Geometry of the proper asymmetric norm

A. B. Németh
Faculty of Mathematics and Computer Science
Babeș Bolyai University, Str. Kogălniceanu nr. 1-3
RO-400084 Cluj-Napoca, Romania
email: nemab@math.ubbcluj.ro

Abstract

The convex geometric approach in the study of asymmetric norms can be useful in their deeper investigation. The note illustrates this in the case of so called proper asymmetric norms, notion revealed in analytical context in [3] and [4].

1. Introduction

While searching for asymmetric vector norms [3] and mutually polar retractions on convex cones [4], retractions on one dimensional cones play a special role. These retractions are intrinsically related to some special, so called proper asymmetric norms. We will show in this note that the analytic conditions on these asymmetric norms have simple interpretation in convex geometric setting.

Definition 1 [1] Let $X$ be a real vector space. The functional $p : X \to \mathbb{R}_{+} = [0, +\infty)$ is said an asymmetric norm if the following conditions hold:

1. $p$ is positive homogeneous, i.e., $q(tx) = tq(x), \forall t \in \mathbb{R}_{+}, \forall x \in X$;
2. $p$ is subadditive, i.e., $p(x + y) \leq p(x) + p(y), \forall x, y \in X$;
3. If $p(x) = p(-x) = 0$ then $x = 0$.

The theory of asymmetrically normed spaces concerns merely on the investigation and relation of the various $T_0$ topologies induced by asymmetric norms on $X$ and the related functional analytic problems. In our approach in [3] and [4] the difference lies in the fact that we consider a priori a norm on $X$ and that in the Definition [1] we have supposed $p$ continuous.
Definition 2: Adapting the terminology from [3], an asymmetric norm \( q : X \to \mathbb{R}_+ \) is said proper if there exists an element \( u \in X \) with \( q(u) = 1 \), such that

\[
q(x - q(x)u) = 0 \quad \forall \ x \in X.
\]  

The note is organized as follows:

After the definition of cones and ordering in a short Section 2, in Section 3 we give the complete geometric characterization of the proper asymmetric norm. In Section 4 we use the result of the preceding section in the study of mutually polar retractions on convex cones of a normed space. The obtained results generalize some results in [3] and [4].

2. Cones and orderings

Let \( X \) be a real vector space. The set \( K \subset X \) is called a pointed convex cone if (i) \( x, y \in K \Rightarrow x + y \in K \), (ii) \( x \in K, t \in \mathbb{R}_+ = [0, +\infty) \Rightarrow tx \in K \), and (iii) \( K \cap (-K) = \{0\} \).

The relation

\[
x \leq_K y \iff y - x \in K
\]

defines a reflexive, transitive and anti-symmetrical order relation on \( K \).

In the convex geometry a cone in a normed space is called proper if it is convex, pointed, closed and possesses interior points. (In [2] the proper cone means pointed cone.)

3. The geometry of asymmetric norms

We shall be in keeping with the terminology in [1] and [2].

Let \( X \) be a real vector space.

The set \( A \subset X \) is called absorbent if

\[
\forall x \in X \ \exists t \in \mathbb{R}_+ = (0, +\infty) \text{ such that } x \in tA.
\]

The set \( A \) is absorbent with respect to \( a \in A \), if \( A - a \) is absorbent.

In the presence of a locally convex topology on \( X \), every non-empty convex set \( C \) in \( X \) is absorbent with respect to every its interior point.

If \( A \) is absorbent, then the functional \( g : X \to \mathbb{R}_+ \) defined with

\[
g(x) = \inf\{t \in \mathbb{R}_+ : x \in tA\}
\]

is called the gauge of \( A \) and is denoted gauge \( A \).

If \( A - a \) with \( a \in A \) is absorbent, then gauge(\( A - a \)) is said the gauge of \( A \) with respect to its point \( a \).

If \( p : X \to \mathbb{R}_+ \) is the asymmetric norm in Definition 1, then we put

\[
B'_{p} = \{x \in X : p(x) < 1\},
\]

\[
B_{p} = \{x \in X : p(x) \leq 1\},
\]

2
and
\[ \Sigma_p = \{ x \in X : p(x) = 1 \} . \]

Obviously, \( B'_p \) and \( B_p \) are both convex and absorbent and \( p \) is the gauge of both \( B'_p \) and \( B_p \).

The family
\[ \{ tB'_p : t \in \mathbb{R}^+ \} \]
form the neighborhood basis of a \( T_0 \) topology in \( X \) which we shall call the \( p \)-topology.

The functional \( p^* : X \to \mathbb{R}_+ \) defined by
\[ p^*(x) = \max \{ p(x), p(-x) \} \]
is a norm on \( X \) called the \( p^* \)-norm.

Since \( p \leq p^* \), the functional \( p \) is \( p^* \)-continuous.

The set
\[ K_p = \{ x \in X : p(x) = 0 \} \]
is a pointed convex cone. Indeed \( x, y \in K_p \Rightarrow x + y \in K_p, \ x \in K_p, \ t \in \mathbb{R}_+ \Rightarrow tx \in K_p \) and \( x, -x \in K_p \Rightarrow x = 0 \). \( K_p \) is also \( p^* \)-closed since \( p \) is \( p^* \)-continuous.

**Theorem 1** Suppose that \( X \) is a real vector space and \( q : X \to \mathbb{R}_+ \) is an asymmetric norm. Then the following assertions are equivalent:

(i) \( q \) is proper, that is there exists \( u \in X \) with \( q(u) = 1 \) such that
\[ q(x - q(x)u) = 0 \ \forall \ x \in X . \]

(ii) \[ q(x - u) = 0 \ \forall \ x \in \Sigma_q. \] (2)

(iii) \( \{ x - q(x)u : x \in X \} = K_q, \) where \( K_q = \{ x \in X : q(x) = 0 \} \).

(iv) There exists a norm on \( X \) such that \( q \) is the gauge with respect to an interior point of a proper cone.

**Proof.**

The relations (i) \( \Leftrightarrow \) (ii) \( \Leftrightarrow \) (iii) are obvious.

(ii) \( \Rightarrow \) (iv).

From (2) we have that \( q \) is zero on the set \( \Sigma_q - u \). This set is nothing else as the translation of the boundary of \( B_q \) with \(-u\). Since \( B_q - u \) is a convex set and the non-negative convex functional \( q \) vanishes on the boundary of \( B_q - u \), it must be zero on this convex set. Hence
\[ B_q - u \subset K_q. \] (3)

\( B'_q - u \) is \( q^* \) open and it is contained in the \( q^* \)-closed pointed cone \( K_q \), whereby this cone is \( q^* \)-proper.
Let \( k \in K_q \) be arbitrary. Then \( y = k + u \in B_q \) since \( q(y) = q(k + u) \leq q(k) + q(u) = q(u) = 1 \). Hence \( k = y - u \), where \( y \in B_q \). It follows that \( K_q \subset B_q - u \). Together with (3) we have

\[
B_q - u \subset K_q \subset B_q - u.
\]

Hence \( B_q \) is the translation with \(-u\) of the \( q^*\)-proper cone \( K_q \), and

\[
q = \text{gauge}(K_q + u).
\]

\(-u\) is a \( q^*\)-interior point of \( B_q - u = K_q \). Thus \( q \) is the gauge of the \( q^*\)-proper cone \( K_q \) in a normed space with respect to its interior point \(-u\). This is nothing else as (iv).

(iv) \( \Rightarrow \) (ii).

Suppose that \( X \) is a normed vector space and \( K \subset X \) is a proper cone.

Take \(-u \in \text{int} \, K \) and put

\[
q = \text{gauge}(K + u),
\]

that is let \( q \) be the gauge of \( K \) with respect to its interior point \(-u\). Then \( q \) is an asymmetric norm,

\[
B_q = K + u,
\]

and

\[
K_q = K.
\]

Then \( \Sigma_q = \text{bdr}(K + u) = \text{bdr} \, K + u \), hence \( u \in \Sigma_q \) and \( q(u) = 1 \).

\( \forall x \in \Sigma_q \), \( x - u \in \text{bdr} \, K \subset K \), hence \( q(x - u) = 0 \), which is nothing else as (2), that is, we have condition (ii) fulfilled for \( q \).

\( \square \)

In conclusion an asymmetric norm is proper if and only if it is the gauge of a proper cone with respect to its interior point.

4. Application to mutually polar retractions

Let \( X \) be a real normed space. The mapping \( S : X \to X \) is called \textit{idempotent} if \( S^2 = S \).

**Definition 3** The mapping \( T : X \to X \) is called \textit{retraction} if:

(i) It is a continuous idempotent mapping;

(ii) It is positive homogeneous, that is, \( T(tx) = tTx \) for every \( x \in X \) and every \( t \in \mathbb{R}_+ = [0, +\infty) \);

(iii) \( T(X) \) is a non-empty, non-zero, closed pointed convex cone.

(iv) \( Tx \in \text{bdr} \, T(X) \) for any \( x \in X \setminus T(X) \).
If $S : X \to X$ is a retract with $K = S(X)$, then it is called \textit{subadditive} if
\[ S(x + y) \leq_K Sx + Sy. \]
We will call a retract subadditive, if it is subadditive with respect to the ordering its range endows.

\textbf{Definition 4} Let $X$ be a real normed space, $0$ its zero mapping and $I$ its identity mapping. The mappings $Q, R : X \to X$ are called \textit{mutually polar retractions} if
\begin{enumerate}[(i)]
  \item $Q$ and $R$ are retractions,
  \item $Q + R = I$,
  \item $QR = RQ = 0$.
\end{enumerate}

Next we use the notations and definitions in the preceding section.

\textbf{Theorem 2} Let $Q, R : X \to X$ be a pair of mutually polar retractions with $Q(X) = M$ and $R(X) = N$. If $\dim N = 1$ and $R$ is subadditive then
\begin{enumerate}[(i)]
  \item The functional $q$ from the representation $Rx = q(x)u$ with $q(u) = 1$, is a proper asymmetric norm.
  \item $M = K_q = \{x \in X : q(x) = 0\}$ is a proper cone, $-u \in \text{int } M$.
  \item $q = \text{gauge}(M + u)$.
  \item $Q$ is subadditive.
\end{enumerate}

\textbf{Proof.}
\begin{enumerate}[(i)]
  \item Obviously, $R$ can be represented in the form $Rx = q(x)u$ with $u \in N$ and $q(u) = 1$, where $q$ is subadditive and positively homogeneous since $R$ is so.
  \hspace{0.5cm} If $q(x) = q(-x) = 0$, then by $Q + R = I$ we have $x = Qx \in M - x = Q(-x) \in M$, hence $x = 0$ since $M$ is pointed by definition.
  \hspace{0.5cm} Since $Q = I - R$ is idempotent, we have
  \[ x - q(x)u = (I - R)x = (I - R)^2x = (I - R)(I - R)x = (I - R)x - R(I - R)x = x - q(x)u - q(x - q(x)u)u, \forall x. \]
  \hspace{0.5cm} Hence
  \[ q(x - q(x)u) = 0, \forall x \in X, \]
  \hspace{0.5cm} and thus $q$ is a proper asymmetric norm.
  \item If $x \in K_q$, then $x = Qx + q(x)u = Qx \in M$. Thus $K_q \subset M$. Since $RQ = 0$ we have $R(Qx) = q(Qx)u = 0 \forall x \in X$, that is, $Qx \in K_q, \forall x \in X$. Hence $M = Q(X) \subset K_q$.
  \hspace{0.5cm} From Theorem $[?]$ $K_q = M$ is a proper cone and $-u \in \text{int } M$.\end{enumerate}
(iii) From the same Theorem, $q = \text{gauge}(u + K_q) = \text{gauge}(M + u)$.

(iv) We have to see that

$$Qx + Qy - Q(x + y) \in M, \quad \forall \, x, y \in X.$$  \hfill (4)

From $Q = I - R$ the left hand side of this relation rewrites as

$$x - Rx + y - Ry - (x + y - R(x + y)) = -Rx - Ry + R(x + y).$$

Now, $-Rx - Ry + R(x + y) \in -N$ as $R$ is subadditive. Since by (ii) $-N = \{-tu : t \in \mathbb{R}_+\} \subset M$, we have (4) fulfilled.

$\blacksquare$

**Theorem 3** If $M \subset X$ is a proper cone with $-u \in \text{int} M$ and $q = \text{gauge}(M + u)$, then

(i) $$Rx = q(x)u, \quad x \in X, \quad \text{and} \quad Q = I - R$$

are mutually polar retractions with $R(X) = \{tu : t \in \mathbb{R}_+\}$ and $Q(X) = M$.

(ii) $Q$ and $R$ are subadditive.

**Proof.**

(i) From Theorem $\blacksquare \hspace{1cm} \square$ $q$ is a proper asymmetric norm with $K_q = M$.

Hence

$$(I - R)^2x = (I - R)(I - R)x = (I - R)x - R(I - R)x = x - q(x)u - q(x - q(x))u = x - q(x)u = (I - R)x, \quad \forall \, x \in X,$$

and it follows that $Q = I - R$ is idempotent.

$Qx = x - q(x)u = x$ if and only if $q(x) = 0$ i.e., if $x \in M$. Thus $Q(X) = M$.

(ii) $R$ is subadditive and $-R(X) = \{-tu : t \in \mathbb{R}_+\} \subset M$ and repeating the argument in the proof of item (iv) of Theorem $\blacksquare \hspace{1cm} \square$ it follows that $Q$ is subadditive too.

$\blacksquare$

In conclusion: Two mutually polar retractions $Q, R : X \to X$ with $\dim R(X) = 1$, $R(x) = q(x)u$, $q(u) = 1$ are subadditive, if and only if $Q(X)$ is a proper cone, $-u \in \text{int} Q(X)$ and $q = \text{gauge}(M + u)$.
References

[1] S. Cobzas. *Functional Analysis in Asymmetric Normed Spaces*. Birkhauser, 2013.

[2] J. Conradie. Asymmetric norms, cones and partial orders. *Topology and Appl.*, 193:100–115, 2015.

[3] A. B. Németh and S. Z. Németh. *Subadditive retractions on cones and asymmetric vector norms*. ArXiv: 2005.10508v, 2020.

[4] A. B. Németh. *Mutually polar retractions on convex cones*. ArXiv: 2012.4530v1, 2020.