ON THE ALGEBRAIC FUNDAMENTAL GROUP OF SMOOTH VARIETIES IN CHARACTERISTIC $p > 0$

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Abstract. We define an analog in characteristic $p > 0$ of the proalgebraic completion of the topological fundamental group of a complex manifold.

1. Introduction

Let $X$ be a smooth algebraic variety defined over a field $k$ endowed with a rational point $x \in X(k)$.

If $k$ is the field of complex numbers $\mathbb{C}$, the proalgebraic completion $\pi_{\text{alg,rs}}^\text{top}(X, x)$ of the topological fundamental group $\pi_1^{\text{top}}(X, x)$ is defined as the prosystem $\varprojlim H$, where $H \subset GL(n, \mathbb{C})$ runs over the Zariski closures of the monodromy groups $\rho(\pi_1^{\text{top}}(X, x))$ of complex linear representations $\rho : \pi_1^{\text{top}}(X, x) \to GL(n, \mathbb{C})$. The profinite completion $\varprojlim H$, where $H$ runs over the finite quotients of $\pi_1^{\text{top}}(X, x)$, is, via the Riemann existence theorem, identified with Grothendieck’s étale fundamental group $\pi_1^\text{ét}(X, x)$. Since any finite group is embeddable in $GL(n, \mathbb{C})$ for some $n$, this defines, thinking of $\pi_1^\text{ét}(X, x)$ as a complex (constant) proalgebraic group, a surjective homomorphism $\varphi_{\text{rs}} : \pi_{\text{alg,rs}}(X, x) \to \pi_1^\text{ét}(X, x)$, and in fact $\pi_1^\text{ét}(X, x)$ is the profinite quotient of $\pi_{\text{alg,rs}}(X, x)$. By the Riemann-Hilbert correspondence, $\pi_{\text{alg,rs}}(X, x)$ is the Tannaka group-scheme of the category of $O_X$-coherent regular singular $D_X$-modules, which is a full subcategory of the category of $O_X$-coherent $D_X$-modules. We denote by $\pi_{\text{alg}}(X, x)$ the corresponding Tannaka group-scheme, and by $\varphi_{\mathbb{C}} : \pi_{\text{alg}}(X, x) \to \pi_{\text{alg,rs}}(X, x)$ the composite morphism. It is surjective as well, and since any flat connection with finite monodromy is regular singular, $\pi_1^\text{ét}(X, x)$ is the profinite quotient of $\pi_{\text{alg}}(X, x)$.

If $k$ is a characteristic 0 field, $\pi_{\text{alg}}(X, x)$ is defined as the Tannaka group-scheme of the $k$-linear tensor category of $O_X$-coherent $D_X$-modules equipped with the fiber functor defined as the restriction of the module on $x$. The full subcategory of finite objects, that is objects with finite monodromy group-scheme, or said differently, objects which have the property that the full Tannaka subcategory which is spanned by it has a finite Tannaka group-scheme, defines a pro-finite $k$-group-scheme $\pi_1^\text{ét}(X, x)$. Since $\pi_1^\text{ét}(X, x)(\bar{k}) = \pi_1^\text{ét}(X, x)(\bar{k})$ ([5, Remark 2.10]), and both $\pi_{\text{alg}}(X, x)$ and $\pi_1^\text{ét}(X, x)$ satisfy base change for finite extensions $k \subset L$ ([6], [7]).

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Property 2.54]), we see that the surjection \( \varphi : \pi_{\text{alg}}(X, x) \to \pi_{\text{et}}(X, x) \) is a \( k \)-form of \( \varphi_\mathbb{C} \) for any complex embedding \( k \subset \mathbb{C} \). Moreover, by definition, \( \varphi \) induces the pro-finite quotient homomorphism.

If \( k \) is a characteristic \( p > 0 \) field, the category of \( \mathcal{O}_X \)-coherent \( \mathcal{D}_X \)-modules is again a \( k \)-linear abelian tensor rigid category. It is part of Katz’ theorem asserting that this category is equivalent to the category of stratified \( \mathcal{O}_X \)-coherent sheaves (see [9] Theorem 1.3, [3] Theorem 8, where it is shown over \( k = \bar{k} \)). If \( k = \bar{k} \), its Tannaka group-scheme \( \pi_{\text{alg}}(X, x) \) is shown to be pro-smooth in [3, Corollary 12] (strictly speaking, it is shown there only for the profinite part, but dos Santos’ proof applies more generally as mentioned in [4, Corollary 7]). The homomorphism \( \varphi \) is then defined by the full embedding of the subcategory of objects with finite monodromy group-scheme. So by definition, \( \varphi \) induces the pro-finite quotient homomorphism.

On the other hand, if \( X \) is a reduced connected scheme over a characteristic \( p > 0 \) field \( k \), endowed with a rational point \( x \in X(k) \), Nori [10, Chapter II] constructed a fundamental group-scheme \( \pi^N(X, x) \) as the projective system of finite \( k \)-group-schemes \( G \) for which there is a \( G \)-torsor \( h : Y \to X \) under \( G \) with trivialization at \( x \). The pro-étale quotient of \( \pi^N(X, x) \) is precisely \( \pi_{\text{et}}(X, x) \).

Summarizing, one has a diagram

\[
\begin{array}{ccc}
\pi_{\text{alg}}(X, x) & \overset{\text{surj}}{\longrightarrow} & \pi_{\text{et}}(X, x) \\
\downarrow \text{surj} & & \downarrow \text{surj} \\
\pi^N(X, x) & & \end{array}
\]

The aim of our article is to define a Tannaka category \( \text{Strat}(X, \infty) \) over a perfect field \( k \), which contains the category of \( \mathcal{O}_X \)-coherent \( \mathcal{D}_X \)-modules as a full subcategory, in such a way that its Tannaka group-scheme \( \pi_{\text{alg}, \infty}(X, x) \), which thus surjects onto \( \pi_{\text{alg}}(X, x) \), also surjects onto \( \pi^N(X, x) \). In other words, we complete \((1.1)\) to

\[
\begin{array}{ccc}
\pi_{\text{alg}}(X, x) & \overset{\text{surj}}{\longrightarrow} & \pi_{\text{et}}(X, x) \\
\downarrow \text{surj} & & \downarrow \text{surj} \\
\pi_{\text{alg}, \infty}(X, x) & \overset{\text{surj}}{\longrightarrow} & \pi^N(X, x) \\
\end{array}
\]

As a byproduct, we obtain a purely tannakian geometric description of \( \pi^N(X, x) \) (see Corollary 4.9). Recall that we assume that \( X \) is smooth. If in addition \( X \) is proper, Nori himself described his fundamental group-scheme \( \pi^N(X, x) \) as the Tannaka group-scheme of the category of essentially finite bundles [10], Chapter I. He extends in [10, Chapter III] his construction to non-proper curves by using parabolic bundles.

Lacking desingularization in characteristic \( p > 0 \) makes it difficult to generalize his construction to the higher dimensional case. If \( k \) has
characteristic 0, then, as already mentioned, \( \pi^N(X, x) = \pi^\text{et}(X, x) \) is the Tannaka group-scheme of the category of finite flat connections \([6, \text{Section 2}]\), or, equivalently, of the category of \( \mathcal{O}_X \)-coherent \( \mathcal{D}_X \)-modules with finite monodromy group-scheme.

Our construction (see Section 3, most particularly Definition 3.2) generalizes on a smooth variety defined over a perfect characteristic \( p > 0 \) field \( k \) the construction of the category of flat connections (loc. cit) in characteristic 0, and the construction of the stratified bundles (loc. cit.) in characteristic \( p > 0 \). We now explain the main idea.

For \( i \in \mathbb{N} \), let us define inductively the relative Frobenius \( F^{(i)} : X^{(i)} \to X^{(i+1)} \) over \( k \) in the usual manner. As \( k \) is assumed to be perfect, one defines \( X^{(-1)} = X \otimes_{k,k^{-1}} k \) where \( F_k : \text{Spec} k \to \text{Spec} k \) is the absolute Frobenius of \( k \), together with the relative Frobenius \( F^{(-1)} : X^{(-1)} \to X^{(0)} \). Then one iterates to define inductively \( F^{(i)} : X^{(i)} \to X^{(i+1)} \) for \( i \in \mathbb{Z}, i < 0 \). For \( a, b \in \mathbb{Z}, a < b \) we define \( F^{(a,b)} : X^{(a)} \xrightarrow{F^{(a)} \circ \ldots \circ F^{(b-1)}} X^{(b)} \).

Recall that a stratified bundle is a sequence \( (E^{(i)}, \sigma^{(i)}, i \in \mathbb{N}) \), where \( E^{(i)} \) is a bundle on \( X^{(i)} \), \( \sigma^{(i)} : E^{(i)} \xrightarrow{\cong} F^{(i)*}E^{(i+1)} \) is a \( \mathcal{O}_{X^{(i)}} \)-isomorphism. For \( t \in \mathbb{N}, t \neq 0 \), we define an object of \( \text{Strat}(X, t) \) to be a sequence \( (E^{(i)}, \sigma^{(i)}, i \in \mathbb{N}) \), where \( E^{(i)} \) is a bundle on \( X^{(i)} \), \( \sigma^{(i)} : E^{(i)} \xrightarrow{\cong} F^{(i)*}E^{(i+1)} \) is a \( \mathcal{O}_{X^{(i)}} \)-isomorphism for all \( i \geq 1 \), but for \( i = 0 \), \( \sigma_0 : F^{(-t,0)*}E^{(0)} \xrightarrow{\cong} F^{(-t,1)*}E^{(1)} \) is a \( \mathcal{O}_{X^{(-1)}} \)-isomorphism. The morphisms are the ones between the bundles which respect all the structures. We show (Theorem 3.4) that the obvious functor \( \text{Strat}(X, t) \subset \text{Strat}(X, t+1) \), which assigns \( (E_t, F^{(-t-1)*}\sigma_0, \sigma_i, i \geq 1) \) to \( (E_t, \sigma_0, \sigma_i, i \geq 1) \), induces a full embedding of Tannaka categories, where the fiber functor is simply the restriction of \( E^{(0)} \) to the rational point \( x \). Then \( \text{Strat}(X, \infty) \) is defined as the inductive limit over \( t \to \infty \) of the categories \( \text{Strat}(X, t) \) (Corollary 3.5). In order to show that the Tannaka group-scheme \( \pi^{\text{alg},\infty}(X, x) \) of \( \text{Strat}(X, \infty) \) surjects onto \( \pi^N(X, x) \), we use a slight modification of Nori’s reconstruction theorem [10, Chapter I, Proposition 2.9] of a torsor \( h : Y \to X \) under a finite group scheme \( G \) out of the induced functor \( h^# : \text{Rep}_k(G) \to \text{Coh}(X) \) which assigns to a finite dimensional \( k \)-linear representation \( V \) of \( G \) the vector bundle on \( X \) which is defined by flat descent for \( h \) on \( \mathcal{O}_Y \times_k V \) (Theorem 2.4).

This allows us to define the group-scheme homomorphism \( \pi^{\text{alg},\infty}(X, x) \to \pi^N(X, x) \) (Theorem 4.5). In order to show that this map induces the profinite quotient, we in particular use the categorical translation of injectivity and surjectivity of homomorphisms of Tannaka group-schemes (\([2, \text{Proposition 2.12}]\)).

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2. Nori’s fundamental group-scheme

Let $k$ be a field of characteristic $p > 0$ and $X$ be a $k$-scheme. Let $x \in X(k)$ be a rational point and $i_x : x \to X$ be the closed embedding.

Nori [10, Chapter II] defines the category $\mathcal{N}(X,x)$ of triples $(Y \xrightarrow{f} X, G, y)$ where

(a) $G/k$ is a finite group scheme,
(b) $f : Y \to X$ is a $G$-torsor,
(c) $y$ is a $k$-point of $Y$ lying above $x$.

A morphism between two such triples $(Y_i \xrightarrow{f_i} X, G_i, y_i)$ is a pair $(\phi : G_1 \to G_2, \psi : Y_1 \to Y_2)$ such that $\psi$ an $X$-morphism which is $\phi$-equivariant and $\psi(y_1) = y_2$. Nori shows [10, Chapter II, Proposition 2] that if $X$ is reduced and geometrically connected, then the projective limit $\varprojlim G$ exits. He defines

**Definition 2.1.** Let $X$ be a reduced geometrically connected $k$-scheme, then its Nori fundamental group-scheme is the profinite $k$-group-scheme

$$\pi^N(X,x) = \varprojlim_{\mathcal{N}(X,x)} G.$$ 

Since giving a rational point $y \in f^{-1}(x)$ is the same as giving a trivialization $f^{-1}(x) \cong_k G$, $\mathcal{N}(X,x)$ is equivalent to the category of triples $(h : Y \to X, G, f^{-1}(x) \cong_k G)$, where the morphisms between two such objects are defined by $X$-torsor morphisms which respect the trivialization. We will not need this slightly different phrasing.

**Definition 2.2.** Let $G$ be a finite $k$-group-scheme, and let $h : Y \to X$ be a $G$-torsor. Then it induces a functor $h^\#: \text{Rep}_k(G) \to \text{Coh}(X)$ which assigns to a finite dimensional $k$-representation $V$ the bundle on $X$ which comes by flat descent from $O_Y \otimes_k V$.

**Properties 2.3.**

1) The functor $h^\#$ defined in Definition 2.2 is exact, $k$-linear and compatible with the tensor structure. Thus it is a fiber functor in the sense of Deligne [11, 1.9]. Since $\text{Rep}_k(G)$ is a Tannaka category, it follows [11, Corollaire 2.10] that $h^\#$ is faithful.

2) The functor $i_x^* : \text{Coh}(X) \to \text{Vec}_k$ defined as the restriction to the rational point, with values in the category of finite dimensional $k$-vector spaces, is a fiber functor on the subcategory of vector bundles. The composite functor $i_x^* \circ h^\# : \text{Rep}_k(G) \to \text{Vec}_k$ is a fiber functor.

3) Let $h_i : Y_i \to X$ be $G_i$ torsors where $i = 1, 2$. Let $\phi : G_1 \to G_2$ be a group homomorphism and $\psi : Y_1 \to Y_2$ be an equivariant map with respect to $\phi$. We denote by $\phi^*$ the induced functor $\text{Rep}_k(G_2) \to \text{Rep}_k(G_1)$. Then
one has the equality $h_2^# = h_1^# \circ \phi^*$ of functors. Indeed, if $V$ is a $G$-representation, $\psi^* : \mathcal{O}_Y \otimes_k V \to \psi_*(\mathcal{O}_Y \otimes_k \phi^*(V))$ induces a $\mathcal{O}_X$-linear map $h_2^*(V) \to h_1^*(V)$ between those two vector bundles, which, after composing with $i_x^*$, is the identity on $V$. So $h_2^*(V) = h_1^# \circ \phi^*(V)$.

4) Let $h : Y \to X$ be a $G$-torsor, let $b : X' \to X$ be a morphism, and let $x' \in X'(k)$ be a rational point with $b(x') = x$. Let $Y' = Y \times_X X' \to X'$ and $h' : Y' \to X'$ denote the projection. Then one has the equality $b^* \circ h^# = h'^#$ of functors. Indeed, denoting by $b' : Y' \to Y$ the induced morphism, if $V$ is a $G$-representation, $(b')^* : \mathcal{O}_Y \otimes_k V \to (b')_*\mathcal{O}_{Y'} \otimes_k V$ induces $\mathcal{O}_{X'}$-linear map $b^* \circ h^#(V) \to (h'^#)(V)$ between vector bundles, which is the identity on $V$ after composing with $i_{x'}$. So $b^* \circ h^# = (h')^#$.

The following is a direct consequence of [10, Proposition 2.9].

**Theorem 2.4.** Let $G$ be a finite $k$-group-scheme and let $F : \text{Rep}_k(G) \to \text{Coh}(X)$ be a fiber functor such that $i_x^* \circ F$ is the forgetful functor $F_G : \text{Rep}_k(G) \to \text{Vec}_k$.

Then there exists a unique object $(Y \xrightarrow{h} X, G, y)$ of $N(X, x)$ such that $F = h^#$ and $(h^{-1}(x), y) = (G, 1)$. For any other object $(Y' \xrightarrow{h'} X, G, y') \in N(X, x)$ such that $F = h'^#$, there exists a unique isomorphism in $N(X, x)$ between $(Y \xrightarrow{h} X, G, y)$ and $(Y' \xrightarrow{h'} X, G, y')$.

**Proof.** By Nori’s reconstruction theorem [10, Proposition 2.9], $F(k[G])$, where $k[G]$ is the regular representaton of $G$, is a finite $\mathcal{O}_X$-algebra. The $G$-torsor $h : Y \to X$ is defined to be $\text{Spec}_X F(k[G])$. By Property (2.3, 2), $i_x^* \circ F(k[G]) = F_G(k[G]) = k[G]$. Said differently, $h^{-1}(x) = \text{Spec}_x k[G] = G$. Then $y$ is the rational point of $h^{-1}(x)$ which is $1 \in G$. By the unicity in loc. cit., $h$ is uniquely defined. If $y' = g \in h^{-1}(x)(k)$ is another rational point, then multiplication $g : Y \to Y$ by $g$, together with the conjugation $G \to G, h \mapsto ghg^{-1}$ defines an isomorphism $(h : Y \to X, G, y) \to (h : Y \to X, G, y')$ in $N(X, x)$. □

### 3. The category of generalized stratified bundles

The aim of this section is to define the category of generalized stratified bundles. We start with some notations.

**Notations 3.1.** Let $k$ be a perfect field of characteristic $p > 0$, $X$ be a smooth scheme over $k$ which is geometrically irreducible.

For $i \in \mathbb{N}$, we define inductively the relative Frobenius $F^{(i)} : X^{(i)} \to X^{(i+1)}$ over $k$ in the usual manner, by defining $X^{(0)} = X$, $X^{(i+1)}$ to be the fiber product of $X^{(i)} \otimes_{k, F_k} k$ over the absolute Frobenius $F_k : \text{Spec} k \to \text{Spec} k$ of $k$, and $F^{(i)}$ to be the factorization of the absolute Frobenius $F_{X^{(i)}} : X^{(i)} \to X^{(i)}$ morphism.

For $i \in \mathbb{Z}, i < 0$, we define inductively $F^{(i)} : X^{(i)} \to X^{(i+1)}$ over $k$ as follows. First we set $X^{(-1)} = X \otimes_{F_k} k$. Then we define $F^{(-1)} : X^{(-1)} \to X$ to be the
relative Frobenius. Similarly, we define \( X^{(-i-1)} = X^{(-i)} \otimes_{F_k^{-1}} k \) together with the relative Frobenius \( F^{(-i-1)} : X^{(-i-1)} \rightarrow X^{(-i)} \) over \( k \).

For \( a, b \in \mathbb{Z}, a < b \) we define \( F^{(a,b)} : X^{(a)} \xrightarrow{F^{(a)} \circ \ldots \circ F^{(b-1)}} X^{(b)} \).

Recall that a stratified bundle (see [9 Section 1]) is a sequence \((E^{(i)}, \sigma^{(i)}), i \in \mathbb{N}\), where \( E^{(i)} \) is a \( O_X \)-coherent sheaf on \( X^{(i)} \), \( \sigma^{(i)} : E^{(i)} \xrightarrow{\cong} F^{(i)} E^{(i+1)} \) is a \( O_X^{(i)} \)-isomorphism. One defines the category \( \text{Strat}(X) \) of stratified bundles by defining

\[
\text{Hom}((D^{(i)}, \tau^{(i)}), (E^{(i)}, \sigma^{(i)}))
\]
to be set of sequences \( f_i : D^{(i)} \rightarrow E^{(i)} \) of morphisms of \( O_X^{(i)} \)-coherent sheaves, which commute with all the \( \sigma_i \) and \( \tau_i \). It is a fact (loc. cit.) that if \( (E^{(i)}, \sigma^{(i)}), i \in \mathbb{N}\) is a stratified sheaf, the \( E^{(i)} \) are all locally free, and if \( f = (f)_i, i \in \mathbb{N} \) is a morphism of stratified sheaves, then \( f_i \) are vector bundle maps (i.e. locally split), so the category is abelian, rigid, and monoidal. Moreover, the Hom-sets are finite dimensional \( k \)-vector spaces. As \( X \) is geometrically irreducible, the unit object \( \mathbb{I} = (O_X, \text{Id}), i \in \mathbb{N} \) fulfills \( \text{End}(\mathbb{I}) = k \). If now \( X \) is endowed with a rational point \( x \in X(k) \), then \( \omega_x : \text{Strat}(X) \rightarrow \text{Vec}_k, (E^{(i)}, \sigma^{(i)}) \mapsto E_{0|x} \) is a fiber functor in the sense of Deligne [11, 1.9], and thus yields the structure of a Tannaka category on \( \text{Strat}(X) \). A fundamental property due to dos Santos is that the corresponding Tannaka \( k \)-group-scheme \( \text{Aut}_{\otimes}^G(\omega_x) \) is pro-smooth ([3, Corollary 12], [4, Corollary 7]).

**Definition 3.2.** Let \( t \geq 0 \) be an integer. A \( t \)-stratified bundle is a sequence \((E^{(i)}, \sigma^{(i)}), i \in \mathbb{N})\), where \( E^{(i)} \) is a \( O_X \)-coherent sheaf on \( X^{(i)} \),

\[
\sigma^{(i)} : E^{(i)} \xrightarrow{\cong} F^{(i)} E^{(i+1)}
\]
is a \( O_X^{(i)} \)-isomorphism for \( i \geq 1 \) and for \( i = 0 \),

\[
\sigma^{(0)} : F^{(-t,0)} E^{(0)} \xrightarrow{\cong} F^{(-t,1)} E^{(1)}
\]
is a \( O_X^{(-t)} \)-isomorphism.

One defines the category \( \text{Strat}(X,t) \) of \( t \)-stratified bundles by defining

\[
\text{Hom}((D^{(i)}, \tau^{(i)}), (E^{(i)}, \sigma^{(i)}))
\]
to be set of sequences \( f_i : D^{(i)} \rightarrow E^{(i)} \) of morphisms of \( O_X \)-coherent sheaves, which commute with all the \( \sigma_i \) and \( \tau_i \).

In particular, \( \text{Strat}(X,0) = \text{Strat}(X) \).

**Example 3.3.** We now give an example of a non-trivial 1-stratified bundle on \( X = \mathbb{A}_k^1 = \text{Spec}(k[[x]]) \). Thus \( X^{(i)} = \text{Spec}(k[x_i]) \) where the relative Frobenius \( X^{(i)} \rightarrow X^{(i+1)} \) is induced by \( x_{i+1} \rightarrow x_i^p \). For simplicity let us assume \( p = \text{char}(k) = 2 \). Let \( V \) be a 2-dimensional vector space over \( k \) with basis \( e_1, e_2 \). Define

\[
E^{(i)} = O_{X^{(i)}} \otimes_k V \quad \forall \ i \geq 0
\]
and
\[ \sigma^{(i)} : E^{(i)} \rightarrow F^{(i)*}E^{(i+1)}, \ i \geq 1 \]
to be the isomorphism induced by the identity on \( V \). We define
\[ \sigma^{(0)} : F^{(-1,0)*}E^{(0)} \rightarrow F^{(-1,1)*}E^{(1)} \]
to be the isomorphism defined by sending
\[ e_1 \rightarrow e_1, \ e_2 \rightarrow x_{-1}e_1 + e_2. \]
We claim that the \(-1\)-stratified bundle thus defined is not isomorphic to the trivial stratified bundle of rank 2. If indeed this were the case, then we would have a \( k[x] \)-module automorphism \( \phi : k[x] \otimes_k V \rightarrow k[x] \otimes_k V \), such that
\[ \phi \otimes k[x]k[x^{-1}] = \sigma^{(0)}. \]
This is impossible since \( x_{-1} \) is not contained in \( k[x] \). It can be shown (see (4.3)) that this \(-1\)-stratified bundle “arises” from the non-trivial \( \alpha \) torsor on \( \mathbb{A}_k^1 \) defined by the relative Frobenius on \( \mathbb{A}_k^1 \).

**Theorem 3.4.** The notations are as in 3.1.

1) For every integer \( t \geq 0 \), \( \text{Strat}(X,t) \) is a \( k \)-linear, abelian, rigid, tensor category.
2) The functor
\[ (+) : \text{Strat}(X,t) \subseteq \text{Strat}(X,t+1) \]
\[ (E_i,\sigma_0,\sigma_i, i \geq 1) \mapsto (E_i, F^{(-t-1)*}\sigma_0, \sigma_i, i \geq 1), \]
induces a full faithful embedding of \( k \)-linear, abelian, rigid, tensor categories.
3) If \( x \in X(k) \) is a rational point, the functor
\[ \omega_x : \text{Strat}(X,t) \rightarrow \text{Vec}_k \]
\[ (E^{(i)},\sigma^{(i)}) \mapsto E_0|x \]
is a fiber functor, which makes \((\text{Strat}(X,t),\omega_x)\) a Tannaka category.

**Proof.** We show 1). Since \( \text{Strat}(X,0) = \text{Strat}(X) \), we assume \( t > 0 \). If \( (E^{(i)},\sigma^{(i)}, i \in \mathbb{N}) \) is an object in \( \text{Strat}(X,t) \), then \( (E^{(i)}_+, \sigma^{(i)}_+, i \in \mathbb{N}) \) is an object \( \text{Ver}(E^{(i)},\sigma^{(i)}, i \in \mathbb{N}) \in \text{Strat}(X^{(1)}) \). Since \( E^{(i)} \) is locally free, by the isomorphism \( \sigma^{(0)}, F^{(-1,0)*}E^{(0)} \) is locally free. Since \( X \) is smooth, the relative Frobenius is flat, thus by flat descent, \( E^{(0)} \) is locally free as well. So \( \text{Strat}(X) \) is rigid and monoidal. On the other hand,

\[ \text{Hom}((D^{(i)},\tau^{(i)}, i \in \mathbb{N}),(E^{(i)},\sigma^{(i)}, i \in \mathbb{N})) \]
\[ \subseteq \text{Hom}((D^{(i)},\tau^{(i)}, i \in \mathbb{N}),(E^{(i)},\sigma^{(i)}, i \in \mathbb{N})) \]
and is obviously a \( k \)-vector space. So the Hom-sets are finite dimensional \( k \)-vector spaces. Moreover, any morphism \( f = (f^{(i)}, i \in \mathbb{N}) \) is such that \( f^i, i \geq 1 \)
is a morphism of vector bundles. Thus by the isomorphisms \( \tau (0), \sigma^0, \text{Ker, Im and Coker of } f^{(0)} \) are pulled back to vector bundles on \( X^{(-t)} \) via \( F^{(-t,0)} \), thus by flat descent again, there are vector bundles on \( X \). We conclude that \( \text{Strat}(X,t) \) is an abelian category. This shows 1).

2) follows immediately from the factorization of \( \text{Strat}(X,t) \) through \( (+) \).

We show 3): the point \( x \in X(k) \) maps to \( x^{(1)} \in X^{(1)}(k) \), and the map \( x \to x^{(1)} \) is the identity on the residue fields \( k(x) = k(x^{(1)}) = k \). If \( 0 \to A \to B \to C \to 0 \) is an exact sequence in \( \text{Strat}(X,t) \), then \( 0 \to \text{Ver}(A) \to \text{Ver}(B) \to \text{Ver}(C) \to 0 \) is an exact sequence in \( \text{Strat}(X^{(1)}) \), thus \( 0 \to \omega_{x^{(1)}}(\text{Ver}(A)) \to \omega_{x^{(1)}}(\text{Ver}(B)) \to \omega_{x^{(1)}}(\text{Ver}(C)) \to 0 \) is an exact sequence in \( \text{Vec}_k \). But

\[
\omega_{x^{(1)}}(\text{Ver}(A)) = \omega_x(A).
\]

This shows that \( \omega_x \) is exact. Furthermore, \( \omega_x \) is obviously \( k \)-linear and compatible with the tensor structure. This finishes the proof.

\[\square\]

**Corollary 3.5.** Let the notations be as in **Theorem 3.4**. The category

\[
\text{Strat}(X,\infty) = \lim_{+t\in \mathbb{N}} \text{Strat}(X,t)
\]

is a \( k \)-linear, abelian, rigid tensor category, on which, if \( X \) has a rational point \( x \in X(k) \), the functor \( \omega_x \) is a fiber functor.

**Definition 3.6.** The notations are as in **Theorem 3.4**

1) We define \( \pi_{\text{alg}}(X,x) \) to be the Tannaka \( k \)-group scheme \( \text{Aut}^\otimes(\omega_x) \) of \( \text{Strat}(X),\omega_x \).

2) We define \( \pi_{\text{alg},\infty}(X,x) \) to be the Tannaka \( k \)-group scheme \( \text{Aut}^\otimes(\omega_x) \) of \( \text{Strat}(X,\infty),\omega_x \).

The functor \( (+): \text{Strat}(X) \to \text{Strat}(X,\infty) \) defines the homomorphism

\[
(+)^*: \pi_{\text{alg},\infty}(X,x) \to \pi_{\text{alg}}(X,x).
\]

**Lemma 3.7.** The homomorphism \( (+)^* \) in (3.3) is faithfully flat.

*Proof.* We apply [2, Proposition 2.21]. As \( (+) \) is fully faithful, the lemma is equivalent to saying that if \( A \) is an object on \( \text{Strat}(X) \), and \( B \subset (+)A \) is a subobject in \( \text{Strat}(X,\infty) \), then there is a subobject \( B' \subset A \) in \( \text{Strat}(X) \) such that \( B = (+)B' \). One has that \( \text{Ver}(B) \subset \text{Ver}(A) \) is a subobject in \( \text{Strat}(X^{(1)}) \), thus \( F^{(0)*}B^{(1)} \subset A^{(0)} \) is a subvector bundle with the property that \( F^{(-t,0)*}F^{(0)*}B^{(1)} = F^{(-t,1)*}B^{(1)} = F^{(-t,0)*}B^{(0)} \). Thus \( B' = (F^{(0)*}B^{(1)}, B^{(i)}, i \geq 1, F^{(0)*}, \sigma^{(i)}, i \geq 1) \subset A \) is a subobject of \( A \) such that \( (+)B' = B \). This finishes the proof. \[\square\]

4. **Comparison of** \( \pi_{\text{alg},\infty}(X,x) \) **with** \( \pi_1^N(X,x) \)

In order to achieve the comparison, we start with a construction.
Construction 4.1. The notations are as in 3.1 and \( x \in X(k) \) is a rational point. Let \((h : Y \to X, G, y)\) be an object of \( \mathbb{N}(X, x) \). Using this object, we construct a tensor functor
\[
h^* : \text{Rep}_k(G) \to \text{Strat}(X, \infty)
\]
together with a factorization of functors
\[
\begin{array}{ccc}
\text{Rep}_k(G) & \xrightarrow{h^*} & \text{Strat}(X, \infty) \\
\downarrow \quad F_G & & \downarrow \omega_x \\
\quad \quad \text{Vec}_k
\end{array}
\]
Here \( F_G : \text{Rep}_k(G) \to \text{Vec}_k \) is the forgetful functor.

Recall that if \( G \) is a finite \( k \)-group-scheme, there is an exact sequence of finite \( k \)-group schemes \( 1 \to G_0 \to G \to G_{\text{ét}} \to 1 \), where \( G_0 \) is the 1-component of \( G \) and \( G_{\text{ét}} \) is étale. Furthermore, as \( k \) is perfect, \( G_{\text{red}} \subset G \) is a closed subgroup-scheme and the composite \( G_{\text{red}} \hookrightarrow G \to G_{\text{ét}} \) is an isomorphism. Thus \( \iota \) yields on \( G \) the structure of a semi-direct product of \( G_{\text{ét}} \) by \( G_0 \). The construction of \( h^* \) will be such that the image of \( h^* \) is contained in \( \text{Strat}(X, t) \), where \( t \) is a natural number such that the image of the \( k \)-group-scheme homomorphism \( G^{(t)} \to G \) is equal to \( G_{\text{ét}} \).

Let \( V \) be a finite dimensional \( k \)-representation of \( G \). We set
\[
E^{(0)} = h^\#(V).
\]
For \( i \in \mathbb{N} \setminus \{0\} \), the relative Frobenius is an isomorphism of the étale \( k \)-group-schemes
\[
F^{(0,i)} : G_{\text{ét}} \xrightarrow{\cong} G_{\text{ét}}^{(i)}.
\]
Thus \( \iota(G) \circ F^{(0,i)-1} : G_{\text{ét}}^{(i)} \subset G \) is a closed embedding and composing with it defines a \( G_{\text{ét}}^{(i)} \)-action on \( V \). Since \( h : Y \to X \) is a \( G \)-torsor, for \( i \geq 0 \), \( h^{(i)} : Y^{(i)} \to X^{(i)} \) is also a \( G^{(i)} \)-torsor. Let \( h_{\text{ét}}^{(i)} : Y_{\text{ét}}^{(i)} \to X^{(i)} \) be the induced \( G_{\text{ét}}^{(i)} \)-torsor obtained by modding out by \( G_0^{(i)} \). We define
\[
E^{(i)} = (h_{\text{ét}}^{(i)})^\#(V).
\]
One has
\[
\sigma^{(i)} : E^{(i)} \cong F^{(i)*}E^{(i+1)}, \quad i \in \mathbb{N} \setminus \{0\}.
\]
The object \( h^*(V) \in \text{Strat}(X, t) \) which we wish to construct will have the property
\[
\text{Ver}(h^*(V)) = (E^{(i)}, \sigma^{(i)}, i \geq 1).
\]
It remains to define \( \sigma^{(0)} \). By definition,
\[
F^{(0)*}E^{(1)} = (h_{\text{ét}}^{(0)})^\# (V) = (h_{\text{ét}})^\# (V).
\]
Let $t$ be a natural number such that the image of $G^{(-t)} \to G$ is equal to $G_{\text{ét}}$. One has the following commutative diagram of $k$-varieties.

\[(4.8)\]

\[
\begin{array}{c}
\begin{tikzpicture}
\node (Y) at (0,0) {$Y^{(-t)}$};
\node (Yet) at (0,-2) {$Y_{\text{ét}}^{(-t)}$};
\node (Y) at (2,0) {$Y$};
\node (Yet) at (2,-2) {$Y_{\text{ét}}$};
\node (X) at (0,-4) {$X^{(-t)}$};
\node (X) at (2,-4) {$X$};
\draw[->] (Y) to node[above]{$F^{(-t,0)}$} (Y); \\
\draw[->] (Yet) to node[above]{$F^{(-t,0)}$} (Y); \\
\draw[->] (Y) to node[above]{$\exists ! \lambda$} (Yet); \\
\draw[->] (Yet) to node[above]{$h$} (X); \\
\draw[->] (Y) to node[above]{$h$} (X);
\end{tikzpicture}
\end{array}
\]

The morphism $F^{(-t,0)} : Y^{(-t)} \to Y$ is equivariant under $F^{(-t,0)} : G^{(-t)} \to G$. Likewise, the morphism $F^{(-t,0)} : Y_{\text{ét}}^{(-t)} \to Y_{\text{ét}}$ is equivariant under $F^{(-t,0)} : G_{\text{ét}}^{(-t)} \to G_{\text{ét}}$. The commutativity of the diagram implies that

\[(4.9)\]

\[
\lambda^* (\mathcal{O}_Y \otimes_k V) = F^{(-t,0)*} (\mathcal{O}_{Y_{\text{ét}}} \otimes_k V) = F^{(-t,1)*} (\mathcal{O}_{Y_{\text{ét}}^{(-1)}} \otimes_k V)
\]

equivariantly for the action of $G_{\text{ét}}^{(-t)}$. Thus

\[(4.10)\]

\[
(h_{\text{ét}}^{(-t)})# (V) = F^{(-t,0)*} E^{(0)} = F^{(-t,1)*} E^{(1)}.
\]

We define $\sigma^{(i)} : F^{(-t,0)*} E^{(0)} = F^{(-t,1)*} E^{(1)}$ to be the equality of (4.10).

Thus, starting with $V \in \text{Rep}_k(G)$, we have constructed an object $h^*(V) = (E^{(i)}, \sigma^{(i)}, i \in \mathbb{N}) \in \text{Strat}(X, t)$. Clearly, any $\phi \in \text{Hom}_{\text{Rep}_k(G)}(V, W)$ induces $h^*(\phi) \in \text{Hom}_{\text{Strat}(X, \text{ét})}(h^*(V), h^*(W))$. This defines the functor

\[(4.11)\]

\[
h^* : \text{Rep}_k(G) \to \text{Strat}(X, \infty).
\]

by composing with $(+).$ Moreover, one has

\[(4.12)\]

\[
h^*(V)_x = (\mathcal{O}_Y \otimes_k V)_y = V.
\]

This shows the commutativity of (4.11).

**Remark 4.2.** In the above construction we use the fact that for a finite flat group scheme $G$ over a perfect field $k$, the epimorphism $G \to G_{\text{ét}}$ admits a section (necessarily unique). In other words $G_{\text{ét}}$ can be canonically thought of as a subgroup scheme of $G$ via the identification $G_{\text{red}} = G_{\text{ét}}$. When $k$ is not a perfect field, $G_{\text{red}}$ may not be a subgroup scheme, (for example, $G = \text{Spec } k[t]/(t^p - at^p), a \in k \setminus k^p$, see [8, Chapter III, Exercice (3.2)]), and the above construction of $h^*$ does not make sense. This is the reason why we assume throughout $k$ to be perfect perfect. We thank Nguyên Duy Tân for this important remark.

**Example 4.3.** Let $p = \text{char}(k) = 2$ for simplicity and let $G = \alpha_2 = \text{Spec } (k[t]/t^2)$. Let $X = \mathbb{A}_k^1 = \text{Spec } (k[x])$. Let $P = \text{Spec } (k[u])$, and $h : P \to X$ be the relative Frobenius defined by $x \to u^2$. Then $h$ is a $G$-torsor. Thus by Construction (4.11), one has a functor

\[
h^* : \text{Rep}_k(G) \to \text{Strat}(X, -1).
\]
We compute now that $h^*(k[G])$ is nothing but the $-1$-stratified bundle defined in Example 3.3. Here $k[G] = k[v]/(v^2)$ is the regular representation of $G$. As in Example (3.3), let $X^{(i)} = k[x_i]$. Let $h^*(k[G]) = (E^{(i)}, \sigma^{(i)}, i \in \mathbb{N})$. As all schemes are affine, we confuse coherent sheaves with corresponding modules. Since $G_\text{et}$ is trivial, by definition of $h^*$ we see that

$$E^{(i)} = k[x_i] \otimes_k k[v]/(v^2) \quad \forall \ i \geq 1$$

with

$$\sigma^{(i)} : E^{(i)} \to F^{(i)} \otimes E^{(i+1)} \quad i \geq 1$$

induced by the identity map on $k[v]/(v^2)$. Then $E^{(0)}$ is by definition the $k[x]$-module of invariants of $k[u] \otimes_k k[v]/(v^2)$, where the action of $G = \text{Spec} \ k[t]/(t^2)$ is defined by

$$u \mapsto u + t, \ v \mapsto v + t.$$

Since $(u + v)^2 = u^2 = x$, one has $E^{(0)} = k[x] \cdot 1 \oplus k[x] \cdot (u + v)$. On $P$ we have an identification

$$h^* E^{(0)} = k[u] \otimes_k k[v]/(v^2)$$

defined by $\tau : 1 \mapsto 1 \otimes 1, u + v \mapsto u \otimes 1 + 1 \otimes v$. The map $\sigma^{(0)}$ is nothing but the pull back of $\tau$ via the isomorphism $X^{(-1)} \to P$ defined by

$$k[u] \to k[x_{-1}], \ u \to x_{-1}.$$

We thus see that

$$\sigma^{(0)} : k[x_{-1}] \cdot 1 \oplus k[x_1] \cdot (u + v) \to k[x_{-1}] \otimes k[v]/(v^2)$$

is defined by $1 \mapsto 1 \otimes (u + v) \mapsto u \otimes 1 + 1 \otimes v$. It is then an elementary exercise to see that the stratified bundle $h^*(k[G])$ is isomorphic to the $-1$ stratified bundle defined in Example 3.3.

**Lemma 4.4.** The functor $h^*$ defined in (4.11) is $k$-linear, exact, compatible with the tensor structure and faithful.

**Proof.** As already recalled in the Properties 2.3 1), faithfulness follows from the remaining properties. On the other hand, $k$-linearity, and compatibility with the tensor structures are straightforward. Exactness is proven as using $\text{Ver}$ as in Theorem 3.4 3). Indeed, $\text{Ver} \circ h^*$ with values in $\text{Strat}(X^{(1)})$ is obviously exact, while a sequence in $\text{Strat}(X, \infty)$ is exact if and only if it remains exact after applying $\text{Ver}$.

If $(h_i : Y_i \to X, G_i, y_i)$ are objects in $\text{N}(X, x)$ for $i = 1, 2$ and $(\psi : Y_1 \to Y_2, \phi : G_1 \to G_2, y_1 \to y_2)$ is a morphism in $\text{N}(X, x)$, then Property 2.3 3) implies that $h_2^* = h_1^* \circ \phi^*$. On the other hand, the projective system of $\phi$ in $\text{N}(X, x)$ induces an inductive system $\lim \to_{\text{N}(X, x), \phi^*} \text{Rep}_k(G)$ which is a Tannaka category, with the forgetful functor $F_G$ as the fiber functor. The Tannaka $k$-group-scheme
\[ \text{Aut}^\otimes(F_G) \text{ is simply } \lim_{\longleftarrow_{\mathcal{N}(X,x),\phi}} G, \text{ which is precisely Nori's fundamental group-scheme } \pi^N(X,x). \text{ As in addition the construction is obviously functorial in } h, \text{ we conclude:} \]

**Theorem 4.5.** Let the notations be as in Construction 4.1. The functor \( h^* \) defined in (4.1) for one object \( (h : Y \rightarrow X, G, y) \) of \( \mathcal{N}(X,x) \) induces a functor of Tannakian categories

\[ h^* : \left( \lim_{\longleftarrow_{\mathcal{N}(X,x),\phi}} \text{Rep}_k(G), F_G \right) \rightarrow (\text{Strat}(X,\infty),\omega_x), \]

and the Tannaka-dual homomorphism of \( k \)-group-schemes

\[ h^*\nu : \pi_{\text{alg},\infty}(X,x) \rightarrow \pi^N(X,x) \]

which is functorial in \( X \).

The aim of the rest of the section is to show that the homomorphism \( h^*\nu \) is faithfully flat and induces the profinite quotient homomorphism.

**Proposition 4.6.** Let \( (Y \xrightarrow{h} X, G, y) \) be an object of \( \mathcal{N}(X,x) \). The following conditions are equivalent.

1) The induced map \( \pi_{\text{alg},\infty}(X,x) \rightarrow G \) (see (4.1)) is an epimorphism.

2) The induced map \( \pi^N(X,x) \rightarrow G \) is an epimorphism.

3) The functor \( h^* \) in (4.1) is fully faithful and its image is closed under taking subquotients in \( \text{Strat}(X,\infty) \).

**Proof.** The equivalence (1) \( \Leftrightarrow \) (3) follows from [2, Proposition 2.21]. Moreover, since by construction, the map \( \pi_{\text{alg},\infty}(X,x) \rightarrow G \) factors through \( \pi^N(X,x) \), (1) \( \Rightarrow \) (2) is obvious.

We show (2) \( \Rightarrow \) (3). Let \( \mathcal{C} \) denote the full subcategory of \( \text{Strat}(X,\infty) \) generated by subquotients in \( \text{Strat}(X,\infty) \) of objects which are in the image of \( h^* : \text{Rep}_k(G) \rightarrow \text{Strat}(X,\infty) \). The property 3) is equivalent to saying that \( h^* : \text{Rep}_k(G) \rightarrow \mathcal{C} \) is an equivalence of categories. By standard Tannaka formalism, \( \mathcal{C} \) itself is a \( k \)-linear, abelian, rigid tensor subcategory of \( \text{Strat}(X,\infty) \), thus \( (\mathcal{C},\rho_x) \) is a Tannaka subcategory of \( (\text{Strat}(X,\infty),\omega_x) \), where \( \rho_x = \omega_x|_{\mathcal{C}} \).

We show now that \( h^* : \text{Rep}_k(G) \rightarrow \mathcal{C} \) is an equivalence of categories. Let \( H = \text{Aut}(\rho_x) \) be the Tannaka \( k \)-group-scheme of \( (\mathcal{C},\rho_x) \). We claim that the induced homomorphism \( H \rightarrow G \) is a closed immersion. This is equivalent ([2, Proposition 2.21]) to saying that every object of \( \mathcal{C} \) is a subquotient in \( \mathcal{C} \) of an object in \( h^*(\text{Rep}_k(G)) \), which is true since by definition of \( \mathcal{C} \), a subquotient in \( \mathcal{C} \) of objects in \( h^*(\text{Rep}_k(G)) \) is the same as a subquotient in \( \text{Strat}(X,\infty) \) of objects in \( h^*(\text{Rep}_k(G)) \). We conclude in particular that \( H \) is a finite group scheme.

The fiber functor (in the sense of Deligne [1, 1.9], see Properties 2.3 1)) \( \omega_X : \text{Strat}(X,\infty) \rightarrow \text{Coh}(X) \) defined by \( (E_i,\sigma_i, i \in \mathbb{N}) \mapsto E_0 \) restricts to the fiber
functor $\rho_X : C \to \text{Coh}(X)$. One has a commutative diagram of functors

$$
\begin{array}{ccc}
\text{Rep}_k(G) & \xrightarrow{h^*} & C \\
\downarrow \rho_X & & \downarrow \rho_X \\
\text{Coh}(X) & & \\
\end{array}
$$

and, upon applying $i_X$, (4.11) implies that $i_X \circ h^# = F_G$. By applying Theorem 2.4, we obtain a morphism

$$
(h_H : Y_H \to X, H, y_H) \to (h : Y \to X, G, y)
$$

in $\mathcal{N}(X, x)$. This in turn induces a factorization of $\pi^N(X, x) \to G$ as

$$
\begin{array}{ccc}
\pi^N(X, x) & \xrightarrow{\pi} & G \\
\downarrow & & \downarrow \\
H & \xrightarrow{\sim} & G
\end{array}
$$

But $\pi^N(X, x) \to G$ is assumed to be an epimorphism. Thus $H \to G$ must be an epimorphism. Since it is also a closed immersion, we conclude

$$
H \xrightarrow{\sim} G.
$$

In other words

$$
h^* : \text{Rep}_k(G) \xrightarrow{\sim} C.
$$

This finishes the proof. \qed

Recall that $k$ is perfect.

**Lemma 4.7.** Let $G$ be a finite $k$-group-scheme, let $h : Y \to X$ be a $G$-torsor. Then the following conditions are equivalent

(i) $h$ admits a reduction (necessarily unique) of structure group to $G_{\text{red}} = G_{\text{ét}} \subset G$.

(ii) For every natural number $n$, there is a $G$-torsor $h_n : Y_n \to X^{(n)}$ which pulls back via $X \xrightarrow{F^{(0,n)}} X^{(n)}$ to $h$.

**Proof.** We show (i) $\Rightarrow$ (ii). Let $h_{\text{ét}} : Y_{\text{ét}} \to X$ be a $G_{\text{ét}}$-torsor which is a reduction of structure of $h$ for the closed embedding $G_{\text{ét}} \subset G$. Thus $Y = Y_{\text{ét}} \times_{G_{\text{ét}}} G$. The isomorphism (4.3) induces a cartesian diagram

$$
\begin{array}{ccc}
Y_{\text{ét}} & \xrightarrow{F^{(0,n)}} & (Y_{\text{ét}})^{(n)} \\
\downarrow h_{\text{ét}} & & \downarrow \quad (h_{\text{ét}})^{(n)} \\
X & \xrightarrow{F^{(0,n)}} & X^{(n)}
\end{array}
$$

We set $Y_n = (Y_{\text{ét}})^{(n)} \times_{G_{\text{ét}}} G$, $h_n = (h_{\text{ét}})^{(n)} \times_{G_{\text{ét}}} G$. 

Recall that $k$ is perfect.
We show (ii) ⇒ (i). For a large enough positive integer \( n \), we consider the commutative diagram similar to (4.8):

\[ \begin{array}{c}
Y_n^{(-n)} \downarrow \quad \gamma \quad \exists ! \\
\downarrow \gamma \\
\downarrow h_n \\
Y^{(-n)}_{\text{ét}} \\
\downarrow \\
X \rightarrow X^{(n)}
\end{array} \]

(4.19)

We explain the terms in the diagram: with Notations 3.1, one has \( Y_n^{(-n)} = Y_n \otimes_{F_{-n}} k \), thus \( h_n \) induces \( h_n \otimes_{F_{-n}} k : Y_n^{(-n)} \to (X^{(n)})^{(-n)} = X \), which is a principal \( G^{(-n)} \) bundle. The top horizontal map \( \gamma \) is equivariant with respect to \( G^{(-n)} \rightarrow G \). Since \( n \) is large, the image of \( G^{(-n)} \to G \) is precisely \( G_{\text{ét}} \subset G \). Therefore, \( \gamma \) factors uniquely through \( \left( Y_n^{(-n)} \right)_{\text{ét}} \). Via the identification \( G^{(-n)}_{\text{ét}} \rightarrow G_{\text{ét}} \), the morphism \( \left( Y_n^{(-n)} \right)_{\text{ét}} \to X^{(n)} \) is a \( G_{\text{ét}} \)-torsor. The above commutative diagram shows the existence of an equivariant map \( \gamma_{\text{ét}} : Y_{\text{ét}} \rightarrow Y_n \times_{X^{(n)}} X \). We conclude that the \( G \)-torsor \( Y_n \times_{X^{(n)}} X \rightarrow X \) has a reduction of structure group to \( G_{\text{ét}} \).

□

**Theorem 4.8.** Let the notations are as in 3.1 and let \( x \in X(k) \) be a rational point. Then the homomorphism \( h^*: \pi_{\text{alg}, \infty}(X, x) \to \pi_N(X, x) \) is the profinite quotient homomorphism.

**Proof.** We have already shown in Proposition 4.6 that the homomorphism \( h^* \) is surjective. In order to show that \( h^* \) is the profinite completion homomorphism, we need to show that any epimorphism

\[ \phi : \pi_{\text{alg}, \infty}(X, x) \to G, \]

where \( G \) is a \( k \)-finite group-scheme, factors through \( \pi_N(X, x) \). This is equivalent to showing that given any finite Tannaka subcategory \( \mathcal{T} \subset \text{Strat}(X, \infty) \), i.e. with \( G = \text{Aut} \otimes(\mathcal{T}, \rho_x) \) finite, where \( \rho_x = \omega_x|_{\mathcal{T}} \), there exists an object \( (h : Y \to X, G, y) \) in \( \mathcal{N}(X, x) \) such that \( \mathcal{T} \) is the image of the functor \( h^* \) constructed in (4.11). We do this in two steps.

**Step(1):** For each \( n \geq 0 \), we consider the fiber functor

\[ \omega_X^{(n)} : \text{Strat}(X, \infty) \to \text{Coh}(X^{(n)}), \quad (E^{(i)}, \sigma^{(i)}, i \in \mathbb{N}) \mapsto E^{(n)}. \]

(4.20)

It restricts to a fiber functor

\[ P_n : \mathcal{T} \to \text{Coh}(X^{(n)}). \]
Let $\delta : \text{Rep}_k(G) \to \mathcal{T}$ be the equivalence given by Tannaka categories defined by the inverse functor to the equivalence induced by $\rho_x$. Consider 

$$P_n \circ \delta : \text{Rep}_k(G) \to \text{Coh}(X^{(n)}) .$$

By Theorem 2.4, we obtain $G$-torsors $(h_n : Y_n \to X^{(n)})$ for every $n$, such that

$$h_n^\# = P_n \circ \delta .$$

Since the $G$-torsors thus obtained are unique up to isomorphism, the equality

$$P_n = F^{(n)*} \circ P_{n+1}, \quad \forall \ n \geq 1$$

implies that the torsor $h_{n+1}$ pulls back to $h_n$. Thus by Lemma 4.7 each $Y_n$ admits a reduction of structure group to $G_{\text{et}} \subset G$ for all $n \geq 1$.

Step(2): Composing $\delta$ with the inclusion $\mathcal{T} \hookrightarrow \text{Strat}(X, \infty)$ we obtain a functor from $\text{Rep}_k(G) \to \text{Strat}(X, \infty)$. We also have the functor $h_0^* : \text{Rep}_k(G) \to \text{Strat}(X, \infty)$ (see (4.11)) defined by the $G$-torsor $h_0 : Y_0 \to X$. In order to finish the proof we have to show that these two functors coincide. This is equivalent to saying that the following diagram of functors commutes.

$$\begin{array}{ccc}
\text{Rep}_k(G) & \xrightarrow{\delta} & \mathcal{T} \\
\downarrow h_0 & & \downarrow \text{incl.} \\
\text{Strat}(X, \infty) & & \\
\end{array}$$

Let $V$ be an object of $\text{Rep}_k(G)$. We will show that there is an isomorphism between $i(V)$ and $h_0^*(V)$, which is functorial in $V$. This will finish the proof. Let

$$\delta(V) = (\delta(V)^{(n)}, \sigma^{(n)}, n \in \mathbb{N})$$

and $h_0^*(V) = (E^{(n)}, \tau^{(n)}, n \in \mathbb{N})$.

We let $h_{n, \text{et}} : Y_{n, \text{et}} \to X^{(n)}$ be the $G_{\text{et}}$-torsor induced by $h_n$ for $n \geq 1$. Note that by construction 4.4.1 of the functor $h_0^*$, one has

$$E^{(n)} = h_{n, \text{et}}^\#(V) \quad \forall \ n \geq 1 \quad \text{and} \quad E^{(0)} = h_0^\#(V).$$

On the other hand, by definition of the functors $P_n$,

$$P_n(i(V)) = i(V)^{(n)}$$

Thus by (4.21), one has

$$i(V)^{(n)} = h_n^\#(V) \quad \forall \ n \geq 0 .$$

But as explained before, for every $n \geq 1$, $h_n : Y_n \to X^{(n)}$ admits a reduction of structure group to $G_{\text{et}}$. Thus by Proposition 2.3(3),

$$h_n^\#(V) = h_{n, \text{et}}^\#(V) \quad \forall \ n \geq 1 .$$

Thus we conclude

$$i(V) = h_0^*(V).$$

\qed
If $T$ is any $k$-linear, abelian, rigid tensor category, together with a neutral fiber functor $\omega : T \to \text{Vec}_k$, we denote by $T^{\text{fin}}$ the full subcategory spanned by objects $E$ which have the property that the full tensor subcategory $\langle E \rangle \subset T$ spanned by $E$ and its dual $E^\vee$ has a finite Tannaka group scheme $\text{Aut}_{\langle E \rangle}(\omega|_{\langle E \rangle})$. So by construction, Theorem 4.8 has the following consequence:

**Corollary 4.9.** With the notations as in Theorem 4.8, the full embedding

$$\text{Strat}(X, x)^{\text{fin}} \subset \text{Strat}(X, x)$$

induces via the fiber functor $\omega_x$ the quotient homomorphism

$$\pi_{\text{alg}, \infty}^{\text{fin}}(X, x) \to \pi^N(X, x).$$

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