A LYAPUNOV ANALYSIS OF FISTA WITH LOCAL LINEAR CONVERGENCE FOR SPARSE OPTIMIZATION

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Abstract. We conduct a Lyapunov analysis of the Fast Iterative Shrinkage-Thresholding Algorithm (FISTA) and show that it achieves local linear convergence for sparse ($\ell_1$-regularized) optimization, even when the objective is not strictly convex. We use an appropriate multi-step potential function to determine conditions on the parameters which imply weak convergence of the iterates to a minimizer in a real Hilbert space (ordinary convergence in $\mathbb{R}^n$). The Lyapunov analysis applies to a modified version of the momentum sequence proposed by Beck and Teboulle [Siam J. Img. Sci., 2009], for which convergence of the iterates is unknown.

For sparse optimization, we prove finite convergence to the optimal support and sign, and determine the local linear convergence rate which holds thereafter. This result holds for a broad range of parameter choices including the classical ones due to Beck and Teboulle and recent suggestions due to Chambolle and Dossal. Our results show that even though these choices have excellent global properties, they are not optimal for sparse optimization in terms of the local (i.e. asymptotic) convergence rate. We determine the locally optimal parameter choice. Finally we propose a method which inherits the excellent global properties of Beck and Teboulle’s choice and the asymptotic optimality of the locally optimal choice.

Key words. proximal gradient methods, accelerated gradient methods, $\ell_1$-regularization, local linear convergence

AMS subject classifications. 65K05, 65K15, 90C06, 90C25

1. Introduction. An important problem in convex optimization is the following:

$$\min_{x \in \mathcal{H}} F(x) \triangleq f(x) + g(x),$$  \hspace{1cm} (1.1)

where $\mathcal{H}$ is a real Hilbert space. The function $f$ is convex (but not necessarily strictly convex), differentiable, and has a Lipschitz continuous gradient. The function $g$ is convex and lower-semicontinuous (l.s.c), but not necessarily differentiable. Problems of this form have come under considerable attention in recent years in areas such as machine learning [1, 2], compressed sensing [3, 4] and image processing [5, 6].

Of particular interest in this paper will be the special case given below which we call sparse optimization (SO).

$$\text{(Problem SO) } \min_{x \in \mathbb{R}^n} F(x) = f(x) + \rho \|x\|_1,$$

where $\rho \geq 0$, $\|x\|_1 = \sum |x_i|$, and $f$ is smooth and convex. This is informally known as “sparse optimization” because the $\ell_1$-norm encourages sparse solutions. When $f$ is a least-squares term and $g$ is the $\ell_1$-norm, Problem (1.1) is referred to as sparse least-squares (also known as basis pursuit denoising or LASSO). Sparse least-squares is of central importance in compressed sensing and also has applications in machine learning [7] and image processing [8]. Other important instances of Problem (1.1) include least-squares with a total-variation [9] or nuclear-norm [10] regularizer, and convex-set constrained minimization [11].

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1.1. Proximal Splitting Algorithms. The forward-backward algorithm (FB) is a classical iterative approach to solving Problem (1.1) [12, 13]. It involves a “forward” step, which is an explicit gradient step with respect to the differentiable component $f$ and a “backward” step, which is an implicit, proximal step with respect to $g$. For many common instances of $g$ this proximal step is computationally inexpensive [11]. For FB the convergence rate of the objective function to the infimum is $O(1/k)$ which is better than the $O(1/\sqrt{k})$ rate achieved by the “black-box” subgradient method [14]. FB outperforms the subgradient method because it splits the objective function into a sum of two components, thus exploiting the structure of the problem to a higher degree. As a first-order method, FB is often suitable for large scale problems where second-order methods are not [15]. FB is also commonly referred to as the proximal splitting method [11], or the proximal gradient method [15]. For the special case of Problem SO, FB is commonly known as the iterative shrinkage and thresholding algorithm (ISTA) [16, 17, 18].

1.2. Inertial Methods. Inertial methods are multi-step iterative methods for solving minimization problems (or, more generally, monotone inclusion problems) that are inspired by discretizing a differential equation whose trajectory is known to converge to a solution (e.g. see [19, 20, 21, 22, 23]). Such methods can be traced back to Polyak [24], who introduced the heavy-ball with friction method (HBF) for minimizing a strongly convex function (see also [14] p. 65). Polyak’s method for minimizing a strongly convex function $p$ is

$$x^{k+1} = x^k - \lambda \nabla p(x^k) + \alpha(x^k - x^{k-1}), \quad x^0, x^1 \in \mathcal{H},$$

with prescriptions on the “momentum” parameter $\alpha$ and the “step-size” $\lambda$. It is equivalent to the classical gradient descent method for minimizing $p$ with the additional momentum term $\alpha(x^k - x^{k-1})$. (Throughout this paper, we will use the words inertia and momentum interchangeably). Remarkably, for strongly convex quadratics, this extra term, with negligible computational cost, can significantly improve the convergence performance. While both gradient descent and HBF obtain linear convergence to the minimizer for strongly convex quadratics, HBF has a far better rate [14, 24].

In [25] Nesterov developed a method for minimizing a convex function with $L$-Lipschitz-continuous gradient. The iterates of the method can be expressed as

$$x^{k+1} = x^k - \frac{1}{L} \nabla p\left(x^k + \alpha_k(x^k - x^{k-1})\right) + \alpha_k(x^k - x^{k-1}), \quad x^0, x^1 \in \mathcal{H}, \alpha_0 \in (0, 1),$$

with a certain update rule for the momentum parameter $\alpha_k$. (In [25], Nesterov uses $\beta_k$ to denote the momentum parameter and $\alpha_k$ for another variable. In this paper we mostly follow the notation of Alvarez [26] who used $\alpha_k$ for the momentum parameter in the inertial proximal algorithm). Notice that the method differs only from HBF in that the gradient is computed at the extrapolated point rather than the previous iterate. Nevertheless this minor difference leads to improved theoretical guarantees. Nesterov proved that the method obtains optimal performance for the class of first order methods. Specifically, for minimizing a convex function $p$ with Lipschitz-continuous gradient, Nesterov’s method guarantees a convergence rate of $O(1/k^2)$ for the objective function, which is optimal in the sense that there exists a pathological function in the complexity class for which no first-order method can achieve a better rate. In this same sense Nesterov’s method achieves the worst-case optimal iteration complexity when $p$ is strongly convex.
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Beck and Teboulle [16] extended Nesterov’s method to Problem (1.1), allowing for the presence of the possibly non-smooth function $g$. Their method, FISTA, combines Nesterov’s inertial update into a forward-backward proximal splitting framework. Beck and Teboulle showed that FISTA also obtains the optimal $O(1/k^2)$ rate, however they did not prove convergence of the iterates $\{x^k\}$ to a minimizer. Recently, Chambolle and Dossal [27] considered a very similar choice of the parameters to Beck and Teboulle which they showed obtains $O(1/k^2)$ rate in the objective function and also weak convergence of the iterates to a minimizer.

### 1.3. Local Linear Convergence.

It has been observed that for many popular instances of Problem (1.1) such as Problem SO, FB exhibits local linear convergence (see e.g. [17, 18, 28, 29]). This behavior has also been discerned in other iterative approaches to Problem (1.1) under certain conditions (see e.g. [30, 31]). Local linear convergence means that after some finite number of iterations the algorithm identifies a manifold on which the solution lies, and thereafter convergence is linear. This was the main result of [17] and [18] for ISTA. Recently in [28] FB was shown to have this behavior in a more general setting where $g$ satisfies certain properties including partial smoothness and local strong convexity. This setting includes many common instances of $g$ such as the $\ell_1$, $\ell_\infty$, nuclear and TV norms. Unlike FB, it is not known whether FISTA obtains local linear convergence for these problems although empirical results suggest it does [23, 32, 33].

### 1.4. Contributions of this Paper.

In the first part of the paper, we analyze FISTA with an appropriate multi-step Lyapunov function. This approach is inspired by Alvarez’ analysis [20] of the inertial proximal algorithm, and allows us to develop novel conditions on the algorithmic parameters that imply convergence of the iterates to a minimizer (weak convergence in a real Hilbert space, ordinary convergence in $\mathbb{R}^n$). This widens the range of possible parameter choices beyond those proposed in prior art. Furthermore recall that convergence of the iterates is unknown for the classical parameter choices proposed by Beck and Teboulle [16]. Our analysis does not directly apply to these parameter choices because $\alpha_k \to 1$, however the analysis does apply to a slightly modified version with $\alpha_k \to \bar{\alpha} < 1$ with $\bar{\alpha}$ allowed to be arbitrarily close to 1.

In the second part of the paper, we consider the behavior of FISTA applied to Problem SO. An additional benefit of the Lyapunov analysis is that it shows that

$$\sum_{k=1}^{\infty} \|x^k - x^{k-1}\|^2 < \infty. \quad (1.2)$$

(Note that this condition on its own is neither sufficient or necessary for convergence of the iterates). This condition can also be proved for the classical set of parameter choices introduced by Beck and Teboulle [16] and the choice suggested by Chambolle and Dossal [27]. It turns out that this condition is crucial to our analysis for Problem SO and allows us to extend the analysis of ISTA in [17] to FISTA. We show that, after a finite number of iterations FISTA reduces to minimizing a local function on a reduced support subject to an orthant constraint. We determine explicit upper bounds on the number of iterations for this to occur.

We show that a simple “locally optimal” parameter choice for FISTA obtains a local linear convergence rate with the best asymptotic iteration complexity given the step-size constraint imposed by the Lyapunov analysis. The asymptotically optimal iteration complexity is better than that obtained by the choices of [16] and [27] and...
ISTA. The improvement gained by FISTA over ISTA when adding the correct amount of momentum is equivalent to the improvement that Nesterov’s accelerated method \cite{25} achieves over gradient descent for strongly convex functions. As a corollary of our analysis, we show that the adaptive momentum restart scheme proposed in \cite{32} achieves the optimal iteration complexity.

With little effort our analysis of FISTA for Problem SO can be adapted to apply to the splitting inertial proximal method (SIPM) proposed by Moudafi and Oliny \cite{37}. We show that SIPM also achieves local linear convergence for this problem under appropriate parameter constraints.

The paper is organized as follows. In Section 2, notation and assumptions are discussed. In Section 3 we precisely define FISTA and discuss known convergence results in more detail. In Section 4 we apply our Lyapunov analysis to FISTA. In Section 5 we derive convergence results for Problem SO. Finally, numerical experiments are presented in Section 6.

2. Preliminaries.

2.1. Notation and Definitions. Throughout the paper, \( \mathcal{H} \) is a Hilbert space over the field of real numbers and \( \| \cdot \| \) is the associated norm. For any real-valued convex function \( g : \mathcal{H} \to \mathbb{R} \) and point \( x \in \mathcal{H} \), we denote by \( \partial g(x) \) for \( \epsilon \geq 0 \) the \( \epsilon \)-enlargement of the subdifferential, defined as the set

\[
\partial_\epsilon g(x) \triangleq \{ v \in \mathcal{H} : g(y) \geq g(x) + \langle v, y - x \rangle - \epsilon, \forall y \in \mathcal{H} \}
\]

which is always non-empty, convex, closed and bounded (\cite{14}, p. 128). We will use \( \partial g \) to denote \( \partial \epsilon g \). When \( \partial g(x) \) is a singleton we will call it the gradient at \( x \), denoted by \( \nabla g(x) \).

For \( a : \mathbb{R} \to \mathbb{R} \) and \( b : \mathbb{R} \to \mathbb{R} \), the notation \( a(k) = O(b(k)) \) (resp. \( a(k) = \Omega(b(k)) \)) will be taken to mean, there exists a constant \( C \geq 0 \) such that such that \( \lim_{k \to \infty} a(k)/b(k) \leq C \) (resp. \( \lim_{k \to \infty} a(k)/b(k) \geq C \)). The notation \( a(k) = o(b(k)) \) will be taken to mean \( \lim_{k \to \infty} a(k)/b(k) = 0 \). We will say a sequence \( \{ x^k \} \subset \mathcal{H} \) converges linearly to \( x^* \in \mathcal{H} \) with rate of convergence \( q \in (0, 1) \), if \( \| x^k - x^* \| = O(q^k) \). To be precise we will occasionally refer to this as asymptotic or local linear convergence. Note that this is different from non-asymptotic, or global linear convergence with rate \( q \), in which case there exists a \( C \) such that \( \| x^k - x^* \| \leq Cq^k \) for all \( k \geq 1 \).

For a matrix \( A \in \mathbb{R}^{m \times n} \) and a set \( S \subset \{1, 2, \ldots, n\} \), \( A_S \) will denote the matrix in \( \mathbb{R}^{m \times |S|} \) formed by taking the columns corresponding to the elements of \( S \). For a vector \( v \in \mathbb{R}^n \), \( v_S \) will denote the \( |S| \times 1 \) vector with entries given by the entries of \( v \) on the indices corresponding to the elements of \( S \), and \( (v_S, 0) \) will denote the vector in \( \mathbb{R}^n \) equal to \( v \) on the indices corresponding to \( S \) and equal to zero everywhere else. Given \( c \in \mathbb{R} \) and \( x \in \mathbb{R}^n \), \( \text{sgn}(c) \) is defined as \( +1 \) if \( c \geq 0 \) and \( -1 \) else, \( \text{sgn}(x) \) is simply applying \( \text{sgn}(\cdot) \) element-wise. We will use the notation \( [c]_+ = \max(c, 0) \).

Define the optimal value of Problem \( (1.1) \) as

\[
F^* \triangleq \inf_{x \in \mathcal{H}} F(x).
\]

and the solution set as

\[
X^* \triangleq \{ x \in \mathcal{H} : F(x) = F^* \}.
\]

Given a function \( a : \mathbb{R} \to \mathbb{R} \), we say that the iteration complexity of a method for minimizing \( F \) is \( \Omega(a(\epsilon)) \) if \( k = \Omega(a(\epsilon)) \) implies \( F(x^k) - F^* = O(\epsilon) \). To be precise we will occasionally refer to this as the asymptotic iteration complexity.
2.2. Proximity Operators. We will make heavy use of the proximity operator of \( g \), denoted by \( \text{prox}_g \). The proximity operator is implicitly defined by

\[
\text{prox}_g(y) \in y - \partial g(\text{prox}_g(y)),
\]

and explicitly defined by

\[
\text{prox}_g(y) = \arg \min_x \left\{ \frac{1}{2} \|x - y\|^2 + g(x) \right\}.
\]  

(2.2)

Since the function being minimized in (2.2) is strongly convex, and l.s.c., \( \text{prox}_g(y) \) exists and is unique. To be more general we will actually use the \( \epsilon \)-enlarged proximity operator, which is the set

\[
\text{prox}_\epsilon g(y) = \{ v : y - v \in \partial \epsilon g(v) \},
\]

which is not necessarily uniquely defined (except when \( \epsilon = 0 \)). Note that \( \text{prox}_g(y) \in \text{prox}_\epsilon g(y) \) for all \( \epsilon \geq 0 \). The use of \( \text{prox}_\epsilon g \) allows for some approximation error in the computation of the proximity operator.

2.3. Co-coercivity and Convexity. We say that a differentiable and convex function \( f \) has a \( \frac{1}{L} \)-co-coercive gradient with \( L > 0 \), if

\[
\langle \nabla f(y) - \nabla f(x), y - x \rangle \geq \frac{1}{L} \| \nabla f(y) - \nabla f(x) \|^2, \quad \forall x, y \in \mathcal{H}.
\]  

(2.3)

Note this is equivalent ([19] Lemmas 1.4 and 1.5) to the gradient being \( L \)-Lipschitz continuous, i.e.

\[
\| \nabla f(y) - \nabla f(x) \| \leq L \| y - x \|, \quad \forall x, y \in \mathcal{H}.
\]  

(2.4)

We will need the following two standard properties of such a function. For all \( u, v \in \mathcal{H} \):

\[
f(u) - f(v) \leq \langle \nabla f(v), u - v \rangle + \frac{L}{2} \| u - v \|^2;
\]  

(2.5)

and (by convexity)

\[
f(u) - f(v) \leq \langle \nabla f(u), u - v \rangle.
\]  

(2.6)

We are now ready to formerly state our Assumptions for Problem (1.1).

Assumption 1. For Problem (1.1), assume \( f \) is proper, convex, differentiable everywhere, and has a \( 1/L \)-co-coercive gradient with \( L > 0 \). Assume \( g \) is proper, convex and l.s.c., and assume \( F^* > -\infty \).

2.4. Properties of Sparse Optimization. We now outline our assumptions for Problem SO and discuss some of the properties of Problem SO.

Assumption SO. For Problem SO, assume \( f \) is proper, convex, twice differentiable everywhere, and has a \( 1/L \)-co-coercive gradient with \( L > 0 \). Finally, assume \( F^* > -\infty \) and \( X^* \) is non-empty.

The only difference between Assumption SO and Assumption 1 is that we assume that \( f \) is twice differentiable. Let \( H(x) \) denote the Hessian of \( f \) at \( x \). Then the Lipschitz constant \( L \) of the gradient is equal to the supremum of the largest eigenvalue
of $H(x)$ over all $x$. Furthermore note that $\| \cdot \|_1$ is proper and convex. Finally note that for $\rho > 0$ the function $f(x) + \rho \|x\|_1$ is coercive thus $X^*$ is non-empty.

Problem SO includes sparse least-squares, defined as

$$\text{(Problem } \ell_1\text{-LS)} \quad \text{minimize } F(x) = \frac{1}{2} \|b - Ax\|^2 + \rho \|x\|_1,$$

where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. The solution set $X^*$ of Problem $\ell_1$-LS is non-empty and $F^* \geq 0$. The function $f$ has gradient equal to $A^T(Ax - b)$ which is Lipschitz-continuous with Lipschitz constant $L$ equal to the largest eigenvalue of $A^TA$.

The proximity operator associated with $\rho \| \cdot \|_1$ is the soft-thresholding or shrinkage-thresholding operator $S_\rho$, applied element-wise. It is defined as

$$\text{prox}_{\rho \| \cdot \|_1}(z) = S_\rho(z) \triangleq \|z|\rho\|_1 \text{sgn}(z).$$

(2.7)

Note that, $\partial \|x\|_1 = \partial \|x\|_1$ for all $\epsilon \geq 0$, so the approximate and exact proximity operators are identical.

In the analysis of FISTA applied to Problem SO we will need the following result proved in [17].

**Theorem 2.1 (Theorem 2.1 [17]).** For problem SO suppose Assumption SO holds, then there exists a vector $h^* \in \mathbb{R}^n$ such that for all $x^* \in X^*$, $\nabla f(x^*) = h^*$. Furthermore, for all $i \in \{1, 2, \ldots, n\}$,

$$h^*_i \begin{cases} = -1 & \text{if } \exists x \in X^*: x \geq 0 \\ = +1 & \text{if } \exists x \in X^*: x < 0 \end{cases} \in [-1, 1] \text{ else.}$$

The following two sets used in [17] will be crucial to our analysis. Let $D \triangleq \{i : |h_i^*| < \rho\}$ and $E \triangleq \{i : |h_i^*| = \rho\}$. Note that $D \cap E = \emptyset$ and $D \cup E = \{1, 2, \ldots, n\}$. By Theorem 2.1 we can infer that $\text{supp}(x) \subseteq E$ for all $x^* \in X^*$. Finally, define

$$\omega \triangleq \min\{|\rho - |h_i^*| : i \in D\} > 0.$$

We will need the following Lemma proved in [17].

**Lemma 2.2 (Lemma 4.1 [17]).** Under Assumption SO, if $\lambda \in [0, 2/L)$,

$$\|x - \lambda \nabla f(x) - (y - \lambda \nabla f(y))\| \leq \|x - y\|, \forall x, y \in \mathbb{R}^n.$$

Finally, the following properties of $S_\nu$ will be useful.

**Lemma 2.3 (Lemma 3.2 [17]).** Fix any $a, b$ and $\nu$ in $\mathbb{R}$, and $x$ and $z$ in $\mathbb{R}^n$.

- **The function $s_\nu$ is component-wise non-expansive.** That is,

$$|S_\nu(x)_i - S_\nu(z)_i| \leq |x_i - z_i|. \quad (2.8)$$

- **If $|b| \geq \nu$ and $\text{sgn}(a) \neq \text{sgn}(b)$ then**

$$|S_\nu(a) - S_\nu(b)| \leq |a - b| - \nu. \quad (2.9)$$

- **If $S_\nu(a) \neq 0 = S_\nu(b)$ then $|a| > \nu, |b| < \nu$ and**

$$|S_\nu(a) - S_\nu(b)| \leq |a - b| - (\nu - |b|). \quad (2.10)$$
3. FISTA. Throughout the rest of the paper we will refer to the following algorithm as FISTA:

\[
\begin{align*}
    y^{k+1} &= x^k + \alpha_k(x^k - x^{k-1}) \\
    x^{k+1} &\in \text{prox}_{\lambda_k g}(y^{k+1} - \lambda_k \nabla f(y^{k+1})) \tag{3.1}
\end{align*}
\]

initialized at arbitrary \(x^0, x^1 \in H\). The sequences \(\{\alpha_k\}, \{\lambda_k\}, \{\epsilon_k\} \subset \mathbb{R}_{\geq 0}\). For any \(\epsilon_k \geq 0\), the choice

\[
x^{k+1} = \text{prox}_{\lambda_k g}(y^{k+1} - \lambda_k \nabla f(y^{k+1}))
\]

satisfies (3.1). Throughout the paper we will refer to \(\{\alpha_k\}\) as the “momentum” parameters and \(\{\lambda_k\}\) as the “step-size” parameters. The algorithm has the same form as the method proposed by Beck and Teboulle [16] but differs in two ways. First it allows for an arbitrary choice of \(\{\lambda_k\}\) and \(\{\alpha_k\}\) subject to constraints to be determined in Section 4, whereas the analysis of [16] relies on \(\alpha_k\) following a specific sequential relationship. Second our version of FISTA uses the \(\epsilon\)-enlarged sub-differential, allowing for some error in the computation of \(x^{k+1}\).

3.1. Known Convergence Results. Beck and Teboulle [16] proposed the following choice of parameters for FISTA, with \(\epsilon_k = 0\) for all \(k\):

\[
\lambda_k = \frac{1}{L}, \quad \alpha_k = \frac{t_k - 1}{t_{k+1}} \quad \forall k, \quad \text{where} \quad t_{k+1} = \frac{1 + \sqrt{4t_k^2 + 1}}{2}, \quad t_0 = 1. \tag{3.2}
\]

With this choice of parameters, Beck and Teboulle showed that the objective function converges to the minimum at the worst-case optimal rate of \(O(1/k^2)\). In fact the \(O(1/k^2)\) rate holds for a variety of choices of \(\alpha_k\) which all have the form: \(\alpha_k = 1 - O(1/k)\). However the choice in (3.2) guarantees the largest possible decrease in a given upper bound of \(F(x^k)\) at each iteration. Chambolle and Dossal [27] considered FISTA with \(\alpha_k = 0\) and a similar choice of \(\{\alpha_k\}\) to what was proposed by Beck and Teboulle. They investigated the choice

\[
0 \leq \lambda_k \leq \frac{1}{L}, \quad \alpha_k = \frac{t_k - 1}{t_{k+1}}, \quad \text{where} \quad t_{k+1} = \frac{k + a - 1}{a}, \quad a > 2, \quad t_0 = 1. \tag{3.3}
\]

With this choice of parameters, the authors showed that the objective function achieves the optimal \(O(1/k^2)\) convergence rate and in addition \(\{x^k\}\) weakly converges to a minimizer. In contrast to [16] and [27], our analysis establishes weak convergence of the iterates for a wide range of parameter choices. Indeed, the momentum sequence is not constrained to follow a particular sequence relationship, but instead must be constrained to \(\alpha_k \in [0, 1]\) and \(\lim \sup \alpha_k < 1\).

Lorenz and Pock [34] generalized FISTA to the problem of finding a zero of the sum of two maximal monotone operators \(A\) and \(B\), one of which is co-coercive. Setting \(A = \nabla f\) and \(B = \partial g\) recovers Problem (1.1). They also replaced the scalar step-size \(\lambda_k\) with a general positive definite operator \(\lambda_k M^{-1}\). Lorenz and Pock proved weak convergence of the iterates to a solution provided certain restrictions on \(\{\alpha_k\}\) and \(\{\lambda_k\}\). The restrictions on \(\alpha_k\) are stronger than those derived in our Lyapunov analysis. In their analysis, if the step-size \(\lambda_k\) is fixed to \(1/L\), \(\alpha_k\) is restricted to be less than \(\sqrt{5} - 2 \approx 0.24\), whereas, as we shall see in Section 4 our Lyapunov analysis allows \(\alpha_k \in [0, 1]\), so long as \(\lim \sup \alpha_k < 1\). For the step-size, their conditions are less restrictive than ours, allowing for values of \(\lambda_k\) up to \(2/L\), whereas our analysis only allows up to \(1/L\). However in their analysis larger values of \(\lambda_k\) lead to a smaller range of feasible values for \(\alpha_k\) reducing to 0 as \(\lambda_k\) approaches \(2/L\).
3.2. Known Convergence Results for Sparse Optimization. The sequences for \( \{\alpha_k\} \) defined in (3.2) and (3.3) both converge to 1. As we will see in Section 5.5, this may not be desirable for Problem SO. In the language of dynamical systems, when the momentum is too high the iterates move into an “underdamped regime” leading to oscillations in the objective function and slow convergence (see [32]). We will show that for Problem SO the choices given in (3.2) and (3.3) are not optimal from the viewpoint of asymptotic rate of convergence.

In [35] a method was developed for solving Problem (1.1) when \( f \) is strongly convex. The method is equivalent to FISTA with the same prescription for \( \{\alpha_k\} \) as determined by Nesterov [25] for his method for minimizing strongly convex functions (constant scheme 2.2.8 of [25]). However it also includes a backtracking procedure for adjusting \( \{\lambda_k\} \) and \( \{\alpha_k\} \) when the strong convexity and Lipschitz gradient parameters are not known.

The authors of [35] also extended their method to Problem \( \ell_1 \)-LS including the case where \( f \) is not strongly convex. The authors showed that under conditions on the matrix \( A \) related to the Restricted Isometry Property (RIP) used in compressed sensing, their algorithm obtains non-asymptotic (global) linear convergence, so long as the initial vector is sufficiently sparse. However, the authors note that their RIP-like conditions are much stronger than those typically found in the literature. Indeed the conditions are much stronger than those required in our proof of local linear convergence. We establish that FISTA obtains local linear convergence regardless of the initialization point. Furthermore no RIP-like assumptions are necessary. Local linear convergence can be proved under the mild condition that the smallest eigenvalue of the Hessian restricted to the support of a minimum is non-zero. Or if this does not hold, under a common strict-complementarity condition (see Section 5.4).

4. Lyapunov Analysis of FISTA. This section derives conditions on \( \{\epsilon_k\} \), \( \{\alpha_k\} \) and \( \{\lambda_k\} \) which imply weak convergence of the iterates \( \{x_k\} \) of FISTA to a minimizer of Problem (1.1). Throughout the rest of the paper, let \( \Delta_{k+1} \) denote \( x_{k+1} - x_k \). Given \( S,T \subset \mathcal{H} \), define \( d(S,T) \equiv \min_{s \in S,t \in T} \|s - t\| \).

**Theorem 4.1.** Suppose that Assumption 1 holds. Assume \( \{\lambda_k\} \) is non-decreasing and satisfies \( 0 < \lambda_k \leq 1/L \) for all \( k \), and \( \{\epsilon_k\} \) satisfies \( \epsilon_k \geq 0 \) for all \( k \) and \( \sum_k \epsilon_k < \infty \). If \( 0 \leq \alpha_k \leq 1 \) for all \( k \) and \( \limsup \alpha_k < 1 \), then for the iterates of FISTA, we have

(i) \( \sum_{k=1}^{\infty} \|\Delta_k\|^2 < \infty \);

(ii) \( \lim_{k \to \infty} d(0, \nabla f(x_k) + \partial g(x_k)) = \lim_{k \to \infty} d(0, \nabla f(y_k) + \partial g(y_k)) = 0 \).

(iii) If, in addition, \( X^* \) is non-empty, then \( x_k \) converges weakly to some \( \hat{x} \in X^* \).

**Proof.** The proof consists of two parts. In the first, we prove statements (i) and (ii) using arguments inspired by Alvarez’ analysis of the inertial proximal method in [20]. In the second part, we invoke Opial’s Lemma [30] to prove statement (iii). The second part is inspired by the analysis of the splitting inertial proximal algorithm by Moudafi and Oliny in [37].

**Proof of statements (i) and (ii).** Define the Lyapunov-like function\(^1\) or discrete energy, to be

\[
E_k = \frac{\alpha_k}{2\lambda_k} \|\Delta_k\|^2 + f(x_k) + g(x_k).
\]

\(^1\)Since the function is not necessarily positive definite, this is not truly a Lyapunov function
Note that this is the same energy function used by Alvarez [26]. Inequalities (2.5), (2.6) and (2.1) imply

\[ E_{k+1} - E_k = \frac{\alpha_{k+1}}{2\lambda_{k+1}} \|\Delta_{k+1}\|^2 - \frac{\alpha_k}{2\lambda_k} \|\Delta_k\|^2 + f(x^{k+1}) - f(x^k) + g(x^{k+1}) - g(x^k) \]

\[ \leq \frac{\alpha_{k+1}}{2\lambda_{k+1}} \|\Delta_{k+1}\|^2 - \frac{\alpha_k}{2\lambda_k} \|\Delta_k\|^2 + \langle \nabla f(y^{k+1}) + v, \Delta_{k+1} \rangle \]

\[ + \frac{L}{2} \|x^{k+1} - y^{k+1}\|^2 + \epsilon_k, \quad \forall v \in \partial g(x^{k+1}). \quad (4.1) \]

Using (4.1), the \( x^{k+1} \) - update in FISTA and the fact that \( \lambda_k \leq \lambda_{k+1} \), we infer that

\[ E_{k+1} - E_k \leq \frac{\alpha_{k+1}}{2\lambda_{k+1}} \|\Delta_{k+1}\|^2 - \frac{\alpha_k}{2\lambda_k} \|\Delta_k\|^2 - \frac{1}{\lambda_k} \langle x^{k+1} - y^{k+1}, \Delta_{k+1} \rangle \]

\[ + \frac{L}{2} \|\Delta_{k+1}\|^2 - \alpha_k \|\Delta_k\|^2 + \epsilon_k \]

\[ = \frac{\alpha_{k+1}}{2\lambda_{k+1}} \|\Delta_{k+1}\|^2 - \frac{\alpha_k}{2\lambda_k} \|\Delta_k\|^2 - \frac{1}{\lambda_k} \langle \Delta_{k+1} - \alpha_k \Delta_k, \Delta_{k+1} \rangle \]

\[ + \frac{L}{2} \|\Delta_{k+1}\|^2 + \alpha_k^2 \|\Delta_k\|^2 - \alpha_k L \langle \Delta_{k+1}, \Delta_k \rangle + \epsilon_k \]

\[ \leq (\frac{L}{2} - \frac{1}{\lambda_k} + \frac{\alpha_{k+1}^2}{2\lambda_k}) \|\Delta_{k+1}\|^2 + \frac{\alpha_k^2 L}{2} \|\Delta_k\|^2 \]

\[ + \frac{\alpha_k(1 - \lambda_k L)}{\lambda_k} \langle \Delta_{k+1}, \Delta_k \rangle + \epsilon_k \]

\[ = -\frac{\alpha_k(1 - \lambda_k L)}{2\lambda_k} \|\Delta_{k+1} - \Delta_k\|^2 - \frac{2 + \lambda_k L(\alpha_k - 1) - \alpha_k - \alpha_{k+1} \|\Delta_{k+1}\|^2}{2\lambda_k} \]

\[ - \frac{L \alpha_k(1 - \alpha_k)}{2\lambda_k} \|\Delta_k\|^2 + \epsilon_k. \]

Moving terms to the other side and summing implies, for all \( N \in \mathbb{Z}_+ \),

\[ \sum_{k=1}^{N} \left[ \frac{\alpha_k(1 - \lambda_k L)}{2\lambda_k} \|\Delta_{k+1} - \Delta_k\|^2 + \frac{2 + \lambda_k L(\alpha_k - 1) - \alpha_k - \alpha_{k+1} \|\Delta_{k+1}\|^2}{2\lambda_k} \right] \]

\[ + \frac{L \alpha_k(1 - \alpha_k)}{2\lambda_k} \|\Delta_k\|^2 \]

\[ \leq E_1 - E_{N+1} + \sum_{k=1}^{N} \epsilon_k \]

\[ \leq E_1 - F^* + \sum_{k=1}^{N} \epsilon_k = F(x^1) - F^* + \frac{\alpha_1}{2\lambda_1} \|\Delta_1\|^2 + \sum_{k=1}^{N} \epsilon_k < \infty. \quad (4.2) \]

Inequality (4.2) along with the assumptions on \( \{\alpha_k\} \) and \( \{\lambda_k\} \) imply statement (i). Statement (ii) implies \( \|\Delta_{k+1}\| \to 0 \), therefore \( \|x^k - y^{k+1}\| \to 0 \) via the \( y^{k+1} \) - update of FISTA. This implies that \( \|x^k - y^k\| \to 0 \), because \( \|x^k - y^k\| \leq \|x^{k-1} - y^k\| + \|x^{k-1} - x^k\| \). Finally, using the \( x^{k+1} \) - update of FISTA and the fact that \( \epsilon_k \to 0 \) we infer that

\[ \lim_{k \to \infty} d(\nabla f(y^k) + \partial g(y^k), 0) = \lim_{k \to \infty} d(\nabla f(x^k) + \partial g(x^k), 0) = 0. \quad (4.3) \]
**Proof of statement (iii).** If $x^{v_k}$ is a subsequence which weakly converges to $x'$, then the $y^{k+1}$-update of FISTA implies $y^{v_k}$ also weakly converges to $x'$. This, combined with the $x^{k+1}$-update implies that $x' \in X^*$. Suppose that for any $x^* \in X^*$, the sequence $\|x^k - x^*\|$ has a limit. This implies the sequence $x^k$ is bounded and by the Bolzano-Weierstrass Theorem it has at least one weakly-convergent subsequence, $x'^*$ (ordinary convergence in $\mathbb{R}^n$). By the above reasoning the limit of this subsequence, $\tilde{x}$ must be in $X^*$. Furthermore $\lim_k \|x^k - \tilde{x}\|$ exists. Consider another subsequence $x'^{v_k}$ which converges to $\tilde{x}' \in X^*$. By considering the fact that $\lim_k \|x^{v_k} - \tilde{x}\|^2 = \lim_k \|x'^{v_k} - \tilde{x}'\|^2$ and the corresponding statement for $\tilde{x}'$, one can see that $\|\tilde{x} - \tilde{x}'\| = 0$. Therefore the set of weakly convergent subsequences is the singleton $\{\tilde{x}\}$. Thus $x^k$ weakly converges to $\tilde{x} \in X^*$ (This is Opial’s Lemma [36]).

Assume $X^*$ is non-empty. We now proceed to show that, for any $x^* \in X^*$, the sequence $\|x^k - x^*\|$ has a limit. Our proof closely follows Moudafi and Oliny’s analysis [37], and is similar to the later variants [27] [34]. The main difference is we allow for $\alpha_k$ to be 1 for a finite number of iterations. Fix $x^* \in X^*$ and define $\varphi_k = \frac{1}{2}\|x^k - x^*\|^2$. Now

$$\varphi_k - \varphi_{k+1} = \frac{1}{2}\|\Delta_{k+1}\|^2 + \langle x^{k+1} - y^{k+1}, x^* - x^{k+1} \rangle + \alpha_k \langle \Delta_k, x^* - x^{k+1} \rangle. \quad (4.4)$$

Since

$$-x^{k+1} + y^{k+1} - \lambda_k \nabla f(y^{k+1}) \in \lambda_k \partial_{x_k} g(x^{k+1}),$$

$$-\lambda_k \nabla f(x^*) \in \lambda_k \partial_{x_k} g(x^*) \quad \text{and} \quad \langle \partial_{x_k} g(x^{k+1}) - \partial_{x_k} g(x^*), x^{k+1} - x^* \rangle \geq -\epsilon,$$ it follows that

$$\langle x^{k+1} - y^{k+1} + \lambda_k (\nabla f(y^{k+1}) - \nabla f(x^*)), x^* - x^{k+1} \rangle \geq -\lambda_k \epsilon_k. \quad (4.5)$$

Combining (4.4) and (4.5) we obtain

$$\varphi_k - \varphi_{k+1} \geq \frac{1}{2}\|\Delta_{k+1}\|^2 + \lambda_k \langle \nabla f(y^{k+1}) - \nabla f(x^*), x^{k+1} - x^* \rangle - \alpha_k \langle \Delta_k, x^* - x^{k+1} \rangle - \lambda_k \epsilon_k. \quad (4.6)$$

Now

$$\langle \Delta_k, x^{k+1} - x^* \rangle = \langle \Delta_k, x^k - x^* \rangle + \langle \Delta_k, \Delta_{k+1} \rangle$$

$$= \varphi_k - \varphi_{k-1} + \frac{1}{2}\|\Delta_k\|^2 + \langle \Delta_k, \Delta_{k+1} \rangle. \quad (4.7)$$

Combining (4.6) and (4.7) yields

$$\varphi_{k+1} - \varphi_k - \alpha_k (\varphi_k - \varphi_{k-1}) \leq -\frac{1}{2}\|\Delta_{k+1}\|^2 + \alpha_k \langle \Delta_k, \Delta_{k+1} \rangle + \frac{\alpha_k}{2}\|\Delta_k\|^2$$

$$-\lambda_k \langle \nabla f(y^{k+1}) - \nabla f(x^*), x^{k+1} - x^* \rangle$$

$$+ \lambda_k \epsilon_k. \quad (4.8)$$

Now we use the co-coercivity of $\nabla f$ as follows. Inequality (2.3) implies

$$\lambda_k \langle \nabla f(y^{k+1}) - \nabla f(x^*), x^{k+1} - x^* \rangle = \lambda_k \langle \nabla f(y^{k+1}) - \nabla f(x^*), y^{k+1} - x^* \rangle$$

$$+ \langle \nabla f(y^{k+1} - \nabla f(x^*), x^{k+1} - y^{k+1} \rangle$$

$$\geq \frac{\lambda_k}{L} \|\nabla f(y^{k+1}) - \nabla f(x^*)\|^2$$

$$+ \langle \nabla f(y^{k+1} - \nabla f(x^*), x^{k+1} - y^{k+1} \rangle$$

$$\geq -\frac{\lambda_k L}{4}\|x^{k+1} - y^{k+1}\|^2 \quad (4.9)$$
where (4.9) follows by completing the square. Combining (4.8) and (4.9) we infer
\[
\varphi_{k+1} - \varphi_k - \alpha_k (\varphi_k - \varphi_{k-1}) \leq -\frac{1}{2} \|\Delta_{k+1}\|^2 + \alpha_k \langle \Delta_k, \Delta_{k+1} \rangle + \frac{\alpha_k}{2} \|\Delta_k\|^2 \\
+ \frac{\lambda_k L}{4} \|\Delta_{k+1} - \alpha_k \Delta_k\|^2 + \lambda_k \epsilon_k \\
= \frac{\alpha_k}{2} \left( \frac{\lambda L}{2} - 1 \right) \|\Delta_{k+1} - \Delta_k\|^2 \\
+ \left[ \alpha_k \left( 1 - \frac{\lambda L}{4} \right) + \frac{\lambda L}{4} \right] \|\Delta_k\|^2 \\
+ \frac{\lambda L}{4} \left( 1 - \alpha_k \right) + \frac{1}{2} \|\Delta_{k+1}\|^2 + \lambda_k \epsilon_k. \tag{4.10}
\]

Note that the coefficient of \(\|\Delta_{k+1} - \Delta_k\|^2\) is non-positive. Set \(\theta_k \triangleq \varphi_k - \varphi_{k-1}\) and
\[
\delta_k \triangleq \left[ \alpha_k \left( 1 - \frac{\lambda L}{4} \right) + \frac{\lambda L}{4} \right] \|\Delta_k\|^2 + \frac{\lambda L}{4} \left( 1 - \alpha_k \right) + \frac{1}{2} \|\Delta_{k+1}\|^2 + \lambda_k \epsilon_k \tag{4.11}
\]
and note that \(\sum_{k=1}^{\infty} \delta_k < \infty\).

The argument from now on is basically identical to [37] except we allow for sequences \(\alpha_k\) which are equal to 1 for a finite number of \(k\). Restate (4.10) as
\[
\theta_{k+1} \leq \alpha_k \theta_k + \delta_k \\
\leq \alpha_k [\theta_k] + + \delta_k. \tag{4.12}
\]
Since \(\limsup \alpha_k < 1\), there exists an integer \(K \geq 0\) and \(\overline{\sigma} \in [0, 1)\) such that \(\alpha_k \leq \overline{\sigma} < 1\) for all \(k > K\). This and (4.12) imply that, for \(k > K\)
\[
[\theta_{k+1}] \leq \overline{\sigma} [\theta_k] + + \delta_k.
\]

Thus for \(k > K\)
\[
[\theta_{K+1}] \leq \overline{\sigma} K \sum_{j=K} W_j + + \sum_{j=K}^{K} \overline{\sigma}^k \sum_{j=1}^k \delta_j.
\]

Careful examination of this expression yields
\[
\sum_{k=0}^{\infty} [\theta_{k+1}] \leq K \sum_{k=0}^{K} \delta_k + \frac{\overline{\sigma}^K}{1 - \overline{\sigma}} \left( [\theta_1] + + \sum_{k=K}^{\infty} \delta_k \right) < \infty. \tag{4.13}
\]

Set \(w_k \triangleq \varphi_k - \sum_{j=0}^{k} [\theta_j] +\). Since \(\varphi_k \geq 0\) and \(\sum [\phi_j] + < \infty\), \(w_k\) is bounded from below. \(w_k\) is non-increasing, therefore we have it converges. Therefore \(\varphi_k\) converges for every \(x^* \in X^*\). By invoking Opial’s Lemma, statement (vi) is established. \(\square\)

Theorem 4.1 does not apply to Beck and Teboulle’s [16] parameter choice because \(\alpha_k \to 1\). However the Theorem does apply if we make the following modification. Replace the momentum parameter sequence \(\{\alpha_k\}\) with \(\min(\alpha_k, \overline{\sigma})\) where \(\overline{\sigma} < 1\). This parameter choice satisfies the assumptions of the Theorem, and \(\overline{\sigma}\) can be chosen arbitrarily close to 1.

In the following Corollary, we use (4.2) to determine explicit bounds on \(\sum_k \|\Delta_k\|^2\) which will be useful in the analysis of Problem SO.
Corollary 4.2. Suppose that Assumption 1 holds. Assume \( \{\lambda_k\} \) is non-decreasing and satisfies \( 0 < \lambda_k \leq 1/L \) for all \( k \), \( \{\epsilon_k\} \) satisfies \( \epsilon_k \geq 0 \) for all \( k \) and \( \sum_k \epsilon_k < \infty \), there exists \( \bar{\alpha} \in [0,1) \) such that \( \{\alpha_k\} \) satisfies \( 0 \leq \alpha_k \leq \bar{\alpha} \) for all \( k \).

Then for the iterates of FISTA,

\[
\sum_{k=1}^{\infty} \|\Delta_k\|^2 \leq \frac{2}{2L(1-\bar{\alpha})-1} \left( F(x^1) - F^* + \frac{\alpha_1}{2\lambda_1} \|\Delta_1\|^2 + \sum_{k=1}^{\infty} \epsilon_k \right). \tag{4.14}
\]

If, in addition, there exists \( \alpha \in [0,\bar{\alpha}] \) such that \( \alpha_k \geq \alpha \) for all \( k \), then

\[
\sum_{k=1}^{\infty} \|\Delta_k\|^2 \leq \frac{2}{L^2 \alpha (1-\bar{\alpha})} \left( F(x^1) - F^* + \frac{\alpha_1}{2\lambda_1} \|\Delta_1\|^2 + \sum_{k=1}^{\infty} \epsilon_k \right). \tag{4.15}
\]

Proof. Inequality (4.2) implies

\[
F(x^1) - F^* + \frac{\alpha_1}{2\lambda_1} \|\Delta_1\|^2 + \sum_{k=1}^{\infty} \epsilon_k \geq \sum_{k=1}^{\infty} \frac{2 + \lambda_k (\alpha_k - 1) - \alpha_k - \alpha_{k+1}}{2\lambda_k} \|\Delta_{k+1}\|^2
\]

\[
\geq \sum_{k=1}^{\infty} \frac{2 - 2\bar{\alpha} - \lambda_k}{2\lambda_k} \|\Delta_{k+1}\|^2 \tag{4.16}
\]

\[
\geq \sum_{k=1}^{\infty} \frac{2L(1-\bar{\alpha})-1}{2} \|\Delta_{k+1}\|^2 \tag{4.17}
\]

which proves (4.14). To derive (4.16) we used the fact that \( 0 \leq \alpha_k \leq \bar{\alpha} \). To derive (4.17) we used the fact that \( \lambda L \leq 1 \).

Inequality (4.2) also implies

\[
F(x^1) - F^* + \frac{\alpha_1}{2\lambda_1} \|\Delta_1\|^2 + \sum_{k=1}^{\infty} \epsilon_k \geq \sum_{k=1}^{\infty} \frac{L\alpha_k (1 - \alpha_k)}{2\lambda_k} \|\Delta_k\|^2
\]

\[
\geq \sum_{k=1}^{\infty} \frac{L^2 \alpha (1-\bar{\alpha})}{2} \|\Delta_k\|^2
\]

which proves (4.15). \( \square \)

5. Convergence Analysis of FISTA for Sparse Optimization.

5.1. Finite Convergence Results. We now turn our attention to Problem SO.

The following theorem proves finite convergence to 0 for the components in \( D \), and finite convergence to the correct sign for the components in \( E \) (recall the definitions of \( D \) and \( E \) in Section 2.4). Following the terminology of [28], we will refer to this as the “finite manifold identification period.” The manifold in this case is the half-space of vectors with support equal to \( E \) and non-zero components with sign \( -h_i^*/\rho \). This theorem generalizes the result of Theorem 4.5 in [17] from ISTA to FISTA.

Theorem 5.1. Suppose that Assumption SO holds. Assume \( \{\lambda_k\} \) is non-decreasing and satisfies \( 0 < \lambda_k \leq 1/L \), and there exists \( \alpha, \bar{\alpha} \in [0,1) \) such that \( \{\alpha_k\} \) satisfies \( \alpha \leq \alpha_k \leq \bar{\alpha} \) for all \( k \). Then, there exist constants \( K_D > 0 \) and \( K_E > 0 \) such that the iterates of FISTA applied to Problem SO satisfy, for all \( k > K_E \),

\[
\text{sgn} \left( y_i^k - \lambda_k \nabla f(y^k)_i \right) = -\frac{h_i^*}{\rho}, \quad \forall i \in E, \tag{5.1}
\]
and, for all $k > K_D$ 

$$x_i^k = y_i^k = 0, \forall i \in D. \quad (5.2)$$

Furthermore, $K_E \leq \overline{K}_E$ and $K_D \leq \overline{K}_D$, where 

$$\overline{K}_E \triangleq \frac{1}{\rho^2 \lambda^2} \left[ \frac{2\alpha(1+\alpha) \left( F(x^1) - F^* + \frac{\alpha_1}{2\lambda_1^2} \|\Delta_1\|^2 \right)}{\alpha(1-\alpha)L^2} + \|x^1 - x^*\|^2 - \|x^0 - x^*\|^2 \right] + \frac{\alpha}{1-\alpha} \quad (5.3)$$

and 

$$\overline{K}_D \triangleq \frac{1}{\omega^2 \lambda^2} \left[ \frac{2\alpha(1+\alpha) \left( F(x^1) - F^* + \frac{\alpha_1}{2\lambda_1^2} \|\Delta_1\|^2 \right)}{\alpha(1-\alpha)L^2} + \|x^1 - x^*\|^2 - \|x^0 - x^*\|^2 \right] + \frac{\alpha}{1-\alpha} + 2 \quad (5.4)$$

for any $x^* \in X^*$.

**Proof.** Note that this parameter choice satisfies the requirements of Theorem 4.1 and Corollary 4.2. Furthermore, by assumption, $X^*$ is non-empty and $F^* \geq -\infty$, thus all conclusions of Theorem 4.1 and Corollary 4.2 hold. Throughout the proof, fix an arbitrary $x^* \in X^*$.

**Proof of (5.1).** Fix a $\lambda > 0$. Recall from Theorem 2.1 there exists a vector $h^*$ such that $\nabla f(x^*) = h^*$ for all $x^* \in X^*$, and that $\text{supp}(x^*) \subset E$. For $i \in \text{supp}(x^*)$,

$$0 \neq x_i^* = \text{sgn}(x_i^* - \lambda h_i^*) \left[ |x_i^* - \lambda h_i^*| - \rho \lambda \right]. \quad (5.5)$$

Therefore $|x_i^* - \lambda h_i^*| > \rho \lambda$ for all $i \in \text{supp}(x^*)$. On the other hand, if $i \in E \setminus \text{supp}(x^*)$, then

$$|x_i^* - \lambda h_i^*| = \lambda |h_i^*| = \rho \lambda.$$

Therefore

$$|x_i^* - \lambda h_i^*| \geq \rho \lambda, \forall i \in E.$$ 

Looking at (5.5) it can be seen that 

$$\text{sgn}(x_i^*) = \text{sgn}(x_i^* - \lambda h_i^*), \forall i \in \text{supp}(x^*). \quad (5.6)$$

Note by Theorem 2.1 if $i \in \text{supp}(x^*)$, then $\text{sgn}(x_i^*) = -h_i^*/\rho$. Else if $i \in E \setminus \text{supp}(x^*)$ then 

$$\text{sgn}(x_i^* - \lambda h_i^*) = \text{sgn}(-\lambda h_i^*) = -\text{sgn}(h_i^*) = -\frac{h_i^*}{\rho}. \quad (5.7)$$

Combining (5.6) and (5.7) yields 

$$\text{sgn}(x_i^* - \lambda h_i^*) = -\frac{h_i^*}{\rho} \quad \forall i \in E, \lambda > 0. \quad (5.8)$$
Let $\nu_k = \rho \lambda_k$. If
\[
\text{sgn} \left( y_i^{k+1} - \lambda_k \nabla f(y^{k+1}) \right) \neq \text{sgn}(x_i^* - \lambda_k h_i^*) = -\nu_i^*/\rho \quad \text{for some } i \in E,
\] (5.9)
then Lemma 2.3 implies
\[
|x_i^{k+1} - x_i^*|^2 = |S_{\nu_k} \circ (y_i^{k+1} - \lambda_k \nabla f(y^{k+1})) - S_{\nu_k} \circ (x_i^* - \lambda_k h_i^*)|^2 \\
\leq \left| (y_i^{k+1} - \lambda_k \nabla f(y^{k+1})) - (x_i^* - \lambda_k h_i^*) \right|^2 \\
\leq \left| y_i^{k+1} - \lambda_k \nabla f(y^{k+1}) - (x_i^* - \lambda_k h_i^*) \right|^2 - \nu_k^2
\] (5.10)
where (5.10) follows because
\[
|\langle y_i^{k+1} - \lambda_k \nabla f(y^{k+1}) \rangle - (x_i^* - \lambda_k h_i^*)| > |(x_i^* - \lambda_k h_i^*)| \geq \nu_k > 0.
\]
Using (5.10) we can say the following: Condition (5.9) implies that
\[
\|x^{k+1} - x^*\|^2 = \sum_{j \neq i} |x_j^{k+1} - x_j^*|^2 + |x_i^{k+1} - x_i^*|^2 \\
\leq \sum_{j \neq i} \left| y_j^{k+1} - \lambda_k \nabla f(y^{k+1}) - (x_j^* - \lambda_k h_j^*) \right| \\
+ \left| y_i^{k+1} - \lambda_k \nabla f(y^{k+1}) - (x_i^* - \lambda_k h_i^*) \right|^2 - \nu_k^2
\leq \| y^{k+1} - \lambda_k \nabla f(y^{k+1}) - (x^* - \lambda_k h^*) \|^2 - \nu_k^2 \\
\leq \| y^{k+1} - x^* \|^2 - \nu_k^2
\] (5.11)
\[
= \| x^k + \alpha_k \Delta_k - x^* \|^2 - \nu_1^2 \\
= \| x^k - x^* \|^2 + \alpha_k^2 \| \Delta_k \|^2 + 2\alpha_k \langle \Delta_k, x^k - x^* \rangle - \nu_1^2.
\] (5.12)
Inequality (5.11) follows from the element-wise non-expansiveness of $S_{\nu}$ along with (5.10). To deduce (5.12), we used Lemma 2.2. Finally, (5.13) follows because $\{\lambda_k\}$ is non-decreasing and therefore so is $\{\nu_k\}$.

Recall the definition of $\varphi_k \triangleq \frac{1}{2} \| x^k - x^* \|^2$ and $\theta_k \triangleq \varphi_k - \varphi_{k-1}$. Now, moving $\langle \Delta_k, \Delta_{k+1} \rangle$ to the other side of (4.17) reveals
\[
\langle \Delta_k, x^k - x^* \rangle = \varphi_k - \varphi_{k-1} + \frac{1}{2} \| \Delta_k \|^2.
\] (5.14)
Substituting (5.14) into (5.13) yields
\[
2(\varphi_{k+1} - \varphi_k) \leq 2\alpha_k (\varphi_k - \varphi_{k-1}) + \sigma(1 + \sigma) \| \Delta_k \|^2 - \nu_1^2,
\]
therefore
\[
\theta_{k+1} \leq \alpha_k \theta_k + \frac{\sigma(1 + \sigma)}{2} \| \Delta_k \|^2 - \frac{\nu_1^2}{2}.
\]
Repeating the arguments that led to (4.13), we can say the following: if (5.9) is true then
\[
\theta_{k+1} \leq \sigma \theta_1 + \frac{\sigma(1 + \sigma)}{2} \sum_{j=1}^{k} \alpha^{k-j} \| \Delta_j \|^2 - \frac{\nu_1^2}{2} \sum_{j=1}^{k} \alpha^{k-j}.
\] (5.15)
Therefore, for $M \in \mathbb{Z}_+$, if (5.9) holds at iteration $M$, then

$$
\varphi_M - \varphi_0 = \sum_{k=1}^{M} \theta_k \\
\leq \frac{\theta_1 (1 - \alpha^M)}{1 - \alpha} + \frac{\bar{\alpha} (1 + \bar{\alpha})}{2 (1 - \alpha)} \sum_{k=1}^{M} \|\Delta_k\|^2 - \nu \frac{\sum_{k=1}^{M} \sum_{j=0}^{k-1} \alpha^j}{2} \\
\leq \frac{\theta_1}{1 - \alpha} + \frac{\bar{\alpha} (1 + \bar{\alpha})}{2 (1 - \alpha)} M \|\Delta_k\|^2 - \nu \frac{\sum_{k=1}^{M} \sum_{j=0}^{k-1} \alpha^j}{2} \left( \frac{M}{1 - \alpha} - \frac{\alpha}{(1 - \alpha)^2} \right). \tag{5.16}
$$

To derive (5.16) we lower bounded the coefficient of $-\frac{\nu^2}{2}$. Since $\varphi_k \geq 0$, if (5.9) is true at iteration $k$ then

$$
k \leq \frac{2 (1 - \alpha)}{\nu_1^2} \left[ \frac{\bar{\alpha} (1 + \bar{\alpha})}{2 (1 - \alpha)} \sum_{k=1}^{\infty} \|\Delta_k\|^2 + \|x_0 - x^*\|^2 \right] \right] + \frac{\alpha}{1 - \alpha} \\
\leq \frac{1}{\nu_1^2} \left[ \frac{2 \bar{\alpha} (1 + \bar{\alpha})}{\alpha (1 - \alpha) L^2} \left( m(x^*) - m + \frac{\alpha}{2 \lambda^2} \|\Delta_1\|^2 \right) \right] + \|x_1 - x^*\|^2 - \bar{\alpha} \|x_0 - x^*\|^2 \right] \\
\leq \frac{\alpha}{1 - \alpha} \tag{5.17}
$$

To derive (5.18) we used the upper bound on $\sum_k \|\Delta_k\|^2$ in (4.15) from Corollary 4.2. This upper bound is tighter than the other upper bound for $\sum_k \|\Delta_k\|^2$ given in (4.14), so long as $L > 2 / \alpha$.

**Proof of (5.2).** Recall the definition of $\omega$ and note that

$$
\lambda_k \omega = \min \{ \nu_k - \|x_i - \lambda_k h_i^*\| : i \in D \} > 0 \tag{5.19}
$$

Consider $i \in D$ (which implies $i \notin \text{supp}(x^*)$). If $x_i^{k+1} \neq 0$, then Lemma 2.3 implies

$$
|x_i^{k+1}|^2 = |S_{\nu_k} (y_i^{k+1} - \lambda_k \nabla f(y_i^{k+1})) - S_{\nu_k} (x_i - \lambda_k h_i^*)|^2 \\
\leq |(y_i^{k+1} - \lambda_k \nabla f(y_i^{k+1})) - (x_i - \lambda_k h_i^*)|^2 \\
\leq |(y_i^{k+1} - \lambda_k \nabla f(y_i^{k+1})) - (x_i - \lambda_k h_i^*)|^2 \\
\leq |(y_i^{k+1} - \lambda_k \nabla f(y_i^{k+1})) - (x_i - \lambda_k h_i^*)|^2 - \omega^2 \lambda_k^2. \tag{5.21}
$$

To derive (5.20) we used the fact that

$$
|(y_i^{k+1} - \lambda_k \nabla f(y_i^{k+1})) + \lambda_k h_i^*| > \nu_k - \lambda_k |h_i^*|.
$$

To derive (5.21) we used (5.19). Repeating the arguments used to prove (5.13) we can say the following. If there exists $i \in D$, such that $x_i^{k+1} \neq 0$, then $k \leq K_D - 2$, with $K_D$ defined in (5.4). Therefore $|x_i^k| = 0$ for all $k > K_D - 2$. Since $y_i^{k+1} = x_i^k + \alpha_k (x_i^k - x_i^{k-1})$, $y_i^k = 0$ for all $i \in D$ and $k > K_D - 2 + 2 = K_D$, which proves (5.2).

We can recover the result by Hale et al. for ISTA (Theorem 4.5 [17]). To see this, consider (5.17) with $\bar{\alpha} = \alpha = 0$ and then use the upper bound on $\sum_k \|\Delta_k\|^2$ given in (4.14) of Corollary 4.2. Note that we defined the constant $\omega$ in a slightly differently way to Hale et al.
5.2. Alternative Conditions for Finite Convergence. We can prove convergence to the optimal manifold in a finite number of iterations under more general conditions than required in Theorem 5.1, however without explicit bounds on the number of iterations. A corollary of the following theorem is that the parameter choices proposed by Beck and Teboulle [16] and Chambolle and Dossal [27] achieve finite manifold identification.

**Theorem 5.2.** Suppose that Assumption SO holds. Assume \( \lambda_k \) is non-decreasing and satisfies \( 0 < \lambda_k < 2/L \) for all \( k \), and there exists \( \overline{\alpha} \geq 0 \) such that \( \{\alpha_k\} \) satisfies

\[
0 \leq \alpha_k \leq \overline{\alpha},
\]

for all \( k \). If, for the iterates of FISTA applied to Problem SO, it is true that

\[
\sum_{k=1}^{\infty} \|\Delta_k\|_2^2 < \infty
\]

and

\[
\|x_k - x^*\|_2^2
\]

is bounded for some \( x^* \in X^* \) and for all \( k \), then there exists a constant \( K > 0 \) such that (5.1) and (5.2) hold for all \( k > K \).

**Proof.** Inequality (5.13) and the equivalent recursion for when (5.9) holds can be proved in exactly the same way. However, we cannot rely on \( \alpha < 1 \), so we have to modify the proof from that point onwards. Once again, fix \( x^* \in X^* \). Rewriting (5.13), we can say that (5.9) implies that

\[
\|x_{k+1} - x^*\|_2^2 \leq \|x_k - x^*\|_2^2 + 2\overline{\alpha}^2 \|\Delta_k\|_2^2 + 2\overline{\alpha} \langle \Delta_k, x_k - x^* \rangle - \nu_1^2 \leq \|x_k - x^*\|_2^2 + \overline{\alpha}^2 \|\Delta_k\|_2^2 + 2\overline{\alpha}M_1 \|\Delta_k\|_2 - \nu_1^2 \tag{5.22}
\]

where we used the Cauchy-Schwarz inequality to get (5.22). To derive (5.23) we used the assumption that there exists \( M_1 > 0 \) such that

\[
\|x_k - x^*\|_2^2 < M_1.
\]

Now, for (5.23)

\[
\sum_{k=0}^{\infty} \|\Delta_k\|_2^2 < \infty
\]

Substituting (5.24) and (5.25) into (5.23) yields the following: If (5.9) is true then

\[
\|x_{k+1} - x^*\|_2^2 \leq \|x_0 - x^*\|_2^2 + \overline{\alpha}^2 \sum_{k=0}^{\infty} \|\Delta_k\|_2^2 + 2\overline{\alpha}M_1 \|\Delta_k\|_2 - \nu_1^2 \tag{5.26}
\]

The r.h.s. of (5.26) can be non-negative for only a finite number of iterations, which proves (5.1).

For (5.2), the recursion is

\[
\|x_{k+1} - x^*\|_2^2 \leq \|x_k - x^*\|_2^2 + \overline{\alpha}^2 \|\Delta_k\|_2^2 + 2\overline{\alpha} \langle \Delta_k, x_k - x^* \rangle - \omega^2 \lambda_1^2
\]

and the reasoning is the same from this point onwards.

The classical parameter choice in (3.2) due to Beck and Teboulle [16], along with others which guarantee the \( O(1/k^2) \) rate, satisfy the assumptions of Theorem 5.2. The first condition, \( \sum_k \|\Delta_k\|_2^2 < \infty \), can be shown by considering the following facts. The sequence \( b_k \) defined on page 196 of [16] is bounded, which implies the sequence \( u_k \) defined on page
194 of [10] is also bounded. If $F$ is coercive, then $x^k$ is bounded, since $F(x^k) \rightarrow F^\ast$. This implies $\|x^k - x^\ast\|$ is bounded for some $x^\ast \in X^\ast$. It also implies $t_k \Delta_k$ is bounded and since $t_k = O(k)$, $\|\Delta_k\| = O(1/k)$, and the result follows.2

The parameter choice (3.3) due to Chambolle and Dossal [27] satisfies the assumptions of this theorem, even when $F$ is not assumed to be coercive. $\sum_k \|\Delta_k\|^2$ is finite by Corollary 2 of [27] and $\|x^k - x^\ast\|$ is shown to be bounded for all $x^\ast \in X^\ast$ in the proof of Theorem 3 of [27]. (In fact Chambolle and Dossal proved that $\sum_k k\|\Delta_k\|^2$ is finite for their parameter choice.)

5.3. Finite Reduction to Local Minimization. Theorems 5.1 and 5.2 allow us to characterize the behavior of FISTA after a finite manifold identification period. In the following corollary, we show that after a finite number of iterations, FISTA reduces to minimizing a smooth function over $E$ subject to an orthant constraint. The following corollary generalizes the result of Corollary 4.6 in [17] from ISTA to FISTA.

Corollary 5.3. Suppose that Assumption SO holds. Assume $\{\lambda_k\}$ is non-decreasing and satisfies $0 < \lambda_k \leq 1/L$, and there exists $\underline{\alpha}, \overline{\alpha} \in (0, 1)$ such that $\underline{\alpha} \leq \overline{\alpha}$ and $\{\alpha_k\}$ satisfies $\underline{\alpha} \leq \alpha_k \leq \overline{\alpha}$ for all $k$. Then, after finitely many iterations, the iterates of FISTA applied to Problem SO become equivalent to the iterates of FISTA applied to minimizing $\phi : \mathbb{R}^{|E|} \rightarrow \mathbb{R}$, where

$$
\phi(x_E) \triangleq -(h^*_E)^T x_E + f((x_E, 0)), \quad (5.27)
$$

constrained to the orthant $O_E$, where

$$
O_E \triangleq \{x_E \in \mathbb{R}^{|E|} : -\text{sgn}(h^*_x) x_i \geq 0, \forall i \in E\}. \quad (5.28)
$$

Specifically, there exists $K > 0$ such that for all $k > K$,

$$
y^{k+1}_E = x^k_E + \alpha_k (x^k_E - x^{k-1}_E), \quad (5.29)
$$

$$
x^{k+1}_E = P_{O_E} \left( y^{k+1}_E - \lambda_k \nabla \phi(y^{k+1}_E) \right), \quad (5.30)
$$

$x^0_D = y^0_D = 0$, and $F(x^k) = \phi(x^k_E)$. Furthermore $K \leq \max\{K_D, K_E\}$, with $K_D$ and $K_E$ defined in [3.3] and [5.4].

Alternatively, if the conditions of Theorem 5.2 hold, then there exists $K' > 0$ such that (5.29)-(5.30) hold, $x^k_D = y^k_D = 0$, and $F(x^k) = \phi(x^k_E)$, for all $k > K'$.

Proof. From Theorem 5.1, there exists a $K$ such that for all $k > K$, (5.1) and (5.2) hold and $K \leq \max\{K_D, K_E\}$. Take $k > K$. Since $x^k_i = 0$ for all $i \in D$ it suffices to consider $i \in E$. For $i \in E$, $k > K$, using (5.1) we have

$$
x^k_i \geq 0 \text{ if } \text{sgn} \left( y^{k+1}_E - \lambda_k \nabla f(y^{k+1}_E) \right)_i = 1 \text{ equivalently } h^*_i < 0
$$

and

$$
x^k_i \leq 0 \text{ if } \text{sgn} \left( y^{k+1}_E - \lambda_k \nabla f(y^{k+1}_E) \right)_i = -1 \text{ equivalently } h^*_i > 0.
$$

Therefore for any $i \in E$, $-h^*_i x^k_i \geq 0$, thus $x^k_E \in O_E$ for all $k > K$. Next note that, for all $i \in E$, $-h^*_i x^k_i = \rho |x_i|$. Therefore for $k > K$, $-(h^*_i)^T x^k_E = \rho \|x^k\|_1$, thus $F(x^k) = \phi(x^k_E)$.

2We thank Antonin Chambolle and Charles Dossal for pointing this out to us.
Now for \( i \in E, k > K \), we calculate the quantity
\[
\begin{align*}
z^{k+1}_i &= y^{k+1}_i - \lambda_k \nabla \phi (y^{k+1}_i) \\
&= y^{k+1}_i - \lambda_k (-h_i^* + \nabla f (y^{k+1}_i)) \\
&= y^{k+1}_i - \lambda_k \nabla f (y^{k+1}_i) + \rho \lambda_k (h_i^*)/\rho \\
&= \text{sgn} (y^{k+1}_i - \lambda_k \nabla f (y^{k+1}_i)) (|y^{k+1}_i - \lambda_k \nabla f (y^{k+1}_i)| - \rho \lambda_k).
\end{align*}
\]
Therefore, for \( i \in E, k > K \),
\[
\begin{align*}
y^{k+1}_i &= x^k_i + \alpha_k (x^k_i - x^{k-1}_i), \\
x^{k+1}_i &= S_{\rho \lambda_k} (y^{k+1}_i - \lambda_k \nabla f (y^{k+1}_i)) = \left\{ \begin{array}{ll} z^{k+1}_i : -h_i^* z^{k+1}_i \geq 0 \\ 0 : \text{else} \end{array} \right. \\
&= \left\{ \begin{array}{ll} z^{k+1}_i : -h_i^* z^{k+1}_i \geq 0 \\ 0 : \text{else} \end{array} \right.
\end{align*}
\]
Equivalently, for \( k > K \),
\[
\begin{align*}
y^{k+1}_E &= x^k_E + \alpha_k (x^k_E - x^{k-1}_E), \\
x^{k+1}_E &= P_{\lambda E} (y^{k+1}_E - \lambda_k \nabla \phi (y^{k+1}_E)).
\end{align*}
\]
Due to Theorem 5.2, the same arguments hold for parameter choices such that \( \sum_k \| \Delta_k \|^2 \) is finite, \( \| x^k - x^* \| \) is bounded for all \( k \) and some \( x^* \in X^* \), and \( \alpha_k \) is bounded. However there is no explicit upper bound on \( K \).

In principle one could switch to minimizing \( \phi \) directly once the algorithm has reduced to (5.29)-(5.30). This would allow for a larger step-size, since the Lipschitz constant of \( \nabla \phi \) is less than \( L \). However it is not possible to know with certainty that the algorithm has transitioned to the form (5.29)-(3.3) unless the number of iterations exceeds the upper bound \( \max \{ K_D, K_E \} \), although we discuss some heuristics for identifying this transition in Section 5.5. The main drawback of this strategy is that once it switches to minimizing \( \phi \) directly the support of \( x^k \) is fixed. Therefore any mismatch between \( \text{supp}(x^k) \) and \( \text{supp}(x^*) \) is not identified and the algorithm will not necessarily converge to an optimal point. In the next section we discuss a method which uses the optimal momentum for minimizing \( \phi \) yet continues to use a smaller step-size and is therefore guaranteed to converge to a minimizer.

### 5.4. A Simple Locally Optimal Parameter Choice for FISTA
The analysis of the previous three sections shows that, after a finite number of iterations, FISTA (subject to parameter conditions) reduces to minimizing the function \( \phi \) subject to an orthant constraint. Even though \( f \) is not assumed to be strongly convex, \( \phi \) might be. If this function is strongly convex, the asymptotic rate of convergence can be determined by the worst-case condition number of the Hessian. Throughout this section let \( H_{EE}(v) \) be the Hessian of \( \phi \) evaluated at \( v \). In terms of strategies for choosing \( \{ \alpha_k \} \) and \( \{ \lambda_k \} \), one approach is to choose them to obtain the best iteration complexity for minimizing \( \phi \). In the following Corollary, we provide a simple fixed choice which does this and thus optimizes the asymptotic iteration complexity.

**Corollary 5.4.** Suppose that Assumption SO holds, and \( \phi \) is strongly convex. Let \( x^* \) be the unique minimizer of Problem SO and \( l_E \) be the strong convexity parameter of \( \phi \). If \( \lambda \in (0, 1/L) \),
\[
\lambda_k = \lambda \quad \text{and} \quad \alpha_k = \frac{1 - \sqrt{l_E \lambda}}{1 + \sqrt{l_E \lambda}} \quad \forall k \in \mathbb{Z}_+,
\]
(5.31)
then the iterates \( \{x^k\} \) of FISTA converge to \( x^* \) linearly and \( F(x^k) \) converges to \( F^* \) linearly. Indeed

\[
F(x^k) - F^* = O \left( \left( 1 - \sqrt{\frac{1}{L_E \lambda}} \right)^k \right).
\]

(5.32)

**Proof.** The analysis of the previous sections shows that, for the given choice of \( \{\alpha_k\} \) and \( \{\lambda_k\} \) there exists a \( K \) such that, for all \( k > K \), (5.29) and (5.30) hold, and \( F(x^k) = \phi(x^k_E) \). Thus for \( k > K \) the algorithm is equivalent to Nesterov’s method applied to the strongly convex function \( \phi \) subject to the orthant constraint \( O_E \). Therefore we apply Theorem 2.2.1 of [25] with the fixed parameter choice (constant scheme 3, discussed on p. 76 of [25]). The only difference compared to Theorem 2.2.1 is that we allow for step-sizes other than \( 1/L_E \), where \( L_E \) is the Lipschitz constant of \( \nabla \phi \). Note that \( L \geq L_E \). This minor change is discussed on p. 72 of [25]. Setting \( \lambda = 1/L \) (the maximum allowed step-size) gives:

\[
F(x^k) - F^* = O \left( \left( 1 - \sqrt{\frac{1}{L_E \lambda}} \right)^k \right).
\]

Another minor issue to note is that the minimization is constrained to the simple convex set \( O_E \). This does not affect the convergence of Nesterov’s method, as discussed in Constant Step Scheme (2.2.17) of [25].

By the strong convexity of \( \phi \), the sequence \( \{x^k\} \) also achieves linear convergence with the same iteration complexity.

The iteration complexity with this parameter choice is

\[
\Omega \left( \sqrt{\frac{L}{L_E \lambda}} \log \left( \frac{1}{\epsilon} \right) \right).
\]

(5.33)

This is the best asymptotic iteration complexity that can be achieved by FISTA using this step-size [25]. Therefore we will refer to it as the locally optimal choice. Indeed it is better than the iteration complexity of ISTA [17] (which corresponds to FISTA with \( \alpha_k \) equal to 0) which is

\[
\Omega \left( \frac{L}{L_E \lambda} \log \left( \frac{1}{\epsilon} \right) \right).
\]

We will see in the next section that (5.33) is better than the iteration complexity achieved by using the parameter choices of [16] and [27].

In practice the optimal momentum is not known a priori as it depends on the smallest eigenvalue of \( H_{EE} \). The momentum could be estimated periodically based on the smallest eigenvalue of the Hessian corresponding to the current support set, or adapted based on the behavior of \( F(x^k) \) (see Section 5.6 and Section 6).

The authors of [32] proposed an adaptive momentum restart scheme for Nesterov’s method in the case of smooth optimization (i.e. \( g(x) = 0 \)). Corollary 5.3 implies that the scheme can also be used for FISTA applied to Problem SO. This follows because FISTA with parameter choice (3.2) reduces to minimizing a smooth function \( \phi \) after a finite number of iterations, after which the momentum restart scheme can be used. Referring to the analysis of [32], it can be shown that the method will have the same iteration complexity as given in (5.33).
Local linear convergence can also be proved when the local function $\phi$ is not strongly convex, but the limit point of the iterations obeys the strict-complementarity condition $E = \text{supp}(x^*)$, and the Hessian matrix of $\phi$ is invariant in a region containing the limit. In the following corollary, let $x^* = \lim_{k \to \infty} x^k$, which is in $X^*$ by Theorem 4.1.

**Corollary 5.5.** Suppose Assumption SO holds and $E = \text{supp}(x^*)$, where $\lim_{k \to \infty} x^k = x^* \in X^*$. Let $H_{EE}(x)$ be the Hessian of the function $\phi$ defined in (5.27). Let $l_E$ be the smallest non-zero eigenvalue of $H_{EE}(x^*_E)$. Assume the range space of $H_{EE}$ is invariant in some neighborhood $N^*$ around $x^*$. If all eigenvalues of $H_{EE}(x^*_E)$ are zero, $x^k = x^*$ after a finite number of iterations, for any choice of $\{\lambda_k, \alpha_k\}$ satisfying the conditions of Theorem 5.1 or Theorem 5.2. If $l_E > 0$, $\lambda \in (0, 1/L]$, 

$$\lambda_k = \lambda \text{ and } \alpha_k = \frac{1 - \sqrt{l_E \lambda}}{1 + \sqrt{l_E \lambda}} \forall k \in \mathbb{Z}_+,$$

(5.34)

then $x^k$ converges to $x^*$ linearly and $F(x^k)$ converges to $F^*$ linearly. Indeed

$$F(x^k) - F^* = O \left( \left(1 - \sqrt{l_E \lambda}\right)^k \right).$$

**Proof.**

The proof proceeds almost identically to Theorem 4.11 of [17]. Note that if $x^k \to x^*$ than $y^k \to x^*$. Now Lemma 5.3 of [17] can be directly applied to FISTA to say that, after a finite number of iterations,

$$x_i^{k+1} = y_i^{k+1} - \lambda_k (\nabla f(y^{k+1})_i - h_i) \quad \forall i \in \text{supp}(x^*).$$

(5.35)

Assume $k$ is large enough that (5.35) holds, $x^k \in N^*$, and $k > \max\{K_D, K_E\}$. Since $x_i^k$ is 0 for all $i \in D$, it suffices to consider $i \in E = \text{supp}(x^*)$. Recall that $H(x)$ is the Hessian of $f$ evaluated at $x$. Now, let $\mathcal{H}^k$ be defined as

$$\mathcal{H}^k \triangleq \int_0^1 H(x^* + t(x^k - x^*))dt$$

By assumption the range spaces of $\mathcal{H}^k$ are now invariant over $k$. For a matrix $W$, let $W_{EE}$ be the $|E| \times |E|$ submatrix of $W$ with row and column indices given by $E$. Let $P$ be the orthogonal projection onto the range space of $H_{EE}$. Since $E = \text{supp}(x^*)$, equation (5.35) can be used to claim that

$$x_E^{k+1} = y_E^{k+1} - \lambda (\nabla f(y^{k+1})_E - h^*_E) = y_E^{k+1} - \lambda \mathcal{H}^k_{EE}(y_E^{k+1} - x_E^*).$$

(5.36)

This follows from the mean value theorem. At each iteration the term $-\lambda \mathcal{H}^k_{EE}(y_E^{k+1} - x_E^*)$ stays in the range space of $H_{EE}$, which implies that the null-space components of the iterates have already converged. In other words, for $k$ sufficiently large,

$$(I - P)(x_E^k - x_E^*) = 0.$$ 

If $l_E > 0$, it suffices to consider the convergence of $\{P x^k\}$, that is, consider the component in the range space of $H_{EE}$. Since $\phi$ restricted to the range space of $H_{EE}$ is strongly convex, we now simply repeat the arguments of Corollary 5.3 and the result follows. \qed
5.5. “Underdamped” FISTA. In [32], the behavior of FISTA for minimizing strongly convex and differentiable functions was investigated. It was shown that for such functions if $\alpha_k \approx 1$, the algorithm moves into what is known as an “under-damped regime”, which leads to oscillations in the objective function at a predictable frequency, and a sub-optimal iteration complexity. The results of the preceding sections allow us to extend this analysis to Problem SO, despite it being non-smooth and not strictly convex.

The analysis of [32] revealed that if $\alpha_k \approx 1$, the trace of the objective function values will oscillate with a frequency proportional to $1/\sqrt{\kappa}$ where $\kappa$ is the condition number of the Hessian at the minimum. The iteration complexity of the method with step-size $\lambda = 1/L$ is

$$\Omega\left(\kappa \log \left(\frac{1}{\epsilon}\right)\right)$$

More precisely, the behavior $F(x^k) \approx C(1 - \frac{1}{\kappa})^k \cos^2(k/\sqrt{\kappa})$ is observed. Now Corollary 5.3 shows that after a finite number of iterations, FISTA (with parametric constraints) reduces to minimizing $\phi$ (defined in (5.27)) subject to an orthant constraint. Therefore we can apply the analysis of [32] once the algorithm is in this regime. Thus if $l_E > 0$, FISTA obeys the conditions of Theorem 5.1 or 5.2 and $\alpha_k \approx 1$, then the iteration complexity will be $\Omega\left(\frac{L}{l_E} \log \left(\frac{1}{\epsilon}\right)\right)$, which is worse than the iteration complexity achieved with the locally optimal choice given in Corollary 5.4. The trace of the objective function will exhibit oscillations with period $\sqrt{l_E/L}$. This result directly applies to the classical parameter choice of Beck and Teboulle in [32], and to Chambolle and Pock’s choice in (3.3), as well as to other classical choices of the form $\alpha_k = 1 - O(1/k^2)$. The result also applies when $l_E = 0$ under the strict-complementarity condition. In this case replace $l_E$ with $\hat{l}_E$ defined in Corollary 5.5.

5.6. An Adaptive Modification. In numerical experiments we have noticed that it can take many iterations for the optimal manifold to be identified by FISTA. This means that Beck and Teboulle’s choice [16] can outperform FISTA with our locally optimal choice [5.31] before the optimal manifold is identified. This is because Beck and Teboulle’s choice guarantees $O(1/k^2)$ convergence during this phase whereas the locally optimal choice does not have a guaranteed rate until the optimal manifold is identified. However, as was elucidated the analysis of the previous section, Beck and Teboulle’s choice has poor performance once the algorithm is in the optimal manifold. Beck and Teboulle’s choice has excellent global properties but poor local properties.

In light of this we propose the following adaptive heuristic. We use the condition, $F(x^k) > F(x^{k-1})$, as an indication the algorithm is at least approximately operating in the optimal manifold. This is because with Beck and Teboulle’s choice the algorithm will eventually converge to the optimal manifold and then the function values will start to oscillate. So the adaptive modification is the following. Use Beck and Teboulle’s parameter choice of (3.2) until $F(x^k) > F(x^{k-1})$. For all iterations after, use the locally optimal momentum given in (5.31). We call this scheme FISTA-AdOPT. See Experiment 2 for empirical results.
It is worth mentioning that it is better to use the condition \((y^{k+1} - x^{k+1})^T (x^{k+1} - x^k) > 0\) rather than \(F(x^k) > F(x^{k-1})\). It was shown in \[32\] that the two conditions are equivalent however the first avoids computation of \(F\).

Another option is to use the adaptive restart scheme proposed in \[32\]. Thanks to our analysis, this method is guaranteed to achieve the same asymptotic iteration complexity as the locally optimal parameter choice, but will also achieve the \(O(1/k^2)\) performance in the transient regime prior to manifold identification. The practical performance of the momentum restart scheme and FISTA-AdOPT appear to be similar (see next section).

5.7. The Splitting Inertial Proximal Method. In \[37\], Moudafi and Oliny introduced the Splitting Inertial Proximal Method (SIPM):

\[
x^{k+1} \in -\lambda_k \partial g(x^{k+1}) + x^k - \lambda_k \nabla f(x^k) + \alpha_k(x^k - x^{k-1}). \tag{5.37}
\]

It is a direct generalization of Polyak’s heavy-ball with friction method to problems involving the sum of two functions. It differs from FISTA in that the gradient is computed at \(x^k\) rather than the extrapolated point \(x^k + \alpha_k(x^k - x^{k-1})\). Our analysis of FISTA in the case of sparse optimization can be extended easily to Moudafi and Oliny’s method under the condition that \(\sum_k \|\Delta_k\|^2\) is finite.

**Theorem 5.6.** Suppose that Assumption SO holds. Assume \(0 < \lambda_k < 2/L\) for all \(k\), and there exists \(\overline{\alpha} \geq 0\) such that \(0 \leq \alpha_k \leq \overline{\alpha}\) for all \(k\). If \(\sum_k \|\Delta_k\|^2 < \infty\), and \(\|x^k - x^*\|\) is bounded for all \(x^* \in X^*\), then there exists a constant \(K > 0\) such that, for all \(k > K\) the iterates of Algorithm (5.37) applied to Problem SO satisfy

\[
\text{sgn}(x^k_i - \lambda_k \nabla f(x^k)_i + \alpha_k (x^k_i - x_i^{k-1})) = -\frac{h^*_i}{\rho}, \quad \forall i \in E,
\]

and

\[
x^k_i = 0, \quad \forall i \in D.
\]

**Proof.** The proof follows in a similar way to Theorems 5.1 and 5.2. Equation (5.22) is proved by following similar arguments as in the proof of Theorem 5.1. We include the salient differences.

Recall that \(\nu_k = \rho \lambda_k\). If

\[
\text{sgn}(x^k_i - \lambda_k \nabla f(x^k)_i + \alpha_k (x^k_i - x_i^{k-1})) \neq \text{sgn}(x^*_i - \lambda_k h^*_i) = -h^*_i / \rho
\]

for some \(i \in E\),

\[
|y^k_i - x^*_i|^2 = |S_{\nu_k} \circ (x^k_i - \lambda_k \nabla f(x^k)_i + \alpha_k (x^k_i - x_i^{k-1})) - S_{\nu_k} \circ (x^*_i - \lambda_k h^*_i)|^2
\]

\[
\leq |x^k_i - \lambda_k \nabla f(x^k)_i + \alpha_k (x^k_i - x_i^{k-1}) - (x^*_i - \lambda_k h^*_i)|^2 - \nu_k^2 \tag{5.39}
\]

where (5.39) follows for the same reasons as given for (5.10). Continuing to follow the
proof of Theorem 5.1 we say the following: if (5.38) holds, then
\[
\|x^{k+1} - x^*\|^2 = \sum \limits_{j \neq i} |x_j^{k+1} - x^*|^2 + |x_i^{k+1} - x^*|^2 \\
\leq \sum \limits_{j \neq i} |x_j^k - \lambda_k \nabla f(x^k)_j + \alpha_k (x_j^k - x_j^{k-1}) - (x_j^* - \lambda_k h_j^*)|^2 \\
+ |x_i^k - \lambda_k \nabla f(x^k)_i + \alpha_k (x_i^k - x_i^{k-1}) - (x_i^* - \lambda_k h_i^*)|^2 \\
- \nu_k^2 \\
\leq \|x^k - \lambda_k \nabla f(x^k) + \alpha_k \Delta_k - (x^* - \lambda_k h^*)\|^2 - \nu_k^2 \\
\leq \|x^k - x^*\|^2 + \bar{\pi}^2 \|\Delta_k\|^2 + 2\bar{\pi} \|x^k - x^*\|\|\Delta_k\| \\
(5.40)
\]

Equation (5.40) follows from the element-wise non-expansiveness of \(S_\nu\) and (5.39), and (5.41) follows from the non-expansiveness of \(I - \lambda \nabla f\), by application of Cauchy-Schwartz, and by substituting the upper bound \(\bar{\pi}\) for \(\alpha_k\). Equation (5.41) is identical to (5.22). From this point on, the proof is identical to Theorem 5.2. As before we cannot explicitly bound the number of iterations unless we know an upper bound for \(\sum_k \Delta_k^2\).

Moudafi and Oliny provided a condition on the parameters under which
\[
\sum \limits_k \|\Delta_k\|^2 < \infty,
\]
and \(\|x^k - x^*\|\) is bounded for all \(x^* \in X^*\). Choose the sequence \(\{\alpha_k\}\) to be non-decreasing, the constant \(\bar{\pi} < 1/3\), and the sequence \(\{\lambda_k\}\) to be bounded away from \(2/L\). They showed that this implies \(\|x^k - x^*\|\) is bounded for all \(k\) and \(x^* \in X^*\) and
\[
\sum \limits_k \alpha_k \|\Delta_k\|^2 < \infty.
\]

Now using the non-decreasing assumption on \(\{\alpha_k\}\), \(\sum_k \|\Delta_k\|^2 < \infty\) so long as \(\alpha_1\) non-zero. If \(\alpha_k\) is zero for all \(k\) the theorem follows from [17] since the algorithm reduces to ISTA.

We have proved that finite convergence in sign on \(E\) and to 0 on \(D\) holds for SIPM applied to Problem SO. An analogous result to Corollary 5.3 can now be shown. After a finite number of iterations, SIPM reduces to HBF projected onto a quadrant. Because of its similarity to Corollary 5.3 the proof is left to the reader.

**Corollary 5.7.** Assume \(0 < \lambda_k < 2/L\) for all \(k\), and there exists \(\bar{\pi} \geq 0\) such that \(0 \leq \alpha_k \leq \bar{\pi}\) for all \(k\). If \(\sum_k \|\Delta_k\|^2 < \infty\), and \(\|x^k - x^*\|\) is bounded for all \(x^* \in X^*\), then there exists a \(K > 0\) such that for \(k > K\), the iterates of SIPM satisfy
\[
x_{E_k+1} = P_{O_E} (x_k^E - \lambda_k \nabla \phi(x_k^E) + \alpha_k (x_k^E - x_k^{E-1})) \\
x_k^D = 0 \\
F(x_k) = \phi(x_k^E)
\]
where \(\phi\) and \(O_E\) are defined in (5.27) and (5.28) respectively.

Since SIPM reduces to projected HBF, parameter choices could be made to optimize the performance of HBF on the optimal manifold. Such computations have been carried out elsewhere [33] for the special case of Problem \(l_1\)-LS. For strongly convex quadratic functions, HBF obtains linear convergence with a \(\sqrt{\kappa}\)-times lower rate than
gradent descent, where $\kappa$ is the condition number of the Hessian at the minimum. For Problem $\ell_1$-LS, when $l_E > 0$ the optimal asymptotic iteration complexity of HBF turns out to be equal to that of FISTA with our optimal choice given in \((5.33)\). However we can only guarantee local linear convergence for non-decreasing choices of $\alpha_k$ in the range $[0, \bar{\alpha}]$ with $\bar{\alpha}$ less than $1/3$. So the locally optimal value of the momentum must be less than $1/3$ for this to hold. Otherwise convergence is linear with a worse iteration complexity \([33]\).

6. Numerical Simulations. We now compare several choices of parameters for FISTA applied to a random instance of Problem $\ell_1$-LS. To compute $E$ and thus find the locally optimal parameter choice given in \((5.31)\), we use the interior point solver of \([38]\) to find a solution to a target duality gap of $10^{-8}$. We then compute $h^* = \nabla f(x^*)$, and approximate the set $E$ by the set of all entries such that $\rho - |h^*_i|$ is smaller than $10^{-4}$. We also use the interior point solver to find an estimate of $F^*$. Recall that $l_E$ denotes the smallest eigenvalue of $A_E^T A_E$ and note that $l_E$ was greater than 0. The parameter choices under consideration will be referred to by the following designations.

- **FISTA-OPT.** FISTA with our locally optimal parameters derived in Section \(5.4\) given in \((5.31)\), with $\lambda = 1/L$. Recall that this choice optimizes the local convergence rate. Note that this is not a practically implementable algorithm as it depends on the optimal momentum. We include it to verify the theory of Section \(5\).
- **ISTA-OPT \([17]\).** ISTA with parameters chosen to optimize the local convergence rate. Corresponds to FISTA with $\alpha_k = 0$ and step-size $\lambda_k = 2/(L+l_E)$. As with FISTA-OPT this step-size cannot be known in practice as it depends on $l_E$. Using $\lambda_k = 1/L$ is common in practice but gives worse convergence rate.
- **FISTA-BT \([16]\).** FISTA with Beck and Teboulle’s parameters given in \((3.2)\).
- **FISTA-AdOPT.** Our proposal. FISTA with Beck and Teboulle’s choice until $F(x^k) > F(x^{k-1})$ (equivalently $(y^{k+1} - x^{k+1})^T(x^{k+1} - x^k) > 0$), FISTA-OPT for all iterations thereafter. We estimate the locally optimal momentum by using $\text{supp} (x^k)$ as a surrogate for $E$, since $\text{supp}(x^k) \approx E$ for $k$ large enough. We compute the smallest eigenvalue of $A_{\text{supp}(x^k)}^T A_{\text{supp}(x^k)}$ one time when $F(x^k) > F(x^{k-1})$ and compute the locally optimal momentum using \((5.31)\). We use this fixed value of the momentum from then on. This algorithm is practically implementable.
- **FISTA-AdRe \([32]\).** Adaptive momentum restart for FISTA. Beck and Teboulle’s parameter choice given in \((3.2)\) except set $t_k$ to 0 whenever $F(x^k) > F(x^{k-1})$ (equivalently $(y^{k+1} - x^{k+1})^T(x^{k+1} - x^k) > 0$). Our analysis shows this method achieves the optimal asymptotic iteration complexity derived in Corollary \(5.3\) (See the remarks after Corollary \(5.4\)).

All algorithms are initialized to $x^1 = x^0 = 0$.

**Experiment Details.** We create a random instance of Problem $\ell_1$-LS, with $A$ of size $300 \times 2000$ and with entries drawn i.i.d. from $N(0, 0.01)$. The vector $b$ is $A x_0$ with $x_0$ being 50-sparse having non-zero entries drawn i.i.d. $N(0, 1)$. The regularization parameter $\rho$ is set to 1.

The results are shown in Fig. \(6.4\). Both FISTA-AdOPT and FISTA-AdRe inherit the $O(1/k^2)$ convergence rate of FISTA-BT during the transient period. However they also have the optimal asymptotic rate of FISTA-OPT. FISTA-BT begins to oscillate once the optimal manifold is identified and the iteration complexity is worse than
FISTA-OPT, as predicted in Section 5.5. The performances of FISTA-AdOPT and FISTA-AdRe are similar. FISTA-AdOPT has the advantage that it will not oscillate like FISTA-AdRe, which has to be continually reset once the momentum exceeds the optimal value. However FISTA-AdOPT requires computation of an estimate of the locally optimal momentum which depends on smallest eigenvalue of $A$ restricted to the support.

![Graph](image)

**Fig. 6.1.** Experiment results: showing $F(x_k) - F^*$ versus iteration $k$ for several algorithms.

**7. Conclusions.** In this paper, we applied a Lyapunov analysis to FISTA. We have proved weak convergence under the following broad conditions with the standard assumptions on the objective function: for the momentum parameter, $0 \leq \alpha_k \leq 1$ for all $k$ and $\limsup \alpha_k < 1$, and for the step-size, non-decreasing and $0 < \lambda_k \leq 1/L$. These conditions are more general than the specific sequences studied in [16] and [27] and less restrictive than the conditions derived in [34]. We considered in detail the behavior of FISTA applied to sparse optimization problems. With the aid of some results from the Lyapunov analysis we were able to show that FISTA achieves local linear convergence for these problems, with finite convergence of certain quantities.

An interesting direction of future research is to see if this local linear convergence behavior holds for a more general class of problems satisfying certain properties such as partial smoothness and local strong convexity, as considered in [28] for the forward-backward algorithm. Another interesting question is whether the Lyapunov analysis can be extended to parameter choices where $\alpha_k \to 1$.

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**REFERENCES**

[1] Kwangmoo Koh, Seung-Jean Kim, and Stephen P Boyd, “An interior-point method for large-scale $\ell_1$-regularized logistic regression.,” *Journal of Machine Learning Research*, vol. 8, no. 8, pp. 1519–1555, 2007.
[2] Trevor Hastie, Robert Tibshirani, Jerome Friedman, T Hastie, J Friedman, and R Tibshirani, *The elements of statistical learning*, vol. 2, Springer, 2009.

[3] Emmanuel J. Candès and Terence Tao, “Decoding by linear programming,” *IEEE Transactions on Information Theory*, vol. 51, no. 12, pp. 4203–4215, 2005.

[4] David L Donoho, “Compressed sensing,” *Information Theory, IEEE Transactions on*, vol. 52, no. 4, pp. 1289–1306, 2006.

[5] Leonid I Rudin, Stanley Osher, and Emad Fatemi, “Nonlinear total variation based noise removal algorithms,” *Physica D: Nonlinear Phenomena*, vol. 60, no. 1, pp. 259–268, 1992.

[6] Antonin Chambolle, “An algorithm for total variation minimization and applications,” *Journal of Mathematical Imaging and Vision*, vol. 20, no. 1-2, pp. 89–97, 2004.

[7] Bradley Efron, Trevor Hastie, Iain Johnstone, Robert Tibshirani, et al., “Least angle regression,” *The Annals of Statistics*, vol. 32, no. 2, pp. 407–499, 2004.

[8] Cédric Vonesch and Michael Unser, “A fast multilevel algorithm for wavelet-regularized image restoration,” *Image Processing, IEEE Transactions on*, vol. 18, no. 3, pp. 509–523, 2009.

[9] Stanley Osher, Martin Burger, Donald Goldfarb, Jinjun Xu, and Wotao Yin, “An iterative regularization method for total variation-based image restoration,” *Multiscale Modeling & Simulation*, vol. 4, no. 2, pp. 460–489, 2005.

[10] Emmanuel J. Candès and Benjamin Recht, “Exact matrix completion via convex optimization,” *Found. Comput. Math.*, vol. 9, no. 6, pp. 717–772, Dec. 2009.

[11] Patrick L Combettes and Jean-Christophe Pesquet, “Proximal splitting methods in signal processing,” in *Fixed-Point Algorithms for Inverse Problems in Science and Engineering*, pp. 185–212, Springer, 2011.

[12] Pierre-Louis Lions and Bertrand Mercier, “Splitting algorithms for the sum of two nonlinear operators,” *SIAM Journal on Numerical Analysis*, vol. 16, no. 6, pp. 964–979, 1979.

[13] Gregory B Pessy, “Ergodic convergence to a zero of the sum of monotone operators in Hilbert space,” *Journal of Mathematical Analysis and Applications*, vol. 72, no. 2, pp. 383–390, 1979.

[14] Boris T Polyak, *Introduction to Optimization*, Optimization Software Inc., 1987.

[15] Neal Parikh and Stephen Boyd, “Proximal algorithms,” *Foundations and Trends in Optimization*, vol. 1, no. 3, pp. 123–231, 2013.

[16] Amir Beck and Marc Teboulle, “A fast iterative shrinkage-thresholding algorithm for linear inverse problems,” *SIAM J. Img. Sci.*, vol. 2, no. 1, pp. 183–202, Mar. 2009.

[17] Elaine T. Hale, Wotao Yin, and Yin Zhang, “Fixed-point continuation for $\ell_1$-minimization: methodology and convergence,” *SIAM J. on Optimization*, vol. 19, no. 3, pp. 1107–1130, Oct. 2008.

[18] Kristian Bredies and Dirk A Lorenz, “Linear convergence of iterative soft-thresholding,” *Journal of Fourier Analysis and Applications*, vol. 14, no. 5-6, pp. 813–837, 2008.

[19] Boushra Abbas and Hedy Attouch, “Dynamical systems and forward-backward algorithms associated with the sum of a convex subdifferential and a monotone cocoercive operator,” *arXiv preprint arXiv:1403.6512*, 2014.

[20] Felipe Alvarez, “On the minimizing property of a second order dissipative system in Hilbert spaces,” *SIAM Journal on Control and Optimization*, vol. 38, no. 4, pp. 1102–1119, 2000.

[21] Hédy Attouch, Jean Peypouquet, and Patrick Redont, “A dynamical approach to an inertial forward-backward algorithm for convex minimization,” *SIAM Journal on Optimization*, vol. 24, no. 1, pp. 232–256, 2014.

[22] Weijie Su, Stephen Boyd, and Emmanuel Candès, “A differential equation for modeling Nesterov accelerated gradient method: theory and insights,” in *Advances in Neural Information Processing Systems*, 2014, pp. 2510–2518.

[23] Patrick R Johnstone, “Inertial iterative thresholding with applications to sparse and low-rank signal recovery,” M.S. thesis, University of Illinois at Urbana-Champaign, USA, August 2014. Available at: http://hdl.handle.net/2142/50628.

[24] Boris Todorovich Polyak, “Some methods of speeding up the convergence of iteration methods,” *USSR Computational Mathematics and Mathematical Physics*, vol. 4, no. 5, pp. 1–17, 1964.

[25] Yuri Nesterov, *Introductory lectures on convex optimization: a basic course*, Springer, 2004.

[26] Felipe Alvarez and Hedy Attouch, “An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping,” *Set-Valued Analysis*, vol. 9, no. 1-2, pp. 3–11, 2001.

[27] Antonin Chambolle and Charles Dossal, “On the convergence of the iterates of FISTA,” *Preprint hal-01060130*, September 2014.

[28] Jingwei Liang, Jalal Fadili, and Gabriel Peyré, “Local linear convergence of forward–backward under partial smoothness,” in *Advances in Neural Information Processing Systems*, 2014,
pp. 1970–1978.

[29] Alekh Agarwal, Sahand Negahban, and Martin J Wainwright, “Fast global convergence rates of gradient methods for high-dimensional statistical recovery,” in Advances in Neural Information Processing Systems, 2010, pp. 37–45.

[30] Deren Han and Xiaoming Yuan, “Local linear convergence of the alternating direction method of multipliers for quadratic programs,” SIAM Journal on Numerical Analysis, vol. 51, no. 6, pp. 3446–3457, 2013.

[31] Laurent Demanet and Xiangxiong Zhang, “Eventual linear convergence of the Douglas Rachford iteration for basis pursuit,” arXiv preprint arXiv:1301.0542, 2013.

[32] Brendan O’Donoghue and Emmanuel Candes, “Adaptive restart for accelerated gradient schemes,” Foundations of Computational Mathematics, pp. 1–18, 2012.

[33] Patrick R Johnstone and Pierre Moulin, “Convergence of an inertial proximal method for L1-regularized least-squares,” in International Conference on Acoustics, Speech and Signal Processing (ICASSP), available at http://web.engr.illinois.edu/~prjohns2/pdfs/Johnstone.pdf. IEEE, April 2015.

[34] Dirk A. Lorenz and Thomas Pock, “An inertial forward-backward algorithm for monotone inclusions,” Journal of Mathematical Imaging and Vision, pp. 1–15, 2014.

[35] Qihang Lin and Lin Xiao, “An adaptive accelerated proximal gradient method and its homotopy continuation for sparse optimization,” in Proceedings of The 31st International Conference on Machine Learning, 2014, pp. 73–81.

[36] Zdzislaw Opial, “Weak convergence of the sequence of successive approximations for nonexpansive mappings,” Bulletin of the American Mathematical Society, vol. 73, no. 4, pp. 591–597, 1967.

[37] A. Moudafi and M. Oliny, “Convergence of a splitting inertial proximal method for monotone operators,” Journal of Computational and Applied Mathematics, vol. 155, no. 2, pp. 447–454, 2003.

[38] S-J Kim, Kwangmoo Koh, Michael Lustig, Stephen Boyd, and Dimitry Gorinevsky, “An interior-point method for large-scale l1-regularized least squares,” Selected Topics in Signal Processing, IEEE Journal of, vol. 1, no. 4, pp. 606–617, 2007.