A KICKING BASIS FOR THE TWO COLUMN GARSIA-HAIMAN MODULES

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Abstract. In the early 1990s, Garsia and Haiman conjectured that the dimension of the Garsia-Haiman module $R_\mu$ is $n!$, and they showed that the resolution of this conjecture implies the Macdonald Positivity Conjecture. Haiman proved these conjectures in 2001 using algebraic geometry, but the question remains to find an explicit basis for $R_\mu$ which would give a simple proof of the dimension. Using the theory of Orbit Harmonics developed by Garsia and Haiman, we present a "kicking basis" for $R_\mu$ when $\mu$ has two columns.

1. Introduction

In 1988, Macdonald [14] found a remarkable new basis of symmetric functions in two parameters which specializes to Schur functions, complete homogeneous, elementary and monomial symmetric functions and Hall-Littlewood functions, among others. With an appropriate analog of the Hall inner product, the transformed Macdonald polynomials $\tilde{H}_\mu(Z;q,t)$ are uniquely characterized by certain triangularity and orthogonality conditions, from which their symmetry follows. The Kostka-Macdonald polynomials, $\tilde{K}_{\lambda\mu}(q,t)$, are defined by

$$\tilde{H}_\mu(Z;q,t) = \sum_\lambda \tilde{K}_{\lambda\mu}(q,t)s_\lambda(Z).$$

The Macdonald Positivity Conjecture states that $\tilde{K}_{\lambda\mu}(q,t) \in \mathbb{N}[q,t]$.

In 1993, Garsia and Haiman [6] conjectured that the transformed Macdonald polynomials could be realized as the bigraded characters for a diagonal action of $S_n$ on two sets of variables. Moreover, they were able to show that knowing the dimension of this module is enough to determine its character. Therefore the $n!$ Conjecture, which states that the dimension of the Garsia-Haiman module is $n!$, implies the Macdonald Positivity Conjecture.

By analyzing the algebraic geometry of the Hilbert scheme of $n$ points in the plane, Haiman [13] was able to prove the $n!$ Conjecture and consequently establish Macdonald Positivity. However, it remains an important open problem in the theory of Macdonald polynomials to prove the $n!$ Theorem directly by finding an explicit basis for the module. After reviewing Macdonald polynomials and the Garsia-Haiman modules in Section 2, we give an explicit basis for the Garsia-Haiman modules indexed by a partition with at most two columns in Section 3. A new basis for hooks is also given in Section 4.

2. Macdonald polynomials and graded $S_n$-modules

We assume the definitions and notations from [15] of partitions and the classical bases for symmetric functions. So as to avoid confusions when defining various modules, we use the alphabet $Z = z_1, \ldots, z_n$ for symmetric functions. For example, the Schur functions shall be denoted $s_\lambda(Z)$.

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2.1. Macdonald positivity. Departing slightly from Macdonald’s convention of defining
$P_\mu(Z; q, t)$ \cite{14}, we instead use the transformed Macdonald polynomials $\tilde{H}_\mu(Z; q, t)$ as pre-
sented in \cite{6}.

**Definition 2.1.** The transformed Macdonald polynomials $\tilde{H}_\mu(Z; q, t)$ are the unique func-
tions satisfying the following triangularity and orthogonality conditions:

(i) $\tilde{H}_\mu(Z; q, t) \in \mathbb{Q}(q, t)\{s_\lambda[Z/(1 - q)] : \lambda \geq \mu\};$

(ii) $\tilde{H}_\mu(Z; q, t) \in \mathbb{Q}(q, t)\{s_\lambda[Z/(1 - t)] : \lambda \geq \mu'\};$

(iii) $\tilde{H}_\mu[1; q, t] = 1.$

The square brackets in Definition 2.1 stand for *plethystic substitution*. In short, $s_\lambda[A]$ means $s_\lambda$ applied as a $\Lambda$-ring operator to the expression $A$, where $\Lambda$ is the ring of symmetric
functions. For a thorough account of plethysm, see \cite{12}.

The existence of such a family of functions is a theorem, following in large part from
Macdonald’s original proof of existence. Once established, the symmetry of $\tilde{H}_\mu(Z; q, t)$ follows by definition. Of particular importance are the change of basis coefficients from the
transformed Macdonald polynomials to the Schur functions, defined by

\[
\tilde{H}_\mu(Z; q, t) = \sum_\lambda \tilde{K}_{\lambda,\mu}(q, t)s_\lambda(Z).
\]

A priori, the $\tilde{K}_{\lambda,\mu}(q, t)$ are rational functions in $q$ and $t$ with rational coefficients.

**Theorem 2.2** \cite{13}. We have $\tilde{K}_{\lambda,\mu}(q, t) \in \mathbb{N}[q, t]$.

Macdonald originally conjectured Theorem 2.2 when he introduced the polynomials in
1988. The original proof, due to Haiman in 2001, realizes $\tilde{H}_\mu(Z; q, t)$ as the bigraded char-
acter of certain modules for the diagonal action of $\mathcal{S}_n$ on $\mathbb{Q}[X, Y]$; see sections 2.2 and 2.3.

From this it follows that the character can be written as a sum of irreducible representations
of $\mathcal{S}_n$ with coefficients in $\mathbb{N}[q, t]$. Under the Frobenius image, these coefficients exactly give
$\tilde{K}_{\lambda,\mu}(q, t)$. The aim of this paper is to follow this method of proof until it departs the realm
of representation theory for algebraic geometry.

It is worth noting that there are now two additional proofs of Macdonald positivity,
both of which utilize an expansion of Macdonald polynomials in terms of LLT polynomials
conjectured by Haglund \cite{10} and proved along with Haiman and Loehr \cite{11}. Grojnowski
and Haiman \cite{9} have a proof using Kazhdan-Lusztig theory and the first author \cite{3} has a
purely combinatorial proof.

2.2. Garsia-Haiman modules. To define the modules mentioned in Section 2.1 we con-
sider the diagonal action of the symmetric group $\mathcal{S}_n$ on the polynomial ring $\mathbb{Q}[X, Y] =
\mathbb{Q}[x_1, \ldots, x_n; y_1, \ldots, y_n]$ permuting the $x_i$’s and $y_j$’s simultaneously and identically. Let the
coordinates of the diagram of a partition $\mu$ of $n$ be $\{(p_1, q_1), \ldots, (p_n, q_n)\}$, where $p$ gives the
row coordinate and $q$ the column coordinate indexed from zero; see Figure 1.

```
(2,0)  |  (1,1)  |  (1,0)
(1,0)  |  (0,1)  |  (0,2)
```

**Figure 1.** The coordinates for each cell of $\mu = (3, 2, 1)$.  

Define the polynomial $\Delta_\mu \in \mathbb{Q}[X,Y]$ by
\[
\Delta_\mu(X,Y) = \det \begin{pmatrix}
x_1^{p_1} y_1^{q_1} & x_2^{p_1} y_2^{q_1} & \cdots & x_n^{p_1} y_n^{q_1} \\
x_1^{p_2} y_1^{q_2} & x_2^{p_2} y_2^{q_2} & \cdots & x_n^{p_2} y_n^{q_2} \\
\vdots & \vdots & \ddots & \vdots \\
x_1^{p_n} y_1^{q_n} & x_2^{p_n} y_2^{q_n} & \cdots & x_n^{p_n} y_n^{q_n}
\end{pmatrix}.
\]
(2)

Since the bi-exponents are all distinct, $\Delta_\mu$ is a non-zero homogeneous $S_n$-alternating polynomial with top degree $n(\mu) = \sum_i (i-1)\mu_i$ in $X$ and $n(\mu')$ in $Y$. Taking $\mu = (1^n)$ or $\mu = (n)$ gives the Vandermonde determinant in $X$ or $Y$, respectively.

Let $\mathcal{I}_\mu \subset \mathbb{Q}[X,Y]$ be the ideal of polynomials $\varphi$ such that
\[
\varphi(\partial/\partial x_1, \ldots, \partial/\partial x_n; \partial/\partial y_1, \ldots, \partial/\partial y_n) \Delta_\mu = 0.
\]
Clearly this defines an $S_n$ invariant doubly homogeneous ideal. Define the Garsia-Haiman module $\mathcal{H}_\mu$ to be the quotient ring $\mathbb{Q}[X,Y]/\mathcal{I}_\mu$ with its natural structure of a doubly graded $S_n$-module.

Garsia and Haiman [7] proved that if this module has the correct dimension (the $n!$ Conjecture), then the bi-graded character is given by the transformed Macdonald polynomial.

**Theorem 2.3 ([7]).** If $\mathcal{H}_\mu$ affords the regular representation of $S_n$, then the bi-graded Frobenius character, given by
\[
\text{Frob}_{\mathcal{H}_\mu}(Z; q,t) = \sum_{i,j} t^i q^j \psi((\mathcal{H}_\mu)_{i,j}),
\]
where $\psi$ is the usual Frobenius map sending the Specht module $S_\lambda$ to the Schur function $s_\lambda$, is equal to the transformed Macdonald polynomials $\widetilde{H}_\mu(Z; q,t)$. In particular, $K_{\lambda,\mu}(q,t) \in \mathbb{N}[q,t]$.

The following theorem is the famed $n!$ Conjecture of Garsia and Haiman [6], proved by Haiman [13] in 2001.

**Theorem 2.4 ([13]).** The dimension of $\mathcal{H}_\mu$ is $n!$.

By Theorem 2.3, Haiman’s proof of the $n!$ Conjecture provided the first proof of the Macdonald positivity conjecture. Haiman’s proof analyzes the isospectral Hilbert scheme of $n$ points in a plane, ultimately showing that it is Cohen-Macaulay (and Gorenstein). As this proof uses difficult machinery in algebraic geometry, it remains an important open problem to prove Theorem 2.4 directly by finding an explicit basis for the module $\mathcal{H}_\mu$.

2.3. Orbit Harmonics. Let $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ be sequences of distinct rational numbers. Let $(p_1, q_1), \ldots, (p_n, q_n)$ be the coordinates of the cells of $\mu$ taken in some order, recorded by the standard filling $S$ of $\mu$ given by placing the entry $i$ in the cell $(p_i, q_i)$. To each $S$, associate the orbit point of $S$, denoted $p_S$, defined by
\[
p_S = (\alpha_{p_1+1}, \ldots, \alpha_{p_n+1}; \beta_{q_1+1}, \ldots, \beta_{q_n+1}).
\]
(3)

Here the shift in indices is a notational convenience. For example,
\[
p = (\alpha_2, \alpha_1, \alpha_3, \alpha_1, \alpha_3, \alpha_2; \beta_2, \beta_1, \beta_2, \beta_1, \beta_1).
\]

Let $S_n$ act on $\mathbb{Q}^{2n}$ by permuting the first $n$ and second $n$ coordinates simultaneously and identically. Let $[p_S]$ denote the regular orbit of $p_S$ under this action. Regarding $\mathbb{Q}[X,Y]$ as the coordinate ring of $\mathbb{Q}^{2n}$, define $J_\mu \subset \mathbb{Q}[X,Y]$ to be the ideal of polynomials vanishing
on \([p_S]\). Define the module \(R_{\mu}\) to be the coordinate ring of \([p_S]\), i.e. \(\mathbb{Q}[X,Y]/\mathcal{J}_{\mu}\), with its natural \(S_n\) action.

Since \(R_{\mu}\) clearly affords the regular representation, the aim is to relate this module to \(\mathcal{H}_{\mu}\). To do this, construct the associated graded module \(\text{gr}R_{\mu} = \mathbb{Q}[X,Y]/\mathcal{J}_{\mu}\). Garsia and Haiman showed that if \(\mathcal{H}_{\mu}\) and \(\text{gr}R_{\mu}\) have the same Hilbert series, then \(\mathcal{H}_{\mu} = \text{gr}R_{\mu}\). While this would demonstrate the \(n!\) Conjecture, the obvious problem is that one needs first to know the Hilbert series of \(\mathcal{H}_{\mu}\) in which case the dimension can be directly calculated. The way around this problem lies in the theory of Orbit Harmonics developed by Garsia and Haiman. The main result is the following.

**Theorem 2.5** ([5]). Let \(\Phi_{\mu}\) be a basis for \(R_{\mu}\). Let \(F_{\mu}(q,t) = \sum_{\varphi \in \Phi_{\mu}} \hat{\varphi}(t,\ldots,t;q,\ldots,q)\), where \(\hat{\varphi}\) is the leading term of \(\varphi\). If \(F_{\mu}\) is symmetric in the following sense,

\[ [t^iu^j] F_{\mu}(q,t) = [t^{n(\mu)\prime-i}q^{\mu(\mu')-j}] F_{\mu}(q,t), \]

then \(\hat{\Phi}_{\mu} = \{\hat{\varphi} \mid \varphi \in \Phi_{\mu}\}\) is a basis for \(\text{gr}R_{\mu}\). Moreover, \(\text{gr}R_{\mu} \cong \mathcal{H}_{\mu}\) as doubly-graded \(S_n\) modules. In particular, \(\dim \mathcal{H}_{\mu} = n!\).

Theorem 2.5 suggests the following strategy for constructing a basis for the Garsia-Haiman module \(\mathcal{H}_{\mu}\). To each filling \(S\) of \(\mu\), define a polynomial \(\varphi_S \in \mathbb{Q}[X,Y]\) so that the evaluation matrix \((\varphi_S(p_T))\) of polynomials on orbit points is nonsingular and the corresponding degree polynomial \(F_{\mu}(q,t)\) is symmetric in the sense of equation (4). The remainder of this paper is devoted to carrying out this strategy in the cases when \(\mu\) is a two column shape (Section 3).

### 3. Two columns

Throughout this section, we restrict our attention to partitions with at most two columns. Following the procedure laid out in Section 2.3 we will construct a basis for \(R_{\mu}\) such that the degree polynomial is symmetric. Following the idea of the kicking basis for the Garsia-Procesi modules described in [8], we will construct the basis together with a linear order on fillings of \(\mu\) so that the evaluation matrix has nice triangularity properties. While the Garsia-Procesi case results in an upper triangular matrix with nonzero diagonal entries, our matrix will only be block triangular with respect to the largest entry.

#### 3.1. The kicking tree

The kicking tree of \(\mu\) provides a nice visualization of the recursive construction of the proposed basis. Though proving that the resulting collection is a basis with symmetric Hilbert series is better done from the recursive definition, the construction is better motivated from this viewpoint.

To construct the kicking tree, entries will be added to an empty shape one at a time in all possible ways in some specified order, ultimately resulting in a total ordering for the fillings. We begin by recalling the Garsia-Procesi ordering for row-increasing tableaux [8].

Let \(S\) be a partial filling of \(\mu\) with distinct entries. Define a total ordering on the rows of \(S\) containing at least one empty cell, called the row preference order, as follows: empty rows of length 2 from top to bottom followed by (empty) rows of length 1 from top to bottom followed by rows of length 2 with a single occupant beginning with the largest occupant. Given two rows \(i\) and \(j\) of a (partial) filling \(S\), say that \(k\) prefers row \(j\) over row \(i\), denoted \(j \succ_k i\), if \(j\) occurs before \(i\) in the row ordering on the filling obtained by removing entries less than \(k + 1\) from \(S\). For example, Figure 2 shows the ranking of the rows (on the left) for two partial fillings of \((2,2,2,1,1)\).

The row preference order is enough to define a total order on fillings with unsorted rows. The basic construction of the tree is to fill entries into unsorted rows one at a time according to row preference, where a row of length 2 is sorted, increasing then decreasing, as soon as
it is fully occupied. The real power of the kicking tree lies in the weights assigned at each stage which we now describe.

Let \( S \) be a partial, partially sorted filling of \( \mu \) with entries \( n > n - 1 > \cdots > k + 1 \). That is, each entry is assigned a row of \( \mu \), and an entry is assigned a specific column if and only if the row is fully occupied. Below \( S \) with arrows going down, place \( k \) into a row, ordered from left to right by row preference with respect to \( k \). Label the arrow going down from \( S \) to the filling with \( k \) by

\[
\prod_{j \in \text{row}(k)} (x_k - \alpha_j).
\]

If \( k \) completed a row of length 2, say with \( m > k \) already in the row, then below this with arrows going down make two partial fillings: the left one having \( k \) before \( m \) and the right having \( m \) before \( k \). Label the left branch put 1, and label the right branch

\[
(y_k - \beta_1).
\]

If ignoring entries larger than \( m \) does not form a rectangle, then move the label from the arrow going down from \( S \) to the left-hand arrow just added, and add to the right-hand arrow

\[
\prod_{\text{row}(k) \leq i} (x_k - \alpha_i).
\]

The tree so constructed beginning with the empty shape \( \mu \) is called the *kicking tree* for \( \mu \). For example, the kicking tree for \( (2,1) \) is constructed in Figure 3. For this example, we omit vertical lines to indicate an unsorted row.

![Figure 2](image1.jpg)

**Figure 2.** The row preference order for partial fillings.

![Figure 3](image2.jpg)

**Figure 3.** The kicking tree for \( (2,1) \). Here the circled term \( x_2 - \alpha_2 \) indicates that this term is pushed to the leftmost branch below. From left to right, the corresponding polynomials are \( 1, y_1 - \beta_1, x_2 - \alpha_2, y_2 - \beta_1, x_3 - \alpha_1, (x_3 - \alpha_1)(y_1 - \beta_1) \).
From the construction of the kicking tree, the product of the branch labels from a leaf \( S \) back to the empty shape \( \mu \) is clearly a polynomial. The collection of polynomials for each filling of \( \mu \) forms the proposed kicking basis for \( R_\mu \).

### 3.2. A recursive construction.

In order to give an alternative recursive description of the kicking basis, we first need a bit more terminology.

For a standard filling of size \( n \), define \( S \setminus n \) to be the standard filling of size \( n - 1 \) obtained by removing the cell containing \( n \) and straightening the shape as follows. If \( n \) lies in a row of length 2, then move the remaining cell in the same row as \( n \) above rows of length 2 and below rows of length 1 and push it to the left if necessary. Otherwise slide the cells down, preserving their order, to close the gap; see Figure 4. Notice that row dominance order commutes with straightening.

![Figure 4. An illustration of straightening after removing the largest entry.](image)

In Definition 3.1, when the largest entry of a tableau is removed and the remaining shape is straightened, the orbit point of the resulting tableau is defined using the original labelling of the rows and columns. That is, the orbit point of \( S \setminus n \) is the orbit point of \( S \) with the \( n \)th and \( 2n \)th coordinates removed. For example, in Figure 4, the orbit point of the filling of shape \((2,1,1)\) will be \((\alpha_2, \alpha_1, \alpha_3, \alpha_1; \beta_2, \beta_1, \beta_2)\).

**Definition 3.1.** Define \( \phi_\emptyset = 1 \). For \( S \) a standard filling of \( \mu \), \( |\mu| > 1 \), define \( \phi_S \) recursively by

\[
\phi_S = \phi_{S \setminus n} \cdot \prod_{j > n, \text{row}(n)} (x_n - \alpha_j) \cdot \begin{cases} 
1 & \text{if } n \text{ at the end of row}(n) \\
(y_k - \beta_1) \prod_{\text{row}(k) > k} (x_k - \alpha_i) & \text{if } \mu = (2^k) \text{ and col}(n) = 1 \\
\prod_{j > \text{row}(k)} (x_k - \alpha_j) & \text{otherwise}
\end{cases}
\]

where \( k \) is such that \( \text{row}(k) = \text{row}(n) \).

Using the example in Figure 4 we compute

\[
\begin{align*}
\varphi & = (x_6 - \alpha_3)(y_1 - \beta_1) \cdot (y_3 - \beta_1) \cdot \frac{x_3 - \alpha_1}{x_3 - \alpha_2} \cdot 1 \cdot \frac{x_3 - \alpha_2}{x_2 - \alpha_2} \cdot 1,
\end{align*}
\]

where each step in the recursion is indicated by the cell removed to obtain the given terms.

The above formula associates to each standard filling \( S \) of \( \mu \) the same polynomial as the kicking tree from Section 3.1. Notice that the denominator in the last case is precisely the label which is ‘pushed down’ when constructing the kicking tree. Analyzing this statement in terms of the recursive definition yields the following result.

**Proposition 3.2.** For \( S \) a standard filling of \( \mu \), \( \varphi_S \) is a polynomial.
Proof. The result for \( \mu = (1) \) is clear, so we proceed by induction on \( n = |\mu| \). It suffices to assume \( k, n \) reside in the same length 2 row with \( n \) in column 1. We must show that each term in the denominator occurs in the numerator of \( \varphi_{S \setminus n} \). The only terms that ever appear in any denominator are \( x_i - \alpha_j \) where \( i \) lies in the second column and the entry to its left is greater. In particular, if \( x_k - \alpha_j \) ever occurs in a numerator in the construction of \( \varphi \), it remains there through \( \varphi_{S \setminus n} \). Now notice that the product outside of the brace (for \( k \), not \( n \)) is precisely the denominator in question. \( \square \)

To show that these polynomials form a basis for \( R_\mu \), we show that the evaluation matrix of polynomials on orbit points is nonsingular. The argument uses a nested induction to show that the matrix is almost block triangular.

**Theorem 3.3.** The \( n! \times n! \) matrix \( (\varphi_S(p_T)) \), where \( S, T \) range over all fillings of \( \mu \), is nonsingular. In particular, the set \( \{\varphi_S\} \) of polynomials associated to fillings of \( \mu \) forms a basis for \( R_\mu \).

**Proof.** We proceed by induction on \( n = |\mu| \), the case \( n = 1 \) being trivial. The row preference order with respect to \( n \) makes \( (\varphi_S(p_T)) \) block triangular with respect to the row of \( n \). Therefore we must show that each block, corresponding to \( n \) in a particular row, is nonsingular. If \( n \) lies in a row of length 1, this is immediate by induction, so assume \( n \) lies in a row of length 2.

For \( k < n \), let \( T_k \) be a partial, partially sorted filling of \( \mu \) with entries \( k + 1, k + 2, \ldots, n \) (here \( n \) must lie in its designated row of length 2). By partially sorted, we mean that the row of each entry is determined, but the column is determined if and only if the row is fully occupied; see Figure 5 for an example. Let \( T_k \) be the set of standard fillings of \( \mu \) which restrict to \( T_k \) on \( \{k + 1, \ldots, n\} \), where here again the restriction allows the column of an entry to be undetermined exactly when the other occupant of the same row is at most \( k \); again, see Figure 5. We will show that the evaluation matrix for \( T_k \) is nonsingular by induction on \( k \). As usual, the base case, \( k = 1 \), is trivial.

![Figure 5. An illustration of \( T_k \) and \( T_{k-1} \).](image)

Restricting our attention to the set of polynomials and orbit points associated to standard fillings \( S \in T_k \), we put the following block ordering based on the position of \( k \): \( k \) is the largest entry in a row of length 2 from highest row to lowest row; \( k \) lies in a row of length 1 from highest row to lowest; \( k \) lies to the left of a larger entry from largest entry to smallest; and \( k \) lies to the right of a larger entry again from smallest entry to largest. Note that the order for the first three blocks comes from the kicking tree, but the order of the fourth block is the reverse of the kicking order. By the definition of \( \varphi_S \), each of the four blocks is triangular with respect to the row of \( k \), therefore by induction each block is nonsingular since each is a fixed polynomial times the polynomials associated with \( T_{k-1} \) for a fixed partial filling \( T_{k-1} \).

Also from the definition of \( \varphi_S \), the first three blocks are triangular with respect to one another in the given order, and the third and fourth blocks are triangular with respect to each other as well. Moreover, for \( S \) in one of the first three cases, the monomial \( (y_m - \beta_i) \).
does not divide $\varphi_S$ for $i = 1, 2$ and any $m \geq k$ that appears by itself in a row of length two in $T_k$. Therefore the block structure of the evaluation matrix is as depicted in Figure 6. Since the first two blocks are nonsingular, we may perform row reductions to eliminate the nonzero elements in the bottom block-row of the matrix. These reductions will change the bottom block-row 0 into some matrix, say $M$, and so by previous remarks the reductions will alter the fourth block by $M$ as well. Hence using the row reductions from the third block to restore the 0 will also restore the fourth block. Hence the matrix can be made block triangular. Since the determinant of the matrix is the product of the determinants of the blocks, the full matrix is nonsingular.

3.3. Symmetry of the Hilbert series. Now that we have a basis for $R_\mu$, we must show that the associated degree polynomial, denoted $F_\mu(q, t)$, is symmetric. Recall that $F_\mu(q, t)$ is given by

$$F_\mu(q, t) = \sum_{S: \mu \sim \langle n \rangle} \hat{\varphi}_S(t, \ldots, t; q, \ldots, q),$$

where $\hat{\varphi}_S$ is the highest degree term of $\varphi_S$. That is, $F_\mu(q, t)$ is the polynomial in $q$ and $t$ obtained by adding leading terms of the kicking basis and recording the total $x$ degree with $t$ and the total $y$ degree with $q$.

Our aim is to show that $F_\mu(q, t)$ is symmetric, i.e.

$$F_\mu(q, t) = t^{n(\mu)} q^{n(\mu')} F_\mu(1/q, 1/t).$$

For example, from Figure 6 we see that $F_{(2,1)}(q, t) = 1 + 2q + 2t + qt$, which indeed exhibits the desired symmetry.

In order to establish symmetry, we will exploit a recurrence relation that follows naturally from the recursive definition of $\varphi_S$. To do this, we must first define a more general degree polynomial, denoted $J^m_{a,b}$, by

$$J^m_{a,b}(q, t) = \frac{1}{q^m} \sum_{S: \mu \sim \langle n \rangle \text{ s.t.} \begin{array}{c} \text{row}(n-j)=b-j, \\ \text{col}(n-j)=1 \end{array}} \hat{\varphi}_S(t, \ldots, t; q, \ldots, q),$$

where $a \geq b \geq m \geq 0$. Note that $J^m_{a,b}$ is a polynomial with maximum $q$ and $t$ exponents given by $b - m$ and $\binom{a-m}{2} + \binom{b}{2}$, respectively. Pictorially, $J^m_{a,b}$ is the degree polynomial of fillings of $(2^b, 1^{a-b})$ with the top $m$ cells on the left-hand side of the rectangle $(2^b)$ deleted. In particular, we have

$$J^0_{a,b}(q, t) = F_\mu(q, t).$$
Therefore it is enough to show that \( J_{a,b}^m \) is symmetric.

\[
\begin{array}{c}
\text{Figure 7. The cell diagram for which } J_{a,b}^m \text{ is the degree polynomial.}
\end{array}
\]

**Proposition 3.4.** The degree polynomials \( J_{a,b}^m \) satisfy the following recurrence relations

\[
\begin{align*}
(9) \quad J_{a,b}^m &= [m]_t J_{a-1,b-1}^{m-1} + t^{b-m} [a-b]_t J_{a-1,b}^{m-1} + t^m [b-m]_t J_{a,b-1}^{m-1} + q[b-m]_t J_{a,b}^{m+1}, \\
(10) \quad J_{a,b}^m &= t^{b-m} [m]_t J_{a-1,b-1}^{m-1} + [a-b]_t J_{a-1,b}^{m-1} + q[b-m]_t J_{a,b-1}^{m-1} + t^{a-b} [b-m]_t J_{a,b}^{m+1},
\end{align*}
\]

with initial conditions

\[
J_{a,b}^b = \left(\begin{array}{c} a \\ b \end{array}\right) [b]_t! [a-b]_t! \quad \text{and} \quad J_{b,b}^m = J_{b,b}^0 = 0, \quad \text{unless } a \geq b \geq m \geq 0.
\]

**Proof.** The first recurrence relation follows immediately from Definition 3.1 with one term coming from each of the four regions in Figure 7. The second recurrence follows from the first by induction on \( a + b - m \), since applying the recurrences in either order yields the same expression.

If all the leftmost cells in length two columns have been removed, then the stalk (top \( a-b \) rows) and base (bottom \( b \) rows) are independent, giving the first initial condition. The second initial condition is similar, since again the newly shortened rows are independent of the rest. \( \square \)

The above recurrence relations follow from the recursive description in Definition 3.1. Expanding \( J_{a,b}^m \) twice using both recurrence relations in Proposition 3.4 taken in one order followed by the other establishes the desired symmetry.

**Theorem 3.5.** For \( a \geq b \geq m \geq 0 \), we have

\[
J_{a,b}^m(q,t) = t^{(a-m)\underline{+}(b)} q^{b-m} J_{a,b}^m(1/q,1/t).
\]

**Proof.** When \( m = b, b = 0 \) or \( a = b \), the result follows, so we proceed by induction on \( a + b - m \). We will expand \( J_{a,b}^m \) using the recurrence relations from Proposition 3.4. To ease notation, observe that from either recurrence relation we may deduce that \( J_{a,b}^m \) is divisible by \([a-b]_t! [b-m]_t! [m]_t!\). Therefore we may define a new family polynomials, denoted \( \tilde{J}_{a,b}^m \), by

\[
(11) \quad \tilde{J}_{a,b}^m = \frac{J_{a,b}^m}{[a-b]_t! [b-m]_t! [m]_t!}.
\]

Then the recurrence relations in equations (9) and (10) become

\[
\begin{align*}
(12) \quad \tilde{J}_{a,b}^m &= \tilde{J}_{a-1,b-1}^{m-1} + t^{b-m} \tilde{J}_{a-1,b}^{m-1} + t^m [a-b+1]_t \tilde{J}_{a,b-1}^{m-1} + q[a-b+1]_t \tilde{J}_{a,b}^{m+1}, \\
(13) \quad \tilde{J}_{a,b}^m &= t^{b-m} \tilde{J}_{a-1,b-1}^{m-1} + \tilde{J}_{a-1,b}^{m-1} + q[a-b+1]_t \tilde{J}_{a,b-1}^{m-1} + t^{a-b} [b-m+1]_t \tilde{J}_{a,b}^{m+1},
\end{align*}
\]
with initial conditions

\[
J^b_{a,b} = \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{and} \quad J^m_{b,b} = J^0_{b,b-m}.
\]

Expanding each of the four terms in equation (12) using the recurrence relation in equation (13) yields the following expression for \(J^m_{a,b}\):

\[
\begin{align*}
\ell^b - m J^m_{a-2,b-2} + \ell^{m-1}_{a-2,b-1} + q[a-b+1]_t J^m_{a-1,b-2} + t^{a-b} [m]_t J^m_{a-1,b-1} \\
+ \ell^{2(b-m)} J^m_{a-2,b-1} + \ell^b - m J^m_{a-2,b} + q^{b-m} [a-b]_t J^m_{a-1,b-1} + t^{a-m-1}[m+1]_t J^m_{a-1,b} \\
+ t^{b-1}[a-b+1]_t J^m_{a-1,b-1} + t^m[a-b+1]_t J^m_{a-1,b-1} + q t^m[a-b+1]_t[a-b+2]_t J^m_{a,b} \\
+ t^{a-b+m-1}[a-b+1]_t[m+1]_t J^m_{a,b-1} + q t^m[a-b+1]_t J^m_{a,b-1} \\
+ q t^{m-a}[m+1]_t[m+2]_t J^m_{a,b}
\end{align*}
\]

Gathering terms, this expression becomes

\[
\begin{align*}
\ell^{b-m} J^m_{a-2,b-2} + \ell^b - m J^m_{a-2,b} + q t^m[a-b+1]_t[a-b+2]_t J^m_{a,b-2} + q t^{a-b} [m+1]_t J^m_{a,b} \\
+ \left((1 + t^{2(b-m)} J^m_{a-2,b-1} + (q + t^{b-1}) [a-b+1]_t J^m_{a-1,b-2} + (q + t^{a-m-1}) [m+1]_t J^m_{a-1,b} \\
+ \left(q^2 + t^{a-b+m-1} J^m_{a,b-1} + t^{b-1} - m J^m_{a,b-1} + (q t^{b-m} + t^m) [a-b]_t J^m_{a-1,b-1} \\
+ \left(q t^{b-m} + t^a - b]_t [m]_t J^m_{a-1,b-1} + (q t^{b-m} + t^m) [a-b]_t J^m_{a-1,b-1}
\end{align*}
\]

where now each term of the above expression exhibits the desired symmetry, meaning the coefficient of \(t^i q^j\) is equal to the coefficient of \(t^N - i q^M - j\), where \(N = (a-1)(b-m) - \binom{b-m}{2}\) and \(M = b - m\) are the maximum powers of \(t\) and \(q\) in \(J^m_{a,b}\), respectively. Hence \(J^m_{a,b}\) symmetric. Since the product of two symmetric polynomials is again symmetric, the theorem now follows.

In particular, by equation (8), Theorem 3.5 shows that the degree polynomial for the two column kicking basis is indeed symmetric. Therefore by Theorem 2.5 we have the following consequence.

**Corollary 3.6.** For \(\mu\) a two column partition, \(\{\tilde{\varphi}_S \mid S : \mu \xrightarrow{\sim} [n]\}\) is a basis for \(\text{gr} R_\mu\) and so too for \(\mathcal{H}_\mu\). In particular, \(\dim(\mathcal{H}_\mu) = n!\) and \(\tilde{K}_{\lambda,\mu}(q,t) \in \mathbb{N}(q,t)\).

4. Hooks

We next treat the case of hooks, i.e. partitions \(\mu = (n-m, 1^m)\). Though there exist several known bases for Garsia-Haiman modules indexed by hooks, the first in [7] and several more in [16] [4] [2] [1], we present this new construction because it is compatible with our two column case, i.e. the definitions of \(\varphi_S\) will agree on shapes of the form \((2, 1^{n-2})\), and thus suggests how to extend this approach to arbitrary shapes.

As with the two column case, we will construct a basis for \(R_\mu\) such that the degree polynomial is symmetric following the idea of the kicking basis for the Garsia-Procesi modules [8]. In this case, the linear order on fillings of \(\mu\) will have the property that the evaluation matrix is upper triangular with nonzero diagonal entries with respect to this basis.
4.1. The kicking tree. For the case of hooks, the kicking tree is rather straightforward to describe. Let $S$ be a partial, partially sorted filling of a hook, say $(m, 1^{M-m})$, with entries $M > M-1 > \cdots > n+1$. As before, each entry is assigned a row, and an entry is assigned a specific column only if its row is fully occupied. Below $S$ with arrows going down, place $n$ into a row, ordered from left to right by row dominance with respect to $n$. Label the arrow going down from $S$ to the filling with $n$ by

$$\prod_{j > n \text{row}(n)} (x_n - \alpha_j).$$

If $n$ completed the bottom row, then below this will be $M$ copies of $S \cup n$ with one entry of row 1 fixed in a column, followed by $m-1$ copies of that with a second entry fixed in a column, and so on for a total of $m-1$ additional levels of the tree. There are two cases to consider in describing these levels.

First, if $n = 1$ and the bottom row contains the entries 1, 2, $\ldots$, $m$, then at each step the new level fixes the largest unfixed entry from right to left. When $k$ is fixed into column $j$, the preceding edge is labelled

$$\prod_{i \in G_j} (y_k - \beta_i),$$

where the product is over the set $G_j$ of unassigned column indices greater than $j$.

Otherwise, at each step the new level fixes the smallest unfixed letter from left to right. When $k$ is fixed into column $j$, the preceding edge is labelled

$$\prod_{i \in L_j} (y_k - \beta_i),$$

where the product is over the set $L_j$ of unassigned column indices less than $j$. For the smallest entry of the first row, multiply this by the label of the arrow coming from $S$, and then replace the label from $S$ by 1. Finally, once the largest letter, say $M$, of the bottom row is assigned a column, for each $h < M$ that is greater than every other entry of the first row, remove the label from when $h$ was added (necessarily above $S$), replacing it with 1, and starting from $S \cup n$, replace each 1 with the removed label provided this substitution was not already made at a branch above. See, for example, Figure 3.

Remark 4.1. At each node of the tree, the product of the labels from the empty shape down to that node annihilates every orbit point of a filling to the left of the given node. In particular, the matrix $(\varphi_S(p_T))$ is upper triangular with nonzero diagonal entries.

4.2. A recursive basis. Both the row dominance order and straightening procedures are the same for hooks as in the two column case, though we can describe them more simply for this case. For row dominance, take the bottom row until only one empty cell remains, and then take the cells from top to bottom. For straightening, push the cells to the left and then down. As with the two column case, when straightening a shape it is essential to recall the original labelling of the rows and columns in order to determine the associated orbit point.
Definition 4.2. Define $\varphi_1 = 1$. For $S$ a standard filling of $\mu$, $|\mu| = n$, define $\varphi_S$ by

$$
\varphi_S = \varphi_{S\setminus n} \prod_{j > n \text{row}(n)} (x_n - \alpha_j) \cdot \begin{cases} 
\prod_{\text{col}(n) < i \leq \mu_1} (y_n - \beta_i) & \text{if } \mu \text{ is a single row;} \\
\prod_{\text{col}(n) < \text{col}(k)} (y_k - \beta_k) & \text{if row}(n) > 1 \text{ or } \text{col}(n) > 1, \\
\prod_{\text{col}(n) < \text{col}(k)} (y_k - \beta_k) & \text{otherwise,}
\end{cases}
$$

where $l_k$ is the maximum column index of all entries in row 1 larger than and to the left of $k$, and $K$ is the entry in the second column of the bottom row.

For example, we compute

$$
\varphi = (y_2 - \beta_3) \cdot \frac{(y_3 - \beta_1)(y_2 - \beta_2)}{(x_4 - \alpha_1)} \cdot \frac{(x_4 - \alpha_1)(x_4 - \alpha_3)}{(x_2 - \alpha_2)} \cdot \frac{(y_2 - \beta_1)}{(x_2 - \alpha_2)} \cdot (x_2 - \alpha_2) \cdot 1,
$$

where each step in the recursion is indicated by the cell removed to obtain the given terms.

Both Proposition 4.3 and Theorem 4.4 are evident from the kicking tree description and are straightforward from the recursive definition.

Proposition 4.3. For $S$ a standard filling of a hook $\mu$, $\varphi_S$ is a polynomial.

Proof. The result for $\mu = (1)$ is clear, so we proceed by induction on $N = |\mu|$. Based on the cases above, we may assume $N$ lies in the first row and not in the rightmost column. Then each term appearing in the denominator in Definition 4.2 must appear in the numerator at some point in the construction of $\varphi_{S\setminus N}$ based on the leftmost product. Furthermore, this is the only instance when a given term may appear in the denominator. □

Theorem 4.4. The $n! \times n!$ evaluation matrix $(\varphi_S(p_T))$, where $S,T$ range over all fillings of $\mu$, is upper triangular with nonzero diagonal entries. In particular, the set $\{\varphi_S\}$ forms a basis for $R_\mu$.

Proof. Consider the following total order on fillings of $\mu$ given by the kicking tree construction. Let $l$ be the largest entry such that $l$ lies in different rows of $S$ and $T$, or 0 if no such entry exists. If $l$ is greater than some entry in the bottom row of either $S$ or $T$, then order $S$ and $T$ by row dominance with respect to $l$. Otherwise $S$ and $T$ must contain the same set of entries in their bottom rows and all of these are greater than $l$. In this case, if $l > 0$, then order $S$ and $T$ lexicographically by the bottom row, breaking a tie with row dominance with respect to $l$, and if $l = 0$, then order $S$ and $T$ by reverse lexicographic order. We claim that the matrix $(\varphi_S(p_T))$ is upper triangular with respect to this ordering. Since the matrix clearly has nonzero diagonal entries, the proposition will follow from the claim.

We proceed by induction on $N = |\mu|$, the case $N = 1$ being trivial. If $\mu$ is a single row, then ordering the fillings based on the column of $N$ gives a block triangular matrix, so by induction the matrix is upper triangular. In general, ordering the fillings based on the row of $N$ gives a block triangular matrix, and by induction each block corresponding to $N$ not in the first row may be assumed to be upper triangular with nonzero diagonal. Therefore it suffices to consider fillings of an honest hook with $N$ in the first row.
The result would be immediate if not for the terms in the denominator of \( \varphi_S \). The term \( x_j - \alpha_1 \) will evaluate to zero on any filling with \( j \) in the bottom row. However, this term appears in the right hand denominator only if \( y_i - \beta_1 \) occurs in the numerator for each \( i \) in the bottom row. Any filling to the left of this which has \( j \) in the bottom row, necessarily has at least one of these \( i \)'s not in the bottom row, and so the product of the \( y \)-terms will still annihilate these fillings. Similarly, the term \( x_k - \alpha_j \) appears in the denominator only if \( y_k - \beta_1 \) appears in the numerator, so any filling annihilated by \( x_k - \alpha_j \) must have \( k \) in the first column and hence will be annihilated by \( y_k - \beta_1 \). Therefore no triangularity is lost, and the claim is proved. \( \square \)

4.3. **Symmetry of the Hilbert series.** In this section we show that the bi-graded Hilbert series of the space spanned by \( B_\mu \) has the same symmetry as the Hilbert series for \( R_\mu \).

As before, let \( \Phi_\mu \) denote the kicking basis and define

\[
F_\mu(q,t) = \sum_{S: \mu \sim S [n]} \hat{\varphi}_S(t, \ldots, t; q, \ldots, q),
\]

where \( \hat{\varphi}_S \) is the leading term of \( \varphi_S \). Note that for a hook \( \mu = (n-m, 1^m) \), the largest powers of \( q \) and \( t \) are \((n-m)(n-m-1)/2 \) and \( m(m+1)/2 \), which again agree with \( n(\mu') \) and \( n(\mu) \), respectively.

Similar to the two column case, we can show the desired symmetry for \( F_\mu(q,t) \) by defining a more general function \( J_\mu(q,t) \). By deriving suitable recurrence relations for \( F \) and \( J \) in order to establish the following theorem.

**Theorem 4.5.** For \( \mu \) a hook partition, both \( F_\mu(q,t) \) and \( J_\mu(q,t) \) exhibit the desired symmetry. In particular, we have a basis for \( \mathcal{H}_\mu \) of size \( n! \), and so \( K_{\lambda,\mu}(q,t) \in \mathbb{N}[q,t] \).

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