Efficient time stepping for the multiplicative Maxwell fluid including the Mooney-Rivlin hyperelasticity

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Summary
A popular version of the finite-strain Maxwell fluid is considered, which is based on the multiplicative decomposition of the deformation gradient tensor. The model combines Newtonian viscosity with hyperelasticity of the Mooney-Rivlin type; it is a special case of the viscoplasticity model proposed by Simo and Miehe in 1992. A simple, efficient, and robust implicit time-stepping procedure is suggested. Lagrangian and Eulerian versions of the algorithm are available, with equivalent properties. The numerical scheme is iteration free, unconditionally stable, and first order accurate. It exactly preserves the inelastic incompressibility, symmetry, and positive definiteness of the internal variables and w-invariance. The accuracy of the stress computations is tested using a series of numerical simulations involving a nonproportional loading and large strain increments. In terms of accuracy, the proposed algorithm is equivalent to the modified Euler backward method with exact inelastic incompressibility; the proposed method is also equivalent to the classical integration method based on exponential mapping. Since the new method is iteration free, it is more robust and computationally efficient. The algorithm is implemented into MSC.MARC, and a series of initial boundary value problems is solved to demonstrate the usability of the numerical procedures.

KEYWORDS
efficient numerics, finite strain, implicit time stepping, Maxwell fluid, Mooney-Rivlin, multiplicative viscoelasticity

1 | INTRODUCTION

The Maxwell fluid, also known as the Maxwell material (cf. Reiner¹), is the focus of the current research. Because of a special combination of elastic and viscous (inelastic) properties, it is widespread in phenomenological material modelling. Apart from obvious applications to the dynamics of viscous fluids,² different versions of the finite-strain Maxwell model are used for solid materials as well. A parallel connection of the Maxwell fluid with idealized elastic elements allows one to model the behaviour of polymers,³⁴ biological tissues,¹⁰¹¹ explosives,¹⁴ and other types of viscoelastic materials. Plasticity models with a nonlinear kinematic hardening may operate with backstresses; the evolution of the backstresses can be described using the Maxwell element as well.¹⁵¹⁹ A generalization of this approach to plasticity with the yield surface distortion is reported in other works.²⁰²¹ Some advanced models of shape memory alloys²²²³ and anisotropic creep²⁴ include the Maxwell fluid as an important constituent. Interestingly, minor modifications of the Maxwell fluid are also used in
production of mechanics-based computer animations. Applications of the Maxwell fluid to the analysis of geological structures are reported by other authors. 

Because of the high prevalence of the Maxwell concept in the phenomenological material modelling, specialized numerical algorithms with an enhanced robustness and efficiency are becoming increasingly important. Depending on the application, the computational effort that is spent on the Gauss point stress evaluation may become very high; in explicit finite element method (FEM), it is even dominant. High computational efficiency of the used algorithms may become especially important for real-time simulations, like FEM simulations of surgical operations.

Different mathematical formulations of the finite-strain Maxwell fluid are currently in use (cf. Shutov). We advocate here the multiplicative approach of Simo and Miehe, because of its numerous advantages like thermodynamic consistency, exact hyperelasticity, absence of spurious shear oscillations, pure isochoric-volumetric split, w-invariance under the isochoric change of the reference configuration, and exponential stability of the solution with respect to the perturbation of the initial data. Within the approach of Simo and Miehe, the Newtonian viscosity can be combined with different types of isotropic elastic potentials in a thermodynamically consistent manner.

In computational practice, the implemented time step size can be larger than the relaxation time of a Maxwell element. For that reason, implicit integration of the 6-dimensional evolution equation is needed. After the discretization in time, a system of 6 nonlinear equations with respect to 6 unknowns is obtained. With the use of spectral decomposition, the number of unknowns can be reduced to 3. Typically, these nonlinear systems of equations are solved iteratively using the Newton-Raphson method or its modifications. Obviously, such an iterative procedure is less robust and less efficient than a procedure based on the closed-form solution of the discretized system of equations.

In the important case of the neo-Hookean potential, an explicit update formula for the implicit time stepping was proposed by Shutov et al. It was used to model finite-strain viscoelastic behaviour of a rubber-like material. Further extension of this algorithm to the case of the Yeoh hyperelasticity was presented by Landgraf et al. and Gobadi et al. extended the algorithm to the thermomechanical case. Silbermann et al. and Shutov et al. used the algorithm to stabilize numerical simulations within a hybrid explicit/implicit procedure for finite-strain plasticity with a nonlinear kinematic hardening. Schüler et al. used the algorithm to model the viscoelastic behaviour of a bituminous binding agent. The closed-form solution reported by Shutov et al. is implemented as a part of efficient implicit procedures for finite-strain viscoplasticity and finite strain creep.

In the current study, a new explicit update formula is suggested for a more general case of the Mooney-Rivlin hyperelasticity. The explicit solution is obtained by exploiting the properties of the underlying constitutive equations. The previously reported update formula for the neo-Hookean potential is covered by this solution as a special case. The new algorithm is unconditionally stable and first order accurate. (For a discussion of higher-order methods, the reader is referred to other papers.) It exactly preserves the inelastic incompressibility, symmetry, and positive definiteness of the internal tensor-valued variable; it also preserves the w-invariance under the change of the reference configuration. A slight modification of the method is also suggested to enforce the symmetry of the consistent tangent operator. Concerning the accuracy of the stress computation, the new algorithm is equivalent to the modified Euler backward method (MEBM) and exponential method (EM). The symmetry of the consistent tangent operator is tested in a series of computations. To demonstrate the applicability of the numerical procedure, an initial boundary value problem is solved in MSC.MARC.

We conclude the introduction by setting up the notation. A coordinate-free tensor formalism is used here. Bold-faced symbols denote first- and second-rank tensors in \( \mathbb{R}^3 \). For instance, \( \mathbf{1} \) stands for the second-rank identity tensor. The deviatoric part of a tensor is denoted by \( \mathbf{A}^D := \mathbf{A} - \frac{1}{3} \text{tr}(\mathbf{A}) \mathbf{1} \), where \( \text{tr}(\mathbf{A}) \) is the trace. The overline \( (\cdot) \) denotes the unimodular part of a tensor:

\[
\bar{\mathbf{A}} := (\det \mathbf{A})^{-1/3} \mathbf{A}.
\]

2 FINITE-STRAIN MAXWELL FLUID ACCORDING TO SIMO AND MIEHE

The considered Maxwell model is a special case of the finite-strain viscoplasticity theory proposed by Simo and Miehe; it has the same structure as the well-known model of associative elastoplasticity introduced by Simo. A referential (Lagrangian) formulation of this model was considered later by Lion. The spatial (Eulerian) constitutive equations proposed by Simo and Miehe were used by Reese and Govindjee in a fundamental study and by many others.

References: 5, 6, 13, 17, 27, 34, 35, 37, 41.

See, for instance, References 5, 6, 27, 35, 40, 41, 48, 49.
2.1 | Formulation on the reference configuration

Here, we follow the presentation of Lion.³ Let \( \mathbf{F} \) be the deformation gradient that maps the local reference configuration \( \hat{\mathcal{C}} \) to the current configuration \( \mathcal{C} \). We consider the multiplicative decomposition of the deformation gradient \( \mathbf{F} \) into the elastic part \( \hat{\mathbf{F}} \) and the inelastic part \( \mathbf{F}_i \) (in the context of viscoelasticity, this multiplicative split is known as the Sidoroff decomposition⁵³):

\[
\mathbf{F} = \hat{\mathbf{F}} \mathbf{F}_i.
\]

This decomposition implies the so-called stress-free intermediate configuration \( \hat{\mathcal{C}} \). Along with the classical right Cauchy-Green tensor \( \mathbf{C} := \mathbf{F}^T \mathbf{F} \), we consider the inelastic right Cauchy-Green tensor \( \mathbf{C}_i \) and the elastic right Cauchy-Green tensor \( \hat{\mathbf{C}}_e \):

\[
\mathbf{C}_i := \mathbf{F}_i^T \mathbf{F}_i \quad \text{and} \quad \hat{\mathbf{C}}_e := \hat{\mathbf{F}}^T_i \hat{\mathbf{F}}_e.
\]

The inelastic velocity gradient \( \hat{\mathbf{L}}_i \) and the corresponding covariant Oldroyd derivative are defined as follows:

\[
\hat{\mathbf{L}}_i := \hat{\mathbf{F}}_i \mathbf{F}_i^{-1}, \quad \triangle (\cdot) := \frac{d}{dt}(\cdot) + \hat{\mathbf{L}}_i^T(\cdot) + (\cdot) \hat{\mathbf{L}}_i.
\]

Here, the superimposed dot denotes the material time derivative. The inelastic Almansi strain tensor \( \hat{\mathbf{F}}_1 \) and the inelastic strain rate tensor \( \mathbf{D}_i \) are introduced respectively through

\[
\hat{\mathbf{F}}_1 := \frac{1}{2} (\mathbf{I} - \mathbf{F}_i^{-T} \mathbf{F}_i^{-1}) \quad \text{and} \quad \mathbf{D}_i := \frac{1}{2} (\hat{\mathbf{L}}_i + \hat{\mathbf{L}}_i^T).
\]

After some straightforward computations, we arrive at

\[
\mathbf{D}_i = \triangle \hat{\mathbf{L}}_i.
\]

By \( \mathbf{T} \), denote the Cauchy stress tensor (true stresses). The Kirchhoff stress \( \mathbf{S} \), the second Piola-Kirchhoff stress \( \mathbf{S} \) operating on the stress-free configuration \( \hat{\mathcal{C}} \), and the classical second Piola-Kirchhoff stress \( \hat{\mathbf{T}} \) operating on the reference configuration \( \hat{\mathcal{C}} \) are now respectively introduced:

\[
\mathbf{S} := (\det \mathbf{F}) \mathbf{T}, \quad \hat{\mathbf{S}} := \hat{\mathbf{F}}_e^T \hat{\mathbf{S}}_e^{-T}, \quad \text{and} \quad \hat{\mathbf{T}} := \mathbf{F}^{-1} \mathbf{S} \mathbf{F}^{-T}.
\]

Let \( \psi \) be the Helmholtz free energy per unit mass. In this work, it is given by the Mooney-Rivlin potential

\[
\rho_R \psi(\hat{\mathbf{C}}_e) = \frac{c_{10}}{2} \left( \text{tr} \hat{\mathbf{C}}_e - 3 \right) + \frac{c_{01}}{2} \left( \text{tr} \hat{\mathbf{C}}_e^{-1} - 3 \right),
\]

where \( \rho_R \) is the mass density in the reference configuration and \( c_{10} \) and \( c_{01} \) are the shear moduli; the overline \( (\cdot) \) denotes the unimodular part of a tensor (recall (1)). (The neo-Hookean potential is obtained when \( c_{01} = 0 \); this special case was already considered in another work.)⁴²) A hyperelastic stress-strain relation is considered on the stress-free configuration:

\[
\hat{\mathbf{S}} = 2 \rho_R \frac{\partial \psi(\hat{\mathbf{C}}_e)}{\partial \hat{\mathbf{C}}_e} \Rightarrow \hat{\mathbf{S}} = (c_{10} \hat{\mathbf{C}}_e - c_{01} \hat{\mathbf{C}}_e^{-1})^D \hat{\mathbf{C}}_e^{-1}.
\]

The Clausius-Duhem inequality requires that the specific internal dissipation \( \delta_i \) remains nonnegative. For simplicity, we consider isothermal processes here, and the Clausius-Duhem inequality takes the reduced form, also known as the Clausius-Planck inequality:

\[
\delta_i := \frac{1}{\rho_R} \hat{\mathbf{T}} : (\hat{\mathbf{E}} - \hat{\mathbf{C}}) \geq 0,
\]

where \( \mathbf{E} := 1/2(\mathbf{C} - \mathbf{I}) \) stands for the Green strain tensor. With (6) and the isotropy of the free energy function taken into account, this inequality is reduced to

\[
\rho_R \delta_i = (\hat{\mathbf{C}}_e \hat{\mathbf{S}}) : \triangle \hat{\mathbf{F}}_1 \geq 0.
\]

The following flow rule is postulated so that (8) holds for arbitrary mechanical loadings (cf. Lion):⁴³

\[
\triangle \hat{\mathbf{F}}_1 = \frac{1}{2 \eta} (\hat{\mathbf{C}}_e \hat{\mathbf{S}})^D.
\]

where \( \eta > 0 \) is a material parameter (Newtonian viscosity). In view of (3), an equivalent formulation of this flow rule is given by
\[ \dot{\mathbf{D}}_i = \frac{1}{2\eta} (\mathbf{\dot{C}}_i \mathbf{\dot{S}})_D. \]  

Both (9) and (10) imply that \( \text{tr}(\hat{\mathbf{F}}_i) = \text{tr}(\hat{\mathbf{D}}_i) = 0 \). The inelastic flow is thus incompressible: \( \det \mathbf{F}_i = \text{const.} \)

Let us transform the constitutive equations to the reference configuration. First, the free energy (5) is represented as a function of \( \mathbf{CC}_i^{-1} \):

\[ \psi = \psi (\mathbf{CC}_i^{-1}) = \frac{c_{10}}{2\rho_R} \left( \frac{\text{tr}\mathbf{CC}_i^{-1} - 3}{2} \right) + \frac{c_{01}}{2\rho_R} \left( \frac{\text{tr}\mathbf{C}_i\mathbf{C}_i^{-1} - 3}{2} \right). \]  

Using (4) and \((6)_1\), one obtains for the second Piola-Kirchhoff stress tensor

\[ \mathbf{T} = 2\rho_R \frac{\partial \psi (\mathbf{CC}_i^{-1})}{\partial \mathbf{C}} \bigg|_{\mathbf{C} = \text{const.}}. \]  

Substituting (11) into this, we arrive at

\[ \mathbf{T} = \mathbf{C}^{-1} \left( c_{10} \mathbf{CC}_i^{-1} - c_{01} \mathbf{C}_i\mathbf{C}_i^{-1} \right)_D. \]  

Next, because of the isotropy of the free energy function, we have

\[ \text{tr} (\mathbf{\dot{C}}_i \mathbf{\dot{S}}) = \text{tr} (\mathbf{CT}). \]  

Combining this with (4), we obtain

\[ \mathbf{F}_i^T (\mathbf{\dot{C}}_i \mathbf{\dot{S}})_D \mathbf{F}_i = \mathbf{CTC}_i - \frac{1}{3} \text{tr} (\mathbf{\dot{C}}_i \mathbf{\dot{S}}) \mathbf{C}_i = \left( \frac{1}{\eta} (\mathbf{CT})^D \right)_D \mathbf{C}_i. \]  

Applying a pull-back transformation to the evolution equation (9), we obtain

\[ \mathbf{C}_i = 2\mathbf{F}_i^T \mathbf{\hat{F}}_i \mathbf{F}_i = \frac{1}{\eta} \mathbf{F}_i^T (\mathbf{\dot{C}}_i \mathbf{\dot{S}})_D \mathbf{F}_i = \frac{1}{\eta} \left( \frac{1}{\eta} (\mathbf{CT})^D \right)_D \mathbf{C}_i. \]  

This flow rule is valid for arbitrary isotropic free energy functions. Substituting (13) into (16), we obtain the evolution equation pertaining to the Mooney-Rivlin potential:

\[ \mathbf{C}_i = \frac{1}{\eta} \left( c_{10} \mathbf{CC}_i^{-1} - c_{01} \mathbf{C}_i\mathbf{C}_i^{-1} \right)_D \mathbf{C}_i. \]  

Finally, the system of constitutive equations (13) and (17) is closed by specifying initial conditions:

\[ \mathbf{C}_i|_{t=0} = \mathbf{C}_i^0. \]  

The exact solution of (17) exhibits the following geometric property:

\[ \mathbf{C}_i(t) \in \mathcal{M} \quad \text{if} \quad \mathbf{C}_i^0 \in \mathcal{M}, \]

where the manifold \( \mathcal{M} \) is a set of symmetric unimodular tensors:

\[ \mathcal{M} := \{ \mathbf{A} \in \text{Sym} : \det \mathbf{A} = 1 \}. \]

In other words, we are dealing with a differential equation of the manifold. A substantial consequence of (19) is that \( \mathbf{C}_i \) remains positive definite. The positive definiteness is important since \( \mathbf{C}_i \) represents a certain metric in \( \mathbb{R}^3 \).

### 2.2 Spatial formulation

Similar to Shutov et al.,\(^{42}\) we show that the model presented in this section is indeed the model of Simo and Miehe,\(^ {30}\) which was originally formulated in the spatial (Eulerian) description. First, for the inelastic strain rate, we have a purely kinematic relation

\[ 2\hat{\mathbf{D}}_i = \mathbf{F}_i^{-T} \mathbf{C}_i \mathbf{F}_i^{-1}. \]
Since the elastic potential $\psi(\hat{C_e})$ is isotropic, the tensors $\hat{S}$ and $\hat{C_e}$ are coaxial. Thus, the Mandel tensor $\hat{C_e}\hat{S}$ and the tensor $\hat{C_e}$ are coaxial as well. The evolution equation (10) implies that $\hat{D_i}$ is coaxial with $\hat{C_e}$. Therefore, these tensors commute, $\hat{D_i}\hat{C_e} = \hat{C_e}\hat{D_i}$, and we have

$$\hat{D}_i = \hat{C_e}\hat{D_i}\hat{C_e}^{-1}. \tag{21}$$

Taking this into account, we rewrite (10) as follows:

$$2\hat{C_e}\hat{D_i}\hat{C_e}^{-1} = \frac{1}{\eta} (\hat{C_e}\hat{S})^D. \tag{22}$$

Substituting (20) into (21) and taking into account that $\frac{d}{dt}(C_i^{-1}) = -C_i^{-1}\dot{C}_i^{-1}$, we obtain

$$-\hat{F}_e^T F \frac{d}{dt} (C_i^{-1}) F^T \hat{F}_e^{-1} \hat{F}_e^{-1} F^T = \frac{1}{\eta} (\hat{C_e}\hat{S})^D. \tag{23}$$

For the Eulerian description, it is convenient to introduce the elastic left Cauchy-Green tensor $B_e := \hat{F}_e\hat{F}_e^T$. Next, we note that the Kirchhoff stress is related to the Mandel tensor through the similarity relation

$$S^D = \hat{F}_e^{-T} (\hat{C_e}\hat{S})^D \hat{F}_e^T. \tag{24}$$

Premultiplying (22) with $\hat{F}_e^{-T}$, postmultiplying it with $\hat{F}_e^T$, and taking (23) into account, we have

$$-F \frac{d}{dt} (C_i^{-1}) F^T B_e^{-1} = \frac{1}{\eta} S^D. \tag{25}$$

Introducing the contravariant Oldroyd rate (which is effectively a Lie derivative)

$$\mathcal{O}(A) = \mathfrak{g}_v(A) := F \frac{d}{dt} \left( F^{-1} A F^{-T} \right) F^T = \dot{A} - LA - AL^T,$$

we obtain the following kinematic relation:

$$\mathcal{O}(B_e) = \mathfrak{g}_v(B_e) = F \frac{d}{dt} \left( C_i^{-1} \right) F^T. \tag{26}$$

With it, the flow rule (24) takes the well-known form

$$-\mathfrak{g}_v(B_e) B_e^{-1} = \frac{1}{\eta} S^D. \tag{27}$$

This equation coincides with the flow rule considered by Simo and Miehe\(^{30}\) (see their equations (2.19a) and (2.26)); it was also implemented by Reese and Govindjee.\(^{34}\) The Kirchhoff stress can be computed through

$$S = 2\rho R \frac{\partial \psi(B_e)}{\partial B_e}. \tag{28}$$

In the case of the Mooney-Rivlin strain energy (11), we have $S = S^D = c_{10}(\overline{B_e})^D - c_{01} (\overline{B_e}^{-1})^D$; the evolution equation (26) is then specified to

$$-\mathfrak{g}_v(B_e) B_e^{-1} = \frac{1}{\eta} \left[ c_{10} \overline{B_e} - c_{01} \overline{B_e}^{-1} \right]^D. \tag{29}$$

Remark 1. The flow rule (26) can be obtained by other arguments as well (cf. the derivation of equation (81) by Latorre and Montáns\(^{34}\)). Moreover, an alternative spatial formulation of the model (26)-(27) can be derived from the additive decomposition of the strain rate tensor (see Appendix A). The abundance of different formulations of this model proposed by different authors indicates the importance of this particular model.

### 3 | TIME-STEPPING ALGORITHM

#### 3.1 | Explicit update formula in the Lagrangian formulation

Consider a generic time interval $(t_n, t_{n+1})$ with the time step size $\Delta t := t_{n+1} - t_n > 0$. By $^nC_i$ and $^{n+1}C_i$, denote numerical solutions at $t_n$ and $t_{n+1}$, respectively. Assume that the current deformation gradient $^nF$ and the previous inelastic right
Cauchy-Green tensor \(^{n+1}C_i \in \mathbb{M}\) are given. The unknown \(^{n+1}C_i \in \mathbb{M}\) is computed by implicit integration of the evolution equation (17). First, we consider the classical Euler backward discretization of (17):

\[
^{n+1}C_i^{(EBM)} = ^{n}C_i + \frac{\Delta t}{\eta} \left( c_{10} \alpha^{n+1} \left( ^{n+1}C_i^{(EBM)} \right)^{-1} - c_{01}^{n+1}C_i^{(EBM)} \right) \frac{D}{n+1} C_i.
\] (29)

Unfortunately, because of its linear structure, the Euler backward method violates the incompressibility restriction, which leads to error accumulation. To enforce incompressibility, a correction term \(^{n+1}C_i\) is added to the right-hand side of (29) (cf. Shutov\(^{47}\))

\[
^{n+1}C_i = ^{n}C_i + \frac{\Delta t}{\eta} \left( c_{10}^{n+1}C_i\left( ^{n+1}C_i \right)^{-1} - c_{01}^{n+1}C_i \right) \frac{D}{n+1} C_i + \phi^{n+1}C_i,
\] (30)

where the unknown correction \(\phi \in \mathbb{R}\) should be defined from the additional equation \(\det( ^{n+1}C_i ) = 1\). Next, using the definition of the deviatoric part, (30) takes the equivalent form

\[
\varphi^{n+1}C_i = ^{n}C_i + \frac{\Delta t}{\eta} c_{10}^{n+1}C_i - c_{01}^{n+1}C_i \frac{D}{n+1} C_i + \phi^{n+1}C_i,
\] (31)

where \(\varphi \in \mathbb{R}\) should be defined from the incompressibility condition. For brevity, introduce the notation

\[
X := \frac{n+1C_i - ^{n}C_i}{n+1C_i - ^{n}C_i} \quad A := \frac{n+1C_i - ^{n}C_i}{n+1C_i - ^{n}C_i}, \quad \varepsilon := c_{01} \frac{\Delta t}{\eta}.
\] (32)

Multiplying both sides of (31) with \(n+1C_i - ^{n}C_i\) on the left and right, we arrive at the quadratic equation with respect to \(X\):

\[
\varphi X = A - \varepsilon X^2.
\] (33)

Here, \(A\) and \(\varepsilon\) are known; the unknown \(\varphi\) is to be defined using the incompressibility condition

\[
\det(X) = 1.
\] (34)

Recall that, for mechanical reasons, \(^{n+1}C_i\) must be positive definite. Therefore, the physically reasonable \(X\) is positive definite as well. Thus, the correct solution of (33) is given by

\[
X = \frac{1}{2\varepsilon} \left[ -\varphi I + (\varphi^2 I + 4\varepsilon A)^{1/2} \right].
\] (35)

Unfortunately, because of the round-off errors, this relation yields unsatisfactory results if evaluated step by step. (Interestingly, a similar problem appeared in the context of finite-strain plasticity with neo-Hookean potentials, discussed by Shutov\(^{47}\)). Indeed, the relation in the square bracket on the right-hand side of (35) can be computed accurately up to a machine precision. In the case of small \(\varepsilon\), however, the corresponding error is then multiplied with the big number \(1/\varepsilon\).

To resolve this issue, (35) needs to be transformed. Using the well-known identity \((X^{1/2} - I)(X^{1/2} + I) = X - I\), (35) can be rewritten in the form

\[
X = 2A \left( (\varphi^2 I + 4\varepsilon A)^{1/2} + \varphi I \right)^{-1}.
\] (36)

This relation is computationally advantageous since it is free from the previously described error magnification.

Now we need to compute the unknown \(\varphi\), which should be estimated using the incompressibility condition \(\det(X) = 1\). As shown in Appendix B, a simple formula can be obtained using the perturbation method for small \(\varepsilon\):

\[
\varphi = \varphi_0 - \frac{\text{tr} A}{3\varphi_0} \varepsilon + O(\varepsilon^2), \quad \text{where} \quad \varphi_0 := (\det A)^{1/3}.
\] (37)

Neglecting the terms \(O(\varepsilon^2)\), we obtain a reliable procedure, which is exact for \(c_{01} = 0\). Moreover, as will be seen in the next sections, this procedure is accurate and robust even for finite values of \(\varepsilon\). After \(\varphi\) is computed, \(X\) is evaluated through (36). Further, \(C_i\) is updated:

\[
^{n+1}C_i^* := \frac{n+1C_i^{1/2}}{n+1C_i^{1/2}} X \frac{n+1C_i^{1/2}}{n+1C_i^{1/2}}.
\] (38)

Since the variable \(\varphi\) is not computed exactly, the incompressibility condition can be violated. To enforce it, a final correction step is needed:
Obviously, IFEBM and 2IEBM exactly preserve the geometric property (19). As shown by Shutov and Kreißig, this allows one to suppress the error accumulation when working with big time steps. Moreover, the tensor $A$ is positive definite. Thus, the right-hand side of (35) is positive definite as well. With (38) and (39) taken into account, the solution $n+1C_i$ is also positive definite for both methods.

The IFEBM is first order accurate (see Appendix C). Note that IFEBM and 2IEBM exactly preserve the w-invariance property (see Appendix D). For the IFEBM, the solution is well defined for all $\Delta t \geq 0$ and $n+1C_i$ is a smooth function of the time step size $\Delta t$. For $\Delta t \geq 0$, the solution $n+1C_i$ ranges smoothly from $nC_i$ to $n+1C$. The IFEBM is unconditionally stable since the solution remains finite for arbitrary time steps.

Remark 2. In the case of the neo-Hookean potential, we have $c_{01} = 0$. Thus, $\varepsilon = 0$, $\varphi = \varphi_0$, and $X = \varphi^{-1}A$. Therefore, both methods are reduced to the explicit update formula from Shutov et al.:

$$c_{01} = 0 \Rightarrow n+1C_i = nC_i + \frac{\Delta t}{\eta}c_{01}n+1C.$$  \hspace{1cm} (43)

Remark 3. The IFEBM is not the only modification of Euler backward method (EBM), which ensures exact incompressibility. Another modification of that kind was considered by Helm. Furthermore, Vladimirov et al. considered two other versions of the EBM to enforce the inelastic incompressibility by introducing the additional equation $\text{det}C_i = 1$. In contrast to the IFEBM presented here, a local iterative procedure was used by Helm and Vladimirov et al.
Explicit update formula in the Eulerian formulation

The Eulerian formulation (26) of the flow rule is quite common. Since its direct time discretization is not trivial (cf. Simo33), efficient time-stepping algorithms are needed. In this subsection, we transform the previously constructed IFEBM to build an iteration-free method on the current configuration. First, recall that for the elastic left Cauchy-Green tensor $B_e := F_e F_e^T$, the following relation holds:

$$B_e = F C_i^{-1} F^T.$$  \hfill (44)

For the generic time interval $(t_n, t_{n+1})$, we introduce the so-called trial elastic left Cauchy-Green tensor $n+1 B_e^{\text{trial}}$, by assuming that $C_i$ remains constant during the step

$$n+1 B_e^{\text{trial}} := n+1 F C_i^{-1} n+1 F^T.$$  \hfill (45)

Using the so-called relative deformation gradient $F_{\text{rel}} := n+1 F nF^T$, we have

$$n+1 B_e^{\text{trial}} = F_{\text{rel}} nB_e F_{\text{rel}}^T.$$  \hfill (46)

Next, we consider the polar decomposition

$$n+1 F = n+1 R n+1 C_i^{-1/2}, \quad \text{where} \quad n+1 R n+1 R^T = 1, \quad \det (n+1 R) = +1.$$  \hfill (47)

Combining (44) and (47), with the definition of $X$ given by (32), we have

$$n+1 B_e^{-1} = n+1 R X n+1 R^T, \quad \left(\frac{n+1 B_e^{-1}}{n+1 R X^2 n+1 R^T}\right) = n+1 R X n+1 R^T.$$  \hfill (48)

Premultiplying (33) with $n+1 R$ and postmultiplying it with $n+1 R^T$, we obtain the following quadratic equation with respect to unknown $n+1 B_e^{-1}$:

$$\varphi n+1 B_e^{-1} = n+1 R A n+1 R^T - \varepsilon \left(\frac{n+1 B_e^{-1}}{n+1 R X^2 n+1 R^T}\right).$$  \hfill (49)

To simplify this relation, we introduce $\tilde{A} := n+1 R A n+1 R^T$; then

$$\tilde{A} = n+1 R n+1 C_i^{-1/2} \left(\frac{n+1 B_e^{-1}}{\eta C_{10} n+1 C_i} n+1 C_i^{-1/2} n+1 R^T = \frac{n+1 B_e^{\text{trial}}}{n+1 R X^2 n+1 R^T} + \frac{\Delta t}{\eta C_{10}} I.\right)$$  \hfill (50)

Thus, we arrive at the following problem:

$$\varphi n+1 B_e^{-1} = \tilde{A} - \varepsilon \left(\frac{n+1 B_e^{-1}}{n+1 R X^2 n+1 R^T}\right), \quad \det \left(n+1 B_e^{-1}\right) = 1.$$  \hfill (51)

In analogy to (36), the following closed-form solution is valid:

$$n+1 B_e^{-1} = 2 \tilde{A} \left[ (\varphi^2 I + 4 \varepsilon \tilde{A})^{1/2} + \varphi I \right]^{-1}.$$  \hfill (52)

Taking into account that $\det \tilde{A} = \det A$ and $\text{tr} \tilde{A} = \text{tr} A$, we have the spatial counterpart of (37):

$$\varphi = \varphi_0 - \frac{\text{tr} \tilde{A}}{3 \varphi_0} (\varepsilon^2) + O (\varepsilon^3), \quad \text{where} \quad \varphi_0 := \left(\det A\right)^{1/3}.$$  \hfill (53)

Since (53) is not exact, an additional correction is needed:

$$n+1 B_e^{-1} = 2 \tilde{A} \left[ (\varphi^2 I + 4 \varepsilon \tilde{A})^{1/2} + \varphi I \right]^{-1}.$$

The explicit update procedure for the evolution equation (28) is summarized in Table 2. Within a time step, the explicit procedure ((53) and (54)) predicts the same stress response as the previously reported Lagrangian procedure. Note that because of the elastic isotropy $n+1 B_e$ and $n+1 B_e^{\text{trial}}$ are coaxial (cf. Simo33).
4 | NUMERICAL TESTS

4.1 | Nonproportional loading

All quantities in this subsection are nondimensional. Assume the following local deformation history:

\[ F(t) = F'(t) \quad \text{for} \quad t \in [1, 3], \]

where \( F'(t) \) is a piecewise linear function

\[
F'(t) := \begin{cases} 
(1 - t)F_1 + tF_2 & \text{if } t \in [0, 1], \\
(2 - t)F_2 + (t - 1)F_3 & \text{if } t \in (1, 2], \\
(3 - t)F_3 + (t - 2)F_4 & \text{if } t \in [2, 3]
\end{cases}
\]

\[
F_1 := 1, \quad F_2 := 2e_1 \otimes e_1 + \frac{1}{\sqrt{2}}(e_2 \otimes e_2 + e_3 \otimes e_3), \quad F_3 := e_1 \otimes e_2, \quad F_4 := 2e_2 \otimes e_2 + \frac{1}{\sqrt{2}}(e_1 \otimes e_1 + e_3 \otimes e_3).
\]

This nonproportional loading programme is strain driven and volume preserving; abrupt changes of the loading direction appear at \( t = 1 \) and \( t = 2 \). The initial condition is \( C_1|_{t=0} = I \); it means that the reference configuration is stress free at \( t = 0 \).

The numerical solution of the initial value problem (17) and (18) obtained with very small time steps will be referred to as the exact solution. Let \( S^{\text{exact}}(t) \) be the corresponding history of the Kirchhoff stress. Next, large time steps \( \Delta t \) yielding large strain increments are used to test the accuracy of the numerical algorithms. The corresponding computed Kirchhoff stress is denoted by \( S^{\text{num}}(t) \), and the corresponding error in stress prediction is given by the Frobenius norm:

\[ \text{Error}(t) := \|S^{\text{exact}}(t) - S^{\text{num}}(t)\|. \]

The following constants of the Mooney-Rivlin elasticity are used: \( c_{10} = c_{01} = 1 \). The proposed IFEBM is compared with an MEBM and the classical EM. A short summary of these methods is provided in Appendix E. The error is plotted versus time in Figure 1 for \( \Delta t = 0.1 \) and in Figure 2 for \( \Delta t = 0.05 \). The Figures show that IFEBM, MEBM, and EM are equivalent in terms of accuracy. The difference between IFEBM and MEBM is much smaller than the difference between MEBM and EM. (The 2IEBM is tested as well. The corresponding error curve practically merges with the error curve for MEBM. Therefore, it is not depicted in Figures 1 and 2.) Since the considered methods are first order accurate, the error for \( \Delta t = 0.05 \) is approximately 2 times smaller than that for \( \Delta t = 0.1 \).

The Newton-Raphson procedure for finding the numerical solution, pertaining to MEBM and EM, may diverge, if the solution from the previous time step is taken as the initial approximation. A substepping can be used to resolve this problem. On the other hand, the solution for the novel IFEBM is given in a closed form. Thus, the new method is a priori free from any convergence problems, which makes it more robust.

Another aspect is the symmetry of the consistent tangent operator \( \partial^{n+1} \hat{T} / \partial^{n+1} C \). In the current study, the tangent is computed by numerical differentiation using the central difference scheme, which provides at least 10-digit accuracy. For computations, the relevant tensors are represented by the 6-vectors as follows (cf. the Voigt notation):

| TABLE 2 | Iteration-free Euler backward method on the current configuration |
|---|---|
| Input: \( n^{+1}F, n^{+1}F', n^{+1}C \) | Output: \( n^{+1}B^{\text{num}}, n^{+1}S \) |
| 1: \( F'_{\text{rel}} = n^{+1}F'F^{-1} \) | 2: \( n^{+1}B_{e}^{\text{rel}}(t) = (F'_{\text{rel}})^{-T} n^{+1}B_{e}^{-1}(F'_{\text{rel}})^{-T} \) |
| 3: \( \tilde{A} = n^{+1}B_{e}^{\text{rel}}(t) + (\Delta t/n)C_{10}I \) | 4: \( e = c_{01}(\Delta t/n) \) |
| 5: \( \varphi_0 = (\text{det} \tilde{A})^{1/3} \) | 6: \( \varphi = \varphi_0 - (\text{tr} \tilde{A})/3\varphi_0 e \) |
| 7: \( n^{+1}B_{e}^{\text{det}} = 2\tilde{A}(\varphi^2 I + 4e \tilde{A})^{1/2} + \varphi I \) | 8: \( n^{+1}S = c_{10}(n^{+1}B_{e}^{\text{det}})^{\text{D}} - c_{01}(n^{+1}B_{e}^{-1}F^{-1})^{\text{D}} \) |
Figure 1: Error graphs for different integrations methods; \( \Delta t = 0.1 \). EM, exponential method; IFEBM, iteration-free Euler backward method; EBM, Euler backward method.

Figure 2: Error graphs for different integrations methods; \( \Delta t = 0.05 \). EM, exponential method; IFEBM, iteration-free Euler backward method; EBM, Euler backward method.

\[
\hat{T} := (n+1) \mathbf{T}_{11}, \ n+1 \mathbf{T}_{22}, \ n+1 \mathbf{T}_{33}, \ n+1 \mathbf{T}_{12}, \ n+1 \mathbf{T}_{13}, \ n+1 \mathbf{T}_{23}, \quad (59)
\]

\[
\hat{C} := (n+1) \mathbf{C}_{11}, \ n+1 \mathbf{C}_{22}, \ n+1 \mathbf{C}_{33}, \ 2n+1 \mathbf{C}_{12}, \ 2n+1 \mathbf{C}_{13}, \ 2n+1 \mathbf{C}_{23}. \quad (60)
\]

The deviation of the tangent from the symmetry is measured by the following normalized quantity:

\[
\text{Deviation} := \max_{\mathbf{t} \in [0, 1]} \left\| \frac{\partial \hat{T}}{\partial \hat{C}} \right\| - \left( \frac{\partial \hat{T}}{\partial \hat{C}} \right)^T \max_{\mathbf{t} \in [0, 1]} \left\| \frac{\partial \hat{T}}{\partial \hat{C}} \right\|. \quad (61)
\]

The computed deviation of the consistent tangent from the symmetry is summarized in Table 3 for IFEBM and in Table 4 for 2IEBM. As can be seen from the table, IFEBM provides a tangent that is close to a symmetric one. For the 2IEBM, the deviation from the symmetry is barely detectable. Thus, 2IEBM can be used in applications where the symmetry of the tangent becomes crucial.

Remark 4. A popular way to enforce the symmetry of the tangent operator exploits the variational nature of the underlying constitutive equations.\cite{13, 55, 56} However, iterative procedures are used in these references.

| Table 3 Iteration-free Euler backward method: deviation of the consistent tangent operator from the symmetry |
|-------------------------------------------------|-----------------|-------------|-----------------|-----------------|
| \( \eta \) | \( \Delta t = 0.1 \) | \( \Delta t = 0.05 \) |
| 100 | \( < 10^{-9} \) | \( < 10^{-9} \) |
| 10 | \( 8.5 \cdot 10^{-7} \) | \( 1.1 \cdot 10^{-7} \) |
| 1 | \( 1.8 \cdot 10^{-4} \) | \( 3.0 \cdot 10^{-5} \) |
| 0.1 | \( 8.5 \cdot 10^{-5} \) | \( 2.7 \cdot 10^{-5} \) |
| 0.01 | \( 6.0 \cdot 10^{-7} \) | \( 4.9 \cdot 10^{-7} \) |
| 0.001 | \( < 10^{-9} \) | \( < 10^{-9} \) |
TABLE 4 Two-iteration Euler backward method: deviation of the consistent tangent operator from the symmetry

| $\eta = 100$ | $\eta = 10$ | $\eta = 1$ | $\eta = 0.1$ | $\eta = 0.01$ | $\eta = 0.001$ |
|-------------|-------------|-------------|-------------|-------------|-------------|
| $\Delta t = 0.1$ | $< 10^{-9}$ | $< 10^{-9}$ | $1.2 \cdot 10^{-9}$ | $< 10^{-9}$ | $< 10^{-9}$ |
| $\Delta t = 0.05$ | $< 10^{-9}$ | $< 10^{-9}$ | $< 10^{-9}$ | $< 10^{-9}$ | $< 10^{-9}$ |

Remark 5. In a number of FEM applications, the symmetry of the tangent operator is not important. These applications include globally explicit FEM procedures (the stiffness matrix is not assembled at all), geometrically nonlinear computations with follower loads, or computations with nonlinear kinematic hardening (the stiffness matrix is a priori nonsymmetric). Moreover, numerical tests from Section 4.3 show that the global convergence of the FEM procedure with an artificially symmetrized tangent is still very good.

4.2 Uniaxial compression tests

In this subsection, we consider a composite model with 1 equilibrium branch and 4 Maxwell branches connected in parallel. The free energy of the equilibrium branch is given by

$$\rho R \psi_{eq}(C) = \frac{c^{(eq)}_{10}}{2} (\text{tr}C - 3) + \frac{c^{(eq)}_{01}}{2} (\text{tr}C + k \frac{(\det C)^{5/2}}{2} + (\det C)^{-5/2} - 2).$$

(62)

The volumetric part appearing in (62) is taken from Hartmann and Neff; $k$ is the bulk modulus. For the $m$th Maxwell branch, we put

$$\rho R \psi_{ov,m} \left( \hat{C}^{(m)}_e \right) = \frac{c^{(m)}_{10}}{2} (\text{tr}C - 3) + \frac{c^{(m)}_{01}}{2} (\text{tr}(\hat{C}^{(m)}_e)^{-1} - 3), \quad m = 1, 2, 3, 4. \quad (63)$$

For the composite model, the second Piola-Kirchhoff stress equals

$$\hat{T} = \hat{T}_{eq} + \sum_{m=1}^{4} \hat{T}_{ov,m}, \quad \text{where}$$

$$\hat{T}_{eq} = C^{-1} \left( c^{(eq)}_{10} \overline{C} - c^{(eq)}_{01} \overline{C}^{-1} \right) D + \frac{k}{10} \left( (\det C)^{5/2} - (\det C)^{-5/2} \right) C^{-1}, \quad (65)$$

$$\hat{T}_{ov,m} = C^{-1} \left( c^{(m)}_{10} \overline{C}(C^{(m)}_i)^{-1} - c^{(m)}_{01} C^{(m)}_i \overline{C}^{-1} \right) D. \quad (66)$$

Analogous to (17), the behaviour of each Maxwell branch is described by

$$\hat{C}^{(m)}_i = \frac{1}{\eta^{(m)}} \left( c^{(m)}_{10} \overline{C}(C^{(m)}_i)^{-1} - c^{(m)}_{01} C^{(m)}_i \overline{C}^{-1} \right) D C^{(m)}_i, \quad C^{(m)}_i |_{t=0} = 1. \quad (67)$$

The material parameters corresponding to a cartilaginous temporomandibular joint are taken from Koolstra et al; they are summarized in Table 5.

In this subsection, we visualize the stress response of the joint tissue subjected to a nonmonotonic volume-preserving uniaxial loading. Here, the material is assumed to be incompressible. (Formally, this corresponds to $k \to \infty$ in ansatz (62)). The strain-controlled loading is given by

$$F(t) = [1 + \varepsilon(t)] e_1 \otimes e_1 + [1 + \varepsilon(t)]^{-1/2} (e_2 \otimes e_2 + e_3 \otimes e_3). \quad (68)$$

TABLE 5 Material parameters of a cartilaginous temporomandibular joint

| $m$ | 1 | 2 | 3 | 4 | Equilibrium |
|-----|---|---|---|---|-------------|
| $c_{10}$, MPa | 0.25 | 0.25 | 0.36 | 1.25 | 0.2 |
| $c_{01}$, MPa | 0.25 | 0.25 | 0.36 | 1.25 | 0.2 |
| $\eta$, MPa·s | 25.0 | 5.0 | 0.144 | 0.005 | $\infty$ |
where $\varepsilon$ stands for the prescribed engineering strain. Three different oscillation frequencies are analysed: 10, 1, and 0.1 Hz. For each frequency, 2 different tests are simulated: one test with the strain amplitude 0.2 and another one with the strain amplitude 0.4. Within each test, the absolute strain rate is constant: $|\dot{\varepsilon}| = \text{const}$. The simulation results are shown in Figure 3. The Figure reveals that for different loading frequencies IFEBM is sufficiently accurate, even when working with moderate time steps.

### 4.3 Solution of a boundary value problem

To demonstrate the applicability of the IFEBM, a representative initial boundary value problem is solved here. The composite material model (65)-(67) introduced in the previous subsection is implemented into the commercial FEM code MSC.MARC via the Hypela2 user interface. The material parameters correspond to the cartilaginous temporomandibular joint (cf. Table 5) with a bulk modulus of $k = 20$ MPa.

A nonmonotonic force-controlled loading is applied to the Cook membrane (Figure 4). (The preference is given to the force-controlled loading here since such problems are more difficult to solve than the strain-controlled ones.) The membrane is discretized using 213 elements with quadratic approximation of geometry and displacements; full integration is used. Let $T_{\text{total}}$ be the overall process time. For $t \in (0, T_{\text{total}}/3]$, the applied load increases from 0 to the maximum of 0.27 N; after that, the load linearly decreases to $-0.27$ N. Three different loading rates are analysed, leading to different process durations: $T_{\text{total}} = 0.3, 3, \text{and } 30$ seconds. The deformed shapes are shown in Figure 4 for $T_{\text{total}} = 30$ at different time instances.

Constant time steps are implemented in each simulation; simulations with small and large time steps are performed. A strict convergence criterion is used at each time step: for convergence, the relative error in force and displacements should not exceed 0.01. (The MSC.MARC settings are as follows: relative force tolerance $= 0.01$ and relative displacement tolerance $= 0.01$.) Two types of computations are performed: with a general nonsymmetric stiffness matrix and with an artificially symmetrized stiffness matrix. The symmetrized version allows us to use a symmetric matrix solver, which is more efficient. Although the consistent tangent operator for IFEBM is not exactly symmetric, the artificial
symmetrization did not affect the number of the global equilibrium iterations: MSC.MARC required the same number of iterations, with both the symmetric and nonsymmetric matrix solvers. The required number of iterations is listed in Table 6. More iterations are needed for the slow process.

The simulated force-displacement curves are shown in Figure 5. Although the stress response is history dependent, simulations with only 15 steps provide sufficiently accurate results. The maximum displacement increases with decreasing loading rate. Thus, the large number of global equilibrium iterations for the slow process can be explained by pronounced geometric nonlinearities.

### DISCUSSION AND CONCLUSION

A popular version of the finite-strain Maxwell fluid is considered, which is based on the multiplicative decomposition of the deformation gradient, combined with hyperelastic relations between stresses and elastic strains. A new iteration-free method is proposed for the implicit time stepping, provided that the elastic potential is of the Mooney-Rivlin type.

The following properties of the exact solution are inherited by the proposed IFEBM:

1. $^{n+1}C_i$ is symmetric and unimodular: $^{n+1}C_i \in \text{Sym, } \det(^{n+1}C_i) = 1$.
2. $^{n+1}C_i$ is positive definite: $^{n+1}C_i > 0$.
3. $^{n+1}C_i$ is a smooth function of $\Delta t$.
4. Complete stress relaxation: $^{n+1}C_i \rightarrow ^{n+1}\overline{C}$ as $\Delta t \rightarrow \infty$.
5. $w$-invariance under volume-preserving changes of the reference configuration.

For IFEBM, the consistent tangent operator is nearly symmetric. A slight modification, called 2IEBM, allows one to obtain a tangent operator that is even closer to the set of symmetric fourth-rank tensors. In terms of accuracy, the IFEBM is equivalent to the existing methods like the EBM with exact incompressibility of the flow (MEBM) and the EM, but the new method is superior in terms of robustness and computational efficiency.

In future work, the developed explicit update formula will be adjusted to advanced material models. The application of the proposed method to the viscoelasticity with the generalized Mooney-Rivlin energy storage and/or process-dependent viscosities and to the viscoplasticity with nonlinear kinematic and distortional hardening, as well as to shape memory alloys, is promising. Since the considered multiplicative Maxwell fluid is widespread in the phenomenological material modelling, the new method can become a method of choice in various applications, especially in those that require increased robustness and efficiency.

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APPENDIX A: ALTERNATIVE EULERIAN FORMULATION OF THE MODEL

Let us consider an alternative formulation of the finite-strain Maxwell model from Section 2. For simplicity, only incompressible behaviour is analysed here. Consider the velocity gradient $L$, the continuum spin $W$, and the strain rate $D$:

$$
L := \dot{FF}^{-1}, \quad W := (L - LT) / 2, \quad D := (L + LT) / 2.
$$  \hfill (A1)

The formulation is based on the additive decomposition of the strain rate tensor $D$ into the elastic part $D_e$ and the inelastic part $D_i$:

$$
D = D_e + D_i.
$$  \hfill (A2)

Following Donner and Ihlemann,58 assume that $X_e$ is a symmetric positive definite tensor, such that $\det(X_e) = 1$; $X_e$ operates on the current configuration. Let the free energy per unit mass be given by the isotropic function $\psi(X_e)$. Assume that the Kirchhoff stress $S$ is computed through

$$
S = 2\rho R \frac{\partial \psi(X_e)}{\partial X_e} X_e.
$$  \hfill (A3)

The flow rule is postulated on the current configuration as follows:

$$
D_i = \frac{1}{2\eta} S^{D}.
$$  \hfill (A4)

Moreover, the evolution of $X_e$ is described by (cf. Donner and Ihlemann58)

$$
X_e^{(t)} = D_e X_e + X_e D_e, \quad \text{where} \quad X_e^{(t)} := \dot{X}_e + X_e W - W X_e.
$$  \hfill (A5)

Let us show that this model is equivalent to (26)-(27). First, combining (A2) and (A5), we arrive at

$$
\dot{X}_e + X_e W - W X_e = (D - D_i) X_e + X_e (D - D_i).
$$  \hfill (A6)

Next, substituting (A4) into this, we obtain after some rearrangements

$$
\dot{X}_e - X_e L^T - L X_e = -\frac{1}{2\eta} (S^D X_e + X_e S^D).
$$  \hfill (A7)

Relation (A3) implies that $X_e$ and $S^D$ commute. Postmultiplying both sides of (A7) with $X_e^{-1}$ and recalling the definition of the Lie derivative (25), we obtain

$$
\mathcal{L}_v(X_e) X_e^{-1} = -\frac{1}{\eta} S^D.
$$  \hfill (A8)

It remains to be noted that (A2) and (A8) become identical to (26)-(27), if $X_e$ is formally replaced by $B_e$. Thus, we are dealing with two different Eulerian formulations of the same model.

APPENDIX B: ESTIMATION OF $\psi$

Here we estimate the unknown $\psi$; the derivation is similar to that presented in Appendix C of Shutov.47 The equation $\psi X = A - \epsilon X^2$ yields $X$ as an implicit function of $\psi$ and $\epsilon$. Its expansion in the Taylor series for small $\epsilon$ is as follows:

$$
X = \tilde{X}(\psi, \epsilon) = \frac{1}{\psi} A - \frac{\epsilon}{\psi^3} A^2 + O(\epsilon^2), \quad X|_{\epsilon = 0} = \frac{1}{\psi} A.
$$  \hfill (B1)

The unknown $\psi$ is estimated using the incompressibility relation $\det(X) = 1$, which yields

$$
\psi = \tilde{\psi}(\epsilon), \quad \psi_0 := \tilde{\psi}(0) = (\det A)^{1/3}.
$$  \hfill (B2)

Using the implicit function theorem, we have

$$
\frac{d\tilde{\psi}(\epsilon)}{d\epsilon}|_{\epsilon = 0} = -\frac{\partial \det \tilde{X}(\psi, \epsilon)}{\partial \epsilon}|_{\psi = \psi_0, \epsilon = 0} \left( \frac{\partial \det \tilde{X}(\psi, \epsilon)}{\partial \psi}|_{\psi = \psi_0, \epsilon = 0} \right)^{-1}.
$$  \hfill (B3)
Differentiating expansion (B1), we obtain
\[
\frac{\partial \det \mathbf{X}(\varphi, \varepsilon)}{\partial \varepsilon} \bigg|_{\varphi=\varphi_0, \varepsilon=0} = \det \mathbf{X}(\varphi_0, 0) \left( \frac{\partial \mathbf{X}(\varphi_0, 0)}{\partial \varepsilon} \right)^{-1} : \left( \begin{array}{c} \mathbf{A}^2 \\ \varphi_0^3 \end{array} \right),
\]
(B4)
\[
\frac{\partial \det \mathbf{X}(\varphi, \varepsilon)}{\partial \varphi} \bigg|_{\varphi=\varphi_0, \varepsilon=0} = \det \mathbf{X}(\varphi_0, 0) \left( \frac{\partial \mathbf{X}(\varphi_0, 0)}{\partial \varphi} \right)^{-1} : \left( \begin{array}{c} \mathbf{A} \\ \varphi_0^2 \end{array} \right).
\]
(B5)

Substituting this into (B3), we arrive at
\[
\frac{d\varphi(\varepsilon)}{d\varepsilon} \bigg|_{\varepsilon=0} = \frac{-\text{tr} \mathbf{A}}{3\varphi_0}, \quad \varphi(\varepsilon) = \varphi_0 - \frac{\text{tr} \mathbf{A}}{3\varphi_0} \varepsilon + O\left(\varepsilon^2\right).
\]
(B6)

Interestingly, this estimation is exact in the case of isotropic \( \mathbf{A} \). Moreover, a higher-order approximation of \( \varphi \) is possible. However, the higher-order approximation would be even less accurate dealing with finite \( \varepsilon \), which may occur in practice.

**APPENDIX C: ORDER OF APPROXIMATION**

Let us show that IFEBM is first order accurate. Towards that end, we consider a typical time interval \((t_n, t_{n+1})\); \(n+1\mathbf{C} \) and \(n\mathbf{C}_i\) are given and \(\det^n\mathbf{C}_i = 1\). Let \( \mathbf{C}_i^{\text{exact}} \) be the exact solution to (17) with the initial condition \(\mathbf{C}_i\big|_{t=t_n} = n\mathbf{C}_i\) and \(\mathbf{C} \equiv n+1\mathbf{C}_i\); \(n+1\mathbf{C}_i^{\text{IFEBM}}\) is the corresponding IFEBM solution. It is sufficient to show that for small \(\Delta t\)
\[
n+1\mathbf{C}_i^{\text{IFEBM}} - \mathbf{C}_i^{\text{exact}}(t_{n+1}) = O\left( (\Delta t)^2 \right).
\]
(C1)

We rewrite (17) in the compact form
\[
\dot{\mathbf{C}}_i(t) = \mathbf{f}(\mathbf{C}_i(t)) \mathbf{C}_i, \quad \text{tr} \mathbf{f} \equiv 0.
\]
(C2)

Since \( \mathbf{C}_i^{\text{exact}}(\cdot) \) is smooth, the mean value theorem implies that there exists \(t^* \in (t_n, t_{n+1})\) such that
\[
\mathbf{C}_i^{\text{exact}}(t_{n+1}) = n\mathbf{C}_i + \Delta t \mathbf{f} \left( \mathbf{C}_i^{\text{exact}}(t^*) \right) n\mathbf{C}_i + O\left( (\Delta t)^2 \right).
\]
(C3)

Because of the smoothness of \( \mathbf{f} \), we have
\[
n\mathbf{C}_i^{\text{exact}}(t_{n+1}) = n\mathbf{C}_i + \Delta t \mathbf{f} (n\mathbf{C}_i) n\mathbf{C}_i + O\left( (\Delta t)^2 \right).
\]
(C4)

Since \( \det \mathbf{C}_i^{\text{exact}}(t_{n+1}) = 1 \) and \( \det^n\mathbf{C}_i = 1 \), using the Jacobi formula, we obtain
\[
\det^n\mathbf{C}_i^{-1} : \left( \Delta t \mathbf{f} (n\mathbf{C}_i) n\mathbf{C}_i + O\left( (\Delta t)^2 \right) \right) = 0.
\]
(C5)

On the other hand, for the exact solution of the discretized equation (30), we have
\[
n+1\mathbf{C}_i = n\mathbf{C}_i + \Delta t \mathbf{f} (n+1\mathbf{C}_i) n+1\mathbf{C}_i + \tilde{\varphi} n+1\mathbf{C}_i, \quad \det n+1\mathbf{C}_i = 1.
\]
(C6)

Because of the smoothness of \( \mathbf{f} \),
\[
n+1\mathbf{C}_i = n\mathbf{C}_i + \Delta t \mathbf{f} (n\mathbf{C}_i) n\mathbf{C}_i + \tilde{\varphi} n\mathbf{C}_i + O\left( (\Delta t)^2 \right).
\]
(C7)

Similarly to (C5), using the incompressibility condition \( \det n+1\mathbf{C}_i = 1 \), we obtain
\[
n\mathbf{C}_i^{-1} : \left( \Delta t \mathbf{f} (n\mathbf{C}_i) n\mathbf{C}_i + \tilde{\varphi} n\mathbf{C}_i + O\left( (\Delta t)^2 \right) \right) = 0.
\]
(C8)

Subtracting (C5) from (C8), we conclude that \( \tilde{\varphi} = O((\Delta t)^2) \). Having this in mind and subtracting (C4) from (C7), we obtain the estimation
\[
n+1\mathbf{C}_i - \mathbf{C}_i^{\text{exact}}(t_{n+1}) = \tilde{\varphi} + O\left( (\Delta t)^2 \right) = O\left( (\Delta t)^2 \right).
\]
(C9)

Note that the IFEBM solution differs slightly from \( n+1\mathbf{C}_i \), since in IFEBM the parameter \( \varphi \) is not identified exactly but estimated using (37). The error in the estimation of \( \varphi \) is of the order \( O(\varepsilon^2) = O((\Delta t)^2) \). Thus, the corresponding \( \tilde{\varphi} \) is still \( O((\Delta t)^2) \) and (C9) is still valid.
Finally, the projection \( \overline{\cdot} \) (cf. (39)) brings changes that are only \( O((\Delta t)^2) \): \( n+1 \mathbf{C}_i^* - n+1 \mathbf{C}_i^{\text{exact}}(t_{n+1}) = O((\Delta t)^2) \).

\[ (C10) \]

**APPENDIX D: EXACT PRESERVATION OF THE W-INVARINCE**

A general definition of the w-invariance under isochoric change of the reference configuration is presented by Shutov and Ihlemann.\(^{31}\) In a simple formulation, the w-invariance of a material model says that for any volume-preserving change of the reference configuration, there is a transformation of initial conditions ensuring that the predicted Cauchy stresses are not affected by the reference change. The w-invariance represents a generalized symmetry of the constitutive equations, indicating that the analysed material exhibits fluid-like properties.

In the particular case of constitutive equations (12) and (16), the w-invariance property is formulated in the following way. Let \( \mathbf{F}_0 \) be an arbitrary second-rank tensor, such that \( \det \mathbf{F}_0 = 1 \). Consider a prescribed history of the right Cauchy-Green tensor \( \mathbf{C}(t), t \in [0, T] \) and the initial condition \( \mathbf{C}_i|_{t=0} = \mathbf{C}_0^0 \). Let \( \mathbf{C}_i^{\text{new}} \) be a new solution of (12) and (16) corresponding to the new loading programme \( \mathbf{C}_i^{\text{new}}(t) \) and the new initial conditions

\[ \mathbf{C}_i^{\text{new}}(t) := \mathbf{F}_0^{-T} \mathbf{C}(t) \mathbf{F}_0^{-1}, \quad \mathbf{C}_i^{\text{new}}|_{t=0} = \mathbf{F}_0^{-T} \mathbf{C}_0^0 \mathbf{F}_0^{-1}. \]

\[ (D1) \]

System (12) and (16) is w-invariant if and only if the original and new solutions are related through

\[ \mathbf{C}_i^{\text{new}}(t) = \mathbf{F}_0^{-T} \mathbf{C}_i(t) \mathbf{F}_0^{-1}. \]

\[ (D2) \]

Let \( n+1 \mathbf{C}, n\mathbf{C}_i, \) and \( \mathbf{F}_0 \) be given. Denote by \( n+1 \mathbf{C}_i \) and \( n+1 \mathbf{C}_i^{\text{new}} \) the IFEBM solutions pertaining to the original and new inputs, respectively:

\[ (n+1 \mathbf{C}, n\mathbf{C}_i) \overset{\text{IFEBM}}{\mapsto} n+1 \mathbf{C}_i, \quad (\mathbf{F}_0^{-T} n+1 \mathbf{C} \quad \mathbf{F}_0^{-1} \quad \mathbf{F}_0^{-T} n\mathbf{C}_i \quad \mathbf{F}_0^{-1}) \overset{\text{IFEBM}}{\mapsto} n+1 \mathbf{C}_i^{\text{new}}. \]

\[ (D3) \]

In analogy to the continuous case (D2), the algorithm is said to preserve the w-invariance if

\[ n+1 \mathbf{C}_i^{\text{new}} = \mathbf{F}_0^{-T} n+1 \mathbf{C}_i^{\text{new}} \mathbf{F}_0^{-1}. \]

\[ (D4) \]

Algorithms satisfying (D4) are advantageous over algorithms that violate this symmetry restriction (cf. the discussion in Shutov et al\(^{42}\)). A straightforward (but tedious) way of proving (D4) is to substitute the IFEBM equations into (D4) and to check the identity. A more elegant proof is based on the observation that the IFEBM on the reference configuration (cf. Table 1) is equivalent to the IFEBM on the current configuration (cf. Table 2). More precisely, let \( n+1 \mathbf{F} \) be any second-rank tensor, such that \( n+1 \mathbf{F}^T n+1 \mathbf{F} = n+1 \mathbf{C} \). Recall that, according to (50) and (54), \( n+1 \mathbf{B}_e \) is a unique function of the trial value \( n+1 \mathbf{B}_e^{\text{trial}} \). The following computation steps yield exactly the IFEBM solution:

\[ n+1 \mathbf{B}_e^{\text{trial}} = n+1 \mathbf{F} n\mathbf{C}_i^{-1} n+1 \mathbf{F}^T, \quad n+1 \mathbf{B}_e^{\text{trial}}(50,52) n+1 \mathbf{B}_e, \quad n+1 \mathbf{C}_i = n+1 \mathbf{F}^T n+1 \mathbf{B}_e^{-1} n+1 \mathbf{F}. \]

\[ (D5) \]

Now, for the new inputs \( n+1 \mathbf{C}_i^{\text{new}} = \mathbf{F}_0^{-T} n+1 \mathbf{C} \quad \mathbf{F}_0^{-1} \) and \( n\mathbf{C}_i^{\text{new}} = \mathbf{F}_0^{-T} n\mathbf{C}_i \quad \mathbf{F}_0^{-1} \), we may put \( n+1 \mathbf{F}^{\text{new}} = n+1 \mathbf{F} \mathbf{F}_0^{-1} \). Then

\[ (n+1 \mathbf{B}_e^{\text{trial})^{\text{new}}} = n+1 \mathbf{F}^{\text{new}} (n\mathbf{C}_i^{\text{new}})^{-1} n+1 \mathbf{F}^{\text{new}} \mathbf{F}_0^{-1} \]

\[ (D6) \]

Since the trial values for the original and new inputs coincide, we have

\[ n+1 \mathbf{B}_e^{\text{new}} = n+1 \mathbf{B}_e, \quad n+1 \mathbf{C}_i^{\text{new}} = (n+1 \mathbf{F}^{\text{new}})^T n+1 \mathbf{B}_e^{-1} n+1 \mathbf{F}^{\text{new}} = \mathbf{F}_0^{-T} n+1 \mathbf{C} \mathbf{F}_0^{-1}. \]

\[ (D7) \]

That is exactly the required relation between \( n+1 \mathbf{C}_i^{\text{new}} \) and \( n+1 \mathbf{C}_i \). In the same way, the w-invariance can be proved for the 2IEBM.
APPENDIX E: MEBM AND EM

We consider the initial value problem

\[ \dot{C}_i(t) = f(C_i(t), t)C_i(t), \quad C_i(0) = C^0_i, \quad \det(C^0_i) = 1, \quad \text{where} \quad \text{tr} f \equiv 0. \]

Assume that \(^nC_i\) is known. The classical EBM is based on the equation

\[ \begin{align*}
  n+1 C_i^{\text{EBM}} &= nC_i + \Delta t f \left( n+1 C_i^{\text{EBM}}, t_{n+1} \right) n+1 C_i^{\text{EBM}}.
\end{align*} \quad (E1)
\]

In the current study, we use its refined version, called MEBM, which guarantees that \(\det(\(n+1C_i\)) = 1:\)

\[ \begin{align*}
  n+1 C_i^{\text{MEBM}} &= nC_i + \Delta t f \left( n+1 C_i^{\text{MEBM}}, t_{n+1} \right) n+1 C_i^{\text{MEBM}}.
\end{align*} \quad (E2)
\]

Another modification of the EBM was presented by Helm,\(^{38}\) also to enforce the inelastic incompressibility. Finally, the EM corresponds to the equation

\[ \begin{align*}
  n+1 C_i^{\text{EM}} &= \exp \left( \Delta t f \left( n+1 C_i^{\text{EM}}, t_{n+1} \right) \right) nC_i.
\end{align*} \quad (E3)
\]

The MEBM and EM exactly preserve the geometric property (19): \(^{n+1}C_i \in \text{Sym}, \det(\(n+1C_i\)) = 1.\)