THE NUCLEAR SCISSORS MODE IN A SOLVABLE MODEL

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Abstract

The coupled dynamics of the scissors mode and the isovector giant quadrupole resonance is studied in a model with separable quadrupole-quadrupole residual interactions. The method of Wigner function moments is applied to derive the dynamical equations for angular momentum and quadrupole moment. Analytical expressions for energies, B(M1)- and B(E2)-values, sum rules and flow-patterns of both modes are found for arbitrary values of the deformation parameter. Some predictions for the case of superdeformation are given. The subtle nature of the phenomenon and its peculiarities are clarified.
1 Introduction

The low-energy orbital magnetic-dipole excitations of deformed nuclei, commonly called the scissors mode, were predicted about 25 years ago [1, 2]. The prediction was inspired by the geometric picture that the (prolate) deformed neutron and proton distributions counter rotate (like scissors) within a small opening angle $\delta \phi$ in an oscillatory way around their common axis perpendicular to the long symmetry axis of the nucleus. Only a few years later the idea was confirmed experimentally with its detection in $^{156}$Gd [3]. At present, in addition to the rare earth nuclei, this excitation is also known in the actinides and in light nuclei. A complete review of the experimental situation can be found in [4]. The discovery of the scissors mode has initiated a cascade of theoretical studies. An excellent review of the present situation in this field is the one by D. Zawischa [5] (see also [6]). Very briefly the situation can be described in the following way. All microscopic calculations with effective forces reproduce experimental data with respect to the position and the strength of the scissors mode, some of them [7] giving also reasonable fragmentation of its strength. However, the situation is more obscure in regard to simple phenomenological models whose aim is to explain the physics of the phenomenon and to interpret it in the most simple and transparent terms. A noticeable discord of the opinions of various authors must be observed here as has been pointed out in [5]. One is forced to conclude that there is no general agreement in the understanding of the nature of this curious phenomenon. We here will try to shed some light into the confusing situation.

Our model Hamiltonian will be the one of the well served harmonic oscillator plus separable Quadrupole-Quadrupole (Q-Q) residual interaction. The interaction will have isoscalar and isovector parts whose coupling constants are reasonably well known from the literature. Of course, given such a simple schematic Hamiltonian the RPA equations, which are the standard tool to describe the scissors mode, can readily be solved. This, however, does not advance us in its physical interpretation. Namely, numerous calculations, performed during 25 years of investigation of the scissors mode have undoubtedly demonstrated at least one fact: the scissors mode is a very non-trivial, subtle type of motion subject to the influence of many factors. In fact its nature is much more complicated and interesting than the above picture of two counter rotating and oscillating pieces of matter might suggest. First of all one easily imagines that the rotational oscillations of
neutron and proton systems are inevitably accompanied by the (quadrupole) distortion of their shapes. This entails that the scissors mode is intricately entangled with the Iso-Vector Giant Quadrupole Resonance (IVGQR). Thus, if one wants to observe the scissors mode, one has to excite at the same time the IVGQR. The quadrupole distortions give rise to high lying excitations because of the so-called nuclear elasticity (or Fermi surface deformation), the quantum effect discovered by Bertsch [8]. It turns out that without this additional restoring force the scissors mode would actually be a zero energy mode! Hence, the scissors mode is a pure quantum mechanical phenomenon, which can not be explained in the frame of classical mechanics. The above properties of the scissors mode highlight perhaps its most characteristic features. They will obtain a natural, sufficiently simple, and visual explanation in the frame of our approach outlined below.

One further issue, strongly debated in the literature, is whether the neutron and proton fluids really spatially separate at least to a certain extent during the scissors motion. With diffuse surfaces and small amplitude motion this is a not completely trivial question and it only makes sense to speak about the separation of the symmetry axes of neutron and proton distributions. The debate is not without foundation, since there were attempts [5] to construct a model with sharp surfaces: neutron and proton liquids perform out-of-phase rotational oscillations with Steinwedel-Jensen boundary conditions. Our approach allows us to derive the analytic expressions for the current lines and the corresponding figures show unambiguously that indeed a separation of the two fluids occurs. More surprisingly, the separation not only exists in the low energy magnetic dipole excitation $1^+$ (the scissors mode proper) but also in the $K^\pi = 1^+$ branch of the IVGQR, which is named “the high energy scissors mode” and whose existence was guessed by various authors many years ago [9, 6].

From the above discussion it seems clear that in order to elucidate all these subtle features an approach involving macroscopic quantities as dynamical variables is indicated. Most naturally this can be achieved by working in phase space [10, 11]. This can be easily performed by applying the Wigner transform to the Time Dependent Hartree-Fock (TDHF) equations. The method of the Wigner Function Moments (WFM) [11, 12] is then applied. It can be characterized as a link between microscopic and macroscopic approaches: starting from a microscopic Hamiltonian one derives macroscopic dynamic
equations for collective variables. Usually one has to establish the set of such variables with the help of some physical considerations. The WFM method allows one to avoid this non-trivial problem: if one knows at least one collective variable (in our case it is the relative angular momentum of neutrons and protons), the procedure of derivation of dynamical equations will automatically generate all the other variables needed.

In this way an unambiguous set of coupled equations in terms of dynamic physical variables is obtained, which in the small amplitude limit (RPA) allows for analytic solutions. Since our equations are written down in the laboratory frame the total angular momentum \( I \) is, of course, conserved. In this work we study the case without rotation and take \( I = 0 \). These remarks are important, since microscopic calculations [5] have shown that for the results to be reasonable, it is very important to exclude from the wave function the spurious component responsible for the rotation of the nucleus, as a whole. Such problems do not arise in our approach, because there is no necessity to introduce the intrinsic coordinate system.

The analytical form of our results is very convenient to study the deformation (\( \delta \)) dependence of various quantities such as position of resonances and transitions probabilities. In the small \( \delta \) limit we mostly reproduce results already obtained by other authors. However, for large \( \delta \) we obtain predictions for super- and hyper-deformed nuclei. This area is practically not investigated at present. The only investigation within a phenomenological model [6] and the only existing microscopic calculation [13] are in rather good agreement with our results. In [13] pairing is taken into account whereas we have, so far, not considered superfluidity in our approach. However, it is well known that at large deformations superfluidity is of little influence on the dynamics and this is also the conclusion in [13]. On the other hand at small and moderate \( \delta \) the influence of pairing may be appreciable and certainly our approach must be generalized to include superfluidity in the future. It is for example shown that pairing reduces \( B(M1) \) - transition probabilities by important factors and that this yields agreement with the experimental deformation dependence [5].

In spite of the importance of pairing correlations in nuclei we know from other features that it certainly changes results quantitatively but it is too weak in nuclei to yield qualitative changes. For example the moment of inertia usually goes only half way from its rigid body value to its irrotational liquid limit (strong pairing case). We therefore believe
that the physical insights we will develop in this paper will stay qualitatively correct, even if superfluidity is included in a later stage. As we mentioned already, this shall be the subject of future studies.

The paper is organized as follows. In Section 2 the general outline of the WFM formalism is presented. This formalism is applied to the model of a harmonic oscillator potential with Q-Q residual interaction in Section 3: the equations of motion for irreducible tensors are derived and analyzed, the energies of collective isoscalar and isovector excitations are calculated. The method of infinitesimal displacements is used in Section 4 to find the expressions for the nucleon currents of the different modes and to display the respective figures. The magnetic and electric transition probabilities are calculated in Section 5 with the help of the linear response theory. Sum rules are analyzed in Section 6. The most interesting points in the description and understanding of the scissors mode are discussed in section 7. The scissors mode in superdeformed nuclei is considered in section 8. The summary of the main results and concluding remarks are contained in Section 9. To obtain a general impression of the approach and results one can omit the sections 4, 5 and 6 in a first reading.

2 Formulation of the method

The basis of our method is the Time Dependent Hartree-Fock (TDHF) equation for the one-body density matrix \( \rho^\tau(r_1, r_2, t) = \langle r_1 | \hat{\rho}^\tau(t) | r_2 \rangle \):

\[
i\hbar \frac{\partial \hat{\rho}^\tau}{\partial t} = \left[ \hat{H}^\tau, \hat{\rho}^\tau \right],
\]

where \( \hat{H}^\tau \) is the one-body self-consistent Hamiltonian depending implicitly on the density matrix and \( \tau \) is an isotopic index. It is convenient to modify equation (1) introducing the Wigner transform [14] of the density matrix

\[
f^\tau(r, p, t) = \int d^3 s \exp(-i p \cdot s / \hbar) \rho^\tau(r + \frac{s}{2}, r - \frac{s}{2}, t)
\]

and of the Hamiltonian

\[
H^\tau_W(r, p) = \int d^3 s \exp(-i p \cdot s / \hbar) (r + \frac{s}{2} | \hat{H}^\tau | r - \frac{s}{2}).
\]

Using (2,3) one arrives [15] at
\[ \frac{\partial f^\tau}{\partial t} = \frac{2}{\hbar} \sin \left( \frac{\hbar}{2} (\nabla_r^H \cdot \nabla_p^f - \nabla_p^H \cdot \nabla_r^f) \right) H^\tau_W f^\tau, \]  
where the upper index on the nabla operator stands for the function on which this operator acts. If the Hamiltonian is a sum of a kinetic term and a local potential \( V^\tau(r) \), its Wigner transform is just the classical version of the same Hamiltonian

\[ H^\tau_W = p^2/2m + V^\tau(r). \]

Then equation (4) becomes

\[ \frac{\partial f^\tau}{\partial t} + 1 \frac{p}{m} \cdot \nabla_r f^\tau = \frac{2}{\hbar} \sin \left( \frac{\hbar}{2} \nabla_r^V \cdot \nabla_p^f \right) V^\tau f^\tau. \]

Expanding in powers of \( \hbar \) leads to

\[ \frac{\partial f^\tau}{\partial t} + \frac{1}{m} \sum_{i=1}^{3} p_i \nabla_i f^\tau - \sum_{i=1}^{3} \nabla_i V^\tau \nabla_i^p f^\tau + \frac{\hbar^2}{24} \sum_{i,j,k=1}^{3} \nabla_i \nabla_j \nabla_k V^\tau \nabla_i^p \nabla_j^p \nabla_k^p f^\tau - \ldots = 0. \]

Now we apply the WFM method to derive a closed set of dynamical equations for different multipole moments and other integral characteristics of the nucleus. This method is described in detail in Ref. [11, 12]. Its idea is based on the virial theorems of Chandrasekhar and Lebovitz [16]. It is shown in [11, 12], that by integrating equation (7) over the phase space \( \{p, r\} \) with the weights \( x_1 x_2 \ldots x_i p_{i+k} \ldots p_{n-1} p_n \), where \( k \) runs from 0 to \( n \), one can obtain a closed finite set of dynamical equations for Cartesian tensors of the rank \( n \). Taking linear combinations of these equations one is able to represent them through irreducible tensors. However, it is more convenient to derive the dynamical equations directly for irreducible tensors using the technique of tensor products [17]. For this it is necessary to rewrite the Wigner function equation (7) in terms of cyclic variables

\[ \frac{\partial f^\tau}{\partial t} + \frac{1}{m} \sum_{\alpha=-1}^{1} (-1)^{\alpha} \nabla_{\alpha} f^\tau - \sum_{\alpha=-1}^{1} (-1)^{\alpha} V^\tau \nabla_{\alpha}^p f^\tau \]

\[ + \frac{\hbar^2}{24} \sum_{\alpha,\nu,\sigma=-1}^{1} (-1)^{\alpha+\nu+\sigma} \nabla_{-\alpha} \nabla_{-\nu} \nabla_{-\sigma} V^\tau \nabla_{\alpha}^p \nabla_{\nu}^p \nabla_{\sigma}^p f^\tau - \ldots = 0 \]  

with

\[ \nabla_+ = -\frac{1}{\sqrt{2}} (\frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2}), \quad \nabla_0 = \frac{\partial}{\partial x_3}, \quad \nabla_- = \frac{1}{\sqrt{2}} (\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2}), \]

\[ r_+ = -\frac{1}{\sqrt{2}} (x_1 + ix_2), \quad r_0 = x_3, \quad r_- = \frac{1}{\sqrt{2}} (x_1 - ix_2) \]

and the analogous definitions for \( \nabla_{+}^p, \quad \nabla_{0}^p, \quad \nabla_{-}^p \), and \( p_+, \quad p_0, \quad p_- \). The required equations shall be obtained by integrating (8) with different tensor products of \( r_{\alpha} \) and \( p_{\alpha} \).
3 Model Hamiltonian, Equations of motion, Eigenfrequencies

As outlined in the introduction, the model considered here is a harmonic oscillator mean field potential with quadrupole-quadrupole residual interactions. Its microscopic Hamiltonian is

\[ H = \sum_{i=1}^{A} \left( \frac{p_i^2}{2m} + \frac{1}{2} m \omega_i^2 r_i^2 \right) + \kappa \sum_{\mu=-2}^{2} (-1)^{\mu} \sum_{i}^{Z} q_{2\mu}(r_i) q_{2-\mu}(r_j) + \frac{1}{2} \kappa \sum_{\mu=-2}^{2} (-1)^{\mu} \left\{ \sum_{i\neq j}^{Z} q_{2\mu}(r_i) q_{2-\mu}(r_j) + \sum_{i\neq j}^{N} q_{2\mu}(r_i) q_{2-\mu}(r_j) \right\}, \]

(9)

where the quadrupole operator \( q_{2\mu} = \sqrt{16\pi/5} r^2 Y_{2\mu} \) and \( N, Z \) are the numbers of neutrons and protons respectively. The Hamiltonian is of the standard Bohr-Mottelson type [18], however, with an interaction coupling protons and neutrons (constant \( \kappa \)) and, separately, coupling protons and neutrons among themselves (constant \( \kappa \)). This form is appropriate for the description of the scissors mode. The corresponding mean field potentials are

\[ V^p(r,t) = \frac{1}{2} m \omega^2 r^2 + \sum_{\mu=-2}^{2} (-1)^{\mu} (\kappa Q_{2\mu}^p(t) + \bar{\kappa} Q_{2\mu}^n(t)) q_{2-\mu}(r) \]

(10)

for protons and

\[ V^n(r,t) = \frac{1}{2} m \omega^2 r^2 + \sum_{\mu=-2}^{2} (-1)^{\mu} (\kappa Q_{2\mu}^p(t) + \bar{\kappa} Q_{2\mu}^n(t)) q_{2-\mu}(r) \]

(11)

for neutrons. The multipole moments \( Q_{2\mu}^\tau(t) \) are defined as

\[ Q_{2\mu}^\tau(t) = \int d\{p,r\} q_{2\mu}(r) f^\tau(r,p,t). \]

(12)

where \( \int d\{p,r\} \equiv 2(2\pi\hbar)^{-3} \int d^3p \int d^3r \). Introducing the notation

\[ D_{2\mu}^p(t) = \kappa Q_{2\mu}^p(t) + \bar{\kappa} Q_{2\mu}^n(t), \quad D_{2\mu}^n(t) = \kappa Q_{2\mu}^p(t) + \bar{\kappa} Q_{2\mu}^n(t), \]

one can rewrite the mean fields in a more compact way

\[ V^\tau(r,t) = \frac{1}{2} m \omega^2 r^2 + \sum_{\mu=-2}^{2} (-1)^{\mu} D_{2\mu}^\tau(t) q_{2-\mu}(r). \]

(13)

Substituting the spherical functions by the tensor products \( r^2 Y_{2\mu} = \sqrt{3 \cdot 5 / 8\pi} r_{2\mu}^2 \), where

\[ r_{2\mu}^2 \equiv \{ r \otimes r \}_{2\mu} = \sum_{\sigma,\nu} C_{1\sigma,1\nu}^\mu r_{\sigma} r_{\nu}, \]

\[ C_{1\sigma,1\nu}^\mu = \sum_{\alpha,\beta} C_{\alpha,\beta,1\sigma,1\nu}^\mu r_{\alpha} r_{\beta} \]
and \( C_{\lambda \mu}^{\gamma} \) is the Clebsch-Gordan coefficient (let us recall that the vector \( \mathbf{r} \) is a tensor of rank one), one has

\[
V^\tau = \frac{1}{2} m \omega^2 r^2 + \sum_\mu (-1)^\mu Z_{2\mu}^\tau r_{2-\mu}^2.
\]

Here

\[
Z_{2\mu}^n = \chi R_{2\mu}^n + \bar{\chi} R_{2\mu}^n, \quad Z_{2\mu}^p = \chi R_{2\mu}^p + \bar{\chi} R_{2\mu}^p,
\]

\[
\chi = 6 \kappa, \quad \bar{\chi} = 6 \bar{\kappa},
\]

\[
R_{\tau \lambda \mu}(t) = \int d\{p, \mathbf{r}\} r_{\lambda \mu}^2 f^\tau(\mathbf{r}, \mathbf{p}, t). \tag{14}
\]

Integration of the equation (8) with the weights \( r_{\lambda \mu}^2, (rp)_{\lambda \mu} \equiv \{r \otimes p\}_{\lambda \mu} \) and \( p_{\lambda \mu}^2 \) yields the following set of equations (it is important to note, that because (13) is of quadratic form, the WFM break off at second order):

\[
\frac{d}{dt} R_{\lambda \mu}^\tau - \frac{2}{m} L_{\lambda \mu}^\tau = 0, \quad \lambda = 0, 2
\]

\[
\frac{d}{dt} L_{\lambda \mu}^\tau - \frac{1}{m} P_{\lambda \mu}^\tau + m \omega^2 R_{\lambda \mu}^\tau - 2 \sqrt{5} \sum_{j=0}^{2} \frac{2 j + 1}{2 j + 1} \{11j\}_{12\lambda\mu} (Z_{2j}^\tau R_{j\lambda\mu}^\tau) = 0, \quad \lambda = 0, 1, 2
\]

\[
\frac{d}{dt} P_{\lambda \mu}^\tau + 2 m \omega^2 L_{\lambda \mu}^\tau - 4 \sqrt{5} \sum_{j=0}^{2} \frac{2 j + 1}{2 j + 1} \{11j\}_{12\lambda\mu} (Z_{2j}^\tau L_{j\lambda\mu}^\tau) = 0, \quad \lambda = 0, 2 \tag{15}
\]

where \( \{11j\}_{12\lambda\mu} \) is the Wigner 6j-symbol. For the sake of simplicity the time dependence of tensors is not written out. Further the following notation is introduced

\[
P_{\lambda \mu}^\tau(t) = \int d\{p, \mathbf{r}\} p_{\lambda \mu}^2 f^\tau(\mathbf{r}, \mathbf{p}, t), \quad L_{\lambda \mu}^\tau(t) = \int d\{p, \mathbf{r}\} (rp)_{\lambda \mu} f^\tau(\mathbf{r}, \mathbf{p}, t). \tag{16}
\]

It is necessary to say some words about the physical meaning of the collective variables introduced above. By definition \( R_{2\mu}^\tau = Q_{2\mu}^\tau/\sqrt{6} \) and \( Q_{2\mu}^\tau \) is the quadrupole moment of the system of particles and \( R_{00}^\tau = -Q_{00}^\tau/\sqrt{3} \) with \( Q_{00}^\tau = N^\tau < r^2 > \) being the mean square radius of the same system. By analogy with these variables, defined in the coordinate space, we can say that the variables \( P_{2\mu}^\tau \) and \( P_{00}^\tau \) describe the quadrupole moment and the mean square radius of the same system in a momentum space. The variables \( L_{\lambda \mu}^\tau \) describe the coupling of momentum and coordinate spaces. To understand their nature it is useful to recall the definitions [11, 12] of nuclear density and mean velocity:

\[
n^\tau(\mathbf{r}, t) = \int \frac{2d^3p}{(2\pi\hbar)^3} f^\tau(\mathbf{r}, \mathbf{p}, t),
\]

\[
mn^\tau(\mathbf{r}, t) u^\tau_i(\mathbf{r}, t) = \int \frac{2d^3p}{(2\pi\hbar)^3} p_i f^\tau(\mathbf{r}, \mathbf{p}, t). \tag{17}
\]
They enter into the definitions (14,16) of irreducible tensors

\[ R^\tau_{\lambda \mu}(t) = \int d^3r \int \frac{2d^3p}{(2\pi \hbar)^3} r_{\lambda \mu}^2 f^\tau(r, p, t) = \int d^3r r_{\lambda \mu}^2 n^\tau(r, t), \]

\[ L^\tau_{\lambda \mu}(t) = \int d^3r \int \frac{2d^3p}{(2\pi \hbar)^3} (rp)_{\lambda \mu} f^\tau(r, p, t) = m \int d^3r (ru^\tau)_{\lambda \mu} n^\tau(r, t). \] (18)

The last expression for \( L^\tau_{\lambda \mu} \) demonstrates in an obvious way the physical meaning of these variables: being the first order moments of mean velocities they give information about the distribution of these velocities in the nucleus. (“First” means that velocities are weighted with the coordinate \( r \)). Sometimes, if the motion is comparatively simple, this information turns out sufficient to completely determine the velocity field (see section 4).

In the case of more intricate motions higher order moments are required for a complete description of velocities [11]. In any case the moments of velocities are a very convenient tool to describe the collective motion. For example, the zero order moment of velocity is nothing more than the linear momentum describing the nucleus’ center of mass motion.

One of the first order moments corresponds to the very well known angular momentum of a nucleus. It is connected with the variable \( L^\tau_{1 \mu} \) by the following relations:

\[ L^\tau_{10} = \frac{i}{\sqrt{2}} I^\tau_3, \quad L^\tau_{1 \pm 1} = \frac{1}{2}(I^\tau_2 \mp iI^\tau_1). \]

It is convenient to rewrite the equations (15) in terms of the isoscalar and isovector variables

\[ R_{\lambda \mu} = R^n_{\lambda \mu} + R^p_{\lambda \mu}, \quad P_{\lambda \mu} = P^n_{\lambda \mu} + P^p_{\lambda \mu}, \quad L_{\lambda \mu} = L^n_{\lambda \mu} + L^p_{\lambda \mu}, \]

\[ \bar{R}_{\lambda \mu} = R^n_{\lambda \mu} - R^p_{\lambda \mu}, \quad \bar{P}_{\lambda \mu} = P^n_{\lambda \mu} - P^p_{\lambda \mu}, \quad \bar{L}_{\lambda \mu} = L^n_{\lambda \mu} - L^p_{\lambda \mu}. \]

So the equations for the neutron and proton systems are transformed into isoscalar and isovector ones. The equations for the isoscalar system are given by

\[ \dot{R}_{00} - 2L_{00}/m = 0, \]

\[ \dot{L}_{00} - P_{00}/m + m \omega^2 R_{00} - 2\sqrt{5/3}[\chi_0(R_2R_2)_{00} + \chi_1(\bar{R}_2\bar{R}_2)_{00}] = 0, \]

\[ \dot{P}_{00} + 2m \omega^2 L_{00} - 4\sqrt{5/3}[\chi_0(R_2L_2)_{00} + \chi_1(\bar{R}_2\bar{L}_2)_{00}] = 0, \]

\[ \dot{R}_{2\mu} - 2L_{2\mu}/m = 0, \]

\[ \dot{L}_{2\mu} - P_{2\mu}/m + m \omega^2 R_{2\mu} - 2\sqrt{1/3}[\chi_0(R_2R_2)_{2\mu} + \chi_1(\bar{R}_2\bar{R}_2)_{2\mu}] \]

\[- \sqrt{7/3}[\chi_0(R_2R_2)_{2\mu} + \chi_1(\bar{R}_2\bar{R}_2)_{2\mu}] = 0, \]
\[ \dot{P}_\mu + 2m\omega^2L_\mu - 4\sqrt{1/3}[\chi_0(R_2L_0)_{2\mu} + \chi_1(R_2\bar{L}_0)_{2\mu}] \\
-2\sqrt{7/3}[\chi_0(R_2L_2)_{2\mu} + \chi_1(R_2\bar{L}_2)_{2\mu}] \\
+2\sqrt{3}[\chi_0(R_2L_1)_{2\mu} + \chi_1(R_2\bar{L}_1)_{2\mu}] = 0, \]
\[ \dot{L}_{1\nu} = 0. \] (19)

and the ones for the isovector system read:

\[ \dot{R}_{00} - 2\bar{L}_{00}/m = 0, \]
\[ \dot{L}_{00} - \bar{P}_{00}/m + m\omega^2\bar{R}_{00} - 2\sqrt{5/3}\chi(R_2\bar{R}_2)_{00} = 0, \]
\[ \dot{P}_{00} + 2m\omega^2\bar{L}_{00} - 4\sqrt{5/3}[\chi_0(R_2\bar{L}_2)_{00} + \chi_1(R_2\bar{L}_2)_{00}] = 0, \]
\[ \dot{R}_{2\mu} - 2\bar{L}_{2\mu}/m = 0, \]
\[ \dot{L}_{2\mu} - \bar{P}_{2\mu}/m + m\omega^2\bar{R}_{2\mu} - 2\sqrt{1/3}[\chi_0(R_2\bar{R}_0)_{2\mu} + \chi_1(R_2\bar{R}_0)_{2\mu}] \\
-\sqrt{7/3}\chi(R_2\bar{R}_2)_{2\mu} = 0, \]
\[ \dot{P}_{2\mu} + 2m\omega^2\bar{L}_{2\mu} - 4\sqrt{1/3}[\chi_0(R_2\bar{L}_0)_{2\mu} + \chi_1(R_2\bar{L}_0)_{2\mu}] \\
-2\sqrt{7/3}[\chi_0(R_2\bar{L}_2)_{2\mu} + \chi_1(R_2\bar{L}_2)_{2\mu}] \\
+2\sqrt{3}[\chi_0(R_2\bar{L}_1)_{2\mu} + \chi_1(R_2\bar{L}_1)_{2\mu}] = 0, \]
\[ \dot{L}_{1\nu} + \sqrt{5}\chi(R_2\bar{R}_2)_{1\nu} = 0. \] (20)

Here
\[ \chi_0 = (\chi + \bar{\chi})/2 \]
is an isoscalar strength constant and
\[ \chi_1 = (\chi - \bar{\chi})/2 \]
is the corresponding isovector one. The last equation of (19) demonstrates the conservation of the isoscalar angular momentum \( L_{1\nu} \). The dynamical equation for the isovector angular momentum \( \bar{L}_{1\nu} \) (the last equation of (20)) describes the relative (out of phase) motion of the neutron and proton angular momenta; hence it must be responsible for the scissors mode.

We will need the following algebraic relations:
\[ (R_2\bar{R}_2)_{00} = \frac{1}{\sqrt{5}}(R_{22}\bar{R}_{2-2} - R_{21}\bar{R}_{2-1} + R_{20}\bar{R}_{20} - R_{2-1}\bar{R}_{21} + R_{2-2}\bar{R}_{22}), \]

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\[(R_2 \tilde{R}_2)_{20} = \sqrt{\frac{2}{7}}(R_{22} \tilde{R}_{2-2} + R_{21} \tilde{R}_{2-1}/2 - R_{20} \tilde{R}_{20} + R_{2-1} \tilde{R}_{21}/2 + R_{2-2} \tilde{R}_{22}),\]

\[(R_2 \tilde{R}_2)_{22} = \sqrt{\frac{2}{7}}(R_{22} \tilde{R}_{20} - \sqrt{\frac{3}{2}}R_{21} \tilde{R}_{21} + R_{20} \tilde{R}_{22}),\]

\[(R_2 \tilde{R}_2)_{2-2} = \sqrt{\frac{2}{7}}(R_{2-2} \tilde{R}_{20} - \sqrt{\frac{3}{2}}R_{2-1} \tilde{R}_{2-1} + R_{2-0} \tilde{R}_{2-2}),\]

\[(R_2 \tilde{R}_2)_{21} = \sqrt{\frac{3}{7}}(R_{22} \tilde{R}_{2-1} - \frac{1}{\sqrt{6}}R_{21} \tilde{R}_{20} - \frac{1}{\sqrt{6}}R_{20} \tilde{R}_{21} + R_{2-1} \tilde{R}_{22}),\]

\[(R_2 \tilde{R}_2)_{2-1} = \sqrt{\frac{3}{7}}(R_{2-2} \tilde{R}_{20} - \frac{1}{\sqrt{6}}R_{2-1} \tilde{R}_{20} - \frac{1}{\sqrt{6}}R_{20} \tilde{R}_{2-1} + R_{2-2} \tilde{R}_{2-2}),\]

\[(R_2 \tilde{R}_2)_{11} = \sqrt{\frac{3}{5}}(\frac{1}{\sqrt{3}}R_{22} \tilde{R}_{2-2} - \frac{1}{\sqrt{2}}R_{21} \tilde{R}_{20} + \frac{1}{\sqrt{2}}R_{20} \tilde{R}_{21} - \frac{1}{\sqrt{3}}R_{2-1} \tilde{R}_{22}),\]

\[(R_2 \tilde{R}_2)_{10} = \sqrt{\frac{3}{5}}(\frac{2}{3}R_{22} \tilde{R}_{2-2} - \frac{1}{\sqrt{3}}R_{21} \tilde{R}_{20} + \frac{1}{\sqrt{3}}R_{20} \tilde{R}_{2-1} - \frac{2}{3}R_{2-2} \tilde{R}_{22}),\]

\[(R_2 \tilde{R}_2)_{1-1} = \sqrt{\frac{3}{5}}(\frac{1}{\sqrt{3}}R_{21} \tilde{R}_{2-2} - \frac{1}{\sqrt{2}}R_{20} \tilde{R}_{2-1} + \frac{1}{\sqrt{2}}R_{2-1} \tilde{R}_{20} - \frac{1}{\sqrt{3}}R_{2-2} \tilde{R}_{21}),\]

\[(R_2 \tilde{L}_1)_{20} = \frac{1}{\sqrt{2}}(R_{21} \tilde{L}_{1-1} - R_{2-1} \tilde{L}_{11}),\]

\[(R_2 \tilde{L}_1)_{22} = \sqrt{\frac{2}{3}}R_{22} \tilde{L}_{10} - \frac{1}{\sqrt{3}}R_{21} \tilde{L}_{11},\]

\[(R_2 \tilde{L}_1)_{2-2} = \sqrt{\frac{2}{3}}R_{2-2} \tilde{L}_{10} - \frac{1}{\sqrt{3}}R_{2-1} \tilde{L}_{1-1},\]

\[(R_2 \tilde{L}_1)_{21} = \frac{1}{\sqrt{3}}R_{22} \tilde{L}_{1-1} + \frac{1}{\sqrt{6}}R_{21} \tilde{L}_{10} - \frac{1}{\sqrt{2}}R_{20} \tilde{L}_{11},\]

\[(R_2 \tilde{L}_1)_{2-1} = -\frac{1}{\sqrt{3}}R_{2-2} \tilde{L}_{11} - \frac{1}{\sqrt{6}}R_{2-1} \tilde{L}_{10} + \frac{1}{\sqrt{2}}R_{20} \tilde{L}_{1-1}.\]

With the help of (21) one can write out in detail the whole set of 42 coupled equations (including integrals of motion) for the whole set of isoscalar and isovector variables. There exists no problem to solve these equations numerically. However, for the time being we want to simplify the situation as much as possible what will allow us to get the results in analytical form and thus will give us a maximum of insight into the nature of the modes.

1) Let us consider the problem in the small amplitude approximation, i.e. we only will study small deviations of the system from equilibrium. Writing all variables as a sum of their equilibrium value plus a small deviation

\[R_{\lambda \mu}(t) = R_{\lambda \mu}^{eq} + \mathcal{R}_{\lambda \mu}(t), \quad P_{\lambda \mu}(t) = P_{\lambda \mu}^{eq} + \mathcal{P}_{\lambda \mu}(t), \quad L_{\lambda \mu}(t) = L_{\lambda \mu}^{eq} + \mathcal{L}_{\lambda \mu}(t),\]

\[\tilde{R}_{\lambda \mu}(t) = \tilde{R}_{\lambda \mu}^{eq} + \tilde{\mathcal{R}}_{\lambda \mu}(t), \quad \tilde{P}_{\lambda \mu}(t) = \tilde{P}_{\lambda \mu}^{eq} + \tilde{\mathcal{P}}_{\lambda \mu}(t), \quad \tilde{L}_{\lambda \mu}(t) = \tilde{L}_{\lambda \mu}^{eq} + \tilde{\mathcal{L}}_{\lambda \mu}(t),\]
we linearize the equations of motion in $\mathcal{R}_{\lambda\mu}$, $\mathcal{P}_{\lambda\mu}$, $\mathcal{L}_{\lambda\mu}$ and $\bar{\mathcal{R}}_{\lambda\mu}$, $\bar{\mathcal{P}}_{\lambda\mu}$, $\bar{\mathcal{L}}_{\lambda\mu}$.

2) We will consider non rotating nuclei, i.e. nuclei with $L_{1\nu}^\text{eq} = L_{1\nu}^\text{eq} = 0$.

3) Let us consider axially symmetric nuclei, i.e. nuclei with $R_{2\pm 2}^\text{eq} = R_{2\pm 1}^\text{eq} = \bar{R}_{2\pm 2}^\text{eq} = \bar{R}_{2\pm 1}^\text{eq} = 0$.

4) Finally, we take $\bar{R}_{20}^\text{eq} = \bar{R}_{00}^\text{eq} = 0$. This means that equilibrium deformation and mean square radius of neutrons are supposed to be equal to that of protons.

After all these simplifications the set of equations for the isoscalar system (19) is transformed into the following set of linear equations:

\[
\begin{align*}
\dot{R}_{00} & - 2L_{00}/m = 0, \\
\dot{L}_{00} & \mathcal{P}_{00}/m + m \omega^2 R_{00} - 4\sqrt{1/3} \chi_0 R_{20}^\text{eq} \mathcal{R}_{20} = 0, \\
\dot{P}_{00} & + 2m \omega^2 L_{00} - 4\sqrt{1/3} \chi_0 R_{20}^\text{eq} \mathcal{L}_{20} = 0, \\
\dot{R}_{2\pm 2} & - 2L_{2\pm 2}/m = 0, \\
\dot{R}_{2\pm 1} & - 2L_{2\pm 1}/m = 0, \\
\dot{R}_{20} & - 2L_{20}/m = 0, \\
\dot{L}_{2\pm 2} & - \mathcal{P}_{2\pm 2}/m + \left[ m \omega^2 - \sqrt{4/3} \chi_0 (R_{00}^\text{eq} + \sqrt{2} R_{20}^\text{eq}) \right] R_{2\pm 2} = 0, \\
\dot{L}_{2\pm 1} & - \mathcal{P}_{2\pm 1}/m + \left[ m \omega^2 - \sqrt{4/3} \chi_0 (R_{00}^\text{eq} - R_{20}^\text{eq}/\sqrt{2}) \right] R_{2\pm 1} = 0, \\
\dot{L}_{20} & - \mathcal{P}_{20}/m + \left[ m \omega^2 - \sqrt{4/3} \chi_0 (R_{00}^\text{eq} - \sqrt{2} R_{20}^\text{eq}) \right] R_{20} - \sqrt{4/3} \chi_0 R_{20}^\text{eq} R_{00} = 0, \\
\dot{P}_{2\pm 2} & + 2[m \omega^2 - \sqrt{2/3} \chi_0 R_{20}^\text{eq}] L_{2\pm 2} = 0, \\
\dot{P}_{2\pm 1} & + 2[m \omega^2 + \sqrt{1/6} \chi_0 R_{20}^\text{eq}] L_{2\pm 1} = 0, \\
\dot{P}_{20} & + 2[m \omega^2 + \sqrt{2/3} \chi_0 R_{20}^\text{eq}] L_{20} - 4\sqrt{1/3} \chi_0 R_{20}^\text{eq} \mathcal{L}_{00} = 0, \\
\dot{\mathcal{L}}_{1\pm 1} & = 0.
\end{align*}
\]

The corresponding set of equations for the isovector system (20) reads

\[
\begin{align*}
\dot{R}_{00} & - 2\bar{L}_{00}/m = 0, \\
\dot{L}_{00} & \mathcal{P}_{00}/m + m \omega^2 \bar{R}_{00} - \sqrt{4/3} \chi R_{20}^\text{eq} \bar{R}_{20} = 0, \\
\dot{P}_{00} & + 2m \omega^2 \bar{L}_{00} - \sqrt{4/3} \chi R_{20}^\text{eq} \bar{L}_{20} = 0, \\
\dot{R}_{2\pm 2} & - 2\bar{L}_{2\pm 2}/m = 0, \\
\dot{R}_{2\pm 1} & - 2\bar{L}_{2\pm 1}/m = 0.
\end{align*}
\]
\[ \dot{R}_{20} - 2 \dot{L}_{20}/m = 0, \]
\[ \dot{L}_{2 \pm 2} - \dot{P}_{2 \pm 2}/m + \left[ m \omega^2 - \sqrt{2/3} \chi R_{20}^{eq} - \sqrt{4/3} \chi_1 R_{00}^{eq} \right] \dot{R}_{2 \pm 2} = 0, \]
\[ \dot{L}_{2 \pm 1} - \dot{P}_{2 \pm 1}/m + \left[ m \omega^2 + \sqrt{1/6} \chi R_{20}^{eq} - \sqrt{4/3} \chi_1 R_{00}^{eq} \right] \dot{R}_{2 \pm 1} = 0, \]
\[ \dot{L}_{20} - \dot{P}_{20}/m + \left[ m \omega^2 + \sqrt{2/3} \chi R_{20}^{eq} - \sqrt{4/3} \chi_1 R_{00}^{eq} \right] R_{20} - \sqrt{4/3} \chi_0 R_{20}^{eq} R_{00} = 0, \]
\[ \dot{P}_{2 \pm 2} + 2[m \omega^2 - \sqrt{2/3} \chi_0 R_{20}^{eq}] \dot{L}_{2 \pm 2} = 0, \]
\[ \dot{P}_{2 \pm 1} + 2[m \omega^2 + \sqrt{1/6} \chi_0 R_{20}^{eq}] \dot{L}_{2 \pm 1} + \sqrt{6} \chi_0 R_{20}^{eq} \dot{L}_{1 \pm 1} = 0, \]
\[ \dot{P}_{20} + 2[m \omega^2 + \sqrt{2/3} \chi_0 R_{20}^{eq}] \dot{L}_{20} - \sqrt{4/3} \chi_0 R_{20}^{eq} \dot{L}_{00} = 0, \]
\[ \dot{L}_{1 \pm 1} \pm \sqrt{3/2} \chi R_{20}^{eq} \dot{R}_{2 \pm 1} = 0, \]
\[ \dot{L}_{10} = 0. \]  

(23)

Due to the approximation 4) the equations for isoscalar and isovector systems are decoupled. Further, due to the axial symmetry the angular momentum projection is a good quantum number. As a result, every set of equations splits into five independent subsets. For example, the equations (23) can be grouped in the following way:

1) the subset of equations for variables \( \dot{R}_{00}, \dot{L}_{00}, \dot{P}_{00}, \dot{R}_{20}, \dot{L}_{20}, \dot{P}_{20} \) and \( \dot{L}_{10} \) with projections \( \mu = 0 \). The equation for \( \dot{L}_{10} \) gives the integral of motion. The rest of equations describes the isovector giant monopole resonance plus the branch of IVGQR corresponding to \( \mu = 0 \) (\( \beta \)-mode).

2) the subset of equations for variables \( \dot{R}_{22}, \dot{L}_{22}, \dot{P}_{22} \) with projections \( \mu = 2 \) and the subset of equations for variables \( \dot{R}_{2-2}, \dot{L}_{2-2}, \dot{P}_{2-2} \) with projections \( \mu = -2 \) describe two degenerate branches of IVGQR corresponding to \( \mu = |2| \) (\( \gamma \)-mode).

3a) the subset of equations for variables \( \dot{R}_{21}, \dot{L}_{21}, \dot{P}_{21}, \dot{L}_{11} \) with projections \( \mu = 1 \) describe the scissors mode plus the \( \mu = 1 \) branch of IVGQR (transverse shear mode).

3b) the subset of equations for variables \( \dot{R}_{2-1}, \dot{L}_{2-1}, \dot{P}_{2-1}, \dot{L}_{1-1} \) with projections \( \mu = -1 \) describe the same dynamics as the subset with \( \mu = 1 \), because the states with \( \mu = \pm 1 \) are degenerate due to the axial symmetry.

One also should mention that equations (22,23) are equivalent to the RPA equations corresponding to the Hamiltonian (9). The RPA equations have partially been solved in [19, 20].
3.1 Isoscalar eigenfrequencies

The dynamics of the isoscalar angular momentum is trivial - no vibrations, this variable is conserved. However it is necessary to treat this mode carefully because, being the nonvibrational mode with zero eigenfrequency, it gives nevertheless a nonzero contribution to the sum rule (see below). Let us analyze the isoscalar set of equations with $\mu = \nu = 1$ in more detail

\[
\begin{align*}
\dot{R}_{21} - 2\mathcal{L}_{21}/m &= 0, \\
\dot{\mathcal{L}}_{21} - \mathcal{P}_{21}/m + \left[m\omega^2 + \sqrt{4/3}\chi_0\left(R_{20}^{eq}/\sqrt{2} - R_{00}^{eq}\right)\right]\mathcal{R}_{21} &= 0, \\
\dot{\mathcal{P}}_{21} + 2\left[\omega^2 + \sqrt{1/6}\chi_0 R_{20}^{eq}\right]\mathcal{L}_{21} &= 0, \\
\dot{\mathcal{L}}_{11} &= 0.
\end{align*}
\]

(24)

Imposing the time evolution via $e^{i\Omega t}$ for all variables one transforms (24) into a set of algebraic equations with the determinant

\[
\Delta_{is} = \begin{vmatrix}
i\Omega & -2/m & 0 & 0 \\
m\omega^2 + \sqrt{4/3}\chi_0\left(R_{20}^{eq}/\sqrt{2} - R_{00}^{eq}\right) & i\Omega & -1/m & 0 \\
0 & 2m\omega^2 + \sqrt{2/3}\chi_0 R_{20}^{eq} & i\Omega & 0 \\
0 & 0 & 0 & i\Omega
\end{vmatrix}
\]

The eigenfrequencies are found from the characteristic equation $\Delta_{is} = 0$ where

\[
\Delta_{is} = \Omega^2\left[\Omega^2 - 4\omega^2 - \frac{\chi_0}{m}\left(\sqrt{6}R_{20}^{eq} - \frac{4}{\sqrt{3}}R_{00}^{eq}\right)\right].
\]

(25)

Using here the relations $\chi_0 = 6\kappa_0$, $R_{20} = Q_{20}/\sqrt{6}$ and $R_{00} = -Q_{00}/\sqrt{3}$ (where $Q_{00} = A < r^2 >$) we rewrite it in more conventional notation:

\[
\Delta_{is} = \Omega^2\left[\Omega^2 - 4\omega^2 - \frac{6\kappa_0}{m}(Q_{20}^{eq} + \frac{4}{3}Q_{00}^{eq})\right].
\]

(26)

For $\kappa_0$ we take the self-consistent value (see Appendix): $\kappa_0 = -\frac{m\bar{\omega}^2}{4Q_{00}}$, where $\bar{\omega}^2 = \frac{\omega^2}{1 + \frac{\delta}{3}}$.

Using now the standard definition of the deformation parameters

\[
Q_{20} = Q_{00}\sqrt{5/\pi}\beta = Q_{00}\frac{4}{3}\delta
\]

we finally obtain

\[
\Omega^2\left[\Omega^2 - 2\omega^2(1 + \delta/3)\right] = 0.
\]

(27)
The nontrivial solution of this equation gives the frequency of the \( \mu = 1 \) branch of the isoscalar GQR

\[ \Omega^2 = \Omega_{\text{is}}^2 = 2\omega^2(1 + \delta/3). \]  

(28)

In the limit of small deformation this result coincides with that of [19]. The trivial solution \( \Omega = \Omega_0 = 0 \) is characteristic of nonvibrational mode, corresponding to the obvious integral of motion \( \mathcal{L}_{11} = \text{const} \) responsible for the rotational degree of freedom. Another, not so obvious, integral is obtained by a simple combination of the third and first equations of (24):

\[ \mathcal{P}_{21} + 2[m\omega^2 + \sqrt{1/6 \chi_0 R_{20}^{eq}} \frac{m}{2} R_{21}] = \text{const} \quad \rightarrow \quad \mathcal{P}_{21} + m^2 \omega^2 (1 + \delta/3) R_{21} = \text{const}. \]

Assuming here \( \delta = 0 \) we reproduce our result from ref. [12] for spherical nuclei, saying that the nuclear density and the Fermi surface oscillate out of phase.

### 3.2 Isovector eigenfrequencies

The information about the scissors mode is contained in the set of isovector equations with \( \mu = \nu = 1 \). Let us analyze it in detail:

\[ \dot{R}_{21} - 2\mathcal{L}_{21}/m = 0, \]
\[ \dot{\mathcal{L}}_{21} - \mathcal{P}_{21}/m + \left[ m\omega^2 + \sqrt{1/6 \chi_0 R_{20}^{eq}} - \sqrt{4/3 \chi_1 R_{00}^{eq}} \right] R_{21} = 0, \]
\[ \dot{\mathcal{P}}_{21} + 2[m\omega^2 + \sqrt{1/6 \chi_0 R_{20}^{eq}}] \mathcal{L}_{21} - \sqrt{6 \chi_0 R_{20}^{eq}} \mathcal{L}_{11} = 0, \]
\[ \dot{\mathcal{L}}_{11} + \sqrt{3/2 \chi R_{20}^{eq}} R_{21} = 0. \]

(29)

Imposing the time evolution via \( e^{i\Omega t} \) for all variables one transforms (29) into a set of algebraic equations with the determinant

\[
\Delta_{iv} = \begin{vmatrix}
\Omega & -2/m & 0 & 0 \\
\Omega & 0 & -1/m & 0 \\
0 & 2m\omega^2 + \sqrt{2/3 \chi_0 R_{20}^{eq}} & i\Omega & -\sqrt{6 \chi_0 R_{20}^{eq}} \\
\sqrt{3/2 \chi R_{20}^{eq}} & 0 & 0 & i\Omega \\
\end{vmatrix}
\]

Again the eigenfrequencies are found from the characteristic equation \( \Delta_{iv} = 0 \) where

\[
\Delta_{iv} = \Omega^2 \left[ \Omega^2 - \frac{2}{m} (m\omega^2 + \frac{\chi_0}{\sqrt{6} R_{20}^{eq}}) \right] - \frac{2}{m} \left[ \Omega^2 (m\omega^2 + \frac{\chi}{\sqrt{6} R_{20}^{eq}} - \frac{2\chi_1}{\sqrt{3} R_{00}^{eq}}) - \frac{3}{m} \chi_0 (R_{20}^{eq})^2 \right] \quad (30)
\]
or in more conventional notation (see the definitions after eq.(25)): 

\[ \Delta_{iv} = \Omega^4 - \Omega^2 [4\omega^2 + \frac{8}{m}\kappa_1 Q_{00}^g + \frac{2}{m}(\kappa_1 + 2\kappa_0)Q_{20}^g] + \frac{36}{m^2}(\kappa_0 - \kappa_1)\kappa_0 (Q_{00}^g)^2. \]  

(31)

Supposing, as usual, the isovector constant $\kappa_1$ proportional to the isoscalar one, $\kappa_1 = \alpha\kappa_0$, and taking the self-consistent value for $\kappa_0$ we finally obtain 

\[ \Omega^4 - 2\Omega^2\bar{\omega}^2(2 - \alpha)(1 + \delta/3) + 4\bar{\omega}^4(1 - \alpha)\delta^2 = 0. \]  

(32)

The solutions of this equation are 

\[ \Omega_{\pm}^2 = \bar{\omega}^2(2 - \alpha)(1 + \delta/3) \pm \sqrt{\bar{\omega}^4(2 - \alpha)^2(1 + \delta/3)^2 - 4\omega^4(1 - \alpha)\delta^2}. \]  

(33)

This expression coincides with the result of Hamamoto and Nazarewizh [13] found in RPA. The solution $\Omega_+$ gives the frequency $\Omega_{iv}$ of the $\mu = 1$ branch of the isovector GQR. The solution $\Omega_-$ gives the frequency $\Omega_{sc}$ of the scissors mode.

It is worth noticing that in the case $\bar{\mathcal{L}}_{11} = 0$ the set of equations (29) becomes quite similar to (24). Its characteristic equation reduces to the equation 

\[ \Omega^3 - 2\Omega\bar{\omega}^2(2 - \alpha)(1 + \delta/3) = 0, \]  

(34)

implying that there exists an integral of motion analogous to the isoscalar one: 

\[ \bar{P}_{21} + m^2\omega^2(1 + \delta/3)\bar{R}_{21} = \text{const.} \]

The nontrivial solution of (34) gives the IVGQR frequency for the case, when rotational degrees of freedom are neglected:

\[ \Omega^2 = 2\bar{\omega}^2(2 - \alpha)(1 + \delta/3). \]  

(35)

Now let us fix the value of the coefficient $\alpha$. The experimental fact is: the energy of an isovector GQR is practically two times higher than that of an isoscalar one. Assuming $\delta = 0$ we have 

\[ \Omega_{\pm}^2 = \Omega_{iv}^2 = 2\omega^2(2 - \alpha). \]

The simple comparison of this expression with (28) shows that the experimental observation is satisfied by $\alpha = -2$. Then equation (33) gives the following formulae for both energies:

\[ E_{iv}^2 = 4(1 + \delta/3 + \sqrt{(1 + \delta/3)^2 - \frac{3}{4}\delta^2})(h\bar{\omega})^2; \]

\[ E_{sc}^2 = 4(1 + \delta/3 - \sqrt{(1 + \delta/3)^2 - \frac{3}{4}\delta^2})(h\bar{\omega})^2. \]  

(36)
In the limit of small deformations one can write for IVGQR energy

$$E_{iv}^2 \simeq 8(1 - \delta/3)(1 - \frac{3}{16}\delta^2)(\hbar\omega_0)^2. \quad (37)$$

For $\alpha = -2$ formula (35) gives: $E_{iv}^2 \simeq 8(1 - \delta/3)(\hbar\omega_0)^2$. Comparing it with (37) one sees that the influence of rotational degrees of freedom on the IVGQR energy is very small.

The scissors mode energy in the limit of small deformation is

$$E_{sc} \simeq \sqrt{\frac{3}{2}}(1 - \delta/2)\hbar\omega_0 \approx \sqrt{\frac{3}{2}}\delta\hbar\omega_0, \quad (38)$$

which is quite close to the result of Hilton [9]: $E_{sc} \approx \sqrt{1 + 0.66}\delta\hbar\omega_0$. Taking $\hbar\omega_0 = 45.2/A^{1/3}$ MeV (what corresponds to $r_0 = 1.15$ fm used in [21]), one obtains

$$E_{sc} \approx 55.4\delta A^{-1/3}\text{MeV},$$

which practically coincides with the result of Lipparrini and Stringari [21]: $E_{sc} \approx 56\delta A^{-1/3}$ MeV obtained with the help of a microscopic approach based on the evaluation of sum rules. Both results are not very far from the experimental [4] value: $E_{sc} \approx 66\delta A^{-1/3}$ MeV.

We now will present the calculation of the flow patterns but the impatient reader may jump to section 7 for further discussion on various aspects of the eigenfrequencies.

4 Flows

The flow distributions will be calculated with the help of the method of infinitesimal displacements. This method is based on the rules of variation of integral quantities of the object [16]. Its detailed description can be found in [11].

Small variations $R_{\lambda\mu}^\tau \equiv \delta R_{\lambda\mu}^\tau$ and $L_{\lambda\mu}^\tau \equiv \delta L_{\lambda\mu}^\tau$ are naturally expressed in terms of variations of $n^\tau(\mathbf{r}, t)$ and $u_i^\tau(\mathbf{r}, t)$ (see definitions (17, 18)):

$$R_{\lambda\mu}^\tau(t) = \int d^3r r_{\lambda\mu}^2\delta n^\tau(\mathbf{r}, t),$$

$$L_{\lambda\mu}^\tau(t) = m \int d^3r [(ru_{eq}^\tau)_{\lambda\mu}\delta n^\tau + (r\delta u^\tau)_{\lambda\mu}n_{eq}^\tau] = m \int d^3r (r\delta u^\tau)_{\lambda\mu}n_{eq}^\tau. \quad (39)$$

In the last equation we have supposed that $u_{eq}^\tau = 0$, i.e., there is no motion at equilibrium. The variations $\delta n$ and $\delta u_i$ are not independent. A relation between them is obtained by means of the continuity equation [16]

$$\delta n = -\sum_{i=1}^{3} \nabla_i(n\xi_i), \quad \delta u_i = \frac{\partial \xi_i}{\partial t},$$

17
where $\xi_i(r, t) \equiv dx_i$ is an infinitesimal displacement. Let us represent it in the form of the series:

$$\xi_i(r, t) = G_i(t) + \sum_{j=1}^{3} G_{i,j}(t)x_j + \sum_{j,k=1}^{3} G_{i,jk}(t)x_jx_k + \sum_{j,k,l=1}^{3} G_{i,jkl}(t)x_jx_kx_l + \cdots \quad (40)$$

For further use we will conserve only the second term of this infinite series, neglecting the rest. This procedure is well founded as explained in [11], so we repeat the most important arguments very briefly. First of all, it is necessary to notice that due to the triplanar symmetry of the equilibrium shape of the nucleus only the tensors $G_{i,j...}$ with an even number of indices will survive after integration over the volume. Further, the set of dynamic equations (15) for the second rank tensors allows us to describe only rather simple types of motion with $\xi_i \sim x_i$. To describe a more refined motion with $\xi_i \sim x_i^3$, one is forced to consider the dynamic equations for the fourth order moments of the Wigner function (the tensors of rank four). There is a one-to-one correspondence: the more the motion is complicated, the larger is the number of moments which must be considered.

So we take $\xi_i^\tau(r, t) = \sum_{j=1}^{3} G_{i,j}^\tau(t)x_j$. It is convenient to introduce the ”cyclic“ combinations of $\xi_i$ analogously to the cyclic variables in (8):

$$\rho_{+1}^\tau = -\frac{1}{\sqrt{2}}(\xi_1^\tau + i\xi_2^\tau), \quad \rho_0^\tau = \xi_3^\tau, \quad \rho_{-1}^\tau = \frac{1}{\sqrt{2}}(\xi_1^\tau - i\xi_2^\tau)$$

and to write them as $\rho_\mu^\tau(r, t) = \sum_{\nu=-1}^{+1} (-1)^\nu S_{\mu,-\nu}^\tau(t)r_\nu$. Then

$$\delta n^\tau = -3 \sum_{i=1}^{3} \nabla_i (n^\tau \xi_i^\tau) = - \sum_{\nu=-1}^{+1} (-1)^\nu \nabla_\nu (n^\tau \rho_{\nu}^\tau), \quad \delta u_\mu^\tau = \frac{\partial \rho_\mu^\tau}{\partial t} = \sum_{\nu=-1}^{+1} (-1)^\nu \dot{S}_{\mu,-\nu}^\tau(t)r_\nu.$$

Using these expressions one finds

$$R_{\lambda \mu}^\tau(t) = - \int d^3r \sum_{\sigma, \nu} C_{\lambda \sigma, 1 \nu}^{\lambda \mu} r_\sigma r_\nu \sum_{\phi=-1}^{+1} (-1)^\phi \nabla_\phi (n^\tau \rho_{\phi}^\tau)$$

$$= \sum_{\sigma, \nu} C_{1 \sigma, 1 \nu}^{\lambda \mu} \int d^3r \ n_{eq}^\tau (\rho_\sigma^\tau r_\nu + \rho_\nu^\tau r_\sigma) = 2 \sum_{\phi, \sigma, \nu} C_{\phi, \sigma, 1 \nu}^{\lambda \mu} (-1)^\phi \int d^3r \ n_{eq}^\tau S_{\sigma,-\phi}^\tau r_\phi r_\nu$$

$$= 2 \sum_{\nu} \sum_{\phi, \sigma} C_{1 \phi, 1 \nu}^{\lambda \mu} (-1)^\phi S_{\sigma,-\phi}^\tau C_{1 \phi, 1 \nu}^{\lambda \mu} R_{\phi}^\tau(eq).$$

Now taking into account the axial symmetry ($k = 0$) one gets

$$R_{\lambda \mu}^\tau = \frac{2}{\sqrt{3}}[(\sqrt{2} R_{20}^\tau - R_{00}^\tau)C_{1 \mu, 10}^{\lambda \mu} S_{\mu, 0}^\tau - (\sqrt{2} R_{20}^\tau + R_{00}^\tau)(C_{1 \mu+1, 1-1}^{\lambda \mu} S_{\mu+1,-1}^\tau + C_{1 \mu-1, 11}^{\lambda \mu} S_{\mu-1,1}^\tau)]$$

Exactly the same derivation for $L_{\lambda \mu}^\tau$ leads to the following result:

$$L_{\lambda \mu}^\tau = m \sum_{\sigma, \nu} C_{\lambda \sigma, 1 \nu}^{\lambda \mu} \int d^3r \ n_{eq}^\tau \rho_\nu^\tau r_\sigma = \cdots$$
We are interested in $\mathcal{R}_{21}$ and $\mathcal{L}_{11}$. Remembering that $R_{00} = -Q_{00}/\sqrt{3}$, $R_{20} = (\frac{2}{3})^2 Q_{00}$ and $Q_{00}^r = \frac{1}{2} Q_{00}$ (due to approximation 4)) we find

$$\begin{align*}
\mathcal{R}_{2\pm 1} &= \frac{1}{3\sqrt{2}} Q_{00} [(1 - \frac{2}{3} \delta) \dot{S}_{0,\pm 1} + (1 + \frac{4}{3} \delta) \ddot{S}_{\pm 1,0}], \\
\mathcal{L}_{1\pm 1} &= \frac{m}{6\sqrt{2}} Q_{00} [(1 - \frac{2}{3} \delta) \dot{S}_{0,\pm 1} - (1 + \frac{4}{3} \delta) \ddot{S}_{\pm 1,0}],
\end{align*}$$

where $\ddot{S}_{\sigma,\nu} = S^a_{\sigma,\nu} - S^p_{\sigma,\nu}$ (and $S_{\sigma,\nu} = S^a_{\sigma,\nu} + S^p_{\sigma,\nu}$). Having in mind the $e^{i\omega t}$ time dependence (vibrational motion), we can substitute $\dot{S}_{\sigma,\nu}$ by $i\Omega \ddot{S}_{\sigma,\nu}$. Solving these equations with respect to $\ddot{S}_{\sigma,\nu}$, we have

$$\ddot{S}_{0,1} = \frac{3}{\sqrt{2}} [\mathcal{R}_{21} - \frac{2i}{m\Omega} \mathcal{L}_{11}] / [Q_{00} (1 - \frac{2}{3} \delta)], \quad \ddot{S}_{1,0} = \frac{3}{\sqrt{2}} [\mathcal{R}_{21} + \frac{2i}{m\Omega} \mathcal{L}_{11}] / [Q_{00} (1 + \frac{4}{3} \delta)].$$

Now we use the set of equations (29) to find that $\mathcal{L}_{11} = -i m \dot{\omega}^2 \delta (1 - \alpha) \mathcal{R}_{21}$ and, as a result,

$$\mathcal{R}_{21} \mp \frac{2i}{m\Omega} \mathcal{L}_{11} = [1 \mp 2 \ddot{\omega} (1 - \alpha) \ddot{\delta}] \mathcal{R}_{21}.$$ 

Introducing the notation

$$\begin{align*}
A &= \frac{3}{\sqrt{2}} [1 - 2 \ddot{\omega}^2 (1 - \alpha) \delta] / [Q_{00} (1 - \frac{2}{3} \delta)], \\
B &= \frac{3}{\sqrt{2}} [1 + 2 \ddot{\omega}^2 (1 - \alpha) \delta] / [Q_{00} (1 + \frac{4}{3} \delta)],
\end{align*}$$

we finally get

$$\ddot{S}_{0,1} = A \mathcal{R}_{21}, \quad \ddot{S}_{1,0} = B \mathcal{R}_{21}.$$ 

A similar analysis of $\mathcal{R}_{2-1}$ and $\mathcal{L}_{1-1}$ allows us to write immediately

$$\ddot{S}_{0,-1} = A \mathcal{R}_{2-1}, \quad \ddot{S}_{-1,0} = B \mathcal{R}_{2-1}.$$ 

So we have for the "cyclic" displacements:

$$\begin{align*}
\ddot{\rho}_{+1} &= \ddot{S}_{1,0} r_0 = B \mathcal{R}_{21} x_3, \quad \ddot{\rho}_{-1} = \ddot{S}_{-1,0} r_0 = B \mathcal{R}_{2-1} x_3, \\
\ddot{\rho}_0 &= -\ddot{S}_{0,-1} r_{-1} - \ddot{S}_{0,1} r_{+1} = \sqrt{2} A (J_{13} x_1 + J_{23} x_2),
\end{align*}$$

where $J_{13} = (\mathcal{R}_{2-1} - \mathcal{R}_{21})/2$, $J_{23} = i(\mathcal{R}_{2-1} + \mathcal{R}_{21})/2$. By definition, the variable $J^r_{ij}$ is a small variation of the tensor $J^r_{ij} = \int d\{p, r\} x_i x_j f^r(r, p, t)$. Cartesian displacements are found by elementary means:

$$\ddot{\xi}_1 = \frac{1}{\sqrt{2}} (\ddot{\rho}_{-1} - \ddot{\rho}_{+1}) = \sqrt{2} B J_{13} x_3, \quad \ddot{\xi}_2 = \frac{i}{\sqrt{2}} (\ddot{\rho}_{-1} + \ddot{\rho}_{+1}) = \sqrt{2} B J_{23} x_3.$$
\[ \tilde{\xi}_3 = \tilde{\rho}_0 = \sqrt{2}A(\tilde{J}_{13}x_1 + \tilde{J}_{23}x_2). \]

Let us analyze the picture of displacements in the plane \( x_1 = 0 \) (the picture in the plane \( x_2 = 0 \) must be exactly the same due to axial symmetry). Knowing the infinitesimal displacements

\[ \tilde{\xi}_2 \equiv dy = \sqrt{2}B\tilde{J}_{23}x_3, \quad \tilde{\xi}_3 \equiv dz = \sqrt{2}A\tilde{J}_{23}x_2, \quad (42) \]

we can derive the differential equation for the flow

\[ \frac{dy}{dz} = \frac{Bz}{Ay} \quad \rightarrow \quad ydy - \frac{B}{A}zdz = 0. \]

Integrating this equation we find

\[ y^2 + \beta z^2 = \text{const} \equiv c \quad \rightarrow \quad \frac{y^2}{c} + \frac{z^2}{c/\beta} = 1, \]

where \( \beta = -B/A \). Depending on the sign of \( \beta \) this curve will be either an ellipse or a hyperbola. So, it is necessary to study carefully the structure of the coefficient \( \beta \). It is convenient to study the coefficients \( A \) and \( B \) separately.

Let us investigate at first the case of the **scissors mode**. Taking \( \Omega = \Omega_{sc} \) and \( \alpha = -2 \) we have

\[ A = \frac{3(1 + \delta/3 - \sqrt{(1 + \delta/3)^2 - \frac{3}{4}\delta^2 - \frac{3}{2}\delta})}{\sqrt{2}Q_{00}(1 - \frac{2}{3}\delta)(1 + \delta/3 - \sqrt{(1 + \delta/3)^2 - \frac{3}{4}\delta^2})}. \quad (43) \]

The bounds of possible values of \( \delta \) are determined by the natural requirements \( \omega_{x,y,z}^2 > 0 \). They give (see Appendix):

\[ -\frac{3}{4} < \delta < \frac{3}{2}. \quad (44) \]

It is easy to check that inside of these bounds the square root \( \sqrt{(1 + \delta/3)^2 - \frac{3}{4}\delta^2} \) is real and the denominator of expression (43) is always positive. The sign of the numerator depends on the sign of \( \delta \). Elementary analysis of the function

\[ F(\delta) = 1 + \delta/3 - \sqrt{(1 + \delta/3)^2 - \frac{3}{4}\delta^2 - \frac{3}{2}\delta} \]

shows that \( F(\delta) = 0 \) at \( \delta = 0 \) and its derivative \( F'(\delta) \) is negative in this point. This means that \( F(\delta) > 0 \) for \( \delta < 0 \) and \( F(\delta) < 0 \) for \( \delta > 0 \). Hence, \( A > 0 \) for \( \delta < 0 \) and \( A < 0 \) for \( \delta > 0 \). Analogous analysis of \( \delta \)-dependence of \( B \) shows that \( B < 0 \) for \( \delta < 0 \) and \( B > 0 \) for \( \delta > 0 \). So, we can conclude that \( \beta > 0 \) for any \( \delta \) and the currents in the case of the scissors mode are described by ellipses.
Let us consider the limit of small $\delta$ for an illustration. We have

$$\beta = \frac{1 - \frac{2}{3} \delta}{1 + \frac{2}{3} \delta} \frac{1 + \delta/3 - \sqrt{(1 + \delta/3)^2 - \frac{5}{9} \delta^2 + \frac{2}{9} \delta}}{1 + \frac{2}{3} \delta} \approx (1 - 2\delta) \frac{1 + \delta/3 + \frac{1}{2} \delta}{1 + \delta/3 - \frac{1}{2} \delta} \approx (1 - 2\delta)(1 + \frac{1}{2} \delta) \approx (1 - \frac{3}{2} \delta).$$

So, for $\delta > 0$ the short semiaxis of the current ellipse is $Y^2 = c$ and the long semiaxis is $Z^2 = c/\beta \simeq c(1 + \frac{3}{2} \delta)$. The eccentricity of this ellipse is

$$e_{cur}^2 = 1 - \frac{Y^2}{Z^2} = 1 - \beta = \frac{3}{2},$$

which must be compared with the eccentricity of ellipsoid corresponding to the shape of the mean field:

$$e_{body}^2 = 1 - \frac{<y^2>}{<z^2>} = \frac{2\delta}{1 + \frac{3}{5} \delta} \approx 2\delta.$$

Thus, the field of currents and the shape of a nucleus are described by prolate ellipsoïdes, their long (and short) semiaxes being disposed along the same coordinate axis. Figure 1 illustrates the situation schematically. The displacements and the difference in eccentricities are exaggerated on purpose to demonstrate clearly the essential features of the motion corresponding to the scissors mode. One can easily see that its main constituent is a rotation (out of phase rotation of neutrons and protons). It is also seen that the rotation is accompanied by the distortion of the nuclear shape - at least it is evident that the long semiaxis becomes smaller.

Hence, the real motion of the scissors mode is a mixture of rotational and irrotational ones. To get a quantitative measure for the contribution of each kind of motion, it is sufficient to write the displacement $\vec{\xi}$ as the respective superposition [5]:

$$\vec{\xi} = a\vec{e}_x \times \vec{r} + b\nabla(yz) = a(0,-z,y) + b(0,z,y).$$

Comparing the components $\xi_y = (b - a)z$, $\xi_z = (b + a)y$ with $\xi_2$, $\xi_3$ in (42) we immediately find

$$b - a = \sqrt{2} J_{23}B, \quad b + a = \sqrt{2} J_{23}A \quad \rightarrow \quad a = D(1 + \beta), \quad b = D(1 - \beta),$$

where $D = \sqrt{2} / \sqrt{2}$. So in the considered example we have

$$a = 2D(1 - \frac{3}{4} \delta), \quad b = \frac{3}{2} D \delta, \quad b/a \simeq \frac{3}{4} \delta(1 + \frac{3}{4} \delta) \approx \frac{3}{4} \delta.$$
Figure 1: Schematic picture of isovector displacements for the scissors mode. Thin ellipses are the lines of currents. The thick oval is the initial position of the nucleus’ surface (common for protons and neutrons). The dashed oval is the final position of the protons’ (or neutrons’) surface as a result of infinitesimal displacements shown by the arrows.
This value of the ratio \(b/a\) has the same order of magnitude as another measure of an "admixture": the ratio \(B(M1)_{iv}/B(M1)_{sc} \approx \sqrt{3} \delta\) (see the subsections 7.2 and 7.4). We also have to mention the following interesting fact. As we know (see section 7.3), in the absence of the Fermi Surface Deformation (FSD) the scissors mode is a zero frequency mode. Calculating (with the help of formulae (41)) the ratio \(B/A\) for \(\Omega = 0\) we find that 
\[
\beta = \frac{1 - \frac{2}{3} \delta}{1 + \frac{4}{3} \delta} \approx 1 - 2\delta.
\]
Hence, the eccentricity of the current ellipse \(e_{cur}^2 = 2\delta\) coincides with that of the body ellipsoid. As a result, the lines of flows are tangential to the nuclear surface, i.e. the motion goes without any separation of neutron and proton surfaces (in agreement with the results of papers [22, 23]). Looking at Fig. 1 we can conclude that the inclusion of FSD inevitably leads to the separation of neutrons and protons, what justifies the name of scissors mode, independently of how large the separation is.

Let us investigate now the structure of flows for the high-lying mode (IVGQR). Taking in (41) \(\Omega = \Omega_{iv}\) and \(\alpha = -2\) we find
\[
A = \frac{3(1 + \delta/3 + \sqrt{(1 + \delta/3)^2 - \frac{3}{4} \delta^2} - \frac{3}{2} \delta)}{\sqrt{2}Q_{00}(1 - \frac{2}{3} \delta)(1 + \delta/3 + \sqrt{(1 + \delta/3)^2} - \frac{3}{4} \delta^2)}.
\]
It is obvious that the denominator is positive in the above-mentioned bounds: \(-\frac{3}{4} < \delta < \frac{3}{2}\). Elementary calculations show that the numerator is equal to zero at \(\delta = 3/2\), being positive for \(\delta < 3/2\) and negative for \(\delta > 3/2\). Hence, \(A > 0\) for \(\delta < 3/2\) and \(A < 0\) for \(\delta > 3/2\). An analogous analysis of the expression
\[
B = \frac{3(1 + \delta/3 + \sqrt{(1 + \delta/3)^2} - \frac{3}{4} \delta^2 + \frac{3}{2} \delta)}{\sqrt{2}Q_{00}(1 + \frac{4}{3} \delta)(1 + \delta/3 + \sqrt{(1 + \delta/3)^2} - \frac{3}{4} \delta^2)}
\]
shows that \(B\) is equal to zero at \(\delta = -3/4\), being positive for \(\delta > -3/4\) and negative for \(\delta < -3/4\). So we can conclude that \(\beta < 0\) for \(-3/4 < \delta < 3/2\). Hence, the currents in the case of IVGQR are described by a hyperbola. As usual, the situation is illustrated for the case with small \(\delta\). We have
\[
\beta = -\frac{1 - \frac{2}{3} \delta}{1 + \frac{4}{3} \delta} \frac{1 + \delta/3 + \sqrt{(1 + \delta/3)^2} + \frac{3}{4} \delta^2 + \frac{3}{2} \delta}{1 + \frac{4}{3} \delta} \frac{1 + \delta/3 + \sqrt{(1 + \delta/3)^2} - \frac{3}{4} \delta^2 - \frac{3}{2} \delta} \approx -(1 - 2\delta)(1 + \frac{3}{2} \delta) \approx -(1 - \frac{1}{2} \delta).
\]
The family of curves \(y^2 - (1 - \frac{1}{2} \delta)z^2 = c\) is displayed schematically in Fig. 2. The most remarkable property of the current lines is seen with one glance: they are nonclosed, demonstrating the typical sample (see, for example, first page of [24]) of irrotational
Figure 2: Schematic picture of isovector displacements for the high-lying mode (IVGQR). The lines of currents are shown by thin curves (hyperbolae). The thick oval is the initial position of the nucleus’ surface (common for protons and neutrons). The dashed oval is the final position of the protons’ (or neutrons’) surface as a result of infinitesimal displacements shown by the arrows.
motion. Nevertheless, the final position of the surface of the proton (or neutron) system looks, as if this system was rotated on the whole, the length of the big semiaxis being increased. That is, from the outside one again sees practically the same picture, as in the case of the scissors mode: rotation plus distortion. This is a curious property of the shear motion and it justifies the second name of the $K^\pi = 1^+$ branch of IVGQR as "the high energy scissors mode" [9, 6]. The quantitative contributions for the two kinds of motion to the IVGQR are

$$a = \frac{1}{2}D\delta, \quad b = 2D(1 - \frac{1}{4}\delta), \quad a/b \approx \frac{1}{4}\delta(1 + \frac{1}{4}\delta) \approx \frac{1}{4}\delta.$$  

Concluding the comparison of the scissors mode current with that of IVGQR it is worth noticing that two principally different types of infinitesimal displacements result approximately in the same change of the nuclear surface position.

5 Linear response and transition probabilities

A direct way of calculating the reduced transition probabilities is provided by the theory of linear response of a system to a weak external field

$$\hat{O}(t) = \hat{O}\exp(-i\Omega t) + \hat{O}^\dagger\exp(i\Omega t).$$  

(45)

A convenient form of the response theory is e.g. given by Lane [25]. The matrix elements of the operator $\hat{O}$ obey the relationship

$$|\langle \psi_a | \hat{O} | \psi_0 \rangle |^2 = \hbar \lim_{\Omega \rightarrow \Omega_a} (\Omega - \Omega_a) \langle \psi' | \hat{O} | \psi' \rangle \exp(-i\Omega t),$$  

(46)

where $\psi_0$ and $\psi_a$ are the stationary wave functions of unperturbed ground and excited states; $\psi'$ is the wavefunction of the perturbed ground state, $\Omega_a = (E_a - E_0)/\hbar$ are the normal frequencies, the bar means averaging over a time interval much larger than $1/\Omega$, $\Omega$ being the frequency of the external field $\hat{O}(t)$.

To use formula (46) in the frame of our method, one must solve two problems [11]:

(1) to express the matrix element $\langle \psi' | \hat{O} | \psi' \rangle$ in terms of collective variables of the system,

(2) to find the solution of the dynamic equations for these variables in the presence of the external field.
The first problem is solved with the help of the formula for the Wigner transformation of a product of two operators [15]

\[
\langle \psi' | \hat{O} | \psi' \rangle = \int d^3r \int d^3r' \rho(r, r', t) \hat{O}(r', r)
\]

\[
= \int d^3r \int \frac{4d^3p}{(2\pi\hbar)^3} \exp \left( \frac{\hbar}{2i} \left( \nabla_r^O \cdot \nabla_p^f - \nabla_p^O \cdot \nabla_r^f \right) \right) O_W(r, p)f(r, p, t).
\] (47)

To deal with the second problem we add the field (45) to the mean field potential (13). The equation for the Wigner function (4) is then modified by the term

\[
F_{\text{ext}} = \frac{2}{\hbar} \sin \left( \frac{\hbar}{2} \left( \nabla_r^O \cdot \nabla_p^f - \nabla_p^O \cdot \nabla_r^f \right) \right) (O_W \exp(-i\Omega t) + O_W^* \exp(i\Omega t)) f.
\] (48)

Proceeding in the same way as before one obtains equations for all collective variables needed to calculate \( \langle \psi' | \hat{O} | \psi' \rangle \). The only new element now is the presence of the term \( F_{\text{ext}} \) that makes the equations for the moments inhomogeneous.

5.1 B(M1)-factors

To calculate the magnetic transition probability, it is necessary to excite the system with the following external field:

\[
\hat{O}_{\lambda\mu'} = -ie\hbar \frac{1}{mc} \lambda + 1 \nabla (r^\lambda Y_{\lambda\mu'}) \cdot [r \times \nabla].
\]

We are interested in the dipole operator \( \lambda = 1 \). In the cyclic coordinates it looks like

\[
\hat{O}_{1\mu'} = -\frac{e\hbar}{2mc} \sqrt{\frac{3}{2\pi}} \sum_{\nu,\sigma} C^{1\mu'}_{1\nu,1\sigma} r_{\nu} \nabla_{\sigma}, \quad \hat{O}_{1\mu'}^\dagger = -\hat{O}_{1\mu'}^* = (-1)^{\mu'} \hat{O}_{1-\mu'}.
\] (49)

Its Wigner transformation is

\[
(O_{1\mu'})_W = \gamma \sum_{\nu,\sigma} C^{1\mu'}_{1\nu,1\sigma} r_{\nu} p_{\sigma} = \gamma (r^p)_{1\mu'},
\]

where \( \gamma = -i e \sqrt{\frac{3}{2\pi}} \). For its matrix element we have

\[
\langle \psi' | \hat{O}_{1\mu'} | \psi' \rangle = \gamma L_{1\mu'}^p = \frac{\gamma}{2}(L_{1\mu'} + \bar{L}_{1\mu'}) = \frac{\gamma}{2}(\mathcal{L}_{1\mu'} - L_{1\mu'}).
\] (50)

Here we have taken into account that \( L_{1\mu'}^p = \bar{L}_{1\mu'}^p = 0 \). The contribution of \( \hat{O}_{1\mu'}(t) \) to the equation for the Wigner function is

\[
F_{\text{ext}} = \gamma \left( F_{\mu'} \exp(-i\Omega t) + (-1)^{\mu'} F_{-\mu'} \exp(i\Omega t) \right).
\]
with

\[ F_{\mu'} = \sum_{\nu} C_{1\nu,1\nu'}^{1\mu'} [p_\sigma \nabla_\nu - r_\nu \nabla_\sigma] f^p. \]

Integration of \( F_{\mu'} \) with the weights \( r_\mu^2 \), \( (rp)_\lambda \mu \) and \( p_\lambda^2 \) yields

\[ \int d\{p, r\} \, r_\lambda^2 F_{\mu'} = 2\sqrt{3(2\lambda + 1)} \sum_{k, \pi} C_{k\lambda, 1\mu'}^{k\pi} \{11\lambda\} R_{k\pi}^p (eq), \]

\[ \int d\{p, r\} \, (rp)_\lambda F_{\mu'} = \sqrt{3(2\lambda + 1)} \sum_{k, \pi} ((-1)^k (1 - (-1)^k) C_{k\lambda, 1\mu'}^{k\pi} \{11\lambda\} L_{k\pi}^p (eq), \]

\[ \int d\{p, r\} \, p_\lambda^2 F_{\mu'} = 2\sqrt{3(2\lambda + 1)} \sum_{k, \pi} C_{k\lambda, 1\mu'}^{k\pi} \{11\lambda\} F_{k\pi}^p (eq). \]

A simple analysis of these expressions shows that the external field modifies only the proton part of the set of equations (15) with \( \lambda = 2 \):

\[ \frac{d}{dt} R_{2\mu}^p + \cdots = -\gamma \sqrt{3} \left[ C_{2\mu, 1\mu'}^{2\mu'} R_{2\mu+1\mu'}^p (eq) \exp^{-i\mu t} + (-1)^{\mu'} C_{2\mu, 1\mu'}^{2\mu} R_{2\mu-1\mu'}^p (eq) \exp^{i\mu t} \right], \]

\[ \frac{d}{dt} L_{2\mu}^p + \cdots = 0, \]

\[ \frac{d}{dt} P_{2\mu}^p + \cdots = -\gamma \sqrt{3} \left[ C_{2\mu, 1\mu'}^{2\mu'+1\mu'} P_{2\mu+1\mu'}^p (eq) \exp^{-i\mu t} + (-1)^{\mu'} C_{2\mu, 1\mu'}^{2\mu-1\mu'} P_{2\mu-1\mu'}^p (eq) \exp^{i\mu t} \right]. \] (51)

We put here \( L_{2\mu}^p (eq) = 0 \). The modifications of the respective isoscalar and isovector equations are obvious.

The \( \mu' = 0 \) component of the external field does not disturb a nucleus due to its axial symmetry. Let us consider the case of \( \mu' = 1 \). The set of equations (51) reads

\[ \dot{R}_{2\mu}^p + \cdots = \gamma \sqrt{3/8} R_{2\mu}^{eq} \left[ \delta_{\mu, -1} \exp^{-i\mu t} + \delta_{\mu, 1} \exp^{i\mu t} \right], \]

\[ \dot{L}_{2\mu}^p + \cdots = 0, \]

\[ \dot{P}_{2\mu}^p + \cdots = \gamma \sqrt{3/8} P_{2\mu}^{eq} \left[ \delta_{\mu, -1} \exp^{-i\mu t} + \delta_{\mu, 1} \exp^{i\mu t} \right]. \] (52)

We have used herein the relations \( R_{20}^{eq} (eq) = R_{20}^{eq} / 2, P_{20}^{eq} (eq) = P_{20}^{eq} / 2 \) which hold true due to approximation 4).

Now, in accord with formula (50), we have to find the tensors \( \hat{\mathcal{L}}_{11} \) and \( \mathcal{L}_{11} \). The tensor \( \hat{\mathcal{L}}_{11} \) is found by solving the modified (as in (52)) set of equations (29):

\[ \dot{\hat{\mathcal{R}}}_{21} - 2 \mathcal{L}_{21}/m = -\gamma \sqrt{3/8} R_{20}^{eq} \exp^{i\mu t}, \]

\[ \dot{\hat{\mathcal{L}}}_{21} - \hat{\mathcal{P}}_{21}/m + \left[ m \omega^2 + \sqrt{1/6} \chi_0 R_{20}^{eq} - \sqrt{4/3} \chi_1 R_{00}^{eq} \right] \hat{\mathcal{R}}_{21} = 0, \]

\[ \dot{\hat{\mathcal{P}}}_{21} + 2[m \omega^2 + \sqrt{1/6} \chi_0 R_{20}^{eq}] \dot{\hat{\mathcal{L}}}_{21} - \sqrt{6} \chi_0 R_{20}^{eq} \hat{\mathcal{L}}_{11} = -\gamma \sqrt{3/8} P_{20}^{eq} \exp^{i\mu t}, \]

\[ \dot{\hat{\mathcal{L}}}_{11} + \sqrt{3/2} \chi_2 R_{20}^{eq} \hat{\mathcal{R}}_{21} = 0. \] (53)
It is clear that the time dependence of all variables must be $\exp^{i\Omega t}$. The required variable is determined by the ratio of two determinants

$$\mathcal{L}_{11} = \frac{\Delta_{\mathcal{L}}}{\Delta_{iv}} \exp^{i\Omega t},$$

where $\Delta_{iv}$ is defined by (31) and

$$\Delta_{\mathcal{L}} = \frac{3}{4} \gamma \bar{x} R_{20}^{eq} R_{20}^{eq} \left[ \Delta_{iv} \left( 2 \omega^2 + \sqrt{2/3} \frac{\chi_0}{m} R_{20}^{eq} - \Omega^2 \right) + \frac{2}{m^2} P_{20}^{eq} \right].$$

At equilibrium the set of dynamic equations (15) considerably simplify turning into the set of equations of equilibrium. Taking into account one of them

$$\frac{1}{m} P_{20}^{eq} = m \omega^2 R_{20}^{eq} - \frac{2}{\sqrt{3}} \chi_0 R_{20}^{eq} R_{00} - \left( \frac{2}{\sqrt{6}} \chi_0 (R_{20}^{eq})^2 \right)$$

we obtain

$$\Delta_{\mathcal{L}} = \frac{3}{4} \gamma \bar{k} Q_{20}^{2} \left[ 4 \omega^2 + \frac{\kappa_0}{m} (6 Q_{20} + 8 Q_{00}) - \Omega^2 \right].$$

Looking at the isoscalar counterpart of the set of equations (53)

$$\dot{\mathcal{L}}_{21} - 2 \mathcal{L}_{21}/m = \gamma \sqrt{3/8} R_{20}^{eq} \exp^{i\Omega t},$$

$$\dot{\mathcal{L}}_{21} - \mathcal{P}_{21}/m + \left[ m \omega^2 + \sqrt{4/3} \chi_0 (R_{20}^{eq}/\sqrt{2} - R_{00}^{eq}) \right] = 0,$$

$$\dot{\mathcal{L}}_{21} + 2 \left[ m \omega^2 + \sqrt{1/6} \chi_0 R_{20}^{eq} \right] \mathcal{L}_{21} = \gamma \sqrt{3/8} R_{20}^{eq} \exp^{i\Omega t},$$

$$\mathcal{L}_{11} = 0$$

one easily finds that the isoscalar tensor $\mathcal{L}_{11} = 0$.

Writing now the determinant $\Delta_{iv}$ as

$$\Delta_{iv} = (\Omega^2 - \Omega_{iv}^2)(\Omega^2 - \Omega_{sc}^2),$$

we easily can find the limit (46). For the case where $|\psi_a| = |\psi_{sc}|$ we have

$$|<sc| \hat{O}_{11}|0>|^2 = \frac{\gamma}{2} \mathcal{h} \Delta_{\mathcal{L}}(\Omega_{sc})/[(\Omega_{sc}^2 - \Omega_{iv}^2)2 \Omega_{sc}].$$

Applying the standard values of parameters

$$\kappa_1 = \alpha \kappa_0, \quad 4 \kappa_0 Q_{00} = -m \bar{\omega}^2, \quad \kappa_0 Q_{20} = -\frac{\delta}{3} m \bar{\omega}^2$$

we arrive at a rather complicated function of the deformation parameter $\delta$

$$|<sc| \hat{O}_{11}|0>|^2 = \frac{1}{8\pi} m \bar{\omega}^2 Q_{00} \delta^2 [E_{sc} - 2(1 + \delta/3)(\hbar \bar{\omega})^2]/[E_{sc}(E_{sc}^2 - E_{iv}^2)] \mu_N^2,$$
where \( \mu_N = \frac{e\hbar}{2mc} \) and \( E_{sc}^2 \) and \( E_{iv}^2 \) are given by (33). For small values of \( \delta \) this expression is considerably simplified. Assuming \( \alpha = -2 \) one finds the formula

\[
|<sc|\hat{O}_{11}|0>|^2/\mu_N^2 \simeq \sqrt{\frac{3}{2}} \left( \frac{Q_{00}^0 m\omega_0}{\hbar} \right) \frac{\delta}{1 + \delta/6} \simeq \sqrt{\frac{3}{2}} \frac{Q_{00}^0 m\omega_0}{\hbar} \delta,
\]

demonstrating the familiar [5] linear dependence on \( \delta \).

For the case \( |\psi_a\rangle = |\psi_{iv}\rangle \) formula (46) gives

\[
|<iv|\hat{O}_{11}|0>|^2/\mu_N^2 \simeq \frac{3}{64\pi} \frac{Q_{00}^0 m\omega_0}{\hbar} \frac{35^2}{\sqrt{2}} \frac{\delta^2}{2 + \delta/2} \simeq \frac{9}{\sqrt{264\pi}} \frac{Q_{00}^0 m\omega_0}{\hbar} \delta^2.
\]

(58)

Exactly the same results are obtained from the set of equations for the variables \( \bar{R}_{2-1}, \bar{P}_{2-1}, \bar{L}_{2-1} \) perturbed by the operator \( \hat{O}_{1-1} \).

Taking into account the relation \( Q_{00}^0 m\omega_0 / \hbar \simeq \frac{1}{2} \left( \frac{3}{2} A \right)^{4/3} \), which is usually [26] used to fix the value of the harmonic oscillator frequency \( \omega_0 \), we obtain the following estimate for the transition probability of the scissors mode:

\[
B(M1) \uparrow = 2 |<sc|\hat{O}_{11}|0>|^2 = \frac{(3/2)^{11/6}}{16\pi} A^{4/3} \delta \mu_N^2 = 0.042 A^{4/3} \delta \mu_N^2,
\]

which practically coincides with the result of [21]: \( B(M1) \uparrow = 0.043 A^{4/3} \delta \mu_N^2 \), obtained with the help of the microscopic approach based on the evaluation of the sum rules.

### 5.2 B(E2)-factors

Electric transition probabilities can be found exactly in the same way as the magnetic ones. To calculate the B(E2)-factor it is necessary to excite the system with the external field operator

\[
\hat{O}_{2\mu'} = er^2 Y_{2\mu'} = \beta r_{2\mu'}^2, \quad \hat{O}_{2\mu'}^\dagger = \hat{O}_{2\mu'} = (-1)^{\mu'} \hat{O}_{2-\mu'}, \quad \beta = e \sqrt{\frac{16}{8\pi}}.
\]

(59)

Its Wigner transform is identical to (59): \( (O_{2\mu'})_W = \beta r_{2\mu'}^2 \). The matrix element is given by

\[
<\psi'|\hat{O}_{2\mu'}|\psi'> = \beta R_{2\mu'}^0 = \frac{1}{2}\beta (R_{2\mu'} - \bar{R}_{2\mu'}).
\]

(60)
The contribution of $\hat{O}_{2\mu'}(t)$ to the equation for the Wigner function is

$$F_{ext} = 2\beta \left( F_{\mu'} \exp^{-i\Omega t} + (-1)^{\mu'} F_{-\mu'} \exp^{i\Omega t} \right)$$

with

$$F_{\mu'} = \sum_{\nu,\sigma} C^{2\mu'}_{1\nu,1\sigma} r_{\mu} \nabla f^p_{\sigma}.$$ Integration of $F_{\mu'}$ with the weights $r^2_{\lambda\mu}$, $(rp)_{\lambda\mu}$ and $p^2_{\lambda\mu}$ yields

$$\int d\{p, r\} r^2_{\lambda\mu} F_{\mu'} = 0,$$

$$\int d\{p, r\} (rp)_{\lambda\mu} F_{\mu'} = \sqrt{5(2\lambda + 1)} \sum_{k,\pi} C_{\lambda\mu,2\mu'}^{k,\pi} \{11\} R^p_{k\pi}(eq),$$

$$\int d\{p, r\} p^2_{\lambda\mu} F_{\mu'} = [1 + (-1)^\lambda] \sqrt{5(2\lambda + 1)} \sum_{k,\pi} C_{\lambda\mu,2\mu'}^{k,\pi} \{11\} L^p_{k\pi}(eq).$$

The external field modifies the set of equations (15) in the following way:

$$\frac{d}{dt} L^p_{1\mu} + \cdots = -\beta \sqrt{3} \left[ C^{2\mu+\mu'}_{1\nu,2\mu} R^p_{2\nu,2\mu'}(eq) \exp^{-i\Omega t} + (-1)^{\mu'} C^{2\mu'-\mu}_{1\nu,2\mu'} R^p_{2\nu,2\mu'}(eq) \exp^{i\Omega t} \right],$$

$$\frac{d}{dt} L^p_{2\mu} - \cdots = \frac{\beta}{\sqrt{3}} \left[ (2\sqrt{5} C^{00}_{2\mu,2\mu'} R^p_{00}(eq) + \sqrt{7} C^{2\mu+\mu'}_{2\mu,2\mu'} R^p_{2\mu+\mu'}(eq) \exp^{-i\Omega t} + (-1)^{\mu'} \left( 2\sqrt{5} C^{00}_{2\mu,2\mu'-\mu'} R^p_{00}(eq) + \sqrt{7} C^{2\mu'-\mu'}_{2\mu,2\mu'-\mu'} R^p_{2\mu'-\mu'}(eq) \exp^{i\Omega t} \right) \exp^{i\Omega t} \right]. \quad (61)$$

The $\mu' = 0$ component of the external field does not disturb a nucleus with axial symmetry. Let us consider the case of $\mu' = 1$ ($\mu' = -1$ gives the same result). The equations (61) then read

$$\dot{L}^p_{2\mu} - \cdots = \frac{\beta}{3} \left( \frac{Q^eq}{Q_{00}} + \frac{1}{4} \frac{Q^eq}{Q_{20}} \right) \left[ \delta_{\mu,-1} \exp^{-i\Omega t} - \delta_{\mu,1} \exp^{i\Omega t} \right],$$

$$\dot{L}^p_{1\mu} + \cdots = \frac{\beta}{4} \frac{Q^eq}{Q_{20}} \left[ \delta_{\mu,-1} \exp^{-i\Omega t} + \delta_{\mu,1} \exp^{i\Omega t} \right]. \quad (62)$$

Now, according to formula (60), we have to find the tensors $\mathcal{R}_{21}$. The tensor $\mathcal{R}_{21}$ is found by solving the modified (as in (62)) set of equations (29)

$$\dot{\mathcal{R}}_{21} - 2\mathcal{L}_{21}/m = 0,$$

$$\dot{\mathcal{L}}_{21} - \mathcal{P}_{21}/m + \left[ m \omega^2 + \sqrt{1/6} R^q_{20} - \sqrt{4/3} \chi_1 R^q_{00} \right] \mathcal{R}_{21} = \frac{\beta}{3} \left( \frac{Q^eq}{Q_{00}} + \frac{1}{4} \frac{Q^eq}{Q_{20}} \right) \exp^{i\Omega t},$$

$$\dot{\mathcal{P}}_{21} + 2[m \omega^2 + \sqrt{1/6} \chi_0 R^q_{20}] \mathcal{L}_{21} - \sqrt{6} \chi_0 R^q_{20} \mathcal{L}_{11} = 0,$$

$$\dot{\mathcal{L}}_{11} + 3/2 \chi R^q_{20} \mathcal{R}_{21} = -\frac{\beta}{4} \frac{Q^eq}{Q_{20}} \exp^{i\Omega t}. \quad (63)$$
It is obvious that the time dependence of all variables must be \( \exp^{i\Omega t} \). The required variable is determined by the ratio of two determinants

\[
\mathcal{R}_{21} = \frac{\Delta_R}{\Delta_{iv}} \exp^{i\Omega t},
\]

where \( \Delta_{iv} \) is defined by (31) and

\[
\Delta_R = -\frac{\beta}{m} \left[ \frac{2}{3} \Omega^2 (Q_{00}^e + \frac{1}{4} Q_{20}^e) + \frac{1}{m} Q_{20}^e \sqrt{\frac{3}{2} \chi_0 R_{20}^e} \right].
\]

The tensor \( \mathcal{R}_{21} \) is found by solving the modified (as in (62)) set of equations (24):

\[
\dot{\mathcal{R}}_{21} - 2\mathcal{L}_{21}/m = 0,
\]

\[
\dot{\mathcal{L}}_{21} - \mathcal{P}_{21}/m + \left[ m \omega^2 + \sqrt{4/3} \chi_0 (R_{20}/\sqrt{2} - R_{00}) \right] \mathcal{R}_{21} = -\frac{\beta}{3} (Q_{00}^e + \frac{1}{4} Q_{20}^e) \exp^{i\Omega t},
\]

\[
\dot{\mathcal{P}}_{21} + 2[m \omega^2 + \sqrt{1/6} \chi_0 R_{20}^e] \mathcal{L}_{21} = 0,
\]

\[
\dot{\mathcal{L}}_{11} = -\frac{\beta}{4} Q_{20}^e \exp^{i\Omega t}.
\]

Again it is obvious that the time dependence of all variables in these equations must be \( \exp^{i\Omega t} \) and the required variable is determined by the ratio of two determinants

\[
\mathcal{R}_{21} = \frac{\Delta_R}{\Delta_{is}} \exp^{i\Omega t},
\]

where \( \Delta_{is} \) is defined by (26) and \( \Delta_R = -\Delta_R \).

The limit (46) is calculated with the help of expression (55) for \( \Delta_{iv} \) and the analogous expression for \( \Delta_{is} \):

\[
\Delta_{is} = (\Omega^2 - \Omega_{0s}^2)(\Omega^2 - \Omega_{is}^2).
\]

In the case \( |\psi_a\rangle = |\psi_{sc}\rangle \) we find

\[
| < sc | \hat{O}_{21} | 0 \rangle |^2 = -\frac{\hbar}{2} \Delta_R (\Omega_{sc}) / [(\Omega_{sc}^2 - \Omega_{iv}^2) 2\Omega_{sc}]
\]

\[
= -\frac{\beta^2 \hbar^2}{2m} \frac{1}{3} E_{sc}^2 (Q_{00}^e + \frac{1}{4} Q_{20}^e) + \frac{3h^2}{2m} \kappa_0 (Q_{20}^e)^2 / [E_{sc} (E_{sc}^2 - E_{iv}^2)]
\]

\[
= -\frac{\delta^2 \hbar^2}{m} \frac{5}{16\pi} Q_{00}^e (2(h\omega\delta)^2 - (1 + \delta/3)) E_{sc}^2 / [E_{sc} (E_{sc}^2 - E_{iv}^2)].
\]

For small \( \delta \) (and \( \alpha = -2 \))

\[
| < sc | \hat{O}_{21} | 0 \rangle |^2 \approx \frac{\delta^2 \hbar}{m \omega_0} \frac{5}{128\sqrt{6\pi}} Q_{00}^0 \frac{\delta}{1 - \delta/6} \approx \frac{\delta^2 \hbar}{m \omega_0} \frac{5}{128\sqrt{6\pi}} Q_{00}^0 \delta.
\]
For small values of $\delta$ (and $\alpha = -2$) this expression reduces to

$$| < iv| \hat{O}_{21}| 0 > |^2 \approx \frac{e^2 \hbar}{m \omega_0} \frac{5}{32 \sqrt{2\pi}} Q_{00}^0 \frac{1 + \frac{2}{3} \delta}{1 + \delta / 6} \approx \frac{e^2 \hbar}{m \omega_0} \frac{5}{32 \sqrt{2\pi}} Q_{00}^0.$$  

In the case $|\psi_i > = |\psi_{iv} >$ formula (46) gives

$$| < iv| \hat{O}_{21}| 0 > |^2 = -\frac{\hbar^2}{2} \Delta_{\Omega}(\Omega_{is})/[2 \Omega_{is}(\Omega_{is}^2 - \Omega_0^2)]$$

$$= \frac{\beta^2 h^2}{2m} \left[ \frac{1}{3} E_{iv}^2 Q_{00}^{eq} + \frac{1}{4} Q_{20}^{eq} \right] + \frac{3h^2}{2m} \kappa_0(Q_{20}^{eq})^2 | [E_{iv}(E_{iv}^2 - \hat{E}_{sc})]$$

$$= \frac{e^2 h^2}{m} \frac{5}{16\pi} Q_{00}[1 + \frac{\delta}{3}] E_{iv}^2 - 2(h\omega \delta)^2 | [E_{iv}^3]. \quad (66)$$

For small values of $\delta$ this expression reduces to

$$| < iv| \hat{O}_{21}| 0 > |^2 \approx \frac{e^2 \hbar}{m \omega_0} \frac{5}{16\sqrt{2\pi}} Q_{00}^0 (1 + \delta / 3).$$

Formula (46) allows one to calculate the matrix element $| < \psi_a | \hat{O} | \psi_0 > |^2$ also in the case when $|\psi_a > = |\Omega_0 >$, i.e., for the rotational state corresponding to the trivial solution of (27):

$$| < \Omega_0| \hat{O}_{21}| 0 > |^2 = -\frac{\hbar^2}{2} \Delta_{\Omega}(\Omega_0)/[2 \Omega_0(\Omega_0^2 - \Omega_{is}^2)]$$

$$= \frac{\beta^2 h^2}{2m} \left[ \frac{1}{3} h^2 \Omega_0^2 Q_{00}^{eq} + \frac{1}{4} Q_{20}^{eq} \right] + \frac{3h^2}{2m} \kappa_0(Q_{20}^{eq})^2 | [h \Omega_0(h^2 \Omega_0^2 - E_{is}^2)]$$

$$= \frac{e^2 h^2}{m} \frac{5}{8\pi} Q_{00} \delta^2 | [h \Omega_0^2(1 + \delta / 3)]. \quad (68)$$

The value of this matrix element is obviously infinite due to the zero value of $\Omega_0$. However, below this expression will be useful to calculate the energy weighted sum rule.

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6 Sum rules

6.1 Magnetic case

The magnetic dipole operator (49) is not hermitian. By definition it is a linear combination of hermitian operators (components of the angular momentum)

$$\hat{O}_{11} = -\frac{i}{2} \gamma (\hat{I}_x + i\hat{I}_y), \quad \hat{O}_{1-1} = \frac{i}{2} \gamma (\hat{I}_x - i\hat{I}_y).$$

This fact allows one to derive several useful relations:

$$[\hat{O}_{11}, [H, \hat{O}_{1-1}]] = \frac{\gamma^2}{4} ([\hat{I}_x, [H, \hat{I}_x]] + [\hat{I}_y, [H, \hat{I}_y]]),$$

$$<0|\hat{O}_{11}|\nu> <\nu|\hat{O}_{1-1}|0> = \frac{\gamma^2}{2} (|<\nu|\hat{I}_x|0>|^2 + |<\nu|\hat{I}_y|0>|^2)$$

$$= -(|<\nu|\hat{O}_{11}|0>|^2 + |<\nu|\hat{O}_{1-1}|0>|^2).$$

Using these formulae and the standard sum rule for a hermitian operator [18]

$$\sum_{\nu} (E_{\nu} - E_0) |<\nu|\hat{I}_i|0>|^2 = \frac{1}{2} <0|[\hat{I}_i, [H, \hat{I}_i]]|0>,$$

one immediately obtains the sum rule for $\hat{O}_{1\pm1}$:

$$\sum_{\nu} (E_{\nu} - E_0) (|<\nu|\hat{O}_{11}|0>|^2 + |<\nu|\hat{O}_{1-1}|0>|^2) = - <0|[\hat{O}_{11}, [H, \hat{O}_{1-1}]]|0>.$$ (69)

It can also be calculated in a more direct way:

$$<0|[\hat{O}_{11}, [H, \hat{O}_{1-1}]]|0> =$$

$$= \sum_{\nu} (E_{\nu} - E_0) (<0|\hat{O}_{11}|\nu><\nu|\hat{O}_{1-1}|0> + <0|\hat{O}_{1-1}|\nu><\nu|\hat{O}_{11}|0>)$$

$$= \sum_{\nu} (E_{\nu} - E_0) (<0|\hat{O}_{11}|\nu><0|\hat{O}_{1-1}^\dagger|\nu>^* + <0|\hat{O}_{1-1}|\nu><0|\hat{O}_{11}^\dagger|\nu>^*).$$

Using here the hermitian conjugation properties (49) of the operator $\hat{O}_{1\mu}$, one reproduces formula (69).

The double commutator is calculated with the help of (9) and (49):

$$[\hat{O}_{1\phi}, [H, \hat{O}_{1\psi}]] = \frac{15\chi}{2\pi N} \sum_i \sum_{\nu,\sigma,\epsilon} \sum_{\nu',\sigma',\epsilon'} (-1)^\nu C_{2\nu,2\sigma}^{1\phi} C_{2\nu',2\sigma'}^{1\phi'} r_{2\sigma}^2 (i) r_{2\sigma'}^2 (j) \mu_N^2.$$ (71)
Taking into account axial symmetry, one finds the ground state matrix element of (71) (in Hartree-Fock approximation)

\[
<0|\hat{O}_{1\phi},[H,\hat{O}_{1\phi}^\prime]|0>/\mu_N^2 = \frac{15}{2\pi} \chi \sum_{\nu} (-1)^\nu C^{1\phi}_{2\nu,20}C^{1\phi^\prime}_{2\nu,20} R_{20}^\nu R_{20}^{\nu} = \frac{15}{8\pi} \delta_{\phi^\prime,-\phi} \chi (C^{1\phi}_{2\phi,20} R_{20}^{\phi})^2.
\]

It is obvious that this expression is different from zero only for \(\phi = \pm 1\). Hence, the final expression for the right-hand side of (69) is

\[
<0|[\hat{O}_{11},[H,\hat{O}_{11}^\prime]]|0> = \frac{9}{16\pi} \chi (R_{20}^{\phi})^2 \mu_N^2 = -\frac{1-\alpha}{4\pi} Q_{00} m\tilde{\omega}^2 \delta^2 \mu_N^2 \equiv -(1-\alpha)\Sigma_0, \quad (72)
\]

where, for the sake of convenience, the notation \(\Sigma_0 = \frac{m\tilde{\omega}^2}{4\pi} Q_{00} \delta^2 \mu_N^2\) is introduced. The left-hand side of (69) is calculated trivially by multiplying the right-hand side of (56) by \(E_{sc}\) and adding it to the second line of (57) multiplied by \(E_{iv}\):

\[
\Sigma_{tot} = \sum_{\nu} (E_\nu - E_0) \left( |<\nu|\hat{O}_{11}|0>|^2 + |<\nu|\hat{O}_{11}^\prime|0>|^2 \right)
= 2 \left( E_{sc} < sc |\hat{O}_{11}|0>|^2 + E_{iv} |<iv|\hat{O}_{11}|0>|^2 \right)
= \Sigma_{sc} + \Sigma_{iv} = (1-\alpha)\Sigma_0, \quad (73)
\]

where

\[
\Sigma_{sc} = \frac{[E_{sc}^2 - 2(1+\delta/3)(\hbar\tilde{\omega})^2]}{(E_{sc}^2 - E_{iv}^2)} (1-\alpha)\Sigma_0 \quad (74)
\]

and

\[
\Sigma_{iv} = \frac{[E_{iv}^2 - 2(1+\delta/3)(\hbar\tilde{\omega})^2]}{(E_{iv}^2 - E_{sc}^2)} (1-\alpha)\Sigma_0. \quad (75)
\]

So, one sees that the sum rule (69) is fulfilled.

It is useful to estimate the contribution to the sum rule from the scissors mode in the small deformation limit. First of all, with the help of formula (33), one evaluates the difference

\[
E_{iv}^2 - E_{sc}^2 \simeq 2(\hbar\tilde{\omega})^2 (2-\alpha)(1+\frac{\delta}{3}).
\]

With this the contribution of the scissors mode is calculated quite easily:

\[
\Sigma_{sc} = (1-\alpha)\Sigma_0 \frac{[2(1+\delta/3)(\hbar\tilde{\omega})^2 - E_{2sc}^2]}{(E_{iv}^2 - E_{sc}^2)} \simeq \Sigma_0 \frac{1-\alpha}{2-\alpha}. \quad (76)
\]

Neglecting the \(\delta^3\)-term and taking \(\hbar\omega_0 = 41/A^{1/3}\text{MeV}\) (such value was used in the papers [20, 27]) we reproduce the result of these papers

\[
\Sigma_{sc} \simeq \frac{41}{8\pi} \left(\frac{3}{2}\right)^\frac{3}{2} A\delta^2 \frac{1-\alpha}{2-\alpha} \mu_N^2 \text{MeV} = 1.4 \frac{2 - 2\alpha}{2 - \alpha} \delta^2 A\mu_N^2 \text{MeV}. \quad (77)
\]

For further discussion see section 7.5.
6.2 Electric case

The sum rule for $\hat{O}_{2\pm 1}$ can easily be obtained by replacing in formula (70) the operators $\hat{O}_{1\pm 1}$ by the operators $\hat{O}_{2\pm 1}$ and using the hermitian conjugation properties (59) of the operator $\hat{O}$:

$$\sum_{\nu} (E_{\nu} - E_0) |\nu\hat{O}_{21}|0>|^2 + |\nu\hat{O}_{2-1}|0>|^2 = - <0|\{\hat{O}_{21}, [H, \hat{O}_{2-1}]\}|0>.$$ \hspace{1cm} (78)

The double commutator is calculated with the help of (9) and (59):

$$[\hat{O}_{2\phi}, [H, \hat{O}_{2\phi}]] = -20\beta^2 \hbar^2 m \sum_{\lambda, \sigma} C_{2\phi,2\phi}^{\lambda}\{^{112}_{\lambda21}\} r_{\lambda\sigma}^2(i).$$ \hspace{1cm} (79)

Taking into account axial symmetry, one finds the ground state matrix element of (79):

$$<0|[\hat{O}_{2\phi}, [H, \hat{O}_{2\phi}]]|0> = -20\beta^2 \hbar^2 m \sum_{\lambda=0,2} C_{2\phi,2-\phi}^{\lambda0}\{^{112}_{\lambda21}\} R_{\lambda0}^p$$

$$= -2\beta^2 \hbar^2 m \sum_{\phi, -\phi} \left((-1)^\phi \frac{2}{\sqrt{3}} R_{00}^p + \frac{1}{\sqrt{6}} R_{20}^p\right).$$ \hspace{1cm} (80)

Taking here $\phi = 1$ we obtain the final expression for the right-hand side of (78)

$$<0|\{\hat{O}_{21}, [H, \hat{O}_{2-1}]\}|0> = -2\beta^2 \hbar^2 \frac{5}{m \pi} Q_{00}(1 + \delta/3).$$ \hspace{1cm} (81)

Exactly the same result is obtained for isoscalar modes:

$$2 \left(E_{sc} <sc|\hat{O}_{21}|0> + E_{iv} <iv|\hat{O}_{21}|0>\right) = \frac{\beta^2 \hbar^2}{3m} (Q_{00} + \frac{1}{4} Q_{20})$$

$$= e^2 \hbar^2 m \frac{5}{8\pi} Q_{00}(1 + \delta/3).$$ \hspace{1cm} (82)

Hence the sum rule (78) is fulfilled.

It is interesting to compare the contributions of the scissors mode and the rotational mode. The scissors mode (for small $\delta$) yields:

$$2E_{sc} <sc|\hat{O}_{21}|0> \simeq \frac{5}{128\pi} e^2 \hbar^2 m Q_{00}\delta^2.$$ \hspace{1cm} (83)
The rotational mode yields:

\[ 2\hbar \Omega_0 | < \Omega_0 | \hat{O}_{21} | 0 > |^2 = \frac{5}{8\pi} e^2 \frac{\hbar^2}{mQ_{00}} \frac{\delta^2}{1 + \delta/3}. \quad (84) \]

It is seen that the contribution of the rotational mode is approximately 16 times larger than the one of the scissors mode. This is a very significant number demonstrating the importance of excluding the spurious state from the theoretical results. Indeed, to describe correctly such a subtle phenomenon as the scissors mode, it is compulsory to eliminate the errors from spurious motion whose value can be an order of magnitude larger than the phenomenon under consideration.

7 Discussion

7.1 Hierarchy of variables

Let us analyze carefully the set of equations (29). It contains a minimal set of variables required to describe the discussed phenomenon - scissors mode. The information of the first equation is more or less trivial: the tensor \( \tilde{L}_{2\mu} \) is just the time derivative of the quadrupole moment \( \tilde{R}_{2\mu} \). Thus, one can say that equations (29) describe the coupled dynamics of the angular momentum \( \tilde{L}_{11}(t) \), the quadrupole moment \( \tilde{R}_{21}(t) \) and the quadrupole kinetic energy tensor \( \tilde{P}_{21}(t) \). And, what is of principal importance, the angular momentum does not play the key role in this ensemble. It is possible to neglect this variable without any serious consequence for the rest of equations, which will in such a case describe the isovector GQR (see formulae (34, 35)).

The variables \( \tilde{R}_{21}(t) \) and \( \tilde{P}_{21}(t) \) are of considerably more fundamental character. It is obvious that one cannot neglect the quadrupole moment \( \tilde{R}_{21}(t) \) because it is the basis of the ensemble and the whole problem loses any physical meaning without \( \tilde{R}_{21}(t) \). The kinetic energy tensor \( \tilde{P}_{21}(t) \) is responsible for the Fermi surface deformation and must be taken into account to correctly describe the elastic properties of nuclei [8] and, as a result, to get the correct value of the GQR energy. Thus, one arrives at the conclusion: it is impossible to construct a reasonable model of the scissors mode with only one pair of variables, the angular momentum \( \tilde{L}_{11}(t) \) and its canonically conjugate variable. The scissors mode can not exist independently, on its own, without being coupled to the
IVGQR (see below). This conclusion is in absolute contradiction with the original idea of N. Lo Indice and F. Palumbo [2] and especially with their two rotor model (TRM) underlying this idea.

7.2 Rotation due to vibration

Rather soon it was understood [21, 28] that the rotational motion must be accompanied by the isovector quadrupole vibration (shear mode), the second kind of motion being a small admixture to the first one. However, this statement, being true in essence, can be misleading to capture the phenomenon. Indeed, one can easily come to the conclusion that the rotational motion is the principal constituent of the phenomenon, and the vibrational motion is accessory and can be neglected to simplify the description of the problem, if one is interested in qualitative results only. It is easy to see, however, that the real situation is exactly the inverse! Our analysis of the set of equations (29) has shown that the rotational motion (the variable $\bar{L}_{11}(t)$) exists only due to the vibrational one (the variables $\bar{R}_{21}(t), \bar{P}_{21}(t)$). If one wants to observe the scissors mode, one has to excite the IVGQR simultaneously. The IVGQR can exist without the scissors mode, but the scissors mode cannot exist without the IVGQR! Neglecting the coupling to the quadrupole deformation in the last of equations (29) would induce a full free counter rotational motion of neutrons versus protons!

The only characteristic, in which the rotational motion exceeds the vibrational one, is the value of the $B$(M1)-factor (see, however, the section 8). The ratio $\frac{B(M1)_{iv}}{B(M1)_{sc}} \approx \frac{3\sqrt{3}}{4}\delta$ is of the same order of magnitude as the coefficient serving to measure the contribution of the vibrational motion to the scissors mode: $\eta = \frac{\omega_y - \omega_z}{\omega_y + \omega_z} \approx \frac{\delta}{2}$ (see papers [28, 9]) and $\alpha = \delta/(1 + \frac{1}{2}\Omega_{is}^2/\Omega_D^2)$ in paper [21] ($\Omega_D$ is a frequency of a giant dipole resonance).

7.3 Fermi surface deformation

As a matter of fact, the most important ingredient to the scissors mode is the Fermi surface deformation. It can be understood by analyzing formulae for the eigenfrequencies of all three modes. Neglecting the variable $\mathcal{P}_{21}$ in (24) we find the following expression
for the frequency of ISGQR:

\[ \Omega_{is}^2 = \frac{2}{m} [m \omega^2 + \frac{2}{\sqrt{3}} \chi_0 (R_{20}^0 / \sqrt{2} - R_{00}^0)] = 2[\omega^2 + 4 \frac{\kappa_0}{m} Q_{00}(1 + \frac{2}{3} \delta)]. \]

For self-consistent value of the strength constant \( \kappa_0 = \frac{m \omega^2}{4Q_{00}} \) one obtains \( \Omega_{is}^2 = 0 \). This result is quite natural, because the pure geometric distortion corresponding to \( R_{21} \) can be produced by the proper rotation of the nucleus, without any disturbance of its internal structure. Neglecting the variable \( \mathcal{P}_{21}(t) \) in (29) we find that the frequency of IVGQR (being determined mainly by the neutron-proton interaction) is changed not so drastically:

\[ \Omega_{iv}^2 = 2\bar{\omega}^2 (1 - \alpha)(1 + \delta/3). \]

Comparing this formula (for \( \alpha = -2 \)) with (37) one sees, that \( \Omega_{iv}^2 \approx 8\omega_0^2 \) changes to \( \Omega_{iv}^2 \approx 6\omega_0^2 \). One should recall that also for the Isovector Giant Dipole Resonance the distortion of Fermi sphere plays only a minor role.

It is also easy to see that omitting \( \mathcal{P}_{21}(t) \) in (29), one obtains zero energy for the scissors mode independent of the strength of the residual interaction.

Thus, the nuclear elasticity discovered by G.F. Bertsch [8] is the single origin for the restoring force of the scissors mode and also for the ISGQR in our simple Hamiltonian of a harmonic oscillator with Q-Q residual interaction. So one can conclude that this mode is in its essence a pure quantum mechanical phenomenon. This agrees with the conclusion of the papers [22, 23]: classically (i.e., without Fermi surface deformation) the scissors mode is a zero energy mode.

### 7.4 Deformed oscillator and isovector Q-Q interaction

It is known that the deformed harmonic oscillator Hamiltonian can be obtained in a Hartree approximation "by making the assumption that the isoscalar part of the Q-Q force builds the one-body container well" [9]. Thus, neglecting the isovector part of the Q-Q residual interaction, i.e. assuming \( \kappa_1 = 0 \rightarrow \kappa = \bar{\kappa} = \kappa_0 \), we have to reproduce the known results in the deformed harmonic oscillator model. The formulas of other authors are obtained, as a rule, in the small \( \delta \) limit. For the sake of convenient comparison just the same approximation is used here. We have for the energies

\[ \Omega_{iv} \simeq 2\omega_0, \quad \Omega_{sc} \simeq \omega_0 \delta, \]
which coincides with the results of [9] and [29]. The formulae for magnetic transition probabilities are

\[ B(M1)_{sc} \uparrow \approx \frac{1}{8\pi} \frac{m\omega_0}{\hbar} Q_{00}^0 \delta \mu_N^2, \quad B(M1)_{iv} \uparrow \approx \frac{1}{16\pi} \frac{m\omega_0}{\hbar} Q_{00}^0 \delta^2 \mu_N^2, \quad \frac{B(M1)_{iv}}{B(M1)_{sc}} = \frac{\delta}{2}, \]

which coincide with the results of [5]. Possibly they coincide also with that of [29] (as a matter of fact, their values are twice larger, but we suppose it is a misprint). For electric transition probabilities we find

\[ B(E2)_{sc} \uparrow \approx \frac{5}{64\pi} \frac{e^2}{m\omega_0} Q_{00}^0 \delta, \quad B(E2)_{iv} \uparrow \approx \frac{5}{32\pi} \frac{e^2}{m\omega_0} Q_{00}^0, \quad \frac{B(E2)_{sc}}{B(E2)_{iv}} = \frac{\delta}{2}, \]

in perfect agreement with [5]. The ratios of different characteristics, calculated with and without an isovector Q-Q interaction, are

\[ \frac{\Omega_{sc}}{\Omega_{sc}(\kappa_1 = 0)} = \sqrt{\frac{3}{2}}, \quad \frac{\Omega_{iv}}{\Omega_{iv}(\kappa_1 = 0)} = \sqrt{2}, \]

\[ \frac{B(M1)_{sc}(\kappa_1 = 0)}{B(M1)_{sc} (\kappa_1 = 0)} = \sqrt{\frac{3}{2}}, \quad \frac{B(M1)_{iv}(\kappa_1 = 0)}{B(M1)_{iv} (\kappa_1 = 0)} = \frac{9}{2\sqrt{2}}, \]

\[ \frac{B(E2)_{sc}(\kappa_1 = 0)}{B(E2)_{sc} (\kappa_1 = 0)} = \frac{1}{2\sqrt{6}}, \quad \frac{B(E2)_{iv}(\kappa_1 = 0)}{B(E2)_{iv} (\kappa_1 = 0)} = \frac{1}{\sqrt{2}}. \]

As we can see, the inclusion of an isovector Q-Q interaction increases the energies and B(M1)-factors of the scissors mode and IVGQR and decreases their B(E2)-factors. It is also necessary to emphasize the following important result: the ratio

\[ RM \equiv \frac{B(M1)_{iv}(\kappa_1 = 0)}{B(M1)_{sc}(\kappa_1 = 0)} \]

coincides exactly with the "admixture" coefficient \( \eta \) introduced by Hilton [28], what supports our idea that \( RM \) can serve as a measure for an "admixture". It is seen that taking into account the long range correlations (isovector Q-Q forces), one increases \( RM \) by the factor \( 3\sqrt{3}/2 \). And finally, the ratio \( \Omega_{sc}/\Omega_{sc}(\kappa_1 = 0) \) is quite close to the number \( \sqrt{(1 + 0.66)} \) found in [9].

### 7.5 Sum rule

Our discussion of the magnetic sum rules will be based on table 4 of the review of Zawischa [5] where the results of different models and approaches for \(^{164}\text{Dy}\) are listed. For the sake of a convenient comparison it is reproduced here (together with its legend) as table 1.
Table 1. The energy-weighted orbital M1 sum rule $\sum E_x B(M1_{orb}) \uparrow$ (in units of $\mu^2_{N}$MeV) in different models evaluated for $^{164}$Dy as an example, compared with the values obtained from the expressions given by Lipparini and Stringari [21]. The total sum rule strength of the model and the part exhausted by the low-energy mode are given. The schematic RPA results are from Bes and Broglia [20], for the RPA with Migdal force entry the data of Zawischa and Speth [23] are used. In lines 4 and 5 the deformation parameter $\delta_{osc} = 0.258$ has been taken.

| Model                                      | Total  | Low energy |
|--------------------------------------------|--------|------------|
| TRRM                                       | 140.8  |            |
| Classical fluids                          | 140.8  | 0          |
| NFD model                                  | 141.6  | 37.8       |
| Deformed harmonic oscillator               | 31.0   | 15.4       |
| Schematic RPA                             | $>$70  | 24.4       |
| RPA with Migdal force                      | 108.2  | 40.6       |
| IBM-2 (with $E_x = 3$ MeV)                 | 12.2   | 12.2       |
| Lipparini and Stringari [21]               | 145.4  | 35.9       |

To complete the legend of this table, it should be said that the results of the lines 1, 2, 3, 4 and 7 are calculated by Zawischa [5], who used $\hbar \omega_0 = 46.5/A^{1/3}$MeV (what corresponds to $r_0 = 1.13$ fm) and $\delta = 0.302$ (except the fourth line). Taking the same values of parameters one obtains $\Sigma_0 = 42.4 \mu^2_{N}$MeV. In the case of $\alpha = -2$ one finds from (73)

$$\Sigma_{tot} = 3\Sigma_0 = 127.3 \mu^2_{N}\text{MeV}.$$ 

It is seen from Table 1 that this number does not contradict the "Schematic RPA" and is in qualitative agreement with the lines "TRRM" (Two Rigid Rotors Model), "Classical fluids", "NFD model" (Nuclear Fluid Dynamics), "Lipparini and Stringari [21]" and "RPA with Migdal force", being exactly in between the last two results. We have to note that the result of the small $\delta$ approximation ($\bar{\omega} \to \omega_0$, $Q_{00} \to Q_{00}^0$)

$$\Sigma_{tot} = 3\Sigma_0 = 142.6 \mu^2_{N}\text{MeV}$$

is in excellent agreement with "NFD model" and "Lipparini and Stringari [21]" lines whose values were obtained in the small approximation also. Such agreement in the last case is
especially surprising if one takes into account that the $M1$ sum rule is model dependent, being determined by the neutron-proton interaction, which is quite different in the two papers: quadrupole-quadrupole residual interaction in our case and that of the Skyrme type in [21]. This fact confirms that the model Hamiltonian used in this work is realistic enough. The exact contribution of the scissors mode to the sum rule (formula (74)) is

$$\Sigma_{sc} = 30.9 \mu^2_N \text{MeV}.$$ 

The result of the small $\delta$ approximation (formula (77) with $\hbar \omega_0 = 46.5/A^{1/3} \text{MeV}$)

$$\Sigma_{sc} = 35.6 \mu^2_N \text{MeV}$$

is rather close to the exact number, being in good agreement with "NFD model" and "Lipparini and Stringari [21]" lines. It is worth noting that the ratio $\Sigma_{sc}/\Sigma_{iv} \simeq 1/3$ is not so far from the value $\Sigma_{sc}/\Sigma_{high} \simeq 1/4$ predicted in [30] on the basis of some theoretical analysis of experimental data (their notation "high" means high energy scissors mode).

In the case of $\alpha = 0$, which corresponds to the deformed harmonic oscillator, one has (for $\delta = 0.258$, as in [5])

$$\Sigma_{tot} = 31.2 \mu^2_N \text{MeV}, \quad \Sigma_{sc} = \Sigma_{iv} = 15.6 \mu^2_N \text{MeV}$$

reproducing the numbers of the line "Deformed harmonic oscillator" in table 1.

Concluding this subsection we have to say that most of the general observations found earlier [5] (such as, for example, $\Sigma_{sc}/\Sigma_{iv} \simeq 1/3$, $\Sigma_{sc} \sim \delta^2$) are confirmed by our investigation.

7.6 Discussion of other approaches to the scissors mode

After having, as we think, clearly worked out the physics of the low lying scissors mode and its interweaving with the IVGQR, one may ask the question about the status of other approaches. As already mentioned all other RPA-type approaches [13, 19, 20, 27] have from the numerical point of view the same status as our approach (but this concerns mostly only the limit of small deformations). We also have pointed out that the original model of counter rotating rigid rotors [2, 6] does, in our opinion, not at all grasp the salient features of the scissors mode. However, the description of the scissors mode also
has been attempted with other quite different approaches like IBM (or IBA), shell model
calculations, etc. We think that the conclusions concerning IBM (IBA) are very well
formulated in the review by Zawischa [5]. Therefore, to demonstrate our point of view we
will simply comment several citations from [5].

i) "The original aim of the IBA was the microscopic explanation of vibrations, rotations
and transitions between the two bosons. Rotational invariance is maintained throughout.
Starting from a microscopic Hamiltonian, the configuration space is drastically truncated
by dealing only with nucleons (or holes) in the valence shell, assuming an inert core, and
approximating correlated pairs of valence nucleons by bosons [31]. (Mostly monopole and
quadrupole bosons are considered, for special purposes bosons with angular momentum
different from 0 or 2 are needed too.)...

The bosons are treated as elementary units, but the internal structure of the fermion
pairs they represent, reflects itself in the parameters of the model Hamiltonian which
are fitted to the low energy spectra and transition probabilities [32]. By construction,
the model is only suitable to describe low-energy states. To describe the giant dipole
resonance, $p$ and $f$ bosons have been included ([33] and references therein.) The gi-
ant quadrupole resonances are outside of its scope, as they involve $\Delta N_{osc} = 2$ excita-
tions ($N_{osc}$ is the oscillator quantum number). We have seen that (in the semiclassical
model, in the deformed harmonic oscillator model and in microscopic RPA) a considerable
amount of strength is in a high-energy mode—the $|K| = 1$ component of the isovector giant
quadrupole resonance. Due to truncation of the configuration space, this strength is miss-
ing in the IBM-2 sum rule. We have also seen that (in microscopic RPA) a considerable
part of the M1 strength resides in the region between 4 and 10 MeV, in two-quasiparticle
type excitations—all these are not included in the model space of the IBA.... Thus, the
IBA sum rule by the basic assumptions of the model comprises only a small part of the
full sum rule. Therefore, it is not so well suited to assess the collectivity of a state.”

ii) "As already mentioned, among the predictions, the IBA was closest to the strengths
to be detected. The reason is, of course, that the model parameters are extracted from
low-energy data and the model is best adapted just to the energy range up to a few MeV
where the M1 states have been found... The naive assumption of bare orbital g-factors
already gives a good prediction of the low-energy strength. Adjustment of the model
parameters to the low-energy vibrational and rotational states [32] finally has the effect that the average energy is also well reproduced."

iii) "In order to find out the physical interpretation of the lowest $K = 1$ mixed symmetry state of the IBM-2, the (semi)classical limit of the IBM-2 Hamiltonian has been investigated by Balantekin and Barrett [34], Bijker [35] and Walet [36], yielding a Hamiltonian similar to that of TRRM, the potential being a function of the angle $\theta$ between the symmetry axes of protons and neutrons: $V(\theta) = \lambda \theta^2$."

With respect to iii) we must note, that deriving the classical limit, the vibrational degrees of freedom have been neglected to simplify the derivation. Not surprisingly they obtained the TRRM Hamiltonian as a result for which we have already given our opinion above.

Generally speaking these above citations give exaustive characteristic of the status of IBM (IBA) calculations of the scissors mode: they are able to give correct values for the energy and strength of the scissors mode but they do not explain the real physics of the phenomenon.

The situation with shell model calculations is rather complicated. There are the well known difficulties to treat heavy nuclei because of the huge dimension of matrices. Therefore the calculations are usually made for very light nuclei. Even there one must divide them into two groupes. There are qualitative estimations with truncated basis ($\Delta N_{osc} = 0$, see for example [37, 38]). Naturally, those calculations suffer from the same drawbacks as the IBM calculations and then the same comment given above apply. There are also ‘realistic’ calculations (for $^8\text{Be}$ and $^{10}\text{Be}$) with an extended ($\Delta N_{osc} = 0$)\textbf{+}($\Delta N_{osc} = 2$) basis [39]. In principle they have the same status as RPA calculations. Still one can ask the question whether it makes sense to talk about scissors mode in such light nuclei.

We also would like to comment on the neutron-proton deformation (NPD) model. We again cite [5]: "The Bohr-Mottelson model has been generalized to isovector degrees of freedom leading to the neutron-proton deformation (NPD) model... Rohozinski and Greiner [40] applied the NPD model to the orbital magnetic dipole excitations.... The TRRM also has been changed by its authors to the TRM, relaxing the condition of rigid rotation and replacing the rigid-body moment of inertia by a smaller one obtained from some model or a phenomenological one [41]." If their parameters "...are
adjusted to the data, the TRM and NPD model will coincide, even though in their original assumptions—rigid rotation in the one, irrotational flow in the other—the models are contradictory.”

Therefore we agree with the conclusion of Zawischa [5] that the NPD model can not describe the scissors mode. The reasons are also especially well born out in terms of our method: the collective variables in their model are only $\mathcal{R}_{2\mu}$ and $\mathcal{L}_{2\mu}$, there are no rotational degrees of freedom ($\mathcal{L}_{1\nu}$) and there is no Fermi surface deformation ($\mathcal{P}_{2\mu}$). As we know from an earlier discussion, only the isovector giant resonance survives in such conditions (see section 7.3). Therefore, the $\delta^2$-dependence of $B(M1)$-factor which the NPD model predicts and which is in principle in agreement with experiment, must actually be interpreted as the $\delta^2$-dependence of $B(M1)$ for the IVGQR (see eq. (58)). The $\delta$-dependence of the low lying scissors becomes quadratic only after inclusion of pairing [5].

In conclusion we think that above set of citations and argumentations is convincing enough to state that all the models and methodes describing the scissors mode without coupling to IVGQR are pure phenomenological and are therefore of restricted usefulness.

8 Superdeformation

As already mentioned, a certain drawback of our approach is that, so far, we have not included superfluidity into our description. Nevertheless, our formulas (36, 56, 57) can be successfully used for the description of the superdeformed nuclei, where the pairing is very weak [6]. For example, applying them to the superdeformed nucleus $^{152}$Dy ($\delta \simeq 0.6, \hbar \omega_0 = 41/A^{1/3}$MeV), we get

$$E_{iv} = 23.6 \text{ MeV}, \quad B(M1)_{iv} = 15.9 \mu_N^2$$

for the isovector GQR and

$$E_{sc} = 5.4 \text{ MeV}, \quad B(M1)_{sc} = 20.0 \mu_N^2$$

for the scissors mode. There are not so many results of other calculations to compare with. As a matter of fact, there are only two papers considering this problem.
The phenomenological TRM model predicts [6]:

\[ E_{iv} \simeq 26 \text{ MeV}, \quad B(M1)_{iv} \simeq 26 \mu^2_N, \quad E_{sc} \simeq 6.1 \text{ MeV}, \quad B(M1)_{sc} \simeq 22 \mu^2_N. \]

The only existing microscopic calculation [13] in the frame of QRPA with separable forces gives slightly more information:

\[ E_{iv} \simeq 28 \text{ MeV}, \quad B(M1)_{iv} \simeq 37 \mu^2_N, \]

\[ E_{sc} \simeq 5 - 6 \text{ MeV}, \quad B(M1)_{1^+} \simeq 23 \mu^2_N, \quad B(M1)_{sc} \simeq 0.4 \mu^2_N. \]

Here \( B(M1)_{1^+} \) denotes the total \( M1 \) orbital strength carried by the calculated \( K^\pi = 1^+ \) QRPA excitations modes in the energy region below 20 MeV. The \( B(M1)_{sc} \) denotes “the calculated overlap probabilities of the QRPA solutions with the synthetic orbital scissors mode which is defined as

\[ |R> = N^{-1}(\mathcal{L}_{1-1} - q\mathcal{L}_{1-1})|\text{g.s.}>, \]

where \( N \) is a normalisation factor. The parameter \( q \) is determined by the requirement that the mode \( |R> \) is orthogonal to the spurious state \( |S> \sim \mathcal{L}_{1-1}|\text{g.s.}> \)” [13].

It is easy to see that in the case of IVGQR one can speak, at least, about qualitative agreement. Our results for \( E_{sc} \) and \( B(M1)_{sc} \) are in good agreement with that of phenomenological model and with \( E_{sc} \) and \( B(M1)_{1^+} \) of Hamamoto and Nazarewicz. The very small value of \( B(M1)_{\text{syn}} \) is explained quite naturally by the fact, that the synthetic mode does not treat properly two main ingredients of the scissors mode: Fermi surface deformation and coupling with IVGQR.

Examples of \( \delta \)-dependences of energies and \( B(M1) \)-factors are shown in Fig.3.

9 Conclusion

In this work we again have considered the issue of the physics behind the nuclear scissors mode. In spite of 25 years of research and many valuable contributions to this subject the subtleties of the scissors mode are still under debate, and in our opinion erroneous interpretations of the subject continue to appear in the literature. Surprisingly, no systematic study of the mode in the Bohr-Mottelson Hamiltonian has yet been carried out,
Figure 3: Dependences on deformation of energies and $B(M1)$-values of scissors mode, IVGQR and ISGQR.
and our purpose here was to fill that gap. The Bohr-Mottelson Hamiltonian consist of a harmonic single-particle potential together with a separable Q-Q interaction. The Q-Q forces have different couplings in the isoscalar and isovector channels. The isoscalar coupling strength is determined from Bohr and Mottelson self-consistency condition, leaving the isovector strength as a free parameter. We adjust it from the fact that the isovector giant quadrupole resonance is experimentally known to lie practically at twice the energy of the isoscalar giant quadrupole mode. With this our model is entirely fixed and its solution in the small amplitude limit can be found analytically for excitation energies and transition amplitudes.

The physics becomes particularly transparent once the TDHF equations are written down in phase space and the so called Wigner function moments are introduced. This approach allows one to establish the optimum set of macroscopic variables: quadrupole moment, angular momentum, pressure tensor, etc. These variables are, in the scheme of our formalism, absolutely unambiguous and, together with the analytic solution, they allow for a maximum of physical insight. At least, the inevitable coupling of the scissors mode with the isovector giant quadrupole resonance becomes obvious immediately, already at the stage of the formulation of the model. Furthermore, the Fermi surface deformation, whose decisive role in the physics of the scissors mode is difficult to predict employing naive phenomenological models, appears in our approach quite naturally.

The eigenvalue equation in the isovector channel yields two frequencies which are given by \( \Omega_{\pm} = 2\bar{\omega}\sqrt{(1 + \delta/3) \pm \sqrt{(1 + \delta/3)^2 - \frac{3}{4}\delta^2}}. \) They are distributed in a non-symmetric way around twice the harmonic oscillator frequency \( \bar{\omega} \) and they gradually approach one another with increasing \( \delta \). The low lying frequency corresponds to the one of the scissors mode proper, whereas the other is the so-called high lying scissors mode. Our analysis shows that, indeed, the motion of both modes is "scissors"-like in the sense that the long symmetry axes of proton and neutron distributions get tilted by a small angle during their oscillatory motion. Nevertheless both modes are quite distinct what is revealed by looking at the respective flow patterns (Figs. 1,2). The flow lines of the scissors mode are closed ellipses (i.e. mostly rotational flow) leading to a compression of the long axis, while the ones of the high lying mode are open hyperbolas (i.e. mostly irrotational flow) leading to an elongation of the long axis. The frequency of the scissors mode as a function of mass
number turns out to be about 15% too low, compared with the experimental data. In respect to the somewhat crude model we have been employing this may seem a reasonable agreement. One should, however, mention that we completely disregarded pairing in our work. It is generally known that superfluidity makes the mass parameters smaller, i.e. the frequencies of the modes become higher. It will be a further task to study whether this accounts for the missing 15% in energy.

Other quantities we studied in our model are transition probabilities, for instance with respect to their deformation dependence. Though for small deformation most of our results have already been found by other authors [5] with different methods, we make a point here in predicting the behaviour for superdeformed nuclei.

We also want to attract the attention to the potential richness of the set of our equations (15). In a further study one may employ them for the description of the joint dynamics of the isoscalar and isovector giant monopole and quadrupole resonances plus the scissors mode in deformed rotating nuclei, the amplitudes of vibrations being not necessarily small. A large amplitude motion was already treated in the frame of this approach to describe the multiphonon giant quadrupole and monopole resonances [12]. What about two-phonon scissors? The question is not only academic - the first attempt to interpret some numerical results as the multiphonon scissors is already known [42].

One may also think to take into account the spin degrees of freedom - only the number of dynamic equations must be doubled (spin projections up and down). Then, the theory becomes capable of describing spin-flip excitations. As a result, there appears a possibility of considering the orbital and spin components of the scissors mode simultaneously.

It is worth noticing that the set of equations (15) is written in the laboratory coordinate system. It allows one to get rid of any troubles connected with spurious rotation, because the total (i.e., isoscalar) angular momentum is conserved (the last equation of (19)). Moreover, the total angular momentum enters into the dynamic equations - hence one can study the behaviour of all modes in rotating nuclei. This would be especially interesting for the scissors mode in superdeformed (SD) nuclei, because "The SD bands in nuclei around ${}^{152}\text{Dy}$ and ${}^{192}\text{Hg}$ are observed at high spins" [13]. The first interesting results of calculation interpreted as the rotational band built on the scissors excitation appeared in [42]. The above mentioned problems shall be investigated in future work.
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Appendix

It is known that the deformed harmonic oscillator Hamiltonian can be obtained in a Hartree approximation "by making the assumption that the isoscalar part of the Q-Q force builds the one-body container well" [9]. In our case it is obtained quite easy by summing formulas (10) and (11):

\[
V(r, t) = \frac{1}{2} (V^p(r, t) + V^n(r, t)) = \frac{1}{2} m \omega^2 r^2 + \kappa_0 \sum_{\mu=-2}^{2} (-1)^\mu Q_{2\mu}(t) q_{2-\mu}(r). \tag{85}
\]

In the state of equilibrium (i.e. in the absence of external field) \(Q_{2\pm1} = Q_{2\pm2} = 0\). Using the definition [18] \(Q_{20} = Q_{00} 4^2 \delta\) and the formula \(q_{20} = 2z^2 - x^2 - y^2\) we obtain the potential of the anisotropic harmonic oscillator

\[
V(r) = \frac{m}{2} [\omega_x^2 (x^2 + y^2) + \omega_z^2 z^2]
\]

with oscillator frequencies

\[
\omega_x^2 = \omega_y^2 = \omega^2 (1 + \sigma \delta), \quad \omega_z^2 = \omega^2 (1 - 2\sigma \delta),
\]

where \(\sigma = -\kappa_0 \frac{8Q_{00}}{3m\omega^2}\). The definition of deformation parameter \(\delta\) must be reproduced by the harmonic oscillator wave functions, that allows one to fix the value of \(\sigma\). We have:

\[
Q_{00} = \frac{\hbar}{m} \left( \sum_x \frac{\omega_x}{\omega_x} + \sum_y \frac{\omega_y}{\omega_y} + \sum_z \frac{\omega_z}{\omega_z} \right), \quad Q_{20} = 2 \frac{\hbar}{m} \left( \sum_z \frac{\omega_z}{\omega_z} - \sum_x \frac{\omega_x}{\omega_x} \right),
\]

where \(\Sigma_x = \Sigma_{i=1}^A (n_x + \frac{1}{2})_i\) and \(n_x\) is the oscillator quantum number. Using the self consistency condition

\[
\Sigma_x \omega_x = \Sigma_y \omega_y = \Sigma_z \omega_z = \Sigma_0 \omega_0,
\]

where \(\Sigma_0\) and \(\omega_0\) are defined for spherical nucleus, we get

\[
\frac{Q_{20}}{Q_{00}} = 2 \frac{\omega_x^2 - \omega_z^2}{\omega_x^2 + 2\omega_z^2} = \frac{2\sigma \delta}{1 - \sigma \delta} = \frac{4}{3} \delta.
\]
Solving last relation with respect of $\sigma$ we find

$$\sigma = \frac{2}{3 + 2\delta}. \quad (86)$$

Now we can write final expressions for oscillator frequences

$$\omega^2_x = \omega^2_y = \omega^2 1 + \frac{4}{3} \delta, \quad \omega^2_z = \omega^2 1 - \frac{2}{3} \delta$$

and the self consistent value of the strength constant

$$\kappa_0 = -\frac{m\omega^2}{4Q_{00}} \frac{1}{1 + \frac{2}{3} \delta}.$$ 

The condition for volume conservation $\omega_x\omega_y\omega_z = \text{const} = \omega_0^3$ makes $\omega$ $\delta$-dependent:

$$\omega^2 = \omega_0^2 \frac{1 + \frac{2}{3} \delta}{(1 + \frac{2}{3} \delta)^{2/3}(1 - \frac{2}{3} \delta)^{1/3}}.$$ 

$Q_{00}$ depends on $\delta$ as

$$Q_{00} = Q^0_{00} \frac{1}{(1 + \frac{2}{3} \delta)^{1/3}(1 - \frac{2}{3} \delta)^{2/3}},$$

where $Q^0_{00} = A^2 \delta^2 R^2$, $R = r_0 A^{1/3}$.

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