Operator Lie Algebras of Rotations and Transformations in White Noise

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Abstract

The infinitesimal generator of a one-parameter subgroup of the infinite
dimensional rotation group associated with the complex Gelfand triple
\((E) \subset L^2(E^*, \mu) \subset (E)^*\) is of the form

\[ R_\kappa = \int_{T \times T} \kappa(s, t)(a^*_s a_t - a^*_t a_s) \, ds \, dt \]

where \(\kappa \in E \otimes E^*\) is a skew-symmetric distribution. Hence \(R_\kappa\) is twice
the conservation operator associated with a skew-symmetric operator \(S\).
The Lie algebra containing \(R_\kappa\), identity operator, annihilation operator,
creation operator, number operator, (generalized) Gross Laplacian is dis-

cussed. We show that this Lie algebra is associated with the orbit of the
skew-symmetric operator \(S\).

1 Introduction

White noise calculus was initiated by Hida in 1975 [10] and became a useful
tool in many different areas, such as Mathematical Physics and Finance, see e.g. [12] [25] [20] [13].

The root of white noise calculus is to switch a functional of Brownian motion
\(f(B(t); t \in \mathbb{R})\) with one of white noise \(\phi(\dot{B}(t); t \in \mathbb{R})\), where \(\dot{B}(t)\) is a
time derivative of a Brownian motion \(B(t)\). We may thereby regard \(\{\dot{B}(t)\}\)
as a collection of infinitely many independent random variables and hence
Let \( S(\mathbb{R}^n) \) and \( S'(\mathbb{R}^n) \) be the Schwartz space consisting of rapidly decreasing \( C^\infty \)-functions and the space of tempered distributions, respectively. The mathematical framework of white noise calculus is based on an infinite dimensional analogue of Schwartz’ distribution theory, where the roles of the Lebesgue measure on \( \mathbb{R}^n \) and the Gelfand triple \( S(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \subset S'(\mathbb{R}^n) \) are played by the Gaussian measure \( \mu \) on \( E^\ast \) (the topological dual space of a nuclear space \( E \)) and \( (E) \subset (L^2(E) \subset (E)^*, \mu \subset (E)^* \), respectively. Furthermore, in white noise calculus the coordinate differential operators are given by \( a_t \), where \( t \) runs over a time parameter space \( T \).

The finite dimensional Laplacian \( \Delta_n \) on \( \mathbb{R}^n \) admits two expressions:

\[
\Delta_n = \sum_{i=1}^{n} \left( \frac{\partial}{\partial x_i} \right)^2 = -\sum_{i=1}^{n} \left( \frac{\partial}{\partial x_i} \right)^* \frac{\partial}{\partial x_i} ,
\]

when \( \Delta_n \) acts on \( S(\mathbb{R}^n) \). By virtue of a general theory established in [11], the Gross Laplacian and the number operator are expressed as follows:

\[
\Delta_G = \int_T a_t^2 dt, \quad N = \int_T a_t^* a_t dt.
\]

These are infinite dimensional analogues of the finite dimensional Laplacian. However, unlike the finite dimensional case, \( \Delta_G \) and \( N \) are completely different from each other [6]. It was Gross [8] and Piech [28] who initiated the study of the Gross Laplacian and the number operator, as natural infinite-dimensional analogues of a finite-dimensional Laplacian, in connection with the Cauchy problem in infinite-dimensional abstract Wiener space. Based on white noise analysis, Kuo [19] formulated the Gross Laplacian \( \Delta_G \) and the number operator \( N \) as continuous linear operators acting on white noise functionals. Hida, et.al. [11] developed a general theory of operators acting on white noise functionals. As a particular case, they discussed infinite dimensional rotations in detail. It was proved that \( \Delta_G \) and \( N \) are in essence the only operators which are rotation-invariant [24]. Moreover, characterization theorems for \( \Delta_G \), \( N \) and the Euler operator \( \Delta_G + N \) were given in [9]. Chung, et. al. [5] studied their generalizations \( N(f) \) and \( \Delta_G(g) \) which are called second order differential operators of diagonal type. A generalized Gross Laplacian \( \Delta_G(K) \), called the \( K \)-Gross Laplacian, was introduced in [3]. Later on, the \( K \)-Gross Laplacian, the second quantization and the differential second quantization were studied within the framework of nuclear algebras of entire functions [1].

In white noise analysis, Lie algebraic approach in studying the infinite dimensional Laplacians \( N \) and \( \Delta_G \) have been discussed in [7]. In 1998, Chung and Ji [5] constructed a two-parameter transformation group which is the two-dimensional Lie group associated with the Lie algebra \( \mathbb{C}\Delta_G + \mathbb{C}N \). Obata [27] discussed all possible two-dimensional complex Lie algebras containing \( N \) and constructed the associated Lie groups. Chung and Chung [2] studied the Lie algebras of Wick derivations on \( (E)^* \). Furthermore, Hida, et.al. [12] obtained a five-dimensional complex Lie algebra \( \mathfrak{g} \) generated by the identity operator.
Id, $\Delta_G$, $N$, and the infinitesimal generators of some differentiation and multiplication operators. Later on, Chung and Ji [7] explicitly constructed a Lie group associated with $h$. They showed that this Lie algebra is spanned by $\text{Id}, a(\zeta), a^* (\zeta), N$ and $\Delta_G$, where $a(\zeta)$ and $a^* (\zeta)$ are the annihilation and creation operators, respectively. In 2016, Ji and Sinha [18], studied a class of fundamental quantum stochastic processes induced by the generators of a six dimensional non-solvable Lie $*$-algebra consisting of all linear combinations of the identity operator, generalized Gross Laplacian and its adjoint, annihilation operator, creation operator, conservation operator.

The paper is organized as follows: In Section 2 we assemble standard notations used in white noise calculus and in Section 3 we discuss the infinite dimensional rotation group. In Section 4 we review the most basic notions in quantum white noise calculus. In Section 5 we survey some known commutation relations of white noise operators. Section 6 is devoted to a study of the Lie algebra generated by the identity operator, annihilation operator, creation operator, Gross Laplacian, generalized Gross Laplacian, number operator and rotation (conservation) operator. We show that this Lie algebra is associated with the orbits of a skew-symmetric operator.

2 White noise calculus

If $\mathcal{X}$ is a locally convex space over $\mathbb{R}$, its complexification is denoted by $\mathcal{X}_\mathbb{C}$ [24]. The following construction of the complex Gelfand triple is lifted from [6]. The whole discussion is based on the special choice of a real Gelfand triple

$$E = S(\mathbb{R}) \subset H = L^2(\mathbb{R}, dt) \subset E^* = S'(\mathbb{R}).$$

(2)

However, $\mathbb{R}$ can be replaced with $\mathbb{R}^n$ with no essential change (sometimes more interesting for applications). It is noteworthy that (2) is constructed from the differential operator $A = 1 + t^2 - \frac{d^2}{dt^2}$. In fact, $E = S(\mathbb{R})$ is identified (up to null functions) with the space of functions $\xi \in H$ such that $|\xi|_p = |A^p \xi|_0 < \infty$ for any $p \in \mathbb{R}$, where $|\cdot|_p$ stands for the norm of $H$, and the topology of $E$ is given by the norms $|\cdot|_p, p \in \mathbb{R}$. Since $A$ is a positive self-adjoint operator with Hilbert-Schmidt inverse, $E$ becomes a countable Hilbert nuclear space. By definition $E^*$ is the strong dual space of $E$. The canonical bilinear form on $E^* \times E$ and the real inner product of $H$ are denoted by the same symbol $\langle \cdot, \cdot \rangle$ because they are consistent. The Gaussian measure $\mu$ is by definition a unique probability measure on $E^*$ to which characteristic function is

$$\exp \left( -\frac{1}{2} |\xi|^2_0 \right) = \int_{E^*} e^{i \langle x, \xi \rangle} \mu(dx), \quad \xi \in E.$$

The probability space $(E^*, \mu)$ is called the white noise space or the Gaussian space. With each $\xi \in E^*_\mathbb{C}$, we associate a function on $E^*$ defined by

$$\phi_\xi (x) = \exp \left( \langle x, \xi \rangle - \frac{1}{2} \langle \xi, \xi \rangle \right), \quad x \in E^*.$$
which is called an exponential vector. The correspondence
\[ \phi_\xi \mapsto \left( 1, \frac{\xi}{1!}, \frac{\xi^2}{2!}, \frac{\xi^3}{3!}, \ldots \right), \quad \xi \in E_\mathbb{C}, \]
is uniquely extended to a unitary isomorphism between \( L^2(E^*, \mu) \) and the Boson Fock space over \( K(E_\mathbb{C}) \), denoted by \( \Gamma(K(E_\mathbb{C})) \), which is the celebrated Wiener-Ito-Segal isomorphism. If \( \phi \in L^2(E^*, \mu) \) and \( (f_n) \in \Gamma(K(E_\mathbb{C})) \) are related, we write \( \phi \sim (f_n) \) simply. In that case,

\[ ||\Phi||_0^2 = \int_{E^*} |\phi(x)|^2 \mu(dx) = \int_{n=0}^{\infty} n! ||f_n||^2. \]

For any \( p \in \mathbb{R} \) we put

\[ ||\phi||_p^2 = \int_{n=0}^{\infty} n! ||f_n||^2 = \int_{n=0}^{\infty} n! (A_\mu^n)^p_{|0}, \quad \phi \sim (f_n). \]

Let \( (E) \) be the subspace of functions \( \phi \in L^2(E^*, \mu) \) such that \( ||\phi||_p < \infty \) for all \( p \). Then \( (E) \) becomes a nuclear Frechet space with the defining seminorms \( || \cdot ||_p, p \in \mathbb{R} \). The dual space \( (E)^* \) consists of all elements \( \Phi \sim (F_n) \) such that \( F_n \in (E_\mathbb{C}^\otimes n)_{\text{sym}} \) and \( ||\Phi||_{-p} < \infty \) for some \( p \geq 0 \). We thereby obtain a complex Gelfand triple:

\( (E) \subset L^2(E^*, \mu) \subset (E)^*. \)

Elements in \( (E) \) and \( (E)^* \) are called a test (white noise) function and a generalized (white noise) function, respectively.

A continuous linear operator from \( (E) \) into \( (E)^* \) is called a white noise operator. The space of white noise operators is denoted by \( \mathcal{L}((E), (E)^*) \) and is equipped with the bounded convergence topology. It is noted that \( \mathcal{L}((E), (E)) \) is a subspace of \( \mathcal{L}((E), (E)^*) \). With each \( y \in E_\mathbb{C}^* \) we may associate an annihilation operator \( D_y \in L((E), (E)) \) which is uniquely determined by

\[ D_y \phi_\xi = \langle y, \xi \rangle \phi_\xi, \quad \xi \in E_\mathbb{C}. \]

Since \( \delta_t \in E^* \) for any \( t \in \mathbb{R} \),

\[ a_t = D_{\delta_t}, \quad t \in \mathbb{R}, \]

belongs to \( L((E), (E)) \). This is called the annihilation operator at a point \( t \in \mathbb{R} \). The creation operator at a point is by definition the adjoint \( a_t^* \in L((E)^*, (E)^*) \).

It is known that \( a_t \) is a differential operator along the direction \( \delta_t \), namely

\[ a_t \phi(x) = \lim_{\theta \to 0} \frac{\phi(x + \theta \delta_t) - \phi(x)}{\theta}, \quad \phi \in (E), t \in \mathbb{R}, x \in E^*. \]

**Definition 2.1.** [20] With each \( \kappa \in (E_\mathbb{C}^\otimes (l+m))^* \) we may associate an integral kernel operator whose formal expression is given by

\[ \Xi_{l,m}(\kappa) = \int_{T^{l+m}} \kappa(s_1, \ldots, s_l, t_1, \ldots, t_m) a_{s_1}^* \cdots a_{s_l}^* a_{t_1} \cdots a_{t_m} ds_1 \cdots ds_1 dt_1 \cdots dt_m. \]

where \( \kappa \) is called the kernel distribution.
Proposition 2.2. \[25\] Let \( \phi \in (E) \) be given with Wiener-Ito expansion:

\[
\phi(x) = \sum_{n=0}^{\infty} \langle x^{\otimes n} :, f_n \rangle.
\]

Then, for \( \kappa \in (E^{\otimes (l+m)})^* \) we have

\[
\Xi_{l,m}(\kappa) \phi(x) = \sum_{n=0}^{\infty} \frac{(n+m)!}{n!} \langle x^{\otimes (l+n)} :, \kappa \otimes_m f_{n+m} \rangle.
\]

Lemma 2.3. \[6\] Let \( \kappa \in (E^{\otimes (l+m)})^* \). Then \( \Xi_{l,m}(\kappa) \in L((E), (E)) \) if and only if \( \kappa \in (E^{\otimes l}) \otimes (E^{\otimes m})^* \). In particular, \( \Xi_{0,m} \in L((E), (E)) \) for any \( \kappa \in (E^{\otimes m})^* \).

Proposition 2.4. \[25\] For any \( T \in L(E_C, E_C) \), there exists a unique operator \( \Gamma(T) \in L((E), (E)) \) such that

\[
\Gamma(T) \phi(x) = \sum_{n=0}^{\infty} \langle x^{\otimes n} :, T^{\otimes n} f_n \rangle
\]

In general, for \( T \in L(E_C, E_C) \) we define an operator \( d\Gamma(T) \) on \((E)\). Suppose \( \phi \in (E) \) is given as

\[
\phi(x) = \sum_{n=0}^{\infty} \langle x^{\otimes n} :, f_n \rangle, \quad x \in E^*,
\]

as usual. Then we put

\[
d\Gamma(T) \phi(x) = \sum_{n=0}^{\infty} \langle x^{\otimes n} :, \gamma_n(T)f_n \rangle,
\]

where

\[
\begin{cases}
\gamma_n(T) = \sum_{k=0}^{n-1} I^{\otimes k} \otimes T \otimes I^{\otimes (n-1-k)}, \quad n \geq 1, \\
\gamma_0(T) = 0.
\end{cases}
\]

Proposition 2.5. \[25\] \( d\Gamma(T) \in L((E), (E)) \) for any \( T \in L(E_C, E_C) \).

Definition 2.6. \[25\] \( \Gamma(T) \) and \( d\Gamma(T) \) are called the second quantization and differential second quantization of \( T \), respectively.

If \( K \) and \( \kappa \in E_C \otimes E_C^* \) are related by the kernel theorem, i.e.

\[
\langle K \xi, \eta \rangle = \langle \kappa, \eta \otimes \xi \rangle, \quad \xi, \eta \in E_C,
\]

then \( d\Gamma(K) = \Xi_{1,1}(\kappa) \).
2.1 Second order differential operators of diagonal type

Second order differential operators of diagonal type and their generalizations were discussed in [6]. Let \( \tau \in (E_C \otimes E_C)^* \) be defined by

\[
\langle \tau, \eta \otimes \xi \rangle = \langle \xi, \eta \rangle, \quad \xi, \eta \in E_C.
\]

In fact, \( \tau \in E_C \otimes E_C^* \) since \( \tau \) corresponds to the identity operator under the canonical isomorphism \( E_C \otimes E_C^* \cong \mathcal{L}(E_C, E_C) \). Thus, by Lemma 2.3

\[
N = \Xi_{1,1}(\tau) = \int_{\mathbb{R}^2} \tau(s,t)a_s^*a_t dsdt = \left( \int_{\mathbb{R}} a_s^*a_t dt \text{ for simplicity} \right)
\]

belongs to \( \mathcal{L}((E), (E)) \). This is called the number operator. On the other hand,

\[
\Delta_G = \Xi_{0,2}(\tau) = \int_{\mathbb{R}^2} \tau(t_1,t_2)a_{t_1}a_{t_2} dt_1dt_2 = \left( \int_{\mathbb{R}} a_t^2dt \text{ for simplicity} \right)
\]

also belongs to \( \mathcal{L}((E), (E)) \) and is called the Gross Laplacian.

2.2 Convolution

Suppose \( S_1 \in \mathcal{L}(E_C, E_C) \) and \( S_2 \in \mathcal{L}(E_C, E_C^*) \). Let \( f_1 \in E_C \otimes E_C^* \) and \( f_2 \in (E_C \otimes E_C)^* \) be the corresponding elements, respectively, see [3]. Then we denote by \( f_2 * f_1 \) the element of \( (E_C \otimes E_C)^* \) corresponding to \( S_2S_1 \in \mathcal{L}(E_C, E_C^*) \). It is noted in [6] that

\[
f_2 * f_1(s,t) = \int_{\mathbb{R}} f_2(s,u)f_1(u,t)du
\]

in a generalized sense.

**Lemma 2.7.** [6] The convolution \( (f_2, f_1) \mapsto f_2 * f_1 \) gives a separately continuous bilinear map from \( (E_C \otimes E_C)^* \times (E_C \otimes E_C^*) \) into \( (E_C \otimes E_C)^* \), and from \( (E_C \otimes E_C^*) \times (E_C \otimes E_C^*) \) into \( E_C \otimes E_C^* \).

3 Infinite dimensional rotation group

Our discussion is based on a Gelfand triple \( E \subset H \subset E^* \). We denote by \( GL(E) \) the group of all linear homeomorphisms from \( E \) onto itself. Then \( GL(E) \subset L(E, E) \). We now set

\[
O(E; H) = \{ g \in GL(E) : \|g\xi\|_0 = \|\xi\|_0 \text{ for all } \xi \in E \},
\]

which is a subgroup of \( GL(E) \). The group \( O(E; H) \) is called the infinite dimensional rotation group (associated with the Gelfand triple \( E \subset H \subset E^* \)). The Gaussian measure \( \mu \) is invariant under \( O(E; H) \) [25]. Similarly we put

\[
U((E); (L^2)) = \{ U \in GL((E)) : \|U\phi\|_0 = \|\phi\|_0, \text{ for all } \phi \in (E) \}.
\]

With each \( g \in O(E; H) \) we associate its second quantization \( \Gamma(g) \).
Definition 3.1. [25] We say that a continuous operator from \((E)\) into \((E)^*\) is rotation-invariant if
\[
\Gamma(g)\Xi = \Xi \quad \text{for all} \quad g \in O(E; H). \tag{4}
\]

Remark 3.2. [25] The condition (4) for \(\Xi \in \mathcal{L}((E), (E))\) is equivalent to the following:
\[
[\Gamma(g), \Xi] = 0 \quad \text{for all} \quad g \in O(E; H).
\]

In the following discussion let \(X\) be a nuclear Frechet space with defining Hilbertian seminorms \(\{\|\cdot\|\}_{\alpha \in A}\) taking \(X = E\) or \(X = (E)\) into consideration.

Definition 3.3. [25] A one-parameter subgroup \(\{g_\theta\}_{\theta \in \mathbb{R}} \subset GL(X)\) is called differentiable if
\[
\lim_{\theta \to 0} \frac{g_\theta \xi - \xi}{\theta}
\]
converges in \(X\) for any \(\xi \in X\). In that case a linear operator \(X\) from \(X\) into itself is defined by
\[
X\xi = \lim_{\theta \to 0} \frac{g_\theta \xi - \xi}{\theta}, \quad \xi \in X.
\]
This operator \(X\) is called the infinitesimal generator of the differentiable one-parameter subgroup \(\{g_\theta\}_{\theta \in \mathbb{R}} \subset GL(X)\).

Proposition 3.4. [25] Let \(\{g_\theta\}_{\theta \in \mathbb{R}} \subset GL(X)\) be a differentiable one-parameter subgroup. Then its infinitesimal generator \(X\) is always continuous, i.e., \(X \in \mathcal{L}(X)\).

Definition 3.5. [25] A differentiable one-parameter subgroup \(\{g_\theta\}_{\theta \in \mathbb{R}} \subset GL(X)\) with infinitesimal generator \(X\) is called regular if for any \(\alpha \in A\) there exists \(\beta \in A\) such that
\[
\lim_{\theta \to 0} \sup_{\|\xi\| \leq 1} \left\| \frac{g_\theta \xi - \xi}{\theta} - X\xi \right\|_\alpha = 0.
\]

Theorem 3.6. [25] For \(y \in E^*\), let \(D_y\) and \(T_y\) be the differential and translation operators, respectively. Then \(\{T_{\theta y}\}_{\theta \in \mathbb{R}}\) is a regular one-parameter subgroup of \(GL(E)\) with infinitesimal generator \(D_y\).

Remark 3.7. [25] The infinitesimal generator \(X\) of a regular one-parameter subgroup \(\{g_\theta\}_{\theta \in \mathbb{R}} \subset GL(X)\) is skew-symmetric in the sense that
\[
\langle X\xi, \eta \rangle = -\langle \xi, X\eta \rangle, \quad \xi, \eta \in E. \tag{5}
\]
Let \(X \in \mathcal{L}(E_C, E_C^*)\) be skew-symmetric in the sense of [29]. By the canonical isomorphism \(E_C \otimes E_C^* \cong \mathcal{L}(E_C, E_C)\) there exists \(\kappa \in E_C \otimes E_C^*\) such that
\[
\langle \kappa, \eta \otimes \xi \rangle = \frac{1}{2} \langle \eta, X\xi \rangle, \quad \xi, \eta \in E_C.
\]
Since \(X\) is skew-symmetric, we have
\[
\langle \kappa, \eta \otimes \xi \rangle = \frac{1}{2} \langle \eta, X\xi \rangle = -\frac{1}{2} \langle \xi, Y\eta \rangle = -\langle \kappa, \xi \otimes \eta \rangle. \tag{6}
\]
Theorem 3.8. [25] Let \( \{g_\theta\}_{\theta \in \mathbb{R}} \) be a regular one-parameter subgroup of \( O(E; H) \) with infinitesimal generator \( X \). Then \( \{\Gamma(g_\theta)\}_{\theta \in \mathbb{R}} \) is a regular one-parameter subgroup of \( U((E); L^2) \) with infinitesimal generator \( d\Gamma(X) \). Moreover, there exists a skew-symmetric distribution \( \kappa \in E \otimes E^* \) such that
\[
d\Gamma(X) = \int_{T \times T} \kappa(s, t)(a_t^* a_t - a_s^* a_s)dsdt = 2\Xi_{1,1}(\kappa). \quad (7)
\]

4 Quantum white noise calculus

For \( f \in E^* \) we define white noise operators:
\[
a(f) = \Xi_{0,1}(f) = \int_T f(t) a_t dt, \quad a^*(f) = \Xi_{1,0}(f) = \int_T f(t) a_t^* dt, \quad (8)
\]
which are called respectively the annihilation and creation operators associated with \( f \). If \( \zeta \in E \), then \( a(\zeta) \) extends to a continuous linear operator from \( (E)^* \) into itself (denoted by the same symbol) and \( a^*(\zeta) \) (restricted to \( E \)) is a continuous linear operator from \( E \) into itself [15].

Lemma 4.1. [17] For \( \zeta \in E \), both \( a(\zeta) \) and \( a^*(\zeta) \) belong to \( \mathcal{L}((E), (E)^*) \cap \mathcal{L}((E)^*, (E)^*) \).

Thus, for any white noise operator \( \Xi \in \mathcal{L}((E), (E)^*) \) and \( \zeta \in E \), the commutators
\[
[a(\zeta), \Xi] = a(\zeta)\Xi - \Xi a(\zeta), \quad -[a^*(\zeta), \Xi] = \Xi a^*(\zeta) - a^*(\zeta)\Xi,
\]
are well-defined white noise operators, i.e., belongs to \( \Xi \in \mathcal{L}((E), (E)^*) \). We define
\[
D^+_\zeta \Xi = [a(\zeta), \Xi], \quad D^-_\zeta \Xi = -[a^*(\zeta), \Xi].
\]

Definition 4.2. [13] \( D^+_\zeta \Xi \) and \( D^-_\zeta \Xi \) are respectively called the creation derivative and annihilation derivative of \( \Xi \), and both together the quantum white noise derivatives (qwn-derivatives for brevity) of \( \Xi \).

Theorem 4.3. [15] The map
\[
E \times \mathcal{L}((E), (E)^*) \to \mathcal{L}((E), (E)^*), \quad (\zeta, \Xi) \mapsto D^+_\zeta \Xi
\]
is continuous bilinear.

Corollary 4.4. [15] For each \( \zeta \in E \), the qwn-differential operator \( D^+_\zeta \) is a continuous operator from \( \mathcal{L}((E), (E)^*) \) into itself.

The quantum white noise derivatives of the generalized Gross Laplacian and conservation operator are given in [16, 17]. For each \( S \in \mathcal{L}(E, E^*) \), by the kernel theorem there exists a unique \( \tau_S \in (E \otimes E)^* \) such that
\[
\langle \tau_S, \eta \otimes \xi \rangle = \langle S\xi, \eta \rangle, \quad \xi, \eta \in E.
\]
The integral kernel operator
\[
\Delta_G(S) = \Xi_{0,2}(\tau_S) = \int_{T \times T} \tau_S(s, t)a_s a_t dsdt
\]
is called the generalized Gross Laplacian associated with \(S\). Note that \(\Delta_G(S) \in \mathcal{L}(E, (E))\). The usual Gross Laplacian is \(\Delta_G = \Delta_G(I)\). The integral kernel operator
\[
\Lambda(S) = \Xi_{1,1}(\tau_S) = \int_{T \times T} \tau_S(s, t)a_s^* a_t dsdt
\]
is called the conservation operator associated with \(S\). In general, \(\Lambda(S) \in \mathcal{L}(E, (E)^*)\). Note that \(N = \Lambda(I)\) is the number operator.

**Lemma 4.5.** [16] For \(S \in \mathcal{L}(E, E^*)\) and \(\zeta \in E\), we have
\[
D^+_\zeta \Delta_G(S) = 0, \quad D^-_\zeta \Delta_G(S) = a(S\zeta) + a(S^* \zeta), \quad (9)
\]
\[
D^+_\zeta \Lambda(S) = a(S^* \zeta), \quad D^-_\zeta \Lambda(S) = a^*(S\zeta). \quad (10)
\]

**5 Commutation relations**

We have the so-called canonical commutation relation:
\[
[a_s, a_t] = 0, \quad [a_s^*, a_t^*] = 0, \quad [a_s, a_t^*] = \delta_s(t)Id, \quad s, t \in \mathbb{R}, \quad (11)
\]
where the last relation is understood in a generalized sense.

**Proposition 5.1.** [25] For \(\xi \in E_C\) and \(y \in E_C^*\), it holds that
\[
[\Xi_{0,1}(y), \Xi_{1,0}(\xi)] = \langle y, \xi \rangle Id
\]
where \(Id\) is the identity operator on \((E)\).

**Theorem 5.2.** [7] For \(\zeta \in E_C\), we have the following commutation relations:
1. \([a(\zeta), a^*(\zeta)] = \langle \zeta, \zeta \rangle Id,\)
2. \([a(\zeta), N] = a(\zeta),\)
3. \([a(\zeta), \Delta_G] = 0,\)
4. \([a^*(\zeta), N] = -a^*(\zeta),\)
5. \([a^*(\zeta), \Delta_G] = -2a(\zeta),\)
6. \([\Delta_G, N] = 2\Delta_G.\)
7. \([N, \Xi_{0,m}(\kappa)] = -m \Xi_{0,m}(\kappa), \quad \kappa \in (E_C^n)^*.\)

**Lemma 5.3.** [6] It holds that
\[
[\Xi_{1,1}(f_1), \Xi_{1,1}(f_2)] = \Xi_{1,1}(f_1 * f_2 - f_2 * f_1), \quad f_1, f_2 \in E_C \otimes E_C^*.
\]
Remark 5.4. In general, $\Xi_{0,2}(\kappa)$ with $\kappa \in (E_C \otimes E_C)^*$ involves only annihilation operators, and so they commute each other.

Theorem 5.5. For each nonzero $\zeta \in E_C$, let $h = \langle \text{Id}, a(\zeta), a^*(\zeta), N, \Delta_G \rangle$. Then $h$ is a five-dimensional non-nilpotent solvable complex Lie algebra.

Theorem 5.6. Let $g = \langle N, \Xi_{0,m_1}(\kappa_1), \ldots, \Xi_{0,m_n}(\kappa_n) \rangle$ be the complex vector space spanned by $N$ and $\Xi_{0,m_i}(\kappa_i)$, $i = 1, \ldots, n$, where for each $i = 1, \ldots, n, \kappa \in (E^*_C)^{\otimes m_i}$. Then $g$ is a $(n + 1)$-dimensional non-nilpotent solvable complex Lie algebra.

Theorem 5.7. Let $K \in L(H, H)$ be a symmetric Hilbert-Schmidt operator, $L \in L(H, H)$ a self-adjoint operator, and $\zeta \in H$ satisfying that

\[ KL = \mathcal{K}K = K, \quad \mathcal{K}K = L, \quad \mathcal{K}\zeta = \zeta, \quad L\zeta = \zeta. \]

Then, the linear span of $\text{Id}, a(\zeta), a^*(\zeta), \Lambda(L), \Delta_G(K), \Delta^*_G(K)$ becomes a six dimensional non-solvable Lie $*$-algebra.

Remark 5.8. In Theorem 5.7, the operators $K, L \in L(H, H)$ satisfy that $\mathcal{K}K = K$ and $\mathcal{K}K = L$. Therefore, we see that $L\mathcal{K}K = \mathcal{K}K$ and so $L^2 = L$. Since $L$ is a self-adjoint operator, $L$ is a projection.

6 Main Results

Motivated by the result in Theorem 5.8, given a skew-symmetric distribution $\kappa \in E \otimes E^*$, we will study an operator whose expression is given by

\[ R_\kappa = \int_{T \times T} \kappa(s, t)(a_s^*a_t - a_t^*a_s) ds dt = 2\Xi_{1,1}(\kappa). \]

We enlarge the Lie algebra described in Theorem 5.5 by adding the operator $\Xi_{1,1}(\kappa)$ and investigating their commutators.

6.1 Notations

We need notations. For any white noise operator $\Xi \in L((E), (E)^*)$ and $\zeta \in E$, set

\[ D^0_\zeta^+ \Xi = D^+\Xi, \quad D^0_\zeta^- \Xi = D^-\Xi. \]

Moreover, for $k = 1, 2, \ldots$, define

\[ D^k_\zeta^+ \Xi = [D^{(k-1)}_\zeta^+ \Xi, \Xi], \quad D^k_\zeta^- \Xi = -[D^{(k-1)}_\zeta^- \Xi, \Xi]. \]
6.2 Quantum white noise derivatives of an integral kernel operator

**Lemma 6.1.** Let $\zeta \in E$ and $S \in \mathcal{L}(E, E)$ be skew-symmetric in the sense of (4). Then for $k = 0, 1, 2, \ldots$, we have

\[
D_\zeta^{k-} \Lambda(S) = a^*(S^{k+1}\zeta), \quad D_\zeta^{k+} \Lambda(S) = (-1)^{k+1} a(S^{k+1}\zeta),
\]

\[
D_\zeta^{k-} \Delta_G(S) = 0, \quad D_\zeta^{k+} \Delta_G(S) = 0.
\]

**Proof:** Let $\Xi = \Lambda(S) = \Xi_{1,1}(\tau_S)$. Since $S$ is skew-symmetric, then $\tau_S$ is a skew-symmetric distribution. Hence,

\[
a(S^*\zeta) = \int_T S^*\zeta(t)a_t dt
\]

\[
= \int_{T \times T} \tau_S(u, t)\zeta(u)a_t dtdu
\]

\[
= -\int_{T \times T} \tau_S(t, u)\zeta(u)a_t dtdu
\]

\[
= -a(S\zeta).
\]

Applying (10), we have

\[
D_\zeta^{1-}\Xi = -[a^*(\zeta), \Xi] = a^*(S\zeta),
\]

\[
D_\zeta^{1+}\Xi = [a(\zeta), \Xi] = a(S^*\zeta) = -a(S\zeta).
\]

Since $S \in \mathcal{L}(E, E)$, it follows that $S\zeta \in E$. Then applying (12) we have

\[
D_\zeta^{k-}\Xi = -[D_\zeta^{k-}\Xi, \Xi] = -[a^*(S\zeta), \Xi] = a^*(S^{k+1}\zeta),
\]

\[
D_\zeta^{k+}\Xi = [D_\zeta^{k+}\Xi, \Xi] = [a(S^{k-1}\zeta), \Xi] = a(S^{k-1}\zeta),
\]

Moreover, applying (13) we have

\[
D_\zeta^{k+}\Xi = [D_\zeta^{k}\Xi, \Xi] = [-a(S\zeta), \Xi] = a(S^2\zeta),
\]

\[
D_\zeta^{k+}\Xi = [D_\zeta^{k+1}\Xi, \Xi] = [a(S^2\zeta), \Xi] = -a(S^3\zeta),
\]

Furthermore, applying (14), we have

\[
D_\zeta^{k+1}\Xi = [D_\zeta^{(k-1)+}\Xi, \Xi] = [(-1)^ka(S^k\zeta), \Xi] = (-1)^{k+1} a(S^{k+1}\zeta).
\]

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Remark 6.2. From the known commutator identity (see, e.g. [21])

\[ [A, BC] = [A, B]C + B[A, C] \]

we can derive the following identity

\[ [AB, CD] = A[B, C]D + [A, C]BD + CA[B, D] + C[A, D]B. \] (14)

Lemma 6.3. Let \( \lambda, \kappa \in E_C \otimes E_C^* \). Then

\[ [\Xi_{0,2}(\lambda), \Xi_{1,1}(\kappa)] = \Xi_{0,2}(\lambda * \kappa) + \Xi_{0,2}(\lambda^* * \kappa), \]

where \( \lambda^*(s, t) = \lambda(t, s) \) for all \( s, t \in T \).

Proof: Applying (14) and (11), we have

\[ [a_s a_t, a_u^* a_v] = a_s [a_t, a_u^*]a_v + [a_s, a_u^*]a_t a_v + a_u^* a_s [a_t, a_v] + a_u^* [a_s, a_v]a_t \]

\[ = \delta_t(u) a_s a_v + \delta_s(u) a_t a_v. \]

Note that

\[ \lambda^* * \kappa(t, v) = \int_T \lambda^*(t, u) \kappa(u, v) du = \int_T \lambda(u, t) \kappa(u, v) du. \]

Then

\[ [\Xi_{0,2}(\lambda), \Xi_{1,1}(\kappa)] = \int_{T^4} \lambda(s, t) \kappa(u, v) [a_s a_t, a_u^* a_v] dsdtdudv \]

\[ = \int_{T^3} \lambda(s, u) \kappa(u, v) a_t a_v dsdudv + \int_{T^3} \lambda(u, t) \kappa(u, v) a_t a_v dtdudv \]

\[ = \int_{T^2} (\lambda * \kappa)(s, v) a_s a_v dsv + \int_{T^2} (\lambda^* * \kappa)(t, v) a_t a_v dtdv \]

\[ = \Xi_{0,2}(\lambda * \kappa) + \Xi_{0,2}(\lambda^* * \kappa). \]

Lemma 6.4. Let \( \kappa \in E_C \otimes E_C^* \). Then \( [\Delta_G, \Xi_{1,1}(\kappa)] = 2\Xi_{0,2}(\kappa). \)

Proof: Recall that \( \Delta_G = \Xi_{0,2}(\tau) \). Then \( \tau^* = \tau \) and \( \tau * \kappa = \kappa \) since \( \tau \) corresponds to the identity operator in the canonical isomorphism \( E_C \otimes E_C^* \cong \mathcal{L}(E_C, E_C) \).

The conclusion follows by Lemma 6.3. \( \square \)

Lemma 6.5. Let \( \kappa \in E_C \otimes E_C^* \) be skew-symmetric. Then \( [\Xi_{0,2}(\kappa), \Xi_{1,1}(\kappa)] = 0. \)

Proof: Since \( \kappa \) is skew-symmetric, \( \kappa^*(s, t) = -\kappa(s, t) \) for all \( s, t \in T \). Then

\[ \kappa^* * \kappa(t, v) = \int_T \kappa^*(t, u) \kappa(u, v) du = -\int_T \kappa(t, u) \kappa(u, v) du = -\kappa * \kappa(t, v). \]

Applying Lemma 6.3 we have

\[ [\Xi_{0,2}(\kappa), \Xi_{1,1}(\kappa)] = \Xi_{0,2}(\kappa * \kappa) - \Xi_{0,2}(\kappa * \kappa) = 0. \]

We record the following commutation relations which are immediate from Proposition 5.1, Remark 5.4, and Lemmas 5.2, 5.3, 6.1, 6.4 and 6.5.
Theorem 6.6. Let $\zeta \in E_C$ and $S \in \mathcal{L}(E_C, E_C)$ be skew-symmetric. Let $S^0 = Id$. Then for $j, k \in \mathbb{N} \cup \{0\}$, it holds that

1. $[a(S^j \zeta), a^*(S^k \zeta)] = \langle S^j \zeta, S^k \zeta \rangle Id,$
2. $[\Delta_G, \Delta_G(S)] = 0,$
3. $[N, \Delta_G(S)] = -2\Delta_G(S),$
4. $[N, \Lambda(S)] = 0,$
5. $[a^*(S^k \zeta), \Lambda(S)] = a^*(S^{k+1} \zeta),$
6. $[\Lambda(S), \Delta_G(S)] = 0,$
7. $[a^*(S^k \zeta), \Delta_G(S)] = 0,$
8. $[\Delta_G, \Lambda(S)] = 2\Delta_G(S),$
9. $[\Delta_G, \Delta_G(S)] = 0.$

Theorem 6.7. For each nonzero $\zeta \in E_C$ and a skew-symmetric $S \in \mathcal{L}(E_C, E_C)$, let $S^0 = Id$ and $h = \langle Id, a(S^k \zeta), a^*(S^k \zeta), N, \Lambda(S), \Delta_G, \Delta_G(S) : k = 0, 1, 2, \ldots \rangle$

Then $h$ is a (possibly infinite dimensional) non-nilpotent solvable complex Lie algebra.

Proof: From the commutation relations given in Theorems 6.2 and 6.6, we see that $h$ is closed under the Lie bracket. Moreover, we have

$h^{(1)} = [h, h] = \langle Id, a(S^k \zeta), a^*(S^k \zeta), \Delta_G, \Delta_G(S) : k = 0, 1, 2, \ldots \rangle$

$h^{(2)} = h^{(1)} = \langle Id, a(S^k \zeta) : k = 0, 1, 2, \ldots \rangle$

$h^{(3)} = [h^{(2)}, h^{(2)}] = \langle 0 \rangle.$

Thus $h$ is solvable. Moreover, $h_{(1)} = [h, h]$ and

$[h, h_{(1)}] = \langle Id, a(S^k \zeta), a^*(S^k \zeta), \Delta_G, \Delta_G(S) : k = 0, 1, 2, \ldots \rangle = h_{(1)}.$

Thus, $h$ is non-nilpotent. □

Remark 6.8. It is worth noting that $S^k$ is skew-symmetric for an odd $k$ while it is symmetric for an even $k$. The dimension of $h$ depends on the properties of $S \in \mathcal{L}(E_C, E_C)$. Note that for $\zeta \in E_C$, $\{S^k \zeta\}$ is contained in the range of $S$. Therefore, if $S$ is a finite rank operator, i.e., $\dim R(S) < \infty$, then $h$ is finite dimensional. We may also investigate the eigenvectors of $S$. For instance, adding the condition that $S^k \zeta = \zeta$ (e.g. Theorem 5.7) makes $h$ a seven-dimensional Lie algebra. More generally, if $\zeta$ is an eigenvector of $S^k$, i.e. $S^k \zeta = \lambda \zeta$, for some
\( \lambda \in \mathbb{C} \), then \( h \) is finite dimensional. By an orbit of \( S \in \mathcal{L}(E_{\mathbb{C}}, E_{\mathbb{C}}) \) we mean a sequence \( \{S^n \xi : n = 0, 1, 2, \ldots \} \), where \( \xi \in E_{\mathbb{C}} \) is a fixed vector. To read more about orbits of operators, see [22, 23]. A vector \( e \) is supercyclic for an operator \( T \) if the vectors \( T^n e, \xi \) complex, are dense. In [17], Hilden and Wallen proved that no normal operator on a Hilbert space of dimension > 1 can be supercyclic.

Observe that the one-dimensional subalgebra \( (\Delta_G(S)) \) is an ideal of \( h \). We have the following proposition.

**Proposition 6.9.** Let \( \zeta \in E_{\mathbb{C}} \) and \( S \in \mathcal{L}(E_{\mathbb{C}}, E_{\mathbb{C}}) \) be skew-symmetric. Set

\[
S^0 = \text{Id} \quad \text{and} \quad h = \langle \text{Id}, a(S^k \zeta), a^*(S^k \zeta), N, \Lambda(S), \Delta_G(S) : k = 0, 1, 2, \ldots \rangle.
\]

Then every nonzero ideal \( a \) of \( h \) with \( \Delta_G(S) \notin a \) contains the identity operator.

**Proof:** Suppose there exists a nonzero ideal \( a \) of \( h \) with \( \Delta_G(S) \notin a \) such that \( \text{Id} \notin a \). Then there exists \( x \in a, x \neq 0 \).

- **Case 1:** \( x = a(S^k \zeta) \). Then \( [a^*(S^k \zeta), a(S^k \zeta)] = -(S^k \zeta, S^k \zeta)\text{Id} \), a contradiction.
- **Case 2:** \( x = a^*(S^k \zeta) \). Then \( [a(S^k \zeta), a^*(S^k \zeta)] = (S^k \zeta, S^k \zeta)\text{Id} \), a contradiction.
- **Case 3:** \( x = N \). Note that \( [a(\zeta), N] = a(\zeta) \in a \). Then apply Case 1.
- **Case 4:** \( x = \Lambda(S) \). Note that \( [a(\zeta), \Lambda(S)] = -a(S \zeta) \in a \). Then apply Case 1.
- **Case 5:** \( x = \Delta_G \). Note that \( [a^*(\zeta), \Delta_G] = -2a(\zeta) \in a \). Then apply Case 1. \( \square \)

**Theorem 6.10.** The complex Lie algebra \( h \) described in Theorem 6.7 is not semisimple.

**Proof:** The proof of Theorem 6.7 shows that \( h \) contains a nonzero solvable ideal, e.g. \( h^{(1)} \). \( \square \)

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