \times D^2)$.

Indeed, using the Morse function, we get an equivariant handle decomposition of $(N - M_1) \times D^2$, without handles of codimension less than three. Moreover, the restriction of the bundle $\partial(E \times T^n \times D^2) \to S^k$ to $\partial(E) \times T^n \times D^2$ is trivialized by assumption. So the restriction of $g$ to the fibers of this bundle gives a compact family of invariant metrics of positive scalar curvature on $\partial(N) \times D^2$. Therefore, by Theorem 1.1 we can do equivariant surgery on $(\partial N) \times D^2$ to get a fiberwise invariant metric of positive scalar curvature on $\partial((F - (S^k \times M_1)) \times D^2)$.

Note that Berard Bergery’s result [BB83] on the existence of a metric of positive scalar curvature on the orbit space of a free torus action, generalizes directly to a family version. This is because Berard Bergery shows that if $g$ is an invariant metric of positive scalar curvature on a free $S^1$-manifold $M$, then $f^{2/\dim M-2} \cdot g^*$ has positive scalar curvature, where $g^*$ is the quotient metric of $g$ and $f$ is the length of the $S^1$-orbits in $M$. This construction clearly generalizes to families of metrics. Moreover the metrics on the orbit space will be invariant under every Lie group action which is induced on $M/S^1$ from an action on $M$ which commutes with $S^1$ and leaves the metrics on $M$ invariant (see [Wie16] Theorem 2.2) for the case of a single metric).
Moreover, note that $F$ is the orbit space of the free action of the diagonal in $\lambda(M_1) \times S^1$ on $\partial((F - (S^k \times M_1)) \times D^2)$, where $S^1$ is the circle group which acts by rotation on $D^2$.

Hence, with the remarks from above one gets an invariant metric of fibrewise positive scalar curvature on $F$ in the same way as in the case of a single metric (see [Wie16, Proof of Theorem 2.4] for details).

This metric defines an element $\gamma$ in $\pi_k(M^+(M,T)) \otimes \mathbb{Q}$. The image of $\gamma$ in

$$\pi_k(M(M,T)) \otimes \mathbb{Q} \cong \pi_k(BD(M,T)) \otimes \mathbb{Q}$$

is represented by the classifying map for our Hatcher bundle $E$.

Therefore it follows from the lemmas in the previous two sections, that $\gamma$ is non-trivial if $M$ is as in example 2.6 because the classifying map of a Hatcher bundle represents a non-trivial element in the homotopy groups of $B\text{Diff}(D^n, \partial D^n)$.

Therefore we have proved the following theorem:

**Theorem 4.2** Let $M$ be a quasitoric manifold of dimension $2n$ such that $\text{Aut}(\mathcal{P}, \lambda) \rightarrow \text{Aut}(\mathcal{P})$ is trivial. Then for $0 < k < \frac{n}{2} - 7$, $n$ odd and $k \equiv 0 \mod 4$, $\pi_k(M^+) \otimes \mathbb{Q}$ is non-trivial, where $M^+$ is some component of $M^+(M; T^n)$.

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