EXISTENCE AND NON-EXISTENCE OF GLOBAL SOLUTIONS
FOR A DISCRETE SEMILINEAR HEAT EQUATION

KEISUKE MATSUYA AND TETSUJI TOKIHIRO

University of Tokyo
Komaba 3-8-1, Meguro
Tokyo 153-8914, Japan

(Communicated by Kuo-Chang Chen)

Abstract. Existence of global solutions to initial value problems for a discrete
analogue of a d-dimensional semilinear heat equation is investigated. We prove
that a parameter \( \alpha \) in the partial difference equation plays exactly the same role
as the parameter of nonlinearity does in the semilinear heat equation. That
is, we prove non-existence of a non-trivial global solution for \( 0 < \alpha \leq \frac{2}{d} \),
and, for \( \alpha > \frac{2}{d} \), existence of non-trivial global solutions for sufficiently small
initial data.

1. Introduction. The blowing up of solutions to the semilinear heat equation has
been analysed extensively since the pioneering work by Fujita\cite{2,1,5}. Fujita studied
the initial value problem of the equation:
\[
\begin{aligned}
\frac{\partial f}{\partial t} &= \Delta f + f^{1+\alpha} \quad (\alpha > 0) \\
\ f(0, \vec{x}) &= a(\vec{x}) \geq 0 \quad (a(\vec{x}) \neq 0)
\end{aligned}
\]  

(1)

where \( f := f(t, \vec{x}) \) (\( t \geq 0, \vec{x} \in \mathbb{R}^d \)) and \( \Delta \) is the \( d \)-dimensional Laplacian \( \Delta := \sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2} \). When the initial value \( a(\vec{x}) \) is continuous and uniformly bounded, there is
a smooth solution for \( t > 0 \) and whenever the solution is bounded, the solution is
prolongable. Moreover, since \( a(\vec{x}) \geq 0 \), the solution satisfies \( f(t, \vec{x}) \geq 0 \). A feature of (1) is that the solution is not necessarily bounded for all \( t \geq 0 \). This fact is easily
understood if one considers spatially uniform initial condition, \( a(\vec{x}) \equiv a \in \mathbb{R}_+ \). In
this case, \( f(t, \vec{x}) = f(t) \) and (1) becomes an ordinary differential equation,
\[
\begin{aligned}
\frac{df}{dt} &= f^{1+\alpha} \\
\ f(0) &= a > 0
\end{aligned}
\]  

(2)

The solution to this differential equation is
\[
f(t) = \frac{a^{-1/\alpha}}{(a^{-1} - a^{-\alpha} - t)^{1/\alpha}},
\]

2000 Mathematics Subject Classification. Primary: 39A14, 74G25; Secondary:35K58, 39A12.
Key words and phrases. Discretization, semilinear heat equation, global solution.
and we see that it diverges as $t \to \alpha^{-1}a^{-\alpha} - 0$. In general, if there exists a finite time $T \in \mathbb{R}_+$ and if the solution of (1) in $(t, \vec{x}) \in [0, T) \times \mathbb{R}^d$ satisfies
\[
\limsup_{t \to T-0} \|f(t, \cdot)\|_{L^\infty} = \infty,
\]
where
\[
\|f(t, \cdot)\|_{L^\infty} := \sup_{\vec{x} \in \mathbb{R}^d} |f(t, \vec{x})|,
\]
then we say that the solution of (1) blows up at time $T$, and therefore, that it is not a global solution (in time). In 1966, Fujita\cite{2} proved the following theorem

**Theorem 1.1.**

(1) If $0 < \alpha < 2/d$, any solution to (1) is not a global solution in time.

(2) If $2/d < \alpha$, then global solutions to (1) do exist for sufficiently small and smooth initial functions $a(\vec{x})$.

A remarkable point in this theorem is that the parameter $\alpha$ affects the existence of the global solution of (1). The critical value $2/d$ in this theorem is called the Fujita exponent. The case $\alpha = 2/d$ was studied by Hayakawa for $d = 1, 2$, and by Kobayashi, Sirao, Tanaka\cite{4}, and Weissler\cite{8} for general $d$. They proved,

**Theorem 1.2.** For the case $\alpha = 2/d$, there exists no global solution to (1).

In numerical computation of (1), one has to discretize it and consider a partial difference equation. A naive discretization would be to replace the $t$-differential with a forward difference and the Laplacian with a central difference such that (1) turns into
\[
\frac{f^{\tau+1}_{\vec{n}} - f^{\tau}_{\vec{n}}}{\delta} = \sum_{k=1}^d \frac{f^{\tau}_{\vec{n} + \vec{e}_k} - 2f^{\tau}_{\vec{n}} + f^{\tau}_{\vec{n} - \vec{e}_k}}{\xi^2} + (f^{\tau}_{\vec{n}})^{1+\alpha},
\]
where $f(\tau, \vec{n})(=: f^{\tau}_{\vec{n}}) : \mathbb{Z}_{\geq 0} \times \mathbb{Z}^d \to \mathbb{R}$, for positive constants $\delta$ and $\xi$.\textsuperscript{3} And where $\vec{e}_k \in \mathbb{Z}^d$ is the unit vector whose $k$th component is 1 and whose other components are 0. Putting $\lambda := \delta/\xi^2$, we obtain
\[
f^{\tau+1}_{\vec{n}} = 2d\lambda\hat{M}(f^{\tau}_{\vec{n}}) + (1 - 2d\lambda)f^{\tau}_{\vec{n}} + \delta(f^{\tau}_{\vec{n}})^{1+\alpha} \quad (0 > \alpha). \tag{3}
\]
Here
\[
\hat{M}(V_{\vec{n}}) := \frac{1}{2d} \sum_{k=1}^d (V_{\vec{n} + \vec{e}_k} + V_{\vec{n} - \vec{e}_k}). \tag{4}
\]

For a spatially uniform initial condition, (3) becomes an ordinary difference equation
\[
f^{\tau+1} = f^\tau + \delta(f^\tau)^{1+\alpha}.
\]
The above equation is a discretization of (2), but the features of its solutions are quite different. In fact, $f^\tau$ will never blow up at finite time steps. Hence, (3) does not preserve the global nature of the original semilinear heat equation (1).

In this article, we propose and investigate a discrete analogue of (1) which does keep the important characteristic of existence and non-existence of the global solutions in time. In section 2, we present a partial difference equation with a parameter $\alpha$ whose continuous limit equals to (1), and state the main theorem which shows that this difference equation has exactly the same properties as (1) with respect to $\alpha$. This theorem is proved in section 3 for the case $0 < \alpha < 2/d$. In section 4 for $2/d < \alpha$ and in section 5 for $\alpha = 2/d$. 
2. Discretization of the semilinear heat equation. We consider the following initial value problem for the partial difference equation

\[
\begin{aligned}
&f_{\tau+1}^\vec{n} = \frac{g_{\tau}^\vec{n}}{\{1 - (g_{\tau}^\vec{n})^\alpha\}^{1/\alpha}} \quad (\tau \in \mathbb{Z}_{\geq 0}, \, \vec{n} \in \mathbb{Z}^d) \\
f_{\tau}^\vec{n} = a(\vec{n}) \geq 0 \quad (a(\vec{n}) \neq 0),
\end{aligned}
\]

where \(\alpha > 0\) is a parameter and \(g_{\tau}^\vec{n}\) is defined by means of \(\hat{M} (4)\) as \(g_{\tau}^\vec{n} := \hat{M} (f_{\tau}^\vec{n})\).

By scaling \(f\) with a positive parameter \(\delta\) as

\[
F_{\tau}^\vec{n} := (\alpha \delta)^{-1/\alpha} f_{\tau}^\vec{n}, \\
G_{\tau}^\vec{n} := \hat{M} (F_{\tau}^\vec{n}),
\]

we have

\[
F_{\tau+1}^\vec{n} = \frac{G_{\tau}^\vec{n}}{\{1 - \alpha \delta (G_{\tau}^\vec{n})^\alpha\}^{1/\alpha}}.
\]

If there exists a smooth function \(F(t, \vec{x}) (t \in \mathbb{R}_{\geq 0}, \, \vec{x} \in \mathbb{R}^d)\) that satisfies \(F(\tau \delta, \xi \vec{n}) = F_{\tau}^\vec{n}\) with \(\xi := \sqrt{2d \delta}\), we find

\[
F(t + \delta, \vec{x}) = G(t, \vec{x}) \{1 + \delta (G(t, \vec{x}))^\alpha\} + O(\delta^2),
\]

with

\[
G(t, \vec{x}) := \frac{1}{2d} \sum_{k=1}^d (F(t, \vec{x} + \xi \vec{e}_k) + F(t, \vec{x} - \xi \vec{e}_k)),
\]

or

\[
\frac{F(t + \delta, \vec{x}) - F(t, \vec{x})}{\delta} = \sum_{k=1}^d \frac{F(t, \vec{x} + \xi \vec{e}_k) - 2F(t, \vec{x}) + F(t, \vec{x} - \xi \vec{e}_k)}{\xi^2} + (F(t, \vec{x}))^{1+\alpha} + O(\delta).
\]

Taking the limit \(\delta \to +0\), we obtain the semilinear heat equation (1)

\[
\frac{\partial F}{\partial t} = \Delta F + F^{1+\alpha}.
\]

Thus (5) can be regarded as a discrete analogue of (1).

Because of the term \((1 - (g_{\tau}^\vec{n})^{1/\alpha})\), if \(g_{\tau}^\vec{n} \to 1 - 0\), then \(f_{\tau}^\vec{n} \to +\infty\), and (5) cannot be defined when \(g_{\tau}^\vec{n} \geq 1\) for generic \(\alpha\). This behaviour may be regarded as an analogue of the blow up of solutions for the semilinear heat equations. Thus we define a global solution of (5) as follows.

**Definition 2.1.** When a solution \(f_{\tau}^\vec{n}\) of (5) is non-negative and uniquely determined for all \(\tau \in \mathbb{Z}_{\geq 0}\) and \(\vec{n} \in \mathbb{Z}^d\), i.e. \(g_{\tau}^\vec{n} < 1\) for all \(\tau \in \mathbb{Z}_{\geq 0}\) and \(\vec{n} \in \mathbb{Z}^d\), then we say that the solution \(f_{\tau}^\vec{n}\) is a global solution (in time) of (5).

The advantage of using (5) instead of (3) is apparent from the following proposition.
Proposition 2.1. No spatially uniform function can be a global solution of (5).

Proof. When the solution is spatially uniform, then (5) takes the form of an ordinary difference equation

\[ f^{\tau + 1} = \frac{f^\tau}{\left(1 - (f^\tau)^\alpha\right)^{1/\alpha}} \]

\[ = \left(f^\tau\right)^{-\alpha} - 1 \quad \text{for } f \neq 0. \]

Hence, if a non-trivial global solution exists, it satisfies \( 0 < f^\tau < 1 \) for all \( \tau \in \mathbb{Z}_{\geq 0} \) and

\[ (f^{\tau + 1})^{-\alpha} = (f^\tau)^{-\alpha} - 1. \]

However, the above equation is easily solved and we get

\[ (f^\tau)^{-\alpha} = (f^0)^{-\alpha} - \tau. \]

Since \((f^0)^{-\alpha}\) is a positive constant, \(f^\tau\) cannot be defined for \( \tau \geq (f^0)^{-\alpha} \), which is a contradiction.

Furthermore, (5) inherits quite similar properties to those of (1). The following theorem is the main result in this article.

Theorem 2.1.

1. For \( 0 < \alpha < 2/d \), there is no global solution to (5).
2. For \( \alpha = 2/d \), there is no global solution to (5).
3. Let \( \|f^0\|_1 \) be the \( l^1 \) norm of the initial function, i.e., \( \|f^0\|_1 := \sum_{\vec{n}} |f^0_{\vec{n}}| \). For \( 2/d < \alpha \), if \( \|f^0\|_1 \) is sufficiently small, then global solutions to (5) exist.

Remark 1. • In the limit \( \alpha \to +0 \) in (6), we have

\[ \begin{cases} F_{\vec{n}}^{\tau + 1} = e^\delta G_{\vec{n}}^\tau \\ F_{\vec{n}}^0 = a(\vec{n}) \geq 0 \quad (a(\vec{n}) \neq 0) \end{cases} \]

Although \( \|F^\tau\|_1 \) diverges as \( \tau \to +\infty \) as far as \( \delta > 0 \), the solution of this difference equation is a global solution in time. On the other hand, when \( \alpha = 0 \), the partial differential equation (1) becomes linear, and its solution is also a global solution.

• To consider negative solutions or oscillatory solutions, we have only to use a slightly modified partial difference equation

\[ f_{\vec{n}}^{\tau + 1} = \frac{g_{\vec{n}}^\tau}{\left(1 - (|g_{\vec{n}}^\tau|)^\alpha\right)^{1/\alpha}}. \]

3. Proof of theorem 2.1 for \( 0 < \alpha < 2/d \). The idea of the proof in this and the following sections is similar to that adopted by Meier in the case of partial differential equations [6]. We construct a subsolution and a supersolution of \( f_{\vec{n}}^\tau \) and prove the existence and non-existence of the global solutions.

We denote by \( U_{\vec{n}}^\tau \) the solution to the initial value problem of the linear partial difference equation

\[ \begin{cases} U_{\vec{n}}^{\tau + 1} = \hat{M}(U_{\vec{n}}^\tau) \\ U_{\vec{n}}^0 = \delta_{0,\vec{n}} \end{cases}. \]  

Using \( U_{\vec{n}}^\tau \), we define

\[ h_{\vec{n}}^\tau := \sum_{\vec{n}'} U_{\vec{n}' - \vec{n}}^\tau f_{\vec{n}'}^0, \]

(8)
and
\[
\mathcal{F}_n^\tau := \frac{h_n^\tau}{\left(1 - \tau (h_n^\tau)^\alpha\right)^{1/\alpha}},
\]
provided that \(1 - \tau (h_n^\tau)^\alpha > 0\).

**Remark 2.** • Since the support of \(U^\tau\) is finite, the summation in the definition of \(h_n^\tau\) is over a finite number of lattice points \(\vec{n}\).
- Due to the definition of \(U^\tau\), it holds that \(\| h^\tau \|_1 = \| f^0 \|_1\).
- The function \(h_n^\tau\) satisfies the linear partial difference equation
  \[
  \begin{cases}
  h_n^{\tau+1} = \hat{M}(h_n^\tau) \\
  h_n^0 = f_n^0
  \end{cases}
  \]

  \(\hat{M}\) is over a finite number of lattice points \(\vec{n}\).

  \(\hat{M}\) is defined by
  \[
  \hat{M}(h_n^\tau) := \sum_{k \in \mathbb{Z}^d} (\bar{f}_n^\tau + f_\alpha \cdot \vec{e}_k - \bar{h}_n^\tau \cdot \vec{e}_k)^{-\alpha} \geq \hat{M}(h_n^\tau) \geq 1 - \tau (h_n^\tau)^\alpha.
  \]

  The condition \(\hat{M}(h_n^\tau) > 0\) implies that at least one \(f_n^\tau\) is positive and that \(h_n^{\tau+1} = \hat{M}(h_n^\tau) > 0\).

**Remark 3.** • The condition \(\hat{M}(h_n^\tau) > 0\) implies that at least one \(f_n^\tau\) is positive and that \(h_n^{\tau+1} = \hat{M}(h_n^\tau) > 0\).
- When \(\hat{M}(h_n^\tau) = 0\), then \(h_n^{\tau+1} = 0\) and \(f_n^\tau = 0\).

**Proof.** We assume that there are \(m\) \((1 \leq m \leq 2d)\) nonzero values among \(\{h_n^\tau \pm \vec{e}_k\}_{k=1}^d\), and denote them by \(h_1, h_2, \ldots, h_m\). Accordingly we put
  \[
  f_i := \frac{h_i}{(1 - \tau (h_i)^\alpha)^{1/\alpha}} = (h_i^{-\alpha} - \tau)^{-1/\alpha} \quad (i = 1, 2, \ldots, m).
  \]

By definition,
  \[
  \chi(\tau) := \{\hat{M}(h_n^\tau)\}^{-\alpha} - \tau - \{\hat{M}(f_n^\tau)\}^{-\alpha} = \left\{\frac{1}{2d} \sum_{k=1}^d (h_n^\tau + \bar{e}_k - h_n^\tau \cdot \vec{e}_k)^{-\alpha} - \tau - \frac{1}{2d} \sum_{k=1}^d (f_n^\tau + \bar{e}_k + f_n^\tau \cdot \vec{e}_k)^{-\alpha}\right\}^{-\alpha}
  \]
  \[
  = \left\{\frac{1}{2d} \sum_{i=1}^m h_i \right\}^{-\alpha} - \tau - \left\{\frac{1}{2d} \sum_{i=1}^m f_i \right\}^{-\alpha}
  \]
  \[
  = \left\{\frac{1}{2d} \sum_{i=1}^m h_i \right\}^{-\alpha} - \tau - \left\{\frac{1}{2d} \sum_{i=1}^m ((h_i)^{-\alpha} - \tau)^{-1/\alpha}\right\}^{-\alpha}.
  \]
Regarding \( \tau \) as a continuous variable,
\[
\frac{d\chi(\tau)}{d\tau} = -1 + (2d)^\alpha \frac{\sum_{i=1}^{m} \{(h_i)_{i-\alpha - \tau}\}^{-(\alpha+1)/\alpha}}{\sum_{i=1}^{m} \{(h_i)_{i-\alpha - \tau}\}^{1/\alpha}} \geq -1 + (m)^\alpha \frac{\sum_{i=1}^{m} \{(h_i)_{i-\alpha - \tau}\}^{-(\alpha+1)/\alpha}}{\sum_{i=1}^{m} \{(h_i)_{i-\alpha - \tau}\}^{1/\alpha}}.
\]
From the Hölder inequality,
\[
(m)^\alpha \frac{\sum_{i=1}^{m} \{(h_i)_{i-\alpha - \tau}\}^{-(\alpha+1)/\alpha}}{\sum_{i=1}^{m} \{(h_i)_{i-\alpha - \tau}\}^{1/\alpha}} \geq 1,
\]
and we find that \( \frac{d\chi(\tau)}{d\tau} \geq 0 \). Thus \( \chi(\tau) \geq \chi(0) = 0 \), and we obtain
\[
\left\{ \hat{M}(h_{\vec{n}}^\alpha) \right\}^{1-\alpha} - \tau \geq \left\{ \hat{M}(f_{\vec{n}}^\alpha) \right\}^{1-\alpha}.
\]
By the assumption \( \{\hat{M}(f_{\vec{n}}^\alpha)\}^{1-\alpha} - 1 > 0 \) and (9) we have
\[
(h_{\vec{n}}^{\alpha+1})^{1-\alpha} - (\tau + 1) = \{\hat{M}(h_{\vec{n}}^\alpha)\}^{1-\alpha} - \tau - 1 \geq \{\hat{M}(f_{\vec{n}}^\alpha)\}^{1-\alpha} - 1 > 0.
\]
Thus we find
\[
1 - (\tau + 1)(h_{\vec{n}}^{\alpha+1})^{1-\alpha} > 0.
\]
Therefore \( f_{\vec{n}}^{\alpha+1} \) exists and due to (11) it satisfies
\[
(f_{\vec{n}}^{\alpha+1})^{1-\alpha} \geq \{\hat{M}(f_{\vec{n}}^\alpha)\}^{1-\alpha} - 1.
\]

\[\square\]

**Proposition 3.1.** If the solution \( f^\alpha_{\vec{n}} \) of the initial value problem (5) exists at \( \tau \) and for all \( \vec{n} \in \mathbb{Z}^d \), then \( f^{\alpha+1}_{\vec{n}} \) exists at the same \( \tau \) and for all \( \vec{n} \in \mathbb{Z}^d \), and satisfies
\[
f_{\vec{n}}^{\alpha+1} \leq f_{\vec{n}}^{\alpha}.
\]

**Proof.** The proof goes by induction for \( \tau \). Since \( f^0_{\vec{n}} = f^0 \) by definition, \( f^1_{\vec{n}} \) exists and satisfies (13) for \( \tau = 0 \) and for all \( \vec{n} \). Suppose that \( f^\alpha_{\vec{n}} \) exists and satisfies (13) for all \( \vec{n} \) and for all \( \tau \leq s \). If \( f^{s+1}_{\vec{n}} \) exists, either \( f^{s+1}_{\vec{n}} = 0 \) or \( f^{s+1}_{\vec{n}} > 0 \). When \( f^{s+1}_{\vec{n}} = 0 \), it implies
\[
f^{s+1}_{\vec{n}} = 0 \iff \hat{M}(f^s_{\vec{n}}) = 0 \iff f^s_{\vec{n} + \vec{e}_k} = 0 \ (k = 1, 2, \ldots, d) \iff f^s_{\vec{n} + \vec{e}_k} = 0 \ (k = 1, 2, \ldots, d) \iff h^{s+1}_{\vec{n} + \vec{e}_k} = 0 \ (k = 1, 2, \ldots, d) \iff \hat{M}(h^s_{\vec{n}}) = 0 \iff h^{s+1}_{\vec{n}} = 0 \iff f^{s+1}_{\vec{n}} = 0.
\]
Hence \( f^{s+1}_{\vec{n}} \) exists and satisfies \( f^{s+1}_{\vec{n}} \leq f^s_{\vec{n}} \).

When \( f^{s+1}_{\vec{n}} > 0 \), if \( \hat{M}(f^s_{\vec{n}}) = 0 \), then \( \hat{M}(h^s_{\vec{n}}) = 0 \) and \( f^{s+1}_{\vec{n}} = 0 \). Otherwise
\[
\{\hat{M}(f^s_{\vec{n}})\}^{1-\alpha} - 1 \geq \{\hat{M}(f^s_{\vec{n}})\}^{1-\alpha} - 1 = (f^{s+1}_{\vec{n}})^{1-\alpha} > 0.
\]
Then, from lemma 3.1, the existence of $f_{s+1}^{\tau} \mathbf{n}$ follows. Moreover, from (12),

$$
(f_{s+1}^{\tau} \mathbf{n})^{-\alpha} \geq \{M(f_{s}^{\tau} \mathbf{n})\}^{-\alpha} - 1
$$

Thus we find

$$
f_{s+1}^{\tau} \mathbf{n} \leq f_{s}^{\tau} \mathbf{n}.
$$

From the induction hypothesis, (13) holds for arbitrary $\tau \in \mathbb{Z}_+$. □

The following asymptotic evaluation of $U_{\mathbf{n}}^{\tau}$ is well known from the analysis of the transition probability of $d$-dimensional simple random walk. See for example F. Spitzer[7].

**Proposition 3.2.** When the components of $\mathbf{n} = (n_1, \cdots, n_d)$ satisfy $n_1 + \cdots + n_d \equiv \tau \pmod{2}$,

$$
U_{\mathbf{n}}^{\tau} \sim 2 \left( \frac{d}{2\pi} \right)^{d/2} \tau^{-d/2} \exp\left(-\frac{4^{1/d}d|\mathbf{n}|^2}{2\tau}\right) \quad (14)
$$

uniformly in $\mathbf{n}$ ($\tau \to +\infty$).

Note that $U_{\mathbf{n}}^{\tau} = 0$ for $n_1 + \cdots + n_d \not\equiv \tau \pmod{2}$ by definition. Since

$$
\lim_{\tau \to +\infty} \exp\left(-\frac{4^{1/d}d|\mathbf{n}|^2}{2\tau}\right) = 1,
$$

we also have

$$
U_{\mathbf{n}}^{\tau} \sim 2 \left( \frac{d}{2\pi} \right)^{d/2} \tau^{-d/2} \quad (\tau \to +\infty). \quad (15)
$$

Here $A^\tau \sim B^\tau$ ($\tau \to +\infty$) means $\lim_{\tau \to +\infty} (A^\tau/B^\tau) = 1$.

Now, we are ready to give the proof of theorem 2.1 (1).

**Proof of theorem 2.1 (1).**

Suppose that the global solution $f_{\mathbf{n}}^{\tau}$ exists. From (15), $h_{\mathbf{n}}^{\tau}$ is evaluated as

$$
h_{\mathbf{n}}^{\tau} = \sum_{\mathbf{n}'} U_{\mathbf{n}-\mathbf{n}'}^{\tau} f_{\mathbf{n}'}^{0}
$$

$$
\sim \frac{1}{\sqrt{\tau^d}} \sum_{\mathbf{n}} 2 \left( \frac{d}{2\pi} \right)^{d/2} f_{\mathbf{n}}^{0} \quad (\tau \to +\infty). \quad (16)
$$

Putting $C := 2 (d/2\pi)^{d/2} \| f^{0} \|_1$,

$$
\frac{h_{\mathbf{n}}^{\tau}}{C} \sim \frac{1}{1 - C \tau^{1-d\alpha/2}} \quad (\tau \to +\infty). \quad (17)
$$

For $0 < \alpha < 2/d$, the exponent of $\tau$ in the above equation satisfies $1 - d\alpha/2 > 0$. However, since $C$ is a positive constant independent of $\tau$, $1 - 2C\tau^{1-d\alpha/2} < 0$ for sufficiently large $\tau$ and the evaluation (17) does not make sense. In other words, $f_{\mathbf{n}}^{\tau}$ cannot exist for sufficiently large $\tau$, which contradicts proposition 3.1 and the statement (1) of theorem 2.1 is therefore proved. □
4. **Proof of the theorem 2.1 for $2/d < \alpha$**. First, we define a supersolution of (5)

\[
\tilde{f}_n^\tau := \frac{h_n^\tau}{\left(1 - \sum_{k=0}^{\tau} (m_k)^\alpha \right)^{1/\alpha}},
\]

where $m_\tau$ is defined in terms of (8) as

\[
m_\tau := \sup_{\bar{n}} h_{\bar{n}}^\tau.
\]

Of course $\tilde{f}_n^\tau$ is well defined only when $1 - \sum_{k=0}^{\tau} (m_k)^\alpha > 0$.

**Proposition 4.1.** When $\tilde{f}_n^\tau$ exists at $\tau$ and for all $\bar{n} \in \mathbb{Z}^d$, $f_{\bar{n}}^\tau$ exists and satisfies

\[
\tilde{f}_n^\tau \geq f_{\bar{n}}^\tau.
\]

**Proof.** We give the proof by induction on $\tau$. When $\tau = 0$, by definition of the initial value problem, $f_0^\bar{n}$ exists and (20) holds because

\[
\tilde{f}_n^0 = \frac{h_n^0}{\left(1 - (m_0)^\alpha \right)^{1/\alpha}} \geq h_n^0 = f_n^0.
\]

Suppose that the statement is true up to $\tau = s$ and $\tilde{f}_n^{s+1}$ exists. When $\tilde{f}_n^{s+1} = 0$,

\[
\tilde{f}_n^{s+1} = 0 \iff h_n^{s+1} = 0 \iff \hat{M}(h_n^s) = 0 \iff h_n^{s+1} \pm e_k = 0 \ (k = 1, 2, \ldots, d) \iff f_n^{s+1} \pm e_k = 0 \ (k = 1, 2, \ldots, d) \iff g_n^s = 0 \iff f_n^{s+1} = 0.
\]

Hence (20) holds.

When $\tilde{f}_n^{s+1} > 0$, if $g_n^s = 0$, then $f_n^{s+1} = 0$ and the statement is true. Otherwise

\[
0 < (\tilde{f}_n^{s+1})^{-\alpha} = \frac{1 - \sum_{k=0}^{s} (m_k)^\alpha}{(h_n^{s+1})^{\alpha}} = \frac{1 - \sum_{k=0}^{s} (m_k)^\alpha}{(h_n^{s+1})^{\alpha}} - \left(\frac{m_{s+1}}{h_n^{s+1}}\right)^\alpha \leq \frac{1 - \sum_{k=0}^{s} (m_k)^\alpha}{\left\{\hat{M}(h_n^s)\right\}^{\alpha}} - 1 = \frac{1}{\left\{\hat{M}(f_n^s)\right\}^{\alpha}} - 1 \leq (g_n^s)^{-\alpha} - 1.
\]

From (5), $(g_n^s)^{-\alpha} - 1 = (f_n^{s+1})^{-\alpha}$ and we find $(\tilde{f}_n^{s+1})^{-\alpha} \leq (f_n^{s+1})^{-\alpha}$, i.e. $f_n^{s+1} \leq \tilde{f}_n^{s+1}$. Thus, from the induction hypothesis, the statement is true for any non-negative integer $\tau$.

Now we prove the statement(3) of theorem 2.1.

**Proof of theorem 2.1 (3).**
From (16), we obtain the asymptotic behaviour of \( m_{\tau} \) as
\[
m_{\tau} \sim \frac{C}{\sqrt{\tau}} \quad (\tau \to +\infty).
\]

Hence with a fixed \( \tau_0 \in \mathbb{Z}_{\geq 0} \), we can evaluate \( \bar{f}_{\tau}^{\tilde{n}} \) as
\[
\bar{f}_{\tau}^{\tilde{n}} \sim \frac{h_{\tau}^{\tilde{n}}}{\left( 1 - \sum_{k=0}^{\tau_0} (m_k)^{\alpha} - \sum_{k=\tau_0+1}^{\tau} C_{\alpha k}^{d\alpha/2} \right)^{1/\alpha}} \quad (\tau \to +\infty).
\]

Since \( d\alpha/2 > 1 \), we have \( \sum_{k=\tau_0+1}^{\infty} \frac{1}{k^{d\alpha/2}} < +\infty \). Noticing the fact \( \|h^\tau\|_1 = \|f^0\|_1 \), the term \( \sum_{k=0}^{\tau_0} (m_k)^{\alpha} \) and the constant \( C \) can be as small as possible by choosing a small value of \( \|f^0\|_1 \). Thus \( \bar{f}_{\tau}^{\tilde{n}} \) exists at any time step \( \tau \) and any lattice point \( \tilde{n} \), if \( \|f^0\|_1 \) is sufficiently small. Then, from proposition 4.1, we find that \( f_{\tau}^{\tilde{n}} \) is a global solution of (5).

\[\square\]

5. **Proof of theorem 2.1 for \( \alpha = 2/d \).** In this section, we prove the statement (2) of 2.1. The idea of the proof is similar to that adopted by Weissler[8].

We define the discrete Green function \( G \) and the function \( \tilde{f} \) as follows.

\[
G_{\tilde{n}}^\tau := \begin{cases} 0 & (\tau = 0) \\ U_{\tilde{n}}^{\tau-1} & (\tau \geq 1) \end{cases}, \quad \tilde{f}_{\tau}^{\tilde{n}} := \begin{cases} 0 & (\tau = 0) \\ \sum_{s=0}^{\tau-1} \sum_{\tilde{n}'} G_{\tilde{n}'}^{\tau-s} H(g_{\tilde{n}'}^{s}) & (\tau \geq 1) \end{cases},
\]

where
\[
H(g) := \frac{g}{(1 - g^{\alpha})^{1/\alpha}} - g.
\]

Since
\[
f_{\tau+1}^{\tilde{n}} - g_{\tilde{n}}^{\tau} = \frac{g_{\tilde{n}}^{\tau}}{\left( 1 - (g_{\tilde{n}}^{\tau})^{\alpha} \right)^{1/\alpha}} - g_{\tilde{n}}^{\tau} = H(g_{\tilde{n}}^{\tau}),
\]

the initial value problem (5) can be rewritten as
\[
\begin{cases}
(\hat{T} - \hat{M}) f_{\tau}^{\tilde{n}} = H(g_{\tilde{n}}^{\tau}) \\
f_{0}^{\tilde{n}} = a(\tilde{n}) \geq 0 \quad (a(\tilde{n}) \neq 0),
\end{cases}
\]

where \( \hat{T} \) denotes the time shift: \( \hat{T} y^{\tau} := y^{\tau+1} \).

For \( \tau \geq 1 \),
\[
(\hat{T} - \hat{M}) G_{\tilde{n}}^{\tau+1} = G_{\tilde{n}}^{\tau+1} - \hat{M}(G_{\tilde{n}}^{\tau}) = U_{\tilde{n}}^{\tau} - \hat{M}(U_{\tilde{n}}^{\tau-1}) = 0,
\]

and for \( \tau = 0 \),
\[
(\hat{T} - \hat{M}) G_{\tilde{n}}^{0} = G_{\tilde{n}}^{1} - \hat{M}(G_{\tilde{n}}^{0}) = U_{\tilde{n}}^{0} = \delta_{\tilde{n},0}.
\]

Thus we find
\[
(\hat{T} - \hat{M}) G_{\tilde{n}}^{\tau} = \delta_{\tau,0} \delta_{\tilde{n},0}.
\]

We also find
\[
(\hat{T} - \hat{M}) f_{\tilde{n}}^{\tau} = H(g_{\tilde{n}}^{\tau}).
\]
Because
\[(\hat{T} - \hat{M})\hat{f}_n^\tau = \hat{f}_n^{\tau + 1} - \hat{M}(\hat{f}_n^\tau)\]
\[= \sum_{s=0}^{\tau} \sum_{\hat{n}'} G_{\hat{n}' - \hat{n}'}^{\tau + 1 - s} H(g_{\hat{n}'}^s) + \sum_{s=0}^{\tau - 1} \hat{M}(G_{\hat{n}' - \hat{n}'}^{\tau - s} H(g_{\hat{n}'}^s))\]
\[= \sum_{\hat{n}'} G_{\hat{n}' - \hat{n}'}^1 H(g_{\hat{n}'}^\tau) + \sum_{s=0}^{\tau - 1} (\hat{T} - \hat{M})(G_{\hat{n}' - \hat{n}'}^{\tau - s} H(g_{\hat{n}'}^s))\]
\[= \sum_{\hat{n}'} U_{\hat{n}' - \hat{n}'}^0 H(g_{\hat{n}'}^\tau) + \sum_{s=0}^{\tau - 1} \{\delta_{\tau-s,0} \delta_{\hat{n}' - \hat{n}'} H(g_{\hat{n}'}^s)\}\]
\[= H(g_{\hat{n}'}^\tau).\]

Using the above results, we can prove the following proposition.

**Proposition 5.1.** When \(f_n^\tau\) is a global solution of (5)
\[f_n^\tau = h_n^\tau + \sum_{s=1}^{\tau} \sum_{\hat{n}'} U_{\hat{n}' - \hat{n}'}^{\tau - s} H(g_{\hat{n}'}^{s-1}).\] (27)

Here \(h_n^\tau\) is defined in (8).

**Proof.** First we note that \(\hat{f}_n^\tau = \sum_{s=1}^{\tau} \sum_{\hat{n}'} U_{\hat{n}' - \hat{n}'}^{\tau - s} H(g_{\hat{n}'}^{s-1})\), and the right hand side of (27) = \(h_n^\tau + \hat{f}_n^\tau\). By definition \(h_n^0 = f_n^0\) and \(\hat{f}_n^0 = 0\). From (10) and (26),
\[\begin{cases} (\hat{T} - \hat{M})(h_n^\tau + \hat{f}_n^\tau) = H(g_{\hat{n}'}^\tau) \\ h_n^0 + \hat{f}_n^0 = f_n^0. \end{cases}\]

Thus, from (24), \(h_n^\tau + \hat{f}_n^\tau\) satisfies the same initial value problem as that for \(f_n^\tau\). Since the solution of the initial value problem (24) ((5)) is unique, \(f_n^\tau = h_n^\tau + \hat{f}_n^\tau\). \(\square\)

**Proposition 5.2.** Suppose that \(\alpha = 2/d\) and that \(f_n^\tau\) is a global solution of (5). Then there exists a constant \(C_0\) and
\[\|f^\tau\|_1 \leq C_0.\]

for all \(\tau \in \mathbb{Z}_+\).

**Proof.** From proposition 3.1 and the assumption of the proposition, \(f_n^\tau\) also exists for all \((\tau, \hat{n})\) and
\[1 - (\tau^{d/2} h_n^\tau)^\alpha > 0.\] (28)

On the other hand, from proposition 3.2,
\[\tau^{d/2} h_n^\tau \sim \sum_{\hat{n}} 2 \left(\frac{d}{4\pi}\right)^{d/2} f_n^0 = 2 \left(\frac{d}{4\pi}\right)^{d/2} \|f^0\|_1 \quad (\tau \to +\infty).\]

Then from (28),
\[\left\{2 \left(\frac{d}{4\pi}\right)^{d/2} \|f^0\|_1\right\}^\alpha < 1.\]

Therefore,
\[\|f^0\|_1 < \frac{1}{2} \left(\frac{4\pi}{d}\right)^{d/2}.\]
Because \( f^*_{\tau} \) is a global solution, if we take \( f^*_{\tau} \) at any time step \( \tau \) as the initial value of (5), the solution is also a global solution. Therefore we obtain
\[
\|f^\tau\|_1 < \frac{1}{2} \left( \frac{4\pi}{d} \right)^{d/2}.
\]

Now we prove the statement (2) of theorem 2.1.

**Proof of theorem 2.1 (2).**

Suppose that \( f^*_{\tau} \) is a global solution of (5). From proposition 5.1,
\[
\|f^\tau\|_1 \geq \sum_{s=1}^{\tau} \sum_{\vec{n}, \vec{n}' \in \mathbb{Z}^d} U_{\vec{n} - \vec{n}'}^\tau H(g^s_{\vec{n}})
\]
\[
= \sum_{s=1}^{\tau} \sum_{\vec{n}} H(g^s_{\vec{n}}).
\]

Here we applied \( \sum_{\vec{n}} U_{\vec{n}}^\tau = 1 \). By an elementary computation we can easily show
\[
H(g) \geq \frac{1}{\alpha} g^{1+\alpha} \quad (0 \leq g < 1).
\]

Hence
\[
\|f^\tau\|_1 \geq \sum_{s=1}^{\tau} \sum_{\vec{n}} \frac{1}{\alpha} (g^s_{\vec{n}})^{1+\alpha}.
\]

From (27), \( U_{\vec{n}}^\tau \geq 0 \) and \( H(g^s_{\vec{n}}) > 0 \), we obtain that \( f^*_{\tau} \geq h^*_{\tau} \) and that
\[
g^s_{\vec{n}} \geq M(h^s_{\vec{n}}) = h^s_{\vec{n}} = \sum_{\vec{n}'} f^0_{\vec{n}'} U_{\vec{n} - \vec{n}'}^\tau.
\]

There is at least one \( \vec{n}_0 \in \mathbb{Z}^d \) such that \( f^0_{\vec{n}_0} > 0 \), and we put \( c := f^0_{\vec{n}_0} > 0 \). Then
\[
h^s_{\vec{n}} \geq c U_{\vec{n} - \vec{n}_0},
\]
and
\[
\|f^\tau\|_1 \geq \sum_{s=1}^{\tau} \sum_{\vec{n}} \frac{1}{\alpha} (h^s_{\vec{n}})^{1+\alpha}
\]
\[
\geq c^{1+\alpha} \sum_{s=1}^{\tau} \sum_{\vec{n}} (U_{\vec{n}}^\tau)^{1+\alpha}.
\]

Let \( \delta \) and \( \xi \) be \( \delta := \frac{1}{\tau} \) and \( \xi := \sqrt{2d\delta} \) respectively. From (14), we obtain the following evaluation for \( \alpha = 2/d \) and \( \tau \to +\infty \) (\( \delta \to +0 \), \( \xi \to +0 \)).
\[
\sum_{\vec{n}} (U_{\vec{n}}^\tau)^{1+2/d} \sim \sum_{\vec{n}} 2^{1+2/d} \left( \frac{d}{2\pi \tau} \right)^{1+d/2} \exp \left\{ -\frac{4^{1/d} |\vec{n}|^2}{2\tau} \left( 1 + \frac{2}{d} \right) \right\} \quad (\tau \to +\infty)
\]
\[
= 2^{1+2/d} \frac{d}{2\pi \tau} \left( 4\pi \tau \delta \right)^{d/2} \sum_{\vec{n}} \exp \left\{ -\frac{4^{1/d} (\xi |\vec{n}|)^2}{4\tau \delta} \left( 1 + \frac{2}{d} \right) \right\}
\]
\[
\sim 2^{2/d} \frac{d}{\pi \tau} \left( 4\pi \delta \right)^{d/2} \frac{1}{\pi d^{d/2}} \int_{\mathbb{R}^d} \exp \left\{ -\frac{4^{1/d} |\vec{x}|^2}{4} \left( 1 + \frac{2}{d} \right) \right\} d\vec{x}
\]
\[
= \frac{2d^{2/d}}{\pi} \left( 1 + \frac{2}{d} \right)^{-d/2} \frac{1}{\tau}.
\]
Since $\sum_{s=1}^{\infty} 1/s$ diverges, $\sum_{s=1}^{r} (\sum_{n} T_{n}^{s})^{1+2/d}$ can take an arbitrarily large value. From (31), $\|f^{*}\|_{1}$ also becomes arbitrarily large, which contradicts with proposition 5.2. Therefore, when $\alpha = 2/d$, there exists no global solution and we have completed the proof of theorem 2.1. \qed

Acknowledgments. The authors would like to thank Profs. Atsushi Nagai, Ralph Willox and Dr. Mikio Murata for useful comments and discussions.

REFERENCES

[1] J. Bebernes and D. Eberly, “Mathematical Problems from Combustion Theory,” Appl. Math. Sci., 83, Springer-Verlag, New York, 1989.
[2] H. Fujita, On the blowing up of solutions of the Cauchy problem for $u_{t} = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo Sect. A Math., 16 (1966), 109–124.
[3] K. Hayakawa, On nonexistence of global solutions of some semilinear parabolic equations, Proc. Japan Acad., 49 (1973), 503–505.
[4] K. Kobayashi, T. Sirao and H. Tanaka, On the growing up problem for semilinear heat equations, J. Math. Soc. Japan, 29 (1977), 407–424.
[5] Howard A. Levine, The role of critical exponents in blowup theorems, SIAM Review, 32 (1990), 262–288.
[6] P. Meier, On the critical exponent for reaction-diffusion equations, Arch. Rational Mech. Anal., 109 (1990), 63–71.
[7] F. Spitzer, “Principles of Random Walk.” Second edition, Graduate Texts in Mathematics, Vol. 34, Springer-Verlag, New York-Heidelberg, 1976.
[8] F. B. Weissler, Existence and nonexistence of global solutions for a semilinear heat equation, Israel J. Math., 38 (1981), 29–40.

Received April 2010; revised October 2010.

E-mail address: matsuya@ms.u-tokyo.ac.jp
E-mail address: toki@ms.u-tokyo.ac.jp