On the Solutions of the $b$-Family of Novikov Equation

Tingting Wang†, Xuanxuan Han† and Yibin Lu*,†

Faculty of Science, Kunming University of Science and Technology, Kunming 650500, China; wangtingting@stu.kust.edu.cn (T.W.); 20202111001@stu.kust.edu.cn (X.H.)
* Correspondence: luyibin@kust.edu.cn
† All three authors contributed equally to this work.

Abstract: In this paper, we study the symmetric travelling wave solutions of the $b$-family of the Novikov equation. We show that the $b$-family of the Novikov equation can provide symmetric travelling wave solutions, such as peakon, kink and smooth soliton solutions. In particular, the single peakon, two-peakon, stationary kink, anti-kink, two-kink, two-anti-kink, bell-shape soliton and hat-shape soliton solutions are presented in an explicit formula.

Keywords: the $b$-family of Novikov equation; peakon; kink; soliton solutions

1. Introduction

The $b$-family of the Camassa–Holm equation

$$m_t + um_x + bu_xm = 0, \quad m = u - u_{xx}, \quad (1)$$

where $b$ is an arbitrary constant and $u(x,t)$ is fluid velocity. Eq. (1) was first proposed by Holm and Stanley in studying the exchange of stability in the dynamics of solitary waves under changes in the nonlinear balance in a $1 + 1$ evolutionary PDE related to shallow water waves and turbulence [1,2]. In the case of $b \neq 0$, peakon solutions of Equation (1) were discussed in [1,2]. In the case of $b = 0$, Xia and Qiao showed that Equation (1) provides N-kink, bell-shape and hat-shape solitary solutions [3]. For $b = 2$, Equation (1) becomes the well-known Camassa–Holm (CH) equation

$$m_t + um_x + 2u_xm = 0, \quad m = u - u_{xx}, \quad (2)$$

which was originally implied in Fokas and Fuchssteiner in [4], but became well-known when Camassa and Holm [5] derived it as a model for the unidirectional propagation of shallow water over a flat bottom. The CH equation was found to be completely integrable with a Lax pair and associated bi-Hamiltonian structure [4–6]. The famous feature of the CH equation is that it provides peaked soliton (peakon) solutions [4,5], which present an essential feature of the travelling waves of largest amplitude [7–9]. For $b = 3$, Equation (1) becomes the Degasperis–Procesi (DP) equation

$$m_t + um_x + 3u_xm = 0, \quad m = u - u_{xx}, \quad (3)$$

which can be regarded as another model for nonlinear shallow water dynamics with peakons [10,11]. The integrability of the DP equation was shown by constructing a Lax pair, and deriving two infinite sequences of conservation laws in [12].

In this paper, we are concerned with the $b$-family of the Novikov equation

$$m_t + u^2m_x + buu_xm = 0, \quad m = u - u_{xx}, \quad (4)$$

where $b$ is an arbitrary constant. It is easy to see that the $b$-family of the Novikov Equation (4) has nonlinear terms that are cubic, rather than quadratic, of the $b$-family of CH...
Equation (1). The Cauchy problem of the $b$-family of the Novikov Equation (4) was studied in [13].

For $b = 3$, Equation (4) becomes the Novikov equation

$$m_t + u^2 m_x + 3uu_x m = 0, \quad m = u - u_{xx},$$

(5)

which was discovered by Vladimir Novikov [14] in a symmetry classification of nonlocal PDEs with quadratic or cubic nonlinearity. In [15,16], it was shown that the Novikov equation provides peakon solutions such as the CH and DP equations. Additionally, the Novikov Equation (5) has a Lax pair in matrix form and a bi-Hamiltonian structure. Moreover, it has infinitely many conserved quantities.

The purpose of this paper is to investigate the solutions of the $b$-family of the Novikov Equation (4) in the case of $b \neq 0$ and $b = 0$. We will show that Equation (4) possesses symmetric travelling wave solutions, such as peakon, kink and smooth soliton solutions. In particular, the single peakon, two-peakon, stationary kink, anti-kink, two-kink, two-anti-kink, bell-shape soliton and hat-shape soliton solutions are presented in an explicit formula and plotted.

The rest of this paper is organized as follows. In Section 2, we derive the $N$-peakon solutions in the case of $b \neq 0$. In Section 3, we discuss the $N$-kink and smooth soliton solutions in the case of $b = 0$.

### 2. Peakon Solutions

In this section, we derive the $N$-peakon solutions in the case of $b \neq 0$. We assume the $N$-peakon solution as the form

$$u = \sum_{j=1}^{N} p_j(t) e^{-|x-q_j(t)|},$$

(6)

where $p_j(t)$ and $q_j(t)$ are to be determined. The derivatives of (6) do not exist at $x = q_j(t)$, thus (6) cannot satisfy Equation (4) in a classical sense. However, in the distribution, we have

$$u_x = - \sum_{j=1}^{N} p_j(t) \text{sgn}(x-q_j(t)) e^{-|x-q_j(t)|},$$

(7)

$$m = 2 \sum_{j=1}^{N} p_j(t) \delta(x-q_j(t)),$$

(8)

$$m_t = 2 \sum_{j=1}^{N} p_j \delta'(x-q_j(t)) - 2 \sum_{j=1}^{N} p_j q_{jt} \delta'(x-q_j(t)),$$

(9)

$$m_x = 2 \sum_{j=1}^{N} p_j(t) \delta'(x-q_j(t)).$$

(10)

Substituting (6)–(10) into (4) and integrating against the test function with compact support, we obtain that $p_j(t)$ and $q_j(t)$ evolve according to the dynamical system:

$$\begin{cases}
q_{jt} = \left( \sum_{i=1}^{N} p_i e^{-|q_j-q_i|} \right)^2, & 1 \leq j \leq N, \\
p_{j,t} = (b - 2)p_j \left( \sum_{i=1}^{N} p_i e^{-|q_j-q_i|} \right) \left( \sum_{i=1}^{N} p_i \text{sgn}(q_j - q_i) e^{-|q_j-q_i|} \right), & 1 \leq j \leq N.
\end{cases}$$

(11)
For $N = 1$, (11) is reduced to
\[
\begin{align*}
q_{1,t} &= p_1^2, \\
p_{1,t} &= 0.
\end{align*}
\]
Thus, the single peakon solution (See Figure 1) is
\[
u = \pm \sqrt{c} e^{-|x-ct|}, \quad c > 0.
\] (12)

For $N = 2$, (11) is reduced to
\[
\begin{align*}
q_{1,t} &= \left( p_1 + p_2 e^{-|q_1-q_2|} \right)^2, \\
q_{2,t} &= \left( p_1 e^{-|q_2-q_1|} + p_2 \right)^2, \\
p_{1,t} &= (b-2)p_1 p_2 \left( p_1 + p_2 e^{-|q_1-q_2|} \right) \text{sgn}(q_1-q_2) e^{-|q_1-q_2|}, \\
p_{2,t} &= (b-2)p_1 p_2 \left( p_2 + p_1 e^{-|q_1-q_2|} \right) \text{sgn}(q_2-q_1) e^{-|q_2-q_1|}.
\end{align*}
\] (13)

Solving (13), we have
\[
\begin{align*}
q_1(t) - q_2(t) &= C, \\
p_1(t) &= -p_2(t) = -\frac{1}{\sqrt{2b}e^{-2c} - 2be^{-c} - 4te^{-2c} + 4te^{-c}}.
\end{align*}
\] (14)

In particular, for $C = 1, q_2(t) = t, b = 1$, the solution (See Figure 2) becomes
\[
u(x, t) = -\frac{1}{\sqrt{2t}e^{-t} - 2te^{-2}}e^{-|x-t-1|} + \frac{1}{\sqrt{2t}e^{-t} - 2te^{-2}}e^{-|x-t|}.
\] (15)

Figure 1. The positive single peakon solution determined by (12) with $c = 1$ at time $t = 2$. 
3. Kink and Smooth Soliton Solutions

In this section, we discuss the $N$-kink and smooth soliton solutions in the case of $b = 0$, namely

$$m_t + u^2 m_x = 0, \quad m = u - u_{xx}. \quad (16)$$

We suppose that the $N$-kink solution as the form

$$u = \sum_{j=1}^{N} c_j \text{sgn}(x - q_j(t)) \left(e^{-|x-q_j(t)|} - 1\right), \quad (17)$$

where $c_j$ are arbitrary constants and $q_j(t)$ are to be determined. The derivatives of (17) do not exist at $x = q_j(t)$, thus (17) can not satisfy Equation (4) in a classical sense. However, in the distribution, we have

$$u_x = -\sum_{j=1}^{N} c_j e^{-|x-q_j(t)|}, \quad (18)$$

$$m_t = 2 \sum_{j=1}^{N} c_j q_j(t) \delta(x-q_j(t)), \quad (19)$$

$$m_x = -2 \sum_{j=1}^{N} c_j \delta(x-q_j(t)). \quad (20)$$

Substituting (17)–(20) into (16) and integrating against the test function with compact support, we obtain that $q_j(t)$ evolves according to the dynamical system:

$$q_{j,t} = \left(\sum_{i=1}^{N} c_i \text{sgn}(q_j - q_i) \left(e^{-|q_j-q_i|} - 1\right)\right)^2, \quad 1 \leq j \leq N. \quad (21)$$

For $N = 1$, we have $q_{1,t} = 0$, which yields $q_1 = c$, where $c$ is an arbitrary constant. Thus the single kink solution (See Figure 3 and Figure 4) is stationary and it reads

$$u = c_1 \text{sgn}(x - c) \left(e^{-|x-c|} - 1\right). \quad (22)$$
Figure 3. The stationary kink solution determined by (22) with \( c_1 = c = 1 \).

Figure 4. The stationary anti-kink solution determined by (22) with \( c_1 = -1, c = 1 \).

For \( N = 2 \), (21) is reduced to

\[
\begin{align*}
q_{1,t} &= \left[ c_2 \text{sgn}(q_1 - q_2) \left( e^{-|q_1 - q_2|} - 1 \right) \right]^2, \\
q_{2,t} &= \left[ c_1 \text{sgn}(q_2 - q_1) \left( e^{-|q_2 - q_1|} - 1 \right) \right]^2.
\end{align*}
\] (23)

If \( c_1^2 = c_2^2 \), we obtain

\[
\begin{align*}
q_1(t) &= \left[ c_1 \text{sgn}(C_1)(e^{-|C_1|} - 1) \right]^2 t, \\
q_2(t) &= q_1(t) - C_1,
\end{align*}
\] (24)

where \( C_1 \) is an arbitrary constant. The solution (See Figure 5 and Figure 6) becomes

\[
u(x, t) = c_1 \text{sgn}(x - q_1(t)) \left( e^{-|x - q_1(t)|} - 1 \right) + c_2 \text{sgn}(x - q_2(t)) \left( e^{-|x - q_2(t)|} - 1 \right),
\] (25)

where \( q_1 \) and \( q_2 \) are given by (24).
Figure 5. The bell-shape solution determined by (25) with $c_1 = C_1 = 1$, $c_2 = -1$ at time $t = 2$.

Figure 6. The hat-shape solution determined by (25) with $c_1 = 1$, $c_2 = -1$, $C_1 = 15$ at time $t = 2$.

If $c_1^2 \neq c_2^2$, we obtain

$$q_1(t) - q_2(t) = \ln \left( \frac{1 + \text{LambertW}(e^{(c_1^2 - c_2^2) t})}{\text{LambertW}(e^{(c_1^2 - c_2^2) t})} \right) \triangleq g(t).$$

(26)

In particular, for $q_1(t) = \frac{1}{2}g(t)$ and $q_2(t) = -\frac{1}{2}g(t)$, the solution (See Figure 7 and Figure 8) becomes

$$u(x, t) = c_1 \text{sgn}(x - \frac{1}{2}g(t)) \left( e^{-|x - \frac{1}{2}g(t)|} - 1 \right) + c_2 \text{sgn}(x + \frac{1}{2}g(t)) \left( e^{-|x + \frac{1}{2}g(t)|} - 1 \right).$$

(27)
Figure 7. The two kink solution determined by (27) with \( c_1 = 2, c_2 = 1 \) at time \( t = 4 \).

Figure 8. The two anti-kink solution determined by (27) with \( c_1 = -2, c_2 = -1 \) at time \( t = 4 \).

**Author Contributions:** Investigation, T.W., X.H. and Y.L.; writing – review and editing, T.W., X.H. and Y.L.; funding acquisition, T.W., X.H. and Y.L. The authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

**Funding:** This work is supported by National Natural Science Foundation of China (Grant No. 11461037).

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** Not applicable.

**Acknowledgments:** The authors are grateful to the anonymous referees for their constructive comments and suggestions, which have greatly improved this paper.

**Conflicts of Interest:** The authors declare no conflict of interest.

**References**

1. Holm, D.; Staley, M. Nonlinear balance and exchange of stability of dynamics of solitons, peakons, ramps/cliffs and leftons in a 1 + 1 nonlinear evolutionary PDE. *Phys. Lett. A* **2003**, *308*, 437–444.
2. Holm, D.; Staley, M. Wave structure and nonlinear balance in a family of 1 + 1 evolutionary PDE’s. *SIAM J. Appl. Dyn. Syst.* 2003, 2, 323–380.
3. Xia, B.; Qiao, Z. The N-kink, bell-shape and hat-shape solitary solutions of b-family equation in the case of $b = 0$. *Phys. Lett. A* 2013, 377, 2340–2342.
4. Fokas, A.; Fuchssteiner, B. Symplectic structures, their Bäklund transformation and hereditary symmetries. *Physica D* 1981, 4, 47–66.
5. Camassa, R.; Holm, D. An integrable shallow water equation with peaked solitons. *Phys. Rev. Lett.* 1993, 71, 1661–1664.
6. Fisher, M.; Fisher, J. The Camassa-Holm equation: conserved quantities and the initial value problem. *Phys. Lett. A* 1999, 259, 371–376.
7. Constantin, A. The trajectories of particles in Stokes waves. *Invent. Math.* 2006, 166, 523–535.
8. Constantin, A.; Escher, J. Particle trajectories in solitary water waves. *Bull. Am. Math. Soc.* 2007, 44, 423–431.
9. Constantin, A.; Escher, J. Analyticity of periodic traveling free surface water waves with vorticity. *Ann. Math.* 2011, 173, 559–568.
10. Degasperis, A.; Procesi, M. Asymptotic Integrability. In *Symmetry and Perturbation Theory*; World Scientific Publishing: River Edge, NJ, USA; Rome, Italy, 1998; pp. 23–37.
11. Constantin, A.; Lannes, D. The hydrodynamical relevant of the Camassa-Holm and Degasperis-Procesi equations. *Arch. Ration. Mech. Anal.* 2009, 192, 165–186.
12. Degasperis, A.; Holm, D.; Hone, A. A new Integrable equation with peakon solutions. *Theor. Math. Phys.* 2002, 133, 1463–1474.
13. Mi, Y.; Mu, C. On the Cauchy problem for the modified Novikov equation with peakon solutions. *J. Differ. Equ.* 2013, 254, 961–982.
14. Novikov, V. Generalizations of the Camassa-Holm equation. *J. Phys. A Math. Theor.* 2009, 42, 342002.
15. Hone, W.; Wang, J. Integrable peakon equations with cubic nonlinearity. *J. Phys. A Math. Theor.* 2008, 41, 372002.
16. Hone, W.; Lundmark, H.; Szmigielski, J. Explicit multipeakon solutions of Novikov cubically nonlinear integrable Camassa-Holm type equation. *Dyn. Partial Differ. Equ.* 2009, 6, 253–289.