NODAL CURVES ON SURFACES OF GENERAL TYPE

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INTRODUCTION

In this paper we investigate to which extent the theory of Severi on nodal plane  
curves of a given degree $d$ extends to a linear system on a complex projective  
nonsingular algebraic surface. As well known, in [S], Anhang F Severi proved that  
for every $d \geq 3$ and $0 \leq \delta \leq \binom{d-1}{2}$ the family $\mathcal{V}_{d,\delta}$ of plane  
irreducible curves of degree $d$ having exactly $\delta$ nodes and no other  
singularities is non empty and everywhere smooth of codimension $\delta$ in the  
linear system $|\mathcal{O}(d)|$.

If $C \in \mathcal{V}_{d,\delta}$ Severi uses the non speciality of the normal line bundle to the  
composition $\nu : \tilde{C} \to C \to \mathbb{P}^2$, where $\tilde{C}$ is the normalization of $C$, to prove that  
$\mathcal{V}_{d,\delta}$ is smooth of the asserted codimension at the point $C$. This proof can be  
extended to rational, ruled and K3 surfaces with little changes: we discuss this  
point in section 1 (see also [T] for the case of rational surfaces.

In the case of a surface of general type $S$ the approach of Severi fails, and in  
fact it is easy to see that the analogous of Severi’s theorem does not hold in general  
if we impose too many nodes to the curves of a complete linear system $|D|$. One  
may nevertheless look for an upper bound on $\delta$ ensuring that the family $\mathcal{V}_{D,\delta}$ of  
irreducible curves in $|D|$ with $\delta$ nodes is smooth of codimension $\delta$. In section 2 we  
give the following partial answer to this problem:

Theorem 2.2. Let $S$ be a surface such that $|K_S|$ is ample and let $C$ be an irreduc  
ible curve on $S$ such that $C \equiv_{\text{num}} pK_S$, $p \geq 2$, $p \in \mathbb{Q}$ and $|C|$ has smooth  
general member. Assume that $C$ has $\delta \geq 1$ nodes and no other singularities and assume that either $\delta < \frac{p(p-2)}{4} K_S^2$, or $\delta < \frac{(p-1)^2}{4} K_S^2$, $p \in \mathbb{Z}$ odd, and the Neron-Severi  
group of $S$ is $\mathbb{Z}$ generated by $K_S$.

Then $\mathcal{V}_{C,\delta}$ is smooth of codimension $\delta$ at the point corresponding to $C$.

For the proof we consider a curve $C \in \mathcal{V}_{D,\delta}$ where Severi’s theorem fails and we  
associate a rank two vector bundle to the zero-cycle $N$ of nodes of $C$. We then
apply the Bogomolov-Reider method ([Rr]) to deduce the inequality of the theorem from the properties of this vector bundle.

As a special case of theorem 2.2 we obtain the following result on smooth surfaces in $\mathbf{P}^3$:

**Proposition 2.4.** Let $S$ be a smooth surface of degree $d \geq 5$ in $\mathbf{P}^3$ with plane section $H$. If $C \in |nH|$ has $\delta$ nodes and no other singularities and $\delta < \frac{nd(n-2d+8)}{4}$ then $\mathcal{V}_{C,\delta}$ is smooth of codimension $\delta$ at the point $C$.

In the case of quintic surfaces we slightly improve this bound proving that $\mathcal{V}_{C,\delta}$ is smooth of codimension $\delta$ at the point $C$ if

$$\delta < \frac{5(n-1)^2}{4}$$

when $n$ is odd (proposition 2.5). We show that these estimates are sharp, by producing explicit examples of curves in the linear systems $|nH|$ having exactly $\delta$ nodes, where $\delta$ is the upper bound given above and which are obstructed as elements of $\mathcal{V}_{C,\delta}$. These examples are discussed in detail in section 4, where we show in particular that they are not general points of a component of $\mathcal{V}_{C,\delta}$.

The construction of the examples is achieved through the consideration of another problem. We consider a complete intersection curve $C$ in $\mathbf{P}^3$ having $\delta$ nodes; call $C$ geometrically linearly normal if it is not a birational projection of a smooth curve of $\mathbf{P}^4$ of the same degree. Since clearly smooth complete intersections are geometrically linearly normal, one may look for a bound $\delta(d,n)$ such that $C$ is geometrically linearly normal if $\delta \leq \delta(d,n)$. Aiming at this we prove the following:

**Theorem 3.4.** Assume that $C \subset \mathbf{P}^3$ is the complete intersection of a smooth surface $S$ of degree $d$ with a surface of degree $n$. $C$ is geometrically linearly normal if and only if the nodes of $C$ impose independent conditions to the linear system $|(n+d-5)H|$ on $S$.

As a consequence of this criterion we find the upper bound $\delta(d,n) = \frac{nd(n-2)}{4}$ (theorem 3.5). Theorem 3.4 is applied to the case of curves on a general quintic surface to deduce that certain non geometrically linearly normal curves we construct are obstructed in the corresponding Severi variety. These examples also show that the previous bound $\delta(5,n)$ is sharp.

We have not considered the problem of existence for $\mathcal{V}_{D,\delta}$; for results in this direction we refer to [CR] and [X].

The paper consists of four section. Section 1 is devoted to known facts on nodal curves and to Severi theory on rational, ruled and K3 surfaces. In section 2 we prove our main results on surfaces of general type. Section 3 deals with geometric linear normality of nodal complete intersection curves in $\mathbf{P}^3$. In section 4 we construct the examples of nodal curves on a quintic surface which show that the results of section 2 are sharp.

We work in the category of schemes over $\mathbf{C}$, the field of complex numbers. As usual, $\dim(H^i(-))$ will be denoted by $h^i(-)$.

1. **PRELIMINARIES**

We will denote by $S$ a projective nonsingular algebraic surface. Let $|D|$ be a complete linear system on $S$ whose general member is an irreducible nonsingular curve. We will denote by $p_a(D)$ the arithmetic genus of $D$, given by:

$$p_a(D) = \frac{D(D+K_S)}{2} + 1.$$
For every $\delta \geq 0$ there is a locally closed subscheme $V_{D,\delta}$ of $|D|$ which parametrizes a universal family of reduced and irreducible curves belonging to $|D|$ and having exactly $\delta$ nodes (ordinary double points) and no other singularities (see [W] for $S = \mathbb{P}^2$, but the proof extends to any $S$).

The schemes $V_{D,\delta}$ will be called Severi varieties. Let $C \in V_{D,\delta}$ and let $N$ be the scheme of nodes of $C$: it is a closed zero-dimensional subscheme of $S$ of degree $\delta$.

The geometric genus of $C$ is 

$$g = p_a(D) - \delta$$

The Zariski tangent space of $|D|$ at $C$ is

$$T_{|D|,C} = \frac{H^0(S, \mathcal{O}_S(D))}{(C)}$$

and the Zariski tangent space of $V_{D,\delta}$ at $C$ is

$$T_{V_{D,\delta},C} = \frac{H^0(S, \mathcal{I}_N(D))}{(C)}$$

while the obstruction space is a subspace of $H^1(S, \mathcal{I}_N(D))$.

In other words, a first order deformation $C + \epsilon C', \epsilon^2 = 0$, is in $V_{D,\delta}$ if and only if it is in $|D|$ and $N \subset C'$. In particular:

$$\dim(T_{V_{D,\delta},C}) \geq h^0(S, \mathcal{O}_S(D)) - \delta - 1 = h^0(S, \mathcal{O}_S(D)) - (p_a(D) - g) - 1$$

and equality holds iff $N$ imposes independent conditions to $|D|$. In this case $V_{D,\delta}$ is nonsingular of dimension

$$h^0(S, \mathcal{O}_S(D)) - \delta - 1 = \dim(|D|) - \delta$$

at $C$.

We recall the theorem of Severi on the projective plane.

**Theorem 1.1.** (Severi) Let $S = \mathbb{P}^2$, $d \geq 3$ and $D$ any divisor of degree $d$. Let $\delta \geq 1$ be such that

$$\delta \leq p_a(D) = \binom{d-1}{2}.$$

Then the Severi variety $V_{D,\delta}$ is non empty and smooth of pure dimension

$$\dim(|D|) - \delta = \frac{d(d + 3)}{2} - \delta.$$

**Proof.** Let us suppose that $C \in V_{D,\delta}$ and let $N$ be the scheme of nodes of $C$. In view of the exact sequence

$$0 \rightarrow \mathcal{I}_N(d) \rightarrow \mathcal{O}_S(d) \rightarrow \mathcal{O}_N(d) \rightarrow 0$$

and of the fact that $H^1(S, \mathcal{O}_S(d)) = 0$, in order to prove that $V_{D,\delta}$ is smooth of the asserted dimension at $C$ it is necessary and sufficient to prove that $H^1(S, \mathcal{I}_N(d)) = 0$. 


Let \( \sigma := h^1(S, \mathcal{I}_N(d)) \). Since \( h^0(\mathcal{O}_N(d)) = \delta \), from the above sequence we deduce that
\[
h^0(S, \mathcal{I}_N(d)) = h^0(\mathcal{O}_S(d)) - \delta + \sigma = \left( \frac{d+2}{2} \right) - \delta + \sigma.
\]
Let \( \nu : \tilde{C} \to C \) be the normalization of \( C \) and let \( \tilde{N} \) be the pullback of \( N \) to \( \tilde{C} \). We have an injective map:
\[
\frac{H^0(S, \mathcal{I}_N(d))}{(C)} \to H^0(\tilde{C}, \nu^*\mathcal{O}(d)(-\tilde{N}))
\]
Since \( \nu^*\mathcal{O}(d)(-\tilde{N}) \) has degree
\[
d^2 - 2\delta = 2g - 2 + 3d
\]
then it is non-special and we deduce that:
\[
h^0(S, \mathcal{I}_N(d)) - 1 \leq h^0(\tilde{C}, \nu^*\mathcal{O}(d)(-\tilde{N})) = d^2 - 2\delta + 1 - g = \left( \frac{d+2}{2} \right) - \delta - 1,
\]
whence \( \sigma = 0 \).

To prove that \( V_{D,\delta} \neq \emptyset \) for all \( \delta \) we start from the case \( \delta = p_a(D) \) i.e. \( g = 0 \).

The family \( V_{D,p_a(D)} \) is not empty because it contains any general projection of a rational and normal curve of \( \mathbb{P}^d \). Let \( C \in V_{D,p_a(D)} \), let as usual \( N \) denote the scheme of nodes of \( C \), let \( P \in N \) and \( M \) the complement of \( P \) in \( N \). Since
\[
h^1(S, \mathcal{I}_N(d)) = h^1(S, \mathcal{I}_M(d))
\]
we have
\[
h^0(S, \mathcal{I}_M(d)) = h^0(S, \mathcal{I}_N(d)) + 1.
\]
Any element of the vector space \( H^0(S, \mathcal{I}_M(d)) \) not in \( H^0(S, \mathcal{I}_N(d)) \) defines an infinitesimal deformation of \( C \) which smooths the node \( P \) while leaving unsmoothed all the other nodes. This means that \( C \in \overline{V_{D,p_a(D)}}, \) the closure of \( V_{D,p_a(D)} \) in\( -1 \). Therefore \( V_{D,p_a(D)} = \emptyset \). By descending induction on \( \delta \) one proves similarly that
\[
V_{D,\delta} \neq \emptyset
\]
for all \( 1 \leq \delta \leq p_a(D) \).

In example 1.3 we show how the proof of theorem 1.1 can be adapted to K3 surfaces.

**Remark 1.2.** The reason why the proof of 1.1 works is because
\[
\nu^*\mathcal{O}(d)(-\tilde{N}) = \nu^*(\mathcal{O}(d-3)(-\tilde{N}) \otimes \mathcal{O}(3)) = K_{\tilde{C}} \otimes \nu^*\mathcal{O}(3)
\]
and therefore this line bundle is non special. This fact has been applied in (2) to get \( \sigma = 0 \).

It is then clear that if we consider any rational or ruled surface \( S \) and any smooth and irreducible curve \( C \) on \( S \), such that \( |C| \) is base point free and \( K_S C < 0 \), then the first part of the proof of 1.1 (excluding the existence statement) can be repeated to this case word by word; this holds in particular for any Del Pezzo surface. For this result we refer also to [T]. Therefore we get the following:

Let \( S \) be a rational or ruled surface and let \( C \subset S \) be a smooth irreducible curve such that \( |C| \) is base point free and \( K_S C < 0 \). If for some \( \delta \leq p_a(C) \) we have \( V_{C,\delta} \neq \emptyset \), then \( V_{C,\delta} \) is smooth of codimension \( \delta \) in \( |C| \).
Example 1.3. Let $S$ be a K3 surface and $D$ a smooth irreducible curve such that $p_a(D) \geq 2$. Then (see [M]) $|D|$ is base point free and of dimension $p_a(D)$; moreover $H^1(S, \mathcal{O}_S(D)) = 0$. For each $1 \leq \delta \leq p_a(D)$ and for any $C \in \mathcal{V}_{D,\delta}$ we have (with notations as above):

$$h^0(S, \mathcal{I}_N(C)) - 1 = p_a(D) - \delta + h^1(S, \mathcal{I}_N(C)) \leq h^0(\tilde{C}, \nu^*\mathcal{O}(\tilde{N})) = h^0(\tilde{C}, K_{\tilde{C}}) = p_a(D) - \delta.$$ 

It follows that $H^1(S, \mathcal{I}_N(C)) = 0$ and therefore $\mathcal{V}_{D,\delta}$ is smooth and of codimension $\delta$ in $|D|$. 

In [MM] it is shown that $\mathcal{V}_{D,p_a(D)} \neq \emptyset$: therefore as in the proof of 1.1 it follows that $\mathcal{V}_{D,\delta} \neq \emptyset$ for all $1 \leq \delta \leq p_a(D)$.

Note that in particular we have that $\mathcal{V}_{D,p_a(D)}$ is finite, i.e. there are finitely many nodal rational curves in $|D|$.

2. SURFACES OF GENERAL TYPE

If $S$ is a surface of general type then we cannot expect that theorem 1.1 extends without changes to linear systems on $S$. The reason for this is obvious. If $|D|$ is a (say very ample) linear system on $S$, then on a general curve $C \in |D|$ the characteristic linear series is special; this implies that

$$\dim(|D|) \leq g(C) - 1 = p_a(D) - 1$$

therefore $\mathcal{V}_{D,p_a(D)}$ cannot have the expected codimension and we should in fact expect that $\mathcal{V}_{D,p_a(D)} = \emptyset$.

In this case we should ask the following more appropriate:

(2.1) Question. Given a surface of general type $S$ and a linear system $|D|$ on $S$ whose general member is smooth and connected, for which values of $\delta$ is $\mathcal{V}_{D,\delta}$ non empty and smooth of codimension $\delta$?

We will give a partial answer to question (2.1). Our main result is the following:

Theorem 2.2. Let $S$ be a surface such that $|K_S|$ is ample, and let $C$ be an irreducible curve on $S$ such that $|C|$ contains smooth elements and such that

$$C \equiv_{num} pK_S \quad p \geq 2, \ p \in \mathbb{Q}$$

Assume that $C$ has $\delta \geq 1$ nodes and no other singularities and assume that either

$$\delta < \frac{p(p-2)}{4}K_S^2$$

or

$$\delta < \frac{(p-1)^2}{4}K_S^2 \quad p \in \mathbb{Z} \text{ odd, and the Neron Severi group of } S \text{ is generated by } K_S.$$ 

Then the nodes of $C$ impose independent conditions to $|C|$. In particular the Severi variety $\mathcal{V}_{C,\delta}$ is smooth of codimension $\delta$ at $C$.

The proof is based on the study of a rank 2 bundle on $S$, associated with the set of nodes of $C$. To do this, we recall briefly the connections between rank 2 bundles and sets of points on a surface.
Remark 2.3. (see [GH]) Let $N$ be a set of $\delta$ points in $S$. If $N$ does not impose independent conditions to a linear system $|C|$, then the restriction map $H^0(S, \mathcal{O}_S(C)) \to H^0(\mathcal{O}_N)$ is not surjective. Let $N_0 \subset N$ be a minimal subset for which the composition $H^0(S, \mathcal{O}_S(C)) \to H^0(\mathcal{O}_N) \to H^0(\mathcal{O}_{N_0})$ does not surject; then a general element of $H^1(S, \mathcal{I}_{N_0}(C))$ defines an extension:

\[(2)\quad 0 \to \mathcal{O}_S \to E \to \mathcal{I}_{N_0}(C - K_S) \to 0\]

where $E$ is a rank 2 vector bundle on $S$, with Chern classes

\[c_1(E) = \mathcal{O}_S(C - K_S)\]
\[c_2(E) = \deg N_0 \leq \delta\]

with also $c_2(E) > 0$ for $N_0$ cannot be empty.

Proof of Theorem 2.2. Call $N$ the set of nodes of the curve $C$ and assume that $N$ does not impose independent conditions to the curves of $|C|$; we show that we get a contradiction.

Take the subset $N_0 \subset N$ and the rank 2 vector bundle $E$ described in the previous remark and denote by $\delta_0$ the degree of $N_0$.

By assumptions, $c_1(E) = \mathcal{O}_S(C - K_S)$ $\equiv_{\text{num}} (p - 1)K_S$ and

\[c_1(E)^2 - 4c_2(E) = (p - 1)^2K_S^2 - 4\delta_0 \geq (p - 1)^2K_S^2 - 4\delta > 0\]

so that $E$ is Bogomolov unstable (see [B]).

It follows that there exists a divisor $M$ which ‘destabilizes’ $E$ with respect to the ample divisor $K_S$, that is, $h^0(S, E(-M)) > 0$ and

\[(3)\quad (2M - c_1(E))K_S > 0 \quad \text{i.e.} \quad MK_S > \left(\frac{p - 1}{2}\right)K_S^2.\]

Taking $M$ maximal, we may further assume that a general section of $E(-M)$ vanishes in a locus $Z$ of codimension 2 (see [R] th.1). It follows $\deg Z = c_2(E(-M)) \geq 0$; hence:

\[(4)\quad \delta_0 + M^2 - (p - 1)MK_S = c_2(E) + M^2 - Mc_1(E) = c_2(E(-M)) \geq 0.\]

Let us now use (2). $h^0(S, \mathcal{O}_S(-M))$ is 0, for $-MK_S < -(p - 1)K_S^2/2 \leq 0$ by assumptions and $K_S$ is ample; thus $h^0(S, E(-M)) > 0$ implies $h^0(S, \mathcal{I}_{N_0}(C - K_S - M)) > 0$, that is, there exists a divisor $\Delta$ in the linear system $|C - K_S - M|$, which contains $\delta_0$ nodes of the curve $C$. $\Delta$ cannot contain $C$ as a component, for as above $(-K_S - M)K_S < 0$, hence $-K_S - M$ cannot be effective. It follows, by Bezout, $(C - K_S - M)C \geq 2\delta_0$ which yields:

\[(5)\quad ((p - 1)K_S - M)(pK_S) \geq 2\delta_0.\]

Now observe that, since $K_S$ is ample, by Hodge index theorem, we have $M^2K_S^2 \leq (MK_S)^2$; putting this together with (4) and (5), one finally gets:

\[\frac{(MK_S)^2}{K_S^2} - \frac{3p - 2}{2}MK_S + \frac{p(p - 1)}{2}K_S^2 \geq 0\]
which can be solved with respect to $MK_S$. Since by assumption $p \geq 2$ we have $p - 1 \geq p/2$, thus the previous inequality implies either $MK_S \leq pK_S^2/2$ or $MK_S \geq (p - 1)K_S^2$. The last inequality yields $((p - 1)K_S - M)K_S \leq 0$ while $pK_S - M \equiv \text{num} C - K_S - M$ and $|C - K_S - M|$ has the effective divisor $\Delta$ which contains $N_0 \neq \emptyset$: this is impossible for $K_S$ is ample.

It remains to exclude $MK_S \leq pK_S^2/2$. If the Neron Severi group of $S$ is $\mathbb{Z}$, generated by $K_S$ and $p$ is odd, then the intersection of any two divisors on $S$ is an integral multiple of $K_S^2$, so this inequality implies $MK_S \leq (p - 1)K_S^2/2$, which is excluded by (3). Otherwise, use the assumption $\delta_0 \leq \delta < p(p - 2)K_S^2/4$ in (4), together with Hodge inequality; we get:

$$\frac{(MK_S)^2}{K_S^2} - (p - 1)MK_S + \frac{p(p - 2)}{4}K_S^2 > 0$$

from which it follows that either $MK_S < (p - 2)K_S^2/2$, absurd by (3), or $MK_S > \frac{p}{2}K_S^2$, which yields the required contradiction. \qed

We will show that our estimate on $\delta$ for having $\mathcal{V}_{C,\delta}$ smooth, of the expected codimension, is in fact sharp at least in some example.

Let us point out that, for surfaces in $\mathbb{P}^3$ of degree $d \geq 5$, we have $K_S = (d - 4)H$ (very) ample and we may apply the theorem to any curve $C \in |nH|$, for $n \geq 2(d - 4)$, getting:

**Proposition 2.4.** Let $S$ be a smooth surface of degree $d \geq 5$ in $\mathbb{P}^3$ with plane section $H$. If $C \in |nH|$, $n \geq 2d - 8$ has $\delta$ nodes and no other singularities, and $\delta < nd(n - 2d + 8)/4$, then $C$ corresponds to a smooth point of a component of the Severi variety $\mathcal{V}_{C,\delta}$ with the expected codimension $\delta$.

When $S$ is a general quintic surface in $\mathbb{P}^3$ and $C = pK_S = pH$, $p$ odd integer, theorem 2.2 gives:

**Proposition 2.5.** Let $S$ be a smooth surface of degree 5 in $\mathbb{P}^3$ with plane section $H$ and Picard group $\mathbb{Z}$. If $C \in |pH|$ ($p \geq 3$ and odd) has $\delta$ nodes and no other singularities, and $\delta < 5(p - 1)^2/4$, then the Severi variety $\mathcal{V}_{C,\delta}$ is smooth, with the (expected) codimension $\delta$ at $C$.

**Remark 2.6.** One may apply the previous procedure also to K3 or rational surfaces and get estimates on $\delta$ which implies that $\mathcal{V}_{C,\delta}$ is smooth, of the expected codimension. However, for these surfaces we get statements which are weaker than theorem 1.1 or example 1.2.

On the other hand, when $S$ is any smooth 5-ic surface in $\mathbb{P}^3$, we are going to provide examples that show that the numerical bounds for $\delta$ found in theorem 2.2 and proposition 2.5 are sharp.

**Remark 2.7.** Let $S$, $C$, $p$, $\delta$ be as in the statement of theorem 2.2, but assume now:

$$\delta = \frac{p(p - 2)}{4}K_S^2.$$

If the nodes of $C$ do not impose independent conditions to $|C|$, then we may go through the proof of the theorem, finding the rank 2 bundle $E$ associated to a subset $N_0 \subset N$ and the destabilizing divisor $M$. The only difference is that, in
(3), one gets only the weak inequality \(MK_S \geq pK_S^2/2\) so that, at the end of the argument, we cannot exclude the case \(MK_S = pK_S^2/2\).

If the equality holds, one deduces from (4) and (6):

\[
c_2(E(-M)) = \delta_0 - \frac{p(p-2)}{4}K_S^2 = \delta_0 - \delta.
\]

Since \(E(-M)\) has sections vanishing in codimension 2, then \(c_2(E(-M)) = 0\), so \(N_0 = N\) and \(E(-M)\) must split, thus

\[
E = \mathcal{O}_S(M) \oplus \mathcal{O}_S(C - M)
\]

and \(N\) is complete intersection of type \(M, C - M\) on \(S\).

3. A RELATED PROBLEM: GEOMETRICALLY LINEAR NORMALITY

Let \(C\) be a smooth, complete intersection curve in \(\mathbb{P}^3\). It is well known that \(C\) is arithmetically normal, so that, in particular, \(C\) cannot be the birational projection of a non-degenerate curve \(C' \in \mathbb{P}^r\) for \(r > 3\), that is, the embedding \(C \to \mathbb{P}^3\) does not factorize through any non-degenerate map \(C \to \mathbb{P}^r, r > 3\).

When \(C\) has singularities, this is no longer true (as we shall see later): there are complete intersection singular curves \(C \in \mathbb{P}^3\) whose normalization \(\tilde{C} \to C\) factors through a birational non-degenerate map \(\tilde{C} \to \mathbb{P}^r\) for some \(r > 3\). On the other hand, when the geometric genus of \(C\) is close enough to the arithmetic genus of \(C\), this factorization is impossible. So, one may look for bounds, for the number \(p_a(C) - g\), which exclude that \(C\) can be obtained as the birational projection of a non-degenerate curve lying in some higher dimensional projective space.

In fact, we shall look at the case of curves having only nodes for singularities and lying on some fixed smooth surface \(S\).

**Definition 3.1.** Let \(C\) be any reduced curve in \(\mathbb{P}^r\). We say that \(C\) is ’geometrically linearly normal’ if the normalization \(\tilde{C} \to C\) cannot be factored with a birational non-degenerate map \(\tilde{C} \to \mathbb{P}^R, R > r\), followed by a projection.

**(3.2) Problem.** Let \(S\) be a smooth surface of degree \(d\) in \(\mathbb{P}^3\); for any number \(n\) find a sharp bound \(\delta(d, n)\) such that if \(C \subset S\) is a complete intersection curve of type \(d, n\), having only \(\delta\) nodes as singularities and \(\delta \leq \delta(d, n)\), then \(C\) is geometrically linearly normal.

A partial answer to this problem can be still given using Reid’s construction as in the proof of theorem 2.2 and it turns out that, on a quintic surface, question (2.1) and problem (3.2) are in fact closely related.

To begin with, let us recall the following, well-known fact:

**Proposition 3.3.** Let \(S\) be a smooth surface of and let \(H\) be a very ample divisor on \(S\), such that for all \(m\) \(h^1\mathcal{O}_S(mH) = 0\); let \(C \in |nH|\) be an irreducible curve, having only nodes for singularities; call \(N\) the set of nodes of \(C\), \(\nu : \tilde{C} \to C\) the normalization and \(\tilde{N}\) the pull-back of \(N\) on \(\tilde{C}\).

For all integers \(m\) we have an isomorphism:

\[
H^0(S, \mathcal{I}_N(mH + K_S)) \cong H^0(\tilde{C}, \nu^*\mathcal{O}(mH + K_S)(-\tilde{N}))
\]
Proof. Call $\mu : \tilde{S} \to S$ the blowing up of $S$ along $N$ and let $B = \sum E_i$ be the exceptional divisor; then $\tilde{C}$ is isomorphic to (and will be identified with) a divisor on $\tilde{S}$ in the class $\mu^*C - 2B$. Since $\omega_{\tilde{C}}$ is cut by the divisors in $|\mu^*C + \mu^*K_S - B|$, the exact sequence:

$$0 \to O_{\tilde{S}}(\mu^*K_S + B + (m-n)\mu^*H) \to O_{\tilde{S}}(\mu^*C + \mu^*K_S - B + (m-n)\mu^*H) \to \omega_{\tilde{C}}((m-n)\mu^*H) \to 0$$

shows that the statement follows once we know that $h^1O_{\tilde{S}}(\mu^*K_S + B + (m-n)\mu^*H) = h^1O_{\tilde{S}}((n-m)\mu^*H)$ vanishes. One can prove this last vanishing, using the Leray spectral sequence and our assumptions on $S$. \hfill \square

We are going to apply the proposition only for smooth surfaces in $\mathbb{P}^3$ with $H =$ plane divisor, so that the assumptions on $S$ hold. In this case, we get for all $m$ an isomorphism

$$\frac{H^0(S, \mathcal{I}_N(mH))}{H^0(S, \mathcal{I}_C(mH))} \to H^0(\tilde{C}, \nu^*O(mH)(-\tilde{N})).$$

**Theorem 3.4.** Let $S$ be a smooth surface of degree $d$ in $\mathbb{P}^3$ and let $C \subset S$ be a complete intersection curve of type $d,n$, having only $\delta$ nodes as singularities.

Then $C$ is geometrically linearly normal if and only if the nodes $N$ of $C$ impose independent conditions to the linear system $|(n + d - 5)H|$, where $H$ is the plane divisor of $S$.

In particular, for $d = 5$, $C$ is geometrically linearly normal if and only if $N$ imposes independent conditions to $|C|$, i.e., if and only if the Severi variety $\mathcal{V}_{C,\delta}$ is smooth of codimension $\delta = \deg N$.

Proof. We use the notation of proposition 3.3. The canonical divisor of $\tilde{C}$ is $\omega = \nu^*((n + d - 4)H(-\tilde{N})$; on the other hand, it is clear by the definition that $C$ is geometrically linearly normal if and only if $h^0(\tilde{C}, \nu^*(H)) = 4$. Now observe that on $\tilde{C}$, $\nu^*(H)$ is residual to $\nu^*((n + d - 5)H(-\tilde{N}))$. Using the previous remark, by Riemann-Roch one computes:

$$h^0(\tilde{C}, \nu^*(H)) = nd - p_a(nH) - 1 + \delta + h^0(S, \mathcal{I}_N((n + d - 5)H)) - h^0(S, \mathcal{I}_C((n + d - 5)H)).$$

Putting $h^0(S, \mathcal{I}_N((n + d - 5)H)) = h^0(S, \mathcal{O}_S((n + d - 5)H)) - \delta + s$, with some computations one finds:

$$h^0(\tilde{C}, \nu^*(H)) = 4 + s$$

so that $C$ is geometrically linearly normal if and only if $s = 0$, i.e. if and only if $N$ imposes independent conditions to $|(n + d - 5)H|$. \hfill \square

We can use the same argument of theorem 2.2 to give a partial answer to problem (3.2).

**Theorem 3.5.** Let $S$ be a smooth surface of degree $d \geq 5$ in $\mathbb{P}^3$ and let $H$ be its plane divisor; let $C \subset |nH|$, $n \geq 2$, be an irreducible curve, having only $\delta$ nodes for singularities. If

$$\delta < \frac{nd(n-2)}{n-1}$$

then $\nu^*(C)$ is geometrically linearly normal if and only if $N$ imposes independent conditions to $|C|$. \hfill \square
then $C$ is geometrically linearly normal.

Proof. It is very similar to the proof of theorem 2.2. We show that if $C$ is not geometrically linearly normal, we get a contradiction.

Indeed, if this happens, then the set of nodes $N$ of $C$ does not impose independent conditions to the curves of $|(n + d - 5)H|$, by theorem 3.4. Take the subset $N_0 \subset N$ and the rank 2 vector bundle $E$ described in remark 2.3 and put $\delta_0 = \deg(N_0)$; in this case the exact sequence is

$$0 \to \mathcal{O}_S \to E \to \mathcal{I}_{N_0}((n - 1)H) \to 0$$

Here $c_1(E) = \mathcal{O}_S((n - 1)H)$, hence by assumption:

$$c_1(E)^2 - 4c_2(E) = (n - 1)^2d - 4\delta_0 \geq (n - 1)^2d - 4\delta > 0$$

so that $E$ is Bogomolov unstable; it follows that there exists a 'destabilizing' divisor $M$ for which $h^0(S, E(-M)) > 0$ and

$$(2M - c_1(E))H > 0 \quad \text{i.e. } MH > (n - 1 \text{ over } 2)d. \quad (7)$$

Taking $M$ maximal, we may further assume that a general section of $E(-M)$ vanishes in a locus of codimension 2, whose degree $c_2(E(-M))$ must be $\geq 0$; hence:

$$\delta_0 + M^2 - (n - 1)MH = c_2(E) + M^2 - Mc_1(E) = c_2(E(-M)) \geq 0.$$  

Now use again the assumption $\delta_0 \leq \delta < n(n - 2)d/4$ and observe that, by Hodge theorem, $dM^2 \leq (MH)^2$; putting all together, we arrive to the inequality:

$$\frac{(MH)^2}{d} - (n - 1)MH + \frac{n(n - 2)}{4}d > 0 \quad (8)$$

from which, since $MH < (n - 2)d/2$ yields a contradiction, one deduces that $MH > nd/2$.

Let us now go back to the exact sequence above. $h^0(S, \mathcal{O}_S(-M))$ is 0, for $-MH < -(n - 1)d/2 < 0$ by assumptions; thus $h^0(S, E(-M)) > 0$ implies $h^0(S, \mathcal{I}_{N_0}((n - 1)H - M)) > 0$, that is, there exists a divisor $\Delta$ in the linear system $|(n - 1)H - M|$, which contains $\delta_0$ nodes of the curve $C$. $\Delta$ cannot contain $C$ as a component, for $(-H - M)H < 0$. It follows, by Bezout,

$$(n - 1)H - M)(nH) \geq 2\delta_0.$$  

Putting all together, one finally gets:

$$\frac{(MH)^2}{d} - \frac{3n - 2}{2}MH + \frac{n(n - 1)d}{2} \geq 0.$$  

Since by assumption $n \geq 2$, then we get that either $MH \leq nd/2$, which is excluded by (8), or $MH \geq (n - 1)d$; but this last inequality yields $(n - 1)H - M)H \leq 0$ while $|(n - 1)H - M|$ has the effective divisor $\Delta$ which contains $N_0 \neq \emptyset$, a contradiction. □

Using the same arrangement of proposition 2.5, one can improve the previous statement when $S$ is a general smooth quintic surface.
**Proposition 3.6.** Let $S$ be a smooth quintic surface in $\mathbb{P}^3$, with Picard group $\mathbb{Z}$. Let $C \in |nH|$ be a curve with only $\delta$ nodes as singularities. Assume $n$ odd and
\[ \delta < \frac{5(n - 1)^2}{4}. \]
Then $C$ is geometrically linearly normal.

**Remark 3.7.** In the hypothesis of theorem 3.4, if $5 \leq n < d$ and $C$ is also contained in a smooth surface $S'$ of degree $n$, then one may interchange the roles of $n, d$ and prove that $C$ is geometrically linearly normal in the wider range:
\[ \delta < \frac{nd(d - 2)}{4}. \]

4. **THE EXAMPLES**

Here we show the sharpness of the bounds in theorem 2.2 and theorem 3.5, for the case of a general smooth quintic surface $S \subset \mathbb{P}^3$.

From now on, in this section, let $S$ be a general smooth 5-ic surface of $\mathbb{P}^3$, with Picard group $\mathbb{Z}$, generated by the plane divisor $H$. Let $C$ be a curve in the linear system $|nH|$, $n \geq 2$, with $\delta$ nodes for singularities.

From theorem 2.2, we know that when
\[ \delta < \frac{5n(n - 2)}{4} \]
then the Severi variety $\mathcal{V}_{nH,\delta}$ is smooth of codimension $\delta$. When $n$ is odd, by proposition 2.5 the same conclusion holds when
\[ \delta < \frac{5(n - 1)^2}{4}. \]

We show with examples that these bounds are sharp. Thus we are going to produce curves $C$ as above, with $5n(n - 2)/4$ nodes for $n$ even or $5(n - 1)^2/4$ nodes, for $n$ odd, such that the nodes do not impose independent conditions to the linear system $|nH|$.

By theorem 3.4, such a curve $C$ is a birational projection of some curve $C'$ lying in $\mathbb{P}^4$.

**Example 4.1.** $n$ even, $n = 2m$, $m \geq 3$.

Let $X$ be a general complete intersection surface of type $2, m$ in $\mathbb{P}^4$ and let $X'$ be a general projection of $X$ in $\mathbb{P}^3$. $X'$ has a double curve $Y$ of degree $m^2 - m$, as one can see taking general hyperplane sections of $X$ and $X'$.

Let $\tilde{S}$ be a general cone in $\mathbb{P}^4$, with vertex $V$, over our general 5-ic surface $S$ and call $\tilde{C}$ the intersection of $\tilde{S}$ with a general complete intersection $X$ as above. Put $C$ = projection of $\tilde{C}$ from $V$; $C$ has degree $5n$ and it is complete intersection of $S$ and $X'$ in $\mathbb{P}^3$, so it belongs to the linear system $|nH|$ on $S$; moreover $C$ has nodes in the points of $S \cap Y$, so it has a set $N$ of $\delta = 5(m^2 - m) = 5n(n - 2)/4$ nodes and no other singularities.

Since $C$ is not geometrically linearly normal, it follows from theorem 3.4 that $N$ cannot impose independent conditions to $|nH|$, so that $\mathcal{V}_{nH,\delta}$ is not smooth of codimension $\delta$, in a neighbourhood of $C$. 
Proposition 4.2. The curve $C$ constructed in the previous example is a singular point of $\mathcal{V}_{nH,\delta}$, which is generically smooth, of the expected codimension $\delta$.

Proof. The previous construction, in fact, together with the proof of theorem 3.4, shows that the tangent space of $\mathcal{V}_{nH,\delta}$ at $C$, that is $H^0(S,\mathcal{I}_N(nH))/\mathcal{K}$, has codimension $\delta - 1$ in the tangent space of $|nH|$ at $C$: indeed $C$ is the projection of a smooth, arithmetically normal curve in $\mathbb{P}^4$. Hence $h^1(S,\mathcal{I}_N(nH)) = 1$.

Let $C'$ be a curve in a neighbourhood of $C$ in $\mathcal{V}_{nH,\delta}$ for which the set of nodes $N'$ does not impose independent conditions to $|nH|$. Then by semicontinuity $h^1(S,\mathcal{I}_{N'}(nH)) = 1$, so by 3.4 again, $C'$ is the projection of a curve $\tilde{C}'$ in $\mathbb{P}^4$ and $\tilde{C}'$ lives in a neighbourhood of $\tilde{C}$ in the Hilbert scheme of $\mathbb{P}^4$. It follows that also $\tilde{C}'$ must be a smooth complete intersection of the cone $\tilde{S}$ with some complete intersection surface of type $2, m$.

Let us compute, now, the dimension of the subvariety $\mathcal{V}$ of $\mathcal{V}_{nH,\delta}$, formed by curves $C'$ at which the tangent space of $\mathcal{V}_{nH,\delta}$ has codimension $\delta - 1$, in a neighbourhood of $C$; all these curves are projection of a complete intersection $\tilde{C}'$ of a cone $\tilde{S}$ over $S$ with a complete intersection surface of type $2, m$ in $\mathbb{P}^4$. If we fix the cone $\tilde{S}$, then we know that such curves $\tilde{C}'$ fill a variety of dimension at most

$$h^0(\mathcal{N}_{\tilde{C}'},\mathcal{S}) = h^0(\tilde{C},\mathcal{O}_{\tilde{C}}(2) \oplus \mathcal{O}_{\tilde{C}}(m)) = 14 + (5m^2 - 10m + 14) = 5m^2 - 10m + 28,$$

so if we let also $\tilde{S}$ move, varying the vertex, then we get a subvariety of dimension at most $5m^2 - 10m + 32$ in the Hilbert scheme of $\mathbb{P}^4$.

Since $\mathcal{V}_{nH,\delta}$ has dimension at least

$$h^0(S,\mathcal{O}_S(nH)) - 1 - \delta = 5m^2 + 4$$

and $5m^2 + 4 > (5m^2 - 10m + 32)$ in our range, then a general element $C'' \in \mathcal{V}_{nH,\delta}$ does not arise from this construction. It follows that such $C''$ cannot be the projection of a non degenerate curve in $\mathbb{P}^4$, thus by theorem 3.4, its set of nodes $N''$ imposes independent conditions to $|nH|$; then the tangent space of $\mathcal{V}_{nH,\delta}$ at a general point has the expected codimension $\delta$, so that $\mathcal{V}_{nH,\delta}$ is generically smooth of the expected codimension $\delta$ but it is singular at the locus $\mathcal{V}$ constructed above. □

Remark 4.3. In the previous construction, the set of nodes $N$ of $C$ is the intersection of $S$ with the singular locus of $X'$; one can show that, accordingly with remark 2.7, the set $N$ is complete intersection, in $\mathbb{P}^3$, of surfaces of degree $5, m, m - 1$.

Example 4.4. $n$ odd, $n = 2m + 1, m \geq 3$.

We start here with a surface $X \subset \mathbb{P}^4$ which is residue to a plane $\pi$ with respect to a general complete intersection of type $2, m + 1$; $X$ is an arithmetically Cohen-Macaulay surface. Let $X'$ be a general projection of $X$ in $\mathbb{P}^3$. $X'$ has a double curve $Y$ of degree $m^2$, as one can see taking general hyperplane sections of $X$ and $X'$.

Let $\tilde{S}$ be a general cone in $\mathbb{P}^4$, with vertex $V$, over our general 5-ic $S$ and call $\tilde{C}$ the intersection of $S$ with a general surface $X$ as above. Let $C$ be the projection of $\tilde{C}$ from $V$; $C$ has degree $5n$ and it is complete intersection, in $\mathbb{P}^3$, of $S$ and $X'$, so it belongs to the linear system $|nH|$ on $\tilde{S}$; moreover $C$ has nodes in the points of $S \cap Y$, so it has as set $N$ of $\delta = 5m^2 = 5(n - 1)^2/4$ nodes and no other singularities.

Since $C$ comes from a smooth curve in $\mathbb{P}^4$, by theorem 3.4 $N$ cannot impose independent conditions to $|nH|$, so that $\mathcal{V}_{nH,\delta}$ is not smooth, of codimension $\delta$ in a neighbourhood of $C$. 

**Proposition 4.5.** For \( m \geq 5 \) the curve \( C \) constructed in example 4.4 is a singular point of \( V_{nH,\delta} \), which is generically smooth, of the expected codimension \( \delta \).

**Proof.** The previous construction, together with the proof of theorem 3.4, shows, arguing as above, that the tangent space of \( V_{nH,\delta} \) at \( C \) has codimension \( \delta - 1 \) in the tangent space of \( |nH| \) at \( C \).

Let \( C' \) be a curve in a neighbourhood of \( C \) in \( V_{nH,\delta} \) for which the set of nodes \( N' \) does not impose independent conditions to \( nH \). Then by semicontinuity \( h^1(S, I_{N'}(nH)) = 1 \), so by 3.4 again, \( C' \) is the projection of a curve \( \tilde{C}' \) in \( \tilde{S} \) and \( \tilde{C}' \) lives in a neighbourhood of \( \tilde{C} \) in the Hilbert scheme of \( \tilde{S} \). Let \((X \cup \pi) \cap \tilde{S} = \tilde{C} \cup \gamma\), where \( \gamma \) is a plane quintic.

We have an exact sequence on \( \tilde{C} \):

\[
0 \rightarrow N_{\tilde{C},\tilde{S}} \rightarrow O_{\tilde{C}}(2) \oplus O_{\tilde{C}}(m+1) \rightarrow T \rightarrow 0
\]

where \( T \) is a torsion sheaf. Therefore, since \( \tilde{C} \) is arithmetically normal, we get:

\[
h^0 N_{\tilde{C},\tilde{S}} \leq h^0(\tilde{C}, O_{\tilde{C}}(2)) + h^0(\tilde{C}, O_{\tilde{C}}(m+1)) \leq 14 + (m+1)5(2m+1) + 1 - p_a(\tilde{C}) + h^1(\tilde{C}, O_{\tilde{C}}(m+1)).
\]

Now observe that \( h^1(\tilde{C}, O_{\tilde{C}}(m+1)) - 1 \) equals the dimension of the linear system cut on \( \tilde{C} \) by the quadrics through \( \gamma \), therefore \( h^1(\tilde{C}, O_{\tilde{C}}(m+1)) \leq 9 \) and since \( p_a(\tilde{C}) = 5m^2 + 15m + 6 \), we obtain \( h^0 N_{\tilde{C},\tilde{S}} \leq 5m^2 + 23 \).

Therefore \( \tilde{C} \) belongs to a component of the Hilbert scheme of \( \tilde{S} \) of dimension at most \( 5m^2 + 23 \); if we let also \( \tilde{S} \) move, varying the vertex, then we obtain a family of curves of dimension at most \( 5m^2 + 27 \). Thus, this is an upper bound for the dimension of the subvariety \( V \) of \( V_{nH,\delta} \) formed by curves \( C' \) at which the tangent space of \( V_{nH,\delta} \) has codimension \( \delta - 1 \). Since \( V_{nH,\delta} \) has dimension at least

\[
h^0(S, O_S(nH)) - 1 - \delta = 10m^2 + 5m + 3 - \delta = 5m^2 + 5m + 4
\]

and \( 5m^2 + 5m + 4 > 5m^2 + 27 \) in our range, then the conclusion follows. \( \square \)

**Remark 4.6.** The two previous examples can be easily arranged on general smooth surfaces \( S \) of degree \( d \geq 5 \) in \( \mathbb{P}^3 \) to provide irreducible, non geometrically linearly normal curves \( C \subset S \) which are complete intersection of type \( d, n \) and have exactly

\[
\delta = \frac{nd(n-2)}{4}
\]

nodes and no other singularities.

In particular, the bound in theorem 3.5 is sharp for all \( d \geq 5 \).

**Proposition 4.7.** Let \( S \) be a general, smooth surface of degree 5 in \( \mathbb{P}^3 \). Assume:

\[
\delta \leq \begin{cases} 
5n(n-2)/4 & \text{for } n \text{ even} \\
5(n-1)^2/4 & \text{for } n \text{ odd}
\end{cases}
\]

Then \( V_{nH,\delta} \) is non empty and it has at least one generically smooth component of codimension \( \delta \) in \( \mathbb{P}(|nH|) \).

**Proof.** For \( \delta = n(n-2)/4 \), \( n \) even, or \( \delta = (n-1)^2/4 \), \( n \) odd, the statement follows by proposition 4.2 and proposition 4.4.
For smaller $\delta$, one can argue as in the proof of theorem 1.1. For $n$ even, start with a general curve $C \in V_{nH,n(n-2)/4}$; by proposition 4.2 or proposition 4.5, the nodes of $C$ impose independent conditions to the curves of $|nH|$, so we may smooth them independently, one by one, getting at any step curves with only nodes for singularities. The conclusion follows from theorem 2.2.

A similar argument works for $n$ odd. □

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