RELATIVE BGG SEQUENCES;
II. BGG MACHINERY AND INVARIANT OPERATORS

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Abstract. For a real or complex semisimple Lie group \( G \) and two nested parabolic subgroups \( Q \subset P \subset G \), we study parabolic geometries of type \( (G, Q) \). Associated to the group \( P \), we introduce a class of relative natural bundles and relative tractor bundles and construct some basic invariant differential operators on such bundles. We define a (rather weak) notion of “compressibility” for operators acting on relative differential forms with values in a relative tractor bundle. The we develop a general machinery which converts a compressable operator to an operator on bundles associated to completely reducible representations on relative Lie algebra homology groups.

Applying this machinery to a specific compressable invariant differential operator of order one, we obtain a relative version of BGG (Bernstein–Gelfand–Gelfand) sequences. All our constructions apply in the case \( P = G \), producing new and simpler proofs in the case of standard BGG sequences. We characterize cases in which the relative BGG sequences are complexes or even fine resolutions of certain sheaves and describe these sheaves. We show that this gives constructions of new invariant differential operators as well as of new subcomplexes in certain curved BGG sequences. The results are made explicit in the case of generalized path geometries.

1. Introduction

The main motivation for the construction of Bernstein–Gelfand–Gelfand sequences (or BGG sequences) came from questions on differential operators which are naturally associated to certain geometric structures. In particular, conformally invariant differential operators were studied in Riemannian geometry for a long time with rather limited success. Starting from the 1970s, it became clear that questions on invariant differential operators are closely related to questions in representation theory. More precisely, for locally flat conformal structures (with the round sphere as the model example), conformally invariant differential operators are equivalent to homomorphisms of generalized Verma modules. A basic source for such homomorphisms is Lepowsky’s generalization \([20]\) of the BGG resolution of a finite dimensional representation by homomorphisms of Verma modules, see \([3]\).

The generalized BGG resolutions and the Jantzen–Zuckermann translation principle for homomorphisms were used in the pioneering article \([16]\) to construct conformally invariant differential operators. Later on, these ideas were combined with
tractor calculus (see [1] and [7]) and extended to the family of *parabolic geometries* under the name *curved translation principle*.

The basic results on BGG sequences were obtained in [12] and with an improved construction in [4] following a slightly different approach. Rather than using results on generalized Verma modules, these articles gave a direct construction of invariant differential operators, based on tractor bundles and the algebraic setup introduced by Kostant for the proof of his theorem on Lie algebra cohomology in [19], which is commonly known as Kostant’s version of the Bott–Borel–Weil theorem. Kostant’s theorem itself is then used to identify the natural vector bundles which show up in the sequence. This provided a general construction for a large class of differential operators naturally associated to a broad class of geometric structures, which apart from conformal structures also contains other well known examples like projective and quaternionic structures, CR structures and path geometries.

In the applications of BGG sequence that were found during the subsequent years, a certain change of perspective evolved. On the one hand, it turned out to be very important that one not only obtains a construction of higher order operators on (relatively) simple bundles but also a relation to first order operators on more complicated bundles. This gives the possibility to work with the operators in a BGG sequence without knowing explicit formulae for them (which tend to become very complicated if the order gets high). On the other hand, already in the original construction, two possible operators on forms with values in a tractor bundle were used as a starting point for the construction. In [6] and the subsequent generalization [18], the construction was applied to certain modifications of the canonical tractor connection. Corresponding to these developments, the terminology *BGG machinery* started turning up.

The aim of this article is to develop a relative version of BGG sequences and at the same time to convert the vague idea of a “BGG machinery” into precise statements. In the notation we are going to use, the starting point for usual BGG sequences is a pair \((G,Q)\), where \(G\) is a real or complex semi–simple Lie group and \(Q \subset G\) is a parabolic subgroup. The construction then associates each representation of \(G\) a sequence of invariant differential operators on the category of parabolic geometries of type \((G,Q)\). For the relative version we develop, one in addition chooses an intermediate parabolic subgroup \(P\) lying between \(G\) and \(Q\). The construction then starts from a completely reducible representation of \(P\), again producing operators on parabolic geometries of type \((G,Q)\). We emphasize at this point that one may always choose \(P = G\) to obtain a construction for the usual BGG sequences, which contains several new features and strong improvements compared to the constructions in [12] and [4]. This is also crucial for some of the applications discussed in the end of the article.

The relative construction needs an algebraic background, a relative version of Kostant’s harmonic theory and a description of relative Lie algebra homology groups parallel to Kostant’s theorem. These results belong to the realm of finite dimensional representation theory and should be of independent interest, so they have been put into the separate article [15].
Building on this algebraic background, we describe the setup needed for the relative BGG construction in Section 2. It turns out that the intermediate subgroup $P$ can be used to single out a subclass of natural bundles that we call relative natural bundles. These contain all irreducible bundles (i.e. those associated to irreducible representations of $Q$) and the class of relative tractor bundles, which are associated to completely reducible representations of $P$. One obtains natural relative versions of the tangent and cotangent bundle and a relative adjoint tractor bundle. The main results of Section 2 are a construction of a relative version of the fundamental derivative in Proposition 2.2, and a relative version of the curved Casimir operator from [13], see Section 2.3.

The actual relative BGG construction is carried out in two steps. In Section 3, we establish a machinery to construct operators on bundles induced by relative Lie algebra homology groups from operators defined on relative differential forms with values in a relative tractor bundle. This construction can be applied to a single operator mapping $k$–forms to $(k + 1)$–forms, and apart from being linear, the only requirement on this operator is compressability as defined in Definition 3.1. This means that the operator preserves a natural filtration on the space of forms and has a specific induced action on the associated graded. Beyond that, it is not even required to be a differential operator. The main feature of the construction is that it entirely works with (universal) polynomials in the composition of a natural bundle map with the given operator. Hence it always produces operators which are “as nice” as the operator one starts from.

The key step for this is the construction of a splitting operator, for which we give two equivalent descriptions. One is parallel to the general constructions of splitting operators from curved Casimirs in [13] and [9], the other construction is closer to the one used in [4]. We also prove that the kernel of the initial operator naturally corresponds to a subspace in the kernel of the induced operator on Lie algebra homology groups (Proposition 3.6), which is a general version of the concept of “normal solutions” of first BGG equations.

In Section 3.6, we discuss the case that one starts with a sequence of operators on forms of all degrees rather than just a single operator. In particular, we show that if two operators in the sequence have trivial composition then the same is true for the induced operators on Lie algebra homology bundles, and we precisely analyze the relation between the cohomologies, see Theorem 3.14.

The second part of the construction is carried out in Section 4. Using the relative fundamental derivative, we construct a first order invariant differential operator called the relative twisted exterior derivative on relative differential forms with values in any relative tractor bundle. This operator is shown to be compressible, so the machinery of Section 3 leads to a sequence of invariant differential operators on relative homology bundles, see Theorem 4.1. In the course of the further developments, it is shown in Proposition 4.10 that for $P = G$, this operator coincides with the one constructed in [12] via semi–holonomic jet modules.

Next, we treat the question when a relative BGG sequence is a complex or even a fine resolution of some sheaf. Apart from a computation of the square of the relative twisted exterior derivative (which seems to be a new result, even for
$P = G$), this requires an interpretation in terms of a relative analog of tractor connections, see Theorem 4.5 and Proposition 4.10. This is done even in the case where the relative tangent bundle (which always is a smooth subbundle in the tangent bundle) is non-involutive, so the naive way to extend a partial connection to higher order forms fails.

Involutivity of the relative tangent bundle is necessary but not sufficient for BGG sequences being resolutions, additional conditions on the (relative) curvature have to be satisfied. To interpret the sheaves resolved by a BGG resolution, one has to use the theory of correspondence spaces and local twistor spaces for parabolic geometries as developed in [5]. The main general results on BGG resolutions we prove is Theorem 4.11 for the interpretation of the sheaves being resolved also Theorem 4.13 is important.

The last topic in Section 4 are algebraic properties of splitting operators which generalize the results for usual BGG sequences in [5], also giving simpler proofs for those results. The main topic here is to systematically obtain restrictions on the image of splitting operators, given information on the sections that they are applied to and/or on the curvature of the geometry. In particular, these results can be used to obtain information on the curvature of a geometry from information on its harmonic part. A crucial result in this context is the description of the Laplacian determined by the twisted exterior derivative in terms of the relative curved Casimir in Proposition 4.14 which completes and extends partial results in this direction from [13].

In the last Section 5 we discuss some applications of our results and make them explicit for one structure. First, we discuss the case in which the bundles showing up in a relative BGG sequence also arise in a standard BGG sequence. In representation theory terms, this means that the weight determining the relative tractor bundle which gives rise to the sequence is in the affine Weyl orbit of a $g$–dominant integral weight. In this case, we are able to prove in general that the operators in the relative BGG sequence are the same as the operators between the bundles in question that are obtained in the absolute BGG sequence, see Theorem 5.2. Under the appropriate curvature conditions, which are much weaker than local flatness of the geometry, one thus obtains subcomplexes in curved BGG sequences, which are different from those constructed in [14]. These results also show how strong the characterization results relating BGG operators to the (relative) twisted exterior derivative are. Initially, the statement that the bundles occur in both sequences only comes from the fact that they are induced by representations with the same highest weight and already finding an explicit bundle map relating absolute and relative homology bundles is a quite non-trivial problem.

Second, there is the case in which we obtain operators that cannot occur in a standard BGG sequence. In representation theory terms this means that either the representation inducing the relative tractor bundle has singular infinitesimal character or its highest weight is non-integral. The latter is not a rare case at all because there are density weights involved, which can be non–integral without problems. In all these cases, we obtain a systematic and general construction for invariant differential operators, for which up to now there were only construction
principles (which usually need case–by–case verifications, even to decide whether they apply) available in the literature.

We conclude the article by making our results explicit in the case of generalized path geometries. This example of parabolic geometries is of particular interest, since the geometric theory of systems of second order ODEs as developed in [17] is a special case of such structures. This is just one example, however, and we see potential for many further applications of relative BGG sequences. In particular, we hope that relative BGG resolutions provide a starting point for a curved version of the Penrose transform as described in [2].

2. Relative natural bundles

We start by briefly recalling the setup of two nested parabolic subalgebras \( q \subset p \) in a semisimple Lie algebra \( \mathfrak{g} \) with a compatible choice \( Q \subset P \subset G \) of groups as discussed in [15]. The intermediate parabolic \( p \) gives rise to a class of natural bundles on parabolic geometries of type \((G,Q)\), which we call relative natural bundles. We show that there are natural analogs of two of the basic differential operators available for parabolic geometries, the fundamental derivative and the curved Casimir operator, which are adapted to the relative setting. Then we describe the geometric counterpart of the algebraic setup developed in [15], which sets the stage for the relative BGG–machinery we develop in the next section.

2.1. Relative natural bundles. Throughout this article, we consider a real or complex semisimple Lie algebra \( \mathfrak{g} \) endowed with two nested parabolic subalgebras \( q \subset p \subset \mathfrak{g} \). Moreover, we assume that we have chosen a Lie group \( G \) with Lie algebra \( \mathfrak{g} \) and a parabolic subgroup \( P \subset G \) corresponding to \( p \). As discussed in Section 2.1 of [15] the normalizer \( Q \) of \( q \) in \( P \) has Lie algebra \( \mathfrak{q} \), so we obtain closed subgroups \( Q \subset P \subset G \) corresponding to \( q \subset p \subset \mathfrak{g} \). For each of the parabolic subalgebras, we have the nilradical, and we denote these by \( p_+ \subset p \) and \( q_+ \subset q \). It turns out that \( p_+ \subset q_+ \) and that the exponential map restricts to diffeomorphisms from these subalgebras onto closed subgroups \( P_+ \subset Q_+ \subset Q \subset P \) such that \( Q_+ \) is normal in \( Q \) and \( P_+ \) is normal in \( P \).

In this setting, we will study parabolic geometries of type \((G,Q)\), and use \( P \) (or \( p \)) as an additional input. By definition, these are Cartan geometries of type \((G,Q)\) and hence can exist on smooth manifolds of dimension \( \dim(G/Q) \). Explicitly, such a geometry on a smooth manifold \( M \) is given by a principal fiber bundle \( p : \mathcal{G} \to M \) with structure group \( Q \) together with a Cartan connection \( \omega \in \Omega^1(\mathcal{G}, \mathfrak{g}) \). This means that \( \omega \) defines a trivialization \( T\mathcal{G} \cong M \times \mathfrak{g} \) which is \( P \)–equivariant and reproduces the generators of fundamental vector fields, see section 1.5 of [11] for details on Cartan geometries. There is a general theory exhibiting such parabolic geometries as equivalent encodings of underlying structures. For the purposes of this article, we may however simply take the Cartan geometry as a given input.

From this description it is clear, that a representation \( \mathcal{W} \) of the Lie group \( Q \) gives rise to a natural vector bundle on parabolic geometries of type \((G,Q)\). If \((p : \mathcal{G} \to M, \omega)\) is such a geometry, then we simply form the associated bundle
\( \mathcal{G} \times_Q \mathcal{W} \). Via the Cartan connection \( \omega \), one can identify some of these natural bundles with more traditional geometric objects like tensor bundles.

**Definition 2.1.** Suppose that \( Q \subset P \subset G \) are nested parabolic subgroups and let \( \mathcal{W} \) be a representation of \( Q \), and consider the corresponding natural vector bundle \( \mathcal{W} \) on parabolic geometries of type \((G, Q)\).

1. \( \mathcal{W} \) is called a **relative natural bundle** if the subgroup \( P_+ \subset Q \) acts trivially on \( \mathcal{W} \).

2. \( \mathcal{W} \) is called a **relative tractor bundle** if \( \mathcal{W} \) is the restriction to \( Q \) of a representation of \( P \) on which \( P_+ \) acts trivially.

Observe that trivial action of \( P_+ \) is equivalent to trivial action of \( p_+ \) under the infinitesimal representation. Moreover, on irreducible (and hence on completely reducible) representations of any parabolic subgroup, the nilradical always acts trivially. Hence any completely reducible representation of \( Q \) gives rise to a relative natural bundle (these are the usual completely reducible bundles) and any completely reducible representation of \( P \) gives rise to a relative tractor bundle.

Beyond the class of completely reducible natural bundles, we can immediately construct some fundamental examples of relative natural bundles. Recall that for any parabolic geometry \((p : \mathcal{G} \to M, \omega)\), the tangent bundle \( TM \) is the associated bundle \( \mathcal{G} \times_Q (g / q) \). This is not a relative natural bundle in general. However, the additional parabolic subalgebra \( p \subset g \) is a \( Q \)-invariant subspace, which gives rise to a smooth subbundle \( \mathcal{G} \times_Q (p / q) =: T_p M \subset TM \). Since \( p_+ \) is an ideal in \( p \) we get \([p_+, p] \subset p_+ \subset q\). Thus \( p_+ \) acts trivially on \( p / q \), so \( T_p M \) is a relative natural bundle, which we will call the **relative tangent bundle**.

From the definition it is clear, that the class of relative natural vector bundles is closed under forming natural subbundles and quotients and under the usual functorial constructions like sums, tensor products, duals and so on. In particular, the dual \( T^*_p M \) of \( T_p M \) is also a relative natural bundle, which we call the **relative cotangent bundle**. As discussed in Section 2.3 of [13], the Killing form of \( g \) induces dualities between \( g / q \) and \( q_+ \) and between \( g / p \) and \( p_+ \), which implies that it also gives rise to a duality between \( p / q \) and \( q_+ / p_+ \). Thus \( T^*_p M \cong \mathcal{G} \times_Q (q_+ / p_+) \), so in particular, this is naturally a bundle of nilpotent Lie algebras. Having the relative tangent bundle and the relative cotangent bundle at hand, we can of course form relative tensor bundles, and in particular, there is the bundle \( \Lambda^k T^*_p M \) of relative \( k \)-forms, which is the associated bundle corresponding to the \( Q \)-module \( \Lambda^k (q_+ / p_+) \).

### 2.2. Relative adjoint tractor bundle and relative fundamental derivative.

Recall that for a parabolic geometry \((p : \mathcal{G} \to M, \omega)\) of type \((G, P)\), the **adjoint tractor bundle** is the natural bundle \( \mathcal{A}M := \mathcal{G} \times_Q g \). This is a fundamental example of a tractor bundle (since it is induced by the restriction to \( Q \) of a representation of \( G \)), but of course not a relative natural bundle. There is a relative analog of this bundle, however. The group \( P \) acts on its Lie algebra \( p \) by the adjoint representation and the nilradical \( p_+ \subset p \) is invariant under this action. Hence there is an induced action on the quotient \( p / p_+ \). Since \( p_+ \) is an ideal in \( p \), it acts trivially on this quotient, so \( \mathcal{A}_p M := \mathcal{G} \times_Q (p / p_+) \) is a relative tractor bundle called the **relative adjoint tractor bundle**.
The bundle $A_p M$ has properties similar to $\mathcal{A} M$ in many respects. First, $p/p_+$ naturally is a Lie algebra and the bracket is $P$–invariant and hence $Q$–invariant. Thus we get an induced bilinear bundle map $\{,\} : A_p M \times A_p M \to A_p M$. Second, as discussed in Section 2.5 of [15], the Lie algebra $\mathfrak{g}$ carries a natural $Q$–invariant filtration, which restricts to $Q$–invariant filtrations on $p$ and $p_+$. The resulting $Q$–invariant filtration of $p/p_+$ induces a filtration of $A_p M$ by smooth subbundles $A^p_\rho M$. Since the initial filtration is compatible with the Lie bracket, we conclude that $\{A^{p^0}_\rho M, A^{p^1}_\rho M\} \subset A^{p^0+j}_\rho M$, so $A_p M$ is a bundle of filtered Lie algebras. By definition, we further have $p^0 = q$ and $p^1 = q_+$. Passing to associated graded bundles, this implies that $A_p M/A^0_\rho M = T_p M$ and that $A^1_\rho M = T^*_p M$. We will denote by $\Pi_\rho$ the projection $\Gamma(A_p M) \to \Gamma(T_p M) \subset \mathfrak{X}(M)$ induced by the first isomorphism.

Having the relative adjoint tractor bundle at hand, we can construct a relative version of the most basic differential operator available on any Cartan geometry, the so–called fundamental derivative. Recall that via the Cartan connection $\omega$, sections of the adjoint tractor bundle $\mathcal{A} M$ can be identified with $Q$–invariant vector fields on the total space $\mathcal{G}$ of the Cartan bundle. Given any associated bundle $E$ to $\mathcal{G}$, one can identify its sections with $Q$–equivariant functions on $\mathcal{G}$, and differentiating such a function with a $Q$–invariant vector field, the result is $Q$–equivariant again. Thus one obtains a natural bilinear differential operator $D : \Gamma(\mathcal{A} M) \times \Gamma(E) \to \Gamma(E)$, which, to emphasize the analogy to a covariant derivative, is written as $(s, \sigma) \mapsto D_s \sigma$. By construction, this operator is linear over smooth functions in the $\mathcal{A} M$–slot, so it can also be interpreted as a natural linear operator $\Gamma(E) \to \Gamma(A^* M \otimes E)$, and in this form it can evidently be iterated.

To construct a relative version of this operator, we need another property of the fundamental derivative. The $Q$–invariant filtration of $\mathfrak{g}$ induces a filtration of $\mathcal{A} M$ by smooth subbundles $A^0 M$ and in particular $A^0_\rho M = \mathcal{G} \times Q q$. Now if $E = \mathcal{G} \times Q \mathfrak{W}$ for a representation $\mathfrak{W}$ of $Q$, then the infinitesimal representation defines a $Q$–equivariant, bilinear map $q \times \mathfrak{W} \to \mathfrak{W}$. Passing to associated bundles, we get a bilinear bundle map $A^0_\rho M \times E \to E$, which we write as $(s, \sigma) \mapsto s \bullet \sigma$. Now in the above picture of vector fields on $\mathcal{G}$, sections of $A^0_\rho M$ correspond to vertical vector fields, and equivariance implies that $D_s \sigma = -s \bullet \sigma$ for $s \in \Gamma(A^0_\rho M)$ and $\sigma \in \Gamma(E)$.

Now the $Q$–invariant subspaces $\mathfrak{p} \subset \mathfrak{g}$ and $\mathfrak{p}_+ \subset q$ give rise to smooth subbundles $\mathcal{G} \times Q \mathfrak{p}_+ \subset A^0 M \subset \mathcal{G} \times Q \mathfrak{p} \subset \mathcal{A} M$ and the quotient of the third of these bundles by the first one can be identified with $A_p M$. Moreover, if $\mathfrak{W}$ is a representation of $Q$ inducing a relative natural bundle, then the infinitesimal representation $q \otimes \mathfrak{W} \to \mathfrak{W}$ descends to $q/p_+$ in the first factor, and the latter representation induces the subbundle $A^0_{\rho p} M \subset A_p M$. If $E$ is the relative natural bundle determined by $\mathfrak{W}$, then we get an induced bilinear bundle map $\bullet : A^0_{\rho p} M \times E \to E$. Having all that at hand, we can construct the relative fundamental derivative and prove that it has the same strong naturality properties as the fundamental derivative.
Proposition 2.2. For a relative natural bundle $E$, the fundamental derivative induces a well defined operator $D^\rho : \Gamma(A_\rho M) \times \Gamma(E) \to \Gamma(E)$ which has the following properties.

1. For $s \in \Gamma(A_0 \rho M)$ and $\sigma \in \Gamma(E)$, we get $D^\rho s \sigma = -s \bullet \sigma$.
2. For $E = M \times \mathbb{R}$, we get $D_s f = \Pi_\rho(s) \cdot f$ for all $s \in \Gamma(A_\rho M)$ and $f \in C^\infty(M, \mathbb{R})$.
3. The operators $D^\rho$ are compatible with any bundle map which comes from a $Q$–equivariant linear map on the inducing representation. In particular, one obtains Leibniz rules both for the multiplication by functions $s$ and for tensor products and compatibility on dual bundles in the usual sense (c.f. Proposition 1.5.8 in \[11\]).

Proof. We can first restrict the fundamental derivative to an operation $\Gamma(G \times Q p) \times \Gamma(E) \to \Gamma(E)$. Since $p_+ \subset q$, this coincides with the negative of $\bullet$ on $\Gamma(G \times Q p_+) \times \Gamma(E)$ and hence vanishes identically if $E$ is a relative natural bundle. Hence we get a well defined operator as claimed. The claimed properties of $D^\rho$ then follow readily from the analogous properties of the fundamental derivative as proved in Proposition 1.5.8 of \[11\]. \hfill $\square$

2.3. The relative curved Casimir operator. The relative fundamental derivative leads to a relative version of another basic tool for parabolic geometries, the curved Casimir operator originally introduced in \[13\]. It is a general fact that for parabolic subalgebras, the nilradical coincides with the annihilator under the Killing form. Hence the Killing form of $\mathfrak{g}$ descends to a non–degenerate bilinear form $B$ on $\mathfrak{p}/\mathfrak{p}_+$, which of course is $Q$ invariant (and even $P$–invariant). Since $B$ then identifies $\mathfrak{p}/\mathfrak{p}_+$ with its dual, we can view $B^{-1}$ as an invariant, non–degenerate bilinear form on the dual. Given a representation $\mathbb{W}$ of $Q$ inducing a relative natural bundle $E$, we get an induced bundle map

$$B^{-1} \otimes \text{id} : A_\rho^* M \otimes A^*_\rho M \otimes E \to E,$$

and we denote by the same symbol the corresponding tensorial operator on smooth sections.

Definition 2.3. Given a relative natural bundle $E$, we define the relative curved Casimir operator $C_\rho : \Gamma(E) \to \Gamma(E)$ by

$$C_\rho(\sigma) := (B^{-1} \otimes \text{id})(D^\rho D^\rho \sigma).$$

By construction, $C_\rho$ has, in the category of relative natural bundles, analogous naturality properties as proved for the curved Casimir in Proposition 2 of \[13\].

The simplest way to evaluate the relative curved Casimir is via dual frames. Choose a local frame $\{s_l\}$ for $A_\rho M$ and denote by $\{t_\ell\}$ the dual frame with respect to $B$, so $B(s_i, t_j) = \delta_{ij}$. Then by definition, for a section $\sigma \in \Gamma(E)$, one can compute $C_\rho(\sigma)$ on the domain of definition of the frame as

$$\sum_\ell (D^\rho D^\rho \sigma)(t_\ell, s_\ell) = \sum_\ell (D_{t_\ell} D_{s_\ell} \sigma - D_{D^\rho_{t_\ell} s_\ell}^\rho \sigma).$$
As in the case of the ordinary curved Casimir, this expression can be simplified considerably by considering a special class of so-called adapted local frames. Recall that $\mathcal{A}_\rho M$ is filtered by smooth subbundles $\mathcal{A}_{\rho i} M$ and for $i = 0$ and $i = 1$, the filtration components correspond to the subspaces $q/p_+$ and $q_+/p_+$ of $p/p_+$, respectively. Since $B$ is induced by the Killing form of $g$ and the $Q$–invariant filtration of $p/p_+$ is induced by the filtration on $g$, the usual compatibilities between the two structures hold in this case. In particular, for $i > 0$, the degree $i$ filtration component coincides with the annihilator with respect to $B$ of the component of degree $-i + 1$. Moreover, as noted before, the filtration is compatible with the induced Lie bracket.

**Definition 2.4.** An adapted local frame for $\mathcal{A}_\rho M$ is a local frame of the form 
\[\{X_i, A_r, Z^i\}\] with the following properties:

- $Z^i \in \Gamma(\mathcal{A}_{\rho i} M)$ for all $i$ and $A_r \in \Gamma(\mathcal{A}_{\rho r} M)$ for all $r$.
- We have $B(X_i, X_j) = 0$, $B(A_r, X^i) = 0$, and $B(X_i, Z^j) = \delta_i^j$ for all $i, j$, and $r$.
- For all $i$, the algebraic bracket $\{Z^i, X_i\}$ is a section of $\mathcal{A}_{\rho 0} M$.

Note that the last condition in this definition is not explicitly stated in [13] but used afterwards. The proof of existence of such frames and of their fundamental properties is parallel to the case of the usual curved Casimir.

**Lemma 2.5.** Adapted local frames for $\mathcal{A}_\rho M$ exist for each parabolic geometry of type $(G, Q)$. Moreover, if $\{X_i, A_r, Z^j\}$ is such a frame, then there are local sections $A' \in \Gamma(\mathcal{A}_{\rho 0} M)$ such that the dual frame has the form $\{Z^i, A^r, X_i\}$. Finally, $B$ descends to a non–degenerate bilinear form on $\mathcal{A}_{\rho 0} M/\mathcal{A}_{\rho 1} M \cong G \times Q (q/q_+)$ and $\{A_r\}$ and $\{A^r\}$ descend to dual local frames for this quotient bundle.

**Proof.** Consider an open subset over which the Cartan bundle is trivial. Then all natural bundles are trivial and hence admit local frames there. We start by choosing a local frame $\{Z^i\}$ for $\mathcal{A}_{\rho 0} M$ which starts with a local frame for the smallest filtration component, then continues with the next larger filtration component and so on. In particular, this implies that for each $i$ and $j$, the algebraic bracket $\{Z^i, Z^j\}$ can be written as a linear combination of elements $Z_\ell$ where $\ell < i$ and $\ell < j$.

Since $\mathcal{A}_{\rho 0} M$ is the annihilator with respect to $B$ of $\mathcal{A}_{\rho 1} M$, we get a duality between $\mathcal{A}_{\rho 0} M$ and $\mathcal{A}_{\rho 0} M/\mathcal{A}_{\rho 1} M$ induced by $B$. Now we consider the frame $\{X_i\}$ of that bundle which is dual to $\{Z_i\}$ and for each $i$ choose a preimage $\tilde{X}_i \in \Gamma(\mathcal{A}_\rho M)$ of $X_i$. Finally, choose any local frame $\{A_r\}$ for the quotient bundle $\mathcal{A}_{\rho 0} M/\mathcal{A}_{\rho 1} M$ and for each $r$ choose a preimage $\tilde{A}_r \in \Gamma(\mathcal{A}_{\rho 0} M)$ of $A_r$. Then by construction $\{\tilde{X}_i, \tilde{A}_r, Z^j\}$ is a local frame for $\mathcal{A}_\rho M$ and since $B$ vanishes on $\mathcal{A}_{\rho 0} M \times \mathcal{A}_{\rho 1} M$, we have $B(Z_i, Z_j) = 0$, $B(\tilde{A}_r, Z_i) = 0$ and $B(\tilde{X}_i, Z^j) = \delta_i^j$.

Putting $X_i := \tilde{X}_i + \sum_j \frac{1}{2} B(\tilde{A}_r, \tilde{X}_j) Z^j$, we see that we still get $B(X_i, Z^j) = \delta_i^j$ but also $B(X_i, X_j) = 0$ for all $i$ and $j$. Defining $A_r := \tilde{A}_r + \sum_i B(\tilde{A}_r, X_i) Z^i$, we see that $\{X_i, A_r, Z^j\}$ is a local frame which satisfies the first two conditions.
of an adapted frame. But for the algebraic bracket \( \{ Z^i, X_i \} \), invariance of the the Killing form implies that \( B(\{ Z^i, X_i \}, Z_j) = -B(X_i, \{ Z_i, Z_j \}) \). As we have noted above, for any \( \{ Z_i, Z_j \} \) is a linear combination of elements \( Z_i \) with \( \ell < i \), so \( B(X_i, \{ Z_i, Z_j \}) = 0 \). Since this holds for all \( j \), \( \{ Z^i, X_i \} \) lies in the annihilator of \( A^k_p M \), so we have constructed an adapted frame.

From the behavior of \( B \) with respect to our frame, it follows immediately that the dual frame must be of the form \( \{ Z^i, A^r, X_i \} \) for some sections \( A^r \) of \( A_p M \). But by definition \( B(A^r, Z^i) = 0 \) for all \( r \) and \( j \), so the \( A^r \) are sections of \( A^0_p M \). The last claim is then obviously true. □

In terms of an adapted local frame, the action of the relative curved Casimir is easy to compute.

**Proposition 2.6.** In terms of an adapted local frame \( \{ X_i, A_r, Z^i \} \) and the elements \( A^r \) in the dual frame, the relative curved Casimir is given by

\[
C_p(\sigma) = -2 \sum_i Z^i \bullet D^p_{X_i} \sigma - \sum_i \{ Z^i, X_i \} \bullet \sigma + \sum_r A^r \bullet A_r \bullet \sigma.
\]

In particular, the relative curved Casimir has at most order one and it has order zero on relative natural bundles induced by completely reducible representations of \( Q \).

**Proof.** We evaluate the relative curved Casimir with respect to the dual frames \( \{ X_i, A_r, Z^i \} \) and \( \{ Z^i, A^r, X_i \} \) as described above. For the first summands, we use \( D^p_{Z^i} D^p_{X_i} \sigma = -Z^i \bullet D^p_{X_i} \sigma \) and \( -D^p_{Z^i, X_i} \sigma = D^p_{(Z^i, X_i)} \sigma = -\{ Z^i, X_i \} \bullet \sigma \). Similarly, for the second summands, we get \( D^p_{A_r} D^p_{A_r} \sigma = A^r \bullet A_r \bullet \sigma \) and \( -D^p_{A^r, A_r} \sigma = -\{ A^r, A_r \} \bullet \sigma \), so these add up to \( \sum_r A^r \bullet A^r \bullet \sigma \). Finally, for the last summands, we get \( D^p_{X_i} D^p_{Z^i} \sigma = -D^p_{X_i} (Z^i \bullet \sigma) \) and \( -D^p_{Z^i, Z^i} \sigma = (D^p_{X_i} Z^i) \bullet \sigma \), so these add up to \( -\sum_i Z^i \bullet D^p_{X_i} \sigma \), so the claimed formula for \( C_p(\sigma) \) follows.

From this formula, it is evident that \( C_p \) is an operator of order at most one, with the first order part coming only from the terms \( Z^i \bullet D^p_{X_i} \sigma \). But on completely reducible representations of \( Q \), \( q_+ \) acts trivially, so \( \bullet \) vanishes identically on \( A^k_p M \times E \) in this case, and hence \( C_p \) is tensorial. □

In the case of a natural bundle \( E \) induced by a complex irreducible representation of \( Q \), the last property in the proposition readily implies that \( C \) acts by a scalar on \( \Gamma(E) \). The corresponding eigenvalue can be computed in terms of representation theory data in complete analogy to [13]. We do not go into details here, since we will not need this result.

### 2.4. The setup for the relative BGG machinery

We next discuss operations on relative natural bundles coming from the relative version of Kostant’s algebraic harmonic theory from [13]. We start with a representation \( V \) of \( P \), such that \( p_+ \) acts trivially under the infinitesimal representation. Given a parabolic geometry \( (p : G \to M, \omega) \) of type \( (G, Q) \), the associated bundle \( VM := G \times_Q V \) by definition is a relative tractor bundle. The main objects we will study are the bundles \( \Lambda^k T^*_p M \otimes VM \) of relative differential forms with coefficients in \( VM \). By definition, these bundles are induced by the representations \( \Lambda^k(q_+/p_+) \otimes V \) of \( Q \).
Proposition 2.7. The relative Kostant codifferential introduced in Section 2.2 of [15] gives rise to morphisms
\[ \partial^*_\rho : \Lambda^k T^*_\rho M \otimes \mathcal{V}M \to \Lambda^{k-1} T^*_\rho M \otimes \mathcal{V}M. \]
of natural bundles such that \( \partial^*_\rho \circ \partial^*_\rho = 0 \). Hence we obtain smooth subbundles
\[ \text{im}(\partial^*_\rho) \subset \ker(\partial^*_\rho) \subset \Lambda^k T^*_\rho M \otimes \mathcal{V}M. \]
The quotient bundle \( \ker(\partial^*_\rho)/\text{im}(\partial^*_\rho) \) is a completely reducible bundle, which can be identified with the bundle induced by the Lie algebra homology space \( H_k(q_+/p_+, \mathcal{V}) \).

Proof. The relative Kostant codifferential is a \( Q \)-equivariant map
\[ \Lambda^k(q_+/p_+) \otimes \mathcal{V} \to \Lambda^{k-1}(q_+/p_+) \otimes \mathcal{V}, \]
which immediately implies the first statement. The fact that the composition of two codifferentials is zero of course carries over to the induced bundle maps, and hence kernel and image are nested smooth subbundles. The last statement follows by definition of Lie algebra homology and the fact that Lie algebra homology groups are completely reducible representations, see Proposition 2.1 of [15]. □

The Lie algebra homology interpretation can be carried over to the bundle level. As we have observed in Section 2.1, \( T^*_\rho M \) is a bundle of Lie algebras, and by construction \( \mathcal{V}M \) is a bundle of modules over this bundle of Lie algebras. In this language, the bundle maps \( \partial^*_\rho \) are just the point–wise Lie algebra homology differentials as in formula (2.2) of [15]. Hence the quotient bundle \( \ker(\partial_\rho)/\text{im}(\partial_\rho) \) can be interpreted as forming a Lie algebra homology group in each point. Thus we will denote these bundles by \( H_k(T^*_\rho M, \mathcal{V}) \) in what follows.

As discussed in Section 2.5 of [15], the representations \( \Lambda^k(q_+/p_+) \otimes \mathcal{V} \) carry natural filtrations by \( Q \)-invariant subspaces, which induce filtrations of the bundles \( \Lambda^k T^*_\rho M \otimes \mathcal{V}M \) by smooth subbundles. Now one can pass to the associated graded both on the level of representations and on the level of bundles, and this is compatible with forming induced bundles. Moreover, forming the associated graded is compatible (on both sides) with tensorial operations, compare with Section 3.1.1 of [11]. Hence we can view the associated graded bundles as \( \Lambda^k \text{gr}(T^*_\rho M) \otimes \text{gr}(\mathcal{V}M) \) and they are induced by the representations \( \Lambda^k \text{gr}(q_+/p_+) \otimes \text{gr}(\mathcal{V}) \). As a vector space, the latter representation can be identified with \( \Lambda^k(q_+/p_+) \otimes \mathcal{V} \) but \( q_+ \) acts trivially on the associated graded, whence this descends to a representation of \( Q/Q_+ \), see again Section 2.5 of [15]. Hence \( \Lambda^k \text{gr}(T^*_\rho M) \otimes \text{gr}(\mathcal{V}M) \) is a completely reducible natural bundle, and thus a much simpler geometric object than the original bundle of relative forms.

Proposition 2.8. (1) The bundle maps \( \partial^*_\rho \) from Proposition 2.7 are compatible with the natural filtrations on the bundles \( \Lambda^k(T^*_\rho M) \otimes \mathcal{V}M \) and thus induce, for each \( k \), natural bundle maps
\[ \partial^*_\rho : \Lambda^k \text{gr}(T^*_\rho M) \otimes \text{gr}(\mathcal{V}M) \to \Lambda^{k-1} \text{gr}(T^*_\rho M) \otimes \text{gr}(\mathcal{V}M). \]
(2) The relative Lie algebra cohomology differential from Section 2.3 of [15] induces, for each \( k \), natural bundle maps
\[ \partial_\rho : \Lambda^k \text{gr}(T^*_\rho M) \otimes \text{gr}(\mathcal{V}M) \to \Lambda^{k+1} \text{gr}(T^*_\rho M) \otimes \text{gr}(\mathcal{V}M) \]
such that $\partial_\rho \circ \partial_\rho = 0$.

(3) Defining $\Box_\rho := \partial_\rho^* \circ \partial_\rho + \partial_\rho \circ \partial_\rho^*$, we get, for each $k$, a decomposition

$$\Lambda^k \text{gr}(T_\rho^* M) \otimes \text{gr}(\nabla M) = \text{im}(\partial_\rho^*) \oplus \ker(\Box_\rho) \oplus \text{im}(\partial_\rho)$$

as a direct sum of natural subbundles. Moreover, the first two summands add up to $\ker(\partial_\rho^*)$, while the last two summands add up to $\ker(\partial_\rho)$.

Proof. Part (1) follows directly from the algebraic properties of the Kostant codifferential. As discussed in Section 2.5 of [15], the Lie algebra cohomology differential can be viewed as a $Q$–equivariant map on the associated graded representation $\Lambda^* \text{gr}(q_+/p_+) \otimes \text{gr}(\nabla)$, so part (2) follows. Finally, part (3) is a direct consequence of the algebraic Hodge decomposition proved in Lemma 2.2 of [15] and the discussion in Section 2.5 of that reference. \(\square\)

3. The Relative BGG Machinery

Having the necessary setup at hand, we can take the first step towards the construction of relative BGG sequences. We develop a machinery to compress operators (with a certain property) defined on relative differential forms with values in a relative tractor bundle to operators defined on the corresponding Lie algebra homology bundles. This procedure is very general (not even requiring the initial operator to be differential) but set up in such a way that nice properties of the initial operators carry over to the compressed operators.

3.1. Compressable operators. Consider a relative tractor bundle $\nabla M = G \times_\mathcal{Q} \nabla$. Then we denote the spaces of relative differential forms with values in $\nabla M$ by $\Omega^k_\rho(M, \nabla M) := \Gamma(\Lambda^k T_\rho^* M \otimes \nabla M)$. The input needed for the relative BGG machinery is a linear operator $\mathcal{D} = \mathcal{D}_k : \Omega^k_\rho(M, \nabla M) \to \Omega^{k+1}_\rho(M, \nabla M)$, satisfying a certain condition, respectively a sequence of such operators. The condition in question is compatibility with a natural filtration together with a condition on the induced operator on the associated graded. Let us explain the necessary background and at the same time make things more explicit.

We have already noted that $T_\rho M$ and $\nabla M$ are filtered by smooth natural subbundles. Let us denote these filtrations by

$$T_\rho M = T_\rho^{-\mu} M \supset T_\rho^{-\mu+1} M \supset \cdots \supset T_\rho^{-1} M$$

$$\nabla M = \nabla^0 M \supset \nabla^1 M \supset \cdots \supset \nabla^N M.$$ 

Then we call a form $\varphi \in \Omega^k_\rho(M, \nabla M)$ (filtration–)homogeneous of degree $\geq \ell$ if for all $\xi_1, \ldots, \xi_k \in \mathfrak{X}(M)$ such that $\xi_i \in \Gamma(T_\rho^\mu M)$ we have $\varphi(\xi_1, \ldots, \xi_k) \in \nabla^{\mu_1 + \cdots + \mu_k + \ell} M$. Note that this is a purely pointwise condition, so it just means that $\varphi$ is a section of the filtration component of degree $\ell$ of the bundle $\Lambda^k T_\rho^* M \otimes \nabla M$.

Now we say that $\mathcal{D} : \Omega^k(M, \nabla M) \to \Omega^{k+1}(M, \nabla M)$ is compatible with the natural filtration if for each $\ell$ and any $k$–form $\varphi$ which is homogeneous of degree $\geq \ell$, also $\mathcal{D}(\varphi)$ is homogeneous of degree $\geq \ell$. If this holds, then we can take a form $\varphi$, which is homogeneous of degree $\geq \ell$, and form the projection $\text{gr}_\ell(\mathcal{D}(\varphi))$. This is a section of the degree–$\ell$ part $\text{gr}_\ell(\Lambda^{k+1} T_\rho^* M \otimes \nabla M)$ of the associated graded bundle.
If we add to $\varphi$ a form $\psi \in \Omega^k(M, VM)$, which is homogeneous of degree $\geq \ell + 1$, then $\text{gr}_\ell(\mathcal{D}(\varphi + \psi)) = \text{gr}_\ell(\mathcal{D}(\varphi))$.

On the other hand, given a section $\alpha$ of $\text{gr}_\ell(\Lambda^kT^*_\rho M \otimes VM)$, we can choose $\varphi \in \Omega^k_\rho(M, VM)$ which is homogeneous of degree $\geq \ell$ such that $\text{gr}_\ell(\varphi) = \alpha$, and $\varphi$ is unique up to addition of a form $\psi$ which is homogeneous of degree $\geq \ell + 1$. Consequently, we see that $\text{gr}_\ell(\mathcal{D}(\varphi))$ depends only on $\alpha$ and not on the choice of $\varphi$. Thus we conclude that any operator $\mathcal{D} : \Omega^k_\rho(M, VM) \to \Omega^{k+1}_\rho(M, VM)$ which is compatible with the natural filtration induces an operator

$$\text{gr}_0(\mathcal{D}_k) : \Gamma(\text{gr}(\Lambda^kT^*_\rho M \otimes VM)) \to \Gamma(\text{gr}(\Lambda^{k+1}T^*_\rho M \otimes VM))$$

which is homogeneous of degree zero. Using this, we can now formulate a crucial definition.

**Definition 3.1.** A linear operator $\mathcal{D} : \Omega^k_\rho(M, V) \to \Omega^{k+1}_\rho(M, VM)$ is called compressable if and only if it preserves the natural filtration and the induced operator $\text{gr}_0(\mathcal{D})$ is the tensorial operator induced by the Lie algebra cohomology differential $\partial_\rho$ from part (2) of Proposition 2.8.

Once one has found one compressable operator, it is easy to describe all of them. Indeed, if $\mathcal{D}$ and $\mathcal{D}$ both are compressable operators defined on $\Omega^k_\rho(M, VM)$, then consider the difference $\mathcal{D} - \mathcal{D}$. Of course, this preserves the natural filtration on $VM$-valued forms and the induced operator $\text{gr}_0(\mathcal{D} - \mathcal{D})$ is identically zero. But this exactly means that for any $\varphi \in \Omega^k_\rho(M, VM)$ which is homogeneous of degree $\geq \ell$, the form $(\mathcal{D} - \mathcal{D})(\varphi) \in \Omega^{k+1}_\rho(M, VM)$ is homogeneous of degree $\geq \ell + 1$.

Conversely if $\mathcal{D}$ is compressable and $\mathcal{E} : \Omega^k_\rho(M, V) \to \Omega^{k+1}_\rho(M, VM)$ is any linear operator, which strictly increases homogeneous degrees, then $\mathcal{D} + \mathcal{E}$ is again compressable.

3.2. Given a compressable linear operator $\mathcal{D} : \Omega^k_\rho(M, VM) \to \Omega^{k+1}_\rho(M, VM)$, the key point is to study the operator $\partial^*_\rho \circ \mathcal{D}$, which maps $\Omega^k_\rho(M, VM)$ to itself. To simplify notation, we will write this composition as $\partial^*_\rho \mathcal{D}$ from now on. Note that there are the natural subbundles $\text{im}(\partial^*_\rho) \subset \ker(\partial^*_\rho) \subset \Lambda^kT^*_\rho M \otimes VM$, and the restriction of $\partial^*_\rho \mathcal{D}$ defines an operator $\Gamma(\ker(\partial^*_\rho)) \to \Gamma(\text{im}(\partial^*_\rho))$. We next prove a property of this restriction, which is the main technical input for what follows.

**Lemma 3.2.** Let $VM$ be a relative tractor bundle and let $\mathcal{D}$ be a compressable operator defined on $\Omega^k_\rho(M, VM)$. Further let $\pi_H : \Gamma(\ker(\partial^*_\rho)) \to \Gamma(H_k(T^*_\rho M, VM))$ be the tensorial operator induced by the canonical projection to the Lie algebra homology bundle. Then we have

1. The restriction of $\partial^*_\rho \mathcal{D}$ to $\Gamma(\text{im}(\partial^*_\rho))$ is injective.
2. The restriction of $\pi_H$ to $\ker(\partial^*_\rho \mathcal{D}) \cap \Gamma(\ker(\partial^*_\rho))$ is injective.

**Proof.** (1) Suppose that $\varphi \in \Gamma(\text{im}(\partial^*_\rho))$ is such that $\partial^*_\rho \mathcal{D}(\varphi) = 0$, and suppose that $\varphi$ is homogeneous of degree $\geq \ell$ for some $\ell$. From the construction of the natural filtrations it follows that for each $\ell$, the restriction of $\partial^*_\rho$ to the component of degree $\geq \ell$ in $\Lambda^{k+1}T^*_\rho M \otimes VM$ maps onto the component of degree $\geq \ell$ in $\text{im}(\partial^*_\rho)$, compare with Section 2.5 of [15]. Consequently, we can find a form $\psi \in \Omega^{k+1}_\rho(M, VM)$,
which is homogeneous of degree $\geq \ell$ such that $\varphi = \partial_\rho^* \psi$. By Proposition 2.8 this
implies that $\text{gr}_\ell(\varphi) = \mathcal{D}_\ell(\text{gr}_\ell(\psi))$. On the other hand, we get
$$0 = \text{gr}_\ell(\partial_\rho^* \mathcal{D}(\varphi)) = \mathcal{D}_\ell(\text{gr}_\ell(\mathcal{D}(\varphi))) = \mathcal{D}_\ell(\text{gr}_\ell(\varphi)).$$

But since $\text{gr}_\ell(\varphi) \in \Gamma(\text{im}(\mathcal{D}_\ell^*))$, the latter expression coincides with $\square_\rho(\text{gr}_\ell(\varphi))$. Hence the Hodge decomposition in part (3) of Proposition 2.8 implies that $\text{gr}_\ell(\varphi) = 0$ and thus $\varphi$ is homogeneous of degree $\geq \ell + 1$. Iterating this argument finitely many times, we get $\varphi = 0$, which completes the proof of (1).

(2) By definition, the kernel of $\pi_H$ coincides with $\Gamma(\text{im}(\partial_\rho^*)) \subset \Gamma(\ker(\partial_\rho^*))$. But we have just shown that this subspace has zero intersection with $\ker(\partial_\rho^* \mathcal{D})$, which implies the claim. 

\section{The splitting operator}

As a next step, we construct an operator from $\Gamma(H(V_M,\mathcal{V}M))$ to $\ker(\partial_\rho^* \mathcal{D}) \subset \Gamma(\ker(\partial_\rho^*))$, which is right invariant to (the restriction of) $\pi_H$, thus proving that this restriction is a linear isomorphism. This so-called splitting operator can be constructed from polynomials in $\partial_\rho^* \mathcal{D}$, which implies that it inherits nice properties from $\mathcal{D}$. We put
$$\mathcal{W} := \ker(\partial_\rho^*) := \mathcal{W} \subset \Lambda^k(q_+/p_+) \otimes \mathcal{V},$$
and denote the filtration components of degree $\ell$ by $\mathcal{W}^\ell$ and $\mathcal{W}_\ell := \mathcal{W}^\ell \cap \mathcal{W}$, respectively. Consequently, for each $\ell$, the quotient $\text{gr}_\ell(\mathcal{W}) = \mathcal{W}^\ell / \mathcal{W}^{\ell+1}$ is naturally a subspace of $\text{gr}_\ell(\mathcal{W})$. From Sections 2.4 and 2.6 of [15], we see that the relative Kostant Laplacian acts on each of these spaces, it acts diagonalizably, and $\text{gr}_\ell(\mathcal{W}) \subset \text{gr}_\ell(\mathcal{W})$ coincides with the direct sum of the eigenspaces corresponding to non-zero eigenvalues. Let $a^\ell_1,\ldots,a^\ell_j$ be the different non-zero eigenvalues which occur in homogeneity $\ell$.

Now we take the corresponding induced bundles $\mathcal{W}M$ and $\mathcal{W}M$ which are just the subbundles $\ker(\partial_\rho^*)$ and $\text{im}(\partial_\rho^*)$ of $\Lambda^k T^*_\rho M \otimes \mathcal{V}M$. The corresponding filtration components give rise to smooth subbundles $\mathcal{W}^\ell M \subset \mathcal{W}M \subset \mathcal{W}M$. Now for each possible homogeneity $\ell$, we define an operator $S_\ell : \Gamma(\mathcal{W}M) \to \Gamma(\mathcal{W}M)$ by
$$S_\ell := (-1)^{j_\ell} \prod_{r=1}^{j_\ell} a^\ell_r \prod_{r=1}^{j_\ell} (\partial_\rho^* \mathcal{D} - a^\ell_r \text{id}).$$

The basic properties of these operators are now easy to prove.

\begin{lemma}
For each $\ell$, the operator $S_\ell : \Gamma(\mathcal{W}) \to \Gamma(\mathcal{W})$ is compatible with the natural filtration and satisfies $\pi_H \circ S_\ell = \pi_H$, $\partial_\rho^* \mathcal{D} \circ S_\ell = S_\ell \partial_\rho^* \mathcal{D}$, and $S_\ell(\Gamma(\mathcal{W}^\ell)) \subset \Gamma(\mathcal{W}^{\ell+1})$.
\end{lemma}

\begin{proof}
By definition, $S_\ell$ is a polynomial in operator $\partial_\rho^* \mathcal{D}$, so it commutes with $\partial_\rho^* \mathcal{D}$. Moreover, since $\partial_\rho^* \mathcal{D}$ is compatible with the natural filtration, the same holds for $S_\ell$. By definition $\pi_H \circ \partial_\rho^* = 0$, so $\pi_H \circ (\partial_\rho^* \mathcal{D} - a^\ell_r \text{id}) = a^\ell_r \pi_H$ for each $r$, which immediately implies $\pi_H \circ S_\ell = \pi_H$.
\end{proof}
To prove the last property, take \( \varphi \in \Gamma(\tilde{W}^{\ell}) \). Then we have already observed that \( \partial^*_\rho D(\varphi) \in \Gamma(\tilde{W}^{\ell}) \), and we compute

\[
\text{gr}_\ell(\partial^*_\rho D(\varphi)) = \partial^*_\rho \text{gr}_\ell(D(\varphi)) = \partial^*_\rho \partial_\rho \text{gr}_\ell(\varphi).
\]

Since \( \text{gr}_\ell \varphi \) is a section of the subbundle \( \text{im}(\partial^*_\rho) \) the last term coincides with \( \square_\rho(\text{gr}_\ell(\varphi)) \). Now we can write \( \varphi \) as a finite sum of sections \( \varphi_i \) of \( \Gamma(\tilde{W}^{\ell}) \) such that each \( \text{gr}_\ell(\varphi_i) \) is a section of the bundle induced by one of the eigenspaces for \( \square_\rho \). If \( a^\rho_i \) is the corresponding eigenvalue, then the above computation shows that \( \text{gr}_\ell((\partial^*_\rho D - a^\rho_i \text{id})(\varphi_i)) = 0 \). Since the factors in the composition defining \( S_\ell \) can be permuted arbitrarily, we see that \( \text{gr}_\ell(S_\ell(\varphi_i)) = 0 \). This implies \( \text{gr}_\ell(S_\ell(\varphi)) = 0 \) and thus \( S_\ell(\varphi) \in \Gamma(\tilde{W}^{\ell+1}) \).

Now we define the splitting operator \( S : \Gamma(\ker(\partial^*_\rho)) \to \Gamma(\ker(\partial^*_\rho)) \) as the composition of the operators \( S_\ell \) for the finitely many possible homogeneities \( \ell \), which show up in \( \ker(\partial^*_\rho) \subset \Lambda^k T^*_\rho M \otimes VM \). Since the operators \( S_\ell \) all are polynomials in \( \partial^*_\rho D \), the order in which they are composed plays no role.

**Theorem 3.4.** (1) The operator \( S \) satisfies \( \pi_H \circ S = \pi_H \) and \( \partial^*_\rho D \circ S = 0 \), and its restriction to \( \Gamma(\text{im}(\partial^*_\rho)) \) vanishes identically. Thus it descends to an operator

\[
\Gamma(\ker(\partial^*_\rho))/\Gamma(\text{im}(\partial^*_\rho)) \cong \Gamma(\text{ker}(\partial^*_\rho)) \ni \ker(\partial^*_\rho) \cap \Gamma(\ker(\partial^*_\rho)),
\]

which is right inverse to the tensorial projection \( \pi_H \). In particular, \( \pi_H \) restricts to a linear isomorphism on \( \ker(\partial^*_\rho) \cap \Gamma(\ker(\partial^*_\rho)) \).

(2) For \( \alpha \in \Gamma(\ker H_k(T^*_\rho M, VM)) \) the form \( \varphi =: S(\alpha) \) is uniquely determined by \( \partial^*_\rho(\varphi) = 0 \), \( \pi_H(\varphi) = \alpha \), and \( \partial^*_\rho D(\varphi) = 0 \).

(3) If the operator \( D \) is such that \( \partial^*_\rho D \) belongs to a class of linear operators which is stable under forming polynomials, then also \( S \) belongs to this class.

**Proof.** (1) Since \( \pi_H \circ S_\ell = \pi_H \) holds for each \( \ell \) by Lemma 3.3, we see that \( \pi_H \circ S = \pi_H \). If \( \varphi \) is a section of \( \text{im}(\partial^*_\rho) \), then \( \varphi \) is homogeneous of degree \( \geq \ell \) for some \( \ell \). Then Lemma 3.3 shows iteratively that \( S_\ell(\varphi) \in \Gamma(\tilde{W}^{\ell+1}) \), \( S_{\ell+1}(S_\ell(\varphi)) \in \Gamma(\tilde{W}^{\ell+2}) \) and continuing up to the maximal possible homogeneity, we conclude that the composition of the \( S_i \) for \( i \geq \ell \) annihilates \( \varphi \). This of course implies \( S(\varphi) = 0 \), so \( S \) vanishes on \( \Gamma(\text{im}(\partial^*_\rho)) \).

Next Lemma 3.3 iteratively implies that \( S \) commutes with \( \partial^*_\rho D \). But since we have just seen that \( S \circ \partial^*_\rho = 0 \), this implies \( \partial^*_\rho D \circ S = 0 \). From this, the rest of (1) is evident.

(2) By part (1), the form \( \varphi = S(\alpha) \) for \( \alpha \in \Gamma(\ker H_k(T^*_\rho M, VM)) \) has the claimed properties. Conversely, if \( \varphi \in \Omega^k_\rho(M, VM) \) has the three properties, then \( \partial^*_\rho \varphi = 0 \) implies that we can form \( \pi_H(\varphi) \in \Gamma(\ker H_k(T^*_\rho M, VM)) \). But then \( \partial^*_\rho D(\varphi) = 0 \) immediately implies \( S_\ell(\varphi) = \varphi \) for all \( \ell \) and hence \( \varphi = S(\pi_H(\varphi)) \).

(3) This is clear, since \( S \) is given by a universal polynomial in \( \partial^*_\rho D \). \( \square \)

### 3.4. The compressed operator

Having the splitting operator at hand, it is now easy to complete the relative version of the BGG construction.
Definition 3.5. Given a compressable operator $D : \Omega^k_p(M, VM) \to \Omega^{k+1}_p(M, VM)$, the compression of $D$, respectively the BGG–operator induced by $D$, is the operator

$$D : \Gamma(\mathcal{H}_k(T^*_p M, VM)) \to \Gamma(\mathcal{H}_{k+1}(T^*_p M, VM))$$

defined by

$$D(\alpha) := \pi_H(D(S(\alpha))),$$

where $S$ denotes the splitting operator associated to $D$.

Notice that this definition makes sense since by Theorem 3.4, $D(S(\alpha))$ is a section of the bundle $\text{ker}(\partial^*_p)$, so $\pi_H$ can be applied to it.

We can easily prove that nice properties of a compressable operator carry over to the corresponding compressed operator. Moreover, the notion of a normal solution of a first BGG operator (see e.g. [8]) has a nice analog in general.

Proposition 3.6. Let $D : \Omega^k_p(M, VM) \to \Omega^{k+1}_p(M, VM)$ be a compressable operator and $D : \Gamma(\mathcal{H}_k(T^*_p M, VM)) \to \Gamma(\mathcal{H}_{k+1}(T^*_p M, VM))$ the corresponding compressed operator.

(1) If $D$ and $\partial^*_p D$ belong to some class of operators which is stable under forming polynomials then also the compressed operator $D$ belongs to this class.

(2) The projection $\pi_H$ maps $\text{ker}(D) \cap \Gamma(\text{ker}(\partial^*_p))$ bijectively onto a subspace of $\text{ker}(D) \subset \Gamma(\mathcal{H}_k(T^*_p M, VM))$.

Proof. (1) follows directly from part (3) of Theorem 3.4.

(2) Suppose that $\varphi \in \Omega^k_p(M, VM)$ satisfies $\partial^*_p(\varphi) = 0$ and $D(\varphi) = 0$. Then $\partial^*_p D(\varphi) = 0$, which by part (2) of Theorem 3.4 implies $\varphi = S(\pi_H(\varphi))$. On the other hand, we also get $0 = D(S(\pi_H(\varphi)))$, which implies $\pi_H(\varphi) \in \text{ker}(D)$.

Remark 3.7. The construction of the compressed operator as a polynomial in the original operator is a major advantage compared to earlier constructions of BGG sequences. This is expressed by part (1) of the proposition, which shows that, while compressability is the only property of a linear operator required to apply the BGG-machinery, many nice properties of such an operator manifestly carry over to the corresponding BGG–operator.

Most easily, if $D$ is a differential operator, then so is $D$. The condition also applies to the usual definition of an invariant differential operator, which requires a universal formula in terms of certain distinguished connections, their torsion and curvature. In this setting, we obtain the first construction of BGG–operators which are manifestly invariant. There are also notions of “strong invariance” to which this condition applies.

A drawback of the construction is that in the construction of the splitting operator as a polynomial in the compressible operator, there is a lot of cancellation. The degree of the polynomial used to define $S$ roughly equals the number of $q_0$–irreducible components in the representation $\text{im}(\partial^*_p) \subset \Lambda^k(q_+/p_+) \otimes V$.

In the important special case that $D$ is a first order differential operator, one might therefore expect that this number of components is the order of the splitting operator. However, as we shall see in Remark 3.11 below, the order of the splitting operator basically is given by the length of the $q$–invariant filtration on
im(\partial_p^*) \subset \Lambda^k(q_+/p_+) \otimes V$, which is much smaller than the number of irreducible components.

3.5. An alternative construction. There is an alternative construction for the splitting operators, which at the same time leads to a relative analog of a further element of the “BGG–calculus” as developed in \cite{4}. This will be particularly useful in the case of sequences of compressable operators discussed below. We preferred to first present the direct construction from Section 3.3 since it seems more intuitive to us.

We continue using the notation from above, so \( \widetilde{W} = \text{im}(\partial_p^*) \subset \Lambda^k(q_+/p_+) \otimes V \), \( \widetilde{W}^\ell \subset \widetilde{W} \) denotes the filtration component of degree \( \ell \) and the different (non–zero) eigenvalues of \( \Box_p \) on \( \text{gr}_t(\widetilde{W}) \) are denoted by \( a_1^{\ell}, \ldots, a_s^{\ell} \). Now we observe that by construction the operator \( \partial_p^* D \) maps \( \Gamma(\widetilde{W}M) \) to itself and preserves each of the filtration components \( \Gamma(\widetilde{W}^\ell M) \). So for each homogeneity \( \ell \), we can define an operator \( \tilde{Q}^\ell \) on \( \Gamma(\widetilde{W}^\ell M) \) by

\[
\tilde{Q}^\ell := \sum_{r=1}^{j_\ell} \frac{1}{a_r!} \prod_{s \neq r} (\partial_p^* D - a_s^\ell \text{id}).
\]

**Lemma 3.8.** For each homogeneity \( \ell \) and each section \( \varphi \in \Gamma(\widetilde{W}^\ell M) \), we have

\[
\varphi - \partial_p^* D \tilde{Q}^\ell (\varphi) \in \Gamma(\widetilde{W}^{\ell+1} M).
\]

**Proof.** As in the proof of Lemma 3.3 we can write a section \( \varphi \in \Gamma(\widetilde{W}^\ell M) \) as a finite sum \( \varphi = \varphi_1 + \cdots + \varphi_{j_\ell} \) in such a way that for each \( i \) the image \( \text{gr}_x(\varphi_i) \) satisfies \( \Box_p(\text{gr}_x(\varphi_i)) = a_i^\ell \varphi_i \). But this implies that \( \partial_p^* D(\varphi_i) \in \Gamma(\widetilde{W}^\ell M) \) and \( \text{gr}_x(\partial_p^* D(\varphi_i)) = a_i^\ell \text{gr}_x(\varphi_i) \). Now in the definition of \( \tilde{Q}^\ell \), we can permute the factors in the composition of operators in each summand arbitrarily. This shows that \( \prod_{s \neq r} (\partial_p^* D - a_s^\ell \text{id}) \) maps each \( \varphi_s \) for \( s \neq r \) to a section of \( \widetilde{W}^{\ell+1} M \). On the other hand, applying this composition to \( \varphi_r \) and applying \( \text{gr}_x \), we obtain \( \prod_{s \neq r} (a_r^\ell - a_s^\ell) \text{gr}_x(\varphi_r) \). Hence we conclude from the definition that \( \tilde{Q}^\ell (\varphi) \in \Gamma(\widetilde{W}^\ell M) \) and \( \text{gr}_x(\tilde{Q}^\ell (\varphi)) = \sum_{r=1}^{j_\ell} \frac{1}{a_r!} \prod_{s \neq r} (\partial_p^* D - a_s^\ell \text{id}) \text{gr}_x(\varphi_r) \). Together with the above observation on the action of \( \partial_p^* D \), this implies the claim of the lemma. \( \square \)

**Theorem 3.9.** There is an operator \( Q : \Gamma(\text{im}(\partial_p^*)) \to \Gamma(\text{im}(\partial_p^*)) \) which can be written as a universal polynomial in \( \partial_p^* D \) such that \( \partial_p^* D \circ Q = \text{id} \) on \( \Gamma(\text{im}(\partial_p^*)) \).

**Proof.** We construct \( Q \) recursively. Denoting by \( N \) the maximal possible homogeneity occurring in \( \widetilde{W} \), Lemma 3.8 shows that \( Q_N := \tilde{Q}_N \) has the property that \( \varphi - \partial_p^* D Q_N(\varphi) \in \Gamma(\widetilde{W}^{N+1} M) = \{0\} \) for each \( \varphi \in \Gamma(\widetilde{W}^N M) \).

Let us inductively assume that for some \( \ell < N \) we have found an operator \( \tilde{Q}^\ell \) on \( \Gamma(\widetilde{W}^\ell M) \) which is a universal polynomial in \( \partial_p^* D \) and satisfies \( \partial_p^* D \circ \tilde{Q}^\ell = \text{id} \) on \( \Gamma(\widetilde{W}^\ell M) \). Then for \( \varphi \in \Gamma(\widetilde{W}^{\ell-1} M) \), we again invoke Lemma 3.5 to conclude that \( \varphi - \partial_p^* D \tilde{Q}^\ell (\varphi) \in \Gamma(\widetilde{W}^\ell M) \), and hence we can define

\[
Q^{\ell-1}(\varphi) := \tilde{Q}^{\ell-1}(\varphi) - \tilde{Q}^\ell (\varphi - \partial_p^* D \tilde{Q}^{\ell-1}(\varphi)).
\]
Then we immediately conclude that $\partial^*_D Q^{\ell-1}(\varphi) = \varphi$. This leads to an operator $Q$ with the required properties in finitely many steps. \hfill \Box

It is easy to describe the splitting operator $S$ in terms of $Q$:

**Corollary 3.10.** Viewed as an operator on $\Gamma(\ker(\partial^*_p))$, the splitting operator $S$ from Theorem 3.4 is given by $S = \text{id} - Q \partial^*_D \rho D$.

**Proof.** Let us put $\tilde{S} := \text{id} - Q \partial^*_D \rho D$. Since $Q$ has values in $\Gamma(\text{im}(\partial^*_p))$ we see that $\tilde{S}$ maps the space $\Gamma(\ker(\partial^*_p))$ to itself and that $\pi_H \circ \tilde{S} = \pi_H$. Moreover, from Lemma 3.8 we conclude immediately that $\partial^*_D \rho D \circ \tilde{S} = 0$. This already shows that $S - \tilde{S}$ maps $\Gamma(\ker(\partial^*_p))$ to $\Gamma(\text{im}(\partial^*_p)) \cap \ker(\partial^*_D \rho D)$, and this intersection is trivial by part (1) of Lemma 3.2. \hfill \Box

**Remark 3.11.** While the operator $Q$ (in the case $p = g$) was the crucial ingredient for the construction of BGG sequences in [4], our construction as a polynomial in $\partial^*_D$, which gives a manifestly invariant operator in case $D$ is invariant, is new even for this special case. Similarly to the case of $S$ discussed in Remark 3.7, also the construction of $Q$ as a polynomial involves a lot of cancellation. There is an alternative construction for $Q$ (and thus via Corollary 3.10 also for $S$), however, which needs much fewer composition factors. This is a relative version of the construction of [4] for ordinary BGG–sequences.

One first fixes a splitting of the natural filtration on the bundle $\text{im}(\partial^*_p)$ and thus an identification with its associated graded bundle. A conceptual way to obtain such a splitting is via the choice of a Weyl–structure, see section 5.1 of [11] or [10]. Now $\text{gr}(\text{im}(\partial^*_p))$ is a subbundle of $\text{gr}(\Lambda^k T^*_\rho M \otimes VM)$, which is invariant under the bundle map $\Box_{\rho}$ from Proposition 2.8. On this subbundle $\Box_{\rho}$ coincides with $\partial^*_D \circ \Box_{\rho}$ and it is invertible, so we may form $(\Box_{\rho})^{-1}$. Via the chosen isomorphism, we can now define bundle maps $\Box_{\rho}$ and $(\Box_{\rho})^{-1}$ on $\text{im}(\partial^*_p)$ and we use the same symbols to denote the resulting tensorial operators.

In terms of these operators defined on $\Gamma(\text{im}(\partial^*_p))$, we can write

$$\partial^*_D = \Box_{\rho}(\text{id} + \Box_{\rho}^{-1}(\partial^*_D \rho D - \Box_{\rho})).$$

From the construction it is clear that $\partial^*_D \rho D - \Box_{\rho}$ raises the filtration degree by one, i.e. in the notation of Sections 3.3 and 3.5 it maps each of the spaces $\Gamma(\tilde{W}^{\ell+1} M)$ to $\Gamma(\tilde{W}^{\ell+2} M)$. This remains true after composing $\Box_{\rho}^{-1}$ and it of course implies that the resulting operator is nilpotent of degree $N+1$ where $N$ describes the length of the natural filtration of the bundle $\text{im}(\partial^*_p)$. Adding the identity to this nilpotent operator, the result is invertible, and there is a usual Neumann series for the inverse, which actually is a finite sum by nilpotency. Thus, one can construct an inverse of $\partial^*_D$ on $\Gamma(\text{im}(\partial^*_p))$ as

$$\left(\sum_{k=0}^{N+1} (-1)^k (\Box_{\rho}^{-1}(\partial^*_D \rho D - \Box_{\rho}))^k\right) \circ \Box_{\rho}^{-1}.$$

As discussed above, this gives a much smaller number of factors in a composition that the construction in 3.5. For example, if $D$ is a first order differential operator,
we conclude that $Q$ is a differential operator of order at most $N + 1$, so $S$ is a differential operator of order at most $N + 2$.

The disadvantage of this construction is that one has to use a non–natural tensorial operation, which makes it much more difficult to see that nice properties of $\mathcal{D}$ carry over to $D$. In particular, this applies to concepts of strong invariance. One solution to this problem is provided by the original construction in [12] in terms of semi–holonomic jet modules, which however is significantly more complicated.

3.6. Sequences of compressable operators. For the next step we have to assume that rather than a single compressable operator, we have a whole sequence $\mathcal{D}_k : \Omega^k_p(M, \mathcal{V}M) \to \Omega^{k+1}_p(M, \mathcal{V}M)$ of compressable operators. Then we have

Definition 3.12. Given a sequence $\mathcal{D}_k : \Omega^k_p(M, \mathcal{V}M) \to \Omega^{k+1}_p(M, \mathcal{V}M)$ of compressable operators, we define the associated Laplacians

$$\Box^D_k := \Box^D_k : \Omega^k_p(M, \mathcal{V}M) \to \Omega^k_p(M, \mathcal{V}M)$$

by $\Box^D_k := \partial^*_p \circ \mathcal{D}_k + \mathcal{D}_{k-1} \circ \partial^*_p$.

Proposition 3.13. (1) If $\varphi \in \Omega^k_p(M, \mathcal{V}M)$ satisfies $\Box^D_k(\varphi) = 0$, then $\partial^*_p(\varphi) = 0$. Hence $\ker(\Box^D_k) \subset \Omega^k_p(M, \mathcal{V}M)$ coincides with $\ker(\partial^*_p \mathcal{D}_k) \cap \Gamma(\ker(\partial^*_p))$.

(2) On $\Gamma(\ker(\partial^*_p))$, the operator $\Box^D_k$ coincides with $\partial^*_p \mathcal{D}_k$, so in the definitions of the operators $S_\ell$ in Section 3.3 and $\tilde{Q}$ in Section 3.3, in the statements of Theorems 3.4 and 3.9 and of Lemma 3.8, and in the construction of Remark 3.11, one may always replace $\partial^*_p \mathcal{D}_k$ by $\Box^D_k$.

Proof. (1) If $0 = \Box^D_k(\varphi)$, then applying $\partial^*_p$ and using $\partial^*_p \circ \partial^*_p = 0$, we get $0 = \partial^*_p \mathcal{D}_k \partial^*_p(\varphi)$. But from part (1) of Lemma 3.2 we know that $\partial^*_p \mathcal{D}_k$ is injective on $\Gamma(\ker(\partial^*_p))$. Thus $\partial^*_p \varphi = 0$ and hence also $\partial^*_p \mathcal{D}_k(\varphi) = 0$.

(2) now follows immediately from $\partial^*_p \circ \partial^*_p = 0$. \qed

We can now look at conditions related to the operators $\mathcal{D}_k$ forming a complex.

Theorem 3.14. Consider a sequence $\mathcal{D}_k : \Omega^k_p(M, \mathcal{V}M) \to \Omega^{k+1}_p(M, \mathcal{V}M)$ of compressable operators. Then we have

(1) Let $Q$ be the operator on $\Gamma(\ker(\partial^*_p)) \subset \Omega^{k-1}_p(M, \mathcal{V}M)$ constructed in Theorem 3.4. Then for each $\varphi \in \Omega^k_p(M, \mathcal{V}M)$, we have $\partial^*_p(\varphi - \mathcal{D}_{k-1} Q \partial^*_p(\varphi)) = 0$.

(2) If $\mathcal{D}_k \circ \mathcal{D}_{k-1} = 0$, then $\mathcal{D}_k \circ \mathcal{D}_{k-1} = 0$. If in addition $\mathcal{D}_{k+1} \circ \mathcal{D}_k = 0$, then the splitting operator induces a surjective linear map,

$$\ker(\mathcal{D}_k) / \ker(\mathcal{D}_{k-1}) \to \ker(\mathcal{D}_k) / \ker(\mathcal{D}_{k-1}),$$

which is a linear isomorphism provided that also $\mathcal{D}_{k-1} \circ \mathcal{D}_{k-2} = 0$.

Proof. (1) follows immediately from the fact that $\partial^*_p \mathcal{D}_{k-1} Q = \text{id} \circ \Gamma(\ker(\partial^*_p))$ which we proved in Theorem 3.9.

(2) Suppose that $\mathcal{D}_k \circ \mathcal{D}_{k-1} = 0$, take a section $\alpha \in \Gamma(\mathcal{H}_{k-1}(\mathcal{T}^*_p M, \mathcal{V}M))$ and consider $\varphi := \mathcal{D}_{k-1}(\mathcal{S}(\alpha))$. Then by definition, $\partial^*_p(\varphi) = 0$ and $\pi_H(\varphi) = \mathcal{D}_{k-1}(\alpha)$. 

By assumption $D_k(\varphi) = 0$, so part (2) of Theorem 3.4 shows that $\varphi = S(p_H(\varphi))$. Thus we get $D_{k-1} \circ S_{k-1} = S_k \circ D_{k-1}$ and hence

$$D_k \circ D_{k-1} = \pi_H \circ D_k \circ S_k \circ D_{k-1} = \pi_H \circ D_k \circ D_{k-1} \circ S_{k-1} = 0.$$ 

Now suppose that in addition $D_{k+1} \circ D_k = 0$, let $\alpha \in \Gamma(\mathcal{H}_k(T^*_pM, VM))$ be such that $D_k(\alpha) = 0$ and consider $\varphi := S_k(\alpha)$. Then from above we see that $D_k(\varphi) = S(D_k(\alpha)) = 0$. Moreover, if $\alpha = D_{k-1}(\beta)$ for some $\beta \in \Gamma(\mathcal{H}_{k-1}(T^*_pM, VM))$, then $S_k(\alpha) = D_{k-1}(S_{k-1}(\beta))$. This shows that $S_k$ induces a well defined map $\ker(D_k)/\im(D_{k-1}) \to \ker(D_k)/\im(D_{k-1})$ in cohomology.

Supposing that $\varphi \in \Omega^k_p(M, VM)$ satisfies $D_k(\varphi) = 0$, we can use part (1) to find $\psi \in \Omega^{k-1}_p(M, VM)$ such that $\tilde{\psi} := \varphi + D_{k-1}(\psi)$ satisfies $\partial^*_p(\tilde{\psi}) = 0$. By assumption $D_k(\tilde{\psi}) = D_k(\varphi) = 0$ and thus $\tilde{\varphi} = S(\pi_H(\tilde{\varphi}))$ and $D_k(\pi_H(\tilde{\varphi})) = 0$. This implies surjectivity of the map in cohomology.

So let us finally assume that $D_{k-1} \circ D_{k-2} = 0$ and that we have given $\alpha \in \Gamma(\mathcal{H}_{k}(T^*_pM, VM))$ such that $D_k(\alpha) = 0$ and $S_k(\alpha) = D_{k-1}(\varphi)$ for some $\varphi \in \Omega^{k-1}_p(M, VM)$. Then again by part (1), we can find an element $\psi \in \Omega^k_p(M, VM)$ such that $\tilde{\varphi} := \varphi + D_{k-2}\psi$ satisfies $\partial^*_p(\tilde{\varphi}) = 0$. By assumption $D_{k-1}(\tilde{\varphi}) = D_{k-1}(\varphi) = S(\alpha)$, so in particular $\partial^*_pD_{k-1}(\tilde{\varphi}) = 0$ and hence $\tilde{\varphi} = S(\pi_H(\tilde{\varphi}))$. Moreover,

$$D_{k-1}(\pi_H(\tilde{\varphi})) = \pi_H(D_{k-1}(\tilde{\varphi})) = \pi_H(S(\alpha)) = \alpha,$$

which implies injectivity of the map in cohomology induced by $S$. \hfill $\square$

4. The relative twisted exterior derivative

In this section, we construct a sequence of compressable first order differential operators on relative forms with values in an arbitrary relative tractor bundle, which has strong naturality properties. Thus we can run the BGG machinery as developed in Section 3 on this sequence to obtain invariant differential operators defined on the relative Lie algebra homology bundles. In view of the discussion in Section 3.1, this also gives a description of all compressable operators.

4.1. Definition of the relative twisted exterior derivative. Given a relative tractor bundle $VM$, we start by defining an operator

$$\tilde{d}^\rho : \Gamma(\Lambda^k_A^*M \otimes VM) \to \Gamma(\Lambda^{k+1}_A^*M \otimes VM).$$

Here $A^*_pM$ is the dual to the relative adjoint tractor bundle introduced in Section 2.2. By definition, the representation $V$ inducing $VM$ is the restriction to $Q$ of a representation of $P$, which in addition has the property that the ideal $p_- \subset p$ acts trivially in the infinitesimal representation. Thus we can view $V$ as a representation of the Lie algebra $p/\mathfrak{p}_+$. Hence there is the standard Lie algebra cohomology differential, compare with Section 2.3 of [15], which, for each $k$, is a linear map

$$\partial_{p/\mathfrak{p}_+} : \Lambda^k(p/\mathfrak{p}_+) \otimes V \to \Lambda^{k+1}(p/\mathfrak{p}_+) \otimes V.$$
In the picture of multilinear maps, this differential is given by

\[
\partial \varphi(A_0, \ldots, A_k) := \sum_{i=0}^{k} (-1)^i A^i \cdot \varphi(A_0, \ldots, \widehat{A_i}, \ldots, A_k) \\
+ \sum_{i<j} (-1)^{i+j} \varphi([A_i, A_j], A_0, \ldots, \widehat{A_i}, \ldots, \widehat{A_j}, \ldots, A_k),
\]

for \(A_0, \ldots, A_k \in \mathfrak{p}/\mathfrak{p}_+.\) This map is evidently \(Q\)-equivariant, so it induces a bundle map between the corresponding associated bundles. We denote this bundle map as well as the corresponding tensorial operator on sections by the same symbol.

On the other hand, applying the relative fundamental derivative from Section 2.2 to \(\varphi \in \Gamma(\Lambda^k \mathcal{A}_\rho^* M \otimes \mathcal{V} M),\) we obtain

\[
D^\rho \varphi \in \Gamma(\mathcal{A}_\rho^* M \otimes \Lambda^k \mathcal{A}_\rho^* M \otimes \mathcal{V} M).
\]

Then we define \(d_1^\rho \varphi\) by

\[
d_1^\rho \varphi(s_0, \ldots, s_k) := \sum_{i=0}^{k} (-1)^i (D^\rho_i \varphi)(s_0, \ldots, s_i, \ldots, s_k).
\]

Observe that this is alternating in all entries, so \(d_1^\rho \varphi \in \Gamma(\Lambda^{k+1} \mathcal{A}_\rho^* M \otimes \mathcal{V} M).\) Having this at hand, we finally put \(d^\rho \varphi := d_1^\rho \varphi + \partial_{\mathfrak{p}/\mathfrak{p}_+} \varphi.\)

Note that the formula (4.2) for \(d_1^\rho \varphi\) can be further expanded using the naturality properties of \(D^\rho\) derived in Proposition 3.2. These imply that for \(s, t, \ldots, t_k \in \Gamma(\mathcal{A}_\rho M)\) we have

\[
(D^\rho_s \varphi)(t_1, \ldots, t_k) = D^\rho_s(\varphi(t_1, \ldots, t_k)) - \sum_{i=1}^r \varphi(t_1, \ldots, D^\rho_i t_i, \ldots, t_k).
\]

On the other hand, we can explicitly express \(\partial_{\mathfrak{p}/\mathfrak{p}_+} \varphi(s_0, \ldots, s_k)\) using the definition in formula (4.1). We only have to replace the action \(\cdot : (\mathfrak{p}/\mathfrak{p}_+) \times \mathcal{V} \to \mathcal{V}\) by the induced bundle map \(\bullet : \mathcal{A}_\rho M \times \mathcal{V} M \to \mathcal{V} M\) and the Lie bracket on \(\mathfrak{p}/\mathfrak{p}_+\) by the (induced) algebraic bracket \(\{ \cdot, \cdot \}\) on (sections of) \(\mathcal{A}_\rho M.\)

As we have noted in Section 2.2, the relative tangent bundle \(T_r M\) can be identified with the quotient \(\mathcal{A}_\rho M/\mathcal{A}_\rho^0 M,\) so dually \(T_r^* M\) is a subbundle of \(\mathcal{A}_\rho^* M.\) Consequently, we can view \(\Omega^k_r(M, \mathcal{V} M)\) as a subspace of \(\Gamma(\Lambda^k \mathcal{A}_\rho^* M \otimes \mathcal{V} M).\) The elements of this subspace can evidently be characterized by the fact that they vanish upon insertion of a single section of the subbundle \(\mathcal{A}_\rho^0 M \subset \mathcal{A}_\rho M.\) Using this, we can now prove:

**Theorem 4.1.** The operators \(d^\rho\) restrict to first order invariant differential operators

\[
d^\rho : \Omega^k_r(M, \mathcal{V} M) \to \Omega^{k+1}_r(M, \mathcal{V} M),
\]

which are compressable in the sense of Definition 3.7.

Via the construction in Section 3 we thus obtain a relative BGG–sequence

\[
(4.3) \quad D_k : \Gamma(H_k(T_r^* M, \mathcal{V} M)) \to \Gamma(H_{k+1}(T_r^* M, \mathcal{M})) \quad k = 0, \ldots, \dim(\mathfrak{q}_+/\mathfrak{p}_+) - 1
\]

of invariant differential operators.

**Proof.** We first show that if \(\varphi\) vanishes upon insertion of one section of the subbundle \(\mathcal{A}_\rho^0 M \subset \mathcal{A}_\rho M,\) then the same is true for \(d^\rho \varphi.\) Since we know that \(d^\rho \varphi\) is alternating, we may assume that \(s_0 \in \Gamma(\mathcal{A}_\rho^0 M)\) and prove vanishing of \((d^\rho \varphi)(s_0, \ldots, s_k)\) for arbitrary \(s_1, \ldots, s_k \in \Gamma(\mathcal{A}_\rho M).\) By part (3) of Proposition 2.2, \((D^\rho_i \varphi)\) lies in
\(\Omega^k(M, VM)\) for each \(i\) and thus vanishes upon insertion of \(s_0\). Hence we conclude that
\[
d_1^i(s_0, \ldots, s_k) = (D^p_{\rho^i} \varphi)(s_1, \ldots, s_k),
\]
and we can expand this as noted above. Since \(s_0\) is a section of \(A^0_{\rho^i} M \subset A_{\rho^i} M\), part (1) of Proposition 2.2 shows that \(D^p_{\rho^i}\) coincides with the negative of the algebraic action by \(s_0\) on the appropriate bundle. This action is \(\bullet\) on \(VM\) and the adjoint action via \(\{ , \}\) on \(A_{\rho^i} M\), so we see that
\[
d_1^i \varphi(s_0, \ldots, s_k) = -s_0 \bullet \varphi(s_1, \ldots, s_k) + \sum_{i=1}^k \varphi(s_1, \ldots, \{s_0, s_i\}, \ldots, s_k).
\]
By naturality of the fundamental derivative, \(D^0_{\rho^i} \varphi \in \Omega^k(M, VM)\) is homogeneous of degree \(\geq \ell\) and take sections \(s_j \in \Gamma(A^0_{\rho^i} M)\) with \(i_j < 0\) for \(j = 0, \ldots, k\). To prove that \(d^p\) preserves the natural filtration, we have to show that for each such choice we have
\[
(d^p \varphi)(s_0, \ldots, s_k) \in \Gamma(V^{s_0 + \cdots + i_k + \ell} M).
\]
By naturality of the fundamental derivative, \(D^p_{s_j} \varphi \in \Omega^k(M, VM)\) is homogeneous of degree \(\geq \ell\). Hence \((D^p_{s_j} \varphi)(s_0, \ldots, s_j, \ldots, s_k)\) lies in the filtration component of \(VM\) of degree
\[
i_0 + \cdots + i_j + \cdots + i_k + \ell > i_0 + \cdots + i_k + \ell.
\]
So we conclude that \(d_1^i\) is not only filtration preserving but also does not contribute to the action on the associated graded.

On the other hand, both the action \(p/p_+ \times \mathcal{V} \rightarrow \mathcal{V}\) and the Lie bracket on \(p/p_+\) are \(Q\)-homomorphisms and thus are homogeneous of degree zero for the grading element of \(q\). Together with the formula for the Lie algebra differential in (4.1), this implies that \(\partial_{p/p_+} \varphi\) is filtration homogeneous of the same degree as \(\varphi\). This implies that \(d^p\) is filtration preserving and the induced operator on sections of the associated graded bundle coincides with the one induced by \(\partial_{p/p_+}\). From the definition in Section 2.3 of [15] it is evident that this is the bundle map induced by \(\partial_{p_+}\), so compressability follows.

4.2. The square of the relative twisted exterior derivative. Having constructed relative BGG–sequences, we next move to the question when we obtain complexes or even resolutions of some sheaves. The first step towards this is computing the composition \(d^p \circ d^p\). In the case of non–vanishing torsion, this result is new even for standard BGG sequences. It is based on the naturality properties of the fundamental derivative, which are very well understood, but some care is needed in the computations.

From the properties of the inducing Lie algebra cohomology differential we conclude that \(\partial_{p/p_+} \circ \partial_{p/p_+} = 0\), and thus the composition \(d^p \circ d^p\) is induced by
\[
\tilde{d}^p \circ \tilde{d}^p = \tilde{d}^p_1 \circ \tilde{d}^p_1 + \tilde{d}^p_1 \circ \partial_{p/p_+} + \partial_{p/p_+} \circ \tilde{d}^p_1.
\]
We start by computing the sum of the last two terms in this formula:

**Lemma 4.2.** For \( \varphi \in \Gamma(\Lambda^k - 1 A^*_p \otimes \mathcal{V} M) \) and \( s_0, \ldots, s_k \in \Gamma(\Lambda_p M) \), we can express

\[
\left( d^V_1(\partial_{p/p_+} \varphi) + \partial_{p/p_+} (d^V_1 \varphi) \right)(s_0, \ldots, s_k)
= 
\sum_{i < j} (-1)^{i+j} (D_{\{s_i, s_j\},} \varphi)(s_0, \ldots, \hat{s_i}, \ldots, \hat{s_j}, \ldots, s_k).
\]

**Proof.** First we observe that by definition

\[
\partial_{p/p_+}(s_i) \text{ of terms of the form }
\]

\[
\sum_{i=j} (-1)^{i+j} (D_{\{s_i, s_j\},} \varphi)(s_0, \ldots, \hat{s_i}, \ldots, \hat{s_j}, \ldots, s_k).
\]

Now since \( \partial_{p/p_+} \) is a natural bundle map between relative natural bundles, part (3) of Proposition 2.2 implies that \( D^p_{s_i}(\partial_{p/p_+} \varphi) = \partial_{p/p_+}(D^p_{s_i} \varphi) \). Hence we may write \( d^V_1 \partial_{p/p_+} \varphi(s_0, \ldots, s_k) \) as

\[
\sum_{i=0}^k (-1)^i (D^p_{s_i}(\partial_{p/p_+} \varphi))(s_0, \ldots, \hat{s_i}, \ldots, s_k).
\]

On the other hand, we can compute \( (\partial_{p/p_+} (d^V_1 \varphi))(s_0, \ldots, s_k) \) as

\[
\sum_{i=0}^k (-1)^i s_i \bullet (d^V_1 \varphi)(s_0, \ldots, \hat{s_i}, \ldots, s_k) + 
\sum_{i < j} (-1)^{i+j} (d^V_1 \varphi)(\{s_i, s_j\}, s_0, \ldots, \hat{s_i}, \ldots, \hat{s_j}, \ldots, s_k).
\]

Inserting the definition of \( d^V_1 \) in the first sum in (4.4), we get a sum over all \( i \neq j \) of terms of the form \( s_i \bullet ((D^p_{s_j} \varphi)(s_0, \ldots, s_k)) \) with \( s_i \) and \( s_j \) omitted between \( s_0 \) and \( s_k \). The sign of this term is \((-1)^{i+j}\) if \( j < i \) or \((-1)^{i+j+1}\) if \( j > i \). This is exactly the opposite of the sign with which the same terms occur when inserting the definition of \( \partial_{p/p_+} \) in (4.4).

Next, we insert the definition of \( d^V_1 \) in the second sum in (4.5). On the one hand, this gives \( \sum_{i < j} (-1)^{i+j} (D^p_{\{s_i, s_j\},} \varphi)(s_0, \ldots, \hat{s_i}, \ldots, \hat{s_j}, \ldots, s_k) \). On the other hand, for each \( \ell \) different from \( i \) and \( j \), we obtain a summand of the form

\[
(D^p_{s_\ell} \varphi)(\{s_i, s_j\}, s_0, \ldots, s_k),
\]

where between \( s_0 \) and \( s_k \) the entries \( s_i \), \( s_j \), and \( s_\ell \) are omitted. This term comes with a sign \((-1)^{i+j+\ell+1}\) if \( \ell < i \) or \( \ell > j \) and with a sign \((-1)^{i+j+\ell}\) if \( i < \ell < j \). Again, this sign is opposite to the one with which the same term occurs after inserting the definition of \( \partial_{p/p_+} \) in (4.4), and the result follows.

Using this, we can completely compute \((d^V)^2\).

**Theorem 4.3.** Consider \( \varphi \in \Omega^{k-1}_p(M, \mathcal{V} M) \subset \Gamma(\Lambda^k - 1 A^*_p \otimes \mathcal{V} M) \) then \( d^V(d^V \varphi) \) is induced by the section of \( \Lambda^{k+1} A^*_p \otimes \mathcal{V} M \) which maps \( s_0, \ldots, s_k \) to

\[
\sum_{i < j} (-1)^{i+j} (D_{\kappa(\Pi(s_i), \Pi(s_j))} \varphi)(s_0, \ldots, \hat{s_i}, \ldots, \hat{s_j}, \ldots, s_k),
\]

where \( \kappa \in \Omega^2(M, \mathcal{A} M) \) is the curvature of the geometry.

**Proof.** For any \( \ell \) consider the complete alternation defined by

\[
\text{Alt}_\ell \psi(s_1, \ldots, s_\ell) = \frac{1}{\ell!} \sum_{\sigma \in S_\ell} \text{sgn}(\sigma) \psi(s_{\sigma_1}, \ldots, s_{\sigma_\ell}).
\]
This can be viewed as a natural bundle map on various bundles, in particular as mapping $\mathcal{A}_\rho^* M \otimes \Lambda^{k-1} \mathcal{A}_\rho^* M \otimes \mathcal{V} M$ to $\Lambda^k \mathcal{A}_\rho^* M \otimes \mathcal{V} M$. Now our definition of $\tilde{d}_1^p$ can be recast as $\tilde{d}_1^p \varphi = k \text{Alt}_k (D^p \varphi)$ since $\varphi$ has degree $k - 1$ and is already alternating in all its entries. But then by definition

$$\tilde{d}_1^p (\tilde{d}_1^p \varphi)(s_0, \ldots, s_k) = k \sum_{i=0}^k (-1)^i \left( D^p_{s_i} (\text{Alt}_k D^p \varphi) \right)(s_0, \ldots, \hat{s}_i, \ldots, s_k) =
$$

$$k \sum_{i=0}^k (-1)^i \text{Alt}_k (D^p_{s_i} D^p \varphi)(s_0, \ldots, \hat{s}_i, \ldots, s_k) = (k + 1) k \text{Alt}_{k+1}(D^p D^p \varphi)(s_0, \ldots, s_k).$$

Now $D^p D^p \varphi$ is a section of $\otimes^2 \mathcal{A}_\rho^* M \otimes \Lambda^{k-1} \mathcal{A}_\rho^* M \otimes \mathcal{V} M$, so in forming the alternation, we do not have to permute the last $r - 1$ entries. Hence we can express $\tilde{d}_1^p (\tilde{d}_1^p \varphi)(s_0, \ldots, s_k)$ as

$$\sum_{i < j} (-1)^{i+j+1} (D^p D^p \varphi(s_i, s_j) - D^p D^p \varphi(s_j, s_i))(s_0, \ldots, \hat{s}_i, \ldots, \hat{s}_j, \ldots, s_k).$$

Now $D^p \varphi$ is induced by the restriction of $D \varphi$ to a natural subbundle, whose sections are preserved by a fundamental derivative. Thus, the usual Ricci identity from Proposition 1.5.9 of [11] implies that

$$D^p D^p \varphi(s_i, s_j) - D^p D^p \varphi(s_j, s_i) = -D_{\kappa(\Pi(s_i), \Pi(s_j))} \varphi + D^p_{\{s_i, s_j\}} \varphi.$$

Together with Lemma 4.2 this implies the claim. 

4.3. An alternative description. To prove that a standard BGG–sequence is a resolution, one usually relates it to a twisted de–Rham resolution. As a next step towards relative versions of such results, we derive a description of the twisted relative exterior derivative which is closer to the standard analogs of the exterior derivative on bundle valued differential forms. To do this, we first need an analog of tractor connections, which we can obtain using the relative fundamental derivative. Consider the relative twisted exterior derivative on $\Omega^p_0(\mathcal{M}, \mathcal{V} M) = \Gamma(\mathcal{V} M)$. Viewed as an operator $\Gamma(\mathcal{A}_\rho^* M) \times \Gamma(\mathcal{V} M) \rightarrow \Gamma(\mathcal{V} M)$ this is induced by $(s, \sigma) \mapsto D^p \sigma + s \cdot \sigma$. Thus from part (1) of Proposition 2.2 we conclude that the induced operator $\Gamma(T^p \mathcal{M}) \times \Gamma(\mathcal{V} M) \rightarrow \Gamma(\mathcal{V} M)$ also satisfies a Leibniz rule. Hence it defines a partial connection, which is called the (normal) relative tractor connection on the relative tractor bundle $\mathcal{V} M$ and denoted by $\nabla^p, \mathcal{V}$.

A linear connection on a vector bundle can be coupled to the exterior derivative to obtain an operator on differential forms with values in that vector bundle. This has an analog for partial connections, provided that the subbundle of the tangent bundle in question is involutive, see Section 4.4 below. In our setting, the relative tangent bundle $T^p \mathcal{M}$ is not involutive in general, but we can overcome this problem by using a modification of the Lie bracket of vector fields.

**Proposition 4.4.** The bilinear operator $\Gamma(\mathcal{A}_\rho^* M) \times \Gamma(\mathcal{A}_\rho^* M) \rightarrow \Gamma(\mathcal{A}_\rho^* M)$ defined by $(s_1, s_2) \mapsto D^p_{s_1} s_2 - D^p_{s_2} s_1 + \{s_1, s_2\}$ descends to a skew symmetric bilinear operator

$$[\ ] : \Gamma(T^p \mathcal{M}) \times \Gamma(T^p \mathcal{M}) \rightarrow \Gamma(T^p \mathcal{M}).$$

This satisfies a Leibniz rule, i.e. $[\xi, f \eta] = (\xi \cdot f) \eta + f[\xi, \eta]$ holds for any $f \in C^\infty(\mathcal{M}, \mathbb{R})$ and all $\xi, \eta \in \Gamma(T^p \mathcal{M})$. 


Proof. It is evident that the operator on $\Gamma(\mathcal{A}_p M)$ is skew symmetric. If $s_1 \in \Gamma(\mathcal{A}_p^0 M)$, then $D^0_{s_1}s_2 = -\{s_1, s_2\}$, while $D^0_{s_2}s_1 \in \Gamma(\mathcal{A}_p^0 M)$ by naturality of $D^0$. Thus the values lie in $\Gamma(\mathcal{A}_p^0 M)$ for any choice of $s_2$, which implies that the operation descends to sections of $T^*_p M$ as claimed. The Leibniz rule for $D^0$ from Proposition 2.2 together with the fact that the algebraic bracket $\{ , \}$ is bilinear over smooth functions implies the Leibniz rule for $[ [ , ] ]$. \hfill \Box

Now it is easy to derive a formula for the twisted exterior derivative which is analogous to the formula for the covariant exterior derivative induced by a linear connection on a vector bundle.

**Theorem 4.5.** Let $\mathcal{V} M$ be a relative tractor bundle and let $\nabla^\rho\mathcal{V}$ be the associated relative tractor connection. Then for $\varphi \in \Omega^k(\mathcal{V} M, \mathcal{V} M)$ the twisted exterior derivative satisfies

$$d^\lambda \varphi(\xi_0, \ldots, \xi_k) = \sum_{i=0}^k (-1)^i \nabla^\rho_{\xi_i} \varphi(\xi_0, \ldots, \hat{\xi}_i, \ldots, \xi_k) + \sum_{i<j} (-1)^{i+j} \varphi([[\xi_i, \xi_j]], \ldots, \hat{\xi}_i, \ldots, \hat{\xi}_j, \ldots, \xi_k)$$

for all $\xi_0, \ldots, \xi_k \in \Gamma(T^*_p M)$.

**Proof.** Let $\Pi_p : \mathcal{A}_p M \to \mathcal{A}_p M/\mathcal{A}_p^0 M = T^*_p M$ be the natural projection. For each $i$, choose $s_i \in \Gamma(\mathcal{A}_p M)$ such that $\Pi(s_i) = \xi_i$ and view $\varphi$ as a section of $\Lambda^k \mathcal{A}_p^* M \otimes \mathcal{V} M$. Then by definition $d^\rho \varphi(\xi_0, \ldots, \xi_k) = d^\rho \varphi(s_0, \ldots, s_k)$.

Now we expand $(D^\rho_s \varphi)(s_0, \ldots, s_1, \ldots, s_k)$ as discussed in Section 4.2. In each of terms in which $D_s$ acts on $s_j$ for $j \neq i$, we can move $D_s s_j$ to the first entry of $\varphi$, picking up a sign $(-1)^{j-1}$ if $j < i$ and $(-1)^j$ if $j > i$. Using this, we obtain the following alternative expression for $d^\rho_1$:

$$d^\rho_1 \varphi(s_0, \ldots, s_k) = \sum_{i=0}^k (-1)^i D^\rho_s i(\varphi(s_0, \ldots, \hat{s}_i, \ldots, s_k)) + \sum_{i<j} (-1)^{i+j} \varphi(D^\rho_s s_j - D^\rho_s s_i, s_0, \ldots, \hat{s}_i, \ldots, \hat{s}_j, \ldots, s_k).

But now it is evident that the first sum in the right hand side combines with the part $\sum_{i=0}^k (-1)^i s_i \cdot \varphi(s_0, \ldots, \hat{s}_i, \ldots, s_k)$ in $\partial_{p/\mathcal{P}, q} \varphi(s_0, \ldots, s_k)$ to produce the first sum in the claimed formula. Likewise, the second sum in the right hand side combines with the second sum in $\partial_{p/\mathcal{P}, q} \varphi(s_0, \ldots, s_k)$ to the second sum in the claimed formula. \hfill \Box

4.4. **The case of involutive relative tangent bundle.** The considerations in Section 4.3 suggest studying parabolic geometries for which the relative tangent bundle is involutive. In view of the relation to twistor spaces discussed in Section 4.6 below, this property is sometimes phrased as “existence of a twistor space corresponding to $P \subset Q$ as a manifold”. Involutivity of $T^*_p M$ for a given parabolic geometry $(p : \mathcal{G} \to M, \omega)$ is easy to characterize in terms of the curvature $\kappa \in \Omega^2(M, \mathcal{M})$ of the geometry. Indeed, it depends only on its torsion $\tau \in \Omega^2(M, TM)$, which by definition is obtained by projecting the values of $\kappa$ to $\mathcal{M}/\mathcal{A}^0 M \cong TM$. The result in Proposition 2.5 of 3 can be phrased as follows.

**Proposition 4.6.** For a parabolic geometry $(p : \mathcal{G} \to M, \omega)$ of type $(\mathcal{G}, Q)$ the relative tangent bundle is involutive if and only if the curvature $\kappa$ maps $T^*_p M \times T^*_p M$
to sections of the subbundle $\mathcal{G} \times_Q p \subset \mathcal{A}M$ or equivalently if $\tau$ maps $T_\rho M \times T_\rho M$ to $T_\rho M$.

Using this result, we can define relative versions of the curvature and the torsion in the case of involutive relative tangent bundle. Observe that there is a natural projection $\mathcal{G} \times_Q p \to \mathcal{G} \times_Q (p/p_+) = \mathcal{A}_\rho M$.

**Definition 4.7.** Consider a parabolic geometry $\rho : \mathcal{G} \to M, \omega$ of type $(\mathcal{G}, Q)$ for which the relative tangent bundle $T_\rho M \subset TM$ is involutive.

Then we define the **relative torsion** $\tau_\rho \in \Gamma(\Lambda^2 T^*_{\rho M}, T_\rho M)$ as the restriction of the torsion $\tau$ to entries from $T_\rho M \subset TM$. Further, we define the **relative curvature** $\kappa_\rho \in \Gamma(\Lambda^2 T^*_{\rho M}, \mathcal{A}_\rho M)$ to be the image of the restriction of the curvature $\kappa$ (which has values in $\mathcal{G} \times_Q p$ by Proposition [1.6]) under the natural bundle map $\mathcal{G} \times_Q p \to \mathcal{A}_\rho M$ from above.

If the relative tangent bundle $T_\rho M$ is involutive, then one can associate a curvature $R^{\rho,\nu} \in \Omega^2(M, L(VM, VM))$ to the partial tractor connection $\nabla^{\rho,\nu}$. This can simply be defined by the usual formula

$$R^{\rho,\nu}(\xi, \eta)(s) = \nabla^{\rho,\nu}_\xi \nabla^{\rho,\nu}_\eta s - \nabla^{\rho,\nu}_\eta \nabla^{\rho,\nu}_\xi s - \nabla^{\rho,\nu}_{[\xi,\eta]} s$$

for $\xi, \eta \in \Gamma(T_\rho M)$ and the usual proof shows that the right hand side is linear over smooth functions in all arguments. To compute this curvature, we first need a lemma.

**Lemma 4.8.** For any parabolic geometry $\rho : \mathcal{G} \to M, \omega$ of type $(\mathcal{G}, Q)$ and $\xi_1, \xi_2 \in \Gamma(T_\rho M)$, we have

$$[[\xi_1, \xi_2]] = [\xi_1, \xi_2] + \tau(\xi_1, \xi_2),$$

where $\tau$ denotes the torsion of the geometry.

If the relative tangent bundle $T_\rho M$ is involutive, then in the right hand side, we can replace $\tau$ by the relative torsion $\tau_\rho$ and the summands in the right hand side both are sections of $T_\rho M$.

**Proof.** For $i = 1, 2$ choose $s_i \in \Gamma(\mathcal{A}_\rho M)$ such that $\Pi_\rho(s_i) = \xi_i$. Then by definition, (4.7)

$$[[\xi_1, \xi_2]] = \Pi_\rho(D^\rho_{s_1} s_2 - D^\rho_{s_2} s_1 + \{s_1, s_2\}).$$

For $i = 1, 2$, choose a section $\tilde{s}_i$ of the subbundle $\mathcal{G} \times_Q p \subset \mathcal{A}M$ which descends to $s_i$. Then by definition, $D_{\tilde{s}_1} \tilde{s}_2 \in \Gamma(\mathcal{G} \times_Q p)$ descends to $D^\rho_{\tilde{s}_1} \tilde{s}_2$ and similarly for the other terms in the right hand side of (4.7).

On the other hand, as we have noted in Section [2.2] sections of $\mathcal{A}M$ can be identified with $Q$–invariant vector fields on $\mathcal{G}$ and in this picture, $\Pi$ is just the projection of such vector fields to $M$. In particular, since the Lie bracket of $Q$–invariant vector fields is again $Q$–invariant, there is an operation $[\cdot, \cdot] : \Gamma(\mathcal{A}M) \times \Gamma(\mathcal{A}M) \to \Gamma(\mathcal{A}M)$ corresponding to the Lie bracket of vector fields. But by construction, we then have $\Pi([\tilde{s}_1, \tilde{s}_2]) = [\Pi(\tilde{s}_1), \Pi(\tilde{s}_2)] = [\xi_1, \xi_2]$. Finally, by Corollary 1.5.8 of [11], one has

$$[\tilde{s}_1, \tilde{s}_2] = D_{\tilde{s}_1} \tilde{s}_2 - D_{\tilde{s}_2} \tilde{s}_1 + \{\tilde{s}_1, \tilde{s}_2\} - \kappa(\Pi(\tilde{s}_1), \Pi(\tilde{s}_2)),$$

which immediately implies the result. \qed
Proposition 4.9. Let \((p : G \to M, \omega)\) be a parabolic geometry of type \((G, Q)\) such that the relative tangent bundle \(T_p M\) is involutive and let \(\mathcal{V} \to M\) be a relative tractor bundle.

1. For \(\varphi \in \Omega^k_p(M, \mathcal{V})\), \(d^\mathcal{V}(d^\mathcal{V}\varphi)\) is induced by the section of \(\Lambda^{k+1}A^*_p M \otimes \mathcal{V}\) which maps \(s_0, \ldots, s_k\) to
   \[
   \sum_{i<j}(-1)^{i+j}(D^\varphi_{\kappa(p(I(s_i),I(s_j)))})(s_0, \ldots, \widehat{s_i}, \ldots, \widehat{s_j}, \ldots, s_k).
   \]
   If in addition the relative torsion vanishes, then this equals
   \[
   \sum_{i<j}(-1)^{i+j+1}(\kappa(p(I(s_i),I(s_j)))\cdot \varphi)(s_0, \ldots, \widehat{s_i}, \ldots, \widehat{s_j}, \ldots, s_k).
   \]

2. The curvature of the relative tractor connection \(\nabla_{\rho,\mathcal{V}}\) is given by
   \[
   R_{\rho,\mathcal{V}}(\xi, \eta)(s) = \kappa(\xi, \eta)\cdot s.
   \]

Proof. (1) We have noted above that involutivity of \(T_p M\) implies that \(\kappa\) has values in \(\mathcal{G} \times Q p\) if both its entries are from \(T_p M\). Together with the fact that \(\varphi\) is a section of a relative natural bundle, this shows that \(D_{\kappa(p(I(s_i),I(s_j)))}\varphi = D^\varphi_{\kappa(p(I(s_i),I(s_j)))}\varphi\) for all \(i\) and \(j\). Hence the first formula follows directly from Theorem 4.3.

If \(\tau_p\) vanishes, then the values of \(\kappa_p\) lie in \(A^0 p M\) and the second formula follows from part (1) of Proposition 2.2.

(2) For \(s \in \Gamma(\mathcal{V}M)\), part (1) implies that for \(\xi, \eta \in \Gamma(T_p M)\), we get \((d^\mathcal{V})^2 s(\xi, \eta) = -D_{\kappa_p(\xi,\eta)}^\mathcal{V}s\). On the other hand, applying Theorem 4.3 to \(d^\mathcal{V}s\), we get \(\nabla_{\rho,\mathcal{V}}s\) which lies in \(\Omega^k_p(M, \mathcal{V}M)\), we see that
   \[
   (d^\mathcal{V})^2 s(\xi, \eta) = R_{\rho,\mathcal{V}}(\xi, \eta)(s) + \nabla^\mathcal{V}_{[\xi,\eta]}s - \nabla^\mathcal{V}_{[\xi,\eta]}s.
   \]

By Lemma 4.8, the last two terms add up to \(-\nabla^\mathcal{V}_{\tau_p(\xi,\eta)}s\). Since \(\kappa_p(\xi, \eta)\) projects onto \(\tau_p(\xi, \eta)\), we get
   \[
   -\nabla^\mathcal{V}_{\tau_p(\xi,\eta)}s = -D_{\kappa_p(\xi,\eta)}^\mathcal{V}s - \kappa_p(\xi, \eta)\cdot s,
   \]
   and the result follows. \(\square\)

4.5. The relative covariant exterior derivative. If the relative tangent bundle \(T_p M\) is involutive, then as mentioned before, one can follow the standard approach of extending a covariant derivative to an operator on bundle valued differential forms in our setting. Namely, for \(\varphi \in \Omega^k_p(M, \mathcal{V}M)\) and \(\xi_0, \ldots, \xi_k \in \Gamma(T_p M)\), we define
   \[
   (d^\mathcal{V}\varphi)(\xi_0, \ldots, \xi_k) := \sum_{i=0}^k(-1)^i \nabla^\mathcal{V}_{\xi_i}\varphi(\xi_0, \ldots, \widehat{\xi_i}, \ldots, \xi_k) + \sum_{i<j}(-1)^{i+j}\varphi(\xi_i, \xi_j, \xi_0, \ldots, \widehat{\xi_i}, \ldots, \widehat{\xi_j}, \ldots, \xi_k).
   \]
   Note that involutivity of \(T_p M\) is needed for this definition to make sense, since only sections of this subbundle may be inserted into \(\varphi\). The right hand side of (4.8) is obviously alternating in \(\xi_0, \ldots, \xi_k\) and the same argument as for usual connections shows that \(d^\mathcal{V}\varphi\) is linear over smooth functions in each entry. Thus \(d^\mathcal{V}\varphi \in \Omega^{k+1}_p(M, \mathcal{V}M)\) and we have defined an operator
   \[
   d^\mathcal{V} : \Omega^k_p(M, \mathcal{V}M) \to \Omega^{k+1}_p(M, \mathcal{V}M)
   \]
called the relative covariant exterior derivative.

The relation between this operator and the relative twisted exterior derivative can be easily described using Lemma 4.8. There is a natural insertion operator on $\Omega^k(M, VM)$–valued relative forms associated to the relative torsion. Namely, for $\varphi \in \Omega^k(M, VM)$ and $\xi_0, \ldots, \xi_k \in \Gamma(T_pM)$, we define

\begin{equation}
(4.9) \quad (i_{\tau_p} \varphi)(\xi_0, \ldots, \xi_k) := \sum_{i<j} (-1)^{i+j+1} \varphi(\tau_p(\xi_i, \xi_j), \xi_0, \ldots, \widehat{\xi_i}, \ldots, \widehat{\xi_j}, \ldots, \xi_k),
\end{equation}

so this coincides with the complete alternation of the insertion up to a positive factor. Having this at hand, we can now formulate:

**Proposition 4.10.** Let $(p : G \to M, \omega)$ be a parabolic geometry of type $(G, Q)$ such that $T_pM \subset TM$ is involutive. Then the relative twisted exterior derivative defined in Section 4.1 is related to the relative covariant exterior derivative by

\[ d^V \varphi = d\nabla \varphi + i_{\tau_p} \varphi \]

for all $\varphi \in \Omega^p(M, VM)$.

In particular, the relative covariant exterior derivatives define a compressable sequence of operators and in the case that $p = g$ (i.e. for usual BGG sequences), $d^V$ coincides with the twisted exterior derivative as defined in [12].

**Proof.** By the last part of Lemma 4.8 both in the right hand side of the formula for $\| \xi_1, \xi_2 \|$ in that lemma are sections of $T_pM$. Hence they can be inserted individually into a relative form, and using this, the first claim follows immediately from the formula for $d^V$ in Theorem 4.5.

The second claim easily follows from the discussion on top of p. 105 in [12] discussing the relation between the twisted exterior derivative (as defined there) and the covariant exterior derivative.

**4.6. Relative BGG resolutions.** One of the key features of BGG–sequences is that on locally flat geometries, they are complexes defining fine resolutions of certain sheaves. We are next aiming at analogs of this results in the relative setting. There is a model case of this situation provided by so–called correspondence spaces, a special class of parabolic geometries of type $(G, Q)$ associated to the parabolic subalgebra $p \supset q$.

Consider an arbitrary regular normal parabolic geometry $(p : G \to N, \omega)$ of type $(G, P)$. Then we can define $M := CN := G/Q$, the orbit space under the restriction of the principal right action to the subgroup $Q \subset P$. Then $G/Q$ can be identified with $G \times_P (P/Q)$, so it is a smooth manifold and the total space of a natural fiber bundle over $N$ with typical fiber the generalized flag manifold $P/Q$. By construction, $G \to M$ is a $Q$–principal bundle and it is easy to see that the Cartan connection $\omega \in \Omega^1(G, g)$ also defines a Cartan connection on $G \to M$. Hence one obtains a parabolic geometry of type $(G, Q)$ on $M$, which turns out to be automatically normal. In this context, $M = CN$ is called the correspondence space associated to $N$ and the subalgebra $q \subset p$.

The bundle projection $\pi : M \to N$ gives rise to the vertical subbundle $\ker(T\pi) \subset TM$. From the above construction it is clear that this vertical subbundle is given
by $G \times_Q (p/q) \subset G \times_Q (g/q) = TM$. For a correspondence space $CN$, the relative tangent bundle $T_pCN$ thus coincides with the vertical subbundle of $CN \to N$. Hence $T_pCN$ is globally integrable with global leaf–space $N$. Observe that in this setting a completely reducible representation $V$ of $P$ gives rise to both a relative tractor bundle $VM \to M$ and a (completely reducible) natural bundle $VN := G \times_P V \to N$ for the underlying geometry.

Now we are ready to prove a general criterion ensuring that relative BGG sequences are resolutions and an interpretation of the sheaf that gets resolved in the model case of a correspondence space. To formulate this, recall from Section 4.3 that for a relative tractor bundle $VM \to M$, there is the relative tractor connection $\nabla^{\rho, V}$. This can be considered as an operator $\Gamma(VM) \to \Omega^1(\rho(M, VM))$ and clearly the kernel of this operator gives rise to a well defined subsheaf of the sheaf of smooth sections of $VM$.

**Theorem 4.11.** Suppose that $(p : G \to M, \omega)$ is a parabolic geometry of type $(G, Q)$ for which the relative tangent bundle is involutive and that $P/Q$ is connected. Let $V$ be any completely reducible representation of $P$ and let $VM \to M$ be the corresponding relative tractor bundle.

1. If the relative curvature $\kappa_\rho$ of the geometry vanishes identically, then the relative BGG–sequence (4.3) from Theorem 4.7 determined by $V$ is a complex and a fine resolution of the sheaf $\ker(\nabla^{\rho, V})$.

2. If $M$ is the correspondence space $CN$ of a parabolic geometry of type $(G, P)$, then the condition in (1) is always satisfied, and the sheaf $\ker(\nabla^{\rho, V})$ can be (globally) identified with the pullback of the sheaf of smooth sections of $VN \to N$.

**Proof.** From part (1) of Proposition 4.9 we see that vanishing of the relative curvature implies that $(d^V)^2 = 0$, so the twisted de–Rham sequence determined by $VN$ is a complex. Moreover, by Proposition 4.10 the twisted exterior derivative coincides with the covariant exterior derivative associated to the relative tractor connection. Thus on sufficiently small open subsets we are dealing with a twisted de Rham sequence along the fibers of the projection to a local leaf space, and it is a standard result that this is a fine resolution of the kernel of the first operator.

By Theorem 3.14 the fact that the relative twisted de–Rham sequence is a complex implies that the same is true for the corresponding relative BGG sequence and that both sequences compute the same cohomology. Since this can be applied locally, (1) follows.

To prove (2), consider a section $\sigma \in \Gamma(VN)$ and let $f : G \to V$ be the corresponding $Q$–equivariant function. Further, let $s \in \Gamma(G \times_Q p)$ be a smooth section. Then we can compute $\nabla_{\Pi(s)}^{\rho, V}\sigma$ as $D_s\sigma + s \bullet \sigma$. This easily implies that $\sigma$ is a section of $\ker(\nabla^{\rho, V})$ if and only if for each $A \in p$ and $u \in G$ we have $(\omega^{-1}(A) \cdot f)(u) = -A \cdot (f(u))$, where in the right hand side $A$ acts via the infinitesimal representation.

If $M$ is a correspondence space $CN$, then $M = G/Q$ for a principal $P$–bundle $G \to N$ and the Cartan connection $\omega$ actually is a Cartan connection on this $P$–bundle. This shows that for $A \in p$, the vector field $\omega^{-1}(A)$ is the fundamental...
vector field of this $P$–bundle. Since such vector fields insert trivially into the curvature, it follows that the condition from (1) is satisfied. On the other hand, the above discussion shows that $\sigma$ lies in $\ker(\nabla^\rho V)$ if and only if $f$ is $p$–equivariant. Since $P/Q$ is assumed to be connected, equivariancy under both $Q$ and $p$ is equivalent to equivariancy under $P$, so $f$ corresponds to a smooth section of $\nabla N \to N$. The converse direction is obvious. □

This shows that the situation is significantly better than for ordinary BGG sequences, since in order to get a resolution, we only need vanishing of the relative curvature, which is much weaker than local flatness. In particular, for any parabolic geometry of type $(G, P)$, any relative BGG sequence for the induced geometry of type $(G, Q)$ on the correspondence space is a resolution. However, as we shall also see in the example of path geometries in Section 5.6, the class of geometries for which we obtain resolutions is much larger than the class of correspondence spaces. Indeed, Theorem 2.7 of [5] shows that a geometry of type $(G, Q)$ is locally isomorphic to a correspondence space if and only if its curvature $\kappa$ vanishes upon insertion of any element of $T_\rho M$, and this condition is much stronger than vanishing of $\kappa_\rho$.

4.7. Local twistor spaces. We continue working in the setting of vanishing relative curvature so that part (1) of Theorem 4.11 shows that the relative BGG sequence associated to $VM$ is a fine resolution of the sheaf $\ker(\nabla^\rho V)$. We are looking for curvature conditions which are weaker than the ones which locally characterize correspondence spaces but still allow us to obtain an explicit description of the sheaf $\ker(\nabla^\rho V)$.

For a parabolic geometry of type $(G, Q)$ over $M$ such that $T_\rho M$ is involutive, we can consider local leaf spaces for the foliation defined by $T_\rho M$, which are then called local twistor space for $M$ corresponding to $p \supset q$. Of course there is the hope to interpret $\ker(\nabla^\rho V)$ as the pullback of some sheaf on $N$. The proof of Theorem 4.11 suggests that the key question here is, loosely speaking, whether the Cartan bundle $G \to M$ can be viewed as a principal $P$–bundle over a local twistor space. This question has been studied in [5], which suggests the following technical formulation of the concepts and directly provides some results.

Lemma 4.12. Let $(p : G \to M, \omega)$ be a parabolic geometry of type $(G, Q)$, such that $\kappa_\rho = 0$ (so $T_\rho M$ is involutive), and suppose that $U \subset M$ is open and $\psi : U \to N$ is a local leaf space for $T_\rho M$. Suppose further that $\pi : F \to N$ is a $P$–principal bundle and that $W \subset F$ is a $Q$–invariant open subset such that for each $x \in N$ the set $W_x/Q \subset F_x/Q \equiv P/Q$ is non–empty and connected. Finally, suppose that $\varphi$ is a $Q$–equivariant diffeomorphism from $W$ onto a $Q$–invariant subset of $p^{-1}(U)$ such that for each $A \in p$, $\varphi^*(\omega^{-1}(A))$ is the fundamental vector field $\zeta_A \in \mathfrak{X}(F)$.

Then over the open subset $p(\varphi(W)) \subset U$, the sheaf $\ker(\nabla^\rho V)$ is isomorphic to the pullback of the sheaf of smooth sections of $F \times_P \mathcal{V}$.

Proof. From the proof of Theorem 4.11 we see that sections of $\ker(\nabla^\rho V)$ are in bijective correspondence with $Q$–equivariant smooth functions $f$ such that $(\omega^{-1}(A) \cdot f)(u) = -A \cdot f(u)$, where in the left hand side the vector field $\omega^{-1}(A)$ is
used to differentiate the function $f$, while in the right hand side $A$ acts on $\mathcal{V}$ via the infinitesimal representation. Pulling back via $\varphi$, we by assumption get the sheaf of $Q$–equivariant smooth functions $W \to \mathcal{V}$ which in addition are $\mathfrak{p}$–equivariant for the infinitesimal action.

Now we claim that restriction to $W$ defines an isomorphism from the sheaf of $P$–equivariant functions $\mathcal{F} \to \mathcal{V}$ onto the above sheaf. Then the result follows from the standard correspondence between equivariant functions and sections of an associated bundle. By assumption, $W$ meets each fiber of $\mathcal{F} \to N$, so restriction to $W$ is injective (on the level of sheaves on $N$).

Surjectivity of the restriction can be proved locally (on $N$). So we can restrict to an open subset in $N$ over which $\mathcal{F}$ admits a smooth section. Since locally such a section can be assumed to have values in $W$, we may restrict to the case that there is a global section $\tau : N \to W$ of $\mathcal{F}$. But then of course for any smooth function $g : W \to \mathcal{V}$, the smooth function $g \circ \tau$ can be uniquely extended to a $P$–equivariant function $f : \mathcal{F} \to \mathcal{V}$. It suffices to prove that if $g$ is both $Q$–equivariant and $\mathfrak{p}$–equivariant, then it coincides with the restriction of $f$. To do this, consider $\{u \in W : g(u) = f(u)\}$, which by definition is $Q$–invariant and meets each fiber $W_x := W \cap \mathcal{F}_x$ (in $\tau(x)$). Since both $f$ and $g$ are $\mathfrak{p}$–equivariant, the intersection with $W_x$ is open. Hence it projects onto a non–empty open subset of $W_x/Q$. But since the complement evidently is open and $Q$–invariant, too, connectedness of $W_x/Q$ implies that $f = g$ on $W_x$. Since $x$ is arbitrary, the result follows.

Now a result of [5] provides a curvature condition ensuring that Lemma 4.12 can be applied locally. This leads to trivial $P$–bundles and initially does not give a geometric description of the sheaf $\ker(\nabla^p, \mathcal{V})$. However, under a slightly stronger condition, we can prove that under the assumptions of Lemma 4.12, the Cartan connection $\omega$ induces a soldering form on $\mathcal{F}$, which leads to a geometric interpretation of sections of associated bundles.

**Theorem 4.13.** Suppose that $(p : G \to M, \omega)$ is a parabolic geometry of type $(G, Q)$, that $P/Q$ is connected, and that the curvature $\kappa$ of the geometry vanishes on $T_p M \times T_p M$ (so that $T_p M$ is involutive). Let $\mathcal{V}$ be any completely reducible representation of $P$ and let $\mathcal{V}M \to M$ be the corresponding relative tractor bundle.

1. For sufficiently small local leaf spaces $N$ for $T_p M$ the assumptions of lemma 4.12 are satisfied for $\mathcal{F} = N \times P$. Also, Theorem 4.11 applies and the sheaf $\ker(\nabla^p, \mathcal{V})$ resolved by the relative BGG sequence associated to $\mathcal{V}$ can be locally identified with the pullback of the sheaf $C^\infty(N, \mathcal{V})$.

2. Assume in addition that inserting one element from $T_p M$ into $\kappa$ forces the values to lie in $G \times Q \mathfrak{p} \subset AM$. Then whenever the assumptions of Lemma 4.12 are satisfied, the Cartan connection $\omega$ induces a strictly horizontal, $P$–equivariant one–form $\theta \in \Omega^1(\mathcal{F}, g/\mathfrak{p})$.

**Proof.** Part (1) is proved in Proposition 2.6 of [5]. (The connectedness of $W_x/Q$ is not explicitly stated there, but $W$ is constructed as $N \times U$ with $U \subset P$ the pre–image of an open neighborhood of $eQ$ in $P/Q$, and one may choose this neighborhood to be connected.)
For part (2), we consider the form \( \tilde{\theta} \in \Omega^1(W, \mathfrak{g}/\mathfrak{p}) \), which is obtained by projecting the values of \( \varphi^* \omega \) to the quotient by \( \mathfrak{p} \). Since \( \varphi \) and \( \omega \) are \( \mathcal{Q} \)-equivariant, also \( \tilde{\theta} \) is \( \mathcal{Q} \)-equivariant. We claim that \( \tilde{\theta} \) is also \( \mathfrak{p} \)-equivariant in the sense that

\[
L_{\zeta_A} \tilde{\theta} = -\text{ad}(A) \circ \tilde{\theta}.
\]

Here by \( \text{ad}(A) \) we denote the action of \( \mathfrak{p} \) on \( \mathfrak{g}/\mathfrak{p} \) induced by the adjoint action.

By assumption we have \( \varphi^* \omega(\zeta_A) = A \), so \( \tilde{\theta}(\zeta_A) = 0 \). Hence for any vector field \( \eta \), we can compute \( (L_{\zeta_A} \tilde{\theta})(\eta) \) as \( d\tilde{\theta}(\zeta_A, \eta) \), which in turn can be computed as the projection to \( \mathfrak{g}/\mathfrak{p} \) of \( (\varphi^* d\omega)(\zeta_A, \eta) \). But now the assumption on the curvature implies that for any vector field \( X \) on \( \mathfrak{g} \), the expression \( d\omega(\omega^{-1}(A), X) + [A, \omega(X)] \) has values in \( \mathfrak{p} \). This shows that \( (\varphi^* d\omega)(\zeta_A, \eta) \) is congruent to \( -[A, \varphi^* \omega(\eta)] \) modulo \( \mathfrak{p} \), which implies the claim.

Having verified \( \mathfrak{p} \)-equivariancy, one proceeds exactly as in the proof of Lemma \( \text{4.12} \) to show that there is a unique \( \mathcal{P} \)-equivariant one–form \( \theta \in \Omega^1(\mathcal{F}, \mathfrak{g}/\mathfrak{p}) \) which restricts to \( \tilde{\theta} \) on \( W \). By construction, the kernel of \( \tilde{\theta} \) in each point of \( W \) is the vertical subspace of \( \mathcal{F} \to N \), and of course, this continues to hold for the equivariant extension \( \theta \).

Observe that the result of part (2) in particular implies that \( \mathcal{F} \times_\mathcal{P} (\mathfrak{g}/\mathfrak{p}) \cong TN \), which gives a description of all tensor bundles as associated bundles to \( \mathcal{F} \).

### 4.8. The Laplacian associated to the relative twisted exterior derivative.

We conclude this section with a finer analysis of the relative BGG–sequences induced by the relative twisted exterior derivative as obtained in Theorem \( \text{4.11} \). Since we are dealing with a sequence of compressable operators here, Section \( \text{3.6} \) suggests looking at the Laplacians

\[
\Box^\mathcal{V}_k = \partial^\mathcal{V}_k \partial^\mathcal{V}_k + d^\mathcal{V} \partial^\mathcal{V}_k : \Omega^k_p(M, VM) \to \Omega^k_p(M, VM).
\]

In the case of ordinary BGG sequences it was already indicated in \( \text{[13]} \) that these Laplacians should be closely related to curved Casimir operators. In particular, it was proved there that the two operators coincide up to a constant in degree zero. To prove this in all degrees, we need a bit of preparation.

Consider the Lie algebra \( \mathfrak{p}/\mathfrak{p}_+ \) endowed with the non–degenerate invariant bilinear form \( B \) induced by the Killing form of \( \mathfrak{g} \). This bilinear form gives rise to a Casimir operator \( C_0 \) acting on representations of \( \mathfrak{p}/\mathfrak{p}_+ \). Via \( B \), one can view the identity map as defining an element in \( (\mathfrak{p}/\mathfrak{p}_+) \otimes (\mathfrak{p}/\mathfrak{p}_+) \) which is invariant under the natural action of \( \mathfrak{p}/\mathfrak{p}_+ \). Projecting this into the universal enveloping algebra \( \mathcal{U}(\mathfrak{p}/\mathfrak{p}_+) \), one obtains an element in the center. Acting by this element then defines a \( \mathfrak{p}/\mathfrak{p}_+ \)-equivariant map on any representation of \( \mathfrak{p}/\mathfrak{p}_+ \). Of course, on a complex irreducible representation, this must be a multiple of the identity. Passing to associated bundles, we obtain an endomorphism of any relative tractor bundle, which we also denote by \( C_0 \).

**Proposition 4.14.** The Laplacians associated to the relative twisted exterior derivative are given by \( \Box^\mathcal{V}_k = \frac{1}{2} (C_p - \text{id}) \). In particular, if \( \mathcal{V} \) is induced by a complex irreducible representation \( \mathcal{V} \), then \( \Box^\mathcal{V}_k = \frac{1}{2} (C_p - c_0 \text{id}) \) where \( c_0 \in \mathbb{C} \) is the eigenvalue of \( C_0 \) on \( \mathcal{V} \).
Proof. In Section 4.1 we have obtained \( d^V \) as the restriction of an operator \( \partial^V \) defined on sections of the bundle \( \Lambda^k \mathcal{A}^*_p M \otimes \mathcal{V} M \). The bundle \( \Lambda^k T^*_p M \otimes \mathcal{V} M \) can be viewed as a subbundle in there, corresponding to the inclusion

\[
\Lambda^k(q_+/p_+) \otimes \mathcal{V} \hookrightarrow \Lambda^k(p/p_+) \otimes \mathcal{V}.
\]

Moreover, the bundle map \( \partial^*_p \) is induced by a Lie algebra homology differential for the smaller algebra, and thus can be viewed as the restriction of a bundle map \( \partial^*_p/p_+ \), induced by the Lie algebra homology differential for the bigger algebra. Thus we can complete the proof by showing that the operator

\[
\partial^*_p/p_+ \circ d^V + d^V \circ \partial^*_p/p_+
\]

on \( \Gamma(\Lambda^k \mathcal{A}^*_p M \otimes \mathcal{V} M) \) coincides with \( \frac{1}{2}(C_\rho - \text{id} \otimes C_0) \).

Now we write \( \partial^V \) as \( \partial^V_1 + \partial^V_2 \), and accordingly split (4.10) into two parts. For the rest of this proof, we will omit the subscripts and just write \( \partial^* \) for \( \partial^*_p/p_+ \) and \( \partial \) for \( \partial_1/p_+ \). To compute the first part, we observe that by definition, in terms of a local frame \( \{\xi_\alpha\} \) for \( \mathcal{A}^*_p M \) and the dual frame \( \{\eta_\alpha\} \) for \( \mathcal{A}_p^* M \), we can write \( \partial^V_1 \varphi \) as \( \sum_\alpha \eta_\alpha \wedge D^p_{\xi_\alpha} \varphi \) for \( \varphi \in \Gamma(\Lambda^k \mathcal{A}^*_p M \otimes \mathcal{V} M) \). Now naturality of the relative fundamental derivative implies that \( D^p_{\xi_\alpha} \partial \varphi = \partial^* D^p_{\xi_\alpha} \varphi \). On the other hand, we observe that Lemma 3.3.2 of [11] continues to hold for the reductive algebra \( p/p_+ \) without any changes. Using part (1) of this lemma, we immediately conclude that we can write \( \partial^* \partial^V_1 \varphi + d^V \partial^* \varphi \) as

\[
-\sum_\alpha \eta_\alpha \bullet D^V_{\xi_\alpha} \varphi,
\]

where we now view \( \{\xi_\alpha\} \) and \( \{\eta_\alpha\} \) as local frames of \( \mathcal{A}^*_p M \), which are dual with respect to \( B \). Using an adapted frame \( \{X_i, A_r, Z^i\} \) for \( \{\xi_\alpha\} \) (see Definition 2.3), this can be rewritten as

\[
-\sum_i Z^i \bullet D^V_{X_i} \varphi + \sum_r A^\alpha \bullet A_r \bullet \varphi + \sum_i X_i \bullet Z^i \bullet \varphi.
\]

On the other hand, we can compute \( \partial^* \partial + \partial \partial^* \) using the algebraic formulæ from Sections 3.3.2 and 3.3.3 of [11]. Similarly to the notation used there, we write \( \mathcal{L} \) for the algebraic action, so for \( s \in \Gamma(\mathcal{A}^*_p M) \), we write \( \mathcal{L}_s \) for the operator \( s \bullet \cdot \).

Further we write \( \mathcal{L}^V_s \) for the tensor product of the identity on \( \Lambda^* \mathcal{A}^*_p M \) and the action of \( s \) on \( \mathcal{V} M \). Finally, we write \( \epsilon_s \) for the wedge product by \( s \), viewed as a section of \( \mathcal{A}^*_p M \) via \( B \). Then from Section 3.3.3 of [11], we see that for dual frames \( \{\xi_\alpha\} \) and \( \{\eta_\alpha\} \) as above, we can write \( \partial \) as

\[
\sum_\alpha \epsilon_{\eta_\alpha} \circ \mathcal{L}^V_{\xi_\alpha} = \frac{1}{2}(\sum_\alpha \epsilon_{\eta_\alpha} \circ (\mathcal{L}_{\xi_\alpha} - \mathcal{L}^V_{\xi_\alpha})) = \frac{1}{2} \sum_\alpha \epsilon_{\eta_\alpha} \circ \mathcal{L}_{\xi_\alpha} + \frac{1}{2} \sum_\alpha \epsilon_{\eta_\alpha} \circ \mathcal{L}^V_{\xi_\alpha}.
\]

Now \( \partial^* \) commutes with \( \mathcal{L}_s \) for any \( s \), and then part (1) of Lemma 3.3.2 of [11] shows that the anti–commutator of the first sum in the right hand side of (4.12) with \( \partial^* \) can be written as \( -\frac{1}{2} \sum_\alpha \mathcal{L}_{\eta_\alpha} \circ \mathcal{L}_{\xi_\alpha} \), so up to the factor this is just the action of the Casimir element of \( p/p_+ \). Now we can use an adapted frame \( \{X_i, A_r, Z^i\} \) as above for \( \{\xi_\alpha\} \). Acting on \( \varphi \), we obtain terms

\[
-\frac{1}{2} \sum_i (Z^i \bullet X_i \bullet \varphi + X_i \bullet Z^i \bullet \varphi),
\]

which completes the proof.
which add up with the last term in (4.11) to $-\frac{1}{2} \sum_i \{Z^i, X_i\} \cdot \varphi$. On the other hand, we get a term $-\frac{1}{2} \sum_r A^r \cdot A_r \cdot \varphi$, which adds up with the middle sum in (4.11) to $\frac{1}{2} \sum_r A^r \cdot A_r \cdot \varphi$. In view of Proposition 2.6, we have exactly obtained the action of half of the relative curved Casimir so far.

Again by part (1) of Lemma 3.3.2 of [11], the anti–commutator of the second sum in the right hand side of (4.12) with $\partial^*$ can be rewritten as $-\frac{1}{2} \sum_\alpha \mathcal{L}_{\eta_\alpha} \circ \mathcal{L}_V^{\xi_\alpha} - \frac{1}{2} \sum_\alpha \epsilon_{\eta_\alpha} \circ (\partial^* \circ \mathcal{L}_V^{\xi_\alpha} - \mathcal{L}_V^{\xi_\alpha} \circ \partial^*)$.

Using parts (2) and (3) of Lemma 3.3.2 of [11], one immediately computes that the last sum (including the sign) equals $\frac{1}{2} \sum_\alpha (\mathcal{L}_{\eta_\alpha} - \mathcal{L}_V^{\xi_\alpha}) \circ \mathcal{L}_V^{\xi_\alpha}$, so this part just gives $-\frac{1}{2}$ times the tensor product of the action of the Casimir element on $V$ with the identity. □

4.9. Algebraic properties of splitting operators. The final important aspect of the BGG–machinery are results ensuring that the splitting operators have values in certain subbundles of the bundles of differential forms. The corresponding results for usual BGG sequences were proved in [5] and they are a crucial ingredient for the results on subcomplexes in BGG sequences in [14].

Consider a representation $V$ of $P$, let $E \subset \Lambda^k(q_+/p_+) \otimes V$ be a $Q$–submodule for some $k$, and put $E_0 := E \cap \ker(\Box_\rho)$. Since the latter is a $Q_0$–invariant subspace, it corresponds to a smooth subbundle $E_0 \subset \mathcal{H}_k(T^*M, VM)$. The kind of statement we want to prove is that applying a splitting operator to a section of $E_0$, we obtain a section of the bundle $E \subset \Lambda^r T^*_\rho M \otimes VM$ corresponding to $E$. As we shall see soon, this needs only minimal assumption in the case of the splitting operators constructed from the relative twisted exterior derivative. However, for some applications, we have to study the splitting operators corresponding to the relative covariant exterior derivative. To deal with those, an additional concept is needed.

**Definition 4.15.** Suppose we have given $Q$–submodules $E \subset \Lambda^k(q_+/p_+) \otimes V$ and $F \subset \Lambda^2(q_+/p_+) \otimes (p/p_+)$. Then we say that $E$ is stable under $F$–insertions if and only if for any $\varphi \in E$ and $\psi \in F$ (both viewed as alternating multilinear maps on $p/q$) the image under $\partial^*$ of the total alternation of the map

$$(X_0, \ldots, X_k) \mapsto \varphi(\psi(X_0, X_1) + q, X_2, \ldots, X_k)$$

lies again in $E$.

The proofs of the following results are significantly simpler than the proofs for usual BGG sequences in Section 3.2 of [5].

**Proposition 4.16.** Suppose that $E \subset \Lambda^k(q_+/p_+) \otimes V$ and $F \subset \Lambda^2(q_+/p_+) \otimes (p/p_+)$ are $Q$–submodules which are invariant under $\text{id} \otimes C_0$, the action of the Casimir element on $V$, respectively $p/p_+$. Put $E_0 = E \cap \ker(\Box_\rho)$ and likewise for $F$ and let
us denote the corresponding natural subbundles by
\[ \mathcal{E}_0 M \subset \mathcal{H}_k(T^*_p M, VM) \quad \mathcal{E} M \subset \mathcal{N} T^*_p M \otimes VM \]
\[ \mathcal{F}_0 M \subset \mathcal{H}_2(T^*_p M, \mathcal{A}_p M) \quad \mathcal{F} M \subset \mathcal{N} T^*_p M \otimes \mathcal{A}_p M. \]

(1) The splitting operator associated to \( d^V \) maps \( \Gamma(\mathcal{E}_0 M) \) to \( \Gamma(\mathcal{E} M) \).

(2) If \( T^*_p M \subset TM \) is involutive, \( \mathcal{E} \) is stable under \( \mathcal{F} \)-insertions, and the relative curvature \( \kappa_\rho \) of the geometry is a section of \( \mathcal{F} M \), then the splitting operator

\[ \square \]

associated to \( d^V \) maps \( \Gamma(\mathcal{E}_0 M) \) to \( \Gamma(\mathcal{E} M) \).

(3) Suppose that \( \mathfrak{p} = \mathfrak{g} \), so \( \mathcal{F} \subset \Lambda^2 \mathfrak{q}_+ \otimes \mathfrak{g} \), and that the geometry in question is regular and normal. Then if \( \mathcal{F} \) is stable under \( \mathcal{F} \)-insertions and the harmonic curvature \( \kappa^h \) of the geometry is a section of \( \mathcal{F} M \), its curvature \( \kappa \) is a section of \( \mathcal{F} M \).

Proof. If necessary, we can replace \( \mathcal{E} \) and \( \mathcal{F} \) by their intersections with \( \ker(\partial_\rho^* \kappa) \), which does not change the intersection with \( \ker(\square_\rho) \).

(1) Let \( \square \) be the Laplacian associated to \( d^V \). Since the relative curved Casimir preserves sections of any natural subbundle, we conclude from Proposition 4.14 that our assumptions imply that \( \square \) maps sections of \( \mathcal{E} M \) to sections of \( \mathcal{E} M \). Of course, the same is then true for any polynomial in \( \square \). The construction of the splitting operator in Section 3.3 shows that its value on a section of \( \mathcal{E}_0 M \) can be obtained from applying such a polynomial to a representative section of \( \ker(\partial_\rho^* \kappa) \). But by construction, we can choose a representative section in \( \mathcal{E} M \), which implies the result.

(2) Our assumptions imply that the relative covariant exterior derivative \( d^V \) is defined and we denote by \( \square \) the associated Laplacian. Then Proposition 4.10 implies that for \( \varphi \in \Gamma(\ker(\partial_\rho^* \kappa)) \), we have \( \square \varphi = \square^V \varphi - \partial^*(i_\tau \varphi) \). Since \( \tau_\rho \) is just the projection of \( \kappa_\rho \in \Gamma(\mathcal{F} M) \), the fact that \( \mathcal{E} \) is stable under \( \mathcal{F} \)-insertions implies that for \( \varphi \in \Gamma(\mathcal{E} M) \), we have \( \square \varphi \in \Gamma(\mathcal{E} M) \), and we can conclude the proof as in part (1).

(3) Here we deal with the adjoint tractor bundle \( \mathcal{A} M \) and the adjoint tractor connection \( \nabla^A \). By part (2) of Proposition 4.9, the curvature of \( \nabla^A \) is given by \( R^A(\xi, \eta)(s) = \{ \kappa(\xi, \eta), s \} \) for \( \xi, \eta \in \mathfrak{X}(M) \) and \( s \in \Gamma(\mathcal{A} M) \). Together with the Bianchi–identity for linear connections, this easily implies that \( \kappa \in \Omega^2(M, \mathcal{A} M) \) satisfies \( d^V \kappa = 0 \). By normality, we have \( \partial^* \kappa = 0 \) and by definition \( \kappa^h = \pi_H(\kappa) \in \Gamma(\mathcal{H}_2(TM, \mathcal{A} M)) \). Hence denoting by \( S^V \) the splitting operator corresponding to \( d^V \), part (2) of Theorem 3.4 shows that \( \kappa = S^V(\kappa^h) \).

Now let \( S^A \) be the splitting operator constructed from \( d^A \). Then by part (1), \( \varphi := S^A(\kappa^h) \) is a section of \( \mathcal{F} M \subset \mathcal{A} M \). As in the proof of part (2), we get \( \square \varphi = -\partial^*(i_\tau \varphi) \). Denote by \( Q^V \) the operator constructed from \( d^V \) as in Theorem 3.9 and put \( \tilde{\varphi} := \varphi + Q^V \partial^*(i_\tau \varphi) \). Then by construction \( \square(\tilde{\varphi}) = 0 \) and since the image of \( Q^V \) lies in the image of \( \partial^* \), we get \( \pi_H(\tilde{\varphi}) = \pi_H(\varphi) = \kappa^h \). Hence we see that \( \kappa = S^V(\kappa^h) \).

Now for \( i \geq 1 \), assume that \( \kappa \) is congruent to a section of \( \mathcal{F} M \) modulo elements which are homogeneous of degree \( \geq i + 1 \). This is certainly satisfied for \( i = 1 \): By
regularity \( \tau \) and \( \kappa^h \) are of homogeneity \( \geq 1 \), so \( \varphi \) is of homogeneity \( \geq 1 \). Thus \( i_{\tau \circ} \varphi \) is of homogeneity \( \geq 2 \), and \( Q^V \) and \( \partial^* \) preserve homogeneities.

But if we assume that this is satisfied for some \( i \), then we can write \( \kappa = \kappa^1 + \kappa^2 \) with \( \kappa^1 \in \Gamma(\mathcal{F}M) \) and \( \kappa^2 \) homogeneous of degree at least \( i + 1 \). Defining \( \tau^1 \) and \( \tau^2 \) as the corresponding images in \( \Omega^2(M, TM) \), we have \( i_{\tau^1 \circ} \varphi = i_{\tau^1} \varphi + i_{\tau^2} \varphi \). Since \( F \) is stable under \( \mathbb{F} \)-insertions, \( \partial^*(i_{\tau^1} \varphi) \) is a section of \( \mathcal{F}M \), and as in (2), \( Q^V \) preserves the space of sections of this subbundle. On the other hand, \( i_{\tau^2 \circ} \varphi \) is homogeneous of degree at least \( i + 2 \), and again this property is preserved by \( Q^V \partial^* \). Hence we conclude that the property is satisfied for \( i + 1 \) and by induction the claim follows. \( \square \)

**Remark 4.17.** We remark briefly here that there is a relative version of normality and harmonic curvature and a corresponding extension of part (3) of Proposition 4.16. Consider a parabolic geometry \( (p : \mathcal{G} \to M, \omega) \) of type \((G, Q)\) with involutive relative tangent bundle \( T_p M \). Then one has the relative curvature \( \kappa_p \in \Omega^2(M, A_p M) \) as defined in Definition 4.7. The relative version of normality then is to require that \( \partial^*(\kappa_p) = 0 \), which allows one to define a relative harmonic curvature \( \kappa^h_p = \pi_H(\kappa_p) \in \Gamma(H_2(T_p M, A_p M)) \).

Consider the relative tractor connection \( \nabla^{h,A} \) on the relative tractor bundle \( A_p M \) and let \( d^V \) be the induced relative covariant exterior derivative. Then similarly as in the proof of Proposition 4.16, one can use the Bianchi identity for linear connections to prove that \( d^V \kappa^h_p = 0 \). Denoting by \( S^V \) the splitting operator constructed from \( d^V \), this immediately implies \( \kappa_p = S^V(\kappa^h_p) \), which can be viewed as a strong version of the Bianchi identity. In particular, vanishing of \( \kappa^h \) implies vanishing of \( \kappa_p \).

The proof of part (3) of Proposition 4.16 then extends without changes to the general setting. Consider a \( Q \)-submodule \( \mathbb{F} \subset \Lambda^2(\mathfrak{q}_+/p_+) \otimes (p/p_+) \), put \( \mathbb{F}_0 := \mathbb{F} \cap \ker(\Delta_p) \), and consider the associated bundles \( \mathcal{F}_0 M \) and \( \mathcal{F} M \). Assuming that \( \mathbb{F} \) is stable under \( \id \otimes C_0 \) and under \( \mathbb{F} \)-insertions, the fact that \( \kappa^h_p \in \Gamma(\mathcal{F}_0 M) \) then implies \( \kappa_p \in \Gamma(\mathcal{F} M) \).

We have not discussed this topic in more detail, since it is unclear how relevant the concept of relative normality is for interesting examples of parabolic geometries.

### 4.10. Twistor spaces and harmonic curvature

Using Proposition 4.16 we can now give sufficient conditions for some of the properties studied so far in terms of the harmonic curvature. These are then easy to verify, since the harmonic curvature is usually easy to understand explicitly. The harmonic curvature \( \kappa^h \) of a geometry of type \((G, Q)\) is a section of the bundle \( \mathcal{G} \times_p H_2(\mathfrak{q}_+, \mathfrak{g}) \) which can be viewed as a subbundle of the bundle \( L(\Lambda^2 \operatorname{gr}(TM), \operatorname{gr}(A \mathcal{M})) \). So in particular, one can naturally formulate conditions on insertions of one or two elements from \( \operatorname{gr}(T_p M) \subset \operatorname{gr}(TM) \) into the harmonic curvature, respectively on harmonic curvatures being of torsion type (i.e. having values in \( \operatorname{gr}_i(A \mathcal{M}) \) with \( i < 0 \)). Using this, we can now formulate:

**Proposition 4.18.** Let \( (p : \mathcal{G} \to M, \omega) \) be a parabolic geometry of type \((G, Q)\) with curvature \( \kappa \) and harmonic curvature \( \kappa^h \).
(1) Suppose that $\kappa^h$ vanishes upon insertion of two elements of $\text{gr}(T_pM)$ and that the torsion-type components of $\kappa^h$ even vanish, if one of their entries is from $\text{gr}(T_pM)$. Then $T_pM$ is involutive, and $\kappa$ vanishes on $T_pM \times T_pM$. Hence the relative curvature $\kappa^h$ vanishes and we are in the setting of part (1) of Theorem 4.11.

(2) Suppose that $\kappa^h$ vanishes upon insertion of one element of $\text{gr}(T_pM)$. Then the same is true for $\kappa$, so we are in the setting of part (2) of Theorem 4.11.

Proof. We apply part (3) of Proposition 4.16 to modules depending on $p$. So we have to look at a submodule $F \subset \Lambda^2 q_+ \otimes g$, which is stable under the $\text{id} \otimes C_0$, where $C_0$ is the Casimir element of $g$. Provided that $F$ is stable under $F$-insertion we can then conclude that $\kappa \in \Gamma(\mathcal{F}M)$ from $\kappa^h \in \Gamma(\mathcal{F}0M)$.

Now from the $q$-submodule $p_+ \subset q_+$ we get submodules $\Lambda^2 p_+ \subset p_+ \wedge q_+ \subset \Lambda^2 q_+$. Since $p_+$ is the annihilator of $p/q \subset g/q$, these are the spaces of those maps which vanish upon insertion of one, respectively two, elements of $p/q$.

(1) Putting $F := \Lambda^2 p_+ \otimes g + p_+ \wedge q_+ \otimes q$, the assumption in (1) is exactly that $\kappa^h \in \Gamma(\mathcal{F}0M)$, while the conclusion of (1) is that $\kappa \in \Gamma(\mathcal{F}M)$. So we only have to show that $F$ is stable under $\text{id} \otimes C_0$ and under $F$-insertions. For the first property it suffices to show that $q \subset g$ is stable under the action of the Casimir element $C_0$. This follows by taking appropriate dual bases to compute the Casimir. Take a basis $\xi_\ell$ which fills up step by step the $q$-invariant filtration components $g^i \subset g$. Then the dual basis $\eta_\ell$ has the property that for $\xi_\ell \in g^i$ we have $\eta_\ell \in g^{-i}$. Hence for $A \in q = g^0$, we always have $[\xi_\ell, [\eta_\ell, A]] \in g^0 = q$.

On the other hand, taking $\varphi, \psi \in F$, the map $(X_1, X_2, X_3) \mapsto \psi(\varphi(X_1, X_2), X_3)$ vanishes, if $X_1$ or $X_2$ lies in $p/q$ and has values in $q$ if $X_3 \in p/q$. This means that the complete alternation of this map has values in $q$, if one inserts one element from $p/q$ and vanishes upon insertion of two such elements. Hence it lies in

$$\Lambda^2 p_+ \wedge q_+ \otimes q + \Lambda^3 p_+ \otimes g.$$ 

Since $[q_+, p_+] \subset p_+$ it follows directly from the definition that $\partial^*_p$ maps this module into $F$, so $F$ is stable under $F$-insertions.

(2) This is proved in Theorem 3.3 of [5], but since the proof becomes very simple with our tools, we provide the argument. Putting $F := \Lambda^2 p_+ \otimes g$, part (2) exactly says that $\kappa^h \in \Gamma(\mathcal{F}0M)$ implies $\kappa \in \Gamma(\mathcal{F}M)$. But in this case stability under $\text{id} \otimes C_0$ is not an issue and the insertion just produces elements of $\Lambda^2 p_+ \otimes g$, which are mapped to $F$ by $\partial^*$. Thus $F$ is stable under $F$-insertion and the result follows from part (3) of Proposition 4.16. \hfill \Box

5. Examples and Applications

We will mainly discuss applications involving the BGG sequences associated to the relative twisted exterior derivative. These applications naturally split into two different groups, which are distinguished by representation theory data. On the one hand, there are cases in which the bundles occurring in a relative BGG sequence also occur in a standard BGG sequence. For this case, our main result is that the operators on these spaces coming from the two sequences actually agree. Hence the relative BGG sequence is a subsequence in a standard BGG sequence in these
cases. Under weak curvature conditions, these subsequences are subcomplexes and fine resolutions of certain sheaves.

On the other hand, there are cases in which the bundles contained in a relative BGG sequence cannot occur in a standard BGG sequence. In these cases, the constructions of invariant differential operators which are available in the literature, need a lot of case–by–case considerations (even to decide whether they are applicable). Hence in these cases, we obtain the first general uniform construction for invariant differential operators between the bundles in question. Of course, the results on curvature conditions ensuring that relative BGG sequences are complexes or even fine resolutions of sheaves from Section 4 also apply in these cases.

We will make both kinds of examples explicit for generalized path geometries. These form an example of broader interest, because of their relation to systems of second order ODEs, and at the same time they nicely expose the features of relative BGG sequences.

5.1. The two types of examples. As mentioned above, the distinction between the two classes of examples we obtain is related to representation theory, in particular to the distinction between regular and singular infinitesimal character. We will formulate things in a direct way first and then discuss the interpretation in terms of weights.

Given nested parabolic subalgebras \( q \subset p \) in a semi–simple Lie algebra \( g \) and corresponding groups \( Q \subset P \subset G \), there are two ways to apply the theory developed in this article to the construction of invariant differential operators. Either we can construct “absolute” BGG sequences as known from \([12]\) and \([4]\) in the way presented in Sections 3 and 4 (so this corresponds to the pair \( q \subset g \)). For this construction, the starting point is a representation \( \tilde{V} \) of \( G \) and the bundles in the resulting BGG sequence are induced by the summands of the completely reducible representation \( H^*_*(q+\tilde{V}) \) of \( Q \).

On the other hand, we can apply the relative BGG construction for the pair \( q \subset p \). Here we start with a completely reducible representation \( V \) of the group \( P \), and the bundles in the resulting relative BGG sequence are associated to the summands of the completely reducible representation \( H_*(q_+/p_+,\tilde{V}) \) of \( Q \).

Now there obviously is some overlap between the two cases. The simplest example of this is that an irreducible representation \( \widetilde{V} \) of \( G \) has a unique \( P \)-irreducible quotient as well as a unique \( Q \)-irreducible quotient. These can be nicely described as \( V := H_0(p_+\tilde{V}) \) and as \( H_0(q_+\tilde{V}) \), respectively. It is easy to see that \( H_0(q_+/p_+,\tilde{V}) \) is the \( Q \)-irreducible quotient of \( V \) and hence isomorphic to \( H_0(q_+\tilde{V}) \). This is vastly generalized in Theorem 3.3 of \([15]\), where it is shown that

\[
H_k(q_+\tilde{V}) \cong \bigoplus_{i+j=k} H_i(q_+/p_+,H_j(p_+\tilde{V})).
\]

Since \( H_*(p_+\tilde{V}) \) is always a completely reducible representation, we conclude that the absolute BGG sequence is (as far as the bundles are concerned) the union of the relative BGG sequences induced by the components of \( H_*(p_+\tilde{V}) \). Of course, in these cases, the main question is relating the two BGG sequences.
The importance of bundles induced by Lie algebra homology spaces is explained by Kostant’s theorem (see [19]), and its relative analog, Theorem 2.7 of [15]. In the complex setting, irreducible representations of a parabolic subalgebra of \( \mathfrak{g} \) can be described in terms of weights, which are linear functionals on a Cartan subalgebra \( \mathfrak{h} \subset \mathfrak{g} \). Now the relation between invariant differential operators on the homogeneous model of a parabolic geometry and representation theory (in particular homomorphisms between generalized Verma modules) heavily restricts the representations inducing bundles which allow a non–zero invariant differential operator between their spaces of sections. Namely, the negatives of the lowest weights of the inducing representations have to lie in the same orbit for the so–called affine action of the Weyl group \( W \) of \( \mathfrak{g} \) on the space of weights. Since each such orbit is finite, this is a very strong restriction.

Given a finite dimensional irreducible representation \( \widetilde{V} \) of \( \mathfrak{g} \), Kostant’s original theorem first shows that each summand of the homology \( H_*(\mathfrak{q}^+; \widetilde{V}) \) corresponds to a weight in the affine Weyl orbit determined by \( \widetilde{V} \). Moreover, the weights obtained by these summands are the only ones in the orbit that can be realized by finite dimensional irreducible representations of \( \mathfrak{q} \). Hence the BGG sequence contains all candidates for targets and domains of invariant differential operators corresponding to that affine Weyl orbit.

The relative version of Kostant’s theorem from [15] proves a similar statement starting from a finite dimensional, complex irreducible representation \( V \) of \( \mathfrak{p} \). Again the first statement is that all summands of \( H_*(\mathfrak{q}^+ / \mathfrak{p}^+; V) \) correspond to weights in the affine Weyl orbit determined by \( V \). Moreover, they are the only weights in the orbit under a subgroup \( W_\mathfrak{p} \subset W \), which can be realized by finite dimensional irreducible representations of \( \mathfrak{q} \).

The description via weights also shows that there are many \( \mathfrak{P} \)–irreducible representations \( \mathcal{V} \) such that no summand of \( H_*(\mathfrak{q}^+ / \mathfrak{p}^+; \mathcal{V}) \) can occur in \( H_*(\mathfrak{q}^+; \widetilde{V}) \) for a representation \( \widetilde{V} \) of \( \mathfrak{g} \). The point here is that realizability of a weight by a finite dimensional representation of one of the Lie algebras or groups in question depends on dominancy and integrality conditions for the weights. Here we have to consider the three conditions of \( \mathfrak{g} \)–dominancy, \( \mathfrak{p} \)–dominancy, and \( \mathfrak{q} \)–dominancy, which imply each other in that sequence. Now for any irreducible representation \( \mathcal{V} \) of \( \mathfrak{g} \) the corresponding weight is \( \mathfrak{g} \)–dominant and integral. For irreducible representations of \( \mathfrak{p} \) and \( \mathfrak{q} \) (and even of \( \mathfrak{P} \) and \( \mathfrak{Q} \)) both the dominancy and integrality conditions are weaker. In particular, there are many examples of finite dimensional, complex, \( \mathfrak{P} \)–irreducible representations \( \mathcal{V} \), which correspond to a weight whose affine Weyl–orbit does not contain any \( \mathfrak{g} \)–dominant integral weight. In particular, the summands of \( H_*(\mathfrak{q}^+ / \mathfrak{p}^+; \mathcal{V}) \) then give rise to bundles that do not show up in a standard BGG sequence. This is always the case in singular infinitesimal character (compare with Section 3.1 of [15]), but also in regular character there are many examples of representations \( \mathcal{V} \) corresponding to a non–integral weight.

In this second situation we obtain “new” invariant differential operators and a new relation to relative forms and the relative twisted exterior derivative. The main question here is to describe the bundles on which these operators act, the
order of the operators, and the complexes and resolutions obtained by the relative BGG construction.

5.2. The case of subsequences. To discuss the first type of results described in Section 5.1, we consider a representation $V$ of $G$ and compare the absolute BGG sequence determined by $V$ to the relative BGG sequences determined by summands of $H_i(p_+; V)$. Initially, the isomorphism

$$H_k(q_+, V) \cong \oplus_{i+j=k} H_i(q_+; p_+, H_j(p_+, V))$$

is obtained in [15] abstractly from comparing weights. However, in [15] we have also given a more explicit description of the relation between the two homologies, which can be translated to geometry.

Namely, for each $\ell \leq k$ we have constructed there $Q$–invariant subspaces $\tilde F_k^\ell \subset F_k^\ell \subset C_k(q_+, V)$.

The description of $F_k^\ell$ in [15] readily translates to geometry. A $\tilde V M$–valued $k$–form lies in $F_k^\ell \subset M$ if and only if it vanishes upon insertion of $k - \ell + 1$ sections of the subbundle $T_p M \subset T M$. Moreover, there is a $q$–equivariant surjection $\pi : \tilde F_k^\ell \to \Lambda^{k-\ell}(q_+/p_+) \otimes H_\ell(p_+, \tilde V)$. Denoting by $\forall_\ell$ the completely reducible representation $H_\ell(p_+, \tilde V)$ and by $\forall_\ell M$ the corresponding relative tractor bundle, we obtain, for each $\ell \leq k$, an induced bundle map

$$\pi : \tilde F_k^\ell \to \Lambda^{k-\ell} T_p^* M \otimes \forall_\ell M. \tag{5.1}$$

**Proposition 5.1.** For each $\ell \leq k$, the twisted exterior derivative $d\tilde V$ maps the subspace $\Gamma(\tilde F_k^\ell M) \subset \Omega^k(M, \tilde V M)$ to $\Gamma(\tilde F_{k-1}^\ell M) \subset \Omega^{k-1}(M, \tilde V M)$. Moreover, denoting by $d\forall_\ell$ the relative twisted exterior derivative on $\Omega^\ell_\rho(M, \forall_\ell M)$, we get $\pi \circ d\tilde V \varphi = d\forall_\ell (\pi \circ \varphi)$ for all $\varphi \in \Gamma(\tilde F_k^\ell M)$.

**Proof.** Observe first that $d\tilde V$ is the twisted exterior derivative giving rise to an absolute BGG sequence. Hence to follow the construction from Section 4.1 we have to start by viewing $\Lambda^k T^* M \otimes \tilde V M$ as subbundle of $\Lambda^k A^* M \otimes \tilde V M$. Moreover, the Lie algebra cohomology differential used in the construction is the differential $\partial_\varphi$ from the standard complex computing the Lie algebra cohomology $H^\rho(\mathfrak g, \tilde V)$.

Now $\Lambda^k A^* M \otimes \tilde V M$ carries an obvious analog of the filtration $\{F_k^\ell \}$. For $k \geq \ell$, we define $E_k^\ell M \subset \Lambda^k A^* M \otimes \tilde V M$ to consist of those $k$–linear maps which vanish upon insertion of $k - \ell + 1$ entries from the subbundle $\mathcal A_p M := G \times_Q p \subset A M$.

Taking $\varphi \in E_k^\ell M$ and inserting $k - \ell$ elements of $\mathcal A_p M$, we obtain an $\ell$–linear alternating map on $A M$ with values in $\tilde V M$. By construction, the latter map vanishes upon insertion of a single element of $\mathcal A_p M$, so it descends to an element of $\Lambda^\ell (A M/\mathcal A_p M)^* \otimes \tilde V M$. Since $(\mathfrak g/p)^* \cong p_+$, the latter bundle is induced by the representation $\Lambda^\ell (A M/\mathcal A_p M)^* \otimes \tilde V M$. Denoting by $C_\ell$ the bundle $\Lambda^\ell (A M/\mathcal A_p M)^* \otimes \tilde V M$, we have obtained a bundle map

$$\Psi : E_k^\ell M \to \Lambda^{k-\ell} (A_p M)^* \otimes C_\ell. \tag{5.2}$$
whose kernel evidently equals $E^i_{k+1}M$. Now since the bundle $C_1$ is induced by a representation of $p$, there is a well-defined Lie algebra cohomology differential $\partial^p$ mapping $\Lambda^{k-\ell}A_\ell M \otimes C_1$ to $\Lambda^{k-\ell+1}A_\ell M \otimes C_1$. On the other hand, given a section of $\Lambda^{k-\ell}A_\ell M \otimes C_1$ we can form the fundamental derivative, restrict to entries from $A_\ell p$ and then form the complete alternation to obtain a section of $\Lambda^{k-\ell+1}A_\ell M \otimes C_1$. We write this operation as $\tau \mapsto \text{Alt}(D\tau)$.

We next claim that for $\varphi \in \Gamma(E^i_k M)$ we have $\partial\varphi \in \Gamma(E^i_{k+1} M)$ and that
\[
\Psi(\partial\varphi) = \text{Alt}(D(\varphi)) + \partial^p(\varphi).
\]

To verify that $\partial^p \varphi = \text{Alt}(D\varphi)$ lies in $E^i_{k+1}$ take sections $s_0, \ldots, s_{k-\ell} \in \Gamma(A_\ell p)$ and $t_1, \ldots, t_\ell$ in $\Gamma(A_\ell M)$. Inserting these into $\partial^p \varphi$, we obtain
\[
\sum_{i=0}^{k-\ell} (-1)^i (D_{s_i} \varphi)(s_0, \ldots, \hat{s}_i, \ldots, t_\ell) + \sum_{j=1}^\ell (D_{t_j} \varphi)(s_0, \ldots, \hat{t}_j, \ldots, t_\ell).
\]

Naturality of the fundamental derivative implies that for each $s \in \Gamma(A_\ell M)$, we have $D_s \varphi \in \Gamma(E^i_k M)$. This readily implies that each summand in the second sum vanishes, since there are $k - \ell + 1$ entries from $A_\ell p$. For the same reason, the summands in the first sum vanish if one of the $t_j$ is a section of $A_\ell p$. Hence we see that $\partial^p \varphi \in \Gamma(E^i_{k+1})$.

On the other hand, expanding $(D_{s_i} \varphi)(s_0, \ldots, \hat{s}_i, \ldots, t_\ell)$, we obtain a term in which $D_{s_i}$ acts on $\varphi(s_0, \ldots, \hat{s}_i, \ldots, t_\ell) = (\Psi(\varphi)(s_0, \ldots, \hat{s}_i, \ldots, s_k))(t_1, \ldots, t_\ell)$. On the other hand, there are terms in which $D_{s_i}$ acts on one of the $t_i$’s and adding those, we obtain $(D_{s_i}(\Psi(\varphi)(s_0, \ldots, \hat{s}_i, \ldots, s_k))(t_1, \ldots, t_\ell)$, Finally, there are terms in which the $D_{s_i}$ hits another $s$, and we can rewrite those terms in the form $-(\Psi(\varphi)(s_0, \ldots, \hat{s}_i, \ldots, D_{s_i} s_j, \ldots, s_k))(t_1, \ldots, t_\ell)$. Adding these in, one obtains $(D_{s_i}(\Psi(\varphi)))(s_0, \ldots, \hat{s}_i, \ldots, s_k \otimes \Lambda)$ evaluated on the $t_j$. This shows that $\Psi(\partial^p \varphi) = \text{Alt}(D(\varphi))$.

The argument for $\partial_{\partial^p} \varphi$ is similar. Inserting the sections $s_i$ and $t_j$ and expanding the definitions, terms of the form $t_j \bullet (\varphi(\ldots))$ vanish identically, since $k + \ell - 1$ entries from $A_\ell p M$ get inserted into $\varphi$. Moreover, if at least one $t_j$ is a section of $A_\ell M$ then the same applies to the remaining terms. (Here one uses that $p \subset g$ is a subalgebra to deal with the terms involving $\{s_i, t_j\}$ for this fixed $t_j$.) This implies that $\partial^p \varphi$ is a section of $E^i_{k+1} M$, too, hence proving the first part of the claim. To describe $\Psi(\partial^p \varphi)$, we observe that
\[
(-1)^i s_i \bullet (\varphi(s_0, \ldots, \hat{s}_i, \ldots, t_\ell)) + \sum_{j=1}^\ell (-1)^{i+j+k-\ell} \varphi(s_i, t_j, \ldots, s_k))(t_1, \ldots, t_\ell).
\]

Likewise, we can write the term involving the bracket $\{s_i, s_j\}$ as
\[
(-1)^{i+j} \varphi(s_i, s_j, \ldots, t_\ell))(t_1, \ldots, t_j).
\]

Hence we conclude that $\Psi(\partial^p \varphi) = \partial^p(\varphi)$, and this completes the proof of the claim.

Consider the subbundles $A^i_k M := \Lambda^{k-\ell}A^i_\ell M \otimes \ker(\partial^r) \subset \Lambda^{k-\ell}A^i_\ell M \otimes C_1$, which correspond to $q$–invariant subspaces in the inducing representations. From the form of these subspaces it is evident, that $\partial^p$ maps sections of $A^i_k M$ to sections...
of $A^k_{k+1}M$. Moreover, by naturality of the fundamental derivative, we see that for $\psi \in \Gamma(A^k_{k+1}M)$ and $s \in \Gamma(AM)$ we have $D_s\psi \in \Gamma(A^k_{k+1}M)$. Applying this to $s \in \Gamma(A_kM)$ and forming the complete alternation, we conclude that $\operatorname{Alt} \circ D$ maps $\Gamma(A^k_{k+1}M)$ to $\Gamma(A^k_{k+1}M)$.

But putting $\tilde{E}^{\ell}_kM := \Psi^{-1}(A^k_{k+1}M) \subset E^{\ell}_kM$, it is clear from the definitions that $\tilde{F}^{\ell}_kM = E^{\ell}_kM \cap F^{\ell}_kM$. Hence for $\varphi \in \Gamma(\tilde{F}^{\ell}_kM)$, we get $\Psi(\varphi) \in \Gamma(A^k_{k+1}M)$, so from our claim we conclude that $\Psi(\tilde{d} \varphi) \in \Gamma(A^k_{k+1}M)$. Since this implies $\tilde{d} \varphi \in \Gamma(\tilde{F}^{\ell}_{k+1})$, the first part of the proposition is proved.

On the other hand, the bundle map $\pi$ from (5.1) evidently extends to a bundle map $\tilde{E}^{\ell}_kM \to \Lambda^{k-\ell}A^*_p \otimes \tilde{V}_lM$, which we denote by the same symbol. Now for a section $\varphi \in \Gamma(\tilde{F}^{\ell}_kM)$, we know from Theorem 4.1 that $\tilde{d} \varphi$ vanishes upon insertion of a single section of the subbundle $G \times_{Q} q$. Since $p_+ \subset q$, we in particular conclude that $\Psi(\tilde{d} \varphi) \in \Gamma(\Lambda^{k-\ell}A^*_p \otimes \ker(\partial^r))$ vanishes upon insertion of a single section of $G \times_{Q} p_+ \subset A_pM$. Hence $\Psi(\tilde{d} \varphi)$ naturally descends to a section of $\Lambda^{k-\ell}A^*_p \otimes \ker(\partial^r)$, so projecting to homology, we see that $\pi(\tilde{d} \varphi)$ naturally descends to $\Lambda^{k-\ell}A^*_p \otimes V_lM$. Moreover, $\operatorname{Alt} \circ D$ descends to $\operatorname{Alt} \circ D^p$ on that bundle. On the other hand, the Lie algebra cohomology differential $\partial_p$ on $\Lambda^{k-\ell}A^*_p \otimes \ker(\partial^r)$ by construction descends to $\partial_p/p_+$ on $\Lambda^{k-\ell}A^*_p \otimes V_l$. Thus the second part of the proposition follows.

5.3. Absolute vs. relative BGG sequences. We can now complete the general discussion of the first kind of examples. We continue using the notation of Section 5.2 so $\tilde{V}$ is a finite dimensional irreducible representation of $G$ and for some $\ell \leq k$, we denote by $\tilde{V}_l$ the completely reducible representation $H_{\ell}(p_+, \tilde{V})$ of $P$. We want to compare the BGG sequence corresponding to the tractor bundle $\tilde{V}M$ induced by $\tilde{V}$ to the relative BGG sequence corresponding to the relative tractor bundle $V_lM$ induced by $V_l$. To explicitly relate the bundles showing up in the two sequences, we have to use another property of $Q$–invariant subspaces $\tilde{F}^{\ell}_k \subset F^{\ell}_k \subset C_k(q_+, \tilde{V})$ from Section 5.2. Namely, we get $F^{\ell+1}_k \subset \tilde{F}^{\ell}_k$ and by Proposition 3.6 of [15], $\ker(\partial^r_q) \cap F^{\ell}_k \subset \tilde{F}^{\ell}_k$. In that proposition it is also shown that the $q$–equivariant map $\pi$ inducing the bundle map (5.1) vanishes on $F^{\ell+1}_k$ and has the property that, up to sign, $\partial_p^\pi \circ \pi$ coincides with $\pi \circ \partial_q^\pi$.

The obvious consequences of these properties for induced bundles and induced bundle maps imply that $\pi$ is defined on $\ker(\partial^r_q) \cap F^{\ell}_kM$ and its values on that space are contained in $\ker(\partial^r_p) \subset \Lambda^{k-\ell}T^*_pM \otimes V_lM$. Hence we can project to the relative homology bundle and obtain a bundle map

$$(5.3) \quad \Pi : \ker(\partial^r_q) \cap \tilde{F}^{\ell}_kM \to \mathcal{H}_{k-\ell}(T^*_pM, V_lM).$$

By Theorem 3.7 of [15], this bundle map vanishes on $\text{im}(\partial^r_q) \cap \tilde{F}^{\ell}_kM$, so it descends to a bundle map on $\mathcal{H}_k(T^*_M, \tilde{V}M)$. That theorem also implies that the result is a surjection $\mathcal{H}_k(T^*_M, \tilde{V}M) \to \mathcal{H}_{k-\ell}(T^*_pM, V_lM)$. Hence we can use it to identify the copies of the bundles showing up in the relative BGG sequence induced by
\( \mathbb{V}_\ell \) with their counterparts in the absolute BGG sequence determined by \( \tilde{\mathbb{V}} \). We refer to the part of the absolute BGG sequence determined by \( \tilde{\mathbb{V}} \) formed by these bundles and the operators mapping between them, as the \textit{subsequence determined by} \( \mathbb{V}_\ell \).

**Theorem 5.2.** For any \( \ell \leq k \), the identification between the relative BGG sequence determined by \( \mathbb{V}_\ell = H_{\ell}(p_+, \tilde{\mathbb{V}}) \) and the subsequence of the absolute BGG sequence determined by \( \tilde{\mathbb{V}} \) constructed above is compatible with the operators in the two sequences.

In particular, under the curvature conditions from Theorem 4.11 respectively Proposition 4.18, this subsequence is a subcomplex and a fine resolution of a sheaf as described there.

**Proof.** Consider the \( Q \)–submodule \( \tilde{\mathcal{F}}_k^\ell \subset \Lambda^k q_+ \otimes \tilde{\mathbb{V}} \). From the proof of Theorem 3.7 in [15], we see that all irreducible components of \( H_k(q_+, \tilde{\mathbb{V}}) \) which also occur in \( H_{k-\ell}(q_+/p_+, H_{\ell}(p_+, \mathbb{V})) \) are contained in \( \ker(A) \cap \tilde{\mathcal{F}}_k^\ell \). Since \( \tilde{\mathbb{V}} \) is an irreducible representation of \( \mathfrak{g} \), the Casimir acts by a scalar on \( \tilde{\mathbb{V}} \), so part (1) of Proposition 4.16 applies. This shows that for any section \( \alpha \) of \( \mathcal{H}_k(T^*M, \mathbb{V}M) \) which is contained in the subsequence determined by \( \mathbb{V}_\ell \), the image \( S(\alpha) \) under the splitting operator lies in \( \Gamma(\tilde{\mathcal{F}}_k^\ell M) \subset \Omega^k(M, \mathbb{V}M) \). Hence we can apply the bundle map \( \pi \) from (5.1) to obtain \( \varphi := \pi(S(\alpha)) \in \Omega^{k-\ell}_p(M, \mathcal{V}_\ell M) \). Since \( \partial^*_\mathcal{V}_\ell(S(\alpha)) = 0 \), we conclude that \( \partial^*_\mathcal{V}_\ell(\varphi) = 0 \) so we can project to a section of \( \mathcal{H}_{k-\ell}(T^*_p M, \mathcal{V}_\ell M) \). By construction, the resulting section coincides with \( \Pi(S(\alpha)) \) so from the above discussion we see that this is the section corresponding to \( \alpha \) under the identification of the subsequence with the relative BGG sequence.

We next claim that \( \varphi = \pi(S(\alpha)) \) coincides with \( S_\rho(\alpha) \), where

\[
S_\rho : \Gamma(H_{k-\ell}(T^*_p M, \mathcal{V}_\ell M)) \to \Omega^{k-\ell}_p(M, \mathcal{V}_\ell M)
\]

is the splitting operator coming from the relative BGG construction. We already know that \( \partial^*_\rho(\varphi) = 0 \) and the projection of \( \varphi \) to cohomology coincides with \( \alpha \). In view of part (2) of Theorem 3.6, it thus suffices to verify that \( \partial^*_\rho d\tilde{\mathcal{V}}_\ell \varphi = 0 \) to complete the proof of the claim. But since \( \varphi = \pi(S(\alpha)) \), Proposition 5.1 shows that \( d\tilde{\mathcal{V}}_\ell \varphi = \pi(d\tilde{\mathcal{V}} S(\alpha)) \) and the compatibility of \( \pi \) with the Lie algebra homology differentials then implies that \( \partial^*_\rho d\tilde{\mathcal{V}}_\ell \varphi = \pm \pi(\partial^*_\rho d\tilde{\mathcal{V}} S(\alpha)) = 0 \).

Knowing that \( \pi(S(\alpha)) = S_\rho(\alpha) \) we can again use Proposition 5.1 to conclude that \( d\tilde{\mathcal{V}} S(\alpha) \in \Gamma(\tilde{\mathcal{F}}_{\ell+1}^\ell M) \) and that \( \pi(d\tilde{\mathcal{V}} S(\alpha)) = d\tilde{\mathcal{V}}_\ell S_\rho(\alpha) \). This is a section of \( \ker(\partial^*_\rho) \subset \Omega^{k+1}_p(M, \mathcal{V}_\ell M) \) and projecting to cohomology we obtain \( D_\rho(\alpha) \), the relative BGG–operator. But this can be equivalently written as \( \Pi(d\tilde{\mathcal{V}} S(\alpha)) \), so under the identification of the two sequences, it coincides with the components of the projection of \( d\tilde{\mathcal{V}} S(\alpha) \) to cohomology, which lie in the subsequence. But denoting by \( D \) the absolute BGG operator, this is exactly the component of \( D(\alpha) \) contained in the subsequence. \( \square \)

**Remark 5.3.** The proof of the theorem shows how the machineries corresponding to the absolute and relative BGG sequences are related. If \( \alpha \in \Gamma(\mathcal{H}_k(T^*M, \mathbb{V}M)) \)
lies in the subsequence determined by $V_\ell$, then the absolute splitting operator $S$ has the property that $S(\alpha) \in \Gamma(\tilde{F}_\ell M) \subset \Omega^k(M, \tilde{V}M)$. Hence we can apply $\pi$ and $\pi(S(\alpha)) = S_\rho(\alpha) \in \Omega^{k-\ell}(M, \tilde{V}_\rho M)$, where $S_\rho$ is the relative splitting operator.

5.4. **Generalized path geometries.** To obtain explicit examples of relative BGG sequences, we consider the example of generalized path geometries, see Section 4.4.3 of [11]. A path geometry on a smooth manifold $N$ of dimension $n + 1$ is given by a smooth family of 1-dimensional submanifolds in $N$, such that, given any point $x \in N$ and any line $\ell$ in $T_x M$, there is a unique submanifold in the family which contains $x$ and is tangent to $\ell$ in $x$. The importance of path geometries comes from their relation to systems of second order ODEs. Given such a system on an open subset $U \subset \mathbb{R}^n$ (or on some manifold), the graphs of all solutions define a path geometry on (an open subset of) $U \times \mathbb{R}$, so this gives a coordinate-independent way to study such systems.

To encode a path geometry, one passes to the projectivized tangent bundle $M := \mathcal{P}(TN)$ of $N$. Viewing the submanifolds in $N$ as regularly parametrized curves, it is evident that they lift to $M$, and taking tangents, one obtains a smooth line subbundle $E \subset TM$, which is transversal to the vertical subbundle $VM$ of the projection $M \to N$. The sum $H := E \oplus V \subset TM$ is the so-called tautological subbundle in $TM$, whose fiber in a point $\ell$ consist of those tangent vectors which project to elements of $\ell$. One can then recover the submanifolds in the initial family as the projections to $N$ of the leaves of the foliation defined by the line subbundle $E \subset TM$. Hence a path geometry can be equivalently defined as specifying a line subbundle $E$ in the tautological bundle which is complementary to the vertical subbundle.

The concept of a generalized path geometry is then obtained by requiring an abstract version of the properties of the subbundles defining a path geometry. If $M$ is any smooth manifold of dimension $2n + 1$, then a generalized path geometry on $M$ is given by two subbundles $E$ and $V$ in $TM$ of rank 1 and $n$, respectively, which intersect only in zero. Moreover, one requires that the Lie bracket of two sections of $V$ is a section of $H := E \oplus V \subset TM$, while projecting the bracket of a section of $E$ and a section of $V$ to the quotient induces an isomorphism $E \otimes V \to TM/H$ of vector bundles.

It turns out (see again Section 4.4.3 of [11]) that generalized path geometries can be equivalently described as parabolic geometries of type $(G, Q)$, where $G = PGL(n + 2, \mathbb{R})$ and $Q$ is a the stabilizer of a flag in $\mathbb{R}^{n+2}$ consisting of a line contained in a plane. Now there are two obvious intermediate parabolics lying between $Q$ and $G$, namely the stabilizer $P$ of the line and the stabilizer $\tilde{P}$ of the plane, so $Q = P \cap \tilde{P}$. Hence on a generalized path geometry, there are two kinds of relative BGG sequences available, namely the ones corresponding to $p \supset q$ and the ones corresponding to $\tilde{p} \supset q$. Since the latter consist of a single operator, we will focus on describing the former class.

5.5. **Relative BGG sequences on generalized path geometries.** To make our results explicit for generalized path geometries, it mainly remains to connect representation theory data to geometric objects. On the one hand, we have to
ensure that there are sufficiently many relative tractor bundles available to start the construction. On the other hand, we have to discuss how weights are realized in terms of natural bundles. We do the second step in detail only in the case $n = 2$, i.e. for generalized path geometries in dimension 5. Here the representation theory information is available in [15]. Higher dimensions can be dealt with in a similar way.

For the question of existence of relative tractor bundles, we have to construct completely reducible representations of the group $P$, the stabilizer of a line (i.e. of a point in projective space) in $G = PGL(n+2, \mathbb{R})$. As discussed in Section 4.1.5 of [11], the Levi component $P_0 \subset P$ is given by the classes of block diagonal matrices of the form $\begin{pmatrix} 0 & C \\ 0 & C \end{pmatrix} \in GL(n+2, \mathbb{R})$ with $0 \neq c \in \mathbb{R}$ and $C \in GL(n+1, \mathbb{R})$. It is also shown there that the representation of $P_0$ on $\mathbb{R}^{n+1}$ defined by $X \mapsto c^{-1}CX$ can be realized on $\mathfrak{g}/\mathfrak{p}$ via the adjoint representation.

Forming exterior powers of this basic representation, one obtains the fundamental representations of $GL(n+1, \mathbb{R})$ up to a twist by a multiplication by some power of $c$. On the other hand, the center of $P$ is isomorphic to $\mathbb{R} \setminus \{0\}$, and the top exterior power of the representation on $\mathfrak{g}/\mathfrak{p}$ from above gives a non–trivial representation of the center. Forming the square of this representation, the action depends only on the absolute value, so one can take arbitrary real roots of the resulting representation. By tensorizing with such representations, the action of the center can be changed arbitrarily. Hence we conclude that any weight which is $\mathfrak{p}$–dominant and $\mathfrak{p}$–integral can be realized by a finite dimensional representation of $P_0$ and hence by a completely reducible representation of $P$. (Initially weights are considered for complex representations, but there is no problem to use them in a real setting here, since the real Lie algebra $\mathfrak{sl}(n+2, \mathbb{R})$ we are dealing with is a split real form of its complexification.) We will make this more explicit in the case $n = 2$ in Section 5.6 below.

As a second step, let us discuss the bundles induced by representations of $Q_0$ in the case $n = 2$. In the Dynkin diagram notation used in [15], we have to consider weights of the form $\begin{array}{ccc} & \circ & \\ \circ & \times & \circ \end{array}$. Again it is no problem to work with weights in the real setting here. From the Dynkin diagram it is clear that the fundamental representations corresponding to the first two (crossed) nodes will be one–dimensional, while the fundamental representation corresponding to the last node has dimension two. Correspondingly, we obtain a two parameter family of line bundles and one basic rank two bundle. For our purposes, there is no need to discuss existence of representations of $Q$ realizing a given weight. We have discussed existence of representations inducing relative tractor bundles above. These give rise to representations of $Q_0$ on relative Lie algebra homology groups, which induce the completely reducible natural bundles showing up in relative BGG sequences.

Hence we just briefly discuss the relation between weights and the basic bundles available for the geometry. For $w, w' \in \mathbb{R}$, we denote by $E(w, w')$ the bundle corresponding to the weight $\begin{array}{ccc} & \circ & \\ \circ & \times & \circ \end{array}$ (not worrying about existence). The correspondence between natural bundles and the Lie algebra $\mathfrak{sl}(n+2, \mathbb{R})$ (c.f. Section 4.4.3 of [11]) allows us to read off the weights corresponding to the constituents
of the associated graded of the tangent– and cotangent bundle, as well as bundles constructed from those. This shows that \( E = \mathcal{E}(2, -1), \Lambda^2 V = \mathcal{E}(2, 3) \), and \( \Lambda^2(TM/H) = \mathcal{E}(2, 1) \). Next, the bundle \( V \) corresponds to \( -1 \), \( -1 \), its dual \( V^* \) corresponds to \( \frac{1}{2} \), \( \frac{1}{2} \), while \( TM/H \cong E \otimes V \) and its dual correspond to \( \frac{1}{2} \), \( \frac{1}{2} \) and \( \frac{1}{2} \), \( \frac{1}{2} \), respectively. Together with the line bundles \( \mathcal{E}(w, w') \), any of these for bundles can be used to construct a bundle corresponding to any given weight.

It will be most convenient to take \( V^* \) as the basic ingredient, and we will follow the usual convention that adding \("(w, w')"\) to the name of a bundle indicates a tensor product with \( \mathcal{E}(w, w') \). Then for \( a, b \in \mathbb{R} \) and \( c \in \mathbb{N} \), the weight \( \frac{a}{2} - \frac{b}{2} - \frac{c}{2} \) is realized by the bundle \( S^c V^*(a - c, b + 2c) \).

Having this at hand, we can describe the basic form of relative BGG sequences corresponding to \( p \supset q \) on a generalized path geometry in dimension five.

**Theorem 5.4.** Let \( H = E \oplus V \subset TM \) be a generalized path geometry of dimension 5. Then for each \( w \in \mathbb{R} \) and \( k, \ell \in \mathbb{N} \), there is a sequence of invariant differential operators

\[
\Gamma(\mathcal{W}_0) \xrightarrow{D_1} \Gamma(\mathcal{W}_1) \xrightarrow{D_2} \Gamma(\mathcal{W}_2), \quad \text{with} \quad \mathcal{W}_0 = S^k V^*(w, 2k + \ell),
\]

\[
\mathcal{W}_1 = S^{k+\ell+1} V^*(w, 2k + \ell), \quad \text{and} \quad \mathcal{W}_2 = S^\ell V^*(w + 2k + 2, \ell - k - 3).
\]

This sequence is contained in a standard BGG sequence if and only if \( w \in \mathbb{Z} \) and one of the following four mutually exclusive conditions is satisfied:

- \( w + k \geq 0 \)
- \( w + k \leq -2 \) and \( w + k + \ell \geq -1 \)
- \( w + k + \ell \leq -3 \) and \( w + 2k + \ell \geq -2 \)
- \( w + 2k + \ell \leq -4 \)

If either \( w = -1-k \), or \( w = -2-k-\ell \), or \( w = -3-2k-\ell \), then the representations in the sequence have singular infinitesimal character.

**Proof.** We start from the relative tractor bundle \( \mathcal{V}M \) induced by the representation \( \mathcal{V} \) corresponding to the weight \( \frac{a}{2} - \frac{b}{2} - \frac{c}{2} \) with \( a = w + k, b = \ell \) and \( c = k \). Then Theorem 2.7 of [15] shows that the homology groups \( H_i(q_+/p_+, \mathcal{V}) \) for \( i = 0, 1, 2 \) correspond to the weights listed in formula (3.1) in Example 3.2 of [15]. Expressing these weights in terms of \( w, k \) and \( \ell \), the discussion above this theorem then shows that the three bundles in the sequence (5.4) are \( \mathcal{H}_i(T^*_p M, \mathcal{V}M) \) for \( i = 0, 1, 2 \). Thus existence of the sequence of invariant differential operators follows directly from Theorem [14].

The standard BGG sequences on \( M \) are indexed by irreducible representations of \( G \) and thus by \( \mathfrak{g} \)–dominant integral weights. Since all weights in the affine Weyl orbit of an integral weight are integral, too, we see that the condition \( a = w + k \in \mathbb{Z} \) is necessary. The bundles occurring in the standard BGG sequence induced by a representation \( \mathcal{V} \) of \( \mathfrak{g} \) correspond to the representations \( H_i(q_+, \mathcal{V}) \). The relation between absolute and relative homology groups is discussed in detail in Section 3.2 of [15], and in Example 3.2 of that reference, this is made explicit in the case we consider here. Starting from a dominant integral weight \( \frac{a}{2} - \frac{b}{2} - \frac{c}{2} \), the corresponding absolute BGG sequence contains four relative BGG sequences as
subsequences. The initial weights of these four sequences are listed in formula (3.2) of [15]. It is elementary to verify that, for \( w \in \mathbb{Z} \) and hence \( a = w + k \in \mathbb{Z} \), the condition that \( \frac{a}{b} + \frac{c}{d} \) equals one of these four weights is equivalent to \( a \geq 0 \), respectively \( a \leq -2 \) and \( a + b \geq -1 \), respectively \( a + b \leq -3 \) and \( a + b + c \leq -2 \), respectively \( a + b + c \leq -4 \). The conditions in the theorem just express these in terms of \( w, k, 1 \) and \( \ell \).

The cases in which the representations corresponding to the relative homology groups have singular infinitesimal character are also listed in Example 3.2 of [15]. The conditions in the theorem just equivalently express these in terms of \( w, k, 1 \) and \( \ell \).

Remark 5.5. At this stage, it is not clear whether the operators in the sequence actually are non–zero. This will follow from the results that we obtain resolutions in the case of path geometries below. One can actually go much further in that direction and obtain a description of the principal parts of the operators using only representation theory.

In order to do this in the case discussed here, it suffices to verify that the orders of the two operators in the sequences are \( \ell + 1 \) for \( D_1 \) and \( k + 1 \) for \( D_2 \). Now for \( D_1 : \Gamma(W_0) \to \Gamma(W_1) \), the target bundle \( W_1 = S^{k+\ell+1} \otimes W_0 \), since \( W_0 = S^k \). Indeed it corresponds to the highest weight component in the tensor product of the inducing representations. Since \( D_1 \) is an invariant differential operator, on the homogeneous model its symbol must be an equivariant map of homogeneous vector bundles. Hence it is induced by a \( Q_0 \)-equivariant map between the inducing representations.

Taking into account that the operators are constructed from vertical derivatives, it follows that the symbol is defined on \( S^{\ell+1} \otimes W_0 \), so \( Q_0 \)-equivariancy pins it down up to a constant multiple.

For the second operator \( D_2 : W_1 \to W_2 \), the situation is only slightly more complicated. Here the dual bundles are \( W_2^* = S^\ell V(-w-2k-2, k+\ell+3) \) and \( W_1^* = S^{k+\ell+1} V(-w,-\ell-2k) \). From the discussion of the relation between representations and bundles above, we see that \( V^* \cong V(2,-3) \), so \( S^{k+1} V^* \cong S^{k+1} \). Hence \( W_1^* \) is naturally contained in the tensor product \( S^{k+1} \otimes W_2^* \) corresponding to the highest weight component in the tensor product of the inducing representations. Now the unique (up to scale) \( Q_0 \)-homomorphism \( S^{k+1} \otimes W_2^* \to W_1^* \) dualizes to a unique \( Q_0 \)-homomorphism \( S^{k+1} \otimes W_1 \to W_2 \). Knowing the order, one again obtains the symbol on the homogeneous model, up to a constant factor. Finally, one argues that passing to a curved geometry does not change the principal part of the operator.

5.6. Relative BGG resolutions on path geometries. The concepts of correspondence spaces and local twistor spaces as discussed in Sections 4.3 and 4.7 arise very naturally in the case of generalized path geometries. To discuss correspondence spaces, recall that a regular normal parabolic geometry of type \((G,P)\) on a manifold \( N \) of dimension \( n+1 \) is equivalent to a projective structure on \( N \). Such a structure is given by an equivalence class of torsion–free linear connections on \( TN \), which share the same geodesics up to parametrization. The
unparametrized geodesics of the connections in the class define a path geometry on $N$ and the correspondence space $CN$ is the associated geometry on $P(TN)$. In the language of systems of second order ODEs, local isomorphism to a correspondence space thus is related to realizability of a system as a geodesic equation.

On the other hand, let us recall the description of harmonic curvature components for generalized path geometries from Section 4.4.3 of [11]. In all dimensions $n \geq 2$, there is one harmonic torsion $\tau_{T^*} \in \Gamma(T^*N/H*) \otimes \text{End}(V)$ and a curvature, which we denote by $\gamma \in \Gamma(T^*N/H* \otimes \text{End}(V))$. For $n = 2$, there is an additional torsion $\tau_r \in \Gamma(\Lambda^2 T^*N \otimes E)$. Using these, we can formulate the conditions for existence of local twistor spaces and local isomorphism to a correspondence space.

**Lemma 5.5.** Let $(M, E, V)$ be a generalized path geometry of dimension $2n + 1$ with $n \geq 2$. Then we have

1. This relative tangent bundle $T_p M$ is the bundle $V$. If $n > 2$, this bundle is always involutive, for $n = 2$, this is the case if and only if $\tau_r = 0$.
2. Involutivity of $T_p M = V$ is equivalent to the fact that the geometry is locally isomorphic to a path geometry on a local twistor space.
3. If $T_p M = V$ is involutive, then the geometry is locally isomorphic to a correspondence space for a projective structure on a local twistor space if and only if the harmonic curvature component $\gamma$ vanishes identically.

**Proof.** Section 4.4.4 of [11] contains a proof of (1). If $V$ is involutive, then for a local leaf space $\psi : U \to N$, define $\tilde{\psi} : U \to PTN$ by mapping $x \in U$ to the line $T_x \psi \cdot E_x \subset T_{\psi(x)} N$. Then it is shown in Proposition 4.4.4 of [11] that for sufficiently small $U$, the map $\tilde{\psi}$ is an open embedding whose tangent map sends $V$ to the vertical bundle and $E \oplus V$ to the tautological bundle, so (2) follows.

If $\gamma = 0$ and (if $n = 2$), also $\tau_r = 0$, then the harmonic curvature $\kappa_b$ evidently satisfies the assumptions of part (2) of Proposition 4.18 which then implies (3).  

The last ingredient we need to make our results on resolutions explicit is notation for some tensor bundles. For a smooth manifold $N$ of dimension $n$, we define $E[w]$ to be the bundle of densities of weight $\frac{n}{n+1}$ on $N$ (so $E[-n-1]$ is the bundle of volume densities). Adding “$[w]$” to the name of a tensor bundle will indicate a tensor product with $E[w]$. Now for $k, \ell \in \mathbb{N}$, consider the tensor product $S^k T^* N \otimes S^\ell T N$. If both $k$ and $\ell$ are positive, then there is a unique contraction from this bundle to $S^{k-1} T^* N \otimes S^{\ell-1} T N$, and we denote by $T_k^\ell$ the kernel of this contraction. We further define $T_k^0 := S^k T^* N$ and $T_0^\ell := S^\ell T N$.

**Proposition 5.6.** Let $(M, E, V)$ be a generalized path geometry of dimension $2n + 1$ with $n \geq 2$ such that the relative tangent bundle $T_p M = V$ is involutive. Let $\mathcal{V}$ be a completely reducible representation of $P, VM \to M$ the corresponding relative tractor bundle and $\nabla_{\rho^p \mathcal{V}}$ the relative tractor connection on $VM$.

1. The relative BGG sequence induced by $\mathcal{V}$ is a complex and a fine resolution of the sheaf $\ker(\nabla_{\rho^p \mathcal{V}})$. In particular, if $\mathcal{V}$ is an irreducible component of $H_1(p_-, \nabla)$ for a representation $\nabla$ of $\mathfrak{g}$, then one obtains a subcomplex in a curved BGG sequence.
2. If $M$ is the correspondence space $CN$ for a projective structure on a manifold $N$ of dimension $n + 1$, then the sheaf $\ker(\nabla_{\rho^p \mathcal{V}})$ is globally isomorphic to the
pullback of the sheaf of smooth sections of the tensor bundle over \( N \) induced by the representation \( \mathbb{V} \).

3. If \( M \cong P(TN) \) is a path geometry on a manifold \( N \) of dimension \( n + 1 \), then the isomorphism of sheaves from (2) holds locally (with the same tensor bundle).

4. If \( n = 2 \) and \( \tau_V = 0 \), then for the sequence (5.4) of invariant differential operators in Theorem 5.4 the tensor bundle from parts (2) and (3) is \( T^k_{\ell}[w + 2\ell] \).

Proof. If \( V \) is involutive, then \( \tau_V = 0 \), and the harmonic curvature \( \kappa^h \) visibly satisfies the assumptions of part (1) of Proposition 4.18. Hence we conclude that \( \kappa_\rho = 0 \) and we may apply Theorem 4.11 to obtain (1). As noted in the proof of Lemma 5.5 in the case of a correspondence space, we can apply part (2) of Proposition 4.18 so (2) again follows from Theorem 4.11.

Assuming that \( M \) is a path geometry over \( N \) (i.e. that \( N \) is a global twistor space for \( M \)), the fact that \( \kappa_\rho = 0 \) implies that we can apply Lemma 4.12. This shows that locally the Cartan bundle \( G \to M \) is isomorphic to a principal \( P \)-bundle \( F \to N \), and the sheaf \( \ker(\nabla^h) \) can be identified with \( \Gamma(F \times_P V) \).

Finally, we can apply the well known result that lowest non–zero homogeneous component of the curvature of any regular normal parabolic geometry is harmonic. Since \( \tau_V \) vanishes identically, the list of harmonic components above shows that the next lowest possible homogeneity is two, and this is represented by \( \tau_E \). Consequently, only components of homogeneity at least three contribute to values of \( \kappa \) if one of the entries is from \( T_\rho M \). But this immediately implies that for \( \xi \in \Gamma(T_\rho M) \) and \( \eta \in X(M) \), we get \( \kappa(\xi, \eta) \in \mathcal{G} \times_Q q \subset \mathcal{G} \times_Q p \). Hence we can apply part (2) of Theorem 4.11 which shows that the Cartan connection induces a soldering form \( \theta \in \Omega^1(F, g/p) \). This implies that \( F \times_p (g/p) \cong TN \), and hence the correspondence between completely reducible representations of \( P \) and tensor bundles is the same as in the case of a projective structure on \( N \). This completes the proof of (3).

4. The representation \( g/p \) is the \( p \)-irreducible quotient of the adjoint representation, so the two representations have the same lowest weight. Hence \( g/p \) corresponds to the weight \( k \cdot -2 -1 0 \). Similarly, one verifies that the dual \( (g/p)^* \cong p_+ \) corresponds to the weight \( -2 1 0 \). Hence the tensor bundle \( T^k_{\ell} \), which is induced by the highest weight component in \( S^\ell(g/p)^* \otimes S^k(g/p) \) corresponds to the weight \( k -2 \ell k \), which implies the result. \( \square \)

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