Near soliton dynamics and singularity formation for $L^2$ critical problems

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Abstract. This survey reviews the state of the art concerning singularity formation for two canonical dispersive problems: the $L^2$ critical non-linear Schrödinger equation and the $L^2$ critical generalized KdV equation. In particular, the currently very topical question of classifying flows with initial data near a soliton is addressed.

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1. Introduction

The study of singularity formation in non-linear dispersive equations has attracted considerable attention over the past thirty years. Recently, this activity has led to the development of powerful tools for constructing and describing blowup solutions, thereby resolving some of the classical conjectures in this area. The description of blowup solutions typically includes determination of the blowup rate, the blowup profile, and the behaviour of concentration points. We start by introducing the two main equations to be considered below.

1.1. The $L^2$ critical problems (NLS) and (gKdV). Our aim is to survey some recent progress over the past ten years in the question of singularity formation for two canonical problems: the $L^2$ critical non-linear Schrödinger equation

\[
\text{(NLS)} \quad \begin{cases}
    i\partial_t u + \Delta u + u|u|^{4/d} = 0, \\
    u|_{t=0} = u_0,
\end{cases} \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d, \quad u \in \mathbb{C}, \tag{1}
\]

and the $L^2$ critical (one-dimensional) generalized Korteweg–de Vries equation

\[
\text{(gKdV)} \quad \begin{cases}
    \partial_t u + (u_{xx} + u^5)_x = 0, \\
    u|_{t=0} = u_0,
\end{cases} \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \quad u \in \mathbb{R}. \tag{2}
\]

A specific algebraic structure underlies these two models. Solutions of both models preserve the so-called energy

\[
E(u) = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u|^2 - \frac{1}{2 + 4/d} \int_{\mathbb{R}^d} |u|^{2+4/d} = E(u_0)
\]

and the mass

\[
\int_{\mathbb{R}^d} |u|^2 = \int_{\mathbb{R}^d} |u_0|^2
\]

(take $d = 1$ in both expressions in the case of (gKdV)). The notion of $L^2$ criticality means that the $L^2$ norm is unchanged by the scaling symmetry of the equation:

if $u(t, x)$ is a solution, then so is $u_\lambda(t, x) := \begin{cases}
\lambda^{d/2} u(\lambda^2 t, \lambda x) \text{ for (NLS)}, \\
\lambda^{1/2} u(\lambda^3 t, \lambda x) \text{ for (gKdV)}.
\end{cases}$
1.2. Local existence theory and blowup. According to Ginibre and Velo [17] the problem (1) is locally well-posed in $H^1(\mathbb{R}^d)$, and according to Kenig, Ponce, and Vega [24] the problem (2) is locally well-posed in $H^1(\mathbb{R})$. Thus, for any initial data $u_0 \in H^1$ there exist a $T$ with $0 < T \leq +\infty$ and a unique maximal solution $u(t) \in C([0, T), H^1)$ of (1) or (2) with the following alternative:

- either $T = +\infty$ and the solution is globally defined in $H^1$;
- or $T < +\infty$ and then the solution blows up in a finite time,

$$\lim_{t \uparrow T} \|\nabla u(t)\|_{L^2} = +\infty.$$

1.3. Ground state and sharp threshold for global existence. Exceptional solutions propagating without deformation (called travelling waves or solitons) play a distinguished role in this analysis. Indeed, the ansatz

$$u(t, x) = \begin{cases} Q(x)e^{it} & \text{for (NLS)}, \\ Q(x-t) & \text{for (gKdV)} \end{cases}$$

leads to the semilinear elliptic problem

$$\Delta Q - Q + Q^{1+4/d} = 0, \quad x \in \mathbb{R}^d,$$

which admits a unique, up to spatial translation, positive $H^1$ solution known as the ground state solitary wave. The explicit variational characterization of $Q$ connected with sharp Sobolev bounds led in the 1980s to the first derivation of a sharp criterion for global existence versus blowup for (NLS) by Weinstein [70] and by Berestycki and Cazenave [3]. In particular, $H^1$ initial data with

$$\|u_0\|_{L^2} < \|Q\|_{L^2}$$

for both (NLS) and (gKdV) generate a unique global solution $u \in C([0, +\infty, H^1)$. In the case (NLS) it is also known that the solution scatters, that is, behaves like a free wave as $t \to +\infty$:

$$\exists u_{\pm\infty} \in H^1 \text{ such that } \lim_{t \to \pm\infty} \|u(t) - e^{it\Delta} u_{\pm\infty}\|_{L^2} = 0.$$

We refer to [8] and to the recent definitive result [9] (see also references therein). In other words, the ground state solitary wave is the smallest non-dispersive and thus non-linear object.

In many non-linear problems solitary waves are expected to be the building blocks for non-linear dynamics: for large time any solution decomposes into a certain number of solitary waves plus a residual which is a free wave. This fact has been well known for integrable problems since the 1960s, at least on a formal level. In blowup regimes the role played by solitary waves has also been clarified in several directions: $Q$ is the universal blowup profile for blowup solutions initially close
to $Q$, independently of the blowup rate. It is thus a fundamental object to study singularity formation in such situations. Moreover, a complete description of the non-linear flow near the ground state solitary wave has become one of the most relevant and challenging questions in this area.

Our paper is organized as follows. In §2 we summarize results on blowup solutions near $Q$ obtained for (NLS) in [46]–[48], [50], [53], [66]. In §3 we discuss for (gKdV) the description of the flow near the ground state obtained in [39]–[41]. Finally, an overview of some proofs for (gKdV) is given in §4.

2. The $L^2$ critical problem for (NLS)

In this section we review the state of the art in the problem of formation of singularities for the $L^2$ critical equation (NLS), and we present some other results on related problems.

2.1. Minimal-mass blowup. As discussed in the Introduction, initial data $u_0 \in H^1$ with mass $\|u_0\|_{L^2} < \|Q\|_{L^2}$ generate bounded global solutions. For (NLS) this criterion is sharp, as a consequence of the so-called pseudoconformal transformation which is a well-known symmetry of the linear Schrödinger flow and of (NLS) in the $L^2$ critical case: if $u(t, x)$ is a solution of (NLS), then so is

$$v(t, x) = \frac{1}{|t|^{d/2}} u\left(-\frac{1}{t}, \frac{x}{t}\right) e^{i|x|^2/(4t)}. \quad (4)$$

Applied to the solitary wave solution $u(t, x) = Q(x)e^{it}$, this transformation gives an explicit minimal-mass blowup solution:

$$S_{\text{NLS}}(t, x) = \frac{1}{|t|^{d/2}} Q\left(\frac{x}{t}\right) e^{i|x|^2/(4t) - i/t}, \quad \|S_{\text{NLS}}(t)\|_{L^2} = \|Q\|_{L^2}. \quad (5)$$

The dynamics generated by the smooth initial data $S_{\text{NLS}}(-1)$ is explicit: $S_{\text{NLS}}(t)$ scatters as $t \to -\infty$, and blows up as $t \uparrow 0$ at the rate

$$\|\nabla S_{\text{NLS}}(t)\|_{L^2} \sim \frac{1}{|t|}. \quad (6)$$

An essential feature of (5) is that $S_{\text{NLS}}(t)$ is compact up to symmetries of the flow, meaning that all the mass goes into the singularity formation:

$$|S_{\text{NLS}}(t)|^2 \rightharpoonup \|Q\|_{L^2}^2 \delta_{x=0} \quad \text{as } t \uparrow 0. \quad (7)$$

The general intuition is that such behaviour is exceptional in the sense that such minimal elements can be classified.\textsuperscript{1} The first result of this type was proved by Merle using the pseudoconformal symmetry.

\textsuperscript{1}This is a dispersive intuition which, for example, is wrong in the parabolic setting [4].
**Theorem 1** (Classification of minimal-mass blowup solutions, [42]). Let \( u_0 \in H^1 \), with
\[
\|u_0\|_{L^2} = \|Q\|_{L^2}.
\]
Assume that the corresponding solution of (NLS) blows up in a finite time \( 0 < T < +\infty \). Then
\[
u(t) = S_{\text{NLS}}(t)
\]
up to symmetries of (NLS).

The question of the existence of minimal elements in various other settings has been a longstanding open problem, mostly due to the fact that the existence of a minimal element for (NLS) is based entirely on the exceptional pseudoconformal symmetry. In [43] Merle considered the inhomogeneous problem
\[
i\partial_t u + \Delta u + k(x)u|u|^2 = 0, \quad x \in \mathbb{R}^2,
\]
which breaks the full symmetry group, and obtained some results on non-existence of minimal elements for non-smooth \( k \). On the other hand, the recent paper [66] contains a sharp criterion for the existence and uniqueness of minimal solutions in terms of the inhomogeneity \( k(x) \) (on existence see also [1]).

**Theorem 2** (Existence and uniqueness of a critical element for (8), [66]). Let \( x_0 \in \mathbb{R}^2 \), with
\[
k(x_0) = 1 \quad \text{and} \quad \nabla^2 k(x_0) < 0.
\]
Let the energy \( E_0 \) of \( u \) satisfy
\[
E_0 + \frac{1}{8} \int \nabla^2 k(x_0)(y, y)Q^4 > 0.
\]
Then there exists a critical-mass \( H^1 \) blowup solution of (8) with energy \( E_0 \), unique up to a phase shift, which blows up at time \( T = 0 \) and at the point \( x_0 \). Moreover,
\[
\lim_{t \to 0} \text{Im} \left( \int \nabla u \bar{u} \right) = 0.
\]

Under the condition (9), minimal blowup elements with a non-degenerate blowup point are thus completely classified. It is also shown in [66] that (9) is a necessary condition for blowup. Theorem 2 is based on a dynamical construction and new Lyapunov functionals at the minimal-mass level. A further extension to non-local dispersion,
\[
i\partial_t u - (-\Delta)^{1/2}u + u|u|^2 = 0, \quad x \in \mathbb{R},
\]
can be found in [27]; see also [5] for an extension to curved backgrounds. These works show that the existence of a minimal-mass bubble is a general property, independent of the existence of an exceptional pseudoconformal symmetry for the model.
2.2. log-log blowup. The minimal-mass blowup solution (5) is explicit, but the corresponding blowup scenario is obviously unstable, since any subcritical-mass perturbation of $S_{NLS}(t)$ leads to a globally defined solution. The question of the description of stable blowup bubbles has attracted considerable attention beginning in the 1980s with the development of sharp numerical methods and the prediction of the ‘log-log law’ for (NLS) by Landman, Papanicoalou, Sulem, and Sulem [32].

We focus our attention on initial data with mass slightly above the minimal mass required for singularity formation:

$$u_0 \in \mathcal{B}_{\alpha^*} = \{u_0 \in H^1 \text{ and } \|Q\|_{L^2} < \|u_0\|_{L^2} < \|Q\|_{L^2} + \alpha^\ast\}, \quad 0 < \alpha^* \ll 1.$$ (11)

Applying concentration-compactness techniques [33] and using the variational characterization of the ground state, we deduce from the assumption (11) that if $u(t)$ blows up at a time $T < +\infty$, then for $t$ close to $T$ the solution admits a non-linear decomposition

$$u(t, x) = \frac{1}{\lambda(t)^{d/2}} (Q + \varepsilon) \left( t, \frac{x - x(t)}{\lambda(t)} \right) e^{i\gamma(t)},$$ (12)

where

$$\lambda(t) \sim \frac{1}{\|\nabla u(t)\|_{L^2}}, \quad \|\varepsilon(t)\|_{H^1} \leq \delta(\alpha^*), \quad \lim_{\alpha^* \to 0} \delta(\alpha^*) = 0.$$ (13)

This decomposition implies that, independently of the blowup regime, the ground state solitary wave $Q$ is a good approximation of the blowup profile, and this is the starting point of a perturbative analysis for (11). The sharp description of the blowup bubble is now based on the determination of a finite-dimensional dynamical system, for a suitable choice of geometrical parameters $(\lambda(t), x(t), \gamma(t))$, that is coupled to the infinite-dimensional dynamics driving the residual term $\varepsilon(t)$.

Remark 3. For example, one can decompose the minimal-mass blowup solution (5) as follows:

$$\lambda(t) = |t|, \quad \varepsilon(t, y) = Q(y)(e^{ib(t)}|y|^2/4 - 1), \quad b(t) = |t|.$$ 

Not all possible regimes for $(\lambda(t), x(t), \gamma(t))$ are known so far for (NLS), but progress has been made in the understanding of both ‘stable’ and ‘threshold’ dynamics. The following theorem summarizes a series of results obtained in [48], [46], [47], [50], [49], [59] (see also [61]).

**Theorem 4** ([48], [46], [47], [50], [49], [59]). Let $d \leq 5$. There exists a universal constant $\alpha^* > 0$ such that the following holds true. Let $u_0 \in \mathcal{B}_{\alpha^*}$, and let $u \in C([0, T), H^1)$, $0 < T \leq +\infty$, be the corresponding maximal solution of (1).

(i) Sharp $L^2$ concentration. Assume that $T < +\infty$. Then there exist parameters $(\lambda(t), x(t), \gamma(t)) \in C^1([0, T), \mathbb{R}_+^* \times \mathbb{R}^d \times \mathbb{R})$ and an asymptotic profile $u^* \in L^2$ such that

$$u(t) - \frac{1}{\lambda(t)^{d/2}} Q \left( \frac{x - x(t)}{\lambda(t)} \right) e^{i\gamma(t)} \to u^* \text{ in } L^2 \text{ as } t \to T.$$ (14)
Moreover, the blowup point is finite:

\[ x(t) \to x(T) \in \mathbb{R}^d \quad \text{as} \quad t \to T. \]

(ii) Blowup rate. Under the conditions (i) the following alternative holds:

- either the solution satisfies the log-log regime, that is,
  \[ \lambda(t) \sqrt{\frac{\log|\log(T-t)|}{T-t}} \to \sqrt{2\pi} \quad \text{as} \quad t \to T, \]  
  and then the asymptotic profile is not smooth, that is, \( u^* \notin H^1 \) and \( u^* \notin L^p \) for all \( p > 2 \);  
  \[ (15) \]
- or there is a sharp lower bound for \( t \) close to \( T \),
  \[ \lambda(t) \leq C(u_0)(T-t), \quad \text{or equivalently} \quad \|\nabla u(t)\|_{L^2} \geq \frac{C_1(u_0)}{T-t}; \]  
  and then the asymptotic profile satisfies
  \[ (17) \]
  \[ u^* \in H^1. \]

(iii) Sufficient condition for log-log blowup. Assume that \( E_0 \leq 0 \). Then the solution blows up in a finite time \( T < +\infty \) in the log-log regime (15).

(iv) \( H^1 \) stability of the log-log blowup. The set of initial data in \( \mathcal{B}_{\alpha^*} \) such that the corresponding solution of (1) blows up in finite time with the log-log law (15) is open in \( H^1 \).

Comments on this theorem.

1. The log-log law. The stable blowup scenario according to the log-log law (15) was proposed in the pioneering paper [32] on the basis of formal and numerical considerations. The first rigorous construction of a class of solutions with such a blowup rate is due to Perelman [58] in dimension \( d = 1 \). The proof of Theorem 4 involves a mild coercivity property of a linearized operator close to \( Q \), a property proved in dimension \( d = 1 \) in [48] and checked numerically in an elementary way in [16] for \( d \leq 5 \). For dimensions \( d \geq 2 \) the lack of an explicit formula for the ground state makes this property difficult to prove.

2. Upper bound on the blowup rate. No general upper bound on the blowup rate of \( \|\nabla u(t)\|_{L^2} \) is known in the \( L^2 \) critical case, not even for initial data \( u_0 \in \mathcal{B}_{\alpha^*} \).

This is in contrast to supercritical regimes where a sharp upper bound was recently derived (see [54] and §2.5.3). The lower bound (17) is sharp and is attained by the minimal blowup element \( S_{\text{NLS}}(t) \). The derivation of various blowup rates, which is equivalent to the construction of solutions blowing up in infinite time using the pseudoconformal symmetry, is connected with a description of the flow near the ground state, as yet incomplete for (NLS). Some intuition may come
from recent classification results obtained for the $L^2$ critical problem (gKdV) and presented in §3.

3. **Quantization of the blowup mass.** The strong convergence (14) precisely describes the blowup bubble in the scaling-invariant space and implies in particular that the mass put into the singularity formation is quantized:

$$|u(t)|^2 \to \|Q\|^2_{L^2} \delta_{x=x(T)} + |u^*|^2 \text{ as } t \to T, \quad |u^*|^2 \in L^1.$$  

Such quantization and the convergence result (14) are based on the property of asymptotic stability of the solitary wave in the blowup regime, which is written in terms of the decomposition (12) as

$$\varepsilon(t, x) \to 0 \text{ as } t \to T \text{ in } L^2_{\text{loc}}.$$  

In fact, in the proof of Theorem 4 the derivation of either upper or lower bounds on the blowup rate is closely connected with the question of dispersion for the residual $\varepsilon(t, x)$. Such asymptotic theorems started to appear in a dispersive setting in [36], and significant progress was made in a recent classification result for energy-critical wave equations, without any restriction on the size of the initial data [11].

4. **Asymptotic profile.** The regularity of the asymptotic profile $u^*$ depends directly on the regime, because the singular and regular parts of the solution are very much coupled in the stable log-log regime, while they are weakly interacting in any other regime.

### 2.3. Threshold dynamics.

Theorem 4 describes stable log-log blowup in a neighbourhood of the soliton, but it does not complete the study of blowup for (NLS) even for initial data $u_0 \in \mathcal{B}_{\alpha^*}$. In particular, it remains to clarify unstable blowup close to $Q$. The explicit minimal-mass blowup solution $S_{\text{NLS}}(t)$ defined in (4) is clearly unstable, but Bourgain and Wang [6] observed that specific perturbations of the initial data preserve blowup with rate $1/t$. The excess of mass in this regime converts into a *flat and smooth* asymptotic profile at the blowup time, which does not alter the blowup law.

**Theorem 5** (Bourgain–Wang solutions [6]). Let $d = 1, 2$, and let the function $u^*$ be such that

$$u^* \in X_A = \{ f \in H^A \text{ with } (1 + |x|^A)f \in L^2 \}$$  \hspace{1cm} (19)  

and

$$D^\alpha u^*(0) = 0 \quad \text{for } 1 \leq |\alpha| \leq A$$  \hspace{1cm} (20)  

for some sufficiently large $A$. Then there exists a solution $u_{\text{BW}} \in \mathcal{C}((\infty, 0), H^1)$ of (1) which blows up at $t = 0, x = 0$ and satisfies

$$u_{\text{BW}}(t) - S_{\text{NLS}}(t) \to u^* \text{ in } H^1 \text{ as } t \uparrow 0.$$

\hspace{1cm} (21)
See [6] for a more precise statement and [29] for a further discussion on the construction of the manifold of Bourgain–Wang solutions. We note that the Bourgain–Wang blowup solutions saturate the lower bound (17):

\[ \|\nabla u(t)\|_{L^2} \sim \frac{1}{T - t}. \]

We recall that by the Strichartz estimates and the \(L^2\) critical Cauchy theory [7], solutions which scatter are \(L^2\) stable. Also, it follows from Theorem 4 that solutions in \(B_{\alpha}^*\) that blow up in finite time in the log-log regime form an open set in \(H^1\). Bourgain–Wang solutions correspond in a certain sense to an unstable threshold dynamics between these two classes of stable dynamics, as was proved in [53].

**Theorem 6** (Instability of Bourgain–Wang solutions, [53]). Let \(d = 2\), let \(u^*\) satisfy (19) and (20), and let \(u_{BW} \in C((-\infty, 0), H^1)\) be the corresponding Bourgain–Wang solution. Then there exists a continuous map

\[ \eta \in [-1, 1] \to u^{\eta}(-1) \in \Sigma = \{ f \in H^1(\mathbb{R}^d) \text{ with } xf \in L^2(\mathbb{R}^d) \} \]

such that, with \(u^{\eta}(t)\) denoting the solution of (1) with initial data \(u^{\eta}(-1)\) at \(t = -1\),

- \(u^{\eta=0}(t) \equiv u_{BW}(t)\);
- \(\forall \eta \in (0, 1] \) the solution \(u^{\eta} \in C(\mathbb{R}, \Sigma)\) is global in time and scatters;
- \(\forall \eta \in [-1, 0)\), \(u^{\eta} \in C((-\infty, T^{\eta}), \Sigma)\) blows up in the log-log regime at a time \(T^{\eta}\) with \(-1 < T^{\eta} < 0\).

This theorem only describes the flow near the Bourgain–Wang solution along one instability direction. A basic open problem for (NLS) is to completely describe the flow near the ground state \(Q\). Theorem 6 is a first step towards a description of the flow near Bourgain–Wang solutions, which is also an interesting open problem (see §2.7 and [29]).

**2.4. Structural instability of the log-log law.** The Zakharov system in spatial dimensions \(d = 2, 3\) is a model with fundamental physical relevance that is closely related to (NLS) [69], [71]:

\[
\begin{align*}
\text{(Zakharov system)} \quad \begin{cases}
iu_t &= -\Delta u + nu, \\
\frac{1}{c_0^2} n_{tt} &= \Delta n + \Delta |u|^2,
\end{cases}
\end{align*}
\]

for some fixed constant \(0 < c_0 < +\infty\). In the limit as \(c_0 \to +\infty\) one formally recovers (NLS). In dimension \(d = 2\) this system displays a variational structure like the one of (NLS), even though the scaling symmetry is destroyed by the wave coupling. Using bifurcation from the exact solution \(S_{\text{NLS}}(t)\) of the equation (NLS), Glangetas and Merle [18] constructed a one-parameter family of blowup solutions with blowup rate

\[ \|\nabla u(t)\|_{L^2} \sim \frac{C(u_0)}{T - t}. \]
Numerical experiments (Papanicolaou, Sulem, Sulem, and Wang, [57]) indicate that such solutions are stable. We recall also that from Merle [44] all finite-time blowup solutions of (22) satisfy the lower bound

$$\|\nabla u(t)\|_{L^2} \geq \frac{C(u_0)}{T-t}.$$ 

In particular, there are no log-log blowup solutions for (22), which means that in some sense the Zakharov blowup dynamics should be more stable than its asymptotic limit (NLS). This is one more hint that the stable log-log law for (NLS) is closely connected with a specific algebraic structure of (NLS), as suggested by the use of non-linear degeneracy properties in the blowup analysis of Theorem 4. We emphasize that a refined study of singularity formation for the Zakharov system is mostly an open question, though it can be considered as the first step towards an understanding of physically relevant and more complicated systems connected with the Maxwell equations.

2.5. Blowup for the $L^2$ supercritical model (NLS). Consider the $L^2$ supercritical model (NLS)

$$(\text{NLS}) \quad \begin{cases} i\partial_t u + \Delta u + u|u|^{p-1} = 0, \\ u|_{t=0} = u_0, \end{cases} \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d,$$

with the choice

$$p > 1 + \frac{4}{d}.$$ 

The scaling symmetry for (23) is given by

$$u_\lambda(t, x) = \lambda^{2/(p-1)}u(\lambda^2 t, \lambda x),$$

and the homogeneous Sobolev space invariant under this symmetry is $\dot{H}^{s_c}(\mathbb{R}^d)$, where

$$s_c = \frac{d}{2} - \frac{2}{p-1},$$

and $s_c > 0$ corresponds to the $L^2$ supercritical case. The problem is said to be energy subcritical if $s_c < 1$, energy critical if $s_c = 1$, and energy supercritical if $s_c > 1$. We now briefly describe three results for the $L^2$ supercritical (NLS). They were obtained by extending some techniques developed for the results above, thus illustrating the robustness of the methods.

2.5.1. Standing ring solutions. In this section we consider the quintic equation (NLS) (that is, $p = 5$ in (23)), which is $L^2$ critical in dimension $d = 1$ and $L^2$ supercritical in dimensions $d \geq 2$. Radially symmetric solutions blowing up on a (non-trivial) sphere were constructed in [60] for dimension $d = 2$, and in [65] for dimensions $d \geq 3$. We note that this covers the energy-subcritical ($d = 2$),
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energy-critical \((d = 3)\), and energy-supercritical \((d \geq 4)\) cases. Heuristically, in radial variables the problem becomes

\[
i\partial_t u + \partial_r^2 u + \frac{N-1}{r} \partial_r u + u|u|^4 = 0,
\]

and one expects that if singularity formation occurs on the unit sphere, then close to the singularity

\[
\left| \frac{\partial_r u}{r} \right| \sim |\partial_r u| \ll |\partial_r^2 u|,
\]

so that the leading-order blowup dynamics should be given by the one-dimensional quintic equation (NLS). Therefore, solutions blowing up on a sphere in any dimension \(d \geq 2\) can be constructed by perturbation of the log-log dynamics for the \(L^2\) critical one-dimensional (NLS).

See further extensions in [20], [21], [72], where axially symmetric blowup solutions are constructed for the cubic (NLS).

2.5.2. Self-similar solutions. In the \(L^2\) supercritical case the so-called self-similar regime, that is, blowup of the type

\[
\|\nabla u(t)\|_{L^2} \sim \frac{1}{(T-t)^{1/(p-1)+1/2-d/4}} \quad \text{as } t \sim T,
\]

is conjectured to be the stable blowup regime, particularly in view of numerical computations (see, for example, [69]). In [52] this conjecture was actually proved in the ‘slightly’ \(L^2\) supercritical case. More precisely, for the focusing non-linear Schrödinger equation (23) with \(p > 1+4/d\) sufficiently close to \(1+4/d\), existence and stability in \(H^1\) of self-similar finite-time blowup dynamics was proved, and a qualitative description was given of singularity formation near the blowup time. These results were obtained by perturbation-theory methods from the log-log analysis of the \(L^2\) critical case reviewed in §2.2.

2.5.3. Collapsing ring solutions. For the NLS equation (23) with \(d \geq 2\) and \(p\) in the range

\[
1 + \frac{4}{d} < p < \min\left(\frac{d+2}{d-2}, 5\right)
\]

it was proved in [54] that any radially symmetric solution \(u(t)\) blowing up at time \(T\) satisfies the following universal upper bound:

\[
\int_T^T (T-\tau)\|\nabla u(\tau)\|^2_{L^2} d\tau \leq C(u_0)(T-t)^{2\alpha/(1+\alpha)},
\]

where \(\alpha\) is given by

\[
\alpha = \frac{5 - p}{(p-1)(N-1)}.
\]

The upper bound (25) was proved to be sharp, since it is actually achieved by a family of collapsing ring blowup solutions whose existence was first formally
predicted in [15]. The construction of ring solutions in [54] relies on a powerful method developed in [66] for building minimal blowup elements. In particular, these solutions are non-dispersive, that is, they concentrate all their $L^2$ mass at the origin as $t$ tends to the blowup time.

2.6. Energy-critical problems. Energy-critical problems have also attracted considerable attention recently, in particular, wave maps, Schrödinger maps, and the harmonic heat flow:

(Wave maps) $\partial_{tt} u - \Delta u = (|\nabla u|^2 - |\partial_t u|^2) u$,

(Schrödinger maps) $u \wedge \partial_t u = \Delta u + |\nabla u|^2 u$,

(Harmonic heat flow) $\partial_t u = \Delta u + |\nabla u|^2 u$,

$(t, x) = \mathbb{R} \times \mathbb{R}^2$, $u(t, x) \in S^2$. These problems exhibit ground-state stationary solutions which are minimizers of the associated energy. This energy is also left invariant by the scaling symmetry of the problem. The longstanding question of the existence of blowup solutions was solved in [30], [67], [62] for the wave map problem and in [51] for the Schrödinger map problem. New types of dynamics, including ground state-like behaviour or infinite-time blowup, were obtained in [28] and [10] for the associated semilinear wave equations. For the harmonic heat flow the full sequence of stable and unstable blowup regimes was obtained in the two papers [63] and [64], which shed new light on the structure of the flow near the ground state. These papers provided a better understanding of the blowup scenario by the determination of explicit regimes, but a full description of the flow, even for initial data close to the ground state, is far from being complete. Another series of papers [11], [12] aims at classifying all possible behaviours of the solutions, in particular, resolving the so-called ‘soliton resolution conjecture’ for energy-critical focusing non-linear wave equations in dimension 3.

2.7. Open problems. Here is a (non-exhaustive) list of interesting open questions concerning the description of the flow near the ground state solitary wave for the $L^2$ critical equation (NLS).

1. Does the set of initial data corresponding to Bourgain–Wang solutions form a codimension-1 manifold? This is clearly suggested by [53] and [29].

2. In [59] it is shown that for initial data close to the ground state in the energy space, a blowup solution either follows the log-log regime, or blows up at least as fast as the conformal blowup rate (see §2.2). Do there exist blowup solutions in $H^1$ such that the corresponding blowup rate is strictly faster than the conformal one? Another related question in this context concerns the existence of a general upper bound on the blowup rate.

3. Can one obtain a complete classification of the flow behaviour near the solitary wave in the spirit of the recent results for the $L^2$ critical equation (gKdV) in [39]–[41] (see §3)?
3. The $L^2$ critical problem (gKdV)

We now summarize the state of the art for the $L^2$ critical problem (gKdV) in (2), in particular, recent results obtained in [39]–[41]. The model (gKdV) is one-dimensional, and we regard $Q$ as the unique (up to translation) explicit $H^1$ solution of the equation $Q'' + Q^5 = Q$, namely, $Q(x) = 3^{1/4} \cosh^{-1/2}(2x)$.

3.1. Existence of blowup solutions. As discussed in the Introduction, the local $H^1$ Cauchy theory of Kenig, Ponce, and Vega [24], combined with Weinstein’s variational characterization of the ground state [70], ensure that $H^1$ initial data with subcritical mass $\|u_0\|_{L^2} < \|Q\|_{L^2}$ yield global-in-time $H^1$ solutions. Scattering is known only for small (no control) $L^2$ data [25]. One motivation for addressing this model is its similarity to (NLS), but without the specific structure of the problem (NLS), in particular, the pseudoconformal symmetry. We remark that recent advances concerning critical (NLS), (gKdV), and other non-linear dispersive models have been obtained in synergy, any new result or technique for one model being a source of inspiration for the others.

The study of singularity formation for $H^1$ initial data with mass close to the minimal mass, that is,

$$\|Q\|_{L^2} \leq \|u_0\|_{L^2} < \|Q\|_{L^2} + \alpha^* \quad \text{for } \alpha^* \ll 1,$$

was initiated in the series of papers [34], [35], [45], [36], [38], [37]. As for (NLS), variational constraints imply that if (26) holds and the solution blows up in a finite time $T < +\infty$, then near the blowup time it admits a decomposition of the form

$$u(t, x) = \frac{1}{\lambda^{1/2}(t)} (Q + \varepsilon(t, x) - x(t)),$$

where $\|\varepsilon(t)\|_{H^1} \leq \delta(\alpha^*)$,

$$\lim_{\alpha^* \to 0} \delta(\alpha^*) = 0. \text{ Therefore, as in the case of (NLS), the problem reduces to understanding the coupling between the finite-dimensional dynamics governing the solitary wave part, here } (\lambda(t), x(t)), \text{ and the dynamics of the residual part } \varepsilon(t). \text{ Two new tools were introduced in [34], [45], [36], [37] for (gKdV), and later extended to (NLS):}$

- a monotonicity formula and localized virial identities involving $\varepsilon$;
- Liouville-type theorems to classify asymptotic solutions.

The classical conjecture of the existence of a blowup dynamics was resolved in [45], [37]. The original proof in [45] is indirect and based on a classification argument in [34]. Basic information about the blowup structure was obtained in [36], in particular, the asymptotic stability of $Q$ as a blowup profile, that is,

$$\varepsilon(t) \to 0 \text{ in } L^2_{\text{loc}} \text{ as } t \to T.$$

Under the further assumption that

$$\int_{x' > x} u_0^2(x') \, dx' < \frac{C}{x^6} \text{ for } x > 0,$$
the first result on finite-time blowup for negative-energy solutions satisfying (11) was proved in [37], together with a partial result about the blowup rate:

$$\|\nabla u(t_n)\|_{L^2} \lesssim \frac{C_u}{T - t_n}$$

for a subsequence $t_n \to T$.

We will see below that a decay assumption of the type (28) is indeed necessary for finite-time blowup. Finally, it was shown in [38] that $H^1$ solutions with minimal mass $\|u_0\|_{L^2} = \|Q\|_{L^2}$ and decay to the right (28) are global in time ($t \geq 0$), which rules out the possibility of minimal-mass blowup under the assumption (28).

3.2. Blowup description and classification of the flow near the ground state. We now summarize a series of more recent papers [39]–[41] which give a complete description of the flow near the ground state, thus completing the results in [34], [35], [45], [36]–[38]. With respect to these earlier papers, the improved results involve techniques developed for the study of (NLS) [48], [46], [47], [50], [49], [59] and of energy-critical geometrical problems [62], [51], [63].

Let us consider the set of initial data

$$\mathcal{A} = \left\{ u_0 = Q + \varepsilon_0 \text{ with } \|\varepsilon_0\|_{H^1} < \alpha_0 \text{ and } \int_{y > 0} y^{10} \varepsilon_0^2 < 1 \right\} \text{ with } \alpha_0 > 0 \text{ small.}$$

For the equation (gKdV) it is necessary to consider initial data with decay on the right and not simply data in the energy space (see Theorem 11 below). Despite many analogies between the two problems, this is a fundamental difference with respect to analysis of (NLS).

**Theorem 7** (Blowup for non-positive energy solutions in $\mathcal{A}$, [39]). Let $0 < \alpha_0 \ll 1$ and let $u_0 \in \mathcal{A}$. If $E(u_0) \leq 0$ and $u_0$ is not a soliton, then $u(t)$ blows up in a finite time $T$ and there exists a number $\ell_0 = \ell_0(u_0) > 0$ such that

$$\|u_x(t)\|_{L^2} = \frac{\|Q'\|_{L^2} + o(1)}{\ell_0(T - t)} \text{ as } t \to T.$$  \hspace{1cm} (29)

Moreover, there exist $\lambda(t)$, $x(t)$, and $u^* \in H^1$ with $u^* \neq 0$ such that

$$u(t, x) - \frac{1}{\lambda^{1/2}(t)} Q \left( \frac{x - x(t)}{\lambda(t)} \right) \to u^* \text{ in } L^2 \text{ as } t \to T,$$  \hspace{1cm} (30)

$$\lambda(t) = (\ell_0 + o(1))(T - t), \quad x(t) = \left( \frac{1}{\ell_0^2} + o(1) \right) \frac{1}{T - t} \text{ as } t \to T.$$  \hspace{1cm} (31)

**Comments on Theorem 7.**

1. **Blowup rate and stable blowup.** An important feature of Theorem 7 is the derivation of the blowup rate for $u_0 \in \mathcal{A}$ with non-positive energy:

$$\|u_x(t)\|_{L^2} \sim \frac{C(u_0)}{T - t}.$$  \hspace{1cm} (32)
This implies in particular that \( x(t) \to +\infty \) as \( t \to T \). The concentrating soliton and the remainder term \( u^* \) thus split spatially. We note that the blowup rate is far above the scaling blowup law which would hold for \((gKdV)\): \( \|u_x(t)\|_{L^2} \sim c(T-t)^{-1/3} \) (see [51], [63] for a similar gap phenomenon in energy-critical geometrical problems).

To complement Theorem 7, one knows that the set of initial data in \( \mathcal{A} \) which led to the blowup according to (29)–(31) is open in the \( H^1 \) topology (see [39]). Thus, \((T-t)^{-1}\) is the stable blowup behaviour for \((gKdV)\), in contrast to \((NLS)\).

2. **Decay assumption on the right.** We stress the importance of the decay assumption on the right in space for the initial data, which already played a fundamental role in the earlier papers [37], [38]. Indeed, in contrast to the equation \((NLS)\), the universal dynamics cannot be seen now in \( H^1 \), and an additional assumption of decay to the right is required (see Theorem 11 below). Note, however, that we do not assert optimality of the weight \( y^{10} \) in Theorem 7.

3. **Dynamical characterization of \( Q \).** Recall from the variational characterization of \( Q \) that \( E(u_0) \leq 0 \) implies that \( \|u_0\|_{L^2} > \|Q\|_{L^2} \), unless \( u_0 \equiv Q \) up to scaling and translational symmetries. Theorem 7 therefore recovers the dynamical classification of \( Q \) as the unique global zero-energy solution in \( \mathcal{A} \), as in the case of the mass-critical equation \((NLS)\) (see [50]).

The key to a complete description of the flow near \( Q \) is the minimal-mass case, which is solved by the following result.

**Theorem 8** (Existence and uniqueness of the minimal-mass blowup element, [40]).

(i) **Existence.** There exists a solution \( S_{KdV}(t) \in C((0, +\infty), H^1) \) of (2) with minimal mass

\[
\|S_{KdV}(t)\|_{L^2} = \|Q\|_{L^2}
\]

which blows up backwards at the origin:

\[
\|S_{KdV}(t)\|_{H^1} = \|Q\|_{L^2} \frac{o(1)}{t} \quad \text{as} \; t \downarrow 0,
\]

\[
S_{KdV}(t, x) - \frac{1}{t^{1/2}} Q\left(\frac{x + 1/t + \bar{c}t}{t}\right) \to 0 \quad \text{in} \; L^2 \quad \text{as} \; t \downarrow 0,
\]

where \( \bar{c} \) is a universal constant.

(ii) **Uniqueness.** Let \( u_0 \in H^1 \) be such that \( \|u_0\|_{L^2} = \|Q\|_{L^2} \) and assume that the corresponding solution \( u(t) \) of (2) blows up in finite time. Then

\[
u \equiv S_{KdV}
\]

up to symmetries of the flow.

Thus, here one has both the existence of a minimal-mass solution for the critical problem \((gKdV)\) completely analogous to \( S_{NLS}(t) \), and the analogue of Merle’s classification result for \((NLS)\), that is, Theorem 1. In the absence of a pseudoconformal symmetry the uniqueness part in the proof of Theorem 8 is a completely
dynamical result and is closely connected with the analysis of the inhomogeneous model (NLS) in [66]. The existence part is connected with the universality of the ‘exit regime’ (see Theorem 9 below, and §4 for a sketch of the proof).

We remark that \( S_{KdV}(t) \) blows up at the same rate as the stable blowup solutions in Theorem 7. However, the minimal-mass blowup is in essence unstable under perturbations since, for example, the initial data \( S_\varepsilon(-1) = (1 - \varepsilon)S_{KdV}(-1) \) for \( 0 < \varepsilon < 1 \) have subcritical mass and thus lead to global and bounded solutions.

We now recall the main result from [39], [40], which classifies the flow for initial data in \( \mathcal{A} \). An \( L^2 \) modulated tube neighbourhood of the soliton manifold is defined by

\[
\mathcal{T}_{\alpha^*} = \left\{ u \in H^1 \mid \inf_{\lambda_0 > 0, x_0 \in \mathbb{R}} \left\| u - \frac{1}{\lambda_0^{1/2}}Q\left(-\frac{x_0}{\lambda_0}\right)\right\|_{L^2} < \alpha^* \right\},
\]

(34)

where \( \alpha_0 \) (see definition of \( \mathcal{A} \) above Theorem 7) and \( \alpha^* \) are universal constants such that

\[
0 < \alpha_0 \ll \alpha^* \ll 1.
\]

Theorem 9 (Rigidity of the dynamics in \( \mathcal{A} \), [39], [40]). For \( u_0 \in \mathcal{A} \) only three scenarios are possible:

- (Blowup) \( u(t) \in \mathcal{T}_{\alpha^*} \) for all \( t \in [0, T) \), and the solution blows up in a finite time \( T < +\infty \) in the regime described in Theorem 7 ((29), (30), (31)).

- (Soliton) The solution is global, \( u(t) \in \mathcal{T}_{\alpha^*} \) for all \( t > 0 \), and there exist \( \lambda_\infty > 0 \) and \( x(t) \) such that

\[
\frac{\lambda_\infty^{1/2}u(t, \lambda_\infty \cdot + x(t))}{Q} \rightarrow \text{in } H_{loc}^1 \text{ as } t \rightarrow +\infty,
\]

(36)

\[
|\lambda_\infty - 1| \leq o_{\alpha_0 \rightarrow 0}(1), \quad x(t) \sim \frac{t}{\lambda_{\infty}^2} \quad \text{as } t \rightarrow +\infty.
\]

(37)

- (Exit) There exists a \( t^* \in (0, T) \) such that \( u(t^*) \notin \mathcal{T}_{\alpha^*} \). Let \( t_u^* \gg 1 \) be the corresponding exit time:

\[
t_u^* = \sup\{0 < t < T : \forall t' \in [0, t] \quad u(t') \in \mathcal{T}_{\alpha^*}\}.
\]

(38)

Then there exist a number \( \tau^* = \tau^*(\alpha^*) \) (independent of \( u \)) and parameters \( (\lambda_u^*, x_u^*) \) such that

\[
\|(\lambda_u^*)^{1/2}u(t_u^*, \lambda_u^* x + x_u^*) - S_{KdV}(\tau^*, x)\|_{L^2} \leq \delta(\alpha_0),
\]

where \( \delta(\alpha_0) \rightarrow 0 \) as \( \alpha_0 \rightarrow 0 \).

Theorem 9 classifies the possible behaviours of solutions with initial data in \( \mathcal{A} \), including a description of the long-time dynamics in the (Exit) regime. This question turned out to be closely connected with the minimal-mass dynamics, which was unexpected. Indeed, any (Exit) solution at its exit time is close to a universal profile \( S_{KdV}(\tau^*) \), to within a large defocusing scaling \( \lambda_u^* \gg 1 \), under the assumption of dispersive behaviour. In view of the universality of \( S_{KdV} \) as an attractor of all exiting solutions, it is thus an important open problem, in extending Theorem 9,
Singularity formation for critical problems

to understand the behaviour of \( S_{KdV}(t) \) as \( t \to +\infty \). For the mass critical equation (NLS), its solution \( S_{NLS}(t) \) scatters as \( t \to \infty \). Scattering of \( S_{KdV}(t) \) as \( t \to +\infty \) is an unsolved problem.\(^2\) We conjecture that \( S_{KdV}(t) \) really does scatter, and because scattering is an open property in \( L^2 \) (see [25]), we obtain the following corollary.

**Corollary 10** [40]. Assume that \( S_{KdV}(t) \) scatters as \( t \to +\infty \). Then any solution in the (Exit) scenario is global for positive time and scatters as \( t \to +\infty \).

Related theorems on rigidity near solitary waves were also obtained by Nakanishi and Schlag [55], [56] and Krieger, Nakanishi, and Schlag [28] for supercritical wave equations and Schrödinger equations using the invariant-set methods of Berestycki and Cazenave [3], the Kenig–Merle concentration-compactness approach [23], the classification of minimal dynamics [13], [14], and a further ‘no return’ lemma in the (Exit) regime. In the (Exit) regime this lemma shows that the solution cannot come back close to solitons and in fact scatters. (In critical situations such an analysis is more delicate and is not yet complete; see [28].) Both the blowup assertion and the no return lemma in [55], [56] are based on a specific algebraic structure—the virial identity—which does not exist for (gKdV). The above results for (gKdV) are based on an explicit computation of the solution in the various regimes, and not on algebraic virial-type identities. Indeed, by introducing the decomposition (27) one can show that in the leading order, \( \lambda(t) \) satisfies the conditions

\[
\lambda_{tt} = 0, \quad \lambda(0) = 1.
\]

Roughly speaking, the three regimes (Exit), (Blowup), (Soliton) correspond respectively to \( \lambda_t(0) > 0, \lambda_t(0) < 0 \), and the threshold dynamics \( \lambda_t(0) = 0 \) (see the next section for details).

We now indicate a wide range of different blowup rates, including blowup in infinite time, for initial data \( u_0 \notin \mathcal{A} \) having slow decay on the right. Recall that under the decay assumption \( u_0 \in \mathcal{A} \) the blowup rate \( 1/(T - t) \) is universal since the dynamics follows in the main order the ordinary differential equation (39). In contrast, when \( u_0 \notin \mathcal{A} \), the tail of slowly decaying initial data can strongly interact with the solitary wave moving to the right, thus perturbing the main-order dynamics and leading to new exotic singular regimes.

**Theorem 11** (Exotic blowup rates [41]).

(i) Blowup in finite time. For any \( \nu > 11/13 \), there exists a solution \( u \in \mathcal{C}((0,T_0), H^1) \) of (2) blowing up at \( t = 0 \) with

\[
\| u_x(t) \|_{L^2} \sim t^{-\nu} \quad \text{as} \quad t \to 0^+.
\]

(ii) Blowup in infinite time. There exists a solution \( u \in \mathcal{C}([T_0, +\infty), H^1) \) of (2) blowing up to \( +\infty \) with

\[
\| u_x(t) \|_{L^2} \sim e^t \quad \text{as} \quad t \to +\infty.
\]

\(^2\)By scattering for (gKdV), we mean that there exists a solution \( v(t, x) \) of the Airy equation \( \partial_t v + v_{xxx} = 0 \) such that \( \lim_{t \to +\infty} \| S_{KdV}(t) - v(t) \|_{L^2} = 0 \).
For any \( \nu > 0 \) there exists a solution \( u \in \mathcal{C}([T_0, \infty), H^1) \) of (2) blowing up to \( +\infty \) with
\[
\|u_x(t)\|_{L^2} \sim t^{\nu} \text{ as } t \to +\infty.
\]
Moreover, such solutions can be taken arbitrarily close in \( H^1 \) to solitons.

It follows from the proof of Theorem 11 that the blowup rate is directly connected with the behaviour of the initial data on the right. In particular, other types of blowup rates can be produced by changing the tail of the initial data. A similar phenomenon was observed for global-in-time blowup solutions of the energy-critical harmonic heat flow by Gustafson, Nakanishi, and Tsai in [19]. There an explicit formula expressing the growth of the solution at infinity was given directly in terms of the initial data. Continua of blowup rates were presented in the pioneering papers [30], [31] of Krieger, Schlag, and Tataru for energy-critical wave problems (see also Donninger and Krieger [10]). We also refer to stability results [2] and finite-time blowup results [51] for the energy-critical Schrödinger map problem. All these results indicate that the critical topology is not enough by itself to classify the flow near the ground state.

4. An overview of the classification of the flow for (gKdV)

Our aim in this section is to give a simple overview of the proof of the classification theorem for the flow of (gKdV), and in particular the rigidity property formally stated in (39) which readily leads to the three scenarios in Theorem 9.

Notation. Let
\[
Lf = -f'' + f - 5Q^4 f
\]
be the linearized operator close to \( Q \). We recall the rescaling operator
\[
f^\lambda = \frac{1}{\lambda^{1/2}} f \left( \frac{x}{\lambda} \right),
\]
and we introduce the generator of \( L^2 \) scaling
\[
\Lambda f = -\frac{\partial f^\lambda}{\partial \lambda} |_{\lambda=1} = \frac{1}{2} f + y f'.
\]
The \( L^2 \) scalar product is denoted by \( (f, g) = \int fg \, dx \).

4.1. The approximate blowup profile. The starting point of the analysis is to study the leading-order profile of the solution. Knowledge of the instability directions of the flow near \( Q \) is essential, and choosing different directions leads to the scenarios in Theorem 9. Let us look for a solution of the problem (gKdV) in the form of a renormalized bubble
\[
u(t, x) = \frac{1}{\lambda^{1/2}(t)} v\left(s, \frac{x - x(t)}{\lambda(t)} \right), \quad \frac{ds}{dt} = \frac{1}{\lambda^3},
\]
which leads to the evolution equation

\[ \partial_s v + (v_{yy} - v + v^5)_y - \left( \frac{x_s}{\lambda} - 1 \right) v_y - \frac{\lambda_s}{\lambda} \Lambda v = 0. \]

We freeze the modulation equations

\[ \frac{x_s}{\lambda} - 1 = 0, \quad -\frac{\lambda_s}{\lambda} = b \]  \hspace{1cm} (45)

and follow the strategy proposed in [46], [62], [66] for constructing blowup solutions: we look for a slowly modulated profile

\[ v(s, y) = Q_{b(s)}(y), \quad Q_b = Q + bP_1 + b^2P_2 + \cdots. \]

The problem can be expressed as follows: how can we choose the law of variation of the parameter \( b \) to ensure the solvability of the system satisfied by the \((P_i)_{i \geq 1}\)?

In terms of the expansions

\[ b_s = -c_2b^2 - c_3b^3 + \cdots, \quad Q_b = Q + bP_1 + b^2P_2 + \cdots, \quad v(s, y) = Q_{b(s)}(y) \]

we want to solve the following system by expanding in powers of \( b \):

\[
\begin{align*}
0 &= \partial_s v + (v_{yy} - v + v^5)_y - \left( \frac{x_s}{\lambda} - 1 \right) v_y - \frac{\lambda_s}{\lambda} \Lambda v \\
&= b_s \frac{\partial Q_b}{\partial b} + ((Q_b)_{yy} - Q_b + Q_b^5)_y + b\Lambda Q_b.
\end{align*}
\]

- The order \( O(1) \) corresponds to the solitary wave equation

\[ (Q_{yy} - Q + Q^5)_y = 0. \]

- The order \( O(b) \) corresponds to the equation

\[ \Lambda Q + (LP_1)' = 0, \]  \hspace{1cm} (46)

where the operator \( L \) is given by (43). The translation invariance implies the existence of a (unique by the theory of ordinary differential equations) non-trivial element of the kernel: \( LQ' = 0 \). Hence, solvability of (46) requires the cancellation condition

\[ (\Lambda Q, Q) = 0, \]

which indeed holds as a direct consequence of the \( L^2 \) critical invariance of the problem. However, since \( \int_{-\infty}^{+\infty} \Lambda Q \neq 0 \), the solution \( P_1 \) of (46) will have non-trivial growth either to the right or to the left. We choose \( P_1 \) as the unique solution of (46) which decays exponentially on the right as \( y \to +\infty \), but behaves like a non-zero constant as \( y \to -\infty \).

- The order \( O(b^2) \) corresponds to an equation of the form

\[ -c_2P_1 + (LP_2 + N(P_1))' = 0, \]
where \( N(P_1) \) is a certain explicit non-linear term. The solvability condition

\[
( -c_2 P_1 + (N(P_1))', Q ) = 0
\]

leads after some computations to the choice

\[
c_2 = 2.
\]

Therefore, at the formal level, we obtain the following universal dynamical system driving the geometrical parameters:

\[
\begin{align*}
b_s &= -2b^2, & b &= -\frac{\lambda s}{\lambda}, & x_s &= 1, & \frac{ds}{dt} &= \frac{1}{\lambda^3}, \\
\end{align*}
\]

which is equivalent to:

\[
\begin{align*}
\lambda_{tt} &= 0, & x_t &= \frac{1}{\lambda^2}, & b &= -\lambda^2 \lambda_t.
\end{align*}
\]

By scaling invariance, we may always choose \( \lambda(0) = 1 \), and then the phase portrait of (48) is as follows:

- if \( b(0) := b_0 < 0 \), then \( \lambda(t) = 1 - b_0 t \to +\infty \) as \( t \to +\infty \);
- if \( b_0 = 0 \), then \( \lambda(t) = 1 \) for all \( t \geq 0 \);
- if \( b_0 > 0 \), then \( \lambda(t) = 1 - b_0 t \) vanishes at \( T = 1/b_0 \), and integration from \( T \) to \( t \) gives us that \( \lambda(t) = b_0(T - t) \).

The analysis now reduces to showing that for initial data in \( \mathcal{A} \) the above dynamical system governs the leading-order solitonic part of the solution, so that the three regimes in Theorem 9 correspond to perturbations of the above three cases. As it is for the finite-dimensional system (48), the first and third regimes are stable, but the second is the unstable threshold dynamics.

4.2. Non-linear decomposition of the flow. A solution \( u(t, x) \) of (2) close in \( H^1 \) to a soliton decomposes as

\[
u(t, x) = \frac{1}{\lambda^{1/2}(t)}(Qb(t) + \varepsilon)(s, y), \quad y = \frac{x - x(t)}{\lambda(t)}, \quad \frac{ds}{dt} = \frac{1}{\lambda^3(t)},
\]

where \( (s, y) \) are the renormalized space and time variables. The parameters \( (b(t), \lambda(t), x(t)) \) which describe the drift along the soliton family can be uniquely adjusted to obtain the following orthogonality conditions on \( \varepsilon \) for all time:

\[
(\varepsilon, Q) = (\varepsilon, \Lambda Q) = (\varepsilon, y\Lambda Q) = 0.
\]

This choice of orthogonality conditions is justified in Lemma 12 below. The renormalized flow for \( \varepsilon \) becomes:

\[
\varepsilon_s - (L\varepsilon)_y = \left( \frac{\lambda s}{\lambda} + b \right) \Lambda Q + \left( \frac{x_s}{\lambda} - 1 \right) Q' + \frac{\lambda s}{\lambda} \Lambda \varepsilon + O(b^2 + \varepsilon + |\varepsilon|^2).
\]
The dynamical system driving \((b, \lambda, x)\) is obtained from this equation and (49). With respect to the idealized system (47), it contains additional perturbation terms coming from \(b^2\), \(b_s\), and \(\varepsilon\). Typically, the construction of \(Q_b\) and the decay assumption on the initial data (it is necessary at this point) ensure a bound of the form
\[
|b_s + 2b^2| + \left| \frac{\lambda_s}{\lambda} + b \right| \leq \int |\varepsilon|^2 e^{-|y|} + O(b^3). \tag{51}
\]
To justify the dynamics it remains to obtain a uniform control of \(\varepsilon(s)\) in a suitable norm whose choice is essential.

Neglecting for the moment second-order and higher-order terms in the equation for \(\varepsilon(s)\), we focus on the simplified model
\[
\ddot{\varepsilon}_s - (L\varepsilon)_y = \alpha(s)\Lambda Q + \beta(s)Q', \tag{52}
\]
for which we can prove the following monotonicity formula.

**Lemma 12.** Let \(\varepsilon(s, y)\) be a solution of (52) satisfying the orthogonality conditions (49). Then the following statements hold.

1. **Energy conservation at the level \(\varepsilon\):**
   \[
   (L\varepsilon(s), \varepsilon(s)) = (L\varepsilon(0), \varepsilon(0)) \quad \forall \ s. \tag{53}
   \]

2. **Virial estimate:**
   \[
   \frac{d}{ds} \int y\varepsilon^2 = -H(\varepsilon, \varepsilon), \tag{54}
   \]
   where
   \[
   H(\varepsilon, \varepsilon) = \int \left(3\varepsilon_y^2 + \varepsilon^2 - 5Q^4\varepsilon^2 + 20yQ'Q^3\varepsilon^2\right) \geq \mu_0 \|\varepsilon(s)\|_{H^1}^2. \tag{55}
   \]

3. **Mixed virial and energy-monotonicity estimate:** for \(B \gg 1\) and \(\mu_1 > 0\),
   \[
   \frac{d}{ds} \left[ \int \left(\varepsilon_y^2 \varphi \left(\frac{y}{B}\right) + \varepsilon^2 \varphi \left(\frac{y}{B}\right) - 5Q^4\varepsilon^2 \varphi \left(\frac{y}{B}\right)\right) \right] + \mu_1 \int (\varepsilon_y^2 + \varepsilon^2) \varphi' \left(\frac{y}{B}\right) \leq 0, \tag{56}
   \]
   \[
   \int \left(\varepsilon_y^2 \psi \left(\frac{y}{B}\right) + \varepsilon^2 \varphi \left(\frac{y}{B}\right) - 5Q^4\varepsilon^2 \psi \left(\frac{y}{B}\right)\right) \geq \mu_1 \int \left(\varepsilon_y^2 \psi \left(\frac{y}{B}\right) + \varepsilon^2 \varphi \left(\frac{y}{B}\right)\right), \tag{57}
   \]
   where the smooth functions \(\varphi\) and \(\psi\) satisfy
   \[
   \varphi(y) = \begin{cases} \ e^y & \text{for } y < -1, \\ 1 + y & \text{for } -1/2 < y < 1/2, \\ y & \text{for } y > 1, \end{cases} \quad \varphi' \geq 0 \quad \text{on } \mathbb{R}, \tag{58}
   \]
   \[
   \psi(y) = \begin{cases} \ e^{2y} & \text{for } y < -1, \\ 1 & \text{for } y > -1/2, \end{cases} \quad \psi' \geq 0 \quad \text{on } \mathbb{R}. \tag{59}
   \]
Proof. The identities (53) and (54) are obtained by direct computations from the equation for $\varepsilon$ and classical properties of $L$ (see [70]). The coercivity of $H(\varepsilon, \varepsilon)$ was proved in [34]. The estimate (56) is obtained by direct computations and estimates from (55), applied to a suitable localization of the solution $\varepsilon$. It combines in a precise way monotonicity arguments from [34] (reminiscent of the Kato smoothing effect [22]) and localized virial estimates. The coercivity property (57) is a consequence of well-known properties of the operator $L$, namely, for $\mu > 0$\[ (\varepsilon, Q) = (\varepsilon, y\Lambda Q) = (\varepsilon, \Lambda Q) = 0 \implies (L\varepsilon, \varepsilon) \geq \mu \|\varepsilon\|^2_{H^1} \] (see [70]), and localization arguments. $\square$

4.3. The energy-virial Lyapunov functional. The mixed energy-Morawetz estimate (56) provides both a pointwise control of the boundary term and space-time control in a certain local norm. At the non-linear level, this kind of tool may be delicate to handle due to localization in space, the main difficulty being to control the non-linear terms using only the weighted norms in (56). This is a well-known problem, and here the choice of the weights $(\varphi, \psi)$ is essential. The presence of the drift operator $(\lambda_s/\lambda)\Lambda \varepsilon$ in the right-hand side of (50) is an additional difficulty for spatial localization, which requires the assumption of spatial decay on the right, that is, the condition $u_0 \in \mathcal{A}$. For blowup problems, such a strategy based on mixed energy-virial estimates to control the residual term $\varepsilon$ was already used in [62], [66], but in a setting where localization in space is simpler to handle. For (gKdV) the strategy developed in [39] is as follows.

For fixed $B \gg 1$ we define the weighted norms\[ \mathcal{N}(s) = \int \left( \varepsilon^2 y \psi \left( \frac{y}{B} \right) + \varepsilon^2 \varphi \left( \frac{y}{B} \right) \right) \] and the non-linear functional\[ \mathcal{F}_i(s) = \int \left( \varepsilon^2 y \psi \left( \frac{y}{B} \right) + \varepsilon^2 (1 + \mathcal{J}_i) \varphi \left( \frac{y}{B} \right) - \frac{1}{3} (\varepsilon + Q_b)^6 - Q^6_b - 6\varepsilon Q^5_b \right) \psi \left( \frac{y}{B} \right), \] with the non-linear lower-order corrections\[ \mathcal{J}_i = (1 - J_1)^{-4i} - 1, \quad J_1 = (\varepsilon, \rho_1), \quad \rho_1(y) = \frac{4}{(\int Q)^2} \int_{-\infty}^{y} \Lambda Q. \]

Then we obtain the following Lyapunov monotonicity result in the form of a bootstrap bound.$^4$

Proposition 13 [39]. Assume the following conditions on some interval $[0, s_0]$.

(H1) Smallness: \[ \|\varepsilon(s)\|_{L^2} + |b(s)| + \mathcal{N}(s) \leq \kappa^*. \] (60)

---

$^3$Which can be ignored at first reading.

$^4$In order to control the non-linear term.
(H2) Comparison between $b$ and $\lambda$:

$$\frac{|b(s)| + \mathcal{N}(s)}{\lambda^2(s)} \leq \kappa^*. \quad (61)$$

(H3) $L^2$ weighted bound on the right:

$$\int_{y>0} y^{10} \varepsilon^2(s, x) \, dx \leq 10 \left(1 + \frac{1}{\lambda^{10}(s)}\right). \quad (62)$$

Then the following bounds hold on $[0, s_0]$ for $B \gg 1$ and $\mu > 0$.

(i) Scaling-invariant Lyapunov control:

$$\frac{d}{ds} F_1 + \mu \int (\varepsilon_y^2 + \varepsilon^2) \phi' \left(\frac{y}{B}\right) \lesssim |b|^4. \quad (63)$$

(ii) Scaling-weighted $H^1$ Lyapunov control:

$$\frac{d}{ds} \left\{ \frac{F_2}{\lambda^2} \right\} + \frac{\mu}{\lambda^2} \int (\varepsilon_y^2 + \varepsilon^2) \phi' \left(\frac{y}{B}\right) \lesssim \frac{|b|^4}{\lambda^2}. \quad (64)$$

(iii) Pointwise bounds:

$$|J_1| + |J_2| \lesssim \mathcal{N}^{1/2}, \quad \mathcal{N} \lesssim F_j \lesssim \mathcal{N}, \quad j = 1, 2. \quad (65)$$

4.4. Rigidity and selection of the dynamics. Note that the estimate (64) controls the residue $\varepsilon$ independently of the dynamics and thus is valid in all possible regimes: (Blowup), (Soliton), or (Exit). In particular, (64) completely reduces the control of $\varepsilon$ to the sole control of the parameter $b$. Combination of this estimate with the finite-dimensional evolution estimates (51) leads to the following rigidity formula for $b$:

**Lemma 14** (Control of the dynamics for $b$, [39]). Under assumptions (H1)–(H3) of Proposition 13, for all $0 \leq s_1 \leq s_2 < s_0$

$$\left| \frac{b(s_2)}{\lambda^2(s_2)} - \frac{b(s_1)}{\lambda^2(s_1)} \right| \leq \frac{C^*}{10} \left[ \frac{b^2(s_1)}{\lambda^2(s_1)} + \frac{b^2(s_2)}{\lambda^2(s_2)} + \frac{\mathcal{N}(s_1)}{\lambda^2(s_1)} \right], \quad (66)$$

where $C^* > 0$ is a universal constant.

The three scenarios in Theorem 9 are a direct consequence of (66). Indeed, there are only two possibilities.

- Either there exists a time $s_1$ at which $|b(s_1)|$ dominates $\varepsilon$ in the sense that

$$\frac{|b(s_1)|}{\lambda^2(s_1)} \gg \frac{\mathcal{N}(s_1)}{\lambda^2(s_1)}. \quad (67)$$

Then from (66) and the a priori bounds (60), (61), this property is propagated to later times, and
\[
\frac{b(s_2)}{\lambda^2(s_2)} \sim \frac{b(s_1)}{\lambda^2(s_1)} = c_0 \neq 0 \quad \forall s_2 > s_1.
\]
Thus, by (51),
\[
\frac{b}{\lambda^2} \sim \frac{-\lambda_s}{\lambda^3} = -\lambda_t \sim c_0.
\]
If \( c_0 > 0 \), then \( \lambda \) vanishes in a finite time \( T \), \( \lambda(t) \sim c_0(T - t) \), and this corresponds to the (Blowup) case. If \( c_0 < 0 \), then \( \lambda \) is growing and hence so is \( b \), so that \( u(t) \) is moving away in \( L^2 \) from the solitary wave, which is the (Exit) case. We note that (67) is an open condition on the data, and hence both these regimes are stable.

- Or such a time \( s_1 \) does not exist, which means that

\[
\frac{|b(s_1)|}{\lambda^2(s_1)} \ll \mathcal{N}(s_1) \quad \forall s_1, \quad \text{that is,} \quad |b(s_1)| \lesssim \mathcal{N}(s_1).
\]

Then the space-time bounds on \( \varepsilon \) eventually lead to the estimate
\[
\int_0^{+\infty} \left| \frac{\lambda_s}{\lambda} \right| ds \lesssim \int_0^{+\infty} |b(s)| ds \lesssim \int_0^{+\infty} \mathcal{N}(s) ds < +\infty,
\]
and thus
\[
\lambda(s) \to \lambda_\infty > 0 \quad \text{as} \quad s \to +\infty.
\]
This is the (Soliton) dynamics, which is the threshold regime.

### 4.5. Construction of the minimal-mass blowup solution.

The construction of the minimal-mass solution and the determination of the universal behaviour of solutions in the (Exit) regime follow the same compactness strategy. The minimal blowup solution in Theorem 8 is obtained as the limit of a sequence of defocusing solutions. Indeed, we pick a sequence of well-prepared initial data
\[
u_n(0) = Q_{b_n(0)}, \quad b_n(0) = -\frac{1}{n},
\]
which by construction have subcritical mass
\[
\|u_n(0)\|_{L^2} - \|Q\|_{L^2} \sim \frac{c}{n}.
\]
Such solutions are necessarily in the (Exit) regime of Theorem 9, and we denote by \( t^*_n \) the corresponding exit time. Moreover, we have from [39] (see also the formal discussion in §3) a precise description of the flow in the time interval \([0, t^*_n] \); in particular, we know that the solution admits a decomposition of the form
\[
u_n(t, x) = \frac{1}{\lambda_n^{1/2}(t)} (Q_{b_n(t)} + \varepsilon_n) \left( t, \frac{x - x_n(t)}{\lambda_n(t)} \right),
\]
where, in the leading order, the pair \((b_n, \lambda_n)\) behaves as follows:
\[
\frac{b_n(t)}{\lambda_n^2(t)} \sim b_n(0) = -\frac{1}{n}, \quad (\lambda_n)_t \sim -b_n(0),
\]
\[
\lambda_n(t) \sim 1 - b_n(0)t, \quad b_n(t) \sim b_n(0)\lambda_n^2(t).
\]
The exit time $t_n^*$ is the time at which the solution moves strictly away from the solitary wave, which in our notation is equivalent to

$$b_n(t_n^*) \sim -\alpha^*,$$

with $\alpha^*$ independent of $n$. This enables us to compute $t_n^*$ and show using (69) that the solution defocuses:

$$\lambda_n^2(t_n^*) \sim \frac{b_n(t_n^*)}{b_n(0)} \sim n\alpha^* \text{ as } n \to +\infty.$$

Next, we renormalize the flow at time $t_n^*$, considering the solution of (gKdV) defined by

$$v_n(\tau, x) = \frac{1}{\lambda_n^{1/2}(\tau)}(Q_{b_{v_n}} + \varepsilon_{v_n})(\tau, \frac{x - x_{v_n}(\tau)}{\lambda_{v_n}(\tau)}),$$

and from the symmetries of the flow,

$$\lambda_{v_n}(\tau) = \frac{\lambda_n(t_\tau)}{\lambda_n(t_n^*)}, \quad x_{v_n}(\tau) = \frac{x_n(t_\tau) - x_n(t_n^*)}{\lambda_n(t_n^*)}, \quad b_{v_n}(\tau) = b_n(t_\tau), \quad \varepsilon_{v_n}(\tau) = \varepsilon_n(t_\tau).$$

The renormalized parameters can be computed in the main order using (69):

$$\lambda_{v_n}(\tau) \sim \frac{1}{\lambda_n(t_n^*)}[1 - b_n(0)(t_n^* + \tau\lambda_n^3(t_n^*))] \sim 1 - \tau b_n(t_n^*) \sim 1 + \tau\alpha^*.$$

We observe that the law of evolution of $\lambda_{v_n}(\tau)$ in this order does not depend on $n$, which is a remarkable and decisive property in this approach. Letting $n \to +\infty$, we extract a weak limit in $H^1$, $v_n(0) \rightharpoonup v(0)$, such that the corresponding solution $v(\tau)$ of (gKdV) blows up backwards at some finite time $\tau^* \sim -1/\alpha^*$ with the blowup rate $\lambda_v(\tau) \sim \tau - \tau^*$, as expected. The extraction of a weak limit requires uniform estimates on the residue $\varepsilon_{v_n}$. Here it is essential that the set of data $u_n(0)$ be well-prepared, since this implies uniform bounds for $\varepsilon_{v_n}(0) = \varepsilon_{u_n}(t_n^*)$ in $H^1$ and lets us use the $H^1$ weak continuity of the flow in the limiting process. Finally, by the weak convergence we get that $\|v\|_{L^2} \leq \|Q\|_{L^2}$, but since the solution $v(\tau)$ blows up in finite time, $\|v\|_{L^2} = \|Q\|_{L^2}$.

For uniqueness, we refer the reader to [40].
4.6. Solutions in the (Exit) regime. Now we prove the universality of $S_{KdV}$ as an attractor in the (Exit) case. For this we consider a sequence of initial data $(u_0)_n$ with $\left\| (u_0)_n \right\|_{L^2} \to \|Q\|_{L^2}$ as $n \to +\infty$ such that the corresponding solutions of $(gKdV)$ are in the (Exit) regime. We write the solution at the exit time in the form (68), renormalize the flow, and extract a weak limit as $n \to +\infty$ as before. The strategy of the proof is similar to the construction of the minimal-mass solution, except that since the data is not well-prepared, no uniform $H^1$ bound on $v_n(0)$ can be obtained. To get around this, we use two additional ingredients:

1) a concentration-compactness argument on sequences of solutions in the critical $L^2$ space in the spirit of [23], using the tools developed in [68] and [26] for the Airy group, which enables us to extract a non-trivial weak limit with suitable dynamical controls;

2) refined local $H^1$ bounds on $v_n(\tau)$ in order to ensure that the $L^2$ limit of this sequence actually belongs to $H^1$.

Hence, the weak limit is a minimal-mass $H^1$ blowup element, and by the uniqueness statement in Theorem 8, the limit is $S_{KdV}$ up to the symmetries of the equation, which provides the final conclusion of Theorem 9.

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