Abstract

Uniform $L^1$ and lower bounds are obtained for the Green’s function on compact Kähler manifolds. Unlike in the classic theorem of Cheng-Li for Riemannian manifolds, the lower bounds do not depend directly on the Ricci curvature, but only on integral bounds for the volume form and certain of its derivatives. In particular, a uniform lower bound for the Green’s function on Kähler manifolds is obtained which depends only on a lower bound for the scalar curvature and on an $L^q$ norm for the volume form for some $q > 1$. The proof relies on auxiliary Monge-Ampère equations, and is fundamentally non-linear. The lower bounds for the Green’s function imply in turn $C^1$ and $C^2$ estimates for complex Monge-Ampère equations with a sharper dependence on the function on the right hand side.

1 Introduction

A fundamental theorem in Riemannian geometry is the inequality of Cheng-Li [3], which provides conditions for a uniform lower bound for the Green’s function. More precisely, let $(X, g)$ be a compact Riemannian manifold, and define the Green’s function $G(x, y)$ as the unique function $X \times X \rightarrow \mathbb{R} \cup \{\infty\}$ satisfying the conditions

$$\Delta g, x G(x, y) = -\delta_x(y) + \frac{1}{\text{Vol}_g(X)}, \quad \int_X G(x, y) dV_g(y) = 0,$$

(1.1)

where $\delta_x(y)$ denotes the Dirac function at the point $x$. It is well-known that $G(x, y) = G(y, x)$ and $\Delta g, x G(x, y) = \Delta g, y G(x, y)$ for any $x \neq y$. Assume that the Ricci curvature of $(X, g)$ satisfies $\text{Ric}(g) \geq -\kappa$ for some constant $\kappa$. Then Cheng-Li [3] prove that there is a constant $C > 0$ depending only on the dimension of $X$ and $\kappa$ such that

$$G(x, y) \geq -C \frac{\text{diam}_g(X)^2}{\text{Vol}_g(X)}, \quad \forall x, y \in X.$$

(1.2)

Here $\text{diam}_g(X)$ is the diameter of $(X, g)$ and $\text{Vol}_g(X)$ is its volume. The dependence of this inequality on a lower bound $\kappa$ for the Ricci curvature is crucial, and it does not seem possible in Riemannian geometry to lessen this dependence in any significant way.

A first primary goal of the present paper is to show that, in the Kähler setting, lower bounds for the Green’s function can actually be established without assumptions on lower bounds for the Ricci curvature. Rather, we assume integral bounds on the volume form

\footnote{Work supported in part by the National Science Foundation under grant DMS-1855947.}
and some specific derivatives. As we shall explain below, for our purposes, integral bounds are far superior to pointwise bounds. It turns out that several distinct sets of assumptions can guarantee lower bounds for the Green’s function, and we shall describe them in detail later in §2 (Theorem 2). While these sets of assumptions may be difficult to assess at a glance, an easy comparison with the theorem of Cheng-Li can be obtained by observing that Theorem 2 implies in particular that the Green’s function is bounded from below if the scalar curvature is bounded from below, and the $L^q$ norm of the volume form is bounded for some $q > 1$ (see Corollary 2). It may also be worth stressing that, while our goal of establishing lower bounds for the Green’s function is the same as Cheng-Li’s, our method of proof is completely different. It builds repeatedly on the key idea in [9] of comparison with an auxiliary Monge-Ampère equation, and even though the Green’s function is the solution of a linear partial differential equation, our method is fundamentally non-linear.

We come now to the second primary goal of this paper, which is also a major motivation for the first, and which is sharp a priori estimates for general complex Monge-Ampère equations. The first estimates were obtained in 1976 by S.T. Yau in his seminal paper [25] solving the Calabi conjecture. However, a new generation of problems in complex geometry is leading to more complicated Monge-Ampère equations, which can be degenerate or singular in many different senses. Thus ever sharper versions of a priori estimates are needed, as each improvement usually has significant geometric consequences.

The first sharp form of $C^0$ estimates for the complex Monge-Ampère equation was obtained by Kolodziej [16], using pluripotential theory. Kolodziej’s estimates were extended to the important case of degenerating background Kähler metrics by Eyssidieux, Guedj, and Zeriahi [5] and Demailly and Pali [4]. More general $C^0$ estimates using the theory of envelopes can be found in [7, 8]. Another approach to $C^0$ estimates, using PDE methods, was introduced very recently in [9]. This method can also apply to nef classes [12], and lead to many sharp estimates, including stability estimates [11], diameter estimates [13], and non-collapse estimates [14]. We have just seen it applied to lower bounds for Green’s functions in the first part of this paper. On the other hand, while $C^1$ and $C^2$ estimates were extended to the case allowing a divisor, building on ideas of Tsuji [23], Blocki [1], and [20, 19], they still require very restrictive conditions on the right hand side of the equation, such as pointwise lower or upper bounds, and in the case of $C^2$ estimates, also a bound on its Laplacian. A lower or upper bound assumption is a particularly severe constraint, as it may rule out equations which are degenerate or have singularities.

Thus our second primary goal in this paper is to establish sharp $C^1$ and $C^2$ estimates for the complex Monge-Ampère equation which depend essentially only on integral bounds for the right hand side. In order to do so, we cannot apply the standard maximum principle to the elliptic differential inequalities satisfied by the derivatives of the solution of the Monge-Ampère equation. Rather, we apply instead the new lower bounds for the Green’s function obtained in the first part of the paper.

We now state precisely our main results. In view of many applications in complex
geometry, it is important to obtain estimates which remain uniform as the Kähler class may degenerate, so we consider the following set-up, which includes both cases of fixed and degenerating Kähler classes as special cases. Let \((X, \omega_X)\) be a compact Kähler manifold with dimension \(n\). Suppose \(\chi\) is a \(d\)-closed \((1, 1)\)-form on \(X\) such that its cohomology class \([\chi] \in H^{1,1}(X, \mathbb{C})\) is nef and big, i.e. \([\chi]\) lies in the closure of the Kähler cone of \(X\) and \(\int_X \chi^n > 0\). So for each \(t > 0\), \([\chi + t\omega_X]\) is a Kähler class. For any fixed \(t \in (0, 1]\) and any Kähler metric \(\omega_t \in [\chi + t\omega_X]\), we define the function \(F_{\omega_t}\) to be the logarithmic of the relative volume form of \(\omega_t\) with respect to the fixed volume form \(\omega_X^n\), that is

\[
F_{\omega_t} = \log \left( \frac{\omega_t^n}{\omega_X^n} \right)
\]

where \(V_t = \int_X \omega_t^n = \int_X (\chi + t\omega_X)^n > 0\) is the volume of the Kähler class \([\chi + t\omega_X]\), and \(V = \int_X \omega_X^n\) is the volume of the fixed metric \(\omega_X\). Note that \(V_0 = \int_X \chi^n > 0\) under our assumptions.

Fix \(p > n\). We denote the \(p\)-th entropy of the Kähler metric \(\omega_t\) by

\[
\text{Ent}_p(\omega_t) = \frac{1}{V_t} \int_X |F_{\omega_t}|^p \omega_t^n = \frac{1}{V} \int_X |F_{\omega_t}|^p e^{F_{\omega_t}} \omega_X^n.
\]

For any \(N > 0\), we define a subset of the space of Kähler metrics in \([\chi + t\omega_X]\) by

\[
\mathcal{M}_t(N, p) = \{ \omega_t \in [\chi + t\omega_X] | \text{Ent}_p(\omega_t) \leq N \}.
\]

Our first main theorem shows that the lower bound of \(G_t\) is equivalent to the \(L^1(X, \omega_t^n)\)-norm of \(G_t\) for \(\omega_t \in \mathcal{M}_t(N, p)\).

**Theorem 1** Given \(p > n\) and \(N > 0\), there is a constant \(C > 0\) depending only on \(n, p, \chi, \omega_X\) and \(N\), such that for any \(t \in (0, 1]\) and any Kähler metric \(\omega_t \in \mathcal{M}_t(N, p)\), the Green’s function \(G_t\) associated to \(\omega_t\) satisfies

\[
\frac{1}{2V_t} \left\| G_t(x, \cdot) \right\|_{L^1(X, \omega_t^n)} \leq -\inf_X G_t(x, \cdot) \leq C(1 + \left\| G_t(x, \cdot) \right\|_{L^1(X, \omega_t^n)}),
\]

for any \(x \in X\).

We remark that \(G_t(x, y)\) satisfies the asymptotic behavior [21]

\[
G_t(x, y) \sim d_{\omega_t}(x, y)^{-2n+2}, \text{ if } n \geq 2
\]

and \(G_t(x, y) \sim -\log d_{\omega_t}(x, y)\) if \(n = 1\), when \(x\) is close to \(y\). Here \(d_{\omega_t}(x, y)\) denotes the geodesic distance of \(x\) and \(y\) under \(\omega_t\). Thus an upper bound for \(G_t(x, y)\) cannot be expected to hold.
For a smaller class of Kähler metrics than $\mathcal{M}_t(N,p)$, we will show that the $L^1$-norms of the Green’s function are uniformly bounded, hence by Theorem 1, we also have a pointwise lower bound on the Green’s functions.

Henceforth we require that $\chi$ is nonnegative and $V_0 = \int_X \chi^n > 0$. Let $F_{\omega_t}$ be associated with a Kähler metric $\omega_t \in [\chi + t\omega_X]$ as in (1.3). For $\epsilon > 0$, $N > 0$ and $\gamma \geq 1$, we denote for each $t \in (0,1]$:

$$\mathcal{M}_t'(N,\epsilon,\gamma) = \{\omega_t \in [\chi + t\omega_X]| \frac{1}{V} \int_X e^{(1+\epsilon)F_{\omega_t}}\omega_X^n \leq N \text{ and } \sup_X e^{-F_{\omega_t}} \leq \gamma\}.$$  \hfill (1.5)

A second class of metrics for $t \in (0,1]$ is given by:

$$\mathcal{M}_t''(N,\epsilon,\gamma) = \{\omega_t \in [\chi + t\omega_X]| \frac{1}{V} \int_X e^{(1+\epsilon)F_{\omega_t}}\omega_X^n \leq N \text{ and } \int_X (e^{-F_{\omega_t}} + |\Delta \omega_X e^{-F_{\omega_t}}|)\omega_X^n \leq \gamma\}.$$  \hfill (1.6)

The case of Kähler metrics in a fixed Kähler class $\omega \in [\omega_X]$ can be considered as a special case of the above more general set-up by taking $\chi = \omega_X/2$ and $t = 1/2$. We can also consider the class:

$$\tilde{\mathcal{M}}''(N,\epsilon,\gamma) = \{\omega \in [\omega_X]| \frac{1}{V} \int_X e^{(1+\epsilon)\omega_X} \omega_X^n \leq N \text{ and } \int_X (e^{-\omega_X} + |\nabla \omega_X e^{-\omega_X}|^2)\omega_X^n \leq \gamma\}.$$  \hfill (1.7)

Abusing notations, when we write $\omega_t \in \tilde{\mathcal{M}}''(N,\epsilon,\gamma)$, we mean that $t = 1/2$ and $\chi = \omega_X/2$, which corresponds to the case of complex Monge-Ampère equations with the fixed background metric $\omega_X$.

It follows easily from calculus inequalities that the sets of metrics in (1.5), (1.6), (1.7) are contained in $\mathcal{M}_t(\tilde{N},p)$ for suitable $\tilde{N} > N$ and $p > n$. Hence Theorem 1 applies to the metrics in these sets. Our second theorem is:

**Theorem 2** Given any $\epsilon > 0$, $N > 0$ and $\gamma \in (0,1)$, for each $t \in (0,1]$ and each Kähler metric $\omega_t \in \mathcal{M}_t'(N,\epsilon,\gamma) \cup \mathcal{M}_t''(N,\epsilon,\gamma) \cup \tilde{\mathcal{M}}''(N,\epsilon,\gamma)$, let $G_t$ be the Green’s function associated with $(X,\omega_t)$. Then the following hold:

(i) There is a constant $C > 0$ which depends on $n, \epsilon, \omega_X, \chi$ and $N, \gamma$ such that for any $x \in X$:

$$\|G_t(x,\cdot)\|_{L^1(X,\omega_t^n)} = \int_X |G_t(x,\cdot)|\omega_t^n \leq C,$$

and

$$\inf_{y \in X} G_t(x,y) \geq -C.$$  \hfill (1.8, 1.9)
(ii) For any given \( \delta \in (0, \frac{2n}{2n-1}) \), there is a constant \( C_\delta > 0 \) depending additionally on \( \delta \) such that \( G_t \) satisfies
\[
\int_X |G_t(x, \cdot)|^{\frac{2n}{2n-1}} \omega_t^n + \int_X |\nabla G_t(x, \cdot)|^{\frac{2n}{2n-1}} \omega_t^n \leq C_\delta,
\]
for any fixed \( x \in X \).

Moreover, when \( n = 1 \), (i) and (ii) hold for any \( \omega_t \) with \( \|e^{F_{\omega_t}}\|_{L^{1+\epsilon}} \leq N \), and no extra conditions on \( e^{-F_{\omega_t}} \) as in (1.5), (1.6) and (1.7) are needed.

We now turn to the application of the estimates of Green’s functions in Theorem 2. Again we assume \( \chi \) is nonnegative and the class \([\chi]\) is big. Then by Kodaira’s lemma, there is an effective divisor \( D \) on \( X \) such that
\[
\chi - \varepsilon_0 \text{Ric}(h_D) \geq \delta_0 \omega_X
\]
for suitable positive constants \( \varepsilon_0 \) and \( \delta_0 \) which we will fix throughout the paper, where \( h_D \) is a Hermitian metric on the line bundle \([D]\) associated with \( D \). Let \( s_D \in \mathcal{O}_X(D) \) be a holomorphic section defining \( D \) such that
\[
\sup_X |s_D|_{h_D}^2 = 1.
\]
Let \( \varphi_t \) be the Kähler potential of the Kähler metric \( \omega_t \in [\chi + t\omega_X] \), i.e. \( \omega_t = \chi + t\omega_X + i\partial\bar{\partial}\varphi_t \). We first derive an estimate on the gradient of \( \varphi_t \) with respect to the fixed metric \( \omega_X \), for \( \omega_t \) in the sets of Kähler metrics in Theorem 2.

**Theorem 3** Given \( N, \epsilon, \gamma \in (0, 1) \), and \( p > n \), for each \( t \in (0, 1) \) and any \( \omega_t \in \mathcal{M}'(N, \epsilon, \gamma) \cup \mathcal{M}''(N, \epsilon, \gamma) \cup \bar{\mathcal{M}}''(N, \epsilon, \gamma) \), the following estimate on \( |\nabla \varphi_t|_{\omega_X}^2 \) holds. There is a constant \( C > 0 \) depending on \( n, \epsilon, \chi, \omega_X, \gamma, \) and \( p \) and \( \int_X |\nabla F_{\omega_t}|_{\omega_X}^p e^{F_{\omega_t}} \omega_X^n \) such that away from \( D \)
\[
|\nabla \varphi_t|_{\omega_X}^2 \leq \frac{C}{|s_D|_{h_D}^2 A^2},
\]
where \( A > 0 \) depends only on \( n, \epsilon, \chi, \omega_X, \) and \( \gamma \).

We observe that gradient estimates for solutions to certain complex Monge-Ampère equations had been obtained in [1, 20], but they required pointwise bounds on \( |\nabla F_{\omega_t}|_{\omega_X} \). In [2, 10], the gradient is shown to depend on certain \( L^p \) bound of \( |\nabla F_{\omega_t}|_{\omega_X} \) for \( p \geq 2n \). Theorem 3 implies that the gradient estimate depends only on integral bound of \( e^{F_{\omega_t}} \) and \( L^p \)-bound of \( |\nabla F_{\omega_t}|_{\omega_X} \) for smaller \( p \) which turns out to be sharp. In particular, the gradient estimate continues to hold in some situations even when \( e^{F_{\omega_t}} \) has zeros or is unbounded. Theorem 3 also provides immediately a gradient estimate for solutions to complex Monge-Ampère equations with a fixed background metric (i.e. when \( \omega \in \bar{\mathcal{M}}''(N, \epsilon, \gamma) \), cf. Corollary 3 in §4 below), in which case we can take \( s_D \equiv 1, h_D \equiv 1 \) and
\( D \) to be trivial. Example 3.1 shows that the gradient estimate may fail if \( p < n \), so the assumption on \( p > n \) in Theorem 3 is sharp.

With regard to the second-order derivatives, we have the following counterpart of Yau’s \( C^2 \) estimate [25].

**Theorem 4** Under the same setup as in Theorem 3 with \( p > 2n \), there is a constant \( C > 0 \) depending on \( n, \epsilon, \chi, \omega_X, N, \gamma, p \) and \( \int_X |\nabla F_{\omega_t}|_{\omega_X}^p e^{F_{\omega_t}} \omega^n_X \) such that away from \( D \) the following holds

\[
|i\partial\bar{\partial}\varphi|^2_{\omega_X} \leq \frac{C}{|s_D|_{h_D}^2 B},
\]

where \( B > 0 \) depends only on \( n, \epsilon, \chi, \omega_X, N \) and \( \gamma \).

We stress that the above \( C^2 \)-estimate of \( \varphi_t \) is independent of the pointwise estimates on the second order derivatives of \( F_{\omega_t} \), and it only depends on certain \( L^p \)-bound of \( |\nabla F_{\omega_t}|_{\omega_X} \). Theorem 4 improves in particular on the known estimates [25] for complex Monge-Ampère equations with a fixed background metric (cf. Corollary 4). We remark that the exponent \( p > 2n \) in Theorem 4 is also sharp, as Example 3.2 shows that the estimates may fail if \( p < 2n \).

By utilizing the estimates of the Green’s functions, we can also improve the \( C^3 \) estimates of complex Monge-Ampère equations (cf. Theorem 5), which generalize the known ones in [25, 18] by a weaker dependence of the function \( e^F \) on the right-hand side.

## 2 Proof of Theorem 1

Given any \( t \in (0, 1] \), we fix an arbitrary Kähler metric \( \omega_t \in \mathcal{M}_t(N, p) \). It is clear from the \( i\partial\bar{\partial} \)-lemma that (1.3) is equivalent to the following complex Monge-Ampère equation with \( \omega_t = \chi + t\omega_X + i\partial\bar{\partial}\varphi_t \)

\[
(\chi + t\omega_X + i\partial\bar{\partial}\varphi_t)^n = c_t e^{F_{\omega_t}} \omega^n_X, \quad \sup_{X} \varphi_t = 0,
\]

where we have normalized \( \varphi_t \) to make it unique, and we denote \( c_t = V_t/V \). The case of a fixed Kähler class \( \omega_X \) corresponds for example to the special case \( \chi = \frac{1}{2}\omega_X \) and \( t = \frac{1}{2} \). For simplicity of notations, we will write \( F_{\omega_t} \) as \( F \).

Since \( \tilde{\omega}_t := \chi + t\omega_X \) may not necessarily be positive, we introduce the following \( \tilde{\omega}_t \)-plurisubharmonic (PSH) function.

**Definition 1** For any \( t \in (0, 1] \), we denote the envelope associated to the \((1,1)\)-form \( \tilde{\omega}_t \)

\[
\mathcal{V}_t = \sup\{v \in PSH(X, \tilde{\omega}_t) \mid v \leq 0 \}.
\]
Note that if $\chi \geq 0$ is a nonnegative $(1,1)$-form, $V_t \equiv 0$ for any $t \in (0,1]$.

The following $L^\infty$ estimate for the family of solutions $\varphi_t$ to the equation (2.1) is proved in [12] (see also [16, 6]).

**Lemma 1 ([12])** There is a uniform constant $C_0 = C_0(n,p,\chi,\omega_X,\text{Ent}_p(\omega_t)) > 0$ such that

$$\sup_X |\varphi_t - V_t| \leq C_0, \quad \forall t \in (0,1].$$

To prove Theorem 1, we need the following mean-value type inequality for functions satisfying certain linear differential inequality.

**Lemma 2** Suppose $v \in L^1(X,\omega^n_t)$ is a function that satisfies $\int_X v \omega^n_t = 0$ and

$$v \in C^2(\Omega_{-1}), \quad \Delta_{\omega_t} v \geq -a \text{ in } \Omega_0$$

for some $a > 0$ and $\Omega_s = \{v > s\}$ is the super-level set of $v$. Then there is a constant $C > 0$ depending only on $n,p,\chi,\omega_X,\text{Ent}_p(\omega_t)$ and $a$ such that

$$\sup_X v \leq C(1 + \|v\|_{L^1(X,\omega^n_t)}).$$

We emphasize that the constant $C > 0$ in the lemma above is independent of $t \in (0,1]$. The proof of Lemma 2 uses the techniques similar to that of the $L^\infty$-estimate for fully nonlinear partial differential equations in [9]. The key idea to introduce an auxiliary equation to compare with $v$. The lemma is trivial if $v \equiv 0$, so we assume $v \not\equiv 0$.

**Proof.** We break the proof into four steps. Since the proof is uniform in $t \in (0,1]$, we fix a $t \in (0,1]$. We may assume $\|v\|_{L^1(X,\omega^n_t)} \leq V_0$, otherwise, replace $v$ by $\hat{v} := V_0 \cdot v/\|v\|_{L^1(X,\omega^n_t)}$ which still satisfies (2.2) with the same $a > 0$. It suffices to show $\sup_X v \leq C$ for some $C > 0$ with the dependence as stated in the lemma.

**Step 1.** We fix a sequence of positive smooth functions $\eta_k : \mathbb{R} \to \mathbb{R}_+ = (0, \infty)$ such that $\eta_k(x)$ converges uniformly and monotonically decreasingly to the function $x \cdot \chi_{\mathbb{R}_+}(x)$, as $k \to \infty$. We may choose $\eta_k(x) \equiv 1/k$ for any $x \leq -1/2$. For $s \geq 0$ and large $k$, we consider the following auxiliary complex Monge-Ampère equations

$$\left(\hat{\omega}_t + i\partial\bar{\partial}\psi_{t,k}\right)^n = c_t \frac{\eta_k(v - s)}{A_{s,k}} e^F \omega^n_X, \quad \sup_X \psi_{t,k} = 0,$$

where

$$A_{s,k} = \frac{c_t}{V_t} \int_X \eta_k(v - s) e^F \omega^n_X \to \frac{1}{V} \int_{\Omega_s} (v - s) e^F \omega^n_X =: A_s \text{ as } k \to \infty.$$  

We remark that the right-hand side of (2.3) is smooth and positive, and $[\hat{\omega}_t]$ is a Kähler class, so by Yau’s theorem [25] this equation admits a unique smooth solution $\psi_{t,k}$. We have also assumed that the open set $\Omega_s \neq \emptyset$ so $A_s > 0$, otherwise the lemma is already
proved. The assumption that \(|v|_{L^1(X, \omega^e_t)} \leq V_0\) implies that \(A_s \leq 1\), hence \(A_{s,k} \leq 2\) for large \(k\).

**Step 2.** We denote \(\Lambda = C_0 + 1\) where \(C_0\) is the constant in Lemma 1. Consider the function

\[
\Phi := -\varepsilon(-\psi_{t,k} + \varphi_t + \Lambda) \frac{1}{n+1} + (v - s),
\]

where \(\varepsilon > 0\) is chosen such that

\[
\varepsilon^{n+1} = \left(\frac{n+1}{n^2}\right)^n (a + \varepsilon n)^n A_{s,k}.
\]

It follows easily from \(A_{s,k} \leq 2\) and equation (2.5) that

\[
\varepsilon \leq C(n, a) A_{s,k}^{1/(n+1)},
\]

for some \(C(n, a) > 0\) depending only on \(n\) and \(a\). \(\Phi\) is a \(C^2\) function on \(\Omega_{-1}\) since \(v - s\) is so and

\[
\Lambda = \left(V_t - \psi_{t,k}\right) + \left(\varphi_t - V_t + C_0\right) + 1 \geq 1.
\]

We claim that \(\Phi \leq 0\) on \(X\). Observe that by the definition of \(\Omega_s\) it is clear that \(\Phi|_{X \setminus \Omega_s} > 0\), so if \(\max_{\Omega_s} \Phi \leq \sup_{X \setminus \Omega_s} \Phi < 0\), we are done. Otherwise, \(\max_{\Omega_s} \Phi > \sup_{X \setminus \Omega_s} \Phi\) and \(\Phi\) achieves its maximum at some point \(x_0 \in \Omega_s\). By maximum principle, \(\Delta \omega_t \Phi(x_0) \leq 0\). Therefore, we calculate (below we denote \(\Omega_t \psi_{t,k} = \tilde{\omega}_t + i \partial \bar{\partial} \psi_{t,k}\))

\[
0 \geq \Delta \omega_t \Phi(x_0) \\
\geq \frac{\varepsilon n}{n+1} (-\psi_{t,k} + \varphi_t + \Lambda) \frac{1}{n+1} \left(\text{tr}_{\omega_t} \omega_t \psi_{t,k} - \text{tr}_{\omega_t} \omega_t\right) + \Delta \omega_t v \\
\geq \frac{\varepsilon n^2}{n+1} (-\psi_{t,k} + \varphi_t + \Lambda) \frac{1}{n+1} \left(\frac{\omega_t^n}{\psi_{t,k}^{n+1}}\right)^{1/n} - \frac{\varepsilon n^2}{n+1} - a \\
\geq \frac{\varepsilon n^2}{n+1} (-\psi_{t,k} + \varphi_t + \Lambda) \frac{1}{n+1} \frac{(v - s)^{1/n}}{A_{s,k}^{1/n}} - a - \varepsilon n,
\]

where in the third line we applied the arithmetic-geometric inequality and in the last line we use the equation (2.3). By the choice of \(\varepsilon\) in (2.5), it follows by a straightforward calculation that \(\Phi(x_0) \leq 0\), as claimed.

**Step 3.** From \(\Phi \leq 0\) and (2.6) in the previous step, we have \((v - s) A_{s,k}^{-1/(n+1)} \leq C_1 (-\psi_{t,k} + \varphi_t + \Lambda)^{n/(n+1)}\) on \(X\), for some \(C_1 > 0\) depending only on \(n\) and \(a\). In particular on \(\Omega_s = \{v - s > 0\}\) we have by taking \((n + 1)/n\)-th power

\[
\frac{(v - s)^{(n+1)/n}}{A_{s,k}^{1/n}} \leq C_1^{(n+1)/n} (-\psi_{t,k} + \varphi_t + \Lambda) \leq C_1^{(n+1)/n} (-\psi_{t,k} + \Lambda)
\]

where the second inequality follows from the normalization \(\sup_X \varphi_t = 0\). Multiply both sides of above by suitable \(0 < \alpha = \alpha(\chi, \omega_X) > 0\) such that \(C_1^{(n+1)/n} \alpha < 1\).
invariant of the Kähler manifold \((X, (C_2 + 1)\omega_X)\) where \(\chi \leq C_2\omega_X\) for some \(C_2 > 0\), and integrate the resulted inequality over \(\Omega_s\). We thus obtain

\[
\int_{\Omega_s} \exp \left( \frac{(v - s)^{n+1}/n}{A_{s,k}^{1/n}} \right) \omega_X^n \leq C \int_X \exp \left( - C_1^{n+1}/n \psi_{t,k} \right) \omega_X^n \leq C_3,
\]

for some uniform constant \(C_3 > 0\) independent of \(t\). In this last inequality we apply the \(\alpha\)-invariant estimate for quasi-PSH functions on compact Kähler manifolds [15, 22]. By Young’s inequality, (2.8) implies that for some \(C_4 > 0\) depending additionally on \(p > n\) and \(\text{Ent}_p(\omega_t)\) (cf. [9])

\[
\int_{\Omega_s} (v - s)^{(n+1)p/n} e^{F \omega_X^n} \leq C_4 A_{s,k}^{p/n} \to C_4 A_s^{p/n},
\]

as \(k \to \infty\), noting that the left-hand side of (2.9) is independent of \(k\). On the other hand, by Hölder inequality and (2.9) we have

\[
A_s \leq \frac{1}{V} \left( \int_{\Omega_s} (v - s)^{\frac{n}{p(n+1)}} e^{F \omega_X^n} \right)^{\frac{p}{p(n+1)}} \left( \int_{\Omega_s} e^{F \omega_X^n} \right)^{\frac{1}{p} - \frac{1}{p'}} \leq C_5 A_s^{\frac{1}{p+1}} \left( \int_{\Omega_s} e^{F \omega_X^n} \right)^{\frac{1}{p'}},
\]

where \(p' > 1\) satisfies \(\frac{n}{p(n+1)} + \frac{1}{p'} = 1\). This implies that \(A_s \leq C_6 \left( \int_{\Omega_s} e^{F \omega_X^n} \right)^{\frac{1+\delta_0}{np}}\), i.e.

\[
\int_{\Omega_s} (v - s) e^{F \omega_X^n} \leq C_6 \left( \int_{\Omega_s} e^{F \omega_X^n} \right)^{1+\delta_0},
\]

for \(1 + \delta_0 = \frac{1+n}{np'}\) with \(\delta_0 = \frac{n-p}{np} > 0\). We denote \(\phi(s) = \int_{\Omega_s} e^{F \omega_X^n}\). Then (2.10) yields easily that

\[
(2.11) \quad r\phi(s + r) \leq C_6 \phi(s)^{1+\delta_0}, \quad \forall s \geq 0, \ r > 0
\]

**Step 4.** By the assumption \(\|v\|_{L^1(X, \omega^n_t)} \leq V_0\) we have

\[
\int_{\Omega_0} v e^{F \omega_X^n} \leq \frac{1}{c_t} \int_X |v| \omega_t^n \leq \frac{V_0 V}{V_t} \leq V,
\]

and this implies that for any \(s > 0\)

\[
(2.12) \quad \phi(s) = \int_{\Omega_s} e^{F \omega_X^n} \leq \frac{1}{s} \int_{\Omega_0} v e^{F \omega_X^n} \leq \frac{V}{s}.
\]

So we can pick \(s_0 = (2C_6)^{1/\delta_0} V\) to guarantee that \(\phi(s_0)^{\delta_0} < 1/2C_6\). Given (2.11), we can apply the De Giorgi type iteration argument of Kolodziej [16] to conclude that \(\phi(s) = 0\) for any \(s > S_\infty\) with

\[
S_\infty = s_0 + \frac{1}{1 - 2^{-\delta_0}} = (2C_6)^{1/\delta_0} V + \frac{1}{1 - 2^{-\delta_0}},
\]
This means that \( v \leq S_\infty \) and we finish the proof of the lemma.

**Proof of Theorem 1.** Fix a point \( x \in X \). We let \( v(y) = -G_t(x, y) \) be the Green’s function of \( \omega_t \) centered at \( x \). This \( v \) satisfies the assumptions in Lemma 2, i.e. \( v \in L^1(X, \omega_t^n) \), \( v \) is smooth on \( X \setminus \{x\} \) and

\[
\int_X v(y) \omega_t^n(y) = 0, \quad \Delta_{\omega_t} v(y) = -\frac{1}{V_t} \geq -\frac{1}{V_0} \text{ for } y \in \{v \geq 0\}
\]

Lemma 2 gives a constant \( C > 0 \) depending only on \( n, p, \omega_X, \chi, N \) such that

\[
v \leq C(1 + \|v\|_{L^1(X, \omega_t^n)})
\]

which implies that \( \inf_{y \in X} G_t(x, y) \geq -C(1 + \|G_t(x, \cdot)\|_{L^1(X, \omega_t^n)}) \).

On the other hand, since \( -Ct = \inf_{y \in X} G_t(x, y) \) is a lower bound of \( G_t(x, \cdot) \), we have

\[
\|G_t(x, \cdot)\|_{L^1(X, \omega_t^n)} \leq \int_X |G_t(x, \cdot) + C_t| \omega_t^n + C_t V_t \leq 2C_t V_t.
\]

### 3 Proof of Theorem 2

Given the parameters \( \epsilon > 0, N > 0 \) and \( \gamma \in (0, 1) \), we fix a Kähler metric \( \omega_t \in \mathcal{M}'(N, \epsilon, \gamma) \cup \mathcal{M}''(N, \epsilon, \gamma) \). We will denote \( G_t \) the associated Green’s function of \((X, \omega_t)\). As in the last section, we let \( \omega_t = \chi + t\omega_X + i\partial\bar{\partial}\phi_t \) be the solution to the following complex Monge-Ampère equation

\[
(\chi + t\omega_X + i\partial\bar{\partial}\phi_t)^n = c_t e^{F_t} \omega_X^n, \quad \sup_X \phi_t = 0.
\]

In this section, we say a constant \( C > 0 \) is uniform if it depends only on \( n, \chi, \omega_X \) and the given parameters \( \epsilon, N, \gamma \).

Since \( \chi \) is assumed to be nonnegative in Theorem 2, by Lemma 1, there is a uniform constant \( C_0 > 0 \) such that for each \( t \in (0, 1] \) and \( \phi_t \) satisfying (3.1)

\[
\sup_X |\phi_t| \leq C_0
\]

We observe the following estimate on \( L^2 \)-norm of \( \nabla \phi \).

**Lemma 3** Suppose \( \omega = \omega_X + i\partial\bar{\partial}\varphi \) is a metric such that \( e^F = \omega^n / \omega_X^n \in L^{1+\epsilon}(X, \omega_X^n) \), then

\[
\int_X |\nabla \phi|^2 X \omega_X^n \leq C;
\]

for some \( C > 0 \) depending only on \( n, \epsilon, \omega_X \) and \( \|e^F\|_{L^{1+\epsilon}} \).
Proof. If we normalize \( \varphi \) such that \( \sup_X \varphi = 0 \), by the \( L^\infty \)-estimates of Kolodziej [16, 9], we have \( \sup_X |\varphi| \leq C \) for some \( C \) depending on \( n, \omega_X \) and \( \|e^F\|_{L^{1+}} \). Then we calculate

\[
\int_X (-\varphi)(e^F - 1)\omega^n_X = \int_X (-\varphi)(\omega^n - \omega^X_n) = \int_X i\partial\varphi \wedge \bar{\partial}\varphi \wedge (\omega^{n-1} + \cdots + \omega^{n-1}_X) \geq \frac{1}{n} \int_X |\nabla \varphi|^2 \omega^X_X.
\]

The lemma follows straightforwardly from this.

From Lemma 2, we easily get that

Lemma 4 Suppose \( v \in C^2(X) \) satisfies

\[
|\Delta_{\omega_t} v| \leq 1, \quad \text{and} \quad \int_X v \omega^n_t = 0,
\]

then there is a uniform constant \( C > 0 \) such that

\[
\sup_X |v| \leq C(1 + \|v\|_{L^1(X, \omega^n_t)}).
\]

We further have the following lemma which asserts that the function \( v \) satisfying (3.4) is in fact bounded uniformly in \( L^\infty \)-norm.

Lemma 5 Under the same assumptions as in Lemma 4, we have \( \|v\|_{L^1(X, \omega^n_t)} \leq C \) for some uniform constant \( C > 0 \), in particular by Lemma 4 this implies \( \sup_X |v| \leq C \).

Assuming Lemma 5, we see how it yields the \( L^1 \)-bound on the Green’s function \( G_t \) of \((X, \omega_t)\). For any fixed point \( x \in X \), we view the Green’s function \( G_t(x, \cdot) = G_t(x, y) \) as a function of \( y \). Consider the equation

\[
\Delta_{\omega_t} v = -\chi_{\{G_t \leq 0\}} + \frac{1}{V_t} \int_{\{G_t \leq 0\}} \omega^n_t, \quad \text{and} \quad \int_X v \omega^n_t = 0.
\]

Take a sequence of smooth and bounded functions \( f_k \) that converge pointwise (in fact uniformly) to the bounded function \( -\chi_{\{G_t \leq 0\}} + \frac{1}{V_t} \int_{\{G_t \leq 0\}} \omega^n_t \) and satisfies \( \int_X f_k \omega^n_t = 0 \). Let \( v_k \) be the smooth solution to \( \Delta_{\omega_t} v_k = f_k \) with \( \int_X v_k \omega^n_t = 0 \). It follows from standard elliptic theory that for fixed \( t > 0 \), \( v_k \) converges uniformly to \( v \), which is the solution to (3.5). By Green’s formula, we have

\[
v_k(x) = \frac{1}{V_t} \int_X v_k \omega^n_t + \int_X G_t(x, \cdot)(-\Delta_{\omega_t} v_k) \omega^n_t = \int_X G_t(x, \cdot)(-f_k) \omega^n_t.
\]

From Lemma 5 we have \( |v_k(x)| \leq C \), for each \( k \), since \( |f_k| \leq 2 \), say. Letting \( k \to \infty \) and we get by the choice of \( f_k \) and the normalization \( \int_X G_t(x, \cdot) \omega^n_t = 0 \) that

\[
|\int_{\{G_t(x, \cdot) \leq 0\}} G_t(x, \cdot) \omega^n_t| \leq C.
\]
Since $|\int_{G_t(x,\cdot) \leq 0} G_t(x,\cdot) \omega^n_t| = \int_{G_t(x,\cdot) \geq 0} G_t(x,\cdot) \omega^n_t$, this easily gives the $L^1(X,\omega^n_t)$-bound of $G_t(x,\cdot)$. This finishes the proof of (i) in Theorem 2, assuming Lemma 5.

Now we turn to the proof of Lemma 5. We argue by contradiction. Suppose there is a sequence of Kähler metrics $\omega_j = \omega t_j \in M_t' \cup M''_t \cup \tilde{M}'_t \cup M''_t \cup \tilde{M}'_t$ and $F_j = F_{\omega_j}$ as defined in (1.3), and a sequence of $C^2(X)$ functions $v_j$ satisfying

$$\Delta \omega_j v_j = h_j, \quad \text{and} \quad \int_X v_j^\omega_j = 0,$$

for some function $h_j$ with $\sup_X |h_j| \leq 1$, for which Lemma 5 fails, i.e. they satisfy

$$\|v_j\|_{L^1(X,\omega^n_j)} = \int_X |v_j| \omega^n_j \to \infty \quad \text{as} \quad j \to \infty. \quad (3.6)$$

We normalize each $v_j$ by

$$\hat{v}_j = \frac{v_j}{\|v_j\|_{L^1(X,\omega^n_j)}}, \quad \text{so} \quad \|\hat{v}_j\|_{L^1(X,\omega^n_j)} = 1.$$

It is clear that $\hat{v}_j \in C^2(X)$ satisfies

$$\Delta \omega_j \hat{v}_j = \frac{h_j}{\|v_j\|_{L^1(X,\omega^n_j)}}, \quad \text{and} \quad \int_X \hat{v}_j^\omega_j = 0. \quad (3.7)$$

We can apply Lemma 4 to conclude that

$$\sup_X |\hat{v}_j| \leq C(1 + \|\hat{v}_j\|_{L^1(X,\omega^n_j)}) \leq C, \quad (3.8)$$

for some uniform constant $C > 0$ independent of $j$. Multiplying both sides of (3.7) by $\hat{v}_j$ and applying integration by parts, we get

$$\int_X |\nabla \hat{v}_j|_{\omega^n_j}^2 \omega^n_j = \int_X -\frac{h_j}{\|v_j\|_{L^1(X,\omega^n_j)}} \omega^n_j \leq \frac{1}{\|v_j\|_{L^1(X,\omega^n_j)}} \to 0, \quad (3.9)$$

as $j \to \infty$ by the hypothesis (3.6). By the Hölder inequality we have

$$\int_X |\nabla \hat{v}_j|_{\omega^n_j} \omega^n_X \leq \int_X (|\nabla \hat{v}_j|_{\omega^n_j} \omega_X)^{1/2} \omega^n_X \leq (\int_X |\nabla \hat{v}_j|_{\omega^n_j} e^{F_j} \omega^n_X)^{1/2} \left( \int_X \omega_X \omega_j \right)^{1/2} \left( \int_X e^{-F_j} \omega^n_X \right)^{1/2}. \quad (3.10)$$

Note that the first factor in (3.10) satisfies

$$\int_X |\nabla \hat{v}_j|_{\omega^n_j} e^{F_j} \omega^n_X = c_j^{-1} \int_X |\nabla \hat{v}_j|_{\omega^n_j} \omega^n_j \to 0$$

by (3.9). For the second factor in (3.10), we consider different cases of $\omega_j$.\textsuperscript{12}
(a) If \( \omega_j \in \mathcal{M}_{t_j}'(N, \epsilon, \gamma) \), then it holds that
\[
\int_X (\text{tr}_{\omega_X} \omega_j) e^{-F_j} \omega^n_X \leq \frac{1}{n\gamma} \int_X \omega_j \wedge \omega_X^{n-1} \leq C.
\]

(b) If \( \omega_j \in \mathcal{M}_{t_j}''(N, \epsilon, \gamma) \), then we have
\[
\int_X (\text{tr}_{\omega_X} \omega_j) e^{-F_j} \omega^n_X = \frac{1}{n} \int_X e^{-F_j}(\chi + t_j \omega_X + i \partial \bar{\partial} \varphi_{t_j}) \wedge \omega_X^{n-1}
\leq C \int_X e^{-F_j} \omega^n_X + \int_X |\varphi_{t_j}| |\Delta_{\omega_X} e^{-F_j}| \omega^n_X
\leq C,
\]
by the definition of \( \mathcal{M}_{t_j}''(N, \epsilon, \gamma) \) and (3.2).

(c) If \( \omega_j \in \tilde{\mathcal{M}}''(N, \epsilon, \gamma) \), then by a similar calculation we have
\[
\int_X (\text{tr}_{\omega_X} \omega_j) e^{-F_j} \omega^n_X = \frac{1}{n} \int_X e^{-F_j}(\chi + t_j \omega_X + i \partial \bar{\partial} \varphi_{t_j}) \wedge \omega_X^{n-1}
\leq C \int_X e^{-F_j} \omega^n_X + \int_X |\nabla \varphi_{t_j}| |\nabla e^{-F_j}| \omega^n_X
\leq C \int_X e^{-F_j} \omega^n_X + (\int_X |\nabla \varphi_{t_j}|^2 \omega_X^{n-1})^{1/2} (\int_X |\nabla e^{-F_j}|^2 \omega_X^n)^{1/2}
\leq C,
\]
by Lemma 3 and the definition of the set \( \tilde{\mathcal{M}}''(N, \epsilon, \gamma) \).

(d) If \( n = 1 \), we observe that from (3.9), as \( j \to \infty \)
\[
\int_X |\nabla \hat{v}_j|^2 \omega_X = \int_X i \partial \bar{\partial} \hat{v}_j \wedge \bar{\partial} \hat{v}_j = \int_X |\nabla \hat{v}_j|^2 \omega_j \to 0.
\]

Combining all cases discussed above, (3.10) or (3.11) entail that
\[
\int_X |\nabla \tilde{v}_j|^2 \omega_X \to 0.
\]

From (3.8) and (3.12) we see that the sequence of functions \( \{ \hat{v}_j \} \) is uniformly bounded in the Sobolev space \( W^{1,1}(X, \omega_X) \) (under the fixed metric \( \omega_X \)). By the Sobolev embedding theorem, there is an embedding
\[
W^{1,1}(X, \omega_X) \hookrightarrow L^q(X, \omega_X^n)
\]
which is compact for any \( 1 \leq q < \frac{2n}{2n-1} \). Therefore, up to a subsequence we have \( \hat{v}_j \to \hat{v}_\infty \) in \( L^q(X, \omega_X^n) \). In particular \( \hat{v}_j \) also converge to \( \hat{v}_\infty \) in \( L^1(X, \omega_X^n) \) and in the a.e. sense up to a further subsequence if necessary.
We now claim that $\hat{v}_\infty$ is constant in the a.e. sense. Indeed, for any fixed $C^2$ function $\rho$ on $X$, we have
\[
\left| \int_X \hat{v}_j \Delta_{\omega_X} \rho \omega^n_X \right| = \left| \int_X (\nabla \hat{v}_j, \nabla \rho) \omega^n_X \right| \leq \| \nabla \rho \|_\infty \int_X |\nabla \hat{v}_j| \omega^n_X \rightarrow 0
\]
as $j \rightarrow \infty$. By the dominated convergence theorem and $\hat{v}_j \rightarrow \hat{v}_\infty$ a.e. we conclude that
\[
\int_X \hat{v}_\infty \Delta_{\omega_X} \rho \omega^n_X = 0
\]
which holds for any $\rho \in C^2(X)$. By Weyl's lemma, this implies that $\hat{v}_\infty$ is $\Delta_{\omega_X}$-harmonic, hence $\hat{v}_\infty = \alpha_0$ in the a.e. sense for some constant $\alpha_0 \in \mathbb{R}$. We next claim that $\alpha_0 \neq 0$. Indeed, from the normalization $\|\hat{v}_j\|_{L^1(X, \omega^n_X)} = 1$ and (3.8), we get
\[
1 = c_t \int_X |\hat{v}_j| e^{F_j} \omega^n_X \leq C \int_X |\hat{v}_j|^\eta e^{F_j} \omega^n_X
\]
(3.13)
\[
\leq C \left( \int_X |\hat{v}_j|^{1+\epsilon} \omega^n_X \right)^{\epsilon/(1+\epsilon)} \left( \int_X e^{(1+\epsilon)F_j} \omega^n_X \right)^{1/(1+\epsilon)}
\]
and here we take $\eta = \epsilon/(1+\epsilon)$. From (3.13) we obtain
\[
(3.14) \quad \int_X |\hat{v}_j| \omega^n_X \geq c_0 > 0
\]
for some uniform constant $c_0 > 0$. Taking limit and applying the dominated convergence theorem again, we get $\int_X |\hat{v}_\infty| \omega^n_X \geq c_0 > 0$, and this implies that $|\alpha_0| > 0$.

However, this will contradict the second equation in (3.7). To see this, we assert that $\lim_{j \rightarrow \infty} \int_X \hat{v}_\infty \omega^n_X = 0$. In fact, by (3.7) we have
\[
|\int_X \hat{v}_\infty \omega^n_X| = |\int_X (\hat{v}_j - \hat{v}_\infty) \omega^n_X| \\
\leq c_d \int_X |\hat{v}_j - \hat{v}_\infty| e^{F_j} \omega^n_X \\
\leq C \int_X |\hat{v}_j - \hat{v}_\infty|^\eta e^{F_j} \omega^n_X \quad \text{here } \eta = \frac{\epsilon}{1+\epsilon} \\
\leq C \left( \int_X |\hat{v}_j - \hat{v}_\infty| \omega^n_X \right)^{\epsilon/(1+\epsilon)} \left( \int_X e^{(1+\epsilon)F_j} \omega^n_X \right)^{1/(1+\epsilon)} \\
\leq C \left( \int_X |\hat{v}_j - \hat{v}_\infty| \omega^n_X \right)^{\epsilon/(1+\epsilon)} \rightarrow 0
\]
since $\hat{v}_j \rightarrow \hat{v}_\infty$ in $L^1(X, \omega^n_X)$. But this is absurd since $\int_X \hat{v}_\infty \omega^n_X = \alpha_0 V_t$ which is strictly away from zero. This finishes the proof of Lemma 5.

Once the $L^1(X, \omega^n_X)$-norm of $G_t$ is achieved, Theorem 1 provides a lower bound of $G_t$. (1.9) is thus proved. Let $C_t > 0$ be the constant in (1.9), i.e. $G_t \geq -C_t$. For notational simplicity, we will denote the positive Green's function
\[
G_t(x, \cdot) = G_t(x, \cdot) + C_t + 1 \geq 1
\]
We are ready to prove (ii) in Theorem 2. To begin with, we show that the $L^q(X, \omega^n_t)$-norm of $G_t$ is uniformly bounded, for any $q < \frac{2n}{2n-2}$, which is optimal in view of the asymptotic behavior of $G_t$ in (1.4).

**Lemma 6** For any $q \in (1, \frac{2n}{2n-2})$, there is a uniform constant $C > 0$ depending on $q$ such that $G_t$ satisfies
\[ \int_X |G_t(x, y)|^q \omega^n_t(y) \leq C, \quad \forall x \in X. \]  

**Proof.** We break the proof into two steps. We fix a point $x \in X$ and consider the Green’s function $G_t(x, y)$ as a function of $y$. It suffices to show (3.16) for $G_t(x, \cdot)$, since $C_t > 0$ in (3.15) is uniform. The first step is to show the $L^q(X, \omega^n_t)$ bound of $G_t(x, \cdot)$ for any $q < 1 + 1/n$, then we can apply an iteration argument similar to the Moser iteration process to improve the exponent $q$.

**Step 1.** We will show the $L^{1+\frac{1}{r}}(X, \omega^n_t)$-norm of $G_t(x, \cdot)$ is uniformly bounded, for any $r_0 > n$. The argument is based on the $L^1$-bound of $G_t(x, \cdot)$ in (i) of Theorem 2.

Fix a large $k \gg 1$ and consider the function $H_k(y) = \min\{G_t(x, y), k\}$. By smoothing $H_k$ if necessary we may assume it is a smooth function and $H_k(y)$ converges monotonically increasingly to $G_t(x, y)$ as $k \to \infty$. We solve the following equation
\[ \begin{align*}
\Delta_{\omega_t} u_k &= -H_k^{1/r_0} + \frac{1}{t} \int_X H_k^{1/r_0} \omega_t^n, \\
\frac{1}{t} \int_X u_k \omega_t^n &= 0.
\end{align*} \]  
Equation (3.17) admits a unique smooth solution since the smooth function on the right-hand side of the first equation has integral 0. To deal with the unbounded term $-H_k^{1/r_0}$ on the right-hand side of (3.17) and non-uniform ellipticity of the linear operator $\Delta_{\omega_t}$, we consider again an auxiliary complex Monge-Ampère equation
\[ (\chi + t \omega_X + i \partial \bar{\partial} \psi_k)^n = \frac{H_k^{1/r_0} + 1}{V_t^{-1} \int_X (H_k^{1/r_0} + 1) \omega_t^n} \omega_t^n = c_t \frac{H_k^{1/r_0} + 1}{B_k} e^{F_{\omega_t} \omega_t^n}, \]  
with $\sup_X \psi_k = 0$ and $B_k = \int_X (H_k^{1/r_0} + 1) e^{F_{\omega_t} \omega_t^n}$. We stress that this auxiliary Monge-Ampère equation plays a very different role from the auxiliary Monge-Ampère equation introduced in the proof of Lemma 2. We note that
\[ V \leq B_k \leq V + \left( \int_X H_k e^{F_{\omega_t} \omega_t^n} \right)^{n/r_0} \left( \int_X e^{F_{\omega_t} \omega_t^n} \right)^{(r_0 - n)/r_0} \leq C(V), \]  
and the upper bound holds because of $0 < H_k \leq G_t$ for any $k$ and the integral bound of $G_t$ in (i) of Theorem 2. We note that the $p$-th (for some $p > n$) entropy of the function on the right-hand side of (3.18) satisfies
\[ \frac{1}{B_k} \int_X (H_k^{n/r_0} + 1) \mid - \log B_k + F_{\omega_t} + \log (1 + H_k^{n/r_0}) \mid^p e^{F_{\omega_t} \omega_t^n}. \]
\begin{align*}
(3.20) & \leq \frac{|\log B_k|^p}{B_k} \int_X (H_k^{n/r_0} + 1)e^{F_{\omega t}}\omega^n_X + \frac{1}{B_k} \int_X (H_k^{n/r_0} + 1)|\log (H_k^{n/r_0} + 1)|p e^{F_{\omega t}}\omega^n_X \\
& + \frac{1}{B_k} \int_X (H_k^{n/r_0} + 1)|F_{\omega t}|^p e^{F_{\omega t}}\omega^n_X \leq C
\end{align*}

for some uniform constant $C > 0$ depending on $n, p, \epsilon, \chi, \omega_X$ and $\|e^{F_{\omega t}}\|_{L^{1+}(\omega_X^2)}$. Here the first term in (3.20) is bounded due to the estimate of the constant $B_k$ in (3.19) and Hölder inequality along with the uniform $L^1(X, \omega^n)$-bound of $H_k$; the second term is bounded because of $\log (1 + x) \leq C\delta x^\delta$ for any $\delta > 0$ and the $L^1(X, \omega^n)$-bound of $H_k$; and the last term is bounded again by Hölder inequality. We can now apply Lemma 1 to conclude that

$$\sup_X |\psi_k| \leq C,$$

We know $\varphi_t$ also satisfies (3.2), so it follows that

$$\sup_X |\psi_k - \varphi_t| \leq C,$$

for some uniform constant $C > 0$, which is independent of $k$. We now consider the function

$$v := (\psi_k - \varphi_t) - \frac{1}{V_t} \int_X (\psi_k - \varphi_t)\omega^n_t + \varepsilon' u_k,$$

where $\varepsilon' > 0$ is a suitable constant to be chosen later. By definition it follows $\frac{1}{V_t} \int_X v\omega^n_t = 0$ and $v$ is a smooth function. We then calculate the Laplacian of $v$ in (3.22)

$$\Delta_{\omega_t} v = \text{tr}_{\omega_t} \omega_t \psi_k - n + \varepsilon' \Delta_{\omega_t} u_k$$

$$\geq n \left(\frac{\omega^n_t}{\omega_t^n}\right)^{1/n} - n - \varepsilon' H_k^{1/r_0} + \frac{\varepsilon'}{V_t} \int_X H_k^{1/r_0} \omega^n_t$$

$$= n B_k^{-1/n} (H_k^{n/r_0} + 1)^{1/n} - n - \varepsilon' H_k^{1/r_0} + \frac{\varepsilon'}{V_t} \int_X H_k^{1/r_0} \omega^n_t$$

$$\geq n B_k^{-1/n} H_k^{1/r_0} - n - \varepsilon' H_k^{1/r_0} \geq -n,$$

if we choose $\varepsilon' = nC(V)^{-1/n}$, where $C(V)$ is the upper bound of $B_k$ in (3.19). We apply the Green’s formula to the function $v$ at $x$

$$v(x) = \frac{1}{V_t} \int_X v\omega^n_t + \int_X G_t(x, \cdot)(-\Delta_{\omega_t} v)\omega^n_t = \int_X G_t(x, \cdot)(-\Delta_{\omega_t} v)\omega^n_t$$

$$\leq \int_X G_t(x, \cdot)n\omega^n_t \leq C,$$

where the last inequality follows from the uniform $L^1(X, \omega^n_t)$-bound of $G_t(x, \cdot)$, as proved in (i) of Theorem 2. It then follows from (3.21) and (3.22) that $u_k(x) \leq C$ for a uniform constant $C > 0$. 
We apply again the Green’s formula to the function $u_k$ at $x$ to get
\[
C \geq u_k(x) = \frac{1}{V_t} \int_X u_k \omega_t^n + \int_X G_t(x, \cdot)(-\Delta \omega_t u_k) \omega_t^n \\
= \int_X G_t(x, y) \left( H_k(y)^{1/r_0} - \frac{1}{V_t} \int_X H_k^{1/r_0} \omega_t^n \right) \omega_t^n(y),
\]
which yields that
\[
\int_X G_t(x, y) H_k(y)^{1/r_0} \omega_t^n(y) \leq C + C \int_X H_k^{1/r_0} \omega_t^n \leq C
\]
where we apply the Hölder inequality and the $L^1(X, \omega_t^n)$-bound of $G_t(x, \cdot)$ and $H_k$. Finally letting $k \to \infty$ we conclude from monotone convergence theorem and (3.23) that
\[
\int_X G_t(x, y) H_k(y)^{1/r_0} \omega_t^n(y) \leq C(r_0).
\]

Here we write the uniform constant as $C(r_0)$ to emphasize its addintional dependence on $r_0 > n$.

**Step 2.** We now iterate the arguments in **Step 1**, given the $L^{1+1/r_0}(X, \omega_t^n)$-bound (3.24) of $G_t$ for any $r_0 > n$. We take an arbitrary $r_1 \in (0, r_0)$ such that $\frac{n}{r_1} < 1 + \frac{1}{r_0}$, and replace $r_0$ by $r_1$ in the equations (3.17) and (3.18). Using the better $L^{1+1/r_0}(X, \omega_t^n)$-bound (3.24) of $G_t$ instead of just the $L^1(X, \omega_t^n)$-bound as in **Step 1**, we can repeat the arguments above to obtain the $L^{1+1/r_1}(X, \omega_t^n)$-bound of $G_t$. Iterating this process we eventually get for any $r_l \in (0, r_{l-1})$ with $\frac{n}{r_l} < 1 + \frac{1}{r_{l-1}}$, the $L^{1+1/r_l}(X, \omega_t^n)$-bound of $G_t$ is achieved, which depends in addition on $l$. A simple combinatorial argument shows that one can take any $r_l > 0$ such that
\[
\frac{1}{r_l} < \frac{1}{n - 1} - \frac{1}{n^l n(n - 1)}.
\]
For $l$ large enough, this implies the bound on $L^q(X, \omega_t^n)$-norm of $G_t(x, \cdot)$ for any $q < 1 + \frac{1}{n - 1} = \frac{n}{n - 1}$. As we mentioned at the beginning, the $L^q(X, \omega_t^n)$-bound of $G_t(x, \cdot)$ follows from the definition of $G_t(x, \cdot)$ in (3.15).

Now we derive the $L^q(X, \omega_t^n)$ bound on the gradient of $G_t(x, \cdot)$ for any $q \in (1, \frac{2n}{2n - 1})$. First we observe the following elementary estimate which follows easily from the Green’s formula.

**Lemma 7** For any $\beta > 0$ we have
\[
\int_X \frac{\left| \nabla_y G_t(x, y) \right|^2 \omega_t^n(y)}{G_t(x, y)^{1+\beta}} \omega_t^n(y) \leq 1/\beta.
\]
Proof. The function $u(y) := G_t(x, y)^{-\beta}$ is a continuous function with $u(x) = 0$ and $u \in C^\infty(X \setminus \{x\})$. By the Green’s formula we have
\[
0 = u(x) = \frac{1}{V_t} \int_X u \omega^n_t + \int_X G_t(x, \cdot)(-\Delta_{\omega_t} u) \omega^n_t
= \frac{1}{V_t} \int_X u \omega^n_t - \beta \int_X \frac{|\nabla G_t(x, \cdot)|^2}{G_t(x, \cdot)^{1+\beta}} \omega^n_t.
\]
Here we have applied the integration by parts, which can be justified by the asymptotic behavior of $G_t(x, y)$ as in (1.4). The lemma then follows easily from the fact that $0 \leq u \leq 1$.

Lemma 8 For any $\delta \in (0, 2/n)$ and $\beta > 0$, there is a uniform constant $C > 0$ depending on $\delta$ and $\beta$ such that
\[
(3.26) \quad \int_X G_t(x, y)^{1-\frac{(1+\beta)(2-n)}{2n}} |\nabla G_t(x, y)|^2 \omega^n_t(y)^{-\delta} \leq C.
\]
Proof. Given Lemma 7, the proof of Lemma 8 is essentially the same as that in Step 1 of Lemma 6. We use similar notations as in Lemma 6. Let
\[
H_k(y) = \min \{ \frac{\nabla_y G_t(x, y)^2}{G_t(x, y)^{(1+\beta)}}, k \}
\]
where $\min$ denotes a suitable smoothing of the min function. We can solve equations (3.17) and (3.18) with this $H_k$ and any $r_0 > n$. With the estimate (3.25) in Lemma 7, the same argument gives
\[
\int_X G_t(x, y) H_k(y)^{1/r_0} \omega^n_t(y) \leq C.
\]
Letting $k \to \infty$, this yields
\[
\int_X G_t(x, y)^{1-\frac{(1+\beta)}{r_0}} |\nabla_y G_t(x, y)|^{2/r_0} \omega^n_t(y) \leq C.
\]
The lemma follows by setting $\delta = \frac{2}{n} - \frac{2}{r_0}$.

With Lemmas 7 and 8, we are ready to derive the $L^s(X, \omega^n_t)$-bound of $\nabla G_t(x, \cdot)$ for $s \in [1, \frac{2n}{2n-1})$.

Lemma 9 For any given $s \in [1, \frac{2n}{2n-1})$, there is a uniform constant $C > 0$ depending on $s$ such that
\[
(3.27) \quad \int_X |\nabla G_t(x, y)|^s \omega^n_t(y) \leq C.
\]
Proof. We fix a constant $1 \leq s < \frac{2n}{2n-1}$ and a point $x \in X$. The Green’s function $G_t(x, y)$ is viewed as a function of $y$ and all the integrals below are integrated over $y \in X$. Then we calculate
\[
(3.28) \quad f_X |\nabla G_t(x, y)|^s \omega^n_t(y) = f_X (|\nabla G_t|^{s\delta} G_t)^{\frac{\nabla G_t(x, \cdot)}{G_t} G_t^{k-a} \omega^n_t}
\leq \left( f_X |\nabla G_t|^{s\delta} G_t^{k-a} \omega^n_t \right)^{1/p} \left( f_X \left( |\nabla G_t|^{1-\delta} a \right)^{1/q} \right)^{1/q} \left( f_X G_t^{(b-a)r} \omega^n_t \right)^{1/r}
\]
18
where the inequality follows from the generalized Hölder inequality and the numbers involved are chosen as follows.

\[
\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1, \ 0 < \delta \ll 1, \ 0 < a < b, \ (b - a)r < \frac{n}{n - 1}.
\]

Furthermore for a small \( \beta > 0 \) we can pick these numbers as

\[
(1 - \delta)sq = 2, \ bq = 1 + \beta, \ s\delta p = \frac{2}{n} - \beta, \ ap = 1 - \frac{(1 + \beta)(2 - n\beta)}{2n}.
\]

With these choices of parameters, Lemma 7 and Lemma 8 imply the first two factors in (3.28) are bounded. It only remains to verify the last factor in (3.28) is also bounded, and by Lemma 6, it suffices to make sure that \((b - a)r < \frac{2n}{2n - 2} = \frac{n}{n - 1}\). From (3.29), we derive

\[
q = \frac{2}{(1 - \delta)s}, \ p = \frac{2 - n\beta}{ns\delta}, \ b = \frac{(1 + \beta)(1 - \delta)s}{2}, \ a = \frac{ns\delta}{2 - n\beta} \left(1 - \frac{(1 + \beta)(2 - n\beta)}{2n}\right).
\]

Therefore, we have

\[
(b - a)r =: \Psi(\delta) = \frac{(1 + \beta)(1 - \delta)s}{2} - \frac{ns\delta}{2 - n\beta} \left(1 - \frac{(1 + \beta)(2 - n\beta)}{2n}\right) \frac{1}{1 - \frac{(1 - \delta)s}{2} - \frac{ns\delta}{2 - n\beta}}.
\]

We note that as \( s < \frac{2n}{2n - 1} \)

\[
\Psi(0) = \frac{(1 + \beta)s}{2} \frac{1}{1 - \frac{s}{2}} = (1 + \beta) \frac{s}{2 - s} < \frac{2n}{2n - 2},
\]

if \( \beta > 0 \) is chosen small enough. Since \( \Psi(\delta) \) is continuous in \( \delta \geq 0 \), \( \Psi(\delta) < \frac{2n}{2n - 2} \) if \( \delta > 0 \) is sufficiently small. This verifies the desired inequality \((b - a)r < \frac{n}{n - 1}\). The proof of Lemma 9 is completed.

As a corollary of Lemma 9, we have the following Sobolev-Morrey type inequality for the metric \( \omega_t \).

**Corollary 1** For any \( p > 2n \) there is a uniform constant \( C > 0 \) depending on \( p \) such that

\[
\sup_X \left| u - \frac{1}{V_t} \int_X u \omega_t^n \right| \leq C \left( \int_X \left| \nabla u \right|_{\omega_t, \omega_t^n}^p \right)^{1/p}, \ \forall u \in C^1(X).
\]

**Proof.** This follows immediately from the Green’s formula below, Lemma 9 and Hölder inequality

\[
u(x) - \frac{1}{V_t} \int_X u \omega_t^n = \int_X \langle \nabla_y G_t(x, y), \nabla u(y) \rangle \omega_t(y) \omega_t^n(y),
\]

noting that the conjugate exponent \( p^* = \frac{p}{p - 1} < \frac{2n}{2n - 1} \).
Proof of (ii) in Theorem 2. This follows from Lemma 6 and Lemma 9.

We conclude this section by comparing the lower bound of the Green’s function obtained in Theorem 2 with the classical one in Cheng-Li [3]. Let \( \omega = \omega_X + i\partial \bar{\partial} \phi \in [\omega_X] \) be a Kähler metric with \( \|e^{F_\omega}\|_{L^{1+\epsilon}(X,\omega_X^n)} \leq N \). Suppose Ric(\( \omega \)) \( \geq -\kappa' \omega \) for some \( \kappa' \geq 0 \), then from [13, 6] we know \( \text{diam}(X, \omega) \leq C(n, \omega_X, N) \). Then Cheng-Li’s estimate (1.2) implies the Green’s function associated with \( \omega \) is bounded below.

We show now that, in the Kähler setting, under a Kolodziej type [16] condition on the volume form, Theorem 2 implies the lower bound of the Green’s function under the less restrictive assumption of a lower bound of the scalar curvature:

**Corollary 2** For any Kähler metric \( \omega \in [\omega_X] \), if its relative volume form \( \|e^{F_\omega}\|_{L^{1+\epsilon}(X,\omega_X^n)} \leq N \) for some \( \epsilon > 0 \), \( N > 0 \), and its scalar curvature \( R(\omega) \geq -\kappa \) for some \( \kappa \geq 0 \), the Green’s function \( G \) of \((X, \omega)\) satisfies

\[
\inf_{y \in X} G(x, y) \geq -C, \quad \forall x \in X,
\]

for some constant \( C > 0 \) depending on \( n, \omega_X, \epsilon, N \) and \( \kappa \).

**Proof.** We claim that under the assumption of a scalar curvature lower bound, the relative volume form satisfies \( \inf_X e^{F_\omega} \geq \delta' \) for some \( \delta' > 0 \). Hence it follows that \( \omega \in \mathcal{M}'(N, \epsilon, \gamma) \) with \( \gamma = 1/\delta' \) and (i) in Theorem 2 implies a lower bound on the Green’s function \( G \).

To see the claim, note that by definition of the Ricci curvature, \( \text{Ric}(\omega) = \text{Ric}(\omega_X) + i\partial \bar{\partial}(-F_\omega) \), so the scalar curvature of \( \omega \) satisfies \( R(\omega) = \text{tr}_\omega(\text{Ric}(\omega_X)) + \Delta_\omega(-F_\omega) \). The assumption that \( R(\omega) \geq -\kappa \) implies \( \Delta_\omega(-F_\omega) \geq -\kappa - \text{tr}_\omega(\text{Ric}(\omega_X)) \). By Kolodziej’s \( L^\infty \) estimate [16] (see also [9]), the Kähler potential \( \varphi \) of \( \omega = \omega_X + i\partial \bar{\partial} \phi \) is bounded, i.e. \( ||\varphi||_{L^\infty} \leq C \) for some \( C = C(\epsilon, N) > 0 \), if we normalize \( \sup_X \varphi = 0 \). For a constant \( A > 0 \) to be determined, we can calculate

\[
\Delta_\omega(-F_\omega - A\varphi) \geq -\kappa - \text{tr}_\omega(\text{Ric}(\omega_X)) + A\text{tr}_\omega \omega_X - An
\]

\[
\geq \text{tr}_\omega \omega_X - \kappa - An
\]

\[
\geq n\left(\frac{\omega_X^n}{\omega^n}\right)^{1/n} - \kappa - An = ne^{-F_\omega + n} - \kappa - An
\]

where we take \( A = C' + 1 \) and \( C' > 0 \) is an upper bound of the Ricci curvature \( \text{Ric}(\omega_X) \). Applying maximum principle, we get at the maximum point of \( -F_\omega - A\varphi, e^{-F_\omega} \leq (A + \kappa/n)^n \). Combined with the \( L^\infty \) bound of \( \varphi \), this easily shows the upper bound of \( -F_\omega \).

We observe that typically, the assumption of lower bound for the scalar curvature is much more difficult to work with than the assumption of lower bound for the Ricci curvature. There seem to be far fewer results under this assumption. One of which, in a very different direction, is the recent result of Munteanu-Wang [17] on the decay of the Green’s function on a real three-dimensional complete manifold with scalar curvature bounded from below.
4 Applications

In this section we discuss some applications of the estimates on the Green’s functions in Theorem 2, and provide the proof of Theorems 3 and 4. Given the parameters $\epsilon, N,$ and $\gamma,$ for each $t \in (0, 1],$ we fix a Kähler metric $\omega_t \in M^s_t(N, \epsilon, \gamma) \cup M''_t(N, \epsilon, \gamma) \cup \tilde{M}''(N, \epsilon, \gamma)$.

Recall that $\chi$ a closed and nonnegative $(1,1)$-form such that its class $[\chi]$ is big. By the definition of $F_{\omega_t}$ in (1.3), the metric $\omega_t = \chi + t\omega_X + i\partial\overline{\partial}\varphi_t$ satisfies the following complex Monge-Ampère equation

\begin{equation}
(\chi + t\omega_X + i\partial\overline{\partial}\varphi_t)^n = c_t e^{F_{\omega_t}}\omega_X^n,
\end{equation}

where as before $c_t = V_t/V$ is a normalizing constant. By the assumptions on $\omega_t$, the Green’s function $G_t$ associated with $\omega_t$ satisfies the estimates stated in Theorem 2.

Since $[\chi]$ is assumed to be big, Kodaira’s lemma implies that there is an effective divisor $D$ such that

\begin{equation}
\chi - \epsilon_0 \text{Ric}(h_D) \geq \delta_0 \omega_X,
\end{equation}

where $h_D$ is a Hermitian metric on the line bundle $[D]$ associated to $D$ and $\epsilon_0 > 0$ and $\delta_0 > 0$ are fixed constants depending only on $\chi, \omega_X$. Let $s_D \in \mathcal{O}_X(D)$ be a holomorphic section of $[D]$ defining $D$ and by rescaling $h_D$ if necessary we assume $\sup_X |s_D|^2_{h_D} \leq 1$.

To ease the notations, throughout this section we will denote by $\tilde{g}$ and $g$ (omitting the subscript $t$ in $g_t$) the associated metrics of $\omega_X$ and $\omega_t$, respectively.

We will omit the subscript $t$ in $\varphi_t$ which solves (4.1) and simply write it as $\varphi$. The function $F_{\omega_t}$ will be simply written as $F$, since $\omega_t$ is a fixed metric.

Recall that we denote $G_t$ the positive Green’s function in (3.15), which differs from $G_t$ by a uniform constant.

4.1 Gradient estimates

We will prove Theorem 3 in this subsection. Fix a constant $p > n$.

The lemma below follows from straightforward calculations, so we omit the proof.

**Lemma 10** Suppose $\varphi$ satisfies (4.1). We have

\begin{equation}
\Delta_g |\nabla\varphi|^2_{\tilde{g}} \geq 2\text{Re}(\nabla F, \tilde{\nabla}\varphi)_{\tilde{g}} + g^{ij}\tilde{g}^{kl}(\varphi_{ki}\varphi_{jl} + \varphi_{kj}\varphi_{il}) - 2K\text{tr}_{\tilde{g}}\tilde{g}|\nabla\varphi|^2_{\tilde{g}},
\end{equation}

where $-K$ is a lower bound of the bisectional curvature of the fixed metric $\omega_X$ and $\varphi_{ki}$ denote the second-order covariant derivatives of $\varphi$ with respect to $\omega_X$.

**Lemma 11** The following inequality holds on $X$:

\begin{equation}
\Delta_g H \geq e^{-\lambda\hat{\varphi}} \text{Re}(\nabla F, \tilde{\nabla}\varphi)_{\tilde{g}} + H\text{tr}_{\tilde{g}}\tilde{g} - (2 + n)\lambda H - CH^{1/2} - C_\lambda H^{1/2} \text{tr}_{\tilde{g}}\tilde{g},
\end{equation}

where $\hat{\varphi} = \varphi - \epsilon_0 \log |s_D|^2_{h_D}$, and $H = e^{-\lambda\hat{\varphi}}|\nabla\varphi|^2_{\tilde{g}}$ for suitable $\lambda > 0$ depending only on $\epsilon_0, \delta_0$ and $\omega_X$. The constants $C > 0$ and $C_\lambda > 0$ in (4.4) are both uniform.
Proof. As above, we denote by \( H = e^{-\lambda\phi}|\nabla\varphi|^2_g \) for some \( \lambda > 0 \) to be determined later. We calculate on \( X \setminus D \)

\[
\Delta_g H = e^{-\lambda\phi}\Delta_g|\nabla\varphi|^2_g + 2e^{-\lambda\phi}Re(-\lambda\nabla\phi, \bar{\nabla}|\nabla\varphi|^2_g)_{\bar{g}} + H(-\lambda\Delta_g\phi + \lambda^2|\nabla\varphi|^2_g) \\
\geq e^{-\lambda\phi}\left(2Re(\nabla F, \nabla\varphi)_{\bar{g}} + g^{ij}\bar{g}^{kl}(\varphi_{ki}\varphi_{lj} + \varphi_{kj}\varphi_{il}) - 2Ktr_g\bar{g}|\nabla\varphi|^2_g\right) \\
+ 2e^{-\lambda\phi}Re(-\lambda\nabla\phi, \bar{\nabla}|\nabla\varphi|^2_g)_{\bar{g}} + H(-\lambda\Delta_g\phi + \lambda^2|\nabla\varphi|^2_g).
\]

(4.5)

We now perform the calculation at a fixed point \( x_0 \in X \setminus D \) and choose normal coordinates at \( x_0 \) relative to \( \bar{g} \), such that \( \bar{g}_{ij}(x_0) = \delta_{ij} \), \( d\bar{g}_{ij}(x_0) = 0 \) and \( g_{ij}(x_0) = g_{ij} \delta_{ij} \) is diagonal. Then at \( x_0 \) we have

\[
2e^{-\lambda\phi}Re(-\lambda\nabla\phi, \bar{\nabla}|\nabla\varphi|^2_g)_{\bar{g}} = -2\lambda e^{-\lambda\phi}Re(g^{ii}(\hat{\varphi}_i\varphi_{ki}\varphi_{ki} + \hat{\varphi}_i\varphi_{ki}\varphi_{ki})) \\
\geq -2\lambda e^{-\lambda\phi}Re(g^{ii}\hat{\varphi}_i\varphi_{ki}\varphi_{ki}) - \lambda^2 e^{-\lambda\phi}|\nabla\hat{\varphi}|^2_{\bar{g}} - e^{-\lambda\phi}g^{ii}\varphi_{ki}\varphi_{ki} \\
= -2\lambda e^{-\lambda\phi}Re(g^{ii}(\hat{\varphi}_i(\delta_{ii} - 1)\varphi_{ki}) - \lambda^2 H|\nabla\hat{\varphi}|^2_{\bar{g}} - e^{-\lambda\phi}g^{ii}\varphi_{ki}\varphi_{ki} \\
= -2\lambda e^{-\lambda\phi}\langle\nabla\phi, \nabla\varphi\rangle_{\bar{g}} + 2\lambda e^{-\lambda\phi}\langle\nabla\phi, \nabla\varphi\rangle_{\bar{g}} - \lambda^2 H|\nabla\hat{\varphi}|^2_{\bar{g}} - e^{-\lambda\phi}g^{ii}\varphi_{ki}\varphi_{ki}.
\]

And by (4.2) we have

\[
-\Delta_g\hat{\varphi} = tr_g(\chi + t\omega_X - \omega_t - \epsilon_0Ric(\bar{h}_D)) \geq \delta_0 tr_g\bar{g} - n
\]

Substituting the above two (in)equalities to (4.5), we obtain that at \( x_0 \)

\[
\Delta_g H \geq e^{-\lambda\phi}\left(2Re(\nabla F, \nabla\varphi)_{\bar{g}} + g^{ij}\bar{g}^{kl}(\varphi_{ki}\varphi_{lj} + \varphi_{kj}\varphi_{il}) - 2Ktr_g\bar{g}|\nabla\varphi|^2_g\right) + \lambda H(\delta_0 tr_g\bar{g} - n) \\
-2\lambda e^{-\lambda\phi}\langle\nabla\phi, \nabla\varphi\rangle_{\bar{g}} + 2\lambda e^{-\lambda\phi}\langle\nabla\phi, \nabla\varphi\rangle_{\bar{g}} \\
\geq 2e^{-\lambda\phi}Re(\nabla F, \nabla\varphi)_{\bar{g}} + (\lambda\delta_0 - 2K)(tr_g\bar{g})H - \lambda nH - 2\lambda e^{-\lambda\phi}|\nabla\varphi|^2_{\bar{g}} + 2\lambda e^{-\lambda\phi}|\nabla\varphi|^2_{\bar{g}} \\
-2\epsilon_0\lambda e^{-\lambda\phi}|s_{Dh_D}^{2\lambda\phi-1}|\nabla_{h_D}s_{\bar{g},h_D}\nabla\varphi|_{\bar{g}} - 2\epsilon_0\lambda e^{-\lambda\phi}|s_{Dh_D}^{2\lambda\phi-1}|\nabla_{h_D}^{s_{D}}|g_{h_D}|\nabla\varphi|_g,
\]

where we denote by \( \nabla_{h_D} \) the Chern connection of the hermitian metric \( h_D \) on the line bundle \( [D] \). We choose \( \lambda > 1 \) large enough so that \( \lambda\delta_0 > 2K + 10 \) and \( 2\lambda\epsilon_0 > 10 \). Then the first term in (4.6) satisfies

\[
-2\epsilon_0\lambda e^{-\lambda\phi}|s_{Dh_D}^{2\lambda\phi-1}|\nabla_{h_D}s_{\bar{g},h_D}\nabla\varphi|_{\bar{g}} \geq -Ce^{-\lambda\phi/2}|s_{Dh_D}^{\lambda\phi-1}H^{1/2} \geq -CH^{1/2},
\]

because \( |\nabla_{h_D}s_{D}|_{g,h_D} \leq C \) and \( ||\varphi||_{L^\infty} \leq C \). Similarly the last term in (4.6) satisfies

\[
-2\epsilon_0\lambda e^{-\lambda\phi}|s_{Dh_D}^{2\lambda\phi-1}|\nabla_{h_D}^{s_{D}}|g_{h_D}|\nabla\varphi|_g \geq -C_sH^{1/2}tr_g\bar{g},
\]

for some uniform constant \( C_s > 0 \). Plugging (4.7) and (4.8) into (4.6), we obtain that at \( x_0 \in X \setminus D \)

\[
\Delta_g H \geq 2e^{-\lambda\phi}Re(\nabla F, \nabla\varphi)_{\bar{g}} + Htr_g\bar{g} - (2 + n)\lambda H - CH^{1/2} - C_sH^{1/2}tr_g\bar{g}.
\]

(4.9)
Proof of Theorem 3. Let $C_s > 0$ be the constant in Lemma 11 and denote $\Lambda = C_s^2 + 1$. From Lemma 11, we have for some uniform constant $C > 0$

\begin{equation}
\Delta_g H \geq -C|\nabla F|_g H^{1/2} + (H^{1/2} - C_s)H^{1/2} \text{tr}_g \tilde{g} - CH - C
\end{equation}

We consider the convex and monotonically increasing function $\Phi_\delta(x) = \frac{1}{2}(\sqrt{x^2 + \delta} + x) + \Lambda$ for $\delta > 0$. It is clear that

$$
\Phi_\delta(H - \Lambda) \to \max(H, \Lambda), \text{ as } \delta \to 0.
$$

We also have $0 \leq \Phi'_\delta(x) \leq 1$, $\Phi''_\delta(x) \geq 0$ for all $x \in \mathbf{R}$ and

\begin{equation}
0 \leq \Phi'_\delta(x) = \frac{1}{2} \frac{\delta}{\sqrt{x^2 + \delta} - x} \leq \frac{\delta}{2}, \text{ when } x \leq -1.
\end{equation}

We denote $\hat{H} = H - \Lambda$ and calculate

\begin{align*}
\Delta_g \Phi_\delta(\hat{H}) &= \Phi'_\delta \Delta_g H + \Phi''_\delta |\nabla H|^2 \\
&\geq \Phi'_\delta(\hat{H}) \left(-C|\nabla F|_g H^{1/2} + (H^{1/2} - C_s)H^{1/2} \text{tr}_g \tilde{g} - CH - C\right) \chi_{\{\hat{H} \geq -1\}} + \Delta_g H \cdot \Phi'_\delta(\hat{H}) \chi_{\{\hat{H} < -1\}} \\
&\geq \Phi'_\delta(\hat{H}) \left(-C|\nabla F|_g H^{1/2} - CH - C\right) \chi_{\{\hat{H} \geq -1\}} - \delta |\Delta_g H| \chi_{\{\hat{H} < -1\}}
\end{align*}

where the second inequality follows since on $\{\hat{H} \geq -1\}$, $H \geq \Lambda - 1 \geq C_s^2$, and on $\{\hat{H} < -1\}$, we have $\Phi'_\delta(\hat{H}) \leq \delta$ by (4.11). Applying Green’s formula to the smooth function $\Phi_\delta(\hat{H})$ we obtain for any $x \in X$

\begin{equation}
\Phi_\delta(\hat{H})(x) = \frac{1}{V_t} \int_X \Phi_\delta(\hat{H}) \omega^n_t + \int_X G_t(x, \cdot)(-\Delta_g \Phi_\delta(\hat{H})) \omega^n_t \\
\leq \frac{1}{V_t} \int_X \Phi_\delta(\hat{H}) \omega^n_t + \delta \int_{\{\hat{H} < -1\}} G_t(x, \cdot) |\Delta_g H| \omega^n_t + \int_{\{\hat{H} \geq -1\}} G_t(x, \cdot) \Phi'_\delta(\hat{H}) \left(C|\nabla F|_g H^{1/2} + CH + C\right) \omega^n_t.
\end{equation}

Letting $\delta \to 0$ in (4.12), we get (denoting $H_\Lambda = \max\{H, \Lambda\}$)

\begin{equation}
H_\Lambda(x) \leq \frac{1}{V_t} \int_X H_\Lambda \omega^n_t + \int_X G_t(x, \cdot) \left(C|\nabla F|_g H^{1/2} + CH + C\right) \omega^n_t.
\end{equation}

Let $x_1 \in X$ be a maximum point of $H_\Lambda$. Then we obtain from (4.13) that

$$
H_{\Lambda, \max} = H_\Lambda(x_1) \leq \frac{c_t H_{\Lambda, \max}^{1-\eta}}{V_t} \int_X H_\Lambda^n e^F \omega^n_x.
$$
Observe that on $E$ we have:

\[
\begin{align*}
&+ CH_{\Lambda, \text{max}}^{1/2} \int_X \mathcal{G}_t(x_1, \cdot) \nabla F \| \tilde{\omega}_t^n + CH_{\Lambda, \text{max}}^{1-\eta} \int_X \mathcal{G}_t(x_1, \cdot) H^n \tilde{\omega}_t^n + C \\
&\leq \frac{c_0 H_{\Lambda, \text{max}}^{1-\eta}}{V_t} \left( \int_X e^{(1+\epsilon)F} \omega_X^n \right)^{1/\eta} \left( \int_X H^{\eta(1+\epsilon)/\epsilon} \omega_X^n \right)^{1/\epsilon} + C \\
&+ CH_{\Lambda, \text{max}} \left( \int_X \mathcal{G}_t(x_1, \cdot) \rho \omega_t^n \right)^{1/p} \left( \int_X \nabla F \| \rho \omega_X^n \right)^{1/p} + C \\
&+ CH_{\Lambda, \text{max}} \left( \int_X \mathcal{G}_t(x_1, \cdot) \rho_0 \omega_t^n \right)^{1/p} \left( \int_X H^{\rho_0^0} \omega_t^n \right)^{1/p_0} .
\end{align*}
\]

(4.14)

where we fix a number $p_0 \in (1, \frac{n}{n-1})$. By assumption $p > n$ so we have $p^* < \frac{n}{n-1}$. If furthermore we choose $\eta > 0$ such that $\eta(1+\epsilon)/\epsilon \leq C$ and $\eta p^*_0 \leq \epsilon/(1+\epsilon)$, then by Theorem 2, and Lemma 12 below it follows that all the integrals involved in (4.14) are bounded uniformly from above. It then follows that

\[
H_{\Lambda, \text{max}} \leq CH_{\Lambda, \text{max}}^{1-\eta} + CH_{\Lambda, \text{max}}^{1/2} + C.
\]

By Young’s inequality we immediately derive the uniform upper bound of $H_{\Lambda, \text{max}}$. Hence the upper bound of $H = e^{-\lambda \phi} |s_D|_{h_D}^{2\lambda \phi} |\nabla \phi|_{\omega_X}^2$. The proof of Theorem 3 is completed.

It only remains to show the integrals of $H$ are bounded.

**Lemma 12** There is a uniform constant $C > 0$ such that

\[
\int_X H \omega_X^n = \int_X e^{-\lambda \phi} |s_D|_{h_D}^{2\lambda \phi} |\nabla \phi|_{\omega_X}^2 \omega_X^n \leq C,
\]

and when integrated against $\omega_t^n$ we have

\[
\int_X H \tilde{\omega}_t^n \leq C.
\]

**Proof.** Recall we write $\varphi = \varphi - \varepsilon_0 \log |s_D|_{h_D}^2$. For any small $\delta > 0$, we denote the super-level set of $|s_D|_{h_D}$, $E_\delta = \{ |s_D|_{h_D} \geq \delta \}$. We write $\tilde{\omega} = \chi - \varepsilon_0 \text{Ric}(h_D) \leq \delta_0 \omega_X$ as in (4.2). Observe that on $E_\delta$ the following equation holds

\[
\omega_t^n - (\tilde{\omega} + t\omega_X)^n = (\tilde{\omega} + t\omega_X + i\partial \bar{\partial} \tilde{\phi})^n - (\tilde{\omega} + t\omega_X)^n = i\partial \bar{\partial} \tilde{\phi} \wedge (\omega_t^{n-1} + \cdots + (\tilde{\omega} + t\omega_X)^{n-1}).
\]

Multiplying both sides by $e^{-\lambda \phi}$ and integrating over $E_\delta$, we obtain by integration by parts

\[
\begin{align*}
\frac{\lambda}{n} \int_{E_\delta} e^{-\lambda \phi} |\nabla \tilde{\phi} |_{\omega_t}^{2} \omega_t^n &\leq \lambda \int_{E_\delta} e^{-\lambda \phi} i\partial \bar{\partial} \tilde{\phi} \wedge (\omega_t^{n-1} + \cdots + (\tilde{\omega} + t\omega_X)^{n-1}) \\
&= \int_{\partial E_\delta} e^{-\lambda \phi} i\partial \bar{\partial} \tilde{\phi} \wedge (\omega_t^{n-1} + \cdots + (\tilde{\omega} + t\omega_X)^{n-1}) + \int_{E_\delta} e^{-\lambda \phi} (\omega_t^n - (\tilde{\omega} + t\omega_X)^n) \\
&\leq \int_{\partial E_\delta} e^{-\lambda \phi} i\partial \bar{\partial} \tilde{\phi} \wedge (\omega_t^{n-1} + \cdots + (\tilde{\omega} + t\omega_X)^{n-1}) + C,
\end{align*}
\]

24
since the function $e^{-\lambda \hat{\varphi}} = e^{-\lambda \varphi} |s_D|^{2\lambda \varepsilon_0}_{h_D}$ is uniformly bounded.

We rewrite the boundary integral as follows.

$$
\int_{\partial E_\delta} e^{-\lambda \hat{\varphi}} i \bar{\partial} (\varphi - \varepsilon_0 \log |s_D|^{2}_{h_D}) \wedge (\omega_t^{n-1} + \cdots + (\hat{\varphi} + t \omega_X)^{n-1})
$$

$$
= \int_{\partial E_\delta} e^{-\lambda \varphi} |s_D|^{2\lambda \varepsilon_0-2}_{h_D} \eta \wedge (\omega_t^{n-1} + \cdots + (\hat{\varphi} + t \omega_X)^{n-1})
$$

$$
= \delta^{2\lambda \varepsilon_0-2} \int_{E_\delta} d(e^{-\lambda \varphi} \eta) \wedge (\omega_t^{n-1} + \cdots + (\hat{\varphi} + t \omega_X)^{n-1})
$$

Here $\eta = i|s_D|^{2}_{h_D} \bar{\partial} (\varphi - \varepsilon_0 \log |s_D|^{2}_{h_D})$ is a smooth $(n, n)$ form on $X$ and in the last line we have integrated by parts again. Now the last integrand is a smooth $(n, n)$ form on $X$ which is independent of $\delta$ and thus the integral remains bounded as $\delta \to 0$. On the other hand, $2\lambda \varepsilon_0 - 2 \geq 10 - 2 > 0$ so the last line tends to zero as $\delta \to 0$.

Letting $\delta \to 0$ we get from the equivalence of the fixed metrics $\omega_X$ and $\hat{\omega}$ that

$$
\int_X e^{-\lambda \hat{\varphi}} |\nabla \hat{\varphi}|_{\omega_X}^2 \omega_X^n \leq C.
$$

The first inequality in the lemma then follows from the following triangle inequality

$$
\int_X e^{-\lambda \hat{\varphi}} |\nabla \hat{\varphi}|_{\omega_X}^2 \omega_X^n \leq 2 \int_X e^{-\lambda \varphi} |\nabla \varphi|^2_{\omega_X} \omega_X^n + 2\varepsilon_0^2 \int e^{-\lambda \varphi} |s_D|^{2\varepsilon_0\lambda}_{h_D} \nabla \log |s_D|^{2}_{h_D} \omega_X^n \omega_X \omega_X^n \leq C.
$$

The second inequality follows from Hölder inequality

$$
\int_X H^{1/2} \omega_X^n \leq c_t \left( \int_X e^{(1+\epsilon)F} \omega_X^n \right)^{1/(1+\epsilon)} \left( \int_X H \omega_X^n \right)^{\epsilon/(1+\epsilon)} \leq C.
$$

The proof of Lemma 12 is completed.

We recall the gradient estimate proved in Theorem 3 for complex Monge-Ampère equations with a fixed background metric.

(4.15) \[ (\omega_X + i \bar{\partial} \partial \varphi)^n = e^F \omega_X^n, \quad \sup_X \varphi = 0. \]

The conditions on the sets $\mathcal{M}'(N, \epsilon, \gamma), \mathcal{M}''(N, \epsilon, \gamma)$ or $\tilde{\mathcal{M}}''(N, \epsilon, \gamma)$ state that

(4.16) \[ \|e^F\|_{L^{1+(\gamma, \omega_X^n)}} \leq N, \quad \text{and} \]

(4.17) \[ \sup_X e^{-F} \leq \gamma, \text{ or } \int_X (e^{-F} + |\Delta_X e^{-F}|) \omega_X^n \leq \gamma, \text{ or } \int_X (e^{-F} + |\nabla e^{-F}|^2_{\omega_X}) \omega_X^n \leq \gamma \]

A corollary of Theorem 3 on equation (4.15) states that

**Corollary 3** Fix a constant $p > n$. Let $\varphi$ solve the equation (4.15). Suppose $F$ satisfies (4.16) and (4.17), then the following gradient estimate of $\varphi$ w.r.t. $\omega_X$ holds

(4.18) \[ \sup_X |\nabla \varphi|^2_{\omega_X} \leq C, \]

where $C > 0$ depends on $n, p, \omega_X, N, \epsilon, \gamma$, and additionally $\int_X |\nabla F|^p_{\omega_X} e^F \omega_X^n$. 25
Corollary 3 follows from Theorem 3 as a particular case if we choose \( \gamma = \omega_X/2, \ t = 1/2, \ D = 0 \) (the trivial divisor), \( s_D \equiv 1 \) and \( h_D \equiv 1 \).

We remark that besides the integral bounds on \( e^{+F} \) and \( |\nabla F| \), the gradient estimate of \( \varphi \) can be made to be independent of the pointwise bounds \( \sup_X e^F \) and \( \inf_X e^F \). The exponent \( p > n \) in the \( L^p(X, e^F \omega_X^n) \)-bound of \( |\nabla F|_{\omega_X} \) is also sharp in the sense that the gradient estimate may fail if \( p < n \) and all other conditions on \( F \) are still valid, as the following example shows. We do not know whether the statement holds or not when \( p = n \).

**Example 3.1.** Let \( 0 \in C^n \subset \mathbb{CP}^n \) and \( z = (z_1, \ldots, z_n) \) be the natural coordinates on \( C^n \). Fix a number \( a \in (0, 1) \) and a small \( \delta \in (0, 1/100) \). We consider the function

\[
\varphi_\delta = \begin{cases} 
(\delta^2 + \delta)^a, & \text{if } |z| < \zeta \\
\max\{(\delta^2 + \delta)^a, 2\log(1 + |z|^2)\}, & \text{if } \zeta \leq |z| \\n2\log(1 + |z|^2), & \text{if } |z| \geq 1,
\end{cases}
\]

where \( \zeta > 0 \) is a constant (independent of \( \delta \)) such that \( (\delta^2 + \delta)^a = 2\log(1 + |z|^2) \) for some \( |z| \in (\zeta, 1) \). The metric \( \omega_\delta = i\partial\bar{\partial}\varphi_\delta \) on \( C^n \) can be naturally extended to a smooth Kähler metric on \( X = \mathbb{CP}^n \). We express the metric \( \omega_\delta \) locally near 0

\[
\omega_\delta = \sum_{i,j} a_{ij} \frac{\delta_{ij} + (a - 1)\delta_{i,j}}{|z|^2 + \delta} \sqrt{-1}dz_i \wedge d\bar{z}_j
\]

so near 0 we have

\[
\omega_\delta^n = a^n \frac{(a|z|^2 + \delta)/(|z|^2 + \delta)}{(|z|^2 + \delta)^{an}} \omega_C^n =: e^F \omega_C^n.
\]

If \( a < 1/2 \), we can choose \( 0 < p < 2an < n \). By straightforward calculations, we see that near 0

\[
(4.19) \quad |\nabla F|_{\omega_C^n}^p e^F \leq C \frac{|z|^p}{|z|^{2p+2n-2an}} = C |z|^{-p+2an-2n}
\]

and the function on the right-hand side is integrable near \( z = 0 \). Since \( \omega_C^n \) is equivalent to \( \omega_{FS} \) (the Fubini-Study metric on \( \mathbb{CP}^n \)) near 0, it follows easily that \( \int_X |\nabla F|_{\omega_{FS}}^p e^F \omega_{FS}^n \) is uniformly bounded above (independent of \( \delta > 0 \)). Moreover, the other conditions \((4.16)\) and \((4.17)\) of \( F \) are satisfied for certain \( \epsilon, N \) and \( \gamma \). Here we have viewed \( F \) as a smooth function on the whole manifold \( \mathbb{CP}^n \). However, near 0 (e.g. at \( |z|^2 = \delta \))

\[
|\nabla \varphi_\delta|_{\omega_C^n} \sim \frac{|z|}{(|z|^2 + \delta)^{1-a}} \sim \delta^{a-\frac{1}{2}}
\]

blows up as \( \delta \to 0 \). Therefore when \( p < n \), the integral \( \int_X |\nabla F|_{\omega_X}^p e^F \omega_X^n \) is not enough to conclude the gradient estimate of \( \varphi \) which satisfies \((4.15)\).
4.2 $C^2$ estimate

We consider the $C^2$ estimates in this subsection and give the proof of Theorem 4. We continue to use the same notations as in the previous subsection. Fix a number $p > 2n$.

Let $\varphi_t$ be the solution to the equation (4.1). We again omit the subscript $t$ in $\varphi_t$.

**Lemma 13** The following holds for any $t \in (0, 1]$:

\[
\Delta g \tr g \geq \Delta g F - K \tr g \cdot \tr g + g^{kl} \nabla_k \bar{g} \nabla_l \bar{g} \cdot g^{pq} \bar{g} \bar{g} \\
\geq \Delta g F - K \tr g \cdot \tr g + \frac{\|\nabla \tr g\|^2_g}{\tr g},
\]

where as before $-K$ is a lower bound of the bisectional curvature of $\bar{g} = e_{\varphi} X$, and $\nabla \bar{g}$ denotes the covariant derivatives with respect to $\bar{g}$.

Lemma 13 follows from standard calculations in [25], so we omit the proof.

**Lemma 14** For some $\mu > 0$ depending only on $\chi$, and $e_{\varphi} X$, we have

\[
\Delta_g Q \geq e^{-\mu \varphi} \Delta g F - n \mu Q,
\]

where we have written $Q = e^{-\mu \varphi} \tr g$ and $\varphi = \varphi - \epsilon_0 \log |s D|_{h_D}^2$.

**Proof.** We fix a constant $\mu > 0$ to be determined later. We calculate using Lemma 13

\[
\Delta_g Q = e^{-\mu \varphi} \Delta_g \tr g - 2 \mu e^{-\mu \varphi} \text{Re} \langle \nabla \varphi, \nabla \tr g \rangle_g + Q(- \mu \Delta_g \varphi + \mu^2 |\nabla \varphi|^2_g)
\]

\[
\geq e^{-\mu \varphi}(\Delta g F - K \tr g \cdot \tr g + \frac{\|\nabla \tr g\|^2_g}{\tr g}) - 2 \mu e^{-\mu \varphi} \text{Re} \langle \nabla \varphi, \nabla \tr g \rangle_g
\]

\[
+ \mu Q\left[ - n + t \cdot \tr g + \tr g (\chi - \epsilon_0 \text{Ric}(h_D)) \right] + \mu^2 e^{-\mu \varphi} \tr g \cdot |\nabla \varphi|^2_g.
\]

Applying Cauchy-Schwarz inequality the following holds

\[
-2 \mu e^{-\mu \varphi} \text{Re} \langle \nabla \varphi, \nabla \tr g \rangle_g \geq - \mu^2 e^{-\mu \varphi} \tr g \cdot |\nabla \varphi|^2_g - e^{-\mu \varphi} \frac{\|\nabla \tr g\|^2_g}{\tr g},
\]

plugging this into (4.21) and applying (4.2) we obtain

\[
\Delta_g Q \geq e^{-\mu \varphi} \Delta g F - n \mu Q + (\mu \delta_0 - K) e^{-\mu \varphi} \tr g \cdot \tr g
\]

\[
\geq e^{-\mu \varphi} \Delta g F - n \mu Q,
\]

if we choose $\mu > 0$ such that $\mu \delta_0 - K \geq 1$.

**Proof of Theorem 4.** By Lemma 14 and Green’s formula we have for any $x \in X$

\[
Q(x) = \frac{1}{V_t} \int_X Q \omega^n_t + \int_X G_t(x, \cdot)(-\Delta g Q) \omega^n_t
\]

\[
\leq \frac{1}{V_t} \int_X Q \omega^n_t + \int_X G_t(x, \cdot)(-e^{-\mu \varphi} \Delta g F + n \mu Q) \omega^n_t.
\]
Let $x_0$ be a maximum point of $Q$. We apply (4.22) at the point $x_0$, then

$$Q_{\text{max}} = \frac{1}{V_t} \int_X Q \omega_t^n + \int_X \mathcal{G}_t(x_0, \cdot)(- e^{-\mu \hat{\varphi}} \Delta_q F + n \mu Q) \omega_t^n$$

(4.23)\[ \leq \frac{Q_{\text{max}}^{1-\eta}}{V_t} \int_X Q^n \omega_t^n + n \mu Q_{\text{max}}^{1-\eta} \int_X \mathcal{G}_t(x_0, \cdot) Q^n \omega_t^n + c_t \int_X \mathcal{G}_t(x_0, \cdot)(- e^{-\mu \hat{\varphi}} \Delta_q F) e^F \omega_t^n.

We observe that by Hölder inequality

$$\int_X Q^{1+\epsilon} \omega_t^n = c_t \int_X Q^{1+\epsilon} e^F \omega_t^n \leq c_t \|e^F\|_{L^{1+\epsilon}} \left( \int_X e^{-\mu \hat{\varphi}} (\text{tr}_X \omega_t) \omega_t^n \right)^{(1+\epsilon)/\epsilon} \leq C,$$

since $e^{-\mu \hat{\varphi}} = e^{-\mu \varphi} |s_D| H_D^{1/2} \leq C$. By Hölder inequality and (4.24), the integral in the first term in (4.23) is bounded if $0 < \eta \leq \frac{\epsilon}{1+\epsilon}$. The integral in the second term in (4.23) satisfies

$$\int_X \mathcal{G}_t(x_0, \cdot) Q^n \omega_t^n \leq \left( \int_X \mathcal{G}_t(x_0, \cdot)^{p_0} \omega_t^n \right)^{1/p_0} \left( \int_X Q^{p_0 \eta} \omega_t^n \right)^{1/p_0} \leq C$$

if we choose $p_0 \in (1, \frac{n}{n-1})$ and $\eta > 0$ small so that $p_0^* \eta \leq \frac{\epsilon}{1+\epsilon}$. The inequality above then follows from (4.24) and Theorem 2.

To deal with the last integral in (4.23), we apply integration by parts to obtain

$$2c_t \int_X \mathcal{G}_t(x_0, \cdot) e^{-\mu \hat{\varphi}} |\nabla F|_{g}^2 e^F \leq C \left( \int_X \mathcal{G}_t(x_0, \cdot)^{(p/2)^*} \omega_t^n \right)^{1/(p/2)^*} \left( \int_X |\nabla F|_{g}^{p/2} e^F \omega_t^n \right)^{2/p}$$

which is bounded above uniformly since $(p/2)^* = \frac{p/2}{p/2-1} < \frac{n}{n-1}$; we can apply Theorem 2 and the assumption on $F$ in Theorem 4. The second integral in (4.25) is also bounded because of Theorem 2 and $e^{-\mu \hat{\varphi}} |\nabla \hat{\varphi}|_{g}^2 \leq C$ which follows from Theorem 3 (we may assume $\mu > A$, where $A > 0$ is the constant in Theorem 3). We now deal with the last integral in (4.25). We calculate

$$\int_X e^{-\mu \hat{\varphi}} |\nabla F|_{g} |\nabla \mathcal{G}_t(x_0, \cdot)|_g e^F \omega_t^n \leq \int_X e^{-\mu \hat{\varphi}} |\nabla F|_{g} (\text{tr}_g g)^{\frac{1}{2}} |\nabla \mathcal{G}_t(x_0, \cdot)|_g e^F \omega_t^n$$

$$= \int_X e^{-\mu \hat{\varphi}/2} |\nabla F|_{g} Q^{\frac{1}{2}} |\nabla \mathcal{G}_t(x_0, \cdot)|_g e^F \omega_t^n$$

$$\leq C Q_{\text{max}}^{1/2} \left( \int_X |\nabla F|_{g}^{p/2} e^F \omega_t^n \right)^{1/p} \cdot \left( \int_X |\nabla \mathcal{G}_t(x_0, \cdot)|_g^{p^*} \omega_t^n \right)^{1/p^*}$$

$$\leq C Q_{\text{max}}^{1/2}.$$

28
where we have used $p^* = \frac{p}{p-1} < \frac{2n}{2n-1}$, so the integral on $L^{p^*}(X, \omega^n)$ of $|\nabla G_t(x_0, \cdot)|_g$ is bounded uniformly by Theorem 2. Plugging these into (4.23), we get

$$Q_{\text{max}} \leq CQ_{\text{max}}^{1-\eta} + CQ_{\text{max}}^{1/2} + C,$$

from which we conclude $Q_{\text{max}} \leq C$ by Young’s inequality, that is

$$\sup_X (e^{-\mu \varphi} s_D^{2\mu \epsilon}) \leq C.$$

The proof of Theorem 4 is complete by noting that $e^{-\mu \varphi}$ is uniformly bounded.

We note that Theorem 4 also applies to the complex Monge-Ampère equations with a fixed background metric.

**Corollary 4** Let $\varphi$ be the solution to (4.15). Suppose $F$ satisfies the conditions (4.16) and (4.17). Given $p > 2n$, the following holds

$$\sup_X |i\partial \bar{\partial} \varphi|_{\omega_X}^2 \leq C,$$

for some constant $C > 0$ depending only $n, p, \omega_X, \epsilon, N, \gamma$ and $\int_X |\nabla F|_{\omega_X}^p \omega_X^n$.

The example below shows that exponent $p > 2n$ in Corollary 4 is also sharp since the estimate may not hold if $p < 2n$, even when other conditions are valid for $F$. The case when $p = 2n$ is not clear to us.

**Example 3.2.** We take the same metrics and notations as in Example 3.1. Fix an $a > 0$ close but smaller than 1. We can pick $1 < p < 2an - 2n$, and near $0 \in \mathbb{C}^n$, (4.19) tells that

$$|\nabla F|_{\omega_X}^p e^F \leq C|z|^{-p+2an-2n},$$

which shows that the $L^1(\mathbb{C}P^n, \omega_X^n)$-norm of $(|\nabla F|_{\omega_X}^p e^F)$ is uniformly bounded (i.e. independent of $\delta > 0$), while conditions (4.16) and (4.17) on $F$ hold for some $\epsilon, N, \gamma$ which are independent of $\delta$. However at the points $z$ with $|z|^2 = \delta$

$$\text{tr}_{\omega_X} (i\partial \bar{\partial} \varphi_\delta) \sim \delta^{a-1} \to \infty \quad \text{as} \ \delta \to 0.$$

Finally we mention an application of Corollary 4 to the regularity of complex Monge-Ampère equations when $e^F$ satisfies weaker regularity than being $C^2$. For this we need a theorem from [24]. Corollary 5 below may be known to experts, but we cannot find a reference in the literature, so we include the statement and a sketched proof.

**Corollary 5** Let $\varphi$ be the solution to (4.15) with $F$ a Lipschitz function (i.e. $|F(x) - F(y)| \leq Ld_{\omega_X}(x, y)$ for some $L > 0$). Then there is an $\alpha \in (0, 1)$ depending on $n, \omega_X$ such that

$$\|\varphi\|_{C^{2, \alpha}(X, \omega_X)} \leq C,$$

for some constant $C > 0$ depending only $n, \omega_X$, and $L$. 

29
Proof of Corollary 5. By smoothing out $F$ and taking limits if necessary, we may assume $F$ is a smooth function with Lipschitz constant $\leq L$. Since $\int_X e^{F} \omega_X^n = V$, and $F$ is Lipschitz, it follows that sup $e^{F}$ and inf $e^{F} > 0$ are both bounded depending on $\omega_X$ and $L$, and so is the $L^p(e^{F} \omega_X^n)$-norm of $|\nabla F|\omega_X$. It then follows from the $C^2$ estimate in Corollary 4 and the equation (4.15) that $\omega_X + i\partial\bar{\partial}\varphi$ is equivalent to $\omega_X$. We can now invoke the main theorem in [24] to conclude the proof of Corollary 5.

4.3 $C^3$ estimates

To keep the notations lighter, we only consider the $C^3$ estimates for the equation (4.15) with a fixed background metric. We follow closely the approach in [18]. We continue to denote $\tilde{g}$ and $g$ the associated metrics of $\omega_X$ and $\omega = \omega_X + i\partial \bar{\partial} \varphi$, respectively. Furthermore we assume there is a $\theta > 1$ such that

$$\theta^{-1}g \leq \tilde{g} \leq \theta g. \quad (4.26)$$

By the $C^2$ estimates in Corollary 4, (4.26) holds with $\theta$ depending additionally on inf $e^{F}$. We remark that Theorem 5 below has been known with the constant $C > 0$ depending on $n,p,\omega_X$ (cf. [18, 25]).

**Theorem 5** Fix $p > 2n$. The following estimate holds:

$$\sup_X |\nabla g_i \partial \bar{\partial} \varphi|^2 \leq C$$

for some $C > 0$ depending on $n,p,\theta,\omega_X, \int_X |\nabla F|^p \omega_X^n$ and $\int_X |i\partial \bar{\partial} F|^p \omega_X^n$.

**Proof of Theorem 5.** As in [18], we denote

$$S_{jk}^i = \Gamma^i_{jk} - \tilde{\Gamma}^i_{jk}$$

to be the difference of the Christoffel symbols of $g$ and $\tilde{g}$. Note that $S_{jk}^i$ is indeed a tensor. We write

$$|S|^2_g = S_{jk}^i S_{pq}^{ik} g^{jk} g^{pq}.$$

Under the assumption (4.26), it is easy to see $|S|^2_g$ is equivalent to $|\nabla g_i \partial \bar{\partial} \varphi|^2$, so it suffices to estimate $|S|^2_g$. By the calculations in [18], we have

$$\Delta_g |S|^2_g = |\nabla S|^2_g + |\nabla \tilde{S}|^2_g - 2Re(S_{jk}^i R_{ip}^{kj} g^{ij} g^{kl})$$

$$+ S * S * \text{Ric}(g) + S * S * \text{Rm}(\tilde{g}) + S * \nabla \tilde{g}^\ast \text{Ric}(\tilde{g}), \quad (4.27)$$

where $T * S$ means certain linear combinations of the tensors $T$ and $S$ contracted by $g$ or $\tilde{g}$, and $R_{ip}^{kj}$ denotes the covariant derivative of $\text{Ric}(g)_{ip}$ with respect to $g$. From the equation (4.15), we have $\text{Ric}(g) = \text{Ric}(\tilde{g}) - i\partial \bar{\partial} F$. Let $G$ be the Green’s function of $g$ and
as before $G$ be the positive Green’s function defined in (3.15). By Green’s formula, for any $x \in X$, we have

$$
|S|^2_g(x) - \frac{1}{V} \int_X |S|^2_g \omega^n = \int_X G(x, \cdot) \left( - |\nabla S|^2_g - |\nabla S|^2_g + 2 \text{Re}(\overline{S^j_k} R_{ip,q} g^{pj} g^{ik}) \
+ S * S * \text{Ric}(g) + S * S * Rm(\tilde{g}) + S * \nabla \tilde{g} \text{Ric}(\tilde{g}) \right) \omega^n
$$

$$
\leq \int_X G(x, \cdot) \left( - |\nabla S|^2_g - |\nabla S|^2_g + C|S|^2_g \right) \omega^n + \int_X 2|\nabla G||S|_g |\text{Ric}|_g \omega^n
$$

$$
(4.28) \quad \leq C \int_X G(x, \cdot) \left( |S|^2_g + |S|^2_g i\partial \bar{\partial} F|_g + |i\partial \bar{\partial} F|^2|_g + 1 \right) \omega^n + \int_X 2|\nabla G||S|_g |\text{Ric}|_g \omega^n,
$$

where in the last line we apply Cauchy-Schwarz inequality. Recall we assume a bound on $\int_X |i\partial \bar{\partial} F|_g^p \omega^n$ for $p > 2n$. We integrate (4.20) against $\omega^n$ over $X$. We obtain by Hölder inequality

$$
(4.29) \quad \int_X |S|^2_g \omega^n \leq C + \int_X |\Delta \tilde{g} F| \omega^n \leq C + \int_X |i\partial \bar{\partial} F|_g \omega^n \leq C.
$$

Let $x_0 \in X$ be a maximum point of the function $|S|^2_g$ and denote $M = |S|^2_g(x_0)$. We apply (4.28) at $x_0$. It follows that (denote $q = p/2 > n$)

$$
M \leq C + CM^{1-\eta} \left( \int \mathcal{G}^q \right)^{1/q} \left( \int |S|^{2q} \right)^{1/q} + C \left( \int \mathcal{G}^q \right)^{1/q} \left( \int |i\partial \bar{\partial} F|_g^p \right)^{1/p} + CM^{1/2}
$$

$$
+ CM^{1/2} \left( \int |\nabla \mathcal{G}^q | \right)^{1/\rho} \left( \int |i\partial \bar{\partial} F|_g^p \right)^{1/p} + CM^{1-\eta} \left( \int \mathcal{G}^{\frac{4n}{2n-1}} \right)^{\frac{2n-1}{2n}} \left( \int |i\partial \bar{\partial} F|_g^p \right)^{\frac{1}{p}} \left( \int |S|^{2\eta b} \right)^{\frac{1}{b}}
$$

where $b = (p - 2n)/2np$. We can choose $\eta > 0$ such that $\max(2n\eta, 2nb) = 2$. All the integrals above are bounded, due to (4.29) and Theorem 2. Hence we have

$$
M \leq C + CM^{1-\eta} + CM^{1/2},
$$

from which we conclude $M \leq C$. This finishes the proof of Theorem 5.

References

[1] Z. Blocki, “A gradient estimate in the Calabi-Yau theorem”, Math. Ann. 344 (2009), no. 2, 317 - 327.

[2] X. Chen and W. He, “The complex Monge-Ampère equation on compact Kähler manifolds”, Math. Ann. 354 (2012), 1583 - 1600.

[3] S. Y. Cheng and P. Li, “Heat kernel estimates and lower bound of eigenvalues”, Comment. Math. Helv., 56 (1981), 327-338.

[4] J.P. Demailly and N. Pali, “Degenerate complex Monge-Ampère equations over compact Kähler manifolds”, Intern. J. Math. 21 (2010) no. 3, 357–405.
[5] P. Eyssidieux, V. Guedj, and A. Zeriahi, “A priori $L^\infty$-estimates for degenerate complex Monge-Ampère equations”. International Mathematics Research Notices 2008 (2008)

[6] X. Fu, B. Guo and J. Song, “Geometric estimates for complex Monge-Ampère equations”, J. Reine Angew. Math. 765 (2020), 69–99

[7] V. Guedj and H.C. Lu, “Quasi-plurisubharmonic envelopes 1: Uniform estimates on Kähler manifolds”, arXiv:2106.04273

[8] V. Guedj and H.C. Lu, “Quasi-plurisubharmonic envelopes 3: Solving Monge-Ampère equations on hermitian manifolds”, arXiv:2107.01938

[9] B. Guo, D.H. Phong, and F. Tong, “On $L^\infty$ estimates for complex Monge-Ampère equations”, preprint, arXiv:2106.02224

[10] B. Guo, D.H. Phong, and F. Tong, “A new gradient estimate for the complex Monge-Ampère equation”, preprint, arXiv:2106.03308

[11] B. Guo, D.H. Phong, and F. Tong, “Stability estimates for the complex Monge-Ampère and Hessian equations”, preprint, arXiv:2106.03913

[12] B. Guo, D.H. Phong, F. Tong, and C. Wang, “On $L^\infty$ estimates for Monge-Ampère and Hessian equations on nef classes”, preprint, arXiv:2111.14186

[13] B. Guo, D.H. Phong, F. Tong, and C. Wang, “On the modulus of continuity of solutions to complex Monge-Ampère equations”, preprint, arXiv:2112.02354

[14] B. Guo and J. Song, “Local noncollapsing for complex Monge-Ampère equations”, arXiv:2201.02930

[15] L. Hörmander, “An introduction to complex analysis in several variables”, Van Nostrand, Princeton, NJ, 1973

[16] S. Kołodziej, “The complex Monge-Ampère equation”, Acta Math. 180 (1998) 69–117

[17] O. Munteanu and J. Wang, “Comparison theorems for three-dimensional manifolds with scalar curvature bound”, preprint, arXiv:2105.12103

[18] D.H. Phong, N. Sesum, and J. Sturm, “Multiplier ideal sheaves and the Kähler-Ricci flow”, Comm. Anal. Geom. 15 (2007), no. 3, 613 - 632.

[19] D.H. Phong, J. Song, and J. Sturm, “Complex Monge-Ampère equations”, arXiv:1209.2203, Surveys in Differential Geometry, vol. 17, 327-411 (2012).
[20] D.H. Phong and J. Sturm, “The Dirichlet problem for degenerate complex Monge-Ampère equations”, Comm. Anal. Geom. 18 (2010), no. 1, 145 - 170.

[21] R. Schoen and S.T. Yau, “Lectures on differential geometry”, Vol. 2. Cambridge, MA: International press, 1994

[22] G. Tian, “On Kähler-Einstein metrics on certain Kähler manifolds with $C_1(M) > 0$”, Invent. Math. 89 (1987), no. 2, 225–246

[23] H. Tsuji, “Existence and degeneration of Kähler-Einstein metrics on Minimal Algebraic Varieties of General Type”, Math. Ann. 281 (1988) 123-133.

[24] Y. Wang, “On the $C^{2,\alpha}$ regularity of the complex Monge-Ampère equation”, Math. Res. Lett. 19 (2012), no.4, 939 - 946

[25] S.T. Yau, “On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I”, Comm. Pure Appl. Math. 31 (1978) 339–411

Department of Mathematics & Computer Science, Rutgers University, Newark, NJ 07102
bguo@rutgers.edu

Department of Mathematics, Columbia University, New York, NY 10027
phong@math.columbia.edu

Department of Mathematics & Computer Science, Rutgers University, Newark, NJ 07102
sturm@andromeda.rutgers.edu