Research Article
Special Issue: Geometric PDEs and Applications

Neil S. Trudinger*

A note on second derivative estimates for Monge-Ampère-type equations

https://doi.org/10.1515/ans-2022-0036
received July 29, 2022; accepted November 9, 2022

Abstract: In this article, we revisit previous Pogorelov-type interior and global second derivative estimates of N. S. Trudinger, F. Jiang, and J. Liu for solutions of Monge-Ampère-type partial differential equations. Taking account of recent strict convexity regularity results of Guillen-Kitagawa and Rankin and following our earlier work in the optimal transportation case, we remove the monotonicity assumptions in the more general case of generated Jacobian equations and consequently in the subsequent application to classical solvability and global regularity for second boundary value problems.

Keywords: Monge-Ampère-type equations, second derivative estimates, generated Jacobian equations, existence, regularity

MSC 2020: 35J96, 90B06, 78A05

1 Introduction

In this article, we are concerned with Pogorelov-type interior and global second derivative estimates of elliptic solutions of nonlinear partial differential equations of Monge-Ampère-type (MATEs), which amplify and improve earlier results in [2,7] and [18]. Such equations can be written in the general form:

\[
\det [D^2u - A(\cdot, u, Du)] = B(\cdot, u, Du),
\]

where \( A \) and \( B \) are, respectively, \( n \times n \) symmetric matrix valued and scalar functions on a domain \( \mathcal{U} \subset \mathbb{R}^n \times \mathbb{R}^n \) and \( Du \) and \( D^2u \) denote, respectively, the gradient and Hessian matrix of the scalar function \( u \in C^2(\Omega) \), with one jet \( j_1[u](\Omega) \subset \mathcal{U} \), where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \). A solution \( u \in C^2(\Omega) \) of (1.1) is called elliptic (degenerate elliptic), whenever \( w = D^2u - A(\cdot, u, Du) > 0, (\geq 0) \), which implies \( B > 0, (\geq 0) \). We assume throughout that \( A \) and \( B \) are \( C^2 \) smooth, with \( B > 0 \), and that the matrix function \( A \) is regular with respect to the gradient variables, that is, denoting points in \( \mathcal{U} \) by \((x, z, p)\),

\[
A^i_j \xi^i \xi^j \eta^i \eta^j = \left(D_{p_l p_m} A_{ij}\right) \xi^i \xi^j \eta^i \eta^j \geq 0
\]

in \( \mathcal{U} \), for all \( \xi, \eta \in \mathbb{R}^n \) such that \( \xi \cdot \eta = 0 \).

For convenience, we may also assume throughout that \( \mathcal{U} \) is convex in \( p \) for fixed \( x \) and \( u \). Condition (1.2), which originated in its strict form A3 for regularity in the special case of optimal transportation in [12], corresponds to the weak form A3w, introduced for global regularity in [15,19], and was shown to be sharp in [9]. When the matrix function \( A \) is not assumed twice differentiable, we may more generally express its regularity as convexity of the form \( A\zeta \). \( \zeta \) along straight line segments in \( p \), orthogonal to \( \xi \), which also
suffices for the associated optimal transportation convexity theory and the more general convexity theory of generating functions [10,11].

Following our note on optimal transportation regularity [16], we can then remove the monotonicity conditions in our second derivative estimates for classical solutions of generated Jacobian equations in [2], under an appropriate local strict convexity control, thereby providing a corresponding extension of the classical existence result in Theorem 1.1 in [3]. Paralleling [16], the crucial elements in our approach are an extension of the Pogorelov-type estimate in [7], using Lemma 3.3 in [18], and the strict convexity result in [1]. These results have been flagged in Section 3 of [18] and can also be further improved using recent work by Rankin [14].

We will treat the Pogorelov estimates in Section 2, followed by their application to second derivative bounds for solutions of the second boundary value problem for generated Jacobian equations in Section 3. Finally, in Section 4 we consider the application to the existence of globally smooth classical solutions, thereby removing the monotonicity conditions on the matrix function $A$ in [3,14]. We also indicate a different argument for global regularity to that in [14], which does not proceed through local regularity.

## 2 Pogorelov estimates

We begin with an interior Pogorelov estimate which combines those in [2,7,18]. In its formulation we use the linearized operator $L$ defined by

$$
L = L[u] = w^{ii} [D_{ii} - D_{pi} A_{ii} (\cdot, u) D_k],
$$

where $[w^{ii}]$ denotes the inverse of $w = [w_{ij}]$.

**Theorem 2.1.** Let $u \in C^4(\Omega) \cap C^{0,1}(\tilde{\Omega})$ be an elliptic solution of (1.1) in $\Omega$, and let $u_0 \in C^2(\Omega) \cap C^{0,1}(\tilde{\Omega})$ be a degenerate elliptic supersolution such that $1_f[u], 1_f[u_0] \subset \mathcal{U}_0 \subset \subset \mathcal{U}$ and $u = u_0$ on $\partial \Omega$, $u < u_0$ in $\Omega$. Suppose there exists a barrier function $\phi \geq 0$, $\in C^2(\Omega)$ satisfying

$$
L \phi \geq w^{ii} - C',
$$

in $\Omega$, for some constant $C' \geq 0$. Then there exist positive constants $\beta$, $\delta_1$, $\delta_2$, and $C$ depending on $n, \mathcal{U}_0, |A|_{L^1(\mathcal{U}_0)}, |D \log B|_{L^1(\mathcal{U}_0)}$, $C'$, and $|\phi|_{L^1(\Omega)}$ such that, if either (i) $D_u A > -\delta I$ in $\mathcal{U}_0$ or (ii) $\text{diam } \Omega < \delta_2$, then

$$
\sup_{\Omega} (u_0 - u)^\beta |D^2 u| \leq C.
$$

**Proof.** First we note that Case (ii) is proved in [18], Lemma 3.3 and the barrier hypothesis (2.2) is automatically satisfied with $C' = 0$, for suitable $\delta_2$, by taking $\phi(x) = \text{const.} |x - x_0|^2$ for some point $x_0 \in \Omega$. Accordingly, we will concentrate on Case (i). Adapting the proofs of Lemma 3.3 in [18], Theorem 1.2 in [2], and Theorem 2.1 in [7], we first consider an auxiliary function,

$$
v = v(\cdot, \xi) = \log(w_{ij}^{ii} x_i x_j) + \tau |D_u|^2 + e^{\phi} + \beta \log(u_0 - u),
$$

where $|\xi| = 1$ and $\tau, \kappa, \beta$ are positive constants to be chosen. Then, using inequality (3.8) in [18] combined with condition (i), we obtain in place of inequality (3.9) in [18], at a maximum point $x_0$ and vector $\xi = e_1$ of $v$ in $\Omega$,

$$
Lv \geq \tau w_{ij} + \frac{\kappa}{2} (1 + \kappa |D\phi|^2) e^{\phi} w^{ii} - C(\tau + \kappa e^{\phi}) + \frac{1}{2w_{ii}^{ii}} \sum_{i, l, \eta} w^{ii} (D_i w_{ij})^2 - \frac{C \beta}{\eta^2} + \frac{C \beta}{\eta^2} |D| |\eta| (|\eta| + |D| |\eta|),
$$

provided $\tau \geq C$ and $\kappa \geq C(\tau + \beta \delta_1)$, where $C$ is a constant depending on the same quantities as in the estimate (2.3) and $L = L - D_p \log B(\cdot, u) D_k, \eta = u_0 - u$. 


From the condition $Dv(x_0) = 0$, we then have, for each $i = 1, \ldots, n$,
\[
\beta \frac{|D_i v|}{\eta} \leq \frac{|D_i w_1|}{w_1} + C\tau (w_1 + 1) + xe^{\eta \phi} |D\phi|
\]  
so that for each $i > 1$,
\[
\beta \left( \frac{D_i \eta}{\eta} \right)^2 \leq \frac{2}{\beta w_1^2} (D_i w_1)^2 + \frac{C^2}{\beta} (w_1^2 + 1) + \frac{3\kappa^2 e^{2\eta \phi}}{\beta} (D\phi)^2.
\]  
Assuming $\eta w_1(x_0) \geq \beta$ and combining (2.5), (2.6), (2.7), we then obtain $Lv(x_0) > 0$, by choosing $\tau \geq C$, $\kappa \geq C\tau$, $\beta \geq C(\tau^2 + e^{\kappa \max \phi})$, and $\delta_i \leq \frac{1}{\beta}$, for sufficiently large constant $C$ depending on $n, U, \mathcal{U}_0, |A|_{\mathcal{U}_0}$, $|D \log B|_{\mathcal{U}_0}$, and $\phi$ and hence conclude the estimate (2.3), from the corresponding estimate for $w$. \hfill \Box

For generated Jacobian equations, in [2], we construct barriers satisfying (2.2) when the matrix function $A$ is either non-decreasing or non-increasing with respect to $u$ (see also [4]). This then includes the special case of optimal transportation equations in [7], when $A$ is independent of $u$. We also remark that even though we have essentially used the same auxiliary functions as in the proofs of the Pogorelov estimates, Theorem 1.1 in [7] and Theorem 1.2 in [2], the proof details are somewhat different and our proof here can be seen as clarifying the approaches in those papers. Recently, in the proof of Theorem 1.1 in [5], we have also adapted the exponentiation of the barrier $\phi$ in this proof to provide a correction to the proof of second derivative bounds for the Neumann problem in [6]. Moreover, we may also, by adapting the extension to the degenerate case in Theorem 1.2 in [5], express the dependence on $B$ in Theorem 2.1 in terms of $\sup \mathcal{U}_0 B$ and the constants, $C_1, C_2$, and $C_3$, defined respectively, in equations (1.11), (1.12), and (1.13) in [5].

From Case (ii) in Theorem 2.1 or Lemma 3.3 in [18], we can now infer interior and global second derivative bounds elliptic solutions of generated Jacobian equations satisfying appropriate local strict convexity conditions with only condition (1.2) assumed on the function $A$. Let us recall from [2, 17] that equation (1.1) is a generated Jacobian equation if $A$ is determined by a generating function $g \in C^2(\Gamma)$ for some domain $\Gamma \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, whose projections
\[
I(x, y) = \{ z \in \mathbb{R} | (x, y, z) \in \Gamma \}
\]
are open intervals. Denoting points in $\Gamma$ by $(x, y, z)$, and
\[
\mathcal{U} = \{(x, g(x, y, z), g_z(x, y, z)) | (x, y, z) \in \Gamma \},
\]
this means that $g_z < 0$ in $\Gamma$ and there exist unique $C^1$ mappings $Y, Z: \mathcal{U} \to \mathbb{R}^n, \mathbb{R}$ satisfying
\[
g(x, Y, Z) = u, \quad g_z(x, Y, Z) = p,
\]  
with the matrix $A$ given by
\[
A(\cdot, u, p) = g_{xx}(\cdot, Y(\cdot, u, p)), Z(\cdot, u, p)).
\]  
Now suppose $u \in C^2(\Omega)$, $I^i[u](\Omega) \subset \mathcal{U}$ and $u$ is elliptic in $\Omega$. Then $u$ is locally strictly $g$-convex in $\Omega$, in the sense that for each $x_0 \in \Omega$, there exists a $g$-affine function $g_0 = g(\cdot, y_0, z_0)$, with $y_0 = Y(\cdot, u, Du)(x_0)$, $z_0 = Z(\cdot, u, Du)(x_0)$, such that $g(x_0) = u(x_0)$ and $g_0 < u$ in $N_0 - \{x_0\}$, for some neighbourhood $N_0$ of $x_0$. For $h > 0$, we can define the section
\[
S_h(u, g_0) = \{ u \in \Omega | (x, y_0, z_0 - h) \in \Gamma, \ u(x) < g_0(x, y_0, z_0 - h) \},
\]
and let $S_h(u, g_0, x_0)$ denote the component of $S_h(u, g_0)$ containing $x_0$. For $0 < R < \text{dist}(x_0, \partial \Omega)$, we can then define a modulus of strict $g$-convexity of $u$ at $x_0$ by
\[
\omega(R) = \omega[u](x_0, R) = \sup \{ h | S_h(u, g_0, x_0) \subset B_h(x_0) \}.
\]
From Lemma 3.3 in [18], or Theorem 2.1, we then have the following interior second derivative estimate for elliptic solutions of generated Jacobian equations. In its formulation we use the quantity
\[
\omega[u](\Omega, \Omega) = \inf_{x_0 \in \Omega, R < R_0} \omega[u](x_0, R)
\]  
for elliptic solutions of generated Jacobian equations. In its formulation we use the quantity
\[
\omega[u](\Omega, \Omega) = \inf_{x_0 \in \Omega, R < R_0} \omega[u](x_0, R)
\]  
for elliptic solutions of generated Jacobian equations. In its formulation we use the quantity
\[
\omega[u](\Omega, \Omega) = \inf_{x_0 \in \Omega, R < R_0} \omega[u](x_0, R)
\]  
for elliptic solutions of generated Jacobian equations. In its formulation we use the quantity
\[
\omega[u](\Omega, \Omega) = \inf_{x_0 \in \Omega, R < R_0} \omega[u](x_0, R)
\]  
for elliptic solutions of generated Jacobian equations. In its formulation we use the quantity
\[
\omega[u](\Omega, \Omega) = \inf_{x_0 \in \Omega, R < R_0} \omega[u](x_0, R)
\]  
for elliptic solutions of generated Jacobian equations. In its formulation we use the quantity
\[
\omega[u](\Omega, \Omega) = \inf_{x_0 \in \Omega, R < R_0} \omega[u](x_0, R)
\]  
for elliptic solutions of generated Jacobian equations. In its formulation we use the quantity
\[
\omega[u](\Omega, \Omega) = \inf_{x_0 \in \Omega, R < R_0} \omega[u](x_0, R)
\]  
for elliptic solutions of generated Jacobian equations. In its formulation we use the quantity
\[
\omega[u](\Omega, \Omega) = \inf_{x_0 \in \Omega, R < R_0} \omega[u](x_0, R)
\]  
for elliptic solutions of generated Jacobian equations. In its formulation we use the quantity
\[
\omega[u](\Omega, \Omega) = \inf_{x_0 \in \Omega, R < R_0} \omega[u](x_0, R)
\]  
for elliptic solutions of generated Jacobian equations. In its formulation we use the quantity
\[
\omega[u](\Omega, \Omega) = \inf_{x_0 \in \Omega, R < R_0} \omega[u](x_0, R)
to denote a lower bound on the modulus of strict $g$-convexity of $u$ over a subdomain $\Omega' \subset \Omega$, where $R_0 = \text{dist}(\Omega, \partial \Omega)$.

**Theorem 2.2.** Let $u \in C^4(\Omega) \cap C^0(\tilde{\Omega})$ be an elliptic solution of a generated Jacobian equation (1.1) in the domain $\Omega$, with $J[u] \subset U_0 \subset U$. Then, for any strictly contained subdomain $\Omega'$, we have the estimate

$$\sup_{\Omega'} |D^2u| \leq C,$$

(2.11)

where $C$ depends on $n, U, U_0, g, B, R_0$, and $\omega[u](\Omega', \Omega)$.

If some neighbourhood, $N'$, of $\partial \Omega$ is $A$-bounded, we can then infer a global second derivative bound from Theorem 2.2, using the global estimate, Theorem 3.1, in [19]. Here we recall the definition from [7,15] that a domain $\Omega$ is $A$-bounded with respect to $u$ if there exists a barrier function $\phi \in C^2(\tilde{\Omega})$ satisfying

$$[D_\phi \phi - D_\xi A_\xi(\cdot, u, Du), D_\phi \xi \xi] \geq |\xi|^2,$$

(2.12)

in $\Omega$, for all $\xi \in \mathbb{R}^n$. Clearly, if $\Omega$ is $A$-bounded with respect to $u$, then the barrier condition (2.2) is satisfied with $C' = 0$.

**Corollary 2.1.** Let $u \in C^4(\Omega) \cap C^2(\tilde{\Omega})$ be an elliptic solution of a generated Jacobian equation (1.1) in the domain $\Omega$, $J[u] \subset U_0 \subset U$, and suppose $N' = \Omega - \tilde{\Omega}$ is $A$-bounded, with respect to $u$, for some subdomain $\Omega' \subset \subset \Omega$, with barrier $\phi$. Then we have the estimate,

$$\sup_{\Omega} |D^2u| \leq C \left( 1 + \sup_{\partial \Omega} |D^2u| \right),$$

(2.13)

where $C$ depends on $n, U, U_0, g, B, R_0, \omega[u](\Omega', \Omega)$, and $\phi$.

More generally from the proof of Theorem 3.1, in [19], we need to only assume that the barrier estimate (2.2) holds for the domain $N'$, and if $N' = \Omega$ the result reduces to the full global estimate, as used in [2].

We will apply Corollary 2.1 in the next section to obtain global second derivative estimates for solutions of the second boundary value problem for generated Jacobian equations, complementing those in [2]. Here we just note that an appropriate condition for $A$-boundedness of some $N'$ is the uniform $A$-convexity of $\Omega$ with respect to $u$, as defined in [2,15], which is also a critical condition for estimating $D^2u$ on $\partial \Omega$ in [2].

### 3 Second boundary value problem

First, we recall that a generated Jacobian equation is a special case of a prescribed Jacobian equation,

$$\det D_Y(\cdot, u, Du) = \psi(\cdot, u, Du),$$

(3.1)

where $\psi$ is a given scalar function on $U$. In this case, the scalar function $B$ in (1.1) is given by

$$B(\cdot, u, p) = \det E(\cdot, Y(\cdot, u, p), Z(\cdot, u, p))\psi(\cdot, u, p),$$

(3.2)

where the matrix function $E$, given by

$$E = [E_{ij}] = g_{ij} - (g_p)^{-1}g_{ij} \otimes g_p,$$

satisfies $\det E \neq 0$.

The second (or natural) boundary value problem for prescribed Jacobian equations is to prescribe the image

$$Tu(\Omega) = Y(\cdot, u, Du)(\Omega) = \Omega^*,$$

(3.3)

where $\Omega^* \subset \mathbb{R}^n$ is a target domain.
So far our assumptions on the generating function \( g \) correspond to conditions A1, A2, and A3\( w \) in [2]. To prove second derivative estimates for solutions of (1.1) and (3.3), we also assume the dual condition A1\(*\), namely that the mapping \( Q = -g_*/g \) is one-to-one in \( x \), for all \((x, y, z) \in \Gamma\). We also recall the notion of dual generating function \( g^* \), defined on the dual set
\[
\Gamma^* = \{(x, y, g(x, y, z)) | (x, y, z) \in \Gamma\}
\]
by
\[
g(x, y, g^*(x, y, u)) = u.
\]

As in [3], it will be convenient to formulate our second derivative bounds using domain convexity assumptions expressed in terms of the mapping \( Y \). Namely, for an open interval \( J \), satisfying \( \Omega \times \Omega^* \times J \subset \subset \Gamma^* \), we repeat the following definitions from [3].

The \( C^2 \) domain \( \Omega \) is \( Y \)-convex (uniformly \( Y \)-convex) with respect to \( \Omega^* \times J \) if it is connected and
\[
\left[D_\gamma J(x) - D_\nu A_\gamma(x, u_0, p) \right] \eta \geq 0, \quad (\delta_0),
\]
for all \( x \in \partial \Omega, \ u_0 \in J, \ Y(x, u_0, p) \in \Omega^* \), unit outer normal \( \gamma \), and unit tangent vector \( \tau \), (for some constant \( \delta_0 > 0 \)).

The domain \( \Omega^* \) is \( Y^* \)-convex (uniformly \( Y^* \)-convex) with respect to \( \Omega \times J \) if the images
\[
P(x, u_0, \Omega^*) = \{ p \in \mathbb{R}^n | (x, u_0, p) \in \Omega, \ Y(x, u_0, p) \in \Omega^* \}
\]
are convex for all \((x, u_0) \in \Omega \times J \) (uniformly convex for all \( x \in \overline{\Omega}, \ u \in J \)).

As remarked in [3], \( Y^* \)-convexity is equivalent to the notion of \( g^* \)-convexity, introduced in [17], while \( Y \)-convexity is implied by \( g \)-convexity with respect to \( y \in \Omega^* \) and \( z = g^*(x, y, u_0) \) for all \( x \in \Omega \) and \( u_0 \in J \). Moreover, under this stronger condition, elliptic solutions \( u \) of the boundary value problem (1.1), (3.3), satisfying \( u(\Omega) \subset \subset J \), will be globally strictly \( g \)-convex in \( \Omega \) [18].

By combining Corollary 2.2 with Lemma 3.2 in [2] and Theorem 2.3 in [1], or more specifically Theorem 1 in [14], we now obtain the following complimentary estimate to Theorem 3.1 in [2]. In its formulation and applications, it will be convenient to fix an open interval \( J \), satisfying \( \Omega \times \Omega^* \times J \subset \subset \Gamma^* \).

**Theorem 3.1.** Let \( u \in C^4(\Omega) \), with \( J_0[u] \subset \mathcal{U}_0 \subset \subset \mathcal{U}_0 \), \( u(\Omega) \subset \subset J_0 \), for some open interval \( J_0 \subset J \), be an elliptic solution of the second boundary value problem (1.1), (3.3) in \( \Omega \), where \( A \in C^2(\mathcal{U}) \) is given by (2.9) with generating function \( g \in C^2(\Gamma) \), satisfying conditions A1, A2, A1\(*\), and A3\( w \), \( B > 0 \), \( \epsilon C^2(\mathcal{U}) \) and the domains \( \Omega, \Omega^* \subset C^2(\Omega) \) are, respectively, uniformly \( Y \)-convex and uniformly \( Y^* \)-convex with respect to \( \Omega^* \times J_0 \) and \( \Omega \times J_0 \).

Suppose additionally that (i) \( \Omega \subset \subset \partial \Omega^* \) for a domain \( \Omega^* \), also satisfying \( \Omega^* \times J_0 \subset \subset \Gamma^* \), which is \( g \)-convex with respect to \( y \in \Omega^* \) and \( z = g^*(x, y, u_0) \) for all \( x \in \Omega, \ u_0 \in J_0 \), and (ii) \( Tu \) is one-to-one and \( u \) is \( g \)-convex in \( \Omega \), with any \( g \)-support, \( g_0 \), satisfying \( g_0(\Omega) \subset \subset J \). Then we have the estimate,
\[
\sup_{\Omega} |D^2 u| \leq C,
\]
where the constant \( C \) depends on \( n, \mathcal{U}, \mathcal{U}_0, g, B, \Omega, \Omega^*, \Omega_0, J_0, \) and \( J \).

**Proof.** Since the proof is a straightforward extension of that of optimal transportation case in Theorem 2.1 in [16], we just describe it briefly here. From Lemma 3.2 in [2] and Theorem 2.2, it is enough to prove an estimate from below for the modulus of strict \( g \)-convexity of \( u \) over any subdomain \( \Omega^* \subset \subset \Omega \), as formulated in (2.10). Defining \( \Psi \) through equation (3.2), we first note that a \( g \)-convex, elliptic solution \( u \in C^4(\Omega) \cap C^4(\mathcal{U}) \) of the second boundary value problem (1.1), (3.3), for which \( Tu \) is one-to-one, will be a generalized solution, as defined in Section 4 of [17], with density \( f = |\Psi|, u, Du) \), satisfying \( \int_{\Omega} f = |\Omega^*| \), (and target density \( f^* \equiv 1 \)). Furthermore, there exist positive constants \( c_0 \) and \( C \), depending on \( \Psi, \mathcal{U}, \mathcal{U}_0 \) such that \( c_0 \leq f \leq C \). Consequently, if there does not exist a positive lower bound for \( w(\Omega^*, \Omega) \), by virtue of the weak continuity of the associated measures and the Radon-Nikodym theorem, there would exist a sequence of such solutions converging uniformly to a generalized solution \( u \), with \( L^\infty \) density \( f \) satisfying the same
bounds, which is not strictly $g$-convex at some point $x_0 \in \tilde{\Omega}'$, thereby contradicting Theorem 1 in [14] and Theorem 2.3 in [1].

Alternatively, we remark that we can obtain an explicit estimate for $\omega[u]$ from the Hölder gradient estimate in Section 8 of [1] corresponding to Theorem 2.4 there, by using duality. $\square$

From Theorem 3.1 in [2], we note that the additional conditions (i) and (ii) are not needed if any of the conditions A3, A4w, or A4*W are also satisfied, or more generally either A3 or the existence of a barrier $\phi$ satisfying (2.2). As in [2], we also have a stronger version of Theorem 3.1, using the full strengths of Lemma 2.2 in [2] and Theorem 1 in [14]. Namely, we need to only assume that in our domain convexity conditions that $\Omega$ and $\Omega'$ are, respectively, uniformly $Y$-convex and $Y^*$-convex with respect to $(\Omega^*, u)$ and $(\Omega, u)$, as defined in [2], while $\Omega_0$ is $g$-convex with respect to $y \in \Omega'$ and $z = g^*(x, y, u(x))$, for all $x \in \Omega$, $y \in \Omega'$. Moreover, taking account of the injectivity of $Tu$, the $Y$-convexity conditions are equivalent to the uniform $g$-convexity of $\tilde{\Omega}$ with respect to the dual function $v = u_0^*$ on $\Omega^*$ and the uniform $g^*$-convexity of $\Omega'$ with respect to the function $u$ itself on $\Omega$. The reader is referred to [17] and [2] for further details concerning our notions of domain convexity.

4 Application to existence and regularity

For our applications to existence, we will assume that the function $\psi$ is separable in the sense that

$$|\psi|(x, u, p) = \frac{f(x)}{f^* \circ Y(x, u, p)},$$

for positive intensities $f \in L^1(\Omega)$ and $f^* \in L^1(\Omega^*)$. Then a necessary condition for the existence of an elliptic solution with the mapping $Tu$ being a diffeomorphism, to the second boundary value problem (1.1), (3.3), is the conservation of energy

$$\int_\Omega f = \int_{\Omega^*} f^*.$$  (4.2)

To fit our previous conditions on $B$ we will assume that the functions $f$ and $f^*$ are both $C^2$ smooth, with positive lower and upper bounds.

It then follows from Theorem 3.1 that our classical existence result, Theorem 1.1 in [3], holds without assuming any of the conditions A3, A4w, and A4*W. Moreover, from [13,14], this result can be refined in the sense that for any $x_0 \in \Omega$ and $u_0 \in J_0$, sufficiently far from the boundary of the gradient control interval $J_0$ in condition A5, there exists a unique $g$-convex, uniformly elliptic solution $u$, of the second boundary value problem (3.1), (3.3), satisfying $u(x_0) = u_0$. We will now formulate these extensions more explicitly, with the interval $J_0$ also permitted to be finite. First we repeat the formulation of condition A5 to include the finite case.

A5: There exists an open interval $J_0 = (m_0, M_0) \subset J$ and a positive constant $K_0$, such that

$$|g_\psi(x, y, z)| < K_0,$$

for all $x \in \tilde{\Omega}$, $y \in \tilde{\Omega}'$, $g(x, y, z) \in J_0$.

Then we have the following extension of the classical existence results in [3,14]. Taking account of condition (i) in Theorem 3.1, it will be convenient in its formulation to use domain convexity with respect to the generating function $g$, as in [14], rather than the mapping $Y$, and treat the slightly more general situation in a remark.

**Theorem 4.1.** Let $g \in C^4(\Gamma)$ be a generating function satisfying conditions A1, A2, A1*, A3w, and A5, for $C^4$ bounded domains $\Omega, \Omega'$ in $\mathbb{R}^n$, with $\Omega$ uniformly $g$-convex with respect to $y \in \Omega'$, $z \in g^*(\tilde{\Omega}, y, J_0)$, and $\Omega'$ uniformly $g^*$-convex with respect to $x \in \tilde{\Omega}, u \in J_0$. Also assume there exists a $g$-affine function, $g_0 = g(\cdot, y_0, z_0)$, on $\tilde{\Omega}$ satisfying $Tg_0 = y_0 \in \Omega^*$,
\[ g_\ell(\hat{\Omega}) \subset (m_0 + K_\ell d, M_0 - K_\ell d), \quad (4.3) \]

and

\[ g(\Omega, y, z) \subset J, \quad \text{for all } y \in \Omega^*, \quad z \in g^*(\Omega, y, (m_0, m_0 + K_\ell d)), \quad (4.4) \]

where \( d = \text{diam}(\Omega) \). Suppose also the function \( \psi \) satisfies (4.1), (4.2). Then for any \( x_0 \in \Omega \), there exists a unique \( g \)-convex elliptic solution \( u \in C^3(\hat{\Omega}) \) of the second boundary value problem (3.1), (3.3), satisfying \( u(x_0) = g_\ell(x_0) \). Furthermore, the mapping \( Tu \) is a \( C^2 \) smooth diffeomorphism from \( \hat{\Omega} \) to \( \hat{\Omega}^* \).

**Remark 4.1.** To avoid possible confusion, we point out that here, as in condition (2.3) in [2] and Theorem 1.1 in [18], we are using the following meaning of the diameter of a domain \( \Omega \subset \mathbb{R}^n \),

\[ \text{diam} (\Omega) = \sup_{x, y \in \Omega} d_{\Omega}(x, y), \]

where \( d_{\Omega}(x, y) = \text{dist}(x, y) \) is the geodesic distance in \( \Omega \) between \( x \) and \( y \), that is, the infimum of the lengths of \( C^1 \) curves in \( \Omega \) joining the points \( x \) and \( y \). Moreover, we may also refine condition (4.3) by replacing \( d \) by \( d' = \text{diam}(\hat{\Omega}) \), where \( \hat{\Omega} \) is any larger domain than \( \Omega \), also satisfying condition A5. If \( \hat{\Omega}' = \hat{\Omega} \) is the convex hull of \( \hat{\Omega} \), we obtain the usual notion of diameter in \( \mathbb{R}^n \). In fact, the above confusion goes back to the statement of the existence result for generalized solutions in Theorem 4.2 in [17], where the corresponding geodesic distance in \( \hat{\Omega} \) is also intended.

**Proof.** Substituting the second derivative estimate in Theorem 3.1 for that in Theorem 3.1 in [2], we would infer from the proof of Theorem 1.1 and Remark 3.2 in [3], the existence of a solution \( u \) whose graph intersects that of \( g_\ell \). For this we need to observe that the assumed uniform \( g \)-convexity conditions on \( \Omega \) would imply the corresponding uniform \( Y \)-convexity of \( \hat{\Omega} \) as well as the supplementary condition (i) and the \( g \)-convexity of an elliptic solution \( u \), from Lemma 2.1 in [18]. Then the rest of the supplementary condition (ii) now follows from the mass balance condition (4.2) and our assumed condition (4.4). To obtain the full strength of Theorem 4.1 we then need to adjust the homotopy family (3.4) in [3], as done by Rankin in equation (93) in [14], to conclude the existence of a classical solution \( u \), with graph intersecting that of \( g_\ell \) at \( x_0 \), the uniqueness of which then follows from [13].

**Remark 4.2.** We can write a cleaner but slightly weaker version of Theorem 4.1 by replacing the uniform \( g \)-convexity condition on \( \hat{\Omega} \) by uniform \( Y \)-convexity as in Theorem 1.1 of [3] with our condition on \( g_\ell \), (4.3), strengthened to

\[ g_\ell(\hat{\Omega}) \subset (m_0 + 2K_\ell d, M_0 - 2K_\ell d), \quad (4.5) \]

which also implies condition (4.4). As mentioned above, we also have a messier, more general statement if we replace the uniform \( g \)-convexity of \( \hat{\Omega} \) by uniform \( Y \)-convexity, but still assume the corresponding \( g \)-convexity of \( \hat{\Omega} \) and supplementary condition (i) in Theorem 3.1. Note that when we use Theorem 3.1 in [2] in the proof of Theorem 4.1, we do not need this last condition when any of the conditions A3, A4w, and A4* hold.

From the proof of Theorem 2 in [14], which combines the existence of classical solutions with the local regularity of strictly convex generalized solutions from [18] and the uniqueness from [13], we then have the following global regularity result, which extends Theorem 2 in [14] to the case when A4w is not assumed.

**Corollary 4.1.** Let \( u \) be a \( g \)-convex generalized solution of the second boundary value problem, (3.1), (3.3), where \( g, \Omega, \Omega^* \) and \( \psi \) satisfy the hypotheses of Theorem 4.1, with \( g_\ell \) a \( g \)-support of \( u \) at some point \( x_0 \in \Omega \). Then \( u \in C^3(\hat{\Omega}) \) is a classical elliptic solution of (3.1), (3.3) and \( Tu \) a \( C^2 \) smooth diffeomorphism from \( \hat{\Omega} \) to \( \hat{\Omega}^* \).

**Remark 4.3.** The modified hypotheses of Theorem 4.1 in Remark 4.2 are also applicable to Corollary 4.1. Moreover, if any of conditions A3, A4w, and A4* hold, we need to only assume, in accordance with
Theorem 2 in [14], that (4.4) only holds for any g-support of u. By adapting the uniqueness argument for global optimal transportation regularity in Section 6 of [19], we may remove condition (4.4) completely in these cases if also

$$\sup_{\Omega} u < M_0 - 2K_0 d.$$  \hspace{1cm} (4.6)

The overall proof is also much simpler in that we do not need to use the strict convexity of generalized solutions and their local regularity. Using condition (4.6), we then obtain, from Theorem 4.1, a classical solution v of (3.1), (3.3) whose graph touches that of the generalized solution u from above so that the uniqueness argument of [19] is applicable in a sufficiently small ball $B \subset \Omega' = \{u < v\}$, with boundary $\partial B$ intersecting $\partial \Omega'$.

A similar remark applies to the general case in Corollary 4.1, except we still need condition (4.4) for the strict convexity control used in our proof of the second derivative estimates in Theorem 3.1.

**Funding information:** This research was supported by Australian Research Council Grant DP180100431.

**Conflict of interest:** Prof. Neil Trudinger, who is the author of this article, is a current Editorial Board member of Advanced Nonlinear Studies. This fact did not affect the peer-review process. The author declare no other conflict of interest.

**References**

[1] N. Guillen and J. Kitagawa, *Pointwise inequalities in geometric optics and other generated Jacobian equations*, Comm. Pure Appl. Math. **70** (2017), 1146–1220.

[2] F. Jiang and N. S. Trudinger, *On Pogorelov estimates in optimal transportation and geometric optics*, Bull. Math. Sci. **4** (2014), 407–431.

[3] F. Jiang and N. S. Trudinger, *On the second boundary value problem for Monge-Ampère type equations and geometric optics*, Arch. Rat. Mech. Anal. **229** (2018), 547–567.

[4] F. Jiang and N. S. Trudinger, *Oblique boundary value problems for augmented Hessian equations II*, Nonlinear Anal. **154** (2017), 148–173.

[5] F. Jiang and N. S. Trudinger, *On the Neumann problem for Monge-Ampère type equations revisited*, New Zealand J. Math. **52** (2021), 671–689.

[6] F. Jiang, N. S. Trudinger, and N. Xiang, *On the Neumann problem for Monge-Ampère type equations*, Canadian J. Math. **68** (2016), 1334–1361.

[7] J.-K. Liu and N. S. Trudinger, *On Pogorelov estimates for Monge-Ampère type equations*, Discrete Contin. Dyn. Syst. **28** (2010), 1121–1135.

[8] J.-K. Liu and N. S. Trudinger, *On classical solutions of near field reflection problems*, Discrete Contin. Dyn. Syst. **36** (2016), 895–916.

[9] G. Loeper, *On the regularity of solutions of optimal transportation problems*. Acta Math. **202** (2009), 241–283.

[10] G. Loeper and N. S. Trudinger, *Weak formulation of the MTW condition and convexity properties of potentials*, Methods Appl. Anal. **28** (2021), 53–60.

[11] G. Loeper and N. S. Trudinger, *On the Convexity Theory of Generating Functions*, 2021, preprint, arXiv: 2109.04585.

[12] X-N. Ma, N. S. Trudinger, and X-J. Wang, *Regularity of potential functions of the optimal transportation problem*, Arch. Ration. Mech. Anal. **177** (2005), 151–183.

[13] C. Rankin, *Distinct solutions to generated Jacobian equations cannot intersect*, Bull. Aust. Math. Soc. **102** (2020), 462–470.

[14] C. Rankin, *Strict g-convexity for generated Jacobian equations with applications to global regularity*, 2021, arXiv:2111.00448.

[15] N. S. Trudinger, *Recent developments in elliptic partial differential equations of Monge-Ampère type*, ICM. Madrid **3** (2006), 291–302.

[16] N. S. Trudinger, *A note on global regularity in optimal transportation*, Bull. Math. Sci. **3** (2013), 551–557.

[17] N. S. Trudinger, *On the local theory of prescribed Jacobian equations*, Discrete Contin. Dyn. Syst. **34** (2014), 1663–1681.

[18] N. S. Trudinger, *On the local theory of prescribed Jacobian equations revisited*, Math. Eng. **3** (2021), 1–17.

[19] N. S. Trudinger and X-J. Wang, *On the second boundary value problem for Monge-Ampère type equations and optimal transportation*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **VIII** (2009), 143–174.