Improving the sequential update algorithm for computing the stationary distribution vector in upper block-Hessenberg Markov chains

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Abstract
This paper improves the sequential update algorithm proposed in the paper (Masuyama, Queueing Syst., 2019, DOI: 10.1007/s11134-019-09599-x), which computes the stationary distribution vector in continuous time upper block-Hessenberg Markov chains (upper BHMCs). The improved algorithm assumes only the ergodicity of upper BHMCs, and it does not assume such additional conditions as the Foster-Lyapunov drift condition required by the original algorithm. Thus, the improved algorithm does not require to find a Lyapunov function or to compute a certain vector expressed with the function, unlike the original algorithm. This shows that the improved algorithm is more efficient and more flexible than the original one.

Keywords: Upper block-Hessenberg Markov chain (upper BHMC); Level-dependent M/G/1-type Markov chain; Matrix-infinite-product (MIP) form; Last-block-column-linearly-augmented truncation (LBCL-augmented truncation); Foster-Lyapunov drift condition; Lyapunov function

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1 Introduction

The purpose of this paper is to improve the sequential update algorithm, proposed in [9], for computing the stationary distribution vector in continuous time upper block-Hessenberg Markov chains (upper BHMCs).

We first provide the definition of upper BHMCs. Let \( \mathbb{N} = \{1, 2, 3, \ldots\} \), \( \mathbb{Z}_+ = \{0, 1, 2, \ldots\} \), and
\[
\mathcal{S} = \bigcup_{k=0}^{\infty} \{k\} \times \mathbb{M}_k,
\]

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where $M_k = \{1, 2, \ldots, M_k\} \subset \mathbb{N}$ for $k \in \mathbb{Z}_+$. Let $Q := (q(k, i; \ell, j))_{(k, i; \ell, j) \in \mathcal{S}^2}$ denote a proper $Q$-matrix (see Definition A.1), i.e.,

$$q(k, i; k, i) \in (-\infty, 0], \quad (k, i) \in \mathcal{S},$$

$$q(k, i; \ell, j) \in [0, \infty), \quad (k, i) \in \mathcal{S}, \ (\ell, j) \in \mathcal{S} \setminus \{(k, i)\},$$

$$\sum_{(\ell, j) \in \mathcal{S}} q(k, i; \ell, j) = 0, \quad (k, i) \in \mathcal{S}.$$ 

Assume that $Q$ is in an upper block-Hessenberg form:

$$Q = \begin{pmatrix}
L_0 & L_1 & L_2 & L_3 & \cdots \\
L_0 & Q_{0,0} & Q_{0,1} & Q_{0,2} & Q_{0,3} & \cdots \\
L_1 & Q_{1,0} & Q_{1,1} & Q_{1,2} & Q_{1,3} & \cdots \\
L_2 & \ddots & \ddots & \ddots & \ddots & \cdots \\
L_3 & \ddots & \ddots & \ddots & \ddots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \cdots
\end{pmatrix},$$

where $L_k = \{k\} \times M_k$ for $k \in \mathbb{Z}_+$ and $L_k$ is called level $k$. A Markov chain on state space $\mathcal{S}$ is said to be an upper block-Hessenberg Markov chain (upper BHMC) if its (infinitesimal) generator is given by $Q$. Note that an upper BHMC may be called a level-dependent $M/G/1$-type Markov chain.

Throughout the paper, unless otherwise stated, we assume that generator $Q$ is ergodic (i.e., irreducible and positive recurrent). We then define $\pi := (\pi(k, i))_{(k, i) \in \mathcal{S}}$ as a unique and positive stationary distribution vector of the ergodic generator $Q$ (see, e.g., [1, Chapter 5, Theorems 4.4 and 4.5]). Thus,

$$\pi Q = 0, \quad \pi e = 1, \quad \pi > 0,$$

where $e$ denotes a column vector of ones with an appropriate dimension. For later reference, we also define $\pi(k) = (\pi(k, i))_{i \in M_k}$ for $k \in \mathbb{Z}_+$, which leads to the partition of $\pi$:

$$\pi = \begin{pmatrix}
L_0 & L_1 & \cdots \\
L_0 & L_1 & \cdots \\
\vdots & \ddots & \cdots
\end{pmatrix}. $$

Several researchers have studied the computation of the stationary distributions vectors of upper BHMCs and level-dependent quasi-birth-and-death processes (LD-QBDs). An LD-QBD is a special case of upper BHMCs. All the previous studies [2, 4, 5, 6, 7, 10, 12, 13], except for [9], focus on computing their respective approximations to the stationary distribution vector.

Masuyama [9] proposes a sequential update algorithm for computing $\pi = (\pi_0, \pi_1, \ldots)$ by using the last-block-column-linearly-augmented truncation approximation (LBCL-augmented truncation approximation). More specifically, the sequential update algorithm in [9] constructs a convergent sequence of LBCL-augmented truncation approximations, which yields a matrix-infinite-product (MIP) form of $\pi$ (see also [8]). However, the algorithm requires the two additional conditions below.
Condition 1 of [9] (Foster-Lyapunov drift condition): There exist a constant $b \in (0, \infty)$, a finite set $\mathbb{C} \subseteq \mathbb{S}$, and a positive column vector $v := (v(k, i))_{(k,i) \in \mathbb{S}}$ such that $\inf_{(k,i) \in \mathbb{S}} v(k, i) > 0$ and

$$Qv \leq -e + b1_{\mathbb{C}},$$

where $1_{\mathbb{A}} := (1_{\mathbb{A}}(k, i))_{(k,i) \in \mathbb{S}}$, $\mathbb{A} \subseteq \mathbb{S}$, denotes a column vector defined by

$$1_{\mathbb{A}}(k, i) = \begin{cases} 1, & (k, i) \in \mathbb{S}, \\ 0, & (k, i) \in \mathbb{S} \setminus \mathbb{A}. \end{cases}$$

The vector $v$ corresponds to a Lyapunov function in this drift condition.

Condition 2 of [9]:

$$\sum_{(n,i) \in \mathbb{S}} \pi(n, i)|q(n, i; n, i)|v(n, i) < \infty. \tag{1.2}$$

Under these conditions, the sequential update algorithm in [9] computes a certain vector $y_n$ as an “intermediate product” leading to an MIP form of $\pi$. The vector $y_n$ includes an infinite sum (see (3.10) below), which can be time-consuming to compute. To improve this algorithm, we avoid computing $y_n$ and remove Conditions 1 and 2 of [9]. This is our main contribution.

The rest of this paper is divided into two sections. Section 2 contains preliminary results on the LBCL-augmented truncation approximation. Section 3 presents the main theorem and an improved version of the sequential update algorithm in [9].

2 Preliminary results

This section describes the LBCL-augmented truncation approximation. We first truncate the state space $\mathbb{S}$ to $\mathbb{S}_n := \bigcup_{k=0}^{n} \mathbb{L}_k$, where $n \in \mathbb{Z}_+$. We then define $(n)Q := (n)q(k; \ell, j))_{(k,\ell,j) \in (\mathbb{S}_n)^2}$, $n \in \mathbb{Z}_+$, as

$$\begin{pmatrix} Q_{0,0} & Q_{0,1} & Q_{0,2} & \cdots & Q_{0,n-2} & Q_{0,n-1} & Q_{0,n} \\ Q_{1,0} & Q_{1,1} & Q_{1,2} & \cdots & Q_{1,n-2} & Q_{1,n-1} & Q_{1,n} \\ 0 & Q_{2,1} & Q_{2,2} & \cdots & Q_{2,n-2} & Q_{2,n-1} & Q_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & Q_{n-1,n-2} & Q_{n-1,n-1} & Q_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 & Q_{n,n-1} & Q_{n,n} \end{pmatrix}.$$  

We also define $(n)\hat{Q} := (n)\hat{q}(k; \ell, j))_{(k,\ell,j) \in (\mathbb{S}_n)^2}$, $n \in \mathbb{Z}_+$, as

$$(n)\hat{Q} = (n)Q - (n)Q e_{(n)\hat{\alpha}}, \quad n \in \mathbb{Z}_+,$$

where $(n)\hat{\alpha}$ is a probability vector such that

$$(n)\hat{\alpha} = \begin{pmatrix} \mathbb{S}_{n-1} & \mathbb{L}_n \\ 0 & \alpha_n \end{pmatrix}.$$
The matrix \((n)\hat{Q}\) is a proper \(Q\)-matrix, and can be considered a *linearly-augmented truncation* of \(Q\) such that the augmentation distribution \((n)\hat{\alpha}\) has its probability masses only on the last block column (see \cite{9} Section 1). Thus, \((n)\hat{Q}\) is referred to as the *last-block-column-linearly-augmented truncation* (LBCL-augmented truncation).

Let \((n)\hat{\pi} := \{(n)\hat{\pi}(k,i)\}_{(k,i)\in S_n}, n \in \mathbb{Z}_+,\) denote
\[
(n)\hat{\pi} = \frac{(n)\hat{\alpha} \left( - (n)Q \right)^{-1}}{(n)\hat{\alpha} \left( - (n)Q \right)^{-1} e}, \quad n \in \mathbb{Z}_+, \tag{2.1}
\]
which is a stationary distribution vector of the LBCL-augmented truncation \((n)\hat{Q}\). Hence, \((n)\hat{\pi}\) is referred to as the LBCL-augmented truncation approximation to \(\pi\).

For later use, we provide a matrix-product form of \((n)\hat{\pi}_k := \{(n)\hat{\pi}(k,i)\}_{i\in M_n}, n \in \mathbb{Z}_+, k \in \mathbb{Z}_n := \{0,1,\ldots,n\}.\) For this purpose, we define some symbols. Let
\[
U^*_n = \begin{cases} 
(-Q_{0,0})^{-1}, & n = 0, \\
\left(-Q_{n,n} - \sum_{\ell=0}^{n-1} U_{n,\ell} Q_{\ell,n}\right)^{-1}, & n \in \mathbb{N},
\end{cases} \tag{2.2}
\]
where the empty sum is defined as zero (e.g., \(\sum_{\ell=0}^{-1} \cdot = 0\)), and where, for \(n \in \mathbb{Z}_+,\)
\[
U_{n,k} = \begin{cases} 
(Q_{n,n-1} U^*_{n-1})(Q_{n-1,n-2} U^*_{n-2}) \cdots (Q_{k+1,k} U^*_k), & k \in \mathbb{Z}_{n-1}, \\
I, & k = n.
\end{cases}
\]
We then have (see \cite{9}, Remark 2.2)
\[
\pi_\ell = \pi_n U_{n,\ell}, \quad n \in \mathbb{N}, \ \ell \in \mathbb{Z}_n. \tag{2.3}
\]
For \(n \in \mathbb{Z}_+,\) we also define \(U^*_{n,k} (k \in \mathbb{Z}_n)\) and \(u^*_n := (u^*_n(i))_{i\in M_n}\) as
\[
U^*_{n,k} = U^*_n U_{n,k}, \quad k \in \mathbb{Z}_n, \tag{2.4}
\]
\[
u^*_n = \sum_{\ell=0}^{n} U^*_{n,\ell} e = \sum_{\ell=0}^{n} U^*_n U_{n,\ell} e > 0, \tag{2.5}
\]
where \(u^*_n > 0\) follows from \cite{9}, Remark 2.3. Combining (2.4), (2.5), and \cite{9}, Lemma 2.2, we obtain a matrix-product form of \((n)\hat{\pi}_k\).

**Proposition 2.1** (\cite{9}, Equation (2.21)) For \(n \in \mathbb{Z}_+\) and \(k \in \mathbb{Z}_n,\)
\[
(n)\hat{\pi}_k = \frac{\alpha_n U^*_{n,k}}{\alpha_n u^*_n}, \tag{2.6}
\]
or equivalently,
\[
(n)\hat{\pi}_k = \frac{\alpha_n U^*_{n,k} U^*_{n-1} U_{n-2} \cdots U_0}{\sum_{\ell=0}^{n} \alpha_n U^*_{n} U_{n-1} U_{n-2} \cdots U_{\ell} e}
\]
where \(U_k = Q_{k+1,k} U^*_k\) for \(k \in \mathbb{Z}_+.\)
3 Main results

In this section, we first present our main theorem, which yields a matrix-infinite-product (MIP) form of the stationary distribution vector \( \pi \) without any additional condition. Using this result, we then improve the sequential update algorithm proposed in [9].

We fix \( N \in \mathbb{Z}_+ \) arbitrarily and define \( \mathbb{Z}_+ \setminus N = \mathbb{Z}_+ \setminus \mathbb{Z}_N \). We then consider a linear fractional programming (LFP) problem for each \( n \in \mathbb{Z}_N \):

\[
\text{LFP (3.1)}: \quad \text{Maximize} \quad r_n(\alpha_n) := \frac{\alpha_n u_{n,N}^*}{\alpha_n u_n^*}; \quad \text{(3.1a)} \\
\text{Subject to} \quad \alpha_n \geq 0, \quad \text{(3.1b)} \\
\alpha_n e = 1, \quad \text{(3.1c)}
\]

where \( u_{n,N}^* := (u_{n,N}^*(i))_{i \in \mathbb{M}_n}, n \in \mathbb{Z}_N \), denotes

\[
u_{n,N}^* = U_{n,N}^* e = U_n^* U_{n,N} e. \quad \text{(3.1d)}
\]

Let \( \alpha_n^* := (\alpha_n^*(j))_{j \in \mathbb{M}_n}, n \in \mathbb{Z}_N \), denote a probability vector such that

\[
\alpha_n^*(j) = \begin{cases} 
1, & j = j_n^*, \\
0, & j \neq j_n^*,
\end{cases} \quad \text{(3.2)}
\]

where

\[
j_n^* = \arg \max_{j \in \mathbb{M}_n} \frac{u_{n,N}^*(j)}{u_n^*(j)}. \quad \text{(3.3)}
\]

We then have the following lemma.

**Lemma 3.1** For each \( n \in \mathbb{Z}_N \), the probability vector \( \alpha_n^* \) is an optimal solution of LFP (3.1).

**Proof.** It follows from (3.2) and (3.3) that

\[
\xi_n := \frac{\alpha_n u_{n,N}^*}{\alpha_n u_n^*} = \frac{u_{n,N}^*(j_n^*)}{u_n^*(j_n^*)} = \max_{j \in \mathbb{M}_n} \frac{u_{n,N}^*(j)}{u_n^*(j)},
\]

which yields \( u_{n,N}^* \leq \xi_n u_n^* \). Thus, for any \( 1 \times M \) probability vector \( p_n \), we obtain

\[
\frac{p_n u_{n,N}^*}{p_n u_n^*} \leq \xi_n = \frac{\alpha_n^* u_{n,N}^*}{\alpha_n^* u_n^*},
\]

which shows that \( \alpha_n^* \) is an optimal solution of LFP (3.1). \( \square \)

Let \( \hat{\pi}_k^*, k \in \mathbb{Z}_N \), denote a vector obtained by replacing \( \alpha_n \) in (2.6) with \( \alpha_n^* \), i.e.,

\[
(\hat{\pi}_k^*)_{(n)j} = \frac{\alpha_n^* U_{n,k}^*}{\alpha_n^* u_n^*} = \frac{\text{row}\{U_{n,k}^*\}_{j_n^*}}{u_n^*(j_n^*)}, \quad k \in \mathbb{Z}_n, \quad \text{(3.4)}
\]

where \( \text{row}\{\cdot\}_{j} \) denotes the \( j \)-th row of the matrix in the brackets. Note that \( (\hat{\pi}_k^*)_{(n)j}, (\hat{\pi}_j^*)_{(n)j}, \ldots, (\hat{\pi}_n^*)_{(n)j} \) is an LBCL-augmented truncation approximation to \( \pi \).

It follows from (3.1a), (3.1d), and (3.4) that

\[
(\hat{\pi}_N^*)e = \frac{\alpha_n^* u_{n,N}^*}{\alpha_n^* u_n^*} = r_n(\alpha_n^*). \quad \text{(3.5)}
\]

Using this equation, we obtain the following lemma.
Lemma 3.2 For any fixed \( N \in \mathbb{Z}_+ \),

\[
\liminf_{n \to \infty} \left( n \hat{\pi}^*_N e \right) \geq \pi_N e. \tag{3.6}
\]

Proof. Using (2.2), (2.3), and \( \sum_{\ell=0}^{\infty} \pi_{\ell} Q_{\ell,n} = 0 \), we obtain

\[
\pi_n(U_n^*)^{-1} = -\pi_n Q_{n,n} - \pi_n \sum_{\ell=0}^{n-1} U_{n,\ell} Q_{\ell,n}
\]

\[
= -\sum_{\ell=0}^{n} \pi_{\ell} Q_{\ell,n} = \sum_{\ell=n+1}^{\infty} \pi_{\ell} Q_{\ell,n} \geq 0, \quad n \in \mathbb{Z}_+.
\]

In addition, since \( \pi > 0 \) and \( Q \) is ergodic, \( \sum_{\ell=n+1}^{\infty} \pi_{\ell} Q_{\ell,n} \neq 0 \) for \( n \in \mathbb{Z}_+ \). Therefore,

\[
\pi_n(U_n^*)^{-1} \geq 0, \quad n \in \mathbb{Z}_+,
\]

which leads to

\[
\tilde{\alpha}_n := \frac{\pi_n(U_n^*)^{-1}}{\pi_n(U_n^*)^{-1} e} \geq 0, \quad n \in \mathbb{Z}_+.
\tag{3.7}
\]

The probability vector \( \tilde{\alpha}_n \) is a feasible solution of LFP (3.1). Thus, using (3.5), we obtain

\[
(n) \hat{\pi}_N^* e = r_n(\alpha^*_n) \geq r_n(\tilde{\alpha}_n) = \frac{\pi_n(U_n^*)^{-1} u_{n,N}^*}{\pi_n(U_n^*)^{-1} e}, \quad n \in \mathbb{Z}_N, \tag{3.8}
\]

where the last equality is due to (3.1a) and (3.7). Furthermore, from (2.3), (2.5), and (3.1d), we have

\[
\pi_n(U_n^*)^{-1} u_{n,N}^* = \pi_n U_{n,N} e = \pi_N e, \quad n \in \mathbb{Z}_N,
\]

\[
\pi_n(U_n^*)^{-1} u_n^* = \sum_{\ell=0}^{n} \pi_n U_{n,\ell} e = \sum_{\ell=0}^{n} \pi_{\ell} e.
\]

Combining these equations and (3.8) yields

\[
(n) \hat{\pi}_N^* e \geq \frac{\pi_N e}{\sum_{\ell=0}^{\infty} \pi_{\ell} e}, \quad n \in \mathbb{Z}_N,
\]

which implies that (3.6) holds. \( \square \)

To describe our main result, we introduce the total variation distance: For vectors \( h_1 := (h_1(j))_{j \in A} \) and \( h_2 := (h_2(j))_{j \in B} \),

\[
\|h_1 - h_2\| := \sum_{j \in A \cap B} |h_1(j) - h_2(j)| + \sum_{j \in A, j \notin B} |h_1(j)| + \sum_{j \in B, j \notin A} |h_2(j)|,
\]

where \( A \) and \( B \) are subsets of a common countable set.

The following is the main theorem of this paper, which is an immediate consequence of Lemmas 3.2 and A.3(ii).
Theorem 3.1 We have
\[ \lim_{n \to \infty} \| \hat{\pi}^*_n - \pi \| = 0. \]

Theorem 3.1 yields an MIP form of \( \pi = (\pi_0, \pi_1, \ldots) \).

Corollary 3.1 For \( k \in \mathbb{Z}_+ \),
\[ \pi_k = \lim_{n \to \infty} \frac{\alpha_n^* U^*_{n,k}}{\alpha_n^* u^*_n} = \frac{\alpha_n^* U^*_{n,k} U^*_{n-1} U^*_{n-2} \cdots U^*_{0}}{\alpha_n^* \sum_{\ell=0}^{n} U^*_{n-1} U^*_{n-2} \cdots U^*_{\ell}}. \]
Furthermore, this limit converges uniformly for all \( k \in \mathbb{Z}_+ \).

Proof. The proof of this corollary is the same as that of [9, Corollary 3.1].

A similar MIP form of \( \pi = (\pi_0, \pi_1, \ldots) \) is presented in [9, Corollary 3.1], though the additional conditions are assumed (see Conditions 1 and 2 therein) and \( \alpha_n^* \) is replaced with an optimal solution of the following LFP problem (see Section 3.1 therein):

\[
\text{LFP (3.9)} : \quad \text{Minimize} \quad \frac{\alpha_n y_n}{\alpha_n^* u^*_n}; \\
\text{Subject to} \quad \alpha_n \geq 0, \\
\alpha_n^* e = 1,
\]
where \( y_n \) is given by
\[ y_n = v_n + \sum_{k=0}^{n} U^*_{n,k} \sum_{\ell=n+1}^{\infty} Q_{k,\ell} v_\ell > 0. \quad (3.10) \]

With an optimal solution of LFP (3.9), the sequential update algorithm in [9] generates an MIP form of \( \pi \).

Our MIP form in Corollary 3.1 is derived by using \( u^*_{n,N} \) given in (3.1d), instead of \( y_n \). The vector \( y_n \) includes the infinite sum involved with the vector \( v = (v^\top_0, v^\top_1, \ldots)^\top \) satisfying (1.1) and (1.2), whereas \( u^*_{n,N} \) is readily calculated by some components of our MIP form. Furthermore, our MIP form does not require any additional condition such as Conditions 1 and 2 of [9]. Therefore, by using an optimal solution \( \alpha_n^* \) of LFP (3.1) [instead of an optimal solution of LFP (3.9)], we can improve the sequential update algorithm in [9] to be more efficient and more flexible.

We present our improved algorithm below, which requires the recursions of \( \{U^*_{n,k}\} \) and \( \{u^*_n\} \) (see [9, Equations (3.23)–(3.25))):

\[ U^*_{0,0} = U^*_0 = (-Q^*_{0,0})^{-1}, \]
\[ U^*_{n,k} = \begin{cases} 
U^*_n Q_{n,n-1} \cdot U^*_{n-1,k}, & n \in \mathbb{N}, \quad k \in \mathbb{Z}_{n-1}, \\
U^*_n, & n \in \mathbb{N}, \quad k = n,
\end{cases} \]

and

\[ u^*_0 = U^*_0 e = (-Q^*_{0,0})^{-1} e, \]
\[ u^*_n = U^*_n (e + Q_{n,n-1} u^*_{n-1}), \quad n \in \mathbb{N}. \]
with
\[ U_n^* = \left( -Q_{n,n} - Q_{n,n-1} \sum_{\ell=0}^{n-1} U_{n-1,\ell}^* Q_{\ell,n} \right)^{-1}, \quad n \in \mathbb{N}. \] (3.13)

**Algorithm**: An improved version of the sequential update algorithm in [9]

**Input**: \( Q, N \in \mathbb{Z}_+ \), \( \varepsilon \in (0,1) \), and increasing sequence \( \{n_\ell \in \mathbb{Z}_N; \ell \in \mathbb{Z}_+\} \).

**Output**: \((n)\hat{\pi}^* = \left((n)\hat{\pi}_0^*, (n)\hat{\pi}_1^*, \ldots, (n)\hat{\pi}_n^*\right)\), where \( n \in \mathbb{Z}_N \) is fixed when the iteration stops.

1. Set \( n = 0 \) and \( \ell = 1 \).
2. Compute \( U_0^* \) by (3.11a) and \( u_0^* \) by (3.12a).
3. Iterate (a)–(d) below:
   (a) Increment \( n \) by one.
   (b) Compute \( U_n^* = U_{n,n}^* \) by (3.13).
   (c) Compute \( U_{n,k}^* \), \( k = 0, 1, \ldots, n - 1 \), by (3.11b) and \( u_n^* \) by (3.12b).
   (d) If \( n = n_\ell \), then perform the following:
      i. Compute \( u_{n,N}^* \) by (3.1d), and find \( j_n^* \) satisfying (3.3).
      ii. Compute \( (n)\hat{\pi}_k^* \), \( k = 0, 1, \ldots, n \), by (3.4).
      iii. If \( \| (n)\hat{\pi}^* - (n_{\ell-1})\hat{\pi}^* \| < \varepsilon \), then stop the iteration; otherwise increment \( \ell \) by one and return to step (a).

**Remark 3.1** We can set parameter \( N \) to an arbitrary value in \( \mathbb{Z}_+ \). The value of \( N \) would impact on the convergence speed of the above algorithm. However, it would be difficult to discuss theoretically an optimal value of \( N \). The only thing we can say is that too large \( N \) would make it difficult to find an optimal solution \( \alpha_n^* \) of LFP (3.1). This is because the values of \( u_{n,N}^*(j)/u_n^*(j) \)'s, \( j \in \mathbb{M}_n \), can be too small and thus numerical errors can not be negligible. A better choice would be to set \( N = 0 \).

**Remark 3.2** The stopping criterion
\[ \| (n)\hat{\pi}^* - (n_{\ell-1})\hat{\pi}^* \| < \varepsilon \]
can yield an undesirable result. In fact, if \( (n)\hat{\pi}^* \) is enough close to the stationary distribution vector \( \pi \), then this criterion is certainly satisfied. However, the reverse is not necessarily true. More specifically, the iteration may stop with not enough large \( n_\ell \) and thus the output \( (n)\hat{\pi}^* \) may not be enough close to \( \pi \). Such an undesirable situation can be avoided by choosing the sequence \( \{n_\ell; \ell \in \mathbb{Z}_+\} \) such that it is rapidly increasing, e.g., \( n_\ell = 2^\ell + N \).

**A Basic lemmas**

This section presents some definitions and basic lemmas associated with the subinvariant measure of \( Q \)-matrices. They are a foundation of the main results of this paper. Especially, the last lemma (Lemma A.3) is crucial to the main theorem (Theorem 3.1).
We redefine the symbols $S$ and $Q$ introduced in the body of the paper: Let $S$ denote a countable set, and let $Q := (q(i, j))_{i,j \in S}$ denote a $Q$-matrix (see, e.g., [1, page 64]); that is, a diagonally dominant matrix such that

\[
q(i, i) \in [-\infty, 0], \quad i \in S, \\
q(i, j) \in [0, \infty), \quad i \in S, \ j \in S \ \{i\}, \\
\sum_{j \in S} q(i, j) \leq 0, \quad i \in S.
\]

We now define the type of $Q$-matrices on which we focus.

**Definition A.1** A $Q$-matrix $Q$ is **proper** if and only if it is stable and conservative, that is, satisfies the following: For all $i \in S$,

(i) $q(i, i)$ is finite (stability); and

(ii) $\sum_{j \in S} q(i, j) = 0$ (conservativity).

**Remark A.1** A proper $Q$-matrix is not necessarily regular (or equivalently, non-explosive). A proper $Q$-matrix $Q$ is regular if and only if, for any $\lambda > 0$, the system of equations

\[
Qx = \lambda x \quad \text{with} \quad x := (x(i))_{i \in S} \geq 0
\]

has no bounded solutions other than $x = 0$ (see, e.g., [1, Chapter 2, Theorem 2.7] and [3, Chapter 8, Theorem 4.4]).

In what follows, we assume that $Q$ is a proper $Q$-matrix, and then we discuss its subinvariant measures (see Definition A.2 below). Note that a proper $Q$-matrix may be called an (infinitesimal) generator if it is associated with a regular-jump Markov chain (see, e.g., [3, Chapter 8, Definition 2.5]). For later reference, we define $\{\Phi(t); t \geq 0\}$ as a regular-jump Markov chain on state space $S$ with generator $Q$.

**Definition A.2** For a proper $Q$-matrix $Q$, a vector $\mu := (\mu(j))_{j \in S} \geq 0, \neq 0$ is said to be a subinvariant measure if

\[
\mu Q \leq 0.
\]

Furthermore, if $\mu Q = 0$, then $\mu$ is said to be an invariant measure (or stationary measure). Note that if an invariant measure is a probability vector then it may be called a stationary distribution vector (stationary distribution) or an invariant (or stationary) probability measure.

The subinvariant measure of a proper $Q$-matrix is associated with its irreducibility and recurrence. Thus, we provide the definitions of them, after introducing some necessary symbols. Let

\[
P = I + \text{diag}\{-Q\}^{-1}Q,
\]

where $\text{diag}\{-Q\}$ is a diagonal matrix whose diagonal elements are identical to those of $-Q$. The stochastic matrix $P$ is the transition probability matrix of an embedded discrete-time Markov chain for $\{\Phi(t)\}$ with generator $Q$ (see, e.g., [3, Chapter 8, Section 4.2]). Hence, we call $P$ the embedded transition probability matrix of $\{\Phi(t)\}$ with generator $Q$. 

**Definition A.3** The proper $Q$-matrix $Q$ and its Markov chain $\{\Phi(t)\}$ are irreducible (resp. transient, recurrent) if and only if the embedded transition probability matrix $P$ in (A.1) is irreducible (resp. transient, recurrent) (see, e.g., [3, Chapter 8, Definitions 5.1 and 5.2]). Furthermore, the irreducible $Q$-matrix $Q$ and its Markov chain $\{\Phi(t)\}$ are ergodic (i.e., positive recurrent) if and only if there exists a summable invariant measure unique up to constant multiples (see, e.g., [3, Chapter 8, Definitions 5.4, Theorems 5.1–5.3]).

We now define $\eta$ as

$$\eta = \mu \text{diag}\{-Q\} \geq 0, \neq 0. \quad (A.2)$$

It then follows from (A.1) and (A.2) that

$$\eta P = \eta + \mu Q.$$ 

Thus, $\eta$ is a subinvariant (resp. invariant) measure of the embedded transition probability matrix $P$ (see, e.g., [11, Definition 5.3]) if and only if $\mu$ is a subinvariant (resp. invariant) measure of the proper $Q$-matrix $Q$. Therefore, the following lemma is an immediate consequence of Theorem 5.4 (together with its corollary) and Lemmas 5.5 and 5.6 in [11].

**Lemma A.1** Suppose that the proper $Q$-matrix $Q$ is irreducible.

(i) There always exists a subinvariant measure of $Q$.

(ii) Any subinvariant measure of $Q$ is positive, that is, its elements are all positive.

(iii) If $Q$ is recurrent, then it has an invariant measure, which is unique up to constant multiples.

(iv) If $\nu$ is a subinvariant measure of recurrent $Q$, then it is an invariant measure.

(v) The matrix $Q$ has no invariant measures if and only if it is transient.

In the rest of this section, we assume the following.

**Assumption A.1**

(i) The proper $Q$-matrix $Q$ is ergodic with a unique stationary distribution vector $\pi := (\pi(j))_{j \in \mathbb{S}} > 0$, where $\pi > 0$ is due to Lemma A.1 (ii)–(iv).

(ii) For all $n \in \mathbb{Z}^+$, $Q_n := (q_n(i,j))_{i,j \in \mathbb{S}}$ is a proper $Q$-matrix that has at least one stationary distribution vector, denoted by $\pi_n := (\pi_n(j))_{j \in \mathbb{S}}$.

(iii) $\lim_{n \to \infty} q_n(i,j) = q(i,j)$ for all $i, j \in \mathbb{S}$.

Under Assumption A.1 we present two lemmas (Lemmas A.2 and A.3 below). The two lemmas are the $Q$-matrix-versions of Lemma 2.1 and Corollary 2.2 in Wolf [14] for stochastic matrices. Proceeding as in the proofs of Wolf’s results, we can prove our two lemmas. However, Wolf’s proofs are somewhat too concise. Thus, for the reader’s convenience, we provide complete proofs of the two lemmas.
Lemma A.2 Suppose that Assumption A.1 holds. Fix \( j_0 \in S \) arbitrarily, and let \( \mathbb{K} \) denote an infinite subset of \( \mathbb{Z}_+ \) such that \( \{\pi_n(j_0); n \in \mathbb{K}\} \) is a convergent subsequence of \( \{\pi_n(j_0); n \in \mathbb{Z}_+\} \). Furthermore, let

\[
\alpha = \lim_{n \to \infty} \pi_n(j_0).
\]

We then have

\[
\lim_{n \to \infty} \pi_n(j) = \frac{\alpha}{\pi(j_0)} \pi(j) \quad \text{for all } j \in S,
\]

where \( \alpha/\pi(j_0) \leq 1 \).

Remark A.2 Since \( \{\pi_n(j_0); n \in \mathbb{Z}_+\} \) is bounded, it has a convergent subsequence.

Proof of Lemma A.2 We first prove that, for any infinite \( \mathbb{K}' \subseteq \mathbb{K} \),

\[
\liminf_{n \to \infty} \pi_n(j) = \frac{\alpha}{\pi(j_0)} \pi(j) \quad \text{for all } j \in S,
\]

with \( \alpha/\pi(j_0) \leq 1 \).

Let \( \mathbb{K}' \) be an arbitrary infinite subset of \( \mathbb{K} \), and let

\[
\mu(j) = \liminf_{n \to \infty} \pi_n(j), \quad j \in S.
\]

It then follows from (A.3) that

\[
\alpha = \mu(j_0).
\]

Furthermore, using (A.6), Fatou's lemma, and Assumption A.1(ii), we obtain

\[
-\mu(j)q(j, j) = -\liminf_{n \to \infty} \pi_n(i)q_n(j, j) = \liminf_{n \to \infty} \sum_{i \in S, i \neq j} \pi_n(i)q_n(i, j)
\geq \sum_{i \in S, i \neq j} \liminf_{n \to \infty} \pi_n(i)q_n(i, j)
= \sum_{i \in S, i \neq j} \mu(i)q(i, j), \quad j \in S,
\]

which leads to

\[
\sum_{i \in S} \mu(i)q(i, j) \leq 0, \quad j \in S.
\]

Thus, it follows from Lemma A.1(iii) and (iv) together with Assumption A.1(i) that if \( \mu := (\mu(j))_{j \in S} \neq 0 \) then \( \mu \) is a unique (up to constant multiples) invariant measure of \( Q \). Therefore, there exists some \( c \geq 0 \) such that

\[
\mu(j) = c \pi(j) \quad \text{for all } j \in S,
\]

where \( c > 0 \) if \( \mu \neq 0 \); otherwise \( c = 0 \). Applying (A.8) to (A.7) yields \( \alpha = \mu(j_0) = c \pi(j_0) \), which leads to

\[
c = \frac{\alpha}{\pi(j_0)}.
\]
Combining this result with \((A.6)\) and \((A.8)\), we obtain \((A.5)\). In addition, it follows from
\[ \pi_n e = \pi e = 1, \] 
Fatou’s lemma, and \((A.5)\) that
\[ \lim_{n \to \infty} \pi_n(j) = \frac{\alpha}{\pi(j_0)}. \]

We have proved that \((A.5)\), together with \(\alpha/\pi(j_0) \leq 1\), holds for any infinite \(K' \subseteq K\).

By using \((A.5)\), we next prove that
\[ \lim_{n \to \infty} n \in K' \pi_n(j) = \frac{\alpha}{\pi(j_0)} \pi(j) \]
which completes the proof of this lemma. Indeed, \((A.9)\) implies that for each \(j \in S\) there exist some \(K_1(j) \subseteq K\) and \(K_2(j) \subseteq K\) such that
\[ \lim_{n \to \infty} n \in K_1(j) \pi_n(j) = \frac{\alpha}{\pi(j_0)} \pi(j), \]
\[ \lim_{n \to \infty} n \in K_2(j) \pi_n(j) = \frac{\alpha}{\pi(j_0)} \pi(j), \]
and thus \((A.4)\) holds.

To prove \((A.9)\), it suffices to show that, for any infinite \(K' \subseteq K\),
\[ \lim_{n \to \infty} n \in K' \pi_n(j) = \lim_{n \to \infty} n \in K \pi_n(j) \quad \text{for all } j \in S. \quad (A.10) \]

For proof by contradiction, suppose that, for some infinite \(K^\dagger \subseteq K\) and \(j^\dagger \in S\),
\[ \lim_{n \to \infty} n \in K^\dagger \pi_n(j^\dagger) > \lim_{n \to \infty} n \in K^\dagger \pi_n(j^\dagger). \quad (A.11) \]

There must exist some infinite \(K^\dagger \subseteq K^\dagger \subseteq K\) such that
\[ \lim_{n \to \infty} n \in K^\dagger \pi_n(j^\dagger) = \lim_{n \to \infty} n \in K^\dagger \pi_n(j^\dagger). \]

Using this and \((A.11)\), we obtain
\[ \lim_{n \to \infty} n \in K^\dagger \pi_n(j^\dagger) = \lim_{n \to \infty} n \in K^\dagger \pi_n(j^\dagger) \]
\[ \quad = \lim_{n \to \infty} n \in K^\dagger \pi_n(j^\dagger) > \lim_{n \to \infty} n \in K^\dagger \pi_n(j^\dagger). \quad (A.12) \]

Note that \(K^\dagger \subseteq K^\dagger \subseteq K\). Thus, \((A.12)\) contradicts the fact that \((A.5)\) holds for any infinite \(K' \subseteq K\). This implies that \((A.10)\) holds for any infinite \(K' \subseteq K\). \(\square\)
**Lemma A.3** Suppose that Assumption A.1 holds. In the setting of Lemma A.2 we have the following:

(i) If $\alpha > 0$, then

$$\lim_{n \to \infty} \left\| \frac{\pi_n^A}{\pi_n^A e} - \frac{\pi_A^k}{\pi^A e} \right\| = 0 \quad \text{for any finite } A \subset S,$$

where, for any vector $x := (x(j))_{j \in S}$, $x^A := (x^A(j))_{j \in S}$ denotes a vector such that

$$x^A(j) = \begin{cases} x(j), & j \in A, \\ 0, & j \notin A. \end{cases}$$

(ii) $\lim_{n \to \infty} \|\pi_n - \pi\| = 0$ if and only if there exists some finite and nonempty $A \subset S$ such that

$$\liminf_{n \to \infty} \sum_{j \in A} \pi_n(j) \geq \sum_{j \in A} \pi(j).$$

**Proof.** We first prove the statement (i). Let $A$ be an arbitrary finite subset of $S$. Since $\alpha > 0$, it follows from (A.4) and $\pi > 0$ that

$$\lim_{n \in K} \pi_n^A e = \frac{\alpha}{\pi(j_0)} \pi_A^k e > 0,$$

and thus $\pi_n^A e > 0$ for all sufficiently large $n \in K$. Therefore, from (A.4) and (A.15), we obtain

$$\lim_{n \to \infty} \frac{\pi_n^A e}{\pi_n^A e} = \frac{\pi_A^k}{\pi^A e},$$

which results in (A.13).

Next we prove the statement (ii). The “only if” part is obvious. Furthermore, we can obtain $\lim_{n \to \infty} \|\pi_n - \pi\| = 0$ from the dominated convergence theorem and

$$\lim_{n \to \infty} \pi_n(j) = \pi(j) \quad \text{for all } j \in S.$$

Therefore, to complete the proof of the statement (ii), it suffices to prove (A.16) under the assumption that (A.14) holds for some finite and nonempty $A \subset S$.

Fix $j \in S$ arbitrarily. There exists some infinite $K_j \subseteq \mathbb{Z}_+$ such that

$$\lim_{n \in K_j} \pi_n(j) = \limsup_{n \to \infty} \pi_n(j).$$

Thus, Lemma A.2 with $K = K_j$ yields

$$\limsup_{n \to \infty} \pi_n(j) = \frac{\alpha}{\pi(j_0)} \pi(j) \leq \pi(j).$$

This implies that the proof of (A.16) is completed by showing that

$$\liminf_{n \to \infty} \pi_n(j) \geq \pi(j).$$
We prove (A.17) by contradiction. Suppose that \( \alpha := \liminf_{n \to \infty} \pi_n(j_0) < \pi(j_0) \) for some \( j_0 \in S \), which yields
\[
\frac{\alpha}{\pi(j_0)} < 1.
\] (A.18)

Note that there exists some infinite \( K \subseteq \mathbb{Z}_+ \) such that
\[
\alpha = \lim_{n \in K} \pi_n(j_0).
\]

Thus, from Lemma A.2 and (A.18), we have
\[
\lim_{n \to \infty} \pi_n(j) = \frac{\alpha}{\pi(j_0)} \pi(j) < \pi(j) \quad \text{for all} \ j \in S.
\] (A.19)

Now let \( A \) be a finite and nonempty subset of \( S \) such that (A.14) holds. It then follows from the finiteness of \( A \) and (A.19) that
\[
\liminf_{n \to \infty} \sum_{j \in A} \pi_n(j) \leq \liminf_{n \in K} \sum_{j \in A} \pi_n(j) = \sum_{j \in A} \liminf_{n \in K} \pi_n(j) < \sum_{j \in A} \pi(j),
\]
which contradicts (A.14). Therefore, we have proved that (A.17) holds. \( \square \)

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