A half-space theorem for ideal Scherk graphs in $M \times \mathbb{R}$

Ana Menezes

Abstract

We prove a half-space theorem for an ideal Scherk graph $\Sigma \subset M \times \mathbb{R}$ over a polygonal domain $D \subset M$, where $M$ is a Hadamard surface with bounded curvature. More precisely, we show that a properly immersed minimal surface contained in $D \times \mathbb{R}$ and disjoint from $\Sigma$ is a translate of $\Sigma$.

1 Introduction

A well known result in the global theory for proper minimal surfaces in the Euclidean 3-space is the half-space theorem by Hoffman and Meeks [9], which says that if a properly immersed minimal surface $S$ in $\mathbb{R}^3$ lies on one side of some plane $P$, then $S$ is a plane parallel to $P$. Moreover, they also proved the strong half-space theorem, which says that two properly immersed minimal surfaces in $\mathbb{R}^3$ that do not intersect must be parallel planes.

This problem of giving conditions which force two minimal surfaces of a Riemannian manifold to intersect has received considerable attention, and many people have worked on this subject.

Let us observe that there is no half-space theorem in Euclidean spaces of dimensions bigger than 4, since there exist rotational proper minimal hypersurfaces contained in a slab.

Similarly, there exists no half-space theorem for horizontal slices in $\mathbb{H}^2 \times \mathbb{R}$, since rotational minimal surfaces (catenoids) are contained in a slab [11] [12]. However there are half-space theorems for constant mean curvature (CMC) $\frac{1}{2}$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$. In fact, Hauswirth, Rosenberg and Spruck proved the following result.

Theorem 1 ([8]). Let $S$ be a properly immersed CMC $\frac{1}{2}$ surface in $\mathbb{H}^2 \times \mathbb{R}$.

1. If $S$ is contained on the mean convex side of a horocylinder $C$, then $S$ is a horocylinder parallel to $C$. 


2. If $S$ is embedded and contains a horocylinder $C$ on its mean convex side, then $S$ is a horocylinder parallel to $C$.

Other examples of homogeneous manifolds where there are half-space theorems for minimal surfaces are $\text{Nil}_3$ and $\text{Sol}_3$ [1, 2, 3]. For instance, we know that if a properly immersed minimal surface $S$ in $\text{Nil}_3$ lies on one side of some entire minimal graph $\Sigma$, then $S$ is the image of $\Sigma$ by a vertical translation.

In [10], Mazet proved a general half-space theorem for constant mean curvature surfaces. Under certain hypothesis, he proved that in a Riemannian 3-manifold of bounded geometry, a constant mean curvature $H$ surface on one side of a parabolic constant mean curvature $H$ surface $\Sigma$ is an equidistant surface to $\Sigma$.

In this paper we consider the half-space problem for an ideal Scherk graph $\Sigma$ over a polygonal domain $D \subset M \times \mathbb{R}$, where $M$ denotes a Hadamard surface with bounded curvature, that is, $M$ is a complete simply connected Riemannian surface with curvature $-b^2 \leq K_M \leq -a^2 < 0$, for some constants $a, b \in \mathbb{R}$. More precisely, we prove the following result.

**Theorem 2.** Let $M$ denote a Hadamard surface with bounded curvature and let $\Sigma = \text{Graph}(u)$ be an ideal Scherk graph over an admissible polygonal domain $D \subset M$. If $S$ is a properly immersed minimal surface contained in $D \times \mathbb{R}$ and disjoint from $\Sigma$, then $S$ is a translate of $\Sigma$.

We remark that Mazet’s theorem does not apply in our case for Scherk surfaces. In fact, one of his hypothesis is that the equidistant surfaces have mean curvature pointing away from the original surface. However, an end of a Scherk surface is asymptotic to some vertical plane $\gamma \times \mathbb{R}$, where $\gamma$ is a geodesic, so the equidistant surface is asymptotic to $\gamma_s \times \mathbb{R}$, where $\gamma_s$ is an equidistant curve to $\gamma$. Hence, in the case of a Scherk surface, the mean curvature vector of an equidistant surface points toward the Scherk surface.

## 2 Preliminaries

In this section we present some basic properties of Hadamard manifolds and state some previous results which are necessary for our study (see for instance [4, 5, 6, 7] for details).

Let $M$ be a Hadamard manifold, that is, a complete simply connected Riemannian manifold with non positive sectional curvature. As is well known that there is a unique geodesic joining two points of $M$, so the concept of geodesic convexity is naturally defined for sets in $M$. 
We say that two geodesics \( \gamma_1(t), \gamma_2(t) \) of \( M \), parametrized by arc length, are *asymptotic* if there exists a constant \( c > 0 \) such that the distance \( d(\gamma_1(t), \gamma_2(t)) \) is less than \( c \) for all \( t \geq 0 \). Note that to be asymptotic is an equivalence relation on the oriented unit speed geodesics of \( M \). We will call each one of these classes a point at infinity. We will denote by \( M(\infty) \) the set of points at infinity and by \( \gamma(+\infty) \) the equivalence class of the geodesic \( \gamma(t) \).

Let us assume that \( M \) has sectional curvature bounded from above by a negative constant. Then we have two important facts:

1. For any two asymptotic geodesics \( \gamma_1, \gamma_2 \), the distance between the two curves \( \gamma_1|_{[t_0, +\infty)}, \gamma_2|_{[t_0, +\infty)} \) is zero for any \( t_0 \in \mathbb{R} \).

2. Given \( x, y \in M(\infty) \), \( x \neq y \), there exists a unique oriented unit speed geodesic \( \gamma \) such that \( \gamma(+\infty) = x \) and \( \gamma(-\infty) = y \), where \( \gamma(-\infty) \) is the corresponding point at infinity when we change the orientation of \( \gamma \).

For any point \( p \in M \), there is a bijective correspondence between the set of unit vectors in the tangent plane \( T_pM \) and \( M(\infty) \), where a unit vector \( v \) is mapped to the point at infinity \( \gamma_v(\infty) \). Equivalently, given a point \( p \in M \) and a point \( x \in M(\infty) \), there exists a unique oriented unit speed geodesic \( \gamma \) such that \( \gamma(0) = p \) and \( \gamma(+\infty) = x \). In particular, \( M(\infty) \) is bijective to a sphere.

There exists a topology on \( M^* = M \cup M(\infty) \) satisfying that the restriction to \( M \) agrees with the topology induced by the Riemannian distance. This topology is called the cone topology of \( M^* \). For more details, see [7], for instance.

Given a set \( A \subset M \), we denote by \( \partial_\infty A \) the set \( \partial A \cap M(\infty) \), where \( \partial A \) is the boundary of \( A \) for the cone topology.

In order to define horospheres we consider Busemann functions. Given a unit vector \( v \), the Busemann function \( B_v : M \to \mathbb{R} \) associated to \( v \) is defined as

\[
B_v(p) = \lim_{t \to +\infty} (d(p, \gamma_v(t)) - t).
\]

This is a \( C^2 \) convex function on \( M \) and it satisfies the following properties.

**Property 1.** The gradient \( \nabla B_v(p) \) in \( T_pM \) such that \( \gamma_v(\infty) = \gamma_w(-\infty) \).

**Property 2.** If \( w \) is a unit vector such that \( \gamma_v(\infty) = \gamma_w(\infty) \) then \( B_v - B_w \) is a constant function on \( M \).

**Definition 1.** Given a point \( x \in M(\infty) \) and a unit vector \( v \) such that \( \gamma_v(\infty) = x \), we define the horospheres at \( x \) as the level sets of the Busemann function \( B_v \).
We have the following important facts with respect to horospheres.

- By Property 2, the horospheres at \( x \) do not depend on the choice of \( v \).
- The horospheres at a point \( x \in M(\infty) \) give a foliation of \( M \), and as \( B_v \) is a convex function, each one bounds a convex domain in \( M \) called a horoball.
- The intersection between a geodesic \( \gamma \) and a horosphere at \( \gamma(\infty) \) is always orthogonal from Property 1.
- Let \( p \in M, H_x \) be a horosphere at \( x \) and \( \gamma \) be the geodesic passing through \( p \) having \( x \) as a point at infinity, then \( H_x \cap \gamma \) is the closest point on \( H_x \) to \( p \).
- If \( \gamma \) is a geodesic with points at infinity \( x, y \), and \( H_x, H_y \) are disjoint horospheres at these points then the distance between \( H_x \) and \( H_y \) agrees with the distance between the points \( H_x \cap \gamma \) and \( H_y \cap \gamma \).

Now we will restrict \( M \) to be a Hadamard surface with curvature bounded from above by a negative constant, and we will write horocycle and horodisk to mean horosphere and horoball, respectively.

Let \( \Gamma \) be an ideal polygon of \( M \), that is, \( \Gamma \) is a polygon all of whose sides are geodesics and the vertices are at infinity \( M(\infty) \). We assume \( \Gamma \) has an even number of sides \( \alpha_1, \beta_1, \alpha_2, \beta_2, ..., \alpha_k, \beta_k \). Let \( D \) be the interior of the convex hull of the vertices of \( \Gamma \), so \( \partial D = \Gamma \) and \( D \) is a topological disk. We call \( D \) an ideal polygonal domain.

At each vertex \( a_i \) of \( \Gamma \), place a horocycle \( H_i \) so that \( H_i \cap H_j = \emptyset \) if \( i \neq j \).

Each \( \alpha_i \) meets exactly two horodisks. Denote by \( \tilde{\alpha}_i \) the compact arc of \( \alpha_i \) which is the part of \( \alpha_i \) outside the two horodisks, and denote by \( |\alpha_i| \) the length of \( \tilde{\alpha}_i \), that is, the distance between these horodisks. Analogously, we can define \( \tilde{\beta}_i \) and \( |\beta_i| \).

Now define

\[
a(\Gamma) = \sum_{i=1}^{k} |\alpha_i|
\]

and

\[
b(\Gamma) = \sum_{i=1}^{k} |\beta_i|.
\]

Observe that \( a(\Gamma) - b(\Gamma) \) does not depend on the choice of the horocycles. For that, it is sufficient to observe that given two horocycles \( H_1, H_2 \) at a point \( x \in M(\infty) \) and a geodesic \( \gamma \) with \( x \) as a point at infinity, then
the distance between $H_1$ and $H_2$ agrees with the distance between the points $\gamma \cap H_1$ and $\gamma \cap H_2$.

**Definition 2.** An ideal polygon $\mathcal{P}$ is said to be inscribed in $D$ if the vertices of $\mathcal{P}$ are among the vertices of $\Gamma$, so its edges are either interior in $D$ or equal to some $\alpha_i$ or $\beta_j$.

The definition of $a(\Gamma)$ and $b(\Gamma)$ extends to inscribed polygons:

$$a(\mathcal{P}) = \sum_{\alpha_i \in \mathcal{P}} |\alpha_i| \quad \text{and} \quad b(\mathcal{P}) = \sum_{\beta_i \in \mathcal{P}} |\beta_i|.$$  

We denote by $|\mathcal{P}|$ the length of the boundary arcs of $\mathcal{P}$ exterior to the horocycles $H_i$ at the vertices of $\mathcal{P}$. We call this the truncated length of $\mathcal{P}$.

**Definition 3.** An ideal polygon $\Gamma$ is said to be admissible if the two following conditions are satisfied.

1. $a(\Gamma) = b(\Gamma)$;
2. For each inscribed polygon $\mathcal{P}$ in $D$, $\mathcal{P} \neq \Gamma$, and for some choice of the horocycles at the vertices, we have

$$2a(\mathcal{P}) < |\mathcal{P}| \quad \text{and} \quad 2b(\mathcal{P}) < |\mathcal{P}|.$$  

Moreover, an ideal polygonal domain $D$ is said to be admissible if its boundary $\Gamma = \partial D$ is an admissible polygon.

The properties of an admissible polygon are the necessary and sufficient conditions given by Gálvez and Rosenberg for the existence of an ideal Scherk graph over $D \subset M$. An ideal Scherk graph is the graph of a function in $D$ taking the values $+\infty$ on each $\alpha_i$, $-\infty$ on each $\beta_i$, and whose graph is a minimal surface.

An important tool for studying minimal (and more generally, constant mean curvature) surfaces are the formulae for the flux of appropriately chosen ambient vector fields across the surface.

Let $u$ be a function defined in $D$ whose graph is a minimal surface, and consider $X = \frac{\nabla u}{W}$ defined on $D$, where $W^2 = 1 + |
abla u|^2$. For an open domain $A \subset D$, and $\alpha$ a boundary arc of $A$, we define the flux formula across $\alpha$ as

$$F_u(\alpha) = \int_\alpha \langle X, \nu \rangle \, ds;$$  

here $\alpha$ is oriented as the boundary of $A$ and $\nu$ is the outer conormal to $A$ along $\alpha$. 

Theorem 3 (Flux Theorem). Let $A \subset D$ be an open domain. Then

1. If $\partial A$ is a compact cycle, $F_u(\partial A) = 0$.

2. If $\alpha$ is a compact arc of $A$, $F_u(\alpha) \leq |\alpha|$.

3. If $\alpha$ is a compact arc of $A$ on which $u$ diverges to $+\infty$,

   $$F_u(\alpha) = |\alpha|.$$

4. If $\alpha$ is a compact arc of $A$ on which $u$ diverges to $-\infty$,

   $$F_u(\alpha) = -|\alpha|.$$

3 Main Result

In this section we consider a Hadamard surface $M$ with bounded curvature, that is, $M$ is a complete simply connected Riemannian surface with curvature $-b^2 \leq K_M \leq -a^2 < 0$, for some constants $a, b \in \mathbb{R}$. We now can establish our main result.

Theorem 4. Let $M$ denote a Hadamard surface with bounded curvature and let $\Sigma = \text{Graph}(u)$ be an ideal Scherk graph over an admissible polygonal domain $D \subset M$. If $S$ is a properly immersed minimal surface contained in $D \times \mathbb{R}$ and disjoint from $\Sigma$, then $S$ is a translate of $\Sigma$.

To prove this theorem we follow an idea of Rosenberg, Schulze and Spruck [13], by constructing a discrete family of minimal graphs in $D \times \mathbb{R}$.

Let $\Sigma = \text{Graph}(u)$ be an ideal Scherk graph over $D$ with $\Gamma = \partial D$. Given any point $p \in D$, consider the geodesics starting at $p$ and going to the vertices of $\Gamma$. Take the points over each one of these geodesics which are at a distance $n$ from $p$. Now consider the geodesics joining two consecutive points as in Figure 1.

The angle at which two of these geodesics meet is less than $\pi$, hence we can smooth the corners to obtain a convex domain $D_n$ with smooth boundary $\Gamma_n = \partial D_n$ and such that $D_1 \subset D_2 \subset ... \subset D_n \subset ...$ is an exhaustion of $D$.

Denote by $A_n$ the annular-type domain $D_n \setminus \overline{D}_1$ and by $\Sigma_n$ the graph of $u$ restrict to $A_n$. Hence $\Sigma_n$ is a stable minimal surface, and any sufficiently small perturbation of $\partial \Sigma_n$ gives rise to a smooth family of minimal surfaces $\Sigma_{n,t}$ with $\Sigma_{n,0} = \Sigma_n$. We use this fact to the deformation of $\partial \Sigma_n$ which is the graph over $\partial A_n$ given by $\partial_1 \cup \partial_{n,t}$ for $t \geq 0$, where $\partial_1 = (\Gamma_1 \times \mathbb{R}) \cap \Sigma$, $\partial_{n,t} = (\Gamma_n \times \mathbb{R}) \cap T(t)(\Sigma)$ and $T(t)$ is the vertical
translation by height $t$. Then for $t$ sufficiently small, there exists a minimal surface $\Sigma_{n,t}$ which is the graph of a smooth function $u_{n,t}$ defined on $A_n$ with boundary $\partial_1 \cup \partial_{n,t}$. Note that $u_{n,t}$ satisfies the minimal surface equation on $A_n$ and, by the maximum principle, $\Sigma_{n,t}$ stays between $\Sigma$ and $\Sigma(t) = T(t)(\Sigma)$. We will show that there exists a uniform interval of existence for $u_{n,t}$, that is, we will prove that there exists $\delta_0 > 0$ such that for all $n$ and $0 \leq t \leq \delta_0$, the minimal surfaces $\Sigma_{n,t} = \text{Graph}(u_{n,t})$ exist.

Consider $\delta_0 > 0$ sufficiently small so that $u_{2,t}$ exists for any $t \in [0, \delta_0]$. We will show this $\delta_0$ works for all $n \geq 2$. In order to do that we will prove that for $n > 2$ the set $B_n = \{ \tau \in [0, \delta_0]; u_{n,t} \text{ exists for } 0 \leq t \leq \tau \}$ is in fact the interval $[0, \delta_0]$.

**Claim:** The set $B_n$ is open and closed. By stability $B_n$ is an open set. Now consider an increasing sequence $\tau_k \in B_n$ such that $\tau_k \to \tau$ when $k \to \infty$. The family of minimal graphs $\Sigma_{n,\tau_k}$ is contained in the region bounded by $\Sigma$ and $\Sigma(\tau)$, and $\partial_1 \subset \partial \Sigma_{n,\tau_k}$ for all $k$, then there exists a minimal surface $\Sigma_{n,\tau}$ which is the limit
of the surfaces $\Sigma_{n,\tau_k}$ with $\partial_1 \subset \partial \Sigma_{n,\tau}$. It remains to prove $\Sigma_{n,\tau}$ is a graph. As $D_2 \subset D_n$, we already know that for all $k$, $u_{n,\tau_k} \leq u_{2,\delta_0}$ in a neighborhood of $\Gamma_1$, then the gradient of $u_{n,\tau_k}$ is uniformly bounded in a neighborhood of $\Gamma_1$. Suppose there exists a sequence $p_k \in \Gamma_n$ with $u_{n,\tau_k}(p_k) \to p \in \partial_{n,\tau}$ such that $|\nabla u_{n,\tau_k}(p_k)| \to \infty$. This implies the minimal surface $\Sigma_{n,\tau}$ is vertical at $p$. Considering the horizontal geodesic $\gamma$ that passes through $p$ and is tangent to $\partial_{n,\tau}$ (recall $\partial_{n,\tau}$ is convex) we can apply the maximum principle with boundary to $\Sigma_{n,\tau}$ and $\gamma \times (\mathbb{R}, \tau]$ to conclude they coincide, which is impossible. Thus we have uniform gradient estimates for $u_{n,\tau_k}$ in $\Gamma_1 \cup \Gamma_n = \partial A_n$. By Lemma 3.1 in [13], we have uniform gradient estimates for $u_{n,k}$ on $A_n$, and then there exists a function $u_{n,\tau}$ such that $\Sigma_{n,\tau} = \text{Graph}(u_{n,\tau})$ is a minimal graph with boundary $\partial \Sigma_{n,\tau} = \partial_1 \cup \partial_{n,\tau}$, what implies $\tau \in B_n$ and the set $B_n$ is closed.

Therefore, we have proved that for all $n \geq 2$ and $0 \leq t \leq \delta_0$, there exists a function $u_{n,t}$ defined on $A_n$ such that $\Sigma_{n,t} = \text{Graph}(u_{n,t})$ is a minimal surface with boundary $\partial \Sigma_{n,t} = \partial_1 \cup \partial_{n,t}$.

Fix $t \in (0, \delta_0]$. For a fixed $n_o$, consider the sequence $\{u_{n,t}|_{A_{n_o}}\}$ for $n > n_o$. We already know $u_{n,t} \leq u_{n_o,t}$ in a neighborhood of $\Gamma_1$, hence we have uniform gradient estimates in such neighborhood. Moreover, as we have uniform curvature estimates for points far from the boundary (see [13]) and $\Gamma_n \not\subset A_{n_o}$ for all $n > n_o$, we can get uniform curvature estimates for $\Sigma_{n,t}$ on $A_{n_o}$ for all $n > n_o$. Thus there exists a subsequence $\{u_{n_j,t}|_{A_{n_o}}\}$ that converges to a function $\hat{u}_{n_o}$ defined over $A_{n_o}$ whose graph $\hat{\Sigma}_{n_o}$ is a minimal surface with $\partial_1 \subset \partial \hat{\Sigma}_{n_o}$ and $\min u_{n_o} \leq \hat{u}_{n_o} \leq u + t$ over $A_{n_o}$.

Using the same argument above, the sequence $\{u_{n_j,t}|_{A_{2n_o}}\}$ for $n_j > 2n_o$ has a subsequence $\{u_{n_{j,k},t}|_{A_{2n_o}}\}$ that converges to a function $\hat{u}_{2n_o}$ defined over $A_{2n_o}$ whose graph $\hat{\Sigma}_{2n_o}$ is a minimal surface with $\partial_1 \subset \partial \hat{\Sigma}_{2n_o}$ and $\min u_{2n_o} \leq \hat{u}_{2n_o} \leq u + t$ over $A_{2n_o}$.

As $\{u_{n_{j,k},t}|_{A_{2n_o}}\} \subset \{u_{n,t}|_{A_{n_o}}\}$, we conclude that $\hat{u}_{2n_o} = \hat{u}_{n_o}$ in $A_{n_o}$.

Continuing this argument to $A_{kn_o}$ for all $k > 2$ and applying the diagonal process, we prove that there exists a subsequence of $\{u_{n,t}\}$ that converges to a function $\hat{u}_\infty$ defined over $\Omega = D \setminus \hat{\Gamma}_1$ whose graph $\hat{\Sigma}_\infty$ is a minimal surface with $\partial \hat{\Sigma}_\infty = \partial_1$, $\min u \leq \hat{u}_\infty < u + t$ over $\Omega$, and $\hat{u}_\infty = \hat{u}_{kn_o}$ in $A_{kn_o}$ for all $k$.

For simplicity, let us write $\hat{u}$ and $\hat{\Sigma}$ to denote $\hat{u}_\infty$ and $\hat{\Sigma}_\infty$.

Note the minimal surface $\hat{\Sigma} = \text{Graph}(\hat{u})$ assumes the same infinite boundary values at $\Gamma$ as $\Sigma = \text{Graph}(u)$, the ideal Scherk graph over $D$. Consider the restriction of $u$ to $\Omega$ and continue denoting by $\Sigma$ the graph of $u$ restricted to $\Omega$. We will show that $\Sigma$ and $\hat{\Sigma}$ coincide by analysing the flux of the functions $u, \hat{u}$ across the boundary of $\Omega$, which is $\Gamma_1 \cup \Gamma$. 
Let $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_k, \beta_k$ be the geodesic sides of the admissible ideal polygon $\Gamma$ with $u(\alpha_i) = +\infty = \hat{u}(\alpha_i)$ and $u(\beta_i) = -\infty = \hat{u}(\beta_i)$. Consider pairwise disjoint horocycles $H_i(n)$ at each vertex $a_i$ of $\Gamma$ such that the convex horodisk bounded by $H_i(n + 1)$ is contained in the convex horodisk bounded by $H_i(n)$. For each side $\alpha_i$, let us denote by $\alpha_i^n$ the compact arc of $\alpha_i$ which is the part of $\alpha_i$ outside the two horodisks, and by $|\alpha_i^n|$ the length of $\alpha_i^n$, that is, the distance between the two horodisks. Analogously, we define $\beta_i^n$ for each side $\beta_i$. Denote by $c_i^n$ the compact arc of $H_i(n)$ contained in the domain $D$ and let $\mathcal{P}^n$ be the polygon formed by $\alpha_i^n, \beta_i^n$ and $c_i^n$.

As the function $u$ is defined in the interior region bounded by $\mathcal{P}^n$, and $\mathcal{P}^n$ is a compact cycle, then $F_u(\mathcal{P}^n) = 0$, by the Flux theorem. In the other hand, as $u \leq \hat{u}$ we have $F_u(\mathcal{P}^n) \leq F_{\hat{u}}(\mathcal{P}^n)$, and then $F_{\hat{u}}(\mathcal{P}^n) \geq 0$. Moreover, the flux of $\hat{u}$ across $\mathcal{P}^n$ satisfies

$$F_{\hat{u}}(\mathcal{P}^n) = \sum_i F_u(\alpha_i^n) + \sum_i F_u(\beta_i^n) + \sum_i F_u(c_i^n)$$

$$\leq \sum_i (|\alpha_i^n| - |\beta_i^n|) + \sum_i |c_i^n|.$$  

Notice that $|c_i^n| \to 0$ when $n \to \infty$ and, since $\Gamma$ is an admissible polygon, we have $\sum_i |\alpha_i^n| = \sum_i |\beta_i^n|$, for any $n$. Hence we conclude

$$F_{\hat{u}}(\mathcal{P}^n) \to 0 \quad \text{when} \quad n \to \infty.$$

Then $F_u(\Gamma) = \lim_{n \to \infty} F_u(\mathcal{P}^n) = 0 = \lim_{n \to \infty} F_{\hat{u}}(\mathcal{P}^n) = F_{\hat{u}}(\Gamma)$.

In the other hand, as $\mathcal{P}^n$ is homotopic to $\Gamma_1$, we have $F_u(\Gamma_1) = F_{\hat{u}}(\mathcal{P}^n)$ for any $n$, and we conclude that $F_{\hat{u}}(\Gamma_1) = 0$. Analogously (or using the Flux theorem as we did for $\mathcal{P}^n$), we also have $F_u(\Gamma_1) = 0$. Therefore, we have proved that the functions $u$ and $\hat{u}$ have the same flux across the boundary $\partial \Omega = \Gamma_1 \cup \Gamma$.

As $\Sigma = \text{Graph}(u)$ and $\hat{\Sigma} = \text{Graph}(\hat{u})$ are two minimal graphs over $\Omega = D \setminus \bar{D}_1$ such that $u \leq \hat{u}$, $\partial \Sigma = \partial \hat{\Sigma}$ and $F_u(\partial \Omega) = F_{\hat{u}}(\partial \Omega)$, we conclude that necessarily $u \equiv \hat{u}$ over $\Omega$, that is, $\hat{\Sigma}$ is the Scherk graph over $\Omega$ with $\partial \hat{\Sigma} = \partial_1$.

**Remark.** We have proved that for any $t \in (0,\delta_0]$ we can get a subsequence of the minimal surfaces $\Sigma_{n,t}$ that converges to a minimal surface $\hat{\Sigma}$ which is the Scherk graph over $D \setminus \bar{D}_1$ with $\partial \hat{\Sigma} = \partial_1$.

Now we are able to prove the theorem.

**Proof of Theorem.** As $\Sigma \cap S = \emptyset$, we can suppose that $S$ is entirely under $\Sigma$. Pushing down $\Sigma$ by vertical translations, we will have two possibilities: either a translate of $\Sigma$ touches $S$ for the first time in the interior, and then, by the maximum principle, we conclude they coincide, or we have that $S$ is asymptotic at infinity to a translate of $\Sigma$. Let us analyse this last case.
Without loss of generality, we can suppose that $S$ is asymptotic at infinity to $\Sigma$. If $S \neq \Sigma$, then $S$ is proper there is a point $p_0 \in \Sigma$ and a cylinder $C = B_{\Sigma}(p_0, r_0) \times (-r_0, r_0)$ for some $r_0 > 0$ such that $S \cap C = \emptyset$, where $B_{\Sigma}(p_0, r_0)$ is the intrinsic ball centered at $p_0$ with radius $r_0$. We can assume $r_0$ is less than the injectivity radius of $\Sigma$ at $p_0$. In our construction of the surfaces $\Sigma_{n,t}$, we can choose $D_1$ so that $\partial_1 \subset B_{\Sigma}(p_0, \frac{r_0}{2})$, and take $t = \min\{\frac{r_0}{2}, \delta_0\}$.

Observe that when we translate $\Sigma_{n,t}$ vertically downwards by an amount $t$, the boundaries of the translates of $\Sigma_{n,t}$ stay strictly above $S$. Thus, by the maximum principle, all the translates remain disjoint from $S$. We call $\Sigma_{n,t}$ this final translate with boundary $\partial \Sigma'_{n,t} = \partial_1' \cup \partial_n'$, where $T(t)(\partial_1') = \partial_1 \subset \Sigma$ and $\partial_n' \subset \Sigma$. Hence, all the surfaces $\Sigma'_{n,t}$ lie above $S$ and, as we proved before, there exists a subsequence of $\Sigma'_{n,t}$ that converges to the ideal Scherk graph $\Sigma'$ defined over $D \setminus \bar{D_1}$ with $T(t)(\Sigma') = \Sigma$. In particular, we conclude that $S$ lies below $\Sigma'$, which yields a contradiction, since we are assuming that $S$ is asymptotic at infinity to $\Sigma$.

□

References

[1] U. Abresch and H. Rosenberg. Generalized Hopf differentials. Mat. Contemp., 28, 1-28, 2005.

[2] B. Daniel and L. Hauswirth. Half-space theorem, embedded minimal annuli and minimal graphs in the Heisenberg group. Proc. Lond. Math. Soc. (3), 98(2), 445-470, 2009.

[3] B. Daniel, W. H. Meeks and H. Rosenberg. Half-space theorems for minimal surfaces in $\text{Nil}_3$ and $\text{Sol}_3$. J. Differential Geom., 88, 41-59, 2011.

[4] P. Eberlein. Geodesic flows on negatively curved manifolds II. Trans. Am. Math. Soc., 178, 57–82, 1973.

[5] P. Eberlein. Geometry of nonpositively curved manifolds. Chicago Lectures in Mathematics, 1996.

[6] P. Eberlein and B. O’Neil. Visibility manifolds. Pacific J. Math., 46, 45–109, 1973.

[7] J. A. Gálvez and H. Rosenberg. Minimal surfaces and harmonic diffeomorphisms from the complex plane onto certain Hadamard surfaces. Amer. J. Math., 132(5), 1249–1273, 2010.
[8] L. Hauswirth, H. Rosenberg and J. Spruck. *On complete mean curvature $\frac{1}{2}$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$*. Comm. Anal. Geom., 16(5), 989-1005, 2008.

[9] D. Hoffman and W. H. Meeks, III. *The strong halfspace theorem for minimal surfaces*. Invent. Math., 101(2), 373-377, 1990.

[10] L. Mazet. *A general halfspace theorem for constant mean curvature surfaces*. Amer. J. Math., 35, 801–834, 2013.

[11] B. Nelli and H. Rosenberg. *Minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$*. Bull. Braz. Math. Soc. (N.S.), 33(2), 263-292, 2002.

[12] B. Nelli and H. Rosenberg. *Errata: "Minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$"* [Bull. Braz. Math. Soc. (N.S.), 33(2002),no 2, 263-292] Bull. Braz. Math. Soc. (N.S.), 38(4), 661-664, 2007.

[13] H. Rosenberg, F. Schulze and J. Spruck. *The half-space property and entire positive minimal graphs in $M \times \mathbb{R}$*. To appear in J. Differential Geom.

[14] H. Rosenberg, R. Souam and E. Toubiana. *General curvature estimates for stable $H$-surfaces in 3-manifolds and applications*. J. Differential Geom., 84, 623-648, 2010.

**Instituto Nacional de Matemática Pura e Aplicada (IMPA)**  
Estrada Dona Castorina 110, 22460-320, Rio de Janeiro-RJ, Brazil  
*Email adress: anamaria@impa.br*