ON THE $L^p$– THEORY OF ANISOTROPIC SINGULAR PERTURBATIONS OF ELLIPTIC PROBLEMS

OGABI CHOKRI

Académie de Grenoble, Lycée Saint-Marc
Nivolas-Vermelle, 38300, France

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Abstract. In this article we give an extension of the $L^2$–theory of anisotropic singular perturbations for elliptic problems. We study a linear and some non-linear problems involving $L^p$ data ($1 < p < 2$). Convergences in pseudo Sobolev spaces are proved for weak and entropy solutions, and rate of convergence is given in cylindrical domains.

1. Introduction.

1.1. Preliminaries. In this article we shall give an extension of the $L^2$–theory of the asymptotic behavior of elliptic, anisotropic singular perturbations problems. This kind of singular perturbations has been introduced by M. Chipot [6]. From the physical point of view, these problems can modelize diffusion phenomena when the diffusion coefficients in certain directions are going toward zero. The $L^2$ theory of the asymptotic behavior of these problems has been studied by M. Chipot and many co-authors. Before describing the problem, let us begin by a brief discussion on the uniqueness of the weak solution (by a weak solution we mean a solution in the sense of distributions) to the problem

\[
\begin{aligned}
    -\nabla \cdot (A \nabla u) &= f \\
    u &= 0 \text{ on } \partial \Omega,
\end{aligned}
\]

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded Lipschitz domain, and $f \in L^p(\Omega)$ with $1 < p < 2$. The diffusion matrix $A = (a_{ij})$ is supposed to be bounded and satisfies the ellipticity assumption on $\Omega$ (see assumptions (3) and (4) in subsection 1.2). It is well known that (1) has at least a weak solution in $W^{1,p}_0(\Omega)$. Moreover, if $A$ is symmetric and continuous and $\partial \Omega \in C^2$ [2] then (1) has a unique solution in $W^{1,p}_0(\Omega)$. If $A$ is discontinuous the uniqueness assertion is false, in [15] Serrin has given a counterexample when $N \geq 3$. However, if $N = 2$ and if $\partial \Omega$ is sufficiently smooth, and without any continuity assumption on $A$, then (1) has a unique weak solution in $W^{1,p}_0(\Omega)$. The proof is based on the Meyers regularity theorem (see for instance [13]). To treat this pathology, Benilin, Boccardo, Gallouet, et al. have introduced the concept of entropy solution [4], for problems involving $L^1$ data or more generally a Radon measure.

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For every $k > 0$ we define the function $T_k : \mathbb{R} \to \mathbb{R}$ by:

$$T_k(s) = \begin{cases} 
  s & |s| \leq k \\
  k \text{sgn}(s) & |s| \geq k 
\end{cases}$$

And we define the space $T_0^{1,2}$ introduced in [4].

$$T_0^{1,2}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable such that for any } k > 0 \text{ there exists } (\phi_n) \subset H_0^1(\Omega) : \phi_n \to T_k(u) \text{ a.e in } \Omega, \\
  (\nabla \phi_n)_{n \in \mathbb{N}} \text{ is bounded in } L^2(\Omega) \right\}$$

This definition of $T_0^{1,2}$ is equivalent to the original one given in [4]. In fact, this is a characterization of this space [4]. Now, more generally, for $f \in L^1(\Omega)$ we have the following definition of entropy solution [4].

**Definition 1.1.** A function $u \in T_0^{1,2}(\Omega)$ is said to be an entropy solution to (1) if

$$\int_{\Omega} A \nabla u \cdot \nabla T_k(u - \varphi) \, dx \leq \int_{\Omega} f T_k(u - \varphi) \, dx, \quad \varphi \in C_0^\infty(\Omega), \; k > 0$$

We refer the reader to [4] for more details about the sense of this formulation.

The main results of [4] show that (1) has a unique entropy solution which is also a weak solution of (1), moreover since $\Omega$ is bounded then this solution belongs to $\bigcap_{1 \leq r < \frac{N}{N-1}} W^{1,r}_0(\Omega)$.

### 1.2. Description of the problem and functional setting.

Throughout this article we will suppose that $f \in L^p(\Omega), 1 < p < 2$, (we can suppose that $f \notin L^2(\Omega)$). We give a description of the linear problem, some nonlinear problems will be studied later. Consider the following singular perturbations problem:

$$\begin{cases} 
  -\nabla \cdot (A \nabla u_\epsilon) = f \\
  u_\epsilon = 0 \quad \text{on } \partial \Omega 
\end{cases} \tag{2}$$

where $\Omega$ is a bounded Lipschitz domain of $\mathbb{R}^N$ (by Lipschitz we mean strongly Lipschitz). Let $q \in \mathbb{N}^*$, $N - q \geq 2$. We denote by $x = (x_1, \ldots, x_N) = (X_1, X_2) \in \mathbb{R}^q \times \mathbb{R}^{N-q}$ i.e. we split the coordinates into two parts. With this notation we set

$$\nabla = \left( \partial_{x_1}, \ldots, \partial_{x_N} \right)^T = \begin{pmatrix} \nabla X_1 \\ \nabla X_2 \end{pmatrix},$$

where

$$\nabla X_1 = \left( \partial_{x_1}, \ldots, \partial_{x_q} \right)^T \text{ and } \nabla X_2 = \left( \partial_{x_{q+1}}, \ldots, \partial_{x_N} \right)^T$$

Let $A = (a_{ij}(x))$ be a $N \times N$ matrix which satisfies the ellipticity assumption

$$\exists \lambda > 0 : A \xi \cdot \xi \geq \lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^N \text{ for a.e } x \in \Omega, \tag{3}$$

and

$$a_{ij}(x) \in L^\infty(\Omega), \forall i, j = 1, 2, \ldots, N, \tag{4}$$

We have decomposed $A$ into four blocks

$$A = \begin{pmatrix} 
  A_{11} & A_{12} \\
  A_{21} & A_{22} 
\end{pmatrix},$$

where $A_{11}, A_{22}$ are respectively $q \times q$ and $(N-q) \times (N-q)$ matrices. For $0 < \epsilon \leq 1$ we have set

$$A_\epsilon = \begin{pmatrix} 
  \epsilon^2 A_{11} & \epsilon A_{12} \\
  \epsilon A_{21} & A_{22} 
\end{pmatrix}$$
Theorem 2.2. Assume (3), (4) then there exists a unique sequence of the entropy solutions $(u_\varepsilon)_{0<\varepsilon \leq 1} \subset W^{1,p}_0(\Omega)$ of problems (2) and $u_0 \in V_p$ such that $\varepsilon \nabla X_1 u_\varepsilon \to 0$ in $L^p(\Omega)$, $u_\varepsilon \to u_0$ in $V_p$ where $u_0$ satisfies (5) for a.e $X_1 \in \Omega^1$.

Corollary 1. Assume (3), (4) then if $A$ is symmetric and continuous and $\partial \Omega \in C^2$, then there exists a unique $u_0 \in V_p$ such that $u_0(X_1,\cdot)$ is the unique solution to (5) in $W^{1,p}_0(\Omega^1)$ for a.e $X_1$. Moreover, the sequence $(u_\varepsilon)_{0<\varepsilon \leq 1}$ of the unique solutions (in $W^{1,p}_0(\Omega)$) to (2) converges in $V_p$ to $u_0$ and $\varepsilon \nabla X_1 u_\varepsilon \to 0$ in $L^p(\Omega)$.

Proof. This corollary follows immediately from Theorem 2.1 and uniqueness of the solutions of (2) and (5) as mentioned in subsection 1.1 (Notice that $\partial \Omega X_1 \subset C^2$).

Theorem 2.2. Assume (3), (4) then there exists a unique $u_0 \in V_p$ such that $u_0(X_1,\cdot)$ is the unique entropy solution of (5) for a.e $X_1 \in \Omega^1$. Moreover, the sequence of the entropy solutions $(u_\varepsilon)_{0<\varepsilon \leq 1}$ of (2) converges to $u_0$ in $V_p$ and $\varepsilon \nabla X_1 u_\varepsilon \to 0$ in $L^p(\Omega)$. 

2. The linear problem. The main results of this section are the following
2.1. Weak convergence. Let us prove the following primary result

**Theorem 2.3.** Assume (3), (4) then there exist a sequence \((u_{\epsilon_k})_{k \in \mathbb{N}} \subset \text{W}^{1,p}_0(\Omega)\) of weak solutions to (2) \((\epsilon_k \to 0 \text{ as } k \to \infty)\), and \(u_0 \in V_p\) such that \(\nabla X_2 u_{\epsilon_k} \rightharpoonup \nabla X_2 u_0\), \(\epsilon_k \nabla X_1 u_{\epsilon_k} \to 0\), \(u_{\epsilon_k} \rightharpoonup u_0\) in \(L^p(\Omega) - \text{weak}\), and \(u_0\) satisfies (5) for a.e \(X_1 \in \Omega^1\).

**Proof.** By density let \((f_n)_{n \in \mathbb{N}} \subset L^2(\Omega)\) be a sequence such that \(f_n \to f\) in \(L^p(\Omega)\), we can suppose that \(\forall n \in \mathbb{N} : \|f_n\|_{L^p} \leq M, M \geq 0\). Consider the regularized problem:

\[ u_{\epsilon}^n \in H^1_0(\Omega), \quad \int_\Omega A_n \nabla u_{\epsilon}^n \cdot \nabla \varphi dx = \int_\Omega f_n \varphi dx, \quad \varphi \in \mathcal{D}(\Omega) \]  \hspace{1cm} (6)

Assumptions (3) and (4) show that \(u_{\epsilon}^n\) exists and it is unique, by the Lax-Milgram theorem. Notice that \(u_{\epsilon}^n\) belongs to \(W^{1,p}_0(\Omega)\). We introduce the function

\[ \theta(t) = \int_0^t (1 + |s|)^{p-2} ds, \quad t \in \mathbb{R}. \]

This kind of function has been used in [3]. We have \(\theta'(t) = (1 + |t|)^{p-2} \leq 1\) and \(\theta(0) = 0\), therefore we have \(\theta(u) \in H^1_0(\Omega)\) for every \(u \in H^1_0(\Omega)\). Testing with \(\theta(u_{\epsilon}^n)\) in (6) and using the ellipticity assumption we deduce

\[ \lambda \epsilon^2 \int_\Omega (1 + |u_{\epsilon}^n|^p - 2|\nabla X_1 u_{\epsilon}^n|^2) dx + \lambda \int_\Omega (1 + |u_{\epsilon}^n|^p - 2|\nabla X_2 u_{\epsilon}^n|^2) dx \]

\[ \leq \int_\Omega f_n \theta(u_{\epsilon}^n) dx \leq \frac{2}{p-1} \int_\Omega |f_n| (1 + |u_{\epsilon}^n|^p)^{-1} dx, \]

where we have used \(|\theta(t)| \leq \frac{2(1+|t|)^{p-1}}{p-1}\). In the other hand, by Hölder’s inequality we have

\[ \int_\Omega |\nabla X_2 u_{\epsilon}^n|^p dx \leq \left( \int_\Omega (1 + |u_{\epsilon}^n|^p)^{-2} |\nabla X_2 u_{\epsilon}^n|^2 dx \right)^{\frac{p}{2}} \left( \int_\Omega (1 + |u_{\epsilon}^n|^p) dx \right)^{1-\frac{p}{2}}. \]

From the two previous integral inequalities we deduce

\[ \int_\Omega |\nabla X_2 u_{\epsilon}^n|^p dx \leq \left( \frac{2}{\lambda(p-1)} \int_\Omega |f_n| (1 + |u_{\epsilon}^n|^p)^{-1} dx \right)^{\frac{p}{2}} \times \]

\[ \left( \int_\Omega (1 + |u_{\epsilon}^n|^p) dx \right)^{1-\frac{p}{2}}. \]

By Hölder’s inequality we get

\[ \|\nabla X_2 u_{\epsilon}^n\|_{L^p(\Omega)} \leq \left( \frac{2}{\lambda(p-1)} \right)^{\frac{1}{2}} \left( \int_\Omega (1 + |u_{\epsilon}^n|^p) dx \right)^{\frac{1}{p}}. \]  \hspace{1cm} (7)

Using Minkowski inequality we get

\[ \|\nabla X_2 u_{\epsilon}^n\|_{L^p(\Omega)}^2 \leq C(1 + \|u_{\epsilon}^n\|_{L^p(\Omega)}), \]

Thanks to Poincaré’s inequality \(\|u_{\epsilon}^n\|_{L^p(\Omega)} \leq C_\Omega \|\nabla X_2 u_{\epsilon}^n\|_{L^p(\Omega)}\) we obtain

\[ \|\nabla X_2 u_{\epsilon}^n\|_{L^p(\Omega)}^2 \leq C'(1 + \|\nabla X_2 u_{\epsilon}^n\|_{L^p(\Omega)}), \]

where the constant \(C'\) depends on \(p, \lambda, \text{mes}(\Omega), M\) and \(C_\Omega\). Whence, we deduce

\[ \|u_{\epsilon}^n\|_{L^p(\Omega)}, \quad \|\nabla X_2 u_{\epsilon}^n\|_{L^p(\Omega)} \leq C''. \]  \hspace{1cm} (8)

Similarly we obtain

\[ \|\epsilon \nabla X_1 u_{\epsilon}^n\|_{L^p(\Omega)} \leq C''', \]  \hspace{1cm} (9)
where the constants \( C'' \), \( C''' \) are independent of \( n \) and \( \epsilon \), so

\[
\|u^n_\epsilon\|_{W^{1,p}(\Omega)} \leq \frac{\text{Const}}{\epsilon}. \tag{10}
\]

Fix \( \epsilon \), since \( W^{1,p}(\Omega) \) is reflexive then (10) implies that there exist a subsequence \((u^n_{\epsilon_l})_{l \in \mathbb{N}}\) and \( u_\epsilon \in W^{1,p}_0(\Omega) \) such that \( u^n_{\epsilon_l} \rightharpoonup u_\epsilon \in W^{1,p}_0(\Omega) \) (as \( l \to \infty \)) in \( W^{1,p}(\Omega) \)-weak. Now, passing to the limit in (6) as \( l \to \infty \) we deduce

\[
\int_\Omega A_\epsilon \nabla u_\epsilon \cdot \nabla \varphi dx = \int_\Omega f \varphi dx, \, \varphi \in D(\Omega). \tag{11}
\]

Whence \( u_\epsilon \) is a weak solution of (2) \((u_\epsilon = 0 \text{ on } \partial \Omega \text{ in the trace sense of } W^{1,p}-\text{functions, indeed the trace operator is well defined since } \partial \Omega \text{ is Lipschitz}).

Now, from (8) we deduce

\[
\|u_\epsilon\|_{L^p(\Omega)} \leq \liminf_{l \to \infty} \left\|u^n_{\epsilon_l}\right\|_{L^p(\Omega)} \leq C'',
\]

and

\[
\|
abla X_2 u_\epsilon\|_{L^p(\Omega)} \leq \liminf_{l \to \infty} \left\|
abla X_2 u^n_{\epsilon_l}\right\|_{L^p(\Omega)} \leq C''.
\]

and similarly from (9) we obtain

\[
\|\epsilon \nabla X_1 u_\epsilon\|_{L^p(\Omega)} \leq C'''.
\]

Using reflexivity of \( L^p(\Omega) \) and continuity of the derivation operator on \( D'(\Omega) \), one can extract a subsequence \((u_{\epsilon_k})_{k \in \mathbb{N}}\) such that \( \nabla X_2 u_{\epsilon_k} \rightharpoonup \nabla X_2 u_0 \), \( \epsilon_k \nabla X_1 u_{\epsilon_k} \to 0 \), \( u_{\epsilon_k} \to u_0 \) in \( L^p(\Omega) \) – weak. Passing to the limit in (11) we get

\[
\int_\Omega A_\epsilon \nabla X_2 u_0 \cdot \nabla X_2 \varphi dx = \int_\Omega f \varphi dx, \, \varphi \in D(\Omega). \tag{12}
\]

Now, we will prove that \( u_0 \in V_p \). Since \( \nabla X_2 u_{\epsilon_k} \rightharpoonup \nabla X_2 u_0 \) and \( u_{\epsilon_k} \to u_0 \) in \( L^p(\Omega) \) – weak then there exists a sequence \((U_n)_{n \in \mathbb{N}} \subset \text{conv}(\{u_{\epsilon_k}\}_{k \in \mathbb{N}})\) such that \( \nabla X_2 U_n \to \nabla X_2 u_0 \) in \( L^p(\Omega) \) – strong, where \( \text{conv}(\{u_{\epsilon_k}\}_{k \in \mathbb{N}}) \) is the convex hull of the set \( \{u_{\epsilon_k}\}_{k \in \mathbb{N}} \). Notice that we have \( U_n \in W^{1,p}_0(\Omega) \) then \( u_0 \in \text{conv}(\{u_{\epsilon_k}\}_{k \in \mathbb{N}}) \). And we also have \( \nabla X_2 U_n \rightharpoonup \nabla X_2 u_0 \) in \( L^p(\Omega) \) – strong a.e \( X_1 \in \Omega^1 \). Whence \( u_0 \in \text{conv}(\{u_{\epsilon_k}\}_{k \in \mathbb{N}}) \in \text{conv}(\{u_{\epsilon_k}\}_{k \in \mathbb{N}}) \) for a.e \( X_1 \in \Omega^1 \), so \( u_0 \in V_p \).

Finally, we will prove that \( u_0 \) is a solution of (5). Let \( E \) be a Banach space, a family of vectors \( \{e_n\}_{n \in \mathbb{N}} \) in \( E \) is said to be a Banach basis or a Schauder basis of \( E \) if for every \( x \in E \) there exists a family of scalars \( \{\alpha_n\}_{n \in \mathbb{N}} \) such that \( x = \sum_{n=0}^{\infty} \alpha_n e_n \), where the series converges in the norm of \( E \). Notice that Schauder basis does not always exist. In [11] P. Enflo has constructed a separable reflexive Banach space without Schauder basis! However, the Sobolev space \( W^{1,p}_0 (1 < r < \infty) \) has a Schauder basis whenever the boundary of the domain is sufficiently smooth [12]. Now, we are ready to finish the proof. Let \( (U_i \times V_i)_{i \in \mathbb{N}} \) be a countable covering of \( \Omega \) such that \( U_i \times V_i \subset \Omega \) where \( U_i \subset \mathbb{R}^q \), \( V_i \subset \mathbb{R}^{N-q} \) are two bounded open domains, where \( \partial V_i \) is smooth (\( V_i \) are Euclidian balls for example), such a covering always exists. Now, fix \( \psi \in D(V_i) \) then it follows from (12) that for every \( \varphi \in D(U_i) \) we have

\[
\int_{U_i} \varphi dX_1 \int_{V_i} A_{22} \nabla X_2 u_0 \cdot \nabla X_2 \psi dX_2 = \int_{U_i} \varphi dX_1 \int_{V_i} f \psi dX_2.
\]
Whence for a.e \( X_1 \in U \), we have
\[
\int_{V_i} A_{22}(X_1, \cdot) \nabla X_2 u_0(X_1, \cdot) \cdot \nabla X_2 \psi dX_2 = \int_{V_i} f(X_1, \cdot) \psi dX_2.
\]

Notice that by density we can take \( \psi \in W^{1,p'}_0(V_i) \) where \( p' \) is the conjugate of \( p \). Using the same techniques as in [8], where we use a Schauder basis of \( W^{1,p'}_0(V_i) \) and a partition of the unity, one can easily obtain
\[
\int_{\Omega_{X_1}} A_{22}(X_1, \cdot) \nabla X_2 u_0(X_1, \cdot) \cdot \nabla X_2 \phi dx = \int_{\Omega_{X_1}} f(X_1, \cdot) \phi dx, \quad \phi \in D(\Omega),
\]
for a.e \( X_1 \in \Omega^1 \). Finally, since \( u_0(X_1, \cdot) \in W^{1,p}(\Omega_{X_1}) \) (as proved above) then \( u_0(X_1, \cdot) \) is a solution of (5), notice that \( \partial \Omega_{X_1} \) is also Lipschitz so, the trace operator is well defined. \( \square \)

2.2. Strong convergence. Theorem 2.1 will be proved in three steps. the proof is based on the use of the approximated problem (6). First, we shall construct the solution of the limit problem.

Step1: Let \( u^n \in H^1_0(\Omega) \) be the unique solution to (6), existence and uniqueness of \( u^n \) follows from assumptions (3), (4) as mentioned previously. One have the following proposition.

**Proposition 1.** Assume (3), (4) then there exists \( (u^n_0)_{n \in \mathbb{N}} \subset V_2 \) such that \( e_n \to 0 \) in \( L^2(\Omega) \), \( u^n_2 \to u^n_0 \) in \( V_2 \) as \( \epsilon \to 0 \), for every \( n \in \mathbb{N} \), in particular the two convergences holds in \( L^p(\Omega) \) and \( V_p \) respectively, where \( u^n_0 \) is the unique weak solution in \( V_2 \) to the problem

\[
\begin{cases}
\nabla X_2 \cdot (A_{22}(X_1, \cdot) \nabla X_2 u^n_0(X_1, \cdot)) = f_n(X_1, \cdot), \text{ a.e } X_1 \in \Omega^1 \\
u^n_0(X_1, \cdot) = 0 \text{ on } \partial \Omega_{X_1}
\end{cases}
\]

(13)

**Proof.** This result follows from the \( L^2 \)-theory (Theorem 1 in [8]). The convergences in \( V_p \) and \( L^p(\Omega) \) follow from the continuous embedding \( V_2 \hookrightarrow V_p, L^2(\Omega) \hookrightarrow L^p(\Omega), \quad 1 < p < 2 \). \( \square \)

Now, we construct \( u_0 \) the solution of the limit problem (5). Testing with \( \phi = \theta(u^n_0(X_1, \cdot)) \) in the weak formulation of (13) (\( \theta \) is the function introduced in subsection 2.1) and estimating like in proof of Theorem 2.3 we obtain as in (7)

\[
\| \nabla X_2 u^n_0(X_1, \cdot) \|_{L^p(\Omega_{X_1})} \leq \left( \frac{\| f_n(X_1, \cdot) \|_{L^p(\Omega_{X_1})}}{\lambda(p-1)} \right)^{\frac{1}{p}} \times \left( \int_{\Omega_{X_1}} (1 + |u^n_0(X_1, \cdot)|^p) dX_2 \right)^{\frac{1}{p}}
\]

(14)

Integrating over \( \Omega^1 \) and using Cauchy-Schwarz’s inequality in the right hand side we get

\[
\| \nabla X_2 u^n_0 \|_{L^p(\Omega)}^p \leq C \| f_n \|_{L^p(\Omega)}^p \left( \int_{\Omega} (1 + |u^n_0|^p) dX_2 \right)^{\frac{1}{2}}.
\]

and therefore

\[
\| \nabla X_2 u^n_0 \|_{L^p(\Omega)}^2 \leq C'(1 + \| u^n_0 \|_{L^p(\Omega)}).
\]
Using Poincaré’s inequality \( \|u^n_0\|_{L^p(\Omega)} \leq C_\Omega \|\nabla X_2 u^n_0\|_{L^p(\Omega)} \) (which holds since \( u^n_0(X_1,\cdot) \in W^{1,p}(\Omega_{X_1}) \) a.e \( X_1 \in \Omega^1 \)), one can obtain
\[
\|u^n_0\|_{L^p(\Omega)} \leq C'' \text{ for every } n \in \mathbb{N},
\]
where \( C'' \) is independent of \( n \). Now, using the linearity of the problem and (13) with the test function \( \theta(u^n_0(X_1,\cdot) - u^m_0(X_1,\cdot)) \), \( m, n \in \mathbb{N} \), one can obtain like in (14)
\[
\|\nabla X_2 (u^n_0(X_1,\cdot) - u^m_0(X_1,\cdot))\|_{L^p(\Omega_{X_1})} \leq \left( \left( \frac{\|f_n(X_1,\cdot) - f_m(X_1,\cdot)\|_{L^p(\Omega_{X_1})}}{\lambda(p-1)} \right)^\frac{1}{2} \times \left( \int_{\Omega_{X_1}} (1 + |u^n_0(X_1,\cdot) - u^m_0(X_1,\cdot)|)^p dX_2 \right)^\frac{1}{p} \right).
\]
Integrating over \( \Omega^1 \) and using Cauchy-Schwarz inequality and (15) yields
\[
\|\nabla X_2 (u^n_0 - u^m_0)\|_{L^p(\Omega)} \leq C \|f_n - f_m\|_{L^p(\Omega)}^\frac{1}{2},
\]
where \( C \) is independent of \( m \) and \( n \). The Poincaré’s inequality shows that
\[
\|u^n_0 - u^m_0\|_{V_p} \leq C' \|f_n - f_m\|_{L^p(\Omega)}^\frac{1}{2}.
\]
Since \( (f_n)_{n \in \mathbb{N}} \) is a converging sequence in \( L^p(\Omega) \) then this last inequality shows that \( (u^n_0)_{n \in \mathbb{N}} \) is a Cauchy sequence in \( V_p \), consequently there exists \( u_0 \in V_p \) such that \( u^n_0 \to u_0 \) in \( V_p \). Now, passing to the limit in (6) as \( \epsilon \to 0 \), we get according to Proposition 1
\[
\int_{\Omega} A_{22} \nabla X_2 u^n_0 \cdot \nabla X_2 \varphi dX_2 = \int_{\Omega} f_n \varphi dX_2, \quad \varphi \in \mathcal{D}(\Omega).
\]
Passing to the limit as \( n \to \infty \) we deduce
\[
\int_{\Omega} A_{22} \nabla X_2 u_0 \cdot \nabla X_2 \varphi dX_2 = \int_{\Omega} f \varphi dX_2, \quad \varphi \in \mathcal{D}(\Omega).
\]
Then it follows, as proved in Theorem 2.3, that \( u_0 \) satisfies (5). Whence we have proved the following

**Proposition 2.** Under assumptions of Proposition 1 there exists \( u_0 \in V_p \) solution to (5) such that \( u^n_0 \to u_0 \) in \( V_p \) where \( (u^n_0)_{n \in \mathbb{N}} \) is the sequence given in Proposition 1.

**Step 2:** We shall construct the sequence \( (u_\epsilon)_{0<\epsilon \leq 1} \) solutions of (2), one can prove the following

**Proposition 3.** There exists a sequence \( (u_\epsilon)_{0<\epsilon \leq 1} \subset W^{1,p}_{0}(\Omega) \) of weak solutions to (2) such that \( u^n_\epsilon \to u_\epsilon \) in \( W^{1,p}(\Omega) \) as \( n \to \infty \), for every \( \epsilon \) fixed. Moreover, \( u^n_\epsilon \to u_\epsilon \) in \( V_p \) and \( \epsilon \nabla X_2 u^n_\epsilon \to \epsilon \nabla X_2 u_\epsilon \), in \( L^p(\Omega) \) uniformly in \( \epsilon \).

**Proof.** Using the linearity of (6) testing with \( \theta(u^n_\epsilon - u^m_\epsilon) \), \( m, n \in \mathbb{N} \) we obtain as in (7)
\[
\|\nabla X_2 (u^n_\epsilon - u^m_\epsilon)\|_{L^p(\Omega)} \leq \left( \frac{\|f_n - f_m\|_{L^p}}{\lambda(p-1)} \right)^\frac{1}{2} \left( \int_{\Omega} (1 + |u^n_\epsilon - u^m_\epsilon|^p) dX_2 \right)^\frac{1}{p}.
\]
And (8) gives
\[ \| \nabla X_2(u^n - u^m) \|_{L^p(\Omega)} \leq C \| f_n - f_m \|_{L^p}, \]
where \( C \) is independent of \( \epsilon \) and \( n \), whence Poincaré's inequality implies
\[ \| u^n - u^m \|_{V^p} \leq C' \| f_n - f_m \|_{L^p}. \quad (16) \]

Similarly we obtain
\[ \| \epsilon \nabla X_2(u^n - u^m) \|_{L^p(\Omega)} \leq C'' \| f_n - f_m \|_{L^p}. \quad (17) \]

It follows that
\[ \| u^n - u^m \|_{W^{1,p}(\Omega)} \leq \frac{C'}{\epsilon} \| f_n - f_m \|_{L^p}. \]

The last inequality implies that for every fixed \( (u^n)_{n \in \mathbb{N}} \) is a Cauchy sequence in \( W^{1,p}_0(\Omega) \). Then there exists \( u_\epsilon \in W^{1,p}_0(\Omega) \) such that \( u^n \to u_\epsilon \) in \( W^{1,p}(\Omega) \), then the passage to the limit in (6) shows that \( u_\epsilon \) is a weak solution of (2). Finally (16) and (17) show that \( u^n \to u_\epsilon \) (resp \( \epsilon \nabla X_2 u^n \to \epsilon \nabla X_2 u_\epsilon \)) in \( V^p \) (resp in \( L^p(\Omega) \)) uniformly in \( \epsilon \).

\[ \square \]

Step 3: Now, we are ready to conclude. Proposition 1, 2 and 3 combined with the triangular inequality show that \( u_\epsilon \to u_0 \) in \( V^p \) and \( \epsilon \nabla X_2 u_\epsilon \to 0 \) in \( L^p(\Omega) \), and the proof of Theorem 2.1 is finished.

2.3. Convergence of the entropy solutions. In this subsection we prove Theorem 2.2. As mentioned in section 1 the entropy solution \( u_\epsilon \) of (2) exists and it is unique. We shall construct this entropy solution. Using the approximated problem (6), one has a \( W^{1,p} \)-strongly converging sequence \( u^n \to u_\epsilon \) in \( W^{1,p}_0(\Omega) \) as shown in Proposition 3. We will show that \( u_\epsilon \in \mathcal{T}^{1,2}_0(\Omega) \). Clearly we have\( T_k(u^n) \in H^1(\Omega) \) for every \( k > 0 \). Now testing with \( T_k(u^n) \) in (6) we obtain
\[ \int_\Omega A_\epsilon \nabla u^n \cdot \nabla T_k(u^n) \, dx = \int_\Omega f_n T_k(u^n) \, dx. \]

Using the ellipticity assumption we get
\[ \int_\Omega |\nabla T_k(u^n)|^2 \, dx \leq \frac{Mk}{\lambda(1+\epsilon^2)}. \quad (18) \]

Fix \( \epsilon, k \), we have \( u^n_\epsilon \to u_\epsilon \) in \( L^p(\Omega) \) then there exists a subsequence \( (u^{n_i}_\epsilon)_{i \in \mathbb{N}} \) such that \( u^{n_i}_\epsilon \to u_\epsilon \) a.e \( x \in \Omega \) and since \( T_k \) is bounded then it follows that \( T_k(u^{n_i}_\epsilon) \to T_k(u_\epsilon) \) a.e in \( \Omega \) and strongly in \( L^2(\Omega) \) whence \( u_\epsilon \in \mathcal{T}^{1,2}_0(\Omega) \).

It follows by (18) that there exists a subsequence still labelled \( T_k(u^{n_i}_\epsilon) \) such that \( \nabla T_k(u^{n_i}_\epsilon) \to v_{\epsilon,k} \in L^2(\Omega) \). The continuity of \( \nabla \) on \( \mathcal{D}'(\Omega) \) implies that \( v_{\epsilon,k} = \nabla T_k(u_\epsilon) \), whence \( T_k(u^{n_i}_\epsilon) \to T_k(u_\epsilon) \) in \( H^1(\Omega) \). Now, since \( T_k(u^{n_i}_\epsilon) \in H^1_0(\Omega) \) then we deduce that \( T_k(u_\epsilon) \in H^1_0(\Omega) \).

It follows [4] that
\[ \int_\Omega A_\epsilon \nabla u_\epsilon \cdot \nabla T_k(u_\epsilon - \varphi) \, dx \leq \int_\Omega fT_k(u_\epsilon - \varphi) \, dx. \]

Whence \( u_\epsilon \) is the entropy solution of (2). Similarly the function \( u_0 \) (constructed in Proposition 2) is the entropy solution to (5) for a.e \( X_1 \) The uniqueness of \( u_0 \) in \( V^p \) follows from the uniqueness of the entropy solution of problem (5). Finally, the convergences given in Theorem 2.2 follow from Theorem 2.1.

**Remark 1.** Uniqueness of the entropy solution implies that it does not depend on the choice of the approximated sequence \( (f_n)_n \).
2.4. A regularity result for the entropy solution of the limit problem. In this subsection we assume that \( \Omega = \omega_1 \times \omega_2 \) where \( \omega_1, \omega_2 \) are two bounded Lipschitz domains of \( \mathbb{R}^n, \mathbb{R}^{N-q} \) respectively. We introduce the space

\[
W_p = \{ u \in L^p(\Omega) \mid \nabla X_1 u \in L^p(\Omega) \}
\]

We suppose the following

\[
f \in W_p \quad \text{and} \quad A_{22}(x) = A_{22}(X_2) \quad \text{i.e} \quad A_{22} \quad \text{is independent of} \quad X_1.
\]

**Theorem 2.4.** Assume (3), (4), (19) then \( u_0 \in W^{1,p}(\Omega) \), where \( u_0 \) is the entropy solution of (5).

**Proof.** Let \((u_n^0)\) be the sequence constructed in subsection 2.2, we have \( u_n^0 \to u_0 \) in \( V_p \), where \( u_0 \) is the entropy solution of (5).

Let \( \omega_1' \subset \subset \omega_1 \) be an open subset, for \( 0 < h < d(\partial \omega_1, \omega_1') \) and for \( X_1 \in \omega_1' \) we set \( \tau_h X_1 u_n^0 = u_n^0(X_1 + he_i, X_2) \) where \( e_i = (0, \ldots, 1, \ldots, 0) \) then we have by (13)

\[
\int_{\omega_2} A_{22} \nabla X_1 (\tau_h X_1 u_n^0 - u_n^0) \cdot \nabla X_2 \varphi dX_2 = \int_{\omega_2} (\tau_h X_1 f_n - f_n) \varphi dX_2, \quad \varphi \in D(\omega_2),
\]

where we have used \( A_{22}(x) = A_{22}(X_2) \).

We introduce the function \( \theta_\delta(t) = \int_0^t (\delta + |s|)^{p-2} ds, \delta > 0, t \in \mathbb{R} \) we have

\[
0 < \theta_\delta'(t) = (\delta + |t|)^{p-2} \leq \delta^{p-2} \quad \text{and} \quad |\theta_\delta(t)| \leq \frac{2(\delta + |t|)^{p-2}}{p-1}
\]

Testing with \( \varphi = \frac{1}{h} \theta_\delta (\tau_h X_1 u_n^0 - u_n^0) \in H_0^1(\omega_2) \). To make the notations less heavy we set

\[
U = \frac{\tau_h X_1 u_n^0 - u_n^0}{h}, \quad (\tau_h X_1 f_n - f_n) = F
\]

Then we get

\[
\int_{\omega_2} \theta_\delta'(U) A_{22} \nabla X_2 U \cdot \nabla X_2 U dX_2 = \int_{\omega_2} F \theta_\delta(U) dX_2.
\]

Using the ellipticity assumption for the left hand side, and Hölder’s inequality for the right hand side of the above integral equality we deduce

\[
\lambda \int_{\omega_2} \theta_\delta'(U) \|
abla X_2 U \|^2 dX_2 \leq \frac{2}{p-1} \| F \|_{L^p(\omega_2)} \left( \int_{\omega_2} (\delta + |U|)^p dX_2 \right)^{\frac{p-1}{p}}.
\]

Using Hölder’s inequality we derive

\[
\| \nabla X_2 U \|_{L^p(\omega_2)}^p \leq \left( \int_{\omega_2} \theta_\delta'(U) |\nabla X_2 U|^2 dX_2 \right)^{\frac{p}{2}} \left( \int_{\omega_2} \theta_\delta'(U)^{\frac{2p}{p-2}} dX_2 \right)^{\frac{2-p}{2}}
\]

\[
\leq \left( \frac{2}{\lambda(p-1)} \| F \|_{L^p(\omega_2)} \left( \int_{\omega_2} (\delta + |U|)^p dX_2 \right)^{\frac{p-1}{p}} \right)^{\frac{p}{2}} \times \left( \int_{\omega_2} \theta_\delta'(U)^{\frac{2p}{p-2}} dX_2 \right)^{\frac{2-p}{2}}.
\]

Then we deduce

\[
\| \nabla X_2 U \|_{L^p(\omega_2)}^{p} \leq \frac{2}{\lambda(p-1)} \| F \|_{L^p(\omega_2)} \left( \int_{\omega_2} (\delta + |U|)^p dX_2 \right)^{\frac{p}{2}}.
\]
Theorem 3.1. Under the above assumptions we have the following

\[ \| \nabla X_2 U \|^2_{L^p(\omega_2)} \leq \frac{2}{\lambda(p-1)} \| F \|^p_{L^p(\omega_2)} \left( \int_{\omega_2} |U|^p dX_2 \right)^{\frac{1}{p}}, \]

and Poincaré’s inequality gives

\[ \| \nabla X_2 U \|^p_{L^p(\omega_2)} \leq \frac{2C_{\omega_2}}{\lambda(p-1)} \| F \|^p_{L^p(\omega_2)}. \]

Now, integrating over \( \omega'_1 \) yields

\[ \left\| \frac{\tau^1_h u^0_n - u^0_0}{h} \right\|_{L^p(\omega'_1 \times \omega_2)} \leq \frac{2C_{\omega_2}}{\lambda(p-1)} \left\| \frac{(\tau^1_h f_n - f_n)}{h} \right\|_{L^p(\omega'_1 \times \omega_2)}. \]

Passing to the limit as \( n \to \infty \) using the invariance of the Lebesgue measure under translations we get

\[ \left\| \frac{\tau^1_h u_0 - u_0}{h} \right\|_{L^p(\omega'_1 \times \omega_2)} \leq \frac{2C_{\omega_2}}{\lambda(p-1)} \left\| \frac{(\tau^1_h f - f)}{h} \right\|_{L^p(\omega'_1 \times \omega_2)}. \]

Whence, since \( f \in W_p \), then

\[ \left\| \frac{\tau^I h u_0 - u_0}{h} \right\|_{L^p(\omega'_1 \times \omega_2)} \leq C, \]

where \( C \) is independent of \( h \), therefore we have \( \nabla X_1 u_0 \in L^p(\Omega) \). Combining this with \( u_0 \in V_p \) we get the desired result. \( \square \)

3. The rate of convergence theorem. In this section we suppose that \( \Omega = \omega_1 \times \omega_2 \) where \( \omega_1, \omega_2 \) are two bounded Lipschitz domains of \( \mathbb{R}^q \) and \( \mathbb{R}^{N-q} \) respectively. We suppose that \( A_{12}, A_{22} \) and \( f \) depend on \( X_2 \) only i.e \( A_{12}(x) = A_{12}(X_2), A_{22}(x) = A_{22}(X_2) \) and \( f(x) = f(X_2) \in L^p(\omega_2) \ (1 < p < 2), \ f \notin L^2(\omega_2) \).

Let \( u_\epsilon, u_0 \) be the unique entropy solutions of (2), (5) respectively then under the above assumptions we have the following

**Theorem 3.1.** For every \( \omega'_1 \subset \subset \omega_1 \) and \( m \in \mathbb{N}^+ \) there exists \( C \geq 0 \) independent of \( \epsilon \) such that

\[ \| u_\epsilon - u_0 \|_{W^{1,p}(\omega'_1 \times \omega_2)} \leq C\epsilon^m \]

**Proof.** Let \( u_\epsilon, u_0 \) be the entropy solutions of (2), (5) respectively, we use the approximated sequence \( (u^\alpha_\epsilon)_{\epsilon, n}, (u^0_0)_n \) introduced in section 2. Subtracting (13) from (6) we obtain

\[ \int_{\Omega} A_{12} \nabla (u^\alpha_\epsilon - u^0_0) \cdot \nabla \varphi dx = 0, \]

where we have used that \( u^0_0 \) is independent of \( X_1 \) (since \( f \) and \( A_{22} \) are independent of \( X_1 \)) and that \( A_{12} \) is independent of \( X_1 \).

Let \( \omega'_1 \subset \subset \omega_1 \) then there exists \( \omega''_1 \subset \subset \omega''_1 \subset \subset \omega_1 \). We introduce the function \( \rho \in \mathcal{D}(\omega_1) \) such that \( \text{Supp}(\rho) \subset \omega''_1 \) and \( \rho = 1 \) on \( \omega'_1 \). We can choose \( 0 \leq \rho \leq 1 \). Testing with \( \varphi = \rho^2 \delta_\delta (u^\alpha_\epsilon - u^0_0) \in H^1_0(\Omega) \) (we can check easily that this function
belongs to $H_0^1(\Omega)$ using an approximation argument) in the above integral equality we get

$$
\int_{\Omega} \rho^2 \theta_\delta(u^n_\varepsilon - u^n_0) A_\varepsilon \nabla(u^n_\varepsilon - u^n_0) \cdot \nabla(u^n_\varepsilon - u^n_0) dx
$$

$$
= - \int_{\Omega} \rho \theta_\delta(u^n_\varepsilon - u^n_0) A_\varepsilon \nabla(u^n_\varepsilon - u^n_0) \cdot \nabla \rho dx
$$

$$
= - \epsilon^2 \int_{\Omega} \rho \theta_\delta(u^n_\varepsilon - u^n_0) A_{11} \nabla X_1(u^n_\varepsilon - u^n_0) \cdot \nabla X_1 \rho dx
$$

$$
- \epsilon \int_{\Omega} \rho \theta_\delta(u^n_\varepsilon - u^n_0) A_{12} \nabla X_2(u^n_\varepsilon - u^n_0) \cdot \nabla X_1 \rho dx.
$$

where we have used that $\rho$ is independent of $X_2$.

Using the ellipticity assumption for the left hand side and assumption (4) for the right hand side of the previous integral equality we deduce

$$
\epsilon^2 \lambda \int_{\Omega} \theta_\delta'(u^n_\varepsilon - u^n_0) |\rho \nabla X_1(u^n_\varepsilon - u^n_0)|^2 dx
$$

$$
+ \lambda \int_{\Omega} \theta_\delta'(u^n_\varepsilon - u^n_0) |\rho \nabla X_2(u^n_\varepsilon - u^n_0)|^2 dx
$$

$$
\leq \epsilon^2 C \int_{\Omega} \rho |\theta_\delta(u^n_\varepsilon - u^n_0)||\nabla X_1(u^n_\varepsilon - u^n_0)| dx
$$

$$
+ \epsilon C \int_{\Omega} \rho |\theta_\delta(u^n_\varepsilon - u^n_0)||\nabla X_2(u^n_\varepsilon - u^n_0)| dx.
$$

Where $C \geq 0$ depends on $A$ and $\rho$. Using Young’s inequality $ab \leq \frac{a^2}{2\epsilon} + \frac{b^2}{2}$ for the two terms in the right hand side of the previous inequality, we obtain

$$
\epsilon^2 \lambda \int_{\Omega} \theta_\delta'(u^n_\varepsilon - u^n_0) |\rho \nabla X_1(u^n_\varepsilon - u^n_0)|^2 dx + \frac{\lambda}{2} \int_{\Omega} \theta_\delta'(u^n_\varepsilon - u^n_0) |\rho \nabla X_2(u^n_\varepsilon - u^n_0)|^2 dx
$$

$$
\leq \epsilon^2 C' \int_{\omega_1' \times \omega_2} |\theta_\delta(u^n_\varepsilon - u^n_0)|^2 \theta_\delta'(u^n_\varepsilon - u^n_0)^{-1} dx.
$$

Whence

$$
\epsilon^2 \lambda \int_{\Omega} \theta_\delta'(u^n_\varepsilon - u^n_0) |\rho \nabla X_1(u^n_\varepsilon - u^n_0)|^2 dx + \frac{\lambda}{2} \int_{\Omega} \theta_\delta'(u^n_\varepsilon - u^n_0) |\rho \nabla X_2(u^n_\varepsilon - u^n_0)|^2 dx
$$

$$
\leq \frac{4}{(p-1)^2} \epsilon^{2} C'' \int_{\omega_1' \times \omega_2} (\delta + |u^n_\varepsilon - u^n_0|)^p dx,
$$

where $C''$ is independent of $\epsilon$ and $n$. 
Now, using Hölder’s inequality and the above inequality we deduce
\[
e^2 \frac{\lambda}{2} \| \rho \nabla X_1(u^n - u_0^n) \|^2_{L^p(\Omega)} + \frac{\lambda}{2} \| \rho \nabla X_2(u^n - u_0^n) \|^2_{L^p(\Omega)}
\leq \left[ e^2 \left( \int_{\Omega} |\nabla \theta'_\delta(u^n - u_0^n)|^2 \, dx \right) + \frac{\lambda}{2} \left( \int_{\Omega} |\rho \nabla X_2(u^n - u_0^n)|^2 \, dx \right) \right] \times
\left( \int_{\omega'_n \times \omega_2} (\delta + |u^n - u_0^n|)^{p-2} \, dx \right) \leq \frac{4C'\epsilon^2}{(p-1)^2} \left( \int_{\omega'_n \times \omega_2} (\delta + |u^n - u_0^n|)^{p-2} \, dx \right)^{\frac{3}{2}}.
\]

Passing to the limit as \( \delta \to 0 \) using the Lebesgue theorem. Passing to the limit as \( n \to \infty \) we get
\[
e^2 \| \nabla X_1(u - u_0) \|^2_{L^p(\omega'_n \times \omega_2)} + \| \nabla X_2(u - u_0) \|^2_{L^p(\omega'_n \times \omega_2)} \leq C' \epsilon^2 \| (u - u_0) \|^2_{L^p(\omega'_n \times \omega_2)} \tag{20}
\]
Using Poincaré’s inequality
\[
\| (u - u_0) \|^2_{L^p(\omega'_n \times \omega_2)} \leq C_{\omega_2} \| \nabla X_2(u - u_0) \|^2_{L^p(\omega'_n \times \omega_2)},
\]
we obtain
\[
e^2 \| \nabla X_1(u - u_0) \|^2_{L^p(\omega'_n \times \omega_2)} + \| \nabla X_2(u - u_0) \|^2_{L^p(\omega'_n \times \omega_2)} \leq C' \epsilon^2 \| \nabla X_2(u - u_0) \|^2_{L^p(\omega'_n \times \omega_2)}.
\]
Let \( m \in \mathbb{N}^+ \) then there exist \( \omega'_1 \subset \subset \omega''_1 \subset \subset \ldots \subset \omega''_1^{(m+1)} \subset \subset \omega'_1 \). Iterating the above inequality \( m \)–time we deduce
\[
e^2 \| \nabla X_1(u - u_0) \|^2_{L^p(\omega''_1 \times \omega_2)} + \| \nabla X_2(u - u_0) \|^2_{L^p(\omega''_1 \times \omega_2)} \leq C_m \epsilon^{2m} \| \nabla X_2(u - u_0) \|^2_{L^p(\omega''_1 \times \omega_2)}.
\]
Now, from (20) (with \( \omega'_1 \) and \( \omega''_1 \) replaced by \( \omega''_1^{(m)} \) and \( \omega''_1^{(m+1)} \) respectively) we deduce
\[
e^2 \| \nabla X_1(u - u_0) \|^2_{L^p(\omega''_1^{(m)} \times \omega_2)} + \| \nabla X_2(u - u_0) \|^2_{L^p(\omega''_1^{(m)} \times \omega_2)} \leq C' \epsilon^{2(m+1)} \| u - u_0 \|^2_{L^p(\omega''_1^{(m+1)} \times \omega_2)}.
\]
Since \( u \to u_0 \) in \( L^p(\Omega) \) then \( \| u - u_0 \|_{L^p(\Omega)} \) is bounded and therefore we obtain
\[
\| u - u_0 \|_{W^{1,p}(\omega'_1 \times \omega_2)} \leq C_m \epsilon^m.
\]
And the proof of the theorem is finished. \( \Box \)

Can one obtain a better convergence rate? In fact, the anisotropic singular perturbation problem (2) can be seen as a problem in a cylinder becoming unbounded. Indeed the two problems can be connected to each other via a scaling \( \epsilon = \frac{1}{\ell} \) (see [5] for more details). So let us consider the problem:
\[
\begin{align*}
-\nabla \cdot (A \nabla u_\ell) &= f \\
u_\ell &= 0 \quad \text{on } \partial \Omega_\ell
\end{align*}
\tag{21}
\]
Theorem 3.2. Let the iteration technique introduced in [7], let
\[ \exists \lambda > 0 : \tilde{A} \xi \cdot \xi \geq \lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^N \] for a.e \( x \in \mathbb{R}^d \times \omega_2 \),

\[ \Omega_\ell = \ell \omega_1 \times \omega_2 \] a bounded domain where \( \omega_1, \omega_2 \) are two bounded Lipschitz domain with \( \omega_1 \) convex and containing 0.

We assume that \( f \in L^p(\omega_2) \) (1 < \( p < 2 \)) and \( \tilde{A}_{22}(x) = \tilde{A}_{22}(X_2), \tilde{A}_{12}(x) = \tilde{A}_{12}(X_2) \).

We consider the limit problem:
\[
\begin{cases}
-\nabla_{X_2} \cdot (\tilde{A}_{22} \nabla_{X_2} u_\infty) = f \\
u_\infty = 0 \quad \text{on} \quad \partial \omega_2
\end{cases}
\] (24)

Then under the above assumptions we have

\[ \|\nabla (u_\ell - u_\infty)\|_{W^{1,p}(\Omega_{\omega_\ell})} \leq Ce^{-c\ell} \]

Proof. Let \( u_\ell, u_\infty \) be the unique entropy solutions to (21) and (24) respectively, and let \( (u^n_\ell) \) and \( (u^n_\infty) \) the approximation sequences (as in section 2). we have \( u^n_\ell \to u_\ell \) in \( W^{1,p}(\Omega_\ell) \) and \( u^n_\infty \to u_\infty \) in \( W^{1,p}(\omega_2) \). Subtracting the associated approximated problems to (21) and (24) and take the weak formulation we get

\[ \int_{\Omega_\ell} \tilde{A} \nabla (u^n_\ell - u^n_\infty) \nabla \phi dx = 0, \phi \in \mathcal{D}(\Omega). \] (25)

Where we have used that \( \tilde{A}_{22}, \tilde{A}_{12}, u^n_\infty \) are independent of \( X_1 \). Now we will use the iteration technique introduced in [7], let \( 0 < \ell_0 \leq \ell - 1 \), and let \( \rho \in \mathcal{D}(\mathbb{R}^d) \) a bump function such that

\[ 0 \leq \rho \leq 1, \rho = 1 \text{ on } \ell_0 \omega_1 \text{ and } \rho = 0 \text{ on } \mathbb{R}^d \setminus (\ell_0 + 1)\omega_1, |\nabla \rho| \leq c_0. \]

where \( c_0 \) is the universal constant (see [5]). Testing with \( \rho^2 \theta_\delta (u^n_\ell - u^n_\infty) \in H^1_0(\Omega_\ell) \) in (25) we get

\[ \int_{\Omega_\ell} \rho^2 \theta_\delta (u^n_\ell - u^n_\infty) \tilde{A} \nabla (u^n_\ell - u^n_\infty) \cdot \nabla (u^n_\ell - u^n_\infty) dx \]

\[ + \int_{\Omega_\ell} \rho \theta_\delta (u^n_\ell - u^n_\infty) \tilde{A} \nabla (u^n_\ell - u^n_\infty) \cdot \nabla \rho dx = 0. \]

Using the ellipticity assumption (23)

\[ \int_{\Omega_\ell} \rho^2 \theta_\delta (u^n_\ell - u^n_\infty) |\nabla (u^n_\ell - u^n_\infty)|^2 dx \leq 2 \int_{\Omega_\ell} \rho |\theta_\delta (u^n_\ell - u^n_\infty)| \left| \tilde{A} \nabla (u^n_\ell - u^n_\infty) \right| |\nabla \rho| dx. \]
Notice that $\nabla \rho = 0$ on $\Omega_{\ell_0}$, and $\Omega_{\ell_0} \subset \Omega_{\ell_0+1}$ (since $\omega_1$ is convex and containing 0). Then by the Cauchy-Schwarz inequality we get

$$\int_{\Omega_{\ell}} \rho^2 \theta'_0(u^n_{\ell} - u^n_{\infty}) |\nabla (u^n_{\ell} - u^n_{\infty})|^2 \, dx \leq 2c_0 C \int_{\Omega_{\ell_0+1} \setminus \Omega_{\ell_0}} \rho |\theta'_0(u^n_{\ell} - u^n_{\infty})| |\nabla (u^n_{\ell} - u^n_{\infty})| \, dx$$

$$\leq 2c_0 C \left( \int_{\Omega_{\ell}} \rho^2 \theta'_0(u^n_{\ell} - u^n_{\infty}) |\nabla (u^n_{\ell} - u^n_{\infty})|^2 \, dx \right)^{\frac{1}{2}} \times \left( \int_{\Omega_{\ell_0+1} \setminus \Omega_{\ell_0}} |\theta'_0(u^n_{\ell} - u^n_{\infty})|^2 \theta'_0(u^n_{\ell} - u^n_{\infty})^{-1} \, dx \right)^{\frac{1}{2}}.$$ 

where we have used (22). Whence we get (since $\rho = 1$ on $\Omega_{\ell_0}$)

$$\int_{\Omega_{\ell_0}} \theta'_0(u^n_{\ell} - u^n_{\infty}) |\nabla (u^n_{\ell} - u^n_{\infty})|^2 \, dx \leq \int_{\Omega_{\ell}} \rho^2 \theta'_0(u^n_{\ell} - u^n_{\infty}) |\nabla (u^n_{\ell} - u^n_{\infty})|^2 \, dx \leq \left( \frac{4c_0 C}{p-1} \right)^2 \int_{\Omega_{\ell_0+1} \setminus \Omega_{\ell_0}} (\delta + |u^n_{\ell} - u^n_{\infty}|)^p \, dx.$$

From Hölder’s inequality it holds that

$$||\nabla (u^n_{\ell} - u^n_{\infty})||^2_{L^p(\Omega_{\ell_0})} \leq \left( \int_{\Omega_{\ell_0}} \theta'_0(u^n_{\ell} - u^n_{\infty}) |\nabla (u^n_{\ell} - u^n_{\infty})|^2 \, dx \right)^{\frac{2}{p}} \times \left( \int_{\Omega_{\ell_0}} (\delta + |u^n_{\ell} - u^n_{\infty}|)^p \, dx \right)^{\frac{2-p}{p}}$$

$$\leq \left( \frac{4c_0 C}{p-1} \right)^2 \left( \int_{\Omega_{\ell_0+1} \setminus \Omega_{\ell_0}} (\delta + |u^n_{\ell} - u^n_{\infty}|)^p \, dx \right)^{\frac{2-p}{p}} \times \left( \int_{\Omega_{\ell_0}} (\delta + |u^n_{\ell} - u^n_{\infty}|)^p \, dx \right)^{\frac{2-p}{p}}.$$

Passing to the limit as $\delta \to 0$ (using the Lebesgue theorem) we get

$$||\nabla (u^n_{\ell} - u^n_{\infty})||^2_{L^p(\Omega_{\ell_0})} \leq C_1 \left( \int_{\Omega_{\ell_0+1} \setminus \Omega_{\ell_0}} |u^n_{\ell} - u^n_{\infty}|^p \, dx \right)^{\frac{2-p}{p}} \times \left( \int_{\Omega_{\ell_0}} |u^n_{\ell} - u^n_{\infty}|^p \, dx \right)^{\frac{2-p}{p}},$$

where we have used $0 \leq \rho \leq 1$. Using Poincaré’s inequality

$$|u^n_{\ell} - u^n_{\infty}|_{L^p(\Omega_{\ell_0})} \leq C_{\omega_2} \|\nabla (u^n_{\ell} - u^n_{\infty})\|_{L^p(\Omega_{\ell_0})},$$

we get

$$||\nabla (u^n_{\ell} - u^n_{\infty})||^p_{L^p(\Omega_{\ell_0})} \leq C_2 \|u^n_{\ell} - u^n_{\infty}\|_{L^p(\Omega_{\ell_0+1} \setminus \Omega_{\ell_0})}^p.$$

Using Poincaré’s inequality

$$|u^n_{\ell} - u^n_{\infty}|_{L^p(\Omega_{\ell_0+1} \setminus \Omega_{\ell_0})} \leq C_{\omega_2} \|\nabla (u^n_{\ell} - u^n_{\infty})\|_{L^p(\Omega_{\ell_0+1} \setminus \Omega_{\ell_0})},$$

we get

$$||\nabla (u^n_{\ell} - u^n_{\infty})||^p_{L^p(\Omega_{\ell_0})} \leq C_3 \|\nabla (u^n_{\ell} - u^n_{\infty})\|_{L^p(\Omega_{\ell_0+1} \setminus \Omega_{\ell_0})}^p.$$

Whence

$$||\nabla (u^n_{\ell} - u^n_{\infty})||^p_{L^p(\Omega_{\ell_0})} \leq \frac{C_3}{C_3 + 1} \|\nabla (u^n_{\ell} - u^n_{\infty})\|_{L^p(\Omega_{\ell_0+1})}^p.$$
Let $\alpha \in (0, 1)$, iterating this formula starting from $\alpha \ell$ we get
\[||\nabla (u^n_\alpha - u_n^\alpha)||_{L^p(\Omega, \ell)} \leq \left(\frac{C_3}{C_3 + 1}\right)^{\frac{1}{\alpha \ell}} ||\nabla (u^n_\alpha - u_\alpha)||_{L^p(\Omega, (1-\alpha)\ell)} .\]
Whence
\[||\nabla (u^n_\alpha - u_\alpha)||_{L^p(\Omega, \ell)} \leq ce^{-c' \ell} ||\nabla (u^n_\alpha - u_\alpha)||_{L^p(\Omega)} ,\]
where $c, c' > 0$ are independent of $\ell$ and $n$.

Now we have to estimate the right hand side of (26). Testing with $\theta(u^n_\alpha)$ in the approximated problem associated to (21) one can obtain as in subsection 2.1
\[||\nabla u^n_\alpha||_{L^p(\Omega)} \leq C\ell^{\frac{1}{2}} .\]
Similarly testing with $\theta(u^n_\alpha)$ in the approximated problem associated to (24), we get
\[||\nabla u^n_\alpha||_{L^p(\Omega_1)} \leq C'\ell^{\frac{1}{2}} .\]
We apply the triangular inequality to the right hand side of (26), and we use (27), (28). Passing to the limit as $n \to \infty$ we obtain the desired result.

**Corollary 2.** Under the above assumptions we have, for every $\alpha \in (0, 1)$ there exist $C \geq 0$, $c > 0$ independent of $\epsilon$ such that
\[||u_\epsilon - u_0||_{W^{1,p}(\omega_1 \times \omega_2)} \leq Ce^{-\epsilon} .\]
where $u_\epsilon$, $u_0$ are the entropy solutions to (2) and (5) respectively.

**Remark 2.** It is very difficult to prove the rate convergence theorem for general data. When $f(x) = f_1(x_2) + f_2(x)$ with $f_1 \in L^p(\omega_2)$ and $f_2 \in W_2$ we only have the estimates
\[\epsilon ||\nabla X_1(u_\epsilon - u_0)||_{L^p(\omega_1' \times \omega_2)} + ||\nabla X_2(u_\epsilon - u_0)||_{L^p(\omega_1' \times \omega_2)} + ||u_\epsilon - u_0||_{L^p(\omega_1' \times \omega_2)} \leq C\epsilon .\]
This follows from the linearity of the equation, Theorem 3.1 and the $L^2$–theory [8].

4. Some Extensions to nonlinear problems and applications.

4.1. A semilinear monotone problem. We consider the semilinear problem:
\[
\begin{cases}
-\nabla \cdot (A_\epsilon \nabla u_\epsilon) = f + a(u_\epsilon) \\
u_\epsilon = 0 & \text{on } \partial \Omega
\end{cases}
\]
(29)

Where $a : \mathbb{R} \to \mathbb{R}$ is a continuous nonincreasing function which satisfies the growth condition
\[\forall x \in \mathbb{R} : |a(x)| \leq K(1 + |x|), K \geq 0 ,\]
\[f \in L^p(\Omega) \text{ where } 1 < p < 2 , f \notin L^2(\Omega) \text{ and } A \text{ is given as in Subsection 1.2. It is clear that the Nemytskii operator } u \mapsto a(u) \text{ maps } L^p(\Omega) \rightarrow L^p(\Omega) \text{ continuously for every } 1 \leq r < \infty. \]
The passage to the limit (formally) gives the limit problem
\[
\begin{cases}
-\nabla X_1 \cdot (A_{22}(X_1, \cdot) \nabla u_0(X_1, \cdot)) = f(X_1, \cdot) + a(u_0(X_1, \cdot)) \\
u_0(X_1, \cdot) = 0 & \text{on } \partial \Omega X_1
\end{cases}
\]
(31)

We can suppose that $a(0) = 0$. Indeed, in the general case the right hand side of (29) can be replaced by $(a(0) + f) + b(x)$ where $b(x) = a(x) - a(0)$. Clearly $b$ is continuous nonincreasing and satisfies $|b(x)| \leq (K + |a(0)|)(1 + |x|)$.
First of all, suppose that \( f \in L^2(\Omega) \), then we have the following

**Proposition 4.** Assume (3), (4) and \( a(0) = 0 \). Let \( u_\varepsilon \) be the unique weak solution in \( H^1_0(\Omega) \) to (29) then \( \varepsilon \nabla X_1 u_\varepsilon \to 0 \) in \( L^2(\Omega) \) and \( u_\varepsilon \to u_0 \) in \( V_2 \) where \( u_0 \) in the unique solution in \( V_2 \) to the limit problem (31).

**Proof.** Existence of \( u_\varepsilon \) follows directly by a simple application of the Schauder fixed point theorem, for example. The uniqueness follows form monotonicity of \( a \) and the Poincaré’s inequality.

Take \( u_\varepsilon \) as a test function in (29) then one can obtain the estimates

\[
\varepsilon \| \nabla X_1 u_\varepsilon \|_{L^2(\Omega)} \cdot \| \nabla X_2 u_\varepsilon \|_{L^2(\Omega)} \cdot \| u_\varepsilon \|_{L^2(\Omega)} \leq C,
\]

where \( C \) is independent of \( \varepsilon \), we have used that \( \int_\Omega a(u_\varepsilon) u_\varepsilon dx \leq 0 \) (thanks to monotonicity assumption and \( a(0) = 0 \)). And we also have (thanks to assumption (30))

\[
\| a(u_\varepsilon) \|_{L^2(\Omega)} \leq K(\| \Omega \|^{\frac{1}{2}} + C)
\]

So there exist \( v \in L^2(\Omega), \ u_0 \in L^2(\Omega), \ \nabla X_2 u_0 \in L^2(\Omega) \) and a subsequence \( (u_{\varepsilon_k})_{k \in \mathbb{N}} \) such that

\[
a(u_{\varepsilon_k}) \to v, \ \varepsilon_k \nabla X_1 u_{\varepsilon_k} \to 0, \ \nabla X_2 u_{\varepsilon_k} \to \nabla X_2 u_0, \ u_{\varepsilon_k} \to u_0 \text{ in } L^2(\Omega)-\text{weak.}
\]

Passing to the limit in the weak formulation of (29) we get

\[
\int_\Omega A_{22} \nabla X_2 u_0 \cdot \nabla X_2 \varphi dx = \int_\Omega f \varphi dx + \int_\Omega v \varphi dx, \ \varphi \in \mathcal{D}(\Omega).
\]

Take \( \varphi = u_{\varepsilon_k} \) in (33) and passing to the limit we get

\[
\int_\Omega A_{22} \nabla X_2 u_0 \cdot \nabla X_2 u_{\varepsilon_k} dx = \int_\Omega f u_{\varepsilon_k} dx + \int_\Omega v u_{\varepsilon_k} dx.
\]

Let us compute the quantity

\[
0 \leq I_k = \int_\Omega A_{22} \left( \frac{\nabla X_1 u_{\varepsilon_k}}{\nabla X_2 (u_{\varepsilon_k} - u_0)} \right) \cdot \left( \frac{\nabla X_1 u_{\varepsilon_k}}{\nabla X_2 (u_{\varepsilon_k} - u_0)} \right) dx
\]

\[
- \int_\Omega \left( a(u_{\varepsilon_k}) - a(u_0) \right) (u_{\varepsilon_k} - u_0) dx
\]

\[
= \int_\Omega f u_{\varepsilon_k} dx - \varepsilon \int_\Omega A_{21} \nabla X_2 u_0 \cdot \nabla X_1 u_{\varepsilon_k} dx - \varepsilon \int_\Omega A_{21} \nabla X_1 u_{\varepsilon_k} \cdot \nabla X_2 u_{\varepsilon_k} u_0 dx
\]

\[
- \int_\Omega A_{22} \nabla X_2 u_{\varepsilon_k} \cdot \nabla X_2 u_0 dx - \int_\Omega A_{22} \nabla X_2 u_0 \cdot \nabla X_2 u_{\varepsilon_k} dx + \int_\Omega f u_0 dx
\]

\[
+ \int_\Omega v u_0 dx + \int_\Omega a(u_0) u_0 dx + \int_\Omega a(u_{\varepsilon_k}) u_0 dx - \int_\Omega a(u_0) u_{\varepsilon_k} dx.
\]

(This quantity is positive thanks to the ellipticity and monotonicity assumptions).

Passing to the limit as \( k \to \infty \) using (32), (33), (34) we get

\[
\lim I_k = 0
\]

And finally The ellipticity assumption and Poincaré’s inequality show that

\[
\| \varepsilon \nabla X_1 u_{\varepsilon_k} \|_{L^2(\Omega)} \cdot \| \nabla X_2 (u_{\varepsilon_k} - u_0) \|_{L^2(\Omega)} \cdot \| u_{\varepsilon_k} - u_0 \|_{L^2(\Omega)} \to 0.
\]

Whence (33) becomes

\[
\int_\Omega A_{22} \nabla X_2 u_0 \cdot \nabla X_2 \varphi dx = \int_\Omega f \varphi dx + \int_\Omega a(u_0) \varphi dx, \ \varphi \in \mathcal{D}(\Omega).
\]
Proof. 1) The existence of estimates where \(C\) defines the mapping \(\Gamma : X\), entropy solution to (31) for a.e \(f / \Omega = \omega\), above proposition and the following inequality, which holds for every \(u,v\), approximated problem;

Proof of Theorem solutions to (29) and (31) follows from the general result proved in [4]. As in Theorem 4.1. Suppose that \(\exists\) Nonlinear problem without monotonicity assumption. Suppose that \(\exists\) \(\Omega = \omega_1 \times \omega_2\) where \(\omega_1, \omega_2\) are two bounded Lipschitz domains, and consider the following nonlinear problem:

\[
\left\{
\begin{array}{l}
-\nabla \cdot (A_e \nabla u_e) = f + a(u_e) \\
u_e = 0 \quad \text{on } \partial \Omega
\end{array}
\right.
\]

We follow the same arguments as in subsections 2.2 and 2.3 where we use the above proposition and the following inequality, which holds for every \(u,v \in L^p(\Omega)\)

\[
\int_{\Omega} (a(u) - a(v)) \theta(u - v) dx \leq 0,
\]

in fact this follows from the monotonicity of \(a\) and \(\theta\). \(\Box\)

4.2. Nonlinear problem without monotonicity assumption. Suppose that \(\exists\) \(\Omega = \omega_1 \times \omega_2\) where \(\omega_1, \omega_2\) are two bounded Lipschitz domains, and consider the following nonlinear problem:

\[
\left\{
\begin{array}{l}
-\nabla \cdot (A_e \nabla u_e) = f + B(u_e) \\
u_e = 0 \quad \text{on } \partial \Omega
\end{array}
\right.
\]

Where \(f \in L^p(\Omega), 1 < p < 2\) and \(B : L^p(\Omega) \to L^p(\Omega)\) is a continuous nonlinear operator. We suppose that

\[
\exists M \geq 0, \forall u \in L^p(\Omega) : \|B(u)\|_{L^p} \leq M \tag{38}
\]

Proposition 5. Assume (3), (4), and (38) then:

1) There exists a sequence \((u_e)_{0 < \epsilon \leq 1} \subset W_0^{1,p}(\Omega)\) of entropy solutions to (37) which are also weak solutions such that

\[
\epsilon \|\nabla X_1 u_e\|_{L^p(\Omega)} : \|\nabla X_2 u_e\|_{L^p(\Omega)} : \|u_e\|_{L^p(\Omega)} \leq C_0,
\]

where \(C_0 \geq 0\) is independent of \(\epsilon\) (the constant \(C_0\) depends only on \(\Omega, \lambda, f\) and \(M\)).

2) If \((u_e)_{0 < \epsilon \leq 1}\) is a sequence of entropy solutions to (37), then we have the above estimates.

Proof. 1) The existence of \(u_e\) is based on the Schauder fixed point theorem, we define the mapping \(\Gamma : L^p(\Omega) \to L^p(\Omega)\) by

\[
v \in L^p(\Omega) \to \Gamma(v) = v_e \in W_0^{1,p}(\Omega)
\]
where \( v_\epsilon \) is the entropy solution (which is also a weak solution) of the linearized problem:

\[
\begin{aligned}
- \nabla \cdot (A_\epsilon \nabla v_\epsilon) &= f + B(v) \\
v_\epsilon &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\] (39)

Since the entropy solution is unique then \( \Gamma \) is well defined, we can prove easily, by using the approximation method, that \( \Gamma \) is continuous. As in subsection 2.1 we can obtain the estimates

\[
\epsilon \|\nabla X_1 v_\epsilon\|_{L^p(\Omega)}, \|\nabla X_2 v_\epsilon\|_{L^p(\Omega)}, \|v_\epsilon\|_{L^p(\Omega)} \leq C_0
\]

where \( C_0 \) is independent of \( \epsilon \) and \( v \) (thanks to (38))

For \( \epsilon \) fixed we define the subset

\[
K_\epsilon = \{ v \in W^{1,p}_0(\Omega) : \epsilon \|\nabla X_1 v\|_{L^p(\Omega)}, \|\nabla X_2 v\|_{L^p(\Omega)}, \|v\|_{L^p(\Omega)} \leq C_0 \}
\]

The subset \( K_\epsilon \) is convex and compact in \( L^p(\Omega) \) thanks to the Sobolev compact embedding \( W^{1,p}_0(\Omega) \subset L^p(\Omega) \).

Since \( C_0 \) is independent of \( v \) then the subset \( K_\epsilon \) is stable under \( \Gamma \). Whence \( \Gamma \) admits at least a fixed point \( u_\epsilon \in K_\epsilon \), in other words \( u_\epsilon \) is an entropy solution to (37) which is also a weak solution.

2) Let \( (u_\epsilon)_{0 < \epsilon \leq 1} \) be a sequence of entropy solutions to (37), \( u_\epsilon \) is the unique entropy solution (which is also a weak solution) to (39) with \( v \) replaced by \( u_\epsilon \) and therefore we obtain the desired estimates as proved in 1).

**Remark 3.** In the general case the entropy solution \( u_\epsilon \) of (37) is not necessarily unique.

Now, assume that

\[
f(x) = f(X_2), A_{22}(x) = A_{22}(X_2), A_{12}(x) = A_{12}(X_2).
\] (40)

And assume that for every \( E \subset W^1_p \) bounded in \( L^p(\Omega) \) we have

\[
\overline{\text{conv}} \{ B(E) \} \subset W_2,
\] (41)

where \( \overline{\text{conv}} \{ B(E) \} \) is the closed convex-hull of \( B(E) \) in \( L^p(\Omega) \). Assumption (41) has been introduced in our preprint [14]. We shall give later some concrete examples of operators which satisfy this assumption. Let us prove the following

**Theorem 4.2.** Assume (3), (4), (38), (40) and (41). Let \( (u_\epsilon)_{0 < \epsilon \leq 1} \subset W^{1,p}_0(\Omega) \) be a sequence of entropy solutions to (37) then for every \( \Omega' \subset \subset \Omega \) there exists \( C_{\Omega'} \geq 0 \) independent of \( \epsilon \) such that

\[
\forall \epsilon : \|u_\epsilon\|_{W^{1,p}(\Omega')} \leq C_{\Omega'}.
\]

**Proof.** The proof is similar to the one given in our preprint [14]. Let \( (\Omega_j)_{j \in \mathbb{N}} \) be an open covering of \( \Omega \) such that \( \overline{\Omega_j} \subset \Omega_{j+1} \). We equip the space \( Z = W^{1,p}_0(\Omega) \) with the topology generated by the family of seminorms \( (p_j)_{j \in \mathbb{N}} \) defined by

\[
p_j(u) = \|u_\epsilon\|_{W^{1,p}(\Omega_j)}
\]

Equipped with this topology, \( Z \) is a separated locally convex topological vector space. We set \( Y = L^p(\Omega) \) equipped with its natural topology. We define the family of linear continuous mappings

\[
\Lambda_\epsilon : Y \rightarrow Z
\]
by: $g \in Y$, $\Lambda_\epsilon(g) = v_\epsilon$ where $v_\epsilon$ is the unique entropy solution to
\[
\begin{cases}
  -\nabla \cdot (A_\epsilon \nabla v_\epsilon) &= g \\
  v_\epsilon &= 0 \quad \text{on } \partial \Omega
\end{cases}
\]

The continuity of $\Lambda_\epsilon$ follows immediately if we observe $\Lambda_\epsilon$ as a composition of $\Lambda_\epsilon : Y \to Y$ and the canonical injection $Y \to Z$.

Now, we denote $Z_w$, $Y_w$ the spaces $Z, Y$ equipped with the weak topology respectively. Then $\Lambda_\epsilon : Y_w \to Z_w$ is also continuous.

Consider the bounded (in $Y$) subset
\[E_0 = \left\{ u \in W_p \mid \|u\|_{L^p(\Omega)} \leq C_0 \right\},\]
where $C_0$ is the constant introduced in Proposition 5. Consider the subset $G = f + \overline{\text{conv}} \{B(E_0)\}$ where the closure is taken in the $L^p-$topology. $G$ is convex closed in $Y$ and it is bounded thanks to (38). Assume (41) then $G \subset L^p(\omega_2) + W_2$, it follows from Remark 2 that for every $g \in G$ the orbit $\{\Lambda_\epsilon g\}_\epsilon$ is bounded in $Z$, and therefore $\{\Lambda_\epsilon g\}_\epsilon$ is bounded in $Z_w$.

Clearly the set $G$ is compact in $Y_w$. Then it follows by the Banach-Steinhaus theorem (applied on the quadruple $(\Lambda_\epsilon, G, Y_w, Z_w)$) that there exists a bounded subset $F$ in $Z_w$ such that
\[\forall \epsilon : \Lambda_\epsilon(G) \subset F.\]

The boundedness of $F$ in $Z_w$ implies its boundedness in $Z$, i.e. for every $j \in N$ there exists $C_j \geq 0$ independent of $\epsilon$ such that
\[\forall \epsilon : p_j(\Lambda_\epsilon(G)) \leq C_j.\]

Let $(u_\epsilon)_\epsilon$ be a sequence of entropy solutions to (37), then we have $(u_\epsilon)_\epsilon \subset E_0$ as proved in Proposition 5 then $\Lambda_\epsilon(f + B(u_\epsilon)) = u_\epsilon \in F$ for every $\epsilon$, therefore
\[\forall \epsilon : \|u_\epsilon\|_{W^{1,p}(\Omega)} \leq C_j.\]

Whence for every $\Omega' \subset \subset \Omega$ there exists $C_{\Omega'} \geq 0$ independent of $\epsilon$ such that
\[\forall \epsilon : \|u_\epsilon\|_{W^{1,p}(\Omega')} \leq C_{\Omega'}.\]

\[\square\]

Now, we are ready to prove the convergence theorem. Assume that
\[B : (L^p(\Omega), \tau_{L^p_{\text{loc}}}) \to L^p(\Omega) \text{ is continuous} \quad (42)\]
where $(L^p(\Omega), \tau_{L^p_{\text{loc}}})$ is the space $L^p(\Omega)$ equipped with the $L^p_{\text{loc}}(\Omega)$-topology. Notice that (42) implies that $B : L^p(\Omega) \to L^p(\Omega)$ is continuous. Then we have the following

**Theorem 4.3.** Under assumptions of Theorem 4.2, assume in addition (42), suppose that $\Omega$ is convex, then there exist $u_0 \in V_p$ and a sequence $(u_{\epsilon_k})_{k \in N}$ of entropy solutions to (37) such that
\[\epsilon_k \nabla X_{\epsilon_k} u_{\epsilon_k} \to 0, \quad \nabla X_{\epsilon_k} u_{\epsilon_k} \to \nabla X_{\epsilon_k} u_0 \text{ in } L^p(\Omega) - \text{weak}\]
and
\[u_{\epsilon_k} \to u_0 \text{ in } L^p_{\text{loc}}(\Omega) - \text{strong}\]
Moreover $u_0$ satisfies in $\mathcal{D}'(\omega_2)$ the equation
\[-\nabla X_{\epsilon_k} \cdot (A_{22} \nabla X_{\epsilon_k} u_0(X_1, \cdot)) = f + B(u_0)(X_1, \cdot)\]
for a.e $X_1 \in \omega_1$. 
Proof. The estimates given in Proposition 5 show that there exist \( u_0 \in L^p(\Omega) \) and a sequence \((u_{\epsilon_k})_{k \in \mathbb{N}}\) solutions to (37) such that
\[
\epsilon_k \nabla_{X_2} u_{\epsilon_k} \rightarrow 0, \quad \nabla_{X_2} u_{\epsilon_k} \rightarrow \nabla_{X_2} u_0 \quad \text{and} \quad u_{\epsilon_k} \rightarrow u_0 \quad \text{in} \quad L^p(\Omega) - \text{weak} \quad (43)
\]

As we have proved in Theorem 2.3 we have \( u_0 \in V_p \). The particular difficulty is the passage to the limit in the nonlinear term. This assertion is guaranteed by Theorem 4.2. Indeed, since \( \Omega \) is convex and Lipschitz then there exists an open covering \( (\Omega_j)_{j \in \mathbb{N}}, \quad \Omega_j \subset \Omega_{j+1} \) and \( \overline{\Omega_j} \subset \Omega \) such that each \( \Omega_j \) is a Lipschitz domain. (Take an increasing sequence of numbers \( 0 < \beta_j < 1 \) with \( \lim \beta_j = 1 \). Fix \( x_0 \in \Omega \) and take \( \Omega_j = \beta_j(\Omega - x_0) + x_0 \), since \( \Omega \) is open and convex then \( \overline{\Omega_j} \subset \Omega \). The Lipschitzness is conserved for \( \Omega_j \), since homothecies and translations are \( C^\infty \) diffeomorphisms).

Theorem 4.2 shows that for every \( j \in \mathbb{N} \) there exists \( C_j \geq 0 \) such that
\[
\|u_{\epsilon_k}\|_{W^{1,p}(\Omega_j)} \leq C_{\Omega_j}.
\]

Since \( \Omega_j \) is Lipschitz and bounded then the embedding \( W^{1,p}(\Omega_j) \hookrightarrow L^p(\Omega_j) \) is compact [1] and therefore for each \( j \) there exists a subsequence \((u_{\epsilon_k})_k \subset L^p(\Omega_j)\) such that
\[
\|u_{\epsilon_k}\|_{\Omega_j} \rightarrow u_0 \quad \text{in} \quad L^p(\Omega_j).
\]

By the diagonal process one can construct a sequence still labeled \((u_{\epsilon_k})_k\) such that \( u_{\epsilon_k} \rightarrow u_0 \) in \( L^p(\Omega_j) \) for every \( j \), in other words we have
\[
\|u_{\epsilon_k}\|_{L^p_{\text{loc}}(\Omega)} - \text{strong}. \quad (44)
\]

Passing to the limit in the weak formulation of (37) we deduce
\[
-\nabla_{X_2} \cdot (A_{22} \nabla_{X_2} u_0(X_1, \cdot)) = f + B(u_0)(X_1, \cdot), \quad \text{in} \quad \mathcal{D}'(\omega_2),
\]

where we have used (43) for the passage to the limit in the left hand side. For the passage to the limit in the nonlinear term we have used (44) and assumption (42).

Example 1.

We give a concrete example of application. Let \( \Omega \) as in this subsection, and let \( A \) be a bounded \((N-q) \times (N-q)\) matrix defined on \( \omega_2 \) which satisfies the ellipticity assumption. We consider the integro-differential problem:
\[
\begin{cases}
-\nabla_{X_2} \cdot (A(X_2) \nabla_{X_2} u) = f(X_2) + \int_{\omega_1} h(X_1', X_1, X_2) a(u(X_1', X_2))dX_1' \\
u(X_1, \cdot) = 0 \quad \text{on} \quad \partial \omega_2
\end{cases}
\]

where \( h \in L^\infty(\omega_1 \times \Omega) \) and \( f \in L^p(\omega_2), \quad 1 < p < 2 \), and \( a \) is a real-valued bounded continuous function.

This equation is based on the Neutron transport equation (see for instance [10]), with the Hammerstein integral operator type. A solution to (45) is a function \( u \in V_p \) which satisfies (45) in \( \mathcal{D}'(\omega_2) \). Suppose that
\[
\nabla_{X_1} h(X_1', X_1, X_2) \in L^\infty(\omega_1 \times \Omega)
\]

Then we have

**Theorem 4.4.** Under the assumptions of this example, (45) has at least a solution in \( V_p \) in the sense of \( \mathcal{D}'(\omega_2) \) for a.e \( X_1 \in \omega_1 \).
Proof. We introduce the singular perturbation problem:

\[
\begin{cases}
- \nabla \cdot (A \nabla u_\varepsilon) = f(X_2) + \int_{\omega_1} h(X'_1, X_1, X_2) a(u_\varepsilon(X'_1, X_2)) dX'_1 \\
u_\varepsilon = 0 \text{ on } \partial \Omega
\end{cases}
\]

where

\[
A_\varepsilon = \begin{pmatrix} \varepsilon^2 I & 0 \\ 0 & A \end{pmatrix}
\]

Clearly \( A_\varepsilon \) satisfies the ellipticity assumption and it is Clear that the operator

\[
u \rightarrow \int_{\omega_1} h(X'_1, X_1, X_2) a(u(X'_1, X_2)) dX'_1
\]

satisfies assumption (38).

We can prove easily that this operator satisfies assumption (42). Indeed, let \( u_n \to u \) in \( L^p_{\text{loc}}(\Omega) \), then there exists a subsequence \( (u_{n_k}) \) constructed by the diagonal process, such that \( u_{n_k} \to u \) a.e in \( \Omega \). Since \( a \) is bounded then it follows by the Lebesgue theorem that

\[
\int_{\omega_1} h(X'_1, X_1, X_2) a(u_{n_k}(X'_1, X_2)) dX'_1 \to \int_{\omega_1} h(X'_1, X_1, X_2) a(u(X'_1, X_2)) dX'_1,
\]

in \( L^p(\Omega) \). Whence by a contradiction argument we get

\[
\int_{\omega_1} h(X'_1, X_1, X_2) a(u_{n_k}(X'_1, X_2)) dX'_1 \to \int_{\omega_1} h(X'_1, X_1, X_2) a(u(X'_1, X_2)) dX'_1,
\]

in \( L^p(\Omega) \).

We can prove similarly as in [14] that (41) holds for the above integral operator, therefore the assertion of the theorem is a simple application of Theorem 4.3. □

Remark 4. Notice that the compacity of the operator given in this example is not sufficient to prove a such result as in the \( L^2 \) theory [10]. This shows the importance of assumption (41) which holds in this case.

Does operator whose assumption (41) holds admits necessarily an integral representation as in (45)?

Example 2.

We shall replace the integral by a general linear operator. Let us consider the following problem: Find \( u \in V_p \) such that

\[
\begin{cases}
- \nabla_{X_2} \cdot (A \nabla_{X_2} u) = f(X_2) + gP(ha(u)) \\
u(X_1, \cdot) = 0 \text{ on } \partial \omega_2
\end{cases}
\]

where \( a, A \) and \( f \) are defined as in Example 1.

We suppose that \( g, h \in L^\infty(\Omega) \) with \( \text{Supp}(h) \subset \Omega \) compact. Assume \( \nabla_{X_1} g \in L^\infty(\Omega) \) and \( P : L^p(\Omega) \to L^2(\omega_2) \) is a bounded linear operator.

When \( P \) is not compact then the operator \( u \to gP(ha(u)) \) is not necessarily compact, if this is the case then this operator cannot admit an integral representation.

Theorem 4.5. Under the assumptions of this example, there exists at least a solution \( u \in V_p \) to (46) in the sense of \( \mathcal{D}'(\omega_2) \) for a.e \( X_1 \in \omega_1 \)

Proof. Similarly, the proof is a simple application of Theorem 4.3. □
5. Some open questions.

Problem.

1. Suppose that $\infty > p > 2$. Given $f \in L^p(\Omega)$ and consider (2), since $f \in L^2(\Omega)$ then $u_\epsilon \to u_0$ in $V_2$. Assume that $\Omega$ and $A$ are sufficiently regular. Can one prove that $u_\epsilon \to u_0$ in $V_p$?

2. What happens when $f \in L^1(\Omega)$? As mentioned in the introduction there exists a unique entropy solution to (2) which belongs to $\bigcap_{1 \leq r < \frac{N}{N-1}} W^{1,r}_0(\Omega)$. Can one prove that $u_\epsilon \to u_0$ in $V_r$ for some $1 \leq r < \frac{N}{N-1}$? Can one prove at least weak convergence in $L^r$ for some $1 < r < \frac{N}{N-1}$ as given in Theorem 2.3?

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E-mail address: chokri.ogabi@ac-grenoble.fr