REPRESENTING HOMOLOGY CLASSES BY SYMPLECTIC SURFACES

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Abstract. We derive an obstruction to representing a homology class of a symplectic 4-manifold by an embedded, possibly disconnected, symplectic surface.

A natural question concerning symplectic 4-manifolds is the following: Given a closed symplectic 4-manifold \((M, \omega)\) and a homology class \(B \in H_2(M; \mathbb{Z})\), determine whether there exists an embedded, possibly disconnected, closed symplectic surface representing the class \(B\). This question has been studied by H.-V. Lê and T.-J. Li \[8, 9\]. We always assume that the orientation of a symplectic surface is the one induced by the symplectic form. One necessary condition is then, of course, that the symplectic class \([\omega]\) evaluates positively on the class \(B\), meaning that \(<[\omega], B> > 0\). Among other things, it is shown in \[9\] that a class \(B\) with \(<[\omega], B> > 0\) in a symplectic 4-manifold is always represented by a symplectic immersion of a connected surface. It is also noted that an obstruction to representing a homology class \(B\) by an embedded connected symplectic surface comes from the adjunction formula: The (even) integer

\[K_M B + B^2,\]

where \(K_M\) denotes the canonical class of the symplectic 4-manifold \((M, \omega)\), has to be at least \(-2\). This obstruction, however, disappears, if the number of components of the symplectic surface is allowed to grow large. Note that there are examples of classes in symplectic 4-manifolds which are represented by an embedded disconnected symplectic surface, but not by a connected symplectic surface: For example in the twofold blow-up \(X \# 2\mathbb{CP}^2\) of any closed symplectic 4-manifold \(X\) the sum of the classes of the exceptional spheres is not represented by a connected embedded symplectic surface according to the adjunction formula. It is the purpose of this article to derive an obstruction to representing a homology class by an embedded, possibly disconnected, symplectic surface.

In \[9\] it is also shown that for symplectic manifolds \(M\) of dimension at least six, every class in \(H_2(M; \mathbb{Z})\) on which the symplectic class evaluates positively is represented by a connected embedded symplectic surface. In \[8\] there is a conjecture which in the case of symplectic 4-manifolds \(M\) says that if \(\alpha\) is a class in \(H_2(M; \mathbb{Z})\) on which the symplectic class evaluates positively, then there exists a
positive integer $N$ depending on $\alpha$ such that $N\alpha$ is represented by an embedded, not necessarily connected, symplectic surface. In the examples at the end of this article we give counterexamples to this conjecture in the 4-dimensional case.

The non-existence of an embedded symplectic surface in the class $B$ has the following consequence for the Seiberg-Witten invariants, which we only state in the case $b_2^+ > 1$.

**Proposition 1.** Let $(M, \omega)$ be a closed symplectic 4-manifold with $b_2^+(M) > 1$ and $B \neq 0$ an integral second homology class which cannot be represented by an embedded, possibly disconnected, symplectic surface. Then the Seiberg-Witten invariant of the Spin$^c$-structure $s_0 \otimes PD(B)$ is zero, where $s_0$ denotes the canonical Spin$^c$-structure with determinant line bundle $K_M^{-1}$ induced by a compatible almost complex structure.

Here $PD$ denotes the Poincaré dual of a homology class. Note that the first Chern class of the Spin$^c$-structure $s_0 \otimes PD(B)$ is equal to $-K_M + 2PD(B)$. Proposition 1 is a consequence of a theorem of Taubes, relating classes with non-zero Seiberg-Witten invariants to embedded symplectic surfaces [14].

In the following, let $(M, \omega)$ denote a closed symplectic 4-manifold and $\Sigma \subset M$ an embedded, possibly disconnected, closed symplectic surface representing a class $B \in H_2(M; \mathbb{Z})$. We always assume that the orientation of $M$ is given by the symplectic form $(\omega \wedge \omega > 0)$. If the class $B$ is divisible by an integer $d > 1$, in the sense that there exists a class $A \in H_2(M; \mathbb{Z})$ such that $B = dA$, then there exists a $d$-fold cyclic ramified covering $\phi: \overline{M} \to M$, branched along $\Sigma$. The branched covering is again a closed symplectic 4-manifold. This is a well-known fact (the pullback of the symplectic form $\omega$ plus $t$ times a Thom form for the preimage $\Sigma$ of the branch locus is for small positive $t$ a symplectic form on $\overline{M}$; see [3, 11] for a careful discussion). The invariants of $\overline{M}$ are given by the following formulas [4, p. 243], [5]:

$$K_{\overline{M}} = \phi^*(K_M + (d-1)PD(A))$$
$$K_{\overline{M}}^2 = d(K_M + (d-1)PD(A))^2$$
$$w_2(\overline{M}) = \phi^*(w_2(M) + (d-1)PD(A)^2)$$
$$\sigma(\overline{M}) = d \left( \sigma(M) - \frac{d^2 - 1}{3}A^2 \right)$$

Here $PD(A)^2 \in H^2(M; \mathbb{Z}_2)$ is the mod 2 reduction of $PD(A)$. The second equation follows from the first because the branched covering map has degree $d$.

Suppose that the branched covering $\overline{M}$ is symplectically minimal and not a ruled surface over a curve of genus greater than 1. Then theorems of C. H. Taubes and A.-K. Liu [10, 13] imply that $K_{\overline{M}}^2 \geq 0$. With the formula above, we get the following obstruction on the class $A$.
Theorem 2. Let \((M, \omega)\) be a closed symplectic 4-manifold, \(\Sigma \subset M\) an embedded, possibly disconnected, closed symplectic surface and \(d > 1\) an integer such that \(dA = [\Sigma]\) for a class \(A \in H_2(M; \mathbb{Z})\). Consider the \(d\)-fold cyclic branched cover \(\overline{M}\), branched along \(\Sigma\). If \(\overline{M}\) is minimal and not a ruled surface over a curve of genus greater than 1, then

\[(K_M + (d-1)PD(A))^2 \geq 0.\]

It is therefore important to ensure that the branched covering \(\overline{M}\) is minimal and not a ruled surface. First, we have the following lemma.

Lemma 3. Let \(\phi: \overline{M} \to M\) be a cyclic \(d\)-fold branched covering of closed oriented 4-manifolds. Then \(b_2^+(\overline{M}) \geq b_2^+(M)\).

Proof. With our choice of orientations, the map \(\phi: \overline{M} \to M\) has positive degree. By Poincaré duality, the induced map \(\phi^*: H^*(M; \mathbb{R}) \to H^*(\overline{M}; \mathbb{R})\) is injective. It maps classes in the second cohomology of positive square to classes of positive square. This implies the claim. \(\square\)

Proposition 4. In the notation of Theorem 2 each of the following two conditions imply that \(\overline{M}\) is minimal and has \(b_2^+(\overline{M}) > 1\) and hence is not a ruled surface:

(a) If \(d\) is odd assume that \(M\) is spin and if \(d\) is even assume that \(PD(A)\) is characteristic. Also assume that \(3\sigma(M) \neq (d^2 - 1)A^2\).

(b) Assume that \(b_2^+(M) \geq 2\) and there exists an integer \(k \geq 2\) such that the class \(K_M + (d-1)PD(A)\) is divisible by \(k\).

Proof. Consider the \(d\)-fold branched covering \(\overline{M}\), branched along \(\Sigma\). The assumptions in case (a) imply that \(\overline{M}\) is spin and that the signature \(\sigma(\overline{M})\) is non-zero. According to a theorem of M. Furuta \cite{furuta} we have \(b_2^+(\overline{M}) \geq 3\). Also the symplectic manifold \(\overline{M}\) is minimal, because it is spin. In case (b) the lemma implies that \(b_2^+(\overline{M}) \geq 2\). In addition, the symplectic manifold \(\overline{M}\) is minimal, because its canonical class is divisible by \(k\) (a non-minimal symplectic 4-manifold \(Y\) contains a symplectic sphere \(S\) with \(K_Y S = -1\)). \(\square\)

Example 5. Consider \(M = K3\). Then we have \(K_M = 0\). Let \(d \geq 3\) be an integer and \(A \in H_2(M; \mathbb{Z})\) a class with \(A^2 < 0\). Theorem 2 together with Proposition 3 part (b) imply that \(dA\) is not represented by an embedded symplectic surface. Note that \(K3\) contains indivisible classes of negative self-intersection which, for a suitable choice of symplectic structure, are represented by symplectic surfaces, for example symplectic \((-2)\)-spheres. Let \(A\) be the homology class of such a sphere and \(\alpha = 3A\). Then \(\alpha\) is a counterexample to Lê’s Conjecture 1.4 in \cite{l}. \(\square\)

Example 6. Let \(X\) be a closed symplectic spin 4-manifold with \(b_2^+ > 1\) and \(M\) the blow-up \(X \# \mathbb{CP}^2\). Let \(E\) denote the class of the exceptional sphere in \(M\). We have \(K_M = K_X + PD(E)\). For every positive even integer \(d\) with \(d^2 > K_X^2\), the class \(dE\) is not represented by a symplectic surface. Taking for example the blow-up of the \(K3\) surface and \(\alpha = 2E\), we get another counterexample to Lê’s conjecture.
Note that with this method it is impossible to find a counterexample to Lê’s conjecture under the additional assumption that $\alpha^2 > 0$.

In light of the second example, the following conjecture seems natural.

**Conjecture.** Let $M$ be the blow-up $X \# \mathbb{CP}^2$ of a closed symplectic 4-manifold $X$ and $E$ the class of the exceptional sphere. Then $dE$ is not represented by an embedded symplectic surface for all integers $d \geq 2$.

This conjecture holds by a similar argument as above for $X$ the $K3$ surface and the 4-torus $T^4$. Moreover, using positivity of intersections, the conjecture holds in the complex category for the blow-up of a complex surface and embedded complex curves. In fact, in this category it generalizes to multiples of the class of any connected embedded complex curve with negative self-intersection.

**Remark 7.** Branched covering arguments have been used in the past to find lower bounds on the genus of a connected surface representing a divisible homology class in a closed 4-manifold, see [1, 6, 7, 12].

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