Double lacunary sequence spaces of double sequence of interval numbers

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Received : January 2012. Accepted : August 2012

Abstract

In this paper we introduce the concepts of double lacunary strongly convergence and double lacunary statistical convergence of double interval numbers. We prove some inclusion relations and study some of their properties.

Subjclass [2000] : 40C05,40J05, 46A45.

Keywords : Sequence space, Lacunary sequence, interval numbers.
1. Introduction

Interval arithmetic was first suggested by Dwyer [12] in 1951. Development of interval arithmetic as a formal system and evidence of its value as a computational device was provided by Moore [15] in 1959 and Moore and Yang [16] 1962. Furthermore, Moore and others [12], [13], [14], [17] and [18] have developed applications to differential equations.

Chiao in [9] introduced sequence of interval numbers and defined usual convergence of sequences of interval number. Recently Esi [1−3] and Şengönül and Eryilmaz in [11] introduced and studied some sequence spaces of interval numbers.

The idea of statistical convergence for ordinary sequences was introduced by Fast [7] in 1951. Schoenberg [8] studied statistical convergence as a summability method and listed some of elementary properties of statistical convergence. Both of these authors noted that if bounded sequence is statistically convergent, then it is Cesaro summable. Existing work on statistical convergence appears to have been restricted to real or complex sequence, but several authors extended the idea to apply to sequences of fuzzy numbers and also introduced and discussed the concept of statistically sequences of fuzzy numbers.

2. Preliminaries

A double sequence of real numbers is a function $x : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}$. We shall use the notation $x = (x_{k,l})$.

A double sequence $x = (x_{k,l})$ has a Pringsheim limit $L$ (denoted by $P - \lim x = L$) provided that given an $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|x_{k,l} - L| < \varepsilon$ whenever $k, l > N$. We shall describe such an $x = (x_{k,l})$ more briefly as "$P -$ convergent"[4].The double sequence $x = (x_{k,l})$ is bounded if there exists a positive number $M$ such that $|x_{k,l}| < M$ for all $k$ and $l$.

$$||x|| = \sup_{k,l} |x_{k,l}| < \infty.$$ 

Let $p = (p_{k,l})$ be a double sequence of positive real numbers. If $0 < h = \inf_{k,l} p_{k,l} \leq p_{k,l} \leq H = \sup_{k,l} p_{k,l} < \infty$ and $D = \max \left(1, 2^{H-1}\right)$, then for all $a_{k,l}, b_{k,l} \in \mathbb{C}$ for all $k, l \in \mathbb{N}$, we have

$$|a_{k,l} + b_{k,l}|^{p_{k,l}} \leq D \left(|a_{k,l}|^{p_{k,l}} + |b_{k,l}|^{p_{k,l}}\right).$$

We should note that in contrast to the case for single sequences, a convergent double sequence need not be bounded. The concept of statistical
convergence was introduced by Fast [7] in 1951. A sequence \( x = (x_k) \) is said to be statistically convergent to the number \( L \) if for every \( \varepsilon > 0 \)
\[
\lim_{n \to \infty} \frac{1}{n} \{ k \leq n : |x_k - L| \geq \varepsilon \} = 0,
\]
where the vertical bars indicate the number of elements in the enclosed set. Later, Mursaleen and Edely [10] defined the statistical analogue for double sequence \( x = (x_{k,l}) \) as follows: A real double sequence \( x = (x_{k,l}) \) is said to be \( P \)-statistically convergent to \( L \) provided that for each \( \varepsilon > 0 \)
\[
P - \lim_{m,n \to \infty} \frac{1}{mn} \{|(k,l) : k < m, l < n; |x_{k,l} - L| \geq \varepsilon \}| = 0.
\]
In this case, we write \( \text{St}^2 - \lim_{k,l} x_{k,l} = L \) and we denote the set of all \( P \)-statistically convergent double sequences by \( \text{St}^2 \).

By a lacunary \( \theta = (k_r); r = 0, 1, 2, \ldots \) where \( k_0 = 0 \), we shall mean an increasing sequence of non-negative integers with \( h_r = k_r - k_{r-1} \to \infty \) as \( r \to \infty \). The intervals determined by \( \theta \) will be denoted by \( I_r = (k_r-1, k_r] \). The ratio \( \frac{k_r}{k_{r-1}} \) will be denoted by \( q_r \). The space of lacunary strongly convergent sequence space \( N_\theta \) was defined by Freedman et.al. [5] as follows:
\[
N_\theta = \left\{ x = (x_k) : \lim_{r} \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| = 0, \text{ for some } L \right\}.
\]

The double sequence \( \theta_{r,s} = \{(k_r, l_s)\} \) is called double lacunary sequence if there exist two increasing sequences of integers such that
\[
k_0 = 0, \ h_r = k_r - k_{r-1} \to \infty \text{ as } r \to \infty
\]
and
\[
l_0 = 0, \ l_s = l_s - l_{s-1} \to \infty \text{ as } s \to \infty.
\]

**Notations:** \( k_{r,s} = k_r l_s, \ h_{r,s} = h_r l_s \) and \( \theta_{r,s} \) is determined by
\[
I_{r,s} = \{(k,l) : k_{r-1} < k \leq k_r \text{ and } l_{s-1} < l \leq l_s\},
\]
\[
q_r = \frac{k_r}{k_{r-1}}, q_s = \frac{l_s}{l_{s-1}} \text{ and } q_{r,s} = q_r q_s. \text{[6]}
\]
The set of all double lacunary sequences denoted by $N_{θ_{r,s}}$ and defined by Savaş and Patterson [6] as follows:

$$N_{θ_{r,s}} = \left\{ x = (x_{k,l}) : \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} |x_{k,l} - L| = 0, \text{ for some } L \right\}.$$  

We denote the set of all real valued closed intervals by $I\mathbb{R}$. Any elements of $I\mathbb{R}$ is called interval number and denoted by $x = [x_l, x_r]$. Let $x_l$ and $x_r$ be first and last points of interval number, respectively. For $x_1, x_2 \in I\mathbb{R}$, we have

$$x_1 = x_2 \iff x_1 = x_2 \quad \text{and} \quad x_1 + x_2 = \{ x \in \mathbb{R} : x_1 + x_2 \leq x \leq x_1 + x_2 \}.$$  

and if $α ≥ 0$, then $αx = \{ x \in \mathbb{R} : αx_1 \leq x \leq αx_1 \}$, and if $α < 0$, then $αx = \{ x \in \mathbb{R} : αx_1 \leq x \leq αx_1 \}$.

The set of all interval numbers $I\mathbb{R}$ is a complete metric space defined by

$$d(x_1, x_2) = \max\{|x_1 - x_2|, |x_1 - x_2|\} \quad [15].$$

In the special case $x_1 = [a, a]$ and $x_2 = [b, b]$, we obtain usual metric of $\mathbb{R}$.

Now we give the definition of convergence of interval numbers:

**Definition 1.1.** [9] A sequence $x = (x_k)$ of interval numbers is said to be convergent to the interval number $x_o$ if for each $ε > 0$ there exists a positive integer $k_o$ such that $d(x_k, x_o) < ε$ for all $k ≥ k_o$ and we denote it by $\lim_k x_k = x_o$.

Thus, $\lim_k x_k = x_o \iff \lim_k x_{k,l} = x_o$.

Let’s define transformation $x$ from $\mathbb{N} × \mathbb{N}$ to $I\mathbb{R}$ by $k, l \rightarrow x(k, l) = x_{k,l}$. We shall use the notation $x = (x_{k,l})$. Then $x = (x_{k,l})$ is called double sequence of interval numbers. The $x_{k,l}$ is called $(k, l)^{th}$ term of sequence $x = (x_{k,l})$.

In this paper, we introduce and study the concepts of double lacunary strongly convergence and double lacunary statistically convergence for interval numbers.
3. Main Results

In this section we give some definition and prove the results of this paper.

**Definition 3.1.** Let \( \theta_{r,s} = \{(k_r,l_s)\} \) be a double lacunary sequence and \( p = (p_{k,l}) \) be any double sequence of strictly positive real numbers. A double sequence \( (\varpi_k,l) \) of interval numbers is said to be double lacunary strongly convergent if there is a double interval number \( \varpi_o \) such that

\[
P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{k \in I_{r,s}} |d(\varpi_k,l,\varpi_o)|^{p_{k,l}} = 0.
\]

In this case we write \( \varpi_k,l \to \varpi_o \) or \( 2N_{p_{\theta_{r,s}}} \). We denote with \( 2N_{p_{\theta_{r,s}}} \) the set of all lacunary strongly convergent double sequences of interval numbers. In the special case \( \theta_{r,s} = \{(2^r,2^s)\} \), we shall write \( 2N^p \) instead of \( 2N_{p_{\theta_{r,s}}} \).

**Definition 3.2.** Let \( \theta_{r,s} = \{(k_r,l_s)\} \) be a double lacunary sequence. A double sequence \( (\varpi_k,l) \) of interval numbers is said to be double lacunary statistically convergent to interval number \( \varpi_o \) if for every \( \varepsilon > 0 \)

\[
P - \lim_{r,s} \frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s} : d(\varpi_k,l,\varpi_o) \geq \varepsilon\}| = 0.
\]

In this case we write \( \varpi_k,l \to \varpi_o \) or \( s_{\theta_{r,s}} \). The set of all double lacunary statistically convergent sequences of interval number is denoted by \( s_{\theta_{r,s}} \). In the special case \( \theta_{r,s} = \{(2^r,2^s)\} \), we shall write \( s \) instead of \( s_{\theta_{r,s}} \).

**Theorem 3.1.** Let \( \varpi = (\varpi_k,l) \) and \( \varphi = (\varphi_k,l) \) be double sequences of interval numbers.

(i) If \( s_{\theta_{r,s}} \lim \varpi_k,l = \varpi_o \) and \( \alpha \in \mathbb{R} \), then \( s_{\theta_{r,s}} \lim \alpha \varpi_k,l = \alpha \varpi_o \).

(ii) If \( s_{\theta_{r,s}} \lim \varphi_k,l = \varphi_o \) and \( s_{\theta_{r,s}} \lim \varphi_k,l = \varphi_o \), then \( s_{\theta_{r,s}} \lim (\varpi_k,l + \varphi_k,l) = \varpi_o + \varphi_o \).

**Proof.** (i) Let \( \alpha \in \mathbb{R} \). For a given \( \varepsilon > 0 \)

\[
\frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s} : d(\alpha \varpi_k,l,\alpha \varpi_o) \geq \varepsilon\}|
\]
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\[ \frac{1}{h_{r,s}} \left| \left\{ (k,l) \in I_{r,s} : d(\mathbf{x}_{k,l}, \mathbf{x}_o) \geq \frac{\varepsilon}{|\alpha|} \right\} \right| . \]

Hence \( s_{\theta_{r,s}} - \lim \alpha x_{k,l} = \alpha x_o. \)

(ii) Suppose that \( s_{\theta_{r,s}} - \lim x_{k,l} = x_o \) and \( s_{\theta_{r,s}} - \lim y_{k,l} = y_o. \) We have

\[
d \left( x_{k,l} + y_{k,l} ; x_o + y_o \right) \\
\leq d \left( x_{k,l} ; x_o \right) + d \left( y_{k,l} ; y_o \right). 
\]

Therefore given \( \varepsilon > 0, \) we have

\[
\frac{1}{h_{r,s}} \left| \left\{ (k,l) \in I_{r,s} : d(x_{k,l}, x_o) \geq \frac{\varepsilon}{2} \right\} \right| \\
\leq \frac{1}{h_{r,s}} \left| \left\{ (k,l) \in I_{r,s} : d(x_{k,l}, x_o) + d(y_{k,l}, y_o) \geq \varepsilon \right\} \right| \\
\leq \frac{1}{h_{r,s}} \left| \left\{ (k,l) \in I_{r,s} : d(x_{k,l}, x_o) \geq \frac{\varepsilon}{2} \right\} \right| + \frac{1}{h_{r,s}} \left| \left\{ (k,l) \in I_{r,s} : d(y_{k,l}, y_o) \geq \frac{\varepsilon}{2} \right\} \right|.
\]

Thus, \( s_{\theta_{r,s}} - \lim \left( x_{k,l} + y_{k,l} \right) = x_o + y_o. \)

**Theorem 3.2.** Let \( \theta_{r,s} = \{(k_r, l_s)\} \) be a double lacunary sequence and \( \mathbf{x} = (x_{k,l}) \) be a double sequence of interval numbers. Then

(i) \( x_{k,l} \rightarrow x_o \left( N^p_{\theta_{r,s}} \right) \) implies \( x_{k,l} \rightarrow x_o \left( s_{\theta_{r,s}} \right), \)

(ii) \( \mathbf{x} = (x_{k,l}) \in m \) and \( x_{k,l} \rightarrow x_o \left( s_{\theta_{r,s}} \right) \) imply \( x_{k,l} \rightarrow x_o \left( N^p_{\theta_{r,s}} \right), \)

(iii) If \( \mathbf{x} = (x_{k,l}) \in m, \) then \( x_{k,l} \rightarrow x_o \left( N^p_{\theta_{r,s}} \right) \) if and only if \( x_{k,l} \rightarrow x_o \left( s_{\theta_{r,s}} \right), \)

where \( m = \{ x = (x_{k,l}) : \sup_{k,l} d(x_{k,l}, x_o) < \infty \} \).

**Proof.** (i) Let \( \varepsilon > 0 \) and \( x_{k,l} \rightarrow x_o \left( N^p_{\theta_{r,s}} \right). \) Then we write

\[
|\{(k,l) \in I_{r,s} : d(x_{k,l}, x_o) \geq \varepsilon\}| \leq \sum_{(k,l) \in I_{r,s} : d(x_{k,l}, x_o) \geq \varepsilon} d(x_{k,l}, x_o)
\]

and

\[
P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{k \in I_{r,s}} [d(x_{k,l}, x_o)]^{pk,l} = 0.
\]

This implies that

\[
P - \lim_{r,s} \frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s} : d(x_{k,l}, x_o) \geq \varepsilon\}| = 0.
\]
This completes the proof (i).

(ii) Suppose that \( x = (x_{k,l}) \in m \) and \( x_{k,l} \to x_0 (\mathfrak{y}_r^\theta) \). Since \( x = (x_{k,l}) \in m \), there is a constant \( C > 0 \) such that \( d(x_{k,l}, x_0) \leq C \). Given \( \varepsilon > 0 \), we have

\[
\frac{1}{h_{r,s} \sum_{(k,l) \in I_{r,s}} [d(x_{k,l}, x_0)]^{p_k}}
\]

\[
= \frac{1}{h_{r,s} d(x_{k,l}, x_0) \geq \varepsilon (k,l) \in I_{r,s}} + \frac{1}{h_{r,s} d(x_{k,l}, x_0) < \varepsilon (k,l) \in I_{r,s}}
\]

\[
\leq \frac{1}{h_{r,s} d(x_{k,l}, x_0) \geq \varepsilon (k,l) \in I_{r,s}} \max \left(C^h, C^H\right) + \frac{1}{h_{r,s} d(x_{k,l}, x_0) < \varepsilon (k,l) \in I_{r,s}} \varepsilon^{p_k}
\]

\[
\leq \max \left(C^h, C^H\right) \frac{1}{h_{r,s}} \left| \{(k,l) \in I_{r,s} : d(x_{k,l}, x_0) \geq \varepsilon \} \right| + \max \left(\varepsilon^h, \varepsilon^H\right).
\]

Thus we obtain \( x_{k,l} \to x_0 (\mathfrak{y}_r^\theta) \).

(iii) It follows from (i) and (ii).

**Theorem 3.3.** Let \( \theta_{r,s} = \{ (k_r, l_s) \} \) be a double lacunary sequence and \( \mathfrak{x} = (x_{k,l}) \) be a double sequence of interval numbers. Then

(i) For \( \lim \inf_r q_r > 1 \) and \( \lim \inf_s q_s > 1 \) then \( x_{k,l} \to x_0 (\mathfrak{y}) \) implies \( x_{k,l} \to x_0 (\mathfrak{y}_r^\theta) \),

(ii) For \( \lim \sup_r q_r < \infty \) and \( \lim \sup_s q_s < \infty \) then \( x_{k,l} \to x_0 (\mathfrak{y}_{r,s}^\theta) \) implies \( x_{k,l} \to x_0 (\mathfrak{y}) \),

(iii) If \( 1 < \lim \inf_r q_r \leq \lim \sup_r q_r < \infty \) and \( \lim \inf_s q_{r,s} \leq \lim \sup_s q_{r,s} < \infty \), then \( x_{k,l} \to x_0 (\mathfrak{y}) \) if and only if \( x_{k,l} \to x_0 (\mathfrak{y}_{r,s}^\theta) \).

**Proof.** (i) Suppose that \( \lim \inf_r q_r > 1 \) and \( \lim \inf_s q_s > 1 \) then there exists a \( \delta > 0 \) such that \( q_r \geq 1 + \delta \), \( q_s \geq 1 + \delta \) for sufficiently large \( r \) and \( s \) which implies

\[
\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta} \quad \text{and} \quad \frac{T_s}{l_s} \geq \frac{\delta}{1+\delta}.
\]

Since \( h_{r,s} = k_r l_s - k_r l_{s-1} - k_{r-1} l_s - k_{r-1} l_{s-1} \) we granted the following

\[
\frac{k_r l_s}{h_{r,s}} \leq \frac{(1+\delta)^2}{\delta^2} \quad \text{and} \quad \frac{k_{r-1} l_{s-1}}{h_{r,s}} \leq \frac{1}{\delta^2}.
\]
Now, let $\sigma_{k,l} \to \sigma_o(\bar{x})$. We are going to prove $\sigma_{k,l} \to \sigma_o(\bar{x}_{r,s})$. Then for sufficiently large $r$ and $s$, we have

$$
\frac{1}{k_r l_s} \{|(k,l) \in I_{r,s} \cap k \leq k_r \text{ and } l \leq l_s : d(\sigma_{k,l}, \sigma_o) \geq \varepsilon\} \\
\geq \frac{1}{k_r l_s} |\{(k,l) \in I_{r,s} : d(\sigma_{k,l}, \sigma_o) \geq \varepsilon\}| \\
\geq (1 + \delta)^2 \frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s} : d(\sigma_{k,l}, \sigma_o) \geq \varepsilon\}|.
$$

Hence $\sigma_{k,l} \to \sigma_o(\bar{x}_{r,s})$.

(ii) If $\limsup_r q_r < \infty$ and $\limsup_s q_s < \infty$ then there exists $C > 0$ such that $q_r < C$ and $q_s < C$ for all $r, s \geq 1$. Let $\sigma_{k,l} \to \sigma_o(\bar{x}_{r,s})$ and $\varepsilon > 0$. Then there exist $r_o < 0$ and $s_o > 0$ such that for every $i \geq r_o$ and $j \geq s_o$

$$
B_{i,j} = \frac{1}{h_{i,j}} |\{(k,l) \in I_{i,j} : d(\sigma_{k,l}, \sigma_o) \geq \varepsilon\}| < \varepsilon.
$$

Let $M = \max \{B_{i,j} : 1 \leq i \leq r_o \text{ and } 1 \leq j \leq s_o\}$ and $m$ and $n$ be such that $k_r - 1 < m \leq k_r$ and $l_s - 1 < n \leq l_s$. Thus we obtain the following

$$
\frac{1}{mn} |\{(k,l) \in I_{i,j} : k \leq m \text{ and } l \leq n : d(\sigma_{k,l}, \sigma_o) \geq \varepsilon\}| \\
\leq \frac{1}{k_r l_s - 1} |\{(k,l) \in I_{i,j} : k \leq k_r \text{ and } l \leq l_s : d(\sigma_{k,l}, \sigma_o) \geq \varepsilon\}| \\
\leq \frac{1}{k_r l_s - 1} \sum_{l,u=1,1}^{r_o,s_o} h_{l,u} B_{l,u} + \frac{1}{k_r l_s - 1} \sum_{(r_o < t \leq r) \cup (s_o < u \leq s)} h_{t,u} B_{t,u} \\
\leq \frac{M}{k_r l_s - 1} \sum_{l,u=1,1}^{r_o,s_o} h_{l,u} + \frac{1}{k_r l_s - 1} \sum_{(r_o < t \leq r) \cup (s_o < u \leq s)} h_{t,u} B_{t,u} \\
\leq \frac{M k_r l_s r_o s_o}{k_r l_s - 1} + \frac{1}{k_r l_s - 1} \sum_{(r_o < t \leq r) \cup (s_o < u \leq s)} h_{t,u} B_{t,u} \\
\leq \frac{M k_r l_s r_o s_o}{k_r l_s - 1} \left(\sup_{t \geq r_o \cup u \geq s_o} B_{t,u}\right) \frac{1}{k_r l_s - 1} \sum_{(r_o < t \leq r) \cup (s_o < u \leq s)} h_{t,u} B_{t,u}.
$$
\[
\leq \frac{M r_o r_s o_0}{k_{r-1} l_{s-1}} + \frac{\varepsilon}{k_{r-1} l_{s-1}} \sum_{(r_o < t \leq r_l) \cup (s_o < u \leq s)} h_{t,u}
\]
\[
\leq \frac{M r_o r_s o_0}{k_{r-1} l_{s-1}} + \varepsilon C^2.
\]

Since \(k_r\) and \(l_s\) both approach infinity as both \(m\) and \(n\) approach infinity it follows that
\[
\frac{1}{mn} |\{(k,l) \in I_{i,j}; k \leq m \text{ and } l \leq n : d(\pi_{k,l}, \pi_o) \geq \varepsilon\}| \to 0.
\]

This completes the proof.

(iii) It follows from (i) and (ii).

Acknowledgement: The author is indebted to the referee for his/her helpful suggestions.

References

[1] A. Esi, Strongly almost \(\lambda\)--convergence and statistically almost \(\lambda\)--convergence of interval numbers, Scientia Magna, 7 (2), pp. 117-122, (2011).

[2] A. Esi, Lacunary sequence spaces of interval numbers, Thai Journal of Mathematics, 10 (2), pp. 445-451, (2012).

[3] A. Esi, A new class of interval numbers, Journal of Qafqaz University, Mathematics and Computer Science, pp. 98-102, (2012).

[4] A. Pringsheim, Zur theorie der zweifach unendlichen Zahlenfolgen, Mathematische Annalen, 53, pp. 289-321, (1900).

[5] A. R. Freedman, J.J.Sember and M.Raphael, Some Cesaro type summability, Proc.London Math.Soc., 37 (3), pp. 508-520, (1978).

[6] E. Savaş and R.F.Patterson, Lacunary statistical convergence of multiple sequences, Applied Mathematics Letters, 19, pp. 527-534, (2006).

[7] H. Fast, Sur la convergence statistique, Collog.Math.2, pp. 241-244, (1951).
[8] I. J. Schoenberg, The integrability of certain functions and related
summability methods, Amer. Math. Monthly, 66, pp. 361-375, (1959).

[9] Kuo-Ping Chiao, Fundamental properties of interval vector max-norm,
Tamsui Oxford Journal of Mathematics, 18 (2), pp. 219-233, (2002).

[10] M. Mursaleen and O.H.Edely, Statistical convergence of double se-
quenences, J. Math. Anal. Appl. 288 (1), pp. 223-231, (2003).

[11] M. Şengönül and A. Eryılmaz, On the sequence spaces of interval num-
bers, Thai Journal of Mathematics, 8 (3), pp. 503-510, (2010).

[12] P. S. Dwyer, Linear Computation, New York, Wiley, (1951).

[13] P. S. Dwyer, Errors of matrix computation, simultaneous equations
and eigenvalues, National Bureau of Standarts, Applied Mathematics
Series, 29, pp. 49-58, (1953).

[14] P. S. Fischer, Automatic propagated and round-off error analysis, pa-
er presented at the 13th National Meeting of the Association of Com-
puting Machinery, June (1958).

[15] R. E. Moore, Automatic Error Analysis in Digital Computation,
LSMD-48421, Lockheed Missiles and Space Company, (1959).

[16] R. E. Moore and C. T. Yang, Interval Analysis I, LMSD-285875, Lock-
heed Missiles and Space Company, (1962).

[17] R. E. Moore and C. T. Yang, Theory of an interval algebra and its ap-
plication to numeric analysis, RAAG Memories II, Gaukutsu Bunken
Fukeyu-kai, Tokyo, (1958).

[18] S. Markov, Quasilinear spaces and their relation to vector spaces, Elec-
tronic Journal on Mathematics of Computation, 2 (1), (2005).

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