AN $\alpha$-STABLE LIMIT THEOREM UNDER SUBLINEAR EXPECTATION

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Abstract. For $\alpha \in (1,2)$, we present a generalized central limit theorem for $\alpha$-stable random variables under sublinear expectation. The foundation of our proof is an interior regularity estimate for partial integro-differential equations (PIDEs). A classical generalized central limit theorem is recovered as a special case, provided a mild but natural additional condition holds. Our approach contrasts with previous arguments for the result in the linear setting which have typically relied upon tools that are nonexistent in the sublinear framework, e.g., characteristic functions.

1. Introduction

The purpose of this manuscript is to prove a generalized central limit theorem for $\alpha$-stable random variables in the setting of sublinear expectation. Such a result complements the limit theorems for $G$-normal random variables due to Peng and others in this context and answers in the affirmative a question posed by Neufeld and Nutz in [11] (see below).

When working with a sublinear expectation, one is simultaneously considering a potentially uncountably infinite and non-dominated collection of probability measures. A construction of this kind is motivated by the study of pricing under volatility uncertainty. Needless to say, a variety of frequently called upon devices from the classical setting are unavailable. The complications encompass further issues as well: new behaviors are occasionally observed like those outlined in [3].

Analogues of significant theorems from classical probability and stochastic analysis are nevertheless moderately abundant. For instance, versions of the law of large numbers can be found in [16] and [17]; the martingale representation theorem is given in [23], [24], and [20]; and Girsanov’s theorem is obtained in [25], [12], and [5]. To conduct investigations along these lines, standard proofs must often be reimagined. For instance, Peng’s proof of the central limit theorem under sublinear expectation in [16] resorts to interior regularity estimates for fully nonlinear parabolic partial differential equations (PDEs). His idea has since been extended to prove a number of variants of his original result, e.g., see [17], [9], [7], and [26].

We will operate in the sublinear expectation framework unless explicitly indicated otherwise. The objects of our special attention here, the $\alpha$-stable random variables for $\alpha \in (1,2)$, were introduced in [11]. The authors pondered whether or not these could...
be the subject of a generalized central limit theorem. Classical generalized central limit theorems ordinarily come in one of three flavors:

(i) a statement indicating that a random variable has a nonempty domain of attraction if and only if it is $\alpha$-stable such as Theorem 2.1.1 in [S],

(ii) a characterization theorem for the domain of attraction of an $\alpha$-stable random variable such as Theorem 2.6.1 in [S], or

(iii) a characterization theorem for the domain of normal attraction for an $\alpha$-stable random variable such as Theorem 2.6.7 in [S].

Recall that an i.i.d. sequence $(Y_i)_{i=1}^{\infty}$ of random variables is in the domain of attraction of a random variable $X$ if there exist sequences of constants $(A_i)_{i=1}^{\infty}$ and $(B_i)_{i=1}^{\infty}$ so that

$$\frac{1}{B_n} \sum_{i=1}^{n} Y_i - A_n$$

converges in distribution to $X$ as $n \to \infty$. $(Y_i)_{i=1}^{\infty}$ is in the domain of normal attraction of $X$ if

$$B_n = \frac{1}{b n^{1/\alpha}}$$

for some $b > 0$.

We confine our search to the direction suggested by (iii) because of the particular importance classically of results of this type (cf. the central limit theorem). Our main findings are summarized in Theorem 3.1, which details sufficient conditions for membership in the domain of normal attraction of a given $\alpha$-stable random variable. While the initial appearance of our distributional hypotheses is perhaps forbidding, in point of fact, our assumptions are manageable. This is illustrated by the discussion immediately following Theorem 3.1, as well as Examples 4.1 and 4.2.

Example 4.1 establishes that the $\alpha$-stable random variables under consideration are in their own domain of normal attraction. Although one need not apply Theorem 3.1 for this purpose, the writeup serves a clarifying role and any credible result clearly must pass this litmus test.

Example 4.2 is more substantive. Setting aside a few mild “uniformity” conditions which arise due to the supremum, this example can be understood in an intuitive manner.

If the uncertainty subset of distributions for a random variable $Y$ consists of one classical distribution in the domain of normal attraction for each classical $\alpha$-stable distribution characterized by the elements of the subset of Lévy triplets corresponding to an $\alpha$-stable random variable $X$, then $Y$ is in the domain of normal attraction of $X$.

This falls out of our analysis just below Theorem 3.1, where we describe the relationship between our work and the classical result noted in (iii) above. More specifically, Theorem 3.1 detects all classical random variables in this collection with mean zero and a cumulative distribution function (cdf) that satisfies a small differentiability requirement. An extra regularity condition on the cdf is unavoidable, as one must translate its form into properties that can be stated only in terms of expectation.

The strategy of our proof is to reduce demonstrating convergence in distribution to showing that a certain limit involving the solution to the backward version of our generating PIDE is zero. Upon breaking up our domain and summing the corresponding increments of the solution, regularity properties of this function are employed to argue that size of the terms being added together decay rapidly enough in the limit to furnish
the desired conclusion. This general scheme is similar to that initiated in [10], except that the generating equation there is
\[
\partial_t u - \frac{1}{2} \left( \sigma^2 (\partial_{xx} u)^+ - \sigma^2 (\partial_{xx} u)^- \right) = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}
\]
\[
u(0, x) = \psi(x), \quad x \in \mathbb{R}
\]
for some \(0 \leq \sigma^2 \leq \sigma^2\) and appropriate function \(\psi\). Recall that this equation is known as the Barenblatt equation if \(\sigma^2 > 0\) and has been studied in [2] and [1], for instance. Ours is given by (2.1), a difference that leads to a few difficulties as reflected by the increased complexity of our hypotheses. Note that the regularity piece of the argument is not a source of adversity because of the technology from [22].

The work in this paper offers a step toward understanding \(\alpha\)-stability under sublinear expectation. The simple interpretation admitted by Example 4.2 is promising, as developing intuition in this environment is usually a tough undertaking for the reasons mentioned previously.

A brief overview of necessary background material can be found in Section 2. We prove our main result and discuss its connection to the classical case in Section 3. Examples highlighting the applications of our main result are contained in Section 4. The proof of the essential interior regularity estimate for our PIDE is located in Appendix A.

2. Background

We now offer a concise account of those aspects of sublinear expectations, \(\alpha\)-stable random variables, and PIDEs which are required for the sequel. References for more comprehensive treatments of these topics are also included for the convenience of the interested reader.

**Definition 2.1.** Let \(\mathcal{H}\) be a collection of real-valued functions on a set \(\Omega\). A sublinear expectation \(\mathcal{E}\) is an operator \(\mathcal{E}: \mathcal{H} \rightarrow \mathbb{R}\) which is

(i) monotonic: \(\mathcal{E}[X] \leq \mathcal{E}[Y]\) if \(X \leq Y\),

(ii) constant-preserving: \(\mathcal{E}[c] = c\) for any \(c \in \mathbb{R}\),

(iii) sub-additive: \(\mathcal{E}[X + Y] \leq \mathcal{E}[X] + \mathcal{E}[Y]\), and

(iv) positive homogeneous: \(\mathcal{E}[\lambda X] = \lambda \mathcal{E}[X]\) for \(\lambda \geq 0\).

The triple \((\Omega, \mathcal{H}, \mathcal{E})\) is called a sublinear expectation space.

One views \(\mathcal{H}\) as a space of random variables on \(\Omega\). Typically, it is assumed that \(\mathcal{H}\)

(i) is a linear space,

(ii) contains all constant functions, and

(iii) contains \(\psi(X_1, X_2, \ldots, X_n)\) for every \(X_1, X_2, \ldots, X_n \in \mathcal{H}\) and \(\psi \in C_{b, Lip}(\mathbb{R}^n)\),

where \(C_{b, Lip}(\mathbb{R}^n)\) is the set of bounded Lipschitz functions on \(\mathbb{R}^n\);

however, we will expend little attention on either \(\Omega\) or \(\mathcal{H}\). Delicacy needs to be exercised while computing sublinear expectations. A rare instance when a classical technique can be justly employed is the following.
Lemma 2.2. Consider two random variables \( X, Y \in \mathcal{H} \) such that \( \mathcal{E} [Y] = -\mathcal{E} [-Y] \).

Then
\[
\mathcal{E} [X + \alpha Y] = \mathcal{E} [X] + \alpha \mathcal{E} [Y]
\]
for all \( \alpha \in \mathbb{R} \).

This result is notably useful in the case where \( \mathcal{E} [Y] = \mathcal{E} [-Y] = 0 \).

Definition 2.3. A random variable \( Y \in \mathcal{H} \) is said to be independent from a random variable \( X \in \mathcal{H} \) if for all \( \psi \in C_{b.Lip}(\mathbb{R}^2) \), we have
\[
\mathcal{E} [\psi (X, Y)] = \mathcal{E} [\mathcal{E} [\psi (x, Y)]_{x = X}].
\]

Observe the deliberate wording. This choice is crucial, as independence can be asymmetric in our context. Note that this definition reduces to the traditional one if \( \mathcal{E} \) is a classical expectation. The same is true for the next three concepts.

Definition 2.4. Let \( X, Y \), and \( (Y_n)_{n=1}^{\infty} \) be random variables, i.e., \( X, Y \), and \( (Y_n)_{n=1}^{\infty} \in \mathcal{H} \).

(i) \( X \) and \( Y \) are identically distributed, denoted \( X \sim Y \), if
\[
\mathcal{E} [\psi (X)] = \mathcal{E} [\psi (Y)]
\]
for all \( \psi \in C_{b.Lip}(\mathbb{R}) \).

(ii) If \( X \) and \( Y \) are identically distributed and \( Y \) is independent from \( X \), then \( Y \) is an independent copy of \( X \).

(iii) \( (Y_n)_{n=1}^{\infty} \) converges in distribution to \( Y \), which we denote by \( Y_n \overset{d}{\to} Y \), if
\[
\lim_{n \to \infty} \mathcal{E} [\psi (Y_n)] = \mathcal{E} [\psi (Y)]
\]
for all \( \psi \in C_{b.Lip}(\mathbb{R}) \).

Random variables need not be defined on the same space to have appropriate notions of (i) or (iii). In this case, the above definitions require the obvious notational modifications. Further details concerning general sublinear expectation spaces can be found in [15] or [19].

Definition 2.5. Let \( \alpha \in (0, 2] \). A random variable \( X \) is said to be (strictly) \( \alpha \)-stable if for all \( a, b \geq 0 \),
\[
aX + bY
\]
and
\[
(a^{\alpha} + b^{\alpha})^{1/\alpha} X
\]
are identically distributed, where \( Y \) is an independent copy of \( X \).

Two examples of \( \alpha \)-stable random variables exist in the current literature. When \( \alpha = 2 \), we have the G-normal random variables of Peng. Resources on this topic are plentiful and include [15], [18], [19], [11], and [3]. If \( \alpha \in (1, 2) \), we can consider \( X_1 \) for a nonlinear \( \alpha \)-stable Lévy process \( (X_t)_{t \geq 0} \) in the framework of [11]. Our focus shall be restricted to latter situation.

The construction of nonlinear Lévy processes in [11] extends that studied in [6], [21], [14], and [13] and is much more general than our present objectives demand. We limit our presentation to a few key ideas. Let

(i) \( \alpha \in (1, 2) \);
(ii) $K_\pm$ be a bounded measurable subset of $\mathbb{R}_+$;

(iii) $\Theta = \{(0,0,F_{k_\pm}) : k_\pm \in K_\pm\}$; and

(iv) $F_{k_\pm}$ be the $\alpha$-stable Lévy measure

$$F_{k_\pm}(dz) = (k_- \mathbf{1}_{(-\infty,0)} + k_+ \mathbf{1}_{(0,\infty)}) (z) |z|^{-\alpha-1} dz$$

for all $k_\pm \in K_\pm$.

One can then produce a process $(X_t)_{t \geq 0}$ which is a nonlinear Lévy process whose local characteristics are described by the set of Lévy triplets $\Theta$. This means the following.

(i) $(X_t)_{t \geq 0}$ is a real-valued càdlàg process.

(ii) $X_0 = 0$.

(iii) $(X_t)_{t \geq 0}$ has stationary increments, i.e., $X_t - X_s$ and $X_t - X_s$ are identically distributed for all $0 \leq s \leq t$.

(iv) $(X_t)_{t \geq 0}$ has independent increments, i.e., $X_t - X_s$ is independent from $(X_{s_1}, \ldots, X_{s_n})$ for all $0 \leq s_1 \leq \cdots \leq s_n \leq s \leq t$.

(v) If $\psi \in C_{b,Lip}(\mathbb{R})$ and $u$ is defined by

$$u(t,x) = \mathcal{E}[\psi(x+X_t)]$$

for all $(t,x) \in [0,\infty) \times \mathbb{R}$, then $u$ is the unique viscosity solution of

$$\partial_t u(t,x) - \sup_{k_\pm \in K_\pm} \left\{ \int_\mathbb{R} [u(t,x+z) - u(t,x) - \partial_x u(t,x) z] F_{k_\pm}(dz) \right\} = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R}$$

$$u(0,x) = \psi(x), \quad x \in \mathbb{R}.$$  

A critical feature of this setup is that if $\Theta$ is a singleton, $(X_t)_{t \geq 0}$ is a classical Lévy process with triplet $\Theta$.

That $X_1$ actually is an $\alpha$-stable random variable is not immediately obvious. We give a brief argument in Example 4.1, but the core of this observation is a result from [11].

**Lemma 2.6.** For all $\beta > 0$ and $t \geq 0$, $X_{\beta t}$ and $\beta^{1/\alpha} X_t$ are identically distributed.

The dynamic programming principle in Lemma 2.7 and absolute value bound in Lemma 2.8 also play a central role when using our main result to check that $X_1$ is in its own domain of normal attraction.

**Lemma 2.7.** For all $0 \leq s \leq t < \infty$ and $x \in \mathbb{R}$,

$$u(t,x) = \mathcal{E}[u(t-s,x+X_s)].$$

**Lemma 2.8.** We have that

$$\mathbb{E}[|X_1|] < \infty.$$  

The remaining essential ingredients for our purposes describe the regularity of $u$. This initial result was proved in [11].

**Lemma 2.9.** The function $u$ is uniformly bounded by $\|\psi\|_{L^\infty(\mathbb{R})}$ and jointly continuous. More precisely, $u(t,\cdot)$ is Lipschitz continuous with constant $\text{Lip}(\psi)$, the Lipschitz constant
of $\psi$, and $u(\cdot, x)$ is locally 1/2-Hölder continuous with a constant depending only on Lip($\psi$) and

$$\sup_{k_\pm \in K_\pm} \left\{ \int_{\mathbb{R}} |z| \wedge |z|^2 F_{k_\pm}(dz) \right\} < \infty.$$  

While we will make use of these properties, we must call upon PIDE interior regularity estimates as well.

**Proposition 2.10.** Suppose that for some $\lambda$, $\Lambda > 0$, we know $\lambda < k_\pm < \Lambda$ for all $k_\pm \in K_\pm$. Then for any $h > 0$, $u$ classically satisfies

$$\partial_t u(t, x) - \sup_{k_\pm \in K_\pm} \left\{ \int_{\mathbb{R}} [u(t, x + z) - u(t, x) - \partial_x u(t, x) z] F_{k_\pm}(dz) \right\} = 0$$

on $(t, x) \in (h, \infty) \times \mathbb{R}$. Moreover, for any $n$,

$$\frac{\partial^n u}{\partial x^n} \text{ and } \frac{\partial^n u}{\partial t^n}$$

exist and are bounded Lipschitz functions on $(h, \infty) \times \mathbb{R}$.

Theorem 6.1 of [22] is the vital observation. The proposition is an immediate consequence of this result using standard techniques from PIDE theory. Details are included in Appendix A.

3. **Main Result**

To facilitate our discussion in the sequel, we now fix some notation. Compared with Section 2, we make only one alteration to our nonlinear $\alpha$-stable Lévy process $(X_t)_{t \geq 0}$: additionally assume that $K_\pm$ is a subset of $(\lambda, \Lambda)$ for some $\lambda, \Lambda > 0$. We will make use of this aspect in conjunction with Proposition 2.10.

We also consider a sequence $(Y_i)_{i=1}^\infty$ of random variables on some sublinear expectation space. The only aspect of this space that we will invoke directly is the sublinear expectation itself, say $\mathcal{E}'$. While we could consider $(Y_i)_{i=1}^\infty$ and $(X_t)_{t \geq 0}$ to exist on the same space, a distinction will be convenient for Example 4.2. We further ask that $Y_{i+1}$ and $Y_i$ be identically distributed and $Y_{i+1}$ be independent from $(Y_1, Y_2, \ldots, Y_i)$ for all $i \geq 1$. After proper normalization,

$$S_n := \sum_{i=1}^n Y_i$$

will be the sequence attracted to $X_1$.

**Theorem 3.1.** Suppose that

(i) $\mathcal{E}'[Y_1] = \mathcal{E}'[-Y_1] = 0$;

(ii) $\mathcal{E}'[|Y_1|] < \infty$; and

(iii) for any $0 < h < 1$ and $\psi \in C_{b, \mathrm{Lip}}(\mathbb{R})$,

$$n \left| \mathcal{E}' \left[ v \left( t, x + \frac{1}{bn^{1/\alpha}} Y_1 \right) - v(t, x) - \partial_x v(t, x) \left( \frac{1}{bn^{1/\alpha}} Y_1 \right) \right] \right| - \left( \frac{1}{n} \right) \sup_{k_\pm \in K_\pm} \left\{ \int_{\mathbb{R}} [v(t, x + z) - v(t, x) - \partial_x v(t, x) z] F_{k_\pm}(dz) \right\} \to 0 \quad (3.1)$$
uniformly on \([0,1] \times \mathbb{R}\) as \(n \to \infty\), where \(v\) is the unique viscosity solution of

\[
\partial_t v(t,x) + \sup_{k \pm \in K_\pm} \left\{ \int_{\mathbb{R}} [v(t,x+z) - v(t,x)] F_{k_\pm}(dz) \mid \partial_x v(t,x) \right\} = 0, \quad (t,x) \in (-h,1+h) \times \mathbb{R}
\]

where \(v \equiv 1 + h, x = \psi(x)\). (3.2)

Then

\[
\frac{1}{bn^{1/\alpha}} S_n \xrightarrow{d} X_1
\]
as \(n \to \infty\).

Admittedly, a cursory glance over our hypotheses leaves one with the impression that they are intractable. The opposite is true. Before presenting the proof of Theorem 3.1, let us demonstrate that when our attention is confined to the classical case, we are imposing only a mild and natural supplementary restriction on the attracted random variable. In addition to being a significant remark in itself, this work also underlies Example 4.2.

Assume that \(\Theta\) is a singleton. Since \((X_t)_{t \geq 0}\) is the classical Lévy process with triplet \((0,0,F_{k_\pm})\), the characteristic function of \(X_1\), denoted \(\varphi_{X_1}\), is given by

\[
\varphi_{X_1}(t) = \exp \left( (k_- / \alpha) \int_{-\infty}^{0} (\exp(itz) - 1 - itz) \frac{dz}{|z|^{\alpha+1}} + (k_+ / \alpha) \int_{0}^{\infty} (\exp(itz) - 1 - itz) \frac{dz}{|z|^{\alpha+1}} \right)
\]

for all \(t \in \mathbb{R}\). In the case where \(Y_1\) is a classical random variable with mean zero, Theorem 2.6.7 from [8] implies that

\[
\frac{1}{bn^{1/\alpha}} S_n \xrightarrow{d} X_1
\]
as \(n \to \infty\) if and only if the cdf of \(Y_1\), denoted \(F_{Y_1}\), has the form

\[
F_{Y_1}(z) = \begin{cases}
[b^\alpha (k_- / \alpha) + \beta_1(z)] |z|^{1-\alpha} & z < 0 \\
1 - [b^\alpha (k_+ / \alpha) + \beta_2(z)] & z > 0
\end{cases}
\]

for some functions \(\beta_1\) and \(\beta_2\) satisfying

\[
\lim_{z \to -\infty} \beta_1(z) = \lim_{z \to \infty} \beta_2(z) = 0.
\]

As there is no appropriate counterpart of the cdf in the sublinear setting, we must recast this condition using expectation. To do so requires \(F_{Y_1}\) to possess further regularity properties. For convenience, say that after an extension, the \(\beta_i\)'s are continuously differentiable on their respective closed half-lines. This is the lone extra requirement we shall need.

It follows that

\[
\mathbb{E}[|Y_1|] < \infty
\]
since

\[
\int_{0}^{\infty} z \; dF_{Y_1}(z) = -\int_{0}^{1} \frac{\beta_2'(z)}{z^{\alpha-1}} \; dz + \int_{0}^{1} \frac{b^\alpha k_+ + \alpha \beta_2(z)}{z^\alpha} \; dz + \beta_2(1) + \int_{1}^{\infty} \frac{\beta_2(z)}{z^\alpha} \; dz + \int_{1}^{\infty} \frac{b^\alpha k_+}{z^\alpha} \; dz < \infty
\]

(3.3)
and similarly for the integral along the negative half-line. One could have cited Theorem 2.6.4 of [8] instead, but (3.3) will be helpful in Example 4.2. We also get

\[ n \mathbb{E} \left[ v \left( t, x + \frac{1}{bn^{1/\alpha}} Y_1 \right) - v (t, x) - \partial_x v (t, x) \left( \frac{1}{bn^{1/\alpha}} Y_1 \right) \right] \]

\[- \left( \frac{1}{n} \right) \int_{\mathbb{R}} \left[ v (t, x + z) - v (t, x) - \partial_x v (t, x) z \right] F_{k_{b_1}} (dz) \]

\[ = \left( \frac{1}{b^\nu} \right) \int_{\mathbb{R}} \left[ v (t, x + z) - v (t, x) - \partial_x v (t, x) z \right] \left( \frac{\beta'_1 (bn^{1/\alpha} z)}{|z|^{\alpha+1}} + \alpha \beta_1 (bn^{1/\alpha} z) \right) 1_{(-\infty, 0)} (z) + \]

\[ \frac{-\beta'_2 (bn^{1/\alpha} z)}{|z|^{\alpha+1}} + \alpha \beta_2 (bn^{1/\alpha} z) \right) 1_{(0, \infty)} (z) \right] dz \]

(3.4)

for all \((t, x) \in [0, 1] \times \mathbb{R}\) and \(n \geq 1\) by changing variables.

A careful application of elementary estimates shows that this last expression tends to zero uniformly on \([0, 1] \times \mathbb{R}\) as \(n \to \infty\). To see this, note that we can choose a constant \(M_1\) that bounds \(|\partial_x v|, |\partial_v|, \) and \(|v|\) on \([0, 1] \times \mathbb{R}\) by Lemma 2.9 and Proposition 2.10. Then using integration by parts and the dominated convergence theorem,

\[ \left| \int_1^\infty \left[ v (t, x + z) - v (t, x) - \partial_x v (t, x) z \right] \left( \frac{-\beta'_2 (bn^{1/\alpha} z)}{|z|^{\alpha+1}} + \alpha \beta_2 (bn^{1/\alpha} z) \right) dz \right| \]

\[ = \left| [v (t, x + 1) - v (t, x) - \partial_x v (t, x)] \left( \frac{\beta_2 (bn^{1/\alpha} z)}{z^\alpha} \right) + \int_1^\infty \left( \frac{\beta_2 (bn^{1/\alpha} z)}{z^{\alpha}} \right) (\partial_z v (t, x + z) - \partial_z v (t, x)) dz \right| \]

\[ \leq 3M_1 \left| \beta_2 \left( \frac{bn^{1/\alpha}}{z} \right) \right| + 2M_1 \int_1^\infty \left| \beta_2 \left( \frac{bn^{1/\alpha}}{z} \right) \right| dz \]

\[ \to 0 \]

(3.5)

as \(n \to \infty\). The mean value theorem and a change of variables gives

\[ \left| \int_0^{bn^{1/\alpha}} \left[ v (t, x + z) - v (t, x) - \partial_x v (t, x) z \right] \left( \frac{-\beta'_2 (bn^{1/\alpha} z)}{|z|^{\alpha+1}} + \alpha \beta_2 (bn^{1/\alpha} z) \right) dz \right| \]

\[ \leq \int_0^{bn^{1/\alpha}} M_1 \left| z \beta_2 (bn^{1/\alpha} z) (bn^{1/\alpha} z + \alpha \beta_2 (bn^{1/\alpha} z)) \right| dz \]

\[ = \left( \frac{M_1}{b^{2-\alpha} n^{2-1}} \right) \int_0^1 \left| \frac{-\beta'_2 (z) z + \alpha \beta_2 (z)}{z^{\alpha-1}} \right| dz \]

\[ \to 0 \]

(3.6)

as \(n \to \infty\). We have

\[ \left| \int_1^{bn^{1/\alpha}} \left[ v (t, x + z) - v (t, x) - \partial_x v (t, x) z \right] \left( \frac{\alpha \beta_2 (bn^{1/\alpha} z)}{|z|^{\alpha+1}} \right) dz \right| \]

\[ \leq \int_1^{bn^{1/\alpha}} M_1 \left| \frac{\alpha \beta_2 (bn^{1/\alpha} z)}{z^{\alpha-1}} \right| dz \]

\[ \leq M_1 \alpha \int_0^1 \left| \frac{\beta_2 (bn^{1/\alpha} z)}{z^{\alpha-1}} \right| dz \]

\[ \to 0 \]

(3.7)
as \( n \to \infty \) by the mean value theorem and dominated convergence theorem. Finally,
\[
\left| \int_{bn^{1/\alpha}}^{1} \left[ v(t,x+z) - v(t,x) - \partial_x v(t,x) z \right] \left( -\beta_2 \left( bn^{1/\alpha} z \right) \left( bn^{1/\alpha} z \right) \right) \, dz \right|
= \left| - \left[ v(t,x+1) - v(t,x) - \partial_x v(t,x) \right] \beta_2 \left( bn^{1/\alpha} \right) \right|
+ \left[ v(t,x + \frac{1}{bn^{1/\alpha}}) - v(t,x) - \partial_x v(t,x) \left( \frac{1}{bn^{1/\alpha}} \right) \right] \left( \frac{1}{bn^{1/\alpha}} \right)^{-\alpha} \beta_2 (1)
+ \int_{bn^{1/\alpha}}^{1} \left[ \partial_x v(t,x+z) - \partial_x v(t,x) \right] \left( \frac{\beta_2 \left( bn^{1/\alpha} z \right)}{z^\alpha} \right) \, dz
- \alpha \int_{bn^{1/\alpha}}^{1} \left[ v(t,x+z) - v(t,x) - \partial_x v(t,x) z \right] \left( \frac{\beta_2 \left( bn^{1/\alpha} z \right)}{z^{\alpha+1}} \right) \, dz
\leq 3M_1 \left| \beta_2 \left( bn^{1/\alpha} \right) \right| + M_1 \left| \beta_2 (1) \right| \left( \frac{1}{b^{2-\alpha} n^{2\frac{\alpha}{\alpha-1}}} \right) + \int_{bn^{1/\alpha}}^{1} M_1 \left| \beta_2 \left( bn^{1/\alpha} z \right) \right| \, dz
+ \alpha \int_{bn^{1/\alpha}}^{1} M_1 \left| \beta_2 \left( bn^{1/\alpha} z \right) \right| \frac{dz}{z^{\alpha-1}}
\leq 3M_1 \left| \beta_2 \left( bn^{1/\alpha} \right) \right| + M_1 \left| \beta_2 (1) \right| \left( \frac{1}{b^{2-\alpha} n^{2\frac{\alpha}{\alpha-1}}} \right) + 2\alpha M_1 \int_{0}^{1} \left| \beta_2 \left( bn^{1/\alpha} z \right) \right| \, dz
\to 0
\tag{3.8}
\]
as \( n \to \infty \) by integration by parts, the dominated convergence theorem, and the mean value theorem. The integrals along the negative half-line are handled similarly.

Having established the connection between Theorem 3.1 and the classical case, we finally present its proof.

**Proof of Theorem 3.1.** To diminish notational clutter, we write \( B_n \) in place of \( 1/ \left( bn^{1/\alpha} \right) \) here and in Lemma 3.2. We need to show that
\[
\lim_{n \to \infty} \mathcal{E} [\psi \left( B_n S_n \right)] = \mathcal{E} [\psi \left( X_1 \right)]
\tag{3.9}
\]
for all \( \psi \in C_b \text{Lip} (\mathbb{R}) \). Our initial step will be to reduce proving (3.9) to demonstrating that another limit, namely (3.12), holds. The advantage of doing so is that we can then incorporate the regularity properties of this function described in Lemma 2.9 and Proposition 2.10. These properties alone do much of the heavy lifting in the estimates at the heart of the argument, and our distributional assumptions do the rest.

Let \( \psi \in C_b \text{Lip} (\mathbb{R}) \), and define \( u \) by
\[
u \left( t,x \right) = \mathcal{E} [\psi \left( x + X_t \right)]
\tag{3.10}
\]
for all \( (t,x) \in [0,\infty) \times \mathbb{R} \). We know from Section 2 that \( u \) is the unique viscosity solution of
\[
\partial_t u \left( t,x \right) - \sup_{k_\pm \in \mathbb{R}_\pm} \left\{ \int_{\mathbb{R}} \left[ u \left( t,x + z \right) - u \left( t,x \right) - \partial_x u \left( t,x \right) z \right] F_{k_\pm} \left( dz \right) \right\} = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R}
\]
\[
u \left( 0,x \right) = \psi \left( x \right).
\]
It will be more convenient for our purposes to work with the backward equation. Since we will soon rely on the interior regularity results of Proposition 2.10 we also let
0 < h < 1 and define \( v \) by

\[
v(t, x) = u(1 + h - t, x) \tag{3.11}
\]

for \((t, x) \in (-h, 1 + h) \times \mathbb{R}\). Then \( v \) will be the unique viscosity solution of

\[
\partial_t v(t, x) + \sup_{k \in K} \left\{ \int_{\mathbb{R}} [v(t, x + z) - v(t, x) - \partial_x v(t, x) z] F_k \, dz \right\} = 0, \quad (t, x) \in (-h, 1 + h) \times \mathbb{R}
\]

\[
v(1 + h, x) = \psi(x).
\]

Observe that \( v \) inherits key regularity properties from \( u \). At the moment, it is enough to note that for any \((t, x) \in (-h, 1 + h) \times \mathbb{R}\), \( v(\cdot, x) \) is (uniformly) 1/2-Hölder continuous with some constant \( K_1 \) and \( v(t, \cdot) \) is Lipschitz continuous with constant \( \text{Lip}(\psi) \) by Lemma 2.9. Since \( 0 < h < 1 \), we can further specify that \( K_1 \) be independent of \( h \). It follows by (3.10) and (3.11) that

\[
\limsup_{n \to \infty} |\mathcal{E}' [\psi(B_n S_n)] - \mathcal{E} [\psi(X_1)]| \\
\leq \limsup_{n \to \infty} (|\mathcal{E}' [\psi(B_n S_n)] - \mathcal{E}' [v(1, B_n S_n)]| + |\mathcal{E}' [v(1, B_n S_n)] - v(0, 0)| + |v(0, 0) - \mathcal{E} [\psi(X_1)]|)
\]

\[
= \limsup_{n \to \infty} (|\mathcal{E}' [v(1 + h, B_n S_n)] - \mathcal{E}' [v(1, B_n S_n)]| + |\mathcal{E}' [v(1, B_n S_n)] - v(0, 0)| + |v(0, 0) - v(h, 0)|)
\]

\[
\leq \limsup_{n \to \infty} \left( \mathcal{E}' \left[ K_1 \sqrt{h} \right] + |\mathcal{E}' [v(1, B_n S_n)] - v(0, 0)| \right) + K_1 \sqrt{h}
\]

\[
= 2K_1 \sqrt{h} + \limsup_{n \to \infty} |\mathcal{E}' [v(1, B_n S_n)] - v(0, 0)|.
\]

As \( h \) is arbitrary, it is sufficient to show that

\[
\lim_{n \to \infty} \mathcal{E}' [v(1, B_n S_n)] = v(0, 0). \tag{3.12}
\]

The required estimates are intricate, so we will give them in Lemma 3.2 below.

**Lemma 3.2.** In the setup of Theorem 3.7,

\[
\lim_{n \to \infty} \mathcal{E}' [v(1, B_n S_n)] = v(0, 0).
\]

**Proof of Lemma 3.2.** For all \( n \geq 3 \),

\[
v(1, B_n S_n) - v(0, 0)
\]

\[
= v(1, B_n S_n) - v \left( \frac{n - 1}{n}, B_n S_n \right) + \sum_{i=2}^{n-1} \left[ v \left( \frac{i}{n}, B_n S_{i+1} \right) - v \left( \frac{i - 1}{n}, B_n S_i \right) \right]
\]

\[
+ v \left( \frac{1}{n}, B_n S_2 \right) - v(0, 0). \tag{3.13}
\]

Our analysis now becomes delicate. We would like to show that when we apply \( \mathcal{E}' \) to this quantity and let \( n \to \infty \), the result goes to zero. Since the number of terms in this decomposition is also growing with \( n \), we must prove that our \( v \)-increments are decaying quite rapidly. The properties of \( v \) arising from Lemma 2.9 are only enough to manage the first and last terms. By the 1/2-Hölder continuity of \( v(\cdot, x) \),

\[
\mathcal{E}' \left[ v(1, B_n S_n) - v \left( \frac{n - 1}{n}, B_n S_n \right) \right] \leq \mathcal{E}' \left[ K_1 \sqrt{\frac{1}{n}} \right] = K_1 \sqrt{\frac{1}{n}}. \tag{3.14}
\]
If we also use the Lipschitz continuity of \( v(t, \cdot) \) and the independence of \( Y_2 \) from \( Y_1 \), we get

\[
\mathcal{E}' \left[ \left| v \left( \frac{1}{n}, B_n S_2 \right) - v \left( 0, 0 \right) \right| \right] \\
\leq \mathcal{E}' \left[ \left| v \left( \frac{1}{n}, B_n S_2 \right) - v \left( 0, B_n S_2 \right) \right| \right] + \mathcal{E}' \left[ \left| v \left( 0, B_n S_2 \right) - v \left( 0, 0 \right) \right| \right] \\
\leq \mathcal{E}' \left[ K_1 \sqrt{\frac{1}{n}} \right] + \mathcal{E}' \left[ \text{Lip} \left( \psi \right) B_n |S_2| \right] \\
\leq K_1 \sqrt{\frac{1}{n}} + 2\text{Lip} \left( \psi \right) B_n \mathcal{E}' \left[ |Y_1| \right].
\] (3.15)

We remark that although we only referred to \( C_{b,Lip} \left( \mathbb{R} \right) \) in our definition of independence, our manipulations are still valid by Exercise 3.20 in [19].

Controlling the remaining terms demands the interior regularity estimates in Proposition 2.10. Again, this motivates our requirement that \( K_\pm \subset (\lambda, \Lambda) \) for some \( 0 < \lambda < \Lambda \).

We can find a constant \( K_2 > 0 \) such that \( \partial_t v \) exists on \( (-h, 1 + h/2] \times \mathbb{R} \) and

\[
|\partial_t v (t_0, x) - \partial_t v (t_1, x)| \leq K_2 |t_0 - t_1| \\
|\partial_t v (t, x_0) - \partial_t v (t, x_1)| \leq K_2 |x_0 - x_1| \quad (3.16)
\]

for all \((t_0, x), (t_1, x), (t, x_0), \) and \((t, x_1) \in (-h, 1 + h/2] \times \mathbb{R} \). We then break down the rest of (3.13) a bit further. If \( 2 \leq i \leq n - 1 \),

\[
v \left( \frac{i}{n}, B_n S_{i+1} \right) - v \left( \frac{i-1}{n}, B_n S_i \right) \\
= v \left( \frac{i}{n}, B_n S_{i+1} \right) - v \left( \frac{i-1}{n}, B_n S_{i+1} \right) - \partial_t v \left( \frac{i-1}{n}, B_n S_{i+1} \right) \frac{1}{n} \\
+ \partial_t v \left( \frac{i-1}{n}, B_n S_i \right) \frac{1}{n} + v \left( \frac{i-1}{n}, B_n S_{i+1} \right) - v \left( \frac{i-1}{n}, B_n S_i \right).
\]

Let

\[
C_i^n = v \left( \frac{i}{n}, B_n S_{i+1} \right) - v \left( \frac{i-1}{n}, B_n S_i \right) - \partial_t v \left( \frac{i-1}{n}, B_n S_i \right) \frac{1}{n}
\]

and

\[
D_i^n = \partial_t v \left( \frac{i-1}{n}, B_n S_i \right) \frac{1}{n} + v \left( \frac{i-1}{n}, B_n S_{i+1} \right) - v \left( \frac{i-1}{n}, B_n S_i \right).
\]
We can establish an appropriate bound for the $C^n_i$’s using (3.16):

$$|C^n_i| = \frac{1}{n} \int_0^1 \left[ \partial_{v} \left( \frac{i-1+\beta}{n}, B_nS_{i+1} \right) - \partial_{v} \left( \frac{i-1}{n}, B_nS_{i+1} \right) \right] d\beta$$

$$+ \frac{1}{n} \left[ \partial_{v} \left( \frac{i-1}{n}, B_nS_{i+1} \right) - \partial_{v} \left( \frac{i-1}{n}, B_nS_{i} \right) \right]$$

$$\leq \frac{1}{n} \int_0^1 \left[ \partial_{v} \left( \frac{i-1+\beta}{n}, B_nS_{i+1} \right) - \partial_{v} \left( \frac{i-1}{n}, B_nS_{i+1} \right) \right] d\beta$$

$$+ \frac{1}{n} \left[ \partial_{v} \left( \frac{i-1}{n}, B_nS_{i+1} \right) - \partial_{v} \left( \frac{i-1}{n}, B_nS_{i} \right) \right]$$

$$\leq \frac{1}{n} \int_0^1 K_2 \left\| \frac{1}{n} \right\| d\beta + \frac{1}{n} K_2 B_n |Y_{i+1}|$$

$$\leq \frac{K_2}{n} \left( \frac{1}{n} + B_n |Y_{i+1}| \right).$$

Hence, for $2 \leq i \leq n - 1$,

$$\mathcal{E}' \|[C^n_i]\| \leq \frac{K_2}{n} \left[ \frac{1}{n} + B_n \mathcal{E}' |[Y_1]| \right]$$

(3.17)

since $Y_{i+1}$ and $Y_1$ are identically distributed.

While we need (3.16) to bound the $D^n_i$’s, we will at last make use of (3.1) too. Let $\epsilon > 0$. By (3.1), we can find an $N$ such that $n \geq N$ implies

$$n \mathcal{E}' \left[ v(t, x + B_n Y_1) - v(t, x) - \partial_x v(t, x) (B_n Y_1) \right]$$

$$- \left( \frac{1}{n} \right) \sup_{k \pm \in K_{\pm}} \left\{ \int_{\mathbb{R}} \left[ v(t, x + z) - v(t, x) - \partial_x v(t, x) z \right] F_{k_{\pm}} (dz) \right\} < \epsilon$$

on $[0, 1] \times \mathbb{R}$. For these $n$,

$$n \left| v \left( \frac{i-2}{n}, B_n x \right) - \mathcal{E}' \left[ v \left( \frac{i-1}{n}, B_n x + B_n Y_1 \right) \right] \right|$$

$$= n \left| v \left( \frac{i-2}{n}, B_n x \right) - \mathcal{E}' \left[ v \left( \frac{i-1}{n}, B_n x + B_n Y_1 \right) \right] + v \left( \frac{i-1}{n}, B_n x \right) - v \left( \frac{i-1}{n}, B_n x \right) + \left( \frac{1}{n} \right) \partial_{v} \left( \frac{i-1}{n}, B_n x \right) \right.$$  

$$+ \left( \frac{1}{n} \right) \sup_{k \pm \in K_{\pm}} \left\{ \int_{\mathbb{R}} \left[ v \left( \frac{i-1}{n}, B_n x + z \right) - v \left( \frac{i-1}{n}, B_n x \right) - \partial_x v \left( \frac{i-1}{n}, B_n x \right) \right] F_{k_{\pm}} (dz) \right\} \right.$$  

$$\leq \left| - v \left( \frac{i-2}{n}, B_n x \right) - v \left( \frac{i-1}{n}, B_n x \right) + \partial_{v} \left( \frac{i-1}{n}, B_n x \right) \right.$$  

$$+ n \mathcal{E}' \left[ v \left( \frac{i-1}{n}, B_n x + B_n Y_1 \right) - v \left( \frac{i-1}{n}, B_n x \right) - \partial_x v \left( \frac{i-1}{n}, B_n x \right) (B_n Y_1) \right]$$

$$- \left( \frac{1}{n} \right) \sup_{k \pm \in K_{\pm}} \left\{ \int_{\mathbb{R}} \left[ v \left( \frac{i-1}{n}, B_n x + z \right) - v \left( \frac{i-1}{n}, B_n x \right) - \partial_x v \left( \frac{i-1}{n}, B_n x \right) \right] F_{k_{\pm}} (dz) \right\} \right|$$

$$< K_2 \left( \frac{1}{n} \right) + \epsilon$$
by the mean value theorem, (3.10), and hypothesis (i). We can therefore assume that
\( n \geq N \) means

\[
\epsilon \left| n \mathbb{E} \left[ v \left( \frac{i-1}{n}, B_n x + B_n Y_1 \right) \right] - v \left( \frac{i-1}{n}, B_n x \right) \right| < \epsilon
\]

for all \( 2 \leq i \leq n - 1 \) and \( x \in \mathbb{R} \). Then the same techniques give

\[
\left| \partial_x v \left( \frac{i-1}{n}, B_n x \right) \frac{1}{n} + \mathbb{E} \left[ v \left( \frac{i-1}{n}, B_n x + B_n Y_{i+1} \right) \right] - v \left( \frac{i-1}{n}, B_n x \right) \right| \\
\leq \frac{1}{n} \left| \partial_x v \left( \frac{i-1}{n}, B_n x \right) + \frac{v \left( \frac{i-2}{n}, B_n x \right) - v \left( \frac{i-1}{n}, B_n x \right)}{1/n} \right| + \frac{1}{n} \left| \mathbb{E} \left[ v \left( \frac{i-1}{n}, B_n x + B_n Y_1 \right) \right] - v \left( \frac{i-2}{n}, B_n x \right) \right| \\
\leq \frac{K_2}{n^2} + \frac{\epsilon}{n}
\]

(3.18)
By repeatedly applying (3.18) and the independence of $Y_{i+1}$ from $(Y_1, \ldots, Y_i)$, it follows that for $n \geq N$,

$$
\mathcal{E}' \left[ \sum_{i=2}^{n-1} D_i^n \right] \\
= \mathcal{E}' \left[ \mathcal{E}' \left[ \cdots \mathcal{E}' \left[ \sum_{i=2}^{n-2} \left( \partial_t v \left( \frac{i-1}{n}, B_n \bar{x}_i \right) \frac{1}{n} + v \left( \frac{i-1}{n}, B_n \bar{x}_{i+1} \right) - v \left( \frac{i-1}{n}, B_n \bar{x}_i \right) \right) \\
+ \left( \partial_t v \left( \frac{n-2}{n}, B_n \bar{x}_{n-1} \right) \frac{1}{n} + v \left( \frac{n-2}{n}, B_n \bar{x}_{n-1} + B_n Y_n \right) \\
- v \left( \frac{n-2}{n}, B_n \bar{x}_{n-1} \right) \right]_{\bar{x}_{n-1}=S_{n-1}} \cdots \right]_{\bar{x}_2=S_2} \right] \\
= \mathcal{E}' \left[ \mathcal{E}' \left[ \cdots \mathcal{E}' \left[ \sum_{i=2}^{n-2} \left( \partial_t v \left( \frac{i-1}{n}, B_n \bar{x}_i \right) \frac{1}{n} + v \left( \frac{i-1}{n}, B_n \bar{x}_{i+1} \right) - v \left( \frac{i-1}{n}, B_n \bar{x}_i \right) \right) \\
+ \left( \partial_t v \left( \frac{n-2}{n}, B_n \bar{x}_{n-1} \right) \frac{1}{n} + \mathcal{E}' \left[ v \left( \frac{n-2}{n}, B_n \bar{x}_{n-1} + B_n Y_n \right) \\
- v \left( \frac{n-2}{n}, B_n \bar{x}_{n-1} \right) \right]_{\bar{x}_{n-1}=S_{n-1}} \cdots \right]_{\bar{x}_2=S_2} \right] \right] \\
< \mathcal{E}' \left[ \mathcal{E}' \left[ \cdots \mathcal{E}' \left[ \sum_{i=2}^{n-2} \left( \partial_t v \left( \frac{i-1}{n}, B_n \bar{x}_i \right) \frac{1}{n} + v \left( \frac{i-1}{n}, B_n \bar{x}_{i+1} \right) - v \left( \frac{i-1}{n}, B_n \bar{x}_i \right) \right) \\
+ \left( \partial_t v \left( \frac{n-2}{n}, B_n \bar{x}_{n-1} \right) \frac{1}{n} + v \left( \frac{n-2}{n}, B_n \bar{x}_{n-1} + B_n Y_n \right) \\
- v \left( \frac{n-2}{n}, B_n \bar{x}_{n-1} \right) \right]_{\bar{x}_{n-1}=S_{n-1}} \cdots \right]_{\bar{x}_2=S_2} \right] \right] + \frac{K_2}{n^2} + \epsilon \frac{\epsilon}{n} \\
= \mathcal{E}' \left[ \mathcal{E}' \left[ \cdots \mathcal{E}' \left[ \sum_{i=2}^{n-3} \left( \partial_t v \left( \frac{i-1}{n}, B_n \bar{x}_i \right) \frac{1}{n} + v \left( \frac{i-1}{n}, B_n \bar{x}_{i+1} \right) - v \left( \frac{i-1}{n}, B_n \bar{x}_i \right) \right) \\
+ \left( \partial_t v \left( \frac{n-3}{n}, B_n \bar{x}_{n-2} \right) \frac{1}{n} + v \left( \frac{n-3}{n}, B_n \bar{x}_{n-2} + B_n Y_{n-1} \right) \\
- v \left( \frac{n-3}{n}, B_n \bar{x}_{n-2} \right) \right]_{\bar{x}_{n-2}=S_{n-2}} \cdots \right]_{\bar{x}_2=S_2} \right] + \frac{K_2}{n^2} + \epsilon \frac{\epsilon}{n} \\
\vdots \\\n< (n-2) \left( \frac{K_2}{n^2} + \frac{\epsilon}{n} \right) \\
< \frac{K_2}{n} + \epsilon.
$$

Similarly,

$$
\mathcal{E}' \left[ \sum_{i=2}^{n-1} D_i^n \right] > - \frac{K_2}{n} - \epsilon.
$$

(3.20)
We only need to combine our bounds above and invoke hypothesis (ii) to finish the proof. By (3.13), (3.15), (3.17), (3.19), and (3.20),

\[ E'[v(1, B_n S_n)] - v(0, 0) \]
\[
= E'[v(1, B_n S_n)] - v\left(\frac{n-1}{n}, B_n S_n\right) + \sum_{i=2}^{n} C_i^n + \sum_{i=2}^{n-1} D_i^n + v\left(\frac{1}{n}, B_n S_2\right) - v(0, 0) \]
\[
\leq E'[v(1, B_n S_n)] - v\left(\frac{n-1}{n}, B_n S_n\right) + \sum_{i=2}^{n} E'[\|C_i^n\|] + E'[\sum_{i=2}^{n-1} D_i^n] + E'\left[v\left(\frac{1}{n}, B_n S_2\right) - v(0, 0)\right] \]
\[
< \left(K_1 \sqrt{\frac{1}{n}}\right) + \left(K_2 \left(\frac{1}{n} + B_n E'[\|Y_1\|]\right)\right) + \left(\frac{K_2}{n} + \epsilon\right) + \left(K_1 \sqrt{\frac{1}{n}} + 2\text{Lip}(\psi) B_n E'[\|Y_1\|]\right) \]

and similarly

\[ E'[v(1, B_n S_n)] - v(0, 0) \]
\[
> - \left(K_1 \sqrt{\frac{1}{n}}\right) - \left(K_2 \left(\frac{1}{n} + B_n E'[\|Y_1\|]\right)\right) - \left(\frac{K_2}{n} + \epsilon\right) - \left(K_1 \sqrt{\frac{1}{n}} + 2\text{Lip}(\psi) B_n E'[\|Y_1\|]\right) \]

for \( n \geq N \). Since \( \epsilon > 0 \) is arbitrary and \( \lim_{n \to \infty} B_n = 0 \), we have \( \lim_{n \to \infty} E'[v(1, B_n S_n)] = v(0, 0) \).

4. Examples

Example 4.1. Of course, \( X_1 \) is in its own domain of normal attraction. Due to its illustrative value, we demonstrate this using Theorem 3.11, however, note that this result could be obtained more directly using the \( \alpha \)-stability of \( X_1 \), a property which we require in any case.

Let \( \psi \in C_{b, \text{Lip}}(\mathbb{R}) \) and \( u \) be defined by

\[ u(t, x) = E\left[\psi(x + X_t)\right] \]

on \([0, \infty) \times \mathbb{R}\). If \( \tilde{X}_1 \) is an independent copy of \( X_1 \), then for any \( a, b \geq 0 \),

\[ E\left[\psi\left(aX_1 + b\tilde{X}_1\right)\right] = E\left[E\left[\psi\left(ax + (b^\alpha)^{\frac{1}{\alpha}} \tilde{X}_1\right)\right]_{x=X_1}\right] \]
\[
= E\left[u\left(b^\alpha, aX_1\right)\right] \]
\[
= u\left(a^\alpha + b^\alpha, 0\right) \]
\[
= E\left[\psi\left((a^\alpha + b^\alpha)^{\frac{1}{\alpha}} X_1\right)\right] \]

by Lemmas 2.6 and 2.7. Since \( \psi \in C_{b, \text{Lip}}(\mathbb{R}) \) is arbitrary, Exercise 3.20 in [19] implies that the same relation holds for a broader class of maps. In particular,

\[ 2^{\frac{1}{\alpha}} E\left[X_1\right] = E\left[E\left[x + \tilde{X}_1\right]_{x=X_1}\right] \]
\[
= E\left[X_1 + E\left[X_1\right]\right] \]
\[
= 2E\left[X_1\right], \]

so

\[ E\left[X_1\right] = 0. \]

It follows similarly that

\[ E\left[-X_1\right] = 0. \]
We know $\mathcal{E} \left[ |X_1| \right] < \infty$ from Lemma 2.8.

It remains to verify the final hypothesis. Let $0 < h < 1$ and $v$ be the unique viscosity solution of

$$\partial_t v(t, x) + \sup_{k_{\pm} \in K_{\pm}} \left\{ \int_{\mathbb{R}} [v(t, x + z) - v(t, x) - \partial_x v(t, x) z] F_{k_{\pm}} (dz) \right\} = 0, \quad (t, x) \in (-h, 1 + h) \times \mathbb{R}$$

\[ v(1 + h, x) = \psi(x). \]

Then for all $(t, x) \in [0, 1] \times \mathbb{R}$,

$$n \left| \mathcal{E} \left[ v \left( t, x + \frac{1}{n^{1/\alpha}} X_1 \right) - v(t, x) - \partial_x v(t, x) \left( \frac{1}{n^{1/\alpha}} X_1 \right) \right] \right|$$

$$- \left( \frac{1}{n} \right) \sup_{k_{\pm} \in K_{\pm}} \left\{ \int_{\mathbb{R}} [v(t, x + z) - v(t, x) - \partial_x v(t, x) z] F_{k_{\pm}} (dz) \right\}$$

$$= n \left| v \left( t, x + \frac{1}{n^{1/\alpha}} X_1 \right) - v(t, x) + \left( \frac{1}{n} \right) \partial_t v(t, x) \right|$$

$$= n \left| v \left( t - \frac{1}{n}, x \right) - v(t, x) + \left( \frac{1}{n} \right) \partial_t v(t, x) \right|$$

$$= \left| \frac{v(t - \frac{1}{n}, x) - v(t, x)}{1/n} + \partial_t v(t, x) \right|$$

$$\leq \frac{K_2}{n}$$

by (3.11), (3.16), and Lemma 2.7. Slightly abusing notation, Theorem 3.1 yields

$$\frac{1}{n^{1/\alpha}} S_n \overset{d}{\rightarrow} X_1$$

as $n \to \infty$.

**Example 4.2.** Despite its appearance, recall that this example has the straightforward interpretation explained in Section 1. Let $b, M > 0$ and $f$ be a nonnegative function on $\mathbb{N}$ tending to zero as $n \to \infty$. For each $k_{\pm} \in K_{\pm}$, let $W_{k_{\pm}}$ be a classical random variable such that

(i) $W_{k_{\pm}}$ has mean zero;

(ii) $W_{k_{\pm}}$ has a cdf $F_{W_{k_{\pm}}}$ of the form

$$F_{W_{k_{\pm}}} (z) = \begin{cases} \left[ b^\alpha (k_- / \alpha) + \beta_{1,k_{\pm}} (z) \right] \frac{1}{z} & z < 0 \\ 1 - \left[ b^\alpha (k_+ / \alpha) + \beta_{2,k_{\pm}} (z) \right] \frac{1}{z} & z > 0 \end{cases}$$

(4.1)

for some continuously differentiable functions $\beta_{1,k_{\pm}}$ on $(-\infty, 0]$ and $\beta_{2,k_{\pm}}$ on $[0, \infty)$ with

$$\lim_{z \to -\infty} \beta_{1,k_{\pm}} (z) = \lim_{z \to \infty} \beta_{2,k_{\pm}} (z) = 0;$$
and

\[ \int_{-\infty}^{1} \beta_{1,k\pm}(z) \, dz \quad \text{and} \quad \int_{1}^{\infty} \left( -\beta_{1,k\pm}(z) + z + \alpha \beta_{1,k\pm}(z) \right) \, dz, \]

\[ \int_{1}^{\infty} \left( 1 - \beta_{2,k\pm}(z) \right) \, dz \quad \text{and} \quad \int_{0}^{1} \left( 1 + \beta_{2,k\pm}(z) \right) \, dz; \]

and

(iii) the following quantities are all less than \( M \):

\[ \left| \int_{-\infty}^{1} \beta_{1,k\pm}(z) \, dz \right|, \quad \left| \int_{1}^{\infty} \left( -\beta_{1,k\pm}(z) + z + \alpha \beta_{1,k\pm}(z) \right) \, dz \right|, \quad \int_{1}^{\infty} \left( 1 - \beta_{2,k\pm}(z) \right) \, dz \quad \text{and} \quad \int_{0}^{1} \left( 1 + \beta_{2,k\pm}(z) \right) \, dz, \]

(iv) the following quantities are less than \( f(n) \) for all \( n \):

\[ \left| \beta_{2,k\pm}(bn^{1/\alpha}) \right| \quad \text{and} \quad \int_{1}^{\infty} \left| \beta_{2,k\pm}(bn^{1/\alpha}) \right| \, dz, \]

\[ \left| \beta_{1,k\pm}(bn^{1/\alpha}) \right| \quad \text{and} \quad \int_{0}^{1} \left| \beta_{1,k\pm}(bn^{1/\alpha}) \right| \, dz. \]

Note that by (ii) alone, the terms in (iii) are finite and the terms in (iv) approach zero as \( n \to \infty \). In other words, the content of (iii) and (iv) is that uniform bounds and minimum rates of convergence exist.

Define an operator \( E' \) on a space \( \mathcal{H} \) of suitable real-valued functions of a single real variable by

\[ E'[\varphi] = \sup_{k\pm \in K} \int_{\mathbb{R}} \varphi(z) \, dF_{W_{k\pm}}(z) \]

for all \( \varphi \in \mathcal{H} \). The exact composition of \( \mathcal{H} \) is irrelevant for our purposes here. Clearly, \( (\mathbb{R}, \mathcal{H}, E') \) is a sublinear expectation space.

Let \( Y_{i} \) be the random variable on this space defined by

\[ Y_{i}(x) = x \]

for all \( x \in \mathbb{R} \). We will use Theorem 3.1 to show that

\[ \frac{1}{bn^{1/\alpha}} \sum_{i=1}^{n} Y_{i} \]

converges in distribution to \( X_{1} \) as \( n \to \infty \). Most of the difficulties have already been addressed during our discussion of the classical case in Section 3.

Since each \( W_{k\pm} \) has mean zero,

\[ E'[Y_{1}] = \sup_{k\pm \in K} \int_{\mathbb{R}} z \, dF_{W_{k\pm}}(z) = 0 \]

and

\[ E'[-Y_{1}] = \sup_{k\pm \in K} \int_{\mathbb{R}} -z \, dF_{W_{k\pm}}(z) = 0. \]

After recalling that \( K_{\pm} \subset (\lambda, \Lambda) \), (iii) gives

\[ E'[|Y_{1}|] < \infty \]

using (3.3) and (4.1). Observe that we are solving (4.1) for the obvious expressions to obtain uniform bounds on the terms

\[ \left| \beta_{2,k\pm}(1) \right|, \quad \left| \beta_{1,k\pm}(-1) \right|, \quad \left| \int_{0}^{1} \frac{b^{\alpha} k_{\pm} + \alpha \beta_{2,k\pm}(z)}{z^\alpha} \, dz \right| \]

and

\[ \left| \int_{-1}^{0} \frac{b^{\alpha} k_{\pm} + \alpha \beta_{1,k\pm}(z)}{(-z)^{\alpha}} \, dz \right|. \]
To check the remaining hypothesis, let \( 0 < h < 1, \psi \in C_{b,Lip}(\mathbb{R}) \), and \( v \) be the unique viscosity solution of
\[
\partial_t v(t, x) + \int_{\mathbb{R}} \left[ v(t, x + z) - v(t, x) - \partial_x v(t, x) z \right] F_{k_\pm}(dz) = 0, \quad (t, x) \in (-h, 1 + h) \times \mathbb{R}
\]
\[
v(1 + h, x) = \psi(x).
\]
The techniques of (3.3) demonstrate that
\[
\left| n \mathcal{E} \left[ v \left( t, x + \frac{1}{b_{n1/\alpha}} Y_1 \right) - v(t, x) - \partial_x v(t, x) \left( \frac{1}{b_{n1/\alpha}} Y_1 \right) \right] \right|
\]
\[
- \left( \frac{1}{n} \right) \sup_{k_\pm \in K_\pm} \left\{ \int_{\mathbb{R}} \left[ v(t, x + z) - v(t, x) - \partial_x v(t, x) z \right] F_{k_\pm}(dz) \right\}
\]
\[
\leq \left( \frac{1}{b^\alpha} \right) \sup_{k_\pm \in K_\pm} \left\{ \int_{\mathbb{R}} \left[ v(t, x + z) - v(t, x) - \partial_x v(t, x) z \right] \right.
\]
\[
\beta_{1,k_\pm}(bn_{1/\alpha} z) \left[ bn_{1/\alpha} z \right] + \alpha \beta_{1,k_\pm}(bn_{1/\alpha} z) \left\{ \left\langle 1_{(-\infty, 0)} (z) + \right. \right.
\]
\[
\left. \left. - \beta_{2,k_\pm}(bn_{1/\alpha} z) \left[ bn_{1/\alpha} z \right] + \alpha \beta_{2,k_\pm}(bn_{1/\alpha} z) \left\{ \left\langle 1_{(0, \infty)} (z) \right. \right. \right.
\]
\[
\left. \left. \right] \int_{\mathbb{R}} \left\{ \left[ v(t, x + z) - v(t, x) - \partial_x v(t, x) z \right] \right. \right.
\]
\[
\left. \right\} dz \right. \right. \right.
\]
for \((t, x) \in [0, 1] \times \mathbb{R}\) and \( n \geq 1 \). Combining (3.5), (3.6), (3.7), and (3.8) with (iii) and (iv) proves that this last expression approaches zero in the required way.

**APPENDIX A. PROOF OF PROPOSITION 2.10**

Before we commence the argument, we must gather a few definitions and results from [4] and [22].

**Definition A.1.** Define the **parabolic cylinders** \( Q_r \) and \( Q_r(t_0, x_0) \) for \( r > 0 \) and \( (t_0, x_0) \in \mathbb{R}^2 \) by
\[
Q_r = (-r^\alpha, 0) \times (-r, r)
\]
and
\[
Q_r(t_0, x_0) = (t_0, x_0) + Q_r = (t_0 - r^\alpha, t_0) \times (-r + x_0, r + x_0).
\]

**Definition A.2.** Let \( K \) denote the class of kernels \( K \) such that
\[
\frac{\lambda}{|z|^{1+\alpha}} \leq K(z) \leq \frac{A}{|z|^{1+\alpha}}
\]
for \( z \neq 0 \). The **extremal operators** \( M^+ \) and \( M^- \) are defined by
\[
M^+ \varphi(x) = \sup_{K \in K} \left\{ \int_{\mathbb{R}} [\varphi(x + z) - \varphi(x) - \partial_x \varphi(x) z] K(z) \, dz \right\}
\]
and
\[
M^- \varphi(x) = \inf_{K \in K} \left\{ \int_{\mathbb{R}} [\varphi(x + z) - \varphi(x) - \partial_x \varphi(x) z] K(z) \, dz \right\}
\]
for all bounded \( \varphi : \mathbb{R} \rightarrow \mathbb{R} \) which are \( C^{1,1} \) at \( x \).

Parabolic cylinders and extremal operators such as \( M^\pm \) figure prominently in the PIDE literature. We require these concepts due to Proposition A.5. It will also be helpful to have two distinct norms for functions \( w \) on a domain \( D \).
Notation A.3. We define \( \| w \|_{C^{n, \gamma} (D)} \) by

\[
\| w \|_{C^{n, \gamma} (D)} = \sum_{|\beta| \leq n} \sup_{x \in D} |\partial^\beta w (x)| + \sum_{|\beta| = n, x_1 \neq x_2 \in D} \frac{|\partial^\beta w (x_1) - \partial^\beta w (x_2)|}{|x_1 - x_2|^\gamma},
\]

i.e., \( \| \cdot \|_{C^{n, \gamma} (D)} \) is the usual Hölder norm. We drop the superscript \( n \) if \( n = 0 \) and there is no ambiguity. For functions \( w \) of \( t \) and \( x \), we define \( \| \cdot \|_{C^{\gamma} (D)} \) by

\[
\| w \|_{C^{\gamma} (D)} = \| w \|_{L^\infty (D)} + \sup_{(t_1, x_1) \neq (t_2, x_2) \in D} \left\{ \frac{|w (t_1, x_1) - w (t_2, x_2)|}{(|t_1 - t_2|^{1/\alpha} + |x_1 - x_2|)^\gamma} \right\}.
\]

The result below, which is Lemma 5.6 in [31], is often invoked when using a \( C^\beta \) estimate to produce higher regularity estimates. It is indispensable to our development for the same reason.

**Lemma A.4.** Let \( 0 < \beta_1 < 1, 0 < \beta_2 \leq 1, \) and \( L > 0. \) Let \( w \in L^\infty \left( [-1, 1] \right) \) satisfy

\[
\| w \|_{L^\infty \left( [-1, 1] \right)} \leq L.
\]

For \( 0 < |h_1| \leq 1 \), define \( w_{\beta_1, h_1} \) by

\[
w_{\beta_1, h_1} (x) = \frac{w (x + h_1) - w (x)}{|h_1|^\beta_1}
\]

for all \( x \in I_{h_1} \), where \( I_{h_1} = [-1, 1 - h_1] \) if \( h_1 > 0 \) and \( I_{h_1} = [-1 - h_1, 1] \) if \( h_1 < 0. \) Suppose that

\[
w_{\beta_1, h_1} \in C^{\beta_2} (I_{h_1})
\]

and

\[
\| w_{\beta_1, h_1} \|_{C^{\beta_2} (I_{h_1})} \leq L
\]

for any \( 0 < |h_1| \leq 1. \) If \( \beta_1 + \beta_2 < 1 \), then

\[
w \in C^{\beta_1 + \beta_2} \left( [-1, 1] \right)
\]

and

\[
\| w \|_{C^{\beta_1 + \beta_2} \left( [-1, 1] \right)} \leq CL.
\]

If \( \beta_1 + \beta_2 > 1 \), then

\[
w \in C^{0, 1} \left( [-1, 1] \right)
\]

and

\[
\| w \|_{C^{0, 1} \left( [-1, 1] \right)} \leq CL.
\]

In either case, \( C \) depends only on \( \beta_1 + \beta_2. \)

Proposition A.5 is the result which gives life to our demonstration. It is a simplified version of Theorem 6.1 in [22].

**Proposition A.5.** Let \( A \in \mathbb{R} \) and \( w : [-1, 0] \times \mathbb{R} \rightarrow \mathbb{R} \) be a continuous bounded function which satisfies

\[
\partial_t w + \Lambda |\partial_x w| - M^- w \geq -A
\]

and

\[
\partial_t w - \Lambda |\partial_x w| - M^+ w \leq A
\]

in the viscosity sense in \( Q_1. \) Then \( w \in C^{\gamma} \left( Q_{1/2} \right) \) and

\[
\| w \|_{C^{\gamma} \left( Q_{1/2} \right)} \leq C \left( \| w \|_{L^\infty \left( [-1, 0] \times \mathbb{R} \right)} + A \right)
\]

for some \( \gamma \in (0, 1) \) and \( C > 0 \) independent of \( w. \)
We now have at our disposal all of the requisite tools and can present the proof of Proposition 2.10.

**Proof of Proposition 2.10.** The heart of the argument is Proposition A.5. The rest follows from this via well-established routines in PIDE theory. Specifically, having obtained an initial $C^\alpha$ estimate from Proposition A.5, one applies the estimate again to all incremental quotients. A $C^{2\alpha}$ estimate is then achieved by employing Lemma A.4 after appropriately translating and rescaling. The program is iterated until we conclude that $\partial_t u_{\gamma h_{1}}$ and $\partial_t u$ exist and are Hölder continuous. We rerun this procedure as needed starting from the partial derivatives $\partial_x u$ and $\partial_t u$, then from $\partial^2_{xx} u$ and $\partial^2_{tt} u$, etc. None of these maneuvers represent difficulties as (2.1) has constant coefficients.

For all $k_{\pm} \in K_{\pm}$,

$$\frac{\lambda}{|z|^{1+\alpha}} \leq \frac{k_{-} 1_{(-\infty,0)}(z) + k_{+} 1_{(0,\infty)}(z)}{|z|^{1+\alpha}} \leq \frac{\Lambda}{|z|^{1+\alpha}}$$

for any $z \neq 0$. Hence,

$$M^{-} \varphi(x) \leq \sup_{k_{\pm} \in K_{\pm}} \left\{ \int_{\mathbb{R}} \left[ \varphi(x+z) - \varphi(x) - \partial_x \varphi(x) z \right] F_{k_{\pm}}(dz) \right\} \leq M^{+} \varphi(x)$$

for all bounded $\varphi : \mathbb{R} \to \mathbb{R}$ which are $C^{1,1}$ at $x$. Then $u$ satisfies

$$\partial_t u + \Lambda \cdot |\partial_x u| - M^{-} u \geq 0$$

and

$$\partial_t u - \Lambda \cdot |\partial_x u| - M^{+} u \leq 0$$

in the viscosity sense on $(0, \infty) \times \mathbb{R}$ by (2.1). After the necessary translations, Lemma 2.9 and Proposition A.5 imply that for some $C_1 > 0$,

$$u \in \hat{C}^\gamma(Q_{1/2}(t_0, x_0))$$

and

$$\|u\|_{\hat{C}^\gamma(Q_{1/2}(t_0, x_0))} \leq C_1 \|\psi\|_{L^\infty(\mathbb{R})} \quad (A.1)$$

for all $(t_0, x_0)$ with $t_0 \geq 1$.

For any $\beta > 0$ and $0 < |h_1| \leq 1$, define $u_{\beta h_1}$ and $\hat{u}_{\beta h_1}$ by

$$u_{\beta h_1}(t, x) = \frac{u(t + h_1, x) - u(t, x)}{|h_1|^\beta}$$

for all $(t, x) \in (-h_1, \infty) \times \mathbb{R}$ and

$$\hat{u}_{\beta h_1}(t, x) = \frac{u(t, x + h_1) - u(t, x)}{|h_1|^\beta}$$

for all $(t, x) \in (0, \infty) \times \mathbb{R}$. If $0 < |h_1| < 1/2$, then

$$\|\hat{u}_{\gamma h_1}\|_{L^\infty((1-(1/2)^\alpha, \infty) \times \mathbb{R})} \leq C_1 \|\psi\|_{L^\infty(\mathbb{R})}$$

by (A.1). In particular,

$$\|\hat{u}_{\gamma h_1}\|_{L^\infty((1-(1/2)^\alpha - \epsilon, \infty) \times \mathbb{R})} \leq C_1 \|\psi\|_{L^\infty(\mathbb{R})},$$

where $\epsilon > 0$ is arbitrary.

Observe that

$$\partial_t \hat{u}_{\gamma h_1} + \Lambda \cdot |\partial_x \hat{u}_{\gamma h_1}| - M^{-} \hat{u}_{\gamma h_1} \geq 0$$

and

$$\partial_t \hat{u}_{\gamma h_1} - \Lambda \cdot |\partial_x \hat{u}_{\gamma h_1}| - M^{+} \hat{u}_{\gamma h_1} \leq 0$$
in the viscosity sense on \((0, \infty) \times \mathbb{R}\). Another round of translations together with Lemma 2.9 and Proposition A.5 give
\[
\hat{u}_{\gamma, h_1} \in \hat{C}^\gamma (Q_{1/2} (t_0, x_0))
\]
and
\[
\|\hat{u}_{\gamma, h_1}\|_{C^\gamma (Q_{1/2} (t_0, x_0))} \leq C^2 \|\psi\|_{L^\infty (\mathbb{R})} \tag{A.2}
\]
for any \((t_0, x_0)\) with \(t_0 \geq 2 - (1/2)^\alpha - \epsilon\). This implies that for any \(t_0 > 2 - (1/2)^\alpha - \epsilon\), \(x_0 \in \mathbb{R}\), and \(0 < |h_1| < 1/2\),
\[
\hat{u}_{\gamma, h_1} (t_0, \cdot) \in C^\gamma (B_{1/2} (x_0))
\]
and
\[
\|\hat{u}_{\gamma, h_1} (t_0, \cdot)\|_{C^\gamma (B_{1/2} (x_0))} \leq C^2 \|\psi\|_{L^\infty (\mathbb{R})}.
\]
Applying a rescaled variant of Lemma A.4, we see that
\[
u (t_0, \cdot) \in C^{2\gamma} (\tilde{B}_{r_1} (x_0))
\]
and
\[
\|\nu (t_0, \cdot)\|_{C^{2\gamma} (\tilde{B}_{r_1} (x_0))} \leq C (r_1) C_1 \|\psi\|_{L^\infty (\mathbb{R})} \tag{A.3}
\]
for all \(t_0 > 2 - (1/2)^\alpha - \epsilon\) and \(x_0 \in \mathbb{R}\). Here, \(0 < r_1 < 1/2\) and \(C (r_1)\) depends only on \(r_1\). Note that (A.3) yields
\[
\|\hat{u}_{2\gamma, h_1}\|_{L^\infty (2 - 2(1/2)^\alpha - \epsilon, \infty) \times \mathbb{R})} \leq C (r_1) C^2 \|\psi\|_{L^\infty (\mathbb{R})}
\]
for \(0 < |h_1| < r_1\).

Driven by Lemma A.4, assume without loss of generality that neither \(1/\gamma\) nor \(\alpha/\gamma\) are integers. We can then repeatedly use these steps to find \(n, C_n\), and \(0 < r_n < r_1\) such that
\[
u (t_0, \cdot) \in C^{0, 1} (\tilde{B}_{r_n} (x_0))
\]
and
\[
\|\nu (t_0, \cdot)\|_{C^{0, 1} (\tilde{B}_{r_n} (x_0))} \leq C_n \|\psi\|_{L^\infty (\mathbb{R})}
\]
for all \(t_0 > (n + 1) - (n + 1) (1/2)^\alpha - ne\) and \(x_0 \in \mathbb{R}\). A covering argument indicates that
\[
u (t_0, \cdot) \in C^{0, 1} (\mathbb{R})
\]
and
\[
\|\nu (t_0, \cdot)\|_{C^{0, 1} (\mathbb{R})} \leq C_n \|\psi\|_{L^\infty (\mathbb{R})} \tag{A.4}
\]
for all \(t_0 > (n + 1) - (n + 1) (1/2)^\alpha - ne\).

Running through our program once more, we get
\[
\hat{u}_{1, h_1} \in \hat{C}^\gamma (Q_{1/2} (t_0, x_0))
\]
and
\[
\|\hat{u}_{1, h_1}\|_{\hat{C}^\gamma (Q_{1/2} (t_0, x_0))} \leq C_n C_1 \|\psi\|_{L^\infty (\mathbb{R})}
\]
for large \(t_0\), small \(h_1\), and any \(x_0 \in \mathbb{R}\). Consequently,
\[
\partial_2 u \in \hat{C}^\gamma (Q_{1/2} (t_0, x_0))
\]
with
\[
\|\partial_2 u\|_{\hat{C}^\gamma (Q_{1/2} (t_0, x_0))} \leq C_n C_1 \|\psi\|_{L^\infty (\mathbb{R})}. \tag{A.5}
\]

Our work thus far is the blueprint for the proof. The incremental quotient \(u_{\beta, h_1}\) can be handled in the same way to get an estimate corresponding to (A.5) for \(\partial_1 u\). One then begins with \(\partial_2 u\) and \(\partial_1 u\) and calls upon similar arguments until estimates are obtained for \(\partial_2^2 u\) and \(\partial_1^2 u\). The iteration continues as long as desired. On the last cycle, we
conclude with analogues of (A.4). While the resulting interior estimate will not be on $(h, \infty) \times \mathbb{R}$, this issue can be addressed by rescaling. □

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