WAVELET TRANSFORMS ASSOCIATED WITH THE BASIC BESSEL OPERATOR

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ABSTRACT. This paper aims to study the $q$-wavelet and the $q$-wavelet transforms, associated with the $q$-Bessel operator for a fixed $q \in ]0,1[$. As an application, an inversion formulas of the $q$-Riemann-Liouville and $q$-Weyl transforms using $q$-wavelets are given. For this purpose, we shall attempt to extend the classical theory by giving their $q$-analogues.

1. Introduction

Continuous wavelet transforms have been introduced by A. Grossmann and J. Morlet [9] in the beginning of the 1980’s and became an active field of research, due to the fact that applications of wavelet analysis to the diverse subjects of communication, seismic data, signal and image processing... are being uncovered.

In [7], A. Fitouhi and K. Trimèche generalized the theory as presented by T. H. Koornwinder [16] and studied the generalized wavelets and the generalized continuous wavelet transforms associated with a class of singular differential operators. This class contains, in particular, the so called Bessel operator, which was studied extensively by K. Trimèche in [18].

In this paper, we shall try to generalize our results in [4] by studying wavelets and continuous wavelet transforms associated with the $q$-Bessel operator, studied in [6]. The basic tool in this work is some elements of $q$-harmonic analysis related to the just mentioned operator. Next, using the $q$-Riemann-Liouville and the $q$-Weyl operators, we will give some relations between the continuous $q$-wavelet transform, studied in [4], and the continuous $q$-wavelet transform associated with the $q$-Bessel operator, and we deduce other formulas which give the inverse operators of the $q$-Riemann-Liouville and the $q$-Weyl transforms. These formulas are better than those given in [6] and [11] because they are simple and we have a large choice of $q$-wavelets associated with the $q$-Bessel operator, that can be used in these formulas.

We are not in a situation to claim that all our results are new, but the methods used are direct and constructive, and have a good resemblance with the classical ones. Our approach in this paper is very similar to the classical picture developed in [7] and [18].
This paper is organized as follows: in Section 2, we present some preliminaries results and notations that will be useful in the sequel. In Section 3, we establish some $q$-harmonic results associated with the $q$-Bessel operator.

In Section 4, we define the $q$-wavelets and the $q$-wavelet transforms associated with the $q$-Bessel operator, and discuss their properties. Special attention is paid to the $q$-analogues of the Plancherel formula and the Parseval formula, and an inversion formula is proved. In Section 5, we give a characterization of the image set of the $q$-wavelet transform associated with the $q$-Bessel operator. Section 6, is devoted to give some inversion formulas of the $q$-Riemann-Liouville and the $q$-Weyl transforms. Finally, in Section 7, we give some relations between the continuous $q$-wavelet transform and the continuous $q$-wavelet transform associated with the $q$-Bessel operator. We use these relations to derive the inversion formulas of the $q$-Riemann-Liouville and the $q$-Weyl transforms using wavelets.

2. Notation and preliminaries

Throughout this paper, we will fix $q \in ]0,1[$ such that $\frac{\log(1-q)}{\log q} \in \mathbb{Z}$ and $\alpha > -\frac{1}{2}$.

We recall some usual notions and notations used in the $q$-theory (see [8]).

For $a \in \mathbb{C}$, the $q$-shifted factorials are defined by

\[(1) \quad (a; q)_0 = 1; \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad n = 1, 2, \ldots; \quad (a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k).\]

We also denote

\[(2) \quad (a_1, a_2, \ldots, a_p; q)_n = (a_1; q)_n (a_2; q)_n \ldots (a_p; q)_n, \quad n = 0, 1, 2, 3, \ldots \infty,\]

\[(3) \quad \lfloor x \rfloor_q = \frac{1 - q^x}{1 - q}, \quad x \in \mathbb{C} \text{ and } [n]_q! = \frac{(q; q)_n}{(1 - q)^n}, \quad n \in \mathbb{N}.\]

The $q$-derivative $D_q f$ of a function $f$ is given by

\[(4) \quad (D_q f)(x) = \frac{f(x) - f(qx)}{(1 - q)x}, \quad \text{if } x \neq 0,\]

$(D_q f)(0) = f'(0)$ provided $f'(0)$ exists. If $f$ is differentiable then $(D_q f)(x)$ tends to $f'(x)$ as $q$ tends to $1$.

The $q$-Jackson integrals from $0$ to $a$ and from $0$ to $\infty$ are defined by (see [11])

\[(5) \quad \int_0^a f(x) d_q x = (1 - q)a \sum_{n=0}^{\infty} f(aq^n)q^n,\]

\[(6) \quad \int_0^{\infty} f(x) d_q x = (1 - q) \sum_{n=-\infty}^{\infty} f(q^n)q^n,\]
provided the sums converge absolutely.

The $q$-Jackson integral in a generic interval $[a, b]$ is given by (see [11])

\begin{equation}
\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x.
\end{equation}

Jackson [11] defined a $q$-analogue of the Gamma function by

\begin{equation}
\Gamma_q(x) = \frac{(q; q)_\infty}{(q^2; q)_\infty} (1 - q)^{-x}, \quad x \neq 0, -1, -2, \ldots.
\end{equation}

It is well known that it satisfies

\begin{equation}
\Gamma_q(x + 1) = \frac{1 - q^x}{1 - q} \Gamma_q(x), \quad \Gamma_q(1) = 1 \text{ and } \lim_{q \to 1} \Gamma_q(x) = \Gamma(x), \Re(x) > 0.
\end{equation}

We denote by

\begin{equation}
R_q = \{ \pm q^n : n \in \mathbb{Z} \} \cup \{0\}, \quad R_q,+ = \{ q^n : n \in \mathbb{Z} \} \text{ and } \tilde{R}_q,+ = R_q,+ \cup \{0\}.
\end{equation}

\begin{itemize}
  \item $\mathcal{E}_{q}(\mathbb{R}_q)$ the space of the restrictions on $\mathbb{R}_q$ of even infinitely $q$-differentiable functions on $\mathbb{R}$, equipped with the induced topology of uniform convergence on all compact, for all functions and its $q$-derivatives.
  \item $\mathcal{D}_{q}(\mathbb{R}_q)$ the space of the restrictions on $\mathbb{R}_q$ of even infinitely $q$-differentiable functions on $\mathbb{R}$ with compact supports, equipped with the induced topology of uniform convergence, for all functions and its $q$-derivatives.
  \item $\mathcal{C}_{q,0}(\mathbb{R}_q)$ the space of the restrictions on $\mathbb{R}_q$ of even smooth functions, continued in 0 and vanishing at $\infty$, equipped with the induced topology of uniform convergence.
  \item $\mathcal{S}_{q}(\mathbb{R}_q)$ the space of the restrictions on $\mathbb{R}_q$ of infinitely $q$-differentiable, even and fast decreasing functions and all its $q$-derivatives i.e.
  \end{itemize}

\begin{equation}
\forall n, m \in \mathbb{N}, \quad P_{n,m,q}(f) = \sup_{x \in \mathbb{R}, 0 \leq k \leq n} |(1 + x^2)^m D_q^k f(x)| < +\infty.
\end{equation}

$\mathcal{S}_{q}(\mathbb{R}_q)$ is equipped with the induced topology defined by the semi-norms $P_{n,m,q}$.

\begin{itemize}
  \item $L^p(\mathbb{R}_q,+; x^{2\alpha + 1} d_q x), \ p > 0$, the set of all functions defined on $\mathbb{R}_q,+ \text{ such that}$
  \end{itemize}

\begin{equation}
\|f\|_{p,\alpha,q} = \left\{ \int_0^\infty |f(x)|^p x^{2\alpha + 1} d_q x \right\}^{1/p} < \infty.
\end{equation}

3. Preliminaries on $q$-Harmonic Analysis Related to the $q$-Bessel Operator

3.1. Normalized $q$-Bessel function. The $q$-Bessel operator is defined and studied in [6] by

\begin{equation}
\Delta_{q,\alpha} f(z) = \left( \frac{1}{x^{2\alpha + 1}} D_q [x^{2\alpha + 1} D_q f] \right) (q^{-1} z)
= q^{2\alpha + 1} \Delta_q f(z) + \frac{1 - q^{2\alpha + 1}}{(1 - q)q^{-1} z} D_q f(q^{-1} z),
\end{equation}
where
\[
\Delta_q f(z) = D_q^2 f(q^{-1}z).
\]

We recall (see [6]) that for \( \lambda \in \mathbb{C} \), the problem
\[
\begin{cases}
\Delta_{\alpha, q} u(x) = -\lambda^2 u(x), \\
u(0) = 1, \ u'(0) = 0
\end{cases}
\]
has as unique solution the normalized q-Bessel function, given by
\[
j^{(3)}_\alpha(z; q^2) = (1 - q^2)^\alpha \Gamma_q (\alpha + 1) ((1 - q)q^{-1}z)^{-\alpha} J^{(3)}_\alpha ((1 - q)q^{-1}z; q^2),
\]
where
\[
J^{(3)}_\alpha(z; q^2) = \frac{z^\alpha (q^{2\alpha+2}; q^2)_\infty}{(q^2; q^2)_\infty} 1\varphi_1 (0; q^{2\alpha+2}; q^2, q^2z^2)
\]
is the Jackson’s third q-Bessel function. This function is called in some literature the Hahn-Exton q-Bessel function (see [17]).

The following lemma shows some estimations for the normalized q-Bessel function.

**Lemma 1.** For \( x \in \mathbb{R}_{q,+} \), we have
1) \( |j^{(3)}_\alpha(x; q^2)| \leq \frac{1}{(q; q^2)_\infty^2} \);
2) \( |j^{(3)}_\alpha(x; q^2)| \leq \frac{(-q^2; q^2)_\infty (-q^{2(\alpha+1)}; q^2)_\infty}{(q^{2(\alpha+1)}; q^2)_\infty} \left\{ \frac{1}{q \left( \frac{\log(1-q^2)}{\log q} \right)^2} \right\} \), if \( x \leq \frac{q^{2(\alpha+1)}}{1-q} \);
3) For all \( \nu \in \mathbb{R} \), we have \( j^{(3)}_\alpha(x; q^2) = o(x^{-\nu}) \) as \( x \to +\infty \).

In particular, we have \( \lim_{x \to +\infty} j^{(3)}_\alpha(x; q^2) = 0 \).

**Proof.**

1) is proved in [6].

2) From the properties of the basic function \( 1\varphi_1 \) (see [6] or [17]), we have:

For \( x = q^n \in \mathbb{R}_{q,+} \ n \in \mathbb{N} \),
\[
|x^{-\alpha} j^{(3)}_\alpha(x; q^2)| = \frac{1}{(q^2; q^2)_\infty^2} |(q^{2\alpha+2}; q^2)_\infty 1\varphi_1 (0; q^{2\alpha+2}; q^2, q^{2n+2})|
\]
\[
\leq \frac{1}{(q^2; q^2)_\infty^2} \left( -q^{2(n+1)}; q^2)_\infty (-q^{2n+2}; q^2)_\infty \right)
\]
\[
\leq \frac{1}{(q^2; q^2)_\infty^2} \left( -q^2; q^2)_\infty (-q^{2n+2}; q^2)_\infty \right).
• For $x = q^{-n} \in \mathbb{R}_{q^+}$, $n \in \mathbb{N}$,

\[
|x^{-\alpha} J^{(3)}_{\alpha}(x; q^2)| = \frac{1}{(q^2; q^2)_\infty} |(q^{2(1-n)}; q^2)_\infty \varphi_1(0; q^{2(1-n)}; q^2, q^{2\alpha+2})| \\
\leq \frac{1}{(q^2; q^2)_\infty} q^{n(\alpha+1)} (-q^2; q^2)_\infty (-q^{2\alpha+2}; q^2)_\infty \\
\leq \frac{1}{(q^2; q^2)_\infty} q^{n^2} (-q^2; q^2)_\infty (-q^{2\alpha+2}; q^2)_\infty,
\]

since $\alpha > -1/2$.

So,

\[
|x^{-\alpha} J^{(3)}_{\alpha}(x; q^2)| \leq \frac{(-q^2; q^2)_\infty (-q^{2\alpha+1}; q^2)_\infty}{(q^2; q^2)_\infty} \begin{cases} 
1, & \text{if } n \geq 0, \\
q^n, & \text{if } n \leq 0,
\end{cases}
\]

which is equivalent to

\[
|x^{-\alpha} J^{(3)}_{\alpha}(x; q^2)| \leq \frac{(-q^2; q^2)_\infty (-q^{2\alpha+1}; q^2)_\infty}{(q^2; q^2)_\infty} \begin{cases} 
1, & \text{if } x \leq 1, \\
q^n \left( \frac{\log(q)}{\log(q)} \right)^2, & \text{if } x \geq 1.
\end{cases}
\]

The relation 2) follows from this inequality and the relation:

\[
j^{(3)}_{\alpha}(x; q^2) = (1-q^{2\alpha+1}) f(z), \quad (1-q^2)^{-\alpha} \Gamma_q^2(\alpha+1) \left( \frac{1-q}{q} \right) \cdot (1-q^2)^{-\alpha} J^{(3)}_{\alpha}(\frac{1-q}{q} z; q^2).
\]

Relation 3) is a direct consequence of 2). □

**Lemma 2.** For $x, y \in \mathbb{R}_{q^+}$, we have

\[
(xy)^{\alpha+1} \int_0^{\infty} j^{(3)}_{\alpha}(xt, q^2) j^{(3)}_{\alpha}(yt, q^2) t^{2\alpha+1} dt = \frac{(1+q)^{2\alpha} \Gamma_q^2(\alpha+1) q^{2(\alpha+1)}}{1-q} \delta_{x,y}.
\]

**Proof.** The result follows from the definition of $j^{(3)}_{\alpha}$ and the orthogonality relation of $J^{(3)}_{\alpha}$ proved in [17]. □

### 3.2. $q$-Bessel Fourier transform.

The generalized $q$-Bessel translation operator $T^{\alpha}_{q,x}, x \in \mathbb{R}_{q^+}$ was defined in [6] on $\mathcal{D}_{eq}(\mathbb{R})$ by

\[
T^{\alpha}_{q,x}(f)(y) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^2, q^{2\alpha+2}; q^2)_n} \frac{x^{2n}}{y} \sum_{k=-n}^{n} (-1)^{n-k} U_k(n) f(q^k y), y \in \mathbb{R}_{q^+}
\]

and $T^{\alpha}_{q,0}(f) = f$, where

\[
U_k(n) = q^{k(k-1)+2n(k+\alpha)} \sum_{p=0}^{k} \binom{n}{p} q^{n+k-p} = q^{-2p(\alpha+k+p)}
\]
is the $q$-Bessel $q$-Binomial coefficient associated with the $q$-Bessel operator (see [6]). It verifies, in particular

\begin{equation}
\int_0^\infty T_{q,x}^\alpha(f(y)g(y)y^{2\alpha+1}dy = \int_0^\infty f(y)T_{q,x}^\alpha(g(y)y^{2\alpha+1}dy, \quad x \in \mathbb{R}_{q,+},
\end{equation}

and

\begin{equation}
T_{q,x}^\alpha j_\alpha^{(3)}(ty; q^2) = j_\alpha^{(3)}(tx; q^2)j_\alpha^{(3)}(ty; q^2), \quad x, y, t \in \mathbb{R}_{q,+}.
\end{equation}

The $q$-Bessel Fourier transform and the $q$-convolution product are defined (see [6]) for $f, g \in D_{q}(\mathbb{R}_{q})$, by

\begin{equation}
\mathcal{F}_{\alpha,q}(f)(\lambda) = c_{\alpha,q} \int_0^\infty f(x)j_\alpha^{(3)}(\lambda x; q^2)x^{2\alpha+1}dx,
\end{equation}

\begin{equation}
f \ast_B g(x) = c_{\alpha,q} \int_0^\infty T_{q,x}^\alpha f(y)g(y)y^{2\alpha+1}dy,
\end{equation}

where

\begin{equation}
c_{\alpha,q} = \frac{(1 + q^{-1})^{-\alpha}}{\Gamma_{q^2}(\alpha + 1)}.
\end{equation}

Using the proprieties of the $q$-generalized Bessel translation, one can prove easily the following result [6].

**Theorem 1.** For $f, g \in D_{q}(\mathbb{R}_{q})$, we have

\begin{equation}
\mathcal{F}_{\alpha,q}(f \ast_B g) = \mathcal{F}_{\alpha,q}(f)\mathcal{F}_{\alpha,q}(g),
\end{equation}

\begin{equation}
\mathcal{F}_{\alpha,q}(T_{q,x}^\alpha f)(\lambda) = j_\alpha^{(3)}(\lambda x; q^2)\mathcal{F}_{\alpha,q}(f)(\lambda), \quad x \in \mathbb{R}_{q,+}, \quad \lambda \in \mathbb{R}_{q,+}
\end{equation}

and

\begin{equation}
\mathcal{F}_{\alpha,q}(\Delta_{\alpha,q} f)(\lambda) = -\frac{\lambda^2}{q^{2\alpha+1}}\mathcal{F}_{\alpha,q}(f)(\lambda), \quad \lambda \in \mathbb{C}.
\end{equation}

**Theorem 2.** For $f \in L_{q}^{1}(\mathbb{R}_{q,+}, x^{2\alpha+1}dx)$, we have

\begin{equation}
\mathcal{F}_{\alpha,q}(f) \in C_{\alpha,q}(\mathbb{R}_{q})
\end{equation}

and

\begin{equation}
\|\mathcal{F}_{\alpha,q}(f)\|_{C_{\alpha,q}(\mathbb{R}_{q})} \leq \frac{c_{\alpha,q}}{(q; q^2)^{\frac{\alpha}{2}\infty}}\|f\|_{1,\alpha,q}.
\end{equation}

**Proof.** Let $f \in L_{q}^{1}(\mathbb{R}_{q,+}, x^{2\alpha+1}dx)$. From the relation 1) of Lemma [1] we have

\[\forall \lambda, \ x \in \mathbb{R}_{q,+}, \ |f(x)j_\alpha^{(3)}(\lambda x; q^2)x^{2\alpha+1}| \leq \frac{1}{(q; q^2)^{\frac{\alpha}{2}\infty}}|f(x)x^{2\alpha+1}|.\]
Then, the definition of $j^{(3)}_\alpha$, the relation 3) of Lemma 1 and the Lebesgue theorem imply that $F_{\alpha,q}(f) \in C_{eq,0}(\mathbb{R}_q)$. On the other hand, we have for all $\lambda \in \mathbb{R}_{q,+}$,
\[
|F_{\alpha,q}(f)(\lambda)| \leq \frac{c_{\alpha,q}}{(q;q^2)^{2\alpha}} \|f\|_{1,\alpha,q},
\]
which achieves the proof. ■

**Theorem 3.**
$F_{\alpha,q}$ is an isomorphism of $L^2_q(\mathbb{R}_{q,+}, x^{2\alpha+1}d_qx)$ (resp. $S_{eq}(\mathbb{R}_q)$), $F_{\alpha,q}^{-1} = q^{-4\alpha-2}F_{\alpha,q}$ and for $f \in L^2_q(\mathbb{R}_{q,+}, x^{2\alpha+1}d_qx)$, we have
\[
\|F_{\alpha,q}(f)\|_{2,\alpha,q} = q^{2\alpha+1}\|f\|_{2,\alpha,q}.
\]

**Proof.**
The parity of $j^{(3)}_\alpha$ and the relation
\[
F_{\alpha,q}(\Delta_{\alpha,q}f) = -\frac{\lambda^2}{q^{2\alpha+1}}F_{\alpha,q}(f)
\]
show that if $f$ is in $S_{eq}(\mathbb{R}_q)$, then $F_{\alpha,q}(f)$ belongs to $S_{eq}(\mathbb{R}_q)$. Lemma 2 achieves the proof. ■

**Remak 1.**
Using the previous theorem and the relation (22), one can see that, for $f \in L^2_q(\mathbb{R}_{q,+}, x^{2\alpha+1}d_qx)$ (resp. $S_{eq}(\mathbb{R}_q)$), we have for all $x \in \mathbb{R}_{q,+}$, $T^\alpha_{q,x}f$ belongs to $L^2_q(\mathbb{R}_{q,+}, x^{2\alpha+1}d_qx)$ (resp. $S_{eq}(\mathbb{R}_q)$) and
\[
\|T^\alpha_{q,x}f\|_{2,\alpha,q} \leq \frac{1}{(q;q^2)^{2\alpha}} \|f\|_{2,\alpha,q}.
\]

**Proposition 1.** Let $f$ and $g$ be in $L^2_q(\mathbb{R}_{q,+}, x^{2\alpha+1}d_qx)$, then
1) $f \ast_B g \in L^2_q(\mathbb{R}_{q,+}, x^{2\alpha+1}d_qx)$ iff $F_{\alpha,q}(f)F_{\alpha,q}(g) \in L^2_q(\mathbb{R}_{q,+}, x^{2\alpha+1}d_qx)$.
2) $q^{4\alpha+2} \int_0^{\infty} |f \ast_B g(x)|^2 x^{2\alpha+1}d_qx = \int_0^{\infty} |F_{\alpha,q}(f)(x)|^2 |F_{\alpha,q}(g)(x)|^2 x^{2\alpha+1}d_qx,$
where both sides are finite or infinite.

**Proof.** The proof is a direct consequence of Theorem 3 and the fact that $F_{\alpha,q}(f \ast_B g) = F_{\alpha,q}(f)F_{\alpha,q}(f)$.

■

4. **q-Wavelet transforms associated with the q-Bessel operator**

**Definition 1.** A q-wavelet associated with the q-Bessel operator is an even function $g \in L^2_q(\mathbb{R}_{q,+}, x^{2\alpha+1}d_qx)$ satisfying the following admissibility condition:
\[
0 < C_g = \int_0^{\infty} |F_{\alpha,q}(g)(a)|^2 \frac{d_qa}{a} < \infty.
\]
Remarks

1) For all $\lambda \in \mathbb{R}_{q,+}$, we have

$$C_g = \int_0^\infty |\mathcal{F}_{\alpha,q}(g)(a\lambda)|^2 \frac{d_q a}{a}.$$ 

2) Let $f$ be a nonzero function in $S_{eq}(\mathbb{R}_q)$ (resp. $D_{eq}(\mathbb{R}_q)$). Then $g = \Delta_{\alpha,q} f$ is a $q$-wavelet associated with the $q$-Bessel operator, in $S_{eq}(\mathbb{R}_q)$ (resp. $D_{eq}(\mathbb{R}_q)$) and we have

$$C_g = \frac{1}{q^{4\alpha+2}} \int_0^\infty a^3 |\mathcal{F}_{\alpha,q}(f)(a)|^2 d_q a.$$ 

Example

Consider the functions $G(x; q^2) = A_\alpha e^{-\frac{q^{2(2\alpha+1)}x^2}{(1+q)^2}}$ and $g = \Delta_{\alpha,q} G(\cdot; q^2)$, where $A_\alpha = c_{\alpha,q} \int_0^\infty x^{2\alpha+1} e^{-x^2} d_q x$ and $e_{q^2}$ is the $q$-analogue of the exponential function.

We have $x \mapsto G(x; q^2)$ is in $S_{eq}(\mathbb{R}_q)$ and (see [6], Proposition 8)

$$\mathcal{F}_{\alpha,q}(G(\cdot; q^2))(x) = q^{4\alpha+2} e_{q^2}^{-x^2}, \ x \in \mathbb{R}_{q,+}.$$ 

Then, $g$ is in $S_{eq}(\mathbb{R}_q)$ and

$$\mathcal{F}_{\alpha,q}(g)(x) = -\frac{x^2}{q^{2\alpha+1}} \mathcal{F}_{\alpha,q}(G(\cdot; q^2))(x) = -q^{2\alpha+1} x^2 e_{q^2}^{-x^2}, \ x \in \mathbb{R}_{q,+}.$$ 

It is then easy to see that

$$0 < |\mathcal{F}_{\alpha,q}(g)(a)|^2 \leq q^{4\alpha+2} a^4 e_{q^2}^{-a^2}, \ \forall a \in \mathbb{R}_{q,+}.$$ 

Thus

$$0 < \int_0^\infty |\mathcal{F}_{\alpha,q}(g)(a)|^2 \frac{d_q a}{a} \leq q^{4\alpha+2} \int_0^\infty a^3 e_{q^2}^{-a^2} d_q a$$

$$= q^{4\alpha+2} \frac{(-q^4, -q^{-2}; q^2)_\infty}{(1+q)}$$

$$= \frac{q^{4\alpha}}{(1+q)}.$$ 

So $g$ is a $q$-wavelet associated with the $q$-Bessel operator.

**Proposition 2.** Let $g \neq 0$ be a function in $L^2(\mathbb{R}_{q,+}, x^{2\alpha+1} d_q x)$ satisfying:

1) $\mathcal{F}_{\alpha,q}(g)$ is continuous at 0.

2) \( \exists \beta > 0 \) such that $\mathcal{F}_{\alpha,q}(g)(x) - \mathcal{F}_{\alpha,q}(g)(0) = O(x^\beta)$, as $x \to 0$.

Then, (29) is equivalent to

$$\mathcal{F}_{\alpha,q}(g)(0) = 0.$$ 

Proof. We suppose that (29) is satisfied.
If $\mathcal{F}_{\alpha,q}(g)(0) \neq 0$, then from the condition 1) there exist $p_0 \in \mathbb{N}$ and $M > 0$, such that
\[ \forall n \geq p_0, \quad |\mathcal{F}_{\alpha,q}(g)(q^n)| \geq M. \]
Then, the integral in (29) would be equal to $\infty$.

Conversely, we suppose that $\mathcal{F}_{\alpha,q}(g)(0) = 0$.
As $g \neq 0$, we deduce from Theorem 3 that the first inequality in (29) is satisfied.
On the other hand, from the condition 2), there exist $n_0 \in \mathbb{N}$ and $\varepsilon > 0$, such that
\[ \forall n \geq n_0, \quad |\mathcal{F}_{\alpha,q}(g)(q^n)| \leq \varepsilon q^{n\beta}. \]
Then using the definition of the $q$-integral and Theorem 3, we obtain
\[ \int_0^\infty |\mathcal{F}_{\alpha,q}(g)(a)|^2 \frac{dq}{a} = (1-q) \sum_{n=-\infty}^{\infty} |\mathcal{F}_{\alpha,q}(g)(q^n)|^2 \]
\[ = (1-q) \sum_{n=-\infty}^{n_0} |\mathcal{F}_{\alpha,q}(g)(q^n)|^2 + (1-q) \sum_{n=n_0+1}^{\infty} |\mathcal{F}_{\alpha,q}(g)(q^n)|^2 \]
\[ \leq \frac{(1-q)}{q^{(2\alpha+2)n_0}} \sum_{n=-\infty}^{\infty} q^{(2\alpha+2)n} |\mathcal{F}_{\alpha,q}(g)(q^n)|^2 + (1-q)\varepsilon^2 \sum_{n=0}^{\infty} q^{2n\beta} \]
\[ \leq \frac{\|\mathcal{F}_{\alpha,q}(g)\|_{2,\alpha,q}^2}{q^{(2\alpha+2)n_0}} + \frac{1-q}{1-q^{2\beta}} \varepsilon^2 \]
\[ = q^{(4\alpha+2)} \frac{\|g\|_{2,\alpha,q}^2}{q^{(2\alpha+2)n_0}} + \frac{1-q}{1-q^{2\beta}} \varepsilon^2. \]
This proves the second inequality of (29).

Remark 2.
Owing to (24), the continuity assumption in the previous proposition will certainly hold if $g$ is moreover in $L^1_q(\mathbb{R}_{q,+}, x^{2\alpha+1}d_qx)$. Then (30) can be equivalently written as
\[ \int_0^\infty g(x)x^{2\alpha+1}d_qx = 0. \]

Theorem 4. Let $a \in \mathbb{R}_{q,+}$ and $g \in L^2_q(\mathbb{R}_{q,+}, x^{2\alpha+1}d_qx)$. Then, the function $g_a$ defined for $x \in \mathbb{R}_{q,+}$, by
\[ g_a(x) = \frac{1}{a^{2\alpha+2}} g\left(\frac{x}{a}\right), \]
satisfies:
i) the function $g_a$ belongs to $L^2_q(\mathbb{R}_{q,+}, x^{2\alpha+1}d_qx)$ and we have
\[ \|g_a\|_{2,\alpha,q} = \frac{1}{a^{\alpha+1}} \|g\|_{2,\alpha,q}; \]
Proof. The change of variable \( u = \frac{x}{a} \) leads to:

\[
\int_0^\infty |g_a(x)|^2 x^{2\alpha+1} dx = \frac{1}{a^{4\alpha+4}} \int_0^\infty |g(u)|^2 u^{2\alpha+1} du
\]

and for \( \lambda \in \mathbb{R}_{q,+} \),

\[
\mathcal{F}_{\alpha,q}(g_a)(\lambda) = \frac{c_{\alpha,q}}{a^{2\alpha+2}} \int_0^\infty g(x) j_\alpha^{(3)}(\lambda x; q^2)x^{2\alpha+1}\, dx
\]

\[
= c_{\alpha,q} \int_0^\infty g(u) j_\alpha^{(3)}(a\lambda u; q^2)u^{2\alpha+1} du = \mathcal{F}_{\alpha,q}(g)(a\lambda).
\]

\[\blacksquare\]

**Proposition 3.** Let \( g \) be in \( \mathcal{S}_{eq}(\mathbb{R}_q) \) (resp. \( \mathcal{D}_{eq}(\mathbb{R}_q) \)). Then for all \( a \in \mathbb{R}_{q,+} \) the function \( g_a \) given by the relation (34) belongs to \( \mathcal{S}_{eq}(\mathbb{R}_q) \) (resp. \( \mathcal{D}_{eq}(\mathbb{R}_q) \)).

**Theorem 5.** Let \( g \) be a \( q \)-wavelet associated with the \( q \)-Bessel operator in \( L^2_q(\mathbb{R}_{q,+}, x^{2\alpha+1} dx) \) (resp. \( \mathcal{S}_{eq}(\mathbb{R}_q) \)). Then for all \( a \in \mathbb{R}_{q,+} \) and \( b \in \mathbb{R}_{q,+} \), the function

\[
g_{a,b}(x) = \sqrt{a} T_{q,b}^\alpha(g_a),
\]

is a \( q \)-wavelet associated with the \( q \)-Bessel operator in \( L^2_q(\mathbb{R}_{q,+}, x^{2\alpha+1} dx) \) (resp. \( \mathcal{S}_{eq}(\mathbb{R}_q) \)) and we have

\[
C_{g_{a,b}} = a \int_0^\infty (j_\alpha^{(3)}( \frac{xb}{a}; q^2))^2 |\mathcal{F}_{\alpha,q}(g)(x)|^2 \frac{dx}{x}.
\]

Where \( T_{q,b}^\alpha \), \( b \in \mathbb{R}_{q,+} \) are the \( q \)-generalized translations defined by the relation (12).

Proof. As \( g_a \) is in \( L^2_q(\mathbb{R}_{q,+}, x^{2\alpha+1}) \) (resp. \( \mathcal{S}_{eq}(\mathbb{R}_q) \)), Remark 1 shows that the relation (34) defines an element of \( L^2_q(\mathbb{R}_{q,+}, x^{2\alpha+1}) \) (resp. \( \mathcal{S}_{eq}(\mathbb{R}_q) \)). On the other hand, from the relations (33) and (22), we have for all \( \lambda \in \mathbb{R}_{q,+} \),

\[
\mathcal{F}_{\alpha,q}(g_{a,b})(\lambda) = \sqrt{a} j_\alpha^{(3)}(b\lambda; q^2)\mathcal{F}_{\alpha,q}(g)(a\lambda).
\]

This relation implies (35).

Now, we shall prove that the function \( g_{a,b} \) satisfies the admissibility relation (29). As \( g \neq 0 \), we deduce from (35) and Theorem 3 that \( C_{g_{a,b}} \neq 0 \). On the other hand, from the relation (29) and the relation 1) of Lemma 1 we deduce that

\[
C_{g_{a,b}} \leq \frac{a}{(q; q^2)^4} C_g.
\]
Proposition 4. Let $g$ be a $q$-wavelet associated with the $q$-Bessel operator in $L^2_q(\mathbb{R}_{q,+}, x^{2\alpha+1}d_qx)$. Then the mapping

$$F : (a, b) \mapsto g_{a,b}$$

is continuous from $\mathbb{R}_{q,+} \times \tilde{\mathbb{R}}_{q,+}$ into $L^2_q(\mathbb{R}_{q,+}, x^{2\alpha+1}d_qx)$.

Proof. It is clear that $F$ is a mapping from $\mathbb{R}_{q,+} \times \tilde{\mathbb{R}}_{q,+}$ into $L^2_q(\mathbb{R}_{q,+}, x^{2\alpha+1}d_qx)$ and it is continuous at all $(a, b) \in \mathbb{R}_{q,+} \times \tilde{\mathbb{R}}_{q,+}$.

Now, fix $a \in \mathbb{R}_{q,+}$. For $b \in \tilde{\mathbb{R}}_{q,+}$, we have

$$\| F(a, b) - F(a, 0) \|_{2, \alpha, q}^2 = \| T^\alpha_{a,b}(g_a) - g_a \|_{2, \alpha, q}^2 = q^{-4\alpha-2} \| \mathcal{F}_{\alpha,q}(T^\alpha_{a,b}(g_a) - g_a) \|_{2, \alpha, q}^2 = q^{-4\alpha-2} \int_0^\infty | 1 - j_{\alpha}^{(3)}(xb; q^2) | | \mathcal{F}_{\alpha,q}(g_a) |^2 (x) x^{2\alpha+1}d_qx.$$  

However, for all $x \in \mathbb{R}_{q,+}$ and $b \in \tilde{\mathbb{R}}_{q,+}$, we have

$$| 1 - j_{\alpha}^{(3)}(xb; q^2) |^2 | \mathcal{F}_{\alpha,q}(g_a) |^2 (x) \leq (1 + \frac{1}{(q; q^2)_\infty})^2 | \mathcal{F}_{\alpha,q}(g_a) |^2 (x)$$

and $\mathcal{F}_{\alpha,q}(g_a) \in L^2_q(\mathbb{R}_{q,+}, x^{2\alpha+1}d_qx)$. So, the Lebesgue theorem leads to

$$\lim_{b \to 0} \| F(a, b) - F(a, 0) \|_{2, \alpha, q} = 0.$$

Then for all open neighborhood $V$ of $F(a, 0)$ in $L^2_q(\mathbb{R}_{q,+}, x^{2\alpha+1}d_qx)$, there exists an open neighborhood $U$ of $0$ in $\tilde{\mathbb{R}}_{q,+}$ such that

$$\forall b \in U, \ F(a, b) \in V.$$  

Thus $\{a\} \times U$ is an open neighborhood of $(a, 0)$ in $\mathbb{R}_{q,+} \times \tilde{\mathbb{R}}_{q,+}$ and $F(\{a\} \times U) \subset V$. Which proves the continuity of $F$ at $(a, 0)$. 

Definition 2. Let $g$ be a $q$-wavelet associated with the $q$-Bessel operator in $\mathcal{D}_{q}(\mathbb{R}_q)$. We define the continuous $q$-wavelet transform associated with the $q$-Bessel operator by

$$(36) \quad \Psi_{\alpha,g}(f)(a, b) = c_{\alpha,q} \int_0^\infty f(x)g_{a,b}(x)x^{2\alpha+1}d_qx, \ a \in \mathbb{R}_{q,+}, \ b \in \tilde{\mathbb{R}}_{q,+} \text{ and } f \in \mathcal{D}_{q}(\mathbb{R}_q).$$
Remark 3. The relation (36) can also be written in the form
\[ \Psi_{q,g}^\alpha(f)(a, b) = \sqrt{a} f * B_g(a) \]
\[ = \sqrt{a} q^{-4\alpha-2} \mathcal{F}_{\alpha,q}(f * B_g(a))(b) \]
\[ = \sqrt{a} q^{-4\alpha-2} \mathcal{F}_{\alpha,q}[ \mathcal{F}_{\alpha,q}(f) \mathcal{F}_{\alpha,q}(B_g(a))](b) \]
\[ = \sqrt{a} q^{-4\alpha-2} c_{a,q} \int_0^\infty \mathcal{F}_{\alpha,q}(f)(x) \mathcal{F}_{\alpha,q}(B_g(a))(ax) j_{\alpha}^{(3)}(bx; q^2)x^{2\alpha+1}d_qx, \]
where \( c_{a,q} \) is given by (24).
We give some properties of \( \Psi_{q,g}^\alpha \) in the following proposition.

Proposition 5. Let \( g \) be a q-wavelet associated with the q-Bessel operator in \( L^2(\mathbb{R}_{q,+}, x^{2\alpha+1}d_qx) \) and \( f \in L^2(\mathbb{R}_{q,+}, x^{2\alpha+1}d_qx) \), then

i) For all \( a \in \mathbb{R}_{q,+} \) and \( b \in \mathbb{R}_{q,+} \), we have
\[
| \Psi_{q,g}^\alpha(f)(a, b) | \leq \frac{c_{a,q}}{(q; q^2)_\alpha^2 a^{\alpha+1/2}} \| f \|_{2, \alpha,q} \| g \|_{2, \alpha,q}.
\]

ii) For all \( a \in \mathbb{R}_{q,+} \), the function \( b \mapsto \Psi_{q,g}^\alpha(f)(a, b) \) is continuous on \( \mathbb{R}_{q,+} \) and we have
\[
\lim_{b \to \infty} \Psi_{q,g}^\alpha(f)(a, b) = 0.
\]

iii) If \( g \) is in \( S_{eq}(\mathbb{R}_q) \), then for all \( f \) in \( S_{eq}(\mathbb{R}_q) \), the function \( b \mapsto \Psi_{q,g}^\alpha(f)(a, b) \) is in \( S_{eq}(\mathbb{R}_q) \).

Proof. .

i) For \( a \in \mathbb{R}_{q,+} \) and \( b \in \mathbb{R}_{q,+} \), we have
\[
| \Psi_{q,g}^\alpha(f)(a, b) | = c_{a,q} | \int_0^\infty f(x)g_{a,b}(x)x^{2\alpha+1}d_qx |
\]
\[
\leq c_{a,q} \sqrt{a} \int_0^\infty | f(x) | T_{q,b}g_a(x) | x^{2\alpha+1}d_qx
\]
\[
\leq \frac{c_{a,q}}{(q; q^2)_\alpha^2 a^{\alpha+1/2}} \| f \|_{2, \alpha,q} \| g \|_{2, \alpha,q},
\]
by using the relations (27) and (32).

ii) As in Proposition 4, it suffices to prove the continuity at 0. For \( b \in \mathbb{R}_{q,+} \), we have
\[
\Psi_{q,g}^\alpha(f)(a, b) = \sqrt{a} q^{-4\alpha-2} \mathcal{F}_{\alpha,q}[ \mathcal{F}_{\alpha,q}(f) \mathcal{F}_{\alpha,q}(B_g(a))](b)
\]
\[
= \sqrt{a} q^{-4\alpha-2} c_{a,q} \int_0^\infty \mathcal{F}_{\alpha,q}(f)(x) \mathcal{F}_{\alpha,q}(B_g(a))(ax) j_{\alpha}^{(3)}(bx; q^2)x^{2\alpha+1}d_qx
\]
and
\[
\forall x \in \mathbb{R}_{q,+}, \quad | j_{\alpha}^{(3)}(bx; q^2) | \leq \frac{1}{(q; q^2)_\alpha^2}.
\]
Since \( f, g \in L^2(\mathbb{R}_{q,+}, x^{2\alpha+1}d_qx) \), then by Theorem 3, \( \mathcal{F}_{\alpha,q}(f) \) and \( \mathcal{F}_{\alpha,q}(B_g(a)) \) are in \( L^2(\mathbb{R}_{q,+}, x^{2\alpha+1}d_qx) \).
So, the product $\mathcal{F}_{\alpha,q}(f)\mathcal{F}_{\alpha,q}(g)$ is in $L_q^2(\mathbb{R}_{q,+},x^{2\alpha+1}d_qx)$. Thus, by application of the Lebesgue theorem, we obtain

$$\lim_{b \to 0} \sqrt{a}q^{-4\alpha-2}c_{\alpha,q} \int_0^\infty \mathcal{F}_{\alpha,q}(f)(x)\mathcal{F}_{\alpha,q}(g)(x)j_\alpha^3(bx;q^2)x^{2\alpha+1}d_qx = \Psi^\alpha_{q,g}(f)(a,0).$$

Which proves the continuity of $\Psi^\alpha_{q,g}(f)(a,.)$ at 0.

Finally (24) implies that

$$\Psi^\alpha_{q,g}(a,b) = \sqrt{a}q^{-4\alpha-2}\mathcal{F}_{\alpha,q}[\mathcal{F}_{\alpha,q}(f)\mathcal{F}_{\alpha,q}(g)](b)$$

tends to 0 as $b$ tends to $\infty$.

iii) is an immediate consequence of the relation

$$\Psi^\alpha_{q,g}(f)(a,b) = \sqrt{a}f \ast_B g(a,b)$$

and the properties of the $q$-Bessel convolution product. 

\textbf{Theorem 6.} Let $g \in L_q^2(\mathbb{R}_{q,+},x^{2\alpha+1}d_qx)$ a $q$-wavelet associated with the $q$-Bessel operator. 

i) Plancherel formula for $\Psi^\alpha_{q,g}$ 

For $f \in L_q^2(\mathbb{R}_{q,+},x^{2\alpha+1}d_qx)$, we have

$$\frac{1}{C_g} \int_0^\infty \int_0^\infty |\Psi^\alpha_{q,g}(f)(a,b)|^2 b^{2\alpha+1}d_qad_qb \frac{a^2}{a^2} = ||f||_{2,\alpha,q}^2.$$ 

ii) Parseval formula for $\Psi^\alpha_{q,g}$

For $f_1, f_2 \in L_q^2(\mathbb{R}_{q,+},x^{2\alpha+1}d_qx)$, we have

$$\int_0^\infty \int_0^\infty \Psi^\alpha_{q,g}(f_1)(a,b)\Psi^\alpha_{q,g}(f_2)(a,b)b^{2\alpha+1}d_qad_qb \frac{a^2}{a^2} = \frac{1}{C_g} \int_0^\infty \int_0^\infty f_1(x)f_2(x)x^{2\alpha+1}d_qx.$$

Proof. By using Fubini’s theorem, Theorem 3, and the relations (33) and (25), we have

$$q^{4\alpha+2} \int_0^\infty \int_0^\infty |\Psi^\alpha_{q,g}(f)(a,b)|^2 b^{2\alpha+1} \frac{d_qad_qb}{a^2} = q^{4\alpha+2} \int_0^\infty \left( \int_0^\infty |f \ast_B g(a,x)|^2 \frac{d_qb}{a} x^{2\alpha+1}d_qx \right) \frac{d_qa}{a} =$$

$$= \int_0^\infty \left( \int_0^\infty |\mathcal{F}_{\alpha,q}(f)(x)|^2 |\mathcal{F}_{\alpha,q}(g)(x)|^2 x^{2\alpha+1}d_qx \right) \frac{d_qa}{a} =$$

$$= \int_0^\infty |\mathcal{F}_{\alpha,q}(f)(x)|^2 \left( \int_0^\infty |\mathcal{F}_{\alpha,q}(g)(ax)|^2 \frac{d_qa}{a} x^{2\alpha+1}d_qx \right).$$

The relation (32) is then proved.

ii) The result is easily deduced from (32).
Remark 4.
If $g \in L^2_q(\mathbb{R}_{q,+}, x^{2\alpha+1} d_q x)$ is a q-wavelet associated with the q-Bessel operator, then for all $f \in L^2_q(\mathbb{R}_{q,+}, x^{2\alpha+1} d_q x)$, we have $\Psi^\alpha_{q,g}(f) \in L^2_q(\mathbb{R}_{q,+} \times \mathbb{R}_{q,+}, b^{2\alpha+1} \frac{d_q d_a b}{a^2})$ and
\[
\|\Psi^\alpha_{q,g}(f)\|^2_{L^2_q(\mathbb{R}_{q,+} \times \mathbb{R}_{q,+}, b^{2\alpha+1} \frac{d_q d_a b}{a^2})} = C_g \|f\|^2_{2,\alpha,q}.
\]

Theorem 7. Let $g$ be a q-wavelet associated with the q-Bessel operator in $L^2_q(\mathbb{R}_{q,+}, x^{2\alpha+1} d_q x)$, then for all $f \in L^2_q(\mathbb{R}_{q,+}, x^{2\alpha+1} d_q x)$, we have
\[
f(x) = \frac{c_{a,g}}{C_g} \int_0^\infty \int_0^\infty \Psi^\alpha_{q,g}(f)(a,b)g_{a,b}(x) b^{2\alpha+1} \frac{d_q d_a b}{a^2}, \quad x \in \mathbb{R}_{q,+}.
\]

Proof. For $x \in \mathbb{R}_{q,+}$, we have $h = \delta_x$ belongs to $L^2_q(\mathbb{R}_{q,+}, x^{2\alpha+1} d_q x)$. On the other hand, according to the relation (40) of the previous theorem, the definition of $\Psi^\alpha_{q,g}$ and the definition of the q-Jackson integral, we have
\[
(1-q)x^{2\alpha+2} f(x) = \int_0^\infty f(t)\overline{h(t)}t^{2\alpha+1} d_q t = \frac{1}{C_g} \int_0^\infty \int_0^\infty \Psi^\alpha_{q,g}(f)(a,b)\overline{\Psi^\alpha_{q,g}(h)(a,b)} b^{2\alpha+1} \frac{d_q d_a b}{a^2}.
\]
\[
= \frac{c_{a,g}}{C_g} \int_0^\infty \int_0^\infty \Psi^\alpha_{q,g}(f)(a,b) \left( \int_0^\infty \overline{h(t)}g_{a,b}(t) t^{2\alpha+1} d_q t \right) b^{2\alpha+1} \frac{d_q d_a b}{a^2}.
\]
\[
= (1-q)x^{2\alpha+2} \frac{c_{a,g}}{C_g} \int_0^\infty \int_0^\infty \Psi^\alpha_{q,g}(f)(a,b)g_{a,b}(x) b^{2\alpha+1} \frac{d_q d_a b}{a^2}.
\]
Thus
\[
f(x) = \frac{c_{a,g}}{C_g} \int_0^\infty \int_0^\infty \Psi^\alpha_{q,g}(f)(a,b)g_{a,b}(x) b^{2\alpha+1} \frac{d_q d_a b}{a^2}.
\]
Which completes the proof.

5. Coherent states

Theorem 6 shows that the continuous wavelet transform associated with the q-Bessel operator $\Psi^\alpha_{q,g}$ is an isometry from the Hilbert space $L^2_q(\mathbb{R}_{q,+}, x^{2\alpha+1} d_q x)$ into the Hilbert space $L^2_q(\mathbb{R}_{q,+} \times \mathbb{R}_{q,+}, b^{2\alpha+1} \frac{d_q d_a b}{a^2})$ (the space of square integrable functions on $\mathbb{R}_{q,+} \times \mathbb{R}_{q,+}$ with respect to the measure $b^{2\alpha+1} \frac{d_q d_a b}{a^2}$). For the characterization of the image of $\Psi^\alpha_{q,g}$, we consider the vectors $g_{a,b}$, $(a,b) \in \mathbb{R}_{q,+} \times \mathbb{R}_{q,+}$, as a set of coherent states in the Hilbert space $L^2_q(\mathbb{R}_{q,+}, x^{2\alpha+1} d_q x)$ (see [16]).

Definition 3. A set of coherent states in a Hilbert space $\mathcal{H}$ is a subset $\{g_i\}_{i \in \mathcal{L}}$ of $\mathcal{H}$ such that
i) $\mathcal{L}$ is a locally compact topological space and the mapping $l \mapsto g_l$ is continuous
from $\mathcal{L}$ into $\mathcal{H}$.

ii) There is a positive Borel measure $dl$ on $\mathcal{L}$ such that, for $f \in \mathcal{H}$,

$$\|f\|^2 = \int_{\mathcal{L}} |(f, g)|^2 \, dl,$$

where $(.,.)$ and $\| . \|$ are respectively the scalar product and the norm of $\mathcal{H}$.

Let now $\mathcal{H} = L^2_q(\mathbb{R}_{q,+}, x^{2\alpha+1}d_{q}x)$, $\mathcal{L} = \mathbb{R}_{q,+} \times \tilde{\mathbb{R}}_{q,+}$ equipped with the induced topology of $\mathbb{R}^2$.

Choose a nonzero function $g \in L^2_q(\mathbb{R}_{q,+}, x^{2\alpha+1}d_{q}x)$ and let $g_t = g_{a,b}$, $l = (a, b) \in \mathcal{L}$ be given by the relation (3). Then we have a set of coherent states. Indeed, i) of Definition 3 is satisfied, because of Proposition 4, and ii) of Definition 3 is satisfied, for the measure $b^{2\alpha+1} \frac{d_{q}a'd_{q}b'}{a'^2C_q}$ (see Theorem 6). By adaptation of the approach introduced by T. H. Koornwinder in [16], we obtain the following result:

**Theorem 8.** Let $F$ be in $L^2_q(\mathbb{R}_{q,+} \times \tilde{\mathbb{R}}_{q,+}; b^{2\alpha+1} \frac{d_{q}a'd_{q}b'}{a'^2C_q})$. Then $F$ belongs to $\text{Im} \Psi_{q,g}^\alpha$ if and only if

$$F(a, b) = \frac{1}{C_g} \int_0^\infty \int_0^\infty \left( \int_0^\infty g_{a', b'}(x) \overline{g_{a,b}(x)} x^{2\alpha+1}d_{q}x \right) \frac{(b')^{2\alpha+1} \frac{d_{q}a'd_{q}b'}{(a')^2}}{a'^2C_q},$$

6. **Inversion Formulas for the $q$-Riemann-Liouville and the $q$-Weyl Operators**

**Notations.** We denote by

- $\mathcal{S}_{q,\alpha}(\mathbb{R})$ the subspace of $\mathcal{S}_{q}(\mathbb{R})$ constituted of functions $f$ such that

$$\int_0^\infty f(x) x^{2k+2\alpha+1}d_{q}x = 0, \quad k = 0, 1, \ldots.$$

- $\mathcal{S}_{q,\alpha}^0(\mathbb{R})$ the subspace of $\mathcal{S}_{q}(\mathbb{R})$ constituted of functions $f$ such that

$$D_{q}^2 f(0) = 0, \quad k = 0, 1, \ldots.$$

The $q$-Riemann-Liouville transform $R_{\alpha,q}$ is defined on $\mathcal{D}_{q}(\mathbb{R})$ by (see [3])

$$R_{\alpha,q}(f)(x) = \frac{(1 + q)\Gamma_q^2(\alpha + 1)}{\Gamma_q^2(\frac{1}{2})} \int_0^1 \frac{(t^2 q^2; q^2)_\infty}{(t^2 q^{\alpha+1}; q^2)_\infty} f(xt)d_qt.$$

The $q$-Weyl transform is defined on $\mathcal{D}_{q}(\mathbb{R})$ by (see [3])

$$W_{\alpha,q}(f)(x) = \frac{q(1 + q^{-1})^{-\alpha+\frac{1}{2}} \Gamma_q^2(\alpha + 1)}{\Gamma_q^2(\frac{1}{2})} \int_q^\infty \frac{x^{2}(t^2 q^2; q^2)_\infty}{(q^{2\alpha+1}; x^2/t^2; q^2)_\infty} f(t) t^{2\alpha}d_qt.$$

These two operators are isomorphism on $\mathcal{D}_{q}(\mathbb{R})$ and we have (see [3])

$$\Delta_{\alpha,q} \circ R_{\alpha,q} = R_{\alpha,q} \circ \Delta_q.$$

and

$$R_{\alpha,q}(f \ast_q g) = R_{\alpha,q}(f) \ast_{B} R_{\alpha,q}(g), \quad f, g \in \mathcal{D}_{q}(\mathbb{R}),$$
The $q$-Fourier-cosine transform $F_q$ (studied in [5]) and the $q$-Bessel transform are linked by the following relation (see [6]):

**Proposition 6.** For $f \in S_{eq}(\mathbb{R}_q)$, we have

\[
F_{\alpha,q}(f) = F_q \circ W_{\alpha,q}(f).
\]

(45)

We state the following results, useful in the sequel.

**Theorem 9.** The $q$-Fourier-cosine transform $F_q$ is a topological isomorphism from $S_{-1/2}^{eq}(\mathbb{R}_q)$ into $S_{0}^{eq}(\mathbb{R}_q)$.

**Proof.** From the Plancheral formula (see [2]), $F_q$ is a topological isomorphism from $S_{eq}(\mathbb{R}_q)$ into itself. Moreover, using the fact that $D^2_q \cos(x; q^2) = -\cos(qx; q^2)$, one can prove by induction that for $n \in \mathbb{N}$ and $f \in S_{eq}(\mathbb{R}_q)$, there exists a constant $C_{q,n}$, such that

\[
D^2_q F_q(f)(0) = C_{q,n} \int_0^{\infty} f(t)t^2d_q t,
\]

which achieves the proof. □

Similarly, we have the following result.

**Theorem 10.** The $q$-Fourier-Bessel transform $F_{\alpha,q}$ is a topological isomorphism from $S_{eq,\alpha}(\mathbb{R}_q)$ into $S_{0}^{eq}(\mathbb{R}_q)$.

**Corollary 1.** The $q$-Weyl transform $W_{\alpha,q}$ is a topological isomorphism from $S_{eq,\alpha}(\mathbb{R}_q)$ into $S_{-1/2}^{eq}(\mathbb{R}_q)$.

**Proof.** From the relation $F_{\alpha,q} = F_q \circ W_{\alpha,q}$, one can see that

\[
W_{\alpha,q} = F_q^{-1} \circ F_{\alpha,q}.
\]

We deduce the result from this relation and Theorems 9 and 10. ■

**Proposition 7.** For $f$ in $S_{eq,-1/2}(\mathbb{R}_q)$ (resp. $S_{eq,\alpha}(\mathbb{R}_q)$) and $g$ in $S_{eq}(\mathbb{R}_q)$ the function $f \ast_q g$ (resp. $f \ast_B g$) belongs to $S_{eq,-1/2}(\mathbb{R}_q)$ (resp. $S_{eq,\alpha}(\mathbb{R}_q)$).

**Proof.** The proof follows from Theorem 9 (resp. 10) and the fact that $f \ast_q g = F_q(F_q(f) \cdot F_q(g))$ (resp. $f \ast_B g = q^{-4\alpha-2}F_{\alpha,q}(F_{\alpha,q}(f) \cdot F_{\alpha,q}(g))$). ■

**Proposition 8.** The operator $K_{\alpha,q,1}$ defined by

\[
K_{\alpha,q,1}(f) = \frac{\Gamma_q^{2}(1/2)}{q^{3\alpha+3/2}(1+q)^{(\alpha+1/2)}\Gamma_q^{2}(\alpha+1)} F_q^{-1}(|\lambda|^{2\alpha+1} F_q(f))
\]

is a topological isomorphism from $S_{eq,-1/2}(\mathbb{R}_q)$ into itself.
Proof. The multiplication operator
\[ f \mapsto \frac{\Gamma_q^2 (1/2)}{q^{3\alpha+3/2}(1 + q)^{(\alpha+1)/2}\Gamma_q^2(\alpha + 1)} |\lambda|^{2\alpha+1} f \]
is a topological isomorphism from \( S_0^0(R_q) \) into itself. The inverse is given by
\[ f \mapsto \frac{q^{3\alpha+3/2}(1 + q)^{(\alpha+1)/2}\Gamma_q^2(\alpha + 1)}{\Gamma_q^2 (1/2)} |\lambda|^{2\alpha+1} f. \]
The result follows from Theorem 9.

Proposition 9. The operator \( K_{\alpha,q,2} \) defined by
\[ K_{\alpha,q,2}(f)(x) = \frac{\Gamma_q^2 (1/2)}{q^{3\alpha+3/2}(1 + q)^{(\alpha+1)/2}\Gamma_q^2(\alpha + 1)} \mathcal{F}_{\alpha,q}^{-1}(|\lambda|^{2\alpha+1} \mathcal{F}_{\alpha,q}(f))(x) \]
is a topological isomorphism from \( S_{\alpha,q}(R_q) \) into itself.
Proof. From the relation \( \mathcal{F}_{\alpha,q} = \mathcal{F}_q \circ W_{\alpha,q} \) and the definition of \( K_{\alpha,q,1} \), we have for all \( f \in S_{\alpha,q}(R_q) \)
\[ K_{\alpha,q,2} = W_{\alpha,q}^{-1} \circ K_{\alpha,q,1} \circ W_{\alpha,q}. \]
We deduce the result from Proposition 8 and Corollary 1.

Proposition 10. i) For all \( f \in S_{\alpha,-1/2}(R_q) \) and \( g \in S_{\alpha}(R_q) \), we have
\[ K_{\alpha,q,1}(f *_q g) = K_{\alpha,q,1}(f) *_q g. \]
ii) For all \( f \in S_{\alpha,q}(R_q) \) and \( g \in S_{\alpha,q}(R_q) \), we have
\[ K_{\alpha,q,2}(f *_B g) = K_{\alpha,q,2}(f) *_B g. \]
Proof. It suffices to prove one of the two relations. We have
\[ K_{\alpha,q,1}(f *_q g) = \frac{\Gamma_q^2 (1/2)}{q^{3\alpha+3/2}(1 + q)^{(\alpha+1)/2}\Gamma_q^2(\alpha + 1)} \mathcal{F}_q^{-1}(|\lambda|^{2\alpha+1} \mathcal{F}_q(f *_q g)) \]
\[ = \frac{\Gamma_q^2 (1/2)}{q^{3\alpha+3/2}(1 + q)^{(\alpha+1)/2}\Gamma_q^2(\alpha + 1)} \mathcal{F}_q^{-1}(|\lambda|^{2\alpha+1} \mathcal{F}_q(f) \mathcal{F}_q(g)) \]
\[ = \frac{\Gamma_q^2 (1/2)}{q^{3\alpha+3/2}(1 + q)^{(\alpha+1)/2}\Gamma_q^2(\alpha + 1)} \left\{ \mathcal{F}_q^{-1}(|\lambda|^{2\alpha+1} \mathcal{F}_q(f)) \right\} *_q g \]
\[ = K_{\alpha,q,1}(f) *_q g. \]
Theorem 11. For all \( f \in S_{q,a}(\mathbb{R}_q) \), we have the following inversion formulas for the operator \( R_{\alpha,q} \)

\[
f = R_{\alpha,q} \circ K_{\alpha,q,1} \circ W_{\alpha,q}(f)
\]

(47)

\[
f = R_{\alpha,q} \circ W_{\alpha,q} \circ K_{\alpha,q,2}(f).
\]

(48)

**Proof.** Using the properties of the operator \( R_{\alpha,q} \), studied in [6], Theorem 3 and Proposition 6, we obtain for \( x \in \tilde{\mathbb{R}}_q^+ \),

\[
q^{4\alpha+2} f(x) = c_{\alpha,q} \int_0^{\infty} F_{\alpha,q}(f)(\lambda) j_\alpha^{(3)}(\lambda x; q^2) \lambda^{2\alpha+1} d_\lambda
\]

\[
= R_{\alpha,q} \left[ c_{\alpha,q} \int_0^{\infty} F_{\alpha,q}(f)(\lambda) \cos(\lambda); q^2) \lambda^{2\alpha+1} d_\lambda \right] (x)
\]

\[
= R_{\alpha,q} \left[ c_{\alpha,q} \int_0^{\infty} \lambda^{2\alpha+1} F_q \circ W_{\alpha,q}(f)(\lambda) \cos(\lambda); q^2) d_\lambda \right] (x)
\]

\[
= R_{\alpha,q} \left\{ \frac{c_{\alpha,q}}{c_{-1/2,q}} F_q^{-1} \left[ \lambda^{2\alpha+1} F_q \circ W_{\alpha,q}(f) \right] \right\} (x)
\]

\[
= q^{4\alpha+2} R_{\alpha,q} \left\{ F_q^{-1} \left[ \frac{\Gamma_q(1/2) \lambda^{2\alpha+1}}{q^{3\alpha+3/2}(1+q)^{\alpha+1/2}\Gamma_q(\alpha+1)} F_q \circ W_{\alpha,q}(f) \right] \right\} (x).
\]

Thus, \( \forall x \in \tilde{\mathbb{R}}_q^+ \), \( f(x) = R_{\alpha,q} \circ K_{\alpha,q,1} \circ W_{\alpha,q}(f)(x) \).

We deduce the second relation from the first relation and the fact

\[
K_{\alpha,q,2} = W_{\alpha,q}^{-1} \circ K_{\alpha,q,1} \circ W_{\alpha,q}.
\]

\[\blacksquare\]

Corollary 2. The operator \( R_{\alpha,q} \) is a topological isomorphism from \( S_{q,-1/2}(\mathbb{R}_q) \) into \( S_{q,a}(\mathbb{R}_q) \).

**Proof.** We deduce the result from Proposition 8, Corollary 1 and the relation (47).

\[\blacksquare\]

Similarly, we have the following result.

Theorem 12. For all \( f \in S_{q,-1/2}(\mathbb{R}_q) \), we have the following inversion formulas for the operator \( W_{\alpha,q} \)

\[
f = W_{\alpha,q} \circ R_{\alpha,q} \circ K_{\alpha,q,1}(f).
\]

(49)

\[
f = W_{\alpha,q} \circ K_{\alpha,q,2} \circ R_{\alpha,q}(f)
\]

(50)
Proof. For $f \in S_{q, -1/2}(\mathbb{R}_q)$, Corollary 1 (resp. 2) implies that $W_{\alpha,q}^{-1}(f)$ (resp. $R_{\alpha,q}(f)$) belongs to $S_{q, a}(\mathbb{R}_q)$. Then by writing the relation (47) (resp. 48) for $W_{\alpha,q}^{-1}(f)$ (resp. $R_{\alpha,q}(f)$), we obtain the result.

Corollary 3. i) For all $f, g \in S_{q, a}(\mathbb{R}_q)$, we have

\begin{equation}
W_{\alpha,q}(f * B g) = W_{\alpha,q}(f) *_q W_{\alpha,q}(g).
\end{equation}

ii) For all $f, g \in S_{q, -1/2}(\mathbb{R}_q)$ we have

\begin{equation}
R_{\alpha,q}(f * g) = R_{\alpha,q}(f) * B W_{\alpha,q}^{-1}(g).
\end{equation}

Proof. i) From Proposition 6, we have

\begin{align*}
W_{\alpha,q}(f * B g) &= F_{q}^{-1} \circ F_{\alpha,q}(f * B g) \\
&= F_{q}^{-1}(F_{\alpha,q}(f)F_{\alpha,q}(g)) \\
&= F_{q}^{-1} \circ F_{\alpha,q}(f) *_q F_{q}^{-1} F_{\alpha,q}(g) \\
&= W_{\alpha,q}(f) *_q W_{\alpha,q}(g).
\end{align*}

ii) Using Theorem 12 and Proposition 10, we obtain

\begin{align*}
R_{\alpha,q}^{-1}(R_{\alpha,q}(f) * B W_{\alpha,q}^{-1}(g)) &= W_{\alpha,q} \circ K_{\alpha,q,2}(R_{\alpha,q}(f) * B W_{\alpha,q}^{-1}(g)) \\
&= W_{\alpha,q}(K_{\alpha,q,2} \circ R_{\alpha,q}(f) * B W_{\alpha,q}^{-1}(g)) \\
&= W_{\alpha,q} \circ K_{\alpha,q,2} \circ R_{\alpha,q}(f) *_q g.
\end{align*}

On the other hand, we have

\begin{equation}
W_{\alpha,q} \circ K_{\alpha,q,2} \circ R_{\alpha,q}(f) = f.
\end{equation}

So,

\begin{equation}
R_{\alpha,q}^{-1}(R_{\alpha,q}(f) * B W_{\alpha,q}^{-1}(g)) = f *_q g.
\end{equation}

This achieves the proof.

7. Inversion formulas for the $q$-Riemann-Liouville and the $q$-Weyl operators using wavelets

We recall that (see [4]):

• the dilatation operator is defined for $a \in \mathbb{R}_{q,+}$ by

\begin{equation}
H_{a}(f)(x) = \frac{1}{\sqrt{a}} f \left( \frac{x}{a} \right),
\end{equation}

• a $q$-wavelet is an even and square $q$-integrable function $g$ satisfying

\[ 0 < C_g^\infty = \int_0^\infty |F_q(g)|^2(a) \frac{d_q a}{a} < \infty, \]
for a \( q \)-wavelet \( g \), the continuous \( q \)-wavelet transform (i.e., associated with the operator \( \Delta_q \)) is defined on \( \mathbb{R}_{q,+} \times \mathbb{R}_{q,+} \) by
\[
\Phi_{q,g}(f)(a,b) = c_{-1/2,q} \int_0^\infty f(x) \overline{g_{a,b}(x)} \, dq \, x
\]
(54)
\[
\Phi_{q,g}(f)(a,b) = f_q \cdot H(a)(b),
\]
(55)
where \( g_{a,b}^c = T_{q,b}(H_a(g)) \) and \( T_{q,b} = T_{q,b}^{-1/2} \) is the \( q \)-even translation operator studied in [5].

**Proposition 11.** For all \( a \in \mathbb{R}_{q,+} \) and \( g \in L^2_q(\mathbb{R}_{q,+}, x^{2\alpha+1} \, dq \, x) \), we have
1) \( g_a = \frac{1}{a^{2\alpha+3/2}} H_a(g) \);
2) \[
\Psi_{q,g}(f)(a,.) = \frac{q^{-4\alpha-2}}{\sqrt{a}} F_{\alpha,q} \circ H_{a^{-1}} \circ F_{\alpha,q}(g)
\]
(56)
\[
= \frac{1}{\sqrt{a}} W_{a,q}^{-1} \circ H_a \circ W_{a,q}(g).
\]
(57)

**Proof.** 1) is clear.

2) From the facts \( F_{\alpha,q}(g_a)(\lambda) = F_{\alpha,q}(g)(a\lambda) \) and \( F_q \circ H_a = H_{a^{-1}} \circ F_q \) (see [4]), one can see
\[
F_{\alpha,q}(g_a) = \frac{1}{\sqrt{a}} H_{a^{-1}} \circ F_{\alpha,q}(g).
\]
Then, by using Proposition 10 and Theorem 3, we obtain
\[
g_a = \frac{1}{\sqrt{a}} F_{\alpha,q}^{-1} \circ H_{a^{-1}} \circ F_{\alpha,q}(g) = \frac{q^{-4\alpha-2}}{\sqrt{a}} F_{\alpha,q} \circ H_{a^{-1}} \circ F_{\alpha,q}(g)
\]
(58)
\[
= \frac{1}{\sqrt{a}} W_{a,q}^{-1} \circ H_a \circ W_{a,q}(g).
\]

**Proposition 12.** Let \( g \) be a \( q \)-wavelet associated with the \( q \)-Bessel operator in \( \mathcal{S}_{q,\alpha}(\mathbb{R}_q) \). Then for all \( f \) in \( \mathcal{S}_{q,\alpha}(\mathbb{R}_q) \), we have the following relation
(59)
\[
\Psi_{q,g}^\alpha(f)(a,.) = W_{a,q}^{-1} \left[ \Phi_{q,W_{a,q}(g)}(W_{a,q}(f))(a,.) \right], \quad a \in \mathbb{R}_{q,+}.
\]

**Proof.** Let \( a \in \mathbb{R}_{q,+} \), we have from the relations (51), (55) and (57),
\[
\Psi_{q,g}^\alpha(f)(a,.) = \sqrt{a} f_B \ast_B \Phi_{q,g}^\alpha = \sqrt{a} W_{a,q}^{-1} \left[ W_{a,q}(f) \ast_q W_{a,q}(\Phi_{q,g}^\alpha) \right]
\]
(58)
\[
= W_{a,q}^{-1} \left[ W_{a,q}(f) \ast_q H_a \circ W_{a,q}(g) \right]
\]
(59)
\[
= W_{a,q}^{-1} \left[ \Phi_{q,W_{a,q}(g)}(W_{a,q}(f))(a,.) \right].
\]
Theorem 13. Let \( g \) be a \( q \)-wavelet associated with the \( q \)-Bessel operator in \( S_{q,a}(\mathbb{R}_q) \). Then

1) for all \( f \) in \( S_{q,a}(\mathbb{R}_q) \), we have the following relation

\[
\Psi^\alpha_{q,g}(f)(a,b) = R_{\alpha,q} \left[ \Phi_{q,W_{\alpha,q}(g)} \left( R_{\alpha,q}^{-1}(f) \right) (a,.) \right] (b), \quad a \in \mathbb{R}_{q,+}, \; b \in \mathbb{R}_{q,+};
\]

2) for all \( f \) in \( S_{q,-1/2}(\mathbb{R}_q) \), we have

\[
\Phi_{q,W_{\alpha,q}(g)}(f)(a,b) = W_{\alpha,q} \left[ \Psi^\alpha_{q,g} \left( W_{\alpha,q}^{-1}(f) \right) (a,.) \right] (b), \quad a \in \mathbb{R}_{q,+}, \; b \in \mathbb{R}_{q,+}.
\]

Proof. 1) From Corollary 3 and the relations (55) and (57), we obtain for \( a \in \mathbb{R}_{q,+} \) and \( b \in \mathbb{R}_{q,+} \),

\[
\Psi^\alpha_{q,g}(f)(a,b) = \sqrt{a} f \ast_B \overline{g}(b)
\]

\[
= \sqrt{a} R_{\alpha,q} \left[ R_{\alpha,q}^{-1}(f) \ast_q W_{\alpha,q}(g) \right] (b)
\]

\[
= R_{\alpha,q} \left[ R_{\alpha,q}^{-1}(f) \ast_q H_a \circ W_{\alpha,q}(g) \right] (b)
\]

\[
= R_{\alpha,q} \left[ \Phi_{q,W_{\alpha,q}(g)} \left( R_{\alpha,q}^{-1}(f) \right) (a,.) \right] (b).
\]

2) For \( a \in \mathbb{R}_{q,+} \) and \( b \in \mathbb{R}_{q,+} \), we have by the relation (55), Corollary 3 and the relation (57),

\[
\Phi_{q,W_{\alpha,q}(g)}(f)(a,b) = f \ast_q H_a \circ W_{\alpha,q}(g)(b)
\]

\[
= W_{\alpha,q} \left[ W_{\alpha,q}^{-1}(f) \ast_B W_{\alpha,q}^{-1} \circ H_a \circ W_{\alpha,q}(g) \right] (b)
\]

\[
= W_{\alpha,q} \left[ \sqrt{a} W_{\alpha,q}^{-1}(f) \ast_B \overline{g}(b) \right] (b)
\]

\[
= W_{\alpha,q} \left[ \Psi^\alpha_{q,g} \left( W_{\alpha,q}^{-1}(f) \right) (a,.) \right] (b).
\]

Proposition 13. 1) If \( g \) is a \( q \)-wavelet in \( S_{q,-1/2}(\mathbb{R}_q) \), then \( K_{\alpha,q,1}(g) \) is a \( q \)-wavelet in \( S_{q,-1/2}(\mathbb{R}_q) \) and we have

\[
K_{\alpha,q,1} \circ H_a(g) = \frac{1}{a^{2\alpha+1}} H_a \circ K_{\alpha,q,1}(g), \quad a \in \mathbb{R}_{q,+}.
\]

2) If \( g \) is a \( q \)-wavelet associated with the \( q \)-Bessel operator in \( S_{q,a}(\mathbb{R}_q) \), then \( K_{\alpha,q,2}(g) \) is a \( q \)-wavelet in \( S_{q,a}(\mathbb{R}_q) \) and we have

\[
K_{\alpha,q,2}(g_a) = \frac{1}{a^{2\alpha+1}} (K_{\alpha,q,2}(g))_a, \quad a \in \mathbb{R}_{q,+}.
\]

Proof. 1) Let \( g \) be a \( q \)-wavelet in \( S_{q,-1/2}(\mathbb{R}_q) \). From the definition of \( K_{\alpha,q,1} \), we have for \( \lambda \in \mathbb{R}_{q,+} \),

\[
\mathcal{F}_q(K_{\alpha,q,1}(g))(\lambda) = \frac{\Gamma_q^2(1/2)}{a^{3\alpha+3/2}(1+q)(\alpha+1/2)} \left( a^{2\alpha+1} \mathcal{F}_q(g)(\lambda) \right).
\]
Proposition 4 of [4], implies that \( K_{\alpha,q,1}(g) \) is a \( q \)-wavelet. On the other hand, using the fact \( F_q \circ H_a = H_{a^{-1}} \circ F_q \), \( a \in \mathbb{R}_+, \) and the above equality, we obtain

\[
F_q(H_a \circ K_{\alpha,q,1}(g)) = \frac{\Gamma_q^2(1/2)}{q^{3\alpha+3/2}(1 + q)^{(\alpha+1)/2}} H_{a^{-1}} (\lambda^{2\alpha+1} F_q(g)(\lambda))
\]

which gives the result.

2) Let \( q \) be a \( q \)-wavelet associated with the \( q \)-Bessel operator in \( S_{q,a}(\mathbb{R}_q) \). From the definition of \( K_{\alpha,q,2} \), we have for \( \lambda \in \mathbb{R}_+, \)

\[
F_{q,a}(K_{\alpha,q,2}(g))(\lambda) = \frac{\Gamma_q^2(1/2)}{q^{3\alpha+3/2}(1 + q)^{(\alpha+1)/2}} \lambda^{2\alpha+1} F_{q,a}(g)(\lambda).
\]

Proposition \( \square \) implies that \( K_{\alpha,q,2}(g) \) is a \( q \)-wavelet associated with the \( q \)-Bessel operator.

Moreover, for \( \lambda \in \mathbb{R}_+, \) we have

\[
F_{q,a}(K_{\alpha,q,2}(g_a))(\lambda) = \frac{\Gamma_q^2(1/2)}{q^{3\alpha+3/2}(1 + q)^{(\alpha+1)/2}} \lambda^{2\alpha+1} F_{q,a}(g_a)(\lambda)
\]

This achieves the proof.

**Theorem 14.** Let \( g \) be a \( q \)-wavelet associated with the \( q \)-Bessel operator in \( S_{q,a}(\mathbb{R}_q) \).

Then for \( a \in \mathbb{R}_{q,+} \) and \( b \in \mathbb{R}_{q,+} \), we have:

1) for all \( f \) in \( S_{q,a}(\mathbb{R}_q) \),

(63) \[ \Psi_{q,a}^\alpha(f)(a,b) = \frac{1}{a^{2\alpha+1}} R_{a,q} \left[ \Phi_{q,K_{\alpha,q,1}\circ W_{a,q}}(W_{a,q}(f))(a,.) \right] (b); \]

2) for all \( f \) in \( S_{q,-1/2}(\mathbb{R}_q) \),

(64) \[ \Phi_{q,W_{a,q}(g)}(f)(a,b) = \frac{1}{a^{2\alpha+1}} W_{a,q} \left[ \Psi_{q,K_{\alpha,q,2}}^\alpha(R_{a,q}(f))(a,.) \right] (b). \]

**Proof.** 1) Let \( f \) be in \( S_{q,a}(\mathbb{R}_q) \), \( a \in \mathbb{R}_{q,+} \) and \( b \in \mathbb{R}_{q,+} \). Using Corollary \( \square \) we obtain

\[
\Psi_{q,a}^\alpha(f)(a,b) = \sqrt{a} f * R_{a,q}(g)(b) = \sqrt{a} R_{a,q} \left[ W_{a,q}(f) * R_{a,q}^{-1}(g)(a) \right] (b).
\]
Thus, from Theorem 11, Proposition 13 and the relation (54), we have
\[
R_{\alpha,q}(f) = K_{\alpha,q,1} \circ W_{\alpha,q}(f)
\]
\[
= \frac{1}{\sqrt{a}} K_{\alpha,q,1} \circ H_a \circ W_{\alpha,q}(g)
\]
\[
= \frac{1}{a^{2\alpha+3/2}} H_a \circ K_{\alpha,q,1} \circ W_{\alpha,q}(g).
\]

2) Similarly, the result derives from the previous theorem and Theorem 7 of [4].

Proof. 1) is a simple deduction from the previous theorem and Theorem 7.

2) Similarly, the result derives from the previous theorem and Theorem 7 of [4].
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