CALCULUS OF VARIATIONS AND OPTIMAL CONTROL FOR GENERALIZED FUNCTIONS

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Abstract. We present an extension of some results of higher order calculus of variations and optimal control to generalized functions. The framework is the category of generalized smooth functions, which includes Schwartz distributions, while sharing many nonlinear properties with ordinary smooth functions. We prove the higher order Euler-Lagrange equations, the D’Alembert principle in differential form, the du Bois-Reymond optimality condition and the Noether’s theorem. We start the theory of optimal control proving a weak form of the Pontryagin maximum principle and the Noether’s theorem for optimal control. We close with a study of a singularly variable length pendulum, oscillations damped by two media and the Pais–Uhlenbeck oscillator with singular frequencies.

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1. Introduction: singular Lagrangian mechanics

Leibniz’s axiom *Natura non facit saltus* seems to be inconsistent not only at the atomic level, but also for several macroscopic phenomena. Even the simple motion of an elastic bouncing ball seems to be more easily modeled using non-differentiable functions than classical $C^2$ ones, at least if we are not interested to model the non-trivial behavior at the collision times. Clearly, we can always use smooth functions to approximately model the forces acting on the ball, but this would introduce additional parameters (see e.g. [62, 71] for smooth approximations of steep potentials in the billiard problem). Therefore, the motivation to introduce a suitable kind of generalized functions formalism in classical mechanics is clear: is it possible, e.g., to generalize Lagrangian mechanics so that the Lagrangian can be presented as a generalized function? This would undoubtedly be of an applicable advantage, since many relevant systems are described by singular Lagrangians: non-smooth constraints, collisions between two or more bodies, motion in different or in granular media, discontinuous propagation of rays of light, even turning on the switch of an electrical circuit, to name but a few, and only in the framework of classical physics. Indeed, this type of problems is widely studied (see e.g. [6, 30, 36, 39, 44–46, 55, 68, 70]), but sometimes the presented solutions are not general or hold only for special conditions and particular potentials.

From the purely mathematical point of view, an enlarged space of solutions (with distributional derivatives) for variational problems leads to the so-called Lavrentiev phenomenon, see [7, 29]. Variational problems with singular solutions (e.g. non-differentiable on a dense set, or with infinite derivatives at some points) are also studied, see [12, 13, 29, 66].

In this sense, the fact that since J.D. Marsden’s works [51–53] there has not been many further attempt to use Sobolev-Schwartz distributions for the description of Hamiltonian mechanics, can be considered as a clue that the classical distributional framework is not well suited to face this problem in general terms (see also [60] for a similar approach with a greater focus on the generalization in geometry). In [51–53], J.D. Marsden introduces distributions on manifolds based on flows. Since the traditional system of Hamilton’s equations breaks down, Marsden considers the flow as a limit of smooth ones. On the other hand, Kunzinger et al. in [38] (using Colombeau generalized functions) critically analyses the regularization approach put forward by Marsden and constructed a counter example for the main flow theorems of [51]. In this field, a number of problems remains open, see again [51]. For example, in the non-smooth case the variational theorems fail, and the study of the virial equation for hard spheres in a box is still an open question. All this motivates the use of different spaces of nonlinear generalized functions in mathematical physics and in singular calculus of variations, see e.g. [3, 10, 27, 31, 36, 58, 72].

In this paper, we introduce an approach to variational problems involving singularities that allows the extension of the classical theory with very natural statements and proofs. We are interested in extremizing functionals which are either distributional themselves or whose set of extremals includes generalized functions. Clearly, distribution theory, being a linear theory, has certain difficulties when nonlinear problems are in play.

To overcome this type of problems, we are going to use the category of generalized smooth functions (GSF), see [22–25, 43]. This theory seems to be a good candidate, since it is an extension of classical distribution theory which allows to model nonlinear singular problems, while at the same time sharing many nonlinear properties with ordinary smooth functions, like the closure with respect to composition and several non trivial classical theorems of the calculus. One could describe GSF as a methodological restoration of Cauchy-Dirac’s original conception of generalized function, see [14, 35, 41]. In essence, the idea of Cauchy and Dirac (but also of Poisson, Kirchhoff, Helmholtz, Kelvin and Heaviside) was to view generalized functions as suitable types of smooth set-theoretical maps obtained from ordinary smooth maps depending on suitable infinitesimal or infinite parameters. For example, the density of a Cauchy-Lorentz distribution with an infinitesimal scale parameter was used by Cauchy to obtain classical properties which nowadays are attributed to the Dirac delta, cf. [35].

In the present work, the foundation of the calculus of variations is set for functionals defined by arbitrary GSF. This in particular applies to any Schwartz distribution and any Colombeau generalized function (see e.g. [9, 31]).
The main aim of the paper is to start the higher-order calculus of variations and the theory of optimal control for GSF. The structure of the paper is as follows. We start with an introduction into the setting of GSF and give basic notions concerning GSF and their calculus that are needed for the calculus of variations (Sec. 2). After the basic definitions to set the problem in the framework of GSF, we prove the higher order Euler-Lagrange equations, the D’Alembert principle in differential form, the du Bois-Reymond optimality condition and the Noether’s theorem (Sec. 3). In Sec. 4, we start the theory of optimal control proving a weak form of the Pontryagin maximum principle and the Noether’s theorem, and in Sec. 5 we close with a study of a singularly variable length pendulum, oscillations damped by two media and the Pais–Uhlenbeck oscillator with singular frequencies.

The paper is self-contained, in the sense that it contains all the statements required for the proofs of calculus of variations we are going to present. If proofs of preliminaries are omitted, we clearly give references to where they can be found. Therefore, to understand this paper, only a basic knowledge of distribution theory is needed.

2. Basic notions

The new ring of scalars. In this work, $I$ denotes the interval $(0, 1] \subseteq \mathbb{R}$ and we will always use the variable $\varepsilon$ for elements of $I$; we also denote $\varepsilon$-dependent nets $x \in \mathbb{R}^I$ simply by $(x_{\varepsilon})$. By $\mathbb{N}$ we denote the set of natural numbers, including zero.

We start by defining a new simple non-Archimedean ring of scalars that extends the real field $\mathbb{R}$. The entire theory is constructive to a high degree, e.g. neither ultrafilter nor non-standard method are used. For all the proofs of results in this section, see [22–25].

Definition 1. Let $\rho = (\rho_\varepsilon) \in (0, 1]^I$ be a net such that $(\rho_\varepsilon) \to 0$ as $\varepsilon \to 0^+$ (in the following, such a net will be called a gauge), then

(i) $\mathcal{I}(\rho) := \{(\rho_\varepsilon^a) : a \in \mathbb{R}_{>0}\}$ is called the asymptotic gauge generated by $\rho$.

(ii) If $P(\varepsilon)$ is a property of $\varepsilon \in I$, we use the notation $\forall^0\varepsilon : P(\varepsilon)$ to denote $\exists \varepsilon_0 \in I \forall \varepsilon \in (0, \varepsilon_0] : P(\varepsilon)$. We can read $\forall^0\varepsilon$ as for $\varepsilon$ small.

(iii) We say that a net $(x_\varepsilon) \in \mathbb{R}^I$ is $\rho$-moderate, and we write $(x_\varepsilon) \in \mathbb{R}_\rho$ if

$$\exists (J_\varepsilon) \in \mathcal{I}(\rho) : x_\varepsilon = O(J_\varepsilon) \text{ as } \varepsilon \to 0^+,$$

i.e., if

$$\exists N \in \mathbb{N} \forall^0\varepsilon : \|x_\varepsilon\| \leq \rho_\varepsilon^{-N}.$$

(iv) Let $(x_\varepsilon), (y_\varepsilon) \in \mathbb{R}^I$, then we say that $(x_\varepsilon) \sim_\rho (y_\varepsilon)$ if

$$\forall (J_\varepsilon) \in \mathcal{I}(\rho) : x_\varepsilon = y_\varepsilon + O(J_\varepsilon^{-1}) \text{ as } \varepsilon \to 0^+,$$

that is if

$$\forall n \in \mathbb{N} \forall^0\varepsilon : \|x_\varepsilon - y_\varepsilon\| \leq \rho_\varepsilon^n.$$  \hfill (2.1)

This is a congruence relation on the ring $\mathbb{R}_\rho$ of moderate nets with respect to pointwise operations, and we can hence define

$$^{\ast}\mathbb{R} := \mathbb{R}_\rho/\sim_\rho,$$

which we call Robinson-Colombeau ring of generalized numbers. This name is justified by [8,63]: Indeed, in [63] A. Robinson introduced the notion of moderate and negligible nets depending on an arbitrary fixed infinitesimal $\rho$ (in the framework of nonstandard analysis); independently, J.F. Colombeau, cf. e.g. [8] and references therein, studied the same concepts without using nonstandard analysis, but considering only the particular gauge $\rho_\varepsilon = \varepsilon$.

We can also define an order relation on $^{\ast}\mathbb{R}$ by saying that $[x_\varepsilon] \leq [y_\varepsilon]$ if there exists $(z_\varepsilon) \in \mathbb{R}^I$ such that $(z_\varepsilon) \sim_\rho 0$ (we then say that $(z_\varepsilon)$ is $\rho$-negligible) and $x_\varepsilon \leq y_\varepsilon + z_\varepsilon$ for $\varepsilon$ small. Equivalently, we have that $x \leq y$ if and only if there exist representatives $[x_\varepsilon] = x$ and $[y_\varepsilon] = y$ such that $x_\varepsilon \leq y_\varepsilon$ for all $\varepsilon$. Although the order $\leq$ is not total, we still have the possibility to define the infimum $[x_\varepsilon] \wedge [y_\varepsilon] := [\min(x_\varepsilon, y_\varepsilon)]$, the supremum $[x_\varepsilon] \vee [y_\varepsilon] := [\max(x_\varepsilon, y_\varepsilon)]$ of a finite number of generalized numbers. Henceforth, we will also use the customary notation $^{\ast}\mathbb{R}^*$ for the set of
invertible generalized numbers, and we write $x < y$ to say that $x \leq y$ and $x - y \in \tilde{\mathbb{R}}^*$. Our notations for intervals are: $[a, b] := \{ x \in \tilde{\mathbb{R}} \mid a \leq x \leq b \}$, $[a, b]_\mathbb{R} := [a, b] \cap \mathbb{R}$, and analogously for segments $[x, y] := \{ x + r \cdot (y - x) \mid r \in [0, 1] \} \subseteq \tilde{\mathbb{R}}^n$ and $[x, y]_{\mathbb{R}^n} = [x, y] \cap \mathbb{R}^n$.

As in every non-Archimedean ring, we have the following

**Definition 2.** Let $x \in \tilde{\mathbb{R}}^n$ be a generalized number, then

1. $x$ is infinitesimal if $|x| \leq r$ for all $r \in \mathbb{R}^*$. If $x = |x|$ this is equivalent to $\lim_{x \to 0^+} x\varepsilon = 0$.
2. We write $x \approx y$ if $x - y$ is infinitesimal.
3. $x$ is finite if $|x| \geq r$ for all $r \in \mathbb{R}^*$. If $x = |x|$ this is equivalent to $\lim_{x \to 0^+} |x| = +\infty$.
4. $x$ is finite if $|x| \leq r$ for some $r \in \mathbb{R}^*$.

For example, setting $d\rho := [\rho_x] \in \tilde{\mathbb{R}}$, we have that $d\rho^n \in \tilde{\mathbb{R}}^n$, $n \in \mathbb{N}^*$, is an invertible infinitesimal, whose reciprocal is $d\rho^{-n} = [\rho_x^{-n}]$, which is necessarily a positive infinite number. Of course, in the ring $\tilde{\mathbb{R}}$ there exist generalized numbers which are not in any of the three classes of Def. 2, like e.g. $x = \frac{1}{2} \sin \left( \frac{1}{2} \right)$.

The following result is useful to deal with positive and invertible generalized numbers. For its proof, see e.g. [31].

**Lemma 3.** Let $x \in \tilde{\mathbb{R}}$. Then the following are equivalent:

1. $x$ is invertible and $x \geq 0$, i.e. $x > 0$.
2. For each representative $(x_\varepsilon) \in \mathbb{R}^\varepsilon$ of $x$ we have $\forall \varepsilon > 0 \exists x_\varepsilon > 0$.
3. For each representative $(x_\varepsilon) \in \mathbb{R}^\varepsilon$ of $x$ we have $\exists m \in \mathbb{N}^\varepsilon \forall \varepsilon > 0 x_\varepsilon > \rho_\varepsilon^m$.
4. There exists a representative $(x_\varepsilon) \in \mathbb{R}^\varepsilon$ of $x$ such that $\exists m \in \mathbb{N}^\varepsilon x_\varepsilon > \rho_\varepsilon^m$.

**Topologies on $\mathbb{R}^n$.** On the $\mathbb{R}$-module $\mathbb{R}^n$ we can consider the natural extension of the Euclidean norm, i.e. $[x_\varepsilon] := |x_\varepsilon| \in \tilde{\mathbb{R}}$, where $[x_\varepsilon] \in \mathbb{R}^n$. Even if this generalized norm takes values in $\tilde{\mathbb{R}}$, it shares some essential properties with classical norms:

$$
\begin{align*}
|x| &= x \vee (-x) \\
|x| &\geq 0 \\
|x| &= 0 \implies x = 0 \\
|y \cdot x| &= |y| \cdot |x| \\
|x + y| &\leq |x| + |y| \\
|x| - |y| &\leq |x - y|.
\end{align*}
$$

It is therefore natural to consider on $\tilde{\mathbb{R}}^n$ a topology generated by balls defined by this generalized norm and the set of radii $\tilde{\mathbb{R}}^*_{>0}$ of positive invertible numbers:

**Definition 4.** Let $c \in \tilde{\mathbb{R}}^n$ and $x, y \in \tilde{\mathbb{R}}^n$, then:

1. $B(c) := \{ x \in \tilde{\mathbb{R}}^n \mid |x - c| < r \}$ for each $r \in \tilde{\mathbb{R}}^*_{>0}$.
2. $B^\varepsilon(c) := \{ x \in \mathbb{R}^n \mid |x - c| < r \}$ for each $r \in \mathbb{R}^*_{>0}$, denotes an ordinary Euclidean ball in $\mathbb{R}^n$ if $c \in \mathbb{R}^n$.

The relation $<$ has better topological properties as compared to the usual strict order relation $a \leq b$ and $a \neq b$ (that we will never use) because the set of balls $\{ B(c) \mid r \in \tilde{\mathbb{R}}^*_{>0}, c \in \tilde{\mathbb{R}}^n \}$ is a base for a sequentially Cauchy complete topology on $\tilde{\mathbb{R}}^n$ called sharp topology. We will call sharply open set any open set in the sharp topology. The existence of infinitesimal neighborhoods (e.g. $r = d\rho$) implies that the sharp topology induces the discrete topology on $\mathbb{R}$. This is a necessary result when one has to deal with continuous generalized functions which have infinite derivatives. In fact, if $f'(x_0)$ is infinite, we have $f(x) \approx f(x_0)$ only for $x \approx x_0$, see [23]. Also open intervals are defined using the relation $<$, i.e. $(a, b) := \{ x \in \tilde{\mathbb{R}} \mid a < x < b \}$. Note that by Lem. 3.(iii), for all $r \in \tilde{\mathbb{R}}^*_{>0}$ there exists $m \in \mathbb{N}$ such that $r \geq d\rho^m$. Therefore, also the set of balls $\{ B^{d\rho^m}(c) \mid m \in \mathbb{N}, c \in \tilde{\mathbb{R}}^n \}$ is a base for the sharp topology.

We will also need the following result.
Lemma 5. Let \( a, b \in \mathbb{R}^\ast \) such that \( a < b \), then the interior \( \text{int}([a,b]) \) in the sharp topology is dense in \([a,b]\).

The reader can feel uneasy in considering a ring of scalars such as \( \mathbb{R}^\ast \) instead of a field. On the other hand, as mentioned above, we will see that all our generalized functions are continuous in the sharp topology. Therefore, the following result is partly reassuring:

Lemma 6. Invertible elements of \( \mathbb{R}^\ast \) are dense in the sharp topology, i.e.
\[
\forall h \in \mathbb{R}^\ast \forall \varepsilon \in \mathbb{R}^\ast > 0 \exists k \in (h - \delta, h + \delta) : k \text{ is invertible.}
\]

2.1. Open, closed and bounded sets generated by nets. A natural way to obtain sharply open, closed and bounded sets in \( \mathbb{R}^n \) is by using a net \((A_\varepsilon)\) of subsets \(A_\varepsilon \subseteq \mathbb{R}^n\). We have two ways of extending the membership relation \( x_\varepsilon \in A_\varepsilon \) to generalised points \( x_\varepsilon \in \mathbb{R}^n \) (cf. \([24, 57]\)):

Definition 7. Let \((A_\varepsilon)\) be a net of subsets of \(\mathbb{R}^n\), then
(i) \[\{x_\varepsilon \in \mathbb{R}^n \mid \forall \varepsilon : x_\varepsilon \in A_\varepsilon\}\] is called the internal set generated by the net \((A_\varepsilon)\).

(ii) Let \((x_\varepsilon)\) be a net of points of \(\mathbb{R}^n\), then we say that \(x_\varepsilon \in A_\varepsilon\), and we read it as \((x_\varepsilon)\) strongly belongs to \((A_\varepsilon)\), if
(i) \(\forall \varepsilon : x_\varepsilon \in A_\varepsilon\).
(ii) If \((x'_\varepsilon) \sim_\rho (x_\varepsilon)\), then also \(x'_\varepsilon \in A_\varepsilon\) for \(\varepsilon\) small.

Moreover, we set \((A_\varepsilon) : \{x_\varepsilon \in \mathbb{R}^n \mid x_\varepsilon \in A_\varepsilon\}\), and we call it the strongly internal set generated by the net \((A_\varepsilon)\).

(iii) We say that the internal set \(K = [A_\varepsilon]\) is sharply bounded if there exists \(M \in \mathbb{R}^\ast \) such that \(K \subseteq B_M(0)\).

(iv) Finally, we say that the \((A_\varepsilon)\) is a sharply bounded net if there exists \(N \in \mathbb{R}^\ast \) such that \(\forall \varepsilon \exists x_\varepsilon \in A_\varepsilon : |x_\varepsilon| \leq \rho_\varepsilon^{-N}\).

Therefore, \(x \in [A_\varepsilon]\) if there exists a representative \([x_\varepsilon] = x\) such that \(x_\varepsilon \in A_\varepsilon\) for \(\varepsilon\) small, whereas this membership is independent from the chosen representative in case of strongly internal sets. An internal set generated by a constant net \(A_\varepsilon = A \subseteq \mathbb{R}^n\) will simply be denoted by \([A]\).

The following theorem (cf. \([24, 25, 57]\)) shows that internal and strongly internal sets have dual topological properties:

Theorem 8. For \(\varepsilon \in I,\) let \(A_\varepsilon \subseteq \mathbb{R}^n\) and let \(x_\varepsilon \in \mathbb{R}^n\). Then we have
(i) \([x_\varepsilon] \in [A_\varepsilon]\) if and only if \(\forall q \in \mathbb{R}^\ast \forall \varepsilon : d(x_\varepsilon, A_\varepsilon) \leq \rho_\varepsilon^2\). Therefore \([x_\varepsilon] \in [A_\varepsilon]\) if and only if \(d(x_\varepsilon, A_\varepsilon) = 0 \in \mathbb{R}^\ast\).

(ii) \([x_\varepsilon] \in A_\varepsilon\) if and only if \(\exists q \in \mathbb{R}^\ast \forall \varepsilon : d(x_\varepsilon, A_\varepsilon) > \rho_\varepsilon^2\), where \(A_\varepsilon^c := \mathbb{R}^n \setminus A_\varepsilon\). Therefore, if \((d(x_\varepsilon, A_\varepsilon^c)) \in \mathbb{R}_\varepsilon^\ast\), then \([x_\varepsilon] \in A_\varepsilon\) if and only if \(d(x_\varepsilon, A_\varepsilon^c) > 0\).

(iii) \([A_\varepsilon]\) is sharply closed.

(iv) \(A_\varepsilon\) is sharply open.

(v) \([A_\varepsilon] = [\text{cl}(A_\varepsilon)]\), where \(\text{cl}(S)\) is the closure of \(S \subseteq \mathbb{R}^n\).

(vi) \(A_\varepsilon = (\text{int}(A_\varepsilon))\), where \(\text{int}(S)\) is the interior of \(S \subseteq \mathbb{R}^n\).

For example, it is not hard to show that the closure in the sharp topology of a ball of center \(c = [c_\varepsilon]\) and radius \(r = [r_\varepsilon] > 0\) is
\[
B_\varepsilon(c) = \left\{x \in \mathbb{R}^d \mid |x - c| \leq r\right\} = \left[B^\varepsilon_{r_\varepsilon}(c_\varepsilon)\right],
\]
whereas
\[
B_\varepsilon(c) = \left\{x \in \mathbb{R}^d \mid |x - c| < r\right\} = \left(B^\varepsilon_{r_\varepsilon}(c_\varepsilon)\right).
\]

2.2. Generalized smooth functions and their calculus. Using the ring \(\mathbb{R}^\ast\), it is easy to consider a Gaussian with an infinitesimal standard deviation. If we denote this probability density by \(f(x, \sigma)\), and if we set \(\sigma = [\sigma_\varepsilon] \in \mathbb{R}^\ast_\varepsilon\), where \(\sigma \approx 0\), we obtain the net of smooth functions \((f(\cdot, \sigma_\varepsilon))_{\varepsilon \in I}\). This is the basic idea we are going to develop in the following.
**Definition 9.** Let $X \subseteq \tilde{\mathbb{R}}^n$ and $Y \subseteq \tilde{\mathbb{R}}^d$ be arbitrary subsets of generalized points. Then we say that

$$f : X \to Y$$

is a generalized smooth function if there exists a net $f_\varepsilon \in C^\infty(\Omega_\varepsilon, \mathbb{R}^d)$ defining $f$ in the sense that

(i) $X \subseteq \langle \Omega_\varepsilon \rangle$,

(ii) $f([x_\varepsilon]) = [f_\varepsilon(x_\varepsilon)] \in Y$ for all $x = [x_\varepsilon] \in X$,

(iii) $(\partial^\alpha f_\varepsilon(x_\varepsilon)) \in \mathbb{R}^d$ for all $x = [x_\varepsilon] \in X$ and all $\alpha \in \mathbb{N}^n$.

The space of generalized smooth functions (GSF) from $X$ to $Y$ is denoted by $^*\mathcal{GC}^\infty(X,Y)$.

Let us note explicitly that this definition states minimal logical conditions to obtain a set-theoretical map from $X$ into $Y$ and defined by a net of smooth functions of which we can take arbitrary derivatives still remaining in the space of $\rho$-moderate nets. In particular, the following Thm. 10 states that the equality $f([x_\varepsilon]) = [f_\varepsilon(x_\varepsilon)]$ is meaningful, i.e. that we have independence from the representatives for all derivatives $[x_\varepsilon] \in X \mapsto (\partial^\alpha f_\varepsilon(x_\varepsilon)) \in \tilde{\mathbb{R}}^d$, $\alpha \in \mathbb{N}^n$.

**Theorem 10.** Let $X \subseteq \tilde{\mathbb{R}}^n$ and $Y \subseteq \tilde{\mathbb{R}}^d$ be arbitrary subsets of generalized points. Let $f_\varepsilon \in C^\infty(\Omega_\varepsilon, \mathbb{R}^d)$ be a net of smooth functions that defines a generalized smooth map of the type $X \to Y$, then

(i) $\forall \alpha \in \mathbb{N}^n \forall \alpha \varepsilon \in \mathbb{R}_s^\rho : [x_\varepsilon] = [x'_\varepsilon] \in X \Rightarrow (\partial^\alpha f_\varepsilon(x_\varepsilon)) = (\partial^\alpha f_\varepsilon(x'_\varepsilon))$.

(ii) Each $f \in ^*\mathcal{GC}^\infty(X,Y)$ is continuous with respect to the sharp topologies induced on $X, Y$.

(iii) $f : X \to Y$ is a GSF if and only if there exists a net $v_\varepsilon \in C^\infty(\mathbb{R}^n, \mathbb{R}^d)$ defining a generalized smooth map of type $X \to Y$ such that $f = [v_\varepsilon(-)]_X$.

(iv) GSF are closed with respect to composition, i.e. subsets $S \subseteq \tilde{\mathbb{R}}^k$ with the trace of the sharp topology, and GSF as arrows form a subcategory of the category of topological spaces. We will call this category $^*\mathcal{GC}^\infty$, the category of GSF. Therefore, with pointwise sum and product, any space $^*\mathcal{GC}^\infty(X, \tilde{\mathbb{R}})$ is an algebra.

Similarly, we can define generalized functions of class $^*\mathcal{GC}^k$, with $k \leq +\infty$:

**Definition 11.** Let $X \subseteq \tilde{\mathbb{R}}^n$ and $Y \subseteq \tilde{\mathbb{R}}^d$ be arbitrary subsets of generalized points and $k \in \mathbb{N} \cup \{+\infty\}$. Then we say that

$$f : X \to Y$$

is a generalized $C^k$ function if there exists a net $f_\varepsilon \in C^k(\Omega_\varepsilon, \mathbb{R}^d)$ defining $f$ in the sense that

(i) $X \subseteq \langle \Omega_\varepsilon \rangle$,

(ii) $f([x_\varepsilon]) = [f_\varepsilon(x_\varepsilon)] \in Y$ for all $x = [x_\varepsilon] \in X$,

(iii) $(\partial^\alpha f_\varepsilon(x_\varepsilon)) \in \mathbb{R}^d$ for all $x = [x_\varepsilon] \in X$ and all $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leq k$.

(iv) $\forall \alpha \in \mathbb{N}^n \forall [x_\varepsilon], [x'_\varepsilon] \in X : |\alpha| = k, [x_\varepsilon] = [x'_\varepsilon] \Rightarrow (\partial^\alpha f_\varepsilon(x_\varepsilon)) = (\partial^\alpha f_\varepsilon(x'_\varepsilon))$.

(v) For all $\alpha \in \mathbb{N}^n$, with $|\alpha| = k$, the map $[x_\varepsilon] \in X \mapsto (\partial^\alpha f_\varepsilon(x_\varepsilon)) \in \tilde{\mathbb{R}}^d$ is continuous in the sharp topology.

The space of generalized $C^k$ functions from $X$ to $Y$ is denoted by $^*\mathcal{GC}^k(X,Y)$.

Note that properties (iv), (v) are required only for $|\alpha| = k$ because for lower length they can be proved using property (iii) (and the classical mean value theorem for $f_\varepsilon$ (see e.g. [24, 25]). From (i) and (ii) of Thm. 10 it follows that this definition of $^*\mathcal{GC}^k$ is equivalent to Def. 9 if $k = +\infty$. Moreover, properties similar to (iii), (iv) of Thm. 10 can also be proved for $^*\mathcal{GC}^k$.

The differential calculus for $^*\mathcal{GC}^{k+1}$ maps can be introduced by showing existence and uniqueness of another $^*\mathcal{GC}^k$ map serving as incremental ratio (sometimes this is called derivative $\partial f/\partial x$ la Carathéodory, see e.g. [37]).

**Theorem 12.** Let $U \subseteq \tilde{\mathbb{R}}^n$ be a sharply open set, let $v = [v_\varepsilon] \in \tilde{\mathbb{R}}^n$, $k \in \mathbb{N} \cup \{+\infty\}$, and let $f \in ^*\mathcal{GC}^{k+1}(U, \tilde{\mathbb{R}})$ be a $^*\mathcal{GC}^{k+1}$ map generated by the net of functions $f_\varepsilon \in C^{k+1}(\Omega_\varepsilon, \mathbb{R})$. Then

(i) There exists a sharp neighborhood $T$ of $U \times \{0\}$ and a map $r \in ^*\mathcal{GC}^k(T, \tilde{\mathbb{R}})$, called the generalized incremental ratio of $f$ along $v$, such that

$$\forall (x,h) \in T : f(x + hv) = f(x) + h \cdot r(x,h).$$
(ii) Any two generalized incremental ratios coincide on a sharp neighborhood of \( U \times \{0\} \), so that we can use the notation \( \frac{\partial f}{\partial v}[x;h] := v(x, h) \) if \((x, h)\) are sufficiently small.

(iii) We have \( \frac{\partial f}{\partial v}|x;0| = \frac{\partial f}{\partial v}v(x) \) for any \( x \) and we can thus define \( Df(x) \cdot v := \frac{\partial f}{\partial v}(x) := \frac{\partial f}{\partial v}|x;0| \), so that \( \frac{\partial f}{\partial v} \in \mathcal{GC}^{1}(U, \mathbb{R}) \).

Note that this result permits us to consider the partial derivative of \( f \) with respect to an arbitrary generalized vector \( v \in \mathbb{R}^n \) which can be, e.g., infinitesimal or infinite. Using recursively this definition, we can also define subsequent differentials \( D^j f(x) \) as \( j \)-multilinear maps, and we set \( D^j f(x) \cdot h^j := D^j f(x)(h, \ldots, h) \) if \( j \leq k \). The set of all the \( j \)-multilinear maps \( (\mathbb{R}^n)^j \rightarrow \mathbb{R}^d \) over the ring \( \mathbb{R} \) will be denoted by \( L^j(\mathbb{R}^n, \mathbb{R}^d) \). For \( A = [A_{\varepsilon}(\cdot)] \in L^j(\mathbb{R}^n, \mathbb{R}^d) \), we set \( |A| := ||A_{\varepsilon}|| \), the generalized number defined by the operator norms of the multilinear maps \( A_{\varepsilon} \in L^j(\mathbb{R}^n, \mathbb{R}^d) \).

The following result follows from the analogous properties for the nets of smooth functions defining \( f \) and \( g \).

**Theorem 13.** Let \( U \subseteq \mathbb{R}^n \) be an open subset in the sharp topology, let \( v \in \mathbb{R}^n \) and \( f, g \in \mathcal{GC}^{k+1}(U, \mathbb{R}^d) \). Then

(i) \( \frac{\partial (f+g)}{\partial v} = \frac{\partial f}{\partial v} + \frac{\partial g}{\partial v} \)

(ii) \( \frac{\partial (f \cdot g)}{\partial v} = \frac{\partial f}{\partial v} \cdot g + f \cdot \frac{\partial g}{\partial v} \)

(iii) For each \( x \in U \), the map \( \frac{\partial f}{\partial v}(x).v := \frac{\partial f}{\partial v}(x) \) is \( \mathbb{R} \)-linear in \( v \in \mathbb{R}^n \)

(iv) For each \( x \in U \), the map \( \frac{\partial f}{\partial v}(x).v := \frac{\partial f}{\partial v}(x) \) is \( \mathbb{R} \)-linear in \( v \in \mathbb{R}^n \)

Let \( U \subseteq \mathbb{R}^n \) and \( V \subseteq \mathbb{R}^d \) be open subsets in the sharp topology and \( g \in \mathcal{GC}^{k+1}(V, U) \), \( f \in \mathcal{GC}^{k+1}(U, \mathbb{R}^d) \) be generalized maps. Then for all \( x \in V \) and all \( v \in \mathbb{R}^d \), we have \( \frac{\partial (f \cdot g)}{\partial v}(x) = \frac{\partial f}{\partial v}(x) \cdot g(x) + f(x) \cdot \frac{\partial g}{\partial v}(x) \).

Note that the absolute value function \( |\cdot| : \mathbb{R} \rightarrow \mathbb{R} \) is not a GSF because its derivative is not sharply continuous at the origin; clearly, it is a \( \mathcal{GC}^{0} \) function. This is a good motivation to introduce integration of \( \mathcal{GC}^{k} \) functions. One dimensional integral calculus is based on the following

**Theorem 14.** Let \( k \in \mathbb{N} \cup \{+\infty\} \) and \( f \in \mathcal{GC}^{k}([a, b], \mathbb{R}) \) be defined in the interval \([a, b] \subseteq \mathbb{R} \), where \( a < b \). Let \( c \in [a, b] \). Then, there exists one and only one generalized \( \mathcal{C}^{k+1} \) map \( F \in \mathcal{GC}^{k+1}(a, b, \mathbb{R}) \) such that \( F(c) = 0 \) and \( F'(x) = f(x) \) for all \( x \in [a, b] \). Moreover, if \( f \) is defined by the net \( f_{\varepsilon} \in \mathcal{C}^{k}(\mathbb{R}, \mathbb{R}) \) and \( c = [c_{\varepsilon}] \), then \( F(x) = \int_{c_{\varepsilon}}^{x} f_{\varepsilon}(s)ds \) for all \( x = [x_{\varepsilon}] \in [a, b] \).

We can thus define

**Definition 15.** Under the assumptions of Thm. 14, we denote by \( f_{\varepsilon}^{(-)} := \int_{0}^{(-)} f(s)ds \in \mathcal{GC}^{k+1}([a, b], \mathbb{R}) \) the unique generalized \( \mathcal{C}^{k+1} \) map such that:

(i) \( f_{\varepsilon}^{(-)} = 0 \)

(ii) \( \left( f_{\varepsilon}^{(-)} \right)'(x) = \frac{d}{dx} \int_{a}^{x} f(s)ds = f(x) \) for all \( x \in [a, b] \).

All the classical rules of integral calculus hold in this setting:

**Theorem 16.** Let \( f \in \mathcal{GC}^{k}(U, \mathbb{R}) \) and \( g \in \mathcal{GC}^{k}(V, \mathbb{R}) \) be generalized \( \mathcal{C}^{k} \) maps defined on sharply open domains in \( \mathbb{R} \). Let \( a, b \in \mathbb{R} \) with \( a < b \) and \( c, d \in [a, b] \subseteq U \cap V \), then

(i) \( \int_{c}^{d} (f + g) = \int_{c}^{d} f + \int_{c}^{d} g \)

(ii) \( \int_{c}^{d} \lambda f = \lambda \int_{c}^{d} f \) for all \( \lambda \in \mathbb{R} \)

(iii) \( \int_{c}^{d} f = \int_{c}^{e} f + \int_{e}^{d} f \) for all \( e \in [a, b] \)

(iv) \( \int_{c}^{d} f = -\int_{d}^{c} f \)

(v) \( \int_{c}^{d} f' = f(d) - f(c) \)
(vii) Let $\rho$ for this direction is the ODE $y = g(x)$ for all $x \in [a, b]$, then $\int_a^b f \leq \int_a^b g$.

(viii) Let $a, b, c, d \in \mathbb{R}$, with $a < b$ and $c < d$, and $f \in \mathcal{GC}^{k+1}([a, b] \times [c, d], \mathbb{R}^d)$, then
\[
\frac{d}{ds} \int_a^b f(\tau, s) \, d\tau = \int_a^b \frac{\partial}{\partial s} f(\tau, s) \, d\tau \quad \forall s \in [c, d].
\]

**Theorem 17.** Let $f \in \mathcal{GC}^k(U, \mathbb{R}^d)$ and $\varphi \in \mathcal{GC}^{k+1}(V, U)$ be GSF defined on sharply open domains in $\mathbb{R}^d$. Let $a, b \in \mathbb{R}$, with $a < b$, such that $[a, b] \subseteq V$, $\varphi(a) < \varphi(b)$, $|\varphi(a), \varphi(b)| \subseteq U$. Finally, assume that $\varphi([a, b]) \subseteq [\varphi(a), \varphi(b)]$. Then
\[
\int_{\varphi(a)}^{\varphi(b)} f(t) \, dt = \int_a^b f(\varphi(s)) \cdot \varphi'(s) \, ds.
\]

We also have a generalization of Taylor formula:

**Theorem 18.** Let $f \in \mathcal{GC}^k(U, \mathbb{R}^d)$ be a generalized $C^{k+1}$ function defined in the sharply open set $U \subseteq \mathbb{R}^d$. Let $a, b \in \mathbb{R}$ such that the line segment $[a, b] \subseteq U$, and set $h := b - a$. Then, for all $n \in \mathbb{N}_{\leq k}$ we have

(i) $\exists \xi \in [a, b] : f(a + h) = \sum_{j=0}^{n} \frac{d^j f(a)}{j!} \cdot h^j + \frac{d^{n+1} f(\xi)}{(n+1)!} \cdot h^{n+1}.$

(ii) $f(a + h) = \sum_{j=0}^{n} \frac{d^j f(a)}{j!} \cdot h^j + \frac{1}{n!} \cdot \int_0^1 (1-t)^n d^{n+1} f(a + th) \cdot h^{n+1} \, dt.$

Moreover, there exists some $R \in \mathbb{R}_{>0}$ such that
\[
\forall k \in B_R(0) \exists \xi \in [a, a+k] : f(a+k) = \sum_{j=0}^{n} \frac{d^j f(a)}{j!} \cdot k^j + \frac{d^{n+1} f(\xi)}{(n+1)!} \cdot k^{n+1} \quad (2.3)
\]
\[
\frac{d^{n+1} f(\xi)}{(n+1)!} \cdot k^{n+1} = \frac{1}{n!} \cdot \int_0^1 (1-t)^n d^{n+1} f(a + tk) \cdot k^{n+1} \, dt \approx 0. \quad (2.4)
\]

Formulas (i) and (ii) correspond to a plain generalization of Taylor’s theorem for ordinary functions with Lagrange and integral remainder, respectively. Dealing with generalized functions, it is important to note that this direct statement also includes the possibility that the differential $d^{n+1} f(\xi)$ may be infinite at some point. For this reason, in (2.3) and (2.4), considering a sufficiently small increment $k$, we get more classical infinitesimal remains $d^{n+1} f(\xi) \cdot k^{n+1} \approx 0$. We can also define right and left derivatives as e.g. $f'(a) := f'_+(a) := \lim_{a < t \to a} f'(t)$, which always exist if $f \in \mathcal{GC}^{k+1}([a, b], \mathbb{R}^d)$.

Analogously to the classical case, we say that $x_0 \in X$ is a local minimum of $f \in \mathcal{GC}^k(X, \mathbb{R})$ if there exists a sharply open neighbourhood (in the trace topology) $Y \subseteq X$ of $x_0$ such that $f(x_0) \leq f(y)$ for all $y \in Y$. A local maximum is defined accordingly.

**Lemma 19.** Let $X \subseteq \mathbb{R}$ and let $f \in \mathcal{GC}^\infty(X, \mathbb{R}).$

(i) If $x_0 \in X$ is a sharply interior local minimum of $f$ then $f'(x_0) = 0$.

(ii) Let $a, b \in \mathbb{R}$ with $a < b$, $[a, b] \subseteq X$ and $x_0$ be a sharply interior point of $[a, b]$. Assume that $x_0$ is a local minimum of $f$. Then $f''(x_0) \geq 0$. Vice versa, if $f'(x_0) = 0$ and $f''(x_0) > 0$, then $x_0$ is a local minimum of $f$.

2.3. Embedding of Sobolev-Schwartz distributions and Colombeau generalized functions. We finally recall two results that give a certain flexibility in constructing embeddings of Schwartz distributions. Note that both the infinitesimal $\rho$ and the embedding of Schwartz distributions have to be chosen depending on the problem we aim to solve. A trivial example in this direction is the ODE $y' = y/ez$, which cannot be solved for $\rho = (\varepsilon)$, but it has a solution for $\rho = (e^{-1/\varepsilon})$. As another simple example, if we need the property $H(0) = 1/2$, where $H$ is the Heaviside function, then we have to choose the embedding of distributions accordingly. See also [26, 48] for further details.

If $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $r \in \mathbb{R}_{>0}$ and $x \in \mathbb{R}^n$, we use the notations $r \circ \varphi$ for the function $x \in \mathbb{R}^n \mapsto \varphi(x) / \rho(x)$. The embedding of distributions into Colombeau’s space requires suitable choices for the parameter $\rho$, which depend on the problem we aim to solve.
Let \( b \in \mathbb{R}_p \) be a net such that \( \lim_{\varepsilon \to 0^+} b_\varepsilon = +\infty \). Let \( d \in (0, 1) \). There exists a net \( \{ \psi_\varepsilon \} \) of \( D(\mathbb{R}^n) \) with the properties:

(i) \( \text{supp}(\psi_\varepsilon) \subseteq B_1(0) \) and \( \psi_\varepsilon \) is even for all \( \varepsilon \in I \).

(ii) Let \( \omega_n \) denote the surface area of \( S^n_{-1} \) and set \( c_n := \frac{2^n}{\omega_n} \) for \( n > 1 \) and \( c_1 := 1 \), then \( \psi_\varepsilon(0) = c_n \) for all \( \varepsilon \in I \).

(iii) \( \int_0^1 \psi_\varepsilon(x) e^{x} dx = 1 \) for all \( \varepsilon \in I \).

(iv) \( \forall \alpha \in \mathbb{N}^n \exists p \in \mathbb{N} : \sup_{x \in \mathbb{R}^n} |\partial^\alpha \psi_\varepsilon(x)| = O(b_\varepsilon) \) as \( \varepsilon \to 0^+ \).

(v) \( \forall j \in \mathbb{N}, \exists \varepsilon : 1 \leq |\alpha| \leq j \rightarrow \int x^\alpha \cdot \psi_\varepsilon(x) dx = 0 \).

(vi) \( \forall \eta \in \mathbb{R}_0, \exists \varepsilon : \int |\psi_\varepsilon| \leq 1 + \eta \).

(vii) If \( n = 1 \), then the net \( \{ \psi_\varepsilon \} \) can be chosen so that \( \int_{-\infty}^0 \psi_\varepsilon d = d \).

In particular \( \psi_\varepsilon := b_\varepsilon^{-1} \circ \psi_\varepsilon \) satisfies (iii) - (vi).

Concerning embeddings of Schwartz distributions, we have the following result, where \( c(\Omega) := \{[x]\in[\Omega] | \exists K \in \Omega \forall \varepsilon : x \in K \} \) is called the set of compactly supported points in \( \Omega \subseteq \mathbb{R}^n \).

**Theorem 21.** Under the assumptions of Lemma 20, let \( \Omega \subseteq \mathbb{R}^n \) be an open set and let \( \psi_\varepsilon \) be the net defined in 20. Then the mapping

\[
\iota^b_\Omega : T \in \mathcal{E}'(\Omega) \rightarrow [(T * \psi_\varepsilon)(*-)] \in \mathcal{C}^\infty \left( c(\Omega), \mathcal{E} \right)
\]

uniquely extends to a sheaf morphism of real vector spaces

\[
i^b : \mathcal{D}' \rightarrow \mathcal{C}^\infty \left( c(-), \mathcal{E} \right),
\]

and satisfies the following properties:

(i) If \( b \geq d_\rho^{-a} \) for some \( a \in \mathbb{R}_0 \), then \( \iota^b|\mathcal{C}^\infty(-) : \mathcal{C}^\infty(-) \rightarrow \mathcal{C}^\infty \left( c(-), \mathcal{E} \right) \) is a sheaf morphism of algebras and \( \iota^b(f)(x) = f(x) \) for all smooth functions \( f \in \mathcal{C}^\infty(\Omega) \) and all \( x \in \Omega \);

(ii) If \( T \in \mathcal{E}'(\Omega) \) then \( \text{supp}(T) = \text{supp}(\iota^b_\Omega(T)) \);

(iii) \( \lim_{\varepsilon \to 0^+} \iota^b_\Omega(T, \varphi) = \langle T, \varphi \rangle \) for all \( \varphi \in \mathcal{D}(\Omega) \) and all \( T \in \mathcal{D}'(\Omega) \);

(iv) \( \iota^b \) commutes with partial derivatives, i.e. \( \partial^\alpha \left( \iota^b_\Omega(T) \right) = \iota^b_\Omega(\partial^\alpha T) \) for each \( T \in \mathcal{D}'(\Omega) \) and \( \alpha \in \mathbb{N} \).

Concerning the embedding of Colombeau generalized functions, we recall that the special Colombeau algebra \( \mathcal{G}^s(\Omega) := \mathcal{E}_M(\Omega)/\mathcal{N}^s(\Omega) \) of moderate nets over negligible nets, where the former is

\[
\mathcal{E}_M(\Omega) := \{(u_\varepsilon) \in \mathcal{C}^\infty(\Omega)^I | \forall K \in \Omega \forall \alpha \in \mathbb{N}^n \forall N \in \mathbb{N} : \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{-N}) \}
\]

and the latter is

\[
\mathcal{N}^s(\Omega) := \{(u_\varepsilon) \in \mathcal{C}^\infty(\Omega)^I | \forall K \in \Omega \forall \alpha \in \mathbb{N}^n \forall m \in \mathbb{N} : \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{-m}) \}
\]

Using \( \rho = (\varepsilon) \), we have the following compatibility result:

**Theorem 22.** A Colombeau generalized function \( u = (u_\varepsilon) + \mathcal{N}^s(\Omega)^d \in \mathcal{G}^s(\Omega)^d \) defines a GSF \( u : [x] \in c(\Omega) \rightarrow [u_\varepsilon(x)] \in \mathcal{E}^d \). This assignment provides a bijection of \( \mathcal{G}^s(\Omega)^d \) onto \( \mathcal{C}^\infty \left( c(\Omega), \mathcal{E}^d \right) \) for every open set \( \Omega \subseteq \mathbb{R}^n \).

**Example 23.**

(i) Let \( \delta, H \in \mathcal{C}^\infty \left( \mathcal{E}, \mathcal{E}^d \right) \) be the corresponding \( i^b \)-embeddings of the Dirac delta and of the Heaviside function. Then \( \delta(x) = b \cdot \psi(b \cdot x) \), where \( \psi(x) := [\psi_\varepsilon(x)] \). We have that \( \delta(0) = b \geq d_\rho^{-a} \) and \( \delta(x) = 0 \) if \( |x| \geq 1 \) because of Lem. 20(i). The condition \( |x| \geq 1 \) surely holds e.g. if \( |x| > \frac{1}{\varepsilon \log^2 \rho} \approx 0 \) for some \( k \in \mathbb{R}_0 \) because \( b \geq d_\rho^{-a} \); in particular,
it is satisfied if $|x| > r$ for some $r \in \mathbb{R}_{>0}$. By the intermediate value theorem (see [25]),
$\delta$ takes any value in the interval $[0, b] \subseteq \mathbb{R}$. Similar properties can be stated e.g. for
$\delta^2(x) = b^2 \cdot \psi(b \cdot x)^2$.

(ii) Analogously, we have $H(x) = 1$ if $|bx| \geq 1$ and $x > 0$; $H(x) = 0$ if $|bx| \geq 1$ and $x < 0$; finally
$H(0) = \frac{1}{2}$ because of Lem. 20.(i). By the intermediate value theorem, $H$ takes any value in the interval
$[0, 1] \subseteq \mathbb{R}$.

(iii) The composition $\delta \circ \delta \in \mathcal{GC}^\infty(\mathbb{R}, \mathbb{R})$ is given by $(\delta \circ \delta)(x) = b \psi(b^2 \psi(bx))$ and is an even function. If $|bx| \geq 1$, then $(\delta \circ \delta)(x) = b$. Since $(\delta \circ \delta)(0) = 0$, again using the intermediate value theorem, we have that $\delta \circ \delta$ takes any value in the interval $[0, b] \subseteq \mathbb{R}$. Suitably choosing the net $(\psi\varepsilon)$ it is possible to have that if $-\frac{1}{2\delta} \leq x \leq \frac{1}{2\delta}$ (hence $x$ is infinitesimal),
then $(\delta \circ \delta)(x) = 0$. If $x = \frac{k}{b}$ for some $k \in \mathbb{N}_{>0}$, then $x$ is still infinitesimal but $(\delta \circ \delta)(x) = b$.

Analogously, one can deal with compositions such as $H \circ \delta$ and $\delta \circ H$.

See Fig. 2.1 for a graphical representations of $\delta$ and $H$. The infinitesimal oscillations shown in this figure occur only in an infinitesimal neighborhood of the origin because of (i) in example 23 and because $\frac{-1}{\log_{\psi\varepsilon}} \approx 0$. They can be proved to actually occur as a consequence of Lem. 20.(v)
which is a necessary property to prove Thm. 21.(i), see [25, 26]. It is well-known that the latter
property is one of the core ideas to bypass the Schwartz’s impossibility theorem, see e.g. [31].

2.4. Functionally compact sets and multidimensional integration.

2.4.1. Extreme value theorem and functionally compact sets. For GSF, suitable generalizations
of many classical theorems of differential and integral calculus hold: intermediate value theorem,
mean value theorems, suitable sheaf properties, local and global inverse function theorems, Banach
fixed point theorem and a corresponding Picard-Lindelöf theorem both for ODE and PDE,
see [22, 24–26, 48].
Even though the intervals \([a,b] \subseteq \mathbb{R}, a, b \in \mathbb{R}\), are not compact in the sharp topology (see [24]), analogously to the case of smooth functions, a \(\mathcal{G}^k\) map satisfies an extreme value theorem on such sets. In fact, we have:

**Theorem 24.** Let \(f \in \mathcal{G}^k(X, \mathbb{R})\) be a generalized \(C^k\) function defined on the subset \(X\) of \(\mathbb{R}^n\). Let \(\emptyset \neq K = [K_\varepsilon] \subseteq X\) be an internal set generated by a sharply bounded net \((K_\varepsilon)\) of compact sets \(K_\varepsilon \subset \mathbb{R}^n\), then

\[
\exists \alpha, \beta \in K \forall x \in K : f(\alpha) \leq f(x) \leq f(\beta). \tag{2.5}
\]

We shall use the assumptions on \(K\) and \((K_\varepsilon)\) given in this theorem to introduce a notion of “compact subset” which behaves better than the usual classical notion of compactness in the sharp topology.

**Definition 25.** A subset \(K\) of \(\mathbb{R}^n\) is called *functionally compact*, denoted by \(K \in \mathfrak{f} \mathbb{R}^n\), if there exists a net \((K_\varepsilon)\) such that

\begin{itemize}
  \item[(i)] \(K = [K_\varepsilon] \subseteq \mathbb{R}^n\).
  \item[(ii)] \(\exists R \in \mathbb{R}_{>0} : K \subseteq B_R(0)\), i.e. \(K\) is sharply bounded.
  \item[(iii)] \(\forall \varepsilon \in I : K_\varepsilon \subseteq \mathbb{R}^n\).
\end{itemize}

If, in addition, \(K \subseteq U \subseteq \mathbb{R}^n\) then we write \(K \in \mathfrak{f} U\). Finally, we write \([K_\varepsilon] \in \mathfrak{f} U\) if \((\text{ii}), (\text{iii})\) and \([K_\varepsilon] \subseteq U\) hold. Any net \((K_\varepsilon)\) such that \([K_\varepsilon] = K\) is called a *representative of \(K)\.

We motivate the name *functionally compact subset* by noting that on this type of subsets, GSF have properties very similar to those that ordinary smooth functions have on standard compact sets.

**Remark 26.**

\begin{itemize}
  \item[(i)] By Thm. 8.(iii), any internal set \(K = [K_\varepsilon]\) is closed in the sharp topology. In particular, the open interval \((0,1) \subseteq \mathbb{R}\) is not functionally compact since it is not closed.
  \item[(ii)] If \(H \in \mathbb{R}^n\) is a non-empty ordinary compact set, then the internal set \([H]\) is functionally compact. In particular, \([0,1] = [[0,1]_{\mathfrak{f}}]\) is functionally compact.
  \item[(iii)] The empty set \(\emptyset = \emptyset \in \mathfrak{f} \mathbb{R}\).
  \item[(iv)] \(\mathbb{R}^n\) is not functionally compact since it is not sharply bounded.
  \item[(v)] The set of compactly supported points \(\mathfrak{c}(\mathbb{R})\) is not functionally compact because the GSF \(f(x) = x\) does not satisfy the conclusion (2.5) of Thm. 24.
\end{itemize}

In the present paper, we need the following properties of functionally compact sets.

**Theorem 27.** Let \(K \subseteq X \subseteq \mathbb{R}^n, f \in \mathcal{G}^k(X, \mathbb{R}^d)\). Then \(K \in \mathfrak{f} \mathbb{R}^n\) implies \(f(K) \in \mathfrak{f} \mathbb{R}^d\).

As a corollary of this theorem and Rem. (26),(ii) we get

**Corollary 28.** If \(a, b \in \mathbb{R}\) and \(a \leq b\), then \([a,b] \in \mathfrak{f} \mathbb{R}\).

Let us note that \(a, b \in \mathbb{R}\) can also be infinite numbers, e.g. \(a = d\rho^{-N}, b = d\rho^{-M}\) or \(a = -d\rho^{-N}, b = -d\rho^{-M}\) with \(M > N\), so that e.g. \([-d\rho^{-N}, d\rho^M] \subseteq \mathbb{R}\). Finally, in the following result we consider the product of functionally compact sets:

**Theorem 29.** Let \(K \in \mathfrak{f} \mathbb{R}^n\) and \(H \in \mathfrak{f} \mathbb{R}^d\), then \(K \times H \in \mathfrak{f} \mathbb{R}^{n+d}\). In particular, if \(a_i \leq b_i\) for \(i = 1,\ldots,n\), then \(\prod_{i=1}^n [a_i, b_i] \in \mathfrak{f} \mathbb{R}^n\).

A theory of compactly supported GSF has been developed in [22], and it closely resembles the classical theory of LF-spaces of compactly supported smooth functions. It establishes that for suitable functionally compact subsets, the corresponding space of compactly supported GSF contains extensions of all Colombeau generalized functions, and hence also of all Schwartz distributions.

As in the classical case (see e.g. [21]), thanks to the extreme value theorem 24 and the property of functionally compact sets \(K\), we can naturally define a topology on the space \(\mathcal{G}^k(K, \mathbb{R}^d)\):
Definition 30. Let $K \Subset_{\ell} \bar{\mathbb{R}}^n$ be a functionally compact set such that $K = \overline{K}$ (so that partial derivatives at sharply boundary points can be defined as limits of partial derivatives at sharply interior points; such $K$ are called solid sets). Let $l \in \mathbb{N}_{\leq k}$ and $v \in \mathcal{G}^k(K, \bar{\mathbb{R}}^d)$. Then

$$
\|v\|_l := \max_{|\alpha| \leq l} \max_{1 \leq i \leq d} \left( |\partial^\alpha v^i(M_{ni})|, |\partial^\alpha v^i(m_{ni})| \right) \in \bar{\mathbb{R}},
$$

where $M_{ni}, m_{ni} \in K$ satisfy

$$
\forall x \in K : \partial^\alpha v^i(m_{ni}) \leq \partial^\alpha v^i(x) \leq \partial^\alpha v^i(M_{ni}).
$$

The following result (see [43] and [2] for a similar approach) permits us to calculate the (generalized) norm $\|v\|_l$ using any net $(v_e)$ that defines $v$.

Lemma 31. Under the assumptions of Def. 30, let $[K_e] = K \Subset_{\ell} \bar{\mathbb{R}}^n$ be any representative of $K$. Then we have:

(i) If the net $(v_e)$ defines $v$, then $\|v\|_l = \left[ \max_{|\alpha| \leq l} \max_{x \in K_e} |\partial^\alpha v^i(x)| \right] \in \bar{\mathbb{R}}$;

(ii) $\|v\|_l \geq 0$;

(iii) $\|v\|_l = 0$ if and only if $v = 0$;

(iv) $\forall c \in \bar{\mathbb{R}} : \|c \cdot v\|_l = |c| \cdot \|v\|_l$;

(v) For all $u \in \mathcal{G}^k(K, \bar{\mathbb{R}}^d)$, we have $\|u + v\|_l \leq \|u\|_l + \|v\|_l$ and $\|u \cdot v\|_l \leq c_1 \cdot \|u\|_l \cdot \|v\|_l$ for some $c_1 \in \bar{\mathbb{R}}_{>0}$.

Using these $\bar{\mathbb{R}}$-valued norms, we can naturally define a topology on the space $\mathcal{G}^k(K, \bar{\mathbb{R}}^d)$.

Definition 32. Let $K \Subset_{\ell} \bar{\mathbb{R}}^n$ be a solid set. Let $l \in \mathbb{N}_{\leq k}$, $u \in \mathcal{G}^k(K, \bar{\mathbb{R}}^d)$, $r \in \bar{\mathbb{R}}_{>0}$, then

(i) $B^k_l(u) := \left\{ v \in \mathcal{G}^k(K, \bar{\mathbb{R}}^d) \mid \|v - u\|_l < r \right\}$

(ii) If $U \subseteq \mathcal{G}^k(K, \bar{\mathbb{R}}^d)$, then we say that $U$ is a sharply open set if

$$
\forall u \in U \exists v \in \mathbb{N}_{\leq k} \exists r \in \bar{\mathbb{R}}_{>0} : B^k_l(u) \subseteq U.
$$

One can easily prove that sharply open sets form a sequentially Cauchy complete topology on $\mathcal{G}^k(K, \bar{\mathbb{R}}^d)$, see e.g. [23, 49]. The structure $\left( \mathcal{G}^k(K, \bar{\mathbb{R}}^d), (\|-\|)_l \right)$ has the usual properties of a graded Fréchet space if we replace everywhere the field $\mathbb{R}$ with the ring $\bar{\mathbb{R}}$, and for this reason it is called an $\bar{\mathbb{R}}$-graded Fréchet space.

2.4.2. Multidimensional integration. Finally, to deal with higher-order calculus of variations, we have to introduce multidimensional integration of GSF on suitable subsets of $\bar{\mathbb{R}}^n$ (see [25]).

Definition 33. Let $\mu$ be a measure on $\mathbb{R}^n$ and let $K$ be a functionally compact subset of $\bar{\mathbb{R}}^n$. Then, we call $K$ $\mu$-measurable if the limit

$$
\mu(K) := \lim_{m \to \infty} [\mu(B^\mu_{\rho^m}(K_e))] \quad (2.6)
$$

exists for some representative $(K_e)$ of $K$. Here $m \in \mathbb{N}$, the limit is taken in the sharp topology on $\bar{\mathbb{R}}$ since $[\mu(B^\mu_{\rho^m}(K_e))] \in \bar{\mathbb{R}}$, and $B^\mu_{\rho^m}(A) := \{ x \in \mathbb{R}^n : d(x, A) \leq m \}$.

In the following result, we show that this definition generates a correct notion of multidimensional integration for GSF.

Theorem 34. Let $K \subseteq \bar{\mathbb{R}}^n$ be $\mu$-measurable.

(i) The definition of $\mu(K)$ is independent of the representative $(K_e)$.

(ii) There exists a representative $(K_e)$ of $K$ such that $\mu(K) = [\mu(K_e)]$.

(iii) Let $(K_e)$ be any representative of $K$ and let $f \in \mathcal{G}^\infty(K, \bar{\mathbb{R}})$ be a GSF defined by the net $(f_e)$. Then

$$
\int_K f d\mu := \lim_{m \to \infty} \int_{B^\mu_{\rho^m}(K_e)} f_e d\mu \in \bar{\mathbb{R}}
$$

exists and its value is independent of the representative $(K_e)$. 
(iv) There exists a representative \((K_x)\) of \(K\) such that
\[
\int_K f \, d\mu = \left[ \int_{K_x} f_x \, d\mu \right] \in \mathcal{R}
\]
for each \(f \in \mathcal{GC}^\infty(K, \mathcal{R})\).

(v) If \(K = \prod_{i=1}^n [a_i, b_i]\), then \(K\) is \(\lambda\)-measurable (\(\lambda\) being the Lebesgue measure on \(\mathbb{R}^n\)) and for all \(f \in \mathcal{GC}^\infty(K, \mathcal{R})\) we have
\[
\int_K f \, d\lambda = \left[ \int_{a_{1,\epsilon}}^{b_{1,\epsilon}} dx_1 \ldots \int_{a_{n,\epsilon}}^{b_{n,\epsilon}} f(x_1, \ldots, x_n) \, dx_n \right] \in \mathcal{R}
\]
for any representatives \((a_{i,\epsilon}), (b_{i,\epsilon})\) of \(a_i\) and \(b_i\) respectively. Therefore, if \(n = 1\), this notion of integral coincides with that of Thm. 14 and Def. 15.

(vi) Let \(K \subseteq \mathcal{R}^n\) be \(\lambda\)-measurable, where \(\lambda\) is the Lebesgue measure, and let \(\varphi \in \mathcal{GC}^\infty(K, \mathcal{R})\) be such that \(\varphi^{-1} \in \mathcal{GC}^\infty(\varphi(K), \mathcal{R}^n)\). Then \(\varphi(K)\) is \(\lambda\)-measurable and
\[
\int_{\varphi(K)} f \, d\lambda = \int_K (f \circ \varphi) |\det(d\varphi)| \, d\lambda
\]
for each \(f \in \mathcal{GC}^\infty(\varphi(K), \mathcal{R}).\)

3. Higher-order calculus of variations for generalized functions

In this section, we prove for GSF the higher-order Euler-Lagrange equation, the du Bois-Reymond optimality condition and the Noether’s theorem. We start with some definition of basic notions and notations. Using the sharp topology of Def. 32, we can define when a curve is a minimizer of a given functional. Note explicitly that there are no restrictions on the generalized numbers \(a, b \in \mathcal{R}\), \(a < b\), e.g. they can also both be infinite numbers.

Definition 35. Let \(t_1, t_2 \in \mathcal{R}\), with \(t_1 < t_2\), and let \(m \in \mathbb{N}_{>0}\), then
(i) For all \(q_1 := (q_{1,0}, \ldots, q_{1,m-1}), q_2 := (q_{2,0}, \ldots, q_{2,m-1}) \in \mathcal{R}^d \times m\), we set
\[
\mathcal{GC}^\infty_{bd}(q_1, q_2; m) := \left\{ q \in \mathcal{GC}^\infty([t_1, t_2], \mathcal{R}^d) \mid q^{(i)}(t_j) = q_{i,j}^j \forall i = 0, \ldots, m - 1, j = 1, 2 \right\}.
\]
We simply set \(\mathcal{GC}^\infty_{bd}(0, 0; m), \mathcal{GC}^\infty_{bd} := \mathcal{GC}^\infty_{bd}(0, 0; 1)\) and
\[
\mathcal{GC}^\infty_0(t_1, t_2) := \bigcap_{m \in \mathbb{N}} \mathcal{GC}^\infty_0(m) = \left\{ q \in \mathcal{GC}^\infty([t_1, t_2], \mathcal{R}^d) \mid q^{(i)}(t_j) = 0 \forall i \in \mathbb{N}, j = 1, 2 \right\}.
\]
The subscript “bd” stands here for “boundary values”. Note explicitly that both \(\mathcal{GC}^\infty_{bd}(m)\) and \(\mathcal{GC}^\infty_0(t_1, t_2)\) are \(\mathcal{R}\)-submodules of \(\mathcal{R}^d \times [t_1, t_2]\).

(ii) Let \(t_1, t_2 \in \mathcal{R}\) with \(t_1 < t_2\). Let \(q \in \mathcal{GC}^\infty([t_1, t_2], \mathcal{R}^d)\) and \(L \in \mathcal{GC}^\infty([t_1, t_2] \times \mathcal{R}^d \times (m+1), \mathcal{R})\) and define
\[
[q]^m(t) := (t, q(t), q^{(1)}(t), \ldots, q^{(m)}(t)) \quad \forall t \in [t_1, t_2]
\]
\[
L[q]^m(t) := L(t, q(t), q^{(1)}(t), \ldots, q^{(m)}(t)) \quad \forall t \in [t_1, t_2]
\]
\[
J[q] := \int_{t_1}^{t_2} L[q]^m(t) \, dt \in \mathcal{R}.
\]
Note explicitly that Thm. (iv).10 (closure of GSF with respect to composition) and Def. 15 of 1-dimensional integral of GSF (i.e. Thm. 14) allows us to say that \(J(q)\) is a well-defined number in \(\mathcal{R}\).

(iii) We say that \(q\) is a local minimizer of \(J\) in \(\mathcal{GC}^\infty_{bd}(q_1, q_2; m)\) if \(q \in \mathcal{GC}^\infty_{bd}(q_1, q_2; m)\) and
\[
\exists r \in \mathcal{R} > 0 \ \exists \ell \in \mathbb{N} \forall p \in B^e(r)(q) \cap \mathcal{GC}^\infty_{bd}(q_1, q_2; m) \ : \ J(p) \geq I(q)
\]
(iv) Let \( q \in \mathfrak{GC}_\infty^\infty([t_1, t_2], \mathfrak{R}) \). We define the first and second variation of \( J \) in direction \( h \in \mathfrak{GC}_\infty^\infty(m) \) at \( q \) as
\[
\delta J(q; h) := \left. \frac{d}{ds} J(q + sh) \right|_{s=0}, \quad \delta^2 J(q; h) := \left. \frac{d^2}{ds^2} J(q + sh) \right|_{s=0}.
\]
The GSF \( q \) is called weak extremal of \( J \) if \( \delta J(q; h) = 0 \) for all \( h \in \mathfrak{GC}_\infty^\infty(m) \).

(v) More generally, if \( q \in \mathfrak{GC}_\infty^\infty([t_1, t_2], \mathfrak{R}) \) and \( Q \in \mathfrak{GC}_\infty^\infty([t_1, t_2] \times \mathfrak{R}^{d(m+1)}, \mathfrak{R}) \), we say that \( J \) satisfies at \( q \) the D’Alembert’s principle with generalized forces \( Q \) if for all \( h \in \mathfrak{GC}_\infty^\infty(t_1, t_2) \):
\[
\delta J(q; h) = \int_{t_1}^{t_2} h(t) \cdot Q(q(t)) dt.
\]

The following results establish classical necessary and sufficient conditions to decide if a function \( u \) is a local minimizer for the functional (3.3).

**Theorem 36.** Let \( t_1, t_2 \in \mathfrak{R} \) with \( t_1 < t_2 \), let \( L \in \mathfrak{GC}_\infty^\infty([t_1, t_2] \times \mathfrak{R}^{d(m+1)}, \mathfrak{R}) \), let \( q_{t_1}, q_{t_2} \in \mathfrak{R}^d \) and let \( q \) be a local minimizer of \( J \) in \( \mathfrak{GC}_\text{bd}(q_{t_1}, q_{t_2}; m) \). Then
\begin{enumerate}[(i)]  
  \item \( \delta J(q; h) = 0 \) for all \( h \in \mathfrak{GC}_\infty^\infty(m) \);  
  \item \( \delta^2 J(q; h) \geq 0 \) for all \( h \in \mathfrak{GC}_\infty^\infty(m) \).
\end{enumerate}

**Proof.** Let \( r \in \mathfrak{R}_{>0} \) be such that (3.4) holds. Since \( h \in \mathfrak{GC}_\infty^\infty(m) \), the map \( s \mapsto q + sh \in \mathfrak{GC}_\text{bd}(q_{t_1}, q_{t_2}; m) \) is well defined and continuous with respect to the trace of the sharp topology in its codomain. Therefore, we can find \( \tilde{r} \in \mathfrak{R}_{>0} \) such that \( q + sh \in B_{\tilde{r}}(q) \cap \mathfrak{GC}_\text{bd}(q_{t_1}, q_{t_2}; m) \) for all \( s \in B_1(0) \). We hence have \( J(q + sh) \geq J(q) \). This shows that the GSF \( s \in B_1(0) \mapsto J(q + sh) \in \mathfrak{R} \) has a local minimum at \( s = 0 \). Now, we apply Lem. 19 and thus the claims are proven. \( \square \)

**Theorem 37.** Let \( t_1, t_2 \in \mathfrak{R} \) with \( t_1 < t_2 \) and \( q_{t_1}, q_{t_2} \in \mathfrak{R}^d \). Let \( q \in \mathfrak{GC}_\text{bd}(q_{t_1}, q_{t_2}; m) \) be such that
\begin{enumerate}[(i)]  
  \item \( \delta J(q; h) = 0 \) for all \( h \in \mathfrak{GC}_\text{bd}^\infty(m) \);  
  \item \( \delta^2 J(q; h) \geq 0 \) for all \( h \in \mathfrak{GC}_\text{bd}^\infty(m) \) and for all \( v \in B_1^l(q) \cap \mathfrak{GC}_\text{bd}(q_{t_1}, q_{t_2}; m) \), where \( r \in \mathfrak{R}_{>0} \) and \( l \in \mathbb{N} \).  
\end{enumerate}
Then \( q \) is a local minimizer of the functional \( J \) in \( \mathfrak{GC}_\text{bd}(q_{t_1}, q_{t_2}; m) \). Moreover, if \( \delta^2 J(v; h) > 0 \) for all \( q \in \mathfrak{GC}_\text{bd}^\infty(m) \) such that \( \|h\|_1 > 0 \) and for all \( v \in B_1^l(q) \cap \mathfrak{GC}_\text{bd}(q_{t_1}, q_{t_2}; m) \), then \( J(v) > J(q) \) for all \( v \in B_1^l(q) \cap \mathfrak{GC}_\text{bd}(q_{t_1}, q_{t_2}; m) \) such that \( \|v - q\|_1 > 0 \).

**Proof.** For any \( v \in B_1^l(q) \cap \mathfrak{GC}_\text{bd}(q_{t_1}, q_{t_2}; m) \), we set \( \psi(s) := J(q + s(v-q)) \in \mathfrak{R} \) for all \( s \in B_1(0) \), so that \( q + s(v-q) \in B_1(q) \). Since \( (v-q)(t_1) = 0 = (v-q)(t_2) \), we have \( v - q \in \mathfrak{GC}_\text{bd}^\infty(m) \), and properties (i), (ii) yield \( \psi'(0) = \delta J(q; v-q) = 0 \) and \( \psi''(s) = \delta^2 J(q + s(v-q); v-q) \geq 0 \) for all \( s \in B_1(0) \). We claim that \( s = 0 \) is a minimum of \( \psi \). In fact, for all \( s \in B_1(0) \) by Taylor’s Thm. 18
\[
\psi(s) = \psi(0) + s\psi'(0) + \frac{s^2}{2} \psi''(\xi)
\]
for some \( \xi \in [0, s] \). But \( \psi'(0) = 0 \) and hence \( \psi(s) - \psi(0) = \frac{s^2}{2} \psi''(\xi) \geq 0 \). Finally, since by Thm. 8.(iii) the interval \([0, +\infty)\) is closed in the sharp topology
\[
\lim_{s \to 1^-} \psi(s) = J(v) \geq \psi(0) = J(u),
\]
which is our conclusion. Note explicitly that if \( \delta^2 J(v; h) = 0 \) for all \( h \in \mathfrak{GC}_\text{bd}^\infty(m) \) and for all \( v \in B_1^l(u) \cap \mathfrak{GC}_\text{bd}(q_{t_1}, q_{t_2}; m) \), then \( \psi''(\xi) = 0 \) and hence \( J(v) = J(q) \).

Now, assume that \( \delta^2 J(v; h) > 0 \) for all \( h \in \mathfrak{GC}_\text{bd}^\infty(m) \) such that \( \|h\|_1 > 0 \) and for all \( v \in B_1^l(u) \cap \mathfrak{GC}_\text{bd}(q_{t_1}, q_{t_2}; m) \) such that \( \|v - q\|_1 > 0 \). As above set \( \psi(s) := J(q + s(v-q)) \in \mathfrak{R} \) for all \( s \in B_3/2(0) \), so that \( q + s(v-q) \in B_{1/2}(q) \). We have \( \psi'(0) = 0 \) and \( \psi''(s) = \delta^2 J(q + s(v-q); v-q) > 0 \) for all \( s \in B_{3/2}(0) \) because \( \|v - u\|_1 > 0 \). Using Taylor’s theorem, we get \( \psi(1) - \psi(0) = J(v) - J(q) = \frac{1}{2} \psi''(\xi) \) for some \( \xi \in [0, 1] \). Therefore \( \psi(1) - \psi(0) = J(v) - J(q) = \frac{1}{2} \psi''(\xi) > 0 \). \( \square \)
In the following, we always assume the notations of Def. 35. Moreover, we also set
\[ \mathcal{R}_{\mathcal{R}}[t] := \left\{ p \in \mathcal{R}[t] \mid \deg(p) \leq d \right\} \]
for the set of all the polynomials with coefficients in the ring \( \mathcal{R} \) having degree less or equal to \( d \in \mathbb{N} \).

In order to prove the higher-order version of the fundamental lemma, we need the following preliminary results:

**Lemma 38.** Let \( a, b \in \mathcal{R} \) be such that \( a < b \), and let \( f \in \mathcal{RGC}^\infty([a, b], \mathcal{R}_{\mathcal{R}}) \). If \( \int_a^b f(t) \, dt = 0 \), then \( f = 0 \).

**Proof.** By contradiction, assume that \( f(x) \neq 0 \) for some \( x \in [a, b] \). Let \( (f_\varepsilon) \) be a net of smooth functions that define the GSF \( f \) and let \( [x_\varepsilon] = x \) be a representative of \( x \). The property
\[ \forall q \in \mathbb{N} \forall \varepsilon : f_\varepsilon(x_\varepsilon) \leq \rho_{\varepsilon}^q \] (3.5)
implies \( f(x) \leq d\rho^q \) and hence also, letting \( q \to +\infty \), \( f(x) \leq 0 \). Since \( f(x) \geq 0 \), this would imply \( f(x) = 0 \). Therefore, taking the negation of (3.5) we get
\[ \exists q \in \mathbb{N} \exists L \subseteq I : 0 \in L, \forall \varepsilon \in L : f_\varepsilon(x_\varepsilon) > \rho_{\varepsilon}^q, \]
where \( L \) is the closure of \( L \) in \( I \). Set \( \tilde{f}_\varepsilon := f_\varepsilon \) for \( \varepsilon \in L \) and \( \tilde{f}_\varepsilon := \frac{1}{2} (f_\varepsilon + \rho_{\varepsilon}^q) \) otherwise. Directly from Def. 9, it follows that \( \tilde{f} := [f_\varepsilon (-\varepsilon)] \in \mathcal{RGC}^\infty([a, b], \mathcal{R}_{\mathcal{R}}) \) and \( \tilde{f}(x) > d\rho^{q+1} \). The sharp continuity of \( f \) at \( x \) (Thm. 10.(iii)) implies
\[ \exists c, d : c < d, [c, d] \subseteq [a, b], \tilde{f}(c, d) > d\rho^{q+1}. \] (3.6)
Moreover, from \( f \geq 0 \) and Thm. 16 we obtain
\[ 0 = \int_a^b f \geq \int_a^c f + \int_c^d f \geq \int_c^d f. \]
Thereby, we can find a negligible net \( [z_\varepsilon] = 0 \) such that \( z_\varepsilon \geq f_{\varepsilon} \) for all \( \varepsilon \) sufficiently small, where \( [c_\varepsilon] = c, [d_\varepsilon] = d \) with \( c_\varepsilon < d_\varepsilon \) because \( c < d \) (see Lem. 3). In particular, for \( \varepsilon \in L \) sufficiently small \( \tilde{f}_\varepsilon = f_\varepsilon \) and hence \( f_{\varepsilon} \tilde{f}_\varepsilon = f_{\varepsilon_\varepsilon} \tilde{f}_\varepsilon \leq z_\varepsilon. \) But then (3.6) and Thm. 16.(vii) yield
\[ \forall \varepsilon \in L : z_\varepsilon \geq f_{\varepsilon} \tilde{f}_\varepsilon \geq f_{\varepsilon} \tilde{f}_\varepsilon \geq \int_{c_\varepsilon}^{d_\varepsilon} \rho_{\varepsilon}^{q+1} = \rho_{\varepsilon}^{q+1} (d_\varepsilon - c_\varepsilon). \]
Therefore, \( \rho_{\varepsilon}^{q+1} \leq \frac{z_\varepsilon}{d_\varepsilon - c_\varepsilon} \). However, \( \left[ \frac{z_\varepsilon}{d_\varepsilon - c_\varepsilon} \right] = \frac{|z_\varepsilon|}{d - c} = 0 \) and hence \( \rho_{\varepsilon}^{q+1} \leq \rho_{\varepsilon}^{q+2} \) by (2.1), a contradiction.

The second preliminary result introduces the use of approximate identities for convolution with GSF (see also [43, Lem. 4.3]):

**Lemma 39.** Let \( a, b \in \mathcal{R} \) be such that \( a < b \) and let \( f \in \mathcal{RGC}^\infty([a, b], \mathcal{R}_{\mathcal{R}}) \). Let \( x \in [a, b] \) and \( R \in \mathcal{R}_{\mathcal{R}} \) be such that \( B_\delta(x) \subseteq [a, b] \). Only in this statement, we write \( \forall \delta \in \mathcal{R}_{\mathcal{R}} \) to denote \( \exists \varepsilon \in \mathcal{R}_{\mathcal{R}} \forall t \in B_\varepsilon(0) \cap \mathcal{R}_{\mathcal{R}} \), i.e. “for \( t \) sufficiently small (in the sharp topology)”. Assume that \( G_t \in \mathcal{RGC}^\infty(\mathcal{R}, \mathcal{R}) \) satisfy
(i) \[ \forall \delta \in \mathcal{R}_{\mathcal{R}} \exists \varepsilon \in \mathcal{R}_{\mathcal{R}} : \int_R^R G_t = 1. \]
(ii) For \( t \) small, \( (G_t)_{t \in \mathcal{R}_{\mathcal{R}}} \) is zero outside every ball \( B_\delta(0) \), \( 0 < \delta < R \), i.e.
\[ \forall \delta \in \mathcal{R}_{\mathcal{R}} \forall \varepsilon \in \mathcal{R}_{\mathcal{R}} \forall y \in [\frac{\varepsilon}{R} - \frac{\varepsilon}{R} \cup [\delta, R] : G_t(y) = 0. \] (3.7)
(iii) \[ \exists M \in \mathcal{R}_{\mathcal{R}} \forall \delta \in \mathcal{R}_{\mathcal{R}} : \int_R^R |G_t(y)| \, dy \leq M. \]

Then
\[ \lim_{t \to 0} \int_R^R f(x - y) G_t(y) \, dy = f(x). \]
Moreover, \( \int_{-R}^R f(x - y) G_t(y) \, dy = \int_{-R}^{x+R} f(y) G_t(x - y) \, dy. \)
In this section, we first compute the first variation of a functional. The higher-order Euler–Lagrange equation, D’Alembert principle and du Bois–Reymond optimality condition hold for all $|\delta J| < \delta$ for 0 < $t < S$, say for 0 < $t < S \in \mathbb{R}_{>0}$, we get

$$\int_{-R}^{R} f(x-y)G_t(y)dy = \int_{-R}^{x-R} f(y)G_t(x-y)dy$$

so that these integrals exist because $(x - R, x + R) = B_R(x) \subseteq [a, b]$. Using (i), for $t$ small, let’s say for 0 < $t < S \in \mathbb{R}_{>0}$, we get

$$\int_{-R}^{R} (f(x-y)G_t(y)dy - f(x)) = \int_{-R}^{R} (f(x-y) - f(x)) G_t(y)dy$$

$$\leq \int_{-R}^{R} |f(x-y) - f(x)| \cdot |G_t(y)|dy.$$ 

For each $r \in \mathbb{R}_{>0}$, sharp continuity of $f$ at $x$ yields $|f(x-y) - f(x)| < r$ for all $y$ such that $|y| < \delta \in \mathbb{R}_{>0}$, and we can take $\delta < R$. Assuming that (ii) holds for all 0 < $t < s$, for 0 < $t < s \wedge S$, we have

$$\int_{-R}^{R} f(x-y)G_t(y)dy - f(x) \leq r \int_{-R}^{+\delta} |G_t(y)|dy \leq rM, \quad (3.8)$$

where we used (iii). The right hand side of (3.8) can be taken arbitrarily small in $\mathbb{R}_{>0}$ because it holds for all $r \in \mathbb{R}_{>0}$.

3.1. Higher-order Euler–Lagrange equation, D’Alembert principle and du Bois–Reymond optimality condition. In this section, we first compute the first variation $\delta J(q; h)$ and thereby deduce the Euler-Lagrange equations both in integral and differential forms, and the D’Alembert principle in differential form.

**Lemma 40.** If $q \in ^*\mathcal{G}^\infty \left([t_1, t_2], ^*\mathbb{R}\right)$ and $h \in ^*\mathcal{G}^\infty_0 (m)$, then

$$\delta J(q; h) = \int_{t_1}^{t_2} h^{(m)}(s) \sum_{i=0}^{m} (-1)^i \left( \int_{t_1}^{t} \int_{t_1}^{s_1} \cdots \int_{t_1}^{s_{m-1}} \left( \partial_1 + L[q](m)(s_{m-i}) \right) ds_{m-i} \cdots ds_1 \right) dt$$

$$= \int_{t_1}^{t_2} h(t) \left( \sum_{i=0}^{m} (-1)^i \frac{d^i}{dt^i} \partial_1 + L[q](m)(t) \right).$$

**Proof.** According to Def. 35, and to Thm. 16 for any $h \in ^*\mathcal{G}^\infty_0 (m)$ we have

$$\delta J(q; h) = \int_{t_1}^{t_2} \left( \sum_{i=0}^{m} \partial_1 + L[q](m)(t) \cdot h^{(i)}(t) \right) dt. \quad (3.9)$$

By repeated integration by parts (Thm. 16.(vi)) one has

$$\sum_{i=0}^{m} \int_{t_1}^{t_2} \partial_1 + L[q](m)(t) \cdot h^{(i)}(t) dt$$

$$= \sum_{i=0}^{m} \left\{ \left( \sum_{j=0}^{m-j} (-1)^j \partial_1 + L[q](m)(t) \right) \left( \int_{t_1}^{t} \int_{t_1}^{s_1} \cdots \int_{t_1}^{s_{j-1}} \left( \partial_1 + L[q](m)(s_{j}) \right) ds_{j} \cdots ds_1 \right) \right\} \int_{t_1}^{t_2} dt$$

$$+ (-1)^i \int_{t_1}^{t_2} h^{(m)}(t) \left( \int_{t_1}^{t} \int_{t_1}^{s_1} \cdots \int_{t_1}^{s_{m-1}} \left( \partial_1 + L[q](m)(s_{m-i}) \right) ds_{m-i} \cdots ds_1 \right) dt \quad (3.10)$$
Since \( h^{(i)}(t_1) = 0 = h^{(i)}(t_2) \), \( i = 0, \ldots, m - 1 \), the first summands in (3.10) vanish. Therefore, equations (3.9) and (3.10) yield the first conclusion. The second equality simply follows by \( m \) times integration by parts. \( \Box \)

In order to prove the Euler-Lagrange equation in integral form (see below Cor. 44), we first need to show the fundamental lemma and the higher order du Bois–Reymond lemma of the calculus of variations for generalized functions.

**Lemma 41** (Fundamental Lemma of the Calculus of Variations). Let \( a, b \in \hat{\mathbb{R}} \) such that \( a < b \), and let \( f \in \mathcal{G}^\infty_{0}[a, b, \hat{\mathbb{R}}] \). If

\[
\int_a^b f(t)h(t) \, dt = 0 \quad \text{for all } h \in \mathcal{G}^\infty_{0}(a, b),
\]

then \( f = 0 \).

**Proof.** Let \( x \in [a, b] \). Because of Thm. 10.(ii) and Lem. 5, without loss of generality we can assume that \( x \) is a sharply interior point, so that \( B_R(x) \subseteq [a, b] \) for some \( R \in \hat{\mathbb{R}} > 0 \). Let \( \varphi \in \mathcal{D}_{[-1,1]}(\mathbb{R}) \) be such that \( \int \varphi = 1 \). Set \( G_{t,x}(\xi) := \frac{1}{t} \varphi \left( \frac{\xi}{t} \right) \), where \( x \in \mathbb{R} \) and \( t \in \hat{\mathbb{R}} > 0 \), and \( G_t(x) := [G_{t,x}(x)] \) for all \( x \in \hat{\mathbb{R}} \). Then, for \( t \) sufficiently small, we have \( G_t(x - \cdot) \in \mathcal{G}^\infty_{0}(a, b) \) and (3.11) yields \( \int_a^b f(y)G_t(x-y) \, dy = 0 \). For \( t \) small, we both have that \( G_t(x - \cdot) = 0 \) on \( [a, x - R] \cup [x + R, b] \) and the assumptions of Lem. 39 hold. Therefore

\[
0 = \int_a^b f(y)G_t(x-y) \, dy = \int_{x-R}^{x+R} f(y)G_t(x-y) \, dy = \int_{-R}^{R} f(x-y)G_t(y) \, dy,
\]

and Lem. 5 hence yields \( f(x) = 0 \). \( \Box \)

**Lemma 42** (Fundamental higher order du Bois–Reymond lemma). Let \( a, b \in \hat{\mathbb{R}} \) be such that \( a < b \), and let \( f \in \mathcal{G}^\infty(\hat{\mathbb{R}}) \). If

\[
\int_a^b f(t)h^{(m)}(t) \, dt = 0 \quad \text{for all } h \in \mathcal{G}^\infty_{0}(a, b)
\]

Then \( f(t) \in \mathcal{G}^\infty_{m-1}[t] \).

**Proof.** For all \( h \in \mathcal{G}^\infty_{0}(a, b) \), using \( m \) times integration by parts in (3.12), by \( h^{(i)}(a) = h^{(i)}(b) = 0 \) for all \( i = 0, \ldots, m - 1 \) we get \( \int_a^b f^{(m)}(t)h(t) \, dt = 0 \). Thereby, the fundamental Lem. 41 yields \( f^{(m)} = 0 \). Integrating \( m \) times (see Thm. 14), we get the claim \( f(t) \in \mathcal{G}^\infty_{m-1}[t] \). \( \Box \)

As a simple consequence, we have the following

**Corollary 43.** Let \( a, b \in \hat{\mathbb{R}} \) be such that \( a < b \), and let \( f, g \in \mathcal{G}^\infty([a, b], \hat{\mathbb{R}}) \). If

\[
\int_a^b (f(t)h(t) + g(t)h'(t)) \, dt = 0 \quad \text{for all } h \in \mathcal{G}^\infty_{0}(a, b),
\]

then \( g' = f \).

**Proof.** Set \( F(x) := \int_a^x f(t) \, dt \) for \( x \in [a, b] \), so that \( F \in \mathcal{G}^\infty([a, b], \hat{\mathbb{R}}) \) and \( F' = f \) by Def. 15. Integrating by parts (Thm. 16.(vi)), we get

\[
\int_a^b (f(t)h(t) + g(t)h'(t)) \, dt = \int_a^b (g(t) - F(t)) h'(t) \, dt = 0
\]

for all \( h \in \mathcal{G}^\infty_{0}(a, b) \). Therefore, Lem. 42 implies that \( g(t) - F(t) \) is a constant (in \( \hat{\mathbb{R}} \)), and hence \( g' = F' = f \) as claimed. \( \Box \)
Applying the higher-order du Bois–Reymond Lem. 42, we arrive at the first form of the Euler-Lagrange equations:

**Corollary 44.** If \( q \in {}^sGC^\infty \left([t_1, t_2], {}^sR \right) \) is a weak extremal of functional (3.3), then \( q \) satisfies the following higher-order Euler–Lagrange integral equation:

\[
\sum_{i=0}^{m} (-1)^i \left( \int_{t_1}^{t} \int_{t_1}^{s_1} \ldots \int_{t_1}^{s_{m-i-1}} \partial_{i+2} L[q]^m(t) \right) ds_{m-i} \ldots ds_1 \in {}^sR_{m-1}[t]. \tag{3.13}
\]

From the second equality of Lem. 40 and the fundamental Lem. 41, we obtain

**Corollary 45** (Higher-order Euler–Lagrange equations in differential form). If the GSF \( q \in {}^sGC^\infty \left([t_1, t_2], {}^sR \right) \) is a weak extremal of functional (3.3), then

\[
\sum_{i=0}^{m} (-1)^i \frac{d^i}{dt^i} \partial_{i+2} L[q]^m(t) = 0 \quad \forall t \in [t_1, t_2]. \tag{3.14}
\]

Finally, Lem. 40, Def. 35.(v) and the fundamental Lem. 41 directly yield the D’Alembert principle in differential form:

**Corollary 46.** Let \( q \in {}^sGC^\infty \left([t_1, t_2], {}^sR \right), \ Q \in {}^sGC^\infty ([t_1, t_2] \times {}^sR^{d(1+m)}, {}^sR) \) and assume that the functional \( J \) satisfies at \( q \) the D’Alembert’s principle with generalized forces \( Q \), then

\[
\sum_{i=0}^{m} (-1)^i \frac{d^i}{dt^i} \partial_{i+2} L[q]^m(t) = Q[q]^m(t) \quad \forall t \in [t_1, t_2].
\]

Associated to a given function \( q \in {}^sGC^\infty \left([t_1, t_2], {}^sR \right), \) it is convenient to introduce the following quantities (see [20]):

\[
\varphi^j(q)(t) := \sum_{i=0}^{m-j} (-1)^i \frac{d^i}{dt^i} \partial_{i+j+2} L[q]^m(t) \quad \forall j = 0, \ldots, m \forall t \in [t_1, t_2] \tag{3.15}
\]

These operators are useful for our purposes because of the following properties:

\[
\frac{d}{dt} \varphi^j(q) = \partial_{j+1} L[q]^m - \varphi^{j-1}(q) \quad \forall j = 1, \ldots, m. \tag{3.16}
\]

We are now in conditions to prove a higher-order du Bois–Reymond optimality condition.

**Theorem 47.** If \( q \in {}^sGC^\infty \left([t_1, t_2], {}^sR^d \right) \) is a weak extremal of functional (3.3), then

\[
\frac{d}{dt} \left( L[q]^m - \sum_{j=1}^{m} \varphi^j(q) \cdot q^{(j)} \right) = \partial_1 L[q]^m. \tag{3.17}
\]

**Proof.** Using, for simplicity, \( \varphi^j := \varphi^j(q), \) we have

\[
\frac{d}{dt} \left( L[q]^m - \sum_{j=1}^{m} \varphi^j \cdot q^{(j)} \right) = \left( \partial_1 L[q]^m + \sum_{j=0}^{m} \partial_{j+2} L[q]^m \cdot q^{(j+1)} - \sum_{j=1}^{m} \left( \varphi^j \cdot q^{(j)} + \varphi^j \cdot q^{(j+1)} \right) \right). \tag{3.18}
\]
Using (3.16), equation (3.18) becomes

\[
\frac{d}{dt} \left( L[q]^m - \sum_{j=1}^{m} \varphi^j \cdot q^{(j)} \right) = \partial_1 L[q]^m + \sum_{j=0}^{m} \partial_{j+2}L[q]^m \cdot q^{(j+1)} \\
- \sum_{j=1}^{m} \left( (\partial_{j+1}L[q]^m - \varphi^{j-1}) \cdot q^{(j)} + \varphi^j \cdot q^{(j+1)} \right).
\]

(3.19)

We now simplify the second term on the right-hand side of (3.19), which forms a telescopic sum:

\[
\sum_{j=1}^{m} \left( \left( \partial_{j+1}L[q]^m - \varphi^{j-1} \right) \cdot q^{(j)} + \varphi^j \cdot q^{(j+1)} \right) = \\
\sum_{j=0}^{m-1} \left( \left( \partial_{j+2}L[q]^m - \varphi^j \right) \cdot q^{(j+1)} + \varphi^{j+1} \cdot q^{(j+2)} \right) = \\
\sum_{j=0}^{m-1} \left( \partial_{j+2}L[q]^m \cdot q^{(j+1)} - \varphi^0 \cdot \dot{q} + \varphi^m \cdot q^{(m+1)} \right). 
\]

(3.20)

Substituting (3.20) into (3.19) and using the higher-order Euler–Lagrange equations (3.14), and since, by definition,

\[
\varphi^m = \partial_{m+2}L[q]^m
\]

and

\[
\varphi^0 = \sum_{i=0}^{m} (-1)^i \frac{d^i}{dt^i} \partial_{i+2}L[q]^m = 0,
\]

we obtain the intended result, that is,

\[
\frac{d}{dt} \left( L[q]^m - \sum_{j=1}^{m} \varphi^j \cdot q^{(j)} \right) = \\
\partial_1 L[q]^m + \partial_{m+2}L[q]^m \cdot q^{(m+1)} + \varphi^0 \cdot \dot{q} = \partial_1 L[q]^m.
\]

□

3.2. Higher-order Noether’s theorem. In our approach to Noether’s theorem, we can use the non-Archimedean language of \( ^*\mathbb{R} \) to formalize some infinitesimal properties frequently informally used in this topic.

Definition 48. Let \( D \subseteq ^*\mathbb{R}^k \) be a sharply open set. We say that \( \Psi = \{\psi(s, \cdot)\}_{s \in P} \in ^*G^\infty(D, D) \) is a one parameter group of diffeomorphisms of \( D \) if it satisfies:

(i) \( P \) is a sharply open set, \( 0 \in P \) and \( \psi \in ^*G^\infty(P \times D, D) \);

(ii) For each \( s \in P \), the map \( \psi(s, \cdot) \in ^*G^\infty(D, D) \) is invertible, and \( \psi(s, \cdot)^{-1} \in ^*G^\infty(D, D) \);

(iii) \( \psi(0, \cdot) = \text{Id}_D \);

(iv) \( \forall s, s' \in P : s + s' \in P \Rightarrow \psi(s, \cdot) \circ \psi(s', \cdot) = \psi(s + s', \cdot) \).

We will see on Sec. 5.5 the actual usefulness of the case of strict inclusion \( D \subset ^*\mathbb{R}^k \). Note that, because of properties (i) and (iii), from Taylor’s formula Thm. 18, we get

\[
\exists S \in ^*\mathbb{R}_{>0} \forall s \in [0, S] \cap P : \psi(s, q(t)) \approx q(t) + s \frac{\delta \psi}{\delta s}(0, q(t)).
\]

Therefore, for a sufficiently small infinitesimal \( s \), we can always say that \( \psi(s, t) \) is infinitely close to a transformation of the form \( q(t) \rightarrow q(t) + s \eta(q(t)) \), where \( \eta(q) := \frac{\delta \psi}{\delta s}(0, q) \). To write an equality sign instead of the infinitely close sign \( \approx \), we can use Thm. 12 and the notation \( \frac{\delta \psi}{\delta s}(0, q(t); s) \) for
the generalized partial incremental ratio with respect to the variable $s$ (we recall that it is another GSF):

$$\exists S \in \mathbb{R}_{>0} \forall s \in [0,S] \cap P : \psi(s,q(t)) = q(t) + s\frac{\partial \psi}{\partial s}[0,q(t);s]$$

$\frac{\partial \psi}{\partial s}[0,q(t);s] \approx \frac{\partial \psi}{\partial s}(0,q(t))$.

**Definition 49.** Let both $T = \{\tau(s,\cdot)\}_{s \in P} \in \mathcal{GC}^\infty(T',T')$ and $S = \{\sigma(s,\cdot)\}_{s \in P} \in \mathcal{GC}^\infty(S',S')$ be one parameter groups of diffeomorphisms on the open sets $T' \subseteq [t_1,t_2]$ and $S' \subseteq \mathbb{R}^d$ respectively. The functional (3.3) is said to be **invariant under the action of $T$ and $S$**, if for any weak extremal $q \in \mathcal{GC}^\infty([t_1,t_2],S')$ it satisfies

$$L[q]^m(t) = L\left(\tau(s,t),\sigma(s,q(t)),\frac{d\sigma(s,q(t))}{d\tau(s,t)},\ldots,\frac{d^m\sigma(s,q(t))}{d\tau^m(s,t)}\right) \frac{\partial \tau}{\partial t}(s,t)$$

(3.21)

for all $s \in P$ and all $t \in T'$, where the expressions $\frac{d\sigma(s,q(t))}{d\tau(s,t)}$, $i = 1,\ldots,m$ are defined in the following Rem. 50.(ii).

**Remark 50.**

(i) From Def. 48.(ii) we have $\tau(s,\tau(s,\cdot)^{-1}(t)) = t$, thereby the chain rule Thm. 13 yields that $\frac{\partial \tau}{\partial t}(s,t)$ is invertible for all $s \in P$, $t \in T'$.

(ii) Based on the previous remark, in (3.21) the expressions $\frac{d^i\sigma(s,q(t))}{d\tau^i(s,t)}$, $i = 1,\ldots,m$ are defined as

$$\frac{d\sigma(s,q(t))}{d\tau(s,t)} := \frac{\partial \sigma(s,q(t))}{\partial \tau(s,t)}(s,t)$$

(3.22)

and

$$\frac{d^i\sigma(s,q(t))}{d\tau^i(s,t)} := \frac{\partial}{\partial \tau} \left( \frac{d^{i-1}\sigma(s,q(t))}{d\tau^{i-1}(s,t)} \right)(s,t), \quad \forall i = 2,\ldots,m.$$  

(3.23)

As usual, because of Def. 48.(iii), we have $\frac{d\sigma(0,q(t))}{d\tau(0,t)} = q^{(i)}(t)$.

(iii) Using again the generalized incremental ratios, i.e Thm. 12, for all $s \in P$, $t \in T'$ and for $h \in \mathbb{R}$ sufficiently small, we have

$$\sigma(s,q(t+h)) = \sigma(s,q(t)) + h \cdot \left. \frac{\partial \sigma(s,q(t))}{\partial t} \right|_{s,t} [s,t;h]$$

$$\tau(s,t+h) = \tau(s,t) + h \cdot \left. \frac{\partial \tau}{\partial t} \right|_{s,t} [s,t;h].$$

Therefore, the ratio (of differentials):

$$\frac{\sigma(s,q(t+h)) - \sigma(s,q(t))}{\tau(s,t+h) - \tau(s,t)} = \left. \frac{\partial \sigma(s,q(t))}{\partial t} \right|_{s,t} [s,t;h] \approx \left. \frac{\partial \sigma(s,q(t))}{\partial t} \right|_{s,t}$$

for all sufficiently small invertible $h$.

(iv) On the basis of Def. 15, condition (3.21) is equivalent to

$$\int_{t_a}^{t_b} L[q]^m(t) \, dt = \int_{t_a}^{t_b} L\left(\tau(s,t),\sigma(s,q(t)),\frac{d\sigma(s,q(t))}{d\tau(s,t)},\ldots,\frac{d^m\sigma(s,q(t))}{d\tau^m(s,t)}\right) \frac{\partial \tau}{\partial t}(s,t) \, dt$$

for all $s \in P$ and for any subinterval $[t_a,t_b] \subseteq T'$ with $t_a < t_b$, even if $L$ can also be a distribution. Clearly, if $L$ is not a continuous function, this equivalence is due to the regularization process inherent to the embedding $\iota^h : D' \rightarrow \mathcal{GC}^\infty(c(-),\mathbb{R})$ of Sec. 2.3.

The next lemma gives a necessary condition for the functional (3.3) to be invariant under the action of the parameter groups of diffeomorphisms $T = \{\tau(s,\cdot)\}_{s \in P} \in \mathcal{GC}^\infty(T',T')$ and $S = \{\sigma(s,\cdot)\}_{s \in P} \in \mathcal{GC}^\infty(S',S')$. 


Lemma 51. If the functional (3.3) is invariant in the sense of Def. 49, then
\[
\frac{\partial L[q]^m}{\partial t} \frac{\partial \tau}{\partial t}(0, t) + \frac{\partial \tau}{\partial t}(0, t) \sum_{i=0}^{m} \frac{\partial L[q]^m}{\partial q^{(i)}} \cdot \eta^{(i)}(q, t) + L[q]^m \frac{d}{dt} \frac{\partial \tau}{\partial s}(0, t) = 0 \quad \forall t \in T'
\]
for any weak extremal \( q \in \mathcal{GC}_C ([t_1, t_2], S') \). In (3.24), we set
\[
\begin{cases}
\eta^{(i)}(q, \cdot) := \frac{\partial \gamma}{\partial \tau}(0, q, \cdot), \\
\eta^{(i)}(q, \cdot) := \frac{\partial \gamma^{(i)}(q, \cdot)}{\partial \tau} - q^{(i)} \frac{d}{dt} \frac{\partial \tau}{\partial s}(0, \cdot), \quad \forall i = 1, \ldots, m.
\end{cases}
\]

Proof. Differentiating (3.21) with respect to \( s \) at \( s = 0 \in P \) and \( t \in T' \) and using Rem 50.(ii), we obtain
\[
\frac{\partial L[q]^m}{\partial t} \frac{\partial \tau}{\partial t}(0, t) + \frac{\partial \tau}{\partial t}(0, t) \sum_{i=0}^{m} \frac{\partial L[q]^m}{\partial q^{(i)}} \cdot \frac{\partial \sigma(s, q(t))}{\partial \tau}(s, t) \bigg|_{s=0} + L[q]^m \frac{d}{dt} \frac{\partial \tau}{\partial s}(0, t) = 0.
\]
Note explicitly that using two times Thm. 12, for any GSF \( f \) we always have
\[
\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = \left[ \frac{\partial f^2}{\partial y \partial x}(x_0, y_0) \right] = \left[ \frac{\partial f^2}{\partial y \partial x}(x_0, y_0) \right] = \frac{\partial f}{\partial y}(x_0, y_0).
\]
Using Rem. 50.(ii) and (3.25), we have
\[
\left. \frac{\partial}{\partial s} \left( \frac{d \sigma(s, q(t))}{dt}(s, t) \right) \right|_{s=0} = \frac{\partial^2 \sigma(s, q(t))}{\partial s \partial t}(0, t) - \frac{\partial^2 \sigma(s, q(t))}{\partial s \partial t}(0, t) \bigg|_{s=0} - \dot{q}(t) \frac{\partial^2 \tau}{\partial s \partial t}(0, t)
\]
and for all \( i = 2, \ldots, m \)
\[
\left. \frac{\partial}{\partial s} \left( \frac{d \sigma^{(i)}(s, q(t))}{dt}(s, t) \right) \bigg|_{s=0} = \frac{\partial^2 \sigma^{(i)}(s, q(t))}{\partial s \partial t}(0, t) - \dot{q}^{(i)}(t) \frac{\partial^2 \tau}{\partial s \partial t}(0, t)
\]
Substituting (3.30) into (3.26), we obtain the conclusion. \( \square \)

Definition 52. Let \( T' \subseteq [t_1, t_2] \) be a sharply open set. We say that a GSF \( C[q]^m(\cdot) \in \mathcal{GC}_C \left( T', \mathbb{R}^d \right) \) is a constant of motion on \( T', S' \) if
\[
\frac{d}{dt} C[q]^m(t) = 0 \quad \forall t \in T'
\]
along all the weak extremals \( q \in \mathcal{GC}_C ([t_1, t_2], S') \).

Note that Thm. 14 implies that condition (3.32) is equivalent to ask that \( C[q]^m(\cdot) \) is constant on any interval \( J \subseteq T' \). In fact, if \( t' \in J \), \( F(t) := C[q]^m(t) - C[q]^m(t') \) is the unique GSF such that \( F(t') = 0 \) and \( F'(t) = 0 \) on any closed interval contained in \( T' \) (and hence on an arbitrary interval by sharp continuity).

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Theorem 53. If functional (3.3) is invariant in the sense of Def. 49, then the quantity \( C[q]^m(t) \) defined for all \( q \in \mathcal{C}^\infty ([t_1, t_2], S') \) and \( t \in T' \) by

\[
C[q]^m(t) := \sum_{i=1}^m \varphi^i(q, t) \cdot \eta^{i-1}(q, t) + \left( L[q]^m(t) - \sum_{i=1}^m \varphi^i(q, t) \cdot q^{(i)}(t) \right) \frac{\partial \tau}{\partial s}(0, t)
\]  

(3.33)

is a constant of motion on \( T' \) and \( S' \).

Proof. The proof follows directly from the previous results and no specific property of GSF is needed. For simplicity, we use the notations \( \varphi^i := \varphi^i(q, \cdot) \) and \( \eta^i := \eta^i(q, \cdot) \).

\[
C[q]^m(\cdot) = \varphi^1 \cdot \eta^0 + \sum_{i=2}^m \varphi^i \cdot \eta^{i-1} + \left( L[q]^m - \sum_{i=1}^m \varphi^i \cdot q^{(i)} \right) \frac{\partial \tau}{\partial s}(0, \cdot).
\]  

(3.34)

Differentiating (3.34) with respect to \( t \), we obtain

\[
\frac{d}{dt} C[q]^m(t) = \eta^0 \frac{d}{dt} \varphi^1 + \varphi^1 \frac{d}{dt} \eta^0 + \sum_{i=2}^m \left( \eta^{i-1} \cdot \frac{d}{dt} \varphi^1 + \varphi^i \cdot \frac{d}{dt} \eta^{i-1} \right) + \frac{\partial \tau}{\partial s}(0, \cdot) \frac{d}{dt} \left( L[q]^m - \sum_{i=1}^m \varphi^i \cdot q^{(i)} \right) + \left( L[q]^m - \sum_{i=1}^m \varphi^i \cdot q^{(i)} \right) \frac{d}{dt} \frac{\partial \tau}{\partial s}(0, \cdot).
\]  

(3.35)

Using the higher order Euler–Lagrange equation (3.14), the higher order du Bois–Reymond condition (3.17), relations (3.15) and (3.25) in (3.35), one gets

\[
\frac{d}{dt} C[q]^m(t) = \partial_2 L[q]^m \cdot \frac{\partial \sigma}{\partial s}(0, q(\cdot)) + \varphi^1 \cdot \left( \eta^1 + \dot{q} \frac{d}{dt} \frac{\partial \tau}{\partial s}(0, \cdot) \right) + \sum_{i=2}^m \left( \partial_{i+1} L[q]^m - \varphi^{i-1} \cdot \eta^{i-1} + \varphi^i \cdot \left( \eta^i + q^{(i)} \frac{d}{dt} \frac{\partial \tau}{\partial s}(0, \cdot) \right) \right) + \frac{\partial \tau}{\partial s}(0, \cdot) \left( L[q]^m - \sum_{i=1}^m \varphi^i \cdot q^{(i)} \right) \frac{d}{dt} \frac{\partial \tau}{\partial s}(0, \cdot)
\]  

\[
+ \partial_1 L[q]^m \frac{\partial \tau}{\partial s}(0, \cdot) + L[q]^m \frac{d}{dt} \frac{\partial \tau}{\partial s}(0, \cdot) + \partial_2 L[q]^m \frac{\partial \sigma}{\partial s}(0, q(\cdot)) + \varphi^1 \cdot \left( \eta^1 + \dot{q} \frac{d}{dt} \frac{\partial \tau}{\partial s}(0, \cdot) \right) - \varphi^1 \cdot \eta^1 - \varphi^m \cdot \eta^m + \sum_{i=2}^m \partial_{i+1} L[q]^m \cdot \eta^{i-1} = 0.
\]  

(3.36)

\[ \square \]

4. Optimal control problems

Thm. 53 gives a Lagrangian formulation of Noether’s principle extended to the generalized smooth functions setting. In this section, we adopt the Hamiltonian formalism in order to generalize Noether’s principle to the wider context of optimal control, see e.g. [15].

Let us consider the optimal control problem for GSF in Lagrange form, i.e. the minimization of the functional:

\[
\int_{t_1}^{t_2} L(t, q(t), u(t)) \, dt \rightarrow \min
\]  

(4.1)

subject to the Cauchy problem
This solution is given by Theorem 54.

Let $\|\cdot\|_r$ and for all $\alpha \in \mathbb{R}_{>0}$, the state equation $\varphi : [t_1, t_2] \times H \times K \rightarrow \mathbb{R}^d$, the state $q : [t_1, t_2] \rightarrow H$ and the control $u : [t_1, t_2] \rightarrow K$ are assumed to be GSF with respect to all the arguments, where $H \in \mathbb{R}_{>0}$, $K \in \mathbb{R}$ are solid functionally compact sets. Note that a particular case of this assumption is e.g. $H = [-d\rho^{-q}, d\rho^{-q}]^d \supseteq \mathbb{R}^d$ because $d\rho^{-q}$ is an infinite number. Since Sobolev-Schwartz distributions can be embedded as GSF (see Sec. 2.3), this is clearly more general than the usual traditional case, where state and control functions are assumed to be piecewise smooth.

In particular, we will always consider state and control functions in the following spaces

$$q \in \mathcal{Q} := \{ q \in \mathcal{C}^{\infty}([t_1, t_2], H) \mid \|q - q_1\|_0 \leq r \} \quad (4.2)$$

$$u \in \mathcal{A} := \mathcal{C}^{\infty}([t_1, t_2], K), \quad (4.3)$$

where $r \in \mathbb{R}_{>0}$ is a fixed generalized radius such that $\overline{B_r(q_1)} \subseteq H$. In this section, we use simplified notations, e.g., of the form $L(\cdot, q, \varphi)$ or $\varphi(\cdot, q, u)$ to denote compositions $t \in [t_1, t_2] \mapsto L(t, q(t), \varphi(t, q(t), u(t))) \in \mathbb{R}^d$ and $t \in [t_1, t_2] \mapsto \varphi(t, q(t), u(t)) \in \mathbb{R}^d$.

In developing this topic, it is therefore essential to already have suitable results about solutions of ODE with GSF. The Banach fixed point theorem can be easily generalized to spaces of generalized continuous functions with the sup-norm $||\cdot||_0$ (see Def. 30). As a consequence, we have the following Picard-Lindelöf theorem for ODE in the $\mathcal{C}^k$ setting, see also [16, 49], and [67] for an updated reference about solution of (fractional) ODE in the framework of Colombeau’s algebra.

**Theorem 54.** Let $t_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}^d$, $\alpha, r \in \mathbb{R}_{>0}$. Let $F \in \mathcal{C}^k([t_0 - \alpha, t_0 + \alpha] \times \overline{B_r(y_0)}, \mathbb{R}^d$). Set $M := \max_{t_0 - \alpha \leq t \leq t_0 + \alpha} |F(t, y)|$, $L := \max_{t_0 - \alpha \leq t \leq t_0 + \alpha} |\partial_y F(t, y)| \in \mathbb{R}$ and assume that

$$\alpha \cdot M \leq r,$$

$$\lim_{n \rightarrow +\infty} \alpha^n L^n = 0,$$

(4.4)

where the limit in (4.4) is clearly taken in the sharp topology. Then there exists a unique solution $y \in \mathcal{C}^{k+1}([t_0 - \alpha, t_0 + \alpha], \mathbb{R}^d)$ of the Cauchy problem

$$\begin{cases} y'(t) = F(t, y(t)) \\ y(t_0) = y_0. \end{cases} \quad (4.5)$$

This solution is given by

$$y = \lim_{n \rightarrow +\infty} P^n(y_0)$$

$$P(y)(t) := y_0 + \int_{t_0}^t F(s, y(s)) \, ds \quad \forall t \in [t_0 - \alpha, t_0 + \alpha],$$

and for all $n \in \mathbb{N}$ satisfies $\|y - P^n(y_0)\|_0 \leq \alpha M \sum_{k=n}^{+\infty} \frac{\alpha^n L^n}{n!}$ and $\|y - y_0\|_0 \leq r$.

Finally, we have the following Grüssenwall-Bellman inequality in integral form:

**Theorem 55.** Let $\alpha \in \mathbb{R}_{>0}$. Let $u, a, b \in \mathcal{C}^k ([0, \alpha], \mathbb{R})$ and assume that $\|a\|_0 \cdot \alpha < N \cdot \log (d\rho^{-1})$ for some $N \in \mathbb{N}$. Assume that $a(t) \geq 0$ for all $t \in [0, \alpha]$, and that $u(t) \leq b(t) + \int_0^t a(s) u(s) \, ds$. Then
Therefore, Thm. 54 allows us to state that we assume that 

\[ q \]

Note that Thm. 54 in particular yields both 

\[ L_K := \max_{t_1 \leq t \leq t_2} |\varphi(t, q, k)|, \]

\[ M_K := \max_{t_1 \leq t \leq t_2} \max_{|q - q_1| \leq r} |\varphi(t, q, k)|, \]  

(4.6)

we assume that

\[ (t_2 - t_1)M_K \leq r \]

\[ \lim_{n \to +\infty} (t_2 - t_1)^n L_K^n = 0. \]  

(4.7)

Therefore, Thm. 54 allows us to state that

\[ \forall u \in A \exists! q^u \in Q : (\text{CP}) \text{ holds for all } t \in [t_1, t_2]. \]  

(4.8)

Note that Thm. 54 in particular yields both \( q^u \in \mathcal{GC}^\infty([t_1, t_2], \mathbb{R}^d) \) and \( \|q^u - q_1\|_0 \leq r \), so that \( q^u(t) \in \overline{B}_r(q_1) \subseteq H \) for all \( t \in [t_1, t_2] \), and thereby \( q^u \in Q \) (see (4.2)). Therefore, (4.8) holds for all solid functionally compact sets \( H \in \mathbb{R}^d \) and \( K \subseteq \mathbb{R}^d \) such that both \( \overline{B}_r(q_1) \subseteq H \) and (4.7) hold. Note also explicitly, the importance in this deduction of the closure of GSF (of which, we recall, Sobolev-Schwartz distributions are a particular case) with respect to composition (Thm. 10.(iv)).

Finally, observe that the constant \( L_K \in \mathbb{R} \) defined in (4.6) and Taylor Thm. 18 yield the Lipschitz condition

\[ \forall u \in A \forall t \in [t_1, t_2] \forall q, \tilde{q} \in \overline{B}_r(q_1) : |\varphi(t, q, u(t)) - \varphi(t, \tilde{q}, u(t))| \leq L_K |q - \tilde{q}|. \]

(4.9)

4.1. Weak Pontryagin Maximum Principle. Based on (4.8), we can introduce the notation and the optimization problem

\[ \forall u \in A : I[u] := \int_{t_1}^{t_2} L(t, q^u(t), u(t)) \, dt \]

(4.10)

Find \( v \in A : (\text{CP}) \) and \( \exists r \in \mathbb{R}_{>0} \exists l \in \mathbb{N} \forall u \in A \cap B^l_r(v) : I[v] \leq I[u]. \)

(4.11)

In order to develop the necessary optimality condition for Problem (4.10), we first assume that \( u \in \hat{A} \) in the sharp topology of the \( \mathbb{R} \)-graded Fréchet space \( (A, \|\cdot\|_0) \), see Sec. 2.4.1, i.e.

\[ \exists r \in \mathbb{R}_{>0} : B_r(u) = \left\{ v \in A \mid \max_{t_1 \leq t \leq t_2} |v(t) - u(t)| < r \right\} \subseteq A. \]

Thereby, for each direction \( \tilde{u} \in \mathcal{GC}^\infty([t_1, t_2], \mathbb{R}^d) \), for some \( \delta = \delta(u, \tilde{u}) \in (0, 1) \) sufficiently small

\[ \forall h \in (-\delta, \delta) : u + h\tilde{u} \in A, \]

(4.12)

and hence we can evaluate \( I[u + h\tilde{u}] \in \mathbb{R} \). Note explicitly that the control \( u \in A = \mathcal{GC}^\infty([t_1, t_2], K) \), whereas the direction \( \tilde{u} \) lies in the \( \mathbb{R} \)-module \( \mathcal{GC}^\infty([t_1, t_2], \mathbb{R}^d) \); this is indispensable to apply the fundamental Lem. 41 (see e.g. the proof of Thm. 63).

To define the first variation of \( I[-] : A \to \mathbb{R} \) at \( u \in A \) in the direction \( \tilde{u} \in \mathcal{GC}^\infty([t_1, t_2], \mathbb{R}^d) \), we can intuitively think at a sort of first order Taylor sum of \( I[u + h\tilde{u}] \) at \( u \):

**Definition 56.** Let \( \delta \in \mathbb{R}_{>0} \) be such that (4.12) holds. We say that \( R : (-\delta, \delta) \to \mathbb{R} \) is an incremental ratio of \( I[-] \) at \( u \in A \) in the direction \( \tilde{u} \in \mathcal{GC}^\infty([t_1, t_2], \mathbb{R}^d) \), if there exists a function \( \tilde{R} : (-\delta, \delta) \to \mathbb{R} \) (called remainder) such that:
Then, there exist constants \( \bar{\rho} \) such that

\[
A \in \mathbb{R}_{>0} \forall h \in (-\delta, \delta): \quad \left| \hat{R}(h) \right| \leq A \cdot h^2.
\]

We first prove that \( R(0) \in \mathbb{R} \) is unique:

**Theorem 57.** Let \( u \in \hat{A} \) and \( \bar{u} \in ^\delta \mathcal{GC}_0^\infty(t_1, t_2) \). If \( R, S \) are incremental ratios of \( I[-] \) at \( u \) in the direction \( \bar{u} \), then \( R(0) = S(0) \).

Proof. We have \( R : (-\delta_1, \delta_1) \rightarrow \mathbb{R} \) and \( S : (-\delta_2, \delta_2) \rightarrow \mathbb{R} \) from Def. 56. Let \( \bar{R} \) and \( \bar{S} \) be remainders of \( R, S \) respectively. Take an invertible \( h \in (-\delta, \delta) \); from Def. 56(i), we get

\[
R(h) - S(h) = h^{-1} \left( \hat{R}(h) - \hat{S}(h) \right).
\]

Therefore, \( |R(h) - S(h)| \leq (A + \bar{A})h \) from Def. 56(iii), and hence \( \lim_{h \to 0} |R(h) - S(h)| = 0 \). But invertible elements are dense in the sharp topology (Lem. 6), and thus \( \lim_{h \to 0} |R(h) - S(h)| = 0 = R(0) - S(0) \) using Def. 56(ii).

**Definition 58.** Assume that there exists an incremental ratio of \( I[-] \). Then, we define the first variation of \( I[-] \) at \( u \in \hat{A} \) in the direction \( \bar{u} \in ^\delta \mathcal{GC}_0^\infty(t_1, t_2) \) as \( \delta I^u(u; \bar{u}) := R(0) \), where \( R \) is any incremental ratio of \( I[-] \). Moreover, we say that \( u \in A \) is a weak extremal of \( I[-] \) if for any \( \bar{u} \in ^\delta \mathcal{GC}_0^\infty(t_1, t_2) \), \( \delta I(u; \bar{u}) = 0 \).

Using Def. 56 and similarly to Thm. 36, we can prove the following

**Theorem 59.** If \( v \in \hat{A} \) solves problem (4.11) and there exists an incremental ratio of \( I[-] \), then \( v \) a weak extremal of \( I[-] \).

Proof. Let \( \bar{u} \in ^\delta \mathcal{GC}_0^\infty(t_1, t_2) \) and let \( \delta \in (0, 1) \) be such that (4.12) holds and such that \( v + h\bar{u} \in B^r(v) \) for all \( h \in (-\delta, \delta) \). From Def. 56(i) of incremental ratio, we have \( I[v + h\bar{u}] = I[v] + hR(h) + \hat{R}(h) \geq I[v] \). Therefore, for all \( k \in (0, \delta) \), we get \( kR(k) \geq -\hat{R}(k) \) and hence \( -R(k) \leq Ak = A|k| \) by Def. 56(iii). Similarly, for all \( k \in (-\delta, 0) \), we get \( R(k) \leq -Ak = A|k| \). Thereby, \( |R(k)| = R(k) \vee -R(k) \leq A|k| \) for all invertible \( k \in (-\delta, \delta) \), which implies \( R(0) = \delta I(v; \bar{u}) = 0 \).

We now prove that, under sufficiently weak conditions, an incremental ratio of \( I[-] \) always exists and \( \delta I(u; \cdot) \) is an \( \mathbb{R} \)-linear continuous map. We first show the following continuity conditions for the map \( u \in A \rightarrow q^u \in Q \) (see [11] for a similar proof):

**Theorem 60 (Stability of order 1 and 2).** Let \( u \in \hat{A}, \bar{u} \in ^\delta \mathcal{GC}_0^\infty(t_1, t_2) \) and \( \delta \in (0, 1) \) as in (4.12). Assume that

\[
\exists N \in \mathbb{N}: \quad L_K \cdot (t_2 - t_1) \leq N \log(d \rho^{-1}).
\]

Then, there exist constants \( A, \bar{A} \in \mathbb{R} \) (depending only on \( u, \bar{u} \) and clearly on \( \varphi \)) such that for all \( |h| < \delta \), we have

\[
\begin{align*}
\left\| q^{u+\bar{u}} - q^u \right\|_0 & \leq |h| A \\
\left\| q^{u+\bar{u}} - q^u - h\bar{q} \right\|_0 & \leq \bar{A} h^2,
\end{align*}
\]

where \( \bar{q} \in ^\delta \mathcal{GC}^\infty([t_1, t_2], \mathbb{R}^d) \) is the unique global solution of the following Cauchy’s problem

\[
\begin{cases}
\begin{align*}
\dot{\bar{q}} &= \frac{\partial \varphi}{\partial q}(t, q^u, u) \cdot \bar{q} + \frac{\partial \varphi}{\partial u}(t, q^u, u) \cdot \bar{u} \\
\bar{q}(t_1) &= 0
\end{align*}
\end{cases}
\] (LCP1)
Proof. We prove in detail (4.14); the proof of (4.15) is similar. Let \( h \in (-\delta, \delta) \). From the evolution ODE (CP), for all \( t \in [t_1, t_2] \), we have

\[
|q^{u+h\bar{u}}(t) - q^u(t)| \leq \int_{t_1}^{t} (\varphi(s, q^{u+h\bar{u}}(s), u(s) + h\bar{u}(s)) - \varphi(s, q^u(s), u(s))) \, ds \leq \\
\leq \int_{t_1}^{t} |\varphi(s, q^{u+h\bar{u}}(s), u(s) + h\bar{u}(s)) - \varphi(s, q^u(s), u(s))| \, ds
\]

(4.16)

for all \( s \in [t_1, t] \), we have

\[
|\varphi(s, q^{u+h\bar{u}}(s), u(s) + h\bar{u}(s)) - \varphi(s, q^u(s), u(s))| \leq \\
\leq |\varphi(s, q^{u+h\bar{u}}(s), u(s) + h\bar{u}(s)) - \varphi(s, q^u(s), u(s) + h\bar{u}(s))| + \\
+ |\varphi(s, q^u(s), u(s) + h\bar{u}(s)) - \varphi(s, q^u(s), u(s))|.
\]

(4.17)

Now, we apply the Lipschitz condition (4.9) to the first summand, and a first order Taylor expansion with Lagrange remainder (Thm. 18.(i)) to the second one, obtaining

\[
|\varphi(s, q^{u+h\bar{u}}(s), u(s) + h\bar{u}(s)) - \varphi(s, q^u(s), u(s))| \leq L_K \cdot |q^{u+h\bar{u}}(s) - q^u(s)|
\]

(4.18)

where \( \xi = \xi(u, h, \bar{u}, s) \in [u(s), u(s) + h\bar{u}(s)] \subseteq \mathbb{R}^l \). The extreme value Thm. 24 applied to \( u, \bar{u} \) on the functionally compact set \([t_1, t_2]\) yields the existence of constants \( B_1, B_2 \in \mathbb{R}_{>0} \) such that \([u(s), u(s) + h\bar{u}(s)] \subseteq [B_1, B_2]^l \); since \(|h| < \delta\), we can always assume that these constants do not depend on \( h \) but only on \( u \) and \( \bar{u} \). In the same way, applying the extreme value theorem with \( \frac{\partial \varphi}{\partial u} \) on the functionally compact set \([t_1, t_2] \times B_r(q_1) \times [B_1, B_2]^l \), we get the existence of a constant \( C \in \mathbb{R} \) (depending only on \( u, \bar{u} \)) such that

\[
\forall s \in [t_1, t_2] : \left| \frac{\partial \varphi}{\partial u}(s, q^u(s), \xi) \cdot \bar{u}(s) \right| \leq C.
\]

Thereby, considering (4.19):

\[
\int_{t_1}^{t} |\varphi(s, q^u(s), u(s) + h\bar{u}(s)) - \varphi(s, q^u(s), u(s))| \, ds \leq |h|C \cdot (t_2 - t_1).
\]

(4.20)

And, using inequalities (4.16), (4.17), (4.18), and (4.20), we have:

\[
\forall t \in [t_1, t_2] : |q^{u+h\bar{u}}(t) - q^u(t)| \leq L_K \cdot \int_{t_1}^{t} |q^{u+h\bar{u}}(s) - q^u(s)| \, ds + |h|C \cdot (t_2 - t_1).
\]

(4.21)

Finally, we apply Grönwall-Bellman Thm. 55 to (4.21) and use assumption (4.13) to obtain

\[
\|q^{u+h\bar{u}} - q^u\|_0 \leq |h|C \cdot (t_2 - t_1) e^{L_K(t_2 - t_1)} \leq |h|C \cdot (t_2 - t_1) d^N.
\]

(4.22)

Setting \( A := C \cdot (t_2 - t_1) d^N \), (4.22) proves the claim, because also \( N \) does not depend on \( h \) but only on \( K, \varphi \) and \( q_1 \). \( \square \)

Note that the existence of a solution \( \bar{q} \) of the Cauchy problem (LCP) can be directly proved by setting for all \( t \in [t_1, t_2] \)

\[
a(t) := \frac{\partial \varphi}{\partial q}(t, q^u(t), u(t))
\]

(4.23)

\[
b(t) := \frac{\partial \varphi}{\partial u}(t, q^u(t), u(t))
\]

(4.24)

\[
\bar{q}(t) := \exp \left( \int_{t_1}^{t} a \right) \cdot \int_{t_1}^{t} b(s) \cdot \exp \left( -\int_{t_1}^{s} a \right) \, ds.
\]

In fact, \( \bar{q} \in \mathcal{GC}^\infty([t_1, t_2], \mathbb{R}^d) \) because primitives of GSF are GSF (Thm. 14), because of the closure with respect to composition (Thm. 10.(iv)), and because of condition (4.13), which implies
that \( \exp \left( \int_{t_1}^{t_2} a \right) \in \mathbb{R}. \) The uniqueness of the solution follows from the Picard-Lindelöf theorem. Finally, note that assumption (4.13) always holds if \( L_K \cdot (t_2 - t_1) \) is a finite number or if \( t_2 - t_1 \) is sufficiently small.

**Theorem 61.** In the assumptions of Thm. 60, there always exists an incremental ratio of \( I[-] \), and the following equality holds:

\[
\delta I(u; \bar{u}) = \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q}(s, q^u(s), u(s)) \cdot \bar{q}(s) + \frac{\partial L}{\partial u}(t, q^u(s), u(s)) \cdot \bar{u}(s) \right) ds \tag{4.25}
\]

where \( \bar{q} = \bar{q}(\bar{u}) \in Q \) is the unique global solution of problem \( (LCP_1) \). Moreover, \( \bar{u} \in \mathcal{GC}^\infty(t_1, t_2) \mapsto \delta I(u; \bar{u}) \in \mathbb{R} \) is an \( \mathbb{R} \)-continuous linear map, i.e. it is \( \mathbb{R} \)-linear and satisfies

\[
\exists J \in \mathbb{R}_{>0} \forall r \in \mathbb{R}_{>0} \forall v, \bar{v} \in B_r(0) : |\delta I(u; \bar{u}) - \delta I(u; \bar{v})| \leq J \cdot \|u - \bar{v}\|_0. \tag{4.26}
\]

**Proof.** Since \( u \in \hat{A} \), there always exists \( \delta \in (0, 1) \) such that (4.12) holds. Set \( v^h := q^{u+h\bar{u}} - q^u - h\bar{q} \) for all \( h \in (-\delta, \delta) \), so that

\[
\|v^h\|_0 \leq Ah^2 \tag{4.27}
\]

by Thm. 60. For \( t \in [t_1, t_2] \) and \( k = (0, k_q, k_u) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^l \), let

\[
R_L(t, q^u(t), u(t); k) := L(t, q^u(t) + k_q, u(t) + k_u) - L(t, q^u(t), u(t)) - \partial_q L(t, q^u(t), u(t)) \cdot k_q - \partial_u L(t, q^u(t), u(t)) \cdot k_u
\]

be the remainder of the first order Taylor formula of \( L \) at the point \( (t, q^u(t), u(t)) \) with increment \( k = (0, k_q, k_u) \) (see Thm. 18). Thereby, for all \( h \in (-\delta, \delta) \), we get

\[
I[u + h\bar{u}] - I[u] = \int_{t_1}^{t_2} \{ L(\cdot, q^{u+h\bar{u}}, u + h\bar{u}) - L(\cdot, q^u, u) \}
= \int_{t_1}^{t_2} \{ L(\cdot, q^u + h\bar{q} + v^h, u + h\bar{u}) - L(\cdot, q^u, u) \}
= \int_{t_1}^{t_2} \{ \partial_q L(\cdot, q^u, u) \cdot (h\bar{q} + v^h) + \partial_u L(\cdot, q^u, u) \cdot h\bar{u} + \}
\]

\[
+ R_L(\cdot, q^u, u; (\cdot, h\bar{q} + v^h, h\bar{u})) \}
= h \int_{t_1}^{t_2} \{ \partial_q L(\cdot, q^u, u) \cdot \bar{q} + \partial_u L(\cdot, q^u, u) \cdot \bar{u} \} +
\]

\[
+ \int_{t_1}^{t_2} \{ \partial_q L(\cdot, q^u, u) \cdot v^h + R_L(\cdot, q^u, u; (\cdot, h\bar{q} + v^h, h\bar{u})) \}
\]

Setting

\[
R := \int_{t_1}^{t_2} \{ \partial_q L(\cdot, q^u, u) \cdot \bar{q} + \partial_u L(\cdot, q^u, u) \cdot \bar{u} \}
\]

\[
\bar{R}(h) := \int_{t_1}^{t_2} \{ \partial_q L(\cdot, q^u, u) \cdot v^h + R_L(\cdot, q^u, u; (\cdot, h\bar{q} + v^h, h\bar{u})) \}
\]

Taylor Thm. 18 and (4.27) yield the existence of a constant \( D \in \mathbb{R}_{>0} \) such that \( |\bar{R}(h)| \leq Dh^2 \). This proves that \( R = R(0) = \delta J(u; \bar{u}) \) is the incremental ratio of \( I[-] \) with remainder \( \bar{R} \) (Def. 56), which is claim (4.25).

Using the notation \( \bar{q} = \bar{q}(\bar{u}) \) and the uniqueness of solution of problem \( (LCP_1) \), it follows that \( \bar{q}(\alpha\bar{u} + \beta\bar{v}) = \alpha \bar{q}(\bar{u}) + \beta \bar{q}(\bar{v}) \) for all \( \alpha, \beta \in \mathbb{R} \) and \( \bar{u}, \bar{v} \in \mathcal{GC}^\infty([t_1, t_2], K) \). Thereby, (4.25) implies that \( \delta I(u; \cdot) : \mathcal{GC}^\infty(t_1, t_2) \mapsto \mathbb{R} \) is an \( \mathbb{R} \)-linear map. Finally, with the simplified notations (4.23)
Moreover, for any control \( u \), we have
\[
\dot{q}(\bar{u}) - \dot{q}(\bar{v}) = \int_{t_1}^{t_2} a \cdot (\dot{q}(\bar{u}) - \dot{q}(\bar{v})) + \int_{t_1}^{t_2} b \cdot (\bar{u} - \bar{v})
\]
\[
\|\dot{q}(\bar{u}) - \dot{q}(\bar{v})\|_0 \leq \|\dot{q}(\bar{u}) - \dot{q}(\bar{v})\|_0 \cdot (t_2 - t_1) L_K + \|b\|_0(t_2 - t_1)\|\bar{u} - \bar{v}\|_0.
\]
Note that \( \|a\|_0 \leq L_K \) by (4.6). But assumption (4.7) implies that \( (t_2 - t_1)^N L_K < 1 \) for some \( N \in \mathbb{N} \) and hence that \( 1 - (t_2 - t_1)L_K \) is invertible, yielding
\[
\|\dot{q}(\bar{u}) - \dot{q}(\bar{v})\|_0 \leq (1 - (t_2 - t_1)L_K)^{-1}\|b\|_0(t_2 - t_1)\|\bar{u} - \bar{v}\|_0.
\]
Now, the conclusion (4.26) follows from this Lipschitz property and (4.25).

Note that the Lipschitz constant \( J \in \mathbb{R} \) in (4.26) can be an infinite number, e.g. if the Lagrangian \( L \) shows some kind of singularity as a generalized function.

**Definition 62.** Let \( \mathcal{H} \) be the Hamiltonian associated to the Lagrangian \( L \) and the state equation (CP), i.e. the GSF
\[
\mathcal{H} : [t_1, t_2] \times H \times K \times \mathbb{R}^d \rightarrow \mathbb{R}
\]
\[
(t, q, u, p) \mapsto L(t, q, u) + p \cdot \varphi(t, q, u).
\]
Moreover, for any control \( u \in \mathcal{A} \), let \( p^u \in \mathcal{G}^\infty([t_1, t_2], \mathbb{R}^d) \) denote the adjoint variable (generalized momentum), i.e. the unique GSF solution of the Cauchy problem
\[
\begin{cases}
\dot{p}^u = -\frac{\partial \mathcal{H}}{\partial q}(t, q^u, u, p^u) - \frac{\partial L}{\partial q}(t, q^u, u) - \left( \frac{\partial^2 \mathcal{H}}{\partial q^2}(t, q^u, u) \right)^T \cdot p^u \\
p^u(t_2) = 0.
\end{cases}
\]
As we showed above, this problem has a unique solution on \([t_1, t_2]\) because of our assumptions (4.7).

We want to close this section, giving a proof of the weak Pontryagin Maximum Principle, i.e. a theorem where instead of the usual condition
\[
\mathcal{H}(t, q^v(t), v(t), p^v(t)) = \min_{k \in K} \mathcal{H}(t, q^v(t), k, p^v(t)) \quad \forall t \in [t_1, t_2]
\]
(if \( v \) is an optimal control, i.e. it solves problem (4.11); see e.g. [4, 33, 61, 65]) we have instead the necessary condition \( \partial_v \mathcal{H}(t, q^v(t), v(t), p^v(t)) = 0 \) assuming that \( v \in \mathcal{A} \) is a local minimum of the functional \( I[-] \).

Directly from the definition of Hamiltonian, we get that, for any control \( u \in \mathcal{A} \), the pair \((q^u, p^u)\) is a solution of the following Hamiltonian system:
\[
\begin{cases}
\dot{q} = \frac{\partial \mathcal{H}}{\partial p}(t, q, u, p) \\
\dot{p} = -\frac{\partial \mathcal{H}}{\partial q}(t, q, u, p).
\end{cases}
\]

Next, we prove the following optimality condition for the functional (4.10):

**Theorem 63.** In the assumptions of Thm. 60, let \( u \in \hat{\mathcal{A}} \) and \( \bar{u} \in \mathcal{G}^\infty([t_1, t_2], K) \). Then
\[
\delta I(u, \bar{u}) = \int_{t_1}^{t_2} \frac{\partial \mathcal{H}}{\partial u}(\cdot, q^u, u, p^u) \cdot \bar{u}.
\]
Therefore \( u \) is a weak extremal of \( I[-] \) if and only if \((q^u, u, p^u)\) satisfy the equation:
\[
\frac{\partial \mathcal{H}}{\partial u}(t, q^u, u, p^u) = 0.
\]
Proof. Let $\tilde{u} \in ^\ast \mathcal{GC}_0^\infty(t_1, t_2)$. Thm. 61 asserts that

$$\delta I(u; \tilde{u}) = \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q}(\cdot, q^n, u) \cdot \tilde{q} + \frac{\partial L}{\partial u}(\cdot, q^n, u) \cdot \tilde{u} \right).$$

Equation (4.29) can be written as

$$\delta I(u; \tilde{u}) = \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q}(\cdot, q^n, u) \cdot \tilde{q} + \frac{\partial L}{\partial u}(\cdot, q^n, u) \cdot \tilde{u} \right).$$

Equation (4.29) can be written as

$$\delta I(u; \tilde{u}) = \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q}(\cdot, q^n, u) \cdot \tilde{q} + \frac{\partial L}{\partial u}(\cdot, q^n, u) \cdot \tilde{u} \right).$$

By (LCP) and (LCP2), we have

$$\tilde{q} = \int_{t_1}^{t_1} \left( \frac{\partial \varphi}{\partial q}(\cdot, q^n, u) \cdot \tilde{q} + \frac{\partial \varphi}{\partial u}(\cdot, q^n, u) \cdot \tilde{u} \right)$$

and

$$p^n = -\int_{t_1}^{t_1} \left( \frac{\partial L}{\partial q}(\cdot, q^n, u) + \frac{\partial \varphi}{\partial u}(\cdot, q^n, u) \right) \cdot \tilde{p}.$$

Now, using in (4.30) equalities (4.31), (4.32) and integrating by parts (Thm. 16.(vi)) with $\tilde{q}(t_1) = 0$, $p^n(t_2) = 0$, we get

$$\delta I(u; \tilde{u}) = \int_{t_1}^{t_2} \left( p^n \cdot \left( \frac{\partial \varphi}{\partial q}(\cdot, q^n, u) \cdot \tilde{q} + \frac{\partial \varphi}{\partial u}(t, q^n, u) \cdot \tilde{u} \right) \right)$$

and

$$\tilde{p} = \int_{t_1}^{t_1} \left( \frac{\partial \varphi}{\partial u}(\cdot, q^n, u) \cdot \tilde{p} + \frac{\partial L}{\partial u}(\cdot, q^n, u) \cdot \tilde{u} \right) = \int_{t_1}^{t_1} \frac{\partial \varphi}{\partial u}(\cdot, q^n, u) \cdot \tilde{p}.$$

The proof is now completed applying the fundamental lemma Cor. 43.

Summarizing these results yields the weak Pontryagin Maximum Principle:

Corollary 64 (Weak Pontryagin Maximum Principle). In the assumptions of Thm. 60, if $v \in \tilde{A}$ is a local extremal of $I[-]$, i.e. it solves problem (4.11), then $(q^n, v, p^n) \in ^\ast \mathcal{GC}_0^\infty([t_1, t_2], \tilde{\mathbb{R}}^d)$ are solution of the system

$$\begin{cases}
\dot{q} = \frac{\partial \mathcal{H}}{\partial q}(t, q, v, p) \\
\dot{p} = -\frac{\partial \mathcal{H}}{\partial q}(t, q, v, p) \\
\frac{\partial \mathcal{H}}{\partial u}(t, q, v, p) = 0 \\
(q(t_1), p(t_2)) = (0, 0).
\end{cases}$$

Moreover, if $(q, u, p) \in ^\ast \mathcal{GC}_0^\infty([t_1, t_2], \mathbb{R}^d)$ solve (WPS), then necessarily $(q, p) = (q^n, p^n)$.

Proof. By Thm. 59, we get that $v$ is also a weak extremal of the functional $I[-]$. Therefore, the conclusion follows by (HS), Thm. 63 and (LCP1), (LCP2). \qed

Definition 65. Any triple $(q, u, p) \in ^\ast \mathcal{GC}_0^\infty([t_1, t_2], \mathbb{R}^d)$ satisfying the system (WPS) is called a weak Pontryagin extremal.
Theorem 70. is a constant of motion on $q$. $J$ is constant along any interval $t_1, t_2$. \( \forall t \in [t_1, t_2] \). \( \frac{d}{dt}(\mathcal{H}(t, q(t), u(t), p(t))) = \frac{\partial \mathcal{H}}{\partial q}(t, q(t), u(t), p(t)) \forall t \in [t_1, t_2]. \) (4.33)

4.2. Noether’s Principle for optimal control. In classical optimal control, (see e.g. [15,28,64]) the optimal problem (4.11) is equivalent to minimize the augmented functional \( I[q, u, p] \) defined by

\[
I[q, u, p] := \int_{t_1}^{t_2} \left( \mathcal{H}(t, q(t), u(t), p(t)) - p(t) \cdot \dot{q}(t) \right) dt,
\]

with \( \mathcal{H} \) given by (4.28). The notion of variational invariance for problem (4.11) is then defined with the help of the augmented functional (4.34). In the GSF setting, we follow the same approach:

**Definition 67.** Let \( T = \{ (s, \cdot) \}_{s \in P} \subseteq \mathcal{GC}(T', T') \), \( S = \{ (s, \cdot) \}_{s \in P} \subseteq \mathcal{GC}(S, S') \), \( U = \{ (v(s, \cdot)) \}_{s \in P} \subseteq \mathcal{GC}(U, U') \), and \( A = \{ (\pi(s, \cdot)) \}_{s \in P} \subseteq \mathcal{GC}(A, A') \) be one parameter groups of diffeomorphisms of the open sets \( T' \subseteq \mathbb{R} \), \( S' \subseteq \mathbb{R}^d \) and \( U' \subseteq \mathbb{R}^l \). The augmented functional (4.34) is said to be invariant under the action of \( T, S, U, A \), if for any weak Pontryagin extremal \( (q, u, p) \) such that \( q \in \mathcal{GC}(T', S') \), \( u \in \mathcal{GC}(T', U') \) and \( p \in \mathcal{GC}(T', A') \), the following equality holds

\[
\left\{ \mathcal{H}(\tau(s, t), \sigma(s, q(t)), v(s, u(t)), \pi(s, p(t))) - \pi(s, p(t)) \cdot \frac{d\sigma(s, q(t))}{d\tau(s, t)} \right\} \frac{\partial \tau}{\partial t}(s, t) = \\
= \mathcal{H}(t, q(t), u(t), p(t)) - p(t) \cdot \dot{q}(t). \quad (4.35)
\]

for all \( s \in P \) and all \( t \in T' \).

**Theorem 68** (Necessary condition of invariance for problem (4.11)). If the augmented functional (4.34) is invariant in the sense of Def. 67, then for all weak Pontryagin extremals \( (q, u, p) \) such that \( q \in \mathcal{GC}(T', S') \), \( u \in \mathcal{GC}(T', U') \) and \( p \in \mathcal{GC}(T', A') \), we have

\[
\frac{\partial \mathcal{H}}{\partial q}(\cdot, q, u, p) \frac{\partial \tau}{\partial s}(0, \cdot) + \frac{\partial \mathcal{H}}{\partial q}(\cdot, q, u, p) \cdot \frac{\partial \sigma}{\partial s}(0, q) + \mathcal{H}(\cdot, q, u, p) \frac{d}{dt} \frac{\partial \tau}{\partial s}(0, \cdot) = p \cdot \frac{d}{dt} \frac{\partial \sigma}{\partial s}(0, q). \quad (4.36)
\]

Proof. We obtain (4.36) by differentiating (4.35) with respect to \( s \) at \( s = 0 \) and then considering Rem. 50 and (WPS).

Note that for the particular case of calculus of variations \( (\varphi(t, q, u) = u) \) and hence \( \mathcal{H} = L + p \cdot \dot{q} \) one obtains from (4.36) the necessary condition of invariance Lem. 51.

**Definition 69** (Constants of Motion/conservation law for problem (4.11)). We say that a function \( t \in T' \mapsto C(t, q(t), u(t), p(t)) \in \mathbb{R} \) is a constant of motion on \( T' \), \( S' \), \( U' \), \( A' \) if along any weak Pontryagin extremal \( (q, u, p) \) such that \( q \in \mathcal{GC}(T', S') \), \( u \in \mathcal{GC}(T', U') \) and \( p \in \mathcal{GC}(T', A') \), we have

\[
\frac{d}{dt}C(t, q(t), u(t), p(t)) = 0 \quad \forall t \in T'. \quad (4.37)
\]

By the uniqueness of Thm. 14, condition (4.37) implies that

\[
t \in J \mapsto C(t, q(t), u(t), p(t)) \in \mathbb{R}
\]
is constant along any interval \( J \subseteq T' \).

**Theorem 70** (Noether’s theorem for optimal control). If the augmented functional (4.34) is invariant in the sense of Def. 67, then the quantity defined for all \( q \in \mathcal{GC}(T', S') \), \( u \in \mathcal{GC}(T', U') \), \( p \in \mathcal{GC}(T', A') \) and \( t \in T' \) by

\[
C(t, q(t), u(t), p(t)) := \mathcal{H}(t, q(t), u(t), p(t)) \frac{\partial \tau}{\partial s}(0, t) - p(t) \cdot \frac{\partial \sigma}{\partial s}(0, q(t)) \quad \forall t \in T'
\]
is a constant of motion on \( T' \), \( S' \), \( U' \), \( A' \).
Proof. By Thm. 66 we have
\[
\frac{d}{dt} \left( \mathcal{H}(\cdot, q, u, p) \frac{\partial \tau}{\partial s}(0, \cdot) - p \cdot \frac{\partial \sigma}{\partial s}(0, q) \right) = \frac{\partial \mathcal{H}}{\partial t}(\cdot, q, u, p) \frac{\partial \tau}{\partial s}(0, \cdot) + \mathcal{H}(\cdot, q, u, p) \frac{d}{dt} \frac{\partial \tau}{\partial s}(0, \cdot) - \dot{p} \cdot \frac{\partial \sigma}{\partial s}(0, q) - p \cdot \frac{d}{dt} \frac{\partial \sigma}{\partial s}(0, q).
\]
Considering (4.36) and (WPS), we obtain:
\[
\frac{d}{dt} \left( \mathcal{H}(\cdot, q, u, p) \frac{\partial \tau}{\partial s}(0, \cdot) - p \cdot \frac{\partial \sigma}{\partial s}(0, q) \right) = \frac{\partial \mathcal{H}}{\partial q}(\cdot, q, u, p) \cdot \frac{\partial \sigma}{\partial s}(0, q) + \dot{p} \cdot \frac{\partial \sigma}{\partial s}(0, q) = 0.
\]
\[\square\]

Repeating the usual calculations, it is possible to prove Thm. 53 also as a particular case of Thm. 70 (cf. e.g. [19]).

5. Examples and applications

5.1. Modeling singular dynamical systems. In this section we show several applications of the calculus of variations with GSF we introduced above. As we already mentioned in the introduction, we will not consider mathematical models of singular dynamical systems at the times when singularities occur. Indeed, this would clearly require new physical ideas, e.g. in order to consider the nonlinear behavior of objects or materials for the entire duration of the singularity. Like in every mathematical model, the correct point of view concerns J. von Neumann’s reasonably wide area of applicability of a mathematical model:

To begin, we must emphasize a statement which I am sure you have heard before, but which must be repeated again and again. It is that the sciences do not try to explain, they hardly ever try to interpret, they mainly make models. By a model is meant a mathematical construct which, with the addition of some verbal interpretations, describes observed phenomena. The justification of such a mathematical construct is solely and precisely that it is expected to work - that is correctly to describe phenomena from a reasonably wide area. [73, pag. 492]

Therefore, it is not epistemologically correct to use the theory described in the present article to deduce a physical property of our modeled systems when a singularity occurs. Stating it with a language typically used in physics, we consider physical systems where the duration of the singularity is negligible with respect to the durations of the other phenomena that take place in the system. Mathematically, this means to consider as infinitesimal the duration of the singularities. As a consequence, several quantities changing during this infinitesimal interval of time have infinite derivatives. We can hence paraphrase the latter sentence saying that the amplitude (of the derivatives) of these physical quantities is much larger than all the other quantities we can estimate in the system. However, this is a logical consequence of our lacking of interest to include in our mathematical model what happens during the singularity, constructing at the same time a beautiful and sufficiently powerful mathematical model, and not because these quantities really become infinite.

As we will see below for the singularly variable length pendulum (Sec. 5.3), only a rigorous mathematical theory of infinitesimal quantities is able to resolve, e.g., the apparent inconsistency of considering infinitesimal oscillations and at the same time neglecting the infinitesimal duration of the singularity.

On the other hand, the aforementioned “wide area” is now able to include in a single equation the dynamical properties of our modeled systems, without being forced to subdivide into cases of the type “before/after the occurrence of each singularity”. Which is not even reasonable in several cases, e.g. in the motion of a particle in a granular medium or of a ray of light in an optical fiber.
Finally, note that remaining far from the singularity (from the point of view of the physical interpretation), is what allow us to state that in several cases this kind of models are already experimentally validated.

5.2. How to use numerical solutions. The applications we are going to present always end up with an ODE. Existence and uniqueness of the solution is therefore guaranteed by Thm. 54. Clearly, if an explicit analytic solution is possible, this is preferable, but this is a rare event. On the other hand, in numerous cases we have to deal with a differential equation whose singularities are generated by Heaviside’s functions or Dirac’s deltas and linear or nonlinear operations with them. Embeddings of these generalized functions as GSF are studied and quite well-known, see Sec. 2.3. Their pictures, Fig. 2.1, are clearly obtained by numerical methods, but their properties can be fully justified by suitable theorems, see e.g. example 23. In the same way, we can consider numerical solutions of our differential equations as empirical laboratories helping us to guess suitable properties and hence conjectures on the solutions. In principle, these properties must be justified by corresponding theorems. From this point of view, the fact that GSF share with ordinary smooth functions a lot of classical theorems (such as the intermediate value, the extreme value, the mean value, Taylor theorems, etc.) is usually of great help.

5.3. Singularly variable length pendulum. As a first example, we want to study the dynamics of a pendulum with singularly variable length, e.g. because it is wrapping on a parallelepiped (see Fig. 5.1; see [54] for a similar but non-singular case).

The pendulum length function is therefore \( \Lambda(\theta) = H(\theta_0 - \theta)L_1 + L_2 \), where \( H \) is the (embedding of the) Heaviside function. We always assume that \( L_1, L_2 \in \mathbb{R}_{>0} \) are finite and non-infinitesimal numbers. From this it follows that for all \( \theta \), \( H(\theta_0 - \theta) > \frac{d\rho - L_2}{L_1} \approx -\frac{L_2}{L_1} \), and hence that also \( \Lambda(\theta) > d\rho \) is invertible (recall that \( H \) has negative infinitesimal oscillations in an infinitesimal neighborhood of the origin, see Fig. 2.1).

The kinetic energy is given by:

\[
T(\theta, \dot{\theta}) = \frac{1}{2} m \dot{\theta}^2 \Lambda(\theta)^2. \tag{5.1}
\]

The potential energy (the zero level being the suspension point of the pendulum) is:

\[
U(\theta) = -mg \Lambda(\theta) \cos \theta - mg(1 - H(\theta_0 - \theta))L_1 \cos \theta_0. \tag{5.2}
\]

Let us define the Lagrangian \( L \) for this problem as

\[
L(\theta, \dot{\theta}) := T(\theta, \dot{\theta}) - U(\theta). \tag{5.3}
\]
The equation of motion is given by the Euler–Lagrange equation, Cor. 45, and can be written as:
\[ \frac{\partial L}{\partial \theta} = \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}}. \] (5.4)

Thereby
\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} \left( \frac{1}{2} m \dot{\theta}^2 \Lambda(\theta)^2 \right) = \frac{d}{dt} \left( m \dot{\theta} \Lambda(\theta)^2 \right) = m \Lambda(\theta)^2 \ddot{\theta} + 2m \dot{\theta} \Lambda(\theta) \dot{\theta}, \] (5.5)

where \( \dot{\Lambda}(\theta) := \frac{d}{dt} \Lambda(\theta(t)). \) From (5.2), the left side of the Euler–Lagrange equation (5.4) reduces to
\[ \frac{\partial L}{\partial \theta} = \frac{\partial T}{\partial \theta} + \frac{\partial (U)}{\partial \theta} = m \dot{\theta}^2 \Lambda(\theta) \Lambda'(\theta) + mg \Lambda'(\theta) (\cos \theta - \cos \theta_0) - mg \Lambda(\theta) \sin \theta, \] (5.6)

where
\[ \Lambda'(\theta) = \frac{d}{d\theta} (H(\theta_0 - \theta)L_1 + L_2) = -\delta(\theta_0 - \theta)L_1, \] (5.7)

and \( \delta \) is the Dirac delta function. We then obtain the following equation of motion:
\[ m \dot{\theta}^2 \Lambda(\theta) \Lambda'(\theta) + mg \Lambda'(\theta) (\cos \theta - \cos \theta_0) - mg \Lambda(\theta) \sin \theta = m \Lambda(\theta)^2 \ddot{\theta} + 2m \dot{\theta} \Lambda(\theta) \dot{\theta}. \] (5.8)

Taking into account that \( \dot{\Lambda}(\theta) = \Lambda'(\theta) \dot{\theta} \), we finally obtain the equation of motion for the variable length pendulum:
\[ \ddot{\theta} + \dot{\theta} \frac{\dot{\Lambda}(\theta)}{\Lambda(\theta)} - \frac{g}{\dot{\theta}} \frac{\dot{\Lambda}(\theta)}{\Lambda(\theta)^2} (\cos \theta - \cos \theta_0) + \frac{g}{\Lambda(\theta) \dot{\theta}} \sin \theta = 0. \] (5.9)

Note in (5.9) the nonlinear operations on the Sobolev-Schwartz distribution \( \Lambda \), on the GSF \( \theta \) and the composition \( t \mapsto \Lambda(\theta(t)). \). Since the Lagrangian \( L \) is autonomous, in the usual way Noether’s Thm. 53 implies that the mechanical energy of the system is a constant of motion:
\[ E(\theta, \dot{\theta}) = T(\theta, \dot{\theta}) + U(\theta) = \frac{1}{2} m \dot{\theta}^2 \Lambda(\theta)^2 - mg \Lambda(\theta) \cos \theta - mg(1 - H(\theta_0 - \theta))L_1 \cos \theta_0 = \text{constant}. \] (5.10)

Even if local (in time) existence and uniqueness of the solution \( \theta \) of (5.10) is easily granted by Thm. 54 and by the extreme value Thm. 24, the existence of a global solution is not so easy to show (see in [49] the general theory) and is out of the scope of the present work. Before showing the numerical solution of (5.9), let us consider the simplest case of the dynamics far from the singularity and that of small oscillations. The former, as we mentioned above, is the only physically meaningful one.

5.3.1. **Description far from singularity and small oscillations.** For simplicity, let us consider the simplest case \( \theta_0 = 0 \). Furthermore, we consider that the pendulum is initially at rest and starts its movement at \( t_1 \in \mathbb{R} \). The initial conditions we use are hence:
\[ \begin{cases} \theta(t_1) = \theta_1; \\ \dot{\theta}(t_1) = 0, \end{cases} \] (5.11)

with \( \theta_1 < 0 \). Assuming that at some time \( t_3 \in \mathbb{R} \) we have \( \theta(t_3) > 0 \), by the intermediate value theorem for GSF, there exists \( t_2 \in \mathbb{R} \) where we have the singularity, i.e. \( \theta(t_2) = 0 \) and the length of the pendulum smoothly changes from \( L_1 + L_2 \) to \( L_2 \) after the rope touches the parallelepiped. By example 23, it follows that this change happens in an infinitesimal interval, because by contradiction it is possible to prove that if \( \Lambda(\theta) \in (L_2, L_1 + L_2) \), then \( |\theta| \leq \frac{1}{\log \rho} \approx 0. \)

**Definition 71.** Let \( x, y \in \mathbb{R} \). We say that \( x \) is far from \( y \) if \( |x - y| \geq d \rho^a \) for all \( a \in \mathbb{R}_{>0} \).

More generally, we say that \( x \) is far from \( y \) with respect to the class of infinitesimals \( \mathcal{I} \subset \mathbb{R} \), if \( |x - y| \geq i \) for all \( i \in \mathcal{I} \).
For example, if $|x| \geq r$ for some $r \in \mathbb{R}_{>0}$, then $x$ is far from 0, but also the infinitesimal number $x = \frac{-1}{k \log \log \log dp}$ ($k \in \mathbb{R}_{>0}$) is far from 0; similarly, the infinitesimal $x = \frac{1}{k \log \log \log dp}$ if far from 0 with respect to all the infinitesimals of the type $\frac{1}{k \log \log \log dp}$ for $h \in \mathbb{R}_{>0}$.

If $\theta$ is far from 0 and $b \geq d\rho^{-a}$, $a \in \mathbb{R}_{>0}$, then $|b\theta| \geq d\rho^{-a}|\theta| \geq d\rho^{-a/2} \geq 1$. Therefore, from example 23, it follows that $H(-\theta) \in (0,1)$ and hence $\Lambda(\theta(t)) = 0$. Equation (5.9) becomes

$$\theta(t) \text{ is far from } 0 \Rightarrow \left\{ \begin{array}{ll}
\dot{\theta} + \frac{\rho}{t_1 + L_2} \sin \theta(t) = 0 & \text{if } \theta(t) < 0, \\
\dot{\theta} + \frac{\rho}{t_2} \sin \theta(t) = 0 & \text{if } \theta(t) > 0.
\end{array} \right. \quad (5.12)$$

If we assume that $\theta(t_1) = \theta_1 < 0$ and $\theta(t_3) > 0$ are far from 0, the sharp continuity of $\theta$ yields the existence of $\delta_1, \delta_3 \in \mathbb{R}_{>0}$ such that

$$\forall t \in [t_1, t_1 + \delta_1) \cup (t_3 - \delta_3, t_3] : \theta(t) \text{ is far from } 0$$

$$\forall t \in [t_1, t_1 + \delta_1), \theta(t) < 0 \quad (5.13)$$

$$\forall t \in (t_3 - \delta_3, t_3], \theta(t) > 0$$

and hence $t_2 \notin [t_1, t_1 + \delta_1) \cup (t_3 - \delta_3, t_3]$ because $\theta(t_2) = 0$. Assuming that $t_1, t_3$ are far from $t_2$, without loss of generality we can also assume to have taken $\delta_1$ so small that also $t_1 + \delta_1$ and $t_3 - \delta_3$ are far from $t_2$.

We now employ the non Archimedean framework of $\mathbb{R}$ in order to formally consider small oscillations, i.e. $\theta_1 \approx 0$. We first note that we cannot only assume $\theta_1$ infinitesimal, because if $\theta_1$ is not far from 0 then our solution will not be physically meaningful. However, we already have seen that we can take $\theta_1$ far from 0 and infinitesimal at the same time, e.g. $\theta_1 = \frac{1}{\log \log dp}$. In other words, $\theta_1$ is a “large” infinitesimal with respect to all the infinitesimals of the form $d\rho^\alpha$. Let $\vartheta_1, \vartheta_2$ be the solution of the linearized problems

$$\begin{cases}
\dot{\vartheta}_1 + \frac{\rho}{t_1 + L_2} \vartheta_1 = 0, & t_1 \leq t < t_1 + \delta_1 \\
\dot{\vartheta}_1(t_1) = 0, \quad \vartheta_1(t_1) = \theta_1
\end{cases}, \quad (5.14)$$

i.e. $\vartheta_1(t) = \theta_1 \cos (\omega(t - t_1)), \omega := \sqrt{\frac{\rho}{L_1 + L_2}}$, and $\vartheta_3(t) = \theta(t_3) \cos (\omega'(t_3 - t) - \frac{\partial \theta(t_3)}{\omega'} \sin (\omega'(t_3 - t)))$

$$\omega' = \sqrt{\frac{\rho}{L_2}}. \quad \text{We want to show that } \theta(t) \approx \vartheta_1(t) \text{ at least in an infinitesimal neighborhood of } t_1 \text{ and } t_3 \text{ exactly because } \theta_1 \approx 0. \text{ For simplicity, we proceed only for } \vartheta_1, \text{ the other case being similar. For any } t \in [t_1, t_1 + \delta_1), \text{ we have that } \theta(t) < 0 \text{ is far from 0 from (5.13), and hence } \theta + \frac{\rho}{t_1 + L_2} \sin \theta(t) = 0 \text{ from (5.12). Recalling the initial conditions, we obtain}

$$\theta(t_1 + h) - \theta_1 = -\omega^2 \int_{t_1}^{t_1 + h} \sin \theta(s) \, ds \quad \forall h \in (0, \delta_1).$$

Similarly, integrating (5.14), we get

$$\vartheta_1(t_1 + h) - \theta_1 = -\omega^2 \int_{t_1}^{t_1 + h} \vartheta_1(s) \, ds \quad \forall h \in (0, \delta_1).$$

Using Taylor Thm. 18 at $t_1$ with increment $h$ of these integral GSF, we obtain

$$\theta(t_1 + h) - \vartheta_1(t_1 + h) = -\omega^2 \left\{ \sin \theta_1 - \theta_1 + h \cos \theta_1 \cdot \dot{\theta}(t_1) - h \dot{\vartheta}_1(t_1) + h^2 R(h) \right\} =$$

$$= -\omega^2 \left\{ \sin \theta_1 - \theta_1 + h^2 R(h) \right\},$$

where $R(-)$ is a suitable GSF. Thereby, $\theta(t_1 + h) - \vartheta_1(t_1 + h) \approx -\omega^2 h^2 R(h) \approx 0$ for all $h \approx 0$ sufficiently small because $\sin \theta_1 \approx \theta_1$ since $\theta_1 \approx 0$.

Since each $t \in [t_1, t_1 + \delta_1) \cup (t_3 - \delta_3, t_3]$ is far from $t_2$, we can also formally join the two solutions $\vartheta_1$ using the Heaviside’s function:

$$\theta(t) \approx \vartheta_1(t) + H(t_2 - t) \left( \vartheta_3(t) - \vartheta_1(t) \right) \quad \forall t \in [t_1, t_1 + h) \cup (t_3 - h, t_3]. \quad (5.15)$$

For the motivations previously stated, this infinitesimal approximation cannot be extended to a neighborhood of $t_2$. 34 GASTÃO S.F. FREDERICO, PAOLO GIORDANO*, ALEKSANDR A. BRYZGALOV, AND MATHEUS J. LAZO
Figure 5.2. 8 times re-scaled solution (violet line) in radians and its derivative in rad/s (at $\theta = \theta_0 = \pi/40$ rad we can see a corner point). Parameters used: $L_1 = 0.4$ m, $L_2 = 0.2$ m, $g = 9.8$ m/s$^2$.

We close this section noting that all these deductions can be repeated using any GSF $H \in \mathcal{GC}^\infty(\mathbb{R}, \mathbb{R})$ satisfying for all $x$ far from zero $H(x) = 1$ if $x > 0$ and $H(x) = 0$ if $x < 0$. This allows us to consider e.g. a GSF that does not show the infinitesimal oscillations of the Heaviside’s function in an infinitesimal neighborhood of the origin.

5.3.2. Numerical Solution. The numerical solution of equation (5.9) has been computed using Mathematica Solver NDSolve (see [74]). Initial conditions we used are:

$$\begin{cases}
\theta(0) = 0 \text{ rad}, \\
\dot{\theta}(0) = 1 \text{ rad/s}.
\end{cases} \tag{5.16}$$

The graph of $\theta(t)$, and its derivative $\dot{\theta}(t)$, based on the Mathematica definitions of $H(x)$ and $\delta(x)$ (see [75]) are shown in Figure 5.2.

In Figure 5.3 we show the second derivative graph. Directly from (5.9) and (5.7) we can prove that when $\theta(t) = \theta_0$, $\ddot{\theta}(t)$ is an infinite number and hence $\dot{\theta}(t)$ has a corner point. Note finally that no infinitesimal oscillations are represented in an infinitesimal neighborhood of the singularities. This is due to the Mathematica implementation of $H$ and $\delta$: we could hence say that these graphs represent the solution far from the singularities.

5.4. Oscillations damped by two media. The second example concerns oscillations of a pendulum in the interface of two media. Since we have to neglect the dynamics occurring at singular times (i.e. at the changing of the medium), this can be considered only a toy model approximating the case of a very small but sufficiently heavy moving particle.

We hence want to model the system employing a “jump” in the damping coefficient $\beta$, i.e. a finite change occurring in an infinitesimal interval of time, see Fig. 5.4. Since the frictional forces acting in this case are not conservative, it is well-known that the Euler-Lagrange equations cannot be assumed to describe the dynamics of the system and we have to use the D’Alembert principle Cor. 46; see [42] for a deeper study.

The kinetic energy is given by:

$$T(\dot{\theta}) = \frac{1}{2}m\dot{\theta}^2\Lambda^2, \quad \tag{5.17}$$
and the potential energy (the zero level is the suspension point of the pendulum) is:

\[ U(\theta) = -mg\Lambda \cos \theta. \quad (5.18) \]

In case of fluid resistance proportional to the velocity, we can introduce the generalized forces \( Q \) as:

\[ Q(\dot{\theta}) = -r\Lambda^2 \dot{\theta}, \quad (5.19) \]

where \( r \) is a proportional coefficient depending on the media. Let’s define the Lagrangian \( L \) as

\[ L(\theta, \dot{\theta}) := T(\dot{\theta}) - U(\theta). \quad (5.20) \]

We hence assume that the equation of motion for this non-conservative system is given by the D'Alembert’s principle, so that Cor. 46 gives

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = Q. \quad (5.21) \]

Inserting (5.17), (5.18) and (5.19) into (5.21) we obtain the following equation of motion:

\[ m\Lambda^2 \ddot{\theta} + mg\Lambda \sin \theta = -r\Lambda^2 \dot{\theta}. \quad (5.22) \]

By introducing the damping coefficient \( \beta(\theta) := r(\theta)/(2m) \) (we clearly assume that the mass \( m \in \mathbb{R} > 0 \) is invertible) we obtain the classical form of the equation of motion for damped
oscillations:
\[ \ddot{\theta} + 2 \beta(\theta) \dot{\theta} + \frac{g \sin \theta}{\Lambda} = 0. \]  
(5.23)

If the pendulum crosses the boundary between two media (see Fig. 5.4) with damping coefficients \( \beta_1 \) and \( \beta_2 \), we can model the system using the Heaviside function \( H \):

\[ \beta(\theta) = \beta_1 + (H(\theta + \theta_0) - H(\theta - \theta_0)) (\beta_2 - \beta_1), \]  
(5.24)

where \( \theta = \pm \theta_0 \) are the angles at which we have the changing of the medium (singularities).

The numerical solution of (5.23) with \( \beta \) defined by (5.24) and initial conditions (5.16) is presented in Fig. 5.5. The numerical solution has been computed using Mathematica Solver NDSolve, but with an implementation of the Heaviside’s function \( H \) corresponding to Thm. 21, i.e. as represented in Fig. 2.1.

We also include the graphs of the angular frequency \( \dot{\theta} \) (which shows corner points) and of the angular acceleration \( \ddot{\theta} \) (which shows “jumps”, i.e. infinite derivatives at singular times, as we can directly see from (5.23) and (5.24)).

Note that, to simplify our analysis, we considered a fixed length \( \Lambda \) for the pendulum. However, in the framework of the suggested model, we could also consider a singularly variable length pendulum dumped by two media.

Once again, we note that the model can be refined by considering, instead of the Heaviside’s function, any GSF \( H \in \mathcal{GC}^\infty(\overline{\mathbb{R}}, \mathbb{R}) \) satisfying for all \( x \) far from zero \( H(x) = 1 \) if \( x > 0 \) and \( H(x) = 0 \) if \( x < 0 \); e.g. this refinement could allow one to consider different “large” infinitesimal neighborhoods of the origin in order to take into account a better modeling near the singularities.

5.5. Pais–Uhlenbeck oscillator with singular frequencies. An interesting higher order system with several applications in Physics is the Pais–Uhlenbeck (PU) oscillator [59]. The PU oscillator is usually studied as a toy model of recent higher-derivative theories like gravity, quantum-mechanical field theories and others (see for example [1, 5, 32, 50]), but it can also describe simple real word systems such as, for example, electrical circuits [5]. Actually, the PU oscillator is a bi-harmonic oscillator with two natural frequencies and is described by a fourth-order equation of motion. An example of PU GSF oscillator would be a PU electric circuit with singularly variable frequencies. We can think at an (exogenous or endogenous) force acting on the system and rapidly changing these frequencies.

The oscillator we consider is given by the following Lagrangian function

\[ L(t, q, \dot{q}, \ddot{q}) = \frac{m}{2} \left[ \dot{q}^2 - (\omega_1^2(t) + \omega_2^2(t)) q^2 + \omega_1^2(t) \omega_2^2(t) \ddot{q}^2 \right] \quad \forall t \in [t_1, t_3] \]  
(5.25)
Figure 5.6. First derivative $\dot{\theta}$ of the solution of (5.23) (blue line). The case with $\beta = \text{const} = \beta_1$ is also shown for comparison (violet line). Note the corner points at the singular moments, for example at $t = 0.083$ s (scaled in the right figure).

Figure 5.7. Second derivative $\ddot{\theta}$ of the solution of (5.23) (blue line). The case with $\beta = \text{const} = \beta_1$ is also shown for comparison (violet line). Note the “jumps” at the singular moments, for example at $t = 0.083$ s (scaled in the right figure). The infinitesimal oscillations are caused by the embedding as GSF of the Heaviside function.

where the natural frequencies

$$
\begin{align*}
\omega_1(t) &= \omega'_1 + H(t_s - t) (\hat{\omega}_1 - \omega'_1), \\
\omega_2(t) &= \omega'_2 + H(t_s - t) (\hat{\omega}_2 - \omega'_2)
\end{align*}
$$

(5.26)

changes from $\omega'_i$ to $\hat{\omega}_i$ ($i = 1, 2$) at $t = t_s \in (t_1, t_2)$, and $m, \omega'_1, \omega'_2, \hat{\omega}_1, \hat{\omega}_2 \in \mathbb{R} > 0$ are constants. The resulting equation of motion for the PU oscillator can be obtained from the Euler-Lagrange equation Cor. 45. We have:

$$q^{(4)} + (\omega'^2_1 + \omega'^2_2) \ddot{q} + 2 (\omega_1 \omega'_1 + \omega_2 \omega'_2) \dot{q} + \omega'^2_1 \omega'^2_2 q = 0,$$

(5.27)

where

$$\dot{\omega}_i(t) = -\delta(t_s - t) (\hat{\omega}_i - \omega'_i) \quad (i = 1, 2).$$

(5.28)

Once again, note the nonlinear operations in (5.27) on the Sobolev-Schwartz distributions $\omega_i$. We now consider an open set of the form $T' := (t_1, t_s - \delta) \cup (t_s + \delta, t_2)$ and assume that

$$\exists b \in \mathbb{R} > 0 \forall t \in T' \forall s \in (-d \rho^b, d \rho^b) : \ t + s \text{ is far from } t_s$$
(in particular any \( t \in T' \) is far from \( t_s \)). For example, it suffices to take \( \delta \geq -\frac{1}{h \log d_P}, \ h \in \mathbb{R}_{>1} \), so that if \( t \in (t_1, t_s - \delta) \), then \( t_s - t - s \geq \delta - s \geq -\frac{1}{h \log d_P} - d_P^b \geq -(h-1) \frac{\log d_P}{\log d_P} \geq d_P^a \) for any \( a, b \in \mathbb{R}_{>0} \); similarly we can proceed if \( t \in (t_s + \delta, t_2) \). Then \( \omega_i(t + s) = \omega_i(t) \) for all \( t \in T' \) and all \( s \in (-d_P^b, d_P^b) =: P \), and hence the Lagrangian (5.25) is invariant for the translation \( \tau(t, s) := t + s \) defined only for \( t \in T' \) and with parameter \( s \in P \). From the Noether’s Thm. 53, we hence obtain the following conserved quantity on the intervals \( (t_1, t_s - \delta) \) and \( (t_s + \delta, t_2) \):

\[
E = \frac{m}{2} \left[ 2\dot{q}q^{(3)} - \ddot{q}^2 + (\omega_1^2 + \omega_2^2) \dot{q}^2 + \omega_1^2 \omega_2^2 \dot{q}^2 \right] \tag{5.29}
\]

that coincides with the total energy for the PU oscillator [1,59]. As it is shown in Fig. 5.10, the total energy (5.29) is not preserved on the entire \( T' = (t_1, t_s - \delta) \cup (t_s + \delta, t_2) \) and we have a decreasing of energy switching from \( \omega'_i \) to \( \omega'_j \).

For the numerical solution of the PU oscillator’s equation of motion (5.27), we used Mathematica Solver NDSolve (see [74]) and Mathematica implementation of \( H \) and \( \delta \) (see [75]), and considered the initial conditions:

\[
\begin{align*}
q(0) &= 1 \text{ rad}, \\
\dot{q}(0) &= 2 \text{ rad/s}, \\
\ddot{q}(0) &= 0 \text{ rad/s}^2, \\
q^{(3)}(0) &= 1 \text{ rad/s}^3. 
\end{align*} \tag{5.30}
\]

See Fig. 5.8 and Fig. 5.9, where we can note the corner point for the third derivative \( q^{(3)} \) and the jump for the fourth derivative \( q^{(4)} \) at \( t = t_s \). The analytical solution of (5.27) for constant \( \omega_i \) is given by \( q(t; \omega_1, \omega_2) := A_1 \sin(\omega_1 t + \varphi_1) + A_2 \sin(\omega_2 t + \varphi_2) \), where \( A_i \) and \( \varphi_i \) are integration constants that for our initial conditions (5.30) are given by \( A_1 = \pm 6.02827, \ A_2 = \pm 1.81181, \ \varphi_1 = -2.88742 \) or \( \varphi_1 = 0.254175, \ \varphi_2 = 0.288674 \) or \( \varphi_2 = -2.85292 = 0.288674 - \pi \). Exactly as we did in Sec. 5.3, we can hence prove that \( q(t) = q(t; \omega_1', \omega_2') \) for \( t \in (t_1, t_s - \delta) \) and \( q(t) = q(t; \hat{\omega}_1, \hat{\omega}_2) \) for \( t \in (t_s + \delta, t_2) \). The non-singular solution \( q(-; \omega_1', \omega_2') \) is also presented as a violet line in Fig. 5.8.

6. Conclusions

The present paper concretely shows the possibility to develop calculus of variations and optimal control for a class of generalized functions extending Sobolev-Schwartz distributions. On the one hand, we have clearly only presented basic results, but, on the other hand, this paves the way for a lot of further results in pure and applied mathematics, theoretical physics and engineering.

We have also shown that generalized smooth function theory has features that closely resemble classical smooth functions. In contrast, some differences have to be carefully considered, such as the fact that the new ring of scalars \( \hat{\mathbb{R}} \) is not a field, it is not totally ordered, it is not ordered complete, so that its theory of supremum and infimum is more involved (see [56]), and its intervals are not connected in the sharp topology because the set of all the infinitesimals is a clopen set. Almost all these properties are necessarily shared by other non-Archimedean rings because their opposite are incompatible with the existence of infinitesimal numbers.

Conversely, the ring of Robinson-Colombeau generalized numbers \( \hat{\mathbb{R}} \) is a framework where the use of infinitesimal and infinite quantities is available, it is defined using elementary mathematics, and with a strong connection with infinitesimal and infinite functions of classical analysis (see Def. 2). As we have shown in Sec. 5.3 with the singularly variable length pendulum, this leads to a better understanding and opens the possibility to define new models of physical systems.

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Figure 5.8. Numerical solution of the PU oscillator and its derivatives $\dot{q}$, $\ddot{q}$.
Switching time is at $t_s = 15$ s, $m = 1$ kg, $\omega_1' = 0.5$ rad/s, $\dot{\omega}_1 = 0.7$ rad/s, $\omega_2' = 1$ rad/s, $\dot{\omega}_2 = 1.2$ rad/s.

Figure 5.9. Third and fourth derivatives of the numerical solution of the PU oscillator.

Figure 5.10. Energy of the PU oscillator.

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