Some Aspects of Quantum Entanglement for CAR Systems

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Abstract

We show some distinct features of quantum entanglement for bipartite CAR systems such as the failure of triangle inequality of von Neumann entropy and the possible change of our entanglement degree under local operations. Those are due to the nonindependence of CAR systems and never occur in any algebraic independent systems. We introduce a new notion half-sided entanglement.

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1 Introduction

In this Letter, we investigate bipartite CAR systems from the viewpoint of quantum entanglement. Quantum entanglement of states refers to the quantum correlations at separated regions which cannot be reduced to the classical (probability) theory.

A pair of quantum systems $A$ and $B$ are given. Here $A$ and $B$ are the algebras representing quantum subsystems. For the usual cases which have been studied extensively in quantum information theory, these $A$ and $B$ are assumed to be algebraically independent, that is, $A$ and $B$ commute elementwise and the total system $C$ is given by the tensor product $A \otimes B$. We are interested in quantum entanglement between pairs of subsystems which are coupled by different kind of algebraic relations other than the tensor product.

We treat a bipartite CAR system which is a typical example of nonindependent systems. Let $\mathcal{A}_{1,2}^{\text{car}}$ and $\mathcal{A}_{2}^{\text{car}}$ be finite-dimensional CAR systems representing a pair of disjoint subsystems. The total system $\mathcal{A}_{1,2}^{\text{car}}$ is given by $\mathcal{A}_{1,2}^{\text{car}} \vee \mathcal{A}_{2}^{\text{car}}$, the algebra algebraically generated by $\mathcal{A}_{1}^{\text{car}}$ and $\mathcal{A}_{2}^{\text{car}}$. For the sake of simplicity, we consider a spinless one-particle Fermion in each subsystem (i.e., one degree of freedom for each region).

The bipartite CAR pair $(\mathcal{A}_{1,2}^{\text{car}}, \mathcal{A}_{2}^{\text{car}})$ are not algebraically independent. Furthermore, we show that $(\mathcal{A}_{1,2}^{\text{car}}, \mathcal{A}_{2}^{\text{car}})$ are not statistically independent.

We give an entanglement degree which makes sense for any pair of finite-dimensional subsystems $(\mathcal{A}, \mathcal{B})$ of a general $\mathbb{C}^*$-algebra $\mathcal{C}$ which represents the total system. (In [1], other kind of entanglement degrees are defined for general finite-dimensional algebraically independent pairs $(\mathcal{A}, \mathcal{B})$ sitting in $\mathcal{C}$.)

There is some difference on quantum entanglement between the CAR systems and general tensor-product systems. We show that the triangle inequality of von Neumann entropy does not hold for the bipartite CAR systems.
We introduce a new notion which we call \textit{half-sided entanglement} in terms of the asymmetry of marginal entropies. For tensor-product systems, it is trivially 0 for any state, but it can take strictly positive value for CAR systems. We compute the degree of the half-sided entanglement for some states of the bipartite CAR system.

We study how \textit{local} operations on a half-sided region affect the quantum entanglement for the bipartite CAR system. We show that the local automorphisms can change the entanglement degree in the bipartite CAR system in contrast to the tensor-product systems.

2 Preliminaries

2.1 Bipartite CAR Systems

Let \( a_i^* \) and \( a_i \) be creation and annihilation operators, respectively, satisfying the canonical anticommutation relations (CAR):

\[
\{a_i^*, a_j\} = \delta_{i,j} \ 1, \quad \{a_i^*, a_j^*\} = \{a_i, a_j\} = 0, \tag{1}
\]

where \( \{A, B\} = AB + BA \) (anticommutator), \( i, j = 1 \) or 2, \( \delta_{i,j} = 1 \) for \( i = j \) and \( \delta_{i,j} = 0 \) for \( i \neq j \).

Let \( \mathcal{A}_{1,2}^{\text{car}} \) be a \( C^* \)-algebra generated by \( \{a_i^*, a_i\mid i = 1, 2\} \). Let \( \mathcal{A}_{1}^{\text{car}} \) be a \( C^* \)-subalgebra of \( \mathcal{A}_{1,2}^{\text{car}} \) generated by \( a_1^* \) and \( a_1 \), and \( \mathcal{A}_{2}^{\text{car}} \) be a \( C^* \)-subalgebra generated by \( a_2^* \) and \( a_2 \). Each \( \mathcal{A}_i^{\text{car}} \) is imbedded in \( \mathcal{A}_{1,2}^{\text{car}} \) and \( \mathcal{A}_{1}^{\text{car}} \lor \mathcal{A}_{2}^{\text{car}} = \mathcal{A}_{1,2}^{\text{car}} \).

We define

\[
e_1^{1,1} \equiv a_1^* a_1, \quad e_1^{1,2} \equiv a_1^* \quad e_1^{2,1} \equiv a_1, \quad e_1^{1,2} \equiv a_1^* a_1^*, \tag{2}
\]

and

\[
e_2^{1,1} \equiv a_2^* a_2, \quad e_2^{1,2} \equiv a_2^* \quad e_2^{2,1} \equiv a_2, \quad e_2^{2,2} \equiv a_2 a_2^*. \tag{3}
\]

Then \( \{e_{i,j}^{1}\}_{i,j} \) is a system of matrix units of \( \mathcal{A}_{1}^{\text{car}} \) (\( \cong M_2(\mathbb{C}) \)), and \( \{e_{i,j}^{2}\}_{i,j} \) is that of \( \mathcal{A}_{2}^{\text{car}} \) (\( \cong M_2(\mathbb{C}) \)). Let \( \mathcal{A}_{2}^{\text{pin}} \) be a relative commutant of \( \mathcal{A}_i^{\text{car}} \) in \( \mathcal{A}_{1,2}^{\text{car}} \), that is,

\[
\mathcal{A}_{2}^{\text{pin}} \equiv \{\mathcal{A}_i^{\text{car}}\}' \cap \mathcal{A}_{1,2}^{\text{car}}.
\]

We also define \( \mathcal{A}_i^{\text{spin}} \equiv \{\mathcal{A}_2^{\text{car}}\}' \cap \mathcal{A}_{1,2}^{\text{car}} \).

The algebraic extension of the map

\[
\Theta(a_i^*) = -a_i^*, \quad \Theta(a_i) = -a_i \quad (i = 1, 2) \tag{4}
\]

is a \( * \)-automorphism of \( \mathcal{A}_{1,2}^{\text{car}} \) and will be denoted by the same symbol \( \Theta \). The even and odd parts of \( \mathcal{A}_{1,2}^{\text{car}} \) are given by

\[
\mathcal{A}_{1,2}^{\text{car}}_+ \equiv \{A \in \mathcal{A}_{1,2}^{\text{car}} \mid \Theta(A) = A\}, \quad \mathcal{A}_{1,2}^{\text{car}}_- \equiv \{A \in \mathcal{A}_{1,2}^{\text{car}} \mid \Theta(A) = -A\}.
\]

In the same way, we define

\[
\mathcal{A}_{1}^{\text{car}}_+ \equiv \{A \in \mathcal{A}_{1}^{\text{car}} \mid \Theta(A) = A\}, \quad \mathcal{A}_{1}^{\text{car}}_- \equiv \{A \in \mathcal{A}_{1}^{\text{car}} \mid \Theta(A) = -A\},
\]

and

\[
\mathcal{A}_{2}^{\text{car}}_+ \equiv \{A \in \mathcal{A}_{2}^{\text{car}} \mid \Theta(A) = A\}, \quad \mathcal{A}_{2}^{\text{car}}_- \equiv \{A \in \mathcal{A}_{2}^{\text{car}} \mid \Theta(A) = -A\}.
\]
2.2 Pure States and their Marginals

Let \( \mathcal{A}^\text{car}_2 = \{ A \in \mathcal{A}^\text{car}_2 | \Theta(A) = A \} \), \( \mathcal{A}^\text{car}_2^- = \{ A \in \mathcal{A}^\text{car}_2 | \Theta(A) = -A \} \).

We introduce the so-called Klein-Wigner transformation on \( \mathcal{A}^\text{car}_2 \) as
\[
a_2^* \mapsto U_1 a_2^* = b_2^*, \quad a_2 \mapsto U_1 a_2 = b_2,
\]
where \( U_1 \equiv a_1^* a_1 - a_1 a_1^* (\in \mathcal{A}^\text{car}_1) \). In the same manner, we introduce the Klein-Wigner transformation on \( \mathcal{A}^\text{car}_1 \) as
\[
a_1^* \mapsto U_2 a_1^* = b_1^*, \quad a_1 \mapsto U_2 a_1 = b_1,
\]
where \( U_2 \equiv a_2^* a_2 - a_2 a_2^* (\in \mathcal{A}^\text{car}_2) \).

Obviously, \( U_1 U_2^* = \mathbf{1}, U_1 U_2^* = \mathbf{1}, \) and \( U_i \in \mathcal{A}^\text{car}_i \). It follows from (5) that \( b_1^* \) and \( b_1 \) satisfy the CAR:
\[
\{ b_i^*, b_j \} = \delta_{i,j} \mathbf{1}, \quad \{ b_i^*, b_i^* \} = \{ b_i, b_j \} = 0 \quad (i, j = 1, 2).
\]

It is easy to see that \( \mathcal{A}^\text{spin}_1 \) is algebraically generated by \( b_1^* \) and \( b_1 \), and is isomorphic to \( \mathcal{A}^\text{car}_1 \) by (6) and (7) for \( i = 1, 2 \). We have
\[
\mathcal{A}^\text{car}_{1,2} = \mathcal{A}^\text{car}_1 \otimes \mathcal{A}^\text{spin}_2 = \mathcal{A}^\text{spin}_1 \otimes \mathcal{A}^\text{car}_2.
\]

From now on, we shall be mainly concerned with the former tensor product structure \( \mathcal{A}^\text{car}_{1,2} \). We express \( \mathcal{A}^\text{car}_{1,2} \) as
\[
\mathcal{A}^\text{car}_{1,2} = M_4(\mathbb{C}) = M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \quad \text{on} \quad \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2,
\]
where \( M_2(\mathbb{C}) \otimes \mathbf{1} \) is identified with the subsystem \( \mathcal{A}^\text{car}_1 \), and \( \mathbf{1} \otimes M_2(\mathbb{C}) \) with \( \mathcal{A}^\text{spin}_2 \). Then \( \mathcal{A}^\text{car}_1 \) acts on \( \mathcal{H}_1 (\cong \mathbb{C}^2) \) while \( \mathcal{A}^\text{spin}_2 \) acts on \( \mathcal{H}_2 (\cong \mathbb{C}^2) \); they are algebraically independent.

We note that
\[
\epsilon_{(1,1)}^{(2,\text{spin})} = b_2^* b_2, \quad \epsilon_{(1,2)}^{(2,\text{spin})} = b_2^*, \quad \epsilon_{(2,1)}^{(2,\text{spin})} = b_2, \quad \epsilon_{(2,2)}^{(2,\text{spin})} = b_2 b_2^*,
\]
give a system of matrix units of \( \mathcal{A}^\text{spin}_2 \). It is easy to see
\[
\epsilon_{(1,1)}^{(2,\text{spin})} = a_2^* a_2, \quad \epsilon_{(1,2)}^{(2,\text{spin})} = U_1 a_2^*, \quad \epsilon_{(2,1)}^{(2,\text{spin})} = U_1 a_2, \quad \epsilon_{(2,2)}^{(2,\text{spin})} = a_2 a_2^*.
\]

Let \( \xi_1^1 \) be an eigenvector of \( \epsilon_{(1,1)}^{(1,1)} \) in \( \mathcal{H}_1 \) belonging to the eigenvalue 1, and let \( \xi_1^2 \equiv a_1 \xi_1^1 \). Then \( \{ \xi_1^1, \xi_1^2 \} \) is a CONS of \( \mathcal{H}_1 \). Let \( \xi_2^1 \) be an eigenvector of \( \epsilon_{(2,2)}^{(2,2)} \) in \( \mathcal{H}_2 \) belonging to the eigenvalue 1, and let \( \xi_2^2 \equiv b_2 \xi_2^1 \). Then \( \{ \xi_1^1, \xi_2^1 \} \) is a CONS of \( \mathcal{H}_2 \). We denote \( \xi_{i,j} \equiv \xi_{i} \otimes \xi_{j} (\in \mathcal{H}) \) by \( \xi_{i,j} \). Then \( \{ \xi_{i,j} \}_{i,j=1,2} \) is a CONS of \( \mathcal{H} \). They are fixed once and for all, but their choice is not essential for all our discussions.

2.2 Pure States and their Marginals

Let \( \rho \) be an arbitrary pure state of \( \mathcal{A}^\text{car}_{1,2} \). It is represented by a unit vector \( \xi \) in \( \mathcal{H} \), \( \xi \) being fixed by \( \rho \) up to a phase factor. For \( A \in \mathcal{A}^\text{car}_{1,2} \), its expectation value is given by
\[
\rho(A) = (A \xi, \xi)_{\mathcal{H}}.
\]
This $\xi$ can be decomposed by the CONS $\{\xi_{i,j}\}$ as

$$\xi = \sum_{i,j=1,2} c_{i,j} \xi_{i,j}, \quad (10)$$

where $c_{i,j} \in \mathbb{C}$. Due to $\|\xi\| = 1$,

$$\sum_{i,j} |c_{i,j}|^2 = 1$$

We calculate the density matrices of the reduced states of $\rho$ to subsystems of $A_{1,2}^{\text{car}}$. $\rho|_{A_{1}^{\text{car}}}$ and $\rho|_{A_{2}^{\text{car}}}$ The restrictions of $\rho$ to $A_{1}^{\text{car}}$ and to $A_{2}^{\text{car}}$ have the following density matrices:

$$\rho|_{A_{1}^{\text{car}}} = \begin{pmatrix} |c_{1,1}|^2 + |c_{1,2}|^2 & c_{1,1}c_{2,1}^* + c_{1,2}c_{2,2}^* \\ c_{1,1}c_{2,1} + c_{1,2}c_{2,2} & |c_{2,1}|^2 + |c_{2,2}|^2 \end{pmatrix}, \quad (11)$$

$$\rho|_{A_{2}^{\text{car}}} = \begin{pmatrix} |c_{2,1}|^2 + |c_{2,2}|^2 & c_{1,1}c_{1,2}^* + c_{2,1}c_{2,2}^* \\ c_{1,1}c_{2,1} + c_{2,1}c_{2,2} & |c_{1,1}|^2 + |c_{1,2}|^2 \end{pmatrix}, \quad (12)$$

where the $(i, j)$ element in $[11]$ is given by the expectation value of $e_{(i,j)}^1$ in the state $\rho$, while the $(i, j)$ element in $[12]$ is given by that of $e_{(j,i)}^{2(\text{spin})}$.

Furthermore, the density matrix of $\rho$ restricted to $A_{2}^{\text{car}}$ is given by

$$\rho|_{A_{2}^{\text{car}}} = \begin{pmatrix} |c_{1,1}|^2 + |c_{2,1}|^2 & c_{1,1}c_{1,2}^* - c_{2,1}c_{2,2}^* \\ c_{1,1}c_{1,2} - c_{2,1}c_{2,2} & |c_{1,1}|^2 + |c_{1,2}|^2 \end{pmatrix}, \quad (13)$$

where the $(i, j)$ element is the expectation value of $e_{(j,i)}^{2}$ in the state $\rho$.

### 3 Failure of Triangle Inequality of von Neumann Entropy

Let $\mathcal{H}$ be an $n$-dimensional Hilbert space and $\mathcal{L}(\mathcal{H})$ be the set of linear operators on $\mathcal{H}$. The von Neumann entropy of a state $\omega$ on $\mathcal{L}(\mathcal{H})$ is given as usual by

$$S(\omega) \equiv -\text{Tr}(D_\omega \log D_\omega),$$

where $\text{Tr}$ is the matrix trace which takes the value 1 on each minimal projection and $D_\omega$ denotes the density matrix of $\omega$ with respect to $\text{Tr}$. It is well-known and easy to see that $\log n \geq S(\omega) \geq 0$, $S(\omega) = \log n$ if and only if $\omega$ is the (unique) tracial state $\tau(\cdot) = \frac{1}{n}\text{Tr}(\cdot)$ of $\mathcal{L}(\mathcal{H})$, and $S(\omega) = 0$ if and only if $\omega$ is a pure state of $\mathcal{L}(\mathcal{H})$.

We introduce the so-called triangle inequality of von Neumann entropy in a general setting. Let $A$ and $B$ be a pair of subalgebras of a finite-dimensional $\mathbb{C}^*$-algebra $C$. Let $\omega$ be a state of $C$. Let $\omega_A$ and $\omega_B$ be its restrictions to $A$ and $B$, respectively. The following entropy inequality is referred to as the triangle inequality:

$$|S(\omega_A) - S(\omega_B)| \leq S(\omega). \quad (14)$$
For any finite-dimensional bipartite tensor-product system where $A = \mathcal{L}(H_A) \otimes 1_B$ and $B = 1_A \otimes \mathcal{L}(H_B)$, and $C = A \otimes B$, the above inequality holds for any state $\omega$ of $C$.

We now give a counterexample of the triangle inequality for our CAR system where $A_{\text{car}}^1$ and $A_{\text{car}}^2$ are $A$ and $B$, respectively, and $A_{\text{car}}^{1,2}$ is $C$ in the above formula.

If we take $c_{i,j} = \frac{1}{2}$ for all $i, j$ in (10), then the (pure) state of the total system $A_{\text{car}}^{1,2}$ is uniquely determined by (9) and will be denoted by $\hat{\rho}$. By substituting $\frac{1}{2}$ into each $c_{i,j}$ in (11), (12), and (13), we have the following explicit formulae for the density matrices:

$$\hat{\rho}|_{A_{\text{car}}^1} = \left( \begin{array}{ccc} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{array} \right),$$

(15)

$$\hat{\rho}|_{A_{\text{spin}}^2} = \left( \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{array} \right),$$

(16)

and

$$\hat{\rho}|_{A_{\text{car}}^2} = \left( \begin{array}{ccc} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{array} \right).$$

(17)

Therefore $\hat{\rho}|_{A_{\text{car}}^1}$ and $\hat{\rho}|_{A_{\text{spin}}^2}$ are pure states with entropy 0 and $\hat{\rho}|_{A_{\text{car}}^2}$ is a tracial state with the maximal entropy $\log 2$. Hence we obtain

$$\log 2 = \left| S(\hat{\rho}|_{A_{\text{car}}^1}) - S(\hat{\rho}|_{A_{\text{car}}^2}) \right| > S(\hat{\rho}) = 0,$$

(18)

yielding a counterexample of the triangle inequality of von Neumann entropy.

**Remark 1** The so-called strong subadditivity (SSA) of von Neumann entropy (which was proved for tensor-product systems [8]) is also shown to hold for our CAR systems [2]. This result will be used in the proof of Proposition 1.

### 4 Nonindependence of CAR Systems

#### 4.1 States with Pure Marginal States

We show a formula of states on $A_{\text{car}}^{1,2}$ such that their restrictions to $A_{\text{car}}^1$ and to $A_{\text{car}}^2$ are both pure states.

**Proposition 1** Let $\omega$ be a state of $A_{\text{car}}^{1,2}$. Suppose that its restrictions to $A_{\text{car}}^1$ and to $A_{\text{car}}^2$ are both pure states. Then $\omega$ is a pure state of $A_{\text{car}}^{1,2}$ and has the following product property over $A_{\text{car}}^1$ and $A_{\text{spin}}^2$:

$$\omega(AB) = \omega(A)\omega(B),$$

(19)

for every $A \in A_{\text{car}}^1$ and $B \in A_{\text{spin}}^2$. The restriction of $\omega$ to $A_{\text{spin}}^2$ is a pure state.

**Proof.** Let $\omega_1$ be the restriction of $\omega$ to $A_{\text{car}}^1$ and $\omega_2$ be that to $A_{\text{car}}^2$. By the assumption that $\omega_1$ and $\omega_2$ are pure states, both von Neumann entropies vanish:

$$S(\omega_1) = S(\omega_2) = 0$$

(20)
It follows from (20) and the subadditivity property of entropy for CAR systems proved in [2] that

\[ S(\omega_{\mathcal{A}_{12}^{\text{car}}}) \leq S(\omega_1) + S(\omega_2) = 0 + 0 = 0. \]

Thus the positivity of entropy implies

\[ S(\omega_{\mathcal{A}_{12}^{\text{car}}}) = 0. \]

By this vanishing result of entropy of \( \omega \), we conclude that \( \omega \) is a pure state of \( \mathcal{A}_{12}^{\text{car}} \). Thus there exists a unique normalized vector \( \eta(\omega) \) in \( \mathcal{H} \) up to a phase factor satisfying

\[ \omega(A) = (A\eta(\omega), \eta(\omega))_\mathcal{H}, \quad A \in \mathcal{A}_{12}^{\text{car}}. \]

The product property (19) follows from the lemma below. By (19), the purity of \( \omega \) implies that of the restriction of \( \omega \) to \( \mathcal{A}_{2}^{\text{spin}} \). □

**Lemma 2** Let \( H_1 \) and \( H_2 \) be (arbitrary dimensional) Hilbert spaces, and \( \mathcal{H} = H_1 \otimes H_2 \). If a state \( \omega \) of \( \mathcal{L}(\mathcal{H}) \) has a pure state restriction to \( \mathcal{L}(H_1) \otimes 1_{H_2} \), then \( \omega \) has the following product property:

\[ \omega(AB) = \omega(A)\omega(B) \]

for \( A \in \mathcal{L}(H_1) \otimes 1_{H_2} \) and \( B \in 1_{H_1} \otimes \mathcal{L}(H_2) \).

This lemma is a well-known fact, see e.g., Lemma IV.4.11 of [12].

**Remark 2** For the \( \omega \) in Proposition 1, the same result holds for the pair \( \mathcal{A}_{2}^{\text{car}} \) and \( \mathcal{A}_{1}^{\text{spin}} \) as that for \( \mathcal{A}_{1}^{\text{car}} \) and \( \mathcal{A}_{2}^{\text{spin}} \).

**Remark 3** The purity of the both restrictions of \( \omega \) to \( \mathcal{A}_{1}^{\text{car}} \) and \( \mathcal{A}_{2}^{\text{spin}} \) does not imply the purity of that to \( \mathcal{A}_{2}^{\text{car}} \). \( \hat{\rho} \) in the preceding section gives an example; it is a product of a pure state on \( \mathcal{A}_{1}^{\text{car}} \) and a pure state on \( \mathcal{A}_{2}^{\text{spin}} \), but has a non-pure marginal state (tracial state) on \( \mathcal{A}_{2}^{\text{car}} \).

### 4.2 Showing Nonindependence

We recall the definition of \( C^* \)-independence [3].

**Definition 1** Let \( \mathcal{A} \) and \( \mathcal{B} \) be subalgebras of a \( C^* \)-algebra \( \mathcal{C} \). The pair \( (\mathcal{A}, \mathcal{B}) \) (or \( \mathcal{A} \) and \( \mathcal{B} \)) are said to be \( C^* \)-independent if and only if for every state \( \varpi_1 \) of \( \mathcal{A} \) and every state \( \varpi_2 \) of \( \mathcal{B} \) there exists a state \( \varpi \) of \( \mathcal{C} \) such that \( \varpi|_\mathcal{A} = \varpi_1 \) and \( \varpi|_\mathcal{B} = \varpi_2 \).

We note that this definition does not exclude noncommuting pairs of algebras. In fact, there are several examples which are noncommuting \( C^* \)-independent pairs, see, e.g., [6], [11] and references therein.

We now show that a pair of \( C^* \)-subalgebras \( (\mathcal{A}_{1}^{\text{car}}, \mathcal{A}_{2}^{\text{car}}) \) of \( \mathcal{A}_{12}^{\text{car}} \) are not \( C^* \)-independent. Let \( \rho_1 \) be an arbitrary pure state of \( \mathcal{A}_{1}^{\text{car}} \) and \( \rho_2 \) be an arbitrary pure state of \( \mathcal{A}_{2}^{\text{car}} \). Let us assume that there exists a state \( \rho \) of \( \mathcal{A}_{12}^{\text{car}} \) such that
\( \rho \big|_{A_{2}^{\operatorname{ar}}} = \rho_1 \) and \( \rho \big|_{A_{2}^{\operatorname{ar}}} = \rho_2 \). Our aim is to derive the inconsistency of this assumption for some pair of states \( \rho_1 \) and \( \rho_2 \) which leads to the proof of the non-existence of such \( \rho \).

Since both \( \rho_1 \) and \( \rho_2 \) are pure states, they are represented by the following density matrices with some positive numbers \( \vartheta, \vartheta_2, \varphi_2 \) such that \( 0 \leq \vartheta, \vartheta_2 < 2\pi \) and \( 0 \leq \varphi, \varphi_2 \leq \frac{\pi}{2} \):

\[
\rho_1 = \left( \begin{array}{cc}
\cos^2(\varphi) & e^{i\vartheta} \cos(\varphi) \sin(\varphi) \\
e^{-i\vartheta} \cos(\varphi) \sin(\varphi) & \sin^2(\varphi)
\end{array} \right),
\tag{21}
\]

\[
\rho_2 = \left( \begin{array}{cc}
\cos^2(\varphi_2) & e^{i\vartheta_2} \cos(\varphi_2) \sin(\varphi_2) \\
e^{-i\vartheta_2} \cos(\varphi_2) \sin(\varphi_2) & \sin^2(\varphi_2)
\end{array} \right).
\tag{22}
\]

Let us denote the restriction of the state \( \rho \) to \( A_{2}^{\operatorname{spin}} \) by \( \rho_2^{\operatorname{spin}} \). By Proposition 6, the assumption on this \( \rho \) implies that \( \rho_2^{\operatorname{spin}} \) is a pure state and \( \rho \) is a pure product state over \( A_{1}^{\operatorname{car}} \) and \( A_{2}^{\operatorname{car}} \) in the form of \( \rho_1 \otimes \rho_2^{\operatorname{spin}} \).

Since \( \rho_2^{\operatorname{spin}} \) is a pure state, it is represented by the following density matrix with \( \vartheta' (0 \leq \vartheta' < 2\pi) \) and \( \varphi' (0 \leq \varphi' \leq \frac{\pi}{2}) \):

\[
\rho_2^{\operatorname{spin}} = \left( \begin{array}{cc}
\cos^2(\varphi') & e^{i\vartheta'} \cos(\varphi') \sin(\varphi') \\
e^{-i\vartheta'} \cos(\varphi') \sin(\varphi') & \sin^2(\varphi')
\end{array} \right).
\tag{23}
\]

By calculating the expectation values of the matrix units of \( A_{2}^{\operatorname{car}} \) given by \( e_{(1,1)}^2 = e_{(1,2)}^2 = U_1 e_{(1,2)}^{2(\operatorname{spin})}, e_{(2,1)}^2 = U_1 e_{(2,1)}^{2(\operatorname{spin})}, e_{(2,2)}^2 = e_{(2,2)}^{2(\operatorname{spin})} \)

for \( \rho = \rho_1 \otimes \rho_2^{\operatorname{spin}} \), we express the density matrix of \( \rho_2 \) in terms of \( \vartheta', \varphi' \) and \( \varphi \) as follows:

\[
\rho_2 = \left( \begin{array}{cc}
\cos^2(\varphi') & g(\varphi) \cdot e^{i\vartheta'} \cos(\varphi') \sin(\varphi') \\
g(\varphi) \cdot e^{-i\vartheta'} \cos(\varphi') \sin(\varphi') & \sin^2(\varphi')
\end{array} \right),
\tag{24}
\]

where \( g(\varphi) = \cos^2(\varphi) - \sin^2(\varphi) \).

In order that (22) coincides with (24), we must have

\[
\cos^2(\varphi_2) = \cos^2(\varphi'), \quad \sin^2(\varphi_2) = \sin^2(\varphi'),
\]

\[
\cos^2(\varphi_2) \sin^2(\varphi_2) = g(\varphi)^2 \cos^2(\varphi') \sin^2(\varphi'),
\]

and, hence,

\[
(g(\varphi)^2 - 1) \cos^2(\varphi_2) \sin^2(\varphi_2) = 0.
\]

Then the one of the following must hold:

(1) \( \varphi_2 = 0 \), \quad (2) \( \varphi_2 = \frac{\pi}{2} \), \quad (3) \( g(\varphi)^2 = 1 \).

In the case (1) and (2), \( \rho_2 \) is diagonal. In the case (3), either \( \varphi = 0 \) or \( \varphi = \frac{\pi}{2} \) must hold, and hence \( \rho_1 \) is diagonal. Therefore, if both \( \rho_1 \) and \( \rho_2 \) are not diagonal, there does not exist the state \( \rho \) whose restrictions to \( A_{1}^{\operatorname{car}} \) and \( A_{2}^{\operatorname{car}} \) are \( \rho_1 \) and \( \rho_2 \), respectively.

In conclusion, we have shown the following theorem.

**Theorem 3** \( A_{1}^{\operatorname{car}} \) and \( A_{2}^{\operatorname{car}} \) are not C*-independent.
5 Half-sided Entanglement

5.1 Definition of Entanglement Degree

We first give a simple definition of entanglement degree for rather general situations including our CAR systems and finite-dimensional tensor-product systems as special cases.

**Definition 2** Let \( \mathcal{C} \) be a C*-algebra and \( \mathcal{A} \) be a finite-dimensional subalgebra of \( \mathcal{C} \). Let \( \omega \) be a state of \( \mathcal{C} \). The quantum entanglement degree of \( \omega \) on \( \mathcal{A} \) is defined by

\[
E(\omega, \mathcal{A}, \mathcal{C}) = \inf_{\omega = \sum \lambda_i \omega_i} \sum \lambda_i S(\omega_i | \mathcal{A}),
\]

where the infimum is taken over all convex decompositions of \( \omega \) in the state space of \( \mathcal{C} \).

By definition, for any pure state \( \omega \) of \( \mathcal{C} \),

\[
E(\omega, \mathcal{A}, \mathcal{C}) = S(\omega | \mathcal{A}).
\]

If \( \mathcal{A} \) and \( \mathcal{B} \) are finite-dimensional matrix algebras and \( \mathcal{C} = \mathcal{A} \otimes \mathcal{B} \), then

\[
\left| S(\omega | \mathcal{A}) - S(\omega | \mathcal{B}) \right| = 0
\]

for any pure state \( \omega \) of \( \mathcal{C} \). Hence,

\[
E(\omega, \mathcal{A}, \mathcal{C}) = \inf_{\omega = \sum \lambda_i \omega_i} \lambda_i S(\omega_i | \mathcal{A}) = \inf_{\omega = \sum \lambda_i \omega_i} \lambda_i S(\omega_i | \mathcal{B}) = E(\omega, \mathcal{B}, \mathcal{C}).
\]

for any state \( \omega \) of \( \mathcal{C} \). For this case, entanglement \( E \) is symmetric in \( \mathcal{A} \) and \( \mathcal{B} \). However, it is not true in general as we will see.

**Remark 4** Different entanglement degrees are known for finite-dimensional tensor-product systems. For this case, uniqueness theorems of entanglement degrees on pure states have been shown, asserting that all possible entanglement degrees satisfying some basic postulates are equal to the von Neumann entropy of the marginal states (of the pure states). (For details, see e.g. [4], [10], [7] and references therein.)

We give our entanglement degree (25) as a straightforward generalization of “entanglement of formation” (28) which was defined for finite-dimensional tensor-product systems in [3]. We leave its justification as a natural entanglement degree for the CAR case as an open problem, although it is a crucial matter.

5.2 Asymmetry of Entanglement

We introduce a new notion named ‘half-sided entanglement’ in this Section. The contrast between CAR systems and tensor-product systems as seen in (18) and (27) leads us to an intuitive understanding: asymmetry of quantum entanglement is caused by the nonindependence of the pairs of subsystems. We now give the following definitions.
Definition 3 Let $\mathcal{C}$ be a $\mathcal{C}^*$-algebra and $\mathcal{A}$ and $\mathcal{B}$ be (a pair of) finite-dimensional subalgebras of $\mathcal{C}$. Let $\omega$ be a state of $\mathcal{C}$. The degree of $S$-asymmetric entanglement of $\omega$ between $\mathcal{A}$ and $\mathcal{B}$ is given by

$$
\tilde{E}(\omega, \mathcal{A}, \mathcal{B}, \mathcal{C}) \equiv \inf_{\omega = \sum \lambda_i \omega_i} \sum \lambda_i |S(\omega_i|_A) - S(\omega_i|_B)|,
$$

(29)

where the infimum is taken over all convex decompositions of $\omega$ in the state space of $\mathcal{C}$.

Definition 4 If $\tilde{E}(\omega, \mathcal{A}, \mathcal{B}, \mathcal{C})$ is nonzero, $\omega$ is said to be an $S$-asymmetrically entangled state with respect to $(\mathcal{A}, \mathcal{B})$.

Let $\{\lambda_i, \omega_i\}$ be a state-decomposition of $\omega$ attaining $\tilde{E}(\omega, \mathcal{A}, \mathcal{B}, \mathcal{C})$, that is, $\omega = \sum \lambda_i \omega_i$ and

$$
\tilde{E}(\omega, \mathcal{A}, \mathcal{B}, \mathcal{C}) = \sum \lambda_i |S(\omega_i|_A) - S(\omega_i|_B)|.
$$

(30)

If each $\omega_i|_A$ is a pure state, and hence $\sum \lambda_i S(\omega_i|_A) = 0$, then $\omega$ is said to have $S$-half-sided entanglement $\tilde{E}(\omega, \mathcal{A}, \mathcal{B}, \mathcal{C})$ on $\mathcal{B}$ with respect to $(\mathcal{A}, \mathcal{B})$.

If $\omega$ takes the maximal value of $\tilde{E}(\cdot, \mathcal{A}, \mathcal{B}, \mathcal{C})$ when it exists, that is,

$$
\tilde{E}(\omega, \mathcal{A}, \mathcal{B}, \mathcal{C}) = \sup_{\omega': \text{state of } \mathcal{C}} \tilde{E}(\omega', \mathcal{A}, \mathcal{B}, \mathcal{C}),
$$

then $\omega$ is said to have maximal $S$-asymmetric entanglement.

If $\omega$ has maximal $S$-asymmetric entanglement and at the same time is $S$-half-sided entangled, it is said to be a maximal $S$-half-sided entangled state.

‘$S$-’ in the above definitions refers to the von Neumann entropy. Obviously,

$$
\tilde{E}(\omega, \mathcal{A}, \mathcal{B}, \mathcal{C}) = \tilde{E}(\omega, \mathcal{B}, \mathcal{A}, \mathcal{C}),
$$

(31)

for any state $\omega$ of $\mathcal{C}$, and

$$
\tilde{E}(\omega, \mathcal{A}, \mathcal{B}, \mathcal{C}) = \left| S(\omega|_A) - S(\omega|_B) \right|,
$$

$$
= \left| E(\omega, \mathcal{A}, \mathcal{C}) - E(\omega, \mathcal{B}, \mathcal{C}) \right|,
$$

(32)

for any pure state $\omega$ of $\mathcal{C}$. Since we take the infimum over all the possible convex decompositions of $\omega$ in the state space $\mathcal{C}$, $\tilde{E}(\omega, \mathcal{A}, \mathcal{B}, \mathcal{C})$ is a convex function of $\omega$.

Remark 5 Let $\mathcal{C} = \mathcal{A} \otimes \mathcal{B}$ and both $\mathcal{A}$ and $\mathcal{B}$ be finite-dimensional matrix algebras. It follows from (27) and (32) that

$$
\tilde{E}(\omega, \mathcal{A}, \mathcal{B}, \mathcal{C}) = 0,
$$

(33)

for any state $\omega$ of $\mathcal{C}$.

Remark 6 When $\mathcal{C} \supset \mathcal{A} \otimes \mathcal{B}$, (33) does not hold in general. A counterexample is given as follows. Let $\mathcal{A}$ and $\mathcal{B}$ be finite-dimensional $\mathcal{C}^*$-algebras. Consider $\mathcal{C} = M_2(\mathbb{C}) \otimes \mathcal{A} \otimes \mathcal{B}$. Let $\omega_1$ be a pure state on $M_2(\mathbb{C}) \otimes \mathcal{A}$ and $\omega_2$ be a pure state on $\mathcal{B}$. Let us give the (total) state $\omega$ on $\mathcal{C}$ as the product state of $\omega_1$ and $\omega_2$. $\omega$ has the only trivial decomposition over $\mathcal{C}$ because of its purity. $S(\omega|_B)$ vanishes, but $S(\omega|_A) \neq 0$ unless $\omega_1$ has the product property over $M_2(\mathbb{C})$ and $\mathcal{A}$.
6 Entanglement in CAR Systems

6.1 S-asymmetric Entanglement in CAR Systems

We calculate \( \tilde{E}(\cdot, A^\text{car}_1, A^\text{car}_2) \), \( E(\cdot, A^\text{car}_1, A^\text{car}_2) \) and \( E(\cdot, A^\text{car}_2, A^\text{car}_1) \) for some states of the following specific form. The state \( \varrho \) of \( A^\text{car}_{1,2} \) to be considered is a pure state of the form

\[
\varrho = \varrho_1 \otimes \varrho_2^{\text{spin}},
\]

(34)

where \( \varrho_1 \) is a pure state of \( A^\text{car}_1 \) given by the density matrix \( (21) \), and \( \varrho_2^{\text{spin}} \) is a pure state of \( A^\text{spin}_2 \) given by the density matrix \( (23) \). By our choice, \( \varrho \) is a vector state whose representative vector \( \eta_{(\varrho)} \in \mathcal{H} \) has the product form

\[
\eta_{(\varrho)} = \eta^1_{(\varrho)} \otimes \eta^2_{(\varrho)},
\]

where \( \eta^1_{(\varrho)} \in \mathcal{H}_1 \) and \( \eta^2_{(\varrho)} \in \mathcal{H}_2 \) can be taken as

\[
\begin{align*}
\eta^1_{(\varrho)} & \equiv e^{i\varphi} \cos(\varphi) \xi^1_1 + \sin(\varphi) \xi^1_2, \\
\eta^2_{(\varrho)} & \equiv e^{i\varphi'} \cos(\varphi') \xi^2_1 + \sin(\varphi') \xi^2_2.
\end{align*}
\]

Let \( H(\cdot, \cdot) \) be an entropy function given by the following formula,

\[
H(a, b) \equiv -a \log a - b \log b,
\]

for two positive numbers \( a, b \). Let \( \varrho_2 \) be a restriction of the state \( \varrho \) to \( A^\text{car}_2 \). Its density matrix is given as \( (23) \). The eigenvalues \( \varrho^\pm_2(\varphi, \varphi') \) of \( \varrho_2 \) are given by

\[
1 \pm \sqrt{1 - 4\left(1 - \{g(\varphi)\}^2\right) \cdot \cos^2(\varphi') \sin^2(\varphi')} \over 2.
\]

Since \( \varrho_1 \) is a pure state of \( A^\text{car}_1 \), we have

\[
S(\varrho | A^\text{car}_1) = S(\varrho_1) = 0.
\]

Hence

\[
E(\varrho, A^\text{car}_1, A^\text{car}_{1,2}) = 0.
\]

(35)

We have also

\[
S(\varrho | A^\text{car}_2) = S(\varrho_2) = H\left( \varrho^+_2(\varphi', \varphi), \varrho^-_2(\varphi', \varphi) \right).
\]

Hence

\[
E(\varrho, A^\text{car}_2, A^\text{car}_{1,2}) = H\left( \varrho^+_2(\varphi', \varphi), \varrho^-_2(\varphi', \varphi) \right).
\]

(36)

Thus we obtain

\[
\tilde{E}(\varrho, A^\text{car}_1, A^\text{car}_2, A^\text{car}_{1,2}) = H\left( \varrho^+_2(\varphi', \varphi), \varrho^-_2(\varphi', \varphi) \right).
\]

(37)

For any fixed \( \varphi \), \( H\left( \varrho^+_2(\varphi', \varphi), \varrho^-_2(\varphi', \varphi) \right) \) increases with \( \varphi' \) from \( \varphi' = 0 \) until \( \varphi' = {\pi \over 4} \), and then decreases from \( \varphi' = {\pi \over 4} \) until \( \varphi' = {\pi \over 2} \). Unless \( \varphi = 0 \) or \( \varphi = {\pi \over 2} \),
namely unless \( \{g(\varphi)\}^2 = 1 \), it first increases strictly until \( \varphi' = \frac{\pi}{4} \) and then decreases strictly.

On the other hand, for any fixed \( \varphi' \), \( H(g_2^2(\varphi', \varphi), \varphi_2^2(\varphi', \varphi)) \) increases with \( \varphi \) from \( \varphi = 0 \) until \( \varphi = \frac{\pi}{4} \) and then decreases from \( \varphi = \frac{\pi}{4} \) until \( \varphi = \frac{\pi}{2} \). Unless \( \varphi' = 0 \) or \( \varphi' = \frac{\pi}{4} \), it first increases strictly until \( \varphi = \frac{\pi}{4} \) and then decreases strictly. Unless \( \varphi' = 0 \) or \( \varphi' = \frac{\pi}{2} \), and at the same time unless \( \varphi = 0 \) or \( \varphi = \frac{\pi}{2} \), \( \varphi \) has strictly positive \( S \)-half-sided entanglement \( H(g_2^2(\varphi', \varphi), \varphi_2^2(\varphi', \varphi)) \) on \( \mathcal{A}_{2\text{car}} \) with respect to \( (\mathcal{A}_{1\text{car}}, \mathcal{A}_{2\text{car}}) \).

If \( \varphi' = \varphi = \frac{\pi}{2} \), then \( \varphi_2 \) is a tracial state. Hence \( \tilde{\varphi}(p, \mathcal{A}_{1\text{car}}, \mathcal{A}_{2\text{car}}, \mathcal{A}_{1.2}) \) takes the maximal value \( \log 2 \). Therefore, this \( \varphi \) is a maximal \( S \)-half-sided entangled state on \( \mathcal{A}_{2\text{car}} \) with respect to \( (\mathcal{A}_{1\text{car}}, \mathcal{A}_{2\text{car}}) \). We have shown the following.

**Theorem 4** For any positive number \( x \in [0, \log 2] \), there exists an \( S \)-half-sided entangled state of \( \mathcal{A}_{1\text{car}}^{\text{car}} \) for the pair \( (\mathcal{A}_{1\text{car}}, \mathcal{A}_{2\text{car}}) \) with its degree of \( S \)-asymmetric entanglement \( x \).

**Remark 7** We may add a remark on our terminology ‘half-sided’ entanglement to avoid a possible misunderstanding. Entanglement is not something which can be localized or concentrated physically in a single local system (half-sided system). It refers to nonlocal correlations shared by subsystems in an entire system. Half-sided entanglement will describe those asymmetric features of entanglement shared by nonindependent pairs, which cannot be observed in any algebraically independent pairs.

### 6.2 Operations on the Half-sided System

It is natural to expect some operational nonlocality accompanies with nonindependent systems. We show how quantum entanglement between \( \mathcal{A}_{1\text{car}} \) and \( \mathcal{A}_{2\text{car}} \) will be effected by operations done in the half-sided system \( \mathcal{A}_{1\text{car}} \).

By local automorphisms of \( \mathcal{A}_{1\text{car}} \), we mean the automorphisms in the form of \( \alpha_1 \otimes 1_{\mathcal{A}_{1\text{pin}}} \), the tensor product of some automorphism \( \alpha_1 \) of \( \mathcal{A}_{1\text{car}} \) and the identity map of \( \mathcal{A}_{2\text{spin}} \). In general, \( \mathcal{A}_{1\text{car}} \) is not invariant as a set under a local automorphism of \( \mathcal{A}_{1\text{car}} \). We shall see that local automorphisms of \( \mathcal{A}_{1\text{car}} \) can change the entanglement degree between \( \mathcal{A}_{1\text{car}} \) and \( \mathcal{A}_{2\text{car}} \).

We consider the set of states of \( \mathcal{A}_{1,2\text{car}} \) in the form of \( \varrho = \varrho_1 \otimes \varrho_2^{\text{spin}} \), where \( \varrho_1 \) is given by \( (21) \), while \( \varrho_2^{\text{spin}} \) is given by \( (22) \). Fixing the parameters \( \vartheta \) and \( \varphi \) of \( (21) \) and \( \vartheta' \) and \( \varphi' \) of \( (22) \), we have an initial state \( \varrho_0 \) of \( C \). By acting local automorphisms of \( \mathcal{A}_{1\text{car}} \), we can transform this \( \varrho_0 \) to any state in the form \( \varrho = \varrho_1 \otimes \varrho_2^{\text{spin}} \) where \( \varrho_1 \) is given by \( (21) \) with arbitrary \( \vartheta \) and \( \varphi \) while \( \varrho_2^{\text{spin}} \) is kept fixed as \( \varrho_0 |_{\mathcal{A}_{1\text{pin}}} \). Just recalling \( (30) \) and \( (57) \), we have

\[
E(\varrho, \mathcal{A}_{2\text{car}}^{\text{car}}, \mathcal{A}_{1,2\text{car}}^{\text{car}}) = \tilde{E}(\varrho, \mathcal{A}_{1\text{car}}^{\text{car}}, \mathcal{A}_{2\text{car}}^{\text{car}}, \mathcal{A}_{1,2\text{car}}^{\text{car}}) = H(\varrho_2^+(\varphi', \varphi), \varrho_2^-(\varphi', \varphi)).
\]

The above (equivalent) functions vary with \( \varphi \). Consequently, we have shown that the entanglement degree of \( \mathcal{A}_{1\text{car}} \) and the \( S \)-asymmetric entanglement degree of \( \varrho \) on \( \mathcal{A}_{2\text{car}} \) with respect to \( (\mathcal{A}_{1\text{car}}, \mathcal{A}_{2\text{car}}) \) can change under the local automorphisms induced by \( \mathcal{A}_{1\text{car}} \).

**Remark 8** Invariance under local automorphisms and the nonincreasing property under local operations are considered as basic desiderata for natural entanglement degrees. As for the tensor-product case, see e.g. \( [3] \), \( [7] \) for details. For
CAR systems, effects induced by half-sided operations (operations made solely by the subsystem in a half-sided region) are nonlocal, and hence our results in this subsection do not conflict with those desiderata.

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