Corotating dyonic binary black holes

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This paper is devoted to derive and study binary systems of identical corotating dyonic black holes separated by a massless strut—two 5-parametric corotating binary black hole models endowed with both electric and magnetic charges—where each dyonic black hole carries equal/opposite electromagnetic charge in the first/second model, satisfying the extended Smarr formula for the mass which includes the magnetic charge as a fourth conserved parameter.

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I. INTRODUCTION

It is well-known that any black hole (BH) solution fulfilling the Einstein-Maxwell equations in stationary spacetimes can be described by only three conserved parameters: the mass, electric charge, and angular momentum [1]. This statement is called the no-hair conjecture [2–4] and a Kerr-Newman BH solution [5] is the one depicted by these physical parameters. Obviously, one may bear in mind the addition of the magnetic charge as a fourth conserved parameter since it is also conserved in Einstein-Maxwell theory [6]. In this context, a duality rotation (DR) studied long time ago by Carter [7] seems to be the easiest path to add the magnetic charge into the Kerr-Newman solution in order to describe a rotating dyon—a particle with both electric and magnetic charges [8]—, which will satisfy an extended Smarr formula for the mass [9]. It should be mentioned that in this approach there is no Dirac string (DS) (or monopole hair) joined to the BH.

On the other hand, Tomimatsu [10] via Komar integrals [11] provided simple formulas that permits the derivation of the mass formula for multi-connected horizons. In particular, in a binary BH system he found that in the absence of the global net magnetic charge there appears a DS linking the BHs, in the way of magnetic flux. In this physical scenario the global magnetic charge is eliminated once each BH is equipped with an individual magnetic charge opposite in sign but carrying the same magnitude. Naturally that, the DS vanishes if there are no magnetic charges in the solution [10] recovering the Kerr-Newman BH description. It is worthwhile to stress the fact that there exists two approaches that allow us to add the magnetic charge into each BH; the DR [7] and the study of the DS into the mass [10, 12].

Following Tomimatsu’s approach, in [13–15] has been studied the contribution of the DS into the mass formula for counterrotating dyonic BHs held apart by a massless strut [16, 17], founding that each individual angular momentum suffers additional rotation provided by the presence of the DS in the form $J - Q_H B_H$. However, in a later paper published by Clément and Gal’tsov [12] it was shown that Tomimatsu’s formula [10] are incorrect in the presence of both electric and magnetic charges since the DS affects only the horizon mass and not the angular momentum on the horizon as was reported earlier in [13, 14], where it is necessary to adopt a constant (or gauge) in the magnetic potential in order to give equal weights to balance the horizon mass and angular momentum.

The present paper has as main objective to use Carter’s proposal [7] on the DR to derive the dyonic extensions of two corotating Kerr-Newman binary BH models previously studied in Ref. [18]. These extended models are well represented by five physical arbitrary parameters: mass $M_H$, angular momentum $J_H$, electric charge $Q_H$, magnetic charge $B_H$, and a relative distance $R$, where all the thermodynamical properties contained inside of the extended Smarr formula have been derived in a concise form. In addition, it is demonstrated that after the DR is applied, the physical Komar parameters $\{M_H, Q_H, B_H, J_H\}$ are conserved without the need of any specific constant in the magnetic potential, contrary to the claim made in [19]. In this regard, is also added a short description on the correct use of this gauge in order to include the DS into the horizon mass, by using once again the results given in [14].

II. COROTATING DYONIC BINARY BLACK HOLES

Let us begin this section by introducing the following Ernst equations [20]:

\begin{align}
(\text{Re} \mathcal{E} + |\Phi|^2) \Delta \mathcal{E} &= (\nabla \mathcal{E} + 2\Phi \nabla \Phi) \cdot \nabla \mathcal{E}, \\
(\text{Re} \mathcal{E} + |\Phi|^2) \Delta \Phi &= (\nabla \mathcal{E} + 2\Phi \nabla \Phi) \cdot \nabla \Phi,
\end{align}

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where \((E, \Phi)\) are complex potentials given by \(E = f - |\Phi|^2 + i\Psi\) and \(\Phi = -A_4 + iA'_4\). We notice that Eqs. (1) remain invariant under DR, in which \(\Phi\) might be replaced by \(\Phi e^{i\alpha}\), where \(\alpha\) is a constant duality angle intimately related to the magnetic charge. On the other hand, the line element defining stationary axisymmetric spacetimes is given by (2):

\[
ds^2 = f^{-1} \left[ e^{2\gamma} (d\rho^2 + dz^2) + \rho^2 d\varphi^2 \right] - f (dt - \omega d\varphi)^2.
\]

where \(f(\rho, z), \omega(\rho, z)\), and \(\gamma(\rho, z)\) are metric functions. In order to derive the Ernst potentials and metric functions at the entire space \((\rho, z)\) one can make use of Sibgatullin’s method \([23]\), but it is necessary to adopt first a specific form of the Ernst potentials on the symmetry axis (the axis data). In Ref. \([18]\) we have used the following axis data for asymptotically flat spacetimes

\[
\mathcal{E}(0, z) = \frac{z^2 - 2(M + iq)z + 2\Delta - R^2/4 - \sigma^2 - 2q_o(Q/M) + i\delta}{z^2 + (2M - iq)z + 2\Delta - R^2/4 - \sigma^2 + 2q_o(Q/M) - i\delta},
\]

\[
\Phi(0, z) = \frac{2(qz + q_o)}{z^2 + (2M - iq)z + 2\Delta - R^2/4 - \sigma^2 + 2q_o(Q/M) - i\delta},
\]

\[
\sigma = \sqrt{\frac{\Delta - 4[q_o^2 - (Q/M)^2 q_o^2]}{R^2 - 4\Delta}},
\]

\[
\Delta = M^2 - Q^2 - q_o^2, \quad q_o = q_o + i\bar{b}_o.
\]

with the main objective to describe corotating binary systems of identical Kerr-Newman BHs \([5]\). From the aforementioned Eq. (3) after applying the Hoenselaers-Perjés procedure \([22, 23]\), it is possible to obtain the first Simon’s multipoles \([24]\), where the total mass, total electric charge and total angular momentum of the binary system are represented by \(2M\), \(2Q\), and \(4Mq - \delta\), respectively, while the total electromagnetic dipole moment is given as \(2(q_o + i\bar{b}_o + 2q(Q))\). Moreover, \(R\) defines a separation distance between the sources (see Fig. 1), which may be BHs if \(\sigma^2 \geq 0\) or hyperextreme sources when \(\sigma^2 < 0\). In this framework, the Ernst potentials as well as the full metric in the entire spacetime were obtained in Ref. \([18]\), and they permit us to calculate the physical Komar parameters \([11]\) for BHs via the Tomimatsu formulae \([10, 12]\)

\[
M_H = \frac{-1}{8\pi} \int_H \omega \Psi_{,z} d\varphi dz - M_A^S,
\]

\[
Q_H = \frac{1}{4\pi} \int_H \omega A_3_{,z} d\varphi dz, \quad B_H = \frac{1}{4\pi} \int_H \omega A_4_{,z} d\varphi dz,
\]

\[
J_H = \frac{-1}{8\pi} \int_H \omega \left[ 1 + \frac{\omega \Psi_{,z}}{2} - 3A_{A_{,z}} \right] d\varphi dz - \frac{\omega^H M_A^S}{2},
\]

where \(\omega^H\) is the constant value for \(\omega\) at the horizon while \(A_3 = A_3 + \omega A_4\), being \(A_3\) the magnetic potential. Furthermore, \(M_A^S\) is a boundary term related to the presence of the DS connecting the BHs that is computed by virtue of

\[
M_A^S = \frac{-1}{4\pi} \int_H (A_{3,A_{,z}}) d\varphi dz.
\]

The above-mentioned formulas from Eq. (4) can be rearranged to derive the Smarr formula \([3]\) for the horizon mass of each BH \([12]\),

\[
M_H = \frac{\kappa S}{4\pi} + 2\Omega J_H + \Phi_E^H Q_H = \sigma + 2\Omega J_H + \Phi_E^H Q_H,
\]

where \(\Omega = 1/\omega^H\) is the angular velocity and \(\Phi_E^H = -A_4^H - \Omega A_3^H\) is the electric potential evaluated on the horizon. It is worth noting that we have written down the formula for the mass Eq. (6) with two different aspects since the area of the horizon \(S\) and surface gravity \(\kappa\) are related to the half-length horizon \(\sigma\). Due to the fact that each thin rod representing the BH horizon in Fig. (1a) contains the same length, without loss of generality, it is possible to calculate the Komar parameters by using the values within the domain \(R/2 - \sigma \leq z \leq R/2 + \sigma\) and \(0 \leq \varphi \leq 2\pi\) that define the upper BH. After using the Ernst potentials and metric functions derived in Ref. \([18]\) the mass \(M_H\) and...
electromagnetic charge $Q_H + iB_H$ assume the form

$$M_H = M + \frac{2q_o(Q/M)P_0R(R^2 - 4\Delta)}{[(R + 2M)(R^2 - 4\Delta) - 4q^2] + 64q^2(Q/M)^2} - M^S_A,$$

and these formulas are solved together with the axis condition that disconnects the middle region among sources; it reads

$$8qP_0b_o^2 + 2P_0(2Qb_o + M\delta)(R^2 - 4\Delta) - [2qs_o - (R + 2M)\delta](R^2 - 4\Delta)^2 + 4q\left((P_0 - 2s_o)\left[2q_o^2(1 - 2(Q/M)^2) - \delta^2\right]
+ 4s_oq_o^2\right) = 0,$$

$$P_0 = (R + 2M)^2 + 4q^2, \quad s_o = M(R + 2M) - Q^2.$$

At this point, we would like to discuss a little bit on some physical implications of $M^S_A$ within the Smarr formula. As has been proved recently by Clément and Gal’tsov [12], the correctness of Tomimatsu formulas fails at the moment of including magnetic charges due to fact that the horizon mass $M_H$ in Tomimatsu’s approach does not contain the extra component $M^S_A$. For that reason Tomimatsu’s expression for the horizon mass is determined by

$$M_H = \sigma + 2\Omega J_H + \Phi^H_QH + M^S_A,$$

which looks different than the one introduced in Eq. (6). However, if we want to consider the contribution of the DS provided by the extra term $M^S_A$ into the horizon mass we need to include a constant in the magnetic potential, namely

$$A_{3\text{new}} = K_0 + A_3,$$

and it creates changes in the terms $M^S_A$ and $\Phi^H_E$ in the form

$$M^S_{A\text{new}} = -\frac{1}{4\pi} \int_H (A'_3A_{3\text{new}})_{z\phi}d\phi dz = M^S_A - K_0\Omega Q_H,$$

$$\Phi^H_{E\text{new}} = -A^H_{3\text{new}} - \Omega A^H_{3\text{new}} = \Phi^H_E - K_0\Omega.$$

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[30] This constant gives a symmetric weight for each DS joined to the BH, providing a balance of mass and angular momentum, where each BH will contain half of the total angular momentum of the binary configuration. In Ref. [12], the constant $K_0 = 0$ since it is treating with a single dyonic Kerr-Newman BH.
In order to explain how this constant might be used, we are going to appeal to the results of the paper concerning to an oppositely electromagnetically charged two-body system of identical counterrotating BHs, where its thermodynamical properties were explicitly calculated; they read

\[ M_A^S = B_H (B_H \phi^H - Q_H \Omega), \quad \Phi^H_E = Q_H \phi^H - B_H \Omega, \]

\[ \Omega = \frac{\mu}{2} \frac{(R + 2 \sigma) \sqrt{X} - 1}{M[R + 2 \sigma - (R - 2M)X] - \mu |Q|^2}, \quad \phi^H = \frac{\mu}{2} \frac{R + 2 \sigma - (R - 2M)X}{M[R + 2 \sigma - (R - 2M)X] - \mu |Q|^2}, \]

\[ \sigma = \sqrt{X(M^2 - |Q|^2 \mu) + \frac{R^2}{4}(1 - X)}, \quad \mu := \frac{R - 2M}{R + 2M}, \]  

and because the mass formula Eq. (9), now turns out to be

\[ M_H := M - M_A^{\text{new}} = \sigma + 2\Omega J_H + \Phi^H_E Q_H, \]

it is quite natural to observe from Eqs. (11)-(12) that \( K_0 \) may be chosen as \( K_0 = -B_H \). Therefore, Eq. (13) is further simplified as follows

\[ M_H := M - B_H^2 \phi^H = \sigma + 2\Omega J_H + Q_H^2 \phi^H. \]

The substitution of \( \sigma \) from Eq. (12) into Eq. (11) allows us to obtain

\[ X = 1 + \frac{4J_H^2}{[M(R + 2M) + |Q|^2]^2}, \]  

thus having the explicit form of \( \sigma \)

\[ \sigma = \sqrt{M^2 - (|Q|^2 + J_H^2)[(R + 2M)^2 + 4|Q|^2]/[M(R + 2M) + |Q|^2]^2]} \frac{R - 2M}{R + 2M}. \]

Notice that there is a distinction between the horizon mass \( M_H \) and the parameter \( M \) as has been pointed out earlier by Clément and Gal’tsov [12]; both are identical only whether the boundary term \( M_A^S \) is killed due mainly to the fact that magnetic charges are not present. On the other hand, we have that in the limit \( R \to \infty \) the horizon mass \( M_H \) behaves as

\[ M_H = \lim_{R \to \infty} \left[ M - B_H^2 \phi^H \right] = M - B_H^2 \frac{M + \sigma}{(M + \sigma)^2 + (J_H/M)^2}, \]

\[ \sigma = \frac{M^2 - |Q|^2 - J_H^2/M^2}, \]

recovering the expression for one isolated dyonic Kerr-Newman BH joined to a DS [12]. In this scenario, \( J_H \) defines half of the total angular momentum of the system since it does not contain any contribution coming from the DS. It should be mentioned here, that contrary to the statement provided by the authors in Ref. [19], there is no need to add any constant to the magnetic potential \( A_3 \) when a DR procedure has been applied to include the magnetic charge inside the Kerr-Newman solution. These authors [19] have combined two approaches; the DR studied many years ago by Carter [7] and the addition of a constant in the magnetic potential \( A_3 \), with the objective to prove the correctness of Tomimatsu’s formulas in the presence of both electric and magnetic charges. However, their procedure cannot be correct since there was an error in such a formulas that might lead to physical and mathematical inconsistencies. To prove the last statement on the no need of a such a constant in \( A_3 \) one might use once again the expressions of Eq. [12][32] after eliminating the magnetic charge, thus having

\[ M_A^S = 0, \quad \Phi^H_E = Q_H \phi^H, \]

\[ \Omega = \frac{\mu}{2} \frac{(R + 2 \sigma) \sqrt{X} - 1}{M[R + 2 \sigma - (R - 2M)X] - \mu |Q|^2}, \quad \phi^H = \frac{\mu}{2} \frac{R + 2 \sigma - (R - 2M)X}{M[R + 2 \sigma - (R - 2M)X] - \mu |Q|^2}, \]

\[ \sigma = \sqrt{X(M^2 - |Q|^2 \mu) + \frac{R^2}{4}(1 - X)}, \]

[31] The constant value agrees with \( K_0 = -B_H \) \( (b_0 = -B) \) in [19]. However, a constant must be added to \( A_3 \) only when the contribution of the DS into the horizon mass should be taken into account [12].

[32] These formulas were obtained in Ref. [14] without adding a constant in the magnetic potential \( A_3 \).
where now the mass formula Eq. \( (i) \) reads
\[
M_H := M = \sigma + 2\Omega J_H + Q_H^2 \phi^H. \tag{19}
\]

We have then that a DR procedure \((Q_H \rightarrow Q_H + iB_H\) and\( Q_H^2 \rightarrow |Q|^2)\) extends the conventional mass formula by adding the magnetic charge as a fourth conserved parameter as follows
\[
M_H := M = \sigma + 2\Omega J_H + \Phi_{EL}^H Q_H + \Phi_{MAG}^H B_H, \\
\Phi_{EL}^H = Q_H^2 \phi^H, \quad \Phi_{MAG}^H = B_H \phi^H, \tag{20}
\]
with \(\sigma\) having the same aspect as shown in Eq. \((16)\), but the main distinction is that the parameter \(M\) is representing now the horizon mass \(M_H\). Hence, no DS exists in between BHs. Another heuristic point of view on the electromagnetic charge conservation is reached by killing first the magnetic charge, and later on, the DR \(\Phi \rightarrow \Phi e^{i\alpha}\) is applied, where the real and imaginary components of the new potential \(\Phi e^{i\alpha}\) take the aspect
\[
A_{4\text{new}} = A_4 + \frac{B_H}{Q_H} A'_3, \quad A'_{3\text{new}} = -\frac{B_H}{Q_H} A_4 + A'_3, \tag{21}
\]
whereby \(\alpha = \arctan(B_H/Q_H)\). Observe that the case \(B_H = 0\) recovers the original potential \(\Phi\). Thus, we write the electric and magnetic charges as follows:
\[
Q_{H\text{new}} = \frac{1}{4\pi} \int_H \omega A'_{3\text{new}}, dz \quad d\phi dz, \quad B_{H\text{new}} = \frac{1}{4\pi} \int_H \omega A'_{4\text{new}}, dz \quad d\phi dz, \tag{22}
\]
after the substitution of Eq. \((21)\) into Eq. \((22)\) it is possible to restore the contribution of the magnetic charge, namely
\[
Q_{H\text{new}} = \frac{1}{4\pi} \int_H \omega \left( -\frac{B_H}{Q_H} A_4 + A'_3 \right), dz \quad d\phi dz = Q_H, \\
B_{H\text{new}} = \frac{1}{4\pi} \int_H \omega \left( A_4 + \frac{B_H}{Q_H} A'_3 \right), dz \quad d\phi dz = Q_H \left( \frac{B_H}{Q_H} \right) = B_H. \tag{23}
\]

We now turn our attention to consider a DR in the solution \([18]\) in order to derive two corotating dyonic binary BH models.

III. COROTATING DYONIC BHS ENDOWED WITH IDENTICAL ELECTROMAGNETIC CHARGE

As has been shown already in Ref. \([18]\) one gets an absence of the magnetic charge; i.e., \(B_H = 0\), from Eqs. \((7)-(8)\) by establishing first \(Q = Q_H\) and \(q_o = 0\), thus having
\[
\delta = 2q(R^2 - 4\Delta) \left[ MP_0 + Q_H^2 (R + 2M) \right], \\
b_o = -\frac{qQ_H(R^2 - 4\Delta)(P_0 + 2Q_H^2)}{(R^2 + 2MR + 4q^2)P_0 + 8q^2Q_H^2}, \tag{24}
\]
where the extra term \(M_S^2\) has been eliminated and for such a reason \(M_H = M\). In this respect, the DR can be performed by doing only the following changes \(Q_H \rightarrow Q_H + iB_H\) and \(Q_H^2 \rightarrow Q_H^2 + B_H^2\) in the Ernst potentials on the symmetry axis given above in Eq. \((3)\). The result is
\[
E(0, z) = z^2 - 2(M + i\Delta_o)z + 2\Delta_o + R^2/4 - \Delta_o - i\delta, \\
\Phi(0, z) = \frac{2(Qz + q_o)}{z^2 + 2(M - i\Delta_o)z + 2\Delta_o + R^2/4 - \Delta_o - i\delta}, \tag{25}
\]
with
\[
\delta = \frac{2q(R^2 - 4\Delta_o) \left[ MP_0 + |Q|^2 (R + 2M) \right]}{(R^2 + 2MR + 4q^2)P_0 + 8q^2|Q|^2}, \quad q_o = q_o + ib_o = -\frac{iqQ(R^2 - 4\Delta_o)(P_0 + 2|Q|^2)}{(R^2 + 2MR + 4q^2)P_0 + 8q^2|Q|^2}, \\
Q = Q_H + iB_H, \quad \Delta_o = M^2 - |Q|^2 - q^2. \tag{26}
\]
It should be pointed out that this procedure has added the magnetic charge $B_H$ as a fourth conserved parameter into the solution, where now the electromagnetic charge obtainable from Eq. (7) is given by

$$Q_H + iB_H = Q_H + iB_H + 2\frac{P_0(q_o + ib_o) + i(Q_H + iB_H)\left(q(R^2 - 4\Delta_o) + (R + 2M)\delta\right)}{(R + 2M)(R^2 - 4\Delta_o) - 4\delta},$$  \hspace{1cm} (27)$$

which is identically satisfied by the set of variables expressed lines above in Eq. (20). Similar to the upper dyonic BH, the lower constituent contains the same electromagnetic charge $Q_H + iB_H$. Once we have incorporated the magnetic charge $B_H$, it follows that the Ernst potentials, Kimmers potential $\Phi_2$ [20], and metric functions read

$$\mathcal{E} = \frac{\Lambda + \Gamma}{\Lambda - \Gamma}, \quad \Phi = \frac{\chi}{\Lambda - \Gamma}, \quad \Phi_2 = \frac{F}{\Lambda - \Gamma}, \quad f = \frac{|\Lambda|^2 - |\Gamma|^2 + |\chi|^2}{|\Lambda - \Gamma|^2}, \quad \omega = 4q - \frac{\text{Im}[(\Lambda - \Gamma)\mathcal{F} - \chi\mathcal{I}]}{|\Lambda|^2 - |\Gamma|^2 + |\chi|^2},$$

$$c_{27} = \frac{|\mathcal{L}|^2 - |\Gamma|^2 + |\chi|^2}{64\alpha^4R^6\kappa_0^2r_1r_2r_3r_4,} \quad \Lambda = 2\sigma^2 \left[ R^2\kappa_0(r_1 + r_2)(r_3 + r_4) + 4a(r_1 - r_3)(r_2 - r_4) \right]$$

$$+ 2R^2 \left[ \kappa_0(2\Delta_o - \sigma^2 - a)(r_1 - r_2)(r_3 - r_4) + 2iR\left( 2\text{Re}(s_+) + \text{Im}(p_+) \right) [R(r_1 - r_2)(r_3 - r_4) - 2\sigma \left(r_1r_4 - r_2r_3 + 4\sigma r_3 r_4 \right)] + \kappa_0 \left[ r_1 \left( R^2 r_3 - \kappa_0 r_4 \right) - r_2 \left( \kappa_0 r_3 - R^2 r_4 \right) - 8\sigma^2 r_3 r_4 \right] \right) \right],$$

$$\Gamma = 4\sigma \Gamma(M\alpha_0 - b\chi_+), \quad F = (4q + iz)\chi - i\mathcal{I}, \quad \chi = -4\sigma \Gamma(Q\alpha_0 + 2Q\chi_+), \quad \alpha_0 = R\chi_+ - 2\sigma \chi_+ + 2\chi_1, \quad \mathcal{G} = \text{const} \left[ R^2 \left( 2\text{Re}(a - 2|q_o|^2) + |Q|^2 \kappa_0 \left[ r_1r_2 - r_3 r_4 \right] + 2iQ^2 \kappa_0 \left( r_2 r_3 + r_1 r_4 \right) + 2i \left[ R\text{Im}(a) + \mathcal{F} \alpha_0 - 4q|q_o|^2 \right] \right] x (r_1 - r_3)(r_2 - r_4) \right) - 4R^2 \left\{ \sigma \left[ 2a - (R - 2\sigma) \left( 2(R + 2i\sigma) s_+ + p_+ \right) \right] + i \left( \mathcal{F} + 2Q \kappa_0 - 4q|q_o|^2 \right) \right\} x (r_1 - r_2)$$

$$+ 2\sigma R^2 \left\{ 4 \left[ \kappa_0 \Delta_o - \text{Re}(a) \right] r_2 + \left[ |Q|^2 \kappa_0 + 4|q_o|^2 \right] \left[ r_1 r_2 - r_3 r_4 \right] \right\} (r_3 - r_4) + 4M\sigma \Gamma(\kappa_0 + 2R\chi_+ - 4\sigma^2 \chi_\rho)$$

$$- 4b\sigma \Gamma(R\chi_+ - 2\sigma \chi_\rho) - 8\sigma \Gamma(Q(2b + 2M\alpha_0) \left[ 2\alpha_0 (r_1 - r_2 + r_3 - r_4) + \mathcal{F} \kappa_0 (r_1 - r_2 - r_3 + r_4) \right] \right),$$

$$\mathcal{I} = \text{const} \left[ 4\sigma^2 (r_1 - r_3)(r_2 - r_4) - R^2(r_1 - r_2)(r_3 - r_4) + \Gamma \left[ B_+ \kappa_0 r_1 - B_- \Gamma r_2 \right] r_4 - R\kappa_+ \left[ B_+ \Gamma r_1 - B_- \kappa_0 r_2 \right] r_3 \right.$$  

$$- 16\sigma R^2 \left[ \left( M(R + 2\sigma)(\kappa_0 + 2QR) - B_+ q_o \right) r_3 r_4 - R\kappa_0 (2M Q + Q b) \right] + 8\sigma \Gamma(R\chi_+ + \sigma \chi_\rho)$$

$$+ 2\sigma \Gamma \left( R^2 - 4\Delta_o + \kappa_0 \right) + 8Q \mathcal{Q} \chi_\rho + 12\sigma R^2 \mathcal{Q} \chi_+ + 8Q \sigma \Gamma(R\chi_+ - 2\sigma \chi_\rho),$$

$$\chi_\pm = s_\pm + s_\pm r_2 \pm (\bar{s}_3 - \bar{s}_4), \quad \chi_{\pm} = p_\pm + t_\pm \pm (\bar{p}_3 - \bar{p}_4), \quad \xi_{\pm} = s_\pm + s_\pm r_2 + s_\pm - s_\pm r_3 + \bar{s}_\pm r_4,$$

$$\xi_{\pm} = p_\pm + t_\pm + p_\pm - p_\pm r_3 - p_\pm r_4, \quad a = 2(R + 2i\sigma)p_\pm - s_\pm \left[ \bar{s}_\pm - 2(R + 2i\sigma) \right] \kappa_\pm = 2q_o - Q(R \pm 2\sigma),$$

$$A = M \left[ (2Q + Q(R - 2\sigma)) s_\pm + 2Q p_\pm \right] + B \left[ R^2 - 4\Delta_o \right] - 2(R + 2i\sigma) q_o, \quad b = i(\delta - 4M q),$$

$$B_{\pm} = \left[ R^2 \pm p_\pm + 2Q \left( \mathcal{F} \alpha_0 + Q(R \pm 2\sigma) \right) \right] / M, \quad p_\pm = -\sigma(R^2 - 4\Delta_o) \pm i \left[ 2M \delta + 4i \mathcal{F} \alpha_0 - (R + 2i\sigma) \Gamma(s_\pm) \right],$$

$$s_\pm = 2\Delta_o + \sigma R \pm iQ(R + 2\sigma), \quad \xi_\pm = 4Q \left[ M \delta - 2iQ \mathcal{F} + Q(R -\sigma)^2 \right] + (2iQ - Q(R)^2 - 4\Delta_o), \quad Q = q_o + 2iQ \kappa_0,$$

$$\kappa_0 = R^2 - 4\sigma^2, \quad \rho_1 = (R - 2\sigma) r_1, \quad \rho_3 = (R + 2\sigma) r_3, \quad (28)$$

where $r_n = \sqrt{\rho^2 + (z - \alpha_n)^2}$ are the distances from the value $\alpha_n$ defining the location of the source to any point $(\rho, z)$ outside the symmetry axis. Explicitly are

$$r_{1,2} = \sqrt{\rho^2 + (z - R/2 \mp \sigma)^2}, \quad r_{3,4} = \sqrt{\rho^2 + (z + R/2 \mp \sigma)^2}. \hspace{1cm} (29)$$

In addition, the half-length parameter defining the BH horizon can be written as

$$\sigma = \sqrt{\Delta_o - 4|q_o|^2 - \delta^2 / R^2 - 4\Delta_o}, \hspace{1cm} (30)$$
which explicitly is

\[
\sigma = \sqrt{\Delta_o + \frac{4q^2(R^2 - 4\Delta_o)\left[(MP_0 + |Q|^2(R + 2M))^2 - |Q|^2(P_0 + 2|Q|^2)^2\right]}{(R^2 + 2MR + 4q^2)P_0 + 8q^2|Q|^2}}. \tag{31}
\]

On the other hand, the magnetic potential \(A_3\) is computed straightforwardly through

\[
A_3 = \text{Re}(\Phi_2) = -4qA_1 - zA_3' + \text{Im}\left(\frac{\mathcal{I}}{\Lambda - \Gamma}\right). \tag{32}
\]

In this case the components of the extended mass formula Eq. (20), \(\Omega\) and \(\phi^H\) are given by

\[
\Omega = \frac{q[8(R^2 - 4\Delta_o + 2\sigma)(R + 2\sigma)]}{\mathcal{L}^2 + \mathcal{M}^2},
\]

\[
\phi^H = \frac{(R + 2\sigma)\mathcal{L} - 2(b_0/Q_H)\mathcal{M}}{\mathcal{L}^2 + \mathcal{M}^2},
\]

\[
\mathcal{L} = MR + 2\Delta_o + (R + 2M)\sigma, \quad \mathcal{M} = \delta + q(R + 2\sigma), \tag{33}
\]

and their combination with Eq. (31) defines the angular momentum from Eq. (20), to obtain

\[
J_H = 2Mq - \frac{q(R^2 - 4\Delta_o)[MP_0 + |Q|^2(R + 2M)]}{(R^2 + 2MR + 4q^2)P_0 + 8q^2|Q|^2}. \tag{34}
\]

which is nothing less than half of the total angular momentum; i.e., \(2Mq - \delta/2\). In addition, we have that the area of the horizon \(S\) and surface gravity \(\kappa\) are expressed as

\[
\frac{S}{4\pi} = \frac{\sigma}{\kappa} = \frac{\mathcal{L}^2 + \mathcal{M}^2}{R(R + 2\sigma)}. \tag{35}
\]

while the interaction force associated to the conical singularity is computed by using the formula \(\mathcal{F} = (e^{-\gamma_s} - 1)/4\) \cite{7, 28}, where \(\gamma_s\) is the constant value for the metric function \(\gamma\) in the axis region in between sources. The result is

\[
\mathcal{F} = \frac{[(M^2 - |Q|^2)P_0^2 - 4q^2|Q|^4](P_0 - 8q^2) - 16q^2|Q|^2[S_oP_0 - |Q|^4]}{(R^2 - 4\Delta_o)P_0^3},
\]

\[
S_o = M(R + 2M) - |Q|^2. \tag{36}
\]

We end this section by underlining that if \(R \to \infty\), from Eqs. (31) and (33) is recovered the formula \(\sigma = \sqrt{M_H^2 - |Q|^2 - J_H^2/M_H^2}\) defining an isolated dyonic BH free of monopolar sources.

IV. COROTATING DYONIC BHS ENDOWED WITH OPPOSITE ELECTROMAGNETIC CHARGE

Concerning the second charged model, where setting first \(Q = 0\) and \(b_0 = 0\) in Eqs. (7)-(8) permits us to eliminate the magnetic charges by means of

\[
\delta = \frac{2q(R^2 - 4\Delta_1)[MP_0 - Q_H^2(R + 2M)]}{(R^2 + 2MR + 4q^2)P_0 - 8q^2Q_H^2}, \quad q_o = \frac{Q_H R(R^2 - 4\Delta_1)P_0}{2[(R^2 + 2MR + 4q^2)P_0 - 8q^2Q_H^2]}, \quad \Delta_1 = M^2 - q^2, \tag{37}
\]

where now each BH contains an opposite electric charge. Once again we have that \(M = M_H\) since \(M_A^S = 0\). So, in this case the Ernst potentials on the symmetry axis after performing the DR are now

\[
\mathcal{E}(0, z) = \frac{z^2 - 2(M + iq)z + 2\Delta_1 - R^2/4 - \sigma^2 + i\delta}{z^2 + 2(M - iq)z + 2\Delta_1 - R^2/4 - \sigma^2 - i\delta},
\]

\[
\Phi(0, z) = \frac{2q_o}{z^2 + 2(M - iq)z + 2\Delta_1 - R^2/4 - \sigma^2 - i\delta}, \tag{38}
\]
with
\[
\delta = \frac{2q(R^2 - 4\Delta_1)[MP_0 - |Q|^2(R + 2M)]}{(R^2 + 2MR + 4q^2)P_0 - 8q^2|Q|^2}, \quad q_o = q_o + ib_o = \frac{QR(R^2 - 4\Delta_1)P_0}{2[(R^2 + 2MR + 4q^2)P_0 - 8q^2|Q|^2]},
\]
(39)

where now the electromagnetic charge derived from Eq. (7) and satisfied by Eq. (39) reduces to
\[
Q_H + iB_H = \frac{2(q_o + ib_o)P_0}{(R + 2M)(R^2 - 4\Delta_1) - 4q^2}.
\]
(40)

It is worthwhile to mention, that the lower dyonic BH is endowed with oppositely electromagnetic charge; i.e., \(-Q_H - iB_H\). Then, the Ernst and Kinnersley potentials, as well as the metric functions, have now the form
\[
E = \frac{\Lambda + \Gamma}{\Lambda - \Gamma}, \quad \Phi = \frac{\chi}{\Lambda - \Gamma}, \quad \Phi_2 = \frac{F}{\Lambda - \Gamma}, \quad f = \frac{|\Delta|^2 - |\Gamma|^2 + |\chi|^2}{|\Lambda|^2 - |\Gamma|^2}, \quad \omega = 4q - \frac{\text{Im}[(\Lambda + \Gamma)(\chi - \chi\bar{\Gamma})]}{|\Lambda|^2 - |\Gamma|^2 + |\chi|^2},
\]
e\[e^{2\gamma} = \frac{|\Delta|^2 - |\Gamma|^2 + |\chi|^2}{64\sigma^4 R^4 \kappa^2 r^2 r^2 r_3 r_4}, \quad \lambda = 2\sigma^2 \left[ R^2 \kappa_o(r_1 + r_2)(r_3 + r_4) + 4a(r_1 - r_3)(r_2 - r_4) \right]
\]
\[+ 2R^2 \left[ \kappa_o(2\Delta_1 - \sigma^2) - a \right] (r_1 - r_2)(r_3 - r_4) + 2iR \left\{ \left( 2q\text{Re}(s+) + \text{Im}(p+) \right) \right] R(r_1 - r_2)(r_3 - r_4)
\]
\[- 2\sigma(r_1 r_4 - r_4 r_3 + 4\sigma r_1 r_4) + q_o \left[ r_1 (R^2 r_3 - \kappa_o r_4) - r_2 (\kappa_o r_3 - R^2 r_4) - 8\sigma^2 r_1 r_4 \right],
\]
\[\Gamma = 4\sigma R(M \Gamma_o - b\chi_+), \quad F = (4q + iz)\chi - i\mathcal{I}, \quad \chi = -8\sigma R q o \chi_+, \quad \Gamma_o = R \chi_+ - 2\sigma \chi_+ + 2\chi_1 + \mathcal{G} = 2\sigma \chi + 8\sigma^2 \left\{ 2R\left( R(a) - 2|q_o|^2 \right)(r_1 r_2 - r_3 r_4) + 2i q^2 R^2 \kappa_o(r_3 r_4 + r_1 r_4) + 2i \left[ \text{Im}(a) - 4q |q_o|^2 \right] (r_1 r_3)(r_2 r_4)
\]
\[+ 2R^2 \left[ \left( 2a - R - 2\sigma \right) \left( 2(R + 2i q) s_+ + p_+ \right) \right] - 4iq |q_o|^2 \right](r_1 - r_2)(r_3 - r_4)
\]
\[+ 8\sigma^2 R \left[ \left( 2\kappa_o\Delta_1 - R(a) \right) r_4 + |q_o|^2 (r_3 + r_4) \right](r_1 - r_2) + 8\sigma^2 R \left[ \left( 2\kappa_o\Delta_1 - R(a) \right) r_2 + |q_o|^2 (r_1 + r_2) \right](r_3 - r_4)
\]
\[+ 4\sigma R \left\{ M \left[ \kappa_o \chi_+ + 2 R \chi_1 + 4\sigma \chi_+ - 8|q_o|^2 (r_1 - r_2 + r_3 - r_4) - b(R \chi_+ + 2\sigma \chi_+) \right],
\]
\[\mathcal{I} = 4q_o \left\{ A \left[ 2\sigma^2 (r_1 - r_3)(r_2 - r_4) - R^2 (r_1 - r_3)(r_2 - r_4) \right] + R \left[ B_+ \left( (R + 2\sigma) r_4 - R r_3 \right) r_1 + B_+ \left( (R - 2\sigma) r_3 - R r_4 \right) r_2
\]
\[+ 8\sigma^2 R \left[ \left( M(2\sigma - 2\sigma) \right) r_3 r_4 - MR \kappa_o \right] + 2\sigma R(\chi_1 + \sigma \chi_+) + 4i \sigma R \chi_+ + 3\sigma R^2 \chi_-= \right],
\]
\[\chi_\pm = s_+ r_1 - s_+ r_2 \pm (s_- r_3 - s_- r_4), \quad \chi_\pm = s+ r_1 + s+ r_2 \pm (s+ r_3 + s+ r_4), \quad \chi_\pm = s_+ r_1 + s_+ r_2 \pm (s_+ r_3 + s_+ r_4), \quad \kappa_o = R^2 - 4\sigma^2, \quad a = 2(R + 2i q) p_+ - s_+ [s_+ - 2(2R + 2i q)^2],
\]
\[A = 2Ms_+ - (R + 2i q) B_+, \quad B_\pm = M(R \pm 2\sigma) + i \delta, \quad r_{1,2} = (R - 2\sigma) r_{1,2}, \quad r_{3,4} = (R + 2\sigma) r_{3,4},
\]
\[b = i(\delta - 4M q), \quad p_\pm = -\sigma(R^2 - 4\Delta_1) \pm i \left[ 2M \delta - (R + 2i q) \text{Im}(s_\pm) \right], \quad s_\pm = 2\Delta_1 + \sigma R \pm i q(R + 2\sigma),
\]
(41)
with \(\sigma\) having now the aspect
\[
\sigma = \sqrt{M^2 - q^2 + \frac{(R^2 - 4M^2 + 4q^2)^2}{[2q(RP_0 - |Q|^2(R + 2M)]^2 - (|Q|R P_0)^2}. \quad \text{(42)}
\]

On the other hand, the thermodynamical properties of the extended Smarr formula Eq. (20) become
\[
\Omega = \frac{q[R^2 - 4M^2 + 4q^2 + 2\sigma(R + 2\sigma)] - 2M \delta}{N^2 + M^2},
\]
\[
\phi^H = \frac{2(q o/Q_H)N}{N^2 + M^2}, \quad \frac{S}{4\pi} = \frac{\sigma}{\kappa} = \frac{N^2 + M^2}{R(R + 2\sigma)},
\]
\[
N = MR + 2M^2 - 2q^2 + (R + 2M) \sigma,
\]
(43)
where they define the angular momentum of the horizon in the way
\[
J_H = 2Mq - \frac{\delta}{2} = 2Mq - \frac{q(R^2 - 4M^2 + 4q^2)[MP_0 - |Q|^2(R + 2M)]}{(R^2 + 2MR + 4q^2)P_0 - 8q^2|Q|^2}. \tag{44}
\]

Moreover, it is not difficult to show that the interaction force is given by
\[
\mathcal{F} = \frac{(M^2P_0^2 - 4q^2|Q|^4)(P_0 - 8q^2) + |Q|^2[R^2P_0 - 4q^2(R^2 - 4\Delta_1)]P_0}{(R^2 - 4\Delta_1)P_0^3}. \tag{45}
\]

Finally, it is worth commenting that if \(B_H = 0\), all the physical and thermodynamical features in both models are reduced to those ones defining corotating binary systems of identical Kerr-Newman BHs [18].

V. CONCLUDING REMARKS

Following Carter’s approach [3], we have been able to apply a DR in two identical corotating Kerr-Newman binary BH models recently studied in [18] with the purpose to add individual magnetic charges to each BH. Therefore, each corotating BH is endowed with identical/opposite electromagnetic charge in the first/second configuration and satisfying a generalized Smarr formula for dyonic BHs. These models containing a conical singularity in between sources, are well represented by five physical arbitrary parameters \(\{M_H, J_H, Q_H, B_H, R\}\). It is worth remarking that the DR approach describe configurations of dyonic BHs free of monopolar hair. On the other hand, we have also shown in both corotating dyonic models that the mass \(M_H\), angular momentum \(J_H\), and electromagnetic charge \(Q_H + iB_H\) are conserved parameters under DR and there is no necessity to add a gauge in the magnetic potential \(A_3\) as has been claimed in [19]. On the contrary, the addition of a gauge is only needed when the contribution of the DS into the horizon mass \(M_H\) is introduced in order to balance each dyonic BH [12].

We will like to pointed out that the extreme limit case \((\sigma = 0)\) of corotating dyonic BHs has not been considered here since it can be easily derived from the formulas given in [18], where both electrically charged configurations were well defined. In fact, we would like to mention that both dyonic configurations satisfy the Gabach-Clement identity [29] for extreme BHs with struts, namely
\[
\sqrt{1 + 4F_{\text{ext}}} = \frac{\sqrt{(8\pi J_H)^2 + (4\pi Q_H^2 + 4\pi B_H^2)^2}}{S_{\text{ext}}}, \tag{46}
\]
where \(S_{\text{ext}}\) and \(F_{\text{ext}}\) define the horizon area and the force in the extreme case, respectively. Due to the fact that the expression of the force contains the same aspect in both extreme and non-extreme configurations, because it does not depend on \(\sigma\). For instance, in the identical electromagnetically charged model, we have that
\[
S_{\text{ext}} = 4\pi \frac{(MR + \Delta_o)^2 + (\delta + qR)}{R^2}, \tag{47}
\]
and the substitution of Eqs. (46) and (47) into Eq. (46) yields the expression
\[
\Delta_o = 4q^2(R^2 - 4\Delta_o)([MP_0 + |Q|^2(R + 2M)]^2 - |Q|^2(P_0 + 2|Q|^2)^2) - [R^2 + 2MR + 4q^2]P_0 + 8q^2|Q|^2 = 0, \tag{48}
\]
which is exactly the condition \(\sigma = 0\) for extreme dyonic BHs on Eq. (51). Furthermore, one may proceed in the same manner for the oppositely dyonic configuration in order to derive once again the extreme condition \(\sigma = 0\) on Eq. (42).

To conclude, one might speculate that the addition of a gauge to the magnetic potential \(A_3\) balancing the mass and angular momentum over the horizon in the presence of the DS, will have the same expression \(M^2_H = B_H \Phi_{MAG}^H = B_H' \phi^H\) as in [12], where \(\phi^H\) has been obtained in this work for both corotating dyonic models. This speculation must be proven by calculating explicitly the term \(M^2_H\) in the presence of magnetic charges, which seems to be a fairly complicated issue. However, if one is capable to circumvent all the technical details, the dyonic models will always contain magnetic charges of equal magnitude but opposite sign.

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