Iitaka-Viehweg Conjectures $C$ and $C^{++}$

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Abstract

Given a fibre space $X/S$ with the generic geometric fibre of Kodaira dimension $\geq 0$, we shall construct a variety $Y$ ramified over $X$ along such a horizontal hyperplane with respect to $X/S$ that Kollár and Kawamata had proved Viehweg conjecture for $Y/S$ with the generic geometric fibre of general type or of the abundant canonical invertible sheaf where Viehweg dimensions of $X/S$ and $Y/S$ are equal, respectively. We shall show that Viehweg dimension of $X/S$ is not greater than that of $Y/S$ by Mochizuki’s Galois theory.

1 Introduction:

To classify algebraic varieties in the category of birational geometry, Iitaka proposed many conjectures after Kodaira-Enriques classification of surfaces. His key birational invariant is Kodaira dimension. One of his main conjectures is the following:

Conjecture 1. Let $X/S$ be a fibre space over the complex number field and $X_\eta$ the generic geometric fibre of $X/S$. Then $\kappa(X) \geq \kappa(X_\eta) + \kappa(S)$.

Remark 1. Mabuchi suggested that the Griffiths infinitesimal variation Hodge theory is applicable to the proof of the conjecture assuming the abundance conjecture. Kawamata independently proved it in the similar idea under the abundance conjecture. Kollár proved the conjecture in the case when the generic geometric fibre is of general type.

Viehweg conjectures the following:

Conjecture 2. Let $f : X \to S$ be a fibre space $X/S$ with the generic geometric fibre of Kodaira dimension $\geq 0$. Then there exists a number $m$ such that

$$\kappa(\det f^*\omega_{X/S}^{\otimes m}) \geq \text{var}(X/S).$$

This conjecture implies

Conjecture 3. $\kappa(\omega_{X/S}) \geq \kappa(\omega_{X_\eta}) + \text{var}(X/S)$

Iitaka conjecture $C$ follows from the conjecture above.
2 Preliminary

Definition 1.

Let $k$ be a field. A geometrically irreducible, reduced, smooth scheme $X$ over $k$ is said to be a non singular variety over $k$.

Let $X$ be a non singular variety of dimension $d$ and $\Omega_X$ the differential sheaf over $X$. $\omega_X$ denotes $\Omega^d_X$.

A connected proper surjective morphism $f : X \to S$ of non singular varieties $X$ and $S$ is said to be a fibre space $X/S$.

Let $f : X \to S$ be a fibre space $X/S$. $\omega_{X/S}$ denotes $\omega_X \otimes f^*\omega_S^{-1}$.

Let $L$ be an invertible sheaf over $X$. $\kappa(L)$ denotes the maximal dimension of the image variety of the rational map $X \to \mathbf{P}(\Gamma(X, L^{\otimes m}))$ defined by $\Gamma(X, L^{\otimes m}) \otimes O_X \to L^{\otimes m}$. We call $\kappa(L)$ Iitaka dimension of $L$.

$\kappa(\omega_X)$ is said to be Kodaira dimension, which is denoted by $\kappa(X)$.

Let $X/S$ be a fibre space. The minimal dimension of $T$ such that there exists a generically finite morphism $S' \to S$ in which $X \times_S S'$ is birationally equivalent to $S' \times_T X_0$ for some varieties $T$, $X_0$ with $X_0/T$ a fibre space. This dimension denotes $\operatorname{var}(X/S)$, which is called Viehweg dimension of a fibre space $X/S$.

The category of bands of profinite groups is defined in the following. The objects are the profinite groups and the arrows are the homomorphisms of profinite groups modulo inner automorphisms.

A $\mathbb{Q}$-divisor $D$ is said to be effective if $\kappa(D) \geq 0$. Similarly, we say that a cycle of codimension 1 is effective if every coefficient is non negative.
Definition 2. Define a functor $\text{Div}_{X/S}$ from the category of $S$-schemes to that of sets by the formula

$$\text{Div}_{X/S}(T) = \{ \text{relative effective divisors } D \text{ on } X_T/T \}.$$ 

Lemma 1. Assume $X/S$ is projective and flat. Then $\text{Div}_{X/S}$ is representable by an open subscheme of the Hilbert scheme $\text{Hilb}_{X/S}$.

Let $k$ be a field of characteristic $0$. Let $X$ be a projective normal variety over $k$ and let $\mathcal{M}_X$ the sheaf of rational functions of $X$, $\mathcal{M}_X^*$ the sheaf of invertible rational functions of $X$, which is a subsheaf of $\mathcal{M}_X$ and $\mathcal{O}_X^* = \mathcal{O}_X \cap \mathcal{M}_X^*$.

Definition 3. $\text{Div}_X = \mathcal{M}_X / \mathcal{O}_X^*$, $\text{Div}(X) = \Gamma(X, \text{Div}_X)$

An invertible $\mathcal{O}_X$-submodule of $\mathcal{M}_X$ is said to be an invertible fractional sheaf, for example, $\mathcal{O}_X(D)$ for a divisor $D$.

An invertible $\mathcal{O}_X$-module is said to be an invertible sheaf. The set of equivalence classes of couples $(L, s)$ denotes $D(X)$. Here $L$ is an invertible sheaf, $s$ is a non $0$ global section of $L$. $(L, s)$ and $(L', s')$ are equivalent if there exists an isomorphism $u : L \to L'$ such that $u(s) = s'$.

An element of $\text{Pic}(X) \otimes \mathbb{Q}$ is said to be a $\mathbb{Q}$-invertible sheaf.

An element of $\text{Div}(X) \otimes \mathbb{Q}$ is said to be a $\mathbb{Q}$-divisor.

Facts 1. The order preserving homomorphism $\text{cyc} : \text{Div}(X) \to Z^1(X)$ which assignes a divisor a cycle of codimension $1$ is injective and the image $\text{cyc}(\text{Div}(X))$ consists of locally principal divisors.

Let $Z^1(X)$ denote the free group generated by the cycles of codimension one on $X$. For every $x \in X$ $\mathcal{O}_{X,x}$ is factorial if and only if $\text{cyc} : \text{Div}(X) \to Z^1(X)$ is bijective.

Let $X^{(1)}$ denote the set of points $x \in X$ such that $\dim \mathcal{O}_{X,x} = 1$. Let $f : X \to S$ be a finite surjective morphism. Let $D' = \sum_{x' \in X^{(1)}} n_{x'} \{ x' \}$ be a codimension $1$ cycle. For $x \in X^{(1)}$, put $n_x = \sum_{x' \in f^{-1}(x)} n_{x'} [k(x') : k(x)]$. $f_*(D') = \sum_{x \in X^{(1)}} n_x \{ x \}$

Suppose that $f$ is flat and that $Z$ is non singular.

Let $D = \sum_{x \in S^{(1)}} n_x \{ x \}$ be a cycle of codimension $1$. Put $\lambda_{x'} = \text{length}(\mathcal{O}_{X,x'}/\mathcal{M}_X \mathcal{O}_{X,x'})$ and $n_{x'} = \lambda_{x'} n_x$. Then $f^*D = \sum_{x' \in X^{(1)}} n_{x'} \{ x' \}$. 

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3 Horizontal Hypersurface

Our main aim is to show the following theorem.

**Theorem 1.** Let $f : X \to S$ be a fibre space of non-singular varieties. Assume that $\kappa(\omega_{X_S}) \geq 0$ for the generic geometric fibre $X_S$. Then there exists an integer $m > 0$ such that $\kappa(\det f_*\omega_{X/S}^\otimes m) \geq \var(X/S)$.

**Viehweg’s Lemma:**

**Lemma 2.** Let $S' \to S$ be a Kummer-Kawamata covering with respect to an ample divisor. Let $X \times_S S' \to S'$ be the pull-back. Then $X \times_S S'$ has rational singularity. Further take a desingularization $X^v \to X \times_S S'$. Then $\kappa(\det f^v_*\omega_{X^v/S'}^\otimes m) \leq \kappa(\det f_*\omega_{X/S}^\otimes m)$.

We shall prove the theorem above in the following several steps. By Viehweg in order to prove the theorem, we can assume further that $\var(X/S) = \dim S$ and show the theorem.

Let $f : X \to S$ be a fibre space. From the extension of the function fields $R(X)/R(S)$, we have purely transcendental indeterminates $t_1, \ldots, t_r$ over $R(S)$ such that $R(X)/R(S)(t_1, \ldots, t_r)$ is a finite extension of degree $d$. Hence we obtain a dominant rational map $X \to S \times \mathbb{P}^r$. Resolving the indeterminacy of the rational map $X \to S \times \mathbb{P}^r$, we have a birational map $X' \to X$ and a morphism $\phi : X' \to S \times \mathbb{P}^r$. We replace $X'$ by $X$ and let $Z$ denote $S \times \mathbb{P}^r$. Let $X''$ be the integral closure of $Z$ in the function field $R(X)$. Namely, $\mu : X \to X''$ with $\nu : X'' \to Z$ is Stein factorization. Let $\mu : X \to X''$ be the structure morphism.
Recall the next lemma.

**Lemma 3.** \( \omega_{X/S} \) is weakly positive with respect to \( f \).

In other words, given any \( \alpha > 0 \) and any big \( \mathbb{Q} \)-invertible sheaf \( L \) over \( S \), it holds that

\[
\kappa(\omega_{X/S}^\alpha \otimes f^*L) \geq \dim S.
\]

We have a rational map \( X \to \mathbb{P}(\Gamma(X, (\omega_{X/S} \otimes f^*L)^{\otimes m})) \) defined by \( \mathcal{O}_X \otimes \Gamma(X, (\omega_{X/S} \otimes f^*L)^{\otimes m}) \to (\omega_{X/S} \otimes f^*L)^{\otimes m} \) for \( m \gg 0 \). Take a resolution of the indeterminacy of the rational map, which is denoted by \( X^* \to X \). Then replace \( X^* \) by \( X \). Since \( X \) is non-singular, there exists an effective \( \mathbb{Q} \)-cartier divisor \( D \) such that \( \omega_{X/S}^\alpha \otimes f^*L = \mathcal{O}_X(D) \) for any \( \alpha > 0 \).

**Lemma 4.** Let \( D \) be an effective \( \mathbb{Q} \)-divisor on \( X \). There exist an effective \( \mathbb{Q} \)-divisor \( E \) and an effective Weil divisor \( D' \) on \( X'' = \mu^*D + E \) such that \( E \) is a \( \mu \)-exceptional divisor, i.e., the \( \mu \) image of the support of \( E \) in \( X'' \) is of codimension \( \leq 2 \).

**Proof.** Since \( \mu : X \to X'' \) is birational, there exists a locus of codimension \( \leq 2 \) outside which the restriction of \( \mu \) is an isomorphism. Hence we have an effective \( \mathbb{Q} \)-divisor decomposition \( D = \mu^*D' + E \) such that \( E \) is a \( \mu \)-exceptional divisor. \( \square \)

**Lemma 5.** There exist \( \mathbb{Q} \)-ample divisors \( C_1 \) and \( C_2 \) such that \( D' = C_1 - C_2 \) in \( Z^{(1)}(X'' \otimes \mathbb{Q}) \) up to \( \mathbb{Q} \)-linear equivalence.

**Proof.** There exists a \( \mathbb{Q} \)-ample divisor \( C_1 \) such that \( C_1 - D' \) is \( \mathbb{Q} \)-ample, say, \( C_2 \). \( \square \)

**Lemma 6.** There exist \( \mathbb{Q} \)-ample divisors \( D_1 \) and \( D_2 \) over \( Z \) such that \( C_1 = \nu^*D_1 \) and \( C_2 = \nu^*D_2 \) up to \( \mathbb{Q} \)-linear equivalence.

**Proof.** We refer to the next lemma.

**Lemma 7** ([EGA] 4-3). Let \( f : X \to Y \) be a proper flat morphism of finite presentation. The set of \( y \in Y \) such that \( X_y \) is smooth over \( k(y) \) is open.
Since \( f : X \rightarrow S \) is a fibre space of non singular varieties, i.e., a projective connected morphism, a general fibre \( X_s \) for a closed point \( s \in S \) is a non singular variety. Let \( \text{Div}_S \) be a scheme representing a functor \( T \rightarrow \text{Div}_{S/k}(T) \). Its components are quasi-projective. Let \( \Gamma \) be the universal relative effective divisor on \( S \times \text{Div}_S/\text{Div}_S \). Note that a fibre of a closed point of \( S \) for \( f^{-1}\Gamma \subset X \times \text{Div}_S \rightarrow S \times \text{Div}_S \rightarrow \text{Div}_S \) is a non singular variety, i.e., smooth and irreducible over \( k(s) \). Let \( \Xi \) be the universal relative effective divisor on \( Z \times \text{Div}_Z \). Since the pullback of the universal relative effective divisor over \( \Gamma \subset S \times \text{Div}_S \) to \( p^{-1}\Gamma \subset Z \times \text{Div}_S \) gives a morphism \( \text{Div}_S \rightarrow \text{Div}_Z \), a general fibre of a closed point \( t \) of \( \text{Div}_Z \) for \( \phi^{-1}\Xi \subset X \times \text{Div}_Z \rightarrow \text{Div}_Z \) is a non singular variety. Hence there exists a dense open set \( U \) of \( \text{Div}_Z \) such that \( \phi^{-1}\Xi/\text{Div}_Z|U \) is smooth and irreducible. Therefore there exists one point in \( X'' \) over the generic point of an effective divisor on \( Z \) corresponding to a closed point \( t \in U \).

A general member of the linear system of a sufficiently ample divisor on \( X'' \) is irreducible and moves freely. Hence the direct image of a general member by \( \nu \) is able to be a general irreducible divisor on \( Z \), which is associated with a closed point \( t \in U \). Hence there exist a \( \mathbb{Q} \)-ample divisors \( D_1 \) and \( D_2 \) over \( Z \) such that \( C_1 = \nu^*D_1 \) and \( C_2 = \nu^*D_2 \) up to \( \mathbb{Q} \)-linear equivalence.

Note that \( \text{Pic}(Z) = \text{Pic}(S) \times \text{Pic}(\mathbb{P}^r) \) and that \( \text{Pic}(\mathbb{P}^r) \cong \mathbb{Z} \). Let \( p : Z \rightarrow S \) and \( q : Z \rightarrow \mathbb{P}^r \). Let \( \phi = \nu \circ \mu : X \rightarrow Z \). Now put them together. We have

1. \( \omega_{X/S} = \mathcal{O}_X(D - f^*L) \)
2. \( D = \mu^*D' + E \)
3. \( D' = C_1 - C_2 = \nu^*(D_1 - D_2) \)
4. \( D_i = p^*A_i + a_i q^*H \), where \( A_i \) is an ample \( \mathbb{Q} \)-divisor on \( S \), \( H \) is a hyperplane section of \( \mathbb{P}^r \), \( a_i \) is a rational number and \( i = 1, 2 \).
5. \( \omega_{X/S} = \mathcal{O}_X(-f^*L + E + \phi^*(p^*A_1 + a_1 q^*H - p^*A_2 - a_2 q^*H)) = \mathcal{O}_X(E + \phi^*p^*A + a\phi^*q^*H) \), where \( A = -L + A_1 - A_2 \), \( a = a_1 - a_2 \).
6. \( \mu_*\omega_{X/S} = \mu_*\mathcal{O}_X(m(E + \phi^*p^*A + a\phi^*q^*H)) = \mu_*\mathcal{O}_X(m(\phi^*p^*A + a\phi^*q^*H)). \)

Note that \( \mu_*=\mathcal{O}_X, \phi = \nu \circ \mu. \)

**Lemma 8.** \( \phi_*\mathcal{O}_X \subset \oplus^d \mathcal{O}_Z \)

**Proof.** We apply the next lemma to a finite morphism \( X''/Z \). We refer to Weierstrass preparation lemma.
Lemma 9. Let a convergent series \( g \in k\{x_1, \ldots, x_n\} \) such that \( g(0, \ldots, x_n) \neq 0 \) and order \( p \). Then \( B = k\{x_1, \ldots, x_n\}/(g) \) is a free module over the ring \( A = k\{x_1, \ldots, x_{n-1}\} \) and has a basis of classes \((1, x_n, \ldots, x_n^{p-1})\) mod \((g)\).

The convergent series rings are strictly henzelian. Hence \( X''/Z \) is Kummer gerbe, i.e., cyclic cover, with respect to the etale topology. We have \( \nu_* \mathcal{O}_{X''} = \bigoplus_{i=0}^{a-d-1} \mathcal{O}_Z(-[\frac{i}{a}D]) \), where \( D \) is an effective divisor with respect to the etale topology. Note, however, that the composition of the morphisms \( X'' \to Z \) and \( Z \to \mathbb{P}^r \) is a connected morphism.

4 Cyclic Cover

Let \( L = \mathcal{O}_X(\phi^*q^*(bH)) \). Here \( b > 0 \) is taken sufficiently large. Choose a non singular irreducible divisor \( D \) such that \( L^\otimes n = \mathcal{O}_X(D) \). Take a cyclic cover \( Y = \text{Spec} \bigoplus_{0 \leq i \leq n-1} \mathcal{O}_{H_i} \) of \( X \), which denotes \( \tau : X \to X \). Let \( D = \text{diva} \). Here \( a \) is a section of \( \mathcal{O}_X(D) \). Let \( Y \) be \( \text{Spec} \mathcal{O}_X[T]/(T^n - a) \). Note that \( Y \) is a non singular variety and \( Y/S \) is a fibre space. By adjunction formula, \( \mathcal{O}_Y(K_Y) = \mathcal{O}_Y(\tau^*K_X \otimes \tau^*L^\otimes(n-1)) \) since \( K_Y + \tau^*L = \tau^*(K_X + D) \). It is well known \( \tau_* \mathcal{O}_Y = \bigoplus_{0 \leq i \leq n-1}(L^{-1})^i \). Hence by projection formula, \( \tau_* \omega_{Y/S} = \tau_* \mathcal{O}_Y \otimes \omega_{Y/S}^\otimes \otimes L^\otimes(n-1) \). Let \( g = f \circ \tau \). We obtain \( g_* \omega_{Y/S}^\otimes \subset \bigoplus_{0 \leq i \leq n-1} f_* (\omega_{X/S}^\otimes \otimes L^\otimes(m(n-1)-i)) \) and \( \det g_* \omega_{Y/S}^\otimes \subset \bigoplus_{0 \leq i \leq n-1} \det f_* (\omega_{X/S}^\otimes \otimes L^\otimes(m(n-1)-i)) \).

Proposition 1. \( g_* \omega_{Y/S}^\otimes = f_* (\tau_* \mathcal{O}_Y \otimes \mathcal{O}_{X/S} \omega_{X/S}^\otimes \otimes \mathcal{O}_S(mA), \text{ where } r_i = \dim \Gamma(\mathbb{P}^r, \mathcal{O}((ma + b(m(n-1) - i))H)). \)

Proof. From the argument above, we have \( g_* \omega_{Y/S}^\otimes \subset \bigoplus_{0 \leq i \leq n-1} f_* (\omega_{X/S}^\otimes \otimes L^\otimes(m(n-1)-i)) \subset \bigoplus_{0 \leq i \leq n-1} \phi_* (\mathcal{O}_X(mE) \otimes \mathcal{O}_Z(ma^*H + mp^*A + b(m(n-1)-i)q^*H)), \) which is an injection into the following sheaf since \( \phi_* \mathcal{O}_X(mE) = \nu_* \mu_* \mathcal{O}_X(mE) = \nu_* \mu_* \mathcal{O}_X = \phi_* \mathcal{O}_X \subset \bigoplus \mathcal{O}_Z \).

We may assume that if \( b \) is taken sufficiently large, the generic geometric fibre of \( Y/S \) is of general type, if necessary, the canonical invertible sheaf over the generic geometric fibre of \( Y/S \) is abundant with \( R^ig_* \omega_{Y/S}^\otimes = 0 \) for \( i > 0 \). The composite map \( \tau \circ \nu \circ \mu : Y \to Z \) is generically finite and the invertible sheaf \( q^*H \) is relatively ample with respect to \( p : Z \to S \).
Lemma 11. Let S be a non-singular variety. Let L be an invertible sheaf and $E'$ and $E$ locally free sheaves of finite rank over $S$. Given the exact sequence $0 \to E' \to E \otimes L$ and $E \cong \mathcal{O}^n$ for some $n > 0$, then $\kappa(L^{\otimes r} \otimes (\det E')^{-1}) \geq 0$, where $r = \text{rank} E'$.

Proof. Take the dual and we have a homomorphism $(E \otimes L)^* \to (E')^*$ and let the image be $F$ and $K$ the kernel. $F$ and $K$ are torsion free and locally free outside a closed subset of codimension $\geq 2$, which we denote $S^0$. $F$ is of the same rank as $E'$. We have the exact sequences $0 \to (E \otimes L)^* \to F \to 0$ and $0 \to K \otimes L \to (E)^* \to F \otimes L \to 0$ over $S^0$. Thus $F \otimes L$ is globally generated and $F \otimes L \to (E')^* \otimes L$ is an isomorphism over $S^0$. Hence $\det(F \otimes L) \subset \det((E')^* \otimes L)$. Note that $\det((E')^* \otimes L) = L^{\otimes r} \otimes (\det E')^{-1}$, where $r = \text{rank} E'$. Therefore $\kappa(L^{\otimes r} \otimes (\det E')^{-1}) \geq 0$. \qed

Proposition 2. $\kappa(\mathcal{O}_S([mA])) \geq \kappa(\det g_*\omega_{Y/S}^{\otimes m})$

Proof. Apply the lemma above to the following formula, $g_*\omega_{Y/S}^{\otimes m} \subset \oplus d \oplus 0 \leq i \leq n-1 \oplus \mathcal{O}_S^{r_i} \otimes \mathcal{O}_S(mA)$, where $r_i = \dim \Gamma(P^i, \mathcal{O}((ma + b(m(n-1)-i))H))$. \qed

Consider the case when $m = 1$. Let $D$ be a classifying space for a variation of Hodge structure and let $\Gamma$ be the monodromy group, which is a subgroup of the arithmetic group of all linear automorphism group of $H^{\dim X_s}(X_s, \mathbb{C})$ which preserve a certain condition. Let $\Phi : S \to \Gamma \setminus D$ be a holomorphic period mapping satisfying the Griffiths transversality relation. A period mapping $\Phi$ gives rise to a variation of Hodge structure by pulling back the universal family over $\Gamma \setminus D$. Since the generic geometric fibre of $Y/S$ is of general type and $\text{var}(Y/S) \geq \dim S$, the period mapping $\Phi$ is a finite to one mapping. Hence we obtain $\kappa(A) = \dim S$.

Kawamata proved the next theorem under the condition that the generic geometric fibre has the abundant canonical invertible sheaf and Kollár proved it when the generic geometric fibre is of general type.

Lemma 12. $\kappa(\det g_*\omega_{Y/S}^{\otimes m}) \geq \text{var}(Y/S)$

Lemma 13. Given the exact sequence $0 \to E' \to E \to E'' \to 0$. If $E$ is weakly positive and if $\det E'$ is big, then $\det E$ is big

Proof. Since the quotient of a weakly positive sheaf is weakly positive, the exact sequence $0 \to E' \to E \to E'' \to 0$, where $E$ is weakly positive and $\det E'$ is big, gives the conclusion that $\det E = \det E' \otimes \det E''$ is big. \qed

Proposition 3. If $\text{var}(Y/S) \geq \text{var}(X/S) = \dim S$, $\max_{m > 0} \kappa(\det f_*\omega_{X/S}^{\otimes m}) \geq \dim S$.  

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Proof. There exists an exact sequence \( 0 \to \mathcal{O}_S([mA]) \to f_*\omega_{X/S}^m \) over \( S \). Take the dual to get the homomorphism \((f_*\omega_{X/S}^m)^* \to \mathcal{O}_S([-mA])\). Let the image denote \( F \) and let the kernel be \( K \). They and \( f_*\omega_{X/S}^m \) are torsion free and hence there exists an open set \( S^o \) such that \( \dim S - \dim(S \setminus S^o) \geq 2 \) and that \( K \), \( F \) and \( f_*\omega_{X/S}^m \) are locally free. Note that \( F \subset \mathcal{O}_S([-mA]) \) and so \( \mathcal{O}_S([mA]) \to F^* \) is a non zero injective map to a torsion free of rank one and that \( \mathcal{O}_S([mA]) \) is big for infinitely many \( m \) since there exist infinitely many \( m \) such that \([mA] = mA = [mA] \).

Hence we have the exact sequence \( 0 \to K \to (f_*\omega_{X/S}^m)^* \to F \to 0 \) of locally free sheaves of finite rank over \( S^o \). Thus we have the exact sequence \( 0 \to F^* \to f_*\omega_{X/S}^m \to K^* \to 0 \) of locally free sheaves of finite rank over \( S^o \). Let \( E' = F^* \), \( E = f_*\omega_{X/S}^m \) and \( E'' = K^* \). Consider sheaves over \( S^o \). Since the quotient of a weakly positive sheaf is weakly positive, the exact sequence \( 0 \to E' \to E \to E'' \to 0 \), where \( E \) is weakly positive and \( \det E' \) is big, gives the conclusion that \( \det E = \det E' \otimes \det E'' \) is big.

Recall \( \mathcal{O}_S([mA]) \subset f_*\omega_{X/S}^m \), we have \( \det f_*\omega_{X/S}^m = \mathcal{O}_S([mA]) \otimes \det G \), where \( G \) is the cokernel of the monomorphism \( \mathcal{O}_S([mA]) \subset f_*\omega_{X/S}^m \). Since \( \det G \) is weakly positive and \( \mathcal{O}_S([mA]) \) is big for infinitely many \( m \). Therefore \( \mathcal{O}_S([mA]) \otimes \det G \) is big for infinitely many \( m \). Therefore \( \max_{m>0} \kappa(\det f_*\omega_{X/S}^m) = \dim S \). \( \square \)

5 Mochizuki’s Galois theory

Let \( k \) be an algebraically closed field of characteristic 0, say, the complex number field. We investigate the birational algebraic geometry from the point of view of the profinite Galois groups thanks to Mochizuki theory. Let \( X \to S \) be a fibre space of smooth algebraic spaces over \( k \). Let \( \text{Spec}k(\eta) \) denote the generic point of the fibre space and \( k(\bar{\eta}) \) the algebraic closure of \( k(\eta) \). The absolute Galois group of \( R(X) \) is defined to be the Galois group with Kull topology of the Galois etension \( R(\bar{X})/R(X) \), which denotes \( \Gamma_X = \text{Gal}(R(\bar{X})/R(X)) \). This is a profinite group.

Theorem 2. [Mch] Let \( p \) be a prime number. Let \( K \) be a subfield of a finitely generated field extension of \( \mathbb{Q}_p \). Let \( X_K \) be a smooth pro-variety over \( K \) and \( Y_K \) a hyperbolic pro-curve over \( K \). Let \( \text{Hom}_{K}^{\text{dom}}(X_K, Y_K) \) be the set of dominant \( K \)-morphisms from \( X_K \) to \( Y_K \) and \( \text{Hom}_{K}^{\text{open}}(\Pi_{X_K}, \Pi_{Y_K}) \) the set of open continuous group homomorphisms \( \Pi_{X_K} \to \Pi_{Y_K} \) over \( \Gamma_K \), modulo up to inner automorphisms arising from \( \Delta_{Y_K} \). Then the natural map

\[
\text{Hom}_{K}^{\text{dom}}(X_K, Y_K) \to \text{Hom}_{K}^{\text{open}}(\Pi_{X_K}, \Pi_{Y_K})
\]

is bijective.
Here we have a natural homomorphism \( \pi_1(X_K) \to \Gamma_K \). Let \( \Delta_{X_K} \) be the maximal pro-p quotient of the geometric fundamental group \( \pi_1(X_K) \). Let \( \Pi_{X_K} = \pi_1(X_K)/\ker(\pi_1(X_K) \to \Delta_{X_K}) \).

**Theorem 3.** \([Mch]\) Let \( p \) be a prime number. Let \( K \) be a subfield of a finitely generated field extension of \( \mathbb{Q}_p \). Let \( L, M \) be function fields of arbitrary dimension over \( K \). Let \( \text{Hom}_{\text{Spec}(K)}(\text{Spec}(L), \text{Spec}(M)) \) be the set of \( K \)-morphisms from \( M \) to \( L \). Let \( \text{Hom}^{\text{open}}_K(\Gamma_L, \Gamma_M) \) over \( \Gamma_K \), considered up to composition with an inner automorphism arising from \( \ker(\Gamma_M, \Gamma_K) \), where \( \Gamma_L \) and \( \Gamma_M \) are the absolute Galois groups of \( L \) and \( M \), respectively. Then the natural map \( \text{Hom}_K(\text{Spec}(L), \text{Spec}(M)) \to \text{Hom}^{\text{open}}_K(\Gamma_L, \Gamma_M) \) is bijective.

Let \( p \) be a prime number. Let \( K \) be a subfield of a finitely generated field extension of \( \mathbb{Q}_p \). It is called a sub-p-adic field. Note that there exists an isomorphism \( \iota : \bar{K} \cong \mathbb{C} \) when \( K \) is uncountable.

Let \( X_\eta \) be the geometric generic fibre of \( X/S \). Then there exists a variety \( F_{K_0} \) and a finitely generated extension field \( K_0 \) of \( \mathbb{Q} \) such that \( F_{K_0} \times_{K_0} \mathbb{C} \cong X_\eta \).

Note that \( \text{Bir}_C(X_\eta) = \text{Bir}_{p}(F_{K_0} \otimes_{K_0} \mathbb{Q}_p) \).

Let \( \pi : \Gamma_{F_K} \to \Gamma_K \) denote the structure map associated to \( \text{Spec} K(F_K) \to F_K \to \text{Spec}(K) \), which is a surjection since \( K \) is algebraically closed in the rational function field of \( F \). Let \( Z(\Gamma_{F_K}) \) denote the centre of \( \Gamma_{F_K} \). Then \( \pi \) induces \( \pi : Z(\Gamma_{F_K}) \to Z(\Gamma_K) \).

**Lemma 14.** Let \( A \) be an algebraic space in group locally of finite type over \( K \) (i.e. with at most countable components) and let \( \rho : \Gamma_{S_K} \to A \) be a continuous homomorphism as topological groups. Then

1. The image of this homomorphism \( \rho \) is a finite group.
2. Let \( A = \Gamma_{S_K} \). There exist a variety \( S'_{K} \) which is generically finite over \( S_{K} \) and an injective homomorphism \( P' \to P \) with \( (P' : P) < \infty \) such that the representation \( \rho' : P' \to A \) is trivial. Here \( P' \) denotes the absolute Galois group \( \Gamma_{S'_{K}} = \text{Gal}(K(S'_{K})/K(S'_{K})) \).

**Proof.** An algebraic space in group \( A \) is locally of finite type over \( K \). The representation \( \rho : P \to A \) induces \( \overline{\rho} : P \to A/A^0 \), where \( A^0 \) denotes the neutral component of \( A \). Note that there is no countable profinite group. Since \( A/A^0 \) is a countable set, \( \overline{\rho}(P) \) is a finite group. Replace by \( P \) the kernel of \( \overline{\rho} \). We have \( \rho : P \to A^0 \). We have an isomorphism

\[
H^1(K(S_K)/K(S_K), A^0(K(S_K))) \cong H^1(P, A^0) \cong \text{Hom}_{\text{Continuous \ Topological group}}(P, A^0)
\]

\[
H^1_{\text{ét}}(\text{Spec} K(S_K), A^0) \cong \text{TORS}(\text{Spec} K(S_K), A^0).
\]
Let $Q$ be an $A^0$-torsor over $\text{Spec} \, K(S_K)$ associated to $\rho : P \to A^0$. $A^0$ is algebraic (quasi-compact, faithfully flat and of finite type) over $\text{Spec} \, K(S_K)$. There exists an isomorphism $A^0 \times Q \to Q \times Q$ over $\text{Spec} \, K(S_K)$. Thus along $\text{Spec} \, K(Q) \to Q \to \text{Spec} \, K(S_K)$, the pullback of $A^0$-torsor $Q$ becomes trivial. Namely, the $A^0$-torsor $Q$ is trivial over $\text{Spec} \, K(S_K')$. Hence the induced homomorphism $\text{Gal}(\overline{K(Q)}/K(Q)) \to A^0$ is trivial. Let $S'_K \to S_K$ be the Stein factorization of $Q \to S_K$. Then $\Gamma_Q \to \Gamma_{S'_K}$ is a surjective homomorphism. We have

$$\Gamma_Q \to \Gamma_{S'_K} \subset \Gamma_K \to A^0 \to A.$$ 

Since $\Gamma_Q \to A^0$ is trivial, i.e., $\Gamma_Q \to 1$, $\Gamma_{S'_K} \to A$ is trivial. It is obvious that $(\Gamma_{S'_K} : \Gamma_K) < \infty$. Hence $\text{im}(\rho)$ is a finite group. 

Note that a quotient of a scheme by a finite group is in the category of algebraic spaces.

**Proposition 4.** Let $X/S$ be a fibre space. Let $1 \to G \to E \to P \to 1$ be an extension of a profinite group $P$ by a profinite group $G$ associated to a fibre space $X \to S$. Namely $G$, $E$ and $P$ are profinite groups which are the absolute Galois groups associated to the rational function fields of the generic geometric fibre $X_{\bar{\eta}}$, $X$ and $S$, respectively.

**Proof.** Let $X/S$ be a fibre space with the generic geometric fibre $X_{\bar{\eta}}$. To a fibre space the epimorphism $\Gamma_{R(X)} \to \Gamma_{R(S)}$ is associated. Consider a morphism $\text{Spec}(R(X)) \to \text{Spec}(R(S))$ with the generic geometric fibre $\text{Spec}R(X_{\bar{\eta}})$. Grothendieck’s algebraic $\pi_1$ in SGA1[SGA] gives the exact sequence: $1 \to \pi_1(\text{Spec}R(X_{\bar{\eta}})) \to \pi_1(\text{Spec}(R(X))) \to \pi_1(\text{Spec}(R(S))) \to 1$. Here $\pi_1(\text{Spec}R(X_{\bar{\eta}}))$, $\pi_1(\text{Spec}(R(X)))$ and $\pi_1(\text{Spec}(R(S)))$ are the absolute Galois groups $G$, $E$ and $P$ themselves, respectively. See the diagram:

\[
\begin{array}{ccc}
X & \to & X_{\bar{\eta}} \\
\downarrow & & \downarrow \\
S & \to & \text{Spec}k(\bar{\eta}) \\
\end{array}
\begin{array}{ccc}
\Gamma_{R(X)} & \to & \Gamma_{R(X_{\bar{\eta}})} \\
\downarrow & & \downarrow \\
\Gamma_{R(S)} & \to & 1 = \Gamma_k(\bar{\eta}) \\
\end{array}
\]

Thus to a fibre space $X/S$ the extension of a profinite groups $1 \to G \to E \to P \to 1$ is associated. 

We make use of theory of Schreier’s classification of group extensions, Grothendieck-Giraud’s classification of topos extensions or Breen’s classification of 2-gerbes and 2-stacks.[AM, Gir, Breen1, Breen2] Here we take the notion of Breen’s.

**Definition 4.** An extension of groups $1 \to G \to E \to P \to 1$ is said to be neutral if it has a section which is a group homomorphism $\sigma : P \to E$. 

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An extension of groups $1 \to G \to E \to P \to 1$ is said to be central if $G$ is contained in the center of $E$.

$E$ is said to be a semi-direct product of $G$ and $P$ if $G$ is a normal subgroup of $E$ and if the multiplication $(x, y), (u, v) \in G \times P$ is defined by $(xu^y, yv)$, where $u^y = \sigma(y)u\sigma(y)^{-1}$. $E$ is denoted by $G \rtimes P$.

$\text{Inn}(G)$ denotes the inner automorphism group of $G$. $E \to \text{Aut}(G)$ denotes the natural homomorphism $x \in E \mapsto (g \mapsto g^x) \in \text{Aut}(G)$. $\text{Out}(G)$ is defined to be $\text{Aut}(G)/\text{Inn}(G)$. This induces a homomorphism $E/G \to \text{Aut}(G)/\text{Inn}(G)$, i.e., $P \to \text{Out}(G)$.

We call left crossed module a homomorphism of groups $\delta : G \to H$, equipped with a left action of $H$ onto $G$ ($h, g \mapsto h^g$) ([Breen1]):

1. $\delta(h^g) = h\delta(g)h^{-1}$
2. $\delta(g')g = g'gg'^{-1}$

\[ \begin{array}{ccc} G & \delta & \to & H \\ & \downarrow & & \downarrow i \\ & \text{Aut}(G) & & \end{array} \]

$i : G \to \text{Aut}(G)$, where $g \mapsto i_g(x \mapsto gxg^{-1})$, and the natural action $\text{Aut}(G)$ onto $G$ defines a crossed module, which denotes $G \to \text{Aut}(G)$.

To an exact sequence $1 \to \text{Inn}G \to \text{Aut}G \to \text{Out}G \to 1$, we have an exact sequence

$H^1(P, \text{Inn}G) \to H^1(P, \text{Aut}G) \to H^1(P, \text{Out}G)$,

i.e.,

$\text{Hom}(P, \text{Inn}G) \to \text{Hom}(P, \text{Aut}G) \to \text{Hom}(P, \text{Out}G)$.

Here $\text{Out}G$ denotes the outer automorphism group of $G$. Let $G \to \text{Aut}G$ denote the crossed module. The set of the extensions of a profinite group $P$ by a profinite group $G$ denotes $\text{Ext}(P, G)$. A group extension can be defined to be as an element of $H^1(P, (G \to \text{Aut}G))$. There exists an exact sequence $1 \to Z(G)[1] \to (G \to \text{Aut}(G)) \to \text{Out}G \to 1$ ([Breen2]). We have the exact sequence of cohomologies ([Breen2]):

$0 \to H^2(P, Z(G)) \to H^1(P, (G \to \text{Aut}G)) \to H^1(P, \text{Out}G)$. 
Here $Z(G)$ denotes the center of $G$. There exists another sequence $\Aut(G) \to (G \to \Aut(G)) \to G[1]$ in the homotopy category. See the next commutative diagram:

$$
\begin{array}{c}
H^1(P, \Inn(G)) \longrightarrow H^1(P, \Aut(G)) \longrightarrow H^2(P, Z(G)) \\
\downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow \\
H^1(P, \Aut(G)) \longrightarrow H^1(P, G \to \Aut(G)) \\
\downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow \\
H^1(P, \Out(G)) \\
\end{array}
$$

This vertical sequence is nothing but the following exact sequence

$$\Ext(P, Z(G)) \to \Ext(P, G) \to \Hom(P, \Out(G)) \to H^3(P, Z(G))$$

**Proposition 5 ([EGA]).** Let $S$ be a scheme, $(X_\lambda, v_{\lambda \mu})$ a filtered projective system of $S$-schemes; assume that there exists $\alpha$ such that $v_{\alpha \lambda}$ is an affine morphism for every $\lambda \geq \alpha$, so that the projective limit $X = \lim \limits_{\leftarrow} X_\lambda$ exists in the category of $S$-schemes. Let $Y$ be an $S$-scheme and for every $\lambda \geq \alpha$ let $e_\lambda : \Hom_S(X_\lambda, Y) \to \Hom_S(X, Y)$ the map which gives $f = f_\lambda \circ v_\lambda$ to each $S$-morphism $f_\lambda : X_\lambda \to Y$, where $v_\lambda : X \to X_\lambda$ is the canonical morphism. The family $(e_\lambda)$ is an inductive system of maps, which defines the canonical map

$$\lim \Hom_S(X_\lambda, Y) \to \Hom_S(X, Y).$$

Suppose that $X_\alpha$ is quasi-compact and quasi-separated and that the structure morphism $Y \to S$ is locally of finite presentation (resp. locally of finite type). Then the map above is bijective (resp. injective). Furthermore, suppose that $\lim Y_\rho$, where $(Y_\rho, t_{\rho \sigma})$ is a filtered projective system of $S$-schemes such that the structure morphism is locally of finite presentation for every $\rho$. One has a canonical bijection

$$\Hom_S(X, Y) \cong \lim \lim \Hom_S(X_\lambda, Y_\rho).$$

**Proposition 6.** Let $K$ be a sub-$p$-adic field and $X/S$ a fibre space of varieties over $K$. Let $S_\lambda$ be a filtered projective system of $K$-varieties such that

1. $K(S_\lambda)/K(S)$ is a normal extension for every $\lambda$
2. for every $\mu \geq \lambda$ $K(S_\mu)/K(S_\lambda)$ is a normal extension
3. the structure homomorphism $\Gamma_{K(S_\lambda)} \to \Gamma_K$ is surjective.
Let $\Gamma_{\mathcal{S}} = \ker(\Gamma_S \to \Gamma_K)$ and $\Gamma_{\overline{X}} = \ker(\Gamma_X \to \Gamma_K)$. Suppose a sectional homomorphism $\Gamma_{\mathcal{S}} \to \Gamma_{\overline{X}} \subset \Gamma_X$ so that one has a homomorphism $\Gamma_{\mathcal{S}} \to \text{Aut}_{\Gamma_K}(\Gamma_X)$ and $\Gamma_{\mathcal{S}} \to \text{Aut}_{\Gamma_K}(\Gamma_{X_\lambda})$.

Let $A(\overline{S})_\lambda = \text{im}(\Gamma_{\overline{S}} \to \text{Aut}_{\Gamma_K}(\Gamma_{X_\lambda})$. If $\mu \leq \lambda$, one has $A(\overline{S})_\mu \to A(\overline{S})_\lambda$. One has a map

$$\lim_{\lambda \to \infty} A(\overline{S})_\lambda \to \lim_{\mu \to \infty}(\lim_{\lambda \to \infty} \text{Hom}^{\text{open}}_{\Gamma_K}(\Gamma_{X_\lambda}, \Gamma_{X_\mu})/\Gamma_{\overline{X}_\mu}).$$

Let $B(\overline{S}) = \text{im}(\Gamma_{\overline{S}} \to \lim_{\mu \to \infty}(\lim_{\lambda \to \infty} \text{Hom}^{\text{open}}_{\Gamma_K}(\Gamma_{X_\lambda}, \Gamma_{X_\mu})/\Gamma_{\overline{X}_\mu})$. One has a canonical Mochizuki bijection:

$$\lim_{\mu \to \infty}(\lim_{\lambda \to \infty} \text{Hom}^{\text{open}}_{\Gamma_K}(\Gamma_{X_\lambda}, \Gamma_{X_\mu})/\Gamma_{\overline{X}_\mu}) \cong \lim_{\mu \to \infty}(\text{Mor}^\text{dom}_K(\text{Spec}(K(X_\lambda), \text{Spec}(K(X_\mu)))).$$

By the precedent proposition, one obtains

$$\lim_{\mu \to \infty}(\text{Mor}^\text{dom}_K(\text{Spec}(K(X_\lambda), \text{Spec}(K(X_\mu))))) \subset \text{Mor}_K(\lim_{\lambda \to \infty} \text{Spec}(K(X_\lambda)), \lim_{\mu \to \infty} \text{Spec}(K(X_\mu))).$$

One also gets canonical homomorphisms of topological groups:

$$\Gamma_{\mathcal{S}} \to \lim_{\lambda \to \infty} A(\overline{S})_\lambda \to B(\overline{S}) \subset \text{Aut}_{\Gamma_K}(\lim_{\lambda \to \infty} \text{Spec}(K(X_\lambda)) \subset \text{Bir}_K(X_\eta).$$

In the other way, one obtains a canonical homomorphism

$$\Gamma_{\mathcal{S}} \to \lim_{\lambda \to \infty} A(\overline{S})_\lambda \to B(\overline{S}) \to \text{Aut}_{\Gamma_K}(\lim_{\lambda \to \infty} \Gamma_{X_\lambda}, \lim_{\mu \to \infty} \Gamma_{X_\mu})/\lim_{\mu \to \infty}(\Gamma_{\overline{X}_\mu} \to \text{Out}(\Gamma_{X_K}).$$

In particular, if $\Gamma_{\mathcal{S}} \to \text{Bir}_K(\overline{K}(X_\eta)$ is a trivial homomorphism, then so is $\Gamma_{\mathcal{S}} \to \text{Out}(\Gamma_{X_K}).$

Proof. Since $S_\lambda$ be a filtered projective system of $K$-varieties such that

1. $K(S_\lambda)/K(S)$ is a normal extension for every $\lambda$
2. for every $\mu \geq \lambda K(S_\mu)/K(S_\lambda)$ is a normal extension
3. the structure homomorphism $\Gamma_{K(S_\lambda)} \to \Gamma_K$ is surjective,

and by assumption there exists a sectional homomorphism $\Gamma_{\mathcal{S}} \to \Gamma_{\overline{X}} \subset \Gamma_X$, one has a homomorphism $\Gamma_{\mathcal{S}} \to \text{Aut}_{\Gamma_K}(\Gamma_X)$ and $\Gamma_{\mathcal{S}} \to \text{Aut}_{\Gamma_K}(\Gamma_{X_\lambda})$. By Mochizuki’s theorem, one has

$$\lim_{\lambda \to \infty} \text{Hom}^{\text{open}}_{\Gamma_K}(\Gamma_{X_\lambda}, \Gamma_{X_\mu})/\Gamma_{\overline{X}_\mu} \cong \lim_{\lambda \to \infty}(\text{Mor}^\text{dom}_K(\text{Spec}(K(X_\lambda), \text{Spec}(K(X_\mu)))).$$

Hence

$$\lim_{\mu \to \infty}(\lim_{\lambda \to \infty} \text{Hom}^{\text{open}}_{\Gamma_K}(\Gamma_{X_\lambda}, \Gamma_{X_\mu})/\Gamma_{\overline{X}_\mu}) \cong \lim_{\mu \to \infty}(\text{Mor}^\text{dom}_K(\text{Spec}(K(X_\lambda), \text{Spec}(K(X_\mu))))$$
Since $A(\mathcal{S})_\lambda = \text{im}(\Gamma_{x_\lambda} \to \text{Aut}_K(\Gamma_{x_\lambda}))$ and since for $\mu \leq \lambda$, $A(\mathcal{S})_\mu \to A(\mathcal{S})_\lambda$, one has a map for every $\mu$

$$\lim_{\mu \to \lambda} A(\mathcal{S})_\lambda \to \lim_{\mu \to \lambda} \text{Hom}^{\text{open}}_{\Gamma_K}(\Gamma_{x_\lambda}, \Gamma_{x_\mu})/\Gamma_{x_\mu}.$$ 

Hence

$$\lim_{\mu \to \lambda} A(\mathcal{S})_\lambda = \lim_{\mu \to \lambda} A(\mathcal{S})_\lambda \to \lim_{\mu \to \lambda} (\text{Hom}^{\text{open}}_{\Gamma_K}(\Gamma_{x_\lambda}, \Gamma_{x_\mu})/\Gamma_{x_\mu}).$$

Owing to Mochizuki's bijection and the precedent proposition in EGA,

$$\lim_{\lambda \to \mu} A(\mathcal{S})_\lambda \to \lim_{\mu \to \lambda} \text{Mor}^{\text{dom}}_{K}(\text{Spec}(K(X_\lambda)), \text{Spec}(K(X_\mu))) \subset \text{Mor}_{K}(\lim_{\lambda \to \mu} \text{Spec}(K(X_\lambda)), \lim_{\lambda \to \mu} \text{Spec}(K(X_\mu)))$$

Therefore one gets non trivial canonical homomorphisms of topological groups:

$$\Gamma_{\mathcal{S}} \to \lim_{\lambda \to \mu} A(\mathcal{S})_\lambda \to \text{Aut}_K(\lim_{\lambda \to \mu} \text{Spec}(K(X_\lambda))).$$

Note that

$$\text{Aut}_K(\lim_{\lambda \to \mu} \text{Spec}(K(X_\lambda))) \subset \text{Aut}_{\bar{K}}(\text{Spec}(\bar{K}(X_\eta))) = \text{Bir}_{\bar{K}}(X_\eta)$$

Secondly, we show that there exists a canonical homomorphism

$$\text{Aut}_{\Gamma_K} (\lim_{\lambda \to \mu} \Gamma_{x_\lambda}, \lim_{\lambda \to \mu} \Gamma_{x_\mu})/\lim_{\lambda \to \mu} \Gamma_{x_\mu} \to \text{Out}(\Gamma_{x_{\bar{K}}}).$$

Since $\lim_{\lambda \to \mu} \Gamma_{x_\lambda} \to \Gamma_{x_\mu}$, one has a canonical map

$$\text{Hom}_{\Gamma_K}(\Gamma_{x_\lambda}, \Gamma_{x_\mu}) \to \text{Hom}_{\Gamma_K}(\lim_{\lambda \to \mu} \Gamma_{x_\lambda}, \Gamma_{x_\mu}).$$

It follows that

$$\lim_{\mu \to \lambda} (\text{Hom}^{\text{open}}_{\Gamma_K}(\Gamma_{x_\lambda}, \Gamma_{x_\mu})/\Gamma_{x_\mu}) \to \lim_{\mu \to \lambda} (\text{Hom}^{\text{open}}_{\Gamma_K}(\lim_{\lambda \to \mu} \Gamma_{x_\lambda}, \Gamma_{x_\mu})/\Gamma_{x_\mu})$$

and from the definition of the projective limit

$$\lim_{\mu \to \lambda} (\text{Hom}^{\text{open}}_{\Gamma_K}(\Gamma_{x_\lambda}, \Gamma_{x_\mu})/\Gamma_{x_\mu}) \cong \text{Hom}^{\text{open}}_{\Gamma_K}(\lim_{\lambda \to \mu} \Gamma_{x_\lambda}, \lim_{\lambda \to \mu} \Gamma_{x_\mu})/\lim_{\lambda \to \mu} \Gamma_{x_\mu}$$

Since

$$\lim_{\mu \to \lambda} \Gamma_{x_\mu} \leftarrow \Gamma_{x_{\bar{K}}}$$

one has a canonical map

$$\text{Hom}^{\text{open}}_{\Gamma_K}(\lim_{\lambda \to \mu} \Gamma_{x_\lambda}, \lim_{\lambda \to \mu} \Gamma_{x_\mu})/\lim_{\lambda \to \mu} \Gamma_{x_\mu} \to \text{Out}(\Gamma_{x_{\bar{K}}})$$

and a canonical homomorphism

$$\text{Aut}_{\Gamma_K} (\lim_{\lambda \to \mu} \Gamma_{x_\lambda}, \lim_{\lambda \to \mu} \Gamma_{x_\mu})/\lim_{\lambda \to \mu} \Gamma_{x_\mu} \to \text{Out}(\Gamma_{x_{\bar{K}}}).$$

\qed
We consider the following diagrams:

\[
\begin{array}{c}
Y_K \times_K S_K \longrightarrow X_K \longrightarrow S_K \\
\downarrow \quad \downarrow \\
\text{Spec } K
\end{array}
\]

\[
\begin{array}{c}
\Gamma_{Y_K} \times_{\Gamma_K} \Gamma_{S_K} \longrightarrow \Gamma_{X_K} \longrightarrow \Gamma_{S_K} \\
\downarrow \quad \downarrow \\
\Gamma_K
\end{array}
\]

**Proposition 7 ([Mcl]).** Let \( L \) be a sub-p-adic field and a function field. \( \Gamma_L \) is center-free.

It is known by Douady that the absolute Galois group of the function field of \( \mathbb{P}^1_{\mathbb{C}} \) is a free profinite group. Every open subgroup of a free profinite group is also free. A free profinite group is center-free.

**Proposition 8.** Let \( K \) be a sub-p-adic field and \( X_K/S_K \) a fibre space over \( K \). Let \( \Gamma_{X_K}, \Gamma_{S_K} \) and \( \Gamma_K \) be the absolute Galois groups of the sub-p-adic fields \( K(X_K), K(S_K) \) and \( K \), respectively. To a fibre space \( X_K/S_K \) up to birational equivalence, i.e., algebraically closed extension \( K(S_K) \subset K(X_K) \), there corresponds an exact sequence:

\[
1 \rightarrow \Gamma_{X_K} \rightarrow \Gamma_{X_K} \rightarrow \Gamma_{S_K} \rightarrow 1
\]

1. The extension of profinite groups above is expressed by an element of the pointed set \( H^1(\Gamma_{S_K}, (\Gamma_{F_K} \rightarrow \text{Aut}(\Gamma_{F_K}))) \).

2. The following is bijection \( H^1(\Gamma_{S_K}, (\Gamma_{F_K} \rightarrow \text{Aut}(\Gamma_{F_K}))) \cong H^1(\Gamma_{S_K}, \text{Out}(\Gamma_{F_K})) \)

3. \[
\begin{array}{c}
H^1(\Gamma_{S_K}, \text{Aut}(\Gamma_{F_K})) \longrightarrow H^1(\Gamma_{S_K}, (\Gamma_{F_K} \rightarrow \text{Aut}(\Gamma_{F_K}))) \\
\downarrow \\
H^1(\Gamma_{S_K}, \text{Out}(\Gamma_{F_K}))
\end{array}
\]
4. Let $S'_K \to S_K$ be a dominant $K$-rational map and $X'_K/S'_K$ a pull-back of $X_K/S_K$.

\[
\begin{array}{c}
\lim_{\to S'_K} \operatorname{Aut}_{S'_K} (\Gamma X'_K) / \Gamma_{F_{\bar{K}}} \to \operatorname{Aut}(\Gamma_{F_{\bar{K}}}) / \Gamma_{F_{\bar{K}}}
\end{array}
\]

(1)

Proof. 1. There exist a natural restrictions $\operatorname{Aut}_{\Gamma_K}(\Gamma X_K) \to \operatorname{Aut}(\Gamma_X)$. 

2. Let $p : \Gamma_{X_K} \to \Gamma_K$ be the structure map. Let $u$ be a $\Gamma_K$-automorphism $\Gamma_{X_K}$ and $x \in \Gamma_{X_K}$. $p(u(x)) = p(x) = 1 \in \Gamma_K$. Hence $u(x) \in \Gamma_{X_K}$. 

Lemma 15. Assume there exists a dominant $S_K$-rational map $Y_K \times_K S_K \to X_K$. Then there exists a continuous homomorphism $\Gamma_{S_{\bar{K}}} \to \operatorname{Aut}_{\Gamma_K}(\Gamma_{X_K})$, which is an element of $H^1(\Gamma_{S_{\bar{K}}}, \operatorname{Aut}_{\Gamma_K}(\Gamma_{X_K}))$. In other word, we have $\Gamma_{S_{\bar{K}}} \to \operatorname{Aut}(K(X_K))$. The following square is commutative.

\[
\begin{array}{c}
H^1(\Gamma_{S_{\bar{K}}}, \operatorname{Aut}_{\Gamma_K}(\Gamma_{X_K})) \to H^1(\Gamma_{S_{\bar{K}}}, (\Gamma_{F_{\bar{K}}} \to \operatorname{Aut}(\Gamma_{F_{\bar{K}}}))
\end{array}
\]

Proof. Since there exists a dominant $S_K$-rational map $Y_K \times_K S_K \to X_K$, Mochzuki'correspondence implies the commutative diagram of open homomorphisms

\[
\begin{array}{c}
\Gamma_{Y_K} \times_{\Gamma_K} \Gamma_{S_K} \to \Gamma_{X_K} \to \Gamma_{S_K}
\end{array}
\]

There exists an open homomorphism $\Gamma_{Y_K} \times \Gamma_{S_K} \to \Gamma_{X_K} \to \Gamma_{S_K}$. Hence there exists a sectional homomorphism $\Gamma_{S_K} \to \Gamma_{X_K} \subset \Gamma_{X_K}$. By the sectional homomorphism $\Gamma_{S_K} \to$
Let $K(S_K)$ and $X_K \to S_K$ be such a finite extension in a sub-p-adic fields and its pull-back that $\kappa(S_K) = \dim S_K$, then $\kappa(X_K) \geq \kappa(F_K) + \dim(S_K)$.

**Lemma 16.** Let $K$ be a sub-p-adic field. Assume that $\text{Bir}(F_K)$ is an algebraic space in group locally of finite type and that there exist a dominant $S_K$-rational map $Y_K \times_K S_K \to X_K$. Then there exists a generically finite morphism $S'_K \to S_K$ such that the natural map $H^1(\Gamma_{S_K}, (\Gamma_{F_K} \to \text{Aut}_{\Gamma_K}(\Gamma_{F_K}))) \to H^1(\Gamma_{S'_K}, (\Gamma_{F_K} \to \text{Aut}(\Gamma_{F_K})))$ sends the extension class to the trivial extension class, i.e., a distinguished element.

**Proof.** Since there exist a dominant $S_K$-rational map $Y_K \times_K S_K \to X_K$, there exist a section $\Gamma_{S_K} \to \Gamma_{X_K}$ and a $\Gamma_{S_K} \to \text{Inn}(\Gamma_{X_K}) \to \text{Aut}(\Gamma_{X_K})$. Choose a $K$-dominant map $S'_K \to S_K$ such that $\Gamma_{S_K} \supset \Gamma_{S'_K}$ is normal and of finite index. Hence it induces $\Gamma_{S_K} \to \text{Aut}_{\Gamma_K}(\Gamma_{X_K})$. Let $\text{Aut}_{\text{equ},\text{nor}}(\Gamma_{X_K}) \subset \text{Aut}(\Gamma_{X_K})$ be a subgroup such that each automorphism is equivariant with natural maps $\Gamma_{X_K} \to \Gamma_{S_K} \to \Gamma_K$ and keeps invariant normal subgroups.

\[
\begin{array}{cccccccccc}
\Gamma_{X_K} & \longrightarrow & \Gamma_{X'_K} & \longrightarrow & \cdots & \longrightarrow & \lim_{\longrightarrow} & \Gamma_{X''_K} & \longrightarrow & \Gamma_{F_K} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\Gamma_{S_K} & \longrightarrow & \Gamma_{S'_K} & \longrightarrow & \cdots & \longrightarrow & \lim_{\longrightarrow} & \Gamma_{S''_K} & \longrightarrow & \Gamma_{K(K)} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\Gamma_K & & \Gamma_K & & \cdots & & \Gamma_K & & \Gamma_K = 1 \\
\end{array}
\]

\[
\begin{array}{cccccccccc}
\text{Aut}_{\text{equ},\text{nor}}(\Gamma_{X_K})/\Gamma_{X_K} & \longrightarrow & \text{Aut}_{\text{equ},\text{nor}}(\Gamma_{X'_K})/\Gamma_{X'_K} & \longrightarrow & \lim_{\longrightarrow} & \text{Aut}_{\text{equ},\text{nor}}(\Gamma_{X''_K})/\Gamma_{X''_K} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\text{Bir}_K(\text{K}(X_K))^{\text{equ},\text{nor}} & \longrightarrow & \text{Bir}_K(\text{K}(X'_K))^{\text{equ},\text{nor}} & \longrightarrow & \lim_{\longrightarrow} & \text{Bir}_K(\text{K}(X''_K))^{\text{equ},\text{nor}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\text{Aut}_{\text{equ},\text{nor}}(\Gamma_{X'_K})/\Gamma_{X'_K} & \longrightarrow & \lim_{\longrightarrow} & \text{Aut}_{\text{equ},\text{nor}}(\Gamma_{X''_K})/\Gamma_{X''_K} & \longrightarrow & \text{Aut}_{\Gamma_K}(\Gamma_{F_K})/\Gamma_{F_K} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\text{Bir}_K(\text{K}(X'_K))^{\text{equ},\text{nor}} & \longrightarrow & \lim_{\longrightarrow} & \text{Bir}_K(\text{K}(X''_K))^{\text{equ},\text{nor}} & \longrightarrow & \text{Bir}_K(\text{K}(F_K)) \\
\end{array}
\]
Thus by the precedent proposition the image of $\Gamma_{S_K} \to B(S_K) \subset \text{Bir}(X_K)$ is finite and there exists a homomorphism $\Gamma_{S_K} \to B(S_K) \to \text{Out}(\Gamma_{F_K})$. Thus one can obtain there exists a generically finite morphism $S'_K \to S_K$ such that the natural map $H^1(\Gamma_{S_K}, \text{Out}(\Gamma_{F_K})) \to H^1(\Gamma_{S'_K}, \text{Out}(\Gamma_{F_K}))$ sends the extension class to the trivial extension class, i.e., a distinguished element. \( \square \)

**Theorem 4** ([]). There are equivalent expressions of extensions of $\Gamma_{S_K}$ by $G$.

1. 
\[
\text{BiTors}(G) \cong \text{Eq}(G[1]) \cong (G \to \text{Aut}(G))
\]
as monoidal categories.

2. 
\[
H^1(\Gamma_{S_K}, \text{BiTors}(G)) \cong H^1(\Gamma_{S_K}, \text{Eq}(G[1])) \cong H^1(\Gamma_{S_K}, (G \to \text{Aut}(G)))
\]
as pointed sets

3. 
\[
\text{Mon}(\Gamma_{S'_K}, \text{BiTors}(G)) \cong \text{Mon}(\Gamma_{S_K}, \text{Eq}(G[1])) \cong \text{Mon}(\Gamma_{S'_K}, (G \to \text{Aut}(G)))
\]
as morphisms of monoidal categories.

**Proposition 9.** We have the following results.

1.
\[
\text{Hom}^{\text{open,_epi}}(\Gamma_{X_K}, \Gamma_{S_K})/\Gamma_{S_K} \cong \text{Mor}^{\text{alg.closed extension}}_K(\text{Spec } K(X_K), \text{Spec } K(S_K))
\]
, where epi and alg.closed mean epimorphisms and algebraically closed extensions, respectively.

2. (1) is equivalent to a category of fibre spaces between projective varieties

\[
\begin{array}{c}
X_K \\
\downarrow \\
S_K \\
\downarrow \\
\text{Spec } K
\end{array}
\]

up to birational equivalence.
3. there exists a restriction map

\[ \text{Hom}_{\Gamma_K}^{\text{open}}(\Gamma_{X_K}, \Gamma_{S_K}) / \Gamma_{S_K} \rightarrow \text{Hom}_{\Gamma_K}^{\text{open}}(\Gamma_{X_K}, \Gamma_{S_K}) / \Gamma_{S_K} \]

, where \( \Gamma_K = 1 \).

4. the following map is pulling back extensions or "base change".

\[
\begin{array}{ccc}
\text{Ext}(\Gamma_{S_K}, \Gamma_{F_K}) & \rightarrow & \text{Ext}(\Gamma_{S_K}, \Gamma_{F_K}) \\
\downarrow & & \downarrow \\
\Gamma_{F_K} & \rightarrow & \Gamma_{F_K} \\
\downarrow & & \downarrow \\
\Gamma_{X_K} & \rightarrow & \Gamma_{X_K} \\
\downarrow & & \downarrow \\
\Gamma_{S_K} & \rightarrow & \Gamma_{S_K} \\
\downarrow & & \downarrow \\
1 & & 1
\end{array}
\]

5. the following is an exact sequence.

\[ 1 \rightarrow \Gamma_{X_K} \rightarrow \Gamma_{X_K} \rightarrow \Gamma_K \rightarrow 1 \]

\[ 1 \rightarrow \text{Hom}_{\Gamma_K}(\Gamma_K, \Gamma_{S_K}) \rightarrow \text{Hom}_{\Gamma_K}(\Gamma_{X_K}, \Gamma_{S_K}) \rightarrow \text{Hom}_{\Gamma_K}(\Gamma_{X_K}, \Gamma_{S_K}) \]

6. 

\[ 1 \rightarrow \Gamma_{S_K} \rightarrow \Gamma_{S_K} \rightarrow \Gamma_K \rightarrow 1 \]

\[ 1 \rightarrow \text{Mon}_{\Gamma_K}(\Gamma_K, \text{Eq}(G[1])) \rightarrow \text{Mon}_{\Gamma_K}(\Gamma_{S_K}, \text{Eq}(G[1])) \rightarrow \text{Mon}_{\Gamma_K}(\Gamma_{S_K}, \text{Eq}(G[1])) \]

7. 

\[ \text{Hom}_{\Gamma_K}^{\text{open}}(\Gamma_{F_K}, \Gamma_K) / \Gamma_K \cong \text{Mor}_K(\text{Spec } K(F_K), \text{Spec } K) \]

Proof. 1. Applying Mochizuki theory we have (1) and (2) since a fibre space has connected fibres.

2. (3) is obtained by restricting homomorphisms to homomorphisms over \( 1 = \Gamma_K \rightarrow \Gamma_K \).
3. (4) and (6) are pulling back extensions by base change $\Gamma_{S_K} \to \Gamma_{S_K}$.

4. (7) is Mochizuki correspondence.

5. (5) Applying a left-exct functor $\text{Hom}_{\Gamma_K}$ we have an exact sequence.

6. The contravariant functor $\text{Hom}_{\Gamma_K}(-, \Gamma_{S_K})$ is a left-exact. Note that $\text{Hom}_{\Gamma_K}(-, \Gamma_{S_K}) = \text{Hom}_{\Gamma_K}(\Gamma_{S_K}, \Gamma_{S_K})$.

7. The contravariant functor $\text{Mon}_{\Gamma_K}(-, \text{Eq}(G[1]))$ is left-exact. Note that $\text{Mon}_{\Gamma_K}(\Gamma_{S_K}, \text{Eq}(G[1])) = \text{Mon}_{\Gamma_K}(\Gamma_{S_K}, \text{Eq}(G[1]))$.

**Proposition 10.** Let $K$ be a sub-p-adic field. Assume that Bir$(F_K)$ is an algebraic space in group locally of finite type and that there exist a dominant $S_K$-rational map $Y_K \times_K S_K \to X_K$ Then there exists a generically finite morphism $S'_{K} \to S_K$ such that there exists a birational equivalence over $S'_{K}$

$$X_K \times_{S_K} S'_{K} \cong F_K \times_K S'_{K}$$

, i.e., $X_K/S_K$ is birationally isotrivial.

**Proof.** Let $K$ be a sub-p-adic field. Assume that Bir$(F_K)$ is an algebraic space in group locally of finite type and that there exist a dominant $S_K$-rational map $Y_K \times_K S_K \to X_K$ Then there exists a generically finite morphism $S'_{K} \to S_K$ such that the natural map $H^1(\Gamma_{S_K}, (\Gamma_{F_K} \to \text{Aut}_{\Gamma_K}(\Gamma_{F_K})) \to H^1(\Gamma_{S'_{K}}, (\Gamma_{F_K} \to \text{Aut}(\Gamma_{F_K}))$ sends the extension class to the trivial extension class, i.e., a distinguished element. Let $G = \Gamma_{F_K}$. One has the following exact sequence:

$$1 \to \text{Mon}_{\Gamma_K}(\Gamma_K, \text{Eq}(G[1])) \to \text{Mon}_{\Gamma_K}(\Gamma_{S'_{K}}, \text{Eq}(G[1])) \to \text{Mon}_{\Gamma_K}(\Gamma_{S'_{K}}, \text{Eq}(G[1]))$$

The element of $X_K \times_{S_K} S'_{K}$ in $\text{Mon}_{\Gamma_K}(\Gamma_{S'_{K}}, \text{Eq}(G[1]))$ maps to 1 in $\text{Mon}_{\Gamma_K}(\Gamma_{S'_{K}}, \text{Eq}(G[1]))$. Hence choosing a variety $F_K$ one obtains $X_K \times_{S_K} S'_{K} = F_K \times_K S'_{K}$ birationally by pull-back and Mochizuki correspondence.

6 Birational automorphism groups in Algebraic Geometry

**Theorem 5.** Let $X$ be a non singular projective variety of Kodaira dimension $\geq 0$. Bir$(X)$ is a scheme which is locally of finite type.
We shall prove the theorem using the following Lemmas.

**Lemma 17.** Let $X$ be a quasi-projective variety. Then $\text{Aut}(X)$ is a group scheme which is locally of finite type.

**Proof.** Since $X$ is quasi-projective, it suffices to consider some compactification $\bar{X}$ of $X$ and $\text{Hilb}_{\bar{X} \times \bar{X}}$.

Let $\text{Aut}^0(X)$ denote the connected component of $\text{Aut}(X)$ which contains an identity of the group.

We refer to the following A. Weil-M. Rosenlicht’s theorem.

**Theorem 6.** ([Ro]) Let the algebraic group $G$ operate on the variety $V$ and let $k$ be a field of definition for $G$, $V$ and the operation of $G$ on $V$. Then there exists a variety $V'$, birationally equivalent over $k$ to $V$, such that the operation of $G$ on $V'$ that is induced by its operation on $V$ is regular.

**Lemma 18.** Let $X$ be a projective variety. There exists an inductive system of monomorphisms $\text{Aut}^0(X_i) \to \text{Aut}^0(X_{i+1})$ such that $X_0 = X$, $\text{Aut}^0(X_i)$ acts regularly on some quasi-projective variety $X_{i+1}$ which is a quasi-projective variety $X_i \setminus H$ where $H$ is a hypersurface of $X_i$. The inductive limit of the system $(\text{Aut}^0(X_i))_{i \in I}$ is locally compact Lie group and an ind-algebraic space. Hence it is a pro-Lie group. If $X$ is of Kodaira dimension $\geq 0$, the birational automorphism group is locally algebraic.

**Proof.** By Weil-Rosenlicht’s theorem([?]) and the lemma above, we can construct an inductive system of monomorphisms $\text{Aut}^0(X_i) \to \text{Aut}^0(X_{i+1})$ such that $X_0 = X$, $\text{Aut}^0(X_i)$ acts regularly on some quasi-projective variety $X_{i+1}$ which is a quasi-projective variety $X_i \setminus H$ where $H$ is a hypersurface of $X_i$. Since any $\text{Aut}^0(X_i)$ is an algebraic group, the inductive limit is a Baire space in the complex topology, i.e., an inner point in the limit space is also an inner point some $\text{Aut}^0(X_i)$. Thus the inductive limit of the system $(\text{Aut}^0(X_i))_{i \in I}$ is locally compact Lie group and an ind-algebraic space, which is also a pro-Lie group. When $\kappa(X) \geq 0$, there exists a maximal algebraic group birationally acting on $X$ by [Mat]. Hence the inductive limit is an algebraic group which turns out to be an abelian group by [Mat].

Thus the theorem above is proved.

Hence the theorem in the preceded section is equivalent to the following theorem.

**Theorem 7.** Let $X/S$ be a fibre space with the generic geometric fibre of Kodaira dimension $\geq 0$. If $X/S$ is neutral, then $X/S$ is isotrivial.
Proof. The birational automorphism group of the generic geometric fibre of $X/S$ is locally of finite type. Hence the extension $1 \to G \to E \to P \to 1$ associated to a fibre space $X/S$ satisfies the assumption of the theorem in the preceded section. 

7 Iitaka-Viehweg conjecture

Theorem 8. Let $f : X \to S$ be a fibre space $X/S$ with the generic geometric fibre $X_\eta$ of Kodaira dimension $\geq 0$.

$$\kappa(\det f_*\omega_{X/S}^m) \geq \operatorname{var}(X/S)$$

The proof shall be obtained by combining the next two lemmas.

Lemma 19. Let $f : X \to S$ be a fibre space $X/S$ with the generic geometric fibre $X_\eta$ of Kodaira dimension $\geq 0$. There exists a fibre space $g : Y \to S$ such that

1. $h : Y \to X$ is a cover over $X$ such that $g = f \circ h$,

2. the generic geometric fibre $Y_\eta$ of $Y/S$ is of general type, if necessary, the canonical invertible sheaf of $Y_\eta$ is taken to be abundant,

3. $\kappa(\det f_*\omega_{X/S}^m) = \kappa(\det g_*\omega_{Y/S}^m)$.

Proof. Embed $X$ into some projective space $\mathbb{P}$ and $X/S$ into the trivial fibre space $S \times \mathbb{P}$. Let $i : X \to S \times \mathbb{P}$ be the embedding over $S$. Choose a general hyperplane $H$ in $S \times \mathbb{P}$ such that the intersection $X \cap H$ is a non singular variety and $H = H_0 \times S$ is horizontal in $S \times \mathbb{P}$. Take a branch cover $Y$ of $X$ along $H$. Choose a hyperplane $H$ such that $Y/S$ has a general fibre of general type. We have proved $\kappa(\det f_*\omega_{X/S}^m) = \kappa(\det g_*\omega_{Y/S}^m)$. 

By Kollar’s theorem, if necessary, Kawamata’s theorem ([Kaw]),

Theorem 9. $\kappa(\det f_*\omega_{X/S}^m) = \kappa(\det g_*\omega_{Y/S}^m) \geq \operatorname{var}(Y/S)$.

Lemma 20. Let $Y/S$ and $X/S$ be fibre spaces over $K$ and $X/S$ with the generic geometric fibre of Kodaira dimension $\geq 0$. Assume that there exists a dominant $S$-rational map $Y \to X$. Then $\operatorname{var}(Y/S) \geq \operatorname{var}(X/S)$.

Proof. Let $\operatorname{var}(Y/S) = v$. By definition of Viehweg dimension, there exist varieties $S', T$ and $Y_0$ such that $Y \times_S S'$ is birationally equivalent to $Y_0 \times_T S'$ with $Y_0/T$ a fibre space and $T$ of dimension $v$. Hence $Y_0 \times_T S' \to X \times_S S'$ is a dominant $S'$-rational map. Let $\zeta$ be the generic point of $T$ and $\overline{k(\zeta)}$ the algebraic closure of $k(\zeta)$. The induced dominant $S' \times_T \operatorname{Spec}(\overline{k(\zeta)})$-rational map $Y_0 \times_T (S' \times_T \operatorname{Spec}(\overline{k(\zeta)})) \to X \times_S S' \times_T \operatorname{Spec}(\overline{k(\zeta)})$. By the following lemma, $\operatorname{var}(X \times_S S' \times_T \operatorname{Spec}(\overline{k(\zeta)})) = 0$. Hence $\operatorname{var}(X/S) \leq v$. We therefore obtain $\operatorname{var}(Y/S) \geq \operatorname{var}(X/S)$. 

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Lemma 21. Let $Y/S$ and $X/S$ be fibre spaces over $K$ and the generic geometric fibre of $X/S$ with Kodaira dimension $\geq 0$. Assume that there exists a dominant $S$-rational map. Then if $\text{var}(Y/S) = 0$, then $\text{var}(X/S) = 0$.

Proof. To show that if $\text{var}(Y/S) = 0$, then $\text{var}(X/S) = 0$, the statement is valid even if the fibre spaces $Y/S$ and $X/S$ are changed to a base $S'$ on which in the definition Viehweg dimension $Y$ is birationally equivalent to $Y_0 \times S'$ for a variety $Y_0$. Since Kodaira dimension of the generic geometric fibre of $X/S$ is non negative, the birational automorphism group is an algebraic space in group locally of finite type. Hence one obtains $\text{var}(X/S) = 0$ from the precedent proposition.

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Iitaka-Viehwe Conjectures $C$ and $C^{++}$

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Abstract

Given a fibre space $X/S$ with the generic geometric fibre of Kodaira dimension $\geq 0$, we shall construct a variety $Y$ ramified over $X$ along such a horizontal hyperplane with respect to $X/S$ that Kollár and Kawamata had proved Viehweg conjecture for $Y/S$ with the generic geometric fibre of general type or of the abundant canonical invertible sheaf where Viehweg dimensions of $X/S$ and $Y/S$ are equal, respectively. We shall show that Viehweg dimension of $X/S$ is not greater than that of $Y/S$ by Mochizuki's Galois theory.

1 Introduction:

To classify algebraic varieties in the category of birational geometry, Iitaka proposed many conjectures after Kodaira-Enriques classification of surfaces. His key birational invariant is Kodaira dimension. One of his main conjectures is the following:

**Conjecture 1.** Let $X/S$ be a fibre space over the complex number field and $X_0$ the generic geometric fibre of $X/S$. Then $\kappa(X) \geq \kappa(X_0) + \kappa(S)$.

**Remark 1.** Mabuchi suggested that the Griffiths infinitesimal variation Hodge theory is applicable to the proof of the conjecture assuming the abundance conjecture. Kawamata independently proved it in the similar idea under the abundance conjecture. Kollár proved the conjecture in the case when the generic geometric fibre is of general type and Viehweg also proved them ([Kaw], [Ko0], [Vieh2], [Vieh3], [Vieh4]).

Viehweg conjectures the following:

**Conjecture 2.** Let $f : X \to S$ be a fibre space $X/S$ with the generic geometric fibre of Kodaira dimension $\geq 0$. Then there exists a number $m$ such that

$$\kappa(\det f_*\omega_{X/S}^{\otimes m}) \geq \var(X/S).$$

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This conjecture implies

**Conjecture 3.** \( \kappa(\omega_{X/S}) \geq \kappa(\omega_{X_0}) + \text{var}(X/S) \)

Iitaka conjecture C follows from the conjecture above. I thank deeply Prof. Noboru Nakayama for discussion.

## 2 Preliminary

**Definition 1.**

Let \( k \) be a field. A geometrically irreducible, reduced, smooth scheme \( X \) over \( k \) is said to be a non singular variety over \( k \).

Let \( X \) be a non singular variety of dimension \( d \) and \( \Omega_X \) the differential sheaf over \( X \). \( \omega_X \) denotes \( \Omega^d_X \).

A connected proper surjective morphism \( f : X \to S \) of non singular varieties \( X \) and \( S \) is said to be a fibre space \( X/S \).

Let \( f : X \to S \) be a fibre space \( X/S \). \( \omega_{X/S} \) denotes \( \omega_X \otimes f^*\omega_S^{-1} \).

Let \( L \) be an invertible sheaf over \( X \). \( \kappa(L) \) denotes the maximal dimension of the image variety of the rational map \( X \to \mathbf{P}(\Gamma(X, L^m)) \) defined by \( \Gamma(X, L^m) \otimes \mathcal{O}_X \to L^m \). We call \( \kappa(L) \) Iitaka dimension of \( L \).

\( \kappa(\omega_X) \) is said to be Kodaira dimension, which is denoted by \( \kappa(X) \).

Let \( X/S \) be a fibre space. The minimal dimension of \( T \) such that there exists a generically finite morphism \( S' \to S \) in which \( X \times_S S' \) is birationally equivalent to \( S' \times_T X_0 \) for some varieties \( T, X_0 \) with \( X_0/T \) a fibre space. This dimension denotes \( \text{var}(X/S) \), which is called Viehweg dimension of a fibre space \( X/S \).
The category of bands of profinite groups is defined in the following. The objects are the profinite groups and the arrows are the homomorphisms of profinite groups modulo inner automorphisms.

A \( \mathbb{Q} \)-divisor \( D \) is said to be effective if \( \kappa(D) \geq 0 \). Similarly, we say that a cycle of codimension 1 is effective if every coefficient is nonnegative.

**Definition 2.** Define a functor \( \text{Div}_{X/S} \) from the category of \( S \)-schemes to that of sets by the formula

\[
\text{Div}_{X/S}(T) = \{ \text{relative effective divisors } D \text{ on } X_T/T \}.
\]

**Lemma 1.** Assume \( X/S \) is projective and flat. Then \( \text{Div}_{X/S} \) is representable by an open subscheme of the Hilbert scheme \( \text{Hilb}_{X/S} \).

Let \( k \) be a field of characteristic 0. Let \( X \) be a projective normal variety over \( k \) and let \( \mathcal{M}_X \) the sheaf of rational functions of \( X \), \( \mathcal{M}^*_X \) the sheaf of invertible rational functions of \( X \), which is a subsheaf of \( \mathcal{M}_X \) and \( \mathcal{O}^*_X = \mathcal{O}_X \cap \mathcal{M}^*_X \).

**Definition 3.** \( \text{Div}_X = \mathcal{M}_X / \mathcal{O}^*_X \), \( \text{Div}(X) = \Gamma(X, \text{Div}_X) \)

An invertible \( \mathcal{O}_X \)-submodule of \( \mathcal{M}_X \) is said to be an invertible fractional sheaf, for example, \( \mathcal{O}_X(D) \) for a divisor \( D \).

An invertible \( \mathcal{O}_X \)-module is said to be an invertible sheaf. The set of equivalence classes of couples \( (L, s) \) denotes \( D(X) \). Here \( L \) is an invertible sheaf, \( s \) is a non 0 global section of \( L \). \( (L, s) \) and \( (L', s') \) are equivalent if there exists an isomorphism \( u : L \to L' \) such that \( u(s) = s' \).

An element of \( \text{Pic}(X) \otimes \mathbb{Q} \) is said to be a \( \mathbb{Q} \)-invertible sheaf.

An element of \( \text{Div}(X) \otimes \mathbb{Q} \) is said to be a \( \mathbb{Q} \)-divisor.

**Facts 1.** The order preserving homomorphism \( \text{cyc} : \text{Div}(X) \to Z^1(X) \) which assigns a divisor a cycle of codimension 1 is injective and the image \( \text{cyc}(\text{Div}(X)) \) consists of locally principal divisors.

Let \( Z^1(X) \) denote the free group generated by the cycles of codimension one on \( X \). For every \( x \in X \) \( \mathcal{O}_{X,x} \) is factorial if and only if \( \text{cyc} : \text{Div}(X) \to Z^1(X) \) is bijective.

Let \( X^{(1)} \) denote the set of points \( x \in X \) such that \( \dim \mathcal{O}_{X,x} = 1 \). Let \( f : X \to S \) be a finite surjective morphism. Let \( D' = \sum_{x' \in X^{(1)}} n_{x'} \{ x' \} \) be a codimension 1 cycle. For \( x \in X^{(1)} \), put \( n_x = \sum_{x' \in f^{-1}(x)} n_{x'} [k(x') : k(x)] \). Then \( f_*(D') = \sum_{x \in X^{(1)}} n_x \{ x \} \).

Suppose that \( f \) is flat and that \( Z \) is non singular.

Let \( D = \sum_{x \in S^{(1)}} n_x \{ x \} \) be a cycle of codimension 1. Put \( \lambda_{x'} = \text{length}(\mathcal{O}_{X,x'}/\mathcal{M}_X \mathcal{O}_{X,x'}) \) and \( n_{x'} = \lambda_{x'} n_x \). Then \( f^*D = \sum_{x' \in X^{(1)}} n_{x'} \{ x' \} \).
3 Horizontal Hypersurface

Our main aim is to show the following theorem.

**Theorem 1.** Let \( f : X \to S \) be a fibre space of non singular varieties. Assume that \( \kappa(\omega_{X_\eta}) \geq 0 \) for the generic geometric fibre \( X_\eta \). Then there exists an integer \( m > 0 \) such that \( \kappa(\det f_\ast \omega_{X/S}^{\otimes m}) \geq \var(X/S) \).

Viehweg’s Lemma:

**Lemma 2.** Let \( S' \to S \) be a Kummer-Kawamata covering with respect to an ample divisor. Let \( X \times_S S' \to S' \) be the pull-back. Then \( X \times_S S' \) has rational singularity. Further take a desingularization \( X^v \to X \times_S S' \). Then \( \kappa(\det f^v_\ast \omega_{X^v/S'}^{\otimes m}) \leq \kappa(\det f_\ast \omega_{X/S}^{\otimes m}) \).

We shall prove the theorem above in the following several steps. By Viehweg in order to prove the theorem, we can assume further that \( \var(X/S) = \dim S \) and show the theorem.

Let \( f : X \to S \) be a fibre space. From the extension of the function fields \( R(X)/R(S) \), we have purely transcendental indeterminates \( t_1, \cdots, t_r \) over \( R(S) \) such that \( R(X)/R(S)(t_1, \cdots, t_r) \) is a finite extension of degree \( d \). Hence we obtain a dominant rational map \( X \to S \times \mathbb{P}^r \).

Resolving the indeterminacy of the rational map \( X \to S \times \mathbb{P}^r \), we have a birational map \( X' \to X \) and a morphism \( \phi : X' \to S \times \mathbb{P}^r \). We replace \( X' \) by \( X \) and let \( Z \) denote \( S \times \mathbb{P}^r \). Let \( X'' \) be the integral closure of \( Z \) in the function field \( R(X) \). Namely, \( \mu : X \to X'' \) with \( \nu : X'' \to Z \) is Stein factorization. Let \( \mu : X \to X'' \) be the structure morphism.
Recall the next lemma.

**Lemma 3.** \(\omega_{X/S}\) is weakly positive with respect to \(f\).

In other words, given any \(\alpha > 0\) and any big \(\mathbb{Q}\)-invertible sheaf \(L\) over \(S\), it holds that \(\kappa(\omega_{X/S}^{\otimes \alpha} \otimes f^*L) \geq \dim S\).

We have a rational map \(X \to \mathbb{P}(\Gamma(X, (\omega_{X/S} \otimes f^*L)^{\otimes m}))\) defined by \(\mathcal{O}_X \otimes \Gamma(X, (\omega_{X/S} \otimes f^*L)^{\otimes m}) \to (\omega_{X/S} \otimes f^*L)^{\otimes m}\) for \(m >> 0\). Take a resolution of the indeterminacy of the rational map, which is denoted by \(X^* \to X\). Then replace \(X^*\) by \(X\). Since \(X\) is non-singular, there exists an effective \(\mathbb{Q}\)-cartier divisor \(D\) such that \(\omega_{X/S}^{\otimes \alpha} \otimes f^*L = \mathcal{O}_X(D)\) for any \(\alpha > 0\).

**Lemma 4.** Let \(D\) be an effective \(\mathbb{Q}\)-divisor on \(X\). There exist a \(\mathbb{Q}\)-divisor \(E\) and an effective Weil divisor \(D'\) on \(X'\) such that \(D = \mu^*D' + E\) such that \(E\) is a \(\mu\)-exceptional divisor, i.e., the \(\mu\) image of the support of \(E\) in \(X'\) is of codimension \(\leq 2\).

**Proof.** Since \(\mu : X \to X'\) is birational, there exists a locus of codimension \(\leq 2\) outside which the restriction of \(\mu\) is an isomorphism. Hence we have an effective \(\mathbb{Q}\)-divisor decomposition \(D = \mu^*D' + E\) such that \(E\) is a \(\mu\)-exceptional divisor. \(\square\)

**Lemma 5.** There exist \(\mathbb{Q}\)-ample divisors \(C_1\) and \(C_2\) on \(X''\) such that \(D' = C_1 - C_2\) in \(Z^{(1)}(X'' \otimes \mathbb{Q})\) up to \(\mathbb{Q}\)-linear equivalence.

**Proof.** There exists a \(\mathbb{Q}\)-ample divisor \(C_1\) such that \(C_1 - D'\) is \(\mathbb{Q}\)-ample, say, \(C_2\). \(\square\)

**Lemma 6.** Let \(C\) be an ample divisor \(C\) on \(X''\). Then \(\nu_*C\) is ample.

**Proof.** Since \(C\) is ample, every intersection number \((C, \nu^*\ell_\alpha) > 0\) for any curve \(\ell_\alpha\) on \(Z\) and for a pseudo-curve which is a limit of curves, \((C, \nu^*\ell_\alpha) = (\nu_*C, \ell_\alpha) > 0\) by the projection formula. Hence \(\nu_*C\) is ample. \(\square\)
Lemma 7. There exist a $\mathbb{Q}$-ample divisor $D_1$ and a $\mathbb{Q}$-ample divisor $D_2$ over $Z$ such that $C_1 = \nu^*D_1$ and $\nu^*D_2 = C_2 + Q$-effective divisor up to $Q$-linear equivalence. Furthermore, let $D_1 = p^*A_i + a_iq^*H$, where $A_i$ are $Q$-divisors on $S$ for $i = 1, 2$, $H$ is a hyperplane section of $\textbf{P}^r$, $a_i$ is a rational number $\neq 0$ for $i = 1, 2$. $C_2 = \nu^*D_2 - Q$-effective divisor = $\nu^*p^*A_2 + C_2'$, $C_2' \leq a_2\nu^*q^*H$.

Proof. Since a general fibre of $X/S$ is irreducible and smooth, $\phi^*A_i$ are irreducible. On the other hand, $\phi^*H$ is reducible. $C_2$ is a component of $\nu^*D_2 = \nu^*(p^*A_2 + a_2q^*H)$. Hence $C_2$ is in the form $\nu^*p^*A_2 + C_2'$ and $C_2' \leq a_2\nu^*q^*H$. We refer to the next lemma.

Lemma 8 ([ECA] 4-3). Let $f : X \rightarrow Y$ be a proper flat morphism of finite presentation. The set of $y \in Y$ such that $X_y$ is smooth over $k(y)$ is open.

Since $f : X \rightarrow S$ is a fibre space of non singular varieties, i.e., a projective connected morphism, a general fibre $X_s$ for a closed point $s \in S$ is a non singular variety. Let $\text{Div}_S$ be a scheme representing a functor $T \rightarrow \text{Div}_{S/k}(T)$. Its components are quasi-projective. Let $\Gamma$ be the universal relative effective divisor on $S \times \text{Div}_S/\text{Div}_S$. Note that a fibre of a closed point of $S$ for $\Gamma \subset S \times \text{Div}_S \rightarrow S$ is an effective divisor on $S$. A general fibre of a closed point $s \in S$ for $f^{-1}\Gamma \subset X \times \text{Div}_S \rightarrow S \times \text{Div}_S \rightarrow \text{Div}_S$ is a non singular variety, i.e., smooth and irreducible over $k(s)$.

Note that $\text{Pic}(Z) = \text{Pic}(S) \times \text{Pic}(\textbf{P}^r)$ and that $\text{Pic}(\textbf{P}^r) \cong \mathbb{Z}$. Let $p : Z \rightarrow S$ and $q : Z \rightarrow \textbf{P}^r$. Let $\phi = \nu \circ \mu : X \rightarrow Z$. Now put them together. We have

1. $\omega_{X/S} = \mathcal{O}_X(D - f^*A_0)$
2. $D = \mu^*D' + E$
3. $\nu^*(D_1 - D_2) + a_2\nu^*q^*H \geq D' = C_1 - C_2 \geq \nu^*(D_1 - D_2)$
4. $D_i = p^*A_i + a_iq^*H$, where $A_i$ is a $Q$-divisor on $S$, $H$ is a hyperplane section of $\textbf{P}^r$, $a_i$ is a rational number and $i = 1, 2$.
5. $\omega_{X/S} \supset \mathcal{O}_X(-f^*A_0 + E + \phi^*(p^*A_1 + a_1q^*H - p^*A_2 - a_2q^*H)) = \mathcal{O}_X(E + \phi^*p^*A + a\phi^*q^*H)$, where $A = -A_0 + A_1 - A_2$, $a = a_1 - a_2$.
6. $\mu_*\omega_{X/S}^m \supset \mu_*\mathcal{O}_X(m(E + \phi^*p^*A + a\phi^*q^*H)) = \mu_*\mathcal{O}_X(m(\phi^*p^*A + a\phi^*q^*H))$.

Note that $\mu_*\mathcal{O}_X(mE) = \mu_*\mathcal{O}_X$, $\phi = \nu \circ \mu$. We refer to theory of etale cohomology for the following lemma.
**Theorem 2** (Riemann existence Th.5.1 [SGA] SGA1, Ex.XI [SGA] SGA4 T.3). Let $X$ be a $\mathbb{C}$-scheme locally of finite type, $X^{an}$ the analytic space associated to $X$. The functor $\Psi$ which associates $X^{an}$ to every finite etale covering $X'$ over $X$, is an equivalence of the category of finite etale covering of $X$ onto the category of finite etale covering of $X^{an}$.

**Proposition 1** (Prop4.3 Ex.VII SGA4 t.2 [SGA]). Let $F$ a sheaf in the sense of Zariski topology, $F_{et}$ a sheaf over $X_{et}$ and a homomorphism of cohomological functor

$$H^*(X, F) \rightarrow H^*(X_{et}, F_{et})$$

If $F$ is quasi-coherent, the homomorphism above is an isomorphism.

**Lemma 9.** $\phi_*\mathcal{O}_X \subset \oplus^d\mathcal{O}_Z$

**Proof.** We apply the next lemma to a finite morphism $X''/Z$. We refer to Weierstrass preparation lemma.

**Lemma 10.** Let a convergent series $g \in k\{x_1, \cdots, x_n\}$ such that $g(0, \cdots, x_n) \neq 0$ and order $p$. Then $B = k\{x_1, \cdots, x_n\}/(g)$ is a free module over the ring $A = k\{x_1, \cdots, x_{n-1}\}$ and has a basis of classes $(1, x_n, \cdots, x_{n-1})$ mod $(g)$.

The convergent series rings are strictly henzelian. Hence $X''/Z$ is Kummer gerbe, i.e., cyclic cover, with respect to the etale topology. We have $\nu_*\mathcal{O}_{X''} = \oplus^i_{i=0} \mathcal{O}_Z(-[\frac{i}{d}D])$, where $D$ is an effective divisor with respect to the etale topology. Note, however, that the composition of the morphisms $X'' \rightarrow Z$ and $Z \rightarrow \mathbb{P}^r$ is a connected morphism.

\[\square\]

### 4 Cyclic Cover

Let $L = \mathcal{O}_X(\phi^*q^*(bH))$. Here $b > 0$ is taken sufficiently large. Choose a non singular irreducible divisor $D$ such that $L^{\otimes a} = \mathcal{O}_X(D)$. Take a cyclic cover $Y = \text{Spec} \oplus_{0 \leq i \leq n-1} L^{\otimes i}$ of $X$, which denotes $\tau: Y \rightarrow X$. Let $D = \text{div} a$. Here $a$ is a section of $\mathcal{O}_X(D)$. Let $Y$ be $\text{Spec} \mathcal{O}_X[T]/(T^n - a)$. Note that $Y$ is a non singular variety and $Y/S$ is a fibre space. By adjunction formula, $\mathcal{O}_Y(K_Y) = \mathcal{O}_Y(\tau^*K_X \otimes \tau^*L^{\otimes (n-1)})$ since $K_Y + \tau^*L = \tau^*(K_X + D)$. It is well known $\tau_*\mathcal{O}_Y = \oplus_{0 \leq i \leq n-1} (L^{-1})^{\otimes i}$. Hence by projection formula, $\tau_*\omega_{Y/S}^{\otimes m} = \tau_*\mathcal{O}_Y \otimes \omega_{X/S}^{\otimes m} \otimes L^{\otimes (n-1)}$. Let $g = f \circ \tau$. We obtain $g_*\omega_{Y/S}^{\otimes m} = \oplus_{0 \leq i \leq n-1} f_* (\omega_{X/S}^{\otimes m} \otimes L^{\otimes (n-1-i)})$ and $\det g_*\omega_{Y/S}^{\otimes m} = \oplus_{0 \leq i \leq n-1} \det f_* (\omega_{X/S}^{\otimes m} \otimes L^{\otimes (n-1-i)})$.

**Proposition 2.** $g_*\omega_{Y/S}^{\otimes m} = f_* (\tau_*\mathcal{O}_Y \otimes \omega_{X/S}^{\otimes m} \otimes L^{\otimes (n-1)}) \subset \oplus^d \oplus_{0 \leq i \leq n-1} \mathcal{O}_S^{r_i} \otimes \mathcal{O}_S(mA)$, where $r_i = \dim \Gamma(\mathbb{P}^r, \mathcal{O}((ma_1 + b(m(n-1) - i))H)$.
Lemma 12. Let locally free sheaves of finite rank over \( S \) be an invertible sheaf and \( E \) be a holomorphic period mapping satisfying the Griffiths transversality relation. A period mapping \( \Phi \) gives rise to a variation of Hodge structure by pulling back the universal family over \( \Gamma \setminus D \). Since the generic geometric fibre of \( Y/S \) is of general type, if necessary, the canonical invertible sheaf over the generic geometric fibre of \( Y/S \) is abundant with \( R^i g_* \omega_{Y/S}^{\otimes m} = 0 \) for \( i > 0 \). The composite map \( \tau \circ \nu \circ \mu : Y \rightarrow \mathcal{Z} \) is generically finite and the invertible sheaf \( q^*H \) is relatively ample with respect to \( p : \mathcal{Z} \rightarrow S \).

Lemma 11. \( p_* \mathcal{O}_Z((ma_1 + b(m(n-1) - i))q^*H) = \oplus \mathcal{O}_S^i \), where \( r_i = \dim \Gamma(\mathbb{P}^r, \mathcal{O}((ma_1 + b(m(n-1) - i))H) \).

Proof. From the argument above, we have \( g_* \omega_{Y/S}^{\otimes m} \subset \oplus_{0 \leq i \leq n-1} \mathcal{O}_Z((ma_1 q^*H + mp^*A + b(m(n-1) - i)q^*H)) \), which is an injection into the following sheaf since \( \phi_* \mathcal{O}_X(mE) \) is of codimension \( \geq r \) of all linear automorphism group of \( H \). Therefore, \( \dim \Gamma(\mathbb{P}^r, \mathcal{O}((ma_1 + b(m(n-1) - i))H) \).

We may assume that if \( b \) is taken sufficiently large, the generic geometric fibre of \( Y/S \) is of general type, if necessary, the canonical invertible sheaf over the generic geometric fibre of \( Y/S \) is abundant with \( R^i g_* \omega_{Y/S}^{\otimes m} = 0 \) for \( i > 0 \). The composite map \( \tau \circ \nu \circ \mu : Y \rightarrow \mathcal{Z} \) is generically finite and the invertible sheaf \( q^*H \) is relatively ample with respect to \( p : \mathcal{Z} \rightarrow S \).

Proposition 3. \( \kappa(\mathcal{O}_S([mA])) \geq \kappa(\det g_* \omega_{Y/S}^{\otimes m}) \)

Proof. Apply the lemma above to the following formula, \( g_* \omega_{Y/S}^{\otimes m} \subset \oplus \mathcal{O}_S^i \otimes \mathcal{O}_S(mA) \), where \( r_i = \dim \Gamma(\mathbb{P}^r, \mathcal{O}((ma_1 + b(m(n-1) - i))H) \).

Consider the case when \( m = 1 \). Let \( D \) be a classifying space for a variation of Hodge structure and let \( \Gamma \) be the monodromy group, which is a subgroup of the arithmetic group of all linear automorphism group of \( H^{\dim X_s}(X_s, \mathbb{C}) \) which preserve a certain condition. Let \( \Phi : S \rightarrow \Gamma \setminus D \) be a holomorphic period mapping satisfying the Griffiths transversality relation. A period mapping \( \Phi \) gives rise to a variation of Hodge structure by pulling back the universal family over \( \Gamma \setminus D \). Since the generic geometric fibre of \( Y/S \) is of general type, if necessary, the canonical invertible sheaf over the generic geometric fibre of \( Y/S \) is abundant with \( R^i g_* \omega_{Y/S}^{\otimes m} = 0 \) for \( i > 0 \). The composite map \( \tau \circ \nu \circ \mu : Y \rightarrow \mathcal{Z} \) is generically finite and the invertible sheaf \( q^*H \) is relatively ample with respect to \( p : \mathcal{Z} \rightarrow S \).
type and \( \text{var}(Y/S) \geq \dim S \), the period mapping \( \Phi \) is a finite to one mapping. Hence we obtain \( \kappa(A) = \dim S \).

Kawamata proved the next theorem under the condition that the generic geometric fibre has the abundant canonical invertible sheaf and Kollár proved it when the generic geometric fibre is of general type.

**Lemma 13.** \( \kappa(\det g_*\omega_{Y/S}^{\otimes m}) \geq \text{var}(Y/S) \)

**Lemma 14.** Given the exact sequence \( 0 \to E' \to E \to E'' \to 0 \). If \( E \) is weakly positive and if \( \det E' \) is big, then \( \det E \) is big.

**Proof.** Since the quotient of a weakly positive sheaf is weakly positive, the exact sequence \( 0 \to E' \to E \to E'' \to 0 \), where \( E \) is weakly positive and \( \det E' \) is big, gives the conclusion that \( \det E = \det E' \otimes \det E'' \) is big.

**Proposition 4.** If \( \text{var}(Y/S) \geq \text{var}(X/S) = \dim S \), \( \max_{m>0} \kappa(\det f_*\omega_{X/S}^{\otimes m}) \geq \dim S \).

**Proof.** There exists an exact sequence \( 0 \to \mathcal{O}_S([mA]) \to f_*\omega_{X/S}^{\otimes m} \) over \( S \). Take the dual to get the homomorphism \( (f_*\omega_{X/S}^{\otimes m})^* \to \mathcal{O}_S([-mA]) \). Let the image denote \( F \) and let the kernel be \( K \). They and \( f_*\omega_{X/S}^{\otimes m} \) are torsion free and hence there exists an open subset \( S^0 \) such that \( \dim S - \dim(S \setminus S^0) \geq 2 \) and that \( K, F \) and \( f_*\omega_{X/S}^{\otimes m} \) are locally free. Note that \( F \subset \mathcal{O}_S([-mA]) \) and so \( \mathcal{O}_S([mA]) \to F^* \) is a non zero injective map to a torsion free of rank one and that \( \mathcal{O}_S([mA]) \) is big for infinitely many \( m \) since there exist infinitely many \( m \) such that \( [mA] = mA = [mA] \).

Hence we have the exact sequence \( 0 \to K \to (f_*\omega_{X/S}^{\otimes m})^* \to F \to 0 \) of locally free sheaves of finite rank over \( S^0 \). Thus we have the exact sequence \( 0 \to F^* \to f_*\omega_{X/S}^{\otimes m} \to K^* \to 0 \) of locally free sheaves of finite rank over \( S^0 \). Let \( E' = F^* \), \( E = f_*\omega_{X/S}^{\otimes m} \) and \( E'' = K^* \). Consider sheaves over \( S^0 \). Since the quotient of a weakly positive sheaf is weakly positive, the exact sequence \( 0 \to E' \to E \to E'' \to 0 \), where \( E \) is weakly positive and \( \det E' \) is big, gives the conclusion that \( \det E = \det E' \otimes \det E'' \) is big.

Recall \( \mathcal{O}_S([mA]) \subset f_*\omega_{X/S}^{\otimes m} \), we have \( \det f_*\omega_{X/S}^{\otimes m} = \mathcal{O}_S([mA]) \otimes \det G \), where \( G \) is the cokernel of the monomorphism \( \mathcal{O}_S([mA]) \subset f_*\omega_{X/S}^{\otimes m} \). \( \det G \) is weakly positive and \( \mathcal{O}_S([mA]) \) is big for infinitely many \( m \). Therefore \( \mathcal{O}_S([mA]) \otimes \det G \) is big for infinitely many \( m \). Therefore \( \max_{m>0} \kappa(\det f_*\omega_{X/S}^{\otimes m}) = \dim S \).

5 Mochizuki’s Galois theory

Let \( k \) be an algebraically closed field of characteristic 0, say, the complex number field. We investigate the birational algebraic geometry from the point of view of the profinite Galois...
groups thanks to Mochizuki theory. Let $X \to S$ be a fibre space of smooth algebraic spaces over $k$. Let $\text{Spec}(k(\eta))$ denote the generic point of the fibre space and $k(\eta)$ the algebraic closure of $k(\eta)$. The absolute Galois group of $R(X)$ is defined to be the Galois group with Kull topology of the Galois extension $R(X)/R(X)$, which denotes $\Gamma_X = \text{Gal}(\overline{R(X)}/R(X))$. This is a profinite group.

**Theorem 3.** [Mch] Let $p$ be a prime number. Let $K$ be a subfield of a finitely generated field extension of $\mathbb{Q}_p$. Let $X_K$ be a smooth pro-variety over $K$ and $Y_K$ a hyperbolic pro-curve over $K$. Let $\text{Hom}^{\text{dom}}(X_K, Y_K)$ be the set of dominant $K$-morphisms from $X_K$ to $Y_K$ and $\text{Hom}^{\text{open}}_K(\Pi_{X_K}, \Pi_{Y_K})$ the set of open continuous group homomorphisms $\Pi_{X_K} \to \Pi_{Y_K}$ over $\Gamma_K$, modulo up to inner automorphisms arising from $\Delta_{Y_K}$. Then the natural map
\[
\text{Hom}^{\text{dom}}_K(X_K, Y_K) \to \text{Hom}^{\text{open}}_K(\Pi_{X_K}, \Pi_{Y_K})
\]
is bijective.

Here we have a natural homomorphism $\pi_1(X_K) \to \Gamma_K$. Let $\Delta_{X_K}$ be the maximal pro-$p$ quotient of the geometric fundamental group $\pi_1(X_K)$. Let $\Pi_{X_K} = \pi_1(X_K)/\ker(\pi_1(X_K) \to \Delta_{X_K})$.

**Theorem 4.** [Mch] Let $p$ be a prime number. Let $K$ be a subfield of a finitely generated field extension of $\mathbb{Q}_p$. Let $L, M$ be function fields of arbitrary dimension over $K$. Let $\text{Hom}_{\text{Spec}(K)}(\text{Spec}(L), \text{Spec}(M))$ be the set of $K$-morphisms from $M$ to $L$. Let $\text{Hom}^{\text{open}}_K(\Gamma_L, \Gamma_M)$ over $\Gamma_K$, considered up to composition with an inner automorphism arising from $\ker(\Gamma_M, \Gamma_K)$, where $\Gamma_L$ and $\Gamma_M$ are the absolute Galois groups of $L$ and $M$, respectively. Then the natural map $\text{Hom}_K(\text{Spec}(L), \text{Spec}(M)) \to \text{Hom}^{\text{open}}_K(\Gamma_L, \Gamma_M)$ is bijective.

**Theorem 5** ([SGA] SGA1 IX Th.6.1). Let $S$ be the spectre of an Artinian ring $A$ with the residue field $k$ and $\bar{k}$ an algebraic closure of $k$, $X$ an $S$ scheme, $X_0 = X \otimes_A k$, $\bar{X}_0 = X \otimes_A \bar{k}$, $\bar{a}$ a geometric point of $X$, $a$ its image in $X$ and $b$ its image in $S$. Suppose that $X_0$ is quasi-compact and geometrically connected over $k$. Then the sequence of canonical homomorphisms
\[
1 \to \pi_1(\bar{X}_0, \bar{a}) \to \pi_1(X, a) \to \pi_1(S, b) \to 1
\]
is exact and
\[
\pi_1(S, b) \cong \pi_1(k, \bar{k}) = \text{Gal}(\bar{k}/k)
\]

We in fact use the following homotopy exact sequence.
Proposition 5 ([TAM] Prop.5.6.1, [SGA] SGA1, [GG] Lemma 5 182-20). Let $X$ be a quasi-compact and geometrically irreducible and connected scheme over a field $k$. Fix an algebraic closure $\bar{k}$ of $k$ and let $k_s$ the separable closure of $k$ in $\bar{k}$. Let $\bar{X} = X \otimes_k \bar{k}$, $\bar{x}$ a geometric point of $\bar{X}$ with values in $\bar{k}$. The sequence of profinite groups

$$1 \rightarrow \pi_1(\bar{X}, \bar{x}) \rightarrow \pi_1(X, \bar{x}) \rightarrow \text{Gal}(k_s/k) \rightarrow 1$$

is exact.

Proposition 6. Let $k$ be an algebraically closed field of characteristic 0, $k(x)$ and $k(y)$ extension fields of $k$. Suppose $k(x) \otimes_k k(y) \rightarrow k(x, y)$ is injective. Then

$$\pi_1(\text{Spec}k(x, y), (\bar{x}, \bar{y})) \rightarrow \pi_1(\text{Spec}k(x) \otimes_k k(y), (\bar{x}, \bar{y}))$$

is an isomorphism. Furthermore we have an isomorphism:

$$\pi_1(\text{Spec}k(x, y), (\bar{x}, \bar{y})) \rightarrow \pi_1(\text{Spec}k(x), \bar{x}) \times \pi_1(\text{Spec}k(y), \bar{y})$$

Proof. For an inclusion $k(x) \otimes_k k(y) \subset k(x, y)$ there exists a canonical dominant map: $\text{Spec}k(x, y) \rightarrow \text{Spec}k(x) \otimes_k k(y)$ and a group homomorphism: $\pi_1(\text{Spec}k(x) \otimes_k k(y), (\bar{x}, \bar{y})) \rightarrow \pi_1(\text{Spec}k(x) \otimes_k k(y), (\bar{x}, \bar{y}))$. By universality of product there exists a homomorphism of groups $\pi_1(\text{Spec}k(x) \otimes_k k(y), (\bar{x}, \bar{y})) \rightarrow \pi_1(\text{Spec}k(x), \bar{x}) \times \pi_1(\text{Spec}k(y), \bar{y})$ since we have $\pi_1(\text{Spec}k(x) \otimes_k k(y), (\bar{x}, \bar{y})) \rightarrow \pi_1(\text{Spec}k(x), \bar{x})$ and $\pi_1(\text{Spec}k(x) \otimes_k k(y), (\bar{x}, \bar{y})) \rightarrow \pi_1(\text{Spec}k(y), \bar{y})$.

We have the following commutative diagram of three exact sequences.

\begin{align*}
1 & \rightarrow \pi_1(\text{Spec}k(x), \bar{x}) \rightarrow \pi_1(\text{Spec}k(x, y), (\bar{x}, \bar{y})) \rightarrow \pi_1(\text{Spec}k(y), \bar{y}) \rightarrow 1 \\
1 & \rightarrow \pi_1(\text{Spec}k(x), \bar{x}) \rightarrow \pi_1(\text{Spec}k(x) \otimes_k k(y), (\bar{x}, \bar{y})) \rightarrow \pi_1(\text{Spec}k(y), \bar{y}) \rightarrow 1 \\
1 & \rightarrow \pi_1(\text{Spec}k(x), \bar{x}) \rightarrow \pi_1(\text{Spec}k(x), \bar{x}) \times \pi_1(\text{Spec}k(y), \bar{y}) \rightarrow \pi_1(\text{Spec}k(y), \bar{y}) \rightarrow 1
\end{align*}

Four arrows of both sides of exact sequence are isomorphisms. There are two arrows among profinite groups in the center of exact sequences. Hence they are isomorphisms.

\[\square\]

Proposition 7 ([Mch]). Let $L$ be a sub-$p$-adic field and a function field. The absolute Galois group $\Gamma_L$ is center-free.

Proposition 8. Let $\bar{K}$ be an algebraically closed field of characteristic 0 and $L$ a function field. The absolute Galois group $\Gamma_L$ is center-free.
Proof. Let $\bar{K}$ be an algebraically closed field. It is known by Douady that the absolute Galois group of the function field of $\mathbb{P}^1_\mathbb{C}$ is a free profinite group. Every open subgroup of a free profinite group is also free ([EL]). A free profinite group is center-free. Let $X$ be a variety associated to $L$. $X$ is considered to be a fibre space over a projective space $P$ with a general fibre a curve. Let $M$ be the absolute Galois group of the function field of $P$. Note that the absolute Galois group $M$ is center-free since a projective space is birationally equivalent to a product of projective lines. A natural group homomorphism $h : L \to M$ is surjective The image of the center of $L$ by $h$ is contained in the center of $M$, which is an identity. Hence it is contained in $\ker h$, which is the absolute Galois group of the general generic fibre of $X/P$. An absolute Galois group of a curve is open subgroup of the absolute Galois group of the function field of a projective line, which is center-free. Thus $\ker h$ is center-free. Therefore $L$ is center-free. 

Let $p$ be a prime number. Let $K$ be a subfield of a finitely generated field extension of $\mathbb{Q}_p$. It is called a sub-$p$-adic field. Note that there exists an isomorphism $\iota : \bar{K} \cong \mathbb{C}$ when $K$ is uncountable.

Let $X_\bar{\eta}$ be the geometric generic fibre of $X/S$. Then there exists a variety $F_{K_0}$ and a finitely generated extension field $K_0$ of $\mathbb{Q}$ such that $F_{K_0} \times_{K_0} \mathbb{C} \cong X_\bar{\eta}$.

Note that $\text{Bir}_\mathbb{C}(X_\bar{\eta}) = \text{Bir}_{\mathbb{Q}_p}(F_{K_0} \otimes_{K_0} \mathbb{Q}_p)$.

Let $\pi : \Gamma_{F_K} \to \Gamma_K$ denote the structure map associated to $\text{Spec} K(F_K) \to F_K \to \text{Spec}(K)$, which is a surjection since $K$ is algebraically closed in the rational function field of $F$. Let $Z(\Gamma_{F_K})$ denote the centre of $\Gamma_{F_K}$. Then $\pi$ induces $\pi : Z(\Gamma_{F_K}) \to Z(\Gamma_K)$.

Lemma 15. Let $K$ be a field of characteristic 0. Let $A$ be an algebraic space in group locally of finite type over $K$ (i.e. with at most countable components) and let $\rho : \Gamma_{S_K} \to A$ be a continuous homomorphism as topological groups. Then

1. The image of this homomorphism $\rho$ is a finite group.

2. Let $P = \Gamma_{S_K}$. There exist a variety $S'_K$ which is generically finite over $S_K$ and an injective homomorphism $P' \to P$ with $\langle P' : P \rangle < \infty$ such that the representation $\rho' : P' \to A$ is trivial. Here $P'$ denotes the absolute Galois group $\Gamma_{S'_K} = \text{Gal}(K(S_K)/K(S'_K))$.

Proof. An algebraic space in group $A$ is locally of finite type over $K$. The representation $\rho : P \to A$ induces $\overline{\rho} : P \to A/A^0$, where $A^0$ denotes the neutral component of $A$. Note that there is no countable profinite group. Since $A/A^0$ is a countable set, $\overline{\rho}(P)$ is a finite group. Replace by $P$ the kernel of $\overline{\rho}$. We have $\rho : P \to A^0$. We have an isomorphism

$$H^1(K(S_K)/K(S_K), A^0(K(S_K))) \cong H^1(P, A^0) \cong \text{Hom}_{\text{Continuous Topological group}}(P, A^0)$$
Let $Q$ be an $A^0$-torsor over $\text{Spec} K(S_K)$ associated to $\rho : P \to A^0$. $A^0$ is algebraic (quasi-compact, faithfully flat and of finite type) over $\text{Spec} K(S_K)$. There exists an isomorphism $A^0 \times Q \to Q \times Q$ over $\text{Spec} K(S_K)$. Thus along $\text{Spec} K(Q) \to Q \to \text{Spec} K(S_K)$, the pull-back of $A^0$-torsor $Q$ becomes trivial. Namely, the $A^0$-torsor $Q$ is trivial over $\text{Spec} K(S_K')$. Hence the induced homomorphism $\text{Gal}(K(Q)/K(Q)) \to A^0$ is trivial. Let $S_K' \to S_K$ be dominant and $S_K'$ the subvariety of $Q$ of the same dimension as $S_K$. Then $\Gamma_Q \to \Gamma_{S_K'}$ is a surjective homomorphism. We have

$$\Gamma_Q \to \Gamma_{S_K'} \subset \Gamma_{S_K} \to A^0 \to A.$$ 

Since $\Gamma_Q \to A^0$ is trivial, i.e., $\Gamma_Q \to 1$, $\Gamma_{S_K'} \to A$ is trivial. It is obvious that $(\Gamma_{S_K} : \Gamma_{S_K'}) < \infty$. Hence $\text{im}(\rho)$ is a finite group. □

Note that a quotient of a scheme by a finite group is in the category of algebraic spaces.

**Proposition 9.** Let $X/S$ be a fibre space. Let $1 \to G \to E \to P \to 1$ be an extension of a profinite group $P$ by a profinite group $G$ associated to a fibre space $X \to S$. Namely $G$, $E$ and $P$ are profinite groups which are the absolute Galois groups associated to the rational function fields of the generic geometric fibre $X_{\bar{\eta}}$, $X$ and $S$, respectively.

**Proof.** Let $X/S$ be a fibre space with the generic geometric fibre $X_{\bar{\eta}}$. To a fibre space the epimorphism $\Gamma_{R(X)} \to \Gamma_{R(S)}$ is associated. Consider a morphism $\text{Spec}(R(X)) \to \text{Spec}(R(S))$ with the generic geometric fibre $\text{Spec}(R(X))$. Grothendieck’s algebraic $\pi_1$ in SGA1 [1SGA1] gives the exact sequence: $1 \to \pi_1(\text{Spec}(R(S))) \to \pi_1(\text{Spec}(R(X)))$. Grothendieck’s algebraic $\pi_1$ in SGA1 [1SGA1] gives the exact sequence: $1 \to \pi_1(\text{Spec}(S)) \to \pi_1(\text{Spec}(R(X))) \to \pi_1(\text{Spec}(R(S)))$. Here $\pi_1(\text{Spec}(R(X)))$, $\pi_1(\text{Spec}(R(X)))$ and $\pi_1(\text{Spec}(R(S)))$ are the absolute Galois groups $G$, $E$ and $P$ themselves, respectively. See the diagram:

$$
\begin{array}{ccc}
X & \longrightarrow & X_{\bar{\eta}} \\
\downarrow & & \downarrow \\
S & \longrightarrow & \text{Spec}(\bar{k}(\bar{\eta})) \\
\end{array}
\quad
\begin{array}{ccc}
\Gamma_{R(X)} & \longrightarrow & \Gamma_{R(X_{\bar{\eta}})} \\
\downarrow & & \downarrow \\
\Gamma_{R(S)} & \longrightarrow & 1 = \Gamma_{k(\bar{\eta})} \\
\end{array}
$$

Thus to a fibre space $X/S$ the extension of a profinite groups $1 \to G \to E \to P \to 1$ is associated. □

We make use of theory of Schreier’s classification of group extensions, Grothendieck-Giraud’s classification of topos extensions or Breen’s classification of 2-gerbes and 2-stacks. ([AM], [Gir], [Breen1], [Breen2]) Here we take the notion of Breen’s.
Definition 4. An extension of groups $1 \to G \to E \to P \to 1$ is said to be neutral if it has a section which is a group homomorphism $\sigma : P \to E$.

An extension of groups $1 \to G \to E \to P \to 1$ is said to be central if $G$ is contained in the center of $E$.

$E$ is said to be a semi-direct product of $G$ and $P$ if $G$ is a normal subgroup of $E$ and if the multiplication $(x, y), (u, v) \in G \times P$ is defined by $(xu^y, yv)$, where $u^y = \sigma(y)u\sigma(y)^{-1}$. $E$ is denoted by $G \rtimes P$.

$\text{Inn}(G)$ denotes the inner automorphism group of $G$. $E \to \text{Aut}(G)$ denotes the natural homomorphism $x \in E \mapsto (g \mapsto g^x) \in \text{Aut}(G)$. $\text{Out}(G)$ is defined to be $\text{Aut}(G)/\text{Inn}(G)$. This induces a homomorphism $E/G \to \text{Aut}(G)/\text{Inn}(G)$, i.e., $P \to \text{Out}(G)$.

We call left crossed module a homomorphism of groups $\delta : G \to H$, equipped with a left action of $H$ onto $G$ $(h, g) \mapsto h^g([\text{Breen1}])$:

1. $\delta(h^g) = h\delta(g)h^{-1}$
2. $\delta(g')g = g'gg'^{-1}$

\[
\begin{array}{ccc}
G & \xrightarrow{\delta} & H \\
\downarrow{i} & & \downarrow{	ext{Aut}(G)} \\
& \text{Aut}(G) & 
\end{array}
\]

$i : G \to \text{Aut}(G)$, where $g \mapsto i_g(x \mapsto gxg^{-1})$, and the natural action $\text{Aut}(G)$ onto $G$ defines a crossed module, which denotes $G \to \text{Aut}(G)$.

To an exact sequence $1 \to \text{Inn}G \to \text{Aut}G \to \text{Out}G \to 1$, we have an exact sequence

$H^1(P, \text{Inn}G) \to H^1(P, \text{Aut}G) \to H^1(P, \text{Out}G),$ 

i.e.,

$\text{Hom}(P, \text{Inn}G) \to \text{Hom}(P, \text{Aut}G) \to \text{Hom}(P, \text{Out}G).$

Here $\text{Out}G$ denotes the outer automorphism group of $G$. Let $G \to \text{Aut}G$ denote the crossed module. The set of the extensions of a profinite group $P$ by a profinite group $G$ denotes $\text{Ext}(P, G)$. A group extension can be defined to be as an element of $H^1(P, (G \to \text{Aut}G))$. There exists an exact sequence $1 \to Z(G)[1] \to (G \to \text{Aut}(G)) \to \text{Out}G \to 1([\text{Breen2}])$. We have the exact sequence of cohomologies([\text{Breen2}]):

$0 \to H^2(P, Z(G)) \to H^1(P, (G \to \text{Aut}G)) \to H^1(P, \text{Out}G).$
Here \( Z(G) \) denotes the center of \( G \). There exists another sequence \( Aut(G) \to (G \to Aut(G)) \to G[1] \) in the homotopy category. See the next commutative diagram:

\[
\begin{array}{ccc}
H^1(P, Inn(G)) & \to & H^1(P, Aut(G)) \\
\downarrow & & \downarrow \\
H^1(P, Aut(G)) & \to & H^1(P, G \to Aut(G)) \\
\downarrow & & \downarrow \\
H^1(P, Out(G)) & \to & H^3(P, Z(G)) \\
\end{array}
\]

This vertical sequence is nothing but the following exact sequence

\[
\text{Ext}(P, Z(G)) \to \text{Ext}(P, G) \to \text{Hom}(P, \text{Out}(G)) \to H^3(P, Z(G))
\]

When \( Z(G) = 0 \), an extension \( 1 \to G \to E \to P \to 1 \) is determined uniquely by a continuous group homomorphism \( \phi : P \to \text{Out}(G) \) pulling back the exact sequence \( 1 \to G \to \text{Aut}(G) \to \text{Out}(G) \to 1 \) (cf. 22 Th. 4.8 ([AM]). Hence

\[
H^1(P, G \to Aut(G)) \cong H^1(P, Out(G)) \cong \text{Hom}(P, G)
\]

In general we refer to the following theorem.

**Theorem 6** ([Gir], [Breen1], [Breen2], [Rou], [AM]). *There are equivalent expressions of extensions of \( \Gamma_{S_K} \) by \( G \).*

1. \( \text{BiTors}(G) \cong \text{Eq}(G[1]) \cong (G \to \text{Aut}(G)) \) as monoidal categories.

2. \( H^1(\Gamma_{S_K}, \text{BiTors}(G)) \cong H^1(\Gamma_{S_K}, \text{Eq}(G[1])) \cong H^1(\Gamma_{S_K}, (G \to \text{Aut}(G))) \) as pointed sets

3. \( \text{Mon}(\Gamma_{S_K}, \text{BiTors}(G)) \cong \text{Mon}(\Gamma_{S_K}, \text{Eq}(G[1])) \cong \text{Mon}(\Gamma_{S_K}, (G \to \text{Aut}(G))) \) as morphisms of monoidal categories.
Proposition 10. Let $K$ be a field of characteristic 0. Let $K(x)$ and $K(y)$ be function fields over $K$. Assume $K(x) \otimes_K K(y) \to K(x, y)$ is an inclusion of $K$-algebras. Then $\pi_1(K(x, y), (\bar{x}, \bar{y})) \to \pi_1(K(x), \bar{x}) \times_{\pi_1(K, K)} \pi_1(K(y), \bar{y})$ is isomorphic.

Proof. Let $\bar{K}$ be an algebraic closure in an algebraically closed field which is an extension of $K$. Let $G = \pi_1(\bar{K}(x), \bar{x})$, $\Gamma_K = \pi_1(K, K)$. The following diagram in two vertical extensions in the right-side is a base-change of extensions:

\[
\begin{array}{ccc}
1 & \rightarrow & \pi_1(\bar{K}(x), \bar{x}) \\
\downarrow & & \downarrow \\
G & \leftarrow & \pi_1(K(x, y), (\bar{x}, \bar{y}))
\end{array}
\]

There exists an exact sequence: $1 \to \pi_1(\bar{K}(y), \bar{y}) \to \pi_1(K(y), \bar{y}) \to \pi_1(K, \bar{K}) \to 1$.

Consider the following exact sequence of extensions and an element $\xi$ associated to the central vertical extension in the diagram above:

\[
1 \to \text{Mon}_{\Gamma_K}(\Gamma_K, \text{Eq}(G[1])) \to \text{Mon}_{\Gamma_K}(\pi_1(K(y), \bar{y}), \text{Eq}(G[1])) \to \text{Mon}_{\pi_1}(\pi_1(\bar{K}(y), \bar{y}), \text{Eq}(G[1]))
\]

The image of $\xi$ in $\text{Mon}_{\pi_1}(\pi_1(\bar{K}(y), \bar{y}), \text{Eq}(G[1]))$ is a trivial. Hence there exists an extension $\xi_0$ in $\text{Mon}_{\Gamma_K}(\Gamma_K, \text{Eq}(G[1]))$ whose image is $\xi$ in $\text{Mon}_{\Gamma_K}(\pi_1(K(y), \bar{y}), \text{Eq}(G[1]))$. Then $\xi_0$ corresponds to a vertical extension in the left-side. Thus $\xi$ is $\pi_1(K(x), \bar{x})$. Therefore

\[
\pi_1(K(x, y), (\bar{x}, \bar{y})) \cong \pi_1(K(x), \bar{x}) \times_{\pi_1(K, K)} \pi_1(K(y), \bar{y})
\]

\[
\square
\]

From here in several steps we shall prove that when there exists an $S$-dominant rational map $Y \to X$ and when $\text{var}(Y/S) = 0$, then $\text{var}(X/S) = 0$ if the general generic fibre of $X/S$ is of Kodaira dimension $\geq 0$.

Proposition 11 ([EGA]). Let $S$ be a scheme, $(X_\lambda, v_{\lambda\mu})$ a filtered projective system of $S$-schemes; assume that there exists $\alpha$ such that $v_{\alpha\lambda}$ is an affine morphism for every $\lambda \geq \alpha$, so that the projective limit $X = \varprojlim X_\lambda$ exists in the category of $S$-schemes. Let $Y$ be an $S$-scheme and for every $\lambda \geq \alpha$ let $e_\lambda : \text{Hom}_S(X_\lambda, Y) \to \text{Hom}_S(X, Y)$ the map which
gives \( f = f_\lambda \circ v_\lambda \) to each \( S \)-morphism \( f_\lambda : X_\lambda \to Y \), where \( v_\lambda : X \to X_\lambda \) is the canonical morphism. The family \((e_\lambda)\) is an inductive system of maps, which defines the canonical map

\[
\lim \text{Hom}_S(X_\lambda, Y) \to \text{Hom}_S(X, Y).
\]

Suppose that \( X_\alpha \) is quasi-compact and quasi-separated and that the structure morphism \( Y \to S \) is locally of finite presentation (resp. locally of finite type). Then the map above is bijective (resp. injective). Furthermore, suppose that \( \lim Y_\rho \), where \((Y_\rho, t_{\rho\sigma})\) is a filtered projective system of \( S \)-schemes such that the structure morphism is locally of finite presentation for every \( \rho \). One has a canonical bijection

\[
\text{Hom}_S(X, Y) \cong \lim_{\rho} \text{Hom}_S(X_\lambda, Y_\rho).
\]

**Proposition 12.** Let \( K \) be a sub-\( p \)-adic field and \( X/S \) a fibre space of varieties over \( K \). Let \( S_\lambda \) be a filtered projective system of \( K \)-varieties such that

1. \( K(S_\lambda)/K(S) \) is a normal extension for every \( \lambda \)
2. the structure homomorphism \( \Gamma_{K(S_\lambda)} \to \Gamma_K \) is surjective.

Let \( \Gamma_S = \ker(\Gamma_S \to \Gamma_K) \) and \( \Gamma_X = \ker(\Gamma_X \to \Gamma_K) \). Suppose that there exists a sectional homomorphism \( \Gamma_S \to \Gamma_X \subset \Gamma_X \) so that one has a homomorphism \( \Gamma_S \to \text{Aut}_{\Gamma_K}(\Gamma_X) \) and \( \Gamma_S \to \text{Aut}_{\Gamma_K}(\Gamma_{X_\lambda}) \).

Let \( B(S) = \text{im}(\Gamma_S \to \lim \text{Hom}_{\Gamma_K}^\text{open}(\Gamma_{X_\lambda}, \Gamma_{X_\mu})/\Gamma_{X_\mu}) \). One has a canonical Mochizuki bijection:

\[
\text{lim}_{\rho} \text{Hom}_{\Gamma_K}^\text{open}(\Gamma_{X_\lambda}, \Gamma_{X_\mu})/\Gamma_{X_\mu} \cong \text{lim}_{\rho} \text{Mor}_{\Gamma_K}(\text{Spec}(K(X_\lambda), \text{Spec}(K(X_\mu)))).
\]

By the precedent proposition, one obtains

\[
\text{lim}_{\rho} \text{Mor}_{\Gamma_K}(\text{Spec}(K(X_\lambda), \text{Spec}(K(X_\mu)))) \subset \text{Mor}_K(\lim \text{Spec} K(X_\lambda), \lim \text{Spec} K(X_\mu)).
\]

One also gets canonical homomorphisms of topological groups:

\[
\Gamma_S \to B(S) \subset \text{Aut}_K(\lim \text{Spec} K(X_\lambda)) \subset \text{Bir}_K(X_\eta).
\]

In the other way, one obtains a canonical homomorphism

\[
\Gamma_S \to B(S) \to \text{Aut}_K(\lim \Gamma_{X_\lambda}, \lim \Gamma_{X_\mu})/\lim \Gamma_{X_\mu} \to \text{Out}(\Gamma_{X_K}).
\]

In particular, if \( \Gamma_S \to \text{Bir}_K(\bar{K}(X_\eta)) \) is a trivial homomorphism, then so is \( \Gamma_S \to \text{Out}(\Gamma_{X_K}) \).
Proof. Since $S_\lambda$ be a filtered projective system of $K$-varieties such that

1. $K(S_\lambda)/K(S)$ is a normal extension for every $\lambda$

2. for every $\mu \geq \lambda$ $K(S_\mu)/K(S_\lambda)$ is a normal extension

3. the structure homomorphism $\Gamma_K(S_\lambda) \to \Gamma_K$ is surjective,

and by assumption there exists a sectional homomorphism $\Gamma_S \to \Gamma_S \subset \Gamma_X$, one has a homomorphism $\Gamma_S \to \text{Aut}_{\Gamma_K}(\Gamma_X)$ and $\Gamma_S \to \text{Aut}_{\Gamma_K}(\Gamma_X)$.

By Mochizuki’s theorem, one has

$$\lim_{\lambda \to} \text{Hom}_{\Gamma_K}^{\text{open}}(\Gamma_{X_\lambda}, \Gamma_{X_\mu})/\Gamma_{X_\mu} \cong \lim_{\lambda \to} \text{Mor}_{\Gamma_K}^{\text{dom}}(\text{Spec}(K(X_\lambda), \text{Spec}(K(X_\mu))).$$

Hence

$$\lim_{\mu \to} \text{Hom}_{\Gamma_K}^{\text{open}}(\Gamma_{X_\lambda}, \Gamma_{X_\mu})/\Gamma_{X_\mu} \cong \lim_{\mu \to} \text{Mor}_{\Gamma_K}^{\text{dom}}(\text{Spec}(K(X_\lambda), \text{Spec}(K(X_\mu))).$$

Owing to Mochizuki’s bijection and the precedent proposition in EGA,

$$\Gamma_S \to \lim_{\mu \to} \text{Mor}_{\Gamma_K}^{\text{dom}}(\text{Spec}(K(X_\lambda), \text{Spec}(K(X_\mu)))) \subset \text{Mor}_K(\text{Spec} K(X_\lambda), \text{Spec} K(X_\mu))$$

Therefore one gets non trivial canonical homomorphisms of topological groups:

$$\Gamma_S \to \text{Aut}_K(\text{Spec} K(X_\lambda)).$$

Note that

$$\text{Aut}_K(\text{Spec} K(X_\lambda)) \subset \text{Aut}_K(\text{Spec} \tilde{K}(X_\eta)) = \text{Bir}_K(X_\eta).$$

Secondly, we show that there exists a canonical homomorphism

$$\text{Aut}_{\Gamma_K}(\lim_{\lambda \to} \Gamma_{X_\lambda}, \lim_{\mu \to} \Gamma_{X_\mu})/\lim_{\mu \to} \Gamma_{X_\mu} \to \text{Out}(\Gamma_{X_K}).$$

Since $\lim_{\lambda \to} \Gamma_{X_\lambda} \to \Gamma_{X_\mu}$, one has a canonical map

$$\text{Hom}_{\Gamma_K}(\Gamma_{X_\lambda}, \Gamma_{X_\mu}) \to \text{Hom}_{\Gamma_K}(\lim_{\lambda \to} \Gamma_{X_\lambda}, \Gamma_{X_\mu}).$$

It follows that

$$\lim_{\mu \to} \text{Hom}_{\Gamma_K}^{\text{open}}(\lim_{\lambda \to} \Gamma_{X_\lambda}, \Gamma_{X_\mu})/\Gamma_{X_\mu} \to \lim_{\mu \to} \text{Hom}_{\Gamma_K}^{\text{open}}(\Gamma_{X_\lambda}, \Gamma_{X_\mu})/\Gamma_{X_\mu}$$

and from the definition of the projective limit

$$\lim_{\mu \to} \text{Hom}_{\Gamma_K}^{\text{open}}(\lim_{\lambda \to} \Gamma_{X_\lambda}, \Gamma_{X_\mu})/\Gamma_{X_\mu} \cong \text{Hom}_{\Gamma_K}^{\text{open}}(\lim_{\lambda \to} \Gamma_{X_\lambda}, \lim_{\mu \to} \Gamma_{X_\mu})/\lim_{\mu \to} \Gamma_{X_\mu}$$
Since
\[ \lim_{\mu} \Gamma_{X, \mu} \leftarrow \Gamma_{X, K} \]
one has a canonical map
\[ \text{Hom}_{\Gamma_K}^{\text{open}}(\lim_{\lambda} \Gamma_{X, \lambda}, \lim_{\mu} \Gamma_{X, \mu})/\lim_{\mu} \Gamma_{X, \mu} \rightarrow \text{Out}(\Gamma_{X, K}) \]
and a canonical homomorphism
\[ \text{Aut}_{\Gamma_K}(\lim_{\lambda} \Gamma_{X, \lambda}, \lim_{\mu} \Gamma_{X, \mu})/\lim_{\mu} \Gamma_{X, \mu} \rightarrow \text{Out}(\Gamma_{X, K}) \].

We consider the following diagrams:

\[ Y_K \times_K S_K \rightarrow X_K \rightarrow S_K \]
\[ \text{Spec } K \]
\[ \Gamma_K \times_{\Gamma_K} \Gamma_{S_K} \rightarrow \Gamma_{X_K} \rightarrow \Gamma_{S_K} \]

**Proposition 13.** Let \( K \) be a sub-p-adic field and \( X_K/S_K \) a fibre space over \( K \). Let \( \Gamma_{X_K} \), \( \Gamma_{S_K} \), and \( \Gamma_K \) be the absolute Galois groups of the sub-p-adic fields \( K(X_K) \), \( K(S_K) \) and \( K \), respectively. To a fibre space \( X_K/S_K \) up to birational equivalence, i.e., algebraically closed extension \( K(S_K) \subset K(X_K) \), there corresponds an exact sequence:

\[ 1 \rightarrow \Gamma_{X_K} \rightarrow \Gamma_{X_K} \rightarrow \Gamma_{S_K} \rightarrow 1 \]

1. The extension of profinite groups above is expressed by an element of the pointed set \( H^1(\Gamma_{S_K}, (\Gamma_{F_K} \rightarrow \text{Aut}(\Gamma_{F_K}))) \).
2. The following is bijection \( H^1(\Gamma_{S_K}, (\Gamma_{F_K} \rightarrow \text{Aut}(\Gamma_{F_K}))) \cong H^1(\Gamma_{S_K}, \text{Out}(\Gamma_{F_K})) \)
3. \[ H^1(\Gamma_{S_K}, \text{Aut}(\Gamma_{F_K})) \rightarrow H^1(\Gamma_{S_K}, (\Gamma_{F_K} \rightarrow \text{Aut}(\Gamma_{F_K}))) \rightarrow H^1(\Gamma_{S_K}, \text{Out}(\Gamma_{F_K})) \]

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4. Let $S'_K \rightarrow S_K$ be a dominant $K$-rational map and $X'_K/S'_K$ a pull-back of $X_K/S_K$.

![Diagram]

\[
\lim_{\rightarrow S'_K} \text{Aut}_{\Gamma_{S'_K}}(\Gamma_{X'_K})/\Gamma_{F_K} \rightarrow \text{Aut}(\Gamma_{F_K})/\Gamma_{F_K}
\]  

(1)

Proof. 1. There exist a natural restrictions $\text{Aut}_{\Gamma_K}(\Gamma_{X_K}) \rightarrow \text{Aut}(\Gamma_{X_K})$.

2. Let $p : \Gamma_{X_K} \rightarrow \Gamma_K$ be the structure map. Let $u$ be a $\Gamma_K$-automorphism $\Gamma_{X_K}$ and $x \in \Gamma_{X_K}$. $p(u(x)) = p(x) = 1 \in \Gamma_K$. Hence $u(x) \in \Gamma_{X_K}$.

Lemma 16. Assume there exists a dominant $S_K$-rational map $Y_K \times_K S_K \rightarrow X_K$. Then there exists a continuous homomorphism $\Gamma_{S_K} \rightarrow \text{Aut}_{\Gamma_K}(\Gamma_{X_K})$, which is an element of $H^1(\Gamma_{S_K}, \text{Aut}_{\Gamma_K}(\Gamma_{X_K}))$. In other word, we have $\Gamma_{S_K} \rightarrow \text{Aut}(K(X_K))$. The following square is commutative.

\[
\begin{array}{ccc}
H^1(\Gamma_{S_K}, \text{Aut}_{\Gamma_K}(\Gamma_{X_K})) & \rightarrow & H^1(\Gamma_{S_K}, (\Gamma_{F_K} \rightarrow \text{Aut}(\Gamma_{F_K}))) \\
& \downarrow & \\
& H^1(\Gamma_{S_K}, \text{Out}(\Gamma_{F_K})) & \\
\end{array}
\]

Proof. Since there exists a dominant $S_K$-rational map $Y_K \times_K S_K \rightarrow X_K$, Mochizuki correspondence implies the commutative diagram of open homomorphisms

\[
\begin{array}{ccc}
\Gamma_{Y_K} \times_{\Gamma_K} \Gamma_{S_K} & \rightarrow & \Gamma_{X_K} \\
\downarrow & & \downarrow \\
\Gamma_{S_K} & \rightarrow & \Gamma_{S_K}
\end{array}
\]

There exists an open homomorphism $\Gamma_{Y_K} \times_{\Gamma_K} \Gamma_{S_K} \rightarrow \Gamma_{X_K} \rightarrow \Gamma_{S_K}$. Hence there exists a sectional homomorphism $\Gamma_{S_K} \rightarrow \Gamma_{X_K} \subset \Gamma_{X_K}$. By the sectional homomorphism $\Gamma_{S_K} \rightarrow$
\( \Gamma_{X_K} \), we have a continuous homomorphism \( \Gamma_{S_K} \to \text{Aut}_{\mathfrak{K}}(\Gamma_{X_K}) \) by inner automorphisms. Since \( \Gamma_{S_K} \to 1 \subset \Gamma_K \), we have \( \Gamma_{S_K} \to \text{Aut}_{\Gamma_K}(\Gamma_{X_K}) \).

\[ \square \]

Note that letting \( K(S_K) \) and \( X_K \to S_K \) be such a finite extension in a sub-p-adic fields and its pull-back that \( \kappa(S_K) = \dim S_K \), then \( \kappa(X_K) \geq \kappa(F_K) + \dim(S_K) \).

**Lemma 17.** Let \( K \) be a sub-p-adic field. Assume that \( \text{Bir}(F_K) \) is an algebraic space in group locally of finite type and that there exist a dominant \( S_K \)-rational map \( Y_K \times_K S_K \to X_K \). Then there exists a generically finite morphism \( S'_K \to S_K \) such that the natural map \( H^1(\Gamma_{S'_K}, (\Gamma_{F_K} \to \text{Aut}_{\Gamma_K}(\Gamma_{F_K}))) \to H^1(\Gamma_{S'_K}, (\Gamma_{F_K} \to \text{Aut}(\Gamma_{F_K}))) \) sends the extension class to the trivial extension class, i.e., a distinguished element.

**Proof.** Since there exist a dominant \( S_K \)-rational map \( Y_K \times_K S_K \to X_K \), there exist a section \( \Gamma_{S_K} \to \Gamma_{X_K} \) and a \( \Gamma_{S_K} \to \text{Inn}(\Gamma_{X_K}) \to \text{Aut}(\Gamma_{X_K}) \). Choose a \( K \)-dominant map \( S'_K \to S_K \) such that \( \Gamma_{S_K} \supset \Gamma_{S'_K} \) is normal and of finite index. Hence it induces \( \Gamma_{S_K} \to \text{Aut}_{\Gamma_K}(\Gamma_{X_K}) \). Let \( \text{Aut}^{\text{equ,nor}}(\Gamma_{X_K}) \subset \text{Aut}(\Gamma_{X_K}) \) be a subgroup such that each automorphism is equivariant with natural maps \( \Gamma_{X_K} \to \Gamma_{S_K} \to \Gamma_K \) and keeps invariant normal subgroups.

\[
\begin{array}{cccccc}
\Gamma_{X_K} & \longrightarrow & \Gamma_{X'_K} & \longrightarrow & \cdots & \longrightarrow & \lim_{\longrightarrow} \Gamma_{X''_K} & \longrightarrow & \Gamma_{F_K} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Gamma_{S_K} & \longrightarrow & \Gamma_{S'_K} & \longrightarrow & \cdots & \longrightarrow & \lim_{\longrightarrow} \Gamma_{S''_K} & \longrightarrow & \Gamma_{K(K)} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Gamma_K & & \Gamma_K & & \cdots & & \Gamma_K & & \Gamma_K = 1
\end{array}
\]

\[
\begin{array}{cccccc}
\text{Aut}^{\text{equ,nor}}(\Gamma_{X_K})/\Gamma_{X_K} & \longrightarrow & \text{Aut}^{\text{equ,nor}}(\Gamma_{X'_K})/\Gamma_{X'_K} & \longrightarrow & \lim_{\longrightarrow} \text{Aut}^{\text{equ,nor}}(\Gamma_{X''_K})/\Gamma_{X''_K} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{Bir}_K(K(X_K))^{\text{equ,nor}} & \longrightarrow & \text{Bir}_K(K(X'_K))^{\text{equ,nor}} & \longrightarrow & \lim_{\longrightarrow} \text{Bir}_K(K(X''_K))^{\text{equ,nor}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{Aut}^{\text{equ,nor}}(\Gamma_{X'_K})/\Gamma_{X'_K} & \longrightarrow & \lim_{\longrightarrow} \text{Aut}^{\text{equ,nor}}(\Gamma_{X''_K})/\Gamma_{X''_K} & \longrightarrow & \text{Aut}_{\Gamma_K}(\Gamma_{F_K})/\Gamma_K \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{Bir}_K(K(X'_K))^{\text{equ,nor}} & \longrightarrow & \lim_{\longrightarrow} \text{Bir}_K(K(X''_K))^{\text{equ,nor}} & \longrightarrow & \text{Bir}_K(K(F_K))
\end{array}
\]

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Thus by the precedent proposition the image of $\Gamma_{S_K} \to B(S_K) \subset \text{Bir}(X_K)$ is finite and there exists a homomorphism $\Gamma_{S_K} \to B(S_K) \to \text{Out}(\Gamma_{F_K})$. Thus one can obtain there exists a generically finite morphism $S'_K \to S_K$ such that the natural map $H^1(\Gamma_{S_K}, \text{Out}(\Gamma_{F_K})) \to H^1(\Gamma_{S'_K}, \text{Out}(\Gamma_{F_K}))$ sends the extension class to the trivial extension class, i.e., a distinguished element. 

\begin{proposition}
We have the following results.
\begin{enumerate}
\item \[
\text{Hom}_{\text{open, epi}}(\Gamma_{X_K}, \Gamma_{S_K})/\Gamma_{S_K} \cong \text{Mor}_{K}^{\text{alg, closed}}(\text{Spec }K(X_K), \text{Spec }K(S_K))
\]
, where epi and alg.closed mean epimorphisms and $K(X_K)/K(S_K)$ algebraically closed extensions, respectively.
\item (1) is equivalent to a category of fibre spaces between projective varieties
\begin{center}
\begin{tikzcd}
X_K \ar[rrr, no head] \ar[ddd] & & & \ar[ddd] \\
S_K \ar[rrrr, no head] & & & \ar[ddd] \\
\text{Spec }K \ar[rrrr, no head] & & & \\
\end{tikzcd}
\end{center}
up to birational equivalence.
\item there exists a restriction map
\[
\text{Hom}_{\Gamma_K}^{\text{open}}(\Gamma_{X_K}, \Gamma_{S_K})/\Gamma_{S_K} \to \text{Hom}_{\Gamma_K}^{\text{open}}(\Gamma_{X_K}, \Gamma_{S_K})/\Gamma_{S_K}
\]
, where $\Gamma_K = 1$.
\item the following map is pulling back extensions or "base change".
\begin{center}
\begin{tikzcd}
1 \ar[rrr, no head] \ar[ddd] & & & \ar[ddd] \\
\Gamma_{S_K} \ar[rrrr, no head] & & & \ar[ddd] \\
\Gamma_{F_K} \ar[rrrr, no head] & & & \ar[ddd] \\
\Gamma_{X_K} \ar[rrrr, no head] & & & \\
\Gamma_{F_K} \ar[rrrr, no head] & & & \\
\Gamma_{X_K} \ar[rrrr, no head] & & & \\
\Gamma_{S_K} \ar[rrrr, no head] & & & \\
1 \ar[rrr, no head] \ar[ddd] & & & \ar[ddd] \\
\end{tikzcd}
\end{center}
\end{enumerate}
\end{proposition}
5. the following is an exact sequence.

\[ 1 \to \Gamma_{X_K} \to \Gamma_K \to 1 \]

\[ 1 \to \text{Hom}_{\Gamma_K}(\Gamma_K, \Gamma_{S_K}) \to \text{Hom}_{\Gamma_K}(\Gamma_{X_K}, \Gamma_{S_K}) \to \text{Hom}_{\Gamma_K}(\Gamma_{X_{S_K}}, \Gamma_{S_K}) \]

6.

\[ 1 \to \Gamma_{S_K} \to \Gamma_K \to 1 \]

\[ 1 \to \text{Mon}_{\Gamma_K}(\Gamma_K, \text{Eq}(G[1])) \to \text{Mon}_{\Gamma_K}(\Gamma_{S_K}, \text{Eq}(G[1])) \to \text{Mon}_{\Gamma_K}(\Gamma_{S_{S_K}}, \text{Eq}(G[1])) \]

7.

\[ \text{Hom}_{\Gamma_K}^{\text{open}}(\Gamma_{F_K}, \Gamma_K)/\Gamma_K \cong \text{Mor}_K(\text{Spec } K(F_K), \text{Spec } K) \]

Proof. 1. Applying Mochizuki theory we have (1) and (2) since a fibre space has connected fibres.

2. (3) is obtained by restricting homomorphisms to homomorphisms over \( 1 = \Gamma_{X_K} \to \Gamma_K \).

3. (7) is Mochizuki correspondence.

4. (4) and (6) are pulling back extensions by base change \( \Gamma_{S_K} \to \Gamma_{S_K} \).

5. (5) Applying a left-exct functor \( \text{Hom}_{\Gamma_K} \) we have an exact sequence.

6. The contravariant functor \( \text{Hom}_{\Gamma_K}(-, \Gamma_{S_K}) \) is a left-exact. Note that \( \text{Hom}_{\Gamma_K}(-, \Gamma_{S_K}) = \text{Hom}_{\Gamma_{X_K}}(-, \Gamma_{S_{X_K}}) \).

7. The contravariant functor \( \text{Mon}_{\Gamma_K}(-, \text{Eq}(G[1])) \) is left-exact. Note that \( \text{Mon}_{\Gamma_K}(\Gamma_{S_K}, \text{Eq}(G[1])) = \text{Mon}_{\Gamma_{S_K}}(\Gamma_{S_{S_K}}, \text{Eq}(G[1])) \)

\[ \Box \]

Proposition 15. Let \( K \) be a sub-p-adic field. Assume that \( \text{Bir}(F_K) \) is an algebraic space in group locally of finite type and that there exist a dominant \( S_K \)-rational map \( Y_K \times_K S_K \to X_K \) Then there exists a generically finite morphism \( S'_K \to S_K \) such that there exists a birational equivalence over \( S'_K \)

\[ X_K \times_{S_K} S'_K \cong F_K \times_K S'_K \]

, i.e., \( X_K/S_K \) is birationally isotrivial.
Proof. Let $K$ be a sub-$p$-adic field. Assume that $\text{Bir}(F_K)$ is an algebraic space in group locally of finite type and that there exist a dominant $S_K$-rational map $Y_K \times_K S_K \to X_K$. Then there exists a generically finite morphism $S_K' \to S_K$ such that the natural map $H^1(\Gamma_{S_K'}, (\Gamma_{F_K} \to \text{Aut}(\Gamma_{F_K}))) \to H^1(\Gamma_{S_K'}, (\Gamma_{F_K} \to \text{Aut}(\Gamma_{F_K})))$ sends the extension class to the trivial extension class, i.e., a distinguished element. Let $G = \Gamma_{F_K}$. One has the following exact sequence:

$$1 \to \text{Mon}_{\Gamma_K} (\Gamma_K, \text{Eq}(G[1])) \to \text{Mon}_{\Gamma_K} (\Gamma_{S'_K}, \text{Eq}(G[1])) \to \text{Mon}_{\Gamma_{S'_K}} (\Gamma_{S'_K}, \text{Eq}(G[1]))$$

The element of $X_K \times_{S_K} S'_K$ in $\text{Mon}_{\Gamma_K} (\Gamma_{S'_K}, \text{Eq}(G[1]))$ maps to 1 in $\text{Mon}_{\Gamma_{S'_K}} (\Gamma_{S'_K}, \text{Eq}(G[1]))$. Hence choosing a variety $F_K$ one obtains $X_K \times_{S_K} S'_K = F_K \times_{K} S'_K$ birationally by pull-back and Mochizuki correspondence.

\[\square\]

6 Birational automorphism groups in Algebraic Geometry

Theorem 7. Let $X$ be a non singular projective variety of Kodaira dimension $\geq 0$. $\text{Bir}(X)$ is a scheme which is locally of finite type.

We shall prove the theorem using the following Lemmas.

Lemma 18. Let $X$ be a quasi-projective variety. Then $\text{Aut}(X)$ is a group scheme which is locally of finite type.

Proof. Since $X$ is quasi-projective, it suffices to consider some compactification $\bar{X}$ of $X$ and $\text{Hilb}_{\bar{X} \times \bar{X}}$.

Let $\text{Aut}^0(X)$ denote the connected component of $\text{Aut}(X)$ which contains an identity of the group.

We refer to the following A.Weil-M.Rosenlicht’s theorem.

Theorem 8. ([Ro]) Let the algebraic group $G$ operate on the variety $V$ and let $k$ be a field of definition for $G$, $V$ and the operation of $G$ on $V$. Then there exists a variety $V'$, birationally equivalent over $k$ to $V$, such that the operation of $G$ on $V'$ that is induced by its operation on $V$ is regular.

Lemma 19. Let $X$ be a projective variety. There exists an inductive system of monomorphisms $\text{Aut}^0(X_i) \to \text{Aut}^0(X_{i+1})$ such that $X_0 = X$, $\text{Aut}^0(X_i)$ acts regularly on some quasi-projective variety $X_{i+1}$ which is a quasi-projective variety $X_i \setminus H$ where $H$ is a hypersurface.
of $X_i$. The inductive limit of the system $(\text{Aut}^0(X_i))_{i \in I}$ is locally compact Lie group and an ind-algebraic space. Hence it is a pro-Lie group. If $X$ is of Kodaira dimension $\geq 0$, the birational automorphism group is locally algebraic.

**Proof.** By Weil-Rosenlicht’s theorem[?]) and the lemma above, we can construct an inductive system of monomorphisms $\text{Aut}^0(X_i) \rightarrow \text{Aut}^0(X_{i+1})$ such that $X_0 = X$, $\text{Aut}^0(X_i)$ acts regularly on some quasi-projective variety $X_{i+1}$ which is a quasi-projective variety $X_i \setminus H$ where $H$ is a hypersurface of $X_i$. Since any $\text{Aut}^0(X_i)$ is an algebraic group, the inductive limit is a Baire space in the complex topology, i.e., an inner point in the limit space is also an inner point some $\text{Aut}^0(X_i)$. Thus the inductive limit of the system $(\text{Aut}^0(X_i))_{i \in I}$ is locally compact Lie group and an ind-algebraic space, which is also a pro-Lie group. When $\kappa(X) \geq 0$, there exists a maximal algebraic group birationally acting on $X$ by $[\text{Mat}]$. Hence the inductive limit is an algebraic group which turns out to be an abelian group by $[\text{Mat}]$.

Thus the theorem above is proved.

Hence the theorem in the preceded section is equivalent to the following theorem.

**Theorem 9.** Let $X/S$ be a fibre space with the generic geometric fibre of Kodaira dimension $\geq 0$. If $X/S$ is neutral, then $X/S$ is isotrivial.

**Proof.** The birational automorphism group of the generic geometric fibre of $X/S$ is locally of finite type. Hence the extension $1 \rightarrow G \rightarrow E \rightarrow P \rightarrow 1$ associated to a fibre space $X/S$ satisfies the assumption of the theorem in the preceded section. \hfill \qed

### 7 Iitaka-Viehweg conjecture

**Theorem 10.** Let $f : \rightarrow S$ be a fibre space $X/S$ with the generic geometric fibre $X_\eta$ of Kodaira dimension $\geq 0$.

$$\kappa(\det f_*\omega_{X/S}^{\otimes m}) \geq \text{var}(X/S)$$

The proof shall be obtained by combining the next two lemmas.

**Lemma 20.** Let $f : X \rightarrow S$ be a fibre space $X/S$ with the generic geometric fibre $X_\eta$ of Kodaira dimension $\geq 0$. There exists a fibre space $g : Y \rightarrow S$ such that

1. $h : Y \rightarrow X$ is a cover over $X$ such that $g = f \circ h$,

2. the generic geometric fibre $Y_\eta$ of $Y/S$ is of general type, if necessary, the canonical invertible sheaf of $Y_\eta$ is taken to be abundant,
3. \( \kappa(\det f_*\omega_{X/S}^{\otimes m}) = \kappa(\det g_*\omega_{Y/S}^{\otimes m}) \).

Proof. Embed \( X \) into some projective space \( P \) and \( X/S \) into the trivial fibre space \( S \times P \). Let \( i : X \to S \times P \) be the embedding over \( S \). Choose a general hyperplane \( H \) in \( S \times P \) such that the intersection \( X \cap H \) is a non singular variety and \( H = H_0 \times S \) is horizontal in \( S \times P \). Take a branch cover \( Y \) of \( X \) along \( H \). Choose a hyperplane \( H \) such that \( Y/S \) has a general fibre of general type. We have proved \( \kappa(\det f_*\omega_{X/S}^{\otimes m}) = \kappa(\det g_*\omega_{Y/S}^{\otimes m}) \). □

By Kollar’s theorem, if necessary, Kawamata’s theorem ([Kaw]),

Theorem 11. \( \kappa(\det f_*\omega_{X/S}^{\otimes m}) = \kappa(\det g_*\omega_{Y/S}^{\otimes m}) \geq \var(Y/S) \).

Lemma 21. Let \( Y/S \) and \( X/S \) be fibre spaces over \( K \) and \( X/S \) with the generic geometric fibre of Kodaira dimension \( \geq 0 \). Assume that there exists a dominant \( S \)-rational map \( Y \to X \). Then \( \var(Y/S) \geq \var(X/S) \).

Proof. Let \( \var(Y/S) = v \). By definition of Viehweg dimension, there exist varieties \( S', T \) and \( Y_0 \) such that \( Y \times_S S' \) is birationally equivalent to \( Y_0 \times_T S' \) with \( Y_0/T \) a fibre space and \( T \) of dimension \( v \). Hence \( Y_0 \times_T S' \to X \times_S S' \) is a dominant \( S' \)-rational map. Let \( \zeta \) be the generic point of \( T \) and \( \overline{k(\zeta)} \) the algebraic closure of \( k(\zeta) \). The induced dominant \( S' \times_T \text{Spec}(\overline{k(\zeta)}) \)-rational map \( Y_0 \times_T (S' \times_T \text{Spec}(\overline{k(\zeta)})) \to X \times_S S' \times_T \text{Spec}(\overline{k(\zeta)}) \). By the following lemma, \( \var(X \times_S S' \times_T \text{Spec}(\overline{k(\zeta)})) = 0 \). Hence \( \var(X/S) \leq v \). We therefore obtain \( \var(Y/S) \geq \var(X/S) \). □

Lemma 22. Let \( Y/S \) and \( X/S \) be fibre spaces over \( K \) and the generic geometric fibre of \( X/S \) with Kodaira dimension \( \geq 0 \). Assume that there exists a dominant \( S \)-rational map. Then if \( \var(Y/S) = 0 \), then \( \var(X/S) = 0 \).

Proof. To show that if \( \var(Y/S) = 0 \), then \( \var(X/S) = 0 \), the statement is valid even if the fibre spaces \( Y/S \) and \( X/S \) are changed to a base \( S' \) on which in the definition Viehweg dimension \( Y \) is birationally equivalent to \( Y_0 \times S' \) for a variety \( Y_0 \). Since Kodaira dimension of the generic geometric fibre of \( X/S \) is non negative, the birational automorphism group is an algebraic space in group locally of finite type. Hence one obtains \( \var(X/S) = 0 \) from the precedent proposition. □

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