Moving solitons in a one-dimensional fermionic superfluid – an exact solution

Dmitry K. Efimkin$^1$ and Victor Galitski$^{1,2}$

$^1$Joint Quantum Institute and Condensed Matter Theory Center, Department of Physics, University of Maryland, College Park, Maryland 20742-4111, USA
$^2$School of Physics, Monash University, Melbourne, Victoria 3800, Australia

A fully analytical theory of a traveling soliton in a one-dimensional fermionic superfluid is developed within the framework of time-dependent self-consistent Bogoliubov-de Gennes equations, which are solved exactly. The soliton manifests itself in a kink-like profile of the superconducting order parameter and hosts a pair of Andreev bound states. They adjust to soliton’s motion and play an important role in its stabilization. A phase jump across the soliton and its energy decrease with soliton’s velocity and vanish at the critical velocity, corresponding to the Landau criterion, where the soliton starts emitting quasiparticles and becomes unstable. The “inertial” and “gravitational” masses of the soliton are calculated and the former is shown to be orders of magnitude larger than the latter. This results in slow oscillations of the soliton in a harmonic trap. These results may be related to recent experiments in cold fermion gases [T. Yefsah et al., Nature 499, 426, (2013)], which observed “heavy” soliton-like excitations in a paired fermion superfluid.

PACS numbers: 67.85.De, 67.85.Lm, 03.75.Lm

Introduction. Solitons are fascinating non-linear phenomena that occur in a diverse array of classical and quantum systems (see, e.g., Ref. [1] and references therein). In particular, they are known to exist in quantum superfluids, and have been demonstrated experimentally in Bose-Einstein condensates (BECs) using various methods including phase imprinting [2–4], density engineering [5, 6], and matter-wave interference [7] methods. A rich theoretical literature on solitons in BECs has also developed [8–12] and it includes both numerical and analytical solutions of Gross-Pitaevskii equations in excellent agreement with both each other and experiment.

Fermionic superfluids also support solitons - a phase jump in the order parameter field. These are more interesting and complicated objects than “Gross-Pitaevskii solitons,” because they can host and carry localized fermionic excitations - Andreev bound states (ABS). Consequently, a description of these non-linear phase excitations is more complicated: there exists no closed equation for the bosonic order parameter field and to include fermionic degrees of freedom is essential. At the technical level, one has to solve two-component Bogoliubov-de Gennes (BdG) equations supplemented from BEC to BCS (Bardeen-Cooper-Schrieffer) regimes were developed [27–32]. On the experimental side, the Zwierlein group at MIT reported in 2013 an observation of an oscillating solitonic vortex in a strongly-interacting fermionic superfluid in an elongated trap and demonstrated that the motion of the excitation is remarkably slow [33, 34]. These developments, along with potential connections to Majorana fermions (which may be carried by solitons in one-dimensional topological superfluids [35]), make the problem of fundamental understanding of soliton dynamics in one-dimensional paired Fermi systems of significant importance and interest.

Here, we develop an analytic theory of a traveling soliton in a one-dimensional paired superfluid in the BCS regime. We show that the time-dependent BdG equations are exactly solvable to describe a uniformly-moving solitary wave of the BCS order parameter and derive a dependence of the soliton’s energy and phase discontinuity across it on its velocity. The two former quantities are shown to decrease monotonically with velocity and vanish at the Landau’s critical velocity. It is also shown that ABS, carried by the soliton, adjust to its motion and play an important role in its stabilization. When confined to a harmonic trap, a soliton oscillates with a frequency which is considerably smaller than the trap frequency, reminiscent to what has been observed in the relevant experiment [35].

Time-dependent mean-field theory – We start with the BCS model for a one-dimensional superfluid, written in the Heisenberg representation

$$H = \int \, dx \left[ \sum_\alpha \bar{\Psi}_\alpha(x,t) (\hat{\mathcal{H}}_\alpha)(\Psi_\alpha^\dagger(x,t)) + V \Psi_\uparrow^\dagger(\Psi_\downarrow^\dagger \Psi_\downarrow \Psi_\uparrow) \right]. \quad (1)$$

Here $\Psi_\alpha(x,t) \equiv \Psi_\alpha(x,t)$ ($\Psi_\alpha^\dagger \equiv \Psi_\alpha^\dagger(x,t)$) is the annihilation (creation) Heisenberg operator for fermions, which can be written in the Nambu representation...
\( \Psi = (\Psi_\uparrow, \Psi_\downarrow)^T; \) \( \epsilon(\hat{p}) = (\hat{p}^2 - p_F^2)/2m \) is the kinetic energy of fermions and \( V \) is the BCS coupling constant. The operators satisfy the equation of motion, \( i\hbar\partial_t \Psi = [H, \Psi] \). The time-dependent mean-field approach involves introducing a time-dependent order parameter, \( \Delta(x,t) = -V(\Psi_\downarrow(x,t)\Psi_\uparrow(x,t)) \), which reduces the interacting Hamiltonian \( \{1\} \) to a quadratic form, and consequently the equations of motion become \( i\hbar\partial_t \Psi(x,t) = H_{\text{BdG}} \Psi(x,t) \), with \( H_{\text{BdG}} \) being the following time-dependent BdG Hamiltonian (in the laboratory frame of reference)

\[
H_{\text{BdG}} = \begin{pmatrix} \epsilon(\hat{p}_x) & \Delta(x,t) \\ \Delta^*(x,t) & -\epsilon(\hat{p}_x) \end{pmatrix}.
\]

We seek a uniformly-moving solution, where the order parameter and field operators are functions of the single variable, \( z = x + v_\text{s}t \). We also employ the semiclassical (Andreev) approximation \( \{30\} \), which treats separately the left- (\( \alpha = -1 \)) and right-moving (\( \alpha = +1 \)) fermions, and present the field operator in the form \( \Psi(x,t) = \sum_n \psi_n(z)b_n^\dagger \exp[i[p_Fz - \epsilon_n t]/\hbar] \). Here the sum is over time-dependent Bogoliubov quasiparticle’s states, described by the operators \( b_n^\dagger \), with the energies \( \epsilon_n \) and semiclassical wave functions \( \psi_n(z) = \{u_n^\dagger(z), v_n(z)\}^T \). The Ansatz for \( \Psi(x,t) \) satisfies the equation of motion, if the Bogoliubov states satisfy

\[
\begin{pmatrix} \alpha v_F \hat{p}_z + \alpha v_s p_F \\ -\alpha v_F \hat{p}_z + \alpha v_s p_F \end{pmatrix} \Delta(z) = \begin{pmatrix} \Delta(z) \\ \Delta^*(z) \end{pmatrix},
\]

The resulting Hamiltonian \( K_{\text{BdG}}^\alpha \) in Eq. \( \{4\} \) above does not have an explicit time dependence and corresponds to the frame of reference moving together with the soliton. In this co-moving frame, we assume Bogoliubov quasiparticles to be in thermal equilibrium. Hence, the self-consistent equation for the order parameter becomes,

\[
\Delta(z) = -V \sum_n u_n^\dagger(z)[v_n^\dagger(z)]^* n_F(\epsilon_n),
\]

where \( n_F(\epsilon_n) \) is the thermal Fermi-Dirac distribution function. The equation has a uniform solution, corresponding to a BCS superfluid state with the order parameter, \( \Delta_0 \). Note that the Andreev approximation (linearization of the fermion spectrum), used in deriving Eq. \( \{3\} \) and henceforth, corresponds to the condition \( \Delta_0 \ll E_F \) (which is actually satisfied in experiment \( \{33\} \) despite strong coupling). The time dependent-problem has been reduced to a time-independent one with the energy shift \( \Delta \varepsilon = \alpha v_F p_F \) of Bogoliubov quasiparticle’s energies. The shift does not change the general structure of the BdG Hamiltonian, but it modifies the energetics of the solitonic solutions in a non-trivial fashion.

**Solitonic solutions** - In the quasiclassical approximation, the BdG equations map onto the massive one-dimensional Dirac equation, where the superconducting order parameter playing the role of a Dirac mass \( \{15, 37\} \).

These equations are exactly solvable, since they can be reduced to a pair of supersymmetric equations, and contain nontrivial non-uniform solitonic solutions. Particularly, it was shown that the system of BdG equations \( \{4\} \) and the equation \( \{5\} \) for an order parameter, \( \Delta(z) \), can be satisfied simultaneously if the order parameter yields a reflectionless potential in the corresponding supersymmetric Schrödinger equation \( \{13\} \), see Eq. \( \{4\} \) below. A family of reflectionless potentials, corresponding to a single localized soliton, can be parameterized by a phase jump, \( 2\phi \), across it as follows.

\[
\Delta(z) = \Delta_0 \{\cos(\phi) + i \sin(\phi) \tanh[\sin(\phi) \cdot z] \}.
\]

Here \( z \xi = z/\xi_0 \), where \( \xi_0 = h v_F/\Delta_0 \) is the coherence length. The real part of the order parameter \( \Delta_1 \) is coordinate independent, while the imaginary part \( \Delta_2 \) has a kink from \(-\Delta_0 \sin(\phi)\) to \(\Delta_0 \sin(\phi)\). The spatial dependencies of the order parameter’s phase and modulus are presented in Fig. 1. Introducing \( f^\alpha_\pm(z) = [u^\alpha(z) \pm v^\alpha(z)] \), the BdG equations can be reduced to a pair of supersymmetric equations, whose solutions are intimately related,

\[
-h^2 v_F^2 \partial_z^2 + |\Delta(z)|^2 \pm \alpha h v_F \partial_z \Delta(z) f_\pm^\alpha \partial_z f_\pm^\alpha.
\]

Using the explicit profile of the order parameter \( \{5\} \), we
write them as

\[ -\hbar^2 v_F^2 \nabla^2 + \Delta_0^2 \left\{ 1 - \frac{2 \sin^2(\phi)}{\cosh^2[\sin(\phi) z]} \right\} f_n^\alpha = \epsilon^2 f_n^\alpha. \]

(7)

The equation for \( f_n^\alpha \) is trivial and contains only a continuous spectrum with plane-wave solutions, while the equation for \( f_0^\alpha \) has both the continuous states and an extra bound state. The continuous solutions have energy, \( \epsilon_{\gamma k} = \gamma \sqrt{(\hbar v_F k)^2 + \Delta_0^2} = \gamma \epsilon_k \), where \( \gamma = \pm 1 \) corresponds to Bogoliubov electrons and holes, and are given by

\[
\begin{align*}
\psi_{\gamma k}(z) &= \sqrt{\frac{\epsilon_{\gamma k} + \alpha \Delta_1}{4 \epsilon_{\gamma k}}} \left[ 1 + \alpha \frac{\hbar v_F k + i \Delta_2(z)}{\epsilon_{\gamma k} + \alpha \Delta_1} \right] e^{ikz}, \\
\psi_{\gamma k}^*(z) &= \sqrt{\frac{\epsilon_{\gamma k} + \alpha \Delta_1}{4 \epsilon_{\gamma k}}} \left[ \alpha - \frac{\hbar v_F k + i \Delta_2(z)}{\epsilon_{\gamma k} + \alpha \Delta_1} \right] e^{ikz}.
\end{align*}
\]

(8)

Solutions, localized on the soliton, have the energy \( \epsilon_{\text{ABS}}^\alpha = -\alpha \Delta_0 \cos \phi \) and are described by the following wave functions

\[
\psi_{\text{ABS}}^\alpha(z) = \frac{1}{2} \sqrt{\frac{\sin(\phi)}{\xi_0}} \frac{1}{\cosh[\sin(\phi) z]} \left( \frac{1}{1 - \alpha} \right).
\]

(9)

The presence of the soliton does not change the dispersion law of Bogoliubov quasiparticles, but it distorts the boundary conditions, which can no longer be considered as simple periodic. Indeed, if a superfluid with a single soliton with the phase jump \( 2\phi \) is spatially translated, the order parameter would not be periodic function due to the phase jump \( \Delta(x + L/2) = \Delta(x - L/2)e^{2\phi} \), where \( L \) is the system size. We have generalized boundary conditions in the presence of a soliton (see, Sec. A of Supplementary Material) and they are given by

\[
\psi(x + L/2) = [\cos(\phi) + i \sin(\phi) \sigma_z] \psi(x - L/2).
\]

(10)

Using the explicit form of the wave functions (8), we obtain the quantization condition for quasiparticle’s momentum \( k_n L + \theta_n^\alpha(k_n) = 2\pi n \), where \( n \) is integer and

\[
\theta_n^\alpha(k) = \text{arg} [\epsilon_k \cos(\phi) + \alpha \gamma - i \alpha \gamma \hbar v_F k \sin(\phi)]
\]

is a phase shift (the calculations are presented in Sec. B of Supplementary Material). Using these phase shifts, we find the number of states \( N_{\gamma} \), split from the continuous bands, as follows: \( N_{\gamma}^\alpha = \phi/\pi \) and \( N_{\phi}^\alpha = (\pi - \phi)/\pi \). Since there is only one bound state per a Fermi point, the sum of these numbers is \( N_{\gamma}^\alpha + N_{\phi}^\alpha = 1 \).

The energies of the continuous states and ABS should be shifted by \( \delta \epsilon = \alpha v_F \phi \). For the continuous spectrum this shift is unimportant as long as \( v_s \leq v_L \), where \( v_L = \Delta_0/p_F \) is the critical velocity within the Landau criterion. At \( v = v_L \), the continuous bands touch the zero energy level and soliton can lower its energy by emitting Bogoliubov excitations and becomes unstable. The details of this soliton dissociation process however are beyond the scope of our paper and we focus only on \( v_s < v_L \) in what follows. For localized states, the energy shift is crucial in this regime as well, because it governs both the energy and occupation of these states. Using quasiclassical wave functions (8) and (9), the self-consistent equation for order parameter (11) can be rewritten as

\[
\begin{align*}
\Delta(z) &= \frac{V \Delta_0}{4 \hbar v_F} \frac{\sin \phi}{\cosh^2[\sin(\phi) z]} + \\
&+ \frac{V}{2\pi} \int \frac{dk}{4 \hbar v_F} \frac{\sin \phi}{\Delta_0 \pi - 2 \phi} \frac{\sin \phi}{\cosh^2[\sin(\phi) z]}.
\end{align*}
\]

(12)

where \( \delta n = n^+ - n^- = n_F[v_s p_F - \Delta_0 \cos(\phi)] - n_F[-v_s p_F + \Delta_0 \cos(\phi)] \) is a difference between the occupation numbers of the ABS, which are influenced by the soliton’s motion. The latter two terms originate from the continuous Bogoliubov states, and for them we can set the temperature to zero. However, the zero-temperature limit for ABS is delicate, because it implies \( T \ll |v_s p_F - \Delta_0 \cos(\phi)| \), which can not hold when the corresponding energies vanish. The self-consistent equation (12) is satisfied if

\[
\sin(\phi) [\pi - 2 \phi - \pi \delta n] = 0.
\]

(13)

This equation has the trivial solution \( 2\phi = 0 \), which corresponds to a uniform BCS state with no solitons. It also
and its velocity $v_s$ across the soliton, which in the laboratory frame (2) on a time-independent model with the shifted Hamiltonian $H_{BdG}$, where the velocity of the soliton $v_s$ plays the role of an external parameter. The corresponding energy $E^K(\phi, v_s)$, in the comoving frame should achieve an extremum as a function of $\phi$, corresponding to the solution \[14\]. However, the actual energy of the solitonic state in the laboratory frame, $E^K(\phi, v_s)$, differs from $E^K(\phi, v_s)$ as discussed below.

The difference between $E^K(\phi, v_s)$ in the solitonic state and one in uniform BCS state can be presented as the sum $E^K = E^c + E^K_{ABS}$, where $E^c$ comes directly from the non-uniformity of the order parameter

$$E^c = \frac{1}{V} \int dz (|\Delta(z)|^2 - \Delta^2_0).$$

The contribution $E^K_{ABS}$ originates from filled continuous Bogoliubov states and can be calculated using Eq. \[11\].

$$E^K_{ABS} = \sum_{\alpha} \left[ N^\alpha_0 \Delta_0 + \sum_k \sigma^\alpha \frac{\partial c_k}{\partial k} - v_{sPF} (N^+ - N^-) \right],$$

with the last term here coming from the asymmetry between the states split from the continuum at the right and left Fermi points. Finally, the contribution, $E^K_{ABS}$, originates from the ABS and is given by $E^K_{ABS} = \sum_{n \uparrow} P_{n \uparrow} - \Delta_0 \cos(\phi) |\delta n|$. Detailed calculations are presented in Sec. C of the Supplementary Material. Putting all three terms together, we arrive at the following soliton energy in the laboratory frame (see also, Fig. 3)

$$E^K(\phi, v_s) = \frac{2\Delta_0}{\pi} \left[ \sin(\phi) + \left( \frac{\pi}{2} - \phi \right) \cos(\phi) \right] - v_{sPF} \left( 1 - \frac{2\phi}{\pi} \right) - v_{sPF} |\Delta_0 \cos(\phi)|.$$

For a soliton at rest, the energy has a clear maximum at $2\theta = \pi$. At a finite velocity, the energy maximum shifts and follows the curve corresponding to Eq. \[13\]. This however does not imply that the corresponding solution is unstable and/or unphysical. If we fix a phase jump across the soliton, which is a global constraint, the solution found self-consistently from the BdG equations becomes a minimum of the corresponding energy functional \[18\] (e.g., distorting the shape of the solitary wave would always increase the system’s energy, as long as global boundary conditions are preserved). This means that the soliton is stable against local perturbations. This local stability was also observed in numerical simulations of the BdG equations \[29,31\]. A similar situation takes place in the BEC regime, where the soliton is stable against local perturbations, but its energy decreases with velocity. Interestingly, at a finite velocity, there appear additional local minima of $E^K(\phi, v_s)$, gradually splitting

$\phi_s(v_s) = 2\arccos \left( \frac{v_s}{v_L} \right).$ \[14\]

The soliton at rest has a phase jump of $2\phi_s = \pi$ across it, as previously derived \[18,21\]. At a finite velocity, the phase jump gradually decreases with velocity $v_s$ until the critical velocity $v_L$ is reached, as illustrated in Fig. 2.

For a soliton at rest, the energies of the ABS are zero and they are equally occupied. In the state with a traveling soliton, the energies of ABS in the laboratory frame become finite, $\epsilon_{ABS,s} = -\alpha v_{sPF}$. The occupation numbers of the ABS also adjust to soliton’s motion, and can be calculated from Eq. \[13\]:

$$n_{ABS,s}^\pm = \frac{1}{2} \pm \epsilon_{ABS,s} \left( \frac{v_s}{v_L} \right).$$

The total occupation of the ABS is equal to one (i.e., $n_{ABS,s}^+ + n_{ABS,s}^- = 1$), and so it coincides with the number of states split away from the lower Bogoliubov band (i.e., $N^+_s + N^-_s = 1$). It means that there is neither a deficit, nor an excess of fermionic matter in the soliton core compared to the uniform state (in sharp contrast to dark Gross-Pitaevskii solitons \[8\]). This circumstance is ultimately responsible for the light "gravitational" mass of the solitons in the fermion superfluid [see, Eq. (22) below].

**Soliton energetics** – In equilibrium, the self-consistent equation corresponds to an extremum of the free energy of the system (energy in the zero-temperature limit). Our time-dependent approach involves a mapping of the time-dependent Hamiltonian in the laboratory frame \[4\] on a time-independent model \[9\] with the shifted Hamiltonian $K_{BdG}$, where the velocity of the soliton $v_s$ plays the role of an external parameter. The latter holds when the BdG equations and self-consistency equation are satisfied simultaneously.

![FIG. 3: (Color online) Shown is the dependence of the energy, $E^K$, on the phase discontinuity across the soliton, $2\phi_s$, and its velocity $v_s$. The dependence has a clear maximum, corresponding to the relation \[14\]. The latter holds when the BdG equations and self-consistency equation are satisfied simultaneously.](image)
from the trivial solutions $2\phi = 0, 2\pi$ (see, Fig. 3). However, they do not satisfy the self-consistency constraint \(13\), and hence are locally unstable.

The energy of the system in the laboratory frame, \(E^H(\phi, v_s)\), follows from Hamiltonian \(12\) and can be calculated in the same manner as above (see, Sec. C of the Supplementary Material C for details). The difference between the energy of a BCS superfluid with a moving soliton and the uniform BCS state is

\[
E_s(v_s) = E^H(\phi_s(v_s), v_s) = \frac{2\Delta_0}{\pi} \sqrt{1 - \left(\frac{v_s}{v_L}\right)^2}.
\]

(19)

The energy of the soliton at rest is \(E_s(0) = 2\Delta_0/\pi\). It gradually decreases with the velocity \(v_s\) and vanishes at the critical velocity \(v_L\), as presented in Fig. 2. From Eq. (19), we see that the “intertial” mass of the soliton is negative, is considerably larger than a single fermion’s mass, and is given by

\[
m_i^s = -\frac{2\Delta_0}{\pi v_L^2} = -\frac{4m}{\pi} \frac{E_F}{\Delta_0}.
\]

(20)

**Dynamics of the soliton in a trap** – For a superfluid in a trap, the confining potential makes the soliton energy position-dependent and drives its motion. In the local density approximation, the chemical potential of fermions is \(\mu(x) = E_F - U(x)\), where \(U(x) = m_0^2 x^2 / 2\) is a harmonic trapping potential with frequency, \(\omega\). The energy of a soliton with velocity \(v_s\) and coordinate \(x_s\) at \(v_s \ll v_L\) and \(U(x_s) \ll E_F\) can be approximated as follows

\[
E_s(v_s, x_s) = \frac{2\Delta_0}{\pi} + \frac{m_i^s v_s^2}{2} + \frac{m_g^s \omega^2 x_s^2}{2},
\]

(21)

where \(m_g^s\) can be interpreted as the “gravitational” mass of the soliton, which measures the strength of its interaction with the trap. This is also negative, but in sharp contrast to the “inertial” mass, is considerably smaller than the mass of a single fermion and is given by

\[
m_g^s = -\frac{2m}{\pi} \frac{\partial\Delta}{\partial E_F} \approx -\frac{2m}{\pi} \frac{\Delta}{E_F}.
\]

(22)

The equation of motion for the soliton can be derived from the energy balance equation, \(\dot{E} = -\Gamma m_i^s v_s\), where \(\Gamma\) is a phenomenological friction constant. It yields \(\ddot{x}_s - \Gamma \dot{x}_s + \omega^2 x_s = 0\) and the oscillation frequency of the soliton in the trap,

\[
\omega_s = \omega \sqrt{\frac{m_g^s}{m_i^s}} \approx \frac{\omega \Delta}{\sqrt{2} E_F}.
\]

(23)

which is considerably smaller than the trapping frequency, \(\omega\), since \(E_F \gg \Delta\) in the BCS regime. It should be noted that friction leads to self-acceleration of the soliton, since its energy decreases with the velocity. Hence, the soliton oscillates with an increasing amplitude, until it achieves the critical velocity and disappears \(13\).

This research was supported by DOE-BES DESC0001911 (D.E.), US-ARO (V.G.), and Simon's Foundation. The authors are grateful to Victor Yakovenko and Martin Zwierlein for illuminating discussions and a number of useful suggestions.

[1] N. Theodorakopoulos, *Nonlinear Physics: Solitons, Chaos, Discrete Breathers* (University of Konstanz, 2006).
[2] J. Denschlag, J. E. Simsarian, D. L. Feder, C. W. Clark, L. A. Collins, J. Cubizolles, L. Deng, E. W. Hagley, K. Helmerson, W. P. Reinhardt, et al., Science 287, 97 (2000).
[3] S. Burger, K. Bongs, S. Dettmer, W. Ertmer, K. Sengstock, A. Sanpera, G. V. Shlyapnikov, and M. Lewenstein, Phys. Rev. Lett. 83, 5198 (1999).
[4] J. Denschlag, J. E. Simsarian, D. L. Feder, C. W. Clark, L. A. Collins, J. Cubizolles, L. Deng, E. W. Hagley, K. Helmerson, W. P. Reinhardt, et al., 287, 97 (2000).
[5] Z. Dutton, M. Budde, C. Slowé, and L. V. Hau, Science 293, 663 (2001).
[6] N. S. Ginsberg, J. Brand, and L. V. Hau, Phys. Rev. Lett. 94, 040403 (2005).
[7] A. Weller, J. P. Ronzheimer, C. Gross, J. Esteve, M. K. Oberthaler, D. J. Frantzeskakis, G. Theocharis, and P. G. Kevrekidis, Phys. Rev. Lett. 101, 130401 (2008).
[8] P. Kevrekidis, D. J. Frantzeskakis, and R. Carretero-Gonzalez, *Emergent Nonlinear Phenomena in Bose-Einstein Condensates*, Theory and Experiment (Springer, Verlag, 2008).
[9] D. J. Frantzeskakis, J. Phys. A: Math. Theor. 43, 213001 (2010).
[10] L. P. Pitaevskii and S. Stringari, *Bose-Einstein Condensation* (Oxford University Press, Oxford, 2010).
[11] A. Muryshev, G. V. Shlyapnikov, W. Ertmer, K. Sengstock, and M. Lewenstein, Phys. Rev. Lett. 89, 110401 (2002).
[12] V. A. Brazhnyi, V. V. Konotop, and L. P. Pitaevskii, Phys. Rev. A 73, 053601 (2006).
[13] S.-S. Shei, Phys. Rev. D 14, 535 (1976).
[14] G. m. c. Ba¸sar and G. V. Dunne, Phys. Rev. Lett. 100, 200404 (2008).
[15] F. Correa, G. V. Dunne, and M. S. Plyushchay, Annals of Physics 324, 2522 (2009).
[16] D. A. Takahashi and M. Nitta, Phys. Rev. Lett. 110, 131601 (2013).
[17] R. Yoshii, S. Tsuchiya, G. Marmorini, and M. Nitta, Phys. Rev. B 84, 024503 (2011).
[18] H. Takayama, Y. R. Lin-Liu, and K. Maki, Phys. Rev. B 21, 2388 (1980).
[19] S. Kivelson, T.-K. Lee, Y. R. Lin-Liu, I. Peschel, and W. P. Su, Rev. Mod. Phys. 60, 781 (1988).
[20] A. J. Heeger, S. Kivelson, J. R. Schrieffer, and W. P. Su, Rev. Mod. Phys. 60, 781 (1988).
[21] S. Brazovski, Zh. Exp. Teor. Fiz. 78, 677 (1980).
[22] S. Brazovski, Zh. Exp. Teor. Fiz. 28, 656 (1978).
[23] K. Machida and H. Nakanishi, Phys. Rev. B 30, 122 (1984).
[24] H.-J. Kwon and V. M. Yakovenko, Phys. Rev. Lett. 89, 017002 (2002).
[25] R. M. Lutchyn, M. Dzero, and V. M. Yakovenko, Phys. Rev. A 84, 033609 (2011).
[26] A. Zharov, A. Lopatin, A. E. Koshelev, and V. M. Vinokur, Phys. Rev. Lett. 98, 197005 (2007).
[27] M. Antezza, F. Dalfovo, L. P. Pitaevskii, and S. Stringari, Phys. Rev. A 76, 043610 (2007).
[28] R. G. Scott, F. Dalfovo, L. P. Pitaevskii, and S. Stringari, Phys. Rev. Lett. 106, 185301 (2011).
[29] R. Liao and J. Brand, Phys. Rev. A 83, 041604 (2011).
[30] A. Spuntarelli, L. D. Carr, P. Pieri, and G. C. Strinati, New Journal of Physics 13, 035010 (2011).
[31] R. G. Scott, F. Dalfovo, L. P. Pitaevskii, S. Stringari, O. Fialko, R. Liao, and J. Brand, New Journal of Physics 14, 023044 (2012).
[32] A. Cetoli, J. Brand, R. G. Scott, F. Dalfovo, and L. P. Pitaevskii, Phys. Rev. A 88, 043639 (2013).
[33] T. Yefsah, A. T. Sommer, M. J. H. Ku, L. W. Cheuk, W. Ji, W. Bakr, and M. W. Zwierlein, Nature 499, 426 (2013).
[34] M. J. H. Ku, W. Ji, B. Mukherjee, E. Guardado-Sanchez, L. W. Cheuk, T. Yefsah, and M. W. Zwierlein, Phys. Rev. Lett. 113, 065301 (2014).
[35] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma, Rev. Mod. Phys. 80, 1083 (2008).
[36] A. F. Andreev, Sov. Phys. JETP 19, 1228 (1964).
[37] R. Jackiw and C. Rebbi, Phys. Rev. D 13, 3398 (1976).
Supplemental Material: Moving solitons in a one-dimensional fermionic superfluid – an exact solution

A. Generalized periodic boundary conditions

Bogoliubov-de Gennes equations require appropriate boundary conditions. For a uniform superfluid, the simple periodic boundary conditions, \( \psi_{\gamma k}(z + L/2) = \psi_{\gamma k}(z - L/2) \) (with \( L \) being the system size) apply. However, they can not be used in the presence of a soliton, since the order parameter is no longer a periodic function of the coordinate. Indeed, while all local physical observables [e.g., the fermion current \( j(z) \), density \( \rho(z) \), etc.] are periodic functions of the coordinate in the closed system \( j(z + L/2) = j(z - L/2) \), \( \rho(z + L/2) = \rho(z - L/2) \), etc., the order parameter is not periodic, because it has a global phase discontinuity across the soliton, and \( \Delta(z + L/2) = \Delta(z - L/2)e^{2i\phi} \).

Here, we generalize the simple periodic boundary conditions to the the system with a soliton. The general form of boundary conditions is

\[
\psi_{\gamma k}(z + L/2) = \hat{B}_{\gamma k} \psi_{\gamma k}(z - L/2),
\]  

where \( \hat{B}_{\gamma k}(\phi) \) is a matrix (whose explicit form is to be determined) that depends on the phase jump across the soliton. We assume that boundary conditions do not mix states with different quantum numbers and omit the corresponding indices \( \alpha, \gamma, \) and \( k \), that become redundant. First, we require that the fermion current and density

\[
j(z) = \psi^* (z) \psi(z), \quad \rho(z) = 1 + \psi^* (z) \sigma_z \psi(z)
\]

are periodic functions. These conditions lead to the following constrains, \( \hat{B}^+ \hat{B} = 1 \) and \( \hat{B}^+ \sigma_z \hat{B} = \sigma_z \). The former implies that the matrix \( \hat{B} \) is unitary, while the latter allows us to parameterize it by two complex phases, \( \Phi \) and \( \Theta \), as follows

\[
\hat{B} = e^{i\Phi} [\cos(\Theta) + i \sin(\Theta) \sigma_z].
\]

Next, assuming the state \( \psi(z - L/2) \) to be an eigenvector of the BdG Hamiltonian, \( K_{BdG}(z - L/2) \psi(z - L/2) = \epsilon \psi(z - L/2) \), we demand that the spatially-translated state, \( \psi(z + L/2) \), is an eigenvector of the translated BdG Hamiltonian \( K_{BdG}(z + L/2) \psi(z + L/2) = \epsilon \psi(z + L/2) \). Note that due to the presence of the phase jump, \( \Delta(z + L/2) = \Delta(z - L/2)e^{2i\phi} \), the Hamiltonian is not invariant under translation. Using the explicit form of the BdG Hamiltonian \( K \), we arrive at

\[
\hat{B}^+ \sigma_z \hat{B} = \sigma_z \cos(\phi) + \sigma_y \sin(\phi),
\]

\[
\hat{B}^+ \sigma_y \hat{B} = \sigma_y \cos(\phi) - \sigma_x \sin(\phi).
\]

The Anzatz satisfies if \( \Theta = \phi \). Finally, we notice that the superfluid state with the order parameter becomes equivalent to the uniform BCS state at \( \phi = 0 \), since the soliton profile vanishes. Therefore, we must require that \( \hat{B}(\phi = 0) = 1 \), since \( \hat{B} = \hat{1} \) corresponds to the simple periodic boundary conditions. This constraint fixes the remaining parameter \( \Phi = 0 \), and determines the unitary matrix \( \hat{B}(\phi) \) as follows

\[
\hat{B}(\phi) = \cos(\phi) + i \sin(\phi) \sigma_z.
\]

The matrix does not depend on the set of indexes \( \alpha, k \) and \( \gamma \) for a continuous Bogoliubov state.

Let us remark that the boundary condition can be straightforwardly generalized to the presence of a soliton train (not relevant here, but of importance to studies of inhomogeneous superconducting states). There, the boundary conditions would have the same form as Eq. \( S5 \), but with \( 2\phi \) replaced by the whole phase jump across the train.

B. Momentum quantization and phase shifts

The simple periodic boundary conditions, that can be used for a uniform superfluid, determine the standard momentum quantization rule: \( k_n L = 2\pi n \). In the presence of a soliton, momentum quantization is modified and follows from the appropriate boundary conditions \( S5 \).

Let us rewrite the boundary conditions in terms of the functions \( f_{\gamma k,\pm}^{\alpha}(z + L/2) = \alpha \gamma k, \pm \) as follows

\[
f_{\gamma k,\pm}^{\alpha}(z + L/2) = \cos(\phi) f_{\gamma k,\pm}^{\alpha}(z - L/2) + i \sin(\phi) f_{\gamma k,\mp}^{\alpha}(z - L/2).
\]
The dependence of the phase shifts on momentum is presented in Fig. S1. Their asymptotic values at infinite momenta are given by

\[ \theta_{\gamma k, \pm} = \frac{\gamma \epsilon_k + \alpha \Delta_1}{k L \gamma \epsilon_k} e^{ikz}, \]

\[ f_{\gamma k, \pm}^\alpha(z) = \alpha \sqrt{\frac{\gamma \epsilon_k + \alpha \Delta_1}{\gamma \epsilon_k + \Delta_1}} e^{ikz}. \]

For the right Fermi point, and to

\[ |\gamma \epsilon_k - \Delta_0 \cos(\phi)| e^{ikL/2} = |\gamma \epsilon_k + \Delta_0 \cos(\phi)| \cos(\phi) e^{-ikL/2} - i[\hbar v_F k - i \Delta_0 \sin(\phi)] \sin(\phi) e^{-ikL/2}, \]

\[ [\hbar v_F k + i \Delta_0 \sin(\phi)] e^{ikL/2} = [\hbar v_F k - i \Delta_0 \sin(\phi)] \cos(\phi) e^{-ikL/2} - i[\gamma \epsilon_k + \alpha \Delta_0 \cos(\phi)] \sin(\phi) e^{-ikL/2}, \]

for the left Fermi point. Each pair of equations can be reduced to \( e^{i k L} = 1 \), which yield a momentum quantization rule as follows \( k_n L + \theta_{\gamma}^\alpha (k_n) = 2 \pi n \). Here \( \theta_{\gamma}^\alpha (k) \) is the phase shift, which is given by

\[ \theta_{\gamma}^\alpha (k) = \arg [\epsilon_k \cos(\phi) + \alpha \gamma - i \alpha \gamma \hbar v_F k \sin(\phi)]. \]

The dependence of the phase shifts on momentum is presented in Fig. S1. Their asymptotic values at infinite momenta are given by

\[ \theta_{\gamma}^\alpha (\infty) = -\frac{\phi}{2}, \quad \theta_{\gamma}^\alpha (\infty) = \frac{\phi}{2}, \quad \theta_{\gamma}^\alpha (-\infty) = \frac{\phi}{2}, \quad \theta_{\gamma}^\alpha (-\infty) = -\frac{\phi}{2}. \]

The number of states split from the left- and right-moving continuous Bogoliubov bands can be calculated with the help of these phase shifts as follows

\[ N_{\alpha} = -\int_{-\infty}^{\infty} \frac{dk}{2 \pi} \frac{d\theta_{\gamma}^\alpha}{dk} = -\frac{\phi}{\pi}, \quad N_{\alpha} = -\int_{-\infty}^{\infty} \frac{dk}{2 \pi} \frac{d\theta_{\gamma}^\alpha}{dk} = 1 - \frac{\phi}{\pi}. \]

Since there is only one ABS per Fermi point, the total splitting from the continuous bands is equal \( N_{\alpha} + N_{\beta} = 1 \). The total number of states split from the Bogoliubov states with negative energies is also equal to \( N_{-} + N_{+} = 1 \).
For a calculation of the energy of a superfluid, which is presented in the next section, it is useful to introduce the average phase shift \( \bar{\theta} = (\theta^+ + \theta^-)/2 \). Using the relations
\[
\cos(\theta^+_c) = \frac{\epsilon_k \cos(\phi) + \alpha \gamma \Delta_0}{\epsilon_k + \alpha \gamma \Delta_0 \cos(\phi)}, \quad \sin(\theta^+_c) = -\frac{\alpha \gamma \sin(\phi)}{\epsilon_k + \alpha \gamma \Delta_0 \cos(\phi)} \tag{S13}
\]
the average phase shift \( \bar{\theta} \) can be calculated as follows
\[
\bar{\theta} = \arctan \left[ \frac{1 + \cos(\theta^+_c + \theta^-_c)}{1 - \cos(\theta^+_c + \theta^-_c)} \right] = \arctan \left[ \frac{\Delta_0 \sin(\phi)}{\hbar v_F k} \right]. \tag{S14}
\]
The dependence of the average phase shift \( \bar{\theta} \) on the momentum is presented in Fig. S1.

**C. Calculation of the soliton energy in the co-moving and laboratory frames**

The energies of a fermionic superfluid in the co-moving (\( E^K \)) and laboratory (\( E^H \)) frames can be determined from the Hamiltonians \( K_{BdG} \) [defined in Eq. (3)] and \( H_{BdG} \) [defined in Eq. (2)], respectively. The energies of Bogoliubov states of \( K_{BdG} \) and \( H_{BdG} \) differ by the shift \( \delta \epsilon = \alpha \epsilon v_F \), while the occupation numbers are the same and correspond to \( K_{BdG} \), since in the co-moving frame the solitonic texture is time-independent and the superfluid achieves thermal equilibrium. The difference in energy between a superfluid with a soliton and the uniform BCS state can be presented as the sum
\[
E^{K(H)} = E_\Delta + E^{K(H)}_c + E^{K(H)}_{ABS}.
\]
The first term, \( E_\Delta \), in this equation comes directly from the non-uniformity of the order parameter, it does not depend on the energy shift, and is given by
\[
E_\Delta = \int d^2 (\frac{\langle \Delta^2 - \Delta^2_0 \rangle}{V}) = -\sum_k \frac{2 \hbar v_F \Delta_0 \sin \phi}{\epsilon_k}, \tag{S15}
\]
where we have eliminated the coupling constant \( V \) using the self-consistency equation (4) for the uniform BCS state. Contributions \( E^H_c \) and \( E^{K}_{c} \) originate from filled continuous Bogoliubov states, whose occupations are not influenced by the energy shift. Therefore, they can be calculated with the help of phase shifts (11) as follows
\[
E^H_c = \sum_\alpha \left[ N^\alpha_0 \Delta_0 + \sum_k \theta^\alpha \frac{\partial \epsilon_k}{\partial k} \right], \quad E^K_c = \sum_\alpha \left[ N^\alpha_0 \Delta_0 + \sum_k \theta^\alpha \frac{\partial \epsilon_k}{\partial k} \right] - v_sp_F (N^+ - N^-). \tag{S16}
\]
The last term in \( E^K_c \) originates from a difference in the number of states split from the right- and the left-moving filled bands. The energy \( E^H_c \) can be calculated as follows
\[
E^H_c = \Delta_0 + \int_0^{\infty} \frac{dk}{\pi} \hat{\theta}(k) \frac{d \epsilon_k}{dk} = -\int_0^{\infty} \frac{dk}{\pi} \hat{\epsilon}(k) - \int_0^{\infty} \frac{dk}{\pi} \frac{2 \Delta_0 \sin(\phi)}{\epsilon_k} \cos(\phi)\right] + \int_0^{\infty} \frac{dk}{\pi} \frac{2 \hbar v_F \Delta_0 \epsilon_k \sin(\phi)}{\epsilon_k^2 + (\Delta_0 \sin(\phi))^2}. \tag{S17}
\]
Here, we have taken into account that the total number of states split from the Bogoliubov hole bands for the right and left Fermi points is \( N^+ + N^- = 1 \) and introduced the average phase shift \( \bar{\theta} = (\theta^+ + \theta^-)/2 = \arctan[\Delta_0 \sin(\phi)/\hbar v_F k] \), calculated in Sec. B. Combining with (S15) and performing an integration, we arrive at
\[
E_\Delta + E^H_c = \frac{2 \Delta_0}{\pi} \left[ \sin(\phi) + \left( \frac{\pi}{2} - \phi \right) \cos(\phi) \right], \quad E_\Delta + E^K_c = E_\Delta + E^H_c - v_sp_F \left( 1 - \frac{2 \phi}{\pi} \right). \tag{S18}
\]
The last contributions \( E^{H}_{ABS} \) and \( E^K_{ABS} \) originate from the ABS. Both energies and occupations of ABS are influenced by the energy shift, \( \delta \epsilon = \alpha \epsilon v_F \). Hence, it is instructive to consider them separately. In the co-moving frame, the energy is given by \( E^{K}_{ABS} = -|v_sp_F - \Delta_0 \cos(\phi)| \tan(\theta^+ - \theta^-) \). The zero-temperature limit \( T \ll |v_sp_F - \Delta_0 \cos(\phi)| \) is well-defined and the energy at \( T = 0 \) is given by \( E^{H}_{ABS} = -|v_sp_F - \Delta_0 \cos(\phi)| \). Combining all contributions together, we get the energy of a superfluid with a soliton in the co-moving frame to be
\[
E^K(\phi, v_s) = \frac{2 \Delta_0}{\pi} \left[ \sin(\phi) + \left( \frac{\pi}{2} - \phi \right) \cos(\phi) \right] - v_sp_F \left( 1 - \frac{2 \phi}{\pi} \right) - |v_sp_F - \Delta_0 \cos(\phi)|. \tag{S19}
\]
In the laboratory frame, the contribution of ABS is given by $E_{\text{ABS}}^H = \Delta_0 \cos(\phi) \tanh\left(\frac{[v_s p_F - \Delta_0 \cos(\phi)]}{T}\right)$. In the zero temperature limit, it tends to $E_{\text{ABS}}^H = -\Delta_0 \cos(\phi) \Theta_H [\Delta_0 \cos(\phi) - v_s p_F]$ and the energy of the superfluid in the laboratory frame is given by

$$E^H(\phi, v_s) = \frac{2\Delta_0}{\pi} \left[ \sin(\phi) + \left( \frac{\pi}{2} - \phi \right) \cos(\phi) \right] - \Delta_0 \cos(\phi) \Theta_H [\Delta_0 \cos(\phi) - v_s p_F], \quad (S20)$$

where $\Theta_H$ is the Heaviside step function. However, in this case, the zero temperature limit is ill-defined since $E_{\text{ABS}}^H(\phi, v_s)$ [and hence $E^H(\phi, v_s)$ too] is not a smooth function of its arguments. The energy has a jump across the line $\Delta_0 \cos(\phi) - v_s p_F = 0$, which corresponds to the solitonic profile $[13]$. Hence the calculation of the energy of a superfluid in the solitonic state, which has the phase profile $[14]$, requires a more delicate approach. In the solitonic state, both energies $\epsilon_s = -\alpha v_s p_F$ and occupations of ABS [defined in $[15]$] adjust to soliton’s motion. Hence the contribution of ABS is well defined and is given by

$$E_{\text{ABS}}^H = \epsilon^+_{\text{ABS},s} n^+_{\text{ABS},s} + \epsilon^-_{\text{ABS},s} n^-_{\text{ABS},s} = \frac{\hbar v_F v_s}{\pi} \left[ 2 \arccos \left( \frac{v_s}{v_L} \right) - \pi \right], \quad (S21)$$

where $v_L = \Delta_0 / p_F$ is the critical velocity within the Landau criterion. Collecting all other contributions, $E_\Delta(\phi_s(v_s))$ and $E_c^H(\phi_s(v_s), v_s)$, we obtain the energy of the soliton in the laboratory frame as follows

$$E_s(v_s) = E^H(\phi_s(v_s), v_s) = \frac{2\Delta_0}{\pi} \sqrt{1 - \left( \frac{v_s}{v_L} \right)^2}. \quad (S22)$$