Relativistic fluids with spin

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Abstract. We study the behavior of relativistic fluids with internal angular momentum, using the tools of quantum mechanics and statistical mechanics to find out the general properties of macroscopic systems. We found that, if we consider thermodynamics, different couples of stress-energy and spin tensors (previously considered equivalent) give different predictions on the momentum and angular momentum density. This difference remains even in the non-relativistic limit and can in principle be measured experimentally.

1. Introduction
The common relativistic generalization of a fluid involves the stress-energy tensor $T^{\mu\nu}$ which gives us the energy density $T^{00}$, the momentum density $T^{0i}$ and the law of motion $\partial_{\mu}T^{\mu\nu} = 0$.

The law of motion is enough to ensure the conservation of the total four momentum of the system, provided that the flux of the tensor on the boundary of the domain vanishes.

For the total angular momentum density we have the analog of the orbital angular momentum $x^\mu T^{0\nu} - x^\nu T^{0\mu}$ and another rank 3 tensor $S^{0,\mu\nu}$ antisymmetric in the last two indices, which is the internal angular momentum of a fluid cell. So the total angular momentum density is $\mathbf{J}^{0,\mu\nu} = S^{0,\mu\nu} + x^\mu T^{0\nu} - x^\nu T^{0\mu}$. In order to ensure the conservation of the angular momentum of the system, and the conservation of the four-momentum, we require:

\[
\begin{cases}
\partial_{\mu}T^{\mu\nu} = 0 \\
\partial_{\lambda}\mathbf{J}^{\lambda,\mu\nu} = \partial_{\lambda}S^{\lambda,\mu\nu} + T^{\mu\nu} - T^{\nu\mu} = 0 \Rightarrow \partial_{\lambda}S^{\lambda,\mu\nu} = T^{\nu\mu} - T^{\mu\nu}
\end{cases}
\]

\[
\begin{aligned}
\int_{\partial V} dS T^{0\mu} n_i &= 0 \\
\int_{\partial V} dS \mathbf{J}^{i,\mu\nu} n_i &= 0.
\end{aligned}
\]

In the first phenomenological model describing relativistic fluids with spin [1] it was made an assumption on the form of the spin tensor:

$S^{\lambda,\mu\nu} = u^\lambda \sigma^{\mu\nu}$,

where $u^\mu$ is the four-velocity of the fluid cell and $\sigma^{\mu\nu}$, interpreted as a proper internal angular momentum density, i.e. the internal angular momentum in the comoving frame. A further condition, called Frenkel condition, was enforced:

$t^\mu = \sigma^{\mu\nu} u_\nu = 0$.

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The physical validity of the last formula has been questioned [2], however the variational Lagrangian theory which is the base of many subsequent papers [3] still embodies the Frenkel condition.

In our work we found that even the simplest fluid, the ideal Boltzmann gas rotating at full thermodynamical equilibrium, doesn’t fulfill the condition [4]. So we have been looking for a general method to study the behavior of relativistic fluids, and we decided to use the tools of quantum mechanics and statistical mechanics to find the general properties.

2. From quantum mechanics to hydrodynamics

From the correspondence principle we know that the classical quantity $O_{cl}$, corresponding to the quantum operator $\hat{O}$, is given by the mean value:

$$O_{cl} = \text{tr} \left( \hat{\rho} : \hat{O} : \right),$$

where $\hat{\rho}$ is the density matrix, the state (pure or mixed) of the system, and the normal ordering $\vdots$ is introduced in the mean value in order to subtract the zero-point infinities$^1$.

So we would like to use as stress-energy and spin tensor:

$$T^{\mu\nu}(x) = \text{tr} \left( \hat{\rho} : \hat{T}^{\mu\nu}(x) : \right), \quad S^{\lambda,\mu\nu}(x) = \text{tr} \left( \hat{\rho} : \hat{S}^{\lambda,\mu\nu}(x) : \right).$$

The Noether theorem gives us a stress energy tensor (corresponding to the invariance of the action under space-time translation) and an angular momentum tensor (invariance under Lorentz boosts and rotations), and thus the spin tensor$^2$, but in quantum field theory (as long as we don’t consider gravity) the stress-energy $\hat{T}$ and spin tensor $\hat{S}$ are not uniquely defined. Once found a particular couple $(\hat{T}, \hat{S})$, e.g. using the Noether’s theorem to get the canonical tensors, we can always construct a new couple fulfilling the same continuity equations (the quantum analogue of (1)) and maintaining the same Poincaré generators:

$$\hat{P}^\mu = \int_V d^3x \hat{T}^{0\mu}(x), \quad \text{total four-momentum}$$

$$\hat{J}^{\mu\nu} = \int_V d^3x \left[ x^\mu \hat{T}^{0\nu}(x) - x^\nu \hat{T}^{0\mu}(x) + \hat{S}^{0,\mu\nu}(x) \right], \quad \text{total angular momentum}$$

if we make this transformation [6]:

$$\hat{T}^{\eta\mu\nu}(x) = \hat{T}^{\eta\mu\nu}(x) + \frac{1}{2} \partial_\alpha \left( \hat{\Phi}^{\alpha,\mu\nu}(x) - \hat{\Phi}^{\mu,\alpha\nu}(x) - \hat{\Phi}^{\nu,\alpha\mu}(x) \right),$$

$$\hat{S}^{\lambda,\mu\nu}(x) = \hat{S}^{\lambda,\mu\nu}(x) - \hat{\Phi}^{\lambda,\mu\nu}(x), \quad (2)$$

where $\hat{\Phi}^{\lambda,\mu\nu}(x)$ is a rank 3 tensor antisymmetric in the last two indices that fulfills:

$$\int_{\partial V} dS \left( \hat{\Phi}^{0,\iota\nu} - \hat{\Phi}^{i,0\iota} - \hat{\Phi}^{j,\iota 0} \right) n_i = 0$$

$$\int_{\partial V} dS \left[ x^\mu \left( \hat{\Phi}^{0,\iota\nu} - \hat{\Phi}^{i,0\iota} - \hat{\Phi}^{j,\iota 0} \right) - x^\nu \left( \hat{\Phi}^{0,i\mu} - \hat{\Phi}^{i,0\mu} - \hat{\Phi}^{j,0i} \right) \right] n_i = 0.$$  

$^1$ For a discussion of the meaning of normal ordering for interacting fields see, e.g., Ref. [5].

$^2$ Since $\hat{J}^{\lambda,\mu\nu} = \hat{S}^{\lambda,\mu\nu} + x^\rho \hat{T}^{\rho\mu\nu} - x^\nu \hat{T}^{\lambda\rho\mu}$ having two of the operators $\hat{J}, \hat{S}$ or $\hat{T}$ is enough to know all of them.
This class of transformations leaves the Poincaré generators invariant, so the couples of tensors \((\hat{T}', \hat{S}')\) we get through them are usually considered equivalent, but we found that, despite leaving the generators invariant, the new couples are not actually equivalent.

3. Thermodynamical inequivalence of quantum stress-energy and spin tensors

Since the components \(T^00, T^0i\) and \(J^{0,i} = S^{0,i} + x^iT^0i - x^jT^0j\) represent the energy density, the linear, and angular momentum density (measurable quantities) we would expect that the mean values of them would measure:

\[ \langle T_0^0 \rangle = \langle \hat{T}_0^0 \rangle \]
\[ \langle J^{0,i} \rangle = \langle \hat{J}^{0,i} \rangle \]

which in general is different from \(\langle T_0^0 \rangle = \langle \hat{T}_0^0 \rangle\) since we didn’t enforce \(T_0^0 = \hat{T}_0^0\). We can make a similar argument for the angular momentum densities, but since we require an equivalence only on the spatial indices \(i, j\) of \(J^{0,i}\), in this case we can make a looser request than \(\langle J^{\lambda,i} \rangle = \langle \hat{J}^{\lambda,i} \rangle\). We are thus to enforce, in a covariant form:

\[ T^{\lambda\mu}(x) = T^{\lambda\mu}(x) \]
\[ J^{\lambda\mu}(x) = J^{\lambda\mu}(x) \]

If we limit ourselves to spatial indices \(\mu, \nu = 1, 2, 3\), the above equation is enough to ensure that the angular momentum densities, with \(\lambda = 0\), are the same in any inertial frame, since in any of them \(\rho^0 = g^00 = 0\) so the four-vector \(K^\nu(x)\) doesn’t change \(\langle J^{0,i} \rangle\) by construction.

As long as we use (2) to make \(\langle \hat{T}', \hat{S}' \rangle\), and trough their average values \(T'\) and \(J'\), we will have some constrains on the possible form of \(K^\mu\) since every difference between \(T', J'\) and \(T, J\) stems from a specific combination of the components of \(\Phi\) (the average of the operator \(\hat{\Phi}\)). In fact we proved that, taking the average of (2), when \(\Phi^{a,\mu\nu}\) is such that the new fields fulfill the last conditions (3), necessarily \(K\) must fulfill the following:

\[ \partial^\nu K^\mu = 0, \]

and so \(K^\mu\) has to be a constant vector field.

It is important to stress how these conditions are for the mean values, on the quantum level operators don’t have to fulfill the same requirements. Even when the quantum analogue of the above equations are not enforced on the couples \(\langle \hat{T}, \hat{S} \rangle\) and \(\langle \hat{T}', \hat{S}' \rangle\) the density matrix \(\hat{\rho}\) could be such that the classical counterparts are still equivalent. The symmetry properties of \(\hat{\rho}\) are crucial, we found that in the grand-canonical ensamble (a very symmetric system) every couple of tensors gives the same average four-momentum and angular momentum density but in a less symmetric case the situation is very different.

\[ ^3 \] All the calculations of this section are found in Ref.[6]
3.1. A particular case: an axisymmetric system
We studied the case of a rotational grand-canonic system, where the density matrix, in the rest frame, is given by:\(^4\):

\[ \hat{\rho} = \frac{1}{Z_\omega} P_V \exp(-\hat{H}/T + \omega \cdot \hat{J}/T + \mu \hat{Q}/T) \] (5)

where \( Z_\omega \) is the rotational partition function:

\[ Z_\omega = \text{tr} \left( P_V \exp(-\hat{H}/T + \omega \cdot \hat{J}/T + \mu \hat{Q}/T) \right), \]

and \( \omega \) has the physical meaning of an angular velocity; but, in order to consider the (tri-) vector \( \omega \times \mathbf{x} \) as the velocity of a fluid cell, we need \( \| \omega \times \mathbf{x} \| < 1 \) (we are using here the natural units where the speed of light in the vacuum is \( c = 1 \)). For that reason we put the projector \( P_V \) in the density matrix, to ensure that the tangential velocity never exceeds the speed of light. For simplicity we took as the volume \( V \) a cylindrical region centered on the axis \( z \) infinitely long and with a radius \( R < 1/\| \omega \| \).

Due to the symmetry of the system it’s useful to decompose the average values of vectors and tensors along the tetrad (essentially four vector fields that act as a base of the tangent space, see figure 1):

\[ u = (\gamma, \gamma v) \quad \tau = (\gamma v, \gamma \hat{v}) \]
\[ n = (0, \mathbf{r}) \quad k = (0, \mathbf{k}), \]

where \( \mathbf{v} = \omega \times \mathbf{x}, \gamma = 1/\sqrt{1-\mathbf{v}^2}, \mathbf{r} \) is the radial versor in cylindrical coordinates, while \( \mathbf{k} \) is the versor of the \( z \) axis, that is, the axis of the cylinder.

If we decompose along the tetrad a rank three tensor antisymmetric in the last two indices like \( \Phi^{\lambda\mu\nu} \), the mean value of the operator in (2), we obtain:

\[ \Phi^{\lambda\mu\nu} = D(r)(n^\mu n^\nu - n^\nu n^\mu)u^\lambda + E(r)(\tau^\mu u^\nu - \tau^\nu u^\mu)u^\lambda + F(r)(n^\mu u^\nu - n^\nu u^\mu)u^\lambda + \]
\[ + N(r)(n^\lambda \tau^\mu - n^\mu \tau^\lambda)u^\nu - N(r)(n^\lambda \tau^\nu - n^\nu \tau^\lambda)u^\mu + P(r)\tau^\lambda (\tau^\mu u^\nu - \tau^\nu u^\mu) + \]
\[ + Q(r)(n^\lambda \tau^\mu + n^\mu \tau^\lambda)u^\nu - Q(r)(n^\lambda \tau^\nu + n^\nu \tau^\lambda)u^\mu + R(r)n^\lambda (n^\mu u^\nu - n^\nu u^\mu) + \]
\[ + S(r)k^\lambda (k^\mu u^\nu - k^\nu u^\mu) + T(r)(\tau^\mu n^\nu - \tau^\nu n^\mu)n^\lambda + U(r)\tau^\lambda (\tau^\mu n^\nu - \tau^\nu n^\mu) + \]
\[ + V(r)k^\lambda (k^\mu n^\nu - k^\nu n^\mu) + W(r)k^\lambda (k^\mu \tau^\nu - k^\nu \tau^\mu), \]

with \( E(0) = F(0) = Q(0) = T(0) = 0 \).

Differently from the case of the grand-canonical ensemble, different couples of tensor \((\hat{\mathbf{T}}, \hat{\mathbf{S}})\) don’t give the same predictions, in fact the tensor \( \Phi \) that we obtain from the rotational grand-canonic density matrix (5) in order to fulfill the conditions (3) must have:

\[ V(r) = U(r) = F(r) \quad P(r) = R(r) = S(r) \]
\[ E(r) = W(r) = -T(r) \quad D(r) = N(r) = Q(r) = 0, \] (7)

\(^4\) It can be obtained in several fashions: by maximizing the entropy with the constraint of fixed mean value of angular momentum [7], generalizing to the quantum-relativistic case an argument used by Landau for classical systems [8], or as the limiting macroscopic case of a quantum statistical system with finite volume and fixed angular momentum in its rest frame in an exact quantum sense, i.e., belonging to a specific representation of the rotation group [9]
which are not trivial relations. If these are fulfilled, then the mean value of $\Phi$ reduces to:

$$
\Phi_{\lambda,\mu\nu} = (F(r)n^\mu + E(r)\tau^\mu + P(r)u^\mu)g^{\lambda\nu} - (F(r)n^\nu + E(r)\tau^\nu + P(r)u^\nu)g^{\lambda\mu} \equiv K^\mu g^{\lambda\nu} - K^\nu g^{\lambda\mu}
$$

The field $K^\mu = F(r)n^\mu + E(r)\tau^\mu + P(r)u^\mu$ ought to be a constant one, according to (4). Since its divergence vanishes, then $F(r) = 0$ and, by using the definitions (6), we readily obtain the conditions:

$$
F(r) = 0 \quad P(r)/\gamma = \text{const} \quad E(r) = -P(r)\omega r
$$

In conclusion, only if $\Phi$ is such that its mean value, calculated with the density operator (5), fulfills conditions (7) and (8), is the corresponding transformation (3) possible. Otherwise, the original and transformed stress-energy and spin tensors are inequivalent because they imply different values of mean energy, momentum or angular momentum densities.

3.2. Example: Free rotating Dirac field in a cylindrical region

We studied the case of the free Dirac field in a cylindrical region to see if there are interesting cases in which the conditions (7) and (8) are not fulfilled.

From the Lagrangian density of the system:

$$
\mathcal{L} = \frac{i}{2}\overline{\Psi}\gamma^\mu \partial_\mu \Psi - m\overline{\Psi}\Psi,
$$

we get the canonical tensors using the Noether’s theorem [10]:

$$
\hat{T}^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Psi)} \partial^\nu \Psi + \partial^\nu \overline{\Psi} \frac{\partial \mathcal{L}}{\partial (\partial_\mu \overline{\Psi})} - g^{\mu\nu} \mathcal{L} = \frac{i}{2}\overline{\Psi} \gamma^{\mu\nu} \partial^\nu \Psi
$$

$$
\hat{S}^{\lambda,\mu\nu} = -i \frac{\partial \mathcal{L}}{\partial (\partial_\lambda \Psi)} \Sigma^{\mu\nu} \Psi + i \overline{\Psi} \Sigma^{\mu\nu} \frac{\partial \mathcal{L}}{\partial (\partial_\lambda \overline{\Psi})} = \frac{1}{2}\overline{\Psi} \{\gamma^\lambda, \Sigma^{\mu\nu}\} \Psi = \frac{i}{8}\overline{\Psi} \{\gamma^\lambda, [\gamma^\mu, \gamma^\nu]\} \Psi,
$$

where $\Sigma^{\mu\nu} = \frac{i}{4}[[\gamma^\mu, \gamma^\nu]]$ is related to the infinitesimal Lorentz transformation on the fields by:

$$
\delta \Psi = -\frac{i}{2}\omega^{\mu\nu} \Sigma^{\mu\nu} \Psi \quad \delta \overline{\Psi} = -\frac{i}{2}\omega^{\mu\nu} \overline{\Psi} (-\Sigma^{\mu\nu}),
$$

5 $\delta \overline{\Psi} = \delta \Psi^1 \gamma^0 = \frac{i}{2}\omega^{\mu\nu} \overline{\Psi} (\Sigma^{\mu\nu})^1 \gamma^0$, but $(\Sigma^{\mu\nu})^1 = -\frac{1}{4}[[\gamma^\mu], [\gamma^\nu]]$ and $[\gamma^\mu, \gamma^\nu] = \gamma^0 \gamma^\mu$, so $(\Sigma^{\mu\nu})^1 \gamma^0 = \gamma^0 \Sigma^{\mu\nu}$ hence $\delta \overline{\Psi} = -\frac{i}{2}\omega^{\mu\nu} \overline{\Psi} (-\Sigma^{\mu\nu})$. q.e.d.
\( \omega_{\mu\nu} \) in this case is simply an infinitesimal number, it’s not related at all with the angular velocity \( \omega \) of the system.

Because of the properties of the gamma matrices the canonical spin tensor is antisymmetric even in the first two indices, and thus the mean value, much more simple to handle:

\[
S^{\lambda,\mu\nu} = D(r)[(n^\mu \tau^\nu - n^\nu \tau^\mu)u^\lambda + (n^\lambda \tau^\mu - n^\mu \tau^\lambda)u^\nu - (n^\lambda \tau^\nu - n^\nu \tau^\lambda)u^\mu],
\]

depends only on one function:

\[
D(r) = S_{0,12}^{0,12} = \text{tr} \left( \hat{\rho} : \hat{S}_{0,12}^{0,12} : \right).
\]

It turns out that this function is the only one from which depends the difference between the mean momentum and angular momentum density of the canonical tensors and the Belinfante symmetrized tensors\(^6\):

\[
T_{\text{Belinfante}}^0 - T_{\text{canonical}}^0 = -\frac{1}{2} \frac{dD(r)}{dr} \hat{\rho}^i,
\]

\[
\mathcal{J}_{\text{Belinfante}} - \mathcal{J}_{\text{canonical}} = -\left( \frac{1}{2r} \frac{dD(r)}{dr} + D(r) \right) \hat{k}.
\]

The problem of the Dirac field within a cylinder with finite radius has been tackled by several authors in the context of the MIT bag model [11, 12]. Using the complete solution of the free Dirac equation for a massive particle in a longitudinally unlimited cylinder with finite transverse radius (obtained by Bezerra de Mello et al. in Ref.[13]) we found:

\[
D(r) = \text{tr}_V \left[ \hat{\rho} : \Psi^\dagger(0, \mathbf{x}) \Sigma_{\mathbf{z}} \Psi(0, \mathbf{x}) : \right] =
\]

\[
= \sum_M \sum_{\xi = \pm 1} \sum_l \sum_\infty \int_{-\infty}^{\infty} dp_z \left[ \frac{1}{e^{(\varepsilon - M\omega + \mu)/T} + 1} + \frac{1}{e^{(\varepsilon - M\omega - \mu)/T} + 1} \right] \times
\]

\[
\frac{p_{Tl}^2 l_{Tl}^2}{8\pi^2 R^2 (2M^{-1/2})(prR)(2Rm^2_{Tl} + 2\xi Mm_{Tl} + m)},
\]

and we proved that, even being vanishing when the angular velocity is zero\(^7\), \( D(0) \) is non vanishing in a neighborhood of \( \omega = 0 \).

### 3.3. The non relativistic limit

It would be very important to evaluate the function \( D(r) \) numerically. The difference between the Belinfante and the canonical tensors seen in the previous section should be measurable to have a practical meaning, if it was too small or a rapidly oscillating function on a microscopic scale, the macroscopic observation of the difference would be impossible. The numerical evaluation of this difference is very hard in the fully relativistic case, but turned out to be relatively easy in the non relativistic limit \( m/T \gg 1 \) where \( D(r) \) is the sum of the particle and antiparticle contribution, \( D^+(r) \) and \( D^-(r) \):

\(^6\) The ones obtained by the transformation in which \( \hat{\Phi} = \hat{S} \), i.e. \( \hat{S}^{\mu\nu} = \frac{4}{3} \left[ \nabla^\mu \nabla^\nu \Psi + \nabla^\nu \nabla^\mu \Psi \right] \), \( \hat{S}_{\lambda,\mu\nu}^{\lambda,\mu\nu} = 0 \).

\(^7\) Actually for \( \omega = 0 \) the whole function \( D(r) \) must be vanishing because of symmetry reasons. In fact in this case the density operator enjoys an additional symmetry: the rotation of an angle \( \pi \) around any axis orthogonal to the cylinder axis. This transformation corresponds to flipping over the cylinder, which leaves the system invariant provided that \( \omega = 0 \), and has the consequence that any pseudovector field directed along the axis must vanish.
\[ D(r)^\pm = \hbar tr \left( \hat{\rho} \left( \frac{\hat{\Psi}^\dagger \Sigma_z \hat{\Psi}}{\hat{\rho}} \right)^\pm \right) \approx \frac{\hbar \omega}{2KT} \hbar tr \left( \hat{\rho} \left( \frac{\hat{\Psi}^\dagger \hat{\Psi}}{\hat{\rho}} \right)^\pm \right) = \hbar \frac{\hbar \omega}{2KT} \left( \frac{dn}{d^3x} \right)^\pm \] (9)

For the numerical computation to be accurate enough we have to make the series involved quickly convergent at any \( r \). This leads to some requirements: first (in natural units) \( \omega/T \ll 1 \), secondly the radius \( R \) should be such that \( R\sqrt{mT} \) is not too large, then \( m/T \gg 1 \) (the non-relativistic approximation itself); we have to choose the chemical potential \( \mu \) so as to keep far from the degenerate Fermi gas case.

**Figure 2.** Function \( D(r) \), corresponding to the mean value of the canonical spin tensor for the free Dirac field, in a rotating cylinder at thermodynamical equilibrium as a function of radius \( r \), in the non-relativistic limit.

The function \( D(r) \) as a function of \( r \) is shown in figure 2 for \( \mu = 0 \), \( R = 300 \), \( T = 0.01 \), \( m = 1 \) and two different values of \( \omega \), \( 10^{-4} \) and \( 2 \cdot 10^{-4} \); the function \( (r/2)D'(r) + D(r) \), which is the difference between angular momentum densities for the canonical and Belinfante tensors, is shown in figure 3.

The plots in figures 2,3 show that the angular momentum density is larger in the canonical than in the Belinfante case almost everywhere, except for a narrow space near the boundary, whose thickness is plausibly determined by the microscopic scales of the problem (thermal wavelength or Compton wavelength). Thereby, the observable macroscopic value of the differences between angular momentum densities, for a rotating system of free fermions, is the slowly varying positive one in the bulk. While the boundary conditions are needed to ensure the invariance of the total angular momentum, the rapid drop to zero within a microscopic distance from the cylinder surface tells us that the chosen boundary conditions at a macroscopic scale of observation correspond to a discontinuity or a surface effect. Any macroscopic coaxial sub-cylinder of the full cylinder with a radius \( r < R \) will therefore have different total angular momenta whether one chooses the canonical or the Belinfante tensors. Such an ambiguity is physically unacceptable and can be solved only by admitting that these tensors are in fact inequivalent.
4. Conclusions and outlook

In conclusion, starting from the study of relativistic fluids with spin, we found that a previously assumed condition, the Frenkel condition, is not generally justified and that previously considered equivalent couples of spin and energy-momentum tensors are in fact thermodynamically inequivalent.

It is a very important issue what is the right couple of tensors; for instance, if it was found that the quantum energy-momentum tensor is not the Belinfante one, this would have major consequences in hydrodynamics and gravity, because a non-symmetric part could imply a torsion of the spacetime.\footnote{For a recent discussion see e.g. Ref. [14].}

In principle we could decide if a specific stress-energy or a spin tensor is \textit{wrong} by measuring with sufficient accuracy the angular momentum density of a rotating system at full thermodynamical equilibrium kept at fixed temperature $T$ and angular velocity $\omega$. This measurement would not seem an easy one since the difference in the average polarization of particles, proportional to $\bar{\hbar}\omega/KT$, is extremely small for ordinary macroscopic systems.\footnote{According to eq. (9) in the non-relativistic limit the difference between these two tensors is of the order of $\hbar \omega / K T$ times $\hbar$ times the particle density, \textit{i.e.} particles have a polarization of the order of $\hbar \omega / K T$.
\footnote{The argument (written in the conclusions of[6]) basically states that, if we assume that the thermodynamical potential $\log Z_\omega$ has a physically objective density, we can consider a sub-cylinder with radius $r \ll R$ as “the system” and the rest as an angular momentum reservoir at full thermodynamical equilibrium. So, since the thermodynamical potential of the sub-cylinder is greater in the canonical case, if the system could choose between canonical and Belinfante tensors the former would be certainly favored.}}

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But this is precisely the polarization responsible for the observed magneto-mechanical phenomena, the Barnett (magnetization induced by a rotation, see e.g.\cite{15, 16}) and Einstein-De Haas (rotation induced by magnetization) effects. It is therefore possible that with some suitable experiment of this sort one can discriminate between spin tensors, for example this effect could be enhanced lowering the temperature so much to increase the ratio $\hbar \omega / K T$ with cold atom techniques.

We cannot, for the present, determine a thermodynamically “best” couple of stress-energy and spin tensor. We can argue, on the basis of a thermodynamical argument - which can in principle be used to assess any other couple of tensors - that the canonical spin tensor is favored over the Belinfante one\footnote{The argument (written in the conclusions of[6]) basically states that, if we assume that the thermodynamical potential $\log Z_\omega$ has a physically objective density, we can consider a sub-cylinder with radius $r \ll R$ as “the system” and the rest as an angular momentum reservoir at full thermodynamical equilibrium. So, since the thermodynamical potential of the sub-cylinder is greater in the canonical case, if the system could choose between canonical and Belinfante tensors the former would be certainly favored.} but, for the present, does not permit to single out the “best” couple of tensors for a given quantum field theory.

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