Measure and Detection of Genuine Multipartite Entanglement for Tripartite Systems

Ming Li\textsuperscript{1}, Lingxia Jia\textsuperscript{1}, Jing Wang\textsuperscript{1}, Shuqian Shen\textsuperscript{1}, and Shao-Ming Fei\textsuperscript{2,3}
\textsuperscript{1}College of the Science, China University of Petroleum, 266580 Qingdao, China
\textsuperscript{2}School of Mathematical Sciences, Capital Normal University, Beijing 100048, China
\textsuperscript{3}Max-Planck-Institute for Mathematics in the Sciences, 04103 Leipzig, Germany

It is a computationally hard task to certify genuine multipartite entanglement (GME). We investigate the relation between the norms of the correlation vectors and the detection of GME for tripartite quantum systems. A sufficient condition for GME and an effective lower bound for the GME concurrence are derived. Several examples are considered to show the effectiveness of the criterion and the lower bound of GME concurrence.

PACS numbers: 03.67.-a, 02.20.Hj, 03.65.-w

INTRODUCTION

Quantum entanglement is a remarkable resource in the theory of quantum information, with various applications\cite{1,2}. A multipartite quantum state that is not separable with respect to any bi-partition is said to be genuine multipartite entangled\cite{3}. As one of the important type of entanglement, genuine multipartite entanglement (GME) offers significant advantage in quantum tasks comparing with bipartite entanglement\cite{4}. In particular, it is the basic ingredient in measurement-based quantum computation\cite{5}, and is beneficial in various quantum communication protocols\cite{6}, including secret sharing\cite{7} (cf.\cite{8}), extreme spin squeezing\cite{9}, high sensitivity in some general metrology tasks\cite{10}, quantum computing using cluster states\cite{11}, and multi-party quantum network\cite{13,14}. However, detecting and measuring quantum entanglement turn out to be quite difficult. To detect GME, series of linear and nonlinear entanglement witnesses\cite{17,18,19,20,21}, generalized concurrence for GME\cite{22,23}, and Bell-like inequalities\cite{32} were derived and a characterization in terms of semi-definite programs (SDP) was developed\cite{33}. Nevertheless, the problem remains far from being satisfactorily solved.

From the norms of the correlation tensors in the generalized Bloch representation of a quantum state, separable conditions for both bi- and multi-partite quantum states have been presented in\cite{38,39,40}. A multipartite entanglement measure for N-qubit and N-qudit pure states is given in\cite{41,42}. A general framework for detecting genuine multipartite entanglement and non full separability in multipartite quantum systems of arbitrary dimensions has been introduced in\cite{43}. In\cite{44} it has been shown that the norms of the correlation tensors has a close relationship to the maximal violation of a kind of multi Bell inequalities and to the concurrence\cite{45}.

In this paper, we investigate the genuine tripartite entanglement in terms of the norms of the correlation tensors and GME concurrence for tripartite qudit quantum systems. We derive criteria to detect GME. An effective lower bound for GME concurrence is also presented.

CRITERIA FOR DETECTING GME

In this section, we present a criterion to detect GME by using the approach presented in\cite{43} for tripartite qudit systems. We start with some definitions and notations.

Let $H^d_i$, $i=1,2,3$, denote $d$-dimensional Hilbert spaces. A tripartite state $\rho \in H^d_1 \otimes H^d_2 \otimes H^d_3$ can be expressed as $\rho = \sum p_\alpha |\psi_\alpha \rangle \langle \psi_\alpha|$, where $0 < p_\alpha \leq 1$, $\sum p_\alpha = 1$, $|\psi_\alpha \rangle \in H^d_1 \otimes H^d_2 \otimes H^d_3$ are normalized pure states. If all $|\psi_\alpha \rangle$ are biseparable, namely, either $|\psi_\alpha \rangle = |\varphi^1_\alpha \rangle \otimes |\varphi^{23}_\alpha \rangle$ or $|\psi_\beta \rangle = |\varphi^{2}_\beta \rangle \otimes |\varphi^{13}_\beta \rangle$ or $|\psi_\gamma \rangle = |\varphi^{3}_\gamma \rangle \otimes |\varphi^{12}_\gamma \rangle$, where $|\varphi^i_\alpha \rangle$ and $|\varphi^i_\beta \rangle$ denote pure states in $H^d_i$ and $H^d_i \otimes H^d_3$ respectively, then $\rho$ is said to be bipartite separable. Otherwise, $\rho$ is called genuine multipartite entangled.

Let $\lambda_i$, $i=1,\cdots,d^2-1$, denote the generators of the special unitary group $SU(d)$, and $I$ the $d \times d$ identity matrix. Any $\rho \in H^d_1 \otimes H^d_2 \otimes H^d_3$ can be represented as follows:

\begin{equation}
\rho = \frac{1}{d^3}I \otimes I \otimes I + \frac{1}{2d} \left( \sum t^1_{ij} \lambda_i \otimes I \otimes I + \sum t^2_{ij} \lambda_j \otimes I \otimes I + \sum t^3_{ijk} I \otimes \lambda_i \otimes \lambda_j \right) + \frac{1}{4d} \left( \sum t^{12}_{ij} \lambda_i \otimes \lambda_j \otimes I + \sum t^{13}_{ijk} \lambda_i \otimes I \otimes \lambda_k + \sum t^{23}_{ijk} I \otimes \lambda_j \otimes \lambda_k \right) + \frac{1}{8} \sum t^{123}_{ijk} \lambda_i \otimes \lambda_j \otimes \lambda_k,
\end{equation}

where $t^1_{ij} = tr(\rho \lambda_i \otimes I \otimes I)$, $t^2_{ij} = tr(\rho I \otimes \lambda_j \otimes I)$, $t^3_{ijk} = tr(\rho I \otimes I \otimes \lambda_k)$, $t^{12}_{ij} = tr(\rho I \otimes I \otimes I)$. $t^{123}_{ijk} = \frac{1}{d} tr(\rho \lambda_i \otimes \lambda_j \otimes \lambda_k)$. 

The following theorem shows that the norms of the correlation tensors give an upper bound for the GME concurrence of $\rho$.
tr (ρλ ⊗ I ⊗ λk), t_{jk}^{23} = tr (ρ I ⊗ λj ⊗ λk) and t_{i}^{123} = tr (ρλi ⊗ λj ⊗ λk). In the following, we set T^{(1)}, T^{(2)}, T^{(3)}, T^{(12)}, T^{(13)}, T^{(23)} and T^{(123)} to be the vectors (tensors) with entries t_i^1, t_i^2, t_j^3, t_{i}^{12}, t_{i}^{13}, t_{j}^{23} and t_{ijk}^{123}, i, j, k = 1, 2, · · · , d^3 − 1, which are the so called correlation vectors.

Let ||M||_k = \sum_{i=1}^{n} \sigma_i denote the k-norm for an n \times n matrix M, where \sigma_i, i = 1, ..., n, are the singular values of M in decreasing order. ||M||_n = ||M||_{\infty} is just the Key-Fan norm. Denote ||·|| the Frobenius norm of a vector or a matrix. Let T_{223}, T_{224} and T_{212} be the matrices with entries t_i,d(j−1)+k = t_{ijk}, t_j,d(i−1)+k = t_{ijk} and t_k,d(i−1)+j = t_{ijk}, respectively. Set M_k(ρ) = \frac{1}{k} (||T_{223}||_k + ||T_{224}||_k + ||T_{212}||_k).

Lemma: For a pure tripartite qudit state, we have for any k = 1, 2, · · · , d^3 − 1 and jlm = 123, 213, 312 that (i) if the state is fully separable, then

||T_{jlm}||_k ≤ \sqrt{\frac{8(d−1)^3}{d^3}};

(ii) if the state is separable under bipartite partition j | rm, then

||T_{jlm}||_k ≤ \sqrt{\frac{8(d−1)^2(d+1)}{d^3}};

(iii) if the state is entangled under bipartite partition j | rm, then

||T_{jlm}||_k ≤ \sqrt{\frac{8(k−1)^2(d+1)}{d^3}}.

Proof. We shall use repeatedly ||T_{(i)}|| ≤ \sqrt{\frac{2(d−1)}{d}}i = 1, 2, 3, ||T_{(lm)}|| ≤ 2\sqrt{\frac{d−1}{d^2}}, lm = 12, 13, 23 and ||M||_k ≤ \sqrt{k}||M|| for any matrix M. Then we have

(i)

||T_{jlm}||_k = ||(T_{(j)} \cdot (T_{(l)} \otimes T_{(m)}))'||_k

= ||T_{(j)}|| ||T_{(l)} \otimes T_{(m)}|| = ||T_{(j)}|| ||T_{(l)}|| ||T_{(m)}|| ≤ \sqrt{\frac{8(d−1)^3}{d^3}};

(ii)

||T_{jlm}||_k = ||(T_{(j)} \cdot (T_{(lm)}))'||_k

= ||T_{(j)}|| ||T_{(lm)}|| ≤ \sqrt{\frac{8(d−1)^2(d+1)}{d^3}};

(iii)

||T_{jlm}||_k = ||(T_{(j)} \otimes (T_{(m)}))'||_k

= ||T_{(j)}|| ||T_{(m)}|| ≤ \sqrt{\frac{8(k−1)^2(d+1)}{d^3}};

where we have denoted T_{(j)} the matrix with entries t_{xjy}.

Theorem 1: If for a tripartite qudit state ρ, it holds that

M_k(ρ) > \frac{2\sqrt{2}}{3} (2\sqrt{k} + 1) \frac{d−1}{d} \sqrt{d+1} d

for any k = 1, 2, · · · , d^3 − 1, then ρ is genuine multipartite entangled.

Proof. Assume that ρ is bipartite separable. By using the above Lemma, we get

M_k(ρ) = \frac{1}{3} (||T_{223}(ρ)||_k + ||T_{224}(ρ)||_k + ||T_{212}(ρ)||_k)

= \frac{1}{3} (\sum_{α} p_{α} ||T_{223}(|ψ_{α}\rangle)||_k + \sum_{α} p_{α} ||T_{224}(|ψ_{α}\rangle)||_k + \sum_{α} p_{α} ||T_{212}(|ψ_{α}\rangle)||_k)

≤ \frac{1}{3} \sum_{α} p_{α} (||T_{223}(|ψ_{α}\rangle)||_k + ||T_{224}(|ψ_{α}\rangle)||_k + ||T_{212}(|ψ_{α}\rangle)||_k)

≤ \frac{1}{3} \sum_{α} p_{α} (2\sqrt{\frac{8(k−1)^2(d+1)}{d^3}} + \sqrt{\frac{8(d−1)^2(d+1)}{d^3}})

= \frac{2\sqrt{2}}{3} (2\sqrt{k} + 1) \frac{d−1}{d} \sqrt{d+1} d.
Thus once the inequality is violated, the quantum state \( \rho \) must be genuine multipartite entangled.

Remark. For \( d = 2, k = 3 \), Theorem 1 reduces to the Theorem 2 in [19]. If we set \( d = 2 \), and \( k = 1, 2 \) one finds that Theorem 1 is strictly covered by Theorem 2 in [19]. However, our theorem can detect GME for any tripartite qudit systems rather than only tripartite qubit systems.

Example 1: Consider quantum state \( \rho \in H^d_1 \otimes H^d_2 \otimes H^d_3 \), \( \rho = \frac{1}{d^3} I + x |\varphi \rangle \langle \varphi | \), where \( |\varphi \rangle = \frac{1}{\sqrt{d^3}} (|000 \rangle + |111 \rangle + |222 \rangle) \) is the GHZ state. By Theorem 1 in [19] we can detect GME for 0.894427 < \( x \leq 1 \). With our Theorem 1, by setting \( d = 3 \) and \( k = 8 \) (which is the optimal selection) one detects GME for 0.716235 < \( x \leq 1 \) (see Fig. 1).

![Vicente criterion](image)

**FIG. 1: Vicente criterion (dashed line) v.s. the lower bound in this manuscript (solid line).** We have used \( f(x) \) to denote the difference between the left and the right side of the two criteria. From the figure one sees that \( \rho \) is genuine multipartite entangled for 0.894427 < \( x \leq 1 \) by Vicente’s result with \( f(x) = -2.177324 + 2.434322x \). One computes \( f(x) = -4.83138 + 6.74552x \) by our proposition. Thus GME is detected for 0.716235 < \( x \leq 1 \).

**Proof.** We first consider pure states. For pure state \( \rho = |\psi \rangle \langle \psi | \) one has \( tr \rho^2 = 1 \), which implies that

\[
tr \rho^2 = \frac{1}{d^3} + \frac{1}{2d^2} \sum (t^1_i)^2 + \sum (t^2_j)^2 + \frac{1}{4d} \sum (t^3_k)^2 + \sum (t^{13}_{ik})^2 + \sum (t^{23}_{jk})^2 + \frac{1}{8} \sum (t^{123}_{ijk})^2 = 1. \tag{8}
\]

From (8) we have

\[
\frac{1}{d^3} + \frac{1}{2d^2} \left( \| T^{(1)} \|^2 + \| T^{(2)} \|^2 + \| T^{(3)} \|^2 \right) + \frac{1}{4d} \left( \| T^{(12)} \|^2 + \| T^{(13)} \|^2 + \| T^{(23)} \|^2 \right) + \frac{1}{8} \| T^{(123)} \|^2 = 1.
\]

In the following we denote \( \rho_{jk} \) the reduced density matrix for the subsystems \( j \neq k = 1, 2, 3 \). One computes from (8) that

\[
\rho_1 = \frac{1}{d} I + \frac{1}{2} \sum t^1_i \lambda_i, \quad \rho_{23} = \frac{1}{d} I \otimes I + \frac{1}{2d} \left( \sum t^2_j \lambda_j^2 \otimes I + \sum t^3_k \lambda_k \right) + \frac{1}{4} \sum t^{23}_{jk} \lambda_j \otimes \lambda_k.
\]

Thus

\[
tr \rho^2 = \frac{1}{d} + \frac{1}{2d} \| T^{(1)} \|^2, \quad tr \rho^{23} = \frac{1}{d} + \frac{1}{2d} \| T^{(2)} \|^2 + \| T^{(3)} \|^2, \quad tr T^{(12)} = \frac{1}{d} + \frac{1}{4d} \| T^{(23)} \|^2.
\]

Similarly we get

\[
tr \rho^2 = \frac{1}{d} + \frac{1}{2d} \| T^{(2)} \|^2, \quad tr \rho^{13} = \frac{1}{d} + \frac{1}{2d} \| T^{(1)} \|^2 + \| T^{(3)} \|^2, \quad tr T^{(12)} = \frac{1}{d} + \frac{1}{4d} \| T^{(13)} \|^2,
\]

\[
tr \rho^2 = \frac{1}{d} + \frac{1}{2d} \| T^{(3)} \|^2, \quad tr \rho^{12} = \frac{1}{d} + \frac{1}{2d} \| T^{(1)} \|^2 + \| T^{(2)} \|^2, \quad tr T^{(12)} = \frac{1}{d} + \frac{1}{4d} \| T^{(12)} \|^2.
\]

**LOWER BOUND OF GME CONCURRENCE**

The GME concurrence is proved a well-defined measure [28, 29]. For a pure state \( |\psi \rangle \in H^d_1 \otimes H^d_2 \otimes H^d_3 \), the GME concurrence is defined by

\[
C_{GME}(|\psi \rangle) = \min \left\{ 1 - tr(\rho_1^2), 1 - tr(\rho_2^2), 1 - tr(\rho_3^2) \right\}.
\]

where \( \rho_i \) is the reduced matrix for the \( i \)th subsystem. For mixed state \( \rho \in H^d_1 \otimes H^d_2 \otimes H^d_3 \), the GME concurrence is then defined by the convex roof

\[
C_{GME}(\rho) = \min_{\{p_\alpha, |\psi_\alpha \rangle\}} \sum p_\alpha C_{GME}(|\psi_\alpha \rangle). \tag{6}
\]

The minimum is taken over all pure ensemble decompositions of \( \rho \).

Since one has to find the optimal ensemble to carry out the minimization, the GME concurrence is hard to compute. In this section we derive an effective lower bound for GME concurrence in terms of the norms of the correlation tensors.

**Theorem 2:** For a tripartite qudit state \( \rho \), the GME concurrence satisfies the following inequality,

\[
C_{GME}(\rho) \geq \max \left\{ \frac{1}{2 \sqrt{2}} \left\| T^{(123)} \right\| - \frac{(d - 1)}{d}, 0 \right\}. \tag{7}
\]
By noticing that $\rho = \ket{\psi}\bra{\psi}$ is a pure state, we get $tr\rho_i^2 = tr\rho_j^2$ for $i \neq j \neq k$, $i, j, k = 1, 2, 3$. Then we have:

$$\frac{3}{d} + \frac{1}{d^2}(\|T^{(1)}\|^2 + \|T^{(2)}\|^2 + \|T^{(3)}\|^2) = \frac{3}{d^2} + \frac{1}{d^2}(\|T^{1}\|^2 + \|T^{(2)}\|^2 + \|T^{(3)}\|^2) + \frac{1}{d^2}(\|T^{(12)}\|^2 + \|T^{(13)}\|^2 + \|T^{(23)}\|^2).$$

Set $A = \|T^{(1)}\|^2 + \|T^{(2)}\|^2 + \|T^{(3)}\|^2$, $B = \|T^{(12)}\|^2 + \|T^{(13)}\|^2 + \|T^{(23)}\|^2$ and $C = \|T^{(123)}\|^2$. We have

$$\frac{1}{4}B = (\frac{1}{2} - \frac{1}{d})A + \frac{3}{d} - \frac{2}{d^2}.$$

Substituting (9) into (8), one has

$$tr\rho^2 = \frac{1}{d^2} + \frac{1}{2d^2}A + \frac{1}{4d}B + \frac{1}{8}C = \frac{1}{d^2} + \frac{1}{2d^2}A + \frac{1}{d} \left(\frac{1}{2} - \frac{1}{d}\right)A + \frac{3}{d} - \frac{2}{d^2} + \frac{1}{8}C.$$

Thus we get

$$tr\rho^2 - tr\rho^2_1 = \left(\frac{3}{d^2} - \frac{2}{d^2} - \frac{1}{d}\right) + \frac{2d - 2 - d^2}{2d}||T^{(1)}||^2 + \left(\frac{1}{d} - \frac{1}{d^2}\right)(||T^{(2)}||^2 + ||T^{(3)}||^2) + \frac{1}{8}C$$

$$\geq \frac{-d^2 + 3d - 2}{d} + \frac{2d - 2 - 2(2d - 1)}{2d} + \frac{1}{8}C = \frac{1}{8}C - \frac{(d - 1)^2}{d^2},$$

where we have used $||T^{(1)}||^2 \leq \frac{2(d - 1)}{d}$ and set $(||T^{(2)}||^2 + ||T^{(3)}||^2) = 0$ to get the inequality. Similarly, one obtains

$$tr\rho^2 - tr\rho^2_2 \geq \frac{1}{8}C - \frac{(d - 1)^2}{d^2}, \quad tr\rho^2 - tr\rho^2_3 \geq \frac{1}{8}C - \frac{(d - 1)^2}{d^2}. $$

Thus we have

$$C_{GME}^2(\ket{\psi}) = \min\{1 - tr\rho^2_1, 1 - tr\rho^2_2, 1 - tr\rho^2_3\} = \min\{tr\rho^2 - tr\rho^2_1, tr\rho^2 - tr\rho^2_2, tr\rho^2 - tr\rho^2_3\}$$

$$\geq \max\{\frac{1}{8}C - \frac{(d - 1)^2}{d^2}, 0\}.$$  

We now consider mixed quantum state $\rho \in H_1^d \otimes H_2^d \otimes H_3^d$. Let $\rho = \sum p_a |\psi_a\rangle \langle \psi_a| $ be the optimal ensemble decomposition of $\rho$. We obtain

$$C_{GME}(\rho) = \sum p_a C_{GME}(|\psi_a\rangle) \geq \max\{\frac{1}{2\sqrt{2}} ||T^{(123)}|| - \frac{(d - 1)}{d}, 0\} \sum p_a = \max\{\frac{1}{2\sqrt{2}} ||T^{(123)}|| - \frac{(d - 1)}{d}, 0\},$$

where we have used $\sqrt{a - b} \geq \sqrt{a} - \sqrt{b}$ for $a > b > 0$ and the convex property of the Frobenius norm.

**Example 2:** We consider the mixture of the GHZ state and W state in three-qubit quantum systems $\rho = \frac{1}{4} - \frac{1}{2^2}I + x|GHZ\rangle\langle GHZ| + y|W\rangle\langle W|$, where $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ and $|W\rangle = \frac{1}{\sqrt{2}}(|010\rangle + |101\rangle + |001\rangle + |110\rangle)$. The lower bound of GME concurrence can detect GME better than Vicente’s criterion (Theorem 1) and Theorem 1 in this manuscript as shown in Figure 4.

**Conclusions and Discussions**

It is a basic and fundamental question in quantum theory to detect and quantify GME. Since the GME concurrence is defined by optimization over all ensemble decom-
positions of a mixed quantum state, it is a formidable task to derive an analytical formula. We have derived an analytical and experimentally feasible lower bound for GME concurrence of any tripartite quantum state based on the correlation tensors of the density matrix. We have also obtained an effective criterion to detecting GME for any tripartite quantum states by sketching the Vicente's method. Genuine multipartite entanglement can be detected by using this bound. The results presented in this manuscript are experimentally feasible as the elements in the correlation tensors are just the mean values of the Hermitian SU(d) generators. The approach used in this manuscript can also be implemented to investigate the k-separability of multipartite quantum systems. The results can be also generalized to any multipartite qudit systems.

Acknowledgments This work is supported by the NSFC No.11775306, No.11701568, No.11675113; the Fundamental Research Funds for the Central University.

FIG. 2: Lower bound of GME concurrence in Theorem 2 for $\rho$ in Example 2. $g(x,y)$ stands for the lower bound.

FIG. 3: GME Detected by vicente criterion (pink region by Theorem 1 and yellow region by Theorem 2 in [19]) and by the lower bound for GME concurrence in our Theorem 2 (blue region).

FIG. 4: GME detection: dashed line by Theorem 1 in [19], dotted and solid line by our Theorem 1 and Theorem 2 respectively. $f(x)$ denotes the difference between the left and the right side of the inequalities in these criteria. We compute that $f(x) = 2.17732(1+x), -3.628874 + 4.044882x$ and $\max\{0.222222(-3 + 3.4641x), 0\}$ for Theorem 1 in [19], our Theorem 1 and Theorem 2 respectively. One finds from the figure that Theorem 1 in [19] can not detect GME for the whole region of $x$ in this example, Theorem 2 in [19] can not be operated on this states since it only fits for tripartite qubits system, while Theorem 1 and Theorem 2 in this manuscript detect GME for $0.9 < x \leq 1$ and $0.866025 < x \leq 1$ respectively.

[1] M.A. Nielsen and I.L. Chuang, Quantum Computation and Quantum Information. Cambridge: Cambridge University Press, (2000).
[2] See, for example, D.P. Di Vincenzo, Science 270,255 (1995).
[3] O. Gühne, and G. Toth, Physics Reports 474, 1 (2009).
[4] R. Horodecki et al., Rev. Mod. Phys. 81, 865 (2009).
[5] H.J. Briegel, D.E. Browne, W. Dür, R. Raussendorf and M. Van den Nest, Nat. Phys. 5, 19 (2009).
[6] A. Sen(De) and U. Sen, Phys. News 40, 17 (2010). arXiv:1105.2412
[7] N. Gisin, G. Ribordy, W. Tittel, and H. Zbinden, Rev. Mod. Phy. 74, 145 (2002).
[8] R. Cleve et al., Phys. Rev. Lett. 83, 648 (1999); A. Karlsson et al., Phys. Rev. A 59, 162 (1999).
[9] A. S. Srensen and K. Mmler, Phys. Rev. Lett. 86, 4431 (2001).
[10] P. Hyllus, W. Laskowski, R. Krischek, C. Schwemmer, W. Wieczorek, H. Weinifer, L. PezzAe, and A. Smerzi, Phys. Rev. A 85, 022321 (2012).
[11] R. Raussendorf and H.J. Briegel, Phys. Rev. Lett 86, 5188 (2001).
[12] M. Murao, D. Jonathan, M.B. Plenio and V. Vedral, Phys. Rev. A 59, 156-161 (1999).
[13] M. Hillery, V. Buzek, A. Berthiaume, Phys. Rev. A 59,
1829 (1999).
[14] V. Scarani and N. Gisin, Phys. Rev. Lett. 87, 117901 (2001).
[15] Z. Zhao, Y.A. Chen, A.N. Zhang, T. Yang, H.J. Briegel and J.W. Pan, Nature 430, 54 (2004).
[16] Y. Yeo and W.K. Chua, Phys. Rev. Lett. 96, 060502 (2006), P.X. Chen, S.Y. Zhu, and G.C. Guo, Phys. Rev. A 74, 032324 (2006).
[17] M. Huber, F. Mintert, A. Gabriel, and B. C. Hiesmayr, Phys. Rev. Lett. 104, 210501 (2010).
[18] M. Huber and R. Sengupta, Phys. Rev. Lett. 113, 100501 (2014).
[19] J.I. de Vicente, M. Huber, Phys. Rev. A, 84, 062306 (2011).
[20] J.Y. Wu, H. Kampermann, D. Bruß, C. Klockl, and M. Huber, Phys. Rev. A 86, 022319 (2012).
[21] M. Huber, M. Perarnau-Llobet, J.I. de Vicente, Phys. Rev. A 88, 042328 (2013).
[22] J. Sperling, W. Vogel, Phys. Rev. Lett. 111, 110503 (2013).
[23] C. Eltschka and J. Siewert, Phys. Rev. Lett. 108, 020502 (2012).
[24] B. Jungnitsch, T. Moroder, and O. Gühne, Phys. Rev. Lett. 106, 190502 (2011).
[25] V. Coffman, J. Kundu, and W. K. Wootters, Phys. Rev. A 61, 052306 (2000); T.J. Osborne and F. Verstraete, Phys. Rev. Lett. 96, 220503 (2006); B. Regula, S.D. Martino, S. Lee, and G. Adesso, Phys. Rev. Lett. 113, 110501 (2014).
[26] C. Klckl, M. Huber, Phys. Rev. A 91, 042339 (2015).
[27] M. Markiewicz, W. Laskowski, T. Paterek, and M. Żukowski Phys. Rev. A 87, 034301 (2013).
[28] Z.H. Ma, Z.H. Chen, J.L. Chen, C. Spengler, A. Gabriel, and M. Huber, Phys. Rev. A 83, 062325 (2011).
[29] Z.H. Chen, Z.H. Ma, J.L. Chen, and S. Severini, Phys. Rev. A 85, 062320 (2012).
[30] Y. Hong, T. Gao, and F.L. Yan, Phys. Rev. A 86, 062323 (2012).
[31] T. Gao, F.L. Yan, and S.J. van Enk, Phys. Rev. Lett. 112, 180501 (2014).
[32] J.D. Bancal, N. Gisin, Y.C. Liang, and S. Pironio, Phys. Rev. Lett. 106, 250404 (2011).
[33] C. Lancien, O. Guhne, R. Sengupta, M. Huber, J. Phys. A: Math. Theor. 48, 505302 (2015).
[34] W.K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).
[35] L. Aolita and F. Mintert, Phys. Rev. Lett. 97, 050501 (2006).
[36] A.R.R. Carvalho, F. Mintert, and A. Buchleitner, Phys. Rev. Lett. 93, 230501 (2004).
[37] K. Chen, S. Albeverio and S.M. Fei, Phys. Rev. Lett. 95, 040504 (2005).
[38] J.I. de Vicente, Quantum Inf. Comput. 7, 624(2007).
[39] J.I. de Vicente, J. Phys. A: Math. and Theor., 41, 065309 (2008).
[40] A.S. M. Hassan, P. S. Joag, Quant. Inf. Comput. 8, 0773 (2008).
[41] M. Li, J. Wang, S.M. Fei and X.Q. Li-Jost, Phys. Rev. A, 89,022325 (2014).
[42] A.S. M. Hassan, P. S. Joag, Phys. Rev. A, 77, 062334 (2008).
[43] A.S. M. Hassan, P. S. Joag, Phys. Rev. A, 80, 042302 (2009).
[44] M. Li and S.M. Fei, Phys. Rev. A, 86, 052119 (2012).
[45] M. Li, J. Wang, S.M. Fei, X.Q. Li-Jost, and H. Fan, Phys. Rev. A 92, 062338 (2015).
[46] G. Kimura, Phys. Lett. A, 314, 339(2003).