On the Non-diffusive Magneto-Geostrophic Equation

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Abstract. Motivated by an equation arising in magnetohydrodynamics, we address the well-posedness theory for the non-diffusive magneto-geostrophic equation. Namely, an active scalar equation in which the divergence-free drift velocity is one derivative more singular than the active scalar. In Friedlander and Vicol (Nonlinearity 24(11):3019–3042, 2011), the authors prove that the non-diffusive equation is ill-posed in the sense of Hadamard in Sobolev spaces, but locally well posed in spaces of analytic functions. Here, we give an example of a steady state that is nonlinearly stable for periodic perturbations with initial data localized in frequency straight lines crossing the origin. For such well-prepared data, the local existence and uniqueness of solutions can be obtained in Sobolev spaces and the global existence holds under a size condition over the $H^{5/2^+}(T^3)$ norm of the perturbation.

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1. Introduction

The geodynamo is the process by which the Earth’s magnetic field is created and sustained by the motion of the fluid core, which is composed of a rapidly rotating, density stratified, electrically conducting fluid. The full dynamo problem requires the examination of the full 3D partial differential equations governing convective, incompressible magnetohydrodynamics (MHD).

It is therefore reasonable to attempt to gain some insight into the geodynamo by considering a reduction of the full MHD equations to a system that is more tractable, but one that retains many of the essential features relevant to the physics of the Earth’s core.

Recently, Moffatt and Loper [14,15] proposed the magneto-geostrophic equation (MG) as a model for the geodynamo which is a reduction of the full MHD system. The physical postulates of this model are the following: slow cooling of the Earth leads to slow solidification of the liquid metal core onto the solid inner core and releases latent heat of solidification that drives compositional convection in the fluid core.

1.1. Governing Equations

We first present the full coupled three-dimensional MHD equations for the evolution of the velocity vector $\mathbf{U}(x,t)$, the magnetic field vector $\mathbf{B}(x,t)$ and the buoyancy field $\Theta(x,t)$ in the Boussinesq approximation and written in the frame of reference rotating with angular velocity $\omega$. For simplicity, we have assumed that the axis of rotation and the gravity $g$ are aligned in the direction of $e_3$.

Following the notation of Moffatt and Loper [15] we obtain the dimensionless equations

$$N^2[R_0(\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{U}) + e_3 \times \mathbf{U}] = -\nabla P + (e_2 \cdot \nabla) \mathbf{b} + R_m \mathbf{b} \cdot \nabla \mathbf{b} + N^2 \Theta e_3 + \epsilon_\nu \Delta \mathbf{U},$$

$$R_m[\partial_t \mathbf{b} + \mathbf{U} \cdot \nabla \mathbf{b} - \mathbf{b} \cdot \nabla \mathbf{U}] = (e_2 \cdot \nabla) \mathbf{U} + \Delta \mathbf{b},$$

$$\partial_t \Theta + \mathbf{U} \cdot \nabla \Theta = \epsilon_\kappa \Delta \Theta,$$
\[ \nabla \cdot \mathbf{U} = 0, \]
\[ \nabla \cdot \mathbf{b} = 0, \]  

(1)

where \( P \) is the sum of the fluid and magnetic pressures, \( \epsilon_\nu \) is the (non-dimensional) kinematic viscosity and \( \epsilon_\kappa \) is the (non-dimensional) thermal diffusivity. Here \( (e_1, e_2, e_3) \) denote the Cartesian unit vectors.

Following [14], we have assumed in (1) that the magnetic field in the core is of the form
\[ \mathbf{B}(\mathbf{x}, t) = \mathbf{B}_0 + \mathbf{b}(\mathbf{x}, t), \]

where \( \mathbf{B}_0 \) results from dynamo action and can be considered as locally uniform and steady, and a perturbation field \( \mathbf{b}(\mathbf{x}, t) \) induced by the flow \( \mathbf{U}(\mathbf{x}, t) \) across \( \mathbf{B}_0 \). Our choice of \( \mathbf{B}_0 \equiv \beta e_2 \) as the underlying magnetic field is consistent with the models where the magnetic field is believed to be predominantly toroidal due to the strong influence of differential rotation [15].

The dimensionless parameters in (1) are the followings:
\[ N^2 = \frac{2\omega \mu_0 \eta \rho / \beta^2}{\mu_0}, \quad \text{inverse of the Elsasser number,} \]
\[ R_o = \frac{V}{2L\omega}, \quad \text{Rossby number,} \]
\[ R_m = \frac{VL / \eta}{\nu}, \quad \text{magnetic Reynolds number,} \]
\[ \epsilon_\nu = \frac{\nu \eta \mu_0 \rho / \beta^2 L^2}{\nu}, \quad \text{inverse square of the Hartman number,} \]
\[ \epsilon_\kappa = \frac{\kappa / LV}{\kappa}, \quad \text{inverse of the Peclet number.} \]

Here \( \nu \) is the kinematic viscosity, \( \eta \) is the magnetic diffusivity of the fluid, \( \kappa \) is the molecular diffusivity of the compositional variation that creates an ambient density \( \rho \) and \( \mu_0 = 4\pi \times 10^{-7} \text{NA}^{-2} \). We adopt as velocity scale \( V = \Theta_0 g / 2 \omega \) where \( \Theta_0 \) is the typical amplitude of \( \Theta \), and that the length-scale of these variations is \( L \).

The orders of magnitude of the nondimensional parameters are motivated by the physical postulates of the Moffatt and Loper model. For the regions in the Earth’s fluid core modeled in (1), it is argued in [15] that the parameters have the following orders of magnitude:
\[ N^2 \approx 1, \quad R_o \approx 10^{-3}, \quad R_m \approx 1, \quad \epsilon_\nu \approx 10^{-8}, \quad \epsilon_\kappa \approx 10^{-8}. \]

The values of \( \nu \) and \( \kappa \) are speculative, but likely to be extremely small. For a detailed discussion of plausible ranges of the physical parameters that are appropriate for the geodynamo, we refer the reader to [11].

According to Moffat and Loper, the magnetic Reynolds number is relatively small. Then, their model neglects the terms multiplied by \( R_o \) and \( R_m \) in comparison with the remaining terms. However, we will for the moment retain the viscous and diffusive terms since it involve the highest derivatives.

For the reasons given above, we now drop in (1) the terms involving the Rossby number \( R_o \) and the magnetic Reynolds number \( R_m \). Then, we obtain the following reduced system:
\[ N^2 [e_3 \times \mathbf{U}] = -\nabla P + (e_2 \cdot \nabla) \mathbf{b} + N^2 \Theta e_3 + \epsilon_\nu \Delta \mathbf{U}, \]
\[ 0 = (e_2 \cdot \nabla) \mathbf{U} + \Delta \mathbf{b}, \]
\[ \partial_t \Theta + \mathbf{U} \cdot \nabla \Theta = \epsilon_\kappa \Delta \Theta, \]
\[ \nabla \cdot \mathbf{U} = 0, \]
\[ \nabla \cdot \mathbf{b} = 0. \]  

(2)

Essentially this means that the evolution equations for the coupled velocity \( \mathbf{U} \) and magnetic field \( \mathbf{b} \) take a simplified “quasi-static” form. This system encodes the vestiges of the physics in the problem, namely the Coriolis force, the Lorentz force and gravity.

The behavior of the model is dramatically different when the parameters \( \epsilon_\nu \) and \( \epsilon_\kappa \) are present or absent. Since both parameters multiply a Laplacian term, their presence is smoothing. In the present paper we focus our attention in the inviscid case \( (\epsilon_\nu = 0) \). The mathematical properties of the model under the presence of viscosity have been addressed in a recent sequence of different articles [5–7].
1.2. The MG Equation

A linear relationship can be established between the divergence-free vector fields \( U \) and \( b \) and the scalar \( \Theta \), wherein \( \Theta \) will now be regarded as known, thanks to the reduced system:

\[
N^2 [e_3 \times U] = -\nabla P + (e_2 \cdot \nabla) b + N^2 e_3, \\
0 = (e_2 \cdot \nabla) U + \Delta b,
\]

along with the incompressibility condition

\[
\nabla \cdot U = 0, \quad \nabla \cdot b = 0.
\]

We note that the ratio of the Coriolis to Lorentz forces in their model is of order 1, so for notational simplicity we have set this parameter, denoted by \( N^2 \) equal to 1. Following [8, p. 297], manipulations of the linear system (3) gives, in component form

\[
\begin{aligned}
U_1 &= -D^{-1} (\partial_2 P + \Gamma \partial_1 P), \\
U_2 &= D^{-1} (\partial_1 P - \Gamma \partial_2 P), \\
\partial_3 U_3 &= D^{-1} \Gamma \Delta_H P, \\
\partial_3 \Theta &= (\Gamma^2 \Delta_H D^{-1} + \partial_3^2) P,
\end{aligned}
\]

where the operators \( \Gamma, D \) and \( \Delta_H \) are defined as

\[
\Gamma := -(-\Delta)^{-1} \partial_2^2, \quad D := 1 + \Gamma^2, \quad \Delta_H := \partial_1^2 + \partial_2^2.
\]

Although the physically relevant boundary for a model of the Earth’s fluid core is a spherical annulus, for mathematical tractability we considered the system on the domain \( x \in \mathbb{T}^2 \times \mathbb{R} \). This can be seen as a first step before considering the case \( \mathbb{T}^2 \times [0,1) \) with appropriate boundary conditions in the vertical variable.

In order to uniquely determine \( U_3 \) and \( \Theta \) form (4), we restrict the system to the function spaces of zero vertical mean, i.e. \( \int_\mathbb{R} U_3 \, dx = \int_\mathbb{R} \Theta \, dx = 0 \). In fact, without such a restriction the system is not well defined. Integrating the last equation of (4) and applying the zero mean assumption, we obtain the relation

\[
\Theta = A[P],
\]

where the operator \( A \) is formally defined as \( A := \partial_3^{-1} (\Gamma^2 \Delta_H D^{-1} + \partial_3^2) \) in the physical space. Now, we use (4) to represent \( U_1, U_2 \) and \( U_3 \) in terms of \( \Theta \) through the Fourier multiplier operator \( M \equiv (M_1, M_2, M_3) \):

\[
\begin{aligned}
U_1 &= -D^{-1} (\partial_2 + \Gamma \partial_1) (A^{-1}[\Theta]) =: M_1[\Theta], \\
U_2 &= D^{-1} (\partial_1 - \Gamma \partial_2) (A^{-1}[\Theta]) =: M_2[\Theta], \\
U_3 &= D^{-1} \Gamma \Delta_H (D^{-1} \Gamma \Delta_H + \partial_3^2)^{-1}[\Theta] =: M_3[\Theta].
\end{aligned}
\]

Finally, the magnetic vector field \( b \) is computed from the scalar \( \Theta \) thanks to (3) via the operator

\[
b_j = (-\Delta)^{-1} \partial_2 M_j[\Theta], \quad \text{for } j \in \{1, 2, 3\}.
\]

Consequently, the sole remaining nonlinearity in the system comes from the coupling of (3) and the evolution equation for the scalar bouyancy \( \Theta \). The active scalar equation for \( \Theta \) that contains the non-linear process in Moffatt’s model is precisely:

\[
(x, t) \in (\mathbb{T}^2 \times \mathbb{R}, \mathbb{R}^+) \quad \begin{cases}
\partial_t \Theta + U \cdot \nabla \Theta = \epsilon_\kappa \Delta \Theta, \\
\nabla \cdot U = 0,
\end{cases}
\]

where the divergence-free velocity \( U \) is explicitly obtained from the bouyancy as \( U = M[\Theta] \) where \( M \) is the non-local differential operator of order 1 defined in (5). Note that, without loss of generality we may assume that \( \int_{\mathbb{T}^2 \times \mathbb{R}} \Theta(x, t) \, dx = 0 \) for all \( t \geq 0 \), since the mean of \( \Theta \) is conserved by the flow.

In the following, we refer to the evolution Eq. (6) with singular drift velocity \( U \) given by (5) as the magnetico-geostrophic equation (MG). In addition, we will distinguish between diffusive (\( \epsilon_\kappa > 0 \)) and non-diffusive case (\( \epsilon_\kappa = 0 \)). In the Earth’s fluid core the value of the diffusivity \( \epsilon_\kappa \) is very small. Hence it is relevant to address both the diffusive evolution, and the non-diffusive version where \( \epsilon_\kappa = 0 \).
1.3. Diffusive vs. Non-diffusive MG Equation

In order to study this dichotomy, we recall the following: In the theory of differential equations, it is classical to call a Cauchy problem well-posed, in the sense of Hadamard, if given any initial data in a functional space $X$, the problem has a unique solution in $L^\infty(0,T;X)$, with $T$ depending only on the $X$-norm of the initial data, and moreover the solution map $Y \mapsto L^\infty(0,T;X)$ satisfies strong continuity properties, e.g. it is uniformly continuous, Lipschitz, or even $C^\infty$ smooth, for a sufficiently nice space $Y \subset X$. If one of these properties fail, the Cauchy problem is called ill-posed.

Considering this, both systems have contrasting properties:

- Diffusive MG equation: For $\epsilon_\kappa > 0$ the equation is globally well-posed and the solutions are $C^\infty$ smooth for positive times, as it is proved in the papers [8,10].
- Non-diffusive MG equation: For $\epsilon_\kappa = 0$, in [9] the authors prove that the equation is ill-posed in the sense of Hadamard in Sobolev spaces, but locally well-posed in spaces of analytic functions.

Hence, without the Laplacian to control the unbounded operator $M$ the situation is dramatically different from the diffusive case $\epsilon_\kappa > 0$. From the above, the problem of the fractionally diffusive MG equation arise naturally. This is, one can replace the Laplacian by nonlocal operators, such as $-(-\Delta)^\gamma$ for $\gamma \in (0,1)$. This situation, which is non-physical but mathematically interesting, it was addressed in [4].

In the subcritical range $\gamma \in (1/2,1)$ the equation is locally well-posed, while it is Hadamard Lipschitz ill-posed for $\gamma \in (0,1/2)$. At the critical value $\gamma = 1/2$ the problem is globally well-posed for suitably small initial data, but is ill-posed for sufficiently large initial data.

A further feature of interest is that the anisotropy of the symbol $M$ can be explored (see [4]) to obtain an improvement in the regularity of the solutions when the initial data is supported on a plane in the Fourier space. For such well-prepared initial data the local existence and uniqueness of solutions can be obtained for all values $\gamma \in (0,1)$, and the global existence holds for all initial data when $\gamma \in (1/2,1)$.

We emphasize that the mechanism producing ill-posedness is not merely the order one derivative loss in the map $\Theta \mapsto U$. Rather, it is the combination of the derivative loss with the anisotropy of the symbol $M$ and the fact that this symbol is even. We note that the even nature of the symbol of $M$ plays a central role in the proof of non-uniqueness for $L^\infty$-weak solutions to the non-diffusive MG equation proved in [17], via methods from convex integration.

1.4. The Perturbated Non-diffusive MG Equation

The aim of the present paper is to show that the Cauchy problem for the non-diffusive MG equation is well-posed with respect to some periodic perturbation around a specific steady profile, in the topology of a certain Sobolev space.

The kinds of exact solutions we are interested are the simplest possible steady state, namely $U = 0$ and $\Theta = \Omega(x_3)$ for some function $\Omega$ with $\int_\mathbb{R} \Omega(x_3) \, dx_3 = 0$. The basic problem is to consider $\Theta$ a given equilibrium state and to study the dynamics of solutions which are close to it in a suitable sense. Now, we write the scalar and the velocity as

$$\Theta(x,t) = \Omega(x_3) + \theta(x,t),$$
$$U(x,t) = u(x,t),$$

and the pressure term is written in a more convenient way as

$$P(x,t) = \Omega(0) + \int_0^{x_3} \Omega(s) \, ds + p(x,t).$$
Then, putting this ansatz in (4) we obtain
\[
\begin{align*}
\begin{cases}
  u_1 &= -D^{-1}(\partial_3 p + \Gamma \partial_1 p), \\
  u_2 &= D^{-1}(\partial_1 p - \Gamma \partial_2 p), \\
  \partial_3 u_3 &= D^{-1}\Gamma \partial_3 p, \\
  \partial_3 \theta &= (\Gamma^2 \Delta_H D^{-1} + \partial_3^2) p.
\end{cases}
\end{align*}
\]  
(7)

Note that if we impose that \( \theta(x,t) \) and \( p(x,t) \) are periodic functions in the three variables \( x = (x_1, x_2, x_3) \), then the operator \( A \) is invertible on the space of functions with zero \( x_3 \)-mean and has an expression as a Fourier multiplier with symbol given by
\[
\hat{A}(k) = \frac{1}{ik_3} \frac{k_3^2|k| + k_3^4}{|k|^2 + k_3^2},
\]
where the Fourier variable \( k \in \mathbb{Z}_3^2 := \mathbb{Z}^3 \setminus \{ k_3 = 0 \} \), by our vertical mean-free assumption. After that, we can use (7) and (3) to represent \( u \) and \( b \) in terms of \( \theta \) via
\[
u_j = M_j[\theta] \quad \text{and} \quad \mathbf{b}_j = (\Delta)^{-1}\partial_3 M_j[\theta] \quad \text{for} \quad j \in \{1, 2, 3\}.
\]  
(8)

Note that the operators \( \{ M_j \}_{j=1}^3 \) are Fourier multipliers with symbols given explicitly for \( k \in \mathbb{Z}_3^2 \) by
\[
\hat{M}_1(k) := \frac{k_2 k_3 |k|^2 - k_1 k_3^2 k_3}{k_3^2 |k|^2 + k_3^4}, \quad \hat{M}_2(k) := \frac{-k_1 k_3 |k|^2 - k_2^2 k_3}{k_3^2 |k|^2 + k_3^4}, \quad \hat{M}_3(k) := \frac{k_2^2 (k_1^2 + k_3^2)}{k_3^2 |k|^2 + k_3^4}.
\]  
(9)

On \( \{ k_3 = 0 \} \) we let \( \hat{M}_j(k) = 0 \), since for consistency of the model we have that \( \theta \) and \( u_3 \) have zero \( x_3 \)-mean. It can be directly checked that \( k_j \cdot \hat{M}_j(k) = 0 \) and hence the velocity field \( u \) given by (8) is divergence-free.

Finally, introducing the ansatz in (6) for the non-diffusive case \( (\epsilon_n = 0) \), we arrive to the following system:
\[
\begin{align*}
\begin{cases}
  \partial_t \theta(x,t) + u(x,t) \cdot \nabla \theta(x,t) &= -u_3(x,t) \Omega'(x_3), \\
  u(x,t) &= \mathbf{M}[\theta](x,t), \\
  \theta(x,0) &= \theta_0(x),
\end{cases}
\end{align*}
\]  
(10)

where our initial data \( \theta_0 \) has zero vertical mean. Notice that for the particular case \( \Omega \equiv 0 \), the above system (10) is just the one widely studied in [3,4,8–10].

**Remark.** Here \( x = (x_1, x_2, x_3) \in \mathbb{T}^2 \times \mathbb{R} \), however \( \theta(x,t) \) and \( u(x,t) \) are periodic in the three variables.

1.5. **Statement of Main Results**

In this paper, we study the perturbative regime near the special steady state \( \Omega(x_3) := x_3 \). The main achievement of the paper is a local existence result for periodic perturbations localized in a suitable section of the frequency space together with a global existence and asymptotic stability result under an additional size condition over the \( H^s(\mathbb{T}^2) \) of the perturbation.

Following the same idea used initially in [4], we take advantage of the anisotropy of the symbol \( \mathbf{M} \) and observe an interesting phenomenon: when the initial perturbation is localized in the frequency space, it is possible to prove a well-posedness result for the ensuing solution.

To sum up, we consider solutions in \( x \in \mathbb{T}^2 \times \mathbb{R} \) and \( t \geq 0 \) with structure \( \Theta(x,t) := x_3 + \theta(x,t) \), for initial periodic perturbations \( \theta_0(x) \in H^s(\mathbb{T}^2) \) with zero vertical mean and frequency support in \( X \subset \mathbb{Z}^3 \). Then, we prove:

- **Local well-posedness** If \( s > \frac{5}{2} \).
- **Global well-posedness** If \( s > \frac{9}{2} \) and \( \| \theta_0 \|_{H^{s/2+}} \) is small enough.
- **GWP & asymptotic stability** If \( s > \frac{7}{2} \) and \( \| \theta_0 \|_{H^{s/2+}} \) is small enough.

A precise statement of our result is presented as Theorem 4.1, where we also illustrate its proof through a bootstrap argument. Despite the apparent simplicity, understanding the stability of this flow is non-trivial.
1.5.1. The Ideas Behind the Proof. In order to prove this, first we fix our attention in the study of the stability of the problem, when linearized it around the steady state. The main mechanism of decay can be seen directly from the linearized equation:

\[ \partial_t \theta(x, t) = -M_3[\theta](x, t). \]

As \( \hat{M}_3(k) \) is a positive operator for any \( k \in \mathbb{Z}^3 \), there exists a unique positive self-adjoint square root operator. Consequently, the previous linearized equation clearly shows the decay over time of \( \theta(x, t) \), except for the zero mode in \( x_3 \). But we do not have that problem because for self-consistency of the model we restrict to functions that have zero vertical mean.

Hence, the main achievement of the paper is thus to control the nonlinearity, so that it would not destroy the decay provided by the linearized equation. Note that, over the curved frequency regions where \( k_3 = O(1) \) and \( k_2 = O(|k_1|^r) \) with \( 0 < r \leq 1/2 \), we have that \( u(x, t) \approx \Lambda \theta(x, t) \) and \( M_3[\theta](x, t) \approx \Lambda^{2r} \theta(x, t) \) with \( 0 < r \leq 1/2 \) and we can not control and close the estimates at the level of Sobolev spaces.

To overcome this issue, we explore the following observation: if the frequency support of \( \theta(x, t) \) lies on a suitable section of the Fourier space, then the operator \( M \) behaves like an order zero operator and hence the corresponding velocity \( u(x, t) \) is as smooth as \( \theta(x, t) \). This enables us to obtain a well-posedness results over the generic setting when no conditions on the Fourier spectrum of the initial perturbation \( \theta_0 \) are imposed. To be more precise, we consider an appropriate subset \( X \subset \mathbb{Z}^3 \) which we will define later, where we can obtain a local well-posedness result for perturbations \( \theta_0(x) \) such that \( \text{supp}(\hat{\theta}_0(k)) \subset X \).

Under the hypothesis over the initial perturbation, at least morally speaking, our perturbed system behaves like an active scalar of order zero with a damping term:

\[ \partial_t \theta(x, t) + u(x, t) \cdot \nabla \theta(x, t) = -\theta(x, t), \]

where \( u(x, t) = M[\theta](x, t) \). Notice that as \( \lim_{\alpha \to 0} \Lambda^\alpha \theta(x, t) = -\theta(x, t) \), the type of results obtained for the supercritical diffusive MG equation in [4] are expected to have also in our setting.

1.6. Notation & Organization

To avoid clutter in computations, function arguments (time and space) will be omitted whenever they are obvious from context. Finally, we use the notation \( f \lesssim g \) when there exists a constant \( C > 0 \) independent of the parameters of interest such that \( f \leq Cg \).

In Sect. 2 we collect some useful technical lemmas about the behaviour of \( \hat{M} \) on suitable subsets of the frequency domain. In Sect. 3 we embark on the proof of a local existence result for frequency-localized initial data following the ideas of [4]. The core of the article is the proof of the main theorem in Sect. 4. We start by the a priori energy estimates given in Sect. 4.1. This is followed by an explanation of the decay given by the linear semigroup in Sect. 4.2. Finally, in Sect. 4.3 we exploit a bootstrapping argument to prove our theorem.

2. Preliminares

In this section we explore the observation cited above: if the frequency support of \( \theta \) lies on a suitable subset of the frequency space, then the operator \( M \) is mild when it acts on \( \theta \), i.e. it behaves like an order zero operator, and hence the corresponding velocity \( u \) is as smooth as \( \theta \).

This enables us to obtain a well-posedness result over the generic setting when no conditions on the Fourier spectrum of the initial perturbation are imposed. For instance, the local existence and uniqueness of smooth solutions holds for the non-diffusive case, a setting in which we know that for generic initial data the problem is ill-posed in Sobolev spaces.
For the perturbated problem, since we are working on the periodic setting, the frequency space is $\mathbb{Z}^3$. Now, for any fixed number $\mathcal{C} > 0$ we define the cone $K_{\mathcal{C}}$ given by

$$K_{\mathcal{C}} := \{ \mathbf{k} \in \mathbb{Z}^3 \setminus \{(0,0,0)\} : |k_1|, |k_3| \leq \mathcal{C}|k_2| \}.$$  

(11)

The next result states that $M$ behaves like a zero order operator when it acts on functions with frequency support in $K_{\mathcal{C}}$. Note that $K_{\mathcal{C}} \neq \emptyset$ for any $\mathcal{C} > 0$, where $\mathcal{C}$ gives us information about the opening of the cone.

**Corollary 2.1.** Let $\mathcal{C} > 0$. For every smooth periodic function $f : \mathbb{T}^3 \to \mathbb{R}$ with zero vertical mean and frequency support in $K_{\mathcal{C}}$, there exists a universal constant $m^* = m^*(\mathcal{C}) > 0$ such that

$$|\hat{M}[f](\mathbf{k})| \leq m^* |\hat{f}(\mathbf{k})|,$$

for all $\mathbf{k} \in \mathbb{Z}^3$. Moreover, the constant $m^*$ blow-up as $\mathcal{C}$ tends to infinity.

**Proof.** It is clear that the bound has to be proven only for $\mathbf{k} \in \mathbb{Z}^3$, since otherwise we have that $\hat{f}(\mathbf{k}) = 0$ and the statement holds trivially. A simple algebraic computation, using the definition (9), gives us that $|\hat{M}_j(\mathbf{k})| \leq m_j(\mathcal{C})$ for $j \in \{1,2,3\}$ and taking $m^* := \max\{m_1, m_2, m_3\}$ concludes the proof. \hfill $\Box$

Another immediate consequence of definition (11) will be a lower bound for $\hat{M}_3(\mathbf{k})$ in $K_{\mathcal{C}}$ for any $\mathcal{C} > 0$. Then, under the same hypothesis as before we have that the Fourier operator $M_3$ is equivalent to the identity operator in $L^2(\mathbb{T}^3)$. More specifically, there exists a pair of real numbers $0 < m_* \leq m^*$ such that

$$m_* \|f\|_{L^2(\mathbb{T}^3)} \leq \|M_3[f]\|_{L^2(\mathbb{T}^3)} \leq m^* \|f\|_{L^2(\mathbb{T}^3)}.$$

As we will see, this lower bound play a key role in the proof of the local and global existence result. Before proceeding with the proof of the above statement, as $M_3$ is a positive operator, i.e. $\hat{M}_3(\mathbf{k}) > 0$ for any $\mathbf{k} \in \mathbb{Z}^3$, we define the square root and the inverse of $M_3$ via Fourier transform as follows:

$$\sqrt{M_3}(\mathbf{k}) := \sqrt{\frac{k_2^2(k_1^2 + k_3^2)}{k_3^2|\mathbf{k}|^2 + k_2^2}}, \quad M_3^{-1}(\mathbf{k}) := \frac{k_2^2|\mathbf{k}|^2 + k_3^2}{k_2^2(k_1^2 + k_3^2)} \quad \mathbf{k} \in \mathbb{Z}_+^3.$$  

(12)

Note that $\sqrt{M_3}(\mathbf{k})$ and $M_3^{-1}(\mathbf{k})$ are not defined on $k_3 = 0$ since for the self-consistency of the model, we only work with periodic functions with zero vertical mean.

**Corollary 2.2.** Let $\mathcal{C} > 0$. For every smooth periodic function $f : \mathbb{T}^3 \to \mathbb{R}$ with zero vertical mean and frequency support in $K_{\mathcal{C}}$, there exists a universal constant $m_* = m_*(\mathcal{C}) > 0$ such that:

$$m_* |\hat{f}(\mathbf{k})| \leq |\hat{M}_3[f](\mathbf{k})|$$

for all $\mathbf{k} \in \mathbb{Z}^3$. Moreover, the constant $m_*$ goes to zero as $\mathcal{C}$ tends to infinity.

**Proof.** It is clear that the bound has to be proven only for $\mathbf{k} \in \mathbb{Z}_+^3$, since otherwise we have that $\hat{f}(\mathbf{k}) = 0$ and the statement holds trivially. As $M_3$ is a Fourier multiplier operator, we obtain

$$|\hat{f}(\mathbf{k})| \leq \|M_3^{-1}\|_{L^\infty(K_{\mathcal{C}})} |\hat{M}_3[f](\mathbf{k})|,$$

where the above expression (12) gives us that $|\hat{f}(\mathbf{k})| \leq \frac{1}{m_*} |\hat{M}_3[f](\mathbf{k})|$ for a suitable constant $m_*$. \hfill $\Box$

A key point in the result of well-posedness is the fact that we only work with frequency localized initial perturbations. As we will see later in the proof, to prove that the perturbation does not leave the region of the frequency space where the operator $M$ behaves like a zero order operator, the sets closed under addition will play a crucial role.

A set $X$ is closed under addition $+: X \times X \to X$ if for all $a, b \in X$ we have that $a + b \in X$. In other words, performing the binary operation on any two elements of the set always gives you back something that is also in the set.
Let $\mathcal{C} > 0$, the simplest set closed under addition inside the cone region $K_\mathcal{C}$ will be the frequency straight lines $L(q)$ crossing the origin given by

$$L(q) := \{k \cdot (q_1, q_2, q_3) : k \in \mathbb{Z}\},$$

for any $q \in K_\mathcal{C}$. In the rest of the paper, for any fixed $\mathcal{C} > 0$ we will assume that $X_\mathcal{C} \in \{L(q) : q \in K_\mathcal{C}\}$. This is, in the following for $X_\mathcal{C}$ we will understand one of the previously defined frequency straight lines.

**Lemma 2.3.** Let $\mathcal{C} > 0$ and $X_\mathcal{C}$. For every pair of smooth periodic function $f, g : \mathbb{T}^3 \to \mathbb{R}$ with frequency support in $X_\mathcal{C}$ we have that:

- $\text{supp}(\hat{f} \hat{g}) \subset X_\mathcal{C}$.
- $\text{supp}(\hat{f} \pm \hat{g}) \subset X_\mathcal{C}$.
- $\text{supp}\left(\hat{M}_j[f]\right) \subset X_\mathcal{C}$ for all $j \in \{1, 2, 3\}$.

**Proof.** The proof is an immediate consequence of the properties of the Fourier transform:

- Clearly $\text{supp}(\hat{f} \hat{g}) = \text{supp}(\hat{f} \ast \hat{g}) \subset \text{supp}(\hat{f}) + \text{supp}(\hat{g}) \subset X_\mathcal{C}$, since $X_\mathcal{C}$ is closed under addition.
- Note that $\text{supp}(\hat{f} \pm \hat{g}) = \text{supp}(\hat{f} \pm \hat{g}) \subset \text{supp}(\hat{f}) \cup \text{supp}(\hat{g}) \subset X_\mathcal{C}$.
- As $M$ is a Fourier multiplier, we have: $\text{supp}(\hat{M}_j[f]) = \text{supp}(\hat{M}_j \hat{f}) \subset \text{supp}(\hat{M}_j) \cap \text{supp}(\hat{f}) \subset X_\mathcal{C}$. \hfill $\square$

Finally, this section contains a few auxiliary results used in the paper. In particular, we recall the, by now classical, product and commutator estimates, as well as the Sobolev embedding inequalities. Proofs of these results can be found for instance in [12,18] and [19].

**Lemma 2.4.** (Product estimate) If $s > 0$, then for all $f, g \in H^s \cap L^\infty$ we have the estimate

$$\|\Lambda^s(fg)\|_{L^2} \lesssim (\|f\|_{L^\infty} \|\Lambda^s g\|_{L^2} + \|\Lambda^s f\|_{L^2} \|g\|_{L^\infty}).$$

(13)

In the case of a commutator we have the following estimate.

**Lemma 2.5.** (Commutator estimate) Suppose that $s > 0$. Then for all $f, g \in \mathcal{S}$ we have the estimate

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^2} \lesssim (\|\nabla f\|_{L^\infty} \|\Lambda^{s-1} g\|_{L^2} + \|\Lambda^s f\|_{L^p} \|g\|_{L^{p'}})$$

(14)

where $\frac{1}{2} = \frac{1}{p} + \frac{1}{p'}$ and $p \in (1, \infty)$.

Moreover, the following Sobolev embeddings holds:

- $W^{s,p}(\mathbb{T}^d) \subset L^q(\mathbb{T}^d)$ continuously if $s < d/p$ and $p \leq q \leq dp/(d - sp)$.
- $W^{s,p}(\mathbb{T}^d) \subset C^k(\overline{\mathbb{T}^d})$ continuously if $s > k + d/p$.

### 3. Local Existence for Frequency-Localized Initial Data

The main result of this section is:

**Theorem 3.1.** Let $\mathcal{C} > 0$ and $X_\mathcal{C}$. Assume that $\theta_0 \in H^s(\mathbb{T}^3)$ with $s > 5/2$ has zero vertical mean and satisfies that $\text{supp}(\hat{\theta}_0) \subset X_\mathcal{C}$. Then, there exists a time $T > 0$ and a unique smooth solution

$$\theta \in L^\infty(0, T; H^s(\mathbb{T}^3))$$

of the Cauchy problem (10) such that $\text{supp}(\hat{\theta}(t)) \subset X_\mathcal{C}$ for all $t \in [0, T)$.

Before that, the goal is to prove the existence of smooth solutions to the scalar linear equation:

$$\begin{cases}
\partial_t \theta(x, t) + v(x, t) \cdot \nabla \theta(x, t) = -M_3[\theta](x, t) \\
\theta(x, 0) = \theta_0(x)
\end{cases}$$

(15)

where the initial datum $\theta_0$ and the given divergence-free drift velocity field $v$ satisfies that:
• supp(\(\hat{\theta}_0\)) \(\subset X_\epsilon\).
• supp(\(\tilde{\psi}(t)\)) \(\subset X_\epsilon\) for all \(t \in [0, T]\) for a positive time \(T\).

We shall show the following result:

**Theorem 3.2.** Given \(s > 3/2\), let \(\theta_0 \in H^s(\mathbb{T}^3)\) and a divergence-free vector field \(\mathbf{v} \in L^\infty(0, T; H^s(\mathbb{T}^3))\) satisfying the above conditions. Then, there exists a unique smooth solution of (15) such that:

\[
\theta \in L^\infty(0, T; H^s(\mathbb{T}^3)).
\]  

Moreover, we have that supp(\(\hat{\theta}(t)\)) \(\subset X_\epsilon\) for all \(t \in [0, T]\).

**Proof of Theorem 3.2.** Following the arguments of [4], we regularize (15) with hyper-dissipation as:

\[
\left\{ \begin{array}{ll}
\partial_t \theta^\epsilon_t(x, t) + \mathbf{v}(x, t) \cdot \nabla \theta^\epsilon(x, t) - \epsilon \Delta \theta^\epsilon_t(x, t) = -M_3[\theta^\epsilon](x, t) \\
\theta^\epsilon(x, 0) = \theta_0(x)
\end{array} \right.
\]  

for \(\epsilon \in (0, 1]\) and finally we pass to the limit \(\epsilon \to 0\) in order to obtain a solution of the original system.

On one hand, since \(\mathbf{v}\) is smooth and divergence-free, it follows from the De Giorgi techniques (see [4, 16]) that there exists a unique global smooth solution \(\theta^\epsilon\) of (17) with

\[
\theta^\epsilon \in L^\infty(0, T; H^s(\mathbb{T}^3)) \cap L^2(0, T; H^{s+1}(\mathbb{T}^3)).
\]

On the other hand, we proceed to construct a solution \(\theta^\epsilon\) of (17) which has the desired frequency support property and belongs to the smooth category. Then, by the uniqueness of strong solutions, we pass to the limit \(\epsilon \to 0\) and obtain a solution with desired properties. We consider the following iterative scheme:

\[
\left\{ \begin{array}{ll}
\partial_t \theta_{n+1}^\epsilon(x, t) + \mathbf{v}(x, t) \cdot \nabla \theta_n^\epsilon(x, t) - \epsilon \Delta \theta_{n+1}^\epsilon(x, t) = -M_3[\theta_n^\epsilon](x, t) \\
\theta_{n+1}^\epsilon(x, 0) = \theta_0(x)
\end{array} \right.
\]  

and

\[
\left\{ \begin{array}{ll}
\partial_t \theta_{n+1}^\epsilon(x, t) + \mathbf{v}(x, t) \cdot \nabla \theta_{n+1}^\epsilon(x, t) - \epsilon \Delta \theta_{n+1}^\epsilon(x, t) = -M_3[\theta_{n+1}^\epsilon](x, t) \\
\theta_{n+1}^\epsilon(x, 0) = \theta_0(x)
\end{array} \right.
\]

for all \(n \geq 1\). We note that the solutions of (18) and (19) respectively, may be written explicitly using the Duhamel’s formula:

\[
\theta_{n+1}^\epsilon(x, t) = e^{-(\epsilon(\Delta + M_3)t)} \theta_0(x),
\]

\[
\theta_{n+1}^\epsilon(x, t) = e^{-(\epsilon(\Delta + M_3)t)} \theta_0(x) + \int_0^t e^{-(\epsilon(\Delta + M_3)(t-\tau)} \mathbf{v}(x, \tau) \cdot \nabla \theta_n^\epsilon(x, \tau) d\tau.
\]

Since \(e^{-(\epsilon(\Delta + M_3)t)}\) is given explicitly by the Fourier multiplier with non-zero symbol \(e^{-(\epsilon|k| + \tilde{M}_3|k|)t}\), this operator does not alter the frequency support of the function on which it acts. Therefore, it follows directly from our assumption on the frequency support of \(\theta_0\) that supp(\(\hat{\theta}^\epsilon(t)\)) \(\subset X_\epsilon\) for all \(t \in [0, T]\).

Now, we proceed inductively and note that if supp(\(\hat{\theta}_n^\epsilon(t)\)) \(\subset X_\epsilon\) for all \(t \in [0, T]\). Then, by our assumption on the frequency support of \(\mathbf{v}\) and Lemma 2.3 we also have supp(\(\hat{\mathbf{v}} \cdot \nabla \theta_n^\epsilon(t)\)) \(\subset X_\epsilon\) for all \(t \in [0, T]\). Hence, we obtain that supp(\(\hat{\theta}_{n+1}^\epsilon(t)\)) \(\subset X_\epsilon\) for all \(t \in [0, T]\) concluding the proof of the induction step. This proves that the frequency support of all the iterates \(\theta_n^\epsilon(t)\) lies on \(X_\epsilon\) for all \(t \in [0, T]\).

Thus, it is left to prove that the sequence \(\{\theta_n^\epsilon\}_{n \geq 1}\) converges to a function \(\theta^\epsilon\) which lies in the smoothness class (16). Note that there is no cancellation of the highest order term in the nonlinearity. However, since (at least for now) \(\epsilon \in (0, 1]\) is fixed, we may use the full smoothing power of the Laplacian.

To prove it, for all \(n \geq 1\) we define:

\[
\mathcal{R}_n(t) := \sup_{\tau \in [0,t]} ||\Lambda^s \theta_n^\epsilon||_{L^2_x(\tau)}^2 + \int_0^t \left[ ||\sqrt{M_3} [\Lambda^s \theta_n^\epsilon]^2 ||_{L^2_x(\tau)}^2 \right] d\tau + \epsilon \int_0^t ||\Lambda^{s+1} \theta_n^\epsilon||_{L^2_x(\tau)}^2 d\tau.
\]

Moreover, as the frequency support of all the iterates \(\theta_n^\epsilon\) lies on \(X_\epsilon \subset K_\epsilon\), using Corollary 2.2 we have:

\[
||\sqrt{M_3} [\Lambda^s \theta_n^\epsilon]^2 ||_{L^2_x(\tau)}^2 \approx ||\Lambda^s \theta_n^\epsilon||_{L^2_x(\tau)}^2.
\]
In the first step, note that from (18) it follows that for any \( t \in (0, T] \) we obtain that \( \mathcal{R}_1(t) \leq 2 \| \Lambda^s \theta_0 \|^2_{L^2} \). We proceed inductively and assume that there exists a time \( T^* \in (0, T] \) such that \( \mathcal{R}_n(T^*) \leq 2 \| \Lambda^s \theta_0 \|^2_{L^2} \). Here, we show that if \( T^* \) is chosen appropriately, in terms of \( \theta_0, v \) and \( \epsilon \), we have \( \mathcal{R}_{n+1}(T^*) \leq 2 \| \Lambda^s \theta_0 \|^2_{L^2} \) too.

From (19), the divergence-free velocity field \( \nabla \cdot v = 0 \), integration by parts and the fact that \( s > 3/2 \) which makes \( H^s \) an algebra, we obtain:

\[
\frac{1}{2} \partial_t \| \Lambda^s \theta_{n+1} \|^2_{L^2}(t) + \| \sqrt{M_3} [\Lambda^s \theta_{n-1}] \|_{L^2}^2(t) + \epsilon \| \Lambda^{s+1} \theta_{n+1} \|^2_{L^2}(t) \leq \frac{\| \Lambda^s(v \theta_n) \|^2_{L^2}(t)}{2} + \epsilon \frac{\| \Lambda^{s+1} \theta_n \|^2_{L^2}(t)}{2}
\]

and in consequence, as \( \mathcal{R}_n(T^*) \leq 2 \| \Lambda^s \theta_0 \|^2_{L^2} \) we have proved that:

\[
\mathcal{R}_{n+1}(T^*) \leq \| \Lambda^s \theta_0 \|^2_{L^2} + \frac{C_s}{\epsilon} \int_0^{T^*} \| \Lambda^s \nabla \|^2_{L^2}(\tau) \| \Lambda^s \theta_n \|^2_{L^2}(\tau) \, d\tau \leq \| \Lambda^s \theta_0 \|^2_{L^2} + \frac{2C_s T^*}{\epsilon} \| \nabla \|^2_{L^\infty(0, T; H^s)} \| \Lambda^s \theta_0 \|^2_{L^2}.
\]

Hence, if we let

\[
T^* \leq \frac{\epsilon}{2C_s} \| \nabla \|^2_{L^\infty(0, T; H^s)}
\]

we have that \( \mathcal{R}_{n+1}(T^*) \leq 2 \| \Lambda^s \theta_0 \|^2_{L^2} \). Since \( T^* \) is independent of \( n \), it is clear that the inductive argument may be carried through, and hence \( \mathcal{R}_n(T^*) \leq 2 \| \Lambda^s \theta_0 \|^2_{L^2} \) for all \( n \geq 1 \).

The second step is the passage to the limit in \( n \). Taking the difference of two iterates:

\[
\begin{align*}
&\left\{ \begin{array}{l}
\partial_t (\theta_n^\epsilon - \theta_n^\epsilon)(x, t) + v(x, t) \cdot \nabla (\theta_n^\epsilon - \theta_n^\epsilon)(x, t) - \epsilon \Delta (\theta_n^\epsilon - \theta_n^\epsilon)(x, t) = -M_3(\theta_n^\epsilon - \theta_n^\epsilon)(x, t) = 0
\end{array} \right.
\end{align*}
\]

for all \( n \geq 2 \). Similarly to the above, it follows from (21) that

\[
\tilde{\mathcal{R}}_n(t) := \sup_{\tau \in [0, t]} \| \Lambda^s (\theta_n^\epsilon - \theta_n^\epsilon) \|^2_{L^2}(\tau)
\]

\[
\quad + \int_0^{t} \| \sqrt{M_3} [\Lambda^s (\theta_{n+1}^\epsilon - \theta_n^\epsilon)] \|_{L^2}(\tau) \, d\tau + \epsilon \int_0^{t} \| \Lambda^{s+1} (\theta_{n+1}^\epsilon - \theta_n^\epsilon) \|_{L^2}(\tau) \, d\tau
\]

\[
\quad \leq \frac{C_s}{\epsilon} \int_0^{t} \| \Lambda^s v \|^2_{L^2}(\tau) \| \Lambda^s (\theta_n^\epsilon - \theta_{n-1}^\epsilon) \|^2_{L^2}(\tau) \, d\tau
\]

\[
\quad \leq \frac{C_s}{\epsilon} \| v \|^2_{L^\infty(0, T; H^s)} \tilde{\mathcal{R}}_{n-1}(t) \quad \text{for all} \quad n \geq 2.
\]

In particular, due to our choice of \( T^* \in (0, T] \) on (20) we have that \( \tilde{\mathcal{R}}_n(T^*) \leq \frac{1}{2} \tilde{\mathcal{R}}_{n-1}(T^*) \), which implies that the sequence \( \{\theta_n^\epsilon\}_{n \geq 1} \) is not only bounded, we actually have a contraction in

\[
L^\infty(0, T^*; H^s(T^3)) \cap \epsilon L^2(0, T^*; H^{s+1}(T^3)).
\]

(22)

Hence there exists a limiting function \( \theta^\epsilon \to \theta^\epsilon \) in the category (22). In addition, since for every \( n \geq 1 \) we have \( \text{supp}(\hat{\theta}_n^\epsilon(t)) \subseteq X_* \) and the set \( X_* \) is closed, we automatically obtain that \( \text{supp}(\hat{\theta}(t)) \subseteq X_* \).

To show that \( \theta^\epsilon \) may be continued in (22) up to time \( T \), we note that \( \| \Lambda^s \theta^\epsilon \|^2_{L^2}(T^*) \leq 2 \| \Lambda^s \theta_0 \|^2_{L^2} \) thanks to the fact that \( \mathcal{R}_n(T^*) \leq 2 \| \Lambda^s \theta_0 \|^2_{L^2} \) for all \( n \geq 1 \). Hence, repeating the above argument with initial condition \( \theta^\epsilon(T^*) \), we obtain a solution \( \theta^\epsilon \in L^\infty(0, 2 T^*; H^s(T^3)) \cap \epsilon L^2(0, 2 T^*; H^{s+1}(T^3)) \) with the bound \( \| \Lambda^s \theta^\epsilon \|^2_{L^2}(2 T^*) \leq 2 \| \Lambda^s \theta^\epsilon \|^2_{L^2}(T^*) \leq 4 \| \Lambda^s \theta_0 \|^2_{L^2} \).

The above argument may be extended iteratively, thereby concluding the construction of the solution \( \theta^\epsilon \) in the category (16).

In order to close the proof we need to pass to the limit as \( \epsilon \to 0 \). By construction we have that \( \theta^\epsilon \) is uniformly bounded, with respect to \( \epsilon \) in \( L^\infty(0, T; H^s(T^3)) \), and from (17) we obtain that \( \partial_t \theta^\epsilon \) is uniformly bounded, with respect to \( \epsilon \) in \( L^\infty(0, T; H^{s-2}(T^3)) \cap L^2(0, T; H^{s-2}(T^3)) \). In particular, from the uniform bounds (with respect to \( \epsilon \)) of \( \theta^\epsilon \) and \( \partial_t \theta^\epsilon \) in the corresponding norms, one can use the Banach-Alaoglu
theorem and the Aubin-Lions’s compactness lemma (see, e.g. [13, 20]) to justify that one can extract a subsequence of \( \theta^\epsilon \) and \( \partial_t \theta^\epsilon \) (using the same index for simplicity) as \( \epsilon \to 0 \) and elements \( \theta \) and \( \partial_t \theta \), such that:

- \( \theta^\epsilon \to \theta \) strongly in \( C(0, T; H^{s-1}(\mathbb{T}^3)) \).
- \( \partial_t \theta^\epsilon \to \partial_t \theta \) weakly in \( L^2(0, T; H^{s-2}(\mathbb{T}^3)) \).
- \( \partial_t \theta^\epsilon \rightharpoonup \partial_t \theta \) weakly* in \( L^\infty(0, T; H^{s-2}(\mathbb{T}^3)) \).

Now, from (17) we have that \( \partial_t \theta^\epsilon \to -M_3[\theta] + \nabla \theta \) in \( C(0, T; H^{s-2}(\mathbb{T}^3)) \). Moreover, as \( \theta^\epsilon \to \theta \) in \( C(0, T; H^{s-1}(\mathbb{T}^3)) \), the distribution limit of \( \partial_t \theta^\epsilon \) must be \( \partial_t \theta \) for the closed graph theorem [1]. In consequence, since the evolution is linear and \( s \) is large enough, it follows that this limiting function is the unique smooth solution of (15) which lies in \( L^\infty(0, T; H^s(\mathbb{T}^3)) \). Lastly, since for every \( \epsilon \in (0, 1) \) we have \( \text{supp}(\hat{\theta}^\epsilon) \subset X_\epsilon \), and since \( X_\epsilon \) is closed, we obtain that the limiting function also has the desired support property, i.e. \( \text{supp}(\hat{\theta}) \subset X_\epsilon \), which concludes the proof of the theorem.

The main difficulty in the proof of the previous theorem is the construction of an iteration scheme which is both suitable for energy estimates and preserves the feature that in each iteration step the frequency support of the approximation lies on \( X_\epsilon \). Now, we are ready to prove the main result of this section.

**Proof of Theorem 3.1.** In order to construct the local in time solution \( \theta \) with frequency support in \( X_\epsilon \), we consider the sequence of approximations \( \{\theta_n\}_{n \geq 1} \) given by the solutions of

\[
\begin{aligned}
\partial_t \theta_1(x, t) &= -M_3[\theta_1](x, t) \\
\theta_1(x, 0) &= \theta_0(x)
\end{aligned}
\]

and

\[
\begin{aligned}
\partial_t \theta_n(x, t) + u_{n-1}(x, t) \cdot \nabla \theta_n(x, t) &= -M_3[\theta_n](x, t) \\
u_{n-1}(x, t) &= M[\theta_{n-1}](x, t) \\
\theta_n(x, 0) &= \theta_0(x)
\end{aligned}
\]

for all \( n \geq 2 \). One may solve (23) explicitly in the frequency space as

\[
\hat{\theta}_1(k, t) = e^{-\hat{M}_3(k)t} \hat{\theta}_0(k)
\]

for \( k \in \mathbb{Z}^3 \). Hence, it is clear that \( \text{supp}(\hat{\theta}_1(t)) \subset \text{supp}(\hat{\theta}_0) \subset X_\epsilon \) for all \( t \geq 0 \). Moreover, we have that:

\[
||\Lambda^s \theta_1||_{L^2(t)}^2 + 2 \int_0^t ||\sqrt{M_3}[\Lambda^s \theta_1]||_{L^2(\tau)}^2 d\tau = ||\Lambda^s \theta_0||_{L^2}^2 \quad \text{for all} \quad t \geq 0.
\]

In particular, fixed \( T > 0 \), we obtain the bound:

\[
||\Lambda^s \theta_1||_{L^\infty(0, T; L^2)}^2 + ||\sqrt{M_3}[\Lambda^s \theta_1]||_{L^2(0, T; L^2)}^2 \leq 2 ||\Lambda^s \theta_0||_{L^2}^2.
\]

In order to solve (24) we appeal to Theorem 3.2. Indeed, by the inductive assumption we have that \( \theta_{n-1} \in L^\infty(0, T; H^s) \) and also that \( \text{supp}(\hat{\theta}_{n-1}(t)) \subset X_\epsilon \) for all \( t \in [0, T) \). Hence, as \( u_{n-1} \equiv M[\theta_{n-1}] \) by applying Corollary 2.1 we have that \( u_{n-1} \in L^\infty(0, T; H^s) \) and by Lemma 2.3 we have \( \text{supp}(u_{n-1}(t)) \subset X_\epsilon \) for \( t \in [0, T) \). Therefore, all the conditions of Theorem 3.2 are satisfied, by letting \( v = u_{n-1} \), and there exists a unique solution \( \theta_n \in L^\infty(0, T; H^s) \) of (24), such that \( \text{supp}(\hat{\theta}_n(t)) \subset X_\epsilon \) for \( t \in [0, T) \). Moreover, using that \( \theta_0 \) has zero vertical mean on \( \mathbb{T}^3 \) the sequence \( \{\theta_n\}_{n \geq 1} \) satisfies the same by construction.

To prove that the sequence \( \{\theta_n\}_{n \geq 1} \) converges, we first prove that it is bounded. To do it, we assume inductively that the following bound holds for all \( 1 \leq j \leq n - 1 \) and proceed to prove that it holds for \( j = n \).

\[
||\Lambda^s \theta_j||_{L^\infty(0, T; L^2)}^2 + ||\sqrt{M_3}[\Lambda^s \theta_j]||_{L^2(0, T; L^2)}^2 \leq 2 ||\Lambda^s \theta_0||_{L^2}^2.
\]

Applying \( \Lambda^s \) to (24) and taking an \( L^2 \) inner product with \( \Lambda^s \theta_n \) we obtain:

\[
\begin{aligned}
\frac{1}{2} \partial_t ||\Lambda^s \theta_n||_{L^2}^2 + ||\sqrt{M_3}[\Lambda^s \theta_n]||_{L^2}^2 \leq ||\Lambda^s \theta_n||_{L^2} \left( ||\nabla u_{n-1}||_{L^\infty} ||\Lambda^s \theta_n||_{L^2} + ||\Lambda^s u_{n-1}||_{L^2} ||\nabla \theta_n||_{L^\infty} \right).
\end{aligned}
\]
and for $s > 5/2$ we have that:
\[
\frac{1}{2}\partial_t ||\Lambda^s \theta_n||_{L^2}^2(t) + ||\sqrt{M_3} [\Lambda^s \theta_n]||_{L^2}^2(t) \lesssim ||\Lambda^s \theta_n||_{L^2}^2(t) ||\Lambda^s \nu_{n-1}||_{L^2}(t). \tag{26}
\]
Above, we have used the fact that $\nabla \cdot \nu_{n-1} = 0$ in order to write the commutator estimate and the Sobolev embedding $L^\infty(\mathbb{T}^3) \hookrightarrow H^{3/2+}(\mathbb{T}^3)$. Since $\nu_{n-1}$ is obtained from $\theta_{n-1}$ by a bounded Fourier multiplier (cf. Corollary 2.1) there exists a positive constant $m^*$ such that:
\[
||\Lambda^s \nu_{n-1}||_{L^2}(t) \leq m^* ||\Lambda^s \theta_{n-1}||_{L^2}(t). \tag{27}
\]
In consequence, putting together (26) and (27) for $s > 5/2$ we have proved that:
\[
\frac{1}{2}\partial_t ||\Lambda^s \theta_n||_{L^2}^2(t) + ||\sqrt{M_3} [\Lambda^s \theta_n]||_{L^2}^2(t) \leq C_5 m^* ||\Lambda^s \theta_n||_{L^2}^2(t) ||\Lambda^s \nu_{n-1}||_{L^2}(t).
\]
By Corollary (2.2) there exists two positive constants such that $m_* \leq M_3(\kappa) \leq m^*$ for all $k \in \mathbb{Z}_+^3$. Hence, applying Hölder’s inequality we get:
\[
\frac{1}{2}\partial_t ||\Lambda^s \theta_n||_{L^2}^2(t) + ||\sqrt{M_3} [\Lambda^s \theta_n]||_{L^2}^2(t) \leq \frac{C_5 m^*}{\sqrt{m_*}} ||\Lambda^s \theta_n||_{L^2} ||\Lambda^s \nu_{n-1}||_{L^2} + \frac{1}{2} ||\sqrt{M_3} [\Lambda^s \theta_n]||_{L^2}^2.
\]
Using the inductive assumption (25), it follows for $t \in [0, T)$ that:
\[
\partial_t ||\Lambda^s \theta_n||_{L^2}^2(t) + ||\sqrt{M_3} [\Lambda^s \theta_n]||_{L^2}^2(t) \leq \left(\frac{C_5 m^*}{\sqrt{m_*}} ||\Lambda^s \theta_0||_{L^2}\right)^2 ||\Lambda^s \theta_n||_{L^2}^2(t)
\]
and applying Grönwall’s inequality, we arrive to:
\[
||\Lambda^s \theta_n||_{L^2}(t) + \int_0^t ||\sqrt{M_3} [\Lambda^s \theta_n]||_{L^2}(\tau) d\tau \leq \exp \left[\left(\frac{C_5 m^*}{\sqrt{m_*}} ||\Lambda^s \theta_0||_{L^2}\right)^2 t\right] ||\Lambda^s \theta_0||_{L^2}.
\]
Therefore, taking
\[
T \leq \frac{\log 2}{\left(\frac{C_5 m^*}{\sqrt{m_*}} ||\Lambda^s \theta_0||_{L^2}\right)^2}
\]
we obtain that (25) holds for $j = n$ and so by induction it holds for all $j \geq 1$. This shows that the sequence $\{\theta_n\}_{n\geq1}$ is uniformly bounded in $L^\infty(0, T; H^s)$.

Moreover, we may show that the sequence $\{\theta_n\}_{n\geq1}$ is Cauchy in $L^\infty(0, T; H^{s-1})$. To see this, we consider the difference of two iterates $\bar{\theta}_n := \theta_n - \theta_{n-1}$. It follows from (24) that $\bar{\theta}_n$ is a solution of:
\[
\begin{cases}
\partial_t \bar{\theta}_n(x, t) + \nu_{n-1}(x, t) \cdot \nabla \bar{\theta}_n(x, t) + \bar{\nu}_{n-1}(x, t) \cdot \nabla \theta_{n-1}(x, t) = -M_3 \bar{\theta}_n(x, t) \\
\nu_{n-1}(x, t) = M[\theta_{n-1}](x, t) \\
\bar{\theta}_n(x, 0) = 0
\end{cases}
\tag{29}
\]
for all $n \geq 3$, where $\bar{\nu}_n(x, t) := M[\bar{\theta}_n](x, t)$. Applying $\Lambda^{s-1}$ to (29), taking an $L^2$ inner product with $\Lambda^{s-1}$ and using that $\nabla \cdot \nu_{n-1} = 0$, we arrive to:
\[
\frac{1}{2}\partial_t ||\Lambda^{s-1} \bar{\theta}_n||_{L^2}(t) + ||\sqrt{M_3} [\Lambda^{s-1} \bar{\theta}_n]||_{L^2}(t) \leq ||\Lambda^{s-1} \bar{\theta}_n||_{L^2}(t) ||\nu_{n-1} \cdot \nabla, \Lambda^{s-1}|| \bar{\theta}_n||_{L^2}(t)
+ ||\Lambda^{s-1} \bar{\theta}_n||_{L^2}(t) ||\Lambda^{s-1} \bar{\nu}_n \cdot \nabla \theta_{n-1}||_{L^2}(t). \tag{30}
\]
Now, applying the Sobolev embeddings into the product and commutator estimate given by Lemma 2.4 and Lemma 2.5, we obtain for $s > 5/2$ that:
\[
\begin{align*}
||\nu_{n-1} \cdot \nabla, \Lambda^{s-1}|| \bar{\theta}_n||_{L^2} &\lesssim ||\nu_{n-1}||_{L^\infty} ||\Lambda^{s-2} \nu_{n-1}||_{L^2} + ||\Lambda^{s-1} \nu_{n-1}||_{L^6} ||\nabla \bar{\theta}_n||_{L^3} \\
&\lesssim ||\Lambda^{5/2} \nu_{n-1}||_{L^2} ||\Lambda^{s-1} \bar{\theta}_n||_{L^2} + ||\Lambda^s \nu_{n-1}||_{L^2} ||\Lambda^{3/2} \bar{\theta}_n||_{L^2} \\
&\lesssim ||\Lambda^s \nu_{n-1}||_{L^2} ||\Lambda^{s-1} \bar{\theta}_n||_{L^2}.
\end{align*}
\]
\[ ||\Lambda^{s-1}(\mathbf{u}_{n-1} \cdot \nabla \theta_{n-1})||_{L^2} \leq ||\mathbf{u}_{n-1}||_{L^\infty} ||\Lambda^{s-1}\nabla \theta_{n-1}||_{L^2} + ||\Lambda^{s-1}\mathbf{u}_{n-1}||_{L^2} ||\nabla \theta_{n-1}||_{L^\infty} \]
\[ \leq ||\Lambda^{3/2}\mathbf{u}_{n-1}||_{L^2} ||\Lambda^s \theta_{n-1}||_{L^2} + ||\Lambda^{s-1}\mathbf{u}_{n-1}||_{L^2} ||\Lambda^{5/2} \theta_{n-1}||_{L^2} \]
\[ \leq ||\Lambda^{s-1}\mathbf{u}_{n-1}||_{L^2} ||\Lambda^s \theta_{n-1}||_{L^2}. \]

Combining these two inequalities with Corollary 2.1 and Corollary 2.2 we have proved that there exists two positive constants satisfying \( m_* \leq M_3(k) \leq m^* \) for all \( k \in \mathbb{Z}_+^3 \) such that:
\[ ||\mathbf{u}_{n-1} \cdot \nabla, \Lambda^{-1} \tilde{\theta}||_{L^2} + ||\Lambda^{s-1} (\mathbf{u}_{n-1} \cdot \nabla \theta_{n-1})||_{L^2} \leq m^* ||\Lambda^s \theta_{n-1}||_{L^2} (||\Lambda^{s-1}\tilde{\theta}||_{L^2} + ||\Lambda^{s-1}\tilde{\theta}_{n-1}||_{L^2}). \]

Hence, combining this estimate with (30), we arrive to:
\[ \frac{1}{2} \partial_t ||\Lambda^{s-1}\tilde{\theta}||_{L^2}^2 + \sqrt{M_3} ||\Lambda^{s-1}\tilde{\theta}||_{L^2} \leq C_s m^* ||\Lambda^{s-1}\tilde{\theta}||_{L^2} ||\Lambda^s \theta_{n-1}||_{L^2} (||\Lambda^{s-1}\tilde{\theta}||_{L^2} + ||\Lambda^{s-1}\tilde{\theta}_{n-1}||_{L^2}) \]
\[ \leq \frac{C_s m^*}{\sqrt{m_*}} ||\sqrt{M_3} ||\Lambda^{s-1}\tilde{\theta}||_{L^2} ||\Lambda^s \theta_{n-1}||_{L^2} (||\Lambda^{s-1}\tilde{\theta}||_{L^2} + ||\Lambda^{s-1}\tilde{\theta}_{n-1}||_{L^2}) \]
\[ \leq ||\sqrt{M_3} ||\Lambda^{s-1}\tilde{\theta}||_{L^2}^2 + \frac{1}{2} \left( \frac{C_s m^*}{\sqrt{m_*}} \right)^2 ||\Lambda^s \theta_{n-1}||_{L^2}^2 (||\Lambda^{s-1}\tilde{\theta}||_{L^2} + ||\Lambda^{s-1}\tilde{\theta}_{n-1}||_{L^2}) \]

and, as consequence of (25) we get:
\[ \partial_t ||\Lambda^{s-1}\tilde{\theta}||_{L^2}^2 (t) \leq \left( \frac{C_s m^*}{\sqrt{m_*}} \right)^2 ||\Lambda^s \theta_{n-1}||_{L^2}^2 \left( ||\Lambda^{s-1}\tilde{\theta}||_{L^2}^2 (t) + ||\Lambda^{s-1}\tilde{\theta}_{n-1}||_{L^2}^2 \right) \]
\[ \leq 2 \left( \frac{C_s m^*}{\sqrt{m_*}} \right)^2 ||\Lambda^s \theta_0||_{L^2}^2 \left( ||\Lambda^{s-1}\tilde{\theta}||_{L^2}^2 (t) + ||\Lambda^{s-1}\tilde{\theta}_{n-1}||_{L^2}^2 \right). \]

Hence, applying Grönwall’s inequality (see [2, p. 624]) we obtain that:
\[ ||\Lambda^{s-1}\tilde{\theta}||_{L^2}^2 (t) \leq \exp \left[ 2 \left( \frac{C_s m^*}{\sqrt{m_*}} ||\Lambda^s \theta_0||_{L^2} \right)^2 t \right] ||\Lambda^{s-1}\tilde{\theta}||_{L^2}^2 (0) \]
\[ + \exp \left[ 2 \left( \frac{C_s m^*}{\sqrt{m_*}} ||\Lambda^s \theta_0||_{L^2} \right)^2 t \right] \cdot 2 \left( \frac{C_s m^*}{\sqrt{m_*}} ||\Lambda^s \theta_0||_{L^2} \right)^2 \int_0^t ||\Lambda^{s-1}\tilde{\theta}_{n-1}||_{L^2}^2 (\tau) d\tau \]
and as by definition \( \tilde{\theta}_n(x,0) = 0 \), we have for \( 0 \leq t \leq T \) that:
\[ ||\Lambda^{s-1}\tilde{\theta}||_{L^2}^2 (t) \leq ||\Lambda^{s-1}\tilde{\theta}_{n-1}||_{L^\infty(0,T;L^2)} \exp \left[ 2 \left( \frac{C_s m^*}{\sqrt{m_*}} ||\Lambda^s \theta_0||_{L^2} \right)^2 T \right] \cdot 2 \left( \frac{C_s m^*}{\sqrt{m_*}} ||\Lambda^s \theta_0||_{L^2} \right)^2 T. \]

Here, after recalling (28), if we let \( T \) be such that:
\[ T := \min \{ \log 2, 1/6 \} \left( \frac{C_s m^*}{\sqrt{m_*}} ||\Lambda^s \theta_0||_{L^2} \right)^2 \]
we obtain that:
\[ ||\Lambda^{s-1}\tilde{\theta}||_{L^\infty(0,T;L^2)} \leq \frac{1}{2} ||\Lambda^{s-1}\tilde{\theta}_{n-1}||_{L^\infty(0,T;L^2)}. \]

Then, \( \{ \theta_n \}_{n \geq 1} \) is Cauchy in \( L^\infty(0,T;H^{s-1}) \) and hence \( \theta_n \) converges strongly to \( \theta \) in \( L^\infty(0,T;H^{s-1}) \). Nothing that \( s - 1 > 3/2 \), this shows that the strong convergence occurs in a Hölder space, which is sufficient to prove that the limiting function \( \theta \in L^\infty(0,T;H^s) \) is a solution of the initial value problem (10).

To conclude the proof of the theorem we prove the uniqueness. We note that if \( \theta^{(1)} \) and \( \theta^{(2)} \) are two solutions of (10), then \( \theta^2 = \theta^{(1)} - \theta^{(2)} \) solves
\[
\begin{aligned}
\partial_t \theta^2(x,t) + \mathbf{u}^{(1)}(x,t) \cdot \nabla \theta^2(x,t) + \mathbf{u}^{(2)}(x,t) \cdot \nabla \theta^{(2)}(x,t) &= -M_2[\theta^2](x,t) \\
\mathbf{u}^2(x,t) &= M[\theta^2](x,t) \\
\theta^2(x,0) &= 0.
\end{aligned}
\]
An $L^2$ estimate on (31) shows that $\theta^2(x,t) = 0$ for all $t \in [0,T]$, since $\theta^{(2)} \in L^\infty(0,T; H^{5/2+})$ and the frequency support of $\theta^2$ belongs to $X_{C}$, due to the fact that $\text{supp}(\theta^{(i)}(t)) \subset X_{C}$ for $t \in [0,T)$ and $i = 1, 2$. □

4. Global Existence in $H^{5/2+}(\mathbb{T}^3)$ for Frequency-Localized Initial Data

This section is devoted to prove the main result of this paper:

**Theorem 4.1.** Given $C > 0$ and the frequency straight line $X_{C}$. Let $\theta_0 \in H^{s}(\mathbb{T}^3)$ with zero vertical mean and frequency support in $X_{C}$ such that $||\theta_0||_{H^s} \leq \epsilon_0$ where $\epsilon_0$ is given by (44) and $\kappa := \frac{1}{\alpha} + \frac{5}{2}^+$ for $\alpha \in (0,1)$. Then, the solution of the non-diffusive MG Eq. (6) with initial datum $\Theta(x,0) = x_3 + \theta_0(x)$ exists globally in time and satisfies the following exponential decay to the steady state:

$$||\theta||_{H^s(t)} \lesssim ||\theta_0||_{H^s} \exp(-m_\star t).$$

In the next sections we give the proof of this result.

4.1. Energy Methods for the MG Equation

For $s > 5/2$ and initial data $\theta_0 \in H^{s}(\mathbb{T}^3)$ with zero vertical mean and $\text{supp}(\tilde{\theta}_0) \subset X_{C}$, we have proved in the previous section that there exists $T > 0$ such that $\theta(t) \in H^{s}(\mathbb{T}^3)$ and $\text{supp}(\tilde{\theta}(t)) \subset X_{C}$ for all $t \in [0,T)$.

4.1.1. A Priori Energy Estimates. In what follows, we assume that $\theta(t) \in H^{s}(\mathbb{T}^3)$ is a solution of (10) and the frequency support $\text{supp}(\tilde{\theta}(t)) \subset X_{C}$ for any $t \geq 0$. Then, the following estimate holds:

$$\partial_t ||\theta||_{H^s(t)}^2 \leq -[1 - C ||\theta||_{H^{5/2^+}(t)}] ||\theta||_{H^s(t)}^2.$$

First of all, we will perform the basic $H^s$-energy estimate for

$$\partial_t \theta(x,t) + u(x,t) \cdot \nabla \theta(x,t) = -M_3[\theta](x,t)$$

where $u(x,t) = M[\theta](x,t)$ and the initial data $\theta_0(x)$ has zero vertical mean and frequency support in $X_{C}$. $L^2$-estimate: We multiply (32) by $\theta$ and integrate over $\mathbb{T}^3$. Then:

$$\frac{1}{2} \partial_t ||\theta||_{L^2}^2 = -\int_{\mathbb{T}^3} \theta M_3[\theta] dx - \int_{\mathbb{T}^3} \theta(u \cdot \nabla) \theta dx.$$

Therefore, using Plancherel’s theorem and (12), we obtain that:

$$\frac{1}{2} \partial_t ||\theta||_{L^2}^2 = -\sum_{k \in \mathbb{Z}_3^d} M_3(k) |\hat{\theta}(k)|^2 = -||\sqrt{M_3}[\theta]||_{L^2}^2. \quad (33)$$

$H^s$-estimate: Applying $\Lambda^s$ to (32) and taking an $L^2$ inner product with $\Lambda^s \theta$ we obtain:

$$\frac{1}{2} \partial_t ||\Lambda^s \theta||_{H^s}^2 = -\int_{\mathbb{T}^3} \Lambda^s \theta M_3[\Lambda^s \theta] dx - \int_{\mathbb{T}^3} \Lambda^s \theta \Lambda^s [(u \cdot \nabla) \theta] dx = I_1 + I_2.$$

First of all we study $I_1$. As before, by Plancherel’s theorem and the square roor of $M_3$ given by (12) we get:

$$I_1 = -\sum_{k \in \mathbb{Z}_3^d} M_3(k) |\hat{\Lambda^s \theta}(k)|^2 = -||\sqrt{M_3}[\Lambda^s \theta]||_{L^2}^2 = -||\sqrt{M_3}[\theta]||_{H^s}^2. \quad (34)$$

Secondly, we study $I_2$. Below, we use the fact that $\nabla \cdot u = 0$ in order to obtain a commutator operator:

$$I_2 = -\int_{\mathbb{T}^3} \Lambda^s \theta \Lambda^s [(u \cdot \nabla) \theta] dx \pm \int_{\mathbb{T}^3} \Lambda^s \theta (u \cdot \nabla) \Lambda^s \theta dx = -\int_{\mathbb{T}^3} \Lambda^s \theta [\Lambda^s, u \cdot \nabla] \theta dx.$$
Hence, using the commutator estimate (14) and the Sobolev embedding $L^\infty(\mathbb{T}^3) \hookrightarrow H^{3/2+}(\mathbb{T}^3)$ we arrive to:

$$I_2 \leq \|\Lambda^s \theta\|_{L^2} \|\Lambda^s \mathbf{u} \cdot \nabla \theta\|_{L^2} \lesssim \|\Lambda^s \theta\|_{L^2} (\|\nabla \mathbf{u}\|_{L^\infty} \|\Lambda^{s-1} \nabla \theta\|_{L^2} + \|\Lambda^s \mathbf{u}\|_{L^2} \|\nabla \theta\|_{L^\infty})$$

$$\lesssim \|\theta\|_{H^s} \left(\|\mathbf{u}\|_{H^{3/2+}} \|\theta\|_{H^s} + \|\mathbf{u}\|_{H^s} \|\theta\|_{H^{5/2+}}\right).$$

Since $\text{supp}(\hat{\theta}(t)) \subset \mathcal{X}_\varepsilon$ as long as the solution exists, applying Corollary 2.1 we have that the Fourier operator $\hat{\mathcal{M}}(k)$ restricted to $k \in \mathcal{X}_\varepsilon$ behaves like a zero order operator. In particular, for $s > 5/2$ we have that:

$$I_2 \leq C_s \mathbf{m}^* \|\theta\|^2_{H^s} \|\theta\|_{H^{5/2+}} \tag{35}$$

where $\mathbf{m}^*(\mathcal{C})$ blows-up as $\mathcal{C}$ tends to infinity. Putting together (34) and (35), for $s > 5/2$ we have that:

$$\frac{1}{2} \partial_t \|\theta\|^2_{H^s} \leq C_s \mathbf{m}^* \|\theta\|^2_{H^s} \|\theta\|_{H^{5/2+}} - \|\sqrt{M_3} \|\theta\|^2_{H^s}. \tag{36}$$

To sum up, we have proved the next energy estimate.

**Theorem 4.2.** Let $\theta(t) \in H^s(\mathbb{T}^3)$ be a solution of (10) with zero mean and $\text{supp}(\hat{\theta}(t)) \subset \mathcal{X}_\varepsilon$ for any $t \geq 0$. Then, for $s > 5/2$ the following estimate holds:

$$\frac{1}{2} \partial_t \|\theta\|^2_{H^s}(t) \leq -\mathbf{m}_* \left[1 - \left(\frac{C_s \mathbf{m}^*}{\mathbf{m}_*}\right) \|\theta\|^2_{H^{5/2+}(t)}\right] \|\theta\|^2_{H^s}(t). \tag{37}$$

**Proof.**

Putting together (33) with (36) we arrive to:

$$\frac{1}{2} \partial_t \|\theta\|^2_{H^s}(t) \leq C_s \mathbf{m}^* \|\theta\|^2_{H^s}(t) \|\theta\|_{H^{5/2+}}(t) - \|\sqrt{M_3} \|\theta\|^2_{H^s}(t).$$

Using that $\text{supp}(\hat{\theta}(t)) \subset \mathcal{X}_\varepsilon$ for any $t \geq 0$ and Lemma 2.2, we obtain that:

$$\frac{1}{2} \partial_t \|\theta\|^2_{H^s}(t) \leq C_s \mathbf{m}^* \|\theta\|^2_{H^s}(t) \|\theta\|_{H^{5/2+}}(t) - \mathbf{m}_* \|\theta\|^2_{H^s}(t)$$

where $\mathbf{m}_*(\mathcal{C})$ goes to zero as $\mathcal{C}$ tends to infinity. Rewriting it, we have achieved our goal. \qed

So, as consequence, we establish a “small” data global existence result.

**Corollary 4.3.** Given $s > 5/2$, let $\theta_0 \in H^s(\mathbb{T}^3)$ with zero vertical mean and $\text{supp}(\hat{\theta}_0) \subset \mathcal{X}_\varepsilon$ such that $\|\theta_0\|_{H^{5/2+}} \leq \varepsilon$ is small enough. Then, the solution exists globally in time and satisfies a maximum principle:

$$\|\theta\|_{H^s}(t) \leq \|\theta_0\|_{H^s}.$$  

**Proof.** For all time $t \geq 0$, the bound $\|\theta\|_{H^{5/2+}}(t) \leq \|\theta_0\|_{H^{5/2+}}$ follows easily by Gronwall’s lemma, energy inequality (37) and the smallness condition on the initial data. Then, the general case $s > 5/2$ follows by the same type of arguments and the fact that the $H^{5/2+}(\mathbb{T}^3)$-norm of the solution is small for all time. \qed

In the following section we will improve the previous result. Using a perturbative argument, we are able to derive explicit expressions that quantify the decay rates. This leads to an asymptotic stability result of the steady state.

### 4.2. Linear & Non-linear Estimates

The linearized equation gives very good decay properties. Hence, the main achievement of this section is to control the nonlinearity, so that it would not destroy the decay provided by the linearized equation.
4.2.1. Linear Decay. We approach the question of global well-posedness for a small initial data from a perturbative point of view, i.e., we see (10) as a non-linear perturbation of the linear problem. The linearized equation around the trivial solution \((\theta, u) = (0, 0)\) reads as

\[
\begin{aligned}
\frac{\partial \theta(x,t)}{\partial t} + M_3[\theta](x,t) &= 0 \\
\theta(x,0) &= \theta_0(x)
\end{aligned}
\]  

(38)

where the initial data \(\theta_0 \in H^s(\mathbb{T}^3)\) has zero vertical mean and frequency support in \(X_\mathcal{C}\).

As \(M_3\) is a positive operator, we derive the exponential decay in time of solutions to the linear problem with decay rate depending on the frequency support of the initial data.

Corollary 4.4. The solution of (38) with initial data \(\theta_0 \in H^s(\mathbb{T}^3)\) and with zero vertical mean and frequency support in \(X_\mathcal{C}\) satisfies that

\[
||\theta||_{H^s}(t) \leq ||\theta_0||_{H^s} \exp(-m_* t),
\]

where \(m_*(\mathcal{C})\) goes to zero as \(\mathcal{C}\) tends to infinity.

4.2.2. Non-linear Decay. Next, we will show how this decay of the linear solutions can be used to establish the stability of the stationary solution \((\theta, u) = (0, 0)\) for the general problem (10). When perturbing around it, we get the following system:

\[
\begin{aligned}
\frac{\partial \theta(x,t)}{\partial t} + M_3[\theta](x,t) &= -u(x,t) \cdot \nabla \theta(x,t) \\
\theta(x,0) &= \theta_0(x)
\end{aligned}
\]  

(39)

where \(u(x,t) = M[\theta](x,t)\) and the initial data \(\theta_0\) satisfies the same hypothesis. Using Duhamel’s formula, we write the solution of (39) as:

\[
\theta(x,t) = e^{\mathcal{L}(t)\theta(x,0)} - \int_0^t e^{\mathcal{L}(t-\tau)} [u \cdot \nabla \theta](x,\tau) \, d\tau
\]

where \(\mathcal{L}(t)\) denotes the solution operator of the associated linear problem (38). Therefore, we have that:

\[
||\theta||_{H^s}(t) \leq ||\theta_0||_{H^s} \exp(-m_* t) + \int_0^t ||u \cdot \nabla \theta||_{H^s}(\tau) \exp(-m_*(t-\tau)) \, d\tau.
\]

(40)

4.3. The Bootstraping

We now demonstrate the bootstrap argument used to prove our goal. The general approach here is a typical continuity argument that has been used successfully in a plethora of other cases. Theorem 4.2 tells us that the following estimate holds for \(s > 5/2\):

\[
\frac{1}{2} \frac{\partial}{\partial t} ||\theta||_{H^s}^2(t) \leq -m_* \left[1 - \left(\frac{C_\mathcal{C} m_*}{m_*}\right) ||\theta||_{H^{5/2+}}(t) \right] ||\theta||_{H^s}^2(t).
\]

(41)

In the following, let \(\alpha \in (0, 1)\) be a free parameter and \(\kappa := \frac{1}{\alpha} + \frac{5}{2}^+\). We want to prove that \(||\theta||_{H^{5/2+}}\) decays in time. This will allow us to close the energy estimate and finish the proof. We will prove it through a bootstrap argument, where the main ingredient is the estimate (41).

4.3.1. Exponential Decay of \(||\theta||_{H^{5/2+}}\). In order to control \(||\theta||_{H^{5/2+}}(t)\) in time, we have the following result.

Lemma 4.5. Assume that \(||\theta_0||_{H^s} \leq \epsilon\) and \(||\theta||_{H^s}(t) \leq 4\epsilon\) for all \(t \in [0,T]\) where \(\kappa = \frac{1}{\alpha} + \frac{5}{2}^+\) with \(0 < \alpha < 1\). Then, we have:

\[
||\theta||_{H^{5/2+}}(t) \leq 2\epsilon \exp(-m_* t) \quad \text{for all} \quad t \in [0,T].
\]
4.3.2. A New Boostraping Argument.

In order to control Duhamel’s formula (40) give us:

\[ ||\theta||_{H^{\frac{3}{2}}}(t) \leq ||\theta_0||_{H^{\frac{3}{2}}} \exp(-m_\ast t) + \int_0^t ||u \cdot \nabla \theta||_{H^{\frac{3}{2}}} \exp(-m_\ast (t-\tau)) \, d\tau \]

and using the algebraic properties of Sobolev spaces we have that:

\[ ||u \cdot \nabla \theta||_{H^{\frac{3}{2}}} \lesssim ||u||_{H^{\frac{3}{2}}} ||\theta||_{H^{\frac{3}{2}}} \lesssim m^* ||\theta||_{H^{\frac{3}{2}}} ||\theta||_{H^{\frac{5}{2}}}. \]

The last inequality is due to Corollary 2.1 and the fact that supp(\(\hat{\theta}(\tau)\)) \(\subset \mathcal{X}_\varepsilon\) as long as the solution exists. Moreover, due to the well-known Gagliardo-Nirenberg interpolation inequality:

\[ ||\theta||_{H^{\frac{7}{2}}} \lesssim ||\theta||_{H^{\frac{3}{2}}}^{1-\alpha} ||\theta||_{H^{\frac{11}{2}}}^{\alpha} \]

we have that:

\[ \text{with } 0 < \alpha < 1 \]

we arrive to

\[ ||\theta||_{H^{\frac{5}{2}}} \leq ||\theta_0||_{H^{\frac{3}{2}}} \exp(-m_\ast t) + \int_0^t C_\alpha m^* ||\theta||_{H^{\frac{3}{2}}}^{2-\alpha} \exp(-m_\ast (t-\tau)) \, d\tau. \]

By hypothesis, we have that \( ||\theta||_{H^{\frac{1}{\alpha+\frac{3}{2}}} \leq 4\varepsilon \text{ on the interval } [0, T] \). Then, we obtain that:

\[ ||\theta||_{H^{\frac{5}{2}}} \leq \varepsilon \exp(-m_\ast t) + \int_0^t (4\varepsilon)^\alpha C_\alpha m^* ||\theta||_{H^{\frac{3}{2}}}^{2-\alpha} \exp(-m_\ast (t-\tau)) \, d\tau. \]  

(42)

In particular, there exist \( 0 < T^*(\alpha) \leq T \) such that for \( t \in [0, T^*] \) we have that:

\[ ||\theta||_{H^{\frac{5}{2}}} \leq 4\varepsilon \exp(-m_\ast t) \]  

(43)

If we restrict to \( 0 \leq t \leq T^* \) and we apply (43) into (42), we have:

\[ ||\theta||_{H^{\frac{5}{2}}} \leq \varepsilon \exp(-m_\ast t) + (4\varepsilon)^2 C_\alpha m^* \exp(-m_\ast t) \int_0^t \exp(-(1-\alpha)m_\ast \tau) \, d\tau \]

\[ \leq \varepsilon \exp(-m_\ast t) \left[ 1 + \varepsilon \frac{4^2 C_\alpha m^*}{(1-\alpha)m^*} \right]. \]

Taking \( 0 < \varepsilon < \frac{(1-\alpha)m_\ast}{4^2 C_\alpha m^*} \) we have proved that:

\[ ||\theta||_{H^{\frac{5}{2}}} \leq 2\varepsilon \exp(-m_\ast t) \]

for all \( t \in [0, T^*] \) and, by continuity, for all \( t \in [0, T] \).

\[ \square \]

4.3.2. A New Boostraping Argument. In order to control \( ||\theta||_{H^\kappa}(t) \) in time, we have the following result.

**Lemma 4.6.** Assume that \( ||\theta_0||_{H^\kappa} \leq \varepsilon \) and \( ||\theta||_{H^\kappa}(t) \leq 4\varepsilon \) for all \( t \in [0, T] \) where \( \kappa = \frac{1}{\alpha} + \frac{3}{2} \) with \( 0 < \alpha < 1 \). Then, we have that:

\[ ||\theta||_{H^\kappa}(t) \leq 2\varepsilon \text{ for all } t \in [0, T]. \]

**Proof.** Applying Grönwall’s inequality into (41) and Lemma 4.5, for \( t \in [0, T] \) we have that:

\[ ||\theta||_{H^\kappa}(t) \leq ||\theta_0||_{H^\kappa} \exp \left[ -m^* \int_0^t \left( 1 - \left( \frac{C_\alpha m^*}{m^*} \right) ||\theta||_{H^{\frac{5}{2}}} \right) \, d\tau \right] \]

\[ \leq ||\theta_0||_{H^\kappa} \exp \left( 2\varepsilon \frac{C_\alpha m^*}{m^*} \right). \]

Taking \( 0 < \varepsilon < \frac{\log \sqrt{2} m_\ast}{C_\alpha m^*} \) we have proved that \( ||\theta||_{H^\kappa}(t) \leq 2\varepsilon \) for all \( t \in [0, T] \).

\[ \square \]

Therefore, it is natural to define a “smallness” parameter \( \varepsilon_0 \) given by:

\[ \varepsilon_0 := \min \left\{ \frac{(1-\alpha)}{4^2 C_\alpha}, \frac{\log \sqrt{2}}{C_\alpha} \frac{m_\ast}{m^*} \right\} \]

(44)

In consequence, a straightforward combination of Lemmas 4.5 and 4.6 give us:
Corollary 4.7. Let $\theta_0 \in H^s(\mathbb{T}^3)$ such that $\|\theta_0\|_{H^s} \leq \epsilon$ with $0 < \epsilon \leq \epsilon_0$. Then, for all $t \geq 0$ we have that:

$$\|\theta\|_{H^s}(t) \leq 2\epsilon$$

and

$$\|\theta\|_{H^{s/2+}}(t) \leq 2\epsilon \exp(-m_* t).$$

4.3.3. Exponential Decay of $||\theta||_{H^s}$ with $s > \frac{7}{2}$. We have proved the exponential decay in time of $||\theta||_{H^{s/2+}}(t)$. Then, we are in the position to show how the bootstrap can be closed. This is merely a matter of collecting the conditions established above and showing that they can indeed be satisfied.

Lemma 4.8. Let $\theta_0 \in H^s(\mathbb{T}^3)$ with $s \geq \kappa$ such that $||\theta_0||_{H^s} \leq \epsilon$ where $0 < \epsilon \leq \epsilon_0$. Then, for all $t \geq 0$ we have that:

$$||\theta||_{H^s}(t) \lesssim ||\theta_0||_{H^s} \exp(-m^* t).$$

Proof. Applying Grönwall’s inequality into (41) we have:

$$||\theta||_{H^s}(t) \leq ||\theta_0||_{H^s} \exp \left[ -m_* \int_0^t \left( 1 - \left( \frac{C_\epsilon m^*}{m_*} \right) ||\theta||_{H^{s/2+}}(\tau) \right) d\tau \right].$$

The exponential decay of $||\theta||_{H^{s/2+}}$ proved in Corollary 4.7 give us:

$$||\theta||_{H^s}(t) \leq ||\theta_0||_{H^s} \exp(-m_* t) \exp \left( 2\epsilon \frac{C_\epsilon m^*}{m_*} \right)$$

and as $0 < \epsilon \leq \epsilon_0$ there exists a constant $C = C(\alpha, s, \kappa)$ such that $||\theta||_{H^s}(t) \leq C ||\theta_0||_{H^s} \exp(-m_* t)$. □

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Compliance with ethical standards

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References

[1] Brezis, H.: Functional analysis. Sobolev spaces and partial differential equations. Universitext. Springer, New York (2011)
[2] Evans, L.C.: Partial differential equations. Graduate Studies in Mathematics, vol. 19. American Mathematical Society, Providence, RI (1998)
[3] Friedlander, S., Rusin, W., Vicol, V.: The Magneto-Geostrophic equations: a survey. In: Proceedings of the St. Petersburg Mathematical Society, Volume XV: Advances in Mathematical Analysis of Partial Differential Equations
[4] Friedlander, S., Rusin, W., Vicol, V.: On the supercritically diffusive magneto-geostrophic equations. Nonlinearity 25(11), 3071–3097 (2012)
[5] Friedlander, S., Suen, A.: Wellposedness and convergence of solutions to a class offered non-diffusive equations with applications. arXiv:1902.04366
[6] Friedlander, S., Suen, A.: Existence, uniqueness, regularity and instability results for the viscous magneto-geostrophic equation. Nonlinearity 28(9), 3193–3217 (2015)
[7] Friedlander, S., Suen, A.: Solutions to a class of forced drift-diffusion equations with applications to the magneto-geostrophic equations. Ann. PDE 4(2), 34 (2018)
[8] Friedlander, S., Vicol, V.: Global well-posedness for an advection-diffusion equation arising in magneto-geostrophic dynamics. Ann. Inst. H. Poincaré Anal. Non Linéaire 28(2), 283–301 (2011)
[9] Friedlander, S., Vicol, V.: On the ill/well-posedness and nonlinear instability of the magneto-geostrophic equations. Nonlinearity 24(11), 3019–3042 (2011)
[10] Friedlander, S., Vicol, V.: Higher regularity of Hölder continuous solutions of parabolic equations with singular drift velocities. J. Math. Fluid Mech. 14(2), 255–266 (2012)
[11] Ghil, M., Childress, S.: Topics in geophysical fluid dynamics: atmospheric dynamics, dynamo theory, and climate dynamics, volume 60 of Applied Mathematical Sciences. Springer, New York (1987)
[12] Kenig, C.E., Ponce, G., Vega, L.: Well-posedness of the initial value problem for the Korteweg-de Vries equation. J. Am. Math. Soc. 4(2), 323–347 (1991)
[13] Lions, J.-L.: Quelques méthodes de résolution des problèmes aux limites non linéaires. Gauthier-Villars, Paris (1969)
[14] Moffatt, H.K.: Magnetostrophic turbulence and the geodynamo. In: Kaneda, Y. (ed.) IUTAM Symposium on Computational Physics and New Perspectives in Turbulence, pp. 339–346. Springer, Dordrecht (2008)
[15] Moffatt, H.K., Loper, D.E.: The magnetostrophic rise of A buoyant parcel in the earth’s core. Geophys. J. Int. 117, 394–402 (1994)
[16] Seregin, G., Silvestre, L., Šverák, V., Zlatoš, A.: On divergence-free drifts. J. Differ. Equ. 252(1), 505–540 (2012)
[17] Shvydkoy, R.: Convex integration for a class of active scalar equations. J. Am. Math. Soc. 24(4), 1159–1174 (2011)
[18] Stein, E.M.: Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals, volume 43 of Princeton Mathematical Series. Princeton University Press, Princeton, NJ. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III (1993)
[19] Tao, T.: Nonlinear dispersive equations, volume 106 of CBMS Regional Conference Series in Mathematics. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI. Local and global analysis (2006)
[20] Temam, R.: Navier–Stokes equations, volume 2 of Studies in Mathematics and its Applications. North-Holland Publishing Co., Amsterdam, revised edition, . ; Theory and numerical analysis. With an appendix by F. Thomasset (1979)

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