The semiclassical limit of chaotic eigenfunctions

Eduardo G. Vergini

Departamento de Física, Comisión Nacional de Energía Atómica. Av. del Libertador 8250, 1429 Buenos Aires, Argentina.

(March 30, 2022)

A generic chaotic eigenfunction has a non-universal contribution consisting of scars of short periodic orbits. This contribution, which can not be explained in terms of random universal waves, survives the semiclassical limit (when $\hbar$ goes to zero). In this limit, the sum of scarred intensities is a simple function of $\eta \equiv \sqrt{\pi/2} (f-1) h_T^{-1} \lambda_2^{1/2}$, with $f$ the degrees of freedom, $h_T$ the topological entropy and $\{\lambda_i\}$ the set of positive Lyapunov exponents. Moreover, the fluctuations of this representation go to zero as $1/|\ln \hbar|$. For this reason, we will be able to provide a detailed description of a generic chaotic eigenfunction in the semiclassical limit.

PACS numbers: 05.45.+b, 03.65.Sq, 03.20.+i

Berry \cite{1} and Voros \cite{2} proposed a semiclassical description of chaotic eigenfunctions by considering the surface of constant energy as the unique classical invariant able to support them. This would be the case if the time required for the definition of individual eigenfunctions was infinite. However, the required Heisenberg time $T_H$ is finite for finite values of $\hbar$, even though it goes to infinity in the semiclassical limit (when $\hbar$ goes to zero).

Recently, we have derived a semiclassical theory which describes individual eigenfunctions in terms of short periodic orbits (POs) \cite{3–5}. The number of POs is such that the sum of their periods is around $T_H$. Using these invariants, we will give a description of chaotic eigenfunctions with fluctuations going to zero with $\hbar$.

It is worth to emphasize that the fluctuations of the Berry-Voros description go to infinity in the semiclassical limit (SL). But this statement demands explanation because it is common to find in the literature expressions like "...this description is supported by Shnirelman theorem.". For example, the peaks of the Husimi distribution of a chaotic eigenfunction have very strong fluctuations with respect to their classical ergodic measure; by assuming a random wave hypothesis they go logarithmically to infinity as $\hbar \to 0 \ \text{\cite{3}}$. However, each of the peaks occupy a volume (the semiclassical volume of a quantum state) going to zero with $\hbar$. Then, a smoothing on a region independent of $\hbar$, a classical smoothing, is sufficient for washing out fluctuations in the SL. In this context, Shnirelman theorem \cite{6} may be expressed as follows: a classical smoothing is sufficient for the elimination of fluctuations (with respect to the classical ergodic measure) of generic chaotic eigenfunctions in the SL.

We are going to show that the highest peaks of the Husimi are not completely random. On the contrary, they live (within an error of order $\sqrt{\hbar}$ in each direction) on short POs, and their phases (considering in this case the corresponding Bargmann function \cite{7}) are connected semiclassically. That is, high fluctuations correspond with the scars of short POs introduced by Heller \cite{8} (although the meaning of short is other; see Eq. (6)).

In this respect, the following commonly used expression is definitely wrong: "...scarred eigenfunctions do not contradict Shnirelman theorem because they are of null measure.". Actually, scar phenomena are of a semiclassical nature (they manifest on scales going to zero with $\hbar$). Then again, a classical smoothing washes out these very rich structures in the SL.

Bogomolny \cite{9} and Berry \cite{10} incorporated the idea of scars into the description of chaotic eigenfunctions. These authors concluded that chaotic eigenfunctions consist of a dominant universal contribution, decorated by scars of short POs: but this result was obtained after making an average over an energy interval that also eliminates fluctuations. In our study, we consider individual eigenfunctions, without either a classical smoothing nor an energy averaging. Then, we arrive to the conclusion that a generic chaotic eigenfunction is scarred by a set of short POs which characterizes the state, and the sum of their intensities survives the SL. This fact stresses that localization on short POs is the signature of chaos in this limit. Moreover, we will give the mean value and dispersion of the scarred intensities showing that relative fluctuations go to zero as $1/|\ln \hbar|$, and providing a detailed description of a generic chaotic eigenfunction.

We have developed a systematic semiclassical construction of wave functions living in the neighbourhood of unstable POs. Reference \cite{8} provides the construction of resonances without transverse excitations and ref. \cite{11} applies the recipe to the Bunimovich stadium billiard. In ref. \cite{8} we construct resonances of a PO with transverse excitations, all at the same Bohr-Sommerfeld (BS) quantized energy. Then, scar functions are the result of a minimization of the energy dispersion in this basis. Finally, ref. \cite{11} gives the details for the construction of scar functions in the stadium billiard. Scar functions are the objects on which we are going to focus our description. We are thinking about structures living along the manifolds up to the first homoclinic point. Then, it is not necessary to include interference effects, and the description remains essentially simple.
Let $\gamma$ be an unstable PO of the system with period $T_\gamma$ and Lyapunov exponent $\lambda_\gamma$ per unit time (consider for the moment a conservative Hamiltonian system with two degrees of freedom). Let $\phi_\epsilon$ be the corresponding scar function with BS energy $E_\gamma$. Finally, let $\varphi_\mu$ be the set of normalized eigenfunctions of the system with eigenenergies $E_\mu$. In ref. [3] we have shown that the set of intensities $I_\mu = \langle \phi_\epsilon | \phi_\mu \rangle^2$ semiclassically satisfies, $\sum_\mu I_\mu = 1$, $\sum_\mu E_\mu I_\mu = E_\gamma$, and

$$\sigma_\gamma \equiv \sqrt{\sum_\mu (E_\mu - E_\gamma)^2 I_\mu} = h\lambda_\gamma \Gamma/2 \ [12].$$

In the SL, the universal dispersion $\Gamma$ goes to zero as $2\pi/|\ln h|$ (expressions of $\Gamma$ for finite values of $h$ are given in ref. [3]). Then, the life time $\tau_\gamma \equiv h/\sigma_\gamma$ of $\phi_\epsilon$ diverges logarithmically in the same way as the Ehrenfest time does (but they are different times). This extremely low decay [3] is governed by a Gaussian law $\langle (\phi_\epsilon(0) | \phi_\epsilon(t)) \rangle^2 = e^{-t^2/\tau_\gamma^2}$, and the smooth part of the intensities $I_\mu$ (the strength function) results a Gaussian function. Then, defining $\epsilon \equiv (E - E_\gamma)/\rho_E$ (with $\rho_E$ the energy density), the mean value of the intensities at $\epsilon$ is $I(\epsilon) = e^{-\epsilon^2/2n^2}/\sqrt{2\pi n^2}$, where $n \equiv \sigma_\gamma \rho_E$ is the mean number of eigenenergies contained in one energy dispersion.

Another consequence of the Gaussian decay is that Eq. (1) is also valid for systems with $f$ degrees of freedom if we replace $\lambda_\gamma$ by $\sum \lambda_j^{1/2}$, with $\{\lambda_j\}$, the set of $f - 1$ positive Lyapunov exponents of $\gamma$ (for an exponential decay the change would be $\lambda_\gamma$ by the K-S entropy ($\sum \lambda_j$)).

Now, we are able to establish a criterium in order to decide when a PO is short. We propose a relation of the form $T_\gamma < \beta \tau_\gamma$, with $\beta$ a constant to be determined from the theory of short POs. This theory says that in the SL an eigenfunction is defined by POs of period lower than $T_0 = h^{-1}\ln(T_0/h_T)$ [3], with $h_T$ the topological entropy. Then, in the SL $T_0$ would be equal to $\beta$ times $\tau$ (the life time of a generic scar function), with

$$\tau = \pi^{-1}(\sum \lambda_j^{1/2})^{-1/2} \ln h,$$  

(3)

$\{\lambda_j\}_{sys}$ is the set of positive Lyapunov exponents of the system. Finally, we say that a PO $\gamma$ is short when

$$T_\gamma / \tau_\gamma < \sqrt{2\pi} \eta \ "short \ PO \ condition",$$

(4)

where the classical invariant

$$\eta \equiv \sqrt{\pi/2(f-1)}h_T^{-1}(\sum \lambda_j^{1/2})^{1/2},$$

(5)

will play a central role below. Evidently, any PO is short at sufficiently high energies.

Of all the intensities of a given scar function we are actually interested in those with the highest values (they are certainly related to the phenomenon of localization on short POs). We are going to study the properties of these high intensities in terms of a statistical model. The main purpose is to estimate their mean values and dispersions, and to provide a range in the spectrum where they live.

The fluctuations of an intensity $I$ at $\epsilon$, around its mean value $I(\epsilon)$ (see Eq. (2)), will be described as usual [4] by a chi-squared distribution with one degree of freedom for systems with time reversal symmetry (appendix D treats systems without time reversal). Then, the probability of finding an intensity lower than $I$ (the accumulated probability) is given, for $a \equiv I/\Gamma(\epsilon) > 1$, by $F_a(I) = 1 - \sqrt{2/\pi} e^{-a/2} (1 - a^{-1} + 3a^{-2} + \ldots)$ . In order to obtain the distribution for an arbitrary intensity (independent of its position in the spectrum), we make an averaging; if $N$ is the number of intensities contributing to the scar function, $F(I) \equiv (1/N) \int_{-\pi/2}^{\pi/2} F_a(I) de$. This expression is computed in terms of Gaussian integrals after the following expansion: $e^{-a^2} = e^{-y^2} e^{2y^2/2a^2} (1 - y^2/4a^2 + \ldots)$, where $y \equiv \sqrt{\pi/2} n l$ (note that $e^2/a^2 = O(y^{-1})$). Then,

$$F(I) = 1 - \frac{\sqrt{2n}}{N} e^{-y} \left( 1 - \frac{9}{8y} + \frac{305}{128y^2} + \ldots \right).$$

(6)

Now, if $x_1$ is the greatest of the intensities, $x_2$ the second one and so on, the probability density of $x_j$ is given by Eq. (10). A simple estimation of $x_1$ is derived from $1 - F(x_j) = j/N$. With this in mind, we define the random variable $x_j$ by the relation

$$1 - F(x_j) = e^{-\gamma_j}(j/N).$$

(7)

Combining Eqs. (8) and (9), $x_j$ is given by

$$y_j \equiv \sqrt{\pi/2n} x_j \sim \alpha - \ln(\alpha + 9/8) + b + b^2/2,$$

(8)

with $\alpha \equiv x_j + \ln(\sqrt{2n}/j)$ and $b \equiv \ln(\alpha + 287/128)/(\alpha + 17/8)$. Equation (8) works for $\alpha > 1$.

Equation (9) gives the mean value $\sum \sigma_j$ and dispersion $\sigma_{x_j}$ of $x_j$. With these, the mean value of $y_j$ is obtained from Eq. (8) by setting $\sum \equiv \sum \ln(\sqrt{2n}/j)$ in place of $\alpha$ and adding to the rhs the term $c \equiv \sigma_2^2/2(\pi + 9/8)^2$ (because $\ln(\alpha + 9/8) \simeq \ln(\pi/2) - c$). On the other hand, the dispersions of $y_j$ and $x_j$ are equal to the leading order. Then, we arrive to the first conclusion: relative fluctuations go to zero in the SL as follows,

$$\sigma_{x_j}/x_j \sim 1/[\sqrt{j} \ln(\sqrt{2n}/j)] = O(1/|\ln h|).$$

(9)

The next question is to know where $x_j$ can be found. The probability $p(\epsilon)$ of finding $x_j$ near $\epsilon$ is proportional to the probability density $dF_x/d\epsilon$ at $x_j$; that is,

$$p(\epsilon) \propto e^{-(y_j + \epsilon^2/2n^2 + \ldots)}.$$

Then, $x_j$ is restricted to a range $\Delta \epsilon_j \sim \sigma_\epsilon/\sqrt{\epsilon_j + 1/2}$ around $E_\gamma$ which in units of $2\pi h/T_\gamma$ (the distance between consecutive BS quantized energies) goes to zero as $|\ln h|^{-3/2}$. In
In the vicinity of the motion and the energy dispersion results independent of the Lyapunov exponents of the orbits with \( \sigma \) concentrated around \( E \) with eigenenergy \( E \) (the criterium of Eq. (4)). Then, an eigenfunction is represented after an evolution equal to \( \tau \), where each step defines a universal wave, the number of required universal waves is

\[
N_u = T_H h_T . \tag{10}
\]

There is a way of verifying the accuracy of Eq. (10). In billiards, Birkhoff coordinates define the right Poincaré surface of section (because all classical or quantal information is contained on the boundary). The area of the section is \( 2\hbar k\mathcal{L} \), with \( \mathcal{L} \) the length of the boundary and \( k \) the eigenwave number. Then, \( N_u = k\mathcal{L}/\pi \), and assuming that also Eq. (14) is right, we arrive to the following expression for the topological entropy \( h'_T \) per bounce

\[
h'_T = \mathcal{L} \bar{l}/\pi \mathcal{A} , \tag{11}
\]

with \( \mathcal{A} \) the area of the billiard and \( \bar{l} \) the mean length per bounce. We have verified Eq. (14) within an error of 2% in the stadium billiard with radius unity and several different areas.

So far, we have described scar functions in the basis of eigenfunctions. Now according to the theory of short POs, eigenfunctions would be described equivalently in terms of scar functions. That is, imagine the spectrum of scar functions given by all BS energies of all short POs. This means for a given PO, all BS energies in those energy regions where \( \gamma \) is short (following the criterium of Eq. (4)). Then, an eigenfunction \( \varphi_\mu \) with eigenenergy \( E_\mu \) is represented in the basis of scar functions by the intensities \( I_\gamma = |\langle \varphi_\mu | \phi_\gamma \rangle|^2 \), which are concentrated around \( E_\mu \) in the spectrum of BS energies. The dispersion \( \sigma_\mu \equiv (\sum (E_j - E_\mu)^2 I_j)^{1/2} \) depends on the Lyapunov exponents of the orbits with BS energies in the vicinity of \( E_\mu \). However, in the SL there is a uniformization and the energy dispersion results independent of the position in the spectrum and equal to \( h/\tau \), with \( \tau \) given by Eq. (3). The smooth part of the intensities \( I_\gamma \) is given by Eq. (2), with

\[
n = \sigma h E = T_H/2\pi \tau , \tag{12}
\]

and \( \epsilon = (E - E_\mu) h E \). If \( x_1 \) is the highest of the intensities, \( x_3 \) the second one and so on, the mean value of \( x_j \) is given by Eq. (8) (and discussion thereafter), and the relative dispersion by Eq. (4). Of all short POs, those with possibility of having intensity \( x_j \) satisfy \( |E_\mu - E_\gamma| < \sigma/\sqrt{y_j + 1}^{1/2} \).

The following question is to decide a criterium for scarring. We will say that \( \varphi_\mu \) is scarred by \( \gamma \) if \( I_\gamma \) is greater than the greatest of the intensities provided by a random model of universal waves. The highest universal intensity is, to the leading order, \( I^{(u)} = 2 \ln(N_u)/N_u \). Then, using Eqs. (3), (10) and (12), the condition \( x_j > I^{(u)} \) reduces to

\[
y_j > \eta \ "scarring\ criterium" . \tag{13}
\]

The number \( n_{scar} \) of POs satisfying Eq. (13) is given by

\[
\frac{n_{scar}}{n} \simeq e^{-\eta} \sqrt{8 \ln(1 + 1/\delta \eta)} (1 + 4\eta/\delta) , \tag{14}
\]

with \( \delta = (9 + \sqrt{73})/2 \). This formula (and the next too) interpolates the behaviors for large \( \eta \) (obtained from Eqs. (6) and (7)) and \( \eta \) going to zero (see appendix C). Finally, the sum of scarred intensities results

\[
\sum_{j=1}^{n_{scar}} x_j \simeq \frac{2}{\sqrt{\pi}} \sqrt{e^{-\eta} - \left( \frac{2}{\sqrt{\pi}} - 1 \right) e^{-2\eta}} . \tag{15}
\]

We emphasize that the unexpected results of Eqs. (14) and (15) depend decisively on the use of scar functions in order to measure localization on short POs. On the contrary, by using wave packets in the transverse direction to the motion (the so called vacuum states in ref. [5]), there result \( n_{scar}/n = 0 \) and \( \sum x_j = 0 \) in the SL. In this respect, the departure of universal behavior found in ref. [18] has a weight going to zero in the SL.

We stress that all formulae of statistical nature verify an impressive agreement with numerical simulations. Moreover, Eqs. (3) and (5) were extensively verified in the stadium billiard. Finally, we present an example where all the ideas developed in the article have been tested. Figure 1 shows the decomposition of a very high excited chaotic state (plotted in configuration space in ref. [3]). We found 9 scarred short POs \( (n_{scar} \simeq 8.5 \) from Eq. (14)) providing a contribution of the 38% (Eq. (13) predicts 33%). Moreover, scarred intensities are in good agreement with predictions (see Fig. 1d)).

In conclusion, localization on short POs survives the SL and depends exclusively on \( \eta \). Evidently, this localization will be strong in systems with few degrees of freedom. In particular for \( f = 2 \), and assuming that \( \lambda \simeq h_T \), this localization results a universal property.
This work was partially supported by SETCYP-ECOS A98E03.

Appendix A. Let \( I_1, I_2, \ldots, I_N \) be a set of independent random variables, with common probability density \( f(I) \), living in the range \([0, \infty)\). Let \( x_1 \) be the greatest of the variables, \( x_2 \) the second one and so on. The joint probability density is given by \( p(x_1, x_2, \ldots, x_N) = N! f(x_1) f(x_2) \cdots f(x_N) \), for \( x_1 \geq x_2 \geq \ldots \geq x_N \); otherwise \( p = 0 \). Then, the probability density of \( x_j \) results (after integration over the others variables)

\[
p(x_j) = \frac{N! f(x_j) F(x_j)^{N-j} [1 - F(x_j)]^{j-1}}{(N-j)! (j-1)!},
\]

with \( F(x) = \int_0^x f(x') dx' \) the accumulated probability.

Appendix B. We will derive the mean value and dispersion of the random variable \( z_j \) defined in Eq. (6). Assuming that \( z_j = \mathcal{O}(1) \), there results from Eq. (7) \( 1 - F(x_j) = \mathcal{O}(j/N) \), and then \( F(x_j)^{N-j} = \exp \left( \int [-N(1 - F(x_j)) + \mathcal{O}(j^2/N)] \right) \). Moreover, \( N!/(N-j)! = N^j [1 + \mathcal{O}(j^2/N)] \). Using these approximations in Eq. (16), the probability density of \( z_j \) is given by

\[
p(z_j) \simeq j^2 e^{-j(z_j + e^{-z_j})}/(j-1)!
\]

For instance, a numerical computation gives \( \overline{z_j} \simeq 0.577 \) and \( \sigma_{z_j} \simeq 1.28 \). On the other hand, for large values of \( j \) we change to a new variable \( w = \sqrt{j} z_j \), and expanding in powers of \( 1/\sqrt{j} \) there results

\[
p(w) \simeq \frac{e^{-w^2/2}}{\sqrt{2\pi}} \left( 1 - \frac{1}{12j} \right) \left( 1 + a \frac{1}{\sqrt{j}} + b \frac{1}{j} + c \frac{1}{j^{3/2}} \right),
\]

with \( a = w^3/6 \), \( b = w^4(w^2 - 3)/72 \) and \( c = w^5(5w^4 - 45w^2 + 54)/6480 \). Finally, using this density we have

\[
\overline{z_j} \simeq \frac{1}{2j} + \frac{1}{12j^2} \quad \text{and} \quad \sigma_{z_j} \simeq \frac{1}{\sqrt{j}} \left( 1 + \frac{1}{4j} \right),
\]

in excellent agreement with numerical data.

Appendix C. For \( \eta \) going to zero, \( n_{\text{scar}} \) satisfies \( T(n_{\text{scar}}/2) = T(x) = \sqrt{2/\pi \eta}/n \). Then, using Eq. (3) there results \( n_{\text{scar}}/n \simeq 2^{3/2} \sqrt{\ln(1/\eta)} \).

Appendix D. The corresponding formulae for systems without time reversal are the following: \( y_j \equiv \sqrt{2\pi} n x_j \simeq \alpha - \ln(\alpha + 3/4)/2 + b + b^2 \), in place of Eq. (8), with \( \alpha \equiv z_j + \ln(\sqrt{2\pi} n/j) \) and \( b = \ln(\alpha + 31/16)/(4\alpha + 5) \). Equation (9) is the same. Equation (14) is replaced by

\[
n_{\text{scar}}/n \simeq e^{-\eta} \sqrt{2\pi \ln(1 + 1/\eta)},
\]

and Eq. (15) by

\[
\sum_{j=1}^{n_{\text{scar}}} x_j \simeq (1 + \eta) e^{-\eta} - \left( \frac{2}{\sqrt{3}} - 1 \right) e^{-2\eta}.
\]

---

[1] M. V. Berry, J. Phys. A 10, 2083 (1977).
[2] A. Voros, Annales de l’Institut Henri Poincaré A 24, 31 (1976); 26, 343 (1977).
[3] E. G. Vergini, J. Phys. A 33, 4709 (2000).
[4] E. G. Vergini and G. G. Carlo, J. Phys. A 33, 4717 (2000).
[5] E. G. Vergini and G. G. Carlo, J. Phys. A 34, 4525 (2001).
[6] S. Nonnenmacher and A. Voros, J. Stat. Phys. 92, 431 (1998).
[7] A. I. Shnirelman, Usp. Mat. Nauk 29, 181 (1974).
[8] E. J. Heller, Phys. Rev. Lett. 53, 1515 (1984).
[9] E. B. Bogomolny, Physica D 31, 169 (1988).
[10] M. V. Berry, Proc. R. Soc. Lond. A 423, 219 (1989).
[11] G. Carlo, E. Vergini and P. Lustemberg.
[12] Equation (6) is derived from Eq. (40) of ref. [1] by considering that the rate of divergence is uniform in time.
[13] With respect to the characteristic classical decay which is governed by the Lyapunov exponent of the system.
[14] C. E. Porter and R. G. Thomas, Phys. Rev. 104, 483 (1956).
[15] M. V. Berry, Proc. R. Soc. London. A 400, 229 (1985).
[16] In place of \( p_E \) we would set the density of BS energies, but these two densities are semiclassically the same. In fact, Eq. (6) was derived from such requirement.
[17] This result is obtained by the method developed in this article, setting \( T = 1/N_n \) in place of Eq. (4).
[18] L. Kaplan, Phys. Rev. Lett. 80, 2582 (1998).

---

**FIG. 1.** Linear Husimi density plots of: a) the state number 141,755 of the desymmetrized stadium billiard with radius 1 and area 1 + \( \pi/4 \), b) its non-universal contribution consisting of 9 scar functions, and c) its universal contribution consisting of 819 plane waves. d) The set of scarred intensities (dots) and the theoretical estimation curves \( \overline{x_j} \pm \delta x_j \); horizontal line displays the value of the highest universal intensity.