Analytical Aircraft State and IMU Signal Generator From Smoothed Reference Trajectory

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Generating aircraft position, velocity, and attitude along a reference trajectory is useful in many path and mission planning applications. In applications simulating an inertial navigation system the body-frame accelerations and angular rates are also required. This work presents a method for generating a smoothed continuous-curvature reference trajectory from a series of waypoints using line, arc, clothoid, and Fermat spiral segments in a fillet smoothing framework. The geometry of the segments are used to generate the position, velocity, and attitude of an aircraft following the reference trajectory. A primary contribution of this article is the use of the path geometry, coordinated turn model, and curvilinear motion theory to obtain analytical solutions for the body-frame accelerations and angular rates of the aircraft along the reference trajectory.

1. INTRODUCTION

An objective for the mission and path planning of unmanned aerial vehicles (UAVs) is to determine a flyable path that meets mission requirements. The characteristics of a flyable path are determined by the capabilities of the aircraft. The mission requirements vary based on the application, such as path planning in contested environments to avoid detection [1], reconnaissance missions, where the aircraft attitude determines sensor coverage [2], and obstacle avoidance [3].

Candidate paths generated by the path planner are assessed based on mission requirements. The path assessment often consists of evaluating functions of the aircraft pose (i.e., position and orientation) and sensor measurements along the path. This article presents a method called ASG for generating analytical aircraft state and inertial measurement unit (IMU) signals along smoothed candidate paths. The aircraft states and IMU measurements will be calculated using path segment geometry, a coordinated turn assumption [4], and curvilinear motion theory [5].

An important aspect of mission and path planning for UAVs is to account for uncertainty during the operation. Uncertainties exist in three components in planning, namely, vehicle position and orientation, environmental, and vehicle motion uncertainties. Path planning techniques often model some or all of these uncertainties to determine an optimal path [3], [6]. In this work, a fixed-wing UAV with an inertial
navigation system (INS) [7] is considered. The propagation of the state estimates and uncertainty of an INS is dependent on measurements from a strapdown IMU [8]. Thus, a path planning algorithm considering the uncertainty of such a system would require estimating aircraft states and the associated IMU measurements along the candidate path.

IMU measurement generation is used in a variety of applications. Among the most popular applications are trajectory reconstruction [9], [10], trajectory generation [7], [11], [12], and simulation [4], [13]. While there is significant overlap between these classifications, the chosen application typically determines the information that is available to generate the IMU measurements. For example, the trajectory reconstruction application typically reconstructs a trajectory using recorded flight data. In contrast, the simulation application can obtain acceleration and angular rates directly from the integration of aircraft dynamics. The trajectory generation application is the most relevant for the purposes of this work.

Trajectory generation takes a series of aircraft poses and calculates expressions for the accelerations and angular rates experienced by the aircraft. These quantities can then be used to generate IMU signals along the path. Most approaches to trajectory generation rely on numerical derivatives of lower order states to calculate accelerations and angular rates, [7], [11]–[13]. The numerical derivative is sufficiently accurate for small sampling times, however, the accuracy degrades with large sampling times. The large sampling times are often desired in path planning to reduce the computational burden of evaluating path validity. The primary contribution of this work is ASG, a method for analytically generating aircraft states and IMU signals using path segment geometry and a coordinated turn model. The analytical approach avoids numerical derivatives and is stable for large sampling times.

The first stage of the ASG method generates a smoothed reference trajectory from a piecewise linear candidate path (or series of waypoints). The piecewise linear path is not flyable for a fixed-wing UAV due to the discontinuities in course angle. The piecewise linear path can be converted to a flyable path using a path smoothing algorithm [14], [15]. This work will use a fillet corner smoothing algorithm that converts the piecewise linear path into a flyable path that maintains curvature continuity. The corner smoothing approach described in this work is not a novel contribution but a description of the approach is provided in Section III for completeness and reproducibility. The corner smoothing approaches used in this work were chosen because the resulting path consists of a series of segments that have analytical solutions for position, course angle, and curvature.

The path segments generated by the path smoothing algorithm are lines, arcs, and transition segments. The transition segments are used to connect the lines and arcs and maintain continuity in course angle and curvature. The first transition segment is the clothoid, which provides a linearly changing curvature per unit length [16]. A drawback to the clothoid curve is the dependence on the Fresnel Integrals that do not have a closed-form solution, however, accurate approximations exist and can be used to simplify the computation [17]. The second transition segment is the Fermat Spiral [18], which avoids the costly computation of the Fresnel Integrals, while maintaining continuous curvature. This work describes an algorithm for generating three types of smoothed paths (Arc Fillet, Clothoid Fillet, and Fermat Fillet) parameterized by a maximum curvature and maximum curvature rate. The smoothed path is then used to obtain analytical expressions for the aircraft state and IMU measurements along the candidate path.

The ASG method generates and samples the smooth reference trajectory and analytically generates IMU measurements. The novel IMU measurement generation approach is compared with a third party open-source software package based on MATLAB code provided with [7], by Paul Groves. The open-source package provides many utilities for INS implementations. In particular, this work uses a method for generating IMU signals given position, velocity, and attitude samples. The IMU measurements generated by the ASG method are compared with those generated from the Groves method. The comparison shows agreement between the two methods. Furthermore, the ASG method provides greater accuracy as the step time is increased because it has analytical solutions for the path geometry instead of relying on discrete derivatives.

The next section describes background information that is used throughout this article. Section III presents the corner smoothing algorithm that converts a series of waypoints to a path of continuous course angle. The path segment definitions (see Section II-B) are used to determine the position and course angle of the aircraft. Section IV describes a method for determining the roll, pitch, and body-frame accelerations and angular rates of the aircraft assuming straight-and-level flight during straight line segments, and a coordinated turn for transition segments and arcs. Finally, Section V provides results for the ASG method that show the resulting aircraft states and that the accuracy of the IMU measurements is maintained for large time steps.

II. BACKGROUND

This section presents background information used throughout this work. Section II-A describes the notation. Section II-B defines the segment types used in the corner smoothing algorithm. Finally, accelerations and angular rates along the smoothed path are determined by curvilinear motion theory which is discussed in Section II-C.

A. Notation

In this article, curvature rate refers to the derivative of curvature with respect to path length, $s$, thus the following convention will be used for brevity

$$\frac{dk}{ds} = k'$$

when the derivative is taken with respect to time, the standard dot notation will be used, $\dot{k}$.
where \( s \) is the length along the segment, and \( \psi_0 \) is the initial segment angle.

2) Arc: The arc is a continuous curvature segment. In the smoothing application, the arc will be parameterized by the maximum curvature \( (k_{\text{max}}) \) attainable by the aircraft. The expressions that define the arc are parameterized by the polar angle, \( \theta \), and are given by

\[
\begin{align*}
    x(\theta) &= x_0 - r \sin(\psi_0) + r \cos(\theta) \\
    y(\theta) &= y_0 + r \sin(\psi_0) + r \sin(\theta) \\
    \psi(\theta) &= \psi_0 + r \theta \\
    k &= \frac{\rho}{r}
\end{align*}
\]

where \( r \) is the radius of the arc (the inverse of curvature), and \( \rho \) indicates the direction of travel where

\[
\rho = \begin{cases} 
1, & \text{counter-clockwise (left) turn} \\
-1, & \text{clockwise (right) turn}.
\end{cases}
\]

The polar angle, \( \theta \), is related to the length on the path by the arc length formula \( s = r\theta \). Thus, the arc can be sampled at a specified arc length interval by computing the associated polar angle interval and applying the equations above. Note that the convention employed here is that \( \theta = 0 \) is associated with the point on the curve where \( \psi = \frac{\pi}{2} \).

3) Clothoid: A clothoid or Euler spiral is a path segment whose curvature varies linearly by segment length \([19]\). The position along the clothoid segment is determined by computing the Fresnel Integrals given by

\[
\begin{align*}
    x &= x_0 + \int_0^s \cos(0.5\sigma_0 \xi^2 + k_0 \xi + \psi_0) d\xi \\
    y &= y_0 + \int_0^s \sin(0.5\sigma_0 \xi^2 + k_0 \xi + \psi_0) d\xi
\end{align*}
\]

where \( s \) is the length along the clothoid segment, \( \psi_0 \) is the initial segment angle, \( k_0 \) is the initial segment curvature, \( x_0 \) and \( y_0 \) represent the starting point of the segment, and \( \sigma_0 \) is the curvature rate of the clothoid segment. A primary challenge when working with clothoids is that the Fresnel Integrals do not have a closed-form solution so numerical solutions are required to compute positions along the path. However, there are numerous methods \([14], [20], [21]\) for computing accurate approximations to these equations that help mitigate this drawback.

The course angle of a clothoid segment changes quadratically by segment length as given by

\[
\psi(s) = \psi_0 + k_0 s + 0.5\sigma_0 s^2.
\]

Finally, the curvature of the clothoid segment is given by

\[
k(s) = k_0 + \sigma_0 s.
\]

This shows the linear relationship between path length and curvature. An example of an Euler spiral is shown in Fig. 2 for a segment that is 5 m long, has an initial segment angle and initial curvature of zero, and a curvature rate of one.

4) Fermat Spiral: Another spiral that has been investigated for corner smoothing is the Fermat spiral \([18]\). The Fermat spiral is an interesting corner smoothing candidate
as it provides many of the benefits of the clothoid but avoids the integral computation used to compute the positions along the curve. The curve is parameterized in a polar coordinate system as \( r = c \sqrt{\theta} \), where the \( c \) parameter can be used to modify the curve characteristics and \( \theta \) is the polar angle. Converting the curve to Cartesian coordinates results in

\[
\begin{align*}
  x(\theta) &= x_0 + c \sqrt{\theta} \cos(\rho \theta + \psi_0) \\
  y(\theta) &= y_0 + c \sqrt{\theta} \sin(\rho \theta + \psi_0).
\end{align*}
\]

(16) (17)

The curve is shown in Fig. 2, where it can be noted that curvature does not monotonically increase as the polar angle increases, instead, the curvature increases to a maximum and then decreases until settling at a nearly constant value. The curvature is given by the nonlinear equation

\[
k(\theta) = \frac{2 \sqrt{\theta} (4\theta^2 + 3)}{c (4\theta^2 + 1)^2}.
\]

(18)

Fig. 3 shows the curvature of the Fermat spiral for a quarter of a rotation of the polar angle. The figure shows that there is a single maximum for \( \theta > 0 \). The value of the polar angle at which the maximum curvature is achieved at

\[
\theta_{k_{\text{max}}} = \sqrt{\frac{\sqrt{7}}{2} - \frac{5}{4}} \approx 0.26995
\]

(19)

which is independent of the shaping parameter, \( c \). Beyond \( \theta_{k_{\text{max}}} \) the curvature decreases. This may appear to be a drawback for using this type of segment in corner smoothing, however, in this application, the polar angle is restricted to the range of \( 0 < \theta < \theta_{k_{\text{max}}} \) so the curvature is monotonically increasing along the transition segment.

The last curve property of interest is the course angle which is given by

\[
\psi(\theta) = \theta + \arctan(2\theta).
\]

(20)

For the corner smoothing application, it is important to define equations for a reflected segment that starts at an arbitrary point along the segment and tracks back to the start of the segment. This operation is more involved for a Fermat Spiral segment than the other segments defined in this section so it will receive special attention here. The position equation for the reflected Fermat Spiral is parameterized by the curve end point \((x_{\text{end}}, y_{\text{end}})\), and course at the end of the segment \( \psi_{\text{end}} \) and is given by

\[
\begin{align*}
  x(\theta) &= x_{\text{end}} + c \sqrt{\theta_{\text{end}} - \theta} \cos(\rho (\theta - \theta_{\text{end}}) + \psi_{\text{end}}) \\
  y(\theta) &= y_{\text{end}} + c \sqrt{\theta_{\text{end}} - \theta} \sin(\rho (\theta - \theta_{\text{end}}) + \psi_{\text{end}})
\end{align*}
\]

(21) (22)

and \( \theta_{\text{end}} \) is the polar angle at the end of the reflected Fermat Spiral segment.

Lekkas et al. [18] showed how the shaping parameter, \( c \), can be set such that the maximum curvature of the segment is equal to the maximum curvature of the aircraft. This is accomplished by solving (18) and using \( \theta = \theta_{k_{\text{max}}} \) and \( k(\theta_{k_{\text{max}}}) = k_{\text{max}} \). Then, an expression for \( c \) is given by

\[
c = \frac{1}{k_{\text{max}}} \frac{2 \sqrt{\theta_{k_{\text{max}}} (4\theta_{k_{\text{max}}}^2 + 3)}}{(4\theta_{k_{\text{max}}}^2 + 1)^2}
\]

(23)

which will result in a scaled curve that reaches a maximum curvature of \( k_{\text{max}} \). Thus, the Fermat spiral can be parameterized to respect the same maximum curvature constraint used in the clothoid smoothing implementation.

One drawback to using the Fermat spiral for corner smoothing is that the length of the path is given by

\[
s = c \sqrt{2} \int_0^{\theta_{k_{\text{max}}}} \sqrt{1 + 4\theta^2} d\theta
\]

(24)

which does not have a closed-form solution. In [18], the length of a Fermat segment is expressed as a hypergeometric function with guaranteed convergence over the domain of interest. Thus, in implementation, the computational cost of (24) can be mitigated using a lookup table or an iterative
solution to the hypergeometric function with some convergence criteria.

C. Curvilinear Motion Theory

Curvilinear motion theory defines the motion of a particle along a curved path [5]. Properties of this theory will be used to relate the geometry of the path with the aircraft state vector. The velocity and acceleration along a curved path is given by

\[ v = \dot{s} \mathbf{e}_t \]  

(25)

and

\[ a = \ddot{s} \mathbf{e}_t + \dot{s} \psi \mathbf{e}_n \]  

(26)

where \( \mathbf{e}_t \) and \( \mathbf{e}_n \), are the tangent and normal vectors, and \( \psi \) is the angular rate of the path.

III. CORNER SMOOTHING

In the first stage of the aircraft state generator presented in this work, a piece-wise linear path is converted into a smooth path with continuous course angle or curvature. In the case of continuous curvature, a limit on curvature and curvature rate can be imposed to ensure that the path will be flyable by a specific aircraft. In this way, the physical limitations of the aircraft can be considered and the resulting states can more accurately reflect the behavior of the aircraft.

This section describes three methods for smoothing a piece-wise linear path 1) Arc Fillet, 2) Clothoid Fillet, and 3) Fermat Fillet. Each of these methods perform corner smoothing at every way-point in the path. The resulting path will consist of a series of straight line, arc, and transition segments with varying curvature. This section is particularly useful for a person in the navigation field that may be unfamiliar with corner smoothing algorithms because it provides a unified framework for three methods used in the path planning field.

The path smoothing algorithms presented in this section assume that the spacing between the way-points is sufficiently large enough to perform the indicated corner smoothing method. For example, if the start or end way-point in Fig. 1 was closer to the corner point than the connection between the straight path and the smoothing arc, then the algorithms presented here would not be sufficient.

The first path smoothing approach presented in this section is the Arc Fillet method, where a maximum curvature segment is used to connect two straight segments. This method has continuous course angle but has discontinuous path curvature. The second is the Clothoid Fillet method, where clothoid segments are used to smooth the corners or, where needed, transition to a \( k_{\text{max}} \) segment. This method has continuous curvature and course angle and respects limits on curvature and curvature rate. The third is the Fermat Fillet method, which is similar to the Clothoid Fillet method but Fermat spiral segments are used in place of clothoid segments. This method has continuous curvature and course angle and respects limits on curvature, however, it is not guaranteed to respect limits on curvature rate. These properties are summarized in Table I.

The three approaches presented in this section iterate through the corners in the path and compute a series of path segments that represent the resulting smooth path. The sharpness of the corner and positions of the way-points before and after the corner are required to determine the smoothing segments. The position of the way-points is provided to the algorithm. The sharpness of the corner is represented by the change of course angle, \( \psi_{\Delta} \), required to transition from the entry segment to the exit segment. The \( \psi_{\Delta} \) quantity can be computed by expressing the entry and exit segments as vectors \( \mathbf{p}_1 \) and \( \mathbf{p}_2 \) from Fig. 1 and using properties of the dot product to determine the angle between the vectors. Another important quantity for each corner is the turn direction, \( \rho \), which is determined using the sign of the cross product of the two vectors. The sign convention for \( \rho \) is given in (11).  

A. Arc Fillet

The Arc Fillet method smooths a corner at a way-point by adding a circular arc of maximum curvature \( (k_{\text{max}}) \) that transitions from the first straight segment to the other. The arc segment is parameterized by the radius, \( r = 1/k_{\text{max}} \), and the arc length, \( s = r\psi_{\Delta} \), as discussed in Section II-B2. Then (8) is used with \( \theta = \psi_{\Delta} \) to determine the end point, \( (x_t, y_t) \), and the midpoint, \( (x_m, y_m) \), of the arc. The distance along the \( x \)-axis from the midpoint of the arc is computed geometrically as

\[ d_t = \left| \frac{y_m}{\tan^{-1} \left( \frac{\pi - \psi_{\Delta}}{2} \right)} \right| \]  

(27)

and the distance from the way-point to the connection point of the arc on the first straight segment is given by

\[ d = d_t + x_m. \]  

(28)

This distance is used to determine the connection point for the arc segment on the first straight segment. Fig. 4 shows the geometry of the Arc Fillet method and includes a graphical representation of \( x_m \) and \( d_t \).

B. Clothoid Fillet

The Clothoid Fillet method uses clothoid segments to connect the straight segments and arcs while maintaining continuous curvature. Using clothoid segments to generate

| Method          | Continuous | \( k_{\text{max}} \) | Respect Limit on Curvature | Respect Limit on Curvature Rate |
|-----------------|------------|----------------------|----------------------------|-------------------------------|
| Arc Fillet      | ✓          | -                    | ✓                          | -                             |
| Clothoid Fillet | ✓          | ✓                    | ✓                          | ✓                             |
| Fermat Fillet   | ✓          | ✓                    | ✓                          | -                             |

TABLE I

Path Smoothing Method Comparison of Course Angle and Curvature Continuity and Respecting Limits on Curvature and Curvature Rate (\( \psi = \text{Yes}, - \psi = \text{No} \))
orbits, and clothoid transition segments with curvature $k_0$ to determine the segment length at which the arc must be added. The course angle change at which $k_0$ is the change of course angle produced and $\phi$ is the central angle of the arc (see Fig. 6) and is $(0)$. The following paragraphs discuss how each of these parameters are determined for the smoothing segments based on the course angle change of the corner.

The initial position for the first clothoid segment is the attachment point for the smoothing segments to the initial straight segment. The attachment point can only be determined after the smoothing segments are defined so the smoothing segments will be defined for a base-case, where the initial position, course angle, and curvature for the first clothoid segment are $x_0 = y_0 = \psi_0 = k_0 = 0$.

The clothoid curvature rate, $\sigma_c$, for the initial clothoid segment is set to the maximum curvature rate ($\sigma_c = k_{\max}'$) desired for the current application. The second clothoid segment is a reflection of the first clothoid segment so the curvature rate is negated ($\sigma_c = -k_{\max}'$) to reduce the curvature from the transition curvature back to $k = 0$.

The segment length for each of the smoothing segments is determined using the course angle change of the corner and the curvature rate for the clothoid segments. For the case, where the turn can be spanned by two clothoid segments ($\psi_\Delta \leq 2\psi_m$), (14) is solved for $s$ with $\psi = \frac{\psi_\Delta}{2}$, $\psi_0 = 0$, and $\sigma_c = k_{\max}'$ which yields $s = \sqrt{\psi_\Delta/k_{\max}}$. For the case, where a maximum curvature segment is required to smooth the corner, $s = s_{\max}$ [see (29)]. The segment length defined here is the same for both clothoids segments used in the Clothoid Fillet Method. The segment length of the arc segment when one is required is computed using the arc length formula as

$$s = \frac{\phi_a}{k_{\max}}$$

where $\phi_a$ is the central angle of the arc (see Fig. 6) and is expressed as

$$\phi_a = 2\left(\frac{\psi_\Delta}{2} - \psi_i\right).$$

For small turns ($\psi_\Delta \leq 2\psi_m$), two clothoid segments are defined to smooth the corner. Each clothoid segment provides a course angle change of $\psi_\Delta/2$. The initial position

continuous curvature paths is well documented in [22]. This section will describe the specific application of smoothing corners using a clothoid fillet method within the unified framework of the current work.

The Clothoid Fillet method relies on the fact that the scaling factor, $\sigma_c$, of a clothoid segment represents the rate of change of the curvature of the segment. This method uses the maximum change in curvature, $k_{\max}'$, to transition from the initial straight segment ($k = 0$), to a maximum curvature ($k = k_{\max}$) arc. The transition segment is then reflected to obtain a transition from the constant curvature arc to the second straight segment. The resulting smoothed path consists of a series of segments including straight segments, $k_{\max}$ orbits, and clothoid transition segments with curvature changing linearly by $k_{\max}'$. Fig. 5 shows a representation of a series of segments generated by the path smoothing algorithm when transition segments are used. The blue segments are transition segments and the red segment is a $k_{\max}$ arc.

When the turn is small enough two clothoid segments can be used to smooth the corner, otherwise, a $k_{\max}$ arc must be added. The course angle change at which the $k_{\max}$ arc segment is required is determined by solving (15) with $k_0 = 0$ to determine the segment length at which the segment reaches maximum curvature ($k = k_{\max}$) as

$$s_{k_{\max}} = \frac{k_{\max}}{k_{\max}'},$$

Then evaluating (14) with $k_0 = \psi_0 = 0$ and $s = s_{k_{\max}}$ gives

$$\psi_m = \frac{k_{\max}^2}{2k_{\max}'}$$

where $\psi_m$ is the change of course angle produced change by one clothoid segment. Thus, the maximum course angle change that can be spanned with two clothoid segments is $2\psi_m$.

It remains to define the parameters of the line segments used for corner smoothing with the Clothoid Fillet method. The parameters include initial position, $(x_0, y_0)$, initial course angle, $\psi_0$, initial curvature, $k_0$, curvature rate, $\sigma_c$, and segment length, $s$. The following paragraphs discuss how each of these parameters are determined for the smoothing segments based on the course angle change of the corner.

The initial position for the first clothoid segment is the attachment point for the smoothing segments to the initial straight segment. The attachment point can only be determined after the smoothing segments are defined so the smoothing segments will be defined for a base-case, where the initial position, course angle, and curvature for the first clothoid segment are $x_0 = y_0 = \psi_0 = k_0 = 0$.

The clothoid curvature rate, $\sigma_c$, for the initial clothoid segment is set to the maximum curvature rate ($\sigma_c = k_{\max}'$) desired for the current application. The second clothoid segment is a reflection of the first clothoid segment so the curvature rate is negated ($\sigma_c = -k_{\max}'$) to reduce the curvature from the transition curvature back to $k = 0$.

The segment length for each of the smoothing segments is determined using the course angle change of the corner and the curvature rate for the clothoid segments. For the case, where the turn can be spanned by two clothoid segments ($\psi_\Delta \leq 2\psi_m$), (14) is solved for $s$ with $\psi = \frac{\psi_\Delta}{2}$, $\psi_0 = 0$, and $\sigma_c = k_{\max}'$ which yields $s = \sqrt{\psi_\Delta/k_{\max}}$. For the case, where a maximum curvature segment is required to smooth the corner, $s = s_{\max}$ [see (29)]. The segment length defined here is the same for both clothoids segments used in the Clothoid Fillet Method. The segment length of the arc segment when one is required is computed using the arc length formula as

$$s = \frac{\phi_a}{k_{\max}}$$

where $\phi_a$ is the central angle of the arc (see Fig. 6) and is expressed as

$$\phi_a = 2\left(\frac{\psi_\Delta}{2} - \psi_i\right).$$

For small turns ($\psi_\Delta \leq 2\psi_m$), two clothoid segments are defined to smooth the corner. Each clothoid segment provides a course angle change of $\psi_\Delta/2$. The initial position
For large turns ($\psi_\Delta > 2\psi_m$), a $k_{\max}$ arc is added between the clothoid transition segments. In this case, the initial clothoid segment transitions from zero curvature to $k_{\max}$ so the length of the segment is $s_{k_{\max}}$ (see (29)). The initial position ($x_0$, $y_0$, $\psi_0$) and course angle ($\psi_1$) of the arc segment is then determined by evaluating (12)-(15) with $c = k_{\max}^\prime$, $s = \sqrt{\psi_\Delta/k_{\max}}$, and $k_0 = 0$. The length of the arc segment is given in (31). The initial position ($x_1$, $y_1$, $\psi_1$) and course angle ($\psi_2$) of the second clothoid segment can be determined by evaluating (7)-(10) with ($x_0$, $y_0$, $\psi_0$) = ($x_1$, $y_1$, $\psi_1$), $\theta = \phi_o$, and $r = 1/k_{\max}$.

The parameters defined in the preceding paragraphs are summarized in Table II. The table shows the segment parameters required to smoothly large and small corners using the Clothoid Fillet method. The parameters are provided in terms of the course angle change of the corner ($\psi_\Delta$), the maximum curvature of the smoothed corner ($k_{\max}$), and the maximum curvature rate ($k_{\max}^\prime$).

With the smoothing segments defined for the baseline case of the Clothoid Fillet method, it remains to determine the attachment point ($x_\alpha$, $y_\alpha$, $\psi_\alpha$) on the initial straight segment. The attachment point is determined by computing the parameter $d$ in Fig. 5, which represents the distance from the attachment point to the end of the straight segment. Equation (27) can be used to compute the value for $d$, where $y_m$ is the $y$ component of the transition point between the two smoothing clothoid segments ($y_m = y_i$), or the midpoint of the $k_{\max}$ segment, where one is used. Then, the attachment distance $d$ is calculated as

$$d = x_i + d_l$$  \hfill (33)

where $x_i$ is the $x$ component of the transition point between the two clothoid segments, or the midpoint of the $k_{\max}$ arc segment, where one is used. Then, (3) and (4) can be evaluated using the segment parameters of the initial straight segment and with $s = s_o - d$, where $s_o$ is the original length of the initial straight segment.

Finally, the initial position and heading for each of the smoothing segments are adjusted based on the calculated attachment position and course angle. The adjustment is accomplished by adding the attachment position and course angle to the initial position and course angle of each of the segments.

### C. Fermat Fillet

The Fermat spiral was shown to be an effective transition segment for path smoothing applications in [18]. This section reiterates some of the development from [18] to fit in the framework presented in this work and define the segments required to smooth a corner using the Fermat Fillet method. The parameters required to define a Fermat spiral segment are the initial position and course angle ($x_0$, $y_0$, $\psi_0$), initial curvature ($k_0$), and span of the polar angle ($\theta_0$).

The Fermat Fillet method results in a continuous curvature path, where the transition segments are Fermat spirals. Similar to the Clothoid Fillet method, the corner can be smoothed with two Fermat spirals for small turns, or two Fermat spirals and a $k_{\max}$ arc segment for large turns. The largest course angle change that can be achieved with one Fermat spiral segment is determined by combining (19) and (20) as

$$\psi_m = \sqrt{\frac{\sqrt{\frac{7}{2}} - \frac{5}{4}}}{\tan^{-1}\left(\sqrt{\frac{\sqrt{\frac{7}{2}} - \frac{5}{4}}}{2}\right)}$$  \hfill (34)

$$\approx 0.7650 \text{ rad.}$$  \hfill (35)

Then, the largest course angle change ($\psi_\Delta$) that can be spanned by two Fermat spiral segments is $2\psi_m$.

In the case, where a $k_{\max}$ segment is needed ($\psi_\Delta > 2\psi_m$), $k_{\max}$ is defined by (19) and the course angle at the transition between the first Fermat spiral segment and the $k_{\max}$ arc is $\psi_m$. In the case, where two Fermat spiral segments can span $\psi_\Delta$ (where $\psi_\Delta \leq 2\psi_m$), the course angle at the transition is given by $\psi_t = \psi_\Delta/2$. In this case, $\theta_{k_{\max}}$ is more difficult to compute because (20) is not invertible. However, [18] suggests using a root finding method to iteratively solve for the roots of the course angle equation. This can be implemented by defining

$$f(\theta) = \theta + \tan^{-1}(2\theta) - \psi_t$$  \hfill (36)

where subtracting $\psi_t$ moves the root of the course angle equation such that applying the root finding method will result in the $\theta$ associated with the desired course angle $\psi_t$. 

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**TABLE II**

| $\psi_\Delta$ | Segment | ($x_0$, $y_0$, $\psi_0$) | $k_0$ | $\sigma_c$ | $s$ |
|----------------|---------|--------------------------|-------|------------|-----|
| Small          | Clothoid | ($x_0$, $y_0$, $\psi_0$) | 0     | $k_{\max}$ | $\sqrt{\psi_\Delta/k_{\max}}$ |
|                | Clothoid | ($x_1$, $y_1$, $\psi_1$) | $\sqrt{k_{\max}^\prime \psi_\Delta}$ | $-k_{\max}^\prime$ | $\sqrt{\psi_\Delta/k_{\max}}$ |
| Large          | Clothoid | ($x_0$, $y_0$, $\psi_0$) | $k_{\max}$ | 0           | $\sqrt{\psi_\Delta/k_{\max}}$ |
| Large          | Arc      | ($x_1$, $y_1$, $\psi_1$) | $k_{\max}$ | $-k_{\max}^\prime$ | $\sqrt{\psi_\Delta/k_{\max}}$ |
|                | Clothoid | ($x_2$, $y_2$, $\psi_2$) | $k_{\max}$ | $-k_{\max}^\prime$ | $\sqrt{\psi_\Delta/k_{\max}}$ |

---

**Fig. 6.** Definition of key parameters in the development of a $k_{\max}$ segment.
With \( \theta_{k_{\text{max}}} \) and \( \psi_i \) defined, the first Fermat spiral segment can be completely defined. The base case starts at the origin with \( \psi_0 = 0 \), and \( \theta_{k_{\text{max}}} = \theta_{k_{\text{max}}} \) using the value for \( \theta_{k_{\text{max}}} \) determined above. Then, the scale factor, \( c \), in the Fermat spiral can be computed using (23) with \( \theta_{k_{\text{max}}} = \psi_m \), which ensures that \( k_{\text{max}} \) will be achieved at \( \psi_m \) even if the segment ends prior to the curvature reaching \( k_{\text{max}} \). The transition point, \((x_t, y_t)\), can be determined by evaluating (16) and (17) at \( \theta_{k_{\text{max}}} \) and the course angle at the transition is \( \psi_i \).

In the case, where a \( k_{\text{max}} \) segment is required, an arc is added with an initial position of \((x_t, y_t)\), initial course angle of \( \psi_i \), radius of \( 1/k_{\text{max}} \), and arc length given by

\[
s = \frac{\psi_{\Delta} - 2\psi_i}{k_{\text{max}}} \tag{37}
\]

then (8) can be evaluated at \( \theta = \psi_{\Delta} - 2\psi_i \) to get the transition point for the second Fermat spiral segment, \((x_t, y_t)\), and \( \theta = \frac{\psi_{\Delta} - 2\psi_i}{2} \) to get the midpoint of the arc, \((x_m, y_m)\).

The second transition segment is then defined that connects to the first transition segment, or the \( k_{\text{max}} \) arc at \( (x_t, y_t) \). The second transition segment is a reflection of the first transition segment. For the Fermat spiral, this reflection is accomplished by computing the connection point, \((x_c, y_c)\), to the exit segment and flipping the sign of the direction parameter, \( \rho \). The connection point is determined by computing the deviation from the straight line segment, where the cross track deviation, \( h \), and down track deviation, \( l \), are given by

\[
h = c\sqrt{\theta_{k_{\text{max}}} \sin (\theta_{k_{\text{max}}})} \tag{38}
\]
\[
l = c\sqrt{\theta_{k_{\text{max}}} \cos (\theta_{k_{\text{max}}})} \tag{39}
\]

then the magnitude of the deviation is given by

\[
\beta = \sqrt{l^2 + h^2}. \tag{40}
\]

Then determine the angle, \( \phi_f \), that relates \((x_c, y_c)\) and \((x_t, y_t)\) as follows:

\[
\psi_c = \pi - \psi_{\Delta} \tag{41}
\]
\[
\phi_f = \frac{\pi}{2} - \psi_c - \theta_{k_{\text{max}}} \tag{42}
\]

The connection point is then calculated as

\[
x_c = x_t - \beta \sin (\phi_f) \tag{43}
\]
\[
y_c = y_t + \rho \beta \cos (\phi_f). \tag{44}
\]

The geometry used to compute these parameters is shown graphically in Fig. 7. These results are used with (21) and (22) to compute the properties of the reflected segment.

This method calculates a final connection point and uses a reflection parameter so that the terminal point of the final segment is at the transition point, \((x_t, y_t)\). This requires additional computation for the creation and sampling of the segments but provides simple segment parameter definitions. The parameters used to define the segments using the Fermat Fillet method are summarized in Table III.

The final parameter of interest in the Fermat Fillet method is the distance, \( d \), to the connection point on the first straight segment. Similar to the approach in the Clothoid Fillet method, \( d_t \) is computed using (27) where \( y_m \) is either the mid-point of the \( k_{\text{max}} \) arc segment or \( y_t \) if there is no \( k_{\text{max}} \) segment. Then, the distance \( d \) is calculated as

\[
d = x_t + d_t \tag{45}
\]

where \( x_t \) is the \( x \) value of the transition point between the two Fermat segments, or it is the \( x \) value of the mid-point of the \( k_{\text{max}} \) arc, if one is required.

Finally, the initial position and heading for each of the smoothing segments are adjusted based on the calculated attachment position and course angle. The adjustment is accomplished by adding the attachment position and course angle to the initial position and course angle of each of the segments.

**IV. IMU SIGNAL GENERATION**

The purpose of this section is to present a method to efficiently determine the accelerations and angular rates experienced by an aircraft following a path smoothed using the methods described in Section III. The accelerations and angular rates can then be used as nominal error-free measurements from an IMU and be used to propagate the covariance of the state estimate of a navigation system. For applications requiring high fidelity IMU measurements, additional elements such as coning, sculling, biases, and
misalignment and scale-factor errors can be added to the accelerations and angular rates generated in this section [9]. This section will present an approach for determining the acceleration and angular rates for an aircraft using curvilinear motion theory (see Section II-C), vehicle dynamics, and knowledge of the aircraft maneuver (coordinated turn).

Other methods for generating accelerations and angular rates exist, namely, inverting the navigation equations, or 6-DOF aircraft simulations. These methods have potential to be highly accurate, but they are computationally expensive which may be prohibitive depending on the application. The method presented in this section is intended to efficiently provide accelerations and angular rates at high enough fidelity to provide accurate covariance propagation results.

The accelerations experienced by the aircraft in the NED frame can be expressed using curvilinear motion theory as

\[ \ddot{\mathbf{a}} = \ddot{s_e} \mathbf{e}_n + \dot{s} \ddot{s_e} \mathbf{e}_n \]

(46)

where tangent vector in the 2-D plane is computed as a function of the course angle as

\[ \mathbf{e}_n = \begin{bmatrix} \cos(\psi_e) & \sin(\psi_e) \end{bmatrix}^T \]

(47)

and the normal vector is defined to complete the right-handed coordinate frame between the tangent vector and positive-z unit vector as

\[ \mathbf{e}_n = \begin{bmatrix} 0 & 1 \end{bmatrix}^T \times \mathbf{e}_s. \]

(48)

Combining (46)–(48) results in

\[ \ddot{\mathbf{a}} = \begin{bmatrix} \ddot{s} \cos(\psi_e) - \dot{s} \ddot{s_e} \sin(\psi_e) \\ \ddot{s} \sin(\psi_e) + \dot{s} \ddot{s_e} \cos(\psi_e) \end{bmatrix} \]

(49)

as the expression for the acceleration experienced by the aircraft in the NED frame and is valid for all of the segment types presented in this work.

The angular rates experienced by an aircraft are dependent on the path shape and the dynamics of the vehicle maneuver. An aircraft, for example, can perform a variety of maneuvers to negotiate a turn (i.e., skid-to-steer, coordinated turn) and each will result in different angular rates. In this work, the angular rates for an aircraft executing a coordinated turn will be examined. The coordinated turn is a common aircraft turning condition where there is no lateral acceleration in the body frame of the aircraft [23].

The coordinated turn assumption provides a useful relationship between the course angle (\(\psi_e\)) and roll angle (\(\phi_e\)) [4] of the aircraft as

\[ \dot{\psi}_e = \frac{g}{V_a} \tan \phi_e. \]

(50)

The rate of change of the roll angle can be computed by first differentiating (50) as

\[ \ddot{\psi}_e = \frac{g}{V_a} \frac{1}{\cos^2 \phi_e} \dot{\phi}_e \]

(51)

and solving for the roll angle rate, which yields

\[ \dot{\phi}_e = \dot{\psi}_e \frac{V_a}{g} \cos^2 \phi_e. \]

(52)

The roll rate (\(\dot{\phi}_e\)) and course angle rate (\(\dot{\psi}_e\)) need to be converted to body-frame angular rates, (\(p, q, r\)). This conversion requires a frame change from the Euler angle frames to the body-frame which will be completed using a ZYX Euler angle sequence. The Euler angle rates are related to the body-frame angular rates [4] as

\[ \begin{bmatrix} p \\ q \\ r \end{bmatrix} = A_B \begin{bmatrix} \dot{\phi}_e \\ \dot{\psi}_e \cos^2 \phi_e \\ 0 \end{bmatrix} \]

(53)

Assuming no change in pitch angle, substitution of (52) and (59) into (53) yields

\[ \begin{bmatrix} p \\ q \\ r \end{bmatrix} = A_B \begin{bmatrix} \dot{\psi}_e \frac{V_a}{g} \cos^2 \phi_e \\ 0 \\ \dot{\psi}_e \end{bmatrix} \]

(54)

or in component form

\[ p = \dot{\psi}_e \frac{V_a}{g} \cos^2 \phi_e - \psi_e \sin \theta_e \]

(55)

\[ q = \dot{\psi}_e \sin \phi_e \cos \theta_e \]

(56)

\[ r = \psi_e \cos \phi_e \cos \theta_e. \]

(57)

This indicates that to compute the angular rates of an aircraft executing a coordinated turn, \(\dot{\psi}_e\) and \(\dot{\phi}_e\) must be determined for each of the segment type in the path.

The time-derivative of the course angle is computed from (14) utilizing the chain rule

\[ \dot{\psi}_e(t) = \frac{d}{dt} \psi_e(s(t)) = \frac{d \psi_e}{ds} \frac{ds}{dt}. \]

(58)

(59)

The second derivative of heading with respect to time is computed utilizing a combination of multiplication rule and chain rule as

\[ \ddot{\psi}_e(t) = \frac{d}{dt} \psi_e'(s(t)) \dot{s}(t) = \frac{d}{ds} \dot{s}(t) + \psi_e''(s(t)) \dot{s}(t)^2 + \psi_e'(s(t)) \dot{\dot{s}}(t). \]

(60)

(61)

(62)

To generate the angular rates for the aircraft, it remains to define \(\dot{\psi}_e\) and \(\dot{\phi}_e\) for each of the segment types used in the path smoothing algorithm. The line segment is trivially defined as \(\psi_e = \dot{\psi}_e = 0\). The following sections provide the derivation of expressions for \(\dot{\psi}_e\) and \(\dot{\phi}_e\) for the remaining segment types.

A. Arc

The course angle of an arc segment is related linearly to the length along the segment as

\[ \psi_e(s) = \psi_0 + ks \]

(63)
where $k$ is the constant curvature of the arc segment. The derivative of this with respect to path length is given by

$$\psi_a'(s) = k \tag{64}$$

and the second derivative is given by

$$\psi_a'' = 0. \tag{65}$$

Then the time derivatives of the course angle [see (59) and (62)] are given by

$$\dot{\psi}_a = k \dot{s} \tag{66}$$

$$\ddot{\psi}_a = k \ddot{s}. \tag{66}$$

B. Clothoid

The course angle of the clothoid segment is given in (14). The derivative with respect to the path length is given by

$$\psi_c'(s) = k_0 + \sigma_c s \tag{67}$$

and the second derivative is given by

$$\psi_c'' = \sigma_c. \tag{68}$$

Then applying (59) and (62), the time derivatives are given by

$$\dot{\psi}_c = (k_0 + \sigma_c s) \dot{s} \tag{69}$$

$$\ddot{\psi}_c = \sigma_c \ddot{s} + (k_0 + \sigma_c s) \dddot{s}. \tag{70}$$

C. Fermat Spiral

The course angle of a Fermat Spiral segment is given in (20) in terms of the polar angle $\theta$. In [18], a change of variables is presented as $u = \sqrt{\theta}$, which provides consistent sampling per path length. Then, the course angle can be expressed as

$$\psi_f(u) = u^2 + \arctan(2 \, u^2). \tag{71}$$

This change of variables is useful because an expression for the time derivative of $u$ exists and is given by

$$\dot{u} = \frac{\dot{s}}{c \sqrt{1 + 4 \, u^4}}. \tag{72}$$

Then, the time derivative of the path segment can be computed as follows:

$$\dot{\psi}_f = \rho \frac{d \psi_f}{d u} \frac{d u}{d t} \tag{73}$$

$$= \rho \left( 2 \, u + \frac{4 \, u}{4 \, u^4 + 1} \right) \dot{u} \tag{74}$$

where $\rho$ is the direction of the curve defined in (11). The second derivative is computed by applying the quotient rule and chain rule as

$$\ddot{\psi}_f = \frac{d}{d t} \left( \frac{2 \dot{s} \left( 4 \, u^5 + 3u \right) \sqrt{4 \, u^4 + 1}}{c \left( 4 \, u^4 + 1 \right)^{3/2}} \right). \tag{75}$$

The time derivative of the numerator is given by

$$\dot{N} = 2 \left( \dot{s} \left( 4 \, u^5 + 3u \right) \sqrt{4 \, u^4 + 1} + \dot{\delta} N_1 \right) \tag{76}$$

where

$$\dot{N}_1 = \frac{d}{d t} \left( 4 \, u^5 + 3u \right) \sqrt{4 \, u^4 + 1}. \tag{77}$$

V. RESULTS

This section provides results for the path smoothing and aircraft state generation. The first set of results will include a comparison of the corner smoothing techniques presented in Section III. These results include a comparison between the path characteristics and the execution time for the methods presented. The second set of results show a comparison between the generated IMU measurements.

A. Path Smoothing

The algorithm defined in Section III was implemented in MATLAB, where a series of waypoints was converted to a continuous curvature path. Fig. 8 shows a piece-wise linear path smoothed using the three methods presented here. This was generated using $k_{max} = 2.1$, and $k'_{max} = 3$.

The course angle and curvature for the scenario shown in Fig. 8 is shown in Fig. 9. For this scenario, all three methods achieve the maximum curvature but the differences in the shape for the curvature and course angle are apparent. The curvature plot shows a step change for the Arc Fillet method, a linear change for the Clothoid Fillet method, and a smooth transition for the Fermat Fillet method.

The shape of the curvature and course angle graphs should be considered when deciding what corner smoothing
algorithm to use. For example, the arc fillet method has a step change in curvature which is infeasible for a fixed-wing aircraft. This highlights the need to consider the dynamics of the target system and the desired maneuvers when selecting a corner smoothing algorithm and the associated path segment types.

The processing time for the three methods were compared using a series of 12 waypoints. The Clothoid Fillet method is the slowest (due to the Fresnel Integrals), and the Arc Fillet method is the fastest. The Fermat Fillet method provides a 63.3% improvement in execution time when compared with the Clothoid Fillet method, whereas the Arc Fillet method provides an 87.98% improvement. These results provide a framework for deciding which method best suits the target application. For example, if the application does not require continuous curvature, the Arc Fillet method is very efficient and fulfills the objective. However, if continuous curvature is desired, the Fermat Fillet or Clothoid Fillet methods should be considered and if execution time is a primary concern, the preference should be given to the Fermat Fillet method.

B. IMU Measurement Results

The previous section showed results for the three path smoothing methods presented in this work. This section will extend these results and present the specific force and angular rates generated using the IMU signal generation method described in Section IV. Due to the discontinuities in course angle for the Arc Fillet method, it will be omitted from this analysis.

The IMU signals generated using the method shown in Section IV will be compared with the Groves method available as an open-source implementation based on derivations in [7] and used in [24]. Fig. 10 shows a diagram of how the accelerometer, \( \tilde{a} = [a_x, a_y, a_z]^T \), and gyro, \( \tilde{\omega} = [g_x, g_y, g_z]^T \), measurements are generated using the ASG and Groves methods. Two simple scenarios of four waypoints (two turns) were used to compare the methods. The first method will be referred to as the nominal trajectory scenario with \( k_{\text{max}} = 0.005 \, \text{rad}, \text{and} \, k'_{\text{max}} = 0.00005 \, \text{rad/m} \). The second will be referred to as the aggressive trajectory scenario that has waypoints closer together and with \( k_{\text{max}} = 0.01 \, \text{rad}, \text{and} \, k'_{\text{max}} = 0.0002 \, \text{rad/m} \). Fig. 11 shows one of the turns in the nominal trajectory scenario with the Clothoid Fillet and Fermat Fillet methods.

The accelerometer and gyro measurements for the Clothoid and Fermat Fillet methods for the nominal trajectory scenario are provided in Figs. 12 and 13, respectively. The IMU signals for both methods match the baseline results well. The slight differences between the Groves method and ASG method are difficult to see in Figs. 12 and

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1[Online]. Available: https://github.com/jmcanana/MATLAB-Groves
Fig. 12. Accelerometer and gyro measurements for the nominal trajectory scenario using the Clothoid Fillet corner smoother.

Fig. 13. Accelerometer and gyro measurements for the nominal trajectory scenario using the Fermat Fillet corner smoother.

so an error metric will be introduced in the following paragraphs to illustrate the differences.

The Groves method relies on a discrete derivative taken at each time step, \( dt \). For a small time step, this method provides consistent results. However, in some applications, accurate IMU signals are desired with larger time steps. This can be particularly useful in mission and path planning scenarios, where reducing processor usage is desired. One benefit to the ASG method is that the IMU signal generation avoids the derivative and maintains accuracy for large time steps because the IMU measurements have analytical solutions along the path segments.

To quantify the effect of increasing the time step on these two methods, baseline IMU signals were generated using the Groves method with a small time step (\( dt = 0.001 \) s). IMU signals for time steps ranging from \( dt = 0.1 \) s to \( dt = 3.0 \) s were generated using both methods. The integrated norm of the error, \( \Upsilon \), was calculated as the integrated difference between the IMU measurements generated by each method and the baseline Groves method (with \( dt = 0.001 \) s). Fig. 14 provides a graph of \( \Upsilon \) for increasing \( dt \) using the Groves and ASG methods for the Clothoid and Fermat fillet methods. The errors associated with the Groves method grow more quickly than the ASG method as the step size increases for both the accelerometer and gyro measurements. The ASG method results in nearly constant errors for both corner smoothing methods. The ASG method clearly provides more accurate IMU signals for the time steps shown.

The integrated norm of the error, \( \Upsilon \), for the aggressive trajectory scenario is shown in Fig. 15. This trajectory has waypoints that are closer together and \( k_{\text{max}} = 0.01 \) 1/rad,
and \( k'_{\text{max}} = 0.0002 \) l.rad/m. The error plots indicate that the ASG method outperforms the Groves method for this trajectory as well, and the general error characteristics are consistent with the nominal trajectory scenario. It should be observed that the errors are higher for the aggressive trajectory scenario than the nominal trajectory scenario.

VI. CONCLUSION

This article has presented ASG as a method for generating aircraft states and IMU signals along a candidate path. The method includes a corner smoothing algorithm to convert a series of waypoints into a smoothed reference trajectory and uses path segment geometry to generate the aircraft states. The final stage of the method applies curvilinear motion theory and vehicle maneuver dynamics to generate IMU signals along the candidate path. The generated states and IMU signals can be applied to a variety of applications, including general path planning, and covariance propagation for INS.

Three corner smoothing approaches were presented in a unified framework. The Clothoid Fillet and Fermat Fillet algorithms can be parameterized to respect vehicle constraints in curvature and curvature rate. The results showed that the Fermat Fillet requires less processing time than the Clothoid Fillet due to the Fresnel integrals. The Arc Fillet method requires the least processing time but has discontinuities in curvature and is thus not suitable for a fixed wing aircraft.

The results for the IMU signal generation stage show that the ASG method performs better than the Groves method by avoiding a discrete derivative. This allows for large step sizes in sampling when using the ASG method.

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