Distributed super dense coding over noisy channels

Z. Shadman, H. Kampermann, and D. Bruß
Institute für Theoretische Physik III, Heinrich-Heine-Universität Düsseldorf, D-40225 Düsseldorf, Germany

C. Macchiavello
Dipartimento di Fisica “A. Volta” and INFN- Sezione di Pavia, Via Bassi 6, 27100, Pavia, Italy
(Dated: May 5, 2014)

We study multipartite super dense coding in the presence of a covariant noisy channel. We investigate the case of many senders and one receiver, considering both unitary and non-unitary encoding. We study the scenarios where the senders apply local encoding or global encoding. We show that, up to some pre-processing on the original state, the senders cannot do better encoding than local, unitary encoding. We then introduce general Pauli channels as a significant example of covariant maps. Considering Pauli channels, we provide examples for which the super dense coding capacity is explicitly determined.

PACS numbers: 03.67.-a, 03.67.Hk, 03.65.Ud

I. INTRODUCTION

The notion of multipartite super dense coding was introduced by Bose et al. [1] to generalize the Bennett-Wiesner scheme [2] of super dense coding to multiparties. In this scheme it was shown that the use of a multipartite entangled state can allow a single receiver to read messages from more than one source through a single measurement. A generalization of this multipartite super dense coding to higher dimensions was given by Liu et al. [3]. Distributed super dense coding was also widely discussed in [4, 5] in which two scenarios of many senders with either one or two receiver(s) were addressed. For a single receiver, the exact super dense coding capacity was determined and it was shown that the senders do not need to apply global unitaries to reach the optimal capacity, but each sender can perform a local encoding on her side. As a result, it was shown that bound entangled states with respect to a bipartite cut between the senders (Alices) and the receiver (Bob) are not “multi” dense-codeable. Furthermore, a general classification of multipartite quantum states according to their dense-codeability was investigated.

The above multipartite scenarios were discussed for noiseless systems. However, in a realistic super dense coding scheme noise is unavoidably present in the system. We assume here that noise is present only in the transmission channels and the other apparatuses involved are perfect. In [11, 12], the bipartite super dense coding for both correlated and uncorrelated channels was discussed. In the present paper we generalize those schemes to the multipartite case in the presence of covariant noise. We investigate the scenario of more than one sender with a single receiver, considering both unitary and non-unitary encoding. We follow two avenues. First, we will consider the case where the senders are far apart and can only apply local operations. Second, we will assume that the senders are allowed to perform global operations. Since the amount of classical information that can be extracted from an ensemble of quantum states can be measured by the Holevo quantity [7–9], the super dense coding capacity for a given resource state is defined to be the maximal amount of this quantity with respect to the encoding procedure. In the present paper we focus on the optimization problem of the Holevo quantity in order to find the super dense coding capacity, considering local (non)unitary encoding as well as global (non)unitary encoding.

The paper is organized as follows. In Sec. II we first review the mathematical definition of the Holevo quantity for an ensemble of multipartite states when the parties are connected through a completely positive trace preserving map (a noisy channel). Considering unitary encoding, and a covariant channel, for both scenarios of local and global encoding, we then find an expression for the super dense coding capacity. This expression only involves to find a single unitary operator acting on the resource state. In Sec. III we discuss the Pauli channel as a typical example of a covariant map. We then give examples of Pauli channels and initial states for which the single unitary operator is explicitly determined. In Sec. IV, considering non-unitary encoding, we derive the multipartite super dense coding capacity in the presence of covariant channels up to a pre-processing on the resource state. We investigate both local and global encoding. Furthermore, we discuss the Pauli channel as particular map. In Sec. V we summarize the main results. Finally, in the Appendix, we provide proofs for two Lemmas reported in the paper.

II. SUPER DENSE CODING CAPACITY WITH MANY SENDERS AND ONE RECEIVER IN THE PRESENCE OF NOISY CHANNELS

A quantum channel is a communication channel which can transmit a quantum system and can be used to carry classical information. If the transfer is undisturbed the channel is noiseless; if the quantum system interacts with some other external systems (environment), a noisy quantum channel results. Mathematically, a quan-
For an illustration, see Fig. 1.

Consider a noisy transmission channel, the multipartite super dense coding scheme works as follows: a given quantum state \( \rho_{a_1 \ldots a_k b} \) is distributed between \( k \) Alices and a single Bob (in our scenario, Bob’s subsystem experiences noise in this stage). Then, Alices perform with the probability \( p_{(i)} \) a unitary operation \( W_{(i)}^{a_1 \ldots a_k} \) on their side of the state \( \rho_{a_1 \ldots a_k b} \), thus encoding classical information through the state \( \rho_{(i)} = (W_{(i)}^{a_1 \ldots a_k} \otimes \mathbb{I}^b) \rho_{a_1 \ldots a_k b} \) \( W_{(i)}^{a_1 \ldots a_k} \otimes \mathbb{I}^b \), where \( \mathbb{I}^b \) is the identity operator on the Bob’s Hilbert space and \( \{i\} \), is a set of indices for Alices. Subsequently, the Alices send their subsystems of the encoded state through the noisy channel to Bob. We consider \( \Lambda_{a_1 \ldots a_k b} : \rho_{(i)} \to \Lambda_{a_1 \ldots a_k b}(\rho_{(i)}) \) to be the CPTP map (quantum channel) that globally acts on the multipartite encoded state \( \rho_{(i)} \). By this process, Bob receives the ensemble \( \{\Lambda_{a_1 \ldots a_k b}(\rho_{(i)}), p_{(i)}\} \). By performing suitable measurements, Bob can extract the accessible information about this ensemble which is given by the Holevo quantity [9]

\[
\chi_{\text{un}} \left( \{\rho_{(i)}, p_{(i)}\} \right) = S \left( \sum_{(i)} p_{(i)} \Lambda_{a_1 \ldots a_k b} (\rho_{(i)}) \right) - \sum_{(i)} p_{(i)} S \left( \Lambda_{a_1 \ldots a_k b} (\rho_{(i)}) \right),
\]

where \( S(\eta) = -\text{tr}(\eta \log \eta) \) is the von Neumann entropy, and the logarithm is taken to base two. The subscript “un” refers to unitary encoding. The super dense coding capacity \( C_{\text{un}} \) for a given resource state \( \rho_{a_1 \ldots a_k b} \) and the noisy channel \( \Lambda_{a_1 \ldots a_k b} \) is defined to be the maximum of the Holevo quantity \( \chi_{\text{un}} \left( \{\rho_{(i)}, p_{(i)}\} \right) \) with respect to the encoding \( \{W_{(i)}^{a_1 \ldots a_k}, p_{(i)}\} \), i.e.

\[
C_{\text{un}} = \max_{\{W_{(i)}^{a_1 \ldots a_k}, p_{(i)}\}} \chi_{\text{un}} \left( \{\rho_{(i)}, p_{(i)}\} \right). \tag{2}
\]

For an illustration, see Fig. 1.

\[\text{Fig. 1. (Color online) Super dense coding with a distributed quantum state } \rho_{a_1 a_2 a_3 b} \text{ between four parties (three Alices and a single Bob). The straight black lines show the entanglement between the parties and the dashed red curves show the transmission channels between Alices and Bob. The Alices encode with the ensemble } \{W_{(i)}^{a_1 a_2 a_3}, p_{(i)}\} \text{ and in the next step, they send their subsystems of the encoded state through the channel } \Lambda_{a_1 a_2 a_3 b} \text{ to the receiver, Bob. The ensemble that Bob gets is } \{\Lambda_{a_1 a_2 a_3 b}(\rho_{(i)}), p_{(i)}\}. \text{ In this process, based on optimal encoding by the Alices, the maximal amount of classical information which is defined to be the capacity is transferred (see main text).}\]

### II.1. Covariant noisy channels

In this section we determine the super dense coding capacity for a special class of channels, up to a single unitary operator acting on the given state \( \rho_{a_1 \ldots a_k b} \). The channels we consider, denoted by \( \Lambda_{a_1 \ldots a_k b}^\xi \), commute with a complete set of orthogonal unitary operators \( \tilde{V}_{(i)} \), namely they have the property

\[
\Lambda_{a_1 \ldots a_k b}^\xi (\tilde{V}_{(i)} \rho_{(i)} \tilde{V}_{(i)}^\dagger) = \tilde{V}_{(i)} \Lambda_{a_1 \ldots a_k b}^\xi (\rho) \tilde{V}_{(i)}^\dagger, \tag{3}
\]

for the set of unitary operators which satisfy the orthogonality condition \( \text{tr}[\tilde{V}_{i} \tilde{V}_{j}^\dagger] = \delta_{ij} \). According to [13], for this set it is guaranteed that \( \frac{1}{2} \sum_{i} U_{i} \Xi U_{i}^\dagger = \text{tr} \Xi \) where \( \Xi \) is an arbitrary operator. The property (3) is usually referred to as covariance [10]. Here we will consider local unitary operators, namely of the form

\[
\tilde{V}_{(i)} = V_{a_1}^{(i)} \otimes \ldots \otimes V_{a_k}^{(i)}. \tag{4}
\]

In the following we will first discuss the case that the \( k \) Alices are far apart and they are restricted to local unitary operations in the presence of a covariant channel with the property (3). We will then investigate the case where the Alices are allowed to perform entangled unitary encoding.
II.1.1. Senders performing local unitary operators

In this scenario, the $j$th Alice applies a local unitary operator $W_{ij}^{a_j}$ with probability $p_{ij}$ on her subsystem of the shared state $\rho^{a_1\ldots a_k b}$. The optimization of the Holevo quantity is given in the following lemma.

**Lemma 1.** Let

$$\chi_{\text{un}}^{\text{lo}} = S\left(\sum_{\{i\}} p_{\{i\}} \Lambda_{a_1 \ldots a_k b} (\rho_{\{i\}})\right) - \sum_{\{i\}} p_{\{i\}} S\left(\Lambda_{a_1 \ldots a_k b} (\rho_{\{i\}})\right),$$

be the Holevo quantity with

$$\rho_{\{i\}} = (W_{i_1}^{a_1} \otimes W_{i_2}^{a_2} \otimes \ldots \otimes W_{i_k}^{a_k} \otimes 1^b) \rho^{a_1 \ldots a_k b}$$

and $\Lambda_{a_1 \ldots a_k b}$ be a covariant channel with the property $\Lambda_{a_1 \ldots a_k b} = \rho^{a_1 \ldots a_k b}$. The superscript “lo” refers to local encoding. Let

$$U_{\text{min}}^{\text{lo}} := U_{\text{min}}^{a_1} \otimes \ldots \otimes U_{\text{min}}^{a_k}$$

be the unitary operator that minimizes the von Neumann entropy after application of this unitary operator and the channel $\Lambda_{a_1 \ldots a_k b}$ to the initial state $\rho^{a_1 \ldots a_k b}$, i.e. $U_{\text{min}}^{\text{lo}}$ minimizes the expression

$$S\left(\Lambda_{a_1 \ldots a_k b} \left(U_{\text{min}}^{\text{lo}} \otimes 1^b\right) \rho^{a_1 \ldots a_k b} \left(U_{\text{min}}^{\text{lo}} \otimes 1^b\right)\right).$$

Then the super dense coding capacity $C_{\text{un}}^{\text{lo}}$ is given by

$$C_{\text{un}}^{\text{lo}} = \log D_A + S\left(\Lambda_{b} (\rho_{b})\right) - S\left(\Lambda_{a_1 \ldots a_k b} \left(U_{\text{min}}^{\text{lo}} \otimes 1^b\right) \rho^{a_1 \ldots a_k b} \left(U_{\text{min}}^{\text{lo}} \otimes 1^b\right)\right),$$

where $D_A = d_a d_2 \ldots d_k$ is the dimension of the Hilbert space of the $k$ Alices, and $\text{tr} \rho^{a_1 \ldots a_k b} = \Lambda_{b} (\rho_{b})$.

**Proof:** The von Neumann entropy is subadditive. The maximum entropy of a $D_A$-dimensional system is $\log D_A$. Since $U_{\text{min}}^{\text{lo}}$ is a unitary operator that leads to the minimum of the output von Neumann entropy, an upper bound on Holevo quantity $\chi_{\text{un}}^{\text{lo}}$ can be given as

$$\chi_{\text{un}}^{\text{lo}} \leq S\left(\sum_{\{i\}} p_{\{i\}} \Lambda_{a_1 \ldots a_k b} (\rho_{\{i\}})\right) - S\left(\Lambda_{a_1 \ldots a_k b} \left(U_{\text{min}}^{\text{lo}} \otimes 1^b\right) \rho^{a_1 \ldots a_k b} \left(U_{\text{min}}^{\text{lo}} \otimes 1^b\right)\right) \leq \log D_A + S\left(\Lambda_{b} (\rho_{b})\right) - S\left(\Lambda_{a_1 \ldots a_k b} \left(U_{\text{min}}^{\text{lo}} \otimes 1^b\right) \rho^{a_1 \ldots a_k b} \left(U_{\text{min}}^{\text{lo}} \otimes 1^b\right)\right).$$

In the next step, we show that the upper bound is reachable by the ensemble $\{\tilde{U}_{\{i\}} = \tilde{V}_{\{i\}} U_{\text{min}}^{\text{lo}}, \tilde{p}_{\{i\}} = \frac{1}{D_A}\}$ where $\tilde{V}_{\{i\}}$ was defined in Eqs. (3) and (4).

The Holevo quantity for the ensemble $\{\tilde{U}_{\{i\}}, \tilde{p}_{\{i\}}\}$ is denoted by $\chi_{\text{un}}^{\text{lo}}$ and is given by

$$\chi_{\text{un}}^{\text{lo}} = S\left(\sum_{\{i\}} \frac{1}{D_A} \Lambda_{a_1 \ldots a_k b} \left(\tilde{U}_{\{i\}} \rho^{a_1 \ldots a_k b} \tilde{U}_{\{i\}}^\dagger\right)\right) - \sum_{\{i\}} \frac{1}{D_A} S\left(\Lambda_{a_1 \ldots a_k b} \left(\tilde{U}_{\{i\}} \rho^{a_1 \ldots a_k b} \tilde{U}_{\{i\}}^\dagger\right)\right).$$

By using the covariance property, the argument in the first term on the RHS of (10) is given by

$$\sum_{\{i\}} \frac{1}{D_A} \Lambda_{a_1 \ldots a_k b} \left(\tilde{U}_{\{i\}} \otimes 1^b\right) \rho^{a_1 \ldots a_k b} \left(\tilde{U}_{\{i\}}^\dagger \otimes 1^b\right) \rho^{a_1 \ldots a_k b} \left(\tilde{U}_{\{i\}} \otimes 1^b\right) \rho^{a_1 \ldots a_k b} \left(\tilde{U}_{\{i\}}^\dagger \otimes 1^b\right) = 1.$$
unitary transformation, we can write

\[
\sum_{\{i\}} \frac{1}{D_A} S\left( \Lambda_{a_1...a_k} \left( \tilde{U}_{\{i\}} \otimes 1^b \right) \rho^{a_1...a_k} \left( \tilde{U}_{\{i\}}^\dagger \otimes 1^b \right) \right) \\
= \frac{1}{D_A} \sum_{\{i\}} S\left( \tilde{V}_{\{i\}} \otimes 1^b \right) \left[ \Lambda_{a_1...a_k} \left( \left( U_{\text{min}}^{lo} \otimes 1^b \right) \rho^{a_1...a_k} \right) \left( U_{\text{min}}^{lo} \otimes 1^b \right)^\dagger \right] \\
= S\left( \Lambda_{a_1...a_k} \left( \left( U_{\text{min}}^{lo} \otimes 1^b \right) \rho^{a_1...a_k} \left( U_{\text{min}}^{lo} \otimes 1^b \right)^\dagger \right) \right).
\]

(15)

Inserting Eqs. (14) and (15) into Eq. (10), one finds that the Holevo quantity \(\tilde{\chi}_{\text{un}}\) is equal to the upper bound given in Eq. (9) and therefore this is the super dense coding capacity.

As we can see from the capacity expression (5), all the parameters are known except the single unitary operator \(U_{\text{min}}^{io}\). However, for some specific situations like noiseless channels, i.e. for \(\Lambda_{a_1...a_k} = 1\), this unitary operator has already been identified as the identity operator. The capacity for noiseless channels is then given by \(C = \log D_A + S(\rho_b)\). We also provide more examples in the next section.

### II.1.2. Senders may perform entangled unitaries

We will now investigate the case where the Alices are allowed to apply entangled unitary operators. The question we want to address is: can the Alices increase the information transfer by applying entangled unitaries? To answer this question we follow a strategy similar to the case of local encoding, mentioned in the previous part.

The difference is that instead of local unitaries \(W_{\text{lo}}^a\), Alices encode with the global unitary operators \(W_{a_1...a_k}\) with the probabilities \(p_{\{i\}}\). In order to find the optimal encoding and thus the super dense coding capacity, we optimize the Holevo quantity (14). The optimization procedure is similar to Lemma 1. The difference is that we now have a global unitary operator \(U_{\text{min}}^g\) which minimizes the output von Neumann entropy. We can then show that the optimal encoding is given by the ensemble \(\{\tilde{U}_{\{i\}} = V_{\{i\}} U_{\text{min}}^g, \tilde{p}_{\{i\}} = \frac{1}{D_A}\}\), and the super dense coding capacity \(C_{\text{un}}^g\) for this situation is given by

\[
C_{\text{un}}^g = \log D_A + S(\Lambda_b(\rho_b)) \\
- S\left( \Lambda_{a_1...a_k} \left( U_{\text{min}}^g \otimes 1^b \right) \rho^{a_1...a_k} \left( U_{\text{min}}^g \otimes 1^b \right)^\dagger \right).
\]

(16)

The difference between the capacities (8) and (16) is the occurrence of the local and global unitary transformation \(U_{\text{min}}^{io}\) and \(U_{\text{min}}^g\), respectively.
and

\[ C_{\text{in}}^{P} = \log D_A + S \left( \Lambda_P^{b} (\rho_b) \right) - S \left( \Lambda_{a_1 \ldots a_k}^{b} \left( (U_{\text{min}}^R \otimes I^b) \rho_{a_1 \ldots a_k}^{b} (U_{\text{min}}^R \otimes I^b) \right) \right), \]

(22)

where, in both of the above equations, \( \rho_b = \text{tr}_{a_1 \ldots a_k} \rho_{a_1 \ldots a_k}^{b} \) represents Bob’s reduced density operator and \( \Lambda_P^{b} \) is the \( d_b \)-dimensional Pauli channel acting on Bob’s subsystem.

This general model of Pauli channels includes both the case of a memoryless channel, where the Pauli noise acts independently on each of the \( k + 1 \) parties and the probabilities \( q_{mn} \) are products of the single party probabilities \( q_{mn} \), or more generally the case where the action of noise is not independent on consecutive uses but is correlated. For example, for \( k + 1 \) uses of a Pauli channel we can define a correlated Pauli channel in the multipartite scenario as follows

\[ q(m,n) = (1 - \mu_{12}) \ldots (1 - \mu_{k,k+1}) q_{m_1 n_1} \ldots q_{m_{k+1} n_{k+1}} + \mu_{12} (1 - \mu_{13}) \ldots (1 - \mu_{k,k+1}) \delta_{m_1 m_2} \delta_{n_1 n_2} q_{m_3 n_3} \ldots q_{m_{k+1} n_{k+1}}. \]

Here, between every two individual channels we have defined a correlation degree \( \mu_{jl} \) with \( 0 \leq \mu_{jl} \leq 1 \) which correlates the channel \( j \) to the channel \( l \) (\( j \neq l \)). Thus, for \( k + 1 \) parties we have \( \frac{k(k+1)}{2} \) correlation degrees \( \mu_{jl} \). For instance, \( \mu_{12} \) correlates the channel one and two, \( \mu_{k,k+1} \) correlates the channel \( k \) and Bob’s channel, etc. If \( \mu_{jl} = 0 \) for all \( j \) and \( l \), then the \( k + 1 \) channels are independent or, in other words, we are in the memoryless (or uncorrelated) case. As mentioned above, this channel can be expressed as a product of independent \( k + 1 \) channels acting separately on each subsystem. If \( \mu_{jl} = 1 \) for all \( j \) and \( l \), we have a fully correlated Pauli channel. For other values of \( \mu_{jl} \) other than zero and one, the channel is partially correlated. For two uses of a Pauli channel, the expression reduces to \( q_{mn,1} q_{mn,2} = (1 - \mu) q_{mn,1} q_{mn,2} + \mu \delta_{m_1 m_2} \delta_{n_1 n_2} q_{mn,3} \) with a single correlation degree \( \mu \). We considered this situation in [11] for the case of bipartite super dense coding.

In the next section, we give examples for which the unitaries \( U_{\text{min}}^R \) and \( U_{\text{min}}^R \) are determined. For these examples, we show that both capacities are the same. Thus the Alices can reach the optimal information transfer via local encoding.

### IV. EXAMPLES

In this section, we show examples of multipartite systems for which \( U_{\text{min}}^R \) and \( U_{\text{min}}^R \) are determined. One example is a correlated Pauli channel and \( k \) copies of the Bell state. Noise here acts just on the Alices’ subsystem. Another example is a fully correlated Pauli channel and a GHZ state as well as \( k \) copies of a Bell diagonal state, both for \( d = 2 \). The last example will be the depolarizing channel with uncorrelated noise.

#### IV.1. \( k \) copies of a Bell state and a correlated Pauli channel

In this section we discuss the example that the Alices and Bob share \( k \) copies of the Bell state. We consider the situation when there is no noise on Bob’s side, and the Alices’ shares of the Bell states globally experience a correlated Pauli channel \( \Lambda_{a_1 \ldots a_k}^{P} \) (see Fig. 2). This example satisfies the situation discussed in Sec. 11.1.1.

A Bell state in \( d \times d \) dimensions is defined as \( | \Phi_{00} \rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |jj \rangle \). The set of the other maximally entangled Bell states is denoted by \( | \Phi_{mn} \rangle = (V_{mn} \otimes 1) | \Phi_{00} \rangle \), for \( m, n = 0, 1, \ldots, d - 1 \). We prove that the von Neumann entropy is invariant under arbitrary unitary rotation \( U_{a_1 \ldots a_k} \) of the state \( \rho_{00}^{a_1 b_1} \ldots \otimes \rho_{00}^{a_k b_k} \) after application of the channel \( \Lambda_{a_1 \ldots a_k}^{P} \), i.e.

\[ S \left( \Lambda_{a_1 \ldots a_k}^{P} \left( \left( U_{a_1 \ldots a_k}^{b_1 \ldots b_k} \otimes I^{b_1 \ldots b_k} \right) \left( \rho_{00}^{a_1 b_1} \otimes \ldots \otimes \rho_{00}^{a_k b_k} \right) \right) \right) = S \left( \Lambda_{a_1 \ldots a_k}^{P} \left( \rho_{00}^{a_1 b_1} \otimes \ldots \otimes \rho_{00}^{a_k b_k} \right) \right). \]

(24)

To show this claim, we first prove the following lemma.

**Lemma 2.** Let

\[ \rho_{00}^{a_1 b_1} \otimes \ldots \otimes \rho_{00}^{a_k b_k} = | \Phi_{00}^{a_1 b_1} \ldots \Phi_{00}^{a_k b_k} \rangle \langle \Phi_{00}^{a_1 b_1} \ldots \Phi_{00}^{a_k b_k} |, \]

(25)
Let us define
\[
\pi_{\{m,n\}} := (V_{m,n}^{a_k} \otimes \cdots \otimes V_{m,n_k}^{a_k} \otimes 1^{b_1 \cdots b_k})
\]
\[
(U_{a_1 \cdots a_k} \otimes 1^{b_1 \cdots b_k}) \left( \rho_{00}^{a_1 b_1} \otimes \cdots \otimes \rho_{00}^{a_k b_k} \right) (U_{a_1 \cdots a_k}^\dagger \otimes 1^{b_1 \cdots b_k})
\]
\[
\otimes 1^{b_1 \cdots b_k}) (V_{m_1,n_1}^{a_1} \otimes \cdots \otimes V_{m_k,n_k}^{a_k} \otimes 1^{b_1 \cdots b_k})
\]
where \(U_{a_1 \cdots a_k}\) is an arbitrary unitary operator and \(V_{m,n}^{a_k}\) are the operators in Eq. (18). For different states \(\pi_{\{m,n\}}\),
\[
\pi_{\{m,n\}} \pi_{\{m',n'\}} = 0,
\]
holds.

A proof for this Lemma is presented in the Appendix. Using the orthogonality property [27], and the purity of the density operator \(\pi_{\{m,n\}}\), the channel output entropy can be written as
\[
S(\Lambda_{n_1 \cdots n_k}^P ((U_{a_1 \cdots a_k} \otimes 1^{b_1 \cdots b_k}) (\rho_{00}^{a_1 b_1} \otimes \cdots \otimes \rho_{00}^{a_k b_k})
\]
\[
(U_{a_1 \cdots a_k}^\dagger \otimes 1^{b_1 \cdots b_k})) = S \left( \sum_{\{m,n\}} q_{\{m,n\}} \pi_{\{m,n\}} \right)
\]
\[
= H (\{q_{\{m,n\}}\})
\]
where \(H (\{p_i\}) = -\sum_i p_i \log p_i\) is the Shannon entropy. Consequently, the channel output entropy is just determined by the channel probabilities \(q_{\{m,n\}}\) and it is invariant under unitary encoding. Therefore, both local encoding and global encoding leads to the same capacity in Eqs. (21) and (22). That is
\[
C_{k, \text{copy}}^{\text{un,B}} = \log d_1^2 + \log d_2^2 + \cdots + \log d_k^2
\]
\[
- H (\{q_{\{m,n\}}\})
\]
\[
\neq k C_{\text{one-copy}}^{\text{un,B}}.
\]

The subscript “B” refers to a Bell state. As we can see from Eq. (29a), for a correlated Pauli channel, the capacity of \(k\) copies of a Bell state is not additive except when \(\mu_{jl} = 0\) for all \(j\) and \(l\), i.e. the case of an uncorrelated Pauli channel with \(q_{\{m,n\}} = q_{m_1 \cdots q_{m_k}}\). Then the capacity for \(k\) copies is \(k\) times the capacity of a single copy with dimension \(d^2\). That is
\[
C_{k, \text{copy,unco}}^{\text{un,B}} = k (\log d^2 - H (\{q_{m,n}\}))
\]
\[
= k C_{\text{one-copy,unco}}^{\text{un,B}}.
\]

If \(\mu_{jl} = 1\) for all \(j\) and \(l\), i.e. the case of a fully correlated Pauli channel with \(q_{\{m,n\}} = q_{mn}\), by using Eq. (29a), we have
\[
C_{k, \text{copy}}^{\text{un,B,f}} = \log d_1^2 + \cdots + \log d_k^2 - H (\{q_{mn}\})
\]
\[
= k \left( \log d^2 - \frac{H (\{q_{mn}\})}{k} \right)
\]
Since \(H (\{q_{mn}\})\) is a constant value, in the limit of many copies \(k\), by using Eq. (31), we can reach the capacity \(\log d^2\) per single copy. This is the highest capacity that we can reach for a \(d^2\) dimensional system.

IV.2. \(k\) copies of a Bell diagonal state and a fully correlated Pauli channel

Here, we give another example for which the capacity is exactly determined. This is the case of \(k\) copies of a Bell diagonal state and a fully correlated Pauli channel.

As we defined in Sec. III when \(\mu_{jl} = 1\) for all \(j\) and \(l\), the channel is called a fully correlated Pauli channel. For \(d = 2\), the operators \(V_{m,n}\) are either the identity or the Pauli operators \(\sigma_m\), i.e.
\[
\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]
\[
\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]
Thus, the channel, for an arbitrary number of parties, can be written as
\[
\Lambda^f (\xi) = \sum_m q_m (\sigma_m \otimes \cdots \otimes \sigma_m) \xi (\sigma_m \otimes \cdots \otimes \sigma_m),
\]
where \(\sum_{m=1}^{3} q_m = 1\). The superscript “\(f\)” refers to a fully correlated Pauli channel. A Bell diagonal state is a convex combination of the four Bell states. That is \(\rho_{Bd} = \sum_{n=0}^{3} p_n \rho_n\), where \(p_n\) is a Bell state, \(p_n \geq 0\), and \(\sum_{n=0}^{3} p_n = 1\). The subscript “Bd” stands for a Bell diagonal state. We here determine both unitaries \(U_{\text{min}}^B\) and \(U_{\text{min}}^\xi\). To do so, we first show that the von Neumann entropy of \(k\) copies of a Bell diagonal state \(\rho_{Bd}\) after
applying an arbitrary unitary operator $U^{a_1...a_k}$, and a fully correlated Pauli channel \[33\] is lower bounded as
\[
S\left(\Lambda_{a_1...a_k b_1...b_k}^{f}\left(U^{a_1...a_k} \otimes 1_{b_1...b_k}\right)\left(\rho_{Bd}^{a_1 b_1} \otimes ... \otimes \rho_{Bd}^{a_k b_k}\right)\right) \\
\geq \sum_m q_m S\left(\sigma_m \otimes ... \otimes \sigma_m\right)\left(U^{a_1...a_k} \otimes 1_{b_1...b_k}\right)\left(\rho_{Bd}^{a_1 b_1} \otimes ... \otimes \rho_{Bd}^{a_k b_k}\right) \\
= S\left(\rho_{Bd}^{a_1 b_1} \otimes ... \otimes \rho_{Bd}^{a_k b_k}\right), \tag{34}
\]
where we used the concavity property of the von Neumann entropy. The lower bound in Eq. \[34\] is reachable by choosing $U^{a_1...a_k} = 1$:
\[
S\left(\Lambda_{a_1...a_k b_1...b_k}^{f}\left(\rho_{Bd}^{a_1 b_1} \otimes ... \otimes \rho_{Bd}^{a_k b_k}\right)\right) \\
= S\left(\sum_{n_1,...,n_k} p_{n_1}...p_{n_k} \sum_m q_m \left(\sigma_m \otimes \sigma_m\right)\rho_{n_1}^{a_1 b_1} \otimes ... \otimes \rho_{n_k}^{a_k b_k}\right) \\
= S\left(\rho_{Bd}^{a_1 b_1} \otimes ... \otimes \rho_{Bd}^{a_k b_k}\right). \tag{35}
\]
As $U^{a_1...a_k} = 1$, the super dense coding capacities with both local encoding and global encoding \[21\] and \[22\] are the same. Therefore, Alice cannot do better encoding than local encoding on each copy of the Bell diagonal states. According to Eq. \[21\], the capacity of $k$ copies of a Bell diagonal state, when the states are sent through a fully correlated Pauli channel \[33\], is additive, i.e.
\[
C_{un,Bd,f}^{k-copy} = k\left(2 - S(\rho_{Bd})\right) = k C_{un,Bd,f}^{one-copy}. \tag{36}
\]
The capacity \[36\] shows that for fully correlated channels no information at all is lost to the environment and this class of channels behaves like a noiseless one.

For $k$ copies of a Bell state, by using Eq. \[36\], and the purity of a Bell state, we have
\[
C_{un,B,B}^{k-copy} = 2k, \tag{37}
\]
which is the highest amount of information transfer for $2k$ parties where each of them has a two-level system.

IV.3. GHZ state and a fully correlated Pauli channel

Another example for which we can determine both unitaries $U_{\text{min}}^{\text{dep}}$ and $U_{\text{min}}^{g}$, is a $|GHZ\rangle$ state of 2-dimensional subsystems distributed between $2k - 1$ Alices and a single Bob. The channel here is a fully correlated Pauli channel, as defined via Eq. \[33\]. For a system of $2k$ parties, the $|GHZ\rangle$ state can be written as
\[
|GHZ\rangle_{2k} = \frac{1}{\sqrt{2}} \sum_j |j^{(1)}...j^{(2k)}\rangle. \tag{38}
\]
Since the minimum value of a von Neumann entropy is zero, and since a $|GHZ\rangle$ state is invariant under the action of a fully correlated Pauli channel, we have
\[
S\left(\Lambda_{a_1...a_{2k-1}b}^{f}\left(|GHZ\rangle_{2k}\right)\right) \\
= S\left(\sum_m q_m \left(\sigma_m \otimes ... \otimes \sigma_m\right)\right) \left(|GHZ\rangle_{2k}\right) \\
= S\left(|GHZ\rangle_{2k}\right) = 0. \tag{39}
\]
Therefore, by using $U_{\text{min}}^{\text{dep}} = U_{\text{min}}^{g} = 1$, we can reach the zero entropy. Then, the super dense coding capacity, according to Eq. \[21\], reads
\[
C_{un,GHZ}^{f} = 2k. \tag{40}
\]
Here, the fully correlated Pauli channel, for a $|GHZ\rangle$ state, behaves like a noiseless channel and again no information is lost through the channel.

IV.4. $k$ copies of an arbitrary state and an uncorrelated depolarizing channel

The last example for which we determine the capacity exactly is the case of the $k$ copies of an arbitrary state $\rho^p$, where the dimention of both $\rho^p$ and $\rho^b$ is $d$, in the presence of an uncorrelated depolarizing channel.

A $d$-dimensional depolarizing channel is a channel that transmits a quantum system intact with the probability $1 - p$ and randomizes its state with the probability $p$. This channel is a special case of a $d$-dimensional Pauli channel with the probability parameters
\[
q_{mn} = \begin{cases} 
1 - p + \frac{p}{d^2}, & m = n = 0 \\
\frac{p}{d^2}, & \text{otherwise}.
\end{cases} \tag{41}
\]
with $0 \leq p \leq 1$, and $m, n = 0, ..., d - 1$.

In \[11\], for a bipartite state, we showed that the von Neumann entropy of a state that was sent through the depolarizing channel with uncorrelated noise is independent of any local unitary transformations that were performed before the action of the channel, i.e.
\[
S\left(S_{ab}^{\text{dep}}\left(U \otimes 1^b\right)\rho^b\left(U^{\dagger} \otimes 1^b\right)\right) = S\left(S_{ab}^{\text{dep}}\left(\rho^b\right)\right). \tag{42}
\]
In this section, we show that the same result is valid for a given resource state $\rho^{a_1 b_1} \otimes ... \otimes \rho^{a_k b_k}$, and the
uncorrelated depolarizing channel $\Lambda_{a_1,...,a_k b_1,...,b_k}^{\text{dep}}$. A proof for this statement is as follows.

\[
S \left( \Lambda_{a_1 b_1}^{\text{dep} \ b_1} \left( (U_{\text{min}}^{a_1} \otimes \mathbb{I}_{b_1}) \rho^{a_1 b_1} \right) \otimes ... \otimes (U_{\text{min}}^{a_k} \otimes \mathbb{I}_{b_k}) \right)
\]

\[
= S \left( \Lambda_{a_1 b_1}^{\text{dep} \ b_1} \left( (U_{\text{min}}^{a_1} \otimes \mathbb{I}_{b_1}) \rho^{a_1 b_1} \left( U_{\text{min}}^{a_1} \otimes \mathbb{I}_{b_1} \right) \right) \otimes ... \otimes (U_{\text{min}}^{a_k} \otimes \mathbb{I}_{b_k}) \right)
\]

\[
= S \left( \Lambda_{a_1 b_1}^{\text{dep} \ b_1} (\rho^{a_1 b_1}) + ... + S \left( \Lambda_{a_k b_k}^{\text{dep} \ b_k} (\rho^{a_k b_k}) \right) \right),
\]

where in the last equality we used Eq. (42), and the additivity of the von Neumann entropy. This proves our above claim. Therefore, by using (43), and according to Eq. (21), the super dense coding capacity, for $k$ copies of a resource state $\rho^{ab}$, is given by

\[
C_{\text{un,dep}}^{k-\text{copy}} = k \left( \log d + S \left( \Lambda_{ab}^{\text{dep} \ (\rho^{ab})} \right) - S \left( \Lambda_{ab}^{\text{dep} \ (\rho^{ab})} \right) \right)
\]

\[
= k C_{\text{one-copy}}^{\text{un,dep}}.
\] (44)

In Table 1, the above mentioned examples and also some of the unsolved examples are summarized.

| Resource state | Channel | Correlated Pauli channel only on the Alices’ sides (arbitrary dimension) | Fully correlated Pauli channel $(\mu_{jl} = 1)$ and $(d = 2)$ | Uncorrelated depolarizing channel $(\mu_{jl} = 0)$ (arbitrary dimension) |
|---------------|---------|--------------------------------------------------------------------------|-----------------------------------------------------------------|--------------------------------------------------------------------------|
| k copies of a Bell state | Eq. (29a) | Eq. (37) | Eq. (44) | |
| k copies of a Bell diagonal state | open | Eq. (36) | Eq. (44) | |
| GHZ state with 2k parties | open | Eq. (40) | open | |
| k copies of an arbitrary state $\rho^{ab}$ | open | open | Eq. (44) | |

Table 1. A summary of the solved examples of multipartite resource states and channels for super dense coding. Here, some of the unsolved examples are also mentioned.

V. DISTRIBUTED SUPER DENSE CODING WITH NON-UNITARY ENCODING

In a multipartite super dense coding scheme with non-unitary encoding, instead of the unitary operators $W_{a_1,...,a_k}^{si}$ considered in the previous sections, the Alices apply the CPTP maps $\Gamma_{a_1,...,a_k}^{si}$ on their side of the shared state $\rho^{a_1,...,a_k b}$ and thereby perform the encoding via the states $\rho(i) = (\Gamma_{a_1,...,a_k}^{si} \otimes \mathbb{I}_{b}) (\rho^{a_1,...,a_k b})$. The rest of the scheme is similar to the case of unitary encoding. The Alices send the encoded state $\rho(i)$, with the probability $p(i)$, through the covariant channel to Bob. The super dense coding capacity is then the maximum of the Holevo quantity with respect to the CPTP maps $\Gamma_{a_1,...,a_k}^{si}$ and the probabilities $p(i)$. We first consider the case for which the Alices are again restricted to local CPTP maps $\Gamma_{a_i}^{si}$. For this situation, the optimization of the Holevo quantity in the presence of a covariant channel is given in the following lemma.

**Lemma 3.** Let

\[
\chi_{\text{non-un}}^{lo} = S \left( \sum_{\{i\}} p(i) \Lambda_{a_1,...,a_k b}^{C_{i}^{lo} \ b} \left( \rho(i) \right) \right)
\]

\[
- \sum_{\{i\}} p(i) S \left( \Lambda_{a_1,...,a_k b}^{C_{i}^{lo} \ b} \left( \rho(i) \right) \right),
\] (45)

be the Holevo quantity with

\[
\rho(i) = \left( \Gamma_{a_1}^{si} \otimes \mathbb{I}_{s_2} \otimes ... \otimes \mathbb{I}_{s_k} \otimes \mathbb{I}_{b} \right) (\rho^{a_1,...,a_k b}),
\] (46)

and $\Lambda_{a_1,...,a_k b}^{C_{i}^{lo}}$ be a covariant channel. Let

\[
\Gamma_{\text{min}}^{lo} := \Gamma_{\text{min}}^{s_1} \otimes \Gamma_{\text{min}}^{s_2} \otimes ... \otimes \Gamma_{\text{min}}^{s_k}
\] (47)

be the map that minimizes the von Neumann entropy after application of this map and the covariant channel to the initial state $\rho^{a_1,...,a_k b}$. Then the super dense coding capacity $C_{\text{non-un}}^{lo}$ is given by

\[
C_{\text{non-un}}^{lo} = \log D_{A} + S \left( \Lambda_{b}^{C_{\text{min}}^{lo} \ b} \left( \rho_{\text{min}} \right) \right)
\]

\[
- S \left( \Lambda_{a_1,...,a_k b}^{C_{\text{min}}^{lo} \ b} \left( \Gamma_{\text{min}}^{lo} (\rho^{a_1,...,a_k b}) \right) \right),
\] (48)

where $\text{tr} a_1,...,a_k \Lambda_{a_1,...,a_k b}^{C_{i}^{lo} \ b} (\rho^{a_1,...,a_k b}) = \Lambda_{b} (\rho_{i})$ and $D_{A} = d_{a_1} d_{a_2}...d_{a_k}$. A proof for this Lemma is shown in the Appendix.
Now, if the Alices are allowed to perform global operations, with an argument similar to Lemma 3, we can show that the super dense coding capacity is

\[ C_{\text{non-un}}^g = \log D_A + S(A_h (\rho_i)) - S \left( A \left( \Gamma_{\text{min}} \left( \rho_i a_i ... a_k b_i \right) \right) \right). \]  

(49)

Here, \( \Gamma_{\text{min}}^g \) is a pre-processing that the Alices globally perform on the initial state \( \rho_i a_i ... a_k b_i \) before applying the optimal local unitary operators \( \hat{V}_i \). The pre-processing \( \Gamma_{\text{min}}^g \) minimizes the von Neumann entropy after applying it and the channel to the initial state.

The capacity \( \Gamma_{\text{min}}^g \) is reachable by the optimal ensemble \( \{ \hat{V}_i (\xi) = (\hat{V}_i (1) \otimes 1^b) [\Gamma_{\text{min}}^g (\xi)] (\hat{V}_i (1) \otimes 1^b), \hat{P}_i \} \frac{1}{D} \).

Similar to unitary encoding, the capacities \( \Gamma_{\text{min}}^g \) and \( \Gamma_{\text{min}}^g \) can also be written for the special case of a Pauli channel.

VI. CONCLUSION

In summary, we discussed in this paper the multipartite super dense coding scenario of many senders and a single receiver in the presence of a covariant channel. Considering (non)unitary encoding, for both cases of local and global encoding, and up to some pre-processing on the resource state, we found expressions for the capacity. In general, the pre-processing is not determined and it is an open question. For unitary encoding, we found examples for which the pre-processing can be determined and turns out to be the identity operator. For the mentioned examples, Alices cannot do better than local encoding. We also showed that for some of these examples Alices cannot do better than unitary encoding.

These results can be seen as first steps in several directions of future research. For example, it would be interesting to consider other types of channels rather than Pauli channels and also other types of memories. It would be also interesting to consider the case where the Alices have more than one receiver. The case of two receivers, for noisecless channels, is discussed in [4] where some of the Alices send their information to the first Bob while the others send theirs to the second Bob. The two receivers are unrestricted to perform local operations and classical communication among themselves. To the best of our knowledge, for this situation the exact super dense coding capacity is still an open question even for noisecless channels. For bipartite super dense coding, we showed previously that there are examples for which the nonunitary encoding leads to a better capacity than unitary one. It is still an open problem to establish whether this can happen also in the multipartite case.

Acknowledgments: This work was partially supported by Deutsche Forschungsgemeinschaft (DFG) and the EU-project CORNER.

VII. APPENDIX

Proof for Lemma 2: To prove the Lemma, we first show that the statement

\[ \langle \Phi_{00}^{b_1} \ldots \Phi_{00}^{a_k b_k} | (U^{a_1 \ldots a_k}) (V_{m_1 n_1}^{a_1} \otimes \ldots \otimes V_{m_k n_k}^{a_k} | V_{m'_1 n'_1} \otimes \ldots \otimes V_{m'_k n'_k}^{a_k} \rangle = 0, \]  

(50)

holds. By using the definition of a Bell state \( \langle \Phi_{00} \rangle = \frac{1}{\sqrt{2}} \sum_{j=0}^{d-1} | j j \rangle \) for a Bell state, we have

\[ \langle \Phi_{00}^{b_1} \ldots \Phi_{00}^{a_k b_k} | (U^{a_1 \ldots a_k}) (V_{m_1 n_1}^{a_1} \otimes \ldots \otimes V_{m_k n_k}^{a_k} | V_{m'_1 n'_1} \otimes \ldots \otimes V_{m'_k n'_k}^{a_k} \rangle \]

\[ = \sum_{j_1 \ldots j_k} \sum_{j_1 \ldots j_k} \langle j_1 j_1 \ldots j_k | (U^{a_1 \ldots a_k}) (V_{m_1 n_1}^{a_1} \otimes \ldots \otimes V_{m_k n_k}^{a_k} | V_{m'_1 n'_1} \otimes \ldots \otimes V_{m'_k n'_k}^{a_k} \rangle \]

\[ = \delta_{m_1 n_1} (a_1 \ldots a_k | V_{m_1 n_1}^{a_1} \otimes \ldots \otimes V_{m_k n_k}^{a_k} | V_{m'_1 n'_1} \otimes \ldots \otimes V_{m'_k n'_k}^{a_k} \rangle, \]

(51)

where in the last line we have used \( tr V_{m n} V_{n'}^{\dagger} = d \delta_{mn} \delta_{n'n'} \). Different states \( \pi_{m_1 n_1} \) have at least one different index for \( m_1 \) or \( n_1 \). Then by using Eq. (51), the statement of equation (50) is proved. Subsequently, we arrive at

\[ \langle \rho_{00}^{a_1 b_1} \otimes \ldots \otimes \rho_{00}^{a_k b_k} | (U^{a_1 \ldots a_k}) (V_{m_1 n_1}^{a_1} \otimes \ldots \otimes V_{m_k n_k}^{a_k} | V_{m'_1 n'_1} \otimes \ldots \otimes V_{m'_k n'_k}^{a_k} \rangle = 0. \]

(52)

By using Eq. (52), for \( \pi_{m_1 n_1} \pi_{m'_1 n'_1} \) we have

\[ \pi_{m_1 n_1} \pi_{m'_1 n'_1} = (V_{m_1 n_1}^{a_1} \otimes \ldots \otimes V_{m_k n_k}^{a_k} | \mathbb{1}_{b_1 \ldots b_k} | (U^{a_1 \ldots a_k}) \rho_{00}^{a_1 b_1} \otimes \ldots \otimes \rho_{00}^{a_k b_k} | U^{a_1 \ldots a_k} | \mathbb{1}_{b_1 \ldots b_k} \rangle = 0, \]

which completes the proof. \( \square \)

Proof for Lemma 3: With an argument similar to Lemma 1, we first introduce an upper bound on the Holevo quantity (45), and in the next step, we show that
the bound is attainable. By using the subadditivity of the von Neumann entropy, noting that the maximum entropy of a $d$-dimensional system is $\log d$, and since the map $\Gamma_{\text{min}}^{lo}$ gives the minimum output entropy, we have the upper bound

$$\chi_{\text{non-un}}^{lo} \leq \log D_A + S(\Lambda_b(\rho_b)) - S\left(\Lambda_{a_1 \cdots a_k b}(\Gamma_{\text{min}}^{lo}(\rho_{a_1 \cdots a_k b}))\right).$$

(53)

The above bound is reachable by the ensemble\{$\tilde{\Gamma}(\xi) = (\tilde{V}(\xi) \otimes 1^b)\Gamma_{\text{min}}^{lo}(\xi)(\tilde{V}^+(\xi) \otimes 1^b)$, $\tilde{p}(i) = \frac{1}{D^3}$\}. In other words, the optimal encoding consists of a fixed pre-processing $\Gamma_{\text{min}}^{lo}$ and a subsequent unitary encoding $\tilde{V}(\xi)$. We recognize $\chi_{\text{non-un}}^{lo}$ as the Holevo quantity for the ensemble \{$\tilde{\Gamma}(\xi), \tilde{p}(i)$\} which is given by

$$\chi_{\text{non-un}}^{lo} = S\left(\sum_{\{i\}} \frac{1}{D^3} A_{a_1 \cdots a_k b}^{c}(\tilde{\Gamma}(\xi)(\rho_{a_1 \cdots a_k b}))\right) - \sum_{\{i\}} \frac{1}{D^3} S\left(\Lambda_{a_1 \cdots a_k b}^{c}(\tilde{\Gamma}(\xi)(\rho_{a_1 \cdots a_k b}))\right).$$

(54)

In the following, we show that the above quantity (54) is equal to the bound on Eq. (53).

The pre-processing $\Gamma_{\text{min}}^{lo}$ maps the quantum state $\rho_{a_1 \cdots a_k b}$ to another quantum state $\rho_{a_1 \cdots a_k b}$. By using the decomposition (12) for $\Lambda_{a_1 \cdots a_k b}$, Eq. (13), and the covariance property of the channel, we find that the first term on the RHS of (54) is given by

$$\sum_{\{i\}} \frac{1}{D^3} A_{a_1 \cdots a_k b}^{c}(\tilde{V}(\xi) \otimes 1^b)(\rho_{a_1 \cdots a_k b}^{\dagger})(\tilde{V}^+(\xi) \otimes 1^b)$$

$$= \frac{1}{D^3} \sum_{\{i\}} S\left(\left(\tilde{V}(\xi) \otimes 1^b\right)\Lambda_{a_1 \cdots a_k b}(\rho_{a_1 \cdots a_k b}^{\dagger})(\tilde{V}^+(\xi) \otimes 1^b)\right)$$

$$= S\left(\Lambda_{a_1 \cdots a_k b}^{c}(\Gamma_{\text{min}}^{lo}(\rho_{a_1 \cdots a_k b}))\right).$$

(55)

Inserting Eqs. (55) and (56) into Eq. (54), one finds that the Holevo quantity $\chi_{\text{non-un}}^{lo}$ is equal to the upper bound given in Eq. (53) and consequently, this is the super dense coding capacity. $\square$

[1] S. Bose, V. Vedral, and P. L. Knight, Phys. Rev. A 57, 822 (1998).
[2] C. H. Bennett and S. J. Wiesner, Phys. Rev. Lett. 69, 2881 (1992).
[3] X. S. Liu, G. L. Long, D. M. Tong, and Feng Li, Phys. Rev. A 65, 022304 (2002).
[4] D. Bruß, G. M. D’Ariano, M. Lewenstein, C. Macchiavello, A. Sen(De), and U. Sen, Phys. Rev. Lett. 93, 210501 (2004).
[5] D. Bruß, G. M. D’Ariano, M. Lewenstein, C. Macchiavello, A. Sen(De), and U. Sen, Int. J. Quant. Inform. 4, 415 (2006).
[6] P. Badziag et al., Phys. Rev. Lett. 91, 117901 (2003).
[7] J. P. Gordon, in Proc. Int. School. Phys. "Enrico Fermi, Course XXXI", ed. P. A. Miles, 156 (1964).
[8] L. B. Levitin, Inf. Theory, Taskent, pp. 111 (1969).
[9] A. S. Holevo, Information-Theoretical Aspects of Quantum Measurement, Problems Inform. Transmission, 9:2 (1973).
[10] See, for example, A. S. Holevo, Int. J. Q. Inf. 3, 41 (2005). Information-Theoretical Aspects of Quantum Measurement, Problems Inform. Transmission, 9:2, 110 (1973).
[11] Z. Shadman, H. Kampermann, C. Macchiavello, and D. Bruß, New J. Phys. 12, 073042 (2010).
[12] Z. Shadman, H. Kampermann, C. Macchiavello, and D. Bruß, Phys. Rev. A 84, 042309 (2011).
[13] T. Hiroshima, J. Phys. A, Math. Gen. 34, 6907 (2001).
[14] C. Macchiavello and G.M. Palma, Phys. Rev. A 65, 050301(R) (2002).
[15] V. Karimipour and L. Memarzadeh, Phys. Rev. A 74, 062311 (2006).