FIBRE BUNDLES, CONNECTIONS, GENERAL RELATIVITY, AND EINSTEIN-CARTAN THEORY

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Introduction

The main purpose of this article is to present in the most natural way, that is, in the context of the theory of vector and principal bundles and connections in them, fundamental geometrical concepts related to general relativity (GR) and one of its extensions, the Einstein-Cartan theory (EC). Central concepts are the curvature tensor $R_{\mu\nu\rho\sigma}$, the torsion tensor $T_{\mu\nu\rho}$ and the non-metricity tensor $Q_{\mu\nu\rho} = -D_{\mu}g_{\nu\rho}$ as properties of connections in a Riemannian or pseudo-Riemannian manifold, with metric $g_{\mu\nu}$ and affine connection $\Gamma_{\mu
u\rho}$. ($D_{\mu}$ is the covariant derivative with respect to $\Gamma_{\mu\nu\rho}$.) GR has to do with a metric symmetric connection, the Levi-Civita connection, that only allows for $R_{\mu\nu\rho\sigma}$; EC theory involves a metric but not necessarily symmetric connection, that allows also for $T_{\mu\nu\rho}$; while the theory of Weylian manifolds involves a non necessarily metric ($Q_{\mu\nu\rho} \neq 0$) and non necessarily symmetric ($T_{\mu\nu\rho} \neq 0$) connection. (In units of length $[L]$, $[R_{\mu\nu\rho\sigma}] = [L]^{-2}, [T_{\mu\nu\rho}] = [Q_{\mu\nu\rho}] = [\Gamma_{\mu\nu\rho}] = [L]^{-1}$, while $[g_{\mu\nu}] = [L]^0$.)

One of the most beautiful equations of Physics is the equality to zero of the Einstein tensor, that is, the Einstein’s equations in vacuum:

$$G_{\mu\nu} = 0,$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R,$$

with $R = g^{\nu\sigma}R_{\nu\sigma} = g^{\nu\sigma}g^{\mu\rho}R_{\mu\nu\rho\sigma} = g^{\nu\sigma}R_{\rho\nu\rho\sigma}$. $G_{\mu\nu} = 0$ is equivalent to Ricci flatness:

$$R_{\mu\nu} = 0.$$

This however does not imply vanishing curvature; so, in GR, empty space-time can be curved. Instead, in EC theory, torsion must be zero in vacuum.

It is remarkable that $G_{\mu\nu}$ appears naturally when the Bianchi equations for the Levi-Civita connection are expressed in terms of the Ricci tensor $R_{\mu\nu}$ and the scalar curvature $R$. Then, $G_{\mu\nu}$ is a purely geometric object.

The use of the tetrads ($e_{c}$) formalism along with their duals, the coframes or anholonomic coordinates ($e^{a}$), allows us to discover how GR and EC theory have an internal or gauge symmetry (Utiyama, 1956), implemented by a connection that takes values in the Lie algebra of the Lorentz group $\mathcal{L}_4$: the spin connection $\omega_{ab}$. The variations of the Einstein-Hilbert action of pure gravity or gravity coupled to Dirac matter with respect to $\omega_{ab}$ and $e_{c}$ lead, respectively, to the Cartan and Einstein equations, the former involving torsion and the spin of matter, and the latter involving curvature and the energy-momentum of matter.

Later, through a shift of the $e^{a}$’s one finds the translation gauge potential $B^{a}$; together, $B^{a}$ and $\omega_{bc}$, define a Poincaré connection, extending the symmetry group of GR and EC theory to the semidirect product $\mathcal{P}_4 \bowtie \mathcal{D}$, where $\mathcal{P}_4 = \mathcal{T}_4 \bowtie \mathcal{L}_4$ is the Poincaré group, with $\mathcal{T}_4$ the translation group, and $\mathcal{D}$ the group of general coordinate transformations.
Finally, in the last section, we discuss the problem of defining a gauge invariant field strength for the Maxwell field coupled to gravity, and the subsisting problem of the $U(1)$-gauge dependence of torsion in the solution of the Cartan equation.

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1. Connections in smooth real vector bundles

Let $\xi: \mathbb{R}^m - E \xrightarrow{\pi} M^n$ be a smooth $m$ dimensional real vector bundle over $M^n \equiv M$, a differentiable manifold of dimension $n$. Let $\Gamma(TM)$ denote the sections of the tangent bundle of $M$ and $\Gamma(E)$ denote the sections of $E$. $E$ is an $m + n$ dimensional differentiable manifold; this can be easily shown from the local triviality condition.

A connection in $\xi$ is a function

$$\nabla: \Gamma(TM) \times \Gamma(E) \rightarrow \Gamma(E),$$

$$(X, s) \mapsto \nabla(X, s) \equiv \nabla_X s$$
which has the following properties:

i) $\nabla_{X+X'} s = \nabla_X s + \nabla_{X'} s$

ii) $\nabla f X s = f \nabla X s$, where $f \in C^\infty(M, \mathbb{R})$: smooth real valued functions on $M$

iii) $\nabla_X(s+s') = \nabla_X s + \nabla_X s'$

iv) $\nabla_X(fs) = X(f)s + f \nabla_X s$ (Leibnitz rule)

$\Gamma(TM)$ and $\Gamma(E)$ are infinite dimensional vector spaces over $\mathbb{R}$, but modules over $C^\infty(M, \mathbb{R})$ as a ring, with dimensions $n$ and $m$ respectively. The Leibnitz rule shows that $\nabla$ is not $C^\infty(M, \mathbb{R})$-linear in the second entry. As will be shown below, this will be reflected in the fact that under a change of local coordinates, the set of connection coefficients (Christoffel symbols) is not a tensor.

The value of the connection at $(X, s)$ is called the covariant (or invariant) derivative of $s$ in the direction of $X$. $\nabla_X s : M \to E$, $x \mapsto \nabla_X s(x) = (x, (\nabla_X s)_x)$, with $(\nabla_X s)_x \in E_x$: the fibre in $E$ over $x$; $E_x$ is a real $m$ dimensional vector space.

Notice that we can define the operator

$\nabla_X : \Gamma(E) \to \Gamma(E)$, $\nabla_X(s) = \nabla_X s$

$\nabla_X \in \text{Lin}_{\mathbb{R}}(\Gamma(E))$ and obeys the Leibnitz rule.

One summarizes these concepts in the following diagram:

$\begin{array}{c}
\mathbb{R}^n \\
\downarrow \pi \\
E \\
\uparrow \nabla_X s \\
\mathbb{R}^n - TM \xrightarrow{\pi_M} M \\
\downarrow \nabla_X \\
E = \bigcup_{x \in M} \{x\} \times E_x \equiv \bigsqcup_{x \in M} E_x.
\end{array}$

Note: $M$ is a differentiable manifold; as such is a topological space. This global structure is defined in $M$ prior to any connection on $\xi$.

2. Linear connection in a differentiable manifold $M$

A linear connection on $M$ is a connection in its tangent bundle. With $E = TM$ we have:

$\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$,

$(X, Y) \mapsto \nabla(X, Y) \equiv \nabla_X Y$

with

i') $\nabla_{X+X'}(Y) = \nabla_X Y + \nabla_{X'} Y$

ii') $\nabla f X(Y) = f \nabla_X Y$
iii') \( \nabla_X(Y + Y') = \nabla_XY + \nabla_XY' \)

iv') \( \nabla_X(fY) = X(f)Y + f\nabla_XY \)

We shall denote by \( \operatorname{Conn}(\xi) \) the set of connections in the vector (or principal, where appropriate) bundle \( \xi \).

Again, \( \nabla_X \in \operatorname{Lin}_\mathbb{R}(\Gamma(TM)) \) and obeys the Leibnitz rule.

3. Total covariant derivative of a section in \( \xi \)

Let \( \Gamma(T^*M \otimes E) \) be the set of differential 1-forms in \( M \) with values in \( E \). \( T^*M \otimes E \) is a vector bundle on \( M \):

\[ \mathbb{R}^{n \times m} - T^*M \otimes E \to M, \quad \text{with} \quad T^*M \otimes E = \bigoplus_{x \in M} T_x^*M \otimes E_x. \]

The section \( \nabla s \in \Gamma(T^*M \otimes E) \) is defined by

\[ \nabla s : \Gamma(TM) \to \Gamma(E), \quad X \mapsto \nabla s(X) := \nabla_X s. \]

As for any differential form on \( M \), \( \nabla s \) is \( C^\infty(M, \mathbb{R}) \)-linear i.e. \( \nabla s(fX) = f\nabla s(X) \); however,

\[ \nabla(f s) = s f + f \nabla s. \]

In fact, \( \nabla(f s)(X) = \nabla_X (f s) = X(f)s + f \nabla_X s + s X(f) = f \nabla s(X) + s \nabla(f s) = (f \nabla s + s f)(X). \)

\( \nabla s \) is called the total covariant derivative of the section \( s \). In detail,

\[ \nabla s : M \to T^*M \otimes E, \quad x \mapsto (\nabla s)(x) = (x, (\nabla s)_x), \quad (\nabla s)_x = \alpha_x \otimes v_x : T_xM \to E_x, \quad X \mapsto (\nabla s)_x(X_x) = \alpha_x(X_v)v_x = \lambda_x v_x \text{ where } \alpha_x \in T^*_xM, \quad v_x \in E_x \text{ and } \lambda_x \in \mathbb{R}. \]

For a linear connection on \( M \),

\[ \nabla Y : \Gamma(TM) \to \Gamma(TM), \quad X \mapsto (\nabla Y)(X) = \nabla_X Y. \]

4. Local expressions for \( \nabla_X s \) and \( \nabla s \)

Let \( (U_\alpha, \sigma_i)_{\alpha \in \mathcal{J}, i \in \{1, \ldots, m\}} \) be a basis of local sections of \( E \) i.e. \( \sigma_i : U_\alpha \to E_\alpha = \pi^{-1}(U_\alpha), \quad x \mapsto \sigma_i(x) = (x, \sigma_{ix}) \in \{x\} \times \mathbb{R} \), with \( \pi \circ \sigma_i = \text{Id} U_\alpha \), and such that if \( s \in \Gamma(E_\alpha) \), then \( s = \sum_{i=1}^m s^i \sigma_i \) with \( s^i \in C^\infty(U_\alpha, \mathbb{R}) \). Let \( \frac{\partial}{\partial x^i} \equiv \partial_{\mu} \) be a local coordinate basis of \( \Gamma(TU_\alpha) \) i.e. if \( X \in \Gamma(TU) \) then \( X = X^\mu \partial_{\mu} \) with \( X^\mu \in C^\infty(U_\alpha, \mathbb{R}) \). (If the domains of the local trivializations of the bundle \( E \) do not coincide with the domains of the atlas \( \mathcal{U} \) of the manifold \( M \), one can always consider their intersections.) Then, locally,

\[ \nabla_X s = \nabla_X \sigma_i (s^j \sigma_j) = X^\mu \nabla_{\partial_{\mu}} (s^j \sigma_j) = X^\mu ((\partial_{\mu} s^i) \sigma_i + s^j \nabla_{\partial_{\mu}} \sigma_i); \]

and since \( \nabla_{\partial_{\mu}} \sigma_i \in \Gamma(E_\alpha) \) then

\[ \nabla_{\partial_{\mu}} \sigma_i := \Gamma^j_{\mu i} \sigma_j \]

for a unique set of \( n \times m^2 \) functions \( \Gamma^j_{\mu i} : U_\alpha \to \mathbb{R} \), called the Christoffel symbols of the connection \( \nabla \) in the atlas \( \mathcal{U} \). For a linear vector bundle, \( m = 1 \), and then there are only \( n \) symbols: \( \Gamma^j_{\mu i} \equiv \Gamma_\mu \).
We then write

$$\nabla_X s = X^\mu (\partial_\mu s^i + s^i \Gamma^j_{\mu i} \sigma_j) = X^\mu (\partial_\mu s^i + s^i \Gamma^j_{\mu i} \sigma_j) = X^\mu (\partial_\mu s^i + s^i \Gamma^j_{\mu i} \sigma_j) = X^\mu D^j_{\mu i} \sigma_j$$

where we have defined the local covariant derivative operator

$$D^j_{\mu i} := \partial_\mu \delta^j_i + \Gamma^j_{\mu i}.$$

(Units: $[\Gamma^j_{\mu i}] = [x^\mu]^{-1}$; in natural units $[\Gamma^j_{\mu i}] = [mass]$ if $[x^\mu] = [length]$.)

We can also write

$$\nabla_X s = X^\mu s^i j \sigma_j \text{ with } D_\mu s^i \equiv s^i j = D^j_{\mu i} s^i = s^j i + \Gamma^j_{\mu i} s^i$$

and $s^j i = \partial_\mu s^j$.

Notice that the ordinary derivative term $s^j i$ is due to the Leibnitz rule.

Locally, we can write

$$\nabla s = dx^\mu \otimes \nabla_{\partial_\mu} s.$$  

In fact, $\nabla s(X) = (dx^\mu \otimes \nabla_{\partial_\mu} s)X = dx^\mu (X) \nabla_{\partial_\mu} s = dx^\mu (X^\nu \nabla_{\partial_\mu} s) = X^\nu dx^\mu (\partial_\nu) \nabla_{\partial_\mu} s = X^\nu \delta^\nu_\mu \nabla_{\partial_\mu} s = \nabla_X s.$$

Then

$$\nabla s = dx^\mu \otimes (\partial_\mu \delta^j_i + \Gamma^j_{\mu i}) s^i \sigma_j = dx^\mu \otimes (\partial_\mu s^i \sigma_j + \Gamma^j_{\mu i} s^i \sigma_j) = dx^\mu \otimes \partial_\mu s^i \sigma_j + dx^\mu \otimes \Gamma^j_{\mu i} s^i \sigma_j = ds^i \otimes \sigma_j + \Gamma^j_{\mu i} s^i \sigma_j$$

where $\Gamma^j_{\mu i}$ is an $m \times m$ matrix of 1-forms on $U$, given by

$$\Gamma^j_{\mu i} = dx^\mu \Gamma^j_{\mu i}.$$

We can write $\Gamma^j_{\mu i} \otimes s^i \sigma_j = \Gamma^j_{\mu i} s^i \otimes \sigma_j = (\Gamma s)^j \otimes \sigma_j$ and then

$$\nabla s = ds^i \otimes \sigma_j + (\Gamma s)^j \otimes \sigma_j = (ds^i + (\Gamma s)^j) \otimes \sigma_j = ((d + \Gamma) s)^j \otimes \sigma_j$$

i.e.

$$\nabla s = (d + \Gamma) s \in \Gamma(T^* U \otimes E).$$

Then, locally,

$$\nabla = d + \Gamma.$$

Let $s^j i (x) = 0$ i.e.

$$s^j i (x) + \Gamma^j_{\mu i} (x) s^i (x) = 0.$$

Multiplying by $dx^\mu |_x$ we obtain

$$s^j i (x) dx^\mu |_x = -\Gamma^j_{\mu i} (x) s^i (x) dx^\mu |_x \in T^*_x U.$$

The r.h.s.

$$-\Gamma^j_{\mu i} (x) s^i (x) dx^\mu |_x \equiv (\delta_i |s^j)|_x,$$

is called the infinitesimal parallel transport (transfer) (Schroedinger, 1950) of the section $s^i$ by the connection $\nabla$ along the 1-form $dx^i |_x$ (see section 8), and we see that for a covariantly constant section, it coincides with the differential of $s^i$ at $x$, $ds^i |_x$. The transfer of the section, proportional to $dx^\lambda |_x$ and to the section itself, just follows the values of $s^i$ along $dx^\lambda$ (when $s^j i (x) = 0$. When $s^j i (x) \neq 0$, $(\delta_i |s^j)|_x$ still is the parallel transfer of $s^i$ through $dx^\lambda$, but it fails to follow the value of the section. (A more detailed discussion can be found in Cheng, 2010.)

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5. Local expression for $\nabla_X Y$

When $E = TM$, $\sigma_i = \partial_x$ and 
\[
\nabla_{\partial_x} \partial_{x^\mu} = \Gamma^\rho_{\mu \nu} \partial_{x^\nu}
\]
for a unique set of $n^3$ smooth functions $\Gamma^\rho_{\mu \nu} : U_\alpha \to \mathbb{R}$. ($\Gamma^\rho_{\mu \nu}$ is not necessarily symmetric or antisymmetric in $\mu$ and $\nu$.) Then,
\[
\nabla_X Y = X^\mu D^\rho_{\mu \nu} Y^\nu \partial_{x^\rho}, \quad D^\rho_{\mu \nu} = \partial_{x^\mu} \delta^\rho_{\nu} + \Gamma^\rho_{\mu \nu},
\]
\[
D^\rho_{\mu \nu} Y^\nu = \partial_{x^\mu} Y^\rho + \Gamma^\rho_{\mu \nu} Y^\nu, \quad \nabla_X Y = (\nabla_X Y)^\rho \partial_{x^\rho},
\]
\[
(\nabla_X Y)^\rho = X^\mu (\partial_{x^\mu} \delta^\rho_{\nu} + \Gamma^\rho_{\mu \nu}) Y^\nu.
\]
The quantity
\[
D^\rho_{\mu \nu} Y^\nu = \partial_{x^\mu} Y^\rho + \Gamma^\rho_{\mu \nu} Y^\nu \equiv Y^\rho_{\mu \nu} \equiv D^\rho_{\mu} Y^\rho
\]
is the covariant derivative of the local vector field $Y^\rho$ in the direction of $\frac{\partial}{\partial x^\rho}$. Then,
\[
(\nabla_X Y)^\rho = X^\mu D^\rho_{\mu} Y^\rho.
\]

If $V^\mu$ is a local vector field and $A_\mu$ is a local differential 1-form in $M^n$, then $\varphi = V^\mu A_\mu$ is a scalar (0-rank tensor) i.e. $\varphi' = \varphi$ or $V^\nu A'_\mu = V^\mu A_\mu$ under $x^\nu \to x'^\nu$. The covariant derivative of a scalar is naturally defined as
\[
\varphi_{,\nu} = \varphi_{,\nu} 
\]
and the Leibnitz rule is assumed for the covariant derivative of the product of arbitrary tensors $T$ and $S$:
\[
(TS)_{,\mu} = T_{,\mu} S + TS_{,\mu}
\]
. Then,
\[
\varphi_{,\nu} = (V^\mu A_\mu)_{,\nu} = (\partial_{x^\nu} V^\mu) A_\mu + V^\mu \partial_{x^\nu} A_\mu = V^\mu_{,\nu} A_\mu + V^\mu A_{\mu,\nu}
\]
and so
\[
V^\mu A_{\mu,\nu} = V^\mu_{,\nu} A_\mu + V^\mu A_{\mu,\nu} - V^\mu_{,\nu} A_\mu = V^\mu_{,\nu} A_\mu + V^\mu A_{\mu,\nu} - V^\mu_{,\nu} A_\mu - \Gamma^\mu_{\nu \rho} V^\rho A_\mu = V^\mu A_{\mu,\nu} - \Gamma^\mu_{\nu \rho} V^\rho A_\mu = V^\rho A_{\rho,\nu}
\]
where
\[
A_{\rho,\nu} = A_{\rho,\nu} - \Gamma^\mu_{\nu \rho} A_\mu \equiv D_{\nu} A_\rho
\]
is the covariant derivative of the local 1-form $A_\rho$ in the direction of $\frac{\partial}{\partial x^\rho}$.

6. Example: the trivial connection in $\mathbb{R}^n$

Consider $M = \mathbb{R}^n$; then with $X, Y \in \Gamma(TM)$ we define
\[
\nabla_X^0 Y := X(Y^\nu) \partial_{x^\nu} = X^\rho (Y^\rho) \partial_{x^\nu}.
\]
Additivity in both $X$ and $Y$, and $C^\infty(M, \mathbb{R})$-linearity in $X$ are trivial; finally, $\nabla_X^0 (fY) = X^\rho (fY^\rho) \partial_{x^\nu} = X(f)Y + fX(Y^\nu) \partial_{x^\nu} = X(f)Y + f \nabla_X^0 Y$.

In particular,
\[
\nabla_{\partial_x} (\partial_{x^\nu}) = \Gamma^0_{\mu \nu} \partial_{x^\nu} = \nabla_{\partial_x} (\delta^0_{\nu}) \partial_{x^\rho} = \partial_{x^\mu} (\delta^0_{\nu}) \partial_{x^\rho} = 0
\]
and therefore
\[
\Gamma^0_{\mu \nu} = 0.
\]
Note. We can compare with the Lie derivative $\mathcal{L}_X Y$:
$$\mathcal{L}_X Y = [X, Y] = X(Y^\nu)\partial_\nu - Y(X^\nu)\partial_\nu = [X, Y]^\nu\partial_\nu = \nabla_Y^0 X - \nabla_X^0 Y;$$
then
$$\nabla_X^0 Y - \nabla_Y^0 X - [X, Y] = 0.$$ 
As we shall see later, this means that $\nabla$ is torsion free.

For $\nabla^0 Y$ we have:
$$\nabla^0 Y = d(Y^\rho)\partial_\rho.$$ 
In fact, $(\nabla^0 Y)(X) = dx^\nu\partial_\nu(Y^\rho)\partial_\rho(X) = dx^\nu(X)\partial_\nu(Y^\rho)\partial_\rho = X(x^\nu)\partial_\nu(Y^\rho)\partial_\rho = X(Y^\rho)\partial_\rho = \nabla_Y^0 (Y).$

7. Transformation of $\Gamma^\nu_{\mu\rho}$ under a change of local coordinates (charts) in $M$

From the definition of the $\Gamma^\nu_{\mu\rho}$-functions and using
$$\Gamma^\nu_{\mu\rho} = \frac{\partial}{\partial x^\nu} \equiv \frac{\partial x^{\nu}}{\partial x^{\lambda}}$$
one obtains:

$$\nabla_{\Gamma^\nu_{\mu\rho}} \left( \frac{\partial}{\partial x^{\nu}} \frac{\partial x^{\lambda}}{\partial x^{\mu}} \right) = \frac{\partial x^{\nu}}{\partial x^{\lambda}} \nabla_{\Gamma^\nu_{\mu\rho}} \left( \frac{\partial x^{\lambda}}{\partial x^{\nu}} \frac{\partial x^{\nu}}{\partial x^{\lambda}} \right) = \frac{\partial x^{\nu}}{\partial x^{\lambda}} \left( \frac{\partial}{\partial x^{\nu}} (\frac{\partial x^{\lambda}}{\partial x^{\nu}} \frac{\partial x^{\nu}}{\partial x^{\lambda}}) \right) + \frac{\partial x^{\nu}}{\partial x^{\lambda}} \left( \frac{\partial x^{\lambda}}{\partial x^{\nu}} \frac{\partial x^{\nu}}{\partial x^{\lambda}} \right)$$

which implies, by the linear independence of the coordinate tangent fields $\frac{\partial}{\partial x^\nu}$,

$$\Gamma^\nu_{\mu\rho} = \frac{\partial x^{\alpha}}{\partial x^{\mu}} \frac{\partial x^{\beta}}{\partial x^{\nu}} \Gamma^\lambda_{\alpha\beta} = \Gamma^\nu_{\mu\rho},$$

finally, multiplying by the inverse transformation $\frac{\partial x^\gamma}{\partial x^\nu}$ and using $\frac{\partial x^\gamma}{\partial x^\nu} \frac{\partial x^\lambda}{\partial x^\gamma} = \delta^\gamma_{\lambda}$ one obtains

$$\Gamma^\gamma_{\mu\nu} = \frac{\partial x^\gamma}{\partial x^\nu} \frac{\partial x^{\alpha}}{\partial x^{\mu}} \frac{\partial x^{\beta}}{\partial x^{\nu}} \Gamma^\lambda_{\alpha\beta} = \Gamma^\gamma_{\mu\nu}.$$

The second term on the r.h.s., which comes from the fact that $\nabla$ obeys the Leibnitz rule, shows that the $\Gamma^\nu_{\mu\rho}$ functions do not transform as tensors.

Provisional definition of a tensor (previous to the existence of any metric in the manifold)

In a chart $U_\alpha(x^\mu_\alpha \equiv x^\mu)$ of $M^n$, a tensor with $r$ “contravariant” and $s$ “covariant” indices is a set of $n^{r+s}$ functions on $U_\alpha$ with values in $\mathbb{R}$,

$$\{T^\mu_{\nu_1...\nu_s}, \quad \mu_k, \nu_l = 1, \ldots, n, \quad k = 1, \ldots, r, \quad l = 1, \ldots, s\}$$

such that in a chart $U_\beta(x^\nu_\beta \equiv x^\nu)$ which overlaps with $U_\alpha(x^\mu_\alpha)$ becomes the set

$$T^{\mu_1...\mu_r}_{\nu_1...\nu_s} = \frac{\partial x^{\mu_1}}{\partial x^\mu_1} \cdots \frac{\partial x^{\mu_r}}{\partial x^\mu_r} \frac{\partial x^{\nu_1}}{\partial x^\nu_1} \cdots \frac{\partial x^{\nu_s}}{\partial x^\nu_s} T_{\nu_1...\nu_s}^{\mu_1...\mu_r}.$$
for $x' = x'(x)$ in the overlap $U_\alpha \cap U_\beta$.

**Remark:** The concept of covariant and contravariant indices has sense only if there exists a metric in the manifold.

An $r$-contravariant and $s$-covariant tensor can be considered a $C^\infty(M, \mathbb{R})$-multilinear map from the tensor product of $s$ factors of $\Gamma(TM)$ and $r$ factors of $\Gamma(T^*M)$ with values in $C^\infty(U_\alpha, \mathbb{R})$. On a chart $U_\alpha$,

$$T = T^{\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_s} \, dx_\mu \otimes \cdots \otimes dx_\mu \otimes \frac{\partial}{\partial x_\nu} \otimes \cdots \otimes \frac{\partial}{\partial x_\nu} : (\otimes_s (\Gamma(TU_\alpha))) \otimes (\otimes_r (\Gamma(T^*U_\alpha))) \to C^\infty(U_\alpha, \mathbb{R}),$$

$$V_1^{\mu_1} \frac{\partial}{\partial x_\alpha} \otimes \cdots \otimes V_s^{\mu_s} \frac{\partial}{\partial x_\alpha} \otimes A_{1\sigma} \, dx_\sigma \otimes \cdots \otimes A_{r\sigma} \, dx_\sigma \mapsto T^{\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_s} (V_1^{\mu_1} \frac{\partial}{\partial x_\alpha}) \otimes \cdots \otimes (V_s^{\mu_s} \frac{\partial}{\partial x_\alpha}) \otimes \alpha_{\nu_1} \sigma_{\nu_2} \cdots \sigma_{\nu_s} A_{1\sigma_1} \cdots A_{r\sigma_r} \, dx_{\sigma_1} \otimes \cdots \otimes dx_{\sigma_r}$$

$$= T^{\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_s} V_1^{\nu_1} \cdots V_s^{\nu_s} A_{1\mu_1} \cdots A_{r\mu_r}.$$  

We’ll call $\tau_r^s(M)$ to the $C^\infty(M, \mathbb{R})$-module of $r$-contravariant and $s$-covariant tensors on $M^n$. For example, $\tau_1^1(M) = \Gamma(TM)$: vector fields on $M$; $\tau_0^1(M) = \Gamma(T^*M)$: differential 1-forms on $M$. In general, $\tau_r^s(M) = \Gamma(T_r^sM)$: sections of the bundle of $r(s)$ contravariant (covariant) tensors on $M$, with $T_0^0M = TM$ and $T^n_0M = T^*M$.

Locally, given a tensor $T^{\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_s}$ and a connection $\Gamma^0_{\mu \nu}$ in the manifold $M^n$, the **covariant derivative of $T^{\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_s}$ in the direction of $\frac{\partial}{\partial x_\alpha}$** is given by

$$D_\mu T^{\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_s} = T^{\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_s} \frac{\partial}{\partial x_\alpha} + \Gamma^0_{\mu 1 \mu \nu} T^{\nu_1 \mu_2 \ldots \mu_r}_{\nu_1 \ldots \nu_s} + \cdots + \Gamma^0_{\mu r \mu \nu} T^{\mu_1 \ldots \mu_{r-1} \nu}_{\nu_1 \ldots \nu_s} - \Gamma^0_{\mu 1 \nu} T^{\nu_1 \ldots \nu_s}_{\mu_1 \nu_2 \ldots \nu_s} - \cdots$$

$$-\Gamma^0_{\mu s \nu} T^{\nu_1 \ldots \nu_s}_{\mu_1 \nu_2 \ldots \nu_{s-1} \nu}.$$  

It can be verified that $T^{\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_s}$ is an $r$-contravariant and $s+1$-covariant tensor.

**Remark:** Notice that while the operators $\nabla_X$ send tensors (or sections in general) of a given order to tensors of the same order, for both covariant and contravariant indices, the operators $D_\mu$ map $(r, s)$-tensors into $(r, s+1)$-tensors.

**Tensors in arbitrary vector bundles**

In $U_\alpha \cap U_\beta \subset M^n$ consider the change of local coordinates and sections: $\frac{\partial}{\partial x^\mu} = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial x'^\mu}$ and $\sigma_k = f^{-1}j^i\sigma'_j$ with $x^\mu = x'^\mu$, $x'^\nu = x'^\nu$, $\sigma_k = \sigma_k$, and $\sigma'_j = \sigma_j$; $\mu, \nu = 1, \ldots, n$; $j, k = 1, \ldots, m$; at each $x \in U_\alpha \cap U_\beta$, $j$ and $f^{-1}$ take values in $GL_m(\mathbb{R})$ with $\sigma'_j = f^{kj} \sigma_k$. We study the transformation of the Christoffel symbols $\Gamma^i_{\mu j}$ of a connection $\nabla$ in $\xi: \mathbb{R}^m \to M^n$:

$$\nabla_{\mu} \sigma_j = \Gamma^j_{\mu j} \sigma_j = \nabla_{\frac{\partial}{\partial x^\mu}} \sigma_j = \nabla_{\frac{\partial x'^\nu}{\partial x^\alpha}} \sigma_j = \frac{\partial x'^\nu}{\partial x^\mu} \nabla_{\frac{\partial x^\alpha}{\partial x'^\nu}} f^{-1}i_j \sigma'_j = \frac{\partial x'^\nu}{\partial x^\mu} \frac{\partial}{\partial x^\nu} ((\frac{\partial}{\partial x'^\nu} f^{-1}i_j) \sigma'_j + f^{-1}i_j \nabla_{\frac{\partial}{\partial x^\nu}} \sigma'_j)$$

$$= \frac{\partial x'^\nu}{\partial x^\mu} \frac{\partial}{\partial x'^\nu} f^{-1}i_j + f^{-1}i_j \Gamma^j_{\nu j} \sigma'_j = \Gamma^j_{\mu j} f^{-1}i_j \sigma'_j$$

i.e.

$$\Gamma^j_{\mu j} f^{-1}i_j = \frac{\partial x'^\nu}{\partial x^\mu} \frac{\partial}{\partial x'^\nu} f^{-1}i_j + f^{-1}i_j \Gamma^j_{\nu j};$$
multiplying by \( f'_{\text{i}} \) and using \( f^{-1}_{\text{j}} f'_{\text{i}} = \delta'_{\text{j}} \) we obtain

\[
\Gamma^\mu_{\nu \lambda} = \frac{\partial x'^\mu}{\partial x^\nu} f^{-1}_{\text{j}} f'_{\text{i}} \Gamma^\mu_{\nu \lambda} + \frac{\partial x'^\nu}{\partial x^\mu} f^{-1}_{\text{j}} \frac{\partial}{\partial x'^\mu} (f^{-1}_{\text{i}}).
\]

The homogeneous part in the connection coefficients gives the general law for the tensorial transformation of an object with \( r(v) \) contravariant or upper (covariant or lower) internal indices, \( r, v = 1, \ldots, m \), and \( s(t) \) contravariant (covariant) space-time (external) indices, \( s, t = 1, \ldots, n \):

\[
T^{\mu_1 \cdots \mu_r \cdots \nu_1 \cdots \nu_s}_{\nu_{s+1} \cdots \nu_{s+t}} = \frac{\partial x'^\mu}{\partial x^\nu} \cdots \frac{\partial x'^\nu}{\partial x^\mu} T^{\mu_1 \cdots \mu_r \cdots \nu_1 \cdots \nu_s}_{\nu_{s+1} \cdots \nu_{s+t}} + \cdots - \frac{\partial x'^\nu}{\partial x^\mu} \frac{\partial}{\partial x'^\mu} T^{\mu_1 \cdots \mu_r \cdots \nu_1 \cdots \nu_s}_{\nu_{s+1} \cdots \nu_{s+t}}.
\]

For example,

\[
T^{\mu \nu} = \frac{\partial x'^\mu}{\partial x^\nu} \Gamma^\nu_{\mu \lambda} f^{a} T^{\nu \sigma b}, \quad T^{\mu \nu}_{\ ab} = \frac{\partial x'^\mu}{\partial x^\nu} \frac{\partial x'^\nu}{\partial x^\mu} f^{-1}_{\cdots c} f^{a} T^{\nu \sigma b}_{\ \ cd}.
\]

### 8. Directional covariant derivative and parallel transport of tensors; geodesics

If \( c: (a, b) \to M^n \), \( \lambda \to c(\lambda) \), with \( (a, b) \) an open interval in \( \mathbb{R} \), is a smooth path in \( M^n \) locally given by \( c(\lambda) = (x^1(\lambda), \ldots, x^n(\lambda)) \), then the covariant derivative of \( T^{\mu_1 \cdots \mu_r}_{\nu_{1} \cdots \nu_{s}} \) along \( c \) is the tensor defined by

\[
\frac{d}{d\lambda} T^{\mu_1 \cdots \mu_r}_{\nu_{1} \cdots \nu_{s}} := \frac{dx'^\mu}{d\lambda} D_{\mu} T^{\mu_1 \cdots \mu_r}_{\nu_{1} \cdots \nu_{s}}.
\]

We have then defined the **covariant derivative operator along the path** \( c \) through

\[
D_{\mu} = \frac{dx'^\mu}{d\lambda} D_{\mu}.
\]

In detail,

\[
\frac{d}{d\lambda} T^{\mu_1 \cdots \mu_r}_{\nu_{1} \cdots \nu_{s}} = \frac{dx'^\mu}{d\lambda} \left( \frac{\partial}{\partial x^\mu} T^{\mu_1 \cdots \mu_r}_{\nu_{1} \cdots \nu_{s}} + \Gamma^\mu_{\mu \lambda} T^{\alpha_1 \cdots \mu_2 \cdots \cdots \mu_r}_{\nu_{1} \cdots \nu_{s}} + \cdots - \Gamma^\mu_{\nu \lambda} T^{\mu_1 \cdots \mu_r}_{\nu_{1} \cdots \nu_{s}} \right)
\]

\[
= \frac{d}{d\lambda} T^{\mu_1 \cdots \mu_r}_{\nu_{1} \cdots \nu_{s}} + \frac{dx'^\mu}{d\lambda} \Gamma^\mu_{\mu \lambda} T^{\alpha_1 \cdots \mu_2 \cdots \cdots \mu_r}_{\nu_{1} \cdots \nu_{s}} + \cdots - \frac{dx'^\nu}{d\lambda} \Gamma^\nu_{\nu \lambda} T^{\mu_1 \cdots \mu_r}_{\nu_{1} \cdots \nu_{s}}.
\]

Symbolically,

\[
\frac{d}{d\lambda} T = \frac{d}{d\lambda} T + c \Gamma^s T^s - c \Gamma^s T^s
\]

where \(*\) denotes the contractions.

For a vector field \( V^\mu \),

\[
\frac{d}{d\lambda} V^\mu = \frac{d}{d\lambda} V^\mu + \frac{dx'^\nu}{d\lambda} \Gamma^\mu_{\nu \rho} V^\rho
\]

i.e.

\[
\left( \frac{d}{d\lambda} \right)^\mu (V^\mu) = \left( \frac{d}{d\lambda} V^\mu (\lambda) + \frac{dx'^\nu(\lambda)}{d\lambda} \Gamma^\mu_{\nu \rho}(\lambda) V^\rho (\lambda) \right) \frac{\partial}{\partial x'^\mu (\lambda)}
\]

where the dependence of \( \lambda \) is through \( c \), and for a differential 1-form \( A_\mu \),

\[
\frac{d}{d\lambda} A^\mu = \frac{dA_\mu}{d\lambda} - \frac{dx'^\nu}{d\lambda} \Gamma^\mu_{\nu \rho} A^\rho
\]

i.e.

\[
\left( \frac{d}{d\lambda} \right)^\mu (A^\mu (\lambda)) = \left( \frac{dA_\mu (\lambda)}{d\lambda} - \frac{dx'^\nu(\lambda)}{d\lambda} \Gamma^\mu_{\nu \rho}(\lambda) A^\rho (\lambda) \right) dx'^\mu (\lambda).
\]
If, in particular, $V^\mu = \frac{dx^\mu}{d\lambda}$: the tangent vector to the curve $c$ at $\lambda$, i.e. $V = \dot{c}$, then

$$\frac{d\dot{c}}{d\lambda} = \frac{d^2 x^\mu}{d\lambda^2} + \frac{dx^\nu}{d\lambda} \Gamma^\mu_{\nu p} \frac{dx^p}{d\lambda}.$$  

Symbolically $\frac{dV}{d\lambda} = (\frac{d}{d\lambda} + \dot{c} \Gamma)V$ i.e. $\frac{d}{d\lambda} = \frac{d}{d\lambda} + \dot{c} \Gamma$ for vectors, and $\frac{dA}{d\lambda} = (\frac{d}{d\lambda} - \dot{c} \Gamma)A$ i.e. $\frac{d}{d\lambda} = \frac{d}{d\lambda} - \dot{c} \Gamma$ for 1-forms.

A tensor $T$ is said to be parallel transported by the connection $\nabla$ from $c(\lambda_0)$ to $c(\lambda_1)$ along the smooth curve $c$ in $M^n (a < \lambda_0 < \lambda_1 < b)$, if

$$\frac{dT}{d\lambda} = 0 \text{ for all } \lambda \in [\lambda_0, \lambda_1]$$

i.e. if

$$\frac{dT(\lambda)}{d\lambda} = -\epsilon \Gamma_\lambda^\mu(\lambda) T^\mu(\lambda) + \epsilon \Gamma_\lambda^\mu(\lambda) T^\mu(\lambda).$$

This is a system of $n^{r+s}$ ordinary differential equations of first order (ODE-1). By general theorems on ODE-1, if $T_{\lambda_0} \in \tau_0^r(c(\lambda_0))$ then there exists and is unique a parallel transported tensor $T(\lambda)$ along $c$, in particular at $c(\lambda_1)$, such that $T(\lambda_0) = T_{\lambda_0}$.  

The parallel transport of $T$ depends on $\Gamma$ i.e. on $\nabla$ and on the path $c$. There exists a vector space isomorphism

$$P^\nabla_e : \tau_0^r(c(\lambda_0)) \to \tau_0^r(c(\lambda_1)), \quad T_{\lambda_0} \mapsto P^\nabla_e(T_{\lambda_0}) = T_{\lambda_1}$$

with $(P^\nabla_e)^{-1} = P^\nabla_{\lambda_1}$ where $c^{-1}(\lambda) = c(\lambda_1 + \lambda_0 - \lambda)$.  

The equations of parallel transport for vector fields and differential 1-forms are

$$(\frac{dV}{d\lambda})^\mu = 0 \iff \frac{dV^\mu}{d\lambda} = -\frac{dx^\nu}{d\lambda} \Gamma^\mu_{\nu p} V^p$$

and

$$(\frac{dA}{d\lambda})^\mu = 0 \iff \frac{dA^\mu}{d\lambda} = \frac{dx^\nu}{d\lambda} \Gamma^\mu_{\nu p} A^p$$

respectively.

In particular, a curve $c$ is a geodesic in $M^n$ with respect to the connection $\nabla$, if its tangent vector $\dot{c}$ is parallel transported along $c$:

$$(\frac{d\dot{c}}{d\lambda})^\mu = 0 \iff \frac{d^2 x^\mu}{d\lambda^2} = -\Gamma^\mu_{\nu p} \frac{dx^\nu}{d\lambda} \frac{dx^p}{d\lambda}.$$  

Symbolically,

$$\frac{d\dot{c}}{d\lambda} = 0 \iff \dot{c} = -\Gamma^2.$$

In more detail,

$$\ddot{x}^\mu(\lambda) + \Gamma^\mu_{\nu p}(\lambda) \dot{x}^\nu(\lambda) \dot{x}^p(\lambda) = 0,$$

which is a system of $n$ ordinary differential equations of second order (ODE-2) for $c(\lambda)$ $(x^\mu(\lambda))$. Given $(x_0, v_{x_0}) \in TM$, there always exists a unique solution to this system of equations in an interval $(\lambda_0 - \epsilon, \lambda_0 + \epsilon)$, $\epsilon > 0$ with the initial conditions $c(\lambda_0) = x_0$ and $c(\lambda_0) = v_{x_0}$.

The geodesic equation is invariant under the change $\lambda \mapsto a\lambda + b$, with $a,b \in \mathbb{R}$, $a \neq 0$. (See also section 13.)
Notice the arbitrariness of \( v_{x_0} \) at \( x_0 \), and the fact that the whole geodesic is determined in the interval \((\lambda_0 - \epsilon, \lambda_0 + \epsilon)\) ("globally") from the initial data.

For the trivial connection in \( \mathbb{R}^n \) (section 6), \( \Gamma^0_{\nu \rho} = 0 \) and then the solutions of the geodesic equation are straight lines:

\[ \ddot{x}^\mu(\lambda) = 0 \Rightarrow x^\mu(\lambda) = a^\mu \lambda + b^\mu. \]

9. Curvature and torsion of a connection

Let \( \nabla \) be a connection on \( \xi : \mathbb{R}^m \to M^n \). The curvature of \( \nabla \) is defined as follows:

\[
\mathcal{R} : \Gamma(TM) \times \Gamma(TM) \times \Gamma(E) \to \Gamma(E), \quad (X, Y, s) \mapsto \mathcal{R}(X, Y, s) := ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})(s)
\]
i.e.

\[
\mathcal{R}(X, Y, s) = \nabla_X(\nabla_Y(s)) - \nabla_Y(\nabla_X(s)) - \nabla_{[X,Y]}(s).
\]

Clearly, \( \mathcal{R}(X, Y, s) = -\mathcal{R}(Y, X, s) \).

We’ll show that \( \mathcal{R} \) is \( C^\infty(M, \mathbb{R}) \)-linear in its three entries. This will have as a consequence that the set of local components of \( \mathcal{R} \) behaves as a tensor.

i) \( \mathcal{R}(fX, Y, s) = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]}(s) = f \nabla_X \nabla_Y s - \nabla_Y (f \nabla_X s) - \nabla_{[X,Y]-Y(f)X}s = f \nabla_X \nabla_Y s - Y(f) \nabla_X s - f \nabla_Y \nabla_X s + \nabla_{[X,Y]}(s) = f \nabla_X \nabla_Y s - \nabla_Y (f \nabla_X s) - f \nabla_X \nabla_Y s + Y(f) \nabla_X s - f \nabla_Y \nabla_X s - f \nabla_{[X,Y]}(s) = f \mathcal{R}(X, Y, s); \)

ii) \( \mathcal{R}(X, fY, s) = -\mathcal{R}(fY, X, s) = -f \mathcal{R}(Y, X, s) = f \mathcal{R}(X, Y, s); \)

iii) \( \mathcal{R}(X, Y, fs) = \nabla_X \nabla_Y (fs) - \nabla_Y \nabla_X (fs) - \nabla_{[X,Y]}(fs) = \nabla_X (Y(fs) + f \nabla_Y s) - \nabla_Y (X(fs) + f \nabla_X s) - [X, Y](f)s - f \nabla_{[X,Y]}(s) = X(Y(f)s) + f \nabla_Y \nabla_X s + X(f) \nabla_Y s + f \nabla_X \nabla_Y s - Y(X(f)s) - X(f) \nabla_Y s - Y(f) \nabla_X s - f \nabla_Y \nabla_X s - X(f) \nabla_Y s - f \nabla_X \nabla_Y s - f \nabla_{[X,Y]}(s) = f \mathcal{R}(X, Y, s). \)

Locally (in a common chart for \( \xi \) and \( M \)),

\[
\mathcal{R}(\partial_\mu, \partial_\nu, \sigma_j) = [\nabla_{\partial_\mu}, \nabla_{\partial_\nu}](\sigma_j) - [\nabla_{\partial_\nu}, \nabla_{\partial_\mu}](\sigma_j), \quad \text{but } [\partial_\mu, \partial_\nu] = 0 \text{ and } \nabla_0 s = \nabla_0 X s = 0, \text{ then}
\]

\[
\mathcal{R}(\partial_\mu, \partial_\nu, \sigma_j) = [\nabla_{\partial_\mu}, \nabla_{\partial_\nu}](\sigma_j) = \nabla_{\partial_\mu} (\nabla_{\partial_\nu} (\sigma_j)) - \nabla_{\partial_\nu} (\nabla_{\partial_\mu} (\sigma_j)) = \nabla_{\partial_\mu} (\Gamma^i_{\nu j} \sigma_i) - \nabla_{\partial_\nu} (\Gamma^i_{\mu j} \sigma_i) = \partial_\mu (\Gamma^i_{\nu j} \sigma_i) + \Gamma^i_{\nu j} \Gamma^k_{\mu l} \sigma_l - \partial_\nu (\Gamma^i_{\mu j} \sigma_i) + \Gamma^i_{\mu j} \Gamma^k_{\nu l} \sigma_l = \mathcal{R}^k_{\mu j l} \sigma_k \text{ with}
\]

\[
\mathcal{R}^k_{\mu j l} = \partial_\mu \Gamma^k_{\nu j} - \partial_\nu \Gamma^k_{\mu j} + \Gamma^k_{\mu l} \Gamma^l_{\nu j} - \Gamma^k_{\nu l} \Gamma^l_{\mu j}.
\]

Then

\[
\mathcal{R}(X, Y, s) = X^\mu Y^\nu S^j \mathcal{R}(\partial_\mu, \partial_\nu, \sigma_j) = X^\mu Y^\nu S^j \mathcal{R}^i_{\mu j l} \sigma_i.
\]

For a linear connection in a manifold,

\[
\mathcal{R}^\rho_{\mu \nu \sigma} = \partial_\mu \Gamma^\rho_{\nu \sigma} - \partial_\nu \Gamma^\rho_{\mu \sigma} + \Gamma^\rho_{\mu \lambda} \Gamma^\lambda_{\nu \sigma} - \Gamma^\rho_{\nu \lambda} \Gamma^\lambda_{\mu \sigma},
\]

with \( \mathcal{R}(X, Y, Z) = X^\mu Y^\nu Z^\sigma \mathcal{R}(\partial_\mu, \partial_\nu, \partial_\sigma) = X^\mu Y^\nu Z^\sigma \mathcal{R}^\rho_{\mu \sigma \rho \sigma \partial_\rho.} \)

In particular, \( \mathcal{R}(\partial_\mu, \partial_\nu, Z) = [\nabla_{\partial_\mu}, \nabla_{\partial_\nu}](Z) = Z^\sigma \mathcal{R}^\rho_{\mu \sigma \partial_\rho} \) or \( \mathcal{R}(\partial_\mu, \partial_\nu, \partial_\sigma) = [\nabla_{\partial_\mu}, \nabla_{\partial_\nu}](\partial_\sigma) = \mathcal{R}^\rho_{\mu \sigma \partial_\rho}. \)

Defining

\[
\check{\mathcal{R}} : \Gamma(TM) \times \Gamma(TM) \to \text{End}_\mathbb{C} \Gamma(TM), \quad \check{\mathcal{R}}(X, Y) : \Gamma(TM) \to \Gamma(TM), \quad \check{\mathcal{R}}(X, Y)(Z) := \mathcal{R}(X, Y, Z)
\]
one obtains, in particular,
\[ \tilde{R}(\partial_\mu, \partial_\nu) = [\nabla_{\partial_\mu}, \nabla_{\partial_\nu}], \]
which is the usual expression of curvature in terms of a commutator of local covariant derivatives.

If in \( R_\rho^{\mu\nu\sigma} \) we contract \( \rho \) with \( \sigma \) we obtain the antisymmetric tensor
\[ S_{\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\rho} - \partial_\nu \Gamma^\rho_{\mu\rho} = -S_{\nu\mu}, \]
and if we contract \( \rho \) with \( \nu \) we obtain the tensor
\[ R_{\rho\sigma} = \partial_\rho \Gamma^\lambda_{\sigma\lambda} - \partial_\sigma \Gamma^\lambda_{\rho\lambda} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\sigma\mu} - \Gamma^\rho_{\sigma\lambda} \Gamma^\lambda_{\mu\rho}. \]

In general, \( R_{\rho\sigma} \) is non symmetric, not even when \( \Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}. \) In this case, however, its antisymmetric part is the half of \( S_{\mu\nu}: \)
\[ R_{\{\mu\sigma\}} = \frac{1}{2} (R_{\rho\sigma\mu} - R_{\rho\mu\sigma}) = \frac{1}{2} S_{\mu\sigma}. \]
But the definition of \( S_{\mu\nu} \) does not require a symmetric connection.

In GR, where \( \nabla \) is the Levi-Civita connection (section 13) uniquely determined by the metric in a pseudo-riemannian (lorentzian) manifold, it is usual to denote
\[ R^e_\rho_{\mu\nu\sigma} = R^e_{\rho\mu\nu\sigma}, \]
with \( R \equiv R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM), (X,Y,Z) \mapsto R(X,Y,Z) = ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})(Z) = X^\mu Y^\nu Z^\rho R^e_{\rho\mu\nu\sigma} \partial_\sigma. \) This definition holds for any connection, like the Weyl connection (non-metric symmetric), or that corresponding to the Einstein-Cartan theory (metric non-symmetric).

Clearly, \( R^e_{\rho\sigma\mu\nu} = -R^e_{\rho\mu\sigma\nu}. \) Since \( R(\partial_\mu, \partial_\nu, \partial_\sigma) = R^e_{\rho\mu\nu\sigma} \partial_\rho = R^e_{\rho\mu\sigma\nu} \partial_\rho, \)

\[ < dx^\lambda, R(\partial_\mu, \partial_\nu, \partial_\sigma) > = < dx^\lambda, R^e_{\rho\mu\nu\sigma} \partial_\rho > = < dx^\lambda, \partial_\rho > R^e_{\rho\mu\nu\sigma} = \delta^\lambda_\rho R^e_{\rho\mu\nu\sigma} = R^e_{\sigma\mu\nu} \]

where \( < , > \) denotes the 1-form-vector contraction, which is independent of the metric.

\[ (R : \Gamma(TU) \times \Gamma(TU) \times \Gamma(TU) \to \Gamma(TU), (\partial_\mu, \partial_\nu, \partial_\sigma) \mapsto R(\partial_\mu, \partial_\nu, \partial_\sigma)). \]

For a symmetric connection, \( \Gamma^e_{\rho\mu\nu} = \Gamma^e_{\rho\nu\mu} \) (see section 13),
\[ R^e_{\rho\sigma\mu\nu} = \partial_\rho \Gamma^e_{\sigma\mu\nu} - \partial_\nu \Gamma^e_{\rho\mu\sigma} + \Gamma^\lambda_{\sigma\mu} \Gamma^e_{\rho\nu\lambda} - \Gamma^\lambda_{\nu\rho} \Gamma^e_{\rho\mu\lambda}. \ (\ast) \]

The torsion \( T \) of a linear connection \( \nabla \) on a manifold \( M^n \) is defined as follows:
\[ T : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM), (X,Y) \mapsto T(X,Y) : \nabla_X Y - \nabla_Y X - [X,Y]. \]

It holds:

i) \( T(X,Y) = -T(Y,X) \)

ii) \( T^0 = 0 \) for the trivial connection \( \nabla^0 \) in \( \mathbb{R}^n. \)

iii) \( T \) is \( C^\infty(M, \mathbb{R}) \)-linear in \( X \) and \( Y \) i.e. \( T(fX,Y) = T(X,fY) = fT(X,Y). \)
iv) Locally, in a chart $U_\alpha(x^\mu_\alpha), \alpha \in J,$

$$T(X, Y) = (T(X, Y))^\mu \frac{\partial}{\partial x^\mu} = ((\nabla_X Y)^\mu - (\nabla_Y X)^\mu - [X, Y]^\mu) \frac{\partial}{\partial x^\mu} = X^\nu 2T^\mu_{\nu p} Y^p \frac{\partial}{\partial x^\mu}$$

with

$$T^\mu_{\nu p} = \frac{1}{2}(\Gamma^\mu_{\nu r} - \Gamma^\mu_{p r}) = -T^\mu_{p r} = \Gamma^\mu_{p r}.$$  

If $X^\nu = \delta^\nu_X i.e. X = \delta^\nu_X \frac{\partial}{\partial x^\nu}$, and $Y^\rho = \delta^\rho_Y i.e. Y = \delta^\rho_Y \frac{\partial}{\partial x^\rho}$, then

$$T(\frac{\partial}{\partial x^\nu}, \frac{\partial}{\partial x^\rho}) = (\Gamma^\mu_{\nu r} - \Gamma^\mu_{p r}) \frac{\partial}{\partial x^\mu} = 2T^\mu_{\nu p} \frac{\partial}{\partial x^\mu}$$

i.e. $T^\mu_{\nu p} = \frac{1}{2}(\Gamma^\mu_{\nu r} - \Gamma^\mu_{p r})$.

A straightforward calculation leads to:

$$[D_\rho, D_\sigma] V^\mu = D_\rho(D_\sigma V^\mu) - D_\sigma(D_\rho V^\mu) = (V^\mu)_{;\rho} - (V^\mu)_{;\sigma} = R^\mu_{\rho \sigma \lambda} V^\lambda - 2T^\lambda_{\rho \sigma} V^\mu : \lambda.$$  

If $\varphi$ is a scalar, then $[D_\mu, D_\nu] (\varphi) = D_\mu(\partial_\nu \varphi) - D_\nu(\partial_\mu \varphi) = D_\mu(\varphi_\nu) - D_\nu(\varphi_\mu) = \partial_\mu \varphi_\nu - \Gamma^\rho_{\mu \nu} \partial_\rho \varphi_\nu - \partial_\nu \varphi_\mu + \Gamma^\rho_{\nu \mu} \varphi_\rho = -(\Gamma^\rho_{\mu \nu} - \Gamma^\rho_{\nu \mu}) \partial^\rho \varphi$ i.e.

$$[D_\mu, D_\nu] (\varphi) = -2T^\rho_{\mu \nu} \partial^\rho \varphi.$$  

So, $[D_\mu, D_\nu](\varphi) = 0$ if $T^\rho_{\mu \nu} = 0$.

Then, with $\varphi = V^\alpha W_\alpha$ and using the Leibnitz rule, for a covariant vector (1-form) one obtains

$$[D_\mu, D_\nu] W_\alpha = -R^\sigma_{\alpha \mu \nu} W_\sigma - 2T^\rho_{\mu \nu} W_{\alpha ; \rho}.$$  

The generalization for a tensor $T^{\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_s}$ is

$$[D_\mu, D_\nu] (T^{\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_s}) = R^\lambda_{\mu \rho} T^{\lambda \nu_1 \ldots \nu_s}_{\nu_1 \ldots \nu_s} + \ldots + R^\lambda_{\nu_1 \nu_2} T^{\mu_1 \ldots \mu_r \nu_1 \nu_2}_{\nu_1 \nu_2} - R^\lambda_{\nu_3 \nu_4} T^{\mu_1 \ldots \mu_r \nu_3 \nu_4}_{\nu_3 \nu_4} - \ldots - R^\lambda_{\nu_1 \mu} T^{\mu_1 \ldots \mu_r}_{\nu_1 \nu_2 \ldots \nu_s - 1 \lambda} - 2T^\mu_{\nu \rho} T^{\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_s - 1 \lambda}.$$  

For a symmetric connection, $\Gamma^\mu_{\nu p} = \Gamma^\mu_{p \nu}$ and therefore

$$T^\mu_{\nu p} = 0$$

in all charts. I.e. a symmetric connection (like the Levi-Civita connection (section 13)) is torsion free. From the transformation of the $\Gamma^\mu_{\nu p}$'s, it is clear that $T^\mu_{\nu p}$ is a tensor. In particular one has

$$[D_\rho, D_\sigma] V^\mu = R^\mu_{\rho \sigma \lambda} V^\lambda.$$  

The modified torsion tensor is defined as

$$\tilde{T}^\mu_{\nu p} = T^\mu_{\nu p} - \frac{1}{n-1}(\delta^\mu_{\nu} T_\nu - \delta^\mu_{\nu} T_\rho)$$

where $T_\nu = T^\sigma_{\nu \sigma}$ is the torsion "vector" (in fact, it is a 1-form). $\tilde{T}^\mu_{\nu p}$ is traceless i.e. $\tilde{T}^\mu_{\nu p} = 0$.

10. Geometric interpretation of curvature and torsion

Curvature
Consider the infinitesimal parallelogram $pqrs$ in $M^n$ with coordinates $x^\alpha$, $x^\mu + \epsilon^\mu$, $x^\nu + \epsilon^\nu$, $x^\rho + \delta^\rho$, $x^\sigma + \delta^\sigma$, respectively, with $|\epsilon^\mu|, |\delta^\rho| \ll 1$. Let $c$ and $c'$, not necessarily part of geodesics, be curves which join $p$ with $r$ through $q$ and $s$ respectively, and $\nabla$ (locally $\Gamma^\nu_{\alpha\rho}$) be an arbitrary connection in $M^n$. Let $V_p \in T_p M$; its variation from $p$ to $q$ through $c$ is obtained from the covariant derivative of a vector along a curve (section 8), which implies $DV^\mu = dM(\partial_{\alpha}\frac{\partial}{\partial x^\alpha})^\mu = dV^\mu + dx^\nu \Gamma^\mu_{\nu\rho} V^\rho$; if the transport is parallel then $DV^\mu = 0$ i.e. $dV^\mu = -dx^\nu \Gamma^\mu_{\nu\rho} V^\rho$. So through $c$, $dV^\mu|_c = V^\mu_q|_c - V^\mu_p$ and with $dx^\mu = \epsilon^\mu$ one obtains

$$V^\mu_q|_c = V^\mu_p - \epsilon^\nu \Gamma^\mu_{\nu\rho}(p) V^\rho.$$  

Then, 

$$V^\mu_r|_c = V^\mu_q|_c - \delta^\alpha \Gamma^\alpha_{\mu\rho}(q) V^\rho|_c = V^\mu_p - \epsilon^\nu \Gamma^\mu_{\nu\rho}(p) V^\rho - \delta^\alpha \Gamma^\alpha_{\nu\rho}(q)(V^\rho_p - \epsilon^\nu \Gamma^\rho_{\nu\rho}(p) V^\rho_p)$$

$$= V^\mu_p - \epsilon^\nu \Gamma^\mu_{\nu\rho}(p) V^\rho - \delta^\alpha (\Gamma^\alpha_{\nu\rho}(p) + \partial_\rho \Gamma^\alpha_{\alpha\rho}(p)) \epsilon^\lambda (V^\rho_p - \epsilon^\nu \Gamma^\rho_{\nu\rho}(p) V^\rho_p)$$

$$= V^\mu_p - \epsilon^\nu \Gamma^\mu_{\nu\rho}(p) V^\rho - \partial_\rho \Gamma^\rho_{\nu\rho}(p) \delta^\alpha \epsilon^\lambda (V^\rho_p - \epsilon^\nu \Gamma^\rho_{\nu\rho}(p) V^\rho_p)$$

$$= V^\mu_p - \epsilon^\nu \Gamma^\mu_{\nu\rho}(p) V^\rho - \partial_\rho \Gamma^\rho_{\nu\rho}(p) \delta^\alpha \epsilon^\lambda + V^\rho_p \Gamma^\rho_{\nu\rho}(p) \Gamma^\alpha_{\alpha\rho}(p) \delta^\alpha \epsilon^\nu,$$

where we have neglected terms of order higher than two in $\epsilon$’s and $\delta$’s; to obtain $V^\mu_r|_c$, we simply change $\delta \leftrightarrow \epsilon$ which is equivalent to the change $\alpha \leftrightarrow \nu$.

$$V^\mu_r|_c = V^\mu_p - \epsilon^\nu \Gamma^\mu_{\nu\rho}(p) \delta^\rho - \epsilon^\nu \Gamma^\mu_{\nu\rho}(p) \epsilon^\nu - \partial_\rho \Gamma^\rho_{\nu\rho}(p) \delta^\alpha \epsilon^\nu.$$  

Then the difference of the two parallel transports is 

$$V^\mu_r|_c - V^\mu_r|_c = (\partial_\nu \Gamma^\nu_{\mu\rho}(p) - \partial_\rho \Gamma^\nu_{\nu\rho}(p) + \Gamma^\rho_{\nu\rho}(p) \Gamma^\nu_{\alpha\rho}(p) - \Gamma^\rho_{\nu\rho}(p) \Gamma^\nu_{\alpha\rho}(p)) V^\nu_p \delta^\alpha \epsilon^\nu = R^\nu_{\rho\alpha\sigma}(p) V^\nu_p \epsilon^\nu \delta^\alpha$$

$$= R^\nu_{\rho\alpha\sigma}(p) V^\nu_p A^\alpha_{\sigma\nu},$$  

where $A^\alpha_{\sigma\nu}$ is the infinitesimal area $\epsilon^\nu \delta^\alpha$, enclosed by the curves $c$ and $c'$. Clearly,

$$V^\mu_r|_c = V^\mu_r|_c \text{ if and only if } R^\mu_{\rho\alpha\sigma}(p) = 0.$$

Then, the curvature tensor measures the difference between the parallel transport of a vector through the paths $c$ and $c'$, where $c \cup (-c')$ is a loop.

$$V^\mu_r|_c - V^\mu_r|_c$$  

amounts to a rotation, since norms of vectors do not change by parallel transport induced by metric connections (appendix B); then one says that curvature is the rotational part of the connection.

When parallel transport is independent of the path, that is, for a vanishing curvature, the connection is said to be integrable (or flat).

**Torsion**

As before, consider the points $p$, $q$ and $s$ with coordinates $x^\mu$, $x^\nu + \epsilon^\nu$ and $x^\rho + \delta^\rho$, respectively. Consider the infinitesimal vectors at $p$, $\epsilon^\mu$, $\frac{\partial}{\partial x^\mu}|_p$, and $\delta^\rho$, $\frac{\partial}{\partial x^\rho}|_p$ ($\epsilon^\mu = \epsilon^\mu$, $\delta^\rho = \delta^\rho$); regarded as infinitesimal displacements (translations) in $M^n$, they respectively define the points $q$ and $s$. Make the parallel transport of $\epsilon^\mu$ along $\delta^\rho$: we obtain the vector at $s$, $V^\mu_s = \epsilon^\mu_p - \epsilon^\nu \Gamma^\nu_{\mu\rho}(p) \epsilon^\rho_p$; so the total displacement vector from $p$ to $r$ is

$$\delta^\rho + \epsilon^\mu_p - \epsilon^\nu \Gamma^\nu_{\mu\rho}(p) \epsilon^\rho_p;$$

similarly, making the parallel displacement of $\epsilon^\nu$ along $\epsilon^\mu$ one obtains the vector at $q$, $V^\nu_q = \delta^\nu + \epsilon^\nu \Gamma^\nu_{\nu\rho}(p) \delta^\rho_p$; and the total displacement vector from $p$ to $r$ is

$$\epsilon^\mu_p + \delta^\nu_p - \epsilon^\nu \Gamma^\nu_{\mu\rho}(p) \delta^\rho_p.$$  

The difference between the two vectors is

$$-\epsilon^\mu_p \delta^\rho_p \Gamma^\nu_{\mu\rho}(p) + \epsilon^\rho_p \delta^\nu_p \Gamma^\nu_{\rho\alpha}(p) = \epsilon^\rho_p \delta^\nu_p \Gamma^\nu_{\rho\alpha}(p) - \epsilon^\nu \Gamma^\nu_{\nu\rho}(p) \delta^\rho_p = 2 T^\mu_{\beta\alpha}(p) \epsilon^\rho_p \delta^\beta_p.$$
So, the torsion measures the failure of the closure of the parallelogram made of the infinitesimal displacement vectors and their parallel transports.

Since the last expression is a translation, one says that torsion is the translational part of the connection.

11. Exterior covariant derivative and curvature 2-form

Up to here we have considered the vector bundle $\xi : \mathbb{R}^m - E \xrightarrow{\pi} M^n$. Now we shall consider the vector bundle $\xi_k$ whose sections are the $k$-differential forms $\alpha \otimes s$ on $M^n$ with values in $E$:

$$
\mathbb{R}^{\otimes k} \otimes \mathbb{R}^m - \Lambda^k T^* M \otimes E \xrightarrow{\pi} M
$$

with $\Lambda^k T^* M \otimes E = \bigoplus_{x \in M} \Lambda^k T^*_x M \otimes E_x$; clearly, $\xi_0 = \xi$. $\alpha \otimes s \in \Gamma(\Lambda^k T^* M \otimes E)$ with $(\alpha \otimes s)(x) = (x, (\alpha \otimes s)_x)$, $(\alpha \otimes s)_x \in \Lambda^k T^*_x M \otimes E_x$; if, as before, $\{s_1\}_{i=1}^m$ is a basis of sections of $E$ in $U_\alpha \subset M$, $x \in U_\alpha$, and $\{x_\alpha^\mu\}_{\mu=1}^n$ are local coordinates on $U_\alpha$, then

$$
\{dx_\alpha^\mu|_x \wedge \ldots \wedge dx_\alpha^n|_x \otimes \sigma_{ix}, \ i = 1, \ldots, m, \ n \geq \mu_k > \ldots > \mu_1 \geq 1\}
$$

is a basis of $\Lambda^k T^*_x M \otimes E_x$. So,

$$(\alpha \otimes s)_x = \sum_{i=1}^m \sum_{n \geq \mu_k > \ldots > \mu_1 \geq 1} t_{i_1 \ldots i_k}^i dx_\alpha^{i_1}|_x \wedge \ldots \wedge dx_\alpha^{i_k}|_x \otimes \sigma_{ix},$$

$$
t_{i_1 \ldots i_k} \in \mathbb{R}.
$$

We define the set of total exterior covariant derivative operators

$$
\{d_0^\nabla, d_1^\nabla, \ldots, d_{n-1}^\nabla\}, \quad d_k^\nabla : \Gamma(\Lambda^k T^* M \otimes E) \rightarrow \Gamma(\Lambda^{k+1} T^* M \otimes E), \ k = 0, \ldots, n-1,
$$

as the $\mathbb{R}$-linear extension of

$$
d_k^\nabla(\alpha \otimes s) = (d_k \alpha) \otimes s + \alpha \wedge \nabla s
$$

with

$$(d_k \alpha) \otimes s + \alpha \wedge \nabla s : M \rightarrow \Lambda^{k+1} T^* M \otimes E, \ x \mapsto (x, (d_k \alpha)_x \otimes s_x + \alpha_x \wedge (\nabla s)_x).$$

For $k = 0$, $d_0^\nabla = \nabla : \Gamma(E) \rightarrow \Gamma(T^* M \otimes E)$.

Let us study the composition $d_k^\nabla \circ d_0^\nabla$. If $s \in \Gamma(E)$, then

$$
d_1^\nabla \circ d_0^\nabla(s) = d_1^\nabla \circ \nabla(s) \equiv R(s) \in \Gamma(\Lambda^2 T^* M \otimes E),
$$

i.e.

$$
R(s) : \Gamma(TM) \times \Gamma(TM) \rightarrow \Gamma(E), \ (X, Y) \mapsto R(s)(X, Y) : M \rightarrow E, \ x \mapsto R(s)(X, Y)(x) = (x, R(s)_x(X_x, Y_x)).
$$

$R(s)$, also denoted by $\nabla^2(s)$, is called the second total covariant derivative of the section $s$. (In general, $d_{k+1}^\nabla \circ d_k^\nabla \neq 0$; compare with $d_{k+1} \circ d_k = 0$ in De Rham theory.)

Locally,

$$
R(s) = \nabla^2(s) = d_1^\nabla(\nabla(s)) = d_1^\nabla(dx^\mu \otimes \nabla_{\partial \mu}(s)) = d_1^\nabla(dx^\mu \otimes ((\partial_s s^i) \sigma_j + \Gamma_{i \mu}^j s^i \sigma_j))
$$

$$
= d_1(dx^\mu) \otimes ((\partial_s s^i) \sigma_j + \Gamma_{i \mu}^j s^i \sigma_j) + dx^\mu \wedge \nabla((\partial_s s^i) \sigma_j + \Gamma_{i \mu}^j s^i \sigma_j)
$$

$$
= dx^\mu \wedge (\nabla(\partial_s s^j \sigma_j) + \nabla(\Gamma_{i \mu}^j s^i \sigma_j)) = dx^\mu \wedge (d(\frac{\partial s^j}{\partial x^\mu}) \otimes \sigma_j + \Gamma_i^j \otimes \frac{\partial s^j}{\partial x^\mu} \sigma_j + d(\Gamma_{i \mu}^j s^i \sigma_j) \otimes \sigma_j + \Gamma_{i \mu}^j \otimes \Gamma_{i \mu}^k s^i \sigma_j).
$$
\[ \begin{align*}
&= dx^\mu \wedge (dx^\nu \frac{\partial}{\partial x^\nu} (\frac{\partial s^j}{\partial x^\mu}) \otimes \sigma_j + dx^\nu \frac{\partial}{\partial x^\nu} (\Gamma^j_i \otimes s^i) \otimes * \sigma_j + dx^\nu \Gamma^j_i \otimes \frac{\partial s^j}{\partial x^\mu} \sigma_j + dx^\nu \Gamma^j_l \otimes \Gamma^l_k \otimes s^i \sigma_j) \\
&= ds^i \wedge \Gamma^j_i \otimes \sigma_j + ds^i \wedge dx^\nu (\frac{\partial s^j}{\partial x^\nu} (\Gamma^j_i \otimes s^i) \otimes * \sigma_j + \Gamma^j_k \wedge \Gamma^j_l \otimes s^i \otimes \sigma_j)
\end{align*} \]

Also, where (Symbolically, \( R \)) \( \otimes \) \( \sigma_j \) \( \otimes \) \( \sigma_j \).

For each pair \( i, j \) \( \in \{1, \ldots, m\} \), \( (\mathbf{R}_j^i) \) is a local 2-form. Its exterior derivative is a local 3-form. Since \( d^2 = 0 \), we have \( d\mathbf{R}_j^i = d(\Gamma^k_i \wedge d\Gamma^k_j) = d\Gamma^k_i \wedge \Gamma^k_j - \Gamma^k_j \wedge d\Gamma^k_i \), then \( d\mathbf{R}_j^i + \Gamma^k_i \wedge (\mathbf{R}_j^k - \mathbf{R}_k^j) - (\mathbf{R}_j^k - \Gamma^k_j \wedge \Gamma^k_i) \wedge \Gamma^k_j = 0 \) or \( d\mathbf{R}_j^i + \Gamma^k_i \wedge (\mathbf{R}_j^k - \mathbf{R}_k^j) - (\mathbf{R}_j^k - \Gamma^k_j \wedge \Gamma^k_i) \wedge \Gamma^k_j = 0 \) i.e.

\[ d\mathbf{R}_j^i + \Gamma^k_i \wedge \mathbf{R}_j^k - \mathbf{R}_k^j \wedge \Gamma^k_j = 0. \]

These are the Bianchi identities. Symbolically, \( d\mathbf{R} + \Gamma \wedge \mathbf{R} - \mathbf{R} \wedge \Gamma = 0 \).

The l.h.s. is an \( m \times m \) matrix of 3-forms. So, in terms of the curvature form and the connection coefficients one has \( m^2 \) equations. However, when the \( \mathbf{R}_j^i \)'s are written in terms of the \( \Gamma^j_i \)'s one obtains \( m^2 \) identities: \( d(\mathbf{R}_j^i + \Gamma^k_i \wedge \Gamma^k_j) + (\Gamma_k^i \wedge (d\mathbf{R}_j^k + \Gamma^k_j \wedge \Gamma^k_i) - (d\mathbf{R}_j^k + \Gamma^k_j \wedge \Gamma^k_i) \wedge \Gamma^k_j = (d\mathbf{R}_j^k + \Gamma^k_j \wedge d\mathbf{R}_j^k + \Gamma^k_j \wedge \Gamma^k_j \wedge \Gamma^k_j \wedge \Gamma^k_j = 0 \) identically.
From the explicit expression of $R_{ij}$ and $\Gamma^k_{ij}$ in terms of local coordinates, in the Bianchi equations one has: 
\[dR_{ij} = \frac{1}{2}(R_{ij}, dx^\nu \wedge dx^\rho) = \frac{3}{2}R_{ij,\mu} dx^\mu \wedge dx^\nu \wedge dx^\rho, \quad \Gamma^k_{ij} = \Gamma^i_{k\mu} dx^\mu \wedge dx^nu \wedge dx^\rho = \frac{1}{2}\Gamma^i_{\mu k} R_{ij} \wedge dx^\mu \wedge dx^nu \wedge dx^\rho, \quad R^i_k \wedge \Gamma^k_j = \frac{1}{2}R^i_{\mu k} \Gamma^k_{\rho j} dx^\mu \wedge dx^nu \wedge dx^\rho, \quad \Gamma^\nu_k j = \frac{1}{2}R^i_{\mu k} \Gamma^k_{\rho j} dx^\mu \wedge dx^nu \wedge dx^\rho;\]
then
\[\frac{1}{2}(R_{ij,\mu} + \Gamma^i_{\mu k} R_{ij} - R^i_{\mu k} \Gamma^k_{\rho j} ) dx^\mu \wedge dx^nu \wedge dx^\rho = 0\]
i.e.
\[R^i_{ij,\mu} + \Gamma^i_{\mu k} R^k_{ij} - R^i_{\mu k} \Gamma^k_{\rho j} = 0.\]

For a linear connection in $M^n$, with $i, j, k = 0, 1, 2$,
\[R_{ij} \Gamma^k_{j} = \frac{1}{2}(R_{ij}, dx^\rho \wedge dx^\nu \wedge dx^\rho) = \frac{1}{2}R^i_{\mu k} \Gamma^k_{\rho j} dx^\mu \wedge dx^nu \wedge dx^\rho, \quad \Gamma^\nu_k j = \frac{1}{2}R^i_{\mu k} \Gamma^k_{\rho j} dx^\mu \wedge dx^nu \wedge dx^\rho, \quad \Gamma^\nu_k j = \frac{1}{2}R^i_{\mu k} \Gamma^k_{\rho j} dx^\mu \wedge dx^nu \wedge dx^\rho;\]
then
\[\frac{1}{2}(R_{ij,\mu} + \Gamma^i_{\mu k} R_{ij} - R^i_{\mu k} \Gamma^k_{\rho j} ) dx^\mu \wedge dx^nu \wedge dx^\rho = 0\]
i.e.
\[R^i_{ij,\mu} + \Gamma^i_{\mu k} R^k_{ij} - R^i_{\mu k} \Gamma^k_{\rho j} = 0.\]

**Note 1.** In section 22, we’ll see that in the case $E = TM$ and the connection is that of Levi-Civita (section 13), when the Bianchi equations are written in terms of the Ricci tensor (see section 16) and the scalar curvature (section 18), we obtain the vanishing of the covariant divergence of the Einstein’s tensor (see section 19): 
\[G_{\mu\nu} \wedge \nu = 0.\]

**Note 2.** In electromagnetism, $dF = 0$ in terms of the curvature tensor $F$ (field strength), amounts to the homogeneous Maxwell equations. Instead, if $F = dA$ is used, we obtain an identity.

**Remark:** Up to here, all the results have been independent of the existence of a metric $g_{\mu\nu}$ in the manifold $M^n$ i.e. of a non degenerate symmetric scalar product at each tangent space $T_x M^n$. This metric is introduced in the next section.

### 13. The Levi-Civita connection

In Appendix A we shall prove the **Fundamental Theorem of Riemannian or Pseudo-Riemannian Geometry**, which states that in a riemannian or pseudo-riemannian manifold $(M^n, g_{\mu\nu})$ there exists a unique symmetric and metric linear connection, the Levi-Civita connection, given by
\[\Gamma^\nu_{\mu\rho} = \frac{1}{2}g^{\mu\sigma}(\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho}) = g^{\mu\sigma} \Gamma^\sigma_{\nu\rho}\]
with
\[\Gamma^\sigma_{\nu\rho} = \frac{1}{2}(\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho}).\]

Then $g_{\lambda\rho} \Gamma^\mu_{\nu\rho} = g_{\lambda\rho} g^{\mu\sigma} \Gamma^\sigma_{\nu\rho} = \delta^\mu_\lambda \Gamma_{\nu\rho} = \Gamma_{\lambda\nu\rho}$.

It holds:

**i)** $D_\mu g_{\nu\rho} = g_{\nu\rho,\mu} = 0$ (and also $D_\mu g^{\nu\rho} = g^{\nu\rho,\mu} = 0$). A consequence of this is that for any smooth path $c : (a, b) \rightarrow M^n$ the metric tensor $g_{\mu\nu}$ is parallel transported along $c$:
\[\left.\left(D g\right)_{\nu\rho}\right|_\lambda = \left.\frac{dx^\rho}{d\lambda} D_\rho g_{\nu\mu}\right|_\lambda = 0.\]
where $\frac{d\mathbf{x}^\mu}{d\lambda} = \dot{c}^\mu$. Then it follows that the scalar product of two parallel transported vectors along $c$ by the Levi-Civita connection is also parallel transported i.e. covariantly constant:

$$\frac{D}{d\lambda}(g_{\mu\nu}V^\mu W^\nu) = (\frac{Dg}{d\lambda})_{\mu\nu}V^\mu W^\nu + g_{\mu\nu}(\frac{DV}{d\lambda})^\mu W^\nu + g_{\mu\nu}(\frac{DW}{d\lambda})^\nu = 0.$$  

In particular, if $V = W = c$, then $g_{\mu\nu}\dot{c}^\mu\dot{c}^\nu \equiv (\dot{c}, \dot{c}) \equiv ||\dot{c}||^2$ remains constant by parallel transport; if $||\dot{c}||^2 > 0, =0, <0$ the geodesic is respectively called timelike, null or lightlike, and spacelike. Since $\delta^\nu_\nu = g^{\mu\alpha}g_{\alpha\nu}$, then $\delta^\mu_{\nu;\rho} = 0$.

i.e. the covariant derivative commutes with the raising or lowering of the indices.

It can be shown that if $c : (a, b) \to M^n$ is a smooth path that extremizes the proper time (or path length) $\tau = \int_a^b d\lambda f^{1/2}$, with $f = g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}$, then $c$ is a geodesic of the Levi-Civita connection. Also, a change of parameter $\tau \to \lambda = \lambda(\tau)$ preserves the form of the geodesic equation if and only if $\tau \to \lambda$ is an affine transformation, i.e.

$$\lambda = a\tau + b,$$

where $a, b \in \mathbb{R}$ and $a \neq 0$. For an arbitrary transformation one obtains

$$\frac{d^2\mathbf{x}^\mu}{d\tau^2} + \Gamma^\mu_{\rho\varphi} \frac{dx^\rho}{d\lambda} \frac{dx^\varphi}{d\lambda} = -\frac{d^2\lambda}{d\tau^2} (\frac{d\lambda}{d\tau})^{-2} \frac{d\mathbf{x}^\mu}{d\lambda}.$$  

This means that the derivation of the geodesic equation in section 8, forces the parameter $\lambda$ to be an affine parameter i.e. a parameter linearly related, up to an additive constant, to the proper time $\tau$.

It is important to mention that the fact that the connection coefficients $\Gamma^\mu_{\nu\rho}$ depend on the metric function $g_{\mu\nu}$, is the usual argument in the literature for denying to G.R. the character of a gauge theory of gravity. More on this below.

14. Physics 1: Equivalence principle in GR

Massive free point particles move along timelike geodesics. Massless free point particles move along lightlike geodesics; in this case $\lambda$ can not be the proper time since $(d\tau)^2 = 0$, (Dirac, 1975).

15. Covariant components of the curvature tensor

Starting from the expression (\star) in section 9, a long but straightforward calculation leads to the result, for the Levi-Civita connection,

$$R_{\mu\rho\sigma\alpha} = g_{\mu\lambda}R^\lambda_{\nu\rho\sigma} = \frac{1}{2}(\partial_\mu \partial_\alpha g_{\nu\rho} + \partial_\lambda \partial_\rho g_{\mu\sigma} - \partial_\nu \partial_\rho g_{\mu\sigma} - \partial_\sigma \partial_\rho g_{\mu\nu}) + g_{\alpha\beta}(\Gamma^\alpha_{\nu\mu} \Gamma^\beta_{\rho\sigma} - \Gamma^\alpha_{\mu\rho} \Gamma^\beta_{\nu\sigma}).$$

Clearly, $R_{\mu\rho\sigma\alpha}$ is a covariant 4-rank tensor. It has $n^4$ components (e.g. if $n = 4, 4^4 = 256$).

Algebraic properties of $R_{\mu\rho\sigma\alpha}$

i) $R_{\rho\sigma\mu\nu} = \frac{1}{2}(\partial_\rho \partial_\mu g_{\sigma\nu} + \partial_\sigma \partial_\mu g_{\rho\nu} - \partial_\nu \partial_\mu g_{\sigma\rho} - \partial_\rho \partial_\mu g_{\sigma\nu}) + g_{\alpha\beta}(\Gamma^\alpha_{\mu\rho} \Gamma^\beta_{\sigma\nu} - \Gamma^\alpha_{\mu\sigma} \Gamma^\beta_{\rho\nu}) = R_{\mu\rho\sigma\alpha}$

(symmetry under the interchange between the first pair of indices with the second pair of indices i.e. $R_{AB} = R_{BA}$ with $A = \rho\sigma$ and $B = \mu\nu$).
ii) \( R_{\mu\nu\rho\sigma} = -R_{\mu\nu\rho\sigma}, R_{\mu\nu\sigma\rho} = -R_{\mu\nu\rho\sigma} \), then \( R_{\nu\mu\sigma\rho} = R_{\mu\nu\sigma\rho} \) which can be summarized as

\[
R_{\mu\nu\rho\sigma} = -R_{\nu\mu\sigma\rho} = -R_{\mu\nu\rho\sigma}.
\]

iii) \( R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\nu\sigma\rho\mu} = 0 \) (cyclicly).

If one defines

\[
A_{\mu\nu\rho\sigma} := R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\nu\sigma\rho\mu},
\]

it can be proved that it is totally antisymmetric in its four indices. Since \( A_{\mu\nu\rho\sigma} = 0 \), this imposes \( \binom{m}{4} = \frac{m!}{4!(m-4)!} \) conditions on \( R_{\mu\nu\rho\sigma} \). (Number of ways one can take four distinct elements among \( m \); obviously it must be \( m \geq 4 \).)

Let us determine the number of algebraically independent components of \( R_{\mu\nu\rho\sigma} \). Let \( S_{ab} \) and \( A_{ab} \) be respectively a symmetric and antisymmetric tensor in \( m \) dimensions. The corresponding number of independent components are \( N(S_{ab}; m) = \frac{m(m+1)}{2} \) and \( N(A_{ab}; m) = \frac{m(m-1)}{2} \). So, we have

\[
\begin{align*}
m & \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad \ldots \quad 10 \quad \ldots \\
N(S_{ab}; m) & \quad 1 \quad 3 \quad 6 \quad 10 \quad 15 \quad 21 \quad \ldots \quad 55 \quad \ldots \\
N(A_{ab}; m) & \quad 0 \quad 1 \quad 3 \quad 6 \quad 10 \quad 15 \quad \ldots \quad 45 \quad \ldots
\end{align*}
\]

Notice that \( \frac{N(A_{ab}; m)}{N(S_{ab}; m)} \rightarrow 1 \) as \( m \rightarrow \infty \). If we write \( R_{\mu\nu\rho\sigma} = R_{AB} \), since under \( \mu \leftrightarrow \nu \) or \( \rho \leftrightarrow \sigma \) \( R_{\mu\nu\rho\sigma} \) is antisymmetric, each index \( A \) or \( B \) contributes with \( \frac{m(m-1)}{2} \) independent components; but now one has a “two-index” symmetric matrix \( R_{AB} \) with \( A, B \in \{1, \ldots, \frac{m(m-1)}{2} \} \), which gives \( \frac{1}{8}m(m-1)(m(m-1)+2) = \frac{1}{8}m(m-1)(m^2-m+2) \) independent components for \( R_{\mu\nu\rho\sigma} \). But iii) and then the antisymmetry of \( A_{\mu\nu\rho\sigma} \) imposes \( \frac{m!}{4!(m-4)!} = \frac{m(m-1)(m-2)(m-3)}{4!} \) conditions. Then,

\[
N(R_{\mu\nu\rho\sigma}; m) = \frac{1}{8}m(m-1)(m^2-m+2) - \frac{1}{4!}m(m-1)(m-2)(m-3) = \frac{m^2(m^2-1)}{12}.
\]

So, we have

\[
\begin{align*}
m & \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad \ldots \quad 10 \quad 11 \quad \ldots \quad 26 \quad \ldots \\
N(R_{\mu\nu\rho\sigma}; m) & \quad 0 \quad 1 \quad 6 \quad 20 \quad 50 \quad 105 \quad \ldots \quad 825 \quad 1210 \quad \ldots \quad 38025 \quad \ldots
\end{align*}
\]

16. Ricci tensor for the Levi-Civita connection

Define the covariant 2-tensor

\[
R_{\nu\sigma} := g^{\mu\rho} R_{\mu\nu\rho\sigma} = R_{\nu\sigma}^\rho.
\]

We contracted indices 1 and 3; contracting 1-2 and 3-4 gives zero; contracting 1-4, 2-3 and 2-4 gives \( \pm R_{\nu\sigma} \):

\[
g^{\mu\sigma} R_{\mu\nu\rho\sigma} = -g^{\nu\sigma} R_{\mu\nu\rho\sigma} = -R_{\nu\sigma}, \quad g^{\mu\rho} R_{\mu\nu\rho\sigma} = -g^{\nu\rho} R_{\mu\nu\rho\sigma} = -R_{\mu\sigma}, \quad g^{\nu\sigma} R_{\mu\nu\rho\sigma} = g^{\nu\rho} R_{\mu\nu\rho\sigma} = R_{\mu\rho}.
\]

So, up to a sign, the Ricci tensor is uniquely defined from \( R_{\mu\nu\rho\sigma} \) and \( g^{\mu\nu} \).

\( R_{\nu\sigma} \) is symmetric: \( R_{\sigma\nu} = g^{\mu\rho} R_{\mu\sigma\rho\nu} = g^{\mu\rho} R_{\mu\rho\sigma\nu} = g^{\nu\rho} R_{\nu\mu\rho\sigma} = R_{\nu\sigma} \).

Then,

\[
N(R_{\nu\mu}; m) = \frac{m(m+1)}{2}.
\]

We have

\[
\begin{align*}
m & \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad \ldots \quad 10 \quad 11 \quad \ldots
\end{align*}
\]

\[
\begin{align*}
N(R_{\nu\mu}; m) & \quad 1 \quad 3 \quad 6 \quad 10 \quad 15 \quad 21 \quad \ldots \quad 55 \quad 66 \quad \ldots
\end{align*}
\]

We can write

\[
R_{\nu\sigma} = g^{\mu\rho} g_{\mu\lambda} R_{\nu\rho\sigma} = \delta^\rho_\lambda R_{\nu\rho\sigma} = R_{\nu\sigma} = \langle dx^\rho, \mathcal{R}(\partial_\rho, \partial_\sigma, \partial_\nu) \rangle = \langle dx^\sigma, \mathcal{R}(\partial_\rho, \partial_\nu, \partial_\sigma) \rangle.
\]
17. Physics 2: Einstein’s hypothesis: (Local) equations for empty (“vacuum”, without matter, vanishing cosmological constant $\Lambda$, only $g_{\mu\nu}$) space-time

In all charts of the manifold i.e. in all reference frames:

$$R_{\mu\nu} = 0,$$

$\mu, \nu = 1, \ldots, m$ or $0, \ldots, m - 1$. $R_{\mu\nu} \in C^\infty(U_\alpha; \mathbb{R})$, (Dirac, 1975). These are equations at each chart or reference system $(U_\alpha, x^\mu_\alpha)$ of $M^m$. The number of algebraically independent components of $R_{\mu\nu}$ equals the number of independent components of $g_{\mu\nu}$. $R_{\mu\nu} = 0$ does not imply $R_{\mu\nu\rho\sigma} = 0$. In other words, in GR empty space-time can be curved.

18. Ricci (or curvature) scalar (with Levi-Civita connection)

$$R := g^{\mu\nu} R_{\mu\nu} = R^\mu_\mu.$$  

$R \in C^\infty(U_\alpha; \mathbb{R})$. Then, in empty space-time,

$$R = 0.$$

19. The Einstein tensor is defined by

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R.$$  

$G_{\mu\nu} \in C^\infty(U_\alpha; \mathbb{R})$. Also, it is symmetric (10 algebraically independent components in 4 dimensions).

Proposition (Mathematics)  

$$G_{\mu\nu} = 0 \iff R_{\mu\nu} = 0$$

for $m = D \neq 2, D = d + 1$ (space-time dimension).

Proof.

$$\Rightarrow G_{\mu\nu} = 0 \Rightarrow R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R \Rightarrow R^\mu_\mu = R = \frac{1}{2} g^\mu_\mu R = \frac{D}{2} R \Rightarrow (\frac{D}{2} - 1) R = 0.$$  

If $\frac{D}{2} \neq 1$ i.e. $D \neq 2$, then $R = 0$ and then $R_{\mu\nu} = G_{\mu\nu} = 0$. In particular this holds for $D = 3 + 1 = 4$.

$$\Leftarrow R_{\mu\nu} = 0 \Rightarrow R^\mu_\mu = R = 0 \Rightarrow G_{\mu\nu} = 0.$$  

This holds for all $D$’s=$m$’s.

20. Physics 2′: (Local) Einstein equations in empty space-time

In all charts of the manifold i.e. in all reference frames:

$$G_{\mu\nu} = 0$$

for all space-time dimensions $D = m = d + 1$. (For $D = 4$ these are ten equations.)

21. Examples in $m = D = 1, 2, 3$. Generalization to $m \geq 4$ and Weyl tensor

$D = 1$. $N(R_{\mu\nu\rho\sigma}; 1) = 0$; then $R_{1111}$ (or $R_{0000}$) = 0. This reflects the fact that the curvature tensor represents intrinsic properties of the space in question, and not how the space (in this case a line, straight or curved) is embedded in a higher dimensional space. Let $g_{11}(x) \equiv g(x)$ be the metric tensor in $D = 1$. 

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If \( x \mapsto x' = x'(x) \) then \( g'(x') = (\frac{dx}{dx'})^2 g(x) \). Choose \( x'(x) = \int^x dy \sqrt{|g(y)|} \); then \((\frac{dx}{dx'})^2 = \frac{1}{g(x)}\) and so \( g'(x') = \frac{g(x)}{|g(x)|} \) which equals +1 (-1) if \( g(x) > 0 \) (\( < 0 \)). From the constancy of \( g(x) \), \( \Gamma(x) = \frac{1}{g(x)} \frac{d}{dx} g(x) = 0 \).

\[ D = 2 \quad N(R_{\mu\nu\rho\sigma}; 2) = 1. \]

By antisymmetry in \( \mu\nu \) and \( \rho\sigma \), the only possibilities for a non-vanishing \( R_{\mu\nu\rho\sigma} \) are \( R_{1010}, R_{1001}, R_{0110} \) and \( R_{0101} \). We choose \( R_{0101} \) and it is easily verified that the unique \( R_{\mu\nu\rho\sigma} \) which satisfies the algebraic properties of section 15 and gives \( R_{0110} = -R_{1010} = R_{1001} = -R_{0101} \) is

\[ R_{\mu\nu\rho\sigma} = (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) \frac{R_{0101}}{g}, \]

with \( g = \text{det} \begin{pmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{pmatrix} = g_{00} g_{11} - g_{01} g_{10} = g_{00} g_{11} - g_{01}^2 \). In the presence of matter, Einstein’s equations are

\[ G_{\mu\nu} = \text{const.} \times T_{\mu\nu} \]

where \( T_{\mu\nu} \) is the energy-momentum tensor of matter. Then, for \( D = 2 = 1 + 1 \),

\[ T_{\mu\nu} = 0. \]

This means that in \( D = 2 \), the unique solutions to Einstein’s equations are those corresponding to the “vacua” i.e. \( T_{\mu\nu} = 0 \).

\[ D = 3 \quad N(R_{\mu\nu\rho\sigma}; 3) = N(R_{\mu\nu}; 3) = 3, \]

the curvature tensor can be expressed in terms of \( g_{\mu\nu} \) and \( R_{\mu\nu} \); the most general form satisfying the symmetry properties of section 15 is

\[ R_{\mu\nu\rho\sigma} = A(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) + B(g_{\mu\rho} R_{\nu\sigma} - g_{\nu\rho} R_{\mu\sigma} - g_{\mu\sigma} R_{\nu\rho} + g_{\nu\sigma} R_{\mu\rho}) \]

with \( A \) and \( B \) numerical constants. From the definition of the Ricci tensor, \( R_{\nu\sigma} = g^{\mu\rho} R_{\mu\nu\rho\sigma} = (2A + BR)g_{\nu\sigma} + BR_{\nu\sigma} \), where \( R \) is the Ricci scalar; then \( B = 1 \) and \( A = -\frac{1}{2} R \). Therefore,

\[ R_{\mu\nu\rho\sigma} = -\frac{R}{2} (g_{\mu\rho} g_{\nu\sigma} - g_{\nu\rho} g_{\mu\sigma}) + g_{\mu\rho} R_{\nu\sigma} - g_{\nu\rho} R_{\mu\sigma} - g_{\mu\sigma} R_{\nu\rho} + g_{\nu\sigma} R_{\mu\rho}. \]

\[ D = m \geq 4. \] In all these cases

\[ N(R_{\mu\nu\rho\sigma}; m) - N(R_{\mu\nu}; m) := N(C_{\mu\nu\rho\sigma}; m) = \frac{m(m+1)(m+2)(m-3)}{12} > 0 \]

where \( C_{\mu\nu\rho\sigma} \), the Weyl tensor, has the same algebraic properties as \( R_{\mu\nu\rho\sigma} \) but can’t be obtained from \( g_{\mu\nu} \) and \( R_{\mu\nu} \). One writes

\[ R_{\mu\nu\rho\sigma} = C(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) + D(g_{\mu\rho} R_{\nu\sigma} - g_{\nu\rho} R_{\mu\sigma} - g_{\mu\sigma} R_{\nu\rho} + g_{\nu\sigma} R_{\mu\rho}) + C_{\mu\nu\rho\sigma} \]

with all traces of \( C_{\mu\nu\rho\sigma} \) vanishing i.e. \( C^\mu_{\nu\rho\sigma} = 0 \).

Then

\[ R_{\nu\sigma} = g^{\mu\rho} R_{\mu\nu\rho\sigma} = C(m g_{\nu\sigma} - g_{\nu\sigma}) + D(m R_{\nu\sigma} - 2 R_{\nu\sigma} + g_{\nu\sigma} R) = g_{\nu\sigma}((m-1)C + DR) + (m-2)DR_{\nu\sigma}. \]

Then \( D = \frac{1}{m-2} \) and \( C = -\frac{R}{(m-1)(m-2)}. \) Therefore,

\[ R_{\mu\nu\rho\sigma} = -\frac{R}{(m-1)(m-2)} (g_{\mu\rho} g_{\nu\sigma} - g_{\nu\rho} g_{\mu\sigma}) + \frac{1}{m-2} (g_{\mu\rho} R_{\nu\sigma} - g_{\nu\rho} R_{\mu\sigma} - g_{\mu\sigma} R_{\nu\rho} + g_{\nu\sigma} R_{\mu\rho}) + C_{\mu\nu\rho\sigma}. \]
The Weyl tensor in terms of curvature and metric is given by

\[ C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \frac{2R}{(m-1)(m-2)}g_{\rho\sigma}g_{\mu\nu} - \frac{2}{m-2}(g_{\rho\sigma}R_{\mu\nu} - g_{\mu\nu}R_{\rho\sigma}). \]

One has the table

| \( m \) | \( N(C_{\mu\nu\rho\sigma}; m) \) |
|--------|-------------------------------|
| 4      | 10                           |
| 5      | 35                           |
| 6      | 84                           |
| \( \cdots \) | \( \cdots \) |

Clearly \( C_{\mu\nu\rho\sigma} = 0 \) for \( m = 2, 3 \).

22. For the Levi-Civita connection, \( G^{\mu\nu}{}_{;\nu} = 0 \)

From the general expression for the covariant derivative of a tensor \( T_{\mu_1\cdots\mu_r} \) in section 7, the relation between the covariant and the ordinary derivatives of \( R^\sigma_{\lambda\nu\rho} \) is given by

\[ R^\sigma_{\lambda\nu\rho,\mu} = R^\sigma_{\lambda\nu\rho,\mu} - \Gamma^{\sigma}_{\mu\alpha}R_{\lambda\nu\rho} + \Gamma^{\beta}_{\mu\lambda}R^\sigma_{\beta\nu\rho} + \Gamma^{\beta}_{\mu\nu}R^\sigma_{\lambda\beta\rho} + \Gamma^{\beta}_{\mu\rho}R^\sigma_{\lambda\nu\beta}. \]

Then, the last form of the Bianchi equations in section 12 becomes

\[ R^\sigma_{\lambda\nu\rho,\mu} + R^\sigma_{\lambda\mu\nu,\rho} + R^\sigma_{\lambda\mu\nu,\rho} = 0. \]

Since the covariant derivative commutes with the lowering of indices, contracting with \( g_{\alpha\sigma} \) we obtain

\[ R_{\alpha\lambda\nu\rho,\mu} + R_{\alpha\lambda\mu\nu,\rho} + R_{\alpha\lambda\mu\nu,\rho} = 0. \]

Contracting \( \alpha \) and \( \rho \) with \( g^{\alpha\rho} \),

\[ R^\alpha_{\lambda\nu\sigma,\mu} + R^\alpha_{\lambda\sigma\mu,\nu} + R^\alpha_{\lambda\sigma\mu,\nu} = 0, \]

and we obtain, in terms of the Ricci and curvature tensors,

\[ -R_{\lambda\nu,\mu} + R_{\lambda\mu,\nu} + R^\alpha_{\lambda\mu\nu,\alpha} = 0. \]

Finally, contracting \( \lambda \) and \( \nu \) with \( g^{\lambda\nu} \), we obtain, in terms of the Ricci tensor and the scalar curvature,

\[ (R^\nu_{\mu} - \frac{1}{2}g^\nu_{\mu}R)_{;\nu} = 0 \]

i.e.

\[ G^{\mu\nu}{}_{;\nu} = 0. \]

So, the Einstein tensor, apparently arbitrarily defined in section 19, appears naturally when the Bianchi equations for the Levi-Civita connection are expressed in terms of the Ricci tensor and the scalar curvature, and some contractions are done. \( G_{\mu\nu} \) is, then, a purely geometrical object.

23. Physics 3: (Local) Einstein equations in the presence of matter

In all charts of the manifold i.e. in all reference frames:

\[ G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}, \]

(Einstein, 1956), where \( G \) is the Newton gravitational constant, \( c \) is the velocity of light in the vacuum, and \( T_{\mu\nu} \) is the energy momentum tensor of matter: all other fields than \( g_{\mu\nu} \). Clearly, \( T_{\mu\nu} \) is symmetric \((T_{\mu\nu} = T_{\nu\mu})\) and covariantly conserved:

\[ T_{\mu\nu}{}_{;\nu} = 0. \]

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Units: \([G_{\mu\nu}] = [R_{\mu\nu}] = [R_{\mu\nu\rho\sigma}] = [L]^{-2}, \quad \frac{[\mathcal{G}]}{[\mathcal{G}]_{[\mu\nu]}} = \frac{[\mathcal{G}]}{[\mathcal{G}]_{[\rho\sigma]}} = \frac{1}{[\mathcal{G}]} = \frac{1}{[\mathcal{G}]}\]

then \([T_{\mu\nu}] = \frac{[M]}{[\mu]} = \frac{[M]}{[\nu]} = \frac{[M]}{[\mu\nu]} = \frac{[M]}{[\nu\mu]} = \frac{[M]}{[\nu\mu\nu]} = \frac{[M]}{[\nu\mu\nu]} = \frac{[M]}{[\nu\mu\nu]}\]

where \([L], [M],\) and \([t]\) denote the units of length, mass, and time, respectively.

### 24. Tensor bundles as associated bundles to the bundle of frames of \(M^n\)

\(T^r_s M^n\) is the total space of the \((n + r + s)\)-dimensional real vector bundle of \(r\)-contravariant and \(s\)-covariant tensors on \(M^n\), with fibre \(\mathbb{R}^{n+r+s} \cong \{x_{i_1,j_1}, \ldots, x_{i_n,j_n} \in \mathbb{R}, i_k,j_l \in \{1, \ldots, n\}, k = 1, \ldots, r, l = 1, \ldots, s\} \equiv \{\vec{\lambda}\}\). The *bundle of frames of \(M^n, \mathcal{F}_{M^n}\), is the principal bundle with structure group \(GL_n(\mathbb{R})\) (the fibre of the bundle) on \(M^n\) (the base space), and with total space \(FM^n\) consisting of the set of all ordered basis of the tangent space at each point of \(M^n\), namely

\[
FM^n = \bigoplus_{x \in M^n} \{r_x \equiv (v_{1x}, \ldots, v_{nx}), (v_{nx})^n_{k=1} \text{ basis of } T_x M^n\} = \cup_{x \in M^n} \{x\} \times \{(v_{1x}, \ldots, v_{nx})\}
\]

where \((FM^n)_x\) is the fibre over \(x\), with \(dim_{\mathbb{R}}(FM^n)_x = n^2\). The bundles of orthogonal frames, Lorentz frames, restricted Lorentz frames, etc. of \(M^n\), are obtained by reducing the group \(GL_n(\mathbb{R})\) respectively to \(O(n), O(n-1,1), SO^0(n-1,1), \) etc. If \(x \in U_a \equiv U, then v_{nx} = \sum_{k=1}^{n} v_{nk}(x) \frac{\partial}{\partial x_k} |_x\); also, \(dim_{\mathbb{R}}FM^n = n + n^2\). The \(n + n^2\) local coordinates on \(\mathcal{F}_{U_a}\) is the set \((x^\rho, X_\mu^\rho)(x, r_x) = x^\rho(x)\) and \(X_\mu^\rho(x, r_x) = v_{nx}, \rho, \mu, \nu \in \{1, \ldots, n\}\).

One has:

\[
\begin{array}{cccccccccc}
\text{dim}_{\mathbb{R}} FM^n & 2 & 6 & 12 & 20 & 30 & \ldots & 110 & \ldots \\
\text{n} & 1 & 2 & 3 & 4 & 5 & \ldots & 10 & \ldots
\end{array}
\]

The bundle structure of \(\mathcal{F}_{M^n}\) is represented by

\[
\mathcal{F}_{M^n} : GL_n(\mathbb{R}) \rightarrow FM^n \xrightarrow{\pi_F} M^n
\]

where \(\pi_F\) is the projection \(\pi_F(x, (v_{1x}, \ldots, v_{nx})) = x\) and \(GL_n(\mathbb{R}) \rightarrow FM^n\) represents the right action of \(GL_n(\mathbb{R})\) on \(FM^n\) given by

\[
FM^n \times GL_n(\mathbb{R}) \xrightarrow{\psi} FM^n, ((v_{1x}, \ldots, v_{nx}), a) \mapsto (v_{1x}a_{1}^1 + \cdots + v_{nx}a_{n}^n, \ldots, v_{nx}a_{n}^n)
\]

\[
\equiv (v_{1x}, \ldots, v_{nx})a.
\]

The left action of \(GL_n(\mathbb{R})\) on \(\mathbb{R}^{n+r+s}\), given by

\[
GL_n(\mathbb{R}) \times \mathbb{R}^{n+r+s} \xrightarrow{\kappa} \mathbb{R}^{n+r+s}, (a, \vec{\lambda}) \mapsto (a\vec{\lambda})_{j_1 \cdots j_n} = a_{i_1 k_1} \cdots a_{i_r k_r} x_{i_1}^1 \cdots a_{i_r k_r} x_{i_r}^r = \chi_{j_1 \cdots j_n} a_{i_1 k_1} \cdots a_{i_r k_r} x_{i_1}^1 \cdots a_{i_r k_r} x_{i_r}^r
\]

induces the associated bundle \(FM^n \times GL_n(\mathbb{R}) \rightarrow \mathbb{R}^{n+r+s}\) which turns out to be isomorphic (through \(\varphi\), see below) to \(T^r_s M^n\). One has the following commutative diagram:

\[
\begin{array}{cccccc}
FM^n \times GL_n(\mathbb{R}) & \xrightarrow{\varphi} & T^r_s M^n \\
\pi_F \downarrow & & \downarrow \pi_F^r \\
M^n & \xrightarrow{Id_M} & M^n
\end{array}
\]

where:

\[
\varphi(((v_{1x}, \ldots, v_{nx}), \vec{\lambda})) = \sum_{i_k,j_l=1}^{n} \chi_{j_1 \cdots j_n} v_{i_1} x \otimes \cdots \otimes v_{i_r} x \otimes w_{x}^{i_1} \otimes \cdots \otimes w_{x}^{i_r},
\]

\[
\text{with } (((v_{1x}, \ldots, v_{nx}), \vec{\lambda})) = \{(v_{1x}, \ldots, v_{nx})a, a^{-1} \vec{\lambda})\}_{a \in GL_n(\mathbb{R})}; \{w_x^1, \ldots, w_x^n\} \text{ is the dual basis of}
\]

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\{v_{1z}, \ldots, v_{nz}\} \text{ i.e. } w^j_x(v_{jx}) = \delta^j_x; \text{ and } \pi^r_x([((v_{1z}, \ldots, v_{nz}), \vec{x})]) = \pi_F(x, (v_{1z}, \ldots, v_{nz})) = x. \text{ Using the fact that the dual basis vectors } w^j_x \text{ transform with } a^{-1}, \text{ it is easily verified that } \varphi \text{ is well defined i.e. it is independent of the representative element of the class } [((v_{1z}, \ldots, v_{nz}), \vec{x})].

A \textit{section} of the bundle of frames of } M^n \text{ i.e. a smooth function } s : M^n \to FM^n \text{ with } \pi_F \circ s = Id_{M^n}, \text{ trivializes } \mathcal{F}_{M^n} \text{ and therefore all the tensor bundles associated with it (in particular the tangent bundle of } M^n). \text{ The same occurs for any of the reductions of } \mathcal{F}_{M^n} \text{(bundle of Lorentz frames, restricted Lorentz frames, etc.).}

\section{25. \textit{Vertical bundle} of a principal fibre bundle}

Let } \eta \text{ be a principal fibre bundle (p.f.b.), } \eta = (P^{r+s}, B^r, \pi, G^r, \psi, U) : G^r \to P^{r+s} \xrightarrow{\pi} B^r, \text{ where } B^r \equiv B \text{ (base space) and } P^{r+s} \equiv P \text{ (total space) are differentiable manifolds of dimensions } s \text{ and } r+s \text{ respectively, } G^r \equiv G \text{ is an } r\text{-dimensional Lie group with right action on } P, \text{ and } U \text{ is a system of local trivializations } \pi^{-1}(U_\alpha) \xrightarrow{\phi_\alpha} U_\alpha \times G \text{ with } \pi_1 \circ \varphi_\alpha = \pi.

For each } p \in P \text{ there exists a } \textit{canonical vector space isomorphism} \varphi_p \text{ between } \mathcal{G} = \text{ Lie}(G) : \text{ the Lie algebra of } G, \text{ and } V_p = T_pP_{\pi(p)} : \text{ the tangent space to the fibre over } \pi(p) \text{ at } p, \text{ the vertical space at } p:

\varphi_p : \mathcal{G} \to V_p,\text{ } A \mapsto \varphi_p(A) \equiv A^s_p,

\text{with}

\begin{align*}
A^s_p : C^\infty(P, \mathbb{R}) & \to \mathbb{R}, \ f \mapsto \frac{d}{dt}f(p(e^tA))|_{t=0}.
\end{align*}

We used the fact that } T_pP_{\pi(p)} \subset T_pP; \text{ if } A_i, i = 1, \ldots, r \text{ is a basis of } \mathcal{G}, \text{ then } \varphi_p(A_i) \text{ is a basis of } V_p; \text{ in general, neither } A_i \text{ nor } \varphi_p(A_i) \text{ are canonical basis.}

Given } p, p' \in P, \text{ since } \varphi_p : \mathcal{G} \to V_p \text{ and } \varphi_{p'} : \mathcal{G} \to V_{p'} \text{ are isomorphisms, there is a } \textit{canonical vector space isomorphism (absolute teleparallelism)} \text{ between } V_p \text{ and } V_{p'}, \text{ for all } p, p' \in P:

\begin{align*}
V_p \xrightarrow{\varphi_{p'} \circ \varphi^{-1}_p} V_{p'}.
\end{align*}

\textbf{Remark: } This result, namely, the existence of } \varphi_p \text{ at each } p \in P, \text{ is independent of any connection.

This implies the \textit{triviality} of the \textit{vertical bundle} } V_\eta \text{ of the p.f.b. } \eta:

\begin{align*}
V_\eta : \mathbb{R}^r \xrightarrow{V^{2r+s} \xrightarrow{\pi_V} P},
\end{align*}

\text{with } V^{2r+s} = \bigsquare_{p \in P} V_p = \bigcup_{p \in P} \{p\} \times V_p \text{ and } \pi_V(p, v_p) = p.

In fact, } V_\eta \text{ admits } r \text{ independent global sections } \sigma_i : P \to V^{2r+s}, \sigma_i(p) = (p, \varphi_p(A_i)); \text{ then there is the following vector bundle isomorphism:

\begin{align*}
\begin{array}{cc}
\mathbb{R}^r & \mathbb{R}^r \\
\uparrow & \uparrow \\
V^{2r+s} & P \times \mathbb{R}^r \\
\pi_V \downarrow & \downarrow \pi_1 \\
P & \xrightarrow{Id} P
\end{array}
\end{align*}

\text{with } \phi(p, v_p) = \phi(p, \sum_{i=1}^r \lambda^i \varphi_p(A_i)) = (p, (\lambda^1, \ldots, \lambda^r)). \phi \text{ is not canonical since it depends on the basis } A_i \text{ of } \mathcal{G}.
Let $\omega$ be a connection on $\eta$, i.e. $\omega \in \Gamma(T^*P \otimes \mathcal{G})$ with $\omega : P \rightarrow T^*P \otimes \mathcal{G}$, $p \mapsto \omega(p) = (p, \omega_p)$, $\omega_p : T_pP \rightarrow \mathcal{G}$, $v_p \mapsto \omega_p(v_p) = \varphi_p^{-1}(\text{ver}(v_p))$. Since $\ker(\omega_p) = H_p^\perp \equiv H_p$: the horizontal vector space at $p$, then $\omega_p$ is $|H_p| = \infty \rightarrow 1$. However, $\omega_p|_{V_p} : V_p \rightarrow \mathcal{G}$, $\omega_p|_{V_p}(v_p) = \varphi_p^{-1}(v_p)$ is a vector space isomorphism i.e. $\omega_p|_{V_p} = \varphi_p^{-1}$.

In other words, if $\omega$ is a connection on $\eta$, then at each $p \in P$, $\omega$ gives an inverse of $\varphi_p$. Therefore, for the isomorphism between $V_p$ and $V_p'$, one has $\omega_p|_{V_p'} = \omega_p|_{V_p}^{-1}$.

In particular, we are interested in the case $P = F_{M^n}$, the frame bundle of a differentiable manifold $M^n$, where $(P, B, \pi, G, \psi, \mathcal{U}) = (FM^n, M^n, \pi_F, GL_n(\mathbb{R}), \psi, \mathcal{U})$: $p = (x, r_x) \in FM^n$, and $r_x = (v_{1x}, \ldots, v_{nx})$. Its vertical bundle is isomorphic to the product bundle $FM^n \times \mathbb{R}^n^2$:

\[
\begin{array}{ccc}
\mathbb{R}^n^2 & \mathbb{R}^n^2 \\
V_{FM^n} & \phi_c & FM^n \times \mathbb{R}^n^2 \\
\pi_V & \downarrow & \downarrow \pi_1 \\
FM^n & \text{Id} & FM^n
\end{array}
\]

where $\phi_c$ is the canonical isomorphism determined by the canonical basis of $gl_n(\mathbb{R}) = \mathbb{R}(n)$ given by the $n^2$ matrices $(A_{ij})_{kl} = \delta_{ik}\delta_{jl}$. The similar result holds for the reductions of $GL_n(\mathbb{R})$ to $O(n)$, $SO^0(n-1,1)$, etc. mentioned in section 24.

In particular, for the case $n = 4$ and $G = SO^0(3,1)$, with $\dim_{\mathbb{R}}(SO^0(3,1)) = \dim_{\mathbb{R}}(so(3,1)) = \dim_{\mathbb{R}}(o(3,1)) = 6$, case relevant in GR, $FM^4$ is the bundle of Lorentz frames $F_{M^4}$, and we have the vector bundle isomorphism

\[
\begin{array}{ccc}
\mathbb{R}^6 & \mathbb{R}^6 \\
V_{FM^4} & \phi_L & F_LM^4 \times \mathbb{R}^6 \\
\pi_V| & \downarrow & \downarrow \pi_1 \\
F_LM^4 & \text{Id} & F_LM^4
\end{array}
\]

with $\dim_{\mathbb{R}}(F_LM^4) = 4 + 6 = 10$ and $\dim_{\mathbb{R}}(V_{FM^4}) = 16$. In this case, the canonical basis of $o(3,1)$ (or of $o(3,1)$) is the set of matrices

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}, \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

respectively $\{l_{23} \equiv a_1, l_{31} \equiv a_2, l_{12} \equiv a_3, l_{01} \equiv b_1, \ldots, b_3\}$, where the first three matrices generate rotations around the axis $x$, $y$, and $z$, and the second three matrices generate boosts along the same axis, respectively. The derivation of the canonical basis is as follows: one starts from the definition of the Lorentz transformations $\Lambda$: $\eta_L := \Lambda^T \eta_L \Lambda$, with $\eta_L \equiv \eta = (\eta_{00}, \eta_{11}, \eta_{22}, \eta_{33}) = (+1, -1, -1, -1)$ (or $(-1, +1, +1, +1)$ and $\eta_{ab} = 0$ if $a \neq b$; if $\Lambda(\lambda)$ is a smooth path through the identity $\Lambda(0) = I$, the corresponding tangent vector at $I$, $\Lambda'(0) = L$, obeys the equation $L^T \eta = -\eta L$. The generators $a_i$ and $b_i$ obey $[a_i, a_j] = \epsilon_{ijk}a_k$, $[b_i, b_j] = -\epsilon_{ijk}a_k$, $[a_i, b_j] = \epsilon_{ijk}b_k$. If $l = \sum_{i=1}^3(\beta_i b_i + \alpha_i a_i)$ and $l'$ =
26. Soldering form on $\mathcal{F}_M$

Given the differentiable manifold $M^n$, the soldering or canonical form $\theta$ on $\mathcal{F}_M$ is the $\mathbb{R}^n$-valued differential 1-form on $FM^n$ i.e. $\theta \in \Gamma(T^*FM^n \otimes \mathbb{R}^n)$ defined as follows:

$$\theta : FM^n \to T^*FM^n \otimes \mathbb{R}^n, \quad (x, r_x) \mapsto \theta((x, r_x)) = ((x, r_x), \theta(x, r_x)),$$

$$\theta_{(x, r_x)} : T_{(x, r_x)}FM^n \to \mathbb{R}^n, \quad v_{(x, r_x)} \mapsto \theta_{(x, r_x)}(v_{(x, r_x)}) = \tilde{r}_x^{-1} \circ \pi_{F,(x, r_x)} (v_{(x, r_x)})$$

i.e.

$$\theta_{(x, r_x)} = \tilde{r}_x^{-1} \circ d\pi_F|_{(x, r_x)},$$

where $\pi_F$ is the projection in the bundle $\mathcal{F}_M$ (section 24) and $\tilde{r}_x$ is the vector space isomorphism

$$\tilde{r}_x : \mathbb{R}^n \to T_x M, \quad (\lambda^1, \ldots, \lambda^n) \mapsto \tilde{r}_x(\lambda^1, \ldots, \lambda^n) = \sum_{i=1}^n \lambda^i v_{ix}$$

with inverse

$$\tilde{r}_x^{-1}(\sum_{i=1}^n \lambda^i v_{ix}) = (\lambda^1, \ldots, \lambda^n).$$

We have the commutative diagram

$$\begin{array}{ccc}
\mathbb{R}^n & \xrightarrow{Id} & \mathbb{R}^n \\
\theta_{(x, r_x)} \uparrow & & \uparrow \tilde{r}_x^{-1} \\
T_{(x, r_x)}FM^n & \xrightarrow{d\pi_F|_{(x, r_x)}} & T_x M^n
\end{array}$$

Notice that $\dim_{\mathbb{R}}(T_{(x, r_x)}FM^n) = n + n^2$. Since $d\pi_F|_{(x, r_x)}$ is onto, $\theta_{(x, r_x)}$ is a vector space epimorphism, with $\ker(\theta_{(x, r_x)}) = V_{(x, r_x)}$, the vertical space of the bundle $FM^n$ at $(x, r_x)$, with $\dim_{\mathbb{R}} \ker(\theta_{(x, r_x)}) = n^2$. The existence of $\theta$ is independent of any connection. Also, it is clearly a global section of the bundle $T^*FM^n \otimes \mathbb{R}^n$.

$$(\tilde{e}_\mu)_j = \delta_{\mu j}, \quad \mu, j = 1, \ldots, n$$

is the canonical basis of $\mathbb{R}^n$, then

$$\theta_{(x, r_x)} = \sum_{\mu=1}^n \theta^\mu_{(x, r_x)} \otimes \tilde{e}_\mu$$

where $\theta^\mu_{(x, r_x)} \in T^*_{(x, r_x)} FM^n$.

In local coordinates on $\mathcal{F}_U$,

$$\theta^\mu = \sum_{\nu=1}^n (X^{-1})^\mu_\nu dx^\nu$$

where $(X^{-1})^\mu_\nu(x, r_x) = (X^\mu_{(x, r_x)})^{-1}$. In fact, if $v_{(x, r_x)} \in T_{(x, r_x)}FU_\alpha$, then $v_{(x, r_x)} = \sum_{\mu=1}^n \lambda^\mu \frac{\partial}{\partial x^\nu}|_{(x, r_x)} + \sum_{\nu=1}^n \lambda^\nu \frac{\partial}{\partial x^\nu}|_{(x, r_x)}$ with $d\pi_F|_{(x, r_x)}(v_{(x, r_x)}) = \sum_{\mu=1}^n \lambda^\mu \frac{\partial}{\partial x^\nu}|_{(x, r_x)} \in T_{x}\tilde{U}_\alpha$; then $\theta_{(x, r_x)}(v_{(x, r_x)})$

$$= \tilde{r}_x^{-1} \circ d\pi_F|_{(x, r_x)} (v_{(x, r_x)}) = \tilde{r}_x^{-1}(\sum_{\mu=1}^n \lambda^\mu \frac{\partial}{\partial x^\nu}|_{(x, r_x)}) = \sum_{\mu=1}^n \lambda^\mu \frac{\partial}{\partial x^\nu}|_{(x, r_x)}$$

on the other hand, $\theta_{(x, r_x)}(v_{(x, r_x)}) = \sum_{\mu=1}^n \theta^\mu_{(x, r_x)} \otimes \tilde{e}_\mu(\sum_{\nu=1}^n \lambda^\nu \frac{\partial}{\partial x^\nu}|_{(x, r_x)} + \sum_{\nu=1}^n \lambda^\nu \frac{\partial}{\partial x^\nu}|_{(x, r_x)})$

$$= \sum_{\mu=1}^n (X^\mu_{(x, r_x)})^{-1} \delta_{\mu j} \lambda^j \tilde{e}_\mu = \sum_{\mu=1}^n (X^\mu_{(x, r_x)})^{-1} \lambda^\mu \tilde{e}_\mu.$$

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Thus, a local section on $\mathcal{F}_{M^n}$, $s_\alpha : U_\alpha \to FU_\alpha$, $x \to s_\alpha(x) = (x, r_x)$, gives rise to a set of $n$ local differential 1-forms $\theta^\mu_\alpha \equiv \theta^\mu$ on $FU_\alpha$.

If $\omega$ is a connection on $\mathcal{F}_{M^n}$, and $H(x, r_x)$ is the horizontal space at $(x, r_x)$, then

$$\theta(x, r_x)|_{H(x, r_x)} = \tilde{r}_x^{-1} \circ d\pi_F|_{H(x, r_x)}$$

is a vector space isomorphism, since $d\pi_F|_{H(x, r_x)}$ is a canonical isomorphism between $H(x, r_x)$ and $T_x M^n$:

$$\begin{array}{ccc}
\mathbb{R}^n & \xrightarrow{Id} & \mathbb{R}^n \\
\theta(x, r_x)|_{H(x, r_x)} & \uparrow & \uparrow \tilde{r}_x^{-1} \\
H(x, r_x) & \xrightarrow{d\pi_F|_{H(x, r_x)}} & T_x M^n
\end{array}$$

We emphasize that $\theta(x, r_x)|_{H(x, r_x)}$ depends on both the frame at $x$ ($r_x$) and the connection $\omega$.

Any connection $\omega$ on the frame bundle $\mathcal{F}_{M^n}$, together with the canonical soldering form $\theta$, trivializes the tangent bundle of $\mathcal{F}_{M^n}$. This fact is known as absolute parallelism. The canonical bundle isomorphism (only depending on $\omega$) is given through the following diagram:

$$\begin{array}{ccc}
\mathbb{R}^{n+n^2} & \xrightarrow{\phi_c} & (FM^n)^{n+n^2} \times \mathbb{R}^{n+n^2} \\
\pi_{F_0} \downarrow & & \downarrow \pi_1 \\
(FM^n)^{n+n^2} & \xrightarrow{Id} & (FM^n)^{n+n^2}
\end{array}$$

with

$$\phi_c(((x, r_x), v_{(x, r_x)})) = ((x, r_x), (\theta(x, r_x)|_{H(x, r_x)} \times \omega(x, r_x)|_{V(x, r_x)})(\text{hor}(v_{(x, r_x)}), \text{ver}(v_{(x, r_x)})))$$

where $v_{(x, r_x)} \in T_{(x, r_x)} FM^n$ and $gl_n(\mathbb{R}) = \mathbb{R}(n) \cong \mathbb{R}^n$.

Absolute parallelism in the bundle of Lorentz frames $\mathcal{F}_{M^n}^L$ is given by the diagram

$$\begin{array}{ccc}
\mathbb{R}^{\frac{n(n+1)}{2}} & \xrightarrow{\phi_c^L} & (FLM^n)^{\frac{n(n+1)}{2}} \times \mathbb{R}^{\frac{n(n+1)}{2}} \\
\pi_{F_0} \downarrow & & \downarrow \pi_1 \\
(FLM^n)^{\frac{n(n+1)}{2}} & \xrightarrow{Id} & (FLM^n)^{\frac{n(n+1)}{2}}
\end{array}$$

with

$$\phi_c^L(((x, e_x), v(x, e_x))) = ((x, e_x), (\theta(x, e_x)|_{H(x, e_x)} \times \omega^L(x, e_x)|_{V(x, e_x)})(\text{hor}(v(x, e_x)), \text{ver}(v(x, e_x))))$$

$e_x = (e_{1x}, \ldots, e_{nx})$, and $H(x, e_x) = \ker(\omega^L(x, e_x))$.

In particular, for the $n = 4$ case:

$$\begin{array}{ccc}
\mathbb{R}^{10} & \xrightarrow{\phi_c^L} & (FLM^4)^{10} \times \mathbb{R}^{10} \\
\pi_{F_0} \downarrow & & \downarrow \pi_1 \\
(FLM^4)^{10} & \xrightarrow{Id} & (FLM^4)^{10}
\end{array}$$

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27. Linear connection in a manifold $M^n$ on $\mathcal{F}_{M^n}$

A $gl_n(\mathbb{R})$-valued linear connection $\nabla$ on $M^n$ is locally given by

$$\omega_U = \Gamma^\rho_{\mu \nu} dx^\nu \otimes E^\rho_\mu,$$

where $(E^\rho_\mu)_{\alpha \beta} = \delta^\rho_\alpha \delta_{\mu \beta}$ with $\rho, \mu, \alpha, \beta = 1, \ldots, n$ is the canonical basis of $\mathbb{R}^n = gl_n(\mathbb{R}) = \text{Lie}(GL_n(\mathbb{R}))$, and $\Gamma^\rho_{\mu \nu}$ are the Christoffel symbols (section 4).

On $FU$, the connection $\omega_{FU}$ such that $\omega_U = \sigma^*(\omega_{FU})$ with $\sigma : U \to FU$ the local section given by $x^\alpha \mapsto \sigma(x^\alpha) = (x^\mu, \delta^\alpha_\mu)$ is given by

$$\omega_{FU} = (X^{-1})^\rho_\mu (dX^\sigma + \Gamma^\sigma_{\rho \lambda} dX^\lambda) \otimes E^\rho_\mu.$$

(Kobayashi and Nomizu, 1963; pp 140-143) Clearly, $\omega_U \in \Gamma(T^*U \otimes gl_n(\mathbb{R}))$ and $\omega_{FU} \in \Gamma(T^*FU \otimes gl_n(\mathbb{R}))$.

Real-valued connection 1-forms $\omega_U^\mu_\rho$ and $\omega_{FU}^\mu_\rho$ are defined by

$$\omega_U = \omega_U^\mu_\rho \otimes E^\rho_\mu \quad \text{and} \quad \omega_{FU} = \omega_{FU}^\mu_\rho \otimes E^\rho_\mu.$$

The horizontal lift of a local vector field $\frac{\partial}{\partial x^\nu}$ by the connection $\omega$ in $\mathcal{F}_{M^n}$ is then given by

$$\left(\frac{\partial}{\partial x^\nu}\right)^\top = \frac{\partial}{\partial x^\nu} - \Gamma^\rho_{\mu \nu} X^\sigma \frac{\partial}{\partial X^\sigma}.$$ 

In fact, $\omega_{FU}^\rho_{\alpha \beta} ((\frac{\partial}{\partial x^\nu})^\top) = (X^{-1})^\rho_\lambda (dX^\lambda_{\beta} + \gamma^\lambda_{\gamma \xi} X^\xi_{\beta} dx^\gamma)(\frac{\partial}{\partial x^\nu} - \Gamma^\rho_{\mu \nu} X^\sigma \frac{\partial}{\partial X^\sigma})$

$$= (X^{-1})^\rho_\lambda (-\Gamma^\rho_{\mu \nu} X^\sigma_{\beta} \frac{\partial}{\partial X^\sigma} + \gamma^\lambda_{\gamma \xi} X^\xi_{\beta} \frac{\partial}{\partial x^\nu})$$

$$= (X^{-1})^\rho_\lambda (-\Gamma^\rho_{\mu \nu} X^\sigma_{\beta} \delta^\sigma_\nu + \gamma^\lambda_{\gamma \xi} X^\xi_{\beta} \delta^\nu_\gamma) = (X^{-1})^\rho_\lambda (-\Gamma^\rho_{\mu \nu} X^\sigma_{\beta} + \gamma^\lambda_{\gamma \xi} X^\xi_{\beta} \delta^\nu_\gamma) = 0.$$

28. Tetrads and spin connection

1. At each chart $U \subset M^n$ we can take as a basis of $\Gamma(TU)$ the local vector fields (Vielbeine)

$$e_a = e^\mu_a \partial_\mu, \quad a = 1, \ldots, n$$

with $r_x = (e_1, \ldots, e_n) \in (FU)_x$. Since the $n \times n$ matrices $(e^\mu_a(x)) \in GL_n(\mathbb{R})$, there exist the inverse vector fields $e_a^{-1} \equiv e^a = e^a_\mu dx^\mu$: 1-forms with $e^a_\mu = (e^a_\mu)^{-1} \in GL_n(\mathbb{R})$ and

$$e^a_\mu e^b_\nu \partial_\mu = \delta^b_\nu \text{ and } e^a_\mu e^a_\nu = \delta^b_\nu.$$

Then $e^a_\nu e_a = e^a_\nu e^a_\mu \partial_\mu = \delta^b_\nu \partial_\mu$ i.e.

$$\partial_\nu = e^a_\nu e_a.$$

In general, the $e_a$'s are called non-coordinate basis and the $e^a$'s anholonomic coordinates. For $n = 4$, the Vielbeine $e_a$ are called tetrads.

2. While $[\partial_\alpha, \partial_\beta] = 0$, the Vielbeine have non-vanishing Lie brackets. In fact, applying the commutator $[e_a, e_b]$ to a function $f \in C^\infty(U, \mathbb{R})$, one easily obtains

$$[e_a, e_b] = \lambda^c_{ab} e_c$$

with $\lambda^c_{ab} = e^\mu_c (e^a_\nu (\partial_\nu e_b^\mu) - e^b_\nu (\partial_\nu e_a^\mu)) = -\lambda^c_{ba}$. 

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3. For a local vector field, \( V = V^\mu \partial_\mu = V^\mu e_\mu e_a = V^a e_a \), with
\[
V^a = e_\mu^a V^\mu, \quad V^\mu = e_a^\mu V^a.
\]

4. At each \( x \in M^n \), the fibre of the co-frame bundle of \( M^n \),
\[
\mathcal{F}^*_{M^n} : GL_n(\mathbb{R}) \rightarrow (F M^n)^* \xrightarrow{\pi} M^n,
\]
is the set
\[
(F M^n)^* = \{ \text{ordered basis of } T^*_x M \} = \{(f^1_x, \ldots, f^n_x), \ f^a_x = f^a(x) dx^a|_x \}.
\]
Again, \((f^\nu_a(x)) \in GL_n(\mathbb{R})\) and, locally, \((f_\nu^a)^{-1} = f^\nu_a\) with
\[
f^a_a f^b_\nu = \delta^b_\nu \quad \text{and} \quad f^a_\nu f^b_\mu = \delta^b_\mu.
\]
Also,
\[
f^a = f^\nu_a dx^\nu \quad \text{and} \quad dx^\nu = f^\nu_a f_a^\nu.
\]
From the duality relation \(dx^\mu (\partial_\nu) = \delta^\mu_\nu\) we obtain \(\delta^\mu_\nu = f^a_\mu f^\nu_a(e^b_\nu e_b) = f^a_\mu f^\nu_a f^b_\nu\); imposing the duality relation between \(f^a\)’s and \(e^a\)’s,
\[
f^a(e_b) = \delta^a_b
\]
one obtains
\[
f^d_\mu = e^d_\mu \quad \text{and} \quad f^b_\nu = e^b_\nu.
\]
Then,
\[
f^a = e^\nu_a dx^\nu \quad \text{and} \quad dx^\nu = e^\nu_a f^a_\nu.
\]
(Another usual notation for \(f^a\) is \(\theta^a\).

5. Given an \((r, s)\)-tensor in \(M^n\),
\[
T = T^{\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_s} \partial_{\mu_1} \otimes \ldots \otimes \partial_{\mu_r} \otimes dx^{\nu_1} \otimes \ldots \otimes dx^{\nu_s},
\]
we obtain
\[
T = T^{\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_s} e_{\mu_1} e_a \otimes \ldots \otimes e_{\mu_r} e_a \otimes f_{b_1}^{\nu_1} f_{b_1}^{\nu_1} \otimes \ldots \otimes f_{b_s}^{\nu_s} f_{b_s}^{\nu_s} = T^{a_1 \ldots a_r}_{b_1 \ldots b_s} e_{a_1} \otimes \ldots \otimes e_{a_r} \otimes f_{b_1} \otimes \ldots \otimes f_{b_s}
\]
with
\[
T^{a_1 \ldots a_r}_{b_1 \ldots b_s} = e_{\mu_1} a_1 \ldots e_{\mu_r} a_r f_{b_1}^{\nu_1} f_{b_2}^{\nu_2} \ldots f_{b_s}^{\nu_s} T^{\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_s}.
\]

For example,
\[
T = T^\nu_\mu \partial_\mu \otimes dx^\nu = T^\nu_\mu e_\mu e_a \otimes dx^\nu = T^a_\nu e_a \otimes dx^\nu = T^a_\nu \partial_\mu \otimes f^b_\nu f^b_\nu = T^a_\nu e_\mu e_a \otimes f^b_\nu f^b_\nu = T^a_\nu f^b_\nu f^b_\nu = T^a_\nu f^b_\nu f^b_\nu = T^a_\nu e_a \otimes f^b_\nu f^b_\nu.
\]

6. Let \( g = g_{\mu \nu} dx^\mu \otimes dx^\nu \) be a metric in \(M^n\). \(g\) has a signature, given by the diagonal metric \(\eta_{ab}\), equal to \(\delta_{ab}\) in the euclidean case \((\eta = \eta_E)\), or with \(-1\)’s and \(+1\)’s in the general riemannian case; the lorentzian case, relevant for GR, has \(\eta = \eta_L\) (section 25.). \(g_{\mu \nu}\) being a symmetric matrix, at any point \(x \in M^n\) it can be diagonalized to \(\eta_{ab}\). The metric and its signature distinguish the subset of Vielbeine which obey the following orthonormality condition:
\[
g(e_a, e_b) = \eta_{ab}.
\]
In detail,
\[
g_{\mu \nu} dx^\mu \otimes dx^\nu (e_a^\rho \partial_\rho, e_b^\sigma \partial_\sigma) = g_{\mu \nu} e_a^\rho e_b^\sigma dx^\mu (\partial_\rho) dx^\nu (\partial_\sigma) = g_{\mu \nu} e_a^\rho e_b^\sigma \delta^\rho_\mu \delta^\sigma_\nu = g_{\mu \nu} e_a^\mu e_b^\nu.
\]
The unique solution of $g$ with $G$ obstructions, "trivialized" the metric everywhere i.e. globally, at the expense of the $i.e.$ an equivalence relation i.e. it determines a class of Vielbeine, whose elements are related by a group $a$ principal $G$-bundle over $M^n$, so one has the lorentzian bundle $\mathcal{L}_4 \rightarrow F\mathcal{L}_4 \rightarrow M^4$.

Notice that formula (b) is fundamental: since it holds in all charts in $M^n$, we have, up to topological obstructions, “trivialized” the metric everywhere i.e. globally, at the expense of the $x$-dependence of the coframes $e^a$.

It is usual the rough statement that the duals of the Vielbeine are the “square roots” of the metric. In particular, for the lorentzian $n = 4$ case, $\det(g_{\mu\nu}) = -(det(e_\mu^a))^2$. Also, equation (a) allows to interprete the $n \times n$ matrices $e_\rho^a$ as the matrices which diagonalize the metric $g_{\mu\nu}$ to the Lorentz metric $\eta_{ab}$. (b) says that the $e^a\gamma$’s are more fundamental than the metric.

7. Equation (b) in the last subsection appears naturally when describing spinor fields in curved spacetimes. If $\gamma_\mu(x)$ are the Dirac matrices in $M^n$, then

$$\{\gamma_\mu(x), \gamma_\nu(x)\} = 2g_{\mu\nu}(x)I.$$  

The solution

$$\gamma_\mu(x) = E_\mu^a(x)\gamma_a$$

with $\gamma_a$ the “flat” Dirac matrices obeying $\{\gamma_a, \gamma_b\} = 2\eta_{ab}I$, leads to

$$\eta_{ab}E_\mu^a(x)E_\nu^b(x) = g_{\mu\nu}$$

which says that the $E_\mu^a$’s are the duals of the Vielbeine $e_\mu^a$. It can be proved that the solution (*) is unique (O’Raifeartaigh, 1997).

8. It is clear that through (b) in subsection 6., the $n^2$ quantities involved by a Vielbein determine uniquely a metric $g_{\mu\nu}$; however, the set of $\frac{n(n+1)}{2}$ components of a metric determines a Vielbein only up to an equivalence relation i.e. it determines a class of Vielbeine, whose elements are related by a group $G$ of $\frac{n(n-1)}{2}$ elements.

Let $e_\mu^a = h_a^c e_c^\mu$; then

$$\eta_{ab} = g_{\mu\nu}h_a^c e_\nu^\mu h_b^d e_d^\nu = g_{\mu\nu}e_\nu^\mu e_d^\nu h_a^c h_b^d = \eta_{cd}h_a^c h_b^d = h_a^c \eta_{cd} h_b^d \text{ i.e.}$$

$$\eta = h \eta h^T.$$  

So, if $\eta = \eta_L$ then $h \in \mathcal{L}_n = O(1, n - 1)$; if $\eta = \eta_E$ then $h \in O_n = O(n)$; etc.

In the following we shall restrict to the case of orthonormal frames in the sense defined in 6., so one has a principal $G$-bundle over $M^n$:

$$G^{\frac{n(n-1)}{2}} \rightarrow F_G \rightarrow M^n,$$

with $G = \mathcal{L}_n$ or $O_n$, and $F_G \subset FM^n$. The interest in GR is for $G = \mathcal{L}_4$ and one has the lorentzian bundle

$$\mathcal{L}_4 \rightarrow F\mathcal{L}_4 \rightarrow M^4.$$

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and the bundle reduction

\[
\begin{array}{cccc}
\mathcal{L}_4 & \xrightarrow{\iota} & GL_4(\mathbb{R}) \\
\downarrow & & \downarrow \\
F\mathcal{L}_4 & \xrightarrow{\iota} & FM^4 \\
\pi\mathcal{L}_4 & \downarrow & \downarrow \pi_F \\
M^4 & \xrightarrow{id} & M^4
\end{array}
\]

where \(\iota\) is the inclusion.

We want to emphasize that the bundle is trivial, that is, \(F\mathcal{L}_4 \cong M^4 \times \mathcal{L}_4\), if \(M^4\) is contractible i.e. if it is of the same homotopy type as a point.

At each \(U \subset M\) one has a local Lorentz group \(\Gamma_U\) which, at \(x \in U\), “takes the value” \(\mathcal{L}(x)\). Since the Vielbeine are natural basis of \(\Gamma(TU)\), and the metric necessarily has a signature \(\eta_{ab}\), the principal \(\mathcal{L}_n\)-bundles over \(M^n\) are also natural. (We emphasize that here, we do not use the word “natural” in its technical sense, but in a colloquial sense.) We also notice the natural appearance of a new group at each \(x \in M^n\), besides the group of general coordinate transformations in the intersections of open sets: the Lorentz group.

9. In section 4, the covariant derivative of a local section \(\sigma_i\) of an arbitrary vector bundle \(E\) was defined by \(\nabla_\mu \sigma_i = \Gamma^j_{\mu} \sigma_j\). Let \(\sigma_i = e_a\); then

\[
\nabla_\mu e_a = \nabla_\mu (e_a \nu \partial_\nu) = (\partial_\mu e_a \nu \partial_\nu) + e_a \nu \nabla_\mu \partial_\nu = (\partial_\mu e_a \rho + e_a \nu \Gamma^\rho_{\mu \nu}) \partial_\rho.
\]

We now define the spin connection coefficients \(\omega^{ab}_{\mu}\) through

\[
\nabla_\mu e_a := \omega^{ab}_{\mu a} e_b \quad \text{and} \quad \nabla_\mu e^a := \omega^{ab}_{\mu b} e^a.
\]

(The \(\omega^{ab}_{\mu}\)'s are, in the present case, nothing but the \(\Gamma^j_{\mu i}\)'s.) Then, \(\omega^{ab}_{\mu a} e^a \partial_\rho = (\partial_\mu e_a \rho + e_a \nu \Gamma^\rho_{\mu \nu}) \partial_\rho\); from the linear independence of the coordinate basis, and multiplying by \(e_\rho \nu\) we obtain

\[
\omega^c_{\mu a} = e_\rho \nu \partial_\mu e_a \rho + e_\rho \nu e_a \nu \Gamma^\rho_{\mu \nu} \quad (c)
\]

or

\[
\omega^c_{ab} = -e_b \rho \partial_\mu e_a \rho + e_\rho \nu e_b \nu e_a \nu \Gamma^\rho_{\mu \nu} \quad (c')
\]

with

\[
\omega^c_{ab} = e_\mu \nu \omega^\mu_{ab}, \quad \omega^\mu_{ab} = g^{\mu \nu} \omega_{\nu ab}, \quad e_\mu \nu = \partial_\mu e_\rho \nu, \quad \partial_\rho = e_\rho \nu \partial_\nu = e_a \nu \partial_\nu.
\]

Multiplying by \(e_\rho \nu e_\lambda \mu\) we obtain the inverse relation:

\[
\Gamma^\rho_{\mu \lambda} = e_\rho \nu \partial_\mu e_\lambda \nu + e_\rho \nu e_\lambda \nu \omega^\rho_{\mu a} \quad (d)
\]

or

\[
\Gamma^\rho_{\mu \lambda} = \omega^c_{da} e_\mu e_\lambda e_\rho \nu e_\nu d - e_\rho \nu \omega^c_{da} e_\mu e_\lambda \nu e_\rho \nu d \quad (d')
\]

with \(e_\rho \nu \partial_\nu = \partial_\rho\).

Multiplying \((c)\) by \(e_\rho \nu\) and \((d)\) by \(e_\rho \sigma\) we obtain, respectively,

\[
\partial_\mu e_\rho \sigma + \Gamma^\rho_{\mu \nu} e_\nu \sigma - \omega^\rho_{\mu a} e_\rho \nu e_\sigma a = 0 \quad (e)
\]

and

\[
\partial_\mu e_\rho \sigma - \Gamma^\rho_{\mu \sigma} e_\rho \nu + \omega^\rho_{\mu a} e_\rho \sigma a = 0. \quad (f)
\]

The covariant derivative of tensors with upper and lower “internal” (Lorentz) and “external” (space-time) indices is defined by:

\[
\mathcal{D}_\mu T^{a_1 \ldots a_n}_{b_1 \ldots b_n} = \partial_\mu T^{a_1 \ldots a_n}_{b_1 \ldots b_n} + \Gamma^\mu_{\lambda \mu} T^{a_1 \ldots a_n}_{b_1 \ldots b_n} + \ldots + \omega^\mu_{\rho a} T^{a_1 \ldots a_n}_{b_1 \ldots b_n} - \Gamma^\mu_{\rho a} T^{a_1 \ldots a_n}_{b_1 \ldots b_n} - \ldots - \omega^\mu_{b a} T^{a_1 \ldots a_n}_{b_1 \ldots b_n - a_1}.
\]
With this definition, the equations (e) and (f) read
\[ D_\mu e_\sigma = 0 \quad \text{and} \quad D_\mu e_\lambda = 0 \]
respectively. (We also denote \( D_\mu T^{\mu_1 \ldots \mu_n}_{\nu_1 \ldots \nu_n b_1 \ldots b_u} = T^{\mu_1 \ldots \mu_n a_1 \ldots a_u}_{\nu_1 \ldots \nu_n b_1 \ldots b_u} \)).

From the Lorentz transformations of the tetrads \( e_\mu^a \) and their inverses, \( e_\mu^a = e_\mu^{\prime a} h^{-1}_c \) we obtain:
\[
e_\sigma^c \partial_\mu e_\sigma^a = e_\sigma^c r^\prime h^{-1}_c \partial_\mu (h_a^d e_\sigma^d) = e_\sigma^c r^\prime h^{-1}_c \partial_\mu (h_a^d) e_\sigma^d + e_\sigma^c r^\prime h^{-1}_c h_a^d \partial_\mu e_\sigma^d
\]
\[
= \partial_\mu (h_a^d) h^{-1}_d + h_a^d (e_\sigma^c r^\prime \partial_\mu e_\sigma^d) h^{-1}_c,
\]
and
\[
e_\sigma^c e_\alpha^\nu \Gamma^\sigma_{\mu \nu} = e_\sigma^c r^\prime h^{-1}_c h_a^d e_\nu^\nu \Gamma^\sigma_{\mu \nu} = h_a^d (e_\sigma^c r^\prime \nu^\nu \Gamma^\sigma_{\mu \nu}) h^{-1}_c.
\]
Then
\[
\omega_\mu^a = h_a^d (e_\sigma^c r^\prime \partial_\mu e_\sigma^d + e_\sigma^c r^\prime \nu^\nu \Gamma^\sigma_{\mu \nu}) h^{-1}_c + \partial_\mu (h_a^d) h^{-1}_d = h_a^d \omega^r_\mu h^{-1}_r + \partial_\mu (h_a^d) h^{-1}_d.
\]

I.e.
\[
\omega = h^\prime \omega^r + (dh) h^{-1}
\]
or, equivalently,
\[
\omega^r = h^{-1} \omega h - h^{-1} dh.
\]

So, the 1-form \( \omega^a_{\mu} := \omega^a_{\mu b} dx^\mu \) is not a 1-1 \( \mathcal{L} \)-tensor, since its transformation has an inhomogeneous term.

Notice that from (c), \( \omega \) has the structure
\[
\omega = e^{-1} (\partial + \Gamma) e,
\]
and from (d), the structure of \( \Gamma \) is
\[
\Gamma = e (\partial + \omega) e^{-1}.
\]

It can be easily shown that for a \textit{metric connection} \( \nabla \), for which
\[
D_\rho g_{\mu \nu} = 0,
\]
the spin connection with lower Lorentz indices
\[
\omega_{\mu b c} = \omega^a_{\mu c} \eta_{a b}
\]
is antisymmetric in these indices. In fact, using (e) and (f),
\[
0 = g_{\mu \nu : \rho} = (\eta_{a b} e_\mu^a e_\nu^b : \rho) = \eta_{a b} e_\mu^a e_\nu^b + \eta_{a b} e_\mu^a : \rho e_\nu^b + \eta_{a b} e_\mu^a e_\nu^b : \rho
\]
\[
= \eta_{a b} e_\mu^a e_\nu^b = -(\omega^a_{\mu a} \eta_{b c} - \omega^a_{\mu b} \eta_{a c}) e_\mu^a e_\nu^b = -(\omega_{\mu a b} + \omega_{\mu b a}) e_\mu^a e_\nu^b,
\]
then
\[
0 = -(\omega_{\mu a b} + \omega_{\mu b a}) e_\mu^a e_\nu^b e_\nu^\mu e_\nu^d = -(\omega_{\mu a b} + \omega_{\mu b a}) \delta_\mu^a \delta_\nu^b = -(\omega_{\mu d c} + \omega_{\mu c d})
\]
i.e.
\[
\omega_{\mu d c} = -\omega_{\mu c d}.
\]

Thus, we see that it is the condition of metric compatibility which \textit{reduces} the Lie algebra of the gauge group from \( gl_n(\mathbb{R}) \) to \( o(1, n - 1) \) (or \( o(n) \)), where the 1-form \( \omega_{\mu b c} = \omega^a_{\mu b c} dx^\mu \) takes values. The reduced gauge group
can be \(O(1, n - 1)\) (or \(O(n)\)) or, for the case \(n = 4\), \(SL(2, \mathbb{C})\) if \(h_a^c \in L_+^4 = SO^0(1, 3)\) at each \(x \in M^4\) (Randono, 2010).

Up to here the content of this section does not depend on the symmetry properties of \(\Gamma^p_{\rho \sigma}\) in its lower indices. If in particular the Levi-Civita connection of section \(13.\) is inserted in (b), using (c) (valid for any connection \(\nabla\) with local coefficients \(\Gamma^p_{\mu \nu}\)), we obtain the spin connection coefficients \(\omega^c_{\mu a}\) in terms of the vielbeine, their derivatives, and the Lorentz metric \(\eta_{ab}\):

\[
\omega^c_{\mu a} = e^c_p \partial_{\mu} e_{a}^p + \frac{1}{2} e^c_p e_a^\nu \eta^{d \rho} e_{d}^\sigma \eta_{bk}(\partial_{\mu} e_{\sigma}^{h} e_{\nu}^{k}) + \partial_{\nu}(e_{\sigma}^{h} e_{\nu}^{k}) - \partial_{\sigma}(e_{\mu}^{h} e_{\nu}^{k}).
\]

So, we have the result analogous to the dependence of the Levi-Civita connection on the metric: the dependence of the spin connection on the tetrad.

10. Explicitly, on each chart \((U, x^\mu)\) in \(M\), the (metric) spin connection with values in \(so(3, 1)\), is constructed as follows:

\[
\omega = \frac{1}{2} \omega^a_{\mu ab} dx^\mu \otimes l_{ab} \in \Gamma(T^*U \otimes so(3, 1)), \ x \mapsto \omega(x) = (x, \omega_x) \text{ with } \omega_x = \frac{1}{2} \omega^a_{\mu ab} dx^\mu|_x \otimes l_{ab} \in T_{x}^*U \otimes so(3, 1)
\]

i.e.

\[
\omega_x : T_x U \rightarrow so(3, 1), \ v_x \mapsto \omega_x(v_x) = \frac{1}{2} \omega^a_{\mu ab}(x) dx^\mu|_x(v_x) l_{ab} = \frac{1}{2} \omega_{ab}^a(x) v^\mu_x l_{ab} = \frac{1}{2} \omega_{ab}^a(x) l_{ab}
\]

with \(\omega'_{ab}(x) = -\omega_{ba}(x) := \omega_{ab}(x) v^\mu_x\) and \(\omega_x(v_x) = \omega_{01}^0 l_{01} + \ldots + \omega_{31}^0 l_{31}\).

For later use, consider the connected component of the \textit{Poincaré group} \(P_4\), the semidirect sum of the of the translation group \(T_4\) and the connected component of the Lorentz group \(L_4\):

\[P_4 = T_4 \circ SO^0(3, 1), \ (a', \Lambda')(a, \Lambda) = (a + \Lambda' a, \Lambda'\Lambda)\]

with Lie algebra

\[p_4 = \mathbb{R}^4 \circ so(3, 1), \ (\vec{x}', \vec{l}')(\vec{x}, \vec{l}) = (l' \vec{x} - l \vec{x}', [l', l]) = -(\vec{x}, \vec{l})(\vec{x}', \vec{l}').\]

\((P_4\) is a subgroup of the \textit{affine group} \(A_4\); for arbitrary \(n\), \(A_n = \mathbb{R}^n \circ GL_n(\mathbb{R})\) with Lie algebra \(a_n = \mathbb{R}^n \circ \mathbb{R}(n)\).)

Also, on each chart \((U, x^n)\) in \(M\), the tetrad (1-form) \(\tilde{e}^a\) with values in \(\text{Lie}(T_4) = \mathbb{R}^4\) i.e. \(\tilde{e}^a \in \Gamma(T^*U \otimes \mathbb{R}^4)\) is constructed as follows:

\[
\tilde{e}^a = e^a_\mu dx^\mu \otimes \vec{x} : U \rightarrow T^*U \otimes \mathbb{R}^4, \ \tilde{e}^a(x) = (x, \tilde{e}^a_x), \ \tilde{e}^a_x = e^a_\mu(x) dx^\mu|_x \otimes \vec{x} \in T_x^*U \otimes \mathbb{R}^4,
\]

\[
\tilde{e}^a_x : T_x U \rightarrow \mathbb{R}^4, \ e^a_\mu(x) dx^\mu|_x(v_x) \vec{x} = e_x \vec{x} \text{ with } e_x = e^a_\mu(x) v^\mu_x \in \mathbb{R}.
\]

11. The Lorentz bundle \(L_4 \rightarrow F_{L_4} \rightarrow M^4\) in subsection 8. extends the symmetry group of GR, the group of general coordinate transformations of \(M^4, \mathcal{D}\), to the \textit{semidirect sum}

\[G_{GR} = L_4 \circ \mathcal{D},\]

with composition law given by

\[(h', g')(h, g) = (h'(g'h g^{-1}), g' g).\]
In fact, it is easy to verify that $\mathcal{D}$ has a left action on $\mathcal{L}_4$ at each fibre $F_{\mathcal{L}_4}^a$ of the bundle, given by the commutative diagram

\[
\begin{array}{ccc}
F_{\mathcal{L}_4}^a & \xrightarrow{h} & F_{\mathcal{L}_4}^a \\
g & \downarrow & \downarrow g \\
F_{\mathcal{L}_4}^a & \xrightarrow{h'} & F_{\mathcal{L}_4}^a
\end{array}
\]

which defines

\[
h' = ghg^{-1} = L_g(h)
\]

(conjugation action), with $g(x, (e_1(x), \ldots, e_4(x))) = g(e_a(x)) = g(e_a(x) \frac{\partial}{\partial x^a} |_{x}) = e'_a(x) \frac{\partial}{\partial x^a} |_{x}$, $e'_a(x) = \frac{\partial e_a}{\partial x^a} (x)$. The action is left since $h' \mapsto h'' = g'h'g^{-1} = g'(ghg^{-1})g''^{-1} = (g'g)h(g'g)^{-1}$ i.e. $L_{g'}g(h) = L_{g'}L_g(h)$. (See the extension to $G_{GR} = P_4 \otimes \mathcal{D}$ in section 34.)

29. Curvature and torsion in terms of spin connection and tetrads. Cartan structure equations; Bianchi identities

In what follows we shall designate by $\Omega^k(L^*_4)$ the real vector space of $k$ differential forms on $M$ with values in the $(r, s)$-Lorentz tensors.

Given the Vielbeine $e^a_\mu$ and the spin connection $\omega^a_{\mu b}$ on the chart $(U, x^\mu)$ on $M$, we have the differential forms

\[
e^a = e^a_\mu dx^\mu \in \Omega^1(L^1) \text{ and } \omega^a_b = \omega^a_{\mu b} dx^\mu.
\]

($\omega^a_b \notin \Omega^1(L^1_4)$ since $\omega^a_b$ is not an $L^1_4$-tensor, but a connection on the Lorentz bundle $\mathcal{L}_4 \rightarrow F_{\mathcal{L}_4} \rightarrow M^4$.) Then we have the 2-forms

\[
T^a = de^a + \omega^a_b \wedge e^b = \frac{1}{2} T^a_{\mu \nu} dx^\mu \wedge dx^\nu \in \Omega^2(L^1),
\]

(*),

with \(T^a_{\mu \nu} = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu + \omega^a_{\mu b} e^b_\nu - \omega^a_{\nu b} e^b_\mu = -T^a_{\nu \mu},\)

and

\[
R^b_a = d\omega^a_b + \omega^c_a \wedge \omega^b_c = \frac{1}{2} R^a_{b \mu \nu} dx^\mu \wedge dx^\nu \in \Omega^2(L^1_4),
\]

(**),

with \(R^a_{b \mu \nu} = \partial_\mu \omega^a_{\nu b} + \partial_\nu \omega^a_{\mu b} + \omega^a_{\mu c} \omega^c_{\nu b} - \omega^a_{\nu c} \omega^c_{\mu b}.\)

(*) and (**) are known as the Cartan structure equations. As we shall show below, $T^a$ and $R^b_a$ are, respectively, the torsion and curvature 2-forms of section 9.

For $de^a$ one has

\[
de^a = d(e^a_\nu dx^\nu) = \partial_\mu e^a_\nu dx^\mu \wedge dx^\nu = \Omega^a_{\mu \nu} dx^\mu \wedge dx^\nu \equiv \Omega^a
\]

with

\[
\Omega^a_{\mu \nu} = (de^a)_{\mu \nu} = \frac{1}{2} (e_{\nu, \mu}^a - e_{\mu, \nu}^a) = -\Omega^a_{\nu \mu}.
\]

Also,

\[
\Omega^a_{bc} = (de^a)_{bc} = e^a_\mu e^p_\nu \Omega^p_{\mu \nu} = \frac{1}{2} e^p_\nu (e^a_\mu \partial_\mu e^b_\nu - e^b_\mu \partial_\nu e^a_\nu) = \frac{1}{2} e^a_\mu (e^b_\mu - e^c_\nu) = -\Omega^a_{cb}.
\]

Comparing with $\lambda^a_{bc}$ of 28.2, we have

\[
\Omega^a_{bc} = -\frac{1}{2} \lambda^a_{bc}
\]

and therefore

\[
[e_b, e_c] = -2\Omega^a_{bc} e_a = -2(de^a)_{bc} e_a.
\]

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So $\Omega_{bc}^a$ also measures the non-commutativity of the Vielbeine. By the Jacobi identity,

$$\Omega_{ad}^f \Omega_{bc}^d + \Omega_{bd}^f \Omega_{ca}^d + \Omega_{cd}^f \Omega_{ab}^d = 0.$$ 

It is easy to show that for a metric connection, the curvature tensor with lower Lorentz indices

$$R_{ab} = \eta_{ad} R_{db}^c$$

is antisymmetric i.e.

$$R_{ab} = - R_{ba}.$$ 

In fact,

$$R_{ab} = \eta_{bd} (d\omega^d_b + \omega^d_c \wedge \omega^c_b) = \eta_{bd} d\omega^d_b + \eta_{bd} \omega^d_c \wedge \omega^c_b = d\omega_{ab} + \omega_{ac} \wedge \omega^c_b = - (d\omega_{ba} + \omega_{ca} \wedge \omega^c_b) = -(d\omega_{ba} - \omega^c_b \wedge \omega^c_a),$$

while

$$R_{ba} = \eta_{bc} R_{ca}^e = \eta_{bc} (d\omega^e_a + \omega^e_d \wedge \omega^d_a) = d\omega_{ba} + \omega_{bd} \wedge \omega^d_a = d\omega_{ba} - \omega_{db} \wedge \omega^d_a.$$ 

Symbolically we write

$$T = de + \omega \wedge e, \quad R = d\omega + \omega \wedge \omega.$$ 

We notice that torsion is related to the tetrads as curvature is related to the spin connection.

On the other hand, while curvature involves only $\omega$, torsion involves both $e$ and $\omega$ (not only $e$). This is related to the fact that the Poincaré group is the semidirect (not direct) sum of $\mathbb{R}^4$ (translations) and $SO^0(3,1)$ (spacetime rotations).

A manifold equipped with a metric $g_{\mu\nu}$ and a connection $\Gamma_{\mu\nu}^\rho$ compatible with the metric but with non-vanishing torsion, is called an Einstein-Cartan manifold. The metric induces the Levi-Civita connection, $(\Gamma_{\mu\nu})_{\nu\rho}$ (section 13) with $\Gamma_{\nu\rho}^\mu = (\Gamma_{LC})_{\nu\rho}^\mu +$ contortion tensor.

In the Einstein-Cartan (E-C) theory of gravity, the 1-forms

$$\{e^a, \omega_{ab}\}$$

are called gauge or gravitational potentials, respectively translational and rotational, while the 2-forms

$$\{T^a, R^a_{\,\,b}\}$$

are called gauge or gravitational field strengths, respectively translational and rotational. (See, however, section 34.) At a point, it is always possible to set $e_{\mu\rho} = 1$ and $\omega_{\mu\rho} = 0$, i.e. respectively $e^a_{\mu} = \delta^a_{\mu}$ (16 conditions) and $\omega_{a\mu} = 0$ (24 conditions). (Hehl, 1985; Hartley, 1995.) The total number of conditions, 40, coincides with that for making zero the Christoffel symbols in the case of the Levi-Civita connection ($|\{\Gamma_{LC}^\mu_{\nu\rho}\}| = 40$).

Comment. Together with the comments in section 10, we have the following relations:

curvature $\longleftrightarrow$ spin connection $\longleftrightarrow$ spacetime rotations,

torsion $\longleftrightarrow$ tetrads $\longleftrightarrow$ spacetime translations.

On the other hand, from Noether theorem, we have the relations:

spacetime rotations $\longleftrightarrow$ angular momentum,

spacetime translations $\longleftrightarrow$ energy momentum.
Naively, one should then expect the following relations:

\[
\text{curvature} \leftrightarrow \text{angular momentum},
\]
\[
\text{torsion} \leftrightarrow \text{energy momentum}.
\]

However, in Einstein-Cartan theory, based on a non-symmetric metric connection, the sources of curvature and torsion are respectively energy momentum and spin angular momentum, i.e.

\[
\text{curvature} \leftrightarrow \text{energy momentum},
\]
\[
\text{torsion} \leftrightarrow \text{spin angular momentum}.
\]

This “crossing” of relations is due to holonomy theorems (Trautman, 1973).

These facts can be better understood as follows: In (special) relativistic field theory (r.f.t.), fields belong to irreducible representations of the Poincaré group \( \mathbb{P} \), which are characterized by two parameters: mass and spin. Invariance under translations (\( T \)) and rotations (\( \mathcal{L} \)) respectively leads, by Noether theorem, to the conservation of energy-momentum (\( T_{\mu \nu} \)) and angular momentum: orbital + intrinsic (spin, with density \( S^\mu_{\nu} \)). On the other hand, differential geometry (d.g.), through holonomy theorems, relates curvature \( \Gamma(\Lambda) \) with the Lorentz group and torsion \( (T^\mu_{\nu}) \) with translations (section 10). Finally, Einstein (E) equations make energy-momentum the source of curvature, while Cartan (C) equations makes spin the source of torsion. This is summarized in the following diagram:

\[
\begin{array}{c}
d.g. \quad \mathcal{L}_4 \quad r.f.t. \\
R^\rho_{\nu \sigma} \quad \delta_{\omega} \quad S^\mu_{\nu} \\
E \quad C \quad \delta_{\epsilon} \quad T^\rho_{\nu \mu} \\
r.f.t. \quad T_4 \quad d.g.
\end{array}
\]

In a formulation of the Einstein-Cartan theory based on tetrads and spin connection, the Einstein equations are obtained by variation with respect to the tetrads (\( \delta_{\epsilon} \)), related to translations, and the Cartan equations by variation with respect to the spin connection (\( \delta_{\omega} \)), related to rotations. (See section 32.)

Locally, as differential 2-forms with values in \( so(3,1) \) and \( \text{Lie}(T_4) = \mathbb{R}^4 \), \( R \) and \( T^a \) are respectively given as follows:

\[
R = \frac{1}{2} R_{ab\sigma\rho} dx^a \wedge dx^\sigma \otimes l_{ab} \in \Gamma(\Lambda^2 \mathbb{U} \otimes so(3,1)), \quad R_{ab\sigma\rho} = \eta_{ad} R^d_{b\sigma\rho}, \quad R(x) = (x, R_x),
\]

\[
R_x = \frac{1}{2} R_{ab\sigma\rho}(x) dx^a|_x \wedge dx^\sigma|_x \otimes l_{ab} \in \Lambda^2 \mathbb{U} \otimes so(3,1), \quad R_x : T_x U \otimes T_x U \to so(3,1), \quad R_x(v_x, w_x)
\]

\[
= \frac{1}{4} R_{ab\sigma\rho}(x)(v_x^a w_x^\sigma - v_x^\sigma w_x^a) l_{ab};
\]

\[
T^a = T^a_{\mu \nu} dx^\mu \wedge dx^\nu \otimes \vec{t} \in \Gamma(\Lambda^2 \mathbb{U} \otimes \mathbb{R}^4), \quad T^a(x) = (x, T^a_x), \quad T^a_x = T^a_{\mu \nu}(x) dx^\mu|_x \wedge dx^\nu|_x \otimes \vec{t},
\]

\[
T^a_x : T_x U \otimes T_x U \to \mathbb{R}^4, \quad T^a_x(v_x, w_x) = \frac{1}{2} T^a_{\mu \nu}(x)(v_x^a w_x^\nu - v_x^\nu w_x^a) \vec{t}.
\]

From the definition of \( T \), we have, since \( d^2 = 0 \),

\[
dT = d\omega \wedge e - \omega \wedge de = d\omega \wedge e - \omega \wedge (T - \omega \wedge e) =
\]

\[
d\omega \wedge e - \omega \wedge T + \omega \wedge \omega \wedge e, \quad \text{i.e.} \ dT + \omega \wedge T = (d\omega + \omega \wedge \omega) \wedge e, \quad \text{that is}
\]

\[
dT + \omega \wedge T = R \wedge e \in \Omega^3(L^1).
\]

\( (\alpha) \)
In Lorentz components, \[dT^a + \omega^a_b \wedge T^b = R^a_b \wedge e^b.\] (\(\alpha'\))

For \(\mathbf{R}\) one has \(d\mathbf{R} = d\omega \wedge \omega - d\omega = (\mathbf{R} - \omega \wedge \omega) \wedge \omega - \omega \wedge (\mathbf{R} - \omega \wedge \omega) = \mathbf{R} \wedge \omega - \omega \wedge \mathbf{R},\) i.e.

\[d\mathbf{R} + \omega \wedge \mathbf{R} - \mathbf{R} \wedge \omega = 0 \in \Omega^2(L^1_\mathbb{L}).\] (\(\beta\))

In components,

\[dR^a_b + \omega^a_c \wedge R^c_b - R^a_c \wedge \omega^c_b = 0.\] (\(\beta'\))

(\(\alpha\)) and (\(\beta\)) (or (\(\alpha'\)) and (\(\beta'\))) are the so-called Bianchi identities. (Compare (\(\beta\)) with the corresponding equation in section 12.)

Defining the covariant exterior derivative operator acting on Lorentz tensors-valued differential forms \(\mathcal{D}_\omega = d + \omega\wedge\) we have the equations

\[\mathbf{T} = \mathcal{D}_\omega e, \quad \mathcal{D}_\omega \mathbf{T} = \mathbf{R} \wedge e, \quad \mathcal{D}_\omega \mathbf{R} = \mathbf{R} \wedge \omega.\]

Though \(\omega\) is not a Lorentz tensor, one has \(\mathbf{R} = \mathcal{D}_\omega \omega.\)

It is easy to verify that \(T^a\) is nothing but twice the torsion tensor of section 9.

\[e_\alpha^\lambda T^a_{\mu \nu} = e_\alpha^\lambda ((de^a)_{\mu \nu} + (\omega^a_b \wedge e^b)_{\mu \nu}) = e_\alpha^\lambda (\partial_\mu e_\nu^a - \partial_\nu e_\mu^a + \omega^a_{\mu \nu} e^b - \omega^a_{\nu \mu} e^b) = (e_\alpha^\lambda \partial_\mu e_\nu^a + e_\alpha^\lambda e_\nu^b \omega_{\mu \nu}^a)
- (e_\alpha^\lambda \partial_\nu e_\mu^a + e_\alpha^\lambda e_\mu^b \omega_{\mu \nu}^a) = \Gamma^\lambda_{\mu \nu} = \omega_{\mu \nu} - \omega_{\nu \mu} = 2T^\lambda_{\mu \nu}.
\]

A similar calculation leads to

\[e_\alpha^\rho e_\sigma^b R^a_{b \mu \nu} = e_\alpha^\rho e_\sigma^b ((d\omega^a_b + \omega^a_c \wedge \omega^c_b)_{\mu \nu}) = e_\alpha^\rho e_\sigma^b (\partial_\mu \omega^a_{\nu b} - \partial_\nu \omega^a_{\mu b} + \omega^a_{\mu \nu} \omega^c_{\nu b} - \omega^a_{\nu \mu} \omega^c_{\mu b}) = R^a_{\sigma \mu \nu}.
\]

For the Ricci tensor and the Ricci scalar of sections 16 and 18 respectively (but now not restricted to the Levi-Civita connection) we have

\[R_{\sigma \nu} = R^a_{\sigma \mu \nu} = e_\alpha^\mu e_\sigma^b R^a_{b \mu \nu} \quad \text{and} \quad R = R^a_{\sigma \nu} = e_\alpha^\mu e_\sigma^b R^a_{b \mu \nu} g^{\sigma \nu} = e_\alpha^\mu e_\nu^b R^a_{b \mu \nu}.
\]

We summarize the above formulae in the following table:

| \(e\) | \(\Omega^1(L^1)\) | \(\Omega^1(L^1)\) | \(\Omega^2(L^1)\) | \(\Omega^2(L^1)\) | \(\Omega^3(L^1)\) | \(\Omega^3(L^1)\) | Components |
|------|----------------|----------------|----------------|----------------|----------------|----------------|-------------|
| \(e\) | \(\times\) | \(-\) | \(-\) | \(-\) | \(-\) | \(-\) | \(e^a\) |
| \(\omega\) | \(-\) | \(-\) | \(-\) | \(-\) | \(-\) | \(-\) | \(\omega_b^a\) |
| \(\mathbf{T} = de + \omega \wedge e\) | \(-\) | \(-\) | \(-\) | \(\times\) | \(-\) | \(-\) | \(T^a = de^a + \omega_b^a \wedge e^b\) |
| \(\mathbf{R} = d\omega + \omega \wedge \omega\) | \(-\) | \(-\) | \(-\) | \(\times\) | \(-\) | \(-\) | \(R^a_b = d\omega^a_b + \omega_c^a \wedge e^b\) |
| \(d\mathbf{T} + \omega \wedge \mathbf{T} = \mathbf{R} \wedge e\) | \(-\) | \(-\) | \(-\) | \(-\) | \(\times\) | \(-\) | \(dT^a + \omega_b^a \wedge T^b = R^a_b \wedge e^b\) |
| \(d\mathbf{R} + \omega \wedge \mathbf{R} = \mathbf{R} \wedge \omega\) | \(-\) | \(-\) | \(-\) | \(-\) | \(-\) | \(\times\) | \(dR^a_b + \omega_c^a \wedge R^a_b = R^a_c \wedge \omega_b^c\) |

30. Spin connection in non-coordinate basis

The Christoffel symbols for a metric connection with torsion is given in Appendix B:

\[\Gamma^\mu_{\nu \rho} = (\Gamma_{LC})^\mu_{\nu \rho} + K^\mu_{\nu \rho}\]

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where the contortion tensor depends on the metric and the torsion, while \((\Gamma^{\mu}_{\nu\rho})^\nu_{\nu\rho}\) only depends on the metric and its derivatives. Contracting with \(e^\alpha_s\) and \(e^\beta_t\) one has
\[
e_a^{\nu} e_b^{\rho} e_c^{\mu} (\Gamma^{\mu}_{\nu\rho})^\nu_{\nu\rho} + e_a^{\nu} e_b^{\rho} e_c^{\mu} K^{\mu}_{\nu\rho} = e_a^{\nu} e_b^{\rho} e_c^{\mu} \Gamma^{\mu}_{\nu\rho},
\]
and using \((c')\) in 28.9 we obtain
\[
\omega^{\nu}_{ab} + e_b^{\rho} e_c^{\mu} = e_a^{\nu} e_b^{\rho} e_c^{\mu} \Gamma^{\mu}_{\nu\rho} + K^{\mu}_{ab}.
\]
Using the expressions for \(g_{\mu\nu}, \partial_\rho g_{\mu\nu},\) etc. in \(\Gamma^{\mu}_{\nu\rho}\) in terms of \(e^\alpha\)'s and their derivatives, a straightforward calculation leads to
\[
\omega_{dab} = \gamma_{abd} + K_{dab} \quad (U_4 - space) \quad (\ast)
\]
where
\[
\gamma_{abd} = -\Omega_{dab} + \Omega_{abd} - \Omega_{da} = -\gamma_{adb}
\]
are the Ricci rotation coefficients, with \(|\{\gamma_{abd}\}| = n^2(n-1)/2\), (24 for \(n = 4\)), and
\[
X_{abc} = \eta_{ad} X_{bc}^d, \quad X = \omega, \gamma, K; \quad X_{eb} = -X_{be}^d, \quad X = \omega, \gamma, \Omega.
\]
If \(T^a = 0\), then
\[
\omega_{dab} = \gamma_{abd}. \quad (V_4 - space)
\]
We emphasize that the \(\gamma_{abd}\)'s come from the metric, the Vielbeine and their inverses and derivatives. So, the parallel transport and concomitant rotations of vectors by \((\ast)\) has two sources: metric (\(g\)) and torsion \((\gamma)\): from
\[
\omega_{dab} = \gamma_{abd} + K_{dab},
\]
we have
\[
(\delta_{\parallel} A)_a|_{\nu} = -\omega_{dab}(x) A^b(x) dx^\nu|_{x} = (\delta_{\parallel}^g A)_a|_{\nu} + (\delta_{\parallel}^\gamma A)_a|_{\nu}
\]
with
\[
(\delta_{\parallel}^g A)_a|_{\nu} = -\gamma_{abc}(x) A^b(x) dx^\nu|_{x}
\]
and
\[
(\delta_{\parallel}^\gamma A)_a|_{\nu} = -K_{abc}(x) A^b(x) dx^\nu|_{x}.
\]
\((A_a = e^\mu_a A^\mu, \ A_b = \eta^{bc} A_c)\)

31. Locally inertial coordinates

Let \((M^n, g, \Gamma)\) be a \(U^n\)-space (see Appendix B), \(x \in M^n\) and \((U, \varphi = (x^\mu))\) a chart on \(M^n\) with \(x \in U\) and \(x^\mu(x) = 0, \mu = 0, \ldots, n-1\). Let \((U', \varphi' = (x'^\mu))\) be an intersecting chart with \(x'^\mu(x) = 0\) and
\[
x^\mu = x'^\mu - \frac{1}{2} \Gamma^{\mu}_{\nu\rho}(x) x'^\nu x'^\rho,
\]
where \((\nu\rho)\) means symmetrization. The antisymmetric part \(\Gamma^{\mu}_{\nu\rho} = T^{\mu}_{\nu\rho} = -T^{\mu}_{\rho\nu}\) (torsion) does not contribute to the change of coordinates.

The condition of metricity at \(x\),
\[
0 = g_{\mu\nu,\lambda}(x) = g_{\mu\nu,\lambda}(x) - \Gamma^{\nu}_{\lambda\mu}(x) g_{\nu\rho}(x) - \Gamma^{\rho}_{\lambda\nu}(x) g_{\mu\rho}(x),
\]
which, being a tensor also holds in the chart \(U'\), the tensor transformation formula of \(g_{\mu\nu}\) (section 7), and the diagonalization of \(g_{\mu\nu}\) to \(\eta_{\mu\nu}\) at \(x\) (uniquely determined up to a Lorentz transformation (section 28.8)), lead to the equations:
\[
g'^{\mu}_{\nu\rho}(x^\lambda) = \eta_{\mu\nu} + (\eta_{\nu\rho} T^{\rho}_{\lambda\nu}(x) + \eta_{\nu\rho} T^{\rho}_{\lambda\mu}(x)) x^\lambda + O(x^\nu^2) = \eta_{\mu\nu} + \frac{\partial}{\partial x^\lambda} g'^{\mu}_{\nu\rho}(x) x^\lambda + O(x^\nu^2), \quad (a)
\]
39
and

$$(\Gamma'_{LC})^{\alpha}_{\nu\rho}(x) = \frac{1}{2} \eta^\alpha{}_{\sigma}(\partial'_\mu(g'_{\rho\sigma})(x) + \partial'_\rho(g'_{\sigma\nu})(x) - \partial'_\sigma(g'_{\nu\rho})(x)).$$  \quad (b)

So,

$$T'_\nu{}^\mu(x) = 0 \Rightarrow \partial'_\lambda(g'_\mu)(x) = 0 \; \text{and} \; (\Gamma'_{LC})^{\mu}_{\nu\rho}(x) = 0,$$

i.e. the vanishing of the torsion at $x$ is a **sufficient** condition for having a local inertial system at $x$.

However, the condition is not **necessary**; in fact,

$$\eta_{\mu\rho}T'_{\lambda\nu}(p) + \eta_{\nu\rho}T'_{\lambda\mu}(p) = T_{\lambda\nu\mu}(p) + T_{\lambda\mu\nu}(p) = 0$$

implies that $T_{\mu\nu\rho}$ is also antisymmetric in its second and third indices, and then it is totally antisymmetric, since $T_{\mu\nu\lambda} = -T_{\mu\lambda\nu} = T_{\lambda\mu\nu} = -T_{\lambda\nu\mu}$.

A calculation gives:

$n = 2$:

$$T^0_{01} = T^1_{01} = 0$$

$n = 3$:

$$T^0_{01} = T^0_{02} = T^1_{01} = T^1_{12} = T^2_{02} = T^2_{12} = T^3_{03} = T^3_{13} = T^3_{23} = 0,$$

$$T^0_{12} = T^2_{10} = T^1_{02}$$

$n = 4$:

$$T^0_{01} = T^0_{02} = T^0_{03} = T^1_{01} = T^1_{12} = T^2_{12} = T^2_{23} = T^3_{03} = T^3_{13} = T^3_{23} = T^3_{32} = 0,$$

$$T^0_{12} = T^2_{10} = T^1_{02},$$

$$T^0_{13} = T^3_{10} = T^1_{03},$$

$$T^0_{23} = T^3_{20} = T^2_{03},$$

$$T^1_{23} = T^2_{31} = T^3_{12}$$

In each case, the number of independent but not necessarily zero components of the torsion tensor coincides with the number of independent components of the totally antisymmetric torsion tensor with covariant indices, number which results from the condition that the definition of geodesics as “world-lines of particles” (parallel transported velocities, section 8) to coincide with their definition as extremals of length. This last fact can be seen as follows:

As world-lines, geodesics are defined in section 8, the equation being

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{(\nu\mu)} \frac{dx^\nu}{d\lambda} \frac{dx^\mu}{d\lambda} = 0 \quad (*)$$

where only the symmetric part of $\Gamma^\mu_{\nu\rho}$ contributes:

$$\Gamma^\alpha_{(\nu\mu)} = (\Gamma_{LC})^\alpha_{\nu\mu} - g^\alpha\beta(T^\lambda_{\mu\rho}g_{\lambda\nu} + T^\lambda_{\nu\rho}g_{\lambda\mu}) = (\Gamma_{LC})^\alpha_{\nu\mu} - (T^\alpha_{\mu\nu} + T^\alpha_{\nu\mu}) = (\Gamma_{LC})^\alpha_{\nu\mu} - 2T^\alpha_{(\mu\nu)}$$

with $g^{\delta\gamma}T^\alpha_{\beta\gamma}g_{\alpha\sigma} = g^{\delta\gamma}T^\beta_{\gamma\sigma} = T^\delta_{\beta\gamma} \delta_{\sigma}$; notice that the covariant form of the torsion tensor, $T^\alpha_{\beta\gamma}$, is antisymmetric in the first two indices: $T^\alpha_{\beta\gamma} = -T^\alpha_{\gamma\beta}$. (With this definition of $T^\alpha_{\alpha\beta\gamma}$, the covariant form of the contortion tensor is

$$K^\alpha_{\mu\nu\rho} = g_{\rho\alpha}K^\alpha_{\mu\nu} = T^\alpha_{\mu\nu\rho} - T^\alpha_{\nu\rho\mu} + T^\rho_{\mu\nu}.$$
which is antisymmetric in the last two indices i.e. $K_{\mu\nu\rho} = -K_{\rho\mu\nu}$.)

On the other hand, the equation of geodesics defined as extremals of arc-length:

$$0 = \delta \int ds = \delta \int (g_{\mu\nu}dx^\mu dx^\nu)^{\frac{1}{2}},$$

turns out to be (Carroll, 2004, pp.106-109)

$$\frac{d^2x^\alpha}{d\lambda^2} + (\Gamma_{\alpha\beta\gamma})^\alpha_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0. \tag{**}$$

Then, for the definitions (*) and (**) to coincide, $T_{(\mu \alpha \nu)}$ must vanish i.e. $T^\alpha_{\mu \nu} = -T^\nu_{\mu \alpha}$ $\Leftrightarrow$ $g^{\alpha \rho}T_{\mu \rho \nu} = -g^{\alpha \rho}T_{\nu \rho \mu} \Leftrightarrow T_{\mu \sigma \nu} = -T_{\nu \sigma \mu}$ i.e. $T_{\alpha \beta \gamma}$ must be 1-3 antisymmetric; but this implies that $T_{\alpha \beta \gamma}$ is also 2-3 antisymmetric: $T_{\mu \sigma \nu} = -T_{\nu \sigma \mu} = T_{\sigma \nu \mu} = -T_{\mu \nu \sigma}$. Since, by definition, $T_{\mu \rho \nu}$ is antisymmetric in the first two indices, it then turns out to be totally antisymmetric; in $n$ dimensions, its number of independent components is $\binom{n}{3} = n(n-1)(n-2)/6 \equiv N$. Some values are:

$$\begin{array}{cccc}
  n & 2 & 3 & 4 \\
  N & 0 & 1 & 4 \\
\end{array}$$

The set of allowed non-vanishing components of the torsion tensor still leads to “physical” (geometrical) effects in the sense of section 10. The non-closure of a parallelogram with infinitesimal sides $\epsilon^\mu$ and $\delta^\nu$ is measured by the vector

$$\Delta^\nu = 2T^\mu_{\beta \alpha} \delta^\beta \epsilon^\alpha = T^\mu_{\beta \alpha} (\delta^\beta \epsilon^\alpha - \delta^\alpha \epsilon^\beta).$$

For $n = 4$ its components are:

$$\begin{align*}
\Delta^0 &= T^0_{12}(\delta^1 \epsilon^2 - \delta^2 \epsilon^1) + T^0_{13}(\delta^1 \epsilon^3 - \delta^3 \epsilon^1) + T^0_{23}(\delta^2 \epsilon^3 - \delta^3 \epsilon^2), \\
\Delta^1 &= T^1_{23}(\delta^2 \epsilon^3 - \delta^3 \epsilon^2) + T^1_{02}(\delta^0 \epsilon^2 - \delta^2 \epsilon^0) + T^1_{03}(\delta^0 \epsilon^3 - \delta^3 \epsilon^0), \\
\Delta^2 &= T^2_{03}(\delta^0 \epsilon^3 - \delta^3 \epsilon^0) + T^2_{10}(\delta^1 \epsilon^0 - \delta^0 \epsilon^1) + T^2_{13}(\delta^1 \epsilon^3 - \delta^3 \epsilon^1), \\
\Delta^3 &= T^3_{10}(\delta^1 \epsilon^0 - \delta^0 \epsilon^1) + T^3_{20}(\delta^2 \epsilon^0 - \delta^0 \epsilon^2) + T^3_{12}(\delta^1 \epsilon^2 - \delta^2 \epsilon^1),
\end{align*}$$

which can be distinct from zero.

In summary, the necessary and sufficient condition for erecting a locally inertial coordinate system at a point $x$ in a $U^4$-space, is that the symmetric part of the contortion tensor vanish up to terms of order $(x^\mu)^2$, where $x^\mu(x) = 0$. (Socolovsky, 2010.)

In the above sense, the weak equivalence principle, which only refers to the free motion of point-like and therefore classical particles, still holds in a $U^4$-space ($U^4 = (M^4, g, \Gamma)$ with $\Gamma$ a metric connection). (A similar result was recently found by Fabbri (Fabbri, 2011).)

32. Einstein-Cartan equations

(We owe this derivation to L. Fabbri, 2010.)

A. Pure gravitational case ("vacuum")

We start from the curvature 2-form $R^a_{\beta \mu \nu}$ with components $R^a_{\beta \mu \nu}$ given in section 29; we define the Ricci tensor

$$R_{\beta \mu \nu} := R^a_{\beta \mu \nu} e_a^\mu$$
and the Ricci scalar

\[ R = R_{\mu\nu} \eta^{\nu\rho} e^\rho e^\nu. \]

Then

\[ R = \eta^{\nu\rho} R_{\mu\nu} e^\mu e^\nu = R(\omega^{a}_{b}, e^{c}). \]

The gravitational action is given by the Einstein-Hilbert lagrangian density \( eR \),

\[ S_{G} = \int d^{4}x \ eR \]

with \( e = \sqrt{-\det g_{\mu\nu}} \equiv \sqrt{\gamma} \).

A.1. Variation with respect to the spin connection: \( \delta = \delta_{\omega} \) (\( \rightarrow \) Cartan equations)

Varying \( R^{a}_{b\mu\nu} \) w.r.t. \( \omega \), using \( \delta(\omega_{c d}) = \partial_{c}(\omega_{d}) \), and adding and subtracting \( (\omega^{a}_{b})\Gamma^{\rho}_{\mu\nu} \) one obtains

\[ \delta R^{a}_{b\mu\nu} = D_{\mu}(\delta\omega^{a}_{\nu\rho}) - D_{\nu}(\delta\omega^{a}_{\mu\rho}) + 2\delta\omega^{a}_{\mu\rho} T^{\rho}_{\mu\nu} \]

where

\[ D_{\mu}(\delta\omega^{a}_{\nu\rho}) = \partial_{\mu}(\delta\omega^{a}_{\nu\rho}) - \omega^{e}_{\nu\rho}(\delta\omega^{a}_{\mu\rho}) + \omega^{a}_{\mu\rho}(\delta\omega^{e}_{\nu\rho}) - \Gamma^{\rho}_{\nu\rho}(\delta\omega^{a}_{\mu\rho}) \]

and

\[ D_{\nu}(\delta\omega^{a}_{\mu\rho}) = \partial_{\nu}(\delta\omega^{a}_{\mu\rho}) - \omega^{e}_{\nu\rho}(\delta\omega^{a}_{\mu\rho}) + \omega^{a}_{\nu\rho}(\delta\omega^{e}_{\mu\rho}) - \Gamma^{\rho}_{\nu\rho}(\delta\omega^{a}_{\mu\rho}) \]

are covariant derivatives since \( \delta\omega^{a}_{\mu\rho} \) is a tensor (\( \omega^{a}_{\sigma\tau} \) is a connection, not a tensor, but the difference of two connections is a tensor). Then,

\[ \delta_{\omega} S_{G} = \int d^{4}x \ e e_{\alpha} \eta^{\nu\rho} e^{\nu} e^{\rho} \delta R^{a}_{b\mu\nu} = \int d^{4}x \ e(D_{\mu}(\delta\omega^{a}_{\nu\rho}) e_{\mu} \eta^{bc} e_{\nu}) - D_{\nu}(\delta\omega^{a}_{\mu\rho} e_{\nu} \eta^{bc} e_{\nu}) \\
+ 2e_{\alpha} \eta^{bc} e_{\nu} \delta\omega^{a}_{\mu\rho} T^{\rho}_{\mu\nu} \int d^{4}x \ e(D_{\mu} V^{\mu}) + 2(\delta\omega^{a}_{\mu\rho} T^{\rho}_{\mu\nu}) \]

with \( V^{\mu} \) the 4-vector given by

\[ V^{\mu} = (\delta\omega^{a}_{\mu})(e_{\mu} e^{\nu} e_{\nu} - e_{\nu} e^{\nu} e_{\nu}), \]

where we have used the Leibnitz rule for \( D_{\alpha}, D_{\alpha} e_{\mu} = 0, D_{\mu} \eta^{bc} = 0 \), and raised Lorentz indices with \( \eta^{ab} \).

For \( D_{\mu} V^{\nu} \) one has

\[ D_{\mu} V^{\nu} = (D_{LC})_{\mu} V^{\nu} + K^{\nu}_{\rho\mu} V^{\rho} = \partial_{\mu} V^{\nu} + (\Gamma_{LC})^{\nu}_{\rho\lambda} V^{\lambda} + K^{\nu}_{\rho\mu} V^{\rho}, \]

then

\[ D_{\mu} V^{\mu} = (D_{LC})_{\mu} V^{\mu} + K^{\mu}_{\rho\mu} V^{\rho} = e^{-1}(e V^{\rho})_{,\rho} - 2T^{\lambda}_{\lambda} \]

where we have used \( (\Gamma_{LC})^{\mu}_{\rho\lambda} = \sqrt{-\gamma} \partial_{\lambda} \sqrt{\gamma} \) (appendix C), appendix B, and the definition of the torsion 1-form \( T_{\lambda} = T^{a}_{\lambda} \) (section 9).

Neglecting the surface term \( \int d^{4}x \ (e V^{\rho})_{,\rho} \) one obtains

\[ 0 = \delta_{\omega} S_{G} = \int d^{4}x \ e(-T^{\mu}_{\rho} V^{\mu} + \delta\omega^{a}_{\rho\mu} T^{\rho}_{\mu\nu}) = \int d^{4}x \ e(\delta\omega^{a}_{\nu\rho})(-e_{c} V^{\nu} T_{a} + e_{c} T_{c} + T^{\nu}_{ac}) \]

with \( T_{a} = T^{a}_{\mu} e_{\mu} \), which, due to the arbitrariness of \( \delta\omega^{a}_{\mu\rho} \) leads to the Cartan equations for torsion (in “vacuum”):

\[ T^{\nu}_{ac} + e_{c} V^{\nu} T_{c} - e_{c} T_{a} = 0. \]

Multiplying \( (i) \) by \( e_{\rho} a e_{\sigma} e^{c} \) one obtains the Cartan equations in local coordinates:

\[ T^{\nu}_{\rho\sigma} + \delta^{\nu}_{\rho} T_{\sigma} - \delta^{\nu}_{\sigma} T_{\rho} = 0. \]
Proposition 1: Torsion vanishes.

Proof. Taking the $\nu - \sigma$ trace in (ii) leads to $T_\rho + T_\rho - 4T_\rho = -2T_\rho = 0$; then $T_\rho = 0$ and from (ii) again,

$$T_\rho = 0.$$  \textit{qed}

Note: The above result holds in $n$ dimensions for $n \neq 2$: the $\nu - \sigma$ trace gives $(2 - n)T_\rho = 0$. For $n = 2$, $T_\rho$ is arbitrary with independent components $T_{01} = -T_1$ and $T_{01} = T_0$. Also, notice that (i) (or (iii)) is not a differential equation, but an algebraic one; this is the mathematical expression of the fact that in E-C theory torsion does not propagate.

Proposition 2: Let

$$T_\rho + \delta^\nu_\nu T_\sigma - \delta^\nu_\sigma T_\rho = \kappa S^\nu_\rho$$  \textit{(iii)}

with $S^\nu_\rho = -S^\nu_\sigma$, and $\kappa$ a constant. ((iii) corresponds to a non-vacuum case and will be used in part B.) Then,

$$T_\rho = \kappa(S^\nu_\rho - \frac{1}{2 - n} (\delta^\nu_\nu S_\sigma - \delta^\nu_\sigma S_\rho)).$$  \textit{(iv)}

where $n \neq 2$ is the dimension of the manifold and $S_\mu = S^\mu_\nu$. In particular, for $n = 4$,

$$T_\rho = \kappa(S^\nu_\rho + \frac{1}{2} (\delta^\nu_\nu S_\sigma - \delta^\nu_\sigma S_\rho)).$$  \textit{(v)}

Proof. Again taking the trace $\nu - \sigma$ in (iii), $(2 - n)T_\rho = \kappa S_\rho$, then $T_\rho = \frac{1}{2 - n} S_\rho$ and $T_\rho$ is (iv).  \textit{qed}

For $n = 2$, the unique solution of (iii) is $S^\nu_\rho = 0$: in fact, $\delta^\nu_\nu = 2$ and then $S_\rho = 0$; so $S_0 = S^3_{01} = 0$ and

$$S_0 = S^0_{10} = 0.$$  \textit{A.2. Variation with respect to the tetrads: $\delta = \delta_e$ (Einstein equations)}

Again from $S_G$,

$$\delta_e S_G = \int d^4x (\delta R)e + R\delta e) = \int d^4x (R^a_{\mu
u}\eta^{bc}e((\delta e^a) e^c e^\nu + e^a \delta e^c e^\nu) - R e e^d \delta e^d)$$

$$= \int d^4x (2R^a_{\mu} - \frac{1}{2} R e^a) e \delta e^a = 0$$

where we used $\delta e = -e \delta e^b \delta e^c$ (appendix C), and from the arbitrariness of $\delta e^a$, we obtain the Einstein equations for curvature (in "vacuum"):

$$R^a_{\mu} - \frac{1}{2} R e^a = 0$$  \textit{(vi)}

or

$$G^a_{\mu} = 0$$  \textit{(vii)}

with

$$G^a_{\mu} = R^a_{\mu} - \frac{1}{2} R e^a.$$  \textit{(viii)}

($R^a_{\mu} = \eta^{ac} R_{c\mu}$) Since in vacuum $R = 0$ (section 18), (vi) amounts to

$$R^a_{\mu} = 0.$$  \textit{(ix)}

Of course, multiplying (vi) by $e_a e^\nu$ we obtain Einstein equations in local coordinates (section 20).
In summary, for the pure gravitational case, Einstein theory = Einstein-Cartan theory; this is a consequence of the form of the Einstein-Hilbert action $S_G$. From the form of $R$, gravity has been expressed as an interacting gauge theory (see section 33) between the spin connection $\omega^a{}_b$ and the coframes field $e^a$; both $\omega^a{}_b$ and $e^a$ are pure geometric fields, which live in the frame and coframe bundles $\mathcal{L}_4 \to \mathcal{F}_{\mathcal{L}_4} \to M^4$ and $\mathcal{L}_4 \to \mathcal{F}_{\mathcal{L}_4} \to M^4$ respectively.

B. Minimal coupling to Dirac fields

The Dirac-Einstein action is given by

$$S_{D-E} = k \int d^4x \, e L_{D-E} = k \int d^4x \, e (\frac{i}{2} \bar{\psi}(\gamma^a(Da\psi) - (Da\bar{\psi})\gamma^a\psi - m\bar{\psi}\psi)$$

where

$$Da\psi = (e_a - i \frac{1}{4} \omega_{abc} \sigma^{bc})\psi = e_a^\mu (\partial_\mu - i \frac{1}{4} \omega_{\mu bc} \sigma^{bc})\psi = e_a^\mu Da_\mu \psi$$

and

$$Da\bar{\psi} = \bar{e}_a \bar{\psi} + i \frac{1}{4} \omega_{abc} \bar{\sigma}^{bc} = e_a^\mu (\partial_\mu \bar{\psi} + i \frac{1}{4} \omega_{\mu bc} \bar{\sigma}^{bc}) = e_a^\mu Da_\mu \bar{\psi}$$

are the covariant derivatives of the Dirac field $\psi$ and its conjugate $\bar{\psi} = \bar{\psi}\gamma_0$ with respect to the spin connection, which give the minimal coupling between fermions and gravity; they are obtained through the replacement $d_a \psi \to Da_\mu \psi$. i.e. $e_a^\mu = e_a^\mu \partial_\mu \psi \to Da_\mu \psi$

which amounts to the “comma goes to semicolon” rule for tensors but here adapted to spinor fields. $\sigma^{bc} = \frac{i}{2} [\gamma^b, \gamma^c]$, and the $\gamma^a$’s are the usual numerical (constant) Dirac gamma matrices satisfying $\{\gamma^a, \gamma^b\} = 2i\eta^{ab}I$, $\gamma^0 \dagger = \gamma^0$ and $\gamma^i \dagger = -\gamma^i$. $k = -16\pi \rho \frac{\alpha}{c^6}$ ($-16\pi$ in natural units). Then the action is

$$S_{D-E} = k \int d^4x \, e (\frac{i}{2} \bar{\psi}\gamma^a(\partial_\mu \psi + \frac{1}{8} \omega_{abc}[\gamma^b, \gamma^c]\psi) - \frac{i}{2} (\partial_\mu \bar{\psi} - \frac{1}{8} \omega_{abc} \bar{\psi}[\gamma^b, \gamma^c])\gamma^a\psi - m\bar{\psi}\psi)$$

where $\gamma^a = e_a^\mu \gamma^a = \gamma^a(x)$.

B.1. Variation with respect to the spin connection: $\delta = \delta_\omega$

$$\delta_\omega S_{D-E} = \frac{k}{8} \int d^4x \, e \{\bar{\psi}(\gamma^a, \sigma^{bc})\psi, \delta \omega_{abc}\} = \frac{k}{2} \int d^4x \, e S^{abc} \delta \omega_{abc}$$

with $S^{abc} = e_a^\mu S^{abc}$, where

$$S^{abc} = \frac{1}{4} \bar{\psi} \{\gamma^a, \sigma^{bc}\} \psi$$

is the spin density tensor of the Dirac field. $S^{abc}$ is totally antisymmetric and therefore in 4 dimensions it has 4 independent components: $S^{012}$, $S^{123}$, $S^{230}$ and $S^{301}$.

Combining this result with the corresponding variation for the pure gravitational field (part A), we obtain

$$0 = \delta_\omega (S_G + S_{D-E}) = \int d^4x \, e \delta \omega^{ac} (T^a_c + e_a \nu T^\nu c - e_c \nu T^\nu a + \frac{k}{2} S^\nu_{ac})$$

and therefore

$$T^\nu_{ac} + e_a \nu T^\nu c - e_c \nu T^\nu a = -\frac{k}{2} S^\nu_{ac}$$

the Cartan equation. Multiplying by $e_p^a e_\sigma^c$ one obtains

$$T^\nu_{p\sigma} + \delta^\nu_{p\sigma} T^\nu - \delta^\nu_{\sigma p} T^\nu = -\frac{k}{2} S^\nu_{p\sigma}$$
with
\[ S_{\rho\sigma}^\nu = \frac{1}{4} \bar{\psi} \{ \gamma^\mu, \sigma_{\rho\sigma} \} \psi. \]

From (iii) in part A, with \( \kappa = -\frac{k}{2} \), we obtain the torsion in terms of the spin tensor:
\[ T_{\rho\sigma}^\nu = \frac{8\pi G}{c^4} (S_{\rho\sigma}^\nu + \frac{1}{2}(\delta_\rho^\nu S_\sigma - \delta_\sigma^\nu S_\rho)) \]
with \( S_\rho = S_{\rho\mu}^\mu \). In natural units, \( G = c = \hbar = 1 \) and so \( T_{\rho\sigma}^\nu = 8\pi (S_{\rho\sigma}^\nu + \frac{1}{2}(\delta_\rho^\nu S_\sigma - \delta_\sigma^\nu S_\rho)) \).

B.2. Variation with respect to the tetrads: \( \delta = \delta e \)

From \( S_{D-E} - E \) and using appendix C for \( \delta e \), one obtains
\[ \delta e S_{D-E} = k \int d^4 x \left( 2 R^a_\mu - Re^a_\mu + \frac{i}{2} (\bar{\psi} \gamma^a (D_\mu \psi) - (D_\mu \bar{\psi}) \gamma^a \psi) \right) \delta e_a^\mu, \]
and from the arbitrariness of \( \delta e_a^\mu \),
\[ R^a_\mu - \frac{1}{2} Re^a_\mu = -\frac{k}{2} T^a_\mu \] (\*),
with
\[ T^a_\mu = \frac{i}{2} (\bar{\psi} \gamma^a (D_\mu \psi) - (D_\mu \bar{\psi}) \gamma^a \psi) \]
the energy-momentum tensor of the Dirac field. Multiplying (*) by \( e_a^\nu \) one obtains
\[ R^\nu_\mu - \frac{1}{2} R\delta^\nu_\mu = -\frac{k}{2} T^\nu_\mu \text{ or } R_{\lambda\mu} - \frac{1}{2} R g_{\lambda\mu} = -\frac{k}{2} T_{\lambda\mu}, \]
the Einstein equation.

Note: For \( L_{D-E} \) one has
\[ L_{D-E} = e_a^\mu T^a_\mu - m \bar{\psi} \psi \]
i.e. \( T^a_\mu \) couples to the tetrad. On the other hand,
\[ T^a_\mu = \theta^a_\mu + \omega_{abc} S^abc \]
where
\[ \theta^a_\mu = \frac{i}{2} (\bar{\psi} \gamma^a \partial_\mu \psi - (\partial_\mu \bar{\psi}) \gamma^a \psi) \]
is the canonical energy-momentum tensor of the Dirac field. Then,
\[ L_{D-E} = e_a^\mu \theta^a_\mu + e_a^\mu \omega_{abc} S^abc - m \bar{\psi} \psi = e_a^\mu \theta^a_\mu + \omega_{abc} S^abc - m \bar{\psi} \psi. \]
So, $\theta^a{}_{\mu}$ couples to the tetrad while spin couples to the spin connection; moreover, since $S_{abc}$ is totally antisymmetric, the Dirac field only interacts with the totally antisymmetric part of the connection.

### 33. Lorentz gauge invariance of Einstein and Einstein-Cartan theories

Under Lorentz transformations $h_a{}^b(x)$ in the tangent, cotangent and, in general, tensor spaces at each point $x$ of the manifold $M^4$ (for simplicity we restrict the discussion to four dimensions), the frames $e_a$ (or $e_a{}^\mu$), coframes $e^a$ (or $e^a{}_{\mu}$) and spin connection $\omega^a{}_{bc}$ (or $\omega^a{}_{\mu b}$) transform as indicated in sections 28.8 and 28.9. The volume element $d^4x$ in $SG$ (or $SD_E$) is invariant; in fact,

$$g_{\mu \nu}(x) = \eta_{ab} e^a{}_{\mu}(x)e^b{}_{\nu}(x) = \eta_{ab} e^a{}_{\mu} h^{-1a} e^d{}_{\nu} h^{-1b} = e^a{}_{\mu} e^d{}_{\nu} h^{-1a} h^{-1b} = e^a{}_{\mu} e^d{}_{\nu} \eta_{cd} = g_{\mu \nu}'(x)$$

and then $\text{det}(-g_{\mu \nu}(x)) = \text{det}(-g_{\mu \nu}'(x))$ i.e. $e'(x) = e(x)$. Then $d^4x = d^4x'$ $e'$ since $x'' = x'$.

#### A. Pure gravitational case ("vacuum")

Though $\omega^a{}_{bc} = \omega^a{}_{\mu a b} dx^\mu$ is a connection and transforms as

$$\omega^a{}_{bc} = h^c{}_{d} \omega^a{}_{d b} h^{-1c} + (dh_a{}^d)h^{-1c},$$

its curvature $R^a{}_{bc}$ is a Lorentz tensor:

$$R^a{}_{bc} = h_b{}^d h^{-1a} R^c{}_{d} \in \Omega^2(L^1_1)$$

(see table in section 29); then the Ricci scalar is gauge invariant:

$$R = R^a{}_{bc} e_a\eta^{bc} e_c = h_b{}^d R^{ce} d h^{-1a} h_a{}^f h^{-1} h_g{}^l e_l = R^{ce} e^f h^{-1} h^{-1} e_l = R^{ce} e^f h^{-1} h^{-1} e_l = R' \in \Omega^0(L^1_1)$$

(An explicit proof of the gauge invariance of $R$ is given in Appendix D.)

**Remark:** In this section, the tetrads $e_a$ (or their duals $e^a$) are **not** gauge potentials; only the spin connection $\omega^a{}_{bc}$ is a gauge potential, related to the Lorentz group and therefore to space-time rotations (at each point of the manifold). So, if only the Lorentz group is gauged, the metric $g = \eta_{ab} e^a \otimes e^b$ (see eq. (b) in section 28.6) does not come from the connection.

#### B. Minimal coupling to Dirac field

See O’Raifeartaigh (1997), Ch. 5, §3, pp 115-116.

### 34. Poincaré gauge invariance of Einstein and Einstein-Cartan theories

We need the concepts of affine structures: spaces, bundles and connections (§3, chapter III, Kobayashi and Nomizu, 1963).

The general linear group $GL_n(\mathbb{R})$ in $n$ real dimensions acts from the left on the vector space $\mathbb{R}^n$ by simple matrix multiplication: $(g, \lambda) \mapsto g\lambda$, which is a linear operation.

The **affine space**

$$\mathbb{A}^n = \{ \begin{pmatrix} \lambda \\ 1 \end{pmatrix} : \lambda \in \mathbb{R}^n \}$$

is acted by the **affine group** in $n$ real dimensions

$$GA_n(\mathbb{R}) = \{ \begin{pmatrix} g & \xi \\ 0 & 1 \end{pmatrix} : g \in GL_n(\mathbb{R}), \; \xi \in \mathbb{R}^n \}$$
as follows:

\[ GA_n(\mathbb{R}) \times \mathbb{A}^n \to \mathbb{A}^n, \quad \left( \begin{pmatrix} g & \xi \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \lambda \\ 1 \end{pmatrix} \right) \mapsto \begin{pmatrix} g\lambda + \xi \\ 1 \end{pmatrix}, \]

Then, one has the following diagram of short exact sequences (s.e.s.’s) of groups and group homomorphisms:

\[
\begin{array}{ccccccccc}
0 & \to & \mathbb{R}^n & \xrightarrow{\mu} & GA_n(\mathbb{R}) & \xrightarrow{\nu} & GL_n(\mathbb{R}) & \to & 0 \\
 & & Id \uparrow & & \uparrow \iota & & \uparrow \iota & & \\
0 & \to & \mathbb{R}^n & \xrightarrow{\nu|} & P_n & \xrightarrow{\nu|} & \mathcal{L}_n & \to & 0
\end{array}
\]

with \( \mu(\xi) = \begin{pmatrix} I_n & \xi \\ 0 & 1 \end{pmatrix} \) and \( \nu\left( \begin{pmatrix} g & \lambda \\ 0 & 1 \end{pmatrix} \right) = g \cdot \mu(\xi) \). \( \nu \) is 1-1, \( \mu(\xi) \) is onto, and \( \ker(\nu) = \text{Im}(\mu) = \{ \begin{pmatrix} I_n & \xi \\ 0 & 1 \end{pmatrix} \mid \xi \in \mathbb{R}^n \} \).

As a consequence, the factorization of an element of \( GA_n(\mathbb{R}) \) (\( P_n \)) in terms of elements of \( \mathbb{R}^n \) and \( GL_n(\mathbb{R}) \) (\( \mathcal{L}_n \)) is unique: \( \begin{pmatrix} g & \xi \\ 0 & 1 \end{pmatrix} = \mu(\xi) \rho(g) \) (or \( \mu(\xi) \rho(g) \)). The dimensions of \( GA_n(\mathbb{R}), GL_n(\mathbb{R}), P_n \) and \( \mathcal{L}_n \) are, respectively, \( n + n^2, n^2, \frac{n(n+1)}{2} \) and \( \frac{n(n-1)}{2} \) (20, 16, 10 and 6 for \( n = 4 \)).

The above s.e.s.’s pass to s.e.s.’s of the corresponding Lie algebras:

\[
\begin{array}{ccccccc}
0 & \to & \mathbb{R}^n & \xrightarrow{\tilde{\mu}} & g\mathfrak{a}_n(\mathbb{R}) & \xrightarrow{\tilde{\nu}} & g\mathfrak{l}_n(\mathbb{R}) & \to & 0 \\
 & & Id \uparrow & & \uparrow \iota & & \uparrow \iota & & \\
0 & \to & \mathbb{R}^n & \xrightarrow{\tilde{\nu}|} & \mathfrak{p}_n & \xrightarrow{\tilde{\nu}|} & \mathfrak{l}_n & \to & 0
\end{array}
\]

with \( g\mathfrak{l}_n(\mathbb{R}) = \mathbb{R}(n), g\mathfrak{a}_n(\mathbb{R}) = \mathbb{R}^n \circ g\mathfrak{l}_n(\mathbb{R}) \) with Lie product

\[
(\lambda', R')(\lambda, R) = (R'\lambda - R\lambda', [R', R]),
\]

where \([R', R]\) is the Lie product in \( g\mathfrak{l}_n(\mathbb{R}) \) and \([\lambda', \lambda] = 0 \) in \( \mathbb{R}^n, \tilde{\mu}(\xi) = (\xi, 0), \tilde{\nu}(\xi, R) = R, \) and \( \tilde{\rho}(R) = (0, R) \). \( \tilde{\mu}, \tilde{\nu} \) and \( \tilde{\rho} \) (and their corresponding restrictions \( \tilde{\mu}|, \tilde{\nu}| \) and \( \tilde{\rho}| \)) are Lie algebra homomorphisms, with \( \tilde{\nu} \circ \tilde{\rho} = \text{Id}_{g\mathfrak{l}_n(\mathbb{R})} \) and \( \tilde{\nu}| \circ \tilde{\rho}| = \text{Id}_{\mathfrak{l}_n} \).

Let us denote:

- \( M^n \): \( n \)-dimensional differentiable manifold
- \( \mathcal{F}_{M^n} \): frame bundle of \( M^n \): \( GL_n \to FM^n \xrightarrow{\pi_E} M^n \)
- \( \mathcal{A}_{M^n} \): affine frame bundle of \( M^n \): \( GA_n \to AM^n \xrightarrow{\pi_A} M^n \)

\( GL_n \): general linear group in \( n \) dimensions (section 24), \( \dim \mathbb{R}GL_n = n^2 \)
$GA_n$: general affine group in $n$ dimensions, $\dim \mathbb{R}GA_n = n + n^2$

$\mathcal{F}^L_{M^n}$: bundle of Lorentz frames of $M^n$: $\mathcal{L}_n \to F^L M^n \xrightarrow{\pi_F} M^n$ (section 25)

$\mathcal{F}^P_{M^n}$: bundle of Poincaré frames of $M^n$: $\mathcal{P}_n \to A^P M^n \xrightarrow{\pi_A} M^n$

One has the following diagram of bundle homomorphisms:

\[
\begin{array}{cccccccc}
AM^n \times GA_n & \xleftarrow{\times} & AP^n \times \mathcal{P}_n & \xrightarrow{\beta \times \nu} & F^L M^n \times \mathcal{L}_n & \xleftarrow{\times} & FM^n \times GL_n \\
\psi_A \downarrow & & \psi_A \downarrow & & \downarrow \psi_F & & \downarrow \psi_F \\
AM^n & \xleftarrow{\beta} & AP^n & \xrightarrow{\dot{\gamma}} & F^L M^n & \xrightarrow{\iota} & FM^n \\
\pi_A \downarrow & & \pi_A \downarrow & & \downarrow \pi_F & & \downarrow \pi_F \\
M^n & \xrightarrow{Id} & M^n & \xrightarrow{Id} & M^n & & \\
\end{array}
\]

where $\beta$ is the bundle homomorphism

\[
\begin{array}{ccc}
AM^n \times GA_n & \xrightarrow{\times} & FM^n \times GL_n \\
\psi_A \downarrow & & \downarrow \psi_F \\
AM^n & \xrightarrow{\beta} & FM^n \\
\pi_A \downarrow & & \downarrow \pi_F \\
M^n & \xrightarrow{Id} & M^n \\
\end{array}
\]

between the bundle of affine frames and the bundle of linear frames over $M^n$, with

\[
\beta(x,(v_x,r_x)) = (x,r_x), \quad \gamma(x,r_x) = (x,(0_x,r_x)), \quad 0 \in T_x M^n,
\]

\[
AM^n = \bigcup_{x \in M^n} \{x\} \times AM^n, \quad AM^n_x = \{(v_x,r_x), \; v_x \in A_x M^n, \; r_x \in F_x\},
\]

where $A_x M^n$ is the tangent space at $x$ considered as an affine space (Appendix E); and

\[
\psi_A((x,(v_x,r_x)),(\xi,g)) = (x,(v_x + r_x \xi, r_x g))
\]

is the action of $GA_n$ on $AM^n$, with $r_x \xi = \sum_{a=1}^n e_{ax} \xi^a$. $\pi|_F, \ldots, \psi|_A, \ldots, \beta|, \ldots$, etc. are restrictions; in particular $\beta|(a) = \beta(a)$ and $\gamma|(e) = \gamma(e)$.

A **general affine connection** (g.a.c.) on $M^n$ is a connection in the bundle of affine frames $AM^n$. If $\omega_A$ is the 1-form of the connection, then

\[
\omega_A \in \Gamma(T^*AM^n \otimes ga_n)
\]

i.e.

\[
\omega_A : AM^n \to T^*AM^n \otimes ga_n, \quad (v_x,r_x) \mapsto (\{v_x,r_x\}, \omega_A(v_x,r_x)), \quad \omega_A(v_x,r_x) : T_{(v_x,r_x)}AM^n \to ga_n,
\]

\[
V_{(v_x,r_x)} \mapsto \omega_A(v_x,r_x)(V_{(v_x,r_x)}) = (\lambda,R) \equiv \lambda \circ R \in \mathbb{R}^n \otimes gl_n(\mathbb{R}).
\]

Obviously, $\omega_A$ obeys the usual axioms of connections.
From the smoothness of \( \gamma \), the pull-back \( \gamma^*(\omega_A) \) is a \( g_{\mathbb{A}} \)-valued 1-form on \( FM^n \):

\[
\gamma^*(\omega_A) = \psi \circ \omega_F,
\]

where \( \omega_F \) is a connection on \( FM^n \), and \( \psi \) is an \( \mathbb{R}^n \)-valued 1-form. There is a 1-1 correspondence between g.a.c.'s on \( AM^n \) and pairs \((\omega_F, \psi)\) on \( FM^n \):

\[
\{\omega_A\}_{g.a.c.} \leftrightarrow \{\omega_F, \psi\}.
\]

\( \omega_A \) is an affine connection (a.c.) on \( M^n \) if \( \psi \) is the soldering (canonical) form \( \theta \) on \( FM^n \) (section 26). Then, if \( \omega_A \) is an a.c. on \( AM^n \),

\[
\gamma^*(\omega_A) = \theta \circ \omega_F
\]

where \( \omega_F \) is a connection on \( FM^n \). There is then a 1-1 correspondence

\[
\{\omega_A\}_{a.c.} \leftrightarrow \{\omega_F\},
\]

since \( \theta \) is fixed. Also, if \( \Omega_A \) is the curvature of \( \omega_A \), then

\[
\gamma^*(\Omega_A) = D_{\omega_F} \theta \circ \Omega_F.
\]

But \( D_{\omega_F} \theta = T_F \) the torsion of the connection \( \omega_F \) on \( FM^n \): in fact, from section 26, \( \theta^\mu = (X^{-1})_\mu^\nu dx^\nu \), then \( \theta^\mu = e\nu^\mu \theta^\nu = e\mu^\alpha (X^{-1})_\nu^\mu dx^\nu = (X^{-1})_\nu^\alpha dx^\nu = e_\alpha dx^\nu = e_\alpha, \) so \( D_{\omega_F} \theta^\mu = d\theta^\mu + \omega_F e^\mu = T_F^\mu \). Therefore,

\[
\gamma^*(\Omega_A) = T_F \circ \Omega_F.
\]

The facts that \( AP^n M^n \) is a subbundle of \( AM^n \) and \( FL M^n \) is a subbundle of \( FM^n \), with structure groups and Lie algebras the corresponding subgroups and sub-Lie algebras, and the existence of the restrictions \( \beta \mid : AP^n M^n \to FL M^n \) and \( \gamma \mid : FL M^n \to AP^n M^n \), allow us to obtain similar conclusions for the relations between affine connections on the Poincaré bundle and linear connections on the Lorentz bundle:

There is a 1-1 correspondence between affine Poincaré connections \( \omega_P \) on \( AP^n M^n \) and Lorentz connections on \( FL M^n \):

\[
\{\omega_P\} \leftrightarrow \{\omega_L\}
\]

with

\[
\gamma|\ast(\omega_P) = \theta_L \circ \omega_L
\]

where \( \theta_L = \theta_{FM^n|FL M^n} \) is the canonical form on \( FL M^n \). Also,

\[
\gamma|\ast(\Omega_P) = D_{\omega_L} \theta_L \circ \Omega_L = T_L \circ \Omega_L.
\]

So, there is a 1-1 correspondence between curvatures of affine connections on \( AP^n M^n \) and torsion and curvature pairs on \( FL M^n \):

\[
\{\Omega_P\} \leftrightarrow \{(T_L, \Omega_L)\}.
\]

For pure gravity governed by the Einstein-Hilbert action, \( T_L = 0 \), as it was shown in section 32.

The Poincaré gauge invariance of G.R. and E-C theory has been discussed by several authors (Hayashi and Shirafuji, 1980; Ali et al, 2009; Gronwald and Hehl, 1996; Hehl, 1998). To explicitly prove it, we have to consider as gauge transformations both the Lorentz part, already studied in the previous section, and the translational part. This has to be done using the bundle of Poincaré frames \( F_4 : \mathcal{P}_4 \to AP M^4 \overset{\pi_p}{\to} M^4 (\pi_P = \pi_A) \), (D’Olivo and Socolovsky, 2011). The action of \( \mathcal{P}_4 \) over on \( AP M^4 \) is given by

\[
\psi_P : AP M^4 \times \mathcal{P}_4 \to AP M^4, \ (\psi_P = \psi_A), \ \psi_P((x, (v_x, r_x)), (\xi, h)) \equiv (x, (v_x, r_x))(\xi, h) = (x, (v_x + r_x \xi, r_x h))
\]
\( = (x, (v'_x, r'_x)) \), where \( r_x = (e_{ax}) \), \( a = 1, 2, 3, 4 \), is a Lorentz frame, \( h \in \mathcal{L}_4 \), and \( \xi \in \mathbb{R}^4 \cong \mathbb{R}^{1,3} \) is a Poincaré gauge translation. For a pure translation, \( h = I_L \) i.e. \( h^a_b = \delta^a_b \) and therefore

\[
(x, (v_x, r_x))(\xi, I_L) = (x, (v_x + r_x \xi, r_x I_L)) = (x, (v_x + r_x \xi, r_x))
\]
i.e.

\[
r'_x = r_x.
\]

Therefore \( e'_{ax} = e_{ax} \), \( a = 1, 2, 3, 4 \), and then, from (c) or (c') in section 28.9,

\[
\omega^a_{\mu b} = \omega^a_{\mu b}
\]
since \( \Gamma^\mu_{\nu\rho} = (\Gamma_{LC})^\mu_{\nu\rho} + K^\mu_{\nu\rho} \) remains unchanged (in the case of pure gravity \( K^\mu_{\nu\rho} = 0 \)), and so the coordinate Ricci scalar \( R \) of section 32.A is also a gauge scalar and \( S_G \) is invariant.

The Poincaré bundle extends the symmetry group of GR and E-C theory to the semidirect sum

\[
G_{GR} = \mathcal{P}_4 \odot \mathcal{D}
\]
cf. 28.11, with composition law

\[
((\xi', h'), g')((\xi, h), g) = ((\xi', h')g'((\xi, h)g'^{-1}), g'g).
\]

The left action of \( \mathcal{D} \) on \( \mathcal{P}_4 \) is given by the commutative diagram

\[
\begin{array}{ccc}
A^\mu M^4 & \xrightarrow{(\xi, h)} & A^\mu M^4 \\
g \downarrow & & g \downarrow \\
A^\mu M^4 & \xrightarrow{(\xi', h')} & A^\mu M^4
\end{array}
\]

with

\[
g : A^\mu M^4 \rightarrow A^\mu M^4, \quad (x, (v^\mu_x \frac{\partial}{\partial x^\mu}|x, (e_{ax}^\nu \frac{\partial}{\partial x^\nu}|x))) \mapsto (x, (v'^\mu_x \frac{\partial}{\partial x^\mu}|x, (e^{ax}_{ax}^\nu \frac{\partial}{\partial x^\nu}))),
\]

where \( v'^\mu_x = \frac{\partial x^{a\mu}}{\partial x^{a\nu}}|x v^a_x \) and \( e^{ax}_{ax}^\nu = \frac{\partial x^{\alpha}}{\partial x^{\beta}}|x e_{ax}^\beta \).

In section 29, following Hehl (Hehl, 1985; Hehl et al, 1976; Hammond, 2002), we called \( e^a^\nu^\nu's \) the translational gravitational gauge potentials. This is not, however, strictly correct, since the \( e^a^\nu^\nu's \) or their duals, the tetrad fields \( e_a = e_a^\nu \frac{\partial}{\partial x^\nu} \), are not connections, but tensors in both their world (\( \mu \)) and internal (\( a \)) indices (Hayashi, 1977; Leston, 2008; Leston and Socolovsky, 2011). The translational potentials \( B^a_\mu \) (in fact their inverses \( B^a_\mu \)) are given by the 1-form fields locally defined as follows (Hayashi and Nakano, 1967; Aldrovandi and Pereira, 2007):

\[
B^a_\mu = e^a_\mu - \frac{\partial v^a_\mu}{\partial x^\mu} \quad \text{or} \quad B^a_\mu = e^a_\mu - d v^a_\mu,
\]

where \( v_x = \sum_{a=0}^3 v^a_x e_{ax} \in A_2 M^4 \); the \( v^a_x 's \) are here considered the coordinates of the tangent space at \( x \).

The transformation properties of the \( B^a_\mu 's \) are the following:

Internal Lorentz:

\[
B'^a_\mu = h^a_b B^b_\mu - \partial_\mu (h^a_b) v^b_x \quad \text{or} \quad B'^a_\mu = h^a_b B^b_\mu - (dh^a_\mu) v^b_x.
\]

Proof: \( B'^a_\mu = e'^a_\mu - \frac{\partial v'^a_\mu}{\partial x^\mu}, \) with \( e'^a_\mu = h^a_b e^b_\mu \) and \( v'^a_x = h^a_b v^b_x \), then \( B'^a_\mu = h^a_b e^b_\mu - \frac{\partial (h^a_b v^b_x)}{\partial x^\mu} \).
General coordinate transformations:

\[ B^a_{\mu} = \frac{\partial x^\nu}{\partial x^{\prime\nu}} B^a_{\nu}. \]

Proof: \( \epsilon^a_{\mu} = \frac{\partial x^\nu}{\partial x^{\prime\nu}} \epsilon^a_{\nu} \) i.e. \( \epsilon^a_{\mu} \) is a 1-form, and \( \frac{\partial}{\partial x^{\nu}} = \frac{\partial x^{\prime\nu}}{\partial x^{\nu}} \frac{\partial}{\partial x^{\prime\nu}}. \)

Internal translations:

\[ B^a_{\mu} = B^a_{\mu} - \partial_\mu x^a \text{ or } B^a_{\mu} = B^a - dx^a. \]

Proof: \( B^a_{\mu} = \epsilon^a_{\mu} - \frac{\partial x^a}{\partial x^{\prime\nu}} = \epsilon^a_{\mu} - \frac{\partial x^a}{\partial x^{\nu}} = B^a_{\mu} - \frac{\partial (x^a - x^a)}{\partial x^{\nu}} = B^a_{\mu} - \frac{\partial x^a}{\partial x^{\nu}}. \)

Then, \( B = B_{\mu} dx^\mu = B^a_{\mu} dx^\mu b_a \), where \( b_a, a = 0, 1, 2, 3 \), is the canonical basis of \( \mathbb{R}^4 \), is the connection 1-form corresponding to the translations.

Remark: The local \((x^\nu)\) dependence of the internal Lorentz and translational transformations is a consequence of the general definition of a gauge transformation in fibre bundle theory (Appendix F).

In terms of the \( B^a_{\mu} \) fields and the spin connection, the Ricci scalar in section 32 is given by

\[ R = \frac{\partial \epsilon^a_{\mu}}{\partial x^{\nu}} \frac{\partial \epsilon^b_{\nu}}{\partial x^{\mu}} + \frac{\partial \epsilon^a_{\mu}}{\partial x^{\nu}} B^b_{\mu} + \partial \epsilon^a_{\mu} B^b_{\mu} + B^a_{\mu} B^b_{\mu} (\partial \omega^\nu_{\alpha\beta} - \partial \omega^\nu_{\alpha\beta} - \partial \omega^\nu_{\beta\alpha} - \partial \omega^\nu_{\alpha\beta} + \omega^\nu_{\alpha\gamma} \omega^\gamma_{\beta\alpha} - \omega^\nu_{\alpha\gamma} \omega^\gamma_{\beta\alpha}). \] (41)

If one intends to use this Lagrangian density as describing a \((B^a_{\mu}, \omega^\nu_{\alpha\beta})\) (or \((\epsilon^a_{\mu}, \omega^\nu_{\alpha\beta})\)) interaction (Randono, 2010), then immediately faces the problem that the \( B^a_{\mu} \) (or \( \epsilon^a_{\mu} \)) does not have a free part (in particular a kinematical part), since all its powers are multiplied by \( \omega^\nu \)’s or \( \partial \omega^\nu \)’s. So an interpretation in terms of fields interaction seems difficult, and may be, impossible.

35. Torsion and gauge invariance

It is well known the problem of the violation, in the presence of torsion, of the local gauge invariance of theories like Maxwell and Yang-Mills due to the straightforward application of the minimal coupling procedure to introduce the interaction with the gauge fields: the “comma goes to semicolon” rule. In fact, as we shall show below, the “prohibition” of this procedure should only be applied to the definition of the field strengths \( F \), as emphasized by Hammond (Hammond, 2002). For simplicity of the presentation we shall restrict to the abelian case.

In special relativity, for the field strength in terms of the gauge potential one has \( F = dA = d(A_\rho dx^\rho) = (\partial_\nu A_\rho) dx^\mu \wedge dx^\nu = \frac{1}{2} (\partial_\mu A_\nu - \partial_\nu A_\mu) dx^\mu \wedge dx^\nu = F_{\mu\nu} dx^\mu \wedge dx^\nu \), which is clearly gauge independent: \( F(A) = F(A + d\lambda) \). Replacing \( \partial_\mu \) by \( D_\mu \) one obtains

\[ F_{\mu\nu} \rightarrow f_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu = (\partial_\mu - \Gamma^\rho_{\mu\nu} A_\rho) - (\partial_\nu A_\mu - \Gamma^\rho_{\nu\mu} A_\rho) = F_{\mu\nu} - 2\Gamma^\rho_{[\mu\nu]} A_\rho = F_{\mu\nu} - 2T^\rho_{[\mu\nu]} A_\rho. \] (*

When torsion vanishes, \( f_{\mu\nu} = F_{\mu\nu} \) i.e. \( A_{\nu\rho} - A_{\rho\nu} = A_{\nu\rho} - A_{\rho\nu} \); when \( T^\rho_{[\mu\nu]} \neq 0 \), \( f_{\mu\nu} \neq F_{\mu\nu} \) and moreover, \( f_{\mu\nu} \) is not gauge invariant: if \( A_\mu \rightarrow A_\mu + \partial_\mu \lambda \), then \( f_{\mu\nu} \rightarrow f'_{\mu\nu} \) with

\[ f'_{\mu\nu} - f_{\mu\nu} = \delta_\rho_{\mu\nu} (f_{\mu\nu}) = -2T^\rho_{[\mu\nu]} \partial_\rho \lambda. \] (**

The equations (*) and (**) show that the acquired dependence on torsion of the classical electric and magnetic fields, also depends on the chosen gauge (by the presence of \( \lambda \)), what is inadmissible.

At this point, we criticize the “solution” given by some authors (Hehl et al, 1976; de Sabbata, 1997), which consists in the assertion that torsion does not couple to the gauge field. This statement would have sense if also the Levi-Civita part of the connection would not couple, since both \( \Gamma^\rho_{[\mu\nu]} \) and torsion “come
together” in the sum \( \Gamma = \Gamma_{LC} + K \) (appendix B), where torsion is the antisymmetric part of the contortion \( K \). However, \( \Gamma_{LC} \) does couple. Moreover, de Sabbata (de Sabbata, 1997) shows that at the microscopic quantum level photons couple to torsion in a gauge invariant way through virtual pairs \( e^+e^- \) creation. There is no reason to expect that in the transit to the classical limit the coupling should disappear; though, as we shall see below, partly due to the absence of the intermediate fermion field, gauge invariance breaks down.

A partial solution to this problem has been given by Benn, Dereli and Tucker (Benn et al, 1980), leaving, as we show below, \( F = dA \) with \( F' = dA' = F \) if \( A' = A + d\lambda \) in a completely natural way.

Let \( A = A_\mu dx^\mu = A_a e^a \) be the connection 1-form of the electromagnetic field, with \( A_a = e^a_\mu A_\mu \) and \( e^a = e^a_\mu dx^\mu \). If \( \omega^a_{b\mu} = \omega^a_{\mu b} dx^\mu \) is the spin connection with \( \omega_{cb} = \eta_{ca} \omega^a_{b} = -\omega_{bc} \), then the exterior covariant derivative of \( A_a \) with respect to \( \omega^a_{b} \) is given by

\[
DA_a = dA_a - \omega^b_a A_b
\]

with \( dA_a = dx^\mu \partial_\mu A_a \) and \( \omega^b_a A_b = A_b \omega^b_\mu dx^\mu \). \((***)\) gives the minimal coupling of the electromagnetic connection with the space-time connection, i.e. \( dA_a \stackrel{m.c.}{\rightarrow} DA_a \).

Exterior multiplication with \( e^a \) gives

\[
DA_a \wedge e^a = (dA_a - \omega^b_a A_b) \wedge e^a = (dA_a) \wedge e^a - A_b \omega^b_a \wedge e^a,
\]

and using the expression for torsion \( T^a = de^a + \omega^a_\mu e^b \) (section 29), we obtain

\[
DA_a \wedge e^a = (dA_a) \wedge e^a - A_b (T^b - de^b) = (dA_a) \wedge e^a - A_b T^b + A_b de^b
\]

i.e.

\[
DA_a \wedge e^a + A_b T^b = (dA_a) \wedge e^a + A_b de^a = d(A_a e^a) = dA.
\]

Then,

\[
F = dA = DA_a \wedge e^a + A_b T^b.
\]

\((***)\)

\( F \) is closed,

\[
dF = d^2 A = 0
\]

and, most important, \( U(1) \)-gauge invariant:

\[
A \rightarrow A' = A + d\lambda \Rightarrow F \rightarrow F' = F + d^2 \lambda = F.
\]

We notice however that \( DA_a \) is not \( U(1) \)-gauge invariant: in fact, with \( d_a = e^a_\mu \partial_\mu \),

\[
DA_a \rightarrow (DA_a)' = D(A_a + d_a\lambda) = d(A_a + d_a\lambda) - \omega^b_a (A_b + d_b\lambda) = DA_a + d(d_a\lambda) - \omega^b_a d_b\lambda
\]

\[
= DA_a + (\delta^b_a - \omega^b_a) d_b\lambda \text{ i.e.} \quad (DA_a)' = DA_a + D^b_a d_b\lambda
\]

with

\[
D^b_a = \delta^b_a d - \omega^b_a.
\]

Nevertheless, even with a gauge invariant field strength, due to the non gauge invariance of the spin density tensor of the electromagnetic field, the solution of the Cartan equation gives a \( U(1) \)-gauge dependent torsion, which points to a difficult (if not impossible) to cure illness of the EC theory. In fact, the Maxwell-Einstein action describing the interaction between the electromagnetic field and gravity is given by

\[
S_{M-E} = l \int d^4 x \ e L_{M-E} = l \int d^4 x \ e \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right)
\]

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with \( l = \frac{G}{c^4} \). From the r.h.s. of \((****)\) and using \((***)\), we obtain the expression for \( F_{\mu\nu} \):

\[
(dA_\mu - \omega^\mu_\nu)e^\nu_a = ((\partial_\mu A_\nu - \omega^\mu_{\nu a} A_b e^\nu_a)dx^\mu \wedge dx^\nu,
\]

and

\[
A_b T^b = A_b T^b_{\mu\nu} dx^\mu \wedge dx^\nu,
\]

then

\[
F = ((\partial_\mu A_\nu - \omega^\mu_{\nu a} A_b) e^\nu_a + A_b T^b_{\mu\nu})dx^\mu \wedge dx^\nu
\]

and so

\[
F_{\mu\nu} = 2((\partial_\mu A_\nu - \omega^\mu_{\nu a} A_b) e^\nu_a + A_b T^b_{\mu\nu}).
\]

The Cartan equation results from the variation with respect to the spin connection of the total action \( S_G + S_{M-E} \):

\[
0 = \delta \omega S_G + \delta \omega S_{M-E}.
\]

The first term was obtained in section 32.A.1.; for the second term, using the expression for \( T^a_{\mu\nu} \) in section 29,

\[
-\frac{1}{4}\delta \omega (F^\mu\nu F_{\mu\nu}) = -\frac{1}{2} F_{\mu\nu} \delta \omega F^\mu\nu = -F^{\mu\nu} \delta \omega \left( \frac{1}{2}((\partial_\mu A_\nu - \omega^\mu_{\nu a} A_b) e^\nu_a - (\partial_\nu A_\mu - \omega^\mu_{\nu a} A_b) e^\nu_a) \right)
+ A_b (\partial_\mu e^\nu_a - \partial_\nu e^\nu_a + \omega^\mu_{\nu a} e^\nu_a - \omega^\nu_{\mu a} e^\nu_a))
= -F^{\mu\nu} \left( \frac{1}{2}(\delta \omega^a_{\mu a}) A_b e^\nu_a + \frac{1}{2}(\delta \omega^a_{\nu a}) A_b e^\nu_a + A_b ((\delta \omega^a_{\mu a}) e^\nu_a - (\delta \omega^a_{\nu a}) e^\nu_a) \right)
= F^{\mu\nu} (\delta \omega^a_{\mu a}) A_b e^\nu_a - 2F^{\mu\nu} A_b (\delta \omega^a_{\mu a}) e^\nu_a = -F^{\mu\nu} A_b e^\nu_a (\delta \omega^a_{\mu a}) = -F^{\mu\nu} A^b e^\nu_a \delta \omega^a_{\mu b a} = -F^{\mu\nu} A^b e^\nu_a \delta \omega^a_{\mu b a}
\]

where we used the antisymmetry \( \omega^a_{\mu b a} = -\omega^a_{\mu b a} \). Then

\[
0 = -e^\mu_a T_b + e^\mu_a T_a + T^\mu_{ba} - lF^{\nu}_{\mu a} e^\nu_a e^\nu_a.
\]

Multiplying by \( e^\mu_a e^\rho_a \) we obtain

\[
T^\mu_{\rho a} + \delta^\mu_\rho T_a - \delta^\mu_\rho T_a = lF^{\mu}_{\rho a} = \frac{1}{2}(F^{\rho}_{\mu a} A_\rho - F^{\mu}_{\rho a} A_\rho) = -lS^\mu_{\rho a}
\]

where

\[
S^\mu_{\rho a} = F^{\mu}_{\rho a} A_\rho - F^{\mu}_{\rho a} A_\rho
\]

is the canonical spin density tensor of the electromagnetic field obtained from the gauge invariant Lagrangian density \( L_{M-E} = -\frac{1}{4}F^{\mu\nu} F_{\mu\nu} \) through the Noether theorem (Bogoliubov and Shirkov, 1980). \( S^\mu_{\rho a} \) is antisymmetric in its lower indices but it is not gauge invariant:

\[
\delta_{a.tr.}(S^\mu_{\rho a}) = 2F^{\mu}_{\rho a} \delta \rho_a \lambda
\]

if \( \delta_{a.tr.}(A_\mu) = \partial_\mu \lambda \). In contradistinction with the density of energy-momentum \( T^\mu_{\rho} \) of any matter field, which always can be made gauge invariant (and symmetric), there is no known way to construct a gauge invariant spin density tensor for the electromagnetic field. However, at least in the special relativistic classical and quantum field theory context, after space integration of \( S^0_{\rho a} \), all the results for the conserved spin angular momentum tensor of the electromagnetic field are physical (light polarization, helicity states, etc.), and therefore gauge independent. This means that \( S^\mu_{\rho a} \) is not directly observable, and then the same could be concluded for the non gauge invariant torsion tensor produced by the electromagnetic spin.

By the same method of section 32 applied to the Dirac field, for the torsion one obtains

\[
T^\mu_{\rho a} = \frac{1}{2}(S^\mu_{\rho a} + \frac{1}{2}(\delta^\mu_\sigma S_{\rho a} - \delta^\mu_\rho S_{\sigma a}))
\]
with $S_\rho = S^\mu_{\rho \mu} = F^\nu_{\rho \mu} A_\mu$.

Acknowledgements

This work was partially supported by the projects PAPIIT IN 118609, 113607-2, and 101711-2, DGAPA-UNAM, México. The author thanks for the hospitality at the IAFE-UBA-CONICET, Argentina.

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**Appendix A**

**Fundamental theorem of riemannian geometry**
**Definition:** Let \((M, g, \nabla)\) be a riemannian manifold with a linear connection \(\nabla\). \(\nabla\) and \(g\) are compatible or, equivalently, \(\nabla\) is a metric connection in \(M\), if for any smooth path \(c: (a, b) \rightarrow M\), \(\lambda \mapsto c(\lambda)\), and any pair of parallel vector fields \(V\) and \(V'\) along \(c\) (i.e. \(\frac{DV}{d\lambda} = \frac{DV'}{d\lambda} = 0\)), then \(g(V, V') = \langle V, V' \rangle\) is constant along \(c\).

**Theorem:** In any riemannian manifold \((M, g)\) there exists a unique symmetric linear connection \(\nabla\), i.e. a connection in the tangent bundle of \(M\), which is compatible with the metric.

**Proof:**

Let \(X, Y, Z \in \Gamma(TM)\), and let \(\nabla\) be a metric and symmetric linear connection in \(M\). Then:

\[
\langle \nabla_X Y, Z \rangle = \langle X < Y, Z > - < Y, \nabla_X Z > = \langle X < Y, Z > \rangle = \langle X < Y, Z > - < Y, \nabla_Z X + [X, Z] > \rangle = \langle X < Y, Z > - < Y, \nabla_Z X > - < Y, [X, Z] > \rangle;
\]

\[
Z < Y, X > = < \nabla_Z Y, X > + < Y, \nabla_Z X > \text{ implies } < Y, \nabla_Z X > = \langle Y, X > - < \nabla_Z Y, X > \rangle = \langle Z, Y, X > - < \nabla_X Y, Z > - < Z, [Y, X] > \rangle;
\]

then:

\[
\langle \nabla_X Y, Z \rangle = \langle X < Y, Z > - Z < Y, X > + < Y, \nabla_X Y, Z > - < Z, [Y, X] > \rangle + < [Z, Y], X > - < [X, Z], Y > \rangle
\]

and therefore

\[
\langle \nabla_X Y, Z \rangle = \frac{1}{2}(X < Y, Z > - Z < Y, X > + Y < Z, X > - < \nabla_X Y, Z > - < Z, [Y, X] > + < [Z, Y], X > - < [X, Z], Y > \rangle)
\]

which gives an explicit expression for \(\langle \nabla_X Y, Z \rangle\) in terms of \(X, Y, Z\) and \(\langle , \rangle\) = \(g\).

In local coordinates, we choose \(X = \partial_i\), \(Y = \partial_j\), \(Z = \partial_k\); then \([\partial_i, \partial_j] = 0\) and therefore

\[
\langle \nabla_{\partial_i} \partial_j, \partial_k \rangle = \langle \Gamma^l_{ij} \partial_l, \partial_k \rangle = \Gamma^l_{ij} g(\partial_l, \partial_k) = \frac{1}{2}(\partial_l < \partial_j, \partial_k > + \partial_j < \partial_k, \partial_l > - \partial_k < \partial_l, \partial_j >) = \frac{1}{2}(\partial_l g_{jk} + \partial_j g_{lk} - \partial_k g_{lj});\]

multiplying by \(g^{mk} = (g^{-1})_{mk}\), \(\Gamma^l_{ij} g^{mk} g_{lk} = \Gamma^l_{ij} \delta^m_l = \Gamma^m_{ij}\), and therefore

\[
\Gamma^m_{ij} = \frac{1}{2} g^{mk}(\partial_l g_{jk} + \partial_j g_{lk} - \partial_k g_{lj}). \quad (\text{qed})
\]

Clearly, \(\Gamma^m_{ij} = \Gamma^m_{ji}\).

**Remark:** The theorem is also valid in pseudo-riemannian geometry; in particular for lorentzian manifolds.
Note: In \( m \) dimensions, the number of independent components of the metric tensor and the Levi-Civita connection are \( N(g_{\mu\nu}; m) = \frac{m(m+1)}{2} \) and \( N(\Gamma^{\mu}_{\nu\rho}; m) = \frac{m^2(m+1)}{2} \).

Appendix B

General form of the local version of a non (necessarily) metric and non (necessarily) symmetric connection

Given a linear connection in a manifold \( M^n \) (\( L^n \)-space), with local Christoffel symbols \( \Gamma^\alpha_{\nu\rho} \), if in addition the manifold is riemannian or pseudo-riemannian \( (M^n, g) \) (\( V^n \)-space), one has a \( (L^n, g) \)-space. The non-metricity tensor is defined as minus the covariant derivative of the metric:

\[
Q_{\mu\nu\rho} = -D_{\rho}g_{\mu\nu}.
\]

(\( Q_{\mu\nu\rho} = Q_{\mu\nu\rho} \).) Using section 7, by cyclic permutation of indices, one obtains

\[
\Gamma^\alpha_{\nu\mu} = (\Gamma_{LC})^\alpha_{\nu\mu} + K^\alpha_{\nu\mu} + \frac{1}{2} g^{\alpha\rho}(Q_{\mu\nu\rho} + Q_{\nu\rho\mu} - Q_{\rho\mu\nu})
\]

where \((\Gamma_{LC})^\alpha_{\nu\mu}\) is the Levi-Civita connection of section 13, and

\[
K^\alpha_{\nu\mu} = T^\alpha_{\nu\mu} + g^{\alpha\rho}T^\lambda_{\rho\mu}g_{\lambda\nu} + g^{\alpha\rho}T^\lambda_{\rho\nu}g_{\lambda\mu}
\]

is the contortion tensor, with

\[
K^\alpha_{\nu\mu} = (K_A)^\alpha_{\nu\mu} + (K_S)^\alpha_{\nu\mu},
\]

\[
(K_A)^\alpha_{\nu\mu} = T^\alpha_{\nu\mu} = -(K_A)^\alpha_{\mu\nu},
\]

\[
(K_S)^\alpha_{\nu\mu} = (K_S)^\alpha_{\mu\nu} = g^{\alpha\rho}(T^\lambda_{\rho\mu}g_{\lambda\nu} + T^\lambda_{\rho\nu}g_{\lambda\mu}).
\]

A metric connection is one in which \( Q_{\mu\nu\rho} = 0 \) i.e. the connection is compatible with the metric, but non necessarily symmetric:

\[
\Gamma = \Gamma_{LC} + \text{contortion}.
\]

\((U^n\)-space.) In particular, scalar products and lengths of vectors are constant in parallel transport. (In fact, \( ||V||_{\mu}^2 = (g_{\alpha\beta}V^\alpha_{\mu}V^\beta_{\mu}) = g_{\alpha\beta}V^\alpha_{\mu}V^\beta_{\mu} + 2g_{\alpha\beta}V^\alpha_{\mu}V^\beta_{\mu} = 2V^\alpha_{\mu}V^\alpha_{\mu} = 2V^\alpha_{\mu}V^\alpha_{\mu} = 2V^\alpha_{\mu}V^\alpha_{\mu} = 0 \).) A physical case corresponds to the Einstein-Cartan theory of gravity. (Cartan, 1922.)

A symmetric connection is one in which torsion vanishes i.e. \( T^\mu_{\nu\rho} = 0 \):

\[
\Gamma = \Gamma_{LC} + \text{non-metricity}.
\]

A particular case is the Weyl connection (1918):

\[
\text{non-metricity} = Q_{\mu\nu\rho} = g_{\nu\rho}A_{\mu}.
\]

with \( A = A_{\mu}dx^\mu \) a 1-form. (In natural units, \([A_{\mu}] = [\text{mass}] \) if \([x^\mu] = [\text{length}]\).)

Appendix C

1. If \( a = (a_{ij}) \) is an invertible matrix with \( a_{ij} = a_{ij}(x) \), then \( \partial_{\mu}(\det a) = a_{ij,\mu}(\det a)a^{ij} \), where \( a^{ij} = (a^{-1})_{ij}. \) In particular, for the metric tensor,

\[
g_{\mu\rho} = g_{\mu\nu,\rho}g^{\mu\nu}
\]
where \( g = \det g_{\mu\nu} \). Then, for the Levi-Civita connection,

\[
(\Gamma^\mu_{LC})^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\lambda} (\partial_\nu g_{\mu\lambda} + \partial_\lambda g_{\nu\mu} - \partial_\mu g_{\nu\lambda}) = \frac{1}{2} g^{\mu\lambda} g_{\mu,\nu} = \frac{1}{2} g^{\mu\lambda} (gg^{\mu\lambda})_{\nu} = \frac{1}{2} g_{,\nu} = \frac{1}{2} (-g, g). \nu
\]

\[
= \frac{1}{-g} \partial_\nu (-g)^{1/2} = \sqrt{-1} \partial_\nu \sqrt{g}.
\]

2. From 1., \( \partial_\nu \sqrt{g} = \frac{\sqrt{g}}{2} g^{\mu\lambda} g_{\mu,\nu} \); then, \( \delta \sqrt{g} = (\partial_\nu \sqrt{g}) \delta x^\nu = \frac{\sqrt{g}}{2} g^{\mu\lambda} g_{\mu,\nu} \delta x^\nu = \frac{\sqrt{g}}{2} g^{\mu\lambda} \delta g_{\nu}. \) Using \( g^{\mu\lambda} = \eta^{ab} e_a^\nu e_b^\lambda \) (section 28.6), we obtain \( g^{\mu\lambda} \delta g_{\mu\nu} = 2 e_d^\lambda \delta e_\lambda^\nu \) and so \( \delta e = e_d^\lambda \delta e_\lambda^\nu \) with \( e = \sqrt{g}. \) From \( e_d^\lambda e_\lambda^\nu = \delta_f^d \), \( \delta e = -ee_d^\lambda \delta e_\lambda^\nu \).

## Appendix D

**Lorentz gauge invariance of the Ricci scalar**

(This proof is due to G. D’Oliveiro.)

The Ricci scalar is given by

\[
R = \eta^{bd} e_a^b e_a^d (\partial_\mu \omega^a_{\nu b} - \partial_\nu \omega^a_{\mu b} + \omega^a_{\nu \nu} \omega^a_{\nu b} - \omega^a_{\nu b} \omega^a_{\nu b}) \equiv \eta^{bd} e_a^b e_a^d ((3) - (4) + (1) - (2)).
\]

Under the transformation

\[
\omega^a_{\mu c} = h_c^l \omega^r_{\mu l} h^{-1}_{r a} + (\partial h^c_l) h^{-1}_{l a}
\]

we have:

(1) = (a) + (b) + (c) + (d) with

\[
(a) = h_c^l \omega^r_{\mu l} h^{-1}_{r a} h_b^g \omega^a_{\nu g} h^{-1}_{s a}, \quad (b) = h_c^l \omega^r_{\mu l} h^{-1}_{r a} (\partial_b h^g_b) h^{-1}_{g a},
\]

\[
(c) = h_b^g \omega^a_{\nu g} h^{-1}_{s c} (\partial_\mu h^g_l) h^{-1}_{l a}, \quad (d) = (\partial_\mu h^g_l) h^{-1}_{l a} (\partial_b h^g_b) h^{-1}_{g a};
\]

(2) = (e) + (f) + (g) + (h) with

\[
(e) = h_c^g \omega^a_{\nu g} h^{-1}_{s c} h_b^l \omega^r_{\mu l} h^{-1}_{r c}, \quad (f) = h_c^g \omega^a_{\nu g} h^{-1}_{s c} (\partial_\mu h^g_l) h^{-1}_{l c},
\]

\[
(g) = h_b^l \omega^r_{\mu l} h^{-1}_{r c} (\partial_\mu h^g_l) h^{-1}_{g a}, \quad (h) = (\partial_\mu h^g_l) h^{-1}_{l a} (\partial_b h^g_b) h^{-1}_{g a};
\]

(3) = [1] + [2] + [3] + [4] with

\[
[1] = h_b^n h^{-1}_{t a} (\partial_\mu \omega^r_{\nu m}), \quad [2] = \omega^a_{\nu m} \partial_\mu (h^n_l h^{-1}_{t a}), \quad [3] = (\partial_\mu \partial_\nu h^m_n) h^{-1}_{n a}, \quad [4] = (\partial_\mu h^m_n) (\partial_\mu h^{-1}_{n a});
\]

and (4) = [5] + [6] + [7] + [8] with

\[
[5] = h_b^n h^{-1}_{s a} (\partial_\mu \omega^r_{\nu m}), \quad [6] = \omega^a_{\nu m} \partial_\mu (h^n_l h^{-1}_{s a}), \quad [7] = (\partial_\mu \partial_\nu h^m_n) h^{-1}_{n a}, \quad [8] = (\partial_\mu h^m_n) (\partial_\mu h^{-1}_{n a}).
\]

Now,

\[
[3] - [7] = (\partial_\mu \partial_\nu h^m_n) h^{-1}_{n a} - (\partial_\mu \partial_\nu h^m_n) h^{-1}_{n a} = 0,
\]

\[
(b) + (c) = \omega^a_{\nu g} h^{-1}_{s c} \partial_\mu h^g_l - \omega^a_{\nu g} h^{-1}_{s c} \partial_\mu h^g_l;
\]

\[
(f) + (g) = \omega^a_{\nu g} h^{-1}_{s c} \partial_\mu h^g_l - \omega^a_{\nu g} h^{-1}_{s c} \partial_\mu h^g_l.
\]
so

\((b) + (c) - ((f) + (g)) = \omega_{\mu \nu}^s \partial_\nu (h_r^{-1} a h_b^l) - \omega_{\mu \nu}^s \partial_\nu (h_s^{-1} a h_b^g)\); 

also,

\([2] - [6] = \omega_{\nu \mu}^s \partial_\mu (h_b^g h_s^{-1} a) - \omega_{\nu \mu}^s \partial_\nu (h_b^l h_r^{-1} a)\); 

then

\(((b) + (c)) - ((f) + (g)) + ([2] - [6]) = 0.\)

Also,

\([4] - [8] = (\partial_\nu h_b^l) (\partial_\mu h_r^{-1} a) - (\partial_\nu h_b^l) (\partial_\mu h_r^{-1} a)\)

and

\((d) - (h) = (\partial_\nu h_r^{-1} a) (\partial_\mu h_b^l) - (\partial_\nu h_b^l) (\partial_\mu h_r^{-1} a)\);

so

\(((4) - [8]) + ((d) - (h)) = 0.\)

Finally,

\([1] - [5] + (a) - (c) = h_b^l h_r^{-1} a (\partial_\mu \omega_{\nu \mu}^s - \partial_\nu \omega_{\mu \mu}^s + \omega_{\mu \nu}^s \omega_{\mu \mu}^s - \omega_{\nu \nu}^s \omega_{\mu \mu}^s)\).

Therefore,

\[R = \eta^{bd} e_a^d e_c^d h_b^l h_r^{-1} a (\partial_\mu \omega_{\nu \mu}^s - \partial_\nu \omega_{\mu \mu}^s + \omega_{\mu \nu}^s \omega_{\mu \mu}^s - \omega_{\nu \nu}^s \omega_{\mu \mu}^s) = \eta^{lt} e_a^l e_c^l (\partial_\mu \omega_{\nu \mu}^s - \partial_\nu \omega_{\mu \mu}^s + \omega_{\mu \nu}^s \omega_{\mu \mu}^s - \omega_{\nu \nu}^s \omega_{\mu \mu}^s) = R'.\]

(ged)

**Appendix E**

**Affine spaces**

An *affine space* is a triple \((V, \varphi, A)\) where \(V\) is a vector space, \(A\) is a set, and \(\varphi\) is a free and transitive left action of \(V\) as an additive group on \(A\):

\[
\varphi : V \times A \to A, \quad (v, a) \mapsto v + a,
\]

with

\[0 + a = a \text{ and } (v_1 + v_2) + a = v_1 + (v_2 + a), \text{ for all } a \in A \text{ and all } v_1, v_2 \in V.\]

Then, given \(a, a' \in A\), there exists a unique \(v \in V\) such that \(a' = v + a\). Also, if \(v_0\) is fixed in \(V\), \(\varphi_{v_0} : A \to A, \quad \varphi_{v_0}(a) = \varphi(v_0, a)\) is a bijection.

**Example.** \(A = V\): The vector space itself is considered as the set on which \(V\) acts. In particular, when \(V = T_x M^n\) and \(A = T_x M^n\), the tangent space is called *affine tangent space* and denoted by \(A_x M^n\). The points “\(a\)” of \(A_x M^n\) are the tangent vectors at \(x\).
It is clear that to define an action of $GA_{n}(\mathbb{R})$ on $A_{x}M^{n}$, we need a frame at $x$ i.e. we have to consider the bundle of affine frames $AM^{n}$ so that $(v_{x},r_{x})(\xi,g) = (v_{x} + r_{x}\xi,r_{x}g)$.

Appendix F

Gauge transformations in $G$-bundles

A gauge transformation or vertical automorphism of a (smooth) principal $G$-bundle $\xi : G \to P \xrightarrow{\pi} B$ is a diffeomorphism $\alpha : P \to P$ such that the following diagram commutes:

\[
\begin{array}{ccc}
P \times G & \xrightarrow{\alpha \times Id_{G}} & P \times G \\
\psi \downarrow & & \downarrow \psi \\
P & \xrightarrow{\alpha} & P \\
\pi \downarrow & & \downarrow \pi \\
B & \xrightarrow{Id_{B}} & B
\end{array}
\]

($\psi$ is the action of $G$ on $P$.) That is:

\[\alpha \circ \psi = \psi \circ (\alpha \times Id_{G})\]

i.e. $\alpha(pg) = \alpha(p)g$

and

\[\pi \circ \alpha = \pi\]

i.e. $\pi(\alpha(p)) = \pi(p)$.

So, $\alpha(p) = ph$ with $h \in G$.

The set of gauge transformations of $\xi$, $G(\xi)$, is called the gauge group of the bundle.

Local form of $\alpha$

A local trivialization of $\xi$ is given by the commutative diagram

\[
\begin{array}{ccc}
P_U & \xrightarrow{\Phi_U} & U \times G \\
\pi \downarrow & & \downarrow \pi_1 \\
U & \xrightarrow{Id_U} & U
\end{array}
\]

i.e. $\pi_1 \circ \Phi_U = \pi$, where $P_U = \pi^{-1}(U)$, $\Phi_U$ is a diffeomorphism, $U$ is an open subset of $B$, and $\pi_1(p,g) = p$.

$\Phi_U$ defines the local section of $\xi$, $\sigma_U : U \to P_U$, $\sigma_U(b) = \Phi_U^{-1}(b,e)$ where $e$ is the identity in $G$. Then there exists the smooth function

\[\alpha_U : U \to G, \quad b \mapsto \alpha_U(b)\]

which determines $\alpha$ for all $p \in P_U$. In fact, let $p = \sigma_U(b)$; then

\[\alpha(\sigma_U(b)) = \sigma_U(b)g \in P_b = \pi^{-1}(\{b\}),\]

and so

\[\alpha_U(b) = g\]

with $g$ unique since $\psi$ acts freely on $P$ and transitively on fibers. If $p' \in P_b$, then $p' = \sigma_U(b)h$ and $\alpha(p') = \alpha(\sigma_U(b)h) = \alpha(p)h = (\sigma_U(b)\alpha_U(b))h = \sigma_U(b)(\alpha_U(b)h)$. This holds for all $b \in U$.

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