Non-asymptotic bounds for the $\ell_\infty$ estimator in linear regression with uniform noise

YUFEI YI$^{1,a}$ and MATEY NEYKOV$^{1,b}$

$^1$Department of Statistics & Data Science, Carnegie Mellon University, Pittsburgh, USA, $^a$yy544@fb.com, $^b$mneykov@stat.cmu.edu

The Chebyshev or $\ell_\infty$ estimator is an unconventional alternative to the ordinary least squares in solving linear regressions. It is defined as the minimizer of the $\ell_1$ objective function

$$b\beta := \arg\min_{\beta} \max_{i} |Y_i - X_i^T \beta|.$$ (1.1)

Compared to ordinary least squares (OLS), the Chebyshev estimator minimizes the $\ell_\infty$ rather than the $\ell_2$ norm of the estimated residuals. The motivation of (1.1) stems from the fact that this is the MLE when the noise is known to be uniform on a bounded interval $U([-a, a])$ where the value of $a$ is unknown. Formally, we should write $\hat{\beta} := \arg\min_{\beta} \max_{i} |Y_i - X_i^T \beta|$, as the minimizer may not be unique. However since our results are valid for any point in this set we abuse the notation slightly.

1. Introduction

The goal of this paper is to analyze the non-asymptotic behavior of the Chebyshev estimator (and some of its close relatives) in a linear model with uniformly distributed errors. Concretely, suppose we have $n$ independent and identically distributed (i.i.d.) observations of the following model $Y_i = X_i^T \beta^* + \epsilon_i$ where $X_i \in \mathbb{R}^p$ are covariates, and $\epsilon_i \sim U([-a, a])$ for some $a > 0$ which may be either known or unknown. Throughout the paper we will additionally assume that $X_i$ is independent of $\epsilon_i$. A natural (although unconventional) estimator of $\beta^*$ is the Chebyshev (also known as $\ell_\infty$ or minimax) estimator which is defined through:

$$\hat{\beta} := \arg\min_{\beta} \max_{i} |Y_i - X_i^T \beta|.$$ (1.1)

Keywords: Chebyshev estimator; Chebyshev’s LASSO; Linear Model; Uniform distribution
unknown (see also Section 2 for this simple calculation). It is easy to see that (1.1) can be conveniently solved through a linear program. Alternatively, there exist iteratively reweighted least squares schemes, originally due to Lawson (Cline, 1972, Lawson, 1961), which can be shown to converge to the solution of (1.1) at a linear rate. Intuitively, (1.1) will be a good estimator of \( \beta^* \) when the noise has bounded support with non-zero probability mass near the boundary or the noise is concentrated on a bounded interval with very thin tails outside the interval. In contrast, when the noise is of unbounded support or whenever there is a negligible probability of the noise being near the boundary the Chebyshev estimator might have poor performance or may be even inconsistent. Importantly, observe that the \( \ell_\infty \) estimator (1.1) is not a linear estimator of the observations, and therefore Gauss-Markov’s theorem is not applicable — which leaves the door open for the Chebyshev estimator to dominate OLS on some occasions. We will verify that this is indeed the case.

Apart from being a cute mathematical problem, regression with uniform errors can be motivated in problems where the error is naturally bounded. For instance if the observations undergo some physical measurement process (such as measuring weight on a scale) it may be natural to assume that the error has bounded support. Although one may argue that uniform distribution is not necessarily the most natural bounded distribution, we find it enlightening to study this model, in part because the uniform distribution is naturally related to the order statistics of any continuous distribution. As we shall see the order statistics play a big role in our non-asymptotic analysis of the performance of the Chebyshev estimator, and therefore we believe the methods we develop here are (much) more broadly applicable. In fact all of our proofs can readily be extended to continuous, symmetric, bounded noise with almost no efforts (see Remark 2.4). Finally, it is also possible to extend the results to cases with asymmetric noise, at the cost of a slightly cumbersome argument on symmetrizing the observations (and critical inequalities) one considers in the proofs. We avoid doing this here in order to keep our exposition as clean as possible.

1.1. Related work and contributions

Although the Chebyshev estimator is not extensively used in practice, there certainly has been some interest coming from various fields. In particular it has found applications in the physical and environmental sciences (Bertsch, Sabbey and Uusnäkki, 2005, Brenner, 2002, James, 1983a,b, Qi, 2015, Zolghadri and Henry, 2004), finance (Jaschke, 1997, Jaschke and Küchler, 2001), and there is also a considerable literature in signal processing on estimation with bounded noise (see Akçay, Hjalmarsson and Ljung, 1996, Alecu et al., 2006, Beck and Eldar, 2007, Milanese and Belforte, 1982, Tse, Dahleh and Tsitsiklis, 1993, e.g., and references therein). In addition there is a lot of literature on Chebyshev’s estimation, dealing with the computational aspects of these estimators using linear programming and numerical analysis (Appa and Smith, 1973, Armstrong and Kung, 1980, Hand and Sposito, 1980, Sielken Jr and Hartley, 1973, Sklar and Armstrong, 1982). Recent statistical studies of the Chebyshev estimator include (Berenguer-Rico, Johansen and Nielsen, 2019, Castillo et al., 2009, Du et al., 2019, Knight, 2020). Of note Du et al. (2019), present a high-dimensional problem in composite fuselage assembly where a regularized Chebyshev estimator is a natural choice (i.e. the \( \ell_\infty \) loss seems more natural in this problem compared to other standard loss functions such as the \( \ell_2 \) and \( \ell_1 \) losses). Du et al. (2019) provide some statistical guarantees about a dual version of what we call Chebyshev’s LASSO below (see (3.1)), assuming that the noise is sub-Gaussian, but under strong assumptions on the design matrix, and most importantly they fail to recognize that if the noise is uniform this estimator will actually outperform the LASSO in terms of the estimation convergence rate (see Theorem 3.4).

Remarkably, even though the Chebyshev estimator has been around for a long time, partly as folklore knowledge, studies of its behavior have been very limited. We attribute this fact to the relatively complicated form of the loss function which is non-smooth. To our knowledge the first proper attempts at
characterizing the rates of convergence of the Chebyshev estimator (in a regression setting) are due to (Robbins and Zhang, 1986, Schechtman and Schechtman, 1986). These authors used very clever ideas, to derive the rate of convergence of the Chebyshev estimator in a simple linear regression case. Both papers observed the “super-efficient” behavior of the Chebyshev estimator in comparison to the OLS. The case with more than one covariate remained unsolved for nearly 30 years, and in 2020 Keith Knight in a breakthrough preprint (which seems to first have been released in 2010 and later revised in 2017 and 2020) derived the exact asymptotic distribution of the Chebyshev estimator with a fixed (but potentially bigger than one) number of covariates (Knight, 2020). The author showed, that the Chebyshev estimator converges to its target at a $n^{-1}$ rate (in the uniform noise case), which should be contrasted to the much slower $n^{-1/2}$ rate for the OLS. The asymptotic distribution however is complicated, and non-pivotal, which means that it cannot be used to perform inference. While being a landmark, the work of Knight (2020) left a lot to be desired. For one, the rate of convergence to the asymptotic distribution is unknown. This means that finite sample results which hold with high probability cannot be extracted easily from the main result of Knight (2020). In addition, due to the complicated form of the asymptotic distribution, it is not straightforward to derive the dependence on the dimension in the rate of convergence for the estimator. This paper proposes a novel non-asymptotic approach, which is able to derive finite-sample guarantees, and in addition can be used to give a rough upper bound for the dependence on the dimension in the convergence rate. In addition, we formalize and analyze Chebyshev’s LASSO, which extends the Chebyshev estimator to high-dimensional settings by incorporating an $\ell_1$-penalty. We demonstrate that Chebyshev’s LASSO can be much more efficient than the regular LASSO in models where the noise is uniform under certain assumptions.

1.2. Organization

The paper is structured as follows. In Section 2 we record our main results on the Chebyshev estimator (and its relatives). Subsection 2.1 uses a simple analysis which as we argue captures a multitude of random designs, while Subsection 2.2 derives a minimax lower bound for the problem. In Section 3 we state our main result for Chebyshev’s LASSO, which illustrates that Chebyshev’s LASSO can be much more accurate than the regular LASSO under certain assumptions. We provide brief numerical results in support of our theoretical findings in Section 4. Section 5 is dedicated to a brief discussion.

1.3. Notation

We use $\lesssim$ and $\gtrsim$ to mean $\leq$ and $\geq$ up to positive universal constants. By convention for any integer $n \in \mathbb{N}$ we set $[n] = \{1, \ldots, n\}$. We use $\mathbb{B}_p^n$ to denote the $p$-dimensional Euclidean ball, while $\mathbb{S}^{p-1}$ to denote the $p$-dimensional Euclidean sphere. For a vector $v \in \mathbb{R}^p$ we denote $\|v\|_q = \left[\sum_{i \in [p]} |v_i|^q\right]^{1/q}$ its $\ell_q$-th norm (with the usual extension for $q = \infty$), and we use $\|\cdot\|$ as a shorthand for $\|\cdot\|_2$. For two vectors $v, w \in \mathbb{R}^p$ we denote their dot product with either $v^T w$ or with $\langle v, w \rangle$. For a matrix $A \in \mathbb{R}^{n \times p}$ we use $\|A\|_{\text{max}}$ to denote the largest absolute value of all its entries, and use $\|A\|_\infty$ to denote the largest $\ell_1$ norm of all its rows such that $\|A\|_\infty = \max_{i \in [n]} \sum_{j \in [p]} |A_{ij}|$. We denote the operator norm of a matrix $A$ with $\|A\|_{\text{op}}$.

2. Linear model with uniform noise

Consider a linear model

$$Y = X\beta^* + \varepsilon, \quad \varepsilon_i \sim U([-a, a]),$$

(2.1)
where $a > 0$ is a known constant. Here $Y$ is a vector of $n$ outcome values, $X$ is an $n \times p$ matrix whose rows are the covariates and $\varepsilon$ is the vector of the error terms (which we assume is independent of $X$).

OLS is probably the most commonly used method to estimate $\beta^*$ in linear models, with an estimation rate $\|\hat{\beta} - \beta^*\| \sim O(\sqrt{p/n})$ when $p$ is changing with $n$, and $X \sim N(0, I)$ (Mourtada, 2022, see equation (18)). Since in our problem the noise $\varepsilon$ is bounded, by incorporating this information, we expect that the following constrained optimization

$$
\hat{\beta} = \arg\min_{\beta} \frac{1}{n} \sum_{i \in [n]} (Y_i - X_i^T \beta)^2 \quad \text{s.t.} \quad Y_i - a \leq X_i^T \beta \leq Y_i + a \quad \forall i \in [n],
$$

may give a better estimation of $\beta^*$ compared to the OLS. Clearly (2.2) given (2.3) can be solved via quadratic programming. In addition, one could consider the best risk equivariant estimator in this problem, which is given by the centroid of the constraint set in (2.3) (Jurecková and Picek, 2009, see equation (1.5) and also references therein), although it may be hard to calculate it in practice (Rademacher, 2007). Our analysis will simultaneously cover both estimators considered above. In fact our analysis covers any estimator taking values in the set (2.3).

We now consider the situation when $a$ is unknown. In this case, none of the two proposed estimators can be implemented since both of them rely on the knowledge of $a$. A natural approach would be to obtain the MLE. The likelihood function is

$$
L(a, \beta) = 1/(2a)^n \prod_{i \in [n]} \mathbb{1}(|Y_i - X_i^T \beta| \leq a).
$$

Hence the MLE of $a$ and $\beta^*$ is given by the following linear program (where the inequalities are entrywise):

$$
\min a,
Y \leq a + X \beta,
X \beta - a \leq Y.
$$

Clearly, this is equivalent to minimizing the loss function $\|Y - X \beta\|_\infty$. Thus the MLE of $\beta^*$ is given by

$$
\tilde{\beta} := \arg\min_{\beta} \|Y - X \beta\|_\infty,
$$

which is also called the $\ell_\infty$ estimator or the Chebyshev estimator. Consequently, $a$ can be estimated by

$$
\hat{a} = \|Y - X \tilde{\beta}\|_\infty.
$$

Observe that trivially we must have $\hat{a} \leq a$. This implies that when $\tilde{\beta}$ is the Chebyshev estimator, it also satisfies (2.3) even though $a$ is unknown. As we mentioned previously, all results below will be valid for any estimator $\hat{\beta}$ which takes values in the set (2.3), hence they are automatically valid for the Chebyshev estimator as well.

Let $\eta_i = -\text{sign}(e_i)$, $i \in [n]$ be independent Rademacher random variables which are also independent from $X_i$ and $|e_i|$. Let

$$
\tilde{X}_i = \eta_i X_i.
$$

We will now introduce the concept of a critical inequality given in (2.5).
Lemma 2.1. From (2.3) one can deduce the inequality

\[ \tilde{X}_i^T (\hat{\beta} - \beta^*) \leq a - |e_i|. \]  

(2.5)

Although we use the term critical inequality to refer to any inequality of the type (2.5), we will actually only use these inequalities for which the \( |e_i| \) value happens to be close to \( a \). This justifies the term critical, as the right hand side of such an inequality is very close to 0. Hence, if we are lucky enough and \( \tilde{X}_i^T (\hat{\beta} - \beta^*) \) is not too small, a critical inequality will yield that \( \tilde{X}_i^T (\hat{\beta} - \beta^*) = 0 \). While this is not exactly how our analysis proceeds, we hope this gives a good intuition why critical inequalities may be useful. It is also worth stressing the fact that \( \tilde{X}_i \) is sign symmetric regardless of the distribution of \( X_i \).

2.1. A simple non-asymptotic analysis of estimators taking values in (2.3)

The high level intuition of the analysis we give in this section is very simple. First, note that due to the nature of the uniform distribution, there will be a significant proportion of critical inequalities whose right hand side will be close to 0. Suppose now that we are able to establish that there exists a “reasonably large” 0-centered \( \ell_2 \)-ball inside the convex hull of the \( \tilde{X}_i \), for indices \( i \) which correspond to the critical inequalities which are close to 0. This will automatically mean that the \( k \) has to be bounded by the largest deviation from 0 in the considered critical inequalities. Formally, we have:

**Theorem 2.2.** Suppose that the design \( X_i \in \mathbb{R}^p \) is random and is independent of the noise \( e_i \). Let \( f(p, \gamma) \) be a known function of the dimension and the scalar \( \gamma > 0 \). Assume that the design is such that for any integer \( m \geq f(p, \gamma) \), and an i.i.d. sample \( \{X_i\}_{i \in [m]} \) from the design we have

\[ \mathbb{P}(\xi \mathbb{E} \mathbb{E}_{\tilde{X}}^2 \not\subset \text{conv(} \eta_1 X_1, \ldots, \eta_m X_m \text{)}) \leq \gamma, \]

for some \( \xi > 0 \), where \( \eta_i \) are i.i.d. Rademacher random variables which are also independent from the design. Then, for any estimator \( \hat{\beta} \) taking values in the set (2.3), we have that for any \( L > 0 \)

\[ \|\hat{\beta} - \beta^*\| \leq \frac{a(L + 1)[f(p, \gamma)]}{\xi n}, \]

with probability at least \( 1 - \gamma - \exp \left( \frac{-L^2}{4L^2} \right) \).

**Remark 2.3.** One can see that when the constant \( L \) is fixed we do obtain constant probability bounds (which decay exponentially with \( L \)). Perhaps with slight abuse of terminology, throughout the paper we refer to this type of bound as a “high probability” bound, even though it does not decay to 0 as \( n \) goes to \( \infty \). This is similar in spirit to how one can only obtain constant confidence bounds for the expression \( |\sum_{i \in [n]} X_i / n - \mu| < C/\sqrt{n} \) for any constant \( C > 0 \), where \( X_i \sim N(\mu, 1) \) (since the variable \( \sqrt{n}(\sum_{i \in [n]} X_i / n - \mu) \sim N(0, 1) \)).
Proof. Sort the absolute values of the errors $|\varepsilon_i| \sim U([0,a])$ in a decreasing manner $|\tilde{\varepsilon}(i)|$, so that $a \geq |\tilde{\varepsilon}(1)| \geq \ldots \geq |\tilde{\varepsilon}(n)| \geq 0$. Take the first $[f(p,\gamma)]$ many of them. By Lemma B.2 we know that:

$$P\left(\left|\frac{\tilde{\varepsilon}(f(p,\gamma))}{a}\right| < 1 - \frac{(L + 1)[f(p,\gamma)]}{n} \right) \leq \exp\left(-\frac{L^2}{4L + 1}\right).$$

Let $E$ be the complement of the event in the probability above. Now, by Lemma 2.1, on the event $E$ we have:

$$\eta_i X_i^T(\hat{\beta} - \beta^*) \leq a - a\left(1 - \frac{(L + 1)[f(p,\gamma)]}{n}\right) = a\frac{(L + 1)[f(p,\gamma)]}{n},$$

for all $i$ corresponding to the $[f(p,\gamma)]$ largest in magnitude $\varepsilon_i$’s (denote this index set by $S$). Since with probability at least $1 - \gamma$, we have the $\xi \frac{\hat{\beta} - \beta^*}{\|\hat{\beta} - \beta^*\|} \in \text{conv}\{\{\eta_i X_i\}_{i \in S}\}$ we can write

$$\xi \frac{\hat{\beta} - \beta^*}{\|\hat{\beta} - \beta^*\|} = \sum_{i=1}^{n} \alpha_i \eta_i X_i,$$

where $\sum_{i=1}^{n} \alpha_i = 1$ and $\alpha_i \geq 0$. We can now multiply the inequalities (2.6) by $\alpha_i$ and sum them up to obtain the desired conclusion upon rearranging terms, and using the union bound. \hfill \Box

Remark 2.4. The above theorem can be readily generalized to settings where the noise is continuous, symmetric and bounded on an interval $[-a, a]$ but is not necessarily uniform. All that needs to be done is to replace the application of Lemma B.2 with Lemma B.3. In fact, all of our results can be extended to cover this more general case with almost no efforts. We do not pursue this further here to keep the exposition simple. It should be noted however, that while the upper bound results can be extended to the more general setting of symmetric bounded noise, the optimality of the Chebyshev estimator in such a setting is less clear.

Example 2.5. We will now exhibit a simple example of a random design which satisfies the condition imposed in Theorem 2.2. Although this example may appear contrived at this point, it is an important example for assessing the difficulty of estimation of $\beta^*$, as we will see later when we discuss a minimax lower bound. More natural design examples will follow below. Take the random design $X_i \sim U(\sqrt{p}\{v_1, \ldots, v_p\})$, where $v_i, i \in [p]$ denote vectors from any orthonormal basis. We therefore have $X_i \sim U(\sqrt{p}(\pm v_1, \ldots, \pm v_p))$.

First we will show that if all vectors $\{\pm \sqrt{p}v_j\}_{j \in [p]}$ are present within the $m$ considered samples we have a 0-centered $\ell_2$-ball inside. Take any point on $x \in B_2^p$, and write it as $x = \sum_{j=1}^{p} a_j v_j$. We have that $\sum a_j^2 \leq 1$, and hence $\sum_{j \in [p]} |a_j| \leq \sqrt{p}$. This means that we can represent $x = \sum_{j \in [p]} a_j (\text{sign}(a_j) \sqrt{p}v_j)$, where $\sum a_j \leq 1, a_j \geq 0$, where $a_j = |a_j|/\sqrt{p}$. On the other hand since clearly $0 \in \text{conv}(\{\pm \sqrt{p}v_j\}_{j \in [p]})$ this implies that $x \in \text{conv}(\{\pm \sqrt{p}v_j\}_{j \in [p]})$, and since $x$ was arbitrary $B_2^p \subseteq \text{conv}(\{\pm \sqrt{p}v_j\}_{j \in [p]}).

Now, it suffices to show that with high probability the set $\{\eta_i X_i\}_{i \in [m]}$ contains all vectors from the set $\{\pm \sqrt{p}v_j\}_{j \in [p]}$. The probability that a specific vector is not in this set is $(1 - (2p)^m)^m$, hence by a union bound we obtain an upper bound $2p(1 - (2p)^m)^m \leq 2p \exp(-m/(2p))$, for $m \geq 2p \log(2\gamma^{-1}p)$ we have this probability is bounded by $\gamma$. Therefore by Theorem 2.2 we can conclude that with probability at least $1 - \gamma - \exp(-L^2/(8L/3 + 2))$ we have $\|\hat{\beta} - \beta^*\| \leq a(L + 1)(2p \log(2\gamma^{-1}p) + 1)/n$. 

---

Example 2.5. We will now exhibit a simple example of a random design which satisfies the condition imposed in Theorem 2.2. Although this example may appear contrived at this point, it is an important example for assessing the difficulty of estimation of $\beta^*$, as we will see later when we discuss a minimax lower bound. More natural design examples will follow below. Take the random design $X_i \sim U(\sqrt{p}\{v_1, \ldots, v_p\})$, where $v_i, i \in [p]$ denote vectors from any orthonormal basis. We therefore have $X_i \sim U(\sqrt{p}(\pm v_1, \ldots, \pm v_p))$.

First we will show that if all vectors $\{\pm \sqrt{p}v_j\}_{j \in [p]}$ are present within the $m$ considered samples we have a 0-centered $\ell_2$-ball inside. Take any point on $x \in B_2^p$, and write it as $x = \sum_{j=1}^{p} a_j v_j$. We have that $\sum a_j^2 \leq 1$, and hence $\sum_{j \in [p]} |a_j| \leq \sqrt{p}$. This means that we can represent $x = \sum_{j \in [p]} a_j (\text{sign}(a_j) \sqrt{p}v_j)$, where $\sum a_j \leq 1, a_j \geq 0$, where $a_j = |a_j|/\sqrt{p}$. On the other hand since clearly $0 \in \text{conv}(\{\pm \sqrt{p}v_j\}_{j \in [p]})$ this implies that $x \in \text{conv}(\{\pm \sqrt{p}v_j\}_{j \in [p]})$, and since $x$ was arbitrary $B_2^p \subseteq \text{conv}(\{\pm \sqrt{p}v_j\}_{j \in [p]}).

Now, it suffices to show that with high probability the set $\{\eta_i X_i\}_{i \in [m]}$ contains all vectors from the set $\{\pm \sqrt{p}v_j\}_{j \in [p]}$. The probability that a specific vector is not in this set is $(1 - (2p)^m)^m$, hence by a union bound we obtain an upper bound $2p(1 - (2p)^m)^m \leq 2p \exp(-m/(2p))$, for $m \geq 2p \log(2\gamma^{-1}p)$ we have this probability is bounded by $\gamma$. Therefore by Theorem 2.2 we can conclude that with probability at least $1 - \gamma - \exp(-L^2/(8L/3 + 2))$ we have $\|\hat{\beta} - \beta^*\| \leq a(L + 1)(2p \log(2\gamma^{-1}p) + 1)/n$. 

---
Next, we will formalize a sufficient condition under which the design must contain a large $\ell_2$ ball.

**Theorem 2.6.** Let $\overline{X}_1, \ldots, \overline{X}_m$ be i.i.d. random points in $\mathbb{R}^p$, whose distribution is symmetric about $0$. If the distribution of $\overline{X}$ satisfies

$$
\rho := \sup_{v \in \mathbb{S}^{p-1}} \frac{\mathbb{P}(|\langle v, \overline{X} \rangle| \leq 2\xi)}{2} + \mathbb{P}(\|\overline{X}\| \geq Y) < \frac{1}{2},
$$

for some $\xi, Y > 0$, then

$$
\mathbb{P}(\xi \mathbb{B}_2^p \not\subset \text{conv}(\overline{X}_1, \ldots, \overline{X}_m)) \leq \left(1 + \frac{2Y}{\xi}\right)^p \left(\frac{1}{2} + \rho\right)^m.
$$

where $\mathbb{B}_2^p$ is the $\ell_2$ ball centered at $0$.

**Remark 2.7.** By the extended Markov’s inequality condition (2.7) is satisfied if for some monotonically increasing positive function $\phi$, assuming $\mathbb{E}\phi(\|\overline{X}\|) < \infty$, we have

$$
\rho \leq \sup_{v \in \mathbb{S}^{p-1}} \frac{\mathbb{P}(|\langle v, \overline{X} \rangle| \leq 2\xi)}{2} + \frac{\mathbb{E}\phi(\|\overline{X}\|)}{\phi(Y)}.
$$

Therefore the theorem statement continues to hold with

$$
\rho := \sup_{v \in \mathbb{S}^{p-1}} \frac{\mathbb{P}(|\langle v, \overline{X} \rangle| \leq 2\xi)}{2} + \frac{\mathbb{E}\phi(\|\overline{X}\|)}{\phi(Y)}.
$$

One simple instance that we will be using throughout the paper is when $\phi(x) = x$. Assuming that $\mathbb{E}\|\overline{X}\| < \infty$ and setting $Y = c\mathbb{E}\|\overline{X}\|$ in the definition of $\rho$ above we obtain that if

$$
\rho := \sup_{v \in \mathbb{S}^{p-1}} \frac{\mathbb{P}(|\langle v, \overline{X} \rangle| \leq 2\xi)}{2} + c^{-1} \leq \frac{1}{2}, \tag{2.7}
$$

then

$$
\mathbb{P}(\xi \mathbb{B}_2^p \not\subset \text{conv}(\overline{X}_1, \ldots, \overline{X}_m)) \leq \left(1 + \frac{2c\mathbb{E}\|\overline{X}\|}{\xi}\right)^p \left(\frac{1}{2} + \rho\right)^m. \tag{2.8}
$$

The proof of Theorem 2.6 is elementary and is based on a covering argument. Furthermore, the proof can be extended to any $\ell_q, q \geq 1$ norm ball. We do not pursue this here in order to simplify the presentation, and since it is not very useful for our purposes (which are to derive bounds on $\|\hat{\beta} - \beta^*\|$). In passing we would also like to mention a recent reference (Guédon et al., 2022) which studies the geometry of the absolute convex hull of $n$ i.i.d. observations $X_1, \ldots, X_n$, i.e., they study the geometry of $\text{conv}\{\pm X_1, \ldots, \pm X_n\}$, and show that this set contains a deterministic set associated with the law of the random vectors $X_i$. This is result is related to but is of different nature compared to Theorem 2.6.

**Proof of Theorem 2.6.** Let $w \in \xi \mathbb{B}_2^p$ be an arbitrary vector such that $\|w\| \leq \xi$. We are interested when is the point $-w$ in $\text{conv}(\overline{X}_1, \ldots, \overline{X}_m)$, which is equivalent to $0$ belonging to the convex hull $\text{conv}(\overline{X}_1 + w, \ldots, \overline{X}_m + w)$. Note that this happens when there does not exist a $v \neq 0$ such that for all $i \in [n]$

$$
\langle v, \overline{X}_i + w \rangle \geq 0 \quad \Rightarrow \quad \langle v, \overline{X}_i \rangle \geq -\langle v, w \rangle \geq -\|v\| \|w\| \geq -\xi \|v\|.
$$
So if such a \( v \in \mathbb{S}^{p-1} \) satisfying \( \langle v, X_i \rangle \geq -\xi \) for all \( i \) does not exist, then we are guaranteed to have \(-w \in \text{conv}(\overline{X}_1, \ldots, \overline{X}_m)\). Since \( w \) is arbitrary it will follow that \( \xi \mathbb{B}_2^p \subset \text{conv}(\overline{X}_1, \ldots, \overline{X}_m) \).

Now consider the probability

\[
P(\exists v \in \mathbb{S}^{p-1}: \inf_{i \in [n]} \langle v, \overline{X}_i \rangle \geq -\xi).
\]

Construct a minimum \( \delta \)-cover \( \mathcal{N}_\delta \) on \( \mathbb{S}^{p-1} \) such that for each \( v \in \mathbb{S}^{p-1} \), there exists \( v' \in \mathcal{N}_\delta \) such that \( |v - v'| \leq \delta \), and \( \mathcal{N}_\delta \) contains as few points as possible.

If \( \exists v \in \mathbb{S}^{p-1}: -\xi \leq \langle v, \overline{X}_i \rangle \), then for the closest-to-\( v \) point \( v' \) in the \( \delta \)-cover set \( \mathcal{N}_\delta \) we have

\[
\langle v, \overline{X}_i \rangle = \langle v - v', \overline{X}_i \rangle + \langle v', \overline{X}_i \rangle \leq \langle v', \overline{X}_i \rangle + \delta \| \overline{X}_i \|.
\]

Hence it follows that

\[
P(\exists v \in \mathbb{S}^{p-1}: -\xi \leq \langle v, \overline{X}_i \rangle, \forall i) \leq \mathbb{P}(\exists v' \in \mathcal{N}_\delta : \langle v', \overline{X}_i \rangle \geq -\xi - \delta \| \overline{X}_i \|, \forall i)
\leq |\mathcal{N}_\delta|(\sup_{v \in \mathbb{S}^{p-1}} \mathbb{P}(\langle v, \overline{X} \rangle \geq -\xi - \delta) + \mathbb{P}(\| \overline{X} \| \geq Y))^m,
\]

for any \( Y > 0 \). Set \( \delta = \xi / Y \), to obtain

\[
P(\exists v \in \mathbb{S}^{p-1}: -\xi \leq \langle v, \overline{X}_i \rangle, \forall i) \leq (1 + 2Y/\xi)^p (\sup_{v \in \mathbb{S}^{p-1}} \mathbb{P}(\langle v, \overline{X} \rangle \geq -2\xi) + \mathbb{P}(\| \overline{X} \| \geq Y))^m,
\]

where we used that by a standard volumetric argument we have \( |\mathcal{N}_\delta| \leq (1 + 2/\delta)^p \). Now we observe that by sign symmetry for any \( v \): \( \mathbb{P}(\langle v, \overline{X} \rangle \geq -2\xi) = 1/2 + \mathbb{P}(|\langle v, \overline{X} \rangle| \leq 2\xi)/2 \). Hence since \( \rho = \sup_{v \in \mathbb{S}^{p-1}} \mathbb{P}(|\langle v, \overline{X} \rangle| \leq 2\xi)/2 + \mathbb{P}(\| \overline{X} \| \geq Y) < 1/2 \) we conclude:

\[
P(\exists v \in \mathbb{S}^{p-1}: -\xi \leq \langle v, \overline{X}_i \rangle, \forall i) \leq \left(1 + \frac{2c\mathbb{E}\| \overline{X} \|}{\xi}\right)^p \left(1 + \rho\right)^m,
\]

which is what we wanted to show. \( \square \)

We will now give a simple Corollary to Theorem 2.6 which is easy to use, as it only relies on certain moment calculations.

**Corollary 2.8.** Suppose \( \mathbb{E}\| \overline{X} \| < \infty \). For a fixed \( \theta \in [0, 1) \) and \( \alpha > 0, q > 1 \) define

\[
\rho := c^{-1} + \frac{1}{2} \left(1 - \inf_{v \in \mathbb{S}^{p-1}} \frac{(1 - \theta)\mathbb{E}|\langle v, \overline{X} \rangle|^\alpha}{(\mathbb{E}|\langle v, \overline{X} \rangle|^q)^{1/q}}\right)^{q-1}, \quad \text{and} \quad \xi := (\theta \inf_{v \in \mathbb{S}^{p-1}} \mathbb{E}|\langle v, \overline{X} \rangle|^\alpha)^{1/\alpha}/2.
\]

If \( \rho < 1/2 \), then (2.8) continues to hold with this choice of \( \rho \) and \( \xi \).

**Remark 2.9.** In what follows, we will mostly use Corollary 2.8 over Theorem 2.6, and we will be setting \( \alpha = 1 \) or \( 2 \) and \( q = 2 \).

**Proof of Corollary 2.8.** To prove the corollary we note that

\[
P(|\langle v, \overline{X} \rangle| \leq 2\xi) = \mathbb{P}(|\langle v, \overline{X} \rangle|^\alpha \leq (2\xi)^\alpha).
\]
Non-asymptotic bounds for the $l_\infty$ estimator

for any $\alpha > 0$. By the generalized Paley-Zygmund’s inequality (see equation (12) Petrov, 2007, where we instantiate it with $r = 1, s = q$) we have that for any $q > 1$

$$\mathbb{P}(\|\langle v, \vec{X} \rangle\|^\alpha \leq \theta \mathbb{E}(\langle v, \vec{X} \rangle)^\alpha) \leq 1 - \frac{(1 - \theta)\mathbb{E}(\langle v, \vec{X} \rangle)^\alpha}{\mathbb{E}(\langle v, \vec{X} \rangle)^q}.$$

It follows that when we set $\xi = (\theta \inf_{v \in S^{p-1}} \mathbb{E}[v^T \vec{X}]^q)^{1/\alpha}/2$,

$$\rho \leq \mathbb{P}(\|\vec{X}\| \geq Y) + \frac{1}{2} \left(1 - \inf_{v \in S^{p-1}} \frac{(1 - \theta)\mathbb{E}(\langle v, \vec{X} \rangle)^\alpha}{\mathbb{E}(\langle v, \vec{X} \rangle)^q}\right),$$

where $\rho$ is as defined in Theorem 2.6. This completes the proof after an application of Markov’s inequality with $Y = c\mathbb{E}\|\vec{X}\|$ as in the remark after Theorem 2.6.

We will proceed by giving multiple examples applying Theorem 2.6 and Corollary 2.8. We will start with a narrow set of examples which consider popular distributions, and move towards more abstract conditions on the design. We hope to convince the reader that there is a surprising variety of designs which satisfy the condition imposed by Theorem 2.2. Below we present only the final results of the application of Theorem 2.6 and Corollary 2.8 to the different designs that we consider, and defer the explicit constant calculations to Appendix A.

**Example 2.10.** The first application of the above result with $\alpha = 1$ and $q = 2$ is for Gaussian design. Suppose $X_i \sim N(0, \Sigma)$, where $\Sigma$ has smallest eigenvalue $\lambda_{\min} > 0$. It follows that $\vec{X} = \eta_iX_i \sim N(0, \Sigma)$. It can be argued using Theorem 2.2 that with probability at least $1 - \gamma - \exp(-L^2/(8L/3 + 2))$

$$\|\hat{\beta} - \beta^*\| \leq \frac{a(L + 1)(Cp \log(1 + C')/\sqrt{\lambda_{\min}} + C \log \gamma^{-1} + 1)}{\xi n},$$

where $\xi = \sqrt{\lambda_{\min}}/(8\pi)$ and $C$ and $C'$ are absolute constants. For more details see Appendix A. We would like to stress the fact that this bound is nearly optimal when $\Sigma = 1$ as we show in Theorem B.9 in the supplement. There we argue that in the isotropic case, with constant probability we have $\|\hat{\beta} - \beta^*\| \geq a\rho/(n(\log n)^{3/2})$. As we discuss later, there exists a different (computationally expensive) estimator which achieves a better dimension dependence in the Gaussian case (for sufficiently large $p$, e.g., $p = n^\alpha$), which implies that the Chebyshev estimator is sub-optimal.

**Example 2.11.** Our next application includes applying Corollary 2.8 with $\alpha = 1$ and $q = 2$ to Rademacher design. Let $X_{ij}$ be i.i.d. Rademacher random variables. In this example, the first variable can also optionally be an intercept. In any case, it follows that $\vec{X} = \eta_iX_i$ are Rademacher vectors. It can be shown with the help of Theorem 2.2 that:

$$\|\hat{\beta} - \beta^*\| \leq \frac{a(L + 1)(Cp \log(1 + C')/\sqrt{\eta} + \log \gamma^{-1} + 1)}{\xi n},$$

with probability at least $1 - \gamma - \exp(-L^2/(8L/3 + 2))$, where $C, C', \xi$ are absolute constants. For the precise constants see Appendix A.
Example 2.12. Let $X_i$ have a uniform distribution on the unit sphere. Then $\tilde{X}_i \stackrel{d}{=} X_i$. Let $g$ be a standard Gaussian random vector, and observe that $\tilde{X}_i \stackrel{d}{=} g/\|g\|$. Using Theorem 2.2 one can show that

$$\|\tilde{\beta} - \beta^*\| \leq \frac{a(L + 1)(C(p^{3/2} \log(1 + 2\xi^{-1}C\sqrt{p}) + \log \gamma^{-1}) + 1)}{\xi n},$$

with probability at least $1 - \gamma - \exp(-L^2/(8L/3 + 2))$, where $C, \xi$ are absolute constants. For more details see Appendix A.

Example 2.13. In this example we analyze a centered elliptical distribution $X$. This generalizes two of our previous examples where we considered Gaussian and uniform on the unit sphere distributions. By a stochastic representation theorem for centered elliptical distributions (see Proposition 4.1.2 of Tong, 2012, e.g.) we know that one can generate a centered elliptical random variable as $X \stackrel{d}{=} RAU$, where $R \geq 0$ is a non-negative random variable independent of $U, U \stackrel{d}{=} g/\|g\|$ is distributed uniformly over the unit sphere $S^{p-1}$, and $A \in \mathbb{R}^{p \times p}$ is a constant matrix. Suppose $\Sigma = AA^T$ has smallest eigenvalue $\lambda_{\min}$ bounded away from 0 and largest eigenvalue $\lambda_{\max}$ being bounded. We have $\tilde{X} \stackrel{d}{=} \tilde{X}$. In what follows we also assume $\mathbb{E}R > 0$ and $\mathbb{E}R^2 < \infty$.

By Theorem 2.2 it can be shown that we have that with probability at least $1 - \gamma - \exp(-L^2/(8L/3 + 2))$:

$$\|\tilde{\beta} - \beta^*\| \leq \frac{a(L + 1)\left(\frac{8\mathbb{E}R^2}{(\mathbb{E}R)^2} + p \log \left(1 + \frac{C'\sqrt{\mathbb{E}R^2}^{1/2}\sqrt{\lambda_{\min}}}{(\mathbb{E}R)^{1/2}\sqrt{\lambda_{\min}}} + \log \gamma^{-1} + 1\right)\right)}{\xi n},$$

where $\xi = \mathbb{E}R \min^{1/2} \sqrt{\frac{p}{8}}/(4\sqrt{p})$. For more details see Appendix A.

Example 2.14. We now give a general example which only assumes that $\inf_{v \in S^{p-1}} \mathbb{E}v^TXX^Tv = \lambda_{\min} > 0$ and $\sup_{v \in S^{p-1}} \mathbb{E}(v^TX)^4 \leq C < \infty$. The latter happens in the case when the variables $X$ are sub-Gaussian e.g. (in other words we assume that $\mathbb{E}\exp(-t^2(\mathbb{E}X)^2) \leq 2$ for some $t \in \mathbb{R}^+$ for any $v \in S^{p-1}$ (see also Definition 3.3 in Section 3 for a formal definition)). Indeed, this is so by Lemma 5.5 of Vershynin (2012).

Clearly, under these assumptions $\inf_{v \in S^{p-1}} \mathbb{E}v^TXX^Tv = \lambda_{\min} > 0$ and $\sup_{v \in S^{p-1}} \mathbb{E}(v^TX)^4 \leq C < \infty$. By Theorem 2.2 one can argue that

$$\|\tilde{\beta} - \beta^*\| \leq \frac{a(L + 1)\left(C'\min^{5/2} + p \log \left(1 + \frac{C''\min^{5/2}p^{1/2}}{\log \gamma^{-1} + 1}\right)\right)}{\xi n},$$

with probability at least $1 - \gamma - \exp(-L^2/(8L/3 + 2))$, where $C', C''$ are constants that depend on $C$ and $\xi = \min^{1/2}/(2\sqrt{2})$.

One can of course assume even less assumptions in which case the bounds will worsen a bit. For instance, instead of assuming $\sup_{v \in S^{p-1}} \mathbb{E}(v^TX)^4 \leq C < \infty$ one can simply assume that the coordinates $X^{(j)}$ for $j \in [p]$ have bounded 4-th moments by some constant $C_0$. Finally, if one is bothered by 4-th moment assumptions, this too can be relaxed. One needs to use Corollary 2.8 with $\alpha = 2$ and $q = 1 + \epsilon/2$ (so that $q\alpha = 2 + \epsilon$) for some $\epsilon > 0$. In this way, it suffices to assume that $\sup_{v \in S^{p-1}} \mathbb{E}|v^TX|^{2+\epsilon} < \infty$.
which is even weaker than a 4-th moment assumption. For more details see Appendix A.

**Example 2.15.** In our final example we will not impose moment assumptions on the variables (except \( \mathbb{E} \phi(||X||) < \infty \) for some increasing and positive \( \phi \)), but we will impose assumptions on the densities of the variables \( v^T X \) for any \( v \in \mathbb{S}^{p-1} \). To this end we will be applying Theorem 2.6 directly rather than its corollary. Before we do that we state a lemma.

**Lemma 2.16.** Suppose that for any \( v \in \mathbb{S}^{p-1} \) the variables \( v^T X \) have density with respect to the Lebesgue measure (denoted by \( f_k \)), and in addition for some \( q > 1 \) we have \( \sup_{v \in \mathbb{S}^{p-1}} (\int f_k^q(t)dt)^{1/q} \leq C < \infty \). Then for

\[
\xi := \frac{1}{2} \left[ \frac{q}{\pi(q-1)(c_0^{-1}eC)^{2q}} \right]^{1/2},
\]

we have \( \sup_{v \in \mathbb{S}^{p-1}} \mathbb{P}(||v^T X|| \leq 2\xi) \leq c_0 \).

Under the assumptions of Lemma 2.16 with \( c_0 \), say \( c_0 = 1/4 \), we can directly apply Theorem 2.6 with \( \rho = \mathbb{E} \phi(||X||)/\phi(Y) + 1/8 < 1/2 \) for \( Y > \phi^{-1}(8/3\mathbb{E} \phi(||X||)) \) (notice here that \( \sup_{v \in \mathbb{S}^{p-1}} \mathbb{P}(||v^T X|| \leq 2\xi) = \sup_{v \in \mathbb{S}^{p-1}} \mathbb{P}(||v^T X|| \leq 2\xi) \)). Set \( Y = \phi^{-1}(8\mathbb{E} \phi(||X||)) \) so that \( \rho = 1/4 \). Assuming that \( \mathbb{E} \phi(||X||) = \mathbb{E} \phi(||X||) < \infty \) we have that

\[
\mathbb{P}(\xi \mathbb{B}_2 \not\subset \text{conv}(\mathbb{X}_1, \ldots, \mathbb{X}_m)) \leq \left( 1 + \frac{2Y}{\xi} \right)^m \left( 1 - \frac{3}{4} \right)^m,
\]

for \( \xi \) as in Lemma 2.16. Hence for \( m \geq 4/3(p \log(1 + 2Y/\xi) + \log \gamma^{-1}) \) we have

\[
\mathbb{P}(\xi \mathbb{B}_2 \not\subset \text{conv}(\mathbb{X}_1, \ldots, \mathbb{X}_m)) \leq \gamma.
\]

Using Theorem 2.2 we can conclude that

\[
\|\tilde{\beta} - \beta^*\| \leq \frac{a(L + 1)(\frac{4}{3}(p \log(1 + 2Y/\xi) + \log \gamma^{-1}) + 1)}{\xi n},
\]

with probability \( 1 - \gamma - \exp(-L^2/(8L/3 + 2)) \).

We now move on to provide a realistic instance when the assumptions above can be met. Suppose that the covariates \( X = \Sigma^{1/2} Z \), where \( Z \) is a vector whose entries are independent variables with densities in \( L_2 := L_2(\mathbb{R}) \), such that \( \max_{j \in [p]} (\int f_j^{1/2}(t)dt)^{1/2} < U \) for some fixed \( U < \infty \), and \( \Sigma^{1/2} \) is a positive semi-definite symmetric matrix whose minimum and maximum eigenvalues \( \lambda_{\text{min}} \) and \( \lambda_{\text{max}} \) are bounded away from 0 and \( \infty \). Additionally, assume that \( \mathbb{E} \phi(\lambda_{\text{max}}||Z||) \leq C(p) \) for some constant \( C(p) \) which potentially depends on the dimension \( p \).

We will now argue that the densities \( f_k \) of the variables \( v^T X \) for a unit vector \( v \) exist and are in \( L_2 \). To this end let \( w := v^T \Sigma^{1/2} \) (for a unit vector \( v \)) and let \( \ell \) be the index such that \( ||w|| = ||w||_{\infty} \geq \sqrt{\mathbb{E} w^T w} \geq \lambda_{\text{min}}/\sqrt{p} \). Next, we will control the following integral, involving the characteristic function of the variable \( w^T Z = v^T X \):

\[
\frac{1}{2\pi} \int ||\mathbb{E} e^{itw^T Z}||^2 dt = \frac{1}{2\pi} \int \prod_{j \in [p]} ||\mathbb{E} e^{itw(j)} Z(j)||^2 dt \leq \frac{1}{2\pi} \int ||\mathbb{E} e^{itw(j)} Z(j)||^2 dt
\]
where we applied Plancherel’s theorem in the next to last identity. By Lemma 1.1 of Fournier and Printems (2010), we know that the above implies that the variable \( v^\top X \) has density with respect to the Lebesgue measure. Denote, as in Lemma 2.16, that density with \( f_x \). We will now argue that \( f_x \) is in \( L_2 \) and satisfies \( \int f_x^2(t)dt < U^2 \sqrt{p}/\lambda_{\min} \). By another application of Plancherel’s theorem we have

\[
\int f_x^2(t)dt = \frac{1}{2\pi} \int |Ee^{it\omega}Z(r)|^2dy = \frac{1}{\|w\|_{\infty}} \int f_z^2(y)dy < \frac{U^2 \sqrt{p}}{\lambda_{\min}}.
\]

It is also easy to verify that \( \mathbb{E}\phi(\|X\|) = \mathbb{E}\phi(\|X\|) \leq \mathbb{E}\phi(\lambda_{\max}\|Z\|) \leq C(p) \).

We end this example with a concrete instance of variables which do not possess moments, yet the above discussion is applicable. Suppose \( \Sigma^{1/2} = I \), and \( X^{(j)} = Z^{(j)} \sim \text{Cauchy}(0,1) \) for all \( j \in [P] \). Clearly \( Z^{(j)} \) do not even possess a first moment, yet it is easy to see that their densities \( f_{Z^{(j)}}(t) = 1/(\pi(1+t^2)) \) belong to \( L_2 \). Coupled with the fact that \( \mathbb{E}\sqrt{\|Z\|} \leq \mathbb{E}\sum_{j \in [P]} |Z^{(j)}|^{1/2} = \sqrt{2}p < \infty \) shows that our results can be applied even to Cauchy random variables (with \( \phi(x) = \sqrt{x} \)). In the last inequalities we used \( \sqrt{\sum_{j \in [P]} x_j^2} \leq \sum_{j \in [P]} |x_j|^{1/2} \), and the fact that \( \mathbb{E}\sqrt{|Z^{(j)}|} = \sqrt{2} \).

We will conclude this section with a result for the known case, which shows that if one fits least squares (2.2), subject to the constraint (2.3), one attains “the best of both worlds” type of behavior, which will at worst have a standard risk of the least squares. We have the following result:

**Proposition 2.17.** Suppose \( \varepsilon_i \sim U([-c, a]) \) where \( a > 0 \) is a known constant, and \( X_i^{(j)} \) has bounded 4-th moment for each coordinate \( j \). Denote with \( \Sigma := \mathbb{E}XX^\top \). If

\[
p^6 \|\Sigma^{-1}\|_{\text{op}}/n = o(1), \tag{2.9}
\]

for \( \tilde{\beta} \) obtained via (2.2) and (2.3), with probability at least \( 1 - C^{-2} - o(1) \) we have

\[
\|\tilde{\beta} - \beta^*\| \lesssim C \sqrt{\frac{p}{n}} \|\Sigma^{-1}\|_{\text{op}}. \tag{2.10}
\]

In addition, if \( \sup_{x \in S_P} \mathbb{E}|v^\top x|^{2+\alpha} < \infty \) for some \( \alpha \in (0,2] \), and instead of (2.9) we have \( n > C_\alpha p \) for a sufficiently large constant \( C_\alpha \) depending only on \( \alpha \), with probability at least \( 1 - \exp(-p) - C^{-2} \) (2.10) continues to hold.

**Remark 2.18.** An unsatisfactory artifact of the first half of Proposition 2.17 is that it requires \( p^6 \|\Sigma^{-1}\|_{\text{op}}/n = o(1) \). This is because of the proof strategy, which aims to lower bound \( \lambda_{\min}(n^{-1} \sum_{i \in [n]} X_iX_i^\top) \), under the minimal constraint that \( X_i^{(j)} \) has bounded 4-th moment. This is known as the “hard edge” problem in random matrix theory (Mendelson, 2010, Rudelson and Vershynin, 2010, Vershynin, 2011, see, e.g.), and to the best of our knowledge there are currently no reasonable bounds available under such general conditions. One example of a general condition that can be used to lower bound the eigenvalue is \( \sup_{x \in S_P} \mathbb{E}|v^\top x|^{2+\alpha} < \infty \) as observed by Koltchinskii and Mendelson (2015), Srivastava and Vershynin (2013), Yaskov (2014). We are using their result in the second part of this proposition to obtain a much better dependence on \( n \) and \( p \).
2.2. Minimax lower bound

To complement the upper bounds derived in the previous section, we derive a minimax lower bound of the estimation error in a uniform noise setting. The minimax lower bound is derived based on Assouad’s Lemma (Yu, 1997). We add a small extension to this standard method in order to also arrive at bounds in probability and not only in expectation. We do so since throughout the paper we focus on probability bounds, hence this is the more relevant object to us.

**Theorem 2.19.** Suppose \( \varepsilon_i \sim U([-a, a]) \) where \( a > 0 \) is a constant, and \( X_i \) is any random design independent of the errors. Let

\[ R := \frac{a^2 p}{16 \inf_{R \in \mathcal{O}} \max_{j \in [p]} \mathbb{E}(X) \sum_{i \in [n]} |(X_i^T R)_j|^2}, \]

where \( \mathcal{O} \) is the set of all orthogonal matrices. Then the following inequalities hold:

\[ \inf_{\hat{\beta}} \sup_{\beta^*} \mathbb{P}(\|\hat{\beta} - \beta^*\|^2 \geq R), \]

and in addition

\[ \inf_{\hat{\beta}} \sup_{\beta^*} \mathbb{P}(\|\hat{\beta} - \beta^*\| \geq \sqrt{R/2}) \geq \frac{1}{2^8}. \]

We will now look into the specific design we considered in Example 2.5.

**Corollary 2.20.** Take the random design \( X_i \sim U([v_1, \ldots, v_p]) \), where \( v_i, i \in [p] \) denote vectors from any orthonormal basis. Then \( R \) from (2.11) is

\[ R = \frac{a^2 p^2}{16 n^2}. \]

**Proof.** Take \( R = [v_1; \ldots; v_p] \) so as to rotate the basis to a standard basis, and observe that \( \mathbb{E}(|(X_i^T R)_j|) = \sqrt{p}/p = 1/\sqrt{p} \). From here the claim follows.

The above example, coupled with the results of Example 2.5 and Theorem 2.2 illustrate that there exist designs under which the Chebyshev estimator is (nearly) optimal. In the most natural case of standard Gaussian design however, Theorem 2.19 yields a lower bound of the order of \( a p \log p / n \), while the Chebyshev estimator has a guarantee of the form \( a p \log p / n \) by Example 2.10. In Appendix B.1 we argue that the lower bound is sharp in this case. There exists an estimator (although non-computationally tractable one) whose rate of estimation is upper bounded by \( a \sqrt{p}/n \) in the known case under standard Gaussian design when \( p^3 (\log p)^4 \ll n \). On the other hand, Theorem B.9 in the supplement argues that in the Gaussian design case with isotropic covariance, with at least constant probability, the Chebyshev estimator makes error \( \|\hat{\beta} - \beta^*\| \geq a p / (n \log n)^{3/2} \). Moreover, both results extend to the case where the design \( X \) consists of i.i.d. centered sub-Gaussian variables with unit variance, which shows that the Chebyshev estimator is sub-optimal in such situations. This fact also shows that, a general analysis of estimators taking values in the set (2.3) is going to produce sub-optimal results in terms of the dimension dependence in the (sub-)Gaussian case (since the Chebyshev estimator also takes values in the set...
(2.3)). One may wonder what prevents the Chebyshev estimator from being optimal. Our intuition is that it overfits. As the proof of Theorem B.9 shows, the value of $\hat{a}$ is much smaller than the true value of $a$ which is indicative of overfitting. Another related reason in addition to overfitting could be that it is not exploiting the knowledge of $a$ properly, and perhaps one can show that the best risk equivariant estimator (which is the centroid of (2.3)) may work optimally, although this appears difficult to prove. One way of proving such a result could be to follow calculations of Ibragimov and Has’ Minskii (2013) which provide a general theory for Bayesian estimators (and the best risk equivariant estimator is generalized Bayesian with an improper prior), specifically their Theorem 5.2. That result however, does not track the dimension dependence and we failed to prove an optimal result for the best risk equivariant estimator or for other Bayesian estimators using their method. There are of course many other examples of high-dimensional settings where the MLE fails to be minimax optimal. One such recent example is given by Neykov (2022) where it is argued that in general the MLE is suboptimal for the Gaussian sequence model with convex constraint, but there exist differnt minimax optimal estimators. The astute reader would notice that in almost all of our upper bounds examples we assumed the quantity $\lambda_{\min}(\mathbb{E}XX^T)$ is bounded from below. This quantity does not explicitly appear in our lower bound above. Below we will show a separate lower bound based on the proof of Theorem 2.19, which illustrates that the quantity $\lambda_{\min}(\mathbb{E}XX^T)$ cannot be too small if one wants to attain reasonable bounds on the estimation error $||\hat{\beta} - \beta^*||$.

**Proposition 2.21.** Assume the same setting as in Theorem 2.19. Let

$$R := \frac{a^2}{16(\inf_{v \in \mathbb{P}^{-1}} \mathbb{E}|X^Tv|)^2} \geq \frac{a^2}{16\lambda_{\min}(\mathbb{E}XX^T)}.$$

Then the following inequalities hold:

$$\inf_{\hat{\beta}} \sup_{\beta^* \in \mathbb{R}^p} \mathbb{E}_{\beta^*} \|\hat{\beta} - \beta^*\|^2 \geq R,$$

and in addition

$$\inf_{\hat{\beta}} \sup_{\beta^* \in \mathbb{R}^p} \mathbb{P}_{\beta^*}(\|\hat{\beta} - \beta^*\| \geq \sqrt{R/2}) \geq \frac{1}{2^5 p^2}.$$

**Remark 2.22.** Our result above is not entirely satisfactory, since it does not capture any dimension dependence. Furthermore, in some examples such as Example 2.14 the quantity $\lambda_{\min}(\mathbb{E}XX^T)$ appears to the power of $5/2$ in the denominator which is not matched by the lower bound above. The latter can be remedied by imposing a lower bound on $\inf_{v \in \mathbb{P}^{-1}} \mathbb{E}|X^Tv|$ in place of $\lambda_{\min}(\mathbb{E}XX^T)$ in Example 2.14. We do not pursue that further here however.

### 3. The $\ell_1$ penalized $\ell_\infty$ estimator (aka Chebyshev’s LASSO)

Another problem of interest is whether we can extend the $\ell_\infty$ estimator (2.4) to high-dimensional situations where $\beta^* \in \mathbb{R}^p$ is $s$-sparse. Consider the program

$$\min a + \lambda \|eta\|_1 \quad \text{subject to} \quad |Y_i - X_i^T\beta| \leq a, \forall i \in [n] \quad (3.1)$$

Luckily one need not write new software to solve problem (3.1) as it is a linear program. A similar but dual version of program (3.1) has recently been considered by (Du et al., 2019), where it was argued
that the $\ell_\infty$ loss function is the most natural loss for a certain problem in fuselage assembly. Du et al. (2019) also provide some theoretical guarantees on their version of the program, however they failed to recognize that this program will converge at much faster rates than the usual LASSO in the case of uniform errors.

The following Theorem 3.4 shows that under some conditions on the design matrix $X$ and the growth rate of the sparsity $s$ and the ambient dimension $p$ with respect to the sample size $n$, the estimator obtained via (3.1) achieves a rate faster than the LASSO estimation rate $s\sqrt{\log p/n}$ (for the $\ell_1$ norm) (Wainwright, 2019, see Chapter 7). Before presenting the theorem we need to introduce the Restricted Eigenvalue (RE) condition (Bickel et al., 2009), which is the least restrictive eigenvalue condition imposed on the population covariance matrix in order to provide good convergence guarantees for $\ell_1$-based methods. First, let us define a set $C(S, \gamma)$ which is relevant to the RE condition.

**Definition 3.1.** For a given subset $S \subset [p]$ and a constant $\gamma \geq 1$, the set $C(S, \gamma)$ is defined as

$$C(S, \gamma) := \{ v \in \mathbb{R}^p : \|v_S\|_1 \leq \gamma\|v_S\|_1 \}.$$ 

Next, the Restricted Eigenvalue condition of order $s$ with parameters $\kappa, \gamma$ is denoted as $RE(\kappa, \gamma, s)$ and defined as following.

**Definition 3.2.** For a constant $\kappa > 0$, we say that a symmetric matrix $A$ satisfies the condition $RE(\kappa, \gamma, s)$ if

$$v^\top Av \geq \kappa^2\|v\|^2$$

holds uniformly for all sets $S$ with cardinality $s$.

Before we state the main result of this section, we will also formally introduce sub-Gaussian and isotropic random variables.

**Definition 3.3 (Sub-Gaussian and Isotropic Random Vectors).** A random vector $\zeta \in \mathbb{R}^p$ is called isotropic if $E\zeta\zeta^\top = I$. A random vector $\zeta \in \mathbb{R}^p$ is called $\gamma$-sub-Gaussian if for any $v \in \mathbb{S}^{p-1}$

$$\inf \{ t : E\exp(t^{-2}(v^\top \zeta)^2) \leq 2 \} \leq \gamma.$$ 

**Theorem 3.4.** Suppose $X = \Sigma^{1/2}\zeta$ where $\zeta \in \mathbb{R}^p$ is an isotropic $\gamma$ sub-Gaussian vector. Let the predictors $X_i \sim \mathcal{L}(X)$ be i.i.d., where $\mathcal{L}(X)$ denotes the law of the random variable $X$. Additionally we assume that the Gram matrix $\Sigma = EXX^\top$ satisfies the $RE(\kappa, 2, s)$ condition for a constant $\kappa > 0$, $\|\Sigma\|_{\text{op}}$ is bounded from above and $\Sigma_{jj} = 1$ for all $j \in [p]$. If $s \leq p/2$, and $s((\log(5ep/s) \vee \log p) \vee \log p(\log n)^2)) \leq p$ then for $\lambda = \frac{\kappa}{(4+\epsilon)\sqrt{3}\log n}$ for any small $\epsilon > 0$, we have

$$\|\hat{\beta} - \beta^*\|_{1, S} \lesssim_{\gamma, \|\Sigma\|_{\text{op}}, \kappa} \frac{s^{3/2}((\log(5ep/s) \vee \log p) \vee \log p(\log n)^2))}{n},$$

with probability converging to 1, where $\lesssim_{\gamma, \|\Sigma\|_{\text{op}}, \kappa}$ hides constants depending on $\gamma, \|\Sigma\|_{\text{op}}, \kappa$.

**Remark 3.5.** The above theorem shows that Chebyshev’s LASSO can be (much) more accurate than the regular LASSO under certain assumptions. Importantly, note that the optimal choice of $\lambda$ does not seem to depend on the parameter $a$, which will be affecting the LASSO tuning parameter (since the
variance of a uniform distribution is \( \frac{\sigma^2}{2} \). However, it does depend on the (potentially) unknown sparsity \( s \), and hence in practice some tuning will be required. This can be done with cross-validation, e.g. We do not provide a tight upper bound on the \( \ell_2 \) norm, but it should be clear that the \( \| \hat{\beta} - \beta^* \|_{\ell_2} \leq \| \hat{\beta} - \beta^* \|_{\ell_1} \) and hence the above bound is valid in terms of the \( \ell_2 \) norm too. In addition, we would like to mention that the factor \( \log n \) that appears in the upper bound on \( \| \hat{\beta} - \beta^* \|_{\ell_1} \) and in the definition of \( \lambda \) may be replaced by any slowly diverging sequence in \( n \). Finally we give some intuition on why we obtain \( s^{3/2} \) in the upper bound for \( \| \beta - \beta^* \|_{\ell_1} \). Recall that (dropping log factors) the rate of the Chebyshev estimator for Gaussian design is \( p/n \), and we obtain \( s^{3/2}/n \) bound for the \( \ell_1 \) norm in the high-dimensional setting (i.e. it is \( \sqrt{s} \) more than the bound \( s/n \)). This is similar to how in the Gaussian design case in regression with Gaussian errors the upper bound under \( \ell_2 \) loss is \( \sqrt{p/n} \) but the LASSO obtains a rate under \( \ell_1 \) loss equal to \( s/\sqrt{n} \) (dropping log factors), i.e., we multiply by \( \sqrt{s} \). Intuitively, this comes from the bound \( \| v \|_{\ell_1} \leq \sqrt{s} \| v \| \) for any \( v \) with \( \| v \|_0 \leq s \) where \( \| v \|_0 \) is the number of non-zero entries in \( v \).

4. Simulations

In this section we provide several brief numerical experiments in support of our findings.

4.1. Chebyshev estimator

We begin with the Chebyshev estimator. We use three designs to construct our experiments — standard Gaussian design, Rademacher design and uniform on the unit sphere design. We remind the reader that these three designs were considered as examples after Theorem 2.6, and we know from our theorems that for the first two designs \( \| \hat{\beta} - \beta^* \|_{\ell_2} \lesssim (p \log p)/n \) while for the last design we have \( \| \hat{\beta} - \beta^* \|_{\ell_2} \lesssim (p \sqrt{p \log p})/n \). We constructed datasets of multiple sizes, one for each pair \((n, p)\) where \( n \in \{30, 40, 50, 60, 70, 80, 90, 100, 110\} \) and \( p \in \{4, 8, 12, 16, 20\} \). Here we set \( \beta^* \) to have its first \( p/2 \) entries equal to 1 and the remaining entries equal to -1, while \( a = 2 \). For each dataset we computed the Chebyshev estimator 100 times and averaged \( \| \hat{\beta} - \beta^* \|_{\ell_2} \). We then plotted these results against \( p/n \) and \( (p \sqrt{p})/n \) since we believe the extraneous log \( p \) factors that we obtained are artifacts of the proof. Figure 1 illustrates our findings. We see a near perfect linear alignment. This empirical evidence suggests that our simple analysis is nearly tight for those designs. This is also corroborated by Theorem B.9 in the supplementary material.

4.2. Chebyshev’s LASSO

In order to illustrate the superiority of Chebyshev’s LASSO vs the regular LASSO, we constructed examples where the \( \beta^* \) vector is very sparse in comparison to the sample size. This is in order to make the requirement \( \sqrt{s \log p \log n} \ll \sqrt{p} \) hold at least approximately. We considered two possible sample sizes \( n = 600,800 \), two possible values of the sparsity of \( \beta^* \): \( s = 4,10 \) and the ambient dimension is \( p = 1000 + s \). Here \( \beta^* \) has its first \( s/2 \) entries equal to 1, the next \( s/2 \) entries equal to -1 and all remaining entries are 0. We also set \( a = 5 \) throughout the simulations. We tuned both Chebyshev’s LASSO and the regular LASSO. For the tuning of Chebyshev’s LASSO we considered six equispaced values in the range \([1.1 \log p/n, 2 \sqrt{\log p/n}]\), and after we run the analysis we pick the value \( \hat{\beta} \) which is closest to the true \( \beta^* \) in terms of the \( \ell_1 \) norm. Similarly, to tune the LASSO, we considered default \( \lambda \) values given by the \texttt{cv.glmnet} function of the \texttt{glmnet} package in \texttt{R} (which are calculated from the data and are around 100), and used the one that gives the closest \( \hat{\beta} \) to the true \( \beta^* \) in \( \ell_1 \) norm. In
Figure 1: From left to right: Gaussian design, Rademacher design, and uniform on the unit sphere design. We observe near perfect linear patterns which suggests that our analysis is nearly tight.

It is evident from the results of Table 1 that Chebyshev’s LASSO dominates the LASSO in all 4 settings considered.

5. Discussion

In this paper we presented some non-asymptotic bounds on the Chebyshev estimator in linear regression with uniform errors. In addition we demonstrated that under certain assumptions Chebyshev’s LASSO can strictly dominate the usual LASSO. As we remarked our approach is immediately extendible to symmetric bounded noise, and with a little more effort can be extended to asymmetric noise as well.
Table 1. Summary of simulation results

| n  | 4  | 10 |
|----|----|----|
| 600| 1.06| 3.4 |
| 800| 0.83| 2.4 |
|     |     |     |     |
| 600| 1.53| 3.7 |
| 800| 1.32| 3.11 |

(a) Chebyshev’s LASSO \( \| \hat{\beta} - \beta^* \|_1 \) averaged over 100 simulations.
(b) LASSO \( \| \hat{\beta} - \beta^* \|_1 \) averaged over 100 simulations.

There are a lot of interesting open questions. Unlike the asymptotic approach in Knight (2020), our analysis does not rest on the epi-convergence techniques; however, it is interesting whether such epi-convergence techniques could be directly turned into finite sample results.

Next, we discuss the lower bounds. As it stands, our Theorem 2.19 does not produce a matching lower bound to the bound in Theorem 2.2 under standard Gaussian design, e.g. On the other hand in Appendix B.1 we establish that the lower bound is sharp in the i.i.d. standard (sub-)Gaussian design case, at least when \( p \) is not too large compared to \( n \). Hence a question arises: is the Chebyshev estimator truly sub-optimal or the gap is this sub-optimality introduced by our imprecise analysis? This question is closed by Theorem B.9 which argues that the dimension dependence we obtain for the Chebyshev estimator with i.i.d. (sub-)Gaussian design is optimal (up to logarithmic factors). This illustrates the interesting phenomenon that the Chebyshev estimator (which is the MLE in the unknown \( a \) case and is an MLE in the known \( a \) case) is provably sub-optimal in terms of the dimension dependence. One explanation of this is that it overfits, and in addition it does not utilize the knowledge of the constant \( a \); whereas the optimal estimator estimator we develop in Theorem B.6 relies on \( a \) being known. An open question is whether one can improve the lower bound Theorem 2.19 to capture the unknown \( a \) case.

There are also a multitude of questions left in for the high-dimensional Chebyshev estimator. First, it is not clear whether the rate that Theorem 3.4 is optimal. In fact is likely suboptimal given the sub-optimality of the Chebyshev estimator in low dimensional situations. Second, deriving matching lower and upper bounds sounds like a challenging but interesting question for future research.

Finally, if one is interested in inference, a possible approach that works for some non-regular models was recently proposed by Wasserman, Ramdas and Balakrishnan (2020). Unfortunately, this approach has problems with models with uniform distributions (see the uniform distribution example before section 4 (Wasserman, Ramdas and Balakrishnan, 2020)), but there may exist smart ways of tweaking it to make it work in our setting. We defer this to future work.

6. Acknowledgements

The authors would like to thank Sivaraman Balakrishnan for inspiring discussions on the topic, and in particular the lower bounds and his advice on the presentation of this work. The second author is also indebted to Alexandre Tsybakov and Tony Cai for communicating to him their belief that the lower bound is tight, and one should try to improve the upper bound in the Gaussian case. Finally the authors would like to express their gratitude to the AE and two anonymous referees for their insightful suggestions which led to substantial improvements of the manuscript.
References

AKÇAY, H., HIJLMARSSON, H. and LJUNG, L. (1996). On the choice of norms in system identification. *IEEE Trans. Automat. Contr.* **41** 1367–1372.

ALECU, A., MUNTEANU, A., CORNELIS, J. P. and SCHELKENS, P. (2006). Wavelet-based scalable L-infinity-oriented compression. *IEEE Trans. Image Process.* **15** 2499–2512.

APPAS, G. and SMITH, C. (1973). On L 1 and Chebyshev estimation. *Math. Program.* **5** 73–87.

ARMSTRONG, R. D. and KUNG, D. S. (1980). A dual method for discrete Chebyshev curve fitting. *Math. Program.* **19** 186–199.

BECK, A. and EL DAR, Y. C. (2007). Regularization in regression with bounded noise: A Chebyshev center approach. *SIAM J. Matrix Anal. Appl.* **29** 606–625.

BERENGUER-RICO, V., JOHANSEN, S. and NIELSEN, B. (2019). Models where the Least Trimmed Squares and Least Median of Squares estimators are maximum likelihood. *Available at SSRN 3455870*.

BERTSCH, G. F., SABBET, B. and UUSNÄKKI, M. (2005). Fitting theories of nuclear binding energies. *Phys. Rev. C* **71** 054311.

BICKEL, P. J., RITOV, Y., TSYBAKOV, A. B. et al. (2009). Simultaneous analysis of Lasso and Dantzig selector. *Ann. Stat.* **37** 1705–1732.

BRENNER, M. J. (2002). Aeroservoelastic model uncertainty bound estimation from flight data. *J Guid Control Dyn* **25** 748–754.

CASTILLO, E., CASTILLO, C., HADI, A. S. and SARABIA, J. M. (2009). Combined regression models. *Comput. Stat. Stat. Decis. Analysis.* **24** 37–66.

CLINE, A. (1990). Rate of convergence of Lawa’s algorithm. *Math. Comput.* **26** 167–176.

DU, J., CAO, S., HUNT, J. H. and HUO, X. (2019). Optimal Shape Control via $L_\infty$ Loss for Composite Fuselage Assembly. *arXiv preprint arXiv:1911.03592*.

FOURNIER, N. and PRINTEMS, J. (2010). Absolute continuity for some one-dimensional processes. *Bernoulli* **16** 343–360.

GUÉDON, O., KRAHMER, F., KÜMMERLE, C., MENDELSON, S. and RAUHUT, H. (2022). On the geometry of polytopes generated by heavy-tailed random vectors. *Commun. Contemp. Math.* **24** 2150056.

HAAGERUP, U. et al. (1978). The best constants in the Khintchine inequality. *Stud. Math.* **70** 231-283.

HAND, M. and SPOSITO, V. (1980). Using the least squares estimator in Chebyshev estimation: Using the least squares estimator in Chebyshev estimation. *Commun. Stat. Simul. Comput.* **9** 43–49.

IBRAGIMOV, I. A. and HAS’ MINSKII, R. Z. (2013). Statistical estimation: asymptotic theory. *16*. Springer Science & Business Media.

JAMES, F. (1983a). Fitting tracks in wire chambers using the Chebyshev norm instead of least squares. *Nucl. Instrum. Methods Phys. Res.* **211** 145–152.

JAMES, F. (1983b). Probability, statistics, and associated computing techniques. In *Techniques and concepts of high-energy physics II* 189–231. Springer.

JASCHKE, S. R. (1997). Arbitrage bounds for the term structure of interest rates. *Finance Stoch.* **2** 29–40.

JASCHKE, S. and KUCHLER, U. (2001). Coherent risk measures and good-deal bounds. *Finance Stoch.* **5** 181–200.

JURECKOVÁ, J. and PIEČEK, J. (2009). Minimum risk equivariant estimator in linear regression model. *Stat. decis.* **27** 37–54.

KNIGHT, K. (2020). On the asymptotic distribution of the $L_\infty$ estimator in linear regression Technical Report, Mimeo, http://www. utstat. utoronto.ca/keith/home. html.

KOLTCHINKII, V. and MENDELSON, S. (2015). Bounding the smallest singular value of a random matrix without concentration. *Int. Math. Res* **2015** 12991–13008.

LAURENT, B. and MASSART, P. (2000). Adaptive estimation of a quadratic functional by model selection. *Ann. Stat.* **30** 1302–1338.

LAWSON, C. L. (1961). Contribution to the theory of linear least maximum approximation. *Ph. D. dissertation, Univ. Calif.*

MENDELSON, S. (2010). Empirical processes with a bounded $\psi$ 1 diameter. *GAFA* **20** 988–1027.
MILANESE, M. and BELForte, G. (1982). Estimation theory and uncertainty intervals evaluation in presence of unknown but bounded errors: Linear families of models and estimators. *IEEE Trans. Automat. Contr.* **27**, 408–414.

MOURTADA, J. (2022). Exact minimax risk for linear least squares, and the lower tail of sample covariance matrices. *Ann. Stat.* **50**, 2175–2178.

NEYKOV, M. (2022). On the minimax rate of the Gaussian sequence model under bounded convex constraints. *IEEE Trans. Inf. Theory*.

PETROV, V. V. (2007). On lower bounds for tail probabilities. *J Stat Plan Inference* **137**, 2703–2705.

QI, C. (2015). Theoretical uncertainties of the Duflo–Zuker shell-model mass formulae. *JPhysG* **42**, 045104.

RADEMACHER, L. A. (2007). Approximating the centroid is hard. In *Proceedings of the twenty-third annual symposium on Computational geometry* 302–305.

ROBBINS, H. and ZHANG, C.-H. (1986). Maximum likelihood estimation in regression with uniform errors. *Lecture Notes-Monograph Series* 365–385.

RUDelson, M. and VERSHYNIN, R. (2010). Non-asymptotic theory of random matrices: extreme singular values. In *Proceedings of the International Congress of Mathematicians 2010 (ICM 2010) (In 4 Volumes) Vol. I: Plenary Lectures and Ceremonies Vols. II–IV: Invited Lectures* 1576–1602. World Scientific.

SCHECTMAN, E. and SCHECTMAN, G. (1986). Estimating the parameters in regression with uniformly distributed errors. *JSCS* **26**, 269–281.

SIEKENTJAR, R. and HARTLEY, H. (1973). Two linear programming algorithms for unbiased estimation of linear models. *JASA* **68**, 639–641.

SKLAR, M. G. and ARMSTRONG, R. D. (1982). Least absolute value and Chebychev estimation utilizing least squares results. *Math. Program.* **24**, 346–352.

SRIVASTAVA, N. and VERSHYNIN, R. (2013). Covariance estimation for distributions with $2 + \varepsilon$ moments. *Ann. Probab.* **41**, 3081–3111.

TONG, Y. L. (2012). *The multivariate normal distribution*. Springer Science & Business Media.

TSE, D. N., DAHLEH, M. A. and TSITSIKLIS, J. N. (1993). Optimal asymptotic identification under bounded disturbances. *IEEE Trans. Automat. Contr.* **38**, 1176–1190.

VERSHYNIN, R. (2011). Spectral norm of products of random and deterministic matrices. *Probab. Theory Relat. Fields* **150**, 471–509.

VERSHEYNIN, R. (2012). Introduction to the non-asymptotic analysis of random matrices. In *Compressed Sensing: Theory and Applications* (Y. C. ELDar and G. KUTYNIOK, eds.). Cambridge University Press.

VERSHEYNIN, R. (2018). *High-dimensional probability: An introduction with applications in data science* **47**, Cambridge university press.

WAINWRIGHT, M. J. (2019). *High-dimensional statistics: A non-asymptotic viewpoint* **48**, Cambridge University Press.

WASSERMAN, L., RAMDAS, A. and BALAKRISHNAN, S. (2020). Universal inference. *PNAS* **117**, 16880–16890.

YASKOV, F. (2014). Lower bounds on the smallest eigenvalue of a sample covariance matrix. *ECP* **19**, 1–10.

YU, B. (1997). Assouad, fano, and le cam. In *Festschrift for Lucien Le Cam* 423–435. Springer.

Zhou, S. (2009). Restricted eigenvalue conditions on subgaussian random matrices. *arXiv preprint arXiv:0912.4045*.

ZOLGHADRI, A. and HENRY, D. (2004). Minimax statistical models for air pollution time series. Application to ozone time series data measured in Bordeaux. *Environ. Monit. Assess* **98**, 275–294.
Appendix A: Examples

Example A.1. The first application of Corollary 2.8 with $\alpha = 1$ and $q = 2$ is for Gaussian design. Suppose $X_i \sim N(0, \Sigma)$, where $\Sigma$ has smallest eigenvalue $\lambda_{\min} > 0$. It follows that $X_i = \eta_i X_i \sim N(0, \Sigma)$. Since, $v^T X \sim N(0, v^T \Sigma v)$ we have

$$\mathbb{E}[v^T \bar{X}] = \sqrt{2/\pi} \sqrt{v^T \Sigma v} \geq \sqrt{2\lambda_{\min}/\pi}, \quad \mathbb{E}(v^T \bar{X})^2 = v^T \Sigma v.$$  

Set $\theta = 1/2$ (so $\xi = \sqrt{\lambda_{\min}/8\pi}$) to obtain $\rho = c^{-1} + (1 - 1/(2\pi))/2$. Set $c = 8\pi$ to obtain $\rho = 1/2 - 1/(8\pi)$. We have

$$\mathbb{P}(\xi B^T_2 \notin \text{conv}(\bar{X}_1, \ldots, \bar{X}_m)) \leq (1 + 32\sqrt{2}\pi^{3/2}/\sqrt{\text{tr}(\Sigma)})^{2} \leq 1 + 32\sqrt{2}\pi^{3/2}/\sqrt{\lambda_{\min}} \exp(-m/(8\pi)).$$

where we used that by Jensen’s inequality $\mathbb{E}[\|\bar{X}\|] \leq \sqrt{\text{tr}(\Sigma)}$. Hence the design contains a sphere of constant radius with probability at least $1 - \gamma$, so long as $m > 8\pi p \log(1 + 32\sqrt{2}\pi^{3/2}/\sqrt{\lambda_{\min}} + 8\pi \log \gamma^{-1}).$

Applying Theorem 2.2 now gives that with probability at least $1 - \gamma - \exp(-L^2/(8L/3 + 2))$

$$\|\hat{\beta} - \beta^*\| \leq \frac{a(L + 1)(8\pi p \log(1 + 32\sqrt{2}\pi^{3/2}/\sqrt{\lambda_{\min}} + 8\pi \log \gamma^{-1} + 1)}{\xi n},$$

where recall that $\xi = \sqrt{\lambda_{\min}/8\pi}$.

Example A.2. Our next application includes applying Corollary 2.8 with $\alpha = 1$ and $q = 2$ to Rademacher design. Let $X_{ij}$ be i.i.d. Rademacher random variables. In this example, the first variable can also optionally be an intercept. In any case, it follows that $\bar{X}_i = \eta_i X_i$ are Rademacher vectors.

By Khintchine’s inequality (Haagerup et al., 1978) we now have that for any $v \in \mathbb{R}^{p-1}$, $1/\sqrt{2} \leq \mathbb{E}[(v, \bar{X}_i)^2]$. In addition, clearly $\mathbb{E}((v, \bar{X}_i))^2 = 1$. It follows that (plugging in $\theta = 1/2$ hence $\xi = 1/(4\sqrt{2})$)

$$\rho = c^{-1} + (1 - 1/8)/2 = c^{-1} + 7/16.$$  

and hence for $c > 32$ we have $\rho = 15/32 < 1/2$. We conclude that

$$\mathbb{P}(\xi B^T_2 \notin \text{conv}(\bar{X}_1, \ldots, \bar{X}_m)) \leq (1 + 256\sqrt{2}\sqrt{p})^{p} (1 - 1/32)^m \leq (1 + 256\sqrt{2}\sqrt{p})^{p} \exp(-m/32).$$

It follows there will be a $\xi$-sphere with probability at least $1 - \gamma$ as long as $m > 32(p \log(1 + 256\sqrt{2}\sqrt{p}) + \log \gamma^{-1})$. This combined with the result of Theorem 2.2 shows that:

$$\|\hat{\beta} - \beta^*\| \leq \frac{a(L + 1)(32(p \log(1 + 256\sqrt{2}\sqrt{p}) + \log \gamma^{-1} + 1)}{\xi n},$$

with probability at least $1 - \gamma - \exp \left( \frac{-L^2/2}{L+1} \right)$.

Example A.3. Let $X_i$ have a uniform distribution on the sphere. Then $\bar{X}_i \sim \tilde{X}_i$. Let $g$ be a standard Gaussian random vector, and observe that $\tilde{X}_i d \tilde{X}_i = g$. We have $\sqrt{\frac{2}{\pi}} = \mathbb{E}[v^T \frac{g}{\|g\|} \|g\|] = \mathbb{E}[v^T \frac{g}{\|g\|} \|g\|]$, so that $\mathbb{E}[v^T \frac{g}{\|g\|}] \geq \frac{\sqrt{2}}{\pi \sqrt{p}}$. Now use $\mathbb{E}[\|g\| \leq \sqrt{p}$ hence $\mathbb{E}[v^T \frac{g}{\|g\|}] \geq \frac{\sqrt{2}}{\sqrt{p}}$. Next $\mathbb{E}(v^T \frac{g}{\|g\|})^2 = \frac{1}{\|g\|^2} = \frac{1}{\sqrt{p}}$.
We also assume whenever $m \geq 8\pi (p \log (1 + \frac{16\pi}{\xi}) + \log \gamma^{-1})$. By Theorem 2.2 we now have that
\[
\|\hat{\beta} - \beta^*\| \leq \frac{a(L + 1) (8\pi (p \log (1 + \frac{16\pi}{\xi}) + \log \gamma^{-1}) + 1)}{\xi n},
\]
with probability at least $1 - \gamma - \exp \left(\frac{-L^2/2}{\frac{\xi}{L+1}}\right)$.

**Example A.4.** In this example we analyze a centered elliptical distribution $X$. This generalizes two of our previous examples where we considered Gaussian and uniform on the unit sphere distributions. By a stochastic representation theorem for centered elliptical distributions (see Proposition 4.1.2 of Tong, 2012, e.g.) we know that one can generate a centered elliptical random variable as $X^d = RAU$, where $R \geq 0$ is a non-negative random variable independent of $U$, $U \overset{d}{=} \frac{R}{\|U\|}$ is distributed uniformly over the unit sphere $S^{p-1}$, and $A \in \mathbb{R}^{p \times p}$ is a constant matrix. Suppose $\Sigma = AA^T$ has smallest eigenvalue $\lambda_{\min}$ bounded away from 0 and largest eigenvalue $\lambda_{\max}$ being bounded. We have $\bar{X} \neq \bar{X}$. In what follows we also assume $R > 0$ and $\Sigma R^2 < \infty$.

We now evaluate for a vector $v$, $\mathbb{E}[v^T RAU] = \mathbb{E}[R] \mathbb{E}[v^T AU] = \frac{\mathbb{E}[R] \|v\|_2 \sqrt{\mathbb{E}R}}{\|v\|_2} \geq \frac{\mathbb{E}R \lambda_{\min} \sqrt{\frac{\pi}{2}}}{\sqrt{p}}$. On the other hand, $\mathbb{E}(v^T RAU)^2 = \frac{\mathbb{E}[R^2 \|v\|_2^2]}{\|v\|_2^2} = \frac{\mathbb{E}R^2}{\mathbb{E}R^2} \frac{\|v\|_2^2}{\|v\|_2^2}$. Hence $\frac{(\mathbb{E}[v^T RAU])^2}{\mathbb{E}(v^T RAU)^2} \geq \frac{\mathbb{E}[v^T RAU]^2}{\mathbb{E}[v^T RAU]^2} \geq \mathbb{E}R^2 \lambda_{\max}$. Next we upper bound $\mathbb{E}[\|\bar{X}\|] \leq \sqrt{\mathbb{E}[\|\bar{X}\|^2]} \leq \sqrt{\mathbb{E}R^2 \lambda_{\max}}$.

Set $\theta = \frac{1}{2}$, $c = \frac{8\pi \mathbb{E}R^2}{(\mathbb{E}R)^2}$ to obtain $\rho = c^{-1} + \frac{1 - \frac{2}{4\mathbb{E}R^2}}{8\pi \mathbb{E}R^2} = \frac{1}{2} - \frac{16\pi \mathbb{E}R}{8\pi \mathbb{E}R^2}$. Then by Corollary 2.8 with $\rho = 1$, $\alpha = 1$, $q = 2$ we obtain
\[
\mathbb{P}(\bar{X} \notin \text{conv}(\bar{X}^1, \ldots, \bar{X}^m)) \leq \left(1 + \frac{C' \sqrt{\mathbb{E}R^2}^{3/2} \sqrt{\lambda_{\max}}}{(\mathbb{E}R)^2 \sqrt{\lambda_{\min}}}\right)^m (1 - \frac{\mathbb{E}R^2}{8\pi \mathbb{E}R^2})^m.
\]

for an absolute constant $C'$. As before if $m > \frac{8\pi \mathbb{E}R^2}{(\mathbb{E}R)^2}$ then $\frac{\mathbb{E}[v^T RAU]^2}{\mathbb{E}(v^T RAU)^2} \geq \mathbb{E}R^2 \lambda_{\max}$, we have
\[
\mathbb{P}(\bar{X} \notin \text{conv}(\bar{X}^1, \ldots, \bar{X}^m)) \leq \gamma,
\]
and hence by Theorem 2.2 we have that with probability at least $1 - \gamma - \exp \left(\frac{-L^2/2}{\frac{\xi}{L+1}}\right)$:
\[
\|\hat{\beta} - \beta^*\| \leq \frac{a(L + 1) \left(\frac{8\pi \mathbb{E}R^2}{(\mathbb{E}R)^2} \left(1 + \frac{C' \sqrt{\mathbb{E}R^2}^{3/2} \sqrt{\lambda_{\max}}}{(\mathbb{E}R)^2 \sqrt{\lambda_{\min}}}\right) + \log \gamma^{-1}\right) + 1)}{\xi n}.
\]

where $\xi = \frac{\mathbb{E}R \lambda_{\min} \sqrt{\frac{\pi}{2}}}{4\sqrt{p}}$.

**Example A.5.** We now give a general example which only assumes that $\inf_{v \in S^{p-1}} \mathbb{E}[v^T XX^T v = \lambda_{\min}} > 0$ and $\sup_{v \in S^{p-1}} \mathbb{E}[v^T X^4] \leq C < \infty$. The latter happens in the case when the variables $X$ are sub-Gaussian e.g. in other words we assume that $\mathbb{E}[|v|^2 (v^T X)^2] < C \gamma^{-2}$ for some $\gamma \in \mathbb{R}^+$ for any $v \in S^{p-1}$.
(see also Definition 3.3 in Section 3 for a formal definition)). Indeed, this is so by Lemma 5.5 of Vershynin (2012).

Clearly, under these assumptions inf_{v \in \mathbb{S}^{p-1}} \mathbb{E} v^T \tilde{X} \tilde{X}^T v = \lambda_{\min} > 0 and sup_{v \in \mathbb{S}^{p-1}} \mathbb{E} (v^T \tilde{X})^4 \leq C < \infty.

Using Corollary 2.8 with \( \alpha = 2, q = 2, \theta = \frac{1}{2} \) we have that \( \rho = c^{-1} + \frac{1}{16}\frac{\lambda_{\min}^2}{C} \) and \( \xi = \frac{\lambda_{\min}}{2\sqrt{C}} \). Setting \( c^{-1} = \frac{\lambda_{\min}^2}{16} \), gives \( \rho = \frac{1}{2} - \frac{1}{16}\frac{\lambda_{\min}^2}{C} < \frac{1}{2} \). Next we can roughly upper bound \( \mathbb{E} \| \tilde{X} \| \leq \sqrt{\mathbb{E} \| \tilde{X} \|^2} \leq \sqrt{pC^{1/2}} \), where we used that for any random variable \( X \) we have \( \mathbb{E} X^2 \leq \sqrt{\mathbb{E} X^4} \).

By Corollary 2.8 we have

\[
\mathbb{P}(\xi \mathbb{E}_2^p \not\subset \text{conv}(\tilde{X}_1, \ldots, \tilde{X}_m)) \leq \left( 1 + 64\sqrt{2}\frac{C}{\lambda_{\min}^{5/2}} \sqrt{pC^{1/2}} \right)^p (1 - \frac{1}{16}\frac{\lambda_{\min}^2}{C})^m.
\]

Hence whenever \( m > \frac{16C}{\lambda_{\min}^2} \left( \rho \log \left( 1 + 64\sqrt{2}\frac{C^{3/4}}{\lambda_{\min}^{5/2}} p^{1/2} \right) + \log \gamma^{-1} \right) \) we have

\[
\mathbb{P}(\xi \mathbb{E}_2^p \not\subset \text{conv}(\tilde{X}_1, \ldots, \tilde{X}_m)) \leq \gamma.
\]

Hence by Theorem 2.2

\[
\| \tilde{\beta} - \beta^* \| \leq \frac{\alpha (L + 1) \left( \frac{16C}{\lambda_{\min}^2} \left( \rho \log \left( 1 + 64\sqrt{2}\frac{C^{3/4}}{\lambda_{\min}^{5/2}} p^{1/2} \right) + \log \gamma^{-1} \right) + 1 \right)}{\xi n},
\]

with probability at least \( 1 - \gamma - \exp \left( \frac{-L^2/2}{4L+1} \right) \).

One can of course assume even less assumptions in which case the bounds will worsen a bit. For instance, instead of assuming \( \sup_{v \in \mathbb{S}^{p-1}} \mathbb{E}(v^T X)^4 \leq C < \infty \) one can simply assume that the coordinates \( X^{(j)} \) for \( j \in [p] \) have bounded 4-th moments by some constant \( C_0 \). The same analysis as above can be applied in this situation upon noting that for any \( v \in \mathbb{S}^{p-1} \):

\[
\mathbb{E}(v^T X)^4 = \mathbb{E} \sum_{i,j,m,l} v_i v_j v_m v_l X^{(i)} X^{(j)} X^{(m)} X^{(l)}
\]

\[
\leq \mathbb{E} \sum_{i,j,m,l} v_i v_j v_m v_l \|X^{(i)}\| \|X^{(j)}\| \|X^{(m)}\| \|X^{(l)}\|
\]

\[
\leq \|v\|_1^4 C_0 \leq p^2 C_0.
\]

Finally, if one is bothered by 4-th moment assumptions, this too can be relaxed. One needs to use Corollary 2.8 with \( \alpha = 2 \) and \( q = 1 + \frac{1}{2} \) (so that \( q\alpha = 2 + \epsilon \)) for some \( \epsilon > 0 \). In this way, it suffices to assume that \( \sup_{v \in \mathbb{S}^{p-1}} \mathbb{E}|v^T X|^{2+\epsilon} < \infty \) which is even weaker than a 4-th moment assumption.

Appendix B: Proofs

Let \( P \) and \( Q \) be two probability distributions defined on space \((\mathcal{F}, \mu)\). The Total Variation Distance between \( P \) and \( Q \) is given by

\[
\|P - Q\|_{TV} = \sup_{A \in \mathcal{F}} |P(A) - Q(A)| = \frac{1}{2} \int |p(x) - q(x)| d\mu(x).
\]

Non-asymptotic bounds for the \( \ell_\infty \) estimator

Appendix B: Proofs
The Hamming Distance for two vectors \( a \) and \( b \) is defined as the number of coordinates where \( a_i \neq b_i \).

The next result is called Gershgorin’s Disk Theorem, which is used to bound the eigenvalues of a square matrix.

**Theorem B.1.** Let \( A \in \mathbb{R}^{n \times n} \) be a complex matrix with entries \( a_{ij} \), and let \( \lambda \) be an eigenvalue of \( A \). Then at least for one \( i \in [n] \) we have

\[
|\lambda - a_{ii}| \leq \sum_{j \neq i} |a_{ij}|
\]

We will also need the following Lemma.

**Proof of Lemma 2.1.** We can obtain two inequalities from the constraint \( Y_i - a \leq X_i^T \beta \leq Y_i + a \):

\[
-(a - |\varepsilon_i|) \leq X_i^T \beta - X_i^T \beta^* \leq a + |\varepsilon_i|, \quad \text{if } \varepsilon_i > 0
\]

\[
-(a + |\varepsilon_i|) \leq X_i^T \beta - X_i^T \beta^* \leq a - |\varepsilon_i|, \quad \text{if } \varepsilon_i \leq 0.
\]

This implies the following two inequalities:

\[
-\text{sign}(\varepsilon_i) X_i^T (\beta - \beta^*) \leq a - |\varepsilon_i|
\]

\[
\text{sign}(\varepsilon_i) X_i^T (\beta - \beta^*) \leq a + |\varepsilon_i|
\]

We finally only use the first one as a critical inequality since \( a - |\varepsilon_i| \) is stricter and more interesting. Its right hand side is close to zero when \( |\varepsilon_i| \) is close to \( a \). Thus we get the critical inequality

\[
-\text{sign}(\varepsilon_i) X_i^T (\beta - \beta^*) \leq a - |\varepsilon_i|.
\]

Note that the RHS of the critical inequality is actually independent of the LHS, and by the independence of \( \varepsilon_i \) and \( X_i \) we have that the \( -\text{sign}(\varepsilon_i) \) and \( |\varepsilon_i| \) are independent. Hence we can think of the critical inequality as

\[
\eta_i X_i^T (\beta - \beta^*) \leq a - |\varepsilon_i|,
\]

where \( \eta_i \) is a Rademacher random variable. \( \square \)

**Lemma B.2.** Let \( \{\varepsilon_i\}_{i \in [n]} \) be i.i.d. \( U([-a, a]) \) random variables. Sort the errors \( |\varepsilon_i| \sim U([0, a]) \) in decreasing manner \( |\tilde{\varepsilon}(i)| \), so that \( a \geq |\tilde{\varepsilon}(1)| \geq \ldots \geq |\tilde{\varepsilon}(n)| \geq 0 \). Suppose \( K \leq n \) is a fixed positive integer. Then

\[
\mathbb{P}\left[ \frac{|\tilde{\varepsilon}(K)|}{a} < 1 - \frac{K(L + 1)}{n} \right] \leq \exp\left(\frac{-KL^2/2}{4L + 1}\right) \leq \exp\left(\frac{-L^2/2}{4L + 1}\right). \tag{B.1}
\]

By summing up the first bound over \( K \) one may establish that for \( L \) large enough, with at least constant probability (B.1) holds simultaneously for all \( K \).

**Proof of Lemma B.2.** Consider the inequality

\[
a - |\tilde{\varepsilon}(K)| \leq \theta \Leftrightarrow |\tilde{\varepsilon}(K)| \geq a - \theta,
\]

where \( \theta \) is a constant and \( K \) is a positive integer.
for some \( \theta \). Suppose now \( \theta \leq a \). Denote the number of \( |\bar{\varepsilon}_i| \) being in the interval \([a - \theta, a]\) with \( Z \). If \( Z > K \), then clearly \( |\bar{\varepsilon}(K)| \geq a - \theta \), hence the event \( |\bar{\varepsilon}(K)| < a - \theta \) is a subset of the event \( Z \leq K \). Since \( |\varepsilon_i| \sim U([0, a]) \), the probability for an individual \( |\varepsilon| \) falling into the interval \([a - \theta, a]\) is \( \frac{\theta}{a} \). One can see \( Z \) follows a binomial distribution \( \text{Bin}(n, \frac{\theta}{a}) \). By (a one-sided) Bernstein’s inequality (Vershynin, 2018, Theorem 2.8.4) we have

\[
\Pr(Z \leq K) \leq \exp \left( \frac{-t^2/2}{n \theta / (1 - \theta/a) + \frac{t}{3}} \right) \leq \exp \left( \frac{-t^2/2}{n \theta / a + \frac{t}{3}} \right).
\]

Observe that if \( K(L + 1) > n \), there is nothing to prove (since the probability in (B.1) is 0). Hence assuming \( K(L + 1) \leq n \), set \( \theta = \frac{K(L+1)a}{n} \) and \( t = KL \). This yields \( \frac{n\theta}{a} - t = K \), and

\[
\Pr(Z \leq K) \leq \exp \left( \frac{-t^2/2}{n \theta / a + \frac{t}{3}} \right) = \exp \left( \frac{-KL^2/2}{\frac{4}{3}L + 1} \right) \leq \exp \left( \frac{-L^2/2}{\frac{4}{3}L + 1} \right),
\]

which is what we wanted to show.

\[\square\]

**Lemma B.3 (Extension to symmetric bounded distributions).** Let \( \{ \varepsilon_i \}_{i \in [n]} \) be i.i.d. symmetric about 0 random variables, bounded on the interval \([-a, a]\) with continuous distribution and cdf equal to \( F_\varepsilon \). Sort the errors \( \varepsilon_i \) in decreasing manner \( |\varepsilon_i| \), so that \( a \geq |\varepsilon_1| \geq \ldots \geq |\varepsilon_n| \geq 0 \). Suppose \( K \leq n \) is a fixed integer and \( L > 0 \) is also fixed. It then follows that:

\[
\Pr(|\bar{\varepsilon}(K)| < a - a_n(K, L)) \leq \exp \left( \frac{-L^2/2}{\frac{3}{4}L + 1} \right),
\]

where \( a_n(K, L) \) is defined as

\[
a_n(K, L) = \inf\{a \in [0, 2a] : n(1 - F_\varepsilon (a - a_n)) > K + 1\}.
\]

**Proof.** Clearly, the cdf of \( |\varepsilon| \) is \( 2F_\varepsilon(y) - 1 \). We know that \( u(K) = 2F_\varepsilon(|\bar{\varepsilon}(K)|) - 1 \), where \( u(K) \sim \text{Beta}(n - K + 1, K) \). By Lemma B.2 we have

\[
\Pr\left( u(K) < 1 - \frac{K(L + 1)}{n} \right) \leq \exp \left( \frac{-L^2/2}{\frac{4}{3}L + 1} \right).
\]

Hence

\[
\Pr\left( F_\varepsilon(|\bar{\varepsilon}(K)|) < 1 - \frac{K(L + 1)}{2n} \right) \leq \exp \left( \frac{-L^2/2}{\frac{4}{3}L + 1} \right).
\]

Since by the definition of \( a_n(K) \) if \( F_\varepsilon(|\bar{\varepsilon}(K)|) \geq 1 - \frac{K(L+1)}{2n} \) it follows that \( |\bar{\varepsilon}(K)| \geq a - a_n(K, L) \), the conclusion follows.

**Proof of Lemma 2.16.** We are concerned with the object

\[
\Pr(\|v^T X\| \leq 2\xi) = \Pr((v^T X)^2 \leq (2\xi)^2).
\]
We have:
\[ P((\mathbf{v}^\top \mathbf{X})^2 \leq (2\xi)^2) = \mathbb{P}(\exp(-\lambda(\mathbf{v}^\top \mathbf{X})^2) \geq \exp(-\lambda(2\xi)^2)) \leq \exp(\lambda(2\xi)^2) \mathbb{E}\exp(-\lambda(\mathbf{v}^\top \mathbf{X})^2), \]
for \( \lambda > 0 \). By Hölder’s inequality we have
\[
\mathbb{E}\exp(-\lambda(\mathbf{v}^\top \mathbf{X})^2) = \int \exp(-\lambda t^2) f(t) \, dt \leq \left[ \int \exp\left(-\frac{\lambda}{q} t^2\right) \, dt \right]^{\frac{q}{2}} \left[ \int f^q(t) \, dt \right]^{\frac{1}{q}}
\leq \left( \sqrt{\frac{\pi (q-1)}{\lambda q}} \right)^{\frac{q-1}{q}} \mathcal{C}
\]
Pick \( \lambda = \frac{\pi (q-1) (C \epsilon^{-1})^{\frac{1}{2q}}}{q} \) so that the above bound becomes:
\[ \mathbb{E}\exp(-\lambda(\mathbf{v}^\top \mathbf{X})^2) \leq \epsilon \]
Now select \( \xi = \frac{\lambda^{1/2}}{2} \). With this choice we obtain
\[ P(|\mathbf{v}^\top \mathbf{X}| \leq 2\xi) = \mathbb{P}((\mathbf{v}^\top \mathbf{X})^2 \leq (2\xi)^2) \leq \epsilon \epsilon < c_0, \]
for any \( \epsilon < \frac{c_0}{2} \). This completes the proof since \( \mathbf{v} \in \mathbb{S}^{p-1} \) was arbitrary.

**Proof of Proposition 2.17.** Since \( \mathbf{\beta}^* \) is a feasible point of (2.3), and \( \widehat{\mathbf{\beta}} \) minimizes the least squares among all feasible points, we have \( \|\mathbf{Y} - \mathbf{X}\widehat{\mathbf{\beta}}\|^2 \leq \|\mathbf{Y} - \mathbf{X}\mathbf{\beta}^*\|^2 = \|\mathbf{v}\|^2 \), and consequently the following basic inequality:
\[ (\widehat{\mathbf{\beta}} - \mathbf{\beta}^*)^\top n^{-1} \sum_{i \in [n]} \mathbf{X}_i \mathbf{X}_i^\top (\widehat{\mathbf{\beta}} - \mathbf{\beta}^*) \leq 2n^{-1} \sum_{i \in [n]} e_i \mathbf{X}_i^\top (\widehat{\mathbf{\beta}} - \mathbf{\beta}^*). \]

The above inequality can be rewritten as
\[
\inf_{\mathbf{v} \in \mathbb{S}^{p-1}} n^{-1} \sum_{i \in [n]} \mathbf{v}^\top \mathbf{X}_i \mathbf{X}_i^\top \mathbf{v} \|\widehat{\mathbf{\beta}} - \mathbf{\beta}^*\|^2 \leq \left\| 2n^{-1} \sum_{i \in [n]} e_i \mathbf{X}_i \right\| \|\widehat{\mathbf{\beta}} - \mathbf{\beta}^*\|.
\]
\[ \Rightarrow \|\widehat{\mathbf{\beta}} - \mathbf{\beta}^*\| \leq \frac{\|2n^{-1} \sum_{i \in [n]} e_i \mathbf{X}_i\|}{\inf_{\mathbf{v} \in \mathbb{S}^{p-1}} n^{-1} \sum_{i \in [n]} \mathbf{v}^\top \mathbf{X}_i \mathbf{X}_i^\top \mathbf{v}}. \tag{B.2} \]
It remains to upper bound the term \( \|2n^{-1} \sum_{i \in [n]} e_i \mathbf{X}_i\| \) and lower bound the term
\[ \inf_{\mathbf{v} \in \mathbb{S}^{p-1}} n^{-1} \sum_{i \in [n]} \mathbf{v}^\top \mathbf{X}_i \mathbf{X}_i^\top \mathbf{v}. \]
We first consider
\[ S^2 := \mathbb{E}\left\| n^{-1} \sum_{i \in [n]} e_i \mathbf{X}_i \right\|^2 = \mathbb{E} \sum_{j \in [p]} \left( \frac{\sum_{i \in [n]} \text{sign}(e_i)|e_i| \mathbf{X}^{(i)}_{x_j}}{n} \right)^2. \]
Observe that \( \text{sign}(\epsilon_i) \) is a Rademacher random variable independent of \( |\epsilon_i|X_i^{(j)} \). Thus we may apply Khintchine’s inequality (conditionally) to argue that

\[
\mathbb{E} \sum_{j \in [p]} \left( \frac{\sum_{i \in [n]} \text{sign}(\epsilon_i)|\epsilon_i|X_i^{(j)}}{n} \right)^2 \leq n^{-2}K_2^2 \sum_{j \in [p]} \sum_{i \in [n]} \mathbb{E} \xi_i^2 X_i^{2(j)} \lesssim \sqrt{C_0a^2} p/n.
\]

where we used that \( \mathbb{E} X_i^{2(j)} \leq \sqrt{C_0} \) and that \( \mathbb{E} \xi_i^2 = a^2/3 \), and where \( K_2 \) is an absolute constant from Khintchine’s inequality. Next, we will evaluate the variance of this term. We have

\[
\text{Var} \sum_{j \in [p]} \left( \frac{\sum_{i \in [n]} \text{sign}(\epsilon_i)|\epsilon_i|X_i^{(j)}}{n} \right)^2 \leq \left( \mathbb{E} \sum_{j \in [p]} \left( \frac{\sum_{i \in [n]} \text{sign}(\epsilon_i)|\epsilon_i|X_i^{(j)}}{n} \right)^2 \right)^2.
\]

Thus we may apply Khintchine’s inequality once again. We have

\[
p \mathbb{E} \sum_f \left( \frac{\sum_{i \in [n]} \text{sign}(\epsilon_i)|\epsilon_i|X_i^{(j)}}{n} \right)^4 \leq pn^{-4}K_4^2 \sum_{f \in [n]} \mathbb{E} \left( \sum_{i \in [n]} (\epsilon_i)^2 X_i^{2(j)} \right)^2 \leq pn^{-3}K_4^2 \sum_{f \in [n]} \sum_{i \in [n]} (\epsilon_i)^4 X_i^{4(j)} \lesssim C_0p^4n^{-2}a^4.
\]

Hence by Chebyshev inequality

\[
\mathbb{P} \left( S^2 \geq \frac{c\sqrt{C_0}pa^2}{n} + t \right) \leq \mathbb{P}(S^2 \geq \mathbb{E}S^2 + t) \leq \mathbb{P}(|S^2 - \mathbb{E}S^2| \geq t) \leq \frac{\text{Var}S^2}{t^2} \lesssim \frac{C_0a^4p^2}{n^2t^2}.
\]

Thus one can set \( t = C\frac{\sqrt{C_0}pa^2}{n} \) for some large constant \( C \). Hence with probability \( 1 - 1/C^2 \) we will have \( S^2 \lesssim \frac{\sqrt{C_0}pa^2}{n} \). This completes the bound of the first term.

To lower bound the second term, we rewrite it as

\[
\inf_{v \in \mathbb{S}^{p-1}} n^{-1} \sum_{i \in [n]} v^T X_i^T v = \inf_{v \in \mathbb{S}^{p-1}} n^{-1} \sum_{i \in [n]} \sum_{j=1}^{1/2} X_i^T \Sigma^{-1/2} X_j^T \Sigma^{1/2} v.
\]
Let \( A = \frac{1}{n} \sum_{i \in [n]} \Sigma^{-1/2} X_i X_i^\top \Sigma^{-1/2} \). Note that
\[
\mathbb{E}A = I_p
\]

Let \( m \in [p] \), \( l \in [p] \), and \( c > 0 \) be constants. By Chebyshev’s inequality, for the \((m,l)\)-th entry of \(A\) and \(I_p\) we have
\[
\mathbb{P} \left( |A^{(m,l)} - I_p^{(m,l)}| > c p \sqrt{\text{Var}(A^{(m,l)})} \right) \leq \frac{1}{c^2 p^2} \tag{B.3}
\]

Apply union bound to (B.3), given the bound on \(\text{Var}(A^{(m,l)})\), with probability at least 1
\[
\|A - I_p\|_{\max} \leq c p \sqrt{\text{Var}(A^{(m,l)})}
\]

It follows that the \(\infty\) norm of the matrix is bounded as
\[
\|A - I_p\|_{\infty} \leq c p^2 \sqrt{\text{Var}(A^{(m,l)})}
\]

From here one can show that
\[
A^{(m,m)} - \sum_{l \neq m} |A^{(m,l)}| \geq 1 - c p^2 \sqrt{\text{Var}(A^{(m,l)})}
\]

From Gershgorin’s Disk Theorem it follows that the eigenvalues of \(A\) are lower bounded as
\[
\lambda_{\min}(A) \geq \left( 1 - c p^2 \sqrt{\text{Var}(A^{(m,l)})} \right) \lambda_{\min}(\Sigma)
\]

The quantity \(\text{Var}(A^{(m,l)})\) can be upper bounded as \(\text{Var}(A^{(m,l)}) \leq \frac{1}{n} \|\Sigma^{-1}\|_{\text{op}}^2 p^2 C_0\), which we prove in Lemma B.4 below. If \( c \leq \frac{\sqrt{n}}{2 \sqrt{C_0 p^2} \|\Sigma^{-1}\|_{\text{op}}} \), we have \(1 - c p^2 \sqrt{\text{Var}(A^{(m,l)})} \geq \frac{1}{2}\), so that combine with (B.2) to get
\[
\|\beta - \beta^*\| \lesssim \frac{\sqrt{p/n}}{\frac{1}{2} \lambda_{\min}(\Sigma)} \simeq \sqrt{\frac{p}{n} \|\Sigma^{-1}\|_{\text{op}}},
\]

with probability converging to 1 if \( c \to +\infty \) as \( n \to +\infty \).

To complete the second part we will use Corollary 3.1 of Yaskov (2014) (see also (Srivastava and Vershynin, 2013, Theorem 1.5)) which states that assuming \( \mathbb{E}[v^\top X]^{2+\alpha} < \infty \) there exists a constant \( C_\alpha \) such that
\[
\lambda_{\min}(A) \geq 1 - C_\alpha \left( \frac{p}{n} \right)^{2/(2+\alpha)}
\]

with probability at least \( 1 - e^{-p} \). Hence when \( n > C_\alpha p \) the above can be made bigger than \( \frac{1}{2}\), in which case the proof may continue in the same fashion as before. This completes the proof. \( \square \)
Lemma B.4. For $A = \frac{1}{n} \sum_{i \in [n]} \Sigma^{-1/2} X_i X_i^\top \Sigma^{-1/2}$, if the 4-th moment of each coordinate in $X_i$ is bounded by $C_0$, then

$$\text{Var}(A^{(m,l)}) \leq \frac{1}{n} \|\Sigma^{-1}\|^2_{\text{op}} p^2 C_0.$$  

Proof. Since $X_i$ are i.i.d., we have

$$\text{Var}(A^{(m,l)}) = \frac{1}{n} \text{Var}[e_m^\top \Sigma^{-1/2} X_i X_i^\top \Sigma^{-1/2} e_l].$$

Therefore

$$\text{Var}[e_m^\top \Sigma^{-1/2} X_i X_i^\top \Sigma^{-1/2} e_l] \leq \mathbb{E}[(e_m^\top \Sigma^{-1/2} X_i)^2 (X_i^\top \Sigma^{-1/2} e_l)^2] \leq \|e_m^\top \Sigma^{-1/2}\|^2_2 \|e_l^\top \Sigma^{-1/2}\|^2_2 \mathbb{E}[^{2} \|X_i\|^4] \leq \|\Sigma^{-1}\|^2_{\text{op}} \mathbb{E}[\|X_i\|^4].$$

It is now clear that $\mathbb{E}[\|X_i\|^4] = \sum_{k,j \in [p]} \mathbb{E} \bar{X}_i^{(k)} X_i^{(l)} \leq \sum_{k,j \in [p]} [\mathbb{E} \bar{X}_i^{(k)}]^2 [\mathbb{E} X_i^{(l)}]^2 \leq p^2 C_0$. This completes the proof.

Proof of Theorem 2.19. The minimax risk is defined as

$$\inf_{\beta} \sup_{\beta' \in \mathbb{R}^p} \mathbb{E} \|\beta - \beta'\|^2.$$  

Let $R$ be any $p \times p$ orthogonal matrix. Pick some $\delta > 0$ and define $\beta_v = \delta Rv$ where $v = \{+1, -1\}^p$. Define a probability distribution corresponding to $v$ as $P_v = X_i^\top \beta_v + U_i$ where $U_i \sim U([-\delta, \delta])$. This is the distribution of $Y_i$ for a linear model $Y_i = X_i^\top \beta_v + U_i$. Notice that we have a $\delta$-Hamming separation for the loss function $\|\cdot\|^2$

$$\|\bar{\beta} - \beta_v\|^2 = \|RR^\top \bar{\beta} - R\delta v\|^2 = \|R^\top \bar{\beta} - \delta v\|^2 \geq \delta^2 \sum_{j \in [p]} \mathbb{I}(\text{sign}(R^\top \bar{\beta})_j \neq v_j).$$

We now need to repeat the proof of Assouad’s lemma (in order to capture the expectation with respect to the covariates $X$). We have

$$\sup_{\beta'} \mathbb{E}_X \mathbb{E}_Y \|\bar{\beta} - \beta'\|^2 \geq \frac{1}{2^p} \sum_{v} \delta^2 \sum_{j \in [p]} \mathbb{E}_X \mathbb{E}_Y \beta_v |X| \mathbb{I}(\text{sign}(R^\top \bar{\beta})_j \neq v_j)$$

$$\geq \delta^2 \mathbb{E}_X \sum_{j \in [p]} \frac{1}{2^{p-1}} \sum_{v, v_j = 1} \mathbb{E}_Y \beta_v |X| \mathbb{I}(\text{sign}(R^\top \bar{\beta})_j \neq 1)$$

$$+ \frac{1}{2^{p-1}} \sum_{v, v_j = -1} \mathbb{E}_Y \beta_v |X| \mathbb{I}(\text{sign}(R^\top \bar{\beta})_j \neq -1)$$

$$\geq \delta^2 \mathbb{E}_X \sum_{j \in [p]} \frac{1}{2^{p-1}} \frac{1}{2^{p-1}} \sum_{v, v_j = 1} \mathbb{E}_Y \beta_v |X| \mathbb{I}(\text{sign}(R^\top \bar{\beta})_j \neq 1)$$

$$- \frac{1}{2^{p-1}} \sum_{v, v_j = -1} \mathbb{E}_Y \beta_v |X| \mathbb{I}(\text{TV}).$$
Taking inf over all estimators on the LHS concludes that:

\[
\inf_{\tilde{\beta}} \sup_{\beta^* \in \mathbb{R}^p} \mathbb{E}_{\beta^*} \| \tilde{\beta} - \beta^* \|^2 \geq \frac{\delta^2}{2} \sum_{j=1}^{p} [1 - \mathbb{E}_{X} \| P_{\nu j}^{\otimes n} - P_{\nu' j}^{\otimes n} \|_{TV}],
\]

(B.4)

Notice that

\[
P_{\nu j}^{\otimes n} = 2^{-p} \sum_{v \in \{+1, -1\}^p} P_{\nu,vj=+1}^{\otimes n}, \quad P_{\nu' j}^{\otimes n} = 2^{-p} \sum_{v' \in \{+1, -1\}^p} P_{\nu',vj=-1}^{\otimes n},
\]

where by \( P_{\nu,vj=+1}^{\otimes n} \) we mean the probability under \( \nu \) where we set \( v_j = +1 \) regardless of the value of \( v_j \), and similarly for \( P_{\nu',vj=-1}^{\otimes n} \). The total variation can be bounded as

\[
\mathbb{E}_X \| P_{\nu j}^{\otimes n} - P_{\nu' j}^{\otimes n} \|_{TV} \leq 2^{-p} \sum_{v \in \{+1, -1\}^p} \mathbb{E}_X \| P_{\nu,vj=1}^{\otimes n} - P_{\nu',vj=-1}^{\otimes n} \|_{TV}
\]

\[
\leq \max_{j \in [p]} \mathbb{E}_X \| P_{\nu,vj=1}^{\otimes n} - P_{\nu',vj=-1}^{\otimes n} \|_{TV}
\]

\[
\leq \max_{v,v':d_H(v,v')=1} \max_{j \in [a]} \mathbb{E}_X \| P_{\nu}^{\otimes n} - P_{\nu'}^{\otimes n} \|_{TV},
\]

where \( d_H \) is the Hamming distance. Now it is easy to see that if one has two uniforms \( c_1 + U([-a,a]) \) and \( c_2 + U([-a,a]) \) (call those \( P_{c_1}, P_{c_2} \)) we have

\[
\| P_{c_1} - P_{c_2} \|_{TV} = \frac{|c_1 - c_2|}{2a} \wedge 1.
\]

Thus for any fixed \( v \) and \( v' \) with \( d_H(v,v') = 1 \) we have that

\[
\mathbb{E}_X \| P_{\nu}^{\otimes n} - P_{\nu'}^{\otimes n} \|_{TV} \leq \sum_{i \in [n]} \mathbb{E}_X |X_i^T (\beta_{\nu} - \beta_{\nu'})| = \sum_{i \in [n]} \mathbb{E}_X \frac{|(X_i^T R)_{j}|}{a},
\]

where \( j \) is the coordinate where \( v_j \neq v'_j \). Hence

\[
\max_{v,v':d_H(v,v') \leq 1} \mathbb{E}_X \| P_{\nu}^{\otimes n} - P_{\nu'}^{\otimes n} \|_{TV} \leq \max_{j \in [a]} \sum_{i \in [n]} \frac{|(X_i^T R)_{j}|}{a}.
\]

(B.5)

Picking \( \delta = \frac{\alpha}{\inf_{x \in \mathbb{R}^d} \max_{j \in [n]} \mathbb{E}_X \sum_{i \in [n]} |(X_i^T R)_{j}|} \), by (B.4) we have

\[
\inf_{\tilde{\beta}} \sup_{\beta^* \in \mathbb{R}^p} \mathbb{E}_{\beta^*} \| \tilde{\beta} - \beta^* \|^2 \geq p \delta^2 / 4,
\]

which completes the first part of the proof.

For the next part suppose \( R \) achieves the min in the definition of \( \delta \) (if the min cannot be achieved the same argument will go through by taking a sequence that converges to the inf). We let \( S = \{ \delta \mathbb{R} \nu : \nu \in \{\pm 1\}^p \} \) where \( \delta \) is as above. We have

\[
\inf_{\tilde{\beta}} \sup_{\beta^* \in S} \mathbb{P}(\| \tilde{\beta} - \beta^* \| \geq r_n) \geq \inf_{\beta} \sup_{\beta^* \in S} \mathbb{P}(\| \tilde{\beta} - \beta^* \| \geq r_n),
\]
Non-asymptotic bounds for the $\ell_\infty$ estimator

where $r_n := \delta \sqrt{p/(2\sqrt{2})}$. Now observe that in the RHS above, instead of taking inf over all $\beta$ it suffices to take inf over $\beta$ which belong to the set $S' = \{ y : \exists \beta \in S, \|y - \beta\| \leq \text{diam}_{L_2}(S) \}$. This is so since if $\hat{\beta}$ achieves the inf we can always consider $\beta = \hat{\beta}$ if $\beta \in S'$ and a random vector in $S$ otherwise. Clearly, by this definition $\|\hat{\beta} - \beta^*\| \leq \|\hat{\beta} - \beta^*\|$ and hence

$$\mathbb{P}(\|\hat{\beta} - \beta^*\| \geq r_n) \geq \mathbb{P}(\|\hat{\beta} - \beta^*\| \geq r_n).$$

Thus we have established

$$\inf_{\beta} \sup_{\beta^*} \mathbb{P}(\|\hat{\beta} - \beta^*\| \geq r_n) \leq \inf_{\beta \in S'} \sup_{\beta^* \in S} \mathbb{P}(\|\hat{\beta} - \beta^*\| \geq r_n),$$

where $\hat{\beta} \in S'$ means measurable functions which output values in the set $S'$. From Assouad’s lemma above we know that for any $\hat{\beta}$, $\sup_{\beta^* \in S} \mathbb{P}(\|\hat{\beta} - \beta^*\| \geq r_n) \geq 2 r_n^2$. Thus for any $\beta$, there exists a $\beta^* \in S$ such that

$$\sup_{\beta^* \in S} \mathbb{P}(\|\hat{\beta} - \beta^*\| \geq r_n^2) \geq \mathbb{P}(\|\hat{\beta} - \beta^*\| \geq r_n^2) \geq \mathbb{P}(\|\hat{\beta} - \beta^*\| \geq 1/2) \|\hat{\beta} - \beta^*\| \geq 1/4 \|\hat{\beta} - \beta^*\|^2 \geq 4 \mathbb{E} \|\hat{\beta} - \beta^*\|^2 \geq 4.$$ 

provided that $\mathbb{E} \|\hat{\beta} - \beta^*\|^2$ exists, where the above follows by Paley-Zygmund’s inequality. We now need to note that

$$\sup_{\beta \in S'} \sup_{\beta^* \in S} \mathbb{E} \|\hat{\beta} - \beta^*\|^4 \leq 4 \text{diam}_{L_2}(S)^4 = 4(\delta^2 p)^2.$$ 

Observe that this is precisely of the same order as $r_n^4$ hence it shows that the probability

$$\inf_{\beta \in S'} \sup_{\beta^* \in S} \mathbb{P}(\|\hat{\beta} - \beta^*\| \geq r_n),$$

is lower bounded by a constant $(1/2^8)$. 

\[\square\]

**Proof of Proposition 2.21.** We will only indicate where the proof differs from the proof of Theorem 2.19. We set the matrix $R$ to the any orthonormal matrix such that one of its rows contains the vector $v$ which minimizes $\inf_{v \in S_p-1} \mathbb{E} |X^T v|$. We then follow the proof of Theorem 2.19 until equation (B.5):

$$\max_{v,v': \|v - v'\|_1 \leq 1} \mathbb{E} |X^T P_{v}^{\otimes n} - P_{v'}^{\otimes n}| \leq \max_{j \in [p]} \sum_{i \in [n]} \frac{|(X^T_i R_j) \delta|}{a}.$$ 

For the index $j$ corresponding to the vector $v$, we have

$$\mathbb{E} X \sum_{i \in [n]} \frac{|(X^T_i R_j) \delta|}{a} = \mathbb{E} X \sum_{i \in [n]} \frac{|X^T_i v| \delta}{a}.$$ 


Hence if one selects \( \delta = \frac{a}{2n \|X\|} \geq \frac{a}{2n \sqrt{\lambda_{\max}(XX^T)}} \) one would obtain that
\[
\inf_{\beta} \sup_{\beta'} \mathbb{E} \beta' \|\hat{\beta} - \beta'\|^2 \geq \frac{1}{4} \delta^2,
\]
which shows the first part of the claim. For the second part the proof is identical to that of Theorem 2.19, except in the definition of \( r_n, \sqrt{P} \) is equal to 1 (i.e. it is not present).

\[\square\]

**Proof of Theorem 3.4.** Let us sort \( \varepsilon_i \) in a decreasing manner in terms of their magnitude \(|\tilde{\varepsilon}_i|\), and keep the first \( m \) terms. By the sharper bound in Lemma B.2 we know that \( \mathbb{P}(|\tilde{\varepsilon}_{(m)}| \leq a(1 - \frac{m(L + 1)}{n})) = \mathbb{P}(|\tilde{\varepsilon}_{(m)}| \leq a(1 - \frac{(L + 1)m}{n})) \leq \exp\left(-\frac{mL^2/2}{2L+1}\right) \to 0 \) if \( m \to \infty \). Hence \( |\tilde{\varepsilon}_{(m)}| \geq a(1 - (L + 1)m/n) \) with probability at least \( 1 - \exp\left(-\frac{mL^2/2}{2L+1}\right) \), where recall that \( a \) is the parameter of uniform distribution of the noise \( \varepsilon_i \).

By keeping the first \( m \) items of \(|\tilde{\varepsilon}_i|\), we have \( m \) critical inequalities as
\[
\eta_{(i)} X_{(i)}^T (\beta' - \hat{\beta}) \leq \hat{a} - |\tilde{\varepsilon}_{(i)}|,
\]
where \( \eta_{(i)} \) and \( X_{(i)} \) are the concomitant values to \(|\tilde{\varepsilon}_i|\) (and recall that they are independent from \(|\tilde{\varepsilon}_i|\)). With a slight abuse of notation we will drop the () brackets from the sub-indexing, and we will also write \( \varepsilon_i \) for \( \tilde{\varepsilon}_{(i)} \).

Let \( S \) be the support of \( \beta^* \). First, we prove that either \( \tilde{\beta} - \beta^* \in C(S, \gamma) \) for \( \gamma = 1 \) or 2, or else \( \lambda \|\beta^*_S - \hat{\beta}_S\|_1 < 2(L + 1)am/n \). The definition of \( C(S, \gamma) \) can be seen in Definition 3.1. We need the condition \( \beta - \beta^* \in C(S, \gamma) \) in order to apply the RE condition (see Definition 3.2) to bound \( \|\tilde{\beta} - \beta^*\|_1 \) and consequently \( \|\beta - \beta^*\|_1 \). Otherwise if \( \lambda \|\beta^*_S - \hat{\beta}_S\|_1 < 2(L + 1)am/n \), the bound of \( \|\beta - \beta^*\|_1 \) will be obtained immediately without any further derivations. We now consider several cases.

1. \( \hat{a} > a \).

   From the optimization (3.1) we have the inequality
   \[
   \hat{a} + \lambda \|\hat{\beta}\|_1 \leq a + \lambda \|\beta^*\|_1,
   \]
   then by the fact \( \|\beta^*\|_1 = \|\beta^*_S\|_1 \) and \( a - \hat{a} < 0 \), we have
   \[
   \lambda \|\beta^*_S - \hat{\beta}^*_S\|_1 = \lambda \|\hat{\beta}^*_S\|_1 \leq a - \hat{a} + \lambda \|\beta^*_S\|_1 - \lambda \|\hat{\beta}^*_S - \hat{\beta}_S\|_1.
   \]
   Hence in this case \( \beta - \beta^* \in C(S, 1) \).

2. \( \hat{a} \leq a \).

   The critical inequalities can be written as
   \[
   -\eta_i X_i^T (\beta^* - \hat{\beta}) \geq (|\varepsilon_i| - \hat{a})
   \]
   First suppose that \( a - \hat{a} > 2(L + 1)am/n \). It follows that \( |\varepsilon_i| - \hat{a} \geq a(1 - (L + 1)m/n) - \hat{a} \geq (a - \hat{a})/2 \). By Hölder’s inequality we have
   \[
   \left\| \frac{1}{m} \sum_{i \in [m]} \eta_i X_i \right\|_\infty \|\beta^* - \hat{\beta}\|_1 \geq \frac{1}{m} \sum_{i \in [m]} -\eta_i X_i^T (\beta^* - \hat{\beta}) \geq (a - \hat{a})/2
   \]
Now under assumption that $X_i$ is sub-Gaussian (actually a product of a matrix and a sub-Gaussian random vector), by Lemma B.5, $\|\frac{1}{m} \sum_{i \in [m]} \eta_i X_i \|_{\infty} \leq 2C' \sqrt{\gamma \|\Sigma\|_{op}^{1/2} \log p}$ in high probability, where $C'$ is an absolute constant. Denote with $R := \sqrt{\gamma \|\Sigma\|_{op}^{1/2}}$. Hence we conclude that

$$\|\beta_{Sc} - \hat{\beta}_{Sc}\|_1 \geq \frac{1}{4R} \sqrt{\frac{m}{\log p}} (a - \tilde{a}) - \|\beta_{Sc} - \hat{\beta}_{S}\|_1. \quad (B.6)$$

Suppose now $\|\beta_{Sc} - \hat{\beta}_{S}\|_1 > \frac{1}{8R} \sqrt{\frac{m}{\log p}} (a - \tilde{a})$. Hence for $\lambda \geq 8R \sqrt{\log p / m}$ we have $\lambda \|\beta_{Sc} - \hat{\beta}_{S}\|_1 \geq a - \tilde{a}$. Combine with the inequality

$$\lambda \|\hat{\beta}_{Sc}\|_1 \leq a - \tilde{a} + \lambda \|\beta_{Sc} - \hat{\beta}_{S}\|_1,$$

which can be deduced from the optimization (3.1), we conclude

$$\|\beta_{Sc} - \hat{\beta}_{Sc}\|_1 = \|\hat{\beta}_{Sc}\|_1 \leq 2\|\beta_{Sc} - \hat{\beta}_{S}\|_1.$$ 

On the other hand if $\|\beta_{Sc} - \hat{\beta}_{S}\|_1 < \frac{1}{8R} \sqrt{\frac{m}{\log p}} (a - \tilde{a})$, from (B.6) we can deduce that

$$\|\beta_{Sc} - \hat{\beta}_{Sc}\|_1 \geq \frac{1}{8R} \sqrt{\frac{m}{\log p}} (a - \tilde{a}).$$

Again from the optimization (3.1) we have

$$\|\hat{\beta}_{Sc}\|_1 \leq (a - \tilde{a}) / \lambda + \|\beta_{Sc} - \hat{\beta}_{S}\|_1,$$

so that for $\lambda \geq 16R \sqrt{\log p / m}$,

$$\|\beta_{S} - \hat{\beta}_{S}\|_1 \geq \frac{1}{8R} \sqrt{\frac{m}{\log p}} (a - \tilde{a}) - (a - \tilde{a}) / \lambda \geq (a - \tilde{a}) / \lambda.$$ 

This shows that in either case

$$\|\beta_{Sc} - \hat{\beta}_{Sc}\|_1 = \|\beta_{Sc} - \hat{\beta}_{S}\|_1 \leq 2\|\beta_{S} - \hat{\beta}_{S}\|_1.$$ 

Then we will handle the case where $a - \tilde{a} \leq 2(L + 1)am / n$. Suppose first that $\lambda \|\beta_{S} - \hat{\beta}_{S}\|_1 < 2(L + 1)am / n$. We can get the bound of $\|\hat{\beta} - \beta\|_1$ immediately since from (3.1) we can deduct $\lambda \|\hat{\beta}_{Sc}\|_1 \leq a - \tilde{a} + \lambda \|\beta_{Sc} - \hat{\beta}_{S}\|_1 \leq 4(L + 1)am / n$. Adding the two bounds we conclude that $\|\beta - \hat{\beta}\|_1 \leq 6(L + 1)am / (\lambda n)$, which is a very fast rate provided that $\lambda$ is not too small.

Next assume that $\lambda \|\beta_{Sc} - \hat{\beta}_{S}\|_1 \geq 2(L + 1)am / n \geq a - \tilde{a}$. Then again from (3.1) we can deduct

$$\lambda \|\beta_{Sc} - \hat{\beta}_{Sc}\|_1 = \lambda \|\beta_{Sc} - \hat{\beta}_{S}\|_1 \leq (a - \tilde{a}) + \lambda \|\beta_{Sc} - \hat{\beta}_{S}\|_1 \leq 2\lambda \|\beta_{S} - \hat{\beta}_{S}\|_1.$$

From the discussions above, we can conclude that when $\lambda \|\beta_{S} - \hat{\beta}_{S}\|_1 \geq 2(L + 1)am / n$ we will have

$$\|\beta_{Sc} - \hat{\beta}_{Sc}\|_1 \leq 2\|\beta_{S} - \hat{\beta}_{S}\|_1,$$ \quad (B.7)
so that

\[ \mathbf{b}^* - \tilde{\mathbf{b}} \in \mathcal{C}(S, 2), \]

where the definition of \( \mathcal{C}(S, 2) \) can be found in Definition 3.1.

Next we will show that in the case when \( \mathbf{b}^* - \tilde{\mathbf{b}} \in \mathcal{C}(S, 2) \), there are at least \( m - m/\log n \) following inequalities

\[ -C((L + 1)am/n + \lambda\|\mathbf{b}_S - \mathbf{b}_S^*\|_1) \leq \sum_{i \in S} \eta_i \mathbf{x}_i^\top (\tilde{\mathbf{b}} - \mathbf{b}^*) \leq (L + 1)am/n + \lambda\|\mathbf{b}_S - \mathbf{b}_S^*\|_1, \tag{B.8} \]

where \( C = 2\log n \). The upper bound is quite simple. From the optimization (3.1) we have the inequality

\[ \tilde{a} + \lambda\|\mathbf{b}\|_1 \leq a + \lambda\|\mathbf{b}^*\|_1, \]

Then we have \( \tilde{a} \leq a + \lambda\|\mathbf{b}_S\|_1 - \lambda\|\tilde{\mathbf{b}}_S\|_1 \leq a + \lambda\|\mathbf{b}_S - \tilde{\mathbf{b}}_S\|_1 \). Combine with \( |\varepsilon_i| \geq a(1 - (L + 1)m/n) \), our critical inequalities become

\[ \eta_i \mathbf{x}_i^\top (\mathbf{b}^* - \tilde{\mathbf{b}}) \leq \tilde{a} - |\varepsilon_i| \leq a - |\varepsilon_i| + \lambda\|\mathbf{b}_S^* - \tilde{\mathbf{b}}_S\|_1 \]

\[ \leq (L + 1)am/n + \lambda\|\mathbf{b}_S^* - \tilde{\mathbf{b}}_S\|_1. \]

The lower bound \( \eta_i \mathbf{x}_i^\top (\mathbf{b}^* - \tilde{\mathbf{b}}) \geq -C((L + 1)am/n + \lambda\|\mathbf{b}_S^* - \tilde{\mathbf{b}}_S\|_1) \) involves a bit more work. We will prove this by contradiction. Suppose that at least \( m/\log n \) critical inequalities actually satisfy the following

\[ \eta_i \mathbf{x}_i^\top (\mathbf{b}^* - \tilde{\mathbf{b}}) \leq -C((L + 1)am/n + \lambda\|\mathbf{b}_S^* - \tilde{\mathbf{b}}_S\|_1), \]

where we will fix \( C \) later on.

Consider the average

\[ \frac{1}{m} \sum_{i \in [m]} -\eta_i \mathbf{x}_i^\top (\mathbf{b}^* - \tilde{\mathbf{b}}) \geq \frac{1}{\log n} C((L + 1)am/n + \lambda\|\mathbf{b}_S^* - \tilde{\mathbf{b}}_S\|_1) - ((L + 1)am/n + \lambda\|\mathbf{b}_S^* - \tilde{\mathbf{b}}_S\|_1) \]

\[ \geq (L + 1)am/n + \lambda\|\mathbf{b}_S^* - \tilde{\mathbf{b}}_S\|_1. \]

for any \( C \geq 2\log n \). By Hölder’s inequality we have

\[ \left\| \frac{1}{m} \sum_{i \in [m]} \eta_i \mathbf{x}_i \right\|_\infty \|\mathbf{b}^* - \tilde{\mathbf{b}}\|_1 \geq \frac{1}{m} \sum_{i \in [m]} -\eta_i \mathbf{x}_i^\top (\mathbf{b}^* - \tilde{\mathbf{b}}). \]

Now under assumption that \( X_i \) is sub-Gaussian, by Lemma B.5, \( \| \frac{1}{m} \sum_{i \in [m]} \eta_i \mathbf{x}_i \|_\infty \leq 2R \sqrt{\frac{\log p}{m}} \) in high probability, where \( R = C' \sqrt{\gamma\|\Sigma\|_{1/2}/\log p} \). Hence we conclude that

\[ \|\mathbf{b}_S^* - \tilde{\mathbf{b}}_S\|_1 \geq ((L + 1)am/n + \lambda\|\mathbf{b}_S^* - \tilde{\mathbf{b}}_S\|_1) \frac{1}{2R} \sqrt{\frac{m}{\log p}} - \|\tilde{\mathbf{b}}_S - \mathbf{b}_S^*\|_1. \]
For $\lambda \geq 6R\sqrt{\log p/m}$, we have
\[
\|\beta_{S^*} - \hat{\beta}_{S^*}\|_1 \geq \frac{(L+1)am}{2Rn} \sqrt{\frac{m}{\log p}} + 2\|\beta_{S} - \beta_{S}^*\|_1 > 2\|\beta_{S} - \beta_{S}^*\|_1,
\]
which is a contradiction to (B.7). Hence we conclude that at least $m - m/\log n$ inequalities satisfy the bound (B.8), so they also satisfy
\[
[\eta_i X_i^T (\hat{\beta} - \beta^*)]^2 \leq ((L+1)am/n + \lambda\|\beta_{S} - \beta_{S}^*\|_1)^2 4\log^2 n.
\]

Let $\mathcal{J}$ be the set of $i$ for which the above bounds hold. We have showed $|\mathcal{J}| \geq m - m/\log n$ in the case when $\beta^* - \hat{\beta} \in C(S,2)$. Our next goal is to bound $\|\beta^* - \beta^*\|$ by the RE condition. Recall that from the discussion above we either have $\|\beta^* - \beta^*\|_1 \leq 6(L+1)am/(\lambda n) \) or $\beta^* - \beta^* \in C(S,2)$.

According to the RE condition on sub-Gaussian ensemble matrices (Zhou, 2009, Theorem 1.6), given that the population covariance matrix $\Sigma$ satisfies the $RE(\kappa, 2, s)$ condition, if $s \leq p/2$, $m \leq p$ and $m \geq \|\Sigma\|_{op}\gamma^4 \kappa^{-2} (s \log (5p/s) \vee \log p)$, with probability at least $1 - 2\exp( -cm/\gamma^4)$, we have $X^T X/m$ satisfies the $RE(\kappa(1 - \theta), 2, s)$ for some fixed small $0 < \theta < 1$ condition. Here $c > 0$ is an absolute constant. Notice that the $m$ inequalities are chosen in terms of $e_i$, which is independent from $X_i$, so the theorem about RE condition in Zhou (2009) applies to them. Now we need to ensure that if we select at least $m - m/\log n$ observations the sample covariance matrix will still satisfy the RE condition. This is easy since
\[
\sum_{i=0}^{m/\log n} \binom{m}{m-i} \leq (e \log n)^{m/\log n} \ll \exp (c(m - m/\log n)/\gamma^4),
\]
where the first inequality is obtained by the fact $\sum_{i \leq k} \binom{n}{k} \leq \left(\frac{e \log n}{k}\right)^k$, and the second inequality is obtained by taking log on both sides. If we select $m - i$ observations for $i \leq m/\log n$, then with probability at most
\[
2\exp \left( -c(m-i)/\gamma^4 \right) \leq 2\exp \left( -c(m - m/\log n)/\gamma^4 \right)
\]
the matrix $X^T X/(m-i)$ (where with a slight abuse of notation $X$ denotes the selected matrix) doesn’t satisfy the RE condition. Thus, by the union bound, the probability that for any $m-i$ for $i \leq m/\log n$, out of $m$ observations the sample covariance matrix will satisfy the RE is at least
\[
1 - 2\sum_{i=0}^{m/\log n} \binom{m}{m-i} \exp \left( -c(m-m/\log n)/\gamma^4 \right),
\]
which is close to 1 according to the bound in (B.9).

Thus with probability converging to 1
\[
\kappa(1-\theta)\|\beta_S^* - \hat{\beta}_S\| \leq \sqrt{\frac{1}{|\mathcal{J}|} \sum_{i \in \mathcal{J}} (X_i^T (\beta^* - \hat{\beta}))^2}
\leq 2 \log n \sqrt{((L+1)am/n + \lambda\|\beta_{S} - \beta_{S}^*\|_1)^2}
\leq 2 \log n ((L+1)am/n + \lambda \sqrt{5\|\beta_{S} - \beta_{S}^*\|})
where \( s = |S| \). Suppose that \( \lambda \leq \frac{k}{(4+4\theta/(1-\theta))\sqrt{s\log n}} \); we obtain that
\[
\|\beta_S^* - \tilde{\beta}_S\| \leq \frac{4(L+1)am\log n}{(1-\theta)kn}.
\]
From here we immediately get a bound on the \( \ell_1 \) norm
\[
\|\beta_S^* - \tilde{\beta}_S\|_1 \leq \sqrt{s}\|\beta_S^* - \tilde{\beta}_S\| \leq \frac{4(L+1)\sqrt{s}\log n}{(1-\theta)kn}.
\]
And then combining with \( \|\tilde{\beta}_S\|_1 \leq 2\|\beta_S^* - \tilde{\beta}_S\|_1 \) we can get
\[
\|\beta^* - \tilde{\beta}\|_1 \leq 3\|\beta_S^* - \tilde{\beta}_S\|_1 \leq \frac{12a(L+1)m\sqrt{s}\log n}{(1-\theta)kn}.
\]
On the other hand, if \( \|\beta^* - \tilde{\beta}\|_1 \leq 6a(L+1)m/(\lambda n) \) the conclusion directly follows. It remains to select \( m \) as the minimum possible number from our requirements. We have required \( m \geq \log n, m \leq p, m \geq \|\Sigma\|_{\text{op}} \gamma^4 k^{-2}(s \log (5s/p)/\sqrt{s} \log p) \) and \( m \) satisfying \( \kappa \sqrt{m}/\log p \geq \gamma^{1/2}\|\Sigma\|_{\text{op}}^{1/2} \sqrt{s} \log n \), where we mean . This completes the proof.

**Lemma B.5.** If \( X_i \) is sub-Gaussian as specified in Theorem 3.4 we have
\[
\left\| \frac{1}{m} \sum_{i \in [m]} \eta_i X_i \right\|_{\infty} \leq 2C' \sqrt{\gamma\|\Sigma\|_{\text{op}}^{1/2} \log p/m},
\]
where \( C' \) is an absolute constant with high probability.

**Proof.** Observe that \( \eta X \) is a centered sub-Gaussian random variable with sub-Gaussian constant \( \leq \gamma\|\Sigma\|_{\text{op}}^{1/2} \). To see this set \( w = \frac{v}{\|v\Sigma^{1/2}\|} \), and note that
\[
\mathbb{E} \exp \left( \frac{\gamma^{-2}}{\|\Sigma\|_{\text{op}}} (v^T X)^2 \right) \leq \mathbb{E} \exp (\gamma^{-2}(w^T \zeta)^2) \leq 2.
\]
Since \( \eta_i X_i \) is sub-Gaussian, each coordinate \( \eta_i X_i^{(j)} \) is a one-dimensional sub-Gaussian variable with sub-Gaussian constant \( \gamma\|\Sigma\|_{\text{op}}^{1/2} \).

By Lemma 5.9 of Vershynin (2012), we know that the random variable \( \sum_i \eta_i X_i^{(j)} \) is sub-Gaussian with constant at most \( Cm\gamma\|\Sigma\|_{\text{op}}^{1/2} \), where \( C \) is an absolute constant. Next, using Lemma 5.5 (1) of Vershynin (2012) and the union bound we conclude that for a constant \( t > 0 \) we have
\[
\mathbb{P}(\|m^{-1} \sum_i \eta_i X_i\|_{\infty} \geq t) \leq p e^{1-mt^2/(C'\gamma\|\Sigma\|_{\text{op}}^{1/2})},
\]
with \( C' \) being another absolute constant. Putting \( t = 2C'\gamma^{-2}\|\Sigma\|_{\text{op}}^{1/4} \sqrt{\log p/m} \) gives the desired result. \( \square \)
B.1. An optimal upper bound for standard Gaussian design

In this subsection we make the case that when $X_i \sim N(0, I_p)$ the lower bound of the order of $\sqrt{p}$ is tight, assuming that the noise variance $a$ is known. For simplicity let $a = 1$ (otherwise one can rescale $\beta^*$ to $\beta^*/a$). Construct $n!$ estimators which are linear regression based. In detail let $E = ((2k - n)/n)_{k \in [n]}^T$. For each of $n!$ permutations $\Pi$ construct the estimators:

$$\hat{\beta}_\Pi = (X^\top X)^{-1} X^\top (Y - \Pi E).$$

Let $\hat{\beta}_C$ be the Chebyshev estimator. Let

$$B = \left\{ \tilde{\beta} : \exists \Pi \text{ s.t. } \tilde{\beta} = \hat{\beta}_\Pi, \| \tilde{\beta} - \hat{\beta}_C \| \leq \frac{2(L + 1)(8\pi p \log(1 + 32\sqrt{2} \pi^{3/2}/\sqrt{\rho}) + 8\pi \log \gamma^{-1} + 1)}{\xi n} \right\},$$

(B.10)

where the constant in the bound is taken from Example A.1 (it is twice the constant in that example), and the constants $\gamma, \xi$ are specified there. Next consider “playing” a “tournament” for the (at most) $n!$ estimators in the set $B$. Specifically, for any estimator $\tilde{\beta} \in B$, let $B_{\tilde{\beta}} = \{ \bar{\beta} \in B : \| \bar{\beta} - \tilde{\beta} \| \geq 5\xi \sqrt{\rho}/n \}$, for some sufficiently large constant $\xi$. For any $\tilde{\beta} \in B_{\tilde{\beta}}$ construct the numbers

$$\bar{A} = \frac{1}{n} \sum_{i \in [n]} (Y_i - X_i^\top \tilde{\beta})^2 - \frac{1}{n} \sum_{i \in [n]} (Y_i - X_i^\top \hat{\beta})^2 + \frac{1}{n} (\sum_{i} Y_i X_i - X_i X_i^\top \tilde{\beta})^\top \hat{\beta} - \bar{\beta},$$

and

$$\bar{A} = -\frac{1}{n} \sum_{i \in [n]} (Y_i - X_i^\top \tilde{\beta})^2 + \frac{1}{n} \sum_{i \in [n]} (Y_i - X_i^\top \hat{\beta})^2 + \frac{1}{n} (\sum_{i} Y_i X_i - X_i X_i^\top \tilde{\beta})^\top \hat{\beta} - \bar{\beta},$$

which can be estimated from the data. If $\bar{A} > \bar{A}$ say that $\tilde{\beta}$ wins otherwise say that $\bar{\beta}$ wins. Take any $\bar{\beta}$ that wins over all points in $B_{\tilde{\beta}}$. We will prove that this succeeds to produce an estimation rate of $\sqrt{\rho}/n$ under the condition that $p^3(\log p)^4 \ll n$. In more detail we have

**Theorem B.6.** Let $\tilde{\beta}$ be the estimator selected by the above procedure. Suppose that $p^3(\log p)^4 \ll n$. Then such an estimator exists, and in addition it satisfies that

$$\| \tilde{\beta} - \beta^* \| \lesssim \frac{\sqrt{p}}{n},$$

(B.11)

with large probability (i.e. with a probability that can be made arbitrarily close to 1 at the expense of increasing the constant in the bound, and in the algorithm).

**Proof.** By the Dvoretzky-Kiefer-Wolfowitz inequality we know that

$$\mathbb{P}(\|E - e^\top\|_\infty \geq 2\tau) \leq 2e^{-2\tau^2},$$

where $e^\top$ is an increasing rearrangement of the error terms $e$. Taking $\tau = C/\sqrt{n}$ for a large enough $C$ gives us that with high probability $\|E - e^\top\|_\infty \leq C/\sqrt{n}$, implying that $\|E - e^\top\| \leq C$. Next if one fits a
Hence when \( \Pi \) to before we can evaluate with probability at least 1
\( k \)
with probability at least 1
\( b \)
triangle inequality). We will show that with high probability and any other estimator above bound be denoted by
\( b \)
estimator
\( \mathcal{Y} \)
38
in the procedure we compare
\( b \)
We will now compare two estimators and
\( \mathcal{A} \)
more than
\( \mathcal{A} \)
\( 1 - 2 \exp(-r^2/2) \) by Corollary 5.35 of Vershynin (2012) and the \( \chi^2(p) \leq 5p \)
with probability at least 1 \(- \exp(-\sqrt{p})\), by Lemma 1 of Laurent and Massart (2000). Hence at least one estimator \( \tilde{\beta} \) is \( \sqrt{p}/n \) near the true point with high probability. Let the proportionality constant in the above bound be denoted by \( C \) (the bigger the \( \tilde{C} \) the bigger the probability of success can be made).

In addition fit the Chebyshev estimator on the data, and discard any estimator of the \( n! \) ones that is more than \( C p \log p/n \) away from the Chebyshev one (the precise expression for \( C \) is given in (B.10)).

We will now compare two estimators and \( \tilde{\beta} =: \tilde{\beta}_\Pi \in \mathcal{B} \) and \( \tilde{\beta} \in \mathbb{B}_\tilde{\beta} \). Note that we have that \( \tilde{\beta} \in \mathcal{B} \) by the triangle inequality. As described in the procedure we compare \( \tilde{\beta} \) and \( \tilde{\beta} \) by comparing the two real numbers (which can be estimated from the data):

\[
\tilde{A} = \frac{1}{n} \sum_{i \in [n]} (Y_i - X_i^\top \tilde{\beta})^2 - \frac{1}{n} \sum_{i \in [n]} (Y_i - X_i^\top \beta)^2 + \frac{1}{n} \sum_{i} Y_i X_i - X_i X_i^\top \beta^\top (\tilde{\beta} - \beta),
\]

and

\[
\tilde{\lambda} = -\frac{1}{n} \sum_{i \in [n]} (Y_i - X_i^\top \beta)^2 + \frac{1}{n} \sum_{i \in [n]} (Y_i - X_i^\top \beta)^2 + \frac{1}{n} \sum_{i} Y_i X_i - X_i X_i^\top \beta^\top (\tilde{\beta} - \beta)
\]

If \( \tilde{A} > \tilde{\lambda} \) say that \( \tilde{\beta} \) wins otherwise say that \( \tilde{\beta} \) wins. The true estimator \( \tilde{\beta} \) satisfies \( \|\tilde{\beta} - \beta^*\| \leq C \sqrt{p}/n \), and any other estimator \( \tilde{\beta} \) which satisfies \( \|\tilde{\beta} - \beta^*\| \geq \kappa \sqrt{p}/n \) for some sufficiently large \( \kappa \geq 4\tilde{C} \) by the triangle inequality. We will show that with high probability \( \tilde{A} > \tilde{\lambda} \). A simple calculation shows that

\[
\tilde{A} = (\beta^* - \tilde{\beta})^\top \tilde{\Sigma}(\tilde{\beta} - \beta) + (\beta - \beta^*)^\top \tilde{\Sigma}(\tilde{\beta} - \beta^* - (\beta - \beta^*)^\top \tilde{\Sigma} (\tilde{\beta} - \beta^*),
\]

where \( \tilde{\Sigma} = X^\top X/n \). By Corollary 5.35 of Vershynin (2012) we have that with high probability (at least \( 1 - 2 \exp(-p^2/2) \)) we have

\[
1 - c \sqrt{p}/n \leq (1 - 2 \sqrt{p}/n)^2 \leq \lambda_{\min}(\tilde{\Sigma}) \leq \lambda_{\max}(\tilde{\Sigma}) \leq (1 + c \sqrt{p}/n)^2 \leq 1 + c \sqrt{p}/n \text{ for some } c.
\]

It follows by Cauchy-Schwartz that with high probability we have

\[
\tilde{A} \geq -(1 + c \sqrt{p}/n)\|\beta^* - \tilde{\beta}\|\|\tilde{\beta} - \beta\| + (1 - c \sqrt{p}/n)^2 \|\tilde{\beta} - \beta^*\|^2 - (1 + c \sqrt{p}/n)^2 \|\tilde{\beta} - \beta^*\|^2 \geq \|
\]

\[
(\|\beta^* - \tilde{\beta}\| - \|\beta^* - \beta^*\|)\|	ilde{\beta} - \beta\| - c \sqrt{p}/n \|\beta^*\|^2/(n \log p)^2,
\]

where we used that \( \|\tilde{\beta} - \beta^*\| > \|\tilde{\beta} - \beta^*\| \). On the other hand let us upper bound \( \tilde{A} \). By a similar logic to before we can evaluate

\[
\tilde{A} = (\beta^* - \tilde{\beta})^\top \tilde{\Sigma}(\tilde{\beta} - \beta) + (\beta - \beta^*)^\top \tilde{\Sigma}(\beta - \beta^*) - (\beta - \beta^*)^\top \tilde{\Sigma} (\beta - \beta^*)
\]

\[
< (1 + c \sqrt{p}/n)\|\beta^* - \tilde{\beta}\|\|\tilde{\beta} - \beta\| + (1 + c \sqrt{p}/n)\|\beta^*\|^2 - (1 - c \sqrt{p}/n)\|\beta^*\|^2
\]
Consider the inequality

\[ \| \beta^* - \beta \| \leq \sqrt{n} p \cdot \sqrt{C \log n} / \sqrt{n} p^2 / n^2 (\log p)^2. \]

If \( \theta \), \( \beta \) are i.i.d. centered sub-Gaussian random variables with variances equal to 1 and sub-Gaussian parameter bounded by some constant \( C < \infty \). This implies that the rows of \( X \) are i.i.d. sub-Gaussian isotropic random vectors. To see why the theorem extends to this setting, one needs to replace the application of Laurent and Massart (2000)'s Lemma 1 with a general sub-exponential bound such as the one offered by Vershynin (2012)'s Proposition 5.16. In addition, the eigenvalue concentration of the matrix \( \Sigma \) can be deduced from Theorem 4.6.1 (Vershynin, 2018). The bound on the Chebyshev estimator can be taken from Example 2.14.

**B.2. Lower bound for the Chebyshev estimator**

In this subsection we prove a general lower bound for the performance of the Chebyshev estimator. We start with a simple lemma on the order statistics of the error terms.

**Lemma B.8.** Let \( \{ \epsilon_i \}_i \sim U([a, b]) \) be i.i.d. \( U([-a, a]) \) random variables. Sort the errors \( |\epsilon_i| \sim U([0, a]) \) in decreasing manner \( |\bar{\epsilon}_i| \), so that \( a \geq |\bar{\epsilon}_1| \geq \cdots \geq |\bar{\epsilon}_n| \geq 0 \). Suppose \( K \leq n \) is a fixed positive integer. Then

\[ \Pr \left( \frac{|\bar{\epsilon}_K|}{a} \geq 1 - \frac{K}{2n} \right) \leq \exp \left( \frac{-3K}{16} \right). \]

**Proof of Lemma B.8.** Consider the inequality

\[ a - |\bar{\epsilon}_K| \leq \theta \iff |\bar{\epsilon}_K| \geq a - \theta, \]

for some \( \theta \). Suppose now \( \theta \leq a \). Denote the number of \( |\bar{\epsilon}| \) being in the interval \([a - \theta, a]\) with \( Z \). If \( |\bar{\epsilon}_K| \geq a - \theta \) then \( Z \leq K \). Since \( |\bar{\epsilon}_i| \sim U([0, a]) \), the probability for an individual \( |\bar{\epsilon}_i| \) falling into the interval \([a - \theta, a]\) is \( \frac{a - \theta}{a} \). One can see \( Z \) follows a binomial distribution \( Bin(n, \frac{a - \theta}{a}) \). By (a one-sided) Bernstein's inequality (Vershynin, 2018, Theorem 2.8.4) we have

\[ \Pr \left( Z \geq \frac{n\theta}{a} + t \right) \leq \exp \left( \frac{-t^2/2}{\left( \frac{n\theta}{a} \right) \left( 1 - \frac{\theta}{a} \right)} \right) \leq \exp \left( \frac{-t^2/2}{\frac{n\theta}{a} + \frac{\theta}{a}} \right). \]
Set $\theta = \frac{Ku}{2n}$ and $t = K/2$. This yields $\frac{nu}{a} + t = K$, and

$$\mathbb{P}(Z \geq K) \leq \exp\left(\frac{-t^2/2}{\frac{nu}{a} + \frac{t}{3}}\right) = \exp\left(\frac{-K^2/8}{K/2 + K/6}\right) \leq \exp\left(\frac{-3K}{16}\right),$$

which is what we wanted to show. \hfill $\square$

**Theorem B.9.** Suppose the matrix $X$ has i.i.d. standard Gaussian entries. With at least a constant probability we have that the Chebyshev estimator $\hat{\beta}$ satisfies

$$\|\hat{\beta} - \beta^\star\| \gtrsim a p/(n(\log n)^{3/2}),$$

where the inequality $\gtrsim$ hides absolute constant factors.

**Proof of Theorem B.9.** Without loss of generality we assume $a = 1$. We have that $\tilde{a} = \min_{\beta} \|Y - X\beta\|_\infty = \min_{\epsilon} \max_i \left| e_i + X^\top v \right| = \min_{\epsilon} \max_{e: \|e\|_1 \leq 1} \epsilon^\top (\epsilon + Xv)$. We can then write,

$$\tilde{a} \leq \hat{a}(R) := \min_{v: \|v\|_2 \leq R} \max_{e: \|e\|_1 \leq 1} \epsilon^\top (\epsilon + Xv),$$

for some $R > 0$. Applying the minimax theorem gives us that

$$\hat{a}(R) = \max_{e: \|e\|_1 \leq 1} \min_{\|v\|_2 \leq R} \epsilon^\top (\epsilon + Xv) = \max_{e: \|e\|_1 \leq 1} \epsilon^\top \epsilon - R \|\epsilon^\top X\|$$

Taking $R \to \infty$, shows that $\hat{a} \leq \max_{e: \|e\|_1 \leq 1} \epsilon^\top \epsilon$. Next using Theorem 9.1.1 (Vershynin, 2018), we have that

$$\mathbb{E} \sup_{e: \|e\|_1 \leq 1} \|\epsilon^\top X\| - \sqrt{p}\|e\| \lesssim \sqrt{\log n},$$

for some absolute constant, where we used that the Gaussian width of the $\ell_1$ ball is $\sqrt{\log n}$ up to constant factors. Hence by Chebyshev’s inequality we can claim that with probability at least $0.99$ for all $e: \|e\|_1 \leq 1$ we have $\|\epsilon^\top X\| \geq \sqrt{p}\|e\| - C\sqrt{\log n}$ for a sufficiently large absolute constant $C$. It follows that $\tilde{a} \leq \max_{e: \|e\|_1 \leq 1, \epsilon \leq C\sqrt{\log n}/p} \epsilon^\top \epsilon$. Let $s$ be the biggest integer smaller than $s \leq \lceil C\sqrt{\log n}/p \rceil^2/4$. Then $\|e_S\|_1 \leq \sqrt{s}C\sqrt{\log n}/p \leq 1/2$, where $S$ is the support of the maximal $s$ coefficients of $\epsilon$ corresponding to the maximal $s$ values of $\epsilon$ (which always needs to be the case due to the rearrangement inequality). By Lemma B.8 we know that with at least a constant probability,

$$|\tilde{\epsilon}_{(s+1)}| \leq 1 - (s + 1)/(2n) \leq 1 - (p/(4C^2 \log n))/(2n) \leq 1 - \kappa'(p/(n \log n)).$$

Since $\|e\|_\infty \leq 1$ we have $\epsilon^\top \epsilon \leq \|e_S\|_1 + (1 - \|e_S\|_1)|\tilde{\epsilon}_{(s+1)}| \leq 1 - 1/2\kappa'(p/(n \log n)).$ We conclude that

$$\tilde{a} \leq 1 - \kappa''p/(n \log n).$$

Now by Lemma B.2 we know that with constant probability for $L$ large enough,

$$(1 - i(L + 1)/n) - (1 - \kappa''p/(n \log n)) \leq |\tilde{\epsilon}_{(i)}| - \tilde{a} \leq \|X^\top_{(i)} (\hat{\beta} - \beta^\star)\|,$$
Non-asymptotic bounds for the $\ell_1$ estimator

and the LHS is positive for the first $\approx \kappa'' p/((L + 1) \log n)$ entries, and where $X_{(i)}$ are the concomitant $X_i$ values for the top order statistics. Squaring and adding these inequalities yields,

$$
\sum_{i < \kappa'' p/((L+1) \log n)} (\kappa'' p/((L+1) \log n) - i(L + 1)/n)^2 \leq (\hat{\beta} - \beta^*)^\top \sum X_{(i)} X_{(i)}^\top (\hat{\beta} - \beta^*)
$$

$$
\leq (p/((L + 1) \log n) + p + p) \| \hat{\beta} - \beta^* \|^2, \quad (B.13)
$$

with probability at least $1 - 2 \exp(c p^2)$ for some absolute constant $c$, where we used Corollary 7.3.3 of Vershynin (2018). On the other hand we have

$$
\sum_{i < \kappa'' p/((L+1) \log n)} (\kappa'' p/((L+1) \log n) - i(L + 1)/n)^2 \geq \sum_{i=1}^{[\kappa'' p/((L+1) \log n)]} (\bar{i}(L + 1)/n - i(L + 1)/n)^2
$$

$$
= \frac{(L + 1)^2}{n^2} \left[ \sum_{i=1}^{[\kappa'' p/((L+1) \log n)]} (i^2 - \bar{i}^2) \right]
$$

$$
\geq \frac{(L + 1)^2}{n^2} [\kappa'' p/((L + 1) \log n)]^3,
$$

where $\bar{i} = \sum_{i=[\kappa'' p/((L+1) \log n)]}^{[\kappa'' p/((L+1) \log n)]} i/([\kappa'' p/((L+1) \log n)] + 1)/2$. Dividing (B.13) by $3p$ yields that

$$
\| \hat{\beta} - \beta^* \| \gtrsim p/(n \log n)^{3/2},
$$

with at least constant probability.

\[\square\]

**Remark B.10.** The proof remains valid if one substitutes the entries of the design matrix $X$ with i.i.d. centered sub-Gaussian random variables with variances equal to 1 and sub-Gaussian parameter bounded by some $C < \infty$. This implies that the rows and columns of $X$ are i.i.d. sub-Gaussian isotropic random vectors. (B.13) needs to be replaced with the eigenvalue concentration of the matrix $\Sigma$ which can be deduced from Theorem 4.6.1 (Vershynin, 2018).