η-NORMALITY, CR-STRUCTURES, PARA-CR STRUCTURES ON ALMOST CONTACT METRIC AND ALMOST PARACONTACT METRIC MANIFOLDS

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ABSTRACT. For almost contact metric or almost paracontact metric manifolds there is natural notion of η-normality. Manifold is called η-normal if is normal along kernel distribution of characteristic form. In the paper it is proved that η-normal manifolds are in one-one correspondence with Cauchy-Riemann almost contact metric manifolds or para Cauchy-Riemann in case of almost paracontact metric manifolds. There is provided characterization of η-normal manifolds in terms of Levi-Civita covariant derivative of structure tensor.

1. INTRODUCTION

Almost contact metric manifold $\mathcal{M}$ is said to be normal if

$$[\phi, \phi] + 2d\eta \otimes \xi = 0,$$

where $[\phi, \phi]$ denotes Nijenhuis torsion of structure affinor $\phi$. In the natural way can be defined almost complex structure on product $\mathcal{M} \times S^1$ with circle. Now $\mathcal{M} \times S^1$ is complex manifold if and only if $\mathcal{M}$ is normal. Almost contact metric manifolds are extensively studied in recent years and in the past. The study mostly focused on contact metric manifolds however there are other important classes: almost cosymplectic (or almost coKähler) and almost Kenmotsu manifolds. For all these classes were obtained similar results, for example classification of so $(\kappa, \mu)$-spaces of different types. Besides other properties every such $(\kappa, \mu)$-space is Cauchy-Riemann manifold. Contact metric $(\kappa, \mu)$-space carries a structure of strictly pseudo-convex CR-manifold. While almost cosymplectic or almost Kenmotsu $(\kappa, \mu)$-spaces are Levi flat CR-manifolds. These results suggests to extend study to general almost contact metric CR-manifolds with Levi form neither strictly positive nor zero. General literature on almost contact metric manifolds are [3], [7], [10], [20], [21]. For almost contact metric $(\kappa, \mu)$-spaces cf. [1], [5], [10], [14], [15], [17], [23].

In analogy to almost contact metric manifolds theory of almost paracontact metric manifolds was developed. There is defined notion of normal almost paracontact metric manifold and also appear almost paracontact metric $(\kappa, \mu)$-spaces. Although for almost paracontact metric manifolds the problem of classifying $(\kappa, \mu)$-spaces is far more difficult and in fact still there is no such classification in most interesting cases: contact para-metric, almost para-cosymplectic or almost para-Kenmotsu. One of early papers which treated subject in way similar to almost contact metric manifolds are [13], [19]. In [26] the author classifies almost paracontact metric structures into classes determined by the decompositions of particular $\mathcal{G}$-module onto

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irreducible components. For general notion of para-CR manifold, cf. [18]. Particular homogeneous classes of para-CR manifolds are studied in [1], [2]. In framework of almost paracontact metric manifolds in [25] the author obtained several interesting conditions and characterizations for manifold to be para-CR manifold. Recently it was found deep relation between contact metric and paracontact metric \((\kappa, \mu)\)-spaces [6], [8]. General study of paracontact metric \((\kappa, \mu)\)-spaces is provided in [9].

2. Preliminaries

All manifolds in this paper are smooth, connected, without boundary. If not otherwise stated we use \(X, Y, Z, \ldots\) to denote vector fields on manifold.

2.1. Almost contact metric manifolds. Quadruple of tensor fields \((\phi, \xi, \eta, g)\), where \(\phi\) is affinor \((1, 1)\)-tensor field), \(\xi\) a vector field, \(\eta\) is one-form, \(g\) is Riemannian metric, and

\[
\begin{align*}
\phi^2 &= -Id + \eta \otimes \xi, \quad \eta(\xi) = 1, \\
g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y),
\end{align*}
\]

is called almost contact metric structure. The vector field \(\xi\) is characteristic vector field or Reeb vector field, form \(\eta\) is characteristic form. Manifold equipped with fixed almost contact metric structure is called almost contact metric manifold.

From definition tensor field \(\Phi(X, Y) = g(X, \phi Y)\) is skew-symmetric, it is two-form - fundamental form of \(\mathcal{M}\)\(^{1}\). There is \(\eta \wedge \Phi^n \neq 0\), on \(\mathcal{M}\).

Set
\[
\begin{align*}
N^{(1)}(X, Y) &= [\phi, \phi](X, Y) + 2d\eta(X, Y)\xi, \\
N^{(2)}(X, Y) &= (L_{\phi X}\eta)(Y) - (L_{\phi Y}\eta)(X),
\end{align*}
\]

where \(L_{\xi}\) denotes the Lie derivative along vector field \(\xi\). Let \(\nabla\) be covariant derivative with resp. to Levi-Civita connection of the metric. We have identity (cf. [3])

\[
\begin{align*}
2g((\nabla_X \phi)Y, Z) &= 3d\Phi(X, \phi Y, \phi Z) - 3d\Phi(X, Y, Z) + g(N^{(1)}(Y, Z), \phi X) + \\
&\quad N^{(2)}(Y, Z)\eta(X) + 2d\eta(\phi Y, X)\eta(Z) - 2d\eta(\phi Z, X)\eta(Y).
\end{align*}
\]

Let \(\mathcal{D} = \{\eta = 0\}\) denote the kernel distribution of \(\eta\). Complexification \(\mathcal{D}' = \mathcal{D}' \oplus \mathcal{D}''\) splits into direct sum of complex distributions \(\mathcal{D}' \cap \mathcal{D}'' = 0\) and \(\overline{\mathcal{D}} = \mathcal{D}''\). If \(\mathcal{D}'\) is formally involutive pair \((\mathcal{M}, \mathcal{D}')\) is called Cauchy-Riemann or shortly CR-manifold. Equivalently \(\mathcal{D}'\) is formally involutive if and only if for vector fields \(X, Y, \eta(X) = \eta(Y) = 0\), there is vector field \(Z, \eta(Z) = 0\), such that

\[
[X - i\phi X, Y - i\phi Y] = Z - i\phi Z,
\]

The Levi form \(L\) of almost contact metric CR-manifold is a conformal equivalence class of quadratic form on \(\mathcal{D}\),

\[
- d\eta(X, \phi X), \quad \eta(X) = 0.
\]

This means that in the above formula \(\eta\) can be replaced by its multiple \(f\eta\), for some smooth function \(f\) on \(\mathcal{M}\).

\(^{1}\)In literature some authors define fundamental form as \(g(\phi X, Y)\).
It is said that almost contact metric manifold is $\eta$-normal if
\begin{equation}
N^{(1)}(X, Y) = 0, \quad \eta(X) = \eta(Y) = 0.
\end{equation}
So $\eta$-normal manifold is manifold which is normal but only along kernel distribution \{\eta = 0\}.

2.2. Almost paracontact metric manifolds. Almost paracontact metric structure is a quadruple of tensor fields $(\phi, \xi, \eta, g)$, where $\phi$ is affinor, $\xi$ is a vector field, $\eta$ is a one-form and $g$ is a pseudo-Riemannian metric. It is assumed that
\begin{equation}
\phi^2 = \text{Id} - \eta \otimes \xi, \quad \eta(\xi) = 1,
\end{equation}
\begin{equation}
g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y).
\end{equation}
Let $V^\pm$ be distributions defined by non-zero eigenvalues $\pm 1$ of $\phi$. From definition we obtain dimensions are equal $\dim(V^+) = \dim(V^-) = n$, and both $V^\pm$ are totally isotropic $g(V^+, V^+) = g(V^-, V^-) = 0$. In conclusion pseudo-metric $g$ has signature $(n + 1, n)$. The triple $(\phi, \eta, \xi)$ is called almost paracontact structure, tensor field $\Phi(X, Y) = g(X, \phi Y)$ is called a fundamental form, $\eta \wedge \Phi^n \neq 0$, everywhere on $\mathcal{M}$. Manifold equipped with fixed almost paracontact metric structure is called almost paracontact metric manifold. Almost paracontact metric manifold is para-CR manifold if eigendistributions $V^\pm$, are involutive
\begin{equation}
[V^+, V^+] \subset V^+, \quad [V^-, V^-] \subset V^-.
\end{equation}
Almost paracomplex structure $J$ is $(1, 1)$-tensor field satisfying $J^2 = \text{Id}$, and eigendistribution corresponding to eigenvalues $\pm 1$ are of the same dimensions. Almost paracomplex structure is said to be integrable if there is atlas, and in every local chart coefficients of $J$ are constants. It is known that sufficient and necessary condition for paracomplex structure $J$ to be integrable is vanishing Nijenhuis torsion $[J, J] = 0$. In fact it is particular case of general Walker theorem for almost product structures.

Let $N^{(1)} = [\phi, \phi] = 2d\eta \otimes \xi$. On product $\mathcal{M} \times S^1$ with circle there is naturally defined almost paracomplex structure $J$,
\begin{equation}
J(X, f \frac{d}{dt}) = (\phi X + f\xi, \eta(X)\frac{d}{dt}),
\end{equation}
If this structure is paracomplex manifold $\mathcal{M}$ is said to be normal. It is known that $\mathcal{M}$ is normal if and only if $N^{(1)} = 0$. For reader convenience we shall provide the proof of this result, cf. Section 4

Similarly almost paracontact metric manifold is $\eta$-normal if is normal along distribution $\{\eta = 0\}$,
\begin{equation}
N^{(1)}(X, Y) = 0, \quad \eta(X) = \eta(Y) = 0.
\end{equation}

3. Almost contact metric CR-manifolds and $\eta$-normal manifolds

In this section we shall prove following result

**Theorem 1.** For almost contact metric manifold $\mathcal{M}$ the following statements are equivalent:

1. $\mathcal{M}$ is $\eta$-normal;
2. $\mathcal{M}$ is Cauchy-Riemann manifold;
(3) Set \( u(Y, X) = dh(\phi Y, X) + g(hY, X) \). Then
\[
\begin{align*}
g((\nabla_X \phi)Y, Z) &= \frac{3}{2}d\Phi(X, \phi Y, \phi Z) - \frac{3}{2}d\Phi(X, Y, Z) + u(Y, X)\eta(Z) - u(Z, X)\eta(Y),
\end{align*}
\]

Here are some simple useful identities, \( \nabla \) denotes covariant derivative operator
\[
\begin{align*}
(\nabla_X \phi)\phi &= \frac{1}{2}(\nabla_X \phi)\phi + \frac{1}{2}\phi L(\phi X, \phi Y, \phi Z),
\end{align*}
\]

\((\nabla_X \phi)\phi Y &= \phi L(\phi X, \phi Y, \phi Z),
\]

\[
\begin{align*}
3d\Phi(X, Y, Z) &= \phi L(\phi X, \phi Y, \phi Z),
\end{align*}
\]

\[
\begin{align*}
L(\phi X, \phi Y, \phi Z) &= \phi L(\phi X, \phi Y, \phi Z).
\end{align*}
\]

\[
\begin{align*}
\eta(X) &= \eta(Y) = 0.
\end{align*}
\]

**Proposition 1.** Manifold is \( \eta \)-normal if and only if a CR-manifold.

*Proof.* Let \( \eta(X) = 0, \eta(Y) = 0 \). For \( \eta \)-normal manifold
\[
\begin{align*}
[X, Y] - [\phi X, \phi Y] &= -\phi([\phi X, Y] + [X, \phi Y]),
\end{align*}
\]

\[
\begin{align*}
\eta([X, Y] - [\phi X, \phi Y]) &= 0,
\end{align*}
\]

\[
\begin{align*}
\eta([X, \phi Y] + [\phi X, Y]) &= 0,
\end{align*}
\]

therefore
\[
\begin{align*}
\phi([X, Y] - [\phi X, \phi Y]) &= [\phi X, Y] + [X, \phi Y].
\end{align*}
\]

Set \( Z = [X, Y] - [\phi X, \phi Y], \eta(Z) = 0 \) by \((\ref{3.3})\), and the above identity implies
\[
\begin{align*}
[X - i\phi X, Y - i\phi Y] &= Z - i\phi Z,
\end{align*}
\]

so \((\mathcal{M}, \phi|\mathcal{D})\) is a CR-manifold.

Conversely if manifold is CR-manifold \((\ref{3.10})\) is satisfied, consequently \((\ref{3.7})\), so \( \mathcal{M} \) is \( \eta \)-normal. \( \square \)

**Corollary 1.** For \( \eta \)-normal manifold
\[
\begin{align*}
N^{(2)}(X, Y) &= 0, \quad \eta(X) = \eta(Y) = 0,
\end{align*}
\]

\[
\begin{align*}
d\eta(\phi X, Y) - d\eta(\phi Y, X) &= 0, \quad \eta(X) = \eta(Y) = 0.
\end{align*}
\]

*Note these two above identities in virtue of \((\ref{3.3})\) are equivalent.*

**Proposition 2.** Almost contact metric manifold is \( \eta \)-normal if and only if \((\ref{3.1})\) holds

*Proof.* Set \( \bar{Y} = Y - \eta(Y)\xi, \bar{Z} = Z - \eta(Z), \) observe \( \phi\bar{Y} = \phi Y, \phi\bar{Z} = \phi Z, \eta(\bar{Y}) = \eta(\bar{Z}) = 0 \). If manifold is \( \eta \)-normal \( N^{(1)}(\bar{Y}, \bar{Z}) = 0, \) \( N^{(2)}(\bar{Y}, \bar{Z}) = 0 \), by \((\ref{2.5})\)
\[
\begin{align*}
2g((\nabla_X \phi)\bar{Y}, \bar{Z}) &= 3d\Phi(X, \phi Y, \phi Z) - 3d\Phi(X, Y, Z),
\end{align*}
\]

\[
\begin{align*}
2g((\nabla_X \phi)\bar{Y}, \bar{Z}) &= 2g((\nabla_X \phi)Y, Z) + \eta(Y)\{2d\eta(\phi Z, X) - (L_{\xi\phi})(\phi Z, X)\} + \eta(Z)\{2d\eta(\phi Y, X) - (L_{\xi\phi})(\phi Y, X)\},
\end{align*}
\]

\[
\begin{align*}
3d\Phi(X, \phi Y, \phi Z) - 3d\Phi(X, \phi Y, \phi Z) &= 3d\Phi(X, Y, Z) + \eta(Y)(L_{\xi\phi})(X, Y) + \eta(Z)(L_{\xi\phi})(Y, X),
\end{align*}
\]

all together these three above identities in view of \((\ref{3.5})\) yield \((\ref{3.11})\).
Conversely, let \( \eta(Y) = \eta(Z) = 0 \). By
\[
2g(hY, Z) = g((\mathcal{L}_\xi \phi)Y, Z) = -(\mathcal{L}_\xi \Phi)(Y, Z) - (\mathcal{L}_\xi g)(\phi Y, Z),
\]
\[
2g(hZ, Y) = -2g(h\phi Z, \phi Y) = (\mathcal{L}_\xi \Phi)(\phi Z, \phi Y) - (\mathcal{L}_\xi g)(Z, \phi Y)
\]
we obtain
\[
(3.17) \quad g(hY, Z) - g(hZ, Y) = -\frac{1}{2}(\mathcal{L}_\xi \Phi)(Y, Z) + \frac{1}{2}(\mathcal{L}_\xi \Phi)(\phi Y, \phi Z).
\]
To each term of the left hand of the identity
\[
-3d\Phi(\xi, Y, Z) = g((\nabla_\xi \phi)Y, Z) + g((\nabla_Y \phi)Z, \xi) + g((\nabla_Z \phi)\xi, Y),
\]
we apply \( (3.1), \text{next } (3.17) \), in result equation above simplifies to
\[
-3d\Phi(\xi, Y, Z) = -3d\Phi(\xi, Y, Z) + d\eta(\phi Z, Y) - d\eta(\phi Y, Z).
\]
Therefore \( d\eta(\phi Y, Z) - d\eta(\phi Z, Y) = 0 \), equivalently \( N^{(2)}(Y, Z) = 0 \). Let \( X \) be arbitrary vector field. By \( (2.5), (3.1), \text{and } N^{(2)}(Y, Z) = 0 \), we find \( g(N^{(1)}(Y, Z), \phi X) = 0, \text{in consequence } N^{(1)}(Y, Z) = 0, \text{and manifold is } \eta \text{-normal}. \)

**Corollary 2.** For almost contact metric manifold \( M \) the following statements are equivalent:

1. \( M \) is CR-manifold and tensor field \( h \) vanishes, \( h = 0 \);
2. \( M \) is \( \eta \)-normal and tensor field \( h \) vanishes, \( h = 0 \);
3. \( M \) is normal.

With help of \( (3.1) \), we shall find out covariant derivatives of \( \phi \), for some classes of almost metric manifolds to compare with already known results.

**Example 1** (Contact metric CR-manifolds). For contact metric CR-manifold \( d\eta = \Phi, \text{ d}\Phi = 0 \), tensor field \( h \) is symmetric \( g(hX, Y) = g(hY, X) \). By \( (3.1) \)
\[
(\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX),
\]
\[\text{cf. [3], p. 74, Theorem 6.6.}\]

**Example 2** (Almost cosymplectic CR-manifolds). For almost cosymplectic manifold \( d\eta = 0, \text{ d}\Phi = 0, \text{again } h \) is symmetric, let \( AX = -\nabla_X \xi, \text{ A is symmetric, moreover } h = -\phi A \). By \( (3.1) \) for almost cosymplectic CR-manifold
\[
(\nabla_X \phi)Y = -g(\phi AX, Y)\xi + \eta(Y)\phi AX.
\]
Therefore almost cosymplectic manifold is CR-manifold if and only if manifold has Kählerian leaves, \( \text{cf. [22].} \)

**Example 3** (Almost Kenmotsu CR-manifolds). For almost Kenmotsu manifold \( d\eta = 0, \text{ d}\Phi = 2\eta \wedge \Phi, \text{ tensor } h \) is symmetric. By \( (3.1) \)
\[
(\nabla_X \phi)Y = g(\phi X + hX, Y)\xi - \eta(Y)(\phi X + hX).
\]
Therefore almost Kenmotsu manifold is CR-manifold if and only if leaves of the distribution \( \{ \eta = 0 \} \) are Kähler, \( \text{cf. [16], [23].} \)
4. Normal and \( \eta \)-normal almost paracontact metric manifolds

For the sake of completeness of exposition we shall provide here some known concepts and constructions on almost paracontact metric manifolds. Here we obtain necessary and sufficient conditions for vanishing Nijenhuis torsion of almost paracomplex structure \( J \), \( (2.12) \). It suffices to compute \([J, J]((X, 0), (Y, 0))\), and \([J, J]((X, 0), (0, \frac{dt}{dt}))\):

\[
[J, J]((X, 0), (Y, 0)) = (\{X, Y\}, 0) + [(\phi X, \eta(X)\frac{d}{dt}), (\phi Y, \eta(Y)\frac{d}{dt})] -
\]

\[
J[(\phi X, \eta(X)\frac{d}{dt}), (Y, 0)] - J[(X, 0), (\phi Y, \eta(Y)\frac{d}{dt})] =
\]

\[
(\phi^2[X, Y] + \eta([X, Y])\xi, 0) + (\{\phi X, \phi Y\}, (\phi X\eta(Y) - \phi Y\eta(X))\frac{d}{dt}) -
\]

\[
(\phi(\phi X, Y) - \eta(X)\xi, \eta([\phi X, \phi Y] + X\eta(Y))\xi, \eta([X, \phi Y]))\frac{d}{dt}) =
\]

\[
(\{\phi, \phi\}(X, Y) - 2d\eta(X, Y)\xi, (\mathcal{L}_{\phi X}\eta(Y) - \mathcal{L}_{\phi Y}\eta(X))\frac{d}{dt}).
\]

Set \( N^{(1)}(X, Y) = [\phi, \phi](X, Y) - 2d\eta(X, Y)\xi, N^{(2)}(X, Y) = \mathcal{L}_{\phi X}\eta(Y) - \mathcal{L}_{\phi Y}\eta(X) \). Vanishing of \( N^{(1)} \), \( N^{(2)} \) is necessary for \( J \) to be paracomplex. Now

\[
[J, J]((X, 0), (0, \frac{d}{dt})) = [(\phi X, \eta(X)\frac{d}{dt}), (\xi, 0)] - J[(X, 0), (\xi, 0)] =
\]

\[
(\{\phi X, \xi\}, -\xi\eta(\xi)^{\frac{d}{dt}}) - (\phi X, \xi, \eta([X, \xi])\frac{d}{dt}) = -((\mathcal{L}_\xi\phi)X, (\mathcal{L}_\eta)\eta(X)\frac{d}{dt}).
\]

Set \( N^{(3)} = \mathcal{L}_\xi\phi, N^{(4)} = \mathcal{L}_\eta \). Exactly in the same way as for almost contact metric manifold it can be proven that vanishing of \( N^{(1)} \) follows vanishing of \( N^{(i)} \), \( i = 2, 3, 4 \), cf. [3]. So we recall well-known result

**Theorem 2.** Almost paracontact structure is normal if and only if for Nijenhuis torsion \( [\phi, \phi] \) we have

\[
(4.1) \quad [\phi, \phi] - 2d\eta \otimes \xi = 0.
\]

Let \( (\mathcal{M}, \phi, \xi, \eta, g) \) be an almost paracontact metric manifold, \( \nabla \) - covariant derivative operator with resp. to Levi-Civita connection of \( g \).

**Proposition 3.** For almost paracontact metric manifold

\[
2g((\nabla_X \phi)Y, Z) = -3d\Phi(X, \phi Y, \phi Z) - 3d\Phi(X, Y, Z) - g(N^{(1)}(Y, Z), \phi X) +
\]

\[
N^{(2)}(Y, Z)\eta(X) + 2d\eta(\phi Y, X)\eta(Z) - d\eta(\phi Z, X)\eta(Y).
\]

**Proof.** Recall formula for Levi-Civita connection

\[
2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) + g([X, Y], Z) +
\]

\[
g([Z, X], Y) + g([Z, Y], X),
\]

and coboundary formula for exterior derivative \( d\Phi \)

\[
3d\Phi(X, Y, Z) = X\Phi(Y, Z) + Y\Phi(Z, X) + Z\Phi(X, Y) - \Phi([X, Y], Z) -
\]

\[
\Phi([Y, Z], X) - \Phi([Z, X], Y).
\]

\[^2\text{All these results are well-known in the literature.}\]
Now

\[ 2g((\nabla_X \phi)Y, Z) = 2g(\nabla_X \phi Y, Z) + 2g(\nabla_X Y, \phi Z) = Xg(\phi Y, Z) + \phi Y g(X, Z) - Zg(X, \phi Y) + g([X, \phi Y], Z) + g([Z, X], \phi Y) + g([Z, \phi Y], X) + Xg(Y, \phi Z) - Yg(X, \phi Z) - \phi Zg(X, Y) + g([X, Y], \phi Z) + g([\phi Z, X], Y) + g([\phi Z, Y], X) = \]

\[ -3d\Phi(X, Y, Z) - 3d\Phi(X, \phi Y, \phi Z) - g([Y, Z], \phi X) + \phi Y(\eta(X)\eta(Y)) + \eta(Z)\eta([X, \phi Y]) + g([Z, \phi Y], X) - \phi Z(\eta(X)\eta(Y)) + \eta(Y)\eta([\phi Z, X]) + g([\phi Z, Y], X) - \Phi([\phi Y, \phi Z], X) = -3d\Phi(X, Y, Z) - \phi Y([Y, Z], \phi X) - g([\phi Y, \phi Z], \phi X) + g([\phi Y, \phi Z], \phi X) - \eta(X)(\phi Y([Z, \phi Y]) + g([\phi Y, \phi Z], \phi X) - \phi Z([\eta(X), \phi Y]) - \eta(Z)\eta([\phi Z, X]) = -3d\Phi(X, Y, Z) - 3d\Phi(X, \phi Y, \phi Z) - g(N^{(1)}(Y, Z), \phi X) + N^{(2)}(Y, Z)\eta(X) + 2d\eta(\phi Y, X)\eta(Z) - 2d\eta(\phi Z, X)\eta(Y). \]

\[ \Box \]

**Example 4** (Para-Sasakian manifold). Almost paracontact metric manifold is paracontact if \( d\eta = \Phi \). Normal paracontact metric manifold is called para-Sasakian. For para-Sasakian manifold

\[ g((\nabla_X \phi)Y, Z) = d\eta(\phi Y, X)\eta(Y) - d\eta(\phi Z, X)\eta(Y), \]

as \( d\eta = \Phi \), above identity follows \( (\nabla_X \phi)Y = -g(X, Y)\xi + \eta(Y)X \).

**Example 5** (Para-cosymplectic manifold). Almost paracontact metric manifold is almost para-cosymplectic, if \( d\eta = 0 \), \( d\Phi = 0 \). If additionally is normal is said to be para-cosymplectic. For para-cosymplectic manifold \( g((\nabla_X \phi)Y, Z) = 0 \), and

\[ \nabla \phi = 0. \]

**Example 6** (Para-Kenmotsu manifold). Almost paracontact metric manifold is called almost para-Kenmotsu if \( d\eta = 0 \), \( d\Phi = 2\eta \wedge \Phi \). Normal almost para-Kenmotsu manifold is called para-Kenmotsu. For para-Kenmotsu manifold

\[ 3d\Phi(X, Y, Z) = 2\eta(X)\Phi(Y, Z) + 2\eta(Y)\Phi(Z, X) + 2\eta(Z)\Phi(X, Y), \]

\[ 3d\Phi(X, \phi Y, \phi Z) = 2\eta(X)\Phi(\phi Y, \phi Z) = -2\eta(X)\Phi(Y, Z), \]

therefore

\[ g((\nabla_X \phi)Y, Z) = \Phi(Y, X)\eta(Z) - \Phi(Z, X)\eta(Y), \]

and \( (\nabla_X \phi)Y = g(\phi Y, X)\xi - \eta(Y)\phi X \).

**Theorem 3.** For almost paracontact metric manifold following statements are equivalent

1. Manifold is \( \eta \)-normal;
2. Manifold is para-CR manifold;
3. Set \( u(X, Y) = d\eta(\phi Y, X) + g(h X, Y) \). The following identity is satisfied

\[ g((\nabla_X \phi)Y, Z) = -\frac{3}{2}d\Phi(X, \phi Y, \phi Z) - \frac{3}{2}d\Phi(X, Y, Z) + u(Y, X)\eta(Z) - u(Z, X)\eta(Y). \]
Proof. Proof goes the same way as proof of Theorem\textsuperscript{1}. There is no additional difficulties here. We repeat the same steps. In the first part we prove that manifold is $\eta$-normal if is para-CR manifold. In the second part we prove that (4.6) characterizes $\eta$-normal manifolds, exactly in the same way as we have proven Proposition\textsuperscript{2}. □

Example 7 (Paracontact para-CR manifolds). \textit{Manifold is paracontact if $d\eta = \Phi$. For paracontact metric manifold tensor $h$ is symmetric. By (4.6)

\begin{equation}
(\nabla_X\phi)Y = -g(X-hX, Y)\xi + \eta(Y)(X-hX).
\end{equation}

In particular paracontact $(\kappa, \mu)$-space, is para-CR manifold, cf. [9].}

Example 8 (Paracosymplectic para-CR manifolds). For paracosymplectic manifold $d\eta = 0$, $d\Phi = 0$, tensor $h$ is symmetric, set $X \mapsto AX = -\nabla_X\xi$, $A$ is $(1,1)$-tensor field and $h = A\phi = -\phi A$. For paracosymplectic $CR$-manifold

\begin{equation}
(\nabla_X\phi)Y = g(hX, Y)\eta(Z) - \eta(Y)hX = g(A\phi X, Y)\xi - \eta(Y)A\phi X.
\end{equation}

We can state that paracosymplectic manifold is CR-manifold if and only if it has para-Kählerian leaves, cf. [13].

Example 9 (Almost para-Kenmotsu para-CR manifolds). For almost para-Kenmotsu manifold $d\eta = 0$, $d\Phi = 2\eta \wedge \Phi$, tensor field $h$ is symmetric. By (4.6)

\begin{equation}
g((\nabla_X\phi)Y, Z) = g(\phi X + hX, Y)\xi - \eta(Y)(\phi X + hX).
\end{equation}

Manifolds is para-CR if and only if it has para-Kählerian leaves.

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