Schrödinger functional formalism for overlap Dirac operator and domain-wall fermion

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In this proceeding we propose a new procedure to impose the Schrödinger functional Dirichlet boundary condition on the overlap Dirac operator and the domain-wall fermion using an orbifolding projection. With this procedure the zero mode problem with Dirichlet boundary condition can easily be avoided.

Keywords: Schrödinger functional; Ginsparg-Wilson fermion; orbifolding.

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1. Introduction

The Schrödinger functional (SF) is defined as a transition amplitude between two boundary states with finite time separation:

\[ Z = \langle C'; x_0 = T | C; x_0 = 0 \rangle = \int \mathcal{D}\Phi e^{-\mathcal{S}[\Phi]} \]  

and is written in a path integral representation of the field theory with some specific boundary condition. One of applications of the SF is to define a renormalization scheme beyond perturbation theory, where the renormalization scale is given by a finite volume \( T \times L^3 \sim L^4 \) of the system. The formulation is already accomplished for the non-linear \( \sigma \)-model, the non-Abelian gauge theory and the QCD with the Wilson fermion including \( \mathcal{O}(a) \) improvement procedure. (See Ref. [9] for review.)

In this formalism several renormalization quantities like running gauge coupling, \( \mathcal{O}(a) \) improvement factors are extracted conveniently by using a Dirichlet boundary conditions for spatial component of the gauge field

\[ A_k(x)|_{x_0=0} = C_k(\vec{x}), \quad A_k(x)|_{x_0=T} = C'_k(\vec{x}) \]  

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and for the quark fields
\[ P_+ \psi(x)|_{x_0=0} = \rho(x), \quad P_- \psi(x)|_{x_0=T} = \rho'(x), \]  
\[ \psi(x)P_-|_{x_0=0} = \mathcal{P}(x), \quad \psi(x)P_+|_{x_0=T} = \mathcal{P}'(x), \]  
\[ P_\pm = \frac{1 \pm \gamma_0}{2}. \tag{5} \]

One of privilege of this Dirichlet boundary condition is that the system acquire a mass gap and there is no infra-red divergence.

We notice that the boundary condition is not free to set since it generally breaks symmetry of the theory and may affect renormalizability. However the field theory with Dirichlet boundary condition is shown to be renormalizable for the pure gauge theory \cite{1}. And it is also the case for the Wilson fermion \cite{2} by including a shift in the boundary fields.

Although it is essential to adopt Dirichlet boundary condition for a mass gap and renormalizability, it has a potential problem of zero mode in fermion system. For instance starting from a free Lagrangian
\[ \mathcal{L} = \overline{\psi} \left( \gamma_\mu \partial_\mu + m \right) \psi \]  
with positive mass \( m > 0 \) and the Dirichlet boundary condition
\[ P_- \psi|_{x_0=0} = 0, \quad P_+ \psi|_{x_0=T} = 0 \]  
the zero eigenvalue equation \( (\gamma_0 \partial_0 + m) \psi = 0 \) in temporal direction allows a solution
\[ \psi = P_+ e^{-mx_0} + P_- e^{-m(T-x_0)} \]  
in \( T \to \infty \) limit and a similar solution remains even for finite \( T \) with an exponentially small eigenvalue \( \propto e^{-mT} \). In the SF formalism this solution is forbidden by adopting an “opposite” Dirichlet boundary condition \cite{3} and the system has a finite gap even for \( m = 0 \) \cite{3}.

In the SF formalism of the Wilson fermion \cite{3} we cut the Wilson Dirac operator at the boundary and the Dirichlet boundary condition is automatically chosen among
\[ P_\pm \psi|_{x_0=0} = 0, \quad P_\mp \psi|_{x_0=T} = 0 \]  
depending on signature of the Wilson term. For example if we adopt a typical signature of the Wilson term
\[ D_W = \gamma_\mu \frac{1}{2} \left( \nabla_\mu^* + \nabla_\mu \right) - \frac{a}{2} \nabla_\mu^* \nabla_\mu + M \]  
the allowed Dirichlet boundary condition is the same as \( \psi \). In this case the zero mode solution can be forbidden by choosing a proper signature for the mass term; the mass should be kept positive \( M \geq 0 \) to eliminate the zero mode \cite{3}.

However this zero mode problem may become fatal in the Ginsparg-Wilson fermion including the overlap Dirac operator \cite{21, 22} and the domain-wall fermion
The overlap Dirac operator is defined by using the Wilson Dirac operator \( D \) as
\[
D = \frac{1}{a} \left( 1 + D_W \sqrt{D_W^* D_W} \right), \quad \bar{a} = \frac{a}{|M|}.
\]
where the anti-periodic boundary condition is set in temporal direction of length \( 2T \)
\[
\psi(\vec{x}, x_0 + 2T) = -\psi(\vec{x}, x_0), \quad \bar{\psi}(\vec{x}, x_0 + 2T) = -\bar{\psi}(\vec{x}, x_0).
\]
The orbifolding $S^1/Z_2$ in temporal direction is accomplished by identifying the negative time with the positive one $x_0 \leftrightarrow -x_0$. Identification of the fermion field is given by using a symmetry transformation including the time reflection
\[ \psi(x) \rightarrow \Sigma \psi(x), \quad \bar{\psi}(x) \rightarrow \bar{\psi}(x) \Sigma, \quad \Sigma = i \gamma_5 \gamma_0 R, \tag{14} \]
where $R$ is a time reflection operator
\[ R\psi(\vec{x}, x_0) = \psi(\vec{x}, -x_0). \tag{15} \]
$R$ has two fixed points $x_0 = 0, T$, where $x_0 = 0$ is a symmetric and $x_0 = T$ is an anti-symmetric fixed point because of the anti-periodicity
\[ R\psi(\vec{x}, 0) = \psi(\vec{x}, 0), \quad R\psi(\vec{x}, T) = -\psi(\vec{x}, T). \tag{16} \]
It is free to add any internal symmetry transformation for the identification and we use the chiral symmetry of the massless fermion
\[ \psi(x) \rightarrow -i \gamma_5 \psi(x), \quad \bar{\psi}(x) \rightarrow -\bar{\psi}(x)i \gamma_5. \tag{17} \]
Combining (14) and (17) we have the orbifolding symmetry transformation
\[ \psi(x) \rightarrow -\Gamma \psi(x), \quad \bar{\psi}(x) \rightarrow \bar{\psi}(x) \Gamma, \quad \Gamma = \gamma_0 R. \tag{18} \]
The orbifolding of the fermion field is given by selecting the following symmetric sub-space
\[ \Pi_+ \psi(x) = 0, \quad \bar{\psi}(x) \Pi_- = 0, \quad \Pi_\pm = \frac{1 \pm \Gamma}{2}. \tag{19} \]
We notice that this orbifolding projection provides the proper homogeneous SF Dirichlet boundary condition at fixed points $x_0 = 0, T$
\[ P_+ \psi(x)|_{x_0=0} = 0, \quad P_- \psi(x)|_{x_0=T} = 0, \tag{20} \]
\[ \bar{\psi}(x)P_-|_{x_0=0} = 0, \quad \bar{\psi}(x)P_+|_{x_0=T} = 0. \tag{21} \]
The orbifolded action is given by the same projection
\[ S = \frac{1}{2} \int d^4x \bar{\psi}(x) D_{SF} \psi(x), \quad D_{SF} = \Pi_+ \partial \Pi_- \tag{22} \]
where factor $1/2$ is included since the temporal direction is doubled compared to the original SF formalism. We notice that the chiral symmetry is broken explicitly for $D_{SF}$ by the projection.

Now we have two comments. Since the Schrödinger functional of the pure gauge theory is already well established, we treat the gauge field as an external field and adopt a configuration which is time reflection invariant
\[ A_0(\vec{x}, -x_0) = -A_0(\vec{x}, x_0), \quad A_i(\vec{x}, -x_0) = A_i(\vec{x}, x_0) \tag{23} \]
and satisfy the SF boundary condition simultaneously. We set periodic boundary condition for the gauge field
\[ A_\mu(\vec{x}, x_0 + 2T) = A_\mu(\vec{x}, x_0). \tag{24} \]
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Second comment is on the mass term. Although the mass term is not consistent with the chiral symmetry, we can find a symmetric mass term under the orbifolding transformation. A requirement is that the mass matrix \( M \) should anti-commute with orbifolding operator \( \{M, \Gamma\} = 0 \). One of the candidate is a time dependent mass \( M = m \eta(x_0) \) with anti-symmetric and periodic step function

\[
\eta(-x_0) = -\eta(x_0), \quad \eta(x_0 + 2T) = \eta(x_0), \\
\eta(x_0) = 1 \quad \text{for} \quad 0 < x_0 < T.
\] (25)

Now the Dirac operator becomes

\[
D(m) = \gamma_\mu \left( \partial_\mu - iA_\mu(x) \right) + m\eta(x_0),
\] (26)

which has the orbifolding symmetry \( \{D(m), \Gamma\} = 0 \). The orbifolded Dirac operator is defined by the projection

\[
D_{SF}(m) = \Pi_+ D(m)\Pi_-.
\] (27)

Once the orbifold construction gives the Dirac operator with SF boundary condition, the spectrum and the propagator are uniquely determined to be equivalent to those of Ref. 5 and Ref. 8. One can easily check that this is the case at tree level.

For example eigenvalues of the free SF Dirac operator \( D_{SF} \) is derived as follows. We first notice that the SF Dirac operator connects two different Hilbert sub-space

\[
D_{SF} : \mathcal{H}_+ \rightarrow \mathcal{H}_- , \quad D_{SF}^\dagger : \mathcal{H}_- \rightarrow \mathcal{H}_+ ,
\] (28)

where \( \mathcal{H}_\pm = \{ \psi | \Pi_\pm \psi = 0 \} \). As in the original SF formulation \(^5\), it is necessary to introduce a “doubled” Hermitian Dirac operator

\[
\mathcal{D} = \begin{pmatrix} D_{SF}^\dagger & D_{SF} \end{pmatrix},
\] (29)

which connects the same Hilbert space \( \mathcal{D} : \mathcal{H}_- \oplus \mathcal{H}_+ \rightarrow \mathcal{H}_- \oplus \mathcal{H}_+ \) in order to make the eigenvalue problem to be well defined. This Dirac operator acts on a “two component” vector

\[
\Psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}, \quad \psi_- \in \mathcal{H}_-, \quad \psi_+ \in \mathcal{H}_+
\] (30)

and the eigenvalue problem is given in a following form

\[
D_{SF}\psi_+ = \lambda\psi_- , \quad D_{SF}^\dagger\psi_- = \lambda\psi_+.
\] (31)

with a real eigenvalue \( \lambda \). In the following we consider one dimensional problem for simplicity

\[
D_{SF}(m) = \Pi_+ D(m)\Pi_- , \quad D(m) = \gamma_0 \partial_0 + m\eta(x_0)
\] (32)

with an eigenvalue \( \lambda_0 \).
A candidate of the eigen-function $f_\pm \in \mathcal{H}_\pm$ satisfying anti-periodicity in $2T$ is given by

\begin{align}
  f_+(x_0) &= \alpha P_+ S_-(x_0) + \beta P_- S_+(x_0), \\
  f_-(x_0) &= \alpha P_+ S_+(x_0) + \beta P_- S_-(x_0),
\end{align}

where $\alpha, \beta$ are normalization constant and $S_\pm$ are defined for each bulk region with a cusp at the boundary

\begin{align}
  S_-(x_0) &= (-)^n \sin p_0 (x_0 - 2nT) \quad \text{for} \quad (2n-1)T \leq x_0 \leq (2n+1)T, \\
  S_+(x_0) &= (-)^{n+1} \sin p_0 (x_0 - (2n+1)T) \quad \text{for} \quad 2nT \leq x_0 \leq 2(n+1)T.
\end{align}

The eigenvalue equation (31) has solution on these functions only when a quantization condition is satisfied for $p_0$

\begin{equation}
  \tan p_0 T = -\frac{p_0}{m}.
\end{equation}

and the eigenvalue becomes

\begin{equation}
  \lambda_0 = -\frac{m}{\cos p_0 T} = \frac{p_0}{\sin p_0 T}, \quad \lambda_0^2 = p_0^2 + m^2.
\end{equation}

This result agrees with that of Ref.5.

When a spatial momentum is introduced we are to solve an eigenvalue problem of the matrix

\begin{equation}
  C = \begin{pmatrix} C^\dagger & C \end{pmatrix}, \quad C = i\gamma_k p_k + \lambda_0
\end{equation}

and the eigenvalue $\lambda$ of the four dimensional Dirac operator is given by

\begin{equation}
  \lambda^2 = p_0^2 + \vec{p}^2 + m^2.
\end{equation}

3. Orbifolding for overlap Dirac fermion

Application of the orbifolding procedure is straightforward to the Ginsparg-Wilson fermions including the overlap Dirac operator 21,22, the domain-wall fermion 23,24,25,26 and the perfect action 28,29,30 which possess both the time reflection symmetry

\begin{equation}
  [\Sigma, D] = 0
\end{equation}

and the lattice chiral symmetry stemming from the Ginsparg-Wilson relation

\begin{equation}
  \gamma_5 D + D\gamma_5 = \pi D\gamma_5 D.
\end{equation}

In this subsection we concentrate on the overlap Dirac operator (11), for which the time reflection symmetry comes from that of the Wilson Dirac operator $[\Sigma, D_W] = 0$. 

3.1. Orbifolding construction of Dirichlet boundary

As in the continuum case we consider a massless fermion on a lattice $2N_T \times N_L^3$ with anti-periodic boundary condition in temporal direction $^{13}$. We use an orbifolding $S^1/Z_2$ in temporal direction. Identification of the fermion field is given by using the time reflection $^{14}$ and the chiral symmetry of the overlap Dirac fermion $^{31}$

$$\psi(x) \rightarrow -i\gamma_5\psi(x), \quad \overline{\psi}(x) \rightarrow -\overline{\psi}(x)i\gamma_5, \quad \gamma_5 = \gamma_5(1 - \pi D),$$

(41)

where the gauge field is treated as an external field and we adopt a time reflection symmetric configuration

$$U_k(\vec{x}, x_0) = U_k(\vec{x}, -x_0), \quad U_0(\vec{x}, x_0) = U_0^\dagger(\vec{x}, -x_0 - 1),$$

(42)

satisfying the SF Dirichlet boundary condition simultaneously

$$U_k(\vec{x}, 0) = W_k(\vec{x}), \quad U_k(\vec{x}, N_T) = W_k^\dagger(\vec{x}).$$

(43)

Combining (14) and (41) we have the orbifolding symmetry transformation

$$\psi(x) \rightarrow \hat{\Gamma}\psi(x), \quad \overline{\psi}(x) \rightarrow \overline{\psi}(x)\Gamma, \quad \hat{\Gamma} = \Gamma(1 - \pi D),$$

(44)

where $\Gamma$ is the same as the continuum one $^{18}$. We notice that starting from the time reflection symmetry of the Dirac operator $^{30}$ and the Ginsparg-Wilson relation $^{10}$ we have another GW relation for $\Gamma$

$$\Gamma D + D\Gamma = \pi D\Gamma D$$

(45)

and $\Gamma$ Hermiticity

$$\Gamma D\Gamma = D^\dagger.$$  

(46)

The operator $\hat{\Gamma}$ has a property $\hat{\Gamma}^2 = 1$ like $\Gamma$ and can be used to define a projection operator in the following.

The orbifolding identification of the fermion field is given in the same way with slightly different projection operator

$$\hat{\Pi}_+\psi(x) = 0, \quad \overline{\psi}(x)\Pi_- = 0, \quad \hat{\Pi}_\pm = \frac{1 \pm \hat{\Gamma}}{2},$$

(47)

which turn out to be the SF Dirichlet boundary condition $^{20}$ and $^{21}$ at fixed points in the continuum limit. Using the time reflection symmetry $^{30}$ we can easily show that the projection operators $\Gamma$ and $\hat{\Gamma}$ do not have an “index”

$$\text{tr}\Gamma = \text{tr}\hat{\Gamma} = 0$$

(48)

and furthermore we can find a local unitary transformation

$$u = \frac{1 + \Sigma}{2}(1 - \pi D) + \frac{1 - \Sigma}{2}, \quad u' = \gamma_5u\gamma_5,$$

(49)

which connects $\hat{\Gamma}$ and $\Gamma$ as

$$\hat{\Gamma} = u^\dagger\Gamma u, \quad \Gamma = u'\Gamma u'^\dagger.$$  

(50)
The projection operator \( \hat{\Pi}_\pm \) spans essentially the same Hilbert sub-space as \( \Pi_\pm \). We notice that this unitary operator connects \( \hat{\gamma}_5 \) and \( \gamma_5 \) in a similar way:

\[
\hat{\gamma}_5 = u^\dagger \gamma_5 u, \quad \hat{\gamma}_5 = u^\prime \gamma_5 u^\prime\dagger.
\] (51)

The physical quark operator is defined to transform in the same manner as the continuum under chiral rotation,

\[
\delta q(x) = \gamma_5 q(x), \quad \delta \bar{q}(x) = \bar{q}(x) \gamma_5.
\] (52)

Since we have a unitary operator \( u \) and \( u' \) we have several ways to define a physical quark field from GW fermion fields \( \psi \) and \( \bar{\psi} \).

\[
q(x) = (1 - \frac{\Pi_+}{2}) \psi(x), \quad \bar{q}(x) = \bar{\psi}(x),
\] (53)

\[
q(x) = u \psi(x), \quad \bar{q}(x) = \bar{\psi}(x),
\] (54)

\[
q(x) = u'\dagger \psi(x), \quad \bar{q}(x) = \bar{\psi}(x).
\] (55)

These three definitions are not independent but connected with \( u + u'\dagger = (2 - \Pi D) \).

The massless orbifolded action is given by

\[
S = \frac{1}{2} a^4 \sum \bar{\psi} D_{SF} \psi, \quad D_{SF} = \Pi_+ D \hat{\Pi}_-.
\] (57)

We have four comments here. (i) It should be emphasized that the SF Dirac operator \( D_{SF} \) is local since it is constructed by multiplying local objects only. (ii) The massless SF Dirac operator \( D_{SF} \) does not satisfy the chiral Ginsparg-Wilson relation \[40\]. (iii) Although two different projection operators \( \Gamma \) and \( \hat{\Gamma} \) are used from the left and right of \( D_{SF} \) this does not bring the problem we encountered in the chiral gauge theory since these two operators are connected by the unitary transformation \( u \) or \( u' \).

### 3.2. Surface term

When extracting the renormalization factors of fermions it is convenient to consider an operator involving the boundary source fields

\[
\zeta(\vec{x}) = \frac{\delta}{\delta \rho(\vec{x})}, \quad \bar{\zeta}(\vec{x}) = -\frac{\delta}{\delta \rho(\vec{x})},
\] (58)

\[
\zeta'(\vec{x}) = \frac{\delta}{\delta \rho'(\vec{x})}, \quad \bar{\zeta'}(\vec{x}) = -\frac{\delta}{\delta \rho'(\vec{x})}.
\] (59)

where \( \rho, \cdots, \rho' \) are boundary values of the fermion fields given in \[38\] and \[41\]. Coupling of the boundary value to the bulk dynamical fields was naturally introduced.
in the Wilson fermion. However this is not the case for our construction since the boundary value vanishes with the orbifolding projection.

In this paper we regard the boundary value as an external source field and introduce its coupling with the bulk fields according to the criteria: the coupling terms (surface terms) are local and reproduce the same form of the correlation function between the boundary fields in the continuum limit. Here we define the boundary values on the physical quark fields

\[ P_+ q(x)|_{x_0=0} = \rho(\vec{x}), \quad P_- q(x)|_{x_0=N_T} = \rho'(\vec{x}), \]

\[ \overline{\psi}(x) P_- |_{x_0=0} = \overline{\psi}(\vec{x}), \quad \overline{\psi}(x) P_+ |_{x_0=N_T} = \overline{\psi}(\vec{x}). \]

One of candidates of the surface term is

\[ S_{\text{surface}} = a^3 \sum \left( - \rho(\vec{x}) P_- q(x)|_{x_0=0} - \overline{\psi}(\vec{x}) P_+ \rho(\vec{x})|_{x_0=0} \right. \]

\[ - \left. \overline{\psi}(\vec{x}) P_+ q(x)|_{x_0=N_T} - \overline{\psi}(\vec{x}) P_- \rho'(\vec{x})|_{x_0=N_T} \right), \]

where \( q \) and \( \overline{\psi} \) are active dynamical fields on the boundary.

According to Ref. [8], we introduce the generating functional

\[ Z_F \left[ \overline{\psi}', \rho'; \overline{\psi}, \rho; \eta, \eta; U \right] = \int D\psi D\overline{\psi} \exp \left\{ -S_F \left[ U, \overline{\psi}, \psi; \overline{\psi}', \rho', \rho \right] \right. \]

\[ + a^4 \sum_x \left( \overline{\psi}(x) \eta(x) + \overline{\psi}(x) \psi(x) \right) \right\}, \]

where \( \eta(x) \) and \( \overline{\psi}(x) \) are source fields for the fermion fields and the total action \( S_F \) is given as a sum of the bulk action and the surface term. We notice that the fermion fields \( \psi \) and \( \overline{\psi} \) obey the orbifolding condition. The correlation functions between the boundary fields are derived according to ordinary procedures of perturbation theory.

### 3.3. Phase of Dirac determinant

In general, the determinant of the SF Dirac operator is not real for the overlap fermion since there is no \( \gamma_5 \) Hermiticity. Instead we have a following “Hermiticity” relation

\[ \gamma_5 u D_{SF} u^\dagger \gamma_5 = D_{SF}^\dagger, \quad \gamma_5 u^\dagger D_{SF} u' \gamma_5 = D_{SF}^\dagger. \]

However, one cannot conclude reality from this relation since the SF Dirac operator connects different Hilbert sub-space as

\[ D_{SF} : \mathcal{H}_+ \to \mathcal{H}_-, \quad \mathcal{H}_+ = \left\{ \psi | \mathbf{\Pi}_+ \psi = 0 \right\}, \quad \mathcal{H}_- = \left\{ \psi | \mathbf{\Pi}_- \psi = 0 \right\}. \]
define the Dirac determinant. This is accomplished by $u^\dagger \gamma_5$ or $u' \gamma_5$, which turns out to be $\gamma_5$ in the continuum. We define

$$H_{SF} = D_{SF} u^\dagger \gamma_5 = \Pi_+ D u^\dagger \gamma_5 \Pi_+ : \mathcal{H}_- \rightarrow \mathcal{H}_-,$$

(66)

$$H'_{SF} = D_{SF} u' \gamma_5 = \Pi_+ D u' \gamma_5 \Pi_+ : \mathcal{H}_- \rightarrow \mathcal{H}_-.$$  

(67)

The determinant is evaluated on the sub-space $\mathcal{H}_-$

$$\det_{\{\mathcal{H}_-\}} H_{SF} = \det (\Pi_+ D u^\dagger \gamma_5 \Pi_+ + \Pi_-),$$

(68)

$$\det_{\{\mathcal{H}_-\}} H'_{SF} = \det (\Pi_+ D u' \gamma_5 \Pi_+ + \Pi_-),$$

(69)

where the right hand side is understood to be evaluated in the full Hilbert space by filling the opposite sub-space $\mathcal{H}_+$ with unity.

The phase of the determinant is given as follows

$$\left( \det_{\{\mathcal{H}_-\}} H'_{SF} \right)^* = e^{-2i\phi'} \left( \det_{\{\mathcal{H}_-\}} H_{SF} \right),$$

(70)

$$e^{-2i\phi} = \det_{\{\mathcal{H}_-\}} (\gamma_5 u)^2 = \det u,$$

(71)

$$e^{-2i\phi'} = \det_{\{\mathcal{H}_-\}} (\gamma_5 u'^\dagger)^2 = \det u'^\dagger = e^{2i\phi},$$

(72)

which is not real in general. The determinant of the unitary operator $u$ is given by a product of eigenvalues $\lambda_n$ of the overlap Dirac operator

$$\det u = \prod_{n \in \{+\}} (1 - a\lambda_n),$$

(73)

where product is taken over a sub-space in which the eigenvalue of $\Sigma = +1$ and the conjugate eigenvalue $\lambda_n^*$ does not necessarily belongs to this sub-space.

However we notice that this complexity of the Dirac determinant is not an essential problem since the phase is an $O(a)$ irrelevant effect and disappears in the continuum limit. Furthermore if we consider variation of the phase

$$\delta_{\epsilon(x)} \phi = \frac{i}{2} \text{tr} \delta_{\epsilon(x)} uu^{-1} = -\frac{i}{4} \text{tr} \left[ \Sigma \delta_{\epsilon(x)} D \left(1 - aD^\dagger\right) \right]$$

(74)

under a local variation of the link variable

$$\delta_{\epsilon(x)} U_\mu(x) = a\epsilon_\mu(x) U_\mu(x)$$

(75)

we can show that $\delta_{\epsilon(x)} \phi$ is localized at the boundary. Since $\Sigma$ contains time reflection $R$ and both of the operator $\delta D$ and $\left(1 - aD^\dagger\right)$ are local, the trace in $\Sigma$ has a contribution only at the boundary. Contribution from the bulk is suppressed exponentially by the locality property.
3.4. Mass term

The mass term may be introduced with the same procedure as the continuum theory. We consider a mass matrix \( M \) which is consistent with the orbifolding symmetry

\[
\Gamma M + M \Gamma = 0.
\]

(76)

Since the orbifolding transformation is the same as the continuum one on the physical quark fields, a naive candidate is to couple the continuum mass matrix \( m_{\eta}(x_0) \) to the physical scalar density consisting of \( q(x) \) and \( \bar{q}(x) \). Corresponding to various definition of the quark fields (53)-(55) we have several definitions of the mass term

\[
\mathcal{L}_m = m \bar{\psi} \eta \left( 1 - \frac{\pi}{2} D \right) \psi, \quad m \bar{\psi} \eta \psi, \quad m \bar{\psi} \eta \psi',
\]

(77)

where \( \eta \) is an anti-symmetric step function (25) on lattice.

However we encounter a problem with this naive definition of mass term, since the massive Dirac operator does not satisfy the “Hermiticity” relation (64). The phase of the Dirac determinant becomes mass dependent although it is still irrelevant \( O(a) \) term. In order to avoid this unpleasant situation we may need even numbers of flavors.

For two flavors case we define the two by two Dirac operator as

\[
D_{SF}^{(2)}(m) = \left( \begin{array}{cc} D_{SF}(m)_1 & D_{SF}(m)_2 \\ \hat{D}_{SF}(m)_2 & \hat{D}_{SF}(m)_1 \end{array} \right),
\]

(78)

where

\[
D_{SF}(m)_1 = \Pi_+ \left( D + m \eta \left( 1 - \frac{\pi}{2} D \right) \right) \hat{\Pi}_-.
\]

(79)

\[
D_{SF}(m)_2 = \Pi_+ \left( D + m \left( 1 - \frac{\pi}{2} D \right) u' \eta u' \right) \hat{\Pi}_-.
\]

(80)

A “Hermitian” relation can be found for this two flavors Dirac operator as

\[
D_{SF}^{(2)}(m)^\dagger = \tau^1 \gamma_5 U D_{SF}^{(2)}(m) U^\dagger \gamma_5 \tau^1,
\]

(81)

where \( \tau^1 \) and \( U \) are two by two matrix acting on the flavor space

\[
\tau^1 = \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right), \quad U = \left( \begin{array}{cc} u & u' \end{array} \right).
\]

(82)

The Hermitian Dirac operator can be defined to connect the same Hilbert sub-space as

\[
H_{SF}^{(2)}(m) = D_{SF}^{(2)}(m) \gamma_5 \tau^1 : \mathcal{H}_- \oplus \mathcal{H}_- \rightarrow \mathcal{H}_- \oplus \mathcal{H}_-,
\]

(83)

which is re-written in a trivially Hermitian form by a unitary matrix \( V \)

\[
H_{SF}^{(2)}(m) = V \left( \begin{array}{cc} \hat{D}_{SF}(m)_1 & \hat{D}_{SF}(m)_2 \\ D_{SF}(m)_2 & D_{SF}(m)_1 \end{array} \right) V^\dagger, \quad V = \left( \begin{array}{cc} 1 & \gamma_5 u \end{array} \right).
\]

(84)

The determinant of this Dirac operator is evaluated in a single Hilbert sub-space

\[
\det \left( \mathcal{H}_- \oplus \mathcal{H}_- \right) H_{SF}
\]

and becomes real.
4. Orbifolding for domain-wall fermion

We consider the Shamir’s domain-wall fermion \(^{24, 25}\) on a lattice \(2N_T \times N^3_L \times N_5\) with anti-periodic boundary condition in temporal direction

\[
S = \sum_{\vec{x}, \vec{y}} \sum_{x_0, y_0 = -N_T + 1} \sum_{s, t = 1}^{N_5} \overline{\psi}(x, s) D_{dwf}(x, y; s, t) \psi(y, t). \tag{85}
\]

We adopt a notation used in CP-PACS collaboration and the Dirac operator is given as a five dimensional Wilson’s one with conventional Wilson parameter \(r = 1\) and negative mass parameter \(-M\) with \(0 < M < 2\)

\[
D_{dwf}(x, y; s, t) = \left(\frac{-1 + \gamma_\mu U_\mu(x)\delta_{y_\mu, x_\mu + 1} + \frac{-1 - \gamma_\mu U_\mu(y)\delta_{y_\mu, x_\mu - 1}}{2}}{2}\right) \delta_{x_0, y_0} \delta_{s, t}
+ \left(\frac{-1 + \gamma_5}{2} \Omega^+_{s, t} + \frac{1 - \gamma_5}{2} \Omega^-_{s, t}\right) \delta_{x, y} + (5 - M) \delta_{x, y} \delta_{s, t}, \tag{86}
\]

where \(\Omega^\pm\) are hopping operator in fifth direction with Dirichlet boundary condition, whose explicit form is given by

\[
\Omega^+_{s, t} = \delta_{t, s + 1}, \quad \Omega^+_{N_5, t} = 0, \quad \Omega^- = (\Omega^+)^\dagger. \tag{87}
\]

The physical quark field is defined by the fifth dimensional boundary field with chiral projection

\[
q(x) = (P_L \delta_{s, 1} + P_R \delta_{s, N_5}) \psi(x, s), \tag{88}
\]

\[
\overline{q}(x) = \overline{\psi}(x, s) (\delta_{s, N_5} P_L + \delta_{s, 1} P_R), \tag{89}
\]

\[
P_{R/L} = \frac{1 \pm \gamma_5}{2}. \tag{90}
\]

As in the formulation with the overlap Dirac operator the gauge field is treated as an external field.

In order to apply the orbifolding construction of the SF Dirac operator we need two symmetries of time reflection and chiral transformation. The time reversal symmetry of the domain-wall fermion is given by

\[
\psi(\vec{x}, x_0, s) \rightarrow \sum_{y_0, t} \psi(\vec{x}, y_0, t), \quad \overline{\psi}(\vec{x}, x_0, s) \rightarrow \overline{\psi}(\vec{x}, y_0, t) \sum_{y_0, x_0, t} \psi(\vec{x}, y_0, t), \tag{91}
\]

\[
\sum_{y_0, x_0, t} = i \gamma_5 \gamma_0 R_{x_0, y_0} P_{s, t}, \tag{92}
\]

where \(P\) is a parity transformation in fifth direction

\[
P_{s, t} \psi(\vec{x}, x_0, t) = \psi(\vec{x}, x_0, N_5 + 1 - s) \tag{93}
\]

and \(R\) is a time reflection operator acting on the temporal direction. The domain-wall fermion Dirac operator is invariant under the time reflection

\[
[\Sigma, D_{dwf}] = 0 \tag{94}
\]

since the reflection invariant gauge configuration \(^{12}\) and \(^{13}\) is adopted.
The chiral transformation is given according to Ref. [25] by rotating the fermion field vector like but with a different charge for two boundaries in fifth direction

$$\psi(x, s) \rightarrow iQ_{s,t} \psi(x, t), \quad \bar{\psi}(x, s) \rightarrow -\bar{\psi}(x, t)iQ_{t,s},$$

(95)

where $Q$ is an $s$ dependent charge

$$Q_{s,t} = \text{Sgn}(N_5 - 2s + 1).$$

(96)

Here we should notice that this chiral rotation is not an exact symmetry of the domain-wall fermion Dirac operator but we have an explicit breaking term

$$Q D_{dwf} - D_{dwf} = 2X,$$

(97)

where $X$ is a contribution from the middle layer and picks up a charge difference there

$$X = \left( P_L \delta_{s, N_5 / 2} \delta_{t, N_5 / 2 + 1} + P_R \delta_{s, N_5 / 2 + 1} \delta_{t, N_5 / 2} \right) \delta_{x,y}.$$

(98)

However it was discussed in Ref. [25] that if we consider the correlation functions between the bilinear $\bar{\psi}X\psi$ and the physical quark operators the contribution is suppressed exponentially in $N_5$ under the condition that the transfer matrix in fifth direction has a gap from unity. Furthermore the domain-wall fermion with explicitly time reflection invariant Dirac operator (94) does not have index, since the contribution to the index

$$\lim_{N_5 \to \infty} a^4 \sum_x \langle \bar{\psi}(x, s)\gamma_5 X_{s,t}\psi(x, t) \rangle = -\lim_{N_5 \to \infty} \text{tr} \left( \gamma_5 X \frac{1}{D_{dwf}} \right)$$

(99)

can be shown to vanish by using anti-commutativity $\{\gamma_5 X, \Sigma\} = 0$. We can expect no effect from $X$ to the physical theory and we shall ignore this term in the following by constraining that we consider the physical quark Green’s functions only.

Combining (91) and (95) we have the orbifolding symmetry transformation

$$\psi(\vec{x}, x_0, s) \rightarrow A_{x_0,y_0;s,t} \psi(\vec{x}, y_0, t), \quad \bar{\psi}(\vec{x}, x_0, s) \rightarrow \bar{\psi}(\vec{x}, y_0, t) A_{y_0,x_0;t,s},$$

(100)

and

$$A_{x_0,y_0;s,t} = \gamma_0 \gamma_5 (PQ)_{s,t} R_{x_0,y_0},$$

(101)

where we used a relation

$$\{P,Q\} = 0.$$  

(102)

The operator $A$ satisfy a property $A^2 = 1$ and can be used to define a projection operator. The orbifolding identification of the fermion field is given by projecting onto the following symmetric sub-space

$$\Pi_- \psi(x, s) = 0, \quad \bar{\psi}(x, s)\Pi_- = 0, \quad \Pi_{\pm} = \frac{1 \pm A}{2}.$$  

(103)

The orbifolding projection for the physical quark field is given by picking up the boundary components from the projected fermion field, which turns out to be
the same condition as the continuum field. The proper homogeneous SF Dirichlet boundary condition is provided at fixed points. The massless orbifolded action is given by

$$S = \frac{1}{2} a^4 \sum \overline{\psi} D_{SF}^{dWf} \psi, \quad D_{SF}^{dWf} = \Pi_+ D_{dWf} \Pi_+.$$

(104)

We have four comments here. (i) We regard the orbifolding transformation as an exact symmetry of the system by ignoring the explicit breaking term

$$[A, D_{dWf}] = 0.$$

(105)

(ii) The massless SF Dirac operator $D_{SF}^{dWf}$ breaks “chiral symmetry” under explicitly by the projection $\Pi_+$. (iii) We have a Hermiticity relation for this Dirac operator

$$\left(D_{SF}^{dWf}\right)^\dagger = \gamma_5 P D_{SF}^{dWf} \gamma_5 P.$$

(106)

(iv) This Dirac operator connects the same Hilbert sub-space

$$D_{SF}^{dWf} : \mathcal{H}_- \rightarrow \mathcal{H}_-, \quad \mathcal{H}_- = \{\psi | \Pi_- \psi = 0\}.$$

(107)

5. Conclusion

In this paper we propose a new procedure to introduce the SF Dirichlet boundary condition for general fermion fields. Instead of cutting the Dirac operator at the boundary we focus on a fact that the chiral symmetry is broken explicitly in the SF formalism by the boundary condition and adopt it as a criterion of the procedure. We also notice that an orbifolded field theory is equivalent to a field theory with some specific boundary condition. We search for the orbifolding symmetry which is not consistent with the chiral symmetry and reproduces the SF Dirichlet boundary condition on the fixed points. We found that the orbifolding $S^1/Z_2$ in temporal direction including the time reflection, the chiral rotation and the anti-periodicity serves this purpose well.

Application of this procedure to the overlap Dirac operator is straightforward since this system has both the time reflection and the chiral symmetry. We found a technical problem that the Dirac determinant is complex. However this is not essential since the phase of the determinant is an irrelevant $O(a)$ term. The mass term may still have a problem. We did not find a massive Dirac operator which has the same phase as the massless one for a single flavor case. For two flavors we can construct a Hermitian massive Dirac operator, where the phase is absorbed into the fermion field. However we should notice that the flavor symmetry is broken in this two flavors formulation. The SF formalism with the domain-wall fermion can be formulated in the same way since the symmetry is exactly the same as the overlap Dirac operator.
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