A family of meta-Fibonacci sequences
defined by variable order recursions

Nathaniel D. Emerson

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Abstract
We define a family of meta-Fibonacci sequences where the order
of the recursion at stage \( n \) is a variable \( r(n) \), and the \( n^{th} \) term of a sequence is the sum of the previous \( r(n) \) terms. For the terms of any such sequence, we give upper and lower bounds which depend only on \( r(n) \).

California State University Channel Islands
One University Drive
Camarillo, California 93012-8599
Email: Nathaniel.Emerson@csuci.edu

1 Introduction
We consider meta-Fibonacci sequences, that is a sequence given by a Fibonacci-type recursion, where the recursion varies with the index. We describe a new family of meta-Fibonacci sequences defined by variable order recursions and give closed-from upper and lower bounds for the terms of any such sequence. This family is considerably different from previously described families of meta-Fibonacci sequences (see [DGNW], [HT], [CCT] and [JR]) both in terms of its definition and behavior. See [CCT] for a nice history of the subject.

In this paper we denote integer valued sequences by Roman letters, and other sequences by Greek letters.

We define a family of meta-Fibonacci sequences. The regular Fibonacci numbers are of course obtained by adding the previous two terms of a sequence: \( f_n = f_{n-1} + f_{n-2} \). If we add the previous three terms, we obtain
the Tribonacci numbers: \( t_n = t_{n-1} + t_{n-2} + t_{n-3} \). If we add the previous \( r \) terms we obtain the \( r \)-generalized Fibonacci numbers (the “\( r \)-bonacci numbers”): \( f_{r,n} = f_{r,n-1} + \cdots + f_{r,n-r} \). Now let \( r \) vary as a function of \( n \). We call the resulting numbers variable-\( r \) meta-Fibonacci numbers. Let \( \mathbb{N} \) denote the non-negative integers and \( \mathbb{Z}^+ \) denote the positive integers.

**Definition 1.1.** Let \( r : \mathbb{N} \to \mathbb{Z}^+ \) such that \( r(0) = 1 \) and \( r(n) \) is sublinear, that is \( r(n) \leq n \) for all \( n \geq 1 \). Define

\[
b(n) = \sum_{k=1}^{r(n)} b(n-k), \quad n > 1,
\]

with initial condition \( b(0) = 1 \). We call the sequence \( b(n) \) a variable-\( r \)-meta-Fibonacci sequence, and say that \( r(n) \) generates \( b(n) \).

For brevity, we call \( b(n) \) a variable-\( r \)-bonacci sequence. It is clear that any such sequence is a non-decreasing sequence of positive integers. Additionally, it is clear distinct \( r(n) \) generate distinct sequences \( b(n) \). So we have defined an uncountable family of meta-Fibonacci sequences, in one-to-one correspondence with sublinear sequences of positive integers. In this paper, we examine the dependence of \( b(n) \) on \( r(n) \).

**Example 1.2.** If \( r(1) = 1 \) and \( r(n) = 2 \) for all \( n \geq 2 \), then \( b(n) = f_{n+1} \), where \( (f_n) \) is the usual Fibonacci sequence.

**Example 1.3.** If \( r(n) = 1 \) for all \( n \), then \( b(n) = 1 \) for all \( n \). If \( r(n) = 1 \) for all \( n \) large, then \( b(n) \) is eventually constant.

**Example 1.4.** If \( r(n) = n \) for \( n \geq 1 \), then \( b(n) = 2^{n-1} \) for \( n \geq 1 \).

| \( n \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-------|---|---|---|---|---|---|---|---|---|---|
| \( r(n) \) | 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| \( b(n) \) | 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

We give an estimate for \( b(n) \), which depends only on \( n \), \( r(1) \), \ldots, \( r(n) \). That is, a closed-form estimate. We define two quantities which we will use to estimate \( b(n) \).

**Definition 1.5.** Let \( b(n) \) be a variable-\( r \) meta-Fibonacci sequence generated by \( r(n) \). For \( n \geq 1 \) define

\[
\lambda(n) = 1 + \frac{r(n)-1}{r(n-1)}.
\]

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Definition 1.6. Let $b(n)$ be a variable-$r$ meta-Fibonacci sequence generated by $r(n)$. For $n \geq s \geq 1$ define

$$\mu(n, s) = 2 + [r(n) - r(n - 1) - 1] \prod_{k=n-s}^{n-1} 1/r(k).$$

We use $\lambda(n)$ and $\mu(n, s)$ to estimate the growth rate of $b(n)$, that is the ratio of successive terms. We obtain closed-from upper and lower bounds.

Main Theorem. Let $b(n)$ be a variable-$r$ meta-Fibonacci sequence generated by $r(n)$. For all $n \geq 1$

$$\min \{\lambda(n), \mu(n, r(n) - 1)\} \leq \frac{b(n)}{b(n - 1)} \leq \max \{\lambda(n), \mu(n, r(n) - 1)\}.$$

We will prove the Main Theorem in $\S 2$. We obtain the following bounds on $b(n)$.

Corollary 1.7. Let $b(n)$ be a variable-$r$ meta-Fibonacci sequence generated by $r(n)$. For all $n \geq 2$ we have

$$\prod_{k=2}^{n} \min \{\lambda(k), \mu(k, r(k) - 1)\} \leq b(n) \leq \max \prod_{k=2}^{n} \{\lambda(k), \mu(k, r(k) - 1)\}.$$

Proof. Note that $r(1) = 1$, so $b(1) = 1$ and $b(n) = b(n)/b(1)$. Write $b(n)$ as a telescoping product,

$$b(n) = \prod_{k=2}^{n} \frac{b(k)}{b(k - 1)},$$

and apply the Main Theorem. \qed

Variable-$r$ meta-Fibonacci sequences are considerably different than any meta-Fibonacci sequence that the author is familiar with. Hofstadter’s $Q$-sequence ([Ho, p. 137] and [Co]) may be taken as a typical example. Let $Q(1) = Q(2) = 1$ and

$$Q(n) = Q(n - Q(n - 1)) + Q(n - Q(n - 2)), \quad n > 2.$$

The recursion for $Q(n)$ is “self-referential” and the order of the recursion is fixed. The terms we add to obtain $Q(n)$ are not necessarily the immediately
previous terms. So for some $n$, we may add terms that are early in the sequence (and thus small) and $Q(n)$ will be small. In contrast for a sequence $b(n)$, an “external” variable $r(n)$ controls the recursion, and the order of the recursion is not generally fixed. We always add the immediately previous $r(n)$ terms, so $b(n)$ is always the sum of the largest of the previous terms. These differences result in $Q(n)$ having much more complicated behavior than $b(n)$. The behavior of $Q(n)$ has been described as “chaotic,” while $b(n)$ is non-decreasing. They also result in very different rates of growth for the two type of sequences, see [3].

Variable-$r$-bonacci numbers were originally discovered by the author while studying dynamical systems [E1]; specifically the dynamics of complex polynomials. In the study of dynamical systems, one can consider closest return times—most intuitively, the iterates of a given point under some map that are closer to the point than any previous iterate. In [E2] it is shown that certain generalized closest return times of polynomials are extended variable-$r$ meta-Fibonacci numbers (see Definition 4.1). This result generalizes the fact that there exist polynomials whose closest return times are the ordinary Fibonacci sequence [BH, Ex. 12.4].

The remainder of this paper is organized as follows. In §2 we prove the Main Theorem. We give a series of estimates, and then combine them. We show that the growth rate of any variable-$r$-bonacci sequence is at most exponential. For variable-$r$-bonacci sequences with $r(n) > 1$ for all $n$ sufficiently large, we show that the growth rate is at least exponential. In §3 we the study the asymptotics of $b(n)$. We show that wide variety of growth rates occur—exponential, linear, and logarithmic. In contrast, for many other meta-Fibonacci sequences the growth rate is linear. In §4 We define a generalization of $b(n)$, which is defined for all integers and removes restrictions on $r$.

2 Estimates on Growth

Consider the growth rate of the ordinary Fibonacci numbers, that is the ratio of successive Fibonacci numbers. It is well known that the growth rate is exponential, and converges to the Golden Section: $(1 + \sqrt{5})/2$. By a similar argument, the growth rate of the $r$-bonacci numbers converges to $\alpha_r$, the unique real root of the polynomial

$$x^r - x^{r-1} - \cdots - x - 1$$
with $1 < \alpha_r < 2$, (all other roots have complex modulus less than 1), see [Mi].

In this section, we examine the growth rate of variable $r$-bonacci sequences.
We give a series of estimates on the growth of variable-$r$-bonacci numbers, which we will combine to prove our Main Theorem. Throughout this section, let $b(n)$ be a variable-$r$ meta-Fibonacci sequence generated by $r(n)$.

We derive basic information about the limiting behavior of $b(n)$.

**Lemma 2.1.** The sequence $b(n)$ is eventually constant if and only if
\[ \limsup_{n \to \infty} r(n) = 1. \]

**Lemma 2.2.** We have
\[ \lim_{n \to \infty} b(n) = \infty \text{ if and only if } \limsup_{n \to \infty} r(n) > 1. \]

Thus, a variable-$r$-bonacci sequence converges if and only if it is eventually constant. Clearly for a given $n$, the larger $r(n)$ is, the larger $b(n)$ will be. However in many of these estimates, it is $\Delta r(n) = r(n) - r(n-1)$ which most strongly influences the growth rate. The following lemma is the our basic estimate; we give a condition for $b(n)$ to double.

**Lemma 2.3.** If $\Delta r(n) = 1$ for some $n \geq 1$, then $b(n)/b(n-1) = 2$.

**Proof.** We have $r(n) = r(n-1) + 1$ for some $n$. Hence
\[
\begin{align*}
b(n) &= \sum_{k=1}^{r(n)} b(n - k) \\
&= b(n - 1) + \sum_{k=2}^{r(n-1)+1} b(n - k) \\
&= b(n - 1) + \sum_{i=1}^{r(n-1)} b(n - 1 - i) \\
&= 2b(n - 1).
\end{align*}
\]

We extend the above lemma to cover all cases for $\Delta r(n)$. We obtain fairly complete information on the short-term growth of $b(n)$, particularly on the relative magnitude of $b(n)/b(n-1)$ and 2.

**Theorem 2.4.** For all $n \geq 1$ the following hold:
a. \(b(n)/b(n-1) = 1\) if and only if \(\Delta r(n) = 1 - r(n-1)\).

b. \(1 < b(n)/b(n-1) < 2\) if and only if \(1 - r(n-1) < \Delta r(n) < 1\).

c. \(b(n)/b(n-1) = 2\) if and only if \(\Delta r(n) = 1\).

d. \(b(n)/b(n-1) > 2\) if and only if \(\Delta r(n) > 1\).

Proof. We will prove the “if” part of each case. Case a is equivalent to \(r(n) = 1\), so it is clear. Case c is Lemma 2.3. In case b we have \(r(n) < r(n-1) + 1\), so

\[
b(n) = \sum_{k=1}^{r(n)} b(n-k) < \sum_{k=1}^{r(n)-1+1} b(n-k) = 2b(n-1),
\]

by Lemma 2.3. Case d is similar. The “only if” directions follow by considering the above cases.

We can start to examine the long-term behavior of the growth of \(b(n)\).

**Corollary 2.5.** If \(\limsup_{n \to \infty} \frac{b(n)}{b(n-1)} < 2\), then \(r(n)\) is eventually constant.

**Proof.** By Theorem 2.4.b for all \(n\) sufficiently large \(r(n) - 1 < r(n-1)\). It follows that for \(n\) large, \(r(n)\) is a non-increasing sequence of positive integers, so is eventually constant.

We give a universal upper bound for \(b(n)\), one which does not depend on \(r(n)\).

**Lemma 2.6.** For all \(n \geq 1\), we have \(b(n) \leq 2^{n-1}\).

**Proof.** Let \(\hat{r}(n) = n\) for all \(n \geq 1\) and let \(\hat{b}(n)\) be the variable-\(r\)-bonacci sequence generated by \(\hat{r}(n)\). By Lemma 2.3 \(\hat{b}(n) = 2^{n-1}\) for all \(n \geq 1\). Note that \(b(0) = \hat{b}(0) = 1\) and inductively

\[
b(n) = \sum_{k=1}^{r(n)} b(n-k) \leq \sum_{k=1}^{n} b(n-k) \leq \sum_{k=1}^{\hat{r}(n)} \hat{b}(n-k) = \hat{b}(n) = 2^{n-1}.
\]
This bound shows that all variable-$r$-bonacci sequences are $O(2^{n-1})$. That is, at worst exponential order.

The following lemma is the basis for many of our other estimates. We relate the growth of $b(n)$ to $r(n)$.

**Lemma 2.7.** For all $n \geq 1$,
\[
\frac{b(n)}{b(n-1)} \leq r(n).
\]

**Proof.**
\[
b(n) = \sum_{k=1}^{r(n)} b(n-k) \\
\leq \sum_{k=1}^{r(n)} b(n-1) \quad \text{since the } b(n) \text{ are non-increasing,} \\
= r(n)b(n-1).
\]

The above estimate is sharp. For any $n > 1$, let $r(1) = \cdots = r(n-1) = 1$ and $r(n) = n$. Then $b(1) = \cdots = b(n-1) = 1$, and $b(n) = n$, so $b(n)/b(n-1) = n = r(n)$.

**Corollary 2.8.** For all $n, m \geq 1$
\[
\frac{b(n+m)}{b(n)} \leq \prod_{k=n+1}^{n+m} r(k).
\]

**Proof.** Write $b(n+m)/b(n)$ as a telescoping product, and apply Lemma 2.7 $m$ times:
\[
\frac{b(n+m)}{b(n)} = \prod_{k=n+1}^{n+m} \frac{b(k)}{b(k-1)} \leq \prod_{k=n+1}^{n+m} r(k).
\]

From the above estimate, it follows that $b(n) \leq \prod_{k=1}^{n} r(k)$. Which implies $b(n) \leq n!$. However, from Lemma 2.6 we know in fact that $b(n) \leq 2^{n-1}$. So while Lemma 2.7 gives a sharp estimate of the short term growth of $b(n)$, in the long term it is highly inaccurate. However, we only use the above corollary to obtain lower bounds on growth, so the inaccuracy is somewhat reduced. We state the reciprocal of it for reference.
Corollary 2.9. For all \( n, m \geq 1 \)

\[
\frac{b(n)}{b(n + m)} \geq \prod_{k=n+1}^{n+m} \frac{1}{r(k)}.
\]

We need an estimate on the ratio of sums. The proof is trivial.

Lemma 2.10. Let \( \alpha_1, \ldots, \alpha_l \) and \( \beta_1, \ldots, \beta_m \) be positive real numbers. If for all \( i \) and \( j \) we have \( \alpha_i \leq \beta_j \), then

\[
\frac{\sum_{i=1}^{l} \alpha_i}{\sum_{j=1}^{m} \beta_j} \leq \frac{l}{m}.
\]

Lemma 2.11. If \( r > s > 1 \), then for any \( n \geq r \)

\[
\frac{\sum_{k=1}^{r} b(n - k)}{\sum_{k=1}^{s} b(n - k)} \leq \frac{r}{s}.
\]

Proof. We have

\[
\frac{\sum_{k=1}^{r} b(n - k)}{\sum_{k=1}^{s} b(n - k)} = \frac{\sum_{k=1}^{s} b(n - k)}{\sum_{k=1}^{s} b(n - k)} + \frac{\sum_{k=s+1}^{r} b(n - k)}{\sum_{k=1}^{s} b(n - k)}.
\]

Since the \( b(n) \) are non-decreasing, Lemma 2.10 applies to the second term, and

\[
\frac{\sum_{k=1}^{s} b(n - k)}{\sum_{k=1}^{s} b(n - k)} \leq 1 + \frac{r - s}{s} = \frac{r}{s}.
\]

When there are more terms in the denominator, we obtain the following corollary in a similar fashion.

Corollary 2.12. If \( s > r > 1 \), then for any \( n \geq s \)

\[
\frac{\sum_{k=1}^{r} b(n - k)}{\sum_{k=1}^{s} b(n - k)} \geq \frac{s}{r}.
\]

Recall that \( \lambda(n) = 1 + [r(n) - 1]/r(n - 1) \).

Lemma 2.13. For any \( n \geq 0 \), if \( \Delta r(n+1) > 1 \), then

\[
\frac{b(n+1)}{b(n)} \leq \lambda(n+1).
\]
Proof.

\[
\frac{b(n+1)}{b(n)} = \frac{b(n) + b(n-1) + \cdots + b(n + 1 - r(n + 1))}{b(n)} = 1 + \frac{b(n-1) + \cdots + b(n + 1 - r(n + 1))}{b(n-1) + \cdots + b(n - r(n))}.
\]

Note that there are \(r(n + 1) - 1\) terms in the numerator, \(r(n)\) terms in the denominator, and by assumption \(r(n + 1) - 1 > r(n)\). Thus, we can use Lemma 2.11 to obtain

\[
\frac{b(n+1)}{b(n)} \leq 1 + \frac{r(n + 1) - 1}{r(n)} = \lambda(n + 1).
\]

Lemma 2.14. For any \(n \geq 0\), if \(\Delta r(n + 1) < 1\), then

\[
\frac{b(n+1)}{b(n)} \geq \lambda(n + 1).
\]

Proof. Similar to Lemma 2.13, except that we use Corollary 2.12.

Remark. For the \(r\)-generalized Fibonacci numbers \((f_{r,n})\), this estimate shows that \(f_{r,n+1}/f_{r,n} \geq 1 + (r - 1)/r = 2 - 1/r\) for \(n > 2r - 1\).

Notice that \(\lambda(n)\) is either an upper bound or a lower bound depending on \(\Delta r(n)\).

The above lemma gives us information about the asymptotics of \(b(n)\).

Corollary 2.15. If \(m = \liminf_{n \to \infty} r(n)\) and \(M = \limsup_{n \to \infty} r(n)\), then \(b(n)\) is \(\Omega(1 + \frac{m-1}{M})\).

Hence, if \(\liminf r(k) > 1\) and \(\limsup r(k) < \infty\), then the \(b(n)\) grow exponentially fast. In contrast, the growth rate of many meta-Fibonacci sequences is of only linear order. For instance, the Conway sequence

\[
a(n) = a(a(n - 1)) + a(n - a(n - 1)), \quad n \geq 3,
a(1) = a(2) = 1.
\]

It is known that \(\lim_{n \to \infty} a(n)/n = 1/2\) \([Ma]\). We discuss this phenomenon in more detail in \([3]\).

Recall that \(\mu(n, s) = 2 + [\Delta r(n) - 1] \prod_{k=n-s}^{n-1} 1/r(k)\). We give estimates on growth, in terms of \(\mu(n, s)\).
Lemma 2.16. For any \( n \geq 1 \), if \( \Delta r(n) < 1 \), then
\[
\frac{b(n)}{b(n-1)} \leq \mu(n, r(n-1)).
\]

Proof. Using Lemma 2.3 we have
\[
\sum_{k=1}^{r(n-1)+1} b(n-k) \leq 2
\]
\[
\sum_{k=1}^{r(n-1)+1} \frac{b(n-k)}{b(n-1)} + \sum_{k=r(n)+1}^{r(n-1)+1} \frac{b(n-k)}{b(n-1)} = 2
\]
\[
\frac{b(n)}{b(n-1)} = 2 - \frac{\sum_{k=r(n)+1}^{r(n-1)+1} b(n-k)}{b(n-1)}
\]
\[
\frac{b(n)}{b(n-1)} \leq 2 - [r(n-1) - r(n)] \frac{b(n-r(n-1)-1)}{b(n-1)}
\]
since the \( b(n) \) are non-decreasing, so
\[
\frac{b(n)}{b(n-1)} \leq 2 - \{ -[r(n) - r(n-1) - 1] \} \prod_{k=n-r(n-1)}^{n-1} 1/r(k)
\]
by Corollary 2.9. The right-hand side is \( \mu(n, r(n-1)) \), so the lemma is shown. \( \square \)

Remark. For the \( r \)-generalized Fibonacci numbers \( (f_{r,n}) \), this proposition implies that \( f_{r,n+1}/f_{r,n} \leq 2 - r^{-r} \), for \( n > 2r - 1 \).

Lemma 2.17. If \( \Delta r(n) > 1 \), then
\[
\frac{b(n)}{b(n-1)} \geq \mu(n, r(n-1)).
\]

Proof. By Lemma 2.3 we have
\[
\sum_{k=1}^{r(n-1)+1} \frac{b(n-k)}{b(n-1)} = 2.
\]
Thus,

\[
\frac{b(n)}{b(n-1)} = \frac{\sum_{k=1}^{r(n)} b(n-k)}{b(n-1)} = \frac{\sum_{k=1}^{r(n)} b(n-k)}{b(n-1)} + \frac{\sum_{k=r(n)+2}^{r(n)} b(n-k)}{b(n-1)} = 2 + \frac{\sum_{k=r(n)+2}^{r(n)} b(n-k)}{b(n-1)} \geq 2 + [r(n) - r(n-1) - 1] \frac{b(n-r(n))}{b(n-1)}
\]

since the \(b(n)\) are non-decreasing,

\[
\geq 2 + [r(n) - r(n-1) - 1] \prod_{k=n-r(n)+1}^{n-1} 1/r(k)
\]

by Corollary 2.9. The right-hand side is \(\mu(n, r(n) - 1)\), so the lemma is shown.

As with \(\lambda(n)\), we have \(\mu(n, R)\), where \(R = \max \{r(n) - 1, r(n-1)\}\), is either an upper and lower bound for growth depending on \(\Delta r(n)\).

We now compare \(\mu\) to \(\lambda\).

**Lemma 2.18.** If \(\Delta r(n) \leq 1\) for some \(n \geq 1\), then

\[
\mu(n, r(n) - 1) \geq \lambda(n).
\]

**Proof.** We have \([r(n) - r(n-1) - 1] \leq 0\) by assumption. Also \(\prod_{k=n-r(n)+1}^{n-2} 1/r(k) \leq 1\), since \(r(k) \geq 1\) for all \(k\). Thus,

\[
[r(n) - r(n-1) - 1] \left[ -1 + \prod_{k=n-r(n)+1}^{n-2} 1/r(k) \right] \geq 0.
\]

The lemma follows by straightforward algebra. \(\square\)
Lemma 2.19. If $\Delta r(n) \geq 1$ for some $n \geq 1$, then

$$\mu(n, r(n-1)) \leq \lambda(n).$$

Proof. Similar to the above lemma. \hfill \end{proof}

We are now ready to prove the Main Theorem. We consider various cases for $\Delta r(n)$. We then combine appropriate estimates of $b(n)/b(n-1)$ in terms of $\mu$ and $\lambda$.

Proof of Main Theorem. Fix $n \geq 1$. If $\Delta r(n) = 1$, then by Lemma 2.3 we know $b(n)/b(n-1) = 2 = \lambda(n) = \mu(n, \cdot)$ and we are done.

If $\Delta r(n) < 1$, then

$$\lambda(n) \leq \frac{b(n)}{b(n-1)} \leq \mu(n, r(n-1)),$$

with the first inequality by Lemma 2.14 and the second by Lemma 2.16. Additionally, $\lambda(n) \leq \mu(n, r(n-1))$ by Lemma 2.18. Hence

$$\min \{\lambda(n), \mu(n, r(n-1))\} \leq \frac{b(n)}{b(n-1)}.$$

Finally, if $\Delta r(n) > 1$, then

$$\mu(n, r(n-1)) \leq \frac{b(n)}{b(n-1)} \leq \lambda(n),$$

by Lemma 2.17 and Lemma 2.13. Also, $\mu(n, r(n-1)) \leq \lambda(n)$ by Lemma 2.18. So

$$\frac{b(n)}{b(n-1)} \leq \max \{\lambda(n), \mu(n, r(n-1))\}.\$$

Therefore, we can combine the above inequalities in all cases to obtain:

$$\min \{\lambda(n), \mu(n, r(n-1))\} \leq \frac{b(n)}{b(n-1)} \leq \max \{\lambda(n), \mu(n, r(n-1))\}.\$$

\hfill \end{proof}
3 Asymptotic Growth

In this section we examine the asymptotic growth of $b(n)$. We compare the growth rate of $b(n)$ to other families of meta-Fibonacci sequences, which is polynomial order in all known cases. We show that $b(n)$ can have a variety of different growth rates: exponential, linear, and logarithmic. However, the possible asymptotic limits for $b(n)$ are restricted.

To date two other families of meta-Fibonacci sequences have appeared in print, see [DGNW] and [CCT]. Sub-families of the latter family are also studied in [HT] and [JR].

In [DGNW] $(p, q)$-sequences were introduced. A $(p, q)$ sequence $(F_n)$ is defined as follows. For fixed positive integers $p$ and $q$, and values $a_1, \ldots, a_p$, let $F_n = a_n$ with probability one for $n \leq p$ and set $F_{n+1} = \sum_{k=1}^{q} F_{j_k}$ for $n \geq p$, where the $j_k$ are randomly chosen, with replacement, from $(1, 2, \ldots, n)$. They give only probabilistic results. They show that the expected value of $F_n$ grows as a polynomial in $n$ of degree $q - 1$ [DGNW, Thm. 1].

In [CCT] J. Callaghan, J. Chew and S. Tanny studied a family of sequences parameterized by $a > 0, k > 1$:

$$T_{a,k}(n) = \sum_{i=0}^{k-1} T_{a,k}(n - i - a - T_{a,k}(n - i - 1)), \quad n > a + k, \quad k \geq 2$$

with $T_{a,k}(n) = 1$ for $1 \leq n \leq a + k$. For $k$ odd, the growth rate of sequences in this family is linear [CCT, Cor. 5.14]; for all $a$ and all odd $k$

$$\lim_{n \to \infty} \frac{T_{a,k}(n)}{n} = \frac{k - 1}{k}.$$

We now examine the asymptotics of variable-$r$-bonacci sequences. In contrast to the above meta-Fibonacci sequences, but like the $r$-generalized Fibonacci sequences, $b(n)$ can grow exponentially. As previously noted, by Lemma 2.6, all such sequences are $O(2^{n-1})$. By Corollary 2.15, $t\Omega([1 + (m - 1)/M]^{n})$ for any $2 \leq m \leq M$. It is possible that that $b(n) \sim \gamma^n$, but the possible values of $\gamma$ are limited.

**Lemma 3.1.** For any variable-$r$-bonacci sequence $b(n)$, we have

$$\liminf_{n \to \infty} \frac{b(n)}{b(n - 1)} \leq 2.$$
Proof. Suppose not. By Theorem 2.4.d, for all \( n \) sufficiently large \( \Delta r(n) > 1 \). It follows that for \( n \) large \( n - r(n) < (n - 1) - r(n - 1) \). Thus, for \( n \) large \( (n - r(n)) \) is a strictly decreasing sequence of integers. Therefore, \( N - r(N) < 0 \) for some \( N \). Contrary to \( r(n) \leq n \) by Definition 1.1. \( \square \)

The possible limits for the sequence \( (b(n)/b(n-1)) \) are restricted.

**Lemma 3.2.** If \( \lim_{n \to \infty} r(n) = R \), then

\[
\lim_{n \to \infty} \frac{b(n)}{b(n-1)} = \alpha_R,
\]

where \( \alpha_1 = 1 \).

**Proof.** For \( n \) sufficiently large, \( b(n) \) satisfies the \( R \)-bonacci recursion or is eventually constant. \( \square \)

The only other possible limit of \( b(n)/b(n-1) \) is 2.

**Proposition 3.3.** If the sequence \( b(n)/b(n-1) \) converges and \( r(n) \) is not eventually constant, then

\[
\lim_{n \to \infty} \frac{b(n)}{b(n-1)} = 2.
\]

**Proof.** If \( \limsup_{n \to \infty} b(n)/b(n-1) < 2 \), then by Corollary 2.5 \( \lim_{n \to \infty} r(n) = R \) for some \( R \in \mathbb{Z}^+ \), contrary to assumption. Thus \( \limsup_{n \to \infty} b(n)/b(n-1) \geq 2 \). By Lemma 3.1 \( \liminf_{n \to \infty} b(n)/b(n-1) \leq 2 \). Therefore, the only possible limit is 2. \( \square \)

**Corollary 3.4.** If \( \lim_{n \to \infty} b(n)/\gamma^n \) exists for some \( \gamma \in \mathbb{R} \), then \( \gamma = \alpha_R \) for some \( R \geq 1 \), or \( \gamma = 2 \).

From Example 1.4, we know that \( \lim_{n \to \infty} b(n)/b(n-1) = 2 \) occurs. The following example shows that it occurs in a non-trivial case.

**Example 3.5.** For \( n \geq 2 \), let \( r(n) = n \) for \( n \) even, and \( r(n) = n - 1 \) for \( n \) odd.

| \( n \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| \( r(n) \) | 1 | 1 | 2 | 2 | 4 | 4 | 6 | 6 | 8 | 8 |
| \( b(n) \) | 1 | 1 | 2 | 3 | 7 | 13 | 27 | 53 | 107 | 213 |

We claim that if \( n > 2 \), then \( b(n) = 2b(n-1) + 1 \) for \( n \) even, and \( b(n) = 2b(n-1) - 1 \) for \( n \) odd. Hence, \( \lim_{n \to \infty} b(n)/b(n-1) = 2 \). The proof is left as an exercise.
The following example shows that the sequence \( b(n)/b(n-1) \) need not converge.

**Example 3.6.** For \( n \geq 2 \), let \( r(n) = 2 \) for \( n \) even, and \( r(n) = 3 \) for \( n \) odd.

| \( n \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-------|---|---|---|---|---|---|---|---|---|---|
| \( r(n) \) | 1 | 1 | 2 | 3 | 2 | 3 | 2 | 3 | 2 | 3 |
| \( b(n) \) | 1 | 1 | 2 | 4 | 6 | 12 | 18 | 36 | 54 | 108 |

It is left as an exercise to show that \( b(n)/b(n-1) = 2 \) for \( n > 2 \) and odd, and \( b(n)/b(n-1) = 3/2 \) for \( n > 2 \) and even.

By taking \( r(n) = 1 \) fairly often, we can have linear growth for \( b(n) \).

**Example 3.7.** For \( n \geq 2 \), let \( r(n) = 2 \) if \( n = 2^k \) for some \( k \in \mathbb{Z} \), and \( r(n) = 1 \) otherwise.

| \( n \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-------|---|---|---|---|---|---|---|---|---|---|
| \( r(n) \) | 1 | 1 | 2 | 1 | 2 | 1 | 1 | 2 | 1 |
| \( b(n) \) | 1 | 1 | 2 | 2 | 4 | 4 | 4 | 8 | 8 |

It is easy to show that \( n/2 \leq b(n) \leq n \) for \( n \geq 1 \). That is, \( b(n) \) is \( \Theta(n) \).

However in the case of linear growth, we cannot have an asymptotic limit other than zero.

**Proposition 3.8.** If \( \lim_{n \to \infty} b(n)/n = L \), where \( 0 \leq L < \infty \), then \( L = 0 \).

**Proof.** Contrarily, if \( L > 0 \) we can take \( 0 < \varepsilon \ll L \). We can then find some large \( N \) such that

1. \( r(N) = 1 \) (or else the growth is exponential);
2. \( r(N+1) > 1 \) (or else \( b(n) \) is eventually constant);
3. \( N/(N+1) > 1 - \varepsilon \);
4. for all \( n \geq N \), \( |b(n)/n - L| < \varepsilon \).
We have
\[
\frac{b(N+1)}{N+1} = \frac{b(N+1)b(N)}{b(N)} \geq \frac{N}{N+1} \geq (2)(L-\varepsilon)(1-\varepsilon),
\]
\[
= 2L + O(\varepsilon).
\]
by Lemma 2.3, condition 4, and condition 3 respectively. So,
\[
\left| \frac{b(N+1)}{N+1} - L \right| \geq 2L + O(\varepsilon) - L = L + O(\varepsilon).
\]
But by condition 4, we have
\[
\left| \frac{b(N+1)}{N+1} - L \right| < \varepsilon \ll L.
\]
Therefore, \( L = 0. \)

We can have slower than polynomial growth. No other known Fibonacci-type sequence grows so slowly.

**Example 3.9.** For \( n \geq 2, \) let \( r(n) = 2 \) if \( n = 2^k \) for some \( k \in \mathbb{Z}, \) and \( r(n) = 1 \) otherwise.

| \( n \)  | 0  | 1  | 2  | 4  | 16 | 256       |
|--------|----|----|----|----|----|-----------|
| \( r(n) \) | 1  | 1  | 2  | 2  | 2  | 2         |
| \( b(n) \) | 1  | 1  | 2  | 4  | 8  | 16        |

It is easy to show that \( b(n) \) is \( \Theta(\log_2 n). \)

Similarly, we can construct examples that are \( \Theta(\log_2 \log_2 n), \) etc. Thus \( b(n) \) can grow quite slowly indeed.

**4 Generalization**

We define a generalization of \( b(n). \) This generalization allows us to pick different initial conditions for our sequence. It also allows us to remove the restrictions that \( r \) be sublinear.
**Definition 4.1.** We call a double sequence \( \beta(n), \ n \in \mathbb{Z} \), an extended variable-\( r \) meta-Fibonacci sequence if there exists \( r : \mathbb{Z} \to \mathbb{Z}^+ \) such that for all \( n \in \mathbb{Z} \)

\[
\beta(n) = \sum_{k=1}^{r(n)} \beta(n-k).
\]

Note that there is no restriction that \( r \) be sublinear. Provided \( \beta(n) > 0 \) for all \( n \in \mathbb{Z} \), all results in this paper apply to an extended \( \beta(n) \), except Lemma 2.6. Similarly, if \( \beta(n) < 0 \) for all \( n \in \mathbb{Z} \), all results in this paper, except Lemma 2.6, are easily generalized. The behavior of \( \beta(n) \) with both positive and negative terms is an interesting question.

Given \( r : \mathbb{N} \to \mathbb{Z}^+ \), we can define a sequence \( \beta(n) \) generated by \( r \) as follows. Pick any real number \( \beta(-1) \) as an initial condition. For \( n \leq -1 \), let \( \beta(n) = \beta(-1) \) and let \( r(n) = 1 \). For \( n \geq 0 \), define \( \beta(n) \) by the variable-\( r \)-bonacci recursion.

Now let \( r : \mathbb{N} \to \mathbb{Z}^+ \) and let \( M_r = \sup_{n \in \mathbb{N}} r(n) - n < \infty \). We consider \( r \) with \( M_r \) finite. Note that \( r(0) - 0 > 0 \), so \( M_r \geq 1 \). We give a construction for extending \( r \) to a function on all integers, so that it generates an extended variable-\( r \)-bonacci sequence.

**Definition 4.2.** Let \( r : \mathbb{N} \to \mathbb{Z}^+ \) and let \( M_r = \sup_{n \in \mathbb{N}} r(n) - n < \infty \). Choose \( \beta(-1), \ldots, \beta(-M_r) \in \mathbb{R} \). For \( n \geq 0 \) let

\[
\beta(n) = \sum_{k=1}^{r(n)} \beta(n-k).
\]

For \( n = -1, -2, \ldots \) let \( r(n) = M_r \), and let

\[
\beta(n-M_r) = \beta(n) - \sum_{k=1}^{M_r-1} \beta(n-k).
\]

**Proposition 4.3.** Let \( r : \mathbb{N} \to \mathbb{Z}^+ \) with \( M_r = \sup_{n \in \mathbb{N}} r(n) - n < \infty \). The double sequence \( \beta(n) \) constructed in Definition 4.2 is an extended variable-\( r \)-meta-Fibonacci sequence generated by the extension of \( r(n) \) to \( \mathbb{Z} \).

**Proof.** It is straightforward to check that \( \beta(n) \) is well defined for all \( n \in \mathbb{Z} \), and it satisfies the correct recursion relation. \( \square \)
Corollary 4.4. Let $r : \mathbb{N} \rightarrow \mathbb{Z}^+$ with $M_r = \sup_{n \in \mathbb{N}} r(n) - n < \infty$. The set of variable-$r$ meta-Fibonacci sequences generated by $r(n)$ is an $M_r$-dimensional real vector space.

In particular, if $r$ is sublinear, $M_r = 1$, so we can use this construction to get a sequence $\beta(n)$ depending on one initial condition $\beta(-1)$.

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