Single Particle Operators and their Correlators
in Free $\mathcal{N} = 4$ SYM

F. Aprile$^1$, J. M. Drummond$^2$, P. Heslop$^3$
H. Paul$^2$, F. Sanfilippo$^4$, M. Santagata$^2$, A. Stewart$^3$

$^1$ Dipartimento di Fisica, Università di Milano-Bicocca & INFN, Sezione di Milano-Bicocca, I-20126 Milano,

$^2$ School of Physics and Astronomy and STAG Research Centre, University of Southampton, Highfield, SO17 1BJ,

$^3$ Mathematics Department, Durham University, Science Laboratories, South Rd, Durham DH1 3LE,

$^4$ INFN Sezione di Roma 3, Via della Vasca Navale 84, I-00146 Roma.

Abstract

We consider a set of half-BPS operators in $\mathcal{N} = 4$ super Yang-Mills theory which are appropriate for describing single-particle states of superstring theory on AdS$_5 \times S^5$. These single-particle operators are defined to have vanishing two-point functions with all multi-trace operators and therefore correspond to admixtures of single- and multi-traces. We find explicit formulae for all single-particle operators and for their two-point function normalisation. We show that single-particle $U(N)$ operators belong to the $SU(N)$ subspace, thus for length greater than one they are simply the $SU(N)$ single-particle operators. Then, we point out that at large $N$, as the length of the operator increases, the single-particle operator naturally interpolates between the single-trace and the $S^3$ giant graviton. At finite $N$, the multi-particle basis, obtained by taking products of the single-particle operators, gives a new basis for all half-BPS states, and this new basis naturally cuts off when the length of any of the single-particle operators exceeds the number of colours. From the two-point function orthogonality we prove a multipoint orthogonality theorem which implies vanishing of all near-extremal correlators. We then compute all maximally and next-to-maximally extremal free correlators, and we discuss features of the correlators when the extremality is lowered. Finally, we describe a half-BPS projection of the operator product expansion on the multi-particle basis which provides an alternative construction of four- and higher-point functions in the free theory.
# Contents

1 Introduction 3

2 Single-particle half-BPS operators (SPOs) 5
   2.1 Definition and low charge examples .......................... 6
   2.2 General Formulae for SPOs ..................................... 9
      2.2.1 Formula in terms of products of traces ................... 10
      2.2.2 Formula in terms of eigenvalues .......................... 13
      2.2.3 Formula in terms of Schur polynomials ..................... 14
      2.2.4 Examples in the three basis ............................... 15
   2.3 SPOs interpolate between single-trace and giant gravitons ........ 15
   2.4 SPOs in $U(N)$ are SPOs in $SU(N)$ .......................... 16
   2.5 Two-point function normalisation ................................ 18
   2.6 On multi-particle operators .................................... 19

3 Multipoint orthogonality 21
   3.1 Proof of the theorem ........................................... 22
   3.2 Near-Extremal correlators vanish .............................. 24
      3.2.1 More cases: coincident points and multi-trace splitting .... 26

4 Exact results for correlators of SPOs 27
   4.1 Maximally-Extremal correlators ............................... 27
      4.1.1 3-point functions ........................................ 27
      4.1.2 $n$-point functions ...................................... 28
   4.2 Next-to-Maximally-Extremal correlators ......................... 31
      4.2.1 3-point functions ........................................ 31
      4.2.2 $n$-point functions ...................................... 34
   4.3 3-point functions as multi-particle 2-point functions ........... 34
1 Introduction

As the most symmetric QFT in four dimensions, and at the same time the paradigmatic example of the AdS/CFT correspondence, $\mathcal{N}=4$ super Yang-Mills (SYM) has been the subject of a huge amount of interest. The expected flow of information uses semiclassical physics in AdS to reconstruct strong 't Hooft coupling phenomena in the gauge theory. Instead, recently this flow has been reversed, with precise and concrete investigations of perturbative quantum gravity undertaken by means of analytic bootstrap techniques at strong coupling. For example one-loop quantum gravity amplitudes in AdS have been obtained by computing $O(1/N^4)$ corrections to strong coupling correlators in SYM [1–5].

One aspect of these investigations was the need to be very careful about the precise definition of the gauge theory operators dual to single-particle supergravity states, in order to properly account for the OPE at tree level and one loop. These single-particle operators are protected half-BPS operators, but the space of half-BPS operators of given charge is degenerate, and only in the planar limit can the single-particle operators be identified with the single-trace half-BPS operators $\text{Tr}(\phi^p)$. This identification indeed was known to receive $O(1/N)$ multi-trace corrections (see eg [6–8]), and the first order double trace corrections have recently been fully worked out directly from supergravity in [9,10]. On the other hand, the non-perturbative nature of the AdS/CFT correspondence strongly points towards a non-perturbative (i.e. valid to all orders in $N$) definition of the single-particle states. In fact, a deceptively simple non-perturbative definition was formulated in [11]:

- Single-particle operators are half-BPS operators which have vanishing two-point functions with all multi-trace operators.

This definition fixes the single-particle operators uniquely, up to normalisation, in terms of a certain admixture of single- and multi-trace operators. In particular, the leading operator in the planar limit is the single-trace operator, but the admixture is new. Our definition has been already used, and indeed is crucial in correctly determining one-loop $O(1/N^4)$ SUGRA correlators of operators with charges higher than four, for example, arbitrary charge correlators in position space [4], where a number of subtle features have been solved, and $22pp$ correlators in Mellin space [5].

Our first result in this paper is to obtain explicit formulae for the multi-trace components of the single-particle operators and examine some of their nice properties. Then, we will study some of their correlators and show that compared to the single-trace cousins, the single-particle operators have a number of very surprising and nice properties. This is slightly counter-intuitive at first, because now we have to deal with an admixture of single and multi-trace operators, but nevertheless it is true in many ways, as we will demonstrate.

One of the nice properties is that the operators in the $U(N)$ theory and the $SU(N)$ theory are the same. Indeed, in the $U(N)$ theory the single-particle operators of charge
greater equal than two must be orthogonal to all multi-trace operators involving also Tr(φ), and this automatically makes them the SU(N) operators. To formalise this statement we introduce the SU(N) projection on the space of the U(N) operators, and show that the U(N) single-particle operators belong to the SU(N) subspace, which is orthogonal to the span of multi-trace operators in which at least one trace is Tr(φ). It follows that correlators of U(N) single-particle operators are equal to correlators of SU(N) single-particle operators.

Another nice property of the single-particle operators is that they automatically vanish as the charge of the operators exceeds the number of colours N. This should be contrasted for example with the single trace basis which do not vanish, but rather become complicated linear combinations of products of lower trace operators. This property of the operators dual to single particle states has long been expected from AdS/CFT and follows from the string exclusion principle [12]. As the angular momentum of the gravitons increases they become less and less pointlike, eventually growing into giant gravitons, D3 branes wrapping an $S^3 \subset S^5$ [13] which can not grow bigger than the size of the $S^5$ sphere. In [14] (sub)-determinant half-BPS operators were defined as duals to these predicted sphere giants and shortly later the Schur polynomial basis of half-BPS operators was defined and the sphere giant gravitons associated with the completely antisymmetric (single column Young tableau) Schur polynomials [15]. We find that at large N, the single-particle operators with charge close to N indeed approach these (sub)-determinant operators.

In fact these operators have appeared before in the literature, and not identified as operators dual to single particle states. A beautiful orthogonal basis for all half-BPS operators in the U(N) theory was given in [15] in terms of Schur polynomials, labelled by Young tableaux. These also automatically truncate with N since a U(N) Young tableau with height larger than N vanishes. However this basis does not project onto an orthogonal basis for SU(N) (and indeed the operators are not even linearly independent in the SU(N) theory). In [16] a non-orthogonal but linearly independent basis of all SU(N) half-BPS operators was defined. This basis was later identified as the dual (via the metric defined by the two point function) to the trace basis in [17] and a group theoretic expression for this dual basis was given. The single particle operators we discuss here are a subset of the dual basis: the operators dual to single trace operators.

It is the purpose of this paper to unpack the definition of single-particle operators, turn it into an explicit formula, and investigate its properties. The outline will be as follows.

In section 2 we discuss the details of the multi-trace admixture which defines the single-particle operators. We first give explicit examples at low charge, and then we use group theory techniques to obtain a general formula, valid for any single-particle operator of any dimension. This leads us to very explicit formulae for two-point functions of single-particle operators.

In section 3 we uplift the defining two-point function orthogonality to a multi-point orthogonality theorem, which in turn implies vanishing of a large class of diagrams in cor-
relators. We call these near-extremal \( n \)-point functions, where extremality will be defined as a measure of how much the diagram is connected w.r.t. the heaviest operator (see (77)). This is the first instance of hidden simplicity of multi-point single-particle correlators versus the single-trace correlators, and very interestingly, a similar feature was noticed on the (super-)gravity side in [18].

In section 4, we consider the first non vanishing correlators, and we study maximally-extremal (ME) and next-to-maximally extremal (NME) \( n \)-point functions. Both are simple. The ME correlators are computed by trees and two point functions. The NME are mostly computed by weighted sums of ME correlators, which we know in general. When we compute these correlators by using Wick contractions techniques on the trace basis, the combinatorics is hard in the intermediate steps. Instead, the final result is way much simpler. We provide more evidence about this mechanism mentioning also the case of NNME three-point functions.

Finally, in section 5 we discuss the half-BPS OPE on the single-particle basis, and we use it as an alternative computational tool to bootstrap, and provide constraints, on the correlators of interest. We exemplify the procedure at four-points for next-to-next- and next-to-next-to-next extremal correlators.

## 2 Single-particle half-BPS operators (SPOs)

Half-BPS operators in \( \mathcal{N} = 4 \) super Yang-Mills theory can be described via products of single-trace scalar operators in the symmetric traceless representations of \( SO(6) \),

\[
T_p(x) = \text{Tr} \phi(x)^p \quad ; \quad \phi(X,Y) = Y^R \phi_R(X)
\]

(1)

Here \( \phi_R \) is the elementary field of the \( \mathcal{N} = 4 \) supermultiplet, and we have introduced an auxiliary \( SO(6) \) vector \( Y^R \) which is null, \( Y \cdot Y = 0 \), to project onto the symmetric traceless representation. An insertion point \( x_i \) corresponds to both a space time \( X_i \) coordinate and an \( SO(6) \) vector \( Y^R \).

In addition to the single-trace operators \( T_p \) we obtain other half-BPS operators from products of the form

\[
T_{p_1 \ldots p_m}(x) \equiv T_{p_1}(x) \ldots T_{p_m}(x) \quad , \quad p_1 \geq \ldots \geq p_m \geq 1 .
\]

(2)

The scaling dimension of \( T_{p_1 \ldots p_m} \) is given by \( p_1 + \ldots + p_m \). The case \( m = 1 \) reduces obviously to the single trace operator.

We will refer to the basis of half-BPS operators made of all possible \( T_{p_1 \ldots p_m}(x) \) as the trace basis, and we will denote the basis elements with the symbol \( T_p \), where \( p \) stands for a partition of \( p \), so if the partition has length greater than two the operator is multi-trace.
To compute correlation functions in free field theory we use elementary propagators. Here we will take the gauge group to be $U(N)$ or $SU(N)$ so that the propagator takes the form,

$$\langle \phi_r \phi^*(X_1, Y_1) \phi_t \phi^*(X_2, Y_2) \rangle = \delta_r^u \delta^*_t \langle g_{12} \rangle$$

where

$$g_{12} = \frac{Y_1 \cdot Y_2}{(X_1 - X_2)^2}$$

A correlation function of operators in the trace basis will have the schematic form

$$\langle T_{p_1}(x_1) \ldots T_{p_n}(x_n) \rangle = \sum_{\{b_{ij}\}} \prod_{i,j} g_{b_{ij}}^2 C_{\{b_{ij}\}, p_1 \ldots p_n}(N)$$

where $b_{ij}$ counts the number of propagators from insertion point $i$ to $j$, and the collection of these bridges, $\{b_{ij}\}_{i<j}$, thus labels the propagator structure. With $C_{\{b_{ij}\}}(N)$ we denote the corresponding color factor.

### 2.1 Definition and low charge examples

The AdS/CFT correspondence maps the spectrum of operators in $\mathcal{N} = 4$ super Yang-Mills theory to the spectrum of IIB superstring theory on the $\text{AdS}_5 \times S^5$ background. The superstring can be found in unexcited states (giving the IIB supergravity multiplet) or excited states. The half-BPS operators correspond to the supergravity states (and their multi-particle products).

In the natural basis of scattering states, the multi-particle states should be orthogonal to single-particle ones. A key point of our discussion is that the trace basis of half-BPS operators is not an orthogonal basis with respect to the inner product given by the two-point functions. In general

$$\langle T_{p}(x_1) \ldots T_{q_n}(x_n) \rangle = 0, \quad n \geq 2.$$
From the Gram-Schmidt orthogonalisation we immediately obtain a formula for $O_p$ in terms of the (color factor of) two-point functions. In fact, consider the space of operators of charge $p$ and label them by corresponding partitions \( \{ \lambda_i = q_1 \ldots q_n \}_{i=1\ldots P} \), with \( \lambda_P \equiv \{ p \} \).

Then,

$$O_p(x) = \frac{1}{N_p} \det \begin{pmatrix} C_{\lambda_1|\lambda_1} & C_{\lambda_2|\lambda_1} & \ldots & C_{p|\lambda_1} \\ \vdots & \vdots & \ldots & \vdots \\ C_{\lambda_1|\lambda_{P-1}} & C_{\lambda_2|\lambda_{P-1}} & \ldots & C_{p|\lambda_{P-1}} \\ T_{\lambda_1}(x) & T_{\lambda_2}(x) & \ldots & T_p(x) \end{pmatrix}$$

where

$$N_p = \det (C_{\lambda_i\lambda_j})_{1 \leq i,j \leq P-1} ; \quad \langle T_{\lambda_i}(x_1)T_{\lambda_j}(x_2) \rangle = g_{12}^p C_{\lambda_i\lambda_j}$$

The determinant (9) is understood as a Laplace expansion about the last row, which contains the list of all allowed operators of charge $p$. Upon using the determinant for computing \( \langle O_p(x_1)T_{q_1 \ldots q_n}(x_2) \rangle \), it becomes obvious that the last row of the matrix would now coincide with another row, leading to a vanishing result.

In our normalisation \( O_p \) coincides with the single-trace operators \( T_p \) up to multi-trace admixtures. Each multi-trace contribution is suppressed at large $N$, and the single-particle operator reduces to single-trace operators in the strict large $N$ limit. However, the novelty of our SPO is precisely the determination of the multi-trace admixture, which we will now exemplify with a number of computations, for low charge operators, before discussing our general formulas in the next section.

With $SU(N)$ gauge group, \( T_p \) and \( O_p \) coincide for $p = 2, 3$ since there are no multi-trace operators for charges $p < 4$.

$$O_p = T_p \quad \text{for } p = 2, 3 \quad SU(N).$$

In terms of supergravity states the $p = 2$ case corresponds to the superconformal primary for the energy-momentum multiplet which is dual to the graviton multiplet in AdS$_5$. The $p = 3$ case is the first Kaluza-Klein mode arising from reduction of the IIB graviton supermultiplet on $S^5$.

For gauge group $U(N)$ on the other hand there exists a $p = 1$ operator, since the trace of the fundamental scalar no longer vanishes. At weight 1 there is only one operator and thus \( O_1 = T_1 \). However using Wick contractions with the $U(N)$ propagator (3) one can easily verify that for $p > 1$, as well as the single-trace contributions, we have non-trivial multi-trace admixtures. At weight 2 there are two operators, $T_2$ and $T_{11}$ with the single particle operator defined to be orthogonal to the double-trace $T_{11}$. Similarly for $p = 3$ the single particle operators is defined to be orthogonal to both the double-trace $T_{21}$ and the triple trace $T_{111}$. Explicitly we obtain

$$O_2 = T_2 - \frac{1}{N} T_{11} \quad U(N).$$

$$O_3 = T_3 - \frac{3}{N} T_{21} + \frac{2}{N^2} T_{111} \quad U(N).$$
where for example the coefficient of the double-trace contribution to $O_2$ is determined from the orthogonality condition $\langle O_2(x_1)T_{11}(x_2) \rangle = 0$. The additional terms compared to the $SU(N)$ operators (11) in fact simply project out the trace part of the fundamental scalar $\phi$ and so the $SU(N)$ and $U(N)$ operators in fact coincide.

For $p > 3$ we have non-trivial multi-trace admixtures even in the $SU(N)$ case as can be easily verified using $SU(N)$ Wick contractions (i.e. with propagator (11)). This was discussed in [11] (see also previous discussions in [6,8,19]). For example, the single-particle operator for $p = 4$ is given by

$$O_4 = T_4 - \frac{2N^2 - 3}{N(N^2 + 1)} T_{22}$$

where the coefficient of the double-trace contribution determined from the orthogonality condition $\langle O_4 T_{22} \rangle = 0$. It is now interesting to consider the single particle operator in the $U(N)$ theory. It is given by

$$O_4 = T_4 - \frac{(2N^2 - 3)}{N(N^2 + 1)} T_{22} + \frac{10}{N^2 + 1} T_{311} - \frac{4}{N} T_{13} - \frac{5}{N(N^2 + 1)} T_{1111}$$

where the coefficients are determined by demanding orthogonality with all higher trace operators $T_{22}, T_{311}, T_{13}, T_{1111}$.

We see that the $U(N)$ operators given above, explicitly reduce to the $SU(N)$ operators upon imposing $T_1 = 0$. This pattern goes on, and for illustration we give the next few higher weight examples,

$$O_5 = T_5 - \frac{5(N^2 - 2)}{N(N^2 + 5)} T_{32} + U_5$$

$$O_6 = T_6 - \frac{(3N^4 - 11N^2 + 80)}{N(N^4 + 15N^2 + 8)} T_{33} - \frac{6(N - 2)(N + 2)(N^2 + 5)T_{42}}{N(N^4 + 15N^2 + 8)} + \frac{7(N^2 - 7)}{N^4 + 15N^2 + 8} T_{22} + U_6$$

where

$$U_5 = \frac{15(N^2 - 2)T_{2211}}{N^2(N^2 + 5)} + \frac{5(3N^2 + 8)T_{3311}}{N^2(N^2 + 5)} - \frac{35T_{21111}}{N(N^2 + 5)} + \frac{14T_{111111}}{N^2(N^2 + 5)} - \frac{5T_{41}}{N}$$

$$U_6 = \frac{42(N - 1)(N + 1)T_{3211}}{N^4 + 15N^2 + 8} + \frac{21(N^2 + 11)T_{4111}}{N^4 + 15N^2 + 8} - \frac{42(2N^2 - 5)T_{22111}}{N(N^4 + 15N^2 + 8)} + \frac{126T_{211111}}{N(N^4 + 15N^2 + 8)} - \frac{42T_{1111111}}{N} - \frac{56(N^2 + 5)T_{311}}{N(N^4 + 15N^2 + 8)} + \frac{126T_{211111}}{N(N^4 + 15N^2 + 8)} - \frac{42T_{1111111}}{N(N^4 + 15N^2 + 8)} - \frac{6T_{51}}{N}$$

For $SU(N)$ the contributions denoted by $U$ vanish in each case.

The statement that $U(N)$ single particle operators reduce to the $SU(N)$ single particle operator upon imposing $T_1 = 0$ is true in general, as we will prove in section 2.4, and it has
very nice consequences. The correlators of $U(N)$ operators are much simpler to compute, and can be directly related to group theoretic quantities, essentially since the propagator (3) is so simple. For this reason, there is a long history of studying half-BPS operators in the $U(N)$ theory, eg [15]. Here we see that although correlators in the $U(N)$ theory are simpler to compute, the trade-off is that the SPOs themselves are more complicated. In the end the computation of a correlator involving SPOs is equivalent whether using the $U(N)$ or $SU(N)$ theory.

2.2 General Formulae for SPOs

So far we have uniquely defined SPOs (up to normalisation) as operators orthogonal to all multi-trace operators and illustrated with some examples at low charges. We will now give three different formulae for the single particle operators, which precisely capture the multi-trace admixture. The plan will be the following:

In section 2.2.1 we will give a formula in the trace basis, see (33), which is perhaps the most familiar basis. This formula is equivalent to (9) but uses more powerful group theory techniques to resolve for the expansion of the SPO in terms of multi-traces. We quote it here

$$O_p(x) = \sum_{\{q_1, \ldots, q_m\} \cup p} C_{q_1, \ldots, q_m} T_{q_1, \ldots, q_m}(x)$$  \hspace{1cm} (18)

$$C_{q_1, \ldots, q_m} = \frac{||[\sigma_{q_1, q_m}]||}{(p-1)!} \sum_{s \in \mathcal{P}(\{q_1, \ldots, q_m\})} \frac{(-1)^{|s|+1}(N+1-p)^p - \Sigma(s)(N+p - \Sigma(s))\Sigma(s)}{(N)^p - (N+1-p)^p}$$

It is very non trivial. The group theory data consists of $\mathcal{P}(\{q_1, \ldots, q_m\})$, the powerset of the traces $T_{q_1, \ldots, q_m}$, then $|s|$ is the cardinality of $s$ and $\Sigma(s) = \sum_{s_i \in s} s_i$. Finally, $||[\sigma_{q_1, q_m}]||$ is the size of the conjugacy classes of $\sigma$ with length cycles $q_1 \ldots q_m$.

In section 2.2.2 we give a much simpler formula, see (39), directly in terms of the eigenvalues $E_k(z_i)$ of the elementary fields $\phi$,

$$O_p(x) = \sum_{k=1}^p d_k(p, N) E_k(z_i)(x)$$  \hspace{1cm} (19)

$$d_k(p, N) = \frac{(-1)^k p^k(N-p+1)^p - k(p-1)^k}{(N)^p - (N-p+1)^p}$$  \hspace{1cm} (20)

In section 2.2.3 we give another simple formula in terms of the Schur polynomial basis (for operators), see (42), where we note that only hook representations appear.

If we think that half-BPS operators are symmetric functions of the eigenvalues of the scalar matrix $\phi_s^a$, the three basis for gauge invariant operators, that we used to expand
the SPOs above, correspond to three well-known bases for symmetric polynomials: 1) The trace basis corresponds to the power sum symmetric polynomials. 2) The explicit eigenvalue monomials (after summing over permutations) are called the monomial symmetric polynomials. 3) The Schur polynomials are named directly in terms of the corresponding symmetric polynomial basis of the same name.

There are a number of well known formulae relating 1), 2) and 3) to each other, known as Newton identities, and it would be interesting to explore these relations further in this context.

2.2.1 Formula in terms of products of traces

We first observe that the single-particle operators $O_p$ must be proportional to the dual $\xi_p$ field of the single-trace operators $T_p$

$$\langle \xi_p(x_1) T_{q_1 \ldots q_n}(x_2) \rangle = 0, \quad n \geq 2 \quad (21)$$

where the dual fields $\xi_p$ were introduced by Tom Brown in [17].

Indeed, group theoretic formulae for SPOs were given in [17] (albeit without the explicit physical description as single particle operators). He defined a more general basis of operators: the dual of the trace basis. This is given by operators $\xi_{p_1 \ldots p_n}$ (with $p_1 \geq \ldots \geq p_n$) which obey

$$\langle \xi_{p_1 \ldots p_m}(x_1) T_{q_1 \ldots q_n}(x_2) \rangle = \begin{cases} 1 & \text{if } (p_1, \ldots, p_m) = (q_1, \ldots, q_n), \\ 0 & \text{otherwise}. \end{cases} \quad (22)$$

In other words each element of the dual to the trace basis is orthogonal to (i.e. it has vanishing two-point function with) all elements of the trace basis but one, and we then normalise it to have unit two-point coefficient with this element. Note that an orthogonal basis is its own dual. We will refer to the basis given by the $\xi_{p_1 \ldots p_n}$ as the dual basis.

For a single index (i.e. $m = 1$) the definition of the dual basis (22) reduces to (21), which is the defining property of single particle operators (8). So the $\xi_p$ are orthogonal to all multi-trace operators and thus equal to $O_p$ up to normalisation. The normalisation can then be determined from (9), i.e. $O_p = T_p + \text{multi-traces}$, which implies

$$\langle \xi_p O_p \rangle = 1 \quad (23)$$

and hence

$$\xi_p(x) = \frac{O_p(x)}{\langle O_p O_p \rangle} \quad \text{and} \quad O_p(x) = \frac{\xi_p(x)}{\langle \xi_p \xi_p \rangle}. \quad (24)$$

\footnote{In this section we will focus only on two-point functions and we will always drop the trivial space-time dependent part, $(g_{12})^p$, referring the discussion to the normalisation/color factor.}
By definition (as the dual basis with respect to the inner product defined by the two-point function) the change of basis matrix from the dual to the trace basis is simply the two-point function:

$$\xi_{p_1..p_n}(x) = \sum_{\{q_1..q_m\} \vdash p} \langle \xi_{p_1..p_n} \xi_{q_1..q_m} \rangle T_{q_1..q_m}(x) \tag{25}$$

where the sum is over all partitions of \( p \), that is all sets of integers \( q_1 \geq \ldots \geq q_m \) such that \( q_1 + \ldots + q_m = p \). Brown in [17] gives group theoretic expressions for these two-point functions as

$$\langle \xi_{p_1..p_n} \xi_{q_1..q_m} \rangle = \prod_{R \vdash p} \frac{1}{f_R} \chi_R(\sigma_{p_1..p_n}) \chi_R(\sigma_{q_1..q_m}) \tag{26}$$

where

- \( \sigma_{p_1..p_n} \in S_p \) is a permutation \(^2\) made of disjoint cycles of lengths \( p_1, p_2, \ldots \) and \( ||\sigma_{p_1..p_n}|| \) is the size of the corresponding conjugacy class of the permutation (i.e. the number of permutations with that disjoint cycle structure). Similarly for \( \sigma_{q_1..q_m} \).

- the sum is over all partitions \( R \) of \( p \). One can view \( R \) as a Young diagram with \( p \) boxes, labelling a representation of the permutation group \( S_p \).

- the expression \( \chi_R(\sigma) \) is the character for this \( S_p \) representation.

- the sum is weighted by

$$f_R = \prod_{(r,c) \in R} (N - r + c) \tag{27}$$

where \((row, cln)\) are the coordinates of the boxes of the Young diagram.\(^3\)

Since we are focusing on single particle operators we can assume \( n = 1 \) in (25) so \( \xi_{p_1..p_n} = \xi_p \) and the permutation \( \sigma_{p_1..p_n} \) consists of a single cycle of length \( p \) of which there are \((p - 1)!\) possibilities, so \( ||\sigma_p|| = (p - 1)!\). Furthermore we observe that for such a permutation, only Hook reprenations have a non vanishing \( S_p \). Indeed we note that:

$$\chi_R(\sigma_p) = \begin{cases} (-1)^{h_R-1} & R = \text{hook YT of height } h_R \\ 0 & \text{otherwise} \end{cases} \tag{28}$$

\(^2\)For example the one given by \((1, \ldots , p_1)(p_1 + 1, \ldots p_1 + p_2)\ldots (p_1 + \ldots + p_{n-1} + 1, \ldots , p)\).

\(^3\)Group theoretically \( f_R \) is proportional to the ratio between the dimensions of \( R \) as a rep of \( U(N) \) and the dimension of \( R \) as a rep of \( S_p \), i.e. \( f_R = p! \frac{d_R[U(N)]}{d_R[S_p]} \).
where “hook YT” means that \( R \) has a hook-shaped diagram: all non-zero rows but the first have length 1 and all non-zero columns but the first have height 1. Thus the coefficient of a given multi-trace operator in \( \xi_p \) is

\[
\langle \xi_p \xi_{q_1\ldots q_m} \rangle = \frac{1}{p} \frac{||\sigma_{q_1\ldots q_m}||}{p!} \sum_{R \in \text{hooks}} \frac{1}{f_R} (-1)^{1 + k_R} \chi_R(\sigma_{q_1\ldots q_m}) .
\] (29)

We have found that this sum over hook representations is always given by the following explicit formula

\[
\langle \xi_p \xi_{q_1\ldots q_m} \rangle = \frac{1}{p} \frac{||\sigma_{q_1\ldots q_m}||}{p!} \sum_{s \in \mathcal{P}(\{q_1,\ldots, q_m\})} \frac{(-1)^{|s|+1}}{(N+1-\Sigma(s))_{p-1}} .
\] (30)

The sum \( s \in \mathcal{P}(\{q_1,\ldots, q_m\}) \) is over all subsets \( s \) of the set \( \{q_1,\ldots, q_m\} \) including the empty set and the full set \( \{q_1,\ldots, q_m\} \). Note that the set of all subsets of a set \( S \) is known as the powerset of \( S \) and denoted \( \mathcal{P}(S) \). Then \(|s|\) denotes the number of elements of the subset \( s \) and \( \Sigma(s) \) the total of all the elements in subset \( s \), \( \Sigma(s) = \sum_{s \in s} s_i \).

An important special case of (30) is the case \( m = 1 \) giving the two-point function of the dual of the single trace operator. In this case the sum is over just two elements \( s = \{\} \) and \( s = \{p\} \) since \( \mathcal{P}(\{p\}) = \{\{\}, \{p\}\} \). The expression (30) thus simplifies to

\[
\langle \xi_p \xi_p \rangle = \frac{1}{p^2} \frac{1}{p - 1} \left( \frac{1}{(N+1-p)_{p-1}} - \frac{1}{(N+1)_{p-1}} \right) .
\] (31)

Finally inserting (30) into (25) and in turn into (24) together with (31) gives an explicit expression for the single particle operator as a sum of multi-trace operators

\[
\mathcal{O}_p(x) = \sum_{\{q_1,\ldots, q_m\} \subset \mathcal{P}(\{1,\ldots, p\})} C_{q_1,\ldots, q_m} T_{q_1,\ldots, q_m}(x)
\] (32)

with coefficients

\[
C_{q_1,\ldots, q_m} = \frac{\langle \xi_p \xi_{q_1,\ldots, q_m} \rangle}{\langle \xi_p \xi_p \rangle} = \frac{||\sigma_{q_1,\ldots, q_m}||}{(p-1)!} \sum_{s \in \mathcal{P}(\{q_1,\ldots, q_m\})} \frac{(-1)^{|s|+1}}{(N+1-\Sigma(s))_{p-1}} \left( \frac{1}{(N+1-p)_{p-1}} - \frac{1}{(N+1)_{p-1}} \right) .
\] (33)

\[^4\text{For example } \mathcal{P}((3,2,1)) = \{\{\}, \{1\}, \{2\}, \{3\}, \{2,1\}, \{3,1\}, \{3,2\}, \{3,2,1\}\}.\]
The second equality is more useful computationally and is obtained by multiplying and dividing by \((N + 1 - p)^2\) and using the identity

\[
\frac{(N + 1 - p)^{2p-1}}{(N + 1 - \Sigma(s))_{p-1}} = (N + 1 - p)^{p-\Sigma(s)}(N + p - \Sigma(s))^{\Sigma(s)}.
\] (34)

The only final ingredient is the size of the conjugacy classes. If all cycle lengths are distinct, i.e. \(p = q_1 + \ldots + q_m\) with \(q_i \neq q_j\), the size of the conjugacy classes is simply \(p! / \prod q_i\). However if there are multiple cycles of the same length, i.e. some \(q_i = q_j\) in the partition of \(p\), then these cycles are interchangeable and we have to divide by this symmetry in addition. To deal with this case, let \(\lambda_1, \lambda_2, \ldots, \lambda_r\) be distinct so that \(\prod_j \lambda_j^{k_j} = \prod_i q_i\) and \(\sum_j k_j \lambda_j = p\). Then

\[
|\sigma_{q_1\ldots q_m}| = \frac{p!}{\prod_{i=1}^r k_i! \lambda_i^{k_i}}.
\] (35)

We thus have a formula for computing the coefficients \(C_{q_1\ldots q_m}\) which give the representation of the SPOs in the trace basis. Very nicely, this formula is explicit in \(p\) and \(\{q_1, \ldots q_m\}\), and depends only on the group theory data associated to such partition.

Notice that the value of \(m\) counts the splitting of \(O_p\) into \(m\) traces. In appendix A we provide some extra examples for generic \(q_1 \ldots q_m\) when \(m = 2\), i.e. double traces, and \(m = 3\), i.e. triple traces.

### 2.2.2 Formula in terms of eigenvalues

In fact the formula for the SPOs is much simpler when expressed directly in terms of the eigenvalues of the adjoint scalar \(\phi^s\) which we call \(z_1, z_2, \ldots, z_N\). For this purpose consider

\[
m_{[\lambda_1, \ldots, \lambda_N]}(z_1, \ldots, z_N) = \sum_{\sigma \in S_N} z_{\sigma(1)}^{\lambda_1} z_{\sigma(2)}^{\lambda_2} \ldots z_{\sigma(N)}^{\lambda_N}
\] (36)

\[
E_{p,k}(z_1, \ldots, z_N) = \sum_{q_1 + \ldots + q_k = p} \sum_{q_1 \geq q_2 \geq \ldots \geq q_k > 0} m_{[q_1, \ldots, q_k, 0^{N-k}]}(z_1, \ldots, z_N)
\] (37)

which respectively are: \(m_{[\lambda]}\) the monomial symmetric polynomial indexed by \(\lambda\), and \(E_{p,k}\) the sum over all monomials indexed by a partition of \(p\) in \(k\) parts.

So \(E_{p,1} = T_p(x)\) is an obvious. Other examples are

\[
E_{p,p}(z_1 \ldots z_N) = z_1 \ldots z_p + \ldots
\]

\[
E_{4,2}(z_1 \ldots z_N) = (z_1^3 z_2 + \ldots) + (z_1^2 z_2^2 + \ldots)
\] (38)
We find that the single particle operators can be written as
\[
O_p = \sum_{k=1}^{p} d_k(p, N) E_{p,k}(z_i), \tag{39}
\]
where the coefficient \(d_k(p, N)\) is
\[
d_k(p, N) = \frac{(-1)^{k+1}p(N - p + 1)_{p-k}(p-1)_k}{(N)_p - (N - p + 1)_p} \tag{40}
\]

Note that interestingly the coefficient of a monomial in this formula only depends on the number of different eigenvalues appearing in the monomial and not on any other details of the monomial.

### 2.2.3 Formula in terms of Schur polynomials

Finally we consider the SPOs written in terms of the Schur polynomial formula. In [17] a group theoretic formula for the dual of the trace basis was also given in terms of the Schur polynomial basis, \(\chi_R(\phi)\). The Schur polynomial basis is an orthogonal (in \(U(N)\)) basis for all half-BPS operators introduced in [15]. The formula given in [17] is:
\[
\zeta_{p_1...p_n} = \frac{1}{p!} \sum_{R: \lambda} \frac{\chi_R(\sigma_{p_1...p_n}) \chi_R[\phi]}{f_R} . \tag{41}
\]

The relation between the dual basis operators and single particle operators (24) together with the observation about \(S_p\) characters of cycle permutations (28) therefore gives the following explicit formula for SPOs directly in terms of the Schur polynomials of Hook Young tableaux of height \(k\), with \(p\) boxes in total, \(R_k^p\):
\[
O_p = \sum_{k=1}^{p} \tilde{d}_k(p, N) \chi_{R_k^p}[\phi] \\
\tilde{d}_k(p, N) = p(p-1)(-1)^{k-1}\frac{(N-p+1)_{p-k}(N+p-k+1)_{k-1}}{(N)_p - (N+1-p)_p} \tag{42}
\]

\[
R_k^p = [p - k + 1, 1^{k-1}] = \begin{array}{c}
\uparrow \\
\rightarrow \hspace{0.5cm} p-k \\
\downarrow \\
\end{array}
\]

14
2.2.4 Examples in the three basis

We find useful at this point to consider some low charges SPOs, and show their representation in the three basis we just constructed.

For $O_2$,

$$O_2 = T_2 - \frac{1}{N}T_{11}$$
$$O_2 = \frac{(N-1)}{N}E_{2,1} - \frac{2}{N}E_{2,2}$$
$$O_2 = \frac{(N-1)}{N}\chi_{R_1^2} - \frac{(N+1)}{N}\chi_{R_2^2}$$

(43)

For $O_3$,

$$O_3 = \frac{2}{N}T_{111} - \frac{3}{N}T_{21} + T_3$$
$$O_3 = \frac{(N-2)(N-1)}{N^2}E_{3,1} - \frac{3(N-2)}{N^2}E_{3,2} + \frac{12}{N^2}E_{3,3}$$
$$O_3 = \frac{(N-2)(N-1)}{N^2}\chi_{R_1^3} - \frac{(N-2)(N+2)}{N^2}\chi_{R_2^3} + \frac{(N+1)(N+2)}{N^2}\chi_{R_3^3}$$

(44)

For $O_4$

$$O_4 = \frac{(2N^2-3)}{N(N^2+1)}T_{22} + \frac{10}{N^2+1}T_{211} - \frac{5}{N(N^2+1)}T_{1111} - \frac{4}{N}T_{31} + T_4$$
$$O_4 = \frac{(N-3)(N-2)(N-1)}{N(N^2+1)}E_{4,1} - \frac{4(N-3)(N-2)}{N(N^2+1)}E_{4,2} + \frac{20(N-3)}{N(N^2+1)}E_{4,3} - \frac{120}{N(N^2+1)}E_{4,4}$$
$$O_4 = \frac{(N-3)(N-2)(N-1)}{N(N^2+1)}\chi_{R_1^4} - \frac{(N-3)(N-2)(N+3)}{N(N^2+1)}\chi_{R_2^4} + \frac{(N-3)(N+2)(N+3)}{N(N^2+1)}\chi_{R_3^4} - \frac{(N+1)(N+2)(N+3)}{N(N^2+1)}\chi_{R_4^4}$$

(45)

A feature of the expansion of the single-particle operator $O_p$ in the Schur basis is the homogeneous degree in $N$ of its coefficients w.r.t the partitions of $p$, i.e. the different basis elements.

2.3 SPOs interpolate between single-trace and giant gravitons

In the large $N$ limit holding the dimension of the operator $p$ fixed, the single particle operator becomes equivalent to the single trace operator

$$O_p \rightarrow T_p + O(1/N).$$

(46)

This is less obvious from the general formula in terms of traces (32), but upon inspection, the coefficients of multi-trace corrections with $m$ traces, $C_{q_1...q_m}$, are indeed $O(1/N^{m-1})$. 

15
What happens is that each term of the sum in $C_{q_1...q_m}$ is actually $O(N)$, but the alternating sum provides $m$ cancellations, one at each order in $N$, this brings it down to $O(1/N^{m-1})$. It is much more direct to see this from the formula in terms of eigenvalues (39). The coefficients of terms with $k$ different eigenvalues $d_k(p, N)$ is $O(1/N^{k-1})$. More precisely

$$d_k(p, N) \to \frac{(-1)^{k-1} p_{k-1}}{N^{k-1}} \quad \text{as} \quad N \to \infty \quad (47)$$

so the $k = 1$ term ($z_1^p + ... z_N^p$ i.e. the single trace term) dominates.

As the charge of the single-particle operator increases, approaching the number of colours $N$, it is clear that the single-particle operator is very different to the single trace operator. For example for $p > N$ the operator vanishes

$$\mathcal{O}_p = 0 \quad p > N \quad (48)$$

as can be seen directly from the explicit formulae in the previous subsection. Note that this behaviour is very different from the single-trace operators which do not vanish for $p > N$, but rather become linear combinations of multi-trace operators. This explains why the single-particle operators vanish: for $p > N$ all operators are multi-trace, therefore by definition the single-particle operators are orthogonal to all operators and hence must vanish.

In the large $N$ limit, with charge $p$ also growing large but with $N - p = p'$ fixed, we find that the single-particle operators become proportional to (sub)-determinant operators. These operators were proposed in [14] as duals to the sphere giants predicted in [13] as Kaluza Klein states with masses approaching the sphere radius. They are the Schur polynomials associated with totally antisymmetric Young tableau ($R_{p}^{p}$ in the notation of (42)) and in terms of eigenvalues take the form $z_1...z_p + ..$ where the dots denote similar terms obtained by permutations.

We find that at large $N$, the coefficient appearing in the eigenvalue formula (40) has the large $N$ limit

$$d_{N-p'}-k'/(N-p', N) \to (-1)^{N(-2)^{-1-k'-p'}(1+p)k}/N^{k'-1} \quad \text{as} \quad N \to \infty . \quad (49)$$

Thus at large $N$ we obtain:

$$\frac{1}{N} \mathcal{O}_{N-p'} \to 2^{-1-p'} \times (z_1z_2...z_{N-p'} + ..) + O(1/N) . \quad (50)$$

where the dots denote all possible terms with $N-p'$ different $z$s all with coefficient one. This is precisely the subdeterminant operator identified as the sphere giant gravitons in [13].

2.4 SPOs in $U(N)$ are SPOs in $SU(N)$

Single-particle operators can be defined for both $SU(N)$ and $U(N)$ gauge group. As we observed in the low charge examples (12), (14) and (15), the two are closely related. The
SU($N$) elementary fields are of course different from the $U(N)$ elementary field, and the two theories in general are different. However, the fact that $U(N)$ SPOs represented in the trace basis were shown to give the $SU(N)$ SPOs upon setting $\text{Tr}(\phi)$ to zero, suggests that SPOs in $U(N)$ are SPOs in $SU(N)$. In this section we make this statement rigorous.

The elementary field $\hat{\phi}_{rs}^*$ in $SU(N)$ is the traceless part of the elementary field $\phi_{rs}^*$ in $U(N)$, namely

$$\hat{\phi}_{rs}^* = \phi_{rs}^* - \frac{1}{N} \delta_{rs} \phi^t_t,$$  \hspace{1cm} (51)

For any operator $O[\phi]$ in $U(N)$ we define the $SU(N)$ projection as the map,

$$\Pi : O[\phi] \rightarrow \hat{O}[\phi] \equiv O[\hat{\phi}(\phi)]$$ \hspace{1cm} (52)

The operator $\hat{O}[\phi]$ is then a new operator in $U(N)$. For example, the $SU(N)$ projection of $O[\phi] = \text{Tr}(\phi^2)$ is

$$\hat{O}[\phi] = \text{Tr}(\hat{\phi}(\phi) \cdot \hat{\phi}(\phi)) = \text{Tr}(\phi^2) - \frac{1}{N} \text{Tr}(\phi) \text{Tr}(\phi).$$ \hspace{1cm} (53)

More generally, we find that an operator $\hat{O}[\phi]$ has the form

$$\hat{O}[\phi] = O[\phi] - [T_1 \hat{O}[\phi]] ; \quad T_1 = \text{Tr}[\phi]$$ \hspace{1cm} (54)

for some operator $\hat{O}[\phi]$. Notice that $O[\phi]$ is also the leading term in the $1/N$ expansion of $\hat{O}[\phi]$, with the fields $\phi$ and $\hat{\phi}$ treated as formal variables.

The $\hat{O}[\phi]$ operators in $U(N)$ span a subspace, since they are independent of the trace. The map $\Pi : O \rightarrow \hat{O}$ decomposes the space of $U(N)$ operators as $\text{Im}(\Pi) \oplus \text{Ker}(\Pi)$, with the kernel everything of the form $[T_1 \hat{O}[\phi]]$. To show that the $SU(N)$ projection is orthogonal we have to show that

$$\langle \hat{O}(x_1) [T_1 \hat{O}(x_2)] \rangle_{U(N)} = 0.$$

The claim follows from the fact that the operator $\hat{O}[\phi]$ is constructed only from the traceless part $\hat{\phi}_s^*$, then by applying Wick’s theorem to compute this two-point function we always find a contraction between $\hat{\phi}_s^*(x_1)$ within $\hat{O}(x_1)$ and $T_1(x_2)$. In the $U(N)$ theory with propagator (3) we have

$$\langle \hat{\phi}_s^*(x_1) T_1(x_2) \rangle = \langle \phi_s^*(x_1) \phi_t^t(x_2) \rangle - \frac{1}{N} \delta_{st} \langle \phi_u^u(x_1) \phi_t^t(x_2) \rangle = 0$$ \hspace{1cm} (56)

Thus any operator $\hat{O}[\phi]$ constructed only from the traceless part $\hat{\phi}_s^*$ is orthogonal to any operator involving the trace as a factor $[T_1 \hat{O}]$. 

17
Single-particle operators $\mathcal{O}$ in $U(N)$ are precisely defined to be orthogonal to all multi-traces, in particular to the operators of the form $[T_1 \mathcal{O}]$ that we discussed above. Thus single particle operators in $U(N)$ automatically live in the $SU(N)$ subspace. The $SU(N)$ single-particle operator is now candidate to give the $U(N)$ single-particle operator.

Since the two-point orthogonality in $U(N)$ and $SU(N)$ is obtained via Wick’s theorem with two different propagators, the only thing remaining to check is that the $U(N)$ inner product restricted on the $SU(N)$ subspace is the same as the inner product of the $SU(N)$ theory. A short computation shows that,

$$\langle \hat{\phi}_s^* (\phi) \hat{\phi}_t (\phi) \rangle_{U(N)} = \left( \phi_s^* - \frac{1}{N} \delta_s^u \phi_v^v \right) \left( \phi_t^u - \frac{1}{N} \delta_t^u \delta_u^v \right)_{U(N)} = \delta_s^u \delta_t^u - \frac{1}{N} \delta_s^u \delta_u^v \delta_t^u$$

which is precisely the $SU(N)$ propagator (4).

We conclude that $U(N)$ single-particle operators are SPOs in $SU(N)$,

$$\mathcal{O}^U_p[\phi] = \mathcal{O}^{SU}_p[\hat{\phi}] \quad p \geq 2$$

and any correlators of SPOs of charge 2 or higher computed in either $U(N)$ or $SU(N)$ will be identical.

Our result here manifests, purely in the free theory, the well known feature of $\mathcal{N}=4$ SYM in the context of AdS/CFT that the $U(1)$ part of the gauge group $U(N)$ decouples and only $SU(N)$ remains in the interacting theory. Here we find that the sector of single-particle operators with length greater than one in the $U(N)$ theory is that of the $SU(N)$ theory, and furthermore it is orthogonal to anything we could construct with a $T_1$.

### 2.5 Two-point function normalisation

From our study of the SPOs in the trace basis, we can obtain readily their two-point function normalisation. Indeed, it follows from the relation (24) that such a normalisation is just the inverse of the $\xi_p$ normalisation from the dual basis, namely

$$\langle \mathcal{O}_p(x_1) \mathcal{O}_p(x_2) \rangle = R_p g^p_{12} \quad ; \quad \langle \xi_p(x_1) \xi_p(x_2) \rangle = \frac{1}{R_p} g^p_{12}$$

where we use the notation $R_p$ to denote the $N$-dependent color factor. From (31) it takes the form

$$R_p = p^2 (p - 1) \left[ \frac{1}{(N - p + 1)p - 1} - \frac{1}{(N + 1)p - 1} \right]^{-1}.$$  

\(^5\text{Notice that in the examples (12), (14) and (15), we did not use the basis } Im(\Pi) \oplus Ker(\Pi) \text{ that instead we are constructing here.}\)
Note that $R_p$ has zeros at $N = 1, \ldots, p - 1$, reflecting the fact that the operator vanishes when $N < p$, and since it is also symmetric in $N \rightarrow -N$ it must contain explicit factors of the form $(N^2 - r^2)$, for $r = 1, \ldots, p - 1$.

It is useful to make these facts manifest by rewriting

$$R_p = \frac{p}{N^{p-2}} \times \prod_{r=1}^{p-1} (N^2 - r^2) \times \frac{1}{Q_p(1/N^2)}, \quad (61)$$

where $Q_p(1/N^2)$ is a polynomial of degree $\lfloor \frac{p}{2} \rfloor$ in $1/N^2$. The first few cases are given by

$$Q_2(1/N^2) = 1, \quad Q_3(1/N^2) = 1, \quad Q_4(1/N^2) = 1 + \frac{1}{N^2}. \quad (62)$$

whereas the general form of $Q_p(1/N^2)$ is given by

$$Q_{p-2}(N) = \frac{(N+1)_{p-1} - (N-p+1)_{p-1}}{p(p-1)}. \quad (63)$$

One can manifest the $N \rightarrow -N$ symmetry by using rising and lowering factorials instead of always the rising factorial (Pochhammer). With the notation $x^P$ for the Pochhammer (rising factorial) and $x^L$ for the falling factorial, we find

$$R_p = p^2(p-1) \frac{(N-1)^{p-1}(N+1)^{p-1}}{(N+1)^{p-1} - (N-1)^{p-1}} \quad (64)$$

In the form (64) it is clear that $R_p$ is the simplest possible rational function of $N^2$ with the above zeros and of order $O(N^p)$ at large $N$.

### 2.6 On multi-particle operators

A complete basis of half-BPS operators in the theory is obtained by taking arbitrary products of the single-particle operators. We call this basis the multi-particle basis.

The statement above can be true also for single-trace operators, modulo the fact that single-traces overcount. Any $T_p$ with $p > N$ is not an independent operator both in $SU(N)$ and $U(N)$, and yet it has non trivial two point functions with other operators. On the other hand, the multi-particle basis does not have this issue. Indeed, if $p > N$ there are only multi-trace operators, and the single-trace operator is not independent. Then, by definition, the single-particle operators are orthogonal to all operators and hence must vanish. Very remarkably, this feature is automatically implemented in the two-point function normalisation (64).
The multi-particle basis will not be orthogonal, although of course the single-particle sector will be orthogonal to the multi-particle sector. This is to be contrasted with the Schur polynomial basis of [15] which is both complete and orthogonal, for all half-BPS operators. In particular, operators are in 1-1 correspondence with Young tableau of height no more than \( N \).

Unlike the Schur polynomial basis, the multi-particle basis has the advantage of being a basis both for \( SU(N) \) and \( U(N) \), depending simply on whether \( O_1 \) is included or not.

Requiring orthogonality of multiparticle states is however subtle. To obtain an orthogonal basis one can of course simply implement the Gram Schmidt procedure: Start with an ordered list of basis elements, then leave the first element unchanged, then run over \( n > 1 \) by considering the \( n \)-th element and add linear combinations of previous elements such that the new element is orthogonal to all previous ones. So for example at weight 6 we could choose the ordering

\[
(T_{111111}, T_{21111}, T_{22111}, T_{3111}, T_{3211}, T_{411}, T_{33}, T_{42}, T_{51}, T_6) \quad U(N)
\]
\[
(T_{222}, T_{33}, T_{42}, T_6) \quad SU(N)
\]

Then performing Gram-Schmidt orthogonalisation, would provide an orthogonal set of operators. In sum this can be done by cutting appropriately the matrix in (9). However at this point, the single-particle operator is singled out as the last one. In the example above \( O_6 \), is the one with greater admixture, in contrast to \( T_{111111} \) or \( T_{222} \), but then it is not clear if there is a canonical choice for the ordering of the other operators.

As long as the single-particles are fixed, a completely equivalent way of obtaining the same orthogonal basis is to start with the dual basis, list the operators in reverse order, and perform Gram-Schmidt orthogonalisation. This process yields exactly the same orthogonal basis (up to normalisation). Referring to the same example as above

\[
(\xi_6, \xi_51, \xi_42, \xi_33, \xi_411, \xi_321, \xi_222, \xi_3111, \xi_22111, \xi_{1111111}) \quad U(N)
\]
\[
(\xi_6, \xi_42, \xi_33, \xi_222) \quad SU(N)
\]

In the first approach the operator with the most traces \( T_{111111} \) (or \( T_{222} \)) remains unchanged whereas the single-particle operator is the most complicated from the point of view of the trace-basis. In the second approach, this is turned on its head. In terms of the dual basis, the single particle operator is – just \( \xi_\rho \) – whereas an operator labelled by a partition with increasing length becomes more intricate from the point of view of the \( \xi \) basis. In the end, the most intricate of such operators - a linear combination of all dual operators - must equal \( T_{11111} \) (or \( T_{222} \)).

We leave the task of extending the single-particle operators to a full orthogonal basis to a future work. Perhaps an AdS/CFT understanding of multi-particle KK modes will help us figuring out a canonical way to fix the multi-particle states in \( \mathcal{N} = 4 \) SYM.
3 Multipoint orthogonality

In the previous section we obtained explicit expressions for SPOs. From the two-point orthogonality it followed that SPOs are given by a specific admixture of single- and multi-trace operators. In a sense, the SPOs are composites of traces and the richness of this structure would suggest that multipoint correlation functions are rather more complicated than multipoint single-trace correlation functions. However this expectation is too naive and as a first counter-example we show in this section that the defining two-point function orthogonality uplifts to a multipoint orthogonality theorem, which in turn implies vanishing of a large class of diagrams. Let us state our main result.

**Multipoint orthogonality Theorem.** Any propagator structure, which can contribute to any half-BPS correlator, with a single-particle operator $O_p$ connected to two sub-diagrams, themselves disconnected from each other, has a vanishing color factor. The statement holds for both $SU(N)$ and $U(N)$ free theories.

We are considering any propagator structure $F_{p|q_1...q_{n-1}}$, which becomes disconnected upon removing $O_p$, and thus has the shape of a dumbbell:

$$F_{p|q_1...q_{n-1}} = \begin{array}{ccc}
  \bullet & T_{q_1} & \cdots \\
  \cdots & \cdots & \cdots \\
  \bullet & T_{q_{n-1}} & \cdots \\
\end{array}$$

A propagator structure of this sort will have a number of bridges going between points in the two (green) blobs, left and right, and a number of bridges connecting $O_p(x)$ with points belonging to the left and right blobs. No bridges between left and right blobs.

Our theorem states that

$$F_{p|q_1...q_{n-1}} = 0. \tag{68}$$

We can gain some intuition about the multipoint orthogonality by considering the fact that $F_{p|q_1...q_{n-1}}$ can be rearranged as a determinant. This follows directly from the representation of $O_p(x)$ as a determinant, that we mentioned in (9). All rows of this determinant, but the last one, are given by the (color factor of the) two points functions $\langle T_{\lambda_i}T_{\lambda_j} \rangle$, where $\lambda_i$ and $\lambda_j$ denote partitions of $p$. The last row is given by (the color factors of) $\langle T_{\lambda_i}(x)T_{q_1}(x_1)\cdots T_{q_{n-1}}(x_{n-1}) \rangle$ in the given propagator structure. The determinant will vanish as long as this last row reduces to a linear combination of the others. But this linear combination might be $N$-dependent and thus hard to chase. We give some examples in Appendix C. However, a crucial point is that for any two point functions $\langle T_{\lambda_i}T_{\lambda_j} \rangle$ at
least one partition has length greater than two. Therefore we might displace a part of it on a fictitious point, and since there cannot be propagators inside a $T_\lambda$, this configuration is actually a dumbbell. We learn in this way that, for the determinant to have a chance to vanish, the topology of the $(T_\lambda(x)T_{q_1}(x_1)\ldots T_{q_{n-1}}(x_{n-1}))$ has to have the same feature of the two point functions $\langle T_\lambda T_\lambda \rangle$. We can then argue that $\mathcal{F}_{p|q_1\ldots q_{n-1}}$ is a dumbbell, as we draw in (67).

The reasoning above leaves a free parameter. In fact, the assignment $q_1, \ldots, q_{n-1}$ can be such that

$$\frac{1}{2} \left( -p + \sum_{i=1}^{n-1} q_i \right) = k \geq 0$$

and yet the diagram disconnects as soon as we remove $\mathcal{O}_p$. The value of $k$ measures the excess of $\sum q_i$ to be a partition of $p$. This is possible precisely because differently from a two-point function, in a multipoint function there can be $k \geq 0$ Wick contractions distributed among the $T_{q_1}(x_1)\ldots T_{q_{n-1}}(x_{n-1})$, such that when $\mathcal{O}_p$ is removed, the diagram still disconnects.

In general, the combinatorics which leads to $\mathcal{F}_{p|q_1\ldots q_{n-1}} = 0$ is hard. It would be very interesting to have a combinatorial proof of our theorem, by using Wick contractions and double line notation, but our proof in the next section will go through a nice alternative route.

Some readers might wish to jump directly to section 3.2 at this point. There we use multipoint orthogonality to show that near-extremal $n$-point functions, defined by the constraint $k \leq n - 3$, vanish.

## 3.1 Proof of the theorem

As long as the topology of $\mathcal{F}_{p|q_1\ldots q_{n-1}}$ is fixed as in (67), we can work with a fixed propagator structure.

To show that $\mathcal{F}_{p|q_1\ldots q_{n-1}} = 0$, we will first make an argument for the part of the diagram, consisting of $\mathcal{O}_p$ and everything to the right. Take $\mathcal{O}_p(x)T_{q_{r+1}}(x_{r+1})\ldots T_{q_{n-1}}(x_{n-1})$ and consider Wick contracting the fundamental fields in all possible ways consistent with the propagator structure $\mathcal{F}_{p|q_1\ldots q_{n-1}}$. Since the number of bridges going from $\mathcal{O}_p(x)$ to the right is fixed and less than $p$, some of the fundamental fields inside $\mathcal{O}_p(x)$ will remain unlinked, thus resulting in a new half-BPS operator of lower charge, say $R$, inserted at $x$. For each arrangement of Wick contractions, we can write this new half-BPS operator of charge $R$...
in the trace basis, as illustrated below,

$$\prod g_{ij}^{d_i j} \sum_{R \vdash R} C_R T_R(x) = O_p(x)$$

The sum is over operators $T_R(x)$, indexed by partitions of $R$, and multiplied by a coefficient $C_R$. The (partial) propagator structure $\prod g_{ij}^{d_i j}$ is the one we assigned and it is factored out.

The initial propagator structure is now computed as

$$\mathcal{F}_{p|q_1...q_{n-1}} = \prod g_{ij}^{d_i j} \sum_{R \vdash R} C_R \langle T_R'(x') T_R(x) \rangle = O_p(x)$$

This is a vector equation for $C_R$, which we can solve by inverting the matrix of two point functions. It follows that the coefficients $C_R$ are given by computing another dumbbell diagram. The two-point functions $\langle T_R(x') T_R(x) \rangle$ are given by $g_{x'x}$, which we can factor out, times the color factor, $C_R$. For clarity we will keep using the notation $\langle T_R(x') T_R(x) \rangle$.

We can now replace the $C_R$ and rewrite the original $\mathcal{F}_{p|q_1...q_{n-1}}$ as follows,

$$\mathcal{F}_{p|q_1...q_{n-1}} \approx \sum_{R \vdash R \vdash R} \langle T_R T_R' \rangle^{-1}$$

Repeating a similar discussion for the left hand side of $\mathcal{F}_{p|q_1...q_{n-1}}$ we conclude then that the original propagator structure can be rewritten as
\[ \mathcal{F}_{\mu Q_1 \cdots Q_{n-1}} \simeq \sum_{L, L', L R, R'} \sum_{T_{L'}, T_{R'}} \sum_{O_p T_{L'} T_{R'}} \sum_{\langle T_{L'} T_{L} \rangle^{-1}} \sum_{(T_{R'} T_{R})^{-1}} \sum_{T_{L} T_{Q} T_{R}} \]

(74)

Crucially, the central connected diagram in (74) is a three-point function \( \langle O_p T_Q T_R \rangle \) with \( p = Q + R \), thus extremal. To conclude we then have to show that any extremal three-point function vanishes, i.e.

\[ \langle O_p T_Q T_R \rangle = 0 \quad ; \quad p = Q + R . \]

(75)

This is a direct consequence of the two-point function orthogonality, which we used in (8) to define the SPOs, together with the fact that extremal 3-point functions are directly related to the corresponding two-point function obtained by bringing points 2 and 3 together (since there are no propagators between \( T_Q \) and \( T_R \))

\[ \langle O_p T_Q T_R \rangle = \left( \frac{g_{13}}{g_{12}} \right)^R \langle O_p [T_Q T_R] \rangle = 0 , \]

(76)

This concludes our proof. We thus have shown that any propagator structure involving a single \( O_p \), which becomes disconnected on removing \( O_p \) vanishes.

### 3.2 Near-Extremal correlators vanish

When discussing \( n \)-point functions of half-BPS operators it is useful to introduce the degree of extremality \( k \) defined as follows. Let \( p \) be the largest charge, and \( q_i \) the others, then

\[ k = \frac{1}{2} \left( -p + \sum_{i=1}^{n-1} q_i \right) . \]

(77)

This definition of \( k \) should be now familiar from (69).

For \( k < 0 \) correlators vanish purely by \( SU(4) \) symmetry. Extremal (\( k = 0 \)) and next-to-extremal (\( k = 1 \)) correlators, they all were shown to be non-renormalised in [19–23].

In [18] the concept of “near-extremal” correlators was introduced. These are correlators in which the degree of extremality \( k \) is not too close to the number of points. Specifically

\[ k \leq n - 3 . \]

(78)
Notice that for \( n \geq 5 \), the near-extremal correlators go beyond the extremal- and next-to-extremal cases mentioned above.

We claim that any near-extremal \( SU(N) \) correlator where the largest charge operator is a single-particle operator vanishes:

\[
\langle O_p(x) T_{q_1}(x_1) \cdots T_{q_{n-1}}(x_{n-1}) \rangle = 0 \quad k \leq n - 3 .
\]  

(79)

As usual, \( T_{q_i} \) stands for any half-BPS operator with total charge \( q_i \). We remark that any refers to any single- or multi-trace operator or any combination of these. A corollary of this is that any near-extremal correlator involving only SPOs vanishes.

\[
\langle O_p(x) O_{q_1}(x_1) \cdots O_{q_{n-1}}(x_{n-1}) \rangle = 0 \quad k \leq n - 3 .
\]  

(80)

A similar statement can also be made in the \( U(N) \) theory, with the caveat that then it is only true for connected correlators: any connected near-extremal \( U(N) \) correlator where the largest charge operator is a single-particle operator vanishes.

In order to prove (79) we will now show that every propagator structure contributing to this correlator has the dumbbell shape, as in (67), namely it becomes disconnected on removing \( O_p \).

Let us first show that there are two few propagators between the operators \( T_{q_i} \), to connect them all together, and therefore there will be two green blobs as in (67). Indeed, since there are \( p \) legs coming out of \( O_p \), and \( k \) is the excess of \( \sum q_i \), there are \( k \) propagators among the \( T_{q_i} \)s. But \( k \leq n - 3 \) and there are \( n-1 \) operators \( T_{q_i} \). The minimal scenario to link all the \( T_{q_i} \) would be to have a necklace with \( k = n - 1 \) bridges, which is not possible, both for \( SU(N) \) and \( U(N) \). We would need at least two more points.

We understood that the topology of \( O_p(x) T_{q_1}(x_1) \cdots T_{q_{n-1}}(x_{n-1}) \) is such that it is not possible to connect all \( T_{q_i} \). This would imply the diagram is dumbbell, but for the case in which the diagram is actually made of two disconnected parts, i.e. one green blob is actually on its own. For concreteness let’s say \( T_{q_1}(x_1) \cdots T_{q_r}(x_r) \) is disconnected from \( O_p(x) T_{q_{r+1}}(x_1) \cdots T_{q_{n-1}}(x_{n-1}) \), for some value of \( r \). Let us see when this can happen: The number of bridges among the \( T_{q_{r+1}}, \ldots, T_{q_{n-1}} \) which are not connecting with \( O_p \) is

\[
k_R = \frac{1}{2} \left( -p + \sum_{i=r+1}^{n-1} q_i \right)
\]

(81)

We can assume \( O_p(x) T_{q_{r+1}}(x_{r+1}), \ldots, T_{q_{n-1}}(x_{n-1}) \) to remain connected on removing \( O_p \), and since there are \( n - r - 2 \) operators \( T_{q_{r+1}}(x_{r+1}), \ldots, T_{q_{n-1}}(x_{n-1}) \), the minimum scenario would be to have them forming a tree, thus \( k_R \geq n - r - 2 \). If we now recall the inequality on \( k \) we find

\[
k = \frac{1}{2} \left( -p + \sum_{i=r+1}^{n-1} q_i + \sum_{i=1}^{r} q_i \right) \leq n - 3 \quad \Rightarrow \quad \frac{1}{2} \sum_{i=1}^{r} q_i \leq n - 3 - k_R \leq r - 1
\]

(82)
But \( \frac{1}{2} \sum_{i=1}^{r} q_i \) is the total number of bridges among the fields \( T_{q_1}(x_1) \ldots T_{q_r}(x_r) \), which recall are disconnected from the rest of the diagram. Therefore, we learn that at most they form a tree. In a tree, some operators will necessarily be connected to others just with a single bridge. For the \( U(N) \) theory this is possible if a number of \( O_1 \) fields are inserted in the original correlator. For \( SU(N) \) theory there is not such a possibility.

We conclude that all \( SU(N) \) diagrams contributing to near extremal correlators have a dumbbell shape (67) and hence vanish. Similarly for all connected \( U(N) \) diagrams.

### 3.2.1 More cases: coincident points and multi-trace splitting

A useful corollary of the vanishing of near-extremal correlators occurs for lower point correlators, say \( m \)-points, which are not near-extremal but have a number of multi-trace operators,

\[
\langle O_p(x) T_{q_1}(x_1) \ldots T_{q_{m-1}}(x_{m-1}) \rangle ; \quad k = \frac{1}{2} (-p + \sum q_i) \geq m - 3. \tag{83}
\]

The idea is that when a number \( r \) of operators are genuinely multi-trace, we can think of (83) as a correlator \( \mathcal{F} \) with \( n \geq m \) points (with specific propagator structure since there cannot be propagators inside a multi-trace). Namely

\[
\langle O_p(x) T_{q_1}(x_1) \ldots T_{q_{m-1}}(x_{m-1}) \rangle \to \mathcal{F}(p|q'_1 \ldots q'_n) \tag{84}
\]

The number of new points \( n \) depends on the way we want to think of the multi-trace operators. The max we can do is to put all its parts on fictitious points. So if \( l(q_i) \) measures the number of parts of \( q_i \), then

\[
m \leq n \leq (m - r) + \sum_{i=1}^{r} l(q_i) \tag{85}
\]

If there is a value of \( n \) such that \( k \leq n - 3 \), the original correlator vanishes, since it can be thought of as an \( n \)-point near-extremal correlator.

A simple example. For three-point functions we find

if \( \exists \ i \) such that \( l(q_i) \geq 2 \) and \( k \leq 1 \) \( \Rightarrow \langle O_p(x) T_{q_1}(x_1)T_{q_2}(x_2) \rangle = 0 \tag{86}\)

The claim follows because in this case we can always think of the three-point function as a near-extremal four-point function, which vanishes.

Another example is

if \( \exists \ i \) such that \( l(q_i) \geq 2 \) and \( k \leq m - 2 \) \( \Rightarrow \langle O_p(x) T_{q_1}(x_1) \ldots T_{q_{m-1}}(x_{m-1}) \rangle = 0 \tag{87}\)

There are many possibilities involving splitting of more than one operator, but we do not list them all, since it should be clear how to generalise the examples above.
4 Exact results for correlators of SPOs

4.1 Maximally-Extremal correlators

Given the vanishing of all near-extrema correlators of SPOs in (78), the next cases to consider are
\[ \langle O_p(x_1) \cdots O_{q_{n-1}}(x_{n-1}) \rangle ; \quad k = n - 2 ; \quad k = \frac{1}{2} (-p + \sum q_i) \] (88)
which we call Maximally-Extremal (ME).

We will compute all ME correlators, beginning with three-points and generalising the argument to \( n \)-points. From now on we focus on the \( SU(N) \) theory. The final result will have the following flavor,

\[ \langle O_p(x_1) \cdots O_{q_n-1}(x_{n-1}) \rangle_{\text{connected}} = \langle O_p O_p \rangle \left( \sum_{\text{trees } T} |W[T]| T_{d_1 \cdots d_{n-1}} \right) \] (89)

where the sum is over all trees \( T \) on \( n - 1 \) points, each point with associated degree \( d_i \geq 1 \) (number of legs per point) such that \( \sum_{i=1}^{n-1} d_i = 2(n - 2) \). The tree is specified by a sub-propagator structure \( b_{ij} \) which we arrange into a matrix, and finally,

\[ |W[T]| = \prod_{i=1}^{n-1} q_i(q_i - 1) \cdots (q_i - d_i + 1) \] (90)

We will progressively get to this result.

4.1.1 3-point functions

ME three-points functions are also next-to-extremal three-point functions since \( k = 1 \). In order to compute them, notice first that

\[ \langle O_p O_{q_1}(x_1) O_{q_2}(x_2) \rangle = \langle O_p T_{q_1}(x_1) T_{q_2}(x_2) \rangle, \quad p = q_1 + q_2 - 2 \] (91)

In fact, the difference between the two is a sum of three-point functions with at least a multi-trace. By the results in section 3.2.1, they all vanish.

In the three-point function \( \langle O_p T_{q_1} T_{q_2} \rangle \) there is a single propagator between \( T_{q_1} \) and \( T_{q_2} \), and therefore a single Wick contraction to do in between \( x_1 \) and \( x_2 \). If at the same time we bring together these two insertion points, we obtain the following result,

\[ \lim_{x_1 \to x_2} \text{Tr} \left( \phi(x_1) \cdots \phi(x_1) \right) \text{Tr} \left( \phi(x_2) \cdots \phi(x_2) \right) \simeq \text{Tr} \left( \phi(x_2) \cdots \phi(x_2) \right) + \ldots \] (92)
Intuitively, since each trace is cyclic invariant we are always considering a configuration which in double line notation looks like the following drawing (with many more free legs),

![Diagram](image)

But in general we find all the contributions of an OPE of scalars on the r.h.s of (92), however since we will contract that with the half-BPS operator $O_p$, any other term does not survive. Clearly there are $q_1 q_2$ ways to perform this Wick contraction and we arrive at

$$
\langle O_p T_{q_1} T_{q_2} \rangle = q_1 q_2 \langle O_p T_p \rangle
$$

(94)

Finally we have that $\langle O_p T_p \rangle = \langle O_p O_p \rangle$ from the defining orthogonality of $O_p$.

Putting all this together we arrive at the result that next-to-extremal three-point functions are simply expressed in terms of two-point functions,

$$
\langle O_p O_{q_1} O_{q_2} \rangle = q_1 q_2 \langle O_p O_p \rangle ; \quad p = q_1 + q_2 - 2.
$$

(95)

4.1.2 $n$-point functions

Now consider maximally extremal $n$-point functions.

As in the case of three-points in (95), note that we can replace all operators, apart from the one with the largest charge, by single trace operators,

$$
\langle O_p(x) O_{q_1}(x_1) \cdots O_{q_n}(x_{n-1}) \rangle = \langle O_p(x) T_{q_1}(x_1) \cdots T_{q_{n-1}}(x_{n-1}) \rangle ; \quad k = n - 2
$$

(96)

The difference between the two is a sum of $n$-point functions involving at least one multi-trace operator. By the results in section 3.2.1, they all vanish.

A non-vanishing connected diagram contributing to the correlator (96) is such that, upon deleting $O_p$ and all the bridges attached to it, there will be $n - 2$ propagators amongst the $n - 1$ operators $T_{q_i}$, just enough to connect the $T_{q_i}$’s all together in a tree. We draw a five-point example here below for clarity,
In this case the tree consists of $T_{q_1}$ connected to $T_{q_2}$, $T_{q_3}$, and $T_{q_4}$. Notice that in a tree there is one and only one bridge which connects pairs of operators. On each point we can have at most an $n$-pointed star with some $n \geq 1$.

We have understood that a propagator structure contributing to a ME $n$-point correlator (96) contains a sub-propagator structure which is a tree $T$. We can represent it by using our same notation for propagator structures in appendix B, namely

$$
T \simeq \begin{bmatrix}
  d_1 & b_{12} & b_{13} & \ldots & b_{1n-1} \\
  d_2 & b_{23} & \ldots & \vdots \\
  \vdots & \vdots & \ddots & \vdots \\
  d_{n-1} & & & & \\
\end{bmatrix} ; \\
\quad d_i \geq 1 \\
\quad b_{ij} \in \{0, 1\} (98)
$$

where now $d_i \leq p_i$ is the number of legs of $T_{q_i}$ entering the tree, and the number of bridges $b_{ij}$ from point $i$ to point $j$ can only take two possible values, $b_{ij} \in \{0, 1\}$. The latter does not yet guarantees that (98) is a tree (for example necklace configurations can be arranged with $b_{ij} = 1$). A unique way to label a tree is to use a Prüfer sequence $s = (s_1 \ldots)$. We explain how the Prüfer algorithm works in appendix B.2. This point of view allows to reduce the pairwise computation of Wick contractions to a sequence, and in particular, gives us the result

$$
|W[T]| = \prod_{i=1}^{n-1} q_i(q_i - 1) \ldots (q_i - d_i + 1) (99)
$$

More details can be found in appendix B.2.

At this point, recall that the legs of the $T_{q_i}$s which do not enter the tree, form a bridge with the single-particle operator $O_p$. By performing the Wick contraction on the tree, leaf by leaf as in the Prüfer algorithm, each time bringing together the points, we can use the same argument of section 4.1.1 to infer that the net result of the tree is to glue together the $T_{q_i}$s to form a new single trace operator of charge $T_p$, together with higher trace terms. These higher-trace terms will not contribute in the full ME correlator according to the discussion in section 3.2.1.
The entire connected part of the ME correlator can thus be written as a sum over trees, multiplying the two point function \( \langle O_p O_p \rangle \) overall:

\[
\langle O_p(x) O_q(x_1) \cdots O_{q_{n-1}}(x_{n-1}) \rangle_{\text{connected}} = \langle O_p O_p \rangle \left( \sum_{\text{trees } T} |W[T]| T \left[ d_1 \cdots d_i \cdots d_{n-1} \right] \right)
\]

(100)

where the sum is restricted to \( d_i \geq 1 \) such that \( \sum_{i=1}^{n-1} d_i = 2(n-2) \).

Note that the total number of trees contributing to this correlator is well known and given by Cayley’s formula:

\[
(n-1)^{n-3}.
\]

(101)

Note also that many trees correspond to the same arrangement of degrees, \( d_{i=1,\ldots,n-1} \). Therefore many trees have the same value of \( |W[T]| \). (In fact the tree is uniquely specified only once the configuration of bridges is also specified.) This degeneracy is counted by the multinomial coefficient

\[
\frac{(n-3)!}{\prod_{i=1}^{n-1}(d_i-1)!}.
\]

(102)

**Disconnected contributions**

The ME correlator can have disconnected contributions even in the \( SU(N) \) theory. Indeed there can be a disconnected component if and only if at least two of the \( T_{q_i} \) operators are \( O_2 \). These are precisely necklace configurations of \( SU(N) \), and products thereof. ⁶ For example, we might have

\[
\langle \prod_{i=1}^{K} O_2(x_i) O_{q_{k+1}} \cdots O_{q_{n-1}} O_p \rangle_{\text{discon.}} = \langle \prod_{i=1}^{K} O_2(x_i) \rangle_{\text{necklace}} \langle O_{q_{k+1}} \cdots O_{q_{n-1}} O_p \rangle_{\text{conn.}}
\]

(103)

Note that the factored expressions on the r.h.s are themselves maximally extremal. In fact if there are \( K O_2 \), then \( n \to n' \equiv n - K \), and we find

\[
n - 2 = \frac{1}{2}(-p + 2K + \sum_{i=\ell+1}^{n-1} q_i) \quad \to \quad (n' - 2) = \frac{1}{2}(-p + \sum_{i=\ell+1}^{n-1} q_i)
\]

(104)

Any other type of disconnected component will vanish as at least one of the connected pieces will be near extremal.

Let us conclude this section with some examples:

---

⁶Following a discussion similar to the one around (82), we conclude that at least one of the connected components has to be made up entirely of \( O_2 \)s.
4-pt function

Here there are three independent arrangements for the possible values of $d_i$, the number of legs of position $i$ in the tree, we write them as $|d_1d_2d_3⟩ = |211⟩,|121⟩$, and $|112⟩$. There is no degeneracy (102), and the correlator is:

$$\langle O_pO_qO_{q_1}O_{q_2}O_{q_3}\rangle_{\text{connected}} = q_1q_2q_3 \langle O_pO_p⟩ [(q_1-1)|211⟩ + (q_2-1)|121⟩ + (q_3-1)|112⟩] .$$  (105)

5-pt function

The number of trees is $4^2 = 16$. This is a case with degeneracy, since the arrangements of $|d_1d_2d_3d_4⟩$ are the following. Four non-degenerate configurations $|3111⟩,|1311⟩,|1131⟩,|1113⟩$. Then twice the configurations $|2211⟩,|2121⟩,|2112⟩,|1221⟩,|1212⟩,|1122⟩$. In fact, the degeneracy is two, as counted by (102), and there are twelve trees plus the four non-degenerate, in total sixteen.

Let us also notice that the contribution of some tree might vanish for low charges. For example, associated to $|3111⟩$ is $q_1q_2q_3q_4(q_1-1)(q_1-2)$ that vanishes in the case $q_1 = 2$. Then, in the case of $\langle O_2(x)O_2O_2O_2O_2⟩$ we are left with the twelve degenerate trees, all contributing with coefficient $q_1q_2q_3q_4 = 16$.

4.2 Next-to-Maximally-Extremal correlators

Going down in extremality, we consider Next-to-Maximally-Extremal (NME). These are $n$-point correlators with $k = n-1$, i.e.

$$\langle O_p(x)O_{q_1}(x_1)\ldots O_{q_{n-1}}(x_{n-1})\rangle ; \quad k = n-1 ; \quad k = \frac{1}{2}(-p + \sum q_i)$$  (106)

We will first study three-point functions and then move to $n$-point functions.

4.2.1 3-point functions

Starting with $n = 3$ we are studying next-next-to-extremal three-point functions

$$\langle O_p(x)O_{q_1}(x_1)O_{q_2}(x_2)⟩ ; \quad p = q_1 + q_2 - 4.$$  (107)

As for the NE case, these can be related to the corresponding two point function $\langle O_pO_p⟩$, but with a more complicated coefficient containing non-factorisable polynomials.

To evaluate the three-point function, we replace $O_{q_1}$ and $O_{q_2}$ by their respective expansions in the trace basis (32). Differently from what happens in the ME case, see section
4.1.1, this time the expansion truncates for any term involving higher than double-trace operators at a single point. These higher-traces will result in vanishing diagrams with \( n \geq 5 \) points, according to our general results in section 3.2.1. For the same reason we can replace the double trace terms with products of single particle operators. Therefore we conclude that:

\[
\langle O_p(x)O_{q_1}(x_1)O_{q_2}(x_2) \rangle = \langle O_p(x)T_{q_1}(x_1)T_{q_2}(x_2) \rangle
\]  

(108)

\[+
\sum_{p_1=2}^{2} C_{p_1(q_1-p_1)} \langle O_p(x)[O_{p_1}O_{q_1-p_1}](x_1)O_{q_2}(x_2) \rangle
\]  

(109)

\[+
\sum_{p_2=2}^{2} C_{p_2(q_2-p_2)} \langle O_p(x)O_{q_1}(x_1)[O_{p_2}O_{q_2-p_2}](x_2) \rangle
\]  

(110)

where \( C_{(p_1,p_2)} \) is the mixing coefficient between single-particle states and double-trace operators \( T_{p_1}T_{p_2} \). (We have written its explicit form in appendix, see (168)).

The terms involving double-traces are equivalent to the four-point ME diagrams given in the previous section. Precisely in (105). We find,

\[
\langle O_p(x)[O_{p_1}O_{q_1-p_1}](x_1)O_{q_2}(x_2) \rangle = q_2(q_2-1)p_1(q_1-p_1)|112\rangle\langle O_pO_p \rangle
\]  

(111)

\[
\langle O_p(x)O_{q_1}(x_1)[O_{p_2}O_{q_2-p_2}](x_2) \rangle = q_1(q_1-1)p_2(q_2-p_2)|112\rangle\langle O_pO_p \rangle
\]  

(112)

There is one term since \( |211 \rangle \) and \( |121 \rangle \) vanish because there cannot be a bridge within \( [O_{p_1}O_{q_1-p_1}](x_1) \).

The only unknown is therefore \( \langle O_p(x)T_{q_1}(x_1)T_{q_2}(x_2) \rangle \). This consists of diagrams with two propagators between \( T_{q_1} \) and \( T_{q_2} \) which always count \( q_1(q_1-1)q_2(q_2-1)/2 \) Wick contractions, according to the general results in appendix B (see (186)). However the net color factor has \( 2^2 = 1 + 1 + 2 \) contributions for each arrangement of Wick contraction, depending on which part of the \( SU(N) \) propagator in (4) we consider, i.e. whether we use the \( U(N) \) part or the other one, \(-\frac{1}{N}\delta^{m_1}\delta^{m_2}\), which we call the capping part.

If we consider the two Wick contraction between \( T_{q_1}(x_1)T_{q_2}(x_2) \) but we use just the capping part since that caps on the matrix indexes on each end of the propagator, we find effectively a dumbbell diagram, which vanishes.

If we consider the two Wick contraction between \( T_{q_1}(x_1)T_{q_2}(x_2) \) but we use just the \( U(N) \) propagator, we expect to produce effective operators of charge \( q_1 + q_2 - 4 \) labelled by partitions of 4. In \( SU(N) \), we find two, the single trace and a double trace. Consider now that generically we produce an effective double trace operator \( [T_{q_1-2}T_{q_2-2}] \) and thus another dumbbell diagram, which will vanish. We illustrate this mechanism with an example
in double line notation,

\[ \begin{array}{c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 & 13 & 14 & 15 & 16
\end{array} \]

\[ \sim \]

\[ \begin{array}{c}
8 & 3 & 4 & 7 & 9 & 14 & 13 & 10 \\
11 & 12 & 1 & 6 & 5 & 2 & 19 & 17
\end{array} \]

Instead, there are two ways of achieving a non-vanishing result, in which the glued operator looks like the single-trace operator \( T_p \). We can have \( U(N) \) propagators one adjacent to the other, then

\[
\text{Tr}(\ldots \phi_{ab} \phi_{bc} \ldots) \delta_f^a \delta_f^b \delta_e^c \delta_e^d = N \text{Tr}(\phi \ldots \phi)
\]  

In this case only \( pq \) Wick contractions are non-zero. Finally, we can take a connected and a disconnected part, resulting in

\[
\text{Tr}(\ldots \phi^a \ldots \phi^d \ldots) \text{Tr}(\ldots \phi^f \ldots \phi^h \ldots) \left( - \delta_h^a \delta_h^b \frac{1}{N} \delta_d^c \delta_f^e - \frac{1}{N} \delta_h^c \delta_h^d \delta_f^e \delta_f^g \right) = - \frac{2}{N} \text{Tr}(\phi \ldots \phi)
\]  

Inputting (114) and (115) into a vev with \( O_p \) and using that \( \langle T_p O_p \rangle = \langle O_p O_p \rangle \) yields the final result,

\[
\langle T_{q_1} T_{q_2} O_p \rangle = q_1 q_2 \left[ N - \frac{(q_1 - 1)(q_2 - 1)}{N} \right] \langle O_p O_p \rangle.
\]  

We then obtain an explicit formula for the next-next-to-extremal three-point functions of single-particle operators

\[
\langle O_p(x) O_{q_1}(x_1) O_{q_2}(x_2) \rangle = \langle O_p O_p \rangle \left[ q_1 q_2 \left[ N - \frac{(q_1 - 1)(q_2 - 1)}{N} \right] + \sum_{p_1=2}^{\lfloor \frac{q_1}{2} \rfloor} C_{p_1(q_1-p_1)} q_2(q_2-1)p_1(q_1-p_1) + \sum_{p_2=2}^{\lfloor \frac{q_2}{2} \rfloor} C_{p_2(q_2-p_2)} q_1(q_1-1)p_2(q_2-p_2) \right]
\]  

(117)
The two sums are symmetric and can be performed with the explicit knowledge of $C_{p_1(q_1-p_1)}$ given in appendix (see (168)). We find

$$\sum_{p_1=2}^{q_1} C_{p_1(q_1-p_1)} p_1 (q_1 - p_1) = \frac{q_1}{2N(q_1 - 2)} \left[ 2N^2 + (q_1 - 1)_2 N + 2(q_1 - 2)_2 + \frac{2(q_1 - 1)_2 N(N)_{q_1}}{(N - q_1 + 1)_{q_1} - (N)_{q_1}} \right]$$  \hspace{2cm} (118)$$

### 4.2.2 $n$-point functions

NME $n$-point functions can be obtained in a way similar to the three-point functions and in this section we sketch how the computation goes.

The definition of $k = \frac{1}{2}(-p + \sum q_i) = 2$ selects an operator $O_p$, and we call the others “light”. Start by expanding all these “light” operators in terms of the trace basis, and note that almost all terms in the expansion vanish since they produce correlators equivalent to near-extremal higher point diagrams. The result is the following generalisation of the three-point expansion in (108), namely

$$\langle O_{q_1} \ldots O_{q_{n-1}} O_p \rangle = \langle T_{q_1} \ldots T_{q_{n-1}} O_p(x_n) \rangle + \sum_{i=1}^{n-1} \left[ \frac{1}{2} \right] \sum_{p_i=2}^q C_{p_i(q_i-p_i)} \langle O_{q_1} \ldots O_{q_{i-1}} [O_{p_i} O_{q_i-p_i}] O_{q_{i+1}} \ldots O_p(x_n) \rangle.$$  \hspace{2cm} (119)$$

This equation can be thought of as a separate equation for each contributing Feynman diagram independently. Note that the correlators in the sum are all contributions to $n+1$-point $N^{n-1}$-extremal correlators which are maximally extremal and given in the previous section. The first term can then be computed by doing the partial Wick contractions $n-1$ in total on the single trace operators $T_{p_1} \ldots T_{p_{n-1}}$, only keeping the relevant terms just as in (114).

### 4.3 3-point functions as multi-particle 2-point functions

The work done in the previous sections has been to start from ME and NME three-point functions, compute them, and understand how to generalise the technique to $n$-points. In order to deal with the three-point functions we substituted two out of three SPOs with their corresponding expansion in the trace basis, and we reduced part of the job to compute a three point function with one single-particle and two traces. In this section instead we note an interesting feature of three-point function of SPO which does not require passing
to the trace basis, and it is valid for any extremality. The relation works as follows,

\[
\langle \mathcal{O}_p \mathcal{O}_q \mathcal{O}_r \rangle = \frac{1}{2^k k!} \langle [\mathcal{O}_p \mathcal{O}_q] \mathcal{O}_r \mathcal{O}_2 \cdots \mathcal{O}_2 \rangle_{\text{connected}}, \quad p + q - r = 2k .
\]

(120)

where the equality is for the color factor of the l.h.s. being the same as that of a two-point functions of multi-particle operators of SPOs on the r.h.s.

Let us first illustrate the case \( k = 3 \) with the following picture,

\[
\text{The diagram on the left is the single diagram contributing to } \langle \mathcal{O}_p \mathcal{O}_q \mathcal{O}_r \rangle, \text{ whereas the one on the right is the only type of diagram contributing to } \langle [\mathcal{O}_p \mathcal{O}_q] \mathcal{O}_r \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_2 \rangle.
\]

To show (120), consider where the two legs out of an \( \mathcal{O}_2 \) can end. They can not go to \( \mathcal{O}_r \) as they are at the same point. If they both went to \( \mathcal{O}_p \), this would result in a diagram of the form of a dumbbell in (67) (centered around \( \mathcal{O}_p \)) which thus vanishes. Similarly if both legs go to \( \mathcal{O}_q \). The only exception to this is if \( p \) or \( q \) equals two in which case you can have a completely disconnected contribution - which also is absent since we specify the connected component. The only remaining possibility is then that one leg goes to \( \mathcal{O}_p \) and one to \( \mathcal{O}_q \) resulting in the diagram shown on the right. There are clearly \( 2^k k! \) different but equivalent diagrams of this sort, arising from the \( k! \) possibilities of swapping the propagators from \( \mathcal{O}_p \) to the \( \mathcal{O}_2 \)s and from the cyclic symmetry around each \( \mathcal{O}_2 \).

The color factor of \( \langle \mathcal{O}_p \mathcal{O}_q \mathcal{O}_r \rangle \) is the same of one of the equivalent configuration of \( \langle [\mathcal{O}_p \mathcal{O}_q] \mathcal{O}_r \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_2 \rangle \) described above. This follows from the fact that

\[
\phi_b^a \phi^e_d = \delta^{ae} \delta^b_d = \delta^a_b \delta^e_d - \frac{1}{N} \delta^a_b \delta^e_d,
\]

\[
\phi_b^a \phi^e_d \phi^f_c \phi^g_f \phi^h_e \phi^i_d = \delta^{ae} \delta^b_d \delta^c_f \delta^e_d - \frac{1}{N} \delta^a_b \delta^c_f \delta^e_d
\]

(122)

which conclude our proof of (120).\(^7\)

\(7\)The ideas of our proof here can be generalised to multipoint correlators as well. For example those which are equivalent to the l.h.s. of (125).
While the right hand side of (120) is restricted to the connected part of the two point function (thinking of it as a limit of a higher point function), rather than the full two-point function, we need this distinction only when \( p \) or \( q \) equals 2 and \( k = 1 \), where instead we have

\[
\langle O_q O_2 O_q \rangle = \frac{1}{2} \left( \langle [O_q O_2] [O_q O_2] \rangle - \langle O_q O_q \rangle \langle O_2 O_2 \rangle \right).
\]

(123)

Note that the condition giving the value of \( k \) in (120) is dependent on the order of \( O_p \), \( O_q \) and \( O_r \); in particular it distinguishes \( O_r \) from the other two operators. However the color factor of the three-point function does not depend on this ordering. For example, consider the three point function of \( O_3 \), \( O_4 \) and \( O_5 \), we can have three multi-particle two-point functions with \( k = 1, 2, 3 \) respectively,

\[
\langle O_3 O_4 O_5 \rangle = \langle O_3 O_5 O_4 \rangle = \langle O_4 O_5 O_3 \rangle = \frac{60 \prod_{i=1}^{4} (N^2 - i^2)}{N(5 + N^2)}
\]

(124)

All three multi-particle two-point functions are then equal!!.

We conclude that for a triplet of single-particle operators all multi-particle two-point functions which correspond to different dispositions of the three SPOs give the same color factor up to a multiplicity counted by \( 2^k k! \).

Also note that while the discussion above required \( O_p \) and \( O_q \) to be SPOs, it nowhere relied on \( O_r \) to be an SPO. Thus the following more general relation holds for any half-BPS operator \( T_{r_1, \ldots, r_l} \):

\[
\langle O_p O_q T_{r_1, \ldots, r_l} \rangle = \frac{1}{2^k k!} \left( \langle [O_p O_q] [T_{r_1, \ldots, r_l} O_2 \cdots O_2] \rangle \right)_{\text{connected}},
\]

(125)

with \( r_1 + \ldots + r_l = r \) and \( p + q - r = 2k \).

### 4.4 On correlators with lower extremality

We understood how to classify free theory correlators according to their degree of extremality w.r.t. ME correlator. Lowering this degree increases the complexity of the computation. ME are the simplest non vanishing correlators and are computed in terms of tree graph. As function of the charges \( p, q_i = 1, \ldots, n-1 \), the charge dependence is fully factorised for each tree graph, with the factor having a clean interpretation. NME showed more structure. We expect that NNME will have more, and so on so forth.

The complexity of NNME is already evident in the three-point functions. For example \( \langle O_p O_q O_q \rangle \) with \( r = p + q - 6 \) will have a contribution of the form \( \langle O_p T_{q_1} T_{q_2} \rangle \), with three
bridges between \( T_{q_1} \) and \( T_{q_2} \). We show few cases are

\[
\langle O_6 T_6 T_6 \rangle = \left( \frac{300}{N^2} + \frac{7200}{N^2} + 36N^2 \right) \langle O_6 O_6 \rangle \quad (126)
\]

\[
\langle O_7 T_7 T_6 \rangle = \left( \frac{840}{N^2} + \frac{12600}{N^2} + 42N^2 \right) \langle O_7 O_7 \rangle \quad (127)
\]

\[
\langle O_8 T_8 T_6 \rangle = \left( \frac{1680}{N^2} + \frac{20160}{N^2} + 48N^2 \right) \langle O_8 O_8 \rangle \quad (128)
\]

\[
\langle O_9 T_7 T_7 \rangle = \left( \frac{1911}{N^2} + \frac{22050}{N^2} + 49N^2 \right) \langle O_9 O_9 \rangle \quad (129)
\]

\[
\langle O_8 T_7 T_7 \rangle = \left( \frac{3528}{N^2} + \frac{35280}{N^2} + 56N^2 \right) \langle O_9 O_9 \rangle \quad (130)
\]

and we attach an ancillary file to show the reader how complicated are the correlators \( \langle T_{p} T_{q_1} T_{q_2} \rangle \) before we convert those to the single-particle operator \( O_p \). About (126)-(130), we can say a number of things based on the possible scenarios in double-line notation:

1. \( O(1/N^2) \) contributions can only arise from picking the three \( SU(N) \) propagators in the configuration: two capping parts \((-\frac{1}{N} \delta^r \delta^l)\) and a \( U(N) \) part in (4). When the two capping parts are used we are left with \( T_{q_2-2}(x_1)T_{q_2-2}(x_2) \) with no link in between. The \( U(N) \) propagator has the effect of producing an effective \( T_p \) operator.

2. \( O(1/N) \) contributions can only arise from picking the propagators in the configuration: one capping part and two \( U(N) \) parts, and these two \( U(N) \) propagators have to be generic, i.e. not consecutive. When the capping part is used, we reduce the number of legs to \( T_{q_1-1}(x_1)T_{q_1-1}(x_2) \) with no link in between. When we attach the two \( U(N) \) propagators we are in a situation like NME (see (113)) and the result will vanish.

3. \( O(N) \) contributions can only arise from picking the propagators in the configuration: three \( U(N) \) parts, but two \( U(N) \) propagators have to be consecutive and one generic. The consecutive propagators produce a factor of \( N \), reduce the number of legs to \( T_{q_1-2}(x_1)T_{q_2-2}(x_2) \) and link the operators. Considering the other generic \( U(N) \) propagators we are again in a situation like NME (see (113)) and the result will vanish.

4. \( O(N^2) \) contributions can only arise from picking the propagators in the configuration: three \( U(N) \) parts, but all consecutive.

5. \( O(1) \) contributions can arise from two configurations: a) one capping part and two \( U(N) \) parts, but these two have to be consecutive. b) three \( U(N) \) parts, generic. Notice that a) is necessarily negative, whereas b) is positive.
The number of Wick contractions is \((q_1 - 2)_3(q_2 - 2)_3 / 3!\), and for each arrangement we have a multiplicity \(2^3 = 1 + 1 + 3 + 3\), depending on the way we pick the propagators, i.e. whether we consider the \(U(N)\) or the capping part. It is clear that item (1) contributes fully, and item (4) contributes only with \(q_1 q_2\). Thus we have found

\[
\langle O_p T_{q_1} T_{q_2} \rangle = \left[ q_1 q_2 N^2 + 3 \times \frac{(q_1 - 2)_3(q_2 - 2)_3}{3! N^2} + \left( \frac{1}{N} (N c_a + c_b) \right) \right] \langle O_p O_p \rangle
\]

where \(c_a, c_b > 0\) and depend on the charges \(q_1\) and \(q_2\).

It is not straightforward at this point to extract further information from the combinatorics, and in practise the dependence on the charges becomes hidden in the combinatorics. Nevertheless, the lesson we learn from our considerations about NNME three point function above is the surprising simplicity of the final result, that once more points to the possibility of understanding single-particle multipoint correlators in a way that eschews from a brute force computation. We hope to make this observation more concrete in the future.

## 5 The half-BPS OPE

In this section we illustrate an alternative approach to obtaining multi-point correlation functions, the half-BPS OPE, which among many interesting features offers a different understanding of the colour dependence of the correlators. The idea of the half-BPS OPE is simply to bootstrap the free theory correlators by projecting it onto the half-BPS states.\(^9\)

The half-BPS OPE follows directly from the full super OPE in \(\mathcal{N} = 4\) analytic superspace \([25]\), and can be defined as the limit,

\[
\lim_{x_1 \to x_2} g_{12}^P \left[ O_{p_1}(x_1) O_{p_2}(x_2) \right] = \sum_{t \vdash t} C_{p_1 p_2}^t O_{\underline{t}}(x_2), \quad t = p_1 + p_2 - 2P. \tag{131}
\]

The symbol \(g_{12}^P[O_{p_1}(x_1) O_{p_2}(x_2)]\) specifies that out of the full super OPE on the l.h.s. we are picking the term with \(Y\) dependence of the form \(Y_{12}^{2P}\).\(^{10}\) The result on the r.h.s. is intuitive, what happens is that by fusing the \(P\) bridges between \(O_{p_1}\) and \(O_{p_2}\), we find an expansion over the half-BPS operators with \(t = p_1 + p_2 - 2P\) legs, i.e. \(t\) is the twist of the operator. We refer to this projection as the ‘half-BPS OPE at twist \(t\)’.

One can pick any basis of half-BPS operators to express the r.h.s. of (131). For definiteness we have written it in terms of the basis generated by products of SPOs, \(O_{\underline{t}} = [O_{t_1} \ldots O_{t_m}]\) where \(\underline{t} = (t_1, \ldots, t_m)\) is a partition of \(t\).

---

\(^8\)A simple guess is \(c_b - c_a = (q_1 - 1)_2(q_2 - 1)_2(\frac{1}{4}(q_1 - 5)(q_2 - 5) - \frac{1}{4}(q_1 - 6)(q_2 - 6))\).

\(^9\)This approach has been pursued also in \([24]\) at order \(1/N^2\).

\(^{10}\)Multiplying by \(X_{12}^{2P}\) we can take \(X_1 \to X_2\), and this limit will now project on \(O_{\underline{t}}\).
The coefficients $C_{t_1 t_2}^{t_3}$ can be related more precisely to three-point functions, by taking the vev with other half BPS operators. We first find
\[\langle O_{t_1} O_{t_2} O_{t_3}' \rangle = \sum_{t \vdash t'} C_{t_1 t_2}^{t_3} g_{t t'},\]  
(132)
where $g$ is the matrix of two-point functions of half-BPS operators of twist $t$,
\[g_{t t'} = \langle O_{t} O_{t'} \rangle .\]  
(133)
Inverting (132) as a vector equation we obtain
\[C_{t_1 t_2}^{t_3} = \sum_{t \vdash t'} \langle O_{t_1} O_{t_2} O_{t_3}' \rangle (g^{-1})_{t t'} .\]  
(134)

Free theory correlators decompose into propagator structure $s$: If we arrange the operators at the corners of a square and as usual we draw a line between point $i$ and point $j$ to represent a propagator $g_{ij}$, we can in fact represent the the correlator as a sum over all possible propagator structures,
\[\langle O_{p_1} \ldots O_{p_n} \rangle = \sum_{\{b_{ij}\}} \prod_{i<j} b_{ij}^{b_{ij}}, \quad \sum_{i \neq j} b_{ij} = p_i, \quad b_{ji} = b_{ij}, \quad b_{ii} = 0 .\]  
(135)
The color factors $\alpha_{\{b_{ij}\}}$ can in principle be computed by Wick contractions. However, a brute force computation is quite expensive, since the single particle operators are admixture of single and multi-trace operators. Therefore intermediate steps are cumbersome. Nevertheless, we already saw for the NNME three-point functions in section 4.4 that the final result is much simpler than the intermediate steps. Thus, the idea of the half-BPS OPE is to constrain the $\alpha_{\{b_{ij}\}}$. and compute them by using the consistency of the general OPE, projected onto the simpler sector of half-BPS operators. To illustrate the power and simplicity of this procedure we will now consider four-point functions $\langle O_{t_1} O_{t_2} O_{t_3} O_{t_4} \rangle$.

In the following analysis we will always take $p_4$ to be the largest charge. We have three ‘channels’ to perform the OPE, depending on which operator $O_{p_{i=1,2,3}}$ we bring close to $O_{p_4}$. Schematically, $(12) \leftrightarrow (34)$, $(23) \leftrightarrow (14)$ and $(13) \leftrightarrow (24)$. Let us consider the first channel, for illustration. The reasoning will be similar for the others.

If we perform the half-BPS OPE ($O_{p_1} \times O_{p_2}$) we find the twist $t$ to lie in the range $\max(|p_{12}|, |p_{43}|) \leq t \leq \min(p_1 + p_2, p_3 + p_4)$. The extrema are special, and will be discussed separately. For any other value of $t$ in $\max(|p_{12}|, |p_{43}|) < t < \min(p_1 + p_2, p_3 + p_4)$, we obtain a relation for a linear combination of the coefficients $\alpha$, of the form
\[\sum_{b_{12} + b_{13} + b_{23} + b_{24} = t} \alpha_{\{b_{ij}\}} = \sum_{t' \vdash t} \langle O_{t_1} O_{t_2} O_{t_3}' \rangle (g^{-1})_{t t'} \langle O_{t'} O_{p_3} O_{p_4} \rangle .\]  
(136)
The sum on the l.h.s. above is specified by the condition $b_{12} + b_{13} + b_{23} + b_{24} = t$ so that it involves only diagrams with $t$ propagators between the pair $(\mathcal{O}_{p_1}, \mathcal{O}_{p_2})$ and the pair $(\mathcal{O}_{p_3}, \mathcal{O}_{p_4})$.

A feature of using the basis generated by products of SPOs is that the single-particle space is orthogonal to the multiparticle sector (by the definition of SPOs). Therefore the r.h.s of (136) can be split into a single particle contribution and a multiparticle contribution,

$$\sum_{b_{12} + b_{13} + b_{23} + b_{24} = t} \alpha_{\{b_{ij}\}} = \langle \mathcal{O}_{p_1} \mathcal{O}_{p_2} \mathcal{O}_t \rangle \langle \mathcal{O}_t \mathcal{O}_{p_3} \mathcal{O}_{p_4} \rangle^{-1} \langle \mathcal{O}_t \mathcal{O}_{p_3} \mathcal{O}_{p_4} \rangle$$

(137)

$$+ \sum_{\underline{t} \vdash t} \langle \mathcal{O}_{p_1} \mathcal{O}_{p_2} K_{\underline{t}} \rangle (\tilde{g}^{-1})_{\underline{tt}'} \langle K_{\underline{t}'} \mathcal{O}_{p_3} \mathcal{O}_{p_4} \rangle.$$  

(138)

Here the sum in the second term is over only partitions $\underline{t} \vdash t$ of length at least two. We use the notation $K_{\underline{t}}$ to emphasise that there is always more than one factor in the operator $K_{\underline{t}} = [\mathcal{O}_1 \ldots \mathcal{O}_m]$, i.e. $(m \geq 2)$ and we write $\tilde{g}_{\underline{tt}'} = \langle K_{\underline{t}} K_{\underline{t}'} \rangle$ for the metric projected on the multi-particle sector.

Referring to (137) and (138), we can appreciate that considering the lowest possible twist, i.e. $t = \max(|p_{12}|, |p_{43}|) > 0$, we find in any case extremal three point functions, which vanish according to our general discussion in section 3.2. For the highest value instead, $t = \min(p_1 + p_2, p_3 + p_4)$, we find that (137) is again extremal, thus vanishing, but (138) instead gives a sum over three point functions with multi-particle states. However, these three-point functions are of the same kind of the original four-point function we want to bootstrap, therefore do not lead to useful constraints. Rather, we will show that can be used to obtain multi-particle two-point functions.

### 5.1 N$^2$E correlators

Let us fix ideas by re-considering the simplest ME four-point functions first addressed in section 4.1. These are the next-to-next-to-extremal (N$^2$E) correlators, and obey,

$$s = p + q + r - 4,$$

(139)

where we always take $s$ to be the largest charge. The condition (139) implies that all but two propagators are connected to $\mathcal{O}_s$, leaving six possible topologies which can contribute to the correlator, depicted below

$$\langle \mathcal{O}_p \mathcal{O}_q \mathcal{O}_t \mathcal{O}_s \rangle = \alpha_1 + \alpha_2 + \alpha_3$$

$$+ \tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3$$

(140)
The operator \( \mathcal{O}_s \) sits at right corner on the bottom of each square. Here the thin lines correspond to single propagators not connected to \( \mathcal{O}_s \) whereas the thick lines represent multiple propagators (as many as needed to match the charges \( p, q, r, s \)).

Let us consider the half-BPS OPE \((\mathcal{O}_p \times \mathcal{O}_q)\) at twist \( t = (p + q - 4) \). This means we project onto the \( Y \) dependence \( Y_P^{12} \) with \( P = 2 \). The only surviving diagram is the first one on the second line of (140) and we obtain the following equation,

\[
\tilde{\alpha}_1 = \sum_{\mathcal{L}' \vdash t} \langle \mathcal{O}_p \mathcal{O}_q \mathcal{O}_t \rangle (g^{-1})_{\mathcal{L}'}^{12} \langle \mathcal{O}_r \mathcal{O}_s \rangle \quad (141)
\]

If \( t > 0 \), the three-point functions \( \langle \mathcal{O}_t \mathcal{O}_r \mathcal{O}_s \rangle \) all vanish. This is because \( \mathcal{O}_s \) is a SPO of charge \( s = p + q + r - 4 = t + r \) and the three-point function is therefore extremal. We conclude \( \tilde{\alpha}_1 = 0 \) and a similar analysis of the \((\mathcal{O}_p \times \mathcal{O}_r)\) and \((\mathcal{O}_q \times \mathcal{O}_r)\) OPEs reveals that also \( \tilde{\alpha}_2 = \tilde{\alpha}_3 = 0 \). This same conclusion was already reached in \([11]\) where we argued that free theory propagator topologies in all SPO four-point functions are absent when one of the operators is only connected to one other. The assumption \( t > 0 \) excludes the identity operator, and implies that the diagrams we considered are connected.

Now let us perform the half-BPS OPE \((\mathcal{O}_p \times \mathcal{O}_q)\) at twist \( t = (p + q - 2) \). We have to project the \( Y \) dependence onto \( Y_P^{12} \) with \( P = 1 \), giving us

\[
\alpha_1 + \alpha_2 = \langle \mathcal{O}_p \mathcal{O}_q \mathcal{O}_t \rangle \langle \mathcal{O}_t \mathcal{O}_r \rangle^{-1} \langle \mathcal{O}_s \mathcal{O}_s \rangle. \quad (142)
\]

In the second equality we have used the fact that the three-point functions \( \langle \mathcal{K}_r \mathcal{O}_r \mathcal{O}_s \rangle \) can be splitted at least on four-point auxiliary points with \( k = s - r - (p + q + 2) = 1 \), thus are near-extremal and vanish according to the discussion in section 3.2.1. Using our previous computation for \( \langle \mathcal{O}_t \mathcal{O}_r \mathcal{O}_s \rangle \) (see (95)) we arrive at

\[
\alpha_1 + \alpha_2 = pq(r + q + 2) \langle \mathcal{O}_s \mathcal{O}_s \rangle. \quad (143)
\]

Repeating the analysis above in the other two OPE channels we obtain two more similar equations with the unique solution

\[
\alpha_1 = pqr(p - 1) \langle \mathcal{O}_s \mathcal{O}_s \rangle, \quad \alpha_2 = pqr(q - 1) \langle \mathcal{O}_s \mathcal{O}_s \rangle, \quad \alpha_3 = pqr(r - 1) \langle \mathcal{O}_s \mathcal{O}_s \rangle, \quad (144)
\]

This is exactly the same solution we derived in equation (105) with different means.

As we anticipated, performing the half-BPS OPE at twist \( t = p + q \) does not yield further constraints on the coefficients \( \alpha_i \). Instead we obtain the relation

\[
\alpha_3 + \langle \mathcal{O}_p \mathcal{O}_r \rangle \langle \mathcal{O}_q \mathcal{O}_s \rangle + \langle \mathcal{O}_p \mathcal{O}_s \rangle \langle \mathcal{O}_q \mathcal{O}_r \rangle = \sum_{\mathcal{L}' \vdash t} \langle \mathcal{O}_p \mathcal{O}_q \mathcal{K}_r \rangle (g^{-1})_{\mathcal{L}'}^{12} \langle \mathcal{K}_r \mathcal{O}_s \rangle \langle \mathcal{O}_s \rangle. \quad (145)
\]

On the l.h.s. we have been careful to include the disconnected contributions that can be present. The contribution \( \langle \mathcal{O}_p \mathcal{O}_q \mathcal{O}_t \rangle \) is extremal and vanishing, thus only multi-particles
exchanged appear in the second equality above. Moreover, from the fact that there are no bridges between \( \mathcal{O}_p \) and \( \mathcal{O}_q \) due to extremality, we find that the three point function \( \langle \mathcal{O}_p \mathcal{O}_q \mathcal{K}_2 \rangle \) is equal to the two-point function \( \langle [\mathcal{O}_p \mathcal{O}_q] \mathcal{K}_2 \rangle = \tilde{g}_{(p,q)} \). The r.h.s. above thus simplifies and we obtain

\[
\alpha_3 + \langle \mathcal{O}_p \mathcal{O}_r \rangle \langle \mathcal{O}_q \mathcal{O}_s \rangle + \langle \mathcal{O}_p \mathcal{O}_s \rangle \langle \mathcal{O}_q \mathcal{O}_r \rangle = \langle [\mathcal{O}_p \mathcal{O}_q] \mathcal{O}_r \mathcal{O}_s \rangle. \tag{146}
\]

This is equivalent to taking the coincidence limit \( x_1 \to x_2 \) on the original four-point function \( \langle \mathcal{O}_p \mathcal{O}_q \mathcal{O}_r \mathcal{O}_s \rangle \) which yields immediately

\[
\langle [\mathcal{O}_p \mathcal{O}_q] \mathcal{O}_r \mathcal{O}_s \rangle = pqr(r-1)\langle \mathcal{O}_s \mathcal{O}_s \rangle + \langle \mathcal{O}_p \mathcal{O}_r \rangle \langle \mathcal{O}_q \mathcal{O}_s \rangle + \langle \mathcal{O}_p \mathcal{O}_s \rangle \langle \mathcal{O}_q \mathcal{O}_r \rangle. \tag{147}
\]

where \( s = p + q + r - 4 \) as before and we have used the derived result (144) for \( \alpha_3 \). The other coincidence limit \( x_3 \to x_4 \) gives us

\[
\langle [\mathcal{O}_p \mathcal{O}_q] \mathcal{O}_r \mathcal{O}_s \rangle = \delta_{2r} 2pq\langle \mathcal{O}_s \mathcal{O}_s \rangle + \langle \mathcal{O}_p \mathcal{O}_r \rangle \langle \mathcal{O}_q \mathcal{O}_s \rangle + \langle \mathcal{O}_p \mathcal{O}_s \rangle \langle \mathcal{O}_q \mathcal{O}_r \rangle. \tag{148}
\]

The connected part only contributes for \( r = 2 \). The double coincidence limit \( (x_1 \to x_2, x_3 \to x_4) \) then gives two-point functions of product operators.

\[
\langle [\mathcal{O}_p \mathcal{O}_q] \mathcal{O}_r \mathcal{O}_s \rangle = \delta_{2r} 2pq\langle \mathcal{O}_s \mathcal{O}_s \rangle + \langle \mathcal{O}_p \mathcal{O}_r \rangle \langle \mathcal{O}_q \mathcal{O}_s \rangle + \langle \mathcal{O}_p \mathcal{O}_s \rangle \langle \mathcal{O}_q \mathcal{O}_r \rangle. \tag{149}
\]

The case of \( \langle \mathcal{O}_2 \mathcal{O}_q \mathcal{O}_r \mathcal{O}_s \rangle \)

At four points, whenever one of the charges takes (the smallest possible) value \( p = 2 \), the correlator contributes to a single \( su(4) \) representation in each OPE channel. Such four-point functions are also called next-to-next-to extremal, and in principle have six propagator structures. Here we focus on the three topologies shown below,

\[
\langle \mathcal{O}_2 \mathcal{O}_q \mathcal{O}_r \mathcal{O}_s \rangle_{c} = \alpha_1 + \alpha_2 + \alpha_3. \tag{150}
\]

Diagrams where one of the operators is connected to only one are omitted, since the color factor vanish by exactly the same arguments as above. We also omit cases in which diagrams are disconnected.

Performing the half BPS OPE \( (\mathcal{O}_2 \times \mathcal{O}_q) \) at twist \( q \) gives

\[
\alpha_1 + \alpha_2 = \frac{\langle \mathcal{O}_2 \mathcal{O}_q \mathcal{O}_q \rangle \langle \mathcal{O}_q \mathcal{O}_r \mathcal{O}_s \rangle}{\langle \mathcal{O}_q \mathcal{O}_q \rangle} = 2q\langle \mathcal{O}_q \mathcal{O}_r \mathcal{O}_s \rangle, \tag{151}
\]

where we simplified the result using our previous results for maximally extremal three-point functions. Again we obtain two similar equations from the other crossing channels.
and finally we arrive at
\begin{align*}
\alpha_1 &= (q + r - s)\langle O_q O_r O_s \rangle , \\
\alpha_2 &= (q + s - r)\langle O_q O_r O_s \rangle , \\
\alpha_3 &= (r + s - q)\langle O_q O_r O_s \rangle .
\end{align*}

Note that setting \( s = q + r - 2 \) above we find agreement with (144) in the case \( p = 2 \). The coincidence limit \( x_1 \to x_2 \) then gives three-point functions involving products of single-particle operators,
\begin{equation}
\langle [O_2 O_q] O_r O_s \rangle = (r + s - q)\langle O_q O_r O_s \rangle + \langle O_2 O_r \rangle \langle O_q O_s \rangle + \langle O_2 O_s \rangle \langle O_q O_r \rangle .
\end{equation}

5.2 N\(^3\)E correlators

Let us now consider \( N^3 \)-extremal 4-point functions where \( s = p + q + r - 6 \). We focus on the seven connected diagrams (out of ten) depicted below,
\begin{equation}
\langle O_p O_q O_r O_s \rangle_c = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 .
\end{equation}

Again, we are omitting three diagrams where one operator is connected to only one other, which would vanish. Recall that the operator \( O_s \) sits at the right corner on the bottom of each square.

Performing the half-BPS OPE \( (O_p \times O_q) \) at twist \( t = p + q - 4 \), i.e. by projection onto \( Y_{12}^P \) with \( P = 2 \), yields
\begin{equation}
\alpha_1 + \alpha_2 = \frac{\langle O_p O_q O_{p+q-4} O_r O_s \rangle}{\langle O_{p+q-4} O_{p+q-4} \rangle} = (p + q - 4)r \frac{\langle O_p O_q O_{p+q-4} \rangle}{\langle O_{p+q-4} O_{p+q-4} \rangle} \langle O_r O_s \rangle .
\end{equation}

The multi-particle term vanishes by extremality and the contribution \( \langle O_{p+q-4} O_r O_s \rangle \) is ME, thus it simplifies to give the second equality above. We obtain two similar equations from the other OPEs \( (O_p \times O_r) \) and \( (O_q \times O_r) \). These only involve the pairs \( \alpha_3 + \alpha_4 \) and \( \alpha_5 + \alpha_6 \). The color factor \( \alpha_7 \) does not enter any of these constraints.

From the OPE \( (O_p \times O_q) \) at twist \( t = p + q - 2 \) we obtain
\begin{equation}
\alpha_3 + \alpha_5 + \alpha_7 = pq \langle O_{p+q-2} O_r O_s \rangle + \sum_{L' + t} \langle O_p O_q K_{L'} \rangle (\hat{g}^{-1})_{L'} \langle K_{L'} O_r O_s \rangle .
\end{equation}
In general both single-particle and multi-particle terms contribute. The multi-particle contribution have more structure than previous cases, but let us note that the three-point function $\langle \mathcal{K}_g \mathcal{O}_i \mathcal{O}_s \rangle$ is only non-zero if $\mathcal{K}_g = [\mathcal{O}_{t_i}^{\dagger} \mathcal{O}_{t_i-1}^{\dagger}]$, namely a product of two single-particle operators. A contribution with more than two single-particle operators can be understood as a near-extremal five-point function and vanishes according to the discussion in section 3.2.1. Then, we are in a favourable position since the three-point function $\langle [\mathcal{O}_{t_i}^{\dagger} \mathcal{O}_{t_i-1}^{\dagger}] \mathcal{O}_s \mathcal{O}_s \rangle$ can be computed as the coincidence limit of the $N^2E$ four-point functions described in (147). In particular,

$$\langle \mathcal{K}_g^{\dagger} \mathcal{O}_s \mathcal{O}_s \rangle = \langle [\mathcal{O}_{t_i}^{\dagger} \mathcal{O}_{t_i-1}^{\dagger}] \mathcal{O}_s \mathcal{O}_s \rangle = t'_1(t-t'_1)r(r-1)\langle \mathcal{O}_s \mathcal{O}_s \rangle .$$

We obtain two similar equations from the other OPEs ($\mathcal{O}_p \times \mathcal{O}_s$) and ($\mathcal{O}_q \times \mathcal{O}_s$). These involve $\alpha_2 + \alpha_4$ with $\alpha_7$, and, $\alpha_1 + \alpha_6$ with $\alpha_7$.

In total we have six equations with particular structure. If we consider the last three equations that we derived, and we subtract the first three equation, we obtain a single equation for $3\alpha_7$. Thus we determine $\alpha_7$ completely,

$$\alpha_7 = \frac{1}{3} \left( pq \langle \mathcal{O}_{p+q-2} \mathcal{O}_s \mathcal{O}_s \rangle + \sum_{l+1} \langle \mathcal{O}_p \mathcal{O}_q \mathcal{K}_l \rangle (\bar{g}^{-1})_{ll'} \langle \mathcal{K}_{ll'}^{\dagger} \mathcal{O}_s \mathcal{O}_s \rangle \right) - (p+q-4)r \frac{\langle \mathcal{O}_p \mathcal{O}_q \mathcal{O}_{p+q-4} \rangle}{\langle \mathcal{O}_{p+q-4} \mathcal{O}_{p+q-4} \rangle} \langle \mathcal{O}_s \mathcal{O}_s \rangle + \text{cyc}(p, q, r) .$$

The remaining equations determine five of the remaining coefficients in terms of one remaining, $\alpha_1$ say. In specific cases additional conditions can come from crossing symmetry constraints. In Appendix D we discuss various examples with different degree of crossing symmetry.

In all examples discussed in Appendix D, the coefficients $\alpha_1, \ldots, \alpha_6$ are consistent with the following solution

$$\alpha_1 = F(p, q, r) \langle \mathcal{O}_s \mathcal{O}_s \rangle , \quad \alpha_2 = F(q, p, r) \langle \mathcal{O}_s \mathcal{O}_s \rangle , \quad \alpha_3 = F(p, r, q) \langle \mathcal{O}_s \mathcal{O}_s \rangle ,$$

$$\alpha_4 = F(r, p, q) \langle \mathcal{O}_s \mathcal{O}_s \rangle , \quad \alpha_5 = F(q, r, p) \langle \mathcal{O}_s \mathcal{O}_s \rangle , \quad \alpha_6 = F(r, q, p) \langle \mathcal{O}_s \mathcal{O}_s \rangle ,$$

where

$$F(p, q, r) = (p-2)r \frac{\langle \mathcal{O}_s \mathcal{O}_q \mathcal{O}_{p+q-4} \rangle}{\langle \mathcal{O}_{p+q-4} \mathcal{O}_{p+q-4} \rangle} \quad (160)$$

The bootstrap problem allows the deformation $F(p, q, r) \rightarrow F(p, q, r) + \tilde{F}(p, q, r)$, where $\tilde{F}(p, q, r)$ is totally antisymmetric and unconstrained. In all cases we studied we have found that the deformation $\tilde{F} = 0$. 

\footnote{If we consider the correlators $\langle T_p T_q T_r \mathcal{O}_s \rangle$ instead of $\langle \mathcal{O}_p \mathcal{O}_q \mathcal{O}_r \mathcal{O}_s \rangle$ we find that all coefficients are a simple Laurent polynomial in $N$ multiplied by $\langle \mathcal{O}_s \mathcal{O}_s \rangle$. For example, from the results presented in Appendix D with $p = 3 \leq q \leq r$ we can see that we have $\alpha_7 = 6qr(N - 3(q-1)(r-1)/N)$.}
6 Conclusions

We have considered a new basis of half-BPS operators in $\mathcal{N}=4$ SYM, namely the single particle operators (and products of these). These are the natural duals to single particle supergravity states on $AdS_5 \times S^5$. In particular we have seen that they naturally interpolate between point-like gravitons and giant gravitons, as they should. We have also considered the free theory, all $N$, correlators of these operators. Interestingly, all near extremal correlators of SPOs vanish, and this is presumably tied to the conjecture in [18] that the corresponding supergravity couplings vanish. The near extremal correlators are $n$-point correlators with extremality degree strictly less than $n - 2$. Going away from this case, the maximally extrema correlators then have a very simple form, similarly for the next-to-maximally extremal correlators. The complexity then increases by lowering the extremality, but even in such a case we have found additional simplicity, compared to the single-trace correlators.

It is interesting to revisit past discussions involving half-BPS operators, especially concerning large $N$ limits and the relation to string theory computations via AdS/CFT, in the light of this basis. As we have seen in section 2.3 the SPOs correctly interpolate between single trace operators and the operators conjectured to be dual to $S^5$ giant graviton operators. In [26] a comparison of half-BPS three-point functions of two giant graviton operators and one point-like graviton was performed and compared with the analogous computation in gauge theory. The gauge theory was computed using two large Schur polynomial operators and one single trace operator. The results were found to not quite agree and it was conjectured this was to do with the inability of the Schur polynomials to correctly interpolate between giant and point-like gravitons. The SPOs on the other hand do precisely interpolate between the two as show in section 2.3. On the other hand, the extremal correlators of SPOs simply vanish! In [27] this issue was revisited and it was argued that indeed there were subtleties in the extremal case which are not present in the N-extremal case. The NE with two giant gravitons and one point-like graviton were computed in gauge theory (using Schur polynomials for the giant gravitons and single trace operators for the pointlike operator) as well as in string theory and this time found agreement. Since we have explicit formulae for the NE 3-point functions we can check this agrees here also.

We start with the next-to-extremal three-point function of unit normalised single particle operators. From (95) we have

$$\frac{\langle O_p O_q O_r \rangle}{\sqrt{\langle O_p O_p \rangle \langle O_q O_q \rangle \langle O_r O_r \rangle}} = pq \sqrt{\frac{\langle O_p O_r \rangle}{\langle O_p O_p \rangle \langle O_q O_q \rangle \langle O_r O_r \rangle}}, \quad p + q = r + 2. \quad (161)$$

Now consider the limit $N \to \infty$ with $p$ staying finite, but $q, r \to \infty$ such that $q' = q/N, r' = r/N$ are fixed. Taking the appropriate limits of the two point functions (60) we find

$$\frac{\langle O_p O_q O_r \rangle}{\sqrt{\langle O_p O_p \rangle \langle O_q O_q \rangle \langle O_r O_r \rangle}} \to \sqrt{\frac{r}{N}} \left(1 - \frac{r}{N}\right)^{-\frac{1}{2}}, \quad p + q = r + 2 \quad (162)$$
which is in precise agreement with [27].

We can also compute the normalised next-to-next-to-extremal three-point function given by (117)-(118) in the same limit, $N \to \infty$ with $p, q' = q/N, r' = r/N$ fixed

$$\frac{\langle O_p O_q O_r \rangle}{\sqrt{\langle O_p O_p \rangle \langle O_q O_q \rangle \langle O_r O_r \rangle}} \to \sqrt{p' N \left( 1 - \frac{r}{N} \right)^{\frac{2N-4}{2N}} \left( 1 - \frac{(p-1)r}{2N} \right)}, \quad p + q = r + 4.$$  \hspace{1cm} (163)

It would be interesting to compare with the corresponding string theory computation.

There are a number of further avenues one could go down from here. Firstly one could try to generalise our story beyond the half-BPS sector. Bases for more general operators in $\mathcal{N} = 4$ SYM have been given, for example [29–35] and it would be interesting to look again at these from the perspective of single-particle operators.

The same definition of SPOs presumably holds also for orthogonal and symplectic gauge groups in $\mathcal{N} = 4$ SYM which can be obtained via a $\mathbb{Z}_2$ orientifold projection of the standard AdS$_5 \times$ S$^5$ setup [36]. Half-BPS operators in these theories have been studied in [37–40] and it would be interesting to consider single-particle operators in that context. It would also be interesting to study single-particle states for other AdS$\times S$ backgrounds, for example in AdS$_3$, as it was pointed out in [41], ABJM, and for the mysterious six-dimensional (2,0) theory on AdS$_7 \times$ S$^4$.

Finally, it would be very interesting to consider aspects of the dynamics of the single-particle operators that have not been explored yet, and go beyond the computation of the one-loop amplitudes in [4], along the lines suggested in [50]. For instance, the trace basis is widely used in the context of integrability and in turn integrability based techniques allow the computation of exact correlators. The simplest four-point correlator one could study, i.e. the octagon configuration of [42], was firstly obtained by using hexagonalization from weak coupling, and later re-derived in other beautiful ways [43–46]. From the OPE point of view at weak coupling, many properties of the correlators are due to single-trace stringy states acquiring an anomalous dimension with universal features. This mechanism is quite democratic and perhaps the distinction between single-traces and single-particle external states does not matter at weak coupling. On the other hand, the situation in the supergravity regime is quite different, and the half-BPS single-particle operators are properly the dual of the KK modes, beyond the planar limit. It would be very interesting to understand how integrability based techniques [47,48] modify or adapt when correlators of single-particle operators are considered.

\footnote{See [49] for a five-point analog, called the decagon.}
Acknowledgements

FA would like to thank Pedro Vieira, Frank Coronado, and Till Bargheer for stimulating discussions. FA is partially supported by the ERC-STG grant 637844-HBQFTNCER. FS is supported by the Italian Ministry of Research under grant PRIN 20172LNEEZ and by the INFN under GRANT73/CALAT. JD, HP were supported in part by the ERC Consolidator grant 648630 IQFT. MS is supported by a Mayflower studentship from the University of Southampton. AS is supported by an STFC studentship and PH acknowledges support from STFC grant ST/P000371/1 and the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 764850 “SAGEX”.

A Trace Sector Formulae

In section 2.2 and 2.2.1 we obtained the result

\[ \mathcal{O}_p = \sum_{\{q_1, ..., q_m\} \vdash p} C_{q_1, ..., q_m} T_{q_1, ..., q_m} \]  

(164)

\[ C_{q_1, ..., q_m} = \frac{|\sigma_{q_1, ..., q_m}|}{(p-1)!} \sum_{s \in \mathcal{P}(\{q_1, ..., q_m\})} \frac{(-1)^{|s|+1}(N+1-p)p^{-\Sigma(s)}(N+p-\Sigma(s))^{\Sigma(s)}}{(N)_p - (N+1-p)_p} \]  

(165)

which is explicit in \( p \) and \( q_1 \ldots q_m \), and depends on group theory data which we explained in section 2.2.1.

The value of \( m \) distinguishes the splitting of \( \mathcal{O}_p \) in number of traces and below we give explicit examples for the double trace sector \( m = 2 \), and the triple trace sector \( m = 3 \).

A.1 Double Trace Sector

Consider the partition \( q_1 + q_2 = p \). The powerset in the sum is

\[ \mathcal{P}(\{q_1, q_2\}) = \{\{\}, \{q_1\}, \{q_2\}, \{q_1, q_2\}\} \]  

(166)

and the corresponding values of \( \Sigma \) are

\[ \Sigma(\{\}) = 0 \, , \, \Sigma(\{q_1\}) = q_1 \, , \, \Sigma(\{q_2\}) = q_2 \, , \, \Sigma(\{q_1, q_2\}) = q_1 + q_2 = p. \]  

(167)

Furthermore the size of the conjugacy class is \( |q_1, q_2| = p!/(q_1, q_2) \) as long as \( q_1 \neq q_2 \). Otherwise \( q_1 = q_2 = p/2 \) and \( |q_1, q_2| = p!/(2q_1 q_2) = (p-1)!/(2p) \). With these informations, the coefficient of \( T_{q_1} T_{q_2} \) in \( \mathcal{O}_p \), from (165), is
\[ C_{q_1q_2} = \begin{cases} \frac{p}{q_1q_2} \times \frac{-(N-p+1)_p - (N)_p + (N-p+1)_{q_2}(N+q_2)_{p-q_2} + (N-p+1)_{q_1}(N+q_1)_{p-q_1}}{(N)_p - (N-p+1)_p} \\ \frac{2}{p} \times \frac{-(N-p+1)_p - (N)_p + 2(N-p+1)_{p/2}(N+p/2)_{p/2}}{(N)_p - (N-p+1)_p} \end{cases} \] (168)

As we pointed out, the above formula holds for the coefficients of the double trace contributions to the single particle operator of any weight.

### A.2 Triple Trace Sector

Consider the partition \( q_1 + q_2 + q_3 = p \). By making explicit (165) we find,

\[ C_{q_1q_2q_3} = \frac{p}{q_1q_2q_3} \times \frac{-(N-p+1)_p + (N)_p}{(N)_p - (N-p+1)_p} \]

\[ - \sum_{i=1}^{3} (N-p+1)_{q_i}(N+q_i)_{p-q_i} + \sum_{i=1}^{3} (N-p+1)_{p-q_i}(N+p-q_i)_{q_i} \]

\[ \frac{p}{q_1q_2q_3} \times \frac{1}{(N)_p - (N-p+1)_p} \] (169)

The other two possible cases, in which \( q_i = q_j \) and \( q_1 = q_2 = q_3 = p/3 \), only differ compared to the result above by the the size of the conjugacy class. In the first case we have to further divide by 2, and in the second case by 6. This formula thus cover all possible triple trace contributions to single particle operators of any weight.

For any value of \( m \), i.e. trace sector, our function \( C_q \) can be made very explicit, as in the examples discussed above.

### B Wick Contractions

Given the set of all admissible propagator structures for a correlator, say

\[ \text{PropStruct} = \{ P_1, \ldots \} \] (170)

we will now determine the associated Wick contractions.

Recall that an elementary fields of \( \mathcal{N} = 4 \) transforms under \( SO(6) \) of R-symmetry, i.e. \( \phi^I \), but itself is an \( N \times N \) matrix in the adjoint of the gauge group. For the rest of this section we will then assume the replacement

\[ (\phi^I(X))_{\alpha\beta} \rightarrow \sum_i \phi^I_i(X) T^i_{\alpha\beta} \] (171)
\[ T_{\alpha\beta} \] are a basis of generators for the representation under the gauge group. This replacement on the trace basis becomes,

\[
T_p(X) \rightarrow \sum_{i_1,\ldots,i_p} T_{\alpha_1\alpha_2} \cdots T_{\alpha_{2p-1}\alpha_1} \phi_{i_1} \cdots \phi_{i_p}(X) \times T_{\alpha_1\alpha_2} \cdots T_{\alpha_{2p-1}\alpha_1}
\]

\[
T_{\{p_1,p_2\}}(X) \rightarrow \sum_{i_1,\ldots,i_p} T_{\alpha_1\alpha_2} \cdots T_{\alpha_{2p-1}\alpha_1} \phi_{i_1} \cdots \phi_{i_p}(X) \times T_{\alpha_1\alpha_2} \cdots T_{\alpha_{2p-1}\alpha_1}
\]

and so on so forth, i.e. if the operator is multi-trace, the trace is split accordingly.

For what concerns finding the Wick contractions of an assigned propagator structure, the trace structure of the operators can be ignored. (Even though sometimes we can use the cyclic symmetry of the traces.) If we draw a blob to represent the operator, the only relevant information at this stage will be the number of \( SO(6) \) indexes, i.e. legs attached to the blob. Single- or multi-trace, we will draw the same blob. For each Wick contraction, or bridge between legs (belonging to different blobs), say \( \alpha_1 \) and \( \beta_2 \), we insert a \( \delta_{\alpha_1\beta_2} \) which links the generators. For \( SU(N) \) the sum over generators is then replaced by

\[
\sum_{i=1}^{N-1} T_{\alpha_1\alpha_2} T_{\alpha_3\alpha_4} = \delta_{\alpha_1\alpha_4} \delta_{\alpha_2\alpha_3} - \frac{1}{N} \delta_{\alpha_1\alpha_2} \delta_{\alpha_3\alpha_4}
\]

The color factor depends crucially on the trace structure of the operators and the type of propagator, whether \( SU(N) \) or \( U(N) \).

To enumerate the Wick contractions we need two standard combinatorial objects:

1. Define \( \mathcal{C}(k \subseteq n) \) as the combinations of \( k \) integers in the set \( \{1, \ldots, n\} \). These are ordered sets. For example, if \( n = 3 \) and \( k = 2 \) the combinations are 12, 13, 23. The number of the combinations is the binomial coefficient

\[
|\mathcal{C}(k \subseteq n)| = \binom{n}{k} = \frac{n!}{(n-k)!k!}
\]

2. Define \( \mathcal{D}(k \subseteq n) \) as the dispositions of \( k \) integers in the set \( \{1, \ldots, n\} \). These are not ordered sets. For example, if \( n = 3 \) and \( k = 2 \), the dispositions are 12, 13, 23, 21, 31, 32. The number of dispositions is indeed the number of \( k \)-combinations acted with \( k \)-permutations,

\[
|\mathcal{D}(k \subseteq n)| = \binom{n}{k} k! = \frac{n!}{(n-k)!}
\]

To begin with, consider a simple example, the two-point function \( \langle T_3, T_3 \rangle \). There is obviously a single propagator structure, with three bridges, \( b_{(12)} = 3 \). Starting with
\( \phi^1 \phi^2 \phi^3 (X_1) \phi^4 \phi^5 \phi^6 (X_2) \), there are six Wick contractions, and we can rearrange them as follows,

\[
\begin{align*}
\phi^1 (X_1) \phi^4 (X_2) & \quad \phi^2 (X_1) \phi^5 (X_2) & \quad \phi^3 (X_1) \phi^6 (X_2) \\
\vdots & \end{align*}
\]

These six Wick contractions are indexed by \( \mathcal{C}(3 \subseteq 3) \) (legs-out), which has dimension 1, paired with \( \mathcal{D}(3 \subseteq 3) \) (legs-in), which has dimension 6. In other words,

\[
\begin{align*}
1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 \\
1' & 2' & 3' & 1' & 2' & 3' & 2' & 1' & 3' \\
1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 \\
1' & 2' & 3' & 1' & 2' & 3' & 2' & 1' & 3'
\end{align*}
\]

where the \( ' \) alphabet is \{1' \( \rightarrow \) 4, 2' \( \rightarrow \) 5, 3' \( \rightarrow \) 6\}.

Generating Wick contractions for a propagator structure \( \mathcal{P} = \{b_{(12)}, \ldots, b_{(ij)}, \ldots\} \) is slightly more involved but the logic is similar to the example above. First organise \( \mathcal{P} = \{b_{(12)}, \ldots, b_{(ij)}, \ldots\} \) as a matrix of the following form

\[
\begin{bmatrix}
p_1 & b_{(12)} & \cdots & b_{(1i)} & \cdots & \cdots & \cdots & b_{(1j)} & \cdots & \cdots \\
\vdots & & & & & & & & & \\
p_i & \cdots & \cdots & \cdots & b_{(ij)} & \cdots & \cdots & \cdots & \cdots & \cdots \\
p_{i+1} & b_{(i+1,i+2)} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & & & & & & & & & \\
p_{j-1} & b_{(j-1,j)} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
p_j & b_{(jj+1)} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
p_{j+1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & & & & & & & & & \\
p_n & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}
\]

Conservation of charge at the blob \( p_i \) means

\[
p_i = \sum_{k=1}^{i-1} b_{(ki)} + \sum_{k=i+1}^{n} b_{(ik)}
\]

pictorially the r.h.s is the sum over all the numbers on the hook having \( p_i \) at the corner.

We will start enumerating the Wick contractions going along the rows of the matrix, starting from the \( b_{(12)} \). Upon visiting a \( b_{(ij)} \), we define the updated values of \( p_i \) and \( p_j \) given
by

\[ p_{ij} \equiv - \sum_{k=1}^{i-1} b_{ki} + p_i - \sum_{k=i+1}^{j-1} b_{ik} \]  
  \text{# on the rows above } (i,i) 
(180)

\[ p_{ji} \equiv - \sum_{k=1}^{i-1} b_{kj} + p_j \]  
  \text{# on the cln before } (i,j) 
(181)

The idea is that when we visit an entry \((ij)\) we assign \(b_{ij}\) of the free legs of \(p_{ij}\). These are legs out, and we link them with free legs of \(p_{ji}\). The latter are legs in. Thus changing row index we consider legs in, and changing column index we consider legs out.

Define now the Wick contractions \(\mathcal{W}[b_{ij}]\) for the bridges going from blob \(i\) with \(p_{ij}\) legs-out, reaching blob \(j\) with \(p_{ji}\) legs-in. These are enumerated by

\[ \mathcal{W}[b_{ij}] = \mathcal{C}(b_{ij} \subseteq p_{ij}) \otimes \mathcal{D}(b_{ij} \subseteq p_{ji}) \]  
(182)

The dimension of \(\mathcal{W}[b_{ij}]\) is simply

\[ |\mathcal{W}[b_{ij}]| = |\mathcal{C}(b_{ij} \subseteq p_{ij})| \times |\mathcal{D}(b_{ij} \subseteq p_{ji})| \]  
(183)

The set of all possible Wick contractions for \(\mathcal{P} = \{b_{ij}, b_{kj}, \ldots\}\) is given by

\[ \mathcal{W}[\mathcal{P}] = \otimes_{1 \leq i < j \leq n} \mathcal{W}[b_{ij}] \]  
(184)

with dimension \(|\mathcal{W}[\mathcal{P}]| = \prod_{1 \leq i < j \leq n} |\mathcal{W}[b_{ij}]|\). By using the explicit formulas,

\[ |\mathcal{C}(k \subseteq n)| = \frac{n!}{(n-k)! k!} ; \quad |\mathcal{D}(k \subseteq n)| = \frac{n!}{(n-k)!} \]  
(185)

we find

\[ |\mathcal{W}[\mathcal{P}]| = \prod_{1 \leq i < j \leq n} \frac{1}{b_{ij}!} \left( - \sum_{k=1}^{i-1} b_{ki} + p_i - \sum_{k=i+1}^{j-1} b_{ik} + 1 \right)_{b_{ij}} \times \left( p_j - \sum_{k=1}^{i} b_{kj} + 1 \right)_{b_{ij}} \]  
(186)

where \((a)_b\) is the Pochhammer symbol.

Consider some explicit examples to fix ideas.

At three points we find that

\[ |\mathcal{W}[\langle \mathcal{O}_p \mathcal{O}_q \mathcal{O}_r \rangle]| = \frac{p_! q_! r_!}{(\frac{p+r-p}{2})! (\frac{p+r-q}{2})! (\frac{p+r-q}{2})!} \]  
(187)
which is indeed fully symmetric.

At four points we find

\[
\text{Legs}_\text{out} = \sum_{(b_{12}) \subseteq \mathbb{P}(12)} \sum_{(b_{13}) \subseteq \mathbb{P}(13)} \sum_{(b_{14}) \subseteq \mathbb{P}(14)} \sum_{(b_{23}) \subseteq \mathbb{P}(23)} \sum_{(b_{24}) \subseteq \mathbb{P}(24)} \sum_{(b_{34}) \subseteq \mathbb{P}(34)} \sum_{(d_{12}) \subseteq \mathbb{P}(12)} \sum_{(d_{13}) \subseteq \mathbb{P}(13)} \sum_{(d_{14}) \subseteq \mathbb{P}(14)} \sum_{(d_{23}) \subseteq \mathbb{P}(23)} \sum_{(d_{24}) \subseteq \mathbb{P}(24)} \sum_{(d_{34}) \subseteq \mathbb{P}(34)}
\]

\[
\text{Legs}_\text{in} = \sum_{(b_{12}) \subseteq \mathbb{P}(12)} \sum_{(b_{13}) \subseteq \mathbb{P}(13)} \sum_{(b_{14}) \subseteq \mathbb{P}(14)} \sum_{(b_{23}) \subseteq \mathbb{P}(23)} \sum_{(b_{24}) \subseteq \mathbb{P}(24)} \sum_{(b_{34}) \subseteq \mathbb{P}(34)} \sum_{(d_{12}) \subseteq \mathbb{P}(12)} \sum_{(d_{13}) \subseteq \mathbb{P}(13)} \sum_{(d_{14}) \subseteq \mathbb{P}(14)} \sum_{(d_{23}) \subseteq \mathbb{P}(23)} \sum_{(d_{24}) \subseteq \mathbb{P}(24)} \sum_{(d_{34}) \subseteq \mathbb{P}(34)}
\]

Combinations \(c_i\) and dispositions \(d_i\) generated so far do not refer to the specific numbering of the legs of the correlator. What we have done so far has been to enumerate Wick contractions. The final task is to assign them to the indexes from 1 to \(\sum_{i=1}^n p_i\). In the example above the color-code denotes which subset of numbers belong to the same blob.

\section*{B.1 Digits}

Let us introduce the characteristic vector of a combination in \(\mathcal{C}(k \subseteq n)\). This is a digit of \(n\) bits, i.e. 0 or 1, with \(k\) values of 1 distributed in the digit. The positions of the 1 give the combination, for example

\[
\{1, 3, 5\} \in \mathcal{C}(3 \subseteq 5) \quad \longleftrightarrow \quad [10101]
\]

(188)

It is simple to think of the combinations in this way, since the only operation one has to do is to sequenciate the 1s to the right or the 0s to the left, starting from the configuration \([111100\ldots]\) with \(k\) values of 1s at the beginning. The integer corresponding to a given combination is computed by the Gosper’s hack in C++.\(^{13}\) This is the fastest way to do it, and it works well for low charges (since there is an obvious bound provided by the number of bits of the machine.)

The idea of sequenciating is useful to generate dispositions as well. Recall that \(\mathcal{D}(k \subseteq n)\) is obtained by acting with the symmetry group \(S_k\) on each combination in \(\mathcal{C}(k \subseteq n)\). So we pick a \(k\)-combination \(C = \{c_1, \ldots, c_k\}\). (It is ordered, but otherwise it is not necessary). Starting from the subset of length one \(C_1 = \{c_1\}\) we generate the set

\[
C_2 = \{\{c_1, 0\}, \{0, c_1\}\} \cup \{0 \rightarrow c_2\}
\]

(189)

Then for each element in \(C_2\) we generate all sets obtained by sequenciating an extra 0. For example,

\[
C_{3,1} = \{\{c_1, c_2, 0\}, \{c_1, 0, c_2\}, \{0, c_1, c_2\}\} \cup \{0 \rightarrow c_3\}
\]

(190)

\[
C_{3,2} = \{\{c_2, c_1, 0\}, \{c_2, 0, c_1\}, \{0, c_2, c_1\}\} \cup \{0 \rightarrow c_3\}
\]

(191)

\(^{13}\)See for example \url{https://en.wikipedia.org/wiki/Combinatorial_number_system}
and so on. This recursive procedure on the length of the subsets in \( C \) gives all the dispositions of \( C \).

For the Wick contractions, there are no repetitions in the \( \{c_i\} \), since they come from the combinations.

Instead, for generating the sequences \( |d_1, \ldots, d_{n-1}\rangle \), which we encountered in section 4.1.2, it is useful to consider cases with repetitions. Indeed, the possible values of \( |d_1, \ldots, d_{n-1}\rangle \) can be obtained by taking all partitions of \( 2(n-2) = \sum_{i=1}^{n-1} d_i \), because of the constraint on the tree, and then by considering all dispositions of these partitions. The parts can have repetitions.

The minor modification of the algorithm above is the following. The idea is to collect the repetitions in the original sequence as vectors \( \vec{c}_i \) with \( \vec{c}_i \neq \vec{c}_j \) if \( i \neq j \). The recursive procedure will now work upon replacing the 0 with these vectors at each step. For example, start from \( C = \{c_1 c_1, c_2 c_2, c_3 \ldots\} \). Then we generate

\[
C_2 = \{\{c_1, c_1, 0\}, \{c_1, 0, c_1\}, \{0, c_1, c_1\}\}/\{0 \rightarrow \vec{c}_2\} \tag{192}
\]

We go on with

\[
C_{3,1} = \{\{c_1, c_1, c_2, c_2, 0\}, \{c_1, c_1, c_2, 0, c_2\}, \{c_1, c_1, 0, c_2, c_2\}, \{c_1, 0, c_1, c_2, c_2\}, \{0, c_1, c_1, c_2, c_2\}\}/\{0 \rightarrow c_3\} \tag{193}
\]

\section*{B.2 \textbf{Prüfer sequences and Trees}}

A unique way to label a tree is to use a Prüfer sequence \( s = (s_1 \ldots) \), constructed as follows [51]. Consider the tree made of points at positions 1, \ldots, \( n-1 \), each point with \( d_i \) legs attached, as described above. For the example in (97), this is

\[
\begin{tikzpicture}
  \node (1) at (0,0) {2};
  \node (2) at (1,0) {4};
  \node (3) at (-1,1) {3};
  \draw (1) -- (2);
  \draw (2) -- (3);
  \end{tikzpicture} \tag{194}
\]

We define a leaf in the tree as a pair of positions \( \ell_1 - \ell_2 \) in which \( \ell_1 \) is connected to \( \ell_2 \) with one and only one bridge. At step \( k \) of the Prüfer algorithm, remove the leaf \( \ell_1 - \ell_2 \) with the smallest labelled position \( \ell_1 \) and assign \( s_k = \ell_2 \) to the sequence. Then, write

\[
|\mathcal{W}[\ell_1 - \ell_2]| = (q_{\ell_1} - d_{\ell_1} + 1) \times (q_{\ell_2} - \#(\ell_2)) \tag{195}
\]

to count the Wick contractions corresponding to that leaf, where \( \#(\ell_2) \) is the number of times \( \ell_2 \) appeared in the sequence previously (step < \( k \)). We stop when there is only one leaf left. Here below we give another example, (rich enough to illustrate the various steps)
The nice feature of the Pr"ufer algorithm is that it reduces the pairwise computation of Wick contractions, which follows from our results in (186),

\[ |W[T]| = \prod_{i<j} q_i - d_i + \sum_{k=j}^{n-1} b_{ik} \times q_j - \sum_{k=1}^{i-1} b_{kj} \]  

(197)

(where we used conservation of charge on the tree) to a sequence. In particular, it gives a more efficient ordering of the \( b_{ij} \) to count \( |W[T]| \). For instance, the pairs \( (ij) = \{(56), (57), (59)\} \) in the example above would count differently, even though the total result does not change. Thus we can use the Pr"ufer algorithm to rearrange the counting.

In the Pr"ufer algorithm it is clear that the first time two operators \( i \) and \( j \) appear in the sequence, they count with \( q_i q_j \), because necessarily \( d_i = 1 \) or \( d_j = 1 \). The second time one of this operators appears again, it counts with \( q_i - 1 \) or \( q_j - 1 \), and so on. The total number of Wick contractions is then,

\[ |W[T]| = \prod_{i=1}^{n-1} q_i(q_i - 1)\ldots(q_i - d_i + 1) \]  

(198)

which is the result we quoted in (99).

C Low charge examples for multipoint orthogonality

In this section we reconsider the idea of phrasing the multi-point orthogonality theorem as an identity of the form \( det = 0 \), and we discuss some examples to have an idea about how the combinatorics works. Let us recall our setup of section 3. We want to study,

\[ F_{p|q_1\ldots q_{n-1}}(x, x_1 \ldots x_{n-1}) = \langle O_p(x)T_{q_1}(x_1)\ldots T_{q_{n-1}}(x_{n-1}) \rangle \]  

(199)
where \( q_i \) can be a partition of \( q \) and \( n \geq 3 \). We can use the arrangement of \( O_p(x) \) in (9) as a determinant, to rewrite our correlator as another determinant

\[
F_{p|q_1,\ldots,q_{n-1}} = \frac{1}{N_p} \det \left( \begin{array}{cccc}
C_{\lambda_1|\lambda_1} & C_{\lambda_2|\lambda_1} & \cdots & C_{p|\lambda_1} \\
\vdots & \vdots & \ddots & \vdots \\
C_{\lambda_1|\lambda_{P-1}} & C_{\lambda_2|\lambda_{P-1}} & \cdots & C_{p|\lambda_{P-1}} \\
C_{\lambda_1|q_1,\ldots,q_{n-1}} & C_{\lambda_2|q_1,\ldots,q_{n-1}} & \cdots & C_{p|q_1,\ldots,q_{n-1}}
\end{array} \right)
\]  

(200)

Notice that differently from \( F_{p|q_1,\ldots,q_{n-1}} \), the (color factor of the) correlator \( C_{\lambda|q_1,\ldots,q_{n-1}} \) now involves only traces,

\[
C_{\lambda|q_1,\ldots,q_{n-1}} = \langle T_{\lambda}(x) T_{q_1}(x_1) \cdots T_{q_{n-1}}(x_{n-1}) \rangle
\]

(201)

As we noticed in the main text of section 3, a feature of writing the correlator as a determinant is that if a propagator structure is such that the rows of the matrix are not independent, the determinant will vanish. The two-point functions from which we started our story are an obvious example, but also a trivial multipoint generalisation of that. Take the list of \( q_1,\ldots,q_{n-1} \) to coincide with a partition of \( p \), say \( \lambda_I \). Then, passing to the double line notation, it is clear that \( C_{\lambda_j|q_1,\ldots,q_{n-1}} \) has the same color factor of the corresponding two-point function regardless of the number of points.

More generally, the property we want to assign to the propagator structure in such a way that the determinant vanishes is that

\[
C_{\lambda|q_1,\ldots,q_{n-1}} = \sum_{I=1,\ldots,P-1} K_I C_{\lambda_I|\lambda_I} \quad j = 1, 2 \ldots P
\]

(202)

for some \( K_I \). Namely, the vector made by \( C_{\lambda_j|q_1,\ldots,q_{n-1}} \), is a linear combination of the rows in (200). This translates into a requirement about the topology of the diagram, since all partitions \( \lambda_I=1,\ldots,P-1 \) involved in (202) have at least length two, i.e. \( \lambda_P = \{p\} \) is excluded, and there cannot be self-contractions within \( T_{\lambda_I} \) in the two point functions \( C_{\lambda_j|\lambda_I} \). We conclude that a necessary condition for \( F_{p|q_1,\ldots,q_{n-1}} \) to vanish is that the diagram disconnects as soon as we remove \( O_p \).

The reasoning above leaves a free parameter. In fact, the assignment \( q_1,\ldots,q_{n-1} \) can be such that

\[
\frac{1}{2} \left( -p + \sum_{i=1}^{n-1} q_i \right) = k \geq 0
\]

(203)

and yet the diagram disconnects as soon as we remove \( O_p \). The value of \( k \) measures the excess of \( \sum q_i \) to be a partition of \( p \). This is possible precisely because differently from a two-point function, in a multipoint function there can be \( k \geq 0 \) Wick contractions distributed among the \( T_{q_i}(x) \ldots T_{q_{n-1}}(x_{n-1}) \), such that when \( O_p \) is removed, the diagram still disconnects.
Example (I)

Consider the $SU(N)$ single-particle operator $O_4$, which is the SPO with the simplest admixture, $T_{\{22\}}$ and $T_4$, and take the four-point function $\langle O_4(x)T_2(x_1)T_2(x_2)T_2(x_3) \rangle$. This is an example with $k = 1$, and there is only one possible propagator structure,

\[
O_4(x) \nonumber
\]

\[
\text{\begin{tikzpicture}[scale=0.8]
    
    
    
    \draw[fill=green!50!white] (0,0) circle (0.3) node (v1) {}; 
    \draw[fill=green!50!white] (1.5,0) circle (0.3) node (v2) {}; 
    \draw[fill=green!50!white] (-1.5,0) circle (0.3) node (v3) {}; 
    \draw[thick] (v1) -- (v2) node[midway, above] {$T_2(z_1)$}; 
    \draw[thick] (v1) -- (v3) node[midway, below] {$T_2(z_3)$}; 
    \draw[thick] (v2) -- (v3) node[midway, left] {$T_2(z_2)$}; 
\end{tikzpicture}}
\]

\[\text{namely, } F_{4|2,2,2} \simeq g_x^2 g_x g_{x_2} g_{x_3} g_{23} \text{. In formulas we will now find that } F_{4|2,2,2} \text{ vanishes,}
\]

\[F_{4|2,2,2} \simeq \frac{1}{N_4} \text{det} \left( \begin{array}{cc} C_{\{22\}|\{22\}} & C_{4|\{22\}} \\ C_{\{22\}|2,2,2} & C_{4|2,2,2} \end{array} \right) = 0 \] (205)

\[\text{where }
\]

\[C_{\{22\}|\{22\}} = 8(N^4 - 1) \quad ; \quad C_{4|\{22\}} = 8(N^2 - 1)(2N^2 - 3)/N \]

\[C_{\{22\}|2,2,2} = 32(N^4 - 1) \quad ; \quad C_{4|2,2,2} = 32(N^2 - 1)(2N^2 - 3)/N \] (206)

\[\text{We see here that } F_{4|2,2,2} = 0 \text{ is a consequence of the four-point functions in the second line of (206) being proportional by a factor of four to the two-point functions in the first line. The figure below provides some more intuition. We draw } F_{4|2,2,2} \text{, and by looking at a single contribution to the color factor of } C_{4|2,2,2} \text{, we illustrate that its double line notation is the same as that of the two-point function } C_{4|\{22\}} \text{, i.e. in double line notation } T_2(x_1)T_2(x_2)T_2(x_3) \text{ “behaves” like the operator } T_{22},
\]

\[
\text{\begin{tikzpicture}[scale=0.8]
    
    
    
    \draw[fill=green!50!white] (0,0) circle (0.3) node (v1) {}; 
    \draw[fill=green!50!white] (1.5,0) circle (0.3) node (v2) {}; 
    \draw[fill=green!50!white] (-1.5,0) circle (0.3) node (v3) {}; 
    \draw[thick] (v1) -- (v2) node[midway, above] {$T_2(z_1)$}; 
    \draw[thick] (v1) -- (v3) node[midway, below] {$T_2(z_3)$}; 
    \draw[thick] (v2) -- (v3) node[midway, left] {$T_2(z_2)$}; 
\end{tikzpicture}}
\]

\[F_{4|2,2,2} \simeq g_x^2 g_x g_{x_2} g_{x_3} g_{23} \] (207)

\[\text{Notice also that the factor of four in (206) is the number of ways we can contract } T_2(x_2)T_2(x_3) \text{ with } k = 1 \text{ Wick contractions. Precisely because of this Wick contraction, we can see in (207) that the loop counting is the same for both the four- and the two-point functions.}
\]

56
Example (II)

In the next example we consider the four-point function \( \langle O_4(x)T_2(z_1)T_2(z_2)T_4(z_3) \rangle \), which corresponds to a case with \( k = 2 \). There are six propagator structures, but we are interested in the one drawn below, since it will have a vanishing color factor,

\[
\mathcal{F}_{4|22} \simeq g_5^2g_2^2g_2^2
\]

By a mechanism similar to the previous example, we will now see that \( \mathcal{F}_{4|4,2,2} = 0 \). However, this new example shows that the combinatorics in general is subtle/complicated.

In formulas we find that \( \mathcal{F}_{4|4,2,2} \) gives

\[
\mathcal{F}_{4|4,2,2} \simeq \frac{1}{N_4} \det \begin{pmatrix}
\mathcal{C}_{\{22\}} & \mathcal{C}_{\{22\}|4,2,2} \\
\mathcal{C}_{\{22\}|4,2,2} & \mathcal{C}_{4|4,2,2}
\end{pmatrix} = 0
\]

where

\[
\mathcal{C}_{\{22\}} = 8(N^4 - 1) \\
\mathcal{C}_{\{22\}|4,2,2} = 32(N^4 - 1)(2N^2 - 3)/N \\
\mathcal{C}_{4|4,2,2} = 8(N^2 - 1)(2N^2 - 3)/N \\
\mathcal{C}_{4|4,2,2} = 32(N^2 - 1)(2N^2 - 3)^2/N^2
\]

This time the four-point functions in the second line of (210) are proportional to the two-point functions on the first line by an \( N \)-dependent factor. This feature can be seen in the double line diagram (208), in which we can count four loops instead of three, and three was the counting of the previous example (207). From the combinatorial point of view is then non trivial that \( \mathcal{F}_{4|4,2,2} \) vanishes.

In general, the determinant argument we started with, only leads to a relation between rows which eventually can get complicated. It would be very interesting to have a more direct combinatorial argument, alternative to the proof we gave in section 3.1.
D More N$^3$E correlators at 4-pt

Here we give a few more details on the N$^3$E four-point functions analysed with the half-BPS OPE as in section 5.2. Thus focussing on the following diagrams,

$$\langle \mathcal{O}_p \mathcal{O}_q \mathcal{O}_r \mathcal{O}_s \rangle_c = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7.$$  \hspace{1cm} (211)

Recall that the operator $\mathcal{O}_s$ sits at the right corner on the bottom of each square.

Example: $\langle \mathcal{O}_3 \mathcal{O}_3 \mathcal{O}_3 \mathcal{O}_3 \rangle$

In this case crossing symmetry implies

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6$$  \hspace{1cm} (212)

The relevant equations, (155) and (156), respectively, simplify to become

$$\alpha_1 = 18 \frac{\langle \mathcal{O}_3 \mathcal{O}_3 \rangle^2}{\langle \mathcal{O}_2 \mathcal{O}_2 \rangle} = 81 \frac{(N^2 - 1)(N^2 - 4)^2}{N^2}. \hspace{1cm} (213)$$

$$2\alpha_1 + \alpha_7 = 9 \langle \mathcal{O}_3 \mathcal{O}_3 \mathcal{O}_4 \rangle + \frac{\langle \mathcal{O}_3 \mathcal{O}_3 [\mathcal{O}_2 \mathcal{O}_2] \rangle^2}{\langle [\mathcal{O}_2 \mathcal{O}_2] [\mathcal{O}_2 \mathcal{O}_2] \rangle} = 324 \frac{(N^2 - 1)(N^2 - 4)(N^2 - 8)}{N^2}. \hspace{1cm} (214)$$

In the latter, which comes from (156), we find that only $\mathcal{K} = [\mathcal{O}_2 \mathcal{O}_2]$ is relevant, and also that the corresponding three-point function is simply obtained from the results in (153). Combining the above results we find the remaining coefficient,

$$\alpha_7 = 162 \frac{(N^2 - 1)(N^2 - 4)(N^2 - 12)}{N^2} = 54 \frac{N^2 - 12}{N} \langle \mathcal{O}_3 \mathcal{O}_3 \rangle. \hspace{1cm} (215)$$

Taking coincidence limits then gives

$$\langle [\mathcal{O}_3 \mathcal{O}_3] \mathcal{O}_3 \mathcal{O}_4 \rangle = \langle [\mathcal{O}_3 \mathcal{O}_3] [\mathcal{O}_3 \mathcal{O}_3] \rangle = 2\alpha_1 + 2 \langle \mathcal{O}_3 \mathcal{O}_3 \rangle^2 = 18 \frac{(N^2 - 1)(N^2 - 4)^2(N^2 + 8)}{N^2}, \hspace{1cm} (216)$$

where the $\langle \mathcal{O}_3 \mathcal{O}_3 \rangle^2$ term comes from the disconnected contributions to the four-point function.
Example: $\langle O_4 O_4 O_4 O_6 \rangle$

This example obeys the maximal crossing symmetry conditions (212). We have

$$\alpha_1 = 8 \frac{\langle O_4 O_4 O_4 \rangle \langle O_6 O_6 \rangle}{\langle O_4 O_4 \rangle}$$

(217)

and

$$2\alpha_1 + \alpha_7 = 16 \langle O_4 O_6 O_6 \rangle + \sum_{L'=6} \langle O_4 O_4 K_{L'} \rangle \langle \tilde{g}^{-1} \rangle_{L'} \langle K_{L'} O_4 O_6 \rangle.$$  

(218)

The solution for $\alpha_7$ reads

$$\alpha_7 = 64 \frac{(2N^4 - 187N^2 + 81)}{N(N^2 + 1)} \langle O_6 O_6 \rangle.$$  

(219)

For $\langle T_4 T_4 T_4 O_6 \rangle$ we have

$$\alpha_7 \rightarrow 64 \frac{2N^2 - 81}{N} \langle O_6 O_6 \rangle.$$  

(220)

Example: $\langle O_3 O_3 O_4 O_4 \rangle$

In this case crossing symmetry requires

$$\alpha_1 = \alpha_2, \quad \alpha_3 = \alpha_5, \quad \alpha_4 = \alpha_6,$$

(221)

So we only have four independent coefficients. From equation (155) and its crossing we find

$$\alpha_1 = 24 \frac{\langle O_3 O_3 \rangle \langle O_4 O_4 \rangle}{\langle O_2 O_2 \rangle},$$  

(222)

$$\alpha_3 + \alpha_4 = 9 \frac{\langle O_3 O_3 O_3 \rangle \langle O_4 O_4 \rangle}{\langle O_3 O_3 \rangle} = 81 \frac{\langle O_4 O_4 \rangle^2}{\langle O_3 O_3 \rangle}.$$  

(223)

From equation (156) and its crossing we obtain

$$2\alpha_3 + \alpha_7 = 9 \langle O_4 O_4 O_4 \rangle + \frac{\langle O_3 O_3 [O_2 O_2] \rangle \langle |O_2 O_2| O_4 O_4 \rangle}{\langle |O_2 O_2| [O_2 O_2] \rangle}$$  

(224)

$$\alpha_1 + \alpha_4 + \alpha_7 = 34 \langle O_3 O_4 O_5 \rangle + \frac{\langle |O_2 O_3| O_3 O_3 \rangle^2}{\langle |O_2 O_3| [O_2 O_3] \rangle}.$$  

(225)

The two-particle operators entering the above equations are $K = [O_2 O_2]$ and $K = [O_2 O_3]$. Notice also the appereance of $\langle O_4 O_4 O_4 \rangle$ which is NME three point function computed in
section 4.2.1. In sum we have four equations and thus we can determine the independent coefficients,
\[
\alpha_1 = 36 \frac{(N^2 - 4)}{N} \langle O_4 O_4 \rangle = 24 \frac{\langle O_3 O_3 \rangle \langle O_4 O_4 \rangle}{\langle O_3 O_2 \rangle}, \\
2\alpha_3 = \alpha_4 = 72 \frac{N(N^2 - 9)}{N^2 + 1} \langle O_4 O_4 \rangle = 54 \frac{(\langle O_3 O_4 \rangle)^2}{\langle O_3 O_3 \rangle}, \\
\alpha_7 = 72 \frac{N^4 - 25N^2 - 6}{N(N^2 + 1)} \langle O_4 O_4 \rangle.
\]

Having obtained the four-point correlator we then find the coincidence limits,
\[
\langle [O_3 O_3] O_4 O_4 \rangle = 144 \frac{N(N^2 - 9)}{N^2 + 1} \langle O_4 O_4 \rangle. \\
\langle [O_3 O_4] O_3 O_4 \rangle = \frac{72N^4 - 6N^2 - 2}{N(N^2 + 1)} \langle O_4 O_4 \rangle + \langle O_3 O_3 \rangle \langle O_4 O_4 \rangle. \tag{226}
\]

If we consider instead \( \langle T_3 T_3 T_4 O_4 \rangle \) we find \( \alpha_1 \) is unchanged while
\[
2\alpha_3 = \alpha_4 \rightarrow 72 \frac{N^2 - 6}{N} \langle O_4 O_4 \rangle, \quad \alpha_7 \rightarrow 72 \frac{N^2 - 18}{N} \langle O_4 O_4 \rangle. \tag{227}
\]

**Example:** \( \langle O_3 O_3 O_5 O_5 \rangle \)

This is the first case where we need to consider more than one possible exchanged multiparticle operator in an OPE channel. The crossing symmetry conditions (221) still apply but now from equation (155) and crossing we have
\[
\alpha_1 = 30 \langle O_3 O_5 \rangle \langle O_3 O_5 \rangle, \\
\alpha_3 + \alpha_4 = \frac{\langle O_3 O_4 O_5 \rangle^2}{\langle O_4 O_4 \rangle} = 144 \frac{(\langle O_5 O_5 \rangle)^2}{\langle O_4 O_4 \rangle}. \tag{228}
\]

From equation (156) we find,
\[
2\alpha_3 + \alpha_7 = 9 \langle O_4 O_5 O_5 \rangle + \frac{\langle O_3 O_4 \langle O_2 O_2 \rangle \langle O_5 O_5 \rangle \rangle}{\langle [O_3 O_2] [O_2 O_2] \rangle}, \tag{229}
\]

Its crossing transformation has more structure:
\[
\alpha_1 + \alpha_4 + \alpha_7 = 15 \langle O_3 O_5 O_6 \rangle + \sum_{\mathcal{I}, \mathcal{J}} \langle O_3 O_5 \mathcal{K}_{\mathcal{L}} \rangle \langle \mathcal{J}^{-1} \rangle \mathcal{M}_{\mathcal{L} \mathcal{K}_{\mathcal{L}}} \langle \mathcal{K}_{\mathcal{L}} O_3 O_5 \rangle. \tag{230}
\]
Here $K$ runs over the set of twist six multiparticle operators $\{[O_2O_4], [O_3O_3], [O_2O_2O_2]\}$. The matrix of their two-point functions and hence its inverse can be determined from results on coincidence limits presented earlier. We find

$$
\langle K_2K_2' \rangle = \begin{pmatrix}
2(N^2 + 7)\langle O_4O_4 \rangle & 18\langle O_4O_4 \rangle & 0 \\
18\langle O_4O_4 \rangle & 4(N^2 + 8)\langle O_3O_3 \rangle \langle O_2O_2 \rangle & 48\langle O_3O_3 \rangle \\
0 & 48\langle O_3O_3 \rangle & 24(N^2 + 1)(N^2 + 3)\langle O_2O_2 \rangle
\end{pmatrix}.
$$

(231)

In sum, the above equations determine all the coefficients,

$$
\alpha_1 = 30\frac{\langle O_3O_3 \rangle \langle O_5O_5 \rangle}{\langle O_2O_2 \rangle },
\quad 3\alpha_3 = \alpha_4 = 108\frac{\langle O_3O_3 \rangle^2}{\langle O_4O_4 \rangle },
\quad \alpha_7 = 90\frac{(N^4 - 43N^2 - 72)}{N(N^2 + 5)}\langle O_3O_3 \rangle.
$$

(232)

If we consider instead $\langle T_3T_3T_5O_5 \rangle$ we again find $\alpha_1$ is unchanged while

$$
3\alpha_3 = \alpha_4 \rightarrow 135\frac{N^2 - 8}{N}\langle O_5O_5 \rangle, \quad \alpha_7 \rightarrow 90\frac{N^2 - 24}{N}\langle O_5O_5 \rangle.
$$

(233)

**Example:** $\langle O_3O_3O_6O_6 \rangle$

This example is very similar to the previous one with the crossing conditions (221) still valid and three twist seven multiparticle operators participating in the crossing transformation of equation (156). We simply quote the results here,

$$
\alpha_1 = 36\frac{\langle O_3O_4 \rangle \langle O_6O_6 \rangle}{\langle O_2O_2 \rangle },
\quad 4\alpha_3 = \alpha_4 = 180\frac{\langle O_6O_6 \rangle^2}{\langle O_3O_3 \rangle },
\quad \alpha_7 = 108\frac{(N^6 - 65N^4 - 408N^2 - 80)}{N(N^4 + 15N^2 + 8)}\langle O_6O_6 \rangle.
$$

(234)

If we consider instead $\langle T_3T_3T_5O_5 \rangle$ we find

$$
4\alpha_3 = \alpha_4 \rightarrow 216\frac{N^2 - 10}{N}\langle O_6O_6 \rangle, \quad \alpha_7 \rightarrow 108\frac{N^2 - 30}{N}\langle O_6O_6 \rangle.
$$

(235)
Example: $⟨O_3 O_4 O_4 O_5⟩$

In this case crossing symmetry implies a different set of conditions,
\[
\alpha_1 = \alpha_3, \quad \alpha_2 = \alpha_4, \quad \alpha_5 = \alpha_6. \tag{236}
\]
equation (155) and its crossing transformation become
\[
\alpha_1 + \alpha_2 = 108 \frac{⟨O_3 O_4⟩⟨O_3 O_5⟩}{⟨O_3 O_3⟩} 2\alpha_5 = 12 \frac{⟨O_4 O_4 O_4⟩⟨O_5 O_5⟩}{⟨O_4 O_4⟩}. \tag{237}
\]
Equation (156) becomes
\[
\alpha_1 + \alpha_5 + \alpha_7 = 12 \frac{⟨O_4 O_5 O_5⟩}{⟨O_4 O_3⟩} + \sum_{t \sqsubset 6} \langle O_4 O_4 K_t g^{-1} \rangle \langle K_t O_3 O_5 ⟩. \tag{238}
\]
The solution obtained is
\[
\alpha_1 = 36 \frac{⟨O_4 O_4⟩⟨O_5 O_5⟩}{⟨O_3 O_3⟩}, \quad \alpha_2 = 72 \frac{⟨O_4 O_4⟩⟨O_5 O_5⟩}{⟨O_3 O_3⟩}, \\
\alpha_5 = 6 \frac{⟨O_4 O_4 O_4⟩⟨O_5 O_5⟩}{⟨O_4 O_4⟩}, \quad \alpha_7 = 96 \frac{(N^4 - 50 N^2 + 9)}{N(N^2 + 1)} ⟨O_3 O_5⟩. \tag{240}
\]
If we consider instead $⟨T_3 T_4 T_5 O_6⟩$ we find
\[
2\alpha_1 = \alpha_2 \rightarrow 96 \frac{N^2 - 6}{N} ⟨O_5 O_5⟩, \quad \alpha_5 \rightarrow 96 \frac{N^2 - 9}{N} ⟨O_5 O_5⟩, \quad \alpha_7 \rightarrow 96 \frac{N^2 - 27}{N} ⟨O_5 O_5⟩. \tag{241}
\]
Example: $⟨O_3 O_4 O_5 O_6⟩$

This is the first example with no crossing symmetry conditions. We have confirmed with explicit Wick contraction computation that our solution in (159)-(160) is reproduced. We quote the coefficient $\alpha_7$
\[
\alpha_7 = 120 \frac{(N^6 - 82 N^4 - 231 N^2 + 12)}{N(N^2 + 1)(N^2 + 5)} ⟨O_6 O_6⟩. \tag{242}
\]
For $⟨T_3 T_4 T_5 O_6⟩$ we have
\[
\alpha_7 \rightarrow 120 \frac{N^2 - 36}{N} ⟨O_6 O_6⟩. \tag{243}
\]
References

[1] F. Aprile, J. M. Drummond, P. Heslop and H. Paul, JHEP 1801 (2018) 035 doi:10.1007/JHEP01(2018)035 [arXiv:1706.02822 [hep-th]].

[2] L. F. Alday and S. Caron-Huot, JHEP 1812 (2018) 017 doi:10.1007/JHEP12(2018)017 [arXiv:1711.02031 [hep-th]].

[3] F. Aprile, J. M. Drummond, P. Heslop and H. Paul, JHEP 1805 (2018) 056 doi:10.1007/JHEP05(2018)056 [arXiv:1711.03903 [hep-th]].

[4] F. Aprile, J. Drummond, P. Heslop and H. Paul, JHEP 2003 (2020) 190 doi:10.1007/JHEP03(2020)190 [arXiv:1912.01047 [hep-th]].

[5] L. F. Alday and X. Zhou, arXiv:1912.02663 [hep-th].

[6] G. Arutyunov and S. Frolov, Phys. Rev. D 61 (2000) 064009 doi:10.1103/PhysRevD.61.064009 [hep-th/9907085].

[7] G. Arutyunov and S. Frolov, JHEP 0004 (2000) 017 doi:10.1088/1126-6708/2000/04/017 [hep-th/0003038].

[8] L. Rastelli and X. Zhou, JHEP 1804 (2018) 014 doi:10.1007/JHEP04(2018)014 [arXiv:1710.05923 [hep-th]].

[9] G. Arutyunov, R. Klabbers and S. Savin, JHEP 1809 (2018) 023 doi:10.1007/JHEP09(2018)023 [arXiv:1806.09200 [hep-th]].

[10] G. Arutyunov, R. Klabbers and S. Savin, JHEP 1809 (2018) 118 doi:10.1007/JHEP09(2018)118 [arXiv:1808.06788 [hep-th]].

[11] F. Aprile, J. Drummond, P. Heslop and H. Paul, Phys. Rev. D 98 (2018) no.12, 126008 doi:10.1103/PhysRevD.98.126008 [arXiv:1802.06889 [hep-th]].

[12] J. M. Maldacena and A. Strominger, JHEP 9812 (1998) 005 doi:10.1088/1126-6708/1998/12/005 [hep-th/9804085].

[13] J. McGreevy, L. Susskind and N. Toumbas, JHEP 0006 (2000) 008 doi:10.1088/1126-6708/2000/06/008 [hep-th/0003075].

[14] V. Balasubramanian, M. Berkooz, A. Naqvi and M. J. Strassler, JHEP 0204 (2002) 034 doi:10.1088/1126-6708/2002/04/034 [hep-th/0107119].

[15] S. Corley, A. Jevicki and S. Ramgoolam, Adv. Theor. Math. Phys. 5 (2002) 809 doi:10.4310/ATMP.2001.v5.n4.a6 [hep-th/0111222].

[16] R. de Mello Koch and R. Gwyn, JHEP 0411 (2004) 081 doi:10.1088/1126-6708/2004/11/081 [hep-th/0410236].
[17] T. W. Brown, JHEP **0807** (2008) 044 doi:10.1088/1126-6708/2008/07/044 [hep-th/0703202 [HEP-TH]].

[18] E. D’Hoker, J. Erdmenger, D. Z. Freedman and M. Perez-Victoria, Nucl. Phys. B **589** (2000) 3 doi:10.1016/S0550-3213(00)00534-4 [hep-th/0003218].

[19] E. D’Hoker, D. Z. Freedman, S. D. Mathur, A. Matusis and L. Rastelli, In *Shifman, M.A. (ed.): The many faces of the superworld* 332-360 doi:10.1142/9789812793850˙0020 [hep-th/9908160].

[20] M. Bianchi and S. Kovacs, Phys. Lett. B **468** (1999) 102 doi:10.1016/S0370-2693(99)01211-3 [hep-th/9910016].

[21] B. Eden, P. S. Howe, C. Schubert, E. Sokatchev and P. C. West, Phys. Lett. B **472** (2000) 323 doi:10.1016/S0370-2693(99)01442-2 [hep-th/9910150].

[22] J. Erdmenger and M. Perez-Victoria, Phys. Rev. D **62** (2000) 045008 doi:10.1103/PhysRevD.62.045008 [hep-th/9912250].

[23] B. U. Eden, P. S. Howe, E. Sokatchev and P. C. West, Phys. Lett. B **494** (2000) 141 doi:10.1016/S0370-2693(00)01181-3 [hep-th/0004102].

[24] S. Caron-Huot and A. K. Trinh, JHEP **1901** (2019) 196 doi:10.1007/JHEP01(2019)196 [arXiv:1809.09173 [hep-th]].

[25] P. J. Heslop and P. S. Howe, Nucl. Phys. B **626** (2002) 265 doi:10.1016/S0550-3213(02)00023-8 [hep-th/0107212].

[26] A. Bissi, C. Kristjansen, D. Young and K. Zoubos, JHEP **1106** (2011) 085 doi:10.1007/JHEP06(2011)085 [arXiv:1103.4079 [hep-th]].

[27] P. Caputa, R. de Mello Koch and K. Zoubos, JHEP **1208** (2012) 143 doi:10.1007/JHEP08(2012)143 [arXiv:1204.4172 [hep-th]].

[28] A. Hashimoto, S. Hirano and N. Itzhaki, JHEP **0008** (2000) 051 doi:10.1088/1126-6708/2000/08/051 [hep-th/0008016].

[29] T. W. Brown, P. J. Heslop and S. Ramgoolam, JHEP **0802** (2008) 030 doi:10.1088/1126-6708/2008/02/030 [arXiv:0711.0176 [hep-th]].

[30] R. Bhattacharyya, S. Collins and R. de Mello Koch, JHEP **0803** (2008) 044 doi:10.1088/1126-6708/2008/03/044 [arXiv:0801.2061 [hep-th]].

[31] T. W. Brown, P. J. Heslop and S. Ramgoolam, JHEP **0904** (2009) 089 doi:10.1088/1126-6708/2009/04/089 [arXiv:0806.1911 [hep-th]].

[32] R. de Mello Koch, J. Smolic and M. Smolic, JHEP **0706** (2007) 074 doi:10.1088/1126-6708/2007/06/074 [hep-th/0701066].
[33] R. de Mello Koch, J. Smolic and M. Smolic, JHEP 0709 (2007) 049 doi:10.1088/1126-6708/2007/09/049 [hep-th/0701067].

[34] D. Bekker, R. de Mello Koch and M. Stephanou, JHEP 0802 (2008) 029 doi:10.1088/1126-6708/2008/02/029 [arXiv:0710.5372 [hep-th]].

[35] C. Lewis-Brown and S. Ramgoolam, [arXiv:2007.01734 [hep-th]].

[36] E. Witten, JHEP 9807 (1998) 006 doi:10.1088/1126-6708/1998/07/006 [hep-th/9805112].

[37] O. Aharony, Y. E. Antebi, M. Berkooz and R. Fishman, JHEP 0212 (2002) 069 doi:10.1088/1126-6708/2002/12/069 [hep-th/0211152].

[38] P. Caputa, R. de Mello Koch and P. Diaz, JHEP 1303 (2013) 041 doi:10.1007/JHEP03(2013)041 [arXiv:1301.1560 [hep-th]].

[39] P. Caputa, R. de Mello Koch and P. Diaz, JHEP 1306 (2013) 018 doi:10.1007/JHEP06(2013)018 [arXiv:1303.7252 [hep-th]].

[40] C. Lewis-Brown and S. Ramgoolam, JHEP 1811 (2018) 035 doi:10.1007/JHEP11(2018)035 [arXiv:1804.11090 [hep-th]].

[41] M. Taylor, “Matching of correlators in AdS(3) / CFT(2),” JHEP 06 (2008), 010 doi:10.1088/1126-6708/2008/06/010 [arXiv:0709.1838 [hep-th]].

[42] F. Coronado, JHEP 1901 (2019) 056 doi:10.1007/JHEP01(2019)056 [arXiv:1811.00467 [hep-th]].

[43] F. Coronado, Phys. Rev. Lett. 124 (2020) no.17, 171601 doi:10.1103/PhysRevLett.124.171601 [arXiv:1811.03282 [hep-th]].

[44] A. V. Belitsky and G. P. Korchemsky, JHEP 2005 (2020) 070 doi:10.1007/JHEP05(2020)070 [arXiv:1907.13131 [hep-th]].

[45] A. V. Belitsky and G. P. Korchemsky, arXiv:2003.01121 [hep-th].

[46] A. V. Belitsky and G. P. Korchemsky, arXiv:2006.01831 [hep-th].

[47] T. Bargheer, F. Coronado and P. Vieira, JHEP 1908 (2019) 162 [JHEP 2019 (2020) 162] doi:10.1007/JHEP08(2019)162 [arXiv:1904.00965 [hep-th]].

[48] T. Bargheer, F. Coronado and P. Vieira, arXiv:1909.04077 [hep-th].

[49] T. Fleury and V. Goncalves, doi:10.1007/JHEP07(2020)030 [arXiv:2004.10867 [hep-th]].

[50] F. Aprile and P. Vieira, “Large p explorations. From SUGRA to big STRINGS in Mellin space”, to appear
[51] Prüfer, H. (1918). “Neuer Beweis eines Satzes über Permutationen”. Arch. Math. Phys. 27 : 742-744.