Surfaces of revolution satisfying $\Delta^{III}x = Ax$

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Abstract
We consider surfaces of revolution in the three-dimensional Euclidean space which are of coordinate finite type with respect to the third fundamental form $III$, i.e., their position vector $x$ satisfies the relation $\Delta^{III}x = Ax$, where $A$ is a square matrix of order 3. We show that a surface of revolution satisfying the preceding relation is a catenoid or part of a sphere.

Key Words: Surfaces in the Euclidean space, surfaces of coordinate finite type, Beltrami operator

MSC 2010: 53A05, 47A75

1 Introduction

Let $x = x(u^1, u^2)$ be a regular parametric representation of a surface $S$ in the Euclidean space $\mathbb{R}^3$ which does not contain parabolic points. For two sufficient differentiable functions $f(u^1, u^2)$ and $g(u^1, u^2)$ the first Beltrami operator with respect to the third fundamental form $III = e_{ij}du^idu^j$ of $S$ is defined by

$$\nabla^{III}(f, g) = e_{ij} f_{/i}g_{/j},$$

where $f_{/i} := \frac{\partial f}{\partial u^i}$ and $e_{ij}$ denote the components of the inverse tensor of $e_{ij}$. The second Beltrami differential operator with respect to $III$ is defined by

$$\Delta^{III} f = \frac{-1}{\sqrt{e}} (\sqrt{e} e_{ij} f_{/i})_{/j}$$

(1)

where $e := \det(e_{ij})$. In [5] we showed the relation

$$\Delta^{III}x = \nabla^{III}\left(\frac{2H}{K}n\right) - \frac{2H}{K}n,$$

(2)

with sign convention such that $\Delta = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$ for the metric $ds^2 = dx^2 + dy^2$. 

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where \( n \) is the unit normal vectorfield, \( H \) the mean curvature and \( K \) the Gaussian curvature of \( S \). Moreover we proved that a surface satisfying the condition

\[
\triangle^{III} x = \lambda x, \quad \lambda \in \mathbb{R},
\]

i.e., a surface \( S : x = x(u^1, u^2) \) for which all coordinate functions are eigenfunctions of \( \triangle^{III} \) with the same eigenvalue \( \lambda \), is part of a sphere (\( \lambda = 2 \)) or a minimal surface (\( \lambda = 0 \)). Using terms of B.-Y. Chen’s theory of finite type surfaces [1] the above result can be expressed as follows: A surface \( S \) in \( \mathbb{R}^3 \) is of \( \text{III-type} \) 1 (or of \( \text{null III-type} \) 1) if and only if \( S \) is part of a sphere (or a minimal surface).

In general a surface \( S \) is said to be of finite type with respect to the fundamental form \( III \) or, briefly, of finite \( III \)-type, if the position vector \( x \) of \( S \) can be written as a finite sum of nonconstant eigenvectors of the operator \( \triangle^{III} \), that is if

\[
x = c + x_1 + x_2 + \ldots + x_m, \quad \triangle^{III} x_i = \lambda_i x_i, \quad i = 1, \ldots, m,
\]

where \( c \) is a constant vector and \( \lambda_1, \ldots, \lambda_m \) are eigenvalues of \( \triangle^{III} \). When there are exactly \( k \) nonconstant eigenvectors \( x_1, \ldots, x_k \) appearing in (3) which all belong to different eigenvalues \( \lambda_1, \ldots, \lambda_k \), then \( S \) is said to be of \( \text{III-type} \) \( k \); when \( \lambda_i = 0 \) for some \( i = 1, \ldots, k \), then \( S \) is said to be of \( \text{null III-type} \) \( k \).

The only known surfaces of finite \( III \)-type are parts of spheres, the minimal surfaces and the parallel of the minimal surfaces (which are actually of null \( III \)-type 2, see [5]).

In this paper we want to determine the connected surfaces of revolution \( S \) in \( \mathbb{R}^3 \) which are of coordinate finite \( III \)-type, i.e., their position vectorfield \( x(u^1, u^2) \) satisfies the condition

\[
\triangle^{III} x = A x, \quad A \in M(3, 3),
\]

where \( M(m, n) \) denotes the set of all matrices of the type \( (m, n) \).

Coordinate finite type surfaces with respect to the first fundamental form \( I \) were studied in [2] and [3]. In the last paper O. Garay showed that the only complete surfaces of revolution in \( \mathbb{R}^3 \), whose component functions are eigenfunctions of their Laplacian are the catenoids, the spheres and the circular cylinders, while F. Dillen, J. Pas and L. Verstraelen proved in [2] that the only surfaces in \( \mathbb{R}^3 \) satisfying

\[
\triangle^I x = A x + B, \quad A \in M(3, 3), \quad B \in M(3, 1),
\]

are the minimal surfaces, the spheres and the circular cylinders.

Our main result is the following

**Proposition 1** A surface of revolution \( S \) satisfies [4] if and only if \( S \) is a catenoid or part of a sphere.
We first show that the mentioned surfaces indeed satisfy the condition (4).

A. On a catenoid the mean curvature vanishes, so, by virtue of (2), \( \Delta III x = 0 \). Therefore a catenoid satisfies (4), where \( A \) is the null matrix in \( M(3,3) \).

B. Let \( S \) be part of a sphere of radius \( r \) centered at the origin. Then \( H = \frac{1}{r}, \ K = \frac{1}{r^2}, \ n = \frac{1}{r} x \).

So, by (2), it is \( \Delta III x = 2x \). Therefore \( S \) satisfies (4) with \( A = 2I_3 \), where \( I_3 \) is the identity matrix in \( M(3,3) \).

2 Proof of the main theorem

Let \( C \) be the profile curve of a surface of revolution \( S \) of the differentiation class \( C^3 \). We suppose that (a) \( C \) lies on the \((x_1, x_3)\)-plane, (b) the axis of revolution of \( S \) is the \( x_3 \)-axis and (c) \( C \) is parametrized by its arclength \( s \). Then \( C \) admits the parametric representation

\[
r(s) = (f(s), 0, g(s)), \quad s \in J
\]

\((J \subset \mathbb{R} \text{ open interval})\), where \( f(s), g(s) \in C^3(J) \). The position vector of \( S \) is given by

\[
x(s, \theta) = (f(s) \cos \theta, f(s) \sin \theta, g(s)), \quad s \in J, \quad \theta \in [0, 2\pi].
\]

Putting \( f'(s) := \frac{df(s)}{ds} \) we have because of (c)

\[
f' \cdot g' \neq 0,
\]

Furthermore it is \( f' \cdot g' = 1 \) otherwise \( f = \text{const.} \) or \( g = \text{const.} \) and \( S \) would be a circular cylinder or part of a plane, respectively. Hence \( S \) would consist only of parabolic points, which has been excluded. In view of (5) we can put

\[
f' = \cos \varphi, \quad g' = \sin \varphi,
\]

where \( \varphi \) is a function of \( s \). Then the unit normal vector of \( S \) is given by

\[
n = (-\sin \varphi \cos \theta, \ -\sin \varphi \sin \theta, \ \cos \varphi).
\]

The components \( h_{ij} \) and \( e_{ij} \) of the the second and the third fundamental tensors in (local) coordinates are the following

\[
h_{11} = \varphi', \quad h_{12} = 0, \quad h_{22} = f \sin \varphi,
\]

\[
e_{11} = \varphi'^2, \quad e_{12} = 0, \quad e_{22} = \sin^2 \varphi,
\]

hence \( \frac{2H}{K} = h_{ij} e^{ij} = \frac{1}{\varphi} + \frac{f}{\sin \varphi} \). (8)
From (1) and (7) we find for a sufficient differentiable function \( u = u(s, \theta) \) defined on \( J \times [2\pi, 0) \)
\[
\Delta^{III} u = -\frac{u''}{\varphi^2} + \left( \frac{\varphi''}{\varphi^2} - \frac{\cos \varphi}{\sin \varphi} \right) \frac{u'}{\varphi} - \frac{u_{/\theta \theta}}{\sin^2 \varphi}.
\] (9)

Considering the following functions of \( s \)
\[
P_1 = R \sin \varphi - \frac{\cos \varphi}{\varphi} R', \quad P_2 = -R \cos \varphi - \frac{\sin \varphi}{\varphi} R',
\] (10)
where we have put for simplicity \( R := \frac{2H}{K} \), and applying (9) on the coordinate functions \( x_i, i = 1, 2, 3, \) of the position vector \( x \) we find
\[
\Delta^{III} x_1 = P_1 \cos \theta, \quad \Delta^{III} x_2 = P_1 \sin \theta, \quad \Delta^{III} x_3 = P_2.
\] (11)

So we have:

(a) The coordinate functions \( x_1, x_2 \) are both eigenfunctions of \( \Delta^{III} \) belonging to the same eigenvalue if and only if for some real constant \( \lambda \) holds
\[
\lambda f = R \sin \varphi - \frac{\cos \varphi}{\varphi} R'.
\]

(b) The coordinate function \( x_3 \) is an eigenfunction of \( \Delta^{III} \) if and only if for some real constant \( \mu \) holds
\[
\mu g = -R \cos \varphi - \frac{\sin \varphi}{\varphi} R'.
\]

We denote by \( a_{ij}, i, j = 1, 2, 3, \) the entries of the matrix \( A \). By using (11) condition (4) is found to be equivalent to the following system
\[
\begin{cases}
P_1 \cos \theta = a_{11} f \cos \theta + a_{12} f \sin \theta + a_{13} g \\
P_1 \sin \theta = a_{21} f \cos \theta + a_{22} f \sin \theta + a_{23} g \\
P_2 = a_{31} f \cos \theta + a_{32} f \sin \theta + a_{33} g.
\end{cases}
\] (12)

Since \( \sin \theta, \cos \theta \) and 1 are linearly independent functions of \( \theta \), we obtain from \( 12_3 \ a_{31} = a_{32} = 0 \). On differentiating \( 12_1 \) and \( 12_2 \) twice with respect to \( \theta \) we have
\[
\begin{align*}
P_1 \cos \theta &= a_{11} f \cos \theta + a_{12} f \sin \theta \\
P_1 \sin \theta &= a_{21} f \cos \theta + a_{22} f \sin \theta.
\end{align*}
\]
Thus \( a_{13} g = a_{23} g = 0 \), so that \( a_{13} \) and \( a_{23} \) vanish. The system \( 12 \) is equivalent to the following
\[
\begin{cases}
(P_1 - a_{11} f) \cos \theta - a_{12} f \sin \theta = 0 \\
(P_1 - a_{22} f) \sin \theta - a_{21} f \cos \theta = 0 \\
P_2 - a_{33} g = 0.
\end{cases}
\]

But \( \sin \theta \) and \( \cos \theta \) are linearly independent functions of \( \theta \), so we finally obtain \( a_{12} = a_{21} = 0, a_{11} = a_{22} \) and \( P_1 = a_{11} f \). Putting \( a_{11} = a_{22} = \lambda \) and \( a_{33} = \mu \) we see that the system \( 12 \) reduces now to the following equations
\[
P_1 = \lambda f, \quad P_2 = \mu g.
\] (13)
On account of (10) and (13) we are left with the system

\[
\begin{align*}
R &= \lambda f \sin \varphi - \mu g \cos \varphi \\
R' &= -\varphi \left(\lambda f \cos \varphi + \mu g \sin \varphi\right).
\end{align*}
\]  

(14)

On differentiating (14) with respect to \( s \) we find, by virtue of (6),

\[
R' = \frac{\lambda - \mu}{2} \sin \varphi \cos \varphi.
\]  

(15)

We distinguish the following cases:

**Case I.** Let \( \lambda = \mu \).

Then (15) reduces to \( R' = 0 \).

**Subcase Ia.** Let \( \lambda = \mu = 0 \). From (14) we obtain \( R = 0 \), i.e., \( H = 0 \). Consequently \( S \), being a minimal surface of revolution, is a catenoid.

**Subcase Ib.** Let \( \lambda = \mu \neq 0 \).

Then from (6), (14) and \( R' = 0 \) we have \( f \cdot f' + g \cdot g' = 0 \), i.e., \( (f^2 + g^2)' = 0 \). Therefore \( f^2 + g^2 = \text{const.} \) and \( S \) is obviously part of a sphere.

**Case II.** Let \( \lambda \neq \mu \). From (14), (15) we find firstly

\[
\frac{1}{\varphi} = 2 \frac{\lambda f \cos \varphi + \mu g \sin \varphi}{(\mu - \lambda) \sin \varphi \cos \varphi}.
\]  

(16)

From this and (8) we obtain

\[
R = \frac{\lambda + \mu}{(\mu - \lambda) \sin \varphi} f + \frac{2\mu}{(\mu - \lambda) \cos \varphi} g.
\]

Hence, by virtue of (14),

\[
a f + b g = 0,
\]  

(17)

where

\[
a = \lambda \sin \varphi + \frac{\lambda + \mu}{(\lambda - \mu) \sin \varphi}, \quad b = \frac{2\mu}{(\lambda - \mu) \cos \varphi} - \mu \cos \varphi.
\]  

(18)

We note that \( \mu \neq 0 \), since for \( \mu = 0 \) we have

\[
a = \frac{\lambda \sin^2 \varphi + 1}{\sin \varphi}, \quad b = 0,
\]

and relation (17) becomes

\[
\frac{\lambda \sin^2 \varphi + 1}{\sin \varphi} f = 0,
\]

whence it follows \( \lambda \sin^2 \varphi + 1 = 0 \), a contradiction.

On differentiating (17) with respect to \( s \) and taking into account (16) we obtain

\[
a \frac{f}{\sin \varphi} + b \frac{g}{\cos \varphi} = 0,
\]  

(19)
where
\[ a_1 = \lambda(\lambda - \mu)^2 \sin^4 \varphi + (\lambda - \mu)(\lambda \mu - \lambda^2 + 3\lambda + \mu) \sin^2 \varphi - (\lambda + \mu)(3\lambda - \mu), \quad (20) \]
\[ b_1 = \mu \left[ (\lambda - \mu)^2 \sin^4 \varphi + (\lambda - \mu)(\mu - \lambda + 4) \sin^2 \varphi - 2(\lambda + \mu) \right]. \quad (21) \]
By eliminating now the functions \( f \) and \( g \) from (17) and (19) and taking into account (18), (20) and (21) we find
\[ \lambda(\lambda - \mu)^2 \sin^4 \varphi + (\lambda - \mu)(\lambda \mu - \lambda^2 + 5\lambda + \mu - 2) \sin^2 \varphi + (\lambda + \mu)(\mu - 3\lambda + 4) = 0. \]
Consequently
\[ \lambda(\lambda - \mu)^2 = 0, \quad (\lambda - \mu)(\lambda \mu - \lambda^2 + 5\lambda + \mu - 2) = 0, \quad (\lambda + \mu)(\mu - 3\lambda + 4) = 0. \]
From the first equation we have \( \lambda = 0 \). Then, the other two become as follows
\[ \mu - 2 = 0, \quad \mu + 4 = 0, \]
which is a contradiction.

So the proof of the theorem is completed.

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