Localised pruning for data segmentation based on multiscale change point procedures

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Abstract

The segmentation of a time series into piecewise stationary segments is an important problem both in time series analysis and signal processing. It requires the estimation of the total number of change points, which amounts to a model selection problem, as well as their locations. Many algorithms exist for the detection and estimation of multiple change points in the mean of univariate data, from those that find a global minimiser of an information criterion, to multiscale methods that achieve good adaptivity in the localisation of change points via multiple scanning of the data. For the latter, the problem of duplicate or false positives needs to be addressed, for which we propose a localised application of Schwarz information criterion. As a generic methodology, this new approach is applicable with any multiscale change point methods fulfilling mild assumptions to prune down the set of candidate change point estimators. We establish the theoretical consistency of the proposed localised pruning method in estimating the number and locations of multiple change points under general conditions permitting heavy tails and dependence. In particular, we show that it achieves minimax optimality in change point localisation in combination with a multiscale extension of the moving sum-based procedure when there are a finite number of change points. Extensive simulation studies and real data applications confirm good numerical performance of the proposed methodology.

1 Introduction

Change point analysis has a long tradition in statistics since Page (1954). In recent years, there has been a surge of interest for computationally fast and statistically efficient methods for change point analysis due to its importance in time series analysis, signal processing and many other applications where data is routinely collected over time in naturally nonstationary environments. In particular, many papers address the problem of testing for a change point, either retrospectively or sequentially, when at most one change is expected, see Csörgő and Horváth (1997) and Horváth and Rice (2014) for an overview. Based on such tests, estimators for the location of a single change point can be derived with optimal localisation properties.
However, it is often unknown how many structural changes are present in the data and allowing for multiple change points, the goal of change point analysis is to estimate both the total number and locations of the change points. Examples where multiple change point analysis is popularly employed include genomics (detecting chromosomal copy number aberrations, see Olshen et al. (2004); Li et al. (2016); Niu and Zhang (2012); Chan and Chen (2017)), neurophysiology (modelling the instabilities in the rate at which a neuron fires an action potential, Messer et al. (2014)), astronomy (detecting orbiting planets and their periodicity from varying luminosity of a star, Fisch et al. (2018)) and finance (identifying and dating change points in financial time series, Cho and Fryzlewicz (2012)), to name but a few.

Broadly, approaches to retrospective change point analysis in the literature can be categorised into two: One line of research relates to the aforementioned tests, while the other aims at optimising objective functions, constructed on the principle of penalised likelihood or minimum description length, via dynamic programming (Killick et al., 2012; Maidstone et al., 2017) or genetic algorithm (Davis and Yau, 2013). There are also methods based on hidden Markov models with algorithms for estimating the sequence of hidden states (Titsias et al., 2016). Recent algorithmic developments include multiscale methodologies which focus on isolating each change point within an interval sufficiently large for its detection, whereby the tests and the estimators designed for the at-most-one-change alternatives are applicable to detect (possibly) multiple change points. The Wild Binary Segmentation (WBS) algorithm proposed in Fryzlewicz (2014) accomplishes this by drawing a large number of random intervals. Eichinger and Kirch (2018) investigate a moving sum (MOSUM) procedure which tests for at most a single change point over moving windows at a single bandwidth, and briefly discuss its multiscale extension for better adaptivity. On the one hand, such multiscale methods enjoy the near-optimal localisation of change points through scanning the same regions of the data multiple times. On the other, this may result in conflicting estimators detected for the identical change point, as well as false positives spuriously detected without any change points in their vicinity.

While there are methods specifically tailored for individual multiscale procedures to handle conflicting estimators and false positives, there is a lack of a unified approach to this task. Also, when pruning down the candidate change point estimators, it is desirable to exploit the information about the detection intervals in which the estimators are obtained, which is typically retrievable for multiscale change point methods. In this paper, we propose a generic algorithm for this purpose, which utilises the Schwarz criterion (Schwarz, 1978) and performs an exhaustive search for change point estimators in a localised way on a candidate set generated by such multiscale methods. Contrary to the common usage of information criteria in change point problems, the proposed localised pruning algorithm does not require the maximum number of change points as an input, nor does it seek for the global minimiser of the criterion. We show that as a generic tool, the localised pruning algorithm inherits the properties of
the candidate generating method. Therefore, with a suitable candidate generating method, it consistently estimates the total number of change points as well as locating the change points with accuracy while being computationally feasible. In addition, we verify the suitability of two candidate generating multiscale methods based on the MOSUM and cumulative sum (CUSUM) statistics.

The rest of the paper is organised as below. In Section 2, we motivate and propose the localised pruning as a generic methodology applicable with a class of multiscale change point algorithms, and establish its theoretical consistency. Section 3 shows that two candidate generating methods based on the MOSUM and CUSUM statistics meet the requirement for the consistency of the localised pruning and, in particular, combined with the former, the localised pruning achieves the minimax optimality in change point localisation. In Section 4, we conduct extensive comparative simulation studies as well as applying the proposed methodology to Kepler light curve data and array comparative genomic hybridisation data. Section 5 concludes the paper, followed by the main proofs. The rest of the proofs and complete simulation results are provided in the Supplementary Appendix.

2 Localised pruning via Schwarz criterion

We consider the canonical change point model

\[ X_t = f_t + \varepsilon_t = \sum_{j=0}^{q_n} f_j \cdot \mathbb{I}_{\{\theta_j+1 \leq t \leq \theta_{j+1}\}} + \varepsilon_t \]

\[ = f_0 + \sum_{j=1}^{q_n} d_j \cdot \mathbb{I}_{\{t \geq \theta_{j+1}\}} + \varepsilon_t, \quad (1) \]

where \( \theta_1 < \theta_2 < \ldots < \theta_{q_n} \) denote the \( q_n \) change points (with \( \theta_0 = 0 \) and \( \theta_{q_n+1} = n \)) at which the mean of \( X_t \) undergoes changes of size \( |d_j| \). The sequence of noise \( \{\varepsilon_t\}_{t=1}^{n} \) satisfies \( E(\varepsilon_t) = 0 \) and is allowed both serial dependence as well as heavy-tailedness as specified later. Our goal is to estimate both the total number \( q_n \) and the locations of the change points \( \theta_j \).

For this purpose, we introduce a localised pruning methodology which, applicable to a set of candidate change point estimators returned by multiscale change point methodologies, achieves consistent estimation of multiple change points in their total number and locations. Many multiscale change point procedures are based on the principle of isolating each change point for its detection and estimation, and typically attach extra information to change point estimators about their detection intervals. Such examples include the multiscale extension of the MOSUM procedure [Eichinger and Kirch, 2018] and the WBS [Fryzlewicz, 2014]: The
MOSUM procedure scans a series of MOSUM statistics

\[ T_{b,n}(G; X) := \sqrt{\frac{G}{2}} (\bar{X}_{(b-G+1):b} - \bar{X}_{(b+1):b+G}) \quad \text{where} \quad \bar{X}_{s:e} = \frac{1}{e-s+1} \sum_{t=s}^{e} X_t \]  

for a given bandwidth \( G \) and \( G \leq b \leq n - G \), and marks as change point candidates the locations where \( |T_{b,n}(G; X)| \) simultaneously exceeds a critical value and forms local maxima; thus each candidate estimator \( k \) is associated with its natural detection interval \( I_N(k) = (k-G, k+G) \). The WBS examines the CUSUM statistics

\[ X_{s,b,e} \equiv X_{s,b,e}(X) = \sqrt{\frac{(b-s)(e-b)}{e-s}} (\bar{X}_{(s+1):b} - \bar{X}_{(b+1):e}) \quad \text{for} \quad s+1 \leq b \leq e-1, \]  

over a large number of randomly drawn intervals \( (s, e) \subset [1, n] \). The maximiser of the CUSUM statistics \( k = \arg \max_{s < b < e} |X_{s,b,e}| \) can be regarded as a change point candidate if the test statistic \( |X_{s,k,e}| \) exceeds a certain threshold, and the interval \( I_N(k) = (s, e) \) is readily associated with its detection.

In what follows, we describe the proposed localised pruning methodology assuming that a set of candidate estimators \( \mathcal{K} \) is given. Details of the MOSUM- and CUSUM-based candidate generating methods are discussed in Section 3.

### 2.1 Methodology

Let \( \mathcal{K} \) denote the set of all the candidate change point estimators to be pruned down. For each \( k \in \mathcal{K} \), we denote the detection interval of \( k \) by \( I(k) \equiv (k-G_L, k+G_R] \), where the left detection distance \( G_L = G_L(k) \) is the distance from \( k \) to the leftmost point of the interval, and the right detection distance \( G_R = G_R(k) \) is defined analogously.

Information criteria are frequently adopted for model selection in change point problems, and we adopt the Schwarz criterion (Schwarz, 1978, SC) for this purpose. For a given set of change point candidates \( \mathcal{A} = \{\tilde{k}_1 < \ldots < \tilde{k}_m\} \subset \mathcal{K} \), the SC is evaluated as

\[ SC(\mathcal{A}) = \frac{n}{2} \log \left\{ \frac{\text{RSS}(\mathcal{A})}{n} \right\} + |\mathcal{A}| \cdot \xi_n, \]  

where it balances between the goodness-of-fit measured by the residual sum of squares

\[ \text{RSS}(\mathcal{A}) = \sum_{j=0}^{m} \sum_{t=k_j+1}^{k_{j+1}} \left( X_t - \bar{X}_{(k_j+1):k_{j+1}} \right)^2 \text{ with } \tilde{k}_0 = 0 \text{ and } \tilde{k}_{m+1} = n, \]

and the penalty imposed on the model complexity \( |\mathcal{A}| \). The amount of the penalty is determined by the parameter \( \xi_n \) satisfying \( \xi_n \to \infty \); we defer the discussion on the choice of \( \xi_n \) to Section 2.2.
Niu and Zhang (2012) consider pruning down a candidate set $\mathcal{K}$ obtained from a multiscale MOSUM procedure (as a collection of local maximisers of MOSUM statistics at which a certain threshold is exceeded), by selecting the set returning the minimum information criterion value among every possible combination of the candidates in $\mathcal{K}$, while Yau and Zhao (2016) use an information criterion to similarly prune down the candidates from a single-bandwidth MOSUM procedure (local maximisers of MOSUM statistics without thresholding). Such an exhaustive approach may result in a computationally inhibitive search space as its size grows exponentially with $|\mathcal{K}|$. Moreover, it does not utilise the information readily available about the detection intervals of change point estimators. For example, if the detection interval of a candidate $k$ does not overlap with that of any other estimator, there is little to be gained by having $k$ considered alongside other candidates in the evaluation of SC. On the other hand, if $I(k)$ overlaps with the detection interval of another candidate, say $k'$, it is possible that $k$ and $k'$ are conflicting estimators of the identical change point, which justifies the joint consideration of the two.

Based on these observations, we propose the localised pruning methodology consisting of two nested algorithms, where the outer algorithm iteratively selects the local environment on which the inner algorithm performs the pruning.

2.1.1 Outer algorithm: Localisation ($\text{LocAlg}$)

Taking the set of change point candidates $\mathcal{K}$ as an input, the outer algorithm for localisation iteratively selects a subset of candidates to be pruned down by the inner algorithm ($\text{PrunAlg}$) described in Section 2.1.2. For this, the algorithm sorts the candidates in $\mathcal{K}$ according to a sorting function $h$. One possibility is to use the jump size associated with each $k \in \mathcal{K}$, which is readily calculated within the detection interval $I(k) = (k - G_L, k + G_R]$ as

$$h_J(k) = \left| \bar{X}_{(k-G_L+1):k} - \bar{X}_{(k+1):(k+G_R)} \right|. \quad (5)$$

If (asymptotic) null distributions of the test statistics are available, another possibility is to use the inverse of the p-values, say $h_P$, as a sorting function. In practice, the use of $h_P$ may slow down the pruning algorithm by generating many ties when many of the p-values are artificially set to zero by the machine (see Meier et al. (2019b)). Either with $h_J$ or $h_P$, additional tie-breaking rules can be employed, e.g., by preferring the candidates associated with the smallest detection interval according to $G_L + G_R$, $G_L$ or $G_R$; if there are still ties, an arbitrary choice can be made. We note that the theoretical results do not depend on the choice of the sorting function or the tie-breaking rule.

Denote by $\hat{\mathcal{C}}$ the candidates for which no decision has been reached yet, and by $\hat{\Theta}$ the set of already accepted candidates. At the beginning of the algorithm, the active candidate set $\mathcal{C}$ is given by the complete candidate set $\mathcal{K}$ and $\hat{\Theta}$ is set to be empty. Then, the outer algorithm iteratively processes the candidates in the following way.
Step 1: Find the most prominent candidate. According to a sorting function $h$ (and tie-breakers if necessary), find a candidate $k_o \in C$ from the active candidate set that maximises $h$.

Step 2: Define the local search environment. Find $k_L$ that is closest to $k_o$ while being strictly left to $k_o$ from the candidates which either

- have already been accepted (and belong to $\hat{\Theta} \cup \{0\}$), or
- are still to be either accepted or discarded ($C$) whose detection intervals do not overlap with that of $k_o$, i.e., $\mathcal{I}(k_L) \cap \mathcal{I}(k_o) = \emptyset$ or equivalently $|k_o - k_L| \geq G_R(k_L) + G_L(k_o)$.

Identify $k_R$ strictly to the right of $k_o$ from $\hat{\Theta} \cup \{n\} \cup C$ with analogous restrictions. Then, any candidates without decision that fall within $(k_L, k_R)$ are considered as candidates competing with $k_o$. We denote this set of change point candidates by $\mathcal{D}$, i.e., $\mathcal{D} = C \cap (k_L, k_R)$.

Step 3: Pruning Algorithm ($\text{PrunAlg}$). Apply the inner algorithm for pruning, $\text{PrunAlg}$, with the arguments $(\mathcal{D}, C, \hat{\Theta}, k_L, k_R)$. As an output, we yield a subset $\hat{\mathcal{A}} \subset \mathcal{D}$ (possibly empty) which contains candidates to be accepted in the next step.

Step 4: Update the accepted ($\hat{\Theta}$) and active ($C$) candidate sets. We accept all estimators from the output of $\text{PrunAlg}$, $\hat{\mathcal{A}}$, but not all of $\mathcal{D} \setminus \hat{\mathcal{A}}$ are discarded yet. This is because $\mathcal{D}$ may contain acceptable estimators of change points that are too close to the boundaries $k_L$ or $k_R$, for which we cannot guarantee their acceptance at the current iteration (see Theorem 2.1). However, if $k_L$ (resp. $k_R$) has already been accepted, we discard any candidates in $\mathcal{D} \setminus \hat{\mathcal{A}}$ which lie to the left (right) of the leftmost (rightmost) candidate in $\hat{\mathcal{A}}$. Similarly, unaccepted candidates in $\mathcal{D} \setminus \hat{\mathcal{A}}$ that lie between any two elements of $\hat{\mathcal{A}}$ are discarded. In addition, we remove $k_o$ from the future consideration regardless of whether it has been accepted by $\text{PrunAlg}$ or not.

In summary, we denote the set of all the candidates for which a decision has been reached, either because it has been accepted or discarded according to the above consideration, by $\mathcal{R}$. Then, we add $\hat{\mathcal{A}}$ to $\hat{\Theta}$ and remove all the candidates in $\mathcal{R}$ from $C$.

Step 5: Iteration. Repeat Steps 1 to 4 until $C$ is empty. The set $\hat{\Theta}$ is the final set of estimators and the output of the algorithm.

A pseudo-code of the outer algorithm can be found in Algorithm 1 of Appendix A. $\text{LocAlg}$ is guaranteed to terminate since at each iteration, Step 4 discards at least one candidate $k_o$ from the active candidate set. Under a mild condition on $K$, we show that this yields consistent estimation by guaranteeing that at least one suitable estimators remain in $C$ for all the undetected change points, see Assumption 2.5 and the discussion thereafter.
In Step 3 of \texttt{LocAlg}, the inner algorithm \texttt{PrunAlg} makes a decision between competing candidates using SC, which are evaluated at each $\mathcal{A} \subset \mathcal{D} = \mathcal{C} \cap (k_L, k_R)$ as

$$ SC(\mathcal{A}|\mathcal{C}, \hat{\Theta}, k_L, k_R) = \frac{n}{2} \log \left( \frac{\text{RSS}(\mathcal{A} \cup \hat{\Theta} \cup (\mathcal{C} \setminus \mathcal{D}))}{n} \right) + (|\mathcal{A}| + |\hat{\Theta}| + |\mathcal{C} \setminus \mathcal{D}|) \cdot \xi_n. $$  \hspace{1cm} (6)

By construction, it makes a decision which of the candidates in $\mathcal{D}$ to accept while treating all the other currently surviving candidates outside of $(k_L, k_R)$ as given. Therefore, at any iterations of \texttt{LocAlg}, all $X_t$, $1 \leq t \leq n$, enter in the computation of SC. In other words, \texttt{LocAlg} has the interpretation of performing an adaptively selected subset of the exhaustive search over the complete candidate set $\mathcal{K}$ in a localised manner, by utilising the information readily available about the detection intervals of change point candidates.

2.1.2 Inner algorithm: Pruning (\texttt{PrunAlg})

The inner pruning algorithm \texttt{PrunAlg} in Step 3 of the outer algorithm \texttt{LocAlg} takes as its input $(\mathcal{D}, \mathcal{C}, \hat{\Theta}, k_L, k_R)$, and looks for a subset $\hat{\mathcal{A}} \subset \mathcal{D}$ to be added to the finally accepted candidates according to the following rules:

Let $\mathcal{F}$ denote the collection of all subsets $\mathcal{A} \subset \mathcal{D}$ for which it holds:

$$ \text{adding further change point candidates to } \mathcal{A} \text{ monotonically increases } SC, \hspace{1cm} (C1) $$

and denote by $m^* = \min_{\mathcal{A} \in \mathcal{F}} |\mathcal{A}|$. Then, we select $\hat{\mathcal{A}}$ as

$$ \hat{\mathcal{A}} = \arg \min_{\mathcal{A} \subset \mathcal{R}} \mathcal{A}' \text{ with } \mathcal{A}' \in \mathcal{F} \text{ and } m^* \leq |\mathcal{A}'| \leq m^* + 2 : SC(\mathcal{A}|\mathcal{C}, \hat{\Theta}, k_L, k_R) $$ \hspace{1cm} (C2)

where, by $\mathcal{A} \subset \mathcal{R} \mathcal{A}' = \{ \hat{k}_1 < \hat{k}_2 < \ldots < \hat{k}_m \}$, we indicate that $\mathcal{A} \setminus \mathcal{A}' \subset \{ \hat{k}_1, \hat{k}_m \}$, i.e., $\mathcal{A}$ contains all inner elements of $\mathcal{A}'$ (if exist) while the first and the last elements of $\mathcal{A}'$ may or may not be included in $\mathcal{A}$. If there are multiple subsets yielding the minimum SC in $\text{(C2)}$, we choose the one with the minimum cardinality. If there are ties in the cardinality as well, we arbitrarily select one.

Remark 2.1. By performing a top-down search, the condition (C1) typically prunes down the search space quickly: If removing $k \in \mathcal{A}$ from $\mathcal{A}$ leads to an increase in SC, no subset of $\mathcal{A} \setminus \{ k \}$ can be an element of $\mathcal{F}$. For a complete algorithmic description of \texttt{PrunAlg}, see Algorithm 2 in Appendix A and also Meier et al. (2019b) for details about its efficient implementation.

Remark 2.2. It is possible to apply the search criteria (C1)–(C2) to $\mathcal{K}$ directly, without iteratively going through the steps of the outer algorithm. In such a case, (C2) is simplified to

$$ \hat{\mathcal{A}} = \arg \min_{\mathcal{A} \in \mathcal{F}} \{ \mathcal{A} \text{ with } |\mathcal{A}| = m^* : SC(\mathcal{A}|\mathcal{K}, \emptyset, 0, n) \}, \hspace{1cm} (C2') $$

i.e., search for $\hat{\mathcal{A}}$ only among the subsets satisfying (C1). This approach still gains computa-
tionally compared to minimising the SC among all the $2^{\lvert K \rvert}$ subsets of $K$ while, as shown in Corollary 2.2, achieves consistency in multiple change point estimation. However, it is still to be avoided when there are many candidates to be pruned down, and LocAlg greatly reduces the computational cost by breaking down the scope of PrunAlg at each iteration.

Remark 2.3. We highlight the key differences between the use of SC in PrunAlg and the conventional use of information criteria as a model selection tool in the change point literature. A common approach is to evaluate an information criterion at a sequence of nested candidate models with increasing number of change points, which often requires the maximum allowable number of change points, say $q_{\text{max}}$, as an input parameter. However, selection of this tuning parameter is not straightforward especially when $n$ is large, without pre-supposing the frequency or the sparsity of the change points, and some approaches require $q_{\text{max}}$ to be fixed in their theoretical consideration (Fryzlewicz, 2014; Baranowski et al., 2019). In contrast, our localised pruning method bypasses such a requirement by identifying local intervals over which the SC-based search is performed. In the simulation studies, we observe empirical evidence of the sub-optimality of sequential evaluation and minimisation of an information criterion, particularly when there are frequent changes in the signal (see e.g., Table 2), which further supports the search criteria $(C1)$–$(C2)$ adopted by PrunAlg.

2.2 Consistency of the localised pruning algorithm

In this section, we show that the localised pruning algorithm combining LocAlg and PrunAlg consistently estimates the total number of change points when applied to a suitable set of candidates. Furthermore, it ‘almost’ inherits the rate of convergence of the change point estimators from the candidate generating mechanisms, and thus achieves consistency in change point localisation when applied to the candidates generated by MOSUM- or CUSUM-based methods as discussed in Section 3.

For convenience, the assumptions below are formulated with asymptotic arguments. In fact, the proofs work directly with non-asymptotic conditions on the quantities therein on the set $\mathcal{M}_n$ defined in Theorem 2.1 such that non-asymptotic results can readily be derived.

For the following assumptions are imposed on the ‘size’ of changes in terms of the magnitude of the jumps and the distance between adjacent change points.

Assumption 2.1 (Size of changes). Let $\delta_j = \min(\theta_j - \theta_{j-1}, \theta_{j+1} - \theta_j)$ for $j = 1, \ldots, q_n$. Then, $\min_{1 \leq j \leq q_n} d_j^2 \delta_j \geq D_n$ for some $D_n \to \infty$ satisfying $D_n^{-1} \xi_n \to 0$. Also, $\max_{1 \leq j \leq q_n} \lvert d_j \rvert = O(1)$.

In the presence of a single change point ($q_n = 1$) with $\theta_1/n \to \kappa \in (0,1)$, Assumption 2.1 translates to $d_1^2 n \to \infty$, a condition commonly required for asymptotic power of change point tests and localisation consistency of estimators (see e.g., Theorem 1 of Chan and Walther (2013)). Effectively, Assumption 2.1 requires a strengthened version that accounts for the use of an information criterion for the estimation of the number of change points, to be held
locally at each change point, where we allow for $\delta_j$ of order smaller than $n$ provided that the corresponding $|d_j|$ is sufficiently large. [Wang et al. (2018)] investigate a phase transition in the space of model parameters for consistent localisation of change points in terms of $\sigma^{-2} \min_{1 \leq j \leq q_n} d_j^2 \cdot \min_{1 \leq j \leq q_n} \delta_j$ (see their Lemma 1). This formulation however rules out signals that e.g., have both frequent large jumps and small jumps over long stretches of stationarity, which serve as one of main motivating examples for adopting multiscale methods in change point analysis.

Next, we impose the following meta-assumptions on the error distribution.

**Assumption 2.2** (Error distribution).

(i) For some $0 < c_l < c_u < \infty$, let $P(M_n^{(1)}) \to 1$ where

$$M_n^{(1)} = \left\{ c_l \leq \frac{1}{n} \sum_{t=1}^{n} \varepsilon_t^2 \leq c_u \right\}.$$

(ii) For some $\omega_n$ satisfying $\xi_n^{-1} \omega_n^2 \to 0$ and $n^{-1} q_n \omega_n^2 \to 0$, let $P(M_n^{(2)}) \to 1$ where

$$M_n^{(2)} = \left\{ \max_{\theta \leq s \leq n} \frac{1}{\sqrt{n-s}} \sum_{t=s+1}^{n} \varepsilon_t \leq \omega_n \right\}.$$

(iii) For any sequences $1 \leq a_n, b_n \leq D_n$ with $D_n$ defined in Assumption 2.1, let $P(M_n^{(13)} \cap M_n^{(14)}) \to 1$ where

$$M_n^{(13)} = \left\{ \max_{1 \leq j \leq q_n} \max_{d_j^{-2} a_n \leq \ell \leq \theta_j - \ell + 1} \frac{\theta_j}{d_j^{-2} a_n} \sum_{t=\ell-\theta_j+1}^{\theta_j+\ell} \varepsilon_t \leq \omega_n^{(1)} \right\}$$

$$\cup \left\{ \max_{1 \leq j \leq q_n} \max_{d_j^{-2} a_n \leq \ell \leq \theta_j - \ell + 1} \frac{\theta_j + \ell}{d_j^{-2} a_n} \sum_{t=\ell+\theta_j+1}^{\theta_j+\ell+1} \varepsilon_t \leq \omega_n^{(1)} \right\},$$

and

$$M_n^{(14)} = \left\{ \max_{1 \leq j \leq q_n} \max_{1 \leq \ell \leq d_j^{-2} b_n} \frac{1}{d_j^{-2} b_n} \sum_{t=\ell+\theta_j+1}^{\theta_j+\ell+1} \varepsilon_t \leq \omega_n^{(2)} \right\}$$

$$\cup \left\{ \max_{1 \leq j \leq q_n} \max_{1 \leq \ell \leq d_j^{-2} b_n} \frac{1}{d_j^{-2} b_n} \sum_{t=\theta_j+1}^{\theta_j+\ell} \varepsilon_t \leq \omega_n^{(2)} \right\}.$$

**Remark 2.4.** (a) Assumption 2.2 (i) is very weak and follows e.g., under ergodicity and the existence of the second moment of $\varepsilon_t$.

(b) Assumption 2.2 (ii) relates the penalty term to the level of noise present. For i.i.d. innovations whose moment generating function exists, the weakest possible choice of $\omega_n$...
is of order $\sqrt{\log(n)}$ (see Theorem 1 of Shao (1995)), thus allowing for the (strengthened) Schwarz penalty of $\xi_n = \log^{1+\Delta}(n)$ for some $\Delta \geq 0$ (see e.g., Yao (1988) and Fryzlewicz (2014)). Indeed, to yield this rate for $\omega_n$, the existence of the moment generating function is necessary (Corollary 2 of Shao (1995)). For heavier-tailed $\varepsilon_t$, when $E(\varepsilon_t^{2+\Delta}) < \infty$, it is not possible to obtain a rate for $\omega_n$ better than $n^{1/(3+\Delta)}$ (see Theorem 1.1 of Mikosch and Račkauskas (2010)). In particular, this shows that for heavy-tailed errors, a penalty stronger than logarithmic in $n$ is required in order to guarantee consistent estimation of the number of change points by means of the SC, an observation also made by Kühn (2001). For dependent errors, similar results are derived in Mikosch and Moser (2013).

(e) Alternatively, suppose that there exists (possibly after changing the probability space) a standard Wiener process $W(\cdot)$ such that $\sum_{t=1}^\ell \varepsilon_t - W(\ell) = O(\lambda_\ell)$ a.s. with $\lambda_\ell = o(\sqrt{\ell})$ (see also Assumption 3.1 (i) below). Under this invariance principle, one can choose $\omega_n$ of order $\max(\lambda_n, \sqrt{\log n})$ in Assumption 2.2 (ii). This approach has the advantage that such invariance principles exist for a large class of dependent time series satisfying e.g., mixing (Theorem 4 in Kuelbs and Philipp (1980)) or functional dependence measure (Berkes et al. 2014) conditions, to name but a few. On the other hand, it results in a sub-optimal choice of $\omega_n$ for the purpose of bounding the sum in Assumption 2.2 (ii), compared to the rates given in the above (b): For example, in the i.i.d. case, the optimal choice for the invariance principle is given by $\lambda_\ell = \log(\ell)$ if the moment generating function exists, and by $\lambda_\ell = \ell^{1/(2+\Delta)}$ if $E(\varepsilon_t^{2+\Delta}) < \infty$ (see Komlós et al. (1975) and Komlós et al. (1976)).

(d) Together with the requirement that $D_n^{-1}\xi_n \to 0$ in Assumption 2.1, the discussion in (b) shows that for heavy-tailed innovations, stronger assumptions are required on $D_n$ and, consequently, on the magnitude of changes.

(e) The bounds $\omega_n^{(1)}$ and $\omega_n^{(2)}$ in Assumption 2.2 (iii) are related to the rate of convergence for the estimators of the locations of the change points (see Assumption 2.4 and Theorem 2.1). In the special case where $q_n$ is fixed (and thus the rates of the inner maxima over $\ell$ carry over), this stochastic boundedness holds for a large class of errors, a result well-known as Hájek-Rényi-type inequalities. In particular, they are easily derived if the forward as well as backward invariance principles hold (the latter follows if the underlying dependence assumption holds both forward and backward in time).

Moreover, for many stationary error sequences, it holds $E \left( \sum_{t=1}^{r} \varepsilon_t \right)^7 \leq C_\varepsilon (r - l)^{\gamma/2}$ for some constants $C_\varepsilon$ and $\gamma > 2$ (see e.g., Appendix B.1 in Kirch (2006) and also Assumption 3.1 (ii)). If so, the $\gamma$-th moment of the inner maxima, say $M(j)$, of Assumption 2.2 (iii) (e.g., $M(j) = \max_{d^{-1}_j a_n \leq \ell \leq \theta_j - \ell - 1} \ell^{-1/2} \sum_{t=\theta_j - \ell + 1}^{d^{-1}_j a_n} \varepsilon_t$ for the first set) are (uniformly) bounded, see Theorem B.3 in Kirch (2006). From this and the
Markov inequality, we yield
\[
P \left( \max_{1 \leq j \leq q_n} M_n(j) \geq \vartheta_n \right) \leq \frac{q_n}{\vartheta_n} E \{ (M_n(1))^{\gamma} \} = O(1) \left( \frac{q_n^{1/\gamma}}{\vartheta_n} \right)^{\gamma}
\]
which can be made \(o(1)\) if \(q_n^{1/\gamma} = o(\vartheta_n)\). Therefore, for increasing \(q_n\), the bounds \(\omega_n^{(1)}\) and \(\omega_n^{(2)}\) in Assumption 2.2 (iii) can be of arbitrarily small polynomial order of \(q_n\) provided that the error sequence has enough moments.

The following assumption is imposed on the candidate generating methodology.

**Assumption 2.3** (Candidate generating algorithm). Let \(K = K_n\) denote the set of candidates obtained from \(X_1, \ldots, X_n\) and \(Q_n = |K|\) the total number of candidates.

(i) With probability approaching one, each change point has at least one candidate in its \((d_j^{-2} \rho_n)\)-environment, i.e., as \(n \to \infty\),
\[
P(\mathcal{M}_n^{(2)}) \to 1 \quad \text{where} \quad \mathcal{M}_n^{(2)} = \left\{ \max_{1 \leq j \leq q_n} \min_{k \in K} d_j^2 |k - \theta_j| \leq \rho_n \right\}.
\]

(ii) The total number of candidates \(Q_n\) fulfils \(n^{-1} \omega_n^2 Q_n \to 0\) with \(\omega_n\) as in Assumption 2.2 (ii).

The sequence \(\rho_n\) is the precision associated with the candidate generating method. We show that the generic pruning algorithm almost inherits this rate in the sense made more precise in Theorem 2.1. Assumption 2.3 (ii) on the number of candidates is necessary because unlike the literature adopting the information criterion for determining the number of change points (e.g., Yao (1988) or Kühn (2001)), we do not impose an upper bound on the number of candidates that are simultaneously considered in the SC. In particular, this rules out applying the localised pruning algorithm with every possible point as candidate estimators, i.e., \(K = \{1, \ldots, n-1\}\). However, a reasonably good candidate generating method ought not to return too many candidates while meeting Assumption 2.3 (i), and we show that indeed, this is the case with the MOSUM- and CUSUM-based candidate generating methods in Section 3.

The following definitions that categorise the candidate estimators in \(K\) are frequently used throughout the paper.

**Definition 2.1.**

(i) A candidate \(k^* \in K\) that yields \(d_j^2 |k^* - \theta_j| \leq \rho_n\) with \(\rho_n\) as in Assumption 2.3 (i) is referred to as a **strictly valid** estimator for \(\theta_j\), and the set of such candidates is denoted by \(V_j^{(s)}\) for each \(j = 1, \ldots, q_n\).

(ii) For \(\nu_n \to \infty\) as specified in Assumption 2.4 below, a candidate \(k' \in K\) with \(d_j^2 |k' - \theta_j| \leq \rho_n \nu_n\) is referred to as an **acceptable** estimator for \(\theta_j\), and the set of such candidates is denoted by \(V_j^{(a)}\).
(iii) The remaining candidates \( k \in \mathcal{K} \setminus \mathcal{V}_j \) are \textit{unacceptable} candidates for \( \theta_j \).

The gap between the best localisation rate \( \rho_n \) of the candidate generating procedure and what is acceptable for the localised pruning algorithm is unavoidable: For two very close neighbouring candidates, the SC from the one slightly further away from the change point than the other can end up being smaller simply by chance. The following assumption specifies the price to pay in terms of change point localisation for guaranteeing that the localised pruning algorithm will always return acceptable estimators.

**Assumption 2.4.** The rate \( \nu_n \to \infty \) as in Definition 2.1 (ii) satisfies

\[
\frac{\omega_n^{(1)}}{\sqrt{\nu_n \rho_n}} \to 0, \quad \frac{\omega_n^{(2)}}{\nu_n \sqrt{\rho_n}} \to 0 \quad \text{and} \quad \frac{\rho_n \nu_n}{\xi_n} \to 0.
\]

As discussed in Remark 2.4 (e), \( \omega_n^{(1)} \) and \( \omega_n^{(2)} \) can be diverging arbitrarily slow in many situations and so does \( \nu_n \), which allows for the localised pruning algorithm to (almost) preserve the precision in change point localisation originally attained by the candidate generating methodology.

We now show that \texttt{PrunAlg} described in Section 2.1.2, as a generic pruning algorithm, achieves consistent estimation of the number of change points as well as returning acceptable estimators for all \( \theta_j, \ j = 1, \ldots, q_n \). Although the boundary points \((k_L, k_R)\) supplied as input arguments to \texttt{PrunAlg} are always chosen among the change point candidates (including 0 and \( n \)) in Step 2 of \texttt{LocAlg}, our theory below is applicable to any \((s, e]\) with \( 0 \leq s < e \leq n \) as the interval of consideration and \( \mathcal{D} = \mathcal{K} \cap (s, e] \) as the set of local candidates to be pruned down. In this context, it is understood that \( \hat{\Theta} \) contains candidates lying outside \((s, e]\) only.

It may be the case that some change points are too close to either \( s \) or \( e \) and thus may or may not be detectable by \texttt{PrunAlg} within \((s, e]\), which necessitates the pruning criterion (C2) instead of the simpler (C2′). We define the following sets with universal constants \( 0 < c^* < C^* < \infty \) as in Proposition 6.1 below:

\[
\Theta^{(s,e)} = \{ \theta_j : d_j^2 \min(\theta_j - s, e - \theta_j) \geq C^* \xi_n \}, \quad (7)
\]

\[
\tilde{\Theta}^{(s,e)} = \{ \theta_j : d_j^2 \min(\theta_j - s, e - \theta_j) \geq c^* \xi_n \}. \quad (8)
\]

Theorem 2.1 establishes the connection between the output of \texttt{PrunAlg} and the sets defined in (7)–(8).

**Theorem 2.1.** Let Assumptions 2.1–2.4 hold and denote by \( \hat{\Theta}^{(s,e)} \) the output of \texttt{PrunAlg} from applying the criteria (C1)–(C2) to the local candidates \( \mathcal{D} = \mathcal{K} \cap (s, e] \) within an interval \((s, e]\), and by \( \mathcal{P}^{(s,e)}_n \) the following event: The output set \( \hat{\Theta}^{(s,e)} \) contains

(i) exactly one acceptable candidate for each \( \theta_j \in \Theta^{(s,e)} \), i.e., \( |\hat{\Theta}^{(s,e)} \cap \mathcal{V}_j\prime| = 1 \) for \( \theta_j \in \Theta^{(s,e)}, \)
(ii) at most one acceptable candidate for each \( \theta_j \in \Theta^{(s,e)} \setminus \bar{\Theta}^{(s,e)} \), i.e., \(|\hat{\Theta}^{(s,e)} \cap \mathcal{V}'_j| \leq 1\) for \( \theta_j \in \Theta^{(s,e)} \setminus \bar{\Theta}^{(s,e)} \), and

(iii) no other candidates, i.e., \( \hat{\Theta}^{(s,e)} \setminus \bigcup_{j: \theta_j \in \bar{\Theta}^{(s,e)}} \mathcal{V}'_j = \emptyset \).

Then, with \( \mathcal{M}_n := \mathcal{M}_n^{(11)} \cap \mathcal{M}_n^{(12)} \cap \mathcal{M}_n^{(13)} \cap \mathcal{M}_n^{(14)} \cap \mathcal{M}_n^{(2)} \), we have

\[
P \left( \bigcap_{0 \leq s < e \leq n} \mathcal{P}^{(s,e)}_n, \mathcal{M}_n \right) \rightarrow 1 \quad \text{as} \quad n \rightarrow \infty.
\]

In view of Theorem 2.1, we categorise the change points according to their detectability within a given interval in the following definition.

**Definition 2.2.** For any \( 0 \leq s < e \leq n \), we refer to

(i) any change points in \( \Theta^{(s,e)} \) as **surely detectable** within \((s,e]\),

(ii) any change points in \( \bar{\Theta}^{(s,e)} \) as **detectable** within \((s,e]\), and

(iii) any change points in \( \{\Theta \cap (s,e]\} \setminus \bar{\Theta}^{(s,e)} \) as **undetectable** within \((s,e]\).

The following corollary establishes that \texttt{PrunAlg}, when applied to the complete candidate set \( \mathcal{K} \), directly achieves consistency in multiple change point estimation.

**Corollary 2.2.** Under the assumptions of Theorem 2.1, applying the search criteria (C1) and (C2) to the candidate set \( \mathcal{K} \) within \((0,n]\) yields \( \hat{\Theta}^{(0,n)} = \{\hat{\theta}_1 < \ldots < \hat{\theta}_q\} \) which consistently estimates \( \Theta \), i.e.,

\[
P \left( \hat{q}_n = q_n; \max_{1 \leq j \leq q_n} d_j^2 |\hat{\theta}_j - \theta_j| \leq \rho_n \nu \right) \geq P(M_n) + o(1) \rightarrow 1.
\]

As pointed out in Remark 2.2, pruning down \( \mathcal{K} \) according to (C1) and (C2) is computationally more efficient than the exhaustive minimisation of SC over all subsets of \( \mathcal{K} \). Nevertheless, the localisation from the outer algorithm \texttt{LocAlg} results in a considerable computational advantage when a large set of candidates needs to be pruned down.

Next, we establish that the consistency achieved by \texttt{PrunAlg} within local search environments (as in Theorem 2.1), is carried over to the entire data set via the outer localisation algorithm \texttt{LocAlg}.

**Assumption 2.5.** Recall that the detection interval of each \( k \in \mathcal{K} \) is denoted by \( I(k) = (k - G_L(k), k + G_R(k)) \). Then, for each \( j = 1, \ldots, q_n \), there exists at least one acceptable candidate \( \hat{k}_j \in \mathcal{V}'_j \) which is situated well within its own detection interval by satisfying

\[
\frac{\xi_n}{d_j^2 \min\{G_L(k_j), G_R(k_j)\}} \rightarrow 0.
\]
Assumption 2.5 justifies the removal of $k_0$ identified in Step 1 of each iteration from the future consideration, regardless of whether it is accepted by PrunAlg or not: If $k_0$ is an acceptable estimator for some $\theta_j$ while meeting (9), such $\theta_j$ is surely detectable within $(k_L, k_R]$ and either $k_0$ or some $k \in \mathcal{V}'_j$ is accepted by PrunAlg at the current iteration; if not, there still remain at least one acceptable estimators in the active candidate set $\mathcal{C}$ for any undetected change points after removing $k_0$. For a more detailed discussion on how Assumption 2.5 is met by specific candidate generating mechanisms, we refer to Remarks 3.1 and 3.3.

Theorem 2.3 proves that PrunAlg combined with the outer algorithm LocAlg achieves consistency in multiple change point estimation.

**Theorem 2.3.** Under the assumptions of Theorem 2.1 and Assumption 2.5, the localised pruning algorithm LocAlg outputs $\hat{\Theta} = \{\hat{\theta}_1 < \ldots < \hat{\theta}_{q_n}\}$ which consistantly estimates $\Theta$, i.e.,

$$
P\left\{q_n = q_n; \max_{1 \leq j \leq q_n} \rho_n \left| \hat{\theta}_j \mathbb{I}_{\{j \leq q_n\}} - \theta_j \right| \leq \rho_n \nu_n \right\} \geq P(M_n) + o(1) \to 1.$$

Its proof follows from the following two observations:

- When a change point is surely detectable for the first time at some iteration (in the sense of Definition 2.2 (i)), it gets detected by an acceptable estimator by Theorem 2.1 and consequently is no longer detectable in the subsequent iterations thanks to how the local environments are defined in Step 2 of LocAlg.

- On the other hand, those change points which are yet to be detected have corresponding acceptable estimators in the pool of candidates $\mathcal{C}$ due to how $\mathcal{C}$ is reduced in Step 4 of LocAlg.

## 3 Candidate generating procedures

One advantage of adopting test-based approaches to multiple change point analysis, over those formulated as optimisation problems, is that confidence intervals for the locations of change points can be constructed conditional on the consistent estimation of multiple change points similarly as in Hušková and Kirch (2008). In this section, we investigate the application of the localised pruning methodology with two candidate generating methodologies, a multiscale extension of the MOSUM procedure of Eichinger and Kirch (2018), and a CUSUM-based procedure motivated by the Wild Binary Segmentation of Fryzlewicz (2014). In particular, combined with the former, it is shown that the localised pruning achieves minimax optimality in change point localisation in the presence of a fixed (yet unknown) number of change points.
3.1 MOSUM-based candidate generation

In this section, we study the consistency of the localised pruning algorithm with a multiscale MOSUM procedure as the candidate generating method.

For a given bandwidth \( G = G_n \), Eichinger and Kirch \citeyearpar{Eichinger2018} propose to estimate the locations of multiple change points by \( K(G, \alpha) = \{ k_{G,j}, 1 \leq j \leq \hat{q}_G \} \) using two methods:

- **\( \epsilon \)-criterion.** Each \( k_{G,j} \) is the local maximiser of the MOSUM detector (modulus of the MOSUM statistic in (2)) within an interval of length at least \( \epsilon G \) for some \( \epsilon > 0 \) over which \( \hat{\tau}_n^{-1} \min_{|b| \leq k_{G,j}} | T_{b,n}(G; X) | > D_n(G; \alpha) \), where \( \hat{\tau}_n^2 \) is an estimator of the long-run variance of the error sequence, \( D_n(G; \alpha) \) denotes the critical value and \( \alpha \in (0, 1) \) the significance level.

- **\( \eta \)-criterion.** Each \( k_{G,j} \) is the local maximiser of the MOSUM detector within its \( \lfloor \eta G \rfloor \)-radius for some \( \eta > 0 \), and \( \hat{\tau}_n^{-1} | T_{k_{G,j},n}(G; X) | > D_n(G; \alpha) \).

The critical value \( D_n(G; \alpha) \) is chosen such that for a signal with no change points, there are no false positives reported with asymptotic probability \( (1 - \alpha) \). As noted in Meier et al. \citeyearpar{Meier2019b}, the \( \epsilon \)-criterion is more conservative of the two. Since we consider the multiscale MOSUM procedure as a candidate generating mechanism, in the remainder of this paper, we assume that the \( \eta \)-criterion is used to generate the estimators of change point locations.

A slight variation of the arguments given in the proofs of Eichinger and Kirch \citeyearpar{Eichinger2018} shows that the set of candidates generated by the MOSUM procedure can achieve a localisation rate of \( O_P(d_j^{-2}) \) for each change point provided that an appropriate bandwidth is used, see Proposition 3.1 below. To this end, we make the following assumptions.

**Assumption 3.1.**

(i) There exists a standard Wiener process \( \{ W(k) : 1 \leq k \leq n \} \) and \( \lambda_n = o(\sqrt{n}) \) such that (possibly after changing the probability space)

\[
\max_{1 \leq k \leq n} \left| \sum_{t=1}^{k} \varepsilon_t - \tau W(k) \right| = O(\lambda_n) \quad \text{a.s.} \tag{10}
\]

with an existing and strictly positive long-run variance \( \tau^2 = \sigma^2 + 2 \sum_{h>0} \text{Cov}(\varepsilon_0, \varepsilon_h) \).

(ii) For some fixed \( \gamma > 2 \) and \( C_\varepsilon > 0 \), it holds for any \( -\infty < l < r < \infty \),

\[
E \left| \sum_{t=l+1}^{r} \varepsilon_t \right|^\gamma \leq C_\varepsilon (r - l)^{\gamma/2}.
\]

Assumption 3.1 is closely related to Assumption 2.2 required for the consistency of PrunAlg, see Remark 2.4.
In this paper, we suppose that a universal estimator \( \hat{\tau}^2_n \) for \( \tau^2 \) is used for simplicity. Section 2.3 of Eichinger and Kirch (2018) is devoted to showing that a scale-dependent local estimator of \( \tau^2 \) fulfils a relaxed version of Assumption 3.2 (their Assumption A.4) that does not require the estimator to be uniformly consistent.

**Assumption 3.2.** The (long-run) variance estimator is consistent, i.e.,

\[
|\hat{\tau}^2_n - \tau^2| = o_P (\log^{-1}(n)).
\]

The following proposition extends Theorem 3.2 of Eichinger and Kirch (2018) for the change point estimators generated according to the \( \eta \)-criterion with \( 0 < \eta < 1 \).

**Proposition 3.1.** Let Assumptions 3.1–3.2 hold. In addition, suppose that for each \( j = 1, \ldots, q_n \), there exists \( G(j) \) such that

\[
\frac{G(j)}{n} \to 0, \quad \frac{\lambda_n^2 \log(n)}{G(j)} \to 0, \quad \frac{\log(n/G(j))}{d^2_j G(j)} \to 0, \quad (11)
\]

as well as

\[
2G(j) < \delta_j = \min(\theta_j - \theta_{j-1}, \theta_{j+1} - \theta_j). \quad (12)
\]

Then, for any \( \rho_n^{(M)} \to \infty \) with \( \rho_n^{(M)} \leq d^2_j G(j) \) and \( q_n^{2/\gamma}/\rho_n^{(M)} \to 0 \), \( \alpha \in (0,1) \) and a set \( S_n \) (specified in Lemma C.1 of Appendix) fulfilling \( P(S_n) \to 1 \), it holds

\[
P\left( \max_{1 \leq j \leq q_n} \left\{ \min_{k \in K(G(j), \alpha)} d^2_j |k - \theta_j| \geq \rho_n^{(M)}, S_n \right\} \right) \to 0.
\]

The assertion of the proposition remains correct if we use \( \alpha = \alpha_n \to 0 \) as long as it does not converge to zero too fast, such that \( \log \log(1/\sqrt{1 - \alpha_n}) = O(\log(n/\max_{1 \leq j \leq q_n} G(j))) \). For the single-bandwidth procedure proposed by Eichinger and Kirch (2018), this extra condition on \( \alpha_n \) is required in order to avoid false positives completely which, for the purpose of candidate generation, can be dropped.

In establishing the consistency of a single-bandwidth MOSUM procedure, conditions akin to (11)–(12) are imposed on a single bandwidth which limits its applicability to a variety of situations. A multiscale extension of the MOSUM procedure is a natural solution to this lack of adaptivity, which also allows for bypassing the difficult task of selecting a single suitable bandwidth. By performing the MOSUM procedure with a range of bandwidths including asymmetric ones, each \( \theta_j \) becomes detectable at some time scale. At the same time, scanning the same data at multiple scales introduces duplicate estimators and false positives, necessitating the use of a pruning method.

To resolve this issue, Messer et al. (2014) and Messer et al. (2018) propose to prune down the estimators from a multiscale MOSUM procedure in a bottom-up manner. Accepting all
the estimators from the smallest bandwidth, it proceeds to coarser scales and only accepts a change point estimator if its detection interval does not contain any estimators that are already accepted. A similar approach is taken by Chan and Chen (2017) for pruning down the estimators from a multiscale MOSUM procedure with asymmetric bandwidths. While the bottom-up approach is applicable with multiple symmetric bandwidths, there is no canonical ordering when asymmetric bandwidths are used. Moreover, this approach rules out the possibility of removing any spurious estimators including those detected from the finest bandwidth, and thus requires the finest bandwidth to be large relative to \( n \) in order to avoid spurious change point estimators.

The localised pruning algorithm proposed in Section 2.1 is well-suited for pruning down the candidates generated by the multiscale MOSUM procedure. Let \( \mathcal{G} \) denote a set of bandwidths. Each estimator \( k \in \mathcal{K}(G,\alpha) \) for \( G \in \mathcal{G} \) is associated with the natural detection interval \( I_N(k) = (k - G, k + G) \). Asymmetric bandwidths \( G = (G_l, G_r) \) with \( (G_l, G_r) \in \mathcal{H} \subset \mathcal{G} \times \mathcal{G} \) are readily incorporated into the methodology using the MOSUM statistics defined as a correctly scaled difference between \( \bar{X}_{[b-G_l+1]:b} \) and \( \bar{X}_{(b+1):(b+G_r)} \) for \( b = G_l, \ldots, n - G_r \), and the corresponding \( I_N(k) = (k - G_l, k + G_r) \) for \( k \in \mathcal{K}(G,\alpha) \); for more details, we refer to Meier et al. (2019b). Then, the collection of all the estimators from the multiscale MOSUM procedure, \( \mathcal{K}(\mathcal{H},\alpha) = \bigcup_{G \in \mathcal{H}} \mathcal{K}(G,\alpha) \), can serve as the set of candidates \( \mathcal{K} \). For Step 1 of the outer localisation algorithm \text{LocAlg}, we can sort the candidate change points either according to the size of associated jumps (see (5)), or using the \( p \)-values derived from the asymptotic null distribution defined for each pair of bandwidths (although care should be taken in their interpretation across multiple scales), giving precedence to those candidates with smaller \( p \)-values.

**Selection of bandwidths**

We propose to generate the set of bandwidths \( \mathcal{G} \) as follows: Selecting a single parameter \( G_0 \), which should be smaller than the minimal distance between adjacent change points, and setting \( G_1 = G_0 \), we iteratively yield \( G_m, m \geq 2 \), as a Fibonacci sequence, i.e., \( G_m = G_{m-1} + G_{m-2} \). Equivalently, we set \( G_m = F_m G_0 \) where \( F_m = F_{m-1} + F_{m-2} \) with \( F_0 = F_1 = 1 \) are the Fibonacci numbers. This is repeated until for some \( H = H_n \), it holds that \( G_H < \lfloor n / \log(n) \rfloor \) while \( G_{H+1} \geq \lfloor n / \log(n) \rfloor \). When using asymmetric bandwidths, it is advisable to avoid the pairs of bandwidths which are too strongly unbalanced, both in view of the asymptotic theory and the finite sample performance as is well-known from two-sample testing. A similar requirement can also be found in Chan and Chen (2017). For this reason, we only include the pairs of bandwidths \( G = (G_l, G_r) \) in \( \mathcal{H} \) that satisfy

\[
G_l, G_r \in \mathcal{G} = \{G_1, \ldots, G_H\} \quad \text{with} \quad \frac{\max(G_l, G_r)}{\min(G_l, G_r)} \leq C_{\text{asym}} \tag{13}
\]
for some constant $C_{\text{asym}} > 0$. This also guarantees that the number of total candidates meets Assumption 2.3 (ii).

The thus-generated set of (asymmetric) bandwidths $\mathcal{H}$ is expected to lend the adaptivity for detecting different types of changes, while effectively controlling the total number of change point candidates.

**Remark 3.1.** (a) For each $k \in \mathcal{K}(\mathcal{H}, \alpha)$, the natural detection interval $I_N(k)$ can serve as its detection interval $I(k) = (k - G_L, k + G_R]$, whereby the detection distances $(G_L, G_R)$ are given by the set of bandwidths $(G_\ell, G_r)$ with which $k$ has been detected. Then, we have Assumption 2.5 fulfilled by $\mathcal{K}(\mathcal{H}, \alpha)$ provided that there exists a single bandwidth $G(j) \in \mathcal{G}$ satisfying $d_j^2 G(j)/\xi_n \rightarrow \infty$ for each $j = 1, \ldots, q_n$. Since changes are detectable with asymptotic probability one only when $d_j^2 G(j) \rightarrow \infty$, this is a mild assumption at least for logarithmic $\xi_n$. Under Assumptions 2.1 and 2.2, we can find such $G(j)$ that also meets (11)–(12), and thus returns a strictly valid estimator $k^* \in \mathcal{V}_j^*$ (defined with $\rho_n^{(M)}$, see Definition 2.1) of $\theta_j$ thanks to Proposition 3.1.

(b) It may be the case that $\mathcal{K}(\mathcal{H}, \alpha)$ contains identical acceptable candidates $k$ returned at multiple scales, including some $(G_\ell, G_r)$ that does not satisfy $d_j^2 \min(G_\ell, G_r)/\xi_n \rightarrow \infty$. Against such a contingency, we propose to assign as $I(k)$ the natural detection interval that returns the smallest $p$-value for the MOSUM test associated with the detection of $k$. Because the $p$-values decrease with the increase of jump size as well as that of bandwidths, this strategy will recommend a reasonably large natural detection interval as $I(k)$. In simulation studies, we use an implementation of the localised pruning algorithm which simply supposes that Assumption 2.5 is satisfied by the candidate generating mechanism.

We now show that the set of candidates $\mathcal{K}(\mathcal{H}, \alpha)$ generated by the multiscale MOSUM procedure meets Assumption 2.3 on the set of candidate estimators, which leads to the consistency of $\text{LocAlg}$ applied with the multiscale MOSUM procedure.

**Proposition 3.2.**

(i) Let Assumptions 3.1–3.2 hold, and suppose that there exists at least one $G(j) \in \mathcal{G}$ for each $j = 1, \ldots, q_n$ which satisfies (11) and (12). Then, for any $\rho_n^{(M)} \rightarrow \infty$ with $\rho_n^{(M)} \leq \min_j d_j^2 G(j)/\xi_n$ and $q_n^{2/\gamma}/\rho_n^{(M)} \rightarrow 0$, it holds

$$P \left( \max_{1 \leq j \leq q_n} \min_{k \in \mathcal{K}(\mathcal{H}, \alpha)} d_j^2 |k - \theta_j| \leq \rho_n^{(M)} \right) \rightarrow 1.$$ 

(ii) Suppose $\omega_n^2/G_0 \rightarrow 0$ with $\omega_n$ as in Assumption 2.2 (ii). Then, for $\mathcal{H}$ fulfilling (13), we have $n^{-1} \omega_n^2 |\mathcal{K}(\mathcal{H}, \alpha)| \rightarrow 0$. 

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We conjecture that the requirement \( q_n^{2/\gamma} / \rho_n^{(M)} \rightarrow 0 \) can be much relaxed. Further, in the special case of a finite number of change points, it does not impose an additional restriction. The assumption \( \omega_n^2 / G_0 \rightarrow 0 \) is made solely to obtain a crude deterministic upper bound on the number of possible candidates from the smallest bandwidth which, in light of Remark 2.4 (e) and (11), is readily met. We may replace the condition by one that directly limits the number of candidates detected at each bandwidth, or an assumption on \( q_n \) in combination with a stochastic version of Assumption 2.3. The consistency of the localised pruning algorithm in combination with the MOSUM-based candidate generating mechanism immediately follows from Proposition 3.2 and Theorem 2.3.

**Theorem 3.3.** Let Assumptions 2.1, 2.2, 2.4 and 2.5 hold, and suppose that the conditions in Proposition 3.2 are satisfied. Then, the localised pruning algorithm \( \text{LocAlg} \) applied to \( K(H, \alpha) \) outputs \( \hat{\Theta} = \{\hat{\theta}_1 < \ldots < \hat{\theta}_{q_n}\} \) which consistently estimates \( \Theta \), i.e.,

\[
P\left\{ \hat{q}_n = q_n; \max_{1 \leq j \leq q_n} d^2_j(\hat{\theta}_j - \theta_j) \leq \rho_n^{(M)} \nu_n \right\} \rightarrow 1,
\]

with \( \rho_n^{(M)} \) as in Proposition 3.2.

**Remark 3.2 (Minimax optimality in localisation).** Lemma 2 of Wang et al. (2018) provides the asymptotic minimax lower bound on the localisation rate in the at-most-one-change scenario, as \( d_1^2(\hat{\theta}_1 - \theta_1) = O_P(1) \). Remark 2.4 (e) shows that under appropriate assumptions on \( \varepsilon_t \), the sequence \( \nu_n \) can diverge arbitrarily slowly. Then, Theorem 3.3 establishes that \( d_j^2(\hat{\theta}_j - \theta_j) = O_P(1) \), i.e., \( \text{LocAlg} \) applied to the candidates generated by the multiscale MOSUM procedure achieves the minimax optimality in multiple change point localisation.

### 3.2 CUSUM-based candidate generation

The CUSUM statistic in (3) is designed to test the null hypothesis of no change point \( (H_0: q_n = 0) \) against the at-most-one-change alternative \( (H_1: q_n = 1) \). It corresponds to the likelihood ratio statistic under i.i.d. Gaussian errors and as such, is particularly appropriate for single change point estimation.

For multiple change point detection, Vostrikova (1981) and Venkatraman (1992) establish the consistency of the Binary Segmentation algorithm that makes recursive use of CUSUM-based estimation. However, its sub-optimality, both in terms of the conditions required for the consistency and the rate of change point localisation, has been noted in Fryzlewicz (2014). As an alternative, he proposes the Wild Binary Segmentation (WBS) which aims at isolating the change points by drawing a large number of random intervals. When a sufficient number of random intervals are drawn, with large probability, there exists at least one interval which is well-suited for the detection and localisation of each \( \theta_j, j = 1, \ldots, q_n \). Recently, Fryzlewicz (2018a) proposes its variation (WBS2) that draws random intervals in a more systematic
fashion and generates a complete solution path, from which the final model is selected via the SDLL principle introduced therein.

We propose the following version of WBS2 as a candidate generating mechanism. It requires the tuning parameters $R_n$, the maximal number of random intervals to be drawn at each iteration, $\Delta_n$, the minimal distance of each random interval from the current boundary points, and $\tilde{Q}_n$, which relates to the maximal depth of recursion $L_n$ as $L_n = \lfloor \log_2(\tilde{Q}_n + 1) \rfloor$. The step-by-step description of the WBS2 is provided below.

**Step 0:** Initialise the input arguments: The set of candidates $\mathcal{K}(R_n, \Delta_n, \tilde{Q}_n) = \emptyset$, $s = 0$, $e = n$ and the recursion depth $\ell = 1$.

**Step 1:** Quit the routine if $e - s = 2\Delta_n + 1$ or $\ell > L_n$; if not, let $\tilde{R} = \min\{R_n, (e - s - 2\Delta_n + 1)/2\}$. If $\tilde{R} \leq R_n$, let $\mathcal{R}_{s,e} = \{(l, r) \in \mathbb{Z}^2 : s + \Delta_n \leq l < r \leq e - \Delta_n\}$ serve as $[s_m, e_m], m = 1, \ldots, \tilde{R}$. If not, draw $\tilde{R}$ intervals $[s_m, e_m], m = 1, \ldots, \tilde{R}$, uniformly at random from the set $\mathcal{R}_{s,e}$.

**Step 2:** Identify $(m_0, k_0) = \arg\max_{(m,b) : 1 \leq m \leq \tilde{R}, s_m < b < e_m} |\mathcal{X}_{s_m,b,e_m}|$.

**Step 3:** Update $\mathcal{K}(R_n, \Delta_n, \tilde{Q}_n)$ by adding $k_0$ and store its natural detection interval $\mathcal{I}_N(k_0) = (s_o, e_o)$.

**Step 4:** Repeat Steps 1–3 separately with $(s, k_0, \ell + 1)$ and $(k_0, e, \ell + 1)$.

Through implementing the maximal recursion depth into the procedure, it trivially holds that the size of candidate set satisfies $|\mathcal{K}(R_n, \Delta_n, \tilde{Q}_n)| \leq \tilde{Q}_n$. We propose to apply the localised pruning to the thus-generated set of candidates $\mathcal{K}(R_n, \Delta_n, \tilde{Q}_n)$, which satisfies Assumptions 2.3 and 2.5 on the set of candidate estimators.

**Proposition 3.4.**

(i) Let Assumption 2.2 hold, and suppose that there exist some $\beta \in (0, 1]$ and $c_d \in (0, 1)$ satisfying

\[
\min_{1 \leq j \leq q_n} \delta_j \geq c_d n^{\beta} \quad \text{and} \quad \frac{\omega_n^2}{\min_{1 \leq j \leq q_n} d_j^2 n^{\beta - 4}} \to 0 \quad (14)
\]

where, as before $\delta_j = \min(\theta_j - \theta_{j-1}, \theta_{j+1} - \theta_j)$. In addition, suppose that

\[
\frac{n^{2-2\beta} \log(n)}{R_n} \to 0, \quad \frac{q_n}{\tilde{Q}_n} \to 0 \quad (15)
\]

and $\Delta_n = 2(\min_{1 \leq j \leq q_n} d_j)^{-2} \rho_n^{(W)}$ with $\rho_n^{(W)} = c_w n^{4-4\beta} \omega_n^2$ for some $c_w \in (0, \infty)$. Then, it holds

\[
P\left( \max_{1 \leq j \leq q_n} \min_{k \in \mathcal{K}(R_n, \Delta_n, \tilde{Q}_n)} d_j^2 |k - \theta_j| \leq \rho_n^{(W)} \right) \to 1.
\]
(ii) Suppose $\omega^2_n \tilde{Q}_n/n \to 0$ with $\omega_n$ as in Assumption 2.2 (ii). Then (for any realisation of the random intervals), 

$$n^{-1}\omega^2_n |K(R_n, \Delta_n, \tilde{Q}_n)| \to 0.$$ 

(iii) Suppose that conditions in (i) hold. Then, for each $j = 1, \ldots, q_n$, there exists $\hat{k} \in \mathcal{K}(R_n, \Delta_n, \tilde{Q}_n) : d^2_j |k - \theta_j| \leq \rho_n^{(W)}$ such that $\min(\hat{k} - \hat{s}, \hat{e} - \hat{k}) \geq c\delta_j$, where $\mathcal{I}_N(\hat{k}) = (\hat{s}, \hat{e})$ represents the natural detection interval of $\hat{k}$ and $c$ is a universal constant satisfying $c \in (0, 1]$.

Remark 3.3. For each $k \in \mathcal{K}(R_n, \Delta_n, \tilde{Q}_n)$, the natural detection interval $\mathcal{I}_N(k) = (s, e]$ can serve as its detection interval $\mathcal{I}(k)$ in which case the detection distances are given by $G_L(k) = k - s$ and $G_R(k_e) = e - k$. Proposition 3.4 (iii) indicates that $\mathcal{K}(R_n, \Delta_n, \tilde{Q}_n)$ fulfils Assumption 2.5 under Assumption 2.1. Besides, by construction, each estimator in $\mathcal{K}(R_n, \Delta_n, \tilde{Q}_n)$ is distinct and therefore $\mathcal{K}(R_n, \Delta_n, \tilde{Q}_n)$ bypasses the issue discussed in Remark 3.1 (b).

Compared to the condition in (11) on the minimal size of change, measured by both the jump size $d_j$ and spacing $\delta_j$, for the MOSUM-based candidate generating mechanism, the corresponding condition in (14) is considerably stronger. Also, the rate of localisation $\rho_n^{(M)}$ achieved by the former is always tighter than $\rho_n^{(W)}$ given in the above theorem. Furthermore, the maximum number of intervals to be drawn at each iteration, $R_n$, is required to increase as the minimal spacing $\min_{1 \leq j \leq q_n} \delta_j$ decreases (see (15)), thus increasing the total computational complexity of the candidate generating procedure.

The consistency of the localised pruning algorithm in combination with the CUSUM-based candidate generating mechanism follows readily from Proposition 3.4 and Theorem 2.3.

**Theorem 3.5.** Let Assumptions 2.1–2.2 and 2.4 hold, and suppose that the conditions in Proposition 3.4 are satisfied. Then, the localised pruning algorithm $\text{LocAlg}$ applied to $\mathcal{K}(R_n, \Delta_n, \tilde{Q}_n)$ yields $\hat{\Theta} = \{\hat{\theta}_1 < \ldots < \hat{\theta}_q\}$ which consistently estimates $\Theta$, i.e.,

$$\Pr\left\{\hat{q}_n = q_n; \max_{1 \leq j \leq q_n} d^2_j |\hat{\theta}_j \mathcal{I}_j \leq q_n - \theta_j| \leq \rho_n^{(W)}\nu_n\right\} \to 1,$$

with $\rho_n^{(W)}$ as in Proposition 3.4.

In practice, it is not straightforward to select $\tilde{Q}_n$ which effectively imposes an upper bound on the number of candidates. For numerical studies in Section 4, instead of selecting $\tilde{Q}_n$, we choose a weak threshold $\zeta_n$ as a multiple of $\sqrt{\log(n)}$, and keep only those candidates for which the corresponding CUSUM statistics (after standardisation) exceed $\zeta_n$. This approach provides more flexibility to deal with heavy-tailedness or serial dependence present in the error sequence.
4 Numerical results

4.1 Simulation study

We conducted an extensive simulation study comparing the performance of the proposed localised pruning algorithm combined with the MOSUM-based candidate generation (‘MoLP’) and CUSUM-based candidate generation (‘CuLP’) discussed in Sections 3.1–3.2, against that of a large number of competitors whose implementations are readily available in R: The multiscale MOSUM procedure with the ‘bottom-up’ merging (bottom.up); WBS (Fryzlewicz 2014) applied with the strengthened Bayesian information criterion (WBS.sBIC); WBS2.SDLL proposed in Fryzlewicz (2018a); Pruned Exact Linear Time (PELT) algorithm of Killick et al. (2012); a functional pruning algorithm (S3IB) proposed in Rigaill (2015); Tail-Greedy Unbalanced Haar (TGUH) algorithm of Fryzlewicz (2018b); FDRSeg (Li et al., 2016); cumSeg (Muggeo and Adelfio, 2010); Functional Pruning Optimal Partitioning (FPOP) algorithm of Maidstone et al. (2017); Jump segmentation for dependent data (JUSD) of Tecuapetla-Gómez and Munk (2017) and DepSMUCE of Dette et al. (2018).

Here, we present the main findings and defer the complete description of the simulation studies (including the choice of input parameters and the references to the R packages for the competitors) to Section D of Appendix. To this end, we focus on two test signals from Fryzlewicz (2014) referred to as mix and teeth10: The former contains both frequent jumps of large size and small jumps over long stretches of stationarity, while the latter has frequent change points occurring every 10th observation in the shape of teeth. In addition, we include the following test signals extending the original ones in order to investigate the scalability of the localised pruning algorithm: Dense test signals, where each test signal is concatenated until the length of the resultant signal exceeds $2 \times 10^4$, and sparse test signals, where each test signal is embedded in the series of i.i.d random variables of length $n = 2 \times 10^4$ at $t = 500$. For error sequences, we consider i.i.d. random variables following Gaussian (E1) and $t_5$ (E2, an example of heavy-tailedness) distributions, and AR(1) processes with Gaussian innovations and the AR parameter $\varrho \in \{0.3, 0.9\}$ (E3, an example of time series errors). For fair comparison, we keep the signal-to-noise ratio defined by $\{\text{Var} (\varepsilon_t)\}^{-1/2} \min_{1 \leq j \leq q_n} |d_j|$ constant across different error distributions. Also for time series errors, the length of each stationary segment is increased by the factor of $\lfloor 1/(1 - \varrho) \rfloor$ in order to account for the information loss due to the serial dependence.

We report the true positive rate (TPR, the proportion of the correctly identified change points out of the $q_n$ true change points) as well as the false positive rate (FPR, the proportion of the spurious estimators out of the $\hat{q}$ estimated change points). For this, we consider that a change point $\theta_j$ is detected if there exists at least one estimator that falls between $\max\{\theta_j + \theta_{j-1})/2, \theta_j - \delta\}$ and $\min\{\theta_j + \theta_{j+1}/2, \theta_j + \delta\}$, where $\delta = \min_{1 \leq j \leq q_n-1} (\theta_{j+1} - \theta_j)$. In addition, as a measure for the precision of the estimated change point locations, we provide the relative
mean squared error (MSE) of the estimated piecewise constant signal to that of the signals estimated using the true change points. Finally, for the dense and sparse test signals, we report the computation time. Further measures of performance are considered in Appendix D. All simulations are based on 1000 replications.

Overall, the proposed localised pruning performs well throughout varying scenarios according to a variety of criteria, often performing as well as or even better than many competitors both in terms of the total number of estimated change points (indicated by TPR and FPR) and their locations (indicated by MSE and other criteria presented in Appendix D). At the same time, the localised pruning is shown to be scalable to long signals with \( n \geq 2 \times 10^4 \).

4.1.1 Independent Gaussian noise (E1)

Tables 1–3 report the simulation results in the presence of independent Gaussian noise. Different choices of tuning parameters lead to similar results for small values of \( n \), as reported in Table D.1 in Appendix. For longer signals, when change points are dense, a lighter penalty \( \xi_n \) and a generous choice of the critical value for the candidate generation method (larger \( \alpha \) for the MoLP, smaller \( \zeta_n \) for the CuLP which in turn is determined by a constant \( C_\zeta \), see Section D.1 of Appendix for the choice of \( \zeta_n \)) are preferable for some test signals such as teeth10, which ensures that the candidate set contains at least one valid estimator for each \( \theta_j \) (Assumption 2.3 (ii)). On the other hand, when the change points are sparse, a heavier penalty \( \xi_n \) is successful in removing spurious false positives without harming the TPR much. Between the two localised pruning methods equipped with different candidate generating methods, CuLP tends to incur more false positives than the MoLP. Overall, the MoLP produces estimators of better localisation accuracy. This may be accounted for by the systematic approach to candidate generation adopted by the multiscale MOSUM procedure, which also yields computational gain when comparing the execution time of the two methods on dense test signals.

The localised pruning algorithm is preferable to the bottom.up approach, which tends to return many false positives as it accepts all the estimators from the smallest bandwidth, or those without any conflicting estimators which may be spurious. This also reflects the corresponding theoretical requirements on the MOSUM procedure, that the significance level is small (\( \alpha = \alpha_n \to 0 \), Eichinger and Kirch (2018)) and that the bandwidths are in the order of \( n \) (Messer et al., 2014).

There is no single method that outperforms the rest universally for all test signals and evaluation criteria. While S3IB marginally outperforms other competitors in terms of TPR, it is at the price of larger FPR. FPOP, another functional pruning algorithm, is computationally fast and generally performs well, but fails at handling the teeth-like jump structure of teeth10 (see Tables 2–3). WBS2.SDLL shows its strength in handling frequent changes, although returning marginally more false positives compared to other methods achieving comparable TPR. Both
PELT and cumSeg tend to under-estimate the number of change points across all test signals and so does WBS.sBIC. The latter result indicates that global minimisation of an information criterion along a solution path is not as efficient as the pruning criteria (C1)–(C2) adopted by PrunAlg, either computationally or empirical performance-wise. Interestingly, when the frequent changes in teeth10 are repeated over \( n \geq 2 \times 10^4 \) observations, the BIC is minimised at the null model along the solution path (Table 2), further suggesting that the global minimisation of BIC often leads to less favourable results compared to PrunAlg.

In terms of computation time, FPOP, PELT and bottom.up take less than 0.1 seconds to process a long signal. It is followed by the MoLP and TGUH, demonstrating that the localised pruning is scalable to long signals. While CuLP tends to be slower than MoLP, it still surpasses WBS.sBIC and WBS2.SDLL in this respect (except for the dense block signal), which demonstrates the computational gain achievable by the localised exhaustive search adopted in the proposed methodology. FDRSeg and S3IB, while showing good performance for short test signals, are computationally too expensive for long signals and, along with cumSeg, are omitted in these situations.

| Table 1: Summary of change point estimation over 1000 realisations for mix and teeth10 test signals with independent Gaussian noise: we set \( \alpha = 0.2 \) for MoLP, bottom.up and FDRSeg and \( C_\zeta = 0.9 \) (determines \( \zeta_n \), see Section D.1 of Appendix for the details) for CuLP, and use \( h = h_J \) and \( \xi_n = \log^{1.01}(n) \) for the localised pruning. |
| --- |
| **Method** | **mix TPR** | **mix FPR** | **mix MSE** | **teeth10 TPR** | **teeth10 FPR** | **teeth10 MSE** |
| MoLP | 0.93 | 0.009 | 4.083 | 0.97 | 0.001 | 1.986 |
| CuLP | 0.937 | 0.054 | 4.844 | 0.985 | 0.017 | 3.65 |
| bottom.up | 0.951 | 0.064 | 4.326 | 0.983 | 0.004 | 1.813 |
| WBS.sBIC | 0.817 | 0.034 | 9.916 | 0.644 | 0.02 | 9.065 |
| WBS2.SDLL | 0.91 | 0.021 | 4.562 | 0.977 | 0.023 | 0.896 |
| PELT | 0.771 | 0.002 | 6.148 | 0.391 | 0.007 | 13.038 |
| S3IB | 0.96 | 0.074 | 4.774 | 0.997 | 0.101 | 4.039 |
| cumSeg | 0.333 | 0 | 25.195 | 0.001 | 0 | 18.287 |
| TGUH | 0.902 | 0.026 | 5.374 | 0.961 | 0.018 | 4.385 |
| FDRSeg | 0.936 | 0.075 | 4.951 | 0.958 | 0.061 | 4.511 |

4.1.2 Independent heavy-tailed noise (E2)

Table 4 reports the results when \( \varepsilon_t \sim_{iid} t_5 \). The heavy penalty \( \xi_n = n^{2/4.99} \) is a theoretically valid choice conforming to the requirement in Assumption 2.2(ii) in light of the discussion in Remark 2.4(b). When \( n \) is small (as for the mix signal in Table 4), this penalty successfully prevents false positives but the resulting procedure lacks power. The light penalty \( \xi_n = \log^{1.1}(n) \) works reasonably well in not causing false positives while attaining high TPR, yielding the performance comparable to that observed with Gaussian noise (see Table 1). When \( n \) is large and change points are sparse, the localised pruning under-estimates the number of change points.
Table 2: Summary of change point estimation over 1000 realisations for the mix and teeth10 test signals with dense change points and Gaussian noise; we set $\alpha = 0.2$ for bottom.up, $\alpha = 0.4$ for MoLP and $C_\zeta = 0.5$ for CuLP, and use $h = h_J$ and $\xi_n = \log^{1.01}(n)$ for the localised pruning.

| method    | mix TPR | mix FPR | mix MSE | mix speed | teeth10 TPR | teeth10 FPR | teeth10 MSE | teeth10 BIC | teeth10 speed |
|-----------|---------|---------|---------|-----------|-------------|-------------|-------------|-------------|---------------|
| MoLP      | 0.887   | 0.002   | 4.154   | 1.105     | 0.821       | 5.124       | 2138.238    | 1.121       |
| CuLP      | 0.906   | 0.005   | 4.364   | 8.807     | 0.903       | 4.904       | -1962.88    | 4.319       |
| bottom.up | 0.887   | 0.025   | 4.385   | 0.060     | 0.847       | 0           | 5.01        | 1478.86     | 0.091         |
| WBS.sBIC  | 0.676   | 0       | 11.937  | 73.173    | 0           | 0           | 16.854      | -8928.946   | 69.941        |
| WBS2.SDLL | 0.908   | 0.013   | 4.383   | 9.400     | 0.932       | 0.005       | 4.612       | -1831.672   | 10.689        |
| PELT      | 0.625   | 0       | 9.266   | 0.032     | 0           | 0.001       | 16.851      | -8927.371   | 0.018         |
| TGUH      | 0.821   | 0.002   | 5.593   | 1.420     | 0.594       | 0.003       | 9.436       | -3818.443   | 1.389         |
| FPOP      | 0.803   | 0.002   | 5.105   | 0.011     | 0.114       | 0.004       | 15.588      | -7955.71    | 0.012         |

Table 3: Summary of change point estimation over 1000 realisations for the mix and teeth10 test signals with sparse change points and Gaussian noise; we set $\alpha = 0.2$ for MoLP and bottom.up and $C_\zeta = 0.9$ for CuLP, and use $h = h_J$ and $\xi_n = \log^{1.1}(n)$ for the localised pruning.

| method    | mix TPR | mix FPR | mix MSE | mix speed | teeth10 TPR | teeth10 FPR | teeth10 MSE | teeth10 speed |
|-----------|---------|---------|---------|-----------|-------------|-------------|-------------|---------------|
| MoLP      | 0.862   | 0.004   | 4.934   | 0.218     | 0.639       | 0.006       | 8.82        | 0.217         |
| CuLP      | 0.845   | 0.006   | 5.554   | 2.759     | 0.671       | 0.011       | 9.464       | 1.294         |
| bottom.up | 0.905   | 0.372   | 6.041   | 0.043     | 0.848       | 0.386       | 6.646       | 0.041         |
| WBS.sBIC  | 0.638   | 0.002   | 14.56   | 62.190    | 0.42        | 0.006       | 13.294      | 63.106        |
| WBS2.SDLL | 0.847   | 0.021   | 6.169   | 8.049     | 0.825       | 0.038       | 6.99        | 7.996         |
| PELT      | 0.665   | 0       | 9.118   | 0.039     | 0.164       | 0.015       | 16.11       | 0.022         |
| TGUH      | 0.535   | 0.006   | 48.116  | 1.377     | 0.8        | 0.007       | 6.922       | 1.402         |
| FPOP      | 0.834   | 0.002   | 5.133   | 0.017     | 0.444       | 0.01        | 12.511      | 0.013         |

points for some test signals such as teeth10. This, in part, is due to that the candidate generating method fails to produce at least one valid estimator for each true change point, e.g., compare the TPR for MoLP (resp. CuLP) and bottom.up (WBS2.SDLL), thus failing Assumption 2.3 (i). In addition, for theoretical consistency, Assumption 2.1 requires the magnitude of changes to be larger for their detection in the presence of heavy-tailed noise whereas it is kept at the same level as in (E1) with Gaussian noise. Most of the competitors are tailored for sub-Gaussian noise, and they incur considerable false positives, a phenomenon that is amplified in the RHS of Table 4 as $n$ is large and change points sparse.

4.1.3 Serially correlated noise (E3)

Table 5 reports the simulation results obtained from the test signals generated with serially correlated noise following AR(1) processes. In order to account for the serial dependence, we use the (local) variance estimator inflated by the factor of $(1 + \hat{\varrho})/(1 - \hat{\varrho})$ for both MoLP
Table 4: Summary of change point estimation over 1000 realisations for the (original) mix test signal and the sparse teeth10 test signal with $t_5$ noise; we set $\alpha = 0.2$ (for MoLP and bottom.up) and $C_\xi = 0.9$ for CuLP, and use $\xi_n \in \{\log^{1.1}(n), n^{2/4.99}\}$ for the localised pruning.

| penalty | method     | mix (original) | teeth10 (sparse) | speed |
|---------|------------|----------------|------------------|-------|
| light   | MoLP       | 0.916          | 0.165            | 7.109 | 0.188 |
| heavy   | MoLP       | 0.854          | 0.075            | 12.442| 0.191 |
| light   | CuLP       | 0.905          | 0.177            | 9.382 | 15.962|
| heavy   | CuLP       | 0.822          | 0.077            | 12.195| 21.479|
| -       | bottom.up  | 0.941          | 0.286            | 8.533 | 0.022 |
| -       | WBS2.SDLL  | 0.961          | 0.398            | 133.288| 3.851 |
| -       | PELT       | 0.837          | 0.143            | 49.521| 0.007 |

Table 5: Summary of change point estimation over 1000 realisations for the teeth10 test signal with Gaussian AR(1) process as $\epsilon_t$ where $\varrho \in \{0.3, 0.9\}$ is used for the AR parameter; we set $\alpha = 0.2$ for MoLP, bottom.up, JUDS and depSMUCE and $C_\xi = 0.9$ for CuLP, and use $h = h_f$ and $\xi_n \in \{\log^{1.1}(n), n^{2/4.99}\}$ when $\varrho = 0.3$, while $\xi_n \in \{\log^{1.1}(n), n^{2/4.99}\}$ when $\varrho = 0.9$ for the localised pruning.

| method | $\rho = 0.3$ | $\rho = 0.9$ |
|--------|--------------|--------------|
|        | TPR | FPR | MSE | TPR | FPR | MSE |
| MoLP   | 0.873   | 0.002 | 2.611 | 0.898 | 0.001 | 2.901 |
| CuLP   | 0.874   | 0.035 | 3.882 | 0.807 | 0.02 | 4.243 |
| bottom.up | 0.78   | 0.003 | 3.979 | 0.689 | 0.002 | 5.783 |
| WBS.sBIC | 0.8     | 0.07 | 4.737 | 1     | 0.844 | 5.45 |
| WBS2.SDLL | 0.098   | 0.004 | 10.081 | 0.651 | 0.017 | 5.568 |
| cumSeg | 0.01 | 0     | 10.693 | 0.997 | 0.292 | 3.976 |
| JUSD   | 0     | 0     | 10.727 | 0.151 | 0.003 | 10.515 |
| DepSMUCE | 0.002  | 0     | 10.724 | 0.079 | 0.002 | 10.607 |
4.2 Real data analysis

In this section, we illustrate the performance of the proposed methodology using two data examples that have previously been analysed in the literature: The Kepler light curve data and array comparative genomic hybridisation (CGH) data.

4.2.1 Kepler light curve data

Figure 1: Top: Luminosity of Kepler-1132 measured every half an hour (approximately) with change point estimators (vertical lines; longdashed: MoLP, dashed: CuLP) and the beginnings of the anomalous intervals detected by Fisch et al. (2018) periodically repeated every 62.89 days (vertical dotted lines). Where two estimators returned by the same method lie too close to each other to be distinguished, a filled circle is added. Second, third: Change point estimators from the top panel binned using the periodicity of 62.89. Bottom: Kepler-1132 data binned and aggregated using the periodicity of 62.89 days. In the second, third and bottom panels, change point estimators from the aggregated data are also given as vertical lines (longdashed: MoLP, dashed: CuLP, dotted: Fisch et al. (2018)).

Kepler light curve dataset contains regularly measured luminosity of stars. The transit of an orbiting planet results in periodically recurring segments of reduced luminosity, which can be
used for detecting exoplanets via the transit method (Sartoretti and Schneider 1999). Regarding segments of dimmed luminosity as collective anomalies, Fisch et al. (2018) apply their anomaly detection methodology to the light curve data obtained from Kepler-1132 (available in the R package anomaly (Fisch et al., 2018)), which is known to host at least one orbiting planet (Rein 2018). In their paper, the data is pre-processed into equally sized bins aggregating the luminosity from different orbits using the known periodicity (62.89 days) of the orbiting planet. This amplifies the signal and transforms the irregularly sampled time series data into a regular one. From the aggregated data, they detect a short interval of collective anomalies over [649, 660] (at the scale of bins).

We first apply the proposed localised pruning to the raw Kepler-1132 data without aggregation, the result of which is reported in the top panel of Figure 1. Without further information available, we simply ignore the presence of missing observations, which yields \( n = 51405 \). Considering the possible presence of outliers and heavy tails, we set the penalty at \( \xi_n = \log^{1.1}(n) \). MoLP (with \( \alpha = 0.2 \) and \( \eta = 0.4 \)) detects 14 estimators in total, while CuLP (with \( C_\xi = 0.5 \)) returns 16 estimators, out of which there are 10 overlapping estimators in the sense that either they are identical or very close to one another. Unlike Fisch et al. (2018), we do not use the known periodicity to accumulate the information obtained from different orbits, nor do we utilise the knowledge that the changes are of epidemic nature. Nonetheless, as demonstrated in Figure 1, both MoLP and CuLP identify the anomalous interval detected by Fisch et al. (2018) at some orbits. Additionally detected change points may be attributed to the missingness in the data which is not accounted for by our methodology, particularly the pair in the vicinity of 1290 in the observation time scale.

We also analyse the binned and aggregated data of length \( N = 3078 \) with the penalty \( \xi_N = \log^{1.01}(N) \) chosen on the basis of Gaussian-like tail behaviour of the binned data. Both MoLP and CuLP yield 5 estimators including 648 and 660, correctly identifying the anomalous segment reported in Fisch et al. (2018).

In summary, our methodology is able to detect the periodic reduction in luminosity of Kepler-1132 without aggregating the signal using the extra information of periodicity which, in the problem of detecting exoplanets, may not be readily available.

4.2.2 Array CGH data

Microarray-based comparative genomic hybridization (array CGH) provides a means to quantitatively measure DNA copy number aberrations and to map them directly onto genomic sequences (Snijders et al. 2001). We analyse a dataset obtained from a breast tumour specimen (S0034) described in Snijders et al. (2001) (\( n = 2227 \)). A number of algorithms have been proposed which, regarding any gains or losses in the copy number from the normalised copy number ratios between two DNA samples as change points, identify their total number and locations under the model (1), see Olshen et al. (2004), Li et al. (2016) and Niu and Zhang.
Figure 2: Normalised copy number ratios of a comparison of DNA from cell strain S0034. Vertical solid lines indicate the boundaries between chromosomes, long-dashed lines are change points estimated by MoLP and dashed lines are those estimated by CuLP. Change-point estimators from different methods are also plotted (∗).

Olshen et al. (2004) proposed to smooth the array CGH data for outlier removal prior to change point analysis. Noticing that such a step may introduce serial correlations, we choose to analyse the raw data and account for possible outliers by adopting the penalty $\xi_n = \log^{1.1}(n)$ for the localised pruning algorithm, with $\alpha = 0.4$ and $\eta = 0.4$ for MoLP and $C_\zeta = 0.5$ for CuLP. In addition to the methods included in the comparative simulation study in Section 4.1, we consider the circular binary segmentation algorithm of Olshen et al. (2004) (CBS, implemented in Seshan and Olshen (2018)) and the modified screening and ranking algorithm of Xiao et al. (2014) (modSaRa, implemented in Xiao et al. (2016)). It is important to note that the CBS takes all boundary markers between neighbouring chromosomes as an input unlike any other procedures in consideration, and automatically marks all of them as change points.

Figure 2 plots the normalised fluorescence ratios from S0034 and the change point estimators returned by various methods, and Table 6 reports the number of estimated change points. Overall, MoLP and CuLP detect fewer number of change points compared to most of the competitors, and many elements of the two sets of estimators either coincide or lie very close to each other. Also, many change point estimators coincide with the boundary markers although they are detected without knowing their positions unlike the CBS.

The data exhibits heteroscedasticity particularly beyond the genome order 2274 where there is a dramatic increase in the variability. Both candidate generating methods return a large number of candidates (MoLP: 167, CuLP: 93) and our localised approach to pruning manages to reduce the size of the candidate sets reasonably well. On the other hand, WBS.sBIC, WBS2.SDLL, TGUH, PELT, S3IB and FDRSeg are susceptible to returning spurious change point estimators particularly in this region of increased volatility. CumSeg misses some of
the change points commonly detected by many methods, which is consistent with the findings reported in Section 4.1.

Interestingly, CuLP, WBS.sBIC, WBS2.SDLL and FDRSeg are affected by the randomness involved in generating either the candidate estimators or the critical values, and yield different results on different runs when applied to this data set. It may be due to that the underlying signal is not exactly piecewise constant, a phenomenon known as genomic waves (Diskin et al., 2008). The results for these methods reported here were obtained by setting the seed of R’s random number generator to be one.

Table 6: Number of change points estimated from the S0034.

| MoLP | CuLP | CBS | modSaRa | WBS.sBIC | WBS2.SDLL | TGUH | PELT | S3IB | cumSeg | FDRSeg |
|------|------|-----|---------|----------|-----------|------|------|------|--------|--------|
| 18   | 20   | 31  | 17      | 52       | 84        | 65   | 46   | 49   | 12     | 126    |

5 Conclusions

In this paper, we propose a generic localised pruning algorithm which is applicable with a class of multiscale procedures for change point analysis. We prove the consistency of the proposed methodology in multiple change point estimation under mild conditions and, moreover, show that it inherits the localisation property of the candidate generating mechanism for which two examples are discussed in detail: A multiscale MOSUM procedure and a WBS algorithm. In particular, combined with the former, the localised pruning algorithm achieves minimax optimality in change point localisation. Simulation studies and applications to real datasets demonstrate that with either of the candidate generating mechanisms, the localised pruning algorithm performs as well as, or even better than the pruning methods specifically designed for the multiscale MOSUM procedure or the WBS. The algorithm combined with the MOSUM procedure, implemented in the R package mosum available on CRAN (Meier et al., 2019a), performs particularly well in a variety of situations and is scalable to large sample size.

Most notably, our pruning algorithm provides the flexibility to cope with heavy-tailedness and serial dependence as well as the adaptivity to a variety of change point configurations. By adopting meta-assumptions on key elements of the change point structure (such as the distance between change points and jump size) and the error sequence (such as invariance principles and growth rates of the moments of their sums), the influence of each element on our theoretical arguments is made transparent and discussed in details.

While we focus on the mean change problem in this paper, the proposed methodology is applicable to general change point problems. This can be achieved via appropriate transformation of the data, such that change points in the stochastic properties of interest are detectable as change points in the mean of the transformed time series, or by adopting estimating functions with an appropriately modified information criterion, see e.g., Kirch and Kamgaing (2015a), Kirch and Kamgaing (2015b) and Reckrühm (2019). Besides, results from this paper can
be adapted to accommodate the settings discussed in Kühn (2001) including e.g., changes in renewal processes as discussed in Messer et al. (2014). We leave research in these directions for the future work.

6 Main proofs

In this section, we provide the proofs of Theorems 2.1–2.3 which establish the consistency of the localised pruning algorithm combining \texttt{LocAlg} and \texttt{PrunAlg}. They are based on Propositions 6.1–6.3 whose proofs can be found in Section B of the Appendix. Throughout, we assume that Assumptions 2.1–2.3 (and Assumption 2.5 for Theorem 2.3) hold. Also, we impose the following non-asymptotic bound replacing the asymptotic conditions in Assumptions 2.1–2.4.

\[
\max \left( \frac{\omega_n^{(1)}}{\sqrt{\nu_n \rho_n}}, \frac{\omega_n^{(2)}}{\sqrt{\kappa_n}}, \frac{Q_n \omega_n^2}{n}, \frac{1}{\nu_n \rho_n}, \frac{\xi_n}{\nu_n}, \frac{D_n}{\nu_n} \right) \leq \frac{1}{M}
\]  

(16)

for some $M > 0$, which holds for all $n \geq n(M)$ for some large enough $n(M)$. In the proofs of Propositions 6.1–6.3 we state the precise requirement on the ratios in the LHS of (16) each instance they appear; while this allows to make a tighter requirement on them with which non-asymptotic results are readily derived, we omit such a detailed analysis here and simply state that the assertion in (16) holds for $n$ large enough.

We write $\text{SC}(\mathcal{A}) = \text{SC}(\mathcal{A}|\mathcal{C}, \hat{\Theta}, s, e)$ where there is no confusion since, for given $s$ and $e$, the difference between $\text{SC}(\mathcal{A}|\mathcal{C}, \hat{\Theta}, s, e)$ and $\text{SC}(\mathcal{A}'|\mathcal{C}, \hat{\Theta}, s, e)$ does not depend on candidates outside $(s, e)$ for any $\mathcal{A}, \mathcal{A}' \subset \mathcal{K} \cap (s, e)$. For a change point currently under consideration, say $\theta_0$, we write its neighbouring change points as $\theta_{\pm}$ (i.e., $\Theta \cap (\theta_-, \theta_+) = \{\theta_0\}$) allowing for $\theta_- = 0$ and $\theta_+ = n$, and denote the associated jump sizes by $d_0$ and $d_\pm$, respectively.

For any given interval $(s, e)$, Proposition 6.1 establishes the sure detectability of any change point in $\Theta(s,e)$ as defined in (7), as well as the undetectability of any change point not belonging to $\hat{\Theta}(s,e)$ as defined in (8).

**Proposition 6.1.** For any $0 \leq s < e \leq n$ (with $\Theta \cap (s, e) \neq \emptyset$) and $\theta_0 \in \Theta \cap (s, e)$, let $\mathcal{A} \subset \mathcal{D} = \mathcal{K} \cap (s, e)$ denote a set of candidate estimators where $k_\pm \in \mathcal{A} \cup \{s, e\}$ satisfy $\theta_0 \in (k_-, k_\pm)$ as well as $\mathcal{A} \cap (k_-, k_\pm) = \emptyset$. Then, there exist universal constants $c^*, C* \in (0, \infty)$ with $c^* < C^*$, with which the following statements hold on $\mathcal{M}_n$ for $n$ large enough: Let

\[
\max \left\{ d^2_+(k_+ - \theta_+), d^2_-(\theta_- - k_-), I_{\{k_+ \geq \theta_+\}}, I_{\{k_- \leq \theta_-\}} \right\} \leq C^* \xi_n.
\]

(i) If $d^2_\min(\theta_0 - k_+, k_+ - \theta_0) \geq C^* \xi_n$, we have $\text{SC}(\mathcal{A}) > \text{SC}(\mathcal{A} \cup \{k'_0\})$ for all $k'_0 \in \mathcal{V}'$.

(ii) Suppose $\theta_- < k_-$ and $d^2_\max(\theta_0 - k_-) < c^* \xi_n$. Then, if either $\theta_+ > k_+$ or $|k_+ - \theta_0| < (\theta_+ - k)$, we have $\text{SC}(\mathcal{A}) < \text{SC}(\mathcal{A} \cup \{k\})$. 

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Suppose \( k_+ < \theta_+ \) and \( d_0^2(k_+ - \theta_0) < c^* \xi_n \). Then, if either \( k_- > \theta_- \) or \( |k - \theta_0| < (k - \theta_-) \), we have \( \text{SC}(A) < \text{SC}(A \cup \{k\}) \).

Throughout, for any \( k_+, k_o \in K \cup \{0,n\} \) with \( k_- < k_o < k_+ \), we refer to \( k_o \) as detecting \( \theta_o \in \Theta \cap (k_-, k_+) \) if \( \theta_o = \arg \min_{\theta \in \Theta \cap (k_-, k_+)} |k_o - \theta| \), i.e., its nearest change point within \( (k_-, k_+) \) is \( \theta_o \), even though there may be some \( \theta_j \notin (k_-, k_+) \) closer to \( \theta_o \) than \( \theta_o \).

Proposition 6.2 states that when a given set \( A \) already contains an acceptable candidate for a change point in a local environment, SC increases if another candidate detecting the same change point is added to \( A \), as well as that adding spurious candidates increases SC.

**Proposition 6.2.** For any \( 0 \leq s < e \leq n \) and some \( k_o \in D = \Theta \cap (s,e) \), let \( A \subset D \setminus \{k_o\} \) with \( k_o \in A \cup \{s,e\} \) chosen such that \( k_- < k_o < k_+ \) and \( (k_-, k_+) \cap A = \emptyset \). Further, we suppose that \( k_o \) satisfies

(i) \( \Theta \cap (k_-, k_+) = \emptyset \), or

(ii) if \( \Theta \cap (k_-, k_+) \neq \emptyset \), then for any \( \theta_j \in \Theta \cap (k_-, k_+) \), we have \( d_0^2 \min(\theta_j - k_-, k_+ - \theta_j) \leq C^* \xi_n \).

Additionally, for \( \theta_o \in \Theta \cap (k_-, k_+) \) detected by \( k_o \), either

(a) at least one of \( k_o \) is acceptable, i.e., \( d_o^2 \min(\theta_o - k_-, k_+ - \theta_o) \leq \rho_o \nu_o \), or

(b) \( d_o^2|k_o - \theta_o| > \tilde{C} \xi_n \) for \( \tilde{C} > \max(C^*, \tilde{C}(C^*)^2) \) with \( \tilde{C} \) as defined in Lemma B.4 of the supplementary document.

Then, adding \( k_o \) to \( A \) yields an increase of SC, i.e., for \( n \) large enough,

\[ \text{SC}(A) < \text{SC}(A \cup \{k_o\}) \quad \text{on} \quad \mathcal{M}_n. \]

The next proposition asserts that a set containing an unacceptable candidate yields larger SC than the one replacing it with a strictly valid estimator, when the corresponding change point is detectable in the interval of consideration.

**Proposition 6.3.** For any \( 0 \leq s < e \leq n \) (with \( \Theta^{(s,e)} \neq \emptyset \)) and \( \theta_o \in \Theta^{(s,e)} \), let \( A \subset D = \Theta \cap (s,e) \) be any candidate subset with \( k_o \in A \cup \{s,e\} \) satisfying \( \theta_o \in (k_-, k_+) \), \( A \cap (k_-, k_+) = \emptyset \), \( d_o^2|k_o - \theta_o| \geq c^* \xi_n \), as well as

\[ \max \{d_o^2(k_o - \theta_+) \cdot \mathbb{1}_{(k_o \geq \theta_+)}, d_o^2(k_o - \theta_-) \cdot \mathbb{1}_{(k_o \leq \theta_-)}\} \leq C^* \xi_n. \]

Denote by \( k^*_o \in V_o^* \) a strictly valid estimator for \( \theta_o \), and by \( k_o \) an estimator detecting \( \theta_o \) within \( (k_-, k_+) \) which satisfies \( d_o^2|k_o - \theta_o| \leq \tilde{C} \xi_n \) with \( \tilde{C} \) as in Proposition 6.2 while being unacceptable for \( \theta_o \). Then, adding \( k^*_o \) to \( A \) yields a greater reduction in the RSS than adding \( k_o \), i.e., for \( n \) large enough,

\[ \text{SC}(A \cup \{k_o\}) > \text{SC}(A \cup \{k^*_o\}) \quad \text{on} \quad \mathcal{M}_n. \]
6.1 Proof of Theorem 2.1

On $\mathcal{M}_n$, the following arguments hold uniformly in $0 \leq s < e \leq n$ and the corresponding $\mathcal{D} = \mathcal{K} \cap (s, e)$ for $n$ large enough. First, we note that

$$\text{(D1) any set } A \subset \mathcal{D} \text{ fulfilling (C1) contains at least one estimator satisfying } d_j^2 \min_{k \in A} |k - \theta_j| \leq C^* \xi_n \text{ for all } \theta_j \in \Theta(s,e).$$

We prove (D1) by contradiction. Suppose that for some $\theta_0 \in \Theta(s,e)$, the set $A$ does not contain any candidate within its $(C^*d_0^{-2}\xi_n)$-environment. To such $A$, we can add, if necessary, strictly valid candidates until the resultant set contains one strictly valid candidate for each $\theta_j \in \Theta \cap (s,e) \setminus \{\theta_0\}$. Then, the conditions of Proposition 6.1 (i) are met, and adding any $k_0' \in \mathcal{V}_0$ to such a set results in a decrease of SC.

Also, we can always find a subset of $\mathcal{D}$ that fulfills (C1), since

$$\text{(D2) any } A \subset \mathcal{D} \text{ containing exactly one acceptable estimator for all } \tilde{\Theta}(s,e) \text{ with } |A| = |\tilde{\Theta}(s,e)| \text{ satisfies (C1).}$$

To see this, adding candidates detecting $\theta_j \in \tilde{\Theta}(s,e)$ to $A$ incurs monotonic increase of SC by Proposition 6.2 since in each step, either (i) or (ii.a) therein is fulfilled for any candidates $k_0 \in \mathcal{D} \setminus A$ (since $\rho_n \nu_n < C^* \xi_n$ under (16) for $n$ large enough). Similarly, when adding those detecting $\theta_j \in \Theta \cap (s,e) \setminus \tilde{\Theta}(s,e)$ to $A$, Proposition 6.1 (ii)–(iii) applies.

Denoting by $\mathcal{F}_{[m]}$ the collection of the subsets of $\mathcal{D}$ of cardinality $m$ that fulfil (C1). By (D1) we have $|\mathcal{F}_{[m]}| = 0$ for $m < |\Theta(s,e)|$. Also, defining $m^* = \min\{1 \leq m \leq |\mathcal{D}| : |\mathcal{F}_{[m]}| \neq 0\}$, we have $m^* \leq |\tilde{\Theta}(s,e)| \leq |\Theta(s,e)| + 2 \leq m^* + 2$ by (D2).

Suppose now that there exists $A \in \bigcup_{m^* \leq m \leq m^* + 2} \mathcal{F}_{[m]}$ for which

(a) $|A \cap \mathcal{V}_j'| \neq 1$ for $\theta_j \in \Theta(s,e)$,

(b) $|A \cap \mathcal{V}_j'| > 1$ for $\theta_j \in \tilde{\Theta}(s,e) \setminus \Theta(s,e)$, or

(c) $A \setminus \bigcup_{j: \theta_j \in \Theta(s,e)} \mathcal{V}_j' \neq \emptyset$.

We show that such a set $A$ cannot be returned by (C2). To this end, we apply the following operations to $A$. Because the set changes after each operation, we denote the active set by $A'$ in the following which is initially set as $A' = A$.

**Step 1:** If $A'$ contains any estimator of $\Theta \cap (s,e) \setminus \tilde{\Theta}(s,e)$, iteratively remove such estimators from $A'$ one at a time which, by Proposition 6.1 (ii)–(iii) and (D1), strictly reduces the SC monotonically. Also remove any estimator $k_0 \in A'$ which is too far from its nearest change point within $(k_-, k_+)$ (with $k_{\pm} \in A' \cup \{s,e\}$ satisfying $(k_-, k_+) \cap \mathcal{A}' = \{k_0\}$), say $\theta_0$, in the sense that $d_j^2|k_0 - \theta_0| > \tilde{C}\xi_n$; this strictly reduces the SC by Proposition 6.2 (ii.b) and (D1).
Step 2: If $\mathcal{A}' \cap \mathcal{V}' = \emptyset$ for some $\theta_0 \in \Theta^{(s,e)}$, by [D1], we have at least one $k_0 \in \mathcal{A}'$ satisfying $d_0^2 k_0 - \theta_0 \leq c^* \xi_n$. Let $k_0$ be the closest estimator of $\theta_0$ in $\mathcal{A}'$ and identify $k_\pm \in \mathcal{A} \cup \{s, e\}$ such that $(k_-, k_+) \cap \mathcal{A} = \{k_0\}$. When $d_0^2 \min(\theta_0 - k_-, k_+ - \theta_0) < c^* \xi_n$, we can remove one of $k_\pm$ closer to $\theta_0$ while decreasing the SC. To see this, suppose without loss of generality (otherwise consider the time series in reverse) that this is $k_+$. Then, $k_+ > \theta_0$ since $k_0$ is the estimator closest to $\theta_0$ in $\mathcal{A}'$. Denote by $\tilde{k}_0 = k_+$ and define $\tilde{k}_\pm$ analogously as $k_\pm$ with regards to $\tilde{k}_0$ (such that $\tilde{k}_- = k_0$), and let $\tilde{d}_0$ denote the jump size associated with a change point $\theta_0$. Then, one of the followings applies.

- Conditions of Proposition 6.1 (ii) are met by $\tilde{k}_\pm$ if $k_0 = \tilde{k}_- \leq \theta_0 = \tilde{\theta}_0 < \tilde{k}_+ = \tilde{k}_0$.
- Conditions of Proposition 6.2 (i) are met by $\tilde{k}_\pm$ if $\theta_0 < \tilde{k}_- < \tilde{k}_0 < \tilde{k}_+ < \theta_+$. 
- Conditions of Proposition 6.2 (ii.b) hold for $\tilde{k}_\pm$ and $\tilde{k}_0$ if $\theta_0 < \tilde{k}_- < \tilde{k}_0 < \theta_+ = \tilde{\theta}_0 < \tilde{k}_+$, since in this case, $d_0^2 (\tilde{\theta}_0 - \tilde{k}_0) = d_0^2 (\tilde{\theta}_0 - \theta_0) (1 - (\tilde{k}_0 - \theta_0)/\tilde{\theta}_0 - \theta_0) \geq D_n - c^* \xi_n > C \xi_n$ for $n$ large enough.

In all cases, removing $\tilde{k}_0 = k_+$ results in a decrease of SC. Iteratively repeat the removal and re-defining of $k_0$ and $k_\pm$ until $d_0^2 \min(\theta_0 - k_-, k_+ - \theta_0) \geq c^* \xi_n$. Then, the resultant $\mathcal{A}'$ and $k_0$ are such that $\mathcal{A}' \setminus \{k_0\}$ meets the conditions of Proposition 6.3 for $\theta_0$. Therefore, replacing $k_0$ with any of $k_0^* \in \mathcal{V}_0^*$ yields a reduction in the SC. Repeat the above until $|\mathcal{A}' \cap \mathcal{V}_j'| = 1$ for all $\theta_j \in \Theta^{(s,e)}$, which strictly decreases SC($\mathcal{A}'$) monotonically.

Step 3: If $\mathcal{A}' \cap \mathcal{V}_j' = \emptyset$ for some $\theta_j \in \Theta^{(s,e)} \setminus \Theta^{(s,e)}$ yet $\mathcal{A}'$ contains an estimator of $\theta_j$, we take the same steps as in Step 2 for all such $\theta_j$ so that $|\mathcal{A}' \cap \mathcal{V}_j'| = 1$, which strictly decreases SC($\mathcal{A}'$) monotonically.

Step 4: If there exists $\theta_j \in \Theta^{(s,e)}$ for which there are more than one estimator in $\mathcal{A}'$, through Steps 2–3, we have $\mathcal{A}' \cap \mathcal{V}_j' \neq \emptyset$. Remove the duplicate estimators one at a time until all $\theta_j$ with $\mathcal{A}' \cap \mathcal{V}_j' \neq \emptyset$ have exactly one acceptable estimator in $\mathcal{A}'$ which, by Proposition 6.2 (i) and (ii.a), results in a strictly monotonic reduction of SC.

After Steps 1–4, we have $\mathcal{A}'$ that satisfies $\mathcal{A}' \setminus \bigcup_{j: \theta_j \in \Theta^{(s,e)}} \mathcal{V}_j' = \emptyset$, with $|\mathcal{A}' \setminus \mathcal{V}_j'| = 1$ for $\theta_j \in \Theta^{(s,e)}$ and $|\mathcal{A}' \cap \mathcal{V}_j'| \leq 1$ for $\theta_j \in \Theta^{(s,e)} \setminus \Theta^{(s,e)}$, as well as SC($\mathcal{A}'$) < SC($\mathcal{A}$) because under (a)–(c), at least one of Steps 1–4 above has to take place. Further, if necessary, by adding strictly valid candidates to $\mathcal{A}'$ for all those $\theta_j \in \Theta^{(s,e)} \setminus \Theta^{(s,e)}$ with $|\mathcal{A}' \cap \mathcal{V}_j'| = 0$, we yield $\mathcal{A}'' \supset \mathcal{A}'$ fulfilling [C1] by [D2] and of cardinality $|\tilde{\Theta}^{(s,e)}|$, i.e., $\mathcal{A}'' \in \bigcup_{m^* \leq m \leq m^* + 2} \mathcal{F}_m$. Since $\mathcal{A}' \subset \mathcal{A}''$ with $\subset$ defined below [C2] and SC($\mathcal{A}'$) < SC($\mathcal{A}$), this shows that $\mathcal{A}$ with candidates belonging to either of (a)–(c) cannot be returned in [C2].

In conclusion, $\tilde{\Theta}^{(s,e)}$ obtained from [C2] satisfies the assertion of the theorem.

6.2 Proof of Theorem 2.3

Under [10], we make the following observations: For all $j = 1, \ldots, q_n$,
(a) \( d_j^2(\hat{\theta}_j - \theta_j) \leq \rho_n \nu_n < c^* \xi_n \) for any \( \hat{\theta}_j \in \mathcal{V}_j \), and
(b) \( d_j^2 \min(\theta_j - \theta_{j-1}, \theta_{j+1} - \theta_j) \geq D_n > 2 \max(C^* \xi_n, \rho_n \nu_n) \)

for \( n \) large enough.

In iteratively applying Steps 1–4 of \textsc{LocAlg}, Theorem 2.1 guarantees that \( \hat{\Theta} \) contains only acceptable estimators of \( \theta_j \in \Theta \). Also, each change point can belong to \( \Theta^{(s,e)} \) defined by the interval of consideration \( (s, e) = (k_L, k_R] \) at most once: When \( \theta_j \in \Theta^{(s,e)} \) for the first time, it gets detected by some \( \hat{\theta}_j \in \mathcal{V}_j \) by Theorem 2.1. Then, in the following iterations, either \( \theta_j \notin (s, e) \), or some \( k \in (\mathcal{C} \cup \hat{\Theta}) \cap [\min(\theta_j, \hat{\theta}_j), \max(\theta_j, \hat{\theta}_j)] \) defines the endpoints of the local environment by Step 2. In the latter case, \( \theta_j \) cannot be a detectable change point within the interval of consideration of this particular iteration due to (a), which guarantees that no further estimator for \( \theta_j \) is added to \( \hat{\Theta} \).

When there exists \( \theta_j \in \hat{\Theta}^{(s,e)} \) at some iteration, Theorem 2.1 indicates that it may or may not get detected at this iteration. If it does, an acceptable estimator of \( \theta_j \) is added to \( \hat{\Theta} \) and the same argument as above applies. If not, without loss of generality, suppose \( \theta_j - s \leq e - \theta_j \). By construction, \( c^* \xi_n \leq d_j^2(\theta_j - s) < C^* \xi_n \) and thus from (b), we have

\[
d_j^{-1}(s - \theta_j_{-1}) = d_j^2(\theta_j - \theta_{j-1}) \left\{ 1 - \frac{d_j^2(\theta_j - s)}{d_j^2(\theta_j - \theta_{j-1})} \right\} \geq D_n - C^* \xi_n > \rho_n \nu_n,
\]

i.e., the boundary point \( s \) cannot be an acceptable estimator for either \( \theta_j_{-1} \) or \( \theta_j \). Consequently, it cannot have already been added to \( \hat{\Theta} \) in the previous iterations by Theorem 2.1. Therefore, all acceptable estimators for \( \theta_j \), with the possible exception of \( k_o \) identified in Step 1, remain in \( \mathcal{C} \) by (a)–(b) and how it is reduced in Step 4 of \textsc{LocAlg}.

Next, we justify the removal of \( k_o \) from \( \mathcal{C} \) at each iteration. Clearly, if \( k_o \) is not acceptable for any change point, it can be safely removed from the future consideration. Next, suppose that \( k_o \) is an acceptable estimator of \( \theta_j \) and \( \theta_j - s \leq e - \theta_j \).

(i) When \( \theta_j \in \Theta^{(s,e)} \), we have either \( k_o \) or another acceptable estimator of \( \theta_j \) accepted by \textsc{PrunAlg}, and therefore \( k_o \) can be removed.

(ii) When \( \theta_j \in \hat{\Theta}^{(s,e)} \setminus \Theta^{(s,e)} \), if \( \theta_j \) is detected at the current iteration, the same argument as in (i) applies. If not, as shown above, \( s \) has not been added to \( \hat{\Theta} \) yet and by construction of the interval of consideration in Step 2, it follows that

\[
d_j^2 G_L(k_o) = d_j^2(k_o - s) \leq C^* \xi_n + \rho_n \nu_n = C^* \xi_n (1 + o(1)),
\]

which shows that \( k_o \) cannot fulfil (9) for \( \theta_j \) (nor any other change point as it is acceptable for \( \theta_j \)). Consequently, \( k_o \) can safely be removed from \( \mathcal{C} \) since by Assumption 2.5 and the construction of \( \mathcal{R} \) in Step 4, there remains at least one acceptable estimator for \( \theta_j \) that fulfils (9) in \( \mathcal{C} \) after the current iteration.
(iii) When $\theta_j \notin \Theta^{(s,e)}$ (which is not necessarily situated within $(s,e)$), we first consider the case where $s$ has already been accepted. Then by Theorem 2.1, $s$ is acceptable for some change point, say $\theta_{j'}$, such that

$$d^2_{j'+1}(\theta_{j'+1} - s) = d^2_{j'+1}(\theta_{j'+1} - \theta_{j'}) - \frac{1}{\theta_{j'+1} - \theta_{j'}} \left( 1 - \frac{d^2_{j}(s - \theta_{j'})}{d^2_{j}(\theta_{j'+1} - \theta_{j'})} \right) \geq D_n \left( 1 - \frac{\rho_n \nu_n}{D_n} \right) > C^* \xi_n,$$

i.e., $\theta_{j'+1}$ is either surely detectable within $(s,e)$, too close to $e$, or $\theta_{j'+1} \notin (s,e)$ to have been detected by $k_0$. Therefore, $j = j'$ and $k_0$ can safely be removed as in (i) since there already exists an acceptable estimator $s$ in $\Theta$. If $s$ has not been accepted, the argument analogous to that in (ii) applies.

The case when $\theta_j - s > e - \theta_j$ is similarly handled.

The above (ii)–(iii) show that under Assumptions 2.3 and 2.5 for each $j = 1, \ldots, q_n$, acceptable estimators of $\theta_j$ remain in $C$ until its detection and at least one of them, when set as $k_0$ in Step 1 of LocAlg, leads $\theta_j$ to belong to $\Theta^{(s,e)}$ at some iteration, from which we conclude that all $\theta_j \in \Theta$ are eventually detected by acceptable estimators. Finally, $|R| \geq 1$ at all iterations since $R$ contains $k_0$ at least, which ensures that LocAlg terminates eventually.

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Supplementary Appendix

Available from [https://sites.google.com/view/haeran-cho/research](https://sites.google.com/view/haeran-cho/research), it provides the pseudo codes for LocAlg and PrunAlg, remaining proofs of the theoretical results and the complete description of the simulation studies.

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