On the concept of mass point in general relativity

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Summary. – The correct characterization of the concept of mass point in general relativity is a straightforward consequence of the original form of solution given by Schwarzschild to the problem of the Einstein field of a material point.

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1. − It is commonly believed that the concept of mass point in general relativity is implicitly defined by the Kruskal-Szekeres form of solution to the Schwarzschild problem [1]. I think, on the contrary, that the only appropriate definition of the above concept is given by Schwarzschild’s original form of solution to the homonymous problem.

Here are my arguments.

2. − The static solution of the static problem (“Schwarzschild problem”) of the Einstein gravitational field generated by a point mass $M$, at rest, is given by the following expression of the space-time interval:

$$
\text{(2.1)} \quad ds^2 = \left[ 1 - \frac{2m}{f(r)} \right] c^2 dt^2 - \left[ 1 - \frac{2m}{f(r)} \right]^{-1} \left[ df(r) \right]^2 - f^2(r) \left[ d\theta^2 + \sin^2 \theta \, d\phi^2 \right] ;

(r > 0; \ 0 \leq \theta \leq \pi; \ 0 \leq \phi < 2\pi) ,
$$
where: \( m \equiv G M c^2 \); \( G \) is the gravitational constant; \( f(r) \) is any regular function of \( r \) such that \( ds^2 \) is Minkowskian at the spatial infinite \([2]\).

If we put \( f(r) \equiv r \), we obtain the well-known standard form of solution, which is due to Hilbert, Droste, and Weyl. It is generally called “Schwarzschild solution”, but in reality the actual Schwarzschild’s form of solution \([3]\) follows from (2.1) by putting:

\[
(2.2) \quad f(r) \equiv \left[ r^3 + (2m)^3 \right]^{1/3};
\]

Schwarzschild’s \( ds^2 \), which holds for \( r > 0 \), is diffeomorphic to the “exterior” part \( r>2m \) of the standard \( ds^2 \).

Another interesting form, valid for \( r > 0 \), can be obtained with the choice

\[
(2.3) \quad f(r) \equiv r + 2m;
\]

the corresponding \( ds^2 \) is diffeomorphic to Schwarzschild’s.

3. – The elementary interval of Kruskal-Szekeres \([1]\) is, with slight changes of notations,

\[
(3.1) \quad d\sigma^2 = F^2 (-dv^2 + du^2) + r^2 (d\theta^2 + \sin^2 \theta \, d\varphi^2),
\]

where \( u \) and \( v \) are suitable functions of \( r \) and \( t \), and

\[
(3.2) \quad u^2 - v^2 = \left( \frac{r}{2m} - 1 \right) \exp \left( \frac{r}{2m} \right);
\]

\[
(3.3) \quad F^2 = \left( 32m^3 / r \right) \exp \left( - \frac{r}{2m} \right) = \text{a transcendental function of } u^2 - v^2.
\]

From the physical standpoint, formula (3.1) has several drawbacks, as it was pointed out by many authors. I shall emphasize here the following four defects.

First of all, (3.1) yields a non-static Einstein field as a solution of a static problem: this is rather baroque. And also baroque is the circumstance that each point of the
standard form is represented twice by the Kruskal-Szekeres form, while the elliptical reduction would encounter insuperable difficulties. Then, the derivatives $\partial u/\partial r$ and $\partial v/\partial r$ are singular at $r=2m$: in other terms, the singularity $r=2m$ of the standard interval has been “incorporated” in the differentials $du$ and $dv$: this means that, rigorously speaking, eq. (3.1) is not a proper extension of the standard form. Finally, at $r=0$, i.e. for $v^2 - u^2 = 1$, the Kruskal-Szekeres solution is singular, but the locus $v^2 - u^2 = 1$ is space-like, and therefore cannot represent a material structure [4]. Accordingly, eq. (3.1) does not characterize adequately the concept of mass point.

4. – In 1923 Marcel Brillouin proved in a detailed way a result which is actually rather intuitive, i.e. that Schwarzschild’s original form, and the analogous form obtained by substituting (2.3) in (2.1), are maximally extended – and that the standard solution is valid only for $r>2m$ [5].

The final section of his paper is as follows (I change only the notations):

“The conclusion seems to me inescapable: the limit $r=0$ [of the forms obtained with the choices (2.2), (2.3)] is insurmountable; it embodies the material singularity. The distance $d$ from this origin ($r=0$) to a point with co-ordinate radius $r$, calculated along a radius vector ($\theta = \text{const.}; \varphi = \text{const.}$) is [for the form corresponding to (2.3)]

$$d = \sqrt{r(r+2m)} + m \ln \frac{r + m + \sqrt{r(r+2m)}}{m}.$$  

The ratio between the circumference $2\pi(r+2m)$ and the [co-ordinate] radius is everywhere larger than $2\pi$; in particular at the origin ($d=0; r=0$) this ratio becomes
Infinite. [...]. It is this singularity \( r=0 \) that constitutes what physics calls the *material point* [...]. (From the English version quoted in [5]).

We can only add that the curvature invariants of the above singularity at \( r=0 \) have finite values.

5. – There is another and very physical proof that Schwarzschild’s original form [3] – or the form corresponding to (2.3) – imply the only reasonable definition of the concept of point mass in general relativity

As is well known, in a second basic work [6] Schwarzschild solved the problem of the Einstein field generated by a sphere of an incompressible and homogeneous fluid. Now, if we calculate, through a suitable limit procedure, the field of a mass point from the field of the fluid, we find again the result (2.1) – (2.2), see [7]. This is clearly a convincing demonstration of the physical adequacy of Schwarzschild’s *original* concept of material point.

“Man kann nicht immer zusammen stehn, 
Am wenigsten mit großen Haufen.”

J.W. v. Goethe

APPENDIX

**On space-time singularities**

The space-time singularities can be classified in two different ways: according to mathematical or physical criteria.

The mathematical classifications are scarcely interesting for physical aims, and therefore I shall not discuss them. Physically, we have true (physical) and false (nonphysical) singularities. The true singularities are time-like, or light-like, loci in which mass-energy is present.
Examples. — The singularity \( r=2m \) of the standard form \( f(r) \equiv r \) in (2.1) is a nonphysical singularity. The singularity \( \nu^2 - \alpha^2 = 1 \) of the Kruskal-Szekeres form [1] is nonphysical because space-like. The singularities \( r=0 \) of Schwarzschild’s original form [3] and of the form investigated by Brillouin [5] are physical singularities: the loci \( r=\varepsilon \), with \( \varepsilon \) arbitrarily small, are time-like — and the correspondence with Newton’s theory tells us that there is actually matter at \( r=0 \).

I emphasize that the Kruskal-Szekeres solution and the solutions of Schwarzschild – Brillouin belong to two different pseudo-Riemannian manifolds. —

A last remark. The so-called “trapped surfaces” are geometrical loci subordinate to the existence of nonphysical singularities analogous to the surface \( r=2m \) of the standard solution. Consequently, theorems on gravitational collapse and cosmological models which utilize in an essential way the notion of trapped surface are wholly void of physical sense.
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