ISING SPINS ON THE LABYRINTH

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ABSTRACT
We consider a zero-field Ising model defined on a quasiperiodic graph, the so-called Labyrinth tiling. Exact information about the critical behaviour is obtained from duality arguments and the subclass of models which yield commuting transfer matrices. For the latter, the magnetization is independent of the position and the phase transition between ordered and disordered phase belongs to the Onsager universality class. In order to obtain information about the generic case, we calculate the magnetization for a series of couplings by standard Monte-Carlo methods.

1. Introduction

Not too much is known about the critical behaviour of statistical systems on non-periodic graphs. Although certain cases may still be treated by commuting transfer matrices (see Ref. and references therein) using so-called Z-invariance arguments, these may well not be representative. For instance, for the case at hand, the local magnetization turns out to be position-independent, what one certainly does not expect to happen for a general (ev. random) distribution of the coupling constants.

The example considered in this note, the Ising model on the so-called Labyrinth tiling, has recently been investigated in detail by means of duality arguments and commuting transfer matrices. Here, we briefly review the relevant results and supplement these by first numerical investigations.

2. The Model

The silver mean chain is obtained by repeated application of the two-letter substitution rule \(a \rightarrow b, b \rightarrow bab\) to the letter \(a\). From this, the Labyrinth tiling can be constructed by considering an orthogonal Cartesian product of two identical silver mean chains in the proper geometric representation and connecting points on one of the two subgrids of the resulting rectangular grid, see Fig. 1.

In this way, we obtain a tiling which consists of three different tiles. As a graph, it has the topology of the square lattice, but contains edges of three types (or eight if one accounts for orientations). It is therefore natural to assign individual coupling constants to different types of edges, defining in this way a nearest-neighbour coupling for the Ising spins \(\sigma_{ij} \in \{\pm 1\}\) which we place on the vertices of the graph. We denote the ferromagnetic couplings (in units of \(k_BT\)) by \(K_{xy}\) and \(L_{xy}\), where \(xy \in \{aa, ab, ba, bb\}\) labels the abscissa and ordinate of the corresponding rectangle in the underlying grid and the letters \(K\) and \(L\) refer to the two different diagonals.

The proper periodic approximants for finite systems are constructed as above, but from a periodic grid. This is defined by identifying the first and last letter of a word obtained by
applying the substitution rule a certain number of times. This ensures that no additional tiles or vertex configurations are created.

3. Exact Results

Let us restrict to the case of non-vanishing ferromagnetic couplings that are uniformly bounded from above and below by finite constants. Under these assumptions, the Peierls argument guarantees the existence of at least one phase transition.

Some more information about the critical behaviour can be obtained from a duality argument. Assuming that there is only a single transition, it must occur on the self-dual surface in the space of coupling constants, which is given by

\[ S_{xy} := \sinh(2K_{xy}) \sinh(2L_{xy}) = 1 \]  

for all index pairs \( xy \in \{aa, ab, ba, bb\} \).

Even more, the model is exactly solvable in the sense of commuting transfer matrices in a subspace defined by the four equations \( S_{xy} = \Omega \) (for the possible index pairs \( xy \in \{aa, ab, ba, bb\} \)) plus one additional equation, see eq. (4.5) in Ref. 1. The corresponding coupling constants can be parametrized explicitly in terms of elliptic functions. For a given bond, the argument is the difference of two rapidity parameters that are attached to the two lines which intersect on the bond, see Fig. 1. In the three-dimensional solvable subspace,

Fig. 1. A finite patch of the Labyrinth with underlying grid and rapidity lines.
the model shows (in terms of the temperature-like variable $\Omega^2$) a single second-order phase transition at $\Omega = 1$ (i.e., on the intersection of the solvable and the self-dual surface, compare Eq. [1]) which belongs to the Onsager universality class. In particular, the local magnetization $\langle \sigma \rangle$ turns out to be *position-independent* and shows, in the thermodynamic limit, the critical singularity$$\langle \sigma \rangle = \begin{cases} 
abla (1 - \Omega - 2)^{1/8} & \text{if } \Omega^2 > 1 \\
abla 0 & \text{if } \Omega^2 \leq 1 \end{cases}$$at $\Omega = 1$ governed by the magnetic exponent $\beta = 1/8$ of the Ising model. Furthermore, one can calculate the free energy by essentially counting bond frequencies[6]. This is due to the “mobility” of the rapidity lines[7] which is a consequence of the Yang-Baxter equation. Note that the periodic boundary conditions guarantee that moving rapidity lines does not create any surface contributions in our case.

Clearly, the latter result reflects the severe restrictions imposed by integrability. It is an interesting question whether the converse is also true, i.e., whether the position-independence of the local magnetization is sufficient for solvability. For a generic choice of couplings, one certainly expects the local magnetization to depend on the neighbourhood. This poses the question whether the solvable case is representative at all — a partial answer to which can be obtained by numerical investigation of non-integrable cases.

4. Numerical Results

As a first approach, we consider the dependence of the local magnetization on the position of the spin while digressing from the solvable surface. The simplest scenario occurs when going from the periodic case (i.e., all couplings equal) to the case of three different couplings according to the length of the bonds, i.e., $K_{aa} = L_{aa} = J_s/k_B T$, $K_{ab} = K_{ba} = L_{ab} = L_{ba} = J_m/k_B T$ and $K_{bb} = L_{bb} = J_l/k_B T$, where the subscripts $s$, $m$ and $l$ refer to short, medium and long bonds, respectively. We consider the periodic approximant which is defined by the word of length 41 obtained after five applications of the substitution rule to the initial letter $a$. For this patch of 1600 sites, we estimated the magnetization at three different sites (where we chose representatives of the three different vertex configurations, see Fig. 2) by means of the Swendsen-Wang Monte-Carlo algorithm[8]. The result is shown in Fig. 3, where the abscissa displays $T/T_c$ and the ordinate the normalized magnetization. Here, the three sets of coupling constants were chosen as follows. Fig. 3(a) corresponds to $J_s = J_m = J_l$, in Fig. 3(b) we used $J_s/J_m = 6/5$, $J_l/J_m = 4/5$ and Fig. 3(c) displays the results for $J_s/J_m = 7/5$ and $J_l/J_m = 3/5$. In order to keep the critical temperature approximately constant, we adjusted the coupling constants such that the average coupling (per bond) for the three cases is the same.
Fig. 3. Local magnetization at three positions for different couplings.

One can clearly see that the magnetization is site-independent in the periodic case (a), as it must be, but develops pronounced site-dependence as the difference of the three couplings increases (from (b) to (c)). On the other hand, we cannot decide whether we get more than one point of phase transition or a critical region, although our calculations seem to support the expectation that the critical point is unique.

5. Concluding Remarks

To get a better understanding of critical phenomena on non-periodic graphs with long-range order (which are in between the periodic and the random case), we have investigated the classical Ising model on a 2D quasiperiodic tiling. The example chosen, the so-called Labyrinth, can be considered as a quasiperiodic modulation of the square lattice. The Ising model is exactly solvable on a three-dimensional subspace of the coupling space considered, which contains the periodic case. Solvability resulted in site-independence of the local magnetization, while the generic case shows clear dependence on the local neighbourhood. A conclusive statement on the phase structure and the nature of critical behaviour requires further investigation of the model by algebraic and numerical means.

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7. References

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