THE THREE-NEUTRINO MIXING AND OSCILLATIONS

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Abstract

The basics of neutrino oscillations is presented. Existing evidences of neutrino oscillations, obtained in the atmospheric and solar neutrino experiments, are considered. The new CHOOZ bound on the element $|U_{e3}|^2$, obtained from the three-neutrino analysis of the data, is discussed. Decoupling of neutrino oscillations in the solar neutrino range of $\Delta m^2$ is considered.

1 Introduction

There exist at present strong model independent evidences in favor of neutrino oscillations, obtained in the atmospheric [1, 2, 3] and in the solar [4, 5, 6, 7, 8, 9, 10] neutrino experiments. The direct evidence for oscillations of atmospheric neutrinos is the significant up-down asymmetry of the high-energy muon events, observed in the atmospheric Super-Kamiokande (S-K) experiment [1]. The three-sigma proof of the presence of $\nu_\mu$ and $\nu_\tau$ in the flux of the solar neutrinos on the earth that stems from the comparison of the results of the SNO [10] and S-K [9] solar neutrino experiments is the direct evidence for the oscillation of solar neutrinos.

Indications in favor of $\bar{\nu}_\mu \to \bar{\nu}_e$ oscillations were obtained in the accelerator LSND experiment [11]. This result requires confirmation and it will be checked by the MiniBooNE experiment [12], started recently. We will not consider LSND result here.

The data of the S-K atmospheric neutrino experiment are perfectly described if we assume that the two-neutrino $\nu_\mu \to \nu_\tau$ oscillations take place. From the analysis of the data of the S-K experiment the following best-fit values for the neutrino oscillation parameters were found [1].
\[ \Delta m_{\text{atm}}^2 = 2.5 \cdot 10^{-3} \text{eV}^2; \quad \sin^2 2\theta_{\text{atm}} = 1. \quad (1) \]

The data of all solar neutrino experiments can be described if we assume that the probability of solar $\nu_e$ to survive has the two-neutrino form is characterized by two parameters $\Delta m_{\text{sol}}^2$ and $\tan^2 \theta_{\text{sol}}$. From the global analysis of all solar neutrino data several allowed regions in the plane of the oscillation parameters were found \[13, 14, 15, 16\]. For the most favorable LMA MSW region the best-fit values of the parameters are \[15\]

\[ \Delta m_{\text{sol}}^2 = 3.7 \cdot 10^{-5} \text{eV}^2; \quad \tan^2 \theta_{\text{sol}} = 3.7 \cdot 10^{-1}. \quad (2) \]

Let us note that for other allowed regions the values of $\Delta m_{\text{sol}}^2$ are significantly smaller than for the LMA region. Thus, from the analysis of the existing neutrino oscillation data it follows that neutrino mass squared differences, relevant for the oscillations of the solar and atmospheric neutrinos, satisfy the hierarchy relation

\[ \Delta m_{\text{sol}}^2 \ll \Delta m_{\text{atm}}^2. \quad (3) \]

### 2 Neutrino mixing

Neutrino oscillations are driven by the neutrino mixing

\[ \nu_{\alpha L} = \sum_{i=1}^{3} U_{\alpha i} \nu_{i L}, \quad (\alpha = e, \mu, \tau) \quad (4) \]

where $\nu_{\alpha L}$ is the flavor neutrino field, $\nu_i$ is the field of neutrino (Dirac or Majorana) with mass $m_i$ and $U$ is the unitary mixing matrix.

From the data of the SLC and LEP experiments on the measurement of the width of the decay of the $Z$-boson into neutrino-antineutrino pairs it follows that only three flavor neutrinos exist in nature. In the latest LEP experiments, for the number of the flavor neutrinos the value \[17\]

\[ n_{\nu_f} = 3.00 \pm 0.06 \]

was obtained.

The minimal number of massive neutrinos is equal to the number of flavor neutrinos (three). If the number of massive neutrinos $n$ is larger than three, in this case $n - 3$ sterile neutrinos must exist (see, for example, \[18\]).

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The data of the solar and atmospheric neutrino experiments can be described in the framework of the minimal scheme with the number of massive neutrinos being equal to the measured number of flavor neutrinos. Let us note, that if LSND data will be confirmed, it would mean that a third independent mass-squared difference must exist and the minimal number of massive neutrinos must be equal to four.

From (4) for the state of the flavor neutrino with momentum \( \vec{p} \) we have

\[
|\nu_\alpha\rangle = \sum_i U_{\alpha i}^* |\nu_i\rangle,
\]

(5)

where \( |\nu_i\rangle \) is the state of neutrino with momentum \( \vec{p} \) and energy \( E_i = \sqrt{p^2 + m^2_i} \approx p + \frac{m^2_i}{2p} \).

The relation (5) is the basic one. We will make a few relevant comments. Let us consider a decay

\[ a \to b + l^+ + \nu_l, \]

in which together with a lepton \( l^+ \) a flavor neutrino \( \nu_l \) is produced. For the state of neutrinos with momentum \( \vec{p} \) we have from relation (4)

\[
|\nu >_{\vec{p}} = \sum_i |\nu_i\rangle \langle i b l^+ |S| a \rangle,
\]

(7)

where \( |\nu_i\rangle \) is the state of neutrino with momentum \( \vec{p} \) and energy \( E_i \).

Taking into account that neutrino mass-squared differences \( \Delta m^2 \) are much smaller than the square of neutrino energy \( (\Delta m^2/E^2 \lesssim 10^{-15}) \), we have \[19\]

\[
\langle i b l^+ |S| a \rangle \simeq U_{li}^* \langle \nu_l b l^+ |S| a \rangle_{SM}.
\]

(8)

Here \( \langle \nu_l b l^+ |S| a \rangle_{SM} \) is the Standard Model matrix element of the process (4), calculated under the assumption that the mass of \( \nu_l \) is equal to the minimal neutrino mass \( m_1 \). This matrix element does not depend on \( i \).

From Eq. (4) and Eq. (8) we obtain relation (3) for the normalized state of the flavor neutrino.

We will consider next the standard parameterization of the neutrino mixing matrix \( U \). If \( \nu_i \) are Dirac particles, the unitary \( n \times n \) mixing matrix

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2 Due to the uncertainty relation, the emission (absorption) of neutrinos with different masses and very small mass-squared differences can not be resolved. This is the reason, why flavor neutrinos are described by the coherent superposition of states of neutrinos with definite masses (see [21]).
is characterized by \( n_\theta = n(n - 1)/2 \) angles and \( n_\phi = (n - 1)(n - 2)/2 \) phases. If \( \nu_i \) are Majorana particles, the mixing matrix is characterized by \( n_\theta = n(n - 1)/2 \) angles and \( n_\phi = n(n - 1)/2 \) phases. The additional \( n - 1 \) phases in the Majorana case do not enter, however, into the expressions for the neutrino transition probabilities \([21, 22]\). Thus, in the case of three massive neutrinos the neutrino mixing matrix in the expression for the neutrino transition probability both in the Dirac and in the Majorana cases is characterized by three mixing angles and one phase.

In order to parameterize the \( 3 \times 3 \) PMNS (Pontecorvo-Maki-Nakagawa-Sakata) mixing matrix \( \) we will construct three vectors \( |\nu_e\rangle, |\nu_\mu\rangle \) and \( |\nu_\tau\rangle \), that satisfy the condition

\[
\langle \nu_{\alpha'} | \nu_\alpha \rangle = \delta_{\alpha' \alpha}.
\]  

Let us start with \( |\nu_e\rangle \). Taking into account that

\[
\sum_{i=1,2} |U_{ei}|^2 = 1 - |U_{e3}|^2
\]

we will introduce the angle \( \theta_{12} \) in the following way

\[
U_{e1} = \sqrt{1 - |U_{e3}|^2} \cos \theta_{12}; \quad U_{e2} = \sqrt{1 - |U_{e3}|^2} \sin \theta_{12}.
\]

For the vector \( \nu_e \) we have

\[
|\nu_e\rangle = \sqrt{1 - |U_{e3}|^2} \ |n_{12}\rangle + U_{e3} \ |3\rangle,
\]

where

\[
|n_{12}\rangle = \cos \theta_{12} \ |1\rangle + \sin \theta_{12} \ |2\rangle.
\]

The CP violating phase \( \delta \) is introduced as follows

\[
U_{e3} = |U_{e3}| \exp(-i\delta).
\]

Two unit vectors

\[^3 \text{B. Pontecorvo considered neutrino oscillations in 1958 after the } V - A \text{ theory of the weak interaction was proposed. At that time only one type of neutrino was known. In his first pioneering paper} \] 

\[\text{he considered neutrino oscillations as a phenomenon analogous to the oscillations of neutral kaons. Later, after the second neutrino was discovered, it was not difficult for him} \]

\[\text{to generalize the idea of oscillations for the case of two flavor neutrinos} \] 

\[\text{Maki, Nakagawa and Sakata in 1962 proposed the mixing of two massive neutrinos.} \]
\[ |n_{12}^\perp \rangle = -\sin \theta_{12} \ |1\rangle + \cos \theta_{12} \ |2\rangle \]

and

\[ |n_3 \rangle = U_{e3}^* \ |n_{12}\rangle + \sqrt{1 - |U_{e3}|^2} \ |3\rangle \]

are orthogonal to \(|\nu_e\rangle\). We will introduce now the angle \(\theta_{23}\) in the following way

\[ |\nu_\mu \rangle = \cos \theta_{23} \ |n_{12}^\perp \rangle + \sin \theta_{23} \ |n_3 \rangle \]

(13)

and

\[ |\nu_\tau \rangle = -\sin \theta_{23} \ |n_{12}^\perp \rangle + \cos \theta_{23} \ |n_3 \rangle . \]

(14)

Using (11), (13) and (14), we can easily express all elements of the mixing matrix through the mixing angles and the CP-violating phase \([17]\). For the elements of the first row and the third column we have

\[ U_{e1} = \sqrt{1 - |U_{e3}|^2} \ \cos \theta_{12} \ ; U_{e2} = \sqrt{1 - |U_{e3}|^2} \ \sin \theta_{12} . \]

(15)

and

\[ U_{\mu3} = \sqrt{1 - |U_{e3}|^2} \ \sin \theta_{23} \ ; U_{\tau3} = \sqrt{1 - |U_{e3}|^2} \ \cos \theta_{23} . \]

(16)

From the analysis of the atmospheric neutrino data it follows that the angle \(\theta_{23} \simeq \theta_{\text{atm}}\) is large (close to \(\pi/4\)). In the case of the most plausible LMA MSW fit of the solar neutrino data the angle \(\theta_{12} \simeq \theta_{\text{sol}}\) is also large. As we will see later, from the analysis of the data of the long baseline reactor experiments CHOOZ \([27]\) and Palo Verde \([28]\) it follows that the element \(|U_{e3}|^2\) is small.

### 3 Neutrino oscillations

Let us now turn to neutrino oscillations. From the basic relation \([3]\) we obtain for the amplitude of the transition \(\nu_\alpha \rightarrow \nu_{\alpha'}\) in vacuum the following relation (see, for example, \([29, 30]\))

\[ A_{\nu_\alpha \rightarrow \nu_{\alpha'}}(t) = |\nu_{\alpha'}| \ \exp (-iH_0 t) \ |\nu_\alpha \rangle = \sum_i U_{\alpha' i} \ e^{-iE_i t} \ U_{\alpha i}^* . \]

(17)

where \(t\) is the transition time.
We will numerate neutrino masses in such a way that
\[ m_1 < m_2 < m_3. \]

From (17) we have for the probability of the transitions \( \nu_\alpha \rightarrow \nu_{\alpha'} \)
\[ P(\nu_\alpha \rightarrow \nu_{\alpha'}) = \left| \sum_{i=1}^{3} U_{\alpha' i} e^{i \Delta m_{1i}^2 \frac{L}{E}} U_{\alpha i}^* \right|^2, \quad (18) \]
where \( \Delta m_{ik}^2 = m_i^2 - m_k^2 \), \( L \approx t \) is the distance between neutrino source and neutrino detector, \( E \) is the energy of the neutrinos.

Using the unitarity of the mixing matrix, we can rewrite this expression in the following form
\[ P(\nu_\alpha \rightarrow \nu_{\alpha'}) = |\delta_{\alpha\alpha'} + \sum_{i=2,3} U_{\alpha' i} \left( e^{-i \Delta m_{1i}^2 \frac{L}{E}} - 1 \right) U_{\alpha i}^*|^2. \quad (19) \]

We will assume that \( \Delta m_{21}^2 \) is relevant for the oscillations of the solar neutrinos and \( \Delta m_{31}^2 \) is relevant for the oscillations of atmospheric neutrinos.

As we have seen before, from the analysis of the existing solar and atmospheric neutrino data follows that neutrino mass-squared differences satisfy the hierarchy
\[ \Delta m_{21}^2 \ll \Delta m_{31}^2. \quad (20) \]

Let us first consider oscillations in the atmospheric and long baseline (LBL) experiments. In these experiments \( \frac{L}{E} \lesssim 10^3 \frac{\text{km}}{\text{GeV}} \) and the inequality
\[ \Delta m_{21}^2 \frac{L}{2E} \ll 1 \]
is satisfied. Thus, the contribution of the \( i = 2 \) term to the expression (19) for the transition probability can be neglected. For the probability of the transition \( \nu_\alpha \rightarrow \nu_{\alpha'} (\alpha \neq \alpha') \) we have (see, for example, [18])
\[ P(\nu_\alpha \rightarrow \nu_{\alpha'}) = \frac{1}{2} A_{\alpha' ; \alpha} \left( 1 - \cos \Delta m_{31}^2 \frac{L}{2E} \right), \quad (21) \]
where the oscillation amplitude \( A_{\alpha' ; \alpha} \) is given by
\[ A_{\alpha' ; \alpha} = A_{\alpha; \alpha'} = 4 |U_{\alpha' 3}|^2 |U_{\alpha 3}|^2. \quad (22) \]

\(^4\)Another possibility, the so-called inverted hierarchy, we will discuss later.
The $\nu_{\alpha}$ survival probability can be obtained from (21) and the condition of the conservation of the probability

$$P(\nu_{\alpha} \to \nu_{\alpha}) = 1 - \sum_{\alpha' \neq \alpha} P(\nu_{\alpha} \to \nu_{\alpha'}) .$$

We have

$$P(\nu_{\alpha} \to \nu_{\alpha}) = 1 - \frac{1}{2} B_{\alpha;\alpha} \left( 1 - \cos \frac{\Delta m_{31}^2 L}{2E} \right) ,$$

where the oscillation amplitude $B_{\alpha;\alpha}$ is given by

$$B_{\alpha;\alpha} = \sum_{\alpha' \neq \alpha} A_{\alpha';\alpha} = 4 |U_{\alpha3}|^2 \left( 1 - |U_{\alpha3}|^2 \right) (24)$$

It is obvious from (24) that for the oscillation amplitudes we have

$$0 \leq B_{\alpha;\alpha} \leq 1; 0 \leq A_{\alpha';\alpha} \leq 1$$

Thus, due to the hierarchy relation (21), the probabilities of the transition $\nu_{\alpha} \to \nu_{\alpha'}$ in the atmospheric and LBL experiments are determined by the largest neutrino mass-squared difference $\Delta m_{31}^2$ and by the elements $|U_{\alpha3}|^2$, which connect the flavor neutrino fields $\nu_{\alpha L}$ with the field of the heaviest neutrino $\nu_{3 L}$. Every oscillation channel is characterized by its own oscillation amplitude. Oscillation amplitudes in appearance and disappearance channels are connected by the relation (24).

From the unitarity of the mixing matrix follows that $\sum_{\alpha} |U_{\alpha3}|^2 = 1$. Thus, in the leading approximation, we are considering, all oscillation channels are characterized by three parameters. We can choose the parameters

$$\Delta m_{31}^2, \tan^2 \theta_{23}, |U_{\alpha3}|^2$$

The oscillation amplitudes in the appearance and disappearance channels are given by the expressions

$$A_{\tau;\mu} = (1 - |U_{e3}|^2)^2 \sin^2 2\theta_{23} ; \quad A_{e;\mu} = 4 |U_{e3}|^2 \left( 1 - |U_{e3}|^2 \right) \sin^2 \theta_{23} \quad (25)$$

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5Let us notice that in the case of oscillations between two types of neutrinos from unitarity of the mixing matrix it follows that oscillation amplitudes are connected by the relations $A_{\alpha';\alpha} = A_{\alpha;\alpha'} = B_{\alpha;\alpha} = B_{\alpha';\alpha'} = \sin^2 2\theta$ ($\alpha' \neq \alpha$).
and
\[ B_{\mu;\mu} = A_{\tau;\mu} + A_{e;\mu} \quad B_{e;e} = 4 |U_{e3}|^2 (1 - |U_{e3}|^2). \] (26)

Let us note that the following relation exists between the oscillation amplitudes
\[ A_{e;\mu} = B_{e;e} \sin^2 \theta_{23} \]

The CP-violating phase \( \delta \) does not enter into the expressions (21) and (22) for the transition probabilities. Thus, in the leading approximation with the dominance of the \( \Delta m_{31}^2 \) term the relations
\[ P(\nu_\alpha \rightarrow \nu_\alpha') = P(\bar{\nu}_\alpha \rightarrow \bar{\nu}_\alpha') \]
are satisfied independently of the value of the CP-violating phase. This means that the effect of CP violation in the lepton sector can not be revealed if only one neutrino mass squared difference is relevant for neutrino oscillations.

4 Atmospheric neutrinos

We will discuss now the results of the atmospheric S-K experiment [1]. In this experiment a significant up-down asymmetry of the high-energy muon events was observed. If there are no neutrino oscillations, the dependence of the number of the high energy muon (electron) events on \( \cos \theta_z \) must satisfy the relation
\[ N_l(\cos \theta_z) = N_l(-\cos \theta_z) \quad (l = e, \mu), \] (27)
where \( \theta_z \) is the zenith angle.

The measured dependence of the number of electron events on \( \cos \theta_z \) is in good agreement with (27). For the muon events in the Multi-GeV region \( (E \geq 1.3 \text{ GeV}) \) a strong \( \cos \theta_z \) asymmetry was observed. For the ratio of the total number \( U \) of up-going muons \( (\cos \theta_z \leq 0) \) to the total number \( D \) of down-going muons \( (\cos \theta_z \geq 0) \) was obtained
\[ \left( \frac{U}{D} \right)_\mu = 0.54 \pm 0.04 \pm 0.01. \]

The \( \cos \theta_z \) dependence of the number of muon events, observed in the Super-Kamiokande experiment, is in a agreement with the disappearance of
muon neutrinos due to neutrino oscillations. The data are perfectly described if the survival probability has the two-neutrino form

\[ P(\nu_\mu \rightarrow \nu_\mu) = 1 - \frac{1}{2} \sin^2 2\theta_{\text{atm}} \left( 1 - \cos \Delta m_{\text{atm}}^2 \frac{L}{2E} \right) \]  

(28)

From the analysis of the S-K data the best-fit values (1) of the neutrino oscillation parameters were obtained.

These values are directly connected with the observed zenith angle dependence. In fact, for the high-energy neutrinos the distance \( L \) between the region, where neutrinos are produced in the atmosphere, and the detector is determined by the zenith angle \( \theta_z \). Down-going neutrinos with \( \cos \theta_z = 1 \) travel a distance of about 20 km and up-going neutrinos with \( \cos \theta_z = -1 \) travel a distance of about 13000 km. For the down-going neutrinos the argument of the cosine in the expression (28) for the survival probability is small and

\[ P^{\text{down}}(\nu_\mu \rightarrow \nu_\mu) \simeq 1. \]  

(29)

For the up-going neutrinos the argument of the cosine in Eq. (28) is large and due to averaging over neutrino energies and distances the cosine term in Eq. (28) vanishes. For the averaged survival probability we have

\[ P^{\text{up}}(\nu_\mu \rightarrow \nu_\mu) \simeq 1 - \frac{1}{2} \sin^2 2\theta_{\text{atm}}. \]  

(30)

The number of muon events with \( \cos \theta_z \simeq 1 \), observed in the S-K experiment, is about two times larger than the number of the muon events with \( \cos \theta_z \simeq -1 \). Thus, \( P^{\text{up}}(\nu_\mu \rightarrow \nu_\mu) \simeq 0.5 \) and \( \sin^2 2\theta_{\text{atm}} \simeq 1 \).

All data of the S-K experiment are in a good agreement with the assumption of \( |U_{e3}|^2 = 0 \) and two-neutrino \( \nu_\mu \rightarrow \nu_\tau \) oscillations. From the analysis of the data the following ranges were obtained for the oscillation parameters

\[ \sin^2 2\theta_{\text{atm}} \geq 0.88; \quad 1.6 \cdot 10^{-3} \leq \Delta m_{\text{atm}}^2 \leq 4 \cdot 10^{-3} \text{eV}^2. \]

The three-neutrino analysis of S-K atmospheric neutrino data allows to obtain an upper bound for the parameter \( |U_{e3}|^2 \). In Ref. [31] it was found that

\[ |U_{e3}|^2 \leq 0.35. \]
5 The upper bound of the parameter $|U_{e3}|^2$

The most direct and stringent bound on the parameter $|U_{e3}|^2$ can be obtained from the results of the LBL reactor experiments CHOOZ \cite{27} and Palo Verde \cite{28} which are sensitive to the atmospheric range of neutrino mass-squared differences. In these experiments, $\bar{\nu}_e$’s from the reactors at a distance of about 1 km from the detectors were recorded via the observation of the process

$$\bar{\nu}_e + p \rightarrow e^+ + n .$$

No indications in favor of a disappearance of the reactor $\bar{\nu}_e$’s were found. For the ratio $R$ of the total number of the detected and expected events was obtained

$$R = 1.01 \pm 2.8\% (\text{stat}) \pm 2.7\% (\text{syst}) \quad \text{CHOOZ}$$

and

$$R = 1.01 \pm 2.4\% (\text{stat}) \pm 5.3\% (\text{syst}) \quad \text{Palo Verde}.$$

In the leading approximation (neglecting the contribution of the $i = 2$ term in Eq.(19)), we have for the $\bar{\nu}_e$ survival probability

$$P(\bar{\nu}_e \rightarrow \bar{\nu}_e) = 1 - \frac{1}{2} B_{ee} \left( 1 - \cos \frac{\Delta m^2_{31} L}{2E} \right) , \quad (31)$$

where

$$B_{ee} = 4 |U_{e3}|^2 \left( 1 - |U_{e3}|^2 \right) .$$

From the exclusion plot, obtained from the analysis of the data of the CHOOZ (or Palo Verde) experiment at a fixed value of $\Delta m^2_{31}$ for the allowed values of the oscillation amplitude we have the bound

$$B_{ee} \leq B^0_{ee} . \quad (32)$$

This bound depends on the value of $\Delta m^2_{31}$. From (31) and (32) follows

$$|U_{e3}|^2 \leq \frac{1}{2} \left( 1 - \sqrt{1 - B^0_{ee}} \right) , \quad (33)$$

or

$$|U_{e3}|^2 \geq \frac{1}{2} \left( 1 + \sqrt{1 - B^0_{ee}} \right) . \quad (34)$$
We are interested in the region of $\Delta m_{31}^2 \gtrsim 10^{-3}\text{eV}^2$. In this region the amplitude $B_{ee}^0$ is small. Thus, $|U_{e3}|^2$ can be small (inequality (33)) or large, close to one (inequality (34)). This last possibility is excluded by the solar neutrino data. In fact, as we will see later, the heaviest neutrino $\nu_3$ (both in the case of vacuum and in the case of matter) gives an incoherent contribution $|U_{e3}|^4$ to the survival probability of the solar $\nu_e$. If $|U_{e3}|^2$ is large, the probability of solar neutrinos to survive will be close to one. As we will see in the next section this possibility is excluded in a model independent way by S-K and SNO data.

Thus, taking into account solar neutrino data, we can conclude from the results of the LBL reactor experiments that the parameter $|U_{e3}|^2$ is small. From the CHOOZ exclusion curve at $\Delta m_{31}^2 = 2.5 \cdot 10^{-3}\text{eV}^2$ (the S-K best-fit value) we have

$$|U_{e3}|^2 \leq 3.7 \cdot 10^{-2}.$$  \hspace{1cm} (35)

This bound was obtained under the assumption that the contribution of the $i = 2$ term to the expression for the transition probability in Eq. (19) can be neglected in the LBL region. This is a good approximation for the LMA best-fit value of $\Delta m_{\text{sol}}^2$ given by Eq. (2). However, the values of $\Delta m_{\text{sol}}^2$ in the LMA region (which is the preferable fit to the solar neutrino data) can be as large as $\Delta m_{31}^2 \simeq 6 \cdot 10^{-4}\text{eV}^2$ (see, for example, [15]). In the CHOOZ experiment the average value of the parameter $L/E$ is approximately equal to 300 m/MeV. If, for example, At $\Delta m_{\text{sol}}^2 = 2 \cdot 10^{-4}\text{eV}^2$, for example, for the average value of the parameter $L/E$ in the CHOOZ experiment we have

$$\Delta m_{\text{sol}}^2 \frac{L}{2E} \simeq 1.5 \cdot 10^{-1}.$$  

Thus, the $i=2$ term in Eq. (19) at the relatively large values of $\Delta m_{31}^2$, belonging to the LMA region, could give a sizable contribution to the survival probability.

From Eq. (19), taking into account all terms, we obtain the following expression for the $\bar{\nu}_e$ survival probability [32]

$$P(\bar{\nu}_e \rightarrow \bar{\nu}_e) = 1 - 2 |U_{e3}|^2 \left(1 - |U_{e3}|^2\right) \left(1 - \cos \frac{\Delta m_{31}^2 L}{2E}\right)$$
\[-\frac{1}{2} (1 - |U_{e3}|^2)^2 \sin^2 2 \theta \left(1 - \cos \frac{\Delta_{\text{sol}}^2}{2 E} \right) \quad (36)\]

\[+ 2 |U_{e3}|^2 (1 - |U_{e3}|^2) \sin^2 \theta \left( \cos \left( \frac{\Delta_{31}^2}{2 E} - \frac{\Delta_{\text{sol}}^2}{2 E} \right) - \cos \frac{\Delta_{31}^2}{2 E} \right),\]

where we put $\Delta_{i2}^2 = \Delta_{\text{sol}}^2$ and $\theta_{12} = \theta_{\text{sol}}$.

The second term in the right-hand part of this expression comes from the main $i = 3$ term in Eq.(19), the third one from the $i = 2$ term and the fourth one from the interference of the $i = 3$ and $i = 2$ terms.

Let us note that in the LMA region the angle $\theta_{\text{sol}}$ is large. Thus, the coefficient in front of the bracket of the third “solar term” in Eq.(36) is large and the coefficient in front of the bracket of the interference term is of the same order as the coefficient in front of the bracket of the second term.

Up to now we have assumed that the mass-squared difference $\Delta_{i2}^2$ of the lightest neutrinos $\nu_2$ and $\nu_1$ is relevant for the oscillations of the solar neutrinos and hierarchy (20) is valid. The existing data do not exclude, however, the possibility that the mass-squared difference $\Delta_{32}^2$ between the squares of the masses of the heaviest neutrinos $\nu_3$ and $\nu_2$ is relevant for the oscillations of the solar neutrinos and $\Delta_{31}^2$ is relevant for the oscillations of the atmospheric neutrinos. In this case the so-called inverted hierarchy

$$\Delta_{32}^2 \ll \Delta_{31}^2$$

(37)

takes place.\(^6\)

We will consider now the case of the inverted neutrino mass spectrum. The probability of the transition $\nu_\alpha \rightarrow \nu_\alpha'$ in vacuum can be written in the form

$$P(\nu_\alpha \rightarrow \nu_\alpha') = \left| \delta_{\alpha\alpha'} + \sum_{i=1,2} U_{\alpha'i} \left(e^{i \frac{\Delta_{3i}}{2 E} \frac{L}{E}} - 1\right) U_{\alpha i}^* \right|^2.$$  \(38\)

Let us consider neutrino oscillations in the atmospheric and LBL experiments. Neglecting the contribution of the small $i = 2$ term in the expression (38), for the transition probabilities we will obtain expressions (21) and (23). The oscillation amplitudes $A_{\alpha':\alpha}$ and $B_{\alpha'\alpha}$ are given by the expressions (22) and (24) after applying the change $|U_{\alpha 3}|^2 \rightarrow |U_{\alpha 1}|^2$. Thus, in the case of the inverted hierarchy, in the leading approximation, the transition probabilities

\(^6\)Recall that we numerate neutrino masses in such a way that $m_1 < m_2 < m_3$.\]
in the atmospheric and LBL experiments are determined by the largest neutrino mass-squared difference $\Delta m_{31}^2$ and the elements $|U_{\alpha L}|^2$, connecting the field of the flavor neutrino $\nu_{\alpha L}$ with the field of the lightest neutrino $\nu_{1L}$. In order to obtain the three-neutrino expression for the $\bar{\nu}_e$ survival probability in the case of the inverted hierarchy it is necessary to change $|U_{e3}|^2 \to |U_{e1}|^2$ and $\theta_{sol} \to \pi/2 - \theta_{sol}$ in Eq. (36).

The expression (36) and similarly the expression for the $\bar{\nu}_e$ survival probability in the case of the inverted hierarchy were used in Ref. [32] in order to obtain from the CHOOZ data new exclusion plots in the plane of the parameters $|U_{e3}|^2 - \Delta m_{31}^2$ (and $|U_{e1}|^2 - \Delta m_{31}^2$ in the case of inverted hierarchy) at fixed values of $\Delta m_{sol}^2$ and $\sin^2 \theta_{sol}$, belonging to the LMA allowed region.

In Table 1 we present the upper bounds of the parameter $|U_{e3}|^2$ ($|U_{e1}|^2$) at $\Delta m_{31}^2 = 2.5 \cdot 10^{-3}eV^2$ and $\Delta m_{31}^2 = 10^{-2}eV^2$. As can be seen from Table 1, at $\Delta m_{sol}^2 \leq 2 \cdot 10^{-4}eV^2$ the bounds on $|U_{e3}|^2$ ($|U_{e1}|^2$) in the three-neutrino case are practically the same as in the leading approximation which corresponds to the two-neutrino case. At the larger values of $\Delta m_{sol}^2$ the limits on $|U_{e3}|^2$ ($|U_{e1}|^2$) are more stringent in the three-neutrino case than in the two-neutrino case. For example, at $\Delta m_{sol}^2 = 6 \cdot 10^{-4}eV^2$ and $\sin^2 \theta_{sol} = 0.27$ at the S-K best-fit point $\Delta m_{31}^2 = 2.5 \cdot 10^{-3}eV^2$ we have

$$|U_{e3}|^2 \leq 2 \cdot 10^{-2}. \quad (39)$$

This bound is about 2 times smaller that the upper bound (35), obtained in the two-neutrino case.

New data of the solar neutrino experiments (SNO, GNO, BOREXINO [33]) and data of the reactor experiment KamLAND [36] will allow to determine the values of the parameters $\Delta m_{sol}^2$ and $\tan^2 \theta_{sol}$ with better accuracy than today. That will permit to obtain from the results of the CHOOZ and Palo Verde experiments more precise upper bounds of the parameter $|U_{e3}|^2$. Let us stress that the exact value of the parameter $|U_{e3}|^2$ is very important for the future Super Beam (see [32]) and Neutrino Factory (see [34]) experiments, in which $\nu_{\mu} \to \nu_e$ oscillations and effects of the three-neutrino mixing, in particular effects of CP violation in the lepton sector, will be investigated in detail.
Table 1: Upper bounds of the mixing parameter $|U_{e3}|^2$ ($|U_{e1}|^2$ in the case of inverted hierarchy) for different values of neutrino oscillation parameters.

| $\Delta m^2_{31}$ (eV$^2$) | $\Delta m^2_{\text{sol}}$ (eV$^2$) | $|U_{e3}|^2$ (sin$^2\theta_{\text{sol}} = 0.5$) | $|U_{e3}|^2$ (sin$^2\theta_{\text{sol}} = 0.27$) | $|U_{e1}|^2$ (sin$^2\theta_{\text{sol}} = 0.27$) |
|-----------------|-----------------|------------------|------------------|------------------|
| 2.5 · 10$^{-3}$ | 0               | 3.6 · 10$^{-2}$  | 3.5 · 10$^{-2}$  | 3.7 · 10$^{-2}$  |
|                 | 2 · 10$^{-4}$   | 3.6 · 10$^{-2}$  | 3.5 · 10$^{-2}$  | 3.7 · 10$^{-2}$  |
|                 | 4 · 10$^{-4}$   | 2.9 · 10$^{-2}$  | 3.0 · 10$^{-2}$  | 3.4 · 10$^{-2}$  |
|                 | 6 · 10$^{-4}$   | 1.7 · 10$^{-2}$  | 2.0 · 10$^{-2}$  | 2.5 · 10$^{-2}$  |
| 10$^{-2}$       | 0               | 3.6 · 10$^{-2}$  | 3.4 · 10$^{-2}$  | 3.4 · 10$^{-2}$  |
|                 | 2 · 10$^{-4}$   | 3.4 · 10$^{-2}$  | 3.4 · 10$^{-2}$  | 3.4 · 10$^{-2}$  |
|                 | 4 · 10$^{-4}$   | 2.7 · 10$^{-2}$  | 2.9 · 10$^{-2}$  | 2.8 · 10$^{-2}$  |
|                 | 6 · 10$^{-4}$   | 1.7 · 10$^{-2}$  | 2.1 · 10$^{-2}$  | 2.0 · 10$^{-2}$  |

6 Solar neutrinos

We now come to the discussion of solar neutrinos. The energy of the sun is generated in the reactions of the thermonuclear pp and CNO cycles. From a thermodynamical point of view the energy of the sun is produced in the transition of four protons and two electrons into $^4\text{He}$ and two electron neutrinos

$$4p + 2e^- \rightarrow ^4\text{He} + 2\nu_e.$$ (40)

The energy, which is released in this transition, is equal to

$$Q = 4m_p + 2m_e - m_{^4\text{He}} \simeq 26.7\text{MeV}.$$ 

Thus, the production of the energy of the sun is accompanied by the emission of $\nu_e$'s.

From (40) we can obtain a model independent relation that connects the fluxes of neutrinos with the luminosity of the sun $L_\odot$. In fact, from (40) it follows that the luminous energy per one neutrino with energy $E$ is equal to $\frac{1}{2} (Q - 2E)$. Thus, the total luminosity of the sun is given by

$$\int \frac{1}{2}(Q - 2E) \ N(E) dE = L_\odot,$$ (41)

where $N(E)$ is the total number of neutrinos with the energy $E$, produced by the sun in 1 sec.
We have
\[ N(E) = 4\pi R^2 \sum_i \Phi_i(E) \tag{42} \]

where \( R \) is the distance between the sun and the earth and \( \Phi_i(E) \) is the flux of \( \nu_e \) from the source \( i \) (in the case of neutrino oscillations \( \Phi_i(E) \) is the total flux of all types of neutrinos, including sterile neutrinos).

From (41) and (42) we obtain the luminosity relation
\[ \sum_i (1/2)Q - \bar{E}_i)\Phi_i^0 = \frac{L_\odot}{4\pi R^2}, \tag{43} \]

where \( \Phi_i^0 \) is the total flux and \( \bar{E}_i \) is the average energy of neutrinos from the source \( i \).

Let us note that in the derivation of the luminosity relation (43) we assumed that the sun is in a stable state. \(^7\) The main source of solar neutrinos is the \( pp \) reaction
\[ p + p \rightarrow d + e^- + \nu_e. \tag{44} \]

This reaction is the source of the low energy neutrinos with a maximum energy of 0.42 MeV. The total flux of \( pp \) neutrinos, predicted by the Standard Solar Model BP00 (SSM BP00) \[^{38}\] is determined mainly by the luminosity relation (43) and is equal to \( \Phi_{pp} = 5.95 \cdot 10^{10} cm^{-2}s^{-1} \).

The reaction
\[ e^- + ^7 Be \rightarrow ^7 Li + \nu_e. \tag{45} \]

is the source of the monochromatic neutrinos with an energy of 0.86 MeV. The flux of \(^7\text{Be} \) neutrinos, predicted by the SSM, is \( \Phi_{Be} = 4.8 \cdot 10^9 cm^{-2}s^{-1} \).

In the Super-Kamiokande and the SNO experiments, mainly high energy neutrinos from the decay
\[ ^8\text{B} \rightarrow ^8\text{Be}^* + e^+ + \nu_e. \tag{46} \]

are detected (high energy thresholds). The maximum energy of the \(^8\text{B} \) neutrinos is approximately equal to 15 MeV and the flux, predicted by the SSM BP00, is \( \Phi_{8B} = 5.9 \cdot 10^6 cm^{-2}s^{-1} \).

\(^7\) It takes about \( 7 \cdot 10^5 \) years for the photons, produced in the central zone of the sun, to reach the surface (see \[^{37}\] ).
At present results of six solar neutrino experiments (Homestake, GALL-EX-GNO, Kamiokande, Super-Kamiokande and SNO) are available. The event rates, measured in all solar neutrino experiments, are significantly smaller than the rates predicted by the SSM.

In the Homestake experiment solar neutrinos are detected by the radiochemical method through the observation of the reaction

\[ \nu_e + ^{37}\text{Cl} \rightarrow e^- + ^{37}\text{Ar}. \]

The threshold of this process is \( E_{th} = 0.81 \text{MeV} \). Thus, in this experiments mainly \(^8\text{B}\) and \(^7\text{Be}\) neutrinos are detected. The observed event rate is equal to

\[ R_{\text{Cl}} = (2.56 \pm 0.16 \pm 0.16) \text{ SNU}. \]

The rate, predicted by SSM BP00, is

\[ R_{\text{Cl}}^{\text{SSM}} = (8.59 +1.1 \ -1.2) \text{ SNU}. \]

In the GALLEX-GNO and SAGE experiments solar neutrinos are detected through the observation of the reaction

\[ \nu_e + ^{71}\text{Ga} \rightarrow e^- + ^{71}\text{Ge}. \]

The threshold of this reaction is \( E_{th} = 0.23 \text{ MeV} \). Thus, in the Gallium experiments neutrinos from all reactions in the sun are detected. The combined event rate of the GNO and the GALLEX experiment is

\[ R_{\text{Ga}} = (74.1 \pm 6.7 \pm 6.8) \text{ SNU}; \quad \text{GALLEX} - \text{GNO} \]

The event rate measured in the SAGE experiment is

\[ R_{\text{Ga}} = (77 \pm 6 \pm 3) \text{ SNU}; \quad \text{SAGE} \]

The rate, predicted by SSM BP00 for the Gallium experiments, is

\[ R_{\text{Ga}}^{\text{SSM}} = (130 \ +9 \ -7) \text{ SNU}. \]

\(^{81} \text{ SNU} = 10^{-36}\text{events} \ \text{atom}^{-1} \ \text{s}^{-1}.\)

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7 Comparison of the results of the SNO and the S-K experiments

The Kamiokande, Super-Kamiokande and SNO are direct counting experiments. In the Kamiokande and S-K experiments the solar neutrinos are observed through the detection of the recoil electrons in the elastic (ES) neutrino-electron scattering

\[ \nu + e \rightarrow \nu + e. \]  

We will discuss the results of the S-K experiment \[9\]. In this experiment the large 50 kton water Cherenkov detector is used. During 1258 days of running, in the S-K experiment \[18464 \pm 677 - 590\] events with the energy of the recoil electrons in the range 5-20 MeV were observed.

As the energy of the recoil electrons is much larger than the electron mass, the direction of the recoil electron momentum practically coincides with the direction of the neutrino momentum. In the S-K experiment, in the distribution of the events on \( \cos \theta_{\text{sun}} \) (\( \theta_{\text{sun}} \) is the angle between the recoil electron momenta and the direction to the sun) a sharp peak at \( \cos \theta_{\text{sun}} = 1 \) was observed. The observation of such a peak is a clear demonstration that the recorded events are due to solar neutrinos.

All flavor neutrinos \( \nu_e, \nu_\mu \) and \( \nu_\tau \) are detected in the S-K experiment. However, the cross section of the (NC) \( \nu_\mu (\nu_\tau) - e \) scattering is about six times smaller than the cross section of the (CC+NC) \( \nu_e - e \) scattering. Thus, the S-K sensitivity to \( \nu_\mu \) and \( \nu_\tau \) is much lower than the sensitivity to \( \nu_e \).

In the SNO experiment \[10\] a heavy water Cherenkov detector is used (1 kton of D\(_2\)O). Solar neutrinos were detected in the experiment via the observation of the CC reaction

\[ \nu_e + d \rightarrow e^- + p + p \]  

and the elastic scattering reaction

\[ \nu + e \rightarrow \nu + e. \]  

The electron kinetic energy threshold in the SNO experiment is \( T_{\text{th}} = 6.75 \) MeV. Thus, in the SNO experiment, like in the S-K experiment, only high
energy $^8$B neutrinos are detected. From November 1999 till January 2001 $975.4 \pm 39.7$ CC events and $106.1 \pm 15.2$ ES events were observed.

The measured ES event rate is consistent with the more precise S-K event rate. The detection of the solar neutrinos via the observation of the CC reaction (48) allowed to determine the flux of the solar $\nu_e$ on the earth. In fact, the total event rate, measured in the SNO experiment, via the observation of the CC reaction is given by

$$R_{\text{CC}} = \int_{E_0} \sigma_{\nu_e d}(E) \Phi_{\nu_e}(E) dE,$$

(50)

where $\sigma_{\nu_e d}(E)$ is the total cross section of the CC process, $\Phi_{\nu_e}(E)$ is the flux of the $\nu_e$ on the earth and $E_0 = T_{th} + 1.44\text{MeV}$.

The flux of $\nu_e$ on the earth is given by

$$\Phi_{\nu_e}(E) = P(\nu_e \rightarrow \nu_e) \Phi^0_{\nu_e}(E),$$

(51)

where $P(\nu_e \rightarrow \nu_e)$ is the probability of the solar $\nu_e$ to survive and $\Phi^0_{\nu_e}(E)$ is the initial flux of $\nu_e$ (the flux that would be observed if there would be no neutrino oscillations).

The flux $\Phi^0_{\nu_e}(E)$ can be presented in the form

$$\Phi^0_{\nu_e}(E) = X(E) \Phi^0_{\nu_e},$$

(52)

where $\Phi^0_{\nu_e}$ is the total initial flux of the $^8$B neutrinos and $X(E)$ is a normalized function ($\int X(E) dE = 1$). The function $X(E)$ characterizes the spectrum of $\nu_e$ from the decay $^8$B $\rightarrow$ $^8$Be$^* + e^+ + \nu_e$. This function is known.

Let us determine the function $\rho_{\nu_e d}(E)$ by the relation

$$\sigma_{\nu_e d}(E) X(E) = < \sigma_{\nu_e d} > \rho_{\nu_e d}(E),$$

(53)

where

$$< \sigma_{\nu_e d} > = \int_{E_0} \sigma_{\nu_e d}(E) X(E) dE$$

is the averaged cross section of the CC process (48). It is obvious that

$$\int_{E_0} \rho_{\nu_e d} dE = 1.$$
From Eq.(50) and Eq.(53) for the CC event rate we have

\[ R_{CC} = \langle \sigma_{\nu_e d} \rangle \Phi_{\nu_e} \]

(54)

where the average flux of \( \nu_e \) on the earth \( \Phi_{\nu_e} \) is given by

\[ \Phi_{\nu_e} = \langle P(\nu_e \rightarrow \nu_e) \rangle \Phi_0 \]

(55)

In this equation

\[ \langle P(\nu_e \rightarrow \nu_e) \rangle = \int_{E_0} P(\nu_e \rightarrow \nu_e) \rho_{\nu_e d} \ dE \]

(56)

is the averaged survival probability.

In the SNO experiment the spectrum of the produced electrons was measured. No significant deviations from the predicted spectrum were found. Thus the results of the SNO experiment are compatible with the assumption that the \( \nu_e \) survival probability in the SNO energy region is constant. We have in this case

\[ \langle P(\nu_e \rightarrow \nu_e) \rangle \simeq P(\nu_e \rightarrow \nu_e) \]

(57)

From the data of the SNO experiment for the averaged flux of \( \nu_e \) on the earth the following value was found [10]

\[ (\Phi_{\nu_e})_{SNO} = (1.75 \pm 0.07 \pm 0.12 \pm 0.05 \text{ (theor)}) \cdot 10^6 \text{ cm}^{-2}\text{s}^{-1} \]

(58)

Let us now discuss the results of the S-K experiment. In this experiment solar neutrinos are detected via the observation of the ES process [17]. The total S-K event rate is given by

\[ R^{ES} = \int_{E_0} \sigma_{\nu_e l}(E) \Phi_{\nu_e}(E) \ dE + \int_{E_0} \sigma_{\nu_\mu}(E) \sum_{l=\mu,\tau} \Phi_{\nu_l}(E) \ dE, \]

(59)

where \( \sigma_{\nu_e l}(E) \) is the cross section of \( \nu_l - e \) scattering, \( \Phi_{\nu_l}(E) \) is the flux of \( \nu_l \) on the earth \( (l = e, \mu, \tau) \) and \( E_0 \) is given by

\[ E_0 = \frac{T_{th}}{2} \left( 1 + \sqrt{1 + \frac{2m}{T_{th}}} \right). \]

(60)

The total energy threshold in the S-K experiment is \( E_{th} = 5\text{MeV} \).
For the flux of $\nu_l$ on the earth we have
\[
\Phi_{\nu_l}(E) = P(\nu_e \to \nu_l) \Phi_{\nu_e}^0(E),
\] (61)
where $\Phi_{\nu_e}^0(E)$ is the initial flux of $\nu_e$. This flux is given by Eq.(52).

Let us determine the normalized functions $\rho_{\nu_l}(E)$ ($l = e, \mu$) as follows:
\[
\sigma_{\nu_l}(E) X(E) = \langle \sigma_{\nu_e} \rangle \rho_{\nu_l}(E),
\] (62)
where
\[
\int_{E_0}^{E} \sigma_{\nu_l}(E) X(E) \, dE = \langle \sigma_{\nu_e} \rangle,
\] (63)
and
\[
\int_{E_0}^{E} \rho_{\nu_l} \, dE = 1.
\]

From (60)-(62) it follows that the total S-K event rate can be written in the form
\[
R^{ES} = \langle \sigma_{\nu_e} \rangle \Phi_{\nu}^{ES},
\] (64)
where
\[
\Phi_{\nu}^{ES} = \Phi_{\nu_e}^{ES} + \frac{\langle \sigma_{\nu_e} \rangle}{\langle \sigma_{\nu_{\mu,\tau}} \rangle} \sum_{l=\mu,\tau} \Phi_{\nu_l}^{ES}.
\] (65)
Here
\[
\frac{\langle \sigma_{\nu_{\mu,\tau}} \rangle}{\langle \sigma_{\nu_{e}} \rangle} \simeq 0.154
\] (66)
and the averaged fluxes of $\nu_e$, $\nu_\mu$ and $\nu_\tau$ on the earth are given by
\[
\Phi_{\nu_e}^{ES} = \langle P(\nu_e \to \nu_e) \rangle \Phi_{\nu_e}^0; \quad \sum_{l=\mu,\tau} \Phi_{\nu_l}^{ES} = \sum_{l=\mu,\tau} \langle P(\nu_e \to \nu_l) \rangle \nu_{\mu,\tau} \Phi_{\nu_e}^0.
\] (67)

From the data of the S-K experiment for the effective flux $\Phi_{\nu}^{ES}$ it was obtained the value \[9\]
\[
(\Phi_{\nu}^{ES})_{SK} = (2.32 \pm 0.03 \pm 0.08) \cdot 10^6 \text{cm}^{-2}\text{s}^{-1}.
\] (68)

In the S-K experiment the spectrum of the recoil electrons was measured. No deviation from the predicted spectrum was observed. Thus, both the S-K and the SNO data are compatible with the assumption of a constant $\nu_e \to \nu_e$ survival probability. We have in this case
\[< P(\nu_e \rightarrow \nu_e) >_{\nu_e e} \simeq P(\nu_e \rightarrow \nu_e). \quad (69)\]

Using (55), (57), (67), and (69) we conclude that \[\Phi_{\nu_e}^{SNO} \simeq \Phi_{\nu_e}^{ES}. \quad (70)\]

From the comparison of the fluxes, measured in the S-K and SNO experiments we can determine the flux of \(\nu_\mu\) and \(\nu_\tau\) on the earth. From (58), (65), (68) and (70), we have \[\sum_{l=\mu,\tau} \Phi_{\nu_l}^{ES} = (3.69 \pm 1.13) \cdot 10^6 \text{ cm}^{-2} \text{ s}^{-1} \quad (71)\]

Thus, the results of the SNO and the S-K experiments give us the first model independent evidence (at 3\(\sigma\) level) of the presence of \(\nu_\mu\) and \(\nu_\tau\) in the flux of solar neutrinos on the earth. The flux of \(\nu_\mu\) and \(\nu_\tau\) on the earth is approximately two times larger than the flux of \(\nu_e\).

For the total flux of the flavor neutrinos on the earth we obtain from (58), (70) and (71)

\[\sum_{l=e,\mu,\tau} \Phi_{\nu_l}^{ES} = (5.44 \pm 0.99) \cdot 10^6 \text{ cm}^{-2} \text{ s}^{-1}. \quad (72)\]

The SSM BP00 value of the total flux of \(^8\text{B}\) neutrinos is \[\Phi_{\nu_e}^{0,\text{SM}} = (5.93 \pm 0.89) \cdot 10^6 \text{ cm}^{-2}. \quad (73)\]

Thus, the total flux of the flavor neutrinos, obtained from the results of the SNO and S-K experiments, is in a good agreement with the value of the total flux of \(\nu_e\) predicted by the SSM BP00.

Let us note that in the region of energies we are interested in we have

\[\rho_{\nu_\mu e}(E) \simeq \rho_{\nu_e e}(E).\]

Taking this relation into account we find for the total flux of flavor neutrinos on the earth

\[\sum_{l=e,\mu,\tau} \Phi_{\nu_l}^{ES} = \sum_{l=e,\mu,\tau} < P(\nu_e \rightarrow \nu_l) >_{\nu_e e} \Phi_{\nu_e}^{0}, \quad (73)\]

\[\text{It was shown in (33) that it is possible to choose the S-K and the SNO thresholds in such a way that these quantities will be practically equal at any } P(\nu_e \rightarrow \nu_e).\]
where $\Phi^0_{\nu_e}$ is the total initial flux of $\nu_e$. Thus, if there are no transitions of the solar $\nu_e$ into sterile states, we have

$$\sum_{l=e,\mu,\tau} \Phi^E_{\nu_l} = \Phi^0_{\nu_e}. \quad (74)$$

8 Solar neutrino oscillations in the framework of three-neutrino mixing

The data of all solar neutrino experiments are well described by the two-neutrino transition probability, which is characterized by the two parameters $\Delta m^2_{\text{sol}}$ and $\tan^2 \theta_{\text{sol}}$. We will now discuss the oscillations observed in the solar neutrino experiments in the framework of three neutrino mixing.

The probability of the transition $\nu_\alpha \rightarrow \nu_\alpha'$ in vacuum (see (Eq.18)) can be presented in the form

$$P(\nu_\alpha \rightarrow \nu_\alpha') = \left| \sum_{i=1,2} U_{\alpha'i} e^{-i \Delta m^2_{\text{sol}} \frac{E}{m_i}} U_{\alpha'i}^* + U_{\alpha'3} e^{-i \Delta m^2_{\text{sol}} \frac{E}{m_3}} U_{\alpha'3}^* \right|^2. \quad (75)$$

We are interested in the probability, averaged over the region where neutrinos are produced, over the neutrino spectrum, etc. Because of the hierarchy (20), in the expression for the averaged transition probability the interference term does not enter. For the averaged transition probability we have

$$\overline{P}(\nu_\alpha \rightarrow \nu_\alpha') = \left| \sum_{i=1,2} U_{\alpha'i} e^{-i \Delta m^2_{\text{sol}} \frac{E}{m_i}} U_{\alpha'i}^* \right|^2 + |U_{\alpha'3}|^2 |U_{\alpha 3}|^2. \quad (76)$$

Using the unitarity of the mixing matrix, it is easy to show that the averaged transition probabilities satisfy the relation

$$\sum_{\alpha' = e,\mu,\tau} \overline{P}(\nu_\alpha \rightarrow \nu_\alpha') = 1. \quad (77)$$

Using Eq.(10), the averaged probability of solar $\nu_e$ to survive can be presented in the form
\[ P(\nu_e \rightarrow \nu_e) = |U_{e3}|^4 + (1 - |U_{e3}|^2)^2 \left( P^{(1,2)}(\nu_e \rightarrow \nu_e) \right), \]  

(78)

where

\[ P^{(1,2)}(\nu_e \rightarrow \nu_e) = 1 - \frac{1}{2} \sin^2 2\theta_{12} (1 - \cos \frac{\Delta m_{21}^2 L}{2E}). \]  

(79)

The probability \( P^{(1,2)}(\nu_e \rightarrow \nu_e) \) has the same form as the two-neutrino survival probability and depends on the parameters \( \Delta m_{21}^2 \) and \( \sin^2 2\theta_{12} \). Let us stress, however, that in the case of three-neutrino mixing, solar \( \nu_e \) transfer into \( \nu_\mu \) and \( \nu_\tau \).

The expression Eq.(78) is valid also in the case of matter [40]. In this case \( P^{(1,2)}(\nu_e \rightarrow \nu_e) \) is the two-neutrino survival probability in matter. In the calculation of this quantity the density of the electrons \( \rho_e(x) \) in the effective Hamiltonian of the interaction of neutrino with matter must be changed to

\[ (1 - |U_{e3}|^2) \rho_e(x). \]

As we have seen before, from the data of the long baseline reactor experiments CHOOZ and Palo Verde follows that the element \( |U_{e3}|^2 \) is small. Thus we see from Eq. (78) that in the framework of three-neutrino mixing the survival probability of the solar \( \nu_e \) (up to corrections not larger than a few \%) has the two neutrino form

\[ P(\nu_e \rightarrow \nu_e) \simeq P^{(1,2)}(\nu_e \rightarrow \nu_e) \]  

(80)

and is characterized by two parameters \( \Delta m_{21}^2 = \Delta m_{\text{sol}}^2 \) and \( \tan \theta_{12} = \tan \theta_{\text{sol}} \).

Let us stress that the reasons for this approximate decoupling [41] of the oscillations of solar neutrinos from oscillations of neutrinos in atmospheric and LBL experiments are the hierarchy of the neutrino mass-squared differences and the smallness of \( |U_{e3}|^2 \).

If the parameters \( \Delta m_{\text{sol}}^2 \) and \( \tan \theta_{\text{sol}} \) are in the MSW LMA region, the solar range of neutrino mass-squared differences can be reached in reactor experiments. Such an experiment is KamLAND [36]. In this experiment \( \bar{\nu}_e \)'s from several reactors in Japan at an average distance of about 170 km from the detector will be recorded via the observation of the process

\[ \bar{\nu}_e + p \rightarrow e^+ + n. \]

The KamLAND detector is a tank filled with a liquid scintillator (1000m\(^3\)) and covered by PMT’s. About 700 events/kt/year are expected. After three years of running the whole LMA region will be investigated in this experiment.
9 Conclusion

In the last years significant progress in the investigation of the phenomenon of neutrino oscillations, envisaged by B.Pontecorvo many years ago, has been reached. The significant $\cos \theta_z$-asymmetry of the Multi-GeV muon atmospheric neutrino events, observed by the Super-Kamiokande collaboration, constitutes convincing evidence for neutrino oscillations driven by the neutrino mixing.

The comparison of the solar neutrino event rates, measured in the Super-Kamiokande and in the SNO experiments, allowed to conclude in a model independent way that solar $\nu_e$ on the way from the sun to the earth transfer into $\nu_\mu$ and $\nu_\tau$.

There exist, however, several fundamental questions to which future experiments will be addressed:

1. How many massive neutrinos exist in nature?

   The minimal number that corresponds to three flavor neutrinos is equal to three. If, however, the LSND result \[1\] will be confirmed, in order to describe all neutrino oscillation data we need three different neutrino mass-squared differences and (at least) four massive neutrinos. The experiment MiniBooNE \[12\], started in 2002, plans to check the LSND claim in about two years.

2. What is the nature of the massive neutrinos. Are they Dirac or Majorana particles?

   The answer to this question can be obtained from experiments on the search for neutrinoless double $\beta$ decay. The best lower bound for the lifetime of this process was obtained in the $^{76}$Ge Heidelberg-Moscow experiment \[12\] ($T_{1/2} \geq 1.9 \cdot 10^{25}$y; 90\%CL). From this result for the effective Majorana mass $| < m > | = | \sum_i U_{ei}^2 m_i |$ the bound

   $| < m > | \leq (0.2 - 0.6) \text{ eV}$ \hspace{1cm} (81)

   can be obtained. Future experiments on the search for neutrinoless double $\beta$ decay \[16\] will have a sensitivity $| < m > | \simeq 10^{-2}$eV.

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\[12\] The bound on $| < m > |$ depends on the value of the corresponding nuclear matrix element; the bound \[81\] takes into account different calculations.
3. What is the value of the minimal neutrino mass $m_1$?

In the experiments on the investigation of the effects of neutrino masses by the measurement of the high energy part of the $\beta$ spectrum of $^3\text{H}$ it was found that $m_1 \leq 2.2\text{ eV}$ [13] and $m_1 \leq 2.5\text{ eV}$ [14]. In the future KATRIN experiment [15] a sensitivity $m_1 \simeq (0.3 - 0.4)\text{ eV}$ is planned to be reached.

The precise measurement of the parameters $\Delta m^2_{31}$, $\theta_{23}$, $|U_{e3}|^2$, the investigation of the character of the neutrino mass spectrum (hierarchy or reversed hierarchy), the search for effects of CP violation in the lepton sector etc. constitute the programs for long baseline experiments with neutrinos from Super Beam facilities [33] and Neutrino Factories [34].

The small neutrino masses could be a signature of a new large scale at which the lepton number is violated [47]. There exist, however, other possible explanations for the smallness of neutrino masses (e.g., large extra dimensions [48] and others).

It is obvious that a lot of new experimental and theoretical efforts are required in order to reveal the true origin of the newly discovered phenomenon of small neutrino masses and neutrino mixing.

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