Linear vector fields and exponential law

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Abstract

The paper is devoted to vector fields on the spaces $\mathbb{R}^2$ and $\mathbb{R}^3$, their flow and invariants. Attention is plaid on the tensor representations of the group $\text{GL}(2, \mathbb{R})$ and on fundamental vector fields. The rotation group on $\mathbb{R}^3$ is generalized to rotation groups with arbitrary quadrics as orbits.
1. Introduction

In the paper are presented equations for calculation of invariants of a vector field when an infinitesimal symmetry is known. The flow and invariants of a linear vector field, which is a fundamental vector field (group operator) for the general linear group $GL(2, \mathbb{R})$ are governed by exponential law. Representations of $GL(2, \mathbb{R})$ on tensor fields are investigated. If a tensor field is invariant relative to a vector field, on its components act an extension of this field (classical situation); if the components are invariant, the field and its Lie derivatives form linear ordinary differential equation. Some examples are presented.

The flow and invariants of vector fields on $\mathbb{R}^2$ are discussed in section 2. Linear vector fields on $\mathbb{R}^2$, their Lie derivatives and flows are briefly considered in section 3. Section 4 contains a full classification of the flows of the linear vector fields on $\mathbb{R}^2$. In particular, it is recalled that their singularities can of the types focus, saddle point and node; the corresponding phase portraits are drawn. The basic fundamental vector fields of the rotation group of $\mathbb{R}^3$ are investigated in section 5. Section 6 is devoted to tensor representations of the two-dimensional real general linear group.

Section 7 closes the paper.

2. Flow and invariants of vector fields on $\mathbb{R}^2$

Let on $\mathbb{R}^2$, coordinated by $(u, v)$, be given a vector field

$$X = x \frac{\partial}{\partial u} + y \frac{\partial}{\partial v} \quad (2.1)$$

with components $(x, y)$ depending on $u$ and $v$. The action of $X$ on a $C^1$ function $f$ will be denoted via “prime”, viz.

$$f' = f_1 x + f_2 y$$

with $f_1 := \frac{\partial f}{\partial u}$ and $f_2 := \frac{\partial f}{\partial v}$. With a prime will be denoted also the Lie derivatives relative to $X$ of tensor fields, if the field $X$ is clear from the context; otherwise the standard notation $L_X$ will be used. In particular, we have

$$u' = x(u, v)$$
$$v' = y(u, v). \quad (2.2)$$

If we consider these equalities as equations, they define the flow of $X$, viz. a 1-parameter group $a_t$, $t \in \mathbb{R}$, of diffeomorphisms of $\mathbb{R}$ such that

$$a_t: (u, v) \rightarrow (u_t, v_t). \quad (2.3)$$

A function $f$ is dragged by the flow $a_t$ according to the rule

$$f \mapsto f_t = f \circ a_t. \quad (2.4)$$

It it happens that $f_t$ has a $C^\infty$ dependence on $t$, we can write the Maclaurin series

$$f_t = \sum_{k=0}^{\infty} f^{(k)} \frac{t^k}{k!}. \quad (2.5)$$

If between the derivatives of $f$ there is some connection, i.e. an ordinary differential equation relative to $f$, its solution gives some dragged function; e.g. it $f'' + f = 0$, then $f_t = f \cos t + f' \sin t$. 

The differential form
\[ \omega = -ydu + xdv, \tag{2.5} \]
which is annihilated by \( X \), is a useful tool for finding invariants of \( X \). If \( \omega \) is closed, i.e. \( d\omega = 0 \), it is locally exact and equals the differential of an invariant \( I \) of \( X \):
\[ \omega = dI \Rightarrow I' = dI(X) = \omega(X) = 0, \tag{2.6} \]
so that
\[ I = \int \omega. \tag{2.7} \]
If \( \omega \) is not exact, there is an integrating factor \( \mu \) such that \( \mu\omega \) is an exact form, \( 0 = d(\mu\omega) = (\mu' + \text{div } X \cdot \mu)\omega \) with \( \text{div } X = x_1 + y_2 \). Consequently, solution of the equation
\[ \mu' + \text{div } X \cdot \mu = 0 \tag{2.8} \]
is an integrating factor of \( \omega \) and hence the equation
\[ I = \int \mu\omega \tag{2.9} \]
gives an invariant of \( X \).

If a vector field \( P \) is an infinitesimal symmetry of \( X \), i.e.
\[ \mathcal{L}_P X \parallel X \text{ or } \mathcal{L}_P \omega \parallel \omega, \]
where \( \parallel \) means equal up to multiplicative function, then \( \frac{1}{\omega(P)} \) is an integrating factor for \( \omega \) (see (2.8)) and hence
\[ I = \int \frac{\omega}{\omega(P)} \tag{2.10} \]
is an invariant of \( X \). This fact is a consequence of \( \omega(X) = 0 \) and \( \omega' = \text{div } X \cdot \omega \) which imply \( (\omega(P))' = \text{div } X \cdot \omega(P) \).

In particular, if the coordinate functions \( u \) and \( v \) are respectively a canonical parameter and invariant of \( P \), \( P(u) = 1 \) and \( P(v) = 0 \), then we can set \( P = \frac{\partial}{\partial u} \), so that \( \mathcal{L}_P X \parallel X \) and \( \omega(P) = -y = -X(v) \). Therefore \( \frac{\omega}{\omega(P)} = du - f(v)dv \) with \( f(v) := \frac{X(u)}{X(v)} \), due to \( x = X(u) \), and consequently \( f(v) \) is an invariant of \( P \) and
\[ I = u - \int f(v)dv \tag{2.11} \]
is an invariant of \( X \).

3. Linear vector fields on \( \mathbb{R}^2 \)

A linear vector field on \( \mathbb{R}^2 \) coordinated by \( (u, v) \) is of the form
\[ X = u' \frac{\partial}{\partial u} + v' \frac{\partial}{\partial v} \tag{3.1} \]
where its components \( u' \) and \( v' \) are linear homogeneous functions of the coordinates \( u \) and \( v \), i.e.
\[ \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{3.2} \]
Introducing the matrices
\[\begin{align*}
U &:= \begin{pmatrix} u \\ v \end{pmatrix}, \\
R &:= \frac{\partial}{\partial U} := \begin{pmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial v} \end{pmatrix}, \\
C &:= \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}, \\
\theta &:= dU := \begin{pmatrix} du \\ dv \end{pmatrix}
\end{align*}\] (3.3)
we see that the flow of \(X\) is locally governed by an exponential law:
\[U' = CU \Rightarrow U_t = e^{tC}U.\] (3.4)

One can interpret this as follows: from the l.h.s. of the implication is an ordinary differential equation and from its r.h.s. is written its general solution.

Calculating the Lie derivatives of the frame \(\frac{\partial}{\partial U}\) and coframe \(dU\) and their dragging in the flow of \(X\), we get
\[\begin{align*}
\left(\frac{\partial}{\partial U}\right)' &= -\frac{\partial}{\partial U} C \Rightarrow \left(\frac{\partial}{\partial U}\right)_t = \frac{\partial}{\partial U} e^{-tC} \\
(dU)' &= CU \Rightarrow (dU)_t = e^{tC}dU.
\end{align*}\] (3.5a, 3.5b)

From the Hamilton-Cayley formula for \(C\), viz. \(C^2 - \text{tr} C \cdot C + \det C \cdot \mathbb{I} = 0\), and the l.h.s. of (3.4), we see that (each element of) \(U\) satisfies the second-order ordinary differential equation
\[U'' - \text{tr} C \cdot U' + \det C \cdot U = 0.\] (3.6)

Similar equations for \(dU\) and \(\frac{\partial}{\partial U}\) can be found by means of the right hand sides of (3.5):
\[\begin{align*}
\left(\frac{\partial}{\partial U}\right)'' &= \text{tr} C \cdot \left(\frac{\partial}{\partial U}\right)' + \det C \cdot \left(\frac{\partial}{\partial U}\right) = 0 \\
(dU)'' &= \text{tr} C \cdot (dU)' + \det C \cdot dU = 0.
\end{align*}\] (3.7, 3.8)

If \(Y\) and \(\Phi\) are respectively a vector field and one-form with matrix representations \(Y = \frac{\partial}{\partial U} y\) and \(\Phi = \varphi dU\), with \(y = \begin{pmatrix} y^1 \\ y^1 \end{pmatrix}\) and \(\varphi = (\varphi_1, \varphi_2)\), one can easily deduce the implications
\[\begin{align*}
Y' &= 0 \Rightarrow y' = Cy \Rightarrow y'' - \text{tr} C \cdot y' + \det C \cdot y = 0 \Rightarrow y_t = e^{Ct}y \\
\Phi' &= 0 \Rightarrow \varphi' = -\varphi C \Rightarrow \varphi'' + \text{tr} C \cdot \varphi' + \det C \cdot \varphi = 0 \Rightarrow \varphi_t = \varphi e^{-Ct}.
\end{align*}\] (3.9a, 3.9b)

We see here a representation of the group \(\text{GL}(2, \mathbb{R})\) on the space of components \(y\) and/or \(\varphi\). To an element \(C \in \mathfrak{gl}(2, \mathbb{R})\) (here we identify \(\mathfrak{gl}(2, \mathbb{R})\) with the isomorphic to it set of all \(2 \times 2\) matrices) corresponds the one-parameter group \(e^{Ct}\) in \(\text{GL}(2, \mathbb{R})\), which defines the dragging of \(y\) and \(\varphi\) in the flow of \(X\). As explained in section 3 below, this situation can be generalized on the space of tensor fields of arbitrary type.

It is worth writing also the (dual to (3.9)) implications (with Lie derivatives)
\[\begin{align*}
y' &= 0 \Rightarrow Y' = (\frac{\partial}{\partial U})' y \Rightarrow Y'' - \text{tr} C \cdot Y' + \det C \cdot Y = 0 \Rightarrow Y_t = \frac{\partial}{\partial U} e^{-Ct}y \\
\varphi' &= 0 \Rightarrow \Phi' = \varphi (dU)' \Rightarrow \Phi'' - \text{tr} C \cdot \Phi' + \det C \cdot \Phi = 0 \Rightarrow \Phi_t = \varphi e^{Ct}dU
\end{align*}\] (3.10a, 3.10b)

which describe the situation when the components \(y\) and \(\varphi\) are invariant in the flow of \(X\).

An invariant of a linear vector field can easily be found by noticing that the homothety operator
\[P = u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v}\] (3.11)
commutes with $X$ and hence is an infinitesimal symmetry. One can calculate, using the formalism from section 2, that the function

$$ I = \frac{1}{2} \ln |W| - \frac{c_1 + c_2}{2} \int \frac{dp}{c_3 + (c_1 - c_4)p - c_2p^2} $$

(3.12)

with $W = \begin{vmatrix} u & u' \\ v & v' \end{vmatrix}$ being the Wronskian of $X$ and $p = \frac{x}{y}$ is an invariant of $P$.

4. Classification of the flows

The classification of the flows and phase portraits of a linear vector field on $\mathbb{R}^2$ will be presented below by means of the eigenvalues $\lambda_1$ and $\lambda_2$ of the matrix $C$, which are generally complex numbers. These are represented on the complex plane $\mathbb{C}^2$ by two points which form a segment with middle point $\alpha = \frac{1}{2}(\lambda_1 + \lambda_2)$. The following four cases are possible:

1. **(Elliptical flow)** The eigenvalues $\lambda_1$ and $\lambda_2$ are complex conjugate, i.e. $\lambda_{1,2} = \alpha \pm i\beta$ for some $\alpha, \beta \in \mathbb{R}$ with $\beta \neq 0$. An elliptic rotation is observed on the plane, affected by a homothety for $\alpha \neq 0$. At the origin, we have a stable (for $\alpha < 0$) or unstable (for $\alpha > 0$) focus.

2. **(Hyperbolic flow)** The numbers $\lambda_1$ and $\lambda_2$ are different real ones, $\lambda_{1,2} = \alpha \pm \beta$ for some $\alpha, \beta \in \mathbb{R}$ with $\beta \neq 0$. Now a hyperbolic rotation can be seen on the plane which is affected by a homothety for $\alpha \neq 0$. At the origin we have a saddle point for $\lambda_1\lambda_2 < 0$ or hyperbolic node for $\lambda_1\lambda_2 > 0$ which is stable if $\alpha < 0$ or unstable if $\alpha > 0$.

3. **(Parabolic flow)** The eigenvalues $\lambda_1$ and $\lambda_2$ are equal and non-vanishing, $\lambda_1 = \lambda_2 = \alpha$ for some $\alpha \in \mathbb{R} \setminus \{0\}$ (and $\beta = 0$). The flow is degenerate and at the origin is observed a parabolic node which is stable if $\alpha < 0$ or unstable if $\alpha > 0$.

4. **(Degenerate case)** The matrix $C$ is one-time degenerate, so that one of its eigenvalues $\lambda_1$ or $\lambda_2$ vanishes

$$ \det C = 0 \quad \text{with} \quad \lambda_1\lambda_2 = 0 \quad \lambda_1 + \lambda_2 \neq 0. $$

Then there are numbers $a$ and $b$ such that one of them is non-zero and

$$ (a, b) \cdot \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = 0. $$

Since (3.2) implies $au' + bv' = 0$, the linear function

$$ I = au + bv $$

is an invariant of the vector field $X$. The trajectories of $X$ are parallel lines. Along them is observed an exponential movement of the points. For instance, if $a = -1$ and $b = k$, the function

$$ f = c_1u + c_2v $$

is dragged via exponential law,

$$ f' = kf \quad \Leftrightarrow \quad f_t = fe^{kt}. $$
The points situated on the line $f = \text{const}$ move in an identical way and the vector field $X$ is

$$X = f \left( \frac{\partial}{\partial x} + \frac{1}{k} \frac{\partial}{\partial v} \right).$$

We shall summarize the above considerations in the following proposition.

On Figure 4.1 are shown the phase portraits of a linear vector field situated relative to the parabola $\Delta = 0$ with $4\Delta = (\text{tr } C)^2 - 4 \det C = (c_1 - c_4)^2 + 4c_2c_3 = (\lambda_1 - \lambda_2)^2$. The focuses are inside the parabola ($\Delta < 0$), the parabolic nodes ($\det C = 0$) are on it ($\Delta = 0$), and at last, the saddle points ($\det C < 0$) and hyperbolic nodes ($\det C > 0$) are outside that curve ($\Delta > 0$) (see also [1, p. 86], where a similar picture is presented, but without some details).

**Figure 4.1:** Classification of the phase portraits on the plane $(x, y)$, with $x = \text{tr } C$ and $y = \det C$, relative to the parabola $\Delta = x^2/4 - y = 0$.

**Proposition 4.1.** Let the system of inhomogeneous equations

$$u' = c_1 u + c_2 v + b_1$$
$$v' = c_3 u + c_4 v + b_2,$$

or in a matrix form

$$U' = CU + B,$$

corresponds to a vector field $X$ (cf. 3.2). If the matrix $C$ is nondegenerate, the flow of $X$ is described similar to the one of a linear vector field with the only difference that the singularity is not at the coordinate origin but at a point $U_0$ such that

$$CU_0 + B = 0.$$
The movement depends on is the rank of the matrix \((C|B)\), consisting of the blocks \(C\) and \(B\), equal or not to the one of \(C\).

**Proof.** Take \(U_0\) such that \(CU_0 + B = 0\). Then the inhomogeneous system \(U' = CU + B\) is tantamount to the homogeneous system
\[
(U - U_0)' = C(U - U_0)
\]
relative to \((U - U_0)\) with a singularity at the point \(U_0\).

If the rank of \(C\) is 2, the classification of linear vector fields holds. When the rank of \(C\) is 1, we can put without lost of generality \(c_1 = kc_3\) and \(c_2 = kc_4\) for some number \(k\). So that \(X\) takes the form
\[
X = (c_3u + c_4v)(k \frac{\partial}{\partial u} + \frac{\partial}{\partial v}) + b_1 \frac{\partial}{\partial u} + b_2 \frac{\partial}{\partial v}.
\]
The trajectories are exponential curves similar to the one of the graph of the function \(\mathbb{R} \ni x \mapsto e^x\).

When \(c_1 = c_2 = 0\), we have only a uniform movement along straight lines. \(\square\)

**Proposition 4.2.** The straight lines remain straight lines and their parallelism is preserved in the flow of a linear vector field, as shown on figure 4.2.

![Figure 4.2: Preservation of straight lines and parallelism.](image)

**Proof.** Since the level lines of a linear function are straight lines, it suffice to prove that a linear function is dragged by such a vector field into a linear function. The flow is determined via the exponential flow
\[
U' = CU \quad \Rightarrow \quad U_t = e^{Ct}U.
\]
Suppose \(A = (a, b)\) is a matrix-raw with \(a, b \in \mathbb{R}\). An arbitrary linear function \(f\) has a representation \(f = AU\) and is dragged according to the law \(f_t = AU_t = Ae^{Ct}U\). Since the last function is linear for any fixed \(t\), the straight lines remain such and the parallelism is preserved; however, segments of the lines can expand (for \(\text{div} \; X > 0\)) or contract (for \(\text{div} \; X < 0\)). \(\square\)

**Proposition 4.3.** When moving along the trajectories of a linear vector field together with a coordinates system (moving frame), all points, moving along their own trajectories, form in the moving frame the same phase portrait as the one in which they are involved. This situation is illustrated on figures 4.3 and 4.4.
Proof. The point $U$, where we are initially situated, is moving according to $U_t = e^{Ct}U$. At a neighboring point, we observe that it has radius-vector $U + dU$ in the non-moving frame, but in the moving frame its radius-vector is $dU$. From the law

$$(U + dU)_t = e^{Ct}(U + dU),$$

we get $(dU)_t = e^{Ct}dU$, which proofs our assertion. The radius-vectors $U$, relative to the non-moving frame, and $dU$, relative to the moving frame, are dragged by the flow in an identical way.

Remark 4.1. The Jacobi matrix in a flow of a nonlinear vector field is non-constant and can change when passing from one point to other.

$$u' = u - v - u(u^2 + v^2)$$
$$v' = u + v - v(u^2 + v^2)$$

with attractor $u^2 + v^2 = 1$. To the five phase portraits correspond Jacobi matrices at five points when moving from the origin $(0, 0)$ up along the $v$ axes.

If an observer judges on a flow by taking into account only the Jacobi matrix at a given point, the different observes obtain different results and correspondingly they will have different opinions. In a case of a linear vector field, the Jacobi matrix is constant and any local picture is an exact copy of the global one.

\[\text{\footnotesize $\diamond$} \quad \text{\footnotesize 1 The same result follows also from $U' = CU$, viz. since $d(U') = (dU)'$, the Lie derivative commutes with the exterior differentiation, then $(dU)' = CdU$.}\]
5. Rotation groups in $\mathbb{R}^3$

The group of rotation of the space $\mathbb{R}^3$ has three basic fundamental vector fields, say $X$, $Y$ and $Z$, which in a matrix notation can be written as (in standard Cartesian coordinates $(u, v, w)$)

$$(X, Y, Z) = \left( \frac{\partial}{\partial u}, \frac{\partial}{\partial v}, \frac{\partial}{\partial w} \right) \cdot \begin{pmatrix} 0 & -w & v \\ w & 0 & -u \\ -v & u & 0 \end{pmatrix}.$$  

As vector fields, the operators

$$X = w \frac{\partial}{\partial v} - v \frac{\partial}{\partial w}, \quad Y = -w \frac{\partial}{\partial u} + u \frac{\partial}{\partial w}, \quad Z = v \frac{\partial}{\partial u} - u \frac{\partial}{\partial v} \quad (5.1)$$

are (linearly) dependent, viz. $uX + vY + wZ = 0$,

and the function

$$f = \frac{1}{2}(u^2 + v^2 + w^2)$$

is their common invariant. The level surfaces of the function $f$ are concentric spheres and are orbits of the rotation group. The table of commutators of the operators $X$, $Y$ and $Z$ can easily be computed:

| $\mathcal{P}$ | $X$ | $Y$ | $Z$ |
|---------------|-----|-----|-----|
| $X$ | 0 | $Z$ | $-X$ |
| $Y$ | $-Z$ | 0 | $Y$ |
| $Z$ | $X$ | $-Y$ | 0 | (5.2)

A linear combination

$$P = \omega_1 X + \omega_2 Y + \omega_3 Z$$

with constant coefficients $\omega_1, \omega_2, \omega_3 \in \mathbb{R}$ is also a fundamental vector field of the rotation group, precisely of any 1-parameter its subgroup. Besides the function $f$, an invariant of the vector field

$$P = \begin{vmatrix} u & v & w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \end{vmatrix}$$

(5.3)

is the linear function

$$g = \omega_1 u + \omega_2 v + \omega_3 w.$$ 

Indeed, one can easily verify that $P(f) = P(g) = 0$. This means that the trajectories of $P$ are circles obtained as an intersection of the spheres $f = \text{const}$ with the planes $g = \text{const}$; therefore $P$ is the rotation operator around the axis $\omega_1 : \omega_2 : \omega_3$.

The flow of $P$ drags the tensors and forces them to rotate around the axis $\omega_1 : \omega_2 : \omega_3$. To illustrate that, we shall study the dragging of $X$, $Y$ and $Z$ in the flow of $P$. A simple calculation of the Lie derivatives relative to $P$, denoted via primes, gives (in a matrix form):

$$(X, Y, Z)' = (X, Y, Z) \cdot \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

$$(X, Y, Z)'' = (\omega_1, \omega_2, \omega_3)P - \omega(X, Y, Z)$$

$$(X, Y, Z)''' = -\omega^2(X, Y, Z)'$$

$$\omega^2 := \omega_1^2 + \omega_2^2 + \omega_3^2.$$ 

It can be verified that anyone of the vector fields $X$, $Y$ and $Z$ satisfies the equation

$$S''' + \omega^2 S' = 0,$$  

(5.4)
whose general solution

\[ S_t = S + S' \frac{\sin \omega t}{\omega} + S'' \frac{1 - \cos \omega t}{\omega^2} \]  

reveals how these vector fields are dragged by the flow of the field \( P \).

An interesting situation arises when the rotation is around the axis 1:1:1, viz.

\[ \cos = \frac{\sqrt{3}}{3} \quad \omega = 1 \quad P = \frac{\sqrt{3}}{3} (X + Y + Z) \]

\[ (X, Y, Z)_t = \frac{1}{3} (X + Y + Z) \cdot (1, 1, 1) \]

\[ + \frac{2}{3} \left[ (X, Y, Z) \cos t + (Z, X, Y) \cos \left( \frac{2\pi}{3} - t \right) + (Y, Z, X) \cos \left( \frac{2\pi}{3} + t \right) \right]. \]

The three vector fields, occupying at \( t = 0 \) the situation \((X, Y, Z)\), at \( t = \frac{2\pi}{3} \) move to \((Z, X, Y)\), and at \( t = \frac{4\pi}{3} \) they move into \((Y, Z, X)\). Therefore a cyclic permutation is in action.

When the flows of \( X, Y \) and \( Z \) are dragged by the flow of \( P \), one should speak about a representation of the Lie algebra of the rotation group or of the adjoint representation of the rotation group.

Let us generalize the above setting [2]. Consider the following quadratic forms on \( \mathbb{R}^3 \) coordinated by \((u, v, w)\):

\[ f = \frac{1}{2} (a_{11} u^2 + a_{22} v^2 + a_{33} w^2 + 2a_{12} u v + 2a_{13} u w + 2a_{23} v w) \]

\[ \bar{f} = \frac{1}{2} (\bar{a}_{11} u^2 + \bar{a}_{22} v^2 + \bar{a}_{33} w^2 + 2\bar{a}_{12} u v + 2\bar{a}_{13} u w + 2\bar{a}_{23} v w). \]  

(5.6)

The matrices of their coefficients

\[ A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \quad \bar{A} = \begin{pmatrix} \bar{a}_{11} & \bar{a}_{12} & \bar{a}_{13} \\ \bar{a}_{12} & \bar{a}_{22} & \bar{a}_{23} \\ \bar{a}_{13} & \bar{a}_{23} & \bar{a}_{33} \end{pmatrix} \]

are required to be symmetric and mutually inverse, \( A^T = A, \bar{A}^T = \bar{A} \) and \( A \cdot \bar{A} = 1 \), so that \((\det A)(\det \bar{A}) = 1\).

The level surfaces of \( f \) and \( \bar{f} \) are central quadrics, viz. ellipsoids, when \( f \) and \( \bar{f} \) have constant signs, or hyperboloids, when \( f \) and \( \bar{f} \) can change signs. The equation \( \bar{f} = \text{const} \) is called tangent for the surface \( f = \text{const} \).

Let us introduce the following shortcuts for the partial derivatives of \( f \):

\[ f_i = a_{i1} u + a_{i2} v + a_{i3} w, \quad i = 1, 2, 3. \]

**Proposition 5.1.** The vector fields \( X_1, X_2 \) and \( X_3 \) given by

\[ (X_1, X_2, X_3) = \left( \frac{\partial}{\partial u}, \frac{\partial}{\partial v}, \frac{\partial}{\partial w} \right) \cdot \begin{pmatrix} 0 & -f_3 & f_2 \\ f_3 & 0 & -f_1 \\ -f_2 & f_1 & 0 \end{pmatrix} \]

are fundamental vector fields of a 3-dimensional Lie group \( G \) and

\[ \begin{array}{c|ccc} \mathcal{A} & X_1 & X_2 & X_3 \\ \hline X_1 & 0 & \sum_{k=1}^{3} a_{3k} X_k & -\sum_{k=1}^{3} a_{2k} X_k \\ X_2 & -\sum_{k=1}^{3} a_{3k} X_k & 0 & \sum_{k=1}^{3} a_{1k} X_k \\ X_3 & \sum_{k=1}^{3} a_{2k} X_k & -\sum_{k=1}^{3} a_{1k} X_k & 0 \end{array} \]  

(5.8)
is the table of their commutators. The fundamental vector fields given by (5.7) are (linearly) dependent,
\[ f_1X_1 + f_2X_2 + f_3X_3 = 0, \]
and the family of quadrics \( f = \text{const} \) are orbits of the group \( G \).

Proof. The commutation table is obtained via direct calculations like
\[
[X_1, X_2] = \begin{bmatrix}
    f_3 \frac{\partial}{\partial v} - f_2 \frac{\partial}{\partial w} & f_1 \frac{\partial}{\partial w} - f_3 \frac{\partial}{\partial u}
\end{bmatrix}
\]
\[= (a_{33}f_2 - a_{23}f_3) \frac{\partial}{\partial u} + (a_{13}f_3 - a_{33}f_1) \frac{\partial}{\partial v} + (a_{23}f_1 - a_{13}f_2) \frac{\partial}{\partial w}\]
\[= a_{31}X_1 + a_{32}X_2 + a_{33}X_3.\]

According the the third Lie theorem (see [3, p. 283 of the English text] or [4, p. 102]), the coefficients of the quadratic form \( f \) define the structure constants \( c^i_{jk} \) of the group \( G \), viz.
\[
a_{11} = c^1_{23}, \quad a_{12} = c^2_{31}, \quad a_{13} = c^3_{12},
\]
and the fundamental vector fields of \( G \) in \( \mathbb{R}^3 \), coordinatized by \( (u, v, w) \), are the vector fields \( X_1, X_2 \) and \( X_3 \). The orbits of \( G \) are the quadrics \( f = \text{const} \) as \( X_i(f) = 0, \ i = 1, 2, 3 \), and the rank of the matrix
\[
\begin{pmatrix}
    0 & -f_3 & f_2 \\
    f_3 & 0 & -f_1 \\
    -f_2 & f_1 & 0
\end{pmatrix}
\]
equals 2; in the opposite case, we should have \( f_1 = f_2 = f_3 = 0 \) which will mean that the matrix \( A \) is degenerate. \( \square \)

**Proposition 5.2.** The vector field
\[
P = p_1X_1 + p_2X_2 + p_3X_3, \quad (5.9)
\]
which is a linear combination of \( X_1, X_2 \) and \( X_3 \) with constant coefficients \( p_1, p_2, p_3 \in \mathbb{R} \) (and hence is a fundamental field of \( G \)), of \( G \) has the invariant
\[
g = p_1u + p_2v + p_3w
\]
besides the function \( f \).

The trajectories of \( P \) are on the intersection of the quadrics \( f = \text{const} \) with the planes \( g = \text{const} \) and, depending on the sign of the determinant
\[
\varepsilon := \begin{vmatrix}
    a_{11} & a_{12} & a_{13} & p_1 \\
    a_{12} & a_{22} & a_{23} & p_2 \\
    a_{13} & a_{23} & a_{33} & p_3 \\
    p_1 & p_2 & p_3 & 0
\end{vmatrix}, \quad (5.10)
\]
define:
- a family of ellipses, including isolated points, for \( \varepsilon < 0 \).
- a family of hyperbolas, including pairs of intersecting straight lines, for \( \varepsilon > 0 \).
- a family of parabolas, including pairs of parallel or coinciding straight lines, for \( \varepsilon = 0 \).
Remark 5.1. The possible flows on the quadrics $f = \text{const}$ are depicted on figures 5.1 and 5.2.

Proof. Representing the field $P$ as

$$P = \begin{pmatrix} f_1 & f_2 & f_3 \\ u & v & w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \end{pmatrix},$$

we immediately get $P(f) = P(g) = 0$, i.e. $f$ and $g$ are invariants of $P$. Therefore the trajectories of $P$ lie on the intersection of the quadrics $f = \text{const}$ with the planes $g = \text{const}$.

The type of the trajectories is determined by the intersection of the plane $g = 0$ with the cone $f = 0$, i.e. by the system $f = g = 0$. If this system has only the zero solution $u = v = 0$, the cone $f = 0$ can be either imaginary and the surfaces $f = \text{const}$ are ellipsoids or real and the surfaces $f = \text{const}$ are hyperboloids. In the both cases a family of ellipsoids is observed on the planes $g = \text{const}$, as shown on figure 5.1.

If the system defines a pair of different lines, the plane $g = 0$ intersects the real cone $f = 0$ along two generants and, on the plane $g = 0$, one can observe a family of hyperbolas (see figure 5.2 left). If the systems defines two identical lines, the plane $g = 0$ is tangent to the real tangential cone $\tilde{f} = 0$ and on the planes $g = \text{const}$ one can see a family of parabolas (see figure 5.2 right). This is equivalent that the vector $p_1 : p_2 : p_3$, from the origin, to be situated inside the tangent cone $f = 0$, outside this cone, or on the cone.

Analytically the situation is as follows. We suppose that $p_3 \neq 0$ without loss of generality. From $g = 0$, we get $p_3w = -p_1u - p_2v$, so that the substitution into $p_3^2f$ results in

$$(a_{11}p_3^2 - 2a_{13}p_1p_2 + a_{33}p_1^2)u^2 + 2(a_{33}p_1p_2 - a_{13}p_2p_3 - a_{23}p_1p_3 + a_{12}p_3^2)uv + (a_{22}p_3^2 - 2a_{13}p_2p_3 + a_{33}p_2^2)v^2.$$
and the corresponding quadratic equation defines the fraction \( u/v \).

In this way one gets the relation \( u : v : w \) and hence the intersection of the plane \( g = 0 \) with the cone \( f = 0 \). All depends on the discriminant which, up to the positive coefficient \( p_3^2 \), coincides with the quantity \( \varepsilon \), viz.

\[
(a_{33}p_1p_2 - a_{13}p_2p_3 - a_{23}p_1p_3 + a_{12}p_3^2)^2
- (a_{11}p_3^2 - 2a_{13}p_1p_2 + a_{33}p_1^2)(a_{22}p_3^2 - 2a_{13}p_2p_3 + a_{33}p_2^2)
= p_3^2\left(-|a_{11}| a_{12} a_{12} p_3^3 - |a_{22}| a_{23} a_{33} p_3^2|\right) - |a_{11}| a_{12} a_{13} a_{33} p_1 p_2
+ 2 |a_{11}| a_{12} a_{13} a_{23} p_1 p_2 + 2 |a_{11}| a_{12} a_{13} a_{23} p_2 p_3) = p_3^2. \varepsilon.
\]

Consequently, for \( \varepsilon < 0 \) (for \( \varepsilon > 0 \)) the flow of \( P \) is elliptic (hyperbolic) and for \( \varepsilon = 0 \) it is parabolic.

**Remark 5.2.** Generally, we have a class of Lie groups \( G \) each of which has its own table of commutators and structure constants. In particular, this may be the classical rotation group with the coefficients

\[
a_{11} = a_{22} = a_{33} = 1 \quad a_{12} = a_{13} = a_{23} = 0
\]

or the function

\[
f = \frac{1}{2}(u^2 + v^2 + w^2).
\]

A more interesting group is determined via the table of commutators

| \( X_i \) | \( X_0 \) | \( X_1 \) | \( X_2 \) | \( X_3 \) |
|---|---|---|---|---|
| \( X_0 \) | 0 : 0 | 0 | 0 | \ldots |
| \ldots | \ldots | \ldots | \ldots | \ldots |
| \( X_1 \) | 0 : 0 | 2X_3 | 2X_2 |
| \( X_2 \) | 0 : \(-2X_3\) | 0 | \(-2X_1\) |
| \( X_3 \) | 0 : \(-2X_2\) | 2X_1 | 0 |

(5.11)

in which the value of the commutator \([X_i, X_j] := X_i \circ X_j - X_j \circ X_i, i, j = 0, 1, 2, 3\), is situated at the intersection between the \( i \)th row and \( j \)th column, and, more precisely by its part which is related to the factor-group of the group of centroaffine transformations relative to the homothetic subgroup, i.e. to the group of equiaffine transformations of the plane. This group is distinguished by the coefficients

\[
a_{33} = -a_{22} - a_{11} = 2 \quad a_{12} = a_{13} = a_{23} = 0,
\]

or via the function

\[
f = -(u^2 + v^2 - w^2).
\]

Of course, this is the group of pseudo-Euclidean transformations of \( \mathbb{R}^3 \), i.e. the Lorentz group in three dimensions.

**Remark 5.3.** One can observe a circular movements on a sphere which themselves are involved in a rotational movement around some axis; e.g. the dragging of a field \( X \) by the flow of a field \( P \). The notion of a “transformation of a transformation” is not new; see, e.g., [4, p. 26]. Here we can talk of “movements of a movement”, “movements of the movements of a movement”, etc., i.e. of movements of higher orders., which seems to be a good item for a future research. For instance, one can imagine that a given flow undergoes a transformation or that it is dragged by the flow of other field, that this process undergoes a transformation or is dragged by the flow of a third field, etc.
One can easily imagine an elliptic flow on an ellipsoid, which is dragged by other elliptic flow. A hyperboloid can be intersected by a family of parallel planes, the result being a family of ellipses, or hyperbolas, or parabolas. Consequently on a hyperboloid can be observed an elliptic, hyperbolic and parabolic flows. However, it is difficult to be imagined the change on a hyperboloid, when an elliptic flow is continuously deformed and that at some moment it transforms into a parabolic flow and, then, into a hyperbolic; similarly, a stable hyperbolic flow can change into a parabolic and then into a stable elliptic flow. Such bifurcations in a (single or two sided) hyperboloid are admissible.

Consider now how the (flow of the) vector field \( P \) influences the operators \( X_1, X_2 \) and \( X_3 \).

**Proposition 5.3.** The Lie derivatives \( X'_i = [P, X_i] \), \( i = 1, 2, 3 \), of the fundamental vector fields \( X_1, X_2 \) and \( X_3 \) relative to the field \( P \) can be expressed through the same fields via the equation

\[
(X_1, X_2, X_3)' = (X_1, X_2, X_3) \cdot B,
\]

with \( B \) being the following product matrix

\[
B = \begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{12} & a_{22} & a_{23} \\
a_{13} & a_{23} & a_{33}
\end{pmatrix} \cdot \begin{pmatrix}
0 & -p_3 & p_2 \\
p_3 & 0 & -p_1 \\
-p_2 & p_1 & 0
\end{pmatrix}
\]

which is such that

\[
B^3 = \varepsilon B,
\]

where \( \varepsilon \) is given by (5.10). Besides, any one of the fundamental vector fields \( X_1, X_2 \) and \( X_3 \) is a solution of the differential equation (cf. (5.4))

\[
S''' - \varepsilon S' = 0,
\]

which, depending on the sign of \( \varepsilon \), admits the following solutions \( S_t \):

1) If \( \varepsilon < 0 \) and \( \lambda = \sqrt{-\varepsilon} \), then (cf. (5.5))

\[
S_t = S + S' \frac{\sin \lambda t}{\lambda} + S'' \frac{1 - \cos \lambda t}{\lambda^2}.
\]

2) If \( \varepsilon > 0 \) and \( \lambda = \sqrt{\varepsilon} \), then

\[
S_t = S + S' \frac{\sinh \lambda t}{\lambda} + S'' \frac{1 - \cosh \lambda t}{\lambda^2}.
\]

3) If \( \varepsilon = 0 \), then

\[
S_t = S + S't + S'' \frac{t^2}{2}.
\]

**Proof.**
The Lie derivative \( X'_1 \) is calculated by using the commutation table (5.8):

\[
X'_1 := [P, X_1] = [p_1 X_1 + p_2 X_2 + p_3 X_3, X_1] = p_2 [X_2 X_1] + p_3 [X_3 X_1]
\]

\[
= (a_{12} p_3 - a_{13} p_1) X_1 + (a_{22} p_3 - a_{23} p_1) X_2 + (a_{23} p_3 - a_{33} p_1) X_3
\]

\[
= (X_1, X_2, X_3) \cdot \begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{12} & a_{22} & a_{23} \\
a_{13} & a_{23} & a_{33}
\end{pmatrix} \cdot \begin{pmatrix}
0 \\
p_3 \\
-p_2
\end{pmatrix}.
\]

The Lie derivatives \( X'_2 \) and \( X'_3 \) can be calculated similarly.
Since the Hamilton-Cayley formula for the matrix $B$ reads

$$\sigma_0 B^3 + \sigma_1 B^2 + \sigma_2 B + \sigma_3 \mathbb{I} = 0,$$

where $\sigma_0 = 1$, $\sigma_1 = \text{tr} B = 0$ and $\sigma_3 = \det B = 0$, we have

$$B^2 + \sigma_2 B = 0.$$

Calculating $\sigma_2$,

$$\sigma_2 = \begin{vmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{vmatrix} + \begin{vmatrix} b_{22} & b_{23} \\ b_{23} & b_{33} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{13} \\ b_{13} & b_{33} \end{vmatrix}
= (a_{12}p_3 - a_{13}p_2)(a_{23}p_1 - a_{12}p_3) - (a_{23}p_1 - a_{12}p_3)(a_{13}p_2 - a_{23}p_1)
+ (a_{12}p_3 - a_{13}p_2)(a_{13}p_2 - a_{23}p_1) - (a_{13}p_1 - a_{11}p_3)(a_{22}p_3 - a_{23}p_2)
- (a_{11}p_2 - a_{12}p_1)(a_{23}p_3 - a_{33}p_2) - (a_{12}p_2 - a_{22}p_1)(a_{33}p_1 - a_{13}p_3),$$

we find $\sigma_2 = -\varepsilon$.

The establishment of the equation $S'' - \varepsilon S' = 0$ for the fields $X_1$, $X_2$ and $X_3$ is trivial. A simple verification reveals that the written functions $S_i$ are its solutions. Besides, the third solution is obtained from any one of the preceding two in the limit $\lambda \to 0$ due to

$$\lim_{\lambda \to 0} \frac{\sin \lambda t}{\lambda} = 1,$$

$$\lim_{\lambda \to 0} \frac{\sinh \lambda t}{\lambda^2} = \frac{1 - \cos \lambda t}{\lambda^2} = \frac{1 - \cosh \lambda t}{\lambda^2} = \frac{t^2}{2}.$$

\[ \square \]

Remark 5.4. The behavior of the vector fields $X_1$, $X_2$ and $X_3$ in the flow of $P$ is consistent with that flow in a sense that, if the flow is elliptic, hyperbolic or parabolic (see the three cases in proposition 5.2), then the behavior of anyone of these fields is respectively elliptic, hyperbolic or parabolic (see cases 1, 2 and 3 of proposition 5.3). A bifurcation can occur if the sign of $\varepsilon$ can change or, in geometrical language, if the normal to the plane $g = 0$ with direction $p_1 : p_2 : p_3$ is moved from the inner part of the tangent cone $f = 0$ to the outside domain, or vice versa if it moves from the outside part of the cone to its interior.

A similar bifurcation can occur with the flow of a linear vector field $X$ on the plane, i.e. a transition form elliptic to hyperbolic behavior or vice versa, if the quantity $\Delta$ changes its sign (see cases 1, 2 or 3 of proposition 5.3). The equality $\Delta = 0$ in the space $\mathbb{R}^3$ coordinated by $b_1$, $b_2$ and $b_3$ should be thought as an equation of the cone

$$b_1^2 + b_2^2 - b_3^2 = 0$$

and the bifurcation is due to the change of the direction $b_1 : b_2 : b_3$ from inside to outside the cone or from outside to inside of it.

6. Tensor representations of the group $\text{GL}(2, \mathbb{R})$ and the algebra $\mathfrak{gl}(2, \mathbb{R})$

Let on a 2-dimensional manifold be given a frame $R$ and its dual coframe $\Theta$. In them a tensor $S$ of type $(p, q) \in \mathbb{N}^2$ is represented according

$$S = R_{i_1} \otimes \cdots \otimes R_{i_p} \otimes s_{j_1 \cdots j_q} \otimes \theta^{i_1} \otimes \cdots \otimes \theta^{i_q}.$$

(6.1)
whith $\bar{s}_{j_1\ldots j_q}^{i_1\ldots i_p} = S(\theta^{i_1},\ldots,\theta^{i_p}; R_{j_1},\ldots,R_{j_q})$ being the components of $S$ relative to the tensor frame induced by $\{R_i\}$ and $\{\theta^i\}$. A regular $2 \times 2$ matrix $A \in \text{GL}(2, \mathbb{R})$ transforms $S$ into a tensor $\bar{S}$ with local components

$$\bar{s}_{j_1\ldots j_q}^{i_1\ldots i_p} = A_{k_1}^{i_1}\ldots A_{k_p}^{i_p} \bar{s}_{k_1\ldots k_p}^{j_1\ldots j_q},$$

(6.2)

where $A_{i}^{j}$ and $\bar{A}_{j}^{i}$ are the elements of $A$ and its inverse matrix $A^{-1}$, respectively. It is said that the group $\text{GL}(2, \mathbb{R})$ is represented or that it acts via (6.2) on the space of tensors of type $(p,q)$.

An arbitrary matrix $C$ of type $2 \times 2$ is an element of the Lie algebra, which is isomorphic with the Lie algebra $\mathfrak{gl}(2, \mathbb{R})$ of the group $\text{GL}(2, \mathbb{R})$. The exponential $e^{Ct}$, $t \in \mathbb{R}$, defines a $1$-parameter subgroup of $\text{GL}(2, \mathbb{R})$ and equation (6.2), with $A = e^{Ct}$ defines a $1$-parameter family of tensors $S_t$. In this way, in the space of tensors of type $(p,q)$, is defined the flow of some vector field $\bar{X}$ whose local components can be found by putting $A = e^{Ct}$ in (6.2), differentiating the result with respect to $t$ and putting $t = 0$. To the vector field $\bar{X}$ corresponds a matrix $C$, similarly to the correspondence $X \rightarrow C$; the structure of $C$ can be obtained by ordering the components of the tensor $S$. We say that the correspondence $C \rightarrow \bar{X}$ or $X \rightarrow \bar{X}$ determines a representation of the Lie algebra $\mathfrak{gl}(2, \mathbb{R})$ on the space of tensors $S$.

It is important to find the eigenvalues of $C$ provided the ones of $C$ are known. For instance, with their help one can restore the exponential $e^{Ct}$. The flow of $\bar{X}$ is determined directly by (6.2) after the substitution $A = e^{Ct}$.

**Proposition 6.1.** Let the group $\text{GL}(2, \mathbb{R})$ acts on the space of tensors of type $(p,q)$ on a 2-dimensional manifold. If $\lambda_1$ and $\lambda_2$ are the eigenvalues of $C$, then the ones of $C$ are

$$\lambda_{i_1} + \cdots + \lambda_{i_p} - \lambda_{j_1} - \cdots - \lambda_{j_q},$$

(6.3)

where all indices take the values 1 and 2. The number of these eigenvalues equals the number of the components of a tensor of type $(p,q)$, i.e. to $2^{p+q}$ in two dimensions.

**Proof.** Chose a frame $R$ such that $C$ will take in it the diagonal form $C = \text{diag}(\lambda_1, \lambda_2)$ with $\lambda_1, \lambda_2 \in \mathbb{C}$ being the eigenvalues of $C$. Then $e^{Ct} = \text{diag}(e^{\lambda_1 t}, e^{\lambda_2 t})$, so that (6.2) reads

$$e^{\lambda_1 t} \ldots e^{\lambda_2 t} \bar{s}_{j_1\ldots j_q}^{i_1\ldots i_p} e^{-\lambda_1 t} \ldots e^{-\lambda_2 t} = \bar{s}_{j_1\ldots j_q}^{i_1\ldots i_p} e^{(\lambda_1 + \cdots + \lambda_1 - \cdots - \lambda_2)t}.$$  

(6.4)

Differentiating this expression with respect to $t$ and setting $t = 0$, we get the components of $S$ multiplied by the numbers (6.3). Thus we get a diagonal matrix with diagonal elements (6.3), which are the eigenvalues of $C$. \hfill \Box

One can restore the linear ordinary differential equation for the components of the tensor field $S$ (which is invariant in the given flow) via the quantities (6.3). The coefficients are symmetric polynomials from (6.3). Similarly can be restored the differential equation for the tensor field $S$ with invariant components, but via the quantities with inverse signs according to the duality principle, viz.

$$\lambda_{j_1} + \cdots + \lambda_{j_q} - \lambda_{i_1} - \cdots - \lambda_{i_p},$$

(6.5)

The numbers (6.3) and (6.5) are situated on the complex plane on the knots of some lattice determined by the initial values $\lambda_1$ and $\lambda_2$. On figure 6.1 is shown the position of the numbers (6.5) on the base of the complex conjugate eigenvalues $\lambda_{1,2} = \alpha \pm i\beta$.

When $p$ and $q$ are fixed, the numbers (6.5) have identical real parts, equal to $(q-p)\alpha$, and, consequently, are situated on a straight line parallel to the imaginary axis.

Let us consider three particular cases of tensor fields dragged by the flow of a linear vector field $X$:

To illustrate the general situation, below we shall reproduce three propositions from [5, sec. 2.9], where the corresponding proofs can be found.
Example 6.1. For a tensor field of type (1, 1) (an affinor)

\[
S = s^i_j R_i \otimes \theta^j,
\]

(6.6)

where \( s = [s^i_j] \) is a \( 2 \times 2 \) matrix, we have the eigenvalues

\[
\lambda_1 - \lambda_2, \quad \lambda_2 - \lambda_1, \quad 0 \text{ (a double eigenvalue)}.
\]

(6.7)

For it are valid the following implications

\[
s' = 0 \Rightarrow S'' = \Delta \cdot S'
\]

(6.8)

\[
S' = 0 \Rightarrow s'' = \Delta \cdot s',
\]

(6.9)

with \( \Delta \) defined by

\[
4\Delta = (\text{tr} C)^2 - 4 \det C = (c_1 - c_4)^2 + 4c_2c_3 = (\lambda_1 - \lambda_2)^2.
\]

(6.10)

Besides, in a case of \( 6.11 \), the matrix \( \bar{C} \) is

\[
\bar{C} = \begin{pmatrix}
0 & -c_3 & c_2 & 0 \\
-c_2 & c_1 - c_4 & 0 & c_3 \\
c_3 & 0 & c_4 - c_1 & -c_2 \\
0 & c_3 & -c_2 & 0
\end{pmatrix}
\]

(6.11)

and the vector field \( \bar{X} \) is given by

\[
\bar{X} = \left( \frac{\partial}{\partial s_1^1}, \frac{\partial}{\partial s_2^1}, \frac{\partial}{\partial s_1^2}, \frac{\partial}{\partial s_2^2} \right) \cdot \bar{C} \cdot (s_1^1, s_1^2, s_2^1, s_2^2)^\top.
\]
**Example 6.2.** For tensor fields of types \((0, 2)\) and \((0, 3)\), i.e. for quadratic and cubic forms

\[
G = g_{ij}\theta^i \otimes \theta^j \quad (6.12a)
\]

\[
H = h_{ijk}\theta^i \otimes \theta^j \otimes \theta^k \quad (6.12b)
\]

with symmetric with respect to the subscripts components \(g_{ij}\) and \(h_{ijk}\), we have respectively the following eigenvalues

\[
2\lambda_1, \quad \lambda_1 + \lambda_2, \quad 2\lambda_2
\]

\[
3\lambda_1, \quad 2\lambda_1 + \lambda_2, \quad \lambda_1 + 2\lambda_2, \quad 3\lambda_3.
\]

(6.13a)

(6.13b)

For the quadratic form \((6.12a)\) hold the implications

\[
g' = 0 \Rightarrow G'' - 3(\lambda_1 + \lambda_2)G'' + 2(\lambda_1^2 + 4\lambda_1\lambda_2 + \lambda_2^2)G' - 4(\lambda_1 + \lambda_2)\lambda_1\lambda_2G = 0 \quad (6.14)
\]

\[
G' = 0 \Rightarrow g'' + 3(\lambda_1 + \lambda_2)g'' + 2(\lambda_1^2 + 4\lambda_1\lambda_2 + \lambda_2^2)g' + 4(\lambda_1 + \lambda_2)\lambda_1\lambda_2g = 0, \quad (6.15)
\]

which mean that, if the matrix \(g := [g_{ij}]\) is invariant in the flow of \(X\), the form \(G\) then satisfies the differential equation in \((6.14)\), and if the form \(G\) is invariant in the flow of \(X\), its components \(g_{ij}\) are solutions of the r.h.s. of \((6.15)\). Now the vector field \(X\) is

\[
\dot{X} = -\left(\frac{\partial}{\partial g_{ij}}, \frac{\partial}{\partial g_{jk}}, \frac{\partial}{\partial g_{kl}}\right) \cdot \begin{pmatrix} 2c_1 & 2c_3 & 0 & 0 \\ c_2 & c_1 + c_4 & c_3 & 0 \\ 0 & 2c_2 & 2c_4 & 0 \\ 0 & 0 & 2c_1 - c_4 & 2c_3 \\ -c_3 & 0 & c_2 & c_1 + c_3 \\ 0 & 0 & -c_3 & 2c_2 \\ 0 & 0 & c_3 & c_4 \end{pmatrix} \cdot \begin{pmatrix} g_{11} \\ g_{12} \\ g_{22} \end{pmatrix}.
\]

(6.16)

Similar but more complicated results are valid for the cubic form \((6.12b)\).

**Example 6.3.** For a tensor of type \((1, 2)\) (vector-valued quadratic form)

\[
K = k_{ij}R_i \otimes \theta^j \otimes \theta^l
\]

with \(k_{ij} = k_{ji}^*\), we have the eigenvalues

\[
2\lambda_1 - \lambda_2, \quad \lambda_1, \quad \lambda_2, \quad 2\lambda_2 - \lambda_1,
\]

(6.18)

where \(\lambda_1\) and \(\lambda_2\) are double eigenvalues. The matrix \(\tilde{C}\) for it is

\[
\tilde{C} = -\begin{pmatrix} c_1 & 2c_3 & 0 & -c_2 & 0 & 0 \\ c_2 & c_1 + c_4 & c_3 & 0 & -c_2 & 0 \\ 0 & 2c_2 & 2c_4 & 0 & 0 & -c_2 \\ -c_3 & 0 & 0 & 2c_1 - c_4 & 2c_3 & 0 \\ 0 & 0 & 0 & c_2 & c_1 + c_3 & 0 \\ 0 & 0 & -c_3 & 0 & 2c_2 & c_4 \end{pmatrix}
\]

(6.19)

and the vector field \(\dot{X}\) is (see also [6])

\[
\dot{X} = -c_1 \left( k_{ij}^1 \frac{\partial}{\partial k_{ij}^1} - k_{ij}^2 \frac{\partial}{\partial k_{ij}^2} + 2k_{ij}^1 \frac{\partial}{\partial k_{ij}^2} + k_{ij}^2 \frac{\partial}{\partial k_{ij}^2} \right) \\
- c_2 \left[ -k_{ij}^2 \frac{\partial}{\partial k_{ij}^1} + (k_{ij}^1 - k_{ij}^2) \frac{\partial}{\partial k_{ij}^2} + (2k_{ij}^1 - k_{ij}^2) \frac{\partial}{\partial k_{ij}^2} + k_{ij}^1 \frac{\partial}{\partial k_{ij}^2} + 2k_{ij}^2 \frac{\partial}{\partial k_{ij}^2} \right] \\
- c_3 \left[ 2k_{ij}^1 \frac{\partial}{\partial k_{ij}^1} + k_{ij}^2 \frac{\partial}{\partial k_{ij}^2} (2k_{ij}^2 - k_{ij}^1) \frac{\partial}{\partial k_{ij}^2} + (k_{ij}^2 - k_{ij}^1) \frac{\partial}{\partial k_{ij}^2} - k_{ij}^2 \frac{\partial}{\partial k_{ij}^2} \right] \\
- c_4 \left( k_{ij}^1 \frac{\partial}{\partial k_{ij}^1} + 2k_{ij}^1 \frac{\partial}{\partial k_{ij}^2} - k_{ij}^1 \frac{\partial}{\partial k_{ij}^2} + k_{ij}^2 \frac{\partial}{\partial k_{ij}^2} \right).
\]

(6.20)
7. Conclusion

An invariant of a vector field $X$ on $\mathbb{R}^2$ can be calculated via the equations

$$ I = \int \omega \quad I = \int \mu \omega \quad I = \int \frac{\omega}{\omega(P)} \quad I = u - \int f(v) \, dv $$

in the cases considered in sections 2 and 3. They clarify the role of infinitesimal symmetry $P$ and the reason why the result can be expressed via quadratures.

For a linear vector field, defined via a matrix $C$, we have the derivative formulae $R' = -RC$ and $\theta' = C\theta$ and the exponential $e^{Ct}$ determines the flow of $X$ and hence the dragging of tensor fields in that flow. So, the problem for finding the invariants and flows of linear vector fields is solved completely.

The considerations presented in this paper admit generalizations in the multidimensional and infinite dimensional cases.

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