Bending of an Elastically Restrained Elliptical Plate under the Combined Action of Lateral Load and In-Plane Force*

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The study on the bending of an elliptical plate clamped rigidly against deflection and restrained elastically against rotation along its periphery, subjected to the uniform lateral load and in-plane force simultaneously, is performed by introducing the elliptical coordinates. The analytical solution satisfying perfectly the differential equation of deflection and the boundary conditions is exactly derived in the form of Mathieu function series. The expressions for the bending moments are also derived rigorously. The deflection and bending moments obtained here coincide with those for a simply supported elliptical plate and the perfectly clamped elliptical plate when the rotational spring stiffness is zero and infinity, respectively. A limiting case of a circular plate is discussed in detail. The effects of the in-plane force and the rotational spring stiffness on the deflection and the bending moments are calculated numerically and are presented in tables and figures.

Key Words: Bending, Elliptical Plate, Lateral Load, In-Plane Force, Elastic Edge-Restraint, Mathieu Function

1. Introduction

Numerous works on the bending of clamped or simply supported elliptical plates under the action of various external forces have been reported during the past years(1)-(10). A plate-bending problem, in which the plate edge is restrained elastically, is also very important in the fields of engineering. However, such a problem concerning the elliptical plates has been hardly solved.

The present paper studies the bending of an elliptical plate supported rigidly against deflection and subjected to the reaction moment proportional to the rotational angle along its periphery. The plate undergoes a combined action of uniform lateral load distributed over its entire surface and in-plane force distributed in its middle plane. By assuming that the deflection of a plate is very small in comparison with its thickness, the analysis is made in the elliptical coordinate system. The general solution of the differential equation for the deflection surface is obtained in the form of an infinite series of Mathieu functions which are the solutions of the deflection equation expressed in terms of the elliptical coordinates and the final solution satisfying completely the boundary conditions is obtained by making use of the orthogonality of Mathieu functions(9), (10). The intensities of bending moments are also formulated rigorously in the form of infinite series involving the Mathieu and modified Mathieu functions and their derivatives. Upon regarding a circle as a special kind of ellipse, the expressions of the deflection surface and the moment intensities for the bending of a circular plate under the combined action of uniform lateral load and in-plane force are derived as a special case from the present analytical solutions for the elliptical plate.

To illustrate the deflection surface and the bending moment intensities, numerical calculations are performed for various restraint parameters, in-plane force parameters and aspect ratios of elliptical plate. Furthermore, the numerical values of buckling load in the previous paper(11) are corrected.

2. Basic Equations and Boundary Conditions

The problem on the bending of an elliptical plate subjected to the combined action of the uniform lateral load p over its entire surface and the uniform in-plane compressive force T along its middle plane as shown in Fig. 1 can be rigorously analyzed by introducing the elliptical coordinates (ξ, η), which are related to the rectangular coordinates (x, y) by the relations

\[ x = c \cosh \xi \cos \eta, \quad y = c \sinh \xi \sin \eta, \]

(1)
where \( c \) is the semi-focal length as shown in Fig. 2. From the above equations, one obtains a family of confocal ellipses and hyperbolas with the common foci for various values of \( \xi \) and \( \eta \), respectively. The differential equation for the deflection \( w \) can be written, by using the flexural rigidity \( D \) which is equal to \( Eh^3/[12(1-\nu^2)] \) with Young's modulus \( E \), plate thickness \( h \) and Poisson's ratio \( \nu \), as

\[
\nabla^2(D\nabla^2 + T)w = p,
\]

where \( \nabla^2 = (1/q^2)(\partial^2/\partial\xi^2 + \partial^2/\partial\eta^2) \) with \( q^2 = c^2(\cosh 2\xi - \cos 2\eta)/2 \). The bending moment intensities \( M_\xi \) and \( M_\eta \) are given in terms of the elliptical coordinates as

\[
M_\xi = -\frac{Dc^2}{2q^2} \left[ (\cosh 2\xi - \cos 2\eta) \left( \frac{\partial^2 w}{\partial \xi^2} + \nu \frac{\partial^2 w}{\partial \eta^2} \right) \right] - (1-\nu) \left( \sinh 2\xi \frac{\partial w}{\partial \xi} - \sin 2\eta \frac{\partial w}{\partial \eta} \right),
\]

\[
M_\eta = -\frac{Dc^2}{2q^2} \left[ (\cosh 2\xi - \cos 2\eta) \left( \nu \frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \eta^2} \right) \right] + (1-\nu) \left( \sinh 2\xi \frac{\partial w}{\partial \xi} - \sin 2\eta \frac{\partial w}{\partial \eta} \right).
\]

The boundary conditions of the plate under consideration are given, along its periphery \( \xi = \xi_o \), as

\[
w|_{\xi=\xi_o} = 0, \quad M_\xi|_{\xi=\xi_o} = (\beta/q)(\partial w/\partial \xi)|_{\xi=\xi_o},
\]

where the symbol \( \beta \) denotes the rotational spring stiffness.

3. Expressions for the Deflection and the Bending Moments

Since the deflection surface of an elliptical plate under consideration is symmetrical about both axes of ellipse, the general solution of Eq. (2) is obtainable as a Mathieu function series in terms of the Mathieu function \( ce_{2m} \) and the modified Mathieu function \( Ce_{2m} \):

\[
w = \sum_{m=0}^{\infty} \left\{ \frac{pc^2}{8DA} \left[ 2A_{0m}^{(2m)} \cosh 2\xi + A_{2m}^{(2m)} \right] \right\} + \sum_{m=0}^{\infty} C_{2m}^{(2m)} ce_{2m}(\xi, q),
\]

\[
e_u = 2(u = 0), \quad 1(u \neq 0), \quad u = 0, 1, 2, \cdots,
\]

in which

\[
ce_{2m}(\xi, q) = \sum_{n=0}^{\infty} A_{2n}^{(2m)} \cos 2m\eta,
\]

\[
Ce_{2m}(\xi, q) = \sum_{n=0}^{\infty} A_{2n}^{(2m)} \cosh 2m\eta,
\]

\[
q = k^2 = \lambda c^2/4, \quad \lambda = T/D,
\]

and the symbols \( C_{2m} \) and \( C_{2m} \) are the unknown constants. In these series the \( A's \) are functions of \( q \). It is noted that when the in-plane force is tensile, \( T < 0 \) and therefore \( \lambda < 0, \quad k^2 < 0 \) and \( q < 0 \). Detailed discussion for the case of \( T > 0 \) will be given in section 4.

Here, so as to make \( w \) satisfy the second of the boundary conditions (5), let us introduce the formula

\[
\frac{1}{\sqrt{\nu^2 + \rho^2}} = \sum_{n=0}^{\infty} L_n(a, \alpha) \cos nx,
\]

where

\[
L_n(a, \alpha) = 2^{n-2} \mu_{\alpha} e^{-(\alpha + n)/2} F(\alpha, n + \alpha, n + 1; e^{-2a}) \Gamma(\alpha + n)/[\Gamma(\alpha)],
\]

\[
\mu_n = 1(n = 0), \quad 2(n \neq 0), \quad n = 0, 1, 2, \cdots,
\]

with the gamma function \( \Gamma \) and the hypergeometric function \( F \). Substituting Eq. (6) into Eq. (5) and consecutively making use of the orthogonality of Mathieu functions, one obtains the simultaneous equations

\[
\frac{pc^2}{8DA} \left[ 2A_{0m}^{(2m)} \cosh 2\xi + A_{2m}^{(2m)} \right] + \sum_{n=0}^{\infty} C_{2n}^{(2m)} \sinh 2\xi_n e_n \cosh 2m\xi_n = 0, \quad m = 0, 1, 2, \cdots,
\]

\[
\frac{pc^2}{\sqrt{2}D} \left[ \frac{2}{A^{(2m)}} \sinh 2\xi_n + \sum_{n=0}^{\infty} C_{2n}^{(2m)} e_n(2a) \sinh 2m\xi_n \right] + \sum_{j=0}^{\infty} \psi_j(2a) L_j(2\xi_n, 3/2)
\]

\[
\times \left[ \frac{pc^2}{2DA} A_{0m}^{(2m)} \cosh 2\xi_n + \sum_{n=0}^{\infty} C_{2n}^{(2m)} e_n(4a^2) \cosh 2m\xi_n \right] \cosh 2\xi_n
\]

\[
- (1-\nu) \sinh 2\xi_n \left[ \frac{pc^2}{2DA} A_{0m}^{(2m)} \sinh 2\xi_n + \sum_{n=0}^{\infty} C_{2n}^{(2m)} e_n(2a) \sin 2m\xi_n + C_{2n}^{(2m)} e_n(2a) \sinh 2m\xi_n \right] \cosh 2\xi_n
\]

\[
- \sum_{l=0}^{\infty} \frac{\psi_l(2a)}{2a} \left[ \frac{pc^2}{DA} A_{0m}^{(2m)} \cosh 2\xi_n \right] + \sum_{m=0}^{\infty} C_{2m}^{(2m)} ce_{2m}(\xi, q) + \sum_{m=0}^{\infty} C_{2m}^{(2m)} ce_{2m}(\xi, q),
\]

where \( \psi_l(2a) \) are functions of \( l \).
Substitution of Eq. (14) into Eq. (12) yields a system of

denotes the $u^2$ of

$u^2 = \sum_{s} u^2 e_{mu}$

where

$e_{mu} = \frac{p c^2}{8 D 1} g_m, \ m = 0, 1, 2, \cdots$, (15)

Substitution of Eq. (14) into Eq. (12) yields a system of infinite simultaneous linear equations

$\sum_{m=0}^{\infty} C_{2n} e_{mu} = \frac{p c^2}{8 D 1} g_m, \ m = 0, 1, 2, \cdots$, (16)

g_m = \frac{\beta c^2}{(\sqrt{2D})} \left[ 4 A_0^{(2m)} \sinh 2 \xi_0 - (2 A_0^{(2m)} \cosh 2 \xi_0) \right]

The solutions of Eq. (15) can be written, formally, as

$C_{2u} = \left[ \frac{p c^2}{8 D A_1} \right] \left[ G_{2u}(\xi, q) \right], \ G_{2u}(\xi, q) = A^{(u)} / A, \ u = 0, 1, 2, \cdots$ (18)

It is noted that the infinite determinantal equation $A = 0$ gives the buckling condition of an elastically restrained elliptical plate (11). Substituting $C_{2m}$ and $C_{2u}$ into Eq. (6), one obtains

$w = \frac{p c^2}{8 D 1} \sum_{m=0}^{\infty} H_{2m}^{(1)}(\xi, q) e_{mu}(\eta, q)$. (20)

The expression of $H_{2m}^{(1)}(\xi, q)$ will be given together with the expressions of $H_{2m}^{(2)}(\xi, q)$ and $H_{2m}^{(3)}(\xi, q)$, which will appear in the following equations.

Substitution of Eq. (20) into Eqs. (3) and (4) yields

$M_{\xi} = -\frac{p}{4 A (\cosh 2 \xi - \cos 2 \eta)} \sum_{m=0}^{\infty} \left[ H_{2m}^{(3)}(\xi, q) \right]$

$- \nu H_{2m}^{(1)}(\xi, q) (a_m - 2k^2 \cos 2 \eta) \cosh 2 \xi - \cos 2 \eta) \sinh 2 \xi e_{2m}(\eta, q)$

$H_{2m}^{(3)}(\xi, q) = 2 A_0^{(2m)} \sin \xi - 2 A_0^{(2m)} \cosh 2 \xi$

$+ A_2^{(2m)} \left[ C_{2m}(\xi, q) / C_{2m}(\xi, q) \right] - \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \eta^{(j)}(2 \xi_0, 3 / 2) \left[ \left[ A_2^{(2m)} e_{mu} \cosh 2 \xi_0 (4u^2 - a_{2m}) \cosh 2 \xi_0 - 2 \right] \cosh 2 \xi_0 - 2 \right]$

$\sum_{m=0}^{\infty} \eta^{(j)}(2 \xi_0, 3 / 2) \left[ A_2^{(2m)} e_{mu} \cosh 2 \xi_0 (4u^2 - a_{2m}) \cosh 2 \xi_0 - 2 \right]$

$\sum_{m=0}^{\infty} \eta^{(j)}(2 \xi_0, 3 / 2) \left[ A_2^{(2m)} e_{mu} \cosh 2 \xi_0 (4u^2 - a_{2m}) \cosh 2 \xi_0 - 2 \right]$

$\sum_{m=0}^{\infty} \eta^{(j)}(2 \xi_0, 3 / 2) \left[ A_2^{(2m)} e_{mu} \cosh 2 \xi_0 (4u^2 - a_{2m}) \cosh 2 \xi_0 - 2 \right]$

$\sum_{m=0}^{\infty} \eta^{(j)}(2 \xi_0, 3 / 2) \left[ A_2^{(2m)} e_{mu} \cosh 2 \xi_0 (4u^2 - a_{2m}) \cosh 2 \xi_0 - 2 \right]$
then

\[ \sum_{m=0}^{\infty} G_{2m}(\xi, q) A_{2m}(2m) e_{m} \cosh 2u \xi e_{m} \]

\[ \times \left[ 4u^2 \cosh 2u \xi' / \cosh 2u \xi e_{m} \right] \]

\[ - (a_{2m} - 2k^2 \cosh 2\xi e_{m} / \cosh 2u \xi e_{m}) \].

(25)

The prime(\(\prime\)) attached to \(c_{e_{m}}\) represents the \(\eta\)-derivative. It is added here that the expressions derived easily by putting \(\beta = 0\) and \(\beta = \infty\) in Eqs. (20) – (22) coincide with those given in Ref. (10) treating a simply supported elliptical plate and Ref. (9) treating a perfectly clamped elliptical plate, respectively.

4. Bending of the Elliptical Plate under In-plane Tension

When the in-plane force is tensile, namely \(T < 0\), one has \(\lambda < 0\), \(k^2 < 0\) and \(q < 0\). Upon putting \(q = -q(\bar{q} > 0), \quad k^2 = -k^2(k^2 > 0), \quad \lambda = \lambda(\bar{\lambda} > 0)\),

\[ a_{2m}(q) \rightarrow a_{2m}(\bar{q}), \quad A_{2m}(q) \rightarrow A_{2m}(\bar{q}), \]

\[ c_{e_{m}}(\eta, q) \rightarrow c_{e_{m}}(\eta, -\bar{q}), \quad C_{e_{m}}(\xi, q) \rightarrow C_{e_{m}}(\xi, -\bar{q}) \]

and, according to the theory of Mathieu function\(^{(9),(10),(12)}\),

\[ a_{2m}(\bar{q}) = a_{2m}(q), \quad A_{2m}(\bar{q}) = (1)^{\nu} A_{2m}(\bar{q}), \]

\[ c_{e_{m}}(\eta, -\bar{q}) = (1)^{\nu} c_{e_{m}}(\eta, \bar{q}), \quad C_{e_{m}}(\xi, q) \rightarrow C_{e_{m}}(\xi, -\bar{q}) \]

(27)

Therefore the deflection \(w\) and the bending moments \(M_{x}\) and \(M_{y}\) having the negative parameter \(q\) can be rewritten using the characteristic number, the Fourier coefficient, the Mathieu function and the modified Mathieu function, which are the functions of positive parameter \(q\) as given by the above expressions. Theory on the Mathieu function having the positive parameter is detailed in Mclachlan’s book\(^{(12)}\).

5. A Circular Plate

When the two foci of an ellipse become very close, namely, the semi-focal length \(c \rightarrow 0\), one has \(k \rightarrow 0\) and \(q \rightarrow 0\) from Eq. (8). Moreover, as \(c \rightarrow 0, \xi \rightarrow \infty\) and \(2k^2 \cosh 2\xi \rightarrow ab^2\) in the polar coordinates \((b, \theta)\). Consequently\(^{(12)}\),

\[ A_{2m}(2m) \rightarrow 1/\sqrt{2}(m = u = 0), \quad 1(m = u \neq 0), \quad 0(m \neq u), \]

\[ a_{2m} \rightarrow 4m^2. \]

\[ c_{e_{m}}(\eta, q) \rightarrow 1/\sqrt{2}(m = 0), \quad \cos 2m\theta (m \neq 0), \]

\[ C_{e_{m}}(\xi, q) \rightarrow \gamma_{2m} J_{2m}(\sqrt{a} b), \]

\[ C_{e_{m}}(\xi, q) \rightarrow \gamma_{2m} b J_{2m}(\sqrt{a} b), \quad m, u = 0, 1, 2, \ldots, \]

(29)

where \(J_{2m}\) denotes the Bessel function of order 2\(m\)\(^{(14)}\), the prime(\(\prime\)) attached to \(J_{2m}\) denotes the \(b\)-derivative, and \(\gamma_{2m}\) is the constant multiplier. Then, the deflection of a circular plate under the combined action of uniform lateral load and in-plane force is derived from Eq. (20) as

\[ w = \frac{pb^2}{2D\lambda} \times \left\{ \frac{(1 + \nu + \beta b_o/D) J_0(\sqrt{a} b_o) - J_0(\sqrt{a} b_o)}{\sqrt{a} b_o J_0(\sqrt{a} b_o) - (1 - \nu - \beta b_o/D) J_0(\sqrt{a} b_o)} \right\} \]

\[ - \frac{1}{2} \left[ 1 - \frac{b}{b_o} \right]^2 \right\}. \]

(30)

It should be noted that the above limiting equation is independent of \(\theta\), that is, only the first term in the series Eq. (20) is retained. Similarly, the bending moments become, from Eqs. (21) and (22),

\[ M_{y} = -\frac{p}{2\lambda} \left\{ 1 + \nu + \frac{b_o}{b} \right\} \times \frac{(1 + \nu + \beta b_o/D) J_0(\sqrt{a} b_o) - J_0(\sqrt{a} b_o)}{\sqrt{a} b_o J_0(\sqrt{a} b_o) - (1 - \nu - \beta b_o/D) J_0(\sqrt{a} b_o)} \}

(31)

\[ M_{y} = -\frac{p}{2\lambda} \left\{ 1 + \nu + \frac{b_o}{b} \right\} \times \frac{(1 + \nu + \beta b_o/D) J_0(\sqrt{a} b_o) - J_0(\sqrt{a} b_o)}{\sqrt{a} b_o J_0(\sqrt{a} b_o) - (1 - \nu - \beta b_o/D) J_0(\sqrt{a} b_o)} \}

(32)

When the in-plane force is tensile, \(T < 0\) and therefore \(\sqrt{a} = i/\sqrt{\lambda}\), \(J_0(\sqrt{a} b_o) = i I_1(\sqrt{a} b_o)\),

\[ J_1(\sqrt{a} b_o) = i I_1(\sqrt{a} b_o), \quad (i = \sqrt{-1}), \]

(33)

where the symbols \(I_0\) and \(I_1\) denote the modified Bessel functions of orders 0 and 1, respectively\(^{(14)}\). Substitution of Eq. (33) into Eqs. (30) – (32) yields readily the deflection surface and the moment intensities for the in-plane tension.

When the in-plane force \(T \to 0\), application of the approximate formulae

\[ J_0(x) \sim 1 - x^2/2 + x^4/(2^4(2!)), \]

\[ J_1(x) \sim x/2 - x^3/(2^2(2!)) \]

for small \(x\) to Eqs. (30) – (32) yields

\[ w = \left[ \frac{pb^2}{(64D)} \right] \left[ 1 - (b/b_o)^2 \right] \left[ (5 + \nu + \beta b_o/D) \right] \]

\[ + (1 + \nu + \beta b_o/D) - (3 + \nu)(b/b_o)^2 \right\}. \]

(35)

\[ M_{y} = \left( \frac{pb^2}{16} \right) \left[ (1 + \nu)(3 + \nu + \beta b_o/D)/(1 + \nu + \beta b_o/D) \right] \]

\[ - (3 + \nu)(b/b_o)^2 \right\}. \]

(36)

\[ M_{y} = \left( \frac{pb^2}{16} \right) \left[ (1 + \nu)(3 + \nu + \beta b_o/D)/(1 + \nu + \beta b_o/D) \right] \]

\[ - (3 + \nu)(b/b_o)^2 \right\}. \]

(37)

The expressions obtained immediately by putting \(\beta = 0\) and \(\beta = \infty\) in Eqs. (30) – (32) and (35) – (37) coincide with those given in Refs. (10) and (9), respectively.
6. Numerical Calculation and Discussion

In numerical calculation for an elastically restrained elliptical plate having the aspect ratio \( a_0/b_0 \), the deflection \( w \), the bending moments \( M_\xi \) and \( M_\eta \) were nondimensionalized in terms of the semi-minor axial-length \( b_0 \) as

\[
\delta(\xi, \eta) = w(\xi, \eta)/(pb_0^5/D), \quad \alpha(\xi, \eta) = M_\xi(\xi, \eta)/(pb_0^3), \\
\beta(\xi, \eta) = M_\eta(\xi, \eta)/(pb_0^3).
\]

(38)

Upon letting \( a \) and \( b \) be the distances from the center of the plate to any point on its major axis and that on its minor axis, respectively, the semi-major axial-length ratio \( a/a_0 \) and the semi-minor axial-length ratio \( b/b_0 \) can be written in terms of variables \( \xi \) and \( \eta \) as

\[
a/a_0 = \begin{cases} 
(c/a_0)\cos \eta & \text{for } 0 \leq a \leq c \\
(c/a_0)\cos \xi & \text{for } c \leq a \leq a_0 
\end{cases},
\]

and

\[
b/b_0 = (c/b_0)\sinh \xi \quad \text{for } 0 \leq b \leq b_0
\]

with \( c^2 = a_0^2 - b_0^2 \). In case of Poisson’s ratio \( \nu = 0.3 \), the numerical calculations of the following quantities were performed:

\[
\delta_a = \delta(0, \eta) \quad \text{for } 0 \leq a \leq c, \quad \delta(\xi, 0) \quad \text{for } c \leq a \leq a_0,
\]

\[
\delta_b = \delta(\xi, \pi/2); \quad \delta_c = \delta(0, \pi/2),
\]

\[
\zeta_a = \alpha(0, \eta) \quad \text{for } 0 \leq a \leq c, \quad \beta(\xi, 0) \quad \text{for } c \leq a \leq a_0,
\]

\[
\zeta_b = \alpha(\xi, \pi/2); \quad \zeta_c = \alpha(0, \pi/2).
\]

The subscripts \( a, b, \) and \( c \) signify the major axis, the minor axis, the end of axis and the center, respectively. So, it is noted that \( \delta_a \) and \( \delta_b \) represent the deflections along the major axis, that is, \( a \)-axis (\( x \)-axis) and the minor axis, that is, \( b \)-axis (\( y \)-axis), respectively. The \( \zeta_a \) represents the dimensionless bending moment per unit length of a plate section perpendicular to the major axis. The \( \zeta_b \) and \( \zeta_c \) represent the dimensionless bending moments per unit length of a plate section perpendicular to the minor axis, respectively.

As the deflection surface and the moment intensities are of the double series form as given by Eqs. (20)–(22), the convergence was examined by truncating those series at \( m = u = N - 1 \) for a certain positive integer \( N \). Truncation like this produces that the infinite determinants \( \Delta \) and \( \Delta^{(o)} \) given by the expressions (19) degenerate to the finite and computable \( N \times N \) determinants \( \Delta_N \) and \( \Delta_N^{(o)} \) as

\[
\Delta_N = \begin{bmatrix} 
&e_{00} & e_{01} & \cdots & e_{0,N-1} \\
&e_{10} & e_{11} & \cdots & e_{1,N-1} \\
&e_{20} & e_{21} & \cdots & e_{2,N-1} \\
&\cdots & \cdots & \cdots & \cdots \\
&e_{N-1,0} & e_{N-1,1} & \cdots & e_{N-1,N-1} 
\end{bmatrix}.
\]

(40)

\[
\Delta_N^{(o)} = \begin{bmatrix} 
&e_{00} & y & e_{0,N-1} \\
&e_{10} & g_1 & \cdots & e_{1,N-1} \\
&e_{20} & g_2 & \cdots & e_{2,N-1} \\
&\cdots & \cdots & \cdots & \cdots \\
&e_{N-1,0} & g_{N-1} & \cdots & e_{N-1,N-1} 
\end{bmatrix}.
\]

Table 1  Convergence study of the dimensionless deflection \( \delta_c \) at the plate center and the dimensionless bending moment intensities \( \zeta_i \) at the plate center and \( \zeta_c \) at the end of minor axis (\( a_0/b_0 = 2; \beta b_0/D = 1; \nu = 0.3 \))

| \( N \) | \( \delta_c \) | \( \zeta_i \) | \( \zeta_c \) |
|-------|------------|------------|------------|
| 1     | 0.089295   | 0.25412    | -0.07366   |
| 2     | 0.11554    | 0.33601    | -0.13791   |
| 3     | 0.11810    | 0.35016    | -0.15268   |
| 4     | 0.11802    | 0.34981    | -0.15371   |
| 5     | 0.11802    | 0.34980    | -0.15371   |
| 6     | 0.11802    | 0.34980    | -0.15371   |

As an example, the convergence study of the dimensionless deflection \( \delta_c \) and the dimensionless bending moment intensities \( \zeta_i \) and \( \zeta_c \) are presented in Table 1 in the case of in-plane compressive force parameter \( \beta b_0/D = 1 \), restraint parameter \( \beta b_0/D = 1 \) and aspect ratio \( a_0/b_0 = 2 \). Here it is added that the buckling load parameter \( (\beta b_0)c_{cr} \) for the elliptical plate with \( \beta b_0/D = 1 \) and \( a_0/b_0 = 2 \) is 4.739 as shown in Table 3 in Appendix. One can see from Table 1 that the convergence is very rapid.

Figure 3 (a)–(c) shows the numerical values of dimensionless deflections \( \delta \) for six kinds of restraint parameters \( \beta b_0/D = 0, 0.1, 1, 10, 100 \) and \( \infty \) in the case of in-plane force parameter \( \beta b_o/D = 1 \). Although the restraint parameter \( \beta b_0/D = \infty \) means the perfect clamping, the numerical solutions for its parameter can be obtained also by using the extremely large restraint parameter, \( \beta b_0/D = 1 \times 10^{20} \) for example, in the deflection and bending moment expressions derived in the present work. Figure 3 (a) and (b) shows the changes of dimensionless deflections \( \delta_c \) with the semi-major axial-length ratio \( a/a_0 \) and \( \delta_b \) with the semi-minor axial-length ratio \( b/b_0 \), respectively. The abscissas \( a/a_0 = 0 \) and 1 in Fig. 3 (a) give the plate center and the end of major axis, respectively. Similarly, the abscissas \( b/b_0 = 0 \) and 1 in Fig. 3 (b) give the plate center and the end of minor axis, respectively. The length of horizontal line giving the dimensionless minor axis in Fig. 3 (b) is taken intentionally to be a half of the length of horizontal line giving the dimensionless major axis in Fig. 3 (a), because the length of the minor axis of the elliptical plate of aspect ratio \( a_0/b_0 = 2 \) is a half of the length of its major axis. One can see from the figures that with increasing length ratios the deflection along the major axis decreases slowly in comparison with the deflection along the minor axis for each restraint parameter. Figure 3 (c) shows the relation between the dimensionless deflection \( \delta_c \) at the plate center and the aspect ratio \( a_0/b_0 \). From the figure, one can see that as the aspect ratio \( a_0/b_0 \) increases from 1 corresponding to a circular plate, the values of \( \delta_c \) increase monotonously. Figure 3 (a)–(c) represents that all values of \( \delta_c \)’s decrease with
increasing restraint parameter $\beta b_o/D$ and both values of $\delta$’s for $\beta b_o/D = 100$ and $\infty$ are very close.

Figure 4 (a)–(d) shows the changes of dimensionless bending moments $\zeta_a$ with $a/a_o$, $\zeta_b$ with $b/b_o$, $\zeta_{bc}$ with $a_o/b_o$ and $\zeta_{bo}$ with $a_o/b_o$, respectively, for six kinds of $\beta b_o/D = 0, 0.1, 1, 10, 100$ and $\infty$ in the case of $\lambda b_o^2 = 1$. As seen in Fig. 4 (a) and (b), the values of $\zeta_a$ and $\zeta_b$ become small with increasing $a/a_o$ and $b/b_o$. Figure 4 (c) shows the relation between $\zeta_{bc}$ and $a_o/b_o$. One can see from the figure that $\zeta_{bc}$ for each value of $\beta b_o/D$ in-

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{Dimensionless deflections $\delta$ for $\lambda b_o^2 = 1$: (a) $\delta_a$ versus $a/a_o$ for $a_o/b_o = 2$; (b) $\delta_b$ versus $b/b_o$ for $a_o/b_o = 2$; (c) $\delta_{bc}$ versus $a_o/b_o$, $\cdots$, $\beta b_o/D = 0$; $\cdots$, $\beta b_o/D = 0.1$; $\cdots$, $\beta b_o/D = 1$; $\cdots$, $\beta b_o/D = 10$; $\cdots$, $\beta b_o/D = 100$; $\cdots$, $\beta b_o/D = \infty$}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{Dimensionless bending moments $\zeta$ for $\lambda b_o^2 = 1$: (a) $\zeta_a$ versus $a/a_o$ for $a_o/b_o = 2$; (b) $\zeta_b$ versus $b/b_o$ for $a_o/b_o = 2$; (c) $\zeta_{bc}$ versus $a_o/b_o$; (d) $\zeta_{bo}$ versus $a_o/b_o$, $\cdots$, $\beta b_o/D = 0$; $\cdots$, $\beta b_o/D = 0.1$; $\cdots$, $\beta b_o/D = 1$; $\cdots$, $\beta b_o/D = 10$; $\cdots$, $\beta b_o/D = 100$; $\cdots$, $\beta b_o/D = \infty$}
\end{figure}
of comparing the arithmetic means obtained by the above-mentioned way with the exact solutions in the case of a clamped elliptical plate without in-plane force. The table gives the numerical values of $\delta_n$, $\zeta_n$, and $\beta_n$ which increase in magnitude except for $\zeta_n = 0$ at $\beta_n/D = 0$ with gradually increasing $AB_0^2$. The numerical values tabulated will constitute also a valuable contribution to checking very precisely the accuracy of the other approximate numerical techniques.

7. Concluding Remark

About the bending problem of an elastically restrained elliptical plate subjected to the combined action of uniform lateral load and uniform in-plane force, the present paper derived the analytical deflection solution in an infinite Mathieu function series. Moreover, the bending moment intensities were formulated in the forms of infinite series involving the Mathieu and modified Mathieu functions and their derivatives. The deflection surface and the bending moment intensities due to symmetrical bending of a circular plate were derived rigorously from those obtained for an elliptical plate as a limiting case that the semi-focal length tends to zero.

Numerical calculation of the deflection surface and the bending moments for elliptical plates with various aspect ratios and in-plane forces was performed for various edge restraints. The numerical results that approach to the numerical values for a perfectly clamped elliptical plate from those for a simply supported elliptical plate with increasing edge-restraint parameter were obtained.

In addition, the slight correction of the buckling loads obtained previously by the author was made and the corrected buckling loads were tabulated.

Appendix: Buckling of an Elliptical Plate with the Elastically Restrained Edge

As was stated in section 3, the infinite determinantal equation $\Delta = 0$ gives the buckling condition and coincides algebraically with that derived in the previous paper. However it was found in the process of performing the present research that the numerical values of the buckling load in the previous paper have the slight errors owing to the careless computer-programming mistake that $(1/2)^2$ was used in place of $1/2$ appearing in $\Psi_{2n-2m,2}$ given by Eq. (13). In the present paper, the corrected buckling load parameters, which are defined by the square of the eigenvalues of buckling used in the previous paper, are given in Table 3 for three kinds of Poisson’s ratios $\nu = 0, 0.3$ and 0.5 as in the previous paper. The differences between the previous values and the present corrected values are less than 5%. The previous numerical solutions for both special cases of simply supported plate ($\beta = 0$) and perfectly clamped plate ($\beta = \infty$) coincide completely within the given significant figures with those in the present paper.
Table 3  Buckling load parameters $\lambda b_2^o\sigma$ as functions of $a_o/b_o$ and $\beta b_o/D$

| $a_o/b_o$ | $\nu = 0$ | $\beta b_o/D$ |
|-----------|-----------|----------------|
| 0         | 0         | 1              |
| 0.1       | 1         | 10             |
| 1         | 10        | $\infty$       |

(a) $\nu = 0$

(b) $\nu = 0.3$

(c) $\nu = 0.5$

| $a_o/b_o$ | $\nu = 0.3$ | $\beta b_o/D$ |
|-----------|-------------|----------------|
| 1         | 1           | 10             |
| 1.1       | 1.1         | $\infty$       |

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