Representation of SO(3) Group by a Maximally Entangled State

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A representation of the SO(3) group is mapped into a maximally entangled two qubit state according to literatures. To show the evolution of the entangled state, a model is set up on an maximally entangled electron pair, two electrons of which pass independently through a rotating magnetic field. It is found that the evolution path of the entangled state in the SO(3) sphere breaks an odd or even number of times, corresponding to the double connectedness of the SO(3) group. An odd number of breaks leads to an additional π phase to the entangled state, but an even number of breaks does not. A scheme to trace the evolution of the entangled state is proposed by means of entangled photon pairs and Kerr medium, allowing observation of the additional π phase.

It is well known that when the spin of a spin-$\frac{1}{2}$ particle rotates for a whole cycle on the Bloch sphere the wave-function of the particle changes a phase of $\pi$. This π phase has been observed in several experiments. This property is commonly attributed to the topological property, i.e., the double connectedness of the SO(3) group. The path on the manifold of the SO(3) group is categorized into two classes under a continuous deformation, one of which leads to a change of π in phase to the wave function, the other does not. However, it was argued by Milman and Mosseri[3] that this π phase may be shared by the multi-connectedness of both SO(3) and SO(2) groups. They also argued that, in general, the π phase is partly geometric and partly dynamic. Only in the extreme case that the spin precesses on the $xy$ plane in the Bloch sphere the π phase is fully geometric. Especially, the π phase may not be directly related to the SO(3) group.

Milman and Mosseri found a one-to-one correspondence between the representation of the SO(3) group and the evolution of a maximally entangled state of a two-qubit system(MES)[5]. They adopted a discontinuously changing magnetic field, which suddenly jumps from one direction to another. This is hardly possible to be accomplished in reality. In the present paper a rotating magnetic field is used to drive the evolution of a MES. A clearer formalism is presented for the trajectory in SO(3).

A MES finds great application to quantum communication and quantum computation techniques, and also to the study of fundamental problems, e.g., non-locality, of quantum mechanics[4, 5, 6]. Much attention has been paid to MES’s in recent years. It is interesting that MES can be applied to the representation of the SO(3) group.

I. MAPPING BETWEEN A MES AND SO(3)

A two-qubit maximally entangled state (MES) of a two-state system can be written as

$$[(\alpha, \beta)] = \frac{1}{\sqrt{2}}(\alpha|00\rangle + \beta|01\rangle - \beta^*|10\rangle + \alpha^*|11\rangle)$$

where the coefficients $\alpha$ and $\beta$ are normalized to unity

$$\alpha \alpha^* + \beta \beta^* = 1.$$  (2)

It is seen that a MES is defined by a pair of complex numbers $(\alpha, \beta)$. To visualize a MES, $\alpha$ and $\beta$ can be parameterized to

$$\alpha = \cos \frac{a}{2} - ik_z \sin \frac{a}{2},$$  (3)
$$\beta = -(k_y + ik_x) \sin \frac{a}{2}$$  (4)

where $(k_x, k_y, k_z) = k$ is a unit vector, and $a$ is an angle between 0 and $\pi$. Hence a MES can also be written as $|\Psi(k, a)\rangle$ in the parameter space. It is easy to check that $|\Psi(k, \pi + a)\rangle = -\bar{|\Psi(-k, \pi - a)\rangle}$. That is, $(k, \pi + a)$ and $(k, \pi - a)$ correspond to the same state except for a global phase factor. This is just the case of the double-valued representation of the SO(3) group, which is written as

$$D^{1/2}(k, a) = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix},$$  (5)

corresponding to a rotation $R(k, a)$ in real space to a two-state particle. Although $R(k, \pi + a)$ and $R(-k, \pi - a)$ are the same rotation, one has $D^{1/2}(k, \pi + a) =$
\[ -D^{1/2}(-\mathbf{k}, \pi - \alpha). \] Therefore, there is a one-to-one correspondence between the two-qubit MES and the double-valued representation of SO(3). In fact, any MES can be constructed by a rotation from an initial MES, e.g.,

\[ D_1[(1, 0)] = \|{(\alpha, \beta)}\|, \quad D_1 \equiv D^{1/2}(k, \alpha) \]

where \( D_1 \) operates on the first particle. If a rotation operates on the second particle, one has

\[ D_2[(1, 0)] = \|{(\alpha, -\beta^*)}\|, \quad D_2 \equiv D^{1/2}(k, \alpha) \]

One could define a SO(3) sphere with diameter \( \pi \) filled by vectors \( a_k = (ak_x, ak_y, ak_z) \). Due to \( \mathbf{a} \) a MES corresponds to a point in the SO(3) sphere, and an evolution of MES corresponds to a trajectory connecting two points. The initial state \( |(1, 0)\rangle \) locates at the center of the SO(3) sphere.

II. A MODEL HAMILTONIAN

Consider an electron in a rotating magnetic field \( B(t) = B(\sin \theta \cos \omega t, \sin \theta \sin \omega t, \cos \theta) \), where \( \theta \) is the angle between the field and the \( z \)-axis, and \( \omega \) is the rotating frequency of the field. The Hamiltonian of the electron is given by

\[ H(t) = \hat{\mathbf{\delta}} \cdot \mathbf{B}(t) = B(\cos \theta \sin \theta e^{-i\omega t}, \sin \theta e^{i\omega t} - \cos \theta). \]

The two exact solutions of the time-dependent Schrödinger equation are given by

\[ |\psi_{\pm}(t)\rangle = \left( a_{\pm}e^{-i\omega t/2}, b_{\pm}e^{i\omega t/2} \right) e^{i\omega_0 t}, \]

where new arguments have been assigned to the group element for convenience, and

\[ \alpha = \left[ \cos \frac{\omega t}{2} + \sin \frac{\omega t}{2} \frac{\hbar \omega - 2B \cos \theta}{2\hbar \omega_0} \right] e^{-i\omega_0 t}, \quad (14) \]
\[ \beta = \left[ \sin \frac{\omega t}{2} \frac{\hbar (\omega - 2\omega_0) - 2B \cos \theta}{2\hbar \omega_0} \right] e^{i\omega_0 t}. \quad (15) \]

It is seen that a rotating magnetic field leads to an evolution of a MES through a continuous trajectory in the SO(3) sphere. Therefore, a rotating magnetic field is equivalent to a three dimensional rotation in real space to the MES.

It is not surprising that when \( \omega t = 2\pi, \omega_0 = n\omega, n = \text{integers} \), the initial state acquires an additional phase of \( \pi \), i.e.,

\[ D_1(2\pi, \omega)|(1, 0)\rangle = -|(1, 0)\rangle \quad (16) \]

The amazing property is that the above operation can be allocated to two particles of the initial state, i.e.,

\[ D_1(\pi, \omega_0)D_2(\pi, n\omega_0)|(1, 0)\rangle = -|(1, 0)\rangle \quad (17) \]

In general, one does not have such a property for other initial state. If \( \omega_0 = (n + 1/2)\omega, n = \text{integers} \) one will have

\[ D_1(\pi, \omega_0)D_2(\pi, \omega_0)|1, 0\rangle = |1, 0\rangle, \quad (18) \]

acquiring no additional phase. Hence, one has a choice for the additional phase through selecting the value of \( \omega_0 \).

Now we can trace the following evolution

\[ |(1, 0)\rangle \rightarrow \|\psi_+(t)\psi_+(0)\rangle + |\psi_-(t)\psi_-(0)\rangle \rangle \sqrt{2} \quad (19) \]
\[ = D_1(\omega t, \omega_0)D_2(\omega t, \omega_0)|1, 0\rangle \quad (20) \]

Under the choice \( \omega_0 = n\omega \), or \( (n + 1/2)\omega, n = \text{integers} \) this evolution makes a closed trajectory in the SO(3) sphere. An example is shown in Fig.1, where parameters \( \theta \) and \( B \) are set to meet \( \omega_0 = \omega \). The final time is \( t = \pi/\omega \), that is, Both magnetic fields of the two electrons rotate half a cycle. It is seen that this trajectory breaks three times on the surface of the sphere. It is known that two ends of a diameter of the sphere correspond to the same rotation but the group element, \( \mathbf{a} \), changes its sign. In this case, through a whole trajectory, the MES acquires an additional phase of \( \pi \).

With proper parameters, one can have closed trajectories with even numbers of breaks, corresponding to a change of \( 2\pi \) in phase. An example is shown in Fig.2. Hence, one has two classes of trajectories, one of which has an odd number of breaks on the surface of the SO(3) sphere and the other has an even number of breaks, corresponding to the two classes of the double connectedness of the SO(3) group. This is the case that Milman and Mosseri considered, whereas their trajectories are hardly possible to be realized, since their magnetic field
has to jump through a few discrete points in the parameter space.

It can easily checked that the closed trajectories \( A - B - F - D - A \) and \( A - B - F - E - A \) and other ones that Milman and Mosseri considered belong to the two simplest classes of trajectories which have 0 or 1 breaks, respectively, in the SO(3) sphere. Therefore, the present work extends their model to include a great number of closed trajectories with even or odd numbers of breaks.

FIG. 1: Closed trajectory in the SO(3) sphere with \( \theta = \frac{\pi}{5}, B = 1.3603 \). The arrow stands for the beginning state and direction of evolution.

FIG. 2: Closed trajectory in the SO(3) sphere with \( \theta = \frac{\pi}{5}, B = 1.8754 \). The arrow stands for the beginning state and direction of evolution.
as shown in Fig.1 and Fig.2, can be exactly traced by varying the electric fields $E_1$ and $E_2$ on the Kerr medium. With proper values of electric fields such that $\phi_1 = n\phi_2$ one may obtain an additional $\pi$ phase, or zero additional phase if $\phi_1 = (n + \frac{1}{2})\phi_2$.

The $\pi$ phase can be easily observed by various interference experiments. For example, according to the scheme described by Milman and Mosseri considered [3], one arm of the entangled photon pair can be transformed into a Mach-Zender interferometer. The two wave plates in that scheme are replaced by combinations of Kerr medium P1 and P2, and that of P3 and P4.

In summary, the present paper sets up a representation for the SO(3) group by maximally entangled two-qubit states. The evolution of the entangled states showed the double connectedness of the SO(3) group. In the SO(3) sphere the evolution path breaks an odd or even number of times. An odd number of breaks causes an additional $\pi$ phase to the entangled state, but an even number of breaks does not. The additional $\pi$ phase can be observed by interference experiments of entangled photon pairs.

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