A novel extension of Generalized Low-Rank Approximation of Matrices based on multiple-pairs of transformations

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Abstract

Dimension reduction is a main step in learning process which plays a essential role in many applications. The most popular methods in this field like SVD, PCA, and LDA, only can apply to vector data. This means that for higher order data like matrices or more generally tensors, data should be fold to a vector. By this folding, the probability of overfitting is increased and also maybe some important spatial features are ignored. Then, to tackle these issues, methods are proposed which work directly on data with their own format like GLRAM, MPCA, and MLDA. In these methods the spatial relationship among data are preserved and furthermore, the probability of overfitting has fallen. Also the time and space complexity are less than vector-based ones. Having said that, because of the less parameters in multilinear methods, they have a much smaller search space to find an optimal answer in comparison with vector-based approach. To overcome this drawback of multilinear methods like GLRAM, we proposed a new method which is a general form of GLRAM and by preserving the merits of it have a larger search space. We have done plenty of experiments to show that our proposed method works better than GLRAM. Also, applying this approach to other multilinear dimension reduction methods like MPCA and MLDA is straightforward.

Keywords: Machine learning, Matrix data classification, Kronecker product,

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1. Introduction

Machine learning (ML) is one of the most important concepts in Computer Science which has many applications in the real world such as face recognition\cite{1}, image processing\cite{2}, criminal recognition\cite{3}, medical images\cite{4}, computer vision\cite{5}, data mining\cite{6}, etc. ML examines the methods which can learn from data in order to improve the machine’s operating in different aspects. Data are usually considered as a vector in ML so to apply well-known ML algorithms like SVM\cite{7}, LDA\cite{8}, PCA\cite{9}, SVD\cite{10} to data with the matrix or tensor structure, we have to fold these data into a vector format. This folding causes two major problems. At first, by converting a matrix or tensor to a long (wide) vector, the number of free variables will be increased sharply, which can make overfitting in the model. Also in vectorizing on some data like images, the spatial relationship among features are not considered, in other words, each datum treated individually\cite{11}. For example, a grayscale image represented by $m \times n$ matrix in this approach will be reshaped to an $nm$ vector. Therefore, not only we have many free variables but also the local spatial relations among pixels of images are not considered.

After a while in order to tackle vector-based methods drawbacks, another approach considering data in their original format has been proposed which is known as the multi-linear(tensor-based) learning\cite{12}\cite{13}. In this approach, data do not need to be reformulated anymore, and so by considering data in their original format, the spatial relationship between data will be preserved.\cite{11} Furthermore, in contrast with vector-based methods, other methods based on tensor representation have much less free variables which can reduce the computational complexity and also probability of overfitting.\cite{13} For instance, for a $m \times n$ matrix with tensor-based methods, we have only $m + n$ free variables which are less than $nm$ in vector-based viewpoint. By developing of this approach, some tensor-based learning methods like MPCA\cite{15}, MLDA\cite{16}, STM\cite{17}, have been
represented as tensor counterparts of PCA, LDA, SVM, respectively. 
Despite of mentioned appropriate properties, these methods have some 
problems, either. The main problem of tensor-based methods is their limited search 
space which is only a subset of vector-based one. So, the probability of finding 
an optimal answer in such a small search space is less than the large feasible 
region in traditional methods, by far.

The problem of dimension reduction has become an essential tool for re-
moving noise from data, reduce redundancy, and feature extraction. One of the main DR methods that are also related to well-known PCA, is 
SVD. However, since data should be vectorized in SVD and in this case, 
the time and space complexities will be increased dramatically, it does not work 
well in some data like images. For this reason, several algorithms have been pro-
posed in the last decade to deal with the high complexity of SVD computation, 
where the best one of them was GLRAM representing each datum in its 
original format, instead of a vector, and by one pair of left and right projector, 
transfer the data into a smaller subspace.

Furthermore in recent years some variants of this method has been proposed. 
For example since at each iteration of GLRAM two SVD should be computed, 
this increases the time complexity. The authors in show that instead of 
SVD, its approximation by Lanczos could be used which improves its speed.

Although GLRAM has less complexity than SVD, its search space is smaller. 
So, the probability of finding an optimal answer has been reduced. In this pa-
er we proposed a method that by applying k-pair left and right projectors to 
data expand this limited space to find a more accurate answer. This idea is 
inspired by a new approach which Hou et al, have stated in their paper. 
They proposed a multiple-rank multilinear SVM for classification, that expands 
the search space of STM in order to gain a more accurate answer same as SVM.

Theoretically, we show that by this idea the space of search is increased and 
so the quality of approximation in this method will be better than GLRAM. Also 
experiments show the quality of the method based on approximation and classi-
fication results. So, our experiments illustrated that in the proposed method, we
can find a better answer for the low-rank approximation. Not to mention that through our approach, it is easy to show that the same idea could be applied to other tensor based dimension reduction methods like MPCA and MLDA, in the near future.

The rest of this paper is organized as follows. In section 2, we analyze SVD and GLRAM methods and the relationship between them. In section 3, we present our proposed method. Next, the experimental results will be discussed in section 4. And finally, the conclusion stated in section 5.

2. Related works

In real applications, data usually contains some noisy and redundant features which affect the quality of the learning process, especially for high dimensional data. Dimension Reduction (DR) is a process that by transforming data into a lower dimension, tries to eliminate noises and redundancy of data. Therefore, the occurrence of curse of dimensionality and other undesired properties of high-dimensional spaces will be reduced, which has an important role in many applications.

In the last decade, a large number of DR techniques with different viewpoints like PCA, SVD, Fisher LDA and so on, have been investigated. Similar to other ML methods the input of the mentioned DR methods should be in vector format and so data types like images and videos (Matrix or Tensor) should be represented as a vector. This folding of high-order data to vector not only has a high complexity but also can causes losing some important spatial relations of the data. In recent years some multilinear versions of the mentioned DR methods have been proposed which are able to work with high-order data like matrices and tensors directly without reshaping them to vector i.e. MPCA,
GLRAM and Multilinear LDA are the tensor versions of PCA, SVD, and LDA respectively. The main advantage of the multilinear methods could be summarized as follows:

- They maintain the structure and so the spatial relations of data.
- The parameters of the multilinear methods are less than the vector-based ones and so the computational complexity becomes less than linear methods. Also, for multilinear methods, the probability of occurrence of overfitting will be decreased.

However, the multilinear methods are not convex typically, and also their search space is much smaller than linear ones. So the solution obtained might be far from the optimal answer.

In this paper, we expand the search space of GLRAM, which can cause to gain better results. It should be mentioned that this approach could be applied to other multilinear DR methods, too.

2.1. Linear DR methods based on low-rank approximation

PCA and SVD are the main DR methods that are related to each other. Let \( X = [x_1, ..., x_N] \in \mathbb{R}^{n \times N} \) became a centralized data set. The principal component analysis (PCA) project the data from \( \mathbb{R}^n \) to \( \mathbb{R}^k, (k \ll n) \), by orthogonal transformation \( W \) such that the variance of the projected data \( Y = WX \) will be maximized.

It is easy to show that this can be formulated as follows:

\[
\max_W \ trace(W^TXX^T W) \quad (1) \\
\text{s.t.} \ W^TW = I.
\]

It is known that the eigenvectors of corresponding of the first \( k \) eigenvalues of \( XX^T \) are the solution to Eq.\[1\]. Hence, if \( X = U \Sigma V^T \) be the Singular Value Decomposition (SVD) of \( X \), the first left singular vectors \( U_k = [u_1, ..., u_k] \) are
the first eigenvectors of $XX^T$ and so $W = U_k$. Then, $Y_k = U_k^T X$ becomes projected data. By SVD, it is clear that the projected data $Y_k$ becomes

$$Y_k = U_k^T X = \Sigma_k V_k^T,$$

where $\Sigma_k = \text{diag}(\sigma_1, \sigma_2, ..., \sigma_k)$.

In addition, the projection could be interpreted with another viewpoint. Since $X_k = U_k \Sigma_k V_k^T = W Y_k$, is the best rank-$k$ approximation of $X$, so we found that $Y_k$ is a reduced form of original data $X$ such that have the least construction error and, the PCA equals to the following problem

$$\min_{Y_k, W} \|X - W Y_k\|_F^2$$

s.t. $W^T W = I$.

Therefore, the PCA and SVD dimension reduction also could be rewritten as follows:

$$\min_{W, Y} \sum_{i=1}^{N} \|x_i - W y_i\|_F^2$$

s.t. $W^T W = I$.

It should be mentioned that for General data matrix $X$, where is not centralized, the SVD on $\bar{X} = [x_1 - \mu, x_2 - \mu, ..., x_N - \mu]$ is equal to PCA on $A$ where $\mu$ is the mean of the data.

2.2. Generalized low-rank approximations

Nowadays by increasing the usage of matrix datasets, the SVD (PCA) could not be used directly on data. Now, if we have a dataset like $\{A_1, ..., A_N\}$ where
$A_i \in \mathbb{R}^{n_1 \times n_2}$, then to apply SVD (PCA), every matrix $A_i$ should be fold to vector as follows:

$$a_i = \text{Vec}(A_i) = \left[ a_{i1}^T, a_{i2}^T, ..., a_{in_2}^T \right]^T,$$

(5)

where $a_{ij}$ is the $j$-th column of matrix $A_i$.

It is obvious that this folding maybe destroys some spatial relations. Fig.1. shows this phenomenon.

![Figure 1: Vectorizing a grayscale image](image)

Recently, an extension of DR based on low-rank approximation to matrix data named Generalized Low-rank Approximation of Matrices (GLRAM) is investigated. In dimension reduction on $A_i$, GLRAM by unknown orthogonal transformation matrices $L \in \mathbb{R}^{n_1 \times k_1}$ and $R \in \mathbb{R}^{n_2 \times k_2}$ looks for reduced data $D_i \in \mathbb{R}^{k_1 \times k_2}$ where its reconstruction $LD_iR^T$ be the best low-rank approximation of $A_i$. Mathematically this can be modeled as follows

$$\min_{L \in \mathbb{R}^{n_1 \times k_1} : L^T L = I_{k_1}} \sum_{i=1}^{N} \|A_i - LD_iR^T\|_F^2.$$  

(6)

Jieping Ye in his article[25] showed that the optimal values of $L$ and $R$ should
be the solution of the following maximization problem

$$\max_{L \in \mathbb{R}^{n_1 \times k_1}} \sum_{i=1}^{N} \| L^T A_i R \|^2_F,$$

and the optimal value of $D_i$ is $D_i = L^T A_i R$. So instead of solving Eq.6, tried to solve Eq.7. Also, to solve Eq.7, an alternating schema is used and at each step this equation is substituted with the following two subproblems according to $R$ and $L$

$$\max_R \ trace(R^T M_R R)$$
$$s.t \ R^T R = I, \quad (8)$$

and

$$\max_L \ trace(L^T M_L L)$$
$$s.t \ L^T L = I, \quad (9)$$

where $M_R = \sum_{i=1}^{N} A_i^T L L^T A_i$ and $M_L = \sum_{i=1}^{N} A_i R R^T A_i^T$. The optimal value of $L \in \mathbb{R}^{n_1 \times k_1}$ in Eq.8 and $R \in \mathbb{R}^{n_2 \times k_2}$ in Eq.9 are the first $k_1$ and $k_2$ eigenvectors of $M_L$ and $M_R$ matrices respectively.\[25\]

The process of solving the GLRAM problem has been shown in Algorithm[1] in detail.

Due to heavy computation for obtaining each eigenvalue, particularly in a deal with large data, the time and computational complexity will be increased which cause dire problems. Then, another approach has been proposed named Bilinear Lanczos components (BLC)[26] by using Lanczos method, operate faster than getting eigenvalues exactly.

2.3. Analysis of search spaces of SVD and GLRAM for Matrix data

Here we are going to compare the search space of SVD and GLRAM in detail. When SVD is applied to matrix data $\{A_1, ..., A_N\}$, the vectorized data
Algorithm 1 GLRAM

Require: Input: matrices \{A_i\}_{i=1}^N

Ensure: Output: matrices L, R, and \{D_i\}_{i=1}^N

1: Obtain initial \(L_0\) and set \(i \leftarrow 1\)
2: While not convergent
3: \hspace{1em} form the matrix \(M_R = \sum_{j=1}^n A_j^T L_{i-1} L_{i-1}^T A_j\)
4: \hspace{1em} compute the \(k_2\) eigenvectors \(\phi_{j}^{R \ k_2}\) of \(M_R\) corresponding to the largest \(k_2\) eigenvalues
5: \hspace{1em} \(R_i \leftarrow [\phi_1^R, \ldots, \phi_{k_2}^R]\)
6: \hspace{1em} form the matrix \(M_L = \sum_{j=1}^n A_j R_i R_i^T A_j^T\)
7: \hspace{1em} compute the \(k_1\) eigenvectors \(\phi_{j}^{L \ k_1}\) of \(M_L\) corresponding to the largest \(k_1\) eigenvalues
8: \hspace{1em} \(L_i \leftarrow [\phi_1^L, \ldots, \phi_{k_1}^L]\)
9: \hspace{1em} \(i \leftarrow i + 1\)
10: EndWhile
11: \(L \leftarrow L_i - 1\)
12: \(R \leftarrow R_i - 1\)
13: For \(j\) from 1 to \(n\)
14: \hspace{1em} \(D_j \leftarrow L_i^T A_j R\)
15: EndFor

\(a_i = \text{vec}(A_i) \in \mathbb{R}^n\), should be used. So the object function of SVD (best low rank approximation) on these data becomes

\[
\min_W \sum_{i=1}^N \|a_i - Wy_i\|_F^2, \quad n = n_1 n_2, \quad k = k_1 k_2, \quad (10)
\]

\(W \in \mathbb{R}^{n \times k}, \quad W^T W = I\).

\(y_i \in \mathbb{R}^k\).
By using the properties of Kronecker product and vectorization, it is easy to show that GLRAM model Eq.\[3\] is equal to the following form

\[
\min_{L,R,D_i} \sum_{i=1}^{N} \|a_i - (L \otimes R)d_i\|_F^2.
\]

(11)

\[L \in \mathbb{R}^{n_1 \times k_1} : L^TL = I_{k_1}\]

\[R \in \mathbb{R}^{n_2 \times k_2} : R^TR = I_{k_2}.
\]

Now, by comparison the Eq.\[10\] and Eq.\[11\] we want to describe the relationship between SVD and GLRAM. For every two orthogonal matrices like \(L\) and \(R\) in feasible region of Eq.\[11\] their corresponding \(L \otimes R\) satisfies the constraints of Eq.\[10\]. This means that \(L \otimes R\) is a special case of \(W\) in Eq.\[10\]. In fact, the search space of GLRAM is a subset of SVD and it can be considered as a special case of SVD. By this relationship, the following facts summaries the relationship of SVD and GLRAM in 3 main parts.

- Far apart the GLRAM works with data in their original format, it is a special case of SVD.

- GLRAM has \(n_1k_1 + n_2k_2\) parameters that should be estimated, while there are \(n_1n_2k_1k_2\) ones for SVD. So the GLRAM’s complexity is much less than SVD and the possibility of overfitting, is too.

- In GLRAM we are free to choose the reduction in each arbitrary mode.

Therefore by the mentioned properties, the GLRAM seems to be better than SVD (PCA) for matrix data.

3. Proposed method

We can see the summary of a comparison between vector-based methods and tensor-based ones in Table.\[1\]
Table 1: Comparison between vector-based and tensor-based methods

| Methods                  | Vector-Based | Tensor-Based |
|--------------------------|--------------|--------------|
| Search space             | Large        | Small        |
| Complexity (time and space) | High        | Low          |
| Spatial Relationship     | Considered   | Ignored      |

More specifically, from the last section, we found that GLRAM has two main advantages and one drawback in comparison with PCA as we can see in Table 2.

Table 2: GLRAM: Advantages and Disadvantages

| Pros                                                                 | Cons                                      |
|---------------------------------------------------------------------|-------------------------------------------|
| Works directly on data with their own format without folding them into vectors | The small search space of GLRAM because of its fewer parameters |
| Parameters of GLRAM are more less than the SVDs which should be estimated |                                           |

As we had in the previous section, the following equation shows that how GLRAM is a special case of SVD

\[
\sum_{i=1}^{N} \| A_i - LD_iR_i^T \|_F^2 = \sum_{i=1}^{N} \| a_i - (L \otimes R)d_i \|_2^2. \tag{12}
\]

Now, let \( V = L \otimes R \) is a small kind of \( U \in \mathbb{R}^{n \times k} \), in detail:

\[
\Phi = \{ V \mid V = L \otimes R, L \in \mathbb{R}^{n_1 \times k_1} \text{ and } R \in \mathbb{R}^{n_2 \times k_2} \text{ are orthogonal} \}, \tag{13}
\]

\[
\Psi = \{ U \mid U \text{ is orthogonal, } U \in \mathbb{R}^{n \times k}, \text{ where } n = n_1n_2, k = k_1k_2 \}. \]

Therefore, \( \Phi \subset \Psi \), so the search space of GLRAM is a small version of SVD. Eventually, due to limited search space and strict constraints, the answer gained by GLRAM is not accurate enough.
In this section, we try to extend the search domain of GLRAM in order to improve its quality without losing its mentioned advantages shown in Table 2. To design our proposed method, we should have new insight to search region Ψ of SVD method. When SVD is used for matrix data samples $A_i \in \mathbb{R}^{n_1 \times n_2}$, according to Eq.13, the solution will lie on the feasible set Ψ and each feasible solution will be an orthogonal matrix $U \in \mathbb{R}^{n \times k}$, where $n = n_1 n_2$ and $k = k_1 k_2$.

Now we design the following partitioning on matrix $U$

$$U = \begin{pmatrix} U_{11} & \cdots & U_{1k_1} \\ \vdots & \ddots & \vdots \\ U_{n_11} & \cdots & U_{n_1k_1} \end{pmatrix}$$

(14)

where $U$ contains $n_1 k_1$ numbers of block matrices $U_{ij} \in \mathbb{R}^{n_2 \times k_2}$, $i = 1, \ldots, n_1$, $j = 1, \ldots, k_1$.

To have a deep insight into the search space Ψ of GLRAM, we define the following reshaping on matrix $U$ based on mentioned partitioning.

$$\tilde{U} = [Vec(U_{1,1}), \ldots, Vec(U_{n_1,1}), \ldots, Vec(U_{1,k_1}), \ldots, Vec(U_{n_1,k_1})]^T \in \mathbb{R}^{n_1 k_1 \times n_2 k_2}.$$  

(15)

Now if rank($\tilde{U}$) = $l$, the SVD of $\tilde{U}$ will be

$$\tilde{U} = \sum_{i=1}^{l} \sigma_i u_i v_i^T,$$  

(16)

by defining $\bar{u}_i = \sqrt{\sigma_i} u_i \in \mathbb{R}^{n_1 k_1}$ and $\bar{v}_i = \sqrt{\sigma_i} v_i \in \mathbb{R}^{n_2 k_2}$ the Eq.16 becomes

$$\tilde{U} = \sum_{i=1}^{l} \bar{u}_i \bar{v}_i^T.$$  

(17)

By the properties of Kronecker product and definition of reshaping matrix $\tilde{U}$, it is easy to show that

$$U = \sum_{j=1}^{l} L_j \otimes R_j,$$  

(18)

where $\text{Vec}(L_j) = \bar{u}_j$ and $\text{Vec}(R_j) = \bar{v}_j$. This shows that every projection matrix $U \in \Psi$ has a form like Eq.18. So, projection matrix of GLRAM is special case
of Eq.18 when \( l = 1 \).

So, if we set a \( k \) larger than 1 and smaller than \( l \), using the projection matrix like

\[
U = \sum_{j=1}^{k} L_j \otimes R_j, \tag{19}
\]

enables us to use the benefits of GLRAM and SVD at the same time. This means that by the mentioned \( U \) in Eq.19 as projection matrix in GLRAM model, we obtained the following approximation model

\[
\min_{L_j, R_j, D_i} \sum_{i=1}^{N} \|a_i - \sum_{j=1}^{k} (L_j \otimes R_j)d_i\|_F^2 \tag{20}
\]

\[
= \sum_{i=1}^{N} \|A_i - \sum_{j=1}^{k} L_j D_i R_j^T\|_F^2,
\]

where \( d_i = \text{Vec}(D_i) \) and \( D_i \in \mathbb{R}^{k_1 \times k_2} \) which similar to GLRAM works on matrix data with their own format and at the same time its search space is larger than GLRAM method.

So \( \Phi_k = \left\{ \sum_{j=1}^{k} L_j \otimes R_j | L_j \in \mathbb{R}^{n_1 \times k_1}, R_j \in \mathbb{R}^{n_2 \times k_2} \right\} \) denotes the search space of problem Eq.20 and from Eq.13 we can conclude that \( \Phi \subset \Phi_k \). Which shows that by our approach the search space of the problem is increased. Also, from Eq.20 it is clear that our proposed method is applied to the data with their own format without folding to vectors. Furthermore, this method has \((n_1 k_1 + n_2 k_2)k\) parameters that should be estimated. Since we consider \( k \) as a small number, so the complexity of this method is not much higher than GLRAM, and still the probability of occurrence of overfitting is less for this method in comparison with SVD or PCA.

Although this problem is nonconvex, once it is going to solve according to each parameter, where other parameters are considered to be known, it will be a convex one. So we could use coordinate descent approach to deal with it.\[31\]
3.1. Algorithm

In Multiple-Pairs of Generalized Low-Rank Approximation of Matrices (MPGLRAM), we deal with the following problem:

\[
\min_{L_j \in \mathbb{R}^{n_1 \times k_1}, j=1,2,...,k, \quad R_j \in \mathbb{R}^{n_2 \times k_2}, j=1,2,...,k, \quad D_i \in \mathbb{R}^{k_1 \times k_2}, i=1,2,...,N} \sum_{i=1}^{N} \left\| A_i - \sum_{j=1}^{k} L_j D_i R_j^T \right\|_F^2.
\] (21)

To solve Eq. 21 like GLRAM, we use a coordinate descent\[31\] approach. So at each step of the algorithm, we have some subproblems that are solved only according to one variable. So, after \(p\) steps let \(L_j^{(p)}\) and \(R_j^{(p)}\) be the estimations of projections and data matrices. At step \(p+1\) firstly we set the projection matrices be known as the estimated at step \(p\). Now we have to estimate the projection matrices with known \(L_j^{(p)}\) and \(R_j^{(p)}\) from the last steps, which leads to the following subproblem

\[
\{D_i^{(p+1)}\}_{i=1}^{N} = \arg \min_{D_i, i=1,...,N} \sum_{i=1}^{N} \left\| A_i - \sum_{j=1}^{k} L_j^{(p)} D_i R_j^{(p)T} \right\|_F^2
\]

\[
= \arg \min_{d_i, i=1,...,N} \sum_{i=1}^{N} \left\| a_i - \left( \sum_{j=1}^{k} R_j^{(p)} \otimes L_j^{(p)} \right) d_i \right\|_2^2 \quad (22)
\]

So, if we set \(B^{(p)} = \sum_{j=1}^{k} \left( R_j^{(p)} \otimes L_j^{(p)} \right)\), this problem can be reformulated as the following least squares problem\[32\].

\[
\min_{D_{i=1,...,N}} \sum_{i=1}^{N} \left\| a_i - B^{(p)} d_i \right\|_2^2 = \min \| A - B^{(p)} D \|_F^2, \quad a_i = vec(A_i), \quad d_i = vec(D_i),
\] (23)

where \(A = [a_1, ..., a_N] \in \mathbb{R}^{n_1 \times n_2 \times N}\) and \(D = [d_1, ..., d_N] \in \mathbb{R}^{k_1 \times k_2 \times N}\). This is a well-known least square problem and could be solved easily by direct and iterative matrix computation techniques.

After solving the mentioned problem we should find \(L_j\), \(R_j\) parameters successively by coordinate descent approach for \(j=1,...,k\).\[31\] So if we assume
The only variable should be determined is $R_{j'}$ and also regarding to Trace properties we have \(\{33\}\) which by removing the constants leads to

$$
\sum_{i=1}^{N} \|A_i - \sum_{j=1, j \neq j'}^{k} L_j D_i R_j^T - L_{j'} D_i R_{j'}^T\|^2_F. 
$$

(24)

By replacing $\bar{A}_i = A_i - \sum_{j=1, j \neq j'}^{k} L_j D_i R_j^T$ the equation becomes

$$
\min_{L_j', R_{j'}' \in \mathbb{R}^{n \times k_1}: j=1,2, \ldots, k} \sum_{i=1}^{N} \left\| \bar{A}_i - L_{j'} D_i R_{j'}^T \right\|^2_F. 
$$

(25)

For solving Eq.\(\{25\}\) we do alternatively. At first, $L_{j'}$, $\bar{A}_i$ and $D_i$ are given. The only variable should be determined is $R_{j'}$. By replacing $M_i = L_{j'} D_i$ in Eq.\(\{25\}\) and also regarding to Trace properties we have \(\{33\}\)

$$
\|\bar{A}_i - M_i R_{j'}^T\|^2_F = \text{tr} ((\bar{A}_i - M_i R_{j'}^T)^T (\bar{A}_i - M_i R_{j'}^T)) = \text{tr}(\bar{A}_i^T \bar{A}_i - 2 \bar{A}_i^T M_i R_{j'}^T + R_{j'} M_i^T M_i R_{j'}^T),
$$

(26)

which by removing the constants leads to the following problem

$$
\min_{R_{j'}} \sum_{i=1}^{N} \text{tr}(-2 \bar{A}_i^T M_i R_{j'}^T + R_{j'} M_i^T M_i R_{j'}^T)
$$

$$
= \min_{R_{j'}} -2 \sum_{i=1}^{N} \text{tr}(\bar{A}_i^T M_i R_{j'}^T) + \sum_{i=1}^{N} \text{tr}(R_{j'} M_i^T M_i R_{j'}^T)
$$

$$
= \min_{R_{j'}} -2 \text{tr}(\sum_{i=1}^{N} \bar{A}_i^T M_i) R_{j'} + \text{tr}(R_{j'} (\sum_{i=1}^{N} M_i^T M_i) R_{j'}^T),
$$

(27)

In order to release the dependency of data from the optimization problem, insert the sigma operators into traces and by replacing $N_R = \sum_{i=1}^{N} \bar{A}_i^T M_i$ and $B_R = \sum_{i=1}^{N} M_i^T M_i$, this optimization problem becomes

$$
\min_{R_{j'}} -2 \text{tr}(N_R R_{j'}^T) + \text{tr}(R_{j'} B_R R_{j'}^T), \quad R_{j'} \in \mathbb{R}^{n_2 \times k_2} : j = 1, 2, \ldots, k,
$$

(28)
Now, for solving Eq. 28 since the function of this problem is quadratic convex, the derivate of this function according to $R$ in the optimal point should be zero. So by taking, derivative of this objective function with respect to $R_j'$, and regarding properties of Trace, by setting the objective function equal to zero, we have

$$-2N_R + 2R_j' B_R = 0,$$

and consequently, $R_j'$ will be determined.

$$R_j' = N_R B_R^{-1}.$$ (30)

Also, same as obtaining $R_j'$ we can determine $L_j'$ with a little difference. To this purpose, we should solve the transpose of Eq. 25 according to $L_j'$.

Let $M_i = R_j' D_i^T$, then

$$\min_{L_j', L_j \in \mathbb{R}^{n_1 \times k_1}, j=1,2,...,k} \sum_{i=1}^{N} ||\bar{X}_i^T - R_j' D_i^T L_j^T||_F^2.$$ (31)

By replacing $N_L = \sum_{i=1}^{N} \bar{A}_i M_i$ and $B_L = \sum_{i=1}^{N} M_i^T M_i$, we have

$$\min_{L_j'} -2tr(N_L L_j^T) + tr(L_j' B_L L_j^T).$$ (33)

Then, by taking derivative of this function respect to $L_j'$ and after simplification $L_j'$ will be determined.

$$L_j' = N_L B_L^{-1}.$$ (34)

Since we should obtain all $k$ variables, we do previous stages $k$ times to determine $L_j$ and $R_j$, for all amount of $j = 1, ..., k$. Eventually, at each time, $k - 1$
parameters will be assumed fixed to obtain one parameter. And for the next parameter, the updated form of the previous ones will be used. Besides, we can repeat this alternative process more than once. Since each time with updating parameters, the result will be improved. Although, these iterations increase the time and space complexity, the more suitable $L_j$ and $R_j$ can be reached. So we solve the MPGLRAM with alternative schema which obtain $L_j$, $R_j$, and $D_i$ alternatively in each iteration by updating the initial values. The details of the process are shown in Algorithm 2.

**Algorithm 2 MPGLRAM**

**Require:** Input: matrices $\{A_i\}_{i=1}^{N}$, $k$, iter

**Ensure:** Output: matrices $\{D_i\}_{i=1}^{N}$

1: Initialize $\{L_j\}_{j=1}^{k} = L_0 = \begin{pmatrix} I_k, 1 \\ 0 \end{pmatrix}$ and $\{R_j\}_{j=1}^{k} = R_0 = \begin{pmatrix} I_k, 2 \\ 0 \end{pmatrix}$

2: Construct $\bar{B} = \sum_{j=1}^{k} (R_j \otimes L_j)$

3: Obtain initial $D$ by solving Eq.23

4: for i from 1 to iter

5: for each $j', j' = 1, ..., k$ solve Eq.25 alternatively to determine $\bar{L}$ and $\bar{R}$

6: update $\bar{L}$ and $\bar{R}$

7: Obtain $D$ by solving Eq.24

8: EndFor

In our proposed method, by the projection of $k$ pairs of left and right projectors which $1 < k < min(m,n)$, the search space will be $k(m+n)$, while SVD has the largest amount of $k$ and most free parameters $(m \times n)$ and GLRAM has only $(m + n)$ with $k = 1$.

Finally, we use $k$ as a parameter to balance between these two traditional and matrix version of the SVD. Therefore, by using $k$ pairs of projectors we expand the search space merely to find the optimal answer and protect it from tending
4. Experimental Results

As we have seen in the previous section, in our proposed method, we expanded the search space of GLRAM problem by using $k$ pair projector matrices instead of one pair used in GLRAM. Now, we are going to compare MPGLRAM with GLRAM in 4 datasets in 2 aspects. The first is RMSRE, and the second is about classification accuracy.

4.1. Data description

We used 4 different matrix datasets in our experiments, ORL, Yale, YaleB, and PIE. In ORL, there are 400 images of 40 people in 10 different angels and position which each of these images is a $32 \times 32$ matrix. While there are 165 $32 \times 32$ images of 15 people in 11 different position at Yale. Another one is Extended YaleB which contains 2414 images in 38 groups of $32 \times 32$ facial images and the last database is PIE which has 210, $44 \times 44$ images in the first 10 people of 68 people, each person under 13 different poses.

The details of the data are listed in Table 3.

| Data  | Size     | Scale    | Class number |
|-------|----------|----------|--------------|
| ORL   | 400      | $32 \times 32$ | 40           |
| Yale  | 165      | $32 \times 32$ | 11           |
| YaleB | 2414     | $32 \times 32$ | 38           |
| PIE   | 210      | $44 \times 44$ | 10           |

1http://www.cad.zju.edu.cn/home/dengcai/Data/faceData.html
2http://web.mit.edu/emeyers/www/face_databases.html
3http://vision.ucsd.edu/~leekc/ExtYaleDatabase/ExtYaleB.html
4http://featureselection.asu.edu/old/datasets/pixraw10p.mat
4.2. Root Mean Square Reconstruction Error

One way to assess our proposed method is Root Mean Square Reconstruction Error (RMSRE) as a measure to evaluate the accuracy of approximation which can be obtained in this way

$$RMSRE = \sqrt{\frac{1}{n} \sum_{i=1}^{n} \|A_i - \sum_{j=1}^{k} L_j D_i R_j^T\|^2}.$$  \hspace{1cm} (35)

So, as a more accurate approximation achieved the value of RMSRE will be decreased. The value of RMSRE respect to different values of $d$, the dimension of reduced matrices, applied to 4 datasets can be seen in Figure 2. As we can see in Figure 2 in GLRAM and MPGLRAM, as the dimension of reduced matrices is close to the original ones, the value of RMSRE will be decreased. We did our experiments on different value of parameter $d$, from 1 to 32 with the best result in iteration between 1 to 20, respect to parameter $k$ from 2 to 5 applied to Yale dataset.

As we stated in MPGLRAM, Figure 2 shows that our method reduces the
RMSRE, consequently, it can obtain a more accurate approximation of matrices than GLRAM.

4.3. Classification

Now, we want to compare the ability of classification between GLRAM and MPGLRAM. In our experiments, we applied $K$-fold measure with different amount of $K$, $K = 2, 5, 10$ using cross-validation for classification. The results of our experiments on YaleB can be seen by detail in Table 4.

Also, to see GLRAM and MPGLRAM behaviors on YaleB, our results are represented in Figure 4 clearly.

As we can see from Table 4 and Figure 4, our proposed method obtained more accurate answer rather than the GLRAM in all dimensions in 2, 5, and 10-fold classification. For example, in 2-fold classification the maximum accuracy gained by GLRAM is 87.28% in $d = 12$ while in the same dimension we achieved 88.53% with $k = 5$ in MPGLRAM. This trend can be seen also in 5-fold and 10-fold classification. In 5-fold classification, our proposed method have a better operation in terms of accuracy such that GLRAM obtained 88.98% and 89.52% in $d = 12$ which are its maximum value, meanwhile we achieved 90.35% in $k = 3$ and 90.47% in $k = 5$ by the same dimension in MPGLRAM with 5-fold and

Figure 3: Ten samples of four individuals with YaleB database
Table 4: Percentage of classification accuracy in GLRAM and MPGLRAM on YaleB for $d = 6, 7, 8, 9, 10, 11, 12$ with best accuracy in iteration from 1 to 20.

| k-fold | d | GLRAM | MPGLRAM |
|--------|---|-------|---------|
|        |   |       | $k=2$  | $k=3$  | $k=4$  | $k=5$  |
| 2      | 6 | 69.76 | 73.49  | **73.90** | 73.70  | 73.45  |
|        | 7 | 74.11 | 76.76  | 76.01   | **77.01** | 76.55  |
|        | 8 | 78.67 | **81.23** | 80.49  | 81.19  | 80.49  |
|        | 9 | 81.23 | **83.18** | 81.98  | 82.73  | 82.39  |
|        | 10| 84.34 | **85.67** | 85.54  | 84.80  | 84.84  |
|        | 11| 85.92 | 86.74  | 87.12  | 86.74  | **87.70** |
|        | 12| 87.28 | 88.28  | 88.11  | 88.40  | **88.53** |
| 5      | 6 | 71.00 | **76.72** | 74.86  | 75.31  | 75.10  |
|        | 7 | 75.23 | 78.58  | 77.71  | 78.62  | **79.16** |
|        | 8 | 80.07 | 82.19  | 81.36  | **82.52** | 81.65  |
|        | 9 | 82.52 | 83.39  | 83.80  | **84.05** | 83.80  |
|        | 10| 84.22 | **87.16** | 86.41  | 85.92  | 86.12  |
|        | 11| 87.70 | 87.78  | 88.11  | 88.19  | **88.40** |
|        | 12| 88.98 | 89.98  | **90.35** | 89.64  | 89.85  |
| 10     | 6 | 71.33 | **76.59** | 75.56  | 75.35  | 75.10  |
|        | 7 | 75.60 | 78.91  | 78.00  | 78.50  | **79.45** |
|        | 8 | 80.41 | **82.60** | 81.48  | 82.48  | 81.32  |
|        | 9 | 82.56 | **84.42** | 83.72  | 84.22  | 84.34  |
|        | 10| 85.05 | **87.32** | 86.66  | 86.45  | 86.33  |
|        | 11| 87.90 | 88.15  | 88.19  | 88.40  | **88.69** |
|        | 12| 89.52 | 90.18  | 90.10  | 89.85  | **90.47** |
Figure 4: The comparison between GLARM and MPGLRAM in accuracy of classification with different dimensions and 10 exterior iteration respect to $k = 2, \ldots, 5$, on YaleB in a) 2-fold b) 5-fold c) 10-fold 10-fold classification respectively.

Furthermore, we did the same experiments on ORL, Yale, and PIE datasets which are reported more generally in Table 5, Table 6, and Table 7 respectively.

In addition to YaleB, other datasets results in a smaller scale, are illustrated in Table 5, Table 6 and Table 7. For ORL dataset shown in Table 5, our results demonstrate a better classification accuracy in comparison with GLRAM. We obtained 98% in 2-fold, 99.50% in 5-fold, and 99.25% in 10-fold which are greater than 97%, 98.75% and 99% in GLRAM with its maximum value respectively.

Besides, for Yale Table 6 shows that MPGLRAM found a better answer in almost all experiments. The maximum value of GLRAM in our experiments with different folds of classification and dimensions is 89.09 in 5-fold where we obtained the same number with the same parameters. Although we can not achieve a better answer in 5-fold classification on Yale dataset, our method’s results in 2-fold and 10-fold classification are better than GLRAM. Such that, in 2-fold the best result of GLRAM is 81.21% and in 10-fold is 88.48% while in MPGLRAM we obtained 82.42% and 90.30% respectively.
Table 5: Percentage of classification accuracy in GLRAM and MPGLRAM on ORL for $d = 5, 6, 7$ with best accuracy in iteration from 1 to 20.

| k-fold | d      | GLRAM | MPGLRAM | MPGLRAM | MPGLRAM |
|--------|--------|-------|---------|---------|---------|
|        |        |       | $k=2$   | $k=3$   | $k=4$   | $k=5$   |
| 2      | 5      | 96.25 | 95.50   | 96.00   | 96.25   | 96.00   |
|        | 6      | 96.50 | 96.50   | 97.25   | 97.50   | 98.00   |
|        | 7      | 97.00 | 97.75   | 98.00   | 97.75   | 97.75   |
| 5      | 5      | 97.00 | 96.00   | 97.75   | 97.75   | 97.50   |
|        | 6      | 98.25 | 98.25   | 98.75   | 99.25   | 99.00   |
|        | 7      | 98.75 | 99.50   | 99.00   | 99.25   | 99.25   |
| 10     | 5      | 97.00 | 96.50   | 97.50   | 98.50   | 97.50   |
|        | 6      | 97.75 | 98.25   | 98.75   | 99.25   | 98.75   |
|        | 7      | 99.00 | 99.25   | 99.25   | 99.25   | 99.25   |

Table 6: Percentage of classification accuracy in GLRAM and MPGLRAM on Yale for $d = 5, 6, 7$ with best accuracy in iteration from 1 to 20.

| k-fold | d      | GLRAM | MPGLRAM | MPGLRAM | MPGLRAM |
|--------|--------|-------|---------|---------|---------|
|        |        |       | $k=2$   | $k=3$   | $k=4$   | $k=5$   |
| 2      | 5      | 81.21 | 82.42   | 78.18   | 79.39   | 76.97   |
|        | 6      | 81.21 | 76.36   | 82.42   | 81.21   | 78.79   |
|        | 7      | 74.55 | 74.55   | 75.76   | 76.36   | 78.18   |
| 5      | 5      | 84.24 | 84.85   | 84.85   | 83.03   | 84.24   |
|        | 6      | 85.45 | 86.06   | 86.67   | 86.06   | 87.88   |
|        | 7      | 89.09 | 87.88   | 89.09   | 87.27   | 89.09   |
| 10     | 5      | 85.45 | 86.06   | 85.45   | 84.85   | 85.45   |
|        | 6      | 86.06 | 86.06   | 87.27   | 87.88   | 86.67   |
|        | 7      | 88.48 | 89.70   | 90.30   | 89.09   | 90.30   |

And in the last but not the least Table 7, we achieved the best performance on PIE dataset. While the best result for GLRAM is 99.52% we achieved 100%
accuracy in many cases.

Table 7: Percentage of classification accuracy in GLRAM and MPGLRAM on PIE for \(d = 5, 6, 7\) with best accuracy in iteration from 1 to 20.

| k-fold | d | GLRAM | MPGLRAM |
|--------|---|--------|---------|
|        |   | \(k=2\) | \(k=3\) | \(k=4\) | \(k=5\) |
| 2      | 5 | 99.05  | 99.5    | 99.05  | 100    | 100    |
|        | 6 | 99.52  | 99.52   | 100    | 100    | 100    |
|        | 7 | 99.52  | 99.52   | 99.52  | 99.52  | 100    |
| 5      | 5 | 99.05  | 100     | 100    | 100    | 100    |
|        | 6 | 99.05  | 100     | 100    | 100    | 100    |
|        | 7 | 99.52  | 100     | 100    | 99.52  | 100    |
| 10     | 5 | 99.05  | 100     | 100    | 100    | 100    |
|        | 6 | 99.05  | 100     | 100    | 100    | 100    |
|        | 7 | 99.52  | 100     | 100    | 100    | 100    |

4.4. \(k\)-Parameter

In our proposed methods, we expanded the GLRAM search space by using \(k\)-pair projections. So, at the first glance, this seems to play an important role to achieve the best accuracy. While, as we have seen from the experimental results, by increasing the value of \(k\) the RMSRE will be decreased as well, but in classification, the best accuracy occurred in different values of \(k\). Although the classification accuracy should be better theoretically when the amount of RMSRE decreased and the search space expanded, sometimes the classification accuracy is reduced due to overfitting phenomenon.

The value of \(k\) can change from 1 to minimum amount of the size of data. For example for a \(m \times n\) matrix it can be \(1 \leq k \leq \min\{m, n\}\). When \(k = 1\) our method behaves like GLRAM, except its orthogonality constraints, so it has a small search space \(m + n\). While when \(k = \min\{m, n\}\), the search space is equal to vectorized form of matrix. So, in MPGLRAM we use \(k\) to make balance between these two methods. Therefore, in MPGLRAM, the search space will
be \( k(m + n) \) which can be larger than GLRAM and smaller than vectorized
dimension reduction method, SVD.

In our experiments we use \( k = 2, 3, 4, 5 \) to show that the proposed method works
better than GLRAM especially when the dimension reduced to a lower value.

5. Conclusion

In this paper we proposed a new method using advantages of both SVD and
GLRAM simultaneously to find a more accurate answer rather than GLRAM
with lower complexity than SVD. Then, since GLRAM has few parameters and
the search space of it is too small to finding an optimal answer, we proposed an
approach that try to expand the search space of GLRAM in order to find a more
accurate answer. For this purpose, we used k-pair projection matrices instead
of one pair in GLRAM. Therefore by increasing the parameters and making a
larger search space, the probability of finding optimal answer will be increased.
Many experimental results have been represented in section.4 to illustrate that
our proposed method, MPGLRAM, works better than GLRAM in terms of less
RMSRE and therefore, more classification accuracy.

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