Three steps away from Shapiro’s problem: lower bounds for graphic sums with functions ‘max’ or ‘min’ in denominators

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Abstract

Taking Shapiro’s cyclic sums $\sum_{i=1}^{n} x_i/(x_{i+1} + x_{i+2})$ (assuming index addition mod $n$) as a starting point, we introduce a broader class of cyclic sums, called generalized Shapiro-Diananda sums, where the denominators are $p$-th order power means of the sets $\{x_{i+j_1}, \ldots, x_{i+j_k}\}$ with fixed distinct integers $j_1, \ldots, j_k$ and $1 \leq i \leq n$.

Generalizing further, we replace the set of arguments of the power mean in the $i$-th denominator by an arbitrary nonempty subset of $\{1, \ldots, n\}$ interpreted as the set of out-neighbors of the node number $i$ in a directed graph with $n$ nodes. We call such sums graphic power sums since their structure is controlled by directed graphs.

The inquiry, as in the well-researched case of Shapiro’s sums, concerns the greatest lower bound of the given “sum” as a function of positive variables $x_1, \ldots, x_n$. We show that the cases of $p = +\infty$ (max-sums) and $p = -\infty$ (min-sums) are tractable.

For the max-sum associated with a given graph the g.l.b. is always an integer; for a strongly connected graph it equals to graph’s girth.

For the similar min-sum, we could not relate the g.l.b. to a known combinatorial invariant; we only give some estimates and describe a method for finding the g.l.b., which has factorial complexity in $n$.

A satisfactory analytical treatment is available for the secondary minimization — when the g.l.b.’s of min-sums for individual graphs are minimized over the class of strongly connected graphs with $n$ nodes. The result (depending only on $n$) is found to be asymptotic to $e \ln n$.

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1 Introduction

In this paper we investigate lower bounds for the sums of the form

\[ S_\lor(x_1, \ldots, x_n) = \sum_{i=1}^{n} \frac{x_i}{\max_{j \in \Omega_i} x_j}, \]  \hspace{1cm} (1)

which will be called max-sums, and for similarly looking min-sums

\[ S_\land(x_1, \ldots, x_n) = \sum_{i=1}^{n} \frac{x_i}{\min_{j \in \Omega_i} x_j}, \]  \hspace{1cm} (2)

Here \(x_1, \ldots, x_n\) are positive real numbers and \(\Omega_1, \ldots, \Omega_n\) are the given nonempty subsets of the set \(\{1, \ldots, n\}\).

The problem proposed by H.S. Shapiro in 1954 \[28] asked to prove the cyclic inequality

\[ \frac{2}{n} \sum_{i=1}^{n} \frac{x_i}{x_{i+1} + x_{i+2}} \geq C \]

with \(C = 1\) for all \(n \geq 3\) and positive \(x_1, \ldots, x_n\). It is assumed that \(x_{i+n} = x_i\). The elementary case \(n = 3\) (Nesbitt’s inequality) appeared in print in 1903 at the latest [25]. Shapiro’s original conjecture was shown to be generally wrong. The correct best constant, found by Drinfeld in 1971 [14], is \(C \approx 0.989133\).

Details of the story about Shapiro’s inequality can be found, for instance, in [24, Ch. 16], [16] or [9].

Let us describe the “three steps” from the headline that link Shapiro’s problem to the present one.

Step 1 is a blunt generalization: the first summand \(x_1/(x_2 + x_3)\) defining the cyclic pattern of Shapiro’s sum is replaced by some function of a fixed (independent of \(n\)) number of variables. In such a broad setting, statements of general nature may be available, see e.g. [20]; however, hardly anything quantitatively interesting can be proposed.
Step 2 purports to be an intelligent specialization. Consider the family of patterns of the form \( x_1 / (|J|^{-1} \sum_{i \in J} x_i^p)^{1/p} \), where \( p \in \mathbb{R} \) and \( J \) is some fixed set. The corresponding cyclic sums for specific sets \( J \) have been studied in the literature; the problem is difficult and the results are sketchy — see Sec. 2. The situation is different in the limiting cases \( p = \pm \infty \), that is, for patterns of the form \( x_1 / \min_{i \in J} x_i \) or \( x_1 / \max_{i \in J} x_i \). Then the problem becomes very tractable: in essence, complete results can be obtained through an application of the inequality between the arithmetic and geometric means (AGM). We treat these cases from the convenience of a prepared, more general position; see Proposition \( 4 \) for max-sums and Proposition \( 7 \) for min-sums.

The purpose of Step 3 is to partly restore the degree of nontriviality by a bold structural modification of the objective function. For patterns of Step 2 with \( |p| < \infty \), the nature of complications was analytical (intractability of conditions of extremum). Now we make it combinatorial with \( p = \pm \infty \), assigning to every index \( i \) an arbitrary set \( \Omega_i \subset \{1, \ldots, n\} \) that prescribes on which of the \( x_j \)'s the function in the \( i \)-th denominator depends. In the cyclic case, \( \Omega_i = (\Omega_i + i - 1) \mod n \). In general, we define the correspondence \( i \mapsto \Omega_i \) in terms of directed graphs, hence the term “graphic sums” (Sec. 3).

The proposed metamorphosis of Shapiro’s problem, which admittedly looks rather artificial, is motivated by the desire to obtain a setting, quite general, while amenable to analysis, yet nontrivial. It is hoped that the reader will find our choice also justified by the final results (particularly, Theorems 1 and 4), which, in a biased view of this author, are not without elegance. These results are analytically; there is also an open question concerning efficient computation of the greatest lower bound for a given individual min-sum (Sec. 7).

If one compares Shapiro’s problem to a small nice cut in a rough rock, long on display, then our present endeavour is alike to chipping at a different spot aiming to create a new attractive facet — and maybe (not expecting too much) — to get a bit closer to revealing deeper secrets of the entire crystal.

Last but not least: the analysis of the functions (1) and especially (2) prompted the author to examine a class of extremal problems of the type “minimize the sum of \( n \) variables provided products of certain sets of those variables are given” — a generalization of the AM-GM problem parametrized by combinatorial data. This led to a little theory \([27]\) that may present an independent interest. Here we make use of some results of \([27]\).

The results in the narrow sense (more significant, called Theorems and
others, called Propositions) with their proofs occupy only about half of the paper. The author feels that it is desirable to present the subject for the first time in an appropriate, sufficiently broad context even if the generality of the primary definitions is not matched by the generality of theorems.

2 Cyclic sums of Shapiro-Diananda type

The purpose of this section is to fix terminology and notation and to briefly review known results about sums structurally similar to Shapiro’s, with power means in the denominators. Reading the review is not necessary for understanding Section 3 and the sequel; however, it may help readers to get a more holistic perception of the subject.

The cardinality of a finite set Ω is denoted \(|Ω|\).

For integers \(a \leq b\), we denote by \([a, b]\) the set of \(b - a + 1\) consecutive integers \(a, a + 1, \ldots, b\). If \(a = 1\), we use the abbreviation \([n] = [1, n] = \{1, 2, \ldots, n\}\). Reduction modulo \(n\) in our context means, for the given \(a\), finding \(a' \in [n]\) such that \(a \equiv a' \mod n\). (So \(0 \mod n = n\).)

Suppose \(n\) is fixed. The \(n\)-tuple \((x_1, \ldots, x_n)\) will be abbreviated as \(x\).

Let \(τ\) denote the (left) cyclic shift, \(τ : [n] \to [n], i \mapsto i + 1 \mod n\). Put \(x^τ = (x_2, \ldots, x_n, x_1)\).

If \(J = (j_1, \ldots, j_k)\) is an ordered \(k\)-tuple of integers, then we put \((x | J) = (x_{j_1}', \ldots, x_{j_k}')\), where \(j_i'\) means \(j_i\) reduced modulo \(n\).

Given a function \(f\) of \(k = |J|\) variables, we define the \(n\)-ary shift-invariant function, the cyclic sum with local pattern \(f\), by the formula

\[
CS_n[f](x_1, \ldots, x_n) = \sum_{i=0}^{n-1} f(x^{τ^i} | J).
\]

One of the earliest examples with indefinite number of variables involving a function of such a form is Boltzmann’s inequality [6, Ch. 7, § 81]

\[
CS_n[(x_1 - x_2) \log x_1](x) = \log \left(x_1^{x_1-x_2}x_2^{x_2-x_3} \cdots x_n^{x_n-x_1}\right) \geq 0.
\]

If \(ϕ\) is some function of two variables and \(f(x_1, x_2, x_3) = ϕ\left(\frac{x_2}{x_1}, \frac{x_3}{x_1}\right)\), then

\[
CS_n[f](x) = ϕ\left(\frac{x_2}{x_1}, \frac{x_3}{x_1}\right) + ϕ\left(\frac{x_3}{x_2}, \frac{x_4}{x_2}\right) + \cdots + ϕ\left(\frac{x_1}{x_n}, \frac{x_2}{x_n}\right).
\] (3)
Shapiro’s $n$-th sum occurs when $\varphi(x, y) = \frac{2}{x+y}$.

Speaking about domains of the functions being discussed, it is sufficient for our purposes to always assume that all $x_i \geq 0$ and exclude those tuples that cause one or more of the denominators to vanish.

Put

$$M_{k,p}(x_1, \ldots, x_k) = \left(\frac{x_1^p + \cdots + x_k^p}{k}\right)^{1/p}$$

and, more generally, for $J = (j_1, \ldots, j_k)$,

$$M_{n,J,p}(x) = M_{n,J,p}(x_1, \ldots, x_n) = \left(\frac{1}{|J|} \sum_{i=1}^k x_{j_i}^p\right)^{1/p}.$$

Here, as before, $j_i'$ means $j_i$ reduced modulo $n$. The limit cases are

$$M_{n,J,-\infty}(x_{j_1}, \ldots, x_{j_k}) = \min_{j \in J} x_{j'}$$

and

$$M_{n,J,+\infty}(x_{j_1}, \ldots, x_{j_k}) = \max_{j \in J} x_{j'}.$$

Let us call the following functions the cyclic power sums of Shapiro-Diananda type (cyclic $p$-sums, for short):

$$S_{n;k,p}(x_1, \ldots, x_n) = CS_n \left[ \frac{x_1}{M_{k,p}(x_2, \ldots, x_{k+1})} \right](x) = \left(\frac{kx_1^p}{x_2^p + \cdots + x_{k+1}^p}\right)^{1/p} + \cdots$$

(4)

and the generalized cyclic $p$-sums

$$S_{n;J,p}(x_1, \ldots, x_n) = CS_n \left[ \frac{x_1}{M_{n,J,p}(x_1, \ldots, x_n)} \right](x).$$

(5)

Shapiro’s $n$-th sum is $S_{n;2,1}$ or, as a particular case of (5), $S_{n;(2,3),1}$.

It is obvious that for any $p \in [-\infty, +\infty]$

$$S_{n;J,+\infty}(x) \leq S_{n;J,p}(x) \leq S_{n;J,-\infty}(x).$$

Moreover, by the monotonicity of power means [21] §16 it follows that

$$p < p' \quad \Rightarrow \quad S_{n;J,p}(x) \geq S_{n;J,p'}(x).$$

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Putting $x_1 = \cdots = x_n = 1$, we get
\[
\inf \frac{1}{n} S_{n; k; p}(x) \leq 1 \tag{6}
\]
and, more generally,
\[
\inf \frac{1}{n} S_{n; J; p}(x) \leq 1. \tag{7}
\]
for any finite set $J \subset \mathbb{Z}$.

Beyond these trivial observations, minimizing cyclic sums is, for the most part, a subtle game of balancing, only patchily explored. The big question is: when the inequality in (6) or (7) is strict — and, in that case, what is the value of the greatest lower bound.

Under very weak conditions on the pattern $f$, Goldberg [20] proved that
\[
\inf \inf \frac{1}{n} \mathcal{C}S_n[f](x) = \lim_{n \to \infty} \inf \frac{1}{n} \mathcal{C}S_n[f](x).
\]

Following Drinfeld’s analysis [14] of Shapiro’s problem, a satisfactory method to explore the asymptotics of the g.l.b. of sums $\mathcal{C}S_n[f](x)$ exists for pattern functions depending on $x_1, x_2, x_3$. Godunova and Levin [17], [18], [19] applied it to a wide class of sums of the form (3) and to a few separate, special cases. See also [15, §3.1].

No method of strength comparable to Drinfeld’s — that is, capable of providing the best lower bound for a family of suitably normalized rational cyclic sums, — is presently known for patterns that depend on more than three variables or even for a pattern of the form $f(x_1, x_2, x_4)$. Even for a fixed $n$ analytical or numerical minimization of a particular cyclic sum can be difficult; the review [9] and the paper [1] provide good evidence.

The sums $S_{n; k, 1}$ were first considered by Diananda [11]. The best currently known estimate applicable to arbitrarily large values of $k$ and $n$ was obtained, using considerations similar to Drinfeld’s, in the author’s paper [26]:
\[
k(2^{1/k} - 1) \leq \lim_{n \to \infty} \inf \frac{1}{n} S_{n; k; 1}(x) \leq \gamma_k,
\]
where $\gamma_k$ is the root of a certain transcendental equation. The sequence $(\gamma_k)$ monotonely decreases; $\gamma_2 \approx 0.98913$ is Drinfeld’s constant and $\lim_{k \to \infty} \gamma_k \approx 0.93050$. Note that $k(2^{1/k} - 1) \to \ln 2 \approx 0.693$, so there is a significant gap between the lower and upper bounds. A rigorously justified computational
procedure for determination of the true value of the limit \((n \to \infty)\) is not known even for \(k = 3\); neither it is known for the sums \(S_{n;J,1}\) with \(J = \{2, 4\}\), very similar visually to Shapiro’s. Plausible numerical results are presented in [5 § 7]. The mentioned “simplest cases” of the challenge in the explicit form read as follows.

**Open problem.** Find (or estimate numerically, with justification) the limits

\[
\lim \inf_{n \to \infty} \frac{1}{n} \left( \frac{x_1}{x_2 + x_3 + x_4} + \frac{x_2}{x_3 + x_4 + x_5} + \cdots + \frac{x_{n-1}}{x_n + x_1 + x_2} + \frac{x_n}{x_1 + x_2 + x_3} \right)
\]

and

\[
\lim \inf_{n \to \infty} \frac{1}{n} \left( \frac{x_1}{x_2 + x_4} + \frac{x_2}{x_3 + x_5} + \cdots + \frac{x_{n-1}}{x_n + x_2} + \frac{x_n}{x_1 + x_3} \right).
\]

Daykin [10] considered the function

\[
\Phi_k(\nu) = \inf_x S_{n,k,1/\nu}(x)
\]

with \(k = 2\) and showed that it is continuous and convex for \(\nu \in (0, +\infty)\) and that \(\Phi_2(\nu) = n\) for \(\nu \geq 2\). Diananda [12], extending the research of Daykin, found that the functions \(\Phi_k(\nu)\) with any \(k \in \{2, 3, \ldots\}\) are continuous and convex for \(\nu \in (0, +\infty)\), and that

\[
\inf_{\nu > 0} \Phi_k(\nu) = \lim_{\nu \to 0^+} \Phi_k(\nu) = \left\lfloor \frac{n + k - 1}{k} \right\rfloor.
\]

We will revisit this formula in Sec. 6 in connection with Theorem 1, see (19).

Generalizing another Daykin’s observation, Diananda noted that \(\Phi_k(\nu) \geq n\) when \(\nu \leq 0\). In [13] he investigated the threshold value \(\nu_n = \inf\{\nu > 0 \mid \Phi_k(\nu) = n\}\), which bounds the region where the trivial estimate of the type \(6\) is valid, and proved that

\[
1.036 < \sup_n \nu_n \leq \frac{\sqrt{5} + 1}{2}.
\]

The discussion so far pertained to bounding cyclic sums from below. In closing this section let us address the natural question: does the analogous maximization problem: to determine \(\sup_x S_{n;J,p}(x)\) — make sense?
If \( 1 \not\in J \), then, letting \( x_1 \to \infty \) while keeping \( x_2, \ldots, x_n \) fixed, we get \( \sup_x S_{n;1;p}(x) = \infty \).

The case \( p = 1, J = [a,b] \) with \( a, b \in \mathbb{Z}, a \leq 1 \leq b \) was studied by Baston \[3\] who considered cyclic sums with pattern \( M_1,\ell(x_1, \ldots, x_\ell)/M_1,k+\ell(x_a, \ldots, x_b) \), where \( b = a + k + \ell - 1 \) and \( a \leq 1 \leq a + k \) (that is, \([1, \ell] \subset [a, b]\)). Baston’s result in the case \( \ell = 1 \) reads:

If \( a \leq 1 \leq b \) and \( J = [a,b] \), then

\[
\frac{1}{n} \sup_x S_{n;1,1}(x) \leq \frac{|J|}{1 + \min(1 - a, b - 1)}.
\]

The equality occurs for infinitely many \( n \).

Beyond that, hardly anything is known about maximization of cyclic \( p \)-sums with \( 0 < p < \infty \). Yamagami \[30\] solved the maximization problem for a cyclic sum with weighted sum in the denominator of the pattern:

\[
\max_x C S_n \left[ \frac{x_1}{(s-m)x_1 + \sum_{j \in J} x_{1+j}} \right](x) = \frac{n}{s},
\]

where \( J = [\ell, \ell + m - 1] \), \( s \geq n \), \( 1 \leq \ell \leq n - m \). He conjectured that the result remains true for any set \( J \subset [1, n] \) of cardinality \( m \).

Our short review is not exhaustive; however, the author is not aware of any substantial results concerning power sums of Shapiro-Diananda type not mentioned or not referred to in the quoted references. Despite a seemingly elementary character of the questions, the area remains wide open for systematic exploration as well as for casually trying various particular cases, possibly even as students’ projects.\[1\]

### 3 Graphic sums

Let \( \Omega \) be a finite set, \( \mathcal{P}(\Omega) \) its power set, and \( \mathcal{P}_0(\Omega) = \mathcal{P}(\Omega) \setminus \{\emptyset\} \) be the set of all nonempty subsets of \( \Omega \).

By a “function with variable number of arguments” we understand a functional symbol \( f \), such as \( \min, \max, \text{sum} \), that comprises a family of

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\[1\] V.G. Drinfeld was a 10th grade student working under supervision of Prof. V.I. Levin when he proved that \( \lim_{n \to \infty} \inf_x n^{-1} S_{n;2,1}(x) = \gamma_2 \).
functions $f^{(k)}$, one for every arity $k = 0, 1, 2, \ldots$. For example, $f(x_1, x_2) = f^{(2)}(x_1, x_2)$. In the terminology of programming, $f$ can be viewed as a virtual function whose actual instance at the place of occurrence is determined by the signature (here, the number $k$ of scalar variables).

We will consider only symmetric functions. Given a vector $x$ with index set $[n]$ and a subset $\Omega = \{i_1, \ldots, i_k\} \subset [n]$, we write
\[
f(x|\Omega) = f(x_{i_1}, \ldots, x_{i_k}).
\]
This notation includes the case $\Omega = \emptyset$: then the right-hand side is the constant $f^{(0)}()$ — the instance of $f$ of arity zero.

**Remark.** If $f$ is not necessarily symmetric, then the above definition would make sense assuming that $\Omega$ is an ordered subset of $[n]$.

For our purposes it suffices to assume that always $x \geq 0$ (i.e. $x_1 \geq 0, \ldots, x_n \geq 0$) and that $f(x|\Omega)$ takes values in $\mathbb{R}_{\geq 0} = [0, \infty]$. Thus
\[
(x, \Omega) \mapsto f(x|\Omega).
\]
is a map $\mathbb{R}^n_{\geq 0} \times \mathcal{P}([n]) \to \mathbb{R}_{\geq 0}$. The target set $\mathbb{R}_{\geq 0}$ has natural linear order, the set $\mathbb{R}^n_{\geq 0}$ is partially ordered by coordinate-wise comparison: $x \geq \tilde{x} \iff (\forall j x_j \geq \tilde{x}_j)$, and the set $\mathcal{P}([n])$ is partially ordered by set inclusion. Hence it makes sense to inquire whether $f(\cdot|\cdot)$ is monotone (order-preserving or order-reversing) separately in $x$ and $\Omega$.

We call $f$ *ascending*, resp., *descending* if $f(x|\cdot)$ is order-preserving, resp., order-reversing, with respect to the set argument for any fixed $x$.

The functions of primary interest to us will be max (abbreviated as $\lor$) and min (abbreviated as $\land$) with conventions
\[
\lor(x | \emptyset) = 0, \quad \land(x | \emptyset) = \infty.
\]

We will also consider the functions $\sum_p$ (power sums of order $p$) and $\mathcal{M}_p$ (power means of order $p$) defined by the familiar expressions
\[
\sum_p(x_1, \ldots, x_k) = \left(\sum_{i=1}^{k} x_i^p\right)^{1/p}
\]
and
\[
\mathcal{M}_p(x_1, \ldots, x_k) = \left(\frac{1}{k} \sum_{i=1}^{k} x_i^p\right)^{1/p} = k^{-1/p} \sum_p(x_1, \ldots, x_k).
\]
Here \( p \in \mathbb{R} \setminus \{0\} \).

The 0-ary instances are \( \mathcal{M}_p() = 0 \) for \( p > 0 \) and \( \mathcal{M}_p() = +\infty \) for \( p < 0 \).

The functions \( \lor \) and \( \land \) are the limit cases:

\[
\land = \text{sum}_{-\infty} = \mathcal{M}_{-\infty}, \quad \lor = \text{sum}_{+\infty} = \mathcal{M}_{+\infty}.
\]

In the case \( p = 0 \), which was excluded above, \( \mathcal{M}_0(x_1, \ldots, x_k) \) can be naturally defined as the geometric mean \((x_1 \cdots x_k)^{1/k}\) with convention \( \mathcal{M}_0() = 1 \). The monotonicity in \( p \) (implying continuity at \( p = 0 \) if \( k \geq 1 \)) is thus ensured.

All these functions (\( f = \text{sum}_p, \mathcal{M}_p \), including \( \land \) and \( \lor \)) are homogeneous of order one:

\[
f(\lambda \mathbf{x} | \Omega) = \lambda f(\mathbf{x} | \Omega), \quad \forall \lambda > 0.
\]

Also, they all are monotone (increasing) with respect to the \( \mathbf{x} \)-argument:

\[
\mathbf{x} \geq \mathbf{x}' \quad \Rightarrow \quad f(\mathbf{x} | \Omega) \geq f(\mathbf{x}' | \Omega).
\]

Clearly, \( \text{sum}_p \) is ascending if \( p > 0 \) and descending if \( p < 0 \). In particular, \( \lor \) is ascending and \( \land \) descending:

\[
\Omega \supset \Omega' \quad \Rightarrow \quad \lor(\mathbf{x} | \Omega) \geq \lor(\mathbf{x} | \Omega'), \quad \land(\mathbf{x} | \Omega) \leq \land(\mathbf{x} | \Omega').
\]

The power means with \( |p| < \infty \) are neither ascending nor descending.

Given a symmetric function \( f \) with variable number of arguments let us introduce the function of a vector argument \( \mathbf{x} \in \mathbb{R}_n \), or equivalently, of the fixed number \( n \) of nonnegative real variables, whose structure is modelled after the Shapiro-Diananda cyclic sums:

\[
\mathbf{x} \mapsto \sum_{i=1}^{n} \frac{x_i}{f(x_i | \Omega_i)}.
\]

(10)

Here \( i \mapsto \Omega_i \) is a given assignment of nonempty subsets of \([n]\). It is natural to interpret such an assignment in terms of a directed graph (digraph).

A simple digraph is a pair \( \Gamma = (\mathcal{V}, \Gamma^+) \) where \( \mathcal{V} = \mathcal{V}(\Gamma) \) is the set of nodes and the map \( \Gamma^+ : \mathcal{V} \to \mathcal{P}(\mathcal{V}) \) determines the set of arcs (directed edges) outgoing from each node.

The whole set of arcs of \( \Gamma \) is

\[
\mathcal{A}(\Gamma) = \{(v, v') \in \mathcal{V} \times \mathcal{V} \mid v' \in \Gamma^+(v)\}.
\]
We will also use the intuitive notation \( v \rightarrow v' \) to denote the adjacency \( v' \in \Gamma^+(v) \).

The *outdegree* of a node \( v \) is \( d^+(v) = |\Gamma^+(v)| \).

Note that *loops* (i.e. arcs of the form \( v \rightarrow v \)) are allowed in a simple digraph, while multiple arcs are not.

In Section 7 we will need digraphs that are not necessarily simple (multiple arcs are allowed). Such a digraph is defined by specifying its set of arcs \( A \), the set of nodes \( V \), and two adjacency functions \( A \rightarrow V \): \( \alpha \) (the beginning of arc) and \( \beta \) (the end of arc).

From now on, we write “graph” meaning “simple digraph” throughout, except in Sec. 7 where non-simple digraphs will appear in a well-defined context.

In the sum (10), we demanded the sets \( \Omega_i \) to be nonempty. The reason is to avoid expressions that are nowhere finite: indeed, for \( f = \text{sum}_p \) with \( p > 0 \), say, we have \( f(x|\emptyset) = 0 \) and a summand in (10) with \( \Omega_i = \emptyset \) would cause the sum to be infinite for any \( x > 0 \). We want to avoid such a situation in the following definition; hence we introduce the notation \( \mathcal{V}'(\Gamma) = \{ v \in \mathcal{V}(\Gamma) \mid d^+(v) > 0 \} \).

**Definition 1.** Let \( \Gamma \) be a graph and \( x \in \mathbb{R}^\mathcal{V}_\geq(\Gamma) \) be a vector with components indexed by the nodes of \( \Gamma \). The graphic \( f \)-sum associated with \( \Gamma \) is the function with values in \( \mathbb{R}^\geq \) defined by

\[
S^\Gamma_f(x) = \sum_{v \in \mathcal{V}'(\Gamma)} \frac{x_v}{f(x|\Gamma^+(v))}.
\]

(11)

The domain of \( S^\Gamma_f(\cdot) \) consists of vectors \( x \in \mathbb{R}^\mathcal{V}_\geq(\Gamma) (n = |\mathcal{V}|) \), called *admissible*, such that \( x \neq 0 \) and \( f(x|\Gamma^+(v)) \neq 0 \) whenever \( x_v = 0 \).

We allow that \( S^\Gamma_f(x) = \infty \) for some \( x \) but we forbid summands of the form \( 0/0 \). The admissibility condition can be stated as the system of inequalities \( \max(x_v, f(x|\Gamma^+(v))) > 0 \) for all \( v \in \mathcal{V}(\Gamma) \).

For the functions \( f \) considered earlier in this section, every summand in the right-hand side of (11) is a homogeneous function of degree 0. Hence

\[
S^\Gamma_f(\lambda x) = S^\Gamma_f(x), \quad \forall \lambda > 0.
\]

The introduced framework makes it possible to discuss Shapiro-type problems from a wider perspective.
Generalized Shapiro’s problem for graphic sums. Given the graph $\Gamma$, find the greatest lower bound

$$m^\Gamma_f = \inf_{\mathbf{x}} S^\Gamma_f(\mathbf{x})$$

(12)

over the set of admissible vectors.

Apart from some simple transformations (see end of this section and Sec.4), this problem is too broad, and a specialization is needed to enable a meaningful treatment.

Remark concerning maximization of $S^\Gamma_f(\mathbf{x})$. One can similarly state a general maximization problem, replacing ‘inf’ in the right-hand side of (12) by ‘sup’. We cited Baston’s result concerning cyclic sums at the end of section 2. Similarly to the subcase $1 \notin J$ in the case of cyclic sums, the general problem is trivial if $v \notin \Gamma^+(v)$ for some $v \in \mathcal{V}(\Gamma)$: then $\sup_{\mathbf{x}} S^\Gamma_f(\mathbf{x}) = +\infty$. If $v \in \Gamma^+(v)$ for all $v$, then the problem is still trivial for $f = \lor$ or $\land$. Indeed, $x_v/ \lor (\mathbf{x}|_{\Gamma^+(v)}) \leq 1$, so $S^\Gamma_{\lor}(\mathbf{x}) \leq |\mathcal{V}(\Gamma)|$. Taking $\mathbf{x} = (1, 1, \ldots, 1)$, we see that $\max_{\mathbf{x}} S^\Gamma_{\lor}(\mathbf{x}) = |\mathcal{V}(\Gamma)|$. Finally, the only case where $\sup_{\mathbf{x}} S^\Gamma_f(\mathbf{x})$ is finite occurs when $\Gamma^+(v) = \{v\}$ for all $v \in \mathcal{V}(\Gamma)$. Then, obviously, $S^\Gamma_{\land}(\mathbf{x}) \equiv |\mathcal{V}(\Gamma)|$.

“There is nothing new under the sun”. Already Daykin [10] considered functions that in our notation are characterized as $S^\Gamma_{\text{sum1}}(\mathbf{x})$ for an out-regular graph $\Gamma$ (that is, $|\Gamma^+(v)| = d$, the same for all $v \in \mathcal{V}(\Gamma)$).

The cyclic power sums of Shapiro-Diananda type (5) make up a special case of $f^\Gamma_{M_p}(\mathbf{x})$ where $\Gamma$ has an automorphism that cyclically permutes the nodes. Generally, if the graph $\Gamma$ is out-regular, then $f^\Gamma_{M_p}$ and $f^\Gamma_{\text{sum}_p}$ differ just by a constant factor, which tends to 1 as $|p| \to \infty$.

In Definition [11] we have put the function $\mathfrak{f}$ in the denominator following the tradition of Shapiro’s problem. One may want to flip the summed fractions upside down, but the so obtained “graphic sums of the second kind” will not widen the scope of the theory.

Consider the reciprocal function $\tilde{\mathfrak{f}}$ defined (for $\mathbf{x} > 0$) by

$$\tilde{\mathfrak{f}}(x_1, \ldots, x_k) = \frac{1}{\mathfrak{f}(x_1^{-1}, \ldots, x_k^{-1})}.$$ 

For $\mathbf{x} > 0$, writing $\mathbf{x}^{-1} = (x_1^{-1}, \ldots, x_n^{-1})$, we have

$$\sum_{v \in \mathcal{V}(\Gamma)} \frac{\tilde{\mathfrak{f}}(\mathbf{x}|_{\Gamma^+(v)})}{x_v} = S^\Gamma_f(\mathbf{x}^{-1}).$$

2 as defined in [8]
If $f$ is homogeneous of order one, then so is $\tilde{f}$. If $f$ is ascending, then $\tilde{f}$ is descending and conversely. The transformation $f \rightarrow \tilde{f}$ does not change the type of monotonicity with respect to $x$.

Thus we see that $m^\Gamma_f$ can be expressed in terms of the “sum of the second kind” with reciprocal function, viz.,

$$m^\Gamma_f = \inf_{x > 0} \sum_{v \in \mathcal{V}(\Gamma)} \tilde{f}(x | \Gamma^+(v)) \frac{x_v}{x}. \tag{13}$$

Due to the reciprocity relation $\tilde{\wedge} = \vee$ (more generally, $\tilde{\sum}_p = \sum_{-p}$ and $\tilde{\mathcal{M}}_p = \mathcal{M}_{-p}$), the greatest lower bounds of graphic max- and min-sums can be represented in two ways,

$$m^\Gamma_\vee = \inf_{x > 0} \sum_{v \in \mathcal{V}(\Gamma)} x_v \vee(x | \Gamma^+(v)) = \inf_{x > 0} \sum_{v \in \mathcal{V}(\Gamma)} \wedge(x | \Gamma^+(v)) \frac{x}{x_v},$$

$$m^\Gamma_\wedge = \inf_{x > 0} \sum_{v \in \mathcal{V}(\Gamma)} x_v \wedge(x | \Gamma^+(v)) = \inf_{x > 0} \sum_{v \in \mathcal{V}(\Gamma)} \vee(x | \Gamma^+(v)) \frac{x}{x_v}.$$  

## 4 Strong reduction

In this section we show that the problem of determining the lower bound $(12)$ for a general graph $\Gamma$ reduces to the case of a strongly connected graph assuming $f$ satisfies certain natural conditions. The corresponding formulas of strong reduction are stated in Propositions 1 and 2 below.

Let $\Gamma$ be a simple digraph and $\mathcal{V} = \mathcal{V}(\Gamma)$. The binary relation $\succeq \in \mathcal{V} \times \mathcal{V}$ (written customarily as $v \succeq v'$ instead of $\succeq (v, v')$) is defined by:

$$v \succeq v' \text{ if } v = v' \text{ or there exists a (directed) path from } v' \text{ to } v.$$  

The symmetrization of this relation is

$$v \leftrightarrow v' \iff v \succeq v' \text{ and } v' \succeq v.$$  

Let $\mathcal{V}_1, \ldots, \mathcal{V}_k$ be the equivalence classes of $\mathcal{V}$ by $\leftrightarrow$. The induced subgraphs $\Gamma_i$ with $\mathcal{V}(\Gamma_i) = \mathcal{V}_i$ are called the strong components of $\Gamma$.

Another symmetrization of the binary relation $\succeq$ whose equivalence classes are weakly connected components is: $v \leftrightarrow v' \iff v \succeq v' \text{ or } v' \succeq v$.  

---

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The graph $\Gamma$ is *strongly connected* if $k = 1$.
The quotient set $\Gamma/\leftrightarrow$ is partially ordered by the relation $\succ$:

$$\Gamma' \succ \Gamma'' \text{ if } \Gamma' \neq \Gamma'' \text{ and } (v' \in \mathcal{V}(\Gamma'), v'' \in \mathcal{V}(\Gamma'')) \Rightarrow v' \succeq v''.$$  

The graph $\Gamma^c$, called the *condensation of* $\Gamma$, is obtained from $\Gamma$ by collapsing every strong component into a single node. It may contain loops. The graph $\Gamma^c_*$ obtained from $\Gamma^c$ by removing loops contains the Hasse diagram of the partial order $\succ$ as a subgraph.

Strong components that are maximal with respect to $\succ$ are called the *final strong components*. The set of such strong components (a subset of $\mathcal{V}(\Gamma^c)$) will be denoted $\text{Max} \Gamma^c$ and their union (a subgraph of $\Gamma$) will be denoted $\text{lim} \Gamma$.

By definition, there are no arcs between different components of $\text{lim} \Gamma$ and the graph $\Gamma^*_c$ is acyclic.

Let us introduce the notion of height (of a component, a node or a graph), which will be instrumental in the proofs concerning the strong reduction.

The *height of a component* $\Gamma_i \in \mathcal{V}(\Gamma^c)$, to be denoted $h(\Gamma_i)$, is the maximum length of a chain connecting $\Gamma_i$ to $\text{Max} \Gamma^c$. In particular, if $\Gamma_i \in \text{Max} \Gamma^c$, then $h(\Gamma_i) = 0$.

The *height of a node* $v \in \mathcal{V}(\Gamma)$, denoted $h(v)$, is the height of the component containing it.

The *height of the graph* $\Gamma$ is $H(\Gamma) = \max_{v \in \mathcal{V}(\Gamma)} h(v)$.

**Definition 2.** Let $f$ be a function with variable number of arguments, homogeneous of order 1. We say that $f$ is a *function of max-type* if for $0 \leq m \leq k - 1$ and any $x > 0$

$$\sup_{t > 0} f^{(k)}(x_1, \ldots, x_m, tx_{m+1}, \ldots, tx_k) = +\infty.$$  

Equivalently,

$$\sup_{\varepsilon > 0} \varepsilon^{-1} f^{(k)}(\varepsilon x_1, \ldots, \varepsilon x_m, x_{m+1}, \ldots, x_k) = +\infty.$$  

We say that $f$ is a *function of min-type* if $f$ is descending and for $0 \leq m < k$ and any $x > 0$

$$\lim_{t \to +\infty} f^{(k)}(x_1, \ldots, x_m, tx_{m+1}, \ldots, tx_k) = f^{(m)}(x_1, \ldots, x_m).$$  

Equivalently,

$$\lim_{\varepsilon \to 0^+} \varepsilon^{-1} f^{(k)}(\varepsilon x_1, \ldots, \varepsilon x_m, x_{m+1}, \ldots, x_k) = f^{(m)}(x_1, \ldots, x_m).$$  

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Recall that \( f^{(k)} \) denotes the \( k \)-ary representative of the family of functions with common symbol \( f \). Since we assume that \( f \) is symmetric, the \( m \) out of \( k \) distinguished arguments of \( f^{(k)} \) in the definition may occupy any positions.

The terminology is defined so as to naturally describe \( f = \lor \) as a function of max-type and \( f = \land \) as a function of min-type. More generally, the functions \( \text{sum}_p \) and \( \mathcal{M}_p \) are of max-type when \( p > 0 \), as well as \( f = \mathcal{M}_0 \) (the geometric mean). The functions \( \text{sum}_p, p < 0 \), are of min-type.

**Remark.** If there exist positive constants \( c_1, c_2, \ldots \) such that \( f^{(k)} = c_k g^{(k)} \), and one of the functions \( f \) or \( g \) is of max-type, then so is the other (e.g. \( \text{sum}_p \) and \( \mathcal{M}_p, p > 0 \)).

The functions \( \mathcal{M}_p \) with \( -\infty < p < 0 \), not being descending, do not fall in the scope of the above definitions and are thus left out of our analysis of strong reduction.

**Proposition 1.** Let \( f \) be a function of max-type. Then

\[
m_i^f = m_i^\lim = \sum_{\Gamma_i \in \text{Max}^c} m_i^{\Gamma_i}.
\]

**Proof.** We will prove the proposition by induction on the height \( H \) of \( \Gamma \).

1. In the base case \( H = 0 \) we have \( \Gamma = \lim \Gamma \) by definition. If \( \Gamma_1, \ldots, \Gamma_k \) are the components of \( \Gamma \), then for \( v \in \text{V}(\Gamma_i) \) we have \( \Gamma^+(v) = \Gamma_i^+(v) \). Also, the space \( \mathbb{R}^{\text{V}(\Gamma)} \) is the direct sum of the spaces \( \mathbb{R}^{\text{V}(\Gamma_i)} \), so that any vector \( x \in \mathbb{R}^{\text{V}(\Gamma)} \) can be decomposed as \( x = \sum_{i=1}^k x^{(i)} \), \( x^{(i)} = (x_v)_{v \in \text{V}(\Gamma_i)} \). Therefore

\[
S_i^{\Gamma}(x) = \sum_{i=1}^k \sum_{v \in \text{V}(\Gamma_i)} \frac{x_v}{f(x^{(i)}|\Gamma_i^+(v))} = \sum_{i=1}^k S_i^{\Gamma_i}(x^{(i)}).
\]

The minimization over \( x \) amounts to the minimization over each of the \( x^{(i)} \) independently. Hence \( m_i^\Gamma = \sum_i m_i^{\Gamma_i} \).

2. For the induction step, suppose that \( H \geq 1 \) and \( \hat{\Gamma} \) is the graph of height \( H - 1 \) obtained from \( \Gamma \) by deletion of all components of height \( H \). Clearly, \( \lim \hat{\Gamma} = \lim \Gamma \). Hence, by the inductive assumption, \( m_i^{\lim \Gamma} = m_i^\Gamma \).

The inequality \( m_i^\Gamma \geq m_i^\hat{\Gamma} \) is obvious, since \( \text{V}(\hat{\Gamma}) \subset \text{V}(\Gamma) \) and \( \Gamma^+(v) = \hat{\Gamma}^+(v) \) for any \( v \in \text{V}(\hat{\Gamma}) \). We have to prove that \( m_i^\Gamma \leq m_i^\hat{\Gamma} \).

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Let us fix a vector \( x \in \mathbb{R}_+^{\mathcal{V}(\Gamma)} \), introduce the parameter \( \varepsilon > 0 \) and define the vector \( x'(\varepsilon) \) whose components are

\[
x'_v(\varepsilon) = \begin{cases} 
x_v & \text{if } v \in \mathcal{V}(\hat{\Gamma}), \\
\varepsilon x_v & \text{if } v \in \mathcal{V}(\Gamma) \setminus \mathcal{V}(\hat{\Gamma})
\end{cases}
\] (14)

We have \( \Gamma^+(v) = \hat{\Gamma}^+(v) \) for any \( v \in \mathcal{V}(\hat{\Gamma}) \), so

\[
\sum_{v \in \mathcal{V}(\hat{\Gamma})} \frac{x'_v(\varepsilon)}{f(x'_v(\varepsilon) | \Gamma^+(v))} = \sum_{v \in \mathcal{V}(\hat{\Gamma})} \frac{x_v}{f(x | \Gamma^+(v))}.
\] (15)

From the definition of functions of max-type it follows that

\[
\inf_{\varepsilon > 0} \sum_{v \in \mathcal{V}(\hat{\Gamma}) \setminus \mathcal{V}(\hat{\Gamma})} \frac{x'_v(\varepsilon)}{f(x'_v(\varepsilon) | \Gamma^+(v))} = 0.
\]

We conclude that

\[
\inf_{\varepsilon > 0} S_{\Gamma}^f(x'(\varepsilon)) = S_{\hat{\Gamma}}^f(\hat{x}),
\]

where \( \hat{x} \) is the truncation of the vector \( x \) containing only the components with indices from \( \mathcal{V}(\hat{\Gamma}) \). Therefore \( m_{\Gamma}^f \leq m_{\hat{\Gamma}}^f \).

**Proposition 2.** Let \( f \) be a function of min-type.

(a) If \( \hat{\Gamma} \) is a subraph of \( \Gamma \) (that is, \( \mathcal{V}(\hat{\Gamma}) \subset \mathcal{V}(\Gamma) \) and \( \hat{\Gamma}^+(v) \subset \Gamma^+(v) \) for any \( v \in \mathcal{V}(\hat{\Gamma}) \)), then

\[
m_{\Gamma}^f \geq m_{\hat{\Gamma}}^f.
\]

(b) For any graph \( \Gamma \)

\[
m_{\Gamma}^f = \sum_{\Gamma_i \in \mathcal{V}(\Gamma)} m_{\Gamma_i}^f.
\]

**Proof.** (a) By definition, a function of min-type is descending. Therefore \( f(x|\Gamma^+(v)) \leq f(x|\hat{\Gamma}^+(v)) \) for any \( x \in \mathbb{R}_+^{\mathcal{V}(\Gamma)} \). Hence \( S_{\Gamma}^f(x) \geq S_{\hat{\Gamma}}^f(\hat{x}) \), where \( \hat{x} \) is the truncation of the vector \( x \) containing only the components with indices from \( \mathcal{V}(\hat{\Gamma}) \). The claimed inequality between the lower bounds follows.

(b) We will carry out the proof by induction on the number \( N = |\mathcal{V}(\Gamma^c)| \) of strong components. The base case \( N = 1 \) is trivial.

Suppose \( N > 1 \) and let \( \Gamma_1 \) be one of the components of \( \Gamma \) of the maximum height \( H = H(\Gamma) \). Let \( \hat{\Gamma} \) be the induced subgraph of \( \Gamma \) on the set of nodes
In view of the induction hypothesis, we need to prove that \( \text{m}_i^\Gamma = \text{m}_i^\Gamma_1 + \text{m}_i^\Gamma \). By (a), the left-hand side is not less than the right-hand side. The remaining task is to prove the inequality \( \text{m}_i^\Gamma \leq \text{m}_i^\Gamma_1 + \text{m}_i^\Gamma \).

The argument is similar to that used in the proof of Proposition 1. Again, we fix a vector \( \mathbf{x} \) and define \( \mathbf{x}'(\varepsilon) \) by the formula (14).

If \( v \in \mathcal{V}(\Gamma_1) \), then by the definition of a function of min-type we have

\[
\lim_{\varepsilon \to 0^+} \varepsilon^{-1} f(\mathbf{x}'(\varepsilon) | \Gamma^+(v)) \geq f(\mathbf{x} | \Gamma^+_1(v))
\]

Hence

\[
\lim_{\varepsilon \to 0^+} \sum_{v \in \mathcal{V}(\Gamma_1)} \frac{x'_v(\varepsilon)}{f(\mathbf{x}'(\varepsilon) | \Gamma^+(v))} \leq \sum_{v \in \mathcal{V}(\Gamma_1)} \frac{x_v}{f(\mathbf{x} | \Gamma^+_1(v))} = S_{i_1}^{\Gamma_1}(\mathbf{x}^{(1)}),
\]

where \( \mathbf{x}^{(1)} \) is the truncation of the the vector \( \mathbf{x} \) containing only the components with indices from \( \mathcal{V}(\Gamma_1) \).

Taking into account the formula (15), which remains valid here, we get

\[
\lim_{\varepsilon \to 0^+} S_i^\Gamma(\mathbf{x}'(\varepsilon)) \leq S_i^{\Gamma_1}(\mathbf{x}^{(1)}) + S_i^\Gamma(\hat{\mathbf{x}}).
\]

Minimizing over \( \hat{\mathbf{x}} \) and \( \mathbf{x}^{(1)} \), we complete the induction step. \( \square \)

5 Admissible maps and functional graphs

A functional graph is an out-regular graph with \( d^+(v) \equiv 1 \). In other words, in such a graph the set \( \Gamma^+(v) \) is a singleton for every node \( v \). The term functional graph refers to the fact that \( \Gamma^+(v) = \{\sigma(v)\} \) with some function \( \sigma : \mathcal{V}(\Gamma) \to \mathcal{V}(\Gamma) \).

A functional graph is strongly connected if and only if \( \sigma \) is a cyclic permutation. The weaker condition: \( \sigma \) is a bijection (a permutation) — is equivalent to the equality \( \Gamma = \lim \Gamma \).

For the analysis of lower bounds for max-sums and min-sums the notion of an admissible map will be instrumental.

**Definition 3.** Given a graph \( \Gamma \), a map \( \sigma : \mathcal{V}(\Gamma) \to \mathcal{V}(\Gamma) \) is called admissible (or \( \Gamma \)-admissible if one needs to be precise about the underlying graph), if \( \sigma(v) \in \Gamma^+(v) \) for any \( v \in \mathcal{V}(\Gamma) \).
For an arbitrary map $\sigma : V(\Gamma) \to V(\Gamma)$ we can define the functional graph $\Gamma_{\sigma}$ on the set of nodes $V(\Gamma_{\sigma}) = V(\Gamma)$ by putting $\Gamma^+_\sigma(v) = \{\sigma(v)\}$. The map $\sigma$ is $\Gamma$-admissible if and only if $\Gamma_{\sigma}$ is a subgraph of $\Gamma$.

The set of all $\Gamma$-admissible maps will be denoted $\mathcal{F}_\Gamma$.

Clearly, $\mathcal{F}_\Gamma = \emptyset$ if and only if $\min_{v \in V(\Gamma)} d^+(v) = 0$ (never the case for strongly connected graphs with more than one node) and $|\mathcal{F}_\Gamma| = 1$ if and only if $\Gamma$ is a functional graph.

Any function $\sigma \in \mathcal{F}_\Gamma$ is a choice function for the family of sets $\{\Gamma^+(v), v \in V(\Gamma)\}$ in the sense of set theory.

Remark. The set $\mathcal{F}_\Gamma$ can also be characterized as the set of bases of the tail partition matroid of the graph $\Gamma$ (by identifying a function $\kappa \in \mathcal{F}_\Gamma$ with the set of arcs $\{(a, \kappa(a)), a \in V(\Gamma)\} \subset A(\Gamma)$) [29, Example 8.2.22, p. 357].

Let us call surjective (equivalently, bijective) admissible functions admissible bijections. The set of such functions will be denoted $\mathcal{F}_\Gamma^*$. An admissible bijection provides a system of distinct representatives (SDR) for the family of sets $\{\Gamma^+(v)\}$. Thus $\mathcal{F}_\Gamma^* \neq \emptyset$ if and only if the system $\{\Gamma^+(v)\}$ admits an SDR.

By Hall’s theorem, $\mathcal{F}_\Gamma^* \neq \emptyset$ if and only if for any subset $U \subset V(\Gamma)$ there holds $|\bigcup_{v \in U} \Gamma^+(v)| \geq |U|$.

The relevance of admissible functions is explained by the following simple lemma.

Lemma 1. Let $\Gamma$ be a graph with $\min_v d^+(v) > 0$. Let $\mathcal{f} = \lor$ or $\land$. Then for any vector $x \in \mathbb{R}^{V(\Gamma)}$ there exists a function $\sigma \in \mathcal{F}_\Gamma$ such that

$$S^{\Gamma^*}_\mathcal{f}(x) = S^{\Gamma^*_\sigma}_\mathcal{f}(x) = \sum_{v \in V(\Gamma)} \frac{x_v}{x_{\sigma(v)}}. \quad (16)$$

Proof. For $\mathcal{f} = \lor$ or $\land$ there is always some $w \in \Gamma^+(v)$ such that $x_v = \mathcal{f}(x|\Gamma^+(v))$. Put $\sigma(v) = w$. (If there are several possibilities, chose one arbitrarily.) \qed

The function $\sigma$ in (16) will be called $(x, \Gamma)$-admissible or simply $x$-admissible if there is no ambiguity. Of course it may (and usually does) vary with $x$, so in the problem of minimization over $x$ Lemma 1 does not provide a reduction from a given graph $\Gamma$ to a specific functional graph. Yet even the “$x$-dependent reduction” will be useful in both max and min cases. In Sec. 7 we will need to describe the $x$-dependence in (16) more precisely.
In the remaining part of this section we find $m_f^\Gamma$ for $f = \mathcal{M}_p$, $p \in [-\infty, +\infty]$, and a functional graph $\Gamma$.

Let $\sigma$ be the (unique) $\Gamma$-admissible function. We have $\mathcal{M}_p(x|\Gamma^+(v)) = x_{\sigma(v)}$ for any $p$. So min-sums and max-sums are the same, as well as any $p$-sums, and have the form as in the right-hand side of (16).

To determine the greatest lower bound of this sum, it suffices, in view of the results of Sec. 4, to consider the case of a strongly connected graph $\Gamma$. In this case, as we noted earlier, $\sigma$ is a cyclic permutation of the set $V(\Gamma)$. By the AM-GM inequality, the minimum value of the sum is $|V(\Gamma)|$, attained when all $x_v$ are equal.

Using Proposition 1 we can drop the assumption of strong connectedness and come to the following result.

**Proposition 3.** If $\Gamma$ is a functional graph and $f = \mathcal{M}_p$ with any $p \in [-\infty, +\infty]$, then

$$m_f^\Gamma = |V(\lim \Gamma)|.$$ 

In particular, this is true for $f = \lor$ and $f = \land$.

6 Max-sums

It turns out that in the problem of minimization of max-sums the answer is always an integer.

**Theorem 1.** If $\Gamma$ is a strongly connected graph, then $m_\lor^\Gamma$ is equal to $g(\Gamma)$, the girth of $\Gamma$, which is the minimum length of a (directed) cycle in $\Gamma$.

If $\Gamma$ is an arbitrary graph, then $m_\lor^\Gamma$ is equal to $g^*(\Gamma)$, the sum of girths of all components of $\lim \Gamma$.

**Proof.** The general result follows from the result for strongly connected graphs by Proposition 1.

Below, till the end of the proof we assume that $\Gamma$ is strongly connected.

Let $\mathcal{V} = \mathcal{V}(\Gamma)$. Suppose $x \in \mathbb{R}^\mathcal{V}_+$. Let $\sigma : \mathcal{V} \to \mathcal{V}$ be an $x$-admissible function, so that (16) holds. By Proposition 3, $S_{\mathcal{V}^*}(x) \geq |\mathcal{V}(\lim \Gamma_\sigma)|$. Since the map $\sigma$ is bijective on the set $\mathcal{V}(\lim \Gamma_\sigma)$, this set is a union of cycles of $\sigma$. Hence $|\mathcal{V}(\lim \Gamma_\sigma)| \geq g(\Gamma)$. We conclude that $m_\lor^\Gamma \geq g(\Gamma)$.

Let us now take a cycle in $\mathcal{V}(\Gamma)$ of length $g = g(\Gamma)$. We may assume that it consists of the nodes $v_1, \ldots, v_g$ and the arcs $v_1 \to v_2, \ldots, v_g \to v_1$. 

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We will define a special $\Gamma$-admissible map $\sigma$. Denote $V_0 = \{v_1, \ldots, v_g\}$. Put $\sigma(v_1) = v_2, \ldots, \sigma(v_g) = v_1$. Thus $\sigma$ is defined on $V_0$.

If $V_0 \neq \mathcal{V}$, then denote by $V_1 \subset \mathcal{V} \setminus V_0$ the set of nodes $v$ such that there is an arc from $v$ to $V_0$. If $v \to v_i$ is such an arc (arbitrarily chosen, if there is more than one suitable $v_i$), put $\sigma(v) = v_i$. Now $\sigma$ is defined on $V_0 \cup V_1$.

If $V_0 \cup V_1 \neq \mathcal{V}$, then we take $V_2 \subset \mathcal{V} \setminus (V_0 \cup V_1)$ the set of nodes $v$ such that there is an arc to $V_0$. If $v \to v_i$ is such an arc (arbitrarily chosen, if there is more than one suitable $v_i$), put $\sigma(v) = v_i$. Now $\sigma$ is defined on $V_0 \cup V_1 \cup V_2$.

If $V_0 \cup V_1 \cup V_2 \neq \mathcal{V}$, then we take $V_3$ the set of nodes from which there is an arc to $V_2$, and define $\sigma$ on $V_3$ similarly to the above. We continue this process until the set $\mathcal{V}$ is exhausted. It must happen, since from any node of $\Gamma$ there exists a path to $V_0$.

As the result, we obtain the map $\sigma$ such that $\sigma : V_{j+1} \to V_j$ for $j > 0$.

Moreover, if $v \in V_j$, then the shortest path from $v$ to $V_0$ consists of $j$ arcs.

Let $\varepsilon \in (0, 1)$ and define the vector $x = x(\varepsilon) \in \mathbb{R}^V_+$ as follows:

$$x_v = \varepsilon^j \quad \text{whenever } v \in V_j, j \geq 0.$$ 

It is easy to see that for $v \in V_j$, $j > 0$, the set $\Gamma^+(v)$ contains a node from $V_{j-1}$ but does not contain nodes from $V_{j-k}$ with $k > 1$. Hence $\vee(x \mid \Gamma^+(v)) = \varepsilon^{j-1}$.

Obviously, $\vee(x \mid \Gamma(v)) = 1$ for $v \in V_0$.

Thus we have

$$S^\nu_\mathcal{V}(x) = \sum_{v \in V_0} 1 + \sum_{v \in \mathcal{V} \setminus V_0} \varepsilon.$$ 

Letting $\varepsilon \to 0^+$ we see that $m^\nu_\mathcal{V} \leq |V_0| = g$. This completes the proof.

Remark. Let us provide initial directions for an algorithmically minded reader concerning the complexity of computing girth of a strongly connected graph with $n$ nodes. The most obvious method is to compute successive powers of the adjacency matrix until its trace becomes positive [2, Exercise 3.22]. It takes $O(n^4)$ operations in the worst case. Better algorithms exist, see e.g. [22] for an $O(n \log_2 \ln n)$ algorithm.

We take a short recess to pay some attention to graphic power sums with exponents $p < \infty$. The evaluation of $m^\Gamma_{\text{sum}_p}$ or $m^\Gamma_{\text{min}_p}$ for an arbitrary $p < \infty$ can be difficult even for graphs with small number of nodes. In the absence of anything better, we state a simple, rough double-sided estimate.

Proposition 4. For any graph $\Gamma$, we have the estimates

$$g^*(\Gamma) \leq m^\Gamma_{\text{sum}_p} \leq |\mathcal{V}(\Gamma)|, \quad p \in [-\infty, +\infty]$$ (17)

and

$$m^\Gamma_{\text{min}_p} \leq |\mathcal{V}(\lim \Gamma)|, \quad p \in [0, +\infty].$$ (18)
Proof. The left estimate in (17) follows from Theorem 1 and monotonicity of the function \( p \mapsto M_p(x|\Gamma^+(v)) \) for any \( x \in \mathbb{R}^{|\mathcal{V}(\Gamma)|}_+ \) and any \( v \in \mathcal{V}(\Gamma) \).

The right estimate in (17) is obvious: putting \( e = (1,1,\ldots,1) \), we get

\[
m_{m_p}^\Gamma \leq S_{m_p}^\Gamma(e) = |\mathcal{V}(\Gamma)|.
\]

Using Proposition 1 and applying this trivial estimate to the graph \( \lim \Gamma \) in place of \( \Gamma \), we obtain (18).

Returning to max-sums \((p = +\infty)\), note some special cases of Theorem 1.

1. If \( \Gamma \) is a strongly connected graph containing at least one loop (a node \( v \) such that \( v \in \Gamma^+(v) \)), then \( m^\Gamma_{\vee} = 1 \).

2. Let us examine cyclic max-sums \( S_{n,J,+,\infty}(x) \) in light of Theorem 1. For every natural \( n \) the relevant graph \( \Gamma \) is circulant: its set of nodes is \( \mathcal{V}(\Gamma) = [n] \) and the node-arc adjacencies are defined by \( \Gamma^+(i) = (J + i) \pmod{n}, i = 1,\ldots,n \).

a) Suppose that \( 1 \in J \). In view of the special case 1 and of Proposition 1, \( \inf_x S_{n,J,+,\infty}(x) \) is equal to the number of components, \( q \), of the graph \( \Gamma \). Since the cyclic shift \( \tau \) (see the beginning of Sec. 2) is an automorphism of the graph \( \Gamma \), the components are \( \tau \)-congruent and we have \( q = n/m \), where \( m \) is the size of the component. It can be characterized as follows. Let \( \mathbb{Z}_n \) be the abelian group \( \mathbb{Z}/n\mathbb{Z} \). Let \( G_J \) be its subgroup generated by the elements \( \{j - 1 \mid j \in J\} \). Then \( m = |G_J| \). Clearly, if \(|J| > 1\), then \( m = n \) (so \( q = 1 \)) for infinitely many \( n \) (for example, for \( n \) coprime with \( j - 1 \) for some \( j \in J \setminus \{1\} \)).

b) Suppose now that \( 1 \notin J \). Let \( G_J \) be defined the same way as in (a) and let \( t \) denote the minimum number of summands (not necessarily distinct) in a sum \((j_1 - 1) + \cdots + (j_t - 1)\) representing zero in \( \mathbb{Z}_n \) with \( j_i \in J \). The number of \( G_J \)-cosets, \( n/|G_J| \), is equal to the number of strong components of \( \Gamma \) and \( t \) is the girth of each of the components. Hence, by Theorem 1

\[
m^\Gamma_{\vee} = t \cdot \frac{n}{|G_J|}.
\]

If \( J \subset [2,b] \) and \( n > b \), then obviously \( t \geq \lfloor (n - 1)/(b - 1) \rfloor + 1 = \lfloor (n + b - 2)/(b - 1) \rfloor \). We obtain the estimate

\[
\inf_x S_{n,J,+,\infty}(x) = m^\Gamma_{\vee} \geq \left\lfloor \frac{n + b - 2}{b - 1} \right\rfloor.
\]
If, moreover, \( J = [2, k + 1] \) is an interval, then \( G_J = \mathbb{Z}_n \), so \( \Gamma \) is strongly connected. In this case the right-hand side of (19) is equal to the number \( t \) of the summands in the representation \( n = k + \cdots + k + k' \), with \( k' = n - (t - 1)k \). Hence \( t = g(\Gamma) \) and the inequality in (19) turns to equality. Thus we obtain the limit case (\( \nu = +0 \)) of Diananda’s formula (8).

Let us summarize the discussion of the special cases 2(a) and 2(b) in a slightly cruder but more transparent form, emphasizing in (b) the behavior for large \( n \).

**Proposition 5.** Let \( m_\vee(n, J) = \inf_{x > 0} S_{n,J,\mathbb{Z}}(x) \) be the greatest lower bound of the cyclic max-sum with \( n \) terms defined by the pattern \( J \subset \mathbb{Z} \).

(a) Suppose that \( J \ni \{1\}, |J| > 1 \), and \( \gcd(j - 1 \mid 1 \neq j \in J) = s \). Then
\[
m_\vee(n, J) = \gcd(n, s).
\]
In particular, \( m_\vee(n, J) = 1 \) infinitely often as \( n \to \infty \).

(b) If \( J \nmid \{1\} \) and \( r(J) = \max(|j - 1|, j \in J) \), then
\[
m_\vee(n, J) = \frac{n}{r(J)} + O(1), \quad n \to \infty.
\]

As a simple consequence of part (b) and the trivial inequality \( \mathcal{M}_p(x|\Omega) \geq |\Omega|^{-1/p} \cdot \max(x|\Omega) \) for \( p > 0 \), we obtain the following asymptotic upper bound for minimum values of cyclic \( p \)-sums with an arbitrary pattern \( J \nmid \{1\} \):
\[
\inf_{x > 0} S_{n,J,p}(x) \leq \frac{n|J|^{1/p}}{r(J)} + O(1) \leq 2n|J|^{1/p - 1} + O(1)
\]
as \( n \to \infty \). In particular, as the there is no upper bound for \( r(J) \) stipulated by the size of \( J \), we at once reject the proposition that there might exist a pattern-independent (“universal”) positive lower bound \( A \) in the inequality \( n^{-1}S_{n,J,1}(x) \geq A \).

We conclude this section with some further upper estimates for girth \( g(\Gamma) \) and “total girth” \( g^*(\Gamma) \) that appears in Theorem 1.

Caccetta and Häggkvist conjectured the upper bound \( g(\Gamma) \leq \lceil n/k \rceil \) for any strongly connected graph \( \Gamma \) with \( n \) nodes and minimum outdegree \( k \). See [7] and [2, Conjecture 8.4.1 (p. 330)]. In our context the CH conjecture implies that for such a graph \( \Gamma \) and for some \( x > 0 \) the inequality \( S^\vee_\Gamma(x) \leq (n + k - 1)/k \) holds. Moreover, this (obviously) remains true for any graph.
with one final strong component, i.e. for any weakly connected digraph. If \( \Gamma \) is a digraph with weak components \( G_1, \ldots, G_m \) and \( |V(G_j)| = n_j, j = 1, \ldots, m \), then
\[
g^*(\Gamma) \leq \sum_{j=1}^{m} \frac{n_j + k - 1}{k} = \frac{n + m(k - 1)}{k} \leq \left\lceil \frac{n - (m - 1)}{k} \right\rceil + (m - 1).
\]

Clearly, \( n_j \geq k \) for all \( j \), hence \( m \leq n/k \) We get the estimate conditional on the CH conjecture:
\[
m_{\Gamma} = g^*(\Gamma) < \frac{2n}{k}
\]
for any digraph \( \Gamma \) (weakly connected or not). In particular, the \( n \)-dimensional vector \( \mathbf{x} = \mathbf{e} = (1, 1, \ldots, 1) \) is very far from optimal (one that minimizes \( S_{n,k}^\prime(\cdot) \)) if \( k \gg n \). We see a sharp contrast with situation observed in the case \( p = 1 \), at least for Diananda’s sums: the result of [26] shows that \( S_{n,k,1}(\mathbf{e}) \) differs from \( \inf_{\mathbf{x} > 0} S_{n,k,1}(\mathbf{x}) \) only by a moderate constant.

Finally, we mention one unconditional result that bounds girth from above if a lower estimate for the number of arcs is known.

Theorem 8.4.7 in [2, p. 331], due to Bermond, Germa, Heydemann and Sotteau (BGHS), says: if \( \Gamma \) is a strongly connected graph, \( |V(\Gamma)| = n \), and an integer \( t \geq 2 \) is such that
\[
|A(\Gamma)| \geq \frac{(n - t)(n - t + 1)}{2} + n,
\]
then \( g(\Gamma) \leq t \).

The BGHS’s unconditional estimate is in most cases much weaker than the one conjectured by Caccetta and Häggkvist.

**Example 1.** Let \( \Gamma \) be a strongly connected graph with \( n = 40 \) nodes and out-regular with \( d^+ (i) \equiv d = 12 \). The number of arcs in \( G \) is \( A = nd = 480 \), and the assumption of the theorem is met with \( t = 10 \), since \( \binom{40}{2} = 435 < A - n = 440 \). So the BGHS theorem gives the bound \( g(\Gamma) \leq 10 \).

The CH conjecture suggests that \( g(\Gamma) \leq \lceil 40/12 \rceil = 4 \).

By Theorem [11] this means that if \( \Omega_1, \ldots, \Omega_{40} \) are any 12-element subsets of \( [1, 40] \) such that (for instance) \( \Omega_i \ni (i + 1) \mod 40 \) (to guarantee strong connectedness), then there exists \( \mathbf{x} \in \mathbb{R}^{40}_+ \) such that
\[
\sum_{i=1}^{40} \frac{x_i}{\max(x_j \mid j \in \Omega_i)} \leq 4.
\]
7 Min-sums I: The problem for an individual graph

We turn to the minimization problem for min-sums. In this section we discuss how the problem can be solved in principle for the given graph $\Gamma$.

We begin with an estimate for $m^\Gamma_\lambda$ in combinatorial terms with sign opposite to that in (18) (in the present case $p = -\infty$). Recall that the set $\mathcal{F}_\Gamma$ of admissible maps is defined in Sec. 3 Definition 3.

**Proposition 6.** For any graph $\Gamma$

$$m^\Gamma_\lambda \geq \max_{\sigma \in \mathcal{F}_\Gamma} |\mathcal{V}(\lim \Gamma_\sigma)|.$$  

(20)

**Proof.** (a) Since $\Gamma^+_\sigma(v) \subset \Gamma^+(v)$, we have $m^\Gamma_\lambda \geq m^\Gamma_\lambda'$, cf. Proposition 2(a). But $m^\Gamma_\lambda' = |\mathcal{V}(\lim \Gamma_\sigma)|$ by Proposition 3. The result follows. □

The case of cyclic min-sums now looks completely trivial.

**Proposition 7.** For $p = -\infty$, any $n \geq 1$, and any finite nonempty set $J \subset \mathbb{Z}$ the inequality in (7) turns to equality. That is, $\inf_x S_{n,j,-\infty}(x) = n$. □

On the contrary, the general problem of finding $m^\Gamma_\lambda$, to be discussed now, does not appear to be simple at all.

Some preparations are required before we can formulate Theorem 2. Essentially they are aimed at a new level of understanding of the formula (16).

A preferential arrangement, or a ballot, on a nonempty set $\Omega$ is a partition of $\Omega$ into blocks with a linear order on the set of the blocks.

For example, the set $\{1, 2, 3\}$ can be block-partitioned as (123), (12)(3), (13)(2), (1)(23), and (1)(2)(3). A partition with $k = 1, 2, 3$ blocks can be ordered in $k!$ ways, which gives us the total of $1 \cdot 1 + 3 \cdot 2! + 1 \cdot 3! = 13$ preferential arrangements on the 3-element set.

For a given graph $\Gamma$, denote the set of all preferential arrangements of the set $\mathcal{V}(\Gamma)$ by $\text{Arr}_\Gamma$.

To every preferential arrangement $\mathfrak{A} \in \text{Arr}_\Gamma$ we put in correspondence certain subset $W_\mathfrak{A}$ of $\mathbb{R}_{+}^{\mathcal{V}(\Gamma)}$. Namely, $W_\mathfrak{A}$ consists of vectors $x$ such that

(i) $x_v = x_{v'}$ if $v$ and $v'$ lie in the same block of $\mathfrak{A};$

(ii) If $v \in B$, $v' \in B'$, where $B$ and $B'$ are distinct blocks and $B > B'$, then $x_v > x_{v'}$. 

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For example, if the nodes of $\Gamma$ are labelled as $1, 2, 3$ and $\mathfrak{A}$ is the preferential arrangement $(1, 3) > (2)$, then

$$W_{(13)> (2)} = \{ x = (x_1, x_2, x_3) \mid x_1 = x_3 > x_2 > 0 \}.$$  

For any vector $x > 0$ there is a unique preferential arrangement $\mathfrak{A}$ such that $x \in W_{\mathfrak{A}}$. We call $\mathfrak{A}$ the $x$-type arrangement.

**Definition 4.** Let $\mathfrak{A} \in \text{Arr}_\Gamma$ be a preferential arrangement with blocks $B_1 > \cdots > B_k$. Introduce the $k$ variables $y_1, \ldots, y_k$ corresponding to the blocks and put $y = (y_1, \ldots, y_k)$. Define the functions $\alpha$ and $\beta$ from $V(\Gamma)$ to $[k]$ as follows:

- $\alpha(v) = j$ if $v \in B_j$;
- $\beta(v) = j$ if $B_j$ is the $\mathfrak{A}$-minimal block among those that have nonempty intersection with $\Gamma^+(v)$.

The $\mathfrak{A}$-*reduction* of the min-sum $S^\Gamma_{\mathfrak{A}}(x)$ is the function

$$S^\Gamma_{\mathfrak{A}}(y|\mathfrak{A}) = \sum_{v \in V(\Gamma)} \frac{y_{\alpha(v)} y_{\beta(v)}}{y_{\beta(v)}}.$$  \hfill (21)

**Lemma 2.** Suppose $x > 0$ and $\mathfrak{A} \in \text{Arr}_\Gamma$ is an $x$-type arrangement. Let $y$ be the vector defined by $y_{\alpha(v)} = x_v$ (in the notation of Definition 4). Then

$$S^\Gamma_{\mathfrak{A}}(y|\mathfrak{A}) = S^\Gamma_{\mathfrak{A}}(x).$$

**Proof.** Note first that $y_j$ are defined correctly: if $j = \alpha(v) = \alpha(v')$ then $x_v = x_{v'}$ by definition of the set $W_{\mathfrak{A}}$.

It remains to see that $y_{\beta(v)} = \wedge(x|\Gamma^+(v))$; this follows from the definitions of the function $\beta$ and the set $W_{\mathfrak{A}}$. \hfill $\square$

**Remark.** The assumptions of Lemma 2 imply the inequalities $y_1 > \cdots > y_k$. Conversely, if $\mathfrak{A}$ is some preferential arrangement, as in Definition 4 and $y \in \mathbb{R}^k_+$ is a vector satisfying this inequalities, then the vector $x$ defined by $x_v = y_{\alpha(v)}$, $v \in V(\Gamma)$, lies in $W_{\mathfrak{A}}$.

In view of Lemma 2, one would expect that the minimization problem for $S^\Gamma_{\mathfrak{A}}(\cdot)$ can be reduced to the analysis of the sums (21). Or, one may ask, — maybe — just to the analysis of the unary sums $S^\Gamma_{\mathfrak{A}*}(x)$, see (16)?

The latter expectation is too naive and false. However the former can indeed be carried through to the end. This will make up the remaining part
Figure 1: Graph Γ of Example 2 and the reduced graphs corresponding to two different preferential arrangements

of this section. After necessary preliminaries we prove the main reduction theorem (Theorem 2) and then state an algorithm for finding $m_\Lambda^\Gamma$.

We emphasize that the nature of the sum (21) is more general than that of the unary sums: the same $y$-variable may occur in the numerators of several different terms. Let us describe the situation more formally.

Suppose a graph Γ and a preferential arrangement $\mathfrak{A}$ on the set $V = V(\Gamma)$ are fixed. Define a new graph $\Gamma_{\mathfrak{A}}$ as follows.

The set of nodes $V(\Gamma_{\mathfrak{A}})$ is indexed by the blocks of $\mathfrak{A}$. That is, each node corresponds to one of the $y$-variables in Definition 4.

The set of arcs $A(\Gamma_{\mathfrak{A}})$ is in one-to-one correspondence with the set of nodes $V(\Gamma)$: the node $v \in V$ gives rise to the arc $\alpha(v) \rightarrow \beta(v)$ of $\Gamma_{\mathfrak{A}}$. (Multiple arcs were not allowed in $\Gamma$, but they may be present in $\Gamma_{\mathfrak{A}}$. For instance, if $\mathfrak{A}$ consists of a single block, then $\Gamma_{\mathfrak{A}}$ contains a single node and $|V(\Gamma)|$ loops.)

Let us label the nodes of the graph $\Gamma$ with numerical values: $v \mapsto x_v$. The induced labelling of the nodes of $\Gamma_{\mathfrak{A}}$ is: $B_j \mapsto y_j = x_{\alpha(j)}$. Moreover, there is an induced labelling of the arcs of $\Gamma_{\mathfrak{A}}$: the arc $\alpha(v) \rightarrow \beta(v)$ is assigned the label $\xi_v = y_\alpha(v)/y_\beta(v)$.

The sum $S^\Gamma(\mathfrak{A} | \mathfrak{I})$ becomes simply $\sum_{v \in V} \xi_v$. However, the values $\xi_v$ are inherently dependent: to every loop $v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_h$ in $\Gamma$ there corresponds the relation $\xi_{v_1} \xi_{v_2} \cdots \xi_{v_h} = 1$. The problem of minimization of sums under constraints of this type is studied in the author’s work [27], which hereinafter will be referred to several times.

**Example 2.** Consider the graph $\Gamma$ with $V(\Gamma) = [5]$ depicted in Figure 7(a).
The corresponding min-sum is

\[
S_\lambda^\Gamma(x) = \frac{x_1}{x_2} + \frac{x_2}{\wedge(x_3, x_4)} + \frac{x_3}{\wedge(x_2, x_4)} + \frac{x_4}{x_1}.
\]

Consider the preferential arrangement \( \mathfrak{A} = \{(4) > (1) > (2) > (3)\} \). If \( x \in W_\mathfrak{A} \), then \( \wedge(x_3, x_4) = x_3 \) and \( \wedge(x_2, x_4) = x_2 \). The functions \( \alpha \) and \( \beta \) (see Definition [4]) are:

|   | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| \( \alpha(v) \) | 2 | 3 | 4 | 1 |
| \( \beta(v) \)   | 3 | 4 | 3 | 1 |

The relation between the \( x \)-variables and \( y \)-variables as defined in Lemma[2] is: \( y_1 = x_4 \), \( y_2 = x_1 \), \( y_3 = x_2 \), \( y_4 = x_3 \). Hence

\[
S_\lambda^\Gamma(y|\mathfrak{A}) = \frac{y_2}{y_3} + \frac{y_3}{y_4} + \frac{y_4}{y_3} + \frac{y_1}{y_2}.
\]

In this case, the numerical labels of the arcs of the graph \( \Gamma_\mathfrak{A} \) are \( \xi_1 = y_2/y_3 \), \( \xi_2 = y_3/y_4 \), \( \xi_3 = y_4/y_3 \), \( \xi_4 = y_1/y_2 \). The corresponding graph \( \Gamma_\mathfrak{A} \) is shown in Fig. 7(b). Note the cycle of length 2 in \( \Gamma_\mathfrak{A} \) and the relation \( \xi_2 \xi_3 = 1 \).

Consider now the preferential arrangement \( \mathfrak{A} = \{(1) > (3, 4) > (2)\} \). Here we have three blocks; the \( y \)-variables corresponding to a vector \( x \in W_\mathfrak{A} \) are: \( y_1 = x_1 \), \( y_2 = x_3 = x_4 \), \( y_3 = x_2 \). The functions \( \alpha \) and \( \beta \) are:

|   | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| \( \alpha(v) \) | 1 | 3 | 2 | 4 |
| \( \beta(v) \) | 3 | 2 | 3 | 1 |

The relation of Lemma[2] becomes

\[
S_\lambda^\Gamma(y|\mathfrak{A}) = \frac{y_1}{y_3} + \frac{y_3}{y_2} + \frac{y_2}{y_3} + \frac{y_2}{y_1}.
\]

The graph \( \Gamma_\mathfrak{A} \) is shown in Fig. 7(c). It has two independent cycles, \( y_1 \to y_3 \to y_2 \to y_1 \) and \( y_3 \to y_2 \to y_3 \), corresponding to the relations \( \xi_1 \xi_2 \xi_4 = \xi_2 \xi_3 = 1 \).

**Lemma 3.** Suppose \( \Gamma \) is a strongly connected graph. Then the symbol \( \inf \) in the definition (12) of \( m_\lambda^\Gamma \) can be replaced by \( \min \). That is, there exists a vector \( x \in \mathbb{R}_+^V(\Gamma) \) such that \( S_\lambda^\Gamma(x) = m_\lambda^\Gamma \).
Proof. Let \( \mu = m_\Gamma^\Gamma \), \( \mu_0 = \min(\mu, 1) \), and \( n = |\mathcal{V}(\Gamma)| \).

For any \( v \) and \( v' \) such that \( v' \in \Gamma^+(v) \) we have \( S_\Lambda^\Gamma(x) \geq x_v/x_{v'} \), hence \( x_v/x_{v'} \geq \mu \) or, equivalently, \( x_{v'}/x_v \leq \mu^{-1} \).

Since \( \Gamma \) is strongly connected, for any two nodes \( v' \) and \( v'' \) there is a path from \( v'' \) to \( v' \) with at most \( n-1 \) arcs. It follows that \( \mu_0^{1-n} \geq x_{v'}/x_{v''} \geq \mu_0^{n-1} \).

Imposing the normalization condition \( x_v/1 = 1 \) for some \( v' \in \mathcal{V}(\Gamma) \), we see that \( S_\Lambda^\Gamma(x) > m_\Lambda^\Gamma \) outside the cube \( K = \{x \mid \mu_0^{1-n} \leq x_v \leq \mu_0^{n-1}, \forall v \in \mathcal{V}(\Gamma)\} \). Since the function \( S_\Lambda^\Gamma(\cdot) \) is continuous on \( K \), it does attain the minimum value at some \( x \in K \). \( \square \)

Any vector \( x \) such that \( S_\Lambda^\Gamma(x) = m_\Lambda^\Gamma \) is called a minimizer (for \( S_\Lambda^\Gamma(\cdot) \)). Clearly, if \( x \) is a minimizer, then so is \( t x \) for any \( t > 0 \).

**Theorem 2.** Suppose \( \Gamma \) is a strongly connected graph and \( x^* \) is a minimizer for \( S_\Lambda^\Gamma(\cdot) \). Let \( \mathfrak{A} = (B_1 > \cdots > B_k) \) be the preferential arrangement of \( x^* \)-type. Let \( y^* \in \mathbb{R}_+^k \) be the vector defined by \( y^*_a(v) = x^*_v \) (in the notation of Definition 4). Then \( y^* \) is a minimizer for the function \( y \mapsto S_\Lambda^\Gamma(y|\mathfrak{A}) \).

**Proof.** According to Remark after Lemma 2 the vector \( y^* \) lies in the cone \( K = \{y \mid y_1 > \cdots > y_k > 0\} \).

The fact that \( K \) is an open set is crucial for our argument.

Consider two cases (cf. Fig. 7(b) and (c)).

**Case 1:** The graph \( \Gamma_{\mathfrak{A}} \) has a node of indegree 0. Equivalently, some variable \( y_j \) never appears as a denominator in the sum \( S_\Lambda^\Gamma(y|\mathfrak{A}) \) (that is, in the right-hand side of (21)). Let \( \varepsilon > 0 \) be so small that the vector \( y^# \) obtained from \( y^* \) by changing \( y_j^* \) into \( y_j^* - \varepsilon \) still lies in \( K \). Then the vector \( x^# \) with components \( x_v^# = y_v^# \) lies in \( W_{\mathfrak{A}} \). By Lemma 2 we have \( S_\Lambda^\Gamma(y^*|\mathfrak{A}) = S_\Lambda^\Gamma(x^*) \) and \( S_\Lambda^\Gamma(y^#|\mathfrak{A}) = S_\Lambda^\Gamma(x^#) \).

Clearly, the function \( S_\Lambda^\Gamma(y) \) is monotone (increasing) with respect to \( y_j \), so \( S_\Lambda^\Gamma(y^#|\mathfrak{A}) < S_\Lambda^\Gamma(y^*|\mathfrak{A}) \). Hence, \( S_\Lambda^\Gamma(x^#) < S_\Lambda^\Gamma(x^*), \) a contradiction.

**Case 2:** The graph \( \Gamma_{\mathfrak{A}} \) does not have nodes of indegree 0. Equivalently, every variable \( y_j \) appears as the denominator of some term in the right-hand side of (21).

The optimization problem

\[
S_\Lambda^\Gamma(y|\mathfrak{A}) \to \inf, \quad y > 0,
\]

(22)
is equivalent, by putting $\xi_v = y_{\alpha(v)}/y_{\beta(v)}$, to the problem

$$\sum_{v \in V(\Gamma)} \xi_v \to \inf, \quad \xi_v > 0, \quad \prod \xi_{v_j}^{e_j(v)} = 1, \quad j = 1, \ldots, m,$$

(23)

where $m$ is the number of independent cycles in the graph $\Gamma_A$ and $\varepsilon_j(v) = 1$ if the arc $\alpha(v) \to \beta(v)$ belongs to the cycle number $j$, otherwise $\varepsilon_j(v) = 0$. (Cf. [27, Sec. I.4.1, Problem 40].)

By an argument similar to the proof of Lemma 3 we see that $\inf$ in (23) can be replaced by $\min$ and the minimization can be carried over a compact set. Moreover, by [27, Problem 15], the minimizer $\vec{\xi}^{**}$ for the problem (23) is determined as the unique critical point (the solution of a system of polynomial equations obtained through Lagrange’s multipliers method).

Let $\vec{\xi}^{**}$ be the vector corresponding to $y^*$, i.e. $\xi^*_v = y_{\alpha(v)}/y_{\beta(v)}$ for all $v \in V(\Gamma)$. Take some neighborhood $O \subset K$ of $y^*$ (it exists, since $K$ is open). It is easy to see that the map $y \mapsto \vec{\xi}$ is open (in fact, its restriction on some “normalizing” hypersurface, say, $\sum_{j=1}^k y_j = 1$ is a homeomorphism), so the image $\Omega$ of $O$ is an open neighborhood of $\vec{\xi}^*$.

Suppose that $\vec{\xi}^{**} \neq \vec{\xi}^*$. Then, since $\vec{\xi}^*$ is not a critical point (which is unique), there exists some $\vec{\xi}^# \in \Omega$ such that $\sum \xi^#_v < \sum \xi^*_v$. Hence there exists $y^# \in O$ such that $S^\Gamma_A(y^#|A) < S^\Gamma_A(y^*|A)$. Now, as in Case 2, we find the corresponding vector $x^# \in W_A$ such that $S^\Gamma_A(x^#) < S^\Gamma_A(x^*)$ and conclude that $x^*$ is not a minimizer, a contradiction.

The conclusion is: Case 2 takes place and $\vec{\xi}^{**} = \vec{\xi}^*$; so $y^*$ is a minimizer for the optimization problem (22). Q.E.D.

**Remark.** A minimizer for $S^\Gamma_A(\cdot)$ is not necessarily unique up to a scalar multiple. Consider, for example,

$$S^\Gamma_A(x) = \frac{x_1}{\wedge(x_1, x_2)} + \frac{x_2}{x_3} + \frac{x_3}{x_2}.$$  

Clearly, $S^\Gamma_A(x) \geq 3$ for any $x$. The set of $x^*$ such that $S^\Gamma_A(x) = 3$ is two-parametric: $x_1^* = a, x_2^* = x_3^* = b$, where $0 < a \leq b$.

This example also shows that the graph $\Gamma_A$ corresponding to the preferential arrangement $A \ni x^*$ is not necessarily (weakly) connected.

Based on Theorem 2, we formulate a theoretical algorithm for computation of $m^\Gamma_A$ for the given graph $\Gamma$.  

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1. Make the list $\text{Arr}_\Gamma$ of all preferential arrangements of the set $\mathcal{V}(\Gamma)$.

2. Select the subset $\text{Arr}'_\Gamma$ comprising those preferential arrangements $\mathfrak{A}$ for which the graph $\Gamma_{\mathfrak{A}}$ does not have a node of indegree zero. (Cf. Case 1 of proof of Theorem 2)

3. For every $\mathfrak{A} \in \text{Arr}'_\Gamma$ find a minimizer $y^{(\mathfrak{A})}$ (unique up to a positive multiple) for the problem (22) by solving the corresponding system of polynomial equations. (The relevant material in [27] is: Problem 40, Eqs. (11), (7), and Problem 21.) Put $m_{\mathfrak{A}}^{\Gamma,\mathfrak{A}} = S_{\mathfrak{A}}^{\Gamma}(y^{(\mathfrak{A})} | \mathfrak{A})$.

4. Select the subset $\text{Arr}''_\Gamma \subset \text{Arr}'_\Gamma$ comprising those preferential arrangements $\mathfrak{A} = (B_1 > \cdots > B_k)$ ($k$ depends on $\mathfrak{A}$) for which the components $y^{(\mathfrak{A})}_j$, $j = 1, \ldots, k$, form a strictly decreasing sequence.

5. The answer is given by the formula

$$m_{\mathfrak{A}}^{\Gamma} = \min_{\mathfrak{A} \in \text{Arr}''_\Gamma} m_{\mathfrak{A}}^{\Gamma,\mathfrak{A}}. \quad (24)$$

**Remark.** We have shown that the problem of finding $m_{\mathfrak{A}}^{\Gamma}$ can be reduced to solution of finitely many systems of nonlinear algebraic equations. However the above algorithm has little value as a practical method because of its tremendous combinatorial complexity: the number of preferential arrangements on a $n$-element set is asymptotic to $\frac{1}{2}(\log 2)^{-n-1} n!$. It would be interesting to devise a method of lower complexity.

8 Min-sums II: Extremal problems

The optimization problem: find $m_{\mathfrak{A}}^{\Gamma}$ for the given $\Gamma$ — will be solved here only in some special cases. The question that turned out more amenable and will be explored rather satisfactorily is an extremal graph problem understood in the standard sense as “determining the extreme value of some graph parameter over some class of graphs” [29, p. 373]. The parameter in present case is $m_{\mathfrak{A}}^{\Gamma}$ and the class of (directed) graphs consists of graphs of positive minimum outdegree with given number of nodes (and possibly additional constraints).

We denote by $\mathfrak{G}(n)$ the class of (isomorphism classes of) directed graphs with $n$ nodes, positive minimum outdegree, positive minimum indegree (so
that every variable \( x_v \) in the corresponding sum appears in the denominator of at least one term) and no multiple arcs.

We begin with the question about the maximum possible value of \( m^\Gamma_\lambda \) for \( \Gamma \in \mathfrak{G}(n) \). If \( \Gamma \) is a graph such that set system \( \{ \Gamma^+(v) \} \) admits an SDR, or, equivalently, there exists a \( \Gamma \)-admissible bijection of \( \mathcal{V}(\Gamma) \), we will say for short that \( \Gamma \) admits and SDR.

Let us revisit Proposition 6 and consider its simplest case where \( \Gamma \) adm its an SDR. Then the upper bound (20) for \( m^\Gamma_\lambda \) coinsides with lower bound in (17), so \( m^\Gamma_\lambda = n \). But does the latter equality take place only if there exists an SDR? The answer in affirmative is given below.

**Theorem 3.** (a) Let \( \Gamma \in \mathfrak{G}(n) \). The upper bound \( m^\Gamma_\lambda = n \) is attained if and only if \( \Gamma \) admits an SDR.

(b) The maximum value excluding SDR-admitting graphs is

\[
\max \{ m^\Gamma_\lambda \mid \Gamma \in \mathfrak{G}(n), \ \Gamma \text{ does not admit an SDR} \} = n - \Delta_n, \tag{25}
\]

where \( \Delta_{2k-1} = \Delta_{2k} = 2k - 1 - 2\sqrt{k(k-1)} \). (Note: \( \Delta_n \sim (2n)^{-1} \) as \( n \to \infty \).)

**Proof.** (a) We have to prove the “only if” part. Let \( \mathcal{V} = \mathcal{V}(\Gamma) \) and \( n = |\mathcal{V}| \). Suppose, contrary to what the theorem claims, that \( m^\Gamma_\lambda = n \), yet \( \Gamma \) does not admit an SDR. Then by Hall’s theorem there exist sets \( \Omega \subset \mathcal{V} \) and \( \Phi \subset \mathcal{V} \) such that \( \Phi = \bigcup_{v \in \Omega} \Gamma^+(v) \) and \( |\Phi| < |\Omega| \). Clearly,

\[
S^\Gamma_\lambda(x) \leq \sum_{v \in \Omega} \frac{x_v}{\min_{v \in \Phi} x_v} \quad + \quad \sum_{v \in \mathcal{V} \setminus \Omega} \frac{x_v}{\min_{v \in \mathcal{V}} x_v}.
\]

Let us take the vector \( x \) with components

\[
x_v = \begin{cases} 
1, & \text{if } v \in \Phi; \\
t, & \text{if } v \in \mathcal{V} \setminus \Phi,
\end{cases}
\]

where \( 0 < t \leq 1 \). Then

\[
S^\Gamma_\lambda(x) \leq |\Omega \cap \Phi| + t|\Omega \setminus \Phi| + \frac{|\Phi \setminus \Omega|}{t} + |\mathcal{V} \setminus (\Omega \cup \Phi)|.
\]

Put \( a = |\Omega \setminus \Phi|, b = |\Phi \setminus \Omega| \). We have \( a - b = |\Omega| - |\Phi| > 0 \). Also,

\[
\min_{0 < t \leq 1} \left( ta + \frac{b}{t} \right) = 2\sqrt{ab}.
\]
Put $\delta(a,b) = a + b - 2\sqrt{a + b}$. Clearly, $\delta(a,b) > 0$. Hence
\[ S^\Gamma_{\lambda}(x) \leq |\Omega \cap \Phi| + |\Omega \setminus \Phi| + |\Phi \setminus \Omega| + |\nu \setminus (\Omega \cup \Phi)| - \delta(a,b) < n. \]
The assumption that $\Gamma$ does not admit an SDR has led to a contradiction.

(b) To prove (25), note first that the function $\delta(a,b)$ decreases in $b$ when $a$ is fixed. Since $a + b \leq n$ and $b \leq a - 1$, we have $\delta(a,b) \geq \delta(a, \min(n - a, a - 1))$. The function $a \mapsto \delta(a, a - 1)$ decreases for $a \geq 1$, while the function $a \mapsto \delta(a, n - a) = n - \sqrt{a(n - a)}$ increases for $n/2 \leq a \leq n$. It follows that for any fixed $n \geq 2$ and $k = \lceil (n + 1)/2 \rceil$
\[ \min \delta(a,b) = \delta(k, k - 1) = \Delta_n. \]
The obtained lower bounds for $\delta(a,b)$ imply corresponding upper bounds for $m^\Gamma_{\lambda}$. It remains to show that those upper bounds are attainable.

Let $\mathcal{V}(\Gamma) = [1, n]$, $\Omega = [1, k]$, $\Phi = [n - k + 2, n]$ (note: $n - k + 2 > k$).
Define $\Gamma \in \mathcal{G}(n)$ by $\Gamma^+(i) = \Phi$ for $i \in [1, k]$ and $\Gamma^+(i) = [1, n]$ for $i \in [k + 1, n]$.
Denote $\land_{i \in [n]} v_i = p$, $\land_{i \in \Phi} = q$. Clearly, $p = \min(q, x_1, \ldots, x_{n-k+1})$. Hence
\[ S^\Gamma_{\lambda}(x) = \frac{x_1 + \cdots + x_k}{q} + \frac{x_{k+1} + \cdots + x_n}{p} \geq \frac{kp}{q} + \frac{(k-1)q}{p} + (n - 2k + 1) \geq 2\sqrt{k(k-1)} + (n - 2k + 1) = n - \delta(k, k - 1). \]
Therefore $m^\Gamma_{\lambda} \geq n - \delta(k, k - 1)$. In combination with previously proved upper bound it shows that $m^\Gamma_{\lambda} = n - \delta(k, k - 1)$. \[ \square \]

Next comes the question about the minimum possible value of $m^\Gamma_{\lambda}$ for $\Gamma \in \mathcal{G}(n)$. Taken at face value, it is quite trivial.

**Proposition 8.** For any $n \geq 2$
\[ \min_{\Gamma \in \mathcal{G}(n)} m^\Gamma_{\lambda} = 2. \]

*Proof.* Suppose first that there is a strong component in $\Gamma$ with at least 2 nodes. Then it contains a cycle of length $\geq 2$. Hence, by Proposition 2, $m^\Gamma_{\lambda} \geq 2$.
Now, if all strong components of $\Gamma$ are singletons, then at least two of them (one final and another — either also final or one without predecessor in the condensation) contain a loop. Again, it follows that $m^\Gamma_{\lambda} \geq 2$.
The graph depicted on the right provides the lower bound $m^\Gamma_{\lambda} = 2$, by Proposition 2. \[ \square \]
The question becomes more interesting and challenging if we restrict the class $\mathcal{G}(n)$ to its subclass $\tilde{\mathcal{G}}(n)$ of strongly connected graphs.

Problem 74(b,c) in [27] asks, in different terms, to prove that for $n = 2021$

$$20 < \min_{\Gamma \in \mathcal{G}(n)} m_{i,\lambda}^{\Gamma} < 21.$$  

The general result is as follows:\footnote{The combinatorial part of solution of Problem 74 given in [27] can be converted to that of a general proof by changing 2021 into $n$ and making obvious induced replacements. We give an independent proof here, in part because the proof of one of the key lemmas (Problem 71) in [27] ver. 1 is grossly inaccurate.}

**Theorem 4.** For any $\Gamma \in \tilde{\mathcal{G}}(n)$

$$m_{i,\lambda}^{\Gamma} > e \ln(n + 1 - \ln(n + 1)).$$  \hfill (26)

Furthermore,

$$\min_{\Gamma \in \mathcal{G}(n)} m_{i,\lambda}^{\Gamma} = e \ln n + O\left(\frac{1}{\ln n}\right)$$  \hfill (27)

as $n \to \infty$.

**Proof.** Suppose $\mathbf{x}$ is a minimizer for $S_{i,\lambda}^{\Gamma}(\cdot)$. Put $x_* = \min(x_v, v \in \mathcal{V})$, $x^* = \max(x_v, v \in \mathcal{V})$. Let $v_* \neq v^*$ be two nodes such that $x_{v_*} = x_*$ and $x_{v^*} = x^*$.

Consider a simple (not self-intersecting) path $\pi$ from $v_*$ to $v_1 \cdots \pi(k+1)$. In particular, $v_1 = v^*$, $v_{k+1} = v_*$. If $k + 1 < n$, we label the remaining nodes by numbers $k + 2, \ldots, n$ arbitrarily. The node values $x_{v_k}$ will be written as $x_k$ for short.

Obviously, $\wedge(x|\Gamma^+(v_i)) \leq x_{i+1}$ for $i = 1, \ldots, k$. Hence

$$\sum_{i=1}^{k} \frac{x_i}{\wedge(x|\Gamma^+(v_i))} \geq \sum_{i=1}^{k} \frac{x_i}{x_{i+1}} \geq k \left(\frac{x^*}{x_*}\right)^{1/k}.$$  \hfill (27)

(The last step uses the AGM inequality.)

For the remaining nodes we use the trivial estimates $\wedge(x|\Gamma^+(i)) \leq x^*$ and $x_i \geq x_*$. Therefore

$$\sum_{i=k+1}^{n} \frac{x_i}{\wedge(x|\Gamma^+(v_i))} \geq (n - k) \frac{x_*}{x^*}.$$
We conclude that
\[ m^r_\Lambda = S^r_\Lambda(x) \geq \min_{1 \leq k \leq n-1} \min_{t \geq 0} \left( kt^{1/k} + \frac{n-k}{t} \right). \]

The minimum of the inner function equals \((k + 1)(n - k)^{\frac{1}{k+1}}\), attained at \(t = (n - k)^{\frac{1}{k+1}}\).

Let \(z = k + 1\) and \(r = n + 1\). Then
\[ \ln \left( (k + 1)(n - k)^{\frac{1}{k+1}} \right) = f(z, r) \overset{\text{def}}{=} \ln z + \frac{\ln(r - z)}{z}. \]

It can be shown that \(\min_{1 < z \leq r - 1} f(z, r) > 1 + \ln\ln(r - \ln r)\). The calculation is straightforward but not too short; details can be found in [27, Solution of Problem 67B] (an even more precise estimate for \(\min_z f(z, r)\) is given in [23]). The inequality (26) follows.

From our analysis one sees that the equality
\[ S^r_\Lambda(x) = k \left( \frac{x_1^1}{x_{k+1}} \right)^{1/k} + (n - k) \frac{x_{k+1}}{x_1} \]
takes place (at least) for the graph \(\Gamma_k\) with \(V(\Gamma_k) = [n]\),
\[ \Gamma_k^+(i) = \{i + 1\}, \quad i = 1, \ldots, k - 1; \]
\[ \Gamma_k^+(k) = \{k + 1, \ldots, n\}; \]
\[ \Gamma_k^+(i) = \{1\}, \quad i = k + 1, \ldots, n, \]
provided that \(x_1 \geq x_2 \geq \cdots \geq x_{k+1} = \cdots = x_n\).

The corresponding min-sum is
\[ S^r_{\Gamma_k}(x) = \frac{x_1}{x_2} + \cdots + \frac{x_{k-1}}{x_k} + \frac{x_k}{\lambda(x_{k+1}, \ldots, x_n)} + \frac{1}{x_1} \sum_{i=k+1}^{n} x_i. \]

One can find the asymptotics \((r \to \infty)\) of the minimum of the function \(f(z, r)\) (again, we refer to [27] or [23] for details) and obtain
\[ \min_{1 < z \leq r - 1} \exp f(z, r) = e \ln r + O \left( \frac{1}{\ln r} \right). \]

The asymptotic formula remains true if \(z\) is allowed to assume only integer values. Putting \(k = z\) and minimizing over \(k\), we find that in the extremal case \(m^r_\Lambda = e \ln(n + 1) + O((\ln n)^{-1})\), which is equivalent to (27). \(\square\)
References

[1] T. Ando, A new proof of Shapiro inequality, *Math. Ineq. Appl.* 16:3, 611–632 (2013). DOI:10.7153/mia-16-46

[2] J. Bang-Jensen, G.Z. Gutin, *Digraphs: Theory, Algorithms and Applications*, Springer, 2002.

[3] W.J. Baston, Some cyclic inequalities. *Proc. Edinburgh Math. Soc.* 19:2, 115–118 (1974). DOI:10.1017/S0013091500010221

[4] F. Bergeron, G. Labelle, P. Leroux, *Combinatorial species and tree-like structures*, Cambridge Univ. Press, Cambridge, 1998.

[5] J.C. Boarder and D.E. Daykin, Inequalities for certain cyclic sums II. *Proc. Edinburgh Math. Soc.*, 18:3, 209–218 (1973). DOI:10.1017/S0013091500009949

[6] L. Boltzmann, *Vorlesungen über Gastheorie*, Bd. 2., Barth, Leipzig, 1898; English translation: *Lectures on Gas Theory*, v. 2, Univ. of California Press, 1964.

[7] L. Caccetta, R. Häggkvist, On minimal digraphs with given girth. In: Proc. 9th Southeastern Conf. on Combinatorics, Graph Theory, and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1978), *Congr. Numer.*, 21, *Utilitas Math.*, 181–187 (1978). MR83h: 05055.

[8] S.A. Choudum, K.R. Parthasarathy, Semi-regular relations and digraphs, *Indagationes Mathematicae* (Proceedings), 75:4, 326–334 (1972). DOI:10.1016/1385-7258(72)90047-9

[9] A. Clausing. A review of Shapiro’s cyclic inequality. In: *General Inequalities 6* (W. Walter, ed.). Int. Series of Numerical Math., v. 103, Birkhäuser, Basel, 1992, 17–30.

[10] D.E. Daykin, Inequalities for functions of a cyclic nature, *J. London Math. Soc.* (2), 3:3, 453–462 (1971). DOI:10.1112/jlms/s2-3.3.453

[11] P.H. Diananda Extensions of an Inequality of H.S. Shapiro *The American Mathematical Monthly*, 66:6, 489–491 (1959). DOI:10.2307/2310633
[12] P.H. Diananda Some cyclic and other inequalities III, *Math. Proc. of the Cambridge Philos. Soc.* 73:1, 69–71 (1973). DOI:10.1017/s0305004100047484

[13] P.H. Diananda Some cyclic and other inequalities IV, *Math. Proc. of the Cambridge Philos. Soc.* 76:1, 183–186 (1974). DOI:10.1017/s0305004100048842

[14] V.G. Drinfeld, A cyclic inequality. *Math. Notes*, 9:2, 68–71 (1971). DOI:10.1007/BF01316982

[15] S. Finch, *Mathematical constants*, Princeton Univ. Press, Princeton, 2003.

[16] A.M. Fink, Shapiro’s inequality, in: *Recent Progress in Inequalities* (G.V. Milovanović, ed.), Springer, 1998, 241–248.

[17] E.K. Godunova, V.I. Levin, Sharpening certain cyclic inequalities, *Math. Notes*, 14:3, 735–741 (1973). DOI:10.1007/BF01147447

[18] E.K. Godunova, V.I. Levin, Exactness of a nontrivial estimate in a cyclic inequality, *Math. Notes*, 20:2, 673–675 (1976). DOI:10.1007/BF01155872

[19] E.K. Godunova, V.I. Levin, Lower bound for a cyclic sum, *Math. Notes*, 32:1, 481–483 (1982). DOI:10.1007/BF01137219

[20] K. Goldberg, The Minima of Cyclic Sums, *J. London Math. Soc.* 35:3, 262–264 (1960). DOI:10.1112/jlms/s1-35.3.262

[21] G.H. Hardy, J.E. Littlewood, G. Pólya, *Inequalities*, Cambridge Univ. Press, Cambridge, 1934.

[22] A. Itai, M. Rodeh, Finding a minimum circuit in a graph, *SIAM J. Comput.* 7:4 (1978), 413–423. DOI:10.1145/800105.803390

[23] G.V. Kalachev, S.Yu. Sadov, A logarithmic inequality, *Math. Notes* 103:2, 209–220 (2018). DOI:10.1134/S0001434618010224

[24] D.S. Mitrinović, J.E. Pečarić, A.M. Fink, *Classical and New Inequalities in Analysis*, Kluwer, Dodrecht-Boston-London, 1993.
[25] A.M. Nesbitt, Problem 15114. *Educ. Times* (2) 3, 37 (1903).

[26] S. Sadov, Lower bound for cyclic sums of Diananda type, *Arch. Math.* 106, 135–144 (2016). DOI:10.1007/s00013-015-0853-3

[27] S. Sadov, Minimization of the sum under product constraints, arxiv.org/2012.15517 (2020).

[28] H.S. Shapiro, Advanced problem 4603. *Amer. Math. Monthly* 61, 571 (1954).

[29] D.B. West, *Introduction to graph theory*, 2nd ed., Pearson Education, 2001.

[30] S. Yamagami, Cyclic Inequalities, *Proc. Amer. Math. Soc.* 118:2, 521–527 (1993). DOI:10.2307/2160333