Second order perturbations of relativistic membranes in curved spacetime

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A manifestly covariant equation is derived to describe the second order perturbations in topological defects and membranes on arbitrary curved background spacetimes. This, on one hand, generalizes work on macroscopic strings in Minkowski spacetime and introduces a framework for studying in a precise manner membranes behavior near the black hole horizon and on the other hand, introduces a more general framework for examining the stability of topological defects in curved spacetimes.

I. INTRODUCTION

Perturbations have been extensively used in studying the physical properties of topological defects and relativistic membranes. The interest in perturbative approaches has, at least, a twofold motivation. First, topological defects of one form or another are inevitably formed during phase transitions in the early universe. Their cosmological implications, however, appear to depend sensitively on their stability with respect to perturbations. Guven, in the approximation of Garriga and Vilenkin [8], has introduced a framework for examining the stability of topological defects moving in a general curved background spacetime [8]. His approach to perturbation theory was to expand the action describing the evolution of the defect in a manifestly covariant way out to first order around a given classical solution. Secondly, let us consider the motion of a membrane in curved background spacetime. This system is endowed with conformal symmetry on the world sheet and the membrane equations of motion are supplied with corresponding constraints. The symmetry and the constraints underline the fact that only the transverse membrane motion is physical. Therefore the actual physical degrees of freedom of a membrane are less than the dimensions of spacetime. There are two alternatives open: Either work with all degrees of freedom and check at every stage of the calculation that the constraints are satisfied; or work directly with the transverse degrees of freedom. Choosing the second approach there is a price to be paid, the membrane equations of motion become nonlinear. However, that is not really a problem in a perturbative scheme where the equations of motion are to be solved order by order in the expansion around the zeroth order unperturbed membrane.

Until now only first order perturbations around the zeroth order membrane and string in arbitrary curved background have been considered [2, 4, 5]. This is perfectly enough for many purposes, but in certain cases it is necessary to consider also the second order perturbations. For instance, considering small perturbations around a contracting circular string, it is easy to see that there is no contribution to the total conserved energy to first order; the first order contribution simply integrates out. The first nonzero contribution (besides the zeroth order contribution) to the total energy is quadratic in the first order perturbations, but then also second order perturbations must be included for consistency, since they contribute to the same order. One of us (AN) in collaboration with A. Larsen managed to generalize the results for first order perturbations to second order perturbations but in flat backgrounds [10].

The purpose of the present paper is to derive a manifestly covariant equation to describe the second order perturbations in a membrane on arbitrary curved backgrounds. In order to do this we exploit the kinematical framework of Capovilla and Guven [1] for describing deformations of an arbitrary world sheet. This language provides us a geometrical approach to the description of perturbations. Within this framework, the equations of motion and both the equations of motion describing perturbations about classical solutions and equations of motion for the second order perturbations can be constructed in lego block fashion, by assembling the various kinematical ingredients.

The paper is organised as follows. To establish our notation we begin in section II by summarizing the well-known classical kinematical description provided by the Gauss-Weingarten equations of an embedded timelike world sheet of dimension $n$ in a spacetime of dimension $n+p$, in terms of its intrinsic and extrinsic geometries [1]. In this same kinematical spirit, in section III, we describe the deformation of the world sheet. There are analogues of the Gauss-Weingarten equations which are useful for identifying the structures associated with such deformations. In section IV we apply this kinematical framework to Nambu action and reproduce the corresponding equations of motion. In the same section we discuss the equations of motion describing first order perturbations on arbitrary curved backgrounds and improve the perturbative expansion by including second order terms. We conclude in section V with a brief discussion.

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II. MATHEMATICS OF SUBMANIFOLDS AND EMBEDDINGS

In this section we provide an overview of the well-known mathematical description of the world sheet of a membrane viewed as an embedded surface (submanifold) in a fixed background spacetime (manifold).

Let \( M \) be an \( n + p \)-dimensional manifold, \( N \) an \( n \)-dimensional manifold and \( \theta : N \rightarrow M \) is an imbedding. The image \( \theta (N) \) of \( N \) is said to be a submanifold of \( M \). If \( \{ x^a \}, \ a = 1, \cdots, n \), are differentiable coordinates on the manifold \( N \), then the submanifold \( \theta (N) \) can be given locally by the equations

\[
y^\mu = y^\mu (x^a), \quad (\mu = 1, \cdots, n + p)
\]

where \( y^\mu \) are differentiable functions of the variables \( x^a \) and the rank of the matrix \( \left( \frac{\partial y^\mu}{\partial x^a} \right) \) is equal to \( n \). The \( n \) vectors

\[
e_a := \frac{\partial y^\mu}{\partial x^a} \frac{\partial}{\partial y^\mu} \equiv y^\mu_a \frac{\partial}{\partial \mu}
\]

form a basis of tangent vectors to \( \theta (N) \) at each point of \( \theta (N) \). The components of the induced metric on the world sheet is then given by

\[
\gamma_{ab} = g_{\mu\nu} \frac{\partial y^\mu}{\partial x^a} \frac{\partial y^\nu}{\partial x^b} = g(e_a, e_b).
\]

Consider the vector \( N^j = n_i \delta^j_i \), with \( n_i \) the \( i \)th unit, to be orthogonal to all the vectors tangent to \( \theta (N) \). Then if \( \{ e_a \}, \ a = 1, \cdots, n \) is a basis of tangent vectors to \( \theta (N) \), \( \{ e_a, n_i \} \) (where \( \{ n_i \}, \ i = 1, \cdots, p \) will be linearly independent and can be used as a basis for the spacetime vectors of \( M \). The components of \( g \) with respect to this basis will be

\[
g_{\mu\nu} = \begin{pmatrix} g(n^i, n^j) & 0 \\ 0 & g(e_a, e_b) \end{pmatrix}.
\]

As the metric \( g \) is assumed to be non-degenerate, this shows that \( g(n^i, n^j) = g^{\mu\nu} n^i_\mu n^j_\nu \neq 0 \). So one can normalize the normal forms \( \{ n^k \} \) to have unit magnitude, i.e. \( g^{\mu\nu} n^i_\mu n^j_\nu = \delta^i_j \). Thus \( \{ n^k \} \) defined by

\[
g(n^i, n^j) = \delta^i_j, \quad g(e_a, n^i) = 0.
\]

Normal vielbein indices are raised and lowered with \( \delta^i_j \) and \( \delta_{ij} \), respectively, whereas tangential indices are raised and lowered with \( \gamma^{ab} \) and \( \gamma_{ab} \), respectively.

We define the world sheet projections of the spacetime covariant derivatives with \( D_a := e^\mu_a D_\mu \), where \( D_\mu \) is the covariant derivative compatible with \( g_{\mu\nu} \). Let us now consider the world sheet basis of the two vectors \( \{ e_a, n^i \} \), \( D_a e_b \) and \( D_a n^i \). Since \( (D_a e_b)_x \) and \( (D_a n^i)_x \) are defined for each \( x \in N \), we can decompose each one of these gradients into a tangential and a normal component:

\[
(D_a e_b)_x = (\nabla_a e_b)_x + a_x (e_a, e_b), \quad (D_a n^i)_x = a_x (e_a, n^i) + (D_a n^i)_x,
\]

where we denote by \( \nabla_a \) the world sheet covariant derivative, by \( D_a \) the world sheet covariant derivative defined on fields transforming as tensors under normal frame rotations and by \( a_x \) the extrinsic curvature of the world sheet. Writing the quantities \( D_a e_b \) and \( D_a n^i \) with respect to the basis vectors \( \{ e_a, n^i \} \), equations (6) and (7) become:

\[
D_a e_b = \gamma^c_{ab} e_c - K^i_{ab} n_i, \quad D_a n^i = K^c_{a} e^c + \omega_x^{ij} n_j.
\]

These kinematical expressions, generalizing the classical Gauss-Weingarten equations, describe completely the extrinsic geometry of the world sheet.

The \( \gamma_{ab} \) are the connection coefficients compatible with the world sheet metric \( \gamma_{ij} \):

\[
\gamma_{ij} = g(D_a e_b, e^c) = \gamma_{bc}.
\]

The quantity \( K^i_{ab} \) is the \( i \)th extrinsic curvature of the world sheet:

\[
K^i_{ab} = -g(D_a e_b, n^i) = K^i_{ba}.
\]

The normal fundamental form, or extrinsic twist potential, of the world sheet is defined by

\[
\omega_x^{ij} = g(D_a n^i, n^j) = -\omega_x^{ji}.
\]

Equations (8) and (9) will help us find a relationship between the curvature tensor of spacetime \( M \) and the curvature tensor of world sheet \( N \). By definition the Riemann tensor of the spacetime covariant derivative \( D_\alpha \) is given by:

\[
R(e_a, e_\beta) e_\gamma = D_\alpha (D_\beta e_\gamma) - D_\beta (D_\alpha e_\gamma) - D_{[a, \beta]} e_\gamma.
\]

Thus, taking advantage of Gauss - Weingarten equations (8) and (9), we derived the Gauss - Codazzi, Codazzi - Mainardi and Ricci integrability conditions:

\[
g(R(e_b, e_c) e_d, e_d) = R_{abcd} - K^i_{ac} K_{bdi} + K^i_{ad} K_{bc}.
\]

We use the notation \( g(R(Y_1, Y_2) Y_3, Y_4) = R_{abcd} Y_2^a Y_1^b Y_3^c Y_4^d \). \( R_{abcd} \) is the Riemann curvature tensor of spacetime, whereas \( R_{abcd} \) is the Riemannian curvature tensor of Wolf space, whereas \( R_{abcd} \) is the Riemannian curvature tensor of Wolf space.

III. DEFORMATION IN THE IMBEDDING

Let us now consider the neighboring \( n \)-dimensional surface \( N' \) described by a deformation of \( N \):

\[
y^\mu = y^\mu (x^a) + \delta y^\mu (x^a).
\]

According to (8) this situation can be investigated by examining the behaviour of a congruence of curves with
tangent vector $\vec{V}$. These curves could represent the histories of small test particles, in which case they would be geodesics. Suppose $\lambda(t)$ is a curve with tangent vector $\vec{\delta} = (\partial/\partial t)_{\lambda}$. Then one may construct a family $\lambda(t, s)$ of curves by moving each point of the curve $\lambda(t)$ a distance $s$ along the integral curves of $\vec{V}$. If one now defines $\vec{\delta}$ as $(\partial/\partial t)_{\lambda(t,s)}$ it follows that the Lie brackets $[\vec{\delta}, \vec{V}]$ is equal to zero (also the Lie derivative, $L_{\vec{V}}\vec{\delta}$, is zero), so because we deal only with torsion-free connections we take the equality

$$D_{\vec{\delta}}\vec{V} = D_{\vec{V}}\vec{\delta}. \quad (15)$$

One may interpret $\vec{\delta}$ as representing the separation of points equal distances from some arbitrary initial points along two neighbouring curves. If one adds a multiple of $\vec{V}$ to $\vec{\delta}$ then this vector will represent the separation of points on the same two curves but at different distances along the curves.

Let us choose $\lambda(t = 0, s)$ as a curve belonging to $N$, then $\lambda(t, s)$ describe a curve in the neighboring surface $N'$. Thus we can define $\vec{\delta} = (\partial/\partial t)_{\lambda}$ as the deformation vector field ($\vec{\delta} = \delta y$) and decomposing it with respect to the spacetime basis $\{e_a, n^i\}$, we take

$$\vec{\delta} = \Phi^a e_a + \Phi^i n_i. \quad (16)$$

The tangential projection can always be identified with the action of a world-sheet diffeomorphism, $\delta^\mu = \Phi^\mu y^\mu_a$, and so will subsequently be ignored. The physically observable measure of the deformation is therefore provided by the projection of $\vec{\delta}$ orthogonal to $N$, characterized by the $p$ scalar fields $\Phi^i$.

The displacement $\delta^\mu = \delta y^\mu$ in the embedding induces a displacement in both the tangent basis $\{e_a\}$ and normal basis $\{n_i\}$. In light of the discussion above, let $\vec{\delta} = \Phi^i n_i$, and consider the gradients of $\{e_a\}$ and $\{n_i\}$ along the vector field $\delta^\mu$, defined with $D_{\vec{\delta}}$.

We can always expand $D_{\vec{\delta}}e_a$ and $D_{\vec{\delta}}n_i$ with respect to the spacetime basis $\{e_a, n_i\}$, in a way analogous to the Gauss - Weingarten equations as

$$D_{\vec{\delta}}e_a = \beta_{ab} e_b + J_{aj} n^j, \quad (17)$$

$$D_{\vec{\delta}}n_i = -J_{ai} e^a + \gamma_{ij} n^j. \quad (18)$$

Comparison (17) with the Gauss equation (8) shows that the quantity $\beta_{ab}$, defined by

$$\beta_{ab} = g(D_{\vec{\delta}}e_a, e_b) = \beta_{ba}, \quad (19)$$

appears in the same position as $\gamma_{ab}$. The quantities $J_{aj}$ are defined by

$$J_{aj} = g(D_{\vec{\delta}}e_a, n_j), \quad (20)$$

and appear in the same position as $K^i_{ab}$ in (9). We note that $\beta_{ab}$ transforms as a scalar under normal frame rotations, whereas $J_{aj}$ transforms as a vector.

The normal projection of $D_{\vec{\delta}}n_i$,

$$\gamma_{ij} = g(D_{\vec{\delta}}n_i, n_j) = -\gamma_{ji}, \quad (21)$$

is a new structure we have not encountered already. In contrast to $\beta_{ab}$ and $J_{aj}$, however, there is no simple relationship between $\gamma_{ij}$ and deformations of the world sheet. The analogy between (18) and (9) suggests a role for $\gamma_{ij}$ analogous to $\omega_i^j$. For this purpose we introduce a covariant deformation derivative as follows

$$\overline{D}_{\vec{\delta}} \Phi_i = D_{\vec{\delta}} \Phi_i + \gamma^i_j \Phi_j. \quad (22)$$

Equation (18) can then be written in the form

$$\overline{D}_{\vec{\delta}} n_i = -J_{ai} e^a. \quad (23)$$

Using (15), it is easy to show that

$$\beta_{ab} = g(D_{\vec{\delta}}e_a, e_b) = g(D_a \vec{\delta}, e_b) = g(D_a n^i, e_b) \Phi_i = K_{ab} \Phi_i, \quad (24)$$

and

$$J_{aj} = g(D_{\vec{\delta}}e_a, n_j) = g(D_a \vec{\delta}, n_j) = g(D_a n^i, n_j) \Phi_i + \nabla_a \Phi_j = \omega_{ij} \Phi_i + \nabla_a \Phi_j = D_{\vec{\delta}} \Phi_j. \quad (25)$$

The deformation in the induced metric on $N$ is just twice $\beta_{ab}$:

$$D_{\vec{\delta}} \gamma_{ab} = D_{\vec{\delta}} g(e_a, e_b) = 2 g(e_a, D_{\vec{\delta}} e_b) = 2 \beta_{ab} = 2 K^i_{ab} \Phi_i. \quad (26)$$

Let us now evaluate the deformation of the extrinsic curvatures, $\overline{D}_{\vec{\delta}} K^i_{ab}$. Using its definition we have that

$$\overline{D}_{\vec{\delta}} K^i_{ab} = -g(\overline{D}_{\vec{\delta}} n^i, D_a e_b) - g(n^i, D_{\vec{\delta}} D_a e_b). \quad (27)$$

Using Eq. (23) and the Gauss equation (8), the first term on the right-hand side is given by

$$-g(\overline{D}_{\vec{\delta}} n^i, D_a e_b) = \gamma_{ij} \delta^i_j. \quad (28)$$

The second term on the right-hand side can be developed using the Ricci identity, as

\begin{equation}
\end{equation}
where in the last line we have used eqs. (25) and (24). Therefore we find

\[
\overline{D}_a K^{ij} = -D_a D_b \Phi^i + \left[ g \left( n^i, R \left( n_j, e_a \right) e_b \right) + K_{bcj} \cdot K^{ic_a} \right] \Phi^j. 
\] (29)

Note that the change of the extrinsic curvature under an infinitesimal deformation of the world sheet involves second derivatives of the scalar fields \( \Phi^i \).

IV. THE EQUATIONS OF MOTION

In this section we apply the kinematical framework we have presented to the derivation of the equations of motion of relativistic membranes of physical interest. The action that we are going to use is the most simple generalization of relativistic membranes of physical interest. The equations of motion are given by the extrema of \( S \) subject to variations

\[
y^\alpha = y^\alpha (x^m) + \delta y^\alpha (x^m), \quad \delta y^\alpha = \frac{1}{2} \sqrt{\gamma} \delta \gamma.
\]

so,

\[
\delta S = -\sigma \int_N d^m x \left( -\frac{1}{2} \sqrt{-\gamma} \delta \gamma \right) = -\frac{1}{2} \sigma \int_N d^m x \left( \sqrt{-\gamma} \delta \gamma \right).
\]

where we have used Jacobi’s formula, the rule for differentiating a determinant. Eq. (26) tells us that the quantity \( \delta \gamma_{ab} \) is equal to

\[
\delta \gamma_{ab} = 2K^i_{ab} \Phi_i, \quad \gamma^{ab} \cdot K^i_{ab} = 0.
\]

A. First order perturbations

In order to derive the equations of motion for first order perturbations of a membrane, we simply have to consider the linearization of eq. (34). This is done by writing the perturbed quantities \( \tilde{\gamma}_{ab} = \gamma^{ab} + \delta \gamma^{ab} \) and \( \tilde{K}^i_{ab} = K^i_{ab} + \delta K^i_{ab} \) in eq. (34) and holding the first order terms:

\[
\tilde{\gamma}_{ab} \cdot \tilde{K}^i_{ab} = 0
\]

\[
\gamma^{ab} K^i_{ab} + \delta \gamma^{ab} K^i_{ab} + \gamma^{ab} \delta K^i_{ab} = 0
\]

\[
\delta \gamma^{ab} K^i_{ab} + \gamma^{ab} \delta K^i_{ab} = 0
\]

The same result we derive taking the quadratic order of action (30). Thus we should compute the quantity \( \delta \gamma^{ab} \cdot K^i_{ab} + \gamma^{ab} \cdot \delta K^i_{ab} \). This is easy, using eqs. (24) and (29) we have:

\[
\delta \gamma^{ab} \cdot K^i_{ab} + \gamma^{ab} \cdot \delta K^i_{ab} = -2K^{ij}_{ab} K^i_{ab} \Phi_j - \gamma^{ab} \cdot D_a D_b \Phi^j
\]

\[
+ \gamma^{ab} \left[ g \left( n^i, R \left( n_j, e_a \right) e_b \right) + K_{bcj} \cdot K^{ic_a} \right] \Phi^j
\]

\[
= -\Delta \Phi^j + g \left( n^i, R \left( n_j, e_a \right) e^a \right) \cdot \Phi^j - K_{baj} \cdot K^{iba} \cdot \Phi^j
\]

where

\[
\Delta = \gamma^{ab} \cdot D_a D_b
\]

and the equations of motion for the first order perturbations take the form:

\[
\Delta \Phi^j - g \left( n^i, R \left( n_j, e_a \right) e^a \right) \cdot \Phi^j + K_{baj} \cdot K^{iba} \cdot \Phi^j = 0
\]

(37)
B. Second order perturbation

We continue to the derivation of the equations of motion for the second order perturbations in world-sheet, which has not been obtained before. For this case, we write $y^{\mu}$ in the following way:

$$y^{\mu} = y^{\mu}(x^{a}) + \delta y^{\mu}_{(1)}(x^{a}) + \delta y^{\mu}_{(2)}(x^{a}).$$  \hfill (38)

Since we are interested only in physical (transverse) perturbations, we have already said that $\delta y^{\mu} = \Phi^{i} n_{i}^{\mu}$. Thus, by expanding up to second order the deformation vector field $\vec{\delta}$, we have:

$$\vec{\delta} = (\Phi^{i} + D_{\delta}\Phi^{i})(n_{i} + D_{\delta}n_{i})$$

$$= \Phi^{i} n_{i} + (D_{\delta}\Phi^{i})n_{i} + \Phi^{i} D_{\delta}n_{i}$$

$$= \Phi^{i} n_{i} + \Psi^{i} n_{i}^{(1)} + \Phi^{i} D_{\delta}n_{i}.$$ \hfill (39)

so in compare with (38) we take

$$\delta y^{\mu}_{(2)}(x^{a}) = \Psi^{i} n_{i} + \Phi^{i} D_{\delta}n_{i},$$ \hfill (40)

with the new deformation vector to take the form:

$$\vec{\delta}^{(2)} = \Psi^{i} n_{i} + \Phi^{i} D_{\delta}n_{i}.$$ \hfill (41)

In (11) there is the first order perturbation of the normal vector, $D_{\delta}n_{i}$, which is given from (13). We consider now the gradients of $\{e_{a}\}$ and $\{n_{i}\}$ along the vector field $\delta n^{(2)}$, defined with $D_{\delta}(\vec{e})$. We can expand $D_{\delta}(\vec{e})$ and $D_{\delta}(n_{i})$ with respect to the spacetime basis $\{e_{a}, n_{i}\}$, in a way analogous to (17) and (15) respectively as

$$D_{\delta}(\vec{e}) = \beta^{(2)}_{ab} e^{b} + J^{(2)}_{ai} n^{i},$$ \hfill (42)

and

$$D_{\delta}(n_{i}) = -J^{(2)}_{ai} e^{a} + \gamma^{(2)}_{ij} n^{j}.$$ \hfill (43)

Quantity $\beta^{(2)}_{ab} (\neq \beta_{ab})$ is defined by the relation

$$\beta^{(2)}_{ab} = g(D_{\delta}(\vec{e}), e_{a}) = \beta^{(2)}_{ba}$$ \hfill (44)

and $J^{(2)}_{ai} (\neq J_{ai})$ by the relation

$$J^{(2)}_{ai} = g(D_{\delta}(n_{i}), e_{a}).$$ \hfill (45)

The quantities $\beta^{(2)}_{ab}$ and $J^{(2)}_{ai}$ can be expressed in terms of scalar fields $\Psi^{i}$ and $\Phi^{i}$, in a way analogous to the equations (24) and (25), considering the equation

$$D_{\delta}(\vec{e}) = D_{a}\delta^{(2)}_{\mu},$$ \hfill (46)

which is the analogue for deformation vector field $\vec{\delta}$ of equation (15). Using (40), we show that:

$$\beta^{(2)}_{ab} = g(D_{\delta}(\vec{e}), e_{b}) = g(D_{a}\delta^{(2)}_{\mu}, e_{b}) = g(D_{a}(\Psi^{j} n_{j}), e_{b}) + g(D_{a}(\Phi^{i} D_{\delta}n_{i}), e_{b})$$

$$= g(D_{a}(\Psi^{j} n_{j}), e_{b}) + g(D_{a}(\Phi^{i} D_{\delta}n_{i}), e_{b})$$

$$= K_{ab}^{i} \Psi^{j} - \gamma^{i}_{\mu}(\nabla\Phi^{k})(D_{c}\Phi^{i})$$

$$+ \gamma^{i}_{\mu} \nabla\Phi^{k}(D_{c}\Phi^{i}) - g(\gamma^{ad} \epsilon^{d} \Phi^{j} - K_{aj}^{i} \epsilon^{i} n_{b}, e_{b})$$

$$= K_{ab}^{i} \Psi^{j} - \gamma^{i}_{\mu} \nabla\Phi^{k}(D_{c}\Phi^{i}) - g(\gamma^{ad} \epsilon^{d} \Phi^{j} - K_{aj}^{i} \epsilon^{i} n_{b}, e_{b})$$

$$= K_{ab}^{i} \Psi^{j} - \gamma^{i}_{\mu} \nabla\Phi^{k}(D_{c}\Phi^{i}) - g(\gamma^{ad} \epsilon^{d} \Phi^{j} - K_{aj}^{i} \epsilon^{i} n_{b}, e_{b})$$

$$= K_{ab}^{i} \Psi^{j} - \gamma^{i}_{\mu} \nabla\Phi^{k}(D_{c}\Phi^{i}) - g(\gamma^{ad} \epsilon^{d} \Phi^{j} - K_{aj}^{i} \epsilon^{i} n_{b}, e_{b})$$

$$+ K_{ab}^{i} \Psi^{j} - \gamma^{i}_{\mu} \nabla\Phi^{k}(D_{c}\Phi^{i}) - g(\gamma^{ad} \epsilon^{d} \Phi^{j} - K_{aj}^{i} \epsilon^{i} n_{b}, e_{b})$$

and

$$J^{(2)}_{ai} = g(D_{\delta}(n_{i}), e_{a}) = g(D_{a}\delta^{(2)}_{\mu}, n_{i}) = g(D_{a}(\Psi^{j} n_{j}), n_{i}) + g(D_{a}(\Phi^{i} D_{\delta}n_{i}), n_{i})$$

$$= g(D_{a}(\Psi^{j} n_{j}), n_{i}) + g(D_{a}(\Phi^{i} D_{\delta}n_{i}), n_{i}) - g(D_{a}(\Phi^{i} J_{cb} e^{c}), n_{i})$$

$$= \delta_{ki} \cdot \omega^{k} + \nabla_{a} \Psi^{j}$$

$$- g(\gamma^{ad} \epsilon^{d} \cdot \epsilon^{i} \Phi^{j} - K_{aj}^{i} \epsilon^{i} n_{b}, n_{i})$$

$$= D_{a} \Psi^{j} + \delta_{ki} \cdot K_{ai}^{c} J_{cj} \Phi^{j}$$

$$= D_{a} \Psi^{j} + K_{ai}^{c} (D_{c}\Phi^{i}) \Phi^{j},$$ \hfill (48)

In order to derive the equations of motion for second order perturbations of a membrane, we substitute the
perturbed quantities $\tilde{\gamma}^{ab} = \gamma^{ab} + \delta(1)\gamma^{ab} + \delta(2)\gamma^{ab}$ and $\tilde{K}^{ab} = K^{ab} + \delta(1)K^{ab} + \delta(2)K^{ab}$ in eq. (33) and keep the quadratic terms:

$$\delta^{(2)}\gamma^{ab} \tilde{K}^{ab} + \gamma^{ab}\delta^{(2)}\tilde{K}^{ab} + (\delta^{(1)}\gamma^{ab})(\delta^{(1)}K^{ab}) = 0 \quad (49)$$

In order to proceed we have to evaluate the quantities $\delta^{(2)}\gamma^{ab}$ and $\delta^{(2)}K^{ab}$. Let us start with quantity $\delta^{(2)}\gamma^{ab} = D_{\delta^{(2)}}\gamma^{ab}$, of which all we need is eqn (17):

$$D_{\delta^{(2)}}\gamma^{ab} = D_{\delta^{(2)}}g(e_a, e_b) = g(D_{\delta^{(2)}}e_a, e_b) + g(e_a, D_{\delta^{(2)}}e_b)$$

$$= 2\beta^{(2)}_{ab}$$

$$= 2K_{ab}\Psi_i - 2(D_a\Phi^i) \cdot (D_i\Phi)$$

$$= -2[D_a(D_b\Phi_i)]\Phi^i. \quad (50)$$

We need also to obtain the quantity:

$$\delta^{(2)}K^{ab} = D_{\delta^{(2)}}K^{ab} \quad (51)$$

or

$$D_{\delta^{(2)}}K^{ab} = -D_{\delta^{(2)}}g(D_a e_b, n^i)$$

$$= -g(D_{\delta^{(2)}}D_a e_b, n^i) - g(D_a e_b, D_{\delta^{(2)}}n^i). \quad (52)$$

The first term on the right-hand side can be developed, using the Ricci identity, as

$$-g(D_{\delta^{(2)}}D_a e_b, n^i) = -g(R(\delta^{(2)}, e_a)e_b, n^i) - g(D_a D_{\delta^{(2)}}e_b, n^i)$$

$$= -g(R(\delta^{(2)}, e_a)e_b, n^i) - D_a [g(D_{\delta^{(2)}}e_b, n^i)] + g(D_{\delta^{(2)}}e_b, D_a n^i)$$

$$= -g(R(\delta^{(2)}, e_a)e_b, n^i) - \nabla_a J_b^{(2)i} + \beta^{(2)}_{bc} \cdot K_a^{ci} + \omega_a^{ij} \cdot J_b^{(2)j}$$

$$= -g(R(n_j, e_a)e_b, n^i)\Psi^j + g(R(e_b, e_a)e_b, n^i)J_b^{(2)j} + K_{bc} \cdot K_a^{ci}\Psi_k$$

$$- (D_b\Phi^i) \cdot (D_c\Phi) \cdot K_a^{ci} - [D_b(D_c\Phi_a)] \cdot K_a^{ci}\Phi^o$$

$$= -g(R(n_j, e_a)e_b, n^i)\Psi^j + \nabla_a J_b^{(2)i} + \omega_a^{ij} \cdot J_b^{(2)j} \quad (53)$$

Using eq. (33), the second term on the right-hand side is given by

$$-g(D_a e_b, D_{\delta^{(2)}}n^i) = \gamma^{ab} \cdot J_c^{(2)i}. \quad (54)$$

We find then that

$$D_{\delta^{(2)}}K^{ab} = -\nabla_a J_b^{(2)i} + \omega_a^{ij} \cdot J_b^{(2)j} + \gamma^{ab} \cdot J_c^{(2)i}$$

$$+ K_{bc} \cdot K_a^{ci}\Psi_k - (D_b\Phi^i) \cdot (D_c\Phi) \cdot K_a^{ci} - [D_b(D_c\Phi_a)] \cdot K_a^{ci}\Phi^o$$

$$- g(R(n_j, e_a)e_b, n^i)\Psi^j + g(R(e_b, e_a)e_b, n^i)J_b^{(2)j}$$

$$= -D_a D_b\Psi^i - D_a [K_{bc}^{ic}(D_c\Phi)\Phi^f] + K_{bc} \cdot K_a^{ci}\Psi^k - g(R(n_k, e_a)e_b, n^i)\Psi^k$$

$$= -D_a D_b\Psi^i + K_{cb} \cdot K_a^{ci}\Psi^k - g(R(n_k, e_a)e_b, n^i)\Psi^k$$

$$= [D_a K_{bc}^{ic}] (D_c\Phi)\Phi^f - K_b^{ic}[D_a(D_c\Phi)]\Phi^f - K_b^{ic}(D_c\Phi) (D_a\Phi)$$

$$= (D_b\Phi^i) \cdot (D_c\Phi) \cdot K_a^{ci} - [D_b(D_c\Phi_a)] \cdot K_a^{ci}\Phi^o + g(R(e_b, e_a)e_b, n^i) \cdot (D_b\Phi) \cdot J^i \quad (55)$$
So, substituting the results into the eq.(49) we find the second order equation of motion in the form:

\[ \nabla^2 \Psi^i + K_{cb} \cdot K_{a}^{ci} \Psi^k + g(R(n_k, e_b) e_b, n^i) \Psi^k = F^i \]  

where the source \( F^i \) is given in terms of the first order perturbations

\[ F^i = 2(D_n \Phi^i) + 2K^{ab} \Phi^j K_{bc} \Phi^k K_{e_a} - 2K_{ab} \Phi^j g(n^i, R(n_k, e_a) e_b) \Phi^k \]  

It should be mentioned that only the mixed term of the equation (49) has contributed to the source \( F^i \).

Even when the background geometry is flat so that \( R_{\beta\gamma\delta} = 0 \), eq.(56) is extremely complicated, involving scalars in the extrinsic geometry \( (K_{ba} \text{ and } \omega_{\alpha i}^j) \) in combinations which, it appears, cannot be eliminated in favor of intrinsic geometric scalars. If we can choose our normal vectors such that all but one of them, for example, \( n^1 \), are parallel transported along any curve on the world sheet,

\[ D_a n^i = 0, \]  

then \( \omega_{\alpha}^{ij} = 0 \) for all \( i \) and \( j \), because of its antisymmetry with respect to normal indices. In addition, the conditions eq.(58) imply that the only linear combination of extrinsic curvature tensors which is nonvanishing is the one that corresponds to the exceptional normal direction:

\[ K_{ba}^i = 0, \quad i = 2, \ldots, p. \]  

These conditions are consistent with a reasonably large class of geometries. Any defect in Minkowski space which lies in a \( n \)-dimensional plane will satisfy these conditions. A \( n \)-dimensional defect in de Sitter space with \( p - 1 \) fixed meridians will also satisfy eqs.(59). A less trivial example satisfying eqs.(59) is a string in Schwarzschild space on a fixed meridian.

Let us suppose, in addition, one more simplification which occurs whenever \( p = 2 \), an example of which is provided by a string in any four-dimensional manifold. For then, the coupling between the two scalar field components vanishes. Equation (57) then reduces to the form of a Klein-Gordon equation:

\[ \Box \gamma \Psi^i + \gamma^i \Psi^j = F^i, \quad i = 1, 2 \]  

where \( \Box = \gamma_{ab} \nabla_a \nabla_b \) is the dAlambertian, with

\[ \begin{align*}
\gamma^1 &= K_{ac} K^{ac} + \gamma_{ab} y_{a}^{\mu} y_{b}^{\nu} R_{\mu\alpha\beta} n_1^\alpha n_2^\beta \\
\gamma^2 &= \gamma_{ab} y_{a}^{\mu} y_{b}^{\nu} R_{\mu\alpha\beta} n_1^\alpha n_2^\beta
\end{align*} \]  

and the source \( F^i \):

\[ F^1 = 2K^{ab} \Phi^j \left( \nabla_a \nabla_b - K_{bc} K_{ac} - y_{a}^{\mu} y_{b}^{\nu} R_{\mu\alpha\beta} n_1^\alpha n_2^\beta \right) \Phi^1 \]

\[ F^2 = 0 \]

\( \nabla_a \) is the strings world-sheet covariant derivative.

V. DISCUSSION

In this paper we have studied membranes of arbitrary dimension moving in any spacetime, using the perturbative scheme of Garriga and Vilenkin. We adopted the kinematical framework of Capovilla and Guven in order to treat perturbations. Carter have also used kinematical terms in order to study the dynamics of relativistic brane models. The physical measure of the first and second order perturbations of \( y^\mu \) is given by two distinct scalar fields, which both live on the world sheet. Our results for the equations of motion and the first order equations of motion are in agreement with the independently obtained results in a number of papers. The expression we derived for the second order equations of motion is compact and can be seen as nontrivially coupled scalar wave equations for a multiplet of scalar field, with a variable mass that depends on a particular projection of the curvature of spacetime and on the extrinsic geometry. It also has a source term which is given in terms of the first order perturbations.

It is highly interesting to apply our result to strings moving in curved spacetime in order to study, in a precise manner, the string behavior near the black hole horizon, issues first raised by Susskind. Along these lines it has already been studied an oscillating circular string in Schwarzschild background to zeroth and first order. In this work it was calculated both the radial and angular spreading of the string, as the string approaches the black hole horizon. It was found that the radial spreading is suppressed by the Lorentz contraction and the string appears (to an asymptotic observer) as wrapping around the event horizon. We plan to calculate and include the second order terms and thus analyze how the string oscillators spread over the event horizon in a future paper. Notice that the second order perturbations are necessary for a consistent discussion of the energy. Hopefully we might understand the entropy of the black hole in terms of string degrees of freedom.

Appendix

computation of \( \gamma_{ab} D_{\delta\gamma\nu} K_{ab}^i \):
\[ \gamma^{ab} D_{\delta(2)} K_{ab} = \gamma^{ab} \left[ -D_a D_b \Psi^i + K_{bck} \cdot K_{a}^c \Psi^k - g(R(n_k, e_a) e_b, n^i) \Psi^k \right] + \]
\[ \gamma^{ab} \left[ - \left( \mathcal{D}_a \mathcal{K}^{bc} \right) (D_c \Phi_j) \Phi^j - K_b^{ic} \left[ D_a (D_c \Phi_j) \right] \Phi^j - K_b^{ic} (D_c \Phi_j) (D_a \Phi^j) \right] + \]
\[ \gamma^{ab} \left[ - (D_b \Phi^i) \cdot (D_c \Phi_j) \cdot K_{a}^c \cdot - \left[ D_b (D_c \Phi_j) \right] \cdot K_{a}^c \cdot \Phi^i + g(R(e^b, e_a) e_b, n^i) \cdot (D_b \Phi_j) \cdot \Phi^j \right] \]
\[ = - \Delta \Psi^i + K_{c}^a \cdot K_{a}^c \Psi^k - g(R(n_k, e^b) e_b, n^i) \Psi^k \]
\[ - \left[ D_a K^{bic} \right] (D_c \Phi_j) \Phi^j - K^{aic} \left[ D_a (D_c \Phi_j) \right] \Phi^j - K^{aic} (D_c \Phi_j) (D_a \Phi^j) \]
\[ - (D_b \Phi^i) \cdot (D_c \Phi_j) \cdot K^{bci} - \left[ D_b (D_c \Phi_j) \right] \cdot K^{bci} \Phi^i + g(R(e^b, e^c) e_b, n^i) \cdot (D_b \Phi_j) \cdot \Phi^j \]
\[ = - \Delta \Psi^i + K_{c}^a \cdot K_{a}^c \Psi^k - g(R(n_k, e^b) e_b, n^i) \Psi^k \]
\[ - 2K^{d}^{aic} (D_c \Phi_j) (D_a \Phi^j) - 2K^{aicom} (D_a (D_c \Phi_j)) \Phi^j \]
\[ - \left[ D^c K^{\hat{b}i} \right] (D_c \Phi_j) \Phi^j \]
\[ \text{(A.1)} \]

Computation of \((D_{\delta(2)} \gamma^{ab}) K_{ab}^i\):

\[ (D_{\delta(2)} \gamma^{ab}) K_{ab}^i = \left[ -2K^{ab} \Psi^i + 2(D^a \Phi^j) \cdot (D^b \Phi_j) + 2[D^a (D^b \Phi_j)] \Phi^j \right] K_{ab}^i \]
\[ = -2K^{ab} \cdot K_{a}^i \Psi^j + 2(D^a \Phi^j) \cdot (D^b \Phi_j) K_{ab}^i + 2[D^a (D^b \Phi_j)] K_{ab}^i \Phi^j \]
\[ \text{(A.2)} \]

Computation of \((D_{\delta} \gamma^{ab}) (D_{\delta} K_{ab}^i)\):

\[ (D_{\delta} \gamma^{ab}) (D_{\delta} K_{ab}^i) = \left[ -2K^{ab} \Psi^j \right] \left[ -D_a D_b \Psi^i + \left[ g \left( n^i, R \left( n_k, e_a \right) e_b \right) + K_{bck} \cdot K_{a}^c \right] \Phi^k \right] \]
\[ = 2D_a D_b \Phi^j K_{ab}^i \Phi^j - 2K^{ab} \Phi^j g \left( n^i, R \left( n_k, e_a \right) e_b \right) \Phi^k \]
\[ - 2K^{ab} \Phi^j K_{bck} \cdot K_{a}^c \Phi^k \]
\[ \text{(A.3)} \]

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