PROGRESS ON THE STUDY OF THE GINIBRE ENSEMBLES I: GinUE

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ABSTRACT. The Ginibre unitary ensemble (GinUE) consists of \( N \times N \) random matrices with independent complex standard Gaussian entries. This was introduced in 1965 by Ginibre, who showed that the eigenvalues form a determinantal point process with an explicit correlation kernel, and after scaling they are supported on the unit disk with constant density. For some time now it has been appreciated that GinUE has a fundamental place within random matrix theory, both for its applications and for the richness of its theory. Here we review the progress on a number of themes relating to the study of GinUE. These are eigenvalue probability density functions and correlation functions, fluctuation formulas, sum rules and asymptotic behaviours of correlation functions, and normal matrix models. We discuss too applications in quantum many body physics and quantum chaos, and give an account of some statistical properties of the eigenvectors.

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1. Introduction

In a fundamental paper on random matrix theory from the early 1960’s, Dyson [117] isolated three ensembles of Hermitian matrices, and three ensembles of unitary matrices. This was done by seeking the minimal requirement of a quantum Hamiltonian $H$, respectively evolution operator $U$, to exhibit a time reversal symmetry or not, and then imposing a probability measure. First, for a time reversal operator $T$ — defined in general by the requirement that it be anti-unitary — it was first shown that there are two possibilities, either $T^2 = I$ or $T^2 = -I$, with the latter requiring that the Hilbert space be even dimensional. For $T$ to commute with $H$ or $U$ it was then shown that in the case $T^2 = I$ both $H$ and $U$ should be invariant under the transpose operation. In the other possible case, that $T^2 = -I$, an invariance under the so-called quaternion dual $M \mapsto Z_{2N} M^T Z_{2N}^{-1}$ was deduced. Here $Z_{2N}$ is the $2N \times 2N$ anti-symmetric tridiagonal matrix with entries all $-1$ in the leading upper triangular diagonal, and all $1$ in the leading lower triangular diagonal, and moreover in this case it was shown that $T$ has the realisation $T = Z_{2N} K$ where $K$ corresponds to complex conjugation and $2N$ is the dimension of the Hilbert space. In the case of a quantum Hamiltonian $H$, it was then shown that $[H, T] = 0$ with $T^2 = I$ implies a basis can be chosen so that the elements are real, while with $T^2 = -I$ it implies that the elements can be chosen...
to have a $2 \times 2$ block structure

\[(1.1) \quad \begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix}.\]

The $2 \times 2$ matrix (1.1) can be identified with a member of the (real) quaternion number field. Hence for quantum Hamiltonians, Dyson was lead to the requirement that matrices in his sought ensemble theory should have real entries, or have a $2 \times 2$ block structure corresponding to the quaternion number field in the presence of time reversal symmetry, or to be complex without a time reversal symmetry. This requirement is in addition to the matrices being Hermitian and thus having real eigenvalues and a matrix of eigenvectors which can be chosen to be unitary.

The work [117] specifies the eigenvalue probability density function (PDF) for an ensemble of quantum Hamiltonians $H$ modelled as random matrices, and chosen from a Gaussian distribution on the elements proportional to

\[(1.2) \quad \exp(-\beta \text{Tr} H^2/2).\]

The scaling factor $\beta$ is chosen for convenience, and takes on the value of the number of independent parts of the corresponding number field — thus $\beta = 1$ for real entries (time reversal symmetry with $T^2 = I$), $\beta = 2$ for complex entries (no time reversal symmetry), and $\beta = 4$ for quaternion entries (time reversal symmetry with $T^2 = -I$). With this specification, $\beta$ is referred to as the Dyson index. A detail is that Hermitian matrices commuting with the quaternion dual must have doubly degenerate eigenvalues (Kramer’s degeneracy), with the convention in (1.2) that the trace operation relates to the independent eigenvalues only. The result of Dyson, known earlier in the case $\beta = 1$ by Wigner (see the Introduction section of the book edited by Porter [253] for references and moreover reprints of the original works) is that the eigenvalue PDF is given by [117, Eq. (146)]

\[(1.3) \quad \prod_{l=1}^{N} \prod_{1 \leq j < k \leq N} e^{-\beta \lambda_l^2/2} |\lambda_k - \lambda_j|^\beta,\]

up to proportionality. If instead of the distribution on elements being chosen as (1.2), a weighting

\[(1.4) \quad \exp(-\beta \text{Tr} V(H)/2)\]

for some real valued function $V(\lambda)$ is chosen instead, the modification of (1.2) is that it now reads

\[(1.5) \quad \prod_{l=1}^{N} \prod_{1 \leq j < k \leq N} e^{-\beta V(\lambda_l)/2} |\lambda_k - \lambda_j|^\beta.\]
Here it is being assumed that $V(\lambda)$ decays sufficiently fast at infinity for (1.5) to be normalisable. A weighting of the form (1.4) is said to specify an invariant ensemble, since it is invariant under conjugation by a unitary matrix $H \mapsto UHU^\dagger$ (and where too the elements of $U$ are restricted to be real ($\beta = 1$) and quaternion ($\beta = 4$) so that $H$ remains in the same ensemble). A weighting of the form (1.2) is said to specify a Gaussian ensemble.

In 1965 Ginibre [158], motivated by mathematical curiosity [18], §2.2, quoting correspondence with Ginibre], initiated a study of non-Hermitian Gaussian ensembles with either real, complex or quaternion entries. Replacing (1.2) is the joint distribution on elements of the corresponding matrices, now to be denoted $G = [g_{ij}]_{i,j=1}^N$, proportional to

$$
(1.6) \quad \exp(-\beta \text{Tr} G^\dagger G/2) = \prod_{i,j=1}^N \exp(-\beta |g_{ij}|^2/2).
$$

Note that the second form in this expression shows that the real and imaginary parts (there $\beta$ such parts; e.g. in the quaternion case $\beta = 4$ there is one real and three imaginary parts) are all independent, identically distributed Gaussians. The concern of Ginibre was with the functional form of the eigenvalue PDF, and the implied eigenvalue statistics. The most obvious difference with the Hermitian case is that the eigenvalues are now in general complex. Also significant is the fact that the eigenvectors no longer form an orthonormal basis. Notwithstanding these differences, it was found in the complex case (referred to as the complex Ginibre ensemble, or alternatively as GinUE, where in the latter the U stands for unitary refers to the bi-unitary invariance of (1.6) being unchanged by the mapping $G \mapsto UGV$ for $U, V$ unitary matrices) that the eigenvalue PDF is proportional to

$$
(1.7) \quad \prod_{l=1}^N e^{-|z_l|^2} \prod_{1 \leq j < k \leq N} |z_k - z_j|^2,
$$

which is in direct correspondence with the Hermitian result (1.3) with $\beta = 2$, obtained essentially by replacing $\lambda_j$ by $z_j$.

The eigenvalue PDF in the case of real and quaternion entries does not follow this correspondence. First, in both these cases the eigenvalues come in complex conjugate pairs. Appreciating this point, the functional form of the eigenvalue PDF as found by Ginibre in the quaternion case can be obtained from (1.7) (not (1.3) with $\beta = 4$) by first replacing $N$ by $2N$, then identifying $z_{j+N}$ as $\bar{z}_j$ and ignoring terms which involve only $\{z_j\}$ (which are thought of as part of the image system [137]) to obtain

$$
(1.8) \quad \prod_{l=1}^N e^{-2|\text{Im} z_l|^2} |z_l - \bar{z}_l|^2 \prod_{1 \leq j < k \leq N} |z_k - z_j|^2 |z_k - \bar{z}_j|^2, \quad \text{Im} z_l > 0.
$$
The real case is still more complicated. First, the eigenvalue PDF is not absolutely continuous. Rather, it decomposes into sectors depending on the number of real eigenvalues. Its precise functional form was not obtained in Ginibre’s original work, with a further 25 years or so elapsing before this was achieved in a publication by Lehmann and Sommers [223].

It would seem that the first occurrence of a Ginibre ensemble in applications (specially the real Ginibre ensemble of GinOE, where here the “O” stands for orthogonal and refers to the bi-orthogonal invariance of the matrices) arose in the 1972 work of May [240] on the stability of complex ecological webs. Upon linearising about a fixed point, and the modelling of the fluctuations away from an attractor by a real Ginibre matrix $G$, May was led to the first order linear differential equation system for the perturbed populations — an $n \times 1$ column vector $x$ — specified by

\begin{equation}
\frac{d}{dt}x = (-I + \alpha G)x,
\end{equation}

where $\alpha$ is a scalar parameter. The stability is then determined by the maximum of the real part of the spectrum of $G$, the precise determination of which has only recently become available in the literature [50, 95].

At the beginning of the 1980’s the interpretation of (1.7), written in the Boltzmann factor form

\begin{equation}
e^{-\beta U(z_1,\ldots,z_N)}, \quad U = \frac{1}{2} \sum_{j=1}^{N} |z_j|^2 - \sum_{1 \leq j < k \leq N} \log |z_k - z_j|, \quad \beta = 2,
\end{equation}

as a model of charged particles, repelling pairwise via a logarithmic potential, and attracted to the origin in the plane via a harmonic potential, gained attention [22, 72]. This viewpoint was already prominent in the works of Dyson in the context of (1.3) (see too the even earlier work of Wigner [288] as reprinted in [253]), and was noted for (1.8) in Ginibre’s original article [158]. In contrast to (1.3), in (1.10) the domain is two-dimensional, and the logarithmic potential is the solution of the corresponding Poisson equation (in one spacial dimension, the solution of the Poisson equation is proportional to $|x|$), so (1.10) corresponds to a type of Coulomb gas. Broad aspects of the latter have been the subject of the recent reviews [224], [80]. Also in the early 1980’s, for all positive values of $\beta/2$ odd, (1.10) gained attention as the absolute value squared of Laughlin’s trial wave function for the fractional quantum Hall effect [213].

Fast forward 40 years, and there are now a multitude of applications which require knowledge of properties of Ginibre matrices, in particular their eigenvalues and eigenvectors. This is due in no small part due to a resurgence of interest in non-Hermitian quantum mechanics [42]. Moreover, the progression of time as seen a much deeper understanding
of the mathematical structures associated with the Ginibre matrices, and the theoretical progress has been considerable. It is the purpose of this article to review a number of these advances, both in the theory and the applications. Due to space considerations, attention will be focused here on the complex case of the GinUE — a subsequent review article is planned relating specifically too GinOE and GinSE. To the era up to the year 2010, accounts of the progress with emphasis similar to the present article can be found in [133, Ch. 15, with proofs], [199, some proofs sketched], with the latter overlapping mainly with §2. To make the presentation self contained, this material is also part of the present review, albeit with some reordering and additional context. And when practical from the viewpoint of the space required, proofs of a number of the results are presented.

There are four main themes to the review. These form sections two through to five: eigenvalue PDFs and correlation functions, fluctuation formulas, sum rules and asymptotic behaviours, and normal matrix models. There is also a sixth section entitled further theory and applications. Here the topics considered are the analogy between GinUE and the quantum many body system for free Fermions in the plane subject to a perpendicular magnetic field, the relevance of GinUE statistics to studies in quantum chaos, and statistical properties of the eigenvectors of GinUE matrices.

2. Eigenvalue PDFs and correlations

2.1. Eigenvalue PDF. In the original paper of Ginibre [158], the diagonalisation formula $G = VΛV^{-1}$, where $Λ$ is the diagonal matrix of eigenvalues, and $V$ is the matrix of corresponding eigenvectors which are unique up to normalisation, was used as the starting point to derive the eigenvalue PDF (1.7). The matrix $V$ was then further decomposed $V = UTD$, where $U$ is unitary, $T$ is upper triangular with all diagonal elements equal to 1, and $D$ is a diagonal matrix with real positive elements. As a consequence

$$\text{Tr} G^\dagger G = \text{Tr} \bar{Λ}BΛB^{-1}, \quad B = T^\dagger T$$

This is independent of $U$ and $D$, and $B^{-1}$ is a simpler structure than $V^{-1}$, since it has determinant unity. It is necessary to integrate out the variables of $B$, which was done in $N$ steps with each one consisting of integrating out over the last remaining row and column.

A more versatile (equally applicable to the GinOE, for example) method of derivation of (1.7) has since been found. It is due to Dyson, and first appeared in published form in [241, Appendix 35]. Here, instead of using the diagonalisation formula for $G$, the starting point is the Schur decomposition

$$G = UZU^\dagger.$$
Here $U$ is a unitary matrix, unique up to the phase of each column, and $Z$ is an upper triangular matrix with elements on the diagonal equal to the eigenvalues of $G$.

**Proposition 2.1.** For GinUE matrices, specified by the distribution on elements proportional to $(1.6)$ in the case $\beta = 2$ (complex elements), the eigenvalue PDF is equal to $1/C_N$ times $(1.7)$, where upon relaxing the ordering constraint on the eigenvalues implied by $(2.2)$, the normalisation constant $C_N$ is specified by

$$
C_N = \pi^N \prod_{j=1}^N j!.
$$

**Proof.** We have from $(2.2)$ that

$$
\text{Tr } G^\dagger G = \sum_{j=1}^N \lvert z_j \rvert^2 + \sum_{1 \leq j < k \leq N} \lvert Z_{jk} \rvert^2,
$$

where here $\{z_j\}$ denotes the diagonal elements of $Z$ (which are the eigenvalues of $G$), and $\{Z_{jk}\}$ denotes the upper triangular elements. Note the simplification relative to $(2.1)$.

After this brisk start, there is still quite a challenge to compute the Jacobian corresponding to $(2.2)$. The strategy of [231] Appendix 35 is explained in more detail in [179], and repeated in [133] Proof of Proposition 15.1.1]. Here one begins by computing the matrix of differentials $U^\dagger dG U$, with the Jacobian corresponding to the (absolute value of) factor which results from the corresponding wedge product. For the latter task, proceeding in the order of the indices $(j,k)$, with $j$ decreasing from $N$ to 1, and $k$ increasing from 1 to $N$, gives the factor

$$
\prod_{j<k} \lvert z_j - z_k \rvert^2.
$$

However the product of differentials so obtained is not immediately recognisable in the factorised form

$$
\wedge_j dz_j^r dz_j^i \wedge (U^\dagger dU) \wedge_{j<k} dZ_{jk}^r dZ_{jk}^i,
$$

where the superscripts indicate the real and imaginary part. Further arguing involving a count of the number of independent real variables associated with $U$, which implies some apparent differentials contribute zero to the wedge product, is required to make the simplification to this form. With $(2.6)$ established, the integration over $\{Z_{jk}\}$ in $(2.2)$ is immediate.

To deduce the normalisation $(2.3)$ from this calculation requires first that the normalisation of $(2.4)$ be included throughout the calculation, and second knowledge of the integration
formula (see e.g. [112 Eq. (4.4)])

\[ \int_{U} (U^*dU) = \text{vol}(U(N)/(U(1))^N) = 2^{N(N-1)/2} \prod_{i=1}^{N-1} \frac{\pi^i}{\Gamma(i+1)}. \]

Finally, an extra factor of $N!$ is required in $C_N$ to account for relaxing an ordering of the eigenvalues. \(\square\)

2.2. Correlation functions. With the joint eigenvalue PDF denoted $p_N(z_1, \ldots, z_N)$, the $k$-point correlation function $\rho_{(k),N}(z_1, \ldots, z_k)$ is specified by

\[ \rho_{(k),N}(z_1, \ldots, z_k) = N(N-1) \cdots (N-k+1) \int_{C} d^2z_{k+1} \cdots \int_{C} d^2z_N p_N(z_1, \ldots, z_N), \]

where, with $z := x + iy$, $d^2z := dxdy$. In the simplest case $k = 1$ this corresponds to the eigenvalue density. Ginibre [158] showed that

\[ \rho_{(1),N}(z_1, \ldots, z_k) = \det \left[ K_N(z_j, z_l) \right]_{j,l=1}^{k}, \]

for a particular function $K_N(\omega, z)$, referred to as the correlation kernel. The structure (2.9) makes the eigenvalues of GinUE an example of a determinantal point process [62].

**Proposition 2.2.** The kernel function in (2.9) is specified by

\[ K_N(\omega, z) = \frac{1}{\pi} e^{-\left(|\omega|^2 + |z|^2\right)/2} \sum_{j=1}^{N} \frac{(\omega z)^{j-1}}{(j-1)!} = \frac{1}{\pi} e^{-\left(|\omega|^2 + |z|^2\right)/2} e^{i\omega z} \frac{\Gamma(N, \omega z)}{\Gamma(N)}, \]

where $\Gamma(j; x) = \int_{x}^{\infty} t^{j-1} e^{-t} dt$ denotes the (upper) incomplete gamma function.

**Proof.** To deduce the determinantal structure (2.9) with $K_N$ specified by the first equality in (2.10) (the second equality follows from the first by the identity $\Gamma(N, \omega z) = e^{-x} \sum_{j=1}^{N} \frac{x^{j-1}}{(j-1)!}$), the first step is to rewrite the product in (1.6) according to

\[ \prod_{1 \leq j < k \leq N} |z_k - z_j|^2 = \prod_{1 \leq j < k \leq N} (z_k - z_j)(\bar{z}_k - \bar{z}_j), \]

then to rewrite each of the product of differences on the RHS as a Vandermonde determinant,

\[ \prod_{1 \leq j < k \leq N} (z_k - z_j) = \det[z_{j-1}^{k-1}]_{j,k=1}^{N}; \]

see e.g. [133] Exercises 1.9 Q.1 for a derivation. Multiplying (2.12) by its conjugate and taking the transpose of the matrix on the RHS (which leaves the determinant unchanged) shows

\[ \frac{1}{C_N} \prod_{l=1}^{N} e^{-|z_l|^2} \prod_{1 \leq j < k \leq N} |z_k - z_j|^2 = \det \left[ K_N(z_j, z_k) \right]_{j,k=1}^{N}. \]
The significance of the form (2.13) for purposes of computing the integrations as required by (2.8) are the reproducing and normalisation properties of $K_N$,

\[ \int C K_N(w_1, z)K_N(z, w_2) \, d^2z = K_N(w_1, w_2), \quad \int C K_N(z, z) \, d^2z = N. \]

Using these properties, a cofactor expansion along the bottom row can be used to show (2.14)

\[ \int C \det[K_N(z_j, z_k)]_{j,k=1}^m \, d^2z_m = (- (m - 1) + N) \det[K_N(z_j, z_k)]_{j,k=1}^{m-1}; \]

see also [135] Proof of Proposition 5.1.2. Applying this inductively gives (2.9). □

According to (2.9) with $k = 1$ and (2.10), the eigenvalue density is given by the rotationally invariant functional form

\[ \rho_{(1),N}(z) = \frac{1}{\pi} \frac{\Gamma(N; |z|^2)}{\Gamma(N)}. \]

Ginibre [158] identified a sharp transition for $|z| \approx \sqrt{N}$ from the constant value $\frac{1}{\pi}$ for $|z|$ less than this critical value, to a value approaching zero for $|z|$ greater than this critical value. An equivalent statement is the limit law

\[ \lim_{N \to \infty} \rho_{(1),N}(\sqrt{Nz}) = \begin{cases} \frac{1}{\pi}, & |z| < 1, \\ 0, & |z| > 0. \end{cases} \]

This was the first example of what now is termed the circular law, which specifies the (global) scaled limiting eigenvalue density for a wide class of non-Hermitian random matrices, with identically and independently distributed elements to be constant inside a particular circle in the complex plane, and zero outside [61].

**Remark 2.3.**

1. In the sense of probability theory, (2.17) is a statement in the mean, due to the ensemble average. The strong version of the circular law establishes that for large $N$ the eigenvalues of a single GinUE matrix obey the circular law almost surely [61].

2. Define the numerical range of an $N \times N$ matrix $X$ by $W(X) = \{ Xu \cdot u \mid ||u|| = 1 \}$. By the variational characterisation of eigenvalues, for $X$ Hermitian $W(X)$ must be contained in the interval of the real line $[\lambda_{\min}, \lambda_{\max}]$, and in fact is equal to this interval. It is proved in [220] that for $X$ a global scaled Ginibre matrix, and thus with eigenvalue density obeying the circular law (2.17), $W(X)$ converges to the centred disk in the complex plane of radius $\sqrt{2}$.

The correlation kernel (2.10), and thus the correlation functions (2.9), admit distinct scaling limits depending on the centring of the variables being in the bulk region, where the
density is constant, or the edge region, where the density begins to decrease to zero. The first was specified in Ginibre’s original paper [158], whereas the latter was not made explicit until some time later [142].

**Proposition 2.4.** Let $K_N(w, z)$ be specified by (2.10). We have

$$K_{\infty}^b(w, z) := \lim_{N \to \infty} K_N(w, z) = \frac{1}{\pi} e^{-|w|^2 - |z|^2} e^{wz}$$

$$K_{\infty}^c((x_1, y_1), (x_2, y_2)) := \lim_{N \to \infty} K_N(x_1 + i(-\sqrt{N} + y_1), x_2 + i(-\sqrt{N} + y_2))$$

$$= e^{-(x_1 - x_2)^2/2 - (y_1 - y_2)^2/2} h\left(\frac{1}{2} (y_1 + y_2 - i(x_1 - x_2))\right),$$

where

$$h(z) = \frac{1}{2\pi} \left(1 + \text{erf}(\sqrt{2}z)\right).$$

**Proof.** The limit (2.18) is immediate from (2.10), and the fact that for fixed $wz$,

$$\lim_{N \to \infty} \frac{\Gamma(N; w\bar{z})}{\Gamma(N)} = 1.$$ 

The derivation of (2.19) relies on (the first term of) the asymptotic expansion [245]

$$\frac{\Gamma(N; N + \tau \sqrt{N})}{\Gamma(N)} = \frac{1}{2} \left(1 - \text{erf}(\tau / \sqrt{2})\right) + \frac{1}{3\sqrt{2\pi N}} \tau^{-\tau/2} (\tau^2 - 1) + O\left(\frac{1}{N}\right).$$

\[\square\]

Generally (2.9) gives for the appropriately scaled two-point correlation

$$\rho_{(2), \infty}(z_1, z_2) = \rho_{(1), \infty}(z_1) \rho_{(1), \infty}(z_2) - K_{\infty}(z_1, z_2) K_{\infty}(z_2, z_1).$$

For large separation of $z_1$ and $z_2$ the leading order of the RHS is given by the first term which is the product of the densities. The second term $-K_{\infty}(z_1, z_2) K_{\infty}(z_2, z_1)$ must decay sufficiently rapidly for it to be square integrable, since the limiting form of the first integration formula in (2.14) remains valid,

$$\int_{C} K_{\infty}(w_1, z) K_{\infty}(z, w_2) d^2z = K_{\infty}(w_1, w_2).$$

This can be verified from the results of Proposition 2.4 by the evaluation of appropriate Gaussian integrals. To separate off the product of densities, one defines the truncated (or connected) two-point correlation

$$\rho_{(2), \infty}^T(z_1, z_2) := \rho_{(2), \infty}(z_1, z_2) - \rho_{(1), \infty}(z_1) \rho_{(1), \infty}(z_2) = -K_{\infty}(z_1, z_2) K_{\infty}(z_2, z_1).$$
In particular, with bulk scaling, we read off from this and (2.18) that
\begin{equation}
\rho_{(2),\infty}^{b,T}(z_1, z_2) = -\frac{1}{\pi^2} e^{-|z_1 - z_2|^2},
\end{equation}
which thus exhibits a Gaussian decay. With edge scaling, (2.23) and (2.19) give
\begin{equation}
\rho_{(2),\infty}^{e,T}(z_1, z_2) = -e^{-(x_1 - x_2)^2/(y_1 - y_2)^2} \left| h \frac{1}{2}(y_1 + y_2 - i(x_1 - x_2)) \right|^2.
\end{equation}
Use of the asymptotic expansion of the error function [248, Eq. (7.12.1)] gives to leading order
\begin{equation}
\rho_{(2),\infty}^{e,T}(z_1, z_2) \mid_{|z_1 - z_2| \to \infty} \sim \frac{1}{2\pi^3 (y_1 + y_2)^2 + (x_1 - x_2)^2} e^{-2y_1^2 - 2y_2^2}.
\end{equation}
While this decays in all directions, parallel to the boundary of the leading order density (i.e. in the x-direction) we see that the decay is algebraic, as an inverse square.

Remark 2.5.
1. It follows from (2.21) that the edge scaling of the eigenvalue density, in the coordinates of (2.19), has the large $N$ expansion
\begin{equation}
\rho_{\infty}^{(1)}(y) = \frac{1}{2\pi} \left( 1 + \text{erf}(\sqrt{2}y) \right) + \frac{1}{3\pi\sqrt{2\pi}N} e^{-2y^2}(y^2 - 1) + O\left( \frac{1}{N} \right);
\end{equation}
in relation to the $1/\sqrt{N}$ correction term, see [220, 29]. Integrating the leading term of this expansion over $y \in (-\infty, 0]$ and multiplying by $2\pi\sqrt{N}$ (the length of the bounding circle of the leading order support) shows that to leading order the expected number of eigenvalues with modulus greater that $\sqrt{N}$ is $\sqrt{N}/(2\pi)$ [158].
2. The two-dimensional classical Coulomb system interpretation (1.10) of the eigenvalue PDF (1.7) allows for (2.12) to be anticipated. For this, one scales $z_j \to \sqrt{N}z_j$ and introduces a mean field energy functional
\begin{equation}
\frac{N}{2} \sum_{j=1}^{N} |z_j|^2 - \sum_{1 \leq j < k \leq N} \log |z_k - z_j| \\
\sim \frac{N}{2} \left( \int_{\Omega} \rho_{(1)}(z)|z|^2 \, d^2z - \int_{\Omega} d^2w \rho_{(1)}(w) \int_{\Omega} d^2z \rho_{(1)}(z) \log |z - w| \right),
\end{equation}
with the hypothesis that $\rho_{(1)}(z)$ is chosen so that this functional is minimised and furthermore integrates over $\Omega$ to unity. Characterising the minimisation property by the vanishing of the functional upon variation with respect to $\rho_{(1)}(z)$ gives
\begin{equation}
|z|^2 - 2 \int_{\Omega} \rho_{(1)}(w) \log |z - w| \, d^2w = C,
\end{equation}
where $C$ is a constant, valid for $z \in \Omega$. Applying the Laplacian operation $\nabla_z^2$ to this, using the standard fact

$$-\nabla_z^2 \log |z - w| = -2\pi \delta(z - w),$$

it follows

$$\rho_{(1)}(w) = \frac{1}{\pi} \chi_{|w|<1}.$$  

Here the restriction to $|w| < 1$ is implied by the rotational invariance, together with the minimisation and normalisation requirements. The notation $\chi_A$ denotes the indicator function of the condition $A$, taking on the value 1 when $A$ is true, and 0 otherwise. The support $\Omega$ of $\rho_{(1)}(w)$ — which here is the unit disk — in such a Coulomb gas picture is typically referred to as the droplet (see e.g. [4, 175]). Moreover, this potential theoretic reasoning can rigorously be justified; see e.g. [80, §3.1] and references therein, as well as the discussion and references in the paragraph including (5.7) of §5 below.

3. Let $X_0$ have finite rank, and $X$ be a GinUE matrix, scaled so that the leading support of the eigenvalues is the unit disk. Assume too that the eigenvalues of $X_0$ are inside of the unit disk and near the boundary. In the limit $N \to \infty$ it has recently been shown that the edge correlation functions centred on the eigenvalues of $X_0$ form a determinantal point process with kernel involving generalisations of the error function [233].

2.3. Elliptic GinUE. In 1991 Lehmann and Sommers introduced a one parameter generalisation of the non-Hermitian complex Gaussian matrices specifying GinUE. In this generalisation, varying the parameter allows for the Hermitian ensemble of Gaussian matrices known as the GUE (Gaussian unitary ensemble) to be obtained. An Hermitian matrix $H$ from the GUE can be constructed from a scaled non-Hermitian GinUE matrix $\tilde{G} = (1/\sqrt{2})G$ according to

$$(2.30) \quad H = \frac{1}{2}(\tilde{G} + \tilde{G}^\dagger).$$

**Proposition 2.6.** Let the parameters $0 < \tau, \nu < 1$ be related by $\tau = (1 - \nu^2)/(1 + \nu^2)$. For $H_1, H_2$ elements of the GUE, define

$$(2.31) \quad J = \sqrt{1 + \tau}(H_1 + i\nu H_2).$$

The eigenvalue PDF of the ensemble of matrices $\{J\}$ is given by

$$(2.32) \quad \exp \left(-\frac{1}{1 - \tau^2} \sum_{j=1}^N \left(|z_j|^2 - \frac{\tau}{2}(z_j^2 + \bar{z}_j^2)\right)\right) \prod_{1 \leq j < k \leq N} |z_k - z_j|^2.$$
Proof. Following [150] and [133, Exercises 15.1 Q.1], the starting point is to note that the joint element PDF of $\sqrt{1+\tau}H_1$ and $\sqrt{1+\tau}H_2$ is proportional to $\exp(-\frac{1}{1+\tau}\text{Tr}(H_1^2 + H_2^2))$. The definition of $J$ gives

$$\text{Tr} H_1^2 = \frac{1}{2}\left(\text{Tr}(JJ^\dagger) + \text{Re Tr}(J^2)\right), \quad \text{Tr} H_2^2 = \frac{1}{2\tau^2}\left(\text{Tr}(JJ^\dagger) - \text{Re Tr}(J^2)\right).$$

As a consequence, it follows that the joint element PDF of $J$ is proportional to

$$\exp\left(-\frac{1}{1-\tau^2}\text{Tr}(JJ^\dagger - \tau\text{Re } J^2)\right).$$

With this knowledge, the strategy of the proof of Proposition 2.1 leads to (2.32).  

The correlations for (2.32) have, for an appropriate correlation kernel $K_N(w,z;\tau)$, the determinantal form (2.9). This involves the scaled monic Hermite polynomials

$$(2.34) \quad C_n(z) := \left(\frac{\tau}{2}\right)^{n/2}H_n\left(\frac{z}{\sqrt{2\tau}}\right), \quad z \in \mathbb{C}.$$

**Proposition 2.7.** In the notation specified above, we have

$$(2.35) \quad K_N(w,z;\tau) = \frac{1}{\pi}\frac{1}{\sqrt{1-\tau^2}} \times \exp\left(-\frac{1}{2(1-\tau^2)}\left(|w|^2 + |z|^2 - \tau(\text{Re } w^2 + \text{Re } z^2)^2\right)\right) \sum_{l=0}^{N-1} \frac{C_l(w)C_l(\overline{z})}{l!}.$$ 

Proof. Following the same procedure as used to begin the proof of Proposition 2.2, we modify (2.12) so that it reads

$$(2.36) \quad \prod_{1 \leq |j| < k \leq N} (z_k - z_j) = \det[p_{k-1}(z_j)]_{j,k=1}^N,$$

where $\{p_l\}$ is a set of monic polynomials, each $p_l$ of degree $l$. Choosing these polynomials according to (2.34), with this choice being motivated by the orthogonality [120, 111]

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \ e^{-x^2/(1+\tau)-y^2/(1-\tau)}C_m(z)C_n(z) = \pi m! \sqrt{1-\tau^2}\delta_{m,n},$$

the remaining working of the proof of Proposition 2.2 establishes the result.  

Applying the Coulomb gas argument of Remark 2.5, with the scaling $z_l \mapsto \sqrt{N}z_l$, we conclude that within some domain $\Omega$ the density is constant, taking the value $\rho_{(1)}(w) = 1/(\pi(1-\tau^2))$. This domain, or equivalently droplet, can be determined to be an ellipse with semi-axes $A = 1+\tau$, $B = 1-\tau$ and area equal to $\pi(1-\tau^2)$. The shape can be verified directly, by showing that with $\Omega$ so specified, and $|z|^2$ in (2.29) replaced by

$$(2.37) \quad \frac{1}{1-\tau^2}(|z|^2 - \tau\text{Re } z^2),$$
the required minimisation equation is indeed satisfied \[93, 133, \text{Exercises 15.2 q.4]}; see also [111, 143, 220]. This droplet shape explains the terminology elliptic GinUE in relation to (2.32). For complex non-Hermitian matrices (2.31), now with the Hermitian random matrices \( H_1, H_2 \) constructed from (2.30) with general zero mean, unit standard deviation identically distributed entries that are not required to be Gaussian, a constant density in an ellipse was first deduced by Girko \[159 \] upon additional assumptions, and without qualification by Nguyen and O’Rourke \[246 \].

As for the GinUE, as the boundary of the leading support of the elliptic GinUE is approached, there is a transition from a constant density to a density which decays to zero. Beginning with (2.35) in the case \( w = z \), the analysis of the eigenvalue density in this edge regime is more complicated than for the deduction of (2.21). It was carried out by Lee and Riser \[220 \]; see also \[17, 243 \].

**Proposition 2.8.** Consider the elliptic GinUE, with the eigenvalues scaled \( z_j \mapsto \frac{(a + i\beta)n}{\sqrt{N}} \) so that for \( N \) large the leading order support is the ellipse \( \Omega \). Let \( z_0 \) be a point on the boundary of \( \Omega \). Denote the unit vector corresponding to the outer normal at this point by \( n \), and the corresponding curvature by \( \kappa \). We have

\[
\rho_{(1),N}(z_0 + \frac{(a + i\beta)n}{\sqrt{N}}) = \frac{1}{2\pi} \left(1 - \text{erf}(\sqrt{2\alpha})\right) + \frac{\kappa}{\pi \sqrt{2\pi N}} e^{-2\alpha^2} \left(\frac{a^2 - 1}{3} - \beta^2\right) + O\left(\frac{1}{N}\right).
\]

**Remark 2.9.**
1. The leading term in (2.38) is identical to that in (2.27) with the identification \( y = -\alpha \). The universality of this functional form has been established for a wide class of normal matrix models (see \S 5 in relation to this class), and moreover extended to the edge scaled correlation kernel (2.19) \[36, 177 \]. This has recently been proved too for non-Hermitian random matrices constructed according to (2.30) with the elements of \( G \) identically distributed mean zero, finite variance random variables \[94 \].
2. Consider the elliptic shaped domain

\[ \{z \in \mathbb{C} : 1 - \frac{1}{1 - \tau^2}(|z|^2 - \tau \text{Re} z^2) > 0\}. \]

Let the function on the LHS of the inequality be denoted by \( 1 - f(z) \). We see that \( f(z) \) coincides with (2.37). It has been shown in \[20, 244 \] that the polynomials \( \{p_n(z)\} \) satisfying the orthogonality

\[
\int_{\Omega} p_m(z)p_n(\bar{z})(1 - f(z))^\alpha \, d^2z \propto \delta_{m,n},
\]

are simply related to the Gegenbauer polynomials \( \{C_n^{(1+\alpha)}(z)\} \). After scaling, the result (2.34) can be reclaimed by taking the limit \( \alpha \to \infty \).
3. We see from (2.31) that as \( \tau \to 1^- \), \( J \) is proportional to a GUE matrix, and in particular its eigenvalues are then all real. It was found by Fyodorov, Khoruzhenko and Sommers [151] that setting \( \tau = 1 - \pi^2 \alpha^2 / 2N \) and scaling the eigenvalues \( z_j \mapsto \pi z_j / N = \pi (x_j + iy_j) / N \) that

\[
\lim_{N \to \infty} \frac{\pi^2}{N^2} K_N(\pi w / N, \pi z / N; \tau) \bigg|_{\tau = 1 - \pi^2 \alpha^2 / 2N} = \sqrt{\frac{2}{\pi \alpha^2}} \exp \left( - \frac{y_1^2 + y_2^2}{\alpha^2} \right) \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp \left( - \frac{\alpha^2 u^2}{2} + iu (w - z) \right) du.
\]

(For a direct comparison between (2.39) and what has been reported in the literature, see [16, K\text{weak}(z_1, z_2) in Theorem 3] and [28, §1].) This is referred to as the weakly non-Hermitian limit. Scaling of the correlation kernel at the edge in this limit has been considered in [153, 50].

2.4. Induced GinUE. The fact that a complex Gaussian matrix \( G \) is bi-invariant with respect to multiplication by unitary matrices allows for the distribution of the singular values to be related to the eigenvalue distribution [128, 198].

**Proposition 2.10.** Let \( G \) be bi-unitary invariant, and let \( U \) be a Haar distributed unitary matrix. We have that \( G \) and \( (G^\dagger G)^{1/2} U \) have the same joint element distribution, and so in particular have the same distribution of eigenvalues.

*Proof.* By the singular value decomposition, \( G = U_1 \Sigma U_2 \) for some unitary matrices \( U_1, U_2 \) and where \( \Sigma \) is the diagonal matrix of the singular values. We then have \( (G^\dagger G)^{1/2} U = U_2^\dagger \Sigma U_2 U \). By the assumed bi-unitary invariance of \( G \), this matrix and \( G \) have the same distribution. \( \square \)

The matrix

\[
A := (G^\dagger G)^{1/2} U
\]

is well defined for \( G \) rectangular, and moreover the property of \( G \) being bi-unitary invariant can be generalised to this setting. Of interest is the relation between the joint element distributions of \( G \) and \( A \) [128].

**Proposition 2.11.** Let \( G \) be a bi-unitary invariant rectangular \( n \times N \) random matrix with joint element distribution of the functional form \( g(G^\dagger G) \). The joint element distribution of the matrix \( A \) (2.40) with \( U \) a Haar distributed unitary matrix is proportional to

\[
(\det A^\dagger A)^{(n-N)/2} g(A^\dagger A).
\]
As a consequence, for \( G \) a rectangular complex Ginibre matrix, the eigenvalue PDF of \( A \) is proportional to

\[
\prod_{l=1}^{N} |z_l|^{2(n-N)} e^{-|z_l|^2} \prod_{1 \leq j<k \leq N} |z_k - z_j|^2, \tag{2.42}
\]

Proof. Write \( B = G^\dagger G \). With the joint element distribution of \( G \) of the functional form \( g(G^\dagger G) \), it is a standard result (see e.g. [133] Eq. (3.23)) that the joint element distribution of \( B \) is proportional to \( \det B^{n-N} g(B) \). But \( B \) and \( A^\dagger A \) have the same joint element distribution from the bi-unitary invariance of \( G \), and the result just quoted with \( n = N \) tells us that the Jacobian for the joint element distribution of \( A \) and \( B \) is a constant, which implies (2.41). In the particular case that \( G \) is a rectangular complex Ginibre matrix, the function \( g \) in (2.41) is the exponential \( g(X) = e^{-\text{Tr}X} \). Furthermore, with the eigenvalues of \( A \) denoted \( \{z_i\} \), we have \( (\det A^\dagger A)^{(n-N)} = \prod_{i=1}^{N} |z_i|^{2(n-N)} \). Taking these points into consideration, we see (2.42) results by following the proof of Proposition 2.1.

The correlations for (2.42) are of the determinantal form (2.9). Denoting the corresponding correlation by \( K_{NG}^{\text{IG}}(w,z) \), the derivation of (2.10) shows \([7, 128]\)

\[
K_{NG}^{\text{IG}}(w,z) = \frac{1}{\pi} e^{-(|w|^2+|z|^2)/2} \sum_{j=1}^{N} \frac{(wz)^{n-N+j-1}}{(n-N+j-1)!}, \tag{2.43}
\]

We know that the eigenvalue density, \( \rho_{(1),N}^{\text{IG}}(z) \) say, results by setting \( w = z \) in \( K_{NG}^{\text{IG}}(w,z) \). Knowing from (2.21) that for \( N \) large \( \Gamma(N;xN)/\Gamma(N) \) exhibits a transition from the value 1 for \( 0 < x < 1 \) to the value 0 for \( x > 1 \), we see from (2.43) that

\[
\lim_{N \to \infty} \rho_{(1),N}^{\text{IG}}(z)(\sqrt{N}z)_{|n/N = x+1} = \frac{1}{\pi} \left( \chi_{|z|<\sqrt{x+1}} - \chi_{|z|<\sqrt{x}} \right). \tag{2.44}
\]

Here \( \chi_{|z|<\sqrt{x+1}} - \chi_{|z|<\sqrt{x}} \) corresponds to an annulus of inner radius \( \sqrt{x} \) and outer radius \( \sqrt{x+1} \). Generally, for random matrices of the form \( A = UTV \), where \( U, V \) are Haar distributed and the singular values of \( T \) converge to a compactly supported probability measure, it is known that the eigenvalue PDF in the complex plane is either a disk or an annulus. This is referred to as the single ring theorem \([123, 169]\). Scaling of (2.43) in the bulk of the annulus, or at the boundary, is straightforward and leads to the functional forms exhibited in Proposition 2.4 for the GinUE. Furthermore, in the double scaling regime that the spectrum tends to form a thin annulus of width \( O(1/N) \), the scaling of (2.43) in the bulk gives rise to the weakly non-Hermitian limit \([2.39, 76]\). We also stress that the induced Ginibre ensemble was introduced more generally in \([7, 128]\), where the eigenvalue PDF is a
mixture of (2.32) and (2.42). (The induced GinUE model is then obtained by setting the masses to zero \(m_f = 0\) and \(\tau = 0\).) In [7], the finite-\(N\) expression of the correlation functions as well as their scaling limits both at strong and weak non-Hermiticity were obtained.

2.5. **Complex spherical ensemble.** For \(G_1, G_2\) matrices from the GinUE, matrices of the form \(G = G_1^{-1}G_2\) are said to form the complex spherical ensemble [201]. For with \(\alpha, \beta \in \mathbb{C}\) with \(|\alpha|^2 + |\beta|^2 = 1\), introduce the transformed pair of matrices \((C, D)\),

\[
C := -\bar{\beta}G_2 + \bar{\alpha}G_1, \quad D = \alpha G_2 + \beta G_1.
\]

Since \(G_1, G_2\) have independent standard complex elements, it is easy to check that the pair \((G_1, G_2)\) has the same distribution as \((C, D)\). As a consequence, the eigenvalues \(\{z_j\}\) of \(G\) are unchanged by the fractional linear transformation

\[
z \mapsto \frac{z\alpha - \bar{\beta}}{z\beta - \bar{\alpha}}.
\]

This implies that upon a stereographic projection from the complex plane to the Riemann sphere, the eigenvalue distribution of \(\{G\}\) is invariant under rotation of the sphere, giving rise to the name of the ensemble and telling us that on this surface the eigenvalue density is constant.

The explicit eigenvalue PDF for the complex spherical ensemble was calculated by Krishnapur [201].

**Proposition 2.12.** For the complex spherical ensemble, the eigenvalue PDF is proportional to

\[
\prod_{l=1}^{N} \frac{1}{(1 + |z_l|^2)^{N+1}} \prod_{1 \leq j < k \leq N} |z_k - z_j|^2, \quad z_l \in \mathbb{C}.
\]

**Proof.** The joint element distribution of \((G_1, G_2)\) is proportional to \(\exp(-\text{Tr}\ G_1^\dagger G_1 - \text{Tr}\ G_2^\dagger G_2)\). Substituting \(G_2 = GG_1\), (this gives a Jacobian factor \(|\det G_1|^N\) then integrating over \(G_1\) gives that the element distribution of \(G\) is proportional to

\[
\det(\mathbb{I} + G^\dagger G)^{-2N}.
\]

Introducing now the Schur decomposition (2.2), and changing variables as in the proof of Proposition 2.1, we have from this that the element distribution of the upper triangular matrix \(Z\) therein is proportional to

\[
\det(\mathbb{I} + Z^\dagger Z)^{-2N} \prod_{1 \leq j < k \leq N} |z_k - z_j|^2.
\]

Here \(\{z_j\}\) are the eigenvalues of \(G\), and also the diagonal entries of \(Z\).
It remains to integrate over the strictly upper triangular entries of $Z$. For this, denote the leading $n \times n$ sub-block of $Z$ by $Z_n$ and let the product of differentials for the strictly upper entries of $Z_n$ be denoted $(d\tilde{Z}_n)$. For $p \geq n$, define

$$I_{n,p}(z_1, \ldots, z_n) := \int \frac{1}{\det(1 + Z_n Z_n^\dagger)^p} (d\tilde{T}_n).$$

Writing

$$Z_n = \begin{bmatrix} Z_{n-1} & u \\ 0_n & z_n \end{bmatrix}, \quad u = [Z_{jn}]_{j=1}^{n-1},$$

it’s possible to integrate out over the elements of $u$ to obtain the recurrence \[179, 145\]

$$I_{n,p}(z_1, \ldots, z_n) = \frac{C_{n-1,p}}{(1 + |z_n|^2)^{p-n+1}} I_{n-1,p-1}(z_1, \ldots, z_{n-1}), \quad C_{n-1,p} = \int \frac{(dv)}{(1 + v^\dagger v)^p}.$$

Iterating this with $n = 2N$, $p = N$ shows

$$\int \det(1 + Z^\dagger Z)^{-2N} (d\tilde{Z}_N) \propto \prod_{l=1}^{N} \frac{1}{(1 + |z_l|^2)^{N+1}},$$

which when used in (2.46) implies (2.45).

The stereographic projection from the south pole of a sphere with radius 1/2, spherical coordinates $(\theta, \phi)$, to a plane tangent to the north pole is specified by the equation

$$z = e^{i\phi} \tan \frac{\theta}{2}, \quad z = x + iy.$$

Making this change of variables in (2.46) gives the PDF on the sphere proportional to

$$\prod_{1 \leq j < k \leq N} |u_k v_j - u_j v_k|^2, \quad u = \cos(\theta/2) e^{i\phi/2}, \quad v = -i \sin(\theta/2) e^{-i\phi/2}.$$

Here $u, v$ are the Cayley-Klein parameters, and moreover $|u_k v_j - u_j v_k| = |r_k - r_j|$ where $r_j, r_k$ are the vector coordinates on the sphere. It is furthermore the case that minus the logarithm of this distance solves the Poisson equation on the sphere, and so (2.48) has the interpretation of the Boltzmann factor at coupling $\beta = 2$ of the corresponding Coulomb gas; see [133, §15.6] for more details.

After first removing $u_k, u_j$ from the product of differences in (2.48), the Vandermonde determinant identity (2.12) can be used to compute the correlation functions following a strategy analogous to that used in the proof of Proposition 2.2. This calculation was first done in the context of the corresponding two-dimensional one-component plasma, as obtained by the rewrite of (2.48) analogous to (1.10) [72].
Proposition 2.13. The $n$-point correlations for (2.48) are given by

$$\rho_n((\theta_1, \phi_1), \ldots, (\theta_n, \phi_n)) = \left(\frac{N}{\pi}\right)^n \det \left[ (u_j \bar{u}_k + v_j \bar{v}_k)^{N-1} \right]_{j,k=1,\ldots,n}.$$  

Remark 2.14.

1. Changing variables $\theta_j = 2r_j \sqrt{\pi/N}$ in (2.49) and taking $N \to \infty$, the bulk scaled correlation kernel [2.18] results. Here there is no edge regime.

2. Denote by $a(X)$ an $n \times N (N \times M)$, $n \geq N (M \geq N)$ standard complex Gaussian matrix, and set $A = a^\dagger a$, $Y = A^{-1/2}X$. In terms of $Y$ define $Z = U(YY^\dagger)^{1/2}$, which corresponds to the induced ensemble construction of §2.4. It was shown in [129] that the element PDF of $Z$ is proportional to

$$\det(Z^\dagger Z)^{M-N} \frac{1}{\det(I + Z^\dagger Z)^{n+M}}.$$  

The proof of Proposition 2.12 now gives that the corresponding eigenvalue PDF is proportional to

$$\prod_{i=1}^N \frac{|z_i|^{2(M-N)}}{1 + |z_i|^2} \prod_{1 \leq j < k \leq N} |z_k - z_j|^2.$$  

With $M, n$ scaled with $N$, $M/N \to \alpha_1 \geq 1, n/N \to \alpha_2 \geq 1$ the leading order eigenvalue support now occurs in an annulus with inner and outer radii $r_1 = \sqrt{1/(\alpha_1 - 1)}$, $r_2 = \sqrt{\alpha_1/(\alpha_2 - 1)}$. Scaling of the correlation functions at either of these boundaries gives the edge kernel (2.19).

2.6. Sub-block of a unitary matrix. Let $U$ be an $(n + N) \times (n + N)$ unitary matrix, and let $A$ be the top $N \times N$ sub-block. The non-zero eigenvalues of $A$ are then the nonzero eigenvalues of $QD$, where $D$ is the diagonal matrix with the first $N$ diagonal entries 1, and the last $n$ diagonal entries 0. The eigenvalues of this matrix are the same as that for $UD$, since in general for square matrices the eigenvalues of $AB$ and $BA$ coincide, and furthermore $D^2 = D$. For $u$ a normalised eigenvector of $UD$ with eigenvalue $\lambda$, computing the length squared of $UDu$ gives $u^\dagger Du = |\lambda|^2$, where use has been made of the fact $U^\dagger U = 1$. But with the entries of $u$ all nonzero, the action of $D$ reduces the length, implying $|\lambda| < 1$. With $U$ chosen with Haar measure, it was shown by Zyczkowski and Sommers [202] that the exact eigenvalue PDF of $A$ can be calculated.

Proposition 2.15. For $A$ the top $N \times N$ sub-block of an $(n + N) \times (n + N)$ unitary matrix chosen with Haar measure, the eigenvalue PDF is proportional to

$$\prod_{l=1}^N (1 - |z_l|^2)^{n-1} \prod_{1 \leq j < k \leq N} |z_k - z_j|^2, \quad |z_l| < 1.$$
Proof. Let $C$ be the block of $U$ in the first $N$ columns directly below $A$. Then by the unitarity of $U$, $A^\dagger A + C^\dagger C = I_N$. This implies a joint distribution in the space of general $N \times N$ and $n \times N$ complex matrices $A, C$ proportional to the matrix delta function

\begin{equation}
\delta(A^\dagger A + C^\dagger C - I_N) \propto \int e^{\text{Tr}(iH - \mu I_N)(A^\dagger A + C^\dagger C - I_N)} \, (dH), \quad \mu > 0.
\end{equation}

(2.51)

Here in the integral form of the matrix delta function $H$ is Hermitian; see e.g. [133, Eq. (3.27)]. Beginning with this integral form the integration of the complex matrix $C$ can be carried out according to [133, displayed equation below (3.27)] to give that the element PDF of $A =: A_N$ is proportional to

\begin{equation}
F_n(A_N) := \int (\det(iH_N - I_N))^{-\eta} e^{\text{Tr}(iH_N - I_N)(A_N^\dagger A_N - I_N))} \, (dH_N).
\end{equation}

(2.52)

This matrix integral is the starting point for the derivation of (2.51) given in [111 Appendix B], which we follow below.

In (2.52) the integral is unchanged by conjugating $H$ with a unitary matrix $V$ say. Choosing $V$ to be the unitary matrix in the Schur decomposition (2.2) allows us to effectively replace $A_N$ by $Z_N$ throughout (2.52). Doing this, and integrating too over $Z_N$ (i.e. the strictly upper triangular entries of $Z_N$), we denote the matrix integral by $\tilde{F}_n(z_1, \ldots, z_N)$.

Substituting the decomposition (2.47) for $Z_N$ we see that the vector $u$ occurs as a quadratic form, and can be integrated over. To progress further, $H_N$ too is decomposed by a bordering procedure of the leading $(N - 1) \times (N - 1)$ sub-block $H_{N-1}$, with the $N - 1$ column (row) vector $w$ ($w^\dagger$) and entry $h_N$ in the bottom right corner. Key now is a determinant identity for a block matrix

\[
\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(D) \det(A - BD^{-1}C),
\]

familiar from the theory of the Schur complement (see e.g. [251]), applied to the first term in the integrand of (2.52) with $D = h_N$ (a scalar). A shift of the integration domain according to the additive rank 1 perturbation $H_{N-1} \mapsto H_{N-1} - iww^\dagger$ gives a quadratic form in $H_{N-1}$ in $w$, which after computation reduces (2.52) to

\begin{equation}
\tilde{F}_n(z_1, \ldots, z_N) \propto \int (\det(iH_{N-1} - I_{N-1}))^{-\eta} e^{\text{Tr}(iH_{N-1}) - 1 - i|z_N|^2 - 1)(i|z_N|^2 - 1)} \times e^{\text{Tr}(Z_{N-1}^\dagger Z_{N-1}(iH_{N-1} - I_{N-1}))} \, (dH_{N-1})(dZ_{N-1})dh_N.
\end{equation}

(2.53)

Here, using the residue theorem, the integral over $h_N$ can be performed, yielding the recurrence

\[
\tilde{F}_n(z_1, \ldots, z_N) \propto (1 - |z_N|^2)^{n-1} \chi_{|z_N|^2 < 1} \tilde{F}_n(z_1, \ldots, z_{N-1}).
\]

This concludes the proof.
Iterating gives the first factor in (2.50), while the product of differences is due to the Jacobian (2.5) for the change of variables to the Schur decomposition as computed in the proof of Proposition 2.1.

We have seen that the eigenvalue PDF (2.45) can, after a stereographic projection, be identified as the Boltzmann factor for a Coulomb gas model on the sphere. The concept of stereographic projection can also be applied to a pseudosphere, which refers to the two-dimensional hyperbolic space with constant negative Gaussian curvature. Doing so gives the metric specifying the Poincaré disk. Consideration of the solution of the Poisson equation on the latter allows (2.50) to be interpreted as the Boltzmann factor for a Coulomb gas model on the Poincaré disk [145].

The eigenvalue correlations for (2.50) are determinantal with correlation kernel, $K_{N,n}$ say, expressible in terms of the incomplete beta function $I_x(a,b) \propto \int_0^x t^{a-1}(1-t)^{b-1}dt$, normalised to equal unity for $x = 1$. Thus [127, § 3.2.3]

$$K_{N,n}(w,z) = n \frac{1-|w|^2}{\pi} \frac{1-|z|^2}{(1-w^2)^{n+1}} \left(1 - I_w(N,n+1)\right).$$

In the limit $n, N \to \infty$ with $n/N = \alpha > 0$ it follows from this that

$$\rho^{(1),N}(z) \sim \frac{N\alpha}{\pi} \frac{1}{(1-|z|^2)^2} \chi_{|z|^2 < 1/(1+\alpha)}.$$

Hence there is a bulk regime, and an edge regime. The neighbourhood of the origin is typical of the bulk regime, and it follows from (2.54) that

$$\lim_{n,N \to \infty} \frac{1}{n} K_{N,n}(w/\sqrt{n}, w/\sqrt{n}) = K^b_{\infty}(w, z),$$

where $K^b_{\infty}$ is specified by (2.18). And after appropriate scaling about $|z| = 1/(1+\alpha)$, the universal edge density as given by the first term in (2.38) is obtained [127],

$$\lim_{n,N \to \infty} \frac{\alpha}{(1+\alpha)^2} \frac{1}{N} \rho^{(1),N} \left(\sqrt{1 + \frac{\alpha}{\sqrt{N} 1 + \alpha}} - \frac{\xi}{\sqrt{N} 1 + \alpha}\right) = \frac{1}{2\pi} \left(1 - \text{erf}(\sqrt{2}\xi)\right).$$

A distinct scaling regime, referred to as close to unitary [152], or more specifically a multiplicative rank $n$ contraction of a random unitary matrix, is obtained by taking $N \to \infty$ with $n$ fixed in (2.54). Scaling the eigenvalues $z_k = (1 - y_k/N)e^{i\phi_k/N}$ gives [199] [127, Eq. (3.2.140)]

$$\lim_{N \to \infty} \frac{1}{N^2} K_{N,n}(z_1, z_2) = \frac{1}{\pi} \frac{(2\sqrt{y_1y_2})^{n-1}}{(n-1)!} \int_0^1 s^{n-1} e^{-(y_1+y_2+i(\phi_1-\phi_2)s)} ds.$$

This functional form was first obtained for the scaled correlation kernel in the setting of a rank $n$ additive anti-Hermitian perturbation of a GUE matrix [149, Eq. (15)] after setting
\( \nu(x) = 1/\pi \), identifying \( \tilde{z}_k = (\phi_k + iy_k)/2 \) and taking the limit \( g_m \to 1 \) \((m = 1, \ldots, M)\) with \( M \) identified as \( n \).

A generalisation of the setting of Proposition 2.15 is to consider the top \((N + L) \times N\) sub-block of an \((n + N + L) \times (n + N + L)\) unitary matrix chosen with Haar measure. Denote such a rectangular matrix by \( A_{N,L} \), and define from this \( \tilde{A}_{N,L} = V(A_{N,L}^* A_{N,L})^{1/2} \), where \( V \) is an \( N \times N \) random Haar distributed unitary matrix. It was shown in \[127\] that the eigenvalue PDF of \( \tilde{A}_{N,L} \) is proportional to \( (2.45) \) with an additional factor of \( \prod_{i=1}^N |z_i|^{2L} \). Denote the corresponding correlation kernel by \( K_{N,n,L} \). Scaling \( L \) by requiring that \( L/N = \alpha > 0 \) as \( N \to \infty \), scaling the eigenvalues \( z_k \) as in the above paragraph, and keeping \( n \) fixed, it was shown in \[199, 127\] Eq. (3.2.122) that

\[
\lim_{N \to \infty} \frac{1}{N^2} K_{N,n,L}(z_1, z_2) \left|_{L/N=\alpha} \right. = \frac{1}{\pi} \left( \frac{2\sqrt{y_1 y_2}}{(n-1)!} \right) \int_{\alpha}^{\alpha+1} s^n e^{-(y_1 + y_2 + i(\phi_1 - \phi_2))s} \, ds.
\]

Note the consistency with (2.58) in the limit \( \alpha \to 0^+ \).

2.7. Products of complex Ginibre matrices. Consideration of the eigenvalue PDF for the product \( G_1 G_2 \), where \( G_1 \) \((G_2)\) are independent \( N \times p \) \((p \times N)\) rectangular complex Ginibre matrices was first undertaken by Osborn \[250\], in the context of a study relating to quantum chromodynamics. (We also mention that the GinUE has been directly used in the QCD bulk spectrum \[236\]) Later \[181\] it was realised that this eigenvalue PDF must be the same as that for the product \( \tilde{G}_1 \tilde{G}_2 \) \((or \tilde{G}_2 \tilde{G}_1)\), where each \( \tilde{G}_i \) is an \( N \times N \) complex random matrix with element distribution proportional to

\[
| \det \tilde{G}_i \tilde{G}_i^\dagger |^{v_1} e^{-\operatorname{Tr} \tilde{G}_i \tilde{G}_i^\dagger}, \quad v_1 = 0, v_2 = p - N;
\]

note the construction \(2.40\) for random matrices with this element PDF. Hence it suffices to consider the square case. Key for this is the generalised Schur decomposition (equivalent to the so-called QZ decomposition in numerical linear algebra)

\[
\tilde{G}_1 = UZ_1 V, \quad \tilde{G}_2 = V^\dagger Z_2 U^\dagger,
\]

where \( Z_1, Z_2 \) are upper triangular matrices with diagonal entries \( \{z_j^{(1)}\}, \{z_j^{(2)}\} \) such that

\[
z_j^{(1)} z_j^{(2)} = z_j
\]

with \( \{z_j\} \) the eigenvalues of \( \tilde{G}_1 \tilde{G}_2 \).

For the change of variables \(2.61\) the wedge product strategy of the proof of Proposition 2.1 can again be implemented \[250, 133\] proof of Proposition 15.11.2] to give for the Jacobian \(2.5\). Substituting \(2.61\) in \(2.60\) and recalling \(2.62\), it follows that the eigenvalue
The functional form (2.63)
\[
\prod_{l=1}^{N} w^{(2)}(z_l) \prod_{j<k} |z_j - z_k|^2, \quad w^{(2)}(z) := \int_{C} d^2z_1 |z_1|^{2\nu_1} \int_{C} d^2z_2 |z_2|^{2\nu_2} \delta(z - z_1z_2) e^{-|z_1|^2 - |z_2|^2}.
\]
Moreover, it was shown in [250] that \(w^{(2)}(z)\) can be expressed in terms of the modified Bessel function \(K_{\nu_2 - \nu_1}(2|z|)\), assuming \(\nu_2 \geq \nu_1\).

Akemann and Burda [9] (in the case of all matrices square), and soon after Adhikari et al. [3] (the general rectangular case), generalised (2.63) to hold for the product of \(M\) complex Gaussian matrices. Following the work of Ipsen and Kieburg [181], it is now known that the rectangular case can be reduced to the square case, where the square matrices have distribution as in (2.60), with the \(\nu_i\) equal to difference (assumed non-negative) between the number of rows in \(G_i\) and the number of rows in \(G_1 (= N)\). The role of the modified Bessel function, as the special function evaluating the weight \(w^{(2)}(z)\) in (2.63), is now played by the Meijer G function

\[
G_{m,n}^{p,q}(z|a_1,\ldots,a_p;b_1,\ldots,b_q) = \frac{1}{2\pi i} \int_{C} \prod_{j=1}^{m} \Gamma(b_j - s) \prod_{j=1}^{n} \Gamma(1 - a_j + s) \prod_{j=m+1}^{p} \Gamma(1 - b_j + s) \prod_{j=n+1}^{q} \Gamma(a_j - s) z^s \, ds,
\]

where \(C\) is an appropriate contour as occurs in the inversion formula for the corresponding Mellin transform; see e.g. [234] for an extended account.

**Proposition 2.16.** Consider the product \(\tilde{G}_1 \cdots \tilde{G}_M\) (in any order) of complex Gaussian matrices with element PDF as given in (2.60), with each \(\nu_i \geq 0\) but otherwise unrestricted. The eigenvalue PDF is proportional to

\[
\prod_{l=1}^{N} \tilde{w}^{(M)}(z_l) \prod_{j<k} |z_j - z_k|^2, \quad \tilde{w}^{(M)}(z) = C_{M,0}^{0,M} \left(|z|^2 \right|_{\nu_1,\ldots,\nu_M}).
\]

The functional form (2.65) remains valid for the product of rectangular complex Ginibre matrices \(G_1 \cdots G_M\), with \(\nu_i\) specified as in the second sentence above (2.64).

**Proof.** (Sketch) The generalised Schur decomposition (2.61) can be extended to a general number \(M\) of square matrices to give what is referred to as the periodic Schur form [60, 235, Corollary 3.2]

\[
\tilde{G}_i = U_iZ_iV_i, \quad V_i = U_{i+1}^t (i = 1,\ldots,M - 1), \quad V_M = U_1^t;
\]
in [9] this was deduced independently. The key features with respect to computing the eigenvalue PDF of the product \(\tilde{G}_1 \cdots \tilde{G}_M\), as already discussed in the case \(M = 2\), again hold true. In particular the Jacobian factor for the change of variables is given by (2.5), and
the $k$-th diagonal entry of each $Z_i$ multiply together to give the eigenvalue $z_k$ of the product. It follows that the eigenvalue PDF is given by (2.65) with

$$(2.67) \quad w^{(M)}(z) \propto \int_C d^2z_1 |z_1|^{2\nu_1} \cdots \int_C d^2z_M |z_M|^{2\nu_M} \delta(z - z_1 \cdots z_M)e^{-\sum_{i=1}^M |z|^2}.$$  

As noted in [9, 3], taking the Mellin transform of this expression leads to the Meijer G function form in (2.65). □

An easy consequence of (2.65) is that the eigenvalue correlations are determinantal with kernel, $K_{N,M}$ say, given by [9, 3],

$$(2.68) \quad K_{N,M}(w,z) = \left(w^{(M)}(w)w^{(M)}(z)\right)^{1/2} \sum_{j=1}^N \frac{(wz)^{j-1}}{\prod_{m=1}^M \Gamma(j + \nu_m)}.$$

Rigorous asymptotic analysis of (2.68) can be carried out [9, 230]. For example, with $w = z$ and $z \mapsto N^{M/2}z$, one obtains

$$(2.69) \quad \lim_{N \to \infty} N^{M-1} \rho_{(1),N}(N^{M/2}z) = \frac{|z|^{-2+2/M}}{\pi M} \chi_{|z|<1},$$

where it is assumed each $\nu_i$ is fixed. To interpret this result, form the $M$-th power of a single GinUE matrix $G$. The eigenvalues are $\{\tilde{z}_j = z_j^M\}$ with $\{z_j\}$ the the eigenvalues of $G$. Changing variables in the circular law (2.17) $\tilde{z} = z^M$ gives the density (2.69) in the variable $\tilde{z}$. Hence, as anticipated in [66], the global eigenvalue density for the product of $M$ independent GinUE matrices is identical to that of the $M$-th power of a single GinUE matrix. This same global scaling limit is also well defined if some or all of the $\nu_i$ are proportional to $N$. An explicit limit formula has been obtained by Liu and Wang [230]. It is found that the density has the support of an annulus if each $\mu_i := \lim_{N \to \infty} \nu_i/N$ is positive, but otherwise is again supported in a disk, albeit with a density function distinct to that in (2.69).

One observes from (2.69) a singularity at $z = 0$, not present in the circular law global density (2.17) for GinUE. A consequence is that the correlation kernel about this point, simply obtained by setting the upper terminal of the sum in (2.68) equal to infinity, is distinct from the corresponding kernel (2.18) for the bulk scaled GinUE. On the other hand, it is shown in [230] that this latter kernel is obtained by choosing the origin a distance $\alpha\sqrt{N}$ from $z = 0$, for any $0 < \alpha < 1$. Also, from [9, 230] we know that the scaling of (2.68) about $|z| = 1$ reclaims the kernel (2.19) for the edge scaled GinUE.

**Remark 2.17.**

1. Required in the derivation of (2.69) from (2.68) is knowledge of the $z \to \infty$ asymptotic
expansion \[234\]

\[(2.70) \quad C_{0,M}^{0,M}(z | v_1, \ldots, v_M) \sim \frac{1}{\sqrt{M}} \left( \frac{2\pi}{z} \right)^{(M-1)/2} e^{-Mz^2/M} z^{(v_1+\cdots+v_M)/M} \left( 1 + O(z^{-1/M}) \right).\]

Of particular interest is the exponential term herein, which with \(z\) replaced by \(|z|^2\) as required in \((2.68)\) reads \(e^{-M|z|^2/M}\). This suggests a modification of \(U\) in \((1.10)\) to read

\[(2.71) \quad U_M := \frac{M}{2} \sum_{j=1}^{N} |z_j|^{2/M} - \sum_{1 \leq j < k \leq N} \log |z_k - z_j|.\]

Indeed starting from \(U_M\) and repeating the working of Remark \(2.5\) reclaims \((2.69)\). The correlation kernel appearing in such a calculation is called the Mittag-Leffler kernel. This terminology applies too in the more general case that a term \(-c \sum_{j=1}^{N} \log |z_j|\) is included in \((2.71)\) \([35, 39]\). This very case, for \(c = (k - M) / M\), \((k = 1, \ldots, M)\), appears in the calculation of the joint distribution of the eigenvalues of the \(M\)-th power \((M \leq N)\) of a GinUE matrix \([113\] Th 1.5\). We also refer to \([196, 197]\) for the appearance of the Mittag-Leffler point process as a degenerate limit of the elliptic determinantal point process.

2. The eigenvalue PDF for a product of GinUE matrices and inverses of GinUE matrices has been shown to be of the form \((2.65)\), but with the Meijer G function therein replace by a different Meijer G function \([3]\). The corresponding limiting eigenvalue density, in the case of equal numbers of matrices and inverses, is \([166, 291]\)

\[(2.72) \quad \lim_{N \to \infty} \rho_{(1),N}(z) = \frac{1}{\pi M} \frac{|z|^{-2+2/M}}{(1 + |z|^{2/M})^2}.\]

This functional form relates to the \(M = 1\) case in the same way as \((2.69)\) relates to \((2.17)\), being the \(M\)-th power of the so-called spherical law.

3. The case \(M = 2\) of Proposition \(2.16\) permits a generalisation. Thus for the matrices in the product \(\hat{G}_1 \hat{G}_2\) choose

\[\hat{G}_1 = \sqrt{1 + \tau} X_1 + \sqrt{1 - \tau} X_2, \quad \hat{G}_2 = \sqrt{1 + \tau} X_1^\dagger + \sqrt{1 - \tau} X_2^\dagger, \quad 0 < \tau < 1,\]

where \(X_1, X_2\) are \(N \times (N + p)\) rectangular complex Gaussian matrices. The joint element distribution is then proportional to

\[\exp \left( -\frac{1}{1 - \tau^2} \text{Tr} \left( \hat{G}_1^\dagger \hat{G}_1 + \hat{G}_2^\dagger \hat{G}_2 - 2\tau \text{Re} \hat{G}_1 \hat{G}_2^\dagger \right) \right);\]

cf. \((2.33)\). Results in \([250]\) give that the eigenvalue PDF of \(\hat{G}_1 \hat{G}_2\) is of the form in \((2.63)\) but with \(w^{(2)}(z)\) now dependent on \(\tau\). Due to the shape of resulting droplet, this gives rise to the so-called shifted elliptic law \([14\] Th. 1\).

4. The product of \(M\) random matrices in the limit \(M \to \infty\) is of interest from a dynamical
systems viewpoint, as the scaled logarithm of the singular values gives the Lyapunov spectrum. Closely related for product matrices themselves are the stability exponents, defined as $\frac{1}{M} \log |z_k|$, $(k = 1, \ldots, N)$. In fact for bi-unitary invariant random matrices at the least, the Lyapunov and stability exponents are the same \cite{254}. Works relating to the computation of these exponents and their related statistical properties for GinUE include \cite{134,135,136,137,138,139,140,232,143,233,144}.

2.8. **Products of truncated unitary matrices.** The eigenvalue PDF for a truncation of a Haar distributed unitary matrix has been given in Proposition \ref{thm:2.15}. It turns out that knowledge of the evaluation of the matrix integral \eqref{eq:2.53} as implied in the proof of Proposition \ref{thm:2.15}, used in conjunction with the periodic Schur form \eqref{eq:2.66}, is sufficient to allow for the determination of the eigenvalue PDF of the product of $M$ truncated Haar distributed unitary matrices \cite{111}.

**Proposition 2.18.** Consider $M$ independent Haar distributed $(n_j + N) \times (n_j + N)$ unitary matrices, with the top $N \times N$ sub-block of each denoted $A_j$ ($j = 1, \ldots, M$). The eigenvalue PDF of the product $A_1 \cdots A_M$ (in any order) is given by \eqref{eq:2.65} with
\begin{equation}
\rho^{(M)}(z) = G_{M,0}^{M,0} \left( |z|^2 \right)^{n_1, \ldots, n_M} \left( 0, \ldots, 0 \right) \chi_{|z|<1}.
\end{equation}

*Proof. (Sketch)* The method of proof of Proposition \ref{thm:2.15}, which begins with the periodic Schur decomposition \eqref{eq:2.66}, and then integrates out over the upper triangular entries of the matrices $Z_j$ (here this latter step requires knowledge of the evaluation of the matrix integral \eqref{eq:2.53}) gives \eqref{eq:2.65} with
\begin{equation}
\rho^{(M)}(z) \propto \int_C d^2 z_1 \cdots \int_C d^2 z_M \delta(z - z_1 \cdots z_M) \prod_{j=1}^M (1 - |z_j|^2)^{n_j - 1}.
\end{equation}

Taking the Mellin transform of this expression leads to the Meijer G function form in \eqref{eq:2.73}.

Analogous to \eqref{eq:2.68}, it follows from the result of Proposition \ref{thm:2.18} that the eigenvalue correlations are determinantal with kernel (to be denoted $K_{N,M}^U$) given by \cite{111}
\begin{equation}
K_{N,M}^U(z_1, z_2) = \left( \rho^{(M)}(z_1) \rho^{(M)}(z_2) \right)^{1/2} \sum_{j=1}^N (z_1 \bar{z}_2)^j - 1 \prod_{m=1}^M \prod_{m=1}^M \frac{(n_m + j - 1)!}{(j - 1)!}.
\end{equation}

Note here, that in distinction to \eqref{eq:2.68}, the $\rho^{(M)}$ are given by \eqref{eq:2.73}. Suppose all the $n_i$ are equal to $n$, and that for large $N$, $n/N = \alpha > 0$. Using \eqref{eq:2.74} in the case $z_1 = z_2$, it is derived in \cite{111} that
\begin{equation}
\rho^{(1),N}(z) \sim \frac{\alpha N}{\pi M} \frac{|z|^{-2+2/M}}{(1 - |z|^{2/M})^2} \chi_{|z|<(1+\alpha)^{-M/\alpha}}.
\end{equation}
cf. (2.72).

In the same limit, with the scaling $z \mapsto z/N^{M/2}$ (and similarly for $w$), the kernel (2.74) multiplied by $N^{-M}$ has the same limit as the kernel (2.68) for the product of $M$ GinUE matrices about the origin in the case of each matrix square (all $\nu_i = 0$). If instead a point $z_0$ with $0 < z_0 < (1 + \alpha)^{-M/2}$ is chosen before this scaling, one obtains instead the GinUE result (2.18), while at the boundary of support, generalising (2.57) it is found that (2.76)

$$
\lim_{n_i \to \infty} M \frac{\alpha}{N} \frac{\alpha}{(1 + \alpha)^{M+1}} \rho(1, N) \left( \frac{1}{(1 + \alpha)^{M/2}} + \frac{\xi}{\sqrt{N}} \frac{\sqrt{\mathcal{M}^1/2}}{(1 + \alpha)^{(M+1)/2}} \right) = \frac{1}{2\pi} \left( 1 - \text{erf}(\sqrt{2\xi}) \right),
$$

thus again exhibiting the universal functional form seen at the edge scaling of the GinUE (2.27).

Also considered in [14] is a close to unity scaling, with $n_i = n$ all fixed ($i = 1, \ldots, M$) as $N \to \infty$. Scaling the eigenvalues $z_k = (1 - y_k/N)e^{i\phi_k/N}$, it is found that (2.58) again holds but with each occurrence of $n$ replaced by $nM$.

### 2.9. The distribution of the squared eigenvalue moduli.

All the explicit eigenvalue PDFs obtained in the above subsections of §2, excluding the elliptic Ginibre ensemble, have the form

$$
\prod_{l=1}^{N} w(|z_l|^2) \prod_{1 \leq j < k \leq N} |z_k - z_j|^2,
$$

for some weight function $w$. In particular, they are invariant under rotations about the origin. An observation of Kostlan [200] for the GinUE case (1.7) of (2.77) is that the set of squared eigenvalue moduli $\{|z_j|^2\}_{j=1}^N$, appropriately ordered, are independently distributed (specifically as gamma random variables $\{\Gamma[j; 1]\}$); see too [84, Theorem 1.2].

**Proposition 2.19.** Let $F_w(s_1, \ldots, s_N)$ denote the PDF for the distribution of $\{s_j := |z_j|^2\}_{j=1}^N$ for the PDF (2.77). We have

$$
F_w(s_1, \ldots, s_N) = \frac{1}{N!} \text{Sym} \prod_{j=1}^{N} \int_0^{s_j} w(s) s^{j-1} ds,
$$

where $\text{Sym}$ denotes symmetrisation with respect to $\{s_j\}$.

**Proof.** Starting with (2.11), then substituting (2.12) and its complex conjugate, we see

$$
\prod_{l=1}^{N} w(|z_l|^2) \prod_{1 \leq j < k \leq N} |z_k - z_j|^2 = \prod_{l=1}^{N} w(|z_l|^2) \det[f_{k-1}(z_j)]_{j,k=1}^{N} \det[g_{k-1}(z_j)]_{j,k=1}^{N},
$$
where \( f_l(z) = z^l \), \( g_l(z) = z^{2l} \). According to Andréief’s identity (see e.g. [138]), it follows from this that the integral over \( z_l \in \mathbb{C} \) (\( l = 1, \ldots, N \)), \( I_{w,N} \) say, is itself given by a determinant

\[
I_{w,N} = N! \det \left[ \int_{\mathbb{C}} w(|z|^2) f_l(z) g_{k-1}(z) \, d^2z \right]_{l,k=1}^N.
\]

Substituting the explicit form of \( f_l, g_l \) and changing to polar coordinates \( z_l = r_l e^{i\theta_l} \) shows that only the diagonal terms are non-zero. This allows the determinant to be evaluated

\[
I_{w,N} = N! \pi^N \prod_{j=1}^N \int_0^\infty w(s)s^{j-1} \, ds,
\]

where in each integration the change of variables \( r^2 = s \) has been made. Forming now \( I_{w\phi,N} / I_{w,N} \), where \( \phi \) is a suitable test function, and assuming that [2.77] is normalisable, the result [2.78] follows.

3. Fluctuation formulas

3.1. Counting function in general domains. The eigenvalues of a non-Hermitian matrix are examples of point processes in the plane. Statistical quantities characterising the point process are functions of the eigenvalues \( \{z_j\} \) of the form \( \sum_{j=1}^N f(z_j) \) — referred to as linear statistics — for given \( f \). Such statistics are closely related to the correlation functions. Thus

\[
\langle \sum_{j=1}^N f(z_j) \rangle = \int_{\mathbb{C}} f(z) \rho_{(1),N}(z) \, d^2z,
\]

while

\[
\text{Cov} \left( \sum_{j=1}^N f(z_j), \sum_{l=1}^N g(z_l) \right) = \int_{\mathbb{C}} d^2z \int_{\mathbb{C}} d^2z' f(z)g(z') \left( \rho_{(2),N}(z,z') + \rho_{(1),N}(z)\delta(z-z') \right)
\]

\[
= -\frac{1}{2} \int_{\mathbb{C}} d^2z \int_{\mathbb{C}} d^2z' \left( f(z) - f(z') \right) \left( g(z) - g(z') \right) \rho_{(2),N}(z,z'),
\]

see e.g. [141], §2.1. Here \( \rho_{(2),N}(z,z') : = \rho_{(2),N}(z,z') - \rho_{(1),N}(z)\rho_{(1),N}(z') \).

One of the most prominent examples of a linear statistic is the choice \( f(z) = \chi_{z \in \mathcal{D}} \), where \( \mathcal{D} \in \mathbb{C} \). The linear statistic is then the counting function for the number of eigenvalues in \( \mathcal{D} \), \( N(\mathcal{D}) \) say. Let \( E_N(n; \mathcal{D}) \) denote the probability that there are exactly \( n \) eigenvalues in \( \mathcal{D} \), so that \( E_N(n; \mathcal{D}) = \Pr \left( \sum_{j=1}^N \chi_{z_j \in \mathcal{D}} = n \right) \). Denote the corresponding generating function (in the variable \( 1 - \xi \)) by \( \hat{E}_N(\xi; \mathcal{D}) \) so that

\[
\hat{E}_N(\xi; \mathcal{D}) = \sum_{n=0}^N E_N(n; \mathcal{D})(1 - \xi)^n.
\]
Note that with $1 - \xi = e^{i\lambda}$ this corresponds to the characteristic function for the probability mass functions $\{E_N(n; D)\}$. A straightforward calculation (see e.g. [133 Prop. 9.1.1]) shows that $\tilde{E}_N(\xi; D)$ can be expressed in terms of the correlation functions according to

$$
\tilde{E}_N(\xi; D) = 1 + \sum_{n=1}^{N} \frac{(-\xi)^n}{n!} \int_D d^2z_1 \cdots \int_D d^2z_n \rho(n) N(z_1, \ldots, z_n).
$$

It can readily be checked that (3.4) and (3.2) in the special case $f(z) = g(z) = \chi_{z \in D}$ are consistent with (3.4).

Specialising now to the circumstance that the PDF for the eigenvalues is of the form (2.77), a particular product formula for $\tilde{E}_N(\xi; D)$ can be deduced, which was known to Gaudin [137].

**Proposition 3.1.** Let $\{\rho(n)\}$ in (3.4) be given by (2.9) where the correlation kernel $K_N$ corresponds to (2.77). Let $K_{N,D}$ denote the integral operator supported on $z_2 \in D$ with kernel $K_N(z_1, z_2)$. This integral operator has at most $N$ non-zero eigenvalues $\{\lambda_j(D)\}_{j=1}^{N}$, where $0 \leq \lambda_j(D) \leq 1$, and

$$
\tilde{E}_N(\xi; D) = \prod_{j=1}^{N} (1 - \xi \lambda_j(D)).
$$

**Proof.** Let $\{p_s(z)\}_{s=0}^{\infty}$ be a set of orthogonal polynomials with respect to the inner product $\langle f, g \rangle := \int_D w(|z|^2) f(z) g(z) d^2z$ with corresponding normalisation denoted $h_s$. Taking as a basis $\{(w(|z|^2)^{1/2} p_s(z)\}_{s=0}^{\infty}$ it is straightforward to check that

$$
K_N(z_1, z_2) = (w(|z_1|^2) w(|z_2|^2))^{1/2} \sum_{s=0}^{N-1} \frac{p_s(z_1) p_s(z_2)}{h_s},
$$

and so the eigenfunctions of $K_{N,D}$ are of the form $(w(|z|^2)^{1/2} \sum_{s=0}^{N-1} c_s p_s(z))$ (see e.g. [133 proof of Prop. 5.2.2]). Hence there are at most $N$ nonzero eigenvalues, which moreover can be related to an Hermitian matrix and so must be real. In terms of these eigenvalues, the determinantal form (2.9) substituted in (3.4) implies (3.5) --- this is a result from the theory of Fredholm integral operators (see e.g. [287]). Since by definition, each $n$-point correlation is non-negative, we see from the RHS of (3.4) that $\tilde{E}_N(\xi; D) > 0$ for $\xi < 0$, and so $\lambda_j(D) \geq 0$ ($j = 1, \ldots, N$). Also, the definition (3.3) tells us that $\tilde{E}_N(\xi; D) > 0$ for all $\xi < 1$. This would contradict (3.5) if it was to be that any $\lambda_j(D) > 1$, since in this circumstance there would be a $\xi$ is this range such that $\tilde{E}_N(\xi; D)$ vanishes. \hfill $\Box$

Consider $\sum_{j=1}^{N} x_j$ where $x_j \in \{0, 1\}$ is a Bernoulli random variable with $\Pr(x_j = 1) = \lambda_j(D)$. The characteristic function is $\prod_{j=1}^{N} (1 - \lambda_j(D) + e^{i\lambda_j(D)})$. With $e^{i\lambda} = 1 - \xi$ this gives the RHS of (3.5). But it has already been noted that $\tilde{E}_N(\xi; D)$ with $\xi$ related to $e^{i\lambda}$ in this way is the characteristic function for the counting statistic $N(D)$, and hence the equality
in distribution \( \mathcal{N}(D) \overset{d}{=} \sum_{j=1}^{N} \text{Bernoulli}(\lambda_j(D)) \). From this, it follows using the standard arguments (see [124] § XVI.5, Theorem 2) that a central limit theorem holds for \( \mathcal{N}(D_N) \) (here the subscript \( N \) on \( D_N \) is to indicate that the region \( D \) depends on \( N \)),

\[
\lim_{N \to \infty} \frac{\mathcal{N}(D_N) - \langle \mathcal{N}(D_N) \rangle}{\sqrt{\text{Var}\mathcal{N}(D_N)}} \overset{d}{=} \mathcal{N}[0,1],
\]

valid provided \( \text{Var}\mathcal{N}(D_N) \to \infty \) as \( N \to \infty \); see also [99, 277, 274].

A stronger result, extending the central limit theorem (3.7), follows from the fact that in the variable \( z = 1 - \xi \) has all its zeros on the negative real axis [146].

**Proposition 3.2.** In the setting of the applicability of Proposition 3.1 and with \( \sigma_{N_D} := \sqrt{\text{Var}\mathcal{N}(D_N)} \), we then have that \( \{E_N(k; D_N)\} \) satisfy the local central limit theorem

\[
\lim_{N \to \infty} \sup_{x \in (-\infty, \infty)} \left| \sigma_{N_D} E_N([\sigma_{N_D} x + \langle \mathcal{N}(D_N) \rangle]; D_N) - \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right| = 0.
\]

**Proof.** The fact that the zeros of (3.5) with the variable \( z = 1 - \xi \) are on the negative real axis implies, by Newton’s theorem on log-cavity of the sequence of elementary symmetric functions [247], that \( \{E_N(k; D)\} \) is log concave. It is known that log-cavity is a sufficient condition for extending a central limit theorem to a local limit theorem [49]. \( \square \)

For large \( N \), inside the disk of radius \( \sqrt{N} \), the eigenvalue density for GinUE is constant and the full distribution is rotationally invariant. In such circumstances, for a two-dimensional point process in general, it is known [47] that \( \text{Var}\mathcal{N}(D_N) \) cannot grow slower than of order \( |\partial D_N| \), i.e. the length of the boundary of \( D_N \). Thus both (3.4) and (3.5) are valid for any region \( D_N \) constrained strictly inside the disk of radius \( \sqrt{N} \) and with a boundary of length tending to infinity with \( N \). In fact for GinUE more precise asymptotic information is available [87, Eq. (11)], [125, Eq. (2.7)], which gives that for any \( D_0 \subseteq \{z : |z| \leq 1\} \),

\[
\text{Var}\mathcal{N}(\sqrt{N}D_0) = \sqrt{N} \left\lfloor \frac{|\partial D_0|}{2\pi} \right\rfloor \int_{-\infty}^{\infty} \left( \text{Var} \chi_{U \leq (1 + \text{erf}(t/\sqrt{2}))/2} \right) dt + O\left( \frac{1}{N^{1/2}} \right).
\]

Here \( U \) is a Bernoulli random variable. A direct calculation gives that the integral evaluates to \( \sqrt{\frac{1}{\pi}} \) — the advantage of the form (3.9) is that it remains true if the appearance of the variance throughout is replaced by any even cumulant [87, 125]. In particular this shows that the growth of the variance with respect to the region is the smallest order possible — the corresponding point process is then referred to as being hyperuniform [283, 282, 160].

A corollary of the property of being hyperuniform, together with the fast decay of the correlations, is that the bulk scaled GinUE exhibits number rigidity [162, 161]. This means that conditioning on the positions of the (infinite number of) points outside a region \( D \) fully
determines the number of (but not positions of) the points inside $D$, and their centre of mass.

3.2. **Counting function in a disk.** In the special case that $D_N$ is a disk of radius $R$ centred at the origin (we write this as $D_R$), the polynomials in (3.6) are simply the monomials $p_s(z) = z^s$. The eigenfunctions of $K_{N,D}$ are also given in terms of the monomials as \{$(w(|z|^2))^{1/2}z^{-1}$\}_{j=1,...,N}, and hence for the corresponding eigenvalues we have

$$\lambda_j(D_R) = \frac{\int_0^{R^2} r^{j-1}w(r) \, dr}{\int_0^\infty r^{j-1}w(r) \, dr}, \quad j = 1, \ldots, N.$$ 

Substituting in (3.5) and choosing $w(|z|^2) = \exp(-|z|^2)$ then shows that for the GinUE \[3.10\]

$$\tilde{E}_N(\xi; D_R) = \prod_{j=1}^N \left(1 - \xi \frac{\gamma(j; R^2)}{\Gamma(j)} \right),$$

where $\gamma(j; x)$ denotes the (lower) incomplete gamma function. Note that this remains valid for $N \to \infty$ in keeping with the discussion of the previous paragraph. Setting $\xi = 1$, asymptotic expansions for the incomplete gamma function can be used to deduce the $N \to \infty$ asymptotic expansion of $E_N(0; D_{\alpha\sqrt{N}})$,

\[3.11\]

$$E_N(0; D_{\alpha\sqrt{N}}) = \exp \left( C_1 N^2 + C_2 N \log N + C_3 N + C_4 \sqrt{N} + \frac{1}{3} \log N + O(1) \right), \quad 0 < \alpha < 1.$$ 

Here the constants $C_1, \ldots, C_4$ depend on $\alpha$ and are known explicitly (e.g. $C_1 = -\alpha^4/4$), being first given in \[130\]. The first two of these can be deduced from the large $R$ expansion of the quantity $F_\infty(0; D_R)$, defined in Remark 3.3 1 below, given in the still earlier work \[168\]. The log $N$ term was determined recently in \[88\], as too was the explicit form of the next order term, a constant with respect to $N$.

Let us also mention that for any $p \geq 2$, the $p$-th cumulant $\kappa_p(D_R)$ of the number of eigenvalues in $D_R$ can be written as

\[3.12\]

$$\kappa_p(R) = (-1)^{p+1} \sum_{j=0}^{N-1} \text{Li}_{1-p} \left(1 - \frac{1}{\lambda_j(D_R)}\right),$$

where $\text{Li}_s(x) = \sum_{k=1}^\infty k^{-s}x^k$ is the polylogarithm function. The formula (3.12) as well as its large $N$ behaviour both in the bulk and at the edge were obtained in \[209\].

We turn our attention now to the circumstance that a (possibly scaled) large $N$ limit has already been taken, and ask about the fluctuations of the number of particles in a region $N(D)$ for large values of $|D|$, i.e. the volume of $D$. The first point to note is that if the coefficient of $\xi^n$ in (3.4) tends to zero as $N \to \infty$, then the expansion remains valid in this limit \[225\]. The decay is easy to establish in the determinantal case, since then (see e.g. \[133\]...
Eq. (9.13)) $\rho_{(n),N}(z_1, \ldots, z_n) \leq \prod_{j=1}^n \rho_{(1),N}(z_j)$. Hence it is sufficient that $\int_{\Omega} \prod_{j=1}^n \rho_{(1),N}(z_j) \, d^2z$ be bounded for $N \to \infty$. With the limiting form of (3.4) valid, the theory of Fredholm integral operators [287] tells us that the limit of (3.5) is well defined with the RHS identified as the Fredholm determinant $\det(\mathbb{I} - \zeta K_{\infty,D})$. In (3.10) the limit corresponds to simply replacing the upper terminal of the product by $\infty$. We stipulate the further structure that the correlation kernel be Hermitian, as holds for the appropriately scaled form of (3.6). Then the argument of the proof of Proposition (3.1) tells us that the eigenvalues of $K_{\infty,D}$ are all between 0 and 1, which in turn allows the reasoning leading to (3.7) to be repeated. The conclusion is, assuming $\text{Var} N(D) \to \infty$ as $|D| \to \infty$ which as already remarked is guaranteed by the results of [47] that (3.7) remains valid with $D_N$ replaced by $D$, and the limit $N \to \infty$ replaced by the limit $|D| \to \infty$ [99, 277, 179].

It is moreover the case that in the above setting and with these modifications the local central limit theorem of Proposition 3.2 remains valid [146]. Another point of interest is that the expansion (3.11) is uniformly valid in the variable $R = \alpha \sqrt{N}$, provided this quantity grows with $N$, and hence also provides the large $R$ expansion of $E_{\infty}(0; D_R)$. Finally we consider results of [218] as they apply to number fluctuations in the infinite GinUE. The plane is to be divided into squares $\Gamma_j$ of area $L^2$ with centres at $LZ^2$. Define $Y_j = N(\Gamma_j) / \sqrt{\text{Var} N(\Gamma_j)}$. For large $L$, in keeping with (3.9) we have $\text{Var} (\Gamma_j) \sim 2L / \pi^{3/2}$. The question of interest is the joint distribution of $\{Y_j\}$. It is established in [218] that for $L \to \infty$ this distribution is Gaussian, with covariance $\frac{1}{4} [\Delta]_{jk}$, where $\Delta$ is the discrete Laplacian on $Z^2$. Consequently, fluctuations of $N(\Gamma_j)$ induce opposite fluctuations in the regions neighbouring $\Gamma_j$.

Remark 3.3.
1. Closely related to the probability $E_N(N; D)$ is the conditioned quantity $F_N(n; D) := \text{Pr}(\sum_{j=1}^N X_j \in D = n | z_j = 0)$. Denote the corresponding generating function by $\tilde{F}_N(\zeta; D)$. Proceeding as in the derivation of (3.10) shows $\tilde{F}_N(\zeta; D_R) = \tilde{E}_N(\zeta; D_R) / (1 - \zeta (1 - e^{-R^2}))$. Thus in particular $F_N(0; D_R) = e^{R^2} E_N(0; D_R)$ [168]. Note that $-\frac{d}{d \pi} F(0; D_R)$ gives the spacing distribution between an eigenvalue conditioned to be at the origin, and its nearest neighbour at a distance $R$.

2. Let $D_{\alpha \sqrt{N}}$ denote the region $\{z : |z| > \alpha \sqrt{N}\}$, i.e. the region outside the disk of radius $\alpha \sqrt{N}$, where it is assumed $0 < \alpha < 1$. Note that then $E_N(0; D_{\alpha \sqrt{N}}) = E_N(N; D_{\alpha \sqrt{N}})$. The analogue of (3.11) has been calculated in [88], where in particular it is found that

$$C_1 = \alpha^4 / 4 - \alpha^2 + (1/2) \log \alpha^2 + 3/4,$$

the coefficient $C_4$ is unchanged, while the coefficient $\frac{1}{2}$ for $\log N$ seen in (3.11) is to be replaced by $-\frac{1}{4}$. (We also refer to [102] for an earlier work for which the expansion (3.11)
was obtained up to $C_3$.) One sees from (3.13) that $C_1 = 0$ for $\alpha = 1$, and the result of [88] gives that $C_2, C_3$ similarly vanish, giving that $E_N(N; D_{\sqrt{N}}) \sim e^{C_2\sqrt{N}}$. Extending $\alpha$ larger that 1 according to the precise $N$ dependent value

$$\alpha = 1 + \frac{1}{2\sqrt{N}}\left(\sqrt{\gamma_N} + \frac{x}{\sqrt{\gamma_N}}\right), \quad \gamma_N = \log \frac{N}{2\pi} - 2\log\log N,$$

gives the extreme value result $\lim_{N \to \infty} E_N(N; D_{\sqrt{N}}) = \exp(-\exp(-x))$ [255] (see too [84] Th. 1.3 with $\alpha = 2$) for a generalisation to the case of (2.71), considered also in [88] for $\alpha < 1$, which is the Gumbel law. (See also [81 267 38 156 70 63] for fluctuations of the spectral radius under various boundary conditions.) Furthermore, an intermediate fluctuation regime which interpolates between the Gumbel law with the large deviation regime (3.13) was investigated in [207]. Another case considered in [88] is when $D_N$ is specified as the outside of an annulus contained inside of the disk of radius $\sqrt{N}$. Two features of the corresponding asymptotic expansion (3.11) are: (1) the absence of a term proportional to $\log N$, and (2) the presence of oscillations of order 1 that are described in terms of the Jacobi theta function. We also refer to [99 30 31] for further recent studies in this direction in the presence of hard edges.

3. For general $D_N$ with $|D_N| \to \infty$ the coefficient $C_1$ in (3.11) relates to an energy minimisation (electrostatics) problem, and similarly for the $|D| \to \infty$ expansion of $E_\infty(0; D)$ [118 188 102 2 1]. Thus for $D = D_{\sqrt{N}}$, the electrostatics problem is to compute the potential due to a uniform charge density $1/\pi$ inside a disk of radius $\alpha$, with a neutralising uniform surface charge density $-\alpha/2\pi$ on the boundary. The applicability of electrostatics remains true for the asymptotic expansion of $E_N(k; D_{\sqrt{N}})$ (and $E_\infty(k; D)$) in the so-called large deviation regime, when $k \ll Na^2$ or $k \gg Na^2$. For a disk the electrostatics problem can be solved explicitly to give [25]

$$E_N(k; D_{\sqrt{N}}) \sim e^{-N^2\psi_0(\alpha k / N)}, \quad \psi_0(\alpha; x) = \frac{1}{4} |(\alpha^2 - x)(\alpha^2 - 3x) - 2x^2\log(x/\alpha^2)|. \tag{3.14}$$

Note in particular that $\psi_0(\alpha; 0) = \alpha^4 / 4$, which is the value of $-C_1$ in (3.11), while setting $x = 1$ gives the value of $-C_1$ noted in the above paragraph. There is also a scaling regime, where $|k - Na^2| = O(N^{1/2})$, for the asymptotic value of $E_N(k; D_{\sqrt{N}})$ which interpolates between (3.14) and the local central limit theorem result (3.8) [208 125]. In the case of $E_\infty(k; D_R)$, it makes sense to consider $k$ proportional to not only $aR^2$ but also to $aR^\gamma$ with $\gamma > 2$. Then [188 125]

$$E_\infty(aR^\gamma; D_R) \sim e^{-\frac{1}{2}(\gamma - 2)a^2R^{2\gamma}\log R(1+o(1))}.$$
4. For the infinite GinUE the exact result in terms of modified Bessel functions

\[
\text{Var} \mathcal{N}(D_R) = R^2 e^{-2R^2} \left( I_0(2R^2) + I_1(2R^2) \right) = \sum_{j=1}^{\infty} \frac{\gamma(j; R^2)}{\Gamma(j)} \left( 1 - \frac{\gamma(j; R^2)}{\Gamma(j)} \right) \]

is known \[272\] Th. 1.3, \[125\] Appendix B]. The second expression in (3.15) also appears in \[209\] as a large \(N\) limit of the number variance of the finite Ginibre ensemble in the deep bulk regime. This exhibits the leading large \(R\) form \(R/\sqrt{\pi}\) which is consistent with identifying \(\sqrt{N} |\partial D_0|\) as \(2\pi R\) on the RHS of (3.9); see also \[13\].

3.3. Smooth linear statistics. The theory of fluctuation formulas for GinUE in the case that \(f(z)\) in (3.1) is smooth has some different features to the discontinuous case \(f(z) = \chi_{z \in \mathcal{D}}\). This can be seen by considering the bulk scaled limit, and in particular the truncated two-point correlation (2.24). From this we can compute the structure factor

\[
S^\text{GinUE}_\infty(k) := \int_{\mathbb{R}^2} \left( \hat{f}_{(2),\infty}^b(0, r) + \frac{1}{\pi} \delta(r) \right) e^{ik \cdot r} d r = \frac{1}{\pi} \left( 1 - e^{-|k|^2/4} \right).
\]

Knowledge of the structure factor allows the limiting covariance (3.2) to be computed using the Fourier transform

\[
\text{Cov}^\text{GinUE}_\infty \left( \sum f(r_j), \sum g(r_j) \right) = \frac{1}{(2\pi)^2} \frac{1}{4\pi} \int_{\mathbb{R}^2} \hat{f}(k) \hat{g}(-k) \left( 1 - e^{-|k|^2/4} \right) d k,
\]

valid provided the integral converges. Here, with \(z = x + iy\), \(r = (x, y)\) and the Fourier transform \(\hat{f}(k)\) is defined by integrating \(f(r)\) times \(e^{ik \cdot r}\) over \(\mathbb{R}^2\) — thus according to (3.16) \(S^\text{GinUE}_\infty(k)\) is a particular Fourier transform. Now introduce a scale \(R\) so that \(f(r) \mapsto f(r/R)\), \(g(r) \mapsto g(r/R)\). It follows from (3.17) that

\[
\lim_{R \to \infty} \text{Cov}^\text{GinUE}_\infty \left( \sum f(r_j/R), \sum g(r_j/R) \right) = \frac{1}{(2\pi)^2} \frac{1}{4\pi} \int_{\mathbb{R}^2} \hat{f}(k) \hat{g}(-k)|k|^2 d k,
\]

again provided the integral converges. Most noteworthy is that this limiting quantity is \(O(1)\). In contrast, with \(f(r) = g(r) = \chi_{|r| < 1}\) and then introducing \(R\) as prescribed above, we know that (3.17) has the evaluation (3.15). As previously commented, the large \(R\) form of the latter is proportional to \(R\).

Remark 3.4. Consider the linear statistic \(A(x) = -\sum_{j=1}^{N} (\log |x - r_j| - \log |r_j|)\). In the Coulomb gas picture relating to (1.10), this corresponds to the difference in the potential at \(x\) and the origin. For bulk scaled GinUE, one has from (3.17) the exact result [23]

\[
\text{Var}^\text{GinUE}_\infty A(x) = \frac{1}{2} \left( 2 \log |x| + (|x|^2 + 1) \int_{|x|^2}^{\infty} e^{-t} \frac{dt}{t} - e^{-|x|^2} + C + 1 \right),
\]
where here \( C \) denotes Euler’s constant. In particular, for large \(|x|\), \( \text{Var}^{\text{GinUE}} A(x) \sim \log |x| \). This last point shows that the introduction of a scale \( R \) as in (3.18) would give rise to a divergence proportional to \( \log R \). Such a log-correlated structure underlies a relationship between the logarithm of the absolute value of the characteristic polynomial for GinUE and Gaussian multiplicative chaos [211].

The covariance with test functions \( f(r) \rightarrow f(r/\sqrt{N}) \), \( g(r) \rightarrow g(r/\sqrt{N}) \) assumed to take on real or complex values is also an order one quantity for GinUE in the \( N \rightarrow \infty \) limit, upon the additional assumption that \( f, g \) are differentiable and don’t grow too fast at infinity [132, 256, 33, 24].

**Proposition 3.5.** Require that \( f, g \) have the properties as stated above. Let \( f(r)|_{r=(\cos \theta, \sin \theta)} = \sum_{n=-\infty}^{\infty} f_n e^{in\theta} \) and similarly for the Fourier expansion of \( g(r) \) for \( r = (\cos \theta, \sin \theta) \). We have (3.20)

\[
\lim_{N \rightarrow \infty} \text{Cov}^{\text{GinUE}} \left( \sum_{j=1}^{N} f(r_j/\sqrt{N}), \sum_{j=1}^{N} g(r_j/\sqrt{N}) \right) = \frac{1}{4\pi} \int_{|r|<1} \nabla f \cdot \nabla g \, dx dy + \frac{1}{2} \sum_{n=-\infty}^{\infty} |n| f_n \bar{g}_n.
\]

**Proof.** (Sketch) In the method of [256], a direct calculation using (3.2), (2.18) and (2.10) allows (3.20) to be established for \( f, g \) polynomials jointly in \( z = x + iy \) and \( \bar{z} = x - iy \). The required integrations can be computed exactly using polar coordinates. To go beyond the polynomial case, the so-called dbar (Cauchy-Pompeiu) representation is used. This gives that for any once continuously differentiable \( f \) in the unit disk \( D_1 \), and \( z \) contained in the interior of the disk,

\[
f(z) = -\frac{1}{\pi} \int_{D_1} \frac{\partial \bar{w} f(w)}{w - z} \, d^2w + \frac{1}{2\pi i} \int_{\partial D_1} \frac{f(w)}{w - z} \, dw,
\]

where with \( w = a + i\gamma \), \( \partial a = \frac{1}{2} (\frac{\partial \bar{w}}{\partial \gamma} + i \frac{\partial \bar{w}}{\partial \gamma}) \). The covariance problem is thus reduced to the particular class of linear statistics of the functional form \( h(z) = 1/(w - z) \). The required analysis in this case is facilitated by the use of the corresponding Laurent expansion, with only a finite number of terms contributing after integration.

**Remark 3.6.**

1. As predicted in [132], upon multiplying the RHS by \( 2/\beta \), (3.20) remains valid for the Coulomb gas model [110] [217] [269] [46]. In the case of the elliptic GinUE, a simple modification of (3.20) holds true. Thus the domain \(|r| < 1\) in the first term is to be replaced by the appropriate ellipse, and the Fourier components of the second term are now in the variable \( \eta \), where \((A \cos \eta, B \sin \eta), 0 \leq \eta \leq 2\pi\) parametrises the boundary of the ellipse. The results of [132, 24, 217] also cover this case.

2. In the case of an ellipse, major and minor axes \( A, B \) say, there is particular interest in
the linear statistic \( P_x := \sum_{j=1}^N x_j \). Linear response theory gives for the \( xx \) component of the susceptibility tensor \( \chi \) — relating the polarisation density to the applied electric field — the formula \( \chi_{xx} = (\beta/(\pi AB)) \lim_{N \to \infty} \text{Var} P_x \) (and similarly for the \( xy \) and \( yy \) component). This same quantity can be computed by considerations of macroscopic electrostatics, which gives \( \chi_{xx} = (A + B)/(\pi B) \). Using (3.20) modified as in the above paragraph, the consistency of these formulas can be verified.

Considering further the case of elliptic GinUE, dividing by \( N \) and taking the limit \( \tau \to 1 \) gives the GUE with eigenvalues supported on \((-2, 2)\), and similarly for the \( \beta \) generalisation limiting to \( (1, 10) \) restricted to this interval. For this model it is known (see e.g. [141, Eq. (3.2)] with the identification \( x = 2 \cos \theta \))

\[
\lim_{N \to \infty} \text{Cov} \left( \sum_{j=1}^N f(x_j), \sum_{j=1}^N g(x_j) \right) = \frac{2}{\beta} \sum_{n=1}^{\infty} n f^c_n g^c_n,
\]

where \( f(x)|_{x=2 \cos \theta} = f_0^c + 2 \sum_{n=1}^{\infty} f_n^c \cos n \theta \) and similarly for \( g(x) \). We observe that this is identical to the final term in (3.20), modified according to the specifications of point 1. above.

There has been a recent application of Proposition 3.5 in relation to the computation of the analogue of the Page curve for a density matrix constructed out of GinUE matrices [96].

We turn our attention now to the limiting distribution of a smooth linear statistic. By way of introduction, consider the particular linear statistic \( \frac{1}{N} \sum_{j=1}^N |r_j|^2 \) for GinUE. An elementary calculation gives that the corresponding characteristic function, \( \hat{P}_N(k) \) say, has the exact functional form

\[
(3.22) \quad \hat{P}_N(k) = (1 - ik/N)^{-N(N+1)/2}.
\]

It follows from this that after centring by the mean, the limiting distribution is a Gaussian with variance given by (3.20) (which is this specific case evaluates to one). A limiting Gaussian form holds in the general case of the applicability of (3.20), as first proved by Rider and Virág [256].

**Proposition 3.7.** Let \( f \) be subject to the same conditions as in Proposition 3.5 and denote the case \( f = g \) of (3.20) by \( \sigma_f^2 \). For the GinUE, if \( f \) takes on complex values then as \( N \to \infty \)

\[
\sum_{j=1}^N f(r_j/\sqrt{N}) - \left( \sum_{j=1}^N f(r_j/\sqrt{N}) \right) \overset{d}{\to} N[0, \sigma_f] + iN[0, \sigma_f],
\]

while if \( f \) is real valued the RHS of this expression is to be replaced by \( N[0, \sigma_f] \). Moreover this same limit formula holds for the elliptic GinUE [34] and its \( \beta \) generalisation [217] (both subject to further technical restrictions on \( f \)), with the variance modified according to Remark 3.6.1.
Proof. (Comments only) The proof of \([256]\) proceeds by establishing that the higher order cumulants beyond the variance tend to zero as \(N \to \infty\). Essential use is made of the rotation invariance of GinUE. The method of \([34]\) uses a loop equation strategy, while \([217]\) involves energy minimisers and transport maps. For GinUE with \(f\) a function of \(|r|\), a simple derivation based on the proof of Proposition \([219]\) together with a Laplace approximation of the integrals \([132]\) (see also \([68]\), Appendix B).

\[\square\]

3.4. Spatial modelling and the thinned GinUE. The GinuE viewed as a point process in the plane has been used to model geographical regions by way of the corresponding Voronoi tessellation \([215]\), the positions of objects, for example trees in a plantation \([214]\) or the nests of birds of prey \([8]\), and the spatial distribution of base stations in modern wireless networks \([242], [110]\), amongst other examples. The wireless network application has made use of the thinned GinUE, whereby each eigenvalue is independently deleted with probability \((1 - \zeta)\), \(0 < \zeta \leq 1\). The effect of this is simple to describe in terms of the correlation functions, according to the replacement

\[
(3.23) \quad \rho_{(n),N}(z_1, \ldots, z_n) \mapsto \zeta^N \rho_{(n),N}(z_1, \ldots, z_n).
\]

With the bulk density of GinUE uniform and is equal to \(1/\pi\), we can also rescale the position \(z_j \mapsto z_j/\zeta\) so that this remains true in the thinned ensemble. For this \((3.23)\) is to be updated to read

\[
(3.24) \quad \rho_{(n),N}(z_1, \ldots, z_n) \mapsto \rho_{(n),N}(z_1/\sqrt{\zeta}, \ldots, z_n/\sqrt{\zeta}).
\]

Recalling now \((2.9)\) and \((2.18)\), for the bulk scaled limit of the thinned GinUE we have in particular

\[
\rho_{GinUE}^{(1)}(z) = \frac{1}{\pi}, \quad \rho_{GinUE,T}^{(2)}(w, z) = -\frac{1}{\pi^2} e^{-|w-z|^2/\zeta}.
\]

From these functional forms we see

\[
\int_{C} \rho_{GinUE,T}^{(2)}(w, z) d^2z = -\frac{\zeta}{\pi} \neq -\rho_{GinUE}^{(1)}(w) \quad \text{unless } \zeta = 1,
\]

where the superscript “tGinU” denotes the thinned GinUE. Equivalently, in terms of the structure factor \((3.16)\),

\[
S_{GinUE}^{(1)}(0) = \frac{1 - \zeta}{\pi} \neq 0 \quad \text{unless } \zeta = 1.
\]

Due to this last fact, the \(O(1)\) scaled covariance for smooth linear statistics \((3.17)\) is no longer true, and now reads instead

\[
(3.25) \quad \text{Cov}_{GinUE}^{(1)} \left( \sum f(r_i/R), \sum g(r_i/R) \right) \sim R^2 \frac{(1 - \zeta)}{(2\pi)^2} \int_{\mathbb{R}^2} \hat{f}(k) \hat{g}(-k) dk.
\]
This leading dependence on $\mathbb{R}^2$ holds too for the counting function $f(z) = \chi_{|z|<1}$, since in distinction to (3.17) the integral now converges. Hence, in the terminology of the text introduced below (3.9), the statistical state is no longer hyperuniform. There is an analogous change to the $O(1)$ scaled covariance (3.20), which is now proportional to $N$ and reads

\begin{equation}
Cov^{GUE}^{\text{thin}} \left( \sum_{j=1}^{N} f(r_j/\sqrt{N}), \sum_{j=1}^{N} g(r_j/\sqrt{N}) \right) \sim N \frac{(1-\zeta)}{\pi} \int_{|r|<1} f \, g \, dx \, dy.
\end{equation}

Notwithstanding this difference, the corresponding limiting distribution function is still Gaussian. A more subtle limit, also considered in (3.10), is when $N \to \infty$ and $\zeta \to 1^-$ simultaneously, with $N(1-\zeta)$ fixed. The quantity (3.20) returns to being $O(1)$, but consists of a contribution of the form (3.20), and a term characteristic of a Poisson process.

We turn our attention now to the probabilities $\{E^\text{GUE}_{\alpha}(k; D_{\alpha \sqrt{N}})\}$. Upon consideration of the thinning prescription (3.24), the proof of Proposition 3.1 and (3.10) shows that the corresponding generating function is given by

\begin{equation}
E^\text{GUE}_{\alpha}^{\zeta} (\zeta; D_{\alpha \sqrt{N}}) = \prod_{j=1}^{N} \left( 1 - \zeta \zeta_j \right).
\end{equation}

Setting $\zeta = 1$ in this gives the probability $E^\text{GUE}_{\alpha}(0; D_{\alpha \sqrt{N}})$. Note that the implied formula shows $E^\text{GUE}_{\alpha}(0; D_{\alpha \sqrt{N}}) = E^\text{GUE}_{\alpha} (\zeta; D_{\alpha \sqrt{N}})$. The large $N$ asymptotics of $E^\text{GUE}_{\alpha} (\zeta; D_{\alpha \sqrt{N}})$, and various generalisations, are available in the literature [86, 67]. Here we present a self contained derivation of the first two terms (cf. (3.11)).

**Proposition 3.8.** For large $N$ and with $0 < \alpha, \zeta < 1$ we have

\begin{equation}
E^\text{GUE}_{\alpha} (0; D_{\alpha \sqrt{N}}) = \exp \left( \alpha^2 N \log(1-\zeta) + \sqrt{\alpha^2 N} h(\zeta) + O(1) \right),
\end{equation}

where

\begin{equation}
h(\zeta) = \int_{0}^{\infty} \log \left( \frac{1 - (\zeta/2)(1 + \text{erf}(t/\sqrt{2}))}{1 - \zeta} \right) dt + \int_{0}^{\infty} \log \left( 1 - (\zeta/2)(1 - \text{erf}(t/\sqrt{2})) \right) dt.
\end{equation}

**Proof.** Our main tool is the uniform asymptotic expansion (284)

\begin{equation}
\frac{\Gamma(M-j+1; M)}{\Gamma(M-j+1)} \sim \frac{1}{2} \left( 1 + \text{erf} \left( \frac{j}{\sqrt{2M}} \right) \right).
\end{equation}

Here it is known that the error term has the structure $(1/\sqrt{M})g(j/\sqrt{2M})$ where $g(t)$ is integrable on $\mathbb{R}$ and decays rapidly at infinity. This expansion suggests we rewrite the
product in \((3.27)\) with \(\xi = 1\) in the form

\[
(1 - \zeta)^{[M^*]} \left( \prod_{j=1}^{M^*} \frac{1 - \zeta \gamma(j; M^*) / \Gamma(j)}{1 - \zeta} \right) \left( \prod_{j=M^*+1}^{N} \frac{1 - \zeta \gamma(j; M^*) / \Gamma(j)}{1 - \zeta} \right),
\]

where \(M^* = \lfloor a^2 N \rfloor\).

We see that the first term in this expression gives the leading order term in \((3.28)\). In the first product we change labels \(j \mapsto M^* - j + 1\) \((j = 1, \ldots, M^*)\). In the second we change labels \(j \mapsto M^* + j + 1\) \((j = 0, \ldots, N - M^* - 1)\). Now writing both these products as exponentials of sums and applying \((3.30)\) gives, upon recognising the sums as Riemann integrals, the \(O(\sqrt{a^2 N})\) term in \((3.28)\). □

The leading term in \((3.28)\) is consistent with the general form expected for thinned log-gas systems, being of the form of the area of the rescaled excluded region, times the density, times \(\log(1 - \zeta)\) [135, Conj. 10].

Remark 3.9. The topic of spatial modelling using Ginibre eigenvalues naturally leads to questions on the efficient simulation of the point process confined to a compact subset of the \(\mathbb{C}\). Practical algorithms for this task have been given in [103, 104].

4. Sum rules and asymptotic behaviours

Throughout this section we consider the Coulomb gas model \((1.10)\) with general \(\beta > 0\), for which we use the notation OCP, which stands for one-component plasma. The special case \(\beta = 2\) coincides with GinUE.

4.1. Asymptotics associated with the configuration integral. The configuration integral for the Boltzmann factor \((1.10)\) is specified by

\[
Q_N^{\text{OCP}}(\beta) = \int_{\mathbb{C}} d^2 z_1 \cdots \int_{\mathbb{C}} d^2 z_N e^{-\left(\beta/2\right) \sum_{j=1}^{N} |z_j|^2} \prod_{1 \leq j < k \leq N} |z_k - z_j|^{\beta}.
\]

From the OCP viewpoint, it is more natural to consider the renormalised quantity

\[
Z_N^{D_R,\text{OCP}}(\beta) = \frac{1}{N!} A_{N,\beta} Q_N^{\text{OCP}}(\beta), \quad A_{N,\beta} = e^{-\beta N^2 \left( \frac{1}{2} \log N - \frac{3}{8} \right)};
\]

see [133, Eq. (1.12)]. The use of the symbol \(D_R\) denoting a disk of radius \(R\) (specifically \(R = \sqrt{N}\)) as a subscript follows from the derivation of \(A_{N,\beta}\). Thus this quantity has the interpretation as the Boltzmann factor of the electrostatic self energy of a smeared out uniform background, charge density \(-\frac{1}{\pi}\), confined to the disk \(D_R\), and the constant terms of its electrostatic energy when coupled to a particle inside of this disk. The quantity \(Z_N^{D_R,\text{OCP}}(\beta)\) is then the partition function of the charge neutral OCP. Generally in statistical
mechanics for a stable system the dimensionless free energy, \( \beta F_N(\beta) = -\log Z_N(\beta) \), is an extensive quantity, meaning that for large \( N \) it is proportional to \( N \). For the closely related model when the particles are strictly restricted to the disk, the validity of this statement was established in [227, 266], and has been reconsidered recently using more far reaching techniques in [216] (which for example form a platform for the study of fluctuation formulas in [217]). Making use of Proposition 2.1 the large \( N \) form of \( \beta F_{N,\text{OCP}}(\beta) \) can be computed for \( \beta = 2 \) [280, Eq. (3.14)].

**Proposition 4.1.** We have

\[
\beta F_{N,\text{OCP}}(\beta) \bigg|_{\beta=2} = N\beta f(\beta)\bigg|_{\beta=2} + \frac{1}{12} \log N - \frac{1}{720N^2} + O\left( \frac{1}{N^4} \right),
\]

where

\[
\beta f(\beta)\bigg|_{\beta=2} = \frac{1}{2} \log \left( \frac{1}{2\pi^3} \right).
\]

**Proof.** This relies on (2.3), identifying \( \Pi j=1 N j! = G(N+1) \), where \( G(x) \) denotes the Barnes \( G \)-function, and knowledge of the known asymptotic expansion of \( G(N+1) \) (see e.g. [126, Th. 1]).

For the OCP on a sphere of radius \( R \), \( S_R^2 \) say, and with \( R = \frac{1}{\sqrt{N}} \) so that the particle density is \( 1/\pi \), the partition function of the charge neutral system is (see e.g. [280, Eq. (2.1)])

\[
Z_{N,\text{OCP}}^R(\beta) = \frac{1}{N!} N^{-N\beta/2} e^{\beta N^2/4} \int_{S_R^2} d\theta_1 d\phi_1 \cdots \int_{S_R^2} d\theta_N d\phi_N \prod_{1 \leq j < k \leq N} |u_j v_k - u_k v_j|^\beta.
\]

Here the variables \( \{u_j, v_k\} \) are the Cayley-Klein parameters as in (2.48). From the analogue of Proposition 2.1 this can be evaluated exactly at \( \beta = 2 \) [72], implying that for large \( N \) [190, 280, Eq. (3.6)]

\[
\beta F_{N,\text{OCP}}^R(\beta) \bigg|_{\beta=2} = N\beta f(\beta)\bigg|_{\beta=2} + \frac{1}{6} \log N + \frac{1}{12} - 2\zeta'(1) - \frac{1}{180N^2} + O\left( \frac{1}{N^4} \right),
\]

where \( \beta f(\beta)\bigg|_{\beta=2} \) is as in (4.3).

Both expansions (4.3) and (4.5) illustrate a conjectured universal property of the large \( N \) expansion of \( \beta F_{N,\text{OCP}}^\beta \) in the case that the droplet forms a shape with Euler index \( \chi \) [190]

\[
\beta F_{N,\text{OCP}}(\beta) = N\beta f(\beta) + a_\beta \sqrt{N} + \frac{\chi}{12} \log N + \cdots,
\]

valid for general \( \beta > 0 \). To compare against the exact results for \( \beta = 2 \) it should be recalled that \( \chi = 1 \) for a disk and \( \chi = 2 \) for a sphere. An analogous calculation in annulus geometry at \( \beta = 2 \) gives an expansion with a term proportional to \( \log N \) absent, in keeping with \( \chi = 0 \) [129, 69]. A further point of interest is that the large \( N \) expansions for \( E_N(0; D_N) \) from [88]
as reviewed in §3.2 also exhibit simple fractions for the coefficient of $\log N$, and moreover this term is not present in the case the eigenvalues are constrained to a single annulus.

We note that the term proportional to $\sqrt{N}$ in (4.6) has the interpretation as a surface tension, and so is expected not to be present in the case of a sphere, as seen in (4.5) for $\beta = 2$. Also, for the disk geometry, it has been conjectured (see [75, Eq. (3.2)]) that

$$a_\beta = \frac{4 \log(\beta/2)}{3\pi^{1/2}},$$

which in particular vanishes for $\beta = 2$, as is consistent with (4.5).

**Remark 4.2.**

1. Multiplication of the configuration integral $Q_\beta^{\text{OCP}}$ by $A_{N,\beta}$ as in (4.2) effectively shifts the microscopic energy $U$ in (1.10) by a function of $N$. We denote this shifted, charge neutral, energy by $U'$, and similarly in relation to the sphere. For $\beta = 2$ direct calculation of the mean is possible. Thus in the case of the OCP on the plane, one finds [271] equivalent to Eqns. (25) and (29)

$$\langle U' \rangle_{Dk}^{\beta=2} = -\frac{1}{2} \left( \frac{N^2}{2} (\Psi(N) - \log N) + \frac{N+1}{4} + \frac{NC}{2} - \frac{\Gamma(N+3/2)}{\Gamma(N+2)\Gamma(3/2)} \right. \times \left. _3F_2 \left( \begin{array}{c} 1, N-1, N+3/2 \\ N+2, N+1 \end{array} \right) 1 \right) = -\frac{CN}{4} + \frac{2\sqrt{N}}{3\sqrt{\pi}} - \frac{5}{48} + O(N^{-1/2}),$$

where here $C$ denotes Euler’s constant and $\Psi(N)$ denotes the digamma function. The corresponding result for the sphere geometry at $\beta = 2$ gives the simpler formula [73, 24, 201]

$$\langle U' \rangle_{S_k}^{\beta=2} = \frac{N}{4} \left( -H_N + \log N \right) = -\frac{CN}{4} - \frac{1}{8} + O(N^{-1}),$$

where here $H_N$ denotes the harmonic numbers. The common leading order value of the charge neutral energy per particle, $-C/4$, was known to Jancovici [183] through the formula

$$\lim_{N \to \infty} \frac{1}{N} \langle U' \rangle_{\text{OCP}}^{\beta=2} = -\frac{\pi}{2} \int_{\mathbb{R}^2} \log |r| \rho_{b,\text{GinUE}}^{\beta=2}(0, r) \, dr = -\frac{C}{4},$$

where the integral is evaluated from the explicit formula for $\rho_{b,\text{GinUE}}^{(2),\infty}$ given by (2.24). Note as an expansion about $\beta = 2$, $\beta F_q(\beta) = \beta F_q(\beta)|_{\beta=2} + (\beta - 2) \langle U' \rangle + O((\beta - 2)^2)$, so the above results allow (4.5) and (4.6) to be extended to first order in $(\beta - 2)$. In particular consistency is obtained with (4.6).

2. In the low temperature limit $\beta \to \infty$ the OCP particles are expected to form a triangular lattice. Recent works relating to this include [264, 257, 217, 55, 26, 77, 41, 40]. The conjectured exact value of twice the charge neutral energy per particle in this limit, with bulk density
\( 1/(4\pi) \) (not \( 1/\pi \) as is natural for GinUE in the plane, rather the geometry used was a sphere of unit radius) is
\[
2\log 2 + \frac{1}{2} \log \frac{2}{3} + 3 \log \frac{\sqrt{\pi}}{\Gamma(1/3)} = -0.0556053 \ldots
\]
We also refer to [83] and references therein for recent works on the opposite, high temperature limit \( \beta \to 0 \) of the two dimensional Coulomb particles.

4.2. Sum rules and asymptotics for the edge density. First we note that for large \( N \) the leading order support of the density is a disk of radius \( \sqrt{N} \), independent of \( \beta \), as follows from the potential theoretic argument of Remark 2.5. Using the vector coordinate \( r = (x, y) \), as in the second equation in (2.19) we introduce edge scaling coordinates by writing
\[
\rho_{OCP}^{(1, N)}((x, \sqrt{N} - y)) = \rho_{e,OCP}^{(1, \infty)}(y) + \frac{1}{\sqrt{N}} \mu_{e,OCP}^{(1, \infty)}(y) + O\left(\frac{1}{N}\right).
\]
Here the form of the correction terms, known to be correct at \( \beta = 2 \) according to (2.27), are at this stage presented as an ansatz. We seek some integral identities that must be satisfied by \( \rho_{e,OCP}^{(1, \infty)}(y) \) and \( \mu_{e,OCP}^{(1, \infty)}(y) \). Identities of this type are referred to as sum rules.

**Proposition 4.3.** We have
\[
\int_{-\infty}^{\infty} \left( \rho_{e,OCP}^{(1, \infty)}(y) - \frac{1}{\pi} \chi_{y>0} \right) dy = 0.
\]
and
\[
\int_{-\infty}^{\infty} y \left( \rho_{e,OCP}^{(1, \infty)}(y) - \frac{1}{\pi} \chi_{y>0} \right) dy = \int_{-\infty}^{\infty} \mu_{e,OCP}^{(1, \infty)}(y) dy.
\]

**Proof.** Because of the large \( N \) form of the density, we have that \( \rho_{e,OCP}^{(1, N)}(r) - \frac{1}{\pi} \chi_{|r|<\sqrt{N}} \) will be concentrated near \( |r| = \sqrt{N} \). Now, the normalisation condition for the density gives, with the use of polar coordinates
\[
\int_0^{\infty} r \left( \rho_{e,OCP}^{(1, N)}(r) - \frac{1}{\pi} \chi_{|r|<\sqrt{N}} \right) dr = \int_{-\infty}^{\sqrt{N}} (\sqrt{N} - y) \left( \rho_{e,OCP}^{(1, N)}(\sqrt{N} - y) - \frac{1}{\pi} \chi_{y>0} \right) dy = 0.
\]
In the second term we now substitute (4.8) and equate terms of order \( \sqrt{N} \) and of order unity to zero to obtain (4.9) and (4.10). □

From the functional forms for \( \rho_{e,GinUE}^{(1, \infty)}(y) \) and \( \mu_{e,GinUE}^{(1, \infty)}(y) \) as read off from (2.27), we verify both of the above sum rules in this special case. Note that the integral on the LHS of (4.10) has the interpretation of the dipole moment of the excess charge in the edge boundary layer, using the Coulomb gas picture. In fact it is possible to derive a further sum rule which evaluates this dipole moment explicitly [281, Eq. (5.13)].
Proposition 4.4. We have

\[
\int_{-\infty}^{\infty} y \left( \rho_{(1),\infty}^\text{e,OCP}(y) - \frac{1}{\pi} \chi_{y>0} \right) dy = -\frac{1}{2\pi\beta} \left( 1 - \frac{\beta}{4} \right).
\]

Proof. From the definition, we observe

\[
\left\langle \sum_{j=1}^{N} |r_j|^2 \right\rangle = \frac{2N}{\beta} + \frac{1}{2} N(N-1).
\]

By writing the LHS as an average over the density and the use of polar coordinates, we see that this is equivalent to the sum rule

\[
2\pi \int_0^\infty r^3 \left( \rho_{(1),N}^\text{OCP}(r) - \frac{1}{\pi} \chi_{r<\sqrt{N}} \right) dr = \frac{N}{2} \left( \frac{4}{\beta} - 1 \right).
\]

Proceeding now as in the proof of Proposition 4.3 by substituting (4.8), equating terms proportional to \( N \) on both sides, and making use too of (4.10), we deduce (4.11). \( \square \)

We now turn our attention to the large \( N \) form of the global scaled density, \( \rho_{(1),N}^\text{g,OCP}(r) := \rho_{(1),N}^\text{OCP}(\sqrt{N} r) \). From the theory noted at the beginning of the subsection, this will limit to the circular law (2.17). We ask about the leading correction term. A hint is given by (4.13), which after dividing both sides by \( N^2 \) to correspond to global coordinates tells us \( \left\langle \frac{1}{N} \sum_{j=1}^{N} |r_j|^2 \right\rangle_{\text{g,OCP}} = \frac{1}{2} - \frac{1}{\pi} \left( 1 - \frac{\delta}{4} \right) \frac{1}{N} \). The first term \( \frac{1}{2} \) is the average of the function \( g(r) = |r|^2 \) over the disk \( D_{\sqrt{N}} \) with density \( \frac{1}{\pi} \). The second term is a \( \frac{1}{N} \) correction, so we might expect that the leading correction to the circular law is \( O(1/N) \). In fact knowledge of (4.11) is sufficient to compute the leading correction term of the large \( N \) expansion of all the moments \( \left\langle \frac{1}{N} \sum_{j=1}^{N} |r_j|^p \right\rangle_{\text{g,OCP}} (p = 1, 2, \ldots) \), allowing us to conclude [281 Eq. (5.18)]

\[
\rho_{(1),N}^\text{g,OCP}(r) = \frac{1}{\pi} \chi_{|r|<1} + \frac{1}{N} \kappa(r) + O(N^{-3/2}), \quad \kappa(r) = \frac{1}{2\pi\beta} \left( 1 - \frac{\beta}{4} \right) \frac{1}{r} \delta'(r-1).
\]

Thus the correction term is concentrated on the boundary. For GinUE, it can be deduced from (2.10) that inside the droplet the corrections are exponentially small [220 Th. 1.2, 182 Lemma 3.1]. This same formula can also be read off from a more general formula relating to \( \beta \) generalised normal matrix models (see §5.5 obtained by Zabrodin and Wiegmann [250 Eq. (5.16)].
As our final topic under this heading, we consider the $y \to -\infty$ asymptotic form of $\rho_{(1),N}^{e,OCP}(y)$. According to the first term in (2.27), at $\beta = 2$ we have

$$\rho_{(1),N}^{e,OCP}(y) \underset{y \to -\infty}{\sim} e^{-2y^2} \left(2\pi\right)^{3/2} |y|^{3/2}. \tag{4.15}$$

As a first step to extend this to general $\beta > 0$, a large deviation formula for $\rho_{(1),N}^{OCP}(r)$ can be computed, which asks for the asymptotic form of $\rho_{(1),N}^{OCP}(\sqrt{Nr})$ (here polar coordinates are being used), $r > 1$ \cite[Prop. 1]{[75]}. 

**Proposition 4.5.** With $\beta f(\beta)$ the dimensionless free energy per particle as in (4.6), for $r > 1$ we have

$$\rho_{(1),N}^{OCP}(\sqrt{Nr}) = \frac{e^{\beta f(\beta)}}{N^8/4} e^{-\left(N\beta/2\right)(r^2 - 1)} \exp \left(N \beta \log r - \frac{\beta}{2} \log(r^2 - 1) + o(1)\right). \tag{4.16}$$

**Proof.** (Sketch) The starting point is to manipulate the definition of $\rho_{(1),N}^{OCP}$ to obtain its form written as an average

$$\rho_{(1),N+1}^{OCP}(\sqrt{N+1}r) = (N + 1)N^{\beta N/2}e^{-(N+1)\beta r^2/2} \frac{Q_N^{OCP}(\beta)}{Q_{N+1}^{OCP}(\beta)} \langle \prod_{l=1}^N \sqrt{\frac{N + 1}{N}} r_l - \hat{r} \rangle_{OCP}. \tag{4.17}$$

Here $Q_N^{OCP}(\beta)$ is the configuration integral (4.1), and the average is with respect to the global scaled GUE, specified by the Botzmann factor (4.10) but with the factor of $\frac{1}{2}$ multiplying the first sum in $U$ replace by $\frac{N}{2}$. The significance of this is that upon exponentiating the product, the average can be recognised as the characteristic function for a particular linear statistics. For this, with $|r| = r > 1$ Proposition \cite[3.7] applies, telling us the leading two terms in its large $N$ asymptotic expansion, once the corresponding mean and variance have been computed. After calculating these, (4.16) results. \hfill \Box

From (4.16) we compute the limit formula

$$\lim_{N \to \infty} \rho_{(1),N}^{OCP}(\sqrt{Nr})|_{r = 1 - y/\sqrt{N}} = e^{\beta f(\beta)} \frac{e^{-\beta y^2}}{(2|y|)^{\beta/2}}, \tag{4.18}$$

which under the assumption that the large deviation formula connects to the $y \to -\infty$ tail of $\rho_{(1),N}^{e,OCP}(y)$ is the sought $\beta > 0$ generalisation of the $\beta = 2$ result (4.15). The consistency between (4.18) and the latter is immediate upon substituting (4.4).

### 4.3. Sum rules and asymptotics for the two-point correlation

Setting $k = 0$ in (3.16) gives

$$\int_{\mathbb{R}^2} \left(\rho_{(2),\infty}^{b,T}(0, r) + \frac{1}{\pi} \delta(\mathbf{r})\right) d\mathbf{r} = 0. \tag{4.19}$$
This constraint on $\rho_T^{T(2),\infty}$ is an example of a sum rule. In physical terms, using the Coulomb gas picture, it says that the response of the system by the introduction of a charge (corresponding to the delta function) is to create a screening cloud (corresponding to $\rho_T^{T(2),\infty}$) of opposite total charge. Consequently (4.19) is referred to as the perfect screening sum rule. It relates to the point process being hyperuniform or equivalently incompressible — for a state that is compressible the RHS of (4.19) is not zero but rather is given in terms of the second derivative of the pressure with respect to the fugacity; see e.g. [140, Eq. (3.7)]. For states with an underlying long range potential (as in (1.10)) the perfect screening sum rule is expected to be a necessary condition for thermodynamic stability [237]. Of similar general validity is the sum rule which results when the integrand of (4.19) is replaced by

$$ (4.20) \quad q(r_1, \ldots, r_k, r) := \rho^{(k+1),\infty}(r_1, \ldots, r_k, r) - \frac{1}{\pi} \rho^{(k),\infty}(r_1, \ldots, r_k) + \sum_{j=1}^{k} \delta(r - r_j) \rho^{(k),\infty}(r_1, \ldots, r_k). $$

Furthermore the fast decay of the correlations upon truncation (i.e. suitable subtraction by combinations of lower order correlation as in the definition of $\rho_T^{T(2),\infty}$) implies that not only does the total charge associated with $q$ vanish, but in fact so too does all the multipole moments [237].

**Proposition 4.6.** Let $q$ be as in (4.20) and set $r = (x, y)$. For bulk scaled GinUE, specified in terms of the correlation functions by (2.9) with $N \to \infty$ and the correlation kernel (2.18), we have

$$ (4.21) \quad \int_{\mathbb{R}^2} (x - iy)^p q(r_1, \ldots, r_k, r) \, dr = 0, \quad p = 0, 1, \ldots $$

**Proof.** For any $k \geq 1$ integrating over the term involving the sum of delta functions gives \( \left( \sum_{j=1}^{k} (x_j - iy_j)^p \right) \rho^{(k),\infty} \). To then integrate over the first two terms, we expand the determinant specifying $\rho^{(k+1),\infty}$ by the final row. The term coming from the last entry is recognised as \( (1/\pi)\rho^{(k),\infty} \) and so cancels. For each of the $k$ other terms, we multiply the term coming from the $j$-th entry of the final row $K^b_\infty(r, r_j)$ times $(x - iy)^p$ down the final column containing $[K^b_\infty(r_m, r)]_{m=1}^{k}$, and integrate over $r$. For the latter task, polar coordinates can be used to deduce that

$$ \int_{\mathbb{R}^2} (x - iy)^p K^b_\infty(r_m, r) K^b_\infty(r, r_j) \, dr = (x_j - iy_j)^p \rho^{(k),\infty}(r_m, r_j); $$

the case $p = 0$ is (2.22) with bulk scaling. After rearranging the columns, the result of the integration in each case can be identified as $-(x_j - iy_j)^p \rho^{(k),\infty}$, and thus in total cancel out with the integration over the final term. \hfill \Box
Replacing $1/\pi$ in (4.20) by $\rho_{(1),\infty}(1)$ the sum rule (4.21) with $p = 0$ remains valid in the case of edge scaling as a consequence of the validity of (2.22). For $p \geq 1$ the slow decay of the correlations along the direction of the boundary as seen in (2.26) means that the integral in (4.21) is not well defined. Specifically the $p = 0, k = 1$ sum rule (4.21) at the edge reads

\[
\int_{\text{C}} \rho_{(2),\infty}^{\text{e,OCP,T}}(z, z') d^2z' = -\rho_{(1),\infty}^{\text{e,OCP}}(z).
\]

This is the edge counterpart of the bulk perfect screening sum rule (4.19).

**Remark 4.7.** It turns out that in the GinUE case, the determinantal structure together with (4.22) can be used to show that $\rho(z) := \rho_{(1),\infty}^{\text{e,GinUE}}(z)$ (this is the first term in (2.27)) satisfies a non-linear equation of infinite order,

\[
\rho(z) = \sum_{n=0}^{\infty} \frac{d^{(n)}_z \rho(z)}{n!};
\]

see [36] § 3.6], where (4.22) is referred to as a mass-one equation. The non-linear equation is equivalent to the special function function identity

\[
\frac{1}{4} \left( 1 + \text{erf}(\sqrt{2}x) \right) \left( 1 - \text{erf}(\sqrt{2}x) \right) = \frac{e^{-4x^2}}{\pi} \sum_{n=1}^{\infty} \frac{(H_{n-1}(\sqrt{2}x))^2}{2^n n!} \quad (x \in \mathbb{R});
\]

see [36] § 4.5 for the proof of (4.24). Together with the loop equation, the identity (4.23) is used in [36] to study the universality of normal matrix models; see § 5.4.

To deduce (3.18) from (3.17) requires that

\[
\lim_{R \to \infty} R^2 S_{\text{GinUE}}^{\infty}(k/R) = \frac{|k|^2}{4\pi}.
\]

In the general $\beta > 0$ case of the OCP we denote the bulk scaled structure factor by $S_{\infty}^{\text{OCP}}(k)$. A perfect screening argument extending the viewpoint which implies (4.19) (for this see e.g. [133] §15.4.1, [131] §3.2]) predicts

\[
\lim_{R \to \infty} R^2 S_{\text{OCP}}^{\infty}(k/R) = \frac{|k|^2}{2\beta\pi}.
\]

This result, in the slightly different guise of a mesoscopic scaling limit, is implied by the recent work [277] Th. 1 mesoscopic case, formula for the variance]. In the cases of the OCP applied to the anomalous quantum Hall effect ($\beta = 2M$ with $M$ odd), the sum rule (4.26) combined with the Feynman-Bijl formula quantifies the collective mode excitation energy from the ground state [163]. From the definition of the structure factor, this is equivalent to the moment formula

\[
\int_{\mathbb{R}^2} |r|^2 \rho_{(2),\infty}^{\text{OCP,T}}(r, 0) dr = -\frac{2}{\pi \beta}.
\]
known as the Stillinger-Lovett sum rule \([278]\); see \([238]\) for a derivation which makes use of \((4.21)\) for \(p = 1\) and \(k = 1, 2\). Higher order moment formulas can also be derived,

\[
\int_{\mathbb{R}^2} |r|^4 \rho_{(2),\infty}^{\text{OCP},T}(r, 0) \, dr = -\frac{16}{\pi \beta^2} \left(1 - \frac{\beta}{4}\right), \quad \int_{\mathbb{R}^2} |r|^6 \rho_{(2),\infty}^{\text{OCP},T}(r, 0) \, dr = -\frac{18}{\pi \beta^3} \left(\beta - 6\right) \left(\beta - \frac{8}{3}\right);
\]

in distinction to \((4.26)\) the precise statement of these results depends on the underlying bulk particle density, which in keeping with GinUE has been assumed to be \(\frac{1}{\pi}\). A linear response argument can be used to derive the first of these; see e.g. \([133] \S 14.1.1\]. It involves the thermal pressure for the OCP, which by a simple scaling argument takes the value \(\rho^b(1 - \beta/4)\) (here \(\rho^b\) denotes the bulk density; for a discussion of the relation between the thermal and mechanic pressure in the OCP, see \([92]\]) which explains the appearance of this factor. Mayer diagrammatic expansion methods were used to first derive the sixth moment condition in \((4.28)\) \([193]\]. Later a response argument involving variations to the spatial geometry was used to give an alternative derivation \([76]\]. In keeping with the relationship between \((4.26)\) and \((4.27)\), the moment formulas \((4.28)\) can be related to the small \(k\) expansion of \(S_{\text{GinUE}}^{\infty}(k)\), which must therefore read

\[
2\pi \beta \left|\frac{k}{2}\right|^2 S_{\text{OCP}}^{\infty}(k) = 1 + \left(\frac{\beta}{4} - 1\right) \left|\frac{k}{2}\right|^2 + \left(\frac{\beta}{4} - \frac{3}{2}\right) \left(\frac{\beta}{4} - \frac{2}{3}\right) \left(\frac{|k|^2}{2\beta}\right)^2 + O(|k|^6).
\]

Evidence is given in \([193]\) that the polynomial structure in \(\beta/4\) of the coefficients in this expansion breaks down at \(O(|k|^6)\). This is in contrast to the power series expansion of the bulk scaled (density \(1/\pi\) for definiteness) structure factor for the log-gas on a one-dimensional domain, \(S_{\infty}^{(\beta)}(k)\) say. Thus expanding \(\pi \beta S_{\infty}^{(\beta)}(k)/k\) as a power series in \((k/\beta)\), the \(j\)-th coefficient is a monic polynomial of degree \(j\) in \(\beta/2\) \([144] [139]\).

The use of linear response to quantify the change of charge density upon the introduction of a charge \(q\) into the OCP, computing from this the change of charge by integrating, and requiring by screening that this must equal \(-q\) can be used \([186]\) to deduce the Carnie-Chan sum rule \([78] [79]\)

\[
-\beta \int_{\mathbb{R}^2} dr \left(\int_{\mathbb{R}^2} d\mathbf{r}' \log |\mathbf{r}'| \left(\rho_{(2),\infty}^{\text{OCP},T}(r, \mathbf{r}') + \frac{1}{\pi} \delta(\mathbf{r} - \mathbf{r}')\right)\right) = 1.
\]

Here the integral cannot be interchanged, as according to \((4.19)\) integrating over \(\mathbf{r}\) first would give zero. The validity of \((4.30)\) can be checked directly for \(\beta = 2\) in the bulk (i.e. for GinUE in the bulk).

**Proposition 4.8.** The Carnie-Chan sum rule \((4.30)\) is valid for GinUE in the bulk.
Proof. Denote the integral over \( r' \) in (4.30) by \( g(r) \). Assuming only rotation and translation invariance of \( \rho_{(2),\infty}^{\text{OCP},T} \) implies
\[
g(r) = \log |r| \left( \int_{|r'|<|r|} \rho_{(2),\infty}^{\text{OCP},T}(r',0) \, dr' + \frac{1}{\pi} \right) + \int_{|r'|>|r|} \log |r'| \rho_{(2),\infty}^{\text{OCP},T}(r',0) \, dr'.
\]
Specialising now to \( \beta = 2 \) by substituting (2.24), the use of polar coordinates and integration by parts allows the integrals to be simplified, with the result
\[
g(r) = -\frac{1}{\pi} \int_{|r|}^{\infty} \frac{e^{-r'^2}}{r'} \, dr'
\]
The integration over \( r \in \mathbb{R}^2 \) of this can be computed by further use of polar coordinates and integration by parts to confirm (4.30). \( \square \)

Formal use of the convolution theorem in (4.30) shows that it reduces to (4.26) [252]. In the case of the edge geometry, this approach can be used [189] to derive from (4.30) in the edge case the edge dipole moment sum rule
\[
-2\pi\beta \int_{-\infty}^{\infty} dy \left( \int_{-\infty}^{\infty} dy' (y'-y) \int_{-\infty}^{\infty} dx' \rho_{(2),\infty}^{\text{OCP},T}((0,y),(x',y')) \right) = 1,
\]
first derived in [58]. In keeping with the analogous property of (4.30), it is not possible to interchange the integrations over \( y \) and \( y' \) in this expression (if it was the LHS would vanish due to the sign change of \( y'-y \)). In the case \( \beta = 2 \) this is readily verified using the exact result (2.25).

**Proposition 4.9.** The edge dipole moment sum rule (4.31) is valid for edge scaled GinUE.

Proof. Substituting (2.25), integration by parts gives
\[
\int_{-\infty}^{\infty} dy' (y'-y) \int_{-\infty}^{\infty} dx' \rho_{(2),\infty}^{\text{GinUE},T}((0,y),(x',y')) = -\frac{1}{\sqrt{8\pi}} e^{-2y^2}
\]
(this formula can be found in [184, Eq. (2.41)]). Now integrating over \( y \) verifies (4.31) for \( \beta = 2 \). \( \square \)

The exact result (2.26) exhibits a slow decay of \( \rho_{(2),\infty}^{T}((x_1,y_1),(x_2,y_2)) \) parallel to the edge. This was first predicted by Jancovici [185, 187], who on the basis of linear response argument relating to the screening an oscillatory external charge density applied at the edge obtained for general \( \beta > 0 \),
\[
\rho_{(2),\infty}^{T}((x_1,y_1),(x_2,y_2)) \sim \frac{f(y_1,y_2)}{y_1-y_2} \quad \int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{\infty} dy_2 f(y_1,y_2) = -\frac{1}{2\beta\pi^2}.
\]
A derivation of this using the Carnie-Chan sum rule for the OCP confined to a strip geometry is given in [186]. In keeping with this, in [191] the amplitude $f(y, y')$ in (4.32) is related to the dipole moment of the screening cloud at the edge by deriving that

$$
\int_{-\infty}^{\infty} dy' (y' - y) \int_{-\infty}^{\infty} dx' \rho_{(2),\infty}^{\mathrm{OCP,T}}((0, y), (x', y')) = \pi \int_{-\infty}^{\infty} dy' f(y, y').
$$

Then the result for the integral over $y$ on the RHS follows from the dipole moment sum rule (4.31).

We consider now a global scaling, so that the droplet support is the unit disk. For large $N$, define the charge-charge surface correlation by

$$
\langle \sigma(\theta) \sigma(\theta') \rangle_T^N := \int d\mathbf{n} \int d\mathbf{n}' \rho_{(2),N}^{\mathrm{OCP,T}}(r, r').
$$

Here $\rho_{(2),N}^{\mathrm{OCP,T}}$ refers to $\rho_{(2),\infty}^{\mathrm{OCP,T}}$ computed using global scaling (as specified below (4.17)), then considering the large $N$ form with edge coordinates (these can be taken as $(r, \theta)$ with $r \approx 1$. The integration is over the normal direction $\mathbf{n}$ (here the radial direction); the more general notation has been used as this quantity can be defined in other geometries e.g. for the elliptic GinUE [143]. The reasoning leading to (4.32) has been extended [187] to lead to the prediction

$$
\langle \sigma(\theta) \sigma(\theta') \rangle_T^\infty = -\frac{1}{8\pi^2 \beta \sin^2((\theta - \theta')/2)}.
$$

This has been checked at $\beta = 2$ using the exact result for GinUE in [93].

Remark 4.10. For the bulk scaled OCP with $\beta$ an even integer, there is a constraint on the small $r = |r|$ form of $\rho_{(2),\infty}^b(r, 0)$. Thus [263]

$$
\rho_{(2),\infty}^b(r, 0) = r^\beta e^{-\beta r^2/4} f(r^4),
$$

for $f(z)$ analytic. Note from (2.24) that for GinUE, $f(z) = \frac{2z}{\pi^4} \frac{\sinh(\sqrt{z}/2)}{\sqrt{z}}$. Recently consequences have been found in [262].

5. Normal matrix models

5.1. Eigenvalue PDF. A generalisation of the joint element PDF (2.33) is the functional form proportional to

$$
\exp \left( -\frac{N}{t_0} \text{Tr}(JJ^\dagger - 2\text{Re} \sum_{p=2}^{M} t_p \text{Tr} f^p) \right), \quad t_0 > 0;
$$

note the scaling so that there is a factor of $N$ in the exponent. Use of the Schur decomposition (2.2) gives a separation of the eigenvalues and the strictly upper triangular elements in the
exponent analogous to (2.4). This allows the latter to be integrated over, showing that the corresponding eigenvalue PDF is proportional to

\[
\exp \left( - \frac{N}{t_0} \sum_{j=1}^{N} \left( |z_j|^2 - 2 \text{Re} \sum_{p=2}^{M} t_p z_j^p \right) \right).
\]

However for \( t_M \neq 0 \) this is not normalisable for \( M > 2 \).

A simple remedy is to impose a cutoff by restricting the eigenvalues to a disk centred about the origin of radius \( R \), and restricting to small \( t_0 \) and values of \( t_2, \ldots, t_M \) so that the support \( \Omega \) (assumed simply connected) of the density is contained inside this disk [122].

The mean field argument leading to (2.29) tells us that in relation (5.2), the normalised density has the constant value \( 1/\pi t_0 \) in \( \Omega \), where the latter is such that (5.3)

\[
W(z) = \frac{1}{\pi} \int_{\Omega} \log |z - w| \, d^2w, \quad W(z) = \frac{1}{2} \left( |z|^2 - 2 \text{Re} \sum_{p=2}^{M} t_p z^p \right),
\]

for \( z \) contained inside of \( \Omega \). From this equation the coupling constants \( \{t_p\}_{p=2}^M \) can be related to moments associated with \( \Omega \) [4] [289].

**Proposition 5.1.** In the above setting, for \( p \geq 2 \) we have

\[
t_p = \frac{1}{2\pi i p} \int_{\partial \Omega} z^{-p} \, dz = -\frac{1}{\pi p} \int_{C_1 \setminus \Omega} z^{-p} \, d^2z.
\]

**Proof.** By applying \( \partial_z \) to both sides of (5.3), then introducing \( W(w) \) in the integrand via the identity \( 2\partial_w \partial_{\bar{w}} W(w) = 1 \) shows

\[
\partial_z W(z) = \frac{1}{\pi} \int_{\Omega} \frac{\partial_w \partial_{\bar{w}} W(w)}{z - w} \, d^2w.
\]

Now using the Cauchy-Pompeiu formula (3.21) with \( C_1 \) replaced by \( \Omega \), we see from this that

\[
\int_{\partial \Omega} \frac{\partial_w W(w)}{w - z} \, dw = 0.
\]

Simple manipulation then shows

\[
\frac{1}{2\pi i} \int_{\partial \Omega} \frac{\partial \bar{w}}{w - z} \, dw = \sum_{p=2}^{M} p t_p z^{p-1}.
\]

The first equality of (5.4) now follows by power series expanding the LHS with respect to \( z \). The second equality can be deduced from the first by applying the version of (3.21) valid for \( z \) outside \( C_1 \) (chosen as \( C \setminus \Omega \)), for which the LHS is to be replaced by 0. \( \square \)
The simplest case of (5.2) beyond that corresponding to (2.33) is to take $M = 3$. After scaling, the choice of $t_3$ can be fixed. Choosing $t_3 = \frac{1}{3}$, results from [204] give that the support of $\Omega$ is contained in a disk for all $0 < t_0 \leq 1/8$, with the critical value $t_0 = 1/8$ involving three cusp singularities. Then, the boundary $\partial \Omega$ is a 3-cusped hypercycloid. For general $M \geq 2$ and $\Omega$ contained in a disk and simply connected, it is shown in [122] that $\partial \Omega$ can be parametrised by a Laurent polynomial of the form $a_1 w + a_0 + \cdots + a_{M-1} w^{-M+1}$ with $|w| = 1$. Dropping the assumption that $\Omega$ be simply connected requires the theory of quadrature domains; see [219] and references therein.

More general than the joint eigenvalue PDF (5.1) is the functional form

$$
(5.5) \quad \exp \left( - \sum_{j,k=1}^{\infty} c_{jk} \text{Tr} \left( J^j (J^\dagger)^k \right) \right) =: \exp \left( - \text{Tr} W(J, J^\dagger) \right).
$$

However unlike the former, substituting the Schur decomposition (2.2) does not in general lead to a separation of the eigenvalues from the strictly upper triangular variables. To overcome this, attention can be restricted to the subset of $N \times N$ complex matrices having the further structure $[J, J^\dagger] = 0$, which specifies the matrices as being normal. For normal matrices, the eigenvectors can be chosen to form an orthonormal set, and so $J = U D U^\dagger$, for $U$ unitary and $D$ the diagonal matrix of eigenvalues. Moreover, the change of variables from $J$ to $\{U, D\}$ gives a decomposition of measure into separate eigenvalue and eigenvector factors, with the Jacobian given by [2.5][111]. Hence the eigenvalue PDF corresponding to (5.5) in this setting is proportional to

$$
(5.6) \quad \exp \left( - \sum_{j=1}^{N} W(z_j, \bar{z}_j) \right) \prod_{1 \leq j < k \leq N} |z_k - z_j|^2.
$$

Note that for the joint element PDF (5.1), the same eigenvalue PDF (5.2) results for $J$ specified on the full space of $N \times N$ complex matrices, as it does on the restriction to normal matrices as implied by (5.6).

5.2. Equilibrium measure. Analogous to (5.1), to obtain a compact eigenvalue support for large $N$, one considers the case that the weight $e^{-W}$ is exponentially varying in a sense that $W$ is of order $N$, say $W(z, 2) = NQ(z)$ for a fixed potential $Q : \mathbb{C} \to \mathbb{R}$. Thus the eigenvalue PDF (5.6) is written as

$$
(5.7) \quad e^{-H(z_1, \ldots, z_N)}, \quad H(z_1, \ldots, z_N) := \sum_{1 \leq j < l \leq N} \log \frac{1}{|z_j - z_l|^2} + N \sum_{j=1}^{N} Q(z_j).
$$

As in Remark 2.5, the macroscopic behaviour of the system (5.7) can be described using the two-dimensional Coulomb gas interpretation with the help of logarithmic potential theory. Namely, it is well known in the literature [192, 175, 82, 83, 27] that for a general $Q$ under
suitable potential theoretic assumptions, the empirical measure $\frac{1}{N} \sum \delta_{z_j}$ weakly converges to a unique probability measure $\mu_Q$ that minimises

$$I_Q[\mu] := \int_{\mathbb{C}^2} \log \left| \frac{1}{z - w} \right| d\mu(z) d\mu(w) + \int_{\mathbb{C}} Q d\mu;$$

cf. (1.10). One may notice that (5.8) can be interpreted as a continuum limit of the Hamiltonian $H_N$ in (5.7) after normalisation, generalising (2.28). The probability measure $\mu_Q$ is called the equilibrium measure and its support $S_Q := \text{supp}(\mu_Q)$ is called the droplet, as previously remarked. Furthermore, if $Q$ is $C^2$-smooth in a neighbourhood of $S_Q$, by Frostman’s theorem [260], $\mu_Q$ is absolutely continuous with respect to the Lebesgue measure and takes the form

$$d\mu_Q(z) = \frac{\partial_z \partial_{\bar{z}} Q(z)}{\pi} \chi_{z \in S_Q} d^2z.$$ 

In particular, for a rotationally symmetric potential $q(r) = Q(|z| = r)$, the droplet is of the form $S_Q = \{r_1 \leq |z| \leq r_2\}$, where $(r_1, r_2)$ are the unique pair of constants satisfying

$$r_1 q'(r_1) = 0, \quad r_2 q'(r_2) = 2,$$

see [260, § IV.6]. Here, we have assumed that $Q(z)$ is strictly subharmonic in $\mathbb{C}$, which is equivalent to the requirement that $r \mapsto rq'(r)$ is increasing in $(0, \infty)$. Let us mention that all the explicit macroscopic densities with rotation invariance in the above subsections of §2 can be obtained as special cases of the formulas (5.9) and (5.10). For instance, the RHS of (2.44) can be realised as the RHS of (5.9) with $Q(z) = |z|^2 - 2a \log |z|$. Beyond the case when $Q$ is radially symmetric, the determination of the droplet, also known as the two-dimensional equilibrium problem (e.g. the ellipse in § 2.3), is far from being obvious even for some explicit potentials with simple form; see [43, 45, 203, 14, 101] and references therein for recent works in this direction.

5.3. Partition functions. Continuing the discussion in § 4.1, we consider the global scaled, non charge neutral, partition function

$$Z_N(\beta; Q) = \frac{1}{N!} \int_{\mathbb{C}} d^2z_1 \cdots \int_{\mathbb{C}} d^2z_N e^{-\frac{\beta}{2} H(z_1, \ldots, z_N)},$$

where $H(z_1, \ldots, z_N)$ is the Hamiltonian given in (5.7). An explicit formula for the large $N$ expansion of $Z_N(\beta; Q)$ was predicted in [290]. Fairly recently, it was shown by Leblé and Serfaty [216] that for general $\beta > 0$ and $Q$, $Z_N(\beta; Q)$ admits the large $N$ asymptotic expansion of the form

$$\log Z_N(\beta; Q) = -\frac{\beta}{2} N^2 I_Q[\mu_Q] + \left( \frac{\beta}{4} - 1 \right) N \log N - \left( C(\beta) + \left( 1 - \frac{\beta}{4} \right) E_Q[\mu_Q] \right) N + o(N);$$
see also an earlier work \[265\] on \[5.12\] up to the \(O(N \log N)\) term. The term \(I_Q[\mu_Q]\) appearing in the leading order asymptotic is the energy \(5.8\) evaluated at the equilibrium measure \(\mu_Q\). In the case of the OCP, up to a sign it is the quantity appearing in the exponent of \(A_{N,\beta}\) in \(4.2\) at order \(N^2\). To see this, we note that for a radially symmetric potential \(q(r) = Q(|z| = r)\) generally, it is evaluated as

\[
I_Q[\mu_Q] = q(r_1) - \log r_1 - \frac{1}{4} \int_{r_0}^{r_1} rq'(r)^2 \, dr,
\]

where \(r_1\) and \(r_2\) are the constants specified in \(5.10\). For the OCP \(Q(z) = |z|^2\), which gives \(I_Q[\mu_Q] = 3/4\), this indeed being the coefficient of \(N^2\) seen in \(A_{N,\beta}\). The appearance of the term \((\beta/4 - 1)N \log N\) in \(5.12\), whereas there is a term \(\frac{1}{4}\beta N^2 \log N\) in \(A_{N,\beta}\) of \(4.2\) is due to the simple scaling \(z_j \mapsto z_j/\sqrt{N}\) required to go from the OCP with global scaled coordinates (as assumed in \(5.12\)) to the OCP itself (as assumed in \(4.2\)). The terms appearing in the \(O(N)\) term of \(5.12\) are the entropy

\[
E_Q[\mu_Q] := \int_{\mathbb{C}} \mu_Q(z) \log \mu_Q(z) \, d^2 z
\]

associated with \(\mu_Q\) and a constant \(C(\beta)\) independent of the potential \(Q\). Their sum is the free energy per particle, \(\beta f(\beta; Q)\). The expansion \(5.12\) with quantitative error bounds is also available in the literature \[46, 270\].

For the random normal matrix model when \(\beta = 2\), the determinantal structure \(2.9\) allows an explicit expression

\[
Z_N(2; Q) = \prod_{j=1}^{N} h_j,
\]

where \(h_j\) is the orthogonal norm in \(5.18\); cf. Proposition \(2.1\). In particular, if \(Q\) is rotationally symmetric, since \(p_j(z) = z^j\), we have

\[
h_j = 2\pi \int_0^\infty r^{2j+1} e^{-Nq(r)} \, dr.
\]

Based on this knowledge together with a Laplace approximation of the integrals, the precise asymptotic expansion of the free energy up to the \(O(1)\) term was derived in a recent work \[69\]. This in particular shows that

\[
\log Z_N(2; Q) = -N^2 I_Q[\mu_Q] - \frac{1}{2} N \log N + \left(\frac{\log(2\pi)}{2} - \frac{1}{2} E_Q[\mu_Q]\right) N - \frac{\chi}{12} \log N + O(1),
\]

where \(\chi\) is the Euler index; cf. \(4.6\). Let us recall here that \(\chi = 0\) for the annulus \((r_0 > 0)\) and \(\chi = 1\) for the disk \((r_0 = 0)\) geometry. See \[69, \S 4\] and \[129, \S 4\] for some concrete examples of \(5.17\) associated with the matrix models discussed in \(\S 2\).
5.4. Correlation functions and universality. Turning to the correlation functions $\rho_{(k),N}$ of (5.7), the determinantal structure (2.9) remains valid with the correlation kernel

$$K_N(z, w) = e^{-\frac{N}{2}(Q(z) + Q(w))} \sum_{j=0}^{N-1} \frac{p_j(z)p_j(w)}{h_j},$$

where $p_j$ is the monic orthogonal polynomial of degree $j$ with respect to the weighted Lebesgue measure $e^{-NQ(z)} d^2z$ and $h_j$ is its squared orthogonal norm; cf. (3.6).

Let us first discuss the asymptotic behaviours of $\rho_{(k),N}$ in the micro-scale. Given a base point $p \in S_Q$ for which we zoom the point process, we denote by

$$\delta := \left. \frac{\partial_z \partial_{\bar{z}} Q(z)}{\pi} \right|_{z=p}$$

the mean eigenvalue density at $p$; cf. (5.9). From universality principles (see e.g. [202]), one expects that for a general $Q$ and $p$ such that $\delta \in (0, \infty)$ (i.e. the eigenvalue density does not vanish or diverge at $p$), the universal scaling limit in Proposition 204 arises. For the bulk case when $p \in \text{Int}(S_Q)$, such a universality was established by Ameur, Hedenmalm and Makarov in [33] where they showed that for a fairly general potential $Q$ under some mild assumptions,

$$\frac{1}{(N\delta)^k} \rho_{(k),N}(p + \frac{z_1}{\sqrt{N\delta}}, \ldots, p + \frac{z_k}{\sqrt{N\delta}}) \to \det \left[ K_{\infty}^0(z_j, z_l) \right]_{j,l=1}^k,$$

uniformly on compact subsets of $\mathbb{C}$ as $N \to \infty$. See also [31, 52]. In a sequential work [34] the theory of loop equations (or Ward’s identities [194, Appendix 6]) was also used to show the bulk scaling limit (5.20). This approach was further developed in [36] and several related works to study various scaling limits of the random normal matrix models (see e.g. [37, 28] for the bulk scaling limit at weak non-Hermiticity (2.39); [38, 268] for the edge scaling limit with boundary confinements (2.59); [15, 39] for normal matrices with Mittag-Leffler type singularities as in §2.4 and §2.7 and we refer to [15, Remark 2.9] for an expository summary of this strategy. In particular, in [36], the rescaled version of Ward’s identity was introduced and used to show the edge universality: for $p \in \partial S_Q$,

$$\frac{1}{(N\delta)^k} \rho_{(k),N}(p + in \frac{z_1}{\sqrt{N\delta}}, \ldots, p + in \frac{z_k}{\sqrt{N\delta}}) \to \det \left[ K_{\infty}^0(z_j, z_l) \right]_{j,l=1}^k,$$

where $n$ is the outer normal vector at $p$ as in Proposition 2.8. However, the result in [36] has an additional assumption that the limiting correlation function is translation invariant along the real axis. This assumption is intuitively natural but hard to rigorously show in general (except e.g. for the rotationally symmetric potential $Q$). The edge universality was later then shown by Hedenmalm and Wennman for a wide class of potentials $Q$ in [177], where
they developed an asymptotic theory for general planar orthogonal polynomials; see also [174] for an alternative approach to derive the main result in [177] and [176] for a theory developed for the orthogonal polynomials associated with non-exponentially varying weight. The asymptotic results in [177] together with (5.18) then leads to (5.21) by the Riemann sum approximation.

We now turn our attention to the asymptotic behaviours of \( \rho(k, N) \) in the macro-scale. For the Ginibre ensemble, equivalently, for the random normal matrix model (5.7) with \( Q(z) = |z|^2 \), the associated correlation kernel \( K_N \) in (5.18) satisfies

\[
K_N(z, w) = \sqrt{\frac{2N}{\pi}} (z \bar{w})^N e^{N - \frac{2}{\pi}(|z|^2 + |w|^2)} S(z, w) \left( 1 + O\left( \frac{1}{N} \right) \right), \quad (z \neq w)
\]

where \( S(z, w) \) is the exterior Szegő kernel

\[
S(z, w) := \frac{1}{2\pi z^* w - 1}.
\]

From a viewpoint of Proposition 2.2, the asymptotic behaviour (5.22) can be realised as a uniform expansion of the incomplete gamma function \( z \mapsto \Gamma(N; Nz) \), which is available in the literature in some particular domains; see e.g. [284] for \( |\arg(z - 1)| < 3\pi/4 \). In a recent work [32], generalising the classical results, it was shown that the asymptotic behaviour (5.22) remains valid as long as \( z \bar{w} \) is outside the Szegő curve \( \{ z \in \mathbb{C} : |z| \leq 1, |ze^{1-z}| = 1 \} \). Note in particular that if \( |z| = |w| = 1 \), then (5.22) reads

\[
K_N(z, w) \sim \sqrt{\frac{2N}{\pi}} S(z, w) \left( 1 + O\left( \frac{1}{N} \right) \right),
\]

where \( \sim \) means that the asymptotic expansion holds up to a sequence of cocycles (in this case \( (z \bar{w})^N \)), which cancel out when forming a determinant (2.9). From a statistical physics point of view, the behaviour (5.24) indicates that there are strong correlations among the particles on the boundary of the droplet, which also shows the slow decay of correlations at the boundary. For GinUE this is explicit in (4.35). For elliptic GinUE, the phenomenon was studied in [143], as a test of the generalisation to more general shaped droplets, when the RHS of (4.35) is predicted to be given in terms of a certain Green’s function for an electrostatics problem outside of the droplet, which acts as a macroscopic conductor [187], [131], Eq. (3.29). Furthermore, it was obtained by Ameur and Cronvall [32] that for a general class of potentials \( Q \), the associated correlation kernel \( K_N \) satisfies

\[
K_N(z, w) \sim 2N \left( \frac{\partial_z \partial_{\bar{z}} Q(z)}{\pi} \right)^{1/4} \left( \frac{\partial_{w} \partial_{\bar{w}} Q(w)}{\pi} \right)^{1/4} S(z, w)(1 + o(1)).
\]
For this, the use of a general theory on the orthogonal polynomial due to Hedenmalm and Wennman \[177\] was made. We also refer to \[17] \[71] \[29\] for more recent studies in this direction.

Remark 5.2. Let \( \{ p_j^{(N,R)}(z) \}_{j=0,1,...} \) be the orthogonal polynomials with respect to the inner product \( \langle f | g \rangle := \int_{c_k} f(z) \overline{g(z)} e^{-2N W(z)/t_0} \), where \( W(z) \) as in \((5.3)\) and it is assumed that the limiting eigenvalue support \( \Omega \) corresponding to \((5.2)\) is contained in \( D_R \). Consider the probability distribution \( \mathcal{P}_k^{(N,R)} \) specified by the eigenvalue PDF proportional to

\[
(5.26) \quad \prod_{l=1}^k e^{-2NW(z_l)/t_0} \prod_{1 \leq j < l \leq k} |z_l - z_j|^2.
\]

As a result of the underlying determinantal structure, one has the formula for \( p_k^{(N,R)}(z) \) as an expectation with respect to \( \mathcal{P}_k^{(N,R)} \) (see e.g. \[133\] proof of Proposition 5.1.3); in fact such formulae were known to Heine \[279\]

\[
p_k^{(N,R)}(z) = \left\langle \prod_{l=1}^k (z - z_l) \right\rangle \mathcal{P}_k^{(N,R)}.
\]

For \( k/N = x \) as \( k, N \to \infty \) it has been conjectured \[124\] that the zeros of \( p_k^{(N,R)}(z) \) accumulate on certain arcs \( \Sigma \) contained in \( \Omega \), with corresponding measure \( \mu_x^* \). Assuming this, it follows that for \( z \in \mathbb{C} \setminus \Omega \)

\[
(5.27) \quad \frac{1}{\pi t_0} \int_{\Sigma} \log |z - w| d^2w = \int_{\Sigma} \log |z - s| d\mu_x^*.
\]

This identifies \( \Sigma \) as the so-called mother body or potential theoretic skeleton of \( \Omega \). A number of works give further developments along these lines, especially in relation to the strong asymptotics of the planar orthogonal polynomials based on Riemann-Hilbert analysis. In the case of \((5.6)\) with a cubic potential, references include \[56\] \[204\] \[239\] \[57\], while for the induced Gaussian type weight \( 2W(z)/t_0 = |z|^2 - 2c \log |z - a| \) or more generally \( 2W(z)/t_0 = |z|^2 - 2 \sum_{j=1}^M c_j \log |z - a_j| \) in \((5.26)\), see \[43\] \( c = O(1) \), \[44\] \( c = O(1/N) \) and \( |a| \neq 1 \), \[54\] \( c = O(1/N) \) and \( |a| \approx 1 \), \[224\] \( c = O(1/N) \) and \( |a| \neq 1 \). We also refer to \[286\] \[108\] \[211\] for applications of such strong asymptotics in the context of the characteristic polynomials of the Ginibre matrix.

6. Further theory and applications

6.1. Fermi gas wave function interpretation. We have seen that the rewrite of \((1.7)\), written in the exponential form \((1.10)\), allows for the GinUE eigenvalue PDF to be interpreted as the
Fermi quantum many body wave function. Thus inside the absolute value of (6.1) is a Slater determinant of single body wave functions \( \{ \phi_i(z) \}_{i=0,\ldots,N-1} \) with \( \phi_i(z) = e^{-|z|^2/2} z_i^l \). What remains then is to identify the corresponding one body Hamiltonian for quantum particles in the plane which have these single body wave functions for the lowest energy states.

The appropriate setting for this task is a quantum particle confined to the \( xy \)-plane subject to a perpendicular magnetic field, \( (0,0,B), B > 0 \). Fundamental to this setting is the vector potential \( \mathbf{A} \), related to the magnetic field by \( \nabla \times \mathbf{A} = (0,0,B) \). The so-called symmetric gauge corresponds to the particular choice \( \mathbf{A} = (-By, Bx, 0) =: (A_x, A_y, 0) \), which henceforth will be assumed. Physical quantities in this setting are \( m \) (the particle mass), \( e \) (particle charge), \( \hbar \) (Planck’s constant), \( c \) (speed of light), which together with \( B \) are combined to give \( \omega_c := eB/mc \) (cyclotron frequency) and \( \ell := \sqrt{\hbar c/eB} \) (magnetic length).

Defining the generalised momenta and corresponding raising and lowering operators by

\[
\Pi_u = -i\hbar \frac{\partial}{\partial u} + \frac{e}{c} A_u \quad (u = x, y), \quad \mathbf{a} = \frac{\ell}{\sqrt{2\hbar}} (\Pi_x + i\Pi_y), \quad \mathbf{a}^\dagger = \mathbf{a}^\dagger, \]

allows the quantum Hamiltonian to be written in the harmonic oscillator like form \( H_B = \hbar \omega_c (\mathbf{a}^\dagger \mathbf{a} + \frac{1}{2}) \) [97]. Important too are the quantum centre of orbit operators and associated raising and lowering operators

\[
U = \hbar - \frac{\ell^2}{\hbar} \Pi_u \quad (U = X, Y; u = x, y), \quad \mathbf{b}^\dagger = \frac{1}{\sqrt{2\ell}} (X - iY), \quad \mathbf{b} = \mathbf{b}^\dagger, \]

for which \( X^2 + Y^2 = 2\ell^2 (\mathbf{b}^\dagger \mathbf{b} + \frac{1}{2}) \). The operators \( \{\mathbf{a}, \mathbf{a}^\dagger\} \) commute with \( \{\mathbf{b}, \mathbf{b}^\dagger\} \), implying that \( H \) and \( X^2 + Y^2 \) permit simultaneous eigenstates. A complete orthogonal set can be constructed using the raising operators according to

\[
|n, m\rangle = \frac{(\mathbf{a}^\dagger)^n (\mathbf{b}^\dagger)^m}{\sqrt{n!m!}} |0, 0\rangle, \tag{6.2}
\]

with eigenvalues of \( H \) equal to \((n + \frac{1}{2})\hbar \omega_c\) and eigenvalue of \( X^2 + Y^2 \) equal to \((2m + 1)\ell^2\).

The ground state \( |0, 0\rangle \) is characterised by \( \mathbf{a}|0, 0\rangle = \mathbf{b}|0, 0\rangle = 0 \), which can be checked to have the unique solution \( |0, 0\rangle \propto e^{-(x^2 + y^2)/4\ell^2} \). From this, application of \((\mathbf{b}^\dagger)^m\) gives \( |0, m\rangle \propto z^m e^{-|z|^2/4\ell^2}, z = x + iy \). Forming a Slater determinant with respect to the first \( N \) eigenstates of this type gives (6.1) with \( \ell^2 = 1/2 \). Generally states with quantum number
$n = 0$ and thus belonging to the ground state are said to be in the lowest Landau level. One remarks that the largest eigenvalue of $X^2 + Y^2$ is then $N - 1/2$, which is in keeping with the squared radius of the leading order support in the circular law.

The above theory of a quantum particle in the plane subject to a perpendicular magnetic field can be recast to apply to a rotating quantum particle in the plane $[178, 209]$. It is further true that the elliptic GinUE PDF $[2.32]$ admits an interpretation as the absolute value squared of state in the lowest Landau level, and furthermore the corresponding orthogonal polynomials $[2.34]$ can be constructed using a Bogolyubov transformation of $\{b, b^\dagger\}$ $[143]$. Also, the PDF on the sphere $[2.48]$ permits an interpretation as the absolute value squared of the ground state wave function for a free Fermi gas on the sphere subject to a perpendicular magnetic field $[172]$. Another point of interest relates to the $N$-body Fermi ground state corresponding to the quantum Pauli Hamiltonian in the plane with a perpendicular inhomogeneous magnetic field $B(x, y)$. The Hamiltonian $H_B$ defined above then is to be multiplied by the $2 \times 2$ identity matrix, and the spin coupling term $- (g\hbar/2m) B(x, y) \text{diag}(1/2, -1/2)$ added. With $B(x, y) = -\frac{1}{2} \nabla^2 W(x, y)$ for some real valued $W$, and with the assumption $\Phi := \int B(x, y) \, dxdy < \infty$, the ground state for this model (which is spin polarised all spins up) permits an exact solution for $g = 2$ $[5]$. The ground state of normalisable eigenfunctions has degeneracy $[\Phi/2\pi\hbar] =: N$, with basis of eigenfunctions $\{z^j e^{W(x,y)/2\hbar}\}_{j=0}^{N-1}$. This implies the Fermi many body ground state $[6.1]$ with $e^{-|z|^2/2}$ replaced by $e^{W(x,y)/2\hbar}$ $[4]$.

**Remark 6.1.**

1. Upon stereographic projection of the sphere to the plane, it is possible to write the quantum Hamiltonian for a charge particle in a constant perpendicular magnetic field in a form unified with the original planar case $[115]$. This involves the Kähler metric and potential, and permits a viewpoint which carries over to further generalise the space to higher dimensional complex manifolds in $\mathbb{C}^m$. A point of interest is that doing so gives, for the bulk scaling limit of the corresponding $N$-particle lowest Landau level state, the natural higher dimensional analogue of the kernel $[2.19]$ $[52, 53]$.

2. The squared wavefunction for higher Landau levels (say the $r$-th) has been shown to give rise to the determinantal point process with bulk scaled kernel

$$K_r^\infty(w, z) = L_r^0(|w - z|^2)e^{wx}e^{-(|w|^2 + |z|^2)/2};$$

see e.g. $[273]$ Prop. 2.5]. Allowing for mixing between Landau levels up to and including level $r$ leads to squared wave functions giving rise to the same determinantal point process except for the replacement of Laguerre polynomials $L_r^0 \mapsto L_r^1$ in the kernel $[171]$. Extending $[209]$, the precise mapping between the rotating fermions in the higher Landau levels and the polyanalytic Ginibre ensemble was established in $[205]$. Furthermore, its full counting
statistics and generalisations to finite temperature were obtained in [276, 206].
3. In the theory of the fractional quantum Hall effect, constructing an anti-symmetric state with filling fraction of the lowest Landau level \( \nu = 1/m \), for \( m \) an odd integer, plays a crucial role. To accomplish this, Laughlin [213] proposed the ground state wave function proportional to

\[
\prod_{l=1}^{N} e^{-|z_l|^2/4\ell^2} \prod_{1 \leq j < k \leq N} (z_k - z_j)^m;
\]

Note that with the assumption that \( m \) is odd, this is anti-symmetric as required for fermions. Moreover it belongs to the lowest Landau level as follows from the theory in the text below (6.2). The absolute value squared of (6.3) coincides with the Boltzmann factor (1.10) with \( \beta = 2m \), and the scaling \( z_l \to z_l/\sqrt{2m\ell^2} \). From potential theoretic/Coulomb gas reasoning, the bulk density is therefore \( 1/(2m\pi\ell^2) \). The factor of \( m \) in the denominator is in precise agreement with the requirement that the filling fraction be equal to \( 1/m \).
4. The ground state \( N \)-body free spinless Fermi gas in the plane, without a magnetic field but confined by a radial harmonic potential, is also an example of a determinantal point process for which exact calculations are possible; see the recent review [107]. However, its statistical state is distinct from that of GinUE. Thus with a global scaling so that the support is the unit disk, the density profile as the macroscopic fluctuation theorem for the linear statistic \( G = \sum_j g(r_j/R) \), with \( g \) assumed sufficiently smooth and absolutely integrable, in the \( R \to \infty \) limit [109] Th. III.2]

\[
\frac{G - \frac{R^d\omega_d}{(2\pi)^d} \int_{R^d} g(r) d^d r}{\sigma_d R^{(d-1)/2}} \to N[0, \Sigma(g)], \quad (\Sigma(g))^2 = \int_{R^d} |\hat{g}(\xi)|^2 |\xi| d^d \xi.
\]

Here \( \omega_d = \pi^{d/2}/\Gamma(1 + d/2) \) is the volume of the Euclidean ball in \( \mathbb{R}^d \), \( \omega_d/(2\pi)^d \) is the bulk density, \( \sigma_d^2 := \omega_{d-1}/(2\pi)^d \) and the Fourier transform has the definition \( \hat{g}(\xi) = \frac{1}{(2\pi)^d} \int_{R^d} e^{-i\mathbf{r} \cdot \mathbf{\xi}} g(\mathbf{r}) d^d \mathbf{r} \). Note in particular that in contrast to (3.18), the variance of \( G \) now diverges with the scale \( R \).
6.2. **Quantum chaos applications.** The pioneering works of Wigner and Dyson relating to the Hermitian random matrix ensembles was, as noted in §1, motivated by seeking a model for the (highly excited) energy levels of a complex quantum system. Later, in the 1980’s, as a fundamental contribution to the then emerging subject of quantum chaos, Bohigas et al. [59] identified the correct meaning of a complex quantum system not by the number of particles but rather as one for which the underlying classical mechanics is chaotic. To test this prediction on say the numerically generated spectrum of a quantum billiard system, the energy levels (beyond some threshold to qualify as being highly excited) were first unfolded so that their local density became unity, and then their numerically determined statistical properties were compared against random matrix predictions for the appropriate symmetry class; see e.g. [170]. Most popular among the statistical properties have been the variance for the number of eigenvalues in a large interval, and the distribution of the spacing between successive eigenvalues.

A natural extension of these advances is to inquire about the spectrum of a dissipative chaotic quantum system, which due to the loss of energy need not be real. This question was taken up by Grobe, Haake and Sommers [168] for the specific model of a damped periodic kicked top. The quantum dynamics are specified by a subunitary density operator. It is the spectrum of this operator, which after unfolding, and considering only those eigenvalues in the upper half plane away from the real axis (there is a symmetry which requires that the eigenvalues come in complex conjugate pairs — see the recent paper [10] for a discussion of this point in a random matrix context) that were compared in [168] a statistical sense to GinUE. Following from precedents in the Hermitian case, in the statistical quantity measured was the distribution of the radial spacing between closest eigenvalues, to be denoted \( P_{s,GUE}(r) \) with the normalisation \( \int_0^\infty P_{s,GUE}(r) \, dr = 1 \). This is the quantity \( F_\infty(0; D_r) \) of Remark 3.3.1. Recalling (3.10) we therefore have

\[
(6.5) \quad P_{s,GUE}(r) = -\frac{d}{dr} e^{r^2} \prod_{j=1}^{\infty} \left(1 - \frac{\gamma(j; r^2)}{\Gamma(j)}\right).
\]

It follows that for small \( r \), \( P_{s,GUE}(r) \approx 2r^3 \), while it follows from (3.11) and the comment in the sentence immediately above Remark 3.3 that for large \( r \), \( \log P_{s,GUE}(r) \approx -r^4/4 \). A numerical plot can be obtained from the functional form (6.5). For the moments the formula \( \langle r^p \rangle = p \int_0^\infty r^{p-1} e^{r^2} E_\infty(0; r) \, dr \) holds true. In particular, for the mean we calculate \( \langle r \rangle = 1.142929 \ldots \).

A variation of the closest neighbour spacing for an eigenvalue at \( z \) is the complex ratio \( (z^c - z)/(z^{nc} - z) \), where \( z^c \) is the closest neighbour to \( z \), and \( z^{nc} \) is the next closest neighbour
An approximation, with fast convergence properties to the large $N$ form, has been given recently in [116].

Very recently a non-Hermitian Hamiltonian realisation of GinUE has been obtained in the context of a proposed non-Hermitian $q$-body Sachev-Ye-Kitaev (SYK) model, with $N$ Majorana fermions — $N$ large and tuned mod 8 — and $q > 2$ and tuned mod 4 [154]. On another front, again very recently, the emergence of GinUE behaviours in certain model many body quantum chaotic systems in the space direction has been demonstrated [275].

Of interest in both these lines of study is the so-called dissipative (connected) spectral form factor

$$K_c^N(t,s) = \frac{1}{N}\text{Cov}\left(\sum_{j=1}^{N} e^{i(x_j t + y_j s)}, \sum_{j=1}^{N} e^{-i(x_j t + y_j s)}\right).$$

Making use of the first formula in (3.2) and the finite $N$ form of (2.23), this can be evaluated in terms of the hypergeometric function $\text{1F1}[226, \text{Eq. (3)}]$ (corrected in [155, Appendix A]; note too that both those references use global scaled variables, whereas we do not).

**Proposition 6.2.** We have

$$(6.6) \quad K_c^N(t,s) = 1 - \frac{1}{N} \sum_{m,n=0}^{N-1} \frac{(t^2 + s^2)^{|m-n|/2}}{n!m!2^{|m-n|}} \left(\frac{\text{max}(m,n)!}{|m-n|!}\right) \text{1F1}\left(\frac{\text{max}(m,n) + 1, |m-n| + 1; \frac{-t^2 + s^2}{4}}{1}\right)^2.$$

In particular,

$$\lim_{N \to \infty} K_c^N(t,s) = 1 - e^{-(t^2+s^2)/4};$$

cf. (3.16).

Also of interest in the many body quantum chaos application is the GinUE average of $|\text{Tr} X^k|^2$ for positive integer $k$ [275].

**Proposition 6.3.** We have

$$\langle |\text{Tr} X^k|^2 \rangle_{\text{GinUE}} = \frac{1}{(k+1)(N-1)!(N+1)} \left((k+N)! - \frac{N!(N-1)!}{(N-k-1)!}\right).$$

In particular

$$\lim_{k,N \to \infty} \frac{1}{kN^k} \langle |\text{Tr} X^k|^2 \rangle_{\text{GinUE}} = \frac{2\sinh(x^2/2)}{x^2}.$$
We conclude this subsection with a brief account of the use of the GinUE in an ensemble theory of Lindblad dynamics \[74, 105, 259\]. This relates to the evolution of the density matrix $\rho_t$ for an $N$-level dissipative quantum system in the so-called Markovian regime, specified by the master equation $\dot{\rho}_t = -i[H, \rho_t] + L(\rho_t)$. Here the operator $L$ assumes a special structure identified by Lindblad [228], and by Gorini, Kossakowski, and Sudarshan [165]. Specifically $L$ consists of the sum of two terms, the first corresponding to the familiar unitary von Neumann evolution, and the second to a dissipative part, being the sum over operators $D$ (referred to as simple dissipators), represented as $N^2 \times N^2$ matrices according to

$$D_L = 2L \otimes T L^\dagger - L^\dagger L \otimes T \mathbb{1}_N - \mathbb{1}_N \otimes T L^\dagger L,$$

for some $N \times N$ matrix $L$. Here $A \otimes B := A \otimes B^T$, where $\otimes$ is the usual Kronecker product.

In an ensemble theory, there is interest in $F_N(t) := \langle \sum_{j,l=1}^N L^\dagger L \rangle_L$ [74].

**Proposition 6.4.** Let $L$ be chosen from GinUE with global scaling. We have

$$\lim_{N \to \infty} F_N(t) = e^{-4t} \left( I_0(2t) + I_1(2t) \right)^2.$$

**Proof.** (Sketch) Following Can [74], using a diagrammatic calculus, it is first demonstrated that

$$\lim_{N \to \infty} \frac{1}{N^2} \langle D_L^k \rangle = \lim_{N \to \infty} \frac{(-1)^k}{N^2} \langle \left( L^\dagger L \otimes T \mathbb{1}_N + \mathbb{1}_N \otimes T L^\dagger L \right)^k \rangle_{L \in \text{GinUE}}.$$

The average on the RHS, in terms of the eigenvalues $\{x_j\}$ of $L^\dagger L$, reads $\langle \sum_{j,l=1}^N (x_j + x_l)^k \rangle_{L^\dagger L}$ and consequently

$$\lim_{N \to \infty} F_N(t) = \lim_{N \to \infty} \left( \frac{1}{N^2} \sum_{j,l=1}^N e^{-t(x_j + x_l)} \right)_{L^\dagger L} = \lim_{N \to \infty} \left( \frac{1}{N} \sum_{j=1}^N e^{-tx_j} \right)_{L^\dagger L}^2.$$

The latter is the mean of a linear statistic in the ensemble $\{L^\dagger L\}$ (complex Wishart matrices; see e.g. [133, §3.2]). Using the Marchenko-Pastur law for the global density of this ensemble (see e.g. [133, §3.4.1]), the stated result follows. \hfill \Box

6.3. **Eigenvectors.** Associated with the set of eigenvalues $\{\lambda_j\}$ of a Ginibre matrix $G$ are two sets of eigenvectors — the left eigenvectors $\{\ell_j\}$ such that $\ell_j^T G = \lambda_j \ell_j^T$, and the right eigenvectors $\{r_j\}$ such that $Gr_j = \lambda_j r_j$. These are not independent, but rather (upon suitable normalisation), form a biorthogonal set

$$(6.7) \quad \ell_j^T r_i = \delta_{ij}.$$

This property follows from the diagonalisation formula $G = XDX^{-1}$, where $X$ is the matrix of right eigenvectors, $D$ the diagonal matrix of eigenvalues, and $X^{-1}$ identified as the matrix...
of left eigenvectors. For nonzero scalars \( \{c_i\} \) we see that (6.7) is unchanged by the rescalings \( \mathbf{r}_j \mapsto c_j \mathbf{r}_j \) and \( \ell_j \mapsto (1/c_j) \ell_j \).

For \( N \times 1 \) column vectors \( \mathbf{u}, \mathbf{v} \), define the inner product \( \langle \mathbf{u}, \mathbf{v} \rangle := \mathbf{u}^T \mathbf{v} \). The so called overlap matrix has its elements \( O_{ij} \) expressed in terms of this inner product according to

\[
O_{ij} := \langle \ell_i, \ell_j \rangle \langle \mathbf{r}_i, \mathbf{r}_j \rangle.
\]

Note that this is invariant under the mappings noted in the final sentence of the above paragraph, and for fixed \( i \) and summing over \( j \) gives 1. Also, it follows from (6.8) that the diagonal entries relate to the lengths

\[
O_{jj} = ||\ell_j||^2 ||\mathbf{r}_j||^2.
\]

The square root of this quantity is known as the eigenvalue condition number; see the introduction to [64] and [100] §1.1 for further context and references. Significant too is the fact that the overlaps (6.9) appear in the specification of a Dyson Brownian motion extension of GinUE [167] [64].

Statistical properties of \( \{O_{ij}\} \) for GinUE were first considered by Chalker and Mehlig [85,86]. By the Schur decomposition (2.2), instead of a GinUE matrix \( G \), we may consider an upper triangular matrix \( Z \) with the eigenvalues \( \{z_i\} \) of \( G \) on the diagonal, and off diagonal entries standard complex Gaussians. For the eigenvalue \( \lambda_1 \), the triangular structure shows that \( \ell_1 = (1, b_2, \ldots, b_N)^T \) and \( \mathbf{r}_1 = (1, 0, \ldots, 0)^T \) where for \( p \geq 1 \) and \( b_1 = 1 \),

\[
b_p = \frac{1}{z_1 - z_p} \sum_{q=1}^{p-1} b_q Z_{pq}.
\]

From this last relation, it follows that with \( \ell^{(n)}_1 = (1, b_2, \ldots, b_n)^T \) for \( n < N \) we have

\[
||\ell_1^{(n+1)}||^2 = ||\ell_1^{(n)}||^2 \left(1 + \frac{1}{|z_1 - z_{n+1}|^2} \sum_{q=1}^{n} b_q Z_{(n+1)q} \right)^2, \quad \tilde{b}_q := \frac{b_q}{\sqrt{\sum_{q=1}^{n} |b_q|^2}}.
\]

This has immediate consequence in relation to \( O_{11} \) as shown by Bourgade and Dubach [64].

**Proposition 6.5.** Let the eigenvalues \( \{z_i\} \) be given. We have

\[
O_{11} \overset{d}{=} \prod_{n=2}^{N} \left(1 + \frac{|X_n|^2}{|z_1 - z_{n}|^2}\right),
\]

where each \( X_n \) is an independent complex standard Gaussian. Furthermore, it follows from this that

\[
O_{11} \bigg|_{z_1=0} \overset{d}{=} \frac{1}{B[2, N-1]},
\]

where \( B[\alpha, \beta] \) refers to the beta distribution.

**Proof.** A product formula for \( O_{11} \) follows from (6.9), the fact that \( ||\mathbf{r}_1|| = 1 \), and by iterating (6.10). This product formula is identified with the RHS of (6.11) upon noting that a vector of
independent standard complex Gaussians dotted with any unit vector (here \( \tilde{b}_1, \ldots, \tilde{b}_n \)) has distribution equal to a standard complex Gaussian.

In relation to (6.12) a minor modification of the proof of Proposition 2.19 shows that conditioned on \( z_1 = 0 \), the ordered squared moduli \( \{ |z_j|^2 \}_{j=2}^N \) are independently distributed as \( \{ \Gamma[j; 1] \}_{j=2}^N \). Noting too that with each \( X_j \) a standard complex Gaussian, \( |X_j|^2 \sim \Gamma[1; 1] \) it follows

\[
\mathcal{O}_{11} \bigg|_{z_1=0} \sim \prod_{n=2}^N \left( 1 + \frac{\bar{X}_n}{Y_n} \right), \quad \bar{X}_n \sim \Gamma[1; 1], \ Y_n \sim \Gamma[n; 1].
\]

Next, we require knowledge of the standard fact that \( Y_n / (Y_n + \bar{X}_n) \sim B[n, 1] \). Furthermore (see e.g. [133] Exercises 4.3 q.1), for \( x \sim B[\alpha + \beta, \gamma], \ y \sim B[\alpha, \beta] \), we have that \( xy \sim B[\alpha, \beta + \gamma] \), which tells us that with \( b_n \sim B[n, 1] \) we have \( \prod_{n=2}^N b_n \sim B[2, N - 1] \).

We mention that the explicit formulas for \( \mathcal{O}_{11} \) and \( \mathcal{O}_{12} \) are also obtained in [21] Th. 1. This in particular shows that an integrable structure exists also for the eigenvectors.

Dividing both sides of (6.12) by \( N \) we see that the \( N \to \infty \) is well defined since \( NB[2, N - 1] \to \Gamma[2, 1] \). After scaling the GinUE matrix \( G \leftrightarrow G/\sqrt{N} \) so that the leading eigenvalue support is the unit disk, an analogous limit formula has been extended from \( z_1 = 0 \) to any \( z_1 = w, \ |w| < 1 \) in [64]. Thus

\[
\mathcal{O}_{11} \bigg|_{z_1=|w|} \sim \frac{1}{N(1-|w|^2)} \to \frac{1}{\Gamma[2, 1]}.
\]

In words, with \( \mathcal{O}_{11} \) corresponding to the condition number, one has that the instability of the spectrum is of order \( N \) and is more stable towards the edge. Another point of interest is that the PDF for \( 1/\Gamma[2, 1] \) is \( \chi_{t > 0}e^{-1/t}/t^2 \), which is heavy tailed. The \( 1/t^3 \) tail has been exhibited from another viewpoint in the work of Fyodorov [148]. There the joint PDF for the overlap non-orthogonality \( \mathcal{O}_{jj} - 1 \), and the eigenvalue position \( z_j \), was computed for finite \( N \). The global scaled limit of this quantity, \( \mathcal{P}^b(t, w) \) say, was evaluated as [148] Eq. (2.24)]

\[
\mathcal{P}^b(t, w) = \left(1 - |w|^2\right)^{-2} e^{-\left(1 - |w|^2\right)/t}, \quad |w| < 1.
\]

We remark that limit theorems of the universal form (6.13) have been proved in the case of the complex spherical ensemble of §2.5, and for a sub-block of a Haar distributed unitary matrix [114]; see also [249].

The average value of (6.13) was first obtained in [85, 86] as a conjecture, and proved in [285]. An explicit formula for the large \( N \) form of the average of the overlap (6.8) in
the off diagonal case (say \((i,j) = (1,2)\)), with the GinUE matrix scaled \(G \mapsto G / \sqrt{N}\), and conditioned on \(z_1 = w_1, z_2 = w_2\) with \(|w_1|, |w_2| < 1\) is also known \([85, 86, 64, 211, 100]\)

\[
(6.15) \quad \langle O_{12} \mid z_1 = w_1, z_2 = w_2 \rangle \sim \frac{1}{N} \left( 1 - \frac{|w_1 w_2|}{|w_1 - w_2|^2} \right) \left( 1 - \frac{(1 + N|w_1 - w_2|^2) e^{-N|w_1 - w_2|^2}}{1 - e^{-N|w_1 - w_2|^2}} \right).
\]

This formula is uniformly valid down to the scale \(N|w_1 - w_2| = O(1)\).

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