Time Evolution of Decay Spectrum in $K^0, \overline{K^0} \to \pi^+\pi^-e^+e^-$

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Abstract

We consider the decay $K^0 (\overline{K^0}) \to \pi^+\pi^-e^+e^-$ of a neutral $K$ meson prepared in a state of strangeness $+1 (-1)$. The time evolution of the state produces remarkable time-dependent effects in the angular distribution of the $\pi^+\pi^-e^+e^-$ system. These effects are correlated with the time-dependence of the photon polarization in the radiative decay $K^0 (\overline{K^0}) \to \pi^+\pi^-\gamma$. We study, in particular, the $CP$-odd, $T$-odd term in the distribution $d\Gamma/d\phi$ of the angle between the $\pi^+\pi^-$ and the $e^+e^-$ planes. We also give the spectrum in the case that the decaying meson is an incoherent mixture of $K^0$ and $\overline{K^0}$, and discuss the case of $K_S$ regeneration in a $K_L$ beam.

1 Introduction

The KTeV experiment [1] has measured a large $CP$-violating, $T$-odd asymmetry in the decay $K_L \to \pi^+\pi^-e^+e^-$, in quantitative agreement with a prediction made some years ago [2,3]. The origin of this effect lies in the amplitude of the radiative decay $K_L \to \pi^+\pi^-\gamma$, which contains a bremsstrahlung term proportional to $\eta_{+-}$, as well as a $CP$-conserving direct emission term of magnetic dipole character [4]. The interference of the odd electric multipoles $E1, E3, E5 \cdots$ present in the bremsstrahlung amplitude, which all have $CP = +1$, with the magnetic $M1$ multipole of $CP = -1$, produces $CP$-violating components in the polarization state (Stokes vector) of the photon [5]. The Dalitz pair process $K_L \to \pi^+\pi^-e^+e^-$ acts as an analyser of the photon polarization, exposing the $CP$-odd, $T$-odd component of the Stokes vector. The specific distribution that reveals the $CP$-violation is

$$\frac{d\Gamma}{d\phi} \sim 1 - (\Sigma_3 \cos 2\phi + \Sigma_1 \sin 2\phi)$$ (1)
where $\phi$ is the angle between the $\pi^+\pi^-$ and $e^+e^-$ planes. The last term in Eq. (1) is $CP$-odd and $T$-odd, and produces an asymmetry

$$
A_\phi = \frac{(\int_{\pi/2}^{\pi/2} + \int_{\pi/2}^{3\pi/2} - \int_{3\pi/2}^{2\pi})}{(\int_{\pi/2}^{\pi/2} + \int_{\pi/2}^{3\pi/2} + \int_{3\pi/2}^{2\pi})} \frac{d^2r}{d\phi} = -\frac{2}{\pi} \Sigma_1.
$$

(2)

The measured value [1] $|A_\phi| = (13.6 \pm 2.5 \pm 1.2)\%$ is in excellent agreement with the prediction [2,3] of 14%.

In a recent report [6], the NA48 collaboration, while confirming the large $CP$-violating effect in $K_L \rightarrow \pi^+\pi^-e^+e^-$, has also studied the decay $K_S \rightarrow \pi^+\pi^-e^+e^-$, finding no asymmetry in this case. This is entirely consistent with the fact that the amplitude $K_S \rightarrow \pi^+\pi^-\gamma$ is accurately reproduced by bremsstrahlung alone, so that no electric-magnetic interference is expected.

An interesting question raised by the above observations is the following: How does the asymmetry $A_\phi$ evolve with time if the neutral $K$ meson is prepared in an initial state $K^0$ or $\bar{K}^0$ of definite strangeness? How does the value of $A_\phi$ evolve from zero at short times (when the state decays like $K_S$) to the value $-14\%$ at large times (when the state is essentially $K_L$)? Finally, what type of evolution is expected in the decay of an incoherent $K^0 - \bar{K}^0$ mixture, such as an untagged neutral kaon beam originating in the decay $\phi \rightarrow K^0\bar{K}^0$?

This paper answers these questions by analysing in detail the time dependence of the decays $K^0, \bar{K}^0 \rightarrow \pi^+\pi^-e^+e^-$. We study the evolution of the full decay spectrum, especially the terms that are odd under $CP$. A comparison is made between the behaviour of beams that are initially $K^0, \bar{K}^0$ or an incoherent (untagged) mixture.

As a prelude to the discussion of the channel $K^0, \bar{K}^0 \rightarrow \pi^+\pi^-e^+e^-$, we analyse in Section 2 the time-dependence of the photon polarization in $K^0, \bar{K}^0 \rightarrow \pi^+\pi^-\gamma$. This will reveal the behaviour of the $CP$-odd components of the photon Stokes vector, one of which is reflected in the asymmetry $A_\phi$ in $K^0, \bar{K}^0 \rightarrow \pi^+\pi^-e^+e^-$, that we discuss in Section 3.

## 2 Time Evolution of Photon Polarization in $K^0, \bar{K}^0 \rightarrow \pi^+\pi^-\gamma$

The measured branching ratios and photon energy spectra in the decays $K_L \rightarrow \pi^+\pi^-\gamma$ and $K_S \rightarrow \pi^+\pi^-\gamma$ [4] can be well-described by the matrix elements [2]

$$
\mathcal{M}(K_S \rightarrow \pi^+\pi^-\gamma) = ef_{S} \left[ \frac{\epsilon \cdot p_+}{k \cdot p_+} - \frac{\epsilon \cdot p_-}{k \cdot p_-} \right]
$$


\[ \mathcal{M}(K_L \rightarrow \pi^+ \pi^- \gamma) = e f_L \left[ \varepsilon \cdot p_+ \frac{k \cdot p_+}{k \cdot p_-} - \varepsilon \cdot p_- \frac{k \cdot p_-}{k \cdot p_+} \right] + e f_{DE} \frac{e \mu_{\rho \sigma}}{M_K} \epsilon^\mu k^\nu p_+^\rho p_-^\sigma \] (3)

where

\[ f_S = |f_S| e^{i\delta_0(s=M_K^2)}, \quad f_L = \eta_{++} f_S, \quad f_{DE} = |f_S| g_{M1}, \quad g_{M1} = i(0.76)e^{i\delta_1(s)}. \]

Here \( \delta_{0,1} \) are the \( \pi^+ \pi^- \) phase shifts in the \( I = 0 \) s-wave and \( I = 1 \) p-wave channel, respectively. Introducing the notation [5]

\[ \mathcal{M}(K_{S,L} \rightarrow \pi^+ \pi^- \gamma) = \frac{e|f_S|}{M_K} \{ E_{S,L}(\omega, \cos \theta) [\varepsilon \cdot p_+ k \cdot p_- - \varepsilon \cdot p_- k \cdot p_+] \\
+ M_{S,L}(\omega, \cos \theta) \epsilon_{\mu \nu \rho \sigma} \epsilon^\mu k^\nu p_+^\rho p_-^\sigma \} \] (4)

we have for the electric and magnetic amplitudes

\[ E_S = \left( \frac{2M_K}{\omega} \right)^2 \frac{e^{i\delta_0(s=M_K^2)}}{1 - \beta^2 \cos^2 \theta} M_S = 0 \]
\[ E_L = \left( \frac{2M_K}{\omega} \right)^2 \frac{\eta_{++} e^{i\delta_0(s=M_K^2)}}{1 - \beta^2 \cos^2 \theta} M_L = i(0.76)e^{i\delta_1(s)} \] (5)

As in [5], \( \omega \) is the photon energy in the kaon rest frame, and \( \theta \) is the angle of the \( \pi^+ \) relative to the photon in the \( \pi^+ \pi^- \) c.m. frame. Noting that the strangeness eigenstates \( K^0 \) and \( \bar{K}^0 \) can be written as

\[ K^0 = (K_S + K_L) / N \]
\[ \bar{K}^0 = (K_S - K_L) / N, \] (6)

the decay amplitudes for \( K^0 \) and \( \bar{K}^0 \) at time \( t \) are

\[ \mathcal{M}(K^0(t) \rightarrow \pi^+ \pi^- \gamma) \sim \{ E(t, \omega, \cos \theta) [\varepsilon \cdot p_+ k \cdot p_- - \varepsilon \cdot p_- k \cdot p_+] \\
+ M(t, \omega, \cos \theta) \epsilon_{\mu \nu \rho \sigma} \epsilon^\mu k^\nu p_+^\rho p_-^\sigma \} \]
\[ \mathcal{M}(|\bar{K}^0(t) \rightarrow \pi^+ \pi^- \gamma) \sim \{ |\bar{E}(t, \omega, \cos \theta) [\varepsilon \cdot p_+ k \cdot p_- - \varepsilon \cdot p_- k \cdot p_+] \\
+ |\bar{M}(t, \omega, \cos \theta) \epsilon_{\mu \nu \rho \sigma} \epsilon^\mu k^\nu p_+^\rho p_-^\sigma \} \] (7)

where

\[ E = e^{-i\lambda_S t} E_S(\omega, \cos \theta) + e^{-i\lambda_L t} E_L(\omega, \cos \theta) \]
\[ M = e^{-i\lambda_L t} M_L(\omega, \cos \theta) \]
\[ \overline{E} = e^{-i\lambda S t} E_S(\omega, \cos \theta) - e^{-i\lambda L t} E_L(\omega, \cos \theta) \]
\[ \overline{M} = -e^{-i\lambda L t} M_L(\omega, \cos \theta) \] (8)

with \( \lambda_{S,L} = m_{S,L} - \frac{i}{2} \Gamma_{S,L} \), \( t \) being the proper time.

The amplitudes (8) determine the Stokes vector of the photon in \( K^0, K^0 \rightarrow \pi^+\pi^-\gamma \). For \( K^0 \) decay, the components of the Stokes vector \( \vec{S} = (S_1, S_2, S_3) \) at a time \( t \) are

\[ S_1(t) = \frac{2 \text{Re} [E^*(t)M(t)]}{|E(t)|^2 + |M(t)|^2} \]
\[ S_2(t) = \frac{2 \text{Im} [E^*(t)M(t)]}{|E(t)|^2 + |M(t)|^2} \]
\[ S_3(t) = \frac{|E(t)|^2 - |M(t)|^2}{|E(t)|^2 + |M(t)|^2}. \] (9)

The corresponding components for \( \overline{K^0} \) decay are obtained by replacing \( E(t) \rightarrow \overline{E}(t) \), \( M(t) \rightarrow \overline{M}(t) \).

The physical significance of \( S_2(t) \) is that it represents the net circular polarization of the photon in \( K^0 \rightarrow \pi^+\pi^-\gamma \) at time \( t \):

\[ S_2(t) = \frac{d\Gamma(\lambda_\gamma = -1, t) - d\Gamma(\lambda_\gamma = +1, t)}{d\Gamma(\lambda_\gamma = -1, t) + d\Gamma(\lambda_\gamma = +1, t)}. \] (10)

The parameters \( S_1(t) \) and \( S_3(t) \), on the other hand describe the dependence of the decay rate on the orientation of the polarization vector \( \vec{e} \) relative to \( \vec{n}_\pi \), the unit vector normal to the \( \pi^+\pi^- \) plane:

\[ \frac{d\Gamma(t)}{d\phi} \sim 1 - (S_3(t) \cos 2\phi + S_1(t) \sin 2\phi) \] (11)

where the coordinates are chosen such that

\[ \vec{k} = (0, 0, k), \ \vec{e} = (\cos \phi, \sin \phi, 0), \ \vec{n}_\pi = \frac{\vec{p}_+ \times \vec{p}_-}{|\vec{p}_+ \times \vec{p}_-|} = (1, 0, 0). \] (12)

In Figs. 1a and 2a, we show the components \( S_1(t) \) and \( S_2(t) \) as functions of photon energy for the decay of an initial \( K^0 \). The corresponding Stokes vector components for an initial \( \overline{K^0} \) are shown in Figs. 1b and 2b. Notice the intricate interference effect in the time-dependence, particularly in the region \( t \sim 10\tau_S \). Of special interest is the limiting case \( t \rightarrow \infty \), when the beam is essentially pure \( K_L \). The Stokes vector components \( S_1 \) and \( S_2 \) then reduce to those shown
in [5] for the case $K_L \to \pi^+\pi^-\gamma$. For comparison with the $K^0$ and $\bar{K}^0$ cases, we have also considered the case of an untagged initial beam, consisting of an incoherent equal mixture of $K^0$ and $\bar{K}^0$. The Stokes vector is then

$$
\langle S_1(t) \rangle = \frac{2 \text{Re} \left[ E^*(t)M(t) + E^*(t)M^*(t) \right]}{|E(t)|^2 + |M(t)|^2 + |E(t)|^2 + |M(t)|^2}
$$

$$
\langle S_2(t) \rangle = \frac{2 \text{Im} \left[ E^*(t)M(t) + E^*(t)M^*(t) \right]}{|E(t)|^2 + |M(t)|^2 + |E(t)|^2 + |M(t)|^2}
$$

$$
\langle S_3(t) \rangle = \frac{|E(t)|^2 - |M(t)|^2 + |E(t)|^2 - |M(t)|^2}{|E(t)|^2 + |M(t)|^2 + |E(t)|^2 + |M(t)|^2}.
$$

(13)

The time-dependent parameters $\langle S_1(t) \rangle$ and $\langle S_2(t) \rangle$ are plotted in Figs. 1c and 2c. Notice that the strong fluctuations in $S_1(t)$ and $S_2(t)$ for the $K^0$ and $\bar{K}^0$ cases are now smoothed out. For $t \gg \tau_S$, of course, all three cases give the same Stokes vector, namely that corresponding to $K_L \to \pi^+\pi^-\gamma$.

The polarization components $\langle S_1(t) \rangle$ and $\langle S_2(t) \rangle$ in the incoherent case have a special significance: they represent $CP$-violating observables at any time $t$. In the model under discussion, they vanish when $\eta_{+\bar{+}} = 0$, which is not the case for the components $S_1(t)$ and $S_2(t)$ originating from $K^0$ or $\bar{K}^0$. Furthermore, in the “hermitian limit” [5] (i.e. $\delta_0 = \delta_1 = 0$, arg $\eta_{+\bar{+}} = \frac{\pi}{2}$) the parameter $\langle S_1(t) \rangle$ survives, but $\langle S_2(t) \rangle$ vanishes. This is the hallmark that characterises the observable $\langle S_1(t) \rangle$ as being $CP$-odd, $T$-odd, and the observable $\langle S_2(t) \rangle$ as being $CP$-odd, $T$-even [7].

Of particular relevance in our discussion of $K^0$, $\bar{K}^0 \to \pi^+\pi^-e^+e^-$ is the behaviour of $S_1(t, \omega)$ portrayed in Fig. 1 as a function of photon energy $\omega$. This will be found to have a resemblance with the asymmetry $A_0(t, s_\pi)$ as a function of the $\pi^+\pi^-$ invariant mass (recall that for $K \to \pi^+\pi^-\gamma$, $s_\pi = M_K^2 - 2M_K\omega$). We now proceed to a systematic analysis of time dependence in the Dalitz pair reaction.

3 Time Evolution of $K^0$, $\bar{K}^0 \to \pi^+\pi^-e^+e^-$

We begin by recalling the matrix element of the long-lived kaon decay $K_L \to \pi^+\pi^-e^+e^-$, treating the $e^+e^-$ system as an internal conversion pair associated with the radiative decay $K_L \to \pi^+\pi^-\gamma$ [2,3]. Writing the matrix element in the form

$$
\mathcal{M}(K_L \to \pi^+\pi^-e^+e^-) = -2\frac{G_F}{\sqrt{2}} \sin \theta_C \left\{ \frac{1}{M_K} \left[ f(p_+ + p_-)\lambda + g(p_+ - p_-)\bar{\lambda} \right] \right\}
$$

$$
\text{where } f, g = \text{functions of } p_+, p_-, \lambda.
$$
\[ + i \frac{\hbar}{M_K^2} \epsilon_{\mu\nu\sigma} p_K^\mu (p_+ + p_-)^\nu (p_+ - p_-)^\sigma \] \times \bar{u}(k_-) \gamma^\lambda v(k_+), \]  

(14)

the form factors \( f, g \) and \( h \) are related to the parameters \( f_S, \eta_+ \) and \( g_{M1} \) of the radiative decay as follows [3]:

\[
\begin{align*}
    f &= CM_K^4 |\eta_+| e^{i(\delta(M_K^2) + \varphi_{+})} \frac{4\beta \cos \theta_{\pi}}{s_{\tau} s(1 - \beta^2 \cos^2 \theta_{\pi})} \\
    g &= CM_K^4 |\eta_+| e^{i(\delta(M_K^2) + \varphi_{+})} \frac{4}{s_{\tau} s(1 - \beta^2 \cos^2 \theta_{\pi})} \\
    h &= -CM_K^4 \frac{1}{s_{\tau}} (0.76) e^{i\delta_1(s_{\tau})}
\end{align*}
\]  

(15)

Here \( \theta_{\pi} \) is the angle of the \( \pi^+ \) in the \( \pi^+\pi^- \) rest frame, relative to the sum of the \( e^+ \) and \( e^- \) momenta in that frame; \( s_{\tau} \) is the invariant mass of the lepton pair; \( s_{\pi} \) is the \( \pi^+\pi^- \) invariant mass; and \( s \) is defined as \( s = \frac{1}{2}(M_K^2 - s_{\pi} - s_{\tau}) \).

The constant \( C \) is given by

\[
CM_K^4 = \left(-\frac{G_F}{\sqrt{2}} \sin \theta_C \frac{1}{M_K}\right)^{-1} \pi \alpha |f_S|
\]  

(16)

It is now easy to obtain from the amplitude \( \mathcal{M}(K_L \rightarrow \pi^+\pi^- e^+e^-) \) the corresponding time-dependent amplitudes for \( K^0 \) and \( \overline{K^0} \) decay by making use of the following artifice: To obtain \( \mathcal{M}(K^0(t) \rightarrow \pi^+\pi^- e^+e^-) \), we make the replacement

\[
\begin{align*}
    f &\rightarrow f \left( e^{-i\lambda_L t} + \frac{1}{\eta_+} e^{-i\lambda_{\tau} t} \right) /\mathcal{N} \\
    g &\rightarrow g \left( e^{-i\lambda_L t} + \frac{1}{\eta_+} e^{-i\lambda_{\tau} t} \right) /\mathcal{N} \\
    h &\rightarrow h e^{-i\lambda_L t} /\mathcal{N}
\end{align*}
\]  

(17)

in Eq. (14). Similarly, to obtain \( \mathcal{M}(\overline{K^0}(t) \rightarrow \pi^+\pi^- e^+e^-) \), we make the replacement

\[
\begin{align*}
    f &\rightarrow f \left( -e^{-i\lambda_L t} + \frac{1}{\eta_+} e^{-i\lambda_{\tau} t} \right) /\mathcal{N} \\
    g &\rightarrow g \left( -e^{-i\lambda_L t} + \frac{1}{\eta_+} e^{-i\lambda_{\tau} t} \right) /\mathcal{N} \\
    h &\rightarrow -he^{-i\lambda_L t} /\mathcal{N}
\end{align*}
\]  

(18)
The normalization factors $N$ and $\overline{N}$ in Eqs. (17) and (18) are those in the states $K^0$ and $\overline{K}^0$ (Eq. (6)), where $N = 2p$, $\overline{N} = 2q$, with $|p|^2 - |q|^2 = \frac{2 \text{Re} \eta_+}{1 + |\eta_+|^2}$ and $|p|^2 + |q|^2 = 1$. With this prescription in mind, we will continue to use the formalism developed in [3] for $K_L \to \pi^+\pi^-e^+e^-$, with the understanding that in dealing with $K^0$ and $\overline{K}^0$ decay, the form factors $f$, $g$ and $h$ are to be replaced by the time-dependent combinations given in Eqs. (17) and (18).

Following the procedure of Ref. [3], we have calculated the angular distribution of the decays $K^0(t) \to \pi^+\pi^-e^+e^-$ and $\overline{K}^0(t) \to \pi^+\pi^-e^+e^-$ in the form

$$\frac{d\Gamma}{d\cos \theta_t d\phi} = K_1 + K_2 \cos 2\theta_t + K_3 \sin^2 \theta_t \cos 2\phi + K_9 \sin^2 \theta_t \sin 2\phi.$$ (19)

Here $\theta_t$ is the angle of $e^+$ relative to the dipion momentum vector in the $e^+e^-$ frame; and $\phi$ is the angle between the $\pi^+\pi^-$ and $e^+e^-$ planes. The last term in Eq. (19) is odd under the $CP$-transformation $\vec{p}_\pm \to -\vec{p}_\pm$, $\vec{k}_\pm \to -\vec{k}_\pm$, as well as under the $T$-transformation $\vec{p}_\pm \to -\vec{p}_\pm$, $\vec{k}_\pm \to -\vec{k}_\pm$. The time-dependent coefficients $K_2/K_1$, $K_3/K_1$ and $K_9/K_1$ are shown in Fig. 3, where we compare the cases $K^0$ and $\overline{K}^0$, and also show the result for an incoherent $K^0 - \overline{K}^0$ mixture. This figure depicts the manner in which the coefficients of the angular distribution evolve to their asymptotic values appropriate to the decay $K_L \to \pi^+\pi^-e^+e^-$, namely $K_2/K_1 = 0.297$, $K_3/K_1 = 0.180$, $K_9/K_1 = -0.309$ [3].

Integrating Eq. (19) over $\cos \theta_t$, we obtain the $\phi$-distribution

$$\frac{d\Gamma}{d\phi} \sim \left(1 - \frac{1}{3} \frac{K_2}{K_1}\right) + \frac{2}{3} \left(\frac{K_3}{K_1} \cos 2\phi + \frac{K_9}{K_1} \sin 2\phi\right)$$ (20)

which corresponds to an asymmetry

$$A_\phi = \frac{2}{\pi} \frac{\frac{2}{3} \frac{K_9}{K_1}}{1 - \frac{1}{3} \frac{K_2}{K_1}}$$ (21)

The time-dependence of this asymmetry is exhibited in Fig. 4, which is the answer to the question of how this asymmetry evolves from the value zero, appropriate to $K_S$ decay, to the value $-14\%$ observed for $K_L$ decay. We have also studied the time-dependent asymmetry $A_\phi$ as a function of the $\pi^+\pi^-$ invariant mass. As seen in Fig. 5, the function $A_\phi(t, s_\pi)$ has a similarity to the Stokes vector $S_1(t, \omega)$ plotted in Fig. 1, confirming the expectation that the $CP$-odd, $T$-odd term in the $\phi$-distribution of $K^0 \to \pi^+\pi^-e^+e^-$ is correlated with the $CP$-odd, $T$-odd component of the Stokes vector in $K^0 \to \pi^+\pi^-\gamma$. 

7
Finally, the spectrum-integrated decay rate in \( K^0 \rightarrow \pi^+\pi^-e^+e^- \) and \( \overline{K^0} \rightarrow \pi^+\pi^-e^+e^- \) contains the standard \( K_L - K_S \) interference effect proportional to \( \eta_{+-,} \), and a corresponding asymmetry between \( K^0 \) and \( \overline{K^0} \), which is shown in Fig. 6.

4 Additional Remarks

(i) Our analysis of the time-dependence in \( K^0, \overline{K^0} \rightarrow \pi^+\pi^-e^+e^- \) assumed the amplitude to be entirely determined by the radiative decay \( K^0, \overline{K^0} \rightarrow \pi^+\pi^-\gamma \). A non-radiative contribution to the amplitude \( K_L \rightarrow \pi^+\pi^-e^+e^- \) is possible in the form of a “charge-radius” term, with the characteristic feature of producing \( \pi^+\pi^- \) in an \( s \)-wave. Such a configuration is not possible in the radiative decay \( K_L \rightarrow \pi^+\pi^-\gamma \). We have investigated the effects of a charge radius term nominally parametrized by the coefficient \( g_P = 0.15e^{i\delta_1} \) defined in [3]. It has the interesting consequence of inducing a small term of the form \( \sin 2\theta_l \cos \phi \) in the angular distribution \( d\Gamma/d\cos \theta_l d\phi \) given in Eq. (19). Such a term is \( CP \)-odd but \( T \)-even [3,7]. In Fig. 7, we show the magnitude and time-dependence of the coefficient \( K_4/K_1 \) generated by a charge-radius term \( g_P \) with the nominal value given above.

(ii) Since the time-dependent interference effects in the decay spectrum of \( K^0, \overline{K^0} \rightarrow \pi^+\pi^-e^+e^- \) are strongest in the region \( t \sim 10\tau_s \), one can ask whether similar effects could be observed with a beam of the form \( K_L + \rho K_S \), where \( \rho \) is a regeneration amplitude induced by passage of \( K_L \) through material. We have calculated the decay spectrum of \( (K_L + \rho K_S) \rightarrow \pi^+\pi^-e^+e^- \) as a function of time, for some typical regeneration parameters \( \rho = |\rho|e^{i\varphi_\rho} \), such as \( |\rho| = 0.002, 0.02 \) and \( 0.2 \) with \( \varphi_\rho = -\pi/4 \). In particular, the time-dependent asymmetry \( \mathcal{A}_\phi \) for such a beam is shown in Fig. 8. It is clear that a suitable choice of a regenerator would permit a study of the \( CP \)-odd, \( T \)-odd feature in the \( \pi^+\pi^-e^+e^- \) decay of a neutral K meson, as well as other aspects of the decay spectrum.

(iii) Several refinements in the matrix element of \( K^0, \overline{K^0} \rightarrow \pi^+\pi^-e^+e^- \) can be imagined. There is a small short-distance contribution associated with the interaction \( s\overline{d} \rightarrow e^+e^- \) [3], which, however, is highly suppressed on account of the small quark mixing factor \( V_{ts}V_{td}^* \). The magnetic dipole term in the radiative amplitude \( K_L \rightarrow \pi^+\pi^-\gamma \), represented by the coupling constant \( g_{MI} \), actually has a certain dependence on \( s_\pi \) [8], which can be incorporated into the analysis. Finally, the radiative amplitudes \( K_{LS} \rightarrow \pi^+\pi^-\gamma \) could contain additional components such as a direct \( E1 \) multipole in \( K_S \) decay or a direct \( E2 \) term in \( K_L \) decay (see, e.g. [9]). The former would give rise to a departure from pure bremsstrahlung in the photon energy spectrum of \( K_S \rightarrow \pi^+\pi^-\gamma \), while the latter would produce a \( CP \)-violating charge asymmetry in the Dalitz
plot of $K_L \to \pi^+\pi^\mp\gamma$. There is also the possibility of a direct $CP$-violating $E1$ multipole in $K_L$ decay, that would show up as a difference between $\eta_{+-\gamma}$ and $\eta_{++}$ in the time-dependence of $K_L + \rho K_S \to \pi^+\pi^\mp\gamma$.

A specific consequence of introducing direct $E1$ multipoles in $K_L \to \pi^+\pi^\mp\gamma$ is to modify the amplitude $E_L$ in Eq.(5) to

$$E_L = \left(\frac{2M_K}{\omega}\right)^2 \eta_{+-} e^{i\delta_0} + g_{E1} e^{i(\varphi_{+-} + \delta_1)} + i\bar{g}_{E1} e^{i\delta_1}$$

(22)

where $g_{E1}$ is a measure of direct ($CP$-conserving) $E1$ emission in $K_S \to \pi^+\pi^\mp\gamma$, and $\bar{g}_{E1}$ a measure of a direct $CP$-violating $E1$ emission in $K_L \to \pi^+\pi^\mp\gamma$ ($g_{E1}$ and $\bar{g}_{E1}$ being real). The resulting change in the asymmetry $A_\phi$ in $K_L \to \pi^+\pi^\mp e^+e^-$ is \[2\]

$$A_\phi = 15\% \sin (\varphi_{+-} + \delta_0 - \delta_1) + 38\% \left[ \frac{g_{E1}}{|g_{M1}|} \sin \varphi_{+-} + \frac{\bar{g}_{E1}}{|g_{M1}|} \right].$$

(23)

The observed branching ratio for $K_S \to \pi^+\pi^\mp\gamma$ limits $|g_{E1}/g_{M1}|$ to $< 5\%$ \[2\], while typical estimates of $|\bar{g}_{E1}/g_{M1}|$ are of order $10^{-3}$ \[9\].

Needless to say, any incisive study of the channels $K^0, \bar{K}^0 \to \pi^+\pi^- e^+e^-$ and $K_{L,S} \to \pi^+\pi^-\gamma$ should be alert to the possible presence of correction terms of the form described above.

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Fig. 1. (a) Component $S1$ of the Stokes vector of the photon as a function of photon energy and time for the decay $K^0 \rightarrow \pi^+\pi^-\gamma$; (b) same as (a) but for initial $\overline{K^0}$; (c) same as (a) but for an incoherent equal mixture of $K^0$ and $\overline{K^0}$.

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Fig. 2. (a) Component $S_2$ of the Stokes vector of the photon as a function of photon energy and time for the decay $K^0 \rightarrow \pi^+ \pi^- \gamma$; (b) same as (a) but for initial $\bar{K}^0$; (c) same as (a) but for an incoherent equal mixture of $K^0$ and $\bar{K}^0$. 
Fig. 3. Time-dependent coefficients $K_i/K_1$: (a) $K_2/K_1$ for the decays $K^0$ and $\bar{K}^0 \to \pi^+\pi^- e^+e^-$ as well as for an untagged initial beam; (b) same as (a) but for $K_3/K_1$; (c) same as (a) but for $K_9/K_1$. 
Fig. 4. Time-dependent Asymmetry $A_\phi$ for the decays $K^0$ and $\bar{K}^0 \rightarrow \pi^+\pi^-e^+e^-$ as well as for an incoherent equal mixture.
Fig. 5. (a) The asymmetry $A_\phi$ as a function of $s_\pi$ and time for the decay $K^0 \to \pi^+\pi^-e^+e^-$; (b) same as (a) but for initial $\bar{K}^0$; (c) same as (a) but for an incoherent equal mixture of $K^0$ and $\bar{K}^0$. 
Fig. 6. (a) Spectrum integrated decay rate for $K^0$ and $\overline{K^0} \to \pi^+\pi^- e^+ e^-$; (b) asymmetry in the decay rate $A(t) = \left( N(t) - \overline{N}(t) \right) / \left( N(t) + \overline{N}(t) \right)$.

Fig. 7. Coefficient $K4/K1$ generated by a charge-radius term for $K^0 \to \pi^+\pi^- e^+ e^-$, $\overline{K^0} \to \pi^+\pi^- e^+ e^-$ and an untagged initial beam.
Fig. 8. Time-dependent asymmetry $A_\phi$ for a regenerated beam $K_L + \rho K_S$. 