DYNAMICS OF THE INDUCED SHIFT MAP

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Abstract. In this article, we compare the dynamics of the shift map and its induced counterpart on the hyperspace of the shift space. We show that many of the properties of induced shift map can be easily demonstrated by appropriate sequences of symbols. We compare the dynamics of the shift system $(\Omega, \sigma)$ with its induced counterpart $(\mathcal{K}(\Omega), \sigma)$, where $\mathcal{K}(\Omega)$ is the hyperspace of all nonempty compact subsets of $\Omega$. Recently, such comparisons have been studied a lot for general spaces. We continue the same study in case of shift spaces, and bring out the significance of such a study in terms of sequences.

We compare the mixing properties, denseness of periodic points, various forms of sensitivity and expansivity of the shift map and its induced counterpart. In particular, we show their equivalence in case of the full shift. We also look into the special case of subshifts of finite type, and in particular prove that the properties of weakly mixing, mixing and sensitivity are equivalent in both systems. And in the case of any general subshift, we show that the concept of cofinite sensitivity is equivalent in both systems and for transitive subshifts, cofinite sensitivity is equivalent to syndectic sensitivity. In the process we prove that all sturmian subshifts are cofinitely sensitive.

1. INTRODUCTION

Many times dynamical systems are studied by discretizing both time and state space. The basic idea involves in taking a partition of the state space into finite number of regions, each of which can be labelled with some symbol. Time is then discretized by taking iterates of all points in the space. Each itinerary in the state space then corresponds to an infinite sequence of symbols, where the symbols are the labels of the region in the partition given by the trajectory of the point. This 'Symbolic Dynamics' though gives an approximation of the actual orbits, but is very useful in capturing the essence of any dynamics.

Symbolic systems are important classes of dynamical systems and have great applicability to topological dynamics and ergodic theory. Their equivalence with many topological dynamical systems and simple computational structure makes them an important class of dynamical systems. They have also been used to approximate various natural processes and predict their long term behavior. Further, it has been seen that most of the dynamical systems, observed in Nature, are collective(set valued) dynamics of many units of individual systems. In particular, the asymptotic behavior of the iterates of any non-empty subset of the space becomes an important study. Hence, there is a strong need to develop a relation between the dynamics on the base space and the hyperspace(space of subsets). Such a study can help in understanding the combined dynamics on systems, which on an individual basis may not be that interesting. This has lead to the study of 'set-valued dynamics'. Roman Flores [15], Banks [2], Liao, et al [8], Sharma and Nagar [18, 19] have given a comparison of individual dynamics and set-valued dynamics. On the other hand, it has been observed that dynamical systems can be better studied via symbolic dynamics [10, 11, 13]. Also in [16], it is shown that any dynamical system can be realized as a subshift of some shift space.

Some recent studies of dynamical systems, in branches of engineering and physical sciences, have revealed that the underlying dynamics is set valued or collective, instead of the normal individual kind which is usually studied. Some recent studies in Population Dynamics, consider population as local subpopulation in discrete habitat patches, with independent dynamics. This initiates the study of metapopulation dynamics ( see [14]). In Chemical Physics, the individual dynamics of the electron and the nuclei are combined to stimulate the dynamics of large molecular systems containing thousands of atoms ( see [20]). In Atmospheric Sciences, the perturbation of waves is studied as a combined effect of the near-surface, intermediate-level and tropopause-level perturbations upon flow development ( see [7]). In Mechanical Engineering recently, lane keeping controllers have been specifically designed, so that they can be coupled with steering force feedback for better maintenance of lane position in absence of driver steering commands. Artificial damping is further injected to make the combined effect of the system stable, ensuring risk free and safe driving ( see [9]).

With these varieties of dynamics observed, there arises the need of a topological treatment of such collective dynamics. Also, the evolution of trajectories of a chaotic dynamical system is equivalent to symbolic dynamics in an appropriate symbol system. Hence, there is a strong need to develop a relation between the dynamics on the shift system and its induced counterpart on its hyperspace.

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In this article, we study the relations between the dynamical behavior of the shift \((\Omega, \sigma)\) and its induced counterpart \((\mathcal{K}(\Omega), \overrightarrow{\sigma})\). We consequently show that many of the chaotic properties of \((\mathcal{K}(\Omega), \overrightarrow{\sigma})\) can be easily exhibited by the sequences in \(\Omega\).

## 2. THEORETICAL PRELIMINARIES

We now introduce some basics from dynamical systems, hyperspace topologies and symbolic dynamics.

### 2.1. Dynamical Systems.

Let \((X, \tau)\) (resp. \((X, d)\)) be a topological (resp. metric) space and \(f : X \to X\) be a continuous function. The pair \((X, f)\) is referred as a dynamical system. We state some dynamical properties here, though we refer to [1][2][3][4][5][6][7] for more details.

A point \(x \in X\) is called periodic if \(f^n(x) = x\) for some positive integer \(n\), where \(f^n = f \circ f \circ f \circ \ldots \circ f\) (\(n\) times). The least such \(n\) is called the period of the point \(x\). If there exists a \(\delta > 0\) such that for every \(x \in X\) and for each \(\epsilon > 0\) there exists \(y \in X\) and a positive integer \(n\) such that \(d(x, y) < \epsilon\) and \(d(f^n(x), f^n(y)) > \delta\), then \(f\) is said to be sensitive (\(\delta\)-sensitive). The constant \(\delta\) is called the sensitivity constant for \(f\). \(f\) is said to be cofinitely sensitive, if there exists \(\delta > 0\) such that the set of instances \(N_f(U, \delta) = \{n \in \mathbb{N} : \text{there exist } y, z \in U\) with \(d(f^n(y), f^n(z)) > \delta\}\) is cofinite. \(f\) is called syndetically sensitive if there exists \(\delta > 0\) with the property that for every \(\epsilon\)-neighborhood \(U\) of \(x\), \(N_f(U, \delta)\) is syndetic. In general, cofinitely sensitive \(\Rightarrow\) syndetically sensitive \(\Rightarrow\) sensitive

\(f\) is Li-Yorke sensitive if there exists \(\delta > 0\) such that for each \(x \in X\) and for each \(\epsilon > 0\), there exists \(y \in X\) with \(d(x, y) < \epsilon\) such that \(\lim\inf_{n \to \infty} d(f^n(x), f^n(y)) = 0\) but \(\lim\sup_{n \to \infty} d(f^n(x), f^n(y)) > \delta\). A very strong form of sensitivity is expansivity. \(f\) is called expansive (\(\delta\)-expansive) if for any pair of distinct elements \(x, y \in X\), there exists \(k \in \mathbb{N}\) such that \(d(f^k(x), f^k(y)) > \delta\).

\(f\) is called transitive if for any pair of non-empty open sets \(U, V\) in \(X\), there exist a positive integer \(n\) such that \(f^n(U) \cap V \neq \phi\), and is called totally transitive if \(f^n\) is transitive for each \(n \in \mathbb{N}\). \(f\) is called weakly mixing if \(f \times f\) is transitive. \(f\) is called mixing or topologically mixing if for each pair of non-empty open sets \(U, V\) in \(X\), there exists a positive integer \(k\) such that \(f^n(U) \cap V \neq \phi\) for all \(n \geq k\). \(f\) is called locally eventually onto (leo) if for each non-empty open set \(U\), there exists a positive integer \(k \in \mathbb{N}\) such that \(f^k(U) = X\). Among the above topological properties, the following relation holds,

\(\text{leo} \Rightarrow \text{topological mixing} \Rightarrow \text{weakly mixing} \Rightarrow \text{totally transitivity} \Rightarrow \text{transitivity}\)

### 2.2. Hyperspace Topologies.

For a Hausdorff space \((X, \tau)\), a hyperspace \((\mathcal{K}(X), \Delta)\) comprises of all nonempty compact subsets of \(X\) endowed with the topology \(\Delta\), where the topology \(\Delta\) is defined using the topology \(\tau\) of \(X\). The topology \(\Delta\), that we consider here will be either the Vietoris topology or the Hausdorff Metric topology (when \(X\) is a metric space). We briefly describe these topologies.

Define, \(\{U_1, U_2, \ldots, U_k\} = \{E \in \mathcal{K}(X) : E \subseteq \bigcup_{i=1}^{k} U_i \text{ and } E \cap U_i \neq \phi \ \forall i\}\). The topology generated by the collection of all such sets, where \(k\) varies over all possible natural numbers and \(U_i\) varies over all possible open subsets of \(X\), is known as the Vietoris topology.

For a metric space \((X, d)\) and for any two non-empty compact subsets \(A_1, A_2\) of \(X\), define, \(d_H(A_1, A_2) = \inf \{\epsilon > 0 : A_1 \subseteq S_\epsilon(B) \text{ and } B \subseteq S_\epsilon(A)\} \) where \(S_\epsilon(A) = \bigcup_{x \in A} S(x, \epsilon)\) and \(S(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}\). \(d_H\) is a metric on \(\mathcal{K}(X)\) and is known as the Hausdorff metric, which generates the Hausdorff metric topology on \(\mathcal{K}(X)\).

It is known that \(\mathcal{K}(X)\) is compact if and only if \(X\) is compact and in this case, the Hausdorff metric topology is equivalent to the Vietoris topology. Also, it is known that the collection of finite sets is dense in \(\mathcal{K}(X)\). We can talk of these topologies for any subspace \(\Psi\) of \(\mathcal{K}(X)\). See [3][4][5] for details.

### 2.3. Symbolic Dynamics.

We study the sequence space generated by a symbol set \(\mathcal{A}\), where \(|\mathcal{A}|\) may be finite or infinite. In general, we study the sequence space \(\mathcal{A}^\mathbb{N}\) or \(\mathcal{A}^\mathbb{Z}\).

Let \(\mathcal{A}\) be a discrete alphabet set (\(|\mathcal{A}|\) may be finite or infinite). Let \(\Sigma_{\mathcal{A}} = \mathcal{A}^\mathbb{N}\) be the space of all infinite sequences over \(\mathcal{A}\). Define \(D_1 : \Sigma_{\mathcal{A}} \times \Sigma_{\mathcal{A}} \to \mathbb{R}^+\) as,

\[D_1(\bar{x}, \bar{y}) = \sum_{i=0}^{\infty} \frac{d(x_i, y_i)}{2^i}\]

where, \(\bar{x} = (x_i), \bar{y} = (y_i)\) and \(\delta\) is the discrete metric.

Then, \(D_1\) defines a metric which generates the product topology on \(\mathcal{A}^\mathbb{N}\). Similarly, \(D_1(\bar{x}, \bar{y}) = \sum_{i=-\infty}^{\infty} \frac{d(x_i, y_i)}{2^|i|}\)

defines a metric which generates the product topology on \(\mathcal{A}^\mathbb{Z}\), the space of all bifinite sequences over \(\mathcal{A}\).

It can be seen that the set \([i_0 i_1 \ldots i_k] = \{(x_r) : x_r = i_r, 0 \leq r \leq k\}\) is a clopen set in \(\Sigma_{\mathcal{A}}\), and is referred to as a cylinder set. Any open set, in \(\mathcal{A}^\mathbb{N}\), is a countable union of such sets. Consequently, the cylinder sets form a basis for the product topology on \(\Sigma_{\mathcal{A}}\). The shift (left shift) operator, defined as \(\sigma(x_0 x_1 \ldots) = x_1 x_2 x_3 \ldots\) is known to be continuous when the space is equipped with the metric \(D_1\). We refer the system \((\mathcal{A}^\mathbb{N}, \sigma)\) as the full shift (shift) space.
When $\mathcal{A}$ is a finite set, $\mathcal{A}^\mathbb{N}$ (resp. $\mathcal{A}^\mathbb{Z}$) is a compact metrizable space. Let $\Sigma \subseteq \Sigma_{\mathcal{A}}$ be a closed $\sigma$-invariant subset of $\Sigma_{\mathcal{A}}$. If there exists a finite collection of words (finite strings) that are forbidden in any sequence in $\Sigma$, then the subsystem $(\Sigma, \sigma)$ is called a subshift of finite type. Every subshift of finite type can be represented by a $\{0,1\}$ square matrix. In such a case, the matrix $M$ is called the transition matrix for the space $\Sigma_M$.

It may be noted that the subsystems “subshifts of finite type” can be considered also when the symbol set $\mathcal{A}$ is infinite. We will, however, not consider such cases.

Let $M$ be a transition matrix. $M$ is said to be irreducible if for every pair of indices $i$ and $j$ there is an $l > 0$ with $(M^l)_{ij} > 0$. Fix an index $i$ and let $p(i) = \gcd\{l : (M^l)_{ii} > 0\}$. This is called the period of the index $i$. When $M$ is irreducible, period of every index is same and is called the period of $M$. If the matrix has period one, it is said to be aperiodic (see [10] [11] [13]). Also, $\Sigma_M$ is transitive if and only if $M$ is irreducible. $M$ is irreducible and aperiodic if and only if there exists $r \in \mathbb{N}$ such that for all $k \geq r$, $M^k$ is strictly positive. $\Sigma_M$ is topological mixing if and only if $M$ is irreducible and aperiodic. Also, from [17] the following are equivalent.

1. $\Sigma_M$ is totally transitive.
2. $\Sigma_M$ is weakly mixing.
3. $\Sigma_M$ is topological mixing.

We now discuss the case when the alphabets in $\mathcal{A}$ are elements of some metric space, i.e. when $(\mathcal{A}, d)$ is a general metric space. Then, $\mathcal{A}^\mathbb{N}$ equipped with the metric

$$D_2(\bar{x}, \bar{y}) = \sum_{i=0}^{\infty} \frac{1}{2^i} d(x_i, y_i)$$

generates the product topology on $\mathcal{A}^\mathbb{N}$.

For any set of symbols $\mathcal{A}$, let $\mathcal{A}^\mathbb{N}$ be endowed with metric $D$ (which is $D_1$ or $D_2$, depending on the space $\mathcal{A}$). In all such cases, $(\mathcal{A}^\mathbb{N}, \sigma)$ is a dynamical system.

From [10] we see that if $(X, f)$ be a compact dynamical system, and $\Sigma = \{x, f(x), f^2(x), \ldots : x \in X\}$, then $\Sigma$ is a shift invariant subset of $X^\mathbb{N}$ and thus, $(\Sigma, \sigma)$ is a subsystem of the full shift $X^\mathbb{N}$. Also, if we define, $\phi : X \to \Sigma$

$$\phi(x) = (x, f(x), f^2(x), \ldots, f^n(x), \ldots)$$

Then, $\phi$ is one-one, onto and continuous function satisfying the relation $\phi \circ f = \sigma \circ \phi$. Thus, the system $(X, f)$ is conjugate to the system $(\Sigma, \sigma)$. This leads to the observation that for any dynamical system $(X, f)$, there exists $\Sigma \subseteq X^\mathbb{N}$ such that the system $(X, f)$ is conjugate to the system $(\Sigma, \sigma)$.

Also, [17] proves that a point in $\Sigma$ is a point of sensitivity for the system $(\Sigma, \sigma)$ if and only if it is not isolated. Though, in [17], this has been proved for the case of a discrete alphabet set, it can be easily established for the case when the alphabet set is any general metric space.

Hence, to study any dynamical system, it is sufficient to study a subsystem of an appropriate symbolic system. Henceforth, we shall constrain ourselves to the subsystems $(\Omega, \sigma)$ of the symbolic space $(\mathcal{A}^\mathbb{N}, \sigma)$, where the symbol set $\mathcal{A}$ comprises of the points in the metric space $(\mathcal{A}, d)$, where $d$ is the discrete metric in case $\mathcal{A}$ is a discrete alphabet set.

3. MAIN RESULTS

3.1. We first consider the case when $(\Omega, \sigma)$ is the full shift, i.e. $\Omega = \mathcal{A}^\mathbb{N}$. It has been shown that if $(X, f)$ has dense set of periodic points, $(\mathcal{K}(X), \mathcal{T})$ also has the same [2] [18]. We, prove the same result in terms of sequences.

Proposition 3.1. $(\mathcal{K}(\Omega), \mathcal{T})$ has dense set of periodic points.

Proof. Let $\mathcal{U} \subset \mathcal{K}(\Omega)$ be any open set. As the set of finite sequences in $\Omega$ is dense in $\mathcal{K}(\Omega)$, let $\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_s\} \in \mathcal{U}$, where $\bar{x}_j = (x_{n_j})$. Then, there exists $r \in \mathbb{N}$ such that $S(\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_s\}, \frac{1}{2^r}) \subset \mathcal{U}$.

Let $\bar{y}_1 = x_{0_1}x_{1_1} \ldots x_{r_1}x_{0_2} \ldots x_{r_2} \ldots x_{r_s} \ldots$ for $1 \leq j \leq s$. Each $\bar{y}_j$ is periodic under $\sigma$ with period $r + 1$ and $D(\bar{x}_j, \bar{y}_j) < \frac{1}{2^r}$.

Thus, $\{\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_s\}$ is a periodic point in $\mathcal{U}$ with period $r + 1$. \hfill $\Box$

The system $(\Omega, \sigma)$ is locally eventually onto for any alphabet set $\mathcal{A}$. And we also have

Proposition 3.2. $(\mathcal{K}(\Omega), \mathcal{T})$ is locally eventually onto.

Proof. Let $\mathcal{U}$ be a non empty open set in $\mathcal{K}(\Omega)$. As set of finite sequences is dense in $\mathcal{K}(\Omega)$, let $\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_s\} \in \mathcal{U}$ where $\bar{x}_j = (x_{j_i}) \in \mathcal{U}$. Thus, there exists $r \in \mathbb{N}$ such that $S(\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_s\}, \frac{1}{2^r}) \subset \mathcal{U}$.

We shall show that $\mathcal{T}^r(S(\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_s\}, \frac{1}{2^r})) = \mathcal{K}(\Omega)$. Let $K \in \mathcal{K}(\Omega)$. Let $A_i = \{x_{i1}x_{i2} \ldots x_{ir} \bar{a} : \bar{a} \in K\}$. Then, $A_i$ is a compact subset of $\Omega$. Let $M = \bigcup_{i=1}^{s} A_i$. Then, $M \in S(\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_s\}, \frac{1}{2^r})$. Also, $\mathcal{T}^{-1}(M) = K$. As $K \in \mathcal{K}(\Omega)$ was arbitrary, $\mathcal{T}^{-1}(S(\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_s\}, \frac{1}{2^r})) = \mathcal{K}(\Omega)$. Hence the result. \hfill $\Box$

This can also be seen in a general case, but our proof is specialized for sequences. The case of sensitivity is not very simple in general (see [19]). But, we observe that $(\Omega, \sigma)$ is sensitive for any discrete alphabet set $\mathcal{A}$, with sensitivity constant $\frac{1}{2}$. Similarly,
Proposition 3.3. \((K(\Omega), \sigma)\) is sensitive with sensitivity constant \(\frac{1}{2}\).

Proof. To establish sensitivity on the induced system, it is sufficient to prove sensitivity on the collection of all finite subsets, since finite sets are dense in \(K(\Omega)\).

Let \(A = \{x_1, x_2, \ldots, x_m\}\) be a finite subset where \(x^i = x_{i1}x_{i2} \ldots\). Let \(\epsilon > 0\) and let \(S_\epsilon(A)\) be an \(\epsilon\)-neighborhood of \(A\). Let \(n \in \mathbb{N}\) such that \(\frac{1}{2m} < \epsilon\). Pick \(x \in A\) such that \(x \neq x_{n+1}^i\).

Let \(B = \{y_1, y_2, \ldots, y_k\}\) where \(y^i = x_{i1}x_{i2} \ldots x_{in}xx \ldots\)

Then, \(B \in S_\epsilon(A)\) and \(\sigma^{n+1}(B) = \{xx \ldots\}\).

Consequently, \(d_H(\sigma^{n+1}(A), \sigma^{n+1}(B)) > \frac{1}{2}\).

For sensitivity, we use the discrete metric on \(A\) to establish the sensitivity on the hyperspace. However, if \(A\) is equipped with a metric \(d\) and \(diam(A) = r\), a similar proof establishes the sensitivity for the induced map on the hyperspace, with sensitivity constant \(\frac{1}{2(r+1)}\).

Proposition 3.4. For any alphabet set \(A\), containing at least two elements,
1. \((\Omega, \sigma)\) is Li-Yorke sensitive.
2. \((K(\Omega), \sigma)\) is Li-Yorke sensitive.

Proof. 1. We note that when \(A\) is a discrete alphabet set, \((\Omega, \sigma)\) is Li-Yorke sensitive. For the sake of the proof we include the case when \((A, d)\) is a metric space.

Let \(x = (x_0, x_1, \ldots) \in A^\mathbb{N}\) and let \(S_\epsilon(x)\) be an \(\epsilon\)-neighborhood of \(x\). Let \(n_\epsilon \in \mathbb{N}\) such that \(\frac{1}{2n_\epsilon} < \epsilon\). For each \(n \in \mathbb{N}\), choose \(x^i_n \in X\) such that \(d(x^i_n, x^i_{n+1}) < \frac{1}{2n_\epsilon}\) where \(diam(A) > r, r \in \mathbb{R}\). Replace \(2^n\)-th entries (i.e. \(x^i_{2^n}\)) in the sequence \(x\) for \(n \geq n_\epsilon\) by \(x^i_{2^n}\) (keeping all others same) to obtain new sequence \(y\). Consequently, \(y \in S_\epsilon(x)\). Further, \(D(\sigma^{2^n+1}(x), \sigma^{2^n+1}(y)) < \frac{1}{2}\) and \(D(\sigma^{2^n}(x), \sigma^{2^n}(y)) < \frac{r}{2(1+r)}\) for infinitely many \(k\). Thus, the space \((A^\mathbb{N}, \sigma)\) is Li-Yorke sensitive.

2. We now prove that the system \((K(A^\mathbb{N}), \sigma)\) is Li-Yorke sensitive. It suffices to show the same in case of the alphabet set being equipped with a metric \(d\), and \(diam(A) > r, r \in \mathbb{R}\).

We now prove that \((K(\Omega), \sigma)\) is Li-Yorke sensitive. Let \(K_1 \in K(\Omega)\) and let \(S_{\frac{1}{m}}(K_1) (m > 1)\) be a neighborhood of \(K_1\) in the hyperspace. Since \(K_1\) is totally bounded and periodic sequences are dense in \(\Omega\), let \(W_m = \{w_{1,m}, w_{2,m}, \ldots, w_{m,m}\}\) be the finite set of words such that \(\frac{1}{m}\)-ball around the sequences \(\{www \ldots w \in W_m\}\) covers \(K_1\). Let each \(w_{i,m}\) be of length \(L_m^i\) and let \(L_m\) be the least common multiple of all such lengths.

As \(\sigma^{L_m+1}(K_1)\) is compact, we construct the set \(W_{m+1}\) of finitely many words such that \(\frac{1}{m+2}\)-ball around the sequences \(\{www \ldots w \in W_{m+1}\}\) covers the set \(\sigma^{L_m+1}(K_1)\). Let \(L_{m+1}\) be the least common multiple of all lengths of words in \(W_{m+1}\).

Inductively, define the set \(W_{m+i}\) as the set of finitely many words such that \(\frac{1}{m+i+2}\)-ball around the sequences \(\{www \ldots w \in W_{m+i}\}\) covers the set \(\sigma^{L_m+L_{m+1}+\ldots +L_{m+i+1}}(K_1)\).

Let \(y = (y_0, y_1, \ldots, y_n) \in K_1\). For each \(i\), choose \(x_i \in A\) such that \(d(x_i, yL_m+L_m+\ldots+L_{m+i+1}) \geq \frac{1}{2}\).

Construct the set \(K_2 = \{w_0w_0w_0w_0w_1 \ldots w_{i-1}w_i \in W_{m+i}\}\), where the repetitions of \(w_0\) are made till length \(L_{m+i}\) is achieved and \(w_i\) varies over all possible entries in \(W_{m+i}\).

We see that \(K_2 \in K(\Omega)\).

Let \(\bar{z}\) be any limit point of \(K_2\). Then, there exists a sequence \((\bar{z}_n) \in K_2\) such that \(\bar{z}_n \to \bar{z}\). But, in any sequence \(\bar{z}_n\), the first \(L_m\) entries are repetitions of words from \(W_m\), which has finite number of choices. Hence, there exists a subsequence \((\bar{z}_{n_k})\) and a word \(w_{0*} \in W_m\) such that first \(L_m\) entries are repetitions of \(w_{0*}\) for each member of the subsequence \((\bar{z}_{n_k})\).

Also, \(\bar{z}_{n_k} \to \bar{z}\). And so, in \(\bar{z}\), the first \(L_m\) entries are repetitions of \(w_{0*}\) and the \(L_m + 1\) th entry will be \(w_0\) again, the next \(L_{m+1}\) entries in each \(\bar{z}_{n_k}\) are repetition of words from \(W_{m+1}\), which have finite number of choices. Hence, there exists a subsequence \((\bar{z}_{n_k})\) in which the next \(L_{m+1}\) entries are repetitions of a word \(w_{1*} \in W_{m+1}\). Consequently, the next \(L_{m+2}\) entries in \(\bar{z}\) are repetitions of the word \(w_{1*}\). Inductively, we see that \(\bar{z}\) is solely of the form of sequences in \(K_2\), implying that \(\bar{z} \in K_2\) i.e. \(K_2\) is closed.

Then, \(K_2 \subseteq S_{\frac{1}{m}}(K_1)\) with \(\limsup_{n \to \infty} D_H(\sigma^n(K_1), \sigma^n(K_2)) \geq \frac{1}{2(1+r)}\) and \(\liminf_{n \to \infty} D_H(\sigma^n(K_1), \sigma^n(K_2)) = 0\). Thus, \((K(\Omega), \sigma)\) is Li-Yorke sensitive. □

3.2. We now consider the case when \((\Omega, \sigma)\) is a subshift of finite type described via the transition matrix \(M\), where,

\[
M = \begin{pmatrix}
m_{11} & m_{12} & \cdots & m_{1n} \\
m_{21} & m_{22} & \cdots & m_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
m_{n1} & m_{n2} & \cdots & m_{nn}
\end{pmatrix}
\]
with each $m_{ij} = 0$ or $1$.

Let $S_2 = \{(i_1, i_2) : 1 \leq i_1, i_2 \leq n\}$. Let $M^{s_2}$ be a matrix indexed by entries of $S_2$ defined as, $M^{s_2}_{(i_1, i_2), (j_1, j_2)} = m_{i_1j_1}m_{i_2j_2}$.

Then $M^{s_2}$ is a square matrix of order $n^2$, given as,

$$M^{s_2} = \begin{pmatrix}
m_{11}m_{11} & \cdots & m_{1n}m_{1n} & \cdots & m_{1n}m_{1(n-1)} \\
m_{21}m_{21} & \cdots & m_{2n}m_{2n} & \cdots & m_{2n}m_{2(n-1)} \\
\vdots & & \vdots & & \vdots \\
m_{n1}m_{n1} & \cdots & m_{nn}m_{nn} & \cdots & m_{nn}m_{n(n-1)} \\
\end{pmatrix}$$

When $\Omega$ comprises of $n$ elements, the transition matrix $M$ provides the dynamical behavior of the system, giving the details of how two given regions, labelled by distinct alphabets, of the system interact. In the matrix generated above, $M^{s_2}_{(i_1, i_2), (j_1, j_2)}$ gives the details of simultaneous interaction of the $i_1$th, $j_1$th, $i_2$th, and $j_2$th regions of the original system respectively. As the entries vary over all possible combinations, the matrix determines the dynamics of $(\Omega \times \Omega, \sigma \times \sigma)$. Thus, the system $(\Omega \times \Omega, \sigma \times \sigma)$ can be embedded into certain symbolic dynamical system of $n^2$ symbols. This is similar to the concept of “higher block codes” discussed in [11].

Similarly, let $S_k = \{(i_1, i_2, \ldots, i_k) : 1 \leq i_1, i_2, \ldots, i_k \leq n\}$. Let $M^{s_k}$ be a matrix indexed by entries of $S_k$ defined as, $M^{s_k}_{(i_1, i_2, \ldots, i_k), (j_1, j_2, \ldots, j_k)} = m_{i_1j_1}m_{i_2j_2} \cdots m_{i_kj_k}$.

Then $M^{s_k}$ is a square matrix of order $n^k$, given as

$$M^{s_k} = \begin{pmatrix}
m_{11}m_{11} & \cdots & m_{1n}m_{1n} & \cdots & m_{1n}m_{1(n-1)} \\
m_{21}m_{21} & \cdots & m_{2n}m_{2n} & \cdots & m_{2n}m_{2(n-1)} \\
\vdots & & \vdots & & \vdots \\
m_{n1}m_{n1} & \cdots & m_{nn}m_{nn} & \cdots & m_{nn}m_{n(n-1)} \\
\end{pmatrix}$$

Arguing as before, the matrix $M^{s_k}$ determines the simultaneous interaction of a set of $k$ regions of the original space, with another set of $k$ regions of the same space. The matrix $M^{s_k}$ hence determines the dynamics of $(\Omega \times \Omega \times \cdots \times \Omega, \sigma \times \sigma \times \cdots \times \sigma)$, where the cartesian product is taken $k$ number of times.

It can be observed that the first $n \times n$ block in $M^{s_k}$ is just $M$.

The Vietoris topology is generated by sets of the form $< U_1, U_2, \ldots, U_n >$. Hence, while studying the hyperspace under Vietoris topology, it is sufficient to study the simultaneous behavior of finitely many regions of the original space. Consequently, the behavior of the matrices $M^{s_k}$ is sufficient to study the dynamics of the hyperspace $(K(\Omega, \sigma))$.

For a dynamical system $(X, f)$, $f$ is weakly mixing if and only if for every $k \geq 2$, $f^{k} \times f^{k} \cdots \times f^{k}$ is transitive [8]. Thus, $\sigma$ is weakly mixing if and only if $\sigma \times \sigma \times \cdots \times \sigma$ is transitive for each $k \geq 2$, and by the discussion above, if and only if $M^{s_k}$ is irreducible for $k \geq 2$.

This proves that on a subshift of finite type, described via a transition matrix $M$, $\sigma$ is weakly mixing if and only if for every $k \geq 2$, $M^{s_k}$ is irreducible. And also establishes that the following are equivalent:

1. $(\Omega, \sigma)$ is weakly mixing
2. $(K(\Omega, \sigma))$ is weakly mixing
3. $(K(\Omega, \sigma))$ is transitive

Similarly, the following are equivalent:

1. $(\Omega, \sigma)$ is topologically mixing
2. $(K(\Omega, \sigma))$ is topologically mixing

Also $M$ is irreducible and aperiodic if and only if $M^{s_k}$ is irreducible and aperiodic for all $k \in \mathbb{N}$, establishing that $(\Omega, \sigma)$ is topologically mixing if and only if $(K(\Omega, \sigma))$ is topologically mixing.

Transitivity of the induced system $(K(\Omega, \sigma))$ is equivalent to weak mixing of the original system $(\Omega, \sigma)$. But, if the system $(\Omega, \sigma)$ is transitive, the induced system $(K(\Omega, \sigma))$ may fail to be transitive. This can be illustrated by a simple counterexample:

**Example 3.5.** Let $\Sigma_M$ be the subshift of finite type given by the matrix

$$
\begin{pmatrix}
1 & 0 & 1 \\
1 & 1 & 0 \\
0 & 1 & 1 \\
\end{pmatrix}
$$

This matrix is irreducible and aperiodic, but $\Sigma_M$ is not transitive.
Then, $M$ is irreducible and hence generates a transitive subshift. However $M^k$ is not irreducible for $k \geq 2$. Hence, the induced map is not transitive.

**Remark 3.6.** The above results involving transitivity, weakly mixing and mixing are known in the case of any general dynamical system $(X, f)$. See [2] [5] [13].

Whereas for sensitivity we can observe that

**Proposition 3.7.** $(\Omega, \sigma)$ is sensitive if and only if $(K(\Omega), \overline{\sigma})$ is sensitive.

**Proof.** Let $\Omega$ be a subshift of finite type and let $(\Omega, \sigma)$ be sensitive. We now show that $(K(\Omega), \overline{\sigma})$ is sensitive. In order to prove this, we show that the induced map, $\overline{\sigma}$, is sensitive on all finite subsets of $\Omega$, with a uniform sensitivity constant. Without loss of generality, we assume that all forbidden words for $\Omega$ have the same length, say $M$.

Let $A = \{\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^r\}$ be a finite subset of $\Omega$ and let $[x^1_0x^1_0 \ldots x^1_n]$ be a neighborhood of $\bar{x}^i$, $i = 1, 2, \ldots, r$. We show that $\overline{\sigma}$ is sensitive on $A$ with sensitivity constant $\frac{1}{M}$. For each $i$, in the sequence $\bar{x}^i \in [x^1_0x^1_0 \ldots x^1_n]$, we see the possible number of options for the next entry $x^i_{n+1}$. If more than one option is available, we find a sequence $\overline{y}^i$ with this entry replaced, or otherwise move to the next entry. We need to continue this process for only the next $M$ entries, since at least one of the entries $x^i_{n+1}$ and $y^i_{n+1}$ will be different for $l = 1, 2, \ldots, M$. Otherwise, if each entry $x^i_{n+1}, x^i_{n+2}, \ldots, x^i_{n+M}$ has a unique option, it means that there is only one allowed word of length $M$. This will contradict the sensitivity of $\sigma$. Thus, we find sequences $\overline{y}^1, \overline{y}^2, \ldots, \overline{y}^r$ such that for each $i$, at least one of the entries $x^i_{n+1}$ and $y^i_{n+1}$ are different for $l = 1, 2, \ldots, M$.

Therefore, for each $i$, $D(\sigma^n(\overline{x}^i), \sigma^n(\overline{y}^i)) \geq \frac{1}{M}$. Construct the set $C := \{\bar{z}^1, \bar{z}^2, \ldots, \bar{z}^r\}$ as:

$$\bar{z}^i = \begin{cases} \overline{y}^i, & D(\sigma^n(\bar{x}^i), \sigma^n(\overline{y}^i)) \geq \frac{1}{M}, \\ \bar{x}^i, & \text{otherwise}. \end{cases}$$

Therefore, $\sigma^n(\bar{x}^i)$ is at least $\frac{1}{M}$ apart from $\sigma^n(\bar{z}^i)$, $i = 1, 2, \ldots, r$. Thus, $\overline{D_H(\overline{\sigma}^n(\bar{A}), \overline{\sigma}^n(\bar{C}))} \geq \frac{1}{M}$, and hence $(K(\Omega), \overline{\sigma})$ is sensitive.

Conversely, let $(K(\Omega), \overline{\sigma})$ be sensitive with sensitivity constant $\delta$. For $\bar{x} \in \Omega$ and for $\epsilon > 0$, let $S_\epsilon(\{\bar{x}\})$ be an $\epsilon$-neighborhood of $\{\bar{x}\} \in K(\Omega)$. Then, there exists $A \in S_\epsilon(\{\bar{x}\})$ and $n \in \mathbb{N}$ such that $D_H(\overline{\sigma}^n(\{\bar{x}\}), \overline{\sigma}^n(\bar{A})) > \delta$. Therefore, there exists a sequence $\bar{y} \in A \subset S_\epsilon(\bar{x})$ such that $D_H(\overline{\sigma}^n(\{\bar{x}\}), \overline{\sigma}^n(\{\bar{y}\})) > \delta$. Consequently, we get $D(\sigma^n(\bar{x}), \sigma^n(\bar{y})) > \delta$ establishing the sensitivity of $\sigma$. □

**Remark 3.8.** It is known that when the system $(X, f)$ is cofinitely sensitive, then so is $(K(\Omega), \overline{\sigma})$ and vice-versa [13]. And since sensitive subshifts of finite type are cofinitely sensitive, the above result should hold as a special case of the general result. We however prove this in terms of sequences, without referring to the cofinite sensitivity of the space.

3.3. We now consider the case when $(\Omega, \sigma)$ is any subshift of $(A^n, \sigma)$, where $A$ can be any alphabet set.

The system $(\Omega, \sigma)$ induces the subsystem $(K(\Omega), \overline{\sigma})$.

**Proposition 3.9.** If $(\Omega, \sigma)$ has dense set of periodic points, then so does $(K(\Omega), \overline{\sigma})$.

**Proof.** Let $K \subset K(\Omega)$ and let $S_{D_H}(K, \epsilon)$ be a neighborhood of $K$ in the hyperspace. As finite set of points in $\Omega$ are dense in $K(\Omega)$, there exists such a finite set $\{\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^\alpha\} \subset S_{D_H}(K, \epsilon)$, where each $\bar{x}^i = x^i_0x^i_1x^i_2 \ldots$. Further, there exists $\eta > 0$ such that $S_{D_H}(\{\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^\alpha\}, \eta) \subset S_{D_H}(K, \epsilon)$. As periodic points are dense in $(\Omega, \sigma)$, $\bar{y}^\prime = x^1_0x^1_1 \ldots x^1_ny^1_{n+1}y^1_{n+2} \ldots y^1_{n+m}x^1_1 \ldots x^\alpha_ny^\alpha_{n+1}y^\alpha_{n+2} \ldots y^\alpha_n, \ldots$ is a periodic point of period $k_1$ in $S(\bar{x}^1, \eta)$. Let $K = \text{lcm}(k_1, k_2, \ldots, k_n)$. Then, $\{\bar{y}^1, \bar{y}^2, \ldots, \bar{y}^\alpha\}$ is a periodic point of $\overline{\sigma}$ with period $K$, contained in $S_{D_H}(\{\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^\alpha\}, \eta)$. Thus, periodic points are dense in $(K(\Omega), \overline{\sigma})$. □

The result proved above is a manifestation of the known result in the general case. It is known that the system $(\Omega, \sigma)$ can be seen as a subsystem of $(K(\Omega), \overline{\sigma})$. And by expanding the symbol set (of the original system) to an appropriate cardinality the system $(K(\Omega), \overline{\sigma})$ can be embedded into the symbolic space of these new symbols. Then if $(K(\Omega), \overline{\sigma})$ is transitive, under successive iterations, any two regions of the space labelled by distinct symbols interact. The same holds for the original set of symbols. Consequently, any two regions of the original dynamical system $(\Omega, \sigma)$ interact under iterations of the map $\sigma$ making the $(\Omega, \sigma)$ transitive.

Similarly, we observe that

**Proposition 3.10.** $(\Omega, \sigma)$ is weakly mixing if and only if $(K(\Omega), \overline{\sigma})$ is weakly mixing.
Proof. Let \((\Omega, \sigma)\) be weakly mixing. Let \(S_{Dn}(A_1, \epsilon), S_{Dn}(A_2, \epsilon)\) and \(S_{Dn}(A_3, \epsilon)\), \(S_{Dn}(A_4, \epsilon)\) be two pairs of open sets in the hyperspace. As finite sets are dense in the hyperspace, let \(\{\bar{a}^{i_1}, \bar{a}^{i_2}, \ldots, \bar{a}^{i_k}\} \subset S_{Dn}(A_1, \epsilon), 1 \leq i \leq 4\). Further, there exists \(\eta > 0\) such that \(S_{Dn}(\{\bar{a}^{i_1}, \bar{a}^{i_2}, \ldots, \bar{a}^{i_k}\}) \subset S_{Dn}(A_1, \epsilon), 1 \leq i \leq 4\). As \(\sigma\) is weakly mixing, there exists \(b^{i_1} = a_0^{i_1} a_1^{i_1} \ldots a_k^{i_1} b^{i_{k+1}} \ldots, b^{i_j} \in S(\bar{a}^{i_1}, \eta), i = 1, 2, j = 1, 2, \ldots, k\) and \(n \in \mathbb{N}\) such that \(\sigma^n(b^{i_1}) \in S(\bar{a}^{i_{j+2}}, \eta)\). Thus, \(B_i = \{b^{i_1}, b^{i_2}, \ldots, b^{i_k}\} \subset S_{Dn}(A_i, \epsilon)\) such that \(\sigma^L(B_i) \subset S_{Dn}(A_i+2, \epsilon)\) for \(i = 1, 2\). Thus, \((K(\Omega), \pi)\) is weakly mixing.

The converse can be deduced similarly as discussed above. □

Proposition 3.11. \((\Omega, \sigma)\) is topological mixing if and only if \((K(\Omega), \pi)\) is topological mixing.

Proof. Let \((\Omega, \sigma)\) be topologically mixing. Let \(S_{Dn}(A_1, \epsilon), S_{Dn}(A_2, \epsilon)\) be non-empty open sets in the hyperspace. As finite sets are dense in \(K(\Omega)\), let \(\{\bar{a}^{i_1}, \bar{a}^{i_2}, \ldots, \bar{a}^{i_k}\} \subset S_{Dn}(A_1, \epsilon), i = 1, 2\). Further, there exists \(\eta > 0\) such that \(S_{Dn}(\{\bar{a}^{i_1}, \bar{a}^{i_2}, \ldots, \bar{a}^{i_k}\}) \subset S_{Dn}(A_1, \epsilon)\). As \(\sigma\) is topological mixing, for each \(j\), there exists \(n_{j_1} \in \mathbb{N}\) such that for any \(n \geq n_{j_1}\), there exists \(b^{i_j} = (a_0^{j_1} a_1^{j_1} \ldots a_k^{j_1} b^{i_{j+1}} \ldots) \in S(\bar{a}^{i_j}, \eta)\) such that \(\sigma^n(b^{i_j}) \in S(\bar{a}^{i_{j+2}}, \eta)\). Let \(r = \max\{n_{j_1} : 1 \leq j \leq k\}\). Thus, for each \(n \geq r\), there exists \(b^{i_j} = (a_0^{j_1} a_1^{j_1} \ldots a_k^{j_1} b^{i_{j+1}} \ldots) \subset S(\bar{a}^{i_1}, \eta)\) such that \(\sigma^n(b^{i_j}) \in S(\bar{a}^{i_{j+2}}, \eta)\).

As the above can be done for all \(n \geq r\), \((K(\Omega), \pi)\) is topologically mixing.

The converse can be deduced similarly as above.

Remark 3.12. Similar techniques have been used to prove the above result in a much more general setting [2][13][15]. The result below also uses techniques similar to the general form as in [19].

Whereas for sensitivity, Proposition 3.13. \((\Omega, \sigma)\) is infinitesimally sensitive if and only if \((K(\Omega), \pi)\) is infinitesimally sensitive.

Proof. Let \((\Omega, \sigma)\) be infinitesimally sensitive. We prove that \((K(\Omega), \pi)\) is infinitesimally sensitive, by showing that \(\sigma^L\) is infinitesimally sensitive on the set of all finite sets in \(K(\Omega)\).

It will suffice to prove the result in case of the alphabet set \(A\) being discrete. In case \(A\) is equipped with some metric \(d\), the sensitivity constants can be suitably modified.

Without loss of generality, let \(\sigma^L\) be infinitesimally sensitive with sensitivity constant \(\frac{1}{\epsilon}\). This implies that for any neighborhood \([x_0, x_1, x_2, \ldots, x_n]\) of \(\bar{x} = (x_0, x_1, x_2, \ldots, \ldots, x_n)\), there exists \(k \in \mathbb{N}\) such that for each \(l > k\), there exists \(y \in \mathbb{N}\) such that \(x_l \in [x_0, x_1, x_2, \ldots, x_n]\) such that \(x_{l+1}, x_{l+2}, \ldots, x_{l+k} \neq y_{l+1}, y_{l+2}, \ldots, y_{l+k}\).

Let \(A = \{x^1, x^2, \ldots, x^k\} \subset K(\Omega)\). Let \(U = U_1, U_2, \ldots, U_k > 0\) be a neighborhood of \(A\) such that \(U_j = [x_j^1, x_j^2, \ldots, x_j^k]\). For each \(j\) and each neighborhood \(U_j\), there exists \(N_j\) such that for each \(l > N_j\), \(D(\sigma^L(x^j), \sigma^L(y^j)) > \frac{\epsilon}{\mathbb{N}}\), for some \(y^j \in U_j\).

Let \(N = \max\{N_1, N_2, \ldots, N_k\}\). For each \(j\) and for each \(l \geq N\), there exists \(y \in U_j\) such that \(x_{l+1}, x_{l+2}, \ldots, x_{l+k} \neq y_{l+1}, y_{l+2}, \ldots, y_{l+k}\).

Therefore, for each \(j\), for each \(l > N\), \(D(\sigma^L(x^j), \sigma^L(y^j)) > \frac{\epsilon}{\mathbb{N}}\).

Construct the set \(C_i := \{x^1, x^2, \ldots, x^k\}\) as,

\[
\bar{x} = \begin{cases} 
\bar{y}, & D(\sigma^L(x^j), \sigma^L(y^j)) > \frac{\epsilon}{\mathbb{N}}; \\
\bar{x}, & \text{otherwise.}
\end{cases}
\]

Therefore, \(\sigma^L(x)\) is at least \(\frac{1}{\epsilon}\) apart from \(\sigma^L(x^j), i = 1, 2, \ldots, k\). Thus, \(D_{\Omega}(\sigma^L(A), \sigma^L(C_i)) \geq \frac{1}{\epsilon}\). As such sets can be constructed for all \(l > N\), \((\Omega, \sigma)\) is infinitesimally sensitive.

Conversely, let \(\bar{x} \in \Omega\) and let \(\epsilon > 0\) be given. As \((K(\Omega), \pi)\) is infinitesimally sensitive with sensitivity constant \(\delta\) and \(S_i(\bar{x})\) is \(\epsilon\)-neighborhood of \(\bar{x}\), there exists \(n_0 \in \mathbb{N}\) such that for each \(n \geq n_0\), there exists \(A_n \in S_i(\bar{x})\) such that \(D_{\Omega}(\pi(A_n), \pi(\bar{x})) > \frac{1}{\epsilon}\). Consequently, there exists a sequence \(y_n \in A_n \subset \Omega(\bar{x})\) such that \(D_{\Omega}(\pi(y_n), \pi(\bar{x})) > \frac{1}{\epsilon}\). As the existence of the set \(A_n\) is guaranteed for all \(n \geq n_0\), for each \(n\) integer \(n\), we obtain \(\bar{y}_n \in S_i(\bar{x})\) such that \(D(\sigma^L(\bar{x}), \sigma^L(\bar{y}_n)) > \frac{1}{\epsilon}\). Hence \((\Omega, \sigma)\) is infinitesimally sensitive.

In case of transitive subshifts, cofinite sensitivity and syndetic sensitivity turn out to be equivalent. Though, this equivalence holds for any alphabet set, we prove it below only for the case of discrete alphabet.

Proposition 3.14. Let \((\Omega, \sigma)\) be a transitive subshift. Then, \((\Omega, \sigma)\) is cofinitely sensitive if and only if \((\Omega, \sigma)\) is syndetically sensitive.

Proof. Since every cofinitely sensitive map is syndetically sensitive, we only need to prove that \((\Omega, \sigma)\) is cofinitely sensitive whenever \((\Omega, \sigma)\) is syndetically sensitive.

Let \((\Omega, \sigma)\) be syndetically sensitive and let \(\bar{x} = (x_0, x_1, x_2, x_3, \ldots, x_n)\) be the point with dense orbit. Let \(K\) be the bound for syndetic sensitivity of the cylinder \([x_0, x_1, x_2, x_3, \ldots, x_n]\). It can be seen that \(K\) is the bound for syndetic sensitivity of any other cylinder of the form \([x_{m+1}, x_{m+2}, \ldots, x_{m+r}]\).

Let \(y \in Orb(\bar{x})\) and let \([y_0, y_1, y_2, \ldots, y_k]\) be a neighborhood of \(y\). As the bound for syndetic sensitivity for any cylinder around \(\bar{y} \in Orb(\bar{x})\) must be \(K\), there exists some \(N \in \mathbb{N}\) such that for each instant \(l > N\), there exists
Corollary 1. Let \((\Omega, \sigma)\) be a transitive subshift. \((\Omega, \sigma)\) is syndetically sensitive if and only if \((K(\Omega), \sigma)\) is syndetically sensitive.

One of the important examples of subshifts are the Sturmian subshifts (see [17]). They are minimal shifts and it has been shown in [17] that Sturmian subshifts are syndetically sensitive.

Corollary 2. All Sturmian subshifts are cofinitely sensitive. And hence, the corresponding induced map \(\sigma\) on the hyperspace of its compact subsets is also cofinitely sensitive.

The above corollary contradicts the result in [17] which states that no Sturmian subshift is cofinitely sensitive. The error there is a trivial one (follows from the definition there).

There exist subshifts \((\Omega, \sigma)\) such that \((K(\Omega), \sigma)\) is not sensitive [?]_. The example constructed there is not syndetically sensitive, and hence not cofinitely sensitive. Does that mean all subshifts \((\Omega, \sigma)\), for which \((K(\Omega), \sigma)\) is sensitive, need to be either syndetically sensitive or cofinitely sensitive? We answer this in the negative, by giving an example (from [17]) of a subshift \((\Omega, \sigma)\) which is not syndetically sensitive, but for which \((K(\Omega), \sigma)\) happens to be sensitive.

Example 3.15. Let \((n_k)\) be a strictly increasing sequence of natural numbers. Inductively, define a sequence of words \(w_k\) as,

\[
 w_1 = 0, \quad w_2 = 1^{n_1}, \quad w_3 = w_1w_2, \quad \ldots, \quad w_{2k} = 1^{n_k}, \quad w_{2k+1} = w_1w_2w_3 \ldots w_{2k}, \quad \ldots
\]

Put \(\bar{x} = w_1w_2w_3 \ldots w_k \in \Sigma = \{0, 1\}^\mathbb{N}\). Clearly, \(\bar{x}\) is a recurrent point for the dynamical system \((\Sigma, \sigma)\), where \(\sigma\) denotes the left shift on the sequence space of two symbols.

Define \(X := \text{Orb}(\bar{x})\) where, \(\text{Orb}(\bar{x}) = \{\sigma^n(\bar{x}) : n \geq 0\}\).

It has been shown in [17], that \(X\) defined above is sensitive, but not syndetically sensitive. We here prove that \((K(X), \sigma)\) is sensitive.

We first show that elements of the form \(w_1w_2 \ldots w_n1^\infty\) are limit points of the orbit of \(\bar{x}\) under the map \(\sigma\). It is sufficient to show that all elements of the form \(w_1w_2 \ldots w_{2k-1}1^\infty\) are limit points of the orbit.

It can be seen that \(w_{2k+1}\) is the word \(w_1w_2 \ldots w_{2k-1}1^{n_k}\). Similarly \(w_{2k+3}\) contains the word \(w_1w_2 \ldots w_{2k-1}1^{n_k}1^{n_{k+1}}\), \(w_{2k+5}\) contains the word \(w_1w_2 \ldots w_{2k-1}1^{n_k}1^{n_{k+1}}1^{n_{k+2}}\) and so on. Consequently, all the sequences of the form \(w_1w_2 \ldots w_{2k-1}1^\infty\) are limit points of the orbit and hence every sequence of the form \(w_1w_2 \ldots w_n1^\infty\) is a limit point of the orbit.

As \(\text{Orb}(\bar{x})\) is dense in \(X\), the set \(D = \{\bar{x}, \bar{x}^2, \ldots, \bar{x}^r\} : \bar{x}^i \in \text{Orb}(\bar{x})\) for all \(i, r \geq 1\) is dense in \(K(X)\). We shall, equivalently, show that \(\sigma\) is sensitive on \(D\).

Let \(\{\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^r\} \subseteq D\). Let \(U_1, U_2, \ldots, U_k > 0\) be its neighborhood, where each \(U_i\) is a neighborhood of \(\bar{x}^i\). As \(\bar{x}^i = \sigma^n(\bar{x})\), let \([x_0^i, x_1^i, \ldots, x_n^i]\) \(\subseteq U_i\) for some \(n_i\). Let \(N = \max\{n_i : 1 \leq i \leq k\}\). Then, there exists \(\bar{y}^i = \bar{x}^i\bar{x}^{i^2} \ldots \bar{x}^{i^n}1^\infty \in U_i\) for each \(i\). Consider the set \(B = \{\bar{y}^1, \bar{y}^2, \ldots, \bar{y}^r\} \subseteq \bar{U}_1, U_2, \ldots, U_k > 0\). The set \(B\) reduces to the constant sequence \(11111 \ldots\) after some finite iterations. Thus, there exists \(l \in \mathbb{N}\) such that \(D_H(\sigma(A), \sigma(B)) \geq \frac{1}{2}\). Consequently, \(\sigma\) is sensitive on \(D\).

And for expansivity, we have

**Proposition 3.16.** For any alphabet set \(A\), \((K(\Omega), \sigma)\) is expansive implies \((\Omega, \sigma)\) is expansive. However, the converse does not hold in general.

**Proof.** Let \((K(\Omega), \sigma)\) be \(\delta\)-expansive and let \(\bar{x}, \bar{y} \in \Omega\). Then, as \(\sigma\) is expansive, for \(\{x\}, \{y\} \subseteq \Omega\), there exists \(k \in \mathbb{N}\) such that \(D_H(\sigma^k(\bar{x}), \sigma^k(\bar{y})) \geq \delta\). Consequently, \(D(\sigma^k(\bar{x}), \sigma^k(\bar{y})) \geq \delta\). Thus, \((\Omega, \sigma)\) is also \(\delta\)-expansive.

We now provide an example to show that the converse is not true.

Let \(\Sigma\) be the sequence space of two symbols 0 and 1 and let \(K(\Sigma)\) be the hyperspace of all non empty compact subsets of \(\Sigma\). It can be easily observed that \((\Sigma, \sigma)\) is expansive with expansivity constant \(\frac{1}{3}\). However, we prove that the system \((K(\Sigma), \sigma)\) is not expansive.

Let \(S_1\) be the set of all sequences comprising of all 0's except one string of 1's of length \(r\), \(0 \leq r \leq n\). Let \(S_2\) be the set of all sequences comprising of all 0's except one string of 1's of length \(r\), \(0 \leq r \leq n + 1\). Then, \(D_H(S_1, S_2) = \frac{1}{\log_2(n + 1)}\). Also, \(\sigma(S_i) = S_i, i = 1, 2\). Thus, for any \(k \in \mathbb{N}\), \(D_H(\sigma^k(S_1), \sigma^k(S_2)) = D_H(S_1, S_2) = \frac{1}{\log_2(n + 1)} < \delta\) which contradicts the definition of \(\delta\).

Thus, the system \((K(\Sigma), \sigma)\) is not expansive.

A similar result for expansivity holds in case of a general dynamical system \((X, f)\). See [18] for details.
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