Logarithmic Gini mean difference measure for apportionment problem

Hozumi Morohosi

1 National Graduate Institute for Policy Studies, 7-22-1 Roppongi, Minato-ku, Tokyo 106-8677, Japan
E-mail morohosi@grips.ac.jp
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Abstract
A new measure for disproportionality, especially for the case of parliament seat allocation, is proposed. After showing some good properties of the measure, we introduce three kinds of variation which work together for detailed investigation on the subject. Its performance is demonstrated by numerical studies using historical data of Japan.

Keywords apportionment, equity, electoral system

Research Activity Group Mathematical Politics

1. Introduction
This note refers to use of Gini mean difference in logarithmic scale for measuring the disproportionality of an electoral system in which parliamentary seats are allocated to the districts (prefectures or states). The allocation of seats makes a major problem in politics, which is called apportionment problem in the literature [1]. The objective of apportionment is to allocate the seats to each district as proportional to its population as possible, and the central point of the problem has been how to measure the proportionality. A lot of measures are proposed and arouse considerable controversy on proportionality reaching no consensus on the matter.

While Gini mean difference (GMD) is a very popular measure of inequality in social sciences (cf. [2]), it seems to be given less attention to in the context of apportionment problem except shortly mentioned in [3]. Our study will shed light on GMD to show its usefulness in the research of proportional representation.

By combining GMD measure and the disparity of seats per capita we show our measure log GMD can work as another measure of proportionality in apportionment problem via numerical studies.

In section 2, we introduce Gini mean difference in logarithmic scale to measure the disproportionality of seat allocation with respect to the population and show it has some good properties to meet the purpose of measure. We apply it to the seat allocation of Japanese National Diet to prefectures to report its performance compared to another typical disproportionality measure in section 3.

2. Logarithmic Gini mean difference
Assume an electoral system which allocates parliamentary seats to districts. There are s districts, each of which has population \( p_i > 0 \) and is allocated \( a_i > 0 \) seats, \( i = 1, \ldots, s \). The total population is denoted by \( P = \sum_{i=1}^{s} p_i \), and the total number of seats (house size) is denoted by \( h = \sum_{i=1}^{s} a_i \). To measure the proportionality of seats to population, we focus on the seats per capita \( a_i/p_i \). If the allocation of seats is well proportional to the population, every district would get almost the same seats per capita, and its ratio of every pair of districts

\[ x_{ij} = \left( \frac{a_i}{p_i} \right) / \left( \frac{a_j}{p_j} \right) \]

would be close to unity. But it is hardly possible to have a seat allocation perfectly proportional to population, since \( a_i \) is integer and lacks the flexibility to realize \( x_{ij} = 1 \) in general. Compared to so many measures of proportionality having been proposed, our proposed measure has a simple interpretation and can be used in several ways. We introduce it in three aspects.

2.1 Global measure
We propose a total disproportionality measure based on the average of absolute value of \( x_{ij} \) in logarithmic scale.

\[ G = \frac{1}{s(s-1)} \sum_{i \neq j} | \log x_{ij} | . \tag{1} \]

A reason why we take logarithmic scale is explained as follows. Consider two ratios \( x_{ij} = (a_i/p_i)/(a_j/p_j) \) and \( x_{ji} = (a_j/p_j)/(a_i/p_i) \). They apparently has a relation \( x_{ij} = 1/x_{ji} \), which yields \( | \log x_{ij} | = | \log x_{ji} | \) in logarithmic scale. This relation is noteworthy, since it would be preferable both \( x_{ij} \) and \( x_{ji} \) are the same amount of disproportionality, when calculating the arithmetic average.

Since \( | \log x_{ij} | = | \log(a_i/p_i) - \log(a_j/p_j) | \), \( G \) can be seen Gini mean difference of \( \log(a_i/p_i) \), seats per capita in logarithmic scale. It has the following good properties as a measure of disproportionality

First assume \( x_{ij} \neq 1 \) for any \( i, j \), with \( i \neq j \), then use
the relation $x_{ij} = 1/x_{ji}$ to simplify the expression of $G$:

$$G = \frac{2}{s(s-1)} \sum_{i,j: x_{ij} > 1} \log x_{ij}. \tag{2}$$

The expression (2) tells us a simple interpretation of $G$, namely

$$G = \log \left( \prod_{i,j: x_{ij} > 1} x_{ij} \right)^{1/(s-1)}. \tag{2}$$

is the logarithm of geometric mean of seats per capita ratio $x_{ij}$ which is greater than one. We may have $x_{ij} = 1$ for some $i, j$ in general, although they do not contribute to the sum in (2). Replacing the condition for summation (2) by $i, j : x_{ij} > 1$ or $i < j : x_{ij} = 1$, we keep the number of terms in the sum to be $s(s-1)/2$, and can see $G$ as the logarithm of geometric mean of terms $x_{ij} > 1$ and $x_{ij} = 1, i < j$.

Another property of $G$ is shown by a simple calculation.

$$|\log x_{ij}| = |\log a_i - \log p_i - \log a_j + \log p_j|$$
$$\quad = |\log p_i - \log a_i - \log p_j + \log a_j|$$
$$\quad = |\log \left( \frac{p_i}{a_i} \right) / \left( \frac{p_j}{a_j} \right) |.$$ 

So $G$ also can be the measure of disproportionality in terms of the population per seat $p_i/a_i$. This fact would be important in the context of apportionment problem, because there has been a long-standing unsettled controversy on which is more essential, seats per capita or population per seat. Our measure $G$ is free from the dispute.

2.2 Local measure

Partial sum for district $i$ in (1) can work for identifying the advantage or disadvantage of district $i$ in the seat allocation. Define a local measure as

$$g_i = \frac{1}{s-1} \sum_{j \neq i} \log x_{ij},$$

where we do not use absolute value in order to find how advantageous district $i$ is compared to others in average. If $g_i > 0$, district $i$ can be seen allocated more seats per capita in average.

2.3 Composite measure

In the two-house system of parliament, such as Japan, seats in each house are allocated to districts. Every person has two votes, one for each house. It would be of interest to study how to measure the proportionality of such a composite case. Our method could afford to cope with this problem.

Let $a'_{ij}$ be the number of seats of district $i$ for the second house, and $y_{ij} = (a'_{ij}/p_{ij})/(a'_{ji}/p_{ji})$ be the ratio of seats per capita between districts $i$ and $j$ for the second house. Then if $x_{ij}y_{ij} = 1$ holds, we could think districts $i$ and $j$ are on even ground throughout two houses.

Logarithmic Gini mean difference for the two-house system is defined straightforwardly as

$$G_2 = \frac{1}{s(s-1)} \sum_{i \neq j} |\log(x_{ij}y_{ij})|.$$ 

This measure can be applied to mix-member system in one house which consists of members elected in districts and those in blocks, each of which is an aggregate of districts. In that case $y_{ij}$ is the ratio of seats per capita in the pair of blocks which district $i$ and $j$ belong to, and calculated by the number of seats $a'_{ij}$ and the population $p'_{ij}$ of block in which district $i$ belongs to.

3. Numerical studies

We report some numerical studies using historical data of Japanese National Diet and census from 1950 to 2015 to show the performance of our proposed measures. Japanese National Diet consists of two Houses: House of Representatives (Shugiin) and House of Councillors (Sangiin). The house size of them varies during the period, and Japanese national census is conducted every five years. We focus on the seats allocated to districts (prefectures), since we are mostly concerned on the disproportionality in prefectures. Total population and the number of seats for districts in Houses are shown in Table 1. The calculation is based on prefectures’ population and allocated seats on these years. The number of prefectures are $s = 47$ after 1970 (included); before 1970 $s = 46$ (Okinawa is excluded.)

3.1 Global analysis

The computational result of logarithmic Gini mean difference (GMD) $G$ is depicted in Fig. 1.

For comparison, we use Rae index that is a commonly used measure of inequality in apportionment problem, and defined as

$$I = \frac{1}{s} \sum_{i=1}^{s} \left| \frac{a_i}{h} - \frac{p_i}{P} \right|.$$ 

Fig. 2 displays Rae index for Shugiin and Sangiin for the same period. They look basically similar; Sangiin always shows worse proportionality than Shugiin. Shugiin increased the disproportionality until the 1970s and kept high index values to 1990, then made major improve-

Table 1. Population and seats for districts in Japan.

| Year | Population of Japan | Seats for districts |
|------|---------------------|---------------------|
|      | Shugiin | Sangiin |
| 1950 | 84,114,574 | 466 | 150 |
| 1955 | 90,076,594 | 467 | 150 |
| 1960 | 94,301,623 | 467 | 150 |
| 1965 | 99,209,137 | 486 | 150 |
| 1970 | 104,665,171 | 486 | 152 |
| 1975 | 111,939,643 | 511 | 152 |
| 1980 | 117,060,396 | 511 | 152 |
| 1985 | 121,048,923 | 511 | 152 |
| 1990 | 123,611,167 | 512 | 152 |
| 1995 | 125,570,246 | 511 | 152 |
| 2000 | 126,925,843 | 300 | 146 |
| 2005 | 127,767,994 | 300 | 146 |
| 2010 | 128,057,352 | 300 | 146 |
| 2015 | 127,094,745 | 295 | 146 |
ment in 1995, when there was a big reform of electoral system in Shugiin.

One apparent difference can be observed in Sangiin after 2000. Our new measure log GMD indicates that Sangiin does not show the improvement of proportionality, while Rae index shows it slightly decreases disproportionality. We investigate the difference more closely in terms of local analysis.

3.2 Local measure

Local measure $g_i$ for each prefecture is computed for 1995 and 2015. Fig. 3 displays the result, where the horizontal axis shows prefecture number. For comparison, Fig. 4 displays $i$-th term of Rae index $(a_i/h - p_i/P)$ for prefecture $i$. The difference can be observed, for example, at prefecture 9 (Tochigi) and 10 (Gumma), which have positive local Gini measure in 1995, but negative value in 2015, while their Rae components are positive in 1995 and very close to zero in 2015.

In Rae index terms about one third of prefectures are closer to zero in 2015 than in 1995. On the other hand, local Gini measures of some prefectures switch to the opposite sign and keep the almost same absolute value. Actually those prefectures decrease the seats from 1995 to 2015. This decrease improves the proportionality in the sense of Rae index, while it does not contribute to the reduction of Gini measure.

3.3 Composite measure

Fig. 5 plots composite log GMD $G_2$ of Shugiin and Sangiin to the average of their Rae indices $I$ between 1950 and 2015.

The graph tells us that both measures show a similar trend. Before the 1980s, they grew monotonically. Then after the 1980s, they are reduced in a somewhat irregular way. Our argument in the previous section suggests this should be due to the poor improvement in Sangiin.

4. Concluding remarks

We proposed logarithmic Gini mean difference for measuring the disproportionality of seat allocation and demonstrated some numerical studies. Essentially this measure is nothing else but a geometric mean of seats per capita in logarithmic scale, as is mentioned in Section 3. Meanwhile, a transparent exposition given in this note could help us to work for both global and local analysis. Much detailed comparison of our measure to other traditional disproportionality measures would be of great interest and a future research topic.

References

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