The cohomology ring of a tree braid group as an exterior face ring

Jesús González and Teresa Hoekstra-Mendoza

Abstract

For a tree $T$ and a positive integer $n$, let $B_nT$ denote the $n$-strand braid group on $T$. We use discrete Morse theory techniques to show that $H^*(B_nT)$ is the exterior face ring determined by an explicit simplicial complex that measures $n$-local interactions among essential vertices of $T$. In this first version of the paper we work out proof details in the case of a binary tree.

2010 Mathematics Subject Classification: 20F36, 55R80, 57M15, 57Q70.

Keywords and phrases: Tree braid group, cubical cup-product, discrete Morse theory, Farley-Sabalka gradient field.

1 Main result

For a finite graph $\Gamma$ and a positive integer $n$, let $\text{Conf}_n\Gamma$ denote the configuration space of $n$ ordered points on $\Gamma$.

$$\text{Conf}_n\Gamma := \{(x_1, \ldots, x_n) \in \Gamma^n : x_i \neq x_j \text{ for } i \neq j\}.$$ 

The usual right action of the $n$-symmetric group $\Sigma_n$ on $\text{Conf}_n\Gamma$ is given by $(x_1, \ldots, x_n) \cdot \sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$, and $\text{UConf}_n\Gamma$ stands for the corresponding orbit space, the configuration space of $n$ unlabelled points on $\Gamma$. Both $\text{Conf}_n\Gamma$ and $\text{UConf}_n\Gamma$ are known to be aspherical ([1, 10]); their corresponding fundamental groups are denoted by $\text{P}_n\Gamma$ (the pure $n$-braid group on $\Gamma$) and $B_n\Gamma$ (the full $n$-braid group or, simply, the $n$-braid group on $\Gamma$). We focus on the case of a tree $\Gamma = T$.

Besides its central role in geometric group theory, graph braid groups have promising applications in areas outside pure mathematics such as robotics, topological quantum computing and data science. Yet, there is a relatively limited knowledge of the algebraic topology of graph braid groups, particularly concerning its cohomology ring structure.

Using discrete Morse theory techniques on Abrams’ cubical model $\text{UD}_nT$ for $\text{UConf}_nT$ (reviewed below), D. Farley gave in [4] an efficient description of the additive structure of the cohomology of $B_nT$. Later, and in order to get at the multiplicative structure, the Morse theoretic methods were replaced in [5] by the use of a Salvetti complex $\mathcal{S}$ obtained by identifying opposite faces in $\text{UD}_nT$. Being a union of tori, $\mathcal{S}$ has a well understood cohomology ring. Yet more importantly, the projection map $q: \text{UD}_nT \to \mathcal{S}$ induces a surjection in cohomology. Farley’s main result in [5] is a description of a set of generators for $\text{Ker}(q^*)$, which yields a presentation for the cohomology ring of $B_nT$.

Although [5] includes an algorithm for performing computations mod $\text{Ker}(q^*)$, the price of not working at the Morse theoretic level is that Farley’s presentation includes many non-essential generators. As a result, calculations are hard to work with, both in concrete examples, as well as in theoretical developments (cf. Remark 1.7 below). In particular, Farley’s conjecture that $H^*(B_nT)$ is an exterior face ring, suggested on the basis of extensive concrete calculations, was left open.

We combine Farley’s original Morse theoretic approach with Forman’s Morse-theoretic description of cup products to prove Farley’s conjecture:
Theorem 1.1. Let $T$ be a tree. For a commutative ring $R$ with 1, the cohomology ring $H^*(B_nT; R)$ is the exterior face ring $\Lambda_R(K_nT)$ determined by a simplicial complex $K_nT$. Explicitly, $H^*(B_nT; R)$ is the quotient $\Lambda/I$, where $\Lambda$ is the exterior graded $R$-algebra generated by the vertex set of $K_nT$, and $I$ is the ideal generated by monomials corresponding to non-faces of $K_nT$.

We refer to $K_nT$ as the $n$-interaction complex of $T$. As noted in [7, p. 68], its isomorphism type is well determined. A more explicit statement of Theorem 1.1 is given in Theorem 3.3 below.

Theorem 1.1 dismisses an apparent belief that the cohomology of $B_nT$ has a complicated and delicate product structure. It is indeed a combinatorially rich algebraic object, yet calculations are fully accessible because the complex $K_nT$ is described in very concrete terms (see below). For instance, we can easily deduce a right-angled Artin group presentation for $B_nT$ when $T$ is a linear tree (Example 1.6 below). This complements the inductive method in [3] proving that linearity is a sufficient condition for a tree to have right-angled Artin braid groups.

Remark 1.2. Ghrist’s pioneering work led to conjecture that any pure braid group on a graph would be a right-angled Artin group. In the case of full braid groups, there is a satisfactory combinatorial characterization of the right-angled-Artin condition: In short, the cohomology ring should be the exterior face ring of a flag complex. Theorem 1.1 asserts that, in the full braid group setting (and for trees), Ghrist’s conjecture is true after removal of the flag condition.

In addition to potential applications of Theorem 1.1, we believe that the techniques developed in this work (discrete Morse theoretic approach to cup products) will find additional uses in other discrete models, such as those for non-particle configuration spaces, as well as those for generalized (e.g., no-$k$-equal) configuration spaces.

This initial version of the paper is devoted to the case of binary trees, i.e., trees all whose vertices have degree at most three, assumption that will be in force from now on. The case of an arbitrary tree is essentially no harder and is treated in version two of this paper. The value of this initial version of our work is that a reader might find illuminating reading the binary case first.

![Figure 1: The three $x$-directions from an essential vertex $x$](image)

A description of the complex $K_nT$ requires some preparatory constructions. Fix once and for all a planar embedding together with a root (a vertex of degree 1) for $T$. Order the vertices of $T$ as they are first encountered through the walk along the tree that (a) starts at the root vertex, which is assigned the ordinal 0, and that (b) takes the left-most branch at each intersection given by an essential vertex (turning around when reaching a vertex of degree 1). Vertices of $T$ will be denoted by the assigned non-negative integer. An edge of $T$, say with endpoints $r$ and $s$, will be denoted by the ordered pair $(r, s)$, where $r < s$. Furthermore, the ordering of vertices will be transferred to edges by declaring that the ordinal of $(r, s)$ is $s$. The resulting ordering of vertices and edges will be referred as the $T$-order. There are three ‘directions’ at each essential vertex $x$ of $T$: the one that leads to the root is defined to be the $x$-direction 0; $x$-directions 1 and 2 are then chosen following the positive orientation coming from the planar embedding. See Figure 1. For instance, the vertex

\[\text{The condition is known to be necessary and sufficient.}\]
incident to \( x \) in \( x \)-direction 0 is \( x - 1 \), while \( x + 1 \) is the vertex incident to \( x \) in \( x \)-direction 1. The vertex incident to \( x \) in \( x \)-direction 2 will be denoted by \( \overline{x} \).

Fix essential vertices \( x_1 < \cdots < x_m \) of \( T \). The complement in \( T \) of \( \{x_1, \ldots, x_m\} \) breaks into \( 2m + 1 \) components \( C_{i,\varepsilon} \) \((i \in \{0,1,\ldots,m\} \text{ and } \varepsilon \in \{1,2\}, \text{ with } \varepsilon = 1 \text{ if } i = 0)\), the closure of each of which is a subtree of \( T \). \( C_{0,1} \) is the component containing the root 0, while \( C_{i,\varepsilon} \) \((i \in \{1,\ldots,m\} \text{ and } \varepsilon \in \{1,2\}) \) is the component whose closure has \( x_i \) and is located on the \( x_i \)-direction \( \varepsilon \). The set \( B(C_{i,\varepsilon}) \) of “bounding” vertices of a component \( C_{i,\varepsilon} \) is the intersection of the closure of \( C_{i,\varepsilon} \) with \( \{x_1, \ldots, x_m\} \).

**Definition 1.3** (The \( n \)-interaction complex of \( T \)). Consider the simplicial complex \( K_nT \) whose vertices are the 4-tuples \( \langle r,x,p_1,p_2 \rangle \), where \( x \) is an essential vertex of \( T \), and \( r,p_1,p_2 \) are non-negative integer numbers satisfying \( r + p_1 + p_2 = n - 1 \) and \( p_1 > 0 \), and where a set of \( m \) different vertices \( \{\langle r_1,x_1,p_{1,1},p_{1,2} \rangle, \ldots, \langle r_m,x_m,p_{m,1},p_{m,2} \rangle\} \) forms a simplex of \( K_nT \) provided \( x_i \neq x_j \) for \( i \neq j \) and, for each component \( C_{i,\varepsilon} \),

\[
\sum_{j=1}^{m} \ell_{C_{i,\varepsilon}}(\langle r_j,x_j,p_{j,1},p_{j,2} \rangle) \geq n \left( \text{card}(B(C_{i,\varepsilon})) - 1 \right),
\]

with a strict inequality for \( i > 0 \) and \( \varepsilon = 1 \). Here we set

\[
\ell_{C_{i,\varepsilon}}(\langle r_j,x_j,p_{j,1},p_{j,2} \rangle) = \begin{cases} p_{j,\varepsilon}, & \text{if } i = j; \\ r_j, & \text{if } i \neq j \text{ and } x_j \text{ lies in the closure of } C_{i,\varepsilon}; \\ 0, & \text{otherwise}. \end{cases}
\]

Definition 1.3 is dictated by discrete Morse theoretic considerations — reviewed in latter sections of the paper. Our choice for using angle brackets instead of parenthesis for tuples will be clarified later in the paper (Remark 3.4). For now, let us explain the role of \( K_nT \) for measuring interactions of essential vertices of \( T \).

![Figure 2: The local information given by a vertex \( \langle r,x,p_1,p_2 \rangle \) of \( K_nT \)](image)

We refer to a vertex \( \langle r,x,p_1,p_2 \rangle \) of \( K_nT \) as a “local information” around the essential vertex \( x \). Indeed, \( r \) is the local information in \( x \)-direction 0, while \( p_{\varepsilon} \) is the local information in \( x \)-direction \( \varepsilon \). See Figure 2. For instance, \( \langle r \rangle \) gives a way to spell out the information ingredients on a given local information. In these terms, \( \langle r \rangle \) means that the local informations around \( m \) different essential vertices \( x_i \) of \( T \) have a ‘simplex interaction’, i.e., they form a simplex of the \( n \)-interaction complex \( K_nT \), provided the sum of the local informations of vertices bounding each component is suitably large (depending on the number of bounding vertices for each component.) Thus, Theorem 1.1 asserts that cup squares vanish in \( H^*(B_nT;R) \), that a basis for \( H^1(B_nT;R) \) is given by the set of local informations \( \langle r,x,p_1,p_2 \rangle \), and that products \( \langle r_1,x_1,p_{1,1},p_{1,2} \rangle \cdots \langle r_m,x_m,p_{m,1},p_{m,2} \rangle \) that are ordered (i.e., with \( x_1 < \cdots < x_m \)) satisfy the following properties:

(i) An ordered cup product of local informations is non-zero if and only if the factors have a simplex interaction (this requires that the relevant essential vertices be pairwise different).
Two non-zero ordered cup products of local informations agree if any only if they have the same set of factors.

Non-zero ordered cup products of local informations give a graded basis of $H^*(B_nT; R)$.

In particular, linear combinations of non-zero cup products of local informations are formal sums, which makes it as easy as possible to work with them. See Remark 1.7 below.

**Remark 1.4.** It is a simple arithmetic exercise to check that $K_nT$ is a simplicial complex. At any rate, such a fact is contained in Theorem 1.1, in view of item (ii) above.

**Example 1.5.** Figure 3 shows three aspects of the smallest possible non-linear tree $T_0$. The four essential vertices are labelled in the central picture. The non-triviality of the 4-fold product $\langle 0, x_1, 1, 7 \rangle \langle 2, x_2, 4, 2 \rangle \langle 6, x_3, 1, 1 \rangle \langle 7, x_4, 1, 0 \rangle \in H^4(B_9T_0; R)$ follows by inspecting the three non-trivial interactions in the picture on the right. Likewise, interaction analysis in the picture on the left exhibits the well known fact that $K_4T_0$ is not flag ($B_4T$ is not a right-angled Artin group): the basis elements $\langle 0, x_1, 1, 2 \rangle, \langle 2, x_3, 1, 0 \rangle, \langle 2, x_4, 1, 0 \rangle \in H^1(B_4T_0; R)$ have pairwise non-trivial products, but their triple product vanishes.

**Example 1.6.** Let $T$ be a linear binary tree. Choosing the planar embedding shown in Figure 4 we see that $B_nT$ has a right-angled Artin group presentation with generators $\langle r, v, p_1, p_2 \rangle$, where $v$ is an essential vertex of $T$, $r, p_1, p_2$ are non-negative integer numbers with $p_1 > 0$, $r + p_1 + p_2 = n - 1$, and with a commutativity relation $\langle r, v, p_1, p_2 \rangle \langle r', v', p'_1, p'_2 \rangle = \langle r', v', p'_1, p'_2 \rangle \langle r, v, p_1, p_2 \rangle$ whenever $v < v'$ and $p_2 + r \leq n$.

**Remark 1.7.** The exterior face ring description allows us to recover and generalize in a simple way Scheirer’s main technical tool [14, Lemma 3.6] for studying Farber’s topological complexity of $B_nT$. Extensions of Scheirer’s results will be the topic of a future publication.

In the rest of the paper we will omit writing the coefficient ring $R$ in cohomology groups and associated (co)chain complexes.
2 Preliminaries

We start by collecting the ingredients and facts we need: cup-products in the cubical setting ([11, 12]), Forman’s discrete Morse theory ([8, 9]), and Farley-Sabalka’s gradient field on Abrams’ discrete model for (ordered and unordered) graph configuration spaces ([1, 2, 6, 13]). We also set the notation we use in the rest of the paper.

2.1 Cup products in cubical sets

An elementary cube in \( \mathbb{R}^k \) is a cartesian product \( c = I_1 \times \cdots \times I_k \) of intervals \( I_i = [m_i, m_i + \epsilon_i] \), where \( m_i \in \mathbb{Z} \) and \( \epsilon_i \in \{0,1\} \). For simplicity, we write \( [m] := [m, m] \) for a degenerate interval.

The standard product orientation of \( c \) is determined by (a) the orientation (from smaller to larger endpoints) of the non-degenerate intervals \( I_{i_1}, \ldots, I_{i_\ell} \) of \( c \), and (b) the order \( i_1 < \cdots < i_\ell \), i.e., the order of factors in the cartesian product. Under these conditions, and for \( 1 \leq r \leq \ell \), set

\[
\begin{align*}
\delta_{2r}(c) &= I_1 \times \cdots \times I_{i_{r-1}} \times [m_{i_r} + 1] \times I_{i_{r+1}} \times \cdots \times I_k, \\
\delta_{2r-1}(c) &= I_1 \times \cdots \times I_{i_{r-1}} \times [m_{i_r}] \times I_{i_{r+1}} \times \cdots \times I_k.
\end{align*}
\]

Then, for a cubical set \( X \subset \mathbb{R}^k \), i.e., a union of elementary cubes in \( \mathbb{R}^k \), the boundary map in the oriented cubical chain complex \( C_*(X) \) is determined by

\[
\partial(c) = \sum_{r=1}^\ell (-1)^{r-1} \left( \delta_{2r}(c) - \delta_{2r-1}(c) \right).
\]

Cup products in cubical cohomology are fairly similar to their classic simplicial counterparts. At the oriented cubical cochain level, there is a cup product graded map \( C^*(X) \times C^*(X) \to C^*(X) \) that is associative, \( R \)-bilinear and is described on basis elements as follows. Firstly, for intervals \( [a, b] \) and \( [a', b'] \), let

\[
[a, b] \cdot [a', b'] := \begin{cases} [a, b'], & \text{if } b = a' \text{ and either } a = b \text{ or } a' = b' \text{ (or both)}; \\
0, & \text{otherwise.}
\end{cases}
\]

Then, for elementary cubes \( c = I_1 \times \cdots \times I_k \) and \( d = J_1 \times \cdots \times J_k \) in \( X \), the cubical cup product \( c \cdot d \) of the corresponding basis element(s)\(^2\) in \( C^*(X) \) vanishes if either \( I_i \cdot J_i = 0 \) for some \( i \in \{1, \ldots, k\} \) or, else, if \( (I_1 \cdot J_1) \times \cdots \times (I_k \cdot J_k) \) is not a cube in \( X \); otherwise \( c \cdot d \) is, up to a sign \( \epsilon_{c,d} \), the dual

\(^2\)We shall omit the use of an asterisk for dual elements. The intended meaning will be clear from the context.
of the cube \((I_1 \cdot J_1) \times \cdots \times (I_k \cdot J_k)\). Given our product-orientation settings, the sign is given by the usual algebraic-topology convention:

\[
\epsilon_{c,d} = \sum_{j=1}^{k-1} \left( \dim J_j \sum_{i=j+1}^{k} \dim I_i \right).
\]

Particularly agreeable is the fact that a finite cartesian product of cubical sets comes equipped for free with the obvious structure of a cubical set. For instance, let \(T\) be a tree whose vertices and edges have been ordered as described in the previous section. Orient the edges of \(T\) from the smaller to the larger endpoint, and fix an orientation-preserving embedding \(T \subset \mathbb{R}^t\) of cubical sets, where cubes in \(\mathbb{R}^t\) have product orientation. For instance, an oriented edge in \(T\) corresponds in \(\mathbb{R}^t\) to an oriented elementary cube \(I_1 \times \cdots \times I_t\) where all but one of the intervals are degenerate. Then, the cartesian power \(T^n\) is a (product-oriented) cubical set in \(\mathbb{R}^{tn}\). In this setting, an oriented cube \(c = c_1 \times \cdots \times c_n\) in \(T^n\) (where each \(c_i\) is either a vertex or an edge of \(T\)) corresponds in \(\mathbb{R}^{tn}\) to an oriented cube \((I_{1,1} \times \cdots \times I_{1,t}) \times \cdots \times (I_{n,1} \times \cdots \times I_{n,t})\) where, for each \(i = 1, \ldots, n\), at most one of the intervals \(I_{i,1}, \ldots, I_{i,t}\) is non-degenerate. These considerations, coupled with the fact that cubes of a single factor \(T\) are at most one-dimensional, yield an explicit description of cubical cup-products associated to \(T\) and \(T^n\).

**Proposition 2.1.** The cup product in \(C^*(T)\) of the duals of a pair of (oriented) cubes \(c\) and \(d\) in \(T\) is given by the dual of

\[
c \cdot d = \begin{cases} (x, y), & \text{if } c = (x, y), \text{ an edge of } T, \text{ and } d = y, \text{ a vertex of } T; \\ (x, y), & \text{if } c = x, \text{ a vertex of } T, \text{ and } d = (x, y), \text{ an edge of } T; \\ x, & \text{if } c = d = x, \text{ a vertex of } T; \\ 0, & \text{otherwise}. \end{cases}
\]

More generally, let \(D\) be a (product-oriented) cubical subset of \(T^n\). The cup product in \(C^*(D)\) of the duals of a pair of cubes \(c = c_1 \times \cdots \times c_n\) and \(d = d_1 \times \cdots \times d_n\) in \(D\) vanishes provided \(c_i \cdot d_i = 0\) for some \(i \in \{1, \ldots, n\}\) or, else, provided the cube \(c \cdot d := (c_1 \cdot d_1) \times \cdots \times (c_n \cdot d_n)\) is not contained in \(D\). Otherwise, the cup product is the multiple \((-1)^{\epsilon_{c,d}}\) of the dual of \(c \cdot d\), where

\[
\epsilon_{c,d} = \sum_{j=1}^{n-1} \left( \dim(d_j) \sum_{i=j+1}^{n} \dim(c_i) \right).
\]

### 2.2 Discrete Morse theory

Let \(X\) denote a finite regular cell complex with face poset \((\mathcal{F}, \subset)\), i.e., \(\mathcal{F}\) is the set of (closed) cells of \(X\) partially ordered by inclusion. For a cell \(a \in \mathcal{F}\), we sometimes write \(a^{(p)}\) to indicate that \(a\) is \(p\)-dimensional. We think of the Hasse diagram \(H_\mathcal{F}\) of \(\mathcal{F}\) as a directed graph: it has vertex set \(\mathcal{F}\), while directed edges (called also “arrows”) are given by the family of ordered pairs \((a^{(p+1)}, b^{(p)})\) with \(b \subset a\). Such an arrow will also be denoted as \(a^{(p+1)} \searrow b^{(p)}\). Let \(W\) be a partial matching on \(H_\mathcal{F}\), i.e., a directed subgraph of \(H_\mathcal{F}\) whose vertices have degree at most 1. The modified Hasse diagram \(H_\mathcal{F}(W)\) is the directed graph obtained from \(H_\mathcal{F}\) by reversing all arrows of \(W\). A reversed edge is denoted as \(b^{(p)} \nearrow a^{(p+1)}\), in which case \(a\) is said to be \(W\)-collapsible and \(b\) is said to be \(W\)-redundant.

Discrete Morse theory focuses on gradient paths, i.e., directed paths in \(H_\mathcal{F}(W)\) given by an alternate chain of up-going and down-going arrows. Gradient paths of the form

\[
a_0 \nearrow b_1 \searrow a_1 \nearrow \cdots \nearrow b_k \searrow a_k \quad \text{and} \quad c_0 \searrow d_1 \nearrow c_1 \searrow \cdots \searrow d_k \nearrow c_k
\]  

(5)
are called upper and lower paths, respectively, and the gradient path is called elementary when \( k = 1 \), or constant when \( k = 0 \). The sets of upper and lower paths that start on a \( p \)-cell \( a \) and end on a \( p \)-cell \( b \) are denoted by \( \Gamma(a, b) \) and \( \Gamma(b, a) \), respectively. Concatenation of upper/lower paths \( \Gamma(a, b) \times \Gamma(b, c) \rightarrow \Gamma(a, c) \) and \( \Gamma(b, a) \times \Gamma(b, c) \rightarrow \Gamma(a, c) \) is defined in the obvious way; for instance, any upper/lower path is a concatenation of corresponding elementary paths. An upper/lower path is then defined to be the graded \( R \)-module generated in dimension \( p \geq 0 \) by the dual of the oriented critical cells \( A^{(p)} \) of \( X \). The matching \( W \) is said to be a gradient field on \( X \) if \( H_\mathcal{F}(W) \) has no cycles. In such a case, cells of \( X \) that are neither \( W \)-redundant nor \( W \)-collapsible are said to be \( W \)-critical or, simply, critical when \( W \) is clear from the context. We follow Forman’s convention to use capital letters to denote critical cells.

It is well known that a gradient field on \( X \) carries all the homotopy information of \( X \). For our purposes, we only need to recall the way that gradient paths recover (co)homological information.

In the rest of the section we assume \( W \) is a gradient field on \( X \).

Start by fixing an orientation on each cell of \( X \) and, for cells \( a^{(p)} \subset b^{(p+1)} \), consider the incidence number \( \iota_{a,b} \) of \( a \) and \( b \), i.e., the coefficient (\( \pm 1 \), since \( X \) is regular) of \( a \) in the expression of \( \partial(b) \). Here \( \partial \) is the boundary operator in the cellular chain complex \( C_*(X) \). The Morse cochain complex \( \mathcal{M}^*(X) \) is then defined to be the graded \( \mathbb{R} \)-free module generated in dimension \( p \geq 0 \) by the dual of the oriented critical cells \( A^{(p)} \) of \( X \). The definition of the Morse coboundary map in \( \mathcal{M}^*(X) \) requires the concept of multiplicity of upper/lower paths. In the elementary case, multiplicity is given by

\[
\mu(a_0 \mapsto b_1 \searrow a_1) = -\iota_{a_0,b_1} \cdot \iota_{a_1,b_1} \quad \text{and} \quad \mu(c_0 \nwarrow d_1 \mapsto c_1) = -\iota_{d_1,c_0} \cdot \iota_{d_1,c_1},
\]

and, in the general case, it is defined to be a multiplicative function with respect to concatenation of elementary paths. The Morse coboundary is then defined by

\[
\partial(A^{(p)}) = \sum_{B^{(p+1)}} \left( \sum_{b^{(p)} \subset B} \iota_{b,B} \sum_{\gamma \in \Gamma(b,A)} \mu(\gamma) \right) \cdot B.
\]

In other words, the Morse theoretic incidence number of \( A \) and \( B \) is given by the number of upper paths, taking into account multiplicity, from oriented faces of \( B \) to \( A \).

Gradient paths give, in addition, a bridge between \( \mathcal{M}^*(X) \) and the usual cellular cochain complex \( C^*(X) \). Indeed, the formulæ

\[
\Phi(A^{(p)}) = \sum_{a^{(p)}} \left( \sum_{\gamma \in \Gamma(a,A)} \mu(\gamma) \right) a \quad \text{and} \quad \Phi(a^{(p)}) = \sum_{A^{(p)}} \left( \sum_{\gamma \in \Gamma(A,a)} \mu(\gamma) \right) A
\]

define (on generators) cochain maps \( \Phi: \mathcal{M}^*(X) \rightarrow C^*(X) \) and \( \Phi: C^*(X) \rightarrow \mathcal{M}^*(X) \) inducing cohomology isomorphisms \( \Phi^* \) and \( \Phi^* \) with \( (\Phi^*)^{-1} = \Phi^* \).

2.3 Abrams discrete model and Farley-Sabalka’s gradient field

Abrams discrete model for \( \text{Conf}_n T \) is the largest cubical subset \( D_n T \) of \( T^n \) inside \( \text{Conf}_n T \). In other words, \( D_n T \) is obtained by removing open cubes from \( T^n \) whose closure intersect the fat diagonal. As usual, the symmetric group \( \Sigma_n \) acts on the right of \( D_n T \) by permuting factors. The action permutes in fact cubes, and the quotient complex is denoted by \( UD_n T \). Following Farley-Sabalka’s lead, from

---

3Cochain coefficients are taken in ring \( \mathbb{R} \) since we are interested in cup-products.

4Recall we omit the use of an asterisk for dual elements.
now on we use the notation \((a_1, \ldots, a_n)\), and even \((a)\), for a cube \(a_1 \times \cdots \times a_n\) in \(T^n\) (so each \(a_i\) is either a vertex or an open edge of \(T\)), and the notation \(\{a_1, \ldots, a_n\}\), and even \(\{a\}\), for the corresponding \(\Sigma_n\)-orbit. Beware not to confuse the parenthesis notation with a point of \(T^n\), or the braces notation with a set of elements of \(T\) — even if all the \(a_i\)'s are vertices. The coordinates \(a_i\) in a cube \((a)\) of \(D_nT\), or in its \(\Sigma_n\)-orbit \(\{a\}\), are referred to as the ingredients of the cube. Closures of ingredients are therefore pairwise disjoint.

\(D_nT\) is a \(\Sigma_n\)-equivariant strong deformation retract of \(\text{Conf}_n T\) provided the following two conditions hold:

1. Each path in \(T\) between distinct vertices of degree not equal to 2 passes through at least \(n - 1\) edges.
2. Each vertex-based loop in \(T\) whose only repeated vertices are the initial and final ones passes through at least \(n + 1\) edges.

A tree satisfying these conditions is said to be \(n\)-sufficiently subdivided, condition that we assume throughout the paper. This is not a real restriction as \(T\) can be subdivided as needed without altering the homeomorphism type of its configuration spaces. The \(\Sigma_n\)-equivariance of the strong deformation retraction above implies that \(\text{UD}_n T\) is a strong deformation retract of \(\text{UConf}_n T\). Consequently, we will switch attention from \(\text{Conf}_n T\) and \(\text{UConf}_n T\) to their homotopy equivalent discrete models \(D_n T\) and \(\text{UD}_n T\).

For a vertex \(x\) of \(T\) different from the root 0, let \(e_x\) be the unique edge of \(T\) of the form \((y, x)\) — recall this requires \(y < x\). A vertex-ingredient \(x\) of a cube \(c\) (either in \(D_n T\) or \(\text{UD}_n T\)) is said to be blocked in \(c\) if \(x = 0\) or, else, if replacing in \(c\) the ingredient \(x\) by the edge \(e_x\) fails to render a cube in the corresponding discrete model; \(x\) is said to be unblocked in \(c\) otherwise. An edge-ingredient \(e\) of a cube \(c\) is said to be order-disrespectful in \(c\) provided \(e\) is of the form \((x, \overline{x})\) and \(x + 1\) is an ingredient of \(c\); \(e\) is said to be order-respecting in \(c\) otherwise. Blocked vertex-ingredients and order-disrespectful edge ingredients in \(c\) are said to be critical. Farley-Sabalka’s gradient field (on \(D_n T\) and \(\text{UD}_n T\)) then works as follows. Order the ingredients of a cube \(c\) by the \(T\)-ordering (as described in Section \(\S 1\)), and look for non-critical ingredients:

\(i\) If the first such ingredient is an unblocked vertex \(y\) in \(c\), then \(c\) is redundant, and one sets \(c \mapsto c'\), where \(c'\) is the cube obtained from \(c\) by replacing \(y\) by \(e_y\). We say that the pairing \(c \mapsto c'\) creates the edge \(e_y\). In this case \(e_y\) is an order-respecting edge of \(c'\), and all ingredients of \(c'\) smaller than \(e_y\) are critical.

\(ii\) If the first such ingredient is an order-respecting edge \((w, z)\), then \(c\) is collapsible, and one sets \(c'' \mapsto c\), where \(c''\) is the cube obtained from \(c\) by replacing \((w, z)\) by \(z\). Again, we say that the edge \((w, z)\) is created by the pairing \(c'' \mapsto c\). In this case \(z\) is an unblocked vertex of \(c''\), and all ingredients of \(c''\) smaller than \(e_z\) are critical.

\(iii\) If all ingredients of \(c\) are critical, then \(c\) is critical.

The ingredients of a critical \(m\)-dimensional cube are, therefore, given by:

\(a\) \(m\) edges \((x_i, \overline{x}_i), i = 1, \ldots, m,\)

\(b\) a (possibly empty) stack of \(r\) vertices \(0, 1, \ldots,\), and

\(c\) for each \(i \in \{1, \ldots, m\}\), two stacks of vertices \(x_i + 1, \ldots, x_i + p_{i, 1}\) and \(\overline{x}_i + 1, \ldots, \overline{x}_i + p_{i, 2}\), where the latter stack can be empty, but the former one cannot.
Figure 6: Ingredients in the critical 3-cell \( \{3|x_1,0|x_2,2,0|x_3,1,1\} \) for the tree shown

The critical cube \( c \) of the unordered discrete model UD\(_n\)T determined by the above information will be denoted as

\[
c = \{r | x_1, p_{i,1}, p_{i,2} | \cdots | x_m, p_{m,1}, p_{m,2} \}.
\] (9)

Vertical bars are meant to stress the fact that each pair of parameters \( p_{i,1} \) and \( p_{i,2} \) are ordered and attached to \( x_i \). Other than that, \( c \) is indeed a set formed by the triples \((x_i, p_{i,1}, p_{i,2})\) and the singleton \( r \). Figure 6 illustrates a typical critical cube.

**Remark 2.2.** In any arrow \( c \rightarrow d \) of Farley-Sabalka’s modified Hasse diagram, \( c \) is an even face of \( d \), i.e., in the notation of (3), \( c = \delta_{2r}(d) \) for some \( r \in \{1,2,\ldots,\dim(d)\} \).

**Remark 2.3.** By construction, Farley-Sabalka’s gradient field in \( D_nT \) is \( \Sigma_n \)-equivariant and, by passing to the quotient, it yields the corresponding gradient field in UD\(_n\)T. Consequently, gradient paths can equivalently be analyzed in either the order or unordered settings. Indeed, a gradient path in UD\(_n\)T corresponds to a “\( \Sigma_n \)-orbit” of gradient paths in \( D_nT \). We find it more convenient to perform the gradient path analysis in \( D_nT \).

### 3 Gradient-path dynamics

Recall that the product orientation of a \( p \)-dimensional cube \((c_1,\ldots,c_n)\) in \( D_nT \) depends on (the orientation of edges in \( T \) and on) the order \( c_{i_1},\ldots,c_{i_p}, \) \( i_1 < \cdots < i_p \), of the edge-ingredients. In particular, the quotient cube \( \{c_1,\ldots,c_n\} \) in UD\(_n\)T inherits no well defined orientation. The following definition avoids the problem and is well suited for the analysis of gradient paths in \( D_nT \).

**Definition 3.1** (Gradient orientation). We say that the edge-ingredients of a cube \( c^{(p)} \) in \( D_nT \) or UD\(_n\)T are listed in gradient order \((x_1,y_1),\ldots,(x_p,y_p)\) if \( x_1 < \cdots < x_p \), where the latter is the \( T \)-ordering discussed in Section 1. The gradient orientation of \( c \) is defined just as the product orientation, except that the gradient order of the edge-ingredients is used.

In what follows, and unless explicitly noted otherwise, we use gradient orientations. In doing so, the definitions of the cubes \( \delta_{2r}(c) \) and \( \delta_{2r-1}(c) \) in (3) require a corresponding adjustment. Namely, if the edge-ingredients of a \( p \)-cube \( c \) are listed in gradient order as \((x_1,y_1),\ldots,(x_p,y_p)\), then replacing the edge \((x_r,y_r)\) by the vertex \( y_r \) or \( x_r \) yields \( \delta_{2r}(c) \) or \( \delta_{2r-1}(c) \), respectively. Remark 2.2 and the expression in (4) for cubical boundaries then remain unaltered. A first advantage of gradient
orientations is that the map induced at the cochain level by the canonical projection \( \pi: D_n T \to UD_n T \) takes the simple form

\[
\pi^*(\{c\}) = \sum_{\sigma \in \Sigma_n} (c) \cdot \sigma.
\]  

(10)

(Recall we omit asterisks from element duals.) Remark 2.3 then yields:

**Lemma 3.2.** The following diagram is commutative:

\[
\begin{array}{ccc}
\mathcal{M}^*(D_n T) & \xrightarrow{\pi^*} & C^*(D_n T) \\
\mathcal{M}^*(UD_n T) & \xrightarrow{\pi} & C^*(UD_n T)
\end{array}
\]

The Morse differential in \( UD_n T \) is known to be trivial (see [4] or Remark 3.12 below). Therefore, for each \( m \geq 0 \), a graded basis of \( H^m(UD_n T) \) is given by the cohomology classes of the \( \Phi \)-images of the duals of the critical cubes \( \mathbf{9} \). By abuse of notation\(^5\), the \( \pi^* \)-image of the cohomology class so determined will also be denoted by the corresponding expression \( \mathbf{9} \). There is no loss of information because vertical maps in the previous diagram are injective and, more importantly, they induce injections in cohomology (the latter assertion follows from a standard transfer argument and the torsion-freeness of \( H^*(UD_n T) \)). Therefore, Theorem 1.1 is equivalent\(^6\) to the more precise statement:

**Theorem 3.3.** The image \( \text{Im}(\pi^*) \) of the ring monomorphism \( \pi^*: H^*(UD_n T) \hookrightarrow H^*(D_n T) \) is the exterior face ring \( \Lambda R(K_n T) \), where the local-information vertices in Section 1 are given by the 1-dimensional cohomology classes

\[
\langle r, x, p_1, p_2 \rangle := \sum_{i=0}^{r} \{ r - i \mid x, p_1 + i, p_2 \} \in \text{Im}(\pi^*).
\]  

(11)

**Remark 3.4.** We use an angle-bracket notation in \( \mathbf{11} \) since we have reserved the parenthesis notation for cubes in \( D_n T \) (as tuples of their ingredients). Additionally, the angle-bracket notation is intended to stress the change of basis in \( \mathbf{11} \), together with the local-information nature of vertices in \( K_n T \) discussed in Section \( \mathbf{1} \).

It is transparent that the elements \( \mathbf{11} \) yield a basis of \( H^1(UD_n T) \); the crux of the matter in Theorem 3.3 is the assertion about the graded basis of face-ring type for \( H^*(UD_n T) \). As a first step, we need an explicit description of a cocycle in \( C^*(D_n T) \) that represents a given cohomology class \( \{ r \mid x, p_1, p_2 \} \in \text{Im}(\pi^*) \) (Proposition 3.11 below). This requires the following discussion of the dynamics of upper-paths to critical cubes.

**Definition 3.5.** An edge-ingredient \((x, y)\) of a cube \( C \) of \( D_n T \) is said to be

- edge order-respecting in \( C \), written as “\((x, y)\) is eor(\(C\))”, if there are no edge-ingredients \((a, b)\) in \( C \) with \( x < a < b < y \).

- strongly order-respecting in \( C \), written as “\((x, y)\) is sor(\(C\))”, if \((x, y)\) is eor(\(C\)) and there is no vertex-ingredient \( v \) in \( C \) with \( x < v < y \).
Examples 3.6. Any edge-ingredient \((x, x + 1)\) of \(c\) is sor(c). On the other hand, an edge-ingredient \((x, x)\) of \(c\) (with \(x\) an essential vertex) is sor(c) if and only if \(c\) has no ingredient (neither vertex nor edge) in the component of \(T \setminus \{x\}\) that lies in \(x\)-direction 1. Furthermore, if \((x, y)\) is an edge-ingredient of a face \(\delta_j(c)\) of some cube \(c\) of \(D_nT\), then \((x, y)\) is sor(\(\delta_j(c)\)) if and only if \((x, y)\) is sor(c). The latter observation is freely used in the proof of:

Proposition 3.7. 1. If an arrow \(\delta_2i(c) \nearrow c\) in the modified Hasse diagram for \(D_nT\) creates an edge-ingredient that is sor(c) then, for any \(k > 2i\), \(\delta_k(c)\) is collapsible.

2. Let \((x_1, y_1), \ldots, (x_p, y_p)\) be the gradient-order listing of the edge-ingredients of a \(p\)-cube \(c\) in \(D_nT\). If the edge \((x_i, y_i)\) is sor(c), then there is no upper path starting at a face \(\delta_j(c)\) with \(j < 2i - 1\) and ending on a critical cube.

Proof. 1. Say \(\delta_2i(c) \nearrow c\) creates the edge-ingredient \((x, y)\) that is sor(c). Since ingredients of \(\delta_2i(c)\) smaller than \(y\) are critical, \((x, y)\) is in fact sor(c). Thus, for \(k \neq 2i, 2i - 1\), \((x, y)\) is sor(\(\delta_k(x)\)) and, therefore, order-respecting in \(\delta_k(x)\). On the other hand, for \(k > 2i\), \(\delta_k(c)\) and \(c\) have the same ingredients smaller than \(y\), so that all ingredients in \(\delta_k(c)\) smaller than \((x, y)\) are critical. Thus, by definition, \(\delta_k(x)\) is collapsible for \(k > 2i\).

2. Under the stated hypothesis, assume (for a contradiction) there is a gradient path

\[
c \downarrow \delta_j(c) =: c_0 \nearrow d_1 \nearrow c_1 \nearrow \cdots \nearrow d_m \nearrow c_m
\]

with \(j < 2i - 1\), \(m \geq 0\) and \(c_m\) critical. Then \((x_i, y_i)\) is sor(\(c_0\)) and, in particular, \((x_i, y_i)\) is order-respecting in \(c_0\), which forces \(m > 0\). Recursively, if \((x_i, y_i)\) is an edge-ingredient of both \(c_{\ell - 1}\) and \(c_{\ell}\) (and so of \(d_{\ell}\), and \((x_i, y_i)\) is sor(\(c_{\ell - 1}\)), then \((x_i, y_i)\) is forced to be sor(\(d_{\ell}\)) and, thus, sor(\(c_{\ell}\)). It is not possible that \((x_i, y_i)\) is an edge-ingredient of all the \(c_{\ell}'s\), for then \((x_i, y_i)\) would be sor(\(c_m\)), which is impossible as \(c_m\) is critical. Let \(k\) be the first integer \((1 \leq k \leq m)\) for which \((x_i, y_i)\) is not an ingredient of \(c_k\) — so that \((x_i, y_i)\) is sor(\(c_{\ell}\)) for \(0 \leq \ell < k\). In particular, \((x_i, y_i)\) is order-respecting in \(c_{k - 1}\) and, thus, the vertex-ingredient \(v\) of \(c_{k - 1}\) responsible for the pairing \(c_{k - 1} \nearrow d_k\) in (12) satisfies \(v < y_i\) and, in fact, \(v < x_i\), since \((x_i, y_i)\) is sor(\(c_{k - 1}\)). On the other hand, since the edge \((u, v)\) created by \(c_{k - 1} \nearrow d_k\) is order-respecting in \(d_k\), and since \(c_k\) is obtained from \(d_k\) by replacing the edge \((x_i, y_i)\) by either \(x_i\) or \(y_i\), the inequalities \(u < v < x_i < y_i\) yield that

\[
(u, v) \text{ is order-respecting in } c_k \text{ too.}
\]

In particular, \(c_k\) is not critical, so \(k < m\). Let \(w\) be the vertex-ingredient of \(c_k\) responsible for the pairing \(c_k \nearrow d_{k + 1}\). By (13), we get the first inequality in \(w < v < x_i < y_i\), so

\begin{itemize}
  \item \((w)\) is an ingredient of \(c_k\) \(\Rightarrow\) \((w)\) is an ingredient of \(d_k\) and, therefore, of \(c_{k - 1}\);
  \item \((w)\) is unblocked in \(c_k\) \(\Rightarrow\) \((w)\) is unblocked in \(d_k\) and, therefore, in \(c_{k - 1}\).
\end{itemize}

But, by definition, \(v\) is the minimal unblocked vertex in \(c_{k - 1}\), so \(v \leq w\), a contradiction.

Definition 3.8. A Farley-Sabalka pairing \(\delta_{2i}(c) \nearrow c\) that creates an edge-ingredient that is sor(c) is said to be of sor type; otherwise, it is said to be of branch type. An elementary path \(e = (\delta_{2i}(c) \nearrow c \downarrow \delta_j(c))\) is said to be

\begin{itemize}
  \item of falling-vertex type, if \(j = 2i + 1\);
  \item of sor type, if \(\delta_{2i}(c) \nearrow c\) is of sor type;
\end{itemize}
• of branch type, if \( \delta_{2i}(c) \not\supset c \) is of branch type.

Note that, if \( y \) is the vertex-ingredient in \( \delta_{2i}(c) \) that is responsible for a pairing \( \delta_{2i}(c) \not\supset c \), say creating the edge-ingredient \((x, y)\) of \( c \), then \( \delta_{2i-1}(c) \) is obtained from \( \delta_{2i}(c) \) by replacing the vertex \( y \) by \( x \). In other words, in the falling-vertex type path \( \delta_{2i}(c) \not\supset c \setminus \delta_{2i-1}(c) \), the vertex-ingredient \( y \) falls to its predecessor \( x \). Note that elementary paths of falling-vertex type have multiplicity 1.

Proposition 3.7 implies that upper paths ending at critical cubes have a forced behavior most of the time:

**Corollary 3.9.** Let \( \gamma \) be an upper path in \( D_n T \) that ends at a critical cube. Any elementary factor of \( \gamma \) of sor type is of falling-vertex type.

**Example 3.10.** We describe the dynamics of an upper path \( \gamma : c_0 \not\supset d_1 \setminus c_1 \not\supset \cdots \setminus c_m \) that starts at a 1-cube \( c_0 \) and ends at a critical 1-cube \( c_m \). By the \( \Sigma_n \)-equivariance of the gradient field, we can assume \( c_0 = (u_1, \ldots, u_i, v_1, \ldots, v_j, (y, z), w_1, \ldots, w_k) \) with

\[
u_1 < \cdots < u_i < y < v_1 < \cdots < v_j < z < w_1 < \cdots < w_k,
\]

i.e., \( c_0 \) is the representative of its \( \Sigma_n \)-orbit whose ingredients appear in the \( T \)-ordering. Note that there is an integer \( \ell \in \{0, 1, \ldots, k\} \) such that the simple path in \( T \) from the root 0 to \( w_i \) passes through \( z \) if and only if \( i \leq \ell \). By Corollary 3.9, the start of \( \gamma \) is forced to consist of falling-vertex elementary paths. First, the vertices \( u_1, \ldots, u_i \) fall, each at a time, to form a pile blocked by the root. At this point, we see that \( j \) must be positive, for otherwise the path would have reached a collapsible cube (in particular \( y \) must be an essential vertex and \( z = \overline{y} \)). Then it is the turn of the vertices \( v_1, \ldots, v_j \) to fall, each at a time, to form a pile blocked by \( y \). At this point \( \gamma \) reaches \( (y, \overline{y}) \) as a critical edge-ingredient. So, then, it is the turn of the vertices \( w_1, \ldots, w_\ell \) to fall, each at a time, to form a pile blocked by \( \overline{y} \). At this point, \( \gamma \) reaches the 1-cube

\[
(0, 1, \ldots, i - 1, y + 1, \ldots, y + j, (y, \overline{y}), \overline{y} + 1, \ldots, \overline{y} + \ell, w_{\ell + 1}, \ldots, w_k),
\]

and branching can hold from this point on. The explicit options are discussed next.

![Figure 7: Cube \( d_{i+1} \) in the branch-type pairing in Example 3.10 that creates the edge \((x, \overline{y})\)](image)

If \( \ell = k \), then (14) is already the critical ending cube of \( \gamma \). Otherwise, \( w_{\ell+1} \) is forced to fall until \( \gamma \) reaches, via some branch type pairing \( c_i \not\supset d_{i+1} \), the 2-cube \( d_{i+1} \) depicted in Figure 7. At this point there are two options for \( d_{i+1} \setminus c_{i+1} \). In the first option, \( c_{i+1} \) is obtained from \( d_{i+1} \) by replacing the recently created edge \((x, \overline{y})\) by \( x \), i.e., \( c_i \not\supset d_{i+1} \setminus c_{i+1} \) is of falling-vertex type. In such a case, \( \gamma \) is forced to continue with the vertex \( x \) falling until it is added to the stack of vertices blocked by the root 0. This leaves us at a situation similar to the one at the start of this paragraph. In the second option, \( c_{i+1} \) is obtained from \( d_{i+1} \) by replacing the edge \((y, \overline{y})\) by either of its end points. In such a case, \( \gamma \) is forced to continue:
1. with the falling of the vertices that are now unblocked in the neighborhood of $y$ (see Figure 7), until they form a pile of vertices blocked by $x$—thus making a critical situation around $x$—and, then,

2. with the falling of the vertices (if any) in the $x$-direction 2, which form a (possibly empty) pile of vertices blocked by $x$. Again, this leaves us at a situation similar to the one at the start of this paragraph, now with the edge $(x, x)$ playing the role of the edge $(y, y)$. The branching process in this paragraph then repeats (necessarily a finite number of times) until all the vertices $w_{k+1}, \ldots, w_k$ have been considered.

**Proposition 3.11.** A cocycle in $C^*(D_nT)$ representing the cohomology class \( \{ r, x, p_1, p_2 \} \in \text{Im}(\pi^*) \) is given by

\[
\sum (u_1, \ldots, u_r, v_1, \ldots, v_{p_1}, (x, x), w_1, \ldots, w_{p_2}) \cdot \sigma,
\]

where the summation runs over

- all permutations $\sigma \in \Sigma_n$,
- all possible vertices $u_1 < \cdots < u_r$ in the component of $T \setminus \{ x \}$ in $x$-direction 0,
- all possible vertices $v_1 < \cdots < v_{p_1}$ in the component of $T \setminus \{ x \}$ in $x$-direction 1,
- all possible vertices $w_1 < \cdots < w_{p_2}$ in the component of $T \setminus \{ x \}$ in $x$-direction 2.

**Proof.** By construction, the representing cocycle $z$ we need is obtained by chasing, on the left hand-side square of the diagram in Lemma 3.2, the dual of the (unordered) critical cube whose ordered representative is $c := (0, 1, \ldots, r - 1, x + 1, \ldots, x + p_1, (x, x), \bar{x} + 1, \ldots, \bar{x} + p_2)$. By (8) and (10),

\[
z = \Phi \circ \pi^*(z) = \sum_{\gamma \in \mathcal{G}} \mu(\gamma) \cdot S_\gamma,
\]

where $\mathcal{G}$ is the set of upper paths $\gamma$ that start at a 1-cube $S_\gamma$ and finish at a 1-cube of the form $c \cdot \sigma$ with $\sigma \in \Sigma_n$. Let $\mathcal{G}'$ be the set of paths $\gamma \in \mathcal{G}$ all whose elementary factors are of falling-vertex type. Since $\mu(\gamma) = 1$ for $\gamma \in \mathcal{G}'$, the analysis in Example 3.10 shows that the summands in (15) arise from the summands in (16) with $\gamma \in \mathcal{G}'$. It thus suffices to show

\[
\sum_{\gamma \in \mathcal{G} \setminus \mathcal{G}'} \mu(\gamma) \cdot S_\gamma = 0,
\]

which will be done by constructing an involution $\iota: \mathcal{G} \setminus \mathcal{G}' \to \mathcal{G} \setminus \mathcal{G}'$ such that every pair of paths $\gamma$ and $\iota(\gamma)$ have the same origin but opposite multiplicities, i.e.,

\[
\mathcal{S}_{\iota(\gamma)} = \mathcal{S}_\gamma \quad \text{and} \quad \mu(\iota(\gamma)) = -\mu(\gamma)
\]

—thus their contributions to (17) cancel each other out. For a path $\gamma \in \mathcal{G} \setminus \mathcal{G}'$, let $\gamma_{\text{last}} = (c \nearrow d \swarrow e)$ denote the last elementary factor of $\gamma$ that is not of falling-vertex type. In the notation of Example 3.10, $e$ is obtained from $d$ by replacing an edge $(y, \overline{y})$ by either $y$ of $\overline{y}$, and both options are possible. Then $\iota(\gamma)$ is defined so to start with the same factorization of $\gamma$ into elementary paths, except for the elementary factor $\gamma_{\text{last}}$, for which the other end-point of $(y, \overline{y})$ is taken, and after which the rest of the elementary factors are of falling-vertex type—just like for $\gamma$. Note that the ending 1-cubes of $\gamma$ and $\iota(\gamma)$ lie in the same $\Sigma_n$-orbit, so $\iota(\gamma) \in \mathcal{G} \setminus \mathcal{G}'$. The required properties (18) follow from (the construction and) the fact that elementary paths of falling-vertex type have multiplicity 1. \(\square\)
Remark 3.12. The cancelation phenomenon in the previous proof allows us to give an easy gradient-path explanation of the main result in [43]: the vanishing of the Morse differential in $U D_n T$. A variant of the cancellation phenomenon will also play an important role in our evaluation of cup products (Proposition 5.1). Thus, in preparation for that argument, we spell out the gradient proof of:

Proposition 3.13. The Morse differential in $U D_n T$ vanishes.

Proof. By Remark 2.3 we can do the gradient path analysis directly at the level of $U D_n T$. For a pair of critical cubes $c, d$ and $d^{(k-1)}$, let $\Gamma(c, d)$ be the set of gradient paths $c \searrow \nearrow \searrow \cdots \searrow d$. By (7), we only need to construct an involution $\iota: \Gamma(c, d) \to \Gamma(c, d)$ so that, for every $\gamma \in \Gamma(c, d)$, $\mu(\iota(\gamma)) = -\mu(\gamma)$. (Here, the multiplicity of $\gamma \in \Gamma(c, d)$ is the incidence number for $c \searrow \nearrow$ multiplied by the multiplicity of the the remaining upper path $\nearrow \searrow \cdots \searrow d$.) Let $\Gamma(c, d)_{\text{fall}}$ consist of the paths in $\Gamma(c, d)$ all whose elementary factors are of falling-vertex type. The definition of the restricted $\iota_{\text{fall}}: \Gamma(c, d)_{\text{fall}} \to \Gamma(c, d)_{\text{fall}}$ uses the two forms of replacing by a vertex the edge-ingredient at the start of the path. Likewise, for $\Gamma(c, d)_{\text{branch}} := \Gamma(c, d) - \Gamma(c, d)_{\text{fall}}$, the definition of the restricted $\iota_{\text{branch}}: \Gamma(c, d)_{\text{branch}} \to \Gamma(c, d)_{\text{branch}}$ uses the two forms of replacing by a vertex the edge ingredient at the last upper elementary factor that is not of falling-vertex type. \qed

4 Cup products

Propositions 2.1 and 3.11 immediately yield:

Corollary 4.1. The product of two basis elements $\{r, x, p_1, p_2\}, \{r', x', p'_1, p'_2\} \in \text{Im}(\pi^\ast)$ vanishes provided $x = x'$. In particular, squares of 1-dimensional elements in $\text{Im}(\pi^\ast)$ are trivial.

In order to analyze general products of 1-dimensional elements in $\text{Im}(\pi^\ast)$, we start by tuning up the components $C_{i, \varepsilon}$ determined by essential vertices $x_1 < \cdots < x_m$ (Section 1). Set $T_{0,1} := C_{0,1}$ and, for $1 \leq i \leq m$, $T_{i,1} := C_{i,1} \cup \{x_i\}$ and $T_{i,2} := C_{i,2} \setminus \text{Int}(x_i, x_i)$, where $\text{Int}(x_i, x_i)$ stands for the interior of the edge $(x_i, x_i)$. We think of $T_{i,\varepsilon}$ as a possibly pruned tree. Namely, in the notation of Section 1 the set of pruned leaves of $T_{i,\varepsilon}$ is

$$L_{i,\varepsilon} := B(C_{i,\varepsilon}) \setminus \{x_i\},$$

and the root of $T_{i,1}$ is $x_i$ (setting $x_0 := 0$) while, for $i > 0$, the root of $T_{i,2}$ is $x_i$. Note that the trees $T_{i,\varepsilon}$ give a partition of $T \setminus \bigcup_{i=1}^m \text{Int}(x_i, x_i)$, and that the sets of pruned leaves $L_{i,\varepsilon}$ give a partition of $\{x_1, \ldots, x_m\}$. Besides, each vertex of $T$, as well as each semi-open edge $(x, y)$ of $T$, belongs to a unique tree $T_{i,\varepsilon}$.

Next we adapt the expression in (15) for usage with the $T_{i,\varepsilon}$-notation. Consider a product

$$\{r_1 \mid x_1, p_{1,1}, p_{1,2}\} \cdots \{r_m \mid x_m, p_{m,1}, p_{m,2}\}$$

in $\text{Im}(\pi^\ast)$. We can assume $x_1 < \cdots < x_m$ (in view of Corollary 4.1), so we consider the corresponding trees $T_{i,\varepsilon}$ too. In each of the cocycle representatives

$$\sum \left(u_1, \ldots, u_r, v_1, \ldots, v_{p_{1,1}}, (x_i, x_i), w_1, \ldots, w_{p_{2,1}}\right) \cdot \sigma$$

(1 \leq i \leq m), we change each of the $T$-ordered $\Sigma_n$-representatives

$$(u_1, \ldots, u_r, v_1, \ldots, v_{p_{1,1}}, (x_i, x_i), w_1, \ldots, w_{p_{2,1}})$$

by one written in the form $(B_{j_1}^i, \ldots, B_{j_m}^i)$, where each $B_{j}^i$ stands for the block collecting the ingredients of (22) in the trees $T_{j,1}$ and $T_{j,2}$ (the latter only if $j > 0$). In detail:
(a) \( B_0^i = B_{0,1}^i \) is the tuple of vertices of (22) in \( T_{0,1} \) written in T-order;

(b) \( B_i^j = ((x_i, \overline{x_i}), B_{i,1}^j, B_{i,2}^j) \), where \( B_{i,\varepsilon}^j \) is the tuple of vertices of (22) in \( T_{i,\varepsilon} \) written in T-order;

(c) For \( 0 < j \neq i \), \( B_j^i = (B_{j,1}^i, B_{j,2}^i) \), where \( B_{j,\varepsilon}^i \) is the tuple of vertices of (22) in \( T_{j,\varepsilon} \) written in T-order.

A cocycle representative for (20) is then given by the corresponding product expression

\[
\cdots \left( \sum (\ldots, B_i^i, \ldots, B_j^i, \ldots) \cdot \sigma_i \right) \cdots \left( \sum (\ldots, B_i^j, \ldots, B_j^j, \ldots) \cdot \sigma_j \right) \cdots,
\]

where we have written down explicitly two generic factors, the \( i \)-th and the \( j \)-th ones, with \( i < j \).

Note that \( x_i \) is not an ingredient of any \( B_{i,1}^i \), just as \( \overline{x_i} \) is not an ingredient of any \( B_{i,2}^i \). However, Proposition 2.1 and the fact that \((x_\ell, \overline{x_\ell})\) is an ingredient of each \( B_\ell^i \) imply that, in order to have a non-trivial product, \( x_j \) must be an ingredient of every \( B_{j,1}^j \), and that \( \overline{x_i} \) must be an ingredient of every \( B_{i,2}^i \). Likewise, \( x_j \) cannot be an ingredient of any \( B_{j,2}^j \), and \( x_i \) cannot be an ingredient of any \( B_{i,1}^i \). We thus replace the notation in (c) by the following one, which is more in the spirit of (b):

\[
\left\{ \begin{array}{ll}
(x_j, B_{j,1}^i, B_{j,2}^i), & \text{if } i < j; \\
(x_j, B_{j,1}^j, B_{j,2}^j), & \text{if } j < i,
\end{array} \right.
\]

where \( B_{j,\varepsilon}^i \) is the tuple of vertices of (22) in \( T_{j,\varepsilon} \) written in T-order, and none of which has \( x_j \) or \( \overline{x_j} \) as an ingredient. Summands in (21) contributing to a non-vanishing product (23) can thus be written as

\[
\left( B_{0,1}^i \mid \overline{x_1}, B_{1,1}^i, B_{1,2}^i \right) \cdots \left( x_i, \overline{x_i}, B_{i,1}^i, B_{i,2}^i \right) \cdots \left( x_{i+1}, B_{i+1,1}^i, B_{i+1,2}^i \right) \cdots \left( x_m, B_{m,1}^i, B_{m,2}^i \right) \cdot \sigma,
\]

where vertical bars are used interchangeably by commas, and are intended to make reading easier. Product distribution takes (23) into a summation of products of the form

\[
\sum \text{sign}(\overline{\sigma}) \left( B_{0,1}^i \mid (x_1, \overline{x_1}), B_{1,1}^i, B_{1,2}^i \right) \cdots \left( x_m, B_{m,1}^i, B_{m,2}^i \right) \cdot \sigma,
\]

where all cubes are product-oriented and \( \overline{\sigma} \) is the permutation determined by the sequence \( \sigma^{-1}(2), \sigma^{-1}(5), \sigma^{-1}(8), \ldots, \sigma^{-1}(3m-1) \), i.e., the sequence of positions of the edges \((x_1, \overline{x_1}), \ldots, (x_m, \overline{x_m})\) in each tuple \( (B_{0,1}^i \mid (x_1, \overline{x_1}), B_{1,1}^i, B_{1,2}^i \mid \cdots \mid (x_m, \overline{x_m}), B_{m,1}^i, B_{m,2}^i) \cdot \sigma \). Note in addition that the product-oriented cube

\[
\text{sign}(\overline{\sigma}) \left( B_{0,1}^i \mid (x_1, \overline{x_1}), B_{1,1}^i, B_{1,2}^i \mid \cdots \mid (x_m, \overline{x_m}), B_{m,1}^i, B_{m,2}^i \right) \cdot \sigma
\]

agrees with the gradient-oriented cube

\[
\left( B_{0,1}^i \mid (x_1, \overline{x_1}), B_{1,1}^i, B_{1,2}^i \mid \cdots \mid (x_m, \overline{x_m}), B_{m,1}^i, B_{m,2}^i \right) \cdot \sigma,
\]

since \( x_1 < \cdots < x_m \). We have thus proved:
Proposition 4.4. Blocks appearing in (24) have a well-determined number of ingredients: any block $B_0,1$ has $R_0$ ingredients, while any block $B_i,\varepsilon$ has $P_{i,\varepsilon}$ ingredients. In particular, the product (20) vanishes provided any of the interaction parameters is negative (in which case we say that the factors $\varepsilon_j(T)$ in the proposition, and where the blocks $B_j,\varepsilon$ of vertices $T_{j,\varepsilon}$.

Proposition 4.3 below asserts that, for fixed subindices $i$ and $\varepsilon$, the number of ingredients in each block $B_{i,\varepsilon}$ appearing in (24) is fixed, so that Proposition 4.2 fully generalizes Proposition 3.11. We need:

Definition 4.3. Given 1-dimensional classes $\{r_1|x_1,p_{1,1},p_{1,2}\}, \ldots ,\{r_m|x_m,p_{m,1},p_{m,2}\} \in \text{Im}(\pi^*)$ with $x_1 < \cdots < x_m$, consider the interaction parameters $R_0 = R_0(x_1, \ldots , x_m) := n + \sum_{x_j \in L_{0,1}} (r_j - n)$, and $P_{i,\varepsilon} = P_{i,\varepsilon}(x_1, \ldots , x_m) := p_{i,\varepsilon} + \sum_{x_j \in L_{i,\varepsilon}} (r_j - n)$, for $i \in \{1, \ldots , m\}$ and $\varepsilon \in \{1, 2\}$.

For instance, in terms of the notation in Section 1 the $m$ local informations $\langle r_i, x_i, p_{i,1}, p_{i,2}\rangle$ $(1 \leq i \leq m)$ have a simplex interaction if and only if all the interaction parameters in Definition 4.3 are non-negative and all $P_{i,1}$ are positive.

Proposition 4.4. Blocks appearing in (24) have a well-determined number of ingredients: any block $B_{0,1}$ has $R_0$ ingredients, while any block $B_{i,\varepsilon}$ has $P_{i,\varepsilon}$ ingredients. In particular, the product (20) vanishes provided any of the interaction parameters is negative (in which case we say that the factors of (20) do not interact).

Note that $R_0 + \sum_{i,\varepsilon} P_{i,\varepsilon} = n - m$ in Definition 4.3. Together with Proposition 4.3 this is compatible with the fact that cubes in (24) have $n$ ingredients. See also Corollary 4.5.

Proof. By induction on $m$ (using Proposition 3.11 to ground the argument), it suffices to consider a product

$$\left[ (B_{0,1}|(x_1, \overline{x}_1), B_{1,1}B_{1,2} \cdots (x_m, \overline{x}_m), B_{m,1}, B_{m,2}) \cdot \sigma \right] \left[ (U(x_{m+1}, \overline{x}_{m+1}), V_1, V_2) \cdot \sigma' \right] ,$$

where $x_1 < \cdots < x_m < x_{m+1}$, where the number of ingredients of the blocks $B_{i,\varepsilon}$ is as specified in the proposition, and where the blocks $U$ and $V_\varepsilon$ have $r_{m+1}$ and $p_{m+1,\varepsilon}$ ingredients, respectively (assuming, of course, $r_{m+1} + p_{m+1,1} + p_{m+1,2} = n - 1$). Signs and orientations have been carefully considered in the previous discussion and, therefore, are ignored in what follows—we focus only on block sizes. In particular we can safely assume both permutations $\sigma$ and $\sigma'$ in (25) are the identity.

Let $T_{i,\varepsilon}$ and $L_{i,\varepsilon}$ be the trees and corresponding pruned leaves determined by the vertices $x_1, \ldots , x_m$. Likewise, let $T'_{i,\varepsilon}$ and $L'_{i,\varepsilon}$ be the corresponding objects determined by the vertices $x_1, \ldots , x_m, x_{m+1}$. There are three cases, depending on whether the edge $(x_{m+1}, \overline{x}_{m+1})$ belongs to $T_{0,1}$, $T_{j,1}$, or $T_{j,2}$ ($j > 0$), and the argument is virtually identical in each. We consider only the situation depicted in Figure 2 where $(x_{m+1}, \overline{x}_{m+1})$ belongs to $T_{j,1}$ with $j > 0$, in which case:

- $T_{\ell,2} = T'_{\ell,2}$ and $L_{\ell,2} = L'_{\ell,2}$, for $\ell \in \{1, \ldots , m\}$;
• $T_{\ell,1} = T'_{\ell,1}$ and $L_{\ell,1} = L'_{\ell,1}$, for $\ell \in \{0, 1, \ldots, m\} \setminus \{j\}$;
• $T_{j,1} \setminus \text{Int}(x_{m+1}, \overline{x}_{m+1}) = T'_{j,1} \cup T''_{m+1,1} \cup T''_{m+1,2}$;
• $L'_{j,1} = L_{j,1} \cup \{x_{m+1}\}$ and $L''_{m+1,1} = \emptyset = L''_{m+1,2}$.

![Figure 8: $(x_{m+1}, \overline{x}_{m+1})$ belongs to $T_{j,1}$]

By Proposition 2.1, (25) vanishes unless

$$\{\overline{x}_1, \ldots, \overline{x}_m\} \cup B_{0,1} \cup \left( \bigcup_{j \neq i=1}^m (B_{i,1} \cup B_{i,2}) \right) \cup B_{j,2} \subseteq U, \quad \{x_{m+1}\} \cup V_1 \cup V_2 \subseteq B_{j,1}, \quad \text{and}$$

$$U \setminus \left( \{\overline{x}_1, \ldots, \overline{x}_m\} \cup B_{0,1} \cup \left( \bigcup_{j \neq i=1}^m (B_{i,1} \cup B_{i,2}) \right) \cup B_{j,2} \right) = B_{j,1} \setminus \left( \{x_{m+1}\} \cup V_1 \cup V_2 \right), \quad (26)$$

in which case the product (25) takes the form

$$\pm \left( B_{0,1} | (x_1, \overline{x}_1), B_{1,1}, B_{1,2}; \cdots; (x_j, \overline{x}_j), B'_{j,1}, B_{j,2}; \cdots; (x_m, \overline{x}_m), B_{m,1}, B_{m,2}; (x_{m+1}, \overline{x}_{m+1}), V_1, V_2 \right)$$

where $B'_{j,1}$ stands for the set in (26). The induction is complete since

$$|B'_{j,1}| = |B_{j,1}| - (1 + p_{m+1,1} + p_{m+1,2}) = p_{j,1} + \sum_{x_\ell \in L_{j,1}} (r_\ell - n) - (n - r_{m+1}) = p_{j,1} + \sum_{x_\ell \in L'_{j,1}} (r_\ell - n)$$

shows that $B'_{j,1}$ has the prescribed cardinality.

**Corollary 4.5.** For essential vertices $x_1 < \cdots < x_m$, the product

$$\{r_1 | x_1, p_{1,1}, p_{1,2}; \cdots; r_m | x_m, p_{m,1}, p_{m,2}\} = \{R_0 | x_1, P_{1,1}, P_{1,2}; \cdots; x_m, P_{m,1}, P_{m,2}\}$$

holds in $\text{Im}(\pi^*)$ provided the interaction parameters $R_0$ and $P_{i,2}$ ($1 \leq i \leq m$) determined by $x_1, \ldots, x_m$ are non-negative and the interaction parameters $P_{i,1}$ ($1 \leq i \leq m$) are positive (in which case we say that the factors $\{r_i | x_i, p_{i,1}, p_{i,2}\}$, $1 \leq i \leq m$, interact strongly).

**Proof.** Since the Morse differential in $D_n T$ is trivial, the product under consideration is read off directly from the image of (24) under $\Phi$. The hypothesis that all parameters $P_{i,1}$ are positive gives that the only summands in (24) that are not redundant (and, therefore, that can potentially be the target of a lower —perhaps constant— gradient path originating at a critical cube) are already critical. The conclusion then follows from (8) and (10).
Lemma 4.6. For essential vertices $x_1 < \cdots < x_m$, non-negative integer numbers $R_0, P_{1,2}, \ldots, P_{m,2}$ and positive integer numbers $P_{1,1}, \ldots, P_{m,1}$ satisfying $n = R_0 + \sum_{i=1}^{m} (P_{i,1} + P_{i,2} + 1)$, the system of equations

$$n + \sum_{x_j \in L_{0,1}} (r_j - n) = R_0,$$

$$p_{i,\varepsilon} + \sum_{x_j \in L_{1,\varepsilon}} (r_j - n) = P_{i,\varepsilon}, \quad (i = 1, \ldots, m, \varepsilon = 1, 2)$$

has a unique solution of non-negative integer numbers $r_1, p_{1,1}, p_{1,2}, \ldots, r_m, p_{m,1}, p_{m,2}$ satisfying $n = r_1 + p_{1,1} + p_{1,2} + 1$ and $p_{i,1} > 0$ for $1 \leq i \leq m$.

Proof. The two equations with $i = m$ reduce to $p_{m,\varepsilon} = P_{m,\varepsilon}$ ($\varepsilon = 1, 2$), which determine the value $r_m = n - P_{m,1} - P_{m,2} - 1$ since

$$n - P_{m,1} - P_{m,2} - 1 = R_0 + \sum_{j=1}^{m-1} (P_{j,1} + P_{j,2} + 1) \geq 0.$$ 

The rest of the equations can be written as

$$n + \sum_{x_j \in L_{0,1} \setminus \{x_m\}} (r_j - n) = R_0' := R_0 - \begin{cases} n - r_m, & \text{if } x_m \in L_{0,1} \\ 0, & \text{otherwise} \end{cases},$$

$$p_{i,\varepsilon} + \sum_{x_j \in L_{1,\varepsilon} \setminus \{x_m\}} (r_j - n) = P_{i,\varepsilon}' := P_{i,\varepsilon} - \begin{cases} n - r_m, & \text{if } x_m \in L_{1,\varepsilon} \\ 0, & \text{otherwise} \end{cases}, \quad (i = 1, \ldots, m - 1, \varepsilon = 1, 2),$$

where

$$R_0' + \sum_{i=1}^{m-1} (P_{i,1}' + P_{i,2}' + 1) = R_0 + \sum_{i=1}^{m-1} (P_{i,1} + P_{i,2} + 1) + n - r_m = R_0 + \sum_{i=1}^{m} (P_{i,1} + P_{i,2} + 1) = n.$$ 

The result then follows by induction. \hfill \Box

In view of Corollary 4.5 and Lemma 4.6, there is a one-to-one correspondence between the graded basis of $\text{Im}(\pi^*)$ consisting of elements $\{R_0 \mid x_1, P_{1,1}, P_{1,2} \mid x_m, P_{m,1}, P_{m,2}\}$ and the set of tuples $(\{r_1 \mid x_1, p_{1,1}, p_{1,2}\}, \ldots, \{r_m \mid x_m, p_{m,1}, p_{m,2}\})$ whose ingredients are ordered (meaning $x_1 < \cdots < x_m$) and interact strongly. In particular:

**Corollary 4.7.** The cohomology ring $H^*(\text{UD}_n T)$ is generated by 1-dimensional classes and has vanishing squares (by Corollary 4.7).

It is not true that a product in $\text{Im}(\pi^*)$ vanishes when its factors do interact, but non-strongly. The description of such products relies on the dynamics of lower gradient paths.

5 Lower gradient paths

**Proposition 5.1.** Let $\{r \mid x, p_{x,1}, p_{x,2}\}$ and $\{r_i \mid x_i, p_{i,1}, p_{i,2}\}$, $1 \leq i \leq m$, be 1-dimensional basis elements of $\text{Im}(\pi^*)$ with $x < x_1 < \cdots < x_m$. Set $\Pi_1 := \{r_1 \mid x_1, p_{1,1}, p_{1,2}\} \cdots \{r_m \mid x_m, p_{m,1}, p_{m,2}\}$ and
\[ \Pi_2 := \{ r \mid x, p_{x,1}, p_{x,2} \} \cdot \Pi_1. \] Assume that the factors of \( \Pi_1 \) interact strongly and that the factors of \( \Pi_2 \) interact, but non-strongly. Then

\[ \Pi_2 = -\sum_{\ell=1}^{R_0} \{ R_0 - \ell \mid x, \ell, P_{x,2} \mid x_{1,1}, P_{1,2} \mid \cdots \mid x_m, P_{m,1}, P_{m,2} \}, \tag{27} \]

where \( R_0 = R_0(x, x_1, \ldots, x_m) \), \( P_{x,2} = P_{1,2}(x, x_1, \ldots, x_m) \), and \( P_{i,\varepsilon} = P_{i,\varepsilon}(x_1, \ldots, x_m) \) for \( \varepsilon = 1, 2 \) and \( i = 1, \ldots, m \).

Note that all summands on the right of (27) are basis elements. Therefore Proposition 5.1 and the results in the previous section give an algorithmic way to assess products in \( \text{Im}(\pi^*) \).

**Proof.** Start by observing that the condition \( x < x_1 \) forces one of the four configurations in Figure 9.

In any case, vertices \( x_i \) with \( i > 1 \) lie either on the component of \( T \setminus \{x_1\} \) in \( x_1 \)-direction 1 or 2 or, else, “below” the horizontal segment joining the root and \( x_1 \). This gives

\[ P_{i,\varepsilon} = P_{i,\varepsilon}(x_1, \ldots, x_m) = P_{i+1,\varepsilon}(x_1, \ldots, x_m), \] for \( i = 1, \ldots, m \) and \( \varepsilon = 1, 2 \). \tag{28} \]

The configuration on the bottom left of Figure 9 is impossible given the definition of \( \bigwedge \). On the other hand, in the two configurations on the right of Figure 9 we have \( P_{1,1,\varepsilon}(x_1, \ldots, x_m) = p_1 \), which is incompatible with (28) and the hypothesis that the factors of \( \Pi_1 \) and \( \Pi_2 \) interact, strongly in the former case. Thus, the only possible configuration is the one on the top left of Figure 9.

Complementing (28), note again by hypothesis that, in addition to having \( R_0 \geq 0 \leq P_{x,2} \) and \( P_{i,1} \geq 0 \leq P_{i,2} \) for \( i = 1, \ldots, m \) and \( \varepsilon = 1, 2 \), we also have \( P_{x,1} := P_{1,1}(x_1, \ldots, x_m) = 0 \). Propositions 4.2 and 4.4 then give that \( \Pi_2 \) is represented in \( C^*(D_nT) \) by the (gradient-oriented) cocycle

\[ \sum (B_{0,1}(x, \bigwedge, \varnothing, B_{x,2}(x_1, \bigwedge_1), B_{1,1}, B_{1,2}) \cdots (x_m, \bigwedge_m), B_{m,1}, B_{m,2}) \cdot \sigma, \tag{29} \]

where the summation runs over all permutations \( \sigma \in \Sigma_n \) and over all possible blocks \( B_{s,s} \) of vertices written in \( T \)-order and taken from the corresponding tress \( T_{s,s} \) determined by the essential vertices \( x < x_1 < \cdots < x_m \). Furthermore:

(i) Each \( B_{0,1} \) has \( R_0 \) ingredients, each \( B_{x,2} \) has \( P_{x,2} \) ingredients, and each \( B_{i,\varepsilon} \) has \( P_{i,\varepsilon} \) ingredients.

(ii) Ingredients of each \( B_{0,1} \) smaller than \( \bigwedge \) form a stack of vertices blocked by the root of \( T \).
(iii) For any $x_i$ smaller than $x$, all ingredients of each $B_{i,\varepsilon}$ are blocked (this uses that $P_{i,1} > 0$), so each triple $((x_i, x), B_{i,1}, B_{i,2})$ assembles a critical situation around $x_i$.

The last two conditions hold since, as in the proof of Corollary 4.5, we can ignore redundant summands in (29) —keep in mind the configuration in the upper left of Figure 9. It follows that each (unignored) summand in (29) is collapsible by a branch-type pairing of the form:

![Diagram of branch-type pairing]

Using (8), we then see that $\Pi_2$ is given in $\mathcal{M}^*(D_n T)$ by the sum

$$\sum_{\gamma \in \mathcal{G}} \mu(\gamma) : S_{\gamma},$$

where $\mathcal{G}$ is the set of lower paths $\gamma$ that start at an $(m + 1)$-critical cube $S_{\gamma}$ and that finish at a summand $c \cdot \sigma$ of (29) via a pairing (30). Note that each such summand $c \cdot \sigma$ supports a gradient path $\lambda : c \cdot \sigma \cdot \succ \cdot \succ \cdot \succ \cdot \succ \cdot \succ$ that ends at a critical $m$-cube. For instance, replace the edge $(x_1, \overline{x}_1)$ in $c \cdot \sigma$ by $x_1$, and let the rest of the path consist of falling-vertex elementary factors. It follows that the concatenation of $\gamma$ and $\lambda$, and therefore $\gamma$ itself, obey the rule in Corollary 3.9 any upper elementary factor of sort type is of falling-vertex type. Such a fact is used next to analyze of (31), mimicking the cancellation phenomenon in the proof of Proposition 3.11. Actually, as in the proof of Proposition 3.11 it suffices to do the analysis at the level of $C^*(D_n T)$, which means that summands $c \cdot \sigma$ can be replaced by orbits $\{c\}$.

As in the proof of Proposition 3.11 we start by describing the set $\mathcal{L} \subset \mathcal{G}$ of paths whose contribution in (31) gives (27). Roughly, paths in $\mathcal{L}$ have a single “lock” dynamics. Explicitly, for $\ell = 1, \ldots, R_0$, consider the lower path $\gamma_\ell$ that starts from the critical $(m + 1)$-cube

$$\{R_0 - \ell \mid x, \ell, P_{x,2} \mid x_1, P_{1,1}, P_{1,2} \mid \ldots \mid x_m, P_{m,1}, P_{m,2}\}$$

(recall (9)) replacing the edge $(x, \overline{x})$ by $\overline{x}$ —this opens the lock. Then $\gamma_\ell$ (ineluctably) continues with the falling of the $\ell$ vertices blocked by $x$, after which $\gamma_\ell$ ends (again ineluctably) with the pairing that closes the lock by creating the edge $(x, \overline{x})$ as in (30). Since the lock opening and closing are associated to the same face (the gradient-orientated $\delta_2$-face), and since falling-vertex elementary paths have multiplicity 1, we see from (9) that $\mu(\gamma_\ell) = -1$. All together, we get (27).

The set of paths $\mathcal{L}$ is contained in a slightly larger subset $\mathcal{L}_- \subset \mathcal{G}$. Namely, paths in $\mathcal{L}_-$ start by taking the face $\delta_2$ (lock opening) of a critical $(m + 1)$-cube of the form

$$\{R_0 - a - b \mid x, a, P_{x,2} + b \mid x_1, P_{1,1}, P_{1,2} \mid \ldots \mid x_m, P_{m,1}, P_{m,2}\}$$

with $a > 0 \leq b$ and $a + b \leq R_0$, then (ineluctably) the $a$ vertices $x + 1, \ldots, x + a$ that were blocked by $x$ fall, followed by the (ineluctable) falling of the $b$ vertices $\overline{x}, \overline{x} + 1, \ldots, \overline{x} + b - 1$, to finish with the (ineluctable) falling of the vertex $\overline{x} + b$ until it creates the required branch-type pairing (30) —which closes the lock. As in the case of $\mathcal{L}$, paths in $\mathcal{L}_-$ have multiplicity $-1$. Likewise, there is the family $\mathcal{L}_+ \subset \mathcal{G}$ consisting of the paths that start by taking the face $\delta_1$ (inverse lock opening) of a critical $(m + 1)$-cube of the form

$$\{R_0 - a - b \mid x, a - 1, P_{x,2} + b + 1 \mid x_1, P_{1,1}, P_{1,2} \mid \ldots \mid x_m, P_{m,1}, P_{m,2}\}$$

20
with \(a - 1 > 0 \leq b\) and \(a + b \leq R_0\), then (ineluctably) the \(a\) vertices \(x, x + 1, \ldots, x + a - 1\) fall, followed by the (ineluctable) falling of the \(b\) vertices \(\overline{x} + 1, \ldots, \overline{x} + b\), to finish with the (ineluctable) falling of the vertex \(\overline{x} + b + 1\) until it creates the required branch-type pairing \((30)\) — which closes the lock.

Note that paths in \(L_+\) have multiplicity \(+1\). What is important to note is that, for \(a', b' > 0\) with \(a' + b' \leq R_0\), the path \(\gamma_{a', b'}\) in \(L_-\) with parameters \(a = a'\) and \(b = b'\) and the path \(\gamma_{a' + 1, b' - 1}\) in \(L_+\) with parameters \(a = a' + 1\) and \(b = b' - 1\) share origin, \(\delta \gamma_{a', b'} = \delta \gamma_{a' + 1, b' - 1}\), so their contributions in \((31)\) cancel each other out. The only paths that remain unmatched are those in \(L_-\) having parameter \(b = 0\), i.e., the paths in \(L_+\), whose contribution to \((31)\) has been analyzed in the previous paragraph.

Therefore, the proof will be complete once we construct an involution \(\iota: G \setminus (L_- \cup L_+) \rightarrow G \setminus (L_- \cup L_+)\) such that each pair of paths \(\gamma\) and \(\iota(\gamma)\) share origin and have opposite multiplicity.

By construction, \(L_- \cup L_+\) are the paths in \(G\) that

1. (I) start from a critical \((m + 1)\)-cube having edge ingredients \((x, \overline{x}), (x_1, \overline{x}_1), \ldots, (x_m, \overline{x}_m)\),
2. (II) take the face \(\delta_1\) or \(\delta_2\) of that initial cube, and
3. (III) before reaching the ending branch-type pairing \((30)\), evolve by falling-vertex elementary paths.

But condition (II) is forced by conditions (I) and (III): in any gradient path \(e \setminus x e' \setminus \cdots\), all whose upper elementary factors are of falling-vertex type, the edge ingredients of \(e'\) are present in all further steps of the path. Therefore \(G \setminus (L_- \cup L_+)\) is partitioned into two sets, \(G_{\text{fall}}\) and \(G_{\text{branch}}\), where the former set consists of the paths in \(G\) that satisfy (III) without satisfying (I), and the latter set consists of the paths in \(G\) that do not satisfy (III). We actually construct involutions \(\iota_{\text{fall}}: G_{\text{fall}} \rightarrow G_{\text{fall}}\) and \(\iota_{\text{branch}}: G_{\text{branch}} \rightarrow G_{\text{branch}}\) with the required properties.

As noted above, for a path \(a_0 \setminus b_1 \setminus a_1 \setminus \cdots \setminus b_k \setminus a_k\) in \(G_{\text{fall}}\), all edges \((x_i, \overline{x}_i), 1 \leq i \leq m\), must be ingredients of all the cubes \(a_i\) and \(b_i\). So the path must start from \(a_0\) by replacing some edge \(^8\) \((y, \overline{y})\) different \(^9\) from \((x, \overline{x})\) by either \(y\) or \(\overline{y}\). As in the proof of Proposition 3.13, the definition of \(\iota_{\text{fall}}\) is based on the two replacing options. Likewise, the definition of \(\iota_{\text{branch}}\) is based on the two forms of replacing by a vertex the edge ingredient at the last upper elementary factor that is not of falling-vertex type.

We have carefully distinguished between \(\text{Im}(\pi^*)\) and \(H^*(\text{UD}_n T)\) in order to describe proof arguments as clear as possible. Having got a full description of these isomorphic rings, we now transfer the notation and descriptions back to \(H^*(\text{UD}_n T)\).

### 6 Exterior face basis

Recall the basis of \(H^1(\text{UD}_n T)\) given by the elements \(\langle r, x, p_1, p_2 \rangle\) in \((11)\). In this section we analyze the relationship between products

\[
\langle r_1, x_1, p_1, 1, p_1, 2 \rangle \cdots \langle r_m, x_m, p_m, 1, p_m, 2 \rangle \quad \text{and} \quad \{r_1 | x_1, p_1, 1, p_1, 2 \} \cdots \{r_m | x_m, p_m, 1, p_m, 2 \}
\]

when they are ordered in the sense that \(x_1 < \cdots < x_m\), assumption that will be in force from now on. We say that a product in \((32)\) is a strong interaction product if the corresponding braces elements \(\{r_1, x_1, p_1, 1, p_1, 2\}, \ldots, \{r_m, x_m, p_m, 1, p_m, 2\}\) interact strongly. The proof of our main result, Theorem 3.3, is completed by Theorem 6.1 below, whose proof is this section’s goal.

---

8 The asserted form of this edge comes from the fact that \(a_0\) is a critical cube.

9 Actually, \((y, \overline{y})\) must lie in the component of \(T \setminus \{x\}\) in \(x\)-direction 2.
Theorem 6.1. An ordered product \( \langle r_1, x_1, p_{1,1}, p_{1,2} \rangle \cdots \langle r_m, x_m, p_{m,1}, p_{m,2} \rangle \) is non-zero if and only if it is a strong interaction product. Two strong interaction products agree if and only if they have the same factors. A graded basis of \( H^*(\text{UD}_n T) \) is given by the strong interaction products.

Remark 6.2. Corollary 4.5 and Propositions 4.4 and 5.1 show that both products in (32) are linear combinations of basis elements \( \{ \cdot, x_1, \ldots, \cdot | x_m, \ldots, \cdot \} \). Such a linear combination will be written simply as \( \sum \{ \cdot, x_1, \ldots, \cdot | x_m, \ldots, \cdot \} \), i.e., omitting the use of coefficients. Here and below, the symbol \( \cdot \) stands for an unspecified non-negative integer number, which should actually be positive if it appears right after an essential vertex.

The crux of the matter in the proof of Theorem 6.1 is getting at a precise description of the conditions that have to be satisfied by some of the unspecified integer numbers in

\[
\langle r_1, x_1, p_{1,1}, p_{1,2} \rangle \cdots \langle r_m, x_m, p_{m,1}, p_{m,2} \rangle = \sum \{ \cdot, x_1, \ldots, \cdot | x_m, \ldots, \cdot \}.
\]

With this in mind, the product in (33) is denoted by \( \varpi \) and, in connection to it, we let

\[
R_0 = \mathcal{R}_0(x_1, \ldots, x_m) \quad \text{and} \quad P_{i,\varepsilon} = \mathcal{P}_{i,\varepsilon}(x_1, \ldots, x_m)
\]
denote the corresponding interaction parameters. Furthermore, we set

\[
B_i := (x_i, P_{i,1}, P_{i,2}) \quad \text{and} \quad \hat{B}_i := (x_i, \cdot),
\]

where the latter expression is used to represent any tuple with unspecified integer coordinates. Additionally, the \( i \)-th factor on the left hand-side of (33) is denoted by \( \phi_i \). Lastly, for a non-negative integer \( k \), the notation \( \frac{1}{k} \) stands for a generic non-negative integer strictly smaller than \( k \) (if any). For example, \( \phi_i = \{ r_i | x_i, p_{i,1}, p_{i,2} \} + \sum \{ r_i | x_i, \cdot, p_{i,1}, p_{i,2} \} \), whereas the second product in (32) agrees with \( \{ R_0 | B_1 | \ldots | B_m \} \) under the strong-interaction condition.

The following results make use of the description of products in \( H^*(\text{UD}_n T) \) obtained in Sections 4 and 5.

Lemma 6.3. In the notation of (34), assume \( L_{0,1} = \{ x_1, x_2, \ldots, x_m \} \). Then

\[
\varpi = \begin{cases} 
\{ R_0 | B_1 | \cdots | B_m \} + \sum \{ R_0 | \hat{B}_1 | \cdots | \hat{B}_m \}, & \text{if } R_0 \geq 0; \\
0, & \text{otherwise.}
\end{cases}
\]

Proof. This follows by direct inspection of the expression

\[
\left( \{ r_1 | x_1, p_{1,1}, p_{1,2} \} + \sum \{ r_1 | x_1, \cdot, \cdot \} \right) \cdots \left( \{ r_m | x_m, p_{m,1}, p_{m,2} \} + \sum \{ r_m | x_m, \cdot, \cdot \} \right),
\]

noticing that interactions occur only in the tree \( T_{0,1} \) (so that \( P_{i,\varepsilon} = p_{i,\varepsilon} \)). \( \square \)

Lemma 6.4. Assume \( L_{1,2} = \{ x_2, x_3, \ldots, x_m \} \). Then

\[
\varpi = \begin{cases} 
\{ R_0 | B_1 | \cdots | B_m \} + \sum \{ R_0 | x_1, P_{1,1}, P_{1,2} | \hat{B}_2 | \cdots | \hat{B}_m \} + \sum \{ R_0 | \hat{B}_1 | \cdots | \hat{B}_m \}, & \text{if } P_{1,2} \geq 0; \\
0, & \text{otherwise.}
\end{cases}
\]

Proof. Interactions occur only in \( T_{1,2} \), so \( R_0 = r_1, P_{1,1} = p_{1,1} \) and \( P_{i,\varepsilon} = p_{i,\varepsilon} \) for \( i \geq 2 \). Lemma 6.3 (applied to the product \( \phi_2 \cdots \phi_m \)) then gives

\[
\varpi = \phi_1 \cdot (\phi_2 \cdots \phi_m)
\]

\[
= \left( \{ R_0 | x_1, P_{1,1}, p_{1,2} \} + \sum \{ R_0 | x_1, \cdot, p_{1,2} \} \right) \left( \{ R_0 | B_2 | \cdots | B_m \} + \sum \{ R_0 | \hat{B}_2 | \cdots | \hat{B}_m \} \right),
\]

where \( R_0 = \mathcal{R}_0(x_2, \ldots, x_m) \) (so \( P_{1,2} = p_{1,2} + R_0 - n \)). The result follows again by direct inspection. \( \square \)
Lemma 6.5. Assume $L_{1,1} = \{x_{2}, x_{3}, \ldots, x_{m}\}$. Then the product of $\langle r_{1}, x_{1}, p_{1,1}, p_{1,2} \rangle$ with any cohomology class $\{R | x_{2}, p_{2,1}, p_{2,2} \cdots | x_{m}, p_{m,1}, p_{m,2} \}$ vanishes provided $R + p_{1,1} \leq n$.

Proof. We proceed by induction on $R + p_{1,1} - n \leq 0$. For $R + p_{1,1} = n$, Lemma 6.6 and the hypothesis show that $R$ and the various $p_{i,\varepsilon}$ ($1 \leq i \leq m$) determine unique non-negative integer numbers $s_{2}, \ldots, s_{m}$ giving a strong interaction product

$$\{R | x_{2}, p_{2,1}, p_{2,2} \cdots | x_{m}, p_{m,1}, p_{m,2} \} = \{s_{2} | x_{2}, p_{2,1}, p_{2,2} \cdots \cdot \{s_{m} | x_{m}, p_{m,1}, p_{m,2} \}$$

(so $R = n + \sum_{i=2}^{m}(s_{i} - n$). Proposition 5.1 then gives

$$\{r_{1} | x_{1}, p_{1,1}, p_{1,2} \} \cdot \left( \{s_{2} | x_{2}, p_{2,1}, p_{2,2} \cdots \cdot \{s_{m} | x_{m}, p_{m,1}, p_{m,2} \} \right) =$$

$$= - \sum_{\ell=1}^{r_{1}} \{r_{1} - \ell | x_{1}, \ell, p_{1,2} | B_{2} \cdots | B_{m} \}$$

$$= - \sum_{\ell=1}^{r_{1}} \{r_{1} - \ell | x_{1}, p_{1,1} + \ell, p_{1,2} \right) \left( \{s_{2} | x_{2}, p_{2,1}, p_{2,2} \cdots \cdot \{s_{m} | x_{m}, p_{m,1}, p_{m,2} \} \right),$$

where the last equality comes from strong interaction products. This and (35) ground the inductive argument. For $R + p_{1,1} < n$,

$$\langle r_{1}, x_{1}, p_{1,1}, p_{1,2} \rangle \cdot \{R | x_{2}, p_{2,1}, p_{2,2} \cdots | x_{m}, p_{m,1}, p_{m,2} \}$$

$$= \left( \langle r_{1} | x_{1}, p_{1,1}, p_{1,2} \rangle + \langle r_{1} - 1, x_{1}, p_{1,1} + 1, p_{1,2} \rangle \right) \cdot \{R | x_{2}, p_{2,1}, p_{2,2} \cdots | x_{m}, p_{m,1}, p_{m,2} \}$$

$$= 0 + 0,$$

where the former zero comes from Proposition 4.4 and the latter zero (which holds only if $r_{1} > 0$) comes from the inductive hypothesis. \[\square\]

Corollary 6.6. If $\varpi$ is not a strong interaction product, then it vanishes.

Proof. By isolating the factors $\phi_{i}$ that are involved in a faulty interaction parameter, it suffices to consider three cases: $L_{0,1} = \{x_{1}, \ldots, x_{m}\}$, $L_{1,2} = \{x_{2}, \ldots, x_{m}\}$ and $L_{1,1} = \{x_{2}, \ldots, x_{m}\}$. The first two cases are covered by Lemmas 6.3 and 6.4. Likewise, the case $L_{1,1} = \{x_{2}, \ldots, x_{m}\}$ is covered by Proposition 4.4 if $P_{1,1} < 0$. It only remains to consider the case $L_{1,1} = \{x_{2}, \ldots, x_{m}\}$ with $P_{1,1} = 0$, in which case Lemma 6.3 gives

$$\varpi = \langle r_{1}, x_{1}, p_{1,1}, p_{1,2} \rangle \cdot \left( \{R_{0} | B_{2} \cdots | B_{m} \} + \sum_{\ell=1}^{r_{1}} \{R_{0} | B_{2} \cdots | B_{m} \} \right),$$

where $R_{0} = R_{0}(x_{2}, \ldots, x_{m})$ (so $0 = P_{1,1} = p_{1,1} + R_{0} - n$ and, in particular, $R_{0} > 0$). The triviality of $\varpi$ then follows from Lemma 6.3. \[\square\]

The following is an $x_{1}$-direction-1 analog of Lemma 6.4.

Lemma 6.7. Assume $L_{1,1} = \{x_{2}, x_{3}, \ldots, x_{m}\}$. Then

$$\varpi = \begin{cases} \{R_{0} | B_{1} \cdots | B_{m} \} + \sum_{\ell=1}^{r_{1}} \{R_{0} | x_{1}, p_{1,1}, P_{1,2} | B_{2} \cdots | B_{m} \} + \sum_{\ell=1}^{r_{1}} \{R_{0} | B_{1} \cdots | B_{m} \}, \text{ if } P_{1,1} > 0; \\
0, \text{ otherwise.} \end{cases}$$
Proof. The case $P_{1,1} \leq 0$ has just been discussed, so we assume $P_{1,1} > 0$. Interactions occur only in $T_{1,1}$, so $R_0 = r_1$, $P_{1,2} = p_{1,2}$ and $P_{1,\varepsilon} = p_{1,\varepsilon}$ for $i \geq 2$. Lemma 6.3 then gives

$$
\varpi = \phi_1 \cdot (\phi_2 \cdots \phi_m)
= \left( \{ R_0 | x_1, P_{1,1}, P_{1,2} \} + \{ R_0 | x_1, P_{1,1}, P_{1,2} \} \cdot \{ R_0 | B_2 \} \cdots \{ R_0 | \hat{B}_2 \} \right) \cdot \left( \{ R_0 | x_1, P_{1,1}, P_{1,2} \} \right) \cdot \left( \{ R_0 | B_2 \} \cdots \{ R_0 | \hat{B}_2 \} \right),
$$

where $R_0 = R_0(x_2, \ldots, x_m)$ (so $P_{1,1} = p_{1,1} + R_0 - n$). As in the proof of Lemma 6.4, the result follows by direct inspection, though this time Proposition 5.1 needs to be used. \[ \square \]

Lemmas 6.4 and 6.7 are particular cases of:

**Proposition 6.8.** Assume $L_{1,1} = \{ x_2, x_3, \ldots, x_\ell \}$ and $L_{1,2} = \{ x_{\ell+1}, x_{\ell+2}, \ldots, x_m \}$ with $1 \leq \ell \leq m$. If $P_{1,1} > 0 \leq P_{1,2}$, then

$$
\varpi = \{ R_0 | B_1 \} \cdots \{ R_0 | B_m \} + \sum \{ R_0 | x_1, P_{1,1}, P_{1,2} \} \cdot \{ R_0 | \hat{B}_2 \} \cdots \{ R_0 | \hat{B}_\ell \},
$$

where each expression $P_{1,1}, P_{1,2}$ stands for a pair of unspecified integer numbers $q_1, q_2$ such that $q_1 > 0 \leq q_2$ and, in the product ordering, $(q_1, q_2) < (P_{1,1}, P_{1,2})$, i.e., $q_1 \leq P_{1,1}$ and $q_2 \leq P_{1,2}$, without both being equalities.

Proof. By Lemmas 6.4 and 6.7 we can assume $1 < \ell < m$. Then using Lemmas 6.3 and 6.7 we can evaluate $\varpi = (\phi_1 \cdots \phi_m) \cdot (\phi_{\ell+1} \cdots \phi_m)$ as the product of

$$
\{ R_0 | x_1, P_{1,1}, P_{1,2} | B_2 \} \cdots | B_\ell \}
+ \sum \{ R_0 | x_1, P_{1,1}, P_{1,2} | \hat{B}_2 \} \cdots | \hat{B}_\ell \}
+ \sum \{ R_0 | \hat{B}_1 \} \cdot \{ R_0 | \hat{B}_2 \} \cdots | \hat{B}_\ell \}
$$

and

$$
\{ R_0 | B_{\ell+1} \} \cdots | B_m \}
+ \sum \{ R_0 | \hat{B}_{\ell+1} \} \cdots | \hat{B}_m \}
$$

where $R_0 = R_0(x_{\ell+1}, \ldots, x_m)$ (so $P_{1,2} = p_{1,2} + R_0 - n$). The result follows by direct inspection. \[ \square \]

We are now ready to set up the strategy for completing the proof of Theorem 6.1. By Lemma 4.6 Remark 6.2 and Corollary 6.6 the goal reduces to describing a partial ordering on the basis elements $\{ s_0 | x_1, q_{1,1}, q_{1,2} \cdots | x_m, q_{m,1}, q_{m,2} \}$ of $H^m(UD_n T)$ (recall the essential vertices $x_1 < \cdots < x_m$ are fixed) such that (33) can be expressed by a congruence

$$
\langle r_1, x_1, P_{1,1}, P_{1,2} \rangle \cdots \langle r_m, x_m, P_{m,1}, P_{m,2} \rangle \equiv \{ R_0 | B_1 \} \cdots | B_m \}
$$

modulo basis elements that are smaller than $\{ R_0 | B_1 \} \cdots | B_m \}$. The partial ordering we need becomes apparent by writing the conclusion in Proposition 6.8 as

$$
\varpi = \{ R_0 | B_1 \} \cdots | B_m \}
+ \sum \{ R_0 | B_1 \} \cdot \{ R_0 | \hat{B}_2 \} \cdots | \hat{B}_m \}
+ \sum \{ R_0 | \hat{B}_1 \} \cdots | \hat{B}_m \}
$$

**Definition 6.9.** The $\ell$-th level of pruned leaves $\mathcal{L}_\ell$ of the essential vertices $x_1 < \cdots < x_m$ is

$$
\mathcal{L}_\ell := \begin{cases} 
L_{0,1}, & \text{if } \ell = 1; \\
\bigcup_{x_i \in \mathcal{L}_{\ell-1}} (L_{i,1} \cup L_{i,2}), & \text{if } \ell > 1.
\end{cases}
$$

The interaction level of the vertices $x_1 < \cdots < x_m$ is the largest $\ell$ such that $\mathcal{L}_\ell \neq \emptyset$. Extending the notation in (27), let $B^{(\ell)}$ denote the collection of blocks $B_i$ with $x_i \in \mathcal{L}_\ell$, and let $\hat{B}^{(\ell)}$ stand for any collection of blocks $\hat{B}_i$ with $x_i \in \mathcal{L}_\ell$. On the other hand, $\tilde{B}^{(\ell)}$ stands for any collection of blocks $(x_i, q_{i,1}, q_{i,2})$ with $i \in \mathcal{L}_\ell$ satisfying $q_{i,\varepsilon} \leq P_{i,\varepsilon}$ for all $x_i \in \mathcal{L}_\ell$ and all $\varepsilon \in \{1, 2\}$, and with at least one of these inequalities being strict.
The definition of $\overline{B}^{(\ell)}$ is slightly less restrictive than actually requiring $\overline{B}^{(\ell)}$ to be a collection of blocks $B_i$ with $x_i \in \mathcal{L}_\ell$. As in Proposition 6.8, the condition we want for $\overline{B}^{(\ell)}$ is based on a strict product-order inequality. The reason for this will become apparent in the proof of Proposition 6.11 below.

**Example 6.10.** Lemma 6.3 gives $\varpi = \{R_0, B^{(1)}\} + \sum \{R_0, \hat{B}^{(1)}\}$ in interaction level 1 (under a strong condition hypothesis). Likewise, (37) becomes

$$\varpi = \{R_0 | B^{(1)} | B^{(2)}\} + \sum \{R_0 | B^{(1)} | \hat{B}^{(2)}\} + \sum \{R_0 | \hat{B}^{(1)} | \hat{B}^{(2)}\}$$

(38)

in interaction level 2 (with $\mathcal{L}_1 = \{x_1\}$, so $B^{(1)} = B_1$). In full generality:

**Proposition 6.11.** Let $x_1 < \cdots < x_m$ be essential vertices of interaction level $\ell$. If $\varpi$ is a strong interaction product, then

$$\varpi = \{R_0 | B^{(1)} | \cdots | B^{(\ell)}\} + \sum \{R_0 | B^{(1)} | \cdots | B^{(\ell-2)} | B^{(\ell-1)} | \hat{B}^{(\ell)}\}$$

$$+ \cdots + \sum \{R_0 | B^{(1)} | \hat{B}^{(2)} | \cdots | \hat{B}^{(\ell)}\} + \sum \{R_0 | \hat{B}^{(1)} | \cdots | \hat{B}^{(\ell)}\}.$$  

(39)

**Proof.** The argument is by direct computation, as in the proofs of Lemma 6.3 and Proposition 6.8 except that we can now proceed in a straightforward way since we already have Corollary 6.6. We provide details for completeness.

The case $\ell = 1$ has been observed in Example 6.10. Assume $\ell \geq 2$ with the result valid for interaction levels smaller than $\ell$. We check first the situation when $\mathcal{L}_1 = \{x_1\}$ (so $R_0 = r_1$), i.e., the generalization of (38). Let $t \in \{1, \ldots, m\}$ be such that $x_2, \ldots, x_t$ lie in $x_1$-direction 1, while $x_{t+1}, \ldots, x_m$ lie in $x_1$-direction 2. Start with the product $\phi_1 \cdot (\phi_2 \cdots \phi_t)$ which, by induction, is obtained by multiplying \{R_0 | x_1, p_{1,1}, p_{1,2}\} and

$$\{R_0 | B^{(1)}_{[1]} | \cdots | B^{(\ell)}_{[1]}\} + \sum \{R_0 | B^{(2)}_{[1]} | \cdots | B^{(j-2)}_{[1]} | B^{(j-1)}_{[1]} | \hat{B}^{(j)}_{[1]} | \cdots | \hat{B}^{(\ell)}_{[1]}\} + \sum \{R_0 | \hat{B}^{(2)}_{[1]} | \cdots | \hat{B}^{(\ell)}_{[1]}\}.$$  

(40)

In the latter expression:

- $R_0 = R_0(x_2, \ldots, x_t)$ (so that $p_{1,1} = p_{1,1} + R_0 - n$),
- the subindex ‘[1]’ in a collection of blocks means that we are only taking blocks in $x_1$-direction 1,
- level numbering for $x_2 < \cdots < x_t$ starts at 2, i.e., it is compatible with that for $x_1 < \cdots < x_m$,
- first summation is a sum (over $j$) of summations as those in (39); similar situations hold below.

Note that the interaction level in the $x_1$-direction-1 branch might be smaller than $\ell$, in which case the corresponding collections of blocks are empty. By direct inspection (using Proposition 5.1), the product we have just described takes the form

$$\{R_0 | x_1, p_{1,1}, p_{1,2} | B^{(2)}_{[1]} | \cdots | B^{(\ell)}_{[1]}\} + \sum \{R_0 | x_1, p_{1,1}, p_{1,2} | B^{(2)}_{[1]} | \cdots | B^{(j-2)}_{[1]} | B^{(j-1)}_{[1]} | \hat{B}^{(j)}_{[1]} | \cdots | \hat{B}^{(\ell)}_{[1]}\}$$

$$+ \{R_0 | x_1, p_{1,1}, p_{1,2} | \hat{B}^{(2)}_{[1]} | \cdots | \hat{B}^{(\ell)}_{[1]}\} + \{R_0 | \hat{B}_1 | \hat{B}^{(2)}_{[1]} | \cdots | \hat{B}^{(\ell)}_{[1]}\},$$

(41)
which when multiplied with $\phi_{t+1} \cdots \phi_m$, i.e.,

$$\left\{ R_0' B_{[2]}^{(2)} \cdots B_{[2]}^{(t)} \right\} + \sum_{3 \leq j \leq t} \left\{ R_0' B_{[2]}^{(2)} \cdots B_{[2]}^{(j-2)} B_{[2]}^{(j-1)} \hat{B}_{[2]}^{(j)} \cdots \hat{B}_{[2]}^{(t)} \right\} + \sum \left\{ R_0' \hat{B}_{[2]}^{(2)} \cdots \hat{B}_{[2]}^{(t)} \right\},$$

where $R_0'' = R_0(x_{t+1}, \ldots, x_m)$ (so $P_{1,2} = p_{1,2} + R_0'' - n$), yields (39) by direct inspection (this time Proposition 5.1 is not needed). This completes the proof when $L_1$ is a singleton.

In general, $L_1$ consists of, say, vertices $x_1 = x_{i_1} < \cdots < x_{i_k}$, and we evaluate (39) as the length-$k$ product $(\phi_{i_1} \cdots \phi_{i_{j-1}})(\phi_{i_j} \cdots \phi_{i_{j-1}}) \cdots (\phi_{i_k} \cdots \phi_m)$. We have just seen that the $w$-th factor in such a product takes the form

$$\left\{ r_{iw} B_{[w]}^{(1)} \cdots B_{[w]}^{(t)} \right\} + \sum_{3 \leq j \leq t} \left\{ r_{iw} B_{[w]}^{(1)} \cdots B_{[w]}^{(j-2)} B_{[w]}^{(j-1)} \hat{B}_{[w]}^{(j)} \cdots \hat{B}_{[w]}^{(t)} \right\} + \sum \left\{ r_{iw} \hat{B}_{[w]}^{(1)} \cdots \hat{B}_{[w]}^{(t)} \right\},$$

where $B_{[w]}^{(1)} := B_{iw}$, $\hat{B}_{[w]}^{(1)} := \hat{B}_{iw}$ and, more generally, a subindex $'[w]'$ in a collection of blocks now indicates that only blocks in $x_{iw}$-directions 1 and 2 are to be taken. The required form (39) for the product of all these expressions follows again from direct inspection.

$$\square$$

The partial ordering that completes the proof of Theorem 6.1 is now self-evident: Basis elements $\{ s_0, x_1, q_{1,1}, q_{1,2}, \cdots, x_m, q_{m,1}, q_{m,2} \}$ are ordered through a level-wise lexicographical comparison (as in Definition 6.9) of their integer ingredients. In this terms, (39) yields the required expression (36), and completes the proof of Theorem 6.1.

### References

1. Aaron David Abrams. *Configuration spaces and braid groups of graphs.* ProQuest LLC, Ann Arbor, MI, 2000. Thesis (Ph.D.)–University of California, Berkeley.

2. Jorge Aguilar-Guzmán, Jesús González, and Teresa Hoekstra-Mendoza. Farley-Sabalka’s Morse-theory model and the higher topological complexity of ordered configuration spaces on trees. To appear in *Discrete & Computational Geometry.*

3. Francis Connolly and Margaret Doig. On braid groups and right-angled Artin groups. *Geom. Dedicata,* 172:179–190, 2014.

4. Daniel Farley. Homology of tree braid groups. In *Topological and asymptotic aspects of group theory,* volume 394 of *Contemp. Math.,* pages 101–112. Amer. Math. Soc., Providence, RI, 2006.

5. Daniel Farley. Presentations for the cohomology rings of tree braid groups. In *Topology and robotics,* volume 438 of *Contemp. Math.,* pages 145–172. Amer. Math. Soc., Providence, RI, 2007.

6. Daniel Farley and Lucas Sabalka. Discrete Morse theory and graph braid groups. *Algebr. Geom. Topol.,* 5:1075–1109, 2005.

7. Daniel Farley and Lucas Sabalka. On the cohomology rings of tree braid groups. *J. Pure Appl. Algebra,* 212(1):53–71, 2008.

8. Robin Forman. A discrete Morse theory for cell complexes. In *Geometry, topology, & physics,* Conf. Proc. Lecture Notes Geom. Topology, IV, pages 112–125. Int. Press, Cambridge, MA, 1995.
[9] Robin Forman. Discrete Morse theory and the cohomology ring. Trans. Amer. Math. Soc., 354(12):5063–5085, 2002.

[10] Robert Ghrist. Configuration spaces and braid groups on graphs in robotics. In Knots, braids, and mapping class groups—papers dedicated to Joan S. Birman (New York, 1998), volume 24 of AMS/IP Stud. Adv. Math., pages 29–40. Amer. Math. Soc., Providence, RI, 2001.

[11] Tomasz Kaczynski, Konstantin Mischaikow, and Marian Mrozek. Computational homology, volume 157 of Applied Mathematical Sciences. Springer-Verlag, New York, 2004.

[12] Tomasz Kaczynski and Marian Mrozek. The cubical cohomology ring: an algorithmic approach. Found. Comput. Math., 13(5):789–818, 2013.

[13] Jee Hyoun Kim, Ki Hyoung Ko, and Hyo Won Park. Graph braid groups and right-angled Artin groups. Trans. Amer. Math. Soc., 364(1):309–360, 2012.

[14] Steven Scheirer. Topological complexity of $n$ points on a tree. Algebr. Geom. Topol., 18(2):839–876, 2018.

Departamento de Matemáticas
Centro de Investigación y de Estudios Avanzados del I.P.N.
Av. Instituto Politécnico Nacional número 2508, San Pedro Zacatenco
México City 07000, México.
jesus@math.cinvestav.mx
idskjen@math.cinvestav.mx