C\(^1\) Quintic Splines on Domains Enclosed by Piecewise Conics and Numerical Solution of Fully Nonlinear Elliptic Equations

Oleg Davydov\(^*\)        Abid Saeed\(^†\)

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Abstract

We introduce bivariate C\(^1\) piecewise quintic finite element spaces for curved domains enclosed by piecewise conics satisfying homogeneous boundary conditions, construct local bases for them using Bernstein-Bézier techniques, and demonstrate the effectiveness of these finite elements for the numerical solution of the Monge-Ampère equation over curved domains by Böhmer’s method.

1 Introduction

Piecewise polynomials on curved domains bounded by piecewise algebraic curves and surfaces is a promising but little studied tool for data fitting and solution of partial differential equations. Since implicit algebraic surfaces are a well-established modeling technique in CAD [6], we are interested in developing isogeometric schemes [19] for domains with such boundaries, where the geometric models of the boundary are used exactly in the form they exist in a CAD system rather than undergoing a remeshing to fit into the traditional isoparametric finite element approach.

In this paper we continue the work started in [13], where C\(^0\) splines vanishing on a piecewise conic boundary have been introduced. In contrast to both the isoparametric curved finite elements and the isogeometric analysis of [19], our approach does not require parametric patching on curved subtriangles, and therefore does not depend on the invertibility of the Jacobian matrices of the nonlinear geometry mappings. Therefore our finite elements remain piecewise polynomial everywhere in the physical domain.

This approach allows to incorporate conditions of higher smoothness in Bernstein-Bézier form standard for the theory and practice of smooth piecewise polynomials on polyhedral domains [20]. It turns out however that imposing boundary conditions make the otherwise well understood spaces of e.g. bivariate C\(^1\) macro-elements on triangulations significantly more complex. Even in the simplest case of a polygonal domain, the dimension of the space of splines vanishing on the boundary is dependent on its

\(^*\)Department of Mathematics, University of Giessen, Department of Mathematics, Arndtstrasse 2, 35392 Giessen, Germany, oleg.davydov@math.uni-giessen.de

\(^†\)Department of Mathematics, Kohat University of Science and Technology, Kohat, Pakistan, abidsaeed@kust.edu.pk
geometry, with consequences for the construction of stable bases (or stable minimal determining sets) [14, 15].

In this paper we suggest a local basis defined through a minimal determining set for the space of $C^1$ piecewise quintic polynomials vanishing on a piecewise conic boundary and apply the resulting finite element space to the numerical solution of the fully nonlinear Monge-Ampère equation on domains with such boundary. The latter is done within the framework of Böhm’s method [7] which we applied previously on polygonal domains [15]. The results are based in part on the thesis of the second named author [24].

It is important to mention that the isoparametric approach to $C^0$ curved elements is problematic when finite element spaces of $C^1$ or higher smoothness are sought, see the remarks in [10, Section 4.7]. A successful $C^1$ quintic construction of this type developed in [5] seems difficult to extend to higher smoothness or higher polynomial degree.

Remarkably, the standard Bernstein-Bézier techniques for dealing with piecewise polynomials on triangulations [20, 25] as well as recent optimal assembly algorithms [1, 2, 3] for high order elements are carried over to the spaces used here without significant loss of efficiency, see [13].

The paper is organized as follows. The spaces of $C^1$ piecewise polynomials on domains with piecewise conic boundary are introduced in Section 2, whereas Section 3 presents our construction of a local basis for the main space of interest $S_{1,0}(\triangle)$. Section 4 briefly summarizes Böhm’s method for fully nonlinear elliptic equations and presents a number of numerical experiments for the Monge-Ampère equation on smooth domains, including a circular domain, an elliptic domain, and piecewise conic domains with $C^1$ and $C^2$ boundaries.

## 2 $C^1$ piecewise polynomials on piecewise conic domains

We first recall from [13] the assumptions on a domain $\Omega$ and its triangulation $\triangle$ with curved pie-shaped triangles at the boundary.

Let $\Omega \subset \mathbb{R}^2$ be a bounded curvilinear polygonal domain with $\Gamma = \partial \Omega = \bigcup_{j=1}^n \Gamma_j$, where each $\Gamma_j$ is an open arc of an algebraic curve of at most second order (i.e., either a straight line or a conic). For simplicity we assume that $\Omega$ is simply connected. Let $Z = \{z_1, \ldots, z_n\}$ be the set of the endpoints of all arcs numbered counter-clockwise such that $z_j, z_{j+1}$ are the endpoints of $\Gamma_j$, $j = 1, \ldots, n$, with $z_{j+n} = z_j$. Furthermore, for each $j$ we denote by $\omega_j$ the internal angle between the tangents $\tau^+_j$ and $\tau^-_j$ to $\Gamma_j$ and $\Gamma_{j-1}$, respectively, at $z_j$. We assume that $\omega_j > 0$ for all $j$.

Let $\triangle$ be a triangulation of $\Omega$, i.e., a subdivision of $\Omega$ into triangles, where each triangle $T \in \triangle$ has at most one edge replaced with a curved segment of the boundary $\partial \Omega$, and the intersection of any pair of the triangles is either a common vertex or a common (straight) edge if it is non-empty. The triangles with a curved edge are said to be pie-shaped. Any triangle $T \in \triangle$ that shares at least one edge with a pie-shaped triangle is called a buffer triangle, and the remaining triangles are ordinary. We denote by $\triangle_0, \triangle_B$ and $\triangle_P$ the sets of all ordinary, buffer and pie-shaped triangles of $\triangle$, respectively, such that $\triangle = \triangle_0 \cup \triangle_B \cup \triangle_P$ is a disjoint union, see Figure [14]. Let $V, E, V_I, E_I, V_B, E_B$ denote the set of all vertices, all edges, interior vertices, interior edges, boundary vertices and boundary edges, respectively.
For each \( j = 1, \ldots, n \), let \( q_j \in \mathbb{P}_2 \) be a polynomial such that \( \Gamma_j \subset \{ x \in \mathbb{R}^2 : q_j(x) = 0 \} \), where \( \mathbb{P}_d \) denotes the space of all bivariate polynomials of total degree at most \( d \).

By changing the sign of \( q_j \) if needed, we ensure that \( \partial_x q_j < 0 \) for all \( x \) in the interior of \( \Gamma_j \), where \( \nu_x \) denotes the unit outer normal to the boundary at \( x \), and \( \partial_a := a \cdot \nabla \) is the directional derivative with respect to a vector \( a \). Hence, \( q_j(x) \) is positive for points in \( \Omega \) near the boundary segment \( \Gamma_j \). We assume that \( q_j \in \mathbb{P}_1 \) if \( \Gamma_j \) is a straight interval.

Clearly, \( q_j \) is an irreducible quadratic polynomial if \( \Gamma_j \) is a genuine conic arc and in all cases

\[
\nabla q_j(x) \neq 0 \quad \text{if} \quad x \in \Gamma_j.
\]

Let \( Z = \{ z_1, \ldots, z_n \} \subset V_B \).

(b) No interior edge has both endpoints on the boundary.

(c) No pair of pie-shaped triangles shares an edge.

(d) Every \( T \in \Delta_P \) is star-shaped with respect to its interior vertex \( v \).

(e) For any \( T \in \Delta_P \) with its curved side on \( \Gamma_j \), \( q_j(z) > 0 \) for all \( z \in T \setminus \Gamma_j \).

It can be easily seen that (b) and (c) are achievable by a slight modification of a given triangulation, while (d) and (e) hold for sufficiently fine triangulations.

For any \( d \geq 1 \) we set

\[
S^1_d(\triangle) := \{ s \in C^1(\Omega) : s|_T \in \mathbb{P}_{d+i}, \ T \in \triangle_i, \ i = 0,1 \}, \quad \triangle_i := \Delta \cup \Delta_B,
S^{1,2}_d(\triangle) := \{ s \in S^1_d(\triangle) : s \text{ is twice differentiable at any } v \in V_i \},
S^{1,2}_d(\triangle) := \{ s \in S^{1,2}_d(\triangle) : s|_T = 0 \}.
\]

As in [13] we use Bernstein-Bézier techniques to obtain a local basis for \( S^{1,2}_{5,0}(\triangle) \) with the help of a minimal determining set.

Recall (see [20]) that the bivariate Bernstein polynomials with respect to a non-degenerate triangle \( T = \langle v_1, v_2, v_3 \rangle \) with vertices \( v_1, v_2, v_3 \in \mathbb{R}^2 \) are defined by

\[
B_{ijk}^d(v) := \frac{d!}{i!j!k!} b_1^i b_2^j b_3^k, \quad i + j + k = d,
\]
where \( b_1, b_2, b_3 \) are the barycentric coordinates of \( v \), that is the unique coefficients of the expansion \( v = \sum_{i=1}^{3} b_i v_i \) with \( \sum_{i=1}^{3} b_i = 1 \). The Bernstein polynomials form a basis for \( \mathbb{P}_d \), and the coefficients \( c_{ijk} \) in the BB-form expansion

\[
p = \sum_{i+j+k=d} c_{ijk} B^d_{ijk}, \quad p \in \mathbb{P}_d,
\]

are called the BB-coefficients of \( p \). They are conveniently indexed by the elements of the set

\[
D_{d,T} := \left\{ \xi_{ijk} = \frac{iv_1 + jv_2 + kv_3}{d} : \right. \left. i + j + k = d, \ i, j, k \geq 0 \right\}
\]

of so-called domain points, such that \( B^d_\xi := B^d_{ijk} \) and \( c_\xi := c_{ijk} \) when \( \xi = \xi_{ijk} \in D_{d,T} \).

We will also use the notation \( D^d_{2,T}(v) \) for the subset of \( D_{d,T} \) consisting of the six domain points closest to a vertex \( v \) of \( T \), in particular

\[
D^d_{2,T}(v_1) = \{ \xi_{d,0,0}, \xi_{d-1,1,0}, \xi_{d-1,0,1}, \xi_{d-2,2,0}, \xi_{d-2,0,2}, \xi_{d-2,1,1} \}.
\]

The continuity and \( C^1 \)-smoothness of piecewise polynomials are expressed as follows. Given two triangles \( T = \langle v_1, v_2, v_3 \rangle \) and \( \bar{T} = \langle v_4, v_3, v_2 \rangle \) sharing an edge \( e = \langle v_2, v_3 \rangle \), let \( p \) and \( \bar{p} \) be two polynomials of degree \( d \) written in the BB-form

\[
p = \sum_{i+j+k=d} c_{ijk} B^d_{ijk} \quad \text{and} \quad \bar{p} = \sum_{r+s+t=d} \bar{c}_{rst} \bar{B}^d_{rst},
\]

where \( B^d_{ijk} \) and \( \bar{B}^d_{rst} \) are the Bernstein polynomials with respect to \( T \) and \( \bar{T} \), respectively. Then \( p \) and \( \bar{p} \) join continuously along \( e \) if and only if their BB-coefficients over \( e \) coincide, i.e.

\[
\bar{c}_{0jk} = c_{0kj}, \quad \text{for all } j + k = d.
\]

Moreover, the condition for \( C^1 \) smoothness across \( e \) is that (4) holds along with

\[
\bar{c}_{1jk} = b_1 c_{1,k,j} + b_2 c_{0,k+1,j} + b_3 c_{0,k,j+1}, \quad j + k = d - 1,
\]

where \( (b_1, b_2, b_3) \) are the barycentric coordinates of \( v_4 \) relative to \( T \).

A finite set \( \Lambda \) of linear functionals \( \lambda : S^{1,2}_{5,0}(\triangle) \to \mathbb{R} \) is said to be a determining set if

\[
\lambda(s) = 0 \quad \forall \lambda \in \Lambda \quad \implies s = 0,
\]

and \( \Lambda \) is a minimal determining set (MDS) if there is no smaller determining set. In other words, a determining set is a spanning set of the dual space \( (S^{1,2}_{5,0}(\triangle))^* \), and an MDS is a basis of \( (S^{1,2}_{5,0}(\triangle))^* \). Any MDS \( \Lambda \) uniquely determines a basis \( \{ s_\lambda : \lambda \in \Lambda \} \) of \( S \) by duality, such that \( \lambda(s_\mu) = \delta_{\lambda \mu} \), for all \( \lambda, \mu \in \Lambda \), and any spline \( s \in S \) can be uniquely written in the form \( s = \sum_{\lambda \in \Lambda} c_\lambda s_\lambda \), with \( c_\lambda = \lambda(s) \in \mathbb{R} \).

To explain what we mean by a local basis we need some further definitions, compare [11] [13]. The \( \ell \)-star of a set \( A \subset \Omega \) with respect to \( \triangle \) is given by

\[
\star^\ell(A) = \star(A) := \bigcup \{ T \in \triangle : T \cap A \neq \emptyset \}, \quad \star^\ell(A) := \star(\star^{\ell-1}(A)), \ \ell \geq 2.
\]
A set $\omega \subset \Omega$ is said to be a supporting set of a linear functional $\lambda \in (S^{1,2}_{5,0}(\triangle))^*$ if $\lambda(s) = 0$ for all $s \in S^{1,2}_{5,0}(\triangle)$ such that $s|_\omega = 0$. Given an MDS $\Lambda$, we define for each $T \in \Delta$ the set $\Lambda_T := \{ \lambda \in \Lambda : T \subset \text{supp } s_\lambda \}$, where $\{s_\lambda : \lambda \in \Lambda \}$ is the basis of $S^{1,2}_{5,0}(\triangle)$ dual to $\Lambda$. Thus, $\lambda \in \Lambda_T$ if and only if for a spline $s \in S$, $s|_T$ depends on the coefficient $c_\lambda = \lambda(s)$. The covering number $\kappa_\Lambda$ of an MDS $\Lambda$ is the maximum number of elements in $\Lambda_T$ for all $T \in \Delta$.

**Definition 2.1.** A minimal determining set $\Lambda$ for $S^{1,2}_{5,0}(\triangle)$ is said to be $\ell$-local if there is a family of supporting sets $\omega_\lambda$ of $\lambda \in \Lambda$ such that $\omega_\lambda \subset \text{star}^\ell(T)$ for any $T \in \Delta$ such that $\lambda \in \Lambda_T$. If $\Lambda$ is $\ell$-local for some $\ell$, then the dual basis $\{s_\lambda : \lambda \in \Lambda \}$ is said to be local.

It is easy to check, see [13, Lemma 4.3], that if $\Lambda$ is $\ell$-local, then the basis functions $s_\lambda$ are locally supported in the sense that $\text{supp } s_\lambda \subset \text{star}^{2\ell+1}(T)$ for some triangle $T \in \Delta$.

## 3 A local basis for $S^{1,2}_{5,0}(\triangle)$

In this section we describe a minimal determining set $\Lambda$ for $S^{1,2}_{5,0}(\triangle)$, which in turn defines a basis $\{s_\lambda : \lambda \in \Lambda \}$ as explained in the previous section. For the sake of simplicity we describe the basis under the following additional assumption:

(f) All boundary edges are curved.

(g) No pair of buffer triangles shares an edge.

In fact we have implemented our bases also for the case where some boundary edges are straight. (It is used in Test Problem 4 in Section 4.3.) In this case we nevertheless assume that the triangle attached to a straight boundary edge is ordinary, and no pie-shaped triangle shares an edge with it, as in Figure 5. A description of this construction would take too much space because it has to include the handling of the boundary vertices on ordinary polygonal triangulations along the lines of [14,15], and so we avoid this by assuming (f). Similarly, allowing buffer triangles to share edges, or equivalently, allowing more than one buffer triangle attached to a single boundary vertex would produce additional degrees of freedom on and near these edges and around the boundary vertex, also requiring the techniques of [14,15].

We denote by $V^1_B$ the set of those boundary vertices $v \in V_B$ where the boundary $\partial \Omega$ has a well-defined tangent, that is either $v \notin Z$, or $\omega_j = \pi$ if $v = z_j$ for some $j = 1, \ldots, n$. In addition, $E_{P,B}$ denotes the set of all edges shared by a pie-shaped and a buffer triangle. We also set $E^0_I := E_I \setminus E_{P,B}$.

Since splines in $S^{1,2}_{5,0}(\triangle)$ are polynomials of degree $d = 5$ on the triangles $T \in \Delta_0$, we can write these polynomials in BB-form (2),

$$s|_T = \sum_{\xi \in D_{5,T}} c_\xi B^5_\xi, \quad s \in S^{1,2}_{5,0}(\triangle).$$

and define for each $\xi = \xi_{ijk} \in D_{5,T}$ a functional $\lambda_\xi \in (S^{1,2}_{5,0}(\triangle))^*$ that picks the BB-coefficient $c_{ijk}$ in (2). With the usual convention (see [20]) we identify the functional $\lambda_\xi$
with the domain point \( \xi \) and speak of an MDS as a set \( M \subset \mathbb{T} \). Thanks to \([4]\) for domain points \( \xi \) at vertices or on the edges of the subtriangulation \( \triangle_0 \) it does not matter which triangle in \( \triangle_0 \) containing \( \xi \) is used to evaluate the BB-form of a spline \( s \in S^{1,2}_{5,0}(\triangle) \). The union \( D_{5,\triangle_0} = \cup_{T \in \triangle_0} D_{5,T} \) forms the standard set of domain points (and corresponding functionals \( \lambda_\xi \)) associated with \( \triangle_0 \). Following the standard construction of an MDS for the space \( S^{1,2}_{5,0}(\triangle_0) \) with only ordinary triangles \([20]\), we define the following subsets of \( D_{5,\triangle_0} \). For each \( v \in V_I \) we choose a triangle \( T_v = \langle v_1, v_2, v_3 \rangle \in \triangle_0 \) attached to \( v \), such that \( v_1 = v \), and set \( M_v := D^{5,T_v}_5(v) = \{ \xi_{500}, \xi_{410}, \xi_{101}, \xi_{320}, \xi_{302}, \xi_{311} \} \subset D_{5,T_v} \). For each edge \( e \in E_0^0 \), let \( T_e := \langle v_1, v_2, v_3 \rangle \) be a triangle in \( \triangle_0 \) attached to the edge \( e = \langle v_2, v_3 \rangle \) and let \( M_e := \{ \xi_{122} \} \subset D_{5,T_e} \). Clearly, \( \omega_\xi := T_v \) (resp. \( \omega_\xi := T_e \)) is a supporting set for any functional \( \lambda_\xi \) with \( \xi \in M_v \) (resp. \( \xi \in M_e \)).

For each \( T \in \triangle_P \), with its curved edge \( e \) given by the equation \( q(x) = 0 \), where \( q \in \mathbb{P}_2 \setminus \mathbb{P}_1 \) is irreducible and normalized so that \( q \langle v \rangle = 1 \) for the interior vertex \( v \) of \( T \), we notice that by Bézout theorem

\[
\{ s \in \mathbb{P}_6 : s|_e = 0 \} = q\mathbb{P}_4 := \{ qp : p \in \mathbb{P}_4 \}.
\]

Let \( T^* \) denote the triangle obtained by joining the boundary vertices of \( T \) by a straight line segment (see the dashed line in Figure \([2]\)). Since the Bernstein polynomials \( B^3_{ijk} \), \( i + j + k = 4 \), w.r.t. \( T^* \) form a basis for \( \mathbb{P}_4 \) it is obvious that the set

\[
\{ qB^3_{ijk} : i + j + k = 4 \}
\]

is a basis for \( q\mathbb{P}_4 \). The set of domain points of degree 4 over \( T^* \) will be denoted \( D^*_4,T \). Even though the set \( D^*_4,T \) formally coincides with \( D_{4,T^*} \), the linear functionals associated with the domain points are different. Namely, each \( \xi \in D^*_4,T \) represents a linear functional \( \lambda_\xi \) on \( S^{1,2}_{5,0}(\triangle) \) which picks the coefficient \( c_\xi \) in the expansion

\[
s|_T = q \sum_{\xi \in D^*_4,T} c_\xi B^3_{\xi}, \quad s \in S^{1,2}_{5,0}(\triangle). \tag{7}
\]

Assuming that \( v_1, v_2, v_3 \) are the vertices of a pie-shaped triangle \( T \in \triangle_P \), with \( v_1 \in V_I \), we set \( M^P_T := \{ \xi_{130}, \xi_{121}, \xi_{112}, \xi_{103}, \xi_{320}, \xi_{302}, \xi_{311} \} \subset D^*_4,T \), see Figure \([2]\) where the points in \( M^P_T \) are marked as black squares. Clearly, \( \omega_\xi := T \) is a supporting set for \( \lambda_\xi \). The vertices \( v_2, v_3 \) of \( T \) are shared by a pair of pie-shaped triangles and may belong to \( V^1_B \). For each \( v \in V^1_B \) let \( M^P_T := \{ v \} \subset D^*_4,T \), where \( T_v \) is one of the two pie-shaped triangle attached to \( v \), and the corresponding functional is \( \lambda_v \) that picks the respective coefficient \( c_v \) in \( \{ qB^3_{ijk} : i + j + k = 4 \} \) for \( T = T_v \). A supporting set for \( \omega_v := T_v \).

For each \( T = \langle v_1, v_2, v_3 \rangle \in \triangle_B \), where \( v_1 \in V_B \), let \( M^B_T := \{ \xi_{411}, \xi_{222} \} \subset D_{6,T} \), see Figure \([3]\) As usual, the functional \( \lambda_\xi \) identified with \( \xi \in D_{6,T} \) picks the coefficient \( c_\xi \) in the BB-form expansion of \( s|_T \in \mathbb{P}_6 \),

\[
s|_T = \sum_{\xi \in D_{6,T}} c_\xi B^6_\xi, \quad s \in S^{1,2}_{5,0}(\triangle), \tag{8}
\]

and \( \omega_\xi := T \) is a supporting set for any \( \xi \in M^B_T \).
Figure 2: The set $D_{4,T}^*$ for a pie-shaped triangle $T$ and domain points in $M_T^P$ (black squares), and $M_{v_2}^P \cup M_{v_3}^P$ (black triangles) under the assumption that $v_2, v_3 \in V_B^1$ and $T = T_{v_2} = T_{v_3}$.

Figure 3: The domain points in the sets $M_T^B, M_T^B$ for the buffer triangles $T_1, T_2$ are marked with black dots.
Remark 3.1. Let $T := \langle v_1, v_2, v_3 \rangle \in \triangle_P$ with $v_1 \in V$. Then $s|_T = qp \in P_6$ for some $p \in P_4$, where the equation $q(x) = 0$ represents the curved edge of $T$, with an irreducible quadratic polynomial $q$ such that $q(v_1) = 1$. We can write all three polynomials $s|_T, q, p$ in BB-form with respect to $T^*$,

$$q = q_{110}B_{110}^2 + q_{101}B_{101}^2 + q_{011}B_{011}^2 + B_{200}^2$$

(9)

(where we used the fact that $q(v_2) = q(v_3) = 0$),

$$s|_T = \sum_{i+j+k=6} a_{ijk}B_{ijk}^6, \quad p = \sum_{i+j+k=4} c_{ijk}B_{ijk}^4.$$  

If the coefficients $c_{ijk}$ are known, then $a_{ijk}$ can be computed by multiplying the expansions for $p$ and $q$, see the explicit formulas in [13, Eq. (35)], where a different numeration of the vertices of $T$ is used. Moreover, the coefficients $c_{ijk}$ can be obtained from $a_{ijk}$ in a stable way [13, Lemma 4.6]. To compute $c_{ijk}$ we may write down the identity

$$(\sum_{i+j+k=4} c_{ijk}B_{ijk}^4)(\sum_{i+j+k=2} q_{ijk}B_{ijk}^2) = \sum_{i+j+k=6} a_{ijk}B_{ijk}^6$$

(10)

as a linear system with respect to the vector of unknown coefficients $c_{ijk}$, $i+j+k = 4$.

It is easy to check that the matrix of this system has a block structure, and by singling out the six rows of the system corresponding to the domain points in $D_2^{6,T^*}(v_1)$ we obtain a non-singular triangular linear system for the coefficients $c_{ijk}$ corresponding to the domain points in $D_2^{4,T^*}(v_1)$, namely

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2}q_{110} & \frac{8}{15} & 0 & 0 & 0 & 0 \\
\frac{1}{3}q_{101} & 0 & \frac{8}{15} & 0 & 0 & 0 \\
0 & \frac{1}{3}q_{110} & 0 & \frac{2}{3} & 0 & 0 \\
\frac{1}{2}q_{101} & \frac{1}{3}q_{101} & 0 & \frac{1}{3} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{8}{15} & 0
\end{bmatrix}
\begin{bmatrix}
c_{400} \\
c_{310} \\
c_{301} \\
c_{211} \\
c_{202} \\
c_{a402}
\end{bmatrix}
= 
\begin{bmatrix}
 a_{600} \\
a_{510} \\
a_{501} \\
a_{411} \\
a_{420} \\
a_{a402}
\end{bmatrix}.
$$

Thus, we can compute the BB-coefficients $\{c_\xi : \xi \in D_2^{4,T^*}(v_1)\}$ of $p$ by using only the BB-coefficients $\{a_\xi : \xi \in D_2^{6,T^*}(v_1)\}$ of $s|_T$.

Theorem 3.2. The set

$$M := \bigcup_{v \in V} M_v \cup \bigcup_{e \in E} M_e \cup \bigcup_{v \in V_B} M_v^B \cup \bigcup_{T \in \triangle_P} M_T^P \cup \bigcup_{T \in \triangle_B} M_T^B$$

(11)

is a 1-local minimal determining set for the space $S_{5,0}^{1,2}(\triangle)$.

Proof. Following the standard scheme [20] we assign some arbitrary values $c_\xi \in \mathbb{R}$ to $\lambda_\xi(s)$, for all $\xi \in M$, and show that all other coefficients $c_\xi$ of $s \in S_{5,0}^{1,2}(\triangle)$ on all triangles $T \in \triangle$ in the form [14], [17] or [15] depending on the type of $T$, can be determined from them consistently. The success of this process will show that $M$ is an MDS. In the same time we will keep track how far the influence of a coefficient $c_\xi$ for any $\xi \in M$ extends, to check the locality of this MDS.
It is easy to see that the set
\[
M_0 := \bigcup_{v \in V_I} M_v \cup \bigcup_{e \in E^o_I} M_e
\]
is a 1-local MDS for the Argyris finite element space
\[
S^{1,2}_d(\Delta_0) := \{ s \in S^1_d(\Delta_0) : s \text{ is twice differentiable at any vertex } v \text{ of } \Delta_0 \}
\]
as shown in [20, Theorem 6.1].

Let \( v \in V_I \) be shared by some pie-shaped triangle \( T \in \Delta_P \). Then there are also two buffer triangles \( T_1, T_2 \in \Delta_B \) attached to \( v \), see Figures 1 and 3. We know that \( M_v = D^{5,T_v}_2(v) \subset D^{5,T_v}_5 \) for some \( T_v \in \Delta_0 \). By [20, Lemma 5.10] and the degree raising formulas of [20, Theorem 2.39], \( M_v \) consistently determines the BB-coefficients of \( s|_{T \cup T_1 \cup T_2} \) in \( D^{5,T}_2(v) \cup D^{5,T_1}_6(v) \cup D^{5,T_2}_6(v) \). For the pie-shaped triangle \( T \) we need to go one more step and compute the BB-coefficients in \( D^{4,T^*}_2(v) \) of the polynomial \( p \in \mathbb{P}_4 \) such that \( s|_T = pq \), where the equation \( q(x) = 0 \) represents the curved edge of \( T \). This can be done uniquely by solving the triangular linear system described in Remark 3.1.

Let \( e = \langle v_2, v_3 \rangle \in E^0_I \) be shared by an ordinary triangle \( T_e := \langle v_1, v_2, v_3 \rangle \in \Delta_0 \) and a buffer triangle \( T = \langle v_4, v_3, v_2 \rangle \in \Delta_B \). Assuming that the BB-coefficients of \( s|_{T_e} \) for all domain points in \( D_{5,T_e} \) have been computed as described above, we can use degree raising to write \( s|_{T_e} \) as a polynomial of degree six, and obtain its BB-coefficients for all domain points in \( D_{6,T_e} \). By using the continuity and \( C^1 \) smoothness conditions (1), (5) we can then compute the BB-coefficients of \( s|_T \) for all those domain points \( \xi_{ijk} \in D_{6,T} \), for which \( i \in \{0,1\} \). Some of them have already been computed at the previous step, namely those that belong to \( D^{5,T}_2(v_2) \cup D^{5,T}_2(v_3) \). It is known that no inconsistencies arise this way, see for example the proof of [20, Theorem 6.1]. We thus obtain three new BB-coefficients of \( s|_T \) corresponding to the domain points \( \xi_{033}, \xi_{132}, \xi_{123} \in D_{6,T} \).

Let \( v \in V_B \) and let \( T_1, T_2 \in \Delta_P \) be the two pie-shaped triangles attached to \( v \), with the curved edges given by \( q_1(x) = 0 \) and \( q_2(x) = 0 \), respectively. Let \( p_1, p_2 \in \mathbb{P}_4 \) be such that \( s|_{T_i} = p_i q_i \), \( i = 1, 2 \). Since \( s \) is continuously differentiable at \( v \) and \( q_1(v) = q_2(v) = 0 \), we have \( \nabla s(v) = p_1(v) \nabla q_1(v) + p_2(v) \nabla q_2(v) \). If \( v \in V_B \setminus V^1_B \), then the vectors \( \nabla q_1(v) \) and \( \nabla q_2(v) \) are linearly independent, and it follows that \( p_1(v) = p_2(v) = 0 \), that is \( c_v = 0 \) in (7) for both \( T_1 \) and \( T_2 \). We now assume that \( v \in V^1_B \). Then \( \nabla q_1(v) = \alpha \nabla q_2(v) \) for some real \( \alpha \neq 0 \), which implies \( p_2(v) = \alpha p_1(v) \). Let \( T_1 = T_v \) be the triangle in the definition of \( M^P_v \), in particular the functional \( \lambda_v \) is evaluated as \( \lambda_v(s) = p_1(v) \). Thus, the value \( c_v \) in (7) for \( T = T_1 \) is known because \( M^P_v \) is part of the MDS \( M \), and the value of the BB-coefficient of \( p_2 \) at the same point \( v \) is \( \alpha c_v \). To compute \( \alpha \), we just need to compare the components of the vectors \( \nabla q_1(v) \) and \( \nabla q_2(v) \), which is easy to do by using the BB-forms (9) of \( q_1, q_2 \) with respect to \( T_1, T_2 \), respectively.

Let \( T_1 = \langle v_1, v_2, v_3 \rangle \in \Delta_B \) with \( v_1 \in V_B \) and \( e = \langle v_1, v_3 \rangle \in E_{P,B} \), and let \( T_2 := \langle v_2, v_4, v_1 \rangle \in \Delta_P \) share the edge \( e \) with \( T \) and has its curved edge defined by the equation \( q(x) = 0 \). Let us write the polynomials \( s|_{T_1}, s|_{T_2} \) and \( p \in \mathbb{P}_4 \) in \( s|_{T_2} = pq \) in the BB-form as
\[
s|_{T_1} = \sum_{\xi \in D_6,T_1} c_\xi B^6_\xi, \quad s|_{T_2} = \sum_{\xi \in D_6,T_2} a_\xi B^6_\xi, \quad p = \sum_{\xi \in D_4,T_2} e_\xi B^4_\xi.
\]
Since the domain point $\xi_{103} \in D^*_{4,T_2}$ belongs to $M^P_{T_2}$ and the coefficients $c_\xi$ for all other $\xi \in D^*_{4,T_2} \cap e$ have been determined above, $s|_e$ is completely determined, and the BB-coefficients $a_\xi$ for all $\xi \in D^*_{6,T_2} \cap e$ can be found by the multiplication of $p|_e$ by $q|_e$. Hence the smoothness conditions (4) and (5) across $e$ give us in particular the equation

$$a_{114} = b_1 \tilde{c}_{501} + b_2 \tilde{c}_{411} + b_3 \tilde{c}_{402} = b_1 a_{105} + b_2 \tilde{c}_{411} + b_3 a_{204},$$

where $(b_1, b_2, b_3)$ are the barycentric coordinates of $v_4$ w.r.t. $T_1$, which determines $a_{114}$ since $\xi_{411} \in D^*_{6,T_1}$ belongs to $M^B_{T_1}$. Moreover, comparing the coefficients of $B^6_{114}$ on both sides of (10) leads to the equation

$$15a_{114} = q_{110} c_{004} + 4q_{101} c_{013} + 4q_{011} c_{103},$$

and hence $c_{013}$ can be computed from the already known BB-coefficients as

$$c_{013} = \frac{1}{4q_{01}} (15a_{114} - q_{110} c_{004} - 4q_{011} c_{103}).$$

Note that $q_{01} \neq 0$ thanks to (1). Similarly, $c_{031}$ is computed using the same argument involving the buffer triangle attached to $v_4$. This completes the computation of the BB-form of $p$. By multiplying it with $q$ we get the missing coefficients of the BB-form of $s|_{T_2}$, and by the smoothness conditions across $e$ the BB-coefficients $\tilde{c}_{312}$ and $\tilde{c}_{213}$ of $s|_{T_1}$. The remaining unset BB-coefficients of $s|_{T_1}$ are obtained in the same way by using the pie-shaped triangle sharing the edge $(v_1, v_2)$ with $T_1$.

A close inspection of the above argumentation shows that $M$ is 1-local in the sense of Definition 2.1.

An example of the MDS of Theorem 3.2 for the space $S^{1,2}_{5,0}(\triangle)$ over a triangulation of a circular disk is depicted in Figure 4, where the points in the sets $\bigcup_{v \in V} M_v$, $\bigcup_{e \in E} M_e$, $\bigcup_{v \in V_B} M^P_v$, $\bigcup_{T \in T_B} M^P_T$ and $\bigcup_{T \in T_B} M^B_T$ are marked as black dots, diamonds, triangles, squares and downward pointing triangles, respectively. Note that $V^1_B = V_B$ in this example.

### 4 Numerical solution of fully nonlinear elliptic equations

To evaluate the performance of our construction of $C^1$ elements for curved domains we implemented Böhmmer’s method for fully nonlinear equations using $S^{1,2}_{5,0}(\triangle)$ as the finite element approximation space.

#### 4.1 Böhmmer’s method

We consider the Dirichlet problem,

$$\text{find } u : \Omega \to \mathbb{R} \text{ such that } G(u) = 0 \text{ and } u|_{\partial \Omega} = \phi, \quad (12)$$

for a second order differential operator of the form $G(u) = \tilde{G}(\cdot, u, \nabla u, \nabla^2 u)$, where $\tilde{G} = \tilde{G}(w)$, $w = (x, z, p, r) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^{2 \times 2}$ is a real valued function defined on a
domain $\tilde{\Omega} \times \Gamma$ such that $\Omega \subset \tilde{\Omega} \subset \mathbb{R}^2$ and $\Gamma \subset \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^{2 \times 2}$, where $\nabla u, \nabla^2 u$ denote the gradient and the Hessian of $u$, respectively. The operator $G$ is said to be elliptic in a subset $\tilde{\Gamma} \subset \tilde{\Omega} \times \Gamma$ if the matrix $[\partial G_{ij}(w)]_{i,j=1}^{2}$ is well defined and positive definite for all $w \in \tilde{\Gamma}$ [8]. Under certain assumptions, including the exterior sphere condition for $\partial \Omega$, the continuity of $\phi : \partial \Omega \to \mathbb{R}$ and sufficient smoothness of $G$, the problem (12) has a unique solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ if $\tilde{\Gamma} = \tilde{\Omega} \times \Gamma$ [18, Theorem 17.17].

The most famous example of a fully nonlinear elliptic operator which is neither quasilinear nor semilinear [8, p. 80] is the Monge-Ampère operator $G(u) := \det(\nabla^2 u) - g$, where $g : \Omega \to \mathbb{R}$ satisfies $g(x) > 0$ for all $x \in \Omega$. In this case $\tilde{\Gamma} = \tilde{\Omega} \times \mathbb{R} \times \mathbb{R}^2 \times \{r \in \mathbb{R}^{2 \times 2} : r \text{ is positive definite}\}$. Under the assumptions that $\partial \Omega$ is $C^3$ and $g \in C^2(\bar{\Omega})$ there exists a unique convex solution $u$ of (12) such that $u \in C^{2,\alpha}(\bar{\Omega})$ for all $\alpha < 1$ [18, Theorem 17.22]. References to further results about the existence and uniqueness of the solution of (12) can be found in [3] Section 2.5.7.

Many fully nonlinear elliptic operators and corresponding equations $G(u) = 0$ are important for applications, see [5]. Several numerical methods have been proposed in the literature, in particular finite difference [16, 23] and finite element type methods [4, 7, 9, 15, 17, 21, 22]. To the best of our knowledge however, no method has been tested before on non-polygonal domains.

Finite element spaces $S^1_0 \subset C^1(\bar{\Omega})$ satisfying homogenous boundary conditions on $\Omega$, where $h$ is the maximum diameter of the underlying partition $\Delta^h$, can be employed in Böhm’s method [7, 8] for the problem (12). For a fixed $h > 0$, let $u^h_0 : \Omega \to \mathbb{R}$ be an initial guess satisfying the boundary condition $u^h_0|_{\partial \Omega} = \phi$. We generate a sequence of functions $\{u^h_k\}_{k \in \mathbb{N}}$ by the Newton type method

$$u^h_{k+1} = u^h_k - u^h, \quad k = 0, 1, \ldots,$$  \hspace{1cm} (13)
where \( u^h \in S^h_0 \) is the Galerkin approximation of the linear elliptic problem

\[
G'(u^h_k)u = G(u^h_k),
\]

that is \( u^h \in S^h_0 \) is determined by the equations

\[
(G'(u^h_k)u^h, v^h)_{L^2(\Omega)} = (G(u^h_k), v^h)_{L^2(\Omega)} \quad \forall v^h \in S^h_0,
\]

where \((\cdot, \cdot)_{L^2(\Omega)}\) denotes the usual inner product in \( L^2(\Omega) \), and \( G'(u^h_k) \) is the linearization of the operator \( G \) at \( u^h_k \) given by

\[
G'(u^h_k)u = \frac{\partial G}{\partial z}(u^h_k)u + \sum_{i=1}^{2} \frac{\partial G}{\partial p_i}(u^h_k) \frac{\partial u}{\partial x_i} + \sum_{i,j=1}^{2} \frac{\partial G}{\partial r_{ij}}(u^h_k) \frac{\partial^2 u}{\partial x_ix_j},
\]

with \( w^h_k(x) := (x, u^h_k(x), \nabla u^h_k(x), \nabla^2 u^h_k(x)), x \in \Omega \). Clearly, (15) can be reformulated into the standard weak form of the Galerkin method: Find \( u^h \in S^h_0 \) such that for all \( v^h \in S^h_0 \),

\[
\int_{\Omega} \nabla u^h \cdot A \nabla v^h \, dx + \int_{\Omega} v^h b \cdot \nabla u^h \, dx + \int_{\Omega} cu^h v^h \, dx = \int_{\Omega} f v^h \, dx,
\]

where \( A = [\frac{\partial\tilde{G}}{\partial r_{ij}}(w^h_k)]^2_{i,j=1}, b = [\frac{\partial\tilde{G}}{\partial p_i}(w^h_k)]^2_{i=1}, c = \frac{\partial\tilde{G}}{\partial z}(w^h_k) \) and \( f = G(u^h_k) \).

Under some additional assumptions on \( G \), satisfied in particular by the Monge-Ampère operator, it is proved in [8, Theorem 5.2] and [7, Theorem 9.1] that \( u^h_k \) converges quadratically (as \( k \to \infty \)) to a unique function \( \hat{u}^h \) satisfying the nonlinear equations

\[
(G(\hat{u}^h), v^h)_{L^2(\Omega)} = 0 \quad \forall v^h \in S^h_0,
\]

such that \( \hat{u}^h - u^0_0 \in S^h_0 \), if the initial guess \( u^0_0 \) is close enough to \( \hat{u}^h \). Moreover, \( \hat{u}^h \) converges to the solution \( u \) of (12) in \( H^2 \)-norm as \( h \to 0 \) if \( u \in H^r(\Omega) \) for some \( r > 2 \) and the spaces \( S^h_0 \) possess appropriate approximation properties for functions vanishing on \( \partial\Omega \). Note that suitable approximation error bounds for the spaces \( S^h_0 = S_{5,0}^{1,2}(\triangle) \) have yet to be proved, see the results of [13, Section 3] for the spaces of continuous piecewise polynomials vanishing on a piecewise conic boundary. The stability of the MDS of Theorem 3.2 and the dual local basis, related to the approximation power of the space [12], has been addressed in [24].

Note that in the case when \( G \) is only conditionally elliptic (e.g. elliptic only for a convex \( u \) for Monge-Ampère equation) the ellipticity of the linear problem (14) is only guaranteed if \( u^h_k \) satisfies the respective side condition \( (x, u(x), \nabla u(x), \nabla^2 u(x)) \in \bar{\Gamma} \) for all \( x \in \Omega \). For the Monge-Ampère equation the side condition of convexity holds for \( u^h_k \) if its second order derivatives are sufficiently close to those of the exact solution \( \hat{u} \).

### 4.2 Implementation issues

The standard techniques of the finite element method allow efficient computation of the solution \( u^h \) of (17) using the local basis of \( S^h_0 = S_{5,0}^{1,2}(\triangle) \) described in Section 3. Moreover, efficient assembly algorithms for the polynomial Bernstein-Bézier shape functions
4.3 Numerical results

In the experiments we focus on the Dirichlet problem for the prototypical and best studied Monge-Ampère equation,

\[ G(u) = \det(\nabla^2 u) - g = 0, \quad u|_{\partial\Omega} = \phi, \]

with \( g(x) > 0, \ x \in \Omega \), where the solution \( u : \Omega \to \mathbb{R} \) is assumed to be convex for the sake of uniqueness.

We choose a number of test problems with a curved domain \( \Omega \) bounded by piecewise conics, a positive function \( g \) and \( \phi = 0 \). As in [13, Section 6], starting from an initial triangulation of \( \Omega \), we obtain a sequence of quasi-uniform triangulations \( \Delta^h \) by uniform refinement, whereby each triangle is subdivided into four triangles by joining the midpoints of every edge. For each \( h \), we use Böhm's method described above, with \( S_h = S_{5,0}^{1,2}(\Delta^h) \). To solve (17) we use the 1-local basis corresponding to the MDS of Theorem 3.2.

We follow the suggestion of [16, Remark 2.1] to use an approximate solution of the Poisson problem

\[ \Delta u = 2\sqrt{g}, \quad u|_{\partial\Omega} = \phi, \]

as initial guess in the iterative schemes for the Monge-Ampère equation (18). Since \( \phi = 0 \), we choose the initial guess \( u_0^h \) in the same space \( S_{5,0}^{1,2}(\Delta^h) \) and obtain it by the standard Galerkin method. However, as in [15], we get much faster convergence of the Newton iteration (13) by a multilevel approach, where this initial guess is only used on the initial triangulation, whereas on the refined triangulations a quasi-interpolant [20, Section 5.7] of the last iterate from the previous level serves as an initial guess \( u_0^h \). As a stopping criteria for Newton iterations (13) on each level the following condition is employed:

\[ \|u_k^h - u_{k+1}^h\|_{L^2(\Omega)} < 10^{-15}. \]  

**Test Problem 1.** Equation (18) in the unit disk \( \Omega \) centered at the origin with \( g \) chosen such that the exact solution is \( u = e^{0.5(x^2_1+x^2_2)} - e^{0.5} \).

We use the same initial triangulation of the disk as in [13, Example 2], see Figure 12 in [13]. The numerical results for Test Problem 1 are presented in Table 1 which shows the \( L^2 \), \( H^1 \) and \( H^2 \) norms of the error \( e_\ell = u_\ell - u \) of the last iterate \( u_\ell = u_{m\ell}^h \) on level \( \ell \) against the exact solution and the number \( m \) of iterations \( [13] \) for levels \( \ell = 1, \ldots, 6 \), where \( \ell = 1 \) corresponds to the initial triangulation. In addition, the first row of the table contains the errors of the initial guess obtained by solving (19) on the initial triangulation. The rate of convergence between levels is estimated by the usual formula \( \log_2(\|e_{\ell-1}\|/\|e_\ell\|)) \).

The results show the convergence rates approaching \( O(h^6) \), \( O(h^5) \) and \( O(h^4) \) for the \( L^2 \), \( H^1 \) and \( H^2 \) norms, respectively, which is expectable since the solution \( u \) is
Table 1: Errors of the approximate solution and the rate of convergence for Test Problem 1 on the unit disk. \( \ell \) indicates the level of refinement of the initial triangulation, and \( m \) is the number of Newton iterations (13) on the \( \ell \)-th level. The row marked ‘init’ gives the errors of the initial guess on level 1.

\[
\begin{array}{|c|c|c|c|c|}
\hline
\ell & L^2\text{-error rate} & H^1\text{-error rate} & H^2\text{-error rate} & m \\
\hline
\text{init} & 1.04e-2 & 3.20e-2 & 1.85e-1 & \\
1 & 2.12e-6 & 3.84e-5 & 1.25e-3 & 2 \\
2 & 2.98e-7 & 8.47e-6 & 1.25e-3 & 2 \\
3 & 2.98e-7 & 2.8 & 3.35e-4 & 1.9 \\
4 & 6.79e-9 & 3.87e-7 & 1.47e-4 & 1.9 \\
5 & 6.79e-9 & 5.5 & 2.86e-5 & 3.6 & 1 \\
6 & 1.36e-10 & 5.6 & 2.12e-6 & 3.8 & 1 \\
\hline
\end{array}
\]

Test Problem 2. Equation (18) with \( g(x) = e^{x_1} \) and \( \phi = 0 \) in the elliptic disk \( \Omega \) with the boundary given by the equation \( x_1^2 + 6.25x_2^2 = 1 \). The initial triangulation is the same as the one used in [13, Example 1 and Figure 7]. The results are presented in Table 2. Since the exact solution \( u \) is not known, we use alternative measures to estimate the error. One is the residual

\[ R = \|G(u_k^h)\|_{L^2(\Omega)}, \quad (21) \]

and another is the \( L^2, H^1 \) and \( H^2 \) norms of the difference \( \varepsilon_\ell := u_\ell - u_{\ell+1} \) between the approximate solutions \( u_\ell, u_{\ell+1} \) of two consecutive levels. Note that in the case that \( u_\ell \) converges to \( u \) at least linearly in some norm, we may assumes that \( \|u - u_{\ell+1}\| \leq \gamma \|u - u_\ell\| \) for some \( \gamma < 1 \) if \( \ell \) is sufficiently large. The triangular inequality then leads to \( \|u - u_\ell\| \leq \frac{1}{1-\gamma} \|\varepsilon_\ell\| \), so that \( \log_2(\|\varepsilon_{\ell-1}\|/\|\varepsilon_\ell\|) \) may serve as an estimate of the convergence rate as long as it is positive.

We see that the numerical convergence rates in \( L^2, H^1 \) and \( H^2 \) norms are similar to those for Test Problem 1. This indicates that the solution \( u \) lies at least in \( H^6(\Omega) \). In fact it is expectable that \( u \) should be infinitely differentiable because so are the data and the domain boundary. Note that [15] Theorem 17.22 only assures that \( u \in C^{2,\alpha}(\Omega) \) for all \( 0 < \alpha < 1 \), but this theorem only requires \( C^3 \) boundary and \( C^2 \) smoothness of \( g \). The convergence rate of the residual (21) is close to \( O(h^4) \), that is to the rate of the \( H^2 \)-norm of the error, which is plausible because \( R \) is based on the second order derivatives of the approximate solution.

Test Problem 3. Equation (18) with \( g(x) = \sin(\pi |x_1|) + 1.1 \) and \( \phi = 0 \) in the same elliptic disk \( \Omega \) as in Test Problem 2.

The numerical results can be found in Table 3. Now [13] Theorem 17.22 is not applicable because \( g \notin C^2(\Omega) \). Nevertheless, the method converges with approximate...
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
\(\ell\) & \(\|\varepsilon_\ell\|_{L^2}\) & \text{rate} & \(\|\varepsilon_\ell\|_{H^1}\) & \text{rate} & \(\|\varepsilon_\ell\|_{H^2}\) & \text{rate} & \(R\) & \text{rate} & \(m\) \\
\hline
\text{init} & & & & & & & & & \text{6.58e-1} \text{rate} \\
1 & 1.02e-8 & 3.64e-7 & 2.90e-5 & 4.95e-6 & 4.95e-6 & 1 \\
2 & 9.59e-10 & 5.26e-8 & 2.8 & 6.37e-6 & 2.2 & 1.62e-5 & 1.6 & 1 \\
3 & 1.32e-11 & 1.29e-9 & 5.3 & 3.16e-7 & 4.3 & 1.37e-7 & 3.6 & 1 \\
4 & 2.25e-13 & 4.27e-11 & 4.9 & 2.05e-8 & 3.9 & 9.83e-9 & 3.8 & 1 \\
5 & 1.32e-11 & 6.2 & 1.61e-12 & 4.7 & 1.56e-9 & 3.7 & 6.61e-10 & 3.9 & 1 \\
6 & — & — & — & — & — & — & — & 3.9 & 1 \\
\hline
\end{tabular}
\caption{Estimated errors of the approximate solution and the rate of convergence for Test Problem 2 with \(g(x) = e^{x^1}\) on the elliptic disk. The meaning of \(\ell\), \(m\) and \text{‘init’} is the same as in Table 1. \(R\) is the residual error (21) for the last iterate \(u_\ell = u_m^h\) on level \(\ell\), and \(\varepsilon_\ell := u_\ell - u_{\ell+1}\) is the difference between the approximate solutions of two consecutive levels. We left the entries for \(\ell = 6\) related to \(\varepsilon_\ell\) blank because their computation requires the approximate solution \(u_7\) of the next level.}
\end{table}


\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
\(\ell\) & \(\|\varepsilon_\ell\|_{L^2}\) & \text{rate} & \(\|\varepsilon_\ell\|_{H^1}\) & \text{rate} & \(\|\varepsilon_\ell\|_{H^2}\) & \text{rate} & \(R\) & \text{rate} & \(m\) \\
\hline
\text{init} & & & & & & & \text{1.06e-0} & \text{rate} & \\
1 & 2.92e-5 & 9.88e-4 & 9.48e-2 & 1.92e-2 & 1.92e-2 & 3 \\
2 & 5.41e-6 & 6.20e-5 & 4.44e-3 & 6.23e-3 & 6.23e-3 & 1.6 & 2 \\
3 & 1.21e-6 & 1.19e-5 & 1.40e-3 & 2.03e-3 & 2.03e-3 & 1.6 & 1 \\
4 & 6.84e-8 & 2.01e-6 & 4.90e-4 & 7.46e-4 & 7.46e-4 & 1.4 & 1 \\
5 & 1.44e-8 & 3.67e-7 & 1.47e-4 & 2.47e-4 & 2.47e-4 & 1.6 & 1 \\
6 & — & — & — & — & — & — & \text{9.04e-5} & 1.5 & 1 \\
\hline
\end{tabular}
\caption{Estimated errors of the approximate solution and the rate of convergence for Test Problem 3 with \(g(x) = \sin(\pi|x_1|) + 1.1\) on the elliptic disk. The layout is the same as in Table 2.}
\end{table}

Orders \(O(h^{2.5})\), \(O(h^{2.5})\) and \(O(h^{1.5})\) for the \(L^2\), \(H^1\) and \(H^2\) norms, respectively. This indicates that \(u\) should be in \(H^r(\Omega)\) for \(r \approx 3.5\), but the approximation order of the method in \(L^2\) norm is suboptimal.

\textbf{Test Problem 4}. Equation (18) with \(g(x) = 1\) and \(\phi = 0\) in a \(C^1\) domain \(\Omega\) bounded by the straight lines \(x_2 = \pm 1\) and semi-circles

\[x_1 = \pm \left(1 + \sqrt{1 - x_2^2}\right), \quad -1 \leq x_2 \leq 1.\]

The domain is visualized in Figure 5 together with the initial triangulation used in our experiments. The straight line and circular segments are connected with \(C^1\) smoothness at the points \((1,1)\) and \((-1,1)\) indicated with circles.

Similar to the tests with \(g(x) = 1\) on a square domain [15, Section 5.1], our experiments do not show convergence of the method with respect to \(\ell\). This is explained in particular by the fact that the second derivatives of the solution \(u\) of (18) with \(\phi = 0\) may not be continuous along any straight line boundary segment unless \(g\) vanishes on...
this segment. Nevertheless, in contrast to the square domain, the approximate solutions $u_\ell$ keep the convex shape and the Newton iterations converge on each level. Figure 6 shows $u_2$ and its contour plot.

Figure 5: The domain of Test Problem 4 with initial triangulation. The boundary is $C^1$ at the four points marked with circles and $C^\infty$ elsewhere. Its top and bottom pieces are straight line segments.

Figure 6: Approximate solution $u_\ell$ of Test Problem 4 for the level $\ell = 2$ and its contour plot.

**Test Problem 5.** Equation (18) with $g(x) = 1$ and $\phi = 0$ in a centrally symmetric $C^2$ domain $\Omega$ bounded by two elliptic and two circular segments, see Figure 7, where the top elliptic segment is given parametrically by the equations

$$x_1 = 4 \cos t, \quad x_2 = 1.3 \sin t - c_2, \quad 0.15 \pi \leq t \leq 0.85 \pi,$$

and the left circular segment has radius $r$ and center $(c_1, 0)$, with $r$ and $(c_1, c_2)$ being the radius and the center of the osculating circle to the ellipse $x_1 = 4 \cos t, x_2 = 1.3 \sin t$ at the point defined by $t = 0.85\pi$. 

16
Table 4: Estimated errors of the approximate solution and the rate of convergence for Test Problem 5 on a $C^2$ domain. The layout is the same as in Table 2.

It is easy to check that elliptic and circular segments of $\Omega$ join with continuous curvature. We use the initial triangulation shown in Figure 7. The numerical results presented in Table 4 indicate $O(h^4)$, $O(h^3)$ and $O(h^2)$ convergence order in the $L^2$, $H^1$ and $H^2$-norm, respectively, so that the solution $u$ is expected to belong to $H^r(\Omega)$ for $r \approx 4$. Note that [18, Theorem 17.22] is not applicable because the boundary is not $C^3$.

Figure 7: The domain of Test Problem 5 with initial triangulation. The boundary is $C^2$ at the four points marked with circles and $C^\infty$ elsewhere.

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