Supersymmetric $\sigma$-Models on Toric Varieties: 
A Test Case.

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ABSTRACT

In this letter we study supersymmetric $\sigma$-models on toric varieties. These manifolds are generalizations of $\mathbb{P}^n$ manifolds. We examine here $\sigma$-models, viewed as gauged linear $\sigma$-models, on one of the simplest such manifold, the blow-up of $\mathbb{P}^2_{(2,1,1)}$, and determine their properties using the techniques of topological-antitopological fusion. We find that the model contains solitons which become massless at the singular point of the theory where a gauge symmetry remains unbroken.
1 Introduction

We aim in this paper to study $N = 2$ $\sigma$-models on a particular class of toric varieties, weighted projective spaces, which are generalizations of ordinary projective spaces. These spaces have been explored extensively by string theorists, in relation to string compactifications on Calabi-Yau manifolds. From a string theory point of view, one is typically interested in Calabi-Yau manifolds that can be realized as hypersurfaces defined by polynomials in weighted projective spaces. These conformal models form a large class of consistent string vacua. More recently, considerations of mirror symmetry have led physicists and mathematicians to study the ambient space of the Calabi-Yau manifold, or the full toric variety.

These nonlinear $\sigma$-models with Kähler target space can be obtained as the low-energy limit of certain two dimensional $N = 2$ supersymmetric models with abelian gauge symmetry, or gauged linear $\sigma$-models (GLSM), as shown by Witten. These gauged linear $\sigma$-models can be twisted to give topological models, which in turn can be used to calculate instanton expansions for correlation functions in topological nonlinear $\sigma$-models.

Weighted projective spaces contain orbifold singularities. We can however replace each singular point of the singular locus by a $\mathbb{P}^1 \sim S^2$ to give a smooth Kähler manifold. These singularities may cause some of the correlation functions to diverge at these points because the infinite instanton sums needed to calculate the correlators can contain singularities.

Recently there has been a great deal of interest concerning the nature of the appearance of physical singularities at points in moduli spaces and this question has been examined in $N = 2$ susy gauge theories and type II string theory compactifications on various manifolds. The explanation is argued to be that the apparent singularity is due to nonperturbative massive states becoming massless at these singular points.

Here we will explore the physics of field theories which contain similar types of singularities, supersymmetric $\sigma$-models on toric varieties in two spacetime dimensions. One interesting question is whether we can make any predictions regarding the behavior of the theory near the singular points.

For this, we look at $\sigma$-models on one of the simplest such toric variety, the space obtained by resolving the singularity of the weighted projective space $\mathbb{P}^2_{(2,1,1)}$ or $\mathbb{P}^2/\mathbb{Z}_2$. We view these models as gauged linear $\sigma$-models and explore their properties using the powerful techniques of topological-antitopological fusion developed by Cecotti and Vafa. These methods allow to study various characteristics of the model along the whole renormalization group flow. The parameters defining the flow are the two couplings $\beta$ and $\alpha$ of the GLSM Lagrangian. These couplings are related to the two homology cycles corresponding to the $\mathbb{P}^2$ and to the blowing up of its singular point. The correlation functions of the model depend on these two parameters. The one singularity of the model appears as a pole in the correlation function at $\beta = 1/4$. We find, by flowing to the infra-red, that the non-linear $\sigma$-model contains Bogolmony solitons whose masses are determined to be proportional to $(1 - 4\beta)^{1/2}$. This shows that, as we tune the coupling $\beta$ towards its singular value of $1/4$, the solitons of the model become massless. At this point, a con-
tinuous gauge symmetry of the GLSM is restored and gives rise to flat directions. Thus we have another example of a field theory where the apparent physical singularity can be explained by a gauge symmetry enhancement with the appearance of massless states. Here, due to the relative simplicity of the model, we can explicitly see what happens.

The organization of the paper is as follows. We start by reviewing the geometrical data needed to define the toric variety and its classical cohomology. In section three, we review the Lagrangian description of non-linear supersymmetric $\sigma$-models on toric varieties, obtain the quantum cohomology and discuss some of the physical properties of the model. In section four we determine the topological-antitopological fusion equations which we will then solve in particular cases. The first case is a solution of the model when we set $\beta$ to zero (section five). This case might be seen as an interesting Landau-Ginsburg model on its own, but we mainly give the solution as an example of a completely solvable case. Next, in section six, we set out to determine the behavior of the model near the singularity. This turns out to be possible because of a simplification of the equations when we take $\beta$ to be real. We obtain, through the asymptotic behavior of the equations, the masses of the solitons and find their dependence on the parameters. Finally we summarize our results and their consequences.

2 The toric data for the blow-up of $\mathbb{P}^2_{(2,1,1)}$

This section is rather technical and is a review of material found in various papers (see [17, 1, 8, 18, 16, 9]).

Toric varieties can be defined in terms of simple combinatorial data. This geometrical data encodes information about various aspects of the space, such as its cohomology. The toric variety $V$ corresponding to a complex weighted projective space is described as

$$\mathbb{P}^{n-1}(k_1, k_2, \ldots, k_n) = \mathbb{C}^{n+1} - \{0\} / \mathbb{C}^*$$

(2.1)

with $\mathbb{C}^*$ acting by $\lambda(z_1, \ldots, z_n) = (\lambda^{k_1} z_1, \ldots, \lambda^{k_n} z_n)$, for all nonzero $\lambda$, where $(z_1, \ldots, z_n)$ are local holomorphic coordinates on $V$. These spaces have orbifold singularities, due to the identifications $(z_1, \ldots, z_n) \simeq (\lambda^{k_1} z_1, \ldots, \lambda^{k_n} z_n)$, except for the case when all weights are unity, which corresponds to ordinary projective space $\mathbb{P}^n$. In fact,

$$\mathbb{P}^{n-1}_{(k_1, \ldots, k_n)} = \frac{\mathbb{C}^{n-1}}{\mathbb{Z}_{k_1} \times \ldots \times \mathbb{Z}_{k_n}}$$

(2.2)

as can be seen by setting $z_j = (\zeta)^{k_j}$ such that $(\zeta_1, \ldots, \zeta_n) \simeq \lambda(\zeta_1, \ldots, \zeta_n)$, and identifying $\zeta_j \simeq e^{2\pi i k_j} \zeta_j$. These identifications lead to singular sets.

A more general definition of a $n$-dimensional toric variety is as the quotient space

$$V = (Y - F_\Delta)/T_\Delta$$

(2.3)

where $Y = \mathbb{C}^n$, $T_\Delta \sim \mathbb{C}^{(n-d)}$ acts diagonally on the coordinates of $Y$ by

$$g_\alpha(\lambda) : x_i \rightarrow \lambda^{Q_i} x_i \quad a = 1, \ldots, (n-d), \quad i = 1, \ldots, d,$$

(2.4)
and \( F_\Delta \) is a subset of \( Y - C^* \) which is a union of certain intersections of coordinate hyperplanes. The combinatorial data \( \Delta \) defining \( V \) determines the intersection of coordinate hyperplanes and the integers \( Q^*_a \) which specify the representation of the gauge group.

The weighted projective space \( \mathbb{P}^2_{(2,1,1)} \) is identified with \( \mathbb{P}^2 / \mathbb{Z}_2 \). Indeed, consider the three complex homogeneous coordinates \((z_1, z_2, z_3)\) describing the space. One has \((z_1, z_2, z_3) \simeq (\lambda^2 z_1, \lambda z_2, \lambda z_3)\) which, with \( \lambda = -1 \), becomes \((z_1, z_2, z_3) \simeq (z_1, -z_2, -z_3)\). Take now a neighborhood of the point \((1, 0, 0)\); there is a \( \mathbb{Z}_2 \) identification on the space and the action fixes \((1, 0, 0)\). The singular set consists of the point \((1, 0, 0)\). In order to obtain a smooth projective space, this point is blown up to a \( \mathbb{P}^1 \). This process of blowing up introduces new \((1, 1)\) forms which contribute (with the Kähler form) to the cohomology of the smooth toric variety.

The toric data can be described in terms of a ‘fan’ which is a collection of two-dimensional cones \( \sigma_i \). Each cone is spanned by two one-dimensional cones (or simplices), and each one-dimensional cone can be identified with a complex variable \( z_i \). The one-dimensional simplices for the weighted projective space \( \mathbb{P}^2_{(2,1,1)} \) are given by the vectors \( \{ \vec{v}_1 = (1, 0) \equiv z_1, \vec{v}_2 = (0, 1) \equiv z_2, \vec{v}_3 = (-2, -1) \equiv z_3 \} \). The area defined by the triangles formed by two such vectors and the origin are not equal. Removing the singularity involves making these areas equal and this is done by adding the vector \( \vec{v}_4 = (-1, 0) \equiv z_4 \). The vector is called the ‘exceptional divisor’ in the language of mathematicians and is the average \((\vec{v}_1 + \vec{v}_2)/2\). The set \( F \) contains points which are removed so that there are no fixed points. Here

\[
F = \{ z_1 = z_4 = 0 \} \cup \{ z_2 = z_3 = 0 \}, \quad \text{(2.5)}
\]

\((F\) is defined by taking the union of all sets obtained by setting to zero all coordinates corresponding to vectors of a primitive collection. A primitive collection consists of vectors not generating a single cone.\)

We now need to mod out by the action of \( T = (\mathbb{P}^*)^2 \). Let \( D \in \mathbb{Z}^n \) be the sublattice of vectors \( d = (d_1, \ldots, d_n) \) such that \( \sum_i d_i v_i = 0 \). Choosing a basis \( \{Q^1, \ldots, Q^{n-d}\} \) for \( D \) gives the \( T \) action. In our case, \( Q^1 = (1, 0, 0, 1) \) and \( Q^2 = (0, 1, 1, -2) \). Thus

\[
\begin{align*}
g_1(\lambda) & \rightarrow (\lambda z_1, z_2, z_3, \lambda z_4) \quad \text{(2.6)} \\
g_2(\lambda) & \rightarrow (z_1, \lambda z_2, \lambda z_3, \lambda^{-2} z_4) \quad \text{(2.7)}
\end{align*}
\]

We can now obtain the cohomology of the manifold by proceeding as explained. The cohomology \( H^*(V) \) is generated by \( H^2(V) \) under the intersection product. The group \( H^2(V) \) is generated by classes \( \xi_i, \quad i = 1, \ldots, n \), dual to the divisors \( \{x_i = 0\} \), subject to linear relations. The dimension of \( H^2(V) \) is \( n - d \). A basis \( \eta_a \) of \( H^2(V) \) is such that

\[
\xi_i = \sum_{a=1}^{n-d} Q^a_i \eta_a. \quad \text{(2.8)}
\]

In our example, \( \xi_1 = \eta_1, \ \xi_2 = \eta_2, \ \xi_3 = \eta_2 \) and \( \xi_4 = \eta_1 - 2\eta_2 \).
The nonlinear relations in the ring $H^*(V)$ are determined as follows. For each irreducible component of $F$, described as $\{x_a = 0 \; | \; a \in A\}$ for some set $A \in \{1, \ldots, n\}$, there is a relation $\prod_{a \in A} \xi_a = 0$. These produce the following classical ring relations

$$\xi_2 \xi_3 = \eta_2^2 = 0, \quad \xi_1 \xi_4 = (\eta_1 - 2\eta_2)\eta_1 = 0. \quad (2.9)$$

The correlations functions are also determined by the toric data. Take a collection of $d$ distinct coordinate hyperplanes $\{x_i = 0\}, \ldots, \{x_i = 0\}$ which do intersect on $V$, then $\langle \xi_i, \ldots, \xi_i \rangle_V = 1$, if $V$ is smooth. This implies $\langle \eta_1 \eta_2 \rangle = 1, \langle \eta_1^2 \rangle = 2$.

### 3 The nonlinear $\sigma$-model and quantum cohomology

Recall that the $\mathbb{CP}^n$ and Grassmannian $\sigma$-models both have descriptions as gauged $N = 2$ models\cite{13, 19, 20}

$$\mathcal{L} = \int d^4 \theta \left[ \sum_{i,a} \bar{S}_{ia} e^{-V} S_{ia} + \alpha \text{Tr} V \right] \quad (3.1)$$

with $a$ a ‘flavour’ $SU(N)$ index and $i$ a ‘gauge’ $U(k)$ index ($a = 1$ and $\alpha \text{Tr} V = \frac{A}{2\pi} V$ for the $\mathbb{CP}^n$ model). The $S_i$ are chiral superfields, and $V$ is a real vector (or matrix) superfield.

Integrating out the superfields $S_i$ in (3.1) for the $\mathbb{CP}^n$ models, one obtains an effective action which, by gauge invariance, contains only the field-strength superfields $X$ and $\bar{X}$ since $V$ is not gauge invariant:

$$S_{\text{eff}} = \frac{N}{2\pi} \int d^2 x \left\{ \int d^2 \theta W(X) + \int d^2 \theta W(\bar{X}) + \int d^4 \theta [Z(X, \bar{X}, \Delta, \bar{\Delta})] \right\} \quad (3.2)$$

with

$$W(X) = \frac{1}{2\pi} X (\log X^N - N + A(\mu) - i\theta), \quad (3.3)$$

$A$ is a renormalized coupling, $\theta$ the instanton angle, and $X = D_L \bar{D}_R V$, $\bar{X} = D_R \bar{D}_L V$.

This action has the form of a Landau-Ginsburg model. A $N = 2$ supersymmetric theory in two dimensions admits a Landau-Ginsburg description if it has a superspace Lagrangian such that

$$\mathcal{L} = \int d^4 \theta \sum_i \phi_i \bar{\phi}_i + \int d^2 \theta W(\phi_i) + h.c. \quad (3.4)$$

where $\phi_i, \bar{\phi}_i$ are chiral and antichiral superfields and the superpotential $W$ is an analytic function of the complex superfields. The ground states of the theory are $dW(\phi) = 0$.

The chiral ring is the ring of polynomials generated by the $\phi_i$ modulo the relations $dW(\phi)/d\phi_i = D\bar{D}\phi_i \sim 0$. (For a review, see \cite{22, 23}).

For the $\mathbb{CP}^n$ model, the chiral ring is thus the powers of $X$ mod $dW = 0$, or $X^N = e^{-A+i\theta} \equiv \beta$.

The Grassmannian $\sigma$-models admit a LG description as well\cite{13, 14} with $W$ given by

$$W_f(\lambda_1, \lambda_2, \ldots, \lambda_k) = \frac{1}{2\pi} \sum_{j=1}^k \lambda_j (\log \lambda_j^N - N + A(\mu) - i\theta). \quad (3.5)$$
The gauge-invariant fields are now polynomials in the eigenvalues $\lambda_m$ of the field-strengths $\lambda$ and are generated by the elementary symmetric functions

$$X_i(\lambda) \equiv \sum_{1 \leq l_1 < l_2 < \ldots < l_i \leq k} \lambda_{l_1} \lambda_{l_2} \ldots \lambda_{l_i} \quad (i = 1, \ldots, k) \quad (3.6)$$

The ring relations are $\lambda_N = \text{const.}$ and the quantum cohomology of the Grassmannian $\sigma$-models are generated by the elementary symmetric functions $X_i$’s.

Similarly, for the nonlinear $\sigma$-model with target space the toric variety $V$, there exists a manifestly $N = 2$ supersymmetric gauged linear $\sigma$-model with target space $Y$ and gauge group $G = U(1)^{(n-d)}$ with $G_c = T$.

The Lagrangian is

$$L = \int d^4\theta \left[ \sum_i \bar{S}_i e^{i(2 \sum_{a=1}^{n-d} Q_i^a V_a)} S_i - \sum_{a=1}^{n-d} r_a V_a \right] \quad (3.7)$$

where the $n$ chiral matter multiplets $S_i$ with charge $Q_i^a$ under $G$ are coupled to the $n - d$ abelian gauge superfields $V_a$.

Integrating out the chiral superfields, one is left with the superpotential

$$W(\Sigma) = \frac{1}{2\sqrt{2}} \sum_{a=1}^{n-d} \Sigma_a \left( i r_a - \frac{1}{2\pi} \sum_{i=1}^{n} Q_i^a \log(\sqrt{2} \sum_{b=1}^{n-d} Q_i^b \Sigma_b / \Lambda) \right) \quad (3.8)$$

where $\Sigma_a = \frac{1}{\sqrt{2}} D_+ D_- V_a$ are the (twisted chiral) gauge-invariant field strengths associated with the gauge fields $V_a$ and have component expansions

$$\Sigma = \sigma - i\sqrt{2}(\theta^+ \bar{\lambda}_+ + \bar{\theta}^- \lambda_-) + \sqrt{2}\theta^+ \bar{\theta}^- (D - f) + \ldots \quad (3.9)$$

As we will see, the gauge-invariant field strengths will again generate the cohomology of the model, with relations determined by $dW(\Sigma) = 0$.

The coupling $\tau_a \equiv i r_a + \frac{\theta_a}{2\pi}$, where $r_a$ is a renormalized coupling and $\theta_a$ is the instanton angle.

As stressed in [8], the model reduces in the low energy limit to the nonlinear $\sigma$-model with target space $V$ when $r_a$ lie within a certain cone $K_c$ of $V$. This follows from finding the space of classical ground states of the theory by setting the potential for the bosonic fields to zero. Doing so induces a relation on the $r_a$ through the auxiliary fields $D_a$

$$U = \sum_{a=1}^{n-d} \frac{(D_a)^2}{2e^2} + 2 \sum_{a,b=1}^{n-d} \bar{\sigma}_a \sigma_b \sum_{i=1}^{n} Q_i^a Q_i^b |\phi_i|^2, \quad (3.10)$$

where $D_a = -e^2 (\sum_{i=1}^{n} Q_i^a |\phi_i|^2 - r_a) \equiv 0$ is the condition for vanishing energy.

In our case, the condition for the $r_a$ to lie in $K_c$ are $r_1 \geq 0$, $r_2 + 2r_1 \geq 0$. The smooth phase, corresponding to $r_a$ in the Kähler cone $K_V$ is determined by the conditions $r_1 > 0$, $r_2 > 0$.

It appears classically that the space of vacua reduces to a point for $r_a \equiv 0$ and that
supersymmetry is spontaneously broken for negative \( r_a \) as the energy can no longer vanish. However, quantum mechanically, one finds a smooth continuation to negative \( r_a \) with unbroken supersymmetry.\(^{10, 11, 8}\) The interaction (3.8) and the constraints on the chiral fields derived from it are thus valid for values of \( r_a \) outside of \( K_c \), when we analytically continue to other regions in parameter space, i.e. here for \( r_a \) negative. In fact, the formalism of the topological-antitopological equations (which we describe in the next section) does not distinguish between the sign of \( r_a \) and allows naturally to go beyond zero radii and resolve the singularities of classical geometries.\(^{14}\)

The quantum cohomology of the toric variety is obtained by setting \( dW(\Sigma_a) = 0 \). This produces the constraints

\[
\prod_{i=1}^{n} \left( \sum_{b=1}^{n-d} Q_i^b \Sigma_b \right)^{Q_i^a} = e^{2\pi i r_a} \equiv q_a, \quad a = 1, \ldots, n - d, \tag{3.11}
\]

and the classical ring relations (2.9) for \( \mathbb{P}^2_{(211)} \) are changed to

\[
\Sigma_1 (\Sigma_1 - 2\Sigma_2) = \alpha, \quad \Sigma_2^2 = \beta (\Sigma_1 - 2\Sigma_2)^2, \tag{3.12}
\]

in the quantum cohomology ring, with the deformation parameters \( q_1 \equiv \alpha \), \( q_2 \equiv \beta \) functions of the GLSM couplings \( r_1 \equiv ir_1 + \frac{\theta_1}{2\pi}, \quad r_2 \equiv ir_2 + \frac{\theta_2}{2\pi} \). The parameters \( r_1, r_2 \) represent the areas of the two homology cycles corresponding respectively to one on \( \mathbb{P}^2 \) and the exceptional divisor. The correlation functions become, using the relations in the ring,

\[
\langle \Sigma_1 \Sigma_2 \rangle = 1, \quad \langle \Sigma_1^2 \rangle = 2, \quad \langle \Sigma_2^2 \rangle = \frac{-2\beta}{1 - 4\beta}. \tag{3.13}
\]

We notice here a singularity of the model at \( \beta = 1/4 \). As explained in \([10, 8]\), such a singularity arises because the instanton sums contributing to the correlator become infinite with possible singularities if \( \sum_i Q_i^a = 0 \) (here for \( a = 2 \)). This happens because there are then solutions to \( D_a = 0 \) which leave a continuous subgroup of \( G \) (here \( g_2 \)) unbroken and give rise to flat directions \( (r_2 = 0) \) where the space of susy ground-states is non-compact and the theory singular. Quantum corrections will have the effect of shifting \( \tau^a \) to \( \tau^a_{\text{eff}} = \tau^a + \frac{i}{2\pi} \sum_i Q_i^a \ln Q_i^a \) and therefore in our case, these will lead to a singularity at \( \tau_2 = \frac{i}{\pi} \ln 2 \), \( \theta_2 = 0 \), or \( \beta = 1/4 \).

Other toric varieties of the form \( \mathbb{P}^{n-1}_{(2,...,2,1,1)} \) (with \([n - 2] \) 2’s) will also have this single singularity and the following results can be assumed to generalize to these other models.

We are now in a position to study the model using the techniques of topological-antitopological fusion.

4 \ The tt* equations for the model

The \( tt^* \) equations describe the way in which a certain hermitian metric \( g_{ij} \) changes along the renormalization group flow. The topological-antitopological metric \( g_{ij} \equiv (\bar{j}|i) \) is just
the inner product on the supersymmetric ground states $|i\rangle$ and $|\bar{j}\rangle$ of the theory. These Ramond ground states are in one-to-one correspondence with the chiral superfields. The metric can be thought of as a generalization of the Zamolodchikov metric away from the conformal point. This metric and a new index derived from it are helpful for understanding various properties of the model, like the scale and coupling dependence and the soliton spectrum (see [12, 19] for a review).

As a basis for the chiral ring, which is generated by $\{\Sigma_1, \Sigma_2\}$, we take

$$\mathcal{R} = \{1, \Sigma_1, \Sigma_2, \Sigma_1\Sigma_2\}. \quad (4.1)$$

The topological metric or two-point function $\eta_{ij}$ is related to $g_{ij}$ by the ‘reality constraints’

$$\eta^{-1}g(\eta^{-1}g)^* = 1. \quad (4.2)$$

The metric is determined by the following differential equations

$$\bar{\partial}_j(g\partial_i g^{-1}) = [C_i, gC_j^\dagger g^{-1}] \quad (4.3)$$

$$\partial_i C_j - \partial_j C_i + [g(\partial_i g^{-1}), C_j] - [g(\partial_j g^{-1}), C_i] = 0. \quad (4.4)$$

The $C_i$ represent the action on the chiral ring of the operators corresponding to a perturbation by the couplings. In the present case, we have two couplings $\alpha$ and $\beta$, and the matrices $C_\alpha$ and $C_\beta$ are (with $a = -2\beta/(1 - 4\beta)$)

$$C_\alpha = -\frac{1}{\alpha}\begin{pmatrix} 0 & 1 & 0 & 0 \\ \alpha & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 2\alpha \beta & -\alpha 2\beta/a & 0 \end{pmatrix}, \quad C_\beta = -\frac{1}{\beta}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\alpha a/2 & 0 & 0 & a \\ 0 & \alpha \beta & -2\alpha \beta & 0 \end{pmatrix}. \quad (4.9)$$

The two-point function is, in view of (3.13)

$$\eta = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & a & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (3.13)$$

$\Sigma_1$ and $\Sigma_2$ having same (complex) dimension, we expect off-diagonal elements in the metric (with $g_{21} = g_{12}^*$)

$$g = \begin{pmatrix} g_{00} & 0 & 0 & 0 \\ 0 & g_{11} & g_{12} & 0 \\ 0 & g_{21} & g_{22} & 0 \\ 0 & 0 & 0 & g_{33} \end{pmatrix}. \quad (4.10)$$

The diagonal entries $g_{11}$ will be a function only of $(|\alpha|, |\beta|)$ since total chiral charge is zero, and chiral charge non-conservation is proportional to instanton number. The reality...
The \(tt^*\) equations and reality constraints will in general be simplest in the so-called ‘flat basis,’ the basis where the two-point functions are independent of the couplings \(\tau_a\).

We find the flat basis to be \(\mathcal{R} = \{1, \Sigma_1/\sqrt{2}, (\Sigma_1 - 2\Sigma_2)/\sqrt{2\delta}, \Sigma_1\Sigma_2\}\) with \(\delta = \sqrt{2a - 1}\). \(\eta\) becomes

\[
\eta = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\]

and the reality constraints simplify to

\[
g_{11}^2 + g_{12}^2 = 1 \\
g_{21}^2 + g_{22}^2 = 1 \\
g_{21}g_{11} = -g_{12}g_{22}
\]

These imply \(g_{11}^2 = g_{22}^2\) (or \(g_{22} = -(+)g_{11}\) if \(g_{12}\) is real (imaginary)).

Calling \(\overline{g_{ij}}\) this new basis, we can go from one basis to the other

\[
\overline{g_{11}} = \frac{1}{2}g_{11}, \quad \overline{g_{00}} = g_{00} \\
\overline{g_{12}} = \frac{1}{2\delta^*}(g_{11} - 2g_{12}) \\
\overline{g_{22}} = \frac{1}{2|\delta|^2}(g_{11} - 2(g_{12} + g_{21}) + 4g_{22})
\]

Making the appropriate changes in the \(C_i\) matrices for the new basis, we can then easily obtain the \(tt^*\) equations (4.3) in the flat basis.

5 The solution of the model for \(\beta = 0\)

The \(tt^*\) equations determine solutions for the metric \(g_{ij}\) from which one can understand many properties of the model. For example, the asymptotic behavior of the metric in the infra-red limit gives the masses of the solitons of the model. The \(tt^*\) equations are in general very complicated and few analytical solutions are known. We find here two interesting special cases where the equations simplify and the model is solvable. The first case is when we set \(\beta \equiv 0\). This case might be more than just interesting as a boundary condition for the general solution of the metric. It might also correspond to some kind
of Landau-Ginsburg theory. Now, the reality constraints (4.5) can be solved completely (since $a \equiv 0$) and they imply (if $g_{22} \neq 0$)

$$g_{11} = g_{22} + g_{22}^{-1}, \quad g_{12} = g_{22}.$$  (5.1)

The $t \bar{t}^*$ equations reduce to

$$- \partial_\alpha \partial_\bar{\alpha} \ln g_{00} = \frac{1}{|\alpha|^2} g_{00}^{-1} \left[ g_{22}^{-1} + g_{22} \right] - g_{00} g_{22}$$

$$- \partial_\alpha \partial_\bar{\alpha} \ln g_{22} = \frac{1}{|\alpha|^2} g_{00}^{-1} \left[ g_{22}^{-1} - g_{22} \right] - g_{00} g_{22}.$$  (5.2)

But this system of equations is familiar. Calling $q_0 = \ln g_{00}$, $q_2 = \ln g_{22}$, $q_{02} = q_0 + q_2 = \ln g_{00} g_{22}$ and $q_{02}' = q_2 - q_0 = \ln g_{22}/g_{00}$, we have

$$\partial_\alpha \partial_\bar{\alpha} q_{02}' = \frac{2}{|\alpha|^2} e^{q_{02}'}$$

$$\partial_\alpha \partial_\bar{\alpha} q_{02} = 2 \left[ e^{q_{02}} - e^{-q_{02}} \right] |\alpha|^2.$$  (5.3)

The first of (5.3) reduces to the Liouville equation by the transformation

$$\ln g_{22}/g_{00} = 2q_{02}' + \ln |\alpha|^2; \quad \partial_\alpha \partial_\bar{\alpha} q_{02}' = e^{q_{02}'}$$

whose general solution is [24]

$$q_{02}' = \ln \left[ 2 \left| \frac{dg}{d\alpha} \right| \frac{1}{(1 - gg^*)} \right]$$  (5.5)

where $g(\alpha)$ is an analytic function of $\alpha$. If we do not want any angular dependance in the solution so that the metric depends only on $|\alpha|$, we need $g = \alpha^m$, $m$ real. However, this solution is singular along the curve $gg^* = 1$ and the real nonsingular solution is thus for $m \neq 0$. Then

$$g_{22}/g_{00} = \frac{4m^4 |\alpha|^{2m}}{(1 + |\alpha|^{2m})^2}.$$  (5.6)

The second of (5.3) can also be put in a recognizable form. Setting $z = 4\sqrt{\alpha}$ and $\ln g_{00} g_{22} = q_{02}' - \ln |\alpha|$, we get the Sinh-Gordon equation

$$\partial_z \partial_{\bar{z}} q_{02}' = \sinh q_{02}'.$$  (5.7)

The asymptotic limits of this equation are known. We look for solutions which are real, regular and non-zero as $z \to 0$. Following [12], the solutions to the Sinh-Gordon equations are classified by their asymptotic behavior as $|z| \to 0$ (or $r_1 \to \infty$),

$$q_{02}'(|z|) \approx r \log |z| + s + \mathcal{O}(|z|^{2-|r|}) \quad \text{for} \quad |r| < 2$$

$$q_{02}'(|z|) \approx \pm 2 \log |z| \pm 2 \log \left[ - \left( \log \frac{|z|}{2} + \gamma \right) \right] + \mathcal{O}(|z|^{2|\log |z||}) \quad \text{for} \quad r = \pm 2.$$  (5.8)
where \( r \) and \( s \) are related by 
\[
e^{s/2} = \frac{1}{2} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)},
\]
and \( \gamma \) is Euler’s constant.

In order for the metric to have no poles as a function of \( \alpha \), we need \( r = -2 \) (and \( m = 1 \)). So
\[
g_{00}g_{22} = \left[4|\alpha| \left(-\ln 2|\alpha|^{1/2} - \gamma\right)\right]^{-2}
\]
and
\[
g_{22} = \frac{1}{4} \left[(1 + |\alpha|^2) \left(-\ln 2|\alpha|^{1/2} - \gamma\right)\right]^{-2},
g_{00} = (1 + |\alpha|^2)^2 \left[8|\alpha|^2 \left(-\ln 2|\alpha|^{1/2} - \gamma\right)\right]^{-2}.
\]

For \( z \to \infty (r_1 \to -\infty) \), the general solution to (5.7) is
\[
q'_{02}(\alpha) \sim -\frac{2}{\sqrt{\pi}} \sin(\pi r/4) \frac{1}{\sqrt{z}} \exp[-2z] + \ldots
\]
\[
\sim \frac{2}{\sqrt{\pi}} \frac{1}{\sqrt{z}} \exp[-2z]
\]
Since the leading asymptotic behavior of the metric is determined by the one-soliton contributions, we have fundamental solitons of mass \( m = 2|\alpha| \). We get
\[
g_{22} = \frac{4|\alpha|}{(1 + |\alpha|^2)^2} \left[1 + \sqrt{\frac{2}{\pi}} |\alpha|^{-1/4} \exp -4|\alpha|^{1/2}\right],
g_{00} = \frac{|\alpha|}{4} \left[1 + \sqrt{\frac{2}{\pi}} |\alpha|^{-1/4} \exp -4|\alpha|^{1/2}\right].
\]
which suggest that the model contains solitons of mass \( 4|\alpha|^{1/2} \).

6 The behavior of the model near the singularity

For the general case, the \( tt^* \) equations look quite complicated to solve. As we presently see, we now describe a case where we can obtain a result which will allow us to predict the general behavior of the theory. This happens when we take the metric to be diagonal in the flat basis \((g_{12} = 0)\). In this case \( \beta \) is real and the equations are tractable. Since the one singular point of the theory occurs at \( \beta = 1/4 \) (or \( \tau_2 = \frac{i}{\pi} \ln 2, \theta = 0 \)), we can study what happens in the vicinity of the singularity.

Recall that if we take \( g_{12} = 0 \), we can have either \( g_{11} = g_{22} = 1 \) or \( g_{11} = -g_{22} = 1 \) (see 4.6). If we look at the full \( tt^* \) equations in the flat metric, we notice that the condition for a solution to exist when \( g_{12} = 0 \) is \( \delta - \delta^* = 0 \) if \( g_{11} = g_{22} = 1 \) and \( \delta + \delta^* = 0 \) when \( g_{11} = -g_{22} = 1 \). In either case, for \( \beta \) real, the condition \( g_{12} \equiv 0 \) implies from (4.7), (4.6) that \( g_{12} = g_{11}/2 = 1 \).
The $tt^*$ equations reduce to
\[
\begin{align*}
\partial_\alpha \partial_\alpha \ln g_{00} &= -\frac{2}{|\alpha|^2} g_{00}^{-1} + \frac{1}{2|\delta|^2} (|\delta|^2 \pm 1) g_{00} \\
\partial_\beta \partial_\beta \ln g_{00} &= \pm 2|\alpha|^2 \delta^2 g_{00} - \frac{1}{2|\beta|^2} \left( 1 \pm |\delta|^2 \right) g_{00}^{-1} \\
\partial_\beta \partial_\alpha \ln g_{00} &= 2\alpha^* g_{00} - \frac{2}{\alpha \beta} g_{00}^{-1} \quad (6.1)
\end{align*}
\]
where $|\delta|^2 = 1/(4\beta - 1)$ if $\beta > 1/4$ (and the + sign is chosen in $\pm$), and $|\delta|^2 = 1/(1 - 4\beta)$ if $\beta < 1/4$ (and the - sign is chosen).

We now solve for $\beta > 1/4$. The solution for $\beta < 1/4$ is similar and the conclusions the same. Redefining for the first of (6.1)
\[
z = 4\alpha^{1/2}\beta^{1/4}, \quad \ln g_{00} = q_0 - \ln |\alpha| + \ln d, \quad d = 2|\delta|/\sqrt{|\delta|^2 + 1} = \beta^{-1/2} \quad (6.2)
\]
we get
\[
\partial_z \partial_z q_0 = \sinh q_0
\]
The regular solution exists for $r = 2$. Then, we can predict the behavior of the metric for fixed $\beta$. As $z \to 0$ (or $|\alpha| \to 0$, $r_1 \to \infty$),
\[
\begin{align*}
g_{00} &\simeq \frac{d}{|\alpha|} |z|^2 \left( -\log |z|/2 - \gamma \right)^2 f(\beta)|z|^m + \ldots \\
&\simeq 16 \left[ -\ln \sqrt{2}|\alpha|^{1/2}\beta^{1/4} - \gamma \right]^2 f(\beta)|z|^m
\end{align*}
\]
where $f(\beta)$ is some integration function.

For $|\alpha| \to \infty$, ($r_1 \to -\infty$),
\[
g_{00} \simeq \frac{1}{|\alpha|\beta^{1/2}} \left[ 1 - \frac{1}{\pi} \frac{1}{|\alpha|^{1/4}\beta^{1/8}} \exp \left( -8\sqrt{|\alpha|\beta^{1/4}} \right) \right] f(\beta)|z|^m. \quad (6.3)
\]
The second equation of (6.1) will give us the behavior of the metric near the singularity. Redefining $\ln g_{00} = q_0 - \frac{1}{2} \ln |\beta| - \ln |\alpha|$, we can write
\[
\frac{1}{4|\alpha|^{1/4} \beta^{1/4} (4\beta - 1)^{1/2}} \frac{\beta^{1/2}}{4\beta - 1} \partial_\beta \partial_\beta q_0 = \sinh q_0. \quad (6.4)
\]
Now we look for a change of variables such that
\[
\frac{1}{2\alpha^{1/2}} \beta^{1/4} (4\beta - 1)^{1/2} \partial_\beta = \frac{d\beta}{dw} \partial_\beta \equiv \partial_w \quad (6.5)
\]
So we need to find the function
\[
w(\beta) = \int \frac{2\alpha^{1/2} d\beta}{\beta^{1/4} (4\beta - 1)^{1/2}} \quad (6.6)
\]
Now the integral becomes, under the substitution \( z = \sqrt{4\beta - 1} \)

\[
\sqrt{2\alpha^{1/2}} \int \frac{dz}{(1 + z^2)^{1/4}}
\]  

(6.7)

which, upon setting \( z = \sinh x \), transforms to

\[
\sqrt{2\alpha^{1/2}} \int \sqrt{\cosh x} \, dx
\]  

(6.8)

whose solution is in terms of elliptic integrals:

\[
w = \sqrt{2\alpha^{1/2}} \left\{ \sqrt{2} \left[ F(\kappa, 1/\sqrt{2}) - 2E(\kappa, 1/\sqrt{2}) \right] + \frac{2 \sinh x}{\sqrt{\cosh x}} \right\},
\]

(6.9)

where \( F(\kappa, r) \) and \( E(\kappa, r) \) are elliptic integrals of the first and second kind and

\[
\kappa = \arcsin \sqrt{\frac{\cosh x - 1}{\cosh x}}.
\]

(6.10)

We see that when \( \beta \to 1/4 \), \( \kappa \to \frac{x}{\sqrt{2}} \sim \frac{1}{\sqrt{2}}(4\beta - 1)^{1/2} \), \( F(\kappa) \) and \( E(\kappa) \sim \kappa \), and \( w \sim \sqrt{2}\alpha^{1/2}(4\beta - 1)^{1/2} \).

Now, as \( w \to 0 \), the solution of (6.4) for the metric is

\[
g_{00}(|w|) \approx \frac{1}{|\alpha||\beta|^{1/2}|w|^2} \left[ - \left( \frac{\log |w|}{2} + \gamma \right) \right]^2 h(|\alpha||w|^m + \ldots
\]

and for \( |w| \to \infty \),

\[
g_{00}(|w|) \sim \frac{1}{|\alpha||\beta|^{1/2}} \left[ 1 - \frac{2}{\sqrt{\pi|w|}} \exp[-2|w|] \right] q(|\alpha||w|^p)
\]

(6.11)

with \( h(|\alpha|), q(|\alpha|) \) some integration functions. We notice here, with \( |\alpha| \) large, the appearance of solitons of mass \( m = 2|w| \sim 2\sqrt{2}|\alpha|^{1/2}(4\beta - 1)^{1/2} \) which become massless as \( \beta \) approaches the singular point in the moduli, \( \beta \to 1/4 \).

Although we haven’t obtained the full solution of the \( tt^* \) system, the behavior of the metric near the singularity is now clear: for \( \beta \) close to \( 1/4 \), the function \( |w| \sim \sqrt{2}|\alpha|^{1/2}|1 - 4\beta|^{1/2} \) and the metric behaves as \( g \sim |w|^2\left[ - \left( \log |w|+\text{const.} + \gamma \right) \right]^2 \). Near \( |\alpha| \to \infty \), we flow to a conformal model with massless solitons.

### 7 Conclusions

We have studied supersymmetric \( \sigma \)-models on a particularly simple example of a toric variety. However, this example allowed us to explore the physics around the singular locus of the model. By using the description of these models as gauged linear \( N = 2 \)
σ-models and the methods of topological-antitopological fusion, we have shown that the model contains solitons which become massless at the singular point where one of the gauge symmetries is unbroken. At that point, the model consists of both a Higgs phase with massless chiral fields and a Coulomb phase. We have here an example of what can happen in $N = 2$ models in 2D as described in ([13, 15, 27]), i.e. the jumping in the number of Bogomolnyi solitons. These solitons are related to the intersection numbers of the vanishing cycles of the singularity. These intersection numbers undergo a jump and the number of BPS solitons changes as a result of the monodromy of the vanishing cycles. This is also the suggestion of what happens for models in $D = 4$. We can also view our example as an example of flows in massive theories. We note that we are flowing in the infra-red limit to another conformal theory with massless solitons and it would be interesting to determine this CFT. We leave this as an open problem for future work.

We can hope that what we have learned in this simple case can be generalized to explain more sophisticated models. In [8], the authors go on to explore the superconformal nonlinear sigma models with Calabi-Yau target spaces that can be embedded as hypersurfaces in toric varieties. They find there again that the singularities divide up the parameter space in different phases. The present work then also suggests that if one encounters the same types of singularities as the one we have studied, a similar explanation in terms of massless solitons appearing at the boundaries of the different phases of the model is plausible, as has been discussed recently in the litterature.

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References

[1] P. Candelas, M. Lynker and R. Schimmrigk, Nucl. Phys. B341 (1990) 383-402.

[2] P. Candelas and X.C. de la Ossa, Nucl. Phys. B342 (1990) 246-268.

[3] S.-T. Yau, Proc. Argonne Symp. on Anomalies, Geometry, Topology, eds. W. A. Bardeen and A.R. White (World Scientific, Singapore, 1985).

[4] A. Strominger and E. Witten, Comm. Math. Phys. 101 (1985) 341.

[5] J.K. Kim, I.G. Koh and Y. Yoon, Phys. Rev. D33 (1986) 2893.

[6] S. Kalara and J. Lykken, SCIPP 89/05 and UMN-TH-719 preprint.

[7] E. Martinec, Phys. Lett. B217 (1989) 431; B.R. Greene, C. Vafa and N. Warner, HUTP-88/A047; J.I. Latorre and C.A. Lütken, Phys. Lett. B222 (1989) 55.
[8] D.R. Morrison and M.R. Plessor, hep-th/9412230, Nucl. Phys. B440 (1995) 279-354.

[9] V.V. Batyrev, “Quantum cohomology rings of toric manifolds,” in Journées de Géométrie Algébrique d’Orsay (Juillet 1992), Astérisque, vol. 218, Société Mathématique de France, 1993, pp. 9-34, alg-geom/9310004.

[10] E. Witten, Nucl. Phys. B403 (1993) 159-222.

[11] E. Witten, ‘The Verlinde algebra and the cohomology of the Grassmannian’, IASSNS-HEP-93/41, hep-th/9312104.

[12] S. Cecotti and C. Vafa, Nucl. Phys. B367 (1991) 359.

[13] S. Cecotti and C. Vafa, Commun. Math. Phys. 158 (1993) 569-644.

[14] S. Cecotti and C. Vafa, Phys. Rev. Let. 68 (1992) 903.

[15] S. Cecotti, P. Fendley, K. Intriligator and C. Vafa, Nucl. Phys. B386 (1992) 405.

[16] D.A. Cox, ‘The homogeneous coordinate ring of a toric variety’, Amherst preprint, 1992, alg-geom/9210008.

[17] W. Fulton, ‘Introduction to Toric Varieties’, Annals of Math. Studies, vol. 131, Princeton University Press, 1993.

[18] P.S. Aspinwall, B.R. Greene and D.R. Morrison, Nucl. Phys. B416 (1994) 414-480.

[19] M. Bourdeau and M.R. Douglas, Nucl. Phys. B420 (1994) 243-267.

[20] M. Bourdeau, Nucl. Phys. B439 (1995) 421-440.

[21] I.S. Gradshteyn and I.M. Ryzhik, “Table of integrals, Series and Products”, Academic Press inc., 1980.

[22] W. Lerche, C. Vafa and N.P. Warner, Nucl. Phys. B324 (1989) 427-474.

[23] N.P. Warner, “N = 2 Supersymmetric Integrable Models and Topological Field Theories”, Lectures Given at the Summer School on High Energy Physics and Cosmology, Trieste, Italy, 1992, Trieste HEP Cosmol. 1992:143-179, hep-th/9301088.

[24] C.L. Saçıloğlu, J. Math. Phys. 25(11) 1984, P. Ginsparg and G. Moore, ‘Lectures on 2D Gravity and 2D String Theory’, TASI summer school, Boulder, CO, 1992, hep-th/9304011.

[25] A. Strominger, ‘Massless Black Holes and Conifolds in String Theory’, hep-th/9504090, Nucl. Phys. B451 (1995) 96-108.

[26] N. Seiberg and E. Witten, Nucl. Phys. B426 (1994) 19-52.
[27] C. Vafa, ‘A Stringy Test of the Fate of the Conifold’, HUTP-95/A014, \texttt{hep-th/9505023}, Nucl. Phys. B447 (1995) 252-260.

[28] A.C. Avram, P. Candelas, D. Jancic and M. Mandelberg, ‘On the Connectedness of Moduli Spaces of Calabi-Yau Manifolds’, UTTG-2-95, \texttt{hep-th/9511230}.

[29] Ti-Ming Chiang, B.R. Greene, M. Gross and Y. Kanter, ‘Black Hole Condensation and the Web of Calabi-Yau Manifolds’, \texttt{hep-th/9511204}.

[30] B. Greene, D. Morrison and A. Strominger, ‘Black Hole Condensation and the Unification of String Vacua’, \texttt{hep-th/9504148}.