Some problems of the pQCD jet calculus

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Abstract

Some problems of the perturbative quantum chromodynamics (pQCD) jet calculus are discussed. The first one is related to the terminology of the order of calculation. Due to cancelation of LO and NLO terms in the ratio of mean multiplicities in gluon and quark jets \( r \) the nowadays obtained results about it should be called as 4NLO approximation. The second problem reveals itself in calculations where corrections to some values (in particular, to \( r' \)) are larger at present energies than lower order terms. Some characteristics which do not suffer from this deficiency are proposed. Next problem lies in interpretation of the negative values of cumulant moments which are considered as an indication to the replacement of attraction by repulsion in sets with definite particle contents. Finally, the problem of the generalization of QCD equations for generating functions is briefly discussed.

The numerous achievements of pQCD in prediction and description of properties of quark and gluon jets are well known and described in many review papers (see, e.g., [1, 2, 3, 4, 5]). Here, I would like to discuss some problems, related to these calculations and, often, left behind the scene.

First, let me remind some simplest definitions [6, 1] concerning jet multiplicities in QCD. The generating function \( G \) is defined by the formula

\[
G(y, u) = \sum_{n=0}^{\infty} P_n(y) u^n,
\]

where \( P_n(y) \) is the multiplicity distribution at the scale \( y = \ln(p\Theta/Q_0) = \ln(2Q/Q_0) \), \( p \) is the initial momentum, \( \Theta \) is the angle of the divergence of the jet (jet opening angle), assumed here to be fixed, \( Q \) is the jet virtuality, \( Q_0 = \text{const} \), \( u \) is an auxiliary variable.

The moments of the distribution are defined as

\[
F_q = \frac{\sum_n P_n n(n-1) \ldots (n-q+1)}{(\sum_n P_n n)^q} = \frac{1}{\langle n \rangle^q} \left. \frac{d^q G(y, u)}{du^q} \right|_{u=1},
\]

\[
K_q = \frac{1}{\langle n \rangle^q} \left. \frac{d^q \ln G(y, u)}{du^q} \right|_{u=1}.
\]

Here, \( F_q \) are the factorial moments, and \( K_q \) are the cumulant moments, responsible for total and genuine (irreducible to lower ranks) correlations, correspondingly. These moments are not independent. They are connected
by definite relations which can easily be derived from moments definitions in terms of the generating function:

\[ F_q = \sum_{m=0}^{q-1} C_m^{m} K_{q-m} F_m. \]  

(4)

The QCD equations for the generating functions are:

\[ G'_G = \int_0^1 dx K^G_G(x) \gamma_0^2[G_G(y + \ln x)G_G(y + \ln(1 - x)) - G_G(y)] \]
\[ + n_f \int_0^1 dx K^F_G(x) \gamma_0^2[G_F(y + \ln x)G_F(y + \ln(1 - x)) - G_F(y)], \]  

(5)

\[ G'_F = \int_0^1 dx K^F_F(x) \gamma_0^2[G_G(y + \ln x)G_F(y + \ln(1 - x)) - G_F(y)], \]  

(6)

where \( G'(y) = dG/dy \), \( n_f \) is the number of active flavours,

\[ \gamma_0^2 = \frac{2N_c \alpha_S}{\pi}, \]  

(7)

the running coupling constant in the two-loop approximation is

\[ \alpha_S(y) = \frac{2\pi}{\beta_0 y} \left( 1 - \frac{\beta_1}{\beta_0} \cdot \frac{\ln 2y}{y} \right) + O(y^{-3}), \]  

(8)

where

\[ \beta_0 = \frac{11N_c - 2n_f}{3}, \quad \beta_1 = \frac{17N_c^2 - n_f(5N_c + 3C_F)}{3}, \]  

(9)

the labels \( G \) and \( F \) correspond to gluons and quarks, and the kernels of the equations are

\[ K^G_G(x) = \frac{1}{x} - (1 - x)[2 - x(1 - x)], \]  

(10)

\[ K^F_F(x) = \frac{1}{4N_c}[x^2 + (1 - x)^2], \]  

(11)

\[ K^G_F(x) = \frac{C_F}{N_c} \left[ \frac{1}{x} - 1 + \frac{x}{2} \right], \]  

(12)

\( N_c=3 \) is the number of colours, and \( C_F = (N_c^2 - 1)/2N_c = 4/3 \) in QCD.

Herefrom, one can get equations for any moment of the multiplicity distribution both in quark and gluon jets. One should just equate the terms with the same powers of \( u \) in both sides of the equations. In particular, the equations for average multiplicities read

\[ \langle n_G(y) \rangle' = \int dx \gamma_0^2[K^G_G(x)\langle n_G(y + \ln x) \rangle + \langle n_G(y + \ln(1 - x)) - \langle n_G(y) \rangle \rangle \]
\[ + n_f K^F_F(x)\langle n_F(y + \ln x) \rangle + \langle n_F(y + \ln(1 - x)) - \langle n_G(y) \rangle \rangle], \]  

(13)
\begin{equation}
\langle n_F(y) \rangle' = \int dx \gamma_0^G K_F^G(x)(\langle n_G(y+\ln x) \rangle + \langle n_F(y+\ln(1-x)) - \langle n_F(y) \rangle \rangle).
\end{equation} (14)

Their solutions can be looked for as
\begin{equation}
\langle n_{G,F} \rangle \propto \exp(\int y \gamma_{G,F}(y')dy').
\end{equation} (15)

Using the perturbative expansion
\begin{equation}
\gamma_G \equiv \gamma = \gamma_0(1 - a_1 \gamma_0 - a_2 \gamma_0^2 - a_3 \gamma_0^3) + O(\gamma_0^4),
\end{equation} (16)
one gets the solution in the form \([3, 4, 8]\)
\begin{equation}
\langle n_{G,F} \rangle = A_{G,F} y^{-a_1 c^2} \exp(2c\sqrt{y} + \delta_{G,F}(y)),
\end{equation} (17)
where \(c = (4N_c/\beta_0)^{1/2}\),
\begin{equation}
\delta_G(y) = \frac{c}{\sqrt{y}}[2a_2 c^2 + \frac{\beta_1}{\beta_0^2}(\ln 2y + 2)] + \frac{c^2}{y}[a_3 c^2 - \frac{a_1 \beta_1}{\beta_0^2}(\ln 2y + 1)] + O(y^{-3/2}).
\end{equation} (18)

Usually, in place of \(\gamma_F\) the ratio of average multiplicities in gluon and quark jets
\begin{equation}
r = \frac{\langle n_G \rangle}{\langle n_F \rangle} = \frac{A_G}{A_F} \exp(\delta_G(y) - \delta_F(y))
\end{equation} (19)
is introduced, and its perturbative expansion
\begin{equation}\label{r}
r = r_0(1 - r_1 \gamma_0 - r_2 \gamma_0^2 - r_3 \gamma_0^3) + O(\gamma_0^4)
\end{equation} (20)
is used. The analytic expressions and numerical values of the parameters \(a_i, r_i\) for all \(i \leq 3\) have been calculated from the perturbative solutions of the above equations (the review is given in \([4]\)). Within these approximations the experimental data about mean multiplicity in \(e^+e^-\)-annihilation are well described as seen in Fig. 1 where the notation \(K \equiv 2A_F = 2A_G/r_0\) is used.. However the data about the ratio \(r\) can be described with much lower accuracy about 15% in such an analytical approach (see Fig. 2) even though each subsequent perturbative approximation improves the agreement.

However, one should mention here the quantitative fit provided by the computer solution of the equation \([3, 10]\). This poses the question about the accuracy of perturbative approximations for this particular characteristics and indicates that the higher order corrections are still comparatively large for this ratio up to the highest presently available energies. Let us also note that the exact solutions of these equations for fixed coupling constant were given in \([11, 12]\).

The relation between the anomalous dimensions \(\gamma\) of gluon and quark jets is
\begin{equation}
\gamma_F = \gamma - \frac{r'}{r},
\end{equation} (21)
where

\[ r' \equiv \frac{dr}{dy} = B r_0 r_1 \gamma_0^3 \left[ 1 + \frac{2r_2}{r_1} \gamma_0 + \left( \frac{3r_3}{r_1} + B_1 \right) \gamma_0^2 + O(\gamma_0^3) \right] \]  

(22)

with \( r_0 = \frac{N_c}{C_F} = 9/4 \); \( B = \beta_0 / 8N_c \); \( B_1 = \beta_1 / 4N_c \beta_0 \).

Thus

\[ \gamma_F = \gamma_0 [ 1 - a_1 \gamma_0 - (a_2 + Br_1) \gamma_0^2 - (a_3 + 2Br_2 + Br_1^2) \gamma_0^3 - (a_4 + B(3r_3 + 3r_2 r_1 + B_1 r_1 + r_1^3)) \gamma_0^4 ] \]  

(23)

In these expressions we meet with two problems.

- **Terminology.**

The two leading terms in the energy behaviour of quark and gluon jet multiplicities are absolutely the same as seen from Eq. (17) and cancel in their ratio \( r \) (19). Therefore this ratio is given by \( r_0 = 9/4 \) both in the leading (LO) and next-to-leading (NLO) approximations. Thus, the common notation DLA, which is used in Fig. 2 near the value \( r = 9/4 \), should be considered as LO+NLO-prediction of QCD for the ratio \( r \). Therefore, the term \( r_1 \gamma_0 \) in (21) describes 2NLO corrections to the anomalous dimension. However, in the literature, it is often called as a MLLA (NLO) term what is wrong. Nevertheless, namely such notation is commonly used in Figures. Here, in Fig. 2 we have used the notation with the letter \( r \) added at the end. It implies that, e.g., 3NLOr means that the term with \( \gamma_3^3 \) in the perturbative expansion of \( r \) has been taken into account but it corresponds to 4NLO-contribution to the anomalous dimension.

A misuse of the terminology for the anomalous dimensions \( \gamma \)'s and for the ratio \( r \) is clearly displayed in the explicit expression for \( \gamma_F \) (23). Its last 4NLO term contains \( a_4 \) which has not yet been calculated. Together with it, the contribution from \( r \) is present with all terms calculated already and containing \( r_i \) for \( i \leq 3 \) only. Thus, let us stress again that in this sense one should say that ”\( r_3 \)”-term in \( r \) corresponds to 4NLO contribution to the anomalous dimension of the quark jet even though it is proportional to \( \gamma_0^3 \) in the perturbative expansion of \( r \).

- **Calculations.**

The cancellation of two leading terms in the ratio \( r \) reveals itself also in the proportionality of the scale (energy) derivative \( r' \) to \( \gamma_0^3 \). Therefore it can be calculated up to the terms \( O(\gamma_0^5) \). The leading term is very small (about 0.02 at the \( Z^0 \)-resonance). Asymptotically, all corrections
vanish. However, at present energies of $Z_0$, they are still quite important. The second term in the brackets in (22) is larger than 1 since $2r_2/r_1 \approx 4.9$ and $\gamma \approx 0.45 - 0.5$. Even the third term is approximately about 0.4. The problem of convergence of the series at $Z_0$-energies and below becomes crucial. The derivative of the ratio $r$ (its energy slope) is very sensitive to high order perturbative corrections.

Therefore, it is desirable to use at present energies such characteristics which are less sensitive to these corrections. In particular, these corrections partially cancel in the ratio of derivatives (slopes)

$$r^{(1)} = \frac{\langle n_G \rangle'}{\langle n_F \rangle'}.$$  \hfill (24)

The same is true for the ratio of curvatures (or second derivatives)

$$r^{(2)} = \frac{\langle n_G \rangle''}{\langle n_F \rangle''}.$$  \hfill (25)

The QCD predictions for them

$$r < r^{(1)} < r^{(2)} < 2.25$$  \hfill (26)

were recently confirmed in experiment (see Figs. 3, 4 from [13]).

• Interpretation.

Another question I’d like to raise concerns physical interpretation of oscillations of cumulant moments in QCD which is not yet completely clarified. Usually exploited phenomenological distributions of the probability theory do not possess any oscillations. E.g., all cumulant moments of the Poisson distribution are identically zero. One interprets this as the absence of genuine correlations irreducible to the lower-rank correlations. For the negative binomial distribution one easily gets

$$H_q = \frac{K_q}{F_q} = \frac{2}{q(q + 1)} > 0.$$  \hfill (27)

Since $F_q$ are always positive according to their definition, this inequality implies the positive values of $K_q$.

In the leading order approximation, the gluodynamics equation for the generating function

$$[\log G(y)]'' = \gamma_0^2 (G(y) - 1)$$  \hfill (28)
transforms in the relation

\[ q^2 K_q = F_q \quad \text{or} \quad H_q = \frac{1}{q^2}. \quad (29) \]

However already in the next-to-leading order \( H_q \)-moments become negative with a minimum at the rank \( q_{\text{min}} \approx \frac{24}{11 \gamma_0} + 0.5 \approx 5 \) \([4]\). This minimum is rather stable. It slowly moves to higher ranks with energy increase and disappears in asymptotics as is required according to the formula \((29)\). At higher orders of the perturbative expansion, the oscillations of higher rank cumulant moments show up \([5]\). They are confirmed in experiment \([16, 17]\) (see Fig. 5).

Let me mention here that the plots of \( D_q = q^2 H_q \) instead of \( H_q \) would be even more instructive to reveal the oscillations. In this case they can be easily compared to the LO prediction according to which \( D_q^{LO} = 1 \). Also the comparison to results of the negative binomial distribution would become simplified. The plot of NBD results shows monotonic increase of \( D_q^{NBD} \) from 1 at \( q = 1 \) to 2 at \( q \to \infty \) which is significantly different from QCD oscillations.

Both the role of conservation laws and the changing character of the genuine correlations can be blamed as originating these oscillations. If the latter factor is important it would imply that attraction (clustering) is replaced by repulsion (and vice versa) in particle systems with different number of particles. It would be interesting to find other examples of such a behaviour in hadronic systems.

- **Generalization.**

Finally, there exists the problem of possible generalization of the equations for the generating functions. From one side, we understand that even if treated as kinetic equations these equations are limited by our ignorance of non-perturbative effects, simplified treatment of conservation laws etc. Some phenomenological attempts to avoid these limitations were attempted from the very beginning \([18, 19, 20]\). In \([18]\) it was proposed to treat hadronization of partons at the final stage of jet evolution in analogy with the ionization in electromagnetic cascades where it leads to their saturation and to the finite length of the shower. Three different stages of the cascade were considered in the modified kinetic equations proposed in \([19, 20]\). No quantitative results were, however, obtained.

The most successful modification of above equations was recently proposed \([21]\) in the framework of the dipole approach to QCD with more accurate kinematic bounds. It has been shown that the ratio \( r \) can be
obtained in good agreement with experimental data. Nevertheless, further study [22] of higher rank moments of the multiplicity distribution predicted by the modified equations has shown their extremely high sensitivity to higher orders of the perturbative expansion. As shown in Fig. 6, the moments diverge at high orders and the only trace of oscillations can be noticed in the changing signs of the moments of the subsequent ranks. The results become inconclusive. Thus no successful generalization is at work nowadays. Rather, the general trend shifted to the direct calculation of non-perturbative effects in some jet characteristics (see, e.g., [23, 24]).

At the same time, the success of numerical solutions of the existing equations [9, 10] raises the question if the generalization will give any other noticeable contribution and our failure to describe more precisely the ratio $r$ could be just due some defects of the purely perturbative expansion at available energies. More rigorous treatment of the numerical solutions of the equations should be done. Moreover, it was claimed recently [25] that the renormalization group improvement of the perturbative results gives rise to good description of experimental data.

In conclusion, I’d say that, even though some principal questions concerning the calculation of some properties of quark-gluon jets and the validity of QCD equations for the generating functions at higher orders are not yet resolved, the practical accuracy of the pQCD calculations is high enough, especially, in view of the rather large expansion parameter.

This work is supported by the RFBR grant 00-02-16101.

Figure captions.

Fig. 1. The energy dependence of average multiplicity of charged particles in $e^+e^-$-annihilation. The results of different fits according to formulas of perturbative QCD and of the Monte Carlo models are shown (the solid and dotted lines are the fits of formula (18) with one and two adjusted parameters, the dashed line is given by the HERWIG Monte Carlo model; the vertically shaded area indicates the gluon jet data multiplied by the theoretical value of the ratio $r$ [24]).

Fig. 2. The experimentally measured ratio $r$ of multiplicities in gluon and quark jets as a function of energy in comparison with the predictions of analytical QCD and of the Monte Carlo model HERWIG (different QCD approximations, described in this paper, as well as $r(\epsilon)$ with integration limits
$e^{-y}$ and $1-e^{-y}$ in Eqns (5), (6) are indicated at the corresponding lines).

Fig. 3. The ratio of the slopes of the energy dependences of mean multiplicities in gluon and quark jets according to experimental data and some theoretical calculations.

Fig. 4. The ratio of the curvatures of the energy dependences of mean multiplicities in gluon and quark jets according to experimental data and some theoretical calculations.

Fig. 5. The measured ratio $H_q$ of the cumulant and factorial moments oscillates as a function of the rank $q$ according to experimental data on multiplicity distributions of charged particles in $e^+e^-$-annihilation at the $Z^0$ energy (the inset in the upper right corner shows the data for the moments of the ranks 2, 3 and 4).

Fig. 6. The $H_q$-moments in the modified dipole approach [21, 22] drastically diverge at higher orders for large ranks $q$ with changing the sign at subsequent ranks.

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