Moduli spaces for finite-order jets of Riemannian metrics

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Abstract

We construct the moduli space of \( r \)-jets of Riemannian metrics at a point on a smooth manifold. The construction is closely related to the problem of classification of jet metrics via differential invariants.

The moduli space is proved to be a differentiable space which admits a finite canonical stratification into smooth manifolds. A complete study on the stratification of moduli spaces is carried out for metrics in dimension \( n = 2 \).

Introduction

Let \( X \) be an \( n \)--dimensional smooth manifold. Fixed a point \( x_0 \in X \) and an integer \( r \geq 0 \), we will denote by \( J^r_{x_0} M \) the smooth manifold of \( r \)--jets at \( x_0 \) of Riemannian metrics on \( X \). On the manifold \( J^r_{x_0} M \), there exists a natural action of the group \( \text{Diff}_{x_0} \) of germs at \( x_0 \) of local diffeomorphisms leaving \( x_0 \) fixed, so it yields an equivalence relation on \( J^r_{x_0} M \):

\[
j^r_{x_0} g \equiv j^r_{x_0} \bar{g} \iff j^r_{x_0} (\tau^* g) = j^r_{x_0} \bar{g},\quad \text{for some } \tau \in \text{Diff}_{x_0}.
\]

The quotient space \( M^r_n := J^r_{x_0} M / \text{Diff}_{x_0} \) is called moduli space for \( r \)--jets of Riemannian metrics in dimension \( n \). It depends neither on the point \( x_0 \) nor on the \( n \)--dimensional manifold \( X \) chosen.

The purpose of this paper is to study the structure of moduli spaces \( M^r_n \).

Moduli spaces \( M^r_n \) have been studied in the literature through their function algebras \( \mathcal{C}^\infty (M^r_n) := \mathcal{C}^\infty (J^r_{x_0} M)^{\text{Diff}_{x_0}} \). This function algebra \( \mathcal{C}^\infty (M^r_n) \) is nothing but the algebra of differential invariants of order \( \leq r \) of Riemannian metrics. Muñoz and Valdés (\[8,9\]) prove that it is an essentially finitely-generated algebra and they determine the number of its functionally independent generators. In a more general setting, Vinogradov (\[15\]) has pointed out a simple and natural relationship between the algebra of differential invariants of homogeneous geometric structures and their characteristic classes. (See also \[14\].)

Let us also mention that in \[13\] García and Muñoz obtain a moduli space for linear frames, which has structure of smooth manifold.

However, apart from some trivial exceptions, moduli spaces \( M^r_n \) of jet metrics are not smooth manifolds, but they possess a differentiable structure in a more general sense: that of a differentiable space. (The typical example of differentiable space is a closed
subset $Y \subseteq \mathbb{R}^m$ where a function $f: Y \to \mathbb{R}$ is said to be differentiable if it is the restriction to $Y$ of a smooth function on $\mathbb{R}^m$, see [10].)

In addition, the differentiable structure of $M^r_n$ is not too far from a smooth structure, since it admits a stratification by a finite number of smooth submanifolds. Our results can be summed up in the following

**Theorem 0.1.** Every moduli space $M^r_n$ is a differentiable space and it admits a finite canonical stratification

$$M^r_n = S^r_{[H_0]} \sqcup \ldots \sqcup S^r_{[H_s]},$$

for locally closed subspaces $S^r_{[H_i]}$ which are smooth manifolds. Moreover, one of them is an open connected dense subset of $M^r_n$.

Each stratum of this decomposition of the space $M^r_n$ consists of those jet metrics having essentially the same group of automorphisms. To be more precise, let us denote by $[H]$ the conjugacy class of a closed subgroup $H$ of the orthogonal group $O(n)$. Then $S^r_{[H]}$ is the set of equivalence classes of jet metrics $j^r_{x_0}g$ whose group of automorphisms $\text{Aut}(j^r_{x_0}g)$ is conjugate to $H$, viewing $\text{Aut}(j^r_{x_0}g)$ as a subgroup of the orthogonal group $O(T_{x_0}X, g_{x_0}) \simeq O(n)$.

It is convenient to notice that Theorem 0.1 is not valid for semi-Riemannian metrics. For metrics of any signature, the problem lies on the existence of non-closed orbits for the action of $\text{Diff}_{x_0}$ on the space $J^r_{x_0}M$ of $r$-jets of such metrics, which means that the corresponding moduli space $J^r_{x_0}M/\text{Diff}_{x_0}$ is not a $T_1$ topological space, and consequently, it does not admit a structure of differentiable space either.

In dimension $n = 2$, we improve the above theorem by determining exactly all the strata which appear in the decomposition of each moduli space $M^r_2$. Let us consider the only, up to conjugacy, closed subgroups of the orthogonal group $O(2)$: the finite group $K_m$ of rotations of order $m$ ($m \geq 1$), the dihedral group $D_m$ of order $2m$ ($m \geq 1$), the special orthogonal group $SO(2)$ and $O(2)$ itself. The stratification of $M^r_2$ is determined by the following

**Theorem 0.2.** The strata in the moduli space $M^r_{n=2}$ correspond exactly to the following conjugacy classes: $[O(2)], [D_1], \ldots, [D_r-1], [K_1], \ldots, [K_{r-2}].$ (And also $[K_1]$, if $r = 4$.)

Finally, we include two appendices. In the first one, we give a brief discussion of the notion of differential invariant. In the second one, we analyze the equivalence problem for infinite-order jets of Riemannian metrics.

## 1 Preliminaries

### 1.1 Quotient spaces

Throughout this paper, we are going to handle geometric objects of a more general nature than smooth manifolds, which appear when one considers the quotient of a smooth manifold by the action of a Lie group.

**Definition 1.1.** Let $X$ be a topological space. A sheaf of continuous functions on $X$ is a map $\mathcal{O}_X$ which assigns a subalgebra $\mathcal{O}_X(U) \subseteq \mathcal{C}(U, \mathbb{R})$ to every open subset $U \subseteq X$, with the following condition:
For every open subset $U \subseteq X$, every open cover $U = \bigcup U_i$ and every function $f : U \to \mathbb{R}$, it is verified

$$f \in \mathcal{O}_X(U) \iff f|_{U_i} \in \mathcal{O}_X(U_i), \ \forall i.$$  

In particular, if $V \subseteq U$ are open subsets in $X$, then it is verified

$$f \in \mathcal{O}_X(U) \implies f|_V \in \mathcal{O}_X(V).$$

**Definition 1.2.** We will call ringed space the pair $(X, \mathcal{O}_X)$ formed by a topological space $X$ and a sheaf of continuous functions $\mathcal{O}_X$ on $X$.

Although the concept of ringed space in the literature, specially in that concerning Algebraic Geometry, is much broader, the previous definition is good enough for our purposes.

Every open subset $U$ of a ringed space $(X, \mathcal{O}_X)$ is itself, in a very natural way, a ringed space, if we define $\mathcal{O}_U(V) := \mathcal{O}_X(V)$ for every open subset $V \subseteq U$.

Hereinafter, a ringed space $(X, \mathcal{O}_X)$ will usually be denoted just by $X$, dropping the sheaf of functions.

**Definition 1.3.** Given two ringed spaces $X$ and $Y$, a morphism of ringed spaces $\varphi : X \to Y$ is a continuous map such that, for every open subset $V \subseteq Y$, the following condition is held:

$$f \in \mathcal{O}_Y(V) \implies f \circ \varphi \in \mathcal{O}_X(\varphi^{-1}(V)).$$  

A morphism of ringed spaces $\varphi : X \to Y$ is said to be an isomorphism if it has an inverse morphism, that is, there exists a morphism of ringed spaces $\phi : Y \to X$ verifying $\varphi \circ \phi = \text{Id}_Y$, $\phi \circ \varphi = \text{Id}_X$.

**Example 1.4. (Smooth manifolds)** The space $\mathbb{R}^n$, endowed with the sheaf $C^\infty_{\mathbb{R}^n}$ of smooth functions, is an example of ringed space. An $n$–smooth manifold is precisely a ringed space in which every point has an open neighbourhood isomorphic to $(\mathbb{R}^n, C^\infty_{\mathbb{R}^n})$. Smooth maps between smooth manifolds are nothing but morphisms of ringed spaces.

**Example 1.5. (Quotients by the action of a Lie group)** Let $G \times X \to X$ be a smooth action of a Lie group $G$ on a smooth manifold $X$, and let $\pi : X \to X/G$ be the canonical quotient map.

We will consider on the quotient topological space $X/G$ the following sheaf $C^\infty_{X/G}$ of “differentiable” functions:

For every open subset $V \subseteq X/G$, $C^\infty_{X/G}(V)$ is defined to be

$$C^\infty_{X/G}(V) := \{f : V \to \mathbb{R} : f \circ \pi \in C^\infty(\pi^{-1}(V))\}.$$  

Note that there exists a canonical $\mathbb{R}$–algebra isomorphism:

$$C^\infty_{X/G}(V) \xrightarrow{\cong} C^\infty(\pi^{-1}(V))^G$$  

$$f \quad \mapsto \quad f \circ \pi.$$
The pair \((X/G, \mathcal{C}^\infty_{X/G})\) is an example of ringed space, which we will call **quotient ringed space** of the action of \(G\) on \(X\).

As it would be expected, this space verifies the **universal quotient property**: Every morphism of ringed spaces \(\varphi : X \to Y\), which is constant on every orbit of the action of \(G\) on \(X\), factors uniquely through the quotient map \(\pi : X \to X/G\), that is, there exists a unique morphism of ringed spaces \(\tilde{\varphi} : X/G \to Y\) verifying \(\varphi = \tilde{\varphi} \circ \pi\).

**Example 1.6. (Inverse limit of smooth manifolds)** Sometimes we will consider an inverse system

\[\ldots \longrightarrow X_{r+1} \longrightarrow X_r \longrightarrow \ldots \longrightarrow X_1\]

of smooth mappings between smooth manifolds (or, with some more generality, an inverse system of ringed spaces).

The inverse limit \(\lim X_r\) is a ringed space in the following natural way. On \(\lim X_r\) it is considered the inverse limit topology, that is, the initial topology induced by the evident projections \(p_s : \lim X_r \to X_s\). A real function on an open subset of \(\lim X_r\) is said to be “differentiable” if it locally coincides with the composition of a projection \(p_s : \lim X_r \to X_s\) and a smooth function on \(X_s\).

The topological space \(\lim X_r\) endowed with the above sheaf of differentiable functions is a ringed space satisfying the suitable universal property:

For every ringed space \(Z\), there exists the bijection

\[
\text{Hom} (Z, \lim X_r) \cong \lim \text{Hom} (Z, X_r)
\]

\[
\varphi \mapsto (\ldots, p_r \circ \varphi, \ldots).
\]

**Example 1.7.** Let \(Z\) be a locally closed subspace of \(\mathbb{R}^n\). We define the sheaf \(\mathcal{C}^\infty_Z\) of differentiable functions on \(Z\) to be the sheaf of functions locally coinciding with restrictions of smooth functions on \(\mathbb{R}^n\). The pair \((Z, \mathcal{C}^\infty_Z)\) is another example of ringed space.

**Definition 1.8.** A **(reduced) differentiable space** is a ringed space in which every point has an open neighbourhood isomorphic to a certain locally closed subspace \((Z, \mathcal{C}^\infty_Z)\) in some \(\mathbb{R}^n\).

A map between differentiable spaces is called **differentiable** if it is a morphism of ringed spaces.

**Theorem 1.9.** (Schwarz [11],[10] Th. 11.14) Let \(G \to \text{Gl}(V)\) be a finite-dimensional linear representation of a compact Lie group \(G\). The quotient space \(V/G\) is a differentiable space.

More precisely: Let \(p_1, \ldots, p_s\) be a finite set of generators for the \(\mathbb{R}\)-algebra of \(G\)-invariant polynomials on \(V\); these invariants define an isomorphism of ringed spaces

\[
(p_1, \ldots, p_s) : V/G \cong Z \subseteq \mathbb{R}^s,
\]

\(Z\) being a closed subspace of \(\mathbb{R}^s\).
1.2 Normal tensors

Let $X$ be an $n$–dimensional smooth manifold. Fix a point $x_0 \in X$ and a semi-Riemannian metric $g$ on $X$ of fixed signature $(p, q)$, with $n = p + q$. Let us recall briefly some definitions and results:

**Definition 1.10.** A coordinate system $(z_1, \ldots, z_n)$ in a neighbourhood of $x_0$ is said to be a normal coordinate system for $g$ at the point $x_0$ if the geodesics passing through $x_0$ at $t = 0$ are precisely the “straight lines” \( \{ z_1(t) = \lambda_1 t, \ldots, z_n(t) = \lambda_n t \} \), where $\lambda_i \in \mathbb{R}$.

In particular, $x_0$ is the origin of any normal coordinate system for $g$ at $x_0$.

**Remark 1.11.** Observe that we do not require $(\partial_{z_1}, \ldots, \partial_{z_n})$ to be an orthonormal basis of $T_{x_0}X$.

As it is well known, via the exponential map $\exp_g : T_{x_0}X \to X$, normal coordinate systems on $X$ correspond bijectively to linear coordinate systems on $T_{x_0}X$. Therefore, two normal systems differ in a linear coordinate transformation.

**Proposition 1.12.** Let $g$, $\bar{g}$ be two semi-Riemannian metrics on $X$. Let us also consider their corresponding exponential maps $\exp_g, \exp_{\bar{g}} : T_{x_0}X \to X$. For every $r \geq 0$ it is verified:

$$j^r_{x_0} g = j^r_{x_0} \bar{g} \implies j^{r+1}_0(\exp_g) = j^{r+1}_0(\exp_{\bar{g}}).$$

As a consequence of Proposition 1.12 whose proof is routine, normal coordinate systems at $x_0$ for a metric $g$ are determined up to the order $r + 1$ by the jet $j^r_{x_0} g$. This fact will be used later on with no more explicit mention.

**Definition 1.13.** Let $r \geq 1$ be a fixed integer and let $x_0 \in X$. The space of normal tensors of order $r$ at $x_0$, which we will denote by $N_r$, is the vector space of $(r + 2)$–covariant tensors $T$ at $x_0$ having the following symmetries:

- $T$ is symmetric in the first two and last $r$ indices:

$$T_{ijkl\ldots k_r} = T_{ijkl\ldots k_r}, \quad T_{ijlk\ldots k_r} = T_{ijlk\ldots k_r}, \quad \forall \sigma \in S_r;$$

- the cyclic sum over the last $r + 1$ indices is zero:

$$T_{ijk\ldots k_r} + T_{ik\ldots k_rj} + \ldots + T_{ik\ldots j} = 0.$$

If $r = 0$, we will assume $N_0$ to be the set of semi-Riemannian metrics at $x_0$ of a fixed signature $(p, q)$ (which is an open subset of $S^2T^*_{x_0}X$, but not a vector subspace).

A simple computation shows that, in general, $N_1 = 0$. Moreover, in [2] it is proved that $N_r, (r \geq 2)$ is a linear irreducible representation of the linear group $\text{GL}(T_{x_0}X)$.

To show how a semi-riemannian metric $g$ produces a sequence of normal tensors $g^r_{x_0}$ at $x_0$, let us recall this classical result:

**Lemma 1.14.** *Gauss Lemma* Let $(z_1, \ldots, z_n)$ be germs of coordinates centred at $x_0 \in X$. These coordinates are normal for the germ of a semi-Riemannian metric $g$ if and only if the metric coefficients $g_{ij}$ verify the equations

$$\sum_j g_{ij} z_j = \sum_j g_{ij}(x_0) z_j.$$
Let \((z_1, \ldots, z_n)\) be a normal coordinate system for \(g\) at \(x_0 \in X\) and let us denote:

\[
g_{ij,k_1 \ldots k_r} := \frac{\partial^r g_{ij}}{\partial z_{k_1} \cdots \partial z_{k_r}}(x_0).
\]

If we differentiate \(r + 1\) times the identity of the Gauss Lemma, we obtain:

\[
g_{ik_0,k_1 \ldots k_r} + g_{ik_1,k_2 \ldots k_r k_0} + \cdots + g_{ik_r,k_0 \ldots k_{r-1}} = 0.
\]

This property, together with the obvious fact that the coefficients \(g_{ij,k_1 \ldots k_r}\) are symmetric in the first two and in the last \(r\) indices, allows to prove that the tensor

\[
g_{x_0}^r := \sum_{ijk_1 \ldots k_r} g_{ij,k_1 \ldots k_r} dz_i \otimes dz_j \otimes dz_{k_1} \otimes \cdots \otimes dz_{k_r}
\]

is a normal tensor of order \(r\) at \(x_0 \in X\). This construction does not depend on the choice of the normal coordinate system \((z_1, \ldots, z_n)\).

**Definition 1.15.** The tensor \(g_{x_0}^r\) is called the \(r\)-th normal tensor of the metric \(g\) at the point \(x_0\). As a consequence of \(N_1 = 0\), the first normal tensor of a metric \(g\) is always zero, \(g_{x_0}^1 = 0\).

The normal tensors associated to a metric were first introduced by Thomas [13]. The sequence \(\{g_{x_0}, g_{x_0}^2, g_{x_0}^3, \ldots, g_{x_0}^r\}\) of normal tensors of the metric \(g\) at a point \(x_0\) totally determines the sequence \(\{g_{x_0}, R_{x_0}, \nabla_{x_0} R, \ldots, \nabla^{r-1}_{x_0} R\}\) of covariant derivatives at \(x_0\) of the curvature tensor \(R\) of \(g\) and vice versa (see [13]). The main advantage of using normal tensors is the possibility of expressing the symmetries of each \(g_{x_0}^r\) without using the other normal tensors, whereas the symmetries of \(\nabla^s_{x_0} R\) depend on \(R\) (recall the Ricci identities).

**Remark 1.16.** Using the exact sequence

\[
0 \rightarrow N_r \rightarrow S^{2T_{x_0}} X \otimes S^{r^* T_{x_0}^*} X \rightarrow T_{x_0}^* X \otimes S^{r+1} T_{x_0}^* X \rightarrow 0,
\]

where \(s\) stands for the symmetrization on the last \((r + 1)\)-indices, we obtain

\[
\dim N_r = \binom{n + 1}{2} \binom{n + r - 1}{r} - n \binom{n + r}{r + 1}.
\]

**2 Differential invariants of metrics**

In the remainder of the paper, \(X\) will always be an \(n\)-dimensional smooth manifold.

Let us denote by \(J^r M \rightarrow X\) the fiber bundle of \(r\)-jets of semi-Riemannian metrics on \(X\) of fixed signature \((p, q)\), with \(n = p + q\). Its fiber over a point \(x_0 \in X\) will be denoted \(J^r_{x_0} M\).

Let \(\text{Diff}_{x_0}\) be the group of germs of local diffeomorphisms of \(X\) leaving \(x_0\) fixed, and let \(\text{Diff}^r_{x_0}\) be the Lie group of \(r\)-jets at \(x_0\) of local diffeomorphisms of \(X\) leaving \(x_0\) fixed. We have the following exact group sequence:

\[
0 \rightarrow H^r_{x_0} \rightarrow \text{Diff}_{x_0} \rightarrow \text{Diff}^r_{x_0} \rightarrow 0,
\]
$H^r_{x_0}$ being the subgroup of $\text{Diff}_{x_0}$ made up of those diffeomorphisms whose $r-$jet at $x_0$ coincides with that of the identity.

The group $\text{Diff}_{x_0}$ acts in an obvious way on $J^r_{x_0}M$. Note that the subgroup $H^{r+1}_{x_0}$ acts trivially, so the action of $\text{Diff}_{x_0}$ on $J^r_{x_0}M$ factors through an action of $\text{Diff}^{r+1}_{x_0}$.

**Definition 2.1.** Two $r-$jets $j^r_{x_0}g, j^r_{x_0}\tilde{g} \in J^r_{x_0}M$ are said to be **equivalent** if there exists a local diffeomorphism $\tau \in \text{Diff}_{x_0}$ such that $j^r_{x_0}\tilde{g} = j^r_{x_0}(\tau^*g)$.

Equivalence classes of $r-$jets of metrics constitute a ringed space. To be precise:

**Definition 2.2.** We call **moduli space** of $r-$jets of semi-Riemannian metrics of signature $(p, q)$ the quotient ringed space

$$M^r_{p,q} := J^r_{x_0}M/\text{Diff}_{x_0} = J^r_{x_0}M/\text{Diff}^{r+1}_{x_0}.$$ 

In the case of Riemannian metrics, that is $p = n, q = 0$, the moduli space will be denoted $M^r_n$.

It is important to observe that the moduli space depends neither on the point $x_0$ nor on the chosen $n-$dimensional manifold:

Given a point $\bar{x}_0$ in another $n-$dimensional manifold $\bar{X}$, let us consider an arbitrary diffeomorphism

$$X \supset U_{x_0} \xrightarrow{\varphi} U_{\bar{x}_0} \subset \bar{X}$$

between corresponding neighbourhoods of $x_0$ and $\bar{x}_0$, verifying $\varphi(x_0) = \bar{x}_0$. Such a diffeomorphism induces an isomorphism of ringed spaces between the corresponding moduli spaces,

$$J^r_{\bar{x}_0}\bar{M}/\text{Diff}_{\bar{x}_0} \overset{\sim}{\longrightarrow} J^r_{x_0}M/\text{Diff}_{x_0}$$

$$[j^r_{x_0}\bar{g}] \quad \longrightarrow \quad [j^r_{x_0}\varphi^*\bar{g}],$$

which is independent of the choice of the diffeomorphism $\varphi$. So both moduli spaces are canonically identified.

Let us now consider the quotient morphism

$$J^r_{x_0}M \xrightarrow{\pi} J^r_{\bar{x}_0}\bar{M}/\text{Diff}_{\bar{x}_0} = M^r_{p,q}.$$ 

Recall that a function $f$ defined on an open subset $U \subseteq M^r_{p,q}$ is said to be **differentiable** if $f \circ \pi$ is a smooth function on $\pi^{-1}(U)$, that is,

$$C^\infty(U) = C^\infty(\pi^{-1}(U))\text{Diff}_{x_0}.$$ 

Every semi-Riemannian metric $g$ on $X$ of signature $(p, q)$ defines a map

$$X \xrightarrow{m_g} M^r_{p,q}$$

$$x \quad \mapsto \quad [j^r_xg],$$

which is “differentiable”, that is, it is a morphism of ringed spaces.

**Definition 2.3.** A **differential invariant** of order $\leq r$ of semi-Riemannian metrics of signature $(p, q)$ is defined to be a global differentiable function on $M^r_{p,q}$.

Taking into account the ringed space structure of $M^r_{p,q}$, we can simply write:

$$\{\text{Differential invariants of order } \leq r\} = C^\infty(M^r_{p,q}) = C^\infty(J^r_{x_0}M)\text{Diff}_{x_0}.$$
A differential invariant \( h : \mathcal{M}^r_{p,q} \to \mathbb{R} \) associates with every semi-Riemannian metric \( g \) on \( X \) a smooth function on \( X \), denoted by \( h(g) \), through the formula \( h(g) := h \circ m_g \), that is,

\[
h(g)(x) = h(j^r_x g) .
\]

In any local coordinates, \( h(g) \) is a function smoothly depending on the coefficients of the metric and their subsequent partial derivatives up to the order \( r \),

\[
h(g)(x) = h(g_{ij}(x), \frac{\partial g_{ij}}{\partial x_k}(x), \ldots, \frac{\partial^r g_{ij}}{\partial x_{k_1} \ldots \partial x_{k_r}}(x)) ,
\]

which is equivariant with respect to the action of local diffeomorphisms,

\[
h(\tau^* g) = \tau^* (h(g)) .
\]

For a discussion on the concept of differential invariant, see Section 6.

3 A fundamental lemma

The aim of this section is to prove that there exist a certain linear finite-dimensional representation \( V^r \) of the orthogonal group \( O(p,q) \) and an isomorphism of ringed spaces

\[
\mathcal{M}^r_{p,q} \cong V^r / O(p,q) .
\]

This bijection is already known at a set-theoretic level (see [2] and also [7] for \( G \)-structures which posses a linear connection). We just add the fact that this bijection is an isomorphism of ringed spaces.

Let us fix for this entire section a local coordinate system \( (z_1, \ldots, z_n) \) centred at \( x_0 \).

We will denote by \( N^r_{x_0} \) the smooth submanifold of \( J^r_{x_0} M \) formed by \( r \)-jets at \( x_0 \) of metrics of signature \( (p,q) \) for which \( (z_1, \ldots, z_n) \) is a normal coordinate system (that is, Taylor expansions of the coefficients of such metrics with respect to coordinates \( (z_1, \ldots, z_n) \) satisfy the equations of the Gauss Lemma up to the order \( r \)).

Consider the subgroup of \( \text{Diff}_{x_0} \)

\[
H^1_{x_0} := \{ \tau \in \text{Diff}_{x_0} : j^1_{x_0} \tau = j^1_{x_0} (\text{Id}) \} .
\]

Note the following exact group sequence:

\[
0 \longrightarrow H^1_{x_0} \longrightarrow \text{Diff}_{x_0} \longrightarrow \text{Gl}(T_{x_0} X) \longrightarrow 0 ,
\]

where the epimorphism \( \text{Diff}_{x_0} \to \text{Gl}(T_{x_0} X) \) takes every diffeomorphism to its linear tangent map at \( x_0 \).

Lemma 3.1. There exists an isomorphism of ringed spaces

\[
\mathcal{N}^r_{x_0} \cong J^r_{x_0} M / H^1_{x_0} .
\]
Proof. Let us start by constructing a smooth section of the natural inclusion
\[ N_{x_0}^r \hookrightarrow J_{x_0}^r M. \]

Given a jet metric \( j^r_{x_0}g \in J_{x_0}^r M \), consider a metric \( g \) representing it. Let \((\bar{z}_1, \ldots, \bar{z}_n)\) be the only normal coordinate system centred at \( x_0 \) with respect to \( g \) which satisfies \( d_{x_0} \bar{z}_i = d_{x_0} z_i \).

Let \( \tau \) be the local diffeomorphism which transforms one coordinate system into another: \( \tau^*(\bar{z}_i) = z_i \). The condition \( d_{x_0} \bar{z}_i = d_{x_0} z_i \) implies that the linear tangent map of \( \tau \) at \( x_0 \) is the identity, i.e. \( \tau \in H_{x_0}^1 \).

As \((\bar{z}_1, \ldots, \bar{z}_n)\) is a normal coordinate system for \( g \), \((z_1 = \tau^*(\bar{z}_1), \ldots, z_n = \tau^*(\bar{z}_n))\) is a normal coordinate system for \( \tau^*g \); that is, \( j^r_{x_0}(\tau^*g) \in N_{x_0}^r \).

Therefore, the section we were looking for is the following map:
\[ j^r_{x_0} \quad \text{and} \quad \sigma \in H_{x_0}^1, \quad \text{for some} \quad \sigma \in H_{x_0}^1 \]

with \( \sigma \) depending on \( g \).

Let us now see that \( \varphi \) is constant on each orbit of the action of \( H_{x_0}^1 \). Let \( j^r_{x_0} g' \) be another point in the same orbit as \( j^r_{x_0} g \), so we can write \( g' = \sigma^* g \) for some \( \sigma \in H_{x_0}^1 \).

Since \((\bar{z}_1, \ldots, \bar{z}_n)\) are normal coordinates for \( g \), \((z_1 = \sigma^*(\bar{z}_1), \ldots, z_n = \sigma^*(\bar{z}_n))\) is a normal coordinate system for \( \sigma^* g \). Then \( z_i = \tau^*(\bar{z}_i) = \tau^*(\sigma^{-1}(z_i')) \), and, if we apply the definition of \( \varphi \), we get
\[ \varphi(j^r_{x_0} g') = j^r_{x_0} (\tau^* \sigma^{-1} g') = j^r_{x_0} (\tau^* g) = \varphi(j^r_{x_0} g). \]

As \( \varphi \) is constant on each orbit of the action of \( H_{x_0}^1 \), it induces, according to the universal quotient property, a morphism of ringed spaces:
\[ J^r_{x_0} M/H_{x_0}^1 \rightarrow N_{x_0}^r. \]

This map is indeed an isomorphism of ringed spaces, because it has an obvious inverse morphism, which is the following composition:
\[ N_{x_0}^r \hookrightarrow J_{x_0}^r M \rightarrow J_{x_0}^r M/H_{x_0}^1. \]

Let us denote by \( \text{Gl}_n \) the general linear group in dimension \( n \):
\[ \text{Gl}_n := \{ n \times n \text{ invertible matrices with coefficients in } \mathbb{R} \}. \]

Considering every matrix in \( \text{Gl}_n \) as a linear transformation of the coordinate system \((z_1, \ldots, z_n)\), we can think of \( \text{Gl}_n \) as a subgroup of \( \text{Diff}_{x_0} \).

Via the action of the group \( \text{Diff}_{x_0} \) on \( J_{x_0}^r M \), the subgroup \( \text{Gl}_n \), for its part, acts leaving the submanifold \( N_{x_0}^r \) stable, and then we can state the following

Lemma 3.2. There exists an isomorphism of ringed spaces
\[ N_{x_0}^r/\text{Gl}_n \cong J_{x_0}^r M/\text{Diff}_{x_0} = M_{p,q}. \]
Proof. Via the epimorphism

\[ \text{Diff}_{x_0} \longrightarrow \text{Diff}_{x_0}/H^1_{x_0} = \text{Gl}(T_{x_0}X), \]

the subgroup \( \text{Gl}_n \) gets identified with \( \text{Gl}(T_{x_0}X) \). Consequently, the subgroups \( H^1_{x_0} \) and \( \text{Gl}_n \) generate \( \text{Diff}_{x_0} \).

If we consider the isomorphism

\[ N^r_{x_0} \longrightarrow J^r_{x_0} M/H^1_{x_0}^r \]

of Lemma 3.1 and take quotient with respect to the action of \( \text{Gl}_n \), we get the desired isomorphism:

\[ N^r_{x_0}/\text{Gl}_n \approx (J^r_{x_0} M/H^1_{x_0}^r)/\text{Gl}_n = J^r_{x_0} M/\text{Diff}_{x_0}. \]

\[ \square \]

Let us express the previous result in terms of normal tensors by using the following

Lemma 3.3. The map

\[ N^r_{x_0} \longrightarrow N_0 \times N_2 \times \ldots \times N_r, \quad j^r_{x_0} g \longmapsto (g_{x_0}, g^2_{x_0}, \ldots, g^r_{x_0}) \]

is a diffeomorphism.

Proof. The inverse map is defined in the obvious way:

Given \( (T^0, T^2, \ldots, T^r) \in N_0 \times N_2 \times \ldots \times N_r \), consider the jet metric \( j^r_{x_0} g \) which in coordinates \( (z_1, \ldots, z_n) \) is determined by the identities

\[ g_{ij,k_1 \ldots k_s} := \frac{\partial^s g_{ij}}{\partial z_{k_1} \ldots \partial z_{k_s}}(x_0) = T^s_{ij,k_1 \ldots k_s}, \quad s = 0, \ldots, r. \]

The symmetries of tensors \( T^s \) guarantee that the coefficients \( g_{ij} \) of the metric \( g \) verify the equations of the Gauss Lemma up to the order \( r \), that is, \( j^r_{x_0} g \in N^r_{x_0} \).

Combining Lemma 3.2 and Lemma 3.3 we obtain an isomorphism of ringed spaces:

\[ M^r_{p,q} = J^r_{x_0} M/\text{Diff}_{x_0} \approx (N_0 \times N_2 \times \ldots \times N_r)/\text{Gl}(T_{x_0}X) \]

\[ [j^r_{x_0} g] \longmapsto [(g_{x_0}, g^2_{x_0}, \ldots, g^r_{x_0})]. \]

Let us now fix a metric \( g_{x_0} \in N_0 \) at \( x_0 \) and let us consider the orthogonal group \( O(p, q) := O(T_{x_0}X, g_{x_0}) \). As the linear group \( \text{Gl}(T_{x_0}X) \) acts transitively on the space of metrics \( N_0 \), and \( O(p, q) \) is the stabilizer subgroup of \( g_{x_0} \in N_0 \), we obtain the following isomorphism:

\[ (N_0 \times N_2 \times \ldots \times N_r)/\text{Gl}(T_{x_0}X) \approx (N_2 \times \ldots \times N_r)/O(p, q). \]

To sum up, we can state the main result of this section:
Lemma 3.4. (Fundamental Lemma) The moduli space $\mathcal{M}_{p,q}^r$ is isomorphic to the quotient space of a linear representation of the orthogonal group $O(p,q)$, through the following isomorphism of ringed spaces:

$$\mathcal{M}_{p,q}^r \cong \frac{(N_2 \times \ldots \times N_r)}{O(p,q)}.$$ 

This isomorphism takes every class $[g^r_x, \bar{g}] \in \mathcal{M}_{p,q}^r$, with $\bar{g}_x = g_x$, to the sequence of normal tensors $[(\bar{g}_x^2, \ldots, \bar{g}_x^r)] \in (N_2 \times \ldots \times N_r)/O(p,q)$.

4 Structure of the moduli spaces

Let $V$ be a finite-dimensional linear representation of a reductive Lie group $G$. The $\mathbb{R}$-algebra of $G$-invariant polynomials on $V$ is finitely generated (Hilbert-Nagata theorem, see [3]). Let $p_1, \ldots, p_s$ be a finite set of generators for that algebra; by a result of Luna [6], every smooth $G$-invariant function $f$ on $V$ can be written as $f = F(p_1, \ldots, p_s)$, for some smooth function $F \in C^\infty(\mathbb{R}^s)$.

Theorem 4.1. (Finiteness of differential invariants, [8]) There exists a finite number $p_1, \ldots, p_s \in C^\infty(\mathcal{M}_{p,q}^r)$ of differential invariants of order $\leq r$ such that any other differential invariant $f$ of order $\leq r$ is a smooth function of the former ones, i.e. $f = F(p_1, \ldots, p_s)$, for a certain $F \in C^\infty(\mathbb{R}^s)$.

Proof. By the Fundamental Lemma (3.4),

$$C^\infty(\mathcal{M}_{p,q}^r) = C^\infty((N_2 \times \ldots \times N_r)/O(p,q)),$$

and we can conclude by applying the above theorem by Luna to the linear representation $N_2 \times \ldots \times N_r$ of the orthogonal group $O(p,q)$.

Remark 4.2. Using the theory of invariants for the orthogonal group and the fact that the sequence of normal tensors $\{g_{x_0}, g_{x_0}^2, g_{x_0}^3, \ldots, g_{x_0}^r\}$ is equivalent to the sequence $\{g_{x_0}, R_{x_0}, \nabla_{x_0} R, \ldots, \nabla_{x_0}^{r-2} R\}$, it can be proved that the generators $p_1, \ldots, p_s$ of Theorem 4.1 can be chosen to be Weyl invariants, that is, scalar quantities constructed from the sequence $\{g_{x_0}, R_{x_0}, \nabla_{x_0} R, \ldots, \nabla_{x_0}^{r-2} R\}$ by reiteration of the following operations: tensor products, raising and lowering indices, and contractions.

Theorem 4.3. In the Riemannian case, differential invariants of order $\leq r$ separate points in the moduli space $\mathcal{M}_{p,q}^r$.

Consequently, differential invariants of order $\leq r$ classify $r$-jets of Riemannian metrics (at a point).

Proof. For positive definite metrics, the orthogonal group $O(n)$ is compact. It is a well-known fact that, if $V$ is a linear representation of a compact Lie group $G$, then smooth $G$-invariant functions on $V$ separate the orbits of the action of $G$, or, in other words, the algebra $C^\infty(V/G)$ separates the points in $V/G$.

Using this, together with the Fundamental Lemma, we conclude our proof.
Neither assertion in Theorem 4.3 is valid for semi-Riemannian metrics. See Note in Subsection 5.2 for a counterexample. For such metrics, moduli spaces $M_{n,q}^r$ are generally pathological in a topological sense, since they have non-closed points (they are not $T_1$ topological spaces).

In the Riemannian case, Schwarz Theorem 1.9 and the Fundamental Lemma directly provide the following

**Theorem 4.4.** *In the Riemannian case, moduli spaces $M_{n}^r$ are differentiable spaces."

More precisely: Let $p_1, \ldots, p_s$ be the basis of differential invariants of order $\leq r$ mentioned in Theorem 4.1. These invariants induce an isomorphism of differentiable spaces

$$
(p_1, \ldots, p_s) : M_{n}^r \cong Z \subseteq \mathbb{R}^s,
$$

$Z$ being a closed subspace of $\mathbb{R}^s$.

Although the differentiable space $M_{n}^r$ is not in general a smooth manifold, its structure is not so deficient as it could seem at first sight, since we are going to prove that it admits a finite stratification by certain smooth submanifolds.

**Definition 4.5.** Let us consider $V_n = \mathbb{R}^n$ endowed with its standard inner product $\delta$, and the corresponding orthogonal group $O(n) := O(V_n, \delta)$. We will denote by $\mathcal{T}$ the set of conjugacy classes of closed subgroups in $O(n)$.

Given another $n$–dimensional vector space $\bar{V}_n$ with an inner product $\bar{\delta}$, we can also consider the set $\bar{\mathcal{T}}$ of conjugacy classes of closed subgroups in $O(\bar{V}_n, \bar{\delta})$.

Observe that there exists a canonical identification

$$
\mathcal{T} \overset{\sim}{\longrightarrow} \bar{\mathcal{T}} , \quad [H] \mapsto [\varphi \circ H \circ \varphi^{-1}],
$$

where $\varphi$ stands for any isometry $\varphi : V_n \to \bar{V}_n$.

As the identification is canonical (i.e. it does not depend on the choice of the isometry $\varphi$), from now on we will suppose that the set $\mathcal{T}$ is just “the same” for every pair $(V_n, \delta)$.

Note that $\mathcal{T}$ possesses a partial order relation: $[H] \leq [H']$, if there exist some representatives $H$ and $H'$ of $[H]$ and $[H']$ respectively, such that $H \subseteq H'$.

**Definition 4.6.** The group of automorphisms of a Riemannian jet metric $j^r_{x_0}g$ is defined to be the stabilizer subgroup $\text{Aut}(j^r_{x_0}g) \subseteq \text{Diff}^{+1}_{x_0} \times \mathbb{R}$ of $j^r_{x_0}g$:

$$
\text{Aut}(j^r_{x_0}g) := \{ j^{r+1}_{x_0} \tau \in \text{Diff}^{+1}_{x_0} : j^r_{x_0}(\tau^* g) = j^r_{x_0}g \}.
$$

Given $\tau \in \text{Diff}_{x_0}$, let us denote by $\tau_{*,x_0} : T_{x_0}X \to T_{x_0}X$ the linear tangent map of $\tau$ at $x_0$.

**Lemma 4.7.** *The group morphism

$$
\text{Aut}(j^r_{x_0}g) \quad \to \quad O(T_{x_0}X, g_{x_0}) \cong O(n)
$$

is injective.*
Proof. For any \( \tau \in \text{Diff}_{x_0} \) and any metric \( g \) on \( X \) we have the following commutative diagram of local diffeomorphisms:

\[
\begin{array}{ccc}
T_{x_0}X & \xrightarrow{\exp_{r}\tau} & X \\
\downarrow & & \downarrow \\
T_{x_0}X & \xrightarrow{\exp_{g}} & X \\
\end{array}
\]

If \( j^{r+1}_{x_0}\tau \in \text{Aut}(j^{r}_{x_0}g) \), that is, \( j^{r}_{x_0}(\tau^*g) = j^{r}_{x_0}g \), then \( j^{r+1}_{0}(\exp_{r}\tau g) = j^{r+1}_{0}(\exp g) \) because of Proposition 1.12.

Now, taking \((r+1)\)-jets in the above diagram, we obtain:

\[
j^{r+1}_{x_0}\tau = j^{r+1}_{0}(\exp g) \circ j^{r+1}_{0}\tau \circ j^{r+1}_{x_0}(\exp^{-1} g),
\]

hence \( j^{r+1}_{x_0}\tau \) is determined by its linear part \( \tau_* \).

By the previous lemma, the group \( \text{Aut}(j^{r}_{x_0}g) \) can be viewed as a subgroup (determined up to conjugacy) of the orthogonal group \( O(n) \).

Definition 4.8. The **type map** is defined to be the map

\[
t : \mathcal{M}^r \rightarrow T, \quad [j^{r}_{x_0}g] \mapsto [\text{Aut}(j^{r}_{x_0}g)].
\]

For each \([H] \in T\), the **stratum of type** \([H] \) is said to be the subset \( S_{[H]} \subseteq \mathcal{M}^r \) of those points of type \([H] \).

Theorem 4.9. (Stratification of the moduli space) The type map \( t : \mathcal{M}^r \rightarrow T \) verifies the following properties:

1. \( t \) takes a finite number of values \([H_0], \ldots, [H_k]\), one of which, say \([H_0]\), is minimum.

2. **Semicontinuity:** For every type \([H] \in T\), the set of points in \( \mathcal{M}^r \) of type \( \leq [H] \) is an open subset of \( \mathcal{M}^r \). In particular, every stratum \( S_{[H]} \) is a locally closed subspace of \( \mathcal{M}^r \).

3. Every stratum \( S_{[H]} \) is a smooth submanifold of \( \mathcal{M}^r \).

4. The (also called generic) stratum \( S_{[H_0]} \) of minimum type is a dense connected open subset of \( \mathcal{M}^r \).

Proof. Fix a positive definite metric \( g_{x_0} \) on \( T_{x_0}X \) and denote by \( O(n) \) its orthogonal group. The Fundamental Lemma 3.4 tells us that there exists an isomorphism

\[
\mathcal{M}^r = (N_2 \times \ldots \times N_r)/O(n).
\]

This isomorphism takes every class \([j^{r}_{x_0}\tilde{g}] \in \mathcal{M}^r\), with \( \tilde{g}_{x_0} = g_{x_0}\), to the sequence of normal tensors \([\tilde{g}^{2}_{x_0}, \ldots, \tilde{g}^{r}_{x_0}] \in (N_2 \times \ldots \times N_r)/O(n)\).

Let us check that the subgroup \( \text{Aut}(j^{r}_{x_0}\tilde{g}) \rightarrow O(n), j^{r+1}_{x_0}\tau \mapsto \tau_* \), coincides with the subgroup

\[
\text{Aut}(\tilde{g}^{2}_{x_0}, \ldots, \tilde{g}^{r}_{x_0}) := \{ \sigma \in O(n) : \sigma^*(\tilde{g}^{k}_{x_0}) = \tilde{g}^{k}_{x_0}, \forall k \leq r \}.
\]
It is clear that if an automorphism \( j^{r+1}_{x_0} \tau \) leaves \( j^r_{x_0}\check{g} \) fixed, then the sequence of its normal tensors must also remain fixed by the automorphism: \( \tau^*(\check{g}^k_{x_0}) = \check{g}^k_{x_0} \).

Reciprocally, given an automorphism \( \sigma : T_{x_0}X \to T_{x_0}X \) of the sequence of normal tensors \( (\check{g}^2_{x_0}, \ldots, \check{g}^r_{x_0}) \), let us consider a normal coordinate system \( z_1, \ldots, z_n \) for \( \check{g} \) at \( x_0 \).

Via the identification provided by the exponential map \( \exp_x : T_{x_0}X \to X \), the map \( \sigma \) can be viewed as a diffeomorphism of \( X \) (a linear transformation of normal coordinates).

In normal coordinates, the expression of the normal tensor \( \check{g}^k_{x_0} \) corresponds to the expression of the homogeneous part of degree \( k \) of the jet metric \( j^r_{x_0}\check{g} \). Hence it is an immediate consequence that the linear transformation \( \sigma \) leaves \( j^r_{x_0}\check{g} \) fixed, i.e. \( j^{r+1}_{x_0}\sigma \in \text{Aut}(j^r_{x_0}\check{g}) \).

The identity \( \text{Aut}(j^r_{x_0}\check{g}) = \text{Aut}(\check{g}^2_{x_0}, \ldots, \check{g}^r_{x_0}) \) implies that the following diagram is commutative:

\[
\begin{array}{ccc}
\mathcal{M}_r & \xrightarrow{t} & \mathcal{T} \\
\mid & \mid & \\
(N_2 \times \ldots \times N_r)/O(n) & \xrightarrow{t} & \mathcal{T} \\
\mid & \mid & \\
[\check{g}^2_{x_0}, \ldots, \check{g}^r_{x_0}] & \mapsto & [\text{Aut}(\check{g}^2_{x_0}, \ldots, \check{g}^r_{x_0})].
\end{array}
\]

Therefore, our theorem has come down to the case of a linear representation \( V(= N_2 \times \ldots \times N_r) \) of a compact Lie group \( G(= O(n)) \) and the corresponding type map:

\[
\begin{array}{ccc}
V/G & \xrightarrow{t} & \mathcal{T} = \{\text{conjugacy classes of closed subgroups of } G\} \\
\mid & \mid & \\
[v] & \mapsto & [\text{Stabilizer subgroup of } v].
\end{array}
\]

For this type map, the analogous properties to 1 – 4 in the statement are well known (see [1], Chap. IX, §9, Th. 2 and Exer. 9).

\( \square \)

**Remark 4.10.** Except for trivial cases, the generic stratum has type \( H_0 = \{0\} \).

**Remark 4.11.** The dimension of the moduli space \( \mathcal{M}_r^n \) (or rather that of its generic stratum) can be deduced directly from the Fundamental Lemma and the formulae giving the dimensions of spaces \( N_r \) of normal tensors which were presented in Section 1.

The result (due, in a different language, to J. Muñoz and A. Valdés, [9]) is as follows:

\[
\begin{align*}
\dim \mathcal{M}_n^0 &= \dim \mathcal{M}_n^1 = 0, \quad \forall n \geq 1; \\
\dim \mathcal{M}_r^1 &= 0, \quad \forall r \geq 0; \\
\dim \mathcal{M}_2^r &= 1, \quad \dim \mathcal{M}_2^r = \frac{1}{2}(r+1)(r-2), \quad \forall r \geq 3; \\
\dim \mathcal{M}_n^r &= n + \frac{(r-1)n^2 - (r+1)n}{2(r+1)} \binom{n+r}{r}, \quad \forall n \geq 3, \; r \geq 2.
\end{align*}
\]
5 Moduli spaces in dimension $n = 2$

5.1 Stratification

We are going to determine the stratification of moduli spaces $M^r_2$ of $r$-jets of Riemannian metrics in dimension $n = 2$.

Let us consider the vector space $\mathbb{R}^2 = \mathbb{C}$, endowed with the standard Euclidean metric, and its corresponding orthogonal group $O(2)$. We will denote by $(x, y)$ the Cartesian coordinates and by $z = x + iy$ the complex coordinate.

Let us denote by $\sigma_m : \mathbb{C} \to \mathbb{C}$ the rotation of angle $2\pi/m$ (that is, $\sigma_m(z) = \varepsilon_m z$, with $\varepsilon_m = \cos(2\pi/m) + i\sin(2\pi/m)$ a primitive $m$th root of unity) and by $\tau : \mathbb{C} \to \mathbb{C}$, $\tau(z) = \bar{z}$ the complex conjugation.

The only (up to conjugacy) closed subgroups of $O(2)$ are the following ones:

1. $SO(2) := \{ \varphi \in O(2) : \det \varphi = 1 \}$ (special orthogonal group),
2. $K_m := \langle \sigma_m \rangle$ (group of rotations of order $m$) ($m \geq 1$),
3. $D_m := \langle \sigma_m, \tau \rangle$ (dihedral group of order $2m$) ($m \geq 1$),
4. $O(2)$ itself.

All these subgroups are normal but the dihedral $D_m$.

The subgroup $SO(2)$ of rotations is identified with the multiplicative group $S_1 \subset \mathbb{C}$ of complex numbers of modulus 1,

$S_1 \overset{\alpha}{\longrightarrow} SO(2)$

$\rho_{\alpha} \mapsto \rho_{\alpha(z)} := \alpha z$.

Besides, every element in $O(2)$ is either $\rho_{\alpha}$ or $\tau \rho_{\alpha}$, for some $\alpha \in S_1$.

The action of $O(2)$ on $\mathbb{R}^2$ induces an action on the algebra $\mathbb{R}[x, y]$ of the polynomials on $\mathbb{R}^2$, to be more specific: $\varphi \cdot P(x, y) := P(\varphi^{-1}(x, y))$.

The following lemma provides us with the list of all invariant polynomials with respect to each of the subgroups of $O(2)$ above mentioned:

**Lemma 5.1.** The following identities hold:

1. $\mathbb{R}[x, y]^{K_m} = \mathbb{R}[x^2 + y^2, p_m(x, y), q_m(x, y)]$,
2. $\mathbb{R}[x, y]^{D_m} = \mathbb{R}[x^2 + y^2, p_m(x, y)]$,
3. $\mathbb{R}[x, y]^{O(2)} = \mathbb{R}[x^2 + y^2]^{SO(2)} = \mathbb{R}[x^2 + y^2]$,

with $p_m(x, y) = \text{Re}(z^m)$ and $q_m(x, y) = \text{Im}(z^m)$.

**Proof.** 1. Let us consider the algebra of polynomials on $\mathbb{R}^2$ with complex coefficients,

$\mathbb{C}[x, y] = \mathbb{C}[z, \bar{z}] = \bigoplus_{ab} \mathbb{C} z^a \bar{z}^b$.

Every summand is stable under the action of $K_m$, since

$\sigma_m \cdot (z^a \bar{z}^b) = \frac{1}{\varepsilon_m^a \varepsilon_m^b} z^a \bar{z}^b = \varepsilon_m^{b-a} z^a \bar{z}^b$.
This formula also tells us that the monomial \( z^a \bar{z}^b \) is invariant by \( K_m \) if and only if \( b-a \equiv 0 \mod m \), that is, \( b-a = \pm km \) for some \( k \in \mathbb{N} \). Then invariant monomials are of the form

\[
z^a \bar{z}^b = (z\bar{z})^a \bar{z}^{km} \quad \text{or} \quad z^a \bar{z}^b = (z\bar{z})^b \bar{z}^{km},
\]

whence

\[
\mathbb{C}[x,y]^{K_m} = \mathbb{C}[z\bar{z}, \bar{z}^m, z^m].
\]

As \( z\bar{z} = x^2 + y^2 \), \( z^m + \bar{z}^m = 2p_m(x,y) \) and \( z^m - \bar{z}^m = 2iq_m(x,y) \), we can conclude that

\[
\mathbb{C}[x,y]^{K_m} = \mathbb{C}[x^2 + y^2, p_m(x,y), q_m(x,y)],
\]

and particularly,

\[
\mathbb{R}[x,y]^{K_m} = \mathbb{R}[x^2 + y^2, p_m(x,y), q_m(x,y)].
\]

2. As \( D_m = \langle K_m, \tau \rangle \), we get

\[
\mathbb{C}[x,y]^{D_m} = (\mathbb{C}[x,y]^{K_m})^{\langle \tau \rangle} = \mathbb{C}[z\bar{z}, \bar{z}^m, z^m]^{\langle \tau \rangle} = \left( \bigoplus_k \mathbb{C}[z\bar{z}]z^{km} \right) + \left( \bigoplus_k \mathbb{C}[z\bar{z}]\bar{z}^{km} \right)^{\langle \tau \rangle}
\]

(as \( \tau \cdot z = \bar{z} \) and \( \tau \cdot \bar{z} = z \))

\[
= \bigoplus_k \mathbb{C}[z\bar{z}] (z^{km} + \bar{z}^{km}) = \mathbb{C}[z\bar{z}, z^m + \bar{z}^m] = \mathbb{C}[x^2 + y^2, p_m(x,y)],
\]

and, in particular,

\[
\mathbb{R}[x,y]^{D_m} = \mathbb{R}[x^2 + y^2, p_m(x,y)].
\]

3. Every summand in the decomposition

\[
\mathbb{C}[z, \bar{z}] = \bigoplus_{ab} \mathbb{C} z^a \bar{z}^b
\]

is stable under the action of \( SO(2) \), since for every \( \rho_\alpha \in SO(2) \) it is satisfied:

\[
\rho_\alpha \cdot (z^a \bar{z}^b) = \frac{1}{\alpha^a \bar{\alpha}^b} z^a \bar{z}^b.
\]

Moreover, this formula assures us that the only monomials \( z^a \bar{z}^b \) which are \( SO(2) \)-invariant are those verifying \( a = b \). Then,

\[
\mathbb{C}[x,y]^{SO(2)} = \mathbb{C}[z, \bar{z}]^{SO(2)} = \mathbb{C}[z\bar{z}] = \mathbb{C}[x^2 + y^2],
\]

whence

\[
\mathbb{R}[x,y]^{SO(2)} = \mathbb{R}[x^2 + y^2].
\]

Finally, this identity tells us that \( SO(2) \)-invariant polynomials are \( O(2) \)-invariant too, so the obvious inclusion \( \mathbb{R}[x,y]^{O(2)} \subseteq \mathbb{R}[x,y]^{SO(2)} \) is indeed an equality. \( \square \)
Corollary 5.2. With the same notations used in the previous lemma, it is verified:

1. $D_m$ is the stabilizer subgroup of the polynomial $p_m(x, y)$, and there exists no polynomial in $\mathbb{R}[x, y]$ of degree $< m$ whose stabilizer subgroup is $D_m$.

2. $K_m(m \geq 2)$ is the stabilizer subgroup of the polynomial $p_m(x, y)+(x^2+y^2)q_m(x, y)$, and there exists no polynomial in $\mathbb{R}[x, y]$ of degree $< m+2$ whose stabilizer subgroup is $K_m$.

3. $K_1 = \{\text{Id}\}$ is the stabilizer subgroup of the polynomial $x + xy$, and there exists no polynomial in $\mathbb{R}[x, y]$ of degree $< 2$ whose stabilizer subgroup is $K_1$.

Proof. 1. Using that every element in $O(2)$ is either of the form $\rho_\alpha$ or of the form $\rho_\alpha \circ \tau$, it is a matter of routine to check that the stabilizer subgroup of the polynomial $p_m(x, y) = \text{Re}(z^m)$ is $D_m$.

If there were another polynomial $\bar{p}(x, y)$ of degree $< m$ with the same property, $\bar{p}(x, y)$ should be a power of $x^2+y^2$, because of Lemma 5.1(2), and in that case its stabilizer subgroup would be the whole $O(2)$, against our hypothesis.

2. According to Lemma 5.1(1), every $K_m$-invariant polynomial of degree $\leq m$ is of the form $\lambda p_m(x, y) + \mu q_m(x, y)$ (up to addition of a power of $x^2+y^2$). However, a polynomial of such a form does not have $K_m$ as its stabilizer subgroup, but a larger dihedral group: after multiplying by a scalar, we can indeed assume $\lambda^2 + \mu^2 = 1$; if $\alpha = \lambda - i\mu$, then

$$\lambda p_m(x, y) + \mu q_m(x, y) = \text{Re}(\alpha z^m) = \text{Re}((\beta z)^m)$$

(with $\beta^m = \alpha$)

$$= \rho_{\beta^{-1}} \cdot \text{Re}(z^m) = \rho_{\beta^{-1}} \cdot p_m(x, y),$$

whose stabilizer subgroup is the dihedral group $\rho_{\beta^{-1}} \cdot D_m \cdot \rho_{\beta}$, which is conjugate to the stabilizer subgroup $D_m$ of $p_m(x, y)$. (In particular, taking $\lambda = 0$, $\mu = -1$, we get that the stabilizer subgroup of $q_m(x, y)$ is $\rho_{\beta^{-1}} \cdot D_m \cdot \rho_{\beta}$, for $\beta^m = i$).

As no polynomial of degree $\leq m$ has the desired stabilizer subgroup $K_m$, and there are not any $K_m$-invariant polynomials of degree $m+1$ (up to a power of $x^2+y^2$), the following degree to be considered is $m+2$. The stabilizer subgroup of the polynomial $p_m(x, y)+(x^2+y^2)q_m(x, y)$, of degree $m+2$, is the intersection of the stabilizer subgroups of its two homogeneous components, $p_m(x, y)$ and $(x^2+y^2)q_m(x, y)$, that is,

$$D_m \cap (\rho_{\beta^{-1}} \cdot D_m \cdot \rho_{\beta}) = K_m \quad (\beta^m = i).$$

3. This case is trivial.

\[\square\]

Theorem 5.3. The strata in the moduli space $\mathcal{M}_2$ correspond exactly to the following types: $[O(2)], [D_1], \ldots, [D_{r-2}], [K_1], \ldots, [K_{r-4}]$. (And also $[K_1]$, if $r = 4$.)

Proof. It is a classical result (see [2]) that in dimension 2 every Riemannian metric can be written in normal coordinates $(x, y)$ (in a unique way up to an orthogonal transformation) as follows:

$$g = dx^2 + dy^2 + h(x, y)(ydx - xdy)^2,$$

for some smooth function $h(x, y)$.
Observe that the stabilizer subgroup of $O(2)$ for the jet $j^h_{0}\,\!h$ is the same as that for $j^{k+2h}_{0}\,\!g$.

If we take $h(x, y) = 0$, we get a metric (the Euclidean one, i.e. $g = dx^2 + dy^2$) whose group of automorphisms (for any jet order) is $O(2)$.

Choosing $h(x, y) = p_m(x, y)$, we obtain an $r$–jet metric (with $r \geq m + 2$) whose stabilizer subgroup is $D_m$, because of Corollary 5.2 (1).

If we choose $h(x, y) = p_m(x, y) + (x^2 + y^2)q_m(x, y)$, we get an $r$–jet metric (with $r \geq m + 4$) whose stabilizer subgroup is $K_m$, by Corollary 5.2 (2).

If we make $h(x, y) = x + xy$, then we get an $r$–jet metric (with $r \geq 4$) whose stabilizer subgroup is $K_1$, according to Corollary 5.2 (3).

Finally, let us note that no $r$–jet metric can have $SO(2)$ as its stabilizer subgroup, since such a metric would correspond to a jet function $j^r_{0}\,\!h$ whose stabilizer subgroup should be $SO(2)$, which is impossible, because, by Lemma 5.1 (3), every $SO(2)$–invariant polynomial is also $O(2)$–invariant. □

**Corollary 5.4.** Every closed subgroup of $O(2)$, except for $SO(2)$, is the group of automorphisms of a jet metric $j^r_{0}\,\!g$ on $\mathbb{R}^2$ for some order $r$.

**Corollary 5.5.** The number of strata in $M^r_2$ is:

\[
\text{Number of strata in } M^r_2 = \begin{cases} 
1 & \text{for } r = 0, 1, 2 \\
2 & \text{for } r = 3 \\
4 & \text{for } r = 4 \\
2r - 5 & \text{for } r \geq 5
\end{cases}
\]

### 5.2 Examples

Now we describe, without proofs, low order jets in dimension $n = 2$.

For order $r = 0, 1$ (and in any dimension $n$) moduli spaces $M^r_2$ come down to a single point.

**Case** $r = 2$.

The moduli space is a line:

\[
M^2_2 \longrightarrow \mathbb{R} \; , \; [j^2_{x_0}\,\!g] \longrightarrow K_g(x_0) \plane
\]

In other words, the curvature classifies $2$–jets of Riemannian metrics in dimension $n = 2$.

In this case there is just one stratum, the generic one, whose type is $\,\!\{O(2)\}$.

**Case** $r = 3$.

The moduli space is a closed semiplane:

\[
M^3_2 \longrightarrow \mathbb{R} \times [0, +\infty) \; , \; [j^3_{x_0}\,\!g] \longrightarrow (K_g(x_0), |\text{grad}_{x_0}K_g|^2) \plane
\]

That is to say, the curvature and the square of the modulus of the gradient of the curvature classify $3$–jet metrics in dimension $n = 2$.  

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Now we have two different strata:

The generic stratum $S_{[O_1]} = \mathbb{R} \times (0, +\infty)$, with type $[D_1]$. This stratum is the set of all classes of jets $j^3_{x_0}g$ verifying $\nabla_{x_0}K_g \neq 0$ (in this case, the group of automorphisms is the group of order 2 generated by the reflection across the vector $\nabla_{x_0}K_g$).

The non-generic stratum $S_{[O(2)]} = \mathbb{R} \times \{0\}$, with type $[O(2)]$, is the set of all classes of jets $j^3_{x_0}g$ verifying $\nabla_{x_0}K_g = 0$ (which are invariant with respect to every orthogonal transformation of normal coordinates).

Note: If we consider metrics of signature $(+, -, -)$, instead of Riemannian metrics, then the map

$$\mathbb{M}_2^3 \longrightarrow \mathbb{R} \times [0, +\infty)$$

is not injective, that is, differential invariants do not classify 3-jet metrics of signature $(+, -, -)$. To illustrate this, consider two metrics $g, \tilde{g}$ of signature $(+, -, -)$, such that $K_g(x_0) = \tilde{K}_g(x_0)$, $\nabla_{x_0}K_g = 0$ and $\nabla_{x_0}\tilde{K}_g$ is a non-zero isotropic vector with respect to $\tilde{g}_{x_0}$. Both jets $j^3_{x_0}g, j^3_{x_0}\tilde{g}$ cannot be equivalent (because the gradient of the curvature at $x_0$ equals zero for the first metric, whereas it is non-zero for the other one), but its differential invariants coincide: $K_g(x_0) = \tilde{K}_g(x_0)$ and $|\nabla_{x_0}K_g|^2 = |\nabla_{x_0}\tilde{K}_g|^2 = 0$.

Case $r = 4$.

A set of generators for differential invariants of order 4 is given by the following five functions:

$$p_1(j^4_{x_0}g) = K_g(x_0),$$
$$p_2(j^4_{x_0}g) = |\nabla_{x_0}K_g|^2,$$
$$p_3(j^4_{x_0}g) = \text{trace (Hess}_{x_0}K_g),$$
$$p_4(j^4_{x_0}g) = \text{det (Hess}_{x_0}K_g),$$
$$p_5(j^4_{x_0}g) = \text{Hess}_{x_0}K_g(\nabla_{x_0}K_g, \nabla_{x_0}K_g),$$

where $\text{Hess}_{x_0}K_g := (\nabla dK_g)_{x_0}$ stands for the hessian of the curvature function at $x_0$.

These above functions satisfy the following inequalities:

$$p_2 \geq 0, \quad p_3^2 - 4p_4 \geq 0, \quad (2p_5 - p_2p_3)^2 \leq p_2^2(p_3^2 - 4p_4).$$

To say it in other words, these five differential invariants define an isomorphism of differentiable spaces

$$(p_1, \ldots, p_5) : \mathbb{M}_2^4 \longrightarrow Y \subset \mathbb{R}^5$$

$Y$ being the closed subset in $\mathbb{R}^5$ determined by the inequalities

$$x_2 \geq 0, \quad x_2^3 - 4x_4 \geq 0, \quad (2x_5 - x_2x_3)^2 \leq x_2^2(x_3^2 - 4x_4).$$

In this case, the moduli space $\mathbb{M}_2^4$ has the following four strata:

- The generic stratum of all classes of jets $j^4_{x_0}g$ verifying that $\nabla_{x_0}K_g$ is not an eigenvector of $\text{Hess}_{x_0}K_g$ (therefore, the eigenvalues of $\text{Hess}_{x_0}K_g$ are different). The type of this stratum (group of automorphisms of its jets) is $[K_1 = \{\text{Id}\}]$. 

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- The stratum of those classes of jet metrics \( j^4_{x_0} g \) verifying that \( \text{grad}_{x_0} K_g \) is a non-zero eigenvector of \( \text{Hess}_{x_0} K_g \). Its type is \([D_1]\): the group of automorphisms of each jet metric is generated by the reflection across the vector \( \text{grad}_{x_0} K_g \).

- The stratum composed of those classes of jet metrics \( j^4_{x_0} g \) with \( \text{grad}_{x_0} K_g = 0 \) and verifying that the eigenvectors of \( \text{Hess}_{x_0} K_g \) are different. The type of this stratum is \([D_2]\): the group of automorphisms of each jet metric is generated by the reflections across either eigenvector of \( \text{Hess}_{x_0} K_g \).

- The stratum of all classes of jets \( j^4_{x_0} g \) with \( \text{grad}_{x_0} K_g = 0 \) and verifying that the eigenvectors of \( \text{Hess}_{x_0} K_g \) are both equal. The type of the stratum is \([O(2)]\).

6 Appendix A: On the notion of differential invariant of metrics

The aim of this Appendix A is to discuss the notion of differential invariant and to back up the Definition 2.3 given in Section 2.

The notion of differential invariant must be understood as a particular case of the concept of regular and natural operator between natural bundles (see [5] for an exposition of the theory of natural bundles). What follows is an adaptation of this point of view, getting around, though, the concept of natural bundle.

Let \( X \) be an \( n \)-dimensional smooth manifold. Let \( M \to X \) be the bundle of semi-Riemannian metrics of a fixed signature \((p,q)\) and let \( \mathcal{M}_X \) denote its sheaf of smooth sections.

Loosely speaking, the concept of differential invariant refers to a function “intrinsically, locally and smoothly constructed from a metric”. Rigorously, as it is a local construction, a differential invariant is a morphism of sheaves:

\[
f : \mathcal{M}_X \to C^\infty_X,
\]

where \( C^\infty_X \) stands for the sheaf of smooth functions on \( X \).

The intuition of “intrinsic and smooth construction” can be encoded by saying that the morphism \( f \) also satisfies the following two properties:

1.- **Regularity**: If \( \{g_s\}_{s \in S} \) is a family of metrics depending smoothly on certain parameters, the family of functions \( \{f(g_s)\}_{s \in S} \) also depends smoothly on those parameters.

   To be exact, let \( S \) be a smooth manifold (the space of parameters) and let \( U \subseteq X \times S \) be an open set. For each \( s \in S \), consider the open set in \( X \) defined as \( U_s := \{x \in X : (x,s) \in U\} \). A family of metrics \( \{g_s \in \mathcal{M}(U_s)\}_{s \in S} \) is said to be smooth if the fibre map \( U \to S^2T^*X, (x,s) \mapsto (g_s)_x \), is smooth. In the same way, a family of functions \( \{f_s \in C^\infty(U_s)\}_{s \in S} \) is said to be smooth if the function \( U \to \mathbb{R}, (x,s) \mapsto (f_s)(x) \), is smooth.

   In these terms, the regularity condition expresses that for each smooth manifold \( S \), each open set \( U \subseteq X \times S \) and each smooth family of metrics \( \{g_s \in \mathcal{M}(U_s)\}_{s \in S} \), the family of functions \( \{f(g_s) \in C^\infty(U_s)\}_{s \in S} \) is smooth.

2.- **Naturalness**: The morphism of sheaves \( f \) is equivariant with respect to the action of local diffeomorphisms of \( X \).

   That is, for each diffeomorphism \( \tau : U \to V \) between open sets of \( X \) and for each metric \( g \) on \( V \), the following condition must be satisfied:

\[
f(\tau^* g) = \tau^*(f(g)).
\]
Taking into account the previous comments, the suitability of the following definition is now clear:

**Definition 6.1.** A differential invariant associated to semi-Riemannian metrics (of the fixed signature) is a regular and natural morphism of sheaves $f : \mathcal{M}_X \to \mathcal{C}_X^\infty$.

Note that this definition of differential invariant seems to be far too general, since a differential invariant $f(g)$ is not assumed a priori to be constructed from the coefficients of the metric $g$ and their subsequent partial derivatives. As we are going to show below, this question is clarified by a beautiful result by J. Slovák.

For every integer $r \geq 0$, we denote by $J^r M \to X$ the fiber bundle of $r$-jets of semi-Riemannian metrics on $X$ (of the prefixed signature). The fiber bundle $J^\infty M \to X$ of $\infty$-jets of semi-Riemannian metrics is not a smooth manifold, but it can be endowed with the structure of a ringed space as follows. On $J^\infty M \to X$ we consider the inverse limit topology: $J^\infty M = \lim \leftarrow J^r M$; a function on an open set $U \subseteq J^\infty M$ is said to be differentiable if it is locally the composition of one of the natural projections $U \subseteq J^\infty M \to J^r M$ with a smooth function on $J^r M$. This way, $J^\infty M$ is a ringed space, with its sheaf of differentiable functions.

In a similar manner, the structure of a ringed space is defined for the fiber of the bundle $J^\infty M \to X$ over a given point $x_0 \in X$: $J^\infty x_0 M = \lim \leftarrow J^r x_0 M$.

**Theorem 6.2.** (Slovák) There exists the following bijective correspondence:

$$
\begin{array}{ccc}
\{\text{differentiable functions } \tilde{f} : J^\infty M \to \mathbb{R}\} & \xrightarrow{\sim} & \{\text{regular morphisms of sheaves } f : \mathcal{M}_X \to \mathcal{C}_X^\infty\} \\
\downarrow & & \downarrow \\
\{\text{regular morphisms of sheaves } \tilde{f} : \mathcal{M}_X \to \mathcal{C}_X^\infty\} & \xrightarrow{f} & \{\text{regular morphisms of sheaves } f : \mathcal{M}_X \to \mathcal{C}_X^\infty\}
\end{array}
$$

with $f(g)(x) := \tilde{f}(j^\infty x g)$.

The result by Slovák [12] refers, with a bit more of generality, to regular morphisms between sheaves of sections of fiber bundles.

If a regular morphism $\mathcal{M}_X \to \mathcal{C}_X^\infty$ is, furthermore, natural (that is, a differential invariant), then the corresponding smooth function $\tilde{f} : J^\infty M \to \mathbb{R}$ is determined by its restriction to the fiber $J^\infty x_0 M$ of an arbitrary point $x_0 \in X$. This assertion can be expressed more precisely in the following way.

**Corollary 6.3.** Fixed a point $x_0 \in X$, the set of differential invariants $f : \mathcal{M}_X \to \mathcal{C}_X^\infty$ is in bijection with the set of differentiable $\text{Diff}_{x_0}$-invariant functions $\tilde{f} : J^\infty x_0 M \to \mathbb{R}$.

**Definition 6.4.** A differential invariant $f : \mathcal{M}_X \to \mathcal{C}_X^\infty$ is said to be of order $\leq r$ if the corresponding differentiable function $\tilde{f} : J^\infty M \to \mathbb{R}$ factors through the projection $J^\infty M \to J^r M$.

Reformulating Corollary 6.3 for invariants of order $r$, we obtain that Definition 6.4 coincides with that originally given in Section 2 (Definition 2.3).
Corollary 6.5. Fixed a point \( x_0 \in X \), the set of all differential invariants
\[ f : \mathcal{M}_X \to C^\infty_X \]
of order \( \leq r \) is in bijection with the set of all smooth \( \text{Diff}_{x_0} \)-invariant functions
\[ \tilde{f} : J^r_{x_0} M \to \mathbb{R}. \]

7 Appendix B: Classification of \( \infty \)-jets of metrics

In Section 4 we have seen that differential invariants of order \( \leq r \) classify \( r \)-jets of Riemannian metrics at a point (Theorem \[ \text{[3]} \]). We are now going to generalize this result for infinite-order jets.

In the proof of next lemma we will use the following well-known fact ([1], Chap. IX, § 9, Lemma 6):

Let \( G \) be a compact Lie group. Every decreasing sequence of closed subgroups \( H_1 \supseteq H_2 \supseteq H_3 \supseteq \cdots \) stabilizes, that is, there exists an integer \( s \) such that \( H_s = H_{s+1} = H_{s+2} = \cdots \).

Lemma 7.1. Let \( G \) a compact Lie group and let
\[ \cdots \longrightarrow X_{r+1} \longrightarrow X_r \longrightarrow \cdots \longrightarrow X_1 \]
be an inverse system of smooth \( G \)-equivariant maps between smooth manifolds endowed with a smooth action of \( G \). There exists an isomorphism of ringed spaces:
\[ (\lim_{\leftarrow} X_r)/G \cong \lim_{\leftarrow} (X_r/G) \]
\[ \left(\ldots, x_2, x_1 \right) \longmapsto \left(\ldots, [x_2], [x_1] \right). \]

Proof. Because of the universal quotient property, compositions of morphisms
\[ \lim_{\leftarrow} X_r \longrightarrow X_r \longrightarrow X_r/G \]
\[ \left(\ldots, x_2, x_1 \right) \longmapsto \left[ x_r \right] \]
induce morphisms of ringed spaces
\[ (\lim_{\leftarrow} X_r)/G \longrightarrow (X_r/G) \]
\[ \left(\ldots, x_2, x_1 \right) \longmapsto \left[ x_r \right], \]
which, for their part, because of the universal inverse limit property, define a morphism of ringed spaces
\[ (\lim_{\leftarrow} X_r)/G \overset{\varphi}{\longrightarrow} \lim_{\leftarrow} (X_r/G) \]
\[ \left(\ldots, x_2, x_1 \right) \longmapsto \left(\ldots, [x_2], [x_1] \right). \]

It is easy to check that this morphism is surjective. Let us see that it is also injective.

First note that, given a point \( \left(\ldots, x_2, x_1 \right) \in \lim_{\leftarrow} X_r \), we can get the decreasing sequence \( H_{x_1} \supseteq H_{x_2} \supseteq H_{x_3} \supseteq \cdots \) of closed subgroups of \( G \), where \( H_{x_k} \) stands for the
stabilizer subgroup of $x_k$. This chain stabilizes, since $G$ is compact, so for a certain $s$ 
 it is verified $H_{x_s} = H_{x_{s+1}} = H_{x_{s+2}} = \cdots$

Let now $[(\ldots, x_2, x_1)]$ and $[(\ldots, x'_2, x'_1)]$ be two points in $(\operatorname{lim} X_r)/G$ having the 
 same image through $\varphi$, i.e. $[x_k] = [x'_k]$, for each $k \geq 0$. Write $x'_s = g \cdot x_s$ for some 
 $g \in G$. As the morphisms $X_s \to X_k$ (with $s \geq k$) are $G$-equivariant, it is verified that 
 $x'_k = g \cdot x_k$ for every $k \leq s$.

Let us show that the same happens when $k > s$. As $[x_k] = [x'_k]$, we have $x'_k = g_k \cdot x_k$ 
 for a certain $g_k \in G$: applying that $X_k \to X_s$ is equivariant yields $x'_s = g_k \cdot x_s$, and then (comparing with $x'_s = g \cdot x_s$) 
 $g^{-1}g_k \in H_{x_k}$; since $H_{x_k} = H_{x_k}$, it follows that $g^{-1}g_k \in H_{x_k}$, and hence the condition $x'_k = g_k \cdot x_k$ is equivalent to $x'_k = g \cdot x_k$. In conclusion, $x'_k = g \cdot x_k$ for every $k > 0$, and therefore $[(\ldots, x_2, x_1)]$ and $[(\ldots, x'_2, x'_1)]$ 
 are the same point in $(\operatorname{lim} X_r)/G$.

Once we have proved that $\varphi$ is bijective, it is routine to check that $\varphi$ is an isomorphism of ringed spaces.

**Definition 7.2.** Let $x_0 \in X$ and let 

$$J^\infty_{x_0} M := \operatorname{lim} J^r_{x_0} M$$

be the ringed space of $\infty$--jets of Riemannian metrics at $x_0$ on $X$. The quotient ringed space 

$$M^\infty_n := J^\infty_{x_0} M / \Diff x_0$$

is called moduli space of $\infty$--jets of Riemannian metrics in dimension $n$.

In the same fashion as for finite-order jets, the moduli space $M^\infty_n$ depends neither on the choice of the point $x_0$ nor on that of the $n$--dimensional manifold $X$.

For every integer $r > 0$, we have an evident morphism of ringed spaces  

$$M^\infty_n \to \operatorname{lim} \Diff x_0$$

and these morphisms allow us to define another morphism of ringed spaces:

$$M^\infty_n \to \operatorname{lim} \Diff x_0$$

**Theorem 7.3.** There exists an isomorphism of ringed spaces

$$M^\infty_n \to \operatorname{lim} \Diff x_0$$

**Proof.** Fix a local coordinate system $(z_1, \ldots, z_n)$ centered at $x_0$. With the same notations as in Section 3, let us define 

$$\mathcal{N}^\infty := \operatorname{lim} \mathcal{N}^r.$$ 

In other words, $\mathcal{N}^\infty$ is the subspace of $J^\infty_{x_0} M$ formed by all those $\infty$--jets at $x_0$ of Riemannian metrics having $(z_1, \ldots, z_n)$ as a normal coordinate system. All lemmas in
Section 3, with their corresponding proofs, remain valid when substituting the integer $\infty$ for $r$. In particular, our Fundamental Lemma 3.4 when $r = \infty$, gives us the desired isomorphism of ringed spaces:

\[
M_n^\infty = \left( \prod_{k \geq 2} N_k \right) / O(n) = \left( \lim \left( N_2 \times \cdots \times N_r \right) \right) / O(n)
\]

(by Lemma 7.1)

\[
= \lim \left( (N_2 \times \cdots \times N_r) / O(n) \right) = \lim M_n^r.
\]

\[\square\]

**Corollary 7.4.** Differential invariants of finite order classify $\infty$—jets of Riemannian metrics: Two jet metrics $j_{x_0}^\infty g$ and $j_{x_0}^\infty \bar{g}$ are equivalent if and only if for each finite-order differential invariant $h$ it is satisfied $h(g)(x_0) = h(\bar{g})(x_0)$.

**Proof.** According to Theorem 7.3 we get:

\[
j_{x_0}^\infty g \equiv j_{x_0}^\infty \bar{g} \iff j_{x_0}^r g \equiv j_{x_0}^r \bar{g}, \quad \forall r \geq 0.
\]

To complete our proof, it is sufficient to use the fact that differential invariants of order $\leq r$ classify $r$—jet metrics (Theorem 4.3).

\[\square\]

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