1. The WDVV prepotential

In terms of the so-called flat coordinates $x^1, x^2, \ldots, x^n$ a solution to the Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations [13], [6] is given by a prepotential $F(x^1, x^2, \ldots, x^n)$ which satisfies the associativity relations:

$$\sum_{\delta, \gamma=1}^{n} \frac{\partial^3 F(x)}{\partial x^\alpha \partial x^\beta \partial x^\delta} \eta^{\delta \gamma} \frac{\partial^3 F(x)}{\partial x^\gamma \partial x^\omega \partial x^\rho} = \sum_{\delta, \gamma=1}^{n} \frac{\partial^3 F(x)}{\partial x^\alpha \partial x^\omega \partial x^\delta} \eta^{\delta \gamma} \frac{\partial^3 F(x)}{\partial x^\gamma \partial x^\beta \partial x^\rho}$$

(1)

together with a quasi-homogeneity condition:

$$\sum_{\alpha=1}^{n} (1 + \mu_1 - \mu_\alpha) x^\alpha \frac{\partial F}{\partial x^\alpha} = (3 - d)F + \text{quadratic terms}.$$ 

(2)

where $\mu_i, i = 1, \ldots, n$ and $d$ are constants.

Furthermore, expression

$$\frac{\partial^3 F(x)}{\partial x^\alpha \partial x^\beta \partial x^1} = \eta_{\alpha \beta}$$

(3)
HENRIK ARATYN AND JOHAN VAN DE LEUR

defines a constant non-degenerate metric: \( g = \sum_{\alpha,\beta=1}^{n} \eta_{\alpha\beta} dx^\alpha dx^\beta \).

As shown by Dubrovin (e.g. in reference [8]) there is an alternative description of the metric in terms of a special class of orthogonal curvilinear coordinates \( u_1, \ldots, u_n \)

\[
g = \sum_{\alpha,\beta=1}^{n} \eta_{\alpha\beta} dx^\alpha dx^\beta = \sum_{i=1}^{n} h_i^2(u)(du_i)^2
\]
called canonical coordinates. These coordinates allow to reformulate the problem in terms of the Darboux-Egoroff metric systems and corresponding Darboux-Egoroff equations and their solutions. In the Darboux-Egoroff metric the Lamé coefficients \( h_i^2(u) \) are gradients of some potential and this ensures that the so-called “rotation coefficients”

\[
\beta_{ij} = \frac{1}{h_j} \frac{\partial h_i}{\partial u_j}, \quad i \neq j, \quad 1 \leq i, j \leq n,
\]
are symmetric \( \beta_{ij} = \beta_{ji} \). The Darboux-Egoroff equations for the rotation coefficients are:

\[
\frac{\partial}{\partial u_k} \beta_{ij} = \beta_{ik} \beta_{kj}, \quad \text{distinct } i, j, k
\]

\[
\sum_{k=1}^{n} \frac{\partial}{\partial u_k} \beta_{ij} = 0, \quad i \neq j.
\]

In addition to these equations one also assumes the conformal condition:

\[
\sum_{k=1}^{n} u_k \frac{\partial}{\partial u_k} \beta_{ij} = -\beta_{ij}.
\]

The Darboux-Egoroff equations (6)-(7) appear as compatibility equations of a linear system:

\[
\frac{\partial \Phi_{ij}(u,z)}{\partial u_k} = \beta_{ik}(u) \Phi_{kj}(u,z) \quad i \neq k
\]

\[
\sum_{k=1}^{n} \frac{\partial \Phi_{ij}(u,z)}{\partial u_k} = z \Phi_{ij}(u,z)
\]

Define the \( n \times n \) matrices \( \Phi = (\Phi_{ij})_{1 \leq i,j \leq n}, \quad B = (\beta_{ij})_{1 \leq i,j \leq n} \) and \( V_i = [B, E_{ii}], \) where \( (E_{ij})_{k\ell} = \delta_{ik}\delta_{j\ell} \). Then the linear system (9)-(10) acquires the following form:

\[
\frac{\partial \Phi(u,z)}{\partial u_i} = (zE_{ii} + V_i(u)) \Phi(u,z), \quad i = 1, \ldots, n,
\]

\[
\sum_{k=1}^{n} \frac{\partial \Phi(u,z)}{\partial u_k} = z \Phi(u,z).
\]
Let, furthermore $\Phi(u, z)$ have a power series expansion

$$
\Phi(u, z) = \sum_{j=0}^{\infty} z^j \Phi^{(j)}(u) = \Phi^{(0)}(u) + z\Phi^{(1)}(u) + z^2\Phi^{(2)}(u) + \cdots \tag{13}
$$

and satisfy the “twisting” condition:

$$
\Phi(u, z) \eta^{-1} \Phi^T(u, -z) = I \tag{14}
$$

Equation (11) implies

$$
\sum_{k=1}^{n} u_k \frac{\partial \Phi(u, z)}{\partial u_k} = z \frac{\partial \Phi(u, z)}{\partial z} + \Phi(u, z)\mu, \quad \mu = \text{diag}(\mu_1, \ldots, \mu_n) \tag{15}
$$

where $\mu$ is a constant diagonal matrix obtained by a similarity transformation from the matrix $[B, U]$. The constant diagonal elements $\mu_i$ entered the quasi-homogeneity condition (2).

Define

$$
\phi_\alpha(u, z) \equiv \sum_{\beta=1}^{n} \Phi^{(0)}_{\beta 1}(u) \Phi_{\beta \alpha}(u, z) = \phi^{(0)}_\alpha(u) + z\phi^{(1)}_\alpha(u) + z^2\phi^{(2)}_\alpha(u) + z^3\phi^{(3)}_\alpha(u) + \ldots \tag{16}
$$

then, in terms of the flat coordinates $x^1, \ldots, x^n$

$$
\phi^{(1)}_\alpha(u) = \sum_{\beta=1}^{n} \eta_{\alpha \beta} x^\beta(u) \tag{17}
$$

and the prepotential is given by a closed expression (see e.g. [1] or [2]):

$$
F = -\frac{1}{2} \phi^{(3)}_1(u) + \frac{1}{2} \sum_{\delta=1}^{n} x^\delta(u) \phi^{(2)}_\delta(u). \tag{18}
$$

2. The CKP hierarchy

The CKP hierarchy [5] can be obtained as a reduction of the KP hierarchy,

$$
\frac{\partial}{\partial t_n} L = [(L^n)_+, L], \quad \text{for } L = L(t, \partial) = \partial_x + \ell^{-1} \partial_x^{-1} + \ell^{-2} \partial_x^{-2} + \cdots, \tag{19}
$$
where $x = t_1$, by assuming the extra condition

$$L^* = -L.$$  

By taking the adjoint, i.e., $*$ of (19), one sees that $\frac{\partial L}{\partial t_n} = 0$ for $n$ even. Date, Jimbo, Kashiwara and Miwa [5], [9] construct such $L$’s from certain special KP wave functions $\psi(t, z) = P(t, z)e^{\sum_t \xi^i} \psi_0(t, z)$ (recall $L(t, \partial) = P(t, \partial)\partial P(t, \partial)^{-1}$), where one then puts all even times $t_n$ equal to 0. Recall that a KP wave function satisfies

$$L\psi(t, z) = z\psi(t, z),$$

and

$$\text{Res} \psi(t, z)\psi^*(s, z) = 0.$$  

The special wave functions which lead to an $L$ that satisfies (20) satisfy

$$\psi^*(t, z) = \psi(\tilde{t}, -z), \quad \text{where} \quad \tilde{t}_i = (-1)^{i+1}t_i.$$  

We call such a $\psi$ a CKP wave function. Note that this implies that $L(t, \partial)^* = -L(\tilde{t}, \partial)$ and that

$$\text{Res} \psi(t, z)\psi(\tilde{s}, -z) = 0.$$  

One can put all even times equal to 0, but we will not do that here.

The CKP wave functions correspond to very special points in the Sato Grassmannian, which consists of all linear spaces

$$W \subset H_+ \oplus H_- = \mathbb{C}[z] \oplus z^{-1}\mathbb{C}[[z^{-1}]],$$

such that the projection on $H_+$ has finite index. Namely, $W$ corresponds to a CKP wave function if for any $f(z), g(z) \in W$ one has $\text{Res} f(z)g(-z) = 0$. The corresponding CKP tau functions satisfy $\tau(\tilde{t}) = \tau(t)$.

We will now generalize this to the multi-component case and show that a CKP reduction of the multi-component KP hierarchy gives the Darboux-Egoroff system. The $n$ component KP hierarchy [4], [10] consists of the equations in $t^{(i)}_j$, $1 \leq i \leq n$, $j = 1, 2, \ldots$

$$\frac{\partial}{\partial t^{(i)}_j} L = [(L^j C_i)_+, L], \quad \frac{\partial}{\partial t^{(i)}_j} C_k = [(L^j C_i)_+, C_k],$$  

for the $n \times n$-matrix pseudo-differential operators

$$L = \partial_x + L^{(-1)}\partial_x^{-1} + L^{(-2)}\partial_x^{-2} + \cdots, \quad C_i = E_{ii} + C_i^{(-1)}\partial_x^{-1} + C_i^{(-2)}\partial_x^{-2} + \cdots,$$
The CKP hierarchy and the WDVV prepotential

Let $1 \leq i \leq n$, where $x = t^{(1)}_1 + t^{(2)}_1 + \cdots + t^{(n)}_1$. The corresponding wave function has the form

$$
\Psi(t, z) = P(t, z) \exp \left( \sum_{i=1}^{n} \sum_{j=1}^{\infty} t^{(i)}_j z^j E_{ii} \right),
$$

where $P(t, z) = I + P^{(-1)}(t) z^{-1} + \cdots$, and satisfies

$$
L \Psi(t, z) = z \Psi(t, z), \quad C_i \Psi(t, z) = \Psi(t, z) E_{ii}, \quad \frac{\partial \Psi(t, z)}{\partial t^{(i)}_j} = (L^j C_i) + \Psi(t, z)
$$

and

$$
Res \Psi(t, z) \Psi^*(s, z)^T = 0.
$$

From this we deduce that $L = P(t, \partial_x) \partial_x P(t, \partial_x)^{-1}$ and $C_i = P(t, \partial_x) E_{ii} P(t, \partial_x)^{-1}$. Using this, the simplest equations in (26) are

$$
\frac{\partial \Psi(t, z)}{\partial t_1^{(i)}} = (z E_{ii} + V_1(t)) \Psi(t, z),
$$

where $V_1(t) = [B(t), E_{ii}]$ with $B(t) = P^{(-1)}(t)$. In terms of the matrix coefficients $\beta_{ij}(t)$ of $B$ we obtain (6) for $u_i = t^{(i)}_1$.

The Sato Grassmannian becomes vector valued, i.e.,

$$
H_+ \oplus H_- = (C[z])^n \oplus z^{-1}(C[[z^{-1}]])^n.
$$

The same restriction as in the 1-component case (23), viz.,

$$
\Psi(t, z) = \Psi^*(\tilde{t}, -z), \quad \text{where} \quad \tilde{t}_n^{(i)} = (-)^{n+1} t_n^{(i)}.
$$

leads to $L^* \tilde{t} = -L(t)$, $C_i^* \tilde{t} = C_i(t)$ and

$$
Res \Psi(t, z) \Psi(\tilde{s}, -z)^T = 0,
$$

which we call the multi-component CKP hierarchy. But more importantly, it also gives the restriction

$$
\beta_{ij}(t) = \beta_{ji}(\tilde{t}).
$$

Such CKP wave functions correspond to points $W$ in the Grassmannian for which

$$
Res f(z)^T g(-z) = Res \sum_{i=1}^{n} f_i(z) g_i(-z) = 0
$$
for any \( f(z) = (f_1(z), f_2(z), \ldots, f_n(z))^T, \) \( g(z) = (g_1(z), g_2(z), \ldots, g_n(z))^T \in W \).

If we finally assume that \( L = \partial_x \), then \( \Psi, W \) also satisfy

\[
\frac{\partial \Psi(t, z)}{\partial x} = \sum_{i=1}^n \frac{\partial \Psi(t, z)}{\partial t^{(i)}_1} = z \Psi(t, z), \quad zW \subset W \tag{30}
\]

and thus \( \beta_{ij} \) satisfies (7) for \( u_i = t^{(i)}_1 \). Now differentiating (28) \( n \) times to \( x \) for \( n = 0, 1, 2, \ldots \) and applying (30) leads to

\[
\Psi(t, z) \Psi(i, -z)^T = I.
\]

These special points in the Grassmannian can all be constructed as follows [11]. Let \( G(z) \) be an element in \( GL_n(\mathbb{C}[z, z^{-1}]) \) that satisfies

\[
G(z)G(-z)^T = 1, \quad \tag{31}
\]

then \( W = G(z)H_+ \). Clearly, any two \( f(z), g(z) \in W \) can be written as \( f(z) = G(z) a(z), g(z) = G(z) b(z) \) with \( a(z), b(z) \in H_+ \), then \( z f(z) = zG(z)a(z) = G(z)z a(z) \in W \), since \( za(z) \in H_+ \). Moreover,

\[
\text{Res} \ f(z)^T g(-z) = \text{Res} \ a(z)^T G(z)^T G(-z) b(-z) = \text{Res} \ a(z)^T b(-z) = 0.
\]

If we define \( M(t, z) = \Psi(t, z)G(z) \), then one can prove [3], [11] that

\[
M(t, z) = M^{(0)}(t) + M^{(1)}(t)z + M^{(2)}(t)z^2 + \cdots
\]

We want to change \( M(t, z) \) a bit more. However, we only want to do that for very special elements in this twisted loop group, i.e., to certain points of the Grassmannian that have a basis of homogeneous elements in \( z \). Let \( n = 2m \) or \( n = 2m + 1 \), choose non-negative integers \( \mu_i \) for \( 1 \leq i \leq m \) and define \( \mu_{n+1-j} = -\mu_j \) and let \( \mu_{m+1} = 0 \) if \( n \) is odd. Then take \( G(z) \) of the form

\[
G(z) = N(z)S^{-1} = Nz^{-\mu}S^{-1}, \quad \text{where} \quad \mu = \text{diag}(\mu_1, \mu_2, \ldots, \mu_n)
\]

and \( N = (n_{ij})_{1 \leq i, j \leq n} \) a constant matrix that satisfies

\[
N^T N = \sum_{j=1}^{n} (-1)^{\mu_j} E_{j,n+1-j} \quad \tag{32}
\]

and

\[
S = \delta_{n,2m+1} E_{m+1,m+1} + \sum_{j=1}^{m} \frac{1}{\sqrt{2}} (E_{jj} + iE_{n+1-j,j} + E_{j,n+1-j} - iE_{n+1-j,n+1-j}).
\]
Then \[\sum_{i=1}^{n} \sum_{j=1}^{\infty} j t_j^{(i)} \frac{\partial \Psi(t, z)}{\partial t_j^{(i)}} = z \frac{\partial \Psi(t, z)}{\partial z},\]
from which one deduces that
\[\sum_{i=1}^{n} \sum_{j=1}^{\infty} j t_j^{(i)} \frac{\partial \beta_{ij}}{\partial t_j^{(i)}} = -\beta_{ij}.\] (33)

Define \(\eta = (\eta_{ij})_{1 \leq i, j \leq n} = S^T S = \sum_{i=1}^{n} E_{i, n+1-i}\) and denote by \(\Phi(t, z) = M(t, z)S = \Psi(t, z)G(z)S = \Psi(t, z)N(z)\), then \(\Phi(t, z)\) satisfies the following relations:
\[\Phi(t, z) = \Phi(0)(t) + \Phi(1)(t)z + \Phi(2)(t)z^2 + \cdots\]
\[\frac{\partial \Phi(t, z)}{\partial t_1^{(i)}} = (zE_{ii} + V_i(t))\Phi(t, z)\]
\[\sum_{i=1}^{n} \frac{\partial \Phi(t, z)}{\partial t_1^{(i)}} = z\Phi(t, z),\]
\[\sum_{i=1}^{n} \sum_{j=1}^{\infty} j t_j^{(i)} \frac{\partial \Phi(t, z)}{\partial t_j^{(i)}} = z \frac{\partial \Phi(t, z)}{\partial z} + \Phi(t, z)\mu.\]

We next put \(t_j^{(i)} = 0\) for all \(i\) and all \(j > 1\) and \(u_i = t_1^{(i)}\), then we obtain the situation of Section 1. Define \(\phi_\alpha(u, z)\) as in (16), then \(\phi_\alpha^{(1)}(u) = \sum_{\gamma=1}^{n} \eta_{\alpha\gamma} x^\gamma(u)\) and the function \(F(u)\) given by (18) satisfies the WDVV equations.

3. An example

We will now give an example of this construction, viz., the case that \(n = 3\) (for simplicity) and \(\mu_1 = -\mu_3 = 2\) and \(\mu_2 = 0\). Hence, the point of the Grassmannian is given by

\[N(z)H_+ = N \begin{pmatrix} z^{-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^2 \end{pmatrix} H_.\]

More precise, let \(n_i = (n_{1i}, n_{2i}, n_{3i})^T\) and \(e_1 = (1, 0, 0)^T\), \(e_2 = (0, 1, 0)^T\) and \(e_3 = (0, 0, 1)^T\), then this point of the Grassmannian has as basis

\(n_1 z^{-2}, n_1 z^{-1}, n_1, n_2, n_1 z, n_2 z, e_1 z^2, e_2 z^2, e_3 z^2, e_1 z^3, e_2 z^3, \cdots\).
Using this one can calculate in a similar way as in [12] (using the boson-fermion correspondence or vertex operator constructions) the wave function:

\[ \Psi(t, z) = P(t, z) \exp \left( \sum_{i=1}^{n} \sum_{j=1}^{\infty} t_{ij}^{(i)} z^j E_{ii} \right), \text{ where} \]

\[ P_{jj}(t, z) = \frac{\hat{\tau}(t_{j}^{(k)}) - \delta_{kj}(\ell z^\ell)^{-1}}{\hat{\tau}(t)}, \]

\[ P_{ij}(t, z) = z^{-1} \frac{\hat{\tau}_{ij}(t_{j}^{(k)}) - \delta_{kj}(\ell z^\ell)^{-1}}{\hat{\tau}(t)} \]

and where

\[ \hat{\tau}(t) = \det \begin{pmatrix} n_{11} S_2(t^{(1)}) & n_{11} S_1(t^{(1)}) & n_{11} & 0 & n_{12} & 0 \\ n_{21} S_2(t^{(2)}) & n_{21} S_1(t^{(2)}) & n_{21} & 0 & n_{22} & 0 \\ n_{31} S_2(t^{(3)}) & n_{31} S_1(t^{(3)}) & n_{31} & 0 & n_{32} & 0 \\ n_{11} S_3(t^{(1)}) & n_{11} S_2(t^{(1)}) & n_{11} S_1(t^{(1)}) & n_{11} & n_{12} S_1(t^{(1)}) & n_{12} \\ n_{21} S_3(t^{(2)}) & n_{21} S_2(t^{(2)}) & n_{21} S_1(t^{(2)}) & n_{21} & n_{22} S_1(t^{(2)}) & n_{22} \\ n_{31} S_3(t^{(3)}) & n_{31} S_2(t^{(3)}) & n_{31} S_1(t^{(3)}) & n_{31} & n_{32} S_1(t^{(3)}) & n_{32} \end{pmatrix}. \]

The functions \( S_i(x) \) are the elementary Schur polynomials:

\[ S_1(x) = x_1, \quad S_2(x) = \frac{x_1^2}{2} + x_2, \quad S_3(x) = \frac{x_1^3}{6} + x_2 x_1 + x_3. \]

The tau function \( \hat{\tau}_{ij}(t) \) is up to the sign \( \text{sign}(i - j) \) equal to the above determinant where we replace the \( j \)-th row by

\[ (n_{i1} S_1(t^{(i)}) \quad n_{i1} \quad 0 \quad 0 \quad 0 \quad 0) \]

Next we put all higher times \( t_{j}^{(i)} \) for \( j > 1 \) equal to 0 and write \( u_i \) for \( t_{1}^{(i)} \). Then using the orthogonality-like condition (32) of the matrix \( N \) to reduce long expressions, the wave function becomes:

\[ \Psi(u, z) = \left( I + \frac{1}{\tau(u)} \sum_{i,j=1}^{3} \left[ \left( -u_1^{(3)} + u_1^{(2)}(u_i + u_j) - w_1^{(1)} u_i u_j \right) z^{-1} \right. \right. \]

\[ + \left. \left. \left( w_1^{(1)} u_i - w_1^{(2)} \right) z^{-2} \right] n_{i1} n_{j1} E_{ij} \right) e^{zU}, \]

where, for convenience of notation, we have introduced some new ”variables”

\[ w_i^{(k)} = \frac{1}{k} \sum_{\ell=1}^{3} u_{\ell}^{(k)} n_{\ell i} n_{\ell 1}. \]
and where
\[ \tau(u) = w_1^{(3)} - w_1^{(2)}. \]

Note that in this way we also have determined the rotation coefficients
\[ \beta_{ij} = \frac{1}{\tau(u)} \left( -w_1^{(3)} + w_1^{(2)}(u_i + u_j) - w_1^{(1)}u_iu_j \right) n_{i1}n_{j1}, \]
which is a new solution of order 3 of the the Darboux–Egoroff equations.

Recall that \( \eta = \sum_{i=1}^{3} E_{i4-i} \). It is now straightforward but tedious to determine the flat coordinates \( x^\alpha \) and the \( \phi^{(j)}_\alpha \) for \( j > 1 \). One finds that for \( \ell > 0 \) and \( p = 1, 2, 3 \):
\[ \phi^{(\ell-\mu_p)}_p = \frac{\tau w_p^{(\ell+2)} + \tau_1 w_p^{(\ell+1)} + \tau_2 w_p^{(\ell)}}{2(\ell-1)!\tau} \]
(34)
and
\[ \phi^{(-2-\mu_p)}_p = \delta_{p3}, \quad \phi^{(-1-\mu_p)}_p = -\delta_{p3} \frac{\tau_1}{2\tau}, \quad \phi^{(-\mu_p)}_p = \delta_{p3} \frac{\tau_2}{2\tau}, \]
(35)
where
\[ \tau_1 = w_1^{(2)} - w_1^{(1)} \]
\[ \tau_2 = w_1^{(4)} - (w_1^{(3)})^2. \]

Note that (34) also holds for \( p = 1 \) and \( \ell = 1, 2 \), it is easy to verify that \( \phi^{(-1)}_1 = \phi^{(0)}_1 = 0 \). Using (17), one has the following flat coordinates:
\[ x^1 = -\frac{\tau_1}{2\tau}, \]
\[ x^2 = \frac{1}{2\tau} \left( \tau w_2^{(3)} + \tau_1 w_2^{(2)} + \tau_2 w_2^{(1)} \right), \]
(36)
\[ x^3 = \frac{1}{4\tau} \left( \tau w_1^{(5)} + \tau_1 w_1^{(4)} + \tau_2 w_1^{(3)} \right), \]

From all this it is straightforward to determine \( F(u) \), given by (18):
\[ F = \frac{\tau_2}{16\tau^2} \left( \tau w_1^{(5)} + \tau_1 w_1^{(4)} + \tau_2 w_1^{(3)} \right) \]
\[ - \frac{\tau_1}{48\tau^2} \left( \tau w_1^{(6)} + \tau_1 w_1^{(5)} + \tau_2 w_1^{(4)} \right) \]
\[ - \frac{\tau}{96\tau^2} \left( \tau w_1^{(7)} + \tau_1 w_1^{(6)} + \tau_2 w_1^{(5)} \right) \]
\[ + \frac{1}{8\tau^2} \left( \tau w_2^{(3)} + \tau_1 w_2^{(2)} + \tau_2 w_2^{(1)} \right) \left( \tau w_2^{(4)} + \tau_1 w_2^{(3)} + \tau_2 w_2^{(2)} \right). \]
We shall not determine the explicit form of this prepotential in terms of the canonical coordinates here, because it is quite long. However, there is a problem even in this "simple" example. We do not know how to express the canonical coordinates $u_i$ in terms of the flat ones, the $x^a$'s and thus cannot express $F$ in terms of the flat coordinates. Hence we cannot determine the desired form of $F$. This can be solved in the simplest example, see [12], viz. the case that $\mu_1 = -\mu_n = 1$ and all other $\mu_i = 0$. This gives a rational prepotential $F$ (in terms of the flat coordinates).

**Acknowledgements**

H.A. was partially supported by NSF (PHY-9820663).

**References**

1. Akhmetshin, A.A., Krichever, I.M., Volvovski, Y.S. (1999) A generating formula for solutions of associativity equations (Russian), Uspekhi Mat. Nauk **54**, no. 2(326), pp. 167–168. English version: hep-th/9904028

2. Aratyn, H., Gomes, J.F., van de Leur, J.W., Zimerman, A.H. (2003) WDVV equations, Darboux–Egoroff metric and the dressing method, in the Unesp2002 workshop on Integrable Theories, Solitons and Duality, Conference Proceedings of JHEP (electronic journal, see http://jhep.sissa.it/) or math-ph/0210038

3. Aratyn, H. and van de Leur, J. (2003) Integrable structures behind WDVV equations, *Teor. Math. Phys.* **134**, Issue 1, pp. 14–26. [arXiv:hep-th/0111243]

4. Aratyn, H., Nissimov, E. and Pacheva, S. (2001) Multi-component matrix KP hierarchies as symmetry-enhanced scalar KP hierarchies and their Darboux-Bäcklund solutions, in Bäcklund and Darboux transformations. The geometry of solitons (Halifax, NS, 1999), CRM Proc. Lecture Notes, **29**, Amer. Math. Soc., Providence, RI, pp. 109–120

5. Date, E., Jimbo, M., Kashiwara, M. and Miwa, T. (1981) Transformation groups for soliton equations. 6. KP hierarchies of orthogonal and symplectic type, *J. Phys. Soc. Japan* **50** pp. 3813–1818

6. Dijkgraaf, R., Verlinde, E. and Verlinde, H. (1991) Topological strings in $d < 1$, *Nucl. Phys.* **B325**, 59

7. Dubrovin, B. (1993) Integrable systems and classification of 2-dimensional topological field theories, in: Integrable Systems, proceedings of Luminy 1991 conference dedicated to the memory of J.-L. Verdier, eds. O. Babelon, O. Cartier, Y. Kosmann-Schwarzbach, Birkhäuser, pp. 313–359

8. Dubrovin, B. (1996) Geometry on 2D topological field theories, in: Integrable Systems and Quantum Groups (Montecatini Terme, 1993), Lecture Notes in Math. 1620, Springer Berlin, pp. 120–348

9. Jimbo, M. and Miwa, T. (1983) Solitons and Infinite Dimensional Lie Algebras, *Publ. RIMS, Kyoto Univ.* **19**, pp. 943 – 1001

10. Kac, V.G. and van de Leur, J.W. (1993) The $n$-component KP hierarchy and representation theory, in Important developments in soliton theory, eds. A.S. Fokas and V. E. Zakharov, Springer Series in Nonlinear Dynamics, pp. 302–343

11. van de Leur, J.W. (2001) Twisted $GL_n$ Loop Group Orbit and Solutions of WDVV Equations, *Internat. Math. Res. Notices* **2001**, no. 11, pp. 551–573.

12. van de Leur, J.W. and Martini, R. (1999) The construction of Frobenius Manifolds from KP tau-Functions, *Commun. Math. Phys.* **205**, pp. 587–616

13. Witten, E. (1990) On the structure of the topological phase of two-dimensional gravity, *Nucl. Phys. B* **340** pp. 281–332