DIFFERENCE EQUATIONS COMPATIBLE WITH TRIGONOMETRIC KZ DIFFERENTIAL EQUATIONS

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To the memory of Anatoly Izergin

Abstract. The trigonometric KZ equations associated with a Lie algebra $g$ depend on a parameter $\lambda \in h$ where $h \subset g$ is the Cartan subalgebra. We suggest a system of dynamical difference equations with respect to $\lambda$ compatible with the KZ equations. The dynamical equations are constructed in terms of intertwining operators of $g$-modules.

1. Introduction

The trigonometric KZ equations associated with a Lie algebra $g$ depend on a parameter $\lambda \in h$ where $h \subset g$ is the Cartan subalgebra. We suggest a system of dynamical difference equations with respect to $\lambda$ compatible with the trigonometric KZ differential equations. The dynamical equations are constructed in terms of intertwining operators of $g$-modules.

Our dynamical difference equations are a special example of the difference equations introduced by Cherednik. In [Ch1, Ch2] Cherednik introduces a notion of an affine R-matrix associated with the root system of a Lie algebra and taking values in an algebra $F$ with certain properties. Given an affine R-matrix, he defines a system of equations for an element of the algebra $F$.

In this paper we construct an example of an affine R-matrix and call the corresponding system of equations the dynamical equations. In our example, $F$ is the algebra of functions of complex variables $z_1, ..., z_n$ and $\lambda \in h$ taking values in the tensor product of

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n copies of the universal enveloping algebra of \( g \). The fact that our dynamical difference equations are compatible with the trigonometric KZ differential equations is a remarkable property of our affine R-matrix.

There is a similar construction of dynamical difference equations compatible with the qKZ difference equations associated with a quantum group. The dynamical difference equations in that case are constructed in the same way in terms of intertwining operators of modules over the quantum group. We will describe this construction in a forthcoming paper.

There is a degeneration of the trigonometric KZ differential equations to the standard (rational) KZ differential equations. Under this limiting procedure the dynamical difference equations constructed in this paper turn into the system of differential equations compatible with the standard KZ differential equations and described in [FMTV]. In [FMTV] we proved that the standard hypergeometric solutions of the standard KZ equations [SV, V] satisfy also the dynamic differential equations of [FMTV].

The trigonometric KZ differential equations also have hypergeometric solutions, see [Ch3, EFK]. We conjecture that the hypergeometric solutions of the trigonometric KZ differential equations also solve the dynamical difference equations of this paper.

In Section 2 we study relations between intertwining operators of \( g \)-modules and the Weyl group \( W \) of \( g \). For any finite dimensional \( g \)-module \( V \) and \( w \in W \) we construct a rational function \( B_{w,V} : \mathbb{C} \to \text{End}(V) \). The operators \( B_{w,V}(\lambda) \) are used later to construct an affine R-matrix and dynamical equations.

In Section 3 we define the dynamical difference equations for \( g = \mathfrak{sl}_N \) in terms of operators \( B_{w,V}(\lambda) \) directly (without introducing affine R-matrices). For \( g = \mathfrak{sl}_N \), we prove that the dynamical equations are compatible with the trigonometric KZ differential equations. We give a formula for the determinant of a square matrix solution of the combined system of KZ and dynamical equations.

In Section 4 we review [Ch1, Ch2] and construct the dynamical difference equations for any simple Lie algebra \( g \). We show that the dynamical equations are compatible with the trigonometric KZ equations if the Lie algebra \( g \) has minuscle weights, i.e. is not of type \( E_8, F_4, G_2 \). We conjecture that the dynamical difference equations and trigonometric KZ equations are compatible for any simple Lie algebra.

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2. Intertwining Operators

2.1. Preliminaries. Let \( g \) be a complex simple Lie algebra with root space decomposition \( g = \mathfrak{h} \oplus (\oplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha) \) where \( \Sigma \subset \mathfrak{h}^* \) is the set of roots.

Fix a system of simple roots \( \alpha_1, ..., \alpha_r \). Let \( \Gamma \) be the corresponding Dynkin diagram, and \( \Sigma_+ \) — the set of positive (negative) roots. Let \( \mathfrak{n}_\pm = \oplus_{\alpha \in \Sigma_\pm} \mathfrak{g}_\alpha \). Then \( g = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_- \).

Let \( ( , ) \) be an invariant bilinear form on \( g \). The form gives rise to a natural identification \( \mathfrak{h} \to \mathfrak{h}^* \). We use this identification and make no distinction between \( \mathfrak{h} \) and \( \mathfrak{h}^* \).
This identification allows us to define a scalar product on $\mathfrak{h}^*$. We use the same notation $(\ , \ )$ for the pairing $\mathfrak{h} \otimes \mathfrak{h}^* \to \mathbb{C}$.

We use the notation: $Q = \oplus_{i=1}^r \mathbb{Z} \alpha_i$ - root lattice; $Q^+ = \oplus_{i=1}^r \mathbb{Z}_{>0} \alpha_i$; $Q^\vee = \oplus_{i=1}^r \mathbb{Z} \alpha_i^\vee$ - dual root lattice, where $\alpha_i^\vee = 2\alpha_i/(\alpha, \alpha)$; $P = \{ \lambda \in \mathfrak{h} \mid (\lambda, \alpha_i^\vee) \in \mathbb{Z} \}$ - weight lattice; $P^+ = \{ \lambda \in \mathfrak{h} \mid (\lambda, \alpha_i^\vee) \in \mathbb{Z}_{>0} \}$ - cone of dominant integral weights; $\omega_i \in P^+$ - fundamental weights: $(\omega_i, \alpha_j^\vee) = \delta_{ij}$; $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} \alpha = \sum_{i=1}^r \omega_i$; $P^\vee = \oplus_{i=1}^r \mathbb{Z} \omega_i^\vee$ - dual weight lattice, where $\omega_i^\vee$ - dual fundamental weights: $(\omega_i^\vee, \alpha_j) = \delta_{ij}$.

Define a partial order on $\mathfrak{h}$ putting $\mu \prec \lambda$ if $\lambda - \mu \in Q^+$.

Let $s_i : \mathfrak{h} \to \mathfrak{h}$ denote a simple reflection, defined by $s_i(\lambda) = \lambda - (\alpha_i^\vee, \lambda) \alpha_i$; $\mathbb{W}$ - Weyl group, generated by $s_1, \ldots, s_r$. The following relations are defining:

$$s_i^2 = 1, \quad (s_is_j)^m = 1 \quad \text{for} \quad m = 2, 3, 4, 6,$$

where $m = 2$ if $\alpha_i$ and $\alpha_j$ are not neighboring in $\Gamma$, otherwise, $m = 3, 4, 6$ if 1,2,3 lines respectively connect $\alpha_i$ and $\alpha_j$ in $\Gamma$. For an element $w \in \mathbb{W}$, denote $l(w)$ the length of the minimal (reduced) presentation of $w$ as a product of generators $s_1, \ldots, s_r$.

Let $U\mathfrak{g}$ be the universal enveloping algebra of $\mathfrak{g}$; $U\mathfrak{g}^{\otimes n}$ - tensor product of $n$ copies of $U\mathfrak{g}$; $\Delta^{(n)} : U\mathfrak{g} \to U\mathfrak{g}^{\otimes n}$ - the iterated comultiplication (in particular, $\Delta^{(1)}$ is the identity, $\Delta^{(2)}$ is the comultiplication); $U\mathfrak{g}_0^{\otimes n} = \{ x \in U\mathfrak{g}^{\otimes n} \mid [\Delta^{(n)}(h), x] = 0 \text{ for any } h \in \mathfrak{h} \}$ - subalgebra of weight zero elements.

For $\alpha \in \Sigma$ choose generators $e_\alpha \in \mathfrak{g}_\alpha$ so that $(e_\alpha, e_{-\alpha}) = 1$. For any $\alpha$, the triple

$$H_\alpha = \alpha^\vee, \quad E_\alpha = \frac{2}{(\alpha, \alpha)} e_\alpha, \quad F_\alpha = e_{-\alpha}$$

forms an $sl_2$-subalgebra in $\mathfrak{g}$, $[H_\alpha, E_\alpha] = 2E_\alpha$, $[H_\alpha, F_\alpha] = -2F_\alpha$, $[E_\alpha, F_\alpha] = H_\alpha$.

A dual fundamental weight $\omega_i^\vee$ is called minuscule if $(\omega_i^\vee, \alpha) = 0$ for all $\alpha \in \Sigma^+$, i.e. for any positive root $\alpha = \sum_{i=1}^r m_i \alpha_i$, the coefficient $m_i$ is either 0 or 1. For a root system of type $A_r$ all dual fundamental weights are minuscule. There is no minuscule dual fundamental weight for $\mathfrak{g}_8, \mathfrak{g}_4, \mathfrak{g}_2$. For a minuscule dual fundamental weight $\omega_i^\vee$, define an element $w_0 = w_0w_0^i \in \mathbb{W}$ where $w_0$ (respectively, $w_0^i$) is the longest element in $\mathbb{W}$ (respectively, in $\mathbb{W}^i$) generated by all simple reflections $s_j$ preserving $\omega_i^\vee$.

**Lemma 1.** Let $\alpha$ be a positive root. Then $w_0(\alpha) \in \Sigma_+$ if $(\omega_i^\vee, \alpha) = 0$ and $w_0(\alpha) \in \Sigma_-$ if $(\omega_i^\vee, \alpha) = 1$.

Let $G$ be the simply connected complex Lie group with Lie algebra $\mathfrak{g}$, $\mathbb{H} \subset G$ the Cartan subgroup corresponding to $\mathfrak{h}$, $N(\mathbb{H}) = \{ x \in G \mid x\mathbb{H}x^{-1} = \mathbb{H} \}$ the normalizer of $\mathbb{H}$. Then the Weyl group is canonically isomorphic to $N(\mathbb{H})/\mathbb{H}$. The isomorphism sends $x$ to $Ad_{x}|_{\mathfrak{h}}$.

Let $V$ be a finite dimensional $\mathfrak{g}$-module with weight decomposition $V = \oplus_{\mu \in \mathfrak{h}^*} V[\mu]$. $G$ acts on $V$ so that $\mathbb{H}$ acts trivially on $V[0]$. Thus the action of $\mathbb{W}$ on $V[0]$ is well defined. For any $n$, the Weyl group in the same way acts also on $U\mathfrak{g}_0^{\otimes n}$. 
Lemma 2. For \( \alpha \in \Sigma \) and \( k \in \mathbb{Z}_{\geq 0} \), consider \( e^{\alpha}_{\alpha} e^{-\alpha} \in U\mathfrak{g}_0 \) and \( e_{\alpha} \otimes e_{-\alpha} \in U\mathfrak{g}_0^{\otimes 2} \). Then for any \( w \in \mathbb{W} \),

\[
w(e^{\alpha}_{\alpha} e^{-\alpha}) = e^{\alpha}_{w(\alpha)} e^{-\alpha}, \quad w(e_{\alpha} \otimes e_{-\alpha}) = e_{w(\alpha)} \otimes e_{-w(\alpha)}.
\]

Proof. Let \( x \in N(\mathbb{H}) \) be a lifting of \( w \). \( \text{Ad}_x : \mathfrak{g} \to \mathfrak{g} \) is an automorphism of \( \mathfrak{g} \) preserving the invariant scalar product and sending \( \mathfrak{g}_\beta \) to \( \mathfrak{g}_{w(\beta)} \) for all \( \beta \). Thus, \( \text{Ad}_x e_\beta = c_x e_{w(\beta)} \) for suitable numbers \( c_{x, \beta} \) and \( c_{x, \alpha} e_{x, \alpha} = 1 \).

Let \( x_1, \ldots, x_r \) be an orthonormal basis in \( \mathfrak{h} \), set

\[
\Omega^0 = \frac{1}{2} \sum_{i=1}^r x_i \otimes x_i, \quad \Omega^+ = \Omega^0 + \sum_{\alpha \in \Sigma^+} e_{\alpha} \otimes e_{-\alpha}, \quad \Omega^- = \Omega^0 + \sum_{\alpha \in \Sigma^+} e_{-\alpha} \otimes e_{\alpha}.
\]

Define the Casimir operator \( \Omega \) and the trigonometric R-matrix \( r(z) \) by

\[
\Omega = \Omega^+ + \Omega^-,
\]

\[
r(z) = \frac{\Omega^+ z + \Omega^-}{z - 1}.
\]

For any \( x \in U\mathfrak{g} \), we have \( \Delta(x) \Omega = \Omega \Delta(x) \). We will use a more symmetric form of the trigonometric R-matrix: \( r(z_1/z_2) \).

The Weyl group acts on \( r(z_1/z_2), \Omega \in U\mathfrak{g}_0^{\otimes 2} \). \( \Omega \) is Weyl invariant. For any \( w \in \mathbb{W} \),

\[
w(r(z_1/z_2)) = \frac{1}{z_1 - z_2} \left( \frac{z_1 + z_2}{2} \sum_{i=1}^r x_i \otimes x_i + \sum_{\alpha \in \Sigma^+} (z_1 e_{w(\alpha)} \otimes e_{-w(\alpha)} + z_2 e_{w(-\alpha)} \otimes e_{w(\alpha)}) \right).
\]

Lemma 3. For a minuscule dual fundamental weight \( \omega_i^\vee \),

\[
z_1(\omega_i^\vee)^{(1)} z_2(\omega_i^\vee)^{(2)} r(z_1/z_2) z_1(\omega_i^\vee)^{(1)} z_2(\omega_i^\vee)^{(2)} = w^{-1}_i(r(z_1/z_2)).
\]

Proof. Using Lemma [] it is easy to see that both sides of the equation are equal to

\[
\frac{1}{z_1 - z_2} \left( \frac{z_1 + z_2}{2} \sum_{i=1}^r x_i \otimes x_i + \sum_{\alpha \in \Sigma^+, (\alpha, \omega_i^\vee) = 0} (z_1 e_{\alpha} \otimes e_{-\alpha} + z_2 e_{-\alpha} \otimes e_{\alpha}) + \right.
\]

\[
\left. \sum_{\alpha \in \Sigma^+, (\alpha, \omega_i^\vee) = 1} (z_1 e_{\alpha} \otimes e_{-\alpha} + z_2 e_{-\alpha} \otimes e_{\alpha}) \right). \]

2.2. The Trigonometric KZ Equations. Let \( V = V_1 \otimes \ldots \otimes V_n \) be a tensor product of \( \mathfrak{g} \)-modules. For \( \kappa \in \mathbb{C} \) and \( \lambda \in \mathfrak{h} \), introduce the KZ operators \( \nabla_i(\lambda, \kappa) \), \( i = 1, \ldots, n \), acting on functions \( u(z_1, \ldots, z_n) \) of \( n \) complex variables with values in \( V \) and defined by

\[
\nabla_i(\lambda, \kappa) = \kappa z_i \frac{\partial}{\partial z_i} - \sum_{j, j \neq i} r(z_i/z_j)^{(i,j)} - \lambda^{(i)}.
\]

Here \( r^{(i,j)} \), \( \lambda^{(i)} \) denote \( r \) acting in the \( i \)-th and \( j \)-th factors of the tensor product and \( \lambda \) acting in the \( i \)-th factor.
The trigonometric KZ equations are the equations
\[ \nabla_i (\lambda, \kappa) u(z_1, \ldots, z_n, \lambda) = 0, \quad i = 1, \ldots, n, \]
see [EFK]. The KZ equations are compatible, \([\nabla_i, \nabla_j] = 0\).

2.3. Intertwining Operators, Fusion Matrices, [ES, EV]. For \( \lambda \in \mathfrak{h} \), let \( M_\lambda \) be the Verma module over \( \mathfrak{g} \) with highest weight \( \lambda \) and highest weight vector \( v_\lambda \). We have \( n_+ v_\lambda = 0 \), and \( h v_\lambda = (h, \lambda) v_\lambda \) for all \( h \in \mathfrak{h} \). Let \( M_\lambda = \bigoplus_{\mu \leq \lambda} M_\lambda [\mu] \) be the weight decomposition. The Verma module is irreducible for a generic \( n \). The universal fusion matrix \( \Phi_\lambda, \mu \) is a rational function of \( h, \lambda \) depending on \( \mathfrak{g} \) and \( \mathfrak{h} \). Let \( \mathfrak{h} \) be the lowest weight vector of \( M_\lambda \) and highest weight vector \( v_\lambda \). Define the dual Verma module \( M_\lambda^* \) to be the graded dual space \( \bigoplus_{\mu \leq \lambda} M_\lambda^*[\mu] \) equipped with the \( \mathfrak{g} \)-action: \( \langle u, av \rangle = -\langle au, v \rangle \) for all \( a \in \mathfrak{g} \), \( u \in M_\lambda \), \( v \in M_\lambda^* \). Let \( v_\lambda^* \) be the lowest weight vector of \( M_\lambda^* \) satisfying \( \langle v_\lambda^*, v_\lambda^* \rangle = 1 \).

Let \( V \) be a finite dimensional \( \mathfrak{g} \)-module with weight decomposition \( V = \bigoplus_{\mu \in \mathfrak{h}} V[\mu] \). For \( \lambda, \mu \in \mathfrak{h} \) consider an intertwining operator \( \Phi : M_\lambda \rightarrow M_{\mu \otimes V} \). Define its expectation value by \( \langle \Phi \rangle = \langle \Phi(v_\lambda), v_\mu^* \rangle \in V[\lambda - \mu] \). If \( M_{\mu} \) is irreducible, then the map \( \text{Hom}_\mathfrak{g}(M_\lambda, M_{\mu \otimes V}) \rightarrow V[\lambda - \mu], \Phi \mapsto \langle \Phi \rangle \), is an isomorphism. Thus for any \( v \in V[\lambda - \mu] \) there exists a unique intertwining operator \( \Phi_\lambda^v : M_\lambda \rightarrow M_{\mu \otimes V} \) such that \( \Phi_\lambda^v(v_\lambda) \in v_\lambda \otimes v + \bigoplus_{\nu < \mu} M_\mu [\nu] \otimes V \).

Let \( V, W \) be finite-dimensional \( \mathfrak{g} \)-modules and \( v \in V[\mu], \ w \in W[\nu] \). Consider the composition
\[ \Phi_\lambda^{w,v} : M_\lambda \xrightarrow{\Phi_\lambda^v} M_{\lambda - \mu} \otimes V \xrightarrow{\Phi_\lambda^{w,\nu}} M_{\lambda - \mu - \nu} \otimes W \otimes V. \]
Then \( \Phi_\lambda^{w,v} \in \text{Hom}_\mathfrak{g}(M_\lambda, M_{\lambda - \mu - \nu} \otimes W \otimes V) \). Hence, for a generic \( \lambda \) there exists a unique element \( u \in (V \otimes W)[\mu + \nu] \) such that \( \Phi_\lambda^u = \Phi_\lambda^{v,v} \). The assignment \( (w, v) \mapsto u \) is bilinear, and defines an \( \mathfrak{h} \)-linear map
\[ J_{WV}(\lambda) : W \otimes V \rightarrow W \otimes V. \]
The operator \( J_{WV}(\lambda) \) is called the fusion matrix of \( W \) and \( V \). The fusion matrix \( J_{WV}(\lambda) \) is a rational function of \( \lambda \). \( J_{WV}(\lambda) \) is strictly lower triangular, i.e. \( J = 1 + L \) where \( L(W[\nu] \otimes V[\mu]) \subset \bigoplus_{\tau < \nu, \mu < \sigma} W[\tau] \otimes V[\sigma] \). In particular, \( J_{WV}(\lambda) \) is invertible.

If \( V_1, \ldots, V_n \) are \( \mathfrak{h} \)-modules and \( F(\lambda) : V_1 \otimes \cdots \otimes V_n \rightarrow V_1 \otimes \cdots \otimes V_n \) is a linear operator depending on \( \lambda \in \mathfrak{h} \), then for any homogeneous \( u_1, \ldots, u_n, u_i \in V_i[\nu_i] \), we define \( F(\lambda - h_i)(u_1 \otimes \cdots \otimes u_n) = F(\lambda - v_i)(u_1 \otimes \cdots \otimes u_n) \).

There is a universal fusion matrix \( J(\lambda) \in U \frak{g} \otimes \mathbb{C}^2 \) such that \( J_{WV}(\lambda) = J(\lambda)|_{W \otimes V} \) for all \( W, V \). The universal fusion matrix \( J(\lambda) \) is the unique solution of the [ABR] equation
\[ J(\lambda) (1 \otimes (\lambda + \rho - \frac{1}{2} \sum_{i=1}^r x_i^2)) = (1 \otimes (\lambda + \rho - \frac{1}{2} \sum_{i=1}^r x_i^2)) + \sum_{\alpha \in \Sigma_+} e_{-\alpha} \otimes e_{\alpha} J(\lambda), \]
such that \( (J(\lambda) - 1) \in \mathfrak{b}_- (U \mathfrak{b}_-) \otimes (U \mathfrak{b}_+) \mathfrak{b}_+ \) where \( \mathfrak{b}_\pm = \mathfrak{h} \oplus \mathfrak{n}_\pm \).
We transform this equation to a more convenient form. The equation can be written as
\[
J(\lambda) (\lambda + \rho - \frac{1}{2} \sum_{i=1}^{r} x_i^2) = \left((\lambda + \rho - \frac{1}{2} \sum_{i=1}^{r} x_i^2) - \frac{1}{2} \sum_{i=1}^{r} x_i \otimes x_i + \Omega^-\right) J(\lambda).
\]
We make a change of variables: \( \lambda \mapsto \lambda - \rho + \frac{1}{2}(h^{(1)} + h^{(2)}) \). Then the equation takes the form
\[
J(\lambda - \rho + \frac{1}{2}(h^{(1)} + h^{(2)})) (\lambda + \frac{1}{2}(h^{(1)} + h^{(2)}) - \frac{1}{2} \sum_{i=1}^{r} x_i^2) = \left((\lambda + \frac{1}{2}(h^{(1)} + h^{(2)}) - \frac{1}{2} \sum_{i=1}^{r} x_i^2) - \frac{1}{2} \sum_{i=1}^{r} x_i \otimes x_i + \Omega^-\right) J(\lambda - \rho + \frac{1}{2}(h^{(1)} + h^{(2)})).
\]
Notice that \( (h^{(1)} + h^{(2)})^2 = \sum_{i=1}^{r} x_i^2 (x_i^{(1)} + x_i^{(2)}) \). Now the equation takes the form
\[
(2) \quad J(\lambda - \rho + \frac{1}{2}(h^{(1)} + h^{(2)})) (\lambda^2 + \Omega^0) = (\lambda^2 + \Omega^-) J(\lambda - \rho + \frac{1}{2}(h^{(1)} + h^{(2)})).
\]
For \( w \in W \), let \( w(J(\lambda)) \) be the image of \( J(\lambda) \) under the action of \( w \). Let \( x \in N(\mathfrak{h}) \) be a lifting of \( w \). Let \( W, V \) be finite dimensional \( \mathfrak{g} \)-modules. Then
\[
(3) \quad w(J(\lambda)|_{W \otimes V}) = xJ_{W, V}(\lambda)x^{-1},
\]
and RHS does not depend on the choice of \( x \).

2.4. **Main Construction, I.** Introduce a new action of the Weyl group \( W \) on \( \mathfrak{h} \) by
\[
w \cdot \lambda = w(\lambda + \rho) - \rho.
\]
Remind facts from [BGG].

Let \( M_\mu, M_\lambda \) be Verma modules. Two cases are possible: a) \( \text{Hom}_\mathfrak{g}(M_\mu, M_\lambda) = 0 \), b) \( \text{Hom}_\mathfrak{g}(M_\mu, M_\lambda) = \mathbb{C} \) and every nontrivial homomorphism \( M_\mu \to M_\lambda \) is an embedding.

Let \( M_\lambda \) be a Verma module with dominant weight \( \lambda \in P^+ \). Then \( \text{Hom}_\mathfrak{g}(M_\mu, M_\lambda) = \mathbb{C} \) if and only if there is \( w \in W \) such that \( \mu = w \cdot \lambda \).

Let \( w = s_{i_k} \ldots s_{i_1} \) be a reduced presentation. Set \( \alpha^1 = \alpha_{i_1} \) and \( \alpha^j = (s_{i_1} \ldots s_{i_{j-1}})(\alpha_{i_j}) \) for \( j = 2, \ldots, k \). Let \( n_j = (\lambda + \rho, (\alpha^j)^\vee) \). For a dominant \( \lambda \in P^+ \), \( n_j \) are positive integers.

**Lemma 4.** The collection of integers \( n_1, \ldots, n_k \) and the product \( (e_{-\alpha_{i_k}})^{n_k} \ldots (e_{-\alpha_{i_1}})^{n_1} \) do not depend on the reduced presentation.

**Proof.** It is known that \( \alpha^1, \ldots, \alpha^k \) are distinct positive roots and \( \{\alpha^1, \ldots, \alpha^k\} = \{\alpha \in \Sigma_+ \mid w(\alpha) \in \Sigma_- \} \). Hence, the collection \( n_1, \ldots, n_k \) does not depend on the reduced presentation.

The vector \( (e_{-\alpha_{i_k}})^{n_k} \ldots (e_{-\alpha_{i_1}})^{n_1} v_\lambda \) is a singular vector in \( M_\lambda \). If \( w = s_{i_k} \ldots s_{i_1} \) is another reduced presentation, then the vectors \( (e_{-\alpha_{i_k}})^{n_k} \ldots (e_{-\alpha_{i_1}})^{n_1} v_\lambda \) and \( (e_{-\alpha_{i_k}})^{n_k} \ldots (e_{-\alpha_{i_1}})^{n_1} v_\lambda \) are proportional. Since \( M_\lambda \) is a free \( \mathfrak{n}_- \)-module, we have
\[(e_{-\alpha'_2})^{n_2} \cdots (e_{-\alpha'_t})^{n_t} = c (e_{-\alpha_1})^{n_1} \cdots (e_{-\alpha_t})^{n_t}\text{ in } n_-\text{ for a suitable } c \in \mathbb{C}. c = 1\text{ since the monomials are equal when projected to the commutative polynomial algebra generated by } e_{-\alpha_1}, \ldots, e_{-\alpha_t}.\]

Define a singular vector \(v_{w,\lambda}^\lambda \in M_\lambda\) by
\[
(4) \quad v_{w,\lambda}^\lambda = \frac{(e_{-\alpha_1})^{n_1}}{n_1!} \cdots \frac{(e_{-\alpha_t})^{n_t}}{n_t!} v_\lambda.
\]

This vector does not depend on the reduced presentation by Lemma 4.

For all \(\lambda \in P^+\), \(w \in \mathbb{W}\), fix an embedding \(M_{w,\lambda} \hookrightarrow M_\lambda\) sending \(v_{w,\lambda}\) to \(v_{w,\lambda}^\lambda\).

Let \(V\) be a finite dimensional \(g\)-module, \(V = \oplus_{\nu \in \mathfrak{h}} V[\nu]\) the weight decomposition, \(P(V) = \{\nu \in \mathfrak{h} | V[\nu] \neq 0\}\) the set of weights of \(V\). We say that \(\lambda \in P^+\) is generic with respect to \(V\) if

I. For any \(\nu \in P(V)\) there exist a unique intertwining operator \(\Phi^\nu_\lambda : M_\lambda \rightarrow M_{\lambda-\nu} \otimes V\) such that \(\Phi^\nu_\lambda(v_\lambda) = v_{\lambda-\nu} \otimes v + \text{ lower order terms.}\)

II. For any \(w, w' \in \mathbb{W}\), \(w \neq w'\), and any \(\nu \in P(V)\), the vector \(w \cdot \lambda - w' \cdot (\lambda - \nu)\) does not belong to \(P(V)\).

It is clear that all dominant weights lying far inside the cone of dominant weights are generic with respect to \(V\).

Lemma 5. Let \(\lambda \in P^+\) be generic with respect to \(V\). Let \(v \in V[\nu]\). Consider the intertwining operator \(\Phi^\nu_\lambda : M_\lambda \rightarrow M_{\lambda-\nu} \otimes V\). For \(w \in \mathbb{W}\), consider the singular vector \(v_{w,\lambda}^\lambda \in M_\lambda\). Then there exists a unique vector \(A_{w,V}(\lambda)(v) \in V[w(\nu)]\) such that
\[
\Phi^\nu_\lambda(v_{w,\lambda}^\lambda) = v_{\lambda-\nu} \otimes A_{w,V}(\lambda)(v) + \text{ lower order terms.}
\]

Proof. \(\Phi^\nu_\lambda(v_{w,\lambda}^\lambda)\) is a singular vector in \(M_{\lambda-\nu} \otimes V\). It has to have weight components of the form \(v_{w,\lambda-\nu} \otimes u\) for suitable \(w' \in \mathbb{W}\) and \(u \in V\). Since \(\lambda\) is generic, we have \(w = w'\) and \(\Phi^\nu_\lambda(v_{w,\lambda}^\lambda)\) is of the required form for a suitable \(A_{w,V}(\lambda)(v) \in V[w(\nu)]\).

For generic \(\lambda \in P^+\), Lemma 5 defines a linear operator \(A_{w,V}(\lambda) : V \rightarrow V\) such that \(A_{w,V}(\lambda)(V[\nu]) \subset V[w(\nu)]\) for all \(\nu \in P(V)\). It follows from calculations in Section 2.3 that \(A_{w,V}(\lambda)\) is a rational function of \(\lambda\) in \(\mathfrak{h}\).

The following Lemmas are easy consequences of definitions.

Lemma 6. If \(w_1, w_2 \in \mathbb{W}\) and \(l(w_1 w_2) = l(w_1) + l(w_2)\), then
\[
A_{w_1 w_2,V}(\lambda) = A_{w_1,V}(w_2 \cdot \lambda) A_{w_2,V}(\lambda).
\]

Lemma 7. Let \(W, V\) be finite dimensional \(g\)-modules. Let \(w \in \mathbb{W}\). Then
\[
A_{w,W \otimes V}(\lambda) J_{W,V}(\lambda) = J_{W,V}(w \cdot \lambda)(A_{w,W}(\lambda - h^{(2)}) \otimes A_{w,V}(\lambda)).
\]

Let \(x_w \in N(\mathbb{H}) \subset G\) be a lifting of \(w \in \mathbb{W}\). For a finite dimensional \(g\)-module \(V\), define an operator
\[
B_{x_w,V}(\lambda) : V \rightarrow V, \quad v \mapsto x_w^{-1} A_{w,V}(\lambda)v.
\]
Lemma 7 implies
\[ B_{x_w, V} \] preserves the weight of elements of \( V \).

Let \( x_w \) be an element of \( \mathbb{N} \), another lifting of \( w \), then \( B_{x_w, V} \) depends on the choice of \( x_w \). If \( x_w g, g \in \mathbb{N} \), is another lifting of \( w \), then \( B_{x_w g, V} \) is a universal \( B_{w, V} \) corresponding to simple reflections.

2.5. Operators \( B_{x_w, V}(\lambda) \) for \( g = sl_2 \). Consider \( sl_2 \) with generators \( H, E, F \) and relations \( [H, E] = 2E, [H, F] = -2F, [E, F] = H \). Let \( \alpha_1 \) be the positive root. Identifying \( h \) and \( \mathfrak{h}^* \), we have \( \alpha_1 = \alpha_1^\vee = H, \omega_1 = \omega^\vee \) = \( H/2 \), \( \mathbb{W} = \{ 1, s_1 \} \).

Let \( \lambda = l \omega_1, l \in \mathbb{Z}_{\geq 0} \), be a dominant weight. Then \( s_1 \cdot \lambda = -(l + 2) \omega_1 \). For any dominant weight \( \lambda \), fix an embedding
\[ M_{s_1, \lambda} \hookrightarrow M_\lambda, \quad v_{s_1, \lambda} \mapsto v_{s_1, \lambda} = \frac{F(\lambda, \alpha_1) + 1}{((\lambda, \alpha_1) + 1)!}! \]
as in Section 2.4.

For \( m \in \mathbb{Z}_{\geq 0} \), let \( L_m \) be the irreducible \( sl_2 \) module with highest weight \( m \omega_1 \). \( L_m \) has a basis \( v_0^m, v_1^m, \ldots, v_m^m \) such that
\[ H v_k^m = (m - 2k) v_k^m, \quad F v_k^m = (k + 1) v_{k+1}^m, \quad E v_k^m = (m - k + 1) v_{k-1}^m. \]

For \( g = sl_2 \), we have \( G = SL(2, \mathbb{C}) \). Then \( \mathbb{H} \hookrightarrow G \) is the subgroup of diagonal matrices. Fix a lifting \( x \in N(\mathbb{H}) \) of \( s_1 \), set \( x = (x_{ij}) \) where \( x_{11} = x_{22} = 0, x_{12} = -1, x_{21} = 1 \). Then the action of \( x \) in \( L_m \) is given by \( v_k^m \mapsto (-1)^k v_m^{m-k} \) for any \( k \). We have \( x = \exp(-E) \exp(F) \exp(-E) \).

For \( t \in \mathbb{C} \), introduce
\[ p(t; H, E, F) = \sum_{k=0}^{\infty} F^k E^k \frac{1}{k!} \prod_{j=0}^{k-1} \frac{1}{(t - H - j)}. \]
p(\( t; H, E, F \)) is an element of \( U(sl_2)_0 \).

Theorem 8. Let \( \lambda \) be a dominant weight for \( sl_2 \). Let \( L_m \) be as above. Let \( B_{x, L_m}(\lambda) : L_m \to L_m \) be the operator defined in Section 2.4. Then for \( k = 0, \ldots, m \),
\[ B_{x, L_m}(\lambda) v_k^m = \frac{((\lambda, \alpha_1^\vee) + 2)((\lambda, \alpha_1^\vee) + 3) \cdots ((\lambda, \alpha_1^\vee) + k + 1)}{((\lambda, \alpha_1^\vee) - m + k + 1)((\lambda, \alpha_1^\vee) - m + k + 2) \cdots ((\lambda, \alpha_1^\vee) - m + 2k)} v_k^m. \]
and
\[
(7) \quad p((\lambda, \alpha_1^\vee); H, E, F)|_{L_m} = B_{x, L_m}(\lambda).
\]

**Corollary 9.** \( B_{x, L_m}(\lambda) \) is a rational function of \((\lambda, \alpha_1^\vee)\). \( B_{x, L_m}(\lambda) \) tends to 1 as \((\lambda, \alpha_1^\vee)\) tends to infinity.

The Theorem is proved by direct verification. First we calculate explicitly \( \Phi^{m}_{\lambda}(v_{\lambda}) \), \( \Phi^{m}_{\lambda}(\frac{F((\alpha_1^\vee)+1)}{(\alpha_1^\vee)+1}} v_{\lambda}) \), and then get an expression for \( B_{x, L_m}(\lambda)v_{k}^{m} \) as a sum of a hypergeometric type. Using standard formulas from [G&R], we see that \( B_{x, L_m}(\lambda)v_{k}^{m} \) is given by (8). Similarly we check that \( p((\lambda, \alpha_1^\vee); H, E, F)\) gives the same result. Thus we get (8). \( \square \)

Formula (8) becomes more symmetric if \( \lambda \) is replaced by \( \lambda - \rho + \frac{1}{2} \nu \) where \( \nu = m\omega_1 - k\alpha_1 \) is the weight of \( v_{k}^{m} \), then

\[
(8) \quad p((\lambda + \frac{1}{2}, \alpha_1^\vee; H, E, F)v_{k}^{m} = \prod_{j=0}^{k-1} \frac{(\lambda, \alpha_1^\vee) + \frac{m}{2} - j}{(\lambda, \alpha_1^\vee) - \frac{m}{2} + j} v_{k}^{m}.
\]

**Theorem 10.**

\[
p(-t - 2; -H, F, E) \cdot p(t; H, E, F)) = \frac{t - H + 1}{t + 1}.
\]

To prove this formula it is enough to check that RHS and LHS give the same result when applied to \( v_{k}^{m} \in L_m \), which is done using (8). \( \square \)

Notice that \( p(t; -H, F, E) = s_{1}(p(t; H, E, F)) \).

**Remark.** Let \( J(\lambda) = \sum_{i} a_{i} \otimes b_{i} \) be the universal fusion matrix of \( sl_{2} \). Following [EV2] introduce \( S(Q)(\lambda) \in U(sl_{2})_{0} \) as \( S(Q)(\lambda) = \sum_{i} S(a_{i})b_{i} \) where \( S(a_{i}) \) is the antipode of \( a_{i} \). The action of \( S(Q)(\lambda) \) in \( L_{m} \) was computed in [EV2]. Comparing the result with Theorem 3, one sees that \( p((\lambda, \alpha_1^\vee); H, E, F) \) is equal to \( (S(Q)(\lambda))^{-1} \) up to a simple change of argument \( \lambda \).

**Corollary 11.** Let \( A_{s_{1}, L_{m}}(\lambda) : L_{m} \to L_{m} \) be the operator defined in Section 2.4. Then

\[
A_{s_{1}, L_{m}}(\lambda) = x \cdot p((\lambda, \alpha_1^\vee); H, E, F)|_{L_m}. \quad A_{s_{1}, L_{m}}(\lambda) \text{ is a rational function of } (\lambda, \alpha_1^\vee). \quad A_{s_{1}, L_{m}}(\lambda) \text{ tends to } x \text{ as } (\lambda, \alpha_1^\vee) \text{ tends to infinity.}
\]

2.6. **Main Construction, II.** Return to the situation considered in Section 2.4.

For any simple root \( \alpha_{i} \), the triple \( H_{\alpha_{i}}, E_{\alpha_{i}}, F_{\alpha_{i}} \) defines an embedding \( sl_{2} \to g \) and induces an embedding \( SL(2, \mathbb{C}) \to G \). Denote \( x_{i} \in G \) the image under this embedding of the element \( x_{i} \in SL(2, \mathbb{C}) \) defined in Section 2.5.

**Lemma 12.** For \( i = 1, \ldots, r \), we have \( x_{i} \in N(\mathbb{R}) \) and \( Ad_{x_{i}} : g \to g \text{ restricted to } h \) is the simple reflection \( s_{i} : h \to h \).

**Proof.** Since \( x_{i} = \exp(-E_{\alpha_{i}}) \exp(F_{\alpha_{i}}) \exp(-E_{\alpha_{i}}) \), we have that \( Ad_{x_{i}}(H_{\alpha_{i}}) = -H_{\alpha_{i}} \) and \( Ad_{x_{i}}(h) = h \) for any \( h \in h \) orthogonal to \( \alpha_{i} \). Hence \( x_{i} \in N(\mathbb{R}) \) and \( Ad_{x_{i}}|_{h} = s_{i} \). \( \square \)
For $i = 1, \ldots, r$ and $\lambda \in \mathfrak{h}$, set
\[
B_{s_i}(\lambda) = p((\lambda, \alpha_i^\vee) ; H_{\alpha_i}, E_{\alpha_i}, F_{\alpha_i})
\]
where $p(t; H, E, F)$ is defined in (3). Set
\[
A_{s_i}(\lambda) = x_i B_{s_i}(\lambda).
\]

For any $\nu \in P(V)$, we have $A_{s_i}(\nu(V[\nu])) \subset V[s_i(\nu)]$.

Let $V$ be a finite dimensional $\mathfrak{g}$-module. For $w \in \mathbb{W}$, let $w = s_{i_k} \cdots s_{i_1}$ be a reduced presentation. For a generic dominant $\lambda \in P^+$, consider the operator $A_{w,V}(\lambda) : V \to V$ defined in Section 2.4.

**Lemma 13.**

\[
A_{w,V}(\lambda) = A_{s_{i_k}}((s_{i_{k-1}} \cdots s_{i_1}) \cdot \lambda)[V] A_{s_{i_{k-1}}}((s_{i_{k-2}} \cdots s_{i_1}) \cdot \lambda)[V] \cdots A_{s_{i_1}}(\lambda)[V].
\]

**Proof.** See Corollary 11 and Lemma 6. ~

**Corollary 14.** The operator $A_{w,V}(\lambda)$ is a rational function of $\lambda$. $A_{w,V}(\lambda)$ tends to $x_{i_k} \cdots x_{i_1}$ as $\lambda$ tends to infinity in a generic direction. In particular, the product $x_{i_k} \cdots x_{i_1}$ does not depend on the choice of the reduced presentation.

Set $x_w = x_{i_k} \cdots x_{i_1}$; $x_w \in N(\mathbb{H})$ is a lifting of $w$. Consider the operator $B_{x_w,V}(\lambda) : V \to V$ defined in Section 2.4 for this lifting $x_w$. Denote this operator $B_{w,V}(\lambda)$.

**Corollary 15.**

\[
B_{w,V}(\lambda) = (s_{i_{k-1}} \cdots s_{i_1})^{-1}(B_{s_{i_k}}((s_{i_{k-1}} \cdots s_{i_1}) \cdot \lambda))[V] (s_{i_{k-2}} \cdots s_{i_1})^{-1}(B_{s_{i_{k-1}}}((s_{i_{k-2}} \cdots s_{i_1}) \cdot \lambda))[V] \cdots B_{s_{i_1}}(\lambda)[V].
\]

$B_{w,V}(\lambda)$ is a rational function of $\lambda$. $B_{w,V}(\lambda)$ tends to 1 as $\lambda$ tends to infinity in a generic direction.

For any nottrivial element $w \in \mathbb{W}$ and $\lambda \in \mathfrak{h}$, define an element $B_{w}(\lambda) \in U\mathfrak{g}_0$ by
\[
B_{w}(\lambda) = (s_{i_{k-1}} \cdots s_{i_1})^{-1}(B_{s_{i_k}}((s_{i_{k-1}} \cdots s_{i_1}) \cdot \lambda))(s_{i_{k-2}} \cdots s_{i_1})^{-1}(B_{s_{i_{k-2}}}((s_{i_{k-2}} \cdots s_{i_1}) \cdot \lambda)) \cdots B_{s_{i_1}}(\lambda).
\]

Set $B_{w}(\lambda) = 1$ if $w$ is the identity in $\mathbb{W}$. We have $B_{w}(\lambda)[V] = B_{w,V}(\lambda)$, and $B_{w}(\lambda)$ does not depend on the choice of the reduced presentation of $w$.

**Properties of $B_{w}(\lambda)$.**

I. If $w_1, w_2 \in \mathbb{W}$ and $l(w_1 w_2) = l(w_1) + l(w_2)$, then
\[
B_{w_1 w_2}(\lambda) = (w_2)^{-1}(B_{w_1}(w_2 \cdot \lambda)) B_{w_2}(\lambda).
\]

II. Let $i = 1, \ldots, r$, $\omega \in \mathfrak{h}$, and $(\alpha_i, \omega) = 0$, then
\[
B_{s_i}(\lambda + \omega) = B_{s_i}(\lambda).
\]
III. For $i = 1, ..., r$, 

$$s_i(B_{s_i}(s_i \cdot \lambda)) \cdot B_{s_i}(\lambda) = \frac{(\lambda, \alpha_i^\vee) - H_{\alpha_i} + 1}{(\lambda, \alpha_i^\vee) + 1}.$$ 

IV. Every relation $(s_is_j)^m = 1$ for $m = 2, 3, 4, 6$ in $\mathbb{W}$ is equivalent to a homogeneous relation $s_is_j... = s_js_i...$. Every such a homogeneous relation generates a relation for $B_{s_i}(\lambda), B_{s_j}(\lambda)$. Namely, for $m = 2$, the relation is 

$$(s_j)^{-1}(B_{s_i}(s_j \cdot \lambda)) B_{s_j}(\lambda) = (s_i)^{-1}(B_{s_j}(s_i \cdot \lambda)) B_{s_i}(\lambda),$$ 

for $m = 3$, the relation is 

$$(s_j)^{-1}(B_{s_i}((s_is_j) \cdot \lambda)) (s_i)^{-1}(B_{s_j}(s_i \cdot \lambda)) B_{s_i}(\lambda) =$$ 

$$(s_is_j)^{-1}(B_{s_j}((s_is_j) \cdot \lambda)) (s_j)^{-1}(B_{s_j}(s_j \cdot \lambda)) B_{s_j}(\lambda),$$ 

and so on.

V. 

$$\Delta(B_w(\lambda)) J(\lambda) = w^{-1}(J(w \cdot \lambda)) (B_w(\lambda - h^{(2)}) \otimes B_w(\lambda)).$$

The operators $B_w(\lambda)$ are closely connected with extremal projectors of Zhelobenko, see \cite{Zh1, Zh2}.

2.7. Operators $\mathbb{B}_{w,V}$. In order to study interrelations of operators $B_{w,V}(\lambda)$ with KZ operators it is convenient to change the argument $\lambda$.

Let $V$ be a finite dimensional $\mathfrak{g}$-module. For $w_1, w_2 \in \mathbb{W}$ and $\lambda \in \mathfrak{h}$, define $w_1(\mathbb{B}_{w_2,V}(\lambda)) : V \to V$ as follows. For any $\nu \in P(V)$ and $v \in V[\nu]$, set 

$$w_1(\mathbb{B}_{w_2,V}(\lambda)) v = w_1(B_{w_2}(\lambda - \rho + \frac{1}{2} \nu))|_V v.$$ 

In particular, 

$$\mathbb{B}_{w,V}(\lambda)v = B_{w,V}(\lambda - \rho + \frac{1}{2} \nu)v.$$ 

$w_1(\mathbb{B}_{w_2,V}(\lambda))$ is a meromorphic function of $\lambda$, $w_1(\mathbb{B}_{w_2,V}(\lambda))$ tends to 1 as $\lambda$ tends to infinity in a generic direction.

**Properties of $\mathbb{B}_{w,V}(\lambda)$.**

I. If $w_1, w_2 \in \mathbb{W}$ and $l(w_1w_2) = l(w_1) + l(w_2)$, then 

$$\mathbb{B}_{w_1w_2,V}(\lambda) = w_2^{-1}(\mathbb{B}_{w_1,V}(w_2(\lambda))) \mathbb{B}_{w_2,V}(\lambda).$$

II. If $i = 1, ..., r, \ b \in \mathbb{W}$, $v \in V[\nu]$, then 

$$\mathbb{B}_{s_i,V}(\lambda) v = p((\lambda + \frac{1}{2} \nu, \alpha_i^\vee) - 1; H_{\alpha_i}, E_{\alpha_i}, F_{\alpha_i}) v$$

and 

$$w(\mathbb{B}_{s_i,V}(w^{-1}(\lambda))) v = p((\lambda + \frac{1}{2} \nu, w(\alpha_i^\vee)) - 1; H_{w(\alpha_i)}, E_{w(\alpha_i)}, F_{w(\alpha_i)}) v$$

where $p(t; H, E, F)$ is defined in \cite{H}. 


For $\alpha \in \Sigma$, $\lambda \in \mathfrak{h}$, define a linear operator $B^V_\alpha(\lambda) : V \to V$ by

$$B^V_\alpha(\lambda)v = p((\lambda + \frac{1}{2} \nu, \alpha^\vee) - 1; H_\alpha, E_\alpha, F_\alpha)v$$

for any $v \in V[\nu]$.

III.

$$B^\alpha_V(\lambda)B^{-\alpha}_V(\lambda)v = \frac{(\lambda - \frac{1}{2} \nu, \alpha^\vee)}{(\lambda + \frac{1}{2} \nu, \alpha^\vee)}v$$

for any $v \in V[\nu]$.

IV. Let $\alpha \in \Sigma$, $\omega \in \mathfrak{h}$, and $(\alpha, \omega) = 0$, then

$$B^\alpha_V(\lambda + \omega) = B^\alpha_V(\lambda).$$

V. Every relation $(s_is_j)^m = 1$ for $m = 2, 3, 4, 6$ in $W$ is equivalent to a homogeneous relation $s_is_j... = s_is_i...$. Every such a homogeneous relation generates a relation for $B_{s_i,V}(\lambda), B_{s_j,V}(\lambda)$. Namely, for $m = 2$, the relation is

$$(s_j)^{-1}(B_{s_i,V}(s_j(\lambda))) B_{s_j,V}(\lambda) = (s_i)^{-1}(B_{s_j,V}(s_i(\lambda))) B_{s_i,V}(\lambda),$$

for $m = 3$, the relation is

$$(sjs_i)^{-1}(B_{s_i,V}((s_is_j(\lambda)))) (s_i)^{-1}(B_{s_j,V}(s_i(\lambda))) B_{s_i,V}(\lambda) = (sjsi)^{-1}(B_{s_j,V}((s_is_j(\lambda)))) (s_j)^{-1}(B_{s_i}(s_j(\lambda))) B_{s_j}(\lambda),$$

and so on.

These relations can be written in terms of operators $B^\alpha_V(\lambda)$.

VI. For $\alpha, \beta \in \Sigma$, denote $\mathbb{R}\langle \alpha, \beta \rangle$ the subspace $\mathbb{R}\alpha + \mathbb{R}\beta \subset \mathfrak{h}$. Then

$$B^\alpha_V(\lambda)B^\beta_V(\lambda) = B^\beta_V(\lambda)B^\alpha_V(\lambda),$$

$$B^\alpha_V(\lambda)B^{\alpha+\beta}_V(\lambda)B^\beta_V(\lambda) = B^\beta_V(\lambda)B^{\alpha+\beta}_V(\lambda)B^\alpha_V(\lambda),$$

$$B^\alpha_V(\lambda)B^{\alpha+\beta}_V(\lambda)B^{\alpha+2\beta}_V(\lambda)B^\beta_V(\lambda) = B^\beta_V(\lambda)B^{\alpha+2\beta}_V(\lambda)B^{\alpha+\beta}_V(\lambda)B^\alpha_V(\lambda),$$

under the assumption that $\mathbb{R}\langle \alpha, \beta \rangle = \{ \pm \gamma \}$ where $\gamma$ runs over all indices in the corresponding identity.

VII.

$$B_{w,W\otimes V}(\lambda)) = x_w^{-1}(J_{WV}(w(\lambda) - \rho + \frac{1}{2}(h^{(1)} + h^{(2)}))x_w \cdot$$

$$(B_{w,W}(\lambda - \frac{1}{2}h^{(2)}) \otimes B_{w,V}(\lambda + \frac{1}{2}h^{(1)})) J(\lambda - \rho + \frac{1}{2}(h^{(1)} + h^{(2)}))^{-1}$$
Lemma 16. Let $W, V$ be finite dimensional $\mathfrak{g}$-modules, $\lambda \in \mathfrak{h}$, $w \in W$. Then
\[
\Omega \mathbb{B}_{w,W \otimes V}(\lambda) = \mathbb{B}_{w,W \otimes V}(\lambda) \Omega
\]
and
\[
(w^{-1}(\Omega^{-}) + \lambda^{(2)})\mathbb{B}_{w,W \otimes V}(\lambda) = \mathbb{B}_{w,W \otimes V}(\lambda)(\Omega^{-} + \lambda^{(2)}) .
\]

Proof. The first equation holds since $\Omega$ commutes with the comultiplication. Now
\[
\mathbb{B}_{w,W \otimes V}(\lambda)(\Omega^{-} + \lambda^{(2)}) = x_{w}^{-1}(J_{WV}(w(\lambda) - \rho + \frac{1}{2}(h^{(1)} + h^{(2)})))x_{w} .
\]

\[
(\mathbb{B}_{w,W}(\lambda - \frac{1}{2}h^{(2)}) \otimes \mathbb{B}_{w,V}(\lambda + \frac{1}{2}h^{(1)})) J_{WV}(\lambda - \rho + \frac{1}{2}(h^{(1)} + h^{(2)}))^{-1} (\Omega^{-} + \lambda^{(2)}) =
\]
\[
x_{w}^{-1}(J_{WV}(w(\lambda) - \rho + \frac{1}{2}(h^{(1)} + h^{(2)})))x_{w} (\Omega^{0} + \lambda^{(2)}) .
\]

3. Difference Equations Compatible with KZ Equations for $\mathfrak{g} = sl_{N}$

3.1. Statement of Results. Let $e_{i,j}$, $i, j = 1, \ldots, N$, be the standard generators of the Lie algebra $gl_{N}$,
\[
[e_{i,j}, e_{k,l}] = \delta_{j,k} e_{i,l} - \delta_{i,l} e_{j,k} .
\]
$sl_{N}$ is the Lie subalgebra of $gl_{N}$ such that $sl_{n} = n_{+} \oplus \mathfrak{h} \oplus n_{-}$ where
\[
n_{+} = \oplus_{1 \leq i < j \leq N} \mathbb{C} e_{i,j} , \quad n_{-} = \oplus_{1 \leq j < i \leq N} \mathbb{C} e_{i,j} .
\]
and $\mathfrak{h} = \{ \lambda = \sum_{i=1}^{N} \lambda_i e_{i,i} \mid \lambda_i \in \mathbb{C}, \sum_{i=1}^{N} \lambda_i = 0 \}$.

The invariant scalar product is defined by $\langle e_{i,j}, e_{k,l} \rangle = \delta_{i,j} \delta_{j,k}$. The roots are $e_{i,i} - e_{j,j}$ for $i \neq j$. $\alpha^\vee = \alpha$ for any root. For a root $\alpha = e_{i,i} - e_{j,j}$, we have $H_\alpha = e_{i,i} - e_{j,j}$, $E_\alpha = e_{i,j}$, $F_\alpha = e_{j,i}$. The simple roots are $\alpha_i = e_{i,i} - e_{i+1,i+1}$ for $i = 1, ..., N - 1$. $\mathbb{W}$ is the symmetric group $S^N$ permuting coordinates of $\lambda \in \mathfrak{h}$. The (dual) fundamental weights are $\omega_i = \omega_i^\vee = \sum_{j=1}^{i} (1 - \frac{i}{N}) e_{j,j} - \sum_{j=1}^{N} \frac{j}{N} e_{j,j}$ for $i = 1, ..., N - 1$. All dual fundamental weights are minuscule. For $i = 1, ..., N - 1$, the permutation $w_{[i]}^{-1} \in S^N$ is $(1_i 2_i 3_i ... N-i N-i+1 ... N)$.

For any finite dimensional $sl_N$-module $V$ and $w \in S^N$ consider the operators $\mathbb{B}_{w,V}(\lambda) : V \to V$.

Let $V = V_1 \otimes ... \otimes V_n$ be a tensor product of finite dimensional $sl_N$-modules. For $\kappa \in \mathbb{C}$ and $\lambda \in \mathfrak{h}$, consider the trigonometric KZ equations with values in $V$,

$$\nabla_j(\lambda, \kappa) u(z_1, ..., z_n, \lambda) = 0, \quad j = 1, ..., n.$$  

Here $u(z_1, ..., z_n, \lambda) \in V$ is a function of complex variables $z_1, ..., z_n$ and $\lambda \in \mathfrak{h}$.

Introduce the dynamical difference equations on a $V$-valued function $u(z_1, ..., z_n, \lambda)$ as

$$u(z_1, ..., z_n, \lambda + \kappa \omega_i^\vee) = K_i(z_1, ..., z_n, \lambda) u(z_1, ..., z_n, \lambda), \quad i = 1, ..., N - 1$$

where

$$K_i(z_1, ..., z_n, \lambda) = \prod_{k=1}^{n} z_k^{(\omega_i^\vee)(k)} \mathbb{B}_{w_{[i]}^{-1},V}(\lambda).$$

The operator $\prod_{k=1}^{n} z_k^{(\omega_i^\vee)(k)}$ is well defined if the argument of $z_1, ..., z_n$ is fixed. The dynamical difference equations are well defined on functions of $(z, \lambda)$ where $\lambda \in \mathfrak{h}$ and $z$ belongs to the universal cover of $(\mathbb{C}^*)^n$. Notice that the KZ equations are well defined for $V$-valued functions of the same variables.

The KZ operators $\nabla_j(\lambda, \kappa)$ and the operators $K_i(z_1, ..., z_n, \lambda)$ preserve the weight decomposition of $V$.

**Theorem 17.** The dynamical equations (10) together with the KZ equations (9) form a compatible system of equations.

3.2. **Proof.** First prove that

$$\prod_{k=1}^{n} z_k^{(\omega_i^\vee)(k)} \mathbb{B}_{w_{[i]}^{-1},V}(\lambda) \nabla_j(\lambda, \kappa) = \nabla_j(\lambda + \kappa \omega_i^\vee, \kappa) \prod_{k=1}^{n} z_k^{(\omega_i^\vee)(k)} \mathbb{B}_{w_{[i]}^{-1},V}(\lambda)$$

for all $i$ and $j$. Multiplying both sides from the left by $\prod_{k=1}^{n} z_k^{-(\omega_i^\vee)(k)}$ and using Lemma 3, we reduce the equation to

$$\mathbb{B}_{w_{[i]}^{-1},V}(\lambda) \left( \sum_{k, k \neq j} r(z_j/z_k)(j,k) + \lambda^{(j)} \right) = \left( \sum_{k, k \neq j} w_{[i]}^{-1}(r(z_j/z_k)(j,k) + \lambda^{(j)}) \mathbb{B}_{w_{[i]}^{-1},V}(\lambda) \right).$$
Lemma 18. For \( j = 1, \ldots, n \) and \( w \in \mathbb{W} \), we have
\[
\mathbb{B}_{w,V}(\lambda) \left( \sum_{k, k \neq j} r(z_j/z_k)^{(j,k)} + \lambda^{(j)} \right) = \left( \sum_{k, k \neq j} w^{-1}(r(z_j/z_k))^{(j,k)} + \lambda^{(j)} \right) \mathbb{B}_{w,V}(\lambda).
\]

Proof. It is sufficient to check the equation for the residues of both sides at \( z_j = z_k, k \neq j \), and for the limit of both sides as \( z_j \to \infty \). The residue equation \( [\mathbb{B}_{w,V}(\lambda), \Omega^{(j,k)}] = 0 \) is true since the Casimir operator commutes with the comultiplication. The limit equation
\[
\mathbb{B}_{w,V}(\lambda) \left( \sum_{k, k \neq j} (\Omega^+)^{(j,k)} + \lambda^{(j)} \right) = \left( \sum_{k, k \neq j} w^{-1}(\Omega^+)^{(j,k)} + \lambda^{(j)} \right) \mathbb{B}_{w[V](\lambda)}
\]
is a corollary of Lemma 16. \( \square \)

The Theorem is proved for \( sl_N, N = 2 \). For \( N > 2 \), it remains to prove that
\[
K_i(z, \lambda + \kappa \omega_i^\vee) K_j(z, \lambda) = K_j(z, \lambda + \kappa \omega_i^\vee) K_i(z, \lambda)
\]
for all \( i, j, 0 < i < j < N \). We prove (11) for \( N = 3 \). For arbitrary \( N \) the proof is similar. Another proof see in Section 4. For \( N = 3, i = 1, j = 2 \), equation (11) takes the form
\[
\prod_{k=1}^n z_k^{(\omega_i^{(k)})} \mathbb{B}_{V}^{\alpha_1+\alpha_2}(\lambda + \kappa \omega_2^\vee) \mathbb{B}_{V}^{\alpha_1}(\lambda + \kappa \omega_2^\vee) \prod_{k=1}^n z_k^{(\omega_i^{(k)})} \mathbb{B}_{V}^{\alpha_1+\alpha_2}(\lambda) \mathbb{B}_{V}^{\alpha_1}(\lambda) = \prod_{k=1}^n z_k^{(\omega_i^{(k)})} \mathbb{B}_{V}^{\alpha_1+\alpha_2}(\lambda + \kappa \omega_i^\vee) \mathbb{B}_{V}^{\alpha_2}(\lambda + \kappa \omega_i^\vee) \prod_{k=1}^n z_k^{(\omega_i^{(k)})} \mathbb{B}_{V}^{\alpha_1+\alpha_2}(\lambda) \mathbb{B}_{V}^{\alpha_1}(\lambda).
\]
We have \( \mathbb{B}_{V}^{\alpha_1}(\lambda + \kappa \omega_2^\vee) = \mathbb{B}_{V}^{\alpha_1}(\lambda) \) since \( (\omega_2^\vee, \alpha_1) = 0 \). We have \( [\mathbb{B}_{V}^{\alpha_1}(\lambda), \prod_{k=1}^n z_k^{(\omega_i^{(k)})}] = 0 \) since \( \mathbb{B}_{V}^{\alpha_1}(\lambda) \) is a power series in \( E_{\alpha_1}, F_{\alpha_1} \). Similarly, \( \mathbb{B}_{V}^{\alpha_2}(\lambda + \kappa \omega_1^\vee) = \mathbb{B}_{V}^{\alpha_2}(\lambda) \) and \( [\mathbb{B}_{V}^{\alpha_2}(\lambda), \prod_{k=1}^n z_k^{(\omega_i^{(k)})}] = 0 \). Using these remarks and the relation
\[
\mathbb{B}_{V}^{\alpha_2}(\lambda) \mathbb{B}_{V}^{\alpha_1+\alpha_2}(\lambda) \mathbb{B}_{V}^{\alpha_1}(\lambda) = \mathbb{B}_{V}^{\alpha_1}(\lambda) \mathbb{B}_{V}^{\alpha_1+\alpha_2}(\lambda) \mathbb{B}_{V}^{\alpha_2}(\lambda)
\]
we reduce (12) to
\[
\prod_{k=1}^n z_k^{(\omega_i^{(k)} - \omega_i^{(k)})} \mathbb{B}_{V}^{\alpha_1+\alpha_2}(\lambda + \kappa \omega_2^\vee) = \mathbb{B}_{V}^{\alpha_1+\alpha_2}(\lambda + \kappa \omega_1^\vee) \prod_{k=1}^n z_k^{(\omega_i^{(k)} - \omega_i^{(k)})}.
\]
This equation holds since \( \mathbb{B}_{V}^{\alpha_1+\alpha_2}(\lambda + \kappa \omega_1^\vee) = \mathbb{B}_{V}^{\alpha_1+\alpha_2}(\lambda + \kappa \omega_2^\vee) \), each of these operators is a power series in \( E_{\alpha_1+\alpha_2}, F_{\alpha_1+\alpha_2} \), and \( (\omega_i^{\vee} - \omega_2^\vee, \alpha_1 + \alpha_2) = 0 \).

3.3. An Equivalent Form of Dynamical Equations for \( sl_N \). For \( j = 1, \ldots, N \), set \( \delta_j = \omega_j^\vee - \omega_{j-1}^\vee \) where \( \omega_0^\vee = \omega_N^\vee = 0 \). Then the system of equations (10) is equivalent to
the system
\[
\begin{array}{c}
u(z_1, ..., z_n, \lambda + \kappa \delta_i) = \\
\left( \prod_{k=1}^{n} z_k^{(\delta_i)^{(k)}} \right) \prod_{j=1}^{n} \frac{\Gamma(1 - \frac{(\nu + j \alpha, \alpha)}{\kappa})}{\Gamma(1 - \frac{(\lambda + \nu + j \alpha, \alpha)}{\kappa})} \\
\end{array}
\]
where \(i = 1, ..., N\).

Notice that the inverse powers can be eliminated using property III in Section 2.7.

3.4. Application to Determinants. Let \(g\) be a simple Lie algebra, \(V\) a finite dimensional \(g\)-module, \(V[\nu]\) a weight subspace. For a positive root \(\alpha\) fix the \(sl_2\) subalgebra in \(g\) generated by \(H_\alpha, E_\alpha, F_\alpha\). Consider \(V\) as an \(sl_2\)-module. Let \(V[\nu]_\alpha \subset V\) be the \(sl_2\)-submodule generated by \(V[\nu]\),

\[
V[\nu]_\alpha = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} W^\alpha_k \otimes L_{\nu + k \alpha}
\]
the decomposition into irreducible \(sl_2\)-modules. Here \(L_{\nu + k \alpha}\) is the irreducible module with highest weight \(\nu + k \alpha\) and \(W^\alpha_k\) the multiplicity space. Let \(d^\alpha_k = \dim W^\alpha_k\). Set

\[
X_{\alpha, V[\nu]}(\lambda) = \prod_{k \in \mathbb{Z}_{\geq 0}} \left( \prod_{j=1}^{k} \frac{\Gamma(1 - \frac{(\nu + j \alpha, \alpha)}{\kappa})}{\Gamma(1 - \frac{(\lambda + \nu + j \alpha, \alpha)}{\kappa})} \right) d^\alpha_k
\]
cf. formula (8). Here \(\Gamma\) is the standard gamma function.

Let \(V = V_1 \otimes ... \otimes V_n\) be a tensor product of finite dimensional \(g\)-modules. Set \(\Lambda_k(\lambda) = \text{tr}_{V[\nu]} \lambda^{(k)}\), \(\epsilon_{k,l} = \text{tr}_{V[\nu]} \Omega^{(k,l)}\), \(\gamma_k = \sum_{l \neq k} \epsilon_{k,l}\). Set

\[
(13) \quad D_{V[\nu]}(z_1, ..., z_n, \lambda) = \prod_{k=1}^{n} z_k^{\Lambda_k(\lambda)} \prod_{1 \leq k < l \leq n} (z_k - z_l)^{\gamma_k} \prod_{\alpha \in \Sigma_+} X_{\alpha, V[\nu]}(\lambda)
\]

Let \(g = sl_N\), \(V = V_1 \otimes ... \otimes V_n\) a tensor product of finite dimensional \(sl_N\)-modules. Fix a basis \(v_1, ..., v_d\) in a weight subspace \(V[\nu]\). Suppose that \(u_i(z_1, ..., z_n, \lambda) = \sum_{j=1}^{d} u_{i,j} v_j\), \(i = 1, ..., d\), is a set of \(V[\nu]\)-valued solutions of the combined system of KZ equations (7) and dynamical equations (11).

**Corollary 19.**

\[
\det(u_{i,j})_{1 \leq i,j \leq d} = C_{V[\nu]}(\lambda) D_{V[\nu]}(z_1, ..., z_n, \lambda)
\]
where \(C_{V[\nu]}(\lambda)\) is a function of \(\lambda\) (depending also on \(V_1, ..., V_n\) and \(\nu\)) such that

\[
C_{V[\nu]}(\lambda + \kappa \omega) = C_{V[\nu]}(\lambda)
\]
for all \(\omega \in P^\vee\).

**Proof.** The Corollary follows from the following simple Lemma.

**Lemma 20.** For \(i = 1, ..., N - 1\), the operator \(B_{\omega^i, V}(\lambda)\) is the product in a suitable order of all operators \(B^\alpha(\lambda)\) with \(\alpha \in \Sigma_+\) and \((\omega^i, \alpha) > 0\).
Notice that Lemma \[20\] in particular implies that operators \(B_\omega \lambda\) and the dynamical equations are well defined in the tensor product of any highest weight \(sl_N\)-modules.

4. Dynamical Difference Equations

In this section we introduce dynamical difference equations for arbitrary simple Lie algebra. The compatibility of the dynamical equations follows from [Ch 1]. We prove the compatibility of dynamical and KZ equations.

4.1. Affine Root Systems, [Ch1], [Ch2]. Let \(\mathfrak{g}\) be a simple Lie algebra. The vectors \(\alpha = [\alpha, j] \in \mathfrak{h} \times \mathbb{R}\) for \(\alpha \in \Sigma, j \in \mathbb{Z}\) form the affine root system \(\Sigma^a\) corresponding to the root system \(\Sigma \subset \mathfrak{h}\). We view \(\Sigma\) as a subset in \(\Sigma^a\) identifying \(\alpha \in \mathfrak{h}\) with \([\alpha, 0]\). The simple roots of \(\Sigma^a\) are \(\alpha_1, ..., \alpha_r \in \Sigma\) and \(\alpha_0 = [-\theta, 1]\) where \(\theta \in \Sigma\) is the maximal root. The positive roots are \(\Sigma^a_+ = \{[\alpha, j] \in \Sigma^a | \alpha \in \Sigma, j > 0\} \cup \{\alpha \in \Sigma_+, j = 0\}\). The Dynkin diagram and its affine completion with \(\{\alpha_i\}_{0 \leq i \leq n}\) as vertices are denoted \(\Gamma\) and \(\Gamma^a\), respectively. The set of the indices of the images of \(\alpha_0\) with respect to all automorphisms of \(\Gamma^a\) is denoted \(O = \{0\}\) for \(E_8, F_4, G_2\). Let \(O^* = \{i \in O | i \neq 0\}\).

For \(i = 1, ..., r\), the dual fundamental weight \(\omega_i^\vee\) is minuscule if and only if \(i \in O^*\).

Given \(\tilde{\alpha} = [\alpha, j] \in \Sigma^a\) and \(\omega \in P^\vee\), set
\[
s_{\tilde{\alpha}}(\tilde{z}) = \tilde{z} - (z, \alpha^\vee)\tilde{\alpha}, \quad t_\omega(\tilde{z}) = [z, \xi - (z, \omega)]
\]
for \(\tilde{z} = [z, \xi]\).

The affine Weyl group \(\mathbb{W}^a\) is the group generated by reflections \(s_{\tilde{\alpha}}, \tilde{\alpha} \in \Sigma^a_+\). One defines the length of elements of \(\mathbb{W}^a\) taking the simple reflections \(s_i = s_{\alpha_i}, i = 0, ..., r\), as generators of \(\mathbb{W}^a\). The group \(\mathbb{W}^a\) is the semidirect product \(\mathbb{W} \ltimes Q^\vee\) of its subgroups \(\mathbb{W} = \langle s_\alpha | \alpha \in \Sigma_+\rangle\) and \(Q^\vee = \{t_\omega | \omega \in Q^\vee\}\), where for \(\alpha \in \Sigma\) we have \(t_\alpha = s_{\alpha}s_{[\alpha, 1]} = s_{\alpha}^{-1}s_{\alpha}\).

Consider the group \(P_i^\vee = \{t_\omega | \omega \in P_i^\vee\}\). The extended affine Weyl group \(\mathbb{W}^b\) is the group of transformations of \(\mathfrak{h} \times \mathbb{R}\) generated by \(\mathbb{W}\) and \(P_i^\vee\). \(\mathbb{W}^b\) is isomorphic to \(\mathbb{W} \ltimes P_i^\vee\) with action \((w, \omega)([z, \xi]) = [w(z), \xi - (z, \omega)]\).

Notice that for any \(w \in \mathbb{W}^b\) and \(\tilde{\alpha} \in \Sigma^a\), we have \(w(\tilde{\alpha}) \in \Sigma^a\).

The extended affine Weyl group has a remarkable subgroup \(\Pi = \{\pi_i | i \in O\}\), where \(\pi_0 \in \Pi\) is the identity element in \(\mathbb{W}^b\) and for \(i \in O^*\) we have \(\pi_i = t_{\omega_i}^{-1} w_{[i]}\). The group \(\Pi\) is isomorphic to \(P^\vee/Q^\vee\) with the isomorphism sending \(\pi_i\) to the minuscule weight \(\omega_i\). For \(i \in O^*\), the element \(w_{[i]}\) preserves the set \([-\theta, \alpha_1, ..., \alpha_r]\) and \(\pi_i(\alpha_0) = \alpha_i = w_{[i]}^{-1}(-\theta)\). We have
\[
\mathbb{W}^b = \Pi \ltimes \mathbb{W}^a, \quad \text{where} \quad \pi_i s_i \pi_i^{-1} = s_k \text{ if } \pi_i(\alpha_i) = \alpha_k \text{ and } 0 \leq k \leq r.
\]

We extend the notion of length to \(\mathbb{W}^b\). For \(i \in O^*, w \in \mathbb{W}^a\), we set the length of \(\pi_i w\) to be equal to the length of \(w\) in \(\mathbb{W}^a\).
4.2. Affine R-matrices, [Ch1, Ch2]. Fix a $\mathbb{C}$-algebra $F$. A set $G = \{G^\alpha \in F \mid \alpha \in \Sigma\}$ is called a closed R-matrix if

$$
G^\alpha G^\beta = G^\beta G^\alpha, \\
G^\alpha G^{\alpha+\beta} G^\beta = G^\beta G^{\alpha+\beta} G^\alpha, \\
G^\alpha G^{\alpha+2\beta} G^{\alpha+2\beta} = G^\beta G^{\alpha+2\beta} G^{\alpha+2\beta} G^\alpha, \\
G^\alpha G^{\alpha+2\beta} G^{\alpha+2\beta} = G^\beta G^{\alpha+2\beta} G^{\alpha+2\beta} G^\alpha
$$

under the assumption that $\alpha, \beta \in \Sigma$ and $\mathbb{R}(\alpha, \beta) = \{\pm \gamma\}$ where $\gamma$ runs over all indices in the corresponding identity.

A set $G^\alpha = \{\tilde{G}^\alpha \in F \mid \tilde{\alpha} \in \Sigma^a\}$ is called a closed affine R-matrix if $\tilde{G}^\alpha$ satisfy the same relations where $\alpha, \beta$ are replaced with $\tilde{\alpha}, \tilde{\beta}$.

If $G^\alpha$ is an affine R-matrix, for any $w \in \mathbb{W}^b$ define an element $\tilde{G}_w \in F$ as follows. Given a reduced presentation $w = \pi_i s_{j_1} ... s_{j_t}, i \in O, 0 \leq j_1, ..., j_t \leq r$, set $\tilde{G}_w = \tilde{G}^{\tilde{\alpha}_1} ... \tilde{G}^{\tilde{\alpha}_t}$ where $\tilde{\alpha}_1 = \alpha_{j_1}, \tilde{\alpha}_2 = s_{j_1}(\alpha_{j_2}), \tilde{\alpha}_3 = s_{j_1} s_{j_2}(\alpha_{j_3}), ...$. The element $\tilde{G}_w$ does not depend on the reduced presentation of $w$. We set $\tilde{G}_{id} = 1$.

The unordered set $\{\tilde{\alpha}_1, ..., \tilde{\alpha}_t\}$ is denoted $\tilde{A}(w)$. There is a useful formula valid for any (not necessarily minuscule) dual fundamental weight $\omega^\vee_i$, $i = 1, ..., r$,

$$
\tilde{A}(t_{\omega^\vee_i}) = \{[\alpha, j] \mid \alpha \in \Sigma_+, \text{ and } (\omega^\vee_i, \alpha) > j \geq 0\},
$$

Prop. 1.4, [Ch2].

Introduce the following formal notation: for $w \in \mathbb{W}^b$, $\tilde{\alpha}, \tilde{\beta} \in \Sigma^a$, set $w(\tilde{G}^\tilde{\alpha}) = G^{w(\tilde{\alpha})}$, $w(\tilde{G}^{\tilde{\alpha}} \tilde{G}^{\tilde{\beta}}) = G^{w(\tilde{\alpha})} G^{w(\tilde{\beta})}, ...$. Then the elements $\{\tilde{G}_w \mid w \in \mathbb{W}^b\}$ form a 1-cocycle:

$$
\tilde{G}_{xy} = y^{-1} \tilde{G}_x \tilde{G}_y
$$

for all $x, y \in \mathbb{W}^b$ such that $l(xy) = l(x) + l(y)$.

There is a way to construct a closed affine R-matrix if a closed nonaffine R-matrix $G = \{G^\alpha \in F \mid \alpha \in \Sigma\}$ is given. Namely, assume that the group $P^\vee_i$ acts on the algebra $F$ so that $t_\omega(G^\alpha) = G^\alpha$ whenever $\omega, \alpha = 0$, $\omega \in P^\vee$, $\alpha \in \Sigma$. Then for $\tilde{\alpha} = [\alpha, j] \in \Sigma^a$, choose $\omega \in P^\vee$ so that $(\omega, \alpha) = -j$ and set $\tilde{G}^{\tilde{\alpha}} = t_\omega(G^\alpha)$. The set $G^\alpha = \{\tilde{G}^{\tilde{\alpha}} \in F \mid \tilde{\alpha} \in \Sigma^a\}$ is well defined and forms a closed affine R-matrix called the affine completion of the R-matrix $G$.

Assume that a closed affine R-matrix $G^\alpha$ is the affine completion of a closed nonaffine R-matrix $G$. Consider the system of equations for an element $\Phi \in F$:

$$
t_{-\omega^\vee_i}(\Phi) = \tilde{G}_{t_{\omega^\vee_i}} \Phi, \quad i = 1, ..., r,
$$

where $\omega^\vee_1, ..., \omega^\vee_r$ are the dual fundamental weights.

**Theorem 21.** [Ch1] The system of equations (13) is compatible,

$$
t_{-\omega^\vee_i}(\tilde{G}_{t_{\omega^\vee_i}}) \tilde{G}_{t_{\omega^\vee_i}} = t_{-\omega^\vee_j}(\tilde{G}_{t_{\omega^\vee_i}}) \tilde{G}_{t_{\omega^\vee_i}}
$$

for $1 \leq i < j \leq r$. 
Example, [Ch1]. Let $\alpha = \alpha_1$, $\beta = \alpha_2$, $a = -\omega_1^\vee$, $b = -\omega_2^\vee$. Then the system for $A_2$ is

$$t_a(\Phi) = \tilde{G}^{\alpha + \beta} \tilde{G}^\alpha \Phi, \quad t_b(\Phi) = \tilde{G}^{\alpha + \beta} \tilde{G}^\beta \Phi.$$ 

The system for $B_2$ is

$$t_a(\Phi) = \tilde{G}^{\alpha + 2\beta} \tilde{G}^\alpha \Phi, \quad t_b(\Phi) = \tilde{G}^{[\alpha + 2\beta, 1]} \tilde{G}^{\alpha + \beta} \tilde{G}^\beta \Phi.$$ 

The system for $G_2$ is

$$t_a(\Phi) = \tilde{G}^{[3\alpha + 2\beta, 2]} \tilde{G}^{[3\alpha + \beta, 2]} \tilde{G}^{[2\alpha + \beta, 1]} \tilde{G}^{[3\alpha + 2\beta, 1]} \tilde{G}^{[3\alpha + \beta, 1]} \times \tilde{G}^{\alpha + \beta} \tilde{G}^{3\alpha + 2\beta} \tilde{G}^{2\alpha + \beta} \tilde{G}^{3\alpha + \beta} \tilde{G}^{\alpha} \Phi, \quad t_b(\Phi) = \tilde{G}^{[3\alpha + 2\beta, 1]} \tilde{G}^{3\alpha + \beta} \tilde{G}^{2\alpha + \beta} \tilde{G}^{3\alpha + 2\beta} \tilde{G}^{\alpha + \beta} \tilde{G}^\beta \Phi.$$ 

4.3. Affine R-matrix for Dynamical Equations. Fix $\kappa \in \mathbb{C}$ and a natural number $n$. Let $F$ be the algebra of meromorphic functions of $z_1, \ldots, z_n \in \mathbb{C}$ and $\lambda \in \mathfrak{h}$ with values in $U\mathfrak{g}_0^\otimes n$. Define an action of $\mathbb{W}$ on $F$ by

$$w f(z_1, \ldots, z_n, \lambda) = w(f(z_1, \ldots, z_n, w^{-1}(\lambda)))$$ 

and an action of $P^\vee_t$ on $F$ by

$$t_\omega f(z_1, \ldots, z_n, \lambda) = \prod_{k=1}^n z_k^{\omega(k)} f(z_1, \ldots, z_n, \lambda - \kappa \omega) \prod_{k=1}^n z_k^{-\omega(k)}$$

where $w \in \mathbb{W}, \omega \in P^\vee, f \in F$.

Lemma 22. Those actions extend to an action of $\mathbb{W}^b = \mathbb{W} \rtimes P^\vee_t$ on $F$, i.e. $w(t_\omega f) = t_w(\omega)(w f)$ for $w \in \mathbb{W}, \omega \in P^\vee, f \in F$. \square

Define a closed nonaffine $F$-valued R-matrix $G_F = \{G_F^\alpha | \alpha \in \Sigma\}$ by

$$G_F^\alpha(\lambda) = \Delta^{(n)}(p((\lambda, \alpha^\vee) - 1; H_\alpha, E_\alpha, F_\alpha)).$$

Properties of operators $\mathcal{B}_F^\alpha$ described in Section 2.7 ensure that $G_F$ is a closed R-matrix. The action of $P^\vee_t$ on $F$ defined above clearly has the property: $t_\omega(G_F^\alpha) = G_F^{\alpha^\vee}$ whenever $(\omega, \alpha) = 0, \omega \in P^\vee, \alpha \in \Sigma$. This allows us to define a closed affine R-matrix $G_F^\alpha = \{\tilde{G}_F^\alpha \in F | \tilde{\alpha} \in \Sigma^n\}$ as the affine completion of the R-matrix $G_F$. Namely, for $\tilde{\alpha} = [\alpha, f] \in \Sigma^n$, we choose $\omega \in P^\vee$ so that $(\omega, \alpha) = -j$ and set

$$\tilde{G}_F^{[\alpha, j]}(z_1, \ldots, z_n, \lambda) = \lambda^{-j}(G_F^\alpha) = \prod_{k=1}^n z_k^{\omega(k)} G_F^\alpha(\lambda - \kappa \omega) \prod_{k=1}^n z_k^{-\omega(k)}.$$ 

Let $V = V_1 \otimes \ldots \otimes V_n$ be a tensor product of finite dimensional $\mathfrak{g}$-modules. Let $F_V$ be the algebra of meromorphic functions of $z_1, \ldots, z_n \in \mathbb{C}$ and $\lambda \in \mathfrak{h}$ with values in $\text{End}(V)$. The closed affine R-matrix $G_F^\alpha$ induces a closed affine R-matrix $G_V^\alpha = \{\tilde{G}_V^\alpha\}$ where

$$\tilde{G}_V^\alpha(z_1, \ldots, z_n, \lambda) = \tilde{G}_F^{\alpha^\vee}(z_1, \ldots, z_n, \lambda + \frac{1}{2} \sum_{k=1}^n h^{(k)})|_V.$$
In other words,
\[ \tilde{G}^{[\alpha,j]}_V(z_1, \ldots, z_n, \lambda) = \prod_{k=1}^n z_k^{\omega^{(k)}} \mathbb{E}_V^\alpha(\lambda - \kappa \omega) \prod_{k=1}^n z_k^{-\omega^{(k)}} \]
where \((\omega, \alpha) = -j\) and the operators \(\mathbb{E}_V^\alpha\) are defined in Section 2.7. For any \(w \in \mathbb{W}^b\) and \(\tilde{\alpha} \in \Sigma^a\), we have \(w(\tilde{G}^\alpha_V) = \tilde{G}^{w(\tilde{\alpha})}_V\).

Let \(\{\tilde{G}^V_w \in F_V \mid w \in \mathbb{W}^b\}\) be the 1-cocycle associated with the affine R-matrix \(G^a_V\). Consider the system
\[ \prod_{k=1}^n z_k^{-(\omega^{(k)})} \Phi(z_1, \ldots, z_n, \lambda + \kappa \omega^{\vee}_i) \prod_{k=1}^n z_k^{(\omega^{(k)})} = \tilde{G}^V_{\omega^{\vee}_i}(z_1, \ldots, z_n, \lambda) \Phi(z_1, \ldots, z_n, \lambda), \]
i = 1, ..., N - 1, cf. (10).

Introduce the dynamical difference equations on a \(V\)-valued function \(u(z_1, \ldots, z_n, \lambda)\) as
\[ \prod_{k=1}^n z_k^{-(\omega^{(k)})} u(z_1, \ldots, z_n, \lambda + \kappa \omega^{\vee}_i) \prod_{k=1}^n z_k^{(\omega^{(k)})} = \tilde{G}^V_{\omega^{\vee}_i}(z_1, \ldots, z_n, \lambda) u(z_1, \ldots, z_n, \lambda), \]
i = 1, ..., r. Notice that the operators \(\tilde{G}^V_{\omega^{\vee}_i}\) preserve the weight decomposition of \(V\).

Notice also that the operators \(\tilde{G}^V_{\omega^{\vee}_i}\) are well defined on the tensor product of any highest weight \(g\)-modules according to formula (14).

An easy corollary of the compatibility of system (15) is

**Lemma 23.** The dynamical difference equations (16) form a compatible system of equations for a \(V\)-valued function \(u(z_1, \ldots, z_n, \lambda)\).

In particular, for \(g = \mathfrak{sl}_N\), the Lemma says that the system (11) is compatible.

**Theorem 24.** Assume that the Lie algebra \(g\) has a minuscule dual fundamental weight, i.e. \(g\) is not of type \(E_8, F_4, G_2\). Then the dynamical equations (14) together with the KZ equations (7) form a compatible system of equations.

The Theorem is proved in Section 4.4.

We conjecture that the statement of the Theorem holds for any simple Lie algebra.

Let \(g\) be a simple Lie algebra for which the KZ and dynamical equations are compatible. Let \(V = V_1 \otimes \ldots \otimes V_r\) be a tensor product of finite dimensional \(g\)-modules. Fix a basis \(v_1, \ldots, v_d\) in a weight subspace \(V[\nu]\). Suppose that \(u_i(z_1, \ldots, z_n, \lambda) = \sum_{j=1}^d u_{i,j} v_j,\)
$i = 1, \ldots, d$, is a set of $V[\nu]$-valued solutions of the combined system of KZ equations \((14)\) and dynamical equations \((16)\).

**Corollary 25.**

$$\det (u_{i,j})_{1 \leq i, j \leq d} = C_{V[\nu]}(\lambda) \, D_{V[\nu]}(z_1, \ldots, z_n, \lambda)$$

where $C_{V[\nu]}(\lambda)$ is a function of $\lambda$ (depending also on $V_1, \ldots, V_n$ and $\nu$) such that

$$C_{V[\nu]}(\lambda + \kappa \omega) = C_{V[\nu]}(\lambda)$$

for all $\omega \in P^\vee$ and $D_{V[\nu]}(z_1, \ldots, z_n, \lambda)$ is defined in \((13)\).

The Corollary follows from formula \((14)\).

4.4. **Proof of Theorem 24.** Introduce an action of $\mathbb{W}^b$ on the KZ operators $\nabla_j(\lambda, \kappa)$, $j = 1, \ldots, n$. Namely, for any $w \in \mathbb{W}$, set

$$w \nabla_j(\lambda, \kappa) = w(\nabla_j(w^{-1}(\lambda, \kappa))) = \kappa z_j \frac{\partial}{\partial z_j} - \sum_{l, l \neq j} w(r(z_j/z_l))^{(j,l)} - \lambda^{(j)}$$

and for any $\omega \in P^\vee_t$ set

$$t^\omega \nabla_j(\lambda, \kappa) = \prod_{k=1}^n z_k^{\omega(k)} \nabla_j(\lambda - \kappa \omega, \kappa) \prod_{k=1}^n z_k^{-\omega(k)} =$$

$$\kappa z_j \frac{\partial}{\partial z_j} - \prod_{k=1}^n z_k^{\omega(k)} \left( \sum_{l, l \neq j} r(z_j/z_l)^{(j,l)} \right) \prod_{k=1}^n z_k^{-\omega(k)} - \lambda^{(j)}.$$

The compatibility conditions of the dynamical and KZ equations take the form

$$\tilde{G}_{w,\nu}^V(z_1, \ldots, z_n, \lambda) \, \nabla_j(\lambda, \kappa) = t^{-\omega} \nabla_j(\lambda, \kappa) \, \tilde{G}_{w,\nu}^V(z_1, \ldots, z_n, \lambda)$$

for $i = 1, \ldots, r$, $j = 1, \ldots, n$.

The compatibility conditions follow from a more general statement.

**Theorem 26.** Assume that the Lie algebra $\mathfrak{g}$ has a minuscule dual fundamental weight, i.e. $\mathfrak{g}$ is not of type $E_8, F_4, G_2$. Then for any $j = 1, \ldots, n$ and any $w \in \mathbb{W}^b$ we have

$$\tilde{G}_{w}^V(z_1, \ldots, z_n, \lambda) \, \nabla_j(\lambda, \kappa) = w^{-1} \nabla_j(\lambda, \kappa) \, \tilde{G}_{w}^V(z_1, \ldots, z_n, \lambda).$$

We conjecture that the statement of the Theorem holds for any simple Lie algebra. The Theorem follows from the next four Lemmas.

**Lemma 27.** Let $j = 1, \ldots, n$. Assume that

$$\tilde{G}_{s_l}^V \nabla_j(\lambda, \kappa) = s_l^i \nabla_j(\lambda, \kappa) \tilde{G}_{s_l}^V, \quad \pi_i \nabla_j(\lambda, \kappa) = \nabla_j(\lambda, \kappa)$$

for $l = 0, \ldots, r$ and $i \in O^*$. Then

$$\tilde{G}_{w}^V(z_1, \ldots, z_n, \lambda) \, \nabla_j(\lambda, \kappa) = w^{-1} \nabla_j(\lambda, \kappa) \, \tilde{G}_{w}^V(z_1, \ldots, z_n, \lambda)$$

for all $w \in \mathbb{W}^b$. 
Proof. If \( w = \pi_1 s_{m_1} \ldots s_{m_1} \) is a reduced presentation, then
\[
\tilde{G}^V = s_{m_1 \ldots m_{n-1}} (G^V_{s_{m_1}})^{s_{m_1}} (G^V_{s_{m_2}})^{s_{m_2}} \quad \text{and}
\]
\[
\tilde{G}^V \nabla_j (\lambda, \kappa) = s_{m_1 \ldots m_{n-1}} (G^V_{s_{m_1}})^{s_{m_1}} (G^V_{s_{m_2}})^{s_{m_2}} \nabla_j (\lambda, \kappa) =
\]
\[
\sum_{i=1}^{m_1 \ldots m_{n-1}} \frac{\partial}{\partial z_j} (\sum_{l \neq j} z^{(w^{-1})}) = \sum_{k=1}^n z^{(w^{-1})} \nabla_j (\lambda, \kappa).
\]
The last equality follows from Lemma 3. Proof.

Lemma 28. Let \( j = 1, \ldots, n \) and \( w \in \mathbb{W} \). Then
\[
\tilde{G}^V \nabla_j (\lambda, \kappa) = w^{-1} \nabla_j (\lambda, \kappa) \tilde{G}^V.
\]
Proof. For \( w \in \mathbb{W} \) we have \( \tilde{G}^V (z_1, \ldots, z_n \lambda) = B_{w, V} (\lambda) \), and Lemma 28 is equivalent to Lemma 18.

Lemma 29. Let \( j = 1, \ldots, n \) and \( i \in O^* \). Then
\[
\pi_i \nabla_j (\lambda, \kappa) = \nabla_j (\lambda, \kappa).
\]
Proof. We have \( \pi_i = t_{\omega_i} w^{-1}_{[i]} \). Hence
\[
\pi_i \nabla_j (\lambda, \kappa) = t_{\omega_i} (w^{-1}_{[i]} \nabla_j (\lambda, \kappa)) = t_{\omega_i} (w^{-1}_{[i]} \frac{\partial}{\partial z_j} (r(z_j, z_l)) - \lambda(j)) =
\]
\[
\kappa z_j \frac{\partial}{\partial z_j} (\frac{1}{w_{[i]}^{(j)}}) \sum_{k=1}^n z^{(w^{-1})} \nabla_j (\lambda, \kappa).
\]
The last equality follows from Lemma 4.

Lemma 30. Let \( j = 1, \ldots, n \). Assume that the Lie algebra \( g \) has a minuscule dual fundamental weight. Then
\[
\tilde{G}^V_{s_0} \nabla_j (\lambda, \kappa) = s_0 \nabla_j (\lambda, \kappa) \tilde{G}^V_{s_0}.
\]
Proof. Let \( \omega_i \) be a minuscule dual fundamental weight. We have \( s_0 = \pi_i^{-1} s_i \pi_i \) and
\[
\tilde{G}^V_{s_0} = \pi_i^{-1} (G^V_{s_i}) \quad \text{according to the 1-cocycle property. Now}
\]
\[
s_0 \nabla_j (\lambda, \kappa) \tilde{G}^V_{s_0} = \pi_i^{-1} s_i \pi_i \nabla_j (\lambda, \kappa) \pi_i^{-1} (G^V_{s_i}) = \pi_i^{-1} (s_i \pi_i \nabla_j (\lambda, \kappa)) \tilde{G}^V_{s_0} =
\]
\[
\pi_i^{-1} (\pi_i \nabla_j (\lambda, \kappa)) \tilde{G}^V_{s_0} = \pi_i^{-1} (G^V_{s_i} \nabla_j (\lambda, \kappa)) = \pi_i^{-1} (G^V_{s_i}) \pi_i^{-1} (\nabla_j (\lambda, \kappa)) = \tilde{G}^V_{s_0} \nabla_j (\lambda, \kappa).
\]

Theorems 24 and 29 are proved.
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