LOOP EQUATION AND AREA LAW IN TURBULENCE

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Abstract

This is the extended version of the preprint [4], based on the lectures given in Cargese Summer School and Chernogolovka Summer School in 93. The incompressible fluid dynamics is reformulated as dynamics of closed loops $C$ in coordinate space. We derive explicit functional equation for the pdf of the circulation $P_C(\Gamma)$ which allows the scaling solutions in inertial range of spatial scales. The pdf decays as exponential of some power of $\Gamma^3/A^2$ where $A$ is the minimal area inside the loop.
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1 Introduction

Incompressible fluid dynamics underlies the vast majority of natural phenomena. It is described by famous Navier-Stokes equation

$$\dot{v}_\alpha = \nu \partial_\beta^2 v_\alpha - v_\beta \partial_\beta v_\alpha - \partial_\alpha p; \, \partial_\alpha v_\alpha = 0$$ (1)

which is nonlinear, and therefore hard to solve. This nonlinearity makes life more interesting, though, as it leads to turbulence. Solving this equation with appropriate initial and boundary conditions we expect to obtain the chaotic behavior of velocity field.

The simplest boundary conditions correspond to infinite space with vanishing velocity at infinity. We are looking for the translation invariant probability distribution for velocity field, with infinite range of the wavelengths. In order to compensate for the energy dissipation, we add the usual random force to the Navier-Stokes equations, with the short wavelength support, corresponding to large scale energy pumping.

One may attempt to describe this probability distribution by the Hopf generating functional (the angular bracket denote time averaging, or ensemble averaging over realizations of the random forces)

$$Z[J] = \langle \exp \left( \int d^3r J_\alpha(r) v_\alpha(r) \right) \rangle$$ (2)\n
which is known to satisfy linear functional differential equation

$$\dot{Z} = H \left[ J, \frac{\delta}{\delta J} \right] Z$$ (3)

similar to the Schrödinger equation for Quantum Field Theory, and equally hard to solve. Nobody managed to go beyond the Taylor expansion in source \( J \), which corresponds to the obvious chain of equations for the equal time correlation functions of velocity field in various points in space. The same equations could be obtained directly from Navier-Stokes equations, so the Hopf equation looks useless.

In this work\(^1\) we argue, that one could significantly simplify the Hopf functional without loosing information about correlation functions. This simplified functional depends upon the set of 3 periodic functions of one variable

$$C : r_\alpha = C_\alpha(\theta); \, 0 < \theta < 2\pi$$ (4)

which set describes the closed loop in coordinate space. The correlation functions reduce to certain functional derivatives of our loop functional with respect to \( C(\theta) \) at vanishing loop \( C \to 0 \).

The properties of the loop functional at large loop \( C \) also have physical significance. Like the Wilson loops in Gauge Theory, they describe the statistics of large scale structures of vorticity field, which is analogous to the gauge field strength.

\(^1\)see also [4] where this approach was initiated and [5] where its relation with the generalized Hamiltonian dynamics and the Gibbs-Boltzmann statistics was established
In Appendix A we recover the expansion in inverse powers of viscosity by direct iterations of the loop equation.

In Appendix B we study the matrix formulation of the Navier-Stokes equation, which may serve as a basis of the random matrix description of turbulence.

In Appendix C we study the reduced dynamics, corresponding to the functional Fourier transform of the loop functional. We argue, that instead of 3D Navier-Stokes equations one can use the 1D equations for the Fourier loop $P_\alpha(\theta, t)$.

In Appendix D we discuss the relation between the initial data for velocity field and the $P$ field, and we find particular realisation for these initial data in terms of the gaussian random variables.

In Appendix E we introduce the generating functional for the scalar products $P_\alpha(\theta)P_\alpha(\theta')$. The advantage of this functional over the original $\Psi[C]$ functional is the smoother continuum limit.

In Appendix F we discuss the possible numerical implementations of the reduced loop dynamics.

In Appendix G we show uniqueness of the tensor area law within certain class of functionals.

In Appendix H we present the modern view at the old problem of the minimal surface.

In Appendix I we show that the triple Kolmogorov correlation function corresponds to a vanishing correlation of vorticity with two velocity fields.

## 2 The Loop Calculus

We suggest to use in turbulence the following version of the Hopf functional

$$\Psi[C] = \left\langle \exp \left( \frac{i}{\nu} \oint dC_\alpha(\theta) v_\alpha(C(\theta)) \right) \right\rangle$$

which we call the loop functional or the loop field. It is implied that all angular variable $\theta$ run from 0 to $2\pi$ and that all the functions of this variable are $2\pi$ periodic. The viscosity $\nu$ was inserted in denominator in exponential, as the only parameter of proper dimension. As we shall see below, it plays the role, similar to the Planck’s constant in Quantum mechanics, the turbulence corresponding to the WKB limit $\nu \to 0$.

As for the imaginary unit $i$, there are two reasons to insert it in the exponential. First, it makes the motion compact: the phase factor goes around the unit circle, when the velocity field fluctuates. So, at large times one may expect the ergodicity, with well

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2 This parametrization of the loop is a matter of convention, as the loop functional is parametric invariant.

3 One could also insert any numerical parameter in exponential, but this factor could be eliminated by space- and/or time rescaling.
defined average functional bounded by 1 by absolute value. Second, with this factor of 
\( i \), the irreversibility of the problem is manifest. The time reversal corresponds to the complex conjugation of \( \Psi \), so that imaginary part of the asymptotic value of \( \Psi \) at \( t \to \infty \) measures the effects of dissipation.

The loop orientation reversal \( C(\theta) \to C(2\pi - \theta) \) also leads to the complex conjugation, so it is equivalent to the time reversal. This symmetry implies, that any correlator of odd/even number of velocities should be integrated odd/even number of times over the loop, and it must enter with an imaginary/real factor. Later, we shall use this property in the area law.

We shall often use the field theory notations for the loop integrals,

\[
\Psi [C] = \left\langle \exp \left( \frac{i}{\nu} \oint_C dr_\alpha v_\alpha \right) \right\rangle
\]

This loop integral can be reduced to the surface integral of vorticity field

\[
\omega_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu
\]

by the Stokes theorem

\[
\Gamma_C[v] \equiv \oint_C dr_\alpha v_\alpha = \int_S d\sigma_{\mu\nu} \omega_{\mu\nu}; \quad \partial S = C
\]

This is the well-known velocity circulation, which measures the net strength of the vortex lines, passing through the loop \( C \). Would we fix initial loop \( C \) and let it move with the flow, the loop field would be conserved by the Euler equation, so that only the viscosity effects would be responsible for its time evolution. However, this is not what we are trying to do. We take the Euler rather than Lagrange dynamics, so that the loop is fixed in space, and hence \( \Psi \) is time dependent already in the Euler equations. The difference between Euler and Navier-Stokes equations is the time irreversibility, which leads to complex average \( \Psi \) in Navier-Stokes dynamics.

It is implied that this field \( \Psi [C] \) is invariant under translations of the loop \( C(\theta) \to C(\theta) + \text{const} \). The asymptotic behavior at large time with proper random forcing reaches certain fixed point, governed by the translation- and scale invariant equations, which we derive in this paper.

The general Hopf functional (2) reduces for the loop field for the following imaginary singular source

\[
J_\alpha(r) = \frac{i}{\nu} \oint_C dr'_\alpha \delta^3 (r' - r)
\]

The \( \Psi \) functional involves connected correlation functions of the powers of circulation at equal times.

\[
\Psi[C] = \exp \left( \sum_{n=2}^{\infty} i^n \frac{n}{n!} \nu^n \left\langle \langle \Gamma_C^n \rangle \right\rangle \right)
\]
This expansion goes in powers of effective Reynolds number, so it diverges in turbulent region. There, the opposite WKB approximation will be used.

Let us come back to the general case of the arbitrary Reynolds number. What could be the use of such restricted Hopf functional? At first glance it seems that we lost most of information, described by the Hopf functional, as the general Hopf source $J$ depends upon 3 variables $x, y, z$ whereas the loop $C$ depends of only one parameter $\theta$. Still, this information can be recovered by taking the loops of the singular shape, such as two infinitesimal loops $R_1, R_2$, connected by a couple of wires

\[ \text{Fig. 1} \]

The loop field in this case reduces to

\[
\Psi [C] \rightarrow \exp \left( \frac{i}{2\nu} \Sigma_{\mu\nu} \omega_{\mu\nu}(r_1) + \frac{i}{2\nu} \Sigma_{\mu\nu} \omega_{\mu\nu}(r_2) \right) \quad (11)
\]

where

\[
\Sigma_{\mu\nu} = \oint_R dr_\nu r_\mu \quad (12)
\]

is the tensor area inside the loop $R$. Taking functional derivatives with respect to the shape of $R_1$ and $R_2$ prior to shrinking them to points, we can bring down the product of vorticities at $r_1$ and $r_2$. Namely, the variations yield

\[
\delta \Sigma_{\mu\nu} = \oint_R (dr_\nu \delta r_\mu + r_\mu d\delta r_\nu) = \oint_R (dr_\nu \delta r_\mu - dr_\mu \delta r_\nu) \quad (13)
\]

where integration by parts was used in the second term.

One may introduce the area derivative $\frac{\delta}{\delta \sigma_{\mu\nu}(r)}$, which brings down the vorticity at the given point $r$ at the loop.

\[
- \nu^2 \frac{\delta^2 \Psi [C]}{\delta \sigma_{\mu\nu}(r_1) \delta \sigma_{\lambda\rho}(r_2)} \rightarrow \langle \omega_{\mu\nu}(r_1) \omega_{\lambda\rho}(r_2) \rangle \quad (14)
\]

The careful definition of these area derivatives are or paramount importance to us. The corresponding loop calculus was developed in [2] in the context of the gauge theory. Here we rephrase and further refine the definitions and relations established in that work.

The basic element of the loop calculus is what we suggest to call the spike derivative, namely the operator which adds the infinitesimal $\Lambda$ shaped spike to the loop

\[
D_\alpha(\theta, \epsilon) = \int_\theta^{\theta+2\epsilon} d\phi \left( 1 - \frac{|\theta + \epsilon - \phi|}{\epsilon} \right) \frac{\delta}{\delta C_\alpha(\phi)} \quad (15)
\]
The finite spike operator
\[ \Lambda(r, \theta, \epsilon) = \exp \left( r \alpha D_\alpha(\theta, \epsilon) \right) \]
adds the spike of the height \( r \). This is the straight line from \( C(\theta) \) to \( C(\theta + \epsilon) + r \), followed by another straight line from \( C(\theta + \epsilon) + r \) to \( C(\theta + 2\epsilon) \).

Note, that the loop remains closed, and the slopes remain finite, only the second derivatives diverge. The continuity and closure of the loop eliminates the potential part of velocity; as we shall see below, this is necessary to obtain the loop equation.

In the limit \( \epsilon \to 0 \) these spikes are invisible, at least for the smooth vorticity field, as one can see from the Stokes theorem (the area inside the spike goes to zero as \( \epsilon \)). However, taking certain derivatives prior to the limit \( \epsilon \to 0 \) we can obtain the finite contribution.

Let us consider the operator
\[
\Pi(r, r', \theta, \epsilon) = \Lambda \left( r, \theta, \frac{1}{2} \epsilon \right) \Lambda (r', \theta, \epsilon)
\]
By construction it inserts the smaller spike on top of a bigger one, in such a way, that a polygon appears.

Taking the derivatives with respect to the vertices of this polygon \( r, r' \), setting \( r = \)
\( r' = 0 \) and antisymmetrising, we find the tensor operator
\[
\Omega_{\alpha\beta}(\theta, \epsilon) = -i \nu D_\alpha \left( \theta, \frac{1}{2} \epsilon \right) D_\beta (\theta, \epsilon) - \{ \alpha \leftrightarrow \beta \}
\] (18)
which brings down the vorticity, when applied to the loop field
\[
\Omega_{\alpha\beta}(\theta, \epsilon) \Psi [C] \xrightarrow{\epsilon \to 0} \omega_{\alpha\beta}(C(\theta)) \Psi [C]
\] (19)

The quick way to check these formulas is to use formal functional derivatives
\[
\frac{\delta \Psi [C]}{\delta C'_\alpha(\theta)} = C'_\beta(\theta) \frac{\delta \Psi [C]}{\delta \sigma_{\alpha\beta}(C'(\theta))}
\] (20)
Taking one more functional derivative derivative we find the term with vorticity times first derivative of the \( \delta \) function, coming from the variation of \( C'(\theta) \)
\[
\frac{\delta^2 \Psi [C]}{\delta C'_\alpha(\theta) \delta C'_\beta(\theta')} = \delta'(\theta - \theta') \frac{\delta \Psi [C]}{\delta \sigma_{\alpha\beta}(C(\theta))} + C'_\gamma(\theta) C'_\lambda(\theta') \frac{\delta^2 \Psi [C]}{\delta \sigma_{\alpha\gamma}(C(\theta)) \delta \sigma_{\beta\lambda}(C'(\theta'))}
\] (21)
This term is the only one, which survives the limit \( \epsilon \to 0 \) in our relation (19).

So, the area derivative can be defined from the antisymmetric tensor part of the second functional derivative as the coefficient in front of \( \delta'(\theta - \theta') \). Still, it has all the properties of the first functional derivative, as it can also be defined from the above first variation. The advantage of dealing with spikes is the control over the limit \( \epsilon \to 0 \), which might be quite singular in applications.

So far we managed to insert the vorticity at the loop \( C \) by variations of the loop field. Later we shall need the vorticity off the loop, in arbitrary point in space. This can be achieved by the following combination of the spike operators
\[
\Lambda (r, \theta, \epsilon) \Pi (r_1, r_2, \theta + \epsilon, \delta); \quad \delta \ll \epsilon
\] (22)
This operator inserts the \( \Pi \) shaped little loop at the top of the bigger spike, in other words, this little loop is translated by a distance \( r \) by the big spike.

Taking derivatives, we find the operator of finite translation of the vorticity
\[
\Lambda (r, \theta, \epsilon) \Omega_{\alpha\beta}(\theta + \epsilon, \delta)
\] (23)
and the corresponding infinitesimal translation operator
\[
D_\mu(\theta, \epsilon) \Omega_{\alpha\beta}(\theta + \epsilon, \delta)
\] (24)
which inserts \( \partial_\mu \omega_{\alpha\beta}(C(\theta)) \) when applied to the loop field.

Coming back to the correlation function, we are going now to construct the operator, which would insert two vorticities separated by a distance. Let us note that the global \( \Lambda \) spike
\[
\Lambda (r, 0, \pi) = \exp \left( r_\alpha \int_0^{2\pi} d\phi \left( 1 - \frac{|\phi - \pi|}{\pi} \right) \frac{\delta}{\delta C'_\alpha(\phi)} \right)
\] (25)
when applied to a shrunk loop $C(\phi) = 0$ does nothing but the backtracking from 0 to $r$

\[
\begin{align*}
0 & \rightarrow \rightarrow r
\end{align*}
\]

Fig. 4

This means that the operator

\[
\Omega_{\alpha\beta}(0, \delta)\Omega_{\lambda\rho}(\pi, \delta)\Lambda(r, 0, \pi)
\]  \hspace{1cm} (26)

when applied to the loop field for a shrunk loop yields the vorticity correlation function

\[
\Omega_{\alpha\beta}(0, \delta)\Omega_{\lambda\rho}(\pi, \delta)\Lambda(r, 0, \pi) \Psi[0] = \langle \omega_{\alpha\beta}(0)\omega_{\lambda\rho}(r) \rangle
\]  \hspace{1cm} (27)

The higher correlation functions of vorticities could be constructed in a similar fashion, using the spike operators. As for the velocity, one should solve the Poisson equation

\[
\partial^2_{\mu}v_{\alpha}(r) = \partial_{\beta}\omega_{\beta\alpha}(r)
\]  \hspace{1cm} (28)

with the proper boundary conditions, say, $v = 0$ at infinity. Formally,

\[
v_{\alpha}(r) = \frac{1}{\partial^2_{\mu}}\partial_{\beta}\omega_{\beta\alpha}(r)
\]  \hspace{1cm} (29)

This suggests the following formal definition of the velocity operator

\[
V_{\alpha}(\theta, \epsilon, \delta) = \frac{1}{D^2_{\mu}(\theta, \epsilon)}D_{\beta}(\theta, \epsilon)\Omega_{\beta\alpha}(\theta, \delta); \delta \ll \epsilon
\]  \hspace{1cm} (30)

\[
V_{\alpha}(\theta, \epsilon, \delta)\Psi[C] \xrightarrow{\delta, \epsilon \rightarrow 0} v_{\alpha}(C(\theta)) \Psi[C]
\]  \hspace{1cm} (31)

Another version of this formula is the following integral

\[
V_{\alpha}(\theta, \epsilon, \delta) = \int d^3\rho \frac{\rho^3}{4\pi|\rho|^3}\Lambda(\rho, \theta, \epsilon)\Omega_{\alpha\beta}(\theta + \epsilon, \delta)
\]  \hspace{1cm} (32)

where the $\Lambda$ operator shifts the $\Omega$ by a distance $\rho$ off the original loop at the point $r = C(\theta + \epsilon)$. 

\[
\begin{align*}
\text{Fig. 5}
\end{align*}
\]
3 Loop Equation

Let us now derive exact equation for the loop functional. Taking the time derivative of the original definition, and using the Navier-Stokes equation we get in front of exponential

\[ \oint_C dr_\alpha \frac{i}{\nu} \left( \nu \partial^2_\beta v_\alpha - \nu_\beta \partial_\beta v_\alpha - \partial_\alpha p \right) \]  

(33)

The term with the pressure gradient yields zero after integration over the closed loop, and the velocity gradients in the first two terms could be expressed in terms of vorticity up to irrelevant gradient terms, so that we find

\[ \oint_C dr_\alpha \frac{i}{\nu} (\nu \partial_\beta \omega_\beta_\alpha - \nu_\beta \omega_\beta_\alpha) \]  

(34)

Replacing the vorticity and velocity by the operators discussed in the previous Section we find the following loop equation (in explicit notations)

\[ -i \dot{\Psi}[C] = \oint dC_\alpha(\theta) \left( D_\beta(\theta, \epsilon) \Omega_\gamma_\beta_\alpha(\theta, \epsilon) + \frac{1}{\nu} \int d^3 \rho \frac{\rho_\gamma}{4\pi|\rho|^3} \Lambda(\rho, \theta, \epsilon) \Omega_\gamma_\beta(\theta + \epsilon, \delta) \Omega_\beta_\alpha(\theta, \delta) \right) \Psi[C] \]  

(35)

The more compact form of this equation, using the notations of [2], reads

\[ i \nu \dot{\Psi}[C] = \mathcal{H}_C \Psi \]  

(36)

\[ \mathcal{H}_C \equiv \nu^2 \oint_C dr_\alpha \left( i \partial_\beta \frac{\delta}{\delta \sigma_\beta_\alpha(r)} + \int d^3 r' \frac{r'_\gamma - r_\gamma}{4\pi|r - r'|^3} \delta \sigma_\beta_\alpha(r) \delta \sigma_\beta_\gamma(r') \right) \]

Now we observe that viscosity \( \nu \) appears in front of time and spatial derivatives, like the Planck constant \( \hbar \) in Quantum mechanics. Our loop hamiltonian \( \mathcal{H}_C \) is not hermitean, due to dissipation. It contains the second loop derivatives, so it represents a (nonlocal!) kinetic term in loop space.

So far, we considered so called decaying turbulence, without external energy source. The energy

\[ E = \int d^3 r \frac{1}{2} v^2_\alpha \]  

(37)

would eventually all dissipate, so that the fluid would stop. In this case the loop wave function \( \Psi \) would asymptotically approach 1

\[ \Psi[C] \xrightarrow{\ t \to \infty \} 1 \]  

(38)

In order to reach the steady state, we add to the right side of the Navier-Stokes equation the usual gaussian random forces \( f_\alpha(r, t) \) with the space dependent correlation function

\[ \langle f_\alpha(r, t)f_\beta(r', t') \rangle = \delta_{\alpha\beta}\delta(t - t')F(r - r') \]  

(39)
concentrated at small wavelengths, i.e. slowly varying with \( r - r' \).

Using the identity

\[
\langle f_\alpha(r,t)\Phi[v(.)]\rangle = \int d^3r' F(r - r') \frac{\delta\Phi[v(.)]}{\delta v_\alpha(r')}
\]

which is valid for arbitrary functional \( \Phi \) we find the following imaginary potential term in the loop hamiltonian

\[
\delta H_C \equiv i U[C] = \frac{i}{\nu} \oint_C dr_\alpha \oint_C dr'_\alpha F(r - r')
\]

Note, that orientation reversal together with complex conjugation changes the sign of the loop hamiltonian, as it should. The potential part involves two loop integrations times imaginary constant. The first term in the kinetic part has one loop integration, one loop derivative times imaginary constant. The second kinetic term has one loop integration, two loop derivatives and real constant. The left side of the loop equation has no loop integrations, no loop derivatives, but has a factor of \( i \).

The relation between the potential and kinetic parts of the loop hamiltonian depends of viscosity, or, better to say, it depends upon the Reynolds number, which is the ratio of the typical circulation to viscosity. In the viscous limit, when the Reynolds number is small, the loop wave function is close to 1. The perturbation expansion in \( \frac{1}{\nu} \) goes in powers of the potential, in the same way, as in Quantum mechanics. The second (nonlocal) term in kinetic part of the hamiltonian also serves as a small perturbation (it corresponds to nonlinear term in the Navier-Stokes equation). The first term of this perturbation expansion is just

\[
\Psi[C] \to 1 - \int \frac{d^3k}{(2\pi)^3} \frac{\tilde{F}(k)}{2\nu^2 k^2} \left| \oint_C dr_\alpha e^{i k r} \right|^2
\]

with \( \tilde{F}(k) \) being the Fourier transform of \( F(r) \). This term is real, as it corresponds to the two-velocity correlation. The next term comes from the triple correlation of velocity, and this term is purely imaginary, so that the dissipation shows up.

This expansion can be derived by direct iterations in the loop space as in [3], inverting the operator in the local part of the kinetic term in the hamiltonian. This expansion is discussed in Appendix A. The results agree with the straightforward iterations of the Navier-Stokes equations in powers of the random force, starting from zero velocity.

So, we have the familiar situation, like in QCD, where the perturbation theory breaks because of the infrared divergencies. For arbitrarily small force, in a large system, the region of small \( k \) would yield large contribution to the terms of the perturbation expansion. Therefore, one should take the opposite WKB limit \( \nu \to 0 \).

In this limit, the wave function should behave as the usual WKB wave function, i.e. as an exponential

\[
\Psi[C] \to \exp \left( \frac{i S[C]}{\nu} \right)
\]
The effective loop Action \( S[C] \) satisfies the loop space Hamilton-Jacobi equation

\[
\dot{S}[C] = -i U[C] + \oint_C dr \int d^3r' \frac{r'_\beta - r_\gamma}{4\pi |r - r'|^3} \frac{\delta S}{\delta \sigma_{\beta\alpha}(r)} \frac{\delta S}{\delta \sigma_{\beta\gamma}(r')}
\]  

(44)

The imaginary part of \( S[C] \) comes from imaginary potential \( U[C] \), which distinguishes our theory from the reversible Quantum mechanics. The sign of \( \Im S \) must be positive definite, since \( |\Psi| < 1 \). As for the real part of \( S[C] \), it changes the sign under the loop orientation reversal \( C(\theta) \rightarrow C(2\pi - \theta) \).

At finite viscosity there would be an additional term

\[
- \nu \oint_C dr \alpha (\partial_\beta S[C]) \frac{\delta S[C]}{\delta \sigma_{\beta\alpha}(r)}
\]  

(45)
on the right of (44). As for the term

\[
- \oint_C dr \alpha (\partial_\beta S[C]) \frac{\delta S[C]}{\delta \sigma_{\beta\alpha}(r)}
\]  

(46)

which formally arises in the loop equation, this term vanishes, since \( \partial_\beta S[C] = 0 \). This operator inserts backtracking at some point at the loop without first applying the loop derivative at this point. As it was discussed in the previous Section, such backtracking does not change the loop functional. This issue was discussed at length in [2], where the Leibnitz rule for the operator \( \partial_\alpha \frac{f}{\delta \sigma_{\beta\gamma}} \) was established

\[
\partial_\alpha \frac{\delta f(g[C])}{\delta \sigma_{\beta\gamma}(r)} = f'(g[C]) \partial_\alpha \frac{\delta g[C]}{\delta \sigma_{\beta\gamma}(r)}
\]  

(47)

In other words, this operator acts as a first order derivative on the loop functional with finite area derivative (so called Stokes type functional). Then, the above term does not appear.

The Action functional \( S[C] \) describes the distribution of the large scale vorticity structures, and hence it should not depend of viscosity. In terms of the above connected correlation functions of the circulation this corresponds to the limit, when the effective Reynolds number \( \Gamma_{\text{eff}}/\nu \) goes to infinity, but the sum of the divergent series tends to the finite limit. According to the standard picture of turbulence, the large scale vorticity structures depend upon the energy pumping, rather than the energy dissipation.

It is understood that both time \( t \) and the loop size \( |C| \) should be greater then the viscous scales

\[
t \gg t_0 = \nu^{\frac{1}{2}} \mathcal{E}^{-\frac{1}{2}}; \quad |C| \gg r_0 = \nu^{\frac{3}{4}} \mathcal{E}^{-\frac{1}{4}}
\]  

(48)

where \( \mathcal{E} \) is the energy dissipation rate.

It is defined from the energy balance equation

\[
0 = \partial_t \left( \frac{1}{2} v_\alpha^2 \right) = \nu \left( v_\alpha \partial_\alpha v_\alpha \right) + \left( f_\alpha v_\alpha \right)
\]  

(49)

\(^4\)As a measure of the loop size one may take the square root of the minimal area inside the loop.
which can be transformed to
\[ \frac{1}{4} \nu \langle \omega_{\alpha\beta}^2 \rangle = 3F(0) \]  
(50)

The left side represents the energy, dissipated at small scale due to viscosity, and the right side - the energy pumped in from the large scales due to the random forces. Their common value is \( \mathcal{E} \).

We see, that constant \( F(r - r') \), i.e., \( \tilde{F}(k) \propto \delta(k) \) is sufficient to provide the necessary energy pumping. However, such forcing does not produce vorticity, which we readily see in our equation. The contribution from this constant part to the potential in our loop equation drops out (this is a closed loop integral of total derivative). This is important, because this term would have the wrong order of magnitude in the turbulent limit - it would grow as the Reynolds number.

Dropping this term, we arrive at remarkably simple and universal functional equation
\[ \dot{S}[C] = \oint_C dr_\alpha \int d^3 r' \frac{r'_\gamma - r_\gamma}{4\pi |r - r'|^3} \frac{\delta S}{\delta \sigma_{\beta\alpha}(r)} \frac{\delta S}{\delta \sigma_{\beta\gamma}(r')} \]  
(51)

The stationary solution of this equation describes the steady distribution of the circulation in the strong turbulence. Note, that the stationary solutions come in pairs \( \pm S \). The sign should be chosen so, that \( \Im S > 0 \), to provide the inequality \( |\Psi| < 1 \).

4 Scaling law

The ‘Hamilton-Jacobi’ equation without the potential term (51) allows the family of the scaling solutions
\[ S[C] = t^{2\kappa-1} \phi \left[ \frac{C}{t^\kappa} \right] \]  
(52)

with arbitrary index \( \kappa \). The scaling function satisfies the equation
\[ (2\kappa - 1)\phi[C] - \kappa \oint_C d\alpha \frac{\delta \phi[C]}{\delta \sigma_{\beta\alpha}(r)} r_\beta = \oint_C d\alpha \int d^3 r' \frac{r'_\gamma - r_\gamma}{4\pi |r - r'|^3} \frac{\delta \phi[C]}{\delta \sigma_{\beta\alpha}(r)} \frac{\delta \phi[C]}{\delta \sigma_{\beta\gamma}(r')} \]  
(53)

The left side here was computed, using the chain rule differentiation of functional.

Asymptotically, at large time, we expect the fixed point, which is the homogeneous functional
\[ S_\infty[C] = |C|^{2-\frac{\kappa}{2}} f \left[ \frac{C}{|C|} \right] \]  
(54)

zeroing the right side of our ‘kinetic’ functional equation
\[ 0 = \oint_C d\alpha \int d^3 r' \frac{r'_\gamma - r_\gamma}{4\pi |r - r'|^3} \frac{\delta S_\infty[C]}{\delta \sigma_{\beta\alpha}(r)} \frac{\delta S_\infty[C]}{\delta \sigma_{\beta\gamma}(r')} \]  
(55)

The Kolmogorov scaling [1] would correspond to
\[ \kappa = \frac{3}{2} \]  
(56)
in which case one can express the $S$ functional in terms of $E$

$$S[C] = E t^2 \phi \left[ \frac{C}{\sqrt{\mathcal{E} t^3}} \right]$$

(57)

One can easily rephrase the Kolmogorov arguments in the loop space. The relation between the energy dissipation rate and the velocity correlator reads

$$\mathcal{E} = \langle v_\alpha(r_0)v_\beta(0)\partial_\beta v_\alpha(0) \rangle$$

(58)

where the point splitting at the viscous scale $r_0$ is introduced. Such splitting is necessary to avoid the viscosity effects; without the splitting the average would formally reduce to the total derivative and vanish.

Instead of the point splitting one may introduce the finite loop of the viscous scale $|C| \sim r_0$, and compute this correlator in presence of such loop. This reduces to the WKB estimates

$$\omega_{\alpha\beta}(r) \to \frac{\delta S[C]}{\delta \sigma_{\alpha\beta}(r)}; \quad v_\alpha(r) = \int d^3r' \frac{r'_\gamma - r_\gamma}{4\pi |r - r'|^3} \omega_{\alpha\gamma}(r')$$

(59)

Using the generic scaling law for $S$ we find

$$\omega \sim r_0^{-1}; \quad v \sim r_0^{1-\frac{1}{\kappa}}; \quad \mathcal{E} \sim r_0^{2-\frac{3}{\kappa}}$$

(60)

We see, that the energy dissipation rate would stay finite in the limit of the vanishing viscous scale only for the Kolmogorov value of the index. This argument looks rather cheap, but I think it is basically correct. The constant value of the energy dissipation rate in the limit of vanishing viscosity arises as the quantum anomaly in the field theory, through the finite limit of the point splitting in the correspondent energy current.

There is another version of this argument, which I like better. The dynamics of Euler fluid in infinite system would not exist, for the non-Kolmogorov scaling. The extra powers of loop size would have to enter with the size $L$ of the whole system, like $(|C| L)^\epsilon$. So, in the regime with finite energy pumping rate $\mathcal{E}$ the infinite Euler system can exist only for the Kolmogorov index. This must be the essence of the original Kolmogorov reasoning.

The problem is that nobody proved that such limit exists, though. Within the usual framework, based on the velocity correlation functions, one has to prove, that the infrared divergencies, caused by the sweep, all cancel for the observables. Within our framework these problems disappear, as we shall see later.

As for the correlation functions in inertial range, unfortunately those cannot be computed in the WKB approximation, since they involve the contour shrinking to a double line, with vanishing area inside. Still, most of the physics can be understood in loop terms, without these correlation functions. The large scale behavior of the loop functional reflects the statistics of the large vorticity structures, encircled by the loop.

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5I am grateful to A. Polyakov and E. Siggia for inspiring comments on this subject.
5 Loop Equation for the Circulation PDF

The loop field could serve as the generating function for the PDF $P_C(\Gamma)$ for the circulation. The Fourier integral

$$P_C(\Gamma) = \int_{-\infty}^{\infty} \frac{dg}{2\pi\nu} \exp \left( \frac{ig}{\nu} \left( \oint_C dr_\alpha v_\alpha(r) - \Gamma \right) \right)$$

(61)

can be analyzed in the same way as the loop field before. The only difference is that the factors of $g$ appear in front of various terms. These factors can be replaced by

$$g \rightarrow i\nu \frac{\partial}{\partial \Gamma}$$

(62)

acting on $P_C(\Gamma)$.

As a result we find

$$\frac{\partial}{\partial \Gamma} \dot{P}_C(\Gamma) = -\oint_C dr_\alpha \int d^3r' \frac{r'_\gamma - r_\gamma}{4\pi |r - r'|^\beta} \frac{\delta^2 P_C(\Gamma)}{\delta \sigma_\beta (r) \delta \sigma_\gamma (r')} + \nu \frac{\partial}{\partial \Gamma} \oint_C dr_\alpha \frac{\delta P_C(\Gamma)}{\delta \sigma_\beta (r)} - U[C] \frac{\partial^2 P_C(\Gamma)}{\partial \Gamma^3}$$

(63)

All the imaginary units disappear, as they should. As for the viscosity and forcing, these terms can be neglected in inertial range in the same way as before. The only new thing is that one has to assume that $\Gamma \gg \nu$ in inertial range in addition to above assumptions about the size of the loop.

In absence of these terms there are no dimensional parameters so that the following scaling laws hold (with the same index $\kappa$ as before)

$$P_C(\Gamma) = t^{2\kappa - 1} F \left[ \frac{C}{t^\kappa}, \frac{\Gamma}{t^{2\kappa - 1}} \right]$$

(64)

The factor $t^{2\kappa-1}$ came from the normalization of probability density. Note, that this is more general law than before. Here we do not have to use the WKB approximation for the PDF. In other words, the whole PDF rather than just its decay at large $\Gamma$ satisfies the scaling law.

The steady distribution would have the form of

$$P_C(\Gamma) \rightarrow \frac{1}{\Gamma} \Phi \left[ \frac{C}{\Gamma^{2\kappa-1}} \right]$$

(65)

where the scaling functional $\Phi$ satisfies the homogeneous equation

$$\oint_C dr_\alpha \int d^3r' \frac{r'_\gamma - r_\gamma}{4\pi |r - r'|^\beta} \frac{\delta^2 \Phi[C]}{\delta \sigma_\beta (r) \delta \sigma_\gamma (r')} = 0$$

(66)

with the normalization condition

$$1 = \int_{-\infty}^{\infty} \frac{d\Gamma}{\Gamma} \Phi \left[ \frac{C}{\Gamma^{2\kappa-1}} \right]$$

(67)
In principle, there could be different scaling functions for positive and negative $\Gamma$, rather than just absolute value $|\Gamma|$ prescription. This would correspond to above mentioned violation of the time reversal symmetry. However, as we mentioned above, there is no exact relation which would eliminate the symmetric solution.

The Kolmogorov triple correlation function vanishes for vorticities (see Appendix I), so that there is no restriction on the asymmetric part of the circulation PDF. Nevertheless, the Kolmogorov scaling $\kappa = \frac{3}{2}$ seems to me the most likely possibility, by the reasons discussed in the previous section.

The homogeneous loop equation requires some boundary conditions at large loops, to provide a meaningful solution. The asymptotic decrease of PDF

$$P_C(\Gamma) \sim \exp \left(-Q \left[ \frac{C}{\pi \kappa \Gamma} \right] \right), Q \rightarrow \infty$$

would lead to the same WKB equation as before

$$\oint_C dr_\alpha \int d^3r' \frac{r'_\gamma - r_\gamma}{4\pi |r - r'|^3} \frac{\delta Q[C]}{\delta \sigma_{\beta\alpha}(r)} \frac{\delta Q[C]}{\delta \sigma_{\beta\gamma}(r')} = 0$$

We are studying this equation in the next section.

6 Tensor Area law

The Wilson loop in QCD decreases as exponential of the minimal area, encircled by the loop, leading to the quark confinement. What is the similar asymptotic law in turbulence? The physical mechanisms leading to the area law in QCD are absent here. Moreover, there is no guarantee, that $\Psi[C]$ always decreases with the size of the loop.

This makes it possible to look for the simple Anzatz, which was not acceptable in QCD, namely

$$S[C] = s \left( \Sigma_{\mu\nu}^C \right)$$

where

$$\Sigma_{\mu\nu}^C = \oint_C r_\mu dr_\nu$$

is the tensor area encircled by the loop $C$. The difference between this area and the scalar area is the positivity property. The scalar area vanishes only for the loop which can be contracted to a point by removal of all the backtracking. As for the tensor area, it vanishes, for example, for the 8 shaped loop, with opposite orientation of petals.

Thus, there are some large contours with vanishing tensor area, for which there would be no decrease of the $\Psi$ functional. In QCD the Wilson loops must always decrease at large distances, due to the finite mass gap. Here, the large scale correlations are known to exist, and play the central role in the turbulent flow. So, I see no convincing arguments to reject the tensor area Anzatz.
This Ansatz in QCD not only was unphysical, it failed to reproduce the correct short-distance singularities in the loop equation. In turbulence, there are no such singularities. Instead, there are the large-distance singularities, which all should cancel in the loop equation.

It turns out, that for this Ansatz the (turbulent limit of the) loop equation is satisfied automatically, without any further restrictions. Let us verify this important property. The first area derivative yields

$$\omega_{\mu \nu}^C (r) = \frac{\delta S}{\delta \sigma_{\mu \nu} (r)} = 2 \frac{\partial s}{\partial \Sigma_{\mu \nu}^C}$$  \hspace{1cm} (72)

The factor of 2 comes from the second term in the variation

$$\frac{\delta \Sigma_{\alpha \beta}^C}{\delta \sigma_{\mu \nu} (r)} = \delta_{\alpha \mu} \delta_{\beta \nu} - \delta_{\alpha \nu} \delta_{\beta \mu}$$  \hspace{1cm} (73)

Note, that the right side does not depend on \( r \). Moreover, you can shift \( r \) aside from the base loop \( C \), with proper wires inserted. The area derivative would not change, as the contribution of wires drops.

This implies, that the corresponding vorticity \( \omega_{\mu \nu}^C (r) \) is space independent, it only depends upon the loop itself. The velocity can be reconstructed from vorticity up to irrelevant constant terms

$$v_{\beta}^C (r) = \frac{1}{2} r_\alpha \omega_{\alpha \beta}^C$$  \hspace{1cm} (74)

This can be formally obtained from the above integral representation

$$v_{\beta}^C (r) = \int d^3 r' \frac{r_\alpha - r'_\alpha}{4\pi |r - r'|^3} \omega_{\alpha \beta}^C$$  \hspace{1cm} (75)

as a residue from the infinite sphere \( R = |r'| \rightarrow \infty \). One may insert the regularizing factor \( |r'|^{-\epsilon} \) in \( \omega \), compute the convolution integral in Fourier space and check that in the limit \( \epsilon \rightarrow 0^+ \) the above linear term arises. So, one can use the above form of the loop equation, with the analytic regularization prescription.

Now, the \( v \omega \) term in the loop equation reads

$$\oint_C d r_\gamma v_{\beta}^C (r) \omega_{\gamma \beta}^C \propto \Sigma_{\gamma \alpha}^C \omega_{\alpha \beta}^C \omega_{\beta \gamma}^C$$  \hspace{1cm} (76)

This tensor trace vanishes, because the first tensor is antisymmetric, and the product of the last two antisymmetric tensors is symmetric with respect to \( \alpha \gamma \).

So, the positive and negative terms cancel each other in our loop equation, like the "income" and "outcome" terms in the usual kinetic equation. We see, that there is an equilibrium in our loop space kinetics.

From the point of view of the notorious infrared divergencies in turbulence, the above calculation explicitly demonstrates how they cancel. By naive dimensional counting these
terms were linearly divergent. The space isotropy lowered this to logarithmic divergency in (75), which reduced to finite terms at closer inspection. Then, the explicit form of these terms was such, that they all cancelled.

This cancellation originates from the angular momentum conservation in fluid mechanics. The large loop $C$ creates the macroscopic eddy with constant vorticity $\omega^C_{\alpha\beta}$ and linear velocity $v^C(r) \propto r$. This is a well known static solution of the Navier-Stokes equation. The eddy is conserved due to the angular momentum conservation. The only nontrivial thing is the functional dependence of the eddy vorticity upon the shape and size of the loop $C$. This is a function of the tensor area $\Sigma^C_{\mu\nu}$, rather than a general functional of the loop.

Combining this Anzatz with the space isotropy and the Kolmogorov scaling law, we arrive at the tensor area law

$$\Psi[C] \propto \exp \left( -B \left( \frac{\mathcal{E}}{\nu^3} \left( \Sigma^C_{\alpha\beta} \right)^2 \right)^{\frac{1}{3}} \right)$$

(77)

The universal constant $B$ here must be real, in virtue of the loop orientation symmetry. When the orientation is reversed $C(\theta) \rightarrow C(2\pi - \theta)$, the loop integral changes sign, but its square, which enters here, stays invariant. Therefore, the constant in front must be real. The time reversal tells the same, since both viscosity $\nu$ and the energy dissipation rate $\mathcal{E}$ are time-odd. Therefore, the ratio $\frac{\mathcal{E}}{\nu^3}$ is time-even, hence it must enter $\Psi[C]$ with the real coefficient. Clearly, this coefficient $B$ must be positive, since $|\Psi[C]| < 1$.

Note, however, that we did not prove this law. The absence of decay for large twisted loops with zero tensor area is suspicious. Also, the physics seems to be different from what we expect in turbulence. The uniform vorticity, even a random one, as in this solution, contrasts the observed intermittent distribution. Besides, there clearly must be corrections to the asymptotic law, whereas the tensor area law is exact. This is far too simple. We discussed this unphysical solution mostly as a test of the loop technology.

7 Scalar Area law

Let us now study the scalar area law, which is a valid Anzatz for the asymptotic decay of the circulation PDF. The set of equations for the minimal surface is summarized in Appendix A. All we need here is the following representation

$$A \rightarrow \frac{1}{2L_T^2} \int \int d\sigma_{\mu\nu}(x)d\sigma_{\mu\nu}(y) \exp \left( -\pi \frac{(x - y)^2}{L_T^2} \right)$$

(78)

where $L_T = |\Gamma|^\frac{3}{2} \mathcal{E}^{-\frac{1}{4}}$. The distance $(x - y)^2$ is measured in 3-space and integration goes along the minimal surface. It is implied that its size is much larger than $L_T$.

In this limit the integration over, say, $y$ can be performed along the local tangent plane at $x$ in small vicinity $y - x \sim L_T$, after which the factors of $L_T$ cancel. We are left then
with the ordinary scalar area integral

\[ A \to \frac{1}{2} \int d\sigma_{\mu\nu}(x)d\sigma_{\mu\nu}(y)\delta^2(x - y) \to \int d^2x \sqrt{g} \quad (79) \]

In the previous, regularized form the area represents so called Stokes functional \[^2\], which can be substituted into the loop equation. The area derivative of the area reads

\[ \frac{\delta A}{\delta \sigma_{\mu\nu}(x)} = \frac{1}{L_t^2} \int d\sigma_{\mu\nu}(y) \exp \left( -\pi \frac{(x - y)^2}{L_t^2} \right) \quad (80) \]

In the local limit this reduces to the tangent tensor

\[ \frac{\delta A}{\delta \sigma_{\mu\nu}(x)} \to \int d\sigma_{\mu\nu}(y) \delta^2(x - y) = t_{\mu\nu}(x) \quad (81) \]

It is implied that the point \( x \) approaches the contour from inside the surface, so that the tangent tensor is well defined

\[ t_{\mu\nu}(x) = t_{\mu}^{\phantom{\mu} \nu} - t_{\nu}^{\phantom{\nu} \mu} \quad (82) \]

Here \( t_{\mu} \) is the local tangent vector of the loop, and \( n_{\nu} \) is the inside normal to the loop along the surface.

The second area derivative of the regularized area in this limit is just the exponential

\[ \frac{\delta^2 A}{\delta \sigma_{\alpha\beta}(x)\delta \sigma_{\gamma\beta}(y)} = \frac{1}{L_t^2} \exp \left( -\pi \frac{(x - y)^2}{L_t^2} \right) \quad (83) \]

Should we look for the higher terms of the asymptotic expansion at large area we would have to take into account the shape of the minimal surface, but in the thermodynamical limit we could neglect the curvature of the loop and use the planar disk.

Let us use the general WKB form of PDF

\[ P_{C}(\Gamma) = \frac{1}{\Gamma} \exp \left( -Q \left( \frac{A}{t^{2\kappa}}, \frac{\Gamma}{t^{2\kappa - 1}} \right) \right) \quad (84) \]

We shall skip the arguments of effective action \( Q \). We find on the left side of the loop equation

\[ \partial_t Q \partial_\Gamma Q - \partial_t \partial_\Gamma Q \quad (85) \]

On the right side we find the following integrand

\[ \left( (\partial Q)^2 - \partial^2 Q \right) \frac{\delta A}{\delta \sigma_{\alpha\beta}(r)} \frac{\delta A}{\delta \sigma_{\gamma\beta}(r')} - \partial_\Gamma Q \frac{\delta^2 A}{\delta \sigma_{\alpha\beta}(r)\delta \sigma_{\gamma\beta}(r')} \quad (86) \]

The last term drops after the \( r' \) integration in virtue of symmetry. The leading terms in the WKB approximation on both sides are those with the first derivatives. We find

\[ \partial_t Q \partial_\Gamma Q = (\partial Q)^2 \int_C dr_\alpha \frac{\delta A}{\delta \sigma_{\alpha\beta}(r)} \int d^3r' \frac{r_\gamma - r'_\gamma}{4\pi |r - r'|^3} \frac{\delta A}{\delta \sigma_{\gamma\beta}(r')} \quad (87) \]
In the last integral we substitute above explicit form of the area derivatives and perform the $d^3r'$ integration first. In the thermodynamical limit only the small vicinity $r' - y \sim L_\Gamma$ contributes, and we find

$$\int d^3r' \frac{r_\gamma - r'_\gamma}{4\pi |r - r'|^3} \frac{\delta A}{\delta \sigma_{\gamma\beta}(r')} \to L_\Gamma^2 \int d\sigma_{\gamma\beta}(y) \frac{r_\gamma - y_\gamma}{4\pi |r - y|^3}$$

(88)

This integral logarithmically diverges. We compute it with the logarithmic accuracy with the following result

$$\int d\sigma_{\gamma\beta}(y) \frac{r_\gamma - y_\gamma}{4\pi |r - y|^3} \propto \frac{t_\beta}{\pi} \ln \frac{L_\Gamma^2}{A}$$

(89)

The meaning of this integral is the average velocity in the WKB approximation. This velocity is tangent to the loop, up to the next correction terms at large area.

Now, the emerging loop integral vanishes due to symmetry

$$\oint_C dr_\alpha t_\beta t_{\alpha\beta} = 0$$

(90)

as the line element $dr_\alpha$ is directed along the tangent vector $t_\alpha$, and the tangent tensor $t_{\alpha\beta}$ is antisymmetric. Similar mechanism was used in the tensor area solution, only there the cancellations emerged at the global level, after the closed loop integration. Here the right side of the loop equation vanishes locally, at every point of the loop. Anyway, we see, that the scalar area indeed represents the steady solution of the loop equation in the leading WKB approximation.

It might be instructive to compare this solution with another known exact solution of the Euler dynamics, namely the Gibbs solution

$$P[v] = \exp \left( -\beta \int d^3r \frac{1}{2} v_\alpha^2 \right)$$

(91)

For the loop functional it reads

$$\Psi_C(\gamma) = \exp \left( -\frac{\gamma^2}{2\beta} \oint_C dr_\alpha \oint_C dr'_\beta \delta^3(r - r') \right)$$

(92)

The integral diverges, and it corresponds to the perimeter law

$$\oint_C dr_\alpha \oint_C dr'_\beta \delta^3(r - r') \to r_0^{-2} \oint_C |dr|$$

(93)

where $r_0$ is a small distance cutoff. For the PDF it yields

$$P(\Gamma) \propto \exp \left( -\frac{\Gamma^2}{2} \frac{\beta r_0^2}{\oint_C |dr|} \right)$$

(94)

When the Gibbs solution is substituted into the loop equation, we observe the same thing. Average velocity is tangent to the loop, which leads to vanishing integrand in the
loop equation. The difference is that in our case this is true only asymptotically, there are next corrections.

The shape of the function $Q$ is not fixed by this equation in the leading WKB approximation. In a scale invariant theory it is natural to expect the power law

$$Q \left( \frac{A}{t^{2\nu}}, \frac{\Gamma}{t^{2\nu-1}} \right) \to \text{const} \left( t^{2\nu} A^{1-2\nu} \right)^{\mu}$$

There is one more arbitrary index $\mu$ involved. Even for the Kolmogorov law $\nu = \frac{3}{2}$ the $\Gamma$ dependence remains unknown.

8 Discussion

So, we found two asymptotic solutions of the loop equation in the turbulent limit, not counting the Gibbs solution. It remains to be seen, which one (if any) is realized in turbulent flows. The tensor area solution is mathematically cleaner, but its physical meaning contradicts the intermittency paradigm. It corresponds to the uniform vorticity with random magnitude and random direction, rather that the regions of high vorticity interlaced with regions of low vorticity, observed in the turbulent flows.

The recent numerical experiments\cite{6} favor the scalar area rather than the tensor one. Also, the Kolmogorov scaling was observed in these experiments. The Reynolds number was only $\sim 100$ which was too small to make any conclusions. We have to wait for the experiments (real or numerical) with the Reynolds numbers few orders of magnitude larger.

The scalar area is less trivial than the tensor one. The minimal area as a functional of the loop cannot be represented as any explicit contour integral of the Stokes type, therefore it corresponds to infinite number of higher correlation functions present. Moreover, there could be several minimal surfaces for the same loop, as the equations for the minimal surface are nonlinear. Clearly, the one with the least area should be taken.

The natural generalization of this solution is the string Anzatz where the sum over all surfaces bounded by the loop is taken

$$P_C(\Gamma) = \sum_{S: \partial S = C} \exp \left( -Q \left( \frac{A}{t^{2\nu}}, \frac{\Gamma}{t^{2\nu-1}} \right) \right)$$

At large loop the minimal $Q$ terms will remain. The extremum condition

$$\delta Q = \frac{\partial Q}{\partial A} \delta A = 0$$

will be satisfied for the minimal surface.

However, the sum over random surfaces is not well defined. The recent studies\cite{3} indicate that the typical closed surfaces degenerate to branched polymers. For the surface
bounded by a fixed loop this cannot happen, of course. Still nobody knows how to compute such sums. The loop equation in principle allows to systematically compute the corrections to the area law as the WKB expansion.

The WKB solution is incomplete so far. The leading term in the loop equation is annihilated by arbitrary function of the area (scalar or tensor). The similar ambiguity was present in the Gibbs solution, where arbitrary function of the Hamiltonian satisfied the Liouville equation for the velocity PDF. In that case the ambiguity was removed by extra requirement of thermodynamic limit: only the exponential of the hamiltonian would agree with the factorization of the PDF for two remote parts of the system.

What could be a similar requirement here? The area of the minimal surface represents the effective volume of the system at large loop. The circulation can be written as a surface integral of vorticity, which makes the circulation an extensive variable at this surface. The average vorticity

\[ \bar{\omega} = \frac{\Gamma}{A} \]  

represents an intensive quantity. The thermodynamic limit would then correspond to

\[ Q = A q(\bar{\omega}) \]  

Comparing this with the previous formula for \( Q \) we conclude that \( \mu = 1 \). In this case

\[ Q = \text{const} \ A \bar{\omega}^{2\kappa} \]  

In principle, there could be two different laws for positive and negative \( \Gamma \), due to violation of the time reversal invariance

\[ Q \to q_+ A \left| \bar{\omega} \right|^{2\kappa} \]  

Another line of argument might start with an assumption of decorrelated average vorticity \( \omega_i \) at various parts \( S_i \) of the area \( A_0 \) of the minimal surface. The net circulation, adding up from the large number \( n \sim \frac{A}{A_0} \gg 1 \) of independent random terms \( \omega_i A_0 \) would be a gaussian variable as a consequence of the law of large numbers. We would have then

\[ Q \sim \frac{\Gamma^2}{n A_0^2 \omega_i^2} = \frac{\Gamma^2}{A A_0 \omega_i^2} \]  

This would agree with the previous estimate at

\[ \mu = \frac{1}{\kappa}, A_0 \omega_i^2 \sim A^{1-\frac{1}{\kappa}} \]  

so that

\[ Q \to \text{const} \ \Gamma^2 A^{\frac{1}{\kappa}-2} \]  

The natural assumption here would be that the vorticity variance \( \omega_i \) does not scale with the area \( A \), so that

\[ A_0 \sim A^{1-\frac{1}{\kappa}} \]
The Gaussian behavior (with $\kappa = \frac{3}{2}$) was observed in numerical experiments [6], but the Reynolds number was too low to make conclusions at this point. There could be a scaling function

$$Q = q \left( \Gamma^2 A^{\frac{1}{\kappa}-2} \right)$$

which starts linearly and then grows as a power, say, $q(x) = (1 + ax)^\kappa$. I suggest that this function should be studied in real and numerical experiments. This would teach us something new about turbulence.

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A Loop Expansion

Let us outline the method of direct iterations of the loop equation. The full description of the method can be found in [2]. The basic idea is to use the following representation of the loop functional

\[ \Psi[C] = 1 + \sum_{n=2}^{\infty} \frac{1}{n} \left\{ \oint_C dr_1^{\alpha_1} \ldots \oint_C dr_n^{\alpha_n} \right\}_{\text{cyclic}} W_{\alpha_1\ldots\alpha_n}^n(r_1, \ldots r_n) \]  

(107)

This representation is valid for every translation invariant functional with finite area derivatives (so called Stokes type functional). The coefficient functions \( W \) can be related to these area derivatives. The normalization \( \Psi[0] = 1 \) for the shrunk loop is implied.

In general case the integration points \( r_1, \ldots r_n \) in (107) are cyclicly ordered around the loop \( C \). The coefficient functions can be assumed cyclicly symmetric without loss of generality. However, in case of fluid dynamics, we are dealing with so called abelian Stokes functional. These functionals are characterized by completely symmetric coefficient functions, in which case the ordering of points can be removed, at expense of the extra symmetry factor in denominator

\[ \Psi[C] = 1 + \sum_{n=2}^{\infty} \frac{1}{n!} \oint_C dr_1^{\alpha_1} \ldots \oint_C dr_n^{\alpha_n} W_{\alpha_1\ldots\alpha_n}^n(r_1, \ldots r_n) \]  

(108)

The incompressibility conditions

\[ \partial_{\alpha_k} W_{\alpha_1\ldots\alpha_n}^n(r_1, \ldots r_n) = 0 \]  

(109)

does not impose any further restrictions, because of the gauge invariance of the loop functionals. This invariance (nothing to do with the symmetry of dynamical equations!) follows from the fact, that the closed loop integral of any total derivative vanishes. So, the coefficient functions are defined modulo such derivative terms. In effect this means, that one may relax the incompressibility constraints (109), without changing the loop functional.

To avoid confusion, let us note, that the physical incompressibility constrains are not neglected. They are, in fact, present in the loop equation, where we used the integral representation for the velocity in terms of vorticity. Still, the longitudinal parts of \( W \) drop in the loop integrals.

The loop calculus for the abelian Stokes functional is especially simple. The area derivative corresponds to removal of one loop integration, and differentiation of the corresponding coefficient function

\[ \frac{\delta \Psi[C]}{\delta \sigma_{\mu\nu}(r)} = \sum_{n=1}^{\infty} \frac{1}{n!} \oint_C dr_1^{\alpha_1} \ldots \oint_C dr_n^{\alpha_n} \hat{V}_{\mu\alpha}^{\alpha} W_{\alpha_1\ldots\alpha_n}^{n+1}(r, r_1, \ldots r_n) \]  

(110)

where

\[ \hat{V}_{\mu\alpha}^{\alpha} \equiv \partial_{\mu} \delta_{\nu\alpha} - \partial_{\nu} \delta_{\mu\alpha} \]  

(111)
In the nonabelian case, there would also be the contact terms, with \( W \) at coinciding points, coming from the cyclic ordering \( [2] \). In abelian case these terms are absent, since \( W \) is completely symmetric.

As a next step, let us compute the local kinetic term

\[
\hat{L}\Psi[C] \equiv \oint_C \! dr \partial_{(r)} \frac{\delta \Psi[C]}{\delta \sigma_{\mu\nu}(r)}
\]  \hspace{1cm} (112)

Using above formula for the loop derivative, we find

\[
\hat{L}\Psi[C] = \sum_{n=1}^{\infty} \frac{1}{n!} \oint_C \! dr_1 \oint_C \! dr_2 \cdots \oint_C \! dr_n \partial^2 W_{\alpha_1,\alpha_2,\ldots,\alpha_n}(r,r_1,\ldots,r_n) \]  \hspace{1cm} (113)

The net result is the second derivative of \( W \) with respect to one variable. Note, that the second term in \( \hat{V}_{\mu\nu} \) dropped, as the total derivative in the closed loop integral.

As for the nonlocal kinetic term, it involves the second area derivative off the loop, at the point \( r' \), integrated over \( r' \) with the corresponding Green’s function. Each area derivative involves the same operator \( \hat{V} \), acting on the coefficient function. Again, the abelian Stokes functional simplifies the general framework of the loop calculus. The contribution of the wires cancels here, and the ordering does not matter, so that

\[
\frac{\delta^2 \Psi[C]}{\delta \sigma_{\mu\nu}(r) \delta \sigma_{\mu'\nu'}(r')} = \sum_{n=0}^{\infty} \frac{1}{n!} \oint_C \! dr_1 \cdots \oint_C \! dr_n \hat{V}^a_{\mu\nu} \hat{V}^{'a'}_{\mu'\nu'} W_{\alpha_1,\alpha_2,\ldots,\alpha_n}^{n+2}(r,r',r_1,\ldots,r_n) \]  \hspace{1cm} (114)

Using these relations, we can write the steady state loop equation as follows

Here the light dotted lines symbolize the arguments \( \alpha_k, r_k \) of \( W \), the big circle denotes the loop \( C \), the tiny circles stand for the loop derivatives, and the pair of lines with the arrow denote the Green’s function. The sum over the tensor indexes and the loop integrations over \( r_k \) are implied.

The first term is the local kinetic term, the second one is the nonlocal kinetic term, and the right side is the potential term in the loop equation. The heavy dotted line in this term stands for the correlation function \( F \) of the random forces. Note that this term is an abelian Stokes functional as well.
The iterations go in the potential term, starting with \( \Psi[C] = 1 \). In the next approximation, only the two loop correction \( W^2_{\alpha_1 \alpha_2} (r_1, r_2) \) is present. Comparing the terms, we note, that nonlocal kinetic term reduces to the total derivatives due to the space symmetry (in the usual terms it would be \( \langle v \omega \rangle \) at coinciding arguments), so we are left with the local one.

This yields the equation

\[
\nu^3 \partial^2 W_{\alpha \beta}^2 (r - r') = F(r - r') \delta_{\alpha \beta}
\]

modulo derivative terms. The solution is trivial in Fourier space

\[
W_{\alpha \beta}^2 (r - r') = -\int \frac{d^3k}{(2\pi)^3} \exp (i k (r - r')) \delta_{\alpha \beta} \frac{\hat{F}(k)}{\nu^3 k^2}
\]

Note, that we did not use the transverse tensor

\[
P_{\alpha \beta}(k) = \delta_{\alpha \beta} - \frac{k_\alpha k_\beta}{k^2}
\]

Though such tensor is present in the physical velocity correlation, here we may use \( \delta_{\alpha \beta} \) instead, as the longitudinal terms drop in the loop integral. This is analogous to the Feynman gauge in QED. The correct correlator corresponds to the Landau gauge.

The potential term generates the four point correlation \( F W^2 \), which agrees with the disconnected term in the \( W^4 \) on the left side

\[
W^4_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} (r_1, r_2, r_3, r_4) \to W^2_{\alpha_1 \alpha_2} (r_1 - r_2) W^2_{\alpha_3 \alpha_4} (r_3 - r_4) + W^2_{\alpha_1 \alpha_3} (r_1 - r_3) W^2_{\alpha_2 \alpha_4} (r_2 - r_4) + W^2_{\alpha_1 \alpha_4} (r_1 - r_4) W^2_{\alpha_2 \alpha_3} (r_2 - r_3)
\]

In the same order of the loop expansion, the three point function will show up. The corresponding terms in kinetic part must cancel among themselves, as the potential term does not contribute. The local kinetic term yields the loop integrals of \( \partial^2 W^3 \), whereas the nonlocal one yields \( \hat{V} W^2 \hat{V} W^2 \), integrated over \( d^3r' \) with the Greens’s function \( \frac{(r-r')}{4\pi|r-r'|} \).

The equation has the structure

\[
L \to W^3 + V W^2 = 0
\]

Fig. 7

Now it is clear, that the solution of this equation for \( W^3 \) would be the same three point correlator, which one could obtain (much easier!) by direct iterations of the Navier-Stokes equation.
The purpose of this painful exercise was not to give one more method of developing the expansion in powers of the random force. We rather verified that the loop equations are capable of producing the same results, as the ordinary chain of the equations for the correlation functions.

In above arguments, it was important, that the loop functional belonged to the class of the abelian Stokes functionals. Let us check that our tensor area Anzatz

$$\Sigma_{\alpha\beta}^C = \oint_C r_\alpha dr_\beta$$

belongs to the same class. Taking the square we find

$$\left(\Sigma_{\alpha\beta}^C\right)^2 = \oint_C dr_\beta \oint_C dr'_\beta r_\alpha r'_\alpha = -\frac{1}{2} \oint_C dr_\beta \oint_C dr'_\beta (r - r')^2$$

where the last transformation follows from the fact, that only the cross term in $(r - r')^2$ yields nonzero after double loop integration.

Any expansion in terms of the square of the tensor area reduces, therefore to the superposition of multiple loop integral of the product of $(r_i - r_j)^2$, which is an example of the abelian Stokes functional. In the limit of large area, this could reduce to the fractional power. An example could be, say

$$\Psi[C] = \exp \left( B \left( 1 - \left( 1 + \frac{\mathcal{E} (\Sigma_{\alpha\beta}^C)^2}{\nu^3} \right)^{\frac{1}{2}} \right) \right)$$

One could explicitly verify all the properties of the abelian Stokes functional. This example is not realistic, though, as it does not have the odd terms of expansion. In the real world such terms are present at the viscous scales. According to our solution, this asymmetry disappears in inertial range of loops (which does not apply to velocity correlators at inertial range, as those correspond to shrunk loops).

\section*{B Matrix Model}

The Navier-Stokes equation represents a very special case of nonlinear PDE. There is a well known galilean invariance

$$v_\alpha(r, t) \rightarrow v_\alpha(r - u t, t) + u_\alpha$$

which relates the magnitude of velocity field with the scales of time and space. \footnote{At the same time it tells us that the constant part of velocity if frame dependent, so that it better be eliminated, if we would like to have a smooth limit at large times. Most of notorious large scale divergencies in turbulence are due to this unphysical constant part.} Let us make this relation more explicit.
First, let us introduce the vorticity field

$$\omega_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu$$  \hfill (123)

and rewrite the Navier-Stokes equation as follows

$$\dot{v}_\alpha = \nu \partial_\beta \omega_{\beta\alpha} - v_\beta \omega_{\beta\alpha} - \partial_\alpha w; \ w = p + \frac{v^2}{2}$$  \hfill (124)

This \( w \) is the well known enthalpy density, to be found from the incompressibility condition \(\text{div} v = 0\), i.e.

$$\partial^2 w = \partial_\alpha v_\beta \omega_{\beta\alpha}$$  \hfill (125)

As a next step, let us introduce "covariant derivative" operator

$$D_\alpha = \nu \partial_\alpha - \frac{1}{2} v_\alpha$$  \hfill (126)

and observe that

$$2 [D_\alpha D_\beta] = \nu \omega_{\beta\alpha}$$  \hfill (127)

$$2 D_\beta [D_\alpha D_\beta] + \text{h.c.} = \nu \partial_\beta \omega_{\beta\alpha} - v_\beta \omega_{\beta\alpha}$$  \hfill (128)

where \( \text{h.c.} \) stands for hermitean conjugate.

These identities allow us to write down the following dynamical equation for the covariant derivative operator

$$\dot{D}_\alpha = D_\beta [D_\alpha D_\beta] - D_\alpha W + \text{h.c.}$$  \hfill (129)

As for the incompressibility condition, it can be written as follows

$$[D_\alpha D_\alpha^\dagger] = 0$$  \hfill (130)

The enthalpy operator \( W = \frac{w}{\nu} \) is to be determined from this condition, or, equivalently

$$[D_\alpha [D_\alpha W]] = [D_\alpha, D_\beta [D_\alpha D_\beta]]$$  \hfill (131)

We see, that the viscosity disappeared from these equations. This paradox is resolved by extra degeneracy of this dynamics: the antihermitean part of the \( D \) operator is conserved. Its value at initial time is proportional to viscosity.

The operator equations are invariant with respect to the time independent unitary transformations

$$D_\alpha \rightarrow S^\dagger D_\alpha S; \ S^\dagger S = 1$$  \hfill (132)

and, in addition, to the time dependent unitary transformations with

$$S(t) = \exp \left( \frac{1}{2\nu} t u_\beta \left( D_\beta - D_\beta^\dagger \right) \right)$$  \hfill (133)
corresponding to the galilean transformations.

One could view the operator $D_\alpha$ as the matrix

$$
\langle i | D_\alpha | j \rangle = \int d^3r \psi_i^* (r) \nu \partial_\alpha \psi_j (r) - \frac{1}{2} \psi_i^* (r) v_\alpha (r) \psi_j (r)
$$

(134)

where the functions $\psi_j (r)$ are the Fourier of Tchebyshev functions depending upon the geometry of the problem.

The finite mode approximation would correspond to truncation of this infinite size matrix to finite size $N$. This is not quite the same as leaving $N$ terms in the mode expansion of velocity field. The number of independent parameters here is $O(N^2)$ rather then $O(N)$. It is not clear whether the unitary symmetry is worth paying such a high price in numerical simulations!

The matrix model of Navier-Stokes equation has some theoretical beauty and raises hopes of simple asymptotic probability distribution. The ensemble of random hermitean matrices was recently applied to the problem of Quantum Gravity [3], which led to a genuine breakthrough in the field.

Unfortunately, the model of several coupled random matrices, which is the case here, is much more complicated then the one matrix model studied in Quantum Gravity. The dynamics of the eigenvalues is coupled to the dynamics of the "angular" variables, i.e. the unitary matrices $S$ in above relations. We could not directly apply the technique of orthogonal polynomials, which was so successful in the one matrix problem.

Another technique, which proved to be successful in QCD and Quantum Gravity is the loop equations. This method, which we are discussing at length in this paper, works in field theory problems with hidden geometric meaning. The turbulence proves to be an ideal case, much simpler then QCD or Quantum Gravity.

C The Reduced Dynamics

Let us now try to reproduce the dynamics of the loop field by a simpler Anzatz

$$
\Psi[C] = \left\langle \exp \left( \frac{1}{\nu} \oint dC_\alpha(\theta) P_\alpha(\theta) \right) \right\rangle
$$

(135)

The difference with original definition (5) is that our new function $P_\alpha(\theta)$ depends directly on $\theta$ rather then through the function $v_\alpha (r)$ taken at $r_\alpha = C_\alpha (\theta)$. This is the $d \to 1$ dimensional reduction we mentioned before. From the point of view of the loop functional there is no need to deal with field $v(r)$, one could take a shortcut.

Clearly, the reduced dynamics must be fitted to the Navier-Stokes dynamics of original field. With the loop calculus, developed above, we have all the necessary tools to build this reduced dynamics.
Let us assume some unknown dynamics for the $P$ field

\[ \dot{P}_\alpha(\theta) = F_\alpha(\theta, [P]) \]  

and compare the time derivatives of original and reduced Anzatz. We find in (135) instead of (34)

\[ \frac{i}{\nu} \oint dC_\alpha(\theta) F_\alpha(\theta, [P]) \]  

Now we observe, that $P'$ could be replaced by the functional derivative, acting on the exponential in (135) as follows

\[ \frac{\delta}{\delta C_\alpha(\theta)} \leftrightarrow -i \nu P'_\alpha(\theta) \]  

This means, that one could take the operators of the Section 2, expressing velocity and vorticity in terms of the spike operator, and replace the functional derivative as above. This yields the following formula for the spike derivative

\[ D_\alpha(\theta, \epsilon) = -i \nu \int_0^{\theta + 2\epsilon} d\phi \left( 1 - \frac{\theta + \epsilon - \phi}{\epsilon} \right) P'_\alpha(\phi) = -i \nu \int_{-1}^1 d\mu \sgn(\mu) P_\alpha(\theta + \epsilon(1 + \mu)) \]  

This is the weighted discontinuity of the function $P(\theta)$, which in the naive limit $\epsilon \to 0$ would become the true discontinuity. However, the function $P(\theta)$ has in general the stronger singularities, then discontinuity, so that this limit cannot be taken yet.

Anyway, we arrive at the dynamical equation for the $P$ field

\[ \dot{P}_\alpha = \nu D_\beta \Omega_{\beta\alpha} - V_\beta \Omega_\beta\gamma P_\gamma \]  

where the operators $V, D, \Omega$ of the Section 2 should be regarded as the ordinary numbers, with definition (139) of $D$ in terms of $P$.

All the functional derivatives are gone! We needed them only to prove equivalence of reduced dynamics to the Navier-Stokes dynamics.

The function $P_\alpha(\theta)$ would become complex now, as the right side of the reduced dynamical equation is complex for real $P_\alpha(\theta)$.

Let us discuss this puzzling issue in more detail. The origin of imaginary units was the factor of $i$ in exponential of the definition of the loop field. We had to insert this factor to make the loop field decreasing at large loops as a result of oscillations of the phase factors. Later this factor propagated to the definition of the $P$ field.

Our spike derivative $D$ is purely imaginary for real $P$, and so is our $\Omega$ operator. This makes the velocity operator $V$ real. Therefore the $D\Omega$ term in the reduced equation (140) is real for real $P$ whereas the $V\Omega$ term is purely imaginary.

This does not contradict the moments equations, as we saw before. The terms with even/odd number of velocity fields in the loop functional are real/imaginary, but the
moments are real, as they should be. The complex dynamics of $P$ simply doubles the number of independent variables.

There is one serious problem, though. Inverting the spike operator $D_\alpha$ we implicitly assumed, that it was antihermitean, and could be regularized by adding infinitesimal negative constant to $D_\alpha^2$ in denominator. This, indeed, works perturbatively, in each term of expansion in time, or that in size of the loop, as we checked. However, beyond this expansion there would be a problem of singularities, which arise when $D_\alpha^2(\theta)$ vanishes at some $\theta$.

In general, this would occur for complex $\theta$, when the imaginary and real part of $D_\alpha^2(\theta)$ simultaneously vanish. One could introduce the complex variable

\[ e^{i\theta} = z; \quad e^{-i\theta} = \frac{1}{z}; \quad \oint d\theta = \oint \frac{dz}{iz} \]  

where the contour of $z$ integration encircles the origin around the unit circle. Later, in course of time evolution, these contours must be deformed, to avoid complex roots of $D_\alpha^2(\theta)$.

D Initial Data

Let us study the relation between the initial data for the original and reduced dynamics. Let us assume, that initial field is distributed according to some translation invariant probability distribution, so that initial value of the loop field does not depend on the constant part of $C(\theta)$.

One can expand translation invariant loop field in functional Fourier transform

\[ \Psi[C] = \int DQ \delta_3 \left( \oint d\phi Q(\phi) \right) W[Q] \exp \left( i \oint d\theta C_\alpha(\theta) Q_\alpha(\theta) \right) \]  

which can be inverted as follows

\[ \delta_3 \left( \oint d\phi Q(\phi) \right) W[Q] = \int DC \Psi[C] \exp \left( -i \oint d\theta C_\alpha(\theta) Q_\alpha(\theta) \right) \]  

Let us take a closer look at these formal transformations. The functional measure for these integrations is defined according to the scalar product

\[ (A, B) = \oint \frac{d\theta}{2\pi} A(\theta) B(\theta) \]  

which diagonalizes in the Fourier representation

\[ A(\theta) = \sum_{-\infty}^{+\infty} A_n e^{in\theta}; \quad A_{-n} = A_n^* \]  

\[ (A, B) = \sum_{-\infty}^{+\infty} A_n B_{-n} = A_0 B_0 + \sum_{1}^{+\infty} a'_n b'_n + a''_n b''_n; \quad a'_n = \sqrt{2}\Re A_n, a''_n = \sqrt{2}\Im A_n \]
The corresponding measure is given by an infinite product of the Euclidean measures for the imaginary and real parts of each Fourier component

\[ DQ = d^3Q_0 \prod_1^\infty d^3q'_n d^3q''_n \]  

(147)

The orthogonality of Fourier transformation could now be explicitly checked, as

\[
\int DC \exp \left( i \int d\theta C_\alpha(\theta) (A_\alpha(\theta) - B_\alpha(\theta)) \right) \\
= \int d^3C_0 \prod_1^\infty d^3c'_n d^3c''_n \exp \left( 2\pi i \left( C_0 (A_0 - B_0) + \sum_1^\infty c'_n (a'_n - b'_n) + c''_n (a''_n - b''_n) \right) \right) \\
= \delta^3 (A_0 - B_0) \prod_1^\infty \delta^3 (a'_n - b'_n) \delta^3 (a''_n - b''_n)
\]  

(148)

Let us now check the parametric invariance

\[ \theta \to f(\theta); \; f(2\pi) - f(0) = 2\pi; \; f'(\theta) > 0 \]  

(149)

The functions \(C(\theta)\) and \(P(\theta)\) have zero dimension in a sense, that only their argument transforms

\[ C(\theta) \to C (f(\theta)); \; P(\theta) \to P (f(\theta)) \]  

(150)

The functions \(Q(\theta)\) and \(P'(\theta)\) in above transformation have dimension one

\[ P'(\theta) \to f'(\theta)P' (f(\theta)); \; Q(\theta) \to f'(\theta)Q (f(\theta)) \]  

(151)

so that the constraint on \(Q\) remains invariant

\[ \oint d\theta Q(\theta) = \oint df(\theta)Q (f(\theta)) \]  

(152)

The invariance of the measure is easy to check for infinitesimal reparametrization

\[ f(\theta) = \theta + \epsilon(\theta); \; \epsilon(2\pi) = \epsilon(0) \]  

(153)

which changes \(C\) and \((C, C')\) as follows

\[ \delta C(\theta) = \epsilon(\theta)C'(\theta); \; \delta(C, C') = \oint \frac{d\theta}{2\pi} \epsilon(\theta)2C_\alpha(\theta)C'_\alpha(\theta) = -\oint \frac{d\theta}{2\pi} \epsilon'(\theta)C^2_\alpha(\theta) \]  

(154)

The corresponding Jacobian reduces to

\[ 1 - \oint d\theta \epsilon'(\theta) = 1 \]  

(155)

in virtue of periodicity.

This proves the parametric invariance of the functional Fourier transformations. Using these transformations we could find the probability distribution for the initial data of

\[ P_\alpha(\theta) = -\nu \oint_0^\theta d\phi Q_\alpha(\phi) \]  

(156)
The simplest but still meaningful distribution of initial velocity field is the Gaussian one, with energy concentrated in the macroscopic motions. The corresponding loop field reads

$$\Psi_0[C] = \exp \left( -\frac{1}{2} \oint dC_\alpha(\theta) \oint dC_\alpha(\theta') f(C(\theta) - C(\theta')) \right)$$  \hspace{1cm} (157)$$

where $f(r - r')$ is the velocity correlation function

$$\langle v_\alpha(r)v_\beta(r') \rangle = \left( \delta_{\alpha\beta} - \partial_\alpha \partial_\beta \partial_\mu^{-2} \right) f(r - r')$$  \hspace{1cm} (158)$$

The potential part drops out in the closed loop integral.

The correlation function varies at macroscopic scale, which means that we could expand it in Taylor series

$$f(r - r') \rightarrow f_0 - f_1(r - r')^2 + \ldots$$  \hspace{1cm} (159)$$

The first term $f_0$ is proportional to initial energy density,

$$\frac{1}{2} \langle v_\alpha^2 \rangle = \frac{d - 1}{2} f_0$$  \hspace{1cm} (160)$$

and the second one is proportional to initial energy dissipation rate

$$\mathcal{E}_0 = -\nu \langle v_\alpha \partial_\beta^2 v_\alpha \rangle = 2d(d - 1)\nu f_1$$  \hspace{1cm} (161)$$

where $d = 3$ is dimension of space.

The constant term in (159) as well as $r^2 + r'^2$ terms drop from the closed loop integral, so we are left with the cross term $rr'$

$$\Psi_0[C] \rightarrow \exp \left( -f_1 \oint dC_\alpha(\theta) \oint dC_\alpha(\theta') C_\beta(\theta)C_\beta(\theta') \right)$$  \hspace{1cm} (162)$$

This is almost Gaussian distribution: it reduces to Gaussian one by extra integration

$$\Psi_0[C] \rightarrow \text{const} \int d^3\omega \exp \left( -\omega_{\alpha\beta}^2 \right) \exp \left( 2i\sqrt{f_1} \omega_{\mu\nu} \oint dC_\mu(\theta)C_\nu(\theta) \right)$$  \hspace{1cm} (163)$$

The integration here goes over all $\frac{d(d-1)}{2} = 3$ independent $\alpha < \beta$ components of the antisymmetric tensor $\omega_{\alpha\beta}$. Note, that this is ordinary integration, not the functional one. The physical meaning of this $\omega$ is the random constant vorticity at initial moment.

At fixed $\omega$ the Gaussian functional integration over $C$

$$\int DC \exp \left( i \oint d\theta \left( \frac{1}{\nu} C_\beta(\theta)P_\beta'(\theta) + 2\sqrt{f_1} \omega_{\alpha\beta} C'_\alpha(\theta)C_\beta(\theta) \right) \right)$$  \hspace{1cm} (164)$$

can be performed explicitly, it reduces to solution of the saddle point equation

$$P_\beta'(\theta) = 4\nu \sqrt{f_1} \omega_{\alpha\beta} C'_\alpha(\theta)$$  \hspace{1cm} (165)$$
which is trivial for constant \( \omega \)

\[
C_{\alpha}(\theta) = \frac{1}{4\nu \sqrt{f_1}} \omega_{\alpha\beta}^{-1} P_{\beta}(\theta) \tag{166}
\]

The inverse matrix is not unique in odd dimensions, since \( \text{Det} \omega_{\alpha\beta} = 0 \). However, the resulting pdf for \( P \) is unique. This is the Gaussian probability distribution with the correlator

\[
\langle P_{\alpha}(\theta) P_{\beta}(\theta') \rangle = 2\nu \sqrt{f_1} \omega_{\alpha\beta} \text{sign}(\theta' - \theta) \tag{167}
\]

Note, that antisymmetry of \( \omega \) compensates that of the sign function, so that this correlation function is symmetric, as it should be. However, it is antihermitean, which corresponds to purely imaginary eigenvalues. The corresponding realization of the \( P \) functions is complex!

Let us study this phenomenon for the Fourier components. Differentiating the last equation with respect to \( \theta \) and Fourier transforming we find

\[
\langle P_{\alpha,n} P_{\beta,m} \rangle = \frac{4\nu}{m} \delta_{nm} \sqrt{f_1} \omega_{\alpha\beta} \tag{168}
\]

This cannot be realized at complex conjugate Fourier components \( P_{\alpha,-n} = P_{\alpha,n}^* \) but we could take \( \bar{P}_{\alpha,n} \equiv P_{\alpha,-n} \) and \( P_{\alpha,n} \) as real random variables, with correlation function

\[
\langle \bar{P}_{\alpha,n} P_{\beta,m} \rangle = \frac{4\nu}{m} \delta_{nm} \sqrt{f_1} \omega_{\alpha\beta} ; \ n > 0 \tag{169}
\]

The trivial realization is

\[
\bar{P}_{\alpha,n} = \frac{4\nu}{n} \sqrt{f_1} \omega_{\alpha\beta} P_{\beta,n} \tag{170}
\]

with \( P_{\beta,n} \) being Gaussian random numbers with unit dispersion.

As for the constant part \( P_{\alpha,0} \) of \( P_{\alpha}(\theta) \), it is not defined, but it drops from equations in virtue of translational invariance.

### E W-functional

The difficulties of turbulence are hidden in the loop equation, but they show up, if you try to solve it numerically. The main problem is that one cannot get rid of the cutoffs \( \epsilon, \delta \to 0 \) in the definitions of the spike derivatives. These cutoffs are designed to pick up the singular contributions in the angular integrals, but with finite number of modes, such as Fourier harmonics there would be no singularities. We did not find any way to truncate degrees of freedom in the \( P \) equation, without violating the parametric invariance. It very well may be, that this invariance would be restored in the limit of large number of modes, but it looks that there are too much ambiguity in the finite mode approximation.
After some attempts, we found the simpler version of the loop functional, which can be studied analytically in the turbulent region. This is the generating functional for the scalar products $P_\alpha(\theta_1)P_\alpha(\theta_2)$

$$W[S] = \left\langle \exp \left( - \oint d\theta_1 \oint d\theta_2 S(\theta_1, \theta_2) P_\alpha(\theta_1)P_\alpha(\theta_2) \right) \right\rangle \tag{171}$$

where, as before, the averaging goes over initial data for the $P$ field.

The time derivative of this $W$-functional

$$\dot{W} = -2 \left\langle \oint d\theta_1 \oint d\theta_2 S(\theta_1, \theta_2) P_\alpha(\theta_1)\dot{P}_\alpha(\theta_2) \exp \left( - \oint d\theta_1 \oint d\theta_2 S(\theta_1, \theta_2) P_\alpha(\theta_1)P_\alpha(\theta_2) \right) \right\rangle \tag{172}$$

can be expressed in terms of functional derivatives of $W$ by replacing

$$P_\alpha(\phi_1)P_\alpha(\phi_2) \rightarrow \frac{\delta}{\delta S(\phi_1, \phi_2)} \tag{173}$$

for every scalar product of $P$ fields, which arise after expansion of the spike derivatives (139), (18), (30) in the scalar product

$$P_\alpha(\theta_1)\dot{P}_\alpha(\theta_2) = \nu P_\alpha(\theta_1)D_{\beta}(\theta_2)\Omega_{\beta\alpha}(\theta_2) - P_\alpha(\theta_1)V_{\beta}(\theta_2)\Omega_{\beta\alpha}(\theta_2) \tag{174}$$

This equation has the structure

$$\dot{W} = \oint d^2\theta S(\theta_1, \theta_2) \left( A_2 \left[ \frac{\delta}{\delta S} \right] W + A_3 \left[ \frac{\delta}{\delta S} \right] D^{-2}(\theta, \epsilon)W \right) \tag{175}$$

where $A_k [X]$ stands for the $k-$ degree homogenous functional of the function $X(\theta_1, \theta_2)$.

The operator $D^{-2}$ is also the homogeneous functional of the negative degree $k = -1$. It can be written as follows

$$D^{-2}(\theta, \epsilon)W[S] = \int_0^\infty d\tau W[S + \tau U] \tag{176}$$

with

$$U(\theta_1, \theta_2) = \epsilon^{-2} \text{sgn}(\theta + \epsilon - \theta_1) \text{sgn}(\theta + \epsilon - \theta_2) \tag{177}$$

### F Possible Numerical Implementation

The above general scheme is fairly abstract and complicated. Could it lead to any practical computation method? This would depend upon the success of the discrete approximations of the singular equations of reduced dynamics.

The most obvious approximation would be the truncation of Fourier expansion at some large number $N$. With Fourier components decreasing only as powers of $n$ this
approximation is doubtful. In addition, such truncation violates the parametric invariance which looks dangerous.

It seems safer to approximate $P(\theta)$ by a sum of step functions, so that it is piecewise constant. The parametric transformations vary the lengths of intervals of constant $P(\theta)$, but leave invariant these constant values. The corresponding representation reads

$$P_\alpha(\theta) = \sum_{l=0}^{N} (p_\alpha(l+1) - p_\alpha(l)) \Theta(\theta - \theta_l); \ p(N+1) = p(1), \ p(0) = 0$$

It is implied that $\theta_0 = 0 < \theta_1 < \theta_2 \ldots < \theta_N < 2\pi$. By construction, the function $P(\theta)$ takes value $p(l)$ at the interval $\theta_{l-1} < \theta < \theta_l$.

We could take $\dot{P}(\theta)$ at the middle of this interval as approximation to $\dot{p}(l)$.

$$\dot{p}(l) \approx \dot{P}(\bar{\theta}_l); \ \bar{\theta}_l = \frac{1}{2} (\theta_{l-1} + \theta_l)$$

As for the time evolution of angles $\theta_l$, one could differentiate (178) in time and find

$$\dot{P}_\alpha(\theta) = \sum_{l=0}^{N} (\dot{p}_\alpha(l+1) - \dot{p}_\alpha(l)) \Theta(\theta - \theta_l) - \sum_{l=0}^{N} (p_\alpha(l+1) - p_\alpha(l)) \delta(\theta - \theta_l) \dot{\theta}_l$$

from which one could derive the following approximation

$$\dot{\theta}_l \approx \frac{(p_\alpha(l) - p_\alpha(l+1))}{(p_\mu(l+1) - p_\mu(l))^2} \int_{\theta_l}^{\theta_{l+1}} d\theta \dot{P}_\alpha(\theta)$$

The extra advantage of this approximation is its simplicity. All the integrals involved in the definition of the spike derivative (133) are trivial for the stepwise constant $P(\theta)$. So, this approximation can be in principle implemented at the computer. This formidable task exceeds the scope of the present work, which we view as purely theoretical.

G  Uniqueness of the tensor area law

Let us address the issue of the uniqueness of the tensor area solution. Let us take the following Anzatz

$$S[C] = f \left( \int_C dr_\alpha \int_C dr'_\alpha W(r - r') \right)$$

When substituted into the static loop equation (with the area derivatives computed in Appendix A), it yields the following equation for the correlation function $W(r)$

$$0 = \int_C dr_\alpha \int_C dr'_\beta \int_C dr''_\gamma U_{\alpha\beta\gamma}(r, r', r'')$$

$$U_{\alpha\beta\gamma}(r, r', r'') = W(r - r') \delta_{\alpha\beta} W(r' - r'') + \text{permutations}$$

$$\dot{V}_{\mu\nu} = \delta_{\alpha\nu} \dot{\theta}_\mu - \delta_{\alpha\mu} \dot{\theta}_\nu$$
The derivative $f'$ of the unknown function drops from the static equation.

This equation should hold for arbitrary loop $C$. Using the Taylor expansion for the Stokes type functional $[2]$, we can argue, that the coefficient function $U$ must vanish up to the total derivatives. An equivalent statement is that the third area derivative of this functional must vanish. Using the loop calculus (see Appendix A) we find the following equation

$$0 = \hat{V}_{\mu\nu} \hat{V}_{\mu'\nu'} \hat{V}_{\mu''\nu''} U_{\alpha\alpha'}(r, r', r'')$$

which should hold for arbitrary $r, r', r''$. This leads to the overcomplete system of equations for $W(r)$ in general case. However, for the special case $W(r) = r^2$ which corresponds to the square of the tensor area

$$\Sigma_{\alpha\beta} = -\frac{1}{2} \oint_{C} dr_{\alpha} \oint_{C} dr'_{\beta} (r - r')^2$$

the system is satisfied as a consequence of certain symmetry. In this case we find in the loop equation

$$2 \oint_{C} dr_{\alpha} \oint_{C} dr'_{\beta} (r - r')^2 \oint_{C} dr''_{\alpha}(r''_{\beta} - r''_{\alpha}) \propto \Sigma_{\alpha\beta} \oint_{C} dr_{\alpha} \oint_{C} dr'_{\beta} (r - r')^2$$

The last integral is symmetric with respect to permutations of $\alpha, \beta$, whereas the first factor $\Sigma_{\alpha\beta}$ is antisymmetric, hence the sum over $\alpha\beta$ yields zero, as we already saw above.

It was assumed in above arguments, that the loop $C$ consist of only one connected part. Let us now consider the more general situation, with arbitrary number $n$ of loops $C_1, \ldots C_n$. The corresponding Anzatz would be

$$S_n [C_1, \ldots C_n] = s_n (\Sigma^1, \ldots \Sigma^n)$$

where $\Sigma^i$ are tensor areas.

This function should obey the same WKB loop equations in each variable. Introducing the loop vorticities

$$\omega^k_{\mu\nu} = 2 \frac{\partial s_n}{\partial \Sigma^k_{\mu\nu}}$$

which are constant on each loop, we have to solve the following problem. What are the values of $\omega^k_{\mu\nu}$ such that the single velocity field $v_{\alpha}(r)$ could produce them?

We do not see any other solutions, but the trivial one, with all equal $\omega^k_{\mu\nu}$ and linear velocity, as before. This would correspond to

$$s_n (\Sigma^1, \ldots \Sigma^n) = s_1 (\Sigma) ; \Sigma_{\mu\nu} = \sum_{k=1}^{n} \Sigma^k_{\mu\nu} = \oint_{\gamma C_k} r_{\mu} dr_{\nu}$$

The loop equation would be satisfied like before, with $C = \gamma C_k$. This corresponds to the additivity of loops

$$S_n [C_1, \ldots C_n] = S_1 [\gamma C_k]$$
Note, that such additivity is the opposite to the statistical independence, which would imply that
\[
S_n [C_1, \ldots C_n] = \sum S_1 [C_k]
\] (191)
The additivity could also be understood as a statement, that any set of \( n \) loops is equivalent to a single loop for the abelian Stokes functional. Just connect these loops by wires, and note that the contribution of wires cancels. So, if the area law holds for arbitrary single loop, than it must be additive.

This assumption may not be true, though, as it often happens in the WKB approximation. There is no single asymptotic formula, but rather collections of different WKB regions, with quantum regions in between. In our case, this corresponds to the following situation.

Take the large circular loop, for which the WKB approximation holds, and try to split it into two large circles. You will have to twist the loop like the infinity symbol ∞, in which case it intersects itself. At this point, the WKB approximation might break, as the short distance velocity correlation might be important near the self-intersection point. This may explain the paradox of the vanishing tensor area for the ∞ shaped loop. From the point of view of our area law such loop is not large at all.

H Minimal surfaces

Let us present here the modern view at the classical theory of the minimal surfaces. The minimal surface can be described by parametric equation
\[
S: r_\alpha = X_\alpha (\xi_1, \xi_2)
\] (192)
The function \( X_\alpha(\xi) \) should provide the minimum to the area functional
\[
A[X] = \int_S \sqrt{d\sigma_{\mu\nu}} = \int d^2\xi \sqrt{\text{Det} G}
\] (193)
where
\[
G_{ab} = \partial_a X_\mu \partial_b X_\mu,
\] (194)
is the induced metric. For the general studies it is sometimes convenient to introduce the unit tangent tensor as an independent field and minimize
\[
A [X, t, \lambda] = \int d^2\xi \left( e_{ab} \partial_a X_\mu \partial_b X_\nu t_{\mu\nu} + \lambda \left( 1 - t_{\nu}^2 \right) \right)
\] (195)
From the classical equations we will find then
\[
t_{\mu\nu} = \frac{e_{ab}}{2\lambda} \partial_a X_\mu \partial_b X_\nu ; t_{\mu\nu}^2 = 1,
\] (196)
which shows equivalence to the old definition.
For the actual computation of the minimal area it is convenient to introduce the auxiliary internal metric $g_{ab}$

$$A[X, g] = \frac{1}{2} \int_S d^2 \xi \text{tr} g^{-1} G \sqrt{\text{Det} \ g}. \quad (197)$$

The straightforward minimization with respect to $g_{ab}$ yields

$$g_{ab} \text{tr} g^{-1} G = 2G_{ab}, \quad (198)$$

which has the family of solutions.

$$g_{ab} = \lambda G_{ab}. \quad (199)$$

The local scale factor $\lambda$ drops from the area functional, and we recover original definition.

So, we could first minimize the quadratic functional (197) with respect to $X(\xi)$ (the linear problem), and then minimize with respect to $g_{ab}$ (the nonlinear problem).

The crucial observation is the possibility to choose conformal coordinates, with the diagonal metric tensor

$$g_{ab} = \delta_{ab} \rho, \quad g^{-1}_{ab} = \delta_{ab} \rho, \quad \sqrt{\text{Det} \ g} = \rho; \quad (200)$$

after which the local scale factor $\rho$ drops from the integral

$$A[X, \rho] = \frac{1}{2} \int_S d^2 \xi \partial_a X_\mu \partial_a X_\mu. \quad (201)$$

However, the $\rho$ field is implicitly present in the problem, through the boundary conditions.

Namely, one has to allow an arbitrary parametrization of the boundary curve $C$. We shall use the upper half plane of $\xi$ for our surface, so the boundary curve corresponds to the real axis $\xi_2 = 0$. The boundary condition will be

$$X_\mu(\xi_1, +0) = C \left( f(\xi_1) \right), \quad (202)$$

where the unknown function $f(t)$ is related to the boundary value of $\rho$ by the boundary condition for the metric

$$g_{11} = \rho = G_{11} = (\partial_1 X_\mu)^2 = C_\mu^2 f^2 \quad (203)$$

As it follows from the initial formulation of the problem, one should now solve the linear problem for the $X$ field, compute the area and minimize it as a functional of $f(.)$. As we shall see below, the minimization condition coincides with the diagonality of the metric at the boundary

$$[\partial_1 X_\mu \partial_2 X_\mu]_{\xi_2 = +0} = 0 \quad (204)$$

The linear problem is nothing but the Laplace equation $\partial^2 X = 0$ in the upper half plane with the Dirichlet boundary condition (202). The solution is well known

$$X_\mu(\xi) = \int_{-\infty}^{+\infty} dt \frac{C_\mu \left( f(t) \right) \xi_2}{\pi \left( \xi_1 - t \right)^2 + \xi_2^2} \quad (205)$$

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The area functional can be reduced to the boundary terms in virtue of the Laplace equation

\[ A[f] = \frac{1}{2} \int d^2 \xi \partial_a (X_\mu \partial_a X_\mu) = -\frac{1}{2} \int_{-\infty}^{+\infty} d\xi_1 [X_\mu \partial_2 X_\mu]_{\xi_2 = +0} \] (206)

Substituting here the solution for \( X_\mu \) we find

\[ A[f] = -\frac{1}{2} \Re \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dt' \frac{C_\mu(f(t)) C_\mu(f(t'))}{(t - t' - i0)^2} \] (207)

This can be rewritten in a nonsingular form

\[ A[f] = \frac{1}{4\pi} \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dt' \frac{(C_\mu(f(t)) - C_\mu(f(t')))^2}{(t - t')^2} \] (208)

which is manifestly positive.

Another nice form can be obtained by integration by parts

\[ A[f] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} df \int_{-\infty}^{+\infty} df' C_\mu'(f) C_\mu'(f') \log |t - t'| \] (209)

This form allows one to switch to the inverse function \( \tau(f) \) which is more convenient for optimization

\[ A[\tau] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} df \int_{-\infty}^{+\infty} df' C_\mu'(f) C_\mu'(f') \log |\tau(f) - \tau(f')| \] (210)

In the above formulas it was implied that \( C(\infty) = 0 \). One could switch to more traditional circular parametrization by mapping the upper half plane inside the unit circle

\[ \xi_1 + i \xi_2 = \frac{1 - \omega}{1 + \omega}; \omega = re^{i \alpha}; r \leq 1. \] (211)

The real axis is mapped at the unit circle. Changing variables in above integral we find

\[ X_\mu(r, \alpha) = \Re \int_{-\pi}^{\pi} \frac{d\theta}{\pi} C_\mu(\phi(\theta)) \left( \frac{1}{1 - r \exp (i \alpha - i \theta)} - \frac{1}{1 + \exp (-i \theta)} \right) \] (212)

Here

\[ \phi(\theta) = f \left( \tan \frac{\theta}{2} \right). \] (213)

The last term represents an irrelevant translation of the surface, so it can be dropped. The resulting formula for the area reads

\[ A[\phi] = \frac{1}{4\pi} \int_{-\pi}^{\pi} d\theta \int_{-\pi}^{\pi} d\theta' \frac{(C_\mu(\phi(\theta)) - C_\mu(\phi(\theta')))^2}{|e^{i\theta} - e^{i\theta'}|^2} \] (214)

or, after integration by parts and inverting parametrization

\[ A[\theta] = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \int_{-\pi}^{\pi} d\phi' C_\mu'(\phi) C_\mu'(\phi') \log \left| \sin \frac{\theta(\phi) - \theta(\phi')}{2} \right| \] (215)
Let us now minimize the area as a functional of the boundary parameterization $f(t)$ (we shall stick to the upper half plane). The straightforward variation yields

$$0 = \Re \int_{-\infty}^{\infty} dt' \frac{C'_\mu(f(t')) C''_\mu(f(t))}{(t-t'+i0)^2}$$

which duplicates the above diagonality condition (204). Note that in virtue of this condition the normal vector $n_\mu(x)$ is directed towards $\partial_2 X_\mu$ at the boundary. Explicit formula reads

$$n_\mu(C(f(t))) \propto \Re \int_{-\infty}^{\infty} dt' \frac{C'_\mu(f(t'))}{(t-t'+i0)^2}$$

Let us have a closer look at the remaining nonlinear integral equation (216). In terms of inverse parametrization it reads

$$0 = \Re \int_{-\infty}^{\infty} df C'_\mu(f) C'_\mu(f') \tau(f) - \tau(f') + i0$$

Introduce the vector set of analytic functions

$$F_\mu(z) = \int_{-\infty}^{\infty} \frac{df}{\pi} \frac{C'_\mu(f)}{\tau(f) - z}$$

which decrease as $z^{-2}$ at infinity. The discontinuity at the real axis

$$\Im F_\mu(\tau \pm i0) = C'_\mu(f)f'(\tau)$$

Which provides the implicit equation for the parametrization $f(\tau)$

$$\int d\tau \Im F_\mu(\tau + i0) = C_\mu(f)$$

We see, that the imaginary part points in the tangent direction at the boundary. As for the boundary value of the real part of $F_\mu(\tau)$ it points in the normal direction along the surface

$$\Re F_\mu \propto n_\mu$$

Inside the surface there is no direct relation between the derivatives of $X_\mu(\xi)$ and $F_\mu(\xi)$. The integral equation (216) reduces to the trivial boundary condition

$$F_\mu^2(t + i0) = F_\mu^2(t - i0)$$

In other words, there should be no discontinuity of $F_\mu^2$ at the real axis. The solution compatible with analyticity in the upper half plane and $z^{-2}$ decrease at infinity is

$$F_\mu^2(z) = (1 + \omega)^4 P(\omega); \quad \omega = \frac{2 - z}{\tau + z}$$

where $P(\omega)$ defined by a series, convergent at $|\omega| \leq 1$. In particular this could be a polynomial. The coefficients of this series should be found from an algebraic minimization problem, which cannot be pursued forward in general case.
The flat loops are trivial though. In this case the problem reduces to the conformal transformation mapping the loop onto the unit circle. For the unit circle we have simply

\[ C_1 + iC_2 = \omega; \quad F_1 = iF_2 = -\frac{(1 + \omega)^2}{2}; \quad P = 0. \]  

(225)

Small perturbations around the circle or any other flat loop can be treated in a systematic way, by a perturbation theory.

I  Kolmogorov triple correlation and time reversal

Are there any restrictions on the circulation PDF from the known asymmetry of velocity correlations, in particular, the Kolmogorov triple correlation? The answer is that the Kolmogorov correlation does not imply the asymmetry of vorticity correlations.

Taking the tensor version of the $\frac{4}{5}$ law in arbitrary dimension $d$

\[ \langle v_\alpha(0)v_\beta(0)v_\gamma(r) \rangle = \frac{\mathcal{E}}{(d-1)(d+2)} \left( \delta_{\alpha\gamma}r_\beta + \delta_{\beta\gamma}r_\alpha - \frac{2}{d}\delta_{\alpha\beta}r_\gamma \right) \]  

(226)

and differentiating, we find that

\[ \langle v_\alpha(0)v_\beta(0)\omega_{\gamma\lambda}(r) \rangle = 0 \]  

(227)

So, the odd vorticity correlations could, in fact, be absent, in spite of the asymmetry of the velocity distribution.
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