AN OPTIMAL EXTENSION THEOREM FOR 1-FORMS
AND THE LIPMAN–ZARISKI CONJECTURE

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Abstract. Let $X$ be a normal variety with Du Bois singularities. We prove that any 1-form defined on the smooth locus of $X$ extends to a log resolution $\tilde{X} \to X$, as a regular differential form. As an application, we show that the Lipman–Zariski conjecture holds for such $X$, reproving a recent result of Druel.

Contents

1. Introduction 1
2. Technical preparations 2
3. Proof of Theorem 1.2 6
4. Optimality of Theorem 1.2 8
5. Proof of Corollary 1.3 8
References 8

1. Introduction

1.1. Main result. In [GKKP11], Greb–Kebekus–Kovács–Peternell proved the following theorem.

**Theorem 1.1** (see [GKKP11, Thm. 1.5]). Let $X$ be a complex quasi-projective variety of dimension $n$ and let $D$ be a $\mathbb{Q}$-divisor on $X$ such that the pair $(X, D)$ is log canonical. Let $\pi : \tilde{X} \to X$ be a log resolution with $\pi$-exceptional set $E$ and

$$\tilde{D} := \text{largest reduced divisor contained in } \text{supp } \pi^{-1}(\text{non-klt locus}),$$

where the non-klt locus is the smallest closed subset $W \subset X$ such that $(X, D)$ is klt away from $W$. Then the sheaves $\pi_*\Omega^p_{\tilde{X}}(\log \tilde{D})$ are reflexive, for all $p \leq n$.

This means that any logarithmic $p$-form defined on the snc locus of $(X, D)$ can be extended to $\tilde{X}$, acquiring logarithmic poles along $\tilde{D}$ – see [GKKP11, Rem. 1.5.2]. It is natural to ask whether Theorem 1.1 is optimal. Firstly, one might try to weaken the assumptions on $(X, D)$. Secondly, it might be possible to shrink the divisor $\tilde{D}$ along which we have to allow logarithmic poles.

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1
In the present paper, we show that in the case $p = 1$, Theorem 1.1 is not optimal in either sense. The result is as follows.

**Theorem 1.2.** Let $(X, D)$ be a pair of Du Bois spaces (see Definition 2.1). If $\pi : \tilde{X} \to X$ is a log resolution of $(X, D)$, then the sheaf

$$\pi_* \Omega^1_{\tilde{X}}(\log \tilde{D})$$

is reflexive, where $\tilde{D}$ is the largest reduced divisor contained in $\pi^{-1}(D)$.

In particular, if $D = 0$ but $X$ is not klt (e.g. if $X$ is a cone over an elliptic curve), then our $\tilde{D} = 0$ while the $\tilde{D}$ from Theorem 1.1 is nonzero. In Section 4, we discuss why Theorem 1.2 in turn is optimal, both with respect to $\tilde{D}$ and with respect to the degree of the forms considered.

**Remark.** If $(X, D)$ is log canonical and $\pi : \tilde{X} \to X$ is a log resolution of $(X, D)$, then $(X, \lfloor D \rfloor)$ and $\pi$ satisfy the assumptions of Theorem 1.2. This follows from [KK10, Theorem 1.4] and [GKKP11, Lemma 2.15].

1.2. **Application to the Lipman–Zariski conjecture.** This conjecture asserts that a variety $X$ with locally free tangent sheaf $\mathcal{T}_X$ is already smooth. It has been verified in a number of special cases, the most recent ones being [GKKP11, Theorem 6.1], where it was shown for varieties $X$ such that the pair $(X, \emptyset)$ is klt, and [Dru13, Thm. 1.1], where it was shown more generally for log canonical spaces. Here, as an immediate corollary of Theorem 1.2 we prove the conjecture for normal Du Bois spaces $X$. Note, however, that if $\mathcal{T}_X$ is locally free, then $K_X$ is Cartier. In this case, by [Kov99, Thm. K'], the notions of Du Bois and log canonical singularities coincide. So in fact we just reprove [Dru13, Thm. 1.1] by a different method.

**Corollary 1.3 (Dru13 Thm. 1.1).** Let $X$ be a normal variety with Du Bois singularities. If the tangent sheaf $\mathcal{T}_X$ is locally free, then $X$ is smooth.

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2. **Technical preparations**

Throughout this paper, we work over the field of complex numbers $\mathbb{C}$.

**Definition 2.1.** A pair of Du Bois spaces is a pair $(X, D)$ consisting of a normal variety $X$ and a reduced divisor $D$ such that $X$ and $D$ both have Du Bois singularities.

2.1. **Steenbrink-type vanishing results for pairs of Du Bois spaces.** In this section we state a vanishing theorem which we will need in the proof of Theorem 1.2. This theorem was already proved by Greb–Kebekus–Kovács–Peternell in [GKKP11, Theorem 14.1], but the authors only formulated it for log canonical pairs, as they were only interested in these. So all we are going to do here is check that their proof works exactly the same way in the more general case.

**Theorem 2.2 (see GKKP11 Theorem 14.1).** Let $(X, D)$ be a pair of Du Bois spaces, where $X$ has dimension $n \geq 2$, and let $\pi : \tilde{X} \to X$ be a log resolution of $(X, D)$ with exceptional locus $E$. Further set $\tilde{D} = \pi^{-1}(D) + E$. Then

$$R^{n-1} \pi_* (\Omega^n_{\tilde{X}}(\log \tilde{D}) \otimes \mathcal{O}_{\tilde{X}}(-\tilde{D})) = 0$$

for all $0 \leq p \leq n$. 
AN OPTIMAL EXTENSION THEOREM FOR 1-FORMS

Proof. For $p > 1$, the claim was proved in [Ste85, Theorem 2.1b)]. If $p = 0$, Theorem 13.3 of [GKKP11] applies. In order to handle the case $p = 1$, we follow the proof of [GKKP11, Theorem 14.1], except that we need to replace Lemma 14.4 of that paper by our Lemma 2.3 below.

Lemma 2.3 (see [GKKP11, Lemma 14.4]). In the setup of Theorem 2.2 if $j : \tilde{X} \setminus \tilde{D} \hookrightarrow \check{X}$ is the inclusion and $j_!\mathcal{C}_{\tilde{X} \setminus \tilde{D}}$ is the sheaf of locally constant functions on $\tilde{X}$ which vanish along $\tilde{D}$, then $R^k\pi_*(j_!\mathcal{C}_{\tilde{X} \setminus \tilde{D}}) = 0$ for all $k > 0$.

Proof. The proof of [GKKP11, Lemma 14.4] applies verbatim. □

Corollary 2.4 (see [GKKP11, Corollary 14.2]). Let $(X, D)$ be a pair of Du Bois spaces, where $X$ has dimension $n \geq 2$. Let $\pi : \tilde{X} \to X$ be a log resolution of $(X, D)$ with exceptional locus $E$ and set $\tilde{D} = \pi^{-1}(D) + E$. If $x \in X$ is any point, with reduced fibre $F = \pi^{-1}(x)_{\text{red}}$, then $H^1_{\text{red}}(\tilde{X}, \Omega^p_{\tilde{X}}(\log \tilde{D})) = 0$ for all $0 \leq p \leq n$.

Proof. This follows from Theorem 2.2 by applying duality for cohomology with support [GKK10, Theorem A.1]. □

2.2. Log poles along a divisor contracted to a point. The aim of the present section is to show that if $\omega$ is a logarithmic 1-form on a smooth variety, with poles contained in a divisor that can be contracted to a point, then $\omega$ in fact does not have any poles. This was first observed by J. Wahl in the case of surfaces (see [Wah85, Lemma 1.3] and [GKK10, Remark 6.2]). Here we extend his argument to arbitrary dimensions by cutting down.

Notation 2.5. Let $X$ be a smooth variety and $L \in \text{Pic} X$ a line bundle. The first Chern class $c_1(L) \in H^1(X, \Omega^1_X)$ is the image of $L$ under the map $\text{Pic} X = H^1(X, O^*_X) \to H^1(X, \Omega^1_X)$ induced by $d \log : O^*_X \to \Omega^1_X$, which sends $f \mapsto f^{-1}df$.

We will need the following technical lemma.

Lemma 2.6. Let $X$ be a smooth variety and $E \subset X$ an snc divisor, consisting of irreducible components $E_1, \ldots, E_k$. Consider the short exact sequence

$$0 \to \Omega^1_X \to \Omega^1_X(\log E) \to \bigoplus_{i=1}^k O_{E_i} \to 0 \tag{2.7}$$

given by the residue map (see [EV92, 2.3.a)]). The associated connecting homomorphism

$$\delta : \bigoplus_{i=1}^k H^0(E_i, O_{E_i}) \to H^1(X, \Omega^1_X)$$

sends

$$\mathbf{1}_{E_i} \mapsto c_1(O_X(E_i)), \quad 1 \leq i \leq k.$$

Here $\mathbf{1}_{E_i}$ denotes the function which is constant with value 1 on $E_i$ and vanishes on the other components.

Proof. Choose an index $1 \leq i \leq k$ and let $U = \{U_\alpha\}_\alpha$ be an affine open cover of $X$ such that the divisor $E_i$ is locally given by equations $\{(U_\alpha, f_\alpha)\}_\alpha$. By [Har77, Ch. III, Theorem 4.5], the cohomology groups in question can be computed as Čech cohomology with respect to $U$. So we will write down parts of the relevant Čech
complexes and prove the lemma by explicitly chasing the element \(1_E\) through the resulting diagram, according to the definition of the connecting homomorphism.

\[
\begin{array}{c}
0 \\
\downarrow \\
0 \\
\downarrow \\
0 \\
\downarrow \\
0 \\
\hline
\end{array}
\begin{array}{c}
\check{C}^0(\mathcal{U}, \Omega^1_X) \to \check{C}^1(\mathcal{U}, \Omega^1_X) \\
\downarrow \\
\check{C}^0(\mathcal{U}, \Omega^1_X(\log E)) \to \check{C}^1(\mathcal{U}, \Omega^1_X(\log E)) \\
\downarrow \\
\check{C}^0(\mathcal{U}, \bigoplus \mathcal{O}_{E_i}) \to \check{C}^1(\mathcal{U}, \bigoplus \mathcal{O}_{E_i}) \\
\downarrow \\
0
\end{array}
\]

We start with \(\{1_{E_i}\}_\alpha\) in the bottom left corner. Going one step upstairs, a preimage is given by \(\{\text{d} \log f_\alpha\}_\alpha\). The image of this in \(\check{C}^1(\mathcal{U}, \Omega^1_X(\log E))\) is \(\{\text{d} \log (f_\beta f_\alpha^{-1})\}_{\alpha, \beta}\), which even is in \(\check{C}^1(\mathcal{U}, \Omega^1_X)\). So
\[
\delta(1_{E_i}) = \{\text{d} \log (f_\beta f_\alpha^{-1})\}_{\alpha, \beta}.
\]

On the other hand, the line bundle \(\mathcal{O}_X(E_i)\) is described by the transition functions \(\{f_\beta f_\alpha^{-1}\}_{\alpha, \beta}\). The first Chern class \(c_1(\mathcal{O}_X(E_i))\) is obtained from these by applying \(\text{d} \log\), which yields nothing else than \(\delta(1_{E_i})\). □

For a proof of the following fact, see [For77, Paragraph 17].

**Fact 2.8.** Let \(C\) be a smooth projective curve. Then there is a canonically defined linear map \(\text{Res}: H^1(C, \Omega^1_C) \to \mathbb{C}\), which is an isomorphism. □

**Remark 2.9.** If \(\mathcal{L} \in \text{Pic} C\) is a line bundle, then \(\text{Res}(c_1(\mathcal{L})) = \text{deg } \mathcal{L}\). (For \(P \in C\) a point and \(\mathcal{L} = \mathcal{O}_C(P)\), the claim is easily seen to be true from the description of \(\text{Res}\) given in [For77, Satz 17.3]. By linearity, this is enough.)

Now we come to the statement announced at the beginning of this section.

**Proposition 2.10.** Let \(X\) be a normal variety of dimension \(\geq 2\) and \(\pi: \tilde{X} \to X\) a log resolution. Let \(E = E_1 + \cdots + E_k\) be a reduced exceptional divisor which is mapped to a single point by \(\pi\). Then the natural inclusion map
\[
H^0(\tilde{X}, \Omega^1_{\tilde{X}}) \to H^0(\tilde{X}, \Omega^1_{\tilde{X}}(\log E))
\]
is an isomorphism.

**Proof.** We may assume \(X\) to be affine. Consider the short exact sequence (2.7),
\[
0 \to \Omega^1_X \to \Omega^1_X(\log E) \to \bigoplus_{i=1}^k \mathcal{O}_{E_i} \to 0.
\]
By the corresponding long exact sequence, it suffices to show injectivity of the induced map
\[
\delta: \bigoplus_{i=1}^k H^0(E_i, \mathcal{O}_{E_i}) \to H^1(\tilde{X}, \Omega^1_{\tilde{X}}).
\]
Let $H \subset \tilde{X}$ be the intersection of general hyperplanes $H_1, \ldots, H_{\dim X - 2} \subset \tilde{X}$. (If $X$ is a surface, then $H = \tilde{X}$.) We formulate the properties of $H$ in a separate lemma.

**Lemma 2.11.** $(H, E|_H)$ is a log smooth surface pair. For any $i$, $C_i := E_i|_H$ is irreducible (in particular, nonempty). $\pi|_H$ is proper and birational onto its image.

**Proof.** This can be proved inductively, cutting by one hyperplane at a time. First we cut by $H_1$. By Bertini’s theorem, $E + H_1$ is snc and $H_1$ is irreducible. So $(H_1, E|_{H_1})$ is log smooth, and the $E_i|_{H_1}$ are smooth. Since these can also be viewed as $H_1|_{E_i}$, they are irreducible by Bertini again. It is clear that $\pi|_{H_1}$ is proper. And since $H_1$ is general, $\pi|_{H_1}$ is birational.

Now we are in the same situation as before cutting by $H_1$, so we may apply the same argument again and obtain the statement for $H_1 \cap H_2$. After finitely many steps, we arrive at $H$. □

The image $\pi(H)$ need not be normal, but we may normalize it and get a birational morphism $H \to \pi(H)^\nu$, which contracts all the $C_i$. So negative definiteness ([KM98, Lemma 3.40]) asserts that the intersection matrix $A := (C_i \cdot C_j)$ is invertible.

Remember that we need to show the injectivity of $\delta$. To this end, think of the $C_i$ as smooth projective curves in $\tilde{X}$, consider the restriction morphism $r: H^1(\tilde{X}, \Omega^1_{\tilde{X}}) \to \bigoplus_{i=1}^k H^1(C_i, \Omega^1_{C_i})$, and observe that the composition $r \circ \delta$ is an isomorphism: on $\bigoplus_{i=1}^k H^0(E_i, \mathcal{O}_{E_i})$, choose the basis consisting of the functions $1_{E_i}$, and on each summand of $\bigoplus_{i=1}^k H^1(C_i, \Omega^1_{C_i})$, choose the basis canonically determined by the residue map of Fact 2.8. By Lemmas 2.6 and 2.11 and using Remark 2.9 the matrix of $r \circ \delta$ with respect to these bases is simply $A$. We have already seen that this matrix is invertible. □

### 2.3. Extension with logarithmic poles along the exceptional set.

In this section we are in a situation similar to that of Section 2.1. We need an extension theorem for pairs of Du Bois spaces, which was proved in [GKKP11, Theorem 16.1] for log canonical pairs. But the proof also works in the Du Bois case, essentially unchanged.

**Lemma 2.12.** Let $(X, D)$ be a pair of Du Bois spaces, and let $H \in |L|$ be a general member of a basepoint-free linear system. If $D_H := \text{supp}(D \cap H)$, then $(H, D_H)$ is also a pair of Du Bois spaces.

**Proof.** [KS09, Section 12] and [GKKP11, Lemma 2.20]. □

**Theorem 2.13** (see [GKKP11, Theorem 16.1]). Let $(X, D)$ be a pair of Du Bois spaces. If $\pi: \tilde{X} \to X$ is a log resolution of $(X, D)$ with exceptional set $E$, then the sheaves

$$
\pi_* \Omega^p_X (\log \tilde{D}'), \quad 0 \leq p \leq n,
$$

are reflexive, where $\tilde{D}'$ is the reduced divisor $\pi^{-1}_*(D) + E$.

**Remark.** Although we will use Theorem 2.13 only for 1-forms, we formulate it for forms of arbitrary degree, since this causes no extra effort in the proof.
Proof. We follow the proof given in [GKKP11, Section 17]. The start of induction (Section 17.B) is not going to work in our case. But note that since in dimension 2 any exceptional divisor is mapped to a point, we may replace the argument of Section 17.B with that of Section 17.C.3 (see below).

The proof of the inductive step works fine up to Section 17.C.3, where we need to substitute [GKKP11, Corollary 14.2] by our Corollary 2.4. The same needs to be done with all further occurrences of that corollary. We will not mention this explicitly every time.

Claim 17.14 clearly needs to be modified so as to read “If \( t \in T \) is a general point, then \((X_t, D_t)\) is a pair of Du Bois spaces” and so on. For the proof of this, replace [GKKP11, Lemma 2.22] with Lemma 2.12.

The rest of the proof goes through without changes. □

3. Proof of Theorem 1.2

The proof of Theorem 1.2 follows the lines of [GKK10, Section 7.D], except that we use the result of Section 2.2. We will make essential use of the following proposition from [GKK10], which we restate here for the reader’s convenience.

Proposition 3.1 ([GKK10, Proposition 7.5]). Let \( \varphi: \tilde{Y} \to Y \) be a projective birational morphism between normal quasi-projective varieties of dimension \( \geq 2 \), where \( \tilde{Y} \) is smooth. Let \( y \in Y \) be a point whose preimage \( \varphi^{-1}(y) \) has (not necessarily pure) codimension one and let \( F_0, \ldots, F_k \subset \varphi^{-1}(y) \) be the reduced divisorial components. If all the \( F_i \) are smooth and \( \sum k_i F_i \) is a nonzero effective divisor, then there is a \( 0 \leq j \leq k \) such that \( k_j \neq 0 \) and \( h^0(F_j, \mathcal{O}_{\tilde{Y}}(\sum k_i F_i)|_{F_j}) = 0 \). □

Proof (of Theorem 1.2). Since the question is local on \( X \), we may assume \( X \) to be quasi-projective. Let \( E \) denote the exceptional set of \( \pi \). Let \( U \subset X \) be an open subset, and let \( \sigma \in H^0 \left( \pi^{-1}(U) \setminus E, \Omega^1_X(\log \tilde{D}) \right) \) be a logarithmic 1-form defined outside the exceptional set. We may shrink \( X \) and assume \( U = X \), so \( \sigma \) is defined on \( X \setminus E \). Set \( \tilde{D}' = \pi^{-1}(D) + E \) and note that this is obtained from \( \tilde{D} \) by adding an effective exceptional divisor.

By Theorem 2.13 we know that \( \sigma \) may be extended to a form

\[
\tilde{\sigma} \in H^0(\tilde{X}, \Omega^1_{\tilde{X}}(\log \tilde{D}')).
\]

We want to show that in fact, \( \tilde{\sigma} \) has logarithmic poles only along the smaller divisor \( \tilde{D} \), that is, \( \tilde{\sigma} \in H^0(\tilde{X}, \Omega^1_{\tilde{X}}(\log \tilde{D})) \). To this end, we will consider separately each irreducible component of \( E \) which is contained in \( \tilde{D}' \) but not in \( \tilde{D} \), i.e. for any such \( E' \subset E \) we will show that \( \tilde{\sigma} \in H^0(\tilde{X}, \Omega^1_{\tilde{X}}(\log(\tilde{D}' - E'))) \).

An irreducible divisor \( E' \subset E \) is contained in \( \tilde{D}' \) but not in \( \tilde{D} \) if and only if \( \pi(E') \not\subset D \), so by further shrinking \( X \), we can assume that \( D = 0 \).

We proceed by induction on pairs of numbers \( (\dim X, \text{codim}_X \pi(E')) \), ordered as indicated in the following table:

| No. | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | ... |
|-----|---|---|---|---|---|---|---|---|---|----|-----|
| \( \dim X \) | 2 | 3 | 3 | 4 | 4 | 4 | 5 | 5 | 5 | 5 | ... |
| \( \text{codim}_X \pi(E') \) | 2 | 2 | 3 | 2 | 3 | 4 | 2 | 3 | 4 | 5 | ... |
In order to simplify notation, we renumber the irreducible components $E_i$ of $E$ such that $E' = E_0$ and $\pi(E_i) = \pi(E_0)$ if and only if $0 \leq i \leq k$, for some number $k$. Let $k_i$ be the pole orders of $\tilde{\sigma}$ along the $E_i$. These are the minimal non-negative numbers such that

$$\tilde{\sigma} \in H^0(\tilde{X}, \Omega^1_{\tilde{X}} \otimes O_X(\sum k_i E_i)).$$

By (3.2), we already know all the $k_i$ are either 0 or 1, and our aim is to show that $k_0 = 0$.

**Start of induction.** This is the case $\dim X = \dim_X \pi(E_0) = 2$. For surfaces, any exceptional divisor is contracted to a point, so Proposition (2.10) applies.

**Inductive step.** We distinguish two possibilities: the divisor $E_i$ is contracted to a point by $\pi$, or it may be mapped to a positive-dimensional variety.

If $\dim \pi(E_0) = 0$, we can shrink $X$ once more and forget about the exceptional divisors whose image is a point different from $\pi(E_0)$. By the induction hypothesis, $\tilde{\sigma}$ does not have a pole along any exceptional divisor not mapped to a single point. Thus we already know that

$$\tilde{\sigma} \in H^0(\tilde{X}, \Omega^1_{\tilde{X}}(\log E_0 + \cdots + E_k)).$$

But since $E_0 + \cdots + E_k$ is contracted to a point by $\pi$, it follows from Proposition (2.10) that in fact,

$$\tilde{\sigma} \in H^0(\tilde{X}, \Omega^1_{\tilde{X}}).$$

In particular, $\tilde{\sigma}$ does not have a pole along $E_0$, which is what we wanted to show.

If $\dim \pi(E_0) > 0$, choose general hyperplanes $H_1, \ldots, H_{\dim \pi(E_0)} \subset X$, let $H$ be the intersection $H_1 \cap \cdots \cap H_{\dim \pi(E_0)}$ and $H$ the preimage $\pi^{-1}(H_0)$. Applying Lemma (2.12) and [GKKP11, Lemma 2.21], we obtain that $H$ is Du Bois and $\pi|_H$ is a log resolution. The intersection $H \cap \pi(E_0)$ is finite, but nonempty. Shrinking X for the last time, we may assume $H \cap \pi(E_0)$ consists of a single point, say $x$. Now set $F_x = \pi^{-1}(x)$ and

$$F_{x,i} = F_x \cap E_i = (\pi|_{E_i})^{-1}(x).$$

Then $F_x$ is the union of the $F_{x,i}$. We claim that $F_{x,0}, \ldots, F_{x,k}$ are smooth, irreducible, and have codimension one in $\tilde{H}$, while the other $F_{x,i}$ have higher codimension in $\tilde{H}$. In particular, we claim that it is possible to apply Proposition (3.1) to $\pi|_{\tilde{H}}: \tilde{H} \to H$, $x \in H$, and $F_{x,0}, \ldots, F_{x,k}$, which we will do later.

Indeed, if $0 \leq i \leq k$, then being a general fibre of $\pi|_{E_i}$, $F_{x,i}$ is smooth of dimension $\dim E_i - \dim \pi(E_0) = \dim H - 1$. Since $F_{x,i} = \tilde{H} \cap E_i$, it is also irreducible by repeated application of Bertini’s theorem.

On the other hand, if $i > k$, then either $\pi(E_0) \not\subset \pi(E_i)$, in which case $x \not\in \pi(E_i)$ and so $F_{x,i} = \emptyset$, or $\pi(E_0) \subset \pi(E_i)$. In the latter case, $F_{x,i}$ is a general fibre of $\pi|_{E_i \cap \pi^{-1}(\pi(E_0))}$, hence has dimension $\leq \dim H - 2$.

Now consider the dual of the normal bundle sequence,

$$0 \longrightarrow N^*_{\tilde{H}/\tilde{X}} \longrightarrow \Omega^1_{\tilde{X}/\tilde{H}} \longrightarrow \Omega^1_\tilde{H} \longrightarrow 0.$$
twist it with $\mathcal{F} := \mathcal{O}_{\tilde{H}}(\sum k_i E_i|_{\tilde{H}})$, and restrict to $F_{x,j}$, for $0 \leq j \leq k$:

$$\begin{array}{c}
N^*_R/\tilde{X} \otimes \mathcal{F} \xrightarrow{\alpha} \Omega^1_X|_{\tilde{H}} \otimes \mathcal{F} \xrightarrow{\beta} \Omega^1_{\tilde{H}} \otimes \mathcal{F} \\
\downarrow \quad \quad \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \quad \quad \downarrow \\
N^*_R/\tilde{X} \otimes \mathcal{F}|_{F_{x,j}} \xrightarrow{\alpha_j} \Omega^1_X|_{\tilde{H}} \otimes \mathcal{F}|_{F_{x,j}} \xrightarrow{\beta_j} \Omega^1_{\tilde{H}} \otimes \mathcal{F}|_{F_{x,j}}.
\end{array}$$

Since $H$ has smaller dimension than $X$, the induction hypothesis gives us (3.3) $\rho(\sigma|_{\tilde{H}}) \in H^0(\tilde{H}, \Omega^1_{\tilde{H}}) \subset H^0(\tilde{H}, \Omega^1_{\tilde{H}} \otimes \mathcal{F})$.

Recall that we want to show $k_0 = 0$. We will show more generally that $k_j = 0$ for all $0 \leq j \leq k$. So assume there is such an index $j$ with $k_j = 1$. By the definition of the $k_i$, $\sigma|_{\tilde{H}}$ as a section in $\Omega^1_X|_{\tilde{H}} \otimes \mathcal{F}$ does not vanish along $F_{x,j}$. But by (3.3), $\beta(\sigma|_{\tilde{H}})$ does vanish along $F_{x,j}$. So $r_j(\sigma|_{\tilde{H}})$ is a nonzero global section in $\ker \beta_j$, which means $H^0(F_{x,j}, N^*_R/\tilde{X} \otimes \mathcal{F}|_{F_{x,j}}) \neq 0$. Now note that $N^*_R/\tilde{X}$ is trivial, because $N^*_R/\tilde{X}$ is the pullback of $N^H_X$. Hence from $H^0(F_{x,j}, N^*_R/\tilde{X} \otimes \mathcal{F}|_{F_{x,j}}) \neq 0$ it follows that $H^0(F_{x,j}, \mathcal{F}|_{F_{x,j}}) \neq 0$. Since this holds for all $j$ with $k_j = 1$, we have a contradiction to Proposition 3.1 showing in particular that $k_0 = 0$ and thus completing the proof of Theorem 1.2. \hfill $\square$

4. Optimality of Theorem 1.2

4.1. Other values of $p$. One cannot expect an analogue of Theorem 1.2 to hold for $p$-forms with $p \geq 2$. Counterexamples may be obtained by taking a $p$-dimensional normal Gorenstein singularity $0 \in X$ which is log canonical but not klt, and considering the product $X \times \mathbb{C}^n$, for $n \geq 0$ arbitrary. If $\sigma$ is a local generator for $\omega_X$, then $\pi^*_X \sigma$ will not be extendable without poles to a resolution of $X \times \mathbb{C}^n$. Note also that this construction does not work for $p = 1$.

4.2. Shrinking $\tilde{D}$ further. In general, for a divisor $\pi^{-1}_X D \leq \tilde{D} < \tilde{D}$, the sheaf $\pi_\ast \Omega^1_X(\log \tilde{D})$ will not be reflexive any more. For example, this happens whenever some component $E_0 \subset \tilde{D} - \tilde{D}$ is mapped into the support of a $\mathbb{Q}$-Cartier divisor $0 \leq D_0 \leq D$. Namely, let some multiple of $D_0$ be given by the local equation $f = 0$. Then $df/f$ is a local section of $\Omega^1_X(\log D)$ which does not extend to $\Omega^1_X(\log \tilde{D})$, because it has a logarithmic pole along $E_0$.

5. Proof of Corollary 1.3

The proof of [GKKP11] Theorem 6.1] applies verbatim, except that Theorem 1.1 is replaced by Theorem 1.2.

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