MATHEMATICAL MODELING OF AN ARRAY OF NUCLEAR WASTE CONTAINERS

Alain Bourgeat\textsuperscript{1}, Olivier Gipouloux\textsuperscript{2} and Eduard Marušić-Paloka\textsuperscript{4}

1 Introduction

The goal of this paper is to give a mathematical model describing the global behavior of an underground waste repository, once the containers start to leak. The purpose of such a global model is to be used for the full field simulations used in safety assessments. The physical situation can be described as an array made of high number of leaking modules inside a thin low permeable layer (e.g. clay), included between two bigger layers with higher permeability (e.g. limestone or marl). The pollutant is transported both by the convection produced by the water flowing slowly (creeping flow) through the rocks and by the diffusion coming from the dilution in the water. The leaking last the all period of time $[0, t_m]$, that is small compared to the millions of years over which convection and diffusion are active. In a real repository there is a pressure drop producing the flow crossing a large number of disposal modules where each module includes several containers. Herein, for simplicity, the repository consists of a set of modules lying on a hypersurface $\Sigma$ and we represent the leaking of a disposal module by a localized density source inside the domain or by a hole in the domain with a given flux on its boundary. Moreover, without losing of generality we assume the convection velocity field to be given. According to the test case \textsuperscript{8}, the typical size of a module is a hundred of meters for the width, a kilometer for the length and five meters for the height. The distance between two modules is also of order 100 meters and the low permeable layer (the clay layer), in which the repository is embedded, has respectively a height and a length of order 150 and 3000 meters. Since there is a large number of modules, each of them with a small size compared to the layers size, see figure 1, a direct numerical simulations of the full field, based on a \textit{microscopic} model taking in account all the detail, is unrealistic. Considering the ratio between the width of a single module $l$ and the layer length $L$, which is of order $1/30$, as a small parameter, $\varepsilon$, in the \textit{microscopic} model, then the modules, have a height of order $\varepsilon^2$, and are now immersed in a layer of thickness $\varepsilon$. The study of the renormalized model behavior, as $\varepsilon$ tends to 0, by means of the homogenization method and boundary layers, gives an asymptotic model which could be used as a repository global model for numerical simulations.

\textsuperscript{1}MCS-ISTIL, Université Lyon1, Bât. ISTIL, 43 Bd. du 11 novembre, 69622 Villeurbanne Cedex, France
\textsuperscript{2}Faculté de sciences, Université de St-Etienne, 23 Rue Dr.Paul Michelain, St-Etienne Cedex 2, France
\textsuperscript{3}Laboratoire de Mécanique et d’Acoustique, UPR 7051,31 Chemin Joseph Aignier, 13402 Marseille cedex 20, France
\textsuperscript{4}Department of Mathematics, University of Zagreb, Bijenicka 30, 10000 Zagreb, Croatia
We use methods similar to those applied to the fluid flow through a sieve in [3], [7] or [2]. Similar stationary problem with zero source term (i.e. $\Phi = 0$) was treated in [4].

2 Setting the problem

2.1 The Geometry

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $\Sigma \subset \mathbb{R}^{n-1} \times \{0\}$ be such that $\Sigma \subset \Omega$. Let $A \subset \mathbb{R}^{n-1}$ be a periodic set obtained by periodic repetition of bounded closed set $M \subset [-1/2, 1/2]^{n-1}$. More precisely

$$A = \bigcup_{\alpha \in \mathbb{Z}^{n-1}} M_\alpha,$$

where $M_\alpha = \alpha + M$. For small parameter $\varepsilon \ll 1$ and $\beta > 1$ we define $B_\varepsilon = (\varepsilon A \cap \Sigma) \times [\varepsilon^\beta, \varepsilon^\beta]$. (in situation described in the introduction $\beta = 2$). We denote by $J(\varepsilon) = \{\alpha \in \mathbb{Z}^{n-1} : \varepsilon M_\alpha \cap \Sigma \neq \emptyset\}$ and by $\Gamma_\varepsilon^\alpha = \partial(\varepsilon M_\alpha \times ]\varepsilon^\beta, \varepsilon^\beta[)$. Finally $\Omega_\varepsilon = \Omega \setminus B_\varepsilon$, $\Omega_\varepsilon^T = (\Omega \setminus B_\varepsilon) \times ]0, T[\}$, $\Omega_\varepsilon^T = \Omega \times ]0, T[\}$ and $\Gamma_\varepsilon = \partial B_\varepsilon$, $\Gamma_\varepsilon^T = \Gamma_\varepsilon \times ]0, T[\}.

2.2 The Equations

Let $\Phi \in L^\infty([0,T])$ be the function describing the time behaviour of an alveolus. As mentioned before it has a compact support $[0, t_m] \subset [0,T]$. Let $\lambda = \frac{\log 2}{\tau} > 0$, with $\tau$ being the half life of the radioactive element, and let $\varphi_0 \in H^1(\Omega_\varepsilon)$ the initial concentration of the radioactive material in the soil (typically equal to zero). The diffusion is described by $A \in L^\infty(\mathbb{R}; \mathbb{R}^{n\times n})$ a positive definite matrix function. Since layers of soil involved in our model have different properties, we assume that

$$A(y_n) = \begin{cases} A^1, & \text{for } |y_n| < h \\ A^2, & \text{for } |y_n| > h \end{cases}.$$

Now we write our diffusion matrix in the form $A_\varepsilon(x_n) = A(\frac{x_n}{\varepsilon})$. In the above described situation the low permeable layer has a height of 150 m meaning that, in that case, $h = 3/2$. We have the same situation
with the convection velocity \( v \in C([0, T]; H^1(\mathbb{R} \times \Omega)^n) \). The dependence on \( y_n \) is similar as in the case of diffusion matrix:

\[
v(x, y_n, t) = \begin{cases} v^1(x, t) & \text{for } |y_n| < h \\ v^2(x, t) & \text{for } |y_n| > h \end{cases}
\]

For simplicity, we assume that the last component \( v_n \) does not depend on \( y_n \). Next, we suppose that \( \text{div}_x v = 0 \), in order to have the divergence free convection velocity. Finally we pose \( v^\varepsilon(x, t) = v(x, x_n/\varepsilon, t) \).

At last, we define the porosity of the medium as

\[
\omega(y_n) = \begin{cases} \omega^1 & \text{for } |y_n| < h \\ \omega^2 & \text{for } |y_n| > h \end{cases}
\]

and we put

\[
\omega^\varepsilon(x_n) = \omega(x_n/\varepsilon).
\]

The process is governed by the following convection-diffusion type equation:

\[
\begin{align*}
\omega^\varepsilon \frac{\partial \varphi^\varepsilon}{\partial t} - \text{div}(A^\varepsilon \nabla \varphi^\varepsilon) + (v^\varepsilon \cdot \nabla) \varphi^\varepsilon + \lambda \omega^\varepsilon \varphi^\varepsilon &= 0 \quad \text{in } \Omega^T \\
\varphi^\varepsilon(0, x) &= \varphi_0(x) \quad x \in \Omega_e \\
\mathbf{n} \cdot (A^\varepsilon \nabla \varphi^\varepsilon - v^\varepsilon \varphi^\varepsilon) &= \Phi(t) \quad \text{on } \Gamma^T
\end{align*}
\]

We also need to impose some boundary condition on the exterior boundary \( S = \partial \Omega \). Let \( S = S_1 \cup S_2 \), where \( S_i \) are disjoint and connected parts of \( S \). We impose

\[
\begin{align*}
\varphi^\varepsilon &= 0 \quad \text{on } S_1 \\
\mathbf{n} \cdot A^\varepsilon \nabla \varphi^\varepsilon - (v^\varepsilon \cdot \mathbf{n}) \varphi^\varepsilon &= 0 \quad \text{on } S_2.
\end{align*}
\]

### 3 A priori estimates

The main result of this section is:

**Proposition 1** Let \( \varphi^\varepsilon \) be a unique solution of \((1)-(5)\). Then there exists a constant \( C > 0 \) independent of \( \varepsilon \) such that

\[
\begin{align*}
|\varphi^\varepsilon|_{L^\infty(\Omega^T_e)} &\leq C \\
|\varphi^\varepsilon|_{L^2(0, T; H^1(\Omega_e))} &\leq C
\end{align*}
\]

**Proof.** The estimate \((6)\) is the consequence of the maximum principle. To prove \((7)\) we use \( \varphi^\varepsilon \) as the test function in \((1)-(5)\). We obtain

\[
\left| \frac{\sqrt{\omega^\varepsilon}}{2} \varphi^\varepsilon(\cdot, T) \right|_{L^2(\Omega_e)}^2 + (A^\varepsilon \nabla \varphi^\varepsilon, \nabla \varphi^\varepsilon)_{L^2(\Omega^T_e)} + \lambda |\sqrt{\omega^\varepsilon} \varphi^\varepsilon|_{L^2(\Omega^T_e)}^2 =
\]

3
\[
\begin{align*}
\int_0^T \Phi \sum_{\alpha \in J(\varepsilon)} \int_{\Gamma_\alpha} \varphi_\varepsilon + \left| \frac{\sqrt{\omega}}{2} \varphi_0 \right|^2_{L^2(\Omega_\varepsilon)} & \leq \\
\leq C \left( 1 + |\varphi_\varepsilon|_{H^1(\Omega_T^\varepsilon)} \right).
\end{align*}
\]

4 Weak convergence

Our solution \( \varphi_\varepsilon \) is defined on variable domain \( \Omega_T^\varepsilon \). To use the weak convergence methods, we extend it to whole domain \( \Omega_T^\varepsilon \) preserving the estimates (6), (7). In the sequel we assume that \( \varphi_\varepsilon \) is extended using the results from \( \text{[1]} \) and we denote that extension by the same symbol. Due to the proposition \( \text{[1]} \), we can conclude that there exists some \( \varphi \in L^2(0, T; H^1(\Omega)) \cap L^\infty(\Omega_T^\varepsilon) \) such that (up to a subsequence)

\[
\begin{align*}
\varphi_\varepsilon & \rightharpoonup \varphi \quad \text{weak* in } L^\infty(\Omega_T^\varepsilon) \quad (8) \\
\nabla \varphi_\varepsilon & \rightharpoonup \nabla \varphi \quad \text{weakly in } L^2(\Omega_T^\varepsilon) \quad . \quad (9)
\end{align*}
\]

The main goal of this section is to identify the limit \( \varphi \). We prove that:

**Theorem 1** The limit function \( \varphi \) is the unique solution of the problem

\[
\begin{align*}
\omega^2 \frac{\partial \varphi}{\partial t} - \text{div} (A^2 \nabla \varphi) + (v^2 \cdot \nabla) \varphi + \lambda \omega^2 \varphi &= 0 \quad \text{in } \tilde{\Omega}^T = (\Omega \setminus \Sigma) \times [0, T] \quad (10) \\
\varphi(x, 0) &= \varphi_0(x) \quad x \in \tilde{\Omega} = \Omega \setminus \Sigma \quad (11) \\
\varphi &= 0 \quad \text{on } S_1 \quad (12) \\
n \cdot A^2 \nabla \varphi - (v^2 \cdot n) \varphi &= 0 \quad \text{on } S_2 \quad (13) \\
[\varphi] = 0 \quad , \quad [e_n \cdot A^2 \nabla \varphi - (v^2 \cdot e_n) \varphi] = 2 \Phi |M| \quad \text{on } \Sigma, \quad (14)
\end{align*}
\]

where \( [w](x') = w(x', 0+) - w(x', 0-) \), \( x' = (x_1, \ldots, x_{n-1}) \) denotes the jump over \( \Sigma \) and \( |M| \) denotes the area of \( M \).

**Proof.** Let \( \psi \in C^\infty([0, T]; C^\infty_c(\Omega)) \) be such that \( \psi(\cdot, T) = 0 \). Using \( \psi \) as the test function in (1)-(5) we get

\[
0 = -\int_{\Omega_T^\varepsilon} \omega^\varepsilon \varphi_\varepsilon \frac{\partial \psi}{\partial t} - \int_{\Omega_\varepsilon} \varphi_0 \psi(\cdot, 0) + \int_{\Omega_T^\varepsilon} A^\varepsilon \nabla \varphi_\varepsilon \nabla \psi + \\
+ \int_{\Omega_T^\varepsilon} \omega^\varepsilon \lambda \varphi_\varepsilon \psi + \int_0^T \Phi \sum_{i \in J(\varepsilon)} \int_{\Gamma_i^\varepsilon} \psi \, .
\]

Passage to the limit for the first four integrals is straightforward. For the last integral we have

\[
\int_{\Gamma_i^\varepsilon} \psi(x, t) \, d\Gamma_i^\varepsilon = (\psi(x_i^\varepsilon, t) + O(\varepsilon)) |\Gamma_i^\varepsilon| = \psi(x_i^\varepsilon, t) 2 |M| \varepsilon^{n-1} + O(\varepsilon^{n+\beta-2}) ,
\]

where \( x_i^\varepsilon = (x_i^\varepsilon)', 0 \) is an arbitrary point from \( \varepsilon M_\alpha \times \{0\} \). But then

\[
\sum_{i \in J(\varepsilon)} \int_{\Gamma_i^\varepsilon} \psi(x, t) \, d\Gamma_i^\varepsilon \to 2 |M| \int_{\Sigma} \psi(x', 0) \, dx' ,
\]

\[
\int_0^T \Phi \sum_{\alpha \in J(\varepsilon)} \int_{\Gamma_\alpha} \varphi_\varepsilon + \left| \frac{\sqrt{\omega}}{2} \varphi_0 \right|^2_{L^2(\Omega_\varepsilon)} \leq \\
\leq C \left( 1 + |\varphi_\varepsilon|_{H^1(\Omega_T^\varepsilon)} \right). \quad \square
\]
where \( x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} \). \( \square \)

**Remark 1** In fact we did not use the periodicity of distribution of alveoli. The same proof holds in case if each alveolus is randomly placed in a mesh of an \( \varepsilon \) net. The alveoli do not even need to have the same shape, only the areas of their surfaces need to be equal.

5 **Asymptotic expansion**

The above weak limit describes the global long time behaviour of the process in case when the flux \( \Phi \) is not too large. However if we need more accurate information on the behaviour in the near field (i.e. in vicinity of \( \Sigma \)), more precise asymptotics is needed.

To avoid cumbersome computations we simplify the geometry by assuming that \( \Omega = ] - L/2, L/2[^n \). We denote then

\[
\begin{align*}
\Sigma &= ] - L/2, L/2[^{n-1} \times \{0\} \\
S^+ &= ] - L/2, L/2[^{n-1} \times \{L/2\} \\
S^- &= ] - L/2, L/2[^{n-1} \times \{-L/2\}.
\end{align*}
\]

We also suppose that the alveoli are rectangular (which is true in real-world situation). More precisely, we take

\[
M = \prod_{i=1}^{n-1} [ -m_i, m_i[ , \quad 1/2 > m_i > 0.
\]

We impose the Dirichlet condition on the bottom and Neumann condition on the top of the domain:

\[
\begin{align*}
\varphi_\varepsilon &= 0 \quad \text{on } S^- \quad & (15) \\
\mathbf{n} \cdot \mathbf{A}^2 \nabla \varphi_\varepsilon - (\mathbf{v}^\varepsilon \cdot \mathbf{n}) \varphi_\varepsilon &= 0 \quad \text{on } S^+. \quad & (16)
\end{align*}
\]

On the lateral boundary we impose the periodicity condition in order to avoid the lateral boundary layer. More precisely, we impose

\[
\varphi_\varepsilon \quad \text{is } L \text{-periodic in } x' = (x_1, \ldots, x_{n-1}) \quad & (17)
\]

We also assume that \( L/\varepsilon \in \mathbb{N} \). For the sake of compatibility, we suppose that given data \( v^1, v^2 \) and \( \varphi_0 \) are \( L \)-periodic in \( x' \).

We expect some fast changes of solution in vicinity of containers. Therefore, in that region, we introduce the fast variable \( y = x/\varepsilon \) to describe that behaviour. Far from the sources we expect \( \varphi_\varepsilon \), the solution of (1)-(3), (16) and (17), to behave almost like our weak limit \( \varphi \), which, in this case, satisfies (10), (11), (14) plus the conditions

\[
\begin{align*}
\varphi &= 0 \quad \text{on } S^- \quad & (18) \\
\mathbf{n} \cdot \mathbf{A}^2 \nabla \varphi - (\mathbf{v}^2 \cdot \mathbf{n}) \varphi &= 0 \quad \text{on } S^+. \quad & (19) \\
\varphi \quad \text{is } L \text{-periodic in } x' = (x_1, \ldots, x_{n-1}). \quad & (20)
\end{align*}
\]
That suggests the use of method of matched asymptotic expansions (see e.g. [2]). We separate the domain in three parts separated by $\Sigma^+_{\varepsilon}, \Sigma^-_{\varepsilon}$:

$$
\Omega^+_{\varepsilon} = [-L/2, L/2]^{n-1}, L/2 [\frac{n-1}{\varepsilon} \log(1/\varepsilon), L/2 [\frac{n-1}{\varepsilon} \times |d\varepsilon(1/\varepsilon), L/2 [\frac{n-1}{\varepsilon} \times -d\varepsilon(1/\varepsilon), L/2 [\frac{n-1}{\varepsilon} \times -d\varepsilon(1/\varepsilon) [ ) .
$$

$$
\Omega^-_{\varepsilon} = [-L/2, L/2]^{n-1} \times -L/2, -d\varepsilon(1/\varepsilon) [ [\frac{n-1}{\varepsilon} \times -d\varepsilon(1/\varepsilon), L/2 [\frac{n-1}{\varepsilon} \times -d\varepsilon(1/\varepsilon), L/2 [\frac{n-1}{\varepsilon} \times -d\varepsilon(1/\varepsilon) [ ) .
$$

$$
G_{\varepsilon} = [-L/2, L/2]^{n-1} \times |d\varepsilon(1/\varepsilon), L/2 [\frac{n-1}{\varepsilon} \times -d\varepsilon(1/\varepsilon), L/2 [\frac{n-1}{\varepsilon} \times -d\varepsilon(1/\varepsilon) [ ) .
$$

Constant $d > 0$ is to be determined later in order to minimize the error of approximation. As suggested, in $\Omega^\pm_{\varepsilon}$, we approximate $\varphi_{\varepsilon}$ by $\varphi^0_{\varepsilon}$ that satisfies the equation (10) and boundary conditions (12), (13) as well as the initial condition (11). In $G_{\varepsilon}$ we look for the asymptotic expansion of $\varphi_{\varepsilon}$, in the form

$$
\varphi_{\varepsilon}(x,t) \approx \varphi^0_{\varepsilon}(x,t) + \varepsilon [\chi^k_{\varepsilon}(x/\varepsilon) \frac{\partial \varphi^0_{\varepsilon}(x,t)}{\partial x_k} + w_{\varepsilon}(x/\varepsilon) \Phi(t) ] +
$$

$$
+ \varepsilon^2 [\chi^{k\ell}_{\varepsilon}(x/\varepsilon) \frac{\partial^2 \varphi^0_{\varepsilon}(x,t)}{\partial x_k \partial x_\ell} + w^{ij}_{\varepsilon}(x/\varepsilon) \frac{\partial \varphi^0_{\varepsilon}(x,t)}{\partial x_i} v(x,t)_j + \Phi(t) z^k_{\varepsilon}(x/\varepsilon) v(x,t)_k + \cdots (21)
$$

Here and in the sequel we assume the summation from 1 to $n$ over the repeating index. The function $\varphi^0_{\varepsilon}$ mimics the behaviour of $\varphi$ but has two close jumps in stead of one. In fact, that suggests that more accurate approximation of the real situation would be to have two jumps of the flux; one just above and another just below the array of alveoli. However taking the weak limit smears those two jumps into one. Namely $\varphi^0_{\varepsilon}$ is defined by

$$
\omega^\varepsilon \frac{\partial \varphi^0_{\varepsilon}}{\partial t} = \text{div} \left( A^\varepsilon \nabla \varphi^0_{\varepsilon} + (v^\varepsilon \cdot \nabla) \varphi^0_{\varepsilon} + \lambda \omega^\varepsilon \varphi^0_{\varepsilon} = 0 \right) \text{ in } \tilde{\Omega}^T_{\varepsilon} = (\Omega \setminus (\Sigma^+_{\varepsilon} \cup \Sigma^-_{\varepsilon}) ) \times ]0,T[ \\
\varphi^0_{\varepsilon}(x,0) = \varphi_0(x) \quad x \in \tilde{\Omega}_{\varepsilon} = \Omega \setminus (\Sigma^+_{\varepsilon} \cup \Sigma^-_{\varepsilon}) \\
\varphi^0_{\varepsilon} = 0 \quad \text{on } S^+ \\
\n \cdot A^2 \nabla \varphi^0_{\varepsilon} - (v^2 \cdot \nabla) \varphi^0_{\varepsilon} = 0 \quad \text{on } S^- \\
[\varphi^0_{\varepsilon}] = 0 \quad , \quad [e_n \cdot A^2 \nabla \varphi^0_{\varepsilon} - (v \cdot e_n) \varphi^0_{\varepsilon}] = -\frac{1}{2} \Phi |\partial P_{\varepsilon}| \text{ on } \Sigma^\pm_{\varepsilon} ,
$$

$$
\varphi^0_{\varepsilon} \text{ is } L - \text{periodic in } x' = (x_1, \ldots, x_{n-1}) ,
$$

with

$$
\Sigma^+_{\varepsilon} = [-L/2, L/2] \times \{\pm d\varepsilon \log(1/\varepsilon) \} ,
$$

$$
[w](x') = w(x', d\varepsilon \log(1/\varepsilon) +) - w(x', d\varepsilon \log(1/\varepsilon) -) \quad x' \in \Sigma^+_{\varepsilon} \\
[w](x') = w(x', -d\varepsilon \log(1/\varepsilon) +) - w(x', -d\varepsilon \log(1/\varepsilon) -) \quad x' \in \Sigma^-_{\varepsilon}
$$

The functions $\chi^k_{\varepsilon}, \lambda_{\varepsilon}, w^{ij}_{\varepsilon}$ and $w_{\varepsilon}$ are the solutions of the auxiliary problems of the stationary diffusion type posed in an infinite strip

$$
G_{\varepsilon} = ( \{ -1/2, 1/2 \}^{n-1} \times \mathbb{R} ) \setminus P_{\varepsilon} ,
$$

with

$$
P_{\varepsilon} = M \times ] - \varepsilon^{\beta-1}, \varepsilon^{\beta-1} [ ) .
$$
First two problems read

\[-\text{div} (A \nabla \chi^k_\varepsilon) = 0 \text{ in } \mathcal{G}_\varepsilon\]
\[n \cdot A \nabla (\chi^k_\varepsilon + y_k) = 0 \text{ on } \partial P_\varepsilon\]  
(23)

\[\chi^k_\varepsilon \text{ is } 1 - \text{periodic in } y' = (y_1, \ldots, y_{n-1})\]
\[\lim_{y_n \to \infty} \nabla \chi^k_\varepsilon = 0\]

\[-\text{div} (A \nabla w_\varepsilon) = 0 \text{ in } \mathcal{G}_\varepsilon\]
\[n \cdot A \nabla w_\varepsilon = 1 \text{ on } \partial P_\varepsilon\]  
(24)

\[w_\varepsilon \text{ is } 1 - \text{periodic in } y' = (y_1, \ldots, y_{n-1})\]
\[\lim_{y_n \to \pm \infty} A \nabla w_\varepsilon (y) = \mp \frac{1}{2} |\partial P_\varepsilon| e_n\]  
(25)

Solvability of problem (23) is classical (see e.g. [5] or [6]). Due to the symmetry of the domain \(\mathcal{G}_\varepsilon\) we obviously have that
\[\chi^k_\varepsilon (y) = \chi^k_\varepsilon (-y)\]  
(26)

Furthermore, there exists a constant \(c_k(\varepsilon)\), such that
\[|\chi^k_\varepsilon - c_k(\varepsilon)|_{H^1(|y_n| > s)} \leq C e^{-\tau s}\]  
(27)

for some \(C, \tau > 0\).

Since \(\chi^k_\varepsilon\) is determined up to a constant we may assume in the sequel that \(c_k(\varepsilon) = 0\).

Remark 2 In general we should have two stabilisation constants \(c^\pm_k(\varepsilon)\) at \(\pm \infty\). Since (26) holds those two constants are equal. In case of general \(P_\varepsilon\), considered in the first part, this seems not to be the case.

The problem (24) does not admit a solution with decaying gradient (due to the source term on \(\partial P_\varepsilon\)). Therefore we have imposed (25). To see its behaviour at \(\infty\) we need to cut-off that boundary condition first. To do so we take a cut-off function
\[\zeta(y_n) = \begin{cases} 
0 & \text{for } -1/2 < y_n < 1/2 \\
1 & \text{for } |y_n| > 1 \\
\text{smooth otherwise}
\end{cases}\]

Now we take
\[\pi(y_n) = -\zeta(y_n) \left( A_{nn}^2 \right)^{-1} |y_n| \frac{1}{2} |\partial P_\varepsilon|\]

The function \(v_\varepsilon(y) = w_\varepsilon(y) - \pi(y_n)\) satisfies the problem
\[
- \text{div}(A \nabla v_\varepsilon) = (A_{nn} \pi')' \quad \text{in } G_\varepsilon \\
n \cdot A \nabla v_\varepsilon = 0 \quad \text{on } \partial P_\varepsilon \\
v_\varepsilon \text{ is } 1 \text{- periodic in } y' = (y_1, \ldots, y_{n-1}) \\
\lim_{y_n \to \infty} \nabla v_\varepsilon = 0 
\]

and it is obviously pair \( v_\varepsilon(y) = v_\varepsilon(-y) \).

Since the right-hand side is compactly supported, reasoning as in the case of (23), we conclude that such problem admits a unique (up to a constant) solution satisfying

\[
|v_\varepsilon - c(\varepsilon)|_{H^1(|y_n| > s)} \leq C e^{-\tau s},
\]

where the constant \( c(\varepsilon) \) can be chosen equal to zero. Therefore the asymptotic behaviour of \( w_\varepsilon \), for large \(|y_n|\) is

\[
w_\varepsilon(y) \approx -(A_{nn}^2)^{-1}|y_n| \frac{1}{2} |\partial P_\varepsilon| + \text{exponentially decaying part}.
\]

The auxiliary problems for the second corrector are as follows

\[
- \text{div}(A \nabla \chi_{\ell m}^\varepsilon) = A_{\ell k} \frac{\partial \chi_{\ell m}^\varepsilon}{\partial y_k} + \frac{\partial}{\partial y_k}(A_{k\ell} \chi_{\ell m}^\varepsilon) \quad \text{in } G_\varepsilon \\
n \cdot A \nabla \chi_{\ell m}^\varepsilon = 0 \quad \text{on } \partial P_\varepsilon \\
\chi_{\ell m}^\varepsilon \text{ is } 1 \text{- periodic in } y' = (y_1, \ldots, y_{n-1}) \\
\lim_{y_n \to \infty} \nabla \chi_{\ell m}^\varepsilon = 0. 
\]

\[
- \text{div}(A \nabla w_{ij}^\varepsilon) = \frac{\partial \chi_{ij}^\varepsilon}{\partial y_i} \quad \text{in } G_\varepsilon \\
n \cdot A \nabla w_{ij}^\varepsilon = 0 \quad \text{on } \partial P_\varepsilon \\
w_{ij}^\varepsilon \text{ is } 1 \text{- periodic in } y' = (y_1, \ldots, y_{n-1}) \\
\lim_{y_n \to \infty} \nabla w_{ij}^\varepsilon = 0. 
\]

\[
- \text{div}(A \nabla z_k^\varepsilon) = -\frac{\partial w_\varepsilon}{\partial y_k} \quad \text{in } G_\varepsilon \\
n \cdot A \nabla z_k^\varepsilon = 0 \quad \text{on } \partial P_\varepsilon \\
z_k^\varepsilon \text{ is } 1 \text{- periodic in } y' = (y_1, \ldots, y_{n-1}) \\
\lim_{y_n \to \infty} \nabla z_k^\varepsilon = 0, \ k \neq n \\
\lim_{y_n \to \infty} (A \nabla z^n + \frac{1}{2}(A_{nn}^2)^{-1} |\partial P_\varepsilon| |y_n|) = 0.
\]
As before we conclude that \( \chi^{(m)}_\varepsilon, w^{ij}_\varepsilon \) and \( z^k_\varepsilon \), \( k \neq n \) can be chosen to decay exponentially towards zero as \( y_n \to \pm \infty \), while 

\[
z^n_\varepsilon(y) \approx -\frac{1}{4} |\partial P_\varepsilon| (A^2_{nn})^{-\frac{3}{2}} |y_n| y_n + \text{exponentially decaying part}.
\]

Now we still have the term \( \varepsilon w(x/\varepsilon) \Phi \) in the inner approximation that hasn’t been matched by our exterior approximation. Taking into account (30) we need to patch our outer approximation with a term of the following form 

\[
d \varepsilon \log(1/\varepsilon) \varphi^1_\varepsilon.
\]

At the same time we will correct the flux jump created by \( z^n_\varepsilon \). To do that we define the second corrector \( \varphi^1_\varepsilon \) by 

\[
\omega^\varepsilon \frac{\partial \varphi^1_\varepsilon}{\partial t} - \text{div}(A^2 \nabla \varphi^1_\varepsilon) + (v^\varepsilon \cdot \nabla) \varphi^1_\varepsilon + \lambda \omega^\varepsilon \varphi^1_\varepsilon = 0 \text{ in } \Omega^T_\varepsilon
\]

\[
\varphi^1_\varepsilon(x,0) = 0 \text{ for } x \in \tilde{\Omega}_\varepsilon
\]

\[
\varphi^1_\varepsilon = 0 \text{ on } S^+
\]

\[
n \cdot (2 \nabla \varphi^1_\varepsilon - (v^2 \cdot n) \varphi^1_\varepsilon = 0 \text{ on } S^-
\]

\[
[\varphi^1_\varepsilon] = \frac{1}{2} \Phi(A^2_{nn})^{-1} |\partial P_\varepsilon|, \text{ on } \Sigma^\pm
\]

\[
\varphi^1_\varepsilon \text{ is } L - \text{periodic in } x' = (x_1, \ldots, x_{n-1})
\]

For expansion (21) we can prove the following error estimate

**Theorem 2** Let \( m = \frac{11}{2} \) for \( n = 3 \) and \( m < 2 \) for \( n = 2 \). Let \( d \geq 2 \). There exists a constant \( C > 0 \) independent on \( \varepsilon \), such that 

\[
|\varphi_\varepsilon - F_\varepsilon|_{L^2(0,T; H^1(B_\varepsilon))} \leq C (\varepsilon \log(1/\varepsilon))^m.
\]

(34)

where 

\[
F_\varepsilon(x,t) = \begin{cases}
\varphi^0_\varepsilon(x,t) + d \varepsilon \log(1/\varepsilon) \varphi^1_\varepsilon(x,t), \text{ in } \Omega^\pm_\varepsilon
\\
\varphi^0_\varepsilon(x,t) + d \varepsilon \log(1/\varepsilon) \varphi^1_\varepsilon(x,t) + \varepsilon \chi_\varepsilon(x/t) \frac{\partial}{\partial x_k} (\varphi^0_\varepsilon + d \log(1/\varepsilon) \varepsilon \varphi^1_\varepsilon)(x,t) + w_\varepsilon(x/t) \Phi(t) + \\
\varepsilon^2 [\chi_\varepsilon(x/t) \frac{\partial^2 \varphi^0_\varepsilon}{\partial x_k \partial x_j} + w^{ij}_\varepsilon(x/t) \frac{\partial \varphi^0_\varepsilon}{\partial x_i} v(x,t)_j + \Phi(t) z^k_\varepsilon(x/t) v(x,t)_k], \text{ in } G_\varepsilon
\end{cases}
\]

with \( \Sigma^\pm_\varepsilon = [L/2, L/2]^{n-1} \times \{ \pm \varepsilon \log(1/\varepsilon) \}. \) Furthermore the same estimate holds in \( L^\infty(0,T; L^2(\Omega_\varepsilon)) \) norm.

**Proof.** We divide the domain in three parts, \( \Omega^+_\varepsilon, \Omega^-_\varepsilon, G_\varepsilon \).

Let \( R_\varepsilon = \varphi_\varepsilon - F_\varepsilon \). In \( \Omega^\pm_\varepsilon \) we have 

\[
\omega^2 \frac{\partial R_\varepsilon}{\partial t} - \text{div}(A^2 \nabla R_\varepsilon) + (v^2 \cdot \nabla) R_\varepsilon + \lambda \omega^2 R_\varepsilon = 0.
\]
In $G_\varepsilon$ the function $R_\varepsilon$ satisfies
\[
\omega \frac{\partial R_\varepsilon}{\partial t} - \text{div}(A_\varepsilon \nabla R_\varepsilon) + (\mathbf{v}_\varepsilon \cdot \nabla)R_\varepsilon + \lambda \omega R_\varepsilon = E_\varepsilon ,
\]
with $|E_\varepsilon|_{L^\infty(G_\varepsilon)} \leq C \varepsilon \log(1/\varepsilon)$ . Furthermore, on $\Sigma^\pm_\varepsilon$ we have jumps
\[
[R^\varepsilon] = O(\varepsilon^2 \log^2(1/\varepsilon)) , \quad [(A_\varepsilon^2 \nabla R_\varepsilon - R_\varepsilon \mathbf{v}_\varepsilon^2) \cdot \mathbf{n}] = O(\varepsilon^d) .
\]
Now the result follows by the standard a priori estimate.  

It should be noticed that the terms in our expansion $F_\varepsilon$ still depend on $\varepsilon$ implicitly. However it is clear that:

**Lemma 1**

\[
|\varphi_\varepsilon^0 - \varphi|_{L^2(0,T;H^1(\Omega))} \leq C \sqrt{d \varepsilon \log(1/\varepsilon)} \\
|\varphi_\varepsilon^0|_{L^2(0,T;H^1(\Omega^\varepsilon))} \leq C \sqrt{d \varepsilon \log(1/\varepsilon)} .
\]

The proof is straightforward and follows by deducing two problems and estimating the remainder.

We have the following consequences of theorem 2:

**Corollary 1** Let

\[
H_\varepsilon(x) = \begin{cases} 
\varphi_\varepsilon^0(x) , & \text{in } \Omega^\varepsilon \\
\varphi_\varepsilon^0(x) + \varepsilon [\chi^k(x/\varepsilon) \frac{\partial \varphi_\varepsilon^0}{\partial x_k}(x) + w_\varepsilon(x/\varepsilon) \Phi(t)] , & \text{in } G_\varepsilon.
\end{cases}
\]

Then
\[
|\varphi_\varepsilon - H_\varepsilon|_{L^2(0,T;H^1(\Gamma_\varepsilon))} \leq C(\varepsilon \log(1/\varepsilon))^{3/2} .
\]

Furthermore
\[
|\varphi_\varepsilon - \varphi|_{L^2(0,T;H^1(\Gamma_\varepsilon))} \leq C \sqrt{\varepsilon \log(1/\varepsilon)} \\
|\varphi_\varepsilon - \varphi_\varepsilon^0|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} \leq C (\varepsilon \log(1/\varepsilon))^{3/2} .
\]

For the auxiliary problems we have:

**Lemma 2**

\[
|\nabla(\chi_\varepsilon^k - \chi^k)|_{L^2(G_\varepsilon)} \leq C_\varepsilon^{(\beta-1)/2} \\
|\nabla(\chi_\varepsilon^{mk} - \chi^{mk})|_{L^2(G_\varepsilon)} \leq C_\varepsilon^{(\beta-1)/2} \\
|\nabla(z_\varepsilon^k - z^k)|_{L^2(G_\varepsilon)} \leq C_\varepsilon^{(\beta-1)/2} \\
|\nabla(w_\varepsilon - w)|_{L^2(G_\varepsilon)} \leq C_\varepsilon^{(\beta-1)/2} \\
|\nabla(w_\varepsilon^{mk} - w^{mk})|_{L^2(G_\varepsilon)} \leq C_\varepsilon^{(\beta-1)/2} ,
\]

10
where $\chi^k, \chi^{km}, w, z^k, w^{km}$ are the solutions of corresponding auxiliary problems posed on

$$\mathcal{G} = [-1/2, 1/2] \times \mathbb{R} \setminus (M \times \{0\}) .$$

![Figure 2: Strip $\mathcal{G}$](image)

**Remark 3** It should be noticed that the domain $\mathcal{G}$ is not locally placed on one side of its boundary. Furthermore, all the problems (23) defining $\chi^k$ for $k \neq n$ have only trivial solutions. On the contrary $\chi^n$ admits a nontrivial solution and the corresponding auxiliary problem now reads

\begin{align*}
- \text{div} (A \nabla \chi^n) &= 0 \text{ in } \mathcal{G} \\
A_{nk} \frac{\partial (\chi^n + y_n)}{\partial y_k} &= 0 \text{ on } M \times \{\pm 0\} \tag{35}
\end{align*}

$\chi^n$ is 1-periodic in $y' = (y_1, \ldots, y_{n-1})$

$$\lim_{y_n \to \infty} \nabla \chi^n = 0 .$$

\begin{align*}
- \text{div} (A \nabla w) &= 0 \text{ in } \mathcal{G} \\
 \mp A_{nk} \frac{\partial w}{\partial y_k} &= 1 \text{ on } M \times \{\pm 0\} \tag{36}
\end{align*}

$w$ is 1-periodic in $y' = (y_1, \ldots, y_{n-1})$

$$\lim_{y_n \to \pm \infty} A \nabla w = \mp |M| e_n .$$

### 6 Conclusion

The expansion (21) clearly points out two important terms in the asymptotic behaviour of $\varphi_\varepsilon$, zero order term $\varphi^0_\varepsilon$ and first order term $\varepsilon w_\varepsilon(x/\varepsilon) \Phi$. In a real-life situation, that we are trying to model, the containers are leaking intensively for very short time. During that time $\Phi$ is large and the second order term $\varepsilon w_\varepsilon(x/\varepsilon) \Phi$ dominates the behaviour of the solution $\varphi_\varepsilon$ (despite of $\varepsilon$ multiplying it). Indeed, the typical diffusion coefficient in a low permeable layer (clay) is small, compared to the one in the rest of the domain (limestone). Thus, at the beginning of the process, diffusion arround sources is slow. After a short period of time $\Phi$ vanishes and it is only then that the diffusion becomes dominant, i.e. $\varphi^0_\varepsilon$ becomes the most important term.
References

[1] Acerbi E., Chiado Piat V., Dal Masso G., Percivale D., An extension theorem from connected sets, and homogenization in general periodic domains. Nonlinear Anal. 18 (1992), no. 5, 481-496.

[2] Bourgeat A., Gipouloux O., Marušić-Paloka E., Mathematical modelling and numerical simulation of non-Newtonian flow through a thin filter, to appear in SIAM J.Appl.Math.

[3] Conca C., Etude d’un fluide traversant une paroi perforée, I, II, J.Math.Pures et Appl., 66 (1987), 1-69.

[4] Del Vecchio T., The thick Neumann’s sieve, Ann.Math.Pure.Appl. (4) 147 (1987), 363-402.

[5] Lions J.L., Some methods in mathematical analysis of systems and their control, Science Press Beijing and Gordon and Breach, New York, 1981.

[6] Oleinik O.A., Iosif’jan, G.A., On the behavior at infinity of solutions of second order elliptic equations in domains with noncompact boundary, Math.USSR Sbornik, Vol 40, No 4 (1981), 527-548.

[7] Sanchez-Palencia, E., Boundary value problems in domains containing perforated walls, Séminaire Collège de France, Research Notes in Mathematics No 70, Pitman, London.

[8] http://andra.fr/Couplex