Strong ordered Abelian groups and dp-rank

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Abstract

We provide an algebraic characterization of strong ordered Abelian groups: An ordered Abelian group is strong iff it has bounded regular rank and almost finite dimension. Moreover, we show that any strong ordered Abelian group has finite Dp-rank.

We also provide a formula that computes the exact value of the Dp-rank of any ordered Abelian group. In particular characterizing those ordered Abelian groups with Dp-rank equal to $n$. We also show the Dp-rank coincides with the Vapnik-Chervonenkis density.
Contents

1 Introduction 3
\hspace{0.5cm}1.1 notation \hspace{4.6cm} \hspace{0.5cm} 4

2 A Quantifier Elimination for ordered Abelian groups with bounded regular rank \hspace{0.5cm} 6

3 On computing dp-rank \hspace{0.5cm} 11

4 dp-minimal ordered groups \hspace{0.5cm} 17

5 Chain conditions \hspace{0.5cm} 20

6 Strong ordered Abelian groups \hspace{0.5cm} 24

7 Computing the dp-rank in ordered Abelian groups \hspace{0.5cm} 27

8 VC-density of ordered Abelian groups \hspace{0.5cm} 32

9 Gurevich-Schmitt Quantifier Elimination for ordered Abelian groups \hspace{0.5cm} 34
1 Introduction

S. Shelah asked the question of which ordered Abelian groups are strongly dependent (a strong form of NIP). H. Adler introduced the notion of strong theory (a strong form of NTP$_2$) which is a generalization of strongly dependent and coincides with the former in NIP theories. In other words, a theory (or more generally a type) is strongly dependent iff it is strong and NIP. Since it is well-known that all ordered Abelian groups are NIP, the question can be stated as: which ordered Abelian groups are strong?

One of the main results of the paper is Theorem 6.5, a characterization of strong ordered Abelian groups showing that an ordered Abelian group is strong iff it has bounded regular rank and almost finite dimension. See sections 2 and 6 for the definition of bounded regular rank and section 6 for the definition of almost finite dimension.

As a corollary we obtain that any strong ordered Abelian group has finite dp-rank. Moreover we provide a formula that computes the exact value of the dp-rank.

The paper is organized as follows.

In section 2 we state and prove Theorem 2.4, a Quantifier Elimination for ordered Abelian groups with bounded regular rank. This QE will be applied in sections 4 and 6. We use a relative QE result of Gurevich and Schmitt to prove our result. The notation and statement of the relative QE are in section 9.

In section 3 we introduce a variation of ict-patterns by relaxing a little bit the conditions on an ict-pattern. This will be useful to obtain upper bounds of dp-rank in Theorem 6.4. We call this new patterns wict-patterns (weak independence partition pattern). We will see that wict-patterns compute dp-rank. This is the content of Proposition 3.5. The nice point is that when there is QE in the language it is enough to consider wict-patterns constituted by literals (atomic or negation of atomic formulas). This is basically the content of Propositions 3.6 and 3.7. Ict-patterns do not have this property since Proposition 3.7 fails for ict-patterns.

We also show in Proposition 3.13 that directed families of formulas cannot occur in two different rows of the same pattern. This is also very useful in order to provide upper bounds on the dp-rank.

In section 4 we provide a characterization of dp-minimal ordered Abelian groups. The arguments of this section are the same as those in sections 6. Moreover the results here are a corollary as those in section 6. This section is just a warming-up of section 6.

The algebraic characterization of a dp-minimal ordered Abelian group as having finite dimension has been first published in [13] using another terminology.

In section 5 we prove Propositions 5.4 and 5.5. They may be considered as chain conditions Theorems, see [14] for close results. They are useful in proving lower bounds for burden (and dp-rank). In particular 5.5 is useful to obtain consequences from strongness. It is used in 6.1 and 6.2. Propositions 5.4 is used in section 7.
Section 6 contains Theorem 6.5, the characterization of strong ordered Abelian groups. It also contains the proof that all strong ordered Abelian groups have finite dp-rank.

Section 7 is a refinement of the arguments of section 6 to obtain a formula in Theorem 7.2 that computes the exact value of the dp-rank.

The characterisation of Strong ordered Abelian groups as having bounded regular rank and almost finite dimension is obtained independently in [12] using another terminology and different proof techniques. The authors also prove in [12] that any strong ordered Abelian group has finite dp-rank. They also provide a formula similar to 7.2. Corollary 7.6 is also there.

Section 8 contains a proof that for ordered Abelian groups, the VC-density coincides with the dp-rank. More precisely Theorem 8.3 shows that the dp-rank coincides with the VC-density function evaluated at 1.

Most of the results of this paper (sections 2, 3, 4, 5, 6 and 7) were obtained in 2014, but they remained unpublished until now. They were exposed in detail in the Barcelona Model theory seminar along various sessions during March and April 2014 (see http://www.ub.edu/modeltheory/mt.htm). I also gave a talk about the subject in the Mathematical Logic Seminar in Freiburg the 25th. of June 2014 (see http://logik.mathematik.uni-freiburg.de/lehre/archiv/ss14/oberseminar-ss14.html).

1.1 notation
We use $\pi$ to denote a tuple of variables. Most of the time is finite but sometimes could be infinite. We denote by $|\pi|$ the length of the tuple. We use $x$ to denote a single variable. The same conventional notation will be used for tuples of elements.

We will denote by $\mathbb{P}$ the set of all prime numbers.

We always play with the monster model of some complete theory $T$. But in fact a little bit of saturation (a weakly saturated model) is enough.

The notation and terminology specific for ordered Abelian groups is contained in sections 2 and 9.

When we say that a structure is VCA-minimal, dp-minimal, strong (or in general any abstract property of theories) we mean the theory of the structure is VCA-minimal, dp-minimal, strong... In practice we always may assume the structure is as much saturated as needed (by replacing it by another elementarily equivalent). The same applies to invariants of a theory, like dp-rk($T$), bdn($T$)...

In this case we denote dp-rk($M$) = dp-rk(Th($M$)) and similarly by bdn and VC-density.

When dealing with dp-rank and burden there are two possible ways to define them, both common in the literature. The first possibility is to define the dp-rank of a type as the supremum of all depths of all ict-patterns for the type. The second one is Shelah’s style: the dp-rank of a type is the first cardinal $\kappa$ for which there is no ict-patterns for the type of depth $\kappa$ (and $\infty$ if such cardinal does not exists). We choose the first definition, although we do not use subscripts to distinguish whether those supremums are achieved or not.
If $\varphi(\bar{x}, \bar{y})$ is a formula, $M$ is a model and $\bar{b}$ is a tuple in $M$ of the same length as $\bar{y}$, we will use $\varphi(M, \bar{b})$ to denote the set defined by $\varphi(\bar{x}, \bar{b})$ in $M$, namely $\{a \in M \mid M \models \varphi(a, \bar{b})\}$.

If $G$ is an ordered Abelian group we will denote $\Delta \leq G$ to indicate that $\Delta$ is a convex subgroup of $G$ and $\Delta \triangleleft G$ that $\Delta$ is a proper convex subgroup.
2 A Quantifier Elimination for ordered Abelian groups with bounded regular rank

In this section we generalize a quantifier elimination result of W. Weispfenning in [24]. Weispfenning result is precisely our theorem 2.4 in the case when $G$ has finite regular rank, see the comments after proposition 2.3. We deduce our result from a more general QE of Y. Gurevich and P. Schmitt, see [10], [18], [17]. This more general result, the notation and the terminology is explained in section 9 of this paper. If $p$ is a prime number, $G/pG$ is a vector space over the field with $p$ elements. We call the dimension of this vector space the $p$-dimension of $G$ and we denote it by $\dim_p(G)$.

We begin with a definition.

**Definition 2.1.** Let $n \geq 2$. The $n$-regular rank of $G$ is the order-type of the following ordered set:

$$\langle \text{RJ}_n(G), \subseteq \rangle,$$

where $\text{RJ}_n(G)$ denotes $\{A_n(g) \mid g \in G - \{0\}\}$, the set of all $n$-regular jumps.

The $n$-regular rank of an ordered Abelian group $G$ is a linear order. When it is finite we can identify this order with its cardinal. Moreover it can be easily characterized without any reference to the sets $A_n(g)$ as next remark shows.

**Remark 2.2.**

1. $G$ has $n$-regular rank 0 iff $G = \{0\}$.

2. $G$ has $n$-regular rank 1 iff $G$ is $n$-regular and non-trivial.

3. $G$ has $n$-regular rank equal to $m$ iff there are $\Delta_0, \ldots, \Delta_m$ convex subgroups of $G$, such that:

   (a) $\{0\} = \Delta_0 < \Delta_1 < \cdots < \Delta_m = G$

   (b) $\Delta_{i+1}/\Delta_i$ is $n$-regular for $0 \leq i < m$

   (c) $\Delta_{i+1}/\Delta_i$ is not $n$-divisible for $0 < i < m$.

   In this case $\text{RJ}_n(G) = \{\Delta_0, \ldots, \Delta_{m-1}\}$.

4. It $G$ has finite $n$-regular rank and $H \equiv G$ then $H$ has the same $n$-regular rank as $G$. Hence, when finite, the $n$-regular rank is an invariant of the theory of $G$. If $G$ has infinite $n$-regular rank, the regular rank of a $\kappa$-saturated model of the theory of $G$ is a linear order which has cardinality at least $\kappa$. It could be interesting characterize the $n$-regular rank of the monster model (it is not an $\eta_\kappa$-ordered set in general).

5. If $\dim_p(G)$ is finite then the $p$-regular rank of $G$ is at most $\dim_p(G) + 1$.

6. The number of convex subgroups $\Delta$ of $G$ with $G/\Delta$ discrete is bounded by the $n$-regular rank for each $n$. In particular, if $G$ has finite $n$-regular rank for some $n$, then the number of convex subgroups $\Delta$ of $G$ with $G/\Delta$ discrete is finite.
7. If the $n$-regular rank of $G$ is finite then each convex subgroup of the form $F_n(x)$ belongs to $\text{RJ}_n(G)$.

Proof. 1 and 2 are particular cases of 3, and 5 follows from 3.

3. If we have such a chain then $\text{RJ}_n(G) = \{\Delta_0, \ldots, \Delta_{m-1}\}$, since $g \in \Delta_{i+1} - \Delta_i$ implies $A_n(g) = \Delta_i$.

4. $G$ has $n$-regular rank $m$ iff in $G$ holds:

$$\exists y_0, \ldots, y_{m-1} \forall z \left( \bigvee_{j=0}^{m-1} A_n(z) = A_n(y_j) \land \bigwedge_{j=0}^{m-2} A_n(y_j) \subset A_n(y_{j+1}) \right)$$

6. Because $G/\Delta$ discrete with first positive element 1 implies $\Delta = A_n(1_\Delta)$ and thus $\Delta \in \text{RJ}_n(G)$ for any $n \geq 2$.

7. Assume $\text{RJ}_n(G) = \{\Delta_0, \ldots, \Delta_{m-1}\}$ as in 3 and $\Delta_i \triangleleft F_n(x) \triangleleft \Delta_{i+1}$. Then $x \in \Delta_{i+1} + nG$. By regularity of the jump, $\Delta_{i+1}/F_n(x)$ is $n$-divisible, thus $\Delta_{i+1} = F_n(x) + n\Delta_{i+1}$. Then $x \in F_n(x) + nG$, a contradiction. □

We also denote by $\text{RJ}(G)$ the set $\bigcup_{n \geq 2} \text{RJ}_n(G)$ and call it the set of regular jumps of $G$.

**Proposition 2.3.** Let $G$ be an ordered Abelian group. The following are equivalent:

1. $G$ has finite $p$-regular rank for each prime $p$.

2. $G$ has finite $n$-regular rank for each $n \geq 2$.

3. There is some cardinal $\kappa$ such that for any $H \equiv G$, $|\text{RJ}(H)| \leq \kappa$ (\text{RJ}(H) is countable).

4. For any $H \equiv G$, any definable convex subgroup of $H$ has a definition without parameters.

5. There is some cardinal $\kappa$ such that for any $H \equiv G$, $H$ has at most $\kappa$ (countably many) definable convex subgroups.

Moreover, in this case, $\text{RJ}(G)$ is the collection of all proper definable convex subgroups of $G$ and all are definable without parameters.

Proof. 1$\Rightarrow$2. $\text{RJ}_n(G) \subseteq \bigcup_{p \nmid n} \text{RJ}_p(G)$ because $A_n(p) = \bigcup_{p \nmid n} A_p(g)$.

2$\Rightarrow$3. Let $H \equiv G$. By Point 4 in Remark 2.2 $H$ has also finite $n$-regular rank for each $n$ so $\text{RJ}(H)$ is countable.

3$\Rightarrow$4. Let $H \equiv G$. A compactness argument shows that $H$ must have finite $n$-regular rank for each $n$. By Theorem 4.1 of [6] any proper definable convex subgroup of $H$ is an intersection of elements of $\text{RJ}_n(H)$ for some $n$. The finiteness of $\text{RJ}_n(H)$ implies that $\text{RJ}(H)$ is the set of all proper definable convex
subgroups of $H$. Now, if $\text{RJ}_n(H) = \{\Delta_0, \ldots, \Delta_{m-1}\}$ and $\Delta_i = A_n(g_i)$ then $\Delta_i$ can be defined in $H$ by the formula

$$\exists y_0, \ldots, y_{m-1} \left( x \in A_n(y_i) \land \bigwedge_{j=0}^{m-2} A_n(y_j) \subset A_n(y_{j+1}) \right)$$

$4 \Rightarrow 5$ is obvious.

$5 \Rightarrow 1$ If for some $p$, $\text{RJ}_p(G)$ is infinite, a compactness argument allows us to find $H \equiv G$ where $\text{RJ}_p(H)$ is as big as we want.

Finally observe that $\text{RJ}_n(G) = \bigcup_{p \in \mathbb{P}} \text{RJ}_p(G)$.

In view of point 3 of Proposition 2.3 will say that an ordered Abelian group has **bounded regular rank** if it satisfies any of the conditions of Proposition 2.3. By Facts 9.2, $\text{RJ}_n = \text{Sp}_n$ when $\text{RJ}_n$ is finite. Therefore an ordered Abelian group has bounded regular rank iff all the spines $\text{Sp}_n(G)$ are finite. In this case we define the regular rank of $G$ as $|\text{RJ}(G)|$, the cardinality of $\text{RJ}(G)$. It is either finite or $\aleph_0$. Observe that the property of having finite or bounded regular rank depends only in the theory of $G$. Also the value of the regular rank depends only on the theory (we may say that the regular rank is infinite or non-defined when the group has not bounded regular rank).

One can easily check that Weispfenning QE in [24] is just Theorem 2.4 in the case of finite regular rank.

**Theorem 2.4.** Let $G$ be an ordered Abelian group with bounded regular rank. Then $G$ admits QE in the following language:

$$L = \{+, -, 0, \leq\} \cup \{1_\Delta \mid \Delta \in \text{RJ}(G), G/\Delta \text{ discrete}\} \cup \{x \equiv y \mod (\Delta + nG) \mid n \geq 1\}$$

Here, if $G/\Delta$ is discrete, $1_\Delta$ denotes an element of $G$ whose projection to $G/\Delta$ is the smallest positive element.

**Proof.** We begin by remarking that we can add the following predicates for free:

$$\{x \equiv y \mod (\Delta + nG) \mid \Delta \in \text{RJ}(G), n \geq 1\}$$

In fact we could have added any set of predicates of the form

$$x \equiv y \mod (\Delta + nG)$$

even if $\Delta$ is not definable. This is because:

**Claim 2.5.** 1. If $n = p_1^{r_1} \cdots p_k^{r_k}$ then

$$x \equiv y \mod (\Delta + nG) \iff \bigwedge_i x \equiv y \mod (\Delta + p_i^{r_i}G)$$
2. If $\Delta \notin RJ_p(G)$, let $\Delta_1, \Delta_2$ be consecutive elements of $RJ_p(G)$ such that $\Delta_1 \subset \Delta \subset \Delta_2$. Then

$$x \equiv y \mod (\Delta + p'G) \iff x \equiv y \mod (\Delta_2 + p'G)$$

Proof of the claim: 1 follows from the following formula:

$$\bigcap_{i=1}^{t} \Delta + m_i G = \Delta + \text{lcm}(m_1, \ldots, m_t) G.$$ 

To prove this it is enough to prove $(\Delta + nG) \cap (\Delta + mG) = \Delta + \text{lcm}(m, n) G$. The other inclusion being obvious, it suffices to prove $\subseteq$. Assume $a = \delta_1 + mg_1 = \delta_2 + ng_2$ with $\delta_i \in \Delta$ and $g_i \in G$. Denote $d = \gcd(m, n)$, $m = dm'$, $n = dn'$. Let $1 = \lambda m' + \mu n'$ be a Bézout identity for $n', m'$. Then $a = \lambda m'a + \mu n'a = \lambda m'(\delta_2 + ng_2) + \mu n'(\delta_1 + mg_1) = (\lambda m'\delta_2 + \mu n'\delta_1) + \text{lcm}(n, m) (\lambda g_2 + \mu g_1) \in \Delta + \text{lcm}(m, n) G$.

2 follows because $\Delta_2/\Delta$ p-divisible implies $\Delta_2 = \Delta + p^2\Delta$ and thus $\Delta + p^2 G = \Delta_2 + p^2 G$. This ends the proof of the claim.

We will use theorem 9.4. Let $n$ be as in the statement of 9.4. Since $RJ_n(G)$ is the domain of $Sp_n(G)$ and is finite, $Sp_n(G) \models \psi_0(C_1, \ldots, C_m, D_1, \ldots, D_r)$ is equivalent to a boolean combination of the following formulas $A_n(t_i(\overline{g})) = \Delta$ and $F_n(s_i(\overline{g})) = \Delta$ with $\Delta \in RJ_n(G) \subseteq \bigcup_{p|\Delta} RJ_p(G)$. The following claim shows that this has a quantifier-free translation into the language $L$:

Claim 2.6. 1.

$$A_n(x) = \Delta \iff (x \not\equiv 0 \mod \Delta) \land (x \equiv 0 \mod \Delta^{n+})$$

2.

$$F_n(x) = \Delta \iff (x \not\equiv 0 \mod (\Delta + nG)) \land (x \equiv 0 \mod (\Delta^{n+} + nG)),$$

where $\Delta^{n+}$ denotes the successor of $\Delta$ in $RJ_n(G)$.

It remains to prove that the LOG*-predicates $M_k$, $E_{(n,k)}$ and $D_{(p,r,i)}$ are expressible without quantifiers in $L$. This is done in the next claim. This ends the proof of the theorem.

Claim 2.7. 1. $M_k(x) \iff \bigvee_{G/\Delta \text{ discrete}} x \equiv k1_\Delta \mod \Delta$

2. $E_{(n,k)}(x) \iff \bigvee_{G/\Delta \text{ discrete}} (x \equiv 0 \mod (\Delta^{n+} + nG)) \land (x \equiv k1_\Delta \mod (\Delta + nG))$

3. $D_{(p,r,i)}(x) \iff (x \equiv 0 \mod p^r G) \lor \bigvee_{G/\Delta \text{ discrete}} \left( (x \equiv 0 \mod (\Delta + p^r G)) \land (x \equiv 0 \mod (\Delta^{n+} + p^r G)) \land (x \equiv 0 \mod (\Delta + p^r G)) \right)$
Proof of the claim: 1 is easy, let us see 2. By remark 7 in 2.2, assume \( F_n(x) = \Delta \) for some \( \Delta \in RJ_n(G) \). It is easy to see that in this case \( \Gamma_{1,n}(x) = \Delta + nG \) and \( \Gamma_{2,n}(x) = \Delta^{n^+} + nG \). Moreover, if \( G/\Delta \) is discrete, \( [x] = k[1_\Delta] \) in \( \Gamma_n(x) \) iff \( x \equiv k1_\Delta \mod \Delta + nG \). Now, keeping in mind that \( F_n(x) = \Delta \) is equivalent to \( (x \equiv 0 \mod \Delta^{n^+} + nG) \land (x \not\equiv 0 \mod \Delta + nG) \), we get the result. One must bear in mind also that, since \( k > 0 \), \( x \equiv k1_\Delta \mod \Delta + nG \) implies \( x \not\equiv 0 \mod \Delta + nG \).

To prove 3, assume \( F_{p^n}(x) = \Delta \) for some \( \Delta \in RJ_{p^n}(G) \). As before, \( \Gamma_{1,p^n}(x) = \Delta + p^nG \) and \( \Gamma_{2,p^n}(x) = \Delta^{p^n} + p^nG \). Then \( [x] \in p^n\Gamma_{p^n}(x) \) iff \( x \in \Delta + p^n\Delta^{p^n} + p^nG = (\Delta + p^nG) \cap (\Delta^{p^n} + p^nG) \). Moreover, \( \Delta^{p^n} = \Delta^{p^n} \), since for ordered Abelian groups, being \( p \)-regular (\( p \)-divisible) is equivalent to being \( p^n \)-regular (\( p^n \)-divisible). This ends the proof of the claim. \( \square \)
3 On computing dp-rank

Along this section we work again with the monster model of some complete theory $T$. We begin by recalling the definition of ict-pattern and dp-rank.

The following was defined in [23] and [16].

**Definition 3.1.** Let $p(\bar{x})$ be a partial type. An ict-pattern for $p(\bar{x})$ consist in the following data: a sequence of formulas $S := (\varphi(\bar{x}, y_i) \mid i \in k)$ and an array $A := (\bar{a}_i \mid i \in \kappa, j \in O)$ of parameters, where $\kappa$ is a cardinal number and $O$ is an infinite linearly ordered set such that:

for every $f \in O^\kappa$, the following set of formulas is consistent with $p(\bar{x})$:

$$\Gamma^S_f(A) := \{ \varphi_i(\bar{x}, \bar{a}_f(i)) \mid i \in \kappa \} \cup \{ \neg \varphi_i(\bar{x}, \bar{a}_j) \mid i \in \kappa, j \neq f(i) \}$$

The cardinal number $\kappa$ is called the depth of the pattern and we allow $\kappa$ to be finite. We also say that it is an ict-pattern of type $\kappa \times O$ with sequence of formulas $S$ and array $A$.

The dp-rank of $p(\bar{x})$, denoted by $\text{dp-rk}(p(\bar{x}))$ is the supremum of all depths of all ict-patterns for $p(\bar{x})$.

If $T$ denotes a complete theory with a main sort, the dp-rank of $T$, denoted by $\text{dp-rk}(T)$ is $\text{dp-rk}(\bar{x} = \bar{x})$ where $\bar{x}$ is a single variable for the main sort.

Now we introduce wict-patterns.

**Definition 3.2.** Let $p(\bar{x})$ be a partial type. A wict-pattern for $p(\bar{x})$ consist in the following data: a sequence of formulas $S := (\varphi(\bar{x}, y_i) \mid i \in k)$ and an array $A := (\bar{a}_i \mid i \in \kappa, j \in O)$, where $\kappa$ is a cardinal number and $O$ is an infinite linearly ordered set such that:

for every $f \in O^\kappa$, the following set of formulas is consistent with $p(\bar{x})$:

$$\Delta^S_f(A) := \{ \varphi_i(\bar{x}, \bar{a}_f(i)) \mid i \in \kappa \} \cup \{ \neg \varphi_i(\bar{x}, \bar{a}_j) \mid i \in \kappa, j > f(i) \}$$

The cardinal number $\kappa$ is called the depth of the pattern and we allow $\kappa$ to be finite. We also say that it is a wict-pattern of type $\kappa \times O$ with sequence of formulas $S$ and array $A$.

**Remark 3.3.**  
1. By the same arguments as for ict-patterns, we can replace the parameters of a wict-pattern (without changing the sequence of formulas nor the depth) by another array with mutually indiscernible rows over the set of parameters of the type. This means that each row is indiscernible over the set of parameters of the type plus all other rows.

2. When the wict-pattern has mutually indiscernible rows over the set of parameters of $p$, it is enough to check the consistency of a single path: If $p(\bar{x}) \cup \Delta^S_f(A)(\bar{x})$ is consistent for some $f \in O^\kappa$ then the same holds for any $f \in O^\kappa$.

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1 there is a sort, called main sort, such that all other sorts are obtained as sorts of imaginaries of the theory of the main sort alone
3. The previous fact is not true in general for ict-patterns, it depends on the order-type of $O$. For instance, it holds for ict-patterns of kind $\kappa \times \mathbb{Z}$, but not for ict-patterns of kind $\kappa \times \omega$ (as the formula $x > y$ shows in the theory on dense linear order without endpoints).

The following kind of patterns were also first used by Shelah. The name ‘special’ is not standard and is simply used to distinguish them from the other two.

**Definition 3.4.** Let $p(\vec{x})$ be a partial type. A special pattern for $p(\vec{x})$ consists in the following data: a sequence of formulas $S := \{ \varphi_i(\vec{x}, \vec{y}_i) | i \in \kappa \}$ and an array $A := \{ \vec{a}_{ij} | i \in \kappa, j \in \omega \}$ with mutually indiscernible rows, where $\kappa$ is a cardinal number such that the following set of formulas is consistent:

$$p(\vec{x}) \cup \{ \varphi_i(\vec{x}, \vec{a}_{i0}) | i \in \kappa \} \cup \{ \neg \varphi_i(\vec{x}, \vec{a}_{i1}) | i \in \kappa \}$$  \hspace{1cm} (3.1)

The following shows we can replace ict-patterns by wict-patterns or special patterns in order to compute dp-rank.

**Proposition 3.5.** Let $p(\vec{x})$ be a partial type and $\kappa$ a cardinal number. The following are equivalent:

1. There is an ict-pattern for $p$ of depth $\kappa$.
2. There is a wict-pattern for $p$ of depth $\kappa$.
3. There is a special pattern for $p$ of depth $\kappa$.

**Proof.**

1$\Rightarrow$2 is obvious.

2$\Rightarrow$1. By a standard argument we may replace the array by a $\kappa \times \omega$-array with mutually indiscernible rows.

3$\Rightarrow$1. A compactness argument allows us to replace the array by a mutually indiscernible array of type $\kappa \times \mathbb{Z}$. Now we fix some witness $\vec{b}$ of the consistency of (3.1). Moreover we fix a row $i \in \kappa$. By deleting some positions we may assume the truth-value of $\varphi_i(\vec{b}, \vec{y}_i)$ for $j < 0$ is constant. We may also assume the truth-value of $\varphi_i(\vec{b}, \vec{y}_i)$ for $j > 1$ is constant. By replacing the formula $\varphi_i(\vec{x}, \vec{y}_i)$ by $\psi_i(\vec{x}, \vec{y}_i) := \varphi_i(\vec{x}, \vec{y}_i) \iff \neg \varphi_i(\vec{x}, \vec{y}_i)$ one achieves a new pattern with negative values on the left (with a possible exception), negative values on the right (with a possible exception) and a positive value in the 0-th column. More precisely the formula $\psi_i(\vec{x}, \vec{y}_i)$ is $\varphi_i(\vec{x}, \vec{y}_i) \iff \neg \varphi_i(\vec{x}, \vec{y}_i)$ and the tuple is $\overline{c}_i := \overline{a}_{i2} \overline{a}_{i2j+1}$. By deleting at most two positions we obtain a row where $\psi(\vec{b}, \overline{c}_i)$ is false for $j \neq 0$ and true for $j = 0$. As the new array is also mutually indiscernible, by point 3 of Remark 3.3 we have obtained an ict-pattern of type $\kappa \times \mathbb{Z}$ for $p$. \hfill \qed

The following lemma allows us to avoid disjunctions in wict-patterns. It is well-known for ict patterns.
Proposition 3.6. Assume there is a wict-pattern for $p$ with formulas $S = \{ \varphi_i(\bar{x}, \bar{y}_i) \mid i \in \kappa \}$ and for each $i \in \kappa$ $\varphi_i(\bar{x}, \bar{y}_i) = \bigwedge_{k=0}^{n_i} \psi^k_i(\bar{x}, \bar{y}_i)$. Then for some $f \in \omega^\kappa$ with $f(i) \leq n_i$ there is a wict-pattern for $p(x)$ with formulas $(\psi^{f(i)}_i(\bar{x}, \bar{y}_i) \mid i \in \kappa)$. The same is true for ict-patterns.

Proof. Let $(\bar{\pi}^j_i)_{i \in \kappa, j \in \omega}$ be a mutually indiscernible $\kappa \times \omega$ array showing there is a wict-pattern for $p$ with formulas $S$. There is some $\bar{b}$ such that $\models \varphi_i(\bar{b}, \bar{a}^j_0)$ and $\not\models \varphi_i(\bar{b}, \bar{a}^j_j)$ for all $i \in \kappa$ and $j > 0$. Hence, for some $f \in \omega^\kappa$ with $f(i) \leq n_i$, $\models \psi^{f(i)}_i(\bar{b}, \bar{a}^j_0)$ for all $i \in \kappa$. By point 2 in Remark 3.3, $(\bar{\pi}^j_i)_{i \in \kappa, j \in \omega}$ together with $(\psi^{f(i)}_i(\bar{x}, \bar{y}_i) \mid i \in \kappa)$ is a $\kappa \times \omega$ wict-pattern for $p$. The argument for ict-patterns is the same starting with an array of type $\kappa \times Z$. \qed

Now, we see that for wict patterns, the same holds for conjunctions.

Proposition 3.7. Assume there is a wict-pattern for $p(\bar{x})$ with sequence of formulas $S = \{ \varphi_i(\bar{x}, \bar{y}_i) \mid i \in \kappa \}$ and $\varphi_i(\bar{x}, \bar{y}_i) = \bigwedge_{k=0}^{n_i} \psi^k_i(\bar{x}, \bar{y}_i)$ for each $i \in \kappa$. Then for some $f \in \omega^\kappa$ with $f(i) \leq n_i$ there is a wict-pattern for $p(x)$ with formulas $(\psi^{f(i)}_i(\bar{x}, \bar{y}_i) \mid i \in \kappa)$.

Proof. If $(\bar{\pi}^j_i)_{i \in \kappa, j \in \omega}$ is a mutually indiscernible $\kappa \times \omega$ wict-pattern with sequence $S$ there is some $\bar{b}$ such that $\models \varphi_i(\bar{b}, \bar{a}^j_0)$ and $\not\models \varphi_i(\bar{b}, \bar{a}^j_j)$ for all $i \in \kappa$ and $j \neq 0$. Fix some row $i$. Since $\varphi_i(\bar{b}, \bar{a}^j_j)$ is false for $j > 0$, there is some $k \in \{1, \ldots, n_i\}$ such that $\psi^k_i(\bar{b}, \bar{a}^j_j)$ fails for infinitely many $j > 0$. By deleting elements in the row, we may assume that $\psi^k_i(\bar{b}, \bar{a}^j_j)$ fails for all $j > 0$. Hence $\models \psi^k_i(\bar{b}, \bar{a}^j_0)$ and $\not\models \psi^k_i(\bar{b}, \bar{a}^j_j)$ for all $i \in \kappa$ and $j \neq 0$. Since this may be done for any $i$, we get $f \in \omega^\kappa$ such that $\models \psi^{f(i)}(\bar{b}, \bar{a}^j_0)$ and $\not\models \psi^{f(i)}(\bar{b}, \bar{a}^j_j)$ for all $i \in \kappa$ and $j \neq 0$. By point 2 in Remark 3.3, $(\bar{\pi}^j_i)_{i \in \kappa, j \in \omega}$ together with $(\psi^{f(i)}_i(\bar{x}, \bar{y}_i) \mid i \in \kappa)$ forms a $\kappa \times \omega$ wict-pattern for $p$. \qed

Proposition 3.7 does not hold for ict-patterns as the following example shows. In the theory of dense linearly ordered sets without endpoints there is an ict-pattern of depth 1 with formula $y_1 < x < y_2$ but there is no ict-pattern for none of the formulas $y_1 < x$ nor $x < y_2$.

However we can reduce any conjunction to a conjunction of at most two formulas in ict-patterns:

Proposition 3.8. Assume there is an ict-pattern for $p(\bar{x})$ with sequence of formulas $S = \{ \varphi_i(\bar{x}, \bar{y}_i) \mid i \in \kappa \}$ and $\varphi_i(\bar{x}, \bar{y}_i) = \bigwedge_{k=0}^{n_i} \psi^k_i(\bar{x}, \bar{y}_i)$ for each $i \in \kappa$. Then for every $i \in \kappa$ there is some $I_i \subseteq \{0, \ldots, n_i\}$ with at most two elements such that there is an ict-pattern for $p(x)$ with formulas $(\bigwedge_{j \in I_i} \psi^j_i(\bar{x}, \bar{y}_i) \mid i \in \kappa)$.

Proof. If $S$ together $(\bar{\pi}^j_i)_{i \in \kappa, j \in \mathbb{Z}}$ is a mutually indiscernible $\kappa \times \mathbb{Z}$ ict-pattern for $p$, there is some $\bar{b}$ such that $\models \varphi_i(\bar{b}, \bar{a}^j_0)$ and $\not\models \varphi_i(\bar{b}, \bar{a}^j_j)$ for all $i \in \kappa$ and $j \neq 0$.

Fix some row $i$. Since $\varphi_i(\bar{b}, \bar{a}^j_j)$ is false for $j > 0$, there is some $k \in \{1, \ldots, n_i\}$ such that $\psi^k_i(\bar{b}, \bar{a}^j_j)$ fails for infinitely many $j > 0$. By deleting elements in the
row, we may assume that $\psi^k_i(b,\pi_j^i)$ fails for all $j > 0$. The same happens for $j < 0$ with maybe another formula. These two formulas provide the set $I_i$. Hence, we get that $\models \bigwedge_{j \in I_i} \psi^i_j(b,\pi_j^i)$ and $\not\models \bigwedge_{j \in I_i} \psi^i_j(b,\pi_j^i)$ for all $i \in \kappa$ and $j \neq 0$. By point 3 in Remark 3.3 $(\pi_j^i)_{i \in \kappa,j \in \mathbb{Z}}$ is the $\kappa \times \mathbb{Z}$ array of an ict-pattern with formulas $(\bigwedge_{j \in I_i} \psi^i_j(x,y) \mid i \in \kappa)$.

The following definition is standard, see [2] or [4]. We include it here for a better readability of the paper.

**Definition 3.9.** Let $\varphi(x,y)$ be a partitioned formula. The dual-alternation number of $\varphi(x,y)$ is the maximum number of changes of the truth-value of $\varphi(b,\pi_i)$ (when $i$ increases in $O$) for any tuple $b$ and any indiscernible sequence $(\pi_i, i \in O)$.

**Remark 3.10.** A (partitioned) formula $\varphi(x,y)$ has dual-alternation number equal to zero iff $\{\varphi(M,\pi) \mid \pi \in M\}$ is a finite set in any (some) model. We consider that a partitioned formula $\varphi(x,y)$ with $y$ the empty tuple has dual-alternation number zero.

**Proof.** If the set $\{\varphi(M,\pi) \mid \pi \in M\}$ is finite, in an indiscernible sequence $(\pi_i \mid i \in \omega)$ some repetition must occur: $\varphi(M,\pi_i) = \varphi(M,\pi_j)$ for some $i \neq j$ hence $\varphi(M,\pi_i)$ should be constant. This implies that the dual-alternation number of $\varphi(x,y)$ is zero. Conversely, if the set $\{\varphi(M,\pi) \mid \pi \in M\}$ is infinite, by a standard compactness plus Ramsey argument (see for instance Lemma 5.1.3 of [22]), one can construct an indiscernible sequence $(\pi_i \mid i \in \omega)$ where $\varphi(M,\pi_i)$ is not constant. Hence the dual-alternation number of $\varphi(x,y)$ is at least one. 

**Definition 3.11.** We call a formula with dual-alternation number zero non-alternating. We will say that the formula is NA or a NA-formula.

The following definition comes from Adler [3].

**Definition 3.12.** We call a set of partitioned formulas $\Psi(x,y)$ an instantiable directed family (or a directed family for short) if for any two sets $A,B$ defined by instances of formulas in $\Psi$ either $A \subseteq B$, $B \subseteq A$, or $A \cap B = \emptyset$. By an instance of $\varphi(x,y)$ we mean the formula $\varphi(x,b)$ for some $b$ of the appropriate length.

In the previous definition $x$ is a finite tuple of variables, while we allow $y$ to be an infinite tuple. Hence, although we use the same tuple $y$, not all formulas in $\Psi$ must have the same finite tuple of parameter variables.

**Proposition 3.13.** Let $\Psi$ be a directed family of partitioned formulas. Then there is no wict-pattern with two different rows with formulas of $\Psi$ or negation of formulas of $\Psi$ (two formulas of the sequence with different index cannot both belong to $\Psi \cup \neg \Psi$, where $\neg \Psi$ denotes $\{\neg \psi \mid \psi \in \Psi\}$).

Obviously the same holds for ict or special patterns.
Proof. Assume there is a wict-pattern of depth 2 where the two formulas of the sequence or their negations belonging to Ψ. We may assume the pattern is of kind \(2 \times \omega\) with mutually indiscernible rows. Let us denote by \((A_i \mid i \in \omega)\) the sets defined by the first row and \((B_i \mid i \in \omega)\) the sets defined by the second. We have to distinguish three cases depending on whether the formulas of the pattern or their negations belong to Ψ.

Case 1. Both formulas belong to Ψ. By the consistency of the path \((0, 0)\), \(A_0\) and \(B_0\) have nonempty intersection, thus without loss of generality \(A_0 \subseteq B_0\). By mutual indiscernibility \(A_0 \subseteq B_1\), which contradicts the consistency of the path \((0, 0)\).

Case 2. Only one formula belongs to Ψ. We may assume the \(A_i\) are defined by instances of formulas in Ψ while \(B_i = D_i\) is the complement of a set \(D_i\) defined by an instance of a formula in Ψ. Again by the consistency of the path \((0, 0)\), \(A_0 \cap D_1 = A_0 \cap D_1\) is nonempty, hence either \(A_0 \subseteq D_1\) or \(D_1 \subseteq A_0\). In the first case, by mutual indiscernibility, \(A_0 \subseteq D_0\) and thus \(A_0 \cap B_0 = \emptyset\), which contradicts the consistency of the path \((0, 0)\). The second case implies \(B_1 \subseteq A_1\). This contradicts again the consistency of the path \((0, 0)\) because \(A_1 \cap B_1\) should be nonempty.

Case 3. No formula belong to Ψ. In this case both \(A_i\) and \(B_i\) are the complements of \(C_i\) and \(D_i\) respectively, sets defined by instances of formulas in Ψ. Again the consistency of the path \((0, 0)\) implies \(A_1 \cap D_1 = C_1 \cap D_1\) is nonempty. Hence we may assume \(C_1 \subseteq D_1\) and thus \(B_1 \subseteq A_1\). By mutual indiscernibility, \(B_0 \subseteq A_1\) which contradicts again the consistency of the path \((0, 0)\).

\[\square\]

Definition 3.14. Let \(\lambda\) be a cardinal (finite or infinite). We say that a complete theory \(T\) is \(\lambda\)-VCA if there is collection of \(\lambda\) instantiable directed families \(\langle \Psi_i(x, y) \mid i \in \lambda \rangle\) such that each definable 1-set in the single variable \(x\) is a boolean combination of instances of formulas in \(\bigcup_{i \in \lambda} \Psi_i(x, y)\) and instances of NA-formulas of kind \(\psi(x, y)\). We call a complete theory \(T\) VCA-minimal if it is 1-VCA.

Obviously any VC-minimal theory is VCA-minimal. However the converse fails as Proposition 4.4 shows.

Fact 3.15. Let \(\Sigma(x, y)\) be a set of formulas without parameters, where the tuple \(y\) can be infinite. Assume each definable set with parameters with free variables among \(x\) is definable by an instance of some formula in \(\Sigma\). Then, for each formula \(\varphi(x, y)\) without parameters there is a finite subset \(\Theta\) of \(\Sigma\) such that each instance of \(\varphi(x, y)\) is an instance of some formula in \(\Theta\).

Proof. \(T \cup \{\neg \exists \overline{\exists} \forall x(\varphi(x, \overline{y}) \leftrightarrow \psi(x, \overline{z})), \psi(x, \overline{z}, \overline{y}) \in \Sigma\}\) is inconsistent. By compactness \(T \cup \{\neg \exists \overline{\exists} \forall x(\varphi(x, \overline{y}) \leftrightarrow \psi(x, \overline{z})), \psi(x, \overline{z}, \overline{y}) \in \Theta\}\) is inconsistent for some finite \(\Theta \subseteq \Sigma\).

\[\square\]

Corollary 3.16. Any VCA-minimal theory is dp-minimal.
Proof. Let $\Psi(x, \overline{y})$ be a directed family witnessing the VCA-minimality of the theory. We may assume that each formula in any wict-pattern for $x = x$ is a boolean combination of formulas of $\Psi$ and NA-formulas: in each row of the pattern some boolean combination provided by Fact 3.15 will occur infinitely many times. By Propositions 3.6 and 3.7 we can assume each row of a wict-pattern is constituted by a formula of $\Psi$, a negation of a formula of $\Psi$ or a non-alternating formula (the negation of an NA-formula is NA). By Proposition 3.13 $\Psi$ can contribute with at most one row, and no non-alternating formula can occur in a wict-pattern.

We will see in Proposition 4.4 that any dp-minimal ordered Abelian group is VCA-minimal.

In fact, this is more general:

Proposition 3.17. If $T$ is $\lambda$-VCA then the dp-rk($T$) $\leq \lambda$.

Proof. As in Corollary 3.16 using propositions 3.7, 3.6, 3.13 and the fact that NA-formulas cannot occur in a wict-pattern.
4 dp-minimal ordered groups

Here we work again with the monster model of some complete theory.

The following is not the standard definition of the dual VC-density, but equivalent to it. See [4].

Definition 4.1. The dual VC-density of a partitioned formula \( \varphi(x, y) \), denoted by \( \text{vc}^*(\varphi(x, y)) \), is the infimum of all real numbers \( r > 0 \) such that for any finite set \( A \) of \( |y| \)-tuples \( |S^\varphi(A)| = O(|A|^r) \). Here \( S^\varphi(A) \) denotes the set of all maximally consistent sets of formulas of kind \( \varphi(x, \overline{a}) \) or \( \neg\varphi(x, \overline{a}) \) with \( \overline{a} \in A \).

For more details on the dual VC-density, see [4].

In the proof of next proposition we use the following facts:

Facts 4.2. 1. \( |S^\varphi(A)| \) coincides with the number of atoms of the Boolean algebra of sets generated by the definable sets \( \varphi(C, \overline{a}) \) with \( \overline{a} \in A \). Here \( C \) denotes the monster model (or any model) of the theory and \( \varphi(C, \overline{a}) \) denotes the set defined by \( \varphi(x, \overline{a}) \) in \( C \).

2. Let \( B \) be a Boolean algebra and \( B_1, B_2 \) be given finite subalgebras (closed under \( \wedge \) and \( \neg \)) of \( B \). Let us denote by \( \text{at}(B_i) \) the number of atoms of \( B_i \). Then the number of atoms of the subalgebra generated by \( B_1 \cup B_2 \) is at most \( \text{at}(B_1)\text{at}(B_2) \).

3. Let \( A_1, \ldots, A_n \) be a directed family of sets, i.e. if \( A_i \cap A_j \neq \emptyset \) then either \( A_i \subseteq A_j \) or \( A_j \subseteq A_i \). Then the Boolean algebra of sets generated by \( A_1, \ldots, A_n \) has at most \( n+1 \) atoms.

Proposition 4.3. In a VCA-minimal theory any formula \( \varphi(x, \overline{y}) \) has dual VC-density at most 1.

Proof. By fact 3.15 for each formula \( \varphi(x, \overline{y}) \) there is a finite set \( \Theta \) of formulas of kind \( \mu(x, \overline{u}) \) which are Boolean combinations of formulas in \( \Psi \) and NA-formulas, such that each instance of \( \varphi(x, \overline{y}) \) is equivalent to an instance of some formula in \( \Theta \). Let \( \Psi_0 \) denote respectively the set of formulas from \( \Psi \) occurring in the boolean combinations of formulas in \( \Theta \) and let \( \Upsilon \) be the set of NA-formulas occurring in the boolean combinations of formulas in \( \Theta \). Let \( N \) be a common upper bound of the number of different sets each formula in \( \Upsilon \) can define. We may assume all formulas in \( \Psi, \Theta \) and \( \Upsilon \) have the same parameter variables, say \( \overline{u} \).

Given a set of \( \overline{y} \)-parameters \( A \), we can choose a set of \( \overline{u} \)-parameters \( B \) of size at most \( |A| \) such that each instance of \( \varphi(x, \overline{y}) \) with parameters from \( A \) is an instance of some formula in \( \Theta \) with parameters from \( B \). Hence, any definable set \( \varphi(C, \overline{u}) \) with \( \overline{u} \in A \) is a boolean combination of sets of kind \( \psi(C, \overline{b}) \) where \( \psi(x, \overline{u}) \in \Psi \cup \Upsilon \) and \( \overline{b} \in B \).

Now it is not difficult to see that \( |S_\varphi(A)| \leq (|\Psi_0| |A| + 1)2^{N|\Upsilon|} \). This holds because the boolean algebra generated by the sets defined by \( \Psi_0 \)-formulas with parameters from \( B \) has at most \( |\Psi_0| |A| + 1 \) atoms. And the Boolean algebra...
generated by \( \forall \)-formulas with parameters in \( B \) has at most \( 2^{N(|\forall|)} \) atoms because there are at most \( N^{|\forall|} \) such nonequivalent formulas.

Here is the complete characterization of all dp-minimal ordered groups.

We say that an ordered Abelian group has **finite dimension** iff \( \dim_p(G) \) is finite for all prime \( p \).

**Proposition 4.4.** Let \( G \) be an ordered group. The following are equivalent:

1. \( G \) is VCA-minimal.
2. Any formula \( \varphi(x, \overline{y}) \) has dual VC-density at most 1 (in the theory of \( G \)).
3. \( G \) is dp-minimal.
4. \( G \) is Abelian and has finite dimension.

**Proof.**

1 implies 2. By proposition 4.3.

2 implies 3. By [7] Proposition 3.2.

3 implies 4. \( G \) is abelian by Proposition 3.3 of [21]. Assume now \( G \) is Abelian but \( \dim_p(G) \) is infinite for some prime \( p \). As \( |G : pG| \) is infinite, let \((b_j)_{j \in \omega}\) be an infinite set of positive elements non congruent modulo \( pG \). Let \( a \) be an element greater than any \( b_j \) (in the monster model!). Then the following is a wict-pattern of depth 2: \((x > ipa | i \in \omega) \), \((x \equiv b_i \mod p | i \in \omega) \).

4 implies 1. Now assume that for each prime number \( p \), \( \dim_p(G) \) is finite. Then, by Remark 5 in 2.2 each \( p \)-regular rank should be finite. By Theorem 2.4 any formula can be written as a boolean combination of formulas of the following kind:

\[
\begin{align*}
nx & \leq t(y) \\
nx & \equiv t(y) \mod \Delta \\
nx & \equiv t(y) \mod \Delta + p^mG
\end{align*}
\]

where \( n \in \mathbb{Z}, \Delta \in RJ_p(G), p \in \mathbb{P}, m \geq 1 \) and \( t(y) \) is a term.

Since the formulas of kind (4.1) and (4.2) define convex subsets, they are boolean combination of definable initial segments. So we may express any formula as a boolean combination of definable initial segments and formulas of kind (4.3). As the formulas defining initials segments constitute a directed family it only remains to prove that formulas of kind (4.3) are NA.

**Claim 4.5.** Every nonempty instance of the formula \( nx \equiv t(y) \mod \Delta + p^mG \) is a coset of \( \Delta + p^rG \), where \( n = n'p^s \), \( \gcd(n', p) = 1 \) and \( r = \min \{m - s, 0\} \).

**Proof.** Easy.

Finally
Claim 4.6. If \( \dim_p(G) \) is finite, the formula

\[ nx \equiv t(y) \mod \Delta + p^mG \]

is NA.

Proof. By claim 4.5, it suffices to show \([G : \Delta + p^r G]\) is finite.

Since \( G/\Delta \cong G/(\Delta + pG) \) is a free \( \mathbb{F}_p \)-module of rank \( \dim_p(G/\Delta) \)
then \( G/\Delta \cong G/(\Delta + p^r G) \) is a free \( \mathbb{Z}/p^r \mathbb{Z} \)-module of rank \( \dim_p(G/\Delta) \)
(see [15]). This implies \([G : \Delta + p^r G] = p^{r \dim_p(G/\Delta)} \leq p^{r \dim_p(G)}\), which is finite.

Observe that Corollary 3.16 gives another proof that 1 implies 3.

We cannot replace VCA-minimal by VC-minimal. In [8] it is shown that
any convexly orderable ordered Abelian group is divisible. Moreover any VC-
minimal theory is convexly orderable (see [9] for a proof of this). This shows that
there are many VCA-minimal ordered Abelian groups non convexly orderable.
5 Chain conditions

We start with Lemma 5.1. It is a result about pure groups and it is a generalization of the Chinese Remainder Theorem to the non-Abelian case. We assume there is some group \( G \), \( a, b \) are elements of \( G \) and \( H \) is a subgroup of \( G \). We use the congruence notation to work with left cosets: \( a \equiv b \mod H \) means \( aH = bH \).

If \( H_1 \) and \( H_2 \) are subgroups of \( G \), we let \( H_1H_2 \) denotes the following set: \( \{h_1h_2 \mid h_1 \in H_1, h_2 \in H_2\} \). Obviously \( H_1H_2 \) is a subgroup iff \( H_1H_2 = H_2H_1 \).

Observe that the condition (5.1) depends on the order of the sequence of groups, while the conclusion not.

Lemma 5.1. Let \( G \) be a group, \( H_1, \ldots, H_n \) a sequence of subgroups of \( G \) satisfying the following:

\[
\bigcap_{i<r}(H_iH_r) = \left( \bigcap_{i<r} H_i \right) H_r \quad \text{for } r = 2 \ldots n \tag{5.1}
\]

Then, given \( a_1, \ldots, a_n \in G \), the system

\[
\begin{cases}
  x \equiv a_1 \mod H_1 \\
  \quad \vdots \\
  x \equiv a_n \mod H_n
\end{cases}
\]

has a solution iff for any \( i < j \leq n \) it happens that \( a_i^{-1}a_j \in H_iH_j \).

Moreover, if \( b \) is a solution of (5.2), the system (5.2) is equivalent to

\[
x \equiv b \mod \bigcap_{i \leq n} H_i.
\]

Proof. If \( c \) is a solution of (5.2) then \( a_i^{-1}c \in H_i \) and \( a_j^{-1}c \in H_j \), so \( a_i^{-1}a_j \in H_iH_j \). If \( b \) is a particular solution of (5.2), \( c \) is a solution of (5.2) iff \( c \equiv b \mod \bigcap_{i \leq n} H_i \). The proof of the existence of a solution in case the compatibility condition is satisfied, is by induction on \( n \).

Case \( n = 2 \). If \( a_1^{-1}a_2 \in H_1H_2 \) then \( a_1^{-1}a_2 = h_1h_2 \) for some \( h_1 \in H_i \). Hence \( b = a_1h_1 = a_2h_2^{-1} \) is a solution.

Now we prove the inductive step. Assume the compatibility conditions \( a_i^{-1}a_j \in H_iH_j \) holds for any \( i < j \leq n + 1 \). By inductive hypothesis, the system of the first \( n \) congruencies has a solution, say \( b \). It suffices to prove that the system

\[
\begin{cases}
  x \equiv b \mod \bigcap_{i \leq n} H_i \\
  x \equiv a_{n+1} \mod H_{n+1}
\end{cases}
\]

Has a solution. By the case with \( n = 2 \) it suffices to show that \( b^{-1}a_{n+1} \in \bigcap_{i \leq n} H_i \). Since \( b \) is a solution of the first \( n \) congruencies \( b = a_ih_i \) for some \( h_i \in H_i \) and for any \( i \leq n \). Hence \( b^{-1}a_{n+1} = (h_i^{-1})a_i^{-1}a_{n+1} \in H_iH_{n+1} \). Therefore \( b^{-1}a_{n+1} \in \bigcap_{i \leq n} (H_iH_{n+1}) = \left( \bigcap_{i \leq n} H_i \right) H_{n+1} \).
Remark 5.2. In the previous lemma, if moreover for any $1 \leq i < j \leq n$ one gets $H_i H_j = H_j H_i$, then all ‘sets’ in the statement are subgroups: $H_i H_j$, and $(\bigcap_{i < r} H_i) H_r$ for $r = 2 \ldots n$. Moreover $(\bigcap_{i < r} H_i) H_r = H_r (\bigcap_{i < r} H_i)$ for $r = 2 \ldots n$.

The main results of this section are Propositions 5.4, 5.5 and 5.6. They are stated in terms of burden, but one can replace burden by dp-rank and remain true. This is because the burden is always smaller or equal to dp-rank. More precisely, if a certain type has an inp-pattern of a certain depth the same type also has an ict-pattern of the same depth. But in fact, in the proofs, we exhibit patterns which are ict and inp at the same time. For the sake of completeness we include a definition of inp-pattern and burden. Here again we work in the monster model of some complete theory. It is also known that burden and dp-rank coincide for NIP types, see [1].

**Definition 5.3.** Let $p(\bar{x})$ be a partial type. An *inp-pattern* for $p(\bar{x})$ consist in the following data: a sequence of formulas $(\varphi_i(\bar{x}, \bar{y}_i) \mid i \in k)$, a sequence of natural numbers $(k_i \mid i \in k)$ and an array $A := (\bar{a}_j^i \mid i \in \kappa, j \in O)$, where $\kappa$ is a cardinal number and $O$ is an infinite linearly ordered set such that:

- for every $f \in O^{\kappa}$, the following set of formulas is consistent:
  $$p(\bar{x}) \cup \{ \varphi_i(\bar{x}, \bar{a}_f^i) \mid i \in \kappa \}$$
- for each $i \in \kappa$ the set $\{ \varphi(\bar{x}, \bar{a}_j) \mid j \in O \}$ is $k_i$-inconsistent.

The cardinal number $\kappa$ is called the depth of the pattern and we allow $\kappa$ to be finite. We also say that it is an inp-pattern of type $\kappa \times O$.

The **burden** of $p(\bar{x})$ denoted by $\text{bdn}(p(\bar{x}))$ is the supremum of all depths of all inp-patterns for $p(\bar{x})$.

If $T$ denotes a complete theory with a main sort $\mathbb{2}$, the burden of $T$, denoted by $\text{bdn}(T)$ is $\text{bdn}(x = x)$ where $x$ is a single variable for the main sort.

A complete theory $T$ is called **strong** (see [1]) if there is no inp-pattern of infinite depth for the formulas $\bar{x} = \bar{x}$, where $\bar{x}$ is of finite length. By the submultiplicativity of burden (see [5]) it is enough to consider a single variable.

Shelah in [20] calls $T$ strongly dependent if there is no ict-pattern of infinite depth for the formulas $\bar{x} = \bar{x}$ with $\bar{x}$ of finite length. It is enough again to consider a single variable. A theory is strongly dependent if it is NIP and strong, see [1]. As all ordered Abelian groups are NIP (see [11]) strong and strongly dependent are equivalent properties for ordered Abelian groups.

When $\text{dp-rk}(T)$ is finite we say that $T$ has **finite dp-rank** (in [14] it is said that the theory has bounded dp-rank). Obviously any theory of finite dp-rank is strong. In section [6] we will see that any strong ordered Abelian group has finite dp-rank.

---

$\mathbb{2}$there is a sort, called main sort, such that all other sorts are obtained as sorts of imaginaries of the theory of the main sort alone.
Proposition 5.4. Let $G$ be an expansion of a group. Assume there is a sequence of definable subgroups (maybe with parameters) $H_1, \ldots, H_n$ satisfying the following three conditions:

**commutativity:** $H_i H_j = H_j H_i$ for $1 \leq i < j \leq n$.

**distributivity:** $\bigcap_{i < r} (H_i H_r) = (\bigcap_{i < r} H_i) H_r$ for $r = 2 \ldots n$.

**infinity:** $[K_i : H_i]$ is infinite for $r = 1 \ldots n$.

where $K_i$ denotes $\bigcap_{j=1}^{n} H_j H_i$.

Then $\text{bdn}(G) \geq n$.

Proof. By **infinity** we can choose $a^i_j$, for $1 \leq i \leq n$, $j \in \omega$ such that $a^i_j \in K_i$ but $a^i_j \not\equiv a^i_j' \mod H_i$ for $j < j'$. Let's check that the formulas $x \equiv a^i_j \mod H_i$ constitute an inp-pattern with all rows 2-inconsistent. Given $f \in \omega^n$, we apply Lemma 5.1 to check that The following system

$$
\begin{align*}
  x &\equiv a^1_f(1) \mod H_1 \\
  &\vdots \\
  x &\equiv a^n_f(n) \mod H_n
\end{align*}
$$

has a solution. Observe that for $i \not\equiv j$, $a^i_j \in K_i \subseteq H_j H_i$ and $a^j_{f(j)} \in K_j \subseteq H_i H_j$ implies $a^i_j a^j_{f(j)} \in H_i H_j$ because of the commutativity condition. \qed

Proposition 5.5. Let $G$ be an expansion of a group. Assume there is a sequence of definable subgroups (maybe with parameters) $(H_i \mid i \in \omega)$ satisfying the following three conditions:

**commutativity:** $H_i H_j = H_j H_i$ for $i, j \in \omega$.

**distributivity:** $\bigcap_{i < n} (H_i H_n) = (\bigcap_{i < n} H_i) H_n$ for $n \in \omega$.

**infinity:** $[K_i^n : H_i]$ is infinite for $i \leq n \in \omega$.

where $K_i^n$ denotes $\bigcap_{j=1}^{n} H_j H_i$.

Then $G$ is not strong.

Proof. Consider the following sequence of formulas for a pattern of depth $\aleph_0$:

$$
(x \equiv y \mod H_i, i \in \omega).
$$

A compactness argument shows that, in order to find an array of $y$-parameters making this sequences an inp-pattern (with all rows 2-inconsistent), it is enough to find parameters making the first $n + 1$ formulas an inp-pattern for each $n$.
It is interesting to compare this result with Proposition 3.12 in [14].

A complete theory is called NTP2 iff there is no inp-pattern of infinite depth with the same formula in all rows (the sequence of formulas of the pattern is constant) for the type $\bar{x} = \bar{x}$, where $\bar{x}$ is a finite tuple of variables.

**Proposition 5.6.** Let $G$ be an expansion of a group. Assume there is a uniformly definable sequence of subgroups $(H_i \mid i \in \omega)$ satisfying the following three conditions:

- **commutativity:** $H_i H_j = H_j H_i$ for $i, j \in \omega$.
- **distributivity:** $\bigcap_{i<n} (H_i H_n) = (\bigcap_{i<n} H_i) H_n$ for $n \in \omega$.
- **infinity:** $[K^n_i : H_i]$ is infinite for $i \leq n \in \omega$.

where $K^n_i$ denotes $\bigcap_{j=1}^{n} H_j H_i$.

Then $G$ is TP2.

**Proof.** As in Proposition 5.5. Observe that given the infinite family of $(H_i \mid i \in \omega)$ we obtain a fixed formula for a pattern of depth $\aleph_0$. \qed
6 Strong ordered Abelian groups

In this section we characterize strong ordered Abelian groups. As a corollary we will see that a strong ordered Abelian group has finite dp-rank.

When \( \{ p \in \mathbb{P} \mid \dim_p(G) \geq \aleph_0 \} \) is finite we will say that \( G \) has \textit{almost finite dimension}.

We begin by something easy:

**Proposition 6.1.** Any strong ordered Abelian group has almost finite dimension.

**Proof.** If \( \{ p \in \mathbb{P} \mid \dim_p(G) \geq \aleph_0 \} \) is infinite, apply Proposition 5.5 with \( H_i = p_i G \), where \( (p_i \mid i \in \omega) \) denotes the sequence of all \( p \) with \( \dim_p(G) \) infinite. Observe that \( K^n_i = G \). Then \( G \) is not strong. \( \square \)

The proof of this lemma shows that in fact \( \text{dp-rk}(G) \geq |\{ p \in \mathbb{P} \mid \dim_p(G) \geq \aleph_0 \}| \), but we will obtain better bounds in 7.2.

**Theorem 6.2.** Any strong ordered Abelian group has bounded regular rank.

**Proof.** We may assume \( G \) is enough saturated. Assume \( G \) has infinite \( p \)-regular rank for some prime number \( p \).

By saturation, there is an infinite ascending chain \( \Delta_0 \lhd \Delta_1 \lhd \cdots \lhd \Delta_t \lhd \cdots \), where \( \Delta_i = A_p(a_i) \) and \( \langle a_i \mid i \in \omega \rangle \) is a sequence of elements of \( G \). By deleting some elements we may assume \( \dim_p(\Delta_{i+1}/\Delta_i) > N \) for any \( i \in \omega \). This is because \( \dim_p(\Delta_{i+2}/\Delta_i) > 0 \). By saturation again we may assume \( \dim_p(\Delta_{i+1}/\Delta_i) \geq \aleph_0 \).

Now consider the following family of definable groups:

\[ H_i := \Delta_i + p^i G \quad \text{for} \quad i \geq 1. \] (6.1)

Our aim is to show that this family satisfies the hypothesis of Proposition 5.5. This follows from the following claim. Observe that 3 together with 4 imply distributivity and 6 and 7 infinity.

**Claim 6.3.**

1. If \( i < j \) then \( H_i + H_j = \Delta_j + p^i G \).

2. If \( i < j \) then \( H_i \cap H_j = \Delta_i + p^i \Delta_j + p^i G \).

3. \( H_1 \cap \cdots \cap H_r = \Delta_1 + p \Delta_2 + \cdots + p^{r-1} \Delta_r + p^r G \).

4. \( \bigcap_{i<\tau}(H_i + H_r) = \Delta_r + p^{r-1} G \).

5. If \( r < n \), then \( K^n_r := \bigcap_{i=1}^{n}(H_i + H_r) = \Delta_r + p^{r-1} \Delta_{r+1} + p^r G \).
If \( r < n \), then \([K^n_r : H_r]\) is infinite.

If \( K^n_n := \bigcap_{i<n} (H_i + H_n) \) then \([K^n_n : H_n]\) is infinite.

Proof. 1 is because \( \Delta_i \subset \Delta_j \) and \( p^iG \subset p^jG \).

2. If \( \delta + p'g = \delta + p'h \) for some \( \delta \in \Delta_i, \delta \in \Delta_j, g, h \in G \), then \( \delta - \delta \in p^i\Delta_j \) whence \( \delta + p'h \in \Delta_i + p^i\Delta_j + p^jG \).

3 is done by induction on \( r \). Assume \( H_1 \cap \cdots \cap H_r = \Delta_1 + p\Delta_2 + \cdots + p^{r-1}\Delta_r + p^rG \). In order to prove \( H_1 \cap \cdots \cap H_{r+1} = \Delta_1 + p\Delta_2 + \cdots + p^r\Delta_{r+1} + p^{r+1}G \), it suffices to see \((\Delta_1 + p\Delta_2 + \cdots + p^{r-1}\Delta_r + p^rG) \cap H_{r+1} \subseteq \Delta_1 + p\Delta_2 + \cdots + p^r\Delta_{r+1} + p^{r+1}G\), the other inclusion being obvious. If \( \delta + p\delta_2 + \cdots + p^{r-1}\delta_r + p^r = \delta_{r+1} + p^{r+1}h \), then \( \delta_{r+1} - (\delta + p\delta_2 + \cdots + p^{r-1}\delta_r) \in p^r\Delta_{r+1} \) and thus \( \delta_{r+1} + p^{r+1}h \in \Delta_1 + p\Delta_2 + \cdots + p^r\Delta_{r+1} + p^{r+1}G \).

4 follows from \( H_j + H_r = \Delta_r + p^jG \subseteq \Delta_r + p^rG = H_i + H_r \) for \( i < j < r \).

5. Since for \( r < i < j \), \( H_i + H_r = \Delta_i + p^iG \subseteq \Delta_j + p^iG = H_j + H_r \), obviously \( \bigcap_{r<i\leq j} (H_i + H_r) = \Delta_{r+1} + p^rG \). It remains to show \((\Delta_r + p^{r-1}G) \cap (\Delta_{r+1} + p^rG) \subseteq \Delta_r + p^{r-1}\Delta_{r+1} + p^rG\), as the converse inclusion is obvious. If \( \delta_r + p^{r-1}g = \delta_{r+1} + p^r \) then \( \delta_{r+1} - \delta_r \in p^{r-1}\Delta_{r+1} \) and thus \( \delta_{r+1} + p^r \in \Delta_r + p^{r-1}\Delta_{r+1} + p^rG \).

6. We show that \( K^n_r / H_r \simeq \Delta_{r+1} / (\Delta_r + p\Delta_{r+1}) \). Then \([K^n_r : H_r]\) is infinite because \( \Delta_{r+1} / (\Delta_r + p\Delta_{r+1}) \simeq (\Delta_{r+1} / \Delta_r) / p(\Delta_{r+1} / \Delta_r) \) and \( \dim_p(\Delta_{r+1} / \Delta_r) \) is infinite. This is done by checking that the morphism \( \Delta_{r+1} \to K^n_r / H_r \) given by \( x \mapsto p^{r-1}x + H_r \) is surjective and its kernel is \( \Delta_r + p\Delta_{r+1} \).

7. Observe that \( K^n_n = \Delta_n + p^{n-1}G \). We show that \( K^n_n / H_n \simeq G / (\Delta_n + pG) \). This is done by checking that the morphism \( G \to (\Delta_n + p^{n-1}G) / (\Delta_n + pG) \) given by \( x \mapsto p^{n-1}x + H_n \) is surjective and its kernel is \( \Delta_n + pG \). Then \([K^n_n : H_n]\) is infinite because \( G / (\Delta_n + pG) \simeq (G / \Delta_n) / p(G / \Delta_n) \) and \( \dim_p(G / \Delta_n) \) is infinite.

\( \square \)

**Theorem 6.4.** Any ordered Abelian group with bounded regular rank and almost finite dimension has finite dp-rank.

Proof. As in the proof of Proposition 4.4. By 2.4, 3.5, 3.6, and 3.7, we may assume the formulas of the pattern either define initial segments or are of kind (4.3). As the formulas defining initials segments constitute a directed family, by Proposition 3.11, they can only contribute with one row to the pattern.

Given a prime number \( p \) and \( \Delta \in \text{RJ}_p(G) \), there is at most one row with formulas of kind (4.3) or its negation with this \( \Delta \) and \( p \). The reason is that these formulas (with the same \( \Delta \) and \( p \) and maybe different \( m \)), by Claim 4.5, constitute again a directed family.

Moreover, if \( \dim_p(G) \) is finite, the formulas of kind (4.3) for \( \Delta \in \text{RJ}_p(G) \) cannot occur (nor their negations) in the pattern because, by Claim 4.6, they are NA-formulas. Hence, as \( \{ p \in \mathbb{P} \mid \dim_p(G) \geq \aleph_0 \} \) is finite and \( \text{RJ}_p(G) \) are finite for each prime \( p \) we have only a finite number of possible rows. \( \square \)
In fact, the proof of Theorem 6.4 provides us the following upper bound for dp-rk(G):

$$dp-rk(G) \leq 1 + \sum_{\dim_p(G) \geq \aleph_0} |RJ_p(G)|$$

Observe that this formula implies that any ordered Abelian group with finite dimension is dp-minimal. In the next section we will improve this formula in order to get the exact value of dp-rk(G).

The main theorem of this section is the characterization of strong ordered Abelian groups:

**Theorem 6.5.** An ordered Abelian group is strong iff it has bounded regular rank and almost finite dimension.

*Proof.* By Proposition 6.1, Theorem 6.2 and Theorem 6.4. □

**Corollary 6.6.** Any strong ordered Abelian group has finite dp-rank.

*Proof.* By Proposition 6.1, Theorem 6.2 and Theorem 6.4. □
7 Computing the dp-rank in ordered Abelian groups

In this section we push forward the argument of the previous section and compute the precise value of the dp-rank of $G$. This value depends on the number of ‘infinite’ regular jumps. Given a prime number $p$, and a $p$-regular jump $\Delta \in \text{RJ}_p(G)$, we say that $\Delta$ is ‘infinite’, if $\dim_p(\Delta'/\Delta)$ is infinite where $\Delta'$ is the successor of $\Delta$ in $\text{RJ}_p(G)$ (or $\Delta' = G$ in case $\Delta$ has no successor). In fact, we are going to see that for a non-trivial ordered Abelian group $G$, the dp-rank of $G$ equals 1+ plus the number of infinite $p$-regular jumps for all different $p$.

But one must be carefully with this not very precise statement. It can happen that the same convex subgroup $\Delta$ can contribute as an infinite regular jump for different primes. Then we must count the number of primes in which $\Delta$ occurs, i.e. count the the ‘multiplicity’. To be more precise let use introduce a little bit more notation. If $\text{RJ}_p(G) = \{\Delta_0, \ldots, \Delta_{u-1}\}$, then

$$\text{RJ}_p^\infty(G) := \{\Delta_i \in \text{RJ}_p(G) \mid \dim_p(\Delta_{i+1}/\Delta_i) = \infty\},$$

where $\Delta_u$ denotes $G$. We call $\text{RJ}_p^\infty(G)$ the set of infinite $p$-regular jumps.

**Remark 7.1.** $G$ has finite $p$-dimension iff $G$ has finite $p$-regular rank and $\text{RJ}_p^\infty = \emptyset$.

**Proof.** By point 5 in remark 2.2 and the additivity of the $p$-dimension:

$$\dim_p(G) = \dim_p(G/\Delta_{u-1}) + \dim_p(\Delta_{u-1}/\Delta_{u-2}) + \cdots + \dim_p(\Delta_1/\Delta_0).$$

By Proposition 6.1 and Remark 7.1.

**Theorem 7.2.** Let $G$ be a nontrivial ordered Abelian group with bounded regular rank. Then

$$\text{dp-rk}(G) = 1 + \sum_{p \in \mathbb{P}} \left| \text{RJ}_p^\infty(G) \right|$$

(7.1)

**Proof.** We begin by showing the inequality $\text{dp-rk}(G) \geq 1 + \sum_{p \in \mathbb{P}} \left| \text{RJ}_p^\infty(G) \right|$. We may assume that $\sum_{p \in \mathbb{P}} \left| \text{RJ}_p^\infty(G) \right|$ is finite by Proposition 6.1 and Remark 7.1.

Let us denote $\Delta_1, \ldots, \Delta_N$ all convex subgroups occurring in $\bigcup_{p \in \mathbb{P}} \text{RJ}_p^\infty(G)$, $\Delta_1 \triangleleft \Delta_2 \triangleleft \cdots \triangleleft \Delta_N$.

For each $i = 1, \ldots, N$, let $p_{i,1}, \ldots, p_{i,k_i}$ denote all primes $p$ for which $\Delta_i \in \text{RJ}_p^\infty(G)$. Observe that

$$\sum_{p \in \mathbb{P}} \left| \text{RJ}_p^\infty(G) \right| = k_1 + \cdots + k_N.$$

27
For each $i = 1, \ldots, N$ and each $j = 1, \ldots, k_i$ let us denote

$$
e_{(i,j)} := \left\{ i' | i' \leq i \text{ and } \Delta_{i'} \in RJ^\infty_{p(i,j)} \right\},$$

$$H_{(i,j)} := \Delta_i + p_{(i,j)} \cdot G$$

$$n_{(i,j)} := \prod_{1 \leq j' < j} p_{(i,j')}$$

$$n_i := \prod_{j = 1}^{k_i} p_{(i,j)}$$

We want to show that the sequence of $H_{(i,j)}$ lexicographically ordered satisfies Distributivity and infinity of Proposition 5.4.

Claim 7.3. 1. Let $\Delta$, $\Delta_1 < \Delta_2 < \cdots < \Delta_t$ denote convex subgroups and $m, n, m_1, \ldots, m_t$ denote integers. Then

(a) $nG + mG = (n, m)G$

(b) $\bigcap_{i=1}^t \Delta + m_i G = \Delta + [m_1, \ldots, m_t]G$

(c) $\bigcap_{i=1}^t \Delta + m_i G =
\Delta_1 + m_1 \Delta_2 + [m_1, m_2] \Delta_3 + \cdots + [m_1, \ldots, m_{t-1}] \Delta_t + [m_1, \ldots, m_t]G.$

2. Let $1 \leq r \leq N$ and $1 \leq s \leq k_r$ be given. Then

(a) $\bigcap_{(i,j) < (r,s)} H_{(i,j)} =
\Delta_1 + [n_1, n_2] \Delta_3 + \cdots + [n_1, \ldots, n_{r-1}] \Delta_r + [n_1, \ldots, n_{r-1}, n_{(r,s)}]G.$

(b) $\bigcap_{(i,j) < (r,s)} H_{(i,j)} + H_{(r,s)} = \Delta_r + p_{(r,s)}^{-1}G.$

(c) If $\Delta_r$ is not the maximum in $RJ^\infty_{p(r,s)}$, then

$$K_{(r,s)} = \bigcap_{(i,j) \neq (r,s)} (H_{(i,j)} + H_{(r,s)}) = \Delta_r + p_{(r,s)}^{-1} \Delta_+ + p_{(r,s)} \cdot G.$$

Here $\Delta_{(r,s)}^+$ denotes the successor of $\Delta_r$ in $RJ^\infty_{p(r,s)}$.

(d) If $\Delta_r$ is the maximum in $RJ^\infty_{p(r,s)}$, then

$$K_{(r,s)} = \bigcap_{(i,j) \neq (r,s)} (H_{(i,j)} + H_{(r,s)}) = \Delta_r + p_{(r,s)}^{-1}G.$$
Proof. 1.(a). Using a Bézout identity.
1.(b). By induction, it is enough to prove \((\Delta+nG) \cap (\Delta+mG) = \Delta+[n,m]G\). The other inclusion being obvious, it suffices to prove \(\subseteq\). Assume \(a = \delta_1 + m_1g_1 = \delta_2 + m_2g_2\) with \(\delta_i \in \Delta\) and \(g_i \in G\). Denote \(d = (m,n)\), \(m = dm'\), \(n = dn'\). Let \(1 = \lambda m' + \mu n'\) be a Bézout identity for \(n', m'\). Then \(a = \lambda m' a + \mu n' a = \lambda m' (\delta_2 + m_2g_2) + \mu n' (\delta_1 + m_1g_1) = (\lambda m' \delta_2 + \mu n' \delta_1) + [n,m] (\lambda g_2 + \mu g_1) \in \Delta+[n,m]G\).

1.c. Since \(\bigcap_{i=1}^t \Delta_i + m_iG = \bigcap_{1 \leq j \leq t} \Delta_i + m_jG = \bigcap_{i=1}^t \Delta_i + [m_1, \ldots, m_t]G\) by 1), we may assume \(m_i\) divides \(m_{i+1}\). We do the proof by induction on \(t\). For the case \(t = 2\) we must show \((\Delta_1 + m_1G) \cap (\Delta_2 + m_2G) \subseteq \Delta_1 + m_1\Delta_2 + m_2G\), the other inclusion being trivial. Assume \(a = \delta_1 + m_1g_1 = \delta_2 + m_2g_2\) with \(\delta_i \in \Delta_i\) and \(g_i \in G\). Then \(\delta_2 - \delta_1 \delta_1 \in m_1G \cap \Delta_2 = m_1\Delta_2\) and thus \(\delta_2 - \delta_1 = m_1\delta_2\) for some \(\delta_2 \in \Delta_2\). Now \(a = \delta_1 + m_1\delta_2 + m_2G\). For the inductive step assume the formula holds for \(t\). In order to prove it for \(t+1\) it suffices to show

\[
\left( \Delta_1 + m_1\Delta_2 + \cdots + m_{t-1}\Delta_t + m_tG \right) \cap (\Delta_{t+1} + m_{t+1}G) = \Delta_1 + m_1\Delta_2 + m_2\Delta_3 + \cdots + m_t\Delta_{t+1} + m_{t+1}G
\]

The inclusion \(\supseteq\) being obvious, assume \(a = \delta_1 + m_1\delta_2 + m_2\delta_3 + \cdots m_{t-1}\delta_t + m_tg = \delta_{t+1} + m_{t+1}g'\) with \(\delta_i \in \Delta_i\) and \(g', g' \in G\). Then \(\delta_{t+1} - (\delta_1 + m_1\delta_2 + m_2\delta_3 + \cdots m_{t-1}\delta_t) \in m_tG \cap \Delta_{t+1} = m_t\Delta_{t+1}\) and thus \(\delta_{t+1} = \delta_2 + m_2\delta_3 + \cdots m_{t-1}\delta_t\) for some \(\delta_{t+1} \in \Delta_{t+1}\). Hence \(a = \delta_1 + m_1\delta_2 + m_2\delta_3 + \cdots m_{t-1}\delta_t + m_{t+1}\delta_{t+1} + m_{t+1}g'\).

2.(a). By 1.(b), \(\bigcap_{(i,j) \prec (r,s)} H_{(i,j)} = \left( \bigcap_{i \prec r} \Delta_i + n_iG \right) \cap (\Delta_r + n_rG)\). It follows then using 1.(c).

2.(b). By 1.(a), \(H_{(i,j)} + H_{(r,s)}\) is either \(G\) or \(\Delta_r + p^{c_{(r,s)}}G\) for some \(e < c_{(r,s)}\).

2.(c). Assume \(R_{\varepsilon_{(r,s)}}^\infty = \{\Lambda_1, \ldots, \Lambda_m\}\) and \(\Delta_r = \Lambda_{\varepsilon_{(r,s)}}\). By 1.(a), 1.(c), it follows that \(\bigcap_{(i,j) \neq (r,s)} H_{(i,j)} + H_{(r,s)} = (\Delta_r + p^{c_{(r,s)}}G) \cap (\Lambda_{\varepsilon_{(r,s)}+1} + p^{c_{(r,s)}}G) = \Delta_r + p^{c_{(r,s)}}\Lambda_{\varepsilon_{(r,s)}+1} + p^{c_{(r,s)}}G\).

2.(d). By 1.(a), \(\bigcap_{(i,j) \neq (r,s)} H_{(i,j)} + H_{(r,s)} = (\Delta_r + p^{c_{(r,s)}}G)\).

2.(e). The map \(\Delta_r^+ \to K_{(r,s)}(G)\) defined by \(x \mapsto +p^{c_{(r,s)}-1}(x) + H_{(r,s)}\) is surjective and its kernel is \(\Delta_r + p_{(r,s)}\Delta_r^+\).
2. (f). The map $G \to K_{i,j}/H_{i,j}$ defined by $x \mapsto +p_{i,j}^{-1}x + H_{i,j}$ is surjective and its kernel is $\Delta_i + p_{i,j}G$.

Now we check that the sequence of the $H_{i,j}$ satisfies Distributivity. By 2. (a) and 1. (a) and using that $\left([n_1, \ldots, n_{r-1}, n_{r,s}], p_{i,j}^{e_{r,s}}\right) = p_{i,j}^{e_{r,s}}$, it follows that
\[
\bigcap_{(i,j) \in (r,s)} H_{i,j} = \Delta_i + p_{i,j}^{e_{r,s}}G.
\]
Hence, bearing in mind 2. (b) distributivity holds.

Infinity holds by 2. (e), and 2. (f).

So by Proposition 5.4 we have an inp-pattern of depth $\sum_{p \in P} |\text{RJ}_p^\infty(G)| = k_1 + \cdots + k_N$. It remains to show we can add an extra row corresponding to the order. Given a countable set of solutions (which we may suppose all positive) to $RJ_{\omega}$ and using that $\inf\{x \mid x \geq b_i \}$ for some $b_i$ which is bigger to all those solutions. By Claims 7.3 1. (b) (c) and 4.5, the solution set of the system corresponding to any path are cosets of some group of the form $\Delta_1 + m_1 \Delta_2 + [m_1, m_2] \Delta_3 + \cdots + [m_1, \ldots, m_{N-1}] \Delta_N + [m_1, \ldots, m_N]G$ for some $m_1, \ldots, m_N$ positive integers. By saturation again, adding an element of $\left[1, \ldots, m_N\right]G$ big enough, we obtain a set of solutions of all paths, each one $\geq b_i$. Iterating this argument, one sees that we can add an extra row of kind $(x > b_i \mid i \in \omega)$ to the inp-pattern provided by Proposition 5.4.

For the proof of the inequality $d_P-rk(G) \leq 1 + \sum_{p \in P} |\text{RJ}_p^\infty(G)|$ we use Proposition 3.17 and the QE given in Theorem 2.4. In fact, we are going to see that $G$ is $N$-VCA, where $N = 1 + \sum_{p \in P} |\text{RJ}_p^\infty(G)|$. Without loss of generality we may assume $N$ is finite, since the theory is clearly $N_0$-VCA.

By Theorem 2.4 any formula can be written as a boolean combination of formulas of the following kind:

\[
\begin{align*}
nx &\leq t(\overline{y}) \quad (7.2) \\
nx &\equiv t(\overline{y}) \mod \Delta \quad (7.3) \\
nx &\equiv t(\overline{y}) \mod \Delta + p^mG \quad (7.4)
\end{align*}
\]

where $n \in \mathbb{Z}$, $\Delta \in RJ_p(G)$, $p \in P$, $m \geq 1$ and $t(\overline{y})$ a term.

Since the formulas of kind (7.2) and (7.3) define convex subsets, they are boolean combination of definable initial segments. So we may express any formula as a boolean combination of definable initial segments and formulas of kind (7.4). The definable initial segments constitute the first directed family.

Let $N_p$ denote the number of infinite $p$-regular jumps, i.e., $N_p = |\text{RJ}_p(G)|$. We can write the chain $RJ_p(G) = \{\Delta_1, \ldots, \Delta_{u-1}\}$ as follows:

\[
\{0\} = \Delta_0 \triangleleft \Delta_1 \triangleleft \cdots \triangleleft \Delta_{m_1} \triangleleft \Delta_{m_1+1} \triangleleft \cdots \triangleleft \Delta_{m_{N_p}} \triangleleft \Delta_{m_{N_p}+1} \triangleleft \cdots \triangleleft \Delta_u = G \quad (7.5)
\]

where $\text{RJ}_p^\infty(G) = \{\Delta_{m_1}, \Delta_{m_2}, \ldots, \Delta_{m_{N_p}}\}$. Therefore $\text{dim}_p(\Delta_{i+1}/\Delta_i)$ is infinite for each $i = 1 \ldots N_p$, while $\text{dim}_p(\Delta_{i+1}/\Delta_i)$ is finite for $i \notin \{m_1, \ldots, m_{N_p}\}$.
Claim 7.4. If $\Delta \leq \Delta'$ are convex subgroups of $G$ and $\dim_p(\Delta'/\Delta)$ is finite then $[\Delta' + p^r G : \Delta + p^r G]$ is also finite.

Proof. The same arguments as in Claim 4.6. In fact it is equal to $p^{mk}$, where $k = \dim_p(\Delta'/\Delta)$. \qed

Now, by Claim 7.4 any coset of a group of kind $\Delta_i + p^r G$ with $0 \leq i \leq m_1$ is a finite union of cosets of $\Delta_0 + p^r G$. Also any coset coset of kind $\Delta_i + p^r G$ with $m_1 + 1 \leq i \leq m_{i+1}$ is a finite union of cosets of $\Delta_{m_{i+1}} + p^r G$. Hence, in the boolean combination provided by QE, we may assume the formulas of kind (7.4) only occur (at most) for $\Delta \in \{\Delta_0, \Delta_{m_1+1}, \ldots, \Delta_{m_{N_p}+1}\}$.

Moreover, if $\Delta_{m_{N_p}+1} = G$ this convex subgroup does not occur in the Boolean combination, since $G / \in RJ_p$. If $\Delta_{m_{N_p}+1} \neq G$ then $[G : \Delta_{m_{N_p}+1}]$ is finite and, by claim 7.4 we get that the formula $nx \equiv t(y)$ mod $\Delta_{m_{N_p}+1} + p^m G$ is NA. In any case we may assume the formulas of kind (7.4) (which are not NA) only occur (at most) for $\Delta \in \{\Delta_0, \Delta_{m_1+1}, \ldots, \Delta_{m_{N_p}-1+1}\}$.

By Claim 4.5 the formulas of kind (7.4) with the same $\Delta$ and $p$ (and maybe different $n, m$) are again a directed family.

Altogether we have found $1 + \sum_p N_p$ directed families (1 for definable initial segments and another one for each $p$ and $\Delta \in \{\Delta_0, \Delta_{m_1+1}, \ldots, \Delta_{m_{N_p}-1+1}\}$) such that any definable set in a single variable is a Boolean combination of formulas of one of those families and NA-formulas. This shows the theory is $1 + \sum_p N_p$-VCA. \qed

Corollary 7.5. Let $G$ be an ordered Abelian group with bounded regular rank. Let $\lambda$ be cardinal number (finite or infinite). Then the following are equivalent:

1. $G$ has dp-rank at most $\lambda$
2. $G$ is $\lambda$-VCA

Proof. 2 implies 1 is just Proposition 3.17. For the converse, observe that for the proof of the inequality $dp-rk(G) \leq 1 + \sum_{p \in P} |RJ^\infty_p|$ in Theorem 7.2 one shows that $G$ is $N$-VCA, where $N = 1 + \sum_{p \in P} |RJ^\infty_p(G)| = dp-rk(G)$. \qed

We say that an ordered Abelian group has finite dimension iff $\dim_p(G)$ is finite for all prime $p$. Observe that by Remark 7.1 and Theorem 7.2 an ordered Abelian group is dp-minimal iff has finite dimension. This characterization of dp-minimality has been independently obtained in [13].

By Corollary 7.5 an ordered Abelian group is dp-minimal iff it is VCA-minimal.

Corollary 7.6. If $G$ and $H$ are non-trivial, then $dp-rk(G \times H) = dp-rk(H) + dp-rk(H) - 1$

Proof. It is easy to check that $|RJ^\infty_p(G \times H)| = |RJ^\infty_p(G)| + |RJ^\infty_p(H)|$.

Hence $\sum_{p \in P} |RJ^\infty_p(G \times H)| = \sum_{p \in P} |RJ^\infty_p(G)| + \sum_{p \in P} |RJ^\infty_p(H)|$. \qed
8 VC-density of ordered Abelian groups

In this section we show that for any ordered Abelian group $G$, $vc^G(1)$, the VC-density function of $G$ evaluated at 1, coincides with the dp-rank of $G$.

Following [4], the VC-density function of a complete theory $T$ is the function $vc^T: \mathbb{N} \to \mathbb{R} \geq 0 \cup \infty$ defined by

$$vc^T(n) = \sup \{ vc^*(\varphi(x, y)) \mid \varphi(x, y) \text{ is a formula and } |x| = n \}$$

Here $vc^*(\varphi(x, y))$ denotes the dual VC-density of the partitioned formula $\varphi(x, y)$, as defined in [4.1]. For more details on the VC-density function, see [4].

In the proof of next proposition we use Facts [4.2].

Proposition 8.1. In a $n$-VCA theory $T$, any formula $\varphi(x, y)$ has dual VC-density at most $n$. In other words, $vc^T(1) \leq n$.

Proof. Let $\langle \Psi_i(x, y) \mid i < n \rangle$ be a collection of $n$ directed families witnessing the theory is $n$-VCA. By fact [3.15], for each formula $\varphi(x, z)$ there is a finite set $\Theta$ of formulas which are Boolean combinations of formulas in $\bigcup_{i<n} \Psi_i(x, y)$ and NA-formulas, such that each instance of $\varphi(x, z)$ is equivalent to an instance of some formula in $\Theta$. Let $\Psi'_i$ denote respectively the set of formulas from $\Psi_i$ occurring in the boolean combinations of formulas in $\Theta$ and let $\Upsilon$ be the set of NA-formulas occurring in the boolean combinations of formulas in $\Theta$. Let $N$ be a common upper bound of the number of different sets each formula in $\Upsilon$ can define. We may assume all formulas in $\Upsilon, \Theta$ and $\Psi'_i$ have the same parameter variables, say $\pi$.

Given a set of $\pi$-parameters $A$, we can choose a set of $\pi$-parameters $B$ of size at most $|A|$ such that each instance of $\psi_0(x, \pi)$ with parameters from $A$ is an instance of some formula in $\Theta$ with parameters from $B$. Hence, any definable set $\varphi(C, \pi)$ with $\pi \in A$ is a boolean combination of sets of kind $\psi_0(C, \pi)$ where $\psi_0(x, \pi) \in \bigcup_{i<n} \Psi'_i \cup \Upsilon$ and $\pi \in B$.

Now it is not difficult to see that $|S^c_0(A)| \leq 2^{N^{|\Upsilon|}} \prod_{i<n} (|\Psi'_i| |A| + 1)$. This holds because the Boolean algebra generated by $\Upsilon$-formulas with parameters in $B$ has at most $2^{N^{|\Upsilon|}}$ atoms because there are at most $N^{|\Upsilon|}$ nonequivalent formulas. And the boolean algebra generated by the sets defined by $\Psi'_i$-formulas with parameters from $B$ has at most $|\Psi'_i| |A| + 1$ atoms. \hfill $\square$

Proposition 8.2. For any complete theory $T$,

$$dp-rk(T) \leq vc^T(1)$$

Proof. We show that $dp-rk(G) \geq n$ implies $vc^G(1) \geq n$, where $n$ is a natural number. Let $\langle \psi_i(x, y) \mid i = 1 \ldots n \rangle$ and $\langle a_{i,j} \mid i = 1 \ldots n, j \in \omega \rangle$ be an ict-pattern for $x = x$. Fix $M \in \omega$ and consider

$$B := \{ \delta_{i,j} \mid i = 1 \ldots n, j = 1 \ldots M \}$$

$$\psi(x, y) := \psi_1(x, y) \lor \cdots \lor \psi_n(x, y),$$

32
where
\[ \bar{b}_{i,j} := a_{1,M+1}, \ldots, a_{i-1,M+1}, a_{i,j}, a_{i+1,M+1}, \ldots, a_{n,M+1}. \]

Observe that
\[
\begin{align*}
\psi(x, \bar{b}_{i,j}) \land \neg\psi(x, \bar{b}) & \vdash \psi_i(x, \bar{a}_{i,j}) \\
\neg\psi(x, \bar{b}_{i,j}) & \vdash \neg\psi_i(x, \bar{a}_{i,j}),
\end{align*}
\]
where
\[ \bar{b} := a_{1,M+1}, \ldots, a_{i,M+1}, \ldots, a_{n,M+1}. \]

Since there are \( M^n \) different paths within the first \( M \) columns of the \( \text{iect-patttern} \), this implies that \( |S^\psi(B)| \geq M^n \). Keeping in mind that \( |B| \leq nM \), this entails that \( |S^\psi(B)| \geq (1/n)^n |B|^n \). As \( M \) can be arbitrarily large this implies that \( \text{vc}^*(\psi) \geq n \). □

**Theorem 8.3.** Let \( G \) be an ordered Abelian group with bounded regular rank. Then
\[ \text{vc}^G(1) = \text{dp-rk}(G). \]

**Proof.** By Proposition 8.2 it is enough to show that \( \text{vc}^G(1) \leq \text{dp-rk}(G) \). We may assume \( G \) has finite dp-rank, say equal to \( n \). By Corollary 7.5 \( G \) is \( n \)-VCA. By Proposition 8.1 \( \text{vc}^G(1) \leq n \). □
9 Gurevich-Schmitt Quantifier Elimination for ordered Abelian groups

The definitions and statements of this section are taken from [18], [17] and [19]. All the proofs may be found there. We start by giving some definitions.

**Definition 9.1.** Let $G$ be an ordered Abelian group, $g \in G \setminus \{0\}$ and $n \geq 2$.

- $G$ is called $n$-regular if for every convex subgroup $H \neq \{0\}$ of $G$, $G/H$ is $n$-divisible.
- $A(g)$ = the largest convex subgroup of $G$ not containing $g$.
- $B(g)$ = the smallest convex subgroup of $G$ containing $g$.
- $C(g) = B(g)/A(g)$.
- $A_n(g)$ = the smallest convex subgroup $C$ of $G$ such that $B(g)/C$ is $n$-regular.
- $B_n(g)$ = the largest convex subgroup $C$ of $G$ such that $C/A(g)$ is $n$-regular.
- $C_n(g) = B_n(g)/A_n(g)$.
- For $g = 0$ we define $A_n(0) = \emptyset$, $B_n(0) = \{0\}$.
- $F_n(g)$ = the largest convex subgroup $C$ of $G$ such that $C \cap (g + nG) = \emptyset$ if $g \notin nG$, $F_n(g) = \emptyset$ otherwise.
- $\Gamma_1,n(g) = \{ h \in G \mid F_n(h) \subseteq F_n(g) \}$.
- $\Gamma_2,n(g) = \{ h \in G \mid F_n(h) \subseteq F_n(g) \}$.

If $g \notin nG$, $\Gamma_1,n(g)$ and $\Gamma_2,n(g)$ are shown to be subgroups of $G$ (Facts 9.2 ii) below) and we can define:

- $\Gamma_n(g) = \Gamma_2,n(g)/\Gamma_1,n(g)$.

The sets $A_n(g)$, $B_n(g)$, $F_n(g)$, $\Gamma_1,n(g)$ and $\Gamma_2,n(g)$ are shown to be definable in the language $\text{LOG} = \{0, +, -, \leq\}$ by a first-order formula with the only parameter $g$ (see [18] and [10]).

**Facts 9.2.** If $g, h \neq 0$ then:

1. $A_n(g + h) \subseteq A_n(g) \cup A_n(h)$ and if $A_n(g) \subset A_n(h)$ then $A_n(g + h) = A_n(h)$.
2. $F_n(g + h) \subseteq F_n(g) \cup F_n(h)$ and if $F_n(g) \subset F_n(h)$ then $F_n(g + h) = F_n(h)$.
3. $F_n(g + nh) = F_n(g)$, $F_n(g) = \emptyset$ iff $g \in nG$.
4. $A_n(h) \subseteq A_n(g)$ iff $B_n(h) \subseteq B_n(g)$ iff $A_n(h) \subset B_n(g)$ iff $h \in B_n(g)$ iff $g \notin A_n(h)$.
5. $A_n(h) \subset A_n(g)$ iff $B_n(h) \subset B_n(g)$ iff $B_n(h) \subseteq A_n(g)$ iff $g \notin B_n(h)$ iff $h \in A_n(g)$.

6. $A_n(g) = \bigcup \{A_p(g) \mid p \text{ a prime divisor of } n\}$.

7. $F_n(g) = \bigcap_{h \in G} A_n(g + nh)$.

The language LSP of spines contains as non-logical symbols a binary relation symbol $\leq$ and the following monadic relation symbols: $A$, $F$, $D$ and $\alpha(p,k,m)$ for all $k, m \in \mathbb{N} \setminus \{0\}$, and $p$ prime. The $n$-spine of $G$, for $n \geq 2$, is defined as the LSP-structure with universe

$$\{A_n(g) \mid g \in G\} \cup \{F_n(g) \mid g \in G\},$$

and with the following interpretation of the relations:

- $C_1 \leq C_2$ iff $C_1 \subseteq C_2$,
- $A(C)$ iff $C = A_n(g)$ for some $g \in G$,
- $F(C)$ iff $C = F_n(g)$ for some $g \in G$,
- $D(C)$ iff $G/C$ is discrete,
- $\alpha(p,k,m)(C)$ iff $C = F_n(g)$ for some $g \in G \setminus nG$ and $\alpha_p(k)(\Gamma_n(g)) \geq m$,

where $\alpha_p(k)(C)$ denotes the dimension of $(p^{k-1}C[p]/p^kC[p])$ as $\mathbb{F}_p$-vector space if it is finite, and $\alpha_p(k)(C) = \infty$ otherwise. Since $n\Gamma_n(g) = \{0\}$, $\Gamma_n(g)$ is a direct sum of finite cyclic groups of order dividing $n$(see [13]) and $\alpha_p(k)(\Gamma_n(g))$ is the number of cyclic groups of order $p^k$ in this decomposition. Thus $\alpha_p(k)(\Gamma_n(g)) = 0$ if $p^k \nmid n$ and the $\alpha_p(k,m)$ are irrelevant for $p^k \nmid n$.

We will denote this structure by $Sp_n(G)$.

Remark 9.3. The structure $Sp_n(G)$ is interpretable in $G$ for every $n \geq 2$. In particular, given any LSP-formula $\psi(z_1, \ldots, z_r, t_1, \ldots, t_s)$, there is a formula in the language of ordered Abelian groups $\varphi(x_1, \ldots, x_r, y_1, \ldots, y_s)$ such that for every ordered Abelian group $G$, and $g_1, \ldots, g_r, h_1, \ldots, h_s \in G$,

$$Sp_n(G) \models \psi(A_n(g_1), \ldots, A_n(g_r), F_n(h_1), \ldots, F_n(h_s))$$

iff $G \models \varphi(g_1, \ldots, g_r, h_1, \ldots, h_s)$

Let LOG$^*$ be the definitional expansion of LOG by the following unary predicates: $M_k$, $E_{(n,k)}$ and $D_{(p,r,i)}$ for all $n \geq 2$, $r \geq 1$, $0 < i < r$, $k > 0$ and $p$ prime.

For $g \neq 0$ they are defined by:

- $M_k(g)$ iff $C_2(g)$ is discrete with $1_{C_2(g)}$ denoting its first positive element and $[g] = k \cdot 1_{C_2(g)}$ in $C_2(g)$.

\[\text{In } \text{[13] and [17] it is used the notation } M(n,k). \text{ We eliminate } n \text{ since } M(n,k) \text{ does not depend on } n: \text{ if } C(g) \text{ is discrete then } A_n(g) = A(g), \text{ conversely if } C_n(g) \text{ is discrete and } \overline{g} = k\overline{g} \text{ in } C_n(g), \text{ then } A_n(g) = A(g)\]
• \(E_{(n,k)}(g)\) iff there exists \(h \in G\) such that \(F_n(g) = A_n(h), M(1)(h)\) holds and \([g] = k[h]\) in \(\Gamma_n(g)\) iff there exists \(h \in G\) such that \(F_n(g) = A_n(h), M(1)(h)\) holds and \(F_n(g - kh) \subset F_n(g)\).

• \(D_{(p,r,i)}(g)\) iff \(g \in p^rG\) or \([g] \in p^i \Gamma_{p^r}(g)\) iff \(g \in p^rG\) or there exists \(h \in G\) such that \(F_{p^r}(g - p^i h) \subset F_{p^r}(p^i h) = F_{p^r}(g)\) iff \(F_{p^r}(p^{r-1}g) \subset F_{p^r}(g)\).

**Theorem 9.4.** For every LOG-formula \(\varphi(\overline{x})\) there exist \(n \geq 2\), a quantifier free \(\text{LOG}^*\)-formula \(\psi_1(\overline{x})\), an LSP-formula \(\psi_0(y_1, \ldots, y_m, z_1, \ldots, z_r)\), LOG-terms \(t_i(\overline{x})\) for \(i = 1, \ldots, m\) and \(s_i(\overline{x})\) for \(i = 1, \ldots, r\) such that for every ordered Abelian group \(G\) and every \(\overline{g} \in G^\omega\)

\[
G \models \varphi(\overline{g}) \iff \begin{cases} 
G \models \psi_1(\overline{g}) \\
\text{Sp}_n(G) \models \psi_0(C_1, \ldots, C_m, D_1, \ldots, D_r),
\end{cases}
\]

where \(C_i = A_n(t_i(\overline{g}))\) and \(D_i = F_n(s_i(\overline{g}))\).
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