Spherically symmetric black holes in $f(R)$ gravity: is geometric scalar hair supported?

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Abstract

We critically discuss current research on black hole (BH) solutions in $f(R)$ gravity and shed light on its geometrical and physical significance. We also investigate the meaning, existence or lack thereof of Birkhoff's theorem (BT) in this kind of modified gravity. We then focus on the analysis and search for non-trivial (i.e. hairy) asymptotically flat (AF) BH solutions in static and spherically symmetric (SSS) spacetimes in vacuum having the property that the Ricci scalar does not vanish identically in the domain of outer communication. To do so, we provide and enforce regularity conditions at the horizon in order to prevent the presence of singular solutions there. Specifically, we consider several classes of $f(R)$ models like those proposed recently for explaining the accelerated expansion in the Universe and which have been thoroughly tested in several physical scenarios. Finally, we report analytical and numerical evidence about the absence of geometric hair in AFSSSBH solutions in those $f(R)$ models. First, we submit the models to the available no-hair theorems (NHTs), and in the cases where the theorems apply, the absence of hair is demonstrated analytically. In the cases where the theorems do not apply, we resort to a numerical analysis due to the complexity of the non-linear differential equations. With that aim, a code to solve the equations numerically was built and tested using well-known exact solutions. In a future investigation we plan to analyze the problem of hair in de Sitter and anti-de Sitter backgrounds.

Keywords: modified theories of gravity, classical black holes, numerical solutions

(Some figures may appear in colour only in the online journal)
1. Introduction

Modified $f(R)$ gravity has become one of the most popular mechanisms to generate a late accelerated expansion in the Universe without the need of introducing new fields [1]. It was also one of the first consistent models for early inflation [2]. During the past 15 years or so, several specific $f(R)$ models have been thoroughly analyzed in many scenarios, but only a few of them can survive the classical tests (e.g. Solar System, binary pulsar) while predicting the correct accelerating expansion, and in general, a successful cosmological model, both at the background and at the perturbative level. Therefore, it is still unclear to what extent these kind of alternative theories of gravity can recover all the successes of general relativity (GR) while making new testable predictions.

As concerns black hole (BH) solutions, the situation of $f(R)$ gravity can, in some sense, differ from GR, and in another sense be almost the same. The last statement is related to the content of section 2 about the existence of the same kind of vacuum BH solutions found in GR, while the former concerns the existence of hairy solutions, or lack thereof, which we analyze in the rest of the paper.

Perhaps the first and simplest theorem concerning BH solutions in GR was Birkhoff’s theorem (BT). Roughly speaking, this theorem establishes that in vacuum all spherically symmetric (SS) spacetimes are also static, and those that are asymptotically flat (AF) are represented by a one-parameter family of solutions, namely, the ubiquitous Schwarzschild solution, where the parameter is interpreted as the (ADM) mass $M$ of the spacetime (see [3] for a discussion). Remarkably, when including an electric field, the static SS solution can be extended as to include the charge $Q$ of the BH; this is the well-known two-parameter Reissner–Nordström (RN) solution. In the AF case, both the Schwarzschild and the RN solution are $R = 0$ solutions of Einstein’s field equations. When including a cosmological constant $\Lambda$, the Schwarzschild and RN BHs become a two and three-parameter family respectively, and the solutions are asymptotically de Sitter (ADS) or asymptotically anti-de Sitter (AADS), depending on if $\Lambda > 0$ or $\Lambda < 0$, respectively.

In the 1960s, motivated by the discovery of the Kerr solution, several theorems (notably, the uniqueness theorems) were established for stationary and axisymmetric spacetimes both in vacuum and with an electromagnetic field (see [4, 5] for details and reviews). One of the main consequences of those theorems is that in the AF case the solutions of Einstein’s field equations under such symmetries are characterized only by three parameters: the mass $M$, the charge $Q$, and the angular momentum $J$ of the BH. These solutions are known as the Kerr–Newman family, which extends the Schwarzschild and RN BHs to more general spacetimes: stationary and axisymmetric. Due to the apparent simplicity of such solutions, Wheeler established the so-called no-hair conjecture, a statement that ’doomed’ all possible stationary AFBH solutions from having any other parameters than the three previously mentioned ($M$, $Q$ and $J$). This conjecture was thereafter reinforced by the elaboration of several no-hair theorems (NHTs) that forbid the existence of BH solutions with more parameters associated with other kinds of matter fields (see [6, 7] for a review). Among such theorems one can mention those that include several kinds of scalar fields. This conjecture proved to be ’false’ in the static and spherically symmetric (SSS) situation within the Einstein–Yang–Mills system [8], and the Einstein-(real)scalar-field system’ endowed with ’exotic’ potentials that can be negative [9, 10]. Nonetheless, since most (if not all) of the static hairy solutions are unstable like in the Einstein-matter systems we just mentioned3, the community (or at least part of it) considers those solutions as weak counterexamples to Wheeler’s no-hair conjecture. Therefore, it has been tantalizing to extend the conjecture in the following more precise.

3 Sometimes termed the Einstein–Higgs system as opposed to the Einstein-(complex) boson system.
4 In the sense that a perturbation can lead to an eventual loss of the hair.
although still informal, statement: the only stable stationary AFBHs are within the Kerr–Newman family [6]. Still there are some caveats in this extended version of the conjecture since it is well known that the Kerr solution is unstable with respect to perturbations that include scalar fields due to superradiance instability [12]. In fact, very recently Wheeler’s conjecture was proved to be false also in the stationary and axisymmetric scenario, in view of the remarkable hairy solutions found within the Einstein-(complex)bozon-field system [11], which are absent in the SS case. Therefore, it is possible that the stationary and axisymmetric hairy solutions found in [11] might be the final state of the superradiance instability associated with the Kerr solution, and thus, the bosonic hair found in [11] (if stable) might become the first genuine counterexample to the non-hair conjecture. However, these instability-stability issues are something that require further and careful investigations.

The proposal of alternative theories of gravity as a possible solution to the dark-matter and dark-energy problems and to other theoretical problems (e.g. inflation, gravity renormalization) has motivated people to generalize several of the theorems and conjectures concerning BHs in GR, to the framework of BHs in other modified-gravity proposals. While it is out of the scope of the present paper to review all such attempts, we shall simply focus on $f(R)$ metric gravity. Unless otherwise stated, by $f(R)$ gravity we mean a theory that departs from the GR Lagrangian $f(R) = R - 2\Lambda$.

With regard to this kind of theory, a large amount of analysis has been devoted to establish an analogue of BT for the SS situation [13–15]. However, the reality is that no rigorous BT exists today in $f(R)$ gravity, as far as we are aware. In fact, if one such theorem were proved, certainly it should be restricted to some specific $f(R)$ models. Moreover, the theorem should establish at least four things, upon fixing the boundary conditions (i.e. regularity and asymptotic conditions)—these are as follows. (1) Staticity: the only SS solutions in vacuum (i.e. without any matter field associated with the standard model of particle physics or any other field that is not associated with the Lagrangian $f(R)$) are necessarily static (i.e. the existence of a static Killing field should be proved from the SS assumptions). (2) Existence: the existence of an exact SSS solution in vacuum. (3) Uniqueness: the SSS solution found in point ‘(2)’ is the only solution in vacuum (or alternatively, prove non-uniqueness). (4) The conditions under which the solution in point ‘(2)’ matches or not the exterior solution of an SSS extended body.

In vacuum there exists BH exact solutions in $f(R)$ gravity, but those that are genuine AF, ADS or AADS correspond simply to the same kind of solutions found in GR, where $R = R_c = \text{const}$ everywhere in the spacetime (with $R_c = 0, R_c > 0$ or $R_c < 0$, for AF, ADS or AADS, respectively). It is unclear if other solutions exist with the same kind of asymptotics but with a varying $R$ in the domain of outer communication of the BH. We shall elaborate more about this point below to be more precise. Furthermore, in the presence of matter (i.e. a star-like object), it is possible to find SSS solutions where $R$ can vary in space [16, 17, 18]. However, those solutions are not exact, but are given only numerically, and it is unclear if the exterior part (i.e. the vacuum part) of those solutions is the same solution found when matter is totally absent in the spacetime, if it exists at all, as happens in GR where the exterior solution of extended objects under such symmetries is always given by the vacuum Schwarzschild solution5.

5 In GR and in the stationary and axisymmetric case, the exterior solution of rotating extended bodies does not exactly match the Kerr solution. On the other hand, in scalar-tensor theories (STTs) of gravity without a potential, and due to the spontaneous scalarization phenomenon [19], there exist star-like SSS solutions where the exterior solution is not given by the Schwarzschild metric, since the scalar field there is not zero. It turns out, however, that in the absence of ordinary matter, the exterior solution within the same STT is given only by the Schwarzschild solution. These two scenarios show that the exterior solutions in the presence of ordinary matter are not necessarily the same as in the complete absence of it; this illustrates the difficulty of establishing a generalization of BT when the hypotheses change.
Now, despite the absence of such BT, some NHTs have been proved in this kind of theory. In order to do so, people have resorted to the equivalence between a certain class of $f(R)$ models (notably, those where $f_R > 0$ and $f_{RR} > 0$, where the subindex indicates differentiation) with STTs. The point is that one performs a conformal transformation from the original Jordan frame (JF) to the so-called Einstein frame (EF) where the conformal metric appears to be coupled minimally to gravity and a new scalar field $\phi(\chi)$ emerges, where $\chi = f_R$, which is also coupled minimally to the conformal metric but endowed with an ‘exotic’ potential $\mathcal{V}(\phi)$. Thus, the available NHTs constructed for the Einstein-(real)scalar-field system in GR can be applied for these theories as well (see section 5), notably in vacuum, and when the spacetime is AF and the potential satisfies the condition $\mathcal{V}(\phi) \geq 0$ [20, 21].

We stress that the applicability of such NHTs is possible because the non-minimal coupling between the scalar field $\phi(\chi)$ and the matter fields that usually appears under the EF obviously vanishes in the absence of the matter. The only caveat of this method is that the potential $\mathcal{V}(\phi)$ is not given a priori but is the result of the specific $f(R)$ model considered ab initio, and thus, $\mathcal{V}(\phi)$ can be negative or even not well defined (i.e. it can be multivalued if the condition $f_{RR} > 0$ does not hold; moreover the EF transformation becomes ill defined if $f_R \leq 0$ or if $f_R$ diverges), which in turn can jeopardize the use of the NHTs. Consequently, the existing NHTs in $f(R)$ gravity can reduce the kind of AFSSSBH solutions that are available in some specific models, but do not rule out completely the absence of geometric hair. In this context, by (geometric) hairy solutions within $f(R)$ gravity we mean AFSSSBH solutions where the Ricci scalar is not trivial (i.e. constant), but rather a function that interpolates non-trivially between the horizon and spatial infinity.

Thus, when the condition $\mathcal{V}(\phi) \geq 0$ fails and the NHTs are not applicable one can resort to a numerical analysis for evidence about the existence of such hair or its absence thereof. At this respect it is important to stress that regularity conditions have to be imposed at the inner boundary, namely, at the BH horizon $r_h$ in order to prevent the presence of singularities there. In section 4 and appendix B we obtain such regularity conditions and then in sections 5.1 and 5.2 we present analytical and numerical evidence, respectively, showing that hairy solutions are absent in several specific $f(R)$ models proposed as dark-energy alternatives in cosmology. In particular, the models considered in section 5.2 are precisely those for which the NHTs cannot be applied as the corresponding potential $\mathcal{V}(\phi)$ can be negative or is not even well defined. On the other hand, when such hairy solutions are absent, one may still find the trivial solution $R = R_1 = \text{const}$ for which the field equations reduce to Einstein field equations with an effective cosmological constant $\Lambda_{\text{eff}} = R_1/4$, and an effective gravitational constant $G_{\text{eff}} = G_0/\sqrt(R_1)$ where $R_1$ is a solution of an algebraic equation involving $f(R)$ and $f_R$. This includes the case where $R_1 \equiv 0$. Therefore, in such circumstances, all the best known BH solutions found in GR exist also in $f(R)$ gravity simply by replacing the usual cosmological constant $\Lambda$ by $\Lambda_{\text{eff}}$, and Newton’s gravitational constant $G_0$ by $G_{\text{eff}}$. In view of this we shall argue in section 2 that such solutions are so trivial (i.e. trivial in the context of $f(R)$ gravity) that almost nothing new arises from them.

Finally, we mention that some ‘non-trivial’ exact SSSBH solutions have been reported in the literature as a result of very ad hoc $f(R)$ models [14, 25-27]. Notwithstanding such solutions cannot be considered as hairy solutions because they have unusual asymptotics, and therefore, the corresponding ‘hairless’ solution $R = \text{const}$ (including the $R = 0$ solution) does

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6 A more recent proof of the same result [22] adopts a weaker convexity assumption $\mathcal{V}''(\phi) \geq 0$ for the potential. This proof is similar to Bekenstein’s [23] which assumes only stationarity as opposed to staticity.
not even exist with the same kind of asymptotics. We shall discuss one such solution in section 3.1.

The article is organized as follows. In section 2 we discuss in a general setting the conditions for the existence of several trivial BH solutions. In section 3 we focus on SSS spacetimes and provide the corresponding differential equations to find BH solutions. We also discuss some exact solutions that will be used later to test a numerical code constructed to solve the equations. The boundary conditions appropriate to solve these equations with the presence of a BH are given in section 3 in the form of regularity conditions at the horizon. NHTs and the properties of \( f(R) \) gravity formulated in the EF are analyzed in section 5. In that section we also provide strong numerical evidence about the absence of hair for several \( f(R) \) models when the NHTs do not apply. Our conclusions and final remarks are presented in section 6. Several appendices at the end of the article complement the ideas of the main sections.

2. \( f(R) \) theory of gravity

The general action for a \( f(R) \) theory of gravity is given by

\[
I[g_{ab}, \psi] = \int \frac{f(R)}{2\kappa} \sqrt{-g} \; d^4x + I_{\text{matter}}[g_{ab}, \psi],
\]

where \( \kappa \equiv 8\pi G_0 \) (we use units where \( c = 1 \))\(^7\), and \( f(R) \) is a sufficiently smooth (i.e. \( C^3 \)) but otherwise an \textit{a priori} arbitrary function of the Ricci scalar \( R \). The first term corresponds to the modified-gravity action, while the second is the usual action for the matter, where \( \psi \) represents the matter fields schematically.

The field equation arising from the action (1) under the metric approach is

\[
f_R R_{ab} - \frac{1}{2} g_{ab} - (\nabla_a \nabla_b - g_{ab} \Box) f_R = \kappa T_{ab},
\]

where \( f_R \) stands for \( df/dR \) (we shall use similar notation for higher derivatives), \( \Box = g^{cd} \nabla_c \nabla_d \) is the covariant d’Alembertian and \( T_{ab} \) is the energy-momentum tensor of matter resulting from the variation of the matter action in (1). It is a straightforward, though non-trivial result, to show that the conservation equation \( \nabla^a T_{ab} = 0 \) holds also in this case (see appendix A for a proof). In turn, this latter leads to the geodesic equation for free-fall particles \( u^a \nabla_a u^a = 0 \). Therefore, the weak-equivalence principle (for point-test particles) is also incorporated in this theory as well. Actually \( f(R) \) metric gravity preserves all the axioms of GR but the one that assumes that the field equations for the metric \( g_{ab} \) must be of second order. Clearly the only case where this happens is for \( f(R) = R - 2\Lambda \), which leads to GR plus a cosmological constant (hereafter GRA)\(^8\).

Now, taking the trace of equation (2) yields

\[
\Box R = \frac{1}{3f_{RR}} [\kappa T - 3f_{RRR} (\nabla R)^2 + 2f - R f'_R].
\]

where \( T := T^a_a \). When using (3) in (2) and after some elementary manipulations we obtain [18]

\(^7\) In section 3.1, we extend our units so that \( G_0 = 1 \) as well.

\(^8\) A result that is also a corollary (when applied to a four-dimensional spacetime) of a theorem known as \textit{Lovelock’s theorem} [24].
\[ G_{ab} = \frac{1}{f_R} \left[ f_{RR} \nabla_a \nabla_b R + f_{RRR} (\nabla_a R)(\nabla_b R) - \frac{g_{ab}}{6} (R f_R + f + 2 \kappa T) + \kappa T_{ab} \right]. \] (4)

Equations (3) and (4) are the basic equations that we have used systematically in the past to tackle several problems in cosmology and astrophysics [18, 28–30, 58], and that we plan to use in this article as well.

Now, apart from the GR \( \Lambda \) theory for which \( f_R \equiv 1, f_{RR} \equiv 0, \) and \( R = 4 \Lambda - \kappa T, \) for more general models, one imposes the conditions \( f_R > 0, \) for a positive \( G_{\text{eff}}, \) and \( f_{RR} > 0, \) for stability [54]. However, in this paper we shall sometimes relax these two assumptions in order to explore their consequences for the sake of finding hairy BH solutions.

In vacuum, that is when \( T_{ab} \equiv 0, \) or more generally, in the presence of matter fields where \( T \equiv 0, \) like in electromagnetism or the Yang–Mills theory, equation (3) admits in principle the trivial exact solution \( R = R_c = \text{const} \) where \( R_c \) is a solution of the algebraic equation \( [2f(R) - R f_R]/f_{RR} = 0. \) In particular, if \( f_{RR}(R_c) \neq 0 \) and \( 0 < f_{RR}(R_c) < \infty, \) as often happens in potentially-viable \( f(R) \) models, then \( R_c \) is an algebraic solution of \( 2f(R) - R f_R = 0 \) that we call \( R_1. \) For such kind of solutions the field equation (4) reduces to

\[ G_{ab} + \Lambda_{\text{eff}} g_{ab} = 8 \pi G_{\text{eff}} T_{ab}, \] (5)

where

\[ \Lambda_{\text{eff}} = \frac{R_c}{4}, \] (6)

\[ G_{\text{eff}} = \frac{G_0}{f_{R}(R_1)}, \] (7)

and we assume \( 0 < f_R(R_1) < \infty. \) Moreover, the condition \( 0 < f_{RR}(R_1) < \infty \) is a stability condition that is usually imposed in order to avoid exponentially growing modes when perturbing around the value \( R = R_1. \)

On the other hand, if there exists other value \( R = R_2 = \text{const}, \) that we call \( R_2 \) (in order to avoid confusion with \( R_1 \)) such that \( f_{RR}(R) \) blows up at \( R = R_2, \) but \( R_2 \neq R_1 \) then \( R_2 \) can be also a possible trivial solution of equation (3). However, in such a case one must be extremely cautious as in equation (4) products may appear of the sort \( \infty \times 0 \) (or \( \infty/\infty \)). Even if such a product is finite, namely zero, still the interpretation of such a solution would be problematic as \( f_{RR} \) would be singular everywhere. To what extent such ‘singularity’ is physical, is something that one should clarify. In section 3.1, we shall be dealing with a SSS solution where the Ricci scalar is not constant but \( R \rightarrow R_2 \) as \( r \rightarrow \infty. \) Hence, in that example \( f_R \rightarrow \infty, f_{RR} \rightarrow \infty \) and \( f_{RRR} \rightarrow \infty \) as \( r \rightarrow \infty. \) However, these pathologies, as peculiar as they may be, do not concern us too much in this article, since we are mainly interested in situations where they are absent.

Now, in the particular and simplest scenario where \( R = R_1 \) is one of the trivial solutions of equation (3), we see that the field equation (5) corresponds to GR with the usual ‘bare’ cosmological constant \( \Lambda \) and the bare Newton’s constant \( G_0 \) replaced by \( \Lambda_{\text{eff}} \) and \( G_{\text{eff}}, \) respectively. Therefore, in that occurrence all the solutions that exist in GR exist in \( f(R) \) gravity as well when taking into account the above replacements. In particular, the AFBH solutions with \( R = 0 \) that exist in GR, like the Kerr–Newman family and its SSS limit, also exist in \( f(R) \) gravity if \( R_1 = 0, \) i.e. if \( f(0) \equiv 0. \) On the other hand, the de Sitter or anti-de Sitter BH solutions associated with the Kerr–Newman family with a cosmological constant [31, 32] also exist in \( f(R) \) gravity if \( R_1 \neq 0. \)

We emphasize that BH solutions with \( R = R_1 = \text{const}, \) which from the point of view of \( f(R) \) gravity are ‘trivial’, have been systematically reported in the literature as something new
or special (e.g. see [14, 33]). However, as we just showed, the existence of such solutions stems from the fact that \( R_1 \) exists in various \( f(R) \) models, like the ones we consider in section 5.1. In turn, the existence of such a trivial solution is just a standard demand for \( f(R) \) theories to produce a late acceleration expansion: the cosmological constant \( \Lambda_{\text{eff}} \) emerges while the Universe evolves towards the solution \( R \to R_0 \) as the Universe expands and matter dilutes. So nothing exceptional, astonishing or radically different from the already known BH solutions in GR are to be expected in this kind of trivial solution in \( f(R) \) gravity. Furthermore, and as a consequence of these remarks, the Bekenstein–Hawking entropy defined for such BHs, has \( G_{\text{eff}} \) instead of \( G_0 \) in the formula. That is, the modified entropy is \( S = A/(4G_{\text{eff}}) = Af_R(R_0)/(4G_0) \) [34], where \( A \) is the area of the BH event horizon, instead of just \( S = A/(4G_0) \).

Let us consider now the ‘odd’ scenario where \( R_1 \) is such that \( f_R(R_0) = 0 \), and \( R_1 = 0 \) in vacuum (i.e. \( T_{ab} = 0 \)). Then equation (3) with \( T = 0 \) admits \( R \equiv 0 \) as one possible trivial solution. Nonetheless, the possible solutions for the metric that satisfies equation (4) degenerate and a whole spectrum of solutions can emerge, besides the usual AF vacuum solutions mentioned above. Such solutions are \textit{all} the possible solutions of the Einstein equation \( G_{ab} = \kappa T_{ab} \) (where \( T_{ab} \) is any other traceless energy-momentum tensor of matter) compatible with a \textit{null} Ricci scalar \( R \equiv 0 \). More specifically, if one considers an \( f(R) \) model in vacuum such that \( 2f - Rf_R = 0 \) at \( R = 0 \) (implying in turn \( f(0) = 0 \)) but assuming \( f_{RR}(0) \neq 0 \), then equation (3) is solved trivially. Moreover the field equation (4) in vacuum reduces to \( f_R(0)G_{ab} = 0 \) which is trivially satisfied for \( f_R(0) = 0 \) even if \( G_{ab} \neq 0 \). Thus, in that instance the model unexpectedly admits all the possible solutions associated with the non-vacuum Einstein equation \( G_{ab} = \kappa T_{ab} \) compatible with \( G^a_a = -R = -T = 0 \). Now, given such scenario one has to deal with a problematic interpretation of the new global quantities that appear in the BH solutions as integration constants since in reality we are dealing with an \( f(R) \) model in vacuum. In non-vacuum GR the global quantities (other than \( M \) and \( J \)) are ascribed to properties associated with the matter described by \( T_{ab} \) (like the electric charge \( Q \) or the nodes of the Yang–Mills field). Therefore the same quantities would appear in this degenerate scenario, but clearly they cannot have the same interpretation in vacuum. In section 5.1 within the framework of model 1, we shall encounter one such example which is associated with the model \( f(R) = kR^2 \). In this model \( R = 0 \) is a trivial solution and clearly \( f_R(0) = 0 \), thus all possible solutions of Einstein equations with a traceless energy-momentum tensor are in principle allowed, for instance, AFSSS solutions. In fact, one such solution is exactly the analogue of the RN, where the roll of the charge is played by a new quantity that appears as an integration constant. This situation was also analyzed recently in [35] and remarked on before in [14]. We shall elaborate more about this in sections 3.1 and 5.1, notably concerning the issue of \textit{uniqueness} and BT.

Having clarified the fact that \( f(R) \) gravity naturally admits trivial BH solutions where \( R = \text{const} \) everywhere, notably the ones with \( R = 0 \), and which correspond to the same solutions found in GR except for a trivial redefinition of the constants \( G_0 \) and \( \Lambda \), the main goal of this paper is an effort to find AFSSS solutions where \( R(r) \) is a non-trivial solution of the field equations and where \( r \) is some radial coordinate. This means that if such a solution exists, \( R(r) \) should interpolate in a non-trivial manner between the event (Killing) horizon of the BH and the asymptotic region. If it exists, this is what we might call a \textit{hairy} solution. In order to find such a solution we enforce suitable regularity conditions at the horizon. These conditions provide a very specific form of derivatives of several variables there, notably, the first derivative of the Ricci scalar. These regularity conditions are extremely important as they prevent the presence of BH solutions that are pathological at the horizon. In particular, they prevent a singularity in the scalar-degree of freedom at the horizon, a singularity that would...
be otherwise considered as physical as opposed to a coordinate singularity. In section 4 and in appendix B we provide such conditions in their full form and compare them with similar regularity conditions reported first in [38], but amended in [39]. Examples of such pathologies (singularities) are common in STTs when using conformal methods to generate SSS exact solutions without enforcing the regularity condition of the scalar field at the BH horizon [40]. Thus, the fact that the field equations may not be satisfied at the horizon in those examples casts serious doubts about its relevance as a genuine counterexample to the no-hair conjecture.

3. SSS vacuum solution in \( f(R) \) gravity

The problem of describing a SSS spacetime in vacuum \((T_{ab} = 0)\) within the framework of \( f(R) \) gravity reduces to solving field equations (4) and (3) for the following SSS metric:

\[
\text{ds}^2 = -\left(1 - \frac{2M(r)}{r}\right)e^{2\phi(r)}dt^2 + \left(1 - \frac{2M(r)}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2),
\]

where the mass function \( M(r) \) provides the ADM mass in the asymptotic region provided the spacetime is AF\(^9\). The function \( \delta(r) \) indicates the extent to which the equality \( G^t_t = G^r_r \) is satisfied or infringed by the components of the Einstein tensor, or equivalently, by the corresponding components of the effective energy-momentum tensor given by the right-hand side (rhs) of equation (4) taking \( T_{ab} = 0 \) there [41]. So, if \( \delta = \text{const} \), in particular zero, it means that the equality holds exactly everywhere in the spacetime. This situation includes some of the best known SSS spacetimes. The field equations for the metric equation (8) have been obtained in a rather convenient form in [18] with a slightly different but equivalent parametrization. We present their final form based on those equations but without the matter terms:

\[
R^{tt} = \frac{1}{3f''_{RR}} \left[ \frac{(2f - Rf_r)r}{r - 2M} - 3f'_{RR}R^2 \right] + \left[ \frac{2(rM' - M)}{(r - 2M)r} - \delta' - \frac{2}{r} \right] R',
\]

\[
M' = \frac{M}{r} + \frac{1}{2(2f_R + rR'_{fRR})} \left\{ - \frac{4f_R M}{r} + \frac{r^2}{3} (Rf_r + f) + \frac{r^2}{2} (2Rf_R - f) - \frac{4M}{r} f_R + 2r'_{fRR} \left( 1 - \frac{2M}{r} \right) \right\},
\]

\[
\left( 1 - \frac{2M}{r} \right) \delta' = \frac{1}{2(2f_R + rR'_{fRR})r} \left\{ \frac{2r^2}{3} (2f - Rf_R) + \frac{rR'_{fRR}}{f_R} \left[ \frac{r^2}{3} (2Rf_R - f) - 2f_R \right] - 4(r - 2M)R'_{fRR} + \frac{2(f_R + rR'_{fRR})(r - 2M)R'_{fRR}}{f_R} \right\}.
\]

In order to find BH solutions, equations (9)–(11) have to be solved from the BH horizon at \( r = r_h \) to the asymptotic region which in the AF or anti-de Sitter cases correspond to spatial infinity. On the other hand, in a de Sitter background a cosmological horizon can be reached

\(^9\) Certain spacetimes that are not AF possess a well-defined ‘ADM’ mass. In those cases, the Minkowski background is then replaced by a suitable background spacetime with respect to which the mass indicates deviations as one approaches the asymptotic region (the mass is the part of the ‘monopole’ term in the \( g_{tt} \) component). In those spacetimes the equivalent of the ADM mass corresponds typically to a suitable renormalization of \( M(r \to \infty) \) (see footnote 9).
at some \( r'_h > r_h \). In this paper we shall focus only on AF solutions, therefore we deal only with the BH horizon.

Suitable boundary conditions at \( r_h \) are to be imposed. Typically one assumes \textit{regularity conditions} that enforce the verification of the field equations there (see section 4). For instance, the value \( M'(r_h) = r_h/2 \) enforces the existence of the Killing horizon, where \( r_h \) can have, in principle, any arbitrary non-negative value. The condition \( \delta(r_h) \) is also arbitrary as it simply determines the value of \( \delta(r) \) in the asymptotic region which can be redefined by a suitable change of the \( t \)-coordinate. In other words, the field equations are left invariant if one performs the transformation \( \delta(r) \rightarrow \delta(r) + \text{const} \), a property that holds due to the existence of a static Killing vector field. So without loss of generality we can take \( \delta(r_h) = 0 \).

As concerns \( R(r_h) \), hereafter \( R_h \), this value is usually fixed so as to obtain the desired asymptotic value for \( R \). Typically, but not necessarily (see below), a value \( R(r_{\infty}) = 0 \) (where \( r_{\infty} \) stands for \( r \rightarrow \infty \)) gives rise to AF spacetimes\(^{10}\).

Due to the complexity of the field equations (9)–(11) one usually resorts to a numerical solution, in which case, the value \( R_h \) is fixed by a \textit{shooting method} (see section 5.2). This value, which provides the adequate asymptotic behavior for \( R(r) \), is also related with the convergence of \( M(r) \) to the ADM mass. Equation (10) can be written as in GR in the form

\[
M' = 4\pi r^2 \rho_{\text{eff}},
\]

where \( \rho_{\text{eff}} \) can be read off by comparing with equation (10). In these kind of coordinates the ADM mass \( M_{\text{ADM}} \) for AF spacetimes is given by \( M(r_{\infty}) \), also called the Komar mass [5]. In order for \( M \) to converge to \( M_{\text{ADM}} \) one requires \( \rho_{\text{eff}} \sim 1/r^{2+\epsilon} \) asymptotically with \( \epsilon > 1 \). That is, we require that the effective energy-density falls off faster than \( 1/r^3 \), otherwise \( M(r) \) can diverge asymptotically as \( M(r) \sim \ln(r) \) if \( \epsilon = 1 \) or \( M(r) \sim r^{1-\epsilon} \) if \( \epsilon < 1 \)\(^{11}\). As a consequence, if the spacetime containing a BH possesses a non-trivial Ricci scalar \( R(r) \), this must behave as \( T_{\text{eff}} \sim -\rho_{\text{eff}} \), namely \( R \sim 1/r^{2+\epsilon} \) with \( \epsilon > 1 \) in the AF scenario. Therefore, the shooting method is implemented such that \( R(r) \) behaves asymptotically in the previous manner using \( R_h \) as a control parameter. If no such behavior exists for all \( R_h \), this implies that AF spacetimes with a \textit{non-trivial} Ricci scalar do \textit{not} exist. Thus, scalar hair would be absent and the only possible AF solutions are \( R = 0 \).

There exist spacetimes with asymptotic behaviors different from AF or ADS/AADS with interesting properties. One of them corresponds to a spacetime that is AF except for a deficit solid angle \( 0 < \Delta < 1 \). This kind of spacetime is typically generated by topological defects, like strings and global monopoles [44]. The deficit angle is related with the ‘symmetry breaking’ scale \( \eta \) of the Mexican hat potential by \( \Delta = 8\pi\eta^2 \) [44]. The deficit angle typically produces a repulsion instead of an attraction of test bodies in the gravitational field generated by these defects. In this kind of spacetime the mass function \( M(r) \sim r\Delta/2 \) asymptotically. It is somehow remarkable that such spacetimes can have a well-defined mass [45] which, in the case of spherical symmetry, can be computed, in practice, by a suitable renormalization of \( M(r_{\infty}) \). This situation corresponds to \( \epsilon = 0 \). It is then tantalizing to define the ‘ADM’ mass in this case simply by

\(^{10}\) In anti-de Sitter backgrounds one demands \( R(r_h) = R_l \) with \( R_l < 0 \). For de Sitter backgrounds enforcing the boundary conditions can be more involved since a cosmological horizon at \( r'_h \) (\( r'_h < \infty \) is present, and in this case the value \( R_h \) is to be fixed so as to recover the regularity conditions at \( r'_h \) as well.

\(^{11}\) In the case of anti-de Sitter backgrounds the corresponding ‘ADM’ mass is given by \( M_{\text{ADM}} = M_{\text{Kom}}(r_{\infty}) \) where \( M_{\text{Kom}}(r) = M(r) - \lambda r^3/6 \) (\( \lambda < 0 \)). A rigorous definition for the mass in these kind of backgrounds has been given by several authors [42, 43]. This mass is like the Komar mass [43] but specialized to the SSS scenario. For instance, when the mass function is given by \( M(r) = m + \lambda r^3/6 \) where \( m \) is a constant (like in Kottler–Schwarzschild–de Sitter metrics), \( m \) coincides with the mass found in the formal definitions [42, 43]. In the context of \( f(R) \) gravity \( \rho_{\text{eff}} \sim \Lambda_{\text{eff}}/m \) must behave as \( 1/r^{2+\epsilon} \) with \( \epsilon > 1 \) for \( M_{\text{Kom}}(r_{\infty}) \) to converge.
where $M_{\text{ADM}}(r)$ is the ADM mass of the spacetime. In the next section we shall see that in fact such a mass is not exactly $M_{\text{ADM}}(r)$ but rather proportional to it where the constant of proportionality depends on the deficit angle $\Delta$. Furthermore, $R \sim 2\Delta/r^2$ asymptotically. Hence, we have here an explicit example where the Ricci scalar is not trivial and it also vanishes asymptotically, and yet the spacetime is not AF. Below we present an explicit $f(R)$ model for which an exact SSS of this sort can be found, except that the deficit angle is not produced by a topological defect, but by the underlying modified-gravity model itself.

We finish this section by giving explicitly the kind of trivial solution $R = R_1 = \text{const}$ that we alluded in section 2. For SSS spacetimes we have

$$ds^2 = -\left(1 - \frac{2M_0}{r} - \frac{\Lambda_{\text{eff}} r^3}{3}\right)dt^2 + \frac{dr^2}{1 - \frac{2M_0}{r} - \frac{\Lambda_{\text{eff}} r^3}{3}} + r^2 d\Omega^2,$$

with

$$M(r) = M_0 + \frac{\Lambda_{\text{eff}} r^3}{6},$$

$$R(r) = R_1 = 4\Lambda_{\text{eff}},$$

$$\delta(r) \equiv 0,$$

$$2f(R_1) = R_1 f'_R(R_1).$$

This is the Kottler–Schwarzschild–de Sitter solution, where $M_0$ is a constant of the integration of equation (10) which is identified with the ‘ADM’ mass (see [42, 43]). In the cases where $R_1 \equiv 0$, the solution reduces to the Schwarzschild solution. It is a straightforward exercise to check that equations (14)–(17) exactly solve equations (9)–(11), regardless of the $f(R)$ model, provided the model satisfies equation (17) and $f'_R(R_1) = 0$, $f''_R(R_1) = 0$, $f'''_R(R_1) < \infty$ and $f''''_R(R_1) < \infty$. As we show in the next section, when the $f(R)$ model is given explicitly $R_1$ writes in terms of its fundamental parameters (see equation (32)).

### 3.1. Exact solutions

Let us consider the model

$$f(R) = 2a\sqrt{R - \alpha} = 2a^2 \left(\frac{R}{a^2} - \left(\frac{\alpha}{a^2}\right)\right),$$

where $a > 0$ is a parameter with units [distance]$^{-1}$, and $\alpha$ is another parameter of the model which is related to an effective cosmological constant as we show below. This $f(R)$ model and a variant of it was considered in the past by several authors [26, 27]. In the SSS scenario the metric

$$ds^2 = -\frac{1}{2} \left(1 - \frac{\alpha r^2}{6} + \frac{2Q}{r^2}\right)dt^2 + \frac{dr^2}{\frac{1}{2} \left(1 - \frac{\alpha r^2}{6} + \frac{2Q}{r^2}\right)} + r^2 d\Omega^2,$$

with the mass function, Ricci scalar, and $\delta(r)$ given, respectively, by

$$M(r) = \frac{r}{4} + \frac{\alpha r^3}{24} - \frac{Q}{2r},$$

$$R(r) = \alpha + \frac{1}{r^2},$$

$$\delta(r) \equiv 0.$$
solve equations (9)–(11) exactly, as one can verify by straightforward substitutions. Here, $Q$ is an integration constant. Taking into account electromagnetic and Yang–Mills fields, this solution was extended in [27]. When $\alpha = 0$ this solution was part of a more general class of solutions associated with the model $f(R) = kR^n$ [14, 25]. However, for $n = 1/2$ the asymptotic behavior of those solutions was not analyzed by those authors as we do here.

The coordinates are defined such that, $-\infty < t < \infty$, $0 \leq \theta < \pi$, $0 \leq \phi < 2\pi$ with $r_h \leq r$ if $\alpha < 0$ and $r_h \leq r \leq r_0^+$ if $\alpha > 0$, where $r_h$ and $r_0^+$ correspond to the location of the event and cosmological horizons, respectively, which we analyze below.

The metric (19) possesses a deficit angle $\Delta = 1/2$, $M_{\text{ADM}} \equiv 0$, ‘charge’ $Q$ and a cosmological constant $\Lambda_\infty \equiv R(r_\infty)/4 = \alpha/4$ (see appendix D for more details). In the current case $M_{\text{ADM}} = (1 - \Delta)^{-1/2} M_{\text{ren}}^2 (r_\infty)$ [45] where $M_{\text{ren}}^2 (r) = M(r) - \Lambda_\infty r^2/6 - r \Delta/2$. The divergent terms (linear and cubic in $r$) appear in this renormalization of mass since, as we remarked before, the spacetime has a deficit angle (associated with the linear term) and also a cosmological constant (associated with the cubic term). Using (20) we conclude $M_{\text{ren}}^2 (r) = -Q/(2r)$, thus, $M_{\text{ADM}} = M_{\text{ren}}^2 (r_\infty) \equiv 0$. Examples of spacetimes with a deficit angle and with a zero-mass BH are not new [46].

The metric with a deficit angle given by equation (19) is a solution with a cosmological constant, $\Lambda_\infty = \alpha/4$. For instance, taking $\alpha > 0$ we can introduce $l^2 = 6/\alpha$ for convenience. The location of the BH horizon depends on the value of $Q$. There are three possibilities; these are outlined as follows. (a) If $Q > 0$, $Q = q^2$, the event horizon of the BH is located at $r_h = \frac{1}{\sqrt{2}} \sqrt{2l^2 + 2l^2 \sqrt{1 + 8q^2/l^2}}$. (b) If $Q < 0$ ($Q = -q^2$) and $1 - 8q^2/l^2 > 0$, there are two horizons which are located at $r_h^+ = \frac{1}{\sqrt{2}} \sqrt{2l^2 + 2l^2 \sqrt{1 - 8q^2/l^2}}$, and $r_h^- = \frac{1}{\sqrt{2}} \sqrt{2l^2 - 2l^2 \sqrt{1 - 8q^2/l^2}}$. In particular, for an extreme BH with $q^2 = l^2/8$ the horizon is given by $r_h = |l|/\sqrt{2}$. (c) If $Q = 0$, then $r_h^+ = |l|$.

The AADS spacetime with a deficit angle is a solution with a negative cosmological constant $\Lambda_\infty = \alpha/4$ (with $\alpha < 0$) and $l^2_{\text{AADS}} = -6/\alpha$. Notice that for $Q \equiv 0$ an horizon does not exist like in the usual anti-de Sitter (i.e. anti-de Sitter spacetime without the deficit angle). However, for $Q = -q^2 < 0$ an horizon exists and is located at $r_h = \frac{1}{\sqrt{2}} \sqrt{-2l^2_{\text{AADS}} + 2l^2_{\text{AADS}} \sqrt{1 + 8q^2/l^2_{\text{AADS}}}}$. When $\alpha = 0$, the spacetime turns to be AF except for a deficit angle. In this case only for $Q = -q^2 < 0$ there is an horizon at $r_h = \sqrt{2q^2}$. Finally when $\alpha = Q$, the spacetime is simply the Minkowski spacetime with a deficit angle. All other cases correspond to naked singularities.

In the absence of naked singularities a straightforward calculation of other scalars, like $R_{\text{eff}}$ and $R_{\text{abcd}}$ show that the only physical singularity appears at $r = 0$. Even if $\alpha = 0 = Q$, such scalars are $R_{\text{eff}} = 1/2r^4$ and $R_{\text{abcd}} = 1/r^4$ while $R = 1/r^2$, and thus, the singularity at $r = 0$ is entirely due to the deficit angle. Since the coordinates used so far do not cover the entire manifold one can look for analytic extensions using Kruskal-like coordinates. These extensions and the construction of Penrose diagrams are out of the scope of the current paper and will be reported elsewhere.

Now, for $\alpha < 0$, there is no cosmological horizon, and thus, one can analyze the solution as $r \rightarrow \infty$. It is then interesting to note that the asymptotic value of the Ricci scalar $R(r_\infty) = \alpha$ is not an algebraic solution of $2f(R) - R_{\text{eff}} = 0$ which is $R = 4\alpha/3$ but rather a pole of $f_{RR} = -a[R - \alpha]^{-3/2}/2$. The trivial solution $R = 4\alpha/3$ provides a cosmological constant $\Lambda_{\text{eff}} = R_0/4 = \alpha/3$ which is different from the actual one $\Lambda_\infty = \alpha/4$. Now, if $f_R = a[R - \alpha]^{-1/2}$, $f_{RR} = -a[R - \alpha]^{-3/2}/2$ and $f_{RRR} = 3a[R - \alpha]^{-5/2}/4$ blow up at
$R = \alpha$, the quantities that appear in equation (9) behave well asymptotically (i.e. they are finite) as $R \to \alpha$:

$$\frac{f_{RRR}(\nabla R)^2}{f_{RR}} = \frac{f_{RRR}g^{rr}R^2}{f_{RR}} = -\frac{g^{rr}R^2}{R - \alpha} \sim \frac{\alpha}{2r^2}, \quad (23)$$

$$\frac{2f - Rf_R}{f_{RR}} = -2(R - \alpha)(3R - 4\alpha) \sim \frac{2\alpha}{r^2}. \quad (24)$$

Moreover the quantities that appear in the r.h.s of equation (4) also behave well asymptotically and give rise to the cosmological constant $\Lambda_\infty = \alpha/4$ as we show next. Take for instance the $r - r$ component of equation (4) in vacuum. For our purposes it is more convenient to take the mixed components. Then

$$G'_r = g^{rr}\left(\frac{f_{RRR}\nabla\nabla R}{f_R} + \frac{f_{RRR}R^2}{f_R}\right) - \frac{1}{6}\left(R + \frac{f}{f_R}\right). \quad (25)$$

Now let us analyze the asymptotic behavior of each of the terms at the r.h.s of equation (25):

$$g_{rr} \sim -\frac{12}{\alpha r^2}, \quad (26)$$

$$g^{rr} \sim -\frac{\alpha r^2}{12}, \quad (27)$$

$$\frac{f_{RRR}\nabla\nabla R}{f_R} = \frac{\nabla\nabla R}{2(R - \alpha)} = \frac{-r^2}{2}\nabla\nabla R$$

$$= -\frac{r^2}{2}(R'' - R'\Gamma''_{rr}) = -\frac{r^2}{2}\left(6 \frac{R''}{r^4} - R'\Gamma''_{rr}\right) \sim -\frac{r^2}{2}\left(6 \frac{6}{r^4} - \frac{\alpha}{12}g_{rr}R'ight)$$

$$= -\frac{r^2}{2}\left(6 + \alpha g_{rr} \frac{6}{6r^2}\right) \sim -\frac{r^2}{2}\left(6 \frac{6}{r^4} - \frac{2}{r^4}\right) = \frac{2}{r^2}. \quad (28)$$

$$\frac{f_{RRR}R^2}{f_R} = -\frac{3R^2}{4(R - \alpha)^2} = \frac{3}{r^2}, \quad (29)$$

$$R + \frac{f}{f_R} = R + 2(R - \alpha) = \alpha + \frac{3}{r^2} \sim \alpha. \quad (30)$$

Therefore, we obtain

$$G'_r \sim g^{rr}\left(-\frac{2}{r^2} + \frac{3}{r^2}\right) - \frac{\alpha}{6} \sim -\frac{\alpha}{12} - \frac{\alpha}{6} = -\frac{\alpha}{4}, \quad (31)$$

so the r.h.s of equation (4) is well-behaved asymptotically, and it is just a constant which we can precisely identify with effective cosmological constant $\Lambda_\infty = \alpha/4$, with $\alpha < 0$. Notice that this constant emerged not only from the last two terms at the r.h.s of equation (25) but also from the contribution of the first two; one would naively think that they do not contribute as $R \to \alpha$ asymptotically. However, as mentioned in section 2, a closer look shows that one actually has in those two terms something like $\infty \times 0$ asymptotically. This is why it was necessary to perform the correct asymptotic analysis which leads then to contribution $-\alpha/12$ due to the first two terms of the r.h.s of equation (25). Of course one can perform the same asymptotic analysis in the full set of equations (9)–(11) to find that all of them behave well and consistently as $R \to \alpha$, since from both the left-hand side (l.h.s) and the r.h.s one obtains
exactly the same behavior. This is otherwise expected as we have explicitly the exact solution from which one can compute $M'$ and $R'$ and $R''$ to confirm that nothing diverges as $R \to \alpha$.

The definition of this cosmological constant $\Lambda_\infty = \alpha/4$ is consistent with the canonical form that the metric coefficients $g_{\mu\nu}$ and $g_{\mu\nu}$ take in (19) in these coordinates. For instance, in terms of $\Lambda_\infty$ they read $\mathfrak{g}_a = 1/g_{aa} = -\left(1 - \Delta - \frac{\Lambda_\infty}{3} + \frac{Q^2}{\mathfrak{f}^4} \right)$ where $\Delta = 1/2$.

Hence, we conclude that when $f_{RR}$ has a pole precisely at the anti-de Sitter point, the cosmological constant $\Lambda_\infty$ does not arise simply from the last term of equation (25), like in the analysis performed in section 2 where $\Lambda_{\text{eff}} = R_0/4$. That analysis was valid provided that as $R \to R_0$ the following two necessary conditions were satisfied: $R_0 = 2f'(R_0)/f_{RR}(R_0)$ and $f_{RR}(R_0) < \infty$, which as emphasized above, is not the actual case for this exact solution.

Finally, we mention that for $\alpha \equiv 0$, which corresponds to a null cosmological constant, the quantity $(2f - R'_R)/f_{RR} = -6R^2$ that appears in equation (9) vanishes as the solution approaches the asymptotic value $R = 0$, even if $f_{RR} \to \infty$. Moreover, the term $f_{RRR}/f_{RR} = -3R^2/(2R)$ also vanishes asymptotically since $R' \sim 1/r^3$ and $R \sim 1/r^2$. Therefore, a posteriori one can understand why equation (9) is well-behaved asymptotically.

It is important to stress that in the previous works [14, 25, 26] the above physical and geometric interpretation of metric (19) was completely absent and therefore, its meaning was rather unclear. In the non-vacuum case, some but not all of the aspects discussed above for this exact solution were elucidated [27].

So far we have mainly discussed the case $\alpha < 0$. Regarding $\alpha > 0$, the Ricci scalar will tend to $\alpha$ asymptotically but will never reach this value since well before $r \to \infty$ the cosmological horizon is reached by the solution. Therefore, $R'(r_0) = 0$ and $R''(r_0) = 0$. That is, the possible solution $R \to \alpha$ with $R' = 0 = R''$ is never reached asymptotically due to the presence of the cosmological horizon.

Now, as concerns the trivial solution $R = R_0 = \text{const}$, model (18) admits the solution $R_1 = 4\alpha/3$, which solves $R_1 = 2f'(R_1)/f_{RR}(R_1)$. Notice that $f_{RR}(R_0) = -\alpha[3/4]^{3/2}/2$. Therefore the SSS solution is given by the metric (13) where

$$\Lambda_{\text{eff}} = \frac{R_0}{4} = \frac{\alpha}{3}. \quad (32)$$

In this case the cosmological constant is given by $\Lambda_{\text{eff}}$, instead of $\Lambda_\infty = \alpha/4$.

We emphasize that solution (13) is not approached asymptotically by solution (19), since as we mentioned before, the latter has a deficit angle $\Delta = 1/2$, while (13) has $\Delta = 0$. In summary, model (18) admits the two exact solutions (19) and (13), one where the Ricci scalar is constant everywhere and one where it varies with the radial coordinate $r$. However, it is important to remark that both solutions are not two different solutions with the same boundary conditions, but two different solutions with different boundary conditions. Even if we put $M_0 \equiv 0, Q \equiv 0$, one solution still has a deficit angle while the other does not. Moreover, both solutions have different cosmological constants $\Lambda_{\text{eff}}$ and $\Lambda_\infty$, and so the spacetimes are not even the same asymptotically when $\alpha = 0$. On the other hand, one could try to make the inner boundaries (i.e. the event horizons) of both solutions artificially coincide as well as the value $R_0$ by fixing $Q$ and $M_0$, but still both spacetimes would have different global quantities (mass and ‘charge’) besides the respective deficit angles ($\Delta = 0$ and $\Delta = 1/2$) and the respective cosmological constants ($\Lambda_{\text{eff}}$ and $\Lambda_\infty$). Namely, one solution would have zero ‘ADM’ mass and non-zero ‘charge’, while the other solution would have non-zero ‘ADM’ mass and zero ‘charge’.

Two final remarks are necessary. The first one concerns the deficit angle solution (19). It is possible to show that in fact one can obtain the $f(R)$ model (18) by using a kind of ‘reconstruction method’. This consists of imposing a solution in the form of equations (19) with (20)–(22). That is
with \( g_{rr} = -1/g_{tt} \) and with \( M \) and \( R \) written in terms of finite powers of \( r \). The condition \( g_{rr} = -1/g_{tt} \) leads then to \( \delta(r) = \text{const} \), i.e. \( \delta'(r) \equiv 0 \), which in turn provides a differential equation for \( f(R) \) in terms of \( r, R(r) \) and \( M(r) \). However, given such a power expansion, one can invert and write \( r = r(R) \), to obtain a differential equation for \( f(R) \) and \( R \) solely, which when solved provides (18). This kind of ‘trick’, which is frequently used to find exact solutions in GR with exotic sources [47], has been applied in modified theories of gravity in the past [26, 27]. Physically, one usually proceeds in the opposite way we just described. That is, one constructs or proposes an explicit \( f(R) \) model in order to fit some observations, for instance, in cosmology, and then one asks if such a model admits or not an exact solution in one or other scenario. In this regard, and as a second remark related with our comment at the end of the previous section, we mention that almost all physically-viable \( f(R) \) models admit the trivial solutions \( R = R_i = \text{const} \) given by the Schwarzschild–de Sitter/anti-de Sitter solution equations (13)-(16), where the parameter \( \alpha \) is replaced by a different constant according to the different \( f(R) \) models. For instance, in section 5.1 we will see more specific examples. These are the kind of trivial solutions that we alluded to in section 2: several authors have systematically reported them as something ‘special’ in \( f(R) \) gravity, while we see that they are nothing more than the usual solution found in GR with \( \Lambda \to \Lambda_{\text{eff}} = R_i/4 \) and \( G_0 \to G_{\text{eff}} = G_0/R(R_i) \). In particular the mass parameter \( M_0 \) can be redefined as \( M_0 = G_{\text{eff}} M \) where \( M \) can be taken as the fiducial mass associated with the spacetime. Moreover, the RN or the Kerr–Newman BH solutions with or without \( \Lambda_{\text{eff}} \) can also be found in \( f(R) \) gravity when considering matter with a traceless energy-momentum tensor as these also correspond to the same trivial solutions \( R = R_i = \text{const} \), including the AF ones when \( R_i = 0 \).

We remark that there are more exact solutions for other choices of \( n \) in the model \( f(R) = kR^n \) [14, 25], not only for \( n = 1/2 \), which corresponds to the model we have just analyzed taking \( \alpha \equiv 0 \). Nevertheless, most of those solutions are still trivial or have exotic asymptotics. In section 5.1 (see model 1) we discuss the case \( n = 2 \), which was not covered in [14, 25], as their equations become singular precisely for \( n = 2 \).

Finding physically interesting exact BH solutions different from the usual ones with \( R = \text{const} \) proves to be difficult when the \( f(R) \) model is complicated, like the physically-viable models that have passed many cosmological and Solar-System tests (e.g. models 4–5 of section 5.1); this is particularly the case if one is interested in genuine AF or ADS/AADS type of spacetimes. In such an instance, one has then to appeal to a numerical analysis. In this regard, we stress that we have used the two exact solutions presented in this section as a testbed for a Fortran code developed to solve numerically equations (9)-(11) for more complicated and ‘realistic’ \( f(R) \) models, submitted to suitable boundary (regularity) conditions that represent the presence of a BH (i.e. when an horizon is present). These regularity conditions are presented next.

4. Regularity conditions

In order to obtain the regularity conditions at the horizon \( r = r_H \), whether \( r_H \) is the event (inner) horizon (denoted by \( r_h \)) or the (outermost) cosmological horizon (denoted by \( r'_c \)) we expand the variables as follows:

\[
F(r) = F(r_H) + (r - r_H)F'(r_H) + \frac{1}{2}(r - r_H)F''(r_H)
+ \frac{1}{6}(r - r_H)^3F'''(r_H) + \mathcal{O}(r - r_H)^4,
\]

(33)
where \( F(r) \) stands for \( M(r) \), \( R(r) \) and \( \delta(r) \). When replacing these expansions in equations (9)–(11) and demanding that the derivatives of these variables are finite at the horizon one obtains after long but straightforward algebra the following regularity conditions:

\[
R'_{\mid r=r_{H}} = \left. \frac{2r(Rf_{R} - 2f)f_{R}}{r^{2}(2Rf_{R} - f) - 6f_{R} f_{fR}} \right|_{r=r_{H}}, \tag{34}
\]

\[
M'_{\mid r=r_{H}} = \left. \frac{r^{2}(2Rf_{R} - f)}{12f_{R}} \right|_{r=r_{H}}. \tag{35}
\]

where as stressed, all the quantities in the previous two equations are to be evaluated at the horizon \( r_{H} \), which in principle is any positive value. In this paper we will only be interested in finding AF solutions, and thus, the spacetime will contain only the event (inner) horizon.

Hereafter the quantities evaluated at the horizon will be written with a subindex ‘\( H \)’.

It is important to mention that regularity conditions of this kind were proposed first in [38], and then rectified in [39]. Those authors did not use exactly the same differential equations (10) and (11) that we used here, which were a consequence of equations (4) and (3). Instead, they departed from the much more involved field equation (2). Nevertheless our regularity conditions (34) and (35) are equivalent to those of [39].

In order to illustrate the consistency of these two regularity conditions, we take three \( f(R) \) models and their exact solutions. First, \( f(R) = R - 2\Lambda \), which corresponds to GR plus a cosmological constant. The SSS solution is like in (13) but with \( \Lambda \) instead of \( \Lambda_{\text{eff}} \). In this case equation (34), when multiplied by \( f_{RR} \), gives an identity \( 0 \equiv 0 \), since the numerator at the r.h.s \( Rf_{R} - 2f = R - 4\Lambda \equiv 0 \). On the other hand, since \( R = 4\Lambda = \text{const} \) then \( R' \equiv 0 \). Therefore, \( R'_{f_{RR}} \equiv 0 \). In contrast, the r.h.s of equation (35) yields \( M'_{H} = \Lambda r_{H}^{2}/2 \), which corresponds precisely to the de Sitter/anti-de Sitter value obtained from equation (20).

The second case corresponds to the trivial constant solution discussed in previous sections: \( R(r) = R_{t} = \text{const} \), \( R'(r) \equiv 0 \), where \( R_{t} \) is given by \( (Rf_{R} - 2f)_{R_{t}} = 0 \), such that \( f_{RR}(R_{t}) \equiv 0 \), \( f_{RR}(R_{t}) < \infty \) and \( f_{RR}(R_{t}) < \infty \). Then equation (34) gives \( R'_{H} \equiv 0 \), which is clearly compatible with the trivial solution, whereas the r.h.s of equation (35) yields \( M'_{H} = R_{t} r_{H}^{2}/8 \), where \( R_{t} = 4\Lambda_{\text{eff}} \).

A less trivial test to our regularity conditions is provided by the exact solution given by equations (20)–(22). For that solution it is easy to verify that both sides of equation (34) give \( R'_{H} = -\frac{2}{r_{H}^{2}} \) while equation (35) yields \( M'_{H} = \frac{1}{4} + \frac{a r^{2}}{8} + \frac{Q}{4 r_{H}^{4}} \) in both sides.

Now, equations (9)–(11) do not provide the regularity conditions for \( R' \) and \( \delta' \) at the horizon. We need to differentiate equations (9) and (11) with respect to \( r \) one more time. When doing so and when replacing the expansions in the form of equation (33) it is possible to obtain the values \( R'_{H} \) and \( \delta'_{H} \). This is a lengthy but otherwise straightforward calculation. In appendix B we provide the explicit expressions.

12 If one takes \( r_{H} = 0 \), the regularity conditions correspond to a spacetime with a regular origin. In this case, \( R'_{\mid r=0} = 0 \), \( M'_{\mid r=0} = 0 \), and \( M'_{\mid r=0} = 0 \). The so-called solitons are localized field configurations that are globally regular, including the origin. In GR there exist examples of SSS spacetimes with matter that allows for SSSAF solitons and a SSSAFBH with hair, like in the Einstein–Yang–Mills system [8, 36] and in the Einstein-(real)scalar-field system with an asymmetric scalar-field potential [9]. In the axisymmetric and stationary case the Einstein-(complex)scalar-field system admits solitons (i.e. boson stars) [37] and hairy BHs [11]. However, in the spherically symmetric case only globally regular boson stars are allowed.
5. NHTs and scalar-tensor approach to \( f(R) \) theories

As remarked in the introduction, there are very well-known NHTs for the Einstein-scalar-field system, hereafter termed the Einstein-\( \phi \) system (i.e. Einstein–Hilbert gravity minimally coupled to a real scalar field \( \phi \)) when the potential associated with a scalar field \( \phi \) verifies the non-negativity condition \( \mathcal{V}(\phi) \geq 0 \). On the other hand, it is also well known that provided \( f_{R\phi} > 0 \) and \( f_{\phi} > 0 \) one can map \( f(R) \) into a STT by defining the scalar \( \chi = f_R \). This is the so-called JF representation of the \( f(R) \) theory. Under this framework and in the presence of matter, the scalar field \( \chi \) turns out to be coupled non-minimally to the curvature but minimally to matter. A further conformal transformation allows one to write the theory in the so-called EF, where a new scalar field \( \phi \) with a new potential \( \mathcal{V}(\phi) \), is now coupled minimally to the curvature but non-minimally to the matter sector. Notwithstanding, in the absence of matter, and still under the EF, the theory looks exactly like an Einstein–\( \phi \) system, and thus, one can appeal to the NHTs and verify if they apply or not to the \( f(R) \) model at hand. Since we are interested precisely in finding SSS BHs in vacuum, we can thus exploit this equivalence to see if hair is absent or if it can exist. Thus, given a specific \( f(R) \) model, we carry out the following protocol. (1) We check if the model admits a ‘trivial’ solution \( R = 0 \) (leading to the Schwarzschild solution). If the model does, then it means that the model is a priori capable of allowing AF hairy solutions (i.e. solutions where \( R \) would be a non-trivial function of \( r \) that interpolates from the event horizon, with the value \( R_h \), to spatial infinity where \( R = 0 \)). Many \( f(R) \) models used as geometric dark energy also admit the trivial solution \( R = 0 \). Thus, in this paper we focus only on such kind of models and analyze if geometric hair can also exist. In a future work, we shall analyze the existence of hair in ADS or AADS BH solutions. (2) We write the model as a STT in the EF. If the potential \( \mathcal{V}(\phi) \) is not negative, then the NHTs apply and we conclude that AFSSS hairy solutions cannot exist in such model. In fact, if \( \mathcal{V}(\phi) \) turns to be strictly positive, the Schwarzschild solution does not even exist, only the de Sitter type of solutions exist at best. (3) If the potential has negative branches then we proceed to analyze if hairy solutions exist by solving numerically the field equations of section 3 under the regularity conditions provided in section 4 and appendix B.

In order to fix the ideas, let us now briefly review the different scalar-field transformations required to properly analyze an \( f(R) \) model under the STT approach.

Let us consider for simplicity the following gravitational action without matter

\[
I_{\text{grav}} = \int d^4x \sqrt{-g} \frac{1}{2\kappa} [f_Q(Q)(R - Q) + f(Q)],
\]

(36)

where at this point \( Q \) is an auxiliary scalar field that depends on \( R \) and \( f_Q = df/dQ \).\(^{13}\) Now, if \( f_{QQ}(Q) \neq 0 \) (in particular \( f_{QQ}(Q) > 0 \)) the variation of this action with respect to the metric gives rise to field equations equivalent to the original theory (1) in vacuum provided \( Q \equiv R \). Thus the action (36) is equivalent to (1) if we enforce \( Q = R \) (see appendix C for a further discussion).

Furthermore, one can introduce the scalar field \( \chi := f_Q(Q) \) and write a dynamically-equivalent action as

\[
I_{\text{EF}} = \int d^4x \sqrt{-g} \left( \frac{1}{2\kappa} R - \chi^2 V(\chi) \right),
\]

(37)

\(^{13}\) Do not confuse the field \( Q \) with the parameter appearing in the metric (19).
where \( V(\chi) \) is defined as follows:
\[
V(\chi) = \frac{1}{2\kappa\chi} [Q(\chi)\chi - f(Q(\chi))].
\] (38)

By *dynamically equivalent*, we mean that we obtain field equations that are completely equivalent to the original action (see appendix C). To do so, we can treat the scalar field \( \chi \) as metric-independent, and \( R \) as independent of \( \chi \), and thus take the action as a functional of both \( g_{ab} \) and \( \chi \). Thus, variation of this action with respect to the metric leads to field equations equivalent to equation (2), while variation with respect to \( \chi \) simply leads to \( R = Q \). We see then that the action (37) is equivalent to the action of a Brans–Dicke-like theory, with a Brans–Dicke parameter \( \omega_{BD} = 0 \) (implying that the kinetic term associated with the gradients of \( \chi \) is absent) and with a potential \( U(\chi) = 2\kappa\chi^2V(\chi) \) (see appendix C for details). Clearly a STT of this sort (i.e. one with \( \omega_{BD} = 0 \)) but without a potential would be incompatible with the bound \( \omega_{BD} < 4 \times 10^4 \) [48], which is required for the theory to pass the Solar-System tests, and thus, would be automatically ruled out. However, the presence of this potential makes it possible for certain \( f(R) \) models to pass those tests even if \( \omega_{BD} = 0 \). This depends if the potential allows for the emergence of the *chameleon* mechanism (see [49, 50]), but not all the potentials have this property.

While we will consider some \( f(R) \) models that seem to pass such tests thanks to the chameleon mechanism, our main purpose in this paper is to analyze the issue about the existence or absence of hairy BH solutions in such models rather to test their observational viability. Thus, there are models that we use only for that purpose and which may fail the observational tests.

As mentioned before, the energy-momentum tensor of matter is conserved in the JF (see appendix A), and therefore it is with respect to this frame that point-test particles follow geodesics.

Furthermore one can define a new scalar field and a conformal metric as follows:
\[
\phi = \sqrt{\frac{3}{2\kappa}} \ln \chi.
\] (39)
\[
\tilde{g}_{ab} = \chi g_{ab} = e^{\sqrt{2\kappa}\phi} g_{ab}.
\] (40)

In terms of these new variables the gravitational action (37) takes the form [56] \(^{14}\)
\[
I_{EF} = \int d^4x \sqrt{-\tilde{g}} \left[ \frac{1}{2\kappa} \tilde{R} - \frac{1}{2} \tilde{g}^{ab} (\tilde{\nabla}_a \phi)(\tilde{\nabla}_b \phi) - \mathcal{U}(\phi) \right],
\] (41)

where all the quantities with a *tilde* are defined with respect to the metric \( \tilde{g}_{ab} \). This corresponds precisely to the Einstein–Hilbert action coupled minimally to a scalar field \( \phi \). Notice that, unlike the JF action, a kinetic term appears due to the conformal transformation.

The action (41) is written in what is known as the EF, where the potential \( \mathcal{U}(\phi) \) is defined as \( \mathcal{U}(\phi) = V(\chi[\phi]) \), which will be given explicitly when the model \( f(R) \) be provided (see section 5.1). If we include the matter action then the scalar field \( \phi \) will be coupled non-

\(^{14}\) In the action (41) we have omitted in the integral the term \( \sqrt{V} \mathcal{C}^\alpha \mathcal{C}_\alpha \phi \), which can be converted into a surface term, that we assume to vanish.
minimally to the matter fields. In this paper we are only interested in the vacuum case\textsuperscript{15}, so the field equations obtained from the action (41) are simply those of the Einstein-\(\phi\) system:

\[
\tilde{G}_{ab} = \kappa T_{ab}^0, \quad (42)
\]

\[
T_{ab}^0 = (\tilde{\nabla}_a \phi)(\tilde{\nabla}_b \phi) - \tilde{g}_{ab} \left[ \frac{1}{2} \tilde{g}^{cd}(\tilde{\nabla}_c \phi)(\tilde{\nabla}_d \phi) + \mathcal{U}(\phi) \right]. \quad (43)
\]

\[
\Box \phi = \frac{d \mathcal{U}}{d \phi}. \quad (44)
\]

As we emphasized previously, based on this equivalence and in view of our interest in finding SSS and AF non-trivial BHs in \(f(R)\) gravity, we have to take into account the NHTs [20, 21] which are valid when \(\mathcal{U}(\phi) \geq 0\)\textsuperscript{16}. The theorems roughly establish that whenever the condition \(\mathcal{U}(\phi) \geq 0\) holds, given an AFSSS spacetime containing a BH (with a regular horizon) within the Einstein-\(\phi\) system, the only possible solution is the hairless Schwarzschild solution. Here by hairless we mean that the scalar field \(\phi(r) = \phi_0\), i.e. the scalar field is constant everywhere in the domain of outer communication of the BH and it is such that \(\mathcal{U}(\phi_0) \equiv 0\), in order to prevent the presence of a cosmological constant which would spoil the AF condition.

The NHTs can be avoided if the potential has negative branches, notably at the horizon [9, 10]. So in our case, given an \(f(R)\) model, we have only to check if the corresponding potential satisfies or not the condition \(\mathcal{U}(\phi) \geq 0\). In the affirmative case, we conclude that SSS and AF hairy BHs are absent. Nevertheless, when this condition fails, one usually needs to resort to a numerical treatment in order to analyze if a BH can support scalar hair or not.

Before concluding this section, two final remarks need to be outlined. The first one, concerns the transformation from the Jordan to the EF. The equivalence (i.e. one-to-one correspondence) between both approaches is guaranteed provided the transformations (39) and (40) between the corresponding variables are not singular (e.g. if \(f_R > 0\)) and also in the domains where \(f_{RR} > 0\). These conditions are not always met in several \(f(R)\) models, and in the event that this happens, we turn to a numerical analysis in the original JF and avoid the EF approach. We elaborate more about this in the following sections when we describe the specific \(f(R)\) models that we analyze.

The second remark is that there is an important relationship between the critical points of the potential \(\mathcal{U}(\phi)\), notably the extrema, and those of the ‘potential’ \(\varphi(R)\) defined such that [18] (see also appendix C)

\[
\varphi(R) = \frac{d \varphi}{d R} = \frac{1}{3} (2f - R f'_R). \quad (45)
\]

In section 2 we denoted the extrema of \(\varphi(R)\) by \(R_1\). It is straightforward to verify

\[
\frac{d \mathcal{U}}{d \phi} \left( \frac{d \chi}{d \phi} \right) \left( \frac{d V}{d \chi} \right) = \frac{1}{f'_R} \left( \frac{3}{2 R} \right)^{1/2} \left( \frac{2 f - R f'_R}{3} \right)^{1/2} \left( \frac{3}{2 R} \right)^{1/2} \frac{1}{f'_R} \frac{d \varphi}{d R}. \quad (46)
\]

Therefore, provided \(0 < f'_R(R_1) < \infty\) (i.e. the conditions for a well-defined map to the EF), we see that the extrema of \(\mathcal{U}(\phi)\) correspond precisely to \(R_1\). However, care must be taken when \(f'_R(R_1) = 0\) or \(f'_R(R_1) = \infty\), as may happen in some models that we will encounter in the next section. When this happens, the conformal transformation (40) becomes singular or ill defined as \(\chi = f_R\).

\textsuperscript{15} See [57] for a thorough discussion of STT in the EF with the presence of matter.

\textsuperscript{16} The theorems actually account for multiple scalar fields.
Finally, for \( f(R) \) models where \( f_{RR} \) is not strictly positive, notably, where \( f_{RR} \) can vanish at some \( R = R_n \), called weak singularity (see model 5 in section 5.1) it will be useful to introduce the ‘potential’ \( \Psi(R) \) defined via

\[
\Psi(R) = \frac{d\Psi}{dR} = \frac{2f - Rf_R}{3f_{RR}} = \frac{\Psi'(R)}{f_{RR}} .
\]

(47)

The finite or divergent behavior of \( \Psi(R) \) also provides an insight about \( \Psi(R_n) \). For instance, if \( \Psi(R_n) \) is finite, it means that \( \Psi'(R_n) \) vanishes like \( f_{RR} (R_n) \). Furthermore, \( \Psi(R) \) can supply further information about the possible trivial solutions \( R = \text{const} \). As we discussed in sections 2 and 3.1, at \( R = R_2 \) where \( f_{RR} (R_2) = \infty \) and \( \Psi(R_2) \) vanishes, \( R = R_2 = \text{const} \) can be one trivial solution of equation (3) in vacuum or more generally, when the matter has a traceless energy-momentum tensor, a solution that can be different from \( R = R_i \) if \( \Psi'(R_i) \neq 0 \).

5.1. \( f(R) \) models and the NHTs

In this section we focus on some \( f(R) \) models that also admit the trivial solution \( R = 0 \), and check whether or not they satisfy the condition \( \Psi(\phi) \geq 0 \) for which the NHTs apply. The results are summarized in table 1 at the end of this section. In order to obtain the EF potential \( \Psi(\phi) \) from \( V(\chi) \) let us recall the relationships

\[
\chi = e^{\sqrt{2} \phi} ,
\]

(48)

\[
\Psi(\phi) = V(\chi(\phi));
\]

(49)

the first one is obtained from equation (39), while the second one is a definition.

**Model 1**: \( f(R) = \lambda_n \left( \frac{R}{R_n} \right)^n \), where \( n, \lambda_n \) and \( R_n \) are positive parameters of the theory. \( R_n \) fixes the scale for each \( n \) and \( \lambda_n \) can be chosen to be proportional to \( R_n \). This model has been thoroughly analyzed in the past in several scenarios (see [29] and references therein). For this model,

\[
\chi = f_R = \frac{n \lambda_n}{R_n} \left( \frac{R}{R_n} \right)^{n-1} .
\]

(50)

If we focus on the domain \( R \in [0, \infty) \), then for \( n > 1, \chi \in [0, \infty) \). In fact the value \( \chi = 0 \) corresponds to a degenerate situation \( \rho_{\text{eff}} \rightarrow \infty \) which we discuss below. For \( 0 < n < 1, \chi \rightarrow \infty \) as \( R \rightarrow 0 \) and vice versa. So, \( \chi \in (0, \infty) \) for \( 0 < n < 1 \).

By inverting the relationship (50) and using equation (38) followed by the use of equations (48) and (49), the two potentials read

\[
V(\chi) = \frac{(n - 1)}{2n} \left( \frac{R_n}{n \lambda_n} \right)^{1/n} \chi^{2/n + 1} \qquad (n \neq 1) ,
\]

(51)

defined for \( \chi > 0 \), and

\[
\Psi(\phi) = \frac{(n - 1)}{2n} \left( \frac{R_n}{n \lambda_n} \right)^{1/n} e^{\left( \frac{2\phi}{n} \right) + \frac{1}{n^2}} \qquad (n \neq 1) ,
\]

(52)

defined for \( -\infty < \phi < +\infty \). The condition \( \Psi(\phi) > 0 \) holds if \( n > 1 \), while for \( n < 1 \), the potential is \( \Psi(\phi) < 0 \); in both cases, however, the potential does not have minima, at least not for a finite \( \phi \). Moreover, we do not consider \( n < 0 \) because \( f_R < 0 \) which can give rise to an effective negative gravitational constant and the EF transformation is not defined in that
Table 1. \( f(R) \) models and their corresponding scalar-field potentials in the EF. The last column indicates if the NHTs apply (✓) or not (×).

| Model | Potential in EF | Properties | No-hair theorems |
|-------|----------------|------------|------------------|
| \( f(R) = \lambda (\frac{x}{m})^n \) | \( \Psi(\phi) = \frac{a - \beta}{2b} \left( \frac{m}{R} \right)^{n-1} e^{\left( \frac{2x}{e^{m/R}} \right)} \) | \( \Psi(\phi) > 0 \), \( R_+ > 0 \), \( \lambda > 0 \), \( n > 1 \) | ✓ |
| \( f(R) = R + c_2 \left( \frac{x}{m} \right)^2 \) | \( \Psi(\phi) = \frac{c_2}{c_2} \left( 1 - e^{-\frac{2x}{c_2}} \right)^2 \) | \( \Psi(\phi) \geq 0 \), \( c_2 > 0 \) | ✓ |
| \( f(R) = R + c_2 \left( \frac{x}{m} \right)^2 \) | \( \Psi(\phi) = \frac{c_2}{c_2} \left( 1 - e^{-\frac{2x}{c_2}} \right)^2 \) | \( \Psi(\phi) \leq 0 \), \( c_2 < 0 \) | × |
| \( f(R) = R - c_2 \ln \left( 1 + \frac{x}{m} \right) \) | \( \Psi(\phi) = \frac{c_2}{c_2} \left( 1 - e^{-\frac{2x}{c_2}} \right)^2 \) | \( \Psi(\phi) > 0 \), \( 0 < c_2 < 1 \), \( R_+ > 0 \) | ✓ |
| \( f(R) = R - R_+ \lambda_1 \left( 1 - e^{-\frac{x}{m}} \right) \) | \( \Psi(\phi) = \frac{c_2}{c_2} \left( 1 - e^{-\frac{2x}{c_2}} \right)^2 \) | \( \Psi(\phi) > 0 \), \( 0 < \lambda_1 < 1 \), \( R_+ > 0 \) | ✓ |
| \( f(R) = R + \lambda_2 R_+ \left( 1 + \frac{x}{m} \right)^4 - 1 \) | (ill defined: multivalued) | — | — |
| \( f(R) = R - R_+ \left( \frac{x}{m} \right)^2 \) | (ill defined: multivalued) | — | — |

In the last column, ✓ indicates that the NHTs apply, and × indicates that they do not. The potential \( \Psi(\phi) \) is given by the equation in the second column for each model. The properties of the solution \( \Psi(\phi) > 0 \), \( R_+ > 0 \), and \( \lambda > 0 \) are checked in the third column, and the application of the no-hair theorems is indicated in the last column. 

For \( n = 1 \), corresponding to GR, \( V(\chi) \equiv 0 \equiv \Psi(\phi) \) as one can see directly from equation (38). For this model the NHTs \textit{a priori} apply since \( \Psi(\phi) > 0 \). The fact that the potential is strictly positive and have no minima implies that the solution \( R = 0 \) cannot even exist in the EF. The point is that for \( 0 < n < 1 \) the solution \( R = 0 \) corresponds to \( \chi \rightarrow \infty \) (\( \phi \rightarrow -\infty \)), while for \( n > 1 \) the same solution corresponds to \( \chi \equiv 0 \equiv R \) (\( \phi \rightarrow -\infty \)). We see then that in both cases the mapping to the STT in the EF is ill defined precisely at \( \chi = 0 \) where \( \tilde{g}_{\text{ab}} = 0 \), while \( \tilde{g}^{\text{ab}} \rightarrow 0 \) as \( \chi \rightarrow \infty \). This problem at \( \chi = 0 \) exacerbates for \( n = 2 \).
which we discuss below. We conclude that for this \( f(R) \) model AFSSS simply cannot exist under the EF. In the original formulation, the theory also degenerates at \( R = 0 \) for \( n > 1 \) since \( f_{R}(0) = 0 \). AFSSS solutions exist but they are not unique as we are about to see.

As we remarked briefly at the end of section 2, for this class of \( f(R) \) model, a quite degenerate situation may occur. To elaborate, let us focus on the case \( n = 2 \) in the original formulation, since in the STT approach the maps break down at \( R = 0 \) as we just mentioned, given that \( \chi = f_{R} = \text{const} \times R \). So in this case the field equations (4) and (3) in vacuum reduce to

\[
G_{ab} = \frac{1}{R} \left[ \nabla_{a} \nabla_{b} R - \frac{g_{ab}(R^{2})}{4} \right],
\]

\[
\Box R = 0,
\]

where we used \( f_{RR} \equiv 0 \), \((Rf_{R} + f)'/f_{R} = 3R/2 \) and \((2f - Rf_{R})/f_{RR} = 0 \) in equations (53) and (54).

Therefore we see that \( R = 0 \) is a trivial solution of equation (54). On the other hand, for such a trivial solution equation (53) is satisfied for any \( G_{ab} = 0 \) compatible with \( G_{ab}^{\nu} = -R = 0 \). For instance, this can be satisfied for any solution of the metric satisfying the Einstein equation \( G_{ab} = \kappa T_{ab} \), with \( T_{ab}^{\nu} = 0 \). This degeneracy is peculiar as it shows that solutions of the field equations in \( f(R) \) gravity may not be unique, as illustrated by this simple model. In the AFSSS scenario, one BH solution is clearly the Schwarzschild solution, but other solutions are possible [14, 35]; it would be interesting to know to what kind of matter content they correspond in pure GR. In this example, the AFSSSBH solutions are unique as concerns the solution \( R = 0 \), since at the horizon \( R = 0 = R' \), but they are highly non-unique as concerns the metric. That is, to the trivial solution \( R = 0 \) of equation (9), one can associate any solution for the metric satisfying \( G_{ab} = \kappa T_{ab} \), with \( T_{ab}^{\nu} = 0 \), like the Schwarzschild solution, a solution within the Einstein–Yang–Mills system, etc.

As emphasized in [14, 35], this \( f(R) \) model provides a specific example showing that a generalization of BT similar to the one elucidated in the introduction, simply cannot exist in general. It is important to stress that this degenerate situation in vacuum is in a way similar, but opposite, to the ‘degeneracy’ that appears in GR with matter sources: given \( f(R) = R \), equation (2) or equation (4) reduce to the Einstein field equation, whereas (3) reduces to \( f_{RR} \Box R \equiv 0 \), with \( f_{RR} \equiv 0 \). This means that this equation is satisfied identically regardless of the value \( \Box R \), which in general, is not zero because \( R = -\kappa T \). In other words, in GR the metric is constrained to satisfy Einstein’s field equation, but \( R \) is not constrained to satisfy any differential equation like equation (2).

Finally we mention that when \( R = \text{const} = R_{0} \), the model with \( n = 2 \) also admits trivial solution equations (13)–(17). Incidentally, for this model the algebraic condition (17) is satisfied for any \( R_{0} \). That is, \( R_{1} \) emerges as an integration constant independent of the parameters of the model. Therefore the value \( \Lambda_{\text{eff}} = R_{0}/4 \) depends on the assigned value for \( R_{0} \). In particular, taking \( R_{0} = 1 \) we just recover the usual Schwarzschild solution as mentioned above.

For any other \( n > 1 \) the model admits the trivial solution \( R = 0 \), but the solution for the metric is not unique either as the degeneracy emerges as well in a similar way to the case \( n = 2 \).

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18 For this \( f(R) \) model \((2f - Rf_{R})/f_{RR} = (n - 2R^{2})/(n - 1) \), and \((Rf_{R} + f)/f_{RR} = (n + 1)R/n \). Thus we can expect degenerate solutions (in the sense described in the main text) for \( n > 2 \) as well.

19 For \( n = 2 \) the potential \( \phi(\phi) \) is constant in the EF, thus the trivial solution \( \phi = \text{const} \) gives rise to the Schwarzschild–de Sitter solution just like in the original variables.
Model 2: $f(R) = R + c_2 R_l (R/R_l)^2$, where $c_2$ is a positive dimensionless constant and $R_l$ is a positive parameter that fixes the scale. This model was proposed by Starobinsky as an alternative to explain the early inflationary period of the Universe [2]. For this model,
\[
\chi = f_R = 1 + 2c_2 R_l/R_f .
\]
In principle the model is defined for $-\infty < R < \infty$. However, if we impose $\chi = f_R > 0$, then we require $R > -R_l/(2c_2)$. In fact if we allow $\chi \leq 0$ the transformation to STT in the EF is not well defined (see equation (48)), and the model degenerates at $\chi = 0$ in the original variables as $G_{\text{eff}} \rightarrow \infty$. If we focus on solutions with $R \geq 0$ then $\chi \geq 1$. Furthermore, $f_R$ and $f_{RR}$ are both finite at $R = 0$ for any $c_2 \in \mathbb{R}$.

Proceeding like in the previous model, this one has the following potentials associated with it:
\[
V(\chi) = \frac{R_l}{8c_2} \left( \frac{\chi - 1}{\chi} \right)^2 ,
\]
\[
\mathcal{V}(\phi) = \frac{R_l}{8c_2} \left( 1 - e^{-\sqrt{c_2} \phi} \right)^2 .
\]
The potential $\mathcal{V}(\phi)$ is defined for $-\infty < \phi < \infty$. The region $\phi \geq 0$ corresponds to $\chi \geq 1$, while $-\infty < \phi < 0$ corresponds to $0 < \chi < 1$.

Clearly $\mathcal{V}(\phi) \geq 0$, with a global minimum located at $\phi = 0$ where the potential vanishes (see figure 1). The model admits the solution $R = 0$ corresponding to $\chi = 1$ and $\phi = 0$. The trivial solution $R = 0$ is the only root of $f_R(R) = R/3$. For this model the NHTs apply, and therefore AFSSSBH hairy solutions are absent.

Although the model is defined for $c_2 > 0$ in order to be compatible with inflation, in the context of BHs one can in principle consider $c_2 < 0$ as a way to evade the NHTs because then $\mathcal{V}(\phi) \leq 0$ (in that instance the condition $\chi > 0$, implies $R < R_l/(2|c_2|)$ and the global minimum becomes a global maximum). It turns out, however, that potentials that are negative around a maximum but vanish there (in this case $\mathcal{V}(0) = 0$) sometimes admit non-trivial solutions that seem hairy and AF\textsuperscript{20}. We shall discuss a numerical example of this kind later, but suffice to say that the BH solutions that one finds may vanish asymptotically but can have an oscillatory behavior that make them not genuinely AF. In order to illustrate this, suppose that asymptotically the metric component $g_{rr}$ behaves like $g_{rr} \sim 1 + 2C_1 r^2 \sin(C_2 r)/r$ (where $\sigma$ and $C_{1,2}$ are some constants, and $0 \leq \sigma < 1$), then the mass function $M(r) \sim C_1 r^2 \sin(C_2 r)$ oscillates (it may even diverge if $\sigma \neq 0$), and thus, it does not really converge to a finite value in the limit $r \rightarrow \infty$, a value that one would identify with the ADM mass. Yet $g_{rr} \rightarrow 1$ as $r \rightarrow \infty$. Thus, for this kind of solution the spacetime is not authentically AF. These arguments can be justified using the following heuristic analysis. Let us consider equation (44) and neglect the non-flat spacetime contributions from the metric. Moreover, expanding $\mathcal{V}(\phi)$ around its maximum (which is equivalent to expand $-\mathcal{V}(\phi)$ around its minimum) gives $\mathcal{V}(\phi) = -m^2 \phi^2/2$. With these simplifying assumptions it is easy to see that the SSS solution of equation (44) is
\[
\phi = \phi_0 \frac{\sin(x + \phi_0)}{x} .
\]
\textsuperscript{20} In more general scenarios such potentials lead to instabilities because perturbations around the maximum lead to growing modes. Thus the trivial solution $\phi = 0$ would be unstable and never settle into a stationary configuration.
where $x = mr$, and $x_0, \phi_0$ are constants. Now, when $r \to \infty$, we can take the conformal factor $\chi = 1$ in equation (40) and both metrics (the JF and the EF metrics) coincide asymptotically. Thus the energy-density contribution $\mathcal{E}$ is given by

$$
\frac{\widetilde{\mathcal{E}}}{\phi} = \frac{\phi_0^2 \cos(x_0)}{r^2}.
$$

which is not positive definite. As a consequence, the mass function is not positive definite either. In fact, the mass function behaves asymptotically as $M(r) \sim \phi_0^2 \sin(2(x + x_0))/m$, which oscillates with $r$ and does not converge to a definite value (the Komar mass). As we mentioned above, this heuristic analysis confirms the behavior of the full numerical solution for potentials of this kind. One such sort of solution is shown for model 4 in section 5.2 below.

**Model 3:** $f(R) = R - \alpha_1 R_a \ln \left(1 + \frac{\mu}{R_a}\right)$, where $\alpha_1$ is a dimensionless constant, and $R_a$ is a positive parameter that fixes the scale. This model was proposed by Miranda et al [17] as a possible solution for the accelerated expansion of the Universe while being free of singularities during the cosmic evolution. This model is also viable for constructing relativistic extended objects. Notwithstanding, it has problems at the level of the Solar System and cosmic structure formation [51]. Although it may not be a realistic model in all the scenarios, it is worthy of consideration in this context due to its simplicity. The model is defined for $R > -R_a$ and the condition $f_R > 0$ restricts $R$ in the range $R > R_a(\alpha_1 - 1)$. In particular for $R_a$ and $\alpha_1$ positive, as we assume for this model, whenever $f_R > 0$ is satisfied the condition $R > -R_a$ is also satisfied. Furthermore, for $\alpha_1 > 0$ and $R$ in the above range, $f_{RR} = \alpha_1 R_a^{-1} (R/R_a + 1)^2$ is strictly positive and it is finite at $R = 0$, as well as $f$ and $f_R$. For this model

$$
\chi = f_R = 1 - \frac{\alpha_1 R_a}{R_a + R}.
$$

The domain $R_a(\alpha_1 - 1) < R < \infty$ corresponds to $0 < \chi < 1$, and to $-\infty < \phi < 0$. The potentials are
associated with model 3 for different values of \( \alpha_1 \) for \( 0 < \alpha_1 \leq 1 \).

For such values of \( \alpha_1 \), the potential \( \mathcal{V}(\phi) \) is strictly positive. In particular for \( \alpha_1 = 1 \) the potential vanishes when \( \phi \rightarrow -\infty \). For \( \alpha_1 > 1 \) the potential has a global minimum at \( \phi_{1,\text{min}} \) which leads to a de Sitter type of solution \( \phi(r) = \phi_{2,\text{min}} \) in the EF and \( R(r) = \text{const} \) in the JF. Middle panel: potential \( \mathcal{U}(\phi) \) for five values of \( \alpha_1 \) in the interval \( 0 < \alpha_1 \leq 1 \). For \( 0 < \alpha_1 < 1 \), the potential satisfies \( \mathcal{U}(\phi) \geq 0 \) and it vanishes at the global minimum \( \phi_{2,\text{min}} \) leading to the Schwarzschild solution where \( \phi(r) = \phi_{2,\text{min}} \) in the EF and \( R(r) = \text{const} \) in the JF (see the right panel). Right panel: the function \( d\mathcal{U}/dR \) is depicted. The zeros of this function at \( R = 0 \), which are mapped to \( \phi_{2,\text{min}} \) in the EF, lead to the Schwarzschild solution. The zeros at \( R = -1 \) (in units of \( R_{s} \)) in principle lead to a Schwarzschild–anti-de Sitter solution, but \( f \rightarrow \infty \) and \( f_{R} \rightarrow -\infty \), \( f_{RR} \rightarrow \infty \), \( f_{RRR} \rightarrow -\infty \) there.

The solution \( R = 0 \) corresponds to \( \chi = 1 - \alpha_1 \) in this model. However, for \( \alpha_1 \geq 1 \) one is led to \( \chi \leq 0 \), which by construction is not allowed in the EF frame (see equation (48)). Therefore this solution cannot be present in that frame for those values of \( \alpha_1 \). In fact, for such values of \( \alpha_1 \) the potential \( \mathcal{U}(\phi) \) has a minimum at some \( \phi_{1,\text{min}} \), but \( \mathcal{U}(\phi_{1,\text{min}}) \geq 0 \) (see figure 2). According to our protocol, a strictly positive potential cannot allow for the Schwarzschild solution. Therefore, such a solution cannot be recovered from the EF approach. As a consequence, the NHTs also rule out the existence of AFSSS hairy BH solutions. In the original variables the trivial solution can be recovered since \( R = 0 \) is a root of \( \gamma_{R}(R) \), but then the effective gravitational constant \( G_{\text{eff}} \) becomes negative, as the condition \( f_{R} > 0 \) fails at \( R = 0 \).

Now, for \( \alpha_1 = 1 \), \( \mathcal{U}(\phi) \rightarrow 0 \) if \( \phi \rightarrow -\infty \), which is the minimum, a situation similar to model 1. Finally, for \( 0 < \alpha_1 < 1 \), the potential satisfies \( \mathcal{U}(\phi) \geq 0 \) (see the middle panel of figure 2). In this case the NHTs also apply, and the Schwarzschild solution in the EF is associated with a local minimum \( \phi = \phi_{2,\text{min}} \) at which \( \mathcal{U}(\phi_{2,\text{min}}) = 0 \). The trivial solution \( \phi(r) = \phi_{2,\text{min}} \) is associated with the trivial solution \( R = 0 \).

The minimum at \( \phi_{1,\text{min}} \) corresponds to one of the roots \( R_{i} = 0 \) of \( \gamma_{R}(R) \), which leads to a Schwarzschild–de Sitter type of solution.
The overall conclusion for this model is that hair is in general forbidden, i.e. non-trivial AFSSSBH solutions \( R(r) \) attempting to interpolate between \( R_h \) and \( R = 0 \) cannot exist if \( f_R > 0 \). More specifically: (1) for \( \alpha_1 \geq 1 \) the EF representation precludes the presence of hair in a AFSSSBH according to the solution \( R = 0 \) does not even exist as the potential \( \mathcal{U}(\phi) \) never vanishes (the potential is only defined for \( f_R > 0 \)). If in the original JF variables one permits the possibility of having negative values for \( f_R \) (which in turn implies negative values for \( G_{\text{eff}} \)), then the trivial solution \( R = 0 \) may exist. But then when looking for a hairy solution that interpolates from \( R_h \) to \( R = 0 \), the solution may cross the value \( f_R = 0 \) if \( f_R|_{R_h} > 0 \), a value which is associated with the singularity \( G_{\text{eff}} = G_{0|f_R} \to \infty \). Such a hairy solution would be rather pathological if it exists at all. On the other hand, if we impose the condition \( f_R < 0 \), in order to avoid that singularity, then \( R \) turns out to be restricted in the range \( -R_h < R < R_d(\alpha_1 - 1) \). In particular one would require \( \alpha_1 > 1 \) to allow for the solution \( R = 0 \) to exist. This range for \( R \) severely restricts the region in which one can look for an optimal shooting \( R_h \) (see section 5.2), unless \( 1 \ll \alpha_1 \). All in all, the numerical exploration shows that hairy solutions seem to be absent in all these scenarios. (2) For \( 0 < \alpha_1 < 1 \) the NHTs apply straightforwardly, and therefore hairy solutions cannot exist either. Unlike the previous subclass \( (\alpha_1 \geq 1) \), the solution \( R(r) = 0 \) corresponding to \( \phi(r) = \phi_{2,\text{min}} [\mathcal{U}(\phi_{2,\text{min}}) = 0] \) can also be recovered in the EF since for this range of values of \( \alpha_1 \) the condition \( f_R(0) > 0 \) is fulfilled.

We turn now our attention to three \( f(R) \) models that have been analyzed recently in cosmology and which are some of the most successful ones as concerns the cosmological and the Solar-System tests. However, as we show below, for two of these models (i.e. models 5 and 6) the potential \( \mathcal{U}(\phi) \) is not even well defined as it is multivalued. Therefore, the conclusion about the absence or existence of hair in these two models has to be obtained numerically using the original formulation (sections 2 and 3) as opposed to the EFSTT approach.

**Model 4:** \( f(R) = R - R_c \lambda_c (1 - e^{-\frac{R}{R_c}}) \) where \( \lambda_c \) is a dimensionless constant that usually is taken to be positive for a successful phenomenology and \( R_c \) is a positive parameter that fixes the scale. For instance, \( \lambda_c > 0 \) ensures \( f_{RR} > 0 \) and \( f_{RR} \) never vanishes at a finite \( R \) regardless of the sign of \( \lambda_c \). This exponential model has been analyzed in the past by several authors [28, 52]. For this model the scalar field \( \chi \) is

\[
\chi = f_R = 1 - \lambda_c e^{-\frac{R}{R_c}}. \tag{63}
\]

The condition \( f_R > 0 \) holds provided \( R > R_c \ln \lambda_c \), in which case \( 0 < \chi < 1 \), where \( \chi \to 1 \) as \( R \to \infty \). In particular, for \( 0 < \lambda_c \leq 1 \) the condition \( f_R > 0 \) is satisfied if \( R > -R_c[\ln \lambda_c] \). The potential \( \mathcal{V}(R) \) has only a global minimum at \( R = 0 \), i.e. \( \mathcal{V}_R(0) = 0 \) [28], and thus, it leads to the Schwarzschild solution (see the right panel of figure 3). For \( \lambda_c > 1 \) the potential \( \mathcal{V}(R) \) has a local maximum at \( R = 0 \), and the potential develops in addition a local minimum at some \( R < 0 \) and a global minimum at some \( R \geq 0 \) [28]. These extrema correspond to trivial solutions \( R = R_t \) associated with the Schwarzschild–anti-de Sitter and Schwarzschild–de Sitter solutions respectively. Notice, however, that \( f_R(0) < 0 \) since the condition \( R > R_c \ln \lambda_c \) fails at \( R = 0 \). Therefore, in this case the Schwarzschild solution has \( G_{\text{eff}} < 0 \). Since the mapping \( \chi \to \phi \) is defined only for \( \chi > 0 \), we conclude that for \( \lambda_c > 1 \) the condition \( R > R_c \ln \lambda_c \) implies \( R > 0 \) for the EFSTT approach to be well defined and thus, like in the previous model, one cannot recover the solution \( R = 0 \). As a consequence, we require the original formulation of the theory to analyze if a hairy solution can exist for these values of \( \lambda_c \). But, again, a hairy solution, if it exists, can encounter the singularity at \( f_R(0) < 0 \) before approaching the asymptotic value \( f_R(0) > 0 \), notably if \( f_R|_{R_h} > 0 \).
The potentials are
\[
V(\chi) = \frac{R_e}{2\kappa\chi^2} \left[ (\chi - 1)\ln \left( \frac{\lambda_\epsilon}{1 - \chi} \right) + \lambda_\epsilon + \chi - 1 \right].
\]
\[
\mathcal{U}(\phi) = \frac{R_e}{2\kappa} e^{-2\int \sqrt{V}} \left[ e^{\int \sqrt{V} \phi} - 1 \right] \ln \left( \frac{\lambda_\epsilon}{1 - e^{\int \sqrt{V} \phi}} \right) + \lambda_\epsilon + e^{\int \sqrt{V} \phi} - 1.
\]
which are valid in the domain $0 < \chi < 1$ and $-\infty < \phi < 0$, respectively. The potential $\mathcal{U}(\phi) \geq 0$ is depicted in figure 3 (left panel) for various values of $\lambda_\epsilon > 0$. For this model the NHTs apply. As emphasized above, for $\lambda_\epsilon \geq 1$ the potential $\mathcal{U}(\phi)$ is strictly positive, in particular its minimum, and therefore the corresponding solution $R = 0$ cannot be recovered from the EFSTT approach, but only the de Sitter type of solution $R = R_0 = const$ which is associated with the minimum of $\mathcal{U}(\phi)$. On the other hand, for $0 < \lambda_\epsilon < 1$ the potential vanishes at its minimum which leads to the solution $R = 0$ that allows one to recover the Schwarzschild solution.

If for a moment we dismiss the condition $f_{RR} > 0$ and consider $\lambda_\epsilon < 0$ we can evade the NHTs because now $\mathcal{U}(\phi)$, defined for $0 < \phi < +\infty$, is never positive (see the middle panel of figure 3). The new domain is a consequence of equation (63) which yields
\[ \chi = 1 + [\lambda c] e^{-\frac{c}{2}}. \] Thus for \(-\infty < R < +\infty\) the scalar field \(\chi\) is defined in the domain \(1 < \chi < +\infty\), which in turns leads to \(0 < \phi < +\infty\). The potentials depicted in figure 3 (middle panel) suggest that hairy solutions might exist. For instance, in the JF variables, such a solution \(R(r)\) would interpolate between the horizon \(R_h\) and its value \(R = 0\) at spatial infinity. It turns out that indeed such a solution can be found numerically (see the next section) but the spacetime is not authentically AF as discussed above within the framework of model 2. The mass \(M(r_c)\) never converges to a well-defined value as the mass function behaves asymptotically \(M(r) \sim r^2 g(r)\), where \(0 < \sigma \leq 1\) and \(g(r)\) is an oscillating but presumably a bounded function (see figure 8).

**Model 5:** \(f(R) = R + \lambda_5 R^2 \left[ \left(1 + \frac{c}{R} \right)^{-q} - 1 \right]\) where \(\lambda_5\) is a dimensionless constant, \(q\) a dimensionless parameter and \(R_s\) is a positive parameter that fixes the scale. This model was proposed by Starobinsky [53] as a mechanism for generating the late accelerating expansion while satisfying several local observational tests. We analyzed this and model 3 in the past in the cosmological setting [58] and for constructing star-like objects [18] using the approach of section 2. In this paper we take \(\lambda_5 = 1\) and explore several values of \(q\) (see section 5.2). For this model the conditions \(f_{RR} > 0\) (for a positive \(G_{\text{eff}}\) and \(f_{RR} > 0\) do not hold in general. In fact, \(f_{RR}\) vanishes at \(R = R_0^\pm = \pm R_s/\sqrt{2q} + 1\). Since \(f_{RR}\) appears in the denominator of equation (9), the vanishing of \(f_{RR}\) was termed by Starobinsky a weak singularity. One can appreciate these features from figure 4 (right panel) where the ‘potential’ \(\Psi' (R)\) is depicted. We see that \(|\Psi'(R)| = \infty\) at \(R_0^\pm\) where \(f_{RR}\) vanishes. Thus, the weak singularities at \(R_0^\pm\) cannot be ‘cured’ by the term \(2f - R f_{RR}\) in \(\Psi'(R)\) because such term does not vanish there, and which otherwise could have lead to a finite \(\Psi'_R\). Therefore any solution \(R(r)\) intending to interpolate between \(R_0\) and \(R = 0\) such that \(R_0 > R_0^\pm\) or \(R_0 < R_0^\pm\) will irremediably encounter the weak singularities at \(R_0^\pm\) where we expect a singular behavior in equation (9). As a consequence, our search for numerical BH solutions with non-trivial \(R\) was limited mostly in the range \([R_0] < R_0^\pm\) (see section 5.2).

For this model the potential \(\Psi'(R)\) has several extrema (see the middle panel of figure 4), in particular, a global minimum at \(R = 0\) with \(f_{RR}(0) = 1\), \(f_{RR}(0) = -2\lambda_5 q/R_s\), which allows one to recover the Schwarzschild solution. Notice that the global minimum at \(R = 0\) corresponds to the global maximum of \(\Psi' (R)\). This is because \(\frac{d^2 \Psi' }{dR^2} |_{R=0} = f_{RR}^{-1} \frac{d^2 f}{dR^2} |_{R=0} \) and \(\frac{d^2 \Psi' }{dR^2} |_{R=0} \) is positive, whereas \(f_{RR}(0) < 0\), and so \(\frac{d^2 \Psi' }{dR^2} |_{R=0} \) is negative. On the other hand \(\Psi' (R)\) at \(R_0^\pm\) is well behaved there. The other extrema, a local maximum and minimum, lead to two Schwarzschild–de Sitter solutions with positive \(R = \text{const}\).

Now, the inversion \(R = R(\chi)\) required to recover the potential \(V(\chi)\) and then the potential \(\Psi'(\phi)\) demands \(f_{RR} > 0\) or \(f_{RR} < 0\). That is, the inversion is possible when \(\chi = f_{R}\) is a monotonic function of \(R\), which is not the case for this model. In principle one could perform the inversion piecewise in very specific domains of the model but not in all the domains where the model is defined. In view of this drawback the potential \(\Psi'(\phi)\) is not well defined. In fact it is multivalued as we are about to see. Its expression cannot be given in closed form but only in parametric representation through the equations

\[ 27 \]

\[ 22 \] In the cosmological scenario one usually aims at a de Sitter ‘point’ \(R_t = 0\) (as opposed to the Minkowski ‘point’ \(R_t = 0\)) in order to recover an effective cosmological constant asymptotically (in time), and thus, to mimic the dark energy. In that scenario the actual numerical solution \(R(t)\) is always positive and larger than \(R_0^\pm\), thus, the solution never crosses the weak singularity [58]. Something similar takes place for model 6.
The form of the potential $U_f$ is shown in figure 4 (left panel). Given that $U_f$ is not single valued it is a priori unclear how to establish a method to solve the differential equations in the EFSTT approach and decide unambiguously which value of $U_f$ to assign for a given $f$. Hence we conclude that one cannot obtain any rigorous result from this frame using this potential, let alone trying to implement the NHTs. But even if we tried to do so, the lower branch of the potential does not satisfy the condition $U_f'(R) > 0$ required by the theorem to prevent the existence of hair. In view of this, any strong conclusion about the existence or absence of hair must be obtained from the original formulation of the theory that was presented in section 3. Furthermore, due to the complexity of the model itself and of the differential equations, a numerical analysis is in order. In the next section we provide the numerical results that show evidence about the absence of hairy AFSSSBHs in this model.
Model 6: $f(R) = R - R_{HS} \frac{c_1 \left( \frac{R}{R_{HS}} \right)^n}{c_2 \left( \frac{R}{R_{HS}} \right)^{n-1} + 1}$, where $c_1$ and $c_2$ are two dimensionless constants, and like in previous models, $R_{HS}$ fixes the scale. This model was proposed by Hu and Sawicki [49], and it is perhaps one of the most thoroughly studied $f(R)$ models (see [58] for a review). In the cosmological context, $c_1$ and $c_2$ were fixed as to obtain adequate cosmological observables, like the actual dark and matter content in the Universe. For instance taking $n = 4$, their values are $c_1 \approx 1.25 \times 10^{-3}$, and $c_2 \approx 6.56 \times 10^{-5}$ [58]. Notice that model 5 with $q = 1$ and this model with $n = 2$ are essentially the same. Like in the previous model, the conditions $f_R > 0$ and $f_{RR} > 0$ are not met in general, therefore, the potential $\mathcal{V}(\phi)$ is multivalued and has negative branches as well. It can be plotted using a parametric representation as in model 5:

$$\chi(R) = f_R = 1 - \frac{nc_1 \left( R/R_{HS} \right)^{n-1}}{1 + c_2 \left( R/R_{HS} \right)^{n-1}}$$

(69)

$$\phi(R) = \frac{3}{2\kappa} \ln \chi(R)$$

(70)

$$\mathcal{U}(\phi(R)) = V(\chi[\phi(R)])$$

(71)

Figure 5 depicts the potential $\mathcal{U}(\phi)$ (left panel) where one can appreciate the pathological features. In fact, in this model a weak singularity $f_{RR} = 0$ is located precisely at $R = 0$, i.e. the value that $R(r)$ should reach asymptotically in the AF scenario, and it is also the value corresponding to the (hairless) Schwarzschild solution that we should be able to recover. Nevertheless, and unlike model 5, this singularity in equation (3) or in equation (9) disappears for some values of $n$ because $f_R$ is finite or vanishes at $R = 0$ in this model. Namely, $f_R$ vanishes at $R = 0$ for $0 < n \leq 2$. Thus, in this range of $n$, the model admits the trivial solution $R = 0$. We do not consider the case $n = 0$ as this model reduces to $f(R) = R + \text{const}$, which amounts to GR plus a cosmological constant. For $n = 3$ we find $\mathcal{U}(0) = -R_{HS}^3/(18c_1)$, which does not even vanish. Therefore, this means that $R = 0$
does not solve equation (9), either trivially or asymptotically. For \( n > 2 \) and \( n \neq 3 \), there is indeed a weak singularity at \( R = 0 \) where \(|\mathcal{W}_R| = \infty\) (see the right panel of figure 5). For \( n < 1 \) the quantity \( f_R(0) \) blows up, thus we consider only \( n \geq 1 \), notably, for the numerical analysis of section 5.2.

In this model the potential \( \mathcal{V}(R) \) has a minimum at \( R = 0 \) for any \( n > 0 \), which allows for the Schwarzschild solution whenever \( f_{RR} = 0 \) at \( R = 0 \) (i.e. for \(|n| \leq 2\)). However, when \( f_{RR} = 0 \) at \( R = 0 \) and for which \(|\mathcal{W}_R(0)| = \infty\) the Schwarzschild solution may not even exist. In those situations, non-trivial solutions where \( R(r) \) vanishes asymptotically will encounter such singularity (see section 5.2).

Like in model 5, any analysis using the ill-defined potential \( \mathcal{U}(\phi) \) for the Hu–Sawicki model is not robust. We then turn to a numerical analysis using the original formulation of the theory. This is presented in the next section.

5.2. Numerical analysis and the quest for hairy solutions

As we discussed in the previous section, in some circumstances it is possible to formulate the original \( f(R) \) model as a STT in the EF where the scalar field is coupled minimally to the EF metric but is subject to a potential \( \mathcal{U}(\phi) \). If this potential verifies the condition \( \mathcal{U}(\phi) > 0 \), then the NHTs apply and, at least in the region where \( f_R > 0 \) and \( f_{RR} > 0 \), we can assert that hair (where \( \phi(r) \) or equivalently \( R(r) \) are not trivial solutions) is absent, in which case, the only possible AF solutions are at best \( \phi(r) = \text{const} \) and \( R(r) = 0 \). This conclusion follows for models 1–4 in the sectors where their parameters allow for the \( R = 0 \) solution and led to \( \mathcal{U}(\phi) > 0 \). On the other hand, we mentioned that models 2 and 4 can have potentials \( \mathcal{U}(\phi) \) with negative branches if we allow for the parameters \( c_2 \) and \( \lambda_4 \) to be negative. Negative values of such parameters are not usually considered in cosmology, but for the sake of finding hairy solutions, we can contemplate them. Because the NHTs do not apply when \( \mathcal{U}(\phi) \) is negative, notably at the horizon, the problem of hair reopens when this happens. In this regard, several strategies are available to solve it: (1) show an explicitly exact AFSSSSBH solution with hair; (2) prove analytically the absence of it (i.e. extend the NHTs); (3) show numerical evidence about one or the other.

Given that the differential equations presented in section 3 are very involved, strategies 1 or 2 might lead to a dead end, thus we opted for option three. In particular, this strategy seems even the most adequate as concerns models 5 and 6, where the potential \( \mathcal{U}(\phi) \) is not even well defined.

We proceed to solve numerically equations (9)–(11) subject to the regularity conditions at the horizon provided in section 4 and in appendix B. The only free conditions are the values \( r_h \) and \( R_h \). The methodology is roughly as follows. One starts by fixing the size of the BH \( r_h \), and then looks for \( R_h \) so that \( R \to 0 \) as \( r \to +\infty \). This ‘boundary-value’ problem is solved using a shooting method [55] within a Runge–Kutta algorithm. We have implemented a similar methodology for constructing star-like objects in \( f(R) \) gravity in the past [18]. Numerical solutions with non-trivial hair with asymmetric (non-positive definite) potentials have been found previously within the Einstein–\( \phi \) system using similar techniques [9]. As we will see in the next section, for certain \( f(R) \) models it is not even necessary to perform a shooting as the dynamics of \( \dot{R} \) naturally drives \( R \to 0 \) asymptotically for a given \( R_h \).

Now, as we mentioned previously, for the AF solutions to exist, it is not sufficient that \( R \to 0 \) as \( r \to +\infty \). In section 3.1 we analyzed one exact solution where this happens precisely, and yet, the solution is not AF but has a deficit angle. In that case the mass function \( M(r) \) diverges at least linearly with \( r \). It is then crucial to ensure that the mass function
converges to a constant value (that we assume to be the Komar or, equivalently, the ADM mass) in order to claim for a genuinely AF solution.

As a matter of fact, we used that exact solution as a testbed for our code. That is, we took the model equation (18) as input and recovered numerically the exact solution provided by equations (19)–(22), notably for $a = 0$. Notice that in this case $R(r)$ is not trivial but grows linearly with the coordinate $r$ due to the deficit angle. Right panel: metric components $-g_{rr}$, $g_{rr}$ and their product $-g_{rr} \times g_{rr} = e^{2\alpha(r)}$ ($\alpha(r) \equiv 0$ in the exact solution). In the middle and right panels the exact and numerical solutions are superposed as well (see figure 7).

Figure 6. Left panel: ricci scalar $R(r)$ (the exact and numerical solutions are superposed) computed using the model equation (18) for $\alpha = 0$ (i.e. null cosmological constant). The Ricci scalar is not trivial and vanishes asymptotically, however, the spacetime is not exactly AF but has a deficit angle (see equations (19)–(22)). At the horizon $r_h$ the Ricci scalar satisfies the regularity conditions. Middle panel: the mass function $M(r)$ is not constant but grows linearly with the coordinate $r$ due to the deficit angle. Right panel: metric components $-g_{rr}$, $g_{rr}$ and their product $-g_{rr} \times g_{rr} = e^{2\alpha(r)}$ ($\alpha(r) \equiv 0$ in the exact solution). In the middle and right panels the exact and numerical solutions are superposed as well (see figure 7).

As a matter of fact, we used that exact solution as a testbed for our code. That is, we took the model equation (18) as input and recovered numerically the exact solution provided by equations (19)–(22), notably for $\alpha = 0$. Notice that in this case $R(r)$ is not trivial. Figure 6 depicts the analytic and the numerical solutions superposed, showing an excellent agreement between the two. Typical numerical errors are depicted in figure 7.

We also checked that the trivial solutions $R = R_1 = \text{const}$ that exist in several of models 1–5 were recovered numerically when starting with $R_0 = R_1$ and which lead to the hairless Kottler–Schwarzschild–de Sitter solutions, including the plain AF Schwarzschild solution when $R_1 \equiv 0$.

Additionally we devised other internal tests to verify the consistency of our code. These tests are similar to those implemented in our analysis of star-like objects [18], and are independent of the fact that exact solutions are available or not.

Let us turn our attention to the specific models that deserved a detailed numerical exploration.

**Model 4:** We consider model 4 with $\lambda_c < 0$. In this sector of $\lambda_c$ the potential $\Psi(\phi)$ is not positive definite and the model may admit hairy solutions because the NHTs do not apply. Notwithstanding, the only solutions with a non-trivial Ricci scalar $R(r)$ that we find numerically are not exactly AF. The Ricci scalar vanishes asymptotically in an oscillating fashion as $r \to +\infty$, but the mass function $M(r)$ does not converge but oscillates as well and grows unboundedly as $r^\sigma$ with $0 < \sigma < 1$ (see figure 8). This behavior is similar to the one provided by the heuristic analysis within model 2, except that here we take into account the full system of equations. Despite such behavior the metric components, which depend on $M(r)$, remain bounded in the asymptotic region. This can be partially understood by looking to $g_{rr} = 1 - 2M(r)/r$, and realizing that for $r_h \ll r$ the non-oscillating part of this metric component behaves like $r^{\sigma-1}$. So if $\sigma \leq 1$, that part of $g_{rr}$ may converge to 1 very slowly, so slowly that one cannot even notice it by looking at the numerical outcome.
This behavior seems to be generic for any \( \lambda_e < 0 \) and any \( R_h \). The conclusion is that we do not find any genuinely AFSSSBH solution in this model.

Finally, let us focus on models 5 and 6 that led to pathological potentials in the EFSTT description.

**Model 5:** For the Starobinsky model we limit our search for a shooting value \( R_h \) first in the region \( 0 < R < R_h/\sqrt{2q + 1} \) and then in \( -R_h/\sqrt{2q + 1} < R_h < 0 \) in order to avoid crossing the weak singularities at \( \pm R_h/\sqrt{2q + 1} \) when \( R(r) \) tries to reach the asymptotic value \( R = 0 \). We never found a successful shooting parameter leading to an authentic AF solution. Two examples of this kind of solution are depicted in figures 9 and 10. Figure 9 shows that for \( q = 2 \) the solutions are similar to the exponential model 4 with \( \lambda_e = -3 \) depicted in figure 8. Thus, the asymptotic behavior does not correspond to an AF spacetime.

For \( q = 4 \), we find situations where the Ricci scalar decreases monotonically to a constant value without oscillating as one can see in the left panel of figure 10. However, this constant is not
related with the trivial solution $R = R_0 = \text{const}$ which is the solution of the algebraic equation $2f(R_0) - R_0 f'_0(R_0) = 0$. In fact, what happens is that $M(r) \to -\infty$ as $r \to \infty$, as we can see from the middle panel of figure 10, and also $M(r)/r \to -\infty$; therefore, by looking at equation (9), we appreciate that the combination $[2f(R) - R f'_R] / (1 - 2M/r)$ goes to zero even if $2f(R) - R f'_R \neq 0$. In this way, we see that $R \to 0$ and $R'' \to 0$, while $R \to \text{const}$ asymptotically, which solves equation (9). This behavior of $M(r)$ explains why the metric components vanish asymptotically (see the right panel of figure 10). Before vanishing we see that $g_{rr} = 1$ at $\log_{10}(r/r_0) \approx 0.6$ precisely where $M(r) = 0$. At this value of $r$, the component $g_{tt} = -e^{2\chi(r)}$. Thus, we conclude that the AF behavior is not recovered either for this and other values $q > 0$ and different $R_0$. 

Figure 9. Examples of numerical solutions for the Starobinsky model 5 with $q = 2$ and $\lambda_1 = 1$. Left panel: ricci scalar for three values of $R_0$ (including $R_0 = 0$ leading to the trivial solution $R(r) \equiv 0$). $R(r) \to 0$ asymptotically, however, for the non-trivial solutions (dotted lines) the mass function $M(r)$ does not converge (see the middle panel). Middle panel: mass function $M(r)$ associated with the solution of $R$ with $R_0 = 0.2$ as shown in the left panel. Similar plots for $M(r)$ are found, but not depicted, for the other non-trivial solution of $R$, while $M(r) = \text{const} = r_0/2$ when $R(r) \equiv 0$, corresponding to the Schwarzschild solution. Right panel: metric components associated with the solution of middle panel. The metric components $-g_{tt}$ and $g_{rr}$ and their product $-g_{tt} \times g_{rr} = e^{2\chi(r)}$ oscillate as $r \to +\infty$ corroborating that the resulting spacetime is not AF. Here $R$ is given in units of $R_{ST}$, and $M$ and $r$ in units of $1/R_{ST}$ ($G_0 = c = 1$).

Figure 10. Example of a numerical solution for the Starobinsky model 5 with $q = 4$ and $\lambda_1 = 1$. Left panel: the numerical solution shows that the spacetime is not AF as $R(r)$ does not vanish asymptotically. It has rather a spurious de Sitter behavior when $R(r) \to \text{const} > 0$ asymptotically. Middle panel: the mass function $M(r)$ does not converge asymptotically but decreases to a very large negative value. This is even opposite to the growing behavior $M(r) \sim r^3 > 0$ that should be expected if the spacetime were genuinely ADS. Right panel: the metric components $-g_{tt}$, $g_{rr}$ and their product $-g_{tt} \times g_{rr} = e^{2\chi(r)}$ vanish asymptotically. This behavior confirms that the spacetime is not even genuinely ADS where a cosmological horizon is expected at $r_h > r_0$ where $g_{tt}(r_0) = 0$ and $g_{rr}(r_0) = +\infty.$
In summary, we find strong numerical evidence that for the Starobinsky model 5 AFSSSBHs with geometric hair do not exist. This conclusion is obtained by changing the parameters in several combinations as well as the values $R_h$.

**Model 6:** The numerical analysis of the Hu–Sawicki model 6 requires more care because $f_{RR}$ vanishes at $R = 0$ for several values of $n$ ($|n| > 2$). The weak singularity where $\Psi(R) = (2f - R f'_R)/f_{RR}$ diverges can be reached where $f_{RR} = 0$, except if $2f - R f'_R$ vanishes at the same $R$. In fact for $|n| > 2$ and $n \neq 3$ the quantity $\Psi(R)$ always diverges. In figure 5 (right panel), one can appreciate this divergence for $n = 4$. For $n = 3$ the quantity $\Psi(R)$ remains finite, but $\Psi(R) \neq 0$, so $R = 0$ cannot be a possible asymptotic solution of equation (9). Thus, the case $n = 3$ is irrelevant for AF solutions. In consequence, for $|n| > 2$ and $n \neq 3$, the weak singularity is approached as $R$ tries to reach its asymptotic value and at this point equation (9) becomes singular. Therefore, in this range of $n$ we never find a well-behaved solution regardless of the shooting parameter $R_h$ and $R$ systematically showing a divergent behavior in the numerical solutions.

In the interval $1 \leq n \leq 2$ the quantity $\Psi(R)$ vanishes and we do not find any pathologies as $R \rightarrow 0$. The numerical solutions are very similar to the oscillating solutions found in the Starobinsky model 5 and the exponential model 4. Figure 11 shows a prototype of such solutions. As we remarked before, for $n = 2$ the Hu–Sawicki model is essentially the Starobinsky model with $q = 1$, thus it is not surprising to find, at least for these values of $n$, such an oscillating behavior for all of our trial $R_h$ values. Our conclusions seem to be insensitive for several values of the constants $c_1$ and $c_2$.

Hence, the numerical evidence indicates that AF solutions with non-trivial hair are absent as well in the Hu–Sawicki model.

**6. Conclusions**

We have argued that generically $f(R)$ gravity contains trivial solutions $R = \text{const}$. This property allows it to mimic an effective cosmological constant that can produce the observed accelerated expansion in the Universe. In the context of BHs (stationary and axisymmetric, or SSS) the same property allows us to find the same solutions known in GR (under the same symmetries) with or without a cosmological constant. The only difference between both type of solutions (i.e. the solutions found in one or the other theory) is that the two fundamental constants involved (Newton’s gravitational constant and the cosmological constant) are replaced by the effective ones, $G_{\text{eff}}$ and $\Lambda_{\text{eff}}$ in $f(R)$ gravity. Therefore, all solutions of this kind reported in the literature within the framework of $f(R)$ gravity do not provide any deeper knowledge than the ones we already know in GR.
We then focused on the problem of finding non-trivial (hairy) AFSSSBH solutions where the Ricci scalar is not constant in the domain of outer communication of the BH but varies with the radial coordinate \( r \). Within that aim we provided the equations to study this scenario for an arbitrary \( f(R) \) model and derived the conditions for a regular BH. We then proceeded to analyze some specific models. Prior to a thorough numerical analysis, we studied the models under the scalar-tensor approach within the EF, stressing that in vacuum \( f(R) \) gravity takes the same form as in the Einstein-scalar-field system (provided \( f_{RR} > 0 \) and \( 0 < f_R < \infty \)). Therefore, this method makes it possible to check if the available NHTs for AFSSS can be applied, thus sparing an unnecessary numerical effort. Thus, for the models where the NHTs apply, we concluded that geometric hair is absent. In those situations the numerical analysis simply confirms the NHTs.

For the cases where the resulting scalar-field potential \( \Phi(\phi) \) does not satisfy the condition \( \Phi(\phi) \geq 0 \) required by the NHTs, or when the mapping to the EF is not well defined, we turned to a detailed numerical study of the original frame and variables. Our conclusion is that we did not find any such hair in any of the models where the NHTs do not apply.

We also discussed some exact solutions that seem to represent AF hairy BHs, however, we showed that in fact the \( r \) dependence in the Ricci scalar is due to the presence of a deficit angle. Therefore such solutions are not really AF. Similarly we report numerical solutions that are not genuinely AF as the the mass function never converges to the Komar mass, a feature necessary to prove asymptotic flatness. Some of the solutions have an asymptotic oscillatory behavior but without encountering any singularity, while others produce singularities in the equations at a finite \( r \).

It remains thus an open question to determine if hair exists in \( f(R) \) gravity. In order to settle the question in the affirmative it is tantalizing to depart from the Einstein-\( \phi \) system with a potential that has negative branches and that allows for hairy BHs \([9, 10]\), and then perform an ‘inverse’ conformal transformation to obtain an \( f(R) \) model. However, in practice this seems to be difficult in a closed explicit form. Moreover, even if this is possible, the resulting \( f(R) \) model would be rather artificial and would need to be submitted to the usual test (i.e. cosmological, Solar System, binary pulsar, etc) to be better motivated physically, irrespective of the issue of hair. Some of the exact BH solutions that have been reported in the literature (including in GR with exotic energy-momentum tensors) have been obtained using similar ‘tricks’ or ad hoc confections, but presented afterwards in the more logical direction within the aim of enhancing their merit. This is not the exception in \( f(R) \) gravity, where an \( f(R) \) model can be deduced by demanding the existence of some kind of exact solution, most of the time, deprived of actual physical interest. For instance, one can impose \( g_{rr} = \frac{1}{g_{\phi\phi}} \) (in area coordinates), which is valid provided \( T^t_{\text{eff}} = T^r_{\text{eff}} \), and then obtain a differential equation for \( f(R) \) which can be solved if the assumed form for \( g_{rr}(r) \) is simple enough. In fact, by using this method we were able to recover the model \( f(R) = f(R) = 2\alpha \sqrt{R - \alpha} \) given by \([19]\). Notice however that such solution was not AF, and thus, it cannot be used as a counterexample to the no-hair conjecture in \( f(R) \) gravity.

In this paper we limited ourselves to the case of AF spacetimes. In a future investigation we will analyze the case of hair in ADS and anti-de Sitter spacetimes. For such spacetimes the NHTs for the AF scenario require amendments due the different asymptotic conditions, and the numerical treatment, although similar to the one presented in this work, is sufficiently different to require a detailed and separate analysis. Furthermore, a natural extension of this work that is worth pursuing includes rotation.
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Appendix A. Conservation of the energy-momentum tensor of matter in \( f(R) \) gravity

The simplest way to show the conservation equation \( \nabla^a T_{ab} = 0 \) in \( f(R) \) metric gravity is by departing from equation (2) written as:

\[
f_R R_{ab} - \frac{1}{2} f_{ab} - (\nabla_b \nabla_a - g_{ab} \Box) f_R = \kappa T_{ab}. \tag{A1}\]

One can use the commutation relation \( \nabla_a \nabla_b f_R = \nabla_b \nabla_a f_R \) since \( f_R \) is a scalar function (i.e. in \( f(R) \) metric gravity we assume the absence of torsion: the l.h.s of equation (A1) must be a symmetric tensor). So, applying the covariant derivative \( \nabla^a \) to equation (A1) we obtain

\[
f_R \nabla^a R_{ab} + R_{ab} \nabla^a f_R - \frac{1}{2} f_R g_{ab} \nabla^a R - (\nabla^a \nabla_b - \nabla_b \nabla^a) f_R = \kappa \nabla^a T_{ab}, \tag{A2}\]

where \( \Box = \nabla^a \nabla_a \) and \( f_R = df/dR \) were used. According to the Bianchi identities

\[
\nabla^a (R_{ab} - \frac{1}{2} g_{ab} R) = 0, \tag{A3}\]

therefore the first and third terms of the l.h.s of equation (A2) cancel out, yielding

\[
R_{ab} \nabla^a f_R - (\nabla^a \nabla_b - \nabla_b \nabla^a) f_R = \kappa \nabla^a T_{ab}. \tag{A4}\]

Furthermore

\[
(\nabla^a \nabla_b - \nabla_b \nabla^a) f_R = g^{ac} (\nabla^b \nabla_c - \nabla_c \nabla^b) f_R = g^{ac} R_{cbd} \nabla_d f_R = R_{bad} \nabla^d f_R. \tag{A5}\]

This result implies that the l.h.s of equation (A4) vanishes identically, leading then to

\[
\nabla^a T_{ab} = 0. \tag{A6}\]

We can summarize this result as a theorem: in \( f(R) \) metric gravity (under the same basic axioms assumed in GR concerning the manifold and the metric) the generalized tensor

\[
g_{ab} := f_R R_{ab} - \frac{1}{2} f_{ab} - (\nabla_b \nabla_a - g_{ab} \Box) f_R \tag{A7}\]

obeys a generalized Bianchi identity

\[
\nabla^a g_{ab} = 0. \tag{A8}\]

Appendix B. Supplementary regularity conditions

In principle the regularity conditions at the horizon (34), (35) together with \( \delta(\eta_H) = \delta_H \), \( R(\eta_H) = R_H \), \( M(\eta_H) = \eta_H/2 \), seem enough data to solve the system of ordinary differential
equations (9)–(11). Nevertheless, in view of the method that we devised to solve these equations numerically, care must be taken when evaluating at the horizon the r.h.s of equations (9) and (11). Therefore at this place we also need to know the values of $R''_H$ and $\delta''_H$ at the horizon when starting the numerical integration. If we do not impose these regularity conditions a mild numerical error is made at $r_H$, but we want to avoid this and impose also the right conditions on $R''_H$ and $\delta''_H$. As we mentioned in section 4, in order to find these two conditions we need first to obtain an expression for $R'''$ and $\delta'$, then as before, develop the quantities around $r_H$, and finally impose that $R'''$ and $\delta'$ are finite at the horizon.

We simply provide the final outcome of this process, and the regularity conditions are as follows:

$$
\delta'' \big|_{r_H} = 4r \delta' \begin{cases}
\frac{r^2 \delta'}{f_{RR}} \left[ 2f' (10f - 13r f') + \frac{6f'}{r^2} (2f' - R f') \right] + \frac{r^2 \delta'}{f_{RR}} [10f - R (13f' - 4R f_R)]
\end{cases}
$$

$$
R''' \big|_{r_H} = -4r^2 f_R (-R f_R + 2f) \delta' \begin{cases}
\frac{r^2 \delta'}{f_{RR}} \left[ -12r^2 f_{RR}^2 + 3r^2 f_{RR} f_R \left[ -4f' + f_{RR} \left( 9R - \frac{20}{r^2} \right) \right] \right]
\end{cases}
$$

Given that these two regularity conditions are quite involved, we performed minimal tests to prove its validity. Like in section 4 we used the exact solutions provided in section 3. First we used the solution equations (14)–(17), notably for the model $f(R) = k R^2$, and then we considered model (18) with the solution provided by equations (20)–(22). In both solutions $\delta (r) \equiv 0$. This means that the r.h.s of equation (B1) must vanish in the two cases. We verified that indeed this happens.

On the other hand, the exact solution $R(r) = R_0 = \text{const}$ obtained from the model $f(R) = k R^2$ leads simply to $R''_H = 0$, whereas $R''_H = 6 / r_H^2$ for model (18). In both cases we checked that the r.h.s of equation (B2) gives respectively these two values at the horizon. We are therefore confident that our expressions are correct and so we enforced them in the numerical treatment presented in section 5.2.
Appendix C. Properties of the STT approach to f(R) gravity

When introducing the action (36) in this way, we can recognize two things; these are detailed as follows. (1) Let us consider the functions \( H(R, Q) = f'(Q)(R - Q) + f(Q) \) and \( \tilde{H}(R, Q, \chi) = Rf'(Q) - N(Q, \chi) \), where \( N(Q, \chi) = Q\chi - f(Q) \), and in this appendix a prime indicates differentiation with respect to the argument of the corresponding function. So, assuming \( f(Q) \) to be a convex function, i.e. \( f''(Q) > 0 \), then clearly \( \partial_Q N = 0 \) if \( \chi = f'(Q) \), and the inverse of this function \( Q(\chi) \) allows us to define \( L(\chi) = N(Q(\chi), \chi) = Q(\chi)\chi - f(Q(\chi)) \). In this way we see that \( L(\chi) \) is no other than the Legendre transformation of \( f(Q) \), and \( H(R, Q(\chi)) = H(R, Q(\chi), \chi) = R\chi - L(\chi) \). Moreover, \( H(R, Q(\chi)) \) defines in turn the Legendre transformation of \( L(\chi) \) (at least in the region where \( L''(\chi) = Q'(\chi) > 0 \) which in this case the condition \( \partial_\chi H = R - L'(\chi) = R - Q(\chi) = 0 \), simply leads to \( R = Q(\chi) \). (2) The condition for the second Legendre transformation can be imposed on the action by considering \( \chi = f'(Q) \) as a Lagrange multiplier, so that the variation with respect to \( \chi \) leads to \( R = Q \). This is the formal construction when treating \( f(R) \) theories as STTs, and in practice it is achieved by taking the action (37).

In the following we perform explicitly the transformation between the original \( f(R) \) theory and the special class of Brans–Dicke model \( \omega_{BD} = 0 \) supplemented with a potential.

The Brans–Dicke action with a potential \( W(\Phi) \) is given by [59]

\[
I_{BD}[g_{ab}, \Phi] = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left[ \Phi R - \frac{\omega_{BD}(\Phi)}{\Phi} g_{ab}(\nabla_a \Phi)(\nabla_b \Phi) - W(\Phi) \right] + I_{\text{mat}}[g_{ab}, \psi]. \tag{C1}
\]

For comparison with the \( f(R) \) model we shall focus only on the case \( \omega_{BD} = 0 \). Thus, the field equations read [59]

\[
\Phi G_{ab} = \kappa T_{ab} + (\nabla_a \nabla_b - g_{ab} \Box) \Phi - \frac{W}{2} g_{ab}, \tag{C2}
\]

\[
\Box \Phi = \frac{1}{3} (\kappa T + \Phi \partial_a W - 2W). \tag{C3}
\]

On the other hand when introducing \( \chi = f_R \) equation (2) simply reads

\[
\chi R_{ab} - \frac{1}{2} g_{ab} - (\nabla_a \nabla_b - g_{ab} \Box) \chi = \kappa T_{ab} \tag{C4}
\]

where \( f(R) = f(R(\chi)) \). Moreover, this equation can be written as

\[
\chi G_{ab} = \kappa T_{ab} + (\nabla_a \nabla_b - g_{ab} \Box) \chi - \frac{1}{2} g_{ab} [\chi R(\chi) - f(R(\chi))]. \tag{C5}
\]

where we have made explicit the functional dependence \( R(\chi) \), which means that if \( f''(R) > 0 \), one can in principle invert the definition \( \chi := f_R(R) \), and obtain \( R(\chi) \), and thus \( f(R(\chi)) \). Therefore, if we choose the potential \( W(\Phi) = \Phi R(\Phi) - f(R(\Phi)) \equiv U(\Phi) \) and identify \( \Phi = \chi \), then equation (C2) becomes exactly equation (C5). Moreover, the trace of equation (C5) reads

\[
\Box \chi = \frac{1}{3} (\kappa T + 2f - \chi R). \tag{C6}
\]

With the above identification of the fields \( \Phi = \chi \) and the potential \( W \) we appreciate that equation (C3) also becomes equation (C6) where one can easily verify that the expression...
$2f - \chi R$ coincides exactly with $\Phi \partial_\phi W - 2W$. Henceforth, we conclude that $f(R)$ theory is equivalent to a Brans–Dicke theory with $\omega_{BD} = 0$ and a potential $U(\Phi)$.

Appendix D. Examples of AF spacetimes with a deficit angle

SSS spacetimes with zero electric charge $Q$ that are AF except for a deficit angle have the asymptotic form

$$ds^2 \sim -\left(1 - \Delta - \frac{2M}{r}\right)dr^2 + \frac{dr^2}{\left(1 - \Delta - \frac{2M}{r}\right)} + r^2d\Omega^2. \quad (D1)$$

After a redefinition of coordinates $r = (1 - \Delta)^{1/2}\tilde{r}$, $t = (1 - \Delta)^{-1/2}\tilde{t}$, and $M = M_{\text{ADM},\Delta}(1 - \Delta)^{3/2}$ the metric acquires the standard angle-deficit form

$$ds^2 \sim -\left(1 - \frac{2M_{\text{ADM},\Delta}}{\tilde{r}}\right)d\tilde{r}^2 + \frac{d\tilde{r}^2}{\left(1 - \frac{2M_{\text{ADM},\Delta}}{\tilde{r}}\right)} + (1 - \Delta)\tilde{r}^2d\Omega^2. \quad (D2)$$

Under this parametrization the coefficient $M_{\text{ADM},\Delta}$ is then identified with the ADM mass associated with this kind of spacetime [45]. In the same way, an SSS metric which is ADS or AADS with a deficit angle

$$ds^2 \sim -\left(1 - \Delta - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)dr^2 + \frac{dr^2}{\left(1 - \Delta - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)} + r^2d\Omega^2, \quad (D3)$$

can be transformed into

$$ds^2 \sim -\left(1 - \frac{2M_{\text{ADM},\Delta}}{\tilde{r}} - \frac{\Lambda \tilde{r}^2}{3}\right)d\tilde{r}^2 + \frac{d\tilde{r}^2}{\left(1 - \frac{2M_{\text{ADM},\Delta}}{\tilde{r}} - \frac{\Lambda \tilde{r}^2}{3}\right)} + (1 - \Delta)\tilde{r}^2d\Omega^2. \quad (D4)$$

Notice that the cosmological constant $\Lambda$ did not require to be redefined in order to obtain the standard metric (D4). We then need $M_{\text{ADM},\Delta} \equiv 0$, $\Delta = 1/2$ and $\Lambda = \Lambda_\infty$ in order to recover the metric (19) in the standard form when $Q \equiv 0$.

Finally, the metric of an SSS that is ADS or AADS with a deficit angle and endowed with a charge $q$

$$ds^2 \sim -\left(1 - \Delta - \frac{2M}{r} - \frac{\Lambda r^2}{3} + \frac{q^2}{r^2}\right)dr^2 + \frac{dr^2}{\left(1 - \Delta - \frac{2M}{r} - \frac{\Lambda r^2}{3} + \frac{q^2}{r^2}\right)} + r^2d\Omega^2, \quad (D5)$$

is transformed into

$$ds^2 \sim -\left(1 - \frac{2M_{\text{ADM},\Delta}}{\tilde{r}} - \frac{\Lambda \tilde{r}^2}{3} + \frac{Q^2}{\tilde{r}^2}\right)d\tilde{r}^2 + \frac{d\tilde{r}^2}{\left(1 - \frac{2M_{\text{ADM},\Delta}}{\tilde{r}} - \frac{\Lambda \tilde{r}^2}{3} + \frac{Q^2}{\tilde{r}^2}\right)} + (1 - \Delta)\tilde{r}^2d\Omega^2, \quad (D6)$$

taking $q = Q(1 - \Delta)$. The quantity $Q$ is presumably the actual charge when $\Delta = 0$. Again, taking $M_{\text{ADM},\Delta} \equiv 0$, $\Delta = 1/2$ and $\Lambda = \Lambda_\infty$ we recover the metric (19) written in the standard form but now with $Q \equiv 0$ given by $Q = \pm q^2 = \pm Q^2/4$. 

\[39\]
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