Logarithmic corrections to black hole and black ring entropy in tunneling approach

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Abstract – The tunneling approach beyond the semiclassical approximation has been used to calculate the corrected Hawking temperature and entropy for various black holes and FRW universe model. We examine their derivations, and show that the quantity \( H \) in the corrected temperature is the explicit function of the only free parameter \( F \) (if we apply the change of variables \( F = \hbar S_{BH} \)). Our analysis improves previous calculations, and indicates that the leading-order logarithmic correction to entropy is a natural result of the corrected temperature and the first law of thermodynamics. Additionally, we apply the tunneling approach beyond semiclassical approximation to neutral black rings. Based on the analysis, we show that the entropy of neutral black rings also has a logarithmic leading-order correction.

Introduction. – Since the discovery of Hawking radiation and black hole thermodynamics [1–3], it is generally believed that black holes are objects which have temperature and entropy. By adopting the Hawking temperature \( T_H = \frac{k}{2\pi} \) and the Bekenstein-Hawking entropy \( S_{BH} = \frac{A}{4\hbar} \), the first law of black hole thermodynamics is obtained [4]. The understanding of black hole entropy is an important subject in quantum gravity. When quantum effects are considered, the area law of black hole entropy should undergo corrections. These corrections have been extensively studied using many approaches, such as field theory methods [4–6], quantum geometry techniques [7,8], general statistical-mechanical arguments [9–11], Cardy formula [12,13]. In all these derivations, the leading-order correction to black hole entropy takes a logarithmic form

\[
S = S_{BH} + \alpha \ln S_{BH} + \cdots \tag{1}
\]

This result also coincides with that obtained by counting the number of microstates in string theory [14,15] and loop quantum gravity [16,17].

The simplest method of calculating black hole entropy is directly integrating from the first law of thermodynamics \( dM = T dS \), using the expression of Hawking temperature. Several derivations of Hawking radiation were presented [18–20], one of which is the tunneling approach [21,22]. In this approach, the emission rate of a particle tunneling from the black hole is associated with the imaginary part of the action, which in turn is related to the Boltzmann factor for the emission at the Hawking temperature. There are two variants of the tunneling approach, namely, the Parikh-Wilczek radial null geodesic method [22,23], and the Hamilton-Jacobi method [21,24,25]. Recently, based on the Hamilton-Jacobi method, Banerjee and Majhi [26,27] developed a general formalism of tunneling beyond semiclassical approximation. According to their calculations, the Hawking temperature with quantum corrections takes the form

\[
T_{cr} = T_H \left( 1 + \sum \gamma_i \hbar^i \right) = T_H \left( 1 + \sum \beta_i \frac{\hbar^i}{H} \right)^{-1}, \tag{2}
\]

where \( \beta_i \) is a dimensionless parameter, \( H \) is a quantity with the dimension of \( \hbar \), and should be expressed in terms of black hole parameters [28]. We should notice that the \( h \) dependence of \( T_{cr} \) has been well considered by series of \( \hbar^i \) in the numerator in eq. (2), and \( H \) has to be independent of \( h \). This argument is highly related to our conclusion. It is also reasonable to assume that, the first law of thermodynamics still holds in the context of quantum corrections, and the entropy \( S_{cr} \) is a state function.

Then, the entropy can be determined by integrating the relation \( dE = T_{cr} dS_{cr} \), which has the same form as (1). This formalism has been applied on various black holes [28–32] and FRW universe model [33]. However, after examining their papers carefully, we find that these derivations rely on one procedure, i.e., they expressed

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$H$ as a polynomial of selected black hole parameters. In fact, all we know is that $H$ can be expressed in terms of black hole parameters. There is no reason why $H$ “should” be a polynomial, and why $H$ took those different particular forms in various models [28–32]. In order to fill this gap, we propose to take $H$ as a general function of a complete set of black hole parameters, i.e., $H = H(\text{black hole parameters})$. For example, in Kerr-Newman black holes, the complete parameters are $M, J,$ and $Q$, while the entropy $S_{BH}$ in a semiclassical frame can be expressed in terms of them like $S_{BH} = F(M, Q, J)/\hbar$. It is always the case whether the black hole entropy is Bekenstein-Hawking entropy or not. $F$ has an area unit just like $H$, and it will make our demonstration of logarithmic entropy more clear. Next we use the change of variables $F(M, Q, J) = \hbar S_{BH}$, and regard $(J, Q, S_{BH})$, or equivalently $(J, Q, F)$ as a complete set of parameters for Kerr-Newman black hole. So $H$ can be expressed as $H = H(M, Q, J) = H(J, Q, F)$. According to the spirit in [28], by demanding $dS_{cr}$ to be an exact differential, we can get that $\frac{\partial H(J, Q, F)}{\partial J} = 0 = \frac{\partial H(J, Q, F)}{\partial Q}$, and so $H = H(F)$. Performing the dimensional analysis as in [28–32], we can show that $H(F)$ will be reduced to a linear function $H = \eta F$, where $\eta$ is a dimensionless parameter, and the leading-order correction to black hole entropy is logarithmic. Comparing to previous calculations, the treatment here is more natural and general. In particular, this treatment emphasizes the use of exact differential in eliminating the extra dependence in $H$. Another advantage is that the analysis here can be easily extended to the cases of AdS Schwarzschild black holes, Gauss-Bonnet and Lovelock black holes, FRW universe model and neutral black rings. To this context, we can say that our analysis indicates that the leading-order logarithmic correction to black hole entropy may be a general result naturally obtained from the corrected temperature (2) and the first law of thermodynamics.

The 5-dimensional black ring is found as a vacuum solution of general relativity, with the event horizon topology $S^1 \times S^2$ [34]. It is very interesting to study the corrected temperature and entropy of black rings due to quantum effects. The tunneling formalism can be used to examine the Hawking radiation of black rings [35,36]. However, the correction due to quantum effects is not considered. On the other hand, according to [37], the near-horizon dynamics of a scalar field $\phi$ on black ring metric background can be reduced to a $(1 + 1)$-dimensional one. Therefore, the tunneling formalism beyond semiclassical approximation can also be applied to black rings. In this article, we perform the analysis on neutral black ring, and show that its corrected temperature and entropy should also take the form of (2) and (1). As far as we know, it is proposed for the first time that the entropy of neutral black ring should also undergo a logarithmic leading-order correction due to quantum effects. In obtaining the corrected entropy from the temperature (2), the previous treatment in [28–32] does not work on black rings. The parameters for a neutral black ring are $R$, $\nu$ and $\lambda$ (with $\lambda = 2\nu/(1 + \nu^2)$), while only $R$ is dimensional. We are not able to determine (or “guess”) a particular polynomial for $H$. However, we may regard $(R, \nu)$, $(M, J)$ or equivalently $(J, F)$ as a complete set of parameters for neutral black rings. Then our treatment for Kerr-Newman black holes can easily be extended to the case of black rings. Although the black hole and black ring entropy can only be fully determined by counting the microstate in well-defined quantum gravity theories, the study of corrected entropy due to quantum effects would be helpful for the understanding of these black objects.

The paper is organized as follows. In the second section, we show that in the quantity $H$ in the tunneling formalism beyond the semiclassical approximation is an explicit function of the only parameter $F$, more specifically, a linear function of $F$. We illustrate our proof in the context of Kerr-Newman black holes, and show that this analysis can easily be extended to the cases of AdS Schwarzschild black holes, Gauss-Bonnet and Lovelock black holes, and FRW universe model. In the third section, we apply the tunneling formalism to the neutral black ring, and obtain the corrected temperature and entropy. We use the units $G = c = k_{B} = \frac{1}{4\pi\hbar}\equiv 1$, while keeping the Planck constant $\hbar$ explicitly. In fact, the Planck constant is an indication of quantum effects, which can be used to identify the leading-order correction.

**Exact differential and logarithmic correction to black hole entropy.** – In this section, we take the Kerr-Newman black hole as an example to demonstrate that the leading-order correction to the entropy is logarithmic, and then extend our treatment to other models. Before calculating the entropy correction from the corrected Hawking temperature (2) obtained in [26,27], we give a brief review of the first law of thermodynamics on Kerr-Newman black holes. The standard first law holds on the event horizon

$$dS_{BH} = \frac{dM}{T_{H}} + \left(-\frac{\Omega_{H}}{T_{H}}\right)dJ + \left(-\frac{\Phi_{H}}{T_{H}}\right)dQ,$$

where $\Omega_{H}$ and $\Phi_{H}$ are the angular velocity and the electrical potential on the horizon, respectively. In eq. (3), the black hole can be treated as an equilibrium state parametrized by $M, J$ and $Q$, and the entropy $S_{BH}$ is a state function. Integrate eq. (3) by $M, J$ and $Q$, $S_{BH}$ can be expressed as a function of these quantities. For example, the Bekenstein-Hawking entropy of Kerr-Newman black hole takes the form

$$S_{BH} = \frac{\pi}{\hbar} \left(2M + \sqrt{M^{2} - \frac{J^{2}}{M^{2}} - Q^{2}} - Q^{2}\right).$$

The thermodynamical quantities $T_{H}, \Omega_{H}$ and $\Phi_{H}$ can be defined as partial derivatives of $S_{BH}$

$$\frac{\partial S_{BH}}{\partial M} \bigg|_{J,Q} = \frac{1}{T_{H}},$$
\[ \frac{\partial S_{BH}}{\partial J} \bigg|_{M,Q} = - \frac{\Omega_H}{T_H}, \quad (6) \]
\[ \frac{\partial S_{BH}}{\partial Q} \bigg|_{M,J} = - \frac{\Phi_H}{T_H}. \quad (7) \]

Besides, \( S_{BH} \) is a state function and \( dS_{BH} \) is an exact differential. This leads to the following Maxwell’s relations:
\[ \frac{\partial}{\partial J} \left( \frac{1}{T_H} \right) \bigg|_{M,Q} = \frac{\partial}{\partial M} \left( \frac{-\Omega_H}{T_H} \right) \bigg|_{J,Q}, \quad (8) \]
\[ \frac{\partial}{\partial Q} \left( \frac{-\Omega_H}{T_H} \right) \bigg|_{M,J} = \frac{\partial}{\partial J} \left( \frac{-\Phi_H}{T_H} \right) \bigg|_{M,Q}, \quad (9) \]
\[ \frac{\partial}{\partial M} \left( \frac{\Phi_H}{T_H} \right) \bigg|_{J,Q} = \frac{\partial}{\partial Q} \left( \frac{1}{T_H} \right) \bigg|_{J,M}. \quad (10) \]

When \( \hbar \to 0 \), the corrected temperature \( T_{cr} \to T_H \), so it is reasonable to assume that the first law of thermodynamics still holds in the context of quantum correction [28], i.e.
\[ dS_{cr} = \frac{dM}{T_{cr}} + \left( -\frac{\Omega_H}{T_{cr}} \right) dJ + \left( -\frac{\Phi_H}{T_{cr}} \right) dQ. \quad (11) \]

The corrected Hawking temperature calculated from the tunneling formulism beyond the semiclassical approximation is
\[ T_{cr} = T_H \left( 1 + \sum \gamma_i \hbar \right) = T_H \left( 1 + \sum \beta_i \hbar \frac{1}{H_{KN}} \right)^{-1}, \quad (12) \]
where \( H_{KN} \) is a quantity with dimension \( \hbar \). Similar Maxwell’s relations hold by replacing \( T_{BH} \) with \( T_{cr} \). By substituting the expression of corrected temperature (12), we obtain the corresponding constraint equations for \( H_{KN} \)
\[ \frac{\partial H_{KN}}{\partial J} \bigg|_{M,Q} = -\Omega_H \frac{\partial H_{KN}}{\partial M} \bigg|_{J,Q}, \quad (13) \]
\[ \frac{\partial H_{KN}}{\partial Q} \bigg|_{M,J} = \left( \frac{\Phi_H}{\Omega_H} \right) \frac{\partial H_{KN}}{\partial J} \bigg|_{M,Q}, \quad (14) \]
\[ \frac{\partial H_{KN}}{\partial M} \bigg|_{J,Q} = -\left( \frac{\Phi_H}{\partial Q} \right) \bigg|_{J,M}. \quad (15) \]

In Kerr-Newman black hole, the three conserved quantities \( M, J, \) and \( Q \) have totally determined the black hole thermodynamics, so the quantity \( H_{KN} \) should be expressed in terms of these black hole parameters. The next step in [28] is to express \( H_{KN} \) as a particular polynomial of \( M, J, Q \), followed by a tedious analysis using the first law of thermodynamics to determine the expression of \( H_{KN} \). This treatment is not natural and general, and can be improved. Here we take the general form \( H_{KN} = H_{KN}(M, J, Q) \). Next we introduce a change of variables that \( H_{KN} = H_{KN}(F, J, Q) \), where \( F = F(M, J, Q) \). It means we take \( F \) and two of the three conserved quantities \( M, J, Q \) as free parameters, and construct the third conserved quantity as a function of them, e.g., taking \( M = M(F, J, Q) \). We will show that the constraint equations lead to a natural choice of \( F \) and in this case \( H_{KN}(F, J, Q) \) will be simplified to \( H_{KN}(F) \). Now we substitute \( H_{KN}(F, J, Q) \) into the constraint equation (13), and use the chain rule in standard calculus textbooks,
\[ \text{LHS} = \frac{\partial H_{KN}(F, J, Q)}{\partial F} \frac{\partial F}{\partial J} \bigg|_{M,Q} \]
\[ + \frac{\partial H_{KN}(F, J, Q)}{\partial J} \bigg|_{F, Q}, \quad (16) \]
\[ \text{RHS} = -\Omega_H \frac{\partial H_{KN}(F, J, Q)}{\partial F} \bigg|_{J, Q} \frac{\partial F}{\partial M} \bigg|_{J, Q}. \quad (17) \]

Comparing them with eqs. (5) (6), we will get
\[ \frac{\partial H_{KN}(F, J, Q)}{\partial J} \bigg|_{F, Q} = 0, \quad (18) \]
if the change of variables satisfies, \( F = \hbar S_{BH} \). Similarly, using \( F = \hbar S_{BH} \), we can obtain
\[ \frac{\partial H_{KN}(F, J, Q)}{\partial Q} \bigg|_{F, J} = 0, \quad (19) \]
from the constraint equation (15). As a result, once we take \( F \) as a free parameter in describing \( H_{KN} \), it is the only explicit free parameter, i.e.
\[ H_{KN} = H_{KN}(F). \quad (20) \]

In fact, any continuous function \( F(S_{BH}) \) will lead eq. (20). We use \( F = \hbar S_{BH} \) for two reasons. It has the same dimension of \( H_{KN} \), which will simplify the following analysis. And it is a natural choice with significant physical interpretation since a semiclassical black hole entropy always has the form \( S_{bh} = S(M, J, Q)/\hbar \). If we take \( J = J(F, M, Q) \) or \( Q = Q(F, M, J) \), eq. (20) can also be obtained from the constraint equations (13)–(15). This indicates that the result (20) is a natural consequence of the first law of thermodynamics. Now we have completed the proof that the \( H_{KN} \) is the function of the only explicit parameter \( F \).

In D-dimensional spacetime, the units \( G = c = k_B = \frac{\text{vol}}{4\pi M^2} = 1 \), and \( [H_{KN}] = [\hbar] = [F] = L^{D-2} \). A natural guess of \( H_{KN} \) is
\[ H_{KN} = \eta F, \quad (21) \]
where \( \eta \) is a dimensionless parameter. Due to the definition of \( F \) and eq. (4), \( H_{KN} \) can be explicitly written as
\[ H_{KN} = \eta \left( 2M \left[ M + \sqrt{M^2 - \frac{J^2}{M^2} - Q^2} \right] - Q^2 \right). \quad (22) \]

We can see that \( H_{KN} \) does no longer depend on \( \hbar \), which is consistent with the ansatz (2). In sum, \( H \) saturates
the constraint equations (13)–(15), so it is a function of \( F = hS_{\text{BH}} \); both \( H \) and \( F \) have the dimension of area, and are independent of \( h \). As a result, it is the only solution for \( H_{KN} \).

Now we use the first law of thermodynamics

\[
\begin{align*}
\frac{dS}{c_T} &= \frac{1}{T} (dM - \Omega_H dJ - \Phi_H dQ), \\
&= \frac{1}{T} (T_H dS_{\text{BH}}),
\end{align*}
\]

which can be written as

\[
\begin{align*}
dS &= \left(1 + \sum \beta_n \frac{\eta F}{H_{KN}}\right) dS_{\text{BH}} \\
&= \left(1 + \beta_1 \frac{\eta F}{\eta F} + O(h^2)\right) dS_{\text{BH}} \\
&= \left(1 + \frac{\lambda}{S_{\text{BH}}} + O(h^2)\right) dS_{\text{BH}} \\
&= \left(1 + \frac{\alpha}{S_{\text{BH}}} + \ldots\right) dS_{\text{BH}}.  \tag{23}
\end{align*}
\]

Here \( \alpha \) is another dimensionless coefficient, and all other higher-order terms are included in \( \ldots \). Integrate eq. (23) up to the first (leading) order, which yields

\[
S_{cr} = S_{\text{BH}} + \alpha \ln S_{\text{BH}} + \ldots + \text{const}.  \tag{24}
\]

This is the corrected entropy of Kerr-Newman black hole obtained from the temperature (12) and the leading-order correction is logarithmic.

Our analysis can be easily extended to the cases of various black holes and FRW universe model [28–33], without assigning \( H \) as specific polynomials for each case:

- For FRW universe model, the parameter is \( \tilde{r}_A \), the radius of apparent horizon, and \( S_{\text{BH}} = \pi \tilde{r}_A^2 h \equiv \frac{\pi h}{T} \). So \( H(\tilde{r}_A) = H(F) \).

- For AdS Schwarzschild black hole, the situation is a bit more complicated. The mass and entropy of AdS Schwarzschild black hole is

\[
M = \frac{1}{2} (1 + r_+^2/l^2), \quad S = \frac{\pi r_+^2}{h},  \tag{25}
\]

where \( \Lambda = -3l^2 \), \( l \) is the AdS radius, \( r_+ = r_+(M, l) \) is the event horizon radius, and \( l^2 = L^2 = \hbar \). Since the cosmological constant comes into the black hole solution, it modifies the form of first law of black hole thermodynamics. Thus it is reasonable to consider an ensemble where the cosmological constant can fluctuate [38],

\[
dS = \frac{1}{T} (dM - \Theta dl).  \tag{26}
\]

It will lead to \( \frac{\partial H}{\partial F} = -\frac{\partial \eta}{\partial F} \), similar as eq. (13).

Making the change of variables \( F = \pi r_+^2 \) to \( H \), substituting \( H(F, M) \) into the above relation, and using chain rule, we can prove that \( H = H(F) \). This result not only preserves the consistency of the black hole thermodynamics in the AdS case, but also ensures that when \( \Lambda \to 0 \), the quantum corrections of black hole temperature and entropy degenerate to the Schwarzschild case.

- For D-dimensional Gauss-Bonnet black hole,

\[
S_{GB} = \frac{A_{D-2} \lambda^{D-2}}{4h} \left(1 + \frac{D-2}{D-4} \frac{\lambda}{\bar{\lambda}}\right),  \tag{27}
\]

where \( \lambda = \lambda(D-3)(D-4) \), \( [\bar{\lambda}] = L^2 \), \( r_h = r_h(M, \bar{\lambda}) \) is the event horizon radius, and \( \lambda \) is the coupling constant of the Gauss-Bonnet term (see [31]). \( \bar{\lambda} \) should be treated as a variable rather than a constant, which undergoes similar situation as in AdS case, so we can also get \( H = H(F) \). Notice that here by definition, \( F \) depends on both \( r_h \) and \( \bar{\lambda} \).

Dimensional analysis reduces the arbitrariness of \( H \) to a linear function of \( F \), and the quantum-corrected black hole entropy after integration has a logarithmic leading order. Because Schwarzschild, RN, and Kerr black holes can be perceived as special cases of Kerr-Neu- man case, the logarithmic corrections of entropy are naturally consistent with eq. (24). There is one comment on the corrected entropy of AdS Schwarzschild black hole. In the treatment of [26], they set \( H = M^2 \) for the AdS Schwarzschild black hole (see eq. (4.9) and (4.10) therein). Following their treatment, we can write the terms of quantum corrections to the entropy explicitly as

\[
S_{cr} - S_{\text{BH}} = \ln S_{\text{BH}} + \frac{\pi l^2}{h S_{\text{BH}}} - \ln \left(\frac{\pi l^2}{h} + S_{\text{BH}}\right) + \ldots.  \tag{28}
\]

When the cosmological constant vanishes, i.e., as \( \Lambda = -3l^{-2} \to 0 \), the dynamics of AdS Schwarzschild black hole should reduce to the corresponding asymptotically flat Schwarzschild black hole. By adding a constant term \( -1 + \ln(\pi l^2/h) \), as \( \Lambda \to 0 \), the above correction (28) can coincide with the corrected entropy of Schwarzschild black hole. However, the high-order corrections obtained by \( H = M^2 \) do not approach the Schwarzschild case in this limit. So the treatment in [26] cannot be correct. After examining their derivation, we think that the choice of \( H = M^2 \) for AdS Schwarzschild black hole is groundless and dogmatic. Their result in Gauss-Bonnet black hole suffers similar problems. In our paper, we can overcome this arbitrariness, and obtain the corrected leading-order logarithmic correction to black hole entropy.

Now we have examined the derivations of corrected entropy in the cases of these various black holes [28–32], FRW universe model [33], and added the proof of the expression \( H = H(F) \). Our treatment strengthens the validity of the method of obtaining the corrected entropy.
from the tunneling formalism beyond the semiclassical approximation. The analysis indicate that the leading-order entropy correction is a natural result obtained from the corrected temperature (12) and the first law of thermodynamics. In the next section we shall extend the above analysis to neutral black rings, and show that its leading-order entropy correction due to quantum effects is also logarithmic.

In the context, we have not given more comments on \( \eta \) and \( \alpha \) other than dimensionless parameters. If one wants to determine them, further information is required. For example, quantum gravity theories such as string theory [14,15] and loop quantum gravity [16,17] can give values about the coefficient \( \alpha \) in various black holes; in [28], the authors also tried to use gravitational anomaly to determine the coefficient \( \alpha \). These discussions are not included in our paper.

**Corrected entropy of neutral black rings.** The 5-dimensional black ring is a vacuum solution of general relativity, with the event horizon topology \( S^1 \times S^2 \). The tunneling approach has also been used to analyze the black ring radiation [35,36]. However, the influence of quantum effects are not included. In this section we apply the tunneling method beyond semiclassical approximation on neutral black rings. The metric of neutral black ring is

\[
\text{ds}^2 = -\frac{F(y)}{F(x)} \left( dt - CR \frac{1+y}{F(y)} \text{d} \psi \right)^2 + \frac{R^2}{(x-y)^2} F(x) \times \left[ -\frac{G(y)}{F(y)} \text{d} y^2 + \frac{\text{d} x^2}{G(x)} + \frac{G(x)}{F(x)} \text{d} \phi^2 \right]
\]

where \( F(\xi) = 1 + \lambda \xi, \ G(\xi) = (1 - \xi^2)(1 + \nu \xi), \ C(\nu, \lambda) = \sqrt{\lambda(\lambda - \nu)(1 + \lambda)(1 - \nu)}, \lambda, \nu \) are dimensionless parameters taking values in the range \( 0 < \nu \leq \lambda < 1 \). \( x \) is one of the angular coordinates in \( S^2 \), with \(-1 \leq x \leq 1 \). \( y \) is the only “ringlike-radical” coordinate, with \(-\infty \leq y \leq -1 \). To remove the conical singularity at \( x = 1, \lambda \) and \( \nu \) must be related to each other via

\[
\lambda = \frac{2 \nu}{1 + \nu^2}.
\]

The horizon is at \( y = y_h = -1/\nu \) with the topology \( S^1 \times S^2 \).

The thermodynamics quantities of the neutral black ring are

\[
M = \frac{3 \pi R^2}{4} \frac{\lambda}{1 - \nu}, \quad J = \frac{\pi R^3}{2} \sqrt{\lambda(\lambda - \nu)(1 + \lambda)} \frac{\nu}{(1 - \nu)^2}, \quad S_{BR} = \frac{A_h}{4 \hbar} = \frac{2 \pi^2 R^3}{\hbar} \frac{\nu^{3/2} \sqrt{\lambda(1 - \lambda^2)}}{(1 - \nu)^2(1 + \nu)}, \quad T_{BR} = \frac{\hbar}{4 \pi R} (1 + \nu) \sqrt{\frac{1 - \lambda}{\lambda(1 + \lambda)}},
\]

where

\[
\Omega_h = \frac{1}{R} \sqrt{\frac{\lambda - \nu}{\lambda(1 + \lambda)}}.
\]

Here \( R \) describes the scale for the solution, and \( \lambda \) and \( \nu \) are parameters that determine the shape and rotation velocity of the ring [34]. The validity of first law \( T dS = dM - \Omega_h dJ \) also requires that \( \lambda \) and \( \nu \) satisfy the relation (29). As a result, a neutral black ring parametrized by \( M \) and \( J \) can be equivalently described by \( R \) and \( \nu \). Furthermore, according to [37], a scalar field \( \varphi \) in the neutral black ring reduces to a \( (1+1) \)-dimensional free field in the near-horizon limit. So the tunneling formalism beyond the semiclassical approximation can also be applied to a neutral black ring. Now let us turn to the \( (1+1) \)-dimensional metric

\[
ds^2 = -f(y) dt^2 + \frac{1}{f(y)} dy^2,
\]

where,

\[
f(y) = \frac{\sqrt{F(y)}}{G(1 + y)} G(y).
\]

Following the standard treatment, consider a massless particle in spacetime (35) described by the Klein-Gordon equation

\[
-\frac{\hbar^2}{\sqrt{-g}} \partial \mu (g^{\mu \nu} \sqrt{-g} \partial \nu) \phi = 0.
\]

The semiclassical wave function satisfying the above equation is obtained by making the standard WKB ansatz for \( \phi \)

\[
\phi(y, t) = \exp \left[ -\frac{i}{\hbar} S(y, t) \right].
\]

Substitute it in (37), and expanding the action \( S(y, t) \) in the powers of \( \hbar \)

\[
S(y, t) = S_0(y, t) + \sum_i \hbar S_i(y, t).
\]

Following the similar treatment of Kerr-Newman black hole in [28], the final result of the corrected Hawking temperature is

\[
T_{cr} = T_{BR} \left( 1 + \sum \frac{\beta_i}{1 + \left( \frac{\hbar}{H_{BR}} \right)^{2i}} \right)^{-1}.
\]

Derivation of (40) via fermion tunneling is also available, similar to the treatment of other black hole models.

In the context, we have stated that the thermodynamics of neutral black ring can be equivalently parametrized by \( M \) and \( J \), or by \( R \) and \( \nu \). The neutral black ring satisfies the Bekenstein-Hawking law, therefore \( F = \frac{A_h}{4 \hbar} \). Now we follow a similar procedure as in the previous section. The change of variables will deform \( H_{BR}(R, \nu) \) into \( H_{BR}(F, R) \). Maxwell relation in black ring thermodynamics will lead to a constraint equation similar to eq. (13), while \( M \) and \( J \) are replaced by \( R \) and \( \nu \) due to chain rule. Substituting \( H_{BR}(F, R) \) into it, we can
prove that $H_{BR} = H(F)$. Dimensional analysis reduces it to $H_{BR} = \eta F$, i.e. (where eq. (29) is used),

$$H_{BR} = 2\pi \eta^2 R^3 \frac{\nu^2}{1 - \nu} \sqrt{\frac{2}{(1 + \nu^3)^3}}.$$  (41)

Finally, integrating an equation similar as (23), the corrected entropy can be obtained

$$S_{cr} = S_{BR} + \alpha \ln S_{BR} + \cdots + \text{const.}$$  (42)

Now based on the tunneling formalism beyond the semiclassical approximation and the first law of thermodynamics, we have shown that the leading-order entropy correction of the neutral black ring due to quantum effects is also a logarithmic term.

**Summary.** – The standard entropy of black holes and black rings should undergo corrections due to quantum effects. The tunneling formalism beyond the semiclassical approximation provides a useful method to calculate the corrected Hawking temperature and entropy in various models. The entropy corrections obtained using this method have a logarithmic leading-order term. This is a natural result from the corrected temperature and the first law of thermodynamics, and coincides with some result obtained by counting the number of microstates in string theory and loop quantum gravity. Maybe this is a somewhat universal phenomenon. In fact, for a black hole model, if the quantum tunneling beyond the semiclassical approximation can be used and the corrected temperature can be expressed as (2), the leading-order correction to its entropy should undergo a logarithmic form, according to the analysis in this paper. However, the exact expressions of black hole and black ring entropy can only be determined in a well-defined quantum gravity theory. We hope that our discussion will be helpful for further understanding of the black hole entropy.

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