NEwTON-OkoUNkovoB BODIES oF FlAg VARIoTES AND COMBINATORIAL MUTATIONS

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Abstract. A Newton–Okounkov body is a convex body constructed from a polarized variety with a higher rank valuation on the function field, which gives a systematic method of constructing toric degenerations of polarized varieties. Its combinatorial properties heavily depend on the choice of a valuation, and it is a fundamental problem to relate Newton–Okounkov bodies associated with different kinds of valuations. In this paper, we address this problem for flag varieties using the framework of combinatorial mutations which was introduced in the context of mirror symmetry for Fano manifolds. By applying iterated combinatorial mutations, we connect specific Newton–Okounkov bodies of flag varieties including string polytopes, Nakashima–Zelevinsky polytopes, and FFLV polytopes.

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1. Introduction

A Newton–Okounkov body $\Delta(X, L, v)$ is a convex body constructed from a polarized variety $(X, L)$ with a higher rank valuation $v$ on the function field $\mathbb{C}(X)$, which generalizes the notion of Newton polytopes for toric varieties to arbitrary projective varieties. It was introduced by Okounkov [37, 38, 39] and afterward developed independently by Kaveh–Khovanskii [29] and by Lazarsfeld–Mustata [33]. A remarkable fact is that the theory of Newton–Okounkov bodies gives a systematic method of constructing toric degenerations (see [2, Theorem 1] and [23, Corollary 3.14]). Since combinatorial properties of $\Delta(X, L, v)$ heavily depend on the choice of a valuation $v$, it is a fundamental problem to give concrete relations among Newton–Okounkov bodies associated with different kinds of valuations. In the case of flag varieties and Schubert varieties, their Newton–Okounkov bodies realize the following representation-theoretic polytopes:

(i) Berenstein–Littelmann–Zelevinsky’s string polytopes [28],
(ii) Nakashima–Zelevinsky polytopes [19],
(iii) FFLV (Feigin–Fourier–Littelmann–Vinberg) polytopes [13, 31],

where the attached references are the ones giving realizations as Newton–Okounkov bodies. The set of lattice points in every polytope of (i)–(iii) parametrizes a specific basis of an irreducible

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highest weight module of a semisimple Lie algebra. In particular, string polytopes and Nakashima–Zelevinsky polytopes give polyhedral parametrizations of crystal bases (see [5, 34, 35, 36]). Our aim in the present paper is to relate these polytopes by applying iterated combinatorial mutations.

Combinatorial mutations for lattice polytopes were introduced by Akhtar–Coates–Galkin–Kasprzyk [1] in the context of mirror symmetry for Fano manifolds. The original motivation in [1] is to classify Fano manifolds by using combinatorial mutations. A Laurent polynomial \( f \in \mathbb{C}[\mathbf{z}^{-1}^{\pm 1}, \ldots, \mathbf{z}^{-m}^{\pm 1}] \) is said to be a mirror partner of an \( m \)-dimensional Fano manifold \( X \) if the period \( \pi_f \) of \( f \) coincides with the quantum period \( \hat{G}_X \) of \( X \) (see [1, 7] and references therein for more details). Note that Laurent polynomials having the same period are not unique. In order to relate Laurent polynomials having the same period, combinatorial mutations are useful. Indeed, it is proved in [1, Lemma 1] that if Laurent polynomials \( f \) and \( g \) are connected by iterated combinatorial mutations, then the period of \( f \) is equal to that of \( g \). The notion of combinatorial mutations for lattice polytopes just rephrases that for Laurent polynomials in terms of their Newton polytopes.

**Notation** 1.1. We adopt the standard notation in toric geometry. Let \( N \simeq \mathbb{Z}^m \) be a \( \mathbb{Z} \)-lattice of rank \( m \), and \( M := \text{Hom}_\mathbb{Z}(N, \mathbb{Z}) \simeq \mathbb{Z}^m \) its dual lattice. We write \( N_\mathbb{R} := N \otimes \mathbb{R} \) and \( M_\mathbb{R} := M \otimes \mathbb{R} \). Denote by \( \langle \cdot, \cdot \rangle : M_\mathbb{R} \times N_\mathbb{R} \to \mathbb{R} \) the canonical pairing.

The combinatorial mutation in [1] is an operation for lattice polytopes in \( N_\mathbb{R} \). For a rational convex polytope \( \Delta \subseteq M_\mathbb{R} \) with a unique interior lattice point \( a \), we define its dual \( \Delta^\vee \) to be the polar dual of its translation:

\[
\Delta^\vee := (\Delta - a)^* = \{ v \in N_\mathbb{R} \mid \langle u - a, v \rangle \geq -1 \text{ for all } u \in \Delta \}.
\]

A combinatorial mutation on \( \Delta^\vee \) corresponds to a piecewise-linear operation on \( \Delta - a \), which is extended to the whole of \( M_\mathbb{R} \). We call it a combinatorial mutation in \( M_\mathbb{R} \). We consider this framework when \( \Delta \) is a Newton–Okounkov body of a flag variety.

To state our results more explicitly, let \( G \) be a simply-connected semisimple algebraic group over \( \mathbb{C} \), \( B \) a Borel subgroup of \( G \), \( W \) the Weyl group, and \( P_+ \) the set of dominant integral weights. We denote by \( X(w) \subseteq G/B \) the Schubert variety corresponding to \( w \in W \), by \( \mathcal{L}_\lambda \) the globally generated line bundle on \( X(w) \) associated with \( \lambda \in P_+ \), and by \( \rho \in P_+ \) the half sum of the positive roots. Let \( R(w) \) be the set of reduced words for \( w \in W \), and \( w_0 \in W \) the longest element. The Schubert variety \( X(w_0) \) corresponding to \( w_0 \) coincides with the full flag variety \( G/B \). Let \( \Delta_i(\lambda) \) (resp., \( \hat{\Delta}_i(\lambda) \)) denote the string polytope (resp., the Nakashima–Zelevinsky polytope) associated with \( \lambda \in P_+ \). In order to relate string polytopes and Nakashima–Zelevinsky polytopes by combinatorial mutations, we use the theory of cluster algebras. Cluster algebras were introduced by Fomin–Zelevinsky [16, 17] to develop a combinatorial approach to total positivity and to the dual canonical basis. Fock–Goncharov [15] introduced a cluster ensemble \( \{ \mathcal{A}, \mathcal{X} \} \) which gives a more geometric point of view to the theory of cluster algebras. Gross–Hacking–Keel–Kontsevich [22] developed the theory of cluster ensembles using methods in mirror symmetry, and proved that the theory of cluster algebras also can be used to obtain toric degenerations of projective varieties. Let \( U^-_w \subseteq G \) be the unipotent cell associated with \( w \in W \), which is naturally regarded as an open subvariety of \( X(w) \). Berenstein–Fomin–Zelevinsky [1] gave an upper cluster algebra structure on the coordinate ring \( \mathbb{C}[U^-_w] \). When \( G \) is simply-laced, the first named author and Oya [21] constructed a family \( \{ \Delta(X(w), \mathcal{L}_\lambda, \rho) \}_{w \in S} \) of Newton–Okounkov bodies parametrized by the set of seeds for \( \mathbb{C}[U^-_w] \) such that

- this family contains \( \Delta_i(\lambda) \) and \( \hat{\Delta}_i(\lambda) \) for all \( i \in R(w) \) up to unimodular transformations,
- the Newton–Okounkov bodies \( \Delta(X(w), \mathcal{L}_\lambda, \rho) \), \( s \in S \), are all rational convex polytopes,
- the Newton–Okounkov bodies \( \Delta(X(w), \mathcal{L}_\lambda, \rho) \), \( s \in S \), are all related by tropicalized cluster mutations.

If \( w = w_0 \) and \( \lambda = 2\rho \), then the Newton–Okounkov body \( \Delta(G/B, \mathcal{L}_{2\rho}, \rho) \) contains exactly one lattice point \( a_\rho \) in its interior, and the dual \( \Delta(G/B, \mathcal{L}_{2\rho}, \rho)^\vee \) is a lattice polytope (see Theorem 3.3 and Corollary 3.13). Realizing tropicalized cluster mutations as combinatorial mutations in \( M_\mathbb{R} \), we obtain the following.
Theorem 1 (Theorem 4.9). If $G$ is simply-laced, then the following hold.

1. For fixed $w \in W$ and $\lambda \in P_+$, the Newton–Okounkov bodies $\Delta(X(w), \mathcal{L}_\lambda, v_s)$, $s \in S$, are all related by combinatorial mutations in $M_\mathbb{R}$ up to unimodular transformations.

2. For $w = w_0$ and $\lambda = 2\rho$, the translated polytopes $\Delta(G/B, \mathcal{L}_{2\rho}, v_s) - a_s$, $s \in S$, are all related by combinatorial mutations in $M_\mathbb{R}$ up to unimodular transformations. In particular, the dual polytopes $\Delta(G/B, \mathcal{L}_{2\rho}, v_s)^\vee$, $s \in S$, are all related by combinatorial mutations in $N_\mathbb{R}$ up to unimodular transformations.

In order to relate FFLV polytopes with these Newton–Okounkov bodies, we use Ardila–Bliem–Salazar’s transfer map [3] between the Gelfand–Tsetlin polytope $\tilde{\Delta}(\lambda)$ and the string polytope $\Delta^{\text{FFLV}}$ (see Theorems 4.9, 5.5). Combining this with Theorem 1, we obtain the following in type $A$.

Theorem 3 (Theorem 5.6). If $G = \text{SL}_{n+1}(\mathbb{C})$, then the following hold.

1. For fixed $\lambda \in P_+$, the string polytopes $\Delta_i(\lambda)$, $i \in R(w_0)$, the Nakashima–Zelevinsky polytopes $\tilde{\Delta}_i(\lambda)$, $i \in R(w_0)$, and the FFLV polytope $\text{FFLV}(\lambda)$ are all related by combinatorial mutations in $M_\mathbb{R}$ up to unimodular transformations and translations by integer vectors.

2. The polytopes in

$$\{\langle \Delta_i(2\rho)^\vee | i \in R(w_0) \rangle \cup \{\tilde{\Delta}_i(2\rho)^\vee | i \in R(w_0) \} \cup \{\text{FFLV}(2\rho)^\vee\} \}
$$

are all related by combinatorial mutations in $N_\mathbb{R}$ up to unimodular transformations.

Note that Ardila–Bliem–Salazar [3] gave such a transfer map also in type $C_n$. Since their transfer map in type $C_n$ can be also described as a composition of combinatorial mutations in $M_\mathbb{R}$, we obtain the following.

Theorem 2 (see Theorems 1.9 5.5). If $G = \text{SL}_{n+1}(\mathbb{C})$, then the following hold.

1. For fixed $\lambda \in P_+$, the Gelfand–Tsetlin polytope $\text{GT}_\lambda(\lambda)$ and the FFLV polytope $\text{FFLV}(\lambda)$ in type $A_n$, where $\lambda \in P_+$, are all related by combinatorial mutations in $M_\mathbb{R}$ up to unimodular transformations and translations by integer vectors.

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2. Newton–Okounkov bodies arising from cluster structures

In order to relate string polytopes and Nakashima–Zelevinsky polytopes by combinatorial mutations, we use Newton–Okounkov bodies of flag varieties arising from cluster structures. In Sect. 2.1 we recall the definitions of higher rank valuations and Newton–Okounkov bodies. We also review their basic properties. In Sect. 2.2 we define valuations using cluster structures, following [21].

2.1. Basic definitions on Newton–Okounkov bodies. We first recall the definition of Newton–Okounkov bodies, following [23 28 29 30]. Let $R$ be a $\mathbb{C}$-algebra without nonzero zero-divisors, and $m \in \mathbb{Z}_{>0}$. We fix a total order $\leq$ on $\mathbb{Z}^m$ respecting the addition.

Definition 2.1. A map $v : R \setminus \{0\} \to \mathbb{Z}^m$ is called a valuation on $R$ if the following holds: for each $\sigma, \tau \in R \setminus \{0\}$ and $c \in \mathbb{C}^\times := \mathbb{C} \setminus \{0\},$

(i) $v(\sigma \cdot \tau) = v(\sigma) + v(\tau),$
(ii) $v(c \cdot \sigma) = v(\sigma),$
(iii) $v(\sigma + \tau) \geq \min\{v(\sigma), v(\tau)\}$ unless $\sigma + \tau = 0.$
Note that we need to fix a total order on $\mathbb{Z}^m$ whenever we consider a valuation. For $a = (a_1, \ldots, a_m) \in \mathbb{Z}^m$ and a valuation $v$ on $R$ with values in $\mathbb{Z}^m$, we define a $C$-subspace $R_a \subseteq R$ by

$$R_a := \{ \sigma \in R \setminus \{0\} \mid v(\sigma) \geq a \} \cup \{0\}.$$

Then the leaf above $a \in \mathbb{Z}^m$ is defined to be the quotient space $R[a] := R_a / \bigcup_{a < b} R_b$. A valuation $v$ is said to have 1-dimensional leaves if $\dim_C(R[a]) = 0$ or 1 for all $a \in \mathbb{Z}^m$.

**Remark** for all $S \in \mathbb{Z}^m$. Since $\mathcal{L}$ is ample, then it follows from [30, Corollary 3.2] that the real dimension of $\Delta(X, \mathcal{L})$ is defined to be the quotient space $\mathbb{R}^m$.

**Example 2.2**. Fix a total order $\leq$ on $\mathbb{Z}^m$ respecting the addition, and let $\mathbb{C}(z_1, \ldots, z_m)$ be the field of rational functions in $m$ variables. The total order $\leq$ on $\mathbb{Z}^m$ induces a total order (denoted by the same symbol $\leq$) on the set of Laurent monomials in $z_1, \ldots, z_m$ as follows:

$$z_1^{a_1} \cdots z_m^{a_m} \leq z_1^{a'_1} \cdots z_m^{a'_m} \text{ if and only if } (a_1, \ldots, a_m) \leq (a'_1, \ldots, a'_m).$$

Let us define a map $v_{\text{low}}^\leq : \mathbb{C}(z_1, \ldots, z_m) \setminus \{0\} \to \mathbb{Z}^m$ as follows:

- $v_{\text{low}}^\leq(f) := (a_1, \ldots, a_m)$ for

$$f = cz_1^{a_1} \cdots z_m^{a_m} + (\text{higher terms}) \in \mathbb{C}[z_1^{\pm 1}, \ldots, z_m^{\pm 1}] \setminus \{0\},$$

where $c \in \mathbb{C}^\times$, and the summand “(higher terms)” stands for a linear combination of Laurent monomials bigger than $z_1^{a_1} \cdots z_m^{a_m}$ with respect to $\leq$.

- $v_{\text{low}}^\leq(f/g) := v_{\text{low}}^\leq(f) - v_{\text{low}}^\leq(g)$ for $f, g \in \mathbb{C}[z_1^{\pm 1}, \ldots, z_m^{\pm 1}] \setminus \{0\}$. Then this map $v_{\text{low}}^\leq$ is a well-defined valuation with 1-dimensional leaves with respect to the total order $\leq$. We call $v_{\text{low}}^\leq$ the lowest term valuation with respect to $\leq$.

**Definition 2.3** (see [28, Sect. 1.2] and [30, Definition 1.10]). Let $X$ be an irreducible normal projective variety over $\mathbb{C}$, $\mathcal{L}$ a line bundle on $X$ generated by global sections, and $m := \dim_{\mathbb{C}}(X)$. Take a valuation $v : \mathbb{C}(X) \setminus \{0\} \to \mathbb{Z}^m$ with 1-dimensional leaves, and fix a nonzero section $\tau \in H^0(X, \mathcal{L})$. We define a subset $S(X, \mathcal{L}, v, \tau) \subseteq \mathbb{Z}^m$ by

$$S(X, \mathcal{L}, v, \tau) := \bigcup_{k \in \mathbb{Z}_{>0}} \{(k, v(\sigma/\tau^k)) \mid \sigma \in H^0(X, \mathcal{L}^\otimes k) \setminus \{0\}\},$$

and denote by $C(X, \mathcal{L}, v, \tau) \subseteq \mathbb{R}_{\geq 0} \times \mathbb{R}^m$ the smallest real closed cone containing $S(X, \mathcal{L}, v, \tau)$. Let us define a subset $\Delta(X, \mathcal{L}, v, \tau) \subseteq \mathbb{R}^m$ by

$$\Delta(X, \mathcal{L}, v, \tau) := \{a \in \mathbb{R}^m \mid (1, a) \in C(X, \mathcal{L}, v, \tau)\};$$

this is called the Newton–Okounkov body of $(X, \mathcal{L})$ associated with $(v, \tau)$.

The definition of valuations implies that $S(X, \mathcal{L}, v, \tau)$ is a semigroup. Hence it follows that $C(X, \mathcal{L}, v, \tau)$ is a closed convex cone, and that $\Delta(X, \mathcal{L}, v, \tau)$ is a convex set. In addition, we deduce by [30, Theorem 2.30] that $\Delta(X, \mathcal{L}, v, \tau)$ is a convex body, i.e., a compact convex set. If $\mathcal{L}$ is ample, then it follows from [30, Corollary 3.2] that the real dimension of $\Delta(X, \mathcal{L}, v, \tau)$ equals $m$; this is not necessarily the case when $\mathcal{L}$ is not ample. By definition, we have

$$0 = v(\tau/\tau) \in \Delta(X, \mathcal{L}, v, \tau).$$

Since $S(X, \mathcal{L}, v, \tau)$ is a semigroup, the definition of Newton–Okounkov bodies implies that

$$\Delta(X, \mathcal{L}^\otimes k, v, \tau^k) = k\Delta(X, \mathcal{L}, v, \tau)$$

for all $k \in \mathbb{Z}_{>0}$.

**Remark 2.4**. If we take another nonzero section $\tau' \in H^0(X, \mathcal{L})$, then it follows that

$$S(X, \mathcal{L}, v, \tau') \cap (\{k\} \times \mathbb{Z}^m) = (S(X, \mathcal{L}, v, \tau) \cap (\{k\} \times \mathbb{Z}^m)) + (0, kv(\tau/\tau'))$$

for all $k \in \mathbb{Z}_{>0}$. Hence we have

$$\Delta(X, \mathcal{L}, v, \tau') = \Delta(X, \mathcal{L}, v, \tau) + v(\tau/\tau'),$$

which implies that the Newton–Okounkov body $\Delta(X, \mathcal{L}, v, \tau)$ does not essentially depend on the choice of $\tau$. Hence it is also denoted simply by $\Delta(X, \mathcal{L}, v)$. 
2.2. Cluster algebras and valuations. The first named author and Oya [21] constructed valuations using the theory of cluster algebras. In this subsection, we review this construction. We first recall the definition of (upper) cluster algebras of geometric type, following [4, 17]. Note that we use the notation in [15, 22]. Fix a finite set $J$ and a subset $J_{af} \subseteq J$. We write $J_\text{fr} := J \setminus J_{af}$. Let $\mathcal{F} := \mathbb{C}(z_j \mid j \in J)$ be the field of rational functions in $|J|$ variables. Then a seed $s = (A, \varepsilon)$ of $\mathcal{F}$ is a pair of

- a $J$-tuple $A = (A_j)_{j \in J}$ of elements of $\mathcal{F}$, and
- $\varepsilon = (\varepsilon_{i,j})_{i \in J_\text{af}, j \in J} \in \text{Mat}_{J_{\text{af}} \times J}(\mathbb{Z})$

such that

1. The matrix $\varepsilon$ forms a free generating set of $\mathcal{F}$, and
2. the $J_{\text{af}} \times J_{\text{af}}$-submatrix $\varepsilon^0$ of $\varepsilon$ is skew-symmetrizable, that is, there exists $(d_{ij})_{i \in J_{\text{af}}} \in \mathbb{Z}_{\geq 0}^{J_{\text{af}}}$ such that $\varepsilon_{i,j}d_i = -\varepsilon_{j,i}d_j$ for all $i, j \in J_{\text{af}}$.

The matrix $\varepsilon$ is called the exchange matrix of $s$.

Remark 2.5. Our exchange matrix $\varepsilon$ is transposed to the one in [17 Sect. 2].

Let $s = (A, \varepsilon) = ((A_j)_{j \in J}, (\varepsilon_{i,j})_{i \in J_\text{af}, j \in J})$ be a seed of $\mathcal{F}$. We write $[a]_+ := \max\{a, 0\}$ for $a \in \mathbb{R}$. For $k \in J_{\text{af}}$, the mutation $\mu_k(s) = (\mu_k(A), \mu_k(\varepsilon))$ in direction $k$ is defined by

$$
\begin{align*}
\varepsilon'_{i,j} &:= \begin{cases} 
-\varepsilon_{i,j} & \text{if } i = k \text{ or } j = k, \\
\varepsilon_{i,j} + \text{sgn}(\varepsilon_{k,j})[\varepsilon_{i,k} \varepsilon_{k,j}]_+ & \text{otherwise},
\end{cases} \\
A'_{ij} &:= \begin{cases} 
\prod_{l \in J} A_l^{[\varepsilon_{l,j}]} + \prod_{l \in J} A_l^{[-\varepsilon_{l,j}]} & \text{if } j = k, \\
A_{ij} & \text{otherwise}
\end{cases}
\end{align*}
$$

for $i \in J_{\text{af}}$ and $j \in J$, where $\mu_k(A) = (A'_{ij})_{j \in J}$ and $\mu_k(\varepsilon) = (\varepsilon'_{i,j})_{i \in J_\text{af}, j \in J}$. Then $\mu_k(s)$ is again a seed of $\mathcal{F}$, and we have $\mu_k(\mu_k(s)) = s$. Two seeds $s$ and $s'$ are said to be mutation equivalent if there exists a sequence $(k_1, k_2, \ldots, k_\ell)$ in $J_{\text{af}}$ such that $\mu_{k_1} \cdots \mu_{k_\ell}(s) = s'$.

Let $T$ be the $|J_{\text{af}}|$-regular tree whose edges are labeled by $J_{\text{af}}$ so that the $|J_{\text{af}}|$-edges emanating from each vertex receive different labels. If $t, t' \in T$ are joined by an edge labeled by $k \in J_{\text{af}}$, then we write $t \overset{k}{\rightarrow} t'$. A cluster pattern $\mathcal{S} = \{s_t\}_{t \in T} = \{(A_t, \varepsilon_t)\}_{t \in T}$ is an assignment of a seed $s_t = (A_t, \varepsilon_t)$ of $\mathcal{F}$ to each vertex $t \in T$ such that $\mu_k(s_t) = s_{t'}$ whenever $t \overset{k}{\rightarrow} t'$. For a cluster pattern $\mathcal{S} = \{s_t\} = \{(A_t, \varepsilon_t)\}_{t \in T}$, write

$$
A_1 := (A_{j,t})_{j \in J}, \quad \varepsilon_t := (\varepsilon_{i,j}^{(t)})_{i \in J_{\text{af}}, j \in J}.
$$

Definition 2.6 (see [3 Definitions 1.6 and 1.11]). We set

$$
\mathcal{A}(\mathcal{S}) := \bigcap_{t \in T} \mathbb{C}[A_{j,t}^{\pm 1} \mid j \in J] \subseteq \mathcal{F},
$$

which is called an upper cluster algebra of geometric type. The (ordinary) cluster algebra $\mathcal{A}(\mathcal{S})$ of geometric type is defined to be the $\mathbb{C}$-subalgebra of $\mathcal{F}$ generated by $\{A_{j,t} \mid t \in T, j \in J_{\text{af}}\}$ and $\{A_{j,t}^{\pm 1} \mid t \in T, j \in J_{\text{af}}\}$.

We usually fix $t_0 \in T$, and construct a cluster pattern $\mathcal{S} = \{s_t\}_{t \in T}$ from a seed $s_{t_0}$. In this case, $s_{t_0}$ is called the initial seed.

Theorem 2.7 ([33 Theorem 3.1]). Let $\mathcal{S} = \{s_t = (A_t, \varepsilon_t)\}_{t \in T}$ be a cluster pattern. Then it follows that

$$
\mathcal{A}(\mathcal{S}) \subseteq \mathbb{C}[A_{j,t}^{\pm 1} \mid j \in J]
$$

for all $t \in T$; this property is called the Laurent phenomenon. In particular, $\mathcal{A}(\mathcal{S})$ is included in the upper cluster algebra $\mathcal{U}(\mathcal{S})$.

In the rest of this subsection, we assume that
where elements of $\mathbb{Z}^{J}$ (resp., $\mathbb{Z}_{\geq 0}^{J}$) are regarded as $1 \times J$ (resp., $1 \times J_{\text{wt}}$) matrices. This $\preceq_{t}$ defines a partial order on $\mathbb{Z}^{J}$, called the dominance order associated with $\epsilon_{t}$.

Definition 2.9 ([21] Definition 3.8]). Let $\mathcal{S} = \{s_{t} = (A_{t}, \epsilon_{t})\}_{t \in \mathcal{T}}$ be a cluster pattern, and fix $t \in \mathcal{T}$. For $a, a' \in \mathbb{Z}^{J}$, we write

$$a \preceq_{t} a' \text{ if and only if } a = a' + v \epsilon_{t} \text{ for some } v \in \mathbb{Z}_{\geq 0}^{J_{\text{wt}}},$$

where $\mathbb{Z}^{J}$ (resp., $\mathbb{Z}_{\geq 0}^{J}$) are regarded as $1 \times J$ (resp., $1 \times J_{\text{wt}}$) matrices. This $\preceq_{t}$ defines a partial order on $\mathbb{Z}^{J}$, called the dominance order associated with $\epsilon_{t}$.

In this section, we restrict ourselves to the case of flag varieties and Schubert varieties. In Sect. 3.2, we review fundamental properties of these varieties, and recall basic facts on their Newton–Okounkov bodies associated with $w_{B}/B$.

The Schubert variety $X(w)$ is a normal projective variety of complex dimension $\ell(w)$ (see, for instance, [26 Sects. II.13.3, II.14.15]). If $w$ is the longest element $w_{0}$ in $W$, then the Schubert variety $X(w_{0})$ coincides with the full flag variety $G/B$. For $\lambda \in P_{+}$, we define a line bundle $\mathcal{L}_{\lambda}$ on $G/B$ by

$$\mathcal{L}_{\lambda} := (G \times \mathbb{C})/B,$$
where $B$ acts on $G \times \mathbb{C}$ from the right as follows:

$$(g, c) \cdot b := (gb, \lambda(b)c)$$

for $g \in G$, $c \in \mathbb{C}$, and $b \in B$. By restricting this bundle, we obtain a line bundle on $X(w)$, which we denote by the same symbol $\mathcal{L}_\lambda$. By \cite[Proposition 1.4.1]{32}, we see that the line bundle $\mathcal{L}_\lambda$ on $X(w)$ is generated by global sections. Let $\mathcal{O}(K_{G/B})$ denote the canonical bundle of $G/B$. By \cite[Proposition 2.2.7 (ii)]{6}, we have

$$\mathcal{O}(K_{G/B}) \simeq \mathcal{L}_{-2\rho},$$

where $\rho \in P_+$ denotes the half sum of the positive roots. For $\lambda \in P_+$, let $V(\lambda)$ be the irreducible highest weight $G$-module over $\mathbb{C}$ with highest weight $\lambda$. We fix a highest weight vector $v_\lambda$ of $V(\lambda)$. The Demazure module $V_\mu(\lambda)$ corresponding to $\mu \in W$ is defined to be the $B$-submodule of $V(\lambda)$ given by

$$V_\mu(\lambda) := \sum_{b \in B} Cb\tilde{w}v_\lambda,$$

where $\tilde{w} \in N_{G}(H)$ denotes a lift for $w$. From the Borel–Weil type theorem (see, for instance, \cite[Corollary 8.1.26]{32}), we know that the space $H^0(G/B, \mathcal{L}_\lambda)$ (resp., $H^0(X(w), \mathcal{L}_\lambda)$) of global sections is a $G$-module (resp., a $B$-module) which is isomorphic to the dual module $V(\lambda)^* := \text{Hom}_\mathbb{C}(V(\lambda), \mathbb{C})$ (resp., $V_\mu(\lambda)^* \simeq \text{Hom}_\mathbb{C}(V_\mu(\lambda), \mathbb{C})$). We fix a lowest weight vector $\tau_\lambda \in H^0(G/B, \mathcal{L}_\lambda)$. By restricting this section, we obtain a section in $H^0(X(w), \mathcal{L}_\lambda)$, which we denote by the same symbol $\tau_\lambda$. Let $\Delta_i(\lambda)$ (resp., $\tilde{\Delta}_i(\lambda)$) denote the string polytope (resp., the Nakashima–Zelevinsky polytope) associated with $i \in R(w)$ and $\lambda \in P_+$; see \cite[Sect. 1]{13}, \cite[Definition 2.15]{19}, \cite[Definition 3.24]{21}, and \cite[Definition 3.9]{18} for their precise definitions. In the present paper, we do not recall the original definitions of these polytopes, but these polytopes are defined from a representation-theoretic structure on $V(\lambda)$, called the Kashiwara crystal basis; see \cite{27} for a survey on Kashiwara crystal bases.

**Remark 3.2.** The definition of Nakashima–Zelevinsky polytopes $\tilde{\Delta}_i(\lambda)$ in \cite[Definition 3.9]{18} is slightly different from the one in \cite[Definition 3.24 (2)]{21} because the order of coordinates is reversed. In the present paper, we use the definition in \cite[Definition 3.9]{18}.

Kaveh \cite{28} proved that the string polytope $\Delta_i(\lambda)$ is identical to the Newton–Okounkov body $\Delta(X(w), \mathcal{L}_\lambda, v_i^{\text{high}}, \tau_\lambda)$ of $(X(w), \mathcal{L}_\lambda)$ associated with a highest term valuation $v_i^{\text{high}}$. Using a different kind of highest term valuation $v_i^{\text{high}}$, the first named author and Naito \cite{19} showed that the Nakashima–Zelevinsky polytope $\tilde{\Delta}_i(\lambda)$ can be realized as a Newton–Okounkov body $\Delta(X(w), \mathcal{L}_\lambda, v_i^{\text{high}}, \tau_\lambda)$. Afterward, the first named author and Oya \cite{20} proved that the Newton–Okounkov body $\Delta(X(w), \mathcal{L}_\lambda, v_i^{\text{high}}, \tau_\lambda)$ (resp., $\Delta(X(w), \mathcal{L}_\lambda, \tilde{v}_i^{\text{high}}, \tau_\lambda)$) is also identical to the one associated with a valuation given by counting the orders of zeros/poles along a specific sequence of Schubert varieties. This description leads to the realization of $\Delta_i(\lambda)$ (resp., $\tilde{\Delta}_i(\lambda)$) in \cite{21} as a Newton–Okounkov body arising from a cluster structure, which is reviewed in the next subsection.

In the context of mirror symmetry, when $G$ is of type $A_n$, Rusenko \cite[Theorem 7]{41} proved that the polar dual of the (properly translated) string polytope $\Delta_i(2\rho)$ is a lattice polytope for all $i \in R(u_0)$. Using Hibi’s criterion \cite{24} on the integrality of the vertices of the dual polytopes, Steinert \cite{43} generalized this result to all Lie types as follows.

**Theorem 3.3** (see \cite[Sects. 4, 6]{43}). Take a valuation $v : \mathbb{C}(G/B) \setminus \{0\} \rightarrow \mathbb{Z}^{\dim(G/B)}$ with a dimension leaves, and fix a nonzero section $\tau \in H^0(G/B, \mathcal{L}_{2\rho})$. If the semigroup $S(\mathbb{C}(G/B), \mathcal{L}_{2\rho}, v, \tau_{2\rho})$ is finitely generated and saturated, then the Newton–Okounkov body $\Delta(G/B, \mathcal{L}_{2\rho}, v, \tau_{2\rho})$ contains exactly one lattice point in its interior. In addition, the dual $\Delta(G/B, \mathcal{L}_{2\rho}, v, \tau_{2\rho})^\vee$ in the sense of Sect. 7 is a lattice polytope.

**Remark 3.4.** In the paper \cite{43}, the algebraic group $G$ is assumed to be simple. However, the proof of Theorem 3.3 can also be applied to the case that $G$ is semisimple.
Example 3.5. Assume that $G$ is simple and simply-laced. We identify the set $I$ of vertices of the Dynkin diagram with $\{1, 2, \ldots, n\}$ as follows:

$$
\begin{align*}
A_n & \quad 1 \quad 2 \quad \cdots \quad n-1 \quad n, \\
D_n & \quad 1 \quad 3 \quad \cdots \quad n-1 \quad n, \\
E_6 & \quad 5 \quad 4 \quad 3 \quad 2 \quad 6 \quad 1, \\
E_7 & \quad 5 \quad 4 \quad 3 \quad 2 \quad 6 \quad 7, \\
E_8 & \quad 5 \quad 3 \quad 2 \quad 6 \quad 7 \quad 8.
\end{align*}
$$

Let $X_n$ be the Lie type of $G$, and define $i_{X_n} \in R(w_0)$ as follows.

- If $G$ is of type $A_n$, then
  $$i_{A_n} := (1, 2, 1, 3, 2, 1, \ldots, n, n-1, \ldots, 1) \in I^{2n+2}. $$

- If $G$ is of type $D_n$, then
  $$i_{D_n} := (1, 2, 3, 1, 2, 3, 4, 3, 1, 2, 3, 4, \ldots, n, n-1, \ldots, 3, 1, 2, 3, \ldots, n-1, n) \in I^{2n}. $$

- If $G$ is of type $E_6$, then
  $$i_{E_6} := (i_{D_6}, 6, 2, 3, 1, 4, 5, 3, 4, 2, 3, 1) \in I^{36}. $$

- If $G$ is of type $E_7$, then
  $$i_{E_7} := (i_{E_6}, 7, 6, 2, 3, 1, 4, 5, 3, 4, 2, 3, 1, 6) \in I^{53}. $$

- If $G$ is of type $E_8$, then
  $$i_{E_8} := (i_{E_7}, 8, 7, 6, 2, 3, 1, 4, 5, 3, 4, 2, 3, 1, 6, 2, 3, 4, 5, 7, 6, 2, 3, 1, 4, 3, 2, 6, 7, 8) \in I^{120}. $$

Littelmann [34] Sect. 1, Corollaries 4, 8, and Theorems 8.1, 8.2, 9.3] gave a system of explicit linear inequalities defining the string polytope $\Delta_i(\lambda)$ associated with the reduced word $i$ above and $\lambda \in P_\ast$. When $\lambda = 2\rho$, let $a_{X_n}$ denote the unique lattice point of $\Delta_{i_{X_n}}(2\rho)$. Then we see the following by Littelmann’s description.

- If $G$ is of type $A_n$, then
  $$a_{A_n} = (1, 2, 1, 3, 2, 1, \ldots, n, n-1, \ldots, 1) \in Z^{n(n+1)}. $$

- If $G$ is of type $D_n$, then
  $$a_{D_n} = (1, 2, 3, 1, 2, 3, 4, 3, 1, 2, 3, 4, \ldots, 2n-3, 2n-4, \ldots, n, n-1, n-2, \ldots, 2, 1) \in Z^{n(n-1)}. $$

- If $G$ is of type $E_6$, then
  $$a_{E_6} = (a_{D_6}, 11, 10, 9, 8, 8, 7, 6, 6, 5, 4, 5, 4, 3, 2, 1) \in Z^{36}. $$

- If $G$ is of type $E_7$, then
  $$a_{E_7} = (a_{E_6}, 17, 16, 15, 14, 13, 12, 11, 11, 10, 9, 10, 9, 8, 7, 6, 9, 8, 7, 6, 5, 5, 4, 3, 2, 1) \in Z^{63}. $$
If \( G \) is of type \( E_8 \), then
\[
a_{E_8} = (a_{E_8}, 28, 27, 26, 25, 24, 23, 22, 21, 20, 19, 18, 17, 16, 19, 18, 17, 16, 15, 14, 13, 12, 11, 10, 11, 10, 9, 8, 7, 10, 9, 8, 7, 6, 6, 5, 4, 3, 2, 1) \in \mathbb{Z}^{120}.
\]

Example 3.6 ([34 Corollary 5]). Let \( G = SL_{n+1}(\mathbb{C}) \), and \( \lambda \in P_+ \). We consider the reduced word \( i_{A_n} \in R(w_0) \) in Example 3.5. Then the string polytope \( \Delta_{i_{A_n}}(\lambda) \) is unimodularly equivalent to the Gelfand–Tsetlin polytope \( GT(\lambda) \) which is defined to be the set of
\[
(a_1^{(1)}, a_1^{(2)}, a_2^{(1)}, a_2^{(2)}, a_3^{(1)}, \ldots, a_1^{(n)}, \ldots, a_1^{(1)}) \in \mathbb{R}^{n(n+1)/2}
\]
satisfying the following conditions:
\[
\begin{array}{cccccccc}
\lambda_{\geq 1} & \lambda_{\geq 2} & \cdots & \lambda_{\geq n} & 0 \\
\lambda_{\geq 1}^{(1)} & \lambda_{\geq 2}^{(1)} & \cdots & \lambda_{\geq n}^{(1)} & a_n^{(1)} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
a_1^{(n-1)} & a_2^{(n-1)} & \cdots & a_1^{(n-1)} & 0
\end{array}
\]
where \( \lambda_{\geq k} := \sum_{k \leq \ell \leq n} \langle \lambda, h_\ell \rangle \) for \( 1 \leq k \leq n \), and the notation
\[
\begin{array}{c}
a \\
b \\
c
\end{array}
\]
means that \( a \geq b \geq c \).

Example 3.7 ([34 Corollary 7]). Let \( G = Sp_{2n}(\mathbb{C}) \), and \( \lambda \in P_+ \). We identify the set \( I \) of vertices of the Dynkin diagram with \( \{1, 2, \ldots, n\} \) as follows:
\[
C_n \quad 1 \quad 2 \quad n-1 \quad n
\]
Define \( i_{C_n} \in R(w_0) \) by
\[
i_{C_n} := (1, 2, 1, 2, 3, 2, 1, 2, 3, \ldots, n, n-1, n, n-1, \ldots, n-1, n) \in I^{n^2}.
\]
Then the string polytope \( \Delta_{i_{C_n}}(\lambda) \) is unimodularly equivalent to the Gelfand–Tsetlin polytope \( GT_{C_n}(\lambda) \) of type \( C_n \) which is defined to be the set of
\[
(b_1^{(1)}, b_1^{(2)}, b_2^{(1)}, b_2^{(2)}, b_3^{(1)}, \ldots, b_n^{(1)}, \ldots, b_1^{(n)}, \ldots, b_1^{(1)}) \in \mathbb{R}^{n^2}
\]
satisfying the following conditions as in Example 3.6:
\[
\begin{array}{cccccccc}
\lambda_{\geq 1} & \lambda_{\geq 2} & \cdots & \lambda_{\geq n} & 0 \\
\lambda_{\geq 1}^{(1)} & \lambda_{\geq 2}^{(1)} & \cdots & \lambda_{\geq n}^{(1)} & a_n^{(1)} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
b_1^{(n-1)} & b_2^{(n-1)} & \cdots & b_1^{(n-1)} & 0
\end{array}
\]
where \( \lambda_{\geq k} := \sum_{k \leq \ell \leq n} \langle \lambda, h_\ell \rangle \) for \( 1 \leq k \leq n \).
3.2. Newton–Okounkov bodies of Schubert varieties arising from cluster structures.
In this subsection, we assume that $G$ is simply-laced. Let $B^- \subseteq G$ denote the Borel subgroup opposite to $B$, and $U^-$ the unipotent radical of $B^-$. We regard $U^-$ as an affine open subvariety of $G/B$ by the following open embedding:

$$U^- \hookrightarrow G/B, \ u \mapsto u \mod B.$$ 

For $w \in W$, we set

$$U_w^- := U^- \cap \widetilde{B} w \widetilde{B},$$

where $\widetilde{w} \in N_G(H)$ is a lift for $w \in W = N_G(H)/H$. The space $U_w^-$ is called the unipotent cell associated with $w$. Note that the open embedding $U^- \hookrightarrow G/B$ induces an open embedding $U_w^- \hookrightarrow X(w)$. Let $\{w_i \mid i \in I\} \subseteq P_+$ be the set of fundamental weights. For $u, u' \in W$ and $i \in I$, we denote by $\Delta_{u w_i, u' w_i} \in \mathbb{C}[G]$ the corresponding generalized minor (see, for instance, [4 Sect. 2.3] for the definition), and by $D_{u w_i, u' w_i} \in \mathbb{C}[U_w^-]$ the restriction of $\Delta_{u w_i, u' w_i}$ to $U_w^-$. This function $D_{u w_i, u' w_i}$ is called a unipotent minor. Fix $w \in W$ and $i = (i_1, \ldots, i_m) \in R(w)$. For $1 \leq k \leq m$, we write

$$w_{\leq k} := s_{i_1} \cdots s_{i_k}, \text{ and}$$

$$k^+ := \min\{m+1 \cup \{k+1 \leq j \leq m \mid i_j = i_k\}\}.$$

Let us set

$$J := \{1, \ldots, m\}, \quad J_{\text{fr}} := \{j \in J \mid j^+ = m+1\}, \text{ and } J_{\text{af}} := J \setminus J_{\text{fr}}.$$ 

Define a $J_{\text{af}} \times J$-integer matrix $\varepsilon^i = (\varepsilon_{s,t})_{s \in J_{\text{af}}, t \in J}$ by

$$\varepsilon_{s,t} := \begin{cases} -1 & \text{if } s = t^+, \\ -c_{s,t} & \text{if } t < s < t^+ < s^+, \\ 1 & \text{if } s^+ = t, \\ c_{s,t} & \text{if } s < t < s^+ < t^+, \\ 0 & \text{otherwise.} \end{cases}$$

For $s \in J$, we set

$$D(s, i) := D_{w_{\leq \varepsilon s}, \varepsilon_i}.$$ 

Let $S = \{s_t = (A_t, \varepsilon_t)\}_{t \in \mathbb{T}}$ be the cluster pattern whose initial seed is given as $s_{t_0} = ((A_{x,t_0})_{s \in J}, \varepsilon^i)$.

**Theorem 3.8** ([4 Theorem 2.10] see also [21 Theorem B.4]). There exists a $\mathbb{C}$-algebra isomorphism

$$\mathcal{H}(S) \sim \mathbb{C}[U_w^-] \text{ given by } A_s, t_0 \mapsto D(s, i) \text{ for } s \in J.$$ 

Through the isomorphism in Theorem 3.8, we obtain a seed of $\mathbb{C}(U_w^-)$ given as

$$s_i := (D_i := (D(s, i))_{s \in J}, \varepsilon^i).$$

**Proposition 3.9** ([4 Remark 2.14]). For $w \in W$, the cluster pattern associated with $s_i$ does not depend on the choice of $i \in R(w)$. Namely, all $s_i$, $i \in R(w)$, are mutually mutation equivalent.

**Remark 3.10.** This proposition can be extended to non-simply-laced case by [14 Theorem 3.5].

Let $S = \{s_t = (A_t, \varepsilon_t)\}_{t \in \mathbb{T}}$ be the cluster pattern associated with $s_t$.

**Theorem 3.11** ([21 Corollaries 6.6, 6.25 and Theorem 7.1]). If $G$ is simply-laced, then the following hold for all $w \in W$, $\lambda \in P_+$, and $t \in \mathbb{T}$.

1. The Newton–Okounkov body $\Delta(X(w), \mathcal{L}, v_{s_t}, \tau_{\lambda})$ is independent of the choice of a refinement of the opposite dominance order $\leq^{op}_{c_t}$.
2. The Newton–Okounkov body $\Delta(X(w), \mathcal{L}, v_{s_t}, \tau_{\lambda})$ is a rational convex polytope.
3. If $t < t'$, then

$$\Delta(X(w), \mathcal{L}, v_{s_t}, \tau_{\lambda}) = \mu_{t'}^t(\Delta(X(w), \mathcal{L}, v_{s_{t'}}, \tau_{\lambda})).$$
(4) The Newton–Okounkov body $\Delta(X(w), \mathcal{L}_\lambda, v_{s_i}, \tau_{s_i})$ is unimodularly equivalent to the string polytope $\Delta_i(\lambda)$. More strongly, the equality

$$\Delta_i(\lambda) = \Delta(X(w), \mathcal{L}_\lambda, v_{s_i}, \tau_{s_i})M_i$$

holds, where $M_i = (d_{s,t})_{s,t \in J} \in \text{Mat}_{J \times J}(\mathbb{Z})$ is defined by

$$d_{s,t} := \begin{cases} \langle s_{i+1} \cdots s_i, \tau_{s_i}, h_{s_i} \rangle & \text{if } t \leq s, \\ 0 & \text{if } t > s. \end{cases}$$

(5) There exists a seed $\mathbb{S}^{\text{mut}}_{\mathbb{R}^n} = (\mathbb{D}^{\text{mut}}_{\mathbb{R}^n}, \mathbb{E}^{\text{mut}}_{\mathbb{R}^n}) \in \mathbb{S}$ such that the corresponding Newton–Okounkov body $\Delta(X(w), \mathcal{L}_\lambda, v_{s_i}^{\text{mut}}, \tau_{s_i})$ is unimodularly equivalent to the Nakashima–Zelevinsky polytope $\tilde{\Delta}_i(\lambda)$.

Remark 3.12. We can extend Theorems 3.11 (4), (5) to non-simply-laced case by taking refinements of the opposite dominance orders $\preceq_{\mathbb{R}^n} \preceq_{\mathbb{R}^n}^{\text{mut}}$ appropriately (see [21] Proposition 6.4 and Theorem 6.24). In addition, by [21] Corollary 7.7 (3) and Theorem 3.3 we obtain the following.

Corollary 3.13. If $G$ is simply-laced and $w = w_0$, then the following hold for all $t \in \mathbb{T}$.

1. The Newton–Okounkov body $\Delta(G/B, \mathcal{L}_{2\rho}, v_{s_i}, \tau_{2\rho})$ contains exactly one lattice point $a_t$ in its interior.

2. The dual $\Delta(G/B, \mathcal{L}_{2\rho}, v_{s_i}, \tau_{2\rho})^\vee$ is a lattice polytope.

4. COMBINATORIAL MUTATIONS ON NEWTON–OKOUNKOV BODIES

In this section, we recall the notion of combinatorial mutations for lattice polytopes which was developed by Akhtar–Coates–Galkin–Kasprzyk in [1]. There are two kinds of combinatorial mutations: one is the operation in $N_\mathbb{R}$-side and the other one is in $M_\mathbb{R}$-side. Our main interest is the operation in $M_\mathbb{R}$-side (see Definition 4.1) and this is originally defined as a “dual version” of the operation in $N_\mathbb{R}$-side. See Proposition 4.3.

4.1. Basic definitions on combinatorial mutations. We first introduce combinatorial mutations for lattice polytopes in $N_\mathbb{R}$. Let $P \subseteq N_\mathbb{R}$ be a lattice polytope, and take $w \in M$. For $h \in \mathbb{Z}$, write

$$H_{w,h} := \{ v \in N_\mathbb{R} \mid \langle w, v \rangle = h \}, \quad P_{w,h} := P \cap H_{w,h}.$$ We use the notation $w^{\perp}$ instead of $H_{w,0}$. Let $V(P) \subseteq N$ denote the set of vertices of $P$. For each subset $A \subseteq N_\mathbb{R}$, we set $A + \emptyset = \emptyset + A = \emptyset$.

Definition 4.1 ([1] Definition 5). Let $w \in M$ be a primitive vector, and take a lattice polytope $F$ which sits in $w^{\perp}$. Suppose that for every negative integer $h$, there exists a possibly-empty lattice polytope $G_h \subseteq N_\mathbb{R}$ such that the inclusions

$$V(P) \cap H_{w,h} \subseteq G_h + |h|F \subseteq P_{w,h}$$

hold. Then we define the lattice polytope $\text{mut}_w(P, F)$ as follows:

$$\text{mut}_w(P, F) := \text{conv} \left( \bigcup_{h \leq -1} G_h \cup \bigcup_{h \geq 0} (P_{w,h} + hF) \right) \subseteq N_\mathbb{R}.$$ Note that $G_h$ and $P_{w,h} + hF$ are empty except for finitely many $h$’s. We call the lattice polytope $\text{mut}_w(P, F)$ (or the operation $\text{mut}_w(-, F)$) the combinatorial mutation in $N_\mathbb{R}$ of $P$ with respect to $w$ and $F$. When (4.1) is satisfied, we say that $\text{mut}_w(P, F)$ is well-defined.

It is proved in [1] Proposition 1] that $\text{mut}_w(P, F)$ is independent of the choice of $\{G_h\}_h$.

Remark 4.2. In [25], the definition of combinatorial mutations in $N_\mathbb{R}$ has been extended to rational convex polytopes and unbounded polyhedra. See [25] Sect. 2 for more details.
Next, we introduce another operation, which is a piecewise-linear transformation on $M_\mathbb{R}$.

**Definition 4.3** ([I Sect. 3]; see also [25 Definition 3.1]). Let $w \in M$ be a primitive vector, and take a lattice polytope $F$ which sits in $w^\perp$. We define a map $\varphi_{w,F} : M_\mathbb{R} \to M_\mathbb{R}$ by

$$\varphi_{w,F}(u) := u - u_{\text{min}}w$$

for $u \in M_\mathbb{R}$, where $u_{\text{min}} := \min\{(u,v) \mid v \in F\}$. We call the piecewise-linear map $\varphi_{w,F}$ a combinatorial mutation in $M_\mathbb{R}$.

Indeed, the combinatorial mutation $\varphi_{w,F}$ in $M_\mathbb{R}$ is compatible with the one in $N_\mathbb{R}$ through the polar dual. More precisely, we see the following.

**Proposition 4.4** ([I and 25 Proposition 3.2]). Let $P \subseteq N_\mathbb{R}$ be a lattice polytope containing the origin. Take a primitive vector $w \in M$, and fix a lattice polytope $F \subseteq w^\perp$. Assume that $\text{mut}_w(P,F)$ is well-defined. Then it holds that

$$\varphi_{w,F}(P^*) = \text{mut}_w(P,F)^*.$$

**Proposition 4.5** ([25 Proposition 3.4]). Fix a primitive vector $w \in M$ and a lattice polytope $F \subseteq w^\perp$. Let $Q \subseteq M_\mathbb{R}$ be a rational convex polytope containing the origin. Then $\text{mut}_w(Q,F)$ is well-defined if and only if $\varphi_{w,F}(Q)$ is convex.

**Example 4.6.** Consider the lattice polygon

$$P = \text{conv}((1,1), (0,1), (-1,-1), (0,-1)) \subseteq N_\mathbb{R} \cong \mathbb{R}^2.$$

Let $w = (0,-1) \in M$, and $F = \text{conv}((0,0), (1,0)) \subseteq w^\perp$.

By setting $G_{-1} = \{0,1\}$, we see that $V(P) \cap H_{w,-1} \subseteq G_{-1} + F = P_{w,-1}$. Hence $\text{mut}_w(P,F)$ is well-defined and

$$\text{mut}_w(P,F) = \text{conv}(G_{-1} \cup (P_{w,1} + F)) = \text{conv}((0,1), (-1,-1), (1,-1)) \subseteq N_\mathbb{R}.$$

By taking the polar dual of this polytope, we obtain that

$$\text{mut}_w(P,F)^* = \text{conv}((0,1), (-2,-1), (2,-1)) \subseteq M_\mathbb{R}.$$

On the other hand, it holds that

$$P^* = \text{conv}((0,-1), (2,-1), (1,0), (-2,1)) \subseteq M_\mathbb{R}.$$

Now, we apply $\varphi_{w,F}$ to $P^*$. By definition, we have

$$\varphi_{w,F}(x,y) = (x,y) - \min\{((x,y), (0,0)) , ((x,y), (1,0))\}(0,-1)$$

$$= (x,y) - \min\{0, x\}(0,-1)$$

$$= \begin{cases} (x,y) & \text{if } x \geq 0, \\ (x, x+y) & \text{if } x \leq 0, \end{cases}$$

which implies that

$$\varphi_{w,F}(P^* \cap \{(x,y) \in \mathbb{R}^2 \mid x \geq 0\}) = \text{conv}((0,-1), (2,-1), (0,1)), \text{ and}$$

$$\varphi_{w,F}(P^* \cap \{(x,y) \in \mathbb{R}^2 \mid x \leq 0\}) = \text{conv}((0,-1), (-2,-1), (0,1)).$$

Hence it follows that

$$\varphi_{w,F}(P^*) = \text{conv}((0,1), (-2,-1), (2,-1)).$$

Therefore, we see that $\text{mut}_w(P,F)^* = \varphi_{w,F}(P^*)$ as in Proposition 4.4; see also Figure 4.1.

We now introduce the notion of combinatorial mutation equivalence.

**Definition 4.7** (see [23 Definition 3.5]). Two lattice polytopes $P$ and $P'$ in $N_\mathbb{R}$ are said to be combinatorially mutation equivalent in $N_\mathbb{R}$ if there exists a sequence $(w_1, F_1), \ldots, (w_r, F_r)$, where $w_i \in M$ is primitive and $F_i \subseteq w_i^\perp$ is a lattice polytope, such that

$$P' = \text{mut}_{w_1}(\cdots \text{mut}_{w_r}(P,F_1), F_2) \cdots , F_r).$$
Similarly, two rational convex polytopes $Q$ and $Q'$ in $\mathbb{R}^M_+$ are said to be **combinatorially mutation equivalent** in $\mathbb{R}^M_+$ if there exists a sequence $\left( (w_1, F_1), \ldots, (w_\ell, F_\ell) \right)$, where $w_i \in M$ is primitive and $F_i \subseteq w_i^\perp$ is a lattice polytope, such that $Q' = \varphi_{w_\ell, F_\ell} \left( \cdots \left( \varphi_{w_1, F_1}(Q) \right) \cdots \right)$ and the image of each of the intermediate steps is always a rational convex polytope.

### 4.2. Tropicalized cluster mutations as combinatorial mutations

Our main interest is the map $\varphi_{w, F}$ in Definition 4.3. We first prove the following.

**Proposition 4.8.** For $k \in \mathcal{J}_{uf}$, the tropicalized cluster mutation $\mu^T_k : \mathbb{R}^J \to \mathbb{R}^J$ can be described as a composition of a combinatorial mutation in $\mathbb{R}^M_+$ and $f \in \text{GL}_J(\mathbb{Z})$.

**Proof.** Recall that $\mu^T_k : \mathbb{R}^J \to \mathbb{R}^J$, $\left( (g_j)_{j \in J} \right) \mapsto \left( (g'_j)_{j \in J} \right)$, is defined by

$$
g'_j := \begin{cases} 
g_j + [-\varepsilon_{k,j}^{(t)}]_+ + \varepsilon_{k,j}^{(t)} |g_k|_+ & (j \neq k), \\
-\varepsilon_{k,j}^{(t)} & (j = k) \end{cases}
$$

for $j \in J$, where $(\varepsilon_{i,j}^{(t)})_{i \in \mathcal{J}_{uf}, j \in J} \in \text{Mat}_{\mathcal{J}_{uf} \times J}(\mathbb{Z})$ is a full rank matrix whose $\mathcal{J}_{uf} \times \mathcal{J}_{uf}$-submatrix $\varepsilon^o$ is skew-symmetrizable. For $i \in \mathcal{J}_{uf}$, let $\varepsilon_i^{(t)}$ denote the $i$-th row of the matrix $(\varepsilon_{i,j}^{(t)})_{i \in \mathcal{J}_{uf}, j \in J}$.

For $j \in J$, we write the $j$-th unit vector of $\mathbb{R}^J$ as $\mathbf{e}_j \in \mathbb{R}^J$. Define $\mathbf{u}_k = (u_{k,j})_{j \in J} \in \mathbb{Z}^J$ by

$$
u_{k,j} := \begin{cases} 
\min \{ \varepsilon_{k,j}^{(t)}, 0 \} & (j \neq k), \\
2 & (j = k). 
\end{cases}
$$

Let $f : \mathbb{R}^J \to \mathbb{R}^J$ be a linear map defined by the matrix $(f_{i,j})_{i \in J}$ whose $i$-th row is $\mathbf{e}_i$ if $i \neq k$ and $\mathbf{e}_k - \mathbf{u}_k$ if $i = k$, where $f$ acts on $g \in \mathbb{R}^J$ from the right, that is, we regard $g$ as a row vector. Then we notice that $f \in \text{GL}_J(\mathbb{Z})$. Let us write $w := \frac{1}{c_k^{(t)}} \mathbf{e}_k^{(t)} \in \mathbb{Z}^J$, where $c_k^{(t)}$ is the greatest common
This implies the assertion for \( t \). We deduce that
\[
\mu^T_k = \varphi_w \cdot F \quad \text{as maps.}
\]
For \( g = (g_j)_{j \in J} \in \mathbb{R}^J \), the direct computation of
\[
\varphi_{w,F}(g) = g - \min \{ \langle g,v \rangle \mid v \in F \}w = g - \min \{0, -g_k \} \varepsilon^{(t)}_k = \begin{cases} 
g & \text{if } g_k \leq 0, 
g + g_k \varepsilon^{(t)}_k & \text{if } g_k \geq 0. 
\end{cases}
\]
Moreover, we see the following:
\[
f(g) = g - g_k u_k = (g'_j)_{j \in J}, \quad g'_j = \begin{cases} 
g_j - g_k \min \{\varepsilon^{(t)}_{k,j}, 0\} = g_j + [-\varepsilon^{(t)}_{k,j}]_+, g_k \quad (j \neq k), 
g_j - 2g_j = -g_j \quad (j = k),
\end{cases}
\]
which coincides with \( \mu^T_k (g) \) in the case \( g_k \leq 0 \). Similar to this, we obtain the following:
\[
f(g + g_k \varepsilon^{(t)}_k) = g + g_k \varepsilon^{(t)}_k - g_k u_k = (g'_j)_{j \in J}, \quad g'_j = \begin{cases} 
g_j + g_k \varepsilon^{(t)}_{k,j} - g_k \min \{\varepsilon^{(t)}_{k,j}, 0\} = g_j + [-\varepsilon^{(t)}_{k,j}]_+ g_k + \varepsilon^{(t)}_{k,j} g_k \quad (j \neq k), 
g_j + 2g_j = -g_j \quad (j = k),
\end{cases}
\]
which coincides with \( \mu^T_k (g) \) in the case \( g_k \geq 0 \). This proves the proposition. \( \Box \)
Recall from Corollary 3.13 that \( \alpha_t \) denotes the unique interior lattice point of \( \Delta(G/B, \mathcal{L}_{2p}, \nu_a, \tau_{2p}) \). As an application of Theorem 3.11 and Proposition 4.8, let us prove the following.

**Theorem 4.9.** If \( G \) is simply-laced, then the following hold.

1. For fixed \( w \in W \) and \( \lambda \in P_+ \), the Newton–Okounkov bodies \( \Delta(X(w), \mathcal{L}_{\lambda}, \nu_a, \tau_a), \quad t \in \mathbb{R}, \quad t \in \mathbb{T} \), are all combinatorially mutation equivalent in \( M_\mathbb{R} \) up to unimodular transformations.

2. For \( w = w_0 \) and \( \lambda = 2\rho \), the translated polytopes \( \Delta(G/B, \mathcal{L}_{2p}, \nu_a, \tau_{2p}) - \alpha_t \), \( t \in \mathbb{R} \), are all combinatorially mutation equivalent in \( M_\mathbb{R} \) up to unimodular transformations. In particular, the dual polytopes \( \Delta(G/B, \mathcal{L}_{2p}, \nu_a, \tau_{2p})^\vee, \quad t \in \mathbb{R} \), are all combinatorially mutation equivalent in \( M_\mathbb{R} \) up to unimodular transformations.

Theorem 4.9 (1) directly follows from Theorem 3.11 and Proposition 4.8. In order to prove Theorem 4.9 (2), we compute \( \alpha_t \) explicitly.

**Proposition 4.10.** If \( G \) is simply-laced, then the unique interior lattice point \( \alpha_t = (a_j)_{j \in J} \) of \( \Delta(G/B, \mathcal{L}_{2p}, \nu_a, \tau_{2p}) \) is given by
\[
a_j = \begin{cases} 
0 & \text{if } j \in J_{af}, 
1 & \text{if } j \in J_{aT},
\end{cases}
\]
for \( j \in J \). In particular, \( \alpha_t \) is independent of the choice of \( t \in \mathbb{T} \), and fixed under the tropicalized cluster mutations.

**Proof.** Since \( g \) is isomorphic to a direct sum of simply-laced simple Lie algebras as a Lie algebra, there exists \( \mathfrak{i}_g \in R(w_0) \) which is a concatenation of reduced words \( \mathfrak{i}_{X_j} \) defined in Example 3.13. Combining the computation of \( \alpha_{X_j} \), in Example 3.13, with Theorem 3.11 (4), we deduce the assertion for \( s_{i_k} = s_{i'_{k}} \). We proceed by induction on the distance from \( t_0 \) in \( \mathbb{T} \). Take \( t, t' \in \mathbb{T} \) and \( k \in J_{af} \) such that \( t - t' \). We assume that the assertion holds for \( t \). By definition, the tropicalized cluster mutation \( \mu^T_k \) is given by a unimodular transformation on each of the half spaces \( \{ (g_j)_{j \in J} \in \mathbb{R}^J \mid g_k \geq 0 \} \) and \( \{ (g_j)_{j \in J} \in \mathbb{R}^J \mid g_k \leq 0 \} \). In addition, \( \mu^T_k \) is identity on the boundary hyperplane \( \{ (g_j)_{j \in J} \in \mathbb{R}^J \mid g_k = 0 \} \) which includes the interior lattice point \( \alpha_t \) of \( \Delta(G/B, \mathcal{L}_{2p}, \nu_a, \tau_{2p}) \). From these, we deduce that \( \mu^T_k (\alpha_t) \) is an interior lattice point of \( \Delta(G/B, \mathcal{L}_{2p}, \nu_a, \tau_{2p}) \).

This implies the assertion for \( t' \), which proves the proposition. \( \Box \)
By Theorem 3.11(4) and Proposition 4.10 we can compute the unique interior lattice point of the string polytope $\Delta_i(2\rho)$ as follows.

**Corollary 4.11.** Let $i = (i_1, \ldots, i_m) \in R(w_0)$. If $G$ is simply-laced, then the unique interior lattice point $a_i = (a_j)_{j \in J}$ of the string polytope $\Delta_i(2\rho)$ is given by

$$a_j = \sum_{k \in J_i, j \leq k} \langle s_{i_{j+1}} \cdots s_{i_k} \varpi_{i_k}, h_{i_j} \rangle$$

for $j \in J$.

**Proof of Theorem 4.9 (2).** We write $\Delta := \Delta(G/B, \mathcal{E}_{2\rho}, v_{w_0}, \tau_{2\rho})$, and set $\Delta' := \mu_k^T(\Delta)$. Let us consider the unique interior lattice point $a_i = (a_j)_{j \in J}$ of $\Delta$. Recall from Proposition 4.8 that $\mu_k^T = f \circ \varphi_{w,F}$ for specific $w \in M$, $F \subseteq w^+$, and $f \in GL_J(\mathbb{Z})$. Since we have $a_j = 0$ for all $j \in J_{uf}$ by Proposition 4.10, the definitions of $\varphi_{w,F}$ and $f$ imply that $\varphi_{w,F}(a_i) = f(a_i) = a_i$.

In addition, we have

$$\varphi_{w,F}(\Delta - a_i) = f^{-1}(\Delta' - a_i),$$

which implies by Propositions 4.4, 4.5 that

$$\text{mut}_w(\Delta', F) = \varphi_{w,F}(\Delta - a_i) = f^{-1}(\Delta' - a_i);$$

here, we note that $(\Delta')^* = \Delta - a_i$. Hence it follows that

$$\text{mut}_w(\Delta', F) = (f^{-1}(\Delta' - a_i))^*.$$

In general, for $Q \subseteq M_\mathbb{Z}$ containing the origin in its interior and $\gamma \in GL(M_\mathbb{Z})$, it follows from the definition of the polar dual that the equality $\gamma(Q)^* = t^* \gamma^{-1}(Q^*)$ holds, where $t^* \gamma \in GL(N_\mathbb{Z})$ denotes the dual map of $\gamma$. Indeed, we have

$$\gamma(Q)^* = \{ v \in N_\mathbb{Z} \mid \langle \gamma(u), v \rangle \geq 1 \text{ for all } u \in Q \}$$

$$= \{ v \in N_\mathbb{Z} \mid \langle u, t^* \gamma(v) \rangle \geq 1 \text{ for all } u \in Q \}$$

$$= \{ t^* \gamma^{-1}(v) \mid v \in N_\mathbb{Z}, \langle u, v \rangle \geq 1 \text{ for all } u \in Q \}$$

$$= t^* \gamma^{-1}(Q^*).$$

Notice that if $\gamma$ is unimodular, then so is $t^* \gamma$.

Hence we conclude that

$$\text{mut}_w(\Delta', F) = t^* f(\Delta').$$

This implies the required assertion since $t^* f$ is a unimodular transformation. □

### 5. Relation with FFLV polytopes

FFLV polytopes were introduced by Feigin–Fourier–Littelmann [11, 12] and Vinberg [14] to study PBW-filtrations of $V(\lambda)$. Kiritchenko [31] proved that the FFLV polytope $FFLV(\lambda)$ of type $A_n$ coincides with the Newton–Okounkov body of $(G/B, \mathcal{E}_\lambda)$ associated with a valuation given by counting the orders of zeros/poles along a specific sequence of translated Schubert varieties. Feigin–Fourier–Littelmann [13] realized the FFLV polytopes of types $A_n$ and $C_n$ as Newton–Okounkov bodies of $(G/B, \mathcal{E}_\lambda)$ using a different kind of valuation. Ardila–Bliem–Salazar [3] gave an explicit bijective piecewise-affine map from the Gelfand–Tsetlin polytope $GT(\lambda)$ of type $A_n$ (resp., type $C_n$) to the FFLV polytope $FFLV(\lambda)$ of type $A_n$ (resp., type $C_n$) by generalizing Stanley’s transfer map [12] to marked poset polytopes. In this section, we relate Ardila–Bliem–Salazar’s transfer map with combinatorial mutations.
5.1. Marked poset polytopes. In this subsection, we recall the definition of Ardila–Bliem–Salazar’s marked order polytopes and marked chain polytopes together with their transfer map \[3\].

First, we recall what a marked poset is. Let \(\tilde{\Pi}\) be a poset equipped with a partial order \(<\), and \(A \subseteq \tilde{\Pi}\) a subset of \(\tilde{\Pi}\) containing all minimal elements and maximal elements in \(\tilde{\Pi}\). Take a vector \(\lambda = (\lambda_a)_{a \in A} \in \mathbb{R}^A\), called a marking, such that \(\lambda_a \leq \lambda_b\) whenever \(a < b\) in \(\tilde{\Pi}\). We call the triple \((\tilde{\Pi}, A, \lambda)\) a marked poset.

Definition 5.1 (\(\mathbb{R}\) Definition 1.2)). Work with the same notation as above. We set

\[
\mathcal{O}(\tilde{\Pi}, A, \lambda) := \{(x_p)_{p \in \tilde{\Pi} \setminus A} \in \mathbb{R}^{\tilde{\Pi} \setminus A} \mid x_p \leq q\text{ if } p < q, \lambda_a \leq x_p \text{ if } a < p, x_p \leq \lambda_b \text{ if } p < a\},
\]

\[
\mathcal{C}(\tilde{\Pi}, A, \lambda) := \{(x_p)_{p \in \tilde{\Pi} \setminus A} \in \mathbb{R}^{\tilde{\Pi} \setminus A} \mid \sum_{i=1}^k x_{p_i} \leq \lambda_b - \lambda_a \text{ if } a < p_1 \prec \cdots \prec p_k < b\}.
\]

If \(\lambda \in \mathbb{Z}^A\), then those polytopes are lattice polytopes (\(\mathbb{R}\) Lemma 3.5). The polytope \(\mathcal{O}(\tilde{\Pi}, A, \lambda)\) is called the marked order polytope, and \(\mathcal{C}(\tilde{\Pi}, A, \lambda)\) is called the marked chain polytope.

Remark 5.2. Originally, the order polytope \(\mathcal{O}(\Pi)\) and the chain polytope \(\mathcal{C}(\Pi)\) of a poset \(\Pi\) were introduced by Stanley \([\mathbb{R}2]\). The notions of marked poset polytopes generalize those of ordinary poset polytopes. Indeed, given a poset \(\Pi\), by setting \(\tilde{\Pi} := \Pi \cup \{0,1\}\), \(A := \{0,1\}\), and \(\lambda := (\lambda_0, \lambda_1) := (0,1)\), the new minimum (resp., maximum) element not belonging to \(\Pi\), we see that the marked order polytope \(\mathcal{O}(\tilde{\Pi}, A, \lambda)\) (resp., \(\mathcal{C}(\tilde{\Pi}, A, \lambda)\)) coincides with the ordinary order polytope \(\mathcal{O}(\Pi)\) (resp., the ordinary chain polytope \(\mathcal{C}(\Pi)\)).

In \(\mathbb{R}\) Theorem 3.4, a piecewise-affine bijection \(\tilde{\phi}\) from \(\mathcal{O}(\tilde{\Pi}, A, \lambda)\) to \(\mathcal{C}(\tilde{\Pi}, A, \lambda)\) was constructed, which is called a transfer map. The piecewise-affine map \(\phi : \mathbb{R}^{\tilde{\Pi} \setminus A} \to \mathbb{R}^{\tilde{\Pi} \setminus A}, (x_p)_{p \in \tilde{\Pi} \setminus A} \mapsto (x'_p)_{p \in \tilde{\Pi} \setminus A}\), is defined as follows:

\[
x'_p := \min\{\{x_p - x_{p'} \mid p' < p, p' \in \tilde{\Pi} \setminus A\} \cup \{x_p - \lambda_{p'} \mid p' < p, p' \in A\}\}
\]

for \(p \in \tilde{\Pi} \setminus A\), where for \(p, q \in \tilde{\Pi}\), \(q < p\) means that \(p\) covers \(q\), that is, \(q < p\) and there is no \(q' \in \tilde{\Pi} \setminus \{p, q\}\) with \(q < q' < p\).

Now, we recall a key notion which we will use in the proof of Theorem 5.3 called marked chain-order polytopes, introduced in \(\mathbb{R}\) and developed in \([\mathbb{R}]\). We remark that the original notion of marked chain-order polytopes is more general, but we restrict it for our purpose. Take a marked poset \((\tilde{\Pi}, A, \lambda)\) and fix \(\Pi' \subseteq \tilde{\Pi} \setminus A\). We define \(\mathcal{O}_{\Pi'}(\tilde{\Pi}, A, \lambda)\) as follows:

\[
\mathcal{O}_{\Pi'}(\tilde{\Pi}, A, \lambda) := \{(x_p)_{p \in \tilde{\Pi} \setminus A} \in \mathbb{R}^{\tilde{\Pi} \setminus A} \mid x_p \geq 0 \text{ for all } p \in \Pi', \sum_{i=1}^k x_{p_i} \leq y_b - y_a \text{ for } a < p_1 \prec \cdots \prec p_k < b \text{ with } p_i \in \Pi' \text{ and } a, b \in \tilde{\Pi} \setminus \Pi'\},
\]

where for \(c \in \tilde{\Pi} \setminus \Pi'\), we set

\[
y_c := \begin{cases} \lambda_c & \text{if } c \in A, \\ x_c & \text{otherwise.} \end{cases}
\]

We can directly check that \(\mathcal{O}_\emptyset(\tilde{\Pi}, A, \lambda) = \mathcal{O}(\tilde{\Pi}, A, \lambda)\) and \(\mathcal{O}_{\tilde{\Pi} \setminus A}(\tilde{\Pi}, A, \lambda) = \mathcal{C}(\tilde{\Pi}, A, \lambda)\). By taking \(\Pi'\) with \(\emptyset \subseteq \Pi' \subseteq \tilde{\Pi} \setminus A\), we obtain an “intermediate polytope” between a marked order polytope and a marked chain polytope. It is proved in \([\mathbb{R}]\) Proposition 2.4 that if \(\lambda \in \mathbb{Z}^A\), then \(\mathcal{O}_{\Pi'}(\tilde{\Pi}, A, \lambda)\) is a lattice polytope for every \(\Pi' \subseteq \tilde{\Pi} \setminus A\). Define a map \(\tilde{\phi}_{\Pi'} : \mathbb{R}^{\tilde{\Pi} \setminus A} \to \mathbb{R}^{\tilde{\Pi} \setminus A}, (x_p)_{p \in \tilde{\Pi} \setminus A} \mapsto (x'_p)_{p \in \tilde{\Pi} \setminus A}\), by

\[
x'_p := \begin{cases} \min\{\{x_p - x_{p'} \mid p' < p, p' \in \tilde{\Pi} \setminus A\} \cup \{x_p - \lambda_{p'} \mid p' < p, p' \in A\}\} & \text{if } p \in \Pi', \\ x_p & \text{otherwise.} \end{cases}
\]

for \(p \in \tilde{\Pi} \setminus A\). Notice that \(\tilde{\phi}_{\tilde{\Pi} \setminus A} = \tilde{\phi}\) and \(\tilde{\phi}_\emptyset = \operatorname{id}\). In \([\mathbb{R}]\) Theorem 2.1, it is proved that the map \(\tilde{\phi}_{\Pi'}\) gives a piecewise-affine bijection from \(\mathcal{O}(\tilde{\Pi}, A, \lambda)\) to \(\mathcal{O}_{\Pi'}(\tilde{\Pi}, A, \lambda)\).
5.2. Combinatorial mutation equivalence of marked poset polytopes. The second named author proved in [25, Theorem 4.1] that the transfer map between ordinary poset polytopes can be described as a composition of combinatorial mutations in \( M_\mathbb{R} \). We can generalize this result to marked poset polytopes under some conditions.

We say that a poset \( \tilde{\Pi} \) is pure if every maximal chain in \( \tilde{\Pi} \) has the same length. When \( \tilde{\Pi} \) is pure, all chains starting from a minimal element in \( \tilde{\Pi} \) and ending at \( p \) have the same length for each \( p \in \tilde{\Pi} \). We denote by \( r(p) \) the length of such chains.

Let \((\tilde{\Pi}, A, \lambda)\) be a marked poset with \( \lambda \in \mathbb{Z}^A \). Assume that \( \tilde{\Pi} \) is pure, and that \( \lambda \) satisfies \( \lambda_a = \lambda_b \) for all \( a, b \in A \) with \( r(a) = r(b) \). Then there exists \( u = (u_p)_{p \in \tilde{\Pi} \setminus A} \in \mathcal{O}(\tilde{\Pi}, A, \lambda) \cap \mathbb{Z}^{\tilde{\Pi} \setminus A} \) such that

\[
\begin{align*}
  u_p &= u_{p'} \text{ for all } p, p' \in \tilde{\Pi} \setminus A \text{ with } r(p) = r(p'), \text{ and} \\
  u_p &= \lambda_a \text{ for all } p \in \tilde{\Pi} \setminus A \text{ and } a \in A \text{ with } r(p) = r(a).
\end{align*}
\]

(5.2)

Let \( \lambda' \) denote the marking given by \( (\lambda')_a = r(a) \) for \( a \in A \). Then it is proved in [10, Corollary 23] that for a pure poset \( \tilde{\Pi}, \mathcal{O}(\tilde{\Pi}, A, \lambda') \) (resp., \( C(\tilde{\Pi}, A, \lambda') \)) contains a unique interior lattice point. Indeed, the unique interior lattice point \( (r_p)_{p \in \tilde{\Pi} \setminus A} \in \mathcal{O}(\tilde{\Pi}, A, \lambda') \) is given by \( r_p = r(p) \) for all \( p \in \tilde{\Pi} \setminus A \), while the unique interior lattice point \( (r_p')_{p \in \tilde{\Pi} \setminus A} \in C(\tilde{\Pi}, A, \lambda') \) is given by \( r_p' = 1 \) for all \( p \in \tilde{\Pi} \setminus A \). We notice that \( (r_p)_{p \in \tilde{\Pi} \setminus A} \) satisfies (5.2) and \( \varphi((r_p)_p) = (r_p')_p \). Write

\[
\mathcal{O}(\tilde{\Pi}, A, \lambda') := \Omega(\tilde{\Pi}, A, \lambda') - (r_p)_{p \in \tilde{\Pi} \setminus A}, \text{ and } C(\tilde{\Pi}, A, \lambda') := \mathcal{C}(\tilde{\Pi}, A, \lambda') - (r_p')_{p \in \tilde{\Pi} \setminus A}.
\]

Namely, \( \mathcal{O}(\tilde{\Pi}, A, \lambda') \) (resp., \( C(\tilde{\Pi}, A, \lambda') \)) contains the origin as the unique interior lattice point. We regard polytopes appearing below as ones living in \( M_\mathbb{R} \).

**Theorem 5.3.** Let \( \tilde{\Pi} \) be a pure poset.

1. Let \((\tilde{\Pi}, A, \lambda)\) be a marked poset with \( \lambda \in \mathbb{Z}^A \) such that \( \lambda \) satisfies \( \lambda_a = \lambda_b \) for all \( a, b \in A \) with \( r(a) = r(b) \). Take a (not necessarily interior) lattice point \( u = (u_p)_{p \in \tilde{\Pi} \setminus A} \) satisfying (5.2). Then the translated marked order polytope \( \Omega(\tilde{\Pi}, A, \lambda) - u \) and the translated marked chain polytope \( \mathcal{C}(\tilde{\Pi}, A, \lambda) - \varphi(u) \) are combinatorially mutation equivalent in \( M_\mathbb{R} \).

2. Consider the marked poset \((\tilde{\Pi}, A, \lambda')\). Then \( \mathcal{O}(\tilde{\Pi}, A, \lambda') \) and \( \mathcal{C}(\tilde{\Pi}, A, \lambda') \) are combinatorially mutation equivalent in \( M_\mathbb{R} \).

**Proof.** Since the assertion (2) directly follows from (1), we will prove the assertion (1).

**The first step.** For each \( p \in \tilde{\Pi} \setminus A \), set

\[
\begin{align*}
  w_p &= -e_p, \\
  F_p &= \text{conv}\{ -e_{p'} \mid p' \in \tilde{\Pi} \setminus A \} \cup \{ 0 \mid p' < p, p' \in A \}, \text{ and} \\
  \varphi_p &= \varphi_{w_p} F_p.
\end{align*}
\]

Note that \( F_p \subseteq w_p^\perp \). Then the direct computation shows the following:

\[
\varphi_q((x_p)_{p \in \tilde{\Pi} \setminus A}) = (x_p)_{p} - \text{min}\{ ((x_p)_q, v) \mid v \in F_q \} w_q = (x_p)_{p} + \text{min}\{ -x_{p'} \mid p' < q, p' \in \tilde{\Pi} \setminus A \} \cup \{ 0 \mid p' < q, p' \in A \} e_q,
\]

which implies that if we write \( \varphi_q((x_p)_{p \in \tilde{\Pi} \setminus A}) = (x_p')_{p \in \tilde{\Pi} \setminus A} \), then we have

\[
x_p' = \begin{cases}
  \min\{ (x_p - x_{p'} \mid p' < p, p' \in \tilde{\Pi} \setminus A \} \cup \{ x_p \mid p' < q, p' \in A \} & \text{if } p = q, \\
  x_p & \text{otherwise}
\end{cases}
\]

for \( p \in \tilde{\Pi} \setminus A \). We write \( \tilde{\Pi} \setminus A = \{ q_1, \ldots, q_d \} \), and arrange \( q_1, q_2, \ldots, q_d \) such as \( q_i < q_j \) in \( \tilde{\Pi} \) only if \( i > j \). Let

\[
\varphi_i := \varphi_{q_i} \circ \cdots \circ \varphi_{q_2} \circ \varphi_{q_1} \text{ for } i = 1, \ldots, d.
\]
In particular, \( \mathcal{V}_i \) includes all \( \varphi_i \)'s for \( q \in \tilde{1} \setminus A \) arranged in the order “from top to bottom”. Since each \( \varphi_q \) changes only the \( q \)-th entry based on \( p' \)-th entries with \( p' \ll q \), we obtain the following: if we write \( \mathcal{V}_i((x_p)_{p \in \tilde{1} \setminus A}) = (x'_p)_{p \in \tilde{1} \setminus A} \), then it follows that

\[
x'_p = \begin{cases} 
\min\{x_p - x_{p'} | p' < p, \ p' \in \tilde{1} \setminus A \} & \text{if } p \in \{q_1, \ldots, q_i\}, \\
x_p - u_p & \text{otherwise} 
\end{cases}
\]

for \( p \in \tilde{1} \setminus A \).

**The second step.** For \( \Pi' \subseteq \tilde{1} \setminus A \), we define a translation map \( f_{\Pi'} \) as follows:

\[
f_{\Pi'} : \mathbb{R}^{\tilde{1} \setminus A} \to \mathbb{R}^{\tilde{1} \setminus A}, \ (x_p)_{p \in \tilde{1} \setminus A} \mapsto (x_p)_{p \in \tilde{1} \setminus A} - \tilde{\phi}_{\Pi'}(u),
\]

where \( \tilde{\phi}_{\Pi'} \) is the map defined in [5.1]. For simplicity, we write \( f := f_\emptyset \).

In the third step, we will prove that

\[
\mathcal{V}_i = f_{\Pi \setminus A}(O(\tilde{1}, A, \lambda) - u) \text{ is convex for all } i \in \{1, \ldots, d\}.
\]

Once we prove this, we obtain that \( \mathcal{V}_i(O(\tilde{1}, A, \lambda) - u) \) is convex for all \( i \) since

\[
\mathcal{V}_i(O(\tilde{1}, A, \lambda) - u) = f_{\Pi \setminus A}(O(\tilde{1}, A, \lambda) - u) = f_{\Pi \setminus A}(O(\tilde{1}, A, \lambda))(\mathcal{V}_i(O(\tilde{1}, A, \lambda)))
\]

and \( O(q_1, \ldots, q_i)(\tilde{1}, A, \lambda) \) is a lattice polytope by \( \mathfrak{P} \) Proposition 2.4. Moreover, we see that

\[
\mathcal{V}_d(O(\tilde{1}, A, \lambda) - u) = f_{\Pi \setminus A}(O(\tilde{1}, A, \lambda)) = O(\tilde{1}, A, \lambda) - \tilde{\phi}(u),
\]

as required.

**The third step.** We prove [5.3]. Given \( (x_p)_{p \in \tilde{1} \setminus A} \in \mathbb{R}^{\tilde{1} \setminus A} \), we apply the map \( \tilde{\phi}_{(q_1, \ldots, q_i)} \) to \( f^{-1}((x_p)_p) = (x_p + u_p)_p \). If we write \( \tilde{\phi}_{(q_1, \ldots, q_i)}((x_p + u_p)_p) = (x'_p)_p \), then it holds that

\[
x'_p = \begin{cases} 
\min\{x_p - x_{p'} + u_p - u_{p'} | p' < p, \ p' \in \tilde{1} \setminus A \} & \text{if } p \in \{q_1, \ldots, q_i\}, \\
x_p + u_p & \text{otherwise}
\end{cases}
\]

for \( p \in \tilde{1} \setminus A \). Remark that the set \( \{u_p - u_{p'} | p' \ll p, \ p' \in \tilde{1} \setminus A \} \cup \{u_p - \lambda_{p'} | p' < p, \ p' \in A \} \) consists of only one element by the assumption [5.2]. Hence we see that

\[
x'_p = \begin{cases} 
\min\{x_p - x_{p'} | p' < p, \ p' \in \tilde{1} \setminus A \} & \text{if } p \in \{q_1, \ldots, q_i\}, \\
x_p + \tilde{\phi}_{(q_1, \ldots, q_i)}(u)_p & \text{otherwise}
\end{cases}
\]

for \( p \in \tilde{1} \setminus A \). Let us apply \( f_{(q_1, \ldots, q_i)} \) to \( \tilde{\phi}_{(q_1, \ldots, q_i)} \circ f^{-1}((x_p)_p) = (x'_p)_p \). If we write

\[
f_{(q_1, \ldots, q_i)} \circ \tilde{\phi}_{(q_1, \ldots, q_i)} \circ f^{-1}((x_p)_{p \in \tilde{1} \setminus A}) = f_{(q_1, \ldots, q_i)}((x'_p)_{p \in \tilde{1} \setminus A}) = (x''_p)_{p \in \tilde{1} \setminus A},
\]

then it holds for \( p \in \tilde{1} \setminus A \) that

\[
x''_p = \begin{cases} 
\min\{x_p - x_{p'} | p' < p, \ p' \in \tilde{1} \setminus A \} & \text{if } p \in \{q_1, \ldots, q_i\}, \\
x_p & \text{otherwise}
\end{cases}
\]

Combining this with the first step, we conclude the desired equality [5.3].

As the following example shows, the transfer map \( \tilde{\phi} \) is not necessarily described as a composition of combinatorial mutations in \( M_{\mathcal{R}} \) if we drop the assumption [5.2].
Example 5.4. Let us consider the marked poset in Figure 5.1. We regard that the marked poset polytopes live in $\mathbb{R}^3$. In this case, the transfer map $\tilde{\phi} : \mathbb{R}^3 \to \mathbb{R}^3$ is given as follows:

$$
\tilde{\phi}(x, y, z) = (\min\{x - 1, x - z\}, \min\{y - z, y - 2\}, z)
$$

$$
= \begin{cases} 
(x - z, y - z, z) & \text{if } z \geq 2, \\
(x - z, y - 2, z) & \text{if } 1 \leq z \leq 2, \\
(x - 1, y - 2, z) & \text{if } z \leq 1.
\end{cases}
$$

Hence, even if we apply any translation to the marked order polytope, the transfer map never becomes piecewise-linear. This implies that the transfer map $\tilde{\phi}$ cannot be described as a composition of combinatorial mutations in $M_R$.

Figure 5.1. The marked Hasse diagram considered in Example 5.4.

5.3. Type A case. Let $G = SL_{n+1}(\mathbb{C})$, and $\lambda \in P_+$. We write $\lambda_{\geq k} := \sum_{k \leq \ell \leq n} \langle \lambda, h_\ell \rangle$ for $1 \leq k \leq n$. Let $O_\lambda$ (resp., $C_\lambda$) denote the marked order (resp., chain) polytope associated with a marked poset whose Hasse diagram is given in Figure 5.2.

Figure 5.2. The marked Hasse diagram in type $A_n$. 
By the definition, the marked order polytope $O_\lambda$ coincides with the Gelfand–Tsetlin polytope $GT(\lambda)$ (see Example 3.6), and the marked chain polytope $C_\lambda$ coincides with the FFLV polytope $FFLV(\lambda)$ (see [11, equation (0.1)]).

Since the associated marked poset satisfies the assumption in Theorem 5.3, the following theorem is an immediate consequence of Theorem 5.3.

**Theorem 5.5.** The following hold.

1. For all $\lambda \in P_+$, the polytopes $GT(\lambda) - u$ and $FFLV(\lambda) - \tilde{\phi}(u)$ are combinatorially mutation equivalent in $M_\mathbb{R}$, where $u \in GT(\lambda)$ is a lattice point satisfying (5.2).
2. The dual polytopes $GT(2\rho)^\vee$ and $FFLV(2\rho)^\vee$ are combinatorially mutation equivalent in $N_\mathbb{R}$.

4. **Type C case.** Let $G = Sp_{2n}(\mathbb{C})$, and $\lambda \in P_+$. We write $\lambda_{\geq k} := \sum_{k \leq \ell \leq n} \langle \lambda, h_{\ell} \rangle$ for $1 \leq k \leq n$. Let $O_\lambda$ (resp., $C_\lambda$) denote the marked order (resp., chain) polytope associated with a marked poset whose Hasse diagram is given in Figure 5.3.

![Figure 5.3. The marked Hasse diagram in type $C_n$.](image)

By the definition, the marked order polytope $O_\lambda$ coincides with the Gelfand–Tsetlin polytope $GT_{C_n}(\lambda)$ of type $C_n$ (see Example 3.7), and the marked chain polytope $C_\lambda$ coincides with the FFLV polytope $FFLV_{C_n}(\lambda)$ of type $C_n$ (see [12, equation (1.2)]).

Similar to Theorem 5.5, the following theorem holds.

**Theorem 5.6.** The following hold.

1. For all $\lambda \in P_+$, the polytopes $GT_{C_n}(\lambda) - u$ and $FFLV_{C_n}(\lambda) - \tilde{\phi}(u)$ are combinatorially mutation equivalent in $M_\mathbb{R}$, where $u \in GT_{C_n}(\lambda)$ is a lattice point satisfying (5.2).
2. The dual polytopes $GT_{C_n}(2\rho)^\vee$ and $FFLV_{C_n}(2\rho)^\vee$ are combinatorially mutation equivalent in $N_\mathbb{R}$.

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