A NOTE ON A THEOREM OF BOURBAKI

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Abstract

We have recently show that Poincare series of Hyperbolic Lie algebras have the form of a ratio between Poincare series of a chosen finite Lie algebra and a polynomial of finite degree. By the aid of some properly chosen examples, we now give some remarks on a related theorem of Bourbaki.
I. INTRODUCTION

Let $P(G_N) = \sum_s d(s) t^s$ be the Poincare Series of a Kac-Moody Lie algebra $G_N$ of rank $N$. Here and in the following, $t$ denotes an indeterminate. It is known that $d(s)$ is the number of Weyl group elements which are composed out of the products of $s$ number of simple Weyl reflections corresponding to simple roots of $G_N$ [1]. In this work, this will be adopted as the definition of Poincare series of Kac-Moody Lie algebras.

For finite Lie algebras, Poincare polynomials are known in the following form

$$P(G_N) = \prod_{i=1}^{N} \frac{t^{\nu_i} - 1}{t - 1} \quad (I.1)$$

where $\nu_i$’s are the degrees of $N$ basic invariants of $G_N$. For an affine Kac-Moody Lie algebra $\hat{G}_N$ originated from a generic finite Lie algebra $G_n$ ($N \geq n$ in general), Bott theorem [2] states that its Poincare polynomial has the following product form

$$P(\hat{G}_N) = P(G_n) \prod_{i=1}^{N} \frac{1}{1 - t^{\nu_i - 1}} \quad (I.2)$$

To the author’s knowledge, explicit results seem to be lacking beyond affine ones although there is a general theorem which is known to be valid in any case. It is this theorem[3] which in fact can be applied also in obtaining of (I.1) and also (I.2). Theorem states that

$$P(G_N) = P(g_n) R \quad (I.3)$$

where $P(g_n)$ is Poincare polynomial of a sub-algebra $g_n \subset G_N$ with the trivial condition that $g_n$ should be contained inside the Dynkin diagram of $G_N$. A non-trivial assertion of theorem is that $R$ is a rational function.

For a Hyperbolic Lie algebra $H$, our observation is that its Poincare polynomial comes in the form

$$P(H) = \frac{P(g)}{Q(g)} \quad (I.4)$$

where $g$ is a properly chosen finite Lie Algebra and $Q(g)$ is a polynomial of some finite degree in indeterminate $t$. Due to the fact that a rational function or its inverse can only be represented by a polynomial of infinite order, (I.3) and (I.4) seem to be in contradiction. This will be exemplified in the following.
We know only a finite number of Hyperbolic Lie algebras [4]. Let us proceed in the concrete example for H with the Dynkin diagram:

Following 3 examples will be instructive in applications of above mentioned theorem.

Let \( W(H) \) be the Weyl group of \( H \) and \( \sigma_i \)'s be its elements corresponding to simple roots \( \alpha_i \)'s where \( i = 1, \ldots, 6 \). We assume that reduced Weyl group elements can be expressed by

\[
\Sigma(i_1, \ldots, i_k) \equiv \sigma_{i_1} \cdots \sigma_{i_k}. \tag{I.5}
\]

Let \( S = \{\alpha_1, \ldots, \alpha_6\} \) be the set of simple roots of \( H \) for which we use following 2 enumerations of its Dynkin diagram for the first 2 examples:

Among several possible choices, our 2 examples lead us respectively to sub-algebras \( A_4 \) and \( D_5 \) of \( H \), in view of the following choices:

1. \( I_1 = \{\alpha_1, \ldots, \alpha_4\} \subset S \)
2. \( I_2 = \{\alpha_1, \ldots, \alpha_5\} \subset S \)

We note here that there could be no choice for a subset which allows us to get a Lie sub-algebra which is not contained inside the Dynkin diagram of \( H \). For the first case,
above-mentioned theorem leads us to a Poincare polynomial

\[ P(H) = P(A_4) \, R_1 \quad (I.6) \]

where \( P(A_4) \) is the Poincare polynomial of \( A_4 \) Lie algebra with the Dynkin diagram

\[ \begin{array}{cccccc}
4 & \circ & \circ & \circ & \circ \\
\end{array} \]

and \( R_1 \) is a rational function which we calculate here, explicitly. In favor of (I.1), we know that

\[ P(A_4) = 1 + 4 \, t + 9 \, t^2 + 15 \, t^3 + 20 \, t^4 + 22 \, t^5 + 20 \, t^6 + 15 \, t^7 + 9 \, t^8 + 4 \, t^9 + t^{10} \]

and corresponding 120 elements of \( W(H) \) form a subset \( W(A_4) \subset W(H) \). Among infinite number of elements of \( W(H) \), there are only 120 elements originate only from the set \( I_1 \subset S \) of example (1). In (I.6), \( R_1 \) originates from a subset \( R_1(H) \subset W(H) \) for which any element \( w \in W(H) \) can be expressed by the factorized form

\[ w = u \, v \quad , \quad u \in W(A_4) \quad , \quad v \in R_1(H) \quad (I.7) \]

in such a way that

\[ l(w) = l(u) + l(v) \quad (I.8) \]

where \( l \) is the length function of \( H \). Due to space limitation, we exemplify our algorithm by giving elements of \( R_1(H) \) up to 6th order:
1

\( \sigma_5, \sigma_6, \)

\( \Sigma(5, 3), \Sigma(5, 6), \Sigma(6, 3), \)

\( \Sigma(5, 3, 2), \Sigma(5, 3, 4), \Sigma(5, 3, 6), \Sigma(5, 6, 3), \Sigma(6, 3, 2), \Sigma(6, 3, 4), \Sigma(6, 3, 5) \)

\( \Sigma(5, 3, 2, 1), \Sigma(5, 3, 2, 4), \Sigma(5, 3, 2, 6), \Sigma(5, 3, 4, 6), \Sigma(5, 3, 6, 3) \)

\( \Sigma(5, 6, 3, 4), \Sigma(6, 3, 2, 1), \Sigma(6, 3, 2, 4), \Sigma(6, 3, 2, 5), \Sigma(6, 3, 4, 5) \)

\( \Sigma(5, 3, 2, 1, 4), \Sigma(5, 3, 2, 1, 6), \Sigma(5, 3, 2, 4, 3), \Sigma(5, 3, 2, 4, 6) \)

\( \Sigma(5, 3, 2, 6, 3), \Sigma(5, 3, 4, 6, 3), \Sigma(5, 3, 6, 3, 2), \Sigma(5, 3, 6, 3, 4), \Sigma(5, 3, 6, 3, 5) \)

\( \Sigma(5, 6, 3, 2, 1), \Sigma(5, 6, 3, 2, 4), \Sigma(5, 6, 3, 2, 5), \Sigma(5, 6, 3, 4, 5) \)

\( \Sigma(6, 3, 2, 1, 5), \Sigma(6, 3, 2, 4, 3), \Sigma(6, 3, 2, 4, 5), \Sigma(6, 3, 2, 5, 3) \)

\( \Sigma(6, 3, 4, 5, 3) \)

\( \Sigma(5, 3, 2, 1, 4, 3), \Sigma(5, 3, 2, 1, 4, 6), \Sigma(5, 3, 2, 1, 6, 3), \Sigma(5, 3, 2, 4, 3, 5) \)

\( \Sigma(5, 3, 2, 4, 3, 6), \Sigma(5, 3, 2, 4, 6, 3), \Sigma(5, 3, 2, 6, 3, 2), \Sigma(5, 3, 2, 6, 3, 4) \)

\( \Sigma(5, 3, 2, 6, 3, 5), \Sigma(5, 3, 4, 6, 3, 2), \Sigma(5, 3, 4, 6, 3, 4), \Sigma(5, 3, 4, 6, 3, 5) \)

\( \Sigma(5, 3, 6, 3, 2, 1), \Sigma(5, 3, 6, 3, 2, 4), \Sigma(5, 3, 6, 3, 2, 5), \Sigma(5, 3, 6, 3, 4, 5) \)

\( \Sigma(5, 6, 3, 2, 1, 4), \Sigma(5, 6, 3, 2, 1, 5), \Sigma(5, 6, 3, 2, 4, 3), \Sigma(5, 6, 3, 2, 4, 5) \)

\( \Sigma(5, 6, 3, 2, 5, 3), \Sigma(5, 6, 3, 4, 5, 3), \Sigma(6, 3, 2, 1, 4, 3), \Sigma(6, 3, 2, 1, 4, 5) \)

\( \Sigma(6, 3, 2, 1, 5, 3), \Sigma(6, 3, 2, 4, 3, 5), \Sigma(6, 3, 2, 4, 3, 6), \Sigma(6, 3, 2, 4, 5, 3) \)

\( \Sigma(6, 3, 2, 5, 3, 4), \Sigma(6, 3, 2, 5, 3, 6), \Sigma(6, 3, 4, 5, 3, 2), \Sigma(6, 3, 4, 5, 3, 6) \)

The reader could verify order by order that the number of these elements do match with the first 6 terms in the infinite polynomial expansion of the following rational function:

\[
R_1 = \frac{(1 + t)^3(1 + t^2)(1 - t + t^2)(1 + t^4)}{1 - t^2 - 2t^3 - t^4 + t^6 + t^7 + 3t^8 + 2t^9 - t^{13} - 2t^{14} - 2t^{15} - t^{16} + t^{19} + t^{20}}
\]

Our algorithm however allows us to investigate the existence of \((1.6)\) at any order. To this end, let us consider

\[
P(H) \equiv \sum_{n=0}^{\infty} u_n \ t^n
\]

and

\[
P(A_4) \equiv \sum_{n=0}^{10} u_n \ t^n.
\]
Since $R_1$ is a rational function, it could be represented also by a polynomial of infinite order:

$$R_1 \equiv \sum_{n=0}^{\infty} v_n \ t^n$$

The reader could verify now that at any order $M = 0, \ldots, \infty$

$$w_M = \sum_{s=0}^{M} u_s \ v_{M-s}. \quad (I.9)$$

In case of example (2), one finds the following Dynkin diagram

```
5

4       3       2       1
```

which gives us

$$P(H) = P(D_5) \ R_2 \quad (I.10)$$

where $P(D_5)$ is the Poincare polynomial of $D_5$ Lie algebra. As in the first example, $R_2$ is to be calculated in the form of the following rational polynomial:

$$R_2 = \frac{(1 + t)}{1 - t^2 - 2t^3 - t^4 + t^6 + t^7 + 3t^8 + 2t^9 - t^{13} - 2t^{14} - 2t^{15} - t^{16} + t^{19} + t^{20}}$$

and the similar of (I.9) is seen to be valid.

For our last example, the Dynkin diagram of $H$ should be enumerated not in the way defined above but as in the following:

```
6

3

1       4       5       6

2
```
in such a way that the choice

\( I_3 = \{\alpha_1, \ldots, \alpha_3\} \subset S \)

gives us an infinite sub-algebra which is in fact the affine Lie algebra \( \hat{D}_4 \) with the following Dynkin diagram:

\[
\begin{array}{cccc}
1 & 4 & 5 \\
& 3 &
\end{array}
\]

This time, similar to (I.6) and (I.10), one obtains

\[
P(H) = P(\hat{D}_4) R_3 \quad (I.11)
\]

where

\[
R_3 = \frac{(1-t)^3(1+t)(1+t+t^2)^2(1+t^4)(1+t+t^2+t^3+t^4)}{1-t^2-2t^3-t^4+t^5+t^6+t^8+t^9+t^{10}+t^{11}-t^{14}-t^{15}}.
\]

From (I.2), we know that \( P(\hat{D}_4) \equiv \sum_{n=0}^{\infty} u_n t^n \). In result, we see that (I.9) is also valid here.

Let us conclude this section by giving the main motivation behind this work. As the theorem said, in all three of above examples, corresponding polynomials \( R_1, R_2 \) and also \( R_3 \) are rational functions. In the next section, we show by explicit calculation that (I.4) is in fact valid for Poincare polynomial of \( H \). From above discussion, it is seen that theorem can not give a way to obtain (I.4).

**II. CALCULATION OF HYPERBOLIC POINCARE POLYNOMIALS**

We follow Humphreys [5] for Lie algebraic technology and Kac-Moody-Wakimoto [6] for the description of hyperbolic Lie algebras. To explain formal structure of our calculations, we follow the example \( H \) for which simple roots \( \alpha_i \) and fundamental dominant weights \( \lambda_i \) are given by

\[
\kappa(\lambda_i, \alpha_j) = \delta_{i,j} \quad i, j = 1, \ldots, 6.
\]
where \( \kappa(\cdot, \cdot) \) is the symmetric scalar product which we know to be exist, on \( H \) weight lattice, by its Cartan matrix \( A \)

\[
A = \begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & -1 & -1 \\
0 & 0 & -1 & 2 & 0 & 0 \\
0 & 0 & -1 & 0 & 2 & 0 \\
0 & 0 & -1 & 0 & 0 & 2
\end{pmatrix}
\]

so we have

\[
\lambda_i = \sum_{j=1}^{6} (A^{-1})_{i,j} \alpha_j
\]

Although following discussions are valid for any \( G_N \), we still proceed in \( H \) mentioned above. To this end, let \( W(H) \) be the Weyl group and \( \rho \) be the Weyl vector of \( H \). For any \( \Sigma \in W(H) \), let us now consider

\[
\Gamma \equiv \rho - \Sigma(\rho)
\]

which is by definition an element of the positive root lattice of \( H \). We know that \( \Gamma \) is unique in the sense that \( \Gamma \equiv \rho - \Sigma(\rho) \) is different from \( \Gamma' \equiv \rho - \Sigma'(\rho) \) for any two different \( \Sigma, \Sigma' \in W(H) \). Note here that the Weyl vector \( \rho \) is a strictly dominant weight. This is sufficient to propose our simple method to calculate the number of Weyl group elements which are expressed in terms of the same number of simple Weyl reflections \( \sigma_i \) which are defined by

\[
\sigma_i(\Lambda) \equiv \Lambda - 2 \frac{\kappa(\Lambda, \alpha_i)}{\kappa(\alpha_i, \alpha_i)} \alpha_i, \quad i = 1, 2, 3, \ldots
\]

for any element \( \Lambda \) of weight lattice.

As in above, the \( k \)-tuple products \( \Sigma(i_1, \ldots, i_k) \) are the reduced elements which can not be reduced into products consisting less than \( k \)-number of simple Weyl reflections. Out of all these reduced elements, we define a class \( W^k \subset W(G_N) \). Different elements of any class \( W^k \) are determined uniquely by their actions on the Weyl vector. We use a definitive algorithm to choose the ones among the equivalents. The results of this algorithm are given in above examples. The aim of this work, however, doesn’t need to show further this algorithm here.

Now we can formally state that a Weyl group is the formal sum of its classes \( W^k \). One should note that the order \( \vert W^k \vert \) of a class \( W^k \) is always finite though the number of these classes is finite for finite and infinite for infinite Kac-Moody Lie algebras.
Looking back to $H$, we give some of its classes in the following:

\[ W^0 = \{1\} \]
\[ W^1 = \{\sigma_1, \ldots, \sigma_6\} \]
\[ W^2 = \{\Sigma(1,2), \Sigma(1,3), \Sigma(1,4), \Sigma(1,5), \Sigma(1,6), \Sigma(2,1), \Sigma(2,3), \Sigma(2,4), \Sigma(2,5), \Sigma(2,6), \Sigma(3,2), \Sigma(3,4), \Sigma(3,5), \Sigma(3,6), \Sigma(4,3), \Sigma(4,5), \Sigma(4,6), \Sigma(5,3), \Sigma(5,6), \Sigma(6,3)\} \]

As a result, one has a polynomial $\sum_{k=0}^{\infty} W^k | t^k$ which is nothing but the Poincare polynomial of $H$, as is mentioned in sec.I.

By explicit calculation up to 26th order, we obtained the following result

\[ P(H) \equiv 1 + 6t + 20t^2 + 52t^3 + 117t^4 + 237t^5 + 445t^6 + 791t^7 + 1347t^8 + 2216t^9 + 3550t^{10} + 5568t^{11} + 8582t^{12} + 13044t^{13} + 19604t^{14} + 29189t^{15} + 43129t^{16} + 63332t^{17} + 92518t^{18} + 134572t^{19} + 195052t^{20} + 281882t^{21} + 406361t^{22} + 584620t^{23} + 839655t^{24} + 1204232t^{25} + \ldots (II.2) \]

One sees that (II.2) is sufficient to conclude that

\[ P(H) \equiv \frac{P(B_5)}{Q(B_5)} \]

where

\[ Q(B_5) \equiv (1 - 2t^3 + t^4 + t^6 - t^7 + 2t^8 - t^9 + t^{10} + t^{12} + t^{13} - t^{14} - t^{15} - t^{18} - t^{20} + t^{24}) \] (II.4)

and $P(B_5)$ comes from (I.1) for $B_5$ Lie algebra. Note here that the number of positive roots of $B_5$ is equal to $D=25$ and hence $Q(B_5)$ is a polynomial of order $D-1=24$.

### III. CONCLUSION

Let us conclude with the main idea of this work by the aid of a beautiful example. One could say, for instance, that Bott theorem and also the factorization theorem given above say the same thing. Although this theorem proves useful as a calculational tool, Bott theorem gives us a general framework to apply for affine Lie algebras due to the fact...
that explicit calculation of a rational function is in fact quite hard if it is not impossible. We note again that a rational function can be expressed only in the form of a polynomial of infinite order.

In the lack of such a general framework for Lie algebras beyond affine ones, we also use an algorithm for explicit calculations. Against the above-mentioned theorem, explicit calculations are possible here due to the fact that in our formalism we only deal with polynomials of some finite degree.

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