Corona graphs as a model of small-world networks

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Abstract. We introduce recursive corona graphs as a model of small-world networks. We investigate analytically the critical characteristics of the model, including order and size, degree distribution, average path length, clustering coefficient, and the number of spanning trees, as well as Kirchhoff index. Furthermore, we study the spectra for the adjacency matrix and the Laplacian matrix for the model. We obtain explicit results for all the quantities of the recursive corona graphs, which are similar to those observed in real-life networks.

Keywords: exact results, growth processes, network dynamics
1. Introduction

For decades we have wanted to know what a graph looks like. We want to reveal the principles of the networks’ behaviour covered by their complex topology and dynamics. We want to learn about how the network structure evolves over time and how it affects the properties of dynamical processes on it. For the nature of decentrality of real networks, it is hard to observe the networks directly. Instead we observe them by taking snapshots of their network structure and content and keep updating them. Yet this gives little information about the future, since many of them keep growing over time. Thus it is desirable to set up models to fit real networks in both structure and functionality. We can use the models to mimic real-life networks. We also expect that the properties of the models can be proven rigorously, thus we can find the relations between topological and dynamical properties of networks. Even if it is hard to give closed-form expressions for some quantities, it would be nice to make them tractable for convenience of estimation.

Among various network models, the ER graph proposed by Erdös and Rényi [1] is the earliest one. It generates a random graph by choosing a constant probability for joining every pair of vertices in the network. The model exhibit interesting statistical properties and has been well studied by many people. However the model lacks some important properties of real-world networks. For example many real-world networks exhibit the
small-world property [2–4] with their diameters growing logarithmically in the number of vertices, while maintaining a high clustering coefficient. The Watts–Strogatz (WS) model [3] is a typical graph model with the small-world effect. Nevertheless, because of its randomness, many of its properties cannot be derived precisely, for example eigenvalues of the adjacency and Laplacian matrices. Thus deterministic models are often used to mimic complex networks [5], since their structural [6] and spectral [7] characteristics can be determined analytically. In addition to the small-world effect, another important feature of a network is degree distribution. Many real-world networks exhibit a heavy-tailed distribution while some networks have an exponential distribution [8–10].

The famous preference attachment [11–14] scheme successfully described the growing process of networks with heavy-tailed degree distribution. This work, like the WS model [3], leads to a network with an exponential degree distribution.

Recently graph (matrix) products have been applied to modelling graphs with the same properties as real-life networked systems, such as the Cartesian product [15], dot product [16], and Kronecker product [17–21]. A merit of such methods is that the graph/matrix products facilitate estimation of the properties of the generated graphs.

In this paper we introduce a recursive way to generate small-world networks with an exponential degree distribution, based on corona product of graphs. We obtain exact solutions to many structural properties of the networks. Moreover, we derive all the eigenvalues for their adjacency matrix and Laplacian matrix, which are provided in a recursive way. Based on the obtained eigenvalues, we calculate the number of spanning trees, as well as the Kirchhoff index of the networks.

2. Graph construction

In this paper, we use corona product to generate a small-world network model. Literature about the corona product and its related graphs is partly established [22, 23]. Let $G = (V(G), E(G))$ be the embedded graph of a network. Suppose the graph is undirected and has vertex set $V(G) = \{1, 2, ..., N\}$ and edge set $E(G) \subseteq V(G) \times V(G)$. We define the number of vertices $N$ as the order of the graph, and the number of vertices as the size of the graph, denoted as $M = |E(G)|$.

Given two graphs $G_1$ and $G_2$, their corona product $G_1 \circ G_2$ is defined as follows.

**Definition 2.1.** Let $G_1$ and $G_2$ be two graphs with disjoint vertex sets. $G_1$ has $N_1$ vertices and $G_2$ has $N_2$ vertices. Their corona product $G_1 \circ G_2$ is a new graph which consists of one copy of $G_1$ and $N_1$ copies of $G_2$. The $i$th vertex of $G_1$ is joined by a new edge with every vertex in the $i$th copy of $G_2$.

In this paper we investigate the case where $G_2$ is the $q$-complete graph, thus we give the definition of the recursive corona graph.

**Definition 2.2.** Let $K_q$ be the $q$-complete graph ($q \geq 2$), then the $g$th generation of the recursive corona graph (RCG) $C_q(g + 1)$ is defined as the corona of the previous generation of the RCG $C_q(g)$ and $K_q$. More formally, $C_q(g + 1)$ is defined as $C_q(g + 1) = C_q(g) \circ K_q$, $g \geq 0$, with the initial condition $C_q(0) = K_q$.

Figure 1: Illustrates the construction process for a particular network $C_q(g)$.
3. Structural properties

In this section we derive several important quantities of the RCG, showing that it is an appropriate model for the small-world complex networks. Thanks to the deterministic feature of $C_q(g)$, we can give exact expressions for the properties of the graph. We will give the explicit results for its order, size, degree distribution, degree correlation, average distance, clustering coefficient, number of spanning trees and Kirchhoff index.

We denote by $N(g)$ and $M(g)$ respectively the order and the size of $C_q(g)$. Next we show how to derive these quantities. Assume that the number of vertices and the number of edges that are newly generated at step $g$ are denoted as $L_V(g)$ and $L_E(g)$. Then it is obvious that we have $L_V(g) = qN(g-1)$ for $g \geq 1$, which leads to the result of $N(g)$ along with the initial condition $N(0) = q$. With respect to the size of the network, we have $L_E(g) = q(q-1)/2N(g-1) + qN(g-1)$, $g \geq 1$, and the initial condition $M(0) = q(q-1)/2$.

**Proposition 3.1.** The order and size of the graph $C_q(g) = (V(g), E(g))$ are, respectively,

$$N(g) = q(q+1)^g$$

and

$$M(g) = \frac{1}{2} q((q+1)^{g+1} - 2).$$

The average degree is $\bar{\delta}(g) = -2(q+1)^{-g} + q + 1$, which tends to $q + 1$ for large $g$. Note that many real-life networks are sparse and their average degree tends to a constant value.

3.1. Degree distribution

The degree distribution $P(\delta)$ for a network is a function indicating the fraction of vertices with degree $\delta$ over all vertices. The degree distribution is a very important characteristic of a graph. It is essential to the analysis of many other structural properties.

The cumulative degree distribution [8] is defined as

$$P_{\text{cum}}(\delta) = \sum_{\delta'=\delta}^{\infty} P(\delta'),$$

which is often used to analyze the degree distribution of a graph. The quantity gives the fraction of vertices whose degree $\delta'$ is greater than or equal to $\delta$. In addition, networks whose degree distributions are exponential, $P(\delta) \sim e^{-\alpha \delta}$, also have an exponential cumulative distribution with the same exponent:

$$P_{\text{cum}}(\delta) = \sum_{\delta'=\delta}^{\infty} P(\delta') \approx \sum_{\delta'=\delta}^{\infty} e^{-\alpha \delta'} = \left( \frac{e^\alpha}{e^\alpha - 1} \right) e^{-\alpha \delta}. \quad (3)$$

Next we investigate the degree distribution of $C_q(g)$. We find that at time $g = 0$ the network has $q$ vertices of degree $q-1$. Now we study the degree of some vertex $v$ at step $g$. Let the value be $\delta_v(g)$; we look in detail at how the quantity evolves. We assume that vertex $v$ is added to the network at step $g_v$ ($g_v > 0$). For any $g_v > 0$, we have
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\[ \delta_v(g_v) = q, \text{ where } q - 1 \text{ edges link to other vertices in } K_q \text{ and the other edge links to } C_q(g - 1). \text{ At every step every existing vertex increases its degree by } q. \]

**Theorem 3.2.** The cumulative degree distribution of the graph \( C_q(g) \) follows an exponential distribution: \( P_{\text{cum}}(\delta) \sim (q + 1)^{-\frac{\theta}{q} + 1}. \)

**Proof.** The degree of vertex \( v \) at step \( g \), denoted as \( \delta_v(g) \), can be written as

\[ \delta_v(g + 1) = \delta_v(g) + q. \]  

Thus we have

\[ \delta_v(g) = q(g - g_v + 1), (g_v > 0), \text{ and } \delta_v(g) = q(g + 1) - 1, (g_v = 0). \]

This means that the numbers of vertices with the degree equal to \( q, 2q, \ldots, gq, (g + 1)q - 1 \) are, respectively, \( q^2(q + 1)^{q-1}, q^2(q + 1)^{q-2}, \ldots, q^2 \).

For a certain value of degree \( \delta \), we have \( P_{\text{cum}}(\delta) = (N(0) + \sum_{g'=1}^{\theta} L\nu(g'))/N(g) \) where \( \theta = [g - (\delta + q)/q] \). Therefore we can find

\[ P_{\text{cum}}(\delta) = \frac{1}{q(q + 1)^\theta} \left( q + \sum_{g'=1}^{\theta} q^2(q + 1)^{q-1} \right) \]

\[ = \frac{1}{(q + 1)^\theta} + \frac{q}{(q + 1)^\theta} \sum_{g'=1}^{\theta} (q + 1)^{g'-1} \]

\[ = (q + 1)^{\theta - \delta}. \]

For large \( g \) we have

\[ P_{\text{cum}}(\delta) = (q + 1)^{\theta - \delta} \sim (q + 1)^{\frac{\delta}{q} + 1}. \]

**3.2. Degree correlation**

One important parameter for the degree correlation is the average degree of adjacent vertices of \( v(\delta) \), which refers as any vertex with degree \( \delta \). We denote the parameter by \( k_{nn}(\delta) \). If \( k_{nn}(\delta) \) increases with \( \delta \), this means that the vertices have a tendency to connect to vertices with a similar or larger degree. In this case we claim the graph to be assortative. The considered value of a vertex \( v(\delta) \) can be written as

\[ k_{nn}(v) = \frac{1}{q(g - g_v + 1)} \left( q(g - g_v + 2) + (q - 1)q(g - g_v + 1) + \sum_{i=g_v}^{g-1} q \cdot q(g - i) \right) \]

\[ = \frac{1}{2} q(g - g_v + 2) + \frac{1}{g - g_v + 1}. \]

According to equation (5), we can express it as

\[ k_{nn}(\delta) = \frac{1}{2}(q + \delta) + \frac{q}{\delta}. \]
For the initial vertices we have
\[ k_{nm}(v_0) = \frac{1}{q(g+1)-1} \left( (q-1)(q-1) + \sum_{i=0}^{g} q \cdot q(g-i) \right) \]
\[ = \frac{(g+2)r}{2} - \frac{(g+2)q - 2}{2(gq + q - 1)}, \quad (9) \]
which yields
\[ k_{nm}(\delta_0) = \frac{1}{2} \left( \delta_0 + q + \frac{1-q}{\delta_0} \right). \quad (10) \]

By checking the results we can see that the considered graph is assortative.

### 3.3. Average distance

Given a graph \( G = (V, E) \), its average distance or mean distance is defined as
\[ \mu(G) = \frac{1}{|V(G)|(|V(G)|-1)} \sum_{u,v \in V(G)} d(u, v) \]
where \( d(u, v) \) is the distance between the pair of vertices \( u \) and \( v \).

**Theorem 3.3.** The average distance of graph \( C_q(g) \) is
\[ \mu(C_q(g)) = \frac{(q+1)^{-q}(2gq^2(q+1)^{2g-1}+(q+1)q+(q-2)((q+1)^2)^{g})}{q(q+1)^{q}-1}. \quad (11) \]

**Proof.** To begin with, we assume that the summation of distances between all pairs of vertices in \( C_q(g) \) is \( D(C_q(g)) \). The sum of distances between all pairs \( (u, v) \), where \( u \) belongs to vertex set \( U \) and \( v \) belongs to a disjoint vertex set \( V \), is denoted as \( D(U, V) \).

In order to utilize the recursive construction process for the recursive corona graph we classify the vertex pairs in \( C_q(g) \) into four different categories, \( W \), \( X \), \( Y \), and \( Z \). The sum of the distances for the four categories is denoted as \( S_W \), \( S_X \), \( S_Y \), and \( S_Z \), respectively.

Category \( W \) refers to the pairs within the same \( K_q \) that we add to the network at step \( g \). Category \( X \) refers to the pairs \( u, v \) where \( u \) is selected from one of the \( N(g-1) \) \( K_q \)s added at step \( g \) and \( v \) selected from any other \( K_q \) added to the network at the same step. \( Y \) refers to the pairs where \( u \) is a new vertex and \( v \) is a vertex in \( C_q(g-1) \). As for category \( Z \), it indicates the pairs where both \( u \) and \( v \) are from the previous generation of the graph \( C_q(g-1) \). Thus we have the following equations:
\[ D(C_q(g)) = S_W(g) + S_X(g) + S_Y(g) + S_Z(g), \quad (12) \]
\[ S_W(g) = \frac{q(q-1)}{2} N(g-1), \quad (13) \]
\[ S_X(g) = \sum_{i,j \in V(C_q(g-1))} \left( d_{i,j} + 2 \right) q^2 \]
\[ = q^2 S_Z(g) + q^2 N(g-1) \cdot (N(g-1) - 1), \quad (14) \]
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\[ S_Y(g) = \sum_{i,j \in V(C_q(g-1))} (d_{i,j} + 1)q \]

\[ = 2qS_Z(g) + qN(g-1) \cdot (N(g-1) - 1) + N(g-1)q, \]

\[ S_Z(g) = D(C_q(g-1)). \]

Combining these recursive expressions we have

\[ D(C_q(g)) = (q + 1)^2 D(C_q(g-1)) + \frac{1}{2} q^2 (2q(q + 1)^{g-1} - 1)(q + 1)^g \]

with the initial condition \( D(C_q(0)) = q(q - 1)/2 \) we get the result

\[ D(C_q(g)) = \frac{1}{2} q(2gq^2(q + 1)^{2g-1} + (q + 1)^g + (q - 2)((q + 1)^2)^g). \]

Dividing \( D(C_q(g)) \) by \( \frac{N(g)N(g-1)}{2} \) yields equation (11). For large \( g \), we have \( \mu(C_q(g)) \sim 2g \sim 2 \log_q N(g) \), which increases logarithmically with the network order.

**3.4. Clustering coefficient**

Clustering coefficient \([3]\) is another crucial quantity used to characterize network structure. Many works about determining clustering coefficient and its related quantities are carried out on both graph models and graphs in reality \([3, 24–26]\).

The clustering coefficient of vertex \( v \) is defined as the following quantity:

\[ c(v) = \frac{2e_v}{\delta_v(\delta_v - 1)}, \]

where \( e_v \) is the number of edges between the neighbours of vertex \( v \). The network clustering coefficient \( C(G) \) is defined as the average of \( c(v) \) among all vertices. That is,

\[ c(G) = \frac{1}{|V(G)|} \sum_{v \in V(G)} c(v). \]

**Theorem 3.4.** Let \( v(\delta) \) be a vertex in \( C_q(g) \) whose degree is \( \delta \). Except for the initial vertices, its clustering coefficient is

\[ c(v(\delta)) = \frac{q - 1}{\delta - 1} \approx \frac{q - 1}{\delta}. \]

**Proof.** Let us review the intermediate result in calculating the degree distribution that the number of vertices with degree \( q, 2q, \ldots, gq, (g + 1)q - 1 \) are, respectively, \( q^2(q + 1)^{g-1}, q^2(q + 1)^{g-2}, \ldots, q^2, q \). Except for the initial vertices, the clustering coefficients of other vertices follow the same rule: that is, a vertex with degree \( kq \) has \( kq \) neighbours, which are evenly distributed in \( k \) clusters. Each cluster forms a complete graph \( K_q \). Thus the clustering coefficient of vertex \( v \) is derived as

\[ c(v) = \frac{kq(q - 1)/2}{kq(kq - 1)/2} = \frac{q - 1}{kq - 1}, \quad (g_e > 0), \]

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and for initial vertices
\[ c(v) = \frac{(q-1)(q-2) + gq(q-1)}{(q(g+1)-1)(q(g+1)-2)}, \quad (g_v = 0). \] (23)

Theorem 3.4 is naturally gained.

**Theorem 3.5.** The clustering coefficient of RCG network \( C_q(g) \) is
\[ c(C_q(g)) \sim \frac{q-1}{q+1} \Phi \left( \frac{1}{q+1}, 1, \frac{q-1}{q} \right), \] (24)
where \( \Phi \) is the Lerch transcendent function, and the clustering coefficient converges to a non-zero when the network order is high enough. For large \( q \), the clustering coefficient tends to 1.

**Proof.** From equations (23) and (20) we can obtain the clustering coefficient of \( C_q(g) \):
\[ c(C_q(g)) = \frac{1}{N(g)} \left[ \sum_{k=1}^{q} \frac{q-1}{kq-1} q^2(q-1)^{q-2} + \frac{(q-1)(gq+q-2)}{(q(g+1)-1)(q(g+1)-2)} \right], \] (25)
which leads to the result of theorem 3.5. Further we have
\[ c(C_q(g)) \sim \frac{q-1}{q+1} \Phi \left( \frac{1}{q+1}, 1, \frac{q-1}{q} \right), \] (26)
which is high and tends to 1 when \( q \) is large. Figure 2 gives the clustering coefficient of some networks.

3.5. Spanning trees

Next we derive the number of spanning trees in graph \( C_q(g) \).

**Theorem 3.6.** The number of spanning trees of \( C_q(g) \) is
\[ N_{tr} C_q(g) = q^{q-2}(q+1)^{1-q((q+1)^{q-1})^{q+1}}. \] (27)

**Proof.** According to Cayley’s theorem [27], the number of spanning trees of a complete graph \( K_q \) is equal to \( q^{q-2} \). Since all vertices of a \( K_q \) added to the graph are connected to a vertex in the original graph, these \( q+1 \) vertices consist of a new complete graph \( K_{q+1} \). Therefore we have the following recursive relation of the spanning trees of \( C_q(g) \):
\[ N_{tr} C_q(g) = ((q+1)^{q-1})^{N(g)} N_{tr}(g-1). \] (28)
Together with the initial condition \( N_{tr} = q^{q-2} \), we can derive the expression for \( N_{tr} \):
\[ N_{tr} C_q(g) = q^{q-2}(q+1)^{1-q((q+1)^{q-1})^{q+1}}. \] (29)

3.6. Kirchhoff index

Resistance distance is an important character of a graph, which can imply many of its dynamic properties. The Kirchhoff index [28] of a graph refers to the sum of resistance
between all vertex pairs in an associated electrical network obtained from the graph by replacing each edge of the graph by a unit resistance. Denote the effective resistance between vertices \( i \) and \( j \) as \( r(i,j) \), or \( r_{ij} \), then the Kirchhoff index \( R_{Kr} \) of graph \( G \) is defined as

\[
R_{Kr}(G) = \sum_{i,j \in V(G), i < j} r_{ij}.
\]  

We denote the Kirchhoff index of graph \( C_q(g) \) by \( R_{Kr}(C_q(g)) \).

**Theorem 3.7.** The Kirchhoff index of \( C_q(g) \) is

\[
R_{Kr}(C_q(g)) = (q^3(2g + 1) - 2q - 1)(q + 1)^{2g-2} + q(q + 1)^{g-1}.
\]

**Proof.** We denote by \( r(U, V) \) the sum of all effective resistances between pairs \( (u, v) \) in which \( u \) and \( v \) belong to two disjoint vertex sets \( U \) and \( V \) respectively. Similar to the method we used in calculating the average distance, we classify these pairs into
four categories \( W, X, Y, \) and \( Z \), where the definition is exactly the same as used in calculating the average distance. Then the sum of the distances for the four categories is denoted as \( R_W, R_X, R_Y, \) and \( R_Z \). We have the following equations:

\[
R_{K_4}(C_q(g)) = R_W(g) + R_X(g) + R_Y(g) + R_Z(g),
\]

\[
R_W(g) = \frac{q(q-1)}{2} \frac{2}{q+1} N(g-1),
\]

\[
R_X(g) = \sum_{i,j \in V(C_q(g-1))} \left( r_{i,j} + \frac{2}{q+1} \right) q^2
\]
\[
= q^2 R_Z(g) + \frac{2q^2 N(g-1)(N(g-1)-1)}{q+1},
\]

\[
R_Y(g) = \sum_{i,j \in V(C_q(g-1))} \left( r_{i,j} + \frac{1}{q+1} \right) q
\]
\[
= 2q R_Z(g) + q \frac{2}{q+1} N(g-1)(N(g-1)-1)
\]
\[
+ q \frac{2}{q+1} N(g-1),
\]

\[
R_Z(g) = R_{K_4}(C_q(g-1)),
\]

which yields

\[
R_{K_4}(C_q(g+1)) = q^2(2q(q+1)^g - 1)(q+1)^g + (q+1)^2 R_{K_4}(C_q(g)).
\]

Notice that the effective resistance between vertices \( v \) and \( u \) in a complete graph \( K_q \) is \( 2/q \) since the potentials between any other vertices are identical, if we impose a potential difference between \( u \) and \( v \).

Along with the initial condition \( R_{K_4}(C_q(0)) = q - 1 \) we can deduce

\[
R_{K_4}(C_q(g)) = (q^3 - 2q - 1)(q+1)^{2g-2} + q(2q^2 g(q+1)^g + q + 1)(q + 1)^{g-2}
\]
\[
= (q^3 - 2q - 1 + 2gg^2)(q+1)^{2g-2} + (q+1)^{g-1} q
\]
\[
= (q^3(2g+1) - 2q - 1)(q+1)^{2g-2} + g(q+1)^{g-1}.
\]

This completes the proof.

\[ \square \]

4. Spectral analysis

By convention the (unweighted) adjacency matrix \( A(G) \) of a graph \( G \) is defined as an \( N \times N \) matrix with the entry \( a_{i,j} \) representing the number of edges incident with endpoints \( i, j \). The degree matrix \( D(G) \), is defined as a diagonal matrix with its \( i \)th entry
on the main diagonal equal to the degree vertex \(i\). We call \(L(G) = D(G) - A(G)\) the Laplacian matrix of graph \(G\). These matrices determine the structure graph, and the eigenvalues of \(A(G)\) and \(L(G)\) are sensitive to many of the structural properties, which have a remarkable impact on the dynamic processes superimposed upon the network.

**Definition 4.1.** Given \(A(C_q(g))\), the adjacency matrix of \(C_q(g)\), we define the *spectra* of \(C_q(g)\) as

\[
\sigma(C_q(g)) := \sigma(g) = (\lambda_1(g), \lambda_2(g) \ldots \lambda_n(g)).
\]

Similarly, we have the following.

**Definition 4.2.** Given \(L(C_q(g))\), the Laplacian matrix of \(C_q(g)\), we define its Laplacian spectra as

\[
S(C_q(g)) := S(g) = (\gamma_1(g), \gamma_2(g), \ldots, \gamma_n(g)).
\]

### 4.1. Spectra of adjacency matrix

**Theorem 4.1.** The relation between \(\sigma(g)\) and \(\sigma(g - 1)\) is

1. \(\frac{\lambda_q(g - 1) + q - 1 \pm \sqrt{(q - 1 - \lambda g)^2 - 4q}}{2} \in \sigma(g)\) with multiplicity 1 for \(i = 1, \ldots, q(q + 1)^{g-1}\) and
2. \(-1 \in \sigma(g)\) with multiplicity \((q - 1)q(q + 1)^{g-1}\).

The result is a corollary of results in [22, 29].

### 4.2. Spectra of Laplacian matrix

**Theorem 4.2.** The relation between \(S(g)\) and \(S(g - 1)\) is

1. \(\frac{\gamma_q(g - 1) + q + 1 \pm \sqrt{(q + 1)^2 - 4q}}{2} \in S(g)\) with multiplicity 1 for \(i = 1, \ldots, q(q + 1)^{g-1}\) and
2. \(q + 1 \in S(g)\) with extra multiplicity \((q - 1)q(q + 1)^{g-1}\).

Note that in the first part \(\gamma_1(g - 1) = 0\) will generate an eigenvalue equal to \(q + 1\) with multiplicity 1 in iteration \(g\). So the actual multiplicity of \(q + 1\) is \((q - 1)q(q + 1)^{g-1} + 1\) for any \(g \geq 1\). The proof of theorem 4.2 is evident using methods in [22, 23, 29]. For convenience of the following discussion we give a similar proof here:

**Proof.** The Laplacian matrix of \(C_q(g)\) is

\[
L(C_q(g)) = \begin{pmatrix}
L(C_q(g - 1)) + qI_n & -I_n & \cdots & -I_n \\
-I_n & \vdots & \ddots & -I_n \\
& \ddots & \ddots & \ddots \\
& & \ddots & L(K_q) + I_q \otimes I_n \\
-I_n & \cdots & \cdots & -I_n
\end{pmatrix}.
\]

Let \(Y_1, \ldots, Y_{N(g-1)}\) be the Laplacian eigenvectors of \(C_q(g - 1)\) corresponding to the eigenvalues \(\gamma_1(g - 1), \gamma_2(g - 1), \ldots, \gamma_{N(g-1)}(g - 1)\), respectively. For \(i = 1, \ldots, N(g - 1)\), let
\[
\phi_i = \frac{\gamma_i(g-1) + q + 1 + \sqrt{(\gamma_i(g-1) + q + 1)^2 - 4\gamma_i(g-1)}}{2}, \\
\hat{\phi_i} = \gamma_i(g-1) + q + 1 - \sqrt{(\gamma_i(g-1) + q + 1)^2 - 4\gamma_i(g-1)}}{2}.
\]

Note that \(\phi_i, \hat{\phi_i}\) are Laplacian eigenvalues of \(C_q(g)\) corresponding to the eigenvectors
\[
\begin{pmatrix}
Y_i \\
g(\phi_i)Y_i \\
\vdots \\
g(\phi_i)Y_i
\end{pmatrix}
= L(C_q(g))
\begin{pmatrix}
Y_i \\
g(\phi_i)Y_i \\
\vdots \\
g(\phi_i)Y_i
\end{pmatrix},
\]
respectively. In fact \(\phi_i\) is obtained by solving
\[
\begin{pmatrix}
\gamma_i(g-1) + q + qg(\phi_i) \\
-1 + g(\phi_i) \\
\vdots \\
-1 + g(\phi_i)
\end{pmatrix}
Y_i = L(C_q(g-1)) + qI_n
\begin{pmatrix}
Y_i \\
g(\phi_i)Y_i \\
\vdots \\
g(\phi_i)Y_i
\end{pmatrix}
+ \sum_{j=1}^q (L(K_\gamma) + I_n)_{ij}I_n
\begin{pmatrix}
-1 \\
\vdots \\
-1
\end{pmatrix}
Y_i.
\]
Thus we can derive the following equations:
\[
\gamma_i(g+1) + q + qg(\phi_i) = \phi_i, \tag{43}
\]
\[
-1 + g(\phi_i) = \phi_i g(\phi_i). \tag{44}
\]

From equation (44) we can obtain that \(g(\phi_i) \neq 0\). Therefore we can substitute equation (44) into equation (43); we have
\[
(\gamma_i(g-1) + q - \phi_i)(\phi_i - 1) = -q, \tag{45}
\]
which leads to the result of the first part of the theorem.

If the Laplacian eigenvalues \(\nu_1 = 0, \nu_2 = \nu_3 = \ldots = \nu_q = q\) of \(L(K_\gamma)\) are correlated with the eigenvectors \(Z_1, Z_2, \ldots, Z_q\), respectively, then for \(j = 2, \ldots, q\) we have
\[
L(C_q(g))\begin{pmatrix}
0 \\
Z_j \otimes e_i
\end{pmatrix} = (q + 1)\begin{pmatrix}
0 \\
Z_j \otimes e_i
\end{pmatrix}. \tag{46}
\]
This completes the proof. \(\square\)
Next we use the results of the Laplacian spectra to prove theorems 3.6 and 3.7. First we give an alternative proof of theorem 3.6.

**Proof.** It is known that the number of spanning trees of a graph $G$ has the following form \[30, 31\]:

$$N_{tr}(G) = \frac{\prod_{i=2}^{N} \tau_i}{N},$$

(47)

where $N$ is the number of vertices and $\tau_i$ refers to $N$ eigenvalues of the graph. Given that the graph is connected, let $\gamma$ be the unique zero eigenvalue, then $\tau_i, i = 2, ..., N$ are $N - 1$ non-zero eigenvalues of the graph.

Theorem 4.2 tells us that the Laplacian spectrum of $C_q(g)$ consists of two parts. For the first part, we can derive from equation (45) that, in iteration $g$, $\gamma_i$ in $C_q(g - 1)$ generates two eigenvalues $\phi_i$ and $\hat{\phi}_i$, which are subject to the relations $\hat{\phi}_i \phi_i = \gamma_i$ and $\hat{\phi}_i + \phi_i = \gamma_i + q + 1$. In particular, the trivial eigenvalue $\gamma_1 = 0$ generates $\phi_i = q + 1$ and $\hat{\phi}_i = 0$. As for the second part, there is an eigenvalue $\gamma = q + 1$ with multiplicity $(q - 1)q(q + 1)^{q-1}$. We denote by $S(g)$ the sum of all non-zero eigenvalues of $L(C_q(g))$ and by $\Upsilon(g)$ the product of all non-zero eigenvalues of $L(C_q(g))$. Then we can obtain

$$\Upsilon(g) = \prod_{i=2}^{N\text{(q)}} \gamma(g) = (q + 1)^{(q-1)q(q+1)^{q-1}+1} \prod_{i=2}^{N\text{(g-1)}} \phi_i \hat{\phi}_i$$

(48)

$$= (q + 1)^{(q-1)q(q+1)^{q-1}+1} \prod_{i=2}^{N\text{(g-1)}} \gamma(g - 1)$$

$$= \Upsilon(g - 1)q(q + 1)^{(q-1)q(q+1)^{q-1}+1}.$$  

Equation (48) and the initial condition $\Upsilon(0) = q^{q-1}$ yield

$$\Upsilon(g) = q^{q-1}(q + 1)^{(q-1)(q+1)^{q-1}+g}.$$  

(49)

Therefore

$$N_{tr}C_q(g) = q^{q-2}(q + 1)^{(q-1)(q+1)^{q-1}-1}.$$  

□

The result is equivalent to what we derived using the combinatorial method.

In the following we give an alternative proof of theorem 3.7 using the spectral information.

**Proof.** The Kirchhoff index of a graph $(G)$ can be expressed as [32, 33]

$$R_K(G) = N \sum_{i=2}^{N} \frac{1}{\tau_i},$$

(50)

where $N$ and $\tau_i$ are the same as the previous definition. Let

$$S = \sum_{j=2}^{N} \prod_{i=2}^{N} \gamma_i = \prod_{i=2}^{N} \gamma_i \sum_{i=2}^{N} \frac{1}{\gamma_i}.$$  

(51)
We can follow the clue of the previous analysis by separating the eigenvalues of its Laplacian matrix into two parts. Recall that the eigenvalues of the Laplacian consist of two parts $\Gamma^{(1)}$ and $\Gamma^{(2)}$ as defined by theorem 4.2. Assume that $\Gamma^{(1)} = \Gamma^{(1)} \setminus \{0\}$. For the first part of the eigenvalues of $L(C_{\gamma}(g))$, we denote them as $\phi_i$ and $\hat{\phi}_i$, $i = 1, 2, ..., N(g - 1)$. Suppose that the original eigenvalue in $L(C_{\gamma}(g - 1))$, which is correlated with $\phi_i$ and $\hat{\phi}_i$, is $\gamma_i$. Then

$$S(g) = S^{(1)}\Upsilon^{(2)} + S^{(2)}\Upsilon^{(1)},$$

where

$$S^{(1)} = \sum_{\gamma_i \in \Gamma^{(1)}, \gamma_j \in \Gamma^{(1)}} \prod_{i \neq j} \gamma_i,$$  \hspace{1cm} (53)

$$\gamma^{(2)} = \prod_{\gamma_i \in \Gamma^{(2)}} \gamma_i,$$  \hspace{1cm} (54)

$$S^{(2)} = \sum_{\gamma_i \in \Gamma^{(2)}, \gamma_j \in \Gamma^{(2)}} \prod_{i \neq j} \gamma_i,$$  \hspace{1cm} (55)

and

$$\gamma^{(1)} = \prod_{\gamma_i \in \Gamma^{(1)}} \gamma_i.$$  \hspace{1cm} (56)

Accordingly we can obtain

$$S^{(1)} = (q + 1) \prod_{i=2}^{N(g-1)} \phi_i \hat{\phi}_i \left( \sum_{j=1}^{N(g-1)} \left( \frac{1}{\phi_j} + \frac{1}{\hat{\phi}_j} \right) \right) + \prod_{i=2}^{N(g-1)} \phi_i \hat{\phi}_i,$$

$$= (q + 1) \prod_{j=2}^{N(g-1)} \left( \prod_{i \neq j}^{N(g-1)} \phi_i \hat{\phi}_j \right) + \prod_{i=2}^{N(g-1)} \phi_i \hat{\phi}_i,$$

$$= \Upsilon(g - 1) + (q + 1) \prod_{j=2}^{N(g-1)} \gamma_j(g - 1)(\gamma_j(g - 1) + q + 1),$$

$$= \Upsilon(g - 1) + (q + 1) \prod_{j=2}^{N(g-1)} \gamma_j(g - 1)\gamma_j(g - 1) + (q + 1) \prod_{j=2}^{N(g-1)} \gamma_j(g - 1)\gamma_j(g - 1)$$

$$+ (q + 1) \prod_{j=2}^{N(g-1)} \gamma_j(g - 1)(q + 1),$$

$$= (1 + (q + 1)(q(q + 1)^{q-1} - 1))\Upsilon(g - 1) + (q + 1)^2 S(g - 1).$$  \hspace{1cm} (57)

$$\Upsilon^{(2)} = (q + 1)(q^{-1}q(q + 1)^{q-1}),$$  \hspace{1cm} (58)

$$S^{(2)} = (q - 1)q(q + 1)(q(q + 1)^{q-1} + q - 2),$$  \hspace{1cm} (59)
\[ \Upsilon^{(1)} = (q + 1)\Upsilon(g - 1). \]

Therefore we obtain a recursive relation of \( S(g) \). Considering the initial condition \( S(0) = (q - 1)q^{g-2} \), we derive

\[ S(g) = q^{g-2}(q + 1)^{(g-1)(q+1)+g-1}((2g + 1)q^3 - 2q - 1)(q + 1)^g + q(q + 1)), \]

thus

\[ R_{Kt}C_q(g) = (q + 1)^{g-2}((2g + 1)q^3 - 2q - 1)(q + 1)^g + q(q + 1)) = (q^3(2g + 1) - 2q - 1)(q + 1)^{2g-2} + q(q + 1)^{g-1}. \]

Note that equation (62) is consistent with equation (27). For large \( g \), the Kirchhoff index displays the following leading behaviour:

\[ R_{Kt}C_q(g) \sim gN(g)^2 \sim N(g)^2 \log_{q+1} N(g). \]

5. Conclusion

In this paper, we have introduced a deterministically growing model to generate a small-world graph, by using the corona product. The advantage of such a model is that many of its properties can be solved exactly. We have derived explicitly many structural quantities of the small-world model. We have also found the eigenvalues for the adjacency matrix and the Laplacian matrix of the model. In future, the properties of various dynamical processes taking place on the small-world model deserve study.

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