

PERFECT MATCHINGS IN RANDOM SPARSIFICATIONS OF DIRAC HYPERGRAPHS

DONG YEAP KANG, TOM KELLY, DANIELA KÜHN, DERYK OSTHUS, AND VINCENT PFENNINGER

ABSTRACT. For all integers \( n \geq k > d \geq 1 \), let \( m_d(k, n) \) be the minimum integer \( D \geq 0 \) such that every \( k \)-uniform \( n \)-vertex hypergraph \( H \) with minimum \( d \)-degree \( \delta_d(H) \) at least \( D \) has an optimal matching. For every fixed integer \( k \geq 3 \), we show that for \( n \in \mathbb{N} \) and \( p = \Omega(n^{-k+1} \log n) \), if \( H \) is an \( n \)-vertex \( k \)-uniform hypergraph with \( \delta_{k-1}(H) \geq m_{k-1}(k, n) \), then a.a.s. its \( p \)-random subhypergraph \( \mathcal{H}_p \) contains a perfect matching. Moreover, for every fixed integer \( d < k \) and \( \gamma > 0 \), we show that the same conclusion holds if \( H \) is an \( n \)-vertex \( k \)-uniform hypergraph with \( \delta_d(H) \geq m_d(k, n) + \gamma(n^{-d}) \).

Both of these results strengthen Johansson, Kahn, and Vu’s seminal solution to Shamir’s problem and can be viewed as “robust” versions of hypergraph Dirac-type results. In addition, we also show that in both cases above, \( H \) has at least \( \exp((1-1/k)n \log n - \Theta(n)) \) many perfect matchings, which is best possible up to an \( \exp(\Theta(n)) \) factor.

1. Introduction

A hypergraph is an ordered pair \( H = (V, E) \) of a set \( V := V(H) \) of vertices of \( H \) and a set \( E := E(H) \) of subsets of \( V \), where the elements of \( E \) are called the edges of \( H \). If \( E(H) \subseteq \binom{V}{k} \) for some positive integer \( k \), then we call \( H \) \( k \)-uniform. We often identify \( E(H) \) with \( H \) if its set of vertices is clear. A matching of a hypergraph \( H \) is a set of disjoint edges of \( H \). An optimal matching of a \( k \)-uniform hypergraph \( H \) is a matching consisting of \( \lfloor |V(H)|/k \rfloor \) edges. An optimal matching of a \( k \)-uniform hypergraph \( H \) is called perfect if \( k \) divides \( |V(H)| \).

In a seminal paper by Edmonds [13], it is proved that there exists a polynomial-time algorithm to determine whether a given graph has a perfect matching. However, for \( k \geq 3 \), it is NP-complete to decide whether a given \( k \)-uniform hypergraph has a perfect matching (see [27, 43]). Thus, it is natural to consider sufficient conditions which force a perfect matching; a minimum degree condition, which is called a Dirac-type condition because of Dirac’s [12] classical result on Hamilton cycles in graphs, is one of the most intensively studied [61, 72]. Perfect matchings in random graphs and hypergraphs have also attracted considerable interest. The so-called Shamir’s problem (see [15]) of determining the threshold for the existence of a perfect matching in a random \( k \)-uniform hypergraph.

DONG YEAP KANG, EXTREMAL COMBINATORICS AND PROBABILITY GROUP (ECOPRO), INSTITUTE FOR BASIC SCIENCE (IBS), DAEJEON, SOUTH KOREA

TOM KELLY, SCHOOL OF MATHEMATICS, GEORGIA INSTITUTE OF TECHNOLOGY, ATLANTA, GA 30332, USA

DANIELA KÜHN, SCHOOL OF MATHEMATICS, UNIVERSITY OF BIRMINGHAM, EDBGBASTON, BIRMINGHAM, B15 2TT, UNITED KINGDOM

DERYK OSTHUS, SCHOOL OF MATHEMATICS, UNIVERSITY OF BIRMINGHAM, EDBGBASTON, BIRMINGHAM, B15 2TT, UNITED KINGDOM

VINCENT PFENNINGER, INSTITUTE OF DISCRETE MATHEMATICS, GRAZ UNIVERSITY OF TECHNOLOGY, GRAZ, AUSTRIA

E-mail addresses: dykang.math@ibs.re.kr, tom.kelly@gatech.edu, D.Kuhn@bham.ac.uk, D.Osthus@bham.ac.uk, pfenninger@math.tugraz.at.

2020 Mathematics Subject Classification. 05C80, 05C65, 05C70, 05D40.

Key words and phrases. perfect matching, random graph, random hypergraph, threshold, Shamir’s problem, absorbing method, spreadness.

This project has received partial funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement no. 786198, D. Kang, T. Kelly, D. Kühn, D. Osthus, and V. Pfenninger). Dong Yeap Kang was supported by Institute for Basic Science (IBS-R029-Y6).
hypergraph was considered one of the most important problems in probabilistic combinatorics before its resolution by Johansson, Kahn, and Vu [37] in 2008. Our results in this paper connect these two streams of research.

1.1. Perfect matchings in Dirac hypergraphs. For $d \in \mathbb{N}$, the minimum $d$-degree $\delta_d(H)$ of a hypergraph $H$ is the minimum of $\{|e \in H : v_1, \ldots, v_d \in e\}$ among all choices of $d$ distinct vertices $v_1, \ldots, v_d \in V(H)$. If $H$ is $k$-uniform, we also call $\delta_{k-1}(H)$ the minimum codegree. For $n \geq k > d \geq 1$, let $m_d(k, n)$ be the minimum integer $D \geq 0$ such that every $k$-uniform $n$-vertex hypergraph $H$ with $\delta_d(H) \geq D$ has an optimal matching, and for each $s \in \{0, \ldots, k-1\}$, let

$$\overline{\mu}_d(s)(k) := \limsup_{n \to \infty} \frac{m_d(k, n)}{\binom{n-s}{n-d}}.$$ 

Determining the value of $m_d(k, n)$, or even just $\overline{\mu}_d(s)(k)$ in many cases, is a major open problem. Rödl, Ruciński, and Szemerédi [65] first proved that $m_{k-1}(k, n) \leq n/2 + o(n)$ for $n \in k\mathbb{N}$ (in fact, they showed a tight Hamilton cycle exists if this codegree condition holds). This bound was improved by Kühn and Osthus [52] to $n/2 + 3k^2\sqrt{n} \log n$, and Rödl, Ruciński, and Szemerédi [62] improved it further to $n/2 + O(\log n)$. Finally, Rödl, Ruciński, and Szemerédi [63] determined $m_{k-1}(k, n) = n/2 - k + C(k, n)$ for all sufficiently large $n \in k\mathbb{N}$, with $C(k, n) \in \{3/2, 2, 5/2, 3\}$ depending on $k$ and $n$. Rödl, Ruciński, Schacht, and Szemerédi [64] also gave a simple proof for a bound of $n/2 + k/4$, that does not require $n$ to be large.

For $1 \leq d \leq 3k/8$ and $n \in k\mathbb{N}$, both the exact and the asymptotic values of $m_d(k, n)$ are unknown for many cases. The exact value of $m_d(k, n)$ is known for $d \geq 3k/8$ and large $n \in k\mathbb{N}$ by a combination of results [21, 71], where $m_d(k, n) = (1/2 + o(1))(\binom{n}{k-d})$ and the exact bound of $m_d(k, n)$ follows from the obstructions called divisibility barriers. Khan [45] and independently Kühn, Osthus, and Treglown [55] showed that $m_1(3, n) = \binom{n-1}{2} - \binom{2n/3}{2}$ for large $n \in 3\mathbb{N}$. Khan [46] showed that $m_1(4, n) = \binom{n-1}{3} - \binom{3n/4}{3}$ for large $n \in 4\mathbb{N}$. Alon, Frankl, Huang, Rödl, Ruciński, and Sudakov [2] related the asymptotics of $m_d(k, n)$ and $m_d^*(k, n)$, where $m_d^*(k, n)$ is the minimum $D$ such that every $n$-vertex $k$-uniform hypergraph with minimum $d$-degree at least $D$ has fractional matching number $n/k$. Ferber and Kwan [19] showed that the limit of $m_d(k, n)/(\binom{n}{k-d})$ exists as $n \in k\mathbb{N}$ tends to infinity, and it is conjectured [34, 53] that $\overline{\mu}_d^*(0)(k) = \max(1/2, 1 - k^{1-kk^{-d}})$. See [72, Conjecture 1.5] for the exact conjectured value of $m_d(k, n)$ when $n \in k\mathbb{N}$ is large.

For the other case $k \nmid n$, Rödl, Ruciński, and Szemerédi [63] showed that $m_{k-1}(k, n) \leq n/k + O(\log n)$, and Han [35] determined $m_{k-1}(k, n) = \lfloor n/k \rfloor$ for all sufficiently large $n$ (and thus $\overline{\mu}_d^*(s)(k) = 1/k$ for all $s \neq 0$). Han [36, Conjecture 1.10] conjectured an upper bound on $\overline{\mu}_d(s)(k)$ for all $d, s \in [k-1]$, and proved a matching lower bound on $m_d(k, n)$. (Thus, if true, Han’s conjecture implies the limit of $m_d(k, n)/(\binom{n}{k-d})$ exists as $n \in k\mathbb{N} + s$ tends to infinity.) See [8, 36, 57] for more background on the non-divisible case, and for more discussion of this topic, see the surveys [61, 72].

1.2. Perfect matchings in random hypergraphs. A random $k$-uniform $n$-vertex hypergraph $H^k(n, p(n))$ is a $k$-uniform hypergraph on $n$ vertices obtained by choosing each subset of $k$ vertices to be an edge with probability $p(n)$ independently at random. Regarding the existence of a perfect matching in a random hypergraph, it is natural to ask for the threshold for $H^k(kn, p(n))$ to contain a perfect matching.

For $k = 2$, in a seminal paper Erdős and Rényi [14] determined the (sharp) threshold for $H^2(2n, p(n))$ to contain a perfect matching. They showed that the probability that $H^2(2n, p(n))$ has a perfect matching tends to 1 if $p(n) = \frac{\log n + o(1)}{2n}$ and tends to 0 if $p(n) = \frac{\log n - o(1)}{2n}$.

On the other hand, for $k \geq 3$, it is much more difficult to determine the threshold for the appearance of a perfect matching. In 1979, Shamir (see [15, 67]) asked for the threshold for $H^k(kn, p(n))$ to contain a perfect matching (a precise and explicit statement was mentioned in [11]).
Schmidt and Shamir [67] showed that asymptotically almost surely (which we abbreviate as a.a.s.) \( H^k(kn, p(n)) \) has a perfect matching if \( p(n) = \omega(n^{-k+3/2}) \). This was further improved by Frieze and Janson [23] to \( p(n) = \omega(n^{-k+4/3}) \). Finally, Johansson, Kahn, and Vu [37] proved that the threshold for \( H^k(kn, p(n)) \) to contain a perfect matching is \( \Theta(n^{-k+1}\log n) \), matching the threshold for \( H^k(kn, p(n)) \) not to contain an isolated vertex. Recently, Kahn [40] determined the sharp threshold for \( H^k(kn, p(n)) \) to contain a perfect matching, as well as the hitting time result [39], which proves the conjecture in [11] in a stronger form.

1.3. Robust version of Dirac-type theorems. For any hypergraph \( H \) and \( p \in [0, 1] \), let \( H_p \) be a spanning random subhypergraph of \( H \) obtained by choosing each edge \( e \in H \) with probability \( p \) independently at random. The problem of determining whether a certain property of the original hypergraph \( H \) is retained by \( H_p \) has been studied extensively [1, 3, 10, 28, 38, 50, 51, 59], and results of this nature are referred to as robustness results [69]. For example, Krivelevich, Lee, and Sudakov [50] showed a robust version of Dirac’s theorem that for every \( n \)-vertex graph \( G \) with minimum degree at least \( n/2 \), a.a.s. its random subgraph \( G_p \) contains a Hamilton cycle for \( p = p(n) \geq C \log n/n^{k-1} \) for some absolute constant \( C > 0 \), providing a common generalization of Dirac’s theorem [12] (when \( p = 1 \)) and the classic result of Pósa [60] (when \( G \cong K_n \)) on the threshold for the appearance of a Hamilton cycle in a random graph.

Our first result is the following robust version of hypergraph Dirac-type results on \( m_d(k, n) \) in the general case \( 1 \leq d \leq k-1 \). Here, the integer \( n \) is not necessarily divisible by \( k \). The case \( k = 2 \) follows from the result by Krivelevich, Lee, and Sudakov [50] on the robust Hamiltonicity.

**Theorem 1.1.** Let \( d, k, s \in \mathbb{Z} \) such that \( k \geq 3 \), \( 1 \leq d \leq k-1 \), and \( 0 \leq s \leq k-1 \). For every \( \gamma > 0 \), there exists \( C > 0 \) such that the following holds for \( n \in k\mathbb{N} + s \) and \( p = p(n) \in [0, 1] \) with \( p \geq C \log n/n^{k-1} \). If \( H \) is a \( k \)-uniform \( n \)-vertex hypergraph with \( \delta_d(H) \geq (\frac{\mu_d(s)}{k-d})(\frac{n-d}{k-d}) \), then a.a.s. a random subhypergraph \( H_p \) contains an optimal matching.

Combining this result with the aforementioned prior work determining \( m_d(k, n) \) [2, 21, 63, 71], we simultaneously obtain that for every \( \gamma > 0 \), as \( n \to \infty \), for \( p = \Omega(\log n/n^{k-1}) \), \( H_p \) a.a.s. has a perfect matching when \( H \) is a \( k \)-uniform \( n \)-vertex hypergraph satisfying \( \delta_d(H) \geq (1/2 + \gamma)(\frac{n-d}{k-d}) \) for some \( d \geq 3k/8 \) when \( k \mid n \) and that \( H_p \) a.a.s. has an optimal matching when \( H \) is a \( k \)-uniform \( n \)-vertex hypergraph satisfying \( \delta_{k-1}(H) \geq (1/k + \gamma)n \) when \( k \nmid n \). Another interesting feature of this result is that it implies the existence of optimal matchings in random sparsifications of hypergraphs with minimum \( d \)-degree at least \( (\frac{\mu_d(s)}{k-d})(\frac{n-d}{k-d}) \) even in the cases in which the value of \( \frac{\mu_d(s)}{k-d} \) is not known. Since \( \lim_{n \to \infty; k|n} m_d(k, n)/\binom{n-d}{k-d} = \frac{\mu_d(0)}{k-d} \) [19], for \( n \in k\mathbb{N} \), the minimum degree condition in Theorem 1.1 can be replaced by \( \delta_d(H) \geq m_d(k, n) + \gamma(n-d) \).

Our main result is the following robust version of the Dirac-type result by Rödl, Ruciński, and Szemerédi [63].

**Theorem 1.2.** Let \( k \geq 3 \) be an integer. There exists \( C > 0 \) such that the following holds for \( n \in k\mathbb{N} \) and \( p = p(n) \in [0, 1] \) with \( p \geq C \log n/n^{k-1} \). If \( H \) is a \( k \)-uniform \( n \)-vertex hypergraph with \( \delta_{k-1}(H) \geq m_{k-1}(k, n) \), then a.a.s. a random subhypergraph \( H_p \) contains a perfect matching.

The value of \( p \) in both Theorems 1.1 and 1.2 is asymptotically best possible, since it is well known that a.a.s. there are \( \omega(1) \) isolated vertices in a random \( k \)-uniform \( n \)-vertex hypergraph \( H^k(n, p) \) if \( p \leq \frac{(k-1)\log n - \omega(1)}{n^{k-1}} \). In fact, both results generalize Johansson, Kahn, and Vu’s [37] solution to Shamir’s problem that the threshold for the existence of a perfect matching in \( H^k(kn, p(n)) \) is \( \Theta(n^{-k+1}\log n) \).

We remark that Theorems 1.1 and 1.2 are implied by Theorems 1.5 and 1.6 below, respectively.
1.4. **Spreadness and a lower bound on the number of perfect matchings.** To prove Theorems 1.1 and 1.2, we use the fractional version of the Kahn–Kalai conjecture [41] (conjectured by Talagrand [70]), recently resolved by Frankston, Kahn, Narayanan, and Park [22]. The Kahn–Kalai conjecture was recently proved in full by Park and Pham [58], but the fractional version is sufficient for our application. A precursor to these results was the main technical ingredient in Alweiss, Lovett, Wu, and Zhang’s [4] breakthrough on the Erdős–Rado sunflower conjecture [16], and the results have been found to have many additional applications. This paper, and the independent work of Pham, Sah, Sawhney, and Simkin [59] (discussed further in the remark at the end of this subsection), are the first to demonstrate an application of the result to robustness of Dirac-type results.

The Frankston–Kahn–Narayanan–Park theorem implies the Johansson–Kahn–Vu solution to Shamir’s problem. Moreover, it reduces our problem to proving that there exists a probability measure on the set of perfect or optimal matchings that is ‘well-spread’. Roughly speaking, this means that the probability measure chooses a perfect matching at random in such a way that no particular set of edges is very likely to be contained in the matching.

**Definition 1.3** (Spreadness). Let $H$ be a $k$-uniform hypergraph and $q \in [0, 1]$. Let $\nu$ be a probability measure on the set of matchings of $H$, and let $M$ be a matching in $H$ chosen at random according to $\nu$. We say that $\nu$ is $q$-spread if for each $s \geq 1$ and $e_1, \ldots, e_s \in H$, we have

$$\mathbb{P}[e_1, \ldots, e_s \in M] \leq q^s.$$  

The next theorem follows from [22, Theorem 1.6]. More precisely, it follows from the derivation of [22, Theorem 1.1] from [22, Theorem 1.6].

**Theorem 1.4** (Frankston, Kahn, Narayanan, and Park [22]). There exists $K > 0$ such that the following holds. Let $H$ be a $k$-uniform $n$-vertex hypergraph and $q \in [0, 1]$. If there exists a $q$-spread probability measure on the set of optimal matchings of $H$ and $p \geq Kq \log n$, then a.a.s. there exists an optimal matching in $H_p$.

In particular, by Theorem 1.4, it suffices to prove the following results to deduce Theorems 1.1 and 1.2, respectively.

**Theorem 1.5.** Let $d, k, s \in \mathbb{Z}$ such that $k \geq 3$, $1 \leq d \leq k-1$, and $0 \leq s \leq k-1$. For every $\gamma > 0$, there exist $C > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for all $n \in k\mathbb{N} + s$ with $n \geq n_0$. For every $k$-uniform $n$-vertex hypergraph $H$ with $\delta_d(H) \geq (\mu_d^{(k)}(k) + \gamma)(n/k)^{d-1}$, there exists a probability measure on the set of optimal matchings in $H$ which is $(C/n^{k-1})$-spread.

**Theorem 1.6.** Let $k \geq 3$ be an integer. There exist $C > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for all integers $n \geq n_0$ divisible by $k$. For every $k$-uniform $n$-vertex hypergraph $H$ with $\delta_{k-1}(H) \geq m_{k-1}(k,n)$, there exists a probability measure on the set of perfect matchings in $H$ which is $(C/n^{k-1})$-spread.

For a $k$-uniform $n$-vertex hypergraph $H$ with $\delta_{k-1}(H) \geq \delta n$ for some $\delta > 1/2$, there are some earlier results [17, 18, 29] on counting the number of perfect matchings in $H$. Recently, Glock, Gould, Joos, Kühn, and Osthus [29] showed that $H$ has at least $\exp((1-1/k)n \log n - \Theta(n))$ perfect matchings, which is best possible up to an $\exp(\Theta(n))$ factor (this is also implicit in [18]), since this is also an upper bound for the number of perfect matchings in an $n$-vertex $k$-uniform complete hypergraph. Very recently, Ferber, Hardiman, and Mond [17] sharpened the bound further by showing that $H$ has at least $(1 - o(1))n|\mathcal{M}(K^k_n)|\delta^{n/k}$ perfect matchings, where $|\mathcal{M}(K^k_n)|$ is the number of perfect matchings in an $n$-vertex complete $k$-uniform hypergraph.

As a corollary to Theorems 1.5 and 1.6, we extend the above results of [18, 29] to $n$-vertex $k$-uniform hypergraphs with minimum $d$-degree at least $m_d(k,n) + o(n^{k-d})$ or minimum codegree at least $m_{k-1}(k,n)$. Our bounds are best possible up to an $\exp(\Theta(n))$ factor.
Corollary 1.7. Let \( k \geq 3 \) be an integer and \( \gamma \in (0, 1) \). There exist \( c > 0 \) and \( n_0 \in \mathbb{N} \) such that the following holds for all integers \( n \geq n_0 \).

(i) For \( d \in [k-1] \), if \( n \equiv s \pmod{k} \), then every \( k \)-uniform \( n \)-vertex hypergraph \( H \) with \( \delta_d(H) \geq (\pi_d(k)(k + \gamma) \binom{n-2}{d-2} \) has at least \( \exp((1 - 1/k)n \log n - cn) \) optimal matchings.

(ii) If \( k \mid n \), then every \( k \)-uniform \( n \)-vertex hypergraph \( H \) with \( \delta_{k-1}(H) \geq m_{k-1}(k, n) \), has at least \( \exp((1 - 1/k)n \log n - cn) \) perfect matchings.

Proof. Let \( \mathcal{M}(H) \) be the set of optimal matchings of \( H \), and let \( M \) be a \((C/n^{k-1})\)-spread random optimal matching of \( H \) for some \( C > 0 \) given by Theorems 1.5 or 1.6, where \( C \) is a function of \( k \) and \( \gamma \) for (i), and a function of \( k \) for (ii).

For each fixed matching \( M \in \mathcal{M}(H) \) which consists of \( [n/k] \) edges \( e_1, \ldots, e_{[n/k]} \), we have that
\[
\mathbb{P}[e_1, \ldots, e_{[n/k]} \in M] = \mathbb{P}[M = M] \leq (C/n^{k-1})^{[n/k]} \text{ since } M \text{ is } (C/n^{k-1})\text{-spread. Thus,}
\]
\[
1 = \sum_{M \in \mathcal{M}(H)} \mathbb{P}[M = M] \leq |\mathcal{M}(H)|(C/n^{k-1})^{[n/k]},
\]
which implies \(|\mathcal{M}(H)| \geq (n^{k-1}/C)^{[n/k]} = \exp((1 - 1/k)n \log n - (\log C/k)n \pm k \log n)\), as desired. \( \square \)

Remark. In independent work, Pham, Sah, Sawhney, and Simkin [59] also proved Theorem 1.5 and its corollaries Corollary 1.7(i) and Theorem 1.1. In addition, they proved a robust version of the Hajnal–Szemerédi theorem [32] regarding embedding a \( K_r \)-factor into an \( n \)-vertex graph with minimum degree at least \((1 - 1/r)n\) and a robust version of Komlós, Sárközy, and Szemerédi’s [49] proof of Bollobás’ conjecture that an \( n \)-vertex graph with minimum degree at least \((1/2 + o(1))n\) contains any bounded-degree spanning tree. Both of these results are also derived from stronger results concerning spread measures.

1.5. Notation. For \( k \in \mathbb{N} \), we let \([k] := \{1, \ldots, k\}\). For a \( k \)-uniform hypergraph \( H \) and an edge \( e \in H \), we often denote by \( V(e) \) the set of vertices incident to \( e \). For any set \( S \subseteq V(H) \), we denote by \( H[S] \) the subgraph of \( H \) induced by \( S \). For sets \( S \subseteq V(H) \) and \( S' \subseteq \binom{V(H)}{k-1} \), we denote by \( N_H(S; S') \) the set of edges \( e \in H \) such that \( e = S \cup S' \) for some \( S' \subseteq S \). If \( |S| = k - 1 \) and \( U \subseteq V(H) \), we abuse notation and write \( d_H(S; U) \) for \( d_H(S; \{u \in U \}) \). Moreover, for \( v \in V(H) \), we write \( d_H(v; S) \) for \( d_H(S; \{v \}) \).

For disjoint subsets \( W_1, \ldots, W_k \subseteq V(H) \), we denote by \( e_H(W_1, \ldots, W_k) \) the number of edges \( e, e' \in H \) with \( |e \cap e'| = 1 \) for all \( i \in [k] \). We denote by \( \overline{H} \) the complement of a \( k \)-uniform hypergraph \( H \) such that \( V(\overline{H}) := V(H) \) and \( \overline{H} := \binom{V(H)}{k} \setminus H \). We take all asymptotic notations \( o(\cdot), O(\cdot), \Theta(\cdot), \omega(\cdot), \Omega(\cdot) \) to be as \( n \to \infty \), and all of their leading coefficients may depend on parameters other than \( n \). We say that an event \( \mathcal{E} \) holds asymptotically almost surely (a.a.s.) if \( \mathbb{P}[\mathcal{E}] = 1 - o(1) \) as \( n \to \infty \). For real numbers \( x, y, \alpha, \) and \( \beta \) with \( \beta \geq 0 \), we write \( x = (\alpha \pm \beta)y \) for \((\alpha - \beta)y \leq x \leq (\alpha + \beta)y \). We sometimes state a result with a hierarchy of constants which are chosen from right to left. If we state that the result holds whenever \( a \ll b_1, \ldots, b_i \), then this means that there exists a function \( f : (0, 1)^i \to (0, 1) \) such that \( f(b_1, \ldots, b_i) \leq f(b_1, \ldots, b_i, 1) \) for \( 0 < b_i \leq b_i < 1 \) and the result holds for all real numbers \( 0 < a, b_1, \ldots, b_i < 1 \) with \( a \ll f(b_1, \ldots, b_i) \).

If a reciprocal \( 1/m \) appears in such a hierarchy, we implicitly assume that \( m \) is a positive integer. For a set \( U \) and \( p \in [0, 1] \), a \( p \)-random subset of \( U \) is a random subset \( U' \) of \( U \) that contains each element of \( U \) independently with probability \( p \).

1.6. Proof outline. Here we briefly sketch the proofs of Theorems 1.5 and 1.6. One key idea of both proofs is that we can use the weak hypergraph regularity lemma (Theorem 2.7) to find a distribution on almost perfect matchings which has good spreadness. The weak hypergraph regularity lemma gives us a reduced \( k \)-uniform hypergraph \( R \) such that almost all subsets of \( V(R) \) of size \( d \) have large \( d \)-degree. Using Lemma 2.9, we can find an almost perfect matching \( M_R \) of \( R \)
such that each edge of $M_R$ corresponds to a vertex-disjoint pseudorandom $k$-partite subhypergraph in which we can easily construct an almost perfect matching with spreadness.

To obtain a distribution on optimal matchings, we use an approach inspired by the method of "iterative absorption" (introduced in [48, 54] and further developed in [5, 6, 7, 30, 31, 42, 56, 66]). Our iterative-absorption approach, combined with the regularity lemma, allows us to ‘bootstrap’ results on the existence of optimal matchings to construct well-spread distributions on optimal matchings. In this approach, we choose a random partition $(U_1, \ldots, U_t)$ of the vertex-set of the $k$-uniform hypergraph $H$, which we call a vertex vortex (Definition 3.1), which a.a.s. satisfies $|U_{i+1}| \sim |U_i|/2$, $|U_1| = O(n^{1/k})$, and some additional conditions on degrees of the vertices in $H[U_i]$ and $H[U_i, U_{i+1}]$. Using the regularity-lemma approach described above, we can find a well-spread distribution on matchings in $H[U_i]$ which cover almost all vertices. Then we cover the leftover uncovered vertices in $U_i$ using edges which intersect $U_{i+1}$ in $k-1$ vertices. By the degree conditions of the vertex vortex, there are many choices of such edges, so a random greedy approach yields a distribution with good enough spreadness. After iterating this procedure $\ell - 1$ times, it suffices to find an optimal matching in the final subset $U_\ell$ (with a small subset of vertices deleted), in a deterministic way, since $|U_\ell| = O(n^{1/k})$ and $\mathbb{P}[e_1, \ldots, e_\ell \subseteq U_\ell] = (|U_\ell|/n)^k = (O(1)/n^{k-1})^\ell$ for any $t$ disjoint edges $e_1, \ldots, e_\ell \in H$.

In the setting of Theorem 1.5, it is straightforward to find an optimal matching in the final step; the hypergraph induced on the remaining vertices will still be sufficiently dense. For Theorem 1.6, we show that the result of Rödl, Ruciński, and Szemerédi [63] holds ‘robustly’. Roughly speaking, it holds in the hypergraph induced by a random set of vertices, even after deleting a small proportion of the vertices. For this we must consider two cases according to whether the original hypergraph is close to being a ‘critical hypergraph’ (see Definition 4.2) which has minimum codegree $m_{k-1}(k, n) - 1$ and no perfect matching. If the original hypergraph $H$ is close to being a critical hypergraph, then we may choose an ‘atypical edge’ among $\Omega(n^{k-1})$ candidates (Lemma 6.9) and delete its vertices in advance. This ensures that the subhypergraph of $H$ induced by the remaining vertices in $U_\ell$ will meet certain ‘divisibility’ conditions and allow us to apply some technical results proved by Rödl, Ruciński, and Szemerédi to find a perfect matching covering the remaining vertices of $U_\ell$. Moreover, since there are $\Omega(n^{k-1})$ candidates for the atypical edge, we have the desired spreadness property for this edge. In the second case, the original hypergraph $H$ is not close to being a critical hypergraph. In this case, we prove that there are still many ‘absorbers’ inside $U_\ell$ (Corollary 5.4), which we can use to build an ‘absorbing matching’. As long as the vertices of the absorbing matching are not among those removed from $U_\ell$, we can transform an almost perfect matching (which covers most of the remaining vertices of $U_\ell$) into a perfect matching (i.e., one which covers all remaining vertices of $U_\ell$).

2. Tools

2.1. Concentration inequalities. We will use the following well-known version of the Chernoff bound.

Lemma 2.1 (Chernoff bound). If $X$ is the sum of mutually independent Bernoulli random variables, then for all $\delta \in [0, 1]$,

$$\mathbb{P}\left[|X - \mathbb{E}[X]| \geq \delta \mathbb{E}[X]\right] \leq 2e^{-\delta^2 \mathbb{E}[X]/3}.$$  

Definition 2.2 (Typical subset). Let $V$ be a finite set, and let $\mathcal{F} \subseteq 2^V$ be a collection of subsets of $V$. For $p, \varepsilon \in [0, 1]$, a subset $U \subseteq V$ is called $(p, \varepsilon, \mathcal{F})$-typical if the number of elements in $\mathcal{F}$ contained in $U$ is $(1 \pm \varepsilon)\sum_{S \in \mathcal{F}} p^{|S|}$.

We will use the following probabilistic lemma which follows from the Kim–Vu polynomial concentration theorem [47]. For the proof, see Appendix A.
Lemma 2.3. Let $1/n \ll 1/s, \beta, \varepsilon < 1$ and $k \geq 2$. Let $V$ be a set of size $n$. Let $p = p(n) \in [0, 1]$ such that $np \geq \varepsilon n^\beta$. Let $F \subseteq \binom{V}{k}$, and let $U$ be a $p$-random subset of $V$. Then the following holds.

(i) If $|F| \geq \varepsilon n^8(np)^{-1/2}$, then with probability at least $1 - \exp(-n^\beta/(10s))$, the set $U$ is $(p, \varepsilon, F)$-typical.

(ii) If $|F| \leq \varepsilon n^8$, then with probability at least $1 - \exp(-n^\beta/(10s))$, the number of elements of $F$ contained in $U$ is at most $2\varepsilon(np)^8$.

2.2. Weak hypergraph regularity. We now introduce the weak hypergraph regularity lemma, which states that any $k$-uniform hypergraph has a vertex partition into clusters $\{V_i\}_{0 \leq i \leq t}$ so that almost all $k$-tuples of clusters induce $\varepsilon$-regular subhypergraphs. Since the notion of $\varepsilon$-regularity is 'weak', its proof is very similar to the graph version. Readers should not confuse the weak hypergraph regularity lemma with the Frieze–Kannan weak regularity lemma [24].

Definition 2.4 ($\varepsilon$-regular $k$-tuple). Let $\varepsilon > 0$ and let $H$ be a $k$-uniform hypergraph. We say that a $k$-tuple $(V_1, \ldots, V_k)$ of mutually disjoint subsets of $V(H)$ is $(d, \varepsilon)$-regular if $e_H(W_1, \ldots, W_k) = (d \pm \varepsilon)|W_1| \cdots |W_k|$ for every $W_1 \subseteq V_1, \ldots$, and $W_k \subseteq V_k$ with $|W_1| \cdots |W_k| \geq \varepsilon|V_1| \cdots |V_k|$. Moreover, we say that $(V_1, \ldots, V_k)$ is $\varepsilon$-regular if it is $(d, \varepsilon)$-regular for some $d > 0$.

Definition 2.5 ($\varepsilon$-regular partition). Let $\varepsilon > 0$, and let $H$ be a $k$-uniform hypergraph. A partition $(V_0, V_1, \ldots, V_t)$ of $V(H)$ is called an $\varepsilon$-regular partition if

- $|V_0| \leq \varepsilon n$ and $|V_1| = \cdots = |V_t|$. 
- For all but at most $\varepsilon \binom{t}{k}$ $k$-sets $\{i_1, \ldots, i_k\} \in \binom{[t]}{k}$, the tuple $(V_{i_1}, \ldots, V_{i_k})$ is $\varepsilon$-regular.

Definition 2.6 (Reduced hypergraph). Let $H$ be a $k$-uniform hypergraph, and let $(V_0, V_1, \ldots, V_t)$ be an $\varepsilon$-regular partition of $V(H)$. The $(\gamma, \varepsilon)$-reduced hypergraph $R$ with respect to $(V_0, V_1, \ldots, V_t)$ is the $t$-vertex $k$-uniform hypergraph with $V(R) = [t]$ and $\{i_1, \ldots, i_k\} \in R$ if and only if $(V_{i_1}, \ldots, V_{i_k})$ is $\varepsilon$-regular and $e_H(V_{i_1}, \ldots, V_{i_k}) \geq \gamma|V_{i_1}| \cdots |V_{i_k}|$.

Theorem 2.7 (Weak hypergraph regularity lemma [9, 20, 68]). Let $1/n, 1/t_1 \ll \varepsilon, 1/t_0 < 1$. For every $n$-vertex $k$-uniform hypergraph $H$, there exists an $\varepsilon$-regular partition $(V_0, \ldots, V_t)$ of $V(H)$ such that $t_0 \leq t \leq t_1$.

The next lemma can be proved with a straightforward adaptation of the proof of [33, Proposition 16], so we defer the proof to Appendix A.

Lemma 2.8. Let $1/n \ll \eta \ll 1/t \ll \varepsilon \ll \gamma < 1, 1/k \leq 1$ with $k \geq 3$ and $d \in [k - 1]$. Let $H$ be a $k$-uniform $n$-vertex hypergraph which satisfies the following.

- All but at most $\eta n^d$ $d$-sets $S \in \binom{V(H)}{d}$ have $d$-degree at least $c(n-d)\beta/\gamma/k$.
- $H$ admits an $\varepsilon$-regular partition $(V_0, \ldots, V_t)$.

Let $R$ be the $(\gamma/3, \varepsilon)$-reduced hypergraph with respect to $(V_1, \ldots, V_t)$. Then all but at most $\varepsilon^{1/2}\binom{t}{d}$ many $d$-sets $S \in \binom{[t]}{d}$ have $d$-degree at least $(c - \gamma)\binom{t-d}{k-d}$ in $R$.

2.3. Almost perfect matchings. For $1 \leq d \leq k - 1$, recall that $m_d(k, n)$ is the minimum $D$ such that every $n$-vertex $k$-uniform hypergraph with minimum $d$-degree at least $D$ has an optimal matching. Let us define

$$\mu_d(k) := \liminf_{n \to \infty} \frac{m_d(k, n)}{(n-d)/(k-d)}.$$

Note that $\mu_d(k) \leq \mu_d^{(s)}(k)$ for $0 \leq s \leq k - 1$, and $\mu_{k-1}(k, n) = 1/k$, since $m_{k-1}(k, n) = n/2 - O(k)$ for large $n \in k\mathbb{N}$ and $m_{k-1}(k, n) = n[k/k]$ for large $n \notin k\mathbb{N}$, as mentioned in Section 1.1. A well-known lower bound on $\mu_d(k)$ is $1 - (\frac{1}{k})^{k-d}$ (see [72, Construction 1.4]).

Now we prove the following lemma which states that if almost all $d$-tuples satisfy the degree condition for an optimal matching then there exists an almost perfect matching. We also remark that there are also similar results on almost perfect matchings [25, 26, 44].
Lemma 2.9. Let $1/n \ll \varepsilon_1 \ll \varepsilon_2 \ll 1/k \leq 1/3$ with $1 \leq d \leq k-1$. Let $\mathcal{H}$ be an $n$-vertex $k$-uniform hypergraph such that $d_{\mathcal{H}}(S) \geq (\mu_d(k) + \varepsilon_2)(\frac{n-d}{k-d})$ for all but at most $\varepsilon_1 n^d$ many $S \in \binom{V(\mathcal{H})}{d}$. Then $\mathcal{H}$ has a matching which covers all but at most $2\varepsilon_2 n$ vertices.

To prove Lemma 2.9, we use the following lemma [19, Lemma 3.4].

Lemma 2.10 (Ferber and Kwan [19]). Let $1/n \ll \delta \ll 1/m \ll \varepsilon \ll c, 1/k < 1$. Let $1 \leq d \leq k-1$. Let $\mathcal{H}$ be an $n$-vertex $k$-uniform hypergraph such that $d_{\mathcal{H}}(S) \geq (c + \varepsilon)(\frac{n-d}{k-d})$ for all but at most $\delta n^d$ many $S \in \binom{V(\mathcal{H})}{d}$. Let $U$ be a random subset of $V(\mathcal{H})$ of size $m$ uniformly chosen from $\binom{V(\mathcal{H})}{m}$. With probability at least $1 - m^d(\delta + e^{-\varepsilon^3 m})$, we have $\delta_{\mathcal{H}}(\mathcal{H}[U]) \geq (c + \varepsilon/2)(\frac{n-d}{k-d})$.

Proof of Lemma 2.9. Let $1/n \ll \varepsilon_1 \ll 1/m \ll \varepsilon_2 \ll 1/k$ such that $m_d(k, m) \leq (\mu_d(k) + \varepsilon_2/2)(\frac{n-d}{k-d})$. For $t := \lfloor n/m \rfloor$, let $U_1, \ldots, U_t$ be $t$ disjoint random subsets of $V(\mathcal{H})$ of size $m$ such that each $U_i$ has a uniform random distribution from $\binom{V(\mathcal{H})}{m}$. For each $i \in [t]$, let $U_i$ be bad if $\delta_{\mathcal{H}}(\mathcal{H}[U_i]) < (\mu_d(k) + \varepsilon_2/2)(\frac{n-d}{k-d})$, and otherwise good. By Lemma 2.10, for each $i \in [t]$, $\mathbb{P}(U_i \text{ is bad}) \leq m^d(\varepsilon_1 + e^{-\varepsilon_2^3 m}) < \varepsilon_2^2$, so the expected number of bad $U_i$’s is at most $\varepsilon_2^2 t$. By Markov’s inequality, with probability at least $1 - \varepsilon_2$, the number of bad $U_i$’s is at most $\varepsilon_2 t$. Fix a choice of $U_1, \ldots, U_t$ for which this holds. For each of the good $U_i$’s, since $m_d(k, m) \leq (\mu_d(k) + \varepsilon_2/2)(\frac{n-d}{k-d}) \leq \delta_{\mathcal{H}}(\mathcal{H}[U_i])$, there is an optimal matching $M_i$ of $\mathcal{H}[U_i]$. Let $M := \bigcup_{i: \text{good}} M_i$. Then

$$|V(\mathcal{H}) \setminus V(M)| \leq |V(\mathcal{H}) \setminus \bigcup_{i=1}^t U_i| + \sum_{i: \text{bad}} |U_i| + \sum_{i: \text{good}} |U_i \setminus V(M_i)|$$

$$\leq m - 1 + \varepsilon_2 t \cdot m + (k-1)(t - \varepsilon_2 t)$$

$$\leq m + \varepsilon_2 n + nk/m < 2\varepsilon_2 n,$$

as desired. \hfill \square

3. Vortices and Iterative Absorption

The main result of this section is Lemma 3.10, which essentially guarantees a $O(1/n^{k-1})$-spread measure on the set of optimal matchings in a $k$-uniform hypergraph $\mathcal{H}$ in which a $O(1/n^{1-1/k})$-random subset of vertices of $\mathcal{H}$ induces a hypergraph with an optimal matching with high probability. To prove this result, we use an ‘iterative absorption’ approach.

3.1. Vortices. Recall from Section 1.6 that a vertex vortex, formally defined below, is a sequence of vertex sets, which all induce relevant properties of the original hypergraph. The first step in the proof of Lemma 3.10 is to randomly partition the vertices of $\mathcal{H}$, and this partition will be a vertex vortex with high probability.

Definition 3.1 (Vertex vortex). Let $k \geq 2$, and let $\mathcal{H}$ be a $k$-uniform hypergraph on $n$ vertices. For a positive integer $\ell$, a vector $p = (p_1, \ldots, p_\ell)$ of non-negative reals such that $\sum p_i = 1$, an integer $d \in [k-1]$, and $\varepsilon, \alpha_1, \alpha_2 > 0$, we say that a partition $(U_1, \ldots, U_\ell)$ of $V(\mathcal{H})$ is an $(\alpha_1, \alpha_2, d, \varepsilon, p)$-vortex for $\mathcal{H}$ if

(V1) $|U_i| = (1 \pm \varepsilon)p_i n$ for all $i \in [\ell]$,

(V2) $d_{\mathcal{H}[U_i]}(S) \geq (\alpha_1 - \varepsilon)(p_i n)^{k-d}$ for all $i \in [\ell - 1]$, and all but $\varepsilon(p_i n)^d$ many $S \in \binom{U_i}{d}$, and

(V3) $d_{\mathcal{H}[v]}(\binom{U_i \setminus \{v\}}{d-1}) \geq (\alpha_2 - \varepsilon)(p_i n)^{k-1}$ for all $i \in [\ell]$ and $v \in V(\mathcal{H})$.

Definition 3.2 ($(\alpha_1, \alpha_2, d, \varepsilon)$-dense). Let $k \geq 2$, let $d \in \{1, \ldots, k\}$, and let $\alpha_1, \alpha_2, \varepsilon \in [0, 1]$. A $k$-uniform hypergraph $\mathcal{H}$ on $n$ vertices is $(\alpha_1, \alpha_2, d, \varepsilon)$-dense if $d_{\mathcal{H}}(S) \geq \alpha_1 n^{k-d}$ for all but $\varepsilon n^d$ many $S \in \binom{V(\mathcal{H})}{d}$ and $d_{\mathcal{H}}(v) \geq \alpha_2 n^{k-1}$ for all $v \in V(\mathcal{H})$. 
The next lemma can be proved via a straightforward combination of Chernoff bounds and Lemma 2.3 with the union bound, so we defer it to the appendix.

**Lemma 3.3** (Vortex existence lemma). Let $1/n \ll \varepsilon < \alpha_1, \alpha_2, 1/k < 1$ with $k \geq 3$ and $d \in [k-1]$. Let $H$ be a $(\alpha_1, \alpha_2, d, \varepsilon)$-dense $k$-uniform hypergraph on $n$ vertices. Let $\ell := \left\lceil \frac{\log_d(n)}{k} \right\rceil$, $C_\ell := \sum_{i=1}^{\ell} 2^{-i}$, $p_i := \frac{1}{2^{i+1}}$ for each $i \in [\ell]$, and $p := (p_1, \ldots, p_\ell)$. Independently for each vertex $v \in V(H)$, let $X_v$ be a random variable with values in $[\ell]$ such that $\Pr[X_v = i] = p_i$ for each $i \in [\ell]$. For each $i \in [\ell]$, let $U_i := \{v \in V(H) : X_v = i\}$. Then we have that a.a.s. $(U_1, \ldots, U_\ell)$ is an $(\alpha_1, \alpha_2, d, 2\varepsilon, p)$-vortex for $H$.

### 3.2. Matchings inside vortex sets

To prove Lemma 3.10, we will find a well-spread measure on almost perfect matchings in each ‘level’ of the vertex vortex using the weak hypergraph regularity lemma (Theorem 2.7). The following lemma is key for this approach.

**Lemma 3.4** (Random matching in an $\varepsilon$-regular $k$-tuple). Let $1/n \ll \varepsilon \ll d, 1/k < 1$. Let $H$ be a $k$-partite $k$-uniform hypergraph with partition $(V_1, \ldots, V_k)$ such that $|V_i| = \cdots = |V_k| = n$ and $(V_1, \ldots, V_k)$ is $\varepsilon$-regular with density at least $d$. Then there exists a $(1/(\varepsilon^2 n^{k-1}))$-spread probability measure on the set of matchings in $H$ which cover all but at most $2k\varepsilon n^{k-1}$ vertices.

**Proof.** We define a probability measure on the set of matchings in $H$ which cover all but at most $2k\varepsilon n^{k-1}$ vertices by randomly constructing a matching $M$ as follows. Let $u_1, \ldots, u_n$ be an enumeration of the vertices in $V_1$. Let $M_0 := \emptyset$, $W_0 := \emptyset$, and $V_{i,0} := V_i$ for each $i \in [k]$. We define $M_j := M_{j-1}$, $W_j := W_{j-1} \cup \{u_j\}$, and $V_{i,j} := V_{i,j-1}$ for each $2 \leq i \leq k$.

- If $e_H(u_j, V_{2,j-1}, \ldots, V_{k,j-1}) \leq \varepsilon^2 n^{k-1}$, then define $M_j := M_{j-1}$, $W_j := W_{j-1} \cup \{u_j\}$, and $V_{i,j} := V_{i,j-1}$ for each $2 \leq i \leq k$.
- Otherwise, if $e_H(u_j, V_{2,j-1}, \ldots, V_{k,j-1}) \leq \varepsilon^2 n^{k-1}$, then choose $(v_{2,j}, \ldots, v_{k,j}) \in V_{2,j-1} \times \cdots \times V_{k,j-1}$ uniformly at random so that $u_j v_{2,j} \cdots v_{k,j} \in E_H(u_j, V_{2,j-1}, \ldots, V_{k,j-1})$. Define $M_j := M_{j-1} \cup \{v_{2,j} \cdots v_{k,j}\}$, $W_j := W_{j-1}$, and $V_{i,j} := V_{i,j-1} \setminus \{v_{2,j} \cdots v_{k,j}\}$ for each $2 \leq i \leq k$.

Let $t \in [n]$ be the first index such that $|V_{2,t}| = \cdots = |V_{k,t}| < 2\varepsilon n/k$. If such an index does not exist, then let $t := n$. For either of the cases, we have $|V_{2,t}| = \cdots = |V_{k,t}| > 2\varepsilon n/k - 1$.

Let $M := M_t$. Since each edge of $H$ is added to $M$ with probability at most $1/(\varepsilon^2 n^{k-1})$ conditional on all other previous random choices, the resulting measure is $1/(\varepsilon^2 n^{k-1})$-spread.

Now we aim to bound $|W_t|$. Indeed, for each $j \geq 1$ such that $u_j \in W_t$, we have

$$e_H(u_j, V_{2,j}, \ldots, V_{k,j}) \leq e_H(u_j, V_{2,j-1}, \ldots, V_{t,j-1}) \leq \varepsilon^2 n^{k-1},$$

so $e_H(W_t, V_{2,t}, \ldots, V_{k,t}) \leq \varepsilon^2 n^{k-1}|W_t| \leq \varepsilon^2 n^k$. Since $|V_{2,t}| = \cdots = |V_{k,t}| > 2\varepsilon n/k - 1$, if $|W_t| > \varepsilon^{1/h} n$, then $|W_t|/|V_{2,t}| \cdots |V_{k,t}| > \varepsilon n^k = \varepsilon|V_i| \cdots |V_k|$ while $e_H(W_t, V_{2,t}, \ldots, V_{k,t}) \leq \varepsilon^2 n^k < (d - \varepsilon)|V_{2,t}| \cdots |V_{k,t}|$, contradicting the assumption that $(V_1, \ldots, V_k)$ is $\varepsilon$-regular with density at least $d$. Thus, $|W_t| \leq \varepsilon^{1/h} n$. This also implies that $t < n$ and $|V_{2,t}| = \cdots = |V_{k,t}| < 2\varepsilon n/k$; otherwise if $t = n$, then $|V_{2,n}| = \cdots = |V_{k,n}| = n - (n - |W_t|) \leq \varepsilon^{1/h} n$, which contradicts $|V_{2,t}| = \cdots = |V_{k,t}| > 2\varepsilon n/k - 1$. Since $|M| = |M_t| = n - |V_{2,t}| \geq n - 2\varepsilon n/k$, the matching $M$ covers all but at most $2k\varepsilon n^{k-1}$ vertices. 

The next lemma shows that we can find a well-spread measure on almost perfect matchings within a vortex set $U_i$.

**Lemma 3.5** (Random Matching inside a vortex set). Let $1/n \ll \delta \ll \varepsilon_1 \ll \varepsilon_2 \ll 1/k < 1$ with $k \geq 3$ and $d \in [k-1]$. Let $H$ be a $k$-uniform hypergraph on $n$ vertices such that $d_H(S) \geq \left\lceil \frac{\log(k)}{k} \right\rceil + \varepsilon_2 \left\lceil \frac{n}{k-d} \right\rceil$. Then we have that a.a.s. $(U_1, \ldots, U_\ell)$ is an $(\alpha_1, \alpha_2, d, 2\varepsilon_1, p)$-vortex for $H$. 

The proof of this lemma involves using the hypergraph regularity lemma to find a well-spread matching within each vortex set. The details of the proof are beyond the scope of this summary.
for all but at most $\varepsilon_1 n^d$ many $d$-sets $S \in \binom{V(H)}{d}$. Then there exists a $(1/(\delta n^{k-1}))$-spread probability measure on the set of matchings in $\mathcal{H}$ which cover all but at most $2\varepsilon_2 n^d$ vertices.

Proof. We define a probability measure on the set of matchings in $\mathcal{H}$ which cover all but at most $2\varepsilon_2 n$ vertices by randomly constructing a matching $M$ as follows. Fix new constants $t_1, t_0, \varepsilon$, and $\gamma$ such that $\varepsilon_1 \ll 1/t_1 \ll 1/t_0 \ll \varepsilon \ll \gamma \ll \varepsilon_2$. By Theorem 2.7, there exists an $\varepsilon$-regular partition $(V_0, V_1, \ldots, V_t)$ of $V(H)$ with $t_0 \leq t \leq t_1$. Let $R$ be the $(\gamma/3, \varepsilon)$-reduced graph with respect to $(V_0, V_1, \ldots, V_t)$. By Lemma 2.8, all but at most $\varepsilon^{1/2}(\varepsilon_1 t_0)$ many $d$-sets $S \in \binom{[t]}{d}$ satisfy $d_R(S) \geq \left( \frac{\mu_k(k) + \varepsilon_2/2}{k-d} \right)$. Thus, by Lemma 2.9, $R$ has a matching $M_R$ covering all but at most $\varepsilon_2 t$ vertices. Let $n_* := \frac{n - |V_0|}{t} \geq (1 - \varepsilon)\frac{n}{t}$. For each $S = \{i_1, \ldots, i_k\} \in M_R$, by Lemma 3.4, there exists a probability measure $\nu_S$ on the set of matchings in $\mathcal{H}[V_{i_1}, \ldots, V_{i_k}]$ that cover all but at most $2k\varepsilon^{1/k}n_*$ of the vertices in $V_{i_1} \cup \cdots \cup V_{i_k}$ that is $(1/(\varepsilon^{2}n_*^{k-1}))$-spread. Choose $M = \bigcup_{S \in M_R} M_S$ where each $M_S$ is chosen independently at random according to $\nu_S$. Since

$$\frac{1}{\varepsilon^2 n_*^{k-1}} \leq \frac{t^{k-1}}{\varepsilon^2(1 - \varepsilon)^{k-1}n^{k-1}} \leq \frac{1}{\delta n^{k-1}},$$

the probability measure on $M$ is $(1/(\delta n^{k-1}))$-spread. Moreover, $M$ covers all but at most $\varepsilon n + 2k\varepsilon^{1/k}n_* \cdot \frac{t}{k} \leq 2\varepsilon_2 n$ vertices of $\mathcal{H}$, as desired. \hfill \Box

3.3. Covering down. The following lemma will be used to cover the vertices in some vertex set $U_i$ by edges whose other vertices lie in $U_{i+1}$, i.e., we will apply it with $A$ playing the role of the set of uncovered vertices in $U_i$ and $B$ that of $U_{i+1}$.

Lemma 3.6 (Cover-down lemma). Suppose $1/n \ll \eta \ll \delta \ll c, 1/k < 1$. Let $\mathcal{H}$ be a $k$-uniform hypergraph on $n$ vertices, and let $(A, B)$ be a partition of $V(\mathcal{H})$ such that $|A| \leq \eta n$ and for each $v \in A$, $d_H(v; \binom{B}{k-1}) \geq cn^{k-1}$. Then there exists a $(1/(\delta n^{k-1}))$-spread probability measure on the set of matchings $M$ in $\mathcal{H}$ of size $|A|$ that cover $A$ and satisfy $|e \cap A| = 1$ for each $e \in M$.

Proof. We define a probability measure on the set of matchings $M$ in $\mathcal{H}$ of size $|A|$ that cover $A$ and $|e \cap A| = 1$ for each $e \in M$ by randomly constructing a matching $M$ as follows. Let $m := |A|$, and let $u_1, \ldots, u_m$ be an enumeration of the vertices in $A$. Independently for each $i = 1, \ldots, m$ in order, choose $S_i \in \binom{B \cup \bigcup_{j=1}^{i-1} S_j}{k-1}$ such that $e_i := u_i \cup S_i \in \mathcal{H}$ uniformly at random. Let $M := \{e_i : i \in [m]\}$. Note that for each $i \in [m]$, we have

$$d_H\left( u_i; \binom{B \setminus \bigcup_{j=1}^{i-1} S_j}{k-1} \right) \geq d_H\left( u_i; \binom{B}{k-1} \right) - (k-1)|A||B|^{k-2} \geq cn^{k-1} - (k-1)\eta n^{k-1} \geq \frac{cn^{k-1}}{2}.$$ 

Hence each edge $e \in \mathcal{H}$ is added to $M$ with probability at most $2/(cn^{k-1}) \leq 1/(\delta n^{k-1})$ irrespective of all other random choices. It follows that the resulting measure is $(1/(\delta n^{k-1}))$-spread. \hfill \Box

3.4. Spreadness of random matchings. Given a vertex vortex $(U_1, \ldots, U_\ell)$ of a hypergraph $\mathcal{H}$, we can iteratively apply Lemmas 3.5 and 3.6 $\ell - 1$ times to obtain a well-spread measure on matchings of $\mathcal{H}$ which cover all vertices of $\mathcal{H}$ not in $U_\ell$. However, edges in ‘lower levels’ (i.e., $U_i$ for $i$ close to 1), so we need to introduce the following ‘weighted’ version of spreadness. Since edges are less likely to appear in the lower levels of a random vertex, the spreadness ‘balances’.
Definition 3.7. Let $\mathcal{H}$ be a $k$-uniform hypergraph, and let $q = (q_e)_{e \in \mathcal{H}}$ where $q_e \in [0, 1]$ for every $e \in \mathcal{H}$. A probability measure $\nu$ on the set of matchings in $\mathcal{H}$ is $q$-spread if for every $S \subseteq \mathcal{H}$, we have

$$\mathbb{P}[S \subseteq M] \leq \prod_{e \in S} q_e,$$

where $M$ is chosen at random according to $\nu$.

Given a vertex vortex $(U_1, \ldots, U_\ell)$, the following lemma provides a $q$-spread measure for appropriately chosen $q$ on matchings which cover all vertices not in $U_\ell$. For technical reasons discussed later, we also need to control the parity of these matchings, we need these matchings to avoid a small 'protected' set $U_* \subseteq U_\ell$, and we need that these matchings do not cover too many vertices of $U_\ell$.

Lemma 3.8. Suppose $1/n \ll \delta \ll \varepsilon_* \ll \varepsilon \ll c, 1/k < 1$ with $k \geq 3$, and $d \in [k-1]$. Let $\ell := \lceil \frac{k-1}{k} \log_2(n) \rceil$, $C_\ell := \sum_{i=1}^\ell 2^{-i}$, $p_i := \frac{1}{C_\ell^2}$ for each $i \in [\ell]$, and $p := (p_1, \ldots, p_\ell)$. Let $\mathcal{H}$ be a $k$-uniform hypergraph on $n$ vertices, and let $(U_1, \ldots, U_\ell)$ be an $\left(\frac{\mu(d(k)+4\varepsilon)}{k-d}, c, d, \varepsilon_*, p\right)$-vortex for $\mathcal{H}$. Let $U_* \subseteq U_\ell$ with $|U_*| \leq \varepsilon |U_\ell|$ and $s \in \{0, 1\}$. Let $q := (q_e)_{e \in \mathcal{H}}$, where for each $e \in \mathcal{H}$,

$$q_e := \begin{cases} \frac{1}{\delta(p,n)^{k-1}} & \text{if } e \subseteq U_i \text{ for some } i \in [\ell-1], \\ \frac{1}{\delta(p+1,n)^{k-1}} & \text{if } e \subseteq U_i \cup U_{i+1} \text{ and } |e \cap U_i| = 1 \text{ for some } i \in [\ell-1], \\ 1 & \text{if } e \subseteq U_\ell, \text{ and} \\ 0 & \text{otherwise}. \end{cases}$$

Then there exists a $q$-spread probability measure on the set of matchings $M$ in $\mathcal{H}$ which satisfy $|M| \equiv s \pmod{2}$, $U_* \subseteq V(H \setminus V(M)) \subseteq U_\ell$, and $|V(M) \cap U_\ell| \leq \varepsilon^2 p_{n|U_\ell|}$.

Proof. Fix a new constant $\delta_*$ such that $\delta \ll \delta_* \ll \varepsilon_*$. We prove by induction on $j$ that for each $j$ such that $0 \leq j \leq \ell - 1$, there exists a $q|_{\mathcal{H}[U_1 \cup \cdots \cup U_j]}$-spread probability measure $\nu_j$ on the set of matchings $M$ in $\mathcal{H}[U_1 \cup \cdots \cup U_j]$ which satisfy $U_* \subseteq V(\mathcal{H}) \setminus V(M) \subseteq U_\ell$, $|V(M) \cap U_\ell| \leq \varepsilon^2 p_{n|U_\ell|}$, and $e \subseteq U_\ell$ for each $e \in M$. If $|M| \equiv s \pmod{2}$, then let $\nu(M) := \nu_{\ell-1}(\{M\})$. Otherwise, we choose an arbitrary edge $e_M \in \mathcal{H}[U_\ell \setminus V(M)]$ and let $\nu(M \cup \{e_M\}) := \nu_{\ell-1}(\{M\})$. Since $e \subseteq U_\ell$ for each $e \in M$ and that is not covered by $M_\ell$, we have a well-defined $q$-spread probability measure on the set of matchings $N$ in $\mathcal{H}$ which satisfy $|N| \equiv s \pmod{2}$, and $U_* \subseteq V(H \setminus V(N)) \subseteq U_\ell$, $|V(N) \cap U_\ell| \leq \varepsilon^2 p_{n|U_\ell|}$.

We define the desired probability measure by randomly constructing a matching $M$ as follows. For $j = 0$ the statement trivially holds for $M = \emptyset$. Now let $j \geq 1$, and let $\nu_{j-1}$ be a $q|_{\mathcal{H}[U_1 \cup \cdots \cup U_j]}$-spread probability measure on the set of matchings $M_* \in \mathcal{H}[U_1 \cup \cdots \cup U_j]$ which satisfy $U_* \subseteq V(H \setminus V(M_*)) \subseteq U_\ell$, $|V(M_* \cap U_\ell| \leq \varepsilon^2 p_{n|U_\ell|}$, and $e \subseteq U_\ell$ for each $e \in M_*$. Let $U_j'$ be the set of vertices in $U_j$ that are not covered by $M_*$. Since $|U_j'| = (1 \pm \varepsilon_*)p_{n|U_\ell|}$ and by the fact that $M_*$ covers at most $\varepsilon^2 p_{n|U_\ell|}$ vertices in $U_j$, we have $|U_j'| = \frac{1}{1 \pm 2\varepsilon^2} p_{n|U_\ell|}$.

Since $(U_1, \ldots, U_\ell)$ is an $\left(\frac{\mu(d(k)+4\varepsilon)}{k-d}, c, d, \varepsilon_*, p\right)$-vortex for $\mathcal{H}$, for all but $\varepsilon_*(p_{n|U_\ell|})^d$ many $S \in \binom{U_{j-1}}{d}$, we have that $d_{\mathcal{H}[U_j]}(S) \geq \left(\frac{\mu(d(k)+4\varepsilon)}{k-d} - \varepsilon_*\right) (p_{n|U_\ell|})^k \geq \frac{\mu(d(k)+3\varepsilon)}{k-d} (p_{n|U_\ell|})^k$. It follows that for all but
Lemma 3.10. Let 1/n ≤ δ ≤ δ* ≤ ε* ≤ ε ≤ c/k < 1 with k ≥ 3, and d ∈ [k − 1]. Let H be a k-uniform hypergraph on n vertices and H′ a \((\frac{\mu_d(k) + 3\epsilon}{(k − d)!}, c, d, \epsilon_2)\)-dense spanning subhypergraph of \(H\). Let \(\ell := \left\lceil \frac{k-1}{\log_2(n)} \right\rceil\), \(C_\ell := \sum_{i=1}^\ell 2^{-i}\), and \(p_\ell := \frac{1}{C_\ell^{k/2}}\). Suppose that a \(p_\ell\)-random subset of \(V(H)\) is \((H′, \epsilon)\)-OM-stable for \(H\) with probability at least \(\delta\). Then there exists a probability measure on the set of optimal matchings of \(H\) that is \(\frac{1}{\delta n^{k-1}}\)-spread.
Proof. For each $i \in [\ell - 1]$, let $p_i := \frac{1}{C_{\ell i}^k}$, and let $p := (p_1, \ldots, p_\ell)$. Independently for each vertex $v \in V(\mathcal{H})$, let $X_v$ be a random variable with values in $[\ell]$ such that $\mathbb{P}[X_v = i] = p_i$ for each $i \in [\ell]$. For each $i \in [\ell]$, let $U_i := \{v \in V(\mathcal{H}) : X_v = i\}$. Let $\mathcal{E}_1$ be the event that $(U_1, \ldots, U_\ell)$ is a $(\frac{(k+4\varepsilon)}{(k-\delta)}c, d, 2\varepsilon, p)$-vortex for $\mathcal{H}'$, and let $\mathcal{E}_2$ be the event that $U_\ell$ is $(\mathcal{H}', \varepsilon)$-OM-stable for $\mathcal{H}$. By Lemma 3.3, $\mathbb{P}[\mathcal{E}_1] \geq 1 - \delta_*/2$, and by assumption, $\mathbb{P}[\mathcal{E}_2] \geq \delta_*$. Hence, $\mathbb{P}[\mathcal{E}_1 \cap \mathcal{E}_2] \geq \delta_*/2$.

Suppose that the outcome of $X_v$, $v \in V(\mathcal{H})$ is such that $\mathcal{E}_1 \cap \mathcal{E}_2$ holds. Since $U_\ell$ is $(\mathcal{H}', \varepsilon)$-OM-stable for $\mathcal{H}$, there exists $U_* \subseteq U_\ell$ with $|U_*| \leq \varepsilon |U_\ell|$ and $s \in \{0, 1\}$ such that for any matching $M$ of $\mathcal{H}'$ with $|M| = s \pmod{2}$, $U_* \subseteq V(\mathcal{H}) \setminus V(M) \subseteq U_\ell$, and $|V(M) \cap U| \leq \varepsilon |U_\ell|$, we have that $\mathcal{H} - V(M)$ contains an optimal matching. By Lemma 3.8 with $n$, $\delta_*$, $2\varepsilon$, $\varepsilon$, $c$, $k$, $d$, $\mathcal{H}'$, $U_*$ playing the roles of $n$, $\delta_*$, $2\varepsilon$, $\varepsilon$, $c$, $k$, $d$, $\mathcal{H}$, $U_*$, there is a $q$-spread probability measure $\nu_*$ on the set of matchings $M_*$ in $\mathcal{H}'$ which satisfy $|M_*| \equiv s \pmod{2}$, $U_* \subseteq V(\mathcal{H}) \setminus V(M_* \cup U_\ell)$, and $|V(M_*) \cup U_\ell| \leq \varepsilon^2 p n$, where $q$ is as defined in Lemma 3.8. Since $\mathcal{E}_2$ holds, we can complete the matching $M_*$ to an optimal matching $M$ in $\mathcal{H}$. Thus, conditional on the event $\mathcal{E}_1 \cap \mathcal{E}_2$, this procedure defines a probability measure on the set of optimal matchings $M$ in $\mathcal{H}$. (For each optimal matching $M$, the probability of $M$ appearing is given by the probability that this procedure outputs $M$. Note that for fixed $M$, there may be several different ways of arriving at output $M$ via this procedure.)

We claim that the resulting measure is $2q/\delta_*^{\ell-1}$-spread, where $\delta_* := 4/(\delta_* n^{k-1})$. To that end, let $s \geq 1$, and let $e_1, \ldots, e_s$ be distinct edges of $\mathcal{H}$. We show that $\mathbb{P}[e_1, \ldots, e_s \in M] \leq 2q^s/\delta_* \leq (2q/\delta_*)^s$. If the edges $e_1, \ldots, e_s$ do not form a matching in $\mathcal{H}$, then clearly $\mathbb{P}[e_1, \ldots, e_s \in M] = 0$ as $M$ is a matching, so we may assume that the edges $e_1, \ldots, e_s$ form a matching in $\mathcal{H}$. Let $\mathcal{P}$ denote the set of partitions $(S_1, S'_1, \ldots, S_{\ell-1}, S'_{\ell-1}, S_\ell)$ of $\{e_1, \ldots, e_s\}$ into $2\ell - 1$ parts. For each $P = (S_1, S'_1, \ldots, S_{\ell-1}, S'_{\ell-1}, S_\ell) \in \mathcal{P}$, let $\mathcal{E}_P$ be the event that

- $e \subseteq U_i$ for all $i \in [\ell]$ and $e \in S_i$ and
- $|e \cap U_i| = 1$ and $|e \cap U_{i+1}| = k - 1$ for all $i \in [\ell - 1]$ and $e \in S'_i$.

Now

$$\mathbb{P}[e_1, \ldots, e_s \in M] = \sum_{P \in \mathcal{P}} \mathbb{P}[e_1, \ldots, e_s \in M \mid \mathcal{E}_P] \mathbb{P}[\mathcal{E}_P \mid \mathcal{E}_1 \cap \mathcal{E}_2].$$

Since $\{e_1, \ldots, e_s\}$ is a matching, for every $P = (S_1, S'_1, \ldots, S_{\ell-1}, S'_{\ell-1}, S_\ell) \in \mathcal{P}$, we have

$$\mathbb{P}[\mathcal{E}_P \mid \mathcal{E}_1 \cap \mathcal{E}_2] \leq \frac{\mathbb{P}[\mathcal{E}_P]}{\mathbb{P}[\mathcal{E}_1 \cap \mathcal{E}_2]} \leq \frac{2}{\delta_*} \prod_{i=1}^{\ell} p_i^{|S_i|} \prod_{i=1}^{\ell-1} \left( p_i p_{i+1}^{k-1} \right)^{|S'_i|},$$

and by Lemma 3.8,

$$\mathbb{P}[e_1, \ldots, e_s \in M \mid \mathcal{E}_P] \leq \prod_{i=1}^{\ell} q_i^{|S_i|} \prod_{i=1}^{\ell-1} q_i'^{|S'_i|},$$

where $q_i := 1/(\delta_* (p_i n)^{k-1})$ and $q_i' := 1/(\delta_* (p_{i+1} n)^{k-1})$ for $i \in [\ell - 1]$ and $q_\ell := 1$. Since $q = 4/(\delta_* n^{k-1})$, for all $i \in [\ell - 1],

$$q_i q_i' = q_{i+1} p_{i+1}^{k-1} = \frac{q_{i+1}}{4},$$

and since $1/n \ll \delta_* \ll 1/k,$

$$q_i q_i' = \frac{q_{i+1}}{4} \leq \left( \frac{2}{C_{\ell i}^k} \right)^k \leq \frac{2}{\delta_* n^{k-1}} = \frac{q}{2}.$$
Therefore, combining the five equations above, we have
\[
\mathbb{P}[e_1, \ldots, e_\ell \in M] \leq \frac{2}{\delta_*} \sum_{(S_1, S_2, \ldots, S_\ell) \in \mathcal{P}} \left( \frac{q}{2} \right)^{|S_1|} \prod_{i=1}^{\ell-1} \left( \frac{q_{p_i}}{4} \right)^{|S_i|} \left( \frac{q_{p_{\ell-1}}}{4} \right)^{|S_{\ell}|}
\]
\[
= \frac{2}{\delta_*} q^s \left( \frac{1}{2} + \frac{p_1}{4} + \frac{p_1}{4} + \cdots + \frac{p_{\ell-1}}{4} + \frac{p_{\ell-1}}{4} \right)^s \leq \frac{2}{\delta_*} q^s,
\]
so our measure is $2q/\delta_*$-spread, as claimed. Since $\delta \leq \delta_*$, the measure is also $1/(\delta n^{k-1})$-spread, as desired.

\[\square\]

4. OM-stability

In this section, we prove Theorem 1.5, and subject to some lemmas proved in later sections, we also prove Theorem 1.6. Lemma 3.10 essentially reduces these proofs to the problem of proving the hypergraphs under consideration are OM-stable.

4.1. Proof of Theorem 1.5. Together with Lemma 3.10, the next lemma implies spreadness of optimal matchings in the case when we have minimum $d$-degree at least $(\overline{\nu}_d^{(s)}(k) + o(1))(n^{-d}_{k-d})$.

**Lemma 4.1.** Let $1/n < \varepsilon < \gamma < 1/k \leq 1/3$ and $d \in [k-1]$. Let $\mathcal{H}$ be a $k$-uniform hypergraph on $n$ vertices with $\delta_d(\mathcal{H}) \geq (\overline{\nu}_d^{(s)}(k) + \gamma)(n^{-d}_{k-d})$, where $n \equiv s \pmod{k}$ for $0 \leq s \leq k - 1$. Let $\ell := \left\lceil \frac{k-1}{k} \log_2(n) \right\rceil$, $C_\ell := \sum_{i=1}^{\ell} 2^{-i}$, and $p_\ell := \frac{1}{C_\ell 2^{\ell}}$. Then a.a.s. a $p_\ell$-random subset of $V(\mathcal{H})$ is $(\mathcal{H}, \varepsilon)$-OM-stable for $\mathcal{H}$.

**Proof.** By the definition of $\overline{\nu}_d^{(s)}(k)$, there exists $n_0 \in \mathbb{N}$ such that $m_d(k, n') < (\overline{\nu}_d^{(s)}(k) + \gamma/4)(n'^{-d}_{k-d})$ for all $n' \in k\mathbb{N} + s$ with $n' \geq n_0$, and we may assume that $n$ is sufficiently larger than $n_0$ so that $n^{1/k}/8 \geq n_0$, which implies $p_\ell n/2 \geq \varepsilon_0$. Let $U$ be a $p_\ell$-random subset of $V(\mathcal{H})$. Let $\mathcal{E}$ be the event that $|U| = (1 \pm \varepsilon)p_\ell n$ and $\delta_d(\mathcal{H}[U]) \geq \overline{\nu}_d^{(s)}(k) + \gamma/3(p_\ell n)^{k-d}$. We show that $\mathcal{E}$ occurs a.a.s. Note that by a Chernoff bound, we have that
\[
\mathbb{P}[|U| \neq (1 \pm \varepsilon)p_\ell n] \leq 2 \exp \left( -\frac{\varepsilon^2}{3} p_\ell n \right) \leq \exp(-\Omega(n^{1/k})).
\]

Note that for each $S \in \binom{V(\mathcal{H})}{d}$, we have $d_\mathcal{H}(S) \geq (\overline{\nu}_d^{(s)}(k) + \gamma/4)(n^{-d}_{k-d}) \geq \overline{\nu}_d^{(s)}(k) + \gamma/2 n^{k-d}$. By Lemma 2.3 (i) and a union bound, with probability at least $1 - \exp(-n^{1/11k^2})$, we have
\[
\delta_d(\mathcal{H}[U]) \geq \overline{\nu}_d^{(s)}(k) + \gamma/3 \left( \frac{n^{k-d}}{n^{k-d}} \right) \geq \overline{\nu}_d^{(s)}(k) + \gamma/3 \left( \frac{n^{k-d}}{n^{k-d}} \right) \geq \overline{\nu}_d^{(s)}(k) + \gamma/3 \left( \frac{n^{k-d}}{n^{k-d}} \right).
\]

Hence, $\mathcal{E}$ occurs a.a.s. We show that in this case $U$ is $(\mathcal{H}, \varepsilon)$-OM-stable for $\mathcal{H}$. Let $M$ be a matching in $\mathcal{H}$ such that $|V(M) \cap U| \leq \varepsilon|U|$ and $V(\mathcal{H}) \setminus V(M) \subseteq U$. Let $U' := V(\mathcal{H}) \setminus V(M)$. Note that
\[
\delta_d(\mathcal{H}[U']) \geq \overline{\nu}_d^{(s)}(k) + \gamma/3 \left( \frac{n^{k-d}}{n^{k-d}} \right) \geq \overline{\nu}_d^{(s)}(k) + \gamma/3 \left( \frac{n^{k-d}}{n^{k-d}} \right) \geq \overline{\nu}_d^{(s)}(k) + \gamma/3 \left( \frac{n^{k-d}}{n^{k-d}} \right).
\]

and since $|U'| \geq p_\ell n/2 \geq n_0$ and $|U'| \equiv n \pmod{k}$, we have $\delta_d(\mathcal{H}[U']) \geq (\overline{\nu}_d^{(s)}(k) + \gamma/4)(n^{-d}_{k-d}) \geq \overline{\nu}_d^{(s)}(k) + \gamma/4 \left( \frac{|U'|}{k-d} \right)$. Therefore, it follows from the definition of $m_d(k, |U'|)$ that $\mathcal{H}[U'] = \mathcal{H} \setminus V(M)$ contains an optimal matching, as desired.

\[\square\]

We are now ready to prove Theorem 1.5.
Proof of Theorem 1.5. Let $1/n \ll \delta \ll \varepsilon_n \ll \varepsilon \ll \gamma, 1/k$ with $\gamma \in (0, 1)$ and $k \geq 3$. Let $0 \leq s \leq k-1$ be an integer such that $n \in k\mathbb{N} + s$. Note that $\delta_d(H) \geq (\overline{\mu_d^{(s)}}(k) + \gamma)(\frac{n-d}{k-d}) \geq \frac{\overline{\mu_d^{(s)}}(k) + \gamma/2}{(k-d)!} n^{k-d}$. Moreover, we have

$$
\delta_1(H) \geq \frac{1}{(k-1)} \delta_d(H) \geq \frac{\overline{\mu_d^{(s)}}(k) + \gamma/2}{2(k-1)!} n^{k-1}.
$$

Thus, since $\varepsilon \ll \gamma$, $H$ is $(\overline{\mu_d^{(s)}}(k) + \varepsilon/2), \frac{\overline{\mu_d^{(s)}}(k) + \gamma/2}{2(k-1)!}, d, \varepsilon)$-dense. By Lemmas 3.10 and 4.1, there exists a probability measure on the set of optimal matchings of $H$ which is $\frac{1}{\varepsilon n^k}$-spread, as desired. \hfill $\square$

4.2. Proof of Theorem 1.6. Now we briefly describe the following critical example mentioned in [63, Section 3]. Note that the critical example for odd $k$ was introduced in [52].

**Definition 4.2** ($H^0(k,n)$). Let $k, n \geq 2$ be positive integers such that $n$ is divisible by $k$. Let $H^0(k,n)$ be a $k$-uniform $n$-vertex hypergraph with an ordered partition $(A,B)$ of $V(H^0(k,n))$ such that the following holds.

- If $k$ is odd, then $|A|$ is the unique odd integer in $\{\frac{n}{2} - 1, \frac{n}{2} - 1, \frac{n}{2} - 1, \frac{n}{2} - 1\}$ and $E(H^0(k,n))$ is the collection of all subsets of size $k$ in $V(H^0(k,n)) = A \cup B$ which intersect $A$ in an even number of vertices.
- Otherwise if $k$ is even, then

$$
|A| = \begin{cases}
\frac{n}{2}, & \text{if } \frac{n}{2} \text{ is odd and } \frac{n}{2} \text{ is even,} \\
\frac{n}{2} - 1, & \text{otherwise (thus } n/k \equiv n/2 \text{ (mod 2)}),
\end{cases}
$$

and $E(H^0(k,n))$ is the collection of all subsets of size $k$ in $V(H^0(k,n)) = A \cup B$ which intersect $A$ in an odd number of vertices.

Let $\delta^0(k,n) := \delta_{k-1}(H^0(k,n))$. If $k$ is odd, then

$$
\delta^0(k,n) = \begin{cases}
\frac{n}{2} + 1 - k & \text{for } n \equiv 0, 2 \pmod{4} \\
\frac{n}{2} + 1/2 - k & \text{for } n \equiv 1 \pmod{4} \\
\frac{n}{2} + 3/2 - k & \text{for } n \equiv 3 \pmod{4}.
\end{cases}
$$

Otherwise if $k$ is even, then

$$
\delta^0(k,n) = \begin{cases}
\frac{n}{2} + 1 - k & \text{if } n/k \text{ is even} \\
\frac{n}{2} + 1 - k & \text{if } n/k \text{ is odd and } k/2 \text{ is odd} \\
\frac{n}{2} + 2 - k & \text{if } n/k \text{ is odd and } k/2 \text{ is even}.
\end{cases}
$$

Note that $H^0(k,n)$ does not contain a perfect matching (for example, see [63, Section 3]), so $m_{k-1}(k,n) \geq \delta^0(k,n) + 1$ if $k \mid n$. In fact, Rödl, Ruciński, and Szemerédi [63] showed that $m_{k-1}(k,n) = \delta^0(k,n) + 1$ when $k \geq 3$, $k \mid n$, and $n$ is sufficiently large.

We may also use the following definition from [63, Definition 3.3].

**Definition 4.3** ($\varepsilon$-containment). For any $\varepsilon \in (0, 1)$, an $n$-vertex $k$-uniform hypergraph $H$ $\varepsilon$-contains another $n$-vertex $k$-uniform hypergraph $G$ (or $G \subseteq \varepsilon H$) if there exists an isomorphic copy $H'$ of $H$ such that $V(H') = V(G)$ and $|G \setminus H'| \leq \varepsilon n^k$.

In the proof of Theorem 1.6, we must consider two cases according to whether $H$ is close to being critical. The following two lemmas give that a.a.s. a small random subset of vertices is OM-stable in both cases.

**Lemma 4.4.** Let $1/n \ll \varepsilon \ll 1/k \leq 1/3$ such that $k \mid n$. Let $H$ be a $k$-uniform $n$-vertex hypergraph with $\delta_{k-1}(H) \geq (1/2 - 1/\log n)n$ such that $H$ $\varepsilon$-contains neither $H^0(k,n)$ nor $H^0(k,n)$. Let $\ell :=...
Let \( \frac{[k-1] \log_2(n)}{n} \), \( C_\ell := \sum_{i=1}^\ell 2^{-i} \), and \( p_\ell := \frac{1}{C_\ell^2} \). Let \( U \) be a \( p_\ell \)-random subset of \( V(\mathcal{H}) \). Then a.a.s. \( U \) is \((\mathcal{H}, \varepsilon)\)-OM-stable for \( \mathcal{H} \).

**Lemma 4.5.** Let \( 1/n \ll \varepsilon \ll \eta \ll 1/k \leq 1/3 \) such that \( k \mid n \). Let \( \mathcal{H} \) be a \( k \)-uniform \( n \)-vertex hypergraph \( \mathcal{H} \) with \( \delta_{k-1}(\mathcal{H}) \geq m_{k-1}(k, n) = \delta^0(k, n) + 1 \) such that \( \mathcal{H} \) \( \varepsilon \)-contains either \( \mathcal{H}^0(k, n) \) or \( \mathcal{H}^{00}(k, n) \). Let \( \ell := 1 - \frac{\log_2 n}{k} \), \( C_\ell := \sum_{i=1}^\ell 2^{-i} \), and \( p_\ell := 1/(C_\ell^2) \). There are at least \( \varepsilon n^{k-1} \) choices of an edge \( e^* \in \mathcal{H} \) such that for each of the choices of \( e^* \), there exists a spanning subhypergraph \( \mathcal{H}' \) of \( \mathcal{H} - V(e^*) \) such that

- (O1) \( \mathcal{H}' \) is \((1/2 - \eta, 3^{-5}(k-1)!, k - 1, \eta)\)-dense, and
- (O2) a.a.s. a \( p_\ell \)-random subset of \( V(\mathcal{H}) - V(e^*) \) is \((\mathcal{H}', \eta)\)-OM-stable for \( \mathcal{H} - V(e^*) \).

We will prove both lemmas in the next two sections. Subject to these lemmas, we prove Theorem 1.6.

**Proof of Theorem 1.6.** Let \( 1/n_0 \ll \delta \ll \varepsilon \ll \eta \ll 1/k \leq 1/3 \). If \( \mathcal{H} \) \( \varepsilon \)-contains neither \( \mathcal{H}^0(k, n) \) nor \( \mathcal{H}^{00}(k, n) \), then Theorem 1.6 follows by Lemmas 3.10 and 4.4, since \( m_{k-1}(n) = 1/k, \delta_{k-1}(\mathcal{H}) \geq m_{k-1}(k, n) \geq n/2 - O(k) \), and \( \delta_1(\mathcal{H}) \geq 1 - (n/2) \delta_{k-1}(\mathcal{H}) \geq \frac{n^{k-1}}{3(k-1)!} \). Thus, we may assume that \( \mathcal{H} \) \( \varepsilon \)-contains either \( \mathcal{H}^0(k, n) \) or \( \mathcal{H}^{00}(k, n) \).

By Lemma 4.5, there are at least \( \varepsilon n^{k-1} \) choices of an edge \( e^* \in \mathcal{H} \) satisfying (O1) and (O2). We choose one of them uniformly at random and let \( M^* := \{e^*\} \). For each of the choices of \( e^* \), there exists a spanning subhypergraph \( \mathcal{H}' \) of \( \mathcal{H} - V(e^*) \) which is \((1/2 - \eta, 3^{-5}(k-1)!, k - 1, \eta)\)-dense by (O1), so \( \mathcal{H}' \) is \((1/k + 3\eta, 3^{-5}(k-1)!, k - 1, \varepsilon)\)-dense. By (O2) and Lemma 3.10, there exists a probability measure \( \nu \) on the set of perfect matchings \( M' \) of \( \mathcal{H} - V(e^*) \) that is \( \frac{e}{\delta n^{k-1}} \)-spread, conditioning on the choice of \( e^* \). Let \( M' \) be chosen randomly according to \( \nu \), and let \( M := M^* \cup M' \). For any disjoint \( e_1, \ldots, e_t \in \mathcal{H} \),

\[
\Pr[e_1, \ldots, e_t \in M] \leq \Pr[e_1, \ldots, e_t \in M' \mid M^*] + \sum_{i=1}^t \Pr[M^* = \{e_i\}] \Pr[\{e_1, \ldots, e_t\} \setminus \{e_i\} \subseteq M' \mid M^*] \\
\leq \left( \frac{1}{\delta n^{k-1}} \right)^t + t \cdot \frac{1}{\delta n^{k-1}} \cdot \left( \frac{1}{\delta n^{k-1}} \right)^{t-1} \leq \left( \frac{e}{\delta n^{k-1}} \right)^t.
\]

Thus, the distribution of \( M \) is \( \frac{e}{\delta n^{k-1}} \)-spread, as desired. \( \square \)

5. **Proof of Lemma 4.4.**

Roughly speaking the proof of Lemma 4.4 proceeds as follows. We show that \( \mathcal{H} \) contains many small absorbing structures. We then use Lemma 2.3 to show that a \( p_\ell \)-random subset of vertices \( U \) still contains many of these small absorbers. We use these to build a larger absorbing matching \( M \) of size \( O(\log(n)) \) in \( \mathcal{H}[U] \). The vertices of \( M \) will be the set \( U_* \) of protected vertices that is allowed by the definition of \((\mathcal{H}, \varepsilon)\)-OM-stable. We let \( M \) be any matching in \( \mathcal{H} \) such that \( U_* \subseteq V(\mathcal{H}) \setminus V(\tilde{M}) \subseteq U \) and \( |V(\tilde{M}) \cap U| \leq \varepsilon |U| \). Then the minimum codegree of \( \mathcal{H} - V(\tilde{M}) - V(M) \) is still large enough to guarantee a matching that either covers all vertices or all but exactly \( k \) vertices. Finally, we use the absorbing property of \( M \) to complete this matching to a perfect matching in \( V(\mathcal{H}) - V(\tilde{M}) \).

Now we define the absorbing structures that were introduced in [63, Definitions 5.1 and 5.2].

**Definition 5.1** \((S\text{-absorbing} \ k\text{-matchings and} S\text{-absorbing} \ (k+1)\text{-matchings})\). Let \( \mathcal{H} \) be a \( k \)-uniform hypergraph and \( S = \{x_1, \ldots, x_k\} \in \binom{V(\mathcal{H})}{k} \).

A \( k \)-matching \( \{e_1, \ldots, e_k\} \in \mathcal{H} \) is \( S\text{-absorbing} \) if there exists a \((k+1)\)-matching \( \{e'_1, \ldots, e'_k, f\} \) in \( \mathcal{H} \) such that

- (AM1) \( e_i \cap e'_j = \emptyset \) for all \( i \neq j \),
for the definition of $H^0(k, n)$. It shows that in the setting of Lemma 4.4, $H$ has many $S$-absorbing matchings for each set $S$ of $k$ vertices.

**Lemma 5.2 ([63]).** Let $1/n \ll \varepsilon, 1/k \leq 1/3$ such that $k | n$. Let $H$ be a $k$-uniform hypergraph on $n$ vertices with $\delta_{k-1}(H) \geq (1/2 - 1/\log n)n$ such that $H^0(k, n) \not\subseteq H$ and $H^{0\varepsilon}(k, n) \not\subseteq H$. Then at least one of the following holds.

(a) For every $S \subseteq V(H)$ with $|S| = k$, there are $\Omega(n^{k^2/\log^3(n)})$ many $S$-absorbing $k$-matchings in $H$.

(b) For every $S \subseteq V(H)$ with $|S| = k$, there are $\Omega(n^{k^2+k/\log^3(n)})$ many $S$-absorbing $(k + 1)$-matchings in $H$.

The next lemma follows from the proof of [63, Fact 5.4]. It says that if we have many $S$-absorbing matchings for each set $S$ of $k$ vertices in $H$ then we can build an absorbing matching of size $O(\log^4(n))$ that can absorb any set of $k$ vertices.

**Lemma 5.3 ([63]).** Let $1/n \ll 1/k \leq 1/3$. Let $H$ be a $k$-uniform $n$-vertex hypergraph. Suppose that at least one of the following holds.

(a) For every $S \subseteq V(H)$ with $|S| = k$, there are $\Omega(n^{k^2/\log^3(n)})$ many $S$-absorbing $k$-matchings in $H$.

(b) For every $S \subseteq V(H)$ with $|S| = k$, there are $\Omega(n^{k^2+k/\log^3(n)})$ many $S$-absorbing $(k + 1)$-matchings in $H$.

Then $H$ contains a matching $M$ of size $O(\log^4(n))$ such that for each set $S \subseteq V(H) \setminus V(M)$ with $|S| = k$, there exists a perfect matching in $H[V(M) \cup S]$.

The following corollary is a direct application of Lemma 2.3. We use it to show that for a $p_{\ell}$-random subset $U$ of vertices of $H$, the property of $H$ of having many $S$-absorbing matchings is inherited a.a.s. by $H[U]$.

**Corollary 5.4.** Let $1/n \ll 1/s \leq 1/k \leq 1/3$. Let $H$ be a $k$-uniform $n$-vertex hypergraph. Let $\ell := \lceil \frac{k+1}{2} \log_2(n) \rceil$, $C_\ell := \sum_{i=1}^{\ell} 2^{i-1}$, and $p_\ell := \frac{1}{C_\ell 2^{\ell}}$. Let $U$ be a $p_\ell$-random subset of $V(H)$, and let $M$ be a set of $s$-matchings in $H$ with $|M| = \Omega(n^{sk/\log^3(n)})$. Then with probability at least $1 - \exp(-n^{1/6sk^2})$, the number of matchings in $M$ that are contained in $H[U]$ is $\Omega((pn)^{sk/\log^3(pn)})$.

To prove Lemma 4.4, we also need the following result by Han [35].

**Theorem 5.5 ([35, Theorem 1.1]).** Let $1/n \ll 1/k \leq 1/3$ such that $k$ does not divide $n$. Let $H$ be a $k$-uniform hypergraph on $n$ vertices with $\delta_{k-1}(H) \geq \lceil n/k \rceil$. Then $H$ contains an optimal matching.

Now we are ready to prove Lemma 4.4.

**Proof of Lemma 4.4.** By Lemma 5.2, at least one of the following holds.

(a) For every $S \subseteq V(H)$ with $|S| = k$, there are $\Omega(n^{k^2/\log^3(n)})$ many $S$-absorbing $k$-matchings in $H$.

(b) For every $S \subseteq V(H)$ with $|S| = k$, there are $\Omega(n^{k^2+k/\log^3(n)})$ many $S$-absorbing $(k + 1)$-matchings in $H$. 

Suppose that (a) holds (the proof for if (b) holds is similar). Let \( n_\ast := |U| \). We have that a.a.s. \( n_\ast = (1 \pm \varepsilon)p_rn \) and \( \delta_{k-1}(\mathcal{H}[U]) \geq (1/2 - 2\varepsilon)|U| \). By Corollary 5.4 and a union bound, it follows that a.a.s. for every \( S \in \binom{U}{k} \), the number of \( S \)-absorbing \( k \)-matchings in \( \mathcal{H}[U] \) is \( \Omega(n_\ast^{k^2}/\log^3(n_\ast)) \). Suppose that all of these events occur. By Lemma 5.3, there exists a matching \( M \) in \( \mathcal{H}[U] \) of size \( O(\log^4(n_\ast)) \) such that for each set \( S \subseteq U \setminus V(M) \) with \( |S| = k \), there exists a perfect matching in \( \mathcal{H}[V(M) \cup S] \). Let \( U_\ast := V(M) \), and note that \( |U_\ast| \leq \varepsilon|U| \). Let \( \tilde{M} \) be a matching in \( \mathcal{H} \) such that \( U_\ast \subseteq V(\tilde{\mathcal{H}}) \setminus V(M) \subseteq U \) and \( |V(\tilde{\mathcal{H}}) \setminus V(M)| \leq \varepsilon|U| \). Let \( U' := V(\tilde{\mathcal{H}}) \setminus V(M) \), and note that \( |U'| = n \equiv 0 \pmod{k} \). Let \( u \in U' \setminus U_\ast \) and \( U'' := U' \setminus (U_\ast \cup \{u\}) \). Note that \( |U''| \equiv k - 1 \pmod{k} \) and \( \delta_{k-1}(\mathcal{H}[U'']) \geq |U''|/k \). Thus, by Theorem 5.5, \( \mathcal{H}[U''] \) contains a matching \( M_\ast \) covering all but a set \( S_\ast \) of \( k - 1 \) vertices of \( U'' \). Let \( S := S_\ast \cup \{u\} \). By the absorption property of \( M \), \( \mathcal{H}[U_\ast \cup S] \) contains a perfect matching \( M' \). Note that \( M_\ast \cup M' \) is a perfect matching in \( \mathcal{H} - V(M) = \mathcal{H}[U'] \). Hence, \( U \) is \((\mathcal{H}, \varepsilon)\)-OM-stable for \( \mathcal{H} \). \( \square \)

6. Proof of Lemma 4.5

Now we briefly sketch the proof of Lemma 4.5. Since \( \mathcal{H} \) is close to being a critical hypergraph, there are \( \Omega(n^{r-k-1}) \) many ‘atypical’ edges (see Lemma 6.9). After choosing one of them (say \( e^* \)) and deleting the vertices from \( V(e^*) \), the resulting hypergraph \( \mathcal{H} - V(e^*) \) will meet the ‘divisibility condition’ (see Definition 6.7) which ensures a perfect matching even though the minimum codegree is slightly below \( m_{k-1}(k, n) \) (see Theorem 6.8). For a spanning subhypergraph \( \mathcal{H}' \) of \( \mathcal{H} - V(e^*) \) which consists of all typical edges of \( \mathcal{H} - V(e^*) \), by the definition of typical edges, \( \mathcal{H}' \) is also close to being a critical hypergraph. Thus, \( \mathcal{H}' \) is ‘dense’ enough to satisfy (O1). To show (O2), for a \( p_r \)-random subset \( U_\ell \) of \( V(\mathcal{H}') \), we need to make sure that \( \mathcal{H} - V(e^*) - V(M') \) has a perfect matching for any matching \( M' \) of \( \mathcal{H}' \) with \( 2 \mid |M'| \) and \( |V(M') \cap U_\ell| = o(|U_\ell|) \). Using the structural properties of \( \mathcal{H}' \), we can show that \( \mathcal{H} - V(e^*) - V(M') \) is close to being a critical hypergraph and also meets the divisibility condition. Thus, by Theorem 6.8, \( \mathcal{H} - V(e^*) - V(M') \) has a perfect matching.

Since the proof of Lemma 4.5 relies on some structural information of \( \mathcal{H} \), we need to introduce several notations first.

Let \( k \geq 3 \) be an integer, let \( 0 \leq r \leq k \) be an integer, and let \( A \) and \( B \) be disjoint sets. Let \( \mathcal{K}_r(A, B) := \{e \subseteq A \cup B: |e| = k, |e \cap A| = r, |e \cap B| = k - r\} \). For any \( k \)-uniform hypergraph \( \mathcal{H} \) with \( V(\mathcal{H}) = A \cup B \), let \( E^j_{\mathcal{H}}(A, B) := \mathcal{H} \cap \mathcal{K}_j(A, B) = \{e \in \mathcal{H}: |e \cap A| = j\} \). We often omit the subscript \( \mathcal{H} \) if it is clear. Extending the definition of \( \mathcal{H}_0^j(k, n) \), let us define

\[
\mathcal{H}_0^j(k, a, B) := \begin{cases} 
\bigcup_r: \text{even } \mathcal{K}_r(A, B) & \text{for odd } k \\
\bigcup_r: \text{odd } \mathcal{K}_r(A, B) & \text{for even } k.
\end{cases}
\]

Note that

\[
\mathcal{H}_0^j(k, A, B) = \begin{cases} 
\bigcup_r: \text{odd } \mathcal{K}_r(A, B) = \bigcup_r: \text{even } \mathcal{K}_r(A, B) & \text{for odd } k \\
\bigcup_r: \text{even } \mathcal{K}_r(A, B) & \text{for even } k.
\end{cases}
\]

Let \( n \in \mathbb{N} \) divisible by \( k \). If \( k \) is odd, then let \( a(k, n) \) be the unique odd integer from \( \{(n + \ell)/2 : \ell \in \mathbb{Z}, -2 \leq \ell \leq 1\} \). Otherwise if \( k \) is even, then let

\[
a(k, n) := \begin{cases} 
n/2 - 1 & \text{for even } n/k \\
n/2 & \text{for odd } n/k \text{ and odd } n/2.
\end{cases}
\]

**Definition 6.1** (Standard ordered pair). Let \( k, n \geq 3 \) be positive integers such that \( k \mid n \). Let \( A \) and \( B \) be disjoint sets such that \( |A| + |B| = n \). An ordered pair \( (A, B) \) is **standard** if \( |A| = a(k, n) \) and \( |B| = n - a(k, n) \).
Note that $\mathcal{H}^0(k, n)$ is a $k$-uniform $n$-vertex hypergraph isomorphic to $\mathcal{H}^0(k, A, B)$ for a standard ordered pair $(A, B)$.

**Definition 6.2 (Types).** Let $k, n \geq 3$ be positive integers such that $k \mid n$, and let $\mathcal{H}$ be a $k$-uniform $n$-vertex hypergraph. For $\varepsilon \in (0, 1)$ and an ordered partition $(A, B)$ of $V(\mathcal{H})$ such that either $|\mathcal{H}^0(k, A, B) \setminus \mathcal{H}| \leq \varepsilon n^k$ or $|\mathcal{H}^0(k, A, B) \setminus \mathcal{H}| \leq \varepsilon n^k$ holds, we define the following.

(a) If $k$ is odd and $|\mathcal{H}^0(k, A, B) \setminus \mathcal{H}| \leq \varepsilon n^k$, then we say $\mathcal{H}$ belongs to the type (a) with respect to $(\varepsilon, A, B)$.

(b) If $k$ is odd and $|\mathcal{H}^0(k, A, B) \setminus \mathcal{H}| \leq \varepsilon n^k$, then we say $\mathcal{H}$ belongs to the type (b) with respect to $(\varepsilon, A, B)$.

(c) If $k$ is even and $|\mathcal{H}^0(k, A, B) \setminus \mathcal{H}| \leq \varepsilon n^k$, then we say $\mathcal{H}$ belongs to the type (c) with respect to $(\varepsilon, A, B)$.

(d) If $k \equiv 0 \pmod{4}$ and $|\mathcal{H}^0(k, A, B) \setminus \mathcal{H}| \leq \varepsilon n^k$, then we say $\mathcal{H}$ belongs to the type (d) with respect to $(\varepsilon, A, B)$.

(e) If $k \equiv 2 \pmod{4}$ and $|\mathcal{H}^0(k, A, B) \setminus \mathcal{H}| \leq \varepsilon n^k$, then we say $\mathcal{H}$ belongs to the type (e) with respect to $(\varepsilon, A, B)$.

We also say $\mathcal{H}$ belongs to the type $\alpha$ if it belongs to the type $\alpha$ with respect to $(\varepsilon, A, B)$ for some $\varepsilon \in (0, 1)$ and partition $(A, B)$ of $V(\mathcal{H})$.

**Definition 6.3 (Typical indices and edges).** Let $k \geq 3$ be a positive integer. For $\alpha \in \{(a), (b), (c), (d), (e)\}$, an index $r \in \{0, \ldots, k\}$ is called $\alpha$-typical (with respect to $k$) if

\[
\begin{aligned}
    r \equiv 0 \pmod{2} & \quad \text{for } \alpha \in \{(a), (c)\} \\
    1 \pmod{2} & \quad \text{for } \alpha \in \{(b), (d), (e)\}.
\end{aligned}
\]

Otherwise $r$ is called $\alpha$-atypical.

For any $k$-uniform hypergraph $\mathcal{H}$ with an ordered partition $(A, B)$, an edge $e \in \mathcal{H}$ is $\alpha$-typical with respect to $(A, B)$ if $e \in E^r_{\mathcal{H}}(A, B)$ for an $\alpha$-typical index $r$. Otherwise an edge $e$ is called $\alpha$-atypical with respect to $(A, B)$.

**Observation 6.4.** Let $k \geq 3$ be a positive integer, and let $\alpha \in \{(a), (b), (c), (d), (e)\}$. For disjoint sets $A$ and $B$,

\[
\bigcup_{r: \alpha\text{-typical}} K_r(A, B) = \begin{cases} 
\mathcal{H}^0(k, A, B) & \text{if } \alpha \in \{(a), (b), (c), (d), (e)\} \\
\mathcal{H}^0(k, A, B) & \text{if } \alpha \in \{(b), (c)\}.
\end{cases}
\]

In particular, for $\varepsilon \in (0, 1)$, a $k$-uniform hypergraph $\mathcal{H}$ belongs to the type $\alpha$ with respect to $(\varepsilon, A, B)$ if and only if

\[
\sum_{r: \alpha\text{-typical}} |K_r(A, B) \setminus E^r_{\mathcal{H}}(A, B)| \leq \varepsilon n^{k}.
\]

**Definition 6.5 (Special typical index).** Let $k \geq 3$ be a positive integer, and let $\mathcal{H}$ be a $k$-uniform hypergraph which belongs to the type $\alpha \in \{(a), (b), (c), (d), (e)\}$. The special $\alpha$-typical index for $\mathcal{H}$ is $r^* := k - 1, 1, k - 2, k/2 + 1, k/2$ for $\alpha = (a), (b), (c), (d), (e)$ respectively.

**Proposition 6.6.** Let $k \geq 3$ be a positive integer, and let $\mathcal{H}$ be a $k$-uniform hypergraph which belongs to the type $\alpha \in \{(a), (b), (c), (d), (e)\}$. Then the special $\alpha$-typical index for $\mathcal{H}$ is $\alpha$-typical.

**Proof.** By Definitions 6.2 and 6.5, since $\mathcal{H}$ belongs to the type $\alpha$, the following hold.

- If $\alpha = (a)$, then $k$ is odd, so the special index $k - 1$ is even.
- If $\alpha = (b)$, then the special index is 1 which is odd.
- If $\alpha = (c)$, then $k$ is even, so the special index $k - 2$ is even.
- If $\alpha = (d)$, then $4 \mid k$, so the special index $k/2 + 1$ is odd.
- If $\alpha = (e)$, then $k \equiv 2 \pmod{2}$, so the special index $k/2$ is odd.
Thus, by Definition 6.3, the special $\alpha$-typical index for $\mathcal{H}$ is $\alpha$-typical. \hfill \square

**Definition 6.7 (Divisibility condition).** Let $n$ be a positive integer divisible by $k$. Let $(A, B)$ be an ordered pair such that $n = |A| + |B|$. We say $(A, B)$ satisfies the divisibility condition with respect to the type $\alpha \in \{(a), (b), (c), (d), (e)\}$ if the following hold.

- If $\alpha \in \{(a), (c)\}$, then $|A|$ is even.
- If $\alpha = (b)$, then $|B|$ is even.
- If $\alpha = (d)$, then $\frac{|A| - |B|}{2} \equiv \frac{\epsilon}{k} \pmod{2}$.
- If $\alpha = (e)$, then $\frac{|A| - |B|}{2}$ is even.

Now we state two ingredients from [63] which we use in the proof of Lemma 4.5. Here we briefly explain how to deduce the following theorem from the proof of [63, Lemma 3.1]: the hypergraph $\mathcal{H}$ in [63, Lemma 3.1] is only assumed to satisfy $\delta_{k-1}(\mathcal{H}) \geq \delta^0(k, n) + 1$ and that $\mathcal{H}$ $\varepsilon$-contains either $\mathcal{H}^0(k, n)$ or $\overline{\mathcal{H}}^0(k, n)$. In their proof, they began with slightly modifying the standard ordered partition $(A, B)$ to $(A', B')$ to ensure that $d_{E_\mathcal{H}^*(A', B')}(v) > 0.1d_{K_{\alpha}(A', B')}(v)$ for each $v \in V(\mathcal{H})$ and the special $\alpha$-typical index $r^*$ for $\mathcal{H}$. Then they used Facts 4.5–4.8 of [63] which provides an atypical edge $e$, and showed that the partition $(A' \setminus V(e), B' \setminus V(e))$ satisfies the divisibility condition if $(A', B')$ does not satisfy the divisibility condition. Since the rest of their proof works for the hypergraphs with minimum codegree at least $n/2 - o(n)$, the minimum degree condition can be relaxed to $\delta_{k-1}(\mathcal{H}) \geq n/2 - o(n)$ if we further assume that $(A', B')$ satisfies the divisibility condition and that $d_{E_\mathcal{H}^*(A', B')}(v) > 0.1d_{K_{\alpha}(A', B')}(v)$ for each $v \in V(\mathcal{H})$, as we stated as below.

**Theorem 6.8 ([63]).** Let $1/n \ll \varepsilon \ll 1/k \leq 1/3$ such that $k \mid n$. Let $A'$ and $B'$ be disjoint sets such that $n = |A'| + |B'|$ and $|A'| - |B'| \leq \varepsilon n$. If $\mathcal{H}$ is a $k$-uniform $n$-vertex hypergraph with an ordered partition $(A', B')$ of $V(\mathcal{H})$, then $\mathcal{H}$ has a perfect matching if the following hold.

(i) $\delta_{k-1}(\mathcal{H}) \geq n/2 - \varepsilon n$.
(ii) The hypergraph $\mathcal{H}$ belongs to some type $\alpha \in \{(a), (b), (c), (d), (e)\}$ with respect to $(\varepsilon, A', B')$.
(iii) The ordered partition $(A', B')$ satisfies the divisibility condition with respect to the type $\alpha$.
(iv) For each vertex $v \in V(\mathcal{H})$, $d_{E_\mathcal{H}^*(A', B')}(v) > 0.1d_{K_{\alpha}(A', B')}(v)$, where $r^*$ is the special $\alpha$-typical index for $\mathcal{H}$.

The following lemma shows that there are $\Omega(n^{k-1})$ atypical edges, which follows from the proofs of Facts 4.5–4.8 of [63].

**Lemma 6.9 ([63]).** Let $1/n \ll \varepsilon \ll 1/k \leq 1/3$ such that $k \mid n$. Let $\mathcal{H}$ be a $k$-uniform $n$-vertex hypergraph such that $\delta_{k-1}(\mathcal{H}) \geq \delta^0(k, n) + 1$. For any partition $\{A', B'\}$ of $V(\mathcal{H})$ such that $|A'|, |B'| \geq n/10$, the following hold.

(a) If $k$ is odd and $|A'|$ is odd, then $|E_\mathcal{H}^1(A', B') \cup E_\mathcal{H}^{k-2}(A', B')| \geq cn^{k-1}$.
(b) If $k$ is odd and $|B'|$ is odd, then $|E_\mathcal{H}^k(A', B') \cup E_\mathcal{H}^{k-2}(A', B')| \geq cn^{k-1}$.
(c) If $k$ is even, then $|E_\mathcal{H}^1(A', B') \cup E_\mathcal{H}^{k-1}(A', B')| \geq cn^{k-1}$.
(d) If $k \equiv 0 \pmod{4}$ and $\frac{|A'| - |B'|}{2} \not\equiv \frac{n}{k} \pmod{2}$, then $|E_\mathcal{H}^2(A', B') \cup E_\mathcal{H}^{k-2}(A', B')| \geq cn^{k-1}$.
(e) If $k \equiv 2 \pmod{4}$, then $|E_\mathcal{H}^2(A', B') \cup E_\mathcal{H}^{k-2}(A', B')| \geq cn^{k-1}$.

Now we are ready to prove Lemma 4.5.

**Proof of Lemma 4.5.** Let $1/n \ll \delta \ll \epsilon \ll \eta \ll 1/k \leq 1/3$. Since $\mathcal{H}$ $\varepsilon$-contains either $\mathcal{H}^0(k, n)$ or $\overline{\mathcal{H}}^0(k, n)$, there exists a standard partition $(A, B)$ of $V(\mathcal{H})$ such that $\mathcal{H}$ belongs to the type $\alpha$ with respect to $(\varepsilon, A, B)$ for some $\alpha \in \{(a), (b), (c), (d), (e)\}$. Let $r^*$ be the special $\alpha$-typical index for $\mathcal{H}$. By [63, Fact 4.4], there exists an ordered partition $(A', B')$ of $V(\mathcal{H})$ such that the following hold.

(S1) $|A \triangle A'| = |B \triangle B'| \leq \epsilon^{1/2}kn$, and thus $|A'| - |B'| \leq 2\varepsilon^{1/2}kn$.
(S2) For each vertex $v \in V(\mathcal{H})$, $d_{E_\mathcal{H}^*(A', B')}(v) > 0.2d_{K_{\alpha}(A', B')}(v) > 0.2\frac{n^{k-1}}{3^{k-1}(k-1)!}$. 
Claim 1. $\mathcal{H}$ belongs to the type $\alpha$ with respect to $(5k\varepsilon^{3/2}, A', B')$.

Proof of claim: Note that

$$
\sum |\mathcal{K}_r(A', B') \setminus E_\mathcal{H}^r(A', B')| \leq \sum |\mathcal{K}_r(A', B') \setminus \mathcal{K}_r(A, B)|
+ \sum |\mathcal{K}_r(A, B) \setminus E_\mathcal{H}^r(A, B)|
+ \sum |E_\mathcal{H}^r(A, B) \setminus E_\mathcal{H}^r(A', B')|.
$$

where the summations are taken over all $\alpha$-typical indices $r$. By Observation 6.4, since $\mathcal{H}$ belongs to the type $\alpha$ with respect to $(\varepsilon, A, B)$, the second term in this sum is at most $\varepsilon n^k \leq k\varepsilon^{1/2}n^k$. By (S1), the first and third terms in this sum are each at most $2\varepsilon^{1/2}kn^k$. Thus, again by Observation 6.4, $\mathcal{H}$ belongs to the type $\alpha$ with respect to $(5k\varepsilon^{3/2}, A', B')$, as desired. ♦

Claim 2. There are at least $\varepsilon n^{k-1}$ choices of an edge $e^* \in \mathcal{H}$ such that for each of the choices of $e^*$, the subhypergraph $\mathcal{H} - V(e^*)$ belongs to the type $\alpha$ with respect to $(6k\varepsilon^{1/2}, A'', B'')$, where $A'' := A' \setminus V(e^*)$ and $B'' := B' \setminus V(e^*)$, and the ordered partition $(A'', B'')$ satisfies the divisibility condition with respect to the type $\alpha$.

Proof of claim: By Claim 1 and Observation 6.4, $\mathcal{H} - V(e)$ belongs to the type $\alpha$ with respect to $(6k\varepsilon^{1/2}, A \setminus V(e), B \setminus V(e))$ for every $e \in \mathcal{H}$, so it suffices to show that there are at least $\varepsilon n^{k-1}$ choices of an edge $e^*$ such that $(A' \setminus V(e^*), B' \setminus V(e^*))$ satisfies the divisibility condition with respect to $\alpha$.

First, if $(A', B')$ satisfies the divisibility condition for $\alpha$, then by the choice of the special typical index $r^*$, it is easy to see that the ordered partition $(A', V(e^*), B' \setminus V(e^*))$ satisfies the divisibility condition for every $e^* \in E_\mathcal{H}^r(A', B')$. In this case, (S2) implies that there are sufficiently many choices for $e^*$.

Thus, we may assume $(A', B')$ does not satisfy the divisibility condition. Let

$$
E^* := \left\{ \begin{array}{ll}
E_\mathcal{H}^1(A', B') \cup E_\mathcal{H}^{k-2}(A', B') & \text{if } \alpha = (a), \\
E_\mathcal{H}^{k-1}(A', B') \cup E_\mathcal{H}^{k-1}(A', B') & \text{if } \alpha = (b), \\
E_\mathcal{H}^0(A', B') \cup E_\mathcal{H}^1(A', B') & \text{if } \alpha = (c), \\
E_\mathcal{H}^0(A', B') \cup E_\mathcal{H}^{k-2}(A', B') & \text{if } \alpha \in \{(d), (e)\}.
\end{array} \right.
$$

Since $(A', B')$ does not satisfy the divisibility condition, it is also easy to see that in all cases of $\alpha$, the ordered partition $(A' \setminus V(e^*), B' \setminus V(e^*))$ satisfies the divisibility condition for every $e^* \in E^*$. Moreover, by Lemma 6.9, we have $|E^*| \geq \varepsilon n^{k-1}$, so there are sufficiently many choices for $e^*$, as desired. ♦

Now we fix $e^* \in \mathcal{H}$ satisfying Claim 2. Let us define

$$
(6.1) \quad \mathcal{H}' := \left\{ \begin{array}{ll}
\mathcal{H} \cap \mathcal{H}^0(k, A'', B'') & \text{if } \alpha \in \{(a), (d), (e)\}, \\
\mathcal{H} \cap \mathcal{H}^0(k, A'', B'') & \text{if } \alpha \in \{(b), (c)\}.
\end{array} \right.
$$

Thus, the subhypergraph $\mathcal{H}'$ is the collection of the $\alpha$-typical edges in $\mathcal{H} - V(e^*)$ with respect to $(A'', B'')$. Since $\mathcal{H} - V(e^*)$ belongs to the type $\alpha$ with respect to $(6k\varepsilon^{1/2}, A'', B'')$ by Claim 2 and Observation 6.4, $\mathcal{H}'$ also belongs to the type $\alpha$ with respect to $(6k\varepsilon^{1/2}, A'', B'')$.

Claim 3. The hypergraph $\mathcal{H}'$ satisfies the following properties.

- At least a $(1 - \varepsilon^{1/6})$-fraction of $(k - 1)$-sets $S \in (A'' \cup B'')_{k-1}$ satisfy $d_{\mathcal{H}'}(S) \geq n/(2 - 10\varepsilon^{1/6}kn)$.
- For each vertex $v \in A'' \cup B''$, $d_{\mathcal{H}'}(A'', B'')(v) > 0.15d_{\mathcal{K}_r}(A'', B'')(v) \geq 0.15\frac{\eta k^{-1}}{3(k-1)!}$.

In particular, since $\varepsilon \ll \eta$, $\mathcal{H}'$ is $(1/2 - \eta, \frac{0.15}{3(k-1)!}, k-1, \eta)$-dense.
Proof of claim: Without loss of generality, we may assume that \( \alpha \in \{(a), (d), (e)\} \). For the other case \( \alpha \in \{(b), (c)\} \), we can just switch the role of \( \mathcal{H}^0(k, A'', B'') \) and \( \mathcal{H}^0(k, A', B') \).

For any \( k-1 \) distinct vertices \( v_1, \ldots, v_{k-1} \in A'' \cup B'' \), depending on the parity of \( |A'' \cap \{v_1, \ldots, v_{k-1}\}| \), \( d_{\mathcal{H}^0(k,A'',B'')}(v_1, \ldots, v_{k-1}) \) is either \( |A'' \setminus \{v_1, \ldots, v_{k-1}\}| \) or \( |B'' \setminus \{v_1, \ldots, v_{k-1}\}| \). Thus, since by (S1), \( \min\{|A''|,|B''|\} \geq \min\{|A'|,|B'|\} \geq n/2 - 2\varepsilon^{1/2}kn - k \) and \( \max\{|A''|,|B''|\} \leq n/2 + 2\varepsilon^{1/2}kn \), we have

\[
\begin{align*}
\delta_{k-1}(\mathcal{H}^0(k,A'',B'')) &\geq \min\{|A''|,|B''|\} - (k-1) \geq n/2 - 3\varepsilon^{1/2}kn, \\
\Delta_{k-1}(\mathcal{H}^0(k,A'',B'')) &\leq \max\{|A''|,|B''|\} \leq n/2 + 2\varepsilon^{1/2}kn,
\end{align*}
\]

where \( \Delta_{k-1}(\mathcal{H}^0(k,A'',B'')) = \max\{d_{\mathcal{H}^0(k,A'',B'')}(S) : S \in (A'' \cup B'') \} \) is the maximum codegree of \( \mathcal{H}^0(k,A'',B'') \).

Since \( \mathcal{H}' \subseteq \mathcal{H}^0(k,A'',B'') \), every \( (k-1) \)-set \( S \in (A'' \cup B'') \) satisfies \( d_{\mathcal{H}'}(S) \leq \Delta_{k-1}(\mathcal{H}^0(k,A'',B'')) \leq n/2 + 2\varepsilon^{1/2}kn \). Let \( N \) be the number of \((k-1)\)-sets \( S \in (A'' \cup B'') \) such that \( d_{\mathcal{H}'}(S) \geq n/2 - 10\varepsilon^{1/6}kn \). Since \( |\mathcal{H}^0(k,A'',B'') \setminus \mathcal{H}'| \leq 6k\varepsilon^{1/2}n^k \),

\[
ke(\mathcal{H}') \geq ke(\mathcal{H}^0(k,A'',B'')) - 6k^2\varepsilon^{1/2}n^k \geq \left( \binom{|A'' \cup B''|}{k-1} \right) \delta_{k-1}(\mathcal{H}^0(k,A'',B'')) - 6k^2\varepsilon^{1/2}n^k \\
\geq \left( \binom{|A'' \cup B''|}{k-1} \right) \left( n/2 - 3\varepsilon^{1/2}kn - 2(k-1)!6k^2\varepsilon^{1/2}n \right).
\]

On the other hand,

\[
ke(\mathcal{H}') = \sum_{S \in (A'' \cup B'')} d_{\mathcal{H}'}(S) \leq \left( \binom{|A'' \cup B''|}{k-1} - N \right) \left( n/2 - 10\varepsilon^{1/6}kn \right) + N \left( n/2 + 2\varepsilon^{1/2}kn \right) \\
= \left( \binom{|A'' \cup B''|}{k-1} \right) \left( n/2 - 10\varepsilon^{1/6}kn \right) + N \left( 10\varepsilon^{1/6}kn + 2\varepsilon^{1/2}kn \right) n.
\]

Combining both inequalities, since \( \varepsilon \ll 1/k \),

\[
N \geq \left( \binom{|A'' \cup B''|}{k-1} \right) \frac{10\varepsilon^{1/6}kn - 3\varepsilon^{1/2}kn - 2(k-1)!6k^2\varepsilon^{1/2}n}{10\varepsilon^{1/6}kn + 2\varepsilon^{1/2}kn} \geq \left( \binom{|A'' \cup B''|}{k-1} \right) \frac{10\varepsilon^{1/6}kn - \varepsilon^{1/3}kn}{10\varepsilon^{1/6}kn} \\
> (1 - \varepsilon^{1/6}) \left( \binom{|A'' \cup B''|}{k-1} \right),
\]

as desired.

Note that \( r^* \) is the special \( \alpha \)-typical index for the hypergraphs \( \mathcal{H}, \mathcal{H} - V(e^*) \), and \( \mathcal{H}' \). Since \( \mathcal{H}' \) is the subhypergraph of typical edges of \( \mathcal{H} - V(e^*) \), we have \( E_{\mathcal{H}'}(A'',B'') = E_{\mathcal{H} - V(e^*)}(A'',B'') \). Moreover, since \( |A' \setminus A''| + |B' \setminus B''| = |V(e^*)| = k \), we have \( d_{E_{\mathcal{H}'}(A'',B'')}(v) \geq d_{E_{\mathcal{H} - V(e^*)}(A'',B')}(v) - kn^{k-2} \) for each vertex \( v \in A'' \cup B'' \). Thus, by (S1), we have \( d_{E_{\mathcal{H}'}(A'',B'')}(v) > 0.15d_{K_{r^*}}(A'',B'')(v) \) as desired.

Claim 4. Let \( M' \) be a matching in \( \mathcal{H}' \) such that \( |M'| \equiv 0 \pmod{2} \) if \( \alpha \in \{(d), (e)\} \). Then the ordered partition \( (A'' \setminus V(M'), B'' \setminus V(M')) \) satisfies the divisibility condition with respect to \( \alpha \).

Proof of claim: By Claim 2, \( (A'', B'') \) satisfies the divisibility condition with respect to the type \( \alpha \). Now we divide the cases according to the type \( \alpha \).

Case \( \alpha \in \{(a), (c)\} \). Since \( |e \cap A''| \equiv 0 \pmod{2} \) for each \( e \in \mathcal{H}' \), we have \( |A''| \equiv |A'' \setminus V(M')| \pmod{2} \).

Case \( \alpha = (b) \). Since \( |e \cap B''| \equiv 0 \pmod{2} \) for each \( e \in \mathcal{H}' \), we have \( |B''| \equiv |B'' \setminus V(M')| \pmod{2} \).
Case $\alpha \in \{(d), (e)\}$. Let $M' = \{e_1, \ldots, e_t\}$ for some even integer $t$. Let $\ell_i := |e_i \cap A''|$ for each $i \in [t]$. Since $|e \cap A''|$ is odd for each $e \in \mathcal{H}'$, we have $\ell_i \equiv 1 \pmod{2}$ for each $i \in [t]$. Thus,

$$|A'' \setminus V(M')| = |A''| - (\ell_1 + \cdots + \ell_t) \text{ and } |B'' \setminus V(M')| = |B''| - kt + (\ell_1 + \cdots + \ell_t),$$

so \(\frac{|A'' \setminus V(M')| - |B'' \setminus V(M')|}{2} = \frac{|A''| - |B''|}{2} + k\frac{t}{2} - (\ell_1 + \cdots + \ell_t) \equiv \frac{|A''| - |B''|}{2} \pmod{2}\). Thus, \((A'' \setminus V(M'), B'' \setminus V(M'))\) satisfies the divisibility condition with respect to $\alpha$.

Now we have all the ingredients to prove Lemma 4.5. By Claim 3, \((O1)\) holds. To show \((O2)\), it suffices to prove the following claim. Recall that $\ell := \lceil \frac{k-1}{k} \log_2 n \rceil$, $C_\ell := \sum_{i=1}^{\ell} 2^{-i} = 1 - 2^{-\ell}$, and $p_\ell = 1/(C_\ell 2^\ell)$.

Claim 5. Let $U_\ell$ be a $p_\ell$-random subset of $V(\mathcal{H}') = V(\mathcal{H}) \setminus V(e^*)$. With probability $1 - o(1)$, for all matchings $M'$ of $\mathcal{H}'$ satisfying 2 \(\{ |M'|, V(\mathcal{H}') \setminus V(M') \subseteq U_\ell, \text{ and } |U_\ell \cap V(M')| \leq \varepsilon |U_\ell| \}$, the subhypergraph $\mathcal{H}'' := \mathcal{H} \setminus V(e^*) \setminus V(M')$ has a perfect matching.

Proof of claim: First of all, by a Chernoff bound (Lemma 2.1), $|U_\ell| = (1 \pm \varepsilon)p_\ell n$ with probability $1 - o(1)$. We apply Theorem 6.8 to show that $\mathcal{H}''$ has a perfect matching. To do so, we will show that the following assumptions of Theorem 6.8 hold with probability $1 - o(1)$, where $A''' := A'' \setminus V(M')$ and $B''' := B'' \setminus V(M')$.

\begin{enumerate}
\item First of all, Claim 4 shows (3). Now we prove (2). Since $\mathcal{H}'$ belongs to the type $\alpha$ with respect to \((6k\varepsilon^{1/2}, A'', B'')\) (see the discussion below (6.1)), let us define $F \subseteq \binom{A'' \cup B''}{\ell}$ such that
\begin{itemize}
\item $\bigcup_{v \in F} K_{r}(A'', B'') \setminus \mathcal{H}' \subseteq \mathcal{F}$ and
\item $|F| = 6k\varepsilon^{1/2} n^k \pm 1$.
\end{itemize}

In particular, $\mathcal{F}$ contains all possible typical ‘non-edges’ of $\mathcal{H}'$. By Lemma 2.3 (i), $U_\ell$ is $(p_\ell, \varepsilon, \mathcal{F})$-typical with probability $1 - o(1)$, so the number of elements in $\mathcal{F}$ contained in $U_\ell$ is $(1 \pm \varepsilon)p_\ell |F| \leq 7k\varepsilon^{1/2} |U_\ell|^k$ with probability $1 - o(1)$. Note that the number of elements in $\mathcal{F}$ contained in $U_\ell$ is at least the number of all possible typical ‘non-edges’ of $\mathcal{H}'$ contained in $U_\ell$. Thus, $|\bigcup_{v \in \mathcal{F}} K_{r}(A'' \cap U_\ell, B'' \cap U_\ell) \setminus \mathcal{H}'[U_\ell]| \leq 7k\varepsilon^{1/2} |U_\ell|^k$, so $\mathcal{H}'[U_\ell]$ belongs to the type $\alpha$ with respect to $(7k\varepsilon^{1/2}, A'' \cap U_\ell, B'' \cap U_\ell)$. Since $\mathcal{H}' \subseteq \mathcal{H}$ and $\varepsilon \ll \eta \ll 1/k$, $\mathcal{H}'[U_\ell]$ belongs to the type $\alpha$ with respect to $(\eta/2, A'' \cap U_\ell, B'' \cap U_\ell)$. Thus, since $\varepsilon \ll \eta$ and $|U_\ell \cap V(M')| \leq \varepsilon |U_\ell|$ and $(V(\mathcal{H}) \setminus V(e^*)) \setminus V(M') \subseteq U_\ell$, $\mathcal{H}'[U_\ell \setminus V(M')] = \mathcal{H}''[U_\ell] \subseteq \mathcal{F}$ and $|\bigcup_{v \in \mathcal{F}} K_{r}(A'', B'') \setminus \mathcal{H}'[U_\ell]| = 7k\varepsilon^{1/2} |U_\ell|^k$, so $\mathcal{H}'[U_\ell]$ belongs to the type $\alpha$ with respect to $(\eta, A'' \setminus V(M'), B'' \setminus V(M'))$, proving (2).

Now we prove (1). Since $\mathbb{E}[d_{\mathcal{H}}(S; U_\ell)] \geq p_\ell (\delta_{k-1}(\mathcal{H}) - |V(e^*)|)$ for each $S \in \binom{V(\mathcal{H}')}{k-1}$ and $|U_\ell| = (1 \pm \varepsilon)n^k$ with probability $1 - o(1)$, by a Chernoff bound (Lemma 2.1) and a union bound, we have $\delta_{k-1}(\mathcal{H}[U_\ell]) \geq (1 - \eta)|U_\ell|/2$ with probability $1 - o(1)$. Thus, $\delta_{k-1}(\mathcal{H}[U_\ell] \setminus V(M')) \geq (1 - \eta)|U_\ell|/2 - |U_\ell \cap V(M')| > |U_\ell|/2 - \varepsilon |U_\ell|$ with probability $1 - o(1)$, which shows (1).

Finally, we prove (4). For each $v \in A'' \cup B''$, since $U_\ell$ is a $p_\ell$-random subset of $V(\mathcal{H}')$, we have
\begin{itemize}
\item $\mathbb{E}[d_{E'_\eta[U_\ell]}(A'' \cap U_\ell, B'' \cap U_\ell)(v)] = p_\ell^{k-1} d_{E'_\eta(A'', B'')(v)}$ and
\item $\mathbb{E}[d_{K_{r}(A'' \cap U_\ell, B'' \cap U_\ell)}(v)] = p_\ell^{k-1} d_{K_{r}(A'', B'')(v)}$.
\end{itemize}

Let $F_v := \{ v \setminus \{v\} : v \in E'_\eta(A'', B'') \}$, and let $G_v := \{ v \setminus \{v\} : v \in K_{r}(A'', B'') \}$. Applying Lemma 2.3 (i) twice for each $v \in A'' \cup B''$ and taking union bounds, with probability $1 - o(1)$, $U_\ell$ is both $(p_\ell, \varepsilon, F_v)$-typical and $(p_\ell, \varepsilon, G_v)$-typical for all $v \in A'' \cup B''$. Thus, for each
where the first equality and the last inequality follow since $U_\ell$ is $(p_\ell, \varepsilon, \mathcal{F}_v)$-typical and $(p_\ell, \varepsilon, \mathcal{G}_v)$-typical, respectively. On the other hand, since $r^*$ is the special $\alpha$-typical index for $\mathcal{H}'$ and $\mathcal{H}'$ is the subhypergraph of typical edges of $\mathcal{H} - V(e^*)$, we have $E_{\mathcal{H}''}(A^m, B^m) = E_{\mathcal{H}' - V(M')} (A^m, B^m)$. Thus,

\[
d_{E_{\mathcal{H}''}(A^m, B^m)}(v) \geq d_{E_{\mathcal{H}''}(A^m \cap U_\ell, B^m \cap U_\ell)}(v) - |U_\ell \cap V(M')||U_\ell|^{k-2}
\]

(6.2)

In the penultimate inequality we used that $|U_\ell|$ is large enough (it is $(1 \pm \varepsilon)p_{11}$ with probability $1 - o(1)$), and in the final inequality we used $V(\mathcal{H}') \setminus V(M') \subseteq U_\ell$. This proves (4). Thus, by Theorem 6.8, $\mathcal{H}''$ has a perfect matching with probability $1 - o(1)$.

\[\Box\]

\section*{References}

[1] P. Allen, J. Böttcher, J. Corsten, E. Davies, M. Jenssen, P. Morris, B. Roberts, and J. Skokan, A robust Corrádi–Hajnal Theorem, arXiv:2209.01116 (2022).

[2] N. Alon, P. Frankl, H. Huang, V. Rödl, A. Ruciński, and B. Sudakov, Large matchings in uniform hypergraphs and the conjecture of Erdős and Samuels, J. Combin. Theory Ser. A 119 (2012), 1200–1215.

[3] Y. Alon and M. Krivelevich, Hitting time of edge disjoint Hamilton cycles in random subgraph processes on dense base graphs, SIAM J. Discrete Math. 36 (2022), 728–754.

[4] R. Alweiss, S. Lovett, K. Wu, and J. Zhang, Improved bounds for the sunflower lemma, Ann. of Math. (2) 194 (2021), 795–815.

[5] B. Barber, S. Glock, D. Kühn, A. Lo, R. Montgomery, and D. Osthus, Minimalist designs, Random Structures Algorithms 57 (2020), 47–63.

[6] B. Barber, D. Kühn, A. Lo, and D. Osthus, Edge-decompositions of graphs with high minimum degree, Adv. Math. 288 (2016), 337–385.

[7] B. Barber, D. Kühn, A. Lo, D. Osthus, and A. Taylor, Clique decompositions of multipartite graphs and completion of Latin squares, J. Combin. Theory Ser. A 151 (2017), 146–201.

[8] Y. Chang, H. Ge, J. Han, and G. Wang, Matching of given sizes in hypergraphs, SIAM J. Discrete Math. 36 (2022), 2323–2338.

[9] F. R. K. Chung, Regularity lemmas for hypergraphs and quasi-randomness, Random Structures Algorithms 2 (1991), 241–252.

[10] P. Condon, A. Espuny Díaz, A. Girão, D. Kühn, and D. Osthus, Hamiltonicity of random subgraphs of the hypercube, Mem. Amer. Math. Soc., to appear.

[11] C. Cooper, A. Frieze, M. Molloy, and B. Reed, Perfect matchings in random r-regular, s-uniform hypergraphs, Combin. Probab. Comput. 5 (1996), 1–14.

[12] G. A. Dirac, Some theorems on abstract graphs, Proc. London Math. Soc. (3) 2 (1952), 69–81.

[13] J. Edmonds, Paths, trees, and flowers, Canadian J. Math. 17 (1965), 449–467.

[14] P. Erdős and A. Rényi, On the existence of a factor of degree one of a connected random graph, Acta Math. Acad. Sci. Hungar. 17 (1966), 359–368.

[15] P. Erdős, On the combinatorial problems which I would most like to see solved, Combinatorica 1 (1981), 25–42.

[16] P. Erdős and R. Rado, Intersection theorems for systems of sets, J. Lond. Math. Soc. 1 (1960), 85-90.

[17] A. Ferber, L. Hardiman, and A. Mond, Counting Hamiltonian cycles in Dirac hypergraphs, arXiv:2110.15475 (2021).

[18] A. Ferber, M. Krivelevich, and B. Sudakov, Counting and packing Hamilton $\ell$-cycles in dense hypergraphs, J. Comb. 7 (2016), 135–157.

[19] A. Ferber and M. Kwan, Dirac-type theorems in random hypergraphs, J. Combin. Theory Ser. B 155 (2022), 318–357.
P. Frankl and V. Rödl, The uniformity lemma for hypergraphs, Graphs Combin. 8 (1992), 309–312.

P. Frankl and A. Kupavskii, The Erdős matching conjecture and concentration inequalities, J. Combin. Theory Ser. B 157 (2022), 366–400.

K. Frankston, J. Kahn, B. Narayanan, and J. Park, Thresholds versus fractional expectation-thresholds, Ann. of Math. (2) 194 (2021), 475–495.

A. Frieze and S. Janson, Perfect matchings in random s-uniform hypergraphs, Random Structures Algorithms 7 (1995), 41–57.

A. Frieze and R. Kannan, Quick approximation to matrices and applications, Combinatorica 19 (1999), 175–220.

W. Gao and J. Han, Minimum codegree threshold for $C_6^3$-factors in 3-uniform hypergraphs, Combin. Probab. Comput. 26 (2017), 536–559.

W. Gao, J. Han, and Y. Zhao, Codegree conditions for tiling complete k-partite k-graphs and loose cycles, Combin. Probab. Comput. 28 (2019), 840–870.

M. R. Garey and D. S. Johnson, Computers and intractability: A guide to the theory of NP-completeness, W. H. Freeman & Co., USA, 1979.

R. Glebov, Z. Luria, and M. Simkin, Perfect matchings in random subgraphs of regular bipartite graphs, J. Graph Theory 97 (2021), 208–231.

S. Glock, S. Gould, F. Joos, D. Kühn, and D. Osthus, Counting Hamilton cycles in Dirac hypergraphs, Combin. Probab. Comput. 30 (2021), 631–653.

S. Glock, D. Kühn, A. Lo, R. Montgomery, and D. Osthus, On the decomposition threshold of a given graph, J. Combin. Theory Ser. B 139 (2019), 47–127.

S. Glock, D. Kühn, A. Lo, and D. Osthus, The existence of designs via iterative absorption: hypergraph F-designs for arbitrary F, Mem. Amer. Math. Soc. 284 (2023), v+131.

A. Hajnal and E. Szemerédi, Proof of a conjecture of P. Erdős, Combinatorial theory and its applications, II (Proc. Colloq., Balatonfüred, 1969), North-Holland, Amsterdam, 1970, 601–623.

H. Hán and M. Schacht, Dirac-type results for loose Hamilton cycles in uniform hypergraphs, J. Combin. Theory Ser. B 140 (2019), 332–346.

H. Hán, Y. Person, and M. Schacht, On perfect matchings in uniform hypergraphs with large minimum vertex degree, SIAM J. Discrete Math. 23 (2009), 732–748.

J. Han, Near perfect matchings in k-uniform hypergraphs, Combin. Probab. Comput. 24 (2015), 723–732.

J. Han, Near perfect matchings in k-uniform hypergraphs II, SIAM J. Discrete Math. 30 (2016), 1453–1469.

A. Johansson, J. Kahn, and V. Vu, Factors in random graphs, Random Structures Algorithms 33 (2008), 1–28.

T. Johnson, On Hamilton cycles in Erdős-Rényi subgraphs of large graphs, Random Structures Algorithms 57 (2020), 132–149.

J. Kahn, Hitting times for Shamir’s problem, Trans. Amer. Math. Soc. 375 (2022), 627–668.

J. Kahn, Asymptotics for Shamir’s problem, Adv. Math. 422 (2023), Paper No. 109019, 39.

J. Kahn and G. Kalai, Thresholds and expectation thresholds, Combin. Probab. Comput. 16 (2007), 495–502.

D. Y. Kang, T. Kelly, D. Kühn, A. Methuku, and D. Osthus, Thresholds for Latin squares and Steiner triple systems: bounds within a logarithmic factor, Trans. Amer. Math. Soc. 376 (2023), 6623–6662.

R. M. Karp, Reducibility among combinatorial problems, Complexity of computer computations, Springer, 1972, 85–103.

P. Keevash and R. Mycroft, A geometric theory for hypergraph matching, Mem. Amer. Math. Soc. 233 (2015), vi+95.

I. Khan, Perfect matchings in 3-uniform hypergraphs with large vertex degree, SIAM J. Discrete Math. 27 (2013), 1021–1039.

I. Khan, Perfect matchings in 4-uniform hypergraphs, J. Combin. Theory Ser. B 116 (2016), 333–366.

J. H. Kim and V. H. Vu, Concentration of multivariate polynomials and its applications, Combinatorica 20 (2000), 417–434.

F. Knox, D. Kühn, and D. Osthus, Edge-disjoint Hamilton cycles in random graphs, Random Structures Algorithms 46 (2015), 397–445.

J. Komlós, G. N. Sárközy, and E. Szemerédi, Proof of a packing conjecture of Bollobás, Combin. Probab. Comput. 4 (1995), 241–255.

M. Krivelevich, C. Lee, and B. Sudakov, Robust Hamiltonicity of Dirac graphs, Trans. Amer. Math. Soc. 366 (2014), 3095–3130.

M. Krivelevich, C. Lee, and B. Sudakov, Long paths and cycles in random subgraphs of graphs with large minimum degree, Random Structures Algorithms 46 (2015), 320–345.

D. Kühn and D. Osthus, Matchings in hypergraphs of large minimum degree, J. Graph Theory 51 (2006), 269–280.

D. Kühn and D. Osthus, Embedding large subgraphs into dense graphs, Surveys in combinatorics 2009, London Math. Soc. Lecture Note Ser., vol. 365, Cambridge Univ. Press, Cambridge, 2009, 137–167.
[54] D. Kühn and D. Osthus, *Hamilton decompositions of regular expanders: a proof of Kelly’s conjecture for large tournaments*, Adv. Math. 237 (2013), 62–146.

[55] D. Kühn, D. Osthus, and A. Treglown, *Matchings in 3-uniform hypergraphs*, J. Combin. Theory Ser. B 103 (2013), 291–305.

[56] M. Kwan, A. Sah, M. Sawhney, and M. Simkin, *High-girth Steiner triple systems*, arXiv:2201.04554 (2022).

[57] H. Lu, X. Yu, and X. Yuan, *Nearly perfect matchings in uniform hypergraphs*, SIAM J. Discrete Math. 35 (2021), 1022–1049.

[58] J. Park and H. T. Pham, *A proof of the Kahn-Kalai conjecture*, J. Amer. Math. Soc. 37 (2024), 235–243.

[59] H. T. Pham, A. Sah, M. Sawhney, and M. Simkin, *A toolkit for robust thresholds*, J. Combin. Theory Ser. B 157 (2021), 267–316.

[60] L. Pósa, *Hamiltonian circuits in random graphs*, Discrete Math. 14 (1976), 359–364.

[61] V. Rödl and A. Ruciński, *Dirac-type questions for hypergraphs—a survey (or more problems for Endre to solve)*, An irregular mind, Bolyai Soc. Math. Stud., vol. 21, János Bolyai Math. Soc., Budapest, 2010, 561–590.

[62] V. Rödl, A. Ruciński, and E. Szemerédi, *Perfect matchings in uniform hypergraphs with large minimum degree*, European J. Combin. 27 (2006), 1333–1349.

[63] V. Rödl, A. Ruciński, and E. Szemerédi, *Perfect matchings in uniform hypergraphs with large minimum degree*, J. Combin. Theory Ser. A 116 (2009), 613–636.

[64] V. Rödl, A. Ruciński, M. Schacht, and E. Szemerédi, *A note on perfect matchings in uniform hypergraphs with large minimum collective degree*, Comment. Math. Univ. Carolin. 49 (2008), 633–636.

[65] V. Rödl, A. Ruciński, and E. Szemerédi, *An approximate Dirac-type theorem for k-uniform hypergraphs*, Combinatorica 28 (2008), 229–260.

[66] A. Sah, M. Sawhney, and M. Simkin, *Perfect matchings in uniform hypergraphs with large minimum degree*, J. Combin. Theory Ser. B 157 (2016), 267–316.

[67] A. Sah, M. Sawhney, and M. Simkin, *Perfect matchings in uniform hypergraphs with large minimum degree*, J. Combin. Theory Ser. B 157 (2021), 267–316.

[68] A. Steger, *Die Kleitman–Rothschild Methode*, Ph.D. thesis, Rheinische Friedrich-Wilhelms-Universität Bonn, 1990.

[69] B. Sudakov, *Robustness of graph properties*, Surveys in combinatorics 2017, London Math. Soc. Lecture Note Ser., vol. 440, Cambridge Univ. Press, Cambridge, 2017, 372–408.

[70] M. Talagrand, *Are many small sets explicitly small?,* Proceedings of the forty-second ACM symposium on Theory of computing, 2010, 13–36.

[71] A. Treglown and Y. Zhao, *A note on perfect matchings in uniform hypergraphs*, Electron. J. Combin. 23 (2016), Paper 1.16, 14.

[72] Y. Zhao, *Recent advances on Dirac-type problems for hypergraphs*, Recent trends in combinatorics, IMA Vol. Math. Appl., vol. 159, Springer, [Cham], 2016, 145–165.

A. Proofs of Lemmas 2.3, 2.8, and 3.3

In this section, we prove Lemmas 2.3, 2.8, and 3.3.

As mentioned, we prove Lemma 2.3 via the polynomial concentration theorem of Kim and Vu [47]. We first give some definitions and then state the theorem. Let $n$ and $r$ be integers and let $G$ be a hypergraph on $n$ vertices in which each edge has size at most $r$. Suppose $\{X_v : v \in V(G)\}$ is a set of mutually independent Bernoulli random variables. We define the random variable

$$Y_G := \sum_{e \in G} \prod_{v \in e} X_v.$$ 

For a subset $A \subseteq V(G)$, we define $G_A$ to be the hypergraph with $V(G_A) := V(G) \setminus A$ and $E(G_A) := \{S \subseteq V(G_A) : S \cup A \in E(G)\}$. Thus we have

$$Y_{G_A} = \sum_{e \in G \setminus A} \prod_{v \in e \setminus A} X_v.$$ 

Moreover, for each $0 \leq i \leq r$, we let

$$\mathcal{E}_i(G) := \max_{A \subseteq V(G)} \mathbb{E}[Y_{G_A}].$$

Finally, we let $\mathcal{E}(G) := \max_{0 \leq i \leq r} \mathcal{E}_i(G)$ and $\mathcal{E}'(G) := \max_{1 \leq i \leq r} \mathcal{E}_i(G)$. 

For a subset $A \subseteq V(G)$, we define $G_A$ to be the hypergraph with $V(G_A) := V(G) \setminus A$ and $E(G_A) := \{S \subseteq V(G_A) : S \cup A \in E(G)\}$. Thus we have
Theorem A.1 (Kim–Vu polynomial concentration [47]). In the above setting, we have
\[ \mathbb{P} \left[ |Y_G - \mathbb{E}[Y_G]| > a_r (\mathcal{E}(G) \mathcal{E}'(G))^{1/2} \lambda^r \right] \leq 2e^{2}e^{-\lambda n^{r-1}} \]
for any \( \lambda > 1 \) and \( a_r := 8^r r!^{1/2} \).

Proof of Lemma 2.3. We first prove (i). Independently for each \( v \in V \), let \( X_v \in \{0,1\} \) with \( \mathbb{P}[X_v = 1] = p \) and let \( U = \{v \in V: X_v = 1\} \). Define \( G \) to be the hypergraph with \( V(G) = V \) and \( E(G) = \mathcal{F} \). Note that each edge in \( G \) has size \( s \). Since \( Y_G \) is the number of elements of \( \mathcal{F} \) that are contained in \( U \), we have
\[ \mathbb{E}[Y_G] = |\mathcal{F}| p^s \geq \varepsilon (np)^{s-1/2}. \]
Let \( 1 \leq i \leq s \) and \( A \subseteq V(G) = V \) with \( |A| = i \). Note that \( Y_{G_A} \) is the number of \( F \in \mathcal{F} \) such that \( A \subseteq F \) and \( F \setminus A \subseteq U \). It follows that
\[ \mathbb{E}[Y_{G_A}] \leq n^{s-i} p^{s-i} = (np)^{s-i} \leq (np)^{s-1} \leq (np)^{-1/4} \mathbb{E}[Y_G]. \]
Hence \( \mathcal{E}(G) = \mathbb{E}[Y_G] \) and \( \mathcal{E}'(G) \leq (np)^{-1/4} \mathbb{E}[Y_G] \). Now let
\[ \lambda := \left( \frac{\varepsilon \mathbb{E}[Y_G]}{a_s (\mathcal{E}(G) \mathcal{E}'(G))^{1/2}} \right)^{1/s} \geq \left( \frac{\varepsilon (np)^{1/8}}{a_s} \right)^{1/s} \geq n^{\beta/(9s)}. \]
By Theorem A.1, we have
\[ \mathbb{P} \left[ |Y_G - \mathbb{E}[Y_G]| > \varepsilon \mathbb{E}[Y_G] \right] = \mathbb{P} \left[ |Y_G - \mathbb{E}[Y_G]| > a_r (\mathcal{E}(G) \mathcal{E}'(G))^{1/2} \lambda^r \right] \leq 2e^{2}e^{-\lambda n^{r-1}} \leq \exp(-n^{\beta/(10s)}). \]
Thus with probability at least \( 1 - \exp(-n^{\beta/(10s)}) \), we have that the number of elements of \( \mathcal{F} \) contained in \( U \) is \((1 \pm \varepsilon)\mathbb{E}[Y_G] = (1 \pm \varepsilon) |\mathcal{F}| p^s \), which concludes the proof.

Now we show that (ii) follows from (i). Let \( \mathcal{F}' \subseteq \binom{V}{s} \) be such that \( \mathcal{F} \subseteq \mathcal{F}' \) and \( \varepsilon n^s (np)^{1/2} \leq |\mathcal{F}'| \leq n^s \). By (i), with probability at least \( 1 - \exp(-n^{\beta/(10s)}) \), \( U \) is \((p, \varepsilon, \mathcal{F}')\)-typical. It follows that, with probability at least \( 1 - \exp(-n^{\beta/(10s)}) \),
\[ |\{S \in \mathcal{F}: S \subseteq U\}| \leq |\{S \in \mathcal{F}: S \subseteq U\}| \leq (1 + \varepsilon)p^s |\mathcal{F}'| \leq 2\varepsilon (np)^s, \]
as desired. \( \square \)

Proof of Lemma 2.8. Let \( S = \{i_1, \ldots, i_d\} \in \binom{[d]}{s} \) be good if there are at least \((1 - \varepsilon^{1/2}) \binom{t-d}{k-d} \) many \((k-d)\)-sets \( \{i_{d+1}, \ldots, i_k\} \in \binom{[d]}{k-d} \) such that \( (V_{i_1}, \ldots, V_{i_k}) \) is \( \varepsilon \)-regular. Since there are at most \( \varepsilon \binom{d}{k} \) many \( k \)-sets in \( \binom{[d]}{k} \) which are not \( \varepsilon \)-regular, by an averaging argument, all but at most \( \varepsilon \binom{d}{k} \) \( k \)-sets in \( \binom{[d]}{k} \) are good.

Now it suffices to show that every good set in \( \binom{[d]}{s} \) has \( d \)-degree at least \((c - \gamma) \binom{t-d}{k-d} \) in \( \mathcal{R} \). Suppose, for a contradiction, that a good set \( S = \{i_1, \ldots, i_d\} \in \binom{[d]}{s} \) has \( d \)-degree less than \((c - \gamma) \binom{t-d}{k-d} \) in \( \mathcal{R} \). Let \( n_* := |V_i| = \cdots = |V_i| \), which satisfies \( \frac{2n}{d} \leq (1 - \varepsilon)n/t \leq n_* \leq n/t \) since \( \varepsilon \leq 1/3 \). Let \( N_S \) be the set of edges \( e \in \mathcal{H} \) with \( |e \cap V_{i_j}| = 1 \) for all \( j \in [d] \). Since all but at most \( \eta n^d \) many \( d \)-sets in \( \binom{V_i \cup \cdots \cup V_{i_d}}{d} \) have \( d \)-degree at least \( c(n-d)^{k-d} \), we have
\[ |N_S| \geq (n_*^d - \eta n^d) c \binom{n-d}{k-d} - n_*^d \cdot d n_* \binom{n-d-1}{k-d-1} \geq n_*^d \binom{n-d}{k-d} (c - \gamma c/6 - d k n_*/n) \]
\[ \geq n_*^d (c - \gamma/3) \binom{n-d}{k-d}. \]
Let $E(S)$ be the set of $(k-d)$-sets $\{i_{d+1}, \ldots, i_k\} \in \binom{[t]}{k-d}$ such that $(V_{i_1}, \ldots, V_{i_k})$ is not $\varepsilon$-regular. Since $S$ is good, we have $|E(S)| \leq \varepsilon^{1/2}(\frac{t-d}{k-d})$. Since $R$ is the $(\gamma/3, \varepsilon)$-reduced hypergraph, for $\{i_{d+1}, \ldots, i_k\} \in \binom{[t]}{k-d} \setminus (E(S) \cup E(S))$, we have $e_H(V_{i_1}, \ldots, V_{i_k}) \leq \gamma/3 \cdot |V_i| \cdots |V_k| = \gamma/3 \cdot n^k$.

Note that moreover there are at most $\varepsilon n^d \cdot n^{k-d}$ edges $e \in N_S$ with $e \cap V_0 = \emptyset$. Finally, there are at most $t^{k-d-1} n^k$ edges $e \in N_S$ with $e \cap V_i = \emptyset$ that contain more than one vertex from $V_i$ for some $i \in [t]$. Recall that by assumption $|N_R(S)| < (c - \gamma)^{(t-d)/(k-d)}$. Hence we have

$$|N_S| \leq \binom{[t]}{k-d} - |(N_R(S) \cup E(S))| \gamma/3 \cdot n^k + |N_R(S) \cup E(S)| n^k + \varepsilon n^d \cdot n^{k-d} + t^{k-d-1} n^k$$

$$< \left(\frac{t-d}{k-d}\right) \gamma n^k/3 + (c - \gamma + \varepsilon^{1/2}) \left(\frac{t-d}{k-d}\right) n^k + \varepsilon n^d \cdot n^{k-d} + t^{k-d-1} n^k$$

$$< n^d(c - \gamma/3) \left(\frac{n-d}{k-d}\right).$$

This contradicts the bound $N_S \geq n^d(c - \gamma/3)^{(n-d)/(k-d)}$ obtained above. Thus every good set in $\binom{[t]}{d}$ has $d$-degree at least $(c - \gamma)^{(t-d)/(k-d)}$ in $R$.

**Proof of Lemma 3.3.** Note that $p_i n \geq \varepsilon n^{1/k}$ for all $i \in [\ell]$. For each $i \in [\ell]$, since $E[U_i] = p_i n$, by a Chernoff bound and a union bound, with probability at least $1 - \exp(-n^{1/(2k)})$, for all $i \in [\ell]$ we have $|U_i| = (1 \pm \varepsilon)p_i n$. Thus a.a.s. (V1) holds.

We call $S \in \binom{[V(H)]}{d}$ good if $d_H(S) \geq \alpha_1 n^d$, otherwise we call it bad. Since $H$ is $(\alpha_1, \alpha_2, d, \varepsilon)$-dense, there are at most $\varepsilon n^d$ bad $d$-sets in $\binom{[V(H)]}{d}$. By Lemma 2.3 (ii) and a union bound, we have that, with probability at least $1 - \exp(-n^{1/(11k^2)})$, for each $i \in [\ell], U_i$ contains at most $2\varepsilon(p_i n)^d$ bad $d$-sets. By Lemma 2.3 (i) and a union bound, we have that, with probability at least $1 - \exp(-n^{1/(11k^2)})$, for each $i \in [\ell]$ and each good $S \in \binom{[V(H)]}{d}$, we have $d_H(S; (U_i)_{k-d}) \geq (\alpha_1 - 2\varepsilon)(p_i n)^{k-d}$. Hence a.a.s. (V2) holds.

By Lemma 2.3 (i) and a union bound, we have that, with probability at least $1 - \exp(-n^{1/(11k^2)})$, for each $i \in [\ell]$ and each vertex $v \in V(H)$, $d_H(v; (U_i \setminus \{v\})_{k-1}) \geq (\alpha_2 - 2\varepsilon)(p_i n)^{k-1}$. So a.a.s. (V3) holds. \hfill $\square$