Some Aspects of Additive Coalescents

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Abstract

We present some aspects of the so-called additive coalescence, with a focus on its connections with random trees, Brownian excursion, certain bridges with exchangeable increments, Lévy processes, and sticky particle systems.

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1. Additive coalescence in the finite setting

The additive coalescence is a simple Markovian model for random aggregation that arises for instance in the study of droplet formation in clouds [13, 14], gravitational clustering in the universe [19], phase transition for parking [10], ... As long as only finitely many clusters are involved, it can be described as follows. A typical configuration is a finite sequence $x_1 \geq \ldots \geq x_n > 0$ with $\sum x_i = 1$, which may be thought of as the ranked sequence of masses of clusters in a universe with unit total mass. Each pair of clusters, say with masses $x_i$ and $x_j$, merges as a single cluster with mass $x_i + x_j$ at rate $K(x_i, x_j) = x_i + x_j$, independently of the other pairs in the system.

This means that to each pair $(i, j)$ of indices with $1 \leq i < j \leq n$, we associate an exponential variable $e(i, j)$ with parameter $x_i + x_j$, such that to different pairs correspond independent variables. If the minimum $\gamma_1 := \min_{1 \leq i < j \leq n} e(i, j)$ of these variables is reached, say for the pair $(i_0, j_0)$, i.e. $\gamma_1 = e(i_0, j_0)$, then at time $\gamma_1$, we replace the clusters with labels $i_0$ and $j_0$ by a single cluster with mass $x_{i_0} + x_{j_0}$. Then the system keeps evolving with the same dynamics until it is reduced to a single cluster.

An additive coalescent $(X(t), t \geq 0)$ started from a finite number $n$ of masses is a Markov chain in continuous times for which the sequence of jump times $\gamma_1 < \ldots < \gamma_{n-1}$ has a simple structure. Specifically, the increments between consecutive coalescence times, $\gamma_1, \gamma_2 - \gamma_1, \ldots, \gamma_{n-1} - \gamma_{n-2}$, are independent exponential variables with parameters $n-1, n-2, \ldots, 1$, and are independent of the state chain.
(X(γ_k), k = 0, ..., n − 1). This elementary observation enables one to focus on the chain of states, and to derive simple relations with other random processes, as we shall now see.

Pitman [17] pointed at the following connection with random trees. Pick a tree \( \tau^{(n)} \) at random, uniformly amongst the \( n^{n−2} \) trees on \( n \) labelled vertices. Enumerate its \( n − 1 \) edges at random and decide that they are all closed at the initial time. At time \( k = 1, ..., n − 1 \), open the \( k \)-th edge, and say that two vertices belong to the same sub-tree if all the edges of the path connecting those two vertices are open at time \( k \). If we denote by \( nY^{(n)}(k) \) the ranked sizes of sub-trees at time \( k \), then the chain \( Y^{(n)} = (Y^{(n)}(k), k = 0, ..., n − 1) \) has the same law as the chain of states \( (X^{(n)}(γ_k), k = 0, ..., n − 1) \) of the additive coalescent started from the initial configuration \( (1/n, ..., 1/n) \).

Forest derived at time 10 from a tree with 14 vertices

Next, we consider a path transformation (see the picture below) that has been introduced by Takács [21] and used by Vervaat [23] to change a Brownian bridge into a normalized Brownian excursion. Specifically, we set

\[
\epsilon(u) = b(u + \mu \mod 1) - b(\mu), \quad 0 \leq u \leq 1,
\]

bridge \( b \)
where \( \mu \) stands for the location of the infimum of the bridge \( b \).

Finally, for every \( t \geq 0 \), call \( t \)-interval any maximal interval \( [a, b] \subseteq [0, 1] \) on which

\[
tu - \epsilon(u) < \max \left\{ (tv - \epsilon(v))^+, 0 \leq v \leq u \right\}, \quad \text{for all } u \in [a, b].
\]

It is easy to see that the \( t \)-intervals get finer as \( t \) increases and tend to reduce to the jump times of \( \epsilon \) when \( t \to \infty \). Denote by \( F(t) \) the ranked sequence of the sums of the jumps made by \( \epsilon \) on each \( t \)-interval, and by \( 0 < \delta_1 < \ldots < \delta_{n-1} \) the jump times of \( F(\cdot) \). Then the chain \( \langle F(\delta_{n-k-1}), k = 0, \ldots n-1 \rangle \) has the same law as the chain of states \( \langle X(\gamma_k), k = 0, \ldots n-1 \rangle \) of the additive coalescent started from the initial configuration \( \langle x_1, \ldots, x_n \rangle \).

2. **Standard and other eternal coalescents**

Dealing with a finite number of clusters may be useful to give a simple description of the dynamics, however it is a rather inconvenient restriction in practice. In
fact, it is much more natural to work with the infinite simplex
\[ S^↓ = \left\{ x = (x_1, x_2, \ldots) : x_i \geq 0 \text{ and } \sum_{i=1}^{\infty} x_i = 1 \right\} \]
endowed with the uniform distance. In this direction, Evans and Pitman [12] have shown that the semigroup of the additive coalescence enjoys the Feller property on \( S^↓ \). Approximating a general configuration \( x \in S^↓ \) by configurations with a finite number of clusters then enables us to view the additive coalescence as a Markovian evolution on \( S^↓ \). It is interesting in this setting to consider asymptotics when the coalescent starts with a large number of small clusters, which we shall now discuss.

Evans and Pitman [12] have first observed that the so-called standard additive coalescent \((X^{(\infty)}(t), -\infty < t < \infty)\) arises at the limit as \( n \to \infty \) of the additive coalescent process \((X^{(n)}(t), -\frac{1}{2} \log n \leq t < \infty)\) started at time \(-\frac{1}{2} \log n\) with \( n \) clusters, each with mass \( 1/n \). This limit theorem is perhaps better understood if we recall the connection with the uniform random tree \( \tau(n) \) on \( n \) vertices that was presented in the previous section. Indeed, if one puts a mass \( 1/n \) at each vertex and let each edge have length \( n^{-1/2} \), then \( \tau(n) \) converges weakly as \( n \to \infty \) to the so-called continuum random tree \( \tau(\infty) \); see Aldous [1]. More precisely, \( \tau(\infty) \) is a compact metric space endowed with a probability measure (arising as the limit of the masses on vertices) which is concentrated on the leaves of the tree, and a skeleton equipped with a length measure which is used to define the distance between leaves.

This suggests that the standard additive coalescent might be constructed as follows: as time passes, one creates a continuum random forest by logging the continuum random tree along its skeleton and consider the ranked sequence of masses of the subtrees. This yields a fragmentation process, and the standard additive coalescent is finally obtained by time-reversing this fragmentation process. Aldous and Pitman [3] have made this construction rigorous; more precisely they showed that the tree \( \tau(\infty) \) has to be cut at points that appear according to a Poisson point process on the skeleton with intensity given by the length measure. This representation yields a number of explicit statistics for the standard additive coalescent. For instance, for every \( t \in \mathbb{R} \), the distribution of \( X^{(\infty)}_t \) is given by that of the ranked sequence \( \xi_1 \geq \xi_2 \geq \ldots \) of the atoms of a Poisson measure on \([0, \infty[ \) with intensity \( e^{-t} \left( 2\pi x^3 \right)^{-1/2} dx \) and conditioned by \( \xi_1 + \cdots = 1 \).

The continuum random tree bears remarkable connections with the Brownian excursion (cf. for instance Le Gall [14]), and one naturally expects that the standard additive coalescent could also be constructed from the latter. This is indeed feasible (see [7] and also [10]) although its does not seem obvious to relate the following construction with that based on the continuum random tree. Specifically, let \( (\epsilon(s), 0 \leq s \leq 1) \) be a Brownian excursion with unit duration, and for every \( t \geq 0 \), consider the random open set
\[
G(t) = \left\{ s \in [0,1] : ts - \epsilon(s) < \max_{0 \leq u \leq s} (tu - \epsilon(u)) \right\}. \tag{2.1}
\]
Then \( G(t) \) decreases as \( t \) increases, and if we denote by \( F(t) \) the ranked sequence of the lengths of its intervals components (which of course are related to the so-called

The continuum random tree
t-intervals of the preceding section), then \((F(e^{-t}), -\infty < t < \infty)\) is a standard additive coalescent.

More generally, Aldous and Pitman [4] have characterized all the processes that may arise as the limit of additive coalescents started with a large number of small clusters. They are referred to as eternal additive coalescents as these processes are indexed by times in \([-\infty, \infty]\). They can be constructed by the same procedure as in [3] after replacing the continuum random tree \(\tau(\infty)\) by a so-called inhomogeneous continuum random tree.

An alternative construction was proposed in [8] and [15]. Specifically, one may replace the standard Brownian excursion \(\epsilon\) by that obtained by the Takács-Vervaat transformation (1.2) where \((b(s), 0 \leq s \leq 1)\) is now a bridge with exchangeable increments, no positive jumps and infinite variation (which arises as the limit of elementary bridges of the type (1.1), see Kallenberg [16]). The ranked sequence \(F(t)\) of the lengths of the interval components of \(G(t)\) defined by (2.1) then yields a fragmentation process, and by time-reversal, \((F(e^{-t}), -\infty < t < \infty)\) is an eternal additive coalescent.

Roughly, this construction can be viewed as the limit of that presented in Section 1 when the additive coalescent starts from a finite number of clusters.

3. Eternal coagulation and certain Lévy processes

A long time before the notion of stochastic coalescence was introduced, Smoluchowski [20] considered a family of differential equations to model the evolution in the hydrodynamic limit of a particle system in which particles coagulate pairwise as time passes. It bears natural connections with the stochastic coalescence; we refer to the survey by Aldous [2] for detailed explanations, physical motivations, references ... Typically, we are given a symmetric kernel \(K : [0, \infty] \times [0, \infty] \to [0, \infty]\) that specifies the rate at which two particles coagulate as a function of their masses. Here, we take of course \(K(x, y) = x + y\). If we represent the density of particles with mass \(dx\) at time \(t\) by a measure \(\mu_t(dx)\) on \([0, \infty]\), then

\[
\frac{d}{dt} \langle \mu_t, f \rangle = \frac{1}{2} \int_{[0, \infty] \times [0, \infty]} (f(x + y) - f(x) - f(y)) (x + y) \mu_t(dx) \mu_t(dy),
\]

where \(f\) a test function and \(\langle \mu_t, f \rangle = \int f(x) \mu_t(dx)\). Motivated by the preceding section, we are interested in the eternal solutions of (3.1), in the sense that the time parameter \(t\) is real (possibly negative). It is proven in [21] that every eternal solution \((\mu_t)_{t \in \mathbb{R}}\) subject to the normalizing condition \(\int x \mu_t(dx) = 1\) (i.e. the total mass of the system is 1), can be constructed as follows.

First, define the function

\[
\Psi_{\sigma^2, \Lambda}(q) = \frac{1}{2} \sigma^2 q^2 + \int_{[0, \infty]} (e^{-qx} - 1 + qx) \Lambda(dx), \quad q \geq 0,
\]

where \(\sigma^2 > 0\) and \(\Lambda\) is a measure on \([0, \infty]\) with \(\int (x \wedge x^2) \Lambda(dx) < \infty\). We further impose that either \(\sigma^2 > 0\) or \(\int x \Lambda(dx) = \infty\). Next, let \(\Phi(\cdot, s)\) be the inverse of the...
bije cion \( q \to \Psi_{\sigma^2, \Lambda}(sq) + q \). One can check that \( \Phi(q, e^t) \) can be expressed in the form
\[
\Phi(q, e^t) = \int_{[0, \infty]} (1 - e^{-qx}) \mu_t(dx), \quad q \geq 0,
\]
(3.3)
where \( \mu_t \in \mathbb{R} \) is then an eternal solution to Smoluchowski’s coagulation equation. For instance, when \( \sigma^2 = 1 \) and \( \Lambda = 0 \), \( \xi \) is a standard Brownian motion and we recover the well-known solution
\[
\mu_t(dx) = \frac{e^{-t}}{\sqrt{2\pi x^3}} \exp \left( -\frac{xe^{-2t}}{2} \right) dx, \quad t \in \mathbb{R}, x > 0.
\]

This invites a probabilistic interpretation. Indeed, (3.2) is a special kind of Lévy-Khintchine formula; see section VII.1 in [5]. More precisely, there exists a Lévy process with no positive jumps, \( \xi = (\xi_r, r \geq 0) \), such that
\[
E(\exp(q\xi_r)) = \exp(r\Psi_{\sigma^2, \Lambda}(q)), \quad q \geq 0.
\]
It is then well-known (e.g. Theorem VII.1 in [5]) that the first passage process
\[
T_x^{(s)} := \inf \{ r \geq 0 : s\xi_r + r > x \}, \quad x \geq 0
\]
is a subordinator with
\[
E\left( \exp\left( -qT_x^{(s)} \right) \right) = \exp(-x\Phi(q, s)), \quad q, x \geq 0,
\]
where the Laplace exponent \( \Phi(\cdot, s) \) is the inverse bijection of \( q \to \Psi_{\sigma^2, \Lambda}(sq) + q \). Thus (3.3) can be interpreted as the Lévy-Khintchine formula for \( \Phi(\cdot, s) \), and we conclude that the eternal solution \( \mu_t \) can be identified as the Lévy measure of the subordinator \( T^{(s)} \) for \( s = e^t \).

This probabilistic interpretation also points at a simple random model for aggregation of intervals. Indeed, the closed range \( T^{(s)} = \{(T_x^{(s)}, x \geq 0) \} \) of \( T^{(s)} \) induces a partition of \([0, \infty]\) into a family of random disjoint open intervals, namely the interval components of \( G(s) = [0, \infty] \setminus T^{(s)} \). We now make the key observation that
\[
T^{(s)} \subseteq T^{(s')} \quad \text{for } 0 < s' < s,
\]
(3.4)
because an instant at which \( r \to s\xi_r + r \) reaches a new maximum is also an instant at which \( r \to s'\xi_r + r \) reaches a new maximum. Roughly, (3.4) means that the random partitions get coarser as the parameter \( s \) increases; and therefore they induce a process in which intervals aggregate. The latter is closely related to a special class of eternal additive coalescents, and has been studied in [7, 18] in the Brownian case, and in [15] in the general case.

4. Sticky particle systems

Sticky particle systems evolve according to the dynamics of completely inelastic collisions with conservation of mass and momentum, which are also known as the
dynamics of ballistic aggregation. This means that the velocity of particles only changes in case of collision, and in that case, a heavier cluster merges at the location of the shock with mass and momentum given by the sum of the masses and momenta of the clusters involved. This has been proposed as a model for the formation of large scale structures in the universe; see the survey article [22]. We now have two quite different dynamics for clustering: on the one hand the ballistic aggregation which is deterministic, and on the other hand the additive coalescence which is random and may appear much more elementary as it does not take into account significant physical parameters such as distances between clusters and the relative velocities. Nonetheless, there is a striking connection between the two when randomness is introduced in the deterministic model, as we shall now see.

We henceforth focus on dimension one and assume that at the initial time, particles are infinitesimal (i.e. fluid) and uniformly distributed on the line. The evolution of the sticky particle system can then be completely analyzed in terms of the entropy solution to a single PDE, the transport equation

\[ \partial_t u + u \partial_x u = 0. \]  

(4.1)

Here \( u(x,t) \) represents the velocity of the particle located at \( x \) at time \( t \), and the entropy condition imposes that for every fixed \( t > 0 \), the function \( u(\cdot,t) \) has only discontinuities of the first kind and no positive jumps (the latter restriction accounts for the total inelasticity of collisions). Provided that the initial velocity \( u(\cdot,0) \) satisfies some very mild hypothesis on its rate of growth, there is a unique weak solution to the equation (4.1) which fulfills the entropy condition, and which can be given explicitly in terms of \( u(\cdot,0) \).

We assume that the initial velocities in the particle system are random, and more precisely

\[ u(r,0) = 0 \quad \text{for} \quad r < 0 \quad \text{and} \quad (u(r,0), r \geq 0) \overset{d}{=} (\xi_r, r \geq 0), \]

where \( \xi \) denotes the Lévy process with no positive jumps which was used in the preceding section.

Roughly, the dynamics of sticky particles are not only deterministic, but also induce a loss of information as time goes by, in the sense that the initial state of the system entirely determines the state at time \( t > 0 \), but cannot be completely recovered from the latter. In this direction, let us observe the system at some fixed time \( t > 0 \), i.e. we know the locations, masses and velocities of the clusters at this time. Let us pick a cluster located in \([0,\infty[\), using for this only the information available at time \( t \) (for instance, we may choose the heaviest cluster located at time \( t \) in \([0,1]\)). We shall work conditionally on the mass of this cluster, and for simplicity, let us assume it has unit mass. For every \( r \in [0,t] \), denote by \( M(r) = (m_1(r), m_2(r), \ldots) \) the ranked sequence of masses of clusters at time \( r \) which, by time \( t \) have aggregated to form the cluster we picked, so \( M(r) \) can be viewed as a random variable with values in \( S^1 \). Then the time-changed processes

\[ M \left( t \left(1 - \frac{t}{t+e^s}\right) \right), \quad -\infty < s < \infty \]
is an eternal additive coalescent. This was established in [6] in the case of Brownian initial velocity; and the recent developments on eternal additive coalescents made in [4, 8, 15] show that the arguments also applies for Lévy type initial velocities.

References

[1] D. J. Aldous. The continuum random tree III. *Ann. Probab.* 21 (1993), 248–289.
[2] D. J. Aldous. Deterministic and stochastic models for coalescence (aggregation, coagulation): a review of the mean-field theory for probabilists. *Bernoulli* 5 (1999), 3–48.
[3] D. J. Aldous and J. Pitman. The standard additive coalescent. *Ann. Probab.* 26 (1998), 1703–1726.
[4] D. J. Aldous and J. Pitman. Inhomogeneous continuum random trees and the entrance boundary of the additive coalescent. *Probab. Theory Related Fields* 118 (2000), 455–482.
[5] J. Bertoin. *Lévy processes*. Cambridge University Press, Cambridge, 1996.
[6] J. Bertoin. Clustering statistics for sticky particles with Brownian initial velocity. *J. Math. Pures Appl.* 79 (2000), 173–194.
[7] J. Bertoin. A fragmentation process connected to Brownian motion. *Probab. Theory Relat. Fields* 117 (2000), 289–301.
[8] J. Bertoin. Eternal additive coalescents and certain bridges with exchangeable increments. *Ann. Probab.* 29 (2001), 344–360.
[9] J. Bertoin. Eternal solutions to Smoluchowski’s coagulation equation with additive kernel and their probabilistic interpretations. *Ann. Appl. Probab.* (2002) (To appear).
[10] P. Chassaing and G. Louchard. Phase transition for parking blocks, Brownian excursions and coalescence. *Random Structures Algorithms.* (2002) (To appear).
[11] R. L. Drake. A general mathematical survey of the coagulation equation. In G. M. Hidy and J. R. Brock (eds): *Topics in current aerosol research, part 2*. International Reviews in Aerosol Physics and Chemistry, 201–376, Pergamon, 1972.
[12] S. N. Evans and J. Pitman (1998). Construction of Markovian coalescents. *Ann. Inst. H. Poincaré, Probabilités Statistiques* 34, 339–383.
[13] A. M. Golovin (1963). The solution of the coagulation equation for cloud droplets in a rising air current. *Izv. Geophys. Ser.* 5, 482–487.
[14] J.-F. Le Gall (1993). The uniform random tree in a Brownian excursion. *Probab. Theory Relat. Fields* 96, 369–384.
[15] G. Miermont (2001). Ordered additive coalescent and fragmentations associated to Lévy processes with no positive jumps. *Electr. J. Prob.* 6 paper 14, 1–33. [http://math.washington.edu/~ejpecp/ejp6contents.html](http://math.washington.edu/~ejpecp/ejp6contents.html)
[16] O. Kallenberg (1973). Canonical representations and convergence criteria for processes with interchangeable increments. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* 27, 23–36.
[17] J. Pitman (1999). Coalescent random forests. *J. Comb. Theory A.* **85**, 165–193.

[18] J. Schweinsberg (2001). Applications of the continuous-time ballot theorem to Brownian motion and related processes. *Stochastic Process. Appl.* **95**, 151–176.

[19] R. K. Sheth and J. Pitman (1997). Coagulation and branching processes models of gravitational clustering. *Mon. Not. R. Astron. Soc.*

[20] M. von Smoluchowski (1916). Drei Vorträge über Diffusion, Brownsche Bewegung und Koagulation von Kolloidteilchen. *Physik. Z.* **17**, 557–585.

[21] L. Takács (1966). *Combinatorial methods in the theory of stochastic processes.* Wiley, New York.

[22] M. Vergassola, B. Dubrulle, U. Frisch and A. Noullez (1994). Burgers’ equation, devil’s staircases and the mass distribution function for large-scale structures. *Astron. Astrophys.* **289**, 325–356.

[23] W. Vervaat (1979). A relation between Brownian bridge and Brownian excursion. *Ann. Probab.* **7**, 141–149.