AN AVERAGED CHOWLA AND ELLIOTT CONJECTURE ALONG INDEPENDENT POLYNOMIALS

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Abstract. We generalize a result of Matomäki, Radziwiłł, and Tao, by proving an averaged version of a conjecture of Chowla and a conjecture of Elliott regarding correlations of the Liouville function, or more general bounded multiplicative functions, with shifts given by independent polynomials in several variables. A new feature is that we recast the problem in ergodic terms and use a multiple ergodic theorem to prove it; its hypothesis is verified using recent results by Matomäki and Radziwiłł on mean values of multiplicative functions on typical short intervals. We deduce several consequences about patterns that can be found on the range of various arithmetic sequences along shifts of independent polynomials.

1. Introduction

Let $\lambda: \mathbb{N} \to \{-1, +1\}$ be the Liouville function which is defined to be $1$ on integers with an even number of prime factors, counted with multiplicity, and $-1$ elsewhere. A well known conjecture of Chowla [3] asserts that if $n_1, \ldots, n_\ell \in \mathbb{N}$ are distinct, then

$$\lim_{M \to \infty} \frac{1}{M} \sum_{m=1}^{M} \lambda(m) \lambda(m+n_1) \cdots \lambda(m+n_\ell) = 0.$$ 

The conjecture remains open even when $\ell = 1$. Very recently, a version involving logarithmic averages was established for $\ell = 1$ in [28] and an averaged form of Chowla’s conjecture was established in [26]. The latter implies that if $(M_k)_{k \in \mathbb{N}}$ is a subsequence of the positive integers such that for every $n = (n_1, \ldots, n_\ell) \in \mathbb{N}^\ell$ the correlation limit

$$C_{\text{lin}}(n) := \lim_{k \to \infty} \frac{1}{M_k} \sum_{m=1}^{M_k} \lambda(m) \lambda(m+n_1) \cdots \lambda(m+n_\ell)$$

exists, then the sequence $(C_{\text{lin}}(n))$ converges to 0 in uniform density (see Definition [4]). We extend this result to the case where the linear polynomials $n_1, \ldots, n_\ell$ are replaced by any collection of independent polynomials $p_1, \ldots, p_\ell: \mathbb{N}^r \to \mathbb{Z}$ where $\ell, r \in \mathbb{N}$. In particular, we show that the sequence $(C_{\text{pol}}(n))$, defined by

$$C_{\text{pol}}(n) := \lim_{k \to \infty} \frac{1}{M_k} \sum_{m=1}^{M_k} \lambda(m) \lambda(m+p_1(n)) \cdots \lambda(m+p_\ell(n)),$$ 

converges to 0 in uniform density. A particular case of interest is when $r = 1$ and $p_j(n) = n^j$ for $j = 1, \ldots, \ell$.

More generally, Elliott conjectured [1, 5, Conjecture II] that if the multiplicative functions $f_0, \ldots, f_\ell$ take values on the complex unit disc and are aperiodic (see Definition [1]), then for all distinct $n_1, \ldots, n_\ell \in \mathbb{N}$ we have

$$\lim_{M \to \infty} \frac{1}{M} \sum_{m=1}^{M} f_0(m) f_1(m+n_1) \cdots f_\ell(m+n_\ell) = 0.$$ 

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In [26] Theorem B.1] it was shown that in this generality the conjecture is false on a technicality and a modification of the conjecture was suggested where one works under the somewhat more restrictive assumption of strong aperiodicity (see Definition 3). An averaged form of the modified conjecture was proved in [26 Theorem 1.6] and we give a polynomial variant of this result in Theorem 2.1.

As was the case in [26], where correlations along linear shifts were studied, our argument is based on a recent result from [28] on averages of multiplicative functions on typical short intervals. But due to the polynomial nature of our problem, a serious additional difficulty appears, since Fourier transform techniques (and their higher order variants) are not suited for the study of polynomial shifts. To overcome this obstacle, we recast the problem in ergodic terms. Using a rather deep rooted result from ergodic theory (Theorem [3.3]) we deduce that the polynomial correlation sequences are controlled by some simpler linear ones which can be handled using number theoretic tools like those used in [26].

We use the previous results in order to deduce several consequences about patterns that can be found on the range of various arithmetic sequences along shifts of independent polynomials. For instance, we show that:

- For every strongly aperiodic multiplicative function $f$ with values in $\{-1, +1\}$, for all $n \in \mathbb{N}$ outside a set of natural density 0, each sign pattern of size $\ell + 1$ is taken by $f$ along progressions of the form $m, m + n, m + n^2, \ldots, m + n^\ell$, and in fact, with natural density (with respect to the variable $m$) that converges to $2^{-\ell+1}$ as $n \to \infty$.

- For all $b_i \in \mathbb{N}$, $a_i \in \{0, \ldots, b_i - 1\}$, $i = 0, \ldots, \ell$, and all $n \in \mathbb{N}$ outside a set of natural density 0, the set of $m \in \mathbb{N}$ for which the integers $m, m + n, m + n^2, \ldots, m + n^\ell$, have $a_0 \bmod b_0, \ldots, a_\ell \bmod b_\ell$ prime factors respectively is non-empty, and in fact its natural density converges to $(\prod_{j=0}^\ell b_j)^{-1}$ as $n \to \infty$.

In the next section, we give the precise statements of the results alluded to in the previous discussion and also give several relevant refinements and open problems.

2. Main results

2.1. Definitions and notation. In order to facilitate exposition, we introduce some definitions and notation. Throughout this article, for $N \in \mathbb{N}$ we let $[N] := \{1, \ldots, N\}$. If $A$ is a finite non-empty set, we let $E_{a \in A} := \frac{1}{|A|} \sum_{a \in A}$. Furthermore, with $\mathbb{P}$ we denote the set of prime numbers.

**Definition 1.** A function $f : \mathbb{N} \to \mathbb{C}$ is called *multiplicative* if

$$f(mn) = f(m)f(n) \text{ whenever } (m, n) = 1.$$  

It is called *completely multiplicative* if the previous identity holds for every $m, n \in \mathbb{N}$.

For convenience, we extend all multiplicative functions to $\mathbb{Z}$ by letting $f(-n) = f(n)$ and $f(0) = 0$, and let

$$\mathcal{M} := \{f : \mathbb{Z} \to \mathbb{C} \text{ multiplicative such that } |f(n)| \leq 1 \text{ for every } n \in \mathbb{Z}\}.$$ 

A *Dirichlet character*, typically denoted with $\chi$, is a periodic completely multiplicative function such that $\chi(1) = 1$.

We say that $f \in \mathcal{M}$ is *aperiodic* if $\lim_{N \to \infty} \mathbb{E}_{n \in [N]} f(an + b) = 0$ for every $a, b \in \mathbb{N}$, equivalently, if $\lim_{N \to \infty} \mathbb{E}_{n \in [N]} f(n) \chi(n) = 0$ for every Dirichlet character $\chi$.

We define the distance between two multiplicative functions as in [12, 13]:

\footnote{Moreover, the recent concatenation techniques for Gowers norms from [30, 31] are devised to treat different averages that are taken jointly on the parameters $m$ and $n$; they also apply to different classes of polynomials (see Section 2.4).}
Remarks.

• Mialea admits correlations along the sequence of intervals density linearly independent. Our main result is the following: equivalent conditions hold:

(i) For every \( \epsilon > 0 \) the set \( \{ \mathbf{n} \in \mathbb{N}^r : |a(\mathbf{n}) - c| \geq \epsilon \} \) has uniform density 0;

(ii) \( \lim_{N \to \infty} \mathbb{E}_{\mathbf{n} \in I_N} |a(\mathbf{n}) - c| = 0 \) for every sequence of parallelepipeds \( (I_N)_{N \in \mathbb{N}} \) in \( \mathbb{N}^r \) with side lengths tending to \( \infty \).

Definition 2. If \( f, g \in \mathcal{M} \) we let \( \mathbb{D} : \mathcal{M} \times \mathcal{M} \to [0, \infty] \) be given by

\[
\mathbb{D}(f, g)^2 := \sum_{p \in \mathbb{P}} \frac{1}{p} \left(1 - \Re \left( f(p)g(p) \right) \right).
\]

We also let \( \mathbb{D} : \mathcal{M} \times \mathcal{M} \times \mathbb{N} \to [0, \infty) \) be given by

\[
\mathbb{D}(f, g; N)^2 := \sum_{p \in \mathbb{P} \cap [N]} \frac{1}{p} \left(1 - \Re \left( f(p)g(p) \right) \right)
\]

and \( \mathbb{M} : \mathcal{M} \times \mathbb{N} \to [0, \infty) \) be given by \( \mathbb{M}(f; N) := \min_{|t| \leq N} \mathbb{D}(f, n^t; N)^2 \).

A celebrated theorem of Halász [17] states that if \( f \in \mathcal{M} \), then it has zero mean value if and only if for every \( t \in \mathbb{R} \) we either have \( \mathbb{D}(f, n^t) = \infty \) or \( f(2^k) = -2^{kt} \) for all \( k \in \mathbb{N} \).

For our purposes, we need information on averages of multiplicative functions taken on typical short intervals. One such result is Theorem 1.6 below and its assumption motivates the following definition:

Definition 3. The multiplicative function \( f \in \mathcal{M} \) is strongly aperiodic if \( (f \cdot \chi; N) \to \infty \) as \( N \to \infty \) for every Dirichlet character \( \chi \).

Note that strong aperiodicity implies aperiodicity. The converse is not in general true (see [26, Theorem B.1]), but it is true for real valued multiplicative functions (see [26, Appendix C]). In particular, the Liouville and the Möbius function are strongly aperiodic. More examples of strongly aperiodic multiplicative functions are given in Corollary 1.2.

A subset \( Z \) of \( \mathbb{N}^r \) has natural density 0 if \( \lim_{N \to \infty} \frac{|Z \cap [0, N]^r|}{N^r} = 0 \), and uniform density 0 if \( \lim_{(|t|, |s|) \to \infty} \frac{|Z \cap [t, s]^r|}{|t - s|^r} = 0 \), where the last limit is taken over all parallelepipeds in \( \mathbb{N}^r \) with side lengths tending to \( \infty \). We make frequent use of the notion of convergence in uniform density which is defined as follows:

Definition 4. Let \( r \in \mathbb{N} \). We say that the sequence \( a \in \ell^\infty(\mathbb{N}^r) \) converges in uniform density to a constant \( c \in \mathbb{C} \), and write UD-lim_{n \to \infty} a(\mathbf{n}) = c, \] if any of the following two equivalent conditions hold:

(i) For every \( \epsilon > 0 \) the set \( \{ \mathbf{n} \in \mathbb{N}^r : |a(\mathbf{n}) - c| \geq \epsilon \} \) has uniform density 0;

(ii) \( \lim_{N \to \infty} \mathbb{E}_{\mathbf{n} \in I_N} |a(\mathbf{n}) - c| = 0 \) for every sequence of parallelepipeds \( (I_N)_{N \in \mathbb{N}} \) in \( \mathbb{N}^r \) with side lengths tending to \( \infty \).

Remark. If we replace the uniform density with the natural density, then the second condition becomes \( \lim_{N \to \infty} \mathbb{E}_{\mathbf{n} \in I_N} a(\mathbf{n}) = c \). Anyway, \( a(\mathbf{n}) = c \) for some \( Z \subset \mathbb{N}^r \) with natural density 0.

Lastly, if \( \mathcal{M} := \{ (M_k)_{k \in \mathbb{N}} \} \) is a sequence of intervals with \( M_k \to \infty \) and \( a \in \ell^\infty(\mathbb{N}) \), we let \( \mathbb{E}_{m \in \mathcal{M}} a(m) := \lim_{k \to \infty} \mathbb{E}_{m \in M_k} a(m) \) assuming that the limit exists.

2.2. Averaged Chowla-Elliott conjecture along independent sequences. We say that the polynomials \( p_1, \ldots, p_r : \mathbb{N}^r \to \mathbb{Z} \) are independent if the set \( \{1, p_1, \ldots, p_r\} \) is linearly independent. Our main result is the following:

Theorem 2.1. Let \( p_1, \ldots, p_r : \mathbb{N}^r \to \mathbb{Z} \) be independent polynomials and \( f_0, \ldots, f_r \in \mathcal{M} \) be multiplicative functions at least one of which is strongly aperiodic. Then there exists a sequence of intervals \( \mathcal{M} := \{ (M_k)_{k \in \mathbb{N}} \} \) with \( M_k \to \infty \) such that

\[
\text{UD-lim}_{n \to \infty} \left( \mathbb{E}_{m \in \mathcal{M}} f_0(m) f_1(m + p_1(\mathbf{n})) \cdots f_r(m + p_r(\mathbf{n})) \right) = 0.
\]

Remarks. • It suffices to choose \( (M_k)_{k \in \mathbb{N}} \) so that the multiplicative functions \( f_0, \ldots, f_r \in \mathcal{M} \) admit correlations along the sequence of intervals \( \mathcal{M} \) (see Definition 3). Hence, we can choose \( (M_k)_{k \in \mathbb{N}} \) to be a subsequence of any strictly increasing sequence of integers.

• Theorem 1.6 in [26] establishes (a quantitative variant of) this result for the polynomials \( p_1(n) = n_1, \ldots, p_r(n) = n_r \). As noted in the introduction, in order to establish the
polynomial extension we lend tools from ergodic theory, in particular, we use Theorem 3.5 below.

- It is not hard to modify our argument in order to prove a similar conclusion for averages of the form $\mathbb{E}_{m \in \mathcal{M}} f_0(a_0m + c_0) f_1(a_1m + p_1(n)) \cdots f_\ell(a_\ell m + p_\ell(n))$ where the polynomials $p_1, \ldots, p_\ell$ are independent, $a_0, \ldots, a_\ell \in \mathbb{Z}$, and $c_0 \in \mathbb{Z}$.

In a similar fashion, using Theorem 3.5 as our ergodic input we prove the following:

**Theorem 2.2.** Let $c_1, \ldots, c_\ell \in \mathbb{R}^+$ be distinct non-integers and $f_0, \ldots, f_\ell \in \mathcal{M}$. There exists a sequence of intervals $M := \{[M_k]\}_{k \in \mathbb{N}}$ with $M_k \to \infty$ such that:

(i) If for some $j \in \{0, \ldots, \ell\}$ we have $M(f_j; N) \to \infty$ as $N \to \infty$, then

$$\lim_{N \to \infty} \mathbb{E}_{n \in [N]} \mathbb{E}_{m \in \mathcal{M}} f_0(m) f_1(m + [n^{c_1}]) \cdots f_\ell(m + [n^{c_\ell}]) = 0.$$  

(ii) If for some $j \in \{0, \ldots, \ell\}$ we have that $f_j$ is strongly aperiodic, then

$$\lim_{N \to \infty} \mathbb{E}_{n \in [N]} \mathbb{E}_{m \in \mathcal{M}} f_0(m) f_1(m + [n^{c_1}]) \cdots f_\ell(m + [n^{c_\ell}]) = 0.$$  

**Remarks.**
- It is not hard to modify the proof of part (ii) in order to allow some of the exponents $c_1, \ldots, c_\ell$ to be integers.
- Using the ergodic theorems [6, Theorem 2.6] and [7, Theorem 2.3] one can substitute the sequences $n^{c_1}, \ldots, n^{c_\ell}$ in the previous statement with collections of sequences having different growth rates taken from a much larger class of Hardy field sequences.

2.3. **Patterns in arithmetic sequences.** Results regarding sign patterns of arithmetic sequences and in particular of the Liouville function $\lambda$ or the Möbius function $\mu$ have a rich history. For instance, it is known that (consecutive) values of $\lambda$ attain all possible sign patterns with size two [18] and size three [19] infinitely often, and in fact with positive lower density [27]. A similar result is known for $\mu$ but only for patterns of size two [27]. On the other hand, it is not known that four consecutive ones are taken by $\lambda$ infinitely often. A partial result for patterns of longer size is that $\lambda$ takes at least $k + 5$ of the possible $2^k$ sign patterns of size $k$ with positive upper density [27, Proposition 2.9]. Of course, Chowla’s conjecture predicts that $\lambda$ takes all possible sign patterns of size $k$ with natural density $2^{-k}$.

On a different direction, it is known that $\lambda$ takes infinitely often all sixteen sign patterns along four term arithmetic progressions $m, m + n, m + 2n, m + 3n$ where $m, n \in \mathbb{N}$ and $n$ belongs to a finite set [2], and $\lambda$ takes all possible $2^{\ell+1}$ sign patterns along arithmetic progressions $m, m + n, \ldots, m + \ell n$; each pattern for a proportion of $(m, n) \in [N] \times [N]$ that converges to $2^{-\ell+1}$ as $N \to \infty$ [14, Proposition 9.1]. Using [8, Theorem 1.1] we get similar results for any aperiodic multiplicative function with values in $\{-1, +1\}$.

Using Theorem 2.1 we get consequences about sign patterns of $\lambda$ and $\mu$ of arbitrary size along polynomial progressions given by independent polynomials. To state our results it is convenient to introduce some notation. Given a sequence of intervals $\mathcal{M} := \{[M_k]\}_{k \in \mathbb{N}}$ with $M_k \to \infty$ we denote with $d_{\mathcal{M}}$ the natural density induced by the sequence of intervals $\mathcal{M}$, that is, for $\Lambda \subset \mathbb{N}$ we let $d_{\mathcal{M}}(\Lambda) := \lim_{k \to \infty} \frac{|\Lambda \cap [M_k]|}{M_k}$, assuming that the limit exists. Moreover, we define the upper density of a set $\Lambda \subset \mathbb{N}$ as $\overline{d}(\Lambda) := \limsup_{M \to \infty} \frac{|\Lambda \cap [M]|}{M}$.

**Theorem 2.3.** Let $p_1, \ldots, p_\ell : \mathbb{N}^\ell \to \mathbb{Z}$ be independent polynomials and $f_0, \ldots, f_\ell : \mathbb{Z} \to \{-1, +1\}$ be strongly aperiodic multiplicative functions. Furthermore, for $n \in \mathbb{N}^\ell$ and $\epsilon_0, \ldots, \epsilon_\ell \in \{-1, +1\}$, let $\epsilon := (\epsilon_0, \ldots, \epsilon_\ell)$ and

$$\Lambda_{n, \epsilon} := \{n \in \mathbb{N} : f_0(m) = \epsilon_0, f_1(m + p_1(n)) = \epsilon_1, \ldots, f_\ell(m + p_\ell(n)) = \epsilon_\ell\},$$

Then there exists a sequence of intervals $\mathcal{M} := \{[M_k]\}_{k \in \mathbb{N}}$ with $M_k \to \infty$ such that

$$\text{UD-lim}_{n \to \infty}(d_{\mathcal{M}}(\Lambda_{n, \epsilon})) = 2^{-(\ell+1)}.$$
Remark. It follows that for every \( \varepsilon > 0 \) we have \( \overline{d}(\Lambda_{n,a}) \geq 2^{-(\ell+1)} - \varepsilon \) outside a set of \( n \in \mathbb{N}^r \) of uniform density 0.

Note that even when \( f_0 = \cdots = f_\ell = \lambda \), \( r = 1 \), and \( p_j(n) = n^j \), \( j = 1, \ldots, \ell \), it is non-trivial to establish that the set \( \Lambda_{n,a} \) is non-empty for all choices of sign patterns.

Given \( a, b \in \mathbb{N} \), with \([a]\), we denote the unique integer in \( \{0, \ldots, b - 1\} \) which is congruent to \( a \mod b \). We denote with \( \omega(n) \) the number of distinct prime factors of an integer \( n \) and with \( \Omega(n) \) the number of prime factors of \( n \) counted with multiplicity.

Theorem 2.4. Let \( p_1, \ldots, p_\ell : \mathbb{N}^r \to \mathbb{Z} \) be independent polynomials and \( b_0, \ldots, b_\ell \in \mathbb{N} \). For \( n \in \mathbb{N}^r \) and \( a_j \in \{0, \ldots, b_j - 1\} \), \( j = 0, \ldots, \ell \), let \( a := (a_0, \ldots, a_\ell) \) and

\[
\Lambda_{n,a} := \{ m \in \mathbb{N} : [\omega(m)]_{b_0} = a_0, [\omega(m + p_1(n))]_{b_1} = a_1, \ldots, [\omega(m + p_\ell(n))]_{b_\ell} = a_\ell \}.
\]

Then there exists a sequence of intervals \( M := ([M_k])_{k \in \mathbb{N}} \) with \( M_k \to \infty \) such that

\[
\text{UD-lim}_{n \to \infty}(d_M(\Lambda_{n,a})) = \left( \prod_{j=0}^\ell b_j \right)^{-1}.
\]

Remark. It follows that for every \( \varepsilon > 0 \) we have \( \overline{d}(\Lambda_{n,a}) \geq (\prod_{j=0}^\ell b_j)^{-1} - \varepsilon \) outside a set of \( n \in \mathbb{N}^r \) of uniform density 0.

With minor modifications one can prove variants of the previous statement where in the definition of the set \( \Lambda_{n,a} \) one uses the arithmetic function \( \Omega \) in the place of \( \omega \) or uses \( \omega \) and \( \Omega \) in different places.

Moreover, using part (ii) of Theorem 2.1 in place of Theorem 2.4 one can get variants of the previous results (with essentially the same proof) where the independent polynomials are substituted by sequences given by the integer part of different fractional powers and the limit in uniform density is substituted by the limit in natural density, defined by \( \text{D-lim}_{n \to \infty}(a(n)) = c \) if \( \lim_{N \to \infty} \mathbb{E}_{n \in [N]}|a(n) - c| = 0 \).

2.4. Further remarks and open problems. It is natural to ask for extensions of Theorem 2.4 which cover families of polynomials that are dependent and have pairwise non-constant differences. In this direction, even showing that

\[
\lim_{N \to \infty} \mathbb{E}_{n \in [N]}(\mathbb{E}_{m \in M} \lambda(m) \lambda(m + n) \lambda(m + 2n)) = 0
\]

for some sequence of intervals \( M := ([M_k])_{k \in \mathbb{N}} \) with \( M_k \to \infty \) remains a challenge. In order to have access to ergodic methodology one needs to prove a variant of Proposition 5.1 below with an assumption on \( \lambda \) that enables to deduce orthogonality of the function \( F = F_\lambda \) to a factor (the \( \mathcal{Z}_1 \)-factor) which is in general larger than the Kronecker factor of the corresponding system. The number theoretic input on \( \lambda \) needed to establish such claim is the following asymptotic

\[
\lim \text{lim sup}_{M \to \infty} \text{lim sup}_{n \in [N]} |\mathbb{E}_{m \in [M]} \mathbb{E}_{n \in [N]} \lambda(m + n) \cdot e(nt)| = 0.
\]

This is not known yet, and the reader can consult [28] [29] for further consequences in case (2) holds (it implies the logarithmically averaged Chowla conjecture for three point correlations). Extending Theorem 2.4 to general families of polynomials with pairwise non-constant differences would require a higher order variant of (2) with nilsequences of bounded complexity used in place of the linear exponential sequences.

In a rather different direction, one can study correlation sequences by taking averages simultaneously on the parameters \( m \) and \( n \). For instance, it was shown in [14] Proposition 9.1], modulo conjectures which were subsequently verified in [15] [16], that

\[
\lim_{N \to \infty} \mathbb{E}_{n \in [N]} \mathbb{E}_{m \in [N]} f(m) f(m + p_1(n)) \cdots f(m + p_\ell(n)) = 0,
\]
where \( f \) is the Möbius or the Liouville function and \( p_1, \ldots, p_\ell \) are linear polynomials with pairwise non-constant differences. In [8, 9] this is extended to the case where \( f \) is an arbitrary aperiodic multiplicative function.

Moreover, the concatenation results from [30, 31], that allow to control local Gowers norms by global ones, can be used to show that

\[
\lim_{N \to \infty} \frac{1}{|A|} \sum_{a \in A} f(a) f(a + m_1) \cdots f(a + m_\ell) = 0,
\]

where \( f \) is the Möbius or the Liouville function, \( L_N \) is of the order \( N^{1/d} \) where \( d \) is the maximum degree of the polynomials \( p_i \), and the polynomials \( p_i \) have the property that \( p_i - p_j \) have degree exactly \( d \) for all \( i \neq j \) (some other cases were handled as well, see [31, Theorem 2.5]). If one combines this with the fact that aperiodic multiplicative functions are Gowers uniform of arbitrary order [9, Theorem 2.5], it allows to extend this result to the class of all aperiodic multiplicative functions.

Extending (1) or (3) to general families of polynomials with pairwise non-constant differences, even when \( f \) is the Liouville function, \( \ell = 2, r = 1 \), remains a challenge.

2.5. **Notation and conventions.** For reader’s convenience, we gather here some notation that we use throughout the article. We denote by \( \mathbb{N} \) the set of positive integers and by \( \mathbb{P} \) the set of prime numbers. For \( N \in \mathbb{N} \) we let \([N] := \{1, \ldots, N\} \). If \( A \) is a finite non-empty set we let \( E_a \) := \( \frac{1}{|A|} \sum_{a \in A} \). We let \( e(t) := e^{2\pi it} \). We use the letter \( f \) to denote multiplicative functions and the letter \( \chi \) to denote Dirichlet characters.

3. **Key ingredients from number theory and ergodic theory**

In this section we gather the key ingredients from number theory and ergodic theory needed in the proof of our main results.

3.1. **Averages over typical short intervals.** The next result from [26] Theorem A.1] is the key ingredient in the proof of Theorem 4.1 below, which in turn is needed in the proof of Theorem 2.1.

**Theorem 3.1.** Let \( f \in \mathcal{M} \) be a multiplicative function such that \( M(f; N) \to \infty \) as \( N \to \infty \). Then

\[
\lim_{N \to \infty} \limsup_{M \to \infty} E_{m \in [M]} |E_{n \in [N]} f(m + n)| = 0.
\]

**Remarks.** The statement in [26] Theorem A.1] involves averages of the form \( E_{M \leq n < 2M} \), but the two statements are equivalent.

3.2. **An orthogonality criterion.** We will need the following qualitative variant of an orthogonality criterion of Kátai:

**Theorem 3.2** ([21]). Let \( a \in \ell^\infty(\mathbb{N}) \) be a sequence such that

\[
\lim_{N \to \infty} E_{n \in [N]} a(pn) \cdot a(qn) = 0
\]

for all primes \( p, q \) with \( p \neq q \). Then for every multiplicative function \( f \in \mathcal{M} \) we have

\[
\lim_{N \to \infty} E_{n \in [N]} f(n) \cdot a(n) = 0.
\]

3.3. **Ergodic theory, invariant and Kronecker factors.** A measure preserving system, or simply a system, is a quadruple \((X, \mathcal{X}, \mu, T)\), where \((X, \mathcal{X}, \mu)\) is a probability space and the transformation \( T : X \to X \) is invertible, measurable, and measure preserving (that is, \( \mu(T^{-1}A) = \mu(A) \) for every \( A \in \mathcal{X} \)).

Throughout, we denote with \( T^n, n \in \mathbb{N} \), the composition \( T \circ \cdots \circ T \), and for \( f \in L^\infty(\mu) \) we denote with \( TF \) the function \( F \circ T \).

The invariant factor of a system is defined to be the linear subspace of \( L^2(\mu) \) consisting of all \( T \)-invariant functions, that is, functions satisfying \( Tf = f \). The Kronecker
factor of a system is defined to be the closed linear subspace of $L^2(\mu)$ spanned by all $T$-eigenfunctions, that is, functions satisfying $Tf = e(\alpha)f$ for some $\alpha \in \mathbb{R}$. We will use the following two facts about these factors:

- A function $F \in L^2(\mu)$ is orthogonal to the invariant factor if and only if
  \[ \lim_{N \to \infty} \mathbb{E}_{n \in [N]} \int T^nF \cdot F d\mu = 0. \]

This is a direct consequence of the mean ergodic theorem.

- The function $F \in L^2(\mu)$ is orthogonal to the Kronecker factor if and only if its spectral measure is continuous, which, by Wiener’s theorem, happens exactly when
  \[ \lim_{N \to \infty} \mathbb{E}_{n \in [N]} \int T^nF \cdot F d\mu = 0. \]

3.4. Ergodic transference principle. We give a result that allows to recast convergent correlation sequences as ergodic correlation sequences.

**Definition 5.** We say that the sequences $a_1, \ldots, a_\ell \in \ell^\infty(\mathbb{Z})$ admit correlations along the sequence of intervals $M := ([M_k])_{k \in \mathbb{N}}$ with $M_k \to \infty$, if the limit
\[ \lim_{k \to \infty} \mathbb{E}_{n \in [M_k]} b_1(m + n_1) \ldots b_s(m + n_s) \]
exists for every $s \in \mathbb{N}$, all integers $n_1, \ldots, n_s$ (not necessarily distinct), and all sequences $b_1, \ldots, b_s$ that belong to the set $\{ a_1, \ldots, a_\ell, \bar{a}_1, \ldots, \bar{a}_\ell \}$.

**Remark.** If $a_1, \ldots, a_\ell \in \ell^\infty(\mathbb{Z})$, then using a diagonal argument we get that any sequence $(M_k)_{k \in \mathbb{N}}$ of integers with $M_k \to \infty$ has a subsequence $(M'_{k})_{k \in \mathbb{N}}$, such that the sequences $a_1, \ldots, a_\ell$ admit correlations along the intervals $([M'_k])_{k \in \mathbb{N}}$.

We use the following transference principle which can be thought of as a variant of Furstenberg’s correspondence principle for sequences:

**Proposition 3.3.** Suppose that the sequences $a_0, \ldots, a_\ell \in \ell^\infty(\mathbb{Z})$ admit correlations along the sequence of intervals $M := ([M_k])_{k \in \mathbb{N}}$ with $M_k \to \infty$. Then there exist a system $(X, X, \mu, T)$ and functions $F_0, \ldots, F_\ell \in L^\infty(\mu)$, such that
\[ \mathbb{E}_{m \in M} b_0(m) b_1(m + n_1) \ldots b_s(m + n_s) = \int \hat{F}_0 \cdot T^{n_1} \hat{F}_1 \cdots T^{n_s} \hat{F}_s d\mu \]
for every $s \in \mathbb{N}$, $n_1, \ldots, n_s \in \mathbb{Z}$, where for $j = 0, \ldots, s$ the sequence $b_j$ is equal to either $a_i$ or $\bar{a}_i$ for some $i \in \{0, \ldots, \ell\}$ and $\hat{F}_j$ is equal to $F_i$ or $\bar{F}_i$ respectively.

**Proof.** We let $D$ be the closed complex disc with radius $\max_{j=0,\ldots,\ell} \|a_j\|_\infty$, $Y := D^2$, and $X := Y^{\ell+1}$. We equip $Y$ and $X$ with the product topology and let $X$ be the Borel $\sigma$-algebra of $X$. We let $T_{sh} : Y \to Y$ and $T : X \to X$ be defined by $(T_{sh}x)(m) := x(m+1)$, $m \in \mathbb{Z}$, and $T(x_0, \ldots, x_\ell) := (T_{sh}x_0, \ldots, T_{sh}x_\ell)$ where $x_0, \ldots, x_\ell \in Y$. Also, for $j = 0, \ldots, \ell$ we define the functions $F_j \in C(X)$ by
\[ F_j(x_0, \ldots, x_\ell) := x_j(0). \]

Finally, we let
\[ \omega := (a_0, \ldots, a_\ell) \in X \]
and let $\mu$ be the weak$^*$ limit (which exists by our assumptions) of the sequence of probability measures
\[ \mu_k := \mathbb{E}_{m \in [M_k]} \delta_{T^m\omega}, \quad k \in \mathbb{N}. \]

\footnote{Then the limit $\lim_{k \to \infty} \mathbb{E}_{m \in [M_k]} F(b_1(m + n_1), \ldots, b_s(m + n_s))$ exists for every $s \in \mathbb{N}$, $n_1, \ldots, n_s \in \mathbb{Z}$, continuous function $F : D^s \to \mathbb{C}$ where $D$ is the closed complex disc with radius $\max_{j=1,\ldots,\ell} \|a_j\|_\infty$, and $b_j \in \{a_1, \ldots, a_\ell\}$ for $j = 1, \ldots, s$. This follows from the Stone-Weierstrass theorem.}
Note $T \in C(X)$, the measure $\mu$ is $T$-invariant, and for $j = 0, \ldots, \ell$, we have
\[ F_j(T^n \omega) = a_j(n), \quad n \in \mathbb{Z}. \]
Hence, assuming that $b_j$ and $\tilde{F}_j$, $j = 0, \ldots, \ell$, are as in the statement of the result, letting $n_0 = 0$, we have for all $n_1, \ldots, n_s \in \mathbb{Z}$ that
\[ E_{m \in M} \prod_{j=0}^{s} b_j(m + n_j) = E_{m \in M} \prod_{j=0}^{s} T^{n_j} \tilde{F}_j(T^m \omega) = \int \prod_{j=0}^{s} T^{n_j} \tilde{F}_j \, d\mu. \]
This completes the proof. \hfill \Box

3.5. Two multiple ergodic theorems. We will use the following result from ergodic theory:

**Theorem 3.4** ([11] for $r = 1$). Let $p_1, \ldots, p_r : \mathbb{N} \rightarrow \mathbb{Z}$ be independent polynomials, $(X, \mathcal{X}, \mu, T)$ be a system, and $F_0, \ldots, F_\ell \in L^\infty(\mu)$ be functions at least one of which is orthogonal to the Kronecker factor of the system. Then
\[ \text{UD-lim}_{n \to \infty} \left( \int F_0 \cdot T^{p_1(n)} F_1 \cdots T^{p_\ell(n)} F_\ell \, d\mu \right) = 0. \]

This result was proved for $r = 1$ in [11, Lemma 4.3]. The proof was based on the theory of characteristic factors [20, 24] and on qualitative equidistribution results on nilmanifolds from [22]. An intermediate result is the convergence result of [10] which can be proved for general $r \in \mathbb{N}$ in exactly the same way using equidistribution results for multi-variable sequences from [23] in place of the equidistribution results for single-variable sequences from [22]. For general $r \in \mathbb{N}$, the remaining part of the proof is essentially identical to the one given in [11, Lemma 4.3] for $r = 1$.

**Theorem 3.5** ([6]). Let $c_1, \ldots, c_\ell \in \mathbb{R}^+$ be distinct non-integers, $(X, \mathcal{X}, \mu, T)$ be a system, and $F_0, \ldots, F_\ell \in L^\infty(\mu)$ be functions.

(i) If one of the functions is orthogonal to the invariant factor of the system, then
\[ \lim_{N \to \infty} E_{n \in [N]} \int F_0 \cdot T^{(c_1 n)} F_1 \cdots T^{(c_\ell n)} F_\ell \, d\mu = 0. \]
(ii) If one of the functions is orthogonal to the Kronecker factor of the system, then
\[ \lim_{N \to \infty} E_{n \in [N]} \left| \int F_0 \cdot T^{(c_1 n)} F_1 \cdots T^{(c_\ell n)} F_\ell \, d\mu \right| = 0. \]

The first part is a direct consequence of [6, Theorem 2.6]. The second part follows by applying the first part for the product system $(X \times X, \mu \times \mu, T \times T)$ and the functions $F_j \otimes \tilde{F}_j$, $j = 0, \ldots, \ell$, and using the well known fact that if $F$ is orthogonal to the Kronecker factor of a system, then $F \otimes \overline{F}$ is orthogonal to the invariant factor of the product system.

4. A CONSEQUENCE OF STRONG APERIODICITY

The next result records a key asymptotic property of single correlations of strongly aperiodic multiplicative functions.

**Theorem 4.1** ([26]). Let $f \in M$ be a strongly aperiodic multiplicative function that admits correlations along the sequence of intervals $M := ([M_k])_{k \in \mathbb{N}}$ with $M_k \to \infty$. Then
\[ \lim_{N \to \infty} E_{n \in [N]} |E_{m \in M} f(m + n) \cdot \overline{f(m)}| = 0. \]

**Remark.** It follows from [26, Theorem B.1] that strong aperiodicity cannot be replaced by aperiodicity; in particular, there exist an aperiodic multiplicative function $f$, a positive constant $c$, and a sequence of intervals $M := ([M_k])_{k \in \mathbb{N}}$ with $M_k \to \infty$, such that
\[ |E_{m \in M} f(m + n) \cdot \overline{f(m)}| \geq c, \quad \text{for every } n \in \mathbb{N}. \]
A quantitative variant of Theorem 5.1 is implicit in [26]. Building on Theorem 3.1 we give in this section a self contained and rather simple proof of Theorem 5.1.

We start with the following “inverse theorem”:

**Lemma 4.2.** Let \( a \in \ell^\infty(\mathbb{N}) \) be a sequence that admits correlations along the sequence of intervals \( \mathcal{M} := ([M_k])_{k \in \mathbb{N}} \) and suppose that

\[
\limsup_{N \to \infty} \mathbb{E}_{n \in [N]} |\mathbb{E}_{m \in \mathcal{M}} a(m+n) \cdot \overline{a(m)}| > 0.
\]

Then there exists \( t \in \mathbb{R} \) such that

\[
\limsup_{N \to \infty} \limsup_{M \to \infty} \mathbb{E}_{m \in [M]} |\mathbb{E}_{n \in [N]} a(m+n) \cdot e(nt)| > 0.
\]

**Proof.** Using the Cauchy Schwarz inequality, our assumption gives that

\[
\lim_{N \to \infty} \mathbb{E}_{n \in [N]} \left( \mathbb{E}_{m \in \mathcal{M}} a(m+n) \cdot \overline{a(m)} \cdot c(n) \right) > 0
\]

where

\[
c(n) := \mathbb{E}_{m \in \mathcal{M}} \overline{a(m+n)} \cdot a(m).
\]

Hence,

\[
\limsup_{N \to \infty} \limsup_{k \to \infty} \mathbb{E}_{m \in [M_k]} |\mathbb{E}_{n \in [N]} a(m+n) \cdot c(n)| > 0.
\]

The sequence \( (c(n)) \) is positive definite, hence, by Herglotz’s theorem we have

\[
c(n) = \int c(nt) \, d\sigma(t), \quad n \in \mathbb{N},
\]

for some finite positive measure \( \sigma \) on \( \mathbb{T} \). We decompose \( \sigma \) to its discrete and continuous part, and deduce that \( c(n) \) can be expressed in the form

\[
c(n) = \sum_{k=1}^{\infty} c_k e(nt_k) + d(n)
\]

for some \( t_k \in \mathbb{T} \) and \( c_k \in \mathbb{R}^+ \), \( k \in \mathbb{N} \), such that \( \sum_{k=1}^{\infty} c_k < \infty \), and some sequence \( (d(n)) \) which, by Wiener’s theorem, converges to 0 in uniform density.

We deduce from this and (5) that there exists \( t \in \mathbb{R} \) such that

\[
\limsup_{N \to \infty} \limsup_{k \to \infty} \mathbb{E}_{m \in [M_k]} |\mathbb{E}_{n \in [N]} a(m+n) \cdot e(nt)| > 0,
\]

which implies the asserted claim. \( \square \)

Thus, in order to prove Theorem 5.1 it suffices to prove the following result:

**Proposition 4.3.** Let \( f \in \mathcal{M} \) be a strongly aperiodic multiplicative function. Then

\[
\lim_{N \to \infty} \limsup_{M \to \infty} \mathbb{E}_{m \in [M]} |\mathbb{E}_{n \in [N]} f(m+n) \cdot e(nt)| = 0 \quad \text{for every } t \in \mathbb{R}.
\]

**Proof.** The asserted claim is equivalent to

\[
\lim_{N \to \infty} \limsup_{M \to \infty} \mathbb{E}_{m \in [M]} |\mathbb{E}_{n \in [N]} f(m+n) \cdot e((m+n)t)| = 0 \quad \text{for every } t \in \mathbb{R}.
\]

Suppose first that \( t \in \mathbb{Q} \). Then using a standard argument it suffices to show that

\[
\lim_{N \to \infty} \limsup_{M \to \infty} \mathbb{E}_{m \in [M]} |\mathbb{E}_{n \in [N]} f(m+n) \cdot \chi(m+n)| = 0
\]

for every Dirichlet character \( \chi \). Since \( f \) is strongly aperiodic, we have that \( f \cdot \chi \) is also strongly aperiodic; hence, (7) is a direct consequence of Theorem 3.1 applied to \( f \cdot \chi \).

Suppose now that \( t \) is irrational. In this case, we will show that (6) holds for every multiplicative function \( f \in \mathcal{M} \). It is not hard to check (for details see the proof of Theorem 5 in [1]) that in place of (6) it suffices to show the following: If \( f \in \mathcal{M} \), \( t \) is
irrational, and \((b_k)_{k \in \mathbb{N}}\) is an increasing sequence of positive integers with \(b_{k+1} - b_k \to \infty\) as \(k \to \infty\), then
\[
\lim_{K \to \infty} \frac{1}{b_K} \sum_{k \in [K]} \sum_{n \in [b_k, b_{k+1})} f(n) \cdot e(nt) = 0.
\]
Equivalently, it suffices to show that for every sequence of complex numbers \((c_k)_{k \in \mathbb{N}}\) of modulus 1 we have
\[
\lim_{K \to \infty} \frac{1}{b_{K+1}} \sum_{k \in [K]} c_k \sum_{n \in [b_k, b_{k+1})} f(n) \cdot e(nt) = 0,
\]
or equivalently, that
\[
\lim_{K \to \infty} \frac{1}{b_K} \sum_{n \in [b_K]} f(n) \cdot a(n) = 0,
\]
where
\[
a(n) := \sum_{k=1}^{\infty} c_k 1_{[b_k, b_{k+1})}(n) \cdot e(nt).
\]
By Theorem 3.2 it suffices to show that for all distinct primes \(p, q\) we have
\[
\lim_{N \to \infty} \mathbb{E}_{n \in [N]} a(pn) \cdot \overline{a(qn)} = 0,
\]
or equivalently, that
\[
\lim_{N \to \infty} \mathbb{E}_{n \in [N]} e(ns) \cdot C(n) = 0,
\]
where \(s := (p - q)t\) is irrational and for \(k, k' \in \mathbb{N}\) we let
\[
I_{k,k'} := [b_k/p, b_{k+1}/p) \cap [b_{k'}/q, b_{k'+1}/q)
\]
and
\[
C(n) := \sum_{k,k' \in \mathbb{N}} 1_{I_{k,k'}}(n) \cdot c_k \cdot \overline{c_{k'}}.
\]
Note that the sequence of disjoint intervals \((I_{k,k'})_{k,k' \in \mathbb{N}}\) partitions \(\mathbb{N}\) into a set of density 0 and a sequence of intervals \((J_l)_{l \in \mathbb{N}}\) with \(|J_l| \to \infty\) as \(l \to \infty\). Therefore, letting
\[
C'(n) := \sum_{l=1}^{\infty} 1_{J_l}(n) \cdot d_l,
\]
where for \(l \in \mathbb{N}\) the complex number \(d_l\) is equal to \(c_k \cdot \overline{c_k'}\) for appropriate \(k, k' \in \mathbb{N}\), we have
\[
\lim_{N \to \infty} \mathbb{E}_{n \in [N]} |C(n) - C'(n)| = 0.
\]
Since \(s\) is irrational and \(|J_l| \to \infty\), we have
\[
\lim_{l \to \infty} \mathbb{E}_{n \in [J_l]} e(ns) = 0,
\]
\[
\lim_{N \to \infty} \mathbb{E}_{n \in [N]} e(ns) \cdot C'(n) = 0.
\]
This, combined with \((\ast)\), implies \((\ast\ast)\), and completes the proof. \(\square\)

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3 This identity was also proved in [3, Theorem 4] using a disjointness argument for topological models of irrational rotations. We give a simpler direct argument.
5. Averaged Chowla-Elliott conjecture along independent sequences

In this section we prove Theorems 2.1 and 2.2. Our approach is to translate the number theoretic statements to ergodic ones which we then verify using the ergodic Theorems 3.4 and 3.5. In order to show that the hypothesis of the ergodic theorems are satisfied we use the number theoretic feedback of Theorems 3.1 and 4.1.

**Working Assumptions.** Throughout this section, we assume that a given collection \( \{f_0, \ldots, f_\ell\} \) of multiplicative functions in \( \mathcal{M} \) admits correlations along the sequence of intervals \( \mathcal{M} := ([M_k])_{k \in \mathbb{N}} \) (such a sequence always exists), and that the measure preserving system \( (X, \mathcal{X}, \mu, T) \) and the functions \( F_0, \ldots, F_\ell \in L^\infty(\mu) \) are defined as in Proposition 3.3 with \( f_0, \ldots, f_\ell \) in place of the sequences \( a_0, \ldots, a_\ell \).

5.1. Orthogonality to the invariant and the Kronecker factor. The following key facts are direct consequences of the number theoretic results stated in Theorems 3.1 and 4.1:

**Proposition 5.1.** Let \( f_0, \ldots, f_\ell \in \mathcal{M} \) be multiplicative functions and let the system \( (X, \mathcal{X}, \mu, T) \) and the functions \( F_0, \ldots, F_\ell \in L^\infty(\mu) \) be as in our working assumptions. We have the following:

(i) If \( f_j \) satisfies \( M(f_j; N) \to \infty \) as \( N \to \infty \) for some \( j \in \{0, \ldots, \ell\} \), then \( F_j \) is orthogonal to the invariant factor of the system.

(ii) If \( f_j \) is strongly aperiodic for some \( j \in \{0, \ldots, \ell\} \), then \( F_j \) is orthogonal to the Kronecker factor of the system.

**Remark.** We deduce that if \( f_j \) is the Möbius or the Liouville function, then the function \( F_j \) is orthogonal to the Kronecker factor of the corresponding system.

**Proof.** (i) By Theorem 3.1 we have

\[
\lim_{N \to \infty} \limsup_{k \to \infty} \mathbb{E}_{m \in [M_k]} |\mathbb{E}_{n \in [N]} f_j(m + n)| = 0
\]

which implies that

\[
\lim_{N \to \infty} \mathbb{E}_{n \in [N]} \mathbb{E}_{m \in \mathcal{M}} f_j(m + n) \cdot \overline{f_j(m)} = 0.
\]

Using our working assumptions, we deduce that

\[
\lim_{N \to \infty} \mathbb{E}_{n \in [N]} \int T^n F_j \cdot \overline{F_j} \, d\mu = 0.
\]

As mentioned in Section 3.3 this implies that the function \( F_j \) is orthogonal to the invariant factor of the system.

(ii) By Theorem 4.1 we have

\[
\lim_{N \to \infty} \mathbb{E}_{n \in [N]} |\mathbb{E}_{m \in \mathcal{M}} f_j(m + n) \cdot \overline{f_j(m)}| = 0.
\]

Using our working assumptions, we deduce that

\[
\lim_{N \to \infty} \mathbb{E}_{n \in [N]} \left| \int T^n F_j \cdot \overline{F_j} \, d\mu \right| = 0.
\]

As mentioned in Section 3.3 this implies that the function \( F_j \) is orthogonal to the Kronecker factor of the system. \( \square \)
5.2. Proof of Theorems 2.3 and 2.4. We can now prove our main results.

Proof of Theorems 2.3. Let the system \((X, \mathcal{X}, \mu, T)\) and the functions \(F_0, \ldots, F_\ell \in L^\infty(\mu)\) be as in our working assumptions. Then for every \(n \in \mathbb{N}^\ast\) we have that

\[
\mathbb{E}_{m \in \mathbb{M}} f_0(m) f_1(m + p_1(n)) \cdots f_\ell(m + p_\ell(n)) = \int F_0 \cdot T^{p_1(n)} F_1 \cdots T^{p_\ell(n)} F_\ell \, d\mu.
\]

Hence, in order to verify (1) it suffices to show that

\[
\text{UD-lim}_{n \to \infty} \left( \int F_0 \cdot T^{p_1(n)} F_1 \cdots T^{p_\ell(n)} F_\ell \, d\mu \right) = 0.
\]

Using our assumption and part (ii) of Proposition 5.1 we get that one of the functions \(F_0, \ldots, F_\ell\) is orthogonal to the Kronecker factor of the system \((X, \mathcal{X}, \mu, T)\). Hence, Theorem 3.5 implies (10), and completes the proof. \(\square\)

In a similar fashion we prove Theorem 2.4.

Proof of Theorem 2.4. We argue as in the proof of Theorem 2.3 using part (i) and (ii) of Proposition 6.1 and part (i) and (ii) of Theorem 3.5. \(\square\)

6. Patterns in arithmetic sequences

In this section we prove Theorems 2.3 and 2.4. We first establish a preliminary result needed in the proof of Theorem 2.4.

6.1. A class of strongly aperiodic multiplicative functions. We will use the following basic facts about \(\mathbb{D}\) (see Definition 2): For every \(N \in \mathbb{N}\) and \(f, g, h \in \mathcal{M}\) we have (see for example [13])

\[
\mathbb{D}(f, g; N) \leq \mathbb{D}(f, h; N) + \mathbb{D}(h, g; N).
\]

Also for every \(N \in \mathbb{N}\) and all \(f_1, f_2, g_1, g_2 \in \mathcal{M}\) we have (see [12, Lemma 3.1])

\[
\mathbb{D}(f_1 f_2, g_1 g_2; N) \leq \mathbb{D}(f_1, g_1; N) + \mathbb{D}(f_2, g_2; N).
\]

We will also use two facts about \(\mathbb{D}(1, n^t; N)\). The first can be found in the proof of [26, Lemma C.1]. It states that for every \(N \geq 100\) we have

\[
\mathbb{D}(1, n^t; N) = \sqrt{\log(1 + |t| \log N)} + O(1), \text{ if } |t| \leq 1.
\]

The second follows from an estimate proved in [25, Lemma 2] (and is attributed to Granville and Soundararajan) using Vinogradov’s zero free region for the Riemann zeta function. It states that for every \(A > 0\) we have

\[
\min_{1 \leq |t| \leq N^A} \mathbb{D}(1, n^t; N) \to \infty \text{ as } N \to \infty.
\]

The next criterion enables us to prove that certain multiplicative functions used in the proof of Theorem 2.4 are strongly aperiodic:

Proposition 6.1. Let \(d \in \mathbb{N}\) and \(f \in \mathcal{M}\) be a multiplicative function such that \(f(p)\) is a \(d\)-th root of unity for all but finitely many primes \(p\). Suppose that \(\mathbb{D}(f, \chi) = \infty\) for every Dirichlet character \(\chi\). Then \(f\) is strongly aperiodic.

Proof. Let \(\chi\) be a Dirichlet character. There exists \(k \in \mathbb{N}\) such that \((\chi(p))^k = 1\) for all but a finite number of primes \(p\). Then for \(m = dk\), using [12] we get that

\[
m \mathbb{D}(f \cdot \chi, n^{it}; N) \geq \mathbb{D}(f^m \cdot \chi^m, n^{imt}; N) = \mathbb{D}(1, n^{imt}; N) + O(1).
\]

Combining this with (14) we deduce that

\[
\min_{1 \leq |t| \leq N} \mathbb{D}(f \cdot \chi, n^{it}; N) \to \infty \text{ as } N \to \infty.
\]
Next, we get lower bounds for \( D(f \cdot \chi, n^t; N) \) for \(|t| \leq 1\). Using (11) we have that
\[
D(f \cdot \chi, n^t; N) \geq D(f \cdot \chi, 1; N) - D(1, n^t; N) \geq \frac{1}{2} D(f \cdot \chi, 1; N)
\]
unless
\[
\frac{1}{2} D(f \cdot \chi, 1; N) \leq D(1, n^t; N) = D(1, n^{int}; N) + O(1)
\]
\[
= D(f^m \cdot \chi^m, n^{int}; N) + O(1) \leq m D(f \cdot \chi, n^t; N) + O(1),
\]
where the first identity follows for \(|t| \leq 1\) from (13) and the last estimate from (12). In either case, we have
\[
(16) \quad \min_{|t| \leq 1} D(f \cdot \chi, n^t; N) \geq \frac{1}{2m} D(f, \chi; N) + O(1).
\]
Combining (15) and (16), with our assumption \( D(f, \chi) = \infty \), we deduce that
\[
\min_{|t| \leq N} D(f \cdot \chi, n^t; N) \to \infty \quad \text{as} \quad N \to \infty.
\]
Hence, \( f \) is strongly aperiodic.

**Corollary 6.2.** Let \( d \in \mathbb{N} \) and \( f \in \mathcal{M} \) be a multiplicative function such that \( f(p) \) is a non-trivial \( d \)-th root of unity for all but finitely many primes \( p \). Then \( f \) is strongly aperiodic.

**Proof.** Using Proposition 6.1 it suffices to show that \( D(f, \chi) = \infty \) for every Dirichlet character \( \chi \). Suppose that \( \chi \) has period \( m \). Since \( \chi(1) = 1 \), we have \( \chi(n) = 1 \) whenever \( n \equiv 1 \mod m \), and since \( f(p) \) is a non-trivial \( d \)-th root of unity for all but finitely many primes \( p \), we have
\[
D(f, \chi)^2 \geq \left( 1 - \cos \left( \frac{2\pi}{d} \right) \right) \cdot \sum_{p \in \mathcal{P} \cap (m\mathbb{Z}+1)} \frac{1}{p} + O(1) = \infty,
\]
where the divergence of the last series follows from Dirichlet’s theorem. This completes the proof.

**6.2. Proof of Theorem 2.3.** Let \( (M_k)_{k \in \mathbb{N}} \) be a sequence of positive integers with \( M_k \to \infty \) so that the multiplicative functions \( f_0, \ldots, f_\ell \in \mathcal{M} \) admit correlations along the sequence of intervals \( \mathcal{M} = (\{M_k\})_{k \in \mathbb{N}} \).

Note that since for \( j = 0, \ldots, \ell \) the multiplicative function \( f_j \) takes values in \( \{-1, +1\} \) and \( \epsilon_j \in \{-1, +1\} \), we have
\[
1_{f_j=\epsilon_j}(n) = \frac{1 + \epsilon_j f_j(n)}{2}, \quad n \in \mathbb{N}.
\]
Hence,
\[
d_{\mathcal{M}}(\Lambda_{n, \epsilon}) = \frac{1}{2^{\ell+1}} \mathbb{E}_{m \in \mathcal{M}} \prod_{j=0}^{\ell} \left( 1 + \epsilon_j f_j(m + p_j(n)) \right)
\]
where \( p_0 := 0 \). Expanding the product to \( 2^{\ell+1} \) terms and using Theorem 2.4, we deduce that \( \text{UD-lim}_{n \to \infty} (d_{\mathcal{M}}(\Lambda_{n, \epsilon})) = 2^{-(\ell+1)} \), completing the proof of Theorem 2.3.


6.3. **Proof of Theorem 2.4** Let $b \in \mathbb{N}$ and $a \in \{0, \ldots, b-1\}$. We first express the indicator $1_{[\omega]_b = a}$ as a weighted average of multiplicative functions. Let $\zeta$ be a root of unity of order $b$. We define the multiplicative function $f_b$ by

$$f_b(p^j) := \zeta, \quad j \in \mathbb{N}, \quad p \in \mathbb{P}.$$  

Then

$$1_{[\omega]_b = a}(n) = \frac{1}{b} \sum_{r=0}^{b-1} \zeta^{-ar}(f_b(n))^r, \quad n \in \mathbb{N}. \tag{17}$$

Let $(M_k)_{k \in \mathbb{N}}$ be a sequence of positive integers with $M_k \to \infty$ such that the multiplicative functions $f_{b_0}, \ldots, f_{b_0^{-1}}, \ldots, f_{b_\ell}, \ldots, f_{b_\ell^{-1}}$ admit correlations along the sequence of intervals $M = (\{M_k\})_{k \in \mathbb{N}}$. Using (17) we get that

$$d_M(\Lambda_{n,a}) = \mathbb{E}_{m \in M} \prod_{j=0}^\ell \left( \frac{1}{b_j} \sum_{r=0}^{b_j-1} \zeta^{-ar}(f_{b_j}(m + p_j(n)))^r \right) \tag{18}$$

where $p_0 := 0$.

Note that by Corollary 6.2 for $j = 0, \ldots, \ell$ the multiplicative functions $f_{b_j}$ are strongly aperiodic for $r = 1, \ldots, b_j - 1$. Expanding the product in (18) to $\prod_{j=0}^\ell b_j$ terms and using Theorem 2.1 we deduce that $\text{UD-lim}_{n \to \infty}(d_M(\Lambda_{n,a})) = (\prod_{j=0}^\ell b_j)^{-1}$, completing the proof of Theorem 2.4.

In a similar fashion we can prove a modification of Theorem 2.4 for the arithmetic function $\Omega$ in place of $\omega$; the only difference is that we use in place of the multiplicative function $f_b$ the completely multiplicative function $f'_b$ defined by $f'_b(p^j) := \zeta^j$, for all $j \in \mathbb{N}$ and primes $p$.

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