Pair cascades in the magnetospheres of strongly-magnetized neutron stars

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ABSTRACT
We present numerical simulations of electron-positron pair cascades in the magnetospheres of magnetic neutron stars for a wide range of surface fields ($B_p = 10^{12}$–$10^{15}$ G), rotation periods (0.1–10 s), and field geometries. This has been motivated by the discovery in recent years of a number of radio pulsars with inferred magnetic fields comparable to those of magnetars. Evolving the cascade generated by a primary electron or positron after it has been accelerated in the inner gap of the magnetosphere, we follow the spatial development of the cascade until the secondary photons and pairs leave the magnetosphere, and we obtain the pair multiplicity and the energy spectra of the cascade pairs and photons under various conditions. Going beyond previous works, which were restricted to weaker fields ($B < \sim \frac{1}{3} \times 10^{12}$ G), we have incorporated in our simulations detailed treatments of physical processes that are potentially important (especially in the high field regime) but were either neglected or crudely treated before, including photon splitting with the correct selection rules for photon polarization modes, one-photon pair production into low Landau levels for the $e^\pm$, and resonant inverse Compton scattering from polar cap hot spots. We find that even for $B \gg B_Q = 4 \times 10^{13}$ G, photon splitting has a small effect on the multiplicity of the cascade since a majority of the photons in the cascade cannot split. One-photon decay into $e^\pm e^\mp$ pairs at low-Landau levels, however, becomes the dominant pair production channel when $B > \frac{1}{3} \times 10^{12}$ G; this tends to suppress synchrotron radiation so that the cascade can develop only at a larger distance from the stellar surface. Nevertheless, we find that the total number of pairs and their energy spectrum produced in the cascade depend mainly on the polar cap voltage $B_p P^{-2}$, and are weakly dependent on $B_p$ (and $P$) alone. We discuss the implications of our results for the radio pulsar death line and for the hard X-ray emission from magnetized neutron stars.

Key words: radiation mechanisms: non-thermal – stars: magnetic fields – stars: neutron – pulsars: general.

1 INTRODUCTION
The pair cascade in the magnetosphere of a pulsar has long been considered an essential ingredient for the pulsar’s nonthermal emission, from radio to gamma rays (e.g., Sturrock 1971, Ruderman & Sutherland 1973, Melrose 2004, Thompson 2004). More recently it has been suggested that the pair cascade is also necessary for nonthermal emission from magnetars (e.g., Beloborodov & Thompson 2007, Thompson 2008a,b; see Woods & Thompson 2008 for a review of magnetars). The basic pair cascade involves several steps: (i) acceleration of primary particles by an electric field parallel to the magnetic field; (ii) gamma ray emission by the accelerated particles moving along the magnetic field lines (either by curvature radiation or inverse Compton upscattering of surface photons); (iii) field-assisted photon decay into electron-positron pairs as the angle between the photon and the magnetic field line becomes sufficiently large, or pair production by two-photon annihilation in weak-field regimes; (iv) gamma ray emission by the newly-created particles as they lose their transverse energy through synchrotron emission; (v) further pair production and gamma ray emission via steps (iii) and (iv). The dense, relativistic (Lorentz factors $\gamma \gtrsim 100$) electron-positron plasma generated by this cascade is a required input in many models for the pulsar radio emission (e.g., Melrose 1993, 2004, Beskin 1993, Melikidze, Gil, & Pataraya 2000, Lyubarsky 2000, 2001).
The behavior of the pair cascade in the superstrong field regime (magnetic field strengths $B \gtrsim B_Q \equiv 4.414 \times 10^{13}$ G) and its effect on emission from pulsars and magnetars is somewhat puzzling. For example, of the dozen-or-so observed magnetars, only two show pulsed radio emission, and it is of a completely different nature than the emission from “standard” radio pulsars (e.g., the radio pulsars are transient and appear to be correlated with strong X-ray outbursts from the magnetars; see Camilo et al. 2007, 2008). In contrast, several radio pulsars with inferred surface field strengths similar to those of magnetars have been discovered (e.g., Kaspi & McLaughlin 2005; Vranevsevic, Manchester, & Melrose 2007). Why the standard mechanism for pulsed radio emission turns off for magnetars but not for these pulsars is unknown.

There have been only a few publications devoted to numerical simulations of the pair cascade in pulsar magnetospheres. For moderate-strength magnetic fields ($B \lesssim 5 \times 10^{12}$ G), significant progress has been made. Daugherty & Harding (1982) present simulations of the cascade initiated by a single electron injected from the neutron star surface, emitting photons through curvature radiation, for (polar) surface field strengths $B_p$ up to $5 \times 10^{12}$ G and rotation periods $P = 0.033 - 1$ s. In a later paper (Daugherty & Harding 1996) they consider gamma ray emission from the entire open-field-line region of the magnetosphere, using a simplified acceleration model and for Vela-like pulsar parameters ($B_p = 3 \times 10^{12}$ G and $P = 0.089$ s). Sturmer, Dermer, & Michel (1993) present a similar simulation to that of Daugherty & Harding, but for cascades initiated by electrons upscattering photons through the inverse Compton process (again for Vela-like parameters). Hibschman & Arons (2001) develop a semi-analytic model of the inner gap cascade, both for curvature radiation-initiated and inverse Compton scattering-initiated cascades, applicable for $B \lesssim 3 \times 10^{12}$ G (see also Zhang & Harding 2000). Cascades occurring in the outer magnetosphere have also been simulated, by Romani (1996) for Vela- and Crab-like ($B_p = 4 \times 10^{12}$ G and $P = 0.033$ s) parameters (see also Cheng, Ho, & Ruderman 1986a,b; Cheng, Ruderman, & Zhang 2000).

However, for superstrong magnetic fields ($B \gtrsim B_Q \equiv 4.414 \times 10^{13}$ G) only limited aspects of the full cascades have been studied. For example, Arendt & Eilek (2002) simulate the cascade for $B_p \lesssim 10^{13}$ G and $P = 0.033$ s (for both a pure dipole and a more complex field geometry), but with the simplification that all photons radiated by the primary particle are emitted from the surface. Baring & Harding (2001) (see also Harding, Baring, & Gonthier 1997) use this same simplification to study the effects of photon splitting on the cascade for field strengths up to $B = 2 \times 10^{14}$ G (however, they assumed that both photon modes can split, and thus overestimated the effect of photon splitting; see Section 4.2). Baring & Harding (2007) model the process of resonant inverse Compton scattering of photons from the neutron star surface (with the blackbody temperature $T = 6 \times 10^6$ K) in the same field range, but only for single scattering events (see also Dermer 1990). The magnetosphere acceleration zone in the superstrong, twisted field regime of magnetars is investigated analytically by Beloborodov & Thompson (2003) for cascades occurring in the closed field line region of the magnetosphere and by Thompson (2008a,b) in the open field line region.

In this paper we present numerical simulations of the pair cascade from onset to completion. Motivated by the lack of full cascade results for the superstrong field regime, and in light of the unexplained differences between the observed emission properties of high-field radio pulsars and magnetars, we run our simulations in magnetospheres with field strengths up to $10^{15}$ G. We consider several important factors that affect high-field cascades, including photon splitting, pair creation in low Landau levels, photon polarization modes ($\perp$ or $\parallel$ to the magnetic field direction), and resonant inverse Compton scattering. We use our simulations to generate spectra of the high-energy photons and the electron-positron plasma produced by the cascade. Additionally, we use our simulations to comment on the conditions for when the radio emission mechanism no longer operates in the neutron star magnetosphere, the so-called “pulsar death line” (e.g., Ruderman & Sutherland 1977; Cheng & Ruderman 1993; Hibschman & Arons 2001). Harding & Muslimov 2002; Harding, Muslimov, & Zhang 2002 (Medin & Lai 2007). While the results of our simulation are most applicable to cascades occurring in the open field line region of the magnetosphere (since the primary particles are injected into the magnetosphere along open field lines), some of our results are also relevant to cascades occurring in the closed field line region for magnetars, e.g., the products of a cascade initiated by a photon injected into a non-dipole magnetosphere.

A necessary component of any pair cascade simulation is a model of the magnetosphere acceleration zone, or “gap”, where the cascade originates. In real magnetospheres of pulsars and magnetars, the acceleration of primary particles is coupled to the rest of the cascade (e.g., charged particles produced in the cascade can screen out the acceleration potential). However, there is significant uncertainty about the precise nature of the acceleration gap. A number of models have been proposed for the location of the gap, from inner magnetosphere accelerators (both “vacuum” and “space-charge-limited flow” types; see, e.g., Ruderman & Sutherland 1977; Arons & Scharlemann 1979; Muslimov & Tsygan 1992; Hibschman & Arons 2001a; Medin & Lai 2003; Thompson 2008a,b), to outer magnetosphere accelerators (e.g., Cheng et al. 1986a,b; Romani 1996; Cheng et al. 2004; Takata et al. 2006), to hybrid inner-outer magnetosphere accelerators (“slot” gaps and extended outer gaps; e.g., Arons 1983, 1984; Muslimov & Harding 2003, 2004; Hirotani 2004). Non-steady (oscillatory) inner gaps have also been discussed recently (e.g., Sakai & Shibata 2003; Levinson et al. 2004; Beloborodov 2005; Luo & Melrose 2008). Numerical simulations of force-free global magnetospheres including magnetic-field twisting near the light cylinder have been performed (e.g., Contopoulos, Kazanas, & Fendt...
we model the effect of this region on the cascade by giving work to remain as model-independent as possible. Instead, we include an actual acceleration region, since we wish in this 2.1 Primary electrons

2 ESTIMATING THE INITIAL PARAMETERS FOR THE PRIMARY PARTICLES

the range $\Phi \sim (1-2) \times 10^{13}$ V, regardless of the acceleration model (e.g., Hbischman & Arons 2001a; Medin & Lai 2007, hereafter ML07). For the surface field strengths we are considering, $B \geq 10^{12}$ G, the primary electrons are not radiation-reaction limited within these gaps (ML07; cf. the millisecond pulsar models of Harding, Usov, & Muslimov 2003), so we can set $\gamma_0 = \Phi/m_e c^2$. We therefore restrict $\gamma_0$ to the range $(2-4) \times 10^7$ for dipole fields. Note that these large voltage drops do not occur in pulsars where the gap electric field is fully screened due to inverse Compton scattering by the primary electron. We discuss this case in Section 2.2.

The voltage drop across the gap can be no larger than the voltage drop across the entire polar cap of the neutron star (e.g., Ruderman & Sutherland 1973):

$$\Phi_{\text{cap}} \simeq \frac{\Omega B_p R^4}{2c} = 7 \times 10^{12} B_{p,12} P_0^{-2} \text{V},$$

where $R$ is the radius of the star (assumed in this paper to be 10 km), $P_0$ is the spin period in units of 1 s, and $B_p = 10^{12} B_{p,12}$ G is the polar surface magnetic field strength. If the voltage drop, $\Phi$, required to initiate pair cascades is not available, i.e., $\Phi > \Phi_{\text{cap}}$, the magnetosphere should not produce pulsed radio emission; the locus of points where $\Phi = \Phi_{\text{cap}}$ defines the pulsar death line. A typical death line for an inner gap model, plotted in $P-P$ space, is shown on the left panel of Fig. 1. The line was made using three assumptions: (i) The magnetosphere field geometry is dipolar. (ii) The pair cascade occurs primarily above the gap, (through curvature radiation) once the primary electron has reached a large Lorentz factor $\gamma_0 \sim 10^7$. (iii) The spindown power of the pulsar, given by

$$\dot{E} = -I\dot{\Omega} = \frac{4\pi^2 I P}{P^3},$$

is approximately equal to the spindown power of a magnetic dipole with its magnetic field and rotational axes orthogonal to each other:

$$\dot{E} \simeq \frac{B_p^2 \Omega R^6}{6c^4} = \frac{2\Phi_{\text{cap}}}{3}.$$  

The polar magnetic field strength inferred from this frequently-used approximation is

$$B_{p,12} \simeq 2.0 \sqrt{P_0 P_{-15}},$$

where $P_{-15}$ is the period derivative in units of $10^{-15}$ s/s and $I = 10^{45}$ g cm$^2$ is assumed.

A well-known problem with the death line made using these assumptions is that it cuts right through the middle of the main group of pulsars. Ruderman & Sutherland 1973; Hibschman & Arons 2001a; Harding & Muslimov 2002; Medin & Lai 2007; i.e., the model incorrectly predicts that there will be no radio emission from many neutron stars that are observed to be active pulsars.

Several authors have proposed models of the neutron star magnetosphere that shift the theoretical death line closer to the observed death line by altering one
Figure 1. Pulsar death lines. Death lines are shown for pulsars with dipole magnetic fields, dipole fields offset from the center of the star by $\Delta r = 0.95R_c$, and magnetic fields with extended polar caps 100 times larger than the dipole value $\Phi_{\text{cap}} = \sqrt{4\Omega R_c}$ (left panel); and for pulsars with magnetic field curvatures $R_c = R$ at the surface (right panel). Note that these death lines do not apply for the millisecond pulsar population in the lower left corner of the diagram, as their short periods and low magnetic field strengths cause the primary electron to be radiation reaction limited. In each panel, rotation-powered pulsars (ATNF catalog, http://www.atnf.csiro.au/research/pulsar/psrcat) are labeled by crosses, while magnetars (McGill catalog, http://www.physics.mcgill.ca/~pulsar/magnetar/main.html) are labeled by solid circles and the two radio magnetars are labeled by solid triangles.

In some models the pair cascade occurs primarily within the gap, due to efficient inverse Compton scattering by the primary electron, rather than above the gap. These cascades occur at much lower energies of the primary electron, rather than above the gap. How-
ing Eq. (3). Depending on alignment and how close the gap potential drop $\Phi$ is to $\Phi_{\text{cap}}$, many pulsars which were predicted to be dead may actually have a potential drop large enough to generate pair cascades ($\Phi \approx 10^7$ V). According to Contopoulos & Spirokovsky, the standard death shown in Fig. 1 is consistent with the observed $P$-$P$ values for all pulsars if the magnetic inclination angles $\alpha$ of nearly-dead pulsars are weighted towards $\alpha = 0$. If this is the case, we can use $\gamma_0 \simeq (2 - 4) \times 10^7$ for all pulsars and do not need to invoke a strongly-curved magnetic field geometry ($R_c \approx R$) or efficient inverse Compton scattering by the primary electron in order to reproduce the observed pulsar death line.

### 2.2 Primary photons

As the primary electron traverses the acceleration region it can pass through two distinct regimes (see, e.g., Hibschman & Arons 2001a). First, for Lorentz factors $\gamma \sim 10^2$–$10^4$, the electron efficiently upscatters photons through the inverse Compton process. Second, for Lorentz factors $\gamma \gtrsim 10^6$, the electron efficiently emits curvature radiation. The above treatment of the acceleration zone (Section 2.1) is best suited for pulsars where the electrons reach the second regime. In that case we can safely ignore the contributions to the cascade made by photons emitted before the primary electron reaches full energy ($\gamma_0$), since the number and energy of photons emitted through curvature radiation increase strongly with $\gamma$ (i.e., $N_{\text{CR}} \propto \gamma$, $E_{\text{CR}} \propto \gamma^3$). The approximation is poor, however, if inverse Compton scattering and subsequent pair production within the gap is efficient enough to screen the accelerating potential before the electrons can reach the second regime. In that case the photons produced in the gap are critical to the cascade, while the photons produced above the gap have a negligible effect on the cascade (the upscattered photons must travel a finite distance before pair production in order to screen the gap; in that distance the primary electron is accelerated to above resonance and exits the gap with a Lorentz factor between the first and second regimes of efficient photon production).

Because ICS is strongly peaked at resonance, primary electrons traveling through this second type of gap will emit a large number of photons at a characteristic “resonance” energy and very few at other energies. The effect of this type of gap on the cascade is better modeled by $N_0$ photons of energy $\epsilon_0$ emitted from the surface (cf. Arendt & Eilek 2002), rather than one electron of energy $\gamma_0 m_e c^2$. We therefore run a second version of the simulation, this time tracking the cascade initiated by a “primary” photon. The quantitative results of this simulation can be multiplied by $N_0$ to obtain the full cascade results (e.g., the number of electron-positron pairs produced per primary electron).

For a primary electron resonantly upscattering primary photons, we estimate the value of $\epsilon_0$ as follows. When the primary electron reaches a Lorentz factor $\gamma_0$, it upscatters photons to a mean energy (e.g., Beloborodov & Thompson 2007)

$$\epsilon = \gamma \left(1 - \frac{1}{\sqrt{1 + 2\beta_0^2}}\right) m_e c^2,$$

where $\beta_0 = B/B_Q$ is the ratio of the magnetic field strength to the critical quantum field strength, $B_Q = 4.414 \times 10^{13}$ G.

The primary electron is most efficient at scattering photons when

$$\gamma = \gamma_{\text{crit}} \simeq \epsilon_e/kT,$$

where $T$ is the surface temperature of the star and $\epsilon_e = \hbar e B/m_e c$ is the electron cyclotron energy. Therefore, for photons scattered from near the surface, where $B = B_p$, the typical energy of a scattered photon is

$$\epsilon_{\text{RICS}} \simeq 70 B_{p,12} T_6^{-1} f(\beta Q) \text{ MeV},$$

where $T_6$ is the surface temperature in units of $10^6$ K and $f(\beta Q) = 1 - 1/\sqrt{1 + 2\beta Q}$ is evaluated at the surface. Setting $\epsilon_0 = \epsilon_{\text{RICS}}$ and assuming a $T_6$ range of 0.3–3, we obtain $\epsilon_0$ in the range 1–10 MeV at $B_{p,12} = 1$ up to (0.4–4) $\times 10^5$ MeV at $B_{p,12} = 1000$.

The number of resonant ICS photons scattered by the primary electron is more difficult to estimate, since it depends on the acceleration model. Inner gaps with space-charge-limited flows have (e.g., Hibschman & Arons 2001b, Medin 2008)

$$N_0 \simeq 10^5 B_{p,12}^{-1} F_0^{3/4} T_6^{-1/2}.$$

Inner vacuum gaps, with accelerating electric fields on the order of 6 times larger, have $N_0$ values at least 20-100 times smaller (the primary electron is more rapidly accelerated out of resonance; see, e.g., ML07).

Note that we can also use this second cascade simulation as a diagnostic tool for the main simulation. For example, we can study the partial cascade initiated by a single curvature radiation photon emitted at some altitude in the magnetosphere to understand how the strength of the local magnetic field affects the cascade. The characteristic energy of curvature photons is

$$\epsilon_{\text{CR}} = \frac{3\gamma^3 \hbar c}{2 R_c}.$$

For dipole fields the typical curvature photon has an energy $\epsilon_{\text{CR}} \lesssim 10^2$–$10^4$ MeV (for $\gamma \lesssim \gamma_0 \sim 10^5$), while for fields with $R_c = R$ we have $\epsilon_{\text{CR}} \lesssim 10^4$–$10^6$ MeV (for $\gamma \lesssim \gamma_0 \sim 10^8$).

### 3 NUMERICAL SIMULATION OF PAIR CASCADES: PHYSICS INGREDIENTS AND METHODS

The general picture of the pair cascade as modeled by our numerical simulation is sketched in Fig. 2. At the start of the simulation, an electron with initial Lorentz factor $\gamma_0 \sim 10^5$–$10^7$ (Section 2.1) travels outward from the stellar surface along the last open field line. As it travels it emits high-energy photons through curvature radiation or inverse Compton upscattering. The simulation tracks these photons

Note that the actual resonance condition is $\epsilon_i \gamma (1 - \beta \cos \psi) = \epsilon_e$, where $\epsilon_i \sim kT$ is the initial (before scattering) photon energy, $\beta \simeq \sqrt{1 - 1/\gamma^2}$ is the ratio of the electron speed to the speed of light and $\psi$ is the incident angle of the photon with respect to the electron’s trajectory. However, because the scattering rate depends inversely on $\gamma$ (see Appendix A), photons with $\cos \psi \ll 1$ are far more likely to scatter off the electron than photons with $\cos \psi \gtrsim 1$. 

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The input parameters for our simulation are the initial energy of the electron ($\gamma m_c c^2$) or photon ($\epsilon_0$), its initial position (in most cases, the intersection of the last open field line with the stellar surface), the general pulsar parameters (surface magnetic field strength $B_p = 10^{12} - 10^{15}$ G, rotation period $P = 0.33 - 5$ s, and surface temperature $T = 10^{6}$ K or $5 \times 10^{6}$ K), and the geometry of the magnetic field. In each run of the simulation, the magnetic field structure is given by one of two topologies: (i) a pure dipole field geometry; or (ii) a more complex field geometry near the stellar surface which gradually reverts to dipole at higher altitudes (a non-dipole, or “multipole” field geometry). Modeling the dipole field geometry is straightforward (see, e.g., von Hoensbroech, Lesch, & Kunzl 1998), but there is no obviously correct way to model the geometry for the multipole field case (see Section 2.1). Two features of a multipole field geometry have a strong effect on the pair cascade dynamics and must be incorporated into our model: First, the radius of curvature $R_c$ is much smaller than dipole (we choose $R_c = R$, the stellar radius) near the surface of the star. This leads to a much larger number and peak energy of photons emitted through curvature radiation than in the dipole field case. Second, as a photon propagates through the magnetosphere the angle between the photon and the field, which scales like $\Delta \Theta_{ph} \sim s_{ph}/R_c$, where $s_{ph}$ is the distance traveled by the photon from the point of emission, grows much faster than dipole. This leads to a much more rapid decay of photons into pairs than in the dipole case. The integration of these two features into our model is discussed in the relevant subsections below (Section 3.3.1 and Section 3.3.2, respectively). Note that Arentz & Eilek (2002) consider the first aspect of a multipole field geometry in their model (that $R_c = R$) but ignore the second. In all of the simulation runs we assume that the local magnetic field strength varies as in the dipole case,

$$B(r, \theta, \phi) = B_p \left(\frac{R}{r}\right)^3 \frac{\sqrt{3} \cos^2 \theta + 1}{2},$$

where $(r, \theta, \phi)$ are the spherical coordinates (with the magnetic north pole at $r = R$ and $\theta = 0$). Our approximation therefore ignores any amplification of the field strength near the surface caused by the complex topology.

For simplicity we consider a “two-dimensional” cascade model in which all photons are emitted and travel in the plane defined by the local magnetic field line. Both the photons and the electrons/positrons are tracked in the “corotating” frame (the frame rotating with the star), and any bending of the photon path due to rotation is ignored – this is expected to be valid since the cascade takes place far inside the light cylinder. Thus we shall also call this corotating frame the “lab” frame for the remainder of the paper. With this approximation the particle positions and trajectories are defined only in terms of $r$ and $\theta$ in our simulation. We justify this approximation below (Sections 5.1 and 5.2). As an additional simplification we ignore any effects of general relativity on the photon/particle trajectory.

The cascade simulation can naturally be divided into three parts: (i) the propagation and photon emission of the primary electron; (ii) photon propagation, pair production, and splitting; and (iii) the propagation and photon emission as they propagate from the point of emission through the magnetosphere, until they decay into electron-positron pairs through magnetic pair production or escape to infinity. In the superstrong field regime, the photon (if it has the correct polarization; see Section 3.2) also has a finite probability of splitting into two photons before pair production, rect polarization; see Section 3.2) also has a finite probability of splitting into two photons before pair production.

Figure 2. A schematic diagram showing the magnetosphere pair cascade, from initiation by a high-energy electron to completion. Photon splitting is also shown. The inset shows the beginning of a cascade initiated by a photon upscattered through the inverse Compton process.
of the secondary electrons and positrons. Each of these aspects of the simulation is described in a separate subsection below. At the end of this section, cascades initiated by primary photons are discussed.

3.1 Propagation and photon emission of the primary electron

In our cascade simulation, the primary electron starts at the position \((r_0, \theta_0) = (R, \theta_0)\) (i.e., at some angle \(\theta_0\) from the magnetic pole on the neutron star surface) with the initial energy \(\gamma_0 m_e c^2\), and moves outward along the local magnetic field line. The initial position of the primary electron is chosen so that it moves along the open field line, whose location at the surface is given by the polar cap angle: \(\theta_0 = \theta_{\text{cap}} \equiv \sqrt{R}/r_{\text{LC}},\) where \(r_{\text{LC}} = c/\Omega\) is the light cylinder radius.\(^4\)

The primary electron moves outward along the field line in a stepwise fashion. The lengths of the steps \(\Delta s(r)\) are chosen so that a uniform amount of energy \(\Delta \gamma\) (we choose \(\sim 0.001\gamma_0\)) is lost by the electron in each step \((\gamma \rightarrow \gamma - \Delta \gamma)\):

\[
\Delta s(r) \approx -\frac{\Delta \gamma}{d\gamma/dr}. \tag{12}
\]

For an electron emitting curvature radiation,

\[
d\gamma = -\frac{2}{3}\gamma \frac{\alpha_f^2 e_0^5}{R_c^2}, \tag{13}
\]

where \(\alpha_f = e^2/(hc)\) is the fine structure constant and \(e_0\) is the Bohr radius. For a dipole field the radius of curvature is given by

\[
R_c = \frac{r}{\sin \theta} \left(1 + 3 \cos^2 \theta - 3 \cos^2 \theta \right)^{3/2}. \tag{14}
\]

while for a near-surface multipole field we use \(R_c = R\). As discussed in Section 2.3 we do not consider photon emission due to inverse Compton scattering here, since this process is very inefficient once the primary electron has reached the energy \(\gamma_0 m_e c^2\). We do, however, consider in our simulation the photon emission due to ICS by the secondary electrons and positrons (see Section 3.3) which typically have \(\gamma \ll \gamma_0\). Note that we also indirectly include ICS in our second cascade simulation (described in Section 3.4), which models photon-initiated cascades, by choosing photon energies \(e_0\) that are typical of ICS photons.

As the electron moves a distance \(\Delta s\) along the field it emits photons with energies divided into discrete bins (our simulation uses \(\sim 50\) bins). The energy in each bin, \(\epsilon\), is a constant multiple of the characteristic energy of curvature photons \(\epsilon_{\text{CR}} = 3\gamma^2 hc/(2R_c)\), with \(\epsilon/\epsilon_{\text{CR}}\) in the range \(10^{-2}\) to \(10\). The number of photons in a given energy bin emitted in one step is given by the classical spectrum of curvature radiation (e.g., Jackson 1998),

\[
\Delta N_\epsilon \approx \frac{dN}{d\epsilon} \approx \frac{3}{2\pi} \frac{\epsilon}{\epsilon_{\text{CR}}} \frac{\gamma \Delta \epsilon}{\epsilon} F \left( \frac{\epsilon}{\epsilon_{\text{CR}}} \right), \tag{15}
\]

where \(\Delta \epsilon\) is the spacing between energy bins and the values of \(R_c\) and \(\gamma\) used are averages over the interval \(\Delta s\). Here, \(F(x) = x \int_x^\infty K_{3/2}(t) dt\) and \(K_{3/2}(x)\) is the \(n = 5/3\) Bessel function of the second kind. Note that \(F(x) \propto x^{1/3}\) for \(x \ll 1\), and \(F(x) \propto x^{1/3}e^{-x}\) for \(x \gg 1\) (e.g., Erber 1966).

The photons are emitted in the direction nearly tangent to the field line at the current location of the electron \((r, \theta)\). For a dipole field geometry the angle between the local magnetic field and the magnetic dipole axis is given by

\[
\chi(\theta) = \theta + \arctan \left(\frac{\tan \theta}{2}\right), \tag{16}
\]

see Fig. 3. There is an additional contribution to the emission angle of \(\sim 1/\gamma\), due to relativistic beaming. In reality this beaming angle is in a random direction; however, for our two-dimensional approximation it can only be in the plane of the magnetic field. The photon emission angle is given by the (projected) sum of these two angles:

\[
\Theta_{\text{ph}} = \chi + \frac{1}{\gamma} \cos \Pi, \tag{17}
\]

where \(\Pi\) is a random angle between 0 and \(2\pi\). Note that ignoring the three-dimensional aspect of the photon emission introduces an error in the emission angle of order \(1/\gamma\). This affects the location at which the photon decays (into pairs) in our simulation, since photon decay depends strongly on the intersection angle between the photon and the magnetic field (see Section 3.2 below). However, as the photon propagates through the magnetosphere these errors (which are on the order of \(1/\gamma \sim 10^{-7}\) for curvature photons and \(10^{-3}\) for resonant ICS photons) quickly become negligible in comparison to the photon-magnetic field intersection angle, which grows like \(s_{\text{ph}}/R_c\) (and so reaches the angle \(1/\gamma\) by \(s_{\text{ph}} \sim 10^{-5} R\) for curvature radiation and \(\sim 0.1 R\) for RICS).

We also use Eq. 17 for simulation runs with a multipole field geometry. This is obviously a simplification, but we have found that in practice the photon propagation direction has little effect on the overall cascade product (as long as it points generally outward). Far more important for the cascade is how the angle between the photon and the magnetic field changes as the photon travels. As is discussed in Section 3.3 we artificially force this angle to change more rapidly with distance than in the dipole case, to account for the effect of the stronger field line curvature.

The total energy lost over each step is

\[
\sum \epsilon \Delta N_\epsilon \approx \Delta \gamma m_e c^2. \tag{18}
\]

Only one photon is tracked for each energy bin \(\epsilon\) at each step \(\Delta s\), so the photon is given a weighting factor \(\Delta N_\epsilon\). In addition to its initial position (the position of the electron at the point of emission \(r, \theta\)) and propagation direction \((\Theta_{\text{ph}})\), the photon has a polarization direction. For curvature radiation the polarization fraction is between 50% and 100% polarized parallel to the magnetic field curvature, depending on photon frequency (Jackson 1998; see also Rybicki & Lightman).
In our simulation, the photon is emitted/scattered from the point \((r_0, \theta_0, \phi_0)\) with energy \(e\), polarization \(\parallel\) or \(\perp\), and weighting factor \(\Delta N_c\) (to represent multiple photons; Section 3.2). It has an optical depth to pair production, \(\tau\), and to photon splitting, \(\tau_{sp}\), both of which are set to zero at the moment of the photon’s creation. The photon propagates in a straight line from the point of emission, at an angle \(\Theta_{ph}\) with respect to the magnetic dipole axis. Note that in the corotating frame (which is the frame we are working in for most of our simulation; but see Section 3.3) the path of the photon is in reality curved, with the angular deviation from a straight line growing approximately as \(s_{ph}/c = s_{ph}/r_{LC}\) (cf. Harding, Tademaru, & Esposito 1978). Like the beaming angle (Section 3.1), this curved path modifies the growth of the photon-magnetic field intersection angle and the location of photon decay in our simulation. However, the total intersection angle grows much faster with photon distance \(s_{ph}\) than the deviation does (\(\sim s_{ph}/R_c\) versus \(s_{ph}/r_{LC}\), or a factor of \(r_{LC}/R_c \approx 100R_0^{1/2}\) larger for dipole fields), so we can safely ignore this deviation.

In each step the photon travels a short distance through the magnetosphere, \(\Delta s_{ph} < 0.05r_{ph}\), where \((r_{ph}, \theta_{ph}, \phi_{ph})\) refers to the current position of the photon; our method for choosing the value of \(\Delta s_{ph}\) for a given photon is discussed at the end of this section. At the new position the change in the optical depth for pair production, \(\Delta \tau\), and for photon splitting, \(\Delta \tau_{sp}\), are calculated:

\[
\Delta \tau \simeq \Delta s_{ph} R_{\parallel,\perp},
\]

\[
\Delta \tau_{sp} \simeq \Delta s_{ph} R_{sp_{\parallel,\perp}},
\]

where \(R_{\parallel,\perp} = R_{0,\perp} \sin \psi\) is the attenuation coefficient for the \(\parallel\) or \(\perp\) polarized photons, \(\psi\) is the angle of intersection between the photon and the local magnetic field, and \(R'\) is the attenuation coefficient in the “perpendicular” frame (i.e., the frame where the photon propagates perpendicular to the local magnetic field). For a dipole field geometry the intersection angle is given by

\[
\psi = \chi(\theta_{ph}) - \Theta_{ph},
\]

where \(\Theta_{ph}\) is given by Eq. (17) and \(\chi(\theta_{ph})\) is the angle between the magnetic axis and the magnetic field at the current location of the photon [Eq. (19)]; see Fig. 4 for a sketch. For the near-surface multipole field geometry we set

\[
\tan \psi = \frac{s_{ph}}{R_c} = \frac{s_{ph}}{R'}.
\]

This approximation has the advantage of accounting for the effect of a strong field curvature on the photon propagation without requiring knowledge of the actual field topology.

The total attenuation coefficient (in the perpendicular frame) for pair production is given by (suppressing the subscripts \(\parallel, \perp\)) \(R' = \sum_{jk} R'_{jk}\), where \(R'_{jk}\) is the attenuation coefficient for the channel in which the photon produces an electron in Landau level \(j\) and a positron in Landau level \(k\),
and the sum is taken over all possible states for the electron-positron pair. Since pair production is symmetric with respect to the electron and the positron, \( R'_{jk} = R'_{kj} \); for simplicity we hereafter use \( R'_{jk} \) to represent the combined probability of creating the pair in either the state \((jk)\) or \((kj)\) (i.e., \( R'_{jk}^{\text{new}} = R'_{jk}^{\text{old}} + R'_{kj}^{\text{old}} \)). For a given channel \((jk)\), the threshold condition for pair production is

\[
\epsilon' > E_j + E_K,
\]

where \( \epsilon' = \epsilon \sin \psi \) is the photon energy in the perpendicular frame and \( E_n = m_e c^2 \sqrt{1 + 2\beta Q n} \) is the minimum energy of an electron/positron in Landau level \( n \) (the energy of an electron/positron with the momentum along the magnetic field \( p_j = 0 \)). In dimensionless form, the condition [Eq. 23] can be written as

\[
x = \frac{\epsilon'}{2m_e c^2} = \frac{\epsilon}{2m_e c^2} \sin \psi > x_{jk} \equiv \frac{1}{2} \left[ \sqrt{1 + 2\beta Q_j} + \sqrt{1 + 2\beta Q_k} \right].
\]

(24)

Note that \( x_{jk} \) satisfies

\[ x_{00} < x_{01} < x_{02} < x_{11} < x_{03} < \cdots , \quad \beta Q_j < 4; \quad x_{03} < x_{11} < \cdots , \quad \beta Q_j > 4. \]

(25)

The first three attenuation coefficients (corresponding to the three lowest threshold levels \( x_{00}, x_{01}, x_{02} \)) for both \( \parallel \) and \( \perp \) polarizations are given in Appendix B Eqs. 11-15; see also Daugherty & Harding (1983). Note that \( R'_{1,00} \) is zero, and thus the first non-zero attenuation coefficient for \( \perp \) polarized photons is actually \( R'_{1,01} \), not \( R'_{1,00} \).

In our simulation a photon is typically created with \( x \) below the first threshold \( x_{00} \) or \( x_{01} \), depending on the photon polarization. As long as \( x \) remains below the first threshold, \( R' = 0 \) and the optical depth to pair production remains zero. As the photon propagates into the magnetosphere and crosses the first threshold, \( R' > 0 \), \( \Delta \tau > 0 \), and \( \tau \) begins to grow. As it continues to travel outward, both \( \tau \) and the number of Landau levels available for pair production \( j_{\text{max}} \) and \( k_{\text{max}} \) increase. Depending on the local magnetic field strength [Eq. (11)], the photon may reach a large enough optical depth (\( \tau \approx 1 \)) for pair production after crossing only a few thresholds (so that \( j_{\text{max}} \) and \( k_{\text{max}} \) are small) or after crossing many thresholds (so that \( j_{\text{max}} \) and \( k_{\text{max}} \) are very large). For “weak” magnetic fields \((\beta Q \lesssim 0.1)\) the optical depth increases slowly with \( s_p \) and it is valid to use the \( j_{\text{max}}, k_{\text{max}} \geq 1 \) asymptotic attenuation coefficient for pair production (e.g., Erber 1968),

\[
R'_{1,\perp} \approx \frac{0.23}{a_0} \beta Q \exp \left( -\frac{4}{3\beta Q} \right),
\]

(26)

which applies for both polarizations. For stronger fields, however, pairs are produced in low Landau levels, and the more accurate coefficients of Daugherty & Harding must be used. In Appendix B2 we find that the critical magnetic field strength separating these two regimes is

\[
B_{\text{crit}} \approx 3 \times 10^{12} \text{ G}
\]

(27)

[see Eq. (15)]. We also find that the boundary between the two regimes is very sharp: pairs are either created at the first few Landau levels \((n \leq 2)\) for \( B \geq B_{\text{crit}} \) or in very high Landau levels for \( B \lesssim B_{\text{crit}} \), with very few electrons/positrons created in intermediate Landau levels. Therefore, in our simulation we only consider the first three attenuation coefficients for \( \parallel \)-polarized photons \((R'_{1,00}, R'_{1,01}, R'_{1,02})\) and the first two non-zero attenuation coefficients for \( \perp \)-polarized photons \((R'_{1,01}, R'_{1,02})\). If the photon reaches the threshold for the \((03)\) or \((11)\) channel [whichever is reached first; see Eq. (25)], we use the asymptotic formula, Eq. (26). The total attenuation coefficient for pair production (as given by this approximation) is plotted in Fig. 5 for both \( \parallel \) and \( \perp \) polarizations at \( \beta Q = 1 \).

We include photon splitting in our simulations. Based on the kinetic selection rule [Adler 1971; Usov 2002], but see Baring & Harding (2001), only the process \( \perp \rightarrow \parallel \parallel \) is allowed. Therefore, for \( \parallel \)-polarized photons, the attenuation coefficient for photon splitting is zero \((R^s_{\perp \parallel}) = 0\). For \( \perp \)-polarized photons we use the following formula, adapted from the numerical calculation of Baring & Harding (1997):

\[
R^s_{\perp \parallel} \approx \frac{a_0^2}{\sigma_{\text{in}} \sigma_{\text{out}}} \left( \frac{26}{15} \right)^2 \frac{2 \pi}{\beta Q} \left( \frac{2 \pi}{\beta Q} \right)^2 \exp \left( -\frac{2 \pi}{\beta Q} \right),
\]

(28)

\[
R'_{1,\perp} \approx \frac{0.23}{a_0} \beta Q \exp \left( -\frac{4}{3\beta Q} \right),
\]

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\[
R^s_{\perp \parallel} \approx \frac{a_0^2}{\sigma_{\text{in}} \sigma_{\text{out}}} \left( \frac{26}{15} \right)^2 \frac{2 \pi}{\beta Q} \left( \frac{2 \pi}{\beta Q} \right)^2 \exp \left( -\frac{2 \pi}{\beta Q} \right),
\]

(28)

\[
\begin{align*}
\text{Figure 5. Attenuation coefficients in the perpendicular frame (the frame where the photon is traveling perpendicular to the magnetic field), for both photon splitting, labeled by } \perp \rightarrow \parallel \parallel, \text{ and pair production, labeled by } \parallel \rightarrow e^+e^- \text{ and } \perp \rightarrow e^+e^-; \text{ The local magnetic field strength is } B = B_Q = 4.414 \times 10^{13} \text{ G.}
\end{align*}
\]
a refinement has a negligible effect on the cascade result. If $\tau_{\text{ph}} \geq 1$ the photon splits into two. As a simplification we assume that each photon takes half of the energy of the parent photon (cf. Baring & Harding [1997]); therefore, at the point of photon splitting a new photon is created with an energy $0.5\varepsilon_0$ and a weighting factor $2\Delta N_r$ (i.e. the simulation photon represents two actual photons). The new photon is $||$-polarized (for the $\perp$ || process) and is assumed to be traveling in the same direction as the parent photon, $\Theta_{\text{ph}}$. If $\tau \geq 1$ the photon creates an electron-positron pair. For $B \lesssim B_{\text{crit}} \sim 3 \times 10^{12}$ G, the pairs are created in high Landau levels (see above), and we assume that the electron and positron each shares half of the photon energy and travels in the same direction as the photon: thus $\gamma_m c^2 = \varepsilon/2$ and the electron/positron’s magnetic pitch angle is $\Psi = \psi$. This approximation is valid as long as $x\beta c < 0.1$ (see Daugherty & Harding [1983]), which according to ML07 is satisfied for $B \lesssim B_{\text{crit}}$. When $B \gtrsim B_{\text{crit}}$, the electron and positron are created in low Landau levels (we choose the maximum allowed values, $\gamma_{\text{max}}, \kappa_{\text{max}}$, since this channel dominates the total attenuation coefficient), with energies given by Eq. [35] of Appendix [35]

In the simulation we try to find the photon-magnetic field intersection angle at which pair creation occurs, $\psi_{\text{pair}}$, to an error of less than 10%. If the error in $\psi_{\text{pair}}$ is too large, the electron and positron will be created in the wrong Landau level and will emit too many or too few synchrotron photons (see Section 3.3). To accurately determine $\psi_{\text{pair}}$ we use the following procedure in our simulation: The photon’s first full step, $s_0$, should be small enough that the probability of pair production is negligible at $s_0$ but large enough that the probability grows rapidly with subsequent steps. At high fields a good choice for $s_0$ is the location of the first non-zero threshold ($x = x_{00}$ for $||$ polarization or $x = x_{01}$ for $\perp$ polarization), since the attenuation coefficients are large enough to allow pair production in a distance much shorter than 1 cm. At low fields a good choice is the point where $x\beta c = 1/20$. At this point the mean free path for pair production is much larger than the gap height while for $x\beta c \sim 1/10$, e.g., the mean free path is much smaller than the gap height. Therefore, $s_0$ is chosen such that it solves

$$x = \begin{cases} 
  x_{00}, & \beta_Q > 1/20 \text{ and } || \text{ polarization;} \\
  x_{01}, & \beta_Q > 1/20 \text{ and } \perp \text{ polarization;} \\
  1/(20\beta_Q), & \beta_Q < 1/20.
\end{cases}$$

(29)

Note that both $x$ and $\beta_Q$ depend on $s_0$, so the value of $s_0$ must be found numerically. Since $\sin \psi \approx \varepsilon_{\text{ph}}/R_e$ (for small angles), this distance is approximately given by

$$s_0 \approx R_e \frac{2m_e c^2}{\varepsilon} \left( 1 + \frac{1}{20\beta_Q} \right).$$

(30)

In our simulation, for $||$ polarizations (no photon splitting), the photon moves directly to $s_0$ in one step. For $\perp$ polarizations, the photon moves to $s_0$ in 10 steps (with step sizes $0.1s_0$), allowing for the possibility of photon splitting before reaching this point. In either case, once the photon reaches $s_0$ it steps outward in the manner described at the beginning of this section. At high fields ($B \gtrsim B_{\text{crit}}$) we choose the step size to be $\Delta s_{\text{ph}} = 0.1s_{0}(x_{01} - x_{00})/x_{00}$ for $||$ polarizations or $0.1s_{0}(x_{02} - x_{01})/x_{02}$ for $\perp$ polarizations. At low fields ($B \lesssim B_{\text{crit}}$) we choose $\Delta s_{\text{ph}} = 0.1s_{0}$.

Sometimes the photon does not pair produce (or split) before exiting the magnetosphere. Conveniently, we do not have to track the photons out to the light cylinder to know whether pair production will occur. Once a photon reaches the $x_{03}$ or $x_{11}$ threshold, such that the asymptotic expression for pair production Eq. [29] can be used (i.e., when $B \lesssim B_{\text{crit}}$), then the growth in optical depth depends “exponentially” on $x\beta c$ [since $\Delta \tau \propto \exp(-1/(x\beta c))]$. Because $x\beta c \propto s_{\text{ph}}(r_{0,\text{ph}} + s_{\text{ph}})^{-7/2}$ reaches a maximum at $s_{\text{ph}} \approx 0.4r_{0,\text{ph}}$ and then rapidly decreases (cf. Hillschmann & Arons [2001]), we assume in our simulation that if the photon does not pair produce by

$$s_{\text{ph,max}} = 0.5r_{0,\text{ph}},$$

(31)

it will never pair produce and instead escapes the magnetosphere. Here $r_{0,\text{ph}}$ is the altitude of the photon at the emission point. Note that Eq. (31) is also approximately valid for our treatment of non-dipole fields ($R_e = R$ near the stellar surface), since once the photon has traveled a distance $s_{\text{ph}} \approx s_{\text{ph,max}}$ it is in the dipole regime ($r > 2R$).

### 3.3 Propagation and photon emission of the secondary electrons and positrons

#### 3.3.1 Synchrotron radiation

In the corotating frame (the “lab” frame) the secondary electron (or positron) is created with energy $\gamma m_e c^2$, pitch angle $\Psi$ [with the corresponding Landau level $n$; see Eq. [32] below] and weighting factor $\Delta N_r$ (Section 3.3). For the purpose of tracking the synchrotron emission from the electron it is easier to work in the “circular” frame, the frame in which the electron has no momentum along the magnetic field direction and only moves transverse to the field in a circular motion. Note that this frame is in general different from the perpendicular frame (defined in Section 3.2) of the progenitor photon; only if the electron-positron pair is created exactly at threshold [$x = x_{\text{ph}}$; see Eq. [21]] are the two frames the same. The energy of the electron in the circular frame, $E_\perp = \gamma_\perp m_e c^2$, is related to that in the lab frame by

$$\gamma_\perp = \sqrt{\gamma^2 \sin^2 \Psi + \cos^2 \Psi} = \sqrt{1 + 2\beta_Q \sin^2 \Psi}.$$  

(32)

Note this expression also gives a relation between $\gamma_\perp$ and $n$; we shall use $\gamma_\perp$ and $n$ interchangeably to refer to the electron’s energy in the circular frame.

In the circular frame $E_\perp$ is radiated away through synchrotron emission on the timescale

$$t_{\text{synch}} \approx \frac{E_\perp}{P_{\text{synch}}} = \frac{\gamma_\perp m_e c^2}{\frac{1}{2}\gamma^{3/2} \sqrt{\gamma_\perp - 1} c^2 \omega_e^2} \approx 5 \times 10^{-16} B_{12}^{-2} \gamma_{\perp}^{-1} s,$$  

(33)

where $\omega_e = eB/m_e c$ is the electron cyclotron frequency and $B_{12}$ is the local magnetic field strength $B$ in units of $10^{12}$ G. This decay time is much shorter than other relevant cascade timescales (e.g., the timescale for $B$ or $R$, to change significantly, which is of order $r/c \gtrsim 10^{-4}$ s, or the timescale for the emission of resonant ICS photons, discussed later in this section). Therefore, in the simulation the electron is assumed to lose all of its perpendicular momentum $p_\perp$ “instantaneously” due to synchrotron radiation, before moving from its initial position (cf. Daugherty & Harding [1982]).
\[ \gamma_\parallel = \frac{1 - \beta^2 \cos^2 \Psi}{1 - \beta^2} = \gamma_\perp, \]  
(34)

where \( \beta = \sqrt{1 - 1/\gamma^2} \) is the electron velocity.

Since the synchrotron photon may carry an energy comparable to \( E_\perp \) of the parent electron, it is necessary to track the electron energy after each photon is emitted in order to obtain accurate synchrotron spectrum (this is in contrast to the case of curvature radiation discussed in Section 3.3.1 where a large number of curvature photons can be emitted without significantly affecting the energy of the parent electron). As a simplification, in the circular frame the synchrotron photons are assumed to be emitted isotropically in the plane of motion, such that no velocity kick is imparted to the electron; thus the frame corresponding to circular motion of the electron does not change over the course of the synchrotron emission process. In other words, as the electron loses its \( p_\perp \), the Lorentz factors \( \gamma_\parallel \) and \( \gamma_\perp \) decrease but \( \gamma_\parallel \) is constant, and Eq. (34) remains valid during the entire synchrotron emission process.

We adopt the following procedure in our simulation: In the circular frame, the electron Lorentz factor \( \gamma_\perp \) drops from its initial value to \( \gamma_\perp = 1 \) in a series of steps; when \( \gamma_\perp = 1 \) (i.e., \( n = 0 \)) synchrotron emission stops. In each step one synchrotron photon is emitted, with an energy \( \epsilon_\perp \) that depends strongly on the “current” value of \( \gamma_\perp \). After the photon is emitted the energy of the electron is reduced by the amount \( \Delta \gamma_\perp = \epsilon_\perp/m_e c^2 \). In the next step another photon is emitted with a new value of \( \epsilon_\perp \), and so on.

In the simulation the photon energy \( \epsilon_\perp \) of the synchrotron radiation is chosen in one of three ways, depending on the Landau level number \( n \) of the electron. (i) If the electron is created in a high Landau level (\( n \geq 3 \)), the energy of the photon is chosen randomly, but with a weighting based on the asymptotic synchrotron spectrum\(^6\) (e.g., Sokolov & Ternov 1964; Harding & Preces 1987)

\[
\frac{d^2N}{dt \, d\epsilon_\perp} = \frac{\sqrt{3} \alpha \omega_c}{2\pi \epsilon_\perp} \times \left[ f F \left( \frac{\epsilon_\perp}{\epsilon_{\text{SR}}} \right) + \left( \frac{\epsilon_\perp}{\gamma_\perp m_e c^2} \right)^2 G \left( \frac{\epsilon_\perp}{\epsilon_{\text{SR}}} \right) \right] \, , \tag{35}
\]

where

\[ \epsilon_{\text{SR}} = \frac{3}{2} \gamma_{\perp}^2 \hbar \omega_c \, , \tag{36} \]

is the characteristic energy of the synchrotron photons, \( f = 1 - \epsilon_\perp/(\gamma_\perp m_e c^2) \) is the fraction of the electron’s energy remaining after photon emission, \( F(x) = x \int_x^{\infty} K_{2/3}(t) \, dt, \) and \( G(x) = x K_{3/2}(x) \) [cf. Eq. (15)]. (ii) If \( n = 2 \), the energy of the photon is either that required to lower the electron to its ground state (\( n = 2 \rightarrow 0 \)) or the first excited state (\( n = 2 \rightarrow 1 \)), with a probability that depends on the local magnetic field strength. We do not use the exact transition rates for the \( n = 2 \) state here. Instead, we use the following simplified prescription, based on the results of [Herald, Ruder, & Wunder 1983] (see also [Harding & Preece 1984]): If \( \beta_\perp < 1 \) the energy of the photon is chosen to be that required to lower the electron to the first excited state, \( \epsilon_\perp = m_e c^2 \left( \sqrt{1 + 4\beta_\perp} - \sqrt{1 + 2\beta_\perp} \right) \). If \( \beta_\perp > 1 \) the energy of the photon is randomly chosen to be that required to lower the electron to either the ground state \( [\epsilon_\perp = m_e c^2 \left( \sqrt{1 + 4\beta_\perp} - 1 \right) \] 50% of the time, or the first excited state, 50% of the time. (iii) If \( n = 1 \), the energy of the photon is that required to lower the electron to its ground state, \( \epsilon_\perp = m_e c^2 \left( \sqrt{1 + 2\beta_\perp} - 1 \right) \). If the electron is not in the ground state after emission of the synchrotron photon (which could happen for the \( n = 2 \) and \( n \geq 3 \) cases discussed above, but not for the \( n = 1 \) case), \( \gamma_\perp \) is recalculated and a new photon energy is chosen.

The energy of the photon is transformed from the circular frame into the “lab” frame using

\[ \epsilon = \gamma_\parallel \epsilon_\perp \, . \quad (37) \]

The photon carries with it the same weighting factor \( \Delta N_\perp \) as the secondary particle that emitted it. Because the photon is emitted in a random direction perpendicular to the magnetic field in the circular frame, in the lab frame the angle of emission (relative to the dipole axis) is approximately given by

\[ \Theta_{\text{ph}} \simeq \chi + \Psi \cos \Pi \, , \tag{38} \]

where \( \Pi \) is a random angle between 0 and 2\( \pi \), \( \chi \) is the angle between the local magnetic field and the dipole axis and is given by Eq. (16), and the pitch angle is given by Eqs. (22) and (24):

\[ \Psi = \arcsin \left( \frac{\sqrt{\gamma_\parallel^2 - 1}}{\sqrt{\gamma_\parallel^2 - \gamma_\perp^2 - 1}} \right) \, . \tag{39} \]

For synchrotron radiation the polarization fraction is between 50% and 100% polarized perpendicular to the magnetic field (which is the exact opposite of the curvature radiation case; see [Rybicki & Lightman 1979]). Therefore we randomly assign the photon a polarization in the ratio of one \( || \) to every seven \( \perp \) photons (corresponding to a 75% perpendicular polarization).

### 3.3.2 Resonant inverse Compton scattering

Once the electron loses all of its perpendicular scattering, it moves along the magnetic field line in a stepwise fashion while upscattering surface thermal photons through RICS. The step size \( \Delta s \) is related to \( \Delta N_{\text{RICS}} \), the number of photons scattered in each step, by

\[ \Delta s \simeq \frac{\epsilon \Delta N_{\text{RICS}}}{dN_{\text{RICS}}/dt} \, . \tag{40} \]
In our simulation we choose $\Delta N_{\text{RICS}}$ to be
\[
\Delta N_{\text{RICS}} = \min \left( 1, 0.1R \frac{dN_{\text{RICS}}}{dt} \right). \tag{41}
\]
In other words, $\Delta N_{\text{RICS}} = 1$ if the RICS process is efficient enough to produce at least one resonant photon within a distance of $0.1R$; otherwise $\Delta N_{\text{RICS}}$ is chosen so that the electron step size is $\Delta s = 0.1R$. Using Eq. (40) from Appendix A we have
\[
\Delta s \simeq \frac{\Delta N_{\text{RICS}}}{\frac{\partial Q}{\gamma_j^2 E \omega} \ln \frac{1-e^{-e_\perp/\gamma_j(1-\beta_{\parallel})E\omega}}{1-e^{-e_\parallel/\gamma_j(1-\beta_{\parallel})E\omega}}}, \tag{42}
\]
where $\beta_{\parallel} = \sqrt{1 - 1/\gamma_j^2}$ is the speed of the electron after it has completed synchrotron emission (so that $p_\perp = 0$), and $\psi_{\text{crit}}$ is the incidence angle with respect to the electron's trajectory of photons coming from the edge of the surface "hot spot" (see Fig. A1). The mean energy of the scattered photons is (e.g., Beloborodov & Thompson [2007])
\[
\epsilon = \gamma_j \left( 1 - \frac{1}{\sqrt{1 + 2\beta_{\parallel}^2}} \right) m_e c^2, \tag{43}
\]
and the energy loss of the electron in each step is given by
\[
\Delta \gamma_j m_e c^2 = -\epsilon \Delta N_{\text{RICS}}. \tag{44}
\]
In the lab frame the photon’s angle of emission is approximately given by
\[
\Theta_{\text{ph}} \simeq \chi + \frac{1}{\gamma e} \cos \Pi, \tag{45}
\]
where $\Pi$ is a random angle between 0 and $2\pi$ and $\chi$ is the angle between the local magnetic field and the dipole axis (Eq. 16). Here, $\gamma e$ is the final Lorentz factor of the electron after emitting the photon; from Eq. (16), its value is approximately
\[
\gamma e \simeq \frac{\gamma_j}{\sqrt{1 + 2\beta_{\parallel}^2}}. \tag{46}
\]
While we include the $1/\gamma e \cos \Pi$ term in Eq. (45) for completeness, we find that it is not important for our simulation. This is true even at $B > B_Q$, where $\gamma e$ is much smaller than the initial Lorentz factor of the electron and pair production can occur almost immediately after the photon is scattered (Beloborodov & Thompson [2007]). The extra distance traveled by the photons in order to pair produce when the photons are upscattered tangent to the local magnetic field (i.e., when $\Theta_{\text{ph}} = \chi$ is assumed) has a negligible effect on the overall cascade.

In the superstrong field regime, the final polarization state of a photon upscattered through RICS is given by the results of Gonthier et al. [2006]. For $B \lesssim B_Q$, both below and above resonance more $\parallel$ photons are produced than $\perp$ photons, at a ratio of $\approx 3:1$. The same situation occurs for $B \gtrsim B_Q$ below resonance; above resonance, however, the situation reverses and more $\perp$ photons are produced than $\parallel$ photons. We therefore assign the photons a polarization in the ratio of one $\parallel$ to every three $\perp$ photons for $B < B_Q$, and a polarization in the ratio of one $\perp$ to every $\parallel$ photon for $B \geq B_Q$ (based on the assumption that approximately 50% of the photons are slightly below resonance and 50% are slightly above). In practice, however, we find that the cascade does not depend sensitively on the initial photon polarization. At low fields ($B \lesssim 3 \times 10^{12}$ G) the polarization has no effect on the cascade, since the asymptotic attenuation coefficient for pair production is used; at high fields a $\perp$ photon is split into two $\parallel$ photons before it can pair produce, and the resulting cascade is not much different from the cascade of a single $\parallel$ photon with twice the energy.

In our simulation we consider thermal photon emission from three types of surface "hot spots" (see Pons et al. [2007] for a review of neutron star surface temperatures in strong magnetic fields): a large cool spot, $T_0 = 0.3$ and $\theta_{\text{phot}} = \pi/2$, representing emission from the entire surface of a neutron star; a mid-sized warm spot, $T_0 = 1.0$ and $\theta_{\text{phot}} = 0.3$; and a small hot spot, $T_0 = 3.0$ and $\theta_{\text{phot}} = 0.1$, representing emission from a heated polar region 1 km across (which, though small, is still significantly larger than the polar cap region, unless $P \lesssim 0.01 s$). We also consider the case where ICS has no effect on the cascade, which we find occurs for $T_0 \lesssim 0.1$ (neutron stars too cold), $\theta_{\text{phot}} \lesssim 0.01$ (hot spots too small), or $r_0 \gtrsim R(1 + 2\theta_{\text{phot}})$ (particles injected too far away from the surface; this is most relevant for photon-initiated cascades discussed in Section 5.4 below).

### 3.4 Cascades initiated by a primary photon

In the second version of the simulation, a photon is created with energy $\epsilon_0$ at the position $(r_{0,\text{ph}}, \theta_{0,\text{ph}})$. We typically choose $r_{0,\text{ph}} = R$ and $\theta_{0,\text{ph}} = \theta_{\text{cap}}$ (cf. Section 5.1), since the resonant ICS photon density is largest at $r \approx R$; however, we are also interested in photons emitted at a higher altitude (e.g., for surface field strengths $B \gtrsim 10^{13}$ G the behavior of the cascade with $r_{0,\text{ph}} = R$ and with $r_{0,\text{ph}} = 3R$ are very different; see Section 3). We set $\Delta N = 1$, such that each photon in the simulation represents exactly one photon in reality; we can later multiply the simulation results by $N_0$ of Eq. (9) if we wish to compare cascades dominated by RICS and by curvature radiation (see Section 2.2). The photon is injected tangent to the magnetic field [$\Theta_{\text{ph}} = \chi$; cf. Eq. (17)], since we find almost no difference in the final photon or pair spectra if we add a beaming angle $1/\gamma \sim 10^{-7} - 10^{-5}$. As was discussed in Section 3.3.2 resonant ICS photons have a polarization ratio $\parallel$ to $\perp$ of approximately 1:3 for $B < B_Q$ and approximately 1:1 for $B \geq B_Q$. In Section 5.4 results for the photon-initiated cascades, we choose the initial photon to be polarized perpendicular to the magnetic field, to create a cascade with particle multiplicities as large as possible. Our results therefore represent upper bounds on the actual cascade multiplicities. The actual cascade should not differ greatly from that presented in Section 3.1 however, as cascades initiated by photons polarized parallel to the magnetic field are only slightly lower in particle multiplicity and are qualitatively similar in spectral shape. Once the initial parameters of the photon have been chosen, the simulation proceeds in the exact same way as described in Sections 3.3.2 and 5.3. The photon steps outward in a straight line from the point of emission until its optical depth is large enough to pair produce or split, etc.

Note that due to the discrete, random nature of the synchrotron emission and the small number of particles involved in the cascade, photon-initiated cascades will have photon and pair spectra that are coarse and that vary between simulation runs. In order to smooth/average the spec-
tra to some extent, we modify the synchrotron emission procedure of Section 5.3 for secondary particles in high Landau levels ($n \geq 3$). In every step 10 photons are emitted, each with a weighting factor of $0.1 \Delta N_e$ (rather than one photon with a weighting factor of $\Delta N_e$, as before). Each photon has a different energy $\epsilon_{\perp, i}$ [chosen randomly according to Eq. (55)], so that the total energy lost by the secondary particle becomes $\gamma_{\perp} m c^2 = 0.1 \sum_i \epsilon_{\perp, i}$. We do not apply this procedure to the synchrotron emission from secondary particles in Landau levels $n = 1$ or $n = 2$, as it would not gain anything; each of the 10 photons emitted would have the same value of $\epsilon_{\perp, i}$.

### 4 RESULTS

In this section we present the results of our simulations of photon- and electron-initiated cascades (Sections 4.1 and 4.2 respectively), for a variety of different surface field strengths, rotation periods, field geometries, and initial energies of the primary particle. For each type of cascade, we present the “final” spectra of the cascade photons and pairs as they cross the light cylinder and escape from the magnetosphere. For the electron-initiated cascades we also show the spectra at several intermediate stages (i.e., the spectra of all photons and pairs that cross the height $r = 1.2R$, $2R$, $5R$, etc.). The photon spectra are plotted over the energy range 10 keV-1 TeV, since for energies $\lesssim 1$ keV the thermal photons dominate the spectra while above $\sim 1$ TeV fewer than one photon is produced per primary electron. We are particularly interested in the pair multiplicities, i.e., the total number of cascade electrons $+\text{ positrons}$ produced per primary particle. We use $n_E$ to denote the number of electrons and positrons per “primary” photon and $N_E$ to denote the number per primary electron; the two multiplicities are related by

$$N_E = N_0 \times n_E,$$

where $N_0$ is the number of photons produced by the primary electron (see Section 2.2). From our numerical results we infer various empirical relations for each cascade; quantitative arguments for the validity of several of these relations are given in Appendix C.

We first present our results for photon-initiated cascades (see Section 3.1), as they are simpler and aid us in our discussion of the results for the full cascade (initiated by a primary electron).

#### 4.1 Results: photon-initiated cascades

Our results for photon-initiated cascades are presented in Figs. 6-10. We consider primary photons with energies in the range of $10^3-10^8$ MeV; for $B_{p,12} = 1-1000$, the primary electron should emit very few photons (via either resonant ICS or curvature radiation) above this energy range (see Section 2). Unless otherwise stated, the primary photon is emitted from near the surface, in the direction tangent to the last open field line. Thus the radius of curvature near the point of emission is $R_c \simeq 9 \times 10^7 P_0^{1/2}$ cm for dipole fields [Eq. (14)].

We find significant differences in the behavior of the cascades at magnetic field strengths below and above $B_{\text{crit}} \simeq 3 \times 10^{12}$ G [Eq. (B17)]. At low fields $B < B_{\text{crit}}$, the primary photon can pair produce if $|E| > \epsilon_{\text{min}}$, see also Hibschman & Arons (2001a)

$$\epsilon_0 > \epsilon_{\text{min}} \sim 3000 B_{p,12}^{-1} R_8 \text{ MeV},$$

where $R_8$ is the radius of curvature $R_c$ in units of $10^8$ cm, evaluated at the surface along the last open field line. Strong cascades, where more than one electron-positron pair is produced, typically occur at energies $\sim 10^3 \epsilon_{\text{min}}$. For $\epsilon_0$ in the range from $\epsilon_{\text{min}}$ to $\sim 10^5$ MeV, we find that the multiplicities of photons and $e^+e^-$ particles produced in the cascade are

$$n_\gamma \sim \frac{\epsilon_0}{500 \text{ MeV}} R_8^{-1}$$

and

$$n_E \sim \frac{\epsilon_0}{10^4 \text{ MeV}} B_{p,12}^{-1} R_8^{-1},$$

respectively. These results are (largely) independent of the hot spot model used. When ICS is inactive, the cascade electron/positron has final energy (after it has finished radiating synchrotron photons) extending from [Eq. (C3)]

$$E_{\text{max}} \sim 0.1 B_{p,12} \epsilon_0$$

(for the first pair produced) down to $\sim 0.1 B_{p,12} \epsilon_{\text{min}}$ for the lowest-energy pairs, and the total energy of the pairs is [Eq. (C4)]

$$\mathcal{E}_{\text{tot}} \sim 2E_{\text{max}} + 0.1 B_{p,12} \epsilon_{\text{min}} n_E \ln \left( \frac{0.075 \epsilon_0}{\epsilon_{\text{min}}} \right).$$

When ICS is active from a hot spot (Section 3.3), the number of pairs produced does not change, since the photons produced through ICS at these field strengths have energies $\sim B_{p,12} T_{p,0}^{-1} \text{ MeV}$ [Eq. (5)], and can not pair produce. The total pair energy $\mathcal{E}_{\text{tot}}$ decreases, however, since the ICS process transfers energy from the pairs to photons. Although resonant ICS is most important for electrons and positrons at $\gamma_{\text{crit}} \simeq \epsilon_{\text{min}}$, see Section 2.2, we find in these cascades that all electrons and positrons with energies in the range of

$$E_{\text{RICS}} \sim (0.3 - 30) \gamma_{\text{crit}} m_e c^2$$

are strongly affected. Thus hot surface spots with higher $T$ tend to lower $\mathcal{E}_{\text{tot}}$. As expected, we find that photon splitting does not affect the cascade at these field strengths (see Section 3.2). The photon and pair cascade spectra for $B_{p,12} = 1$ are shown in Fig. 6 both when ICS is inactive and when ICS is active from a “warm spot” ($T_8 = 1, \theta_{\text{spot}} = 0.3$). At high fields ($B \gtrsim B_{\text{crit}}$), a primary photon injected from the surface will pair produce when

$$\epsilon_0 > \epsilon_{\text{min}} \sim 200 R_8 \text{ MeV},$$

largely independent of field strength. When ICS is inactive, almost all of the cascade energy resides in the pairs; i.e., $\mathcal{E}_{\text{tot}} \simeq \epsilon_0$. The pair cascade will be very weak regardless of photon energy, with $n_E < 10$ and $n_\gamma = 0$ or 1 (i.e., at most one photon escapes the magnetosphere without pair production). This is because the $e^\pm$ pairs are produced exclusively through the $(jk) = (00)$ or $(01)$ channel (see Section 3.2), so that at most one synchrotron photon is emitted per pair. For $B_{p,12} \gtrsim 20$, photon splitting causes all pairs to be produced
Figure 6. The final photon and pair spectra of photon-initiated cascades for surface magnetic fields $B_{p,12} = 1$. The NS spin period is $P_0 = 1$ and a dipole field geometry is adopted. In the upper panels, ICS is assumed to be inactive, while in the lower panels, ICS from a hot spot with $T_6 = 1, \theta_{\mathrm{spot}} = 0.3$ is included in the simulation. The primary photon is injected from the surface and has an energy of $10^4$ MeV (left panels) or $10^5$ MeV (right panels); for photons with energy $10^3$ MeV, no cascade is initiated. The spike in the pair spectra of each panel represents the electron-positron pair produced by the primary photon. The spectra in the top panels (where ICS is inactive) are nearly identical to the spectra generated, e.g., by a photon injected at $r_{\mathrm{ph}} = 3R$ above a star with surface field $B_{p,12} = 3^3 = 27$, such that the local field strength at the injection point is $B = 10^{12}$ G [Eq. (11)].

with $(jk) = (00)$, such that the cascades are even weaker: $n_E \leq 4$ and $n_\epsilon = 0$. When ICS is active, both $n_\epsilon$ and $n_E$ can be larger, though not as large as would be predicted by an extrapolation of Eqs. (49) and (50) to high fields. In order for ICS to affect the cascade, however, the primary photon must have an energy

$$
\epsilon_0 \gtrsim 70B_{p,12}T_6^{-1} \text{ MeV};
$$

the energy of the electron/positron produced by this photon, $E_{\text{max}} \sim (0.1–0.5)\epsilon_0$ [Eq. (30)], must be larger than the minimum energy at which ICS is effective, $\sim 0.3\gamma_{\text{crit}}m_e c^2$ (see Appendix C1). Note that this energy is approximately equal to $\epsilon_{\text{RICS}}$, the energy of a typical ICS photon upscattered by the primary electron at high fields [Eq. (3)]; therefore, at high fields a typical ICS photon is able to initiate a weak cascade. We find that for $B_p \lesssim 0.5B_Q$ a significant fraction ($\sim 20\%–60\%$) of the total cascade energy $\epsilon_0$ resides in the photons; as in the low field case [Eq. (52)], this fraction decreases as either $\epsilon_0$ or $B_{p,12}$ increases. For $B_p \gtrsim 0.5B_Q$, even ignoring photon splitting, the total photon energy fraction is very low, $<1\%$; but with photon splitting included, it is almost negligible, $<1\%$. The photon and pair cascade spectra for $B_{p,12} = 10$ and 100 when ICS is active from a “warm spot” ($T_6 = 1, \theta_{\mathrm{spot}} = 0.3$) are shown in Fig. 7.
Pair cascades of strongly-magnetized neutron stars

Figure 7. The final photon and pair spectra of photon-initiated cascades with active ICS, for surface magnetic fields $B_{p,12} = 10$ (upper panels) and 100 (lower panels). The pulsar spin period is $P_0 = 1$, a dipole field geometry is adopted, photon splitting $\perp \rightarrow \parallel$ is active, and thermal photons are emitted from a hot surface spot with $T_6 = 1$, $\theta_{\text{spot}} = 0.3$. The primary photon has an energy of $10^3$ MeV (left column of panels), $10^4$ MeV (middle column), or $10^5$ MeV (right column). The spikes in the pair spectra of several panels represent the electron-positron pair produced by the primary photon. Note that there are actually two such pairs in the bottom left panel ($B_{p,12} = 100, \epsilon_0 = 10^3$ MeV), as the primary photon has split in that case; while in the upper right panel ($B_{p,12} = 10, \epsilon_0 = 10^5$ MeV) the electron and positron have significantly different energies from each other and so are represented by two shorter spikes. Also note that photon spectra do not appear in the lower panels. This is due to a combination of weak synchrotron emission and efficient pair production near the surface of a $B_{p,12} \geq 100$ neutron star; very few secondary photons are created, and none of them survive to escape the magnetosphere.

As discussed in Section 3.3, we consider three hot surface spot models for active ICS: a “cool” $T_6 = 0.3, \theta_{\text{spot}} = \pi/2$ spot; a “warm” $T_6 = 1, \theta_{\text{spot}} = 0.3$ spot; and “hot” $T_6 = 3, \theta_{\text{spot}} = 0.1$ spot. At low fields we find that the only effect the various hot spot models have is to lower $E_{\text{tot}}$ relative to the total cascade energy $\epsilon_0$ (see above). At high fields, the cascades due to warm and hot spots are similar in multiplicities $n_\epsilon$ and $n_E$ for energies $\epsilon_0 \lesssim 10^4$ MeV; but for energies $\epsilon_0 \gtrsim 10^5$ MeV, the hot spots give multiplicities $\sim 3$ times larger than warm spots; see Fig. 8 [from Eq. (55), at $B_{p,12} = 1000$ a strong cascade requires $T_6 \gtrsim 3$]. Cool spots give much smaller multiplicities (factors of $>10$ smaller) than warm or hot spots regardless of the primary photon energy.

At low fields, the multiplicities of photons and $e^+e^-$ particles produced per primary photon, $n_\epsilon$ and $n_E$, depend on $R_e$ to the $(-1)$ power [Eqs. (49) and (50)], such that cascades in non-dipole magnetospheres can be much larger
than in dipole magnetospheres. At high fields, we find that this dependence on curvature radius is much weaker: \( n_E \propto R_c^{-1/4} \) at \( B_p \simeq 0.5 B_Q \) and \( n_E \propto R_c^{-1/4} \) power at larger fields. Thus for high fields, the cascades in non-dipole and dipole magnetospheres are of similar sizes. The photon and pair cascade spectra for \( B_{p,12} = 1, 10, \) and 100 and a \( R_c = R \) non-dipole magnetosphere are shown in Fig. 9.

Two aspects of the cascade depend strongly on the altitude \( (r_{0,ph}) \) at which the primary photon is injected: the local magnetic field strength, \( B \propto r^{-3} \), and the effectiveness of ICS, which is completely negligible for \( r_{0,ph} \geq 3R \) (regardless of the temperature and size of the hot spot). The other cascade parameters have a much weaker dependence on altitude (e.g., radius of curvature \( R_c \propto \sqrt{r_{0,ph}} \)). We find that the photon and pair spectra for primary photons injected at \( r_{0,ph} > R \) are very similar to the spectra for photons injected at the surface, as long as ICS is inactive and the local magnetic field strengths are the same in both cases. For example, the spectra for \( r_{0,ph} = 3R \) and \( B_{p,12} = 27 \) [such that the local field strength at the point of injection is \( B_{12} = 1 \); Eq. (11)] is nearly identical to the spectra given in the first row of Fig. 9, where \( r_{0,ph} = R, B_{p,12} = 1 \), and ICS is inactive.

We can use the pair multiplicity per primary photon, \( n_E (\epsilon_0) \), obtained from our numerical simulations to estimate the pair multiplicity per primary electron, \( N_E = N_0 n_E \), when ICS is the dominant cascade emission process. For \( \epsilon_0 \) and \( N_0 \), we use the expressions for the typical energy \( \epsilon_{RICS} \) and total number \( \simeq 10 B_{p,12}^{-1} P_0^{3/4} T_6^{5/2} \) of resonant ICS photons upscattered by the primary electron [Eqs. (8) and (9)]; see Section 2.2. The results are shown in Table 1. Note that although \( \epsilon_{RICS} \) is independent of the acceleration model used, the number of upscattered photons, \( N_0 \), as given by Eq. (10) is applicable for only inner gap accelerators with space-charge-limited flow; inner vacuum gap accelerators, for example, would yield \( N_0 \) about 20–100 times lower. Also note that, for a given acceleration model, an accurate determination of \( N_E \) requires that rather than just setting \( \epsilon_0 = \epsilon_{RICS} \), a distribution of energies be used which takes into account resonant (and possibly non-resonant) scattering away from the thermal peak (i.e., \( \gamma \neq \gamma_{crit} \)); although fewer photons are upscattered at energies greater than \( \epsilon_{RICS} \), these photons can have an important effect on the total multiplicity of pairs produced, since \( n_E \) grows approximately linearly with photon energy \( \epsilon_0 \); the \( \epsilon_0 > \epsilon_{RICS} \) photons are especially important for cascades where \( n_E (\epsilon_{RICS}) = 0 \) (e.g., cascades with \( B_{p,12} = 10 \) and \( B_0 = 1 \)).

4.2 Results: electron-initiated cascades

Our results for electron-initiated cascades are presented in Table 2 and Figs. 10–15. In our simulation, the primary electron is emitted from the surface along the last open field line \( (\theta_0 = \theta_{cap}) \). We consider the cases of \( \gamma_0 = 2 \times 10^7 \) and \( 4 \times 10^7 \) in dipole magnetospheres, as well as the case of \( \gamma_0 = 2 \times 10^6 \) in non-dipole \( R_c = R \) magnetospheres, as discussed in Section 2.1. Although a larger initial primary energy \( \gamma_0 \) gives rise to more cascade particles with a larger total energy, the behavior of the cascade at \( \gamma_0 > 4 \times 10^7 \) is qualitatively similar to that at \( \gamma_0 = (2–4) \times 10^7 \) (or \( \gamma_0 = 2 \times 10^6 \) and \( R_c = R \)) and key quantities such as cascade multiplicities and energies can be extrapolated from our results. For most of this section we “turn off” ICS in our simulation and allow photon splitting only through the \( \perp \rightarrow \parallel \parallel \) mode; at the end of this section we discuss how changing these simulation parameters affects the cascade.

In Table 2 we list some key quantitative results of our simulations: For each cascade (characterized by the spin pe-
For dipole fields or the curvature radius for multipole fields, the surface field strength $B_p$ and the primary electron energy $\gamma_0 m_e c^2$, we give $\gamma f m_e c^2$ (the final energy of the primary electron when it escapes the light cylinder), $\varepsilon_{\text{tot}}$ (the total energy of the cascade photons), $E_{\text{tot}}$ (the total energy of the secondary $e^+e^-$ pairs), and $N_E$ (the multiplicity of $e^+e^-$ pairs produced per primary electron). Note that the total cascade energy must satisfy $\gamma_0 m_e c^2 = \gamma f m_e c^2 + E_{\text{tot}} + \varepsilon_{\text{tot}}$. We find that $N_E$ is largest for cascades with strong surface fields, short rotation periods, multipole geometries, or large initial energies for the primary electron, but that regardless of cascade parameters the particle multiplicity saturates at $N_E \sim 10^4$. Increasing $B_p$ or $\gamma_0$, or decreasing $P$ or $R_c$, tends to increase the ratio $f_E = E_{\text{tot}}/(\varepsilon_{\text{tot}} + E_{\text{tot}})$; i.e., under these conditions a larger fraction of the “secondary” energy, $\gamma_0 m_e c^2 - \gamma f m_e c^2$, is transferred from the photons to the pairs. At low fields only a small fraction of the secondary energy is held by the secondary pairs (e.g., for $B_{p,12} = 1$, $f_E \lesssim 0.05$), but at high fields this fraction is typically above 50% (e.g., for $B_{p,12} = 1000$, $f_E \gtrsim 0.8$). The average energy of a secondary electron/positron, $\bar{E} = E_{\text{tot}}/N_E$, is directly proportional to the total cascade energy (i.e., $\bar{E} \propto \gamma_0$), but depends only weakly on $B_p$, $P$, and $R_c$ (e.g., $\bar{E}$ is approximately the same for $R_c = R$ and $\gamma_0 = 2 \times 10^7$ as in the dipole case for $\gamma_0 = 2 \times 10^7$).

Table 1. Pair multiplicity $N_E$ when resonant ICS is the dominant photon emission process of the primary electron. For simplicity, all photons are assumed to be upscattered from the thermal peak ($\gamma_{\text{crit}}$; see text). Each entry in the table gives $N_E = N_{00} n_E$ for a different surface magnetic field strength $B_p$, radius of curvature (either $R_c = R$ or a dipole field curvature with the pulsar spin period $P_0$ is specified), and hot spot temperature $T$ and size $\theta_{\text{spot}}$. The pair multiplicity is zero for $B_{p,12} = 1$ and the hot spot models used here, even when $R_c = R$; we therefore omitted these entries from the table.

| $T_0 = 1$, $\theta_{\text{spot}} = 0.3$ | $T_0 = 3$, $\theta_{\text{spot}} = 0.1$ |
|---|---|
| $B_{p,12} = 10$ | $B_{p,12} = 10$ |
| $P_0 = 10$ | $P_0 = 1$ |
| $P_0 = 1$ | $P_0 = 1$ |
| $R_c = R$ | $R_c = R$ |
| $220$ | $31$ |
| $0$ | $0$ |
| $0$ | $0$ |
| $14$ | $35$ |
| $0.7$ | $1.2$ |
| $0.1$ | $1.2$ |
| $0.1$ | $35$ |
| $11$ | $6.2$ |
| $1.9$ | $4.4$ |

Figure 9. The final photon and pair spectra of photon-initiated cascades for a non-dipole magnetosphere with local radius of curvature $R_c = R_0$ for surface magnetic fields $B_{p,12} = 1$ (left panel), 10 (middle panel), and 100 (right panel). Here, the primary photon has $\epsilon_0 = 10^4$ MeV and ICS is active from a hot spot with $T_0 = 1$, $\theta_{\text{spot}} = 0.3$.
The value of $\gamma_{\text{death}}$ changes very little if we alter the critical value of $N_E$ in Eq. (56) by a factor of $\sim 10$, because of the steep dependence of $N_E$ on $\gamma_0$ in this region. Therefore, although it is unknown exactly what value $N_E$ must have for a pulsar to be active, $\gamma_0 = \gamma_{\text{death}}$ is a good predictor of pulsar death regardless. Using Fig. 10 we find empirically that

$$N_E (\gamma_0 = \gamma_{\text{death}}) = 1.$$  

(56)

The value of $\gamma_{\text{death}}$ varies with the magnetic field strength, field geometry (curvature radius $R_c$ or spin period in the case of dipole fields), and cascade energy ($\gamma_0 m_e c^2$).

### Table 2. Energies and multiplicities for a cascade initiated by a single electron.

| $P$ (1 s) | $B_p$ (10$^{12}$ G) | $\gamma_0 m_e c^2$ (MeV) | $\gamma_f m_e c^2$ (MeV) | $\varepsilon_{\text{tot}}$ (MeV) | $N_E$ | $E$ (MeV) |
|-----------|---------------------|--------------------------|--------------------------|-----------------------------|-------|-----------|
| 10        | 10                  | 1.622e7                  | 8.1e6                    | 2.1e6                       | 1.3e1 | 5.7e2     |
|           |                     |                          |                          | 2.0e6                       | 1.7e2 | 4.8e2     |
| 1000      |                     |                          |                          | 1.9e6                       | 5.1e2 | 3.5e2     |
| 1         | 1                   | 4.8e6                    | 5.4e6                    | 1.9e4                       | 5.3e1 | 3.6e2     |
|           |                     |                          |                          | 4.0e6                       | 3.0e3 | 4.7e2     |
| 1000      |                     |                          |                          | 3.4e6                       | 5.9e3 | 3.4e2     |
| 0.1       | 1                   | 2.4e6                    | 7.4e6                    | 4.0e5                       | 1.5e3 | 2.7e2     |
|           |                     |                          |                          | 4.4e6                       | 6.0e3 | 5.7e2     |
| 1000      |                     |                          |                          | 2.8e6                       | 5.0e6 | 2.7e2     |
|           |                     |                          |                          | 2.1e6                       | 5.7e6 | 3.4e2     |
| 1         | 1                   | 2.044e7                  | 4.9e6                    | 1.5e7                       | 6.3e5 | 8.2e2     |
|           |                     |                          |                          | 9.5e6                       | 3.8e3 | 1.6e3     |
| 1000      |                     |                          |                          | 6.3e6                       | 9.2e6 | 1.3e3     |
|           |                     |                          |                          | 4.8e6                       | 1.1e4 | 1.6e3     |
| $R_c = R$ | 1                   | 1.022e6                  | 5.8e5                    | 3.9e5                       | 5.2e4 | 6.5e3     |
|           |                     |                          |                          | 8.0e4                       | 3.7e5 | 2.0e4     |
| 1000      |                     |                          |                          | 1.3e4                       | 1.9e4 | 2.3e1     |

For dipole fields, $R_8 = 0.9 P_0^{1/2}$, and we can write

$$\gamma_{\text{death}} \simeq 1.5 \times 10^7 B_p^{1/6} R_8^{2/3}.$$  

(57)

where $\Phi_{\text{cap}}$ is given by Eq. (44) and

$$\Phi_{\text{death}} = 7 	imes 10^{12} \text{ V}.$$  

(58)

Therefore, for dipole fields, the death line is given approxi-
will pair produce through the \((jk) = (00)\) channel and \(\perp\) photons will split into two \(|j\rangle\) photons before pair producing (see Section 3.2). For \(B < 0.5B_0\), pair production dominates over photon splitting, such that a few synchrotron photons are emitted from electrons and positrons in the \(n = 1\) Landau level. For \(B < B_{\text{crit}}\), electrons and positrons are created in higher Landau levels, such that many synchrotron photons are produced by each electron or positron (\(\sim 10\); see Appendix C1). The minimum photon energy for pair production \(\epsilon_{\text{min}} \propto B^{-1}\) [Eq. (8)] grows with radius while the energy of the typical curvature photon \(\epsilon_{\text{CR}} \propto \gamma^2\) [Eq. (10)] and synchrotron photon \(\epsilon_{\text{SR}} \propto \epsilon_{\text{CR}}\) (Appendix C1) fall with radius; therefore the number of electron-positron pairs created per photon decreases rapidly with radius.

Figures 12 and 13 show the final spectra (i.e., the spectra as measured at the light cylinder) of photons and pairs, as well as the spectra of curvature photons emitted by the primary electron, for a variety of magnetic field strengths, spin periods, and cascade energies. Figure 12 shows cumulative photon and pair spectra at various magnetosphere radii. These spectra are generated by recording the energy of any photon, electron, or positron which reaches various spin periods, and cascade energies. Figure 14 shows cumulative photon and final photon spectra if the average electron/positron energy \(\bar{\epsilon}\) lies in this range (which occurs, e.g., for \(B_{p,12} \lesssim 44.14\) when \(P_0 \geq 0.1\) and \(\gamma_0 \leq 4\); see Table 2). In general, we find that including RICS tends to make the cascade pair energy distribution narrower. On the other hand, ICS has only a minor effect on the photon spectra; its effect is moderate when \(\dot{E} \sim 20–2000B_{p,12}T^{-1}_6\) MeV and \(\epsilon_{\text{dot}} < \epsilon_{\text{tot}}\) (which only occurs when \(R_c = R\) and \(B_{p,12} = 10\); see Table 2). Regardless of the cascade parameters, resonant ICS by the secondary particles has very little effect on the multiplicity of photons \(N_\gamma\) or the number of synchrotron photons to the number of curvature photons at low energies is largest for large \(\gamma_0\), small \(P\), or small \(R_c\). The ratio of synchrotron photons to curvature photons is also largest for \(B_p\) close to \(B_{\text{crit}}\), such that the point of maximum synchrotron radiation, \(B \approx B_{\text{crit}}\) (see Fig. 11), occurs at a low altitude. Note that for \(\epsilon \lesssim 0.1\) MeV the cumulative and final photon spectra all have power-law shapes with index \(\Gamma = 2/3\) (where \(dN/\epsilon d\epsilon = \epsilon^{-2/3}\)), regardless of whether synchrotron radiation from the secondary pairs or curvature radiation from the primary electron dominates; this is because both processes have spectra that depend on \(F(x)/\epsilon \propto \epsilon^{-2/3}\) at low energies \([i.e., \epsilon \ll 1];\) see Eqs. (15) and (35).

Figure 13 shows the effect of resonant inverse Compton scattering on the final photon and pair spectra. Both a "warm" hot spot (\(T_6 = 1, \theta_{\text{spot}} = 0.3\)), and a "cool" spot (\(T_6 = 0.3, \theta_{\text{spot}} = \pi/2\)) are considered, as well as the case where ICS is inactive. For the cascade parameters adopted in our simulations, the results for "hot" hot spots (\(T_6 = 3, \theta_{\text{spot}} = 0.1\)) are nearly the same as for warm spots, and so are not shown. This is because the increased cascade activity due to the larger \(T\) almost exactly cancels the decreased activity due to the smaller maximum photon-electron intersection angle \(\psi_{\text{crit}}\). We find due to resonant ICS, a majority of the electrons and positrons with energies in the range \(E_{\text{RICS}} \sim 20–2000B_{p,12}T^{-1}_6\) MeV [Eq. (52)] radiatively cool to below that range. This has a strong effect on the pair spectra if the average electron/positron energy \(\dot{E}\) lies in this range (which occurs, e.g., for \(B_{p,12} \lesssim 44.14\) when \(P_0 \geq 0.1\) and \(\gamma_0 \leq 4\); see Table 2). In general, we find that including RICS tends to make the cascade pair energy distribution narrower.

**Figure 10.** Secondary electron/positron multiplicities \(N_\gamma\) as a function of the initial Lorentz factor of the primary electron \(\gamma_0\), for \(P_0 = 1\) and a dipole geometry at several different surface field strengths (left panel), and for \(B_{p,12} = 100\) at several different periods/field geometries (right panel).
Figure 11. The number of photons and electrons + positrons as a function of the radius where they are created, $r_0$, for $B_{p,12} = 1$ (left panel), 44.14 (middle panel), and 1000 (right panel). The neutron star spin period is assumed to be $P_0 = 1$ and a dipole field geometry is used. The $\perp \rightarrow \parallel$ photon splitting mode is active, and the primary electron has $\gamma_0 = 2 \times 10^7$; the effects of ICS are not included. The curve labeled “$e^+ e^-$ pairs” shows where the secondary electrons and positrons are created, “Photons” shows where the photons that escape the magnetosphere (i.e., that do not split or pair produce) are created, “Curvature” shows where the curvature photons are emitted by the primary electron (which continues in a similar manner beyond the graph out to the light cylinder), and “Synchrotron” shows where the synchrotron photons are emitted by the secondary pairs.

Figure 12. The final photon and pair spectra of electron-initiated cascades. The primary electron has Lorentz factor $\gamma_0 = 2 \times 10^7$ (the left two panels) and $4 \times 10^7$ (right two panels), and the surface magnetic field strengths are $B_{p,12} = 1$, 44.14, and 1000. The neutron star spin period is $P_0 = 1$ and a dipole field geometry is adopted. The secondary photon spectra are shown in the top two panels; the pair spectra are shown in the bottom two panels. The curve labeled “Curvature photons” in each of the top two panels (the dot-dashed line) shows the spectrum of curvature radiation emitted by the primary electron, which is the same for all field strengths.
The final photon and pair spectra of electron-initiated cascades for dipole magnetospheres with \( \mathcal{R}_c = R \) (right panels). The surface field strengths are \( B_{p,12} = 1, 44.14, \) and 1000. The Lorentz factor of the primary electron is \( \gamma_0 = 2 \times 10^7 \) for the dipole magnetospheres (left and middle columns) and \( \gamma_0 = 2 \times 10^6 \) for the non-dipole magnetosphere (right column; see Section 4). The photon spectra are shown in the top three panels; the pair spectra are shown in the bottom three panels. The curve labeled “Curvature photons” in each of the top three panels (the dot-dashed line) shows the non-dipole magnetosphere (right column). The photon spectra when photon splitting is “turned off” completely, the spectra are nearly indistinguishable from the case where photon splitting is allowed only in the \( \perp \to || \) mode. In general, the effect of photon splitting when both polarizations are allowed to split is strongest for large \( B_p \) and at high energies (e.g., large \( \gamma_0 \)), such that the quantity \( x\beta_Q \) is large [see Eq. (23)]. At altitudes where the local field strength is \( B \gtrsim 0.5B_Q \) (which occurs, e.g., at \( r \lesssim 4R \) for \( B_{p,12} = 1000 \)), most photons continue to split until they reach \( \epsilon_{\text{min}} \) [Eq. (18) or (51)], and very few pairs are created. However, once the cascade reaches an altitude such that \( B < 0.5B_Q \), photon splitting has very little effect on the cascade. Because a majority of pairs are produced at altitudes above \( B \gtrsim 0.5B_Q \) (see Figs. 11 and 13), the pair spectrum is not strongly affected by photon splitting even when both polarizations are allowed to split and \( B_{p,12} = 1000 \).

Figure 13 shows the effect of photon splitting on the final photon and pair spectra. We consider both the case where photon splitting is allowed only in the \( \perp \to || \) mode (as discussed in Section 4.2) and in agreement with the selection rule of Adler [1971], Usov [2002], and the case where both \( \perp \) and \( || \) photons are allowed to split (as is suggested in Baring & Harding [2001]). For the cascade parameters adopted in Fig. 16 we find that when photon splitting is “turned off” completely, the spectra are nearly indistinguishable from the case where photon splitting is allowed only in the \( \perp \to || \) mode. In general, the effect of photon splitting when both polarizations are allowed to split is strongest for large \( B_p \) and at high energies (e.g., large \( \gamma_0 \)), such that the quantity \( x\beta_Q \) is large [see Eq. (23)]. At altitudes where the local field strength is \( B \gtrsim 0.5B_Q \) (which occurs, e.g., at \( r \lesssim 4R \) for \( B_{p,12} = 1000 \)), most photons continue to split until they reach \( \epsilon_{\text{min}} \) [Eq. (18) or (51)], and very few pairs are created. However, once the cascade reaches an altitude such that \( B < 0.5B_Q \), photon splitting has very little effect on the cascade. Because a majority of pairs are produced at altitudes above \( B \gtrsim 0.5B_Q \) (see Figs. 11 and 13), the pair spectrum is not strongly affected by photon splitting even when both polarizations are allowed to split and \( B_{p,12} = 1000 \).

5 DISCUSSIONS

We have presented numerical simulations of pair cascades in the open-field line regions of neutron star magnetospheres, for surface magnetic fields ranging from \( B_p = 10^{12} \) G to \( 10^{15} \) G, rotation periods \( P = 0.1–10 \) s, and dipole and more complex field geometries. Compared to previous studies (e.g., Daugherty & Harding [1982], Sturner et al. [1995], Daugherty & Harding [1996], Hilschman & Arons [2001b], Arendt & Eilek [2002]), which were restricted to weaker magnetic fields (\( B \lesssim 10^{12} \) G) and dipole geometry, we have incorporated in our simulations additional physical processes that are potentially important (especially in the high field regime) but were either neglected or crudely treated before, including photon splitting with the correct selection...
Figure 14. The cumulative photon and pair spectra at magnetosphere radii $r = 1.05R, 1.2R, 2R, 5R, 20R$, and $r_{L C}$ (denoted “Light cylinder”), for $B_{p,12} = 1$ (left two panels), 44.14 (middle panels), and 1000 (right panels). The NS spin period is $P_0 = 1$ and a dipole field geometry is used, and $\gamma_0 = 2 \times 10^7$. The photon spectra are shown in the top three panels; the pair spectra are shown in the bottom three panels. For a given magnetosphere radius $r$, every cascade particle that is created below $r$ and survives until reaching $r$ is recorded in the spectrum.

Figure 15. The effect of resonant inverse Compton scattering on the final photon and pair spectra. Two surface hot spot models are considered: a “warm” spot ($T_6 = 1, \theta_{\text{spot}} = 0.3$) and “cool” spot ($T_6 = 0.3, \theta_{\text{spot}} = \pi/2$). The case where ICS is inactive is also shown. The other cascade parameters are: $B_{p,12} = 44.14, P_0 = 1$ (a dipole field geometry is used), and $\gamma_0 = 4 \times 10^7$. The photon spectra are shown in the left panel and the pair spectra are shown in the right panel.
Pair cascades of strongly-magnetized neutron stars

Figure 16. The effect of photon splitting on the final photon and pair spectra. The solid lines show the case where only photons with perpendicular polarizations are allowed to split (\(\perp\rightarrow||\)) and the dashed lines show the case where photons of both polarizations are allowed to split. The other cascade parameters are: \(B_{p,12} = 1000\), \(P_0 = 1\) (a dipole field geometry is adopted), and \(\gamma_0 = 4 \times 10^7\). The photon spectra are shown in the left panel and the pair spectra are shown in the right panel. For typical cascade parameters, the spectra when photon splitting is “turned off” in the simulation are nearly indistinguishable from the spectra when only \(\perp\rightarrow||\)) is allowed; we therefore do not present the turned-off case here (but see Fig. 12).

rules for photon polarization modes, one-photon pair production into low Landau levels for the \(e^\pm\), and resonant inverse Compton scattering from polar cap hot spots [with \(T = (0.3-3) \times 10^6\) K]. Both cascades initiated by a single electron (representing the entire cascade process described in Section 1) and by a single photon (representing one branch of the cascade) are simulated, for a variety of initial energies of the primary particle (\(\gamma_0 m_e c^2\) and \(\epsilon_0\)). We have made an effort to present our numerical results systematically, including empirical relations for the pair multiplicity (i.e., the number of electrons and positrons produced per primary particle).

Locally the cascade behaves very differently above and below the critical QED field strength \(B_Q \equiv 4.414 \times 10^{13}\) G (e.g., pair production is \(\sim 10\) times more efficient for \(B > B_Q\) but synchrotron emission is highly suppressed; see Section 1). Globally, however, the cascade followed from the surface to the light cylinder behaves similarly regardless of surface field strength \(B_p\). For example, we find that the total number of pairs produced in electron-initiated cascades, as well as their energy spectrum, depends on the polar cap voltage \(B_p P^{-2}\) but is weakly dependent on \(B_p\) alone. Additionally, the total photon spectra for \(B > B_Q\) and \(B < B_Q\) have similar shapes over the energy range we consider in our simulation (10 keV-1 TeV); this is because curvature radiation, which dominates the photon spectrum over most of this energy range, is independent of magnetic field strength. Photon splitting, a process that is only active for \(B \gtrsim B_Q\), could potentially distinguish between neutron stars with high and low surface fields, by lowering the multiplicity of electrons and positrons \(N_E\) produced in the cascade to such a degree that the radio emission mechanism can not function (Baring & Harding 2001). However, we find that even if both photons polarized parallel to and perpendicular to the magnetic field are allowed to split, photon splitting lowers \(N_E\) by at most 50% for \(B_{p,12} = 1000\) and by < 10% for \(B_{p,12} \leq 100\). With the correct selection rule (\(\perp\rightarrow||\)) \(\perp\)), the effect of photon splitting is even smaller.

Our results show that a strongly non-dipole magnetic field with radius of curvature \(\sim 10^8\) cm near the stellar surface (\(\gamma\) less than a few stellar radii) can account for pulsars which lie below the standard death line for dipole fields (see Section 2). Whether or not such a strongly-curved geometry could form is another question altogether. It is doubtful that cascades initiated by resonant inverse Compton scattering in the open field region of the magnetosphere can account for the “missing” pulsars, as the multiplicity of pairs produced in such cascades \((\lesssim 1\) per primary electron) are too low for the current models of the pulsar radio emission. High-multiplicity cascades due to resonant ICS may still occur along twisted lines in the closed field region (where the primary electrons never reach Lorentz factors larger than \(\gamma_0 \sim 10^3\)), as they do in magnetars (e.g., Thompson et al. 2002, Beloborodov & Thompson 2007, Thompson 2008, Beloborodov 2009).

At any altitude in the magnetosphere, the photon spectrum at low energies \((\lesssim 1\) MeV) has a power-law shape with spectral index \(\Gamma = 2/3\) (where \(dN/d\epsilon \propto \epsilon^{-2}\)); this is true regardless of whether synchrotron radiation or curvature radiation dominates the spectrum, as both processes have the same low-energy shape. The photon spectrum \(dN/d\epsilon \propto \epsilon^{-2/3}\) is harder than the hard X-ray spectra observed in several pulsars, which typically have photon indices \(\Gamma \sim 1-2\) (see, e.g., Kuiper & Hermsen 2009, Bogdanov & Grindlay 2009). One way to reconcile our results with observations is to include an additional radiative process, cyclotron resonant absorption, in the simulation. Such an approach is taken by Harding et al. (2005) (see also Harding et al. 2008), who find that in addition to the synchrotron radiation emitted immediately upon creation, the \(e^\pm\) particles emit an additional component of synchrotron radiation at large altitudes that dominates the low-energy spectrum. The efficiency of this high-altitude synchrotron emission is due to the large pitch angles of the \(e^\pm\), which they obtain through resonant absorption of radio photons that are beamed from some intermediate height in the magneto-
sphere. The high-altitude synchrotron emission will have a hard X-ray part with photon index $\Gamma_{10-1000\text{ keV}}$ generally different from $2/3$, because the hard X-ray band lies near the peak of the emission (rather than in the low-energy tail as is the case for the low-altitude emission). Harding et al. (2003) find that for typical millisecond parameters the photon index is in the observed range, $1 < \Gamma_{10-1000\text{ keV}} < 2$.

To fully incorporate the effects of cyclotron absorption into our simulation, we would need a model of the radio beam structure and spectrum, since the evolution in the pitch angle of each $e^\pm$ particle depends on the angle at which radio photons of the resonant frequency are incident on the particle and their density. Empirical models of radio beams exist (e.g., Rankin 1993, Arzoumanian, Chernoff, & Cordes 2002, Kijak & Gil 2003); however, inclusion of such a model is beyond the scope of this paper. This means that we can not say anything in detail about the hard X-ray spectra of strongly-magnetized neutron stars, such as whether the high-altitude synchrotron component dominates and what its low-energy cutoff is (but see Harding et al. 2005 for examples of these spectra at several different cascade parameters). Nevertheless, we can use the $e^\pm$ spectra from our simulation to estimate how the $\Gamma_{10-1000\text{ keV}}$ photon index varies as a function of the cascade parameters. For synchrotron emission from a distribution of $e^\pm$ particles, the hardness of the $e^\pm$ spectrum and the hardness of resulting photon spectrum are correlated [for a $dN/dE \propto E^{-p}$ power-law distribution of $e^\pm$ the photon spectrum has an index given by the familiar expression $\Gamma = (p + 1)/2$]. We find that the $e^\pm$ spectrum is harder, and therefore $\Gamma_{10-1000\text{ keV}}$ is lower, for larger $B_\parallel$ or $\gamma_0$ or shorter $P$ (i.e., anything that increases the energy of the cascade); additionally, for two neutron stars with the same polar cap voltage $B_\parallel P^{-2}$, $\Gamma_{10-1000\text{ keV}}$ is lower in the neutron star with the stronger surface field. Therefore, in pulsars where the cyclotron absorption – synchrotron emission mechanism dominates the hard X-ray spectrum, the most-strongly magnetized pulsars will have the hardest spectra (lowest photon indices), all other parameters being equal.

Luminous hard X-ray (from 20 keV to several hundreds of keV) emission has also been detected from a number of Anomalous X-ray Pulsars by INTEGRAL and RXTE (e.g., Kuiper et al. 2006). Possible mechanisms for this emission were discussed by Thompson & Beloborodov (2005) and Beloborodov & Thompson (2007). Since the observed hard X-ray luminosity exceeds the spin-down power by several orders of magnitude, it must be fed by an alternative source of energy such as the dissipation of a superstrong magnetic field. We note, however, that the observed photon indices, $\Gamma \simeq 0.8-1.8$, are similar to (but slightly harder than) those of radio pulsars. Synchrotron radiation by secondary $e^+e^-$ pairs produced in a cascade similar to those studied in this paper could plausibly explain the observations.

In constructing our simulations, we have attempted to rely as little as possible on the precise nature of the acceleration region. However, there are several key assumptions that we have made that are only valid for inner gap accelerators (e.g., that the cascade begins at the neutron star surface). It has become apparent that models where particle acceleration occurs only in the inner gap can not account the observed high-energy gamma ray emission from young pulsars. For example, the gamma-ray pulse profiles (e.g., the widely separated double peaks) of the six pulsars detected by EGRET already suggested that models of high-altitude gamma-ray emission were required. More recently, observations of the Crab pulsar by the Major Atmospheric Gamma-ray Imaging Cherenkov Telescope (MAGIC) and around 50 pulsars with “above average” spin-down powers by the Fermi Gamma-ray Space Telescope have revealed that the high-energy tails of the photon spectra fall off exponentially or more slowly than exponential (Aliu et al. 2008, Abdo et al. 2009a,b), while inner gap models predict that the tails fall off faster than exponential (see, e.g., Fig. 12). The outer gap model and the high-altitude version of the slot gap model have been successful in explaining the $\gamma$-ray pulsar light curves (e.g., Watters et al. 2009, Venter, Harding, & Guillemot 2009), but see Bai & Spitkovsky 2009.

Although an outer gap or slot gap accelerator model is required to explain gamma-ray observations, our results based on the inner gap model still have general applicability. First, even when the acceleration region is located in the outer magnetosphere a significant fraction of the pair creation must occur in the inner magnetosphere (e.g., Cheng et al. 2004). Indeed, pulsar radio emission is strongly constrained to originate from well inside the light cylinder radius; the only way to generate highly coherent radio emission is to have copious $e^\pm$ plasma produced by vigorous pair cascades (e.g., Melrose 2004, Luvbarsky 2008). Second, both inner gap and outer gap accelerators may exist in a neutron star simultaneously, whether as one extended acceleration region (e.g., Hirota 2006) or on different field lines. Radio observations suggest that radio emission can come from intermediate field lines, neither along the pole of the star nor at the edge of the open field region (e.g., Cheng 2004); if this is true, it provides further support for an inner gap origin of the $e^\pm$ plasma, since the plasma generated by an outer gap model is formed on field lines close to the last open field line. In addition, as discussed above, hard X-ray (10-1000 keV) observations of both pulsars and magnetars support a magnetosphere model where the hard X-ray emission is dominated by radiation from the $e^+e^-$ pairs generated by inner gap cascades. Third, it is unlikely that the death lines for the inner gap and outer gap mechanisms overlap completely. There should therefore be regions of the $P$-$P$ diagram where the outer gap mechanism is excluded but the inner gap mechanism still functions; these regions, if they exist, will be located near the inner gap death line, where pulsars have very low spin down powers. Although observations show that the gamma-ray spectra is dominated by slot or outer gap emission for pulsars with moderate to large spin down power, they have not yet ruled out an inner gap origin for pulsars with low spin down power.

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APPENDIX A: RESONANT INVERSE COMPTON SCATTERING

Here we calculate the photon scattering rate for the resonant inverse Compton process, using the simplified model of an electron positioned directly above the magnetic pole (hot spot) and traveling radially outward; see Fig. A1. The resonant cross section for inverse Compton scattering, in the rest frame of the electron before scattering, is

\[ \sigma_{\text{res}} = 2\pi^2 \frac{e^2 \hbar}{m_e c} \delta(\epsilon' - \epsilon) , \]  

(A1)

where \( \epsilon' = \gamma \epsilon_c (1 - \beta \cos \psi) \), \( \epsilon_c \) is the photon energy in the “lab” frame, and \( \psi \) is the incident angle of the photon with respect to the electron’s trajectory. The cross section is appropriate even for \( B > B_Q \), since the resonant condition \( \epsilon' = \epsilon_c \) holds regardless of field strength [though at the highest field strengths the prefactor \( 2\pi^2 e^2 \hbar/m_e c \) in Eq. (A1) serves only as an upper bound to the exact expression; see Gonthier et al. (2000)]. The polar hot spot has a angular size of \( \theta_{\text{spot}} \); this sets a maximum value for \( \psi \) of

\[ \cos \psi_{\text{crit}} = \frac{r - R \cos \theta_{\text{spot}}}{\sqrt{r^2 + R^2 - 2rr \cos \theta_{\text{spot}}}} , \]  

(A2)

where \( r \) is altitude of the electron (from the center of the star). The intensity of emission from the hot spot is

\[ I_{\epsilon_i}(\psi; r) = \begin{cases} \frac{\epsilon_i}{\epsilon_{\text{crit}}} \left( \frac{\gamma_{\text{crit}}}{\epsilon_i} \right), & \psi < \psi_{\text{crit}}; \\
0, & \text{otherwise}. \end{cases} \]  

(A3)

Therefore, the photon scattering rate for the resonant ICS process above a polar hot spot is given by [see, e.g., Eq. (B3) of ML07]

\[ \frac{dN_{\text{ph}}}{dt} = \int d\Omega_i \int d\epsilon_i (1 - \beta \cos \psi) \left( \frac{\epsilon_i}{\epsilon_{\text{crit}}} \right) \sigma_{\text{res}} \]  

(A4)

\[ = \frac{2\pi^2 e^2 \hbar}{m_e c \gamma} \int \left[ \psi < \psi_{\text{crit}} \right] d\Omega_i \left( \frac{B_Q}{\epsilon_i} \right) \]  

\[ = \frac{c}{\gamma^2 a_0} \left( \frac{kT}{m_e c^2} \right) \beta_Q \ln \left( \frac{1}{1 - \epsilon_c/\gamma (1 - \beta \cos \psi)} \right) \]  

(A5)

\[ = \frac{c}{\gamma^2 a_0} \left( \frac{kT}{m_e c^2} \right) \beta_Q \ln \frac{1 - e^{-\epsilon_c/\gamma (1 - \beta \cos \psi)}}{1 - e^{-\epsilon_c/\gamma (1 - \beta \cos \psi_{\text{crit}})kT}} . \]  

(A6)

APPENDIX B: PAIR PRODUCTION

B1 Kinematics

Consider the pair production of a photon with energy \( \epsilon \) and angle \( \psi \) (the photon – magnetic field intersection angle). In the frame where the photon is traveling perpendicular to the local magnetic field direction (the “perpendicular” frame moves at the velocity \( c \cos \psi \) relative to the “lab” frame), the photon has energy \( \epsilon' = \epsilon \sin \psi \), and energy conservation demands

\[ \epsilon' = E_j' + E_k' = \sqrt{p_j'^2 c^2 + m_e c^2 (1 + 2\beta_j Q)} + \sqrt{p_k'^2 c^2 + m_e c^2 (1 + 2\beta_k Q)} , \]  

(B1)

where \( E_j' \) and \( E_k' \) are the energies of the electron and the positron and \( p_i' \) is the momentum along the magnetic field of either particle (\( p_2' = -p_1' \)). From this we find

\[ |p_j'| = m_e c \sqrt{x^2 - 1} \left[ (j + k) \beta_Q + (j - k)^2 \frac{\beta_Q^2}{4x^2} \right] \]  

(B2)

where \( x = \epsilon'/(2m_e c^2) \), and

\[ E_j' = m_e c^2 \left[ x^2 + (j + k) \beta_Q + (j - k)^2 \frac{\beta_Q^2}{4x^2} \right] \]  

(B3)

\[ E_k' = m_e c^2 \left[ x^2 + (k - j) \beta_Q + (j - k)^2 \frac{\beta_Q^2}{4x^2} \right] . \]  

(B4)

In the “lab” frame, the energies of the electron and positron at the moment of pair creation are given by

\[ E = \frac{1}{\sin \psi} \left( E' \pm p'_c \cos \psi \right) . \]  

(B5)

In our simulation one particle is randomly assigned the ‘+’ energy and the other the ‘−’ energy, with equal probability of either outcome.

B2 Photon attenuation coefficients and optical depth

In the perpendicular frame, the first three attenuation coefficients for \( \parallel \) and the first two non-zero attenuation coefficients for \( \perp \) photons are (Daugherty & Harding 1983)

\[ R'_{\parallel,00} = \frac{1}{2a_0} \frac{\beta_Q}{x^2 - 1 - \beta_Q} e^{-2x^2/\beta_Q} , \quad x > 1 ; \]  

(B6)

\[ R'_{\parallel,01} = 2 \times \frac{1}{2a_0} \frac{2 + \beta_Q - \beta_Q^2}{x^2 - 1 - \beta_Q + \beta_Q^2} e^{-2x^2/\beta_Q} , \]  

(B7)

\[ x > (1 + \sqrt{1 + 2\beta_Q})/2 ; \]  

\[ R'_{\parallel,02} = 2 \times \frac{1}{2a_0} \frac{2x^2 + 1 + \beta_Q - \beta_Q^2}{x^2 - 1 - 2\beta_Q + \beta_Q^2} e^{-2x^2/\beta_Q} , \]  

(B8)

\[ x > (1 + \sqrt{1 + 4\beta_Q})/2 ; \]
\[
R'_{\perp,01} = 2 \times \frac{1}{2a_0} \frac{\beta Q}{2x^2} \frac{2x^2 - \beta Q}{\sqrt{x^2 - 1 - \beta Q + \frac{\beta Q}{2}}} e^{-2x^2/\beta Q}, \tag{B9}
\]
\[
R'_{\perp,02} = 2 \times \frac{1}{2a_0} \frac{x^2 - \beta Q}{\sqrt{x^2 - 1 - 2\beta Q + \frac{2\beta Q}{x}}} e^{-2x^2/\beta Q}, \tag{B10}
\]

Note that \(R'_{\perp,00} = 0\). In the above expressions, the attenuation coefficients of Daugherty & Harding (1983) are multiplied by a factor of two for all channels but \((00)\) [i.e., in Eqs. (B7)-(B10)], since we are using the convention \(\tau') = R_{\text{DHH}} + R_{\text{DHH}} = 2R_{\text{DHH}}\) for \(j \neq k\). We now examine the conditions for pair production by a \(\|\)-polarized photon; the analysis is similar for a \(\perp\)-polarized photon and yields the same result. The optical depth for pair production is
\[\tau = \int_{0}^{s_{\text{ph}}} ds R(s) = \int_{0}^{s_{\text{ph}}} ds R'(s) \sin \psi. \tag{B11}\]
We assume \(\psi \ll 1\), which is valid since most photons that can pair produce will do so long before the angle \(\psi\) approaches unity (only photons with energies \(\varepsilon \approx 2m_e c^2\) in the lab frame must wait until \(\psi \sim 1\) to pair produce). In this limit, we have \(\sin \psi \approx s/R_c\), so that \(x\) and \(s\) are related by
\[x \approx \frac{s}{R_c} \frac{\epsilon}{2m_e c^2}. \tag{B12}\]
Let \(s_0\) to be distance traveled by the photon to reach the first threshold \(x = x_{\text{100}} \equiv 1\), and \(s_1\) to be the distance traveled by the photon to reach the second threshold \(x = x_{\text{110}} \equiv (1 + \sqrt{1 + 2\beta Q})/2\). The optical depth to reach the second threshold for pair production is
\[\tau_{\text{110}} = \int_{s_0}^{s_{\text{110}}} ds R_{\text{110}}(s) \tag{B13}\]
\[= \frac{\beta Q}{2a_0} \left(\frac{2m_e c^2}{\epsilon}\right)^2 R_c \int_{x_{\text{100}}}^{x_{\text{110}}} \frac{dx}{x\sqrt{x^2 - 1}} e^{-2x^2/\beta Q} \tag{B14}\]
\[= 9.87 \times 10^{11} \left(\frac{R_c}{10^8 \text{ cm}}\right) \left(\frac{100 \text{ MeV}}{\epsilon}\right)^2 T(\beta Q), \tag{B15}\]
where
\[T(\beta Q) = \beta Q \int_{x_{\text{100}}}^{x_{\text{110}}} \frac{dx}{x\sqrt{x^2 - 1}} e^{-2x^2/\beta Q}. \tag{B16}\]
We plot \(\tau_{\text{110}}\) as a function of magnetic field strength in Fig. [B1] for \(\epsilon = 100 \text{ MeV}\) and \(R_c = 10^6 \text{ cm}\). From Fig. [B1] we see that pair production occurs in the \((jk) = (00)\) channel (\(\tau_{\text{110}} \geq 1\)) when
\[B \geq B_{\text{crit}} \approx 3 \times 10^{12} \text{ G}. \tag{B17}\]
Because of the steep dependence of \(\tau\) on \(B\) for \(B \approx 3 \times 10^{12} \text{ G}\), the value of \(B_{\text{crit}}\) does not change much for different parameters \(\epsilon\) and \(R_c\). For example, \(B_{\text{crit}} \approx 3 \times 10^{12} \text{ G}\) for \(\epsilon = 100 \text{ MeV}\) and \(R_c = 10^6 \text{ cm}\), and \(B_{\text{crit}} \approx 7 \times 10^{12} \text{ G}\) for \(\epsilon = 10^4 \text{ MeV}\) and \(R_c = 10^6 \text{ cm}\). Figure [B1] also shows that for \(B \leq B_{\text{crit}}\), the optical depth \(\tau_{\text{110}}\) is much less than unity. The same result is found for the optical depth from the second to the third threshold, and for higher thresholds.

**APPENDIX C: EMPIRICAL RELATIONS FOR THE NUMERICAL RESULTS**

In this section we justify several of the empirical relations given in Sections 4.1 and 4.2. In the derivations below we treat the radius of the emission point for the primary photon, \(r_6 = r_{0,ph}/(10^6 \text{ cm})\), as a free parameter. This allows the results to be applicable both to Section 4.1 where we assume that \(r_6 = 1\), and to Section 4.2 where \(r_6 \geq 1\).

### C1 Photon-initiated cascades

At low fields \((B \lesssim B_{\text{crit}} \approx 3 \times 10^{12} \text{ G})\), we can use Eq. (20) for the attenuation coefficient, which implies that pair production occurs when \(x\beta Q \sim 1/15-1/10\). In the following we will use \(x\beta Q \simeq 1/10\), appropriate for photon energy \(\epsilon \sim 10^4 \text{ MeV}\). To initiate an effective cascade, a photon must pair produce before traveling a distance \(s_{\text{ph,max}} = 0.5r_{0,ph} \text{ [Eq. (31)]}\). Therefore the minimum photon energy for cascade is given by

\[
\frac{\epsilon_{\text{min}}}{2m_e c^2} \frac{0.5r_{0,ph}}{R_c} \simeq \frac{1}{10\beta Q},
\]

where \(\beta Q\) and \(R_c\) should be evaluated at the pair creation point \(r \simeq s_{\text{ph,max}} + r_{0,ph} = 1.5r_{0,ph}\). Since \(\beta Q = B/B_Q \simeq 0.02B_{p,12}/(r/R_{\text{crit}})^{3/2}\) and \(R_c \simeq 10^8 R_{8}\sqrt{r/R_{\text{crit}}} \text{ cm}\), where \(B_{p,12}\) and \(R_8\) are the magnetic field strength (in units of \(10^{12} \text{ G}\)) and curvature radius (in units of \(10^8 \text{ cm}\)) at the surface, we find

\[
\epsilon_{\text{min}} \sim 3000B_{p,12}^{-1}R_8^{-5/2} \text{ MeV}.
\]

For a photon injected from the surface \((r_6 = 1)\), this reduces to Eq. (8).

In each generation of pair production, a photon of energy \(\epsilon\) and angle \(\psi\) creates an \(\epsilon \pm\) pair, each with energy
γmc^2 \approx 0.5ε and traveling in the same direction as the photon. The electron and positron radiate synchrotron photons until they reach an energy of γmc^2 = mc^2/\sin ψ = \gamma mc^2/x \approx 0.1β_1ε, since x = γ sin ψ \simeq 1/(10β_2). The synchrotron photons have a characteristic energy ϵ_\gamma = 1.5β^2 sin(ψ) β_1mc^2 \approx 0.075ε. Therefore, in each generation, the energy of the photons and pairs drops by a factor \sim 0.075 while the number of particles increases by 1/0.075 (see Hilschman & Arons 2001a). When RICS is inactive (e.g., when T \lesssim 10^7 K), such that the only mechanism for energy loss is synchrotron radiation, the final pair spectrum consists of an e\pm pair, each with energy

E_{max} \sim 0.1B_{p,12}r_6^{-3}\epsilon_0,  \quad (C3)

created by the primary photon, and a power-law distribution of pairs with index p \sim 2 (where dN/dE \propto E^{-p}) extending from \sim 0.075E_{max} down to \sim 0.1B_{12}\epsilon_{min} (see Fig. [5]). Hence the total energy of the cascade pairs produced by the primary photon (of energy \epsilon_0) is

\epsilon_{tot} \sim 2E_{max} + 0.1B_{p,12}r_6^{-3}\epsilon_{min}n_ε \ln \left(\frac{0.075\epsilon_0}{\epsilon_{min}}\right),  \quad (C4)

which reduces to Eq. (52) for r_6 = 1.

For the range of high fields considered in this work (B_{crit} \lesssim B \lesssim 1000), pair production occurs when x = 1 (for \textit{x} = x_{00}) or x \sim 1 (for \textit{x} = x_{01}; see Section [C2]). For a photon emitted from the point (\textit{r}_{0\text{ph}}, \textit{θ}_{0\text{ph}}), the maximum interaction angle between the photon and the magnetic field is given by [Eqs. (19), (17), and (21)]

\sin ψ_{max} \approx \chi [\chi (θ_{0\text{ph}}) - χ (θ_{0\text{ph}})],

(C5)

= \arctan \left\{ \frac{1}{2} \tan \left[ x + \arctan \left( \frac{\tan θ_{0\text{ph}}}{2} \right) \right] \right\};

(C6)

for “small” angles θ_{0\text{ph}} \lesssim 0.6,

\sin ψ_{max} \lesssim \sin θ_{0\text{ph}}.  \quad (C7)

For θ \lesssim 0.6 the local radius of curvature is given by \textit{R}_c \simeq 1.3/r \sin θ; therefore, using x = ε sin ψ/(2mc^2) \sim 2 we have

\epsilon_{min} \sim 200\textit{R}_c r_6^{-1/2} \text{MeV},

(C8)

which at \textit{r}_6 = 1 is Eq. (54).

At high fields, each electron-positron pair is created with energies \gamma mc^2 \approx 0.5ε (for \parallel photons) or \approx 0.5ε/x_{01} and \approx 0.5ε(2 - 1/x_{01}) (for \perp photons). After synchrotron radiation the electron and positron have energies 0.5ε or 0.5ε/x_{01}. Therefore, the final energies of the electron and positron created by the primary photon are given by

E_{max} \approx \frac{ε_0}{2x_{01}} \text{ or } \frac{ε_0}{2x_{00}} \sim 0.1ε_0 \text{ or } 0.5ε_0.  \quad (C9)

In order for resonant ICS to modify the cascade spectrum, the electron (positron) must have an energy (after synchrotron radiation) larger than the minimum energy at which RICS is effective, i.e., \epsilon_{max} > 3\gamma_{\text{crit}}mc^2 [where \gamma_{\text{crit}} = ε_0/kT; see Eq. (25)] which implies

\epsilon_0 \gtrsim 70B_{p,12}T_6^{-1} \text{MeV}.  \quad (C10)

C2 Electron-initiated cascades

The spectrum of curvature photons extends from approximately one photon at

\epsilon_{max} \sim 10\epsilon_{CR}(\gamma_0) = 3 \times 10^3\gamma_0^2R_8^{-1}\text{MeV},  \quad (C11)

where γ_0 is the primary electron’s initial Lorentz factor γ_0 in units of 10^7, up to a maximum of \sim 6 \times 10^4R_8^{-1/2} photons at \text{e}_{\text{peak}} \sim 6R_8^{-1/2} \text{MeV}. For photon energies below e_{\text{peak}} the spectrum is a power law with spectral index Γ = 2/3 (where dN/dE \propto E^{-1}), characteristic of curvature and synchrotron radiation at low energies (e.g., Erber 1966).

At low fields B < B_{crit} the pair spectrum extends from

E_{max} \sim 0.1B_{p,12}\epsilon_{max} \sim 300\gamma_0^2B_{p,12}R_8^{-1} \text{MeV}  \quad (C12)

down to

E_{min} \sim 0.1B_{12}\epsilon_{min} \sim 300R_8r_6^{-1/2} \text{MeV};  \quad (C13)

at high fields

E_{max} \sim 0.5\epsilon_{max} \sim 6 \times 10^3\gamma_0^2R_8^{-1} \text{MeV}  \quad (C14)

and

E_{min} \sim 0.5\epsilon_{min}/x_{01} \sim 20R_8r_6^{-1/2} \text{MeV},  \quad (C15)

(see Section [C1]).

REFERENCES

Abdo A. A. et al. (Fermi Collab.), 2009, ApJ, 696, 1084.
Abdo A. A. et al. (Fermi Collab.), 2009, Science, 325, 848.
Abdo A. A. et al. (Fermi Collab.), 2009, ApJ, 706, 1331.
Abdo A. A. et al. (Fermi Collab.), 2010, ApJ, 708, 1254.
Aliu et al. (MAGIC Collab.), 2008, Science, 322, 1221.
Adler S. L., 1971, Ann. Phys., 67, 599.
Arendt P. N., Eilek J. A., 2002, ApJ, 581, 451.
Arons J., 1996, ApJ, 466, 215.
Arons J., 1996, A&A, 120, 49.
Arons J., 1998, in Shibazaki et al., eds, Proc. Intl. Conf. on Neutron Stars and Pulsars, Neutron Stars and Pulsars: Thirty Years after the Discovery. UAP, Tokyo, p. 339.
Arons J., 2008, in Becker W., ed, Neutron Stars and Pulsars, 40 Years After the Discovery, arXiv:0708.1050.
Arons J., Scharlemann E. T., 1979, ApJ, 231, 854.
Arzoumanian Z., Chernoff D. F., Cordes J. M., 2002, ApJ, 568, 289.
Bai X.-N., Spitkovsky A., 2009, ApJ, submitted. [arXiv:0910.3741]
Baring M. G., Harding A. K., 1997, ApJ, 482, 372.
Baring M. G., Harding A. K., 2001, ApJ, 547, 929.
Baring M. G., Harding A. K., 2007, ApJSS, 308, 109.
Beloborodov A. M., 2008, ApJ, 683, L41.
Beloborodov A. M., 2009, ApJ, 703, 1044.
Beloborodov A. M., Thompson C., 2007, ApJ, 657, 967.
Beskin V. S., 1999, Physics-Uspekhi, 42, 1071.
Bogdanov S., Grindlay J. E., 2009, ApJ, 703, 1557.
Camilo F. et al., 2007, ApJ, 669, 561.
Camilo F. et al., 2008, ApJ, 679, 681.
Chen K., Ruderman M., 1993, ApJ, 402, 264.
Cheng K. S., Ho C., Ruderman, M., 1986, ApJ, 300, 500.
Cheng K. S., Ho C., Ruderman, M., 1986, ApJ, 300, 522.
Cheng K. S., Ruderman, M., Zhang L., 2000, ApJ, 537, 964.
Contopoulos I., 2005, A&A, 442, 579.
Contopoulos I., Kazanas D., Fendt C., 1999, ApJ, 511, 351.
Contopoulos I., Spitkovsky A., 2006, ApJ, 643, 1139.
Daugherty J. K., Harding A. K., 1982, ApJ, 252, 337.
Daugherty J. K., Harding A. K., 1983, ApJ, 273, 761.
