SIZE AND SHAPE OF BARYONS IN
A LARGE $N_C$ QUARK MODEL

A. J. BUCHMANN
University of Tübingen, Institute for Theoretical Physics, Auf der Morgenstelle 14,
D-72076 Tübingen, Germany
E-mail: alfons.buchmann@uni-tuebingen.de

Baryon charge radii and quadrupole moments are calculated in a quark model
generalized to an arbitrary number ($N_c$) of colors. Several relations among the
charge radii and quadrupole moments are found. In particular, the relation
$Q_{p \rightarrow \Delta^+} = \frac{1}{\sqrt{2}} r_n^2$ between the neutron charge radius $r_n^2$ and the $p \rightarrow \Delta^+$ transition
quadrupole moment $Q_{p \rightarrow \Delta^+}$ is shown to hold for physical baryons as well in the
large $N_c$ limit.

1 Introduction

Charge radii ($r_B^2$) and quadrupole moments ($Q_B$) are the lowest moments of
the charge density $\rho$ in a low-momentum expansion. For example, for a member
of the baryon decuplet with unit charge, one has up to $q^2$ contributions

$$\langle B|\rho(q)|B\rangle = 1 - \frac{q^2}{6} r_B^2 - \frac{q^2}{6} Q_B + \cdots,$$  \hspace{1cm} (1)

where $q$ is the photon momentum transfer to the baryon. The first two terms
arise from the spherically symmetric monopole part of $\rho$, while the third term
is obtained from the quadrupole part of the charge density. They characterize
the total charge, spatial extension, and shape of the system.

Baryon charge radii and quadrupole moments are closely related conse-
quences of the quark-gluon dynamics. They have been calculated in different
models of baryon structure, e.g., an $N_c = 3$ constituent quark model with two-
body exchange currents. Several new relations among the different $r_B^2$ and
$Q_B$ have been abstracted from these model calculations, some of which have
later been shown to be general consequences of the underlying symmetries
and dynamics of quantum chromodynamics.

2 Large $N_c$ Quark Model

It is instructive to investigate the charge radii and quadrupole moments of
baryons in a quark model in which the number of colors $N_c$ is not fixed to its
physical value $N_c = 3$ but allowed to take on an arbitrary odd integer value.
Among the advantages of treating $N_c$ as a parameter are:

(i) an expansion in powers of $1/N_c$ of the operator structures contributing to a given observable is obtained, (ii) there is a hierarchy in importance of one-, two-, and three-quark operators.

A multipole expansion of the baryon charge $\rho$ up to quadrupole terms leads to the following invariants in spin-isospin space

$$\rho = A \sum_{i} e_i - \frac{B}{N_c} \sum_{i<j} e_i \left[ 2\sigma_i \cdot \sigma_j - (3\sigma_i z \sigma_j z - \sigma_i \cdot \sigma_j) \right],$$  \hspace{1cm} (2)

where $\sigma_i z$ is the $z$-component of the Pauli spin matrix of quark $i$, and $e_i$ is the quark charge. The constants $A$ and $B$ in front of the one- and two-quark operators parametrize the orbital and color matrix elements. If the two-quark invariants are generated by gluon exchange, there is a fixed ratio ($-2$) of the factors multiplying the monopole (spin-scalar) and quadrupole (spin-tensor) contributions to $\rho$.

Three-quark operators are neglected in the present work.

3 Baryon Charge Radii

According to Witten\cite{4}, baryons have charge radii proportional to $N_c^0$ in leading order so that, e.g., nonstrange baryons have the same size independent of their spin. Here, we consider the $1/N_c$ corrections associated with the spin-dependent two-body operators in Eq. (2). The latter break SU(6) spin-flavor symmetry and lead to a splitting between octet and decuplet charge radii. This is analogous to the octet-decuplet mass splittings by spin-dependent interactions\cite{5}.

The $N$ and $\Delta$ charge radii\cite{3} are the matrix elements of the first two terms in Eq. (2) evaluated between large $N_c$ nucleon and $\Delta$ wave functions\cite{6}.

$$r_B^2 = \frac{1}{Q} \left\{ A Q - B \frac{4J(J+1) + N_c(N_c + 2)(2Q - 1)}{N_c^2} + 6BQ \frac{N_c}{N_c^2} \right\},$$  \hspace{1cm} (3)

where $Q$ is the total charge and $J$ the total angular momentum of the baryon.

Specializing to $J = 1/2$ for the nucleon states, one obtains (for neutral baryons there is no normalization factor $1/Q$)

$$r_p^2 = A - B \frac{(N_c - 1)(N_c - 3)}{N_c^2},$$

$$r_n^2 = B \frac{(N_c - 1)(N_c + 3)}{N_c^2}.$$  \hspace{1cm} (4)
Similarly, from Eq. (3) we obtain for the $\Delta$ states with $J = 3/2$

$$r^2_{\Delta^{++}} = A - \frac{3}{2} B \frac{N_c^2 - 2 N_c + 5}{N_c^2},$$

$$r^2_{\Delta^+} = A - B \frac{N_c^2 - 4 N_c + 15}{N_c^2},$$

$$r^2_{\Delta^0} = B \frac{(N_c - 3)(N_c + 5)}{N_c^2},$$

$$r^2_{\Delta^-} = A - 3 B \frac{N_c^2 - 5}{N_c^2}.$$  \hspace{1cm} \text{(5)}

From these equations one readily derives that

$$r^2_p - r^2_{\Delta^+} = r^2_n - r^2_{\Delta^0}$$ \hspace{1cm} \text{(6)}

is valid for all $N_c$. Eq. (6) contains the SU(6) symmetry-breaking effect due to the spin-isospin dependent $O(1/N_1^2)$ correction in the baryon charge. Further relations and a more complete treatment including the effect of three-body operators can be found in Ref. 3.

4 Baryon Quadrupole Moments

Quadrupole moments measure the deviation of the baryon’s charge density from spherical symmetry. For the diagonal quadrupole moment of a non-strange decuplet baryon with charge $Q$, total angular momentum $J$, and projection $J_z$ one finds

$$Q_B = B \frac{\left[3 J_z^2 - J(J+1)\right]}{J(J+1)} \frac{4 J(J+1) + N_c(N_c+2)(2Q-1)}{N_c^2}. \hspace{1cm} \text{(7)}$$

Eq. (7) is the matrix element of the quadrupole part of $\rho$ evaluated between large $N_c$ quark model wave functions. Specializing to $J = J_z$ one gets for the different charge states of the $\Delta$,

$$Q_{\Delta^{++}} = \frac{2}{5} B \frac{[15 + 3 N_c (N_c + 2)]}{N_c^2},$$

$$Q_{\Delta^+} = \frac{2}{5} B \frac{[15 + N_c (N_c + 2)]}{N_c^2},$$

$$Q_{\Delta^0} = \frac{2}{5} B \frac{[15 - N_c (N_c + 2)]}{N_c^2},$$

$$Q_{\Delta^-} = \frac{2}{5} B \frac{[15 - 3 N_c (N_c + 2)]}{N_c^2}. \hspace{1cm} \text{(8)}$$
The $\Delta$ quadrupole moments satisfy the relations
\begin{align*}
Q_{\Delta^+} + Q_{\Delta^-} &= Q_{\Delta^+} + Q_{\Delta^0}, \\
Q_{\Delta^+} + Q_{\Delta^0} &= 2 Q_{\Delta^0}, \\
\end{align*}
which hold for all $N_c$. They are linear combinations of the quadrupole moment relations obtained with Morpurgo’s general parametrization method. Only very weak assumptions, such as invariance of the strong interactions under isospin rotations, are required to derive them.

The nondiagonal $N \to \Delta$ transition quadrupole moment in the large $N_c$ quark model is
\begin{equation}
Q_{N \to \Delta} = \frac{1}{2} B \sqrt{2} \frac{(N_c - 1)(N_c + 5)}{N_c},
\end{equation}
both for the $p \to \Delta^+$ and the $n \to \Delta^0$ transition. Additional results including an analysis of strange baryons and three-body operators will be published elsewhere.

5 Relations Between Charge Radii and Quadrupole Moments

Comparing Eq. (10) and Eq. (4), a relation between the neutron charge radius $r_{n}^2$ and the $p \to \Delta^+$ transition quadrupole moment $Q_{p \to \Delta^+}$ is obtained
\begin{equation}
Q_{p \to \Delta^+} = \frac{1}{\sqrt{2}} r_{n}^2,
\end{equation}
which is exact for $N_c = 3$ and $N_c \to \infty$. For intermediate values of $N_c$ the difference between left and right hand side is $O(1/N_c^2)$. Eq. (11) was originally derived in an $N_c = 3$ constituent quark model with two-body exchange currents. Recent experiments provide some evidence that it is satisfied within 20%.

The observables $r_{n}^2$ and $Q_{p \to \Delta^+}$ are closely related because (i) in both cases one-quark operators do not contribute, (ii) both observables are dominated by the two-quark ($B$) term, (iii) the relative weight of the monopole (spin-scalar) and quadrupole (spin-tensor) parts in $\rho$ is uniquely determined if these arise from gluon exchange.

The generalization of Eq. (11) to finite momentum transfers is
\begin{equation}
G_{C2}^{p \to \Delta^+}(q^2) = -\frac{3\sqrt{2}}{q^2} G_{C0}^{n}(q^2),
\end{equation}
Together with the result $G_{M1}^{p \to \Delta}(q^2) = -\frac{\sqrt{2}}{q} G_{M1}^{n}(q^2)$, it can be used to predict the quadrupole (C2) over dipole (M1) ratio in $N \to \Delta$ electroexcitation from
the elastic neutron form factor data This agrees very well with C2/M1 data from $N \to \Delta$ electroexcitation.

Finally, the following relation between the charge radii and quadrupole moments holds for all $N_c$:

$$Q_{\Delta^{++}} + Q_{\Delta^+} + Q_{\Delta^0} + Q_{\Delta^-} = r_p^2 - r_{\Delta^+}^2 + r_n^2 - r_{\Delta^0}^2.$$  \hspace{1cm} (13)

6 Summary

The size and shape of baryons are different but related aspects of the underlying quark-gluon dynamics. A large $N_c$ quark model shows that there is a number of relations between baryon charge radii and quadrupole moments that can be put to experimental tests.

Acknowledgments

I thank the organizers for the invitation. Some financial support by the DFG under title BU 813/3-1 and the Institute for Nuclear Theory is gratefully acknowledged.

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