Investigation of nonlinear transformations of the signal spectrum

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Abstract. We consider various echo-processes having different nature and discuss their mathematical models based on the special nonlinear transformations in the spectral domain. It is shown that the presence of a power polynomial in the spectral domain results in a delay of the external perturbation (signal). The correspondence between the degree of nonlinearity and arising delayed signals is established. The agreement between the numerical results and experimental data is demonstrated for several external signals.

1. Introduction
Nonlinear transformation of signal spectrum is a transformation, wherein the output signal depends nonlinearly on the input one. It can be expected that the supplementary signals may occur at time axis in addition to the base (base baud) signals. The echo-signals can be considered as examples of such supplementary pulses [1]. The corresponding interpretation of echo-signals as a result of the nonlinear transformation of input signals spectrum can be found in [2,3].

2. Theory
Let us describe what happens if we apply a nonlinear transformation to a physical signal limited in time. Notice, that physically realizable input signal \( s(t) \) is determined only at positive semi-axis of time, while the Fourier transformation is defined for \(-\infty < t < \infty\). Because of this, let us redefine the function \( s(t) \) along negative time axis by the even reflection. Hence, the spectrum of signal \( S(t) \) defined along whole time axis is denoted by expression \( \hat{S}(\omega) = \hat{s}(\omega) + \hat{s}^*(\omega) \), where \( \hat{s}(\omega) = \int_{-\infty}^{\infty} s(t) e^{-i\omega t} dt \). Let us also designate the spectrum of output signal \( U(t) \) by \( \hat{U}(\omega) \). Let us now suppose that the frequency conversion of input signal by some device (quadrupole) can be expressed by

\[
\hat{U}(\omega) = \sum_{n=0}^{N} \{ a_n [\hat{S}(\omega)]^n \},
\]

where \( a_n \) are factors determining the level of the corresponding components.

The term with \( n = 0 \) in formula (1) can be interpreted as the own thermal noise of our quadrupole. The term with \( n = 1 \) corresponds to the standard linear transformation of input signal (e.g. filtration), and all the subsequent terms with \( n \geq 2 \) are nonlinear frequency transformations.

Let us consider the sequence of \( M \) non-overlapping pulses (see figure 1) as input signal, i.e.,
Here, $\tau_m$ stands for the beginning of the $m$-th pulse.

If all the functions $S_m(t-\tau_m)$ are real ones, we have $\hat{S}(\omega) = \hat{S}^*(\omega)$. It means that the spectrum of the signal in the negative frequency range can be represented by the complex conjugation of spectrum of the same signal in the positive frequency range. So, we obtain the following expansion:

$$\hat{S}(\omega) = \sum_{m=1}^{M} \hat{S}_m(\omega) = \sum_{m=1}^{M} [\hat{s}_m(\omega)e^{-i\omega \tau_m} + \hat{s}_m^*(\omega)e^{i\omega \tau_m}],$$

where $i$ is an imaginary unit.

**Figure 1.** Input signal (see formula (2)).

This signal acts on the quadrupole, which transforms the frequencies in a nonlinear manner. Substituting (3) into (1) we obtain that the spectrum of output signal after $M$ pulses can be expresses by

$$\hat{U}(\omega) = \sum_{n=0}^{N} \left[ \sum_{m=1}^{(M+n-1)!} C_m \prod_{k=1}^{(M-1)!} \hat{S}_k^{q_k} \right].$$

Here, $N \geq 2$, $C_m$ are numerical coefficients, and exponents $q_k$ take the all possible values from 0 to $n$ such that $q_1 + q_2 + \ldots + q_k = n$.

The signal emitted by the quadrupole can be obtained from the formula (4) with a use of the inverse Fourier transform. If there are several quadrupoles, the overall signal is a sum of the responses from all of them taking into account the distribution function of quadrupoles over resonance frequencies. Assuming that this distribution is continuous and described by distribution function $F(\omega_l)$, we can replace summation by integration and obtain the following formula for the signal emitted by material or system of quadrupoles after irradiation by the sequence of $M$ pulses:

$$U(t) = \frac{1}{2\pi} \int_0^{\infty} F(\omega_l) \int_{-\infty}^{\infty} \left[ \sum_{n=0}^{N} \left( \sum_{m=1}^{(M+n-1)!} C_m \prod_{k=1}^{(M-1)!} \hat{S}_k^{q_k} \right) e^{i\omega t} d\omega \right] d\omega_l.$$

The output signal formed by contributions from individual quadrupoles can change its intensity with time if the radiation from individuals becomes in-phase one at certain moments.

Now, using (5) we can show that echo-signals can be generated in the systems of quadrupoles with frequency nonlinearity. Let us consider the simplest case, where the quadrupole is undergone by
action of two pulses, \( s_{1}(t) \) starting at the moment \( \tau_{1} \) and \( s_{2}(t) \) starting at the moment \( \tau_{2} \). It is obvious that there is no echo if we set \( N = 2 \) in formula (4). Otherwise, if we take \( N = 3 \), the output signal emitted by quadrupoles can be expressed by

\[
U(t) = \frac{1}{2\pi} \int_{0}^{\infty} F(\omega_{1}) \int_{-\infty}^{\infty} \left[ a_{0} + a_{1}(\hat{S}_{1} + \hat{S}_{2}) + a_{2}(\hat{S}_{1}^{2} + 2\hat{S}_{1}\hat{S}_{2} + \hat{S}_{2}^{2}) \right] e^{i\omega t} d\omega d\omega_{1}.
\]

(6)

Substituting representation \( \hat{S}_{\mu} = \hat{S}_{\mu 0}e^{-i\omega \tau_{\mu}} + \hat{S}_{\mu 0}^{*}e^{i\omega \tau_{\mu}} \) in (6), we obtain that the echo-signals can be provided only by the following terms:

\[
\frac{1}{2\pi} \int_{0}^{\infty} F(\omega_{1}) \int_{-\infty}^{\infty} 3a_{2}[\hat{S}_{1}\hat{S}_{2}^{*}e^{i\omega(2\tau_{2}-\tau_{1})} + \hat{S}_{1}^{*}\hat{S}_{2}e^{i\omega(2\tau_{2}+\tau_{1})} + \hat{S}_{1}\hat{S}_{2}^{*}e^{i\omega(2\tau_{2}-2\tau_{1})}] d\omega d\omega_{1}.
\]

(7)

The signal arising during the doubled time interval between the first and second exciting pulses gives the second term in (7). Thus, the primary echo can be expressed by

\[
\frac{1}{2\pi} \int_{0}^{\infty} F(\omega_{1}) \int_{-\infty}^{\infty} 3a_{2}\hat{S}_{1}\hat{S}_{2}^{*}e^{i\omega(2\tau_{2}+\tau_{1})} d\omega d\omega_{1}.
\]

(8)

3. Calculation / Experiment

With a use of the above described algorithm, we performed the calculations for echo-signals occurring from NMR after excitation of substance (system of quadrupoles) by two or four radio-frequency pulses. We also compared the results of calculations with the experimental data obtained during the excitation of echoes in thin ferromagnetic cobalt films [4,5] and lithium ferrite [6]. The corresponding results are shown in figures 2 and 3. The simulations and experiments are in good agreement.

4. Discussion

It is of interest to compare the results obtained by our method with the spectra of two-pulse echo-signals of a different nature. These spectra can be determined as approximate solutions of the nonlinear partial differential equations describing the relevant phenomena. It is shown in [5] that the approximate solution of the Bloch equations gives at \( \tau_{1} = 0 \) the following formula for the spectrum of the primary nuclear spin echo arising at the moment \( t = 2\tau_{2} \): \( \hat{S}_{2}\tau_{2} = \hat{S}_{1}\hat{S}_{2}^{*} \). Similar expressions are also found for the spectra of the primary echo-signals generated in high-temperature superconductors by solving the corresponding nonlinear equations (see [7]). So, the suggested algorithm is in good agreement with the theoretical and experimental results obtained earlier.

Based on the above arguments, one can conclude that in the case of excitation by pulses with equal carrier frequencies, the minimum value of parameter \( N \) in formula (4) guaranteeing the existence of the echo-signal is \( N = 3 \). For higher degree of nonlinearity (i.e., \( N \geq 4 \)), the additional or secondary echo-signals can be registered at the moments \( t = k\tau_{2} \). Notice, for the special case, where the frequency of the second pulse is two times more than the frequency of the first one, the so-called parametric echo [8] appears for \( N \geq 2 \).
5. Conclusion
The suggested algorithm can especially be attractive for echo-signal analysis upon application of the complicated and noise external actions since it is extremely difficult to obtain the approximate solutions of nonlinear equations.

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