Tridiagonal pairs and the $\mu$-conjecture

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Abstract

Let $\mathbb{F}$ denote a field and let $V$ denote a vector space over $\mathbb{F}$ with finite positive dimension. We consider a pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy the following conditions: (i) each of $A, A^*$ is diagonalizable; (ii) there exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of $A$ such that $A^*V_i \subseteq V_{i-1} + V_i + V_{i+1}$ for $0 \leq i \leq d$, where $V_{-1} = 0$ and $V_{d+1} = 0$; (iii) there exists an ordering $\{V_i^*\}_{i=0}^\delta$ of the eigenspaces of $A^*$ such that $AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*$ for $0 \leq i \leq \delta$, where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$; (iv) there is no subspace $W$ of $V$ such that $AW \subseteq W$, $A^*W \subseteq W$, $W \neq 0$, $W \neq V$. We call such a pair a tridiagonal pair on $V$. It is known that $d = \delta$ and for $0 \leq i \leq d$ the dimensions of $V_i, V_{d-i}$, $V_i^*, V_{d-i}^*$ coincide. We say the pair $A, A^*$ is sharp whenever $\dim V_0 = 1$. It is known that if $\mathbb{F}$ is algebraically closed then $A, A^*$ is sharp. A conjectured classification of the sharp tridiagonal pairs was recently introduced by T. Ito and the second author. We present a result which supports the conjecture. Given scalars $\{\theta_i\}_{i=0}^d$, $\{\theta_i^*\}_{i=0}^\delta$ in $\mathbb{F}$ that satisfy the known constraints on the eigenvalues of a tridiagonal pair, we define an $\mathbb{F}$-algebra $T$ by generators and relations. We consider the $\mathbb{F}$-algebra $e_0^*Te_0^*$ for a certain idempotent $e_0^* \in T$. Let $\mathbb{F}[x_1, \ldots, x_d]$ denote the polynomial algebra over $\mathbb{F}$ involving $d$ mutually commuting indeterminates. We display a surjective $\mathbb{F}$-algebra homomorphism $\mu : \mathbb{F}[x_1, \ldots, x_d] \to e_0^*Te_0^*$. We conjecture that $\mu$ is an isomorphism. We show that this $\mu$-conjecture implies the classification conjecture, and that the $\mu$-conjecture holds for $d \leq 5$.

Keywords. Tridiagonal pair, Leonard pair, $q$-Racah polynomial.

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1 Tridiagonal pairs

Throughout this paper $\mathbb{F}$ denotes a field.

We begin by recalling the notion of a tridiagonal pair. We will use the following terms. Let $V$ denote a vector space over $\mathbb{F}$ with finite positive dimension. For a linear transformation $A : V \to V$ and a subspace $W \subseteq V$, we call $W$ an eigenspace of $A$ whenever $W \neq 0$ and there exists $\theta \in \mathbb{F}$ such that $W = \{v \in V \mid Av = \theta v\}$; in this case $\theta$ is the eigenvalue of $A$ associated with $W$. We say that $A$ is diagonalizable whenever $V$ is spanned by the eigenspaces of $A$.

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Definition 1.1 [14, Definition 1.1] Let $V$ denote a vector space over $\mathbb{F}$ with finite positive dimension. By a tridiagonal pair on $V$ we mean an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy the following four conditions.

(i) Each of $A$ and $A^*$ is diagonalizable.

(ii) There exists an ordering $\{V_i\}_{i=0}^{d}$ of the eigenspaces of $A$ such that

$$A^*V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d),$$

where $V_{-1} = 0$ and $V_{d+1} = 0$.

(iii) There exists an ordering $\{V_i^*\}_{i=0}^{\delta}$ of the eigenspaces of $A^*$ such that

$$AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \quad (0 \leq i \leq \delta),$$

where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$.

(iv) There does not exist a subspace $W$ of $V$ such that $AW \subseteq W$, $A^*W \subseteq W$, $W \neq 0$, $W \neq V$.

We say the pair $A, A^*$ is over $\mathbb{F}$. We call $V$ the vector space underlying $A, A^*$.

Note 1.2 According to a common notational convention $A^*$ denotes the conjugate-transpose of $A$. We are not using this convention. In a tridiagonal pair $A, A^*$ the linear transformations $A$ and $A^*$ are arbitrary subject to (i)–(iv) above.

We now summarize what is known about tridiagonal pairs. Let $A, A^*$ denote a tridiagonal pair on $V$, as in Definition 1.1. By [14, Lemma 4.5] the integers $d$ and $\delta$ from (ii), (iii) are equal; we call this common value the diameter of the pair. By [14, Theorem 10.1] the pair $A, A^*$ satisfy two polynomial equations called the tridiagonal relations; these generalize the q-Serre relations [39, Example 3.6] and the Dolan-Grady relations [39, Example 3.2]. See [5–11, 24, 39, 43] for results on the tridiagonal relations. An ordering of the eigenspaces of $A$ (resp. $A^*$) is said to be standard whenever it satisfies (i) (resp. (ii)). We comment on the uniqueness of the standard ordering. Let $\{V_i\}_{i=0}^{d}$ denote a standard ordering of the eigenspaces of $A$. By [14, Lemma 2.4], the ordering $\{V_i\}_{i=0}^{d}$ is also standard and no further ordering is standard. A similar result holds for the eigenspaces of $A^*$. Let $\{V_i\}_{i=0}^{d}$ (resp. $\{V_i^*\}_{i=0}^{d}$) denote a standard ordering of the eigenspaces of $A$ (resp. $A^*$).

By [14, Corollary 5.7], for $0 \leq i \leq d$ the spaces $V_i, V_i^*$ have the same dimension; we denote this common dimension by $\rho_i$. By [14, Corollaries 5.7, 6.6] the sequence $\{\rho_i\}_{i=0}^{d}$ is symmetric and unimodal; that is $\rho_i = \rho_{d-i}$ for $0 \leq i \leq d$ and $\rho_{i-1} \leq \rho_i$ for $1 \leq i \leq d/2$. We call the sequence $\{\rho_i\}_{i=0}^{d}$ the shape of $A, A^*$. See [15, 17, 21, 22, 25, 26, 33] for results on the shape. We say $A, A^*$ is sharp whenever $\rho_0 = 1$. By [36, Theorem 1.3], if $\mathbb{F}$ is algebraically closed then $A, A^*$ is sharp. By [36, Theorem 1.4], if $A, A^*$ is sharp then there exists a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $V$ such that $\langle Au, v \rangle = \langle u, Av \rangle$ and $\langle A^*u, v \rangle = \langle u, A^*v \rangle$ for all $u, v \in V$. See [3, 34] for results on the bilinear form. The following special cases of tridiagonal pairs have been studied extensively. In [44] the tridiagonal pairs of shape $(1, 2, 1)$ are classified and described in detail. The tridiagonal pairs of shape $(1, 1, \ldots, 1)$
are called Leonard pairs [38, Definition 1.1], and these are classified in [38, 40]. This classification yields a correspondence between the Leonard pairs and a family of orthogonal polynomials consisting of the q-Racah polynomials and their relatives [4, 41]. This family coincides with the terminating branch of the Askey scheme [23]. See [27–32, 42] and the references therein for results on Leonard pairs. For the above tridiagonal pair \( A, A^* \) and for \( 0 \leq i \leq d \) let \( \theta_i \) (resp. \( \theta_i^* \)) denote the eigenvalue of \( A \) (resp. \( A^* \)) associated with \( V_i \) (resp. \( V_i^* \)). The pair \( A, A^* \) is said to have Krawtchouk type (resp. q-geometric type) whenever \( \theta_i = d - 2i \) (resp. \( \theta_i = q^{d-2i} \)) and \( \theta_i^* = d - 2i \) (resp. \( \theta_i^* = q^{d-2i} \)) for \( 0 \leq i \leq d \). In [13, Theorems 1.7, 1.8] the tridiagonal pairs of Krawtchouk type are classified. By [13, Remark 1.9] these tridiagonal pairs are in bijection with the finite-dimensional irreducible modules for the three-point loop algebra \( \mathfrak{s}\mathfrak{l}_2 \otimes \mathbb{F}[t, t^{-1}, (t - 1)^{-1}] \). See [20, 21] for results on tridiagonal pairs of Krawtchouk type. In [17, Theorems 1.6, 1.7] the q-geometric tridiagonal pairs are classified. By [18, Theorems 10.3, 10.4] these tridiagonal pairs are in bijection with the type 1, finite-dimensional, irreducible modules for the algebra \( \mathbb{E}_d \); this is a q-deformation of \( \mathfrak{s}\mathfrak{l}_2 \otimes \mathbb{F}[t, t^{-1}, (t - 1)^{-1}] \) as explained in [18]. See [1, 2, 15–17, 19] for results on q-geometric tridiagonal pairs.

We now summarize the present paper. A conjectured classification of the sharp tridiagonal pairs was introduced in [21, Conjecture 14.6] and studied carefully in [34]; see Conjecture 3.1 below. In the present paper we obtain two results which clarify the conjecture and provide some more evidence that it is true. To describe these results, we start with a sequence of scalars \( \{ \theta_i \}_{i=0}^d \) and \( \{ \theta_i^* \}_{i=0}^d \) taken from \( \mathbb{F} \) that satisfy the known constraints on the eigenvalues of a tridiagonal pair over \( \mathbb{F} \); these are conditions (i) and (iii) in Conjecture 3.1. We associate with this sequence an \( \mathbb{F} \)-algebra \( T \) defined by generators and relations; \( T \) is reminiscent of an algebra introduced by E. Egge [12, Definition 4.1]. We are interested in the \( \mathbb{F} \)-algebra \( e_0^* T e_0^* \) where \( e_0^* \) is a certain idempotent element of \( T \). Let \( \{ x_i \}_{i=1}^d \) denote mutually commuting indeterminates. Let \( \mathbb{F}[x_1, \ldots, x_d] \) denote the \( \mathbb{F} \)-algebra consisting of the polynomials in \( \{ x_i \}_{i=1}^d \) that have all coefficients in \( \mathbb{F} \). We display a surjective \( \mathbb{F} \)-algebra homomorphism \( \mu : \mathbb{F}[x_1, \ldots, x_d] \to e_0^* T e_0^* \). We conjecture that \( \mu \) is an isomorphism; let us call this the \( \mu \)-conjecture. Our two main results are that the \( \mu \)-conjecture implies the classification conjecture, and that the \( \mu \)-conjecture holds for \( d \leq 5 \). These results are contained in Theorems 10.1 and 12.1.

2 Tridiagonal systems

When working with a tridiagonal pair, it is often convenient to consider a closely related object called a tridiagonal system. To define a tridiagonal system, we recall a few concepts from linear algebra. Let \( V \) denote a vector space over \( \mathbb{F} \) with finite positive dimension. Let \( \text{End}(V) \) denote the \( \mathbb{F} \)-algebra of all linear transformations from \( V \) to \( V \). Let \( A \) denote a diagonalizable element of \( \text{End}(V) \). Let \( \{ V_i \}_{i=0}^d \) denote an ordering of the eigenspaces of \( A \) and let \( \{ \theta_i \}_{i=0}^d \) denote the corresponding ordering of the eigenvalues of \( A \). For \( 0 \leq i \leq d \) define \( E_i \in \text{End}(V) \) such that \( (E_i - I)V_i = 0 \) and \( E_i V_j = 0 \) for \( j \neq i \) \((0 \leq j \leq d) \). Here \( I \) denotes the identity of \( \text{End}(V) \). We call \( E_i \) the primitive idempotent of \( A \) corresponding to \( V_i \) (or \( \theta_i \)). Observe that (i) \( \sum_{i=0}^d E_i = I \); (ii) \( E_i E_j = \delta_{i,j} E_i \) \((0 \leq i, j \leq d) \); (iii) \( V_i = E_i V \).
(0 ≤ i ≤ d); (iv) \( A = \sum_{i=0}^{d} \theta_i E_i \). Moreover
\[
E_i = \prod_{\substack{0 \leq j \leq d \\text{ or } j \neq i}} \frac{A - \theta_j I}{\theta_i - \theta_j}.
\]

Note that each of \( \{A^i\}_{i=0}^{d} \), \( \{E_i\}_{i=0}^{d} \) is a basis for the \( F \)-subalgebra of \( \operatorname{End}(V) \) generated by \( A \). Moreover \( \prod_{i=0}^{d} (A - \theta_i I) = 0 \). Now let \( A, A^* \) denote a tridiagonal pair on \( V \). An ordering of the primitive idempotents or eigenvalues of \( A \) (resp. \( A^* \)) is said to be standard whenever the corresponding ordering of the eigenspaces of \( A \) (resp. \( A^* \)) is standard.

**Definition 2.1** [14, Definition 2.1] Let \( V \) denote a vector space over \( F \) with finite positive dimension. By a tridiagonal system on \( V \) we mean a sequence \( \Phi = (A; \{E_i\}_{i=0}^{d}; A^*; \{E^*_i\}_{i=0}^{d}) \) that satisfies (i)–(iii) below.

(i) \( A, A^* \) is a tridiagonal pair on \( V \).

(ii) \( \{E_i\}_{i=0}^{d} \) is a standard ordering of the primitive idempotents of \( A \).

(iii) \( \{E^*_i\}_{i=0}^{d} \) is a standard ordering of the primitive idempotents of \( A^* \).

We say \( \Phi \) is over \( F \). We call \( V \) the vector space underlying \( \Phi \).

The notion of isomorphism for tridiagonal systems is defined in [34, Definition 3.1].

The following result is immediate from lines (1), (2) and Definition 2.1

**Lemma 2.2** [35, Lemma 2.5] Let \( (A; \{E_i\}_{i=0}^{d}; A^*; \{E^*_i\}_{i=0}^{d}) \) denote a tridiagonal system. Then for \( 0 \leq i, j, k \leq d \) the following (i), (ii) hold.

(i) \( E_i^* A^k E_j = 0 \) if \( k < |i - j| \).

(ii) \( E_i A^k E_j = 0 \) if \( k < |i - j| \).

**Definition 2.3** Let \( \Phi = (A; \{E_i\}_{i=0}^{d}; A^*; \{E^*_i\}_{i=0}^{d}) \) denote a tridiagonal system on \( V \). For \( 0 \leq i \leq d \) let \( \theta_i \) (resp. \( \theta^*_i \)) denote the eigenvalue of \( A \) (resp. \( A^* \)) associated with the eigenspace \( E_i V \) (resp. \( E^*_i V \)). We call \( \{\theta_i\}_{i=0}^{d} \) (resp. \( \{\theta^*_i\}_{i=0}^{d} \)) the eigenvalue sequence (resp. dual eigenvalue sequence) of \( \Phi \). We observe that \( \{\theta_i\}_{i=0}^{d} \) (resp. \( \{\theta^*_i\}_{i=0}^{d} \)) are mutually distinct and contained in \( F \). We say \( \Phi \) is sharp whenever the tridiagonal pair \( A, A^* \) is sharp.

We now recall the split sequence of a tridiagonal system. We will use the following notation.
Definition 2.4 Let $\lambda$ denote an indeterminate and let $\mathbb{F}[\lambda]$ denote the $\mathbb{F}$-algebra consisting of the polynomials in $\lambda$ that have all coefficients in $\mathbb{F}$. Let $d$ denote a nonnegative integer and let $\{\theta_i^d\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d$ denote a sequence of scalars taken from $\mathbb{F}$. Then for $0 \leq i \leq d$ we define the following polynomials in $\mathbb{F}[\lambda]$:

$$
\tau_i = (\lambda - \theta_0)(\lambda - \theta_1) \cdots (\lambda - \theta_{i-1}),
$$
$$
\eta_i = (\lambda - \theta_d)(\lambda - \theta_{d-1}) \cdots (\lambda - \theta_{d-i+1}),
$$
$$
\tau_i^* = (\lambda - \theta_0^*)(\lambda - \theta_1^*) \cdots (\lambda - \theta_{i-1}^*),
$$
$$
\eta_i^* = (\lambda - \theta_d^*)(\lambda - \theta_{d-1}^*) \cdots (\lambda - \theta_{d-i+1}^*).
$$

Note that each of $\tau_i$, $\eta_i$, $\tau_i^*$, $\eta_i^*$ is monic with degree $i$.

The following definition of the split sequence is motivated by [36, Lemma 5.4].

Definition 2.5 Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a sharp tridiagonal system over $\mathbb{F}$, with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$. By [36, Lemma 5.4], for $0 \leq i \leq d$ there exists a unique $\zeta_i \in \mathbb{F}$ such that

$$
E_0^* \tau_i(A) E_0^* = \frac{\zeta_i E_0^*}{(\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_i^*)}.
$$

We note that $\zeta_0 = 1$. We call $\{\zeta_i\}_{i=0}^d$ the split sequence of the tridiagonal system.

Definition 2.6 [34, Definition 6.2] Let $\Phi$ denote a sharp tridiagonal system. By the parameter array of $\Phi$ we mean the sequence $\{\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d\}$ where $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) is the eigenvalue sequence (resp. dual eigenvalue sequence) of $\Phi$ and $\{\zeta_i\}_{i=0}^d$ is the split sequence of $\Phi$.

3 The classification conjecture

In this section we discuss a conjectured classification of the sharp tridiagonal systems.
**Conjecture 3.1** [21, Conjecture 14.6] Let $d$ denote a nonnegative integer and let

$$\left(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d\right)$$

denote a sequence of scalars taken from $\mathbb{F}$. Then there exists a sharp tridiagonal system $\Phi$ over $\mathbb{F}$ with parameter array (4) if and only if (i)–(iii) hold below.

(i) $\theta_i \neq \theta_j$, $\theta_i^* \neq \theta_j^*$ if $i \neq j$ ($0 \leq i, j \leq d$).

(ii) $\zeta_0 = 1$, $\zeta_d \neq 0$, and

$$\sum_{i=0}^{d} \eta_{d-i}(\theta_0)\eta_{d-i}(\theta_0^*)\zeta_i \neq 0.$$ (5)

(iii) The expressions

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \quad \frac{\theta_i^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}$$

are equal and independent of $i$ for $2 \leq i \leq d - 1$.

Suppose (i)–(iii) hold. Then $\Phi$ is unique up to isomorphism of tridiagonal systems.

In [34, Section 8] we proved the “only if” direction of Conjecture 3.1. In [36, Theorem 1.6] we proved the last assertion of Conjecture 3.1. In this paper we consider what is involved in proving the rest of Conjecture 3.1. We are going to define a certain $\mathbb{F}$-algebra $T$ by generators and relations, and consider the $\mathbb{F}$-algebra $e_0^* T e_0^*$ for a certain idempotent $e_0^* \in T$. We will state a conjecture about $e_0^* T e_0^*$ called the $\mu$-conjecture. The $\mu$-conjecture asserts, roughly speaking, that $e_0^* T e_0^*$ is isomorphic to the algebra of all polynomials over $\mathbb{F}$ involving $d$ mutually commuting indeterminates. In Section 10 we show that the $\mu$-conjecture implies Conjecture 3.1. In Section 12 we show that the $\mu$-conjecture holds for $d \leq 5$.

### 4 The algebra $T$

In this section we recall the algebra $T$ from [36]. From now until the end of Section 6 let $d$ denote a nonnegative integer and let $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d)$ denote a sequence of scalars taken from $\mathbb{F}$ that satisfy conditions (i) and (iii) of Conjecture 3.1.

The following algebra is reminiscent of an algebra introduced by E. Egge [12, Definition 4.1].
Definition 4.1 [36, Definition 2.4] Let $T$ denote the associative $F$-algebra with 1, defined by generators $a$, $\{e_i\}_{i=0}^d$, $a^*$, $\{e_i^*\}_{i=0}^d$ and relations

$$e_i e_j = \delta_{i,j} e_i, \quad e_i^* e_j^* = \delta_{i,j} e_i^* \quad (0 \leq i, j \leq d),$$

(7)

$$\sum_{i=0}^d e_i = 1, \quad \sum_{i=0}^d e_i^* = 1,$$

(8)

$$a = \sum_{i=0}^d \theta_i e_i, \quad a^* = \sum_{i=0}^d \theta_i^* e_i^*,$$

(9)

$$e_i^* a^k e_j = 0 \quad \text{if} \quad k < |i-j| \quad (0 \leq i, j, k \leq d),$$

(10)

$$e_i a^k e_j = 0 \quad \text{if} \quad k < |i-j| \quad (0 \leq i, j, k \leq d).$$

(11)

Let $D$ (resp. $D^*$) denote the $F$-subalgebra of $T$ generated by $a$ (resp. $a^*$).

We now give bases for the $F$-vector spaces $D$ and $D^*$.

Lemma 4.2 With reference to Definition 4.1 the following (i), (ii) hold.

(i) Each of $\{a^i\}_{i=0}^d$, $\{e_i\}_{i=0}^d$ is a basis for $D$.

(ii) Each of $\{a^{*i}\}_{i=0}^d$, $\{e_i^*\}_{i=0}^d$ is a basis for $D^*$.

Proof. (i): Observe that for $0 \leq r, s \leq d$ there exists an $F$-algebra homomorphism $T \to F$ that sends $e_i \mapsto \delta_{i,r}$ and $e_i^* \mapsto \delta_{i,s}$ for $0 \leq i \leq d$. This is verified by checking that the defining relations for $T$ are respected. By the observation, $e_i \neq 0$ for $0 \leq i \leq d$. By this and the equation on the left in (7), the elements $\{e_i\}_{i=0}^d$ are linearly independent. Let $D'$ denote the $F$-subspace of $T$ spanned by $\{e_i\}_{i=0}^d$. By the equations on the left in (7), (8) we find $D'$ is an $F$-subalgebra of $T$. By the equation on the left in (9) and since $\{\theta_i\}_{i=0}^d$ are mutually distinct, $a$ generates $D'$ so $D = D'$. The result follows.

(ii): Similar to the proof of (i) above. □

Lemma 4.3 With reference to Definition 4.1

$$ae_i = \theta_i e_i, \quad a^* e_i^* = \theta_i^* e_i^* \quad (0 \leq i \leq d),$$

(12)

$$e_i = \prod_{\substack{0 \leq j \leq d \ \text{or} \ j \neq i}} \frac{a - \theta_j 1}{\theta_i - \theta_j}, \quad e_i^* = \prod_{\substack{0 \leq j \leq d \ \text{or} \ j \neq i}} \frac{a^* - \theta_j^* 1}{\theta_i^* - \theta_j^*} \quad (0 \leq i \leq d),$$

(13)

$$\prod_{i=0}^d (a - \theta_i 1) = 0, \quad \prod_{i=0}^d (a^* - \theta_i^* 1) = 0.$$ 

(14)

Proof. Routinely verified using (7)-(9). □
Lemma 4.4 With reference to Definitions 2.4 and 4.1 the following (i), (ii) hold.

(i) The sequence \( \{ \tau_i(a) \}_{i=0}^{d} \) is a basis for \( D \).

(ii) The sequence \( \{ \tau_i^*(a^*) \}_{i=0}^{d} \) is a basis for \( D^* \).

Proof. (i): The sequence \( \{ a^i \}_{i=0}^{d} \) is a basis for \( D \), and the polynomial \( \tau_i \) has degree exactly \( i \) for \( 0 \leq i \leq d \). The result follows.

(ii): Similar to the proof of (i) above. \( \square \)

Note 4.5 When we introduced \( T \) in [36] we assumed that there exists a tridiagonal system with eigenvalue sequence \( \{ \theta_i \}_{i=0}^{d} \) and dual eigenvalue sequence \( \{ \theta_i^* \}_{i=0}^{d} \). The assumption was natural in the context of [36] but it was not used in any substantial way. Indeed one can check that every proof in [36, Sections 4, 5] is valid verbatim under our present assumption that \( \{ \theta_i \}_{i=0}^{d} \) and \( \{ \theta_i^* \}_{i=0}^{d} \) satisfy conditions (i), (iii) of Conjecture 3.1. With this understanding, later in the paper we will invoke some results from [36, Sections 4, 5].

5 Finite-dimensional \( T \)-modules

In this section we collect some useful facts about finite-dimensional \( T \)-modules. For the most part the proofs are routine and omitted.

Lemma 5.1 Let \( V \) denote a finite-dimensional \( T \)-module. Then (i)–(v) hold below.

(i) \( V \) is a direct sum of the nonzero spaces among \( e_i V \ (0 \leq i \leq d) \).

(ii) For all \( i \) \((0 \leq i \leq d)\) such that \( e_i V \neq 0 \), the space \( e_i V \) is an eigenspace for \( a \) with eigenvalue \( \theta_i \), and \( e_i \) acts on \( V \) as the projection onto \( e_i V \).

(iii) \( V \) is a direct sum of the nonzero spaces among \( e_i^* V \ (0 \leq i \leq d) \).

(iv) For all \( i \) \((0 \leq i \leq d)\) such that \( e_i^* V \neq 0 \), the space \( e_i^* V \) is an eigenspace for \( a^* \) with eigenvalue \( \theta_i^* \), and \( e_i^* \) acts on \( V \) as the projection onto \( e_i^* V \).

(v) Each of \( a, a^* \) is diagonalizable on \( V \).

Lemma 5.2 Let \( V \) denote a finite-dimensional \( T \)-module. Then (i), (ii) hold below.

(i) For \( 0 \leq i \leq d \),

\[ a^* e_i V \subseteq e_{i-1} V + e_i V + e_{i+1} V, \]

where \( e_{-1} = 0 \) and \( e_{d+1} = 0 \).

(ii) For \( 0 \leq i \leq d \),

\[ a e_i^* V \subseteq e_{i-1}^* V + e_i^* V + e_{i+1}^* V, \]

where \( e_{-1}^* = 0 \) and \( e_{d+1}^* = 0 \).

We now consider finite-dimensional irreducible \( T \)-modules.
Lemma 5.3 Let $V$ denote a finite-dimensional irreducible $T$-module. Then (i), (ii) hold below.

(i) There exist nonnegative integers $r$, $\delta$ ($r + \delta \leq d$) such that

$$e_i^*V \neq 0 \quad \text{if and only if} \quad r \leq i \leq r + \delta \quad (0 \leq i \leq d).$$

(ii) There exist nonnegative integers $t$, $\delta^*$ ($t + \delta^* \leq d$) such that

$$e_iV \neq 0 \quad \text{if and only if} \quad t \leq i \leq t + \delta^* \quad (0 \leq i \leq d).$$

Proof. (i): By Lemma 6.1(iii) and since $V \neq 0$, there exists an integer $i$ ($0 \leq i \leq d$) such that $e_i^*V \neq 0$. Define $r = \min \{i \mid 0 \leq i \leq d, \; e_i^*V \neq 0\}$ and $\rho = \max \{i \mid 0 \leq i \leq d, \; e_i^*V \neq 0\}$. For $r + 1 \leq h \leq \rho - 1$ we have $e_h^*V \neq 0$; otherwise $\sum_{i=r}^{h-1} e_i^*V$ is a nonzero $T$-module properly contained in $V$, a contradiction to the irreducibility of $V$. The result follows.

(ii): Similar to the proof of (i) above. \qed

Proposition 5.4 Let $V$ denote a finite-dimensional irreducible $T$-module and let $\delta$, $\delta^*$, $r$, $t$ denote the corresponding parameters from Lemma 5.3. Then $\delta = \delta^*$. Moreover the sequence $(a; \{e_i\}_{i=t}^{l+\delta}; a^*; \{e_i^*\}_{i=r}^{r+\delta})$ acts on $V$ as a tridiagonal system.

Proof. Immediate from Lemmas 5.4, 5.3 and the third sentence below Note 4.2. \qed

Proposition 5.5 Fix integers $\delta, r, t$ such that $0 \leq \delta \leq d$ and $0 \leq r, t \leq d - \delta$. Let $(A; \{E_i\}_{i=t}^{l+\delta}; A^*; \{E_i^*\}_{i=0}^{r+\delta})$ denote a tridiagonal system over $\mathbb{F}$ that has eigenvalue sequence $\{\theta_i\}_{i=t}^{l+\delta}$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=r}^{r+\delta}$. Let $V$ denote the underlying vector space. Then there exists a $T$-module structure on $V$ such that (i)–(iii) hold below.

(i) a (resp. $a^*$) acts on $V$ as $A$ (resp. $A^*$).

(ii) For $0 \leq i \leq d$, $e_i$ acts on $V$ as $E_{i-t}$ if $t \leq i \leq t + \delta$, and zero otherwise.

(iii) For $0 \leq i \leq d$, $e_i^*$ acts on $V$ as $E_{i-r}^*$ if $r \leq i \leq r + \delta$, and zero otherwise.

This $T$-module is irreducible.

6 The $\mu$-conjecture

Observe that $e_0^*Te_0^*$ is an $\mathbb{F}$-algebra with multiplicative identity $e_0^*$. This section contains a general description of $e_0^*Te_0^*$ followed by a conjecture about the precise nature of $e_0^*Te_0^*$. We start by recalling [36, Theorem 2.6] with the wording slightly changed.

Lemma 6.1 [36, Theorem 2.6] The algebra $e_0^*Te_0^*$ is commutative and generated by $e_0^*\tau_i(a)e_0^*$ $(1 \leq i \leq d)$.

Proof. Follows from [36, Theorem 2.6] in view of Lemma 4.4(i) and Note 4.5. \qed
Definition 6.2 Let \( \{x_i\}_{i=1}^d \) denote mutually commuting indeterminates. Let \( \mathbb{F}[x_1, \ldots, x_d] \) denote the \( \mathbb{F} \)-algebra consisting of the polynomials in \( \{x_i\}_{i=1}^d \) that have all coefficients in \( \mathbb{F} \). We abbreviate \( R = \mathbb{F}[x_1, \ldots, x_d] \).

Corollary 6.3 There exists a surjective \( \mathbb{F} \)-algebra homomorphism \( \mu : R \rightarrow e_0^*Te_0^* \) that sends \( x_i \mapsto e_0^*\tau_i(a)e_0^* \) for \( 1 \leq i \leq d \).

Proof. Immediate from Lemma 6.1 \( \blacksquare \)

Conjecture 6.4 The map \( \mu \) from Corollary 6.3 is an isomorphism.

We call Conjecture 6.4 the \( \mu \)-conjecture. In Section 10 we show that the \( \mu \)-conjecture implies Conjecture 3.1.

We finish this section with some notation that is motivated by Definition 2.5 and Corollary 6.3.

Definition 6.5 We define

\[
y_i = (\theta_0^* - \theta_1^*) \cdots (\theta_0^* - \theta_i^*)x_i \quad (1 \leq i \leq d).
\]

(15)

7 The left ideal \( J \) of \( T \)

From now until the end of Section 9 we adopt the following assumption.

Assumption 7.1 We assume Conjecture 6.4 is true. Let \( d \) denote a nonnegative integer and let \((\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d)\) denote a sequence of scalars taken from \( \mathbb{F} \) that satisfies all three conditions (i)–(iii) of Conjecture 3.1. Let \( T \) denote the \( \mathbb{F} \)-algebra from Definition 4.1 that is associated with the sequence \((\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d)\).

With reference to Assumption 7.1 and with an eye towards proving Conjecture 3.1 we will construct a sharp tridiagonal system over \( \mathbb{F} \) with parameter array \((\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d)\). To this end we define a certain left ideal \( J \) of \( T \) and consider the quotient \( T \)-module \( M = T/J \). We will show \( M \) is nonzero, finite-dimensional, and has a unique maximal proper \( T \)-submodule \( M' \). The quotient \( T \)-module \( L = M/M' \) will yield the desired tridiagonal system via Proposition 5.4.

Definition 7.2 Let \( J \) denote the following left ideal of \( T \):

\[
J = T(1 - e_0^*) + \sum_{i=1}^d Tg_i,
\]

(16)

where

\[
g_i = e_0^*\tau_i(a)e_0^* - \frac{\zeta_i e_0^*}{(\theta_0^* - \theta_1^*) \cdots (\theta_0^* - \theta_i^*)} \quad (1 \leq i \leq d).
\]

(17)
Lemma 7.3 We have

\[ Te_0^* \cap J = \sum_{i=1}^{d} Tg_i. \]  \hfill (18)

**Proof.** By (17) we have \( g_i \in Te_0^* \) for \( 1 \leq i \leq d \). Therefore the right-hand side of (18) is contained in \( Te_0^* \). The result follows from this, line (16), and since \( T = T(1 - e_0^*) + Te_0^* \) (direct sum). \( \square \)

Proposition 7.4 We have

\[ e_0^* Te_0^* = F e_0^* + e_0^* Te_0^* \cap J \]  \hfill (direct sum). \hfill (19)

**Proof.** We claim

\[ e_0^* Te_0^* \cap J = \sum_{i=1}^{d} e_0^* Tg_i. \]  \hfill (20)

To obtain (20), observe that \( e_0^* Te_0^* \) contains \( g_i \) for \( 1 \leq i \leq d \), so \( e_0^* Te_0^* \) contains \( \sum_{i=1}^{d} e_0^* Tg_i \). By Definition 7.2 the ideal \( J \) contains \( \sum_{i=1}^{d} e_0^* Tg_i \) so \( e_0^* Te_0^* \cap J \) contains \( \sum_{i=1}^{d} e_0^* Tg_i \). To obtain the reverse inclusion in (20), we fix \( x \in e_0^* Te_0^* \cap J \) and show \( x \in \sum_{i=1}^{d} e_0^* Tg_i \). Since \( x \in e_0^* Te_0^* \) we have \( e_0^* x = x \). By Lemma 7.3 and since \( e_0^* Te_0^* \subseteq Te_0^* \), there exist \( t_i \in T \) \( (1 \leq i \leq d) \) such that \( x = \sum_{i=1}^{d} t_i g_i \). In this equation we multiply each term on the left by \( e_0^* \) and use \( e_0^* g_i = g_i \) to get \( x = \sum_{i=1}^{d} e_0^* t_i g_i \). Therefore \( x \in \sum_{i=1}^{d} e_0^* Tg_i \). We have proved (20). Now we can easily show (19). By Corollary 6.3, Definition 6.5, and line (17), the map \( \mu \) satisfies

\[ \mu(y_i - \zeta_i) = (\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_i^*)g_i \]  \hfill (1 \leq i \leq d). \hfill (21)

Let \( J \) denote the ideal of \( R \) generated by \( \{y_i - \zeta_i\}_{i=1}^{d} \), so that \( J = \sum_{i=1}^{d} R(y_i - \zeta_i) \). By Definition 6.2 and (15) we obtain a direct sum of \( F \)-vector spaces \( R = F 1 + J \). In this equation we apply the isomorphism \( \mu \) to each term. The \( \mu \)-image of \( R \) (resp. \( F 1 \)) is \( e_0^* Te_0^* \) (resp. \( F e_0^* \)). By (20), (21) the \( \mu \)-image of \( J \) is \( e_0^* Te_0^* \cap J \). Line (19) follows. \( \square \)

8 The \( T \)-module \( M \)

**Definition 8.1** Let \( J \) denote the left ideal of \( T \) from Definition 7.2. Observe that \( T/J \) has a natural \( T \)-module structure; we abbreviate this \( T \)-module by \( M \). We abbreviate \( \xi \) for the element \( 1 + J \) of \( M \). We note that

\[ J = \{ t \in T \mid t \xi = 0 \}. \]  \hfill (22)

**Lemma 8.2** The following (i), (ii) hold.

(i) \( M = T \xi \).

(ii) \( \xi \) is a basis for \( e_0^* M \).

**Proof.** (i): Recall \( M = T/J \) and \( \xi = 1 + J \).

...
(ii): Observe $J \neq T$ by (19) so $1 \not\in J$. Now $\xi \neq 0$ in view of Definition 8.1. Since $1 - e_0^* \in J$ we have $(1 - e_0^*)\xi = 0$, so $\xi = e_0^*\xi$. Using this and $M = T\xi$ we find $e_0^*M = e_0^*Te_0^*\xi$. In this equation we evaluate $e_0^*Te_0^*\xi$ using (19), (22) to get $e_0^*M = \mathbb{F}\xi$. The result follows.

Our next goal is to show that the $\mathbb{F}$-vector space $M$ has finite dimension. We will use the following notation. For subsets $X, Y$ of $T$ let $XY$ denote the $\mathbb{F}$-subspace of $T$ spanned by $\{xy \mid x \in X, y \in Y\}$.

Lemma 8.3  [36, Corollary 4.5] For $0 \leq r, s \leq d$ we have

$$e_r^*DD^*De_s^* = \sum_{k=0}^{[(r+s)/2]} e_r^*De_k^*De_s^*,$$

where $[x]$ denotes the greatest integer less than or equal to $x$.

Definition 8.4 Fix an integer $m \geq 1$. A sequence of integers $(k_0, k_1, \ldots, k_m)$ is called convex whenever $k_{i-1} - k_i \geq k_i - k_{i+1}$ for $1 \leq i \leq m - 1$.

Lemma 8.5 For $0 \leq r, s \leq d$ and $n \geq 0$ the space

$$e_r^*DD^*DD^* \cdots DD^*De_s^* \quad (n + 1 \text{ D's}) \quad (23)$$

is equal to

$$\sum e_r^*De_{k_1}^*De_{k_2}^*D \cdots De_{k_n}^*De_s^*, \quad (24)$$

where the sum is over all sequences $(k_1, k_2, \ldots, k_n)$ such that $0 \leq k_i \leq d \ (1 \leq i \leq n)$ and $(r, k_1, k_2, \ldots, k_n, s)$ is convex.

Proof. We assume $n \geq 1$; otherwise there is nothing to prove. Since $\{e_i^*\}_{i=0}^d$ is a basis for $D^*$ it suffices to show that

$$e_r^*De_{k_1}^*De_{k_2}^*D \cdots De_{k_n}^*De_s^* \quad (25)$$

is contained in (24) for all sequences $(k_1, k_2, \ldots, k_n)$ such that $0 \leq k_i \leq d \ (1 \leq i \leq n)$. For each such sequence $(k_1, k_2, \ldots, k_n)$ we define the weight to be $\sum_{i=1}^n k_i$. Suppose there exists a sequence $(k_1, k_2, \ldots, k_n)$ such that (25) is not contained in (24). Of all such sequences, pick one with minimal weight. Denote this weight by $w$. For notational convenience define $k_0 = r$ and $k_{n+1} = s$. The sequence $(k_0, k_1, k_2, \ldots, k_n, k_{n+1})$ is not convex, so there exists an integer $i \ (1 \leq i \leq n)$ such that $k_{i-1} - k_i < k_i - k_{i+1}$. Abbreviate $h = [(k_{i-1} + k_{i+1})/2]$ and note that $h < k_i$. By Lemma 8.3 the space (25) is contained in the space

$$\sum_{\ell=0}^h e_r^*De_{k_1}^*D \cdots De_{k_{i-1}}^*De_{k_i}^*DDe_{k_{i+1}}^* \cdots De_{k_n}^*De_s^*. \quad (26)$$

For $0 \leq \ell \leq h$ the $\ell$-summand in (26) has weight less than $w$, so this summand is contained in (24). Therefore (25) is contained in (24), for a contradiction. The result follows.
Lemma 8.6 For $1 \leq r \leq d$ the $\mathbb{F}$-vector space $e_r^* M$ is equal to

$$
\sum e_r^* De_{k_1} D e_{k_2} D \cdots De_{k_m} D \xi,
$$

where the sum is over all sequences $(k_1, k_2, \ldots, k_m)$ $(m \geq 0)$ such that $r > k_1 > k_2 > \cdots > k_m > 0$ and $(r, k_1, k_2, \ldots, k_m, 0)$ is convex.

Proof. We have $M = T \xi$ and $\xi = e_1^* \xi$ so $e_1^* M = e_1^* T e_0^* \xi$. The algebra $T$ is generated by $D, D^*$. Therefore $e_1^* T e_0^*$ is the sum over $n = 0, 1, 2, \ldots$ of terms (23) (with $s = 0$). We apply these terms to $\xi$ and simplify the result using Lemma 8.2(ii) and Lemma 8.5; this yields terms contained in the sum (27). The result follows. □

Proposition 8.7 The $\mathbb{F}$-vector space $M$ has finite dimension.

Proof. We have $M = \sum_{r=0}^d e_r^* M$. The subspace $e_0^* M$ has dimension 1 by Lemma 8.2(ii). For $1 \leq r \leq d$ the subspace $e_r^* M$ has finite dimension by Lemma 8.6 because in the sum (27) there are only finitely many terms and each term has finite dimension. □

Lemma 8.8 For $0 \leq i \leq d$ the following holds on $M$:

$$
e_0^* \tau_i(a) e_0^* = \frac{\zeta_i e_0^*}{(\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_i^*)}. \tag{28}
$$

Proof. First assume $i = 0$. Then (28) holds since $\tau_0 = 1$, $e_0^* = e_0^*$, and $\zeta_0 = 1$. Next assume $1 \leq i \leq d$. To show that (28) holds on $M$ we show $g_i M = 0$ where $g_i$ is from (17). By (22) and since $g_i \in J$ we have $g_i \xi = 0$. By this and Lemma 8.2(ii) we find $g_i e_0^* M = 0$. Now $g_i M = 0$ since $g_i e_0^* = g_i$. □

Lemma 8.9 The elements $e_0^* e_0^* e_0^*$, $e_0^* e_0^* e_0^*$ are nonzero on $M$.

Proof. By Lemma 8.2(ii) $e_0^*$ is nonzero on $M$. Concerning $e_0^* e_0^* e_0^*$, by the equation on the left in (13) we have $e_d = \tau_d(a) \tau_d(\theta_d)^{-1}$. By Lemma 8.8 (with $i = d$) $e_d^* \tau_d(a) e_0^* = \eta_d^* (\theta_d)^{-1} \zeta_d e_0^*$ on $M$. Therefore $e_0^* e_d^* e_0^* = \tau_d(\theta_d)^{-1} \eta_d^* (\theta_d)^{-1} \zeta_d e_0^*$ on $M$. By this and since $\zeta_d \neq 0$ we find $e_0^* e_0^* e_0^*$ is nonzero on $M$. Concerning $e_0^* e_0^* e_0^*$, by the equation on the left in (13) we have $e_0 = \eta_d(a) \eta_d(\theta_d)^{-1}$. By [30, Proposition 5.5], $\eta_d = \sum_{i=0}^d \eta_{d-i}(\theta_0) \tau_i$. By these comments and Lemma 8.8

$$
e_0^* e_0 e_0^* = e_0^* \eta_d(\theta_0)^{-1} \eta_d^* (\theta_0)^{-1} \sum_{i=0}^d \eta_{d-i}(\theta_0) \eta_{d-i}(\theta_0)^{-1} \zeta_i
$$
on $M$. In the above line the sum is nonzero by (5) so $e_0^* e_0^* e_0^*$ is nonzero on $M$. □
9 The $T$-module $L$

In this section we show that there exists a unique maximal proper $T$-submodule of $M$. We call this module $M'$ and consider the quotient module $L := M/M'$.

Lemma 9.1 Let $V$ denote a proper $T$-submodule of $M$. Then $e_0^*V = 0$.

Proof. Suppose $e_0^*V \neq 0$. By construction $e_0^*V \subseteq e_0^*M$ and $e_0^*M$ has basis $\xi$ so $\xi \in V$. The space $V$ is $T$-invariant and $T\xi = M$ so $M = V$, for a contradiction. We conclude $e_0^*V = 0$. □

Lemma 9.2 Let $V$ and $V'$ denote proper $T$-submodules of $M$. Then $V + V'$ is a proper $T$-submodule of $M$.

Proof. We show $V + V' \neq M$. Note that $e_0^*(V + V') = e_0^*V + e_0^*V'$. By Lemma 9.1 $e_0^*V = 0$ and $e_0^*V' = 0$, so $e_0^*(V + V') = 0$. But $e_0^*M \neq 0$ by Lemma 8.2(ii), so $V + V' \neq M$. The result follows. □

Definition 9.3 Let $V$ denote a proper $T$-submodule of $M$. Then $V$ is called maximal whenever $V$ is not contained in any proper $T$-submodule of $M$, besides itself.

Lemma 9.4 There exists a unique maximal proper $T$-submodule in $M$.

Proof. Concerning existence, consider

$$\sum_{V} V,$$

where the sum is over all proper $T$-submodules $V$ of $M$. The space (29) is a proper $T$-submodule of $M$ by Lemma 9.2, and since $M$ has finite dimension. The $T$-submodule (29) is maximal by the construction. Concerning uniqueness, suppose $V$ and $V'$ are maximal proper $T$-submodules of $M$. By Lemma 9.2 $V + V'$ is a proper $T$-submodule of $M$. The space $V + V'$ contains each of $V$, $V'$, so $V + V'$ is equal to each of $V$, $V'$ by the maximality of $V$ and $V'$. Therefore $V = V'$ and the result follows. □

Definition 9.5 Let $M'$ denote the maximal proper $T$-submodule of $M$. Let $L$ denote the quotient $T$-module $M/M'$. By construction $L$ is nonzero, finite-dimensional, and irreducible.

Proposition 9.6 The sequence $(a;\{e_i\}_{i=0}^d; a^*; \{e_i^*\}_{i=0}^d)$ acts on $L$ as a sharp tridiagonal system with parameter array $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d)$.

Proof. We first show that $(a;\{e_i\}_{i=0}^d; a^*; \{e_i^*\}_{i=0}^d)$ acts on $L$ as a tridiagonal system. This will follow from Proposition 5.4 once we show that the integers $r, t, \delta$ from that proposition are $0, 0, d$ respectively. By construction, for $0 \leq i \leq d$ the dimension of $e_iL$ is equal to the
By Lemma 8.8 and since the canonical map $M$ have

$$
\phi
$$

\{1. By Lemma 5.1 $\Phi$ has eigenvalue sequence
diagonal system which we denote by $\Phi$. Observe that $\Phi$ is sha
to get $e_0^* e_0^* M = 0$. This implies $e_0^* e_0^* M = 0$ which con
day and since $\delta = \delta^*$ by Proposition 5.4, so $\delta^* \neq d$. Now $e_d M = 0$ by Lemma 5.3(ii), so $e_d M \subseteq M'$. In this containment we apply $e_0^*$ to both sides and use $e_0^* M' = 0$ to get $e_0^* e_0^* M = 0$. This implies $e_0^* e_0^* M = 0$ which contradicts Lemma 8.9. Therefore $t = 0$. Next we show $\delta = d$. Suppose $\delta \neq d$. Recall $\delta = \delta^*$ by Proposition 5.4, so $\delta^* \neq d$. Now $e_d L = 0$ by Lemma 5.3(ii), so $e_d M \subseteq M'$. In this containment we apply $e_0^*$ to both sides and use $e_0^* M' = 0$ to get $e_0^* e_0^* M = 0$. This implies $e_0^* e_d e_0^* M = 0$ which contradicts Lemma 8.9. Therefore $\delta = d$. We have shown $(r, t, \delta) = (0, 0, d)$. Therefore $(a; \{e_i\}_{i=0}^d; a^*; \{e_i^*\}_{i=0}^d)$ acts on $L$ as a tridiagonal system which we denote by $\Phi$. Observe that $\Phi$ is sharp since $e_0^* L$ has dimension 1. By Lemma 5.4 $\Phi$ has eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$. By Lemma 8.8 and since the canonical map $M \rightarrow L$ is a $T$-module homomorphism, we have

$$
eq_0^* e_i^* = \frac{\zeta_i e_0^*}{(\theta_0^* - \theta_i^*)(\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_d^*)} \quad (0 \leq i \leq d)
$$
on $L$. By this and Definition 2.5 the sequence $\{\zeta_i\}_{i=0}^d$ is the split sequence for $\Phi$. By these comments $\Phi$ has parameter array $((\theta_i)_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d)$ and the result follows. 

\[\square\]

10 The $\mu$-conjecture and the classification conjecture

In this section we show that the $\mu$-conjecture implies Conjecture 3.1. This is our first main result.

\textbf{Theorem 10.1} Conjecture 6.4 implies Conjecture 3.1.

\textbf{Proof.} We assume Conjecture 6.4 is true and show Conjecture 3.1 is true. Let the scalars $\{\theta_i\}_{i=0}^d$ be given. To prove Conjecture 3.1 in one direction, assume that there exists a sharp tridiagonal system $\Phi$ that has parameter array $\{\theta_i\}_{i=0}^d$. Then Conjecture 3.1(i) holds by the construction, Conjecture 3.1(ii) holds by [34, Corollary 8.3], and Conjecture 3.1(iii) holds by [14, Theorem 11.1]. To prove Conjecture 3.1 in the other direction, assume that the scalars $\{\theta_i\}_{i=0}^d$ satisfy Conjecture 3.1(i)–(iii). Then by Proposition 9.6 there exists a sharp tridiagonal system over $\mathbb{F}$ with parameter array $\{\theta_i\}_{i=0}^d$. By [36, Theorem 1.6] this tridiagonal system is unique up to isomorphism of tridiagonal systems. 

\[\square\]

11 Tridiagonal pairs over an algebraically closed field

In this section we give a variation of Conjecture 3.1 involving tridiagonal systems over an algebraically closed field. We show that this variation follows from the $\mu$-conjecture.
Conjecture 11.1 Assume the field $\mathbb{F}$ is algebraically closed. Let $d$ denote a nonnegative integer and let
\[ (\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d) \] (30)
denote a sequence of scalars taken from $\mathbb{F}$. Then there exists a tridiagonal system $\Phi$ over $\mathbb{F}$ with parameter array (30) if and only if (i)-(iii) hold below.

(i) $\theta_i \neq \theta_j, \theta_i^* \neq \theta_j^*$ if $i \neq j$ $(0 \leq i, j \leq d)$.

(ii) $\zeta_0 = 1, \zeta_d \neq 0$, and
\[ \sum_{i=0}^{d} \eta_{d-i}(\theta_0^*)\eta_{d-i}(\theta_0^*)\zeta_i \neq 0. \]

(iii) The expressions
\[ \frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \quad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*} \]
are equal and independent of $i$ for $2 \leq i \leq d - 1$.

Suppose (i)-(iii) hold. Then $\Phi$ is unique up to isomorphism of tridiagonal systems.

Theorem 11.2 Conjecture 6.4 implies Conjecture 11.1.

Proof. By [36, Theorem 1.3] every tridiagonal system over an algebraically closed field is sharp. The result follows from this and Theorem 10.1. \qed

12 The $\mu$-conjecture is true for $d \leq 5$

In this section we show that the $\mu$-conjecture holds for $d \leq 5$. This is our second main result.

Theorem 12.1 Conjecture 6.4 is true for $d \leq 5$.

Proof. Let $d$ be given. Referring to the Appendix, let $V$ denote the $R$-module consisting of formal $R$-linear combinations of the given basis. We define a $T$-module structure on $V$ as follows. An $\mathbb{F}$-linear transformation $\psi : V \to V$ is said to commute with $R$ whenever $\psi r - r \psi$ is zero on $V$ for all $r \in R$. Let $a : V \to V$ and $a^* : V \to V$ denote the unique $\mathbb{F}$-linear transformations that commute with $R$ and act in the specified way on the basis. For $0 \leq i \leq d$ define $\mathbb{F}$-linear transformations $e_i : V \to V$ and $e_i^* : V \to V$ such that (13) holds. By a laborious computation (or with the aid of Mathematica) one can check that relations (7)-(11) hold on $V$. This gives a $T$-module structure on $V$. Among the basis elements in the Appendix there is one denoted $\phi$. For $1 \leq i \leq d$ we show
\[ e_0^* \tau_i(a)(e_0^*)^* \phi = \frac{y_i}{(\theta_0^* - \theta_1^*)(\theta_0^* - \theta_1^*) \cdots (\theta_0^* - \theta_i^*)} \phi. \] (31)
Assume $d \geq 1$; otherwise there is nothing to show. From the Appendix $a^*.\phi = \theta_0^*\phi$. By this and since $e_0^* = \eta_0^*(a^*)\eta_0^*(\theta_0^*)^{-1}$ we find $e_0^*.\phi = \phi$. Among the basis elements in the Appendix, consider the following elements:

$$\phi \ r \ r^2 \ldots \ r^i \ l r^i \ l^2 r^i \ldots \ l^{i-1} r^i$$

Abbreviate $r^0 = \phi$. By the data in the Appendix, $(a - \theta_h).r^h = r^{h+1}$ for $0 \leq h \leq i-1$ and $(a^* - \theta^*_{i-h}).l^h r^i = l^{h+1} r^i$ for $0 \leq h \leq i-2$. Moreover $(a^* - \theta^*_1).l^{i-1} r^i = y_i \phi$. Therefore

$$(a^* - \theta^*_1)(a^* - \theta^*_2)\ldots(a^* - \theta^*_i)\tau_1(a).\phi = y_i \phi.$$ 

In this equation we multiply both sides on the left by $e_0^*$ and simplify the result using the equation on the right in (12). Evaluating the result further using $\phi = e_0^*.\phi$ yields (31). By (15), (31), and Corollary 6.3

$$\mu(x_i).\phi = x_i \phi \quad (1 \leq i \leq d).$$

By this and since $\mu$ is an $\mathbb{F}$-algebra homomorphism, for $f \in R$ we have $\mu(f).\phi = f \phi$. By construction $f \phi \neq 0$ if $f \neq 0$; therefore $\mu$ in injective and hence an isomorphism. The result follows. 

\[ \square \]

**Corollary 12.2** Conjectures [3.1] and [11.1] are true for $d \leq 5$.

**Proof.** Follows from Theorems [10.1] [11.2] and [12.1] \[ \square \]

### 13 Suggestions for future research

In this section we give some suggestions for future research.

In what follows let $\{\{\theta_i\}^d_{i=0}, \{\theta^*_i\}^d_{i=0}\}$ denote a sequence of scalars taken from $\mathbb{F}$ that satisfy the conditions (i), (iii) of Conjecture 3.1. Let $T$ denote the corresponding algebra from Definition 4.1. We are going to describe a subset of $T$ that we think is a basis. To aid in this description we make a few definitions.

**Definition 13.1** Referring to Definition 4.1, we call $\{e_i\}^d_{i=0}$ and $\{e_i^*\}^d_{i=0}$ the standard generators for $T$. We call $\{e_i^*\}^d_{i=0}$ starred and $\{e_i\}^d_{i=0}$ nonstarred. A pair of standard generators is alternating whenever one of them is starred and the other is nonstarred. For $0 \leq i \leq d$ we call $i$ the index of $e_i$ and $e_i^*$.

**Definition 13.2** For an integer $n \geq 0$, by a word of length $n$ in $T$ we mean a product $u_1u_2\ldots u_n$ such that $\{u_i\}^n_{i=1}$ are standard generators and $u_{i-1}, u_i$ are alternating for $2 \leq i \leq n$. We interpret the word of length $0$ as the identity element of $T$. We call this word trivial.

**Definition 13.3** For $0 \leq i, j, r \leq d$ we say $r$ is between the ordered pair $i, j$ whenever $i \geq r \geq j$ or $i \leq r \leq j$. For standard generators $u, v, w$ we say $u$ is between the ordered pair $v, w$ whenever the index of $u$ is between the indices of $v, w$. 

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Definition 13.4 A word $u_1 u_2 \cdots u_n$ in $T$ is called zigzag (or ZZ) whenever (i), (ii) hold below:

(i) $u_i$ is not between $u_{i-1}, u_{i+1}$ for $2 \leq i \leq n - 1$;

(ii) At least one of $u_{i-1}, u_i$ is not between $u_{i-2}, u_{i+1}$ for $3 \leq i \leq n - 1$.

Conjecture 13.5 Fix integers $r, s$ ($0 \leq r, s \leq d$). Then the $\mathbb{F}$-vector space $T$ has a basis consisting of the ZZ words that do not involve $e_r, e^*_s$.

Conjecture 13.6 The $\mathbb{F}$-vector space $Te^*_0$ has a basis consisting of the nontrivial ZZ words that end in $e^*_0$ and do not involve $e_0, e^*_d$.

Before we state the next conjecture, we have some comments concerning $Te^*_0$ and the algebra $R$ from Definition 6.2. Note that $Te^*_0$ has a (right) $R$-module structure such that $v.r = v\mu(r)$ for all $v \in Te^*_0$ and $r \in R$ (the map $\mu$ is from Corollary 6.3). Let $\text{End}(Te^*_0)$ denote the $\mathbb{F}$-algebra consisting of all $\mathbb{F}$-linear transformations from $Te^*_0$ to $Te^*_0$. An element $f \in \text{End}(Te^*_0)$ is said to commute with $R$ whenever $f(v.r) = f(v).r$ for all $v \in Te^*_0$ and all $r \in R$. Let $\text{End}(Te^*_0)$ denote the subalgebra of $\text{End}(Te^*_0)$ consisting of the elements which commute with $R$. Note that $Te^*_0$ has a (left) $T$-module structure such that $t.v \mapsto tv$ for all $t \in T$ and $v \in Te^*_0$. Observe that the action of $T$ on $Te^*_0$ commutes with $R$ and therefore induces an $\mathbb{F}$-algebra homomorphism $T \to \text{End}_R(Te^*_0)$.

Conjecture 13.7 The above map $T \to \text{End}_R(Te^*_0)$ is an injection.

Definition 13.8 Let $W$ denote a right $R$-module. By an $R$-basis for $W$ we mean a sequence $\{z_i\}_{i=1}^n$ of elements in $W$ such that each element of $W$ can be written uniquely as $\sum_{i=1}^n z_i.r_i$ with $r_i \in R$ for $1 \leq i \leq n$.

Definition 13.9 [37, Section 7.4] Let $W$ denote a right $R$-module. Then $W$ is called free whenever $W$ has at least one $R$-basis. In this case the number of elements in a basis is independent of the $R$-basis. This number is called the rank of $W$.

We are going to describe a subset of $Te^*_0$ which we think is an $R$-basis. To describe the subset we will use the following notation.

Definition 13.10 By a feasible ZZ word in $T$ we mean a nontrivial ZZ word that ends in $e^*_0$ and whose indices are mutually distinct.
Example 13.11 For $d \leq 4$ we list the feasible $ZZ$ words in $T$.

| $d$ | feasible $ZZ$ words in $T$ |
|-----|-----------------------------|
| 0   | $e_0^*$                     |
| 1   | $e_0^*$, $e_1 e_0^*$        |
| 2   | $e_0^*$, $e_1 e_0^*$, $e_2 e_0^*$, $e_1^* e_2 e_0^*$ |
| 3   | $e_0^*$, $e_1 e_0^*$, $e_2 e_0^*$, $e_3 e_0^*$, $e_1^* e_2 e_0^*$, $e_1^* e_3 e_0^*$, $e_2^* e_3 e_0^*$, $e_2^* e_1^* e_3 e_0^*$ |
| 4   | $e_0^*$, $e_1 e_0^*$, $e_2 e_0^*$, $e_3 e_0^*$, $e_4 e_0^*$, $e_1^* e_2 e_0^*$, $e_1^* e_3 e_0^*$, $e_1^* e_4 e_0^*$, $e_2^* e_3 e_0^*$, $e_3^* e_4 e_0^*$, $e_2^* e_3^* e_4 e_0^*$, $e_2^* e_3 e_1^* e_4 e_0^*$ |

Conjecture 13.12 The $R$-module $Te_0^*$ is free with rank $2^d$. Moreover this module has an $R$-basis consisting of the feasible $ZZ$ words.

Conjecture 13.13 For $0 \leq i \leq d$ the $R$-submodules $e_i^* Te_0^*$ and $e_i Te_0^*$ are both free with rank $\binom{d}{i}$.

Problem 13.14 An element of $T$ is called central whenever it commutes with every element of $T$. The center $Z(T)$ is the $\mathbb{F}$-subalgebra of $T$ consisting of the central elements of $T$. Describe $Z(T)$. Find a generating set for $Z(T)$. Find a basis for the $\mathbb{F}$-vector space $Z(T)$.

Problem 13.15 For $0 \leq i, j \leq d$ write the word $e_0^* e_i e_j e_0$ as a linear combination of the words $e_0^* e_r e_0$ ($0 \leq r \leq d$). What are the coefficients in this linear combination? See [41, Lemma 14.5] for a partial answer.
14 Appendix

In this appendix we give some data that is used in the proof of Theorem 12.1. Let $d$ denote a nonnegative integer at most 5 and let $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d)$ denote a sequence of scalars taken from $\mathbb{F}$ that satisfy the conditions (i), (iii) of Conjecture 3.1. For $0 \leq i \leq d - 2$ we define scalars

$$\varepsilon_i = (\theta_{i+1} - \theta_{i+2})(\theta_{i+1}^* - \theta_{i+2}^*) - (\theta_i - \theta_{i+1})(\theta_i^* - \theta_{i+1}^*).$$

**The case $d = 0$**

The basis is $\phi$. The action is $a.\phi = \theta_0 \phi$, $a^*\phi = \theta_0^* \phi$.

**The case $d = 1$**

The basis is $\phi, r$. The action of $a$ is $a.\phi = \theta_0 \phi + r$, $a.r = \theta_1 r$. The action of $a^*$ is $a^*\phi = \theta_0^* \phi$, $a^*.r = \theta_1^* r + y_1 \phi$.

**The case $d = 2$**

The basis is $\phi, r, lr^2, r^2$. The action of $a$ is $a.\phi = \theta_0 \phi + r$, $a.r = \theta_1 r + r^2$, $a.lr^2 = \theta_1 lr^2 + (y_1 - \varepsilon_0) r^2$, $a.r^2 = \theta_2 r^2$. The action of $a^*$ is $a^*\phi = \theta_0^* \phi$, $a^*r = \theta_1^* r + y_1 \phi$, $a^*lr^2 = \theta_1^* lr^2 + y_2 \phi$, $a^*r^2 = \theta_2^* r^2 + lr^2$.

For $d \geq 3$ we define $\beta \in \mathbb{F}$ such that $\beta + 1$ is the common value of (6). We remark that $\beta + 1$ is nonzero; otherwise $\theta_0 = \theta_3$. For $d \geq 4$ the scalar $\beta$ is nonzero; otherwise $\theta_0 = \theta_4$. For $d = 5$ the scalar $\beta^2 + \beta - 1$ is nonzero; otherwise $\theta_0 = \theta_5$. 
The case $d = 3$

The basis is

$$
\begin{align*}
\phi \\
r & r^2 \\
l_{r^2} & l^2 r^3 \\
l_{r^3} & r l^2 r^3 \\
l_{r^3} & r^3
\end{align*}
$$

The action of $a$ is

| $v$ | $a.v$ |
|-----|-------|
| $\phi$ | $\theta_0 \phi + r$ |
| $r$ | $\theta_1 r + r^2$ |
| $l_{r^2}$ | $\theta_1 l_{r^2} + (y_1 - \varepsilon_0)r^2 + (\beta + 1)^{-1}l_{r^3}$ |
| $l^2 r^3$ | $\theta_1 l^2 r^3 + r l^2 r^3$ |
| $r^2$ | $\theta_2 r^2 + r^3$ |
| $l_{r^3}$ | $\theta_2 l_{r^3} + (y_1 + (\theta_0 - \theta_1)(\theta_0^* - \theta_3^*) - (\theta_0 - \theta_3)(\theta_2^* - \theta_3^*))r^3$ |
| $r l^2 r^3$ | see below |
| $r^3$ | $\theta_3 r^3$ |

$a r l^2 r^3$ is the weighted sum involving the following terms and coefficients.

| term | coefficient |
|------|-------------|
| $r l^2 r^3$ | $\theta_2$ |
| $r^3$ | $y_2 + y_1(\beta + 2)((\theta_0 - \theta_1)(\theta_1^* - \theta_2^*) - (\theta_1 - \theta_2)(\theta_2^* - \theta_3^*))$ |
| | $+((\theta_0 - \theta_1)(\theta_0^* - \theta_3^*) - (\theta_0 - \theta_3)(\theta_2^* - \theta_3^*))$ |
| | $\times((\theta_0 - \theta_1)(\theta_0^* - \theta_2^*) - (\theta_1 - \theta_2)(\theta_1^* - \theta_3^*))$ |

The action of $a^*$ is

| $v$ | $a^*v$ |
|-----|-------|
| $\phi$ | $\theta_0^* \phi$ |
| $r$ | $\theta_1^* r + y_1 \phi$ |
| $l_{r^2}$ | $\theta_1^* l_{r^2} + y_2 \phi$ |
| $l^2 r^3$ | $\theta_1^* l^2 r^3 + y_3 \phi$ |
| $r^2$ | $\theta_2^* r^2 + l r^2$ |
| $l_{r^3}$ | $\theta_2^* l_{r^3} + l^2 r^3$ |
| $r l^2 r^3$ | $\theta_2^* l^2 r^3 + y_3(\beta + 1)^{-1}r + (y_1 + (\theta_0 - \theta_1)(\theta_0^* - \theta_2^*) - (\theta_1 - \theta_2)(\theta_2^* - \theta_3^*))l^2 r^3$ |
| $r^3$ | $\theta_3^* r^3 + l r^3$ |
The case $d = 4$

The basis is

$$
\begin{align*}
\phi \\
r & r^2 \\ l^2 & r^2 \\
l^3 & r^3 \\
l^4 & r^4 \\
l^2 & r^3 \\
l^3 & r^4 \\
l^4 & r^4
\end{align*}
$$

The action of $a$ is

| $v$ | $a.v$ |
|-----|-------|
| $\phi$ | $\theta_0 \phi + r$ |
| $r$ | $\theta_1 r + r^2$ |
| $l^2$ | $\theta_1 l^2 r^2 + (y_1 - \varepsilon_0)r^2 + (\beta + 1)^{-1}l r^3$ |
| $l^3$ | $\theta_1 l^3 r^3 + r l^2 r^3$ |
| $l^4$ | $\theta_1 l^4 r^4 + r l^3 r^4$ |
| $l^2 r^3$ | see below |
| $l^3 r^4$ | $\theta_2 l^3 r^4 + r l^2 r^4$ |
| $l^4 r^4$ | $\theta_2 l^4 r^4 + r l^3 r^4$ |
| $l^2 l^3 r^4$ | see below |
| $l^3 l^4 r^4$ | see below |
| $r^3$ | $\theta_3 l^3 r^4 + r^4$ |
| $l^4$ | $\theta_3 l^4 r^4 + (y_1 - \beta(\beta + 1)\varepsilon_1)r^4$ |
| $l^2 r^4$ | $\theta_3 l^2 r^4 + (y_2 - y_1(\beta + 1)(\beta + 2)\varepsilon_1 + (\beta + 1)^2\varepsilon_1(\varepsilon_0 + (\beta + 1)\varepsilon_1))r^4$ |
| $r^2 l^3 r^4$ | see below |
| $r^4$ | $\theta_4 r^4$ |

$a rl^2 r^3$ is the weighted sum involving the following terms and coefficients.

| term | coefficient |
|------|-------------|
| $rl^2 r^3$ | $\theta_2$ |
| $r^3$ | $y_2 - y_1 \beta^{-1}(\beta + 2)(\varepsilon_0 + \varepsilon_1) + \beta^{-2}(\beta + 1)(\varepsilon_0 + \varepsilon_1)((\beta + 1)\varepsilon_0 + \varepsilon_1)$ |
| $l r^4$ | $\beta^{-2}(-\varepsilon_0 + (\beta^2 + \beta - 1)\varepsilon_1)$ |
| $r l^2 r^4$ | $\beta^{-1}$ |
\( a.lr^2l^3r^4 \) is the weighted sum involving the following terms and coefficients.

| term | coefficient |
|------|-------------|
| \( lr^2l^3r^4 \) | \( \theta_2 \) |
| \( r^3 \) | \( y_4 \beta^{-1}(\beta+1)^{-1} \) |
| \( l^4 \) | \( y_3(\beta+1)^{-1} - y_2(\beta+1)^2 \epsilon_1 + y_1 \beta^{-1}(\beta+1)^2 \epsilon_1(\epsilon_0 + (\beta+1)\epsilon_1) \) |
| \( y_1 - \beta^{-1}(\beta+1)\epsilon_0 + (2\beta+1)\epsilon_1 \) |

\( a.r^2l^3r^4 \) is the weighted sum involving the following terms and coefficients.

| term | coefficient |
|------|-------------|
| \( r^2l^3r^4 \) | \( \theta_3 \) |
| \( r^4 \) | \( y_3 - y_2(\beta+1)^2 \epsilon_1 + y_1 \beta^{-1}(\beta+1)^3 \epsilon_1(\epsilon_0 + (\beta+1)\epsilon_1) \) |
| \( -\beta^{-1}(\beta+1)^3 \epsilon_1(\epsilon_0 + \epsilon_1)(\epsilon_0 + (\beta+1)\epsilon_1) \) |

The action of \( a^* \) is

| \( v \) | \( a^*.v \) |
|------|-------------|
| \( \phi \) | \( \theta^*_0 \phi \) |
| \( r \) | \( \theta^*_1 r + y_1 \phi \) |
| \( lr^2 \) | \( \theta^*_2 l^2 + y_2 \phi \) |
| \( l^2r^3 \) | \( \theta^*_3 l^2r^3 + y_3 \phi \) |
| \( lr^3 \) | \( \theta^*_4 l^3r^4 + y_4 \phi \) |
| \( r^2l^3 \) | \( \theta^*_2 r^2l^3 + y_2(\beta+1)^{-1} r + (y_1 - \beta^{-1}((\beta+1)\epsilon_0 + \epsilon_1))l^2r^3 + \beta^{-1}(\beta+1)^{-1}l^3r^4 \) |
| \( l^2r^4 \) | \( \theta^*_3 l^2r^4 + \beta^3r^4 \) |
| \( rl^3r^4 \) | \( \theta^*_2 rl^3r^4 + y_4 \beta^{-1} r + (y_1 - \beta^{-1}(\beta+1)(\epsilon_0 + \epsilon_1))l^3r^4 \) |
| \( lr^2l^3r^4 \) | see below |
| \( r^3 \) | \( \theta^*_3 r^3l^3 + \beta^3r^3 \) |
| \( lr^4 \) | \( \theta^*_3 lr^4 + l^2r^4 \) |
| \( r^2l^4r^4 \) | \( \theta^*_2 r^2l^4r^4 + (y_1 - (\epsilon_0 + (\beta+2)\epsilon_1))l^2r^4 + (\beta + 1)^{-1}rl^3r^4 \) |
| \( r^2l^3r^4 \) | \( \theta^*_3 r^2l^3r^4 + lr^2l^3r^4 \) |
| \( r^4 \) | \( \theta^*_4 r^4 + lr^3 \) |
\[ a^* l r^2 l^3 r^4 \] is the weighted sum involving the following terms and coefficients.

| term      | coefficient                                          |
|-----------|------------------------------------------------------|
| \(l r^2 l^3 r^4\) | \(\theta_2^*\)                                     |
| \(r\)     | \(- y_1 \beta^2 (\varepsilon_0 + (\beta + 1)\varepsilon_1)\) |
| \(l r^2\) | \(y_4 \beta^{-1}\)                                  |
| \(l^3 r^4\) | \(y_2 - y_1 \beta^{-1} (\beta + 2)(\varepsilon_0 + (\beta + 1)\varepsilon_1) + \beta^{-2} (\beta + 1)^2(\varepsilon_0 + \varepsilon_1)(\varepsilon_0 + (\beta + 1)\varepsilon_1)\) |

**The case \(d = 5\)**

The basis is

\[
\begin{align*}
\phi & \quad r & l r^2 & l^2 r^3 & l^3 r^4 & l^4 r^5 \\
\phi & \quad r^2 & l r^3 & r l^2 r^3 & l^2 r^4 & l^3 r^4 & l^4 r^5 & r l^4 r^5 & l r^2 l^4 r^5 & l^2 r^3 l^4 r^5 \\
\phi & \quad r^3 & l r^4 & r l^2 r^4 & r^2 l^3 r^4 & l^2 r^5 & r l^3 r^5 & l l^2 l^3 r^5 & r l^4 r^5 & r^{2} l^1 r^5 & l^{2} l^{2} l^{4} r^5 & r^{2} l^{4} l^4 r^5 \\
\phi & \quad r^4 & l r^5 & r l^2 r^5 & r^2 l^3 r^5 & l^2 r^6 & r l^3 r^6 & l^{2} l^{2} l^{4} r^6 & r l^4 r^6 & r^{2} l^{4} l^4 r^6 & r^{2} l^{2} l^{2} r^6 & l^{2} l^{2} l^{4} r^6 \\
\phi & \quad r^5 & l r^6 & r l^2 r^6 & r^2 l^3 r^6 & l^2 r^7 & r l^3 r^7 & l^{2} l^{2} l^{4} r^7 & r l^4 r^7 & r^{2} l^{4} l^4 r^7 & r^{2} l^{2} l^{2} r^7 & l^{2} l^{2} l^{4} r^7
\end{align*}
\]
The action of $a$ is

| $v$  | $a.v$                                              |
|------|----------------------------------------------------|
| $\phi$ | $\theta_0 \phi + r$                              |
| $r$  | $\theta_1 r + r^2$                                |
| $l_r^2$ | $\theta_1 l_r^2 + (y_1 - \varepsilon_0) r^2 + (\beta + 1)^{-1} l_r^3$ |
| $l_r^2 l_r^3$ | $\theta_1 l_r^2 l_r^3 + r l_r^2 l_r^3$ |
| $l_r^3 l_r^4$ | $\theta_1 l_r^3 l_r^4 + r l_r^3 l_r^4$ |
| $l_r^4 l_r^5$ | $\theta_1 l_r^4 l_r^5 + r l_r^4 l_r^5$ |
| $r^2$ | $\theta_2 r^2 + r^3$                              |
| $l_r^3$ | $\theta_2 l_r^3 + (y_1 - \beta^{-1}(\beta + 1)(\varepsilon_0 + \varepsilon_1)) r^3 + \beta^{-1} l_r^4$ |
| $l_r^2 l_r^4$ | see below                                          |
| $l_r^3 l_r^4$ | $\theta_2 l_r^3 l_r^4 + r l_r^3 l_r^4$ |
| $l_r^4 l_r^5$ | see below                                          |
| $r^2 l_r^4$ | $\theta_2 l_r^2 l_r^4 + r l_r^2 l_r^4$ |
| $l_r^2 l_r^4 l_r^5$ | $\theta_2 l_r^2 l_r^4 l_r^5 + r l_r^2 l_r^4 l_r^5$ |
| $l_r^3 l_r^4 l_r^5$ | $\theta_2 l_r^3 l_r^4 l_r^5 + r l_r^3 l_r^4 l_r^5$ |
| $r^4$ | $\theta_3 r^3 + r^4$                              |
| $l_r^4$ | $\theta_3 l_r^4 + (y_1 - \beta(\beta + 1) \varepsilon_1) r^4 + \frac{\beta + 1}{\beta^2 + \beta - 1} l_r^5$ |
| $l_r^2 l_r^4$ | see below                                          |
| $r^2 l_r^4 l_r^5$ | see below                                          |
| $l_r^3 l_r^4 l_r^5$ | $\theta_3 l_r^3 l_r^4 l_r^5 + r l_r^3 l_r^4 l_r^5$ |
| $l_r^4 l_r^5$ | see below                                          |
| $r^2 l_r^4 l_r^5$ | $\theta_3 l_r^2 l_r^4 l_r^5 + r l_r^2 l_r^4 l_r^5$ |
| $l_r^3 l_r^4 l_r^5$ | see below                                          |
| $r^4 l_r^5$ | see below                                          |
| $l_r^5$ | $\theta_4 l_r^4 + r^5$                            |
| $l_r^2 l_r^5$ | $\theta_4 l_r^2 l_r^5 + (y_1 - (\beta^2 + \beta - 1)(-\varepsilon_0 + (\beta^2 - 1) \varepsilon_1)) r^5$ |
| $r^3 l_r^4 l_r^5$ | see below                                          |
| $r^5$ | $\theta_5 r^5$                                    |
$a.r^{l^2}r^3$ is the weighted sum involving the following terms and coefficients.

| term    | coefficient |
|---------|-------------|
| $r^{l^2}r^3$ | $\theta_2$ |
| $r^3$ | $y_2 - y_1\beta^{-1}((\beta + 2)(\varepsilon_0 + \varepsilon_1) + \beta^{-2}(\beta + 1)(\varepsilon_0 + \varepsilon_1)((\beta + 1)\varepsilon_0 + \varepsilon_1))$ |
| $l^r$ | $\beta^{-2}(\varepsilon_0 + (\beta^2 + \beta - 1)\varepsilon_1)$ |
| $r^{l^2}r^4$ | $\beta^{-1}$ |
| $l^r$ | $-\frac{1}{\beta(\beta^2 + \beta - 1)}$ |

$a.l^2r^3$ is the weighted sum involving the following terms and coefficients.

| term    | coefficient |
|---------|-------------|
| $l^{l^2}r^4$ | $\theta_2$ |
| $r^3$ | $y_4\beta^{-1}(\beta + 1)^{-1}$ |
| $l^r$ | $y_3((\beta + 1)^{-1} - y_2((\beta + 1)\varepsilon_1 + y_1\beta^{-1}(\beta + 1)^2\varepsilon_1(\varepsilon_0 + (\beta + 1)\varepsilon_1))$ |
| $r^{l^2}r^4$ | $\beta^{-1}(\beta + 1)^2\varepsilon_1(\varepsilon_0 + \varepsilon_1)((\beta + 1)\varepsilon_0 + \varepsilon_1)$ |
| $l^r$ | $y_1 - \beta^{-1}(\beta + 1)\varepsilon_0 + (2\beta + 1)\varepsilon_1$ |
| $l^{l^2}r^5$ | $\frac{1}{\beta(\beta^2 + \beta - 1)}$ |
| $l^r$ | $\frac{1}{\beta(\beta^2 + \beta - 1)}$ |

$a.r^{l^2}r^4$ is the weighted sum involving the following terms and coefficients.

| term    | coefficient |
|---------|-------------|
| $r^{l^2}r^4$ | $\theta_3$ |
| $r^4$ | $y_2 - y_1((\beta + 1)(\beta + 2)\varepsilon_1 + (\beta + 1)^2\varepsilon_1(\varepsilon_0 + (\beta + 1)\varepsilon_1)$ |
| $l^r$ | $\frac{(\beta + 1)(\beta + 2)(-\varepsilon_0 + (\beta^2 + 2)\varepsilon_1)}{\beta^2 + \beta - 1}$ |
| $r^{l^2}r^5$ | $\beta + 2$ |
| $l^r$ | $\frac{\beta^2 + \beta - 1}{\beta^2 + \beta - 1}$ |

$a.r^{2l^3}r^4$ is the weighted sum involving the following terms and coefficients.

| term    | coefficient |
|---------|-------------|
| $r^{2l^3}r^4$ | $\theta_3$ |
| $r^4$ | $y_3 - y_2((\beta + 1)^2\varepsilon_1 + y_1\beta^{-1}(\beta + 1)^3\varepsilon_1(\varepsilon_0 + (\beta + 1)\varepsilon_1)$ |
| $l^r$ | $-\beta^{-1}(\beta + 1)^3\varepsilon_1(\varepsilon_0 + \varepsilon_1)(\varepsilon_0 + (\beta + 1)\varepsilon_1)$ |
| $l^{l^2}r^5$ | $\frac{(\beta + 1)^3((\beta + 1)\varepsilon_0^2 - (2\beta + 3)(\beta^2 + 2\beta - 3)\varepsilon_1 + (\beta^2 + 2)(\beta^2 + 2\beta - 1)\varepsilon_2)}{\beta(\beta^2 + \beta - 1)}$ |
| $r^{l^2}r^5$ | $\frac{(\beta + 1)^2(-\varepsilon_0 + (\beta^2 + 2\beta - 1)\varepsilon_1)}{\beta(\beta^2 + \beta - 1)}$ |
| $r^{2l^3}r^5$ | $\beta + 1$ |
| $l^r$ | $\frac{\beta^2 + \beta - 1}{\beta(\beta^2 + \beta - 1)}$ |
\( a_1 r^2 l^3 r^5 \) is the weighted sum involving the following terms and coefficients.

| term | coefficient |
|------|-------------|
| \( l^2 l^3 r^5 \) | \( \theta_3 \) |
| \( l^5 \) | \( y_3 (\beta + 1)^{-1} - y_2 \beta^{-1} (\beta + 1)^2 (-\varepsilon_0 + (\beta^2 - 1)\varepsilon_1) \\ + y_1 \beta^{-1} (\beta + 1)^3 (-\varepsilon_0 + (\beta^2 - 1)\varepsilon_1) (-\varepsilon_0 + (\beta^2 + \beta - 1)\varepsilon_1) \\ - (\beta + 1)^2 (\beta^2 + \beta - 1)\varepsilon_1 (-\varepsilon_0 + (\beta^2 - 1)\varepsilon_1) (-\varepsilon_0 + (\beta^2 + \beta - 1)\varepsilon_1) \) |
| \( r^2 l^3 r^5 \) | \( y_1 - \beta^{-1} (-\varepsilon_0 + (2\beta^3 + 2\beta^2 - 2\beta - 1)\varepsilon_1) \) |
| \( r^3 l^4 r^5 \) | \( \beta^{-1} (\beta + 1)^{-1} \) |

\( a_1 r^3 t^4 r^5 \) is the weighted sum involving the following terms and coefficients.

| term | coefficient |
|------|-------------|
| \( l^3 t^4 r^5 \) | \( \theta_3 \) |
| \( r^4 \) | \( \frac{y_5}{\alpha(\beta^2 + \beta - 1)} \) |
| \( l^5 \) | \( y_4 \beta^{-1} - y_3 (\beta + 2)(-\varepsilon_0 + (\beta^2 - 1)\varepsilon_1) \\ + y_2 \beta^{-1} (\beta + 1)^2 (-\varepsilon_0 + (\beta^2 - 1)\varepsilon_1) (-\varepsilon_0 + (\beta^2 + \beta - 1)\varepsilon_1) \\ - y_1 (\beta + 1)^3 (\beta + 2)\varepsilon_1 (-\varepsilon_0 + (\beta^2 - 1)\varepsilon_1) (-\varepsilon_0 + (\beta^2 + \beta - 1)\varepsilon_1) \\ + \beta^{-1} (\beta + 1)^3 (\beta^2 + \beta - 1)\varepsilon_1 (\varepsilon_0 + (\beta + 1)\varepsilon_1) \times (-\varepsilon_0 + (\beta^2 - 1)\varepsilon_1) (-\varepsilon_0 + (\beta^2 + \beta - 1)\varepsilon_1) \) |
| \( r^3 t^4 r^5 \) | \( y_1 - \beta^{-1} (-\varepsilon_0 + (\beta + 1)(\beta^2 + \beta - 1)\varepsilon_1) \) |
\( a.r^2l^3r^4r^5 \) is the weighted sum involving the following terms and coefficients.

| term | coefficient |
|------|-------------|
| \( r^4l^3r^4l^5 \) | \( \frac{y_2(\beta^2 + \beta - 1)\varepsilon_0 + (\beta + 1)(2\beta^3 + 3\beta - 1)\varepsilon_1}{\beta^3(\beta^2 + \beta - 1)^2} \) |
| \( l^5r^4l^5 \) | \( y_1\beta^{-1}(\beta + 2)(-\varepsilon_0 + (\beta^2 - 1)\varepsilon_1) \) |

\( a.r^2l^3r^5 \) is the weighted sum involving the following terms and coefficients.

| term | coefficient |
|------|-------------|
| \( l^5r^5 \) | \( y_2 \) |

\( a.r^2l^3r^5 \) is the weighted sum involving the following terms and coefficients.

| term | coefficient |
|------|-------------|
| \( r^4l^3r^5 \) | \( \theta_4 \) |

\( a.r^2l^3r^5 \) is the weighted sum involving the following terms and coefficients.

| term | coefficient |
|------|-------------|
| \( r^5r^3l^4r^5 \) | \( \theta_4 \) |
$a_r^3 l^4 r^5$ is the weighted sum involving the following terms and coefficients.

| term     | coefficient |
|----------|-------------|
| $r^3 l^4 r^5$ | $\theta_4$ |
| $r^5$     | $y_4 - y_3 \beta (\beta + 2)(-\varepsilon_0 + (\beta^2 - 1)\varepsilon_1)$  
           | $+ y_2 (\beta + 1)^3(-\varepsilon_0 + (\beta^2 - 1)\varepsilon_1)(-\varepsilon_0 + (\beta^2 + \beta - 1)\varepsilon_1)$  
           | $- y_1 \beta (\beta + 1)^3(\beta + 2)\varepsilon_1(-\varepsilon_0 + (\beta^2 - 1)\varepsilon_1)(-\varepsilon_0 + (\beta^2 + \beta - 1)\varepsilon_1)$  
           | $+ (\beta + 1)^3(\beta^2 + \beta - 1)\varepsilon_1(\varepsilon_0 + (\beta + 1)\varepsilon_1)(-\varepsilon_0 + (\beta^2 - 1)\varepsilon_1)(-\varepsilon_0 + (\beta^2 + \beta - 1)\varepsilon_1)$ |
The action of $a^*$ is

\[
\begin{array}{c|c}
v & a^*.v \\
\hline
\phi & \theta_0^*\phi \\
\theta & \theta_1^*\theta + y_1\phi \\
l^2 & \theta_1^*l^2 + y_2\phi \\
l^2 r & \theta_1^*l^2 r^3 + y_3\phi \\
l^2 r^3 & \theta_1^*l^3 r^4 + y_4\phi \\
l^4 r & \theta_1^*l^4 r^5 + y_5\phi \\
l^2 & \theta_2^*l^2 + l^2 r \\
l^3 & \theta_2^*l^3 + l^3 r \\
l^2 r & \theta_2^*l^2 r^3 + y_6(\beta + 1)^{-1} r + (y_1 - \beta^{-1}(\beta + 1)\varepsilon_0 + \varepsilon_1))l^2 r^3 + \beta^{-1}(\beta + 1)^{-1}l^3 r^4 \\
l^2 r^3 & \theta_2^*l^2 r^4 + l^3 r^4 \\
l^2 & \theta_2^*l^2 r^5 + l^3 r^5 \\
l^2 r^4 & \theta_2^*l^3 r^5 + \frac{y_6(\beta + 1)}{\beta^2 + \beta - 1} r + (y_1 - (\varepsilon_0 + (\beta + 1)\varepsilon_1))l^4 r^5 \\
l^2 r^5 & \theta_2^*l^4 r^5 + y_7(\beta + 1)^{-1} r + (y_1 - \beta^{-1}(\beta + 1)\varepsilon_0 - \varepsilon_1))l^5 r^5 + \beta^{-1}l^6 r^5 \\
l^2 & \theta_3^*l^2 r^3 + l^3 r \\
l^3 & \theta_3^*l^3 r^4 + l^4 r \\
l^2 r & \theta_3^*l^2 r^5 + l^3 r^5 \\
l^2 r^3 & \theta_3^*l^3 r^5 + (y_1 - \beta^{-1}(\beta + 1)^{-1} l^3 r^5 + \beta^{-1}l^4 r^5 \\
l^2 r & \theta_4^*l^2 r^4 + l^3 r^4 \\
l^3 & \theta_4^*l^3 r^5 + l^4 r^5 \\
l^2 r & \theta_4^*l^4 r^5 + l^5 r^5 \\
l^3 & \theta_5^*l^5 + l^5 r \\
\end{array}
\]

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\(a^* lr^2 l^3 r^4\) is the weighted sum involving the following terms and coefficients.

| term | coefficient |
|------|-------------|
| \(lr^2 l^3 r^4\) | \(\theta_2^*\) |
| \(r\) | \(\frac{y_5(\beta+2)}{\beta(\beta^2+\beta-1)} - y_4\beta^{-2}(\epsilon_0 + (\beta + 1)\epsilon_1)\) |
| \(l^2\) | \(y_4\beta^{-1}\) |
| \(l^3 r^4\) | \(y_2 - y_1\beta^{-1}(\beta + 2)(\epsilon_0 + (\beta + 1)\epsilon_1) + \beta^{-2}(\beta + 1)^2(\epsilon_0 + \epsilon_1)(\epsilon_0 + (\beta + 1)\epsilon_1)\) |
| \(l^4 r^5\) | \(\frac{y_3(\beta+2)}{\beta(\beta^2+\beta-1)} - \frac{(\beta+1)^2(\epsilon_0 + (\beta+1)\epsilon_1)}{\beta^2(\beta^2+\beta-1)}\) |

\(a^* lr^2 l^4 r^5\) is the weighted sum involving the following terms and coefficients.

| term | coefficient |
|------|-------------|
| \(lr^2 l^4 r^5\) | \(\theta_2^*\) |
| \(r\) | \(\frac{y_5(\beta+1)(\beta+2)\epsilon_1}{\beta(\beta^2+\beta-1)}\) |
| \(l^2\) | \(\frac{y_5(\beta+2)}{\beta(\beta^2+\beta-1)}\) |
| \(l^4 r^5\) | \(y_2 - y_1(\beta + 1)(\beta + 2)\epsilon_1 + (\beta + 1)^2\epsilon_1(\epsilon_0 + (\beta + 1)\epsilon_1)\) |

\(a^* lr^2 l^3 l^4 r^5\) is the weighted sum involving the following terms and coefficients.

| term | coefficient |
|------|-------------|
| \(l^2 r^3 l^4 r^5\) | \(\theta_2^*\) |
| \(r\) | \(\frac{y_5(\beta+1)^2(\epsilon_0 + (\beta^2+\beta-1)\epsilon_1)}{\beta(\beta^2+\beta-1)}\) |
| \(l^2\) | \(-\frac{y_5(\beta+1)(\beta+2)(\epsilon_0 + (\beta^2+\beta-1)\epsilon_1)}{\beta(\beta^2+\beta-1)}\) |
| \(l^4 r^5\) | \(y_3 - y_2\beta^{-1}(\beta + 1)^2(-\epsilon_0 + (\beta^2 + \beta - 1)\epsilon_1) + y_1\beta^{-1}(\beta + 1)^4\epsilon_1(-\epsilon_0 + (\beta^2 + \beta - 1)\epsilon_1) - (\beta + 1)^3\epsilon_1(\epsilon_0 + (\beta + 1)\epsilon_1)(-\epsilon_0 + (\beta^2 + \beta - 1)\epsilon_1)\) |

\(a^* lr^2 l^3 r^5\) is the weighted sum involving the following terms and coefficients.

| term | coefficient |
|------|-------------|
| \(lr^2 l^3 r^5\) | \(\theta_3^*\) |
| \(r\) | \(-\frac{y_5}{\beta(\beta^2+\beta-1)}\) |
| \(l^3 r^5\) | \(y_2 - y_1\beta^{-1}(\beta + 2)(-\beta + 1)\epsilon_0 + (\beta^3 + 2\beta^2 - \beta - 1)\epsilon_1) + \beta^{-2}(\beta + 1)^2(-\epsilon_0 + (\beta^2 + \beta - 1)\epsilon_1) + \beta^{-2}(\beta + 1)^4(\epsilon_0 + (\beta^2 + \beta - 1)\epsilon_1) + (\beta + 1)^3(\epsilon_0 + (\beta^2 + \beta - 1)\epsilon_1)\) |
| \(l^4 r^5\) | \(-\beta^{-2}(-\beta + 1)\epsilon_0 + (\beta^3 + \beta^2 - 2\beta - 1)\epsilon_1) + (\beta + 1)(\beta^2 + \beta - 1)\epsilon_1\) |
| \(rl^4 r^5\) | \(\beta^{-1}\) |
$a^*, rl^2r^3l^4r^5$ is the weighted sum involving the following terms and coefficients.

| term | coefficient |
|------|-------------|
| $rl^2r^3l^4r^5$ | $\frac{y_3(\beta+1)\epsilon_0+(\beta+1)\epsilon_1}{\beta(\beta^2+\beta-1)}$ |
| $rl^2r^3$ | $-\frac{y_2(\beta+1)(\epsilon_0+\beta^2+\beta-1)\epsilon_1}{\beta(\beta^2+\beta-1)}$ |
| $l^2r^4$ | $-\frac{y_2(\epsilon_0+\beta^2+\beta-1)\epsilon_1}{\beta(\beta^2+\beta-1)}$ |
| $r^3l^5$ | $y_1\beta^{-1}(\beta+1)^{-1}-y_3(\beta+1)^{-1}(\beta+2)(\epsilon_0+(\beta^2-1)\epsilon_1)$ |
| $rl^4r^5$ | $y_3(\beta+1)^{-1}-y_2\beta^{-1}(\beta+1)(\epsilon_0+(\beta^2+\beta-1)\epsilon_1)$ |
| $l^2r^3l^4r^5$ | $y_1(\beta+1)^{-1}(\beta+1)^{-1}y_2(\beta+1)^{-1}(\epsilon_0+(\beta^2+\beta-1)\epsilon_1)$ |

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