Let $p$ be a prime number. We define the notion of $F$-finiteness of homomorphisms of $\mathbb{F}_p$-algebras, and discuss some basic properties. In particular, we prove a sort of descent theorem on $F$-finiteness of homomorphisms of $\mathbb{F}_p$-algebras. As a corollary, we prove the following.

Let $g : B \to C$ be a homomorphism of Noetherian $\mathbb{F}_p$-algebras. If $g$ is faithfully flat reduced and $C$ is $F$-finite, then $B$ is $F$-finite. This is a generalization of Seydi’s result on excellent local rings of characteristic $p$.

1. Introduction

Throughout this paper, $p$ denotes a prime number, and $\mathbb{F}_p$ denotes the finite field with $p$ elements. In commutative algebra of characteristic $p$, $F$-finiteness of rings are commonly used for a general assumption which guarantees the “tameness” of the theory, as well as excellence. Although $F$-finiteness for a Noetherian $\mathbb{F}_p$-algebra is stronger than excellence [Kun], $F$-finiteness is not so restrictive for practical use. A perfect field is $F$-finite. An algebra essentially of finite type over an $F$-finite ring is $F$-finite. An ideal-adic completion of a Noetherian $F$-finite ring is again $F$-finite. See Example 3 and Example 9.

In this paper, replacing the absolute Frobenius map by the relative one, we...
define the $F$-finiteness of homomorphism between rings of characteristic $p$. We say that an $\mathbb{F}_p$-algebra map $A \to B$ is $F$-finite (or $B$ is $F$-finite over $A$) if the relative Frobenius map (Radu–André homomorphism) $\Phi_1(A, B) : B^{(1)} \otimes_{A^{(1)}} A \to B$ is finite (Definition 1, see section 2 for the notation). Thus a ring $B$ of characteristic $p$ is $F$-finite if and only if it is $F$-finite over $\mathbb{F}_p$. Replacing absolute Frobenius by relative Frobenius, we get definitions and results on homomorphisms instead of rings. This is a common idea in [Rad], [And], [And2], [Dum], [Dum2], [Ene], [Has], and [Has2].

In section 2, we discuss basic properties of $F$-finiteness of homomorphisms and rings. Some of well-known properties of $F$-finiteness of rings are naturally generalized to those for $F$-finiteness of homomorphisms. $F$-finiteness of homomorphisms has connections with that for rings. For example, if $A \to B$ is $F$-finite and $A$ is $F$-finite, then $B$ is $F$-finite (Lemma 2).

In section 3, we prove the main theorem (Theorem 19). This is a sort of descent of $F$-finiteness. As a corollary, we prove that for a faithfully flat reduced homomorphism of Noetherian rings $g : B \to C$, if $C$ is $F$-finite, then $B$ is $F$-finite. Considering the case that $f$ is a completion of a Noetherian local ring, we recover Seydi’s result on excellent local rings of characteristic $p$ [Sey].

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2. $F$-finiteness of homomorphisms

Let $k$ be a perfect field of characteristic $p$, and $r \in \mathbb{Z}$. For a $k$-space $V$, the additive group $V$ with the new $k$-space structure $\alpha \cdot v = \alpha^{p^r}v$ is denoted by $V^{(r)}$. An element $v$ of $V$, viewed as an element of $V^{(r)}$ is (sometimes) denoted by $v^{(r)}$. If $A$ is a $k$-algebra, then $A^{(r)}$ is a $k$-algebra with the product $a^{(r)} \cdot b^{(r)} = (ab)^{(r)}$. We denote the Frobenius map $A \to A (a \mapsto a^p)$ by $F$ or $F_A$. Note that $F^e : A^{(r+e)} \to A^{(r)}$ is a $k$-algebra map. Throughout the article, we regard $A^{(r)}$ as an $A^{(r+e)}$-algebra through $F^e (A$ is viewed as $A^{(0)}$).

For an $A$-module $M$, the action $a^{(r)} \cdot m^{(r)} = (am)^{(r)}$ makes $M^{(r)}$ an $A^{(r)}$ module. If $I$ is an ideal of $A$, then $I^{(r)}$ is an ideal of $A^{(r)}$. If $e \geq 0$, then $I^{(e)}A = I^{[p^e]}$, where $I^{[p^e]}$ is the ideal of $A$ generated by $\{a^{p^e} \mid a \in I\}$. In commutative algebra, $A^{(r)}$ is also denoted by $^{-r}A$. We employ the notation more consistent with that in representation theory — the eth Frobenius twist
of $V$ is denoted by $V^{(e)}$, see [Jan]. We use this notation for $k = \mathbb{F}_p$.

Let $A \to B$ be an $\mathbb{F}_p$-algebra map, and $e \geq 0$. Then the relative Frobenius map (or Radu–André homomorphism) $\Phi_e(A, B) : B^{(e)} \otimes_{A^{(e)}} A \to B$ is defined by $\Phi_e(A, B)(b^{(e)} \otimes a) = b^p \cdot a$.

**Definition 1.** An $\mathbb{F}_p$-algebra map $A \to B$ is said to be $F$-finite if $\Phi_1(A, B) : B^{(1)} \otimes_{A^{(1)}} A \to B$ is finite. That is, $B$ is a finitely generated $B^{(1)} \otimes_{A^{(1)}} A$-module through $\Phi_1(A, B)$. We also say that $B$ is $F$-finite over $A$.

**Lemma 2.** Let $f : A \to B$, $g : B \to C$, and $h : A \to \tilde{A}$ be $\mathbb{F}_p$-algebra maps, and $\tilde{B} := \tilde{A} \otimes_A B$.

1. The following are equivalent.
   a. $f$ is $F$-finite. That is, $\Phi_1(A, B)$ is finite.
   b. For any $e > 0$, $\Phi_e(A, B)$ is finite.
   c. For some $e > 0$, $\Phi_e(A, B)$ is finite.

2. If $f$ and $g$ are $F$-finite, then so is $gf$.

3. If $gf$ is $F$-finite, then so is $g$.

4. The ring $A$ is $F$-finite (that is, the Frobenius map $F_A : A^{(1)} \to A$ is finite) if and only if the unique homomorphism $\mathbb{F}_p \to A$ is $F$-finite.

5. If $f : A \to B$ is $F$-finite, then the base change $\tilde{f} : \tilde{A} \to \tilde{B}$ is $F$-finite.

6. If $B$ is $F$-finite, then $f$ is $F$-finite.

7. If $A$ and $f$ are $F$-finite, then $B$ is $F$-finite.

**Proof.** 1 This is immediate, using [Has, Lemma 4.1, 2]. 2 and 3 follow from [Has, Lemma 4.1, 1]. 4 follows from [Has, Lemma 4.1, 5]. 5 follows from [Has, Lemma 4.1, 4]. 6 follows from 3 and 4. 7 follows from 2 and 4. \qed

**Example 3.** Let $e \geq 1$, and $f : A \to B$ be an $\mathbb{F}_p$-algebra map.

1. If $B = A[x]$ is a polynomial ring, then it is $F$-finite over $A$.

2. If $B = A_S$ is a localization of $A$ by a multiplicatively closed subset $S$ of $A$, then $\Phi_e(A, B)$ is an isomorphism. In particular, $B$ is $F$-finite over $A$.
3 If $B = A/I$ with $I$ an ideal of $A$, then

$$B^{(e)} \otimes_{A^{(e)}} A \cong (A^{(e)}/I^{(e)}) \otimes_{A^{(e)}} A \cong A/I^{(e)}A = A/I^{[p^e]}.$$  

Under this identification, $\Phi_e(A, B)$ is identified with the projection $A/I^{[p^e]} \to A/I$. In particular, $B$ is $F$-finite over $A$.

4 If $B$ is essentially of finite type over $A$, then $B$ is $F$-finite over $A$.

**Proof.** 1 The image of $\Phi_1(A, B)$ is $A[x^p]$, and hence $B$ is generated by $1, x, \ldots, x^{p-1}$ over it. 2 Note that $B^{(e)}$ is identified with $(A^{(e)})_{S^{(e)}}$, where $S^{(e)} = \{ s^{(e)} \mid s \in S \}$. So $B^{(e)} \otimes_{A^{(e)}} A$ is identified with $(A^{(e)})_{S^{(e)}} \otimes_{A^{(e)}} A \cong A_{S^{(e)}}$, and $\Phi_e(A, B)$ is identified with the isomorphism $A_{S^{(e)}} \cong A_S$. 3 is obvious. 4 This is a consequence of 1, 2, 3, and Lemma 2. □

**Lemma 4.** Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a sequence of $\mathbb{F}_p$-algebra maps. Then for $e > 0$, the diagram

$$
\begin{array}{ccc}
B^{(e)} \otimes_{A^{(e)}} A & \xrightarrow{\Phi_e(A, B)} & B \\
\downarrow{g^{(e)} \otimes 1} & & \downarrow{g} \\
C^{(e)} \otimes_{A^{(e)}} A & \xrightarrow{\Phi_e(A, C)} & C 
\end{array}
$$

is commutative.

**Proof.** This is straightforward. □

**Lemma 5.** Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a sequence of $\mathbb{F}_p$-algebra maps, and assume that $C$ is $F$-finite over $A$. If $g$ is finite and injective, and $B^{(e)} \otimes_{A^{(e)}} A$ is Noetherian for some $e > 0$, then $B$ is $F$-finite over $A$.

**Proof.** By assumption, $C^{(e)} \otimes_{A^{(e)}} A$ is finite over $B^{(e)} \otimes_{A^{(e)}} A$, and $C$ is finite over $C^{(e)} \otimes_{A^{(e)}} A$. So $C$ is finite over $B^{(e)} \otimes_{A^{(e)}} A$. As $B$ is a $B^{(e)} \otimes_{A^{(e)}} A$-submodule of $C$ and $B^{(e)} \otimes_{A^{(e)}} A$ is Noetherian, $B$ is finite over $B^{(e)} \otimes_{A^{(e)}} A$. □

**Lemma 6.** Let $A \to B$ be a ring homomorphism, and $I$ a finitely generated nilpotent ideal of $B$. If $B/I$ is $A$-finite, then $B$ is $A$-finite.

**Proof.** As $I/I^{i+1}$ is $B/I$-finite for each $i$, it is also $A$-finite. So $B/I^r$ is $A$-finite for each $r$. Taking $r$ large, $B$ is $A$-finite. □

**Lemma 7.** Let $f : A \to B$ be an $\mathbb{F}_p$-algebra map, and $I$ a finitely generated nilpotent ideal of $B$. If $B/I$ is $F$-finite over $A$, then $B$ is $F$-finite over $A$.
Proof. As $B/I$ is $F$-finite over $A$, $B/I$ is $(B^{(1)}/I^{(1)}) \otimes_{A^{(1)}} A$-finite. So $B/I$ is also $B^{(1)} \otimes_{A^{(1)}} A$-finite. By Lemma 6, $B$ is $B^{(1)} \otimes_{A^{(1)}} A$-finite.

For the absolute $F$-finiteness, we have a better result.

Lemma 8. Let $B$ be an $\mathbb{F}_p$-algebra, and $I$ a finitely generated ideal of $B$. If $B$ is $I$-adically complete and $B/I$ is $F$-finite, then $B$ is $F$-finite.

Proof. $B/I$ is $B^{(1)}/I^{(1)}$-finite. So $B/I^{(1)}$ is $B^{(1)}$-finite by Lemma 6. As $\bigcap_i I^i = 0$, we have $\bigcap_i (I^{(1)})^i B = 0$. Moreover, $B^{(1)}$ is $I^{(1)}$-adically complete. Hence $B$ is $B^{(1)}$-finite by [Mat, Theorem 8.4].

Example 9. Let $A$ be an $\mathbb{F}_p$-algebra.

1 If $A$ is $F$-finite, then the formal power series ring $A[[x]]$ is so.

2 Let $J$ be an ideal of $A$. If $A$ is Noetherian and $A/J$ is $F$-finite, then the $J$-adic completion $A^*$ of $A$ is $F$-finite.

3 If $(A, \mathfrak{m})$ is complete local and $A/\mathfrak{m}$ is $F$-finite, then $A$ is $F$-finite.

Proof. For each of 1–3, we use Lemma 8. 1 Set $B = A[[x]]$ and $I = Bx$. Then $B/I \cong A$ is $F$-finite. 2 Set $B = A^*$ and $I = JB$. Then $B/I \cong A/J$ is $F$-finite. 3 is immediate.

Let $A$ be a Noetherian ring and $I$ its ideal. If $A$ is $I$-adically complete and $A/I$ is Nagata, then $A$ is Nagata [Mar]. If $A$ is semi-local, $I$-adically complete, and $A/I$ is quasi-excellent, then $A$ is quasi-excellent [Rot2]. See also [Nis].

Lemma 10. Let $A$ be an $\mathbb{F}_p$-algebra, and $B$ and $C$ be $A$-algebras. If $B$ and $C$ are $F$-finite over $A$, then

1 $B \otimes_A C$ is $F$-finite over $A$.

2 $B \times C$ is $F$-finite over $A$.

Proof. 1 $B$ is $F$-finite over $A$, and $B \otimes_A C$ is $F$-finite over $B$ by Lemma 2. 5. By Lemma 2, $B \otimes_A C$ is $F$-finite over $A$. 2 Both $B$ and $C$ are finite over $(B \times C)^{(1)} \otimes_{A^{(1)}} A$, and so is $B \times C$. 

Lemma 11. Let $A \to B$ be an $\mathbb{F}_p$-algebra map, and assume that $B$ and $B^{(e)} \otimes_{A^{(e)}} A$ are Noetherian for some $e > 0$. Then $B$ is $F$-finite over $A$ if and only if $B/P$ is $F$-finite over $A$ for every minimal prime $P$ of $B$. 

5
Proof. The ‘only if’ part is obvious by Example 3. We prove the converse. Let $\text{Min} B$ be the set of minimal primes of $B$. Then $\prod_{P \in \text{Min} B} B/P$ is $F$-finite over $A$ by Lemma 10. As $B_{\text{red}} \rightarrow \prod_{P \in \text{Min} B} B/P$ is finite injective, and $B_{\text{red}} \otimes_{A(e)} A$ is Noetherian, $B_{\text{red}}$ is $F$-finite over $A$ by Lemma 5. As $B$ is Noetherian, $B$ is $F$-finite over $A$ by Lemma 7.

Remark 12. Fogarty asserted that an $\mathbb{F}_p$-algebra map $A \rightarrow B$ with $B$ Noetherian is $F$-finite if and only if the module of Kähler differentials $\Omega_{B/A}$ is a finite $B$-module [Fog, Proposition 1]. The ‘only if’ part is true and easy. The proof of ‘if’ part therein has a gap. Although $R_1$ in step (iii) is assumed to be Noetherian, it is not proved that $R'$ in step (iv) is Noetherian. The author does not know if this direction is true or not.

3. Descent of $F$-finiteness

In this section, we prove a sort of descent theorem on $F$-finiteness of homomorphisms.

Lemma 13. Let $R$ be a commutative ring, $\varphi : M \rightarrow N$ and $h : F \rightarrow G$ be $R$-linear maps. If $\varphi$ is $R$-pure and $1_N \otimes h : N \otimes F \rightarrow N \otimes G$ is surjective, then $1_M \otimes h : M \otimes F \rightarrow M \otimes G$ is surjective.

Proof. Let $C := \text{Coker} h$. Then by assumption, $N \otimes C = 0$. By the injectivity of $\varphi \otimes 1_C : M \otimes C \rightarrow N \otimes C$, we have that $M \otimes C = 0$.

Corollary 14. Let $A \rightarrow B$ be a pure ring homomorphism, and $h : F \rightarrow G$ an $A$-linear map. If $1_B \otimes h : B \otimes_A F \rightarrow B \otimes_A G$ is surjective, then $h$ is surjective.

Lemma 15. Let $A \rightarrow B$ be a pure ring homomorphism, and $G$ an $A$-module. If $B \otimes_A G$ is a finitely generated $B$-module, then $G$ is finitely generated as an $A$-module.

Proof. Let $\theta_1, \ldots, \theta_r$ be generators of $B \otimes_A G$. Then we can write $\theta_j = \sum_{i=1}^s b_{ij} \otimes g_{ij}$ for some $s > 0$, $b_{ij} \in B$, and $g_{ij} \in G$. Let $F$ be the $A$-free module with the basis $\{f_{ij} | 1 \leq i \leq s, 1 \leq j \leq r\}$, and $h : F \rightarrow G$ be the $A$-linear map given by $f_{ij} \mapsto g_{ij}$. Then by construction, $1_B \otimes h$ is surjective. By Corollary 14, $h$ is surjective, and hence $G$ is finitely generated.
Definition 16 (cf. [Has2 (2.7)]). An $\mathbb{F}_p$-algebra map $A \to B$ is said to be $e$-Dumitrescu if there exists some $e > 0$ such that $\Phi_e(A, B)$ is $A$-pure (i.e., pure as an $A$-linear map).

Lemma 17. Let $e, e' > 0$. If $A \to B$ is both $e$-Dumitrescu and $e'$-Dumitrescu, then it is $(e + e')$-Dumitrescu. In particular, an $e$-Dumitrescu map is er-Dumitrescu map for $r > 0$.

Proof. This follows from [Has, Lemma 4.1, 2].

So a 1-Dumitrescu map is Dumitrescu (that is, $e$-Dumitrescu for all $e > 0$), see [Has2, Lemma 2.9].

Lemma 18. Let $e > 0$.

1 [Has2 Lemma 2.8],

2 [Has2 Lemma 2.12], and

3 [Has2 Corollary 2.13]

hold true when we replace all the ‘Dumitrescu’ therein by ‘$e$-Dumitrescu’.

The proof is straightforward, and is left to the reader.

Theorem 19. Let $f : A \to B$ and $g : B \to C$ be $\mathbb{F}_p$-algebra maps, and $e > 0$. Assume that $g$ is $e$-Dumitrescu, and the image of the associated map $a g : \text{Spec } C \to \text{Spec } B$ contains the set of maximal ideals $\text{Max } B$ of $B$. If $gf$ is $F$-finite, and $B$ and $C^{(e)} \otimes_{A^{(e)}} A$ are Noetherian, then $f$ is $F$-finite.

Proof. Note that $\Phi_e(A, C) : C^{(e)} \otimes_{A^{(e)}} A \to C$ is a finite map. Note also that $C^{(e)} \otimes_{B^{(e)}} B$ is a $C^{(e)} \otimes_{A^{(e)}} A$-submodule of $C$ through $\Phi_e(B, C)$, since $\Phi_e(B, C)$ is $B$-pure and hence is injective. As $C^{(e)} \otimes_{A^{(e)}} A$ is Noetherian, $C^{(e)} \otimes_{B^{(e)}} B$, which is a submodule of the finite module $C$, is a finite $C^{(e)} \otimes_{A^{(e)}} A$-module. Since $g^{(e)} : B^{(e)} \to C^{(e)}$ is pure by Lemma 15, $B^{(e)} \otimes_{A^{(e)}} A \to C^{(e)} \otimes_{A^{(e)}} A$ is also pure. Since

$$C^{(e)} \otimes_{B^{(e)}} B \cong (C^{(e)} \otimes_{A^{(e)}} A) \otimes_{B^{(e)}} A B$$

is a finite $C^{(e)} \otimes_{A^{(e)}} A$-module, $B$ is a finite $B^{(e)} \otimes_{A^{(e)}} A$-module by Lemma 15.

\[ \square \]
A homomorphism $f : A \to B$ between Noetherian rings is said to be reduced if $f$ is flat with geometrically reduced fibers.

**Corollary 20.** Let $g : B \to C$ be a faithfully flat reduced homomorphism between Noetherian $\mathbb{F}_p$-algebras. If $C$ is $F$-finite, then $B$ is $F$-finite.

**Proof.** By [Dum2, Theorem 3], $g$ is Dumitrescu. As $g$ is faithfully flat, $g : \text{Spec } C \to \text{Spec } B$ is surjective. Letting $A = \mathbb{F}_p$ and $f : A \to B$ be the unique map, the assumptions of Theorem [19] are satisfied, and hence $f$ is $F$-finite. That is, $B$ is $F$-finite.

**Corollary 21** (Seydi [Sey]). Let $(B, m)$ be a Nagata local ring with the $F$-finite residue field $k = B/m$. Then $B$ is $F$-finite. In particular, $B$ is excellent, and is a homomorphic image of a regular local ring.

**Proof.** Let $g : B \to C = \hat{B}$ be the completion of $B$. Then $C$ is a complete local ring with the residue field $k$. By Example [9], $C$ is $F$-finite. As $g$ is reduced by [Gro, (7.6.4), (7.7.2)], $B$ is $F$-finite by Corollary 20.

The last assertions follow from [Kun, Theorem 2.5] and [Gab, Remark 13.6].

Even if $A \to B$ is a faithfully flat reduced homomorphism and $B$ is excellent, $A$ need not be quasi-excellent. There is a Nagata local ring $A$ which is not quasi-excellent [Rot, Nis2], and its completion $A \to \hat{A} = B$ is an example.

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