Optimal investment and consumption for pairs trading financial markets on small time interval

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Abstract

In this paper we consider a pairs trading financial market with the spread of risky assets defined by the Ornstein-Uhlenbeck (OU) process. We implement an optimal strategy for power utility functions for investment/consumption problem. Through the Feynman-Kac (FK) method, we study the Hamilton-Jacobi-Bellman (HJB) equation for this problem. Moreover, the existence and uniqueness has been shown for classical solution for the HJB equation. In addition, the numeric approximation for the solution of the HJB equation has been studied and the convergence rate has been established and it is been found that the convergence rate is extremely explosive.

keywords Optimality, Feynman–Kac mapping, Hamilton–Jacobi–Bellman equation, Itô formula, Brownian motion, Ornstein–Uhlenbeck process, Stochastic processes, Financial market, Spread market.

AMS subject classification primary 62P05, secondary 60G05

1 Introduction

This paper deals with an optimal investment/consumption problem during a fixed time interval $[0, T]$ for a financial market generated by risky spread assets defined through the Ornstein–Uhlenbeck (OU) processes. Such problems

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are of prime interest for practical investors such as electricity gas markets. Also in other sectors like the microstructoe level within the airline industry (see, for example, [3]) as well it is known in many hedge funds [3]. Usually, for such model, one uses a dynamical programming method (see, for example, [10]). Unfortunately, we can not use the Hamilton–Jacobi–Bellman (HJB) analysis method developed for the Black–Scholes market since for the OU model in the HJB equation there is an additional variable corresponding to the risky asset. Note that in [2] for the pure investment problem, they found the HJB solution in explicit form. Unfortunately, we can not apply this method to the general investment/consumption problem in view of an additional strongly nonlinear term due to the consumption. Moreover, even in the pure investment problem, (see, for example, [2]) the HJB solution is extremely explosive, i.e. it goes to infinity in a squared exponential power ($e^{x^2}$) rate as the variable corresponding to the risky assets ($s$) goes to infinity in this financial market. By this reason, we can not use the analytical tool of Black–Scholes model to proof the verification theorem for spread market. In this paper we develop a new method for the probabilistic analysis of the parabolic PDE. Similarly to [1], we study the HJB equation through the Feynman–Kac (FK) representation. To this end we introduce a special metric space in which the FK mapping is contracted. Taking this into account we show the fixed-point theorem for this mapping and we show that the fixed-point solution is the classical solution for the HJB equation in our case. Moreover, by using the verification theorem we find the optimal financial strategies. It turned out that the optimal investment and consumption strategies depend on the solution of a nonlinear parabolic partial differential equation. Therefore, to calculate the optimal strategies one needs to study numerical schemes.

The rest of the paper is organized as follows. In Section 2 we introduce the financial market. In Section 3 we define all necessary parameters. In Section 4 we write the HJB equation. In Section 5 we state the main results of the paper. In Section 6 we study the properties of the FK mapping. In Section 7 we study the properties of the fixed-point function. The proofs of the main results are given in Section 8. The corresponding verification theorem is stated in the Appendix with some auxiliary results.

2 Market model

Let $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{0 \leq t \leq T}, P)$ be a standard filtered probability space with $(\mathcal{F}_t)_{0 \leq t \leq T}$ adapted Wiener processes $(W_t)_{0 \leq t \leq T}$. Our financial market consists of one
riskless bond \((\tilde{S}_t)_{0 \leq t \leq T}\) and risky spread stocks \((S_t)_{0 \leq t \leq T}\) governed by the following equations:

\[
\begin{aligned}
    d\tilde{S}_t &= r\tilde{S}_t dt, \quad \tilde{S}_0 = 1, \\
    dS_t &= -\kappa S_t dt + \sigma dW_t, \quad S_0 > 0.
\end{aligned}
\] (2.1)

Here the constant \(\kappa > 0\) is the market mean-reverting parameter from \(\mathbb{R}\) and \(\sigma > 0\) is the market volatility. We assume that the interest rate \(r \leq \kappa\). Let now \(\tilde{\alpha}_t\) be the number of riskless assets \(\tilde{S}\) and \(\alpha_t\) be the number of risky assets in the moment \(0 \leq t \leq T\), and the consumption rate is given by a nonnegative integrated function \((c_t)_{0 \leq t \leq T}\) [10]. Thus the wealth process is

\[X_t = \tilde{\alpha}_t \tilde{S}_t + \alpha_t S_t.\]

Using the self financial principle from [10] we get

\[dX_t = \tilde{\alpha}_t d\tilde{S}_t + \alpha_t dS_t - c_t dt.\] (2.2)

We define the financial strategy as

\[v = (v_t)_{0 \leq t \leq T} = (\alpha_t, c_t)_{0 \leq t \leq T},\]

where \(c_t \geq 0\) is the consumption rate and \(\alpha_t\) is the investment position in the risky asset. So, replacing now in Eq. (2.2) the differentials \(d\tilde{S}_t\) and \(dS_t\) by their definitions in Eq. (2.1) we obtain the differential equation for the wealth process

\[dX^v_t = (rX^v_t - \kappa_1 \alpha_t S_t) dt + \alpha_t \sigma dW_t - c_t dt,\] (2.3)

where \(\kappa_1 = \kappa + r > 0\).

**Definition 2.1.** The financial strategy \(v = (v_t)_{0 \leq t \leq T}\) is called admissible if this process is adapted and the equation Eq. (2.3) has a unique strong nonnegative solution.

We denote by \(\mathcal{V}\) the set of all admissible financial strategies. For initial endowment \(x > 0\), admissible strategy \(v\) in \(\mathcal{V}\) and the state process \(\varsigma_t = (X^v_t, S_t)\), we introduce for \(0 < \gamma < 1\) the following value function

\[J(\varsigma, t, v) := \mathbb{E}_{\varsigma, t} \left( \int_t^T c^\gamma u \, du + \varpi(X^v_T)^\gamma \right),\] (2.4)

where \(\varpi > 0\) is some fixed constant, \(\mathbb{E}_{\varsigma, t}\) is the conditional expectation with respect to \(\varsigma_t = \varsigma = (x, s)\). We set \(J(\varsigma, 0, v) = J(\varsigma, 0, v)\).

Our goal is to maximize the value function Eq. (2.4), i.e.

\[
\sup_{v \in \mathcal{V}} J(\varsigma, v) \quad (2.5)
\]
To do this we use the dynamical programming method. Therefore, we need to study the problem Eq. (2.4) for any $0 \leq t \leq T$.

**Remark 2.1.** The coefficient $0 < \varpi < \infty$, explains the investor preference between consumption and pure investment problem. Therefore, we did not consider the case where $\varpi = 0$, as in reality the trader is more interested in the terminal wealth than consumption.

### 3 Main Parameters

First we introduce the following ordinary differential equation

$$
g'(t) - 2\gamma_2 g(t) + \gamma_1 g^2(t) + \gamma_3 = 0 \quad \text{and} \quad g(T) = 0,
$$

where

$$
\gamma_1 = \frac{\sigma^2}{1 - \gamma}, \quad \gamma_2 = \frac{\gamma \kappa_1}{1 - \gamma} + \kappa, \quad \kappa_1 = \kappa + r \quad \text{and} \quad \gamma_3 = \frac{\gamma \kappa_1^2}{(1 - \gamma)\sigma^2}.
$$

One can check directly that

$$
g(t) = \tilde{\gamma}_2 - \vartheta - \frac{2\vartheta(\tilde{\gamma}_2 - \vartheta)}{e^{\omega(T-t)}(\tilde{\gamma}_2 + \vartheta) - \tilde{\gamma}_2 + \vartheta},
$$

where $\tilde{\gamma}_2 = \gamma_2/\gamma_1$, $\tilde{\gamma}_3 = \gamma_3/\gamma_1$, $\omega = 2\vartheta \gamma_1$ and $\vartheta = \sqrt{\tilde{\gamma}_2^2 - \tilde{\gamma}_3}$.

Note that $g(t)$ is decreasing, i.e., $\max_{0 \leq t \leq T} g(t) = g(0)$. Taking into account that $r \leq \kappa$, we get $\tilde{\gamma}_2^2 \geq \tilde{\gamma}_3$. Furthermore, we set

$$
B_1 = \frac{1}{\gamma_1} \left( \sqrt{\frac{\pi}{2T}} + \sqrt{\frac{|\pi - 4T\sigma^2\varpi^{-1}|}{2T}} \right) \quad \text{and} \quad B_0 = \left( q_1 + \frac{\gamma_1 B_1^2}{2} \right) T,
$$

where $q_1 = g(0)\sigma^2/2 + r\gamma + (1 - \gamma)\varpi^{-1}$. We denote by $C^{1,0}_+([0, T])$ the set of all positive functions from $C^{1,0}([0, T])$, i.e. the set of all $\mathbb{R} \times [0, T] \to \mathbb{R}_+$ continuous partial derivatives with respect to the first variable $s$ and continuous functions in the second variable $t$. Now we introduce the following set

$$
\mathcal{X} = \left\{ h \in C^{1,0}_+([0, T]) : \sup_{s,t} h(s, t) \leq B_0, \sup_{s,t} |h_s(s, t)| \leq B_1 \right\}.
$$
For some $\kappa > 1$, which we will precise later, we introduce the metric in this space
\[ \rho(f, h) = \sup_{s \in \mathbb{R}, 0 \leq t \leq T} e^{-\kappa(T-t)} \Upsilon_{f,h}(s, t), \]
where $\Upsilon_{f,h}(s, t) = |h(s, t) - f(s, t)| + |h_s(s, t) - f_s(s, t)|$. Now for any $0 \leq t \leq T$ and $s \in \mathbb{R}$ we introduce the process $(\eta^{s,t}_u)_{t \leq u \leq T}$ as the solution of the following stochastic differential equation
\[ d\eta^{s,t}_u = g_1(u)\eta^{s,t}_u du + \sigma d\tilde{W}_u, \quad \eta^{s,t}_t = s, \]
where \( g_1(t) = \gamma_1 g(t) - \gamma_2 \) and $(\tilde{W}_u)_{u \geq 0}$ is a standard Brownian motion. It is clear that $\eta^{s,t}_u \sim \mathcal{N}(s \mu(u, t), \sigma^2_0(u, t))$, with
\[ \mu(u, t) = \exp\left\{ \int_t^u g_1(\nu) d\nu \right\} \quad \text{and} \quad \sigma^2_0(u, t) = \int_t^u \mu^2(u, z) dz. \]

Now for any $h \in \mathcal{X}$ we define the FK mapping as
\[ \mathcal{L}_h(s, t) = \int_t^T E\Psi_{\eta^{s,t}_u, u} du, \]
where $\Psi_{\eta^{s,t}_u, u} = \Gamma_0(s, t, h(s, t), h_u(s, t))$ and
\[ \Gamma_0(s, t, y_1, y_2) = \frac{\sigma^2 y_1^2}{2(1 - \gamma)} + \frac{\sigma^2 g(t)}{2} + r\gamma + (1 - \gamma)\overline{w}_1 G(s, t, y_1). \]
Here, the coefficient $\overline{w}_1 = \overline{w}^{-1/(1-\gamma)}$ and
\[ G(s, t, y) = \exp\left\{ -\frac{1}{1-\gamma} \left( \frac{s^2}{2} g(t) + y \right) \right\}. \]

In this paper we assume that $T < T_0$ and
\[ T_0 = \min\left( \frac{\kappa(1 - \gamma)}{2(3 + \gamma)\kappa_2}, \frac{\gamma(1 - \gamma)}{(3 + \gamma)(\gamma + 1)\sigma^2 g(0) \cdot \pi}{4\sigma^2} \right), \]
where $\kappa_2 = \kappa^2 \left( 1/\sigma^2 + 1/2 + g(0)/\kappa_1 \right)$.

**Remark 3.1.** Note that we use the FK mapping to study the HJB equation which will be defined in the next section.
4 Hamilton–Jacobi–Bellman equation

Denoting by $\varsigma_t = (X_t, S_t)$, we can rewrite equations Eq. (2.1) and Eq. (2.3) as,

$$d\varsigma_t = a(\varsigma_t, v_t)dt + b(\varsigma_t, v_t)dW_t,$$

where

$$a(\varsigma, u) = \begin{pmatrix} rx - \kappa_1 \alpha s - c \\ -\kappa s \end{pmatrix}, \quad b(\varsigma, u) = \begin{pmatrix} \alpha \sigma \\ \sigma \end{pmatrix} \quad \text{and} \quad u = (\alpha, c).$$

We introduce the Hamilton function, for any

$$q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \quad M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \quad \text{and} \quad 0 \leq t \leq T,$$

we set,

$$H(\varsigma, t, q, M) := \sup_{u \in \Theta} H_0(\varsigma, t, q, M, u), \quad \Theta \in \mathbb{R} \times \mathbb{R}^+,$$

where

$$H_0(\varsigma, t, q, M, u) := \alpha^2 \sigma^2 \sigma M_{11} + \sigma^2 M_{12} - \kappa_1 sq_1 \alpha + \frac{1}{2} \sigma^2 M_{22} + rxq_1 - \kappa sq_2 - cq_1 + c^\gamma.$$  (4.4)

Moreover, here

$$H_0(\varsigma, t, q, M, u) = \frac{\alpha^2 \sigma^2}{2} M_{11} + (\sigma^2 M_{12} - \kappa_1 sq_1)\alpha + \frac{1}{2} \sigma^2 M_{22} + rxq_1 - \kappa sq_2 - cq_1 + c^\gamma.$$  (4.4)

From Eq. (4.2) we find that for $q_1 > 0$

$$a_0(s, q, M) = \frac{\kappa_1 sq_1}{\sigma^2 M_{11}} - \frac{M_{21}}{M_{11}} \quad \text{and} \quad c_0(s, q, M) = \left(\frac{q_1}{\gamma}\right)^{\frac{1}{\gamma}}.$$  (4.4)
By substituting these conditions into the HJB equation Eq. (4.3) for the value function, we obtain the following nonlinear PDE.

\[ z_t(\varsigma, t) + \frac{1}{2} \left( \frac{\sigma^2 z_{xx} - \kappa_1 s z_x}{\sigma^2 z_x} \right)^2 + \frac{\sigma^2 z_{xx}}{2} + r x z_x - \kappa s z_x + (1 - \gamma) \left( \frac{z_x}{\gamma} \right)^{\frac{1-\gamma}{\gamma}} = 0, \] (4.5)

where \( z(\varsigma, T) = \varpi x^\gamma \).

To study this equation we use the following form for the solution

\[ z(x, s, t) = \varpi x^\gamma U(s, t) \quad \text{and} \quad U(s, t) = \exp \left\{ \frac{s^2}{2} g(t) + Y(s, t) \right\}. \] (4.6)

The function \( g(.) \) is defined in Eq. (3.2), and

\[
\begin{align*}
Y_t(s, t) + \frac{1}{2} \sigma^2 Y_{ss}(s, t) + s g_1(t) Y_x(s, t) + \Psi_Y(s, t) &= 0, \\
Y(s, T) &= 0,
\end{align*}
\] (4.7)

where \( \Psi_Y(s, t) \) is given in Eq. (3.9) and the function \( g_1(.) \) is defined in Eq. (3.6).

As we will see later, that the equation Eq. (4.7) has a solution in \( C^{2,0}(\mathbb{R} \times [0, T]) \) which can be represented as a fixed point for the FK mapping

\[ h(s, t) = \mathbf{E} \int_t^T \Psi_h(\eta_s^{u,t}, u) du = \mathcal{L}_h(s, t). \] (4.8)

Using the solution by equation Eq. (4.6), we define the functions

\[
\begin{align*}
\alpha_0(\varsigma, t) &= \frac{\kappa_1 s z_x(\varsigma, t)}{\sigma^2 z_x(\varsigma, t)} - \frac{z_{xx}(\varsigma, t)}{z_x(\varsigma, t)} = \beta(s, t)x, \\
\epsilon_0(\varsigma, t) &= \left( \frac{z_x(\varsigma, t)}{\gamma} \right)^{\frac{1}{\gamma}} = \bar{G}(s, t)x,
\end{align*}
\] (4.9)

where

\[
\bar{\beta}(s, t) = \frac{1}{1 - \gamma} \left( sg(t) + Y_s(s, t) - \frac{\kappa_1 s}{\sigma^2} \right) \quad \text{and} \quad \bar{G}(s, t) = \varpi^{\frac{1}{\gamma}} G(s, t, Y(s, t)).
\]

Now we set the following stochastic equation to define the optimal wealth process, i.e., we set

\[ dX_t^* = a^*(t)X_t^*dt + b^*(t)X_t^*dW_t, \] (4.10)

7
where \( a^*(t) = r - \kappa_1 S_t \hat{\beta}(S_t, t) - \tilde{G}(S_t, t) \) and \( b^*(t) = \sigma \hat{\beta}(S_t, t) \).

By Itô formula we can obtain that
\[
X^*_t = x \exp \left\{ \int_0^t a^*(u)du \right\} \xi_{0,t}(b^*).
\]

Using the stochastic differential equation Eq. (4.10) we define the optimal strategies:
\[
\alpha^*_t = \hat{\alpha}_0(\varsigma^*_t, t) \quad \text{and} \quad c^*_t = \tilde{c}_0(\varsigma^*_t, t), \quad (4.11)
\]
where \( \varsigma^*_t = (X^*_t, S_t) \) and \( X^*_t \) is defined in Eq. (4.10).

**Remark 4.1.** Note, the main difference in the HJB equation Eq. (4.5) from the one in [2] is the last nonlinear term as we see we can not use the solution method from [2]. One can check that the solution for pure investment problem from [2] can be obtained in Eq. (4.11) as \( \omega \to \infty \).

### 5 Main results

First we study the HJB equation.

**Theorem 5.1.** Assume that \( 0 < T < T_0 \), with \( T_0 \) is given in Eq. (3.11), then equation Eq. (4.3) is the solution defined by Eq. (4.6), where \( Y \) is the unique solution of Eq. (4.7) in \( X \) and is the fixed point for the FK mapping, i.e., \( Y = L_Y \).

**Theorem 5.2.** Assume that \( 0 < T < T_0 \). Then the optimal value of \( J(t, \varsigma, \upsilon) \) is given by
\[
\max_{\upsilon \in \mathcal{V}} J(\varsigma, t, \upsilon) = J(\varsigma, t, \upsilon^*) = \omega x^* U(s, t),
\]
where the optimal control \( \upsilon^* = (\alpha^*, c^*) \) for all \( 0 \leq t \leq T \) is given in Eq. (4.11) with the function \( Y \) defined in Eq. (4.8). The optimal wealth process \( (X^*_t)_{0 \leq t \leq T} \) is the solution to Eq. (4.10).

Let us now define the approximation sequence \( (h_n)_{n \geq 1} \) for \( h \) as \( h_0 = 0 \), and for \( n \geq 1 \), as
\[
h_n = L_{h_{n-1}}. \quad (5.1)
\]

In the following theorems we show that the approximation sequence goes to the fixed function \( h \), i.e. \( h = L_h \).
Theorem 5.3. For any $0 < \delta < 1/2$, the approximation

$$\| h - h_n \| \leq O(n^{-\delta n}) \quad \text{as} \quad n \to \infty.$$  

Remark 5.1. Note that the convergence rate is super geometrical.

Now we define the approximation. We set

$$\alpha_n^*(s, t) = \tilde{\beta}_n(s, t)x \quad \text{and} \quad c_n^*(s, t) = \tilde{G}_n(s, t)x$$

where

$$\tilde{\beta}_n(s, t) = \frac{1}{1 - \gamma} \left( sg(t) + \frac{\partial h_n(s, t)}{\partial s} - \frac{\kappa_1}{\sigma} s \right) \quad \text{and} \quad \tilde{G}_n(s, t) = \omega^{-\frac{1}{1 - \gamma}} G(s, t, h_n(s, t)).$$

Theorem 5.4. For any $0 < \delta < 1/2$

$$\sup_{0 \leq t \leq T} \left( |\alpha^*(s, t) - \alpha_n^*(s, t)| + |c^*(s, t) - c_n^*(s, t)| \right) \leq O(n^{-\delta n}), \quad \text{as} \quad n \to \infty.$$  

Remark 5.2. As it is seen from Theorem 5.1 the approximation scheme for the HJB equation implies the approximation for the optimal strategy with super geometrical rate. i.e. more rapid than any geometrical ones.

6 Properties of the Feynman–Kac mapping

We need to study the properties of the mapping (3.8).

Proposition 6.1. The space $(X, \rho)$ is the completed metrical space.

Proposition 6.2. Assume that $T \leq \pi/16$. Then $L_h \in X$ for any $h \in X$, i.e. $L_h : \mathcal{X} \to \mathcal{X}$.

Proof. The function $L_h(s, t)$ is given in Eq. (3.8) and can be written as

$$L_h(s, t) = \frac{\sigma^2}{2} \int_t^T g(u)du + \frac{\sigma^2}{2(1 - \gamma)} E \int_t^T h^2_s(\eta_{\alpha}^{st}, u)du + r\gamma(T - t)$$

$$+ (1 - \gamma) \omega^{\frac{-1}{1 - \gamma}} E \int_t^T G(\eta_{\alpha}^{st}, u, h(\eta_{\alpha}^{st}, u))du,$$

with $G(s, t, y)$ is given in Eq. (3.10).

Therefore,

$$|L_h(s, t)| \leq \frac{\sigma^2}{2} g(0)(T - t) + \frac{\sigma^2}{2(1 - \gamma)} B_1^2(T - t) + r\gamma(T - t)$$

$$+ (1 - \gamma) \omega^{\frac{-1}{1 - \gamma}}(T - t) \leq B_0,$$  \hspace{1cm} (6.1)
where $B_0$ and $B_1$ are given in Eq. (3.3). Then by taking the derivative with respect to $s$, we get

$$
\frac{\partial}{\partial s} L_h(s, t) = \frac{\sigma^2}{2(1-\gamma)} \frac{\partial}{\partial s} \mathbb{E} \int_t^T h_s^2(\eta_u^{s,t}, u) du
+ (1-\gamma)^{\frac{1}{\omega-1}} \frac{\partial}{\partial s} \mathbb{E} \int_t^T G(\eta_u^{s,t}, u, h(\eta_u^{s,t}, u)) du.
$$

From Lemma A.3 and as $\| G(\eta_u^{s,t}, u, h(\eta_u^{s,t}, u)) \|_{l,\infty} \leq 1$, we have

$$
\left| \frac{\partial}{\partial s} L_h(s, t) \right| \leq \frac{\sigma}{(1-\gamma)} \sqrt{\frac{2(T-t)}{\pi}} B_1^2 + (1-\gamma)^{\frac{1}{\omega-1}} \frac{2}{\sigma} \sqrt{\frac{2(T-t)}{\pi}}.
$$

Then by taking into account the definition of $B_1$ in Eq. (3.4) we obtain,

$$
\left| \frac{\partial}{\partial s} L_h(s, t) \right| \leq \frac{\sigma}{(1-\gamma)} \sqrt{\frac{2T}{\pi}} B_1^2 + (1-\gamma)^{\frac{1}{\omega-1}} \frac{2}{\sigma} \sqrt{\frac{2T}{\pi}} \leq B_1.
$$

So, we get that $L_h \in \mathcal{X}$. Hence Proposition 6.2. □

**Proposition 6.3.** For all $f \in \mathcal{X}$, for all $s$, and $0 \leq t \leq T$,

$$
\frac{\partial}{\partial s} L_f(s, t) = \int_t^T \left( \int_{\mathbb{R}} \Gamma_0(z, t, f(z, u), f(s,z,u)) \varrho(s, t, z, u) dz \right) du,
$$

where $\Gamma_0$ is as in Eq. (3.9) and

$$
\varrho(s, t, z, u) = \frac{\partial}{\partial s} \varphi(s, t, z, u) = K \frac{\mu(u, t)}{\sigma_1(u, t)} \varphi(s, t, z, u),
$$

where

$$
\varphi(s, z, u) = \frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi\sigma_1(u, t)}} \quad \text{and} \quad K(s, z, u) = \frac{z-s\mu(u, t)}{\sigma_1(u, t)}.
$$

**Proposition 6.4.** The mapping $L$ is constraint in $\mathcal{X}$, i.e. for any $0 < \lambda < 1$, there exists $\kappa \geq 1$ in the metric Eq. (3.5) such that for any $h$ and $f \in \mathcal{X}$

$$
\rho(L_h, L_f) \leq \lambda \rho(h, f).
$$
Proof. Using the definition of the mapping $L_h$ in Eq. (3.8) we obtain that for any $h$ and $f$ from $\mathcal{X}$,

$$
L_h - L_f = \frac{\sigma^2}{2(1 - \gamma)} E \int_t^T \left( h^2_s(\eta^{s,t}, u) - f^2_s(\eta^{s,t}, u) \right) du
+ (1 - \gamma) \wedge \int_t^T E \left( \frac{\gamma}{\gamma - 1} \right) \int_t^T \left( G\left( \eta^{s,t}_u, u, h(\eta^{s,t}_u, u) \right) - G\left( \eta^{s,t}_u, u, f(\eta^{s,t}_u, u) \right) \right) du.
$$

Taking into account that the function $G$ is lipschitzian, i.e. for any $y_1 \geq 0$ and $y_2 \geq 0$

$$\left| G(s, t, y_1) - G(s, t, y_2) \right| \leq \frac{1}{1 - \gamma} |y_1 - y_2|,$$

we obtain that

$$
|L_h - L_f| \leq \frac{\sigma^2}{2(1 - \gamma)} \int_t^T E \left( h^2_s(\eta^{s,t}, u) - f^2_s(\eta^{s,t}, u) \right) du
+ \wedge \int_t^T E \left( h(\eta^{s,t}_u, u) - f(\eta^{s,t}_u, u) \right) du. \quad (6.5)
$$

Recall that $f$ and $h$ belong to $\mathcal{X}$, i.e. the difference for the squares of their derivatives can be estimated as $|h^2_s(z, u) - f^2_s(z, u)| \leq 2B_1|h_s(z, u) - f_s(z, u)|$, therefore,

$$
|L_h(s, t) - L_f(s, t)| \leq \left( \frac{\sigma^2 B_1}{1 - \gamma} + \frac{1}{\wedge \gamma - 1} \right) \int_t^T \Upsilon_{h,f}(u)e^{-\kappa(T-u)}e^{\kappa(T-u)} du,
$$

where $\Upsilon_{h,f}(t) = \sup_{y \in \mathbb{R}^p} \Upsilon_{h,f}(y, t)$. In view of the definition Eq. (3.4)

$$
|L_h(s, t) - L_f(s, t)| \leq \left( \frac{\sigma^2 B_1}{1 - \gamma} + \frac{1}{\wedge \gamma - 1} \right) \rho(h, f) \int_t^T e^{-\kappa(T-u)} du
\leq \left( \frac{\sigma^2 B_1}{1 - \gamma} + \frac{1}{\wedge \gamma - 1} \right) \rho(h, f) e^{\kappa(T-t)}.
$$

Therefore for all $0 \leq t \leq T$,

$$
\sup_{s \in \mathbb{R}} |L_h(s, t) - L_f(s, t)| \leq \frac{\tilde{B}_1}{\kappa} \rho(h, f) e^{\kappa(T-t)}, \quad \tilde{B}_1 = \frac{\sigma^2 B_1}{1 - \gamma} + \frac{1}{\wedge \gamma - 1}.
$$
The partial derivative of \( L(s, t) \) with respect to \( s \) is given by

\[
\frac{\partial}{\partial s} L_h(s, t) = \frac{\sigma^2}{2(1 - \gamma)} E \frac{\partial}{\partial s} \int_t^T h_s^2(\eta_u^s, u) du \\
+ (1 - \gamma) \sigma \frac{1}{\omega^{1+tr}} E \frac{\partial}{\partial s} \int_t^T G(\eta_u^s, u, h(\eta_u^s)) du.
\]

By taking the expectation we obtain

\[
\frac{\partial}{\partial s} L_h(s, t) = \frac{\sigma^2}{2(1 - \gamma)} \int_t^T \int_R h_s^2(z, u) \frac{\partial}{\partial s} \varphi(z, u) dz du \\
+ (1 - \gamma) \sigma \frac{1}{\omega^{1+tr}} \int_t^T \int_R G(z, u, h(z, u)) \frac{\partial}{\partial s} \varphi(z, u) dz du,
\]

where \( \varphi(s, t, z, u) = \partial \varphi(s, t, z, u)/\partial s \) and \( \varphi(s, t, z, u) \) is given in Eq. (6.3). Therefore, for \( u > t \) and for some constant \( c^* > 0 \)

\[
\sup_{s \in \mathbb{R}} \int_R |\varphi(s, t, z, u)| dz \leq \frac{c^*}{\sqrt{(u - t)}}.
\] (6.6)

Putting now \( \hat{\alpha}_1 = \sigma^2(2 - 2\gamma)^{-1} \) and \( \hat{\alpha}_2 = (1 - \gamma) \sigma \omega^{1+tr} \), we obtain that

\[
\left| \frac{\partial}{\partial s} L_h(s, t) - \frac{\partial}{\partial s} L_f(s, t) \right| = \left| \int_t^T \int_R \left( \hat{\alpha}_1 (h_s^2(z, u) - f_s^2(z, u)) + \hat{\alpha}_2 (G(z, u, h(z, u)) - G(z, u, f(z, u)) \right) \varphi(s, t, z, u) dz du \right|.
\]

Note here, that

\[
|\hat{\alpha}_1 (h_s^2(z, u) - f_s^2(z, u)) + \hat{\alpha}_2 (G(z, u, h(z, u)) - G(z, u, f(z, u))| \leq \mathbf{B}_2 \Upsilon_{f,h}^*(u),
\]

where \( \mathbf{B}_2 = \left( 2\hat{\alpha}_1 \tilde{\mathbf{B}}_1 + \hat{\alpha}_2 (1 - \gamma) \right) \). Thus,

\[
\left| \frac{\partial}{\partial s} L_h(s, t) - \frac{\partial}{\partial s} L_f(s, t) \right| \leq \mathbf{B}_2 \int_t^T \Upsilon_{f,h}^*(u) \left( \int_R |\varphi(s, t, z, u)| dz \right) du.
\]

Using here the bound Eq. (6.6), we obtain that

\[
\left| \frac{\partial}{\partial s} L_h(s, t) - \frac{\partial}{\partial s} L_f(s, t) \right| \leq \mathbf{B}_2 \sqrt{\frac{2}{\pi}} \int_t^T \frac{1}{\sqrt{u - t}} \Upsilon_{f,h}^*(u) e^{-\kappa(T - u)} e^{\kappa(T - u)} du.
\]
Using again here the definition Eq. (3.4) we get
\[
\left| \frac{\partial}{\partial s} L_h(s, t) - \frac{\partial}{\partial s} L_f(s, t) \right| \leq \sqrt{\frac{2}{\pi}} B_2 \rho(f, h) \int_t^T \frac{e^{\kappa(T-u)}}{\sqrt{u-t}} du 
\]
\[
\leq \frac{2}{\pi} B_2 \rho(f, h) e^{\kappa(T-t)} \int_t^T \frac{e^{-\kappa(u-t)}}{\sqrt{u-t}} du \leq B_2 \rho(f, h) \frac{e^{\kappa(T-t)}}{\sqrt{\kappa}}.
\]
and, therefore,
\[
\left| \frac{\partial}{\partial s} L_h(s, t) - \frac{\partial}{\partial s} L_f(s, t) \right| \leq B_2 \rho(f, h) \frac{e^{\kappa(T-t)}}{\sqrt{\kappa}}.
\]
Thus
\[
\left| L_h(s, t) - L_f(s, t) \right| + \left| \frac{\partial}{\partial s} L_h(s, t) - \frac{\partial}{\partial s} L_f(s, t) \right| 
\]
\[
\leq \left( \frac{\tilde{B}_1 e^{\kappa(T-t)}}{\kappa} + B_2 e^{\kappa(T-t)} \right) \rho(f, h).
\]
So, taking into account that \( \kappa > 1 \), we get
\[
\rho(L_h, L_f) \leq \frac{\tilde{B}_2}{\sqrt{\kappa}} \rho(f, h), \quad \tilde{B}_2 = \tilde{B}_1 + B_2.
\]
Choosing here \( \kappa = (\tilde{B}_2)^2 / \lambda^2 \) we obtain the inequality Eq. (6.4). Hence Proposition 6.4. \( \square \)

**Proposition 6.5.** For the mapping \( L \) there exists a unique fixed point \( h \) from \( X \), i.e. \( L_h = h \), such that for any \( n \geq 1 \) and for any \( \kappa > (\tilde{B}_2)^2 \)
\[
\rho(h, h_n) \leq B^* \lambda^n, \quad \lambda = \frac{\tilde{B}_2}{\sqrt{\kappa}}, \quad (6.7)
\]
where \( B^* = (B_0 + B_1) / (1 - \lambda) \), with \( B_0 \) and \( B_1 \) are defined in Eq. (3.3).

**Proof.** We want to show that the approximation sequence \( (h_n)_{n \geq 1} \) converge to a fixed point \( h \), where \( h_0 = 0 \) and \( h_n = L_{h_{n-1}} \) for \( n \geq 1 \). Using here Proposition 6.3, we obtain that \( \rho(h_n, h_{n+1}) = \rho(L_{h_{n-1}}, L_{h_n}) \leq \lambda \rho(h_{n-1}, h_n) \). Therefore,
\[
\rho(h_n, h_{n+1}) \leq \lambda \rho(L_{h_{n-1}}, L_{h_n}) \leq \lambda^2 \rho(h_{n-2}, h_{n-1}) \leq \cdots \leq \lambda^n \rho(h_0, h_1).
\]
Note that Eq. (3.4) implies directly that \( \rho(h_0, h_1) \leq B_0 + B_1 \). So, for \( m > n \),

\[
\rho(h_m, h_n) \leq (\lambda^n + \lambda^{n+1} + \ldots + \lambda^{m-1})(B_0 + B_1) \leq \sum_{i=n}^{\infty} \lambda^i(B_0 + B_1).
\]

Therefore, there exists \( h \), such that \( \rho(h_n, h) \to 0 \), i.e., for all \( n \), we obtain Eq. (3.5). Hence Proposition 6.5. \( \square \)

7 Properties of the fixed-point function \( h \)

In this section we study some regularity properties for the function \( h \). First we study the smoothness with respect to the variable \( s \).

**Proposition 7.1.** If \( h \in \mathcal{X} \) is a fixed point for \( L \), i.e. \( h = Lh \), then for any \( 0 < \beta < 1 \),

\[
\sup_{0 \leq t \leq T} \sup_{|s_1|,|s_2|} \frac{|h_s(s_1, t) - h_s(s_2, t)|}{|s_1 - s_2|^{\beta}} < +\infty.
\]

**Proof.**

\[
\frac{\partial}{\partial s} h(s, t) = \int_t^T \int_{\mathbb{R}} \Psi_h(z, u) \varrho(s, t, z, u)dzdu,
\]

where \( \Psi_h(z, u) \) and \( \varrho(s, t, z, u) \) are given in Eq. (3.9) and Eq. (6.2) respectively.

Therefore,

\[
\left| \frac{\partial}{\partial s} h(s_1, t) - \frac{\partial}{\partial s} h(s_2, t) \right| = \left| \int_t^T \int_{\mathbb{R}} \left( \int_{\mathbb{R}} (\varrho(s_1, t, z, u) - \varrho(s_2, t, z, u))dz \right)dud\right|.
\]

If \( \delta > 1 \) then,

\[
\frac{1}{\delta^\beta} \left| \frac{\partial}{\partial s} h(s_1, t) - \frac{\partial}{\partial s} h(s_2, t) \right| \leq \int_t^T \left( \int_{\mathbb{R}} |\varrho(s_1, t, z, u)|dz \right)du + \int_t^T \left( \int_{\mathbb{R}} |\varrho(s_2, t, z, u)|dz \right)du < +\infty.
\]

For \( 0 < \delta = |s_1 - s_2| < 1 \), then,

\[
\frac{1}{\delta^\beta} \left| \frac{\partial}{\partial s} h(s_1, t) - \frac{\partial}{\partial s} h(s_2, t) \right| \leq B_0 \int_t^T \left( \int_{\mathbb{R}} |\varrho(s_1, t, z, u) - \varrho(s_2, t, z, u)|dz \right)du.
\]
where from Eq. (6.1), \( \int_t^T \Psi_h(z, u) \, du \leq B_0 \), and \( B_0 \) is given in Eq. (3.3). Let

\[
I(\delta) = \int_t^T \left( \int_{\mathbb{R}} \frac{|\varrho(s_1, t, z, u) - \varrho(s_2, t, z, u)|}{\delta^\beta} \, dz \right) \, du.
\]

Then we can rewrite it as

\[
I(\delta) = \int_t^{t+\delta_1} \int_{\mathbb{R}} \frac{|\varrho(s_1, t, z, u) - \varrho(s_2, t, z, u)|}{\delta^\beta} \, dz \, du
+ \int_{t+\delta_1}^T \int_{\mathbb{R}} \frac{|\varrho(s_1, t, z, u) - \varrho(s_2, t, z, u)|}{\delta^\beta} \, dz \, du.
\]

Putting \( \delta_1 = \delta^{2\beta} \) we obtain that

\[
I(\delta) \leq \frac{1}{\delta^\beta} \int_t^{t+\delta_1} \left( \int_{\mathbb{R}} |\varrho(s_1, t, z, u)| \, dz + \int_{\mathbb{R}} |\varrho(s_2, t, z, u)| \, dz \right) \, du
+ \frac{1}{\delta^\beta} \int_{t+\delta_1}^T \int_{\mathbb{R}} |\varrho(s_1, t, z, u) - \varrho(s_2, t, z, u)| \, dz \, du.
\]

Taking into account the bound Eq. (6.5), we estimate the integral \( I(\lambda) \) as

\[
I(\delta) \leq \frac{\sqrt{2}}{\sqrt{\pi}} + \frac{1}{\delta^\beta} \int_{t+\delta_1}^T \int_{\mathbb{R}} |\varrho(s_1, t, z, u) - \varrho(s_2, t, z, u)| \, dz \, du.
\]

Then

\[
I(\delta) \leq \frac{\sqrt{2}}{\sqrt{\pi}} + \frac{1}{\delta^\beta} \int_{t+\delta_1}^T \int_{s_1}^{s_2} \left( \int_{\mathbb{R}} |\varphi_s(v, t, z, u)| \, dz \right) \, dv \, du,
\]

where

\[
\varphi_s = \frac{\partial}{\partial s} \varrho(v, t, z, u) = \frac{\mu^2}{\sqrt{2\pi\sigma_1^2}} e^{-K^2/2} (K^2 - 1),
\]

where \( K = K(s, z, u) \) is given in Eq. (6.3). Thus

\[
|\varphi_s| \leq \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-K^2/2} (K^2 + 1)
\]

and

\[
\int_{\mathbb{R}} |\varphi_s(v, t, z, u)| \, dz = \frac{1}{\sigma_1^2} \int_{\mathbb{R}} K^2 + 1 e^{-K^2/2} \, dK \leq \frac{C^*}{\sigma_1^2}.
\]
Taking into account that $\sigma_1^{-2} \leq c^*(u - t)^{-1}$ for some $c^* > 0$, we get
\[ I(\delta) \leq \frac{2\sqrt{2}}{\sqrt{\pi}} + c^*\delta^{1-\beta} \int_{t+\delta_1}^{T} \frac{1}{u-t} \, du \leq c^* + \delta^{1-\beta}(|\ln \delta_1| + |\ln T|). \]

Hence Proposition 7.1. \qed

Now we need to study the smoothness property with respect to $t$. We show now that the function $f$ and its derivatives are Hölderians.

**Proposition 7.2.** Let $h = L_h$, with $h \in X$. Therefore, for all $t$, for all $N \geq 1$, and $0 < \beta < 1/2$,
\[ \sup_{0 \leq t_1 \leq t_2 \leq T} \sup_{|s| \leq N} \left( \frac{|h(s, t_1) - h(s, t_2)| + |h_s(s, t_1) - h_s(s, t_2)|}{|t_1 - t_2|^{\beta}} \right) < +\infty. \]

**Proof.** Firstly, note that
\[ h(s, t) = \int_t^{T} \Gamma(s, t, u) \, du \quad \text{and} \quad \Gamma(s, t, u) = \int_{\mathbb{R}} \Omega(z, u) \, dz. \]

Therefore, for any $0 \leq t_1 \leq t_2 \leq T$
\[ h(s, t_2) - h(s, t_1) = \int_{t_1}^{T} \left( \Gamma(s, t_2, u) \, du - \Gamma(s, t_1, u) \right) \, du - \int_{t_1}^{t_2} \Gamma(s, t_1, u) \, du. \]

Let now $\delta = t_2 - t_1$ and $\delta_1 = \delta^{2\beta}$ for some $0 < \beta < 1/2$. Talking into account that $\Gamma$ is bounded, we obtain that for some $c^* > 0$
\[ \frac{1}{\delta^{\beta}} \left| h(s, t_2) - h(s, t_1) \right| \leq c^* I(\delta) + c \delta^{1-\beta}. \tag{7.1} \]

where $I(\delta) = \int_{t_2}^{T} \int_{\mathbb{R}} |\Omega(z, u)| \, dz \, du$ and $\Omega(z, u) = \varphi(s, t_2, z, u) - \varphi(s, t_1, z, u)$.

We reprent this term as $I(\delta) = I_1(\delta) + I_2(\delta)$, where
\[ I_1(\delta) = \int_{t_2}^{t_2+\delta_1} \int_{\mathbb{R}} |\Omega(z, u)| \, dz \, du \quad \text{and} \quad I_2(\delta) = \int_{t_2+\delta_1}^{T} \int_{\mathbb{R}} |\Omega(z, u)| \, dz \, du. \]

It is clear that $I_1(\delta) \leq 2\delta_1$. To estimate the term $I_2(\delta)$ note that
\[ |\Omega(z, u)| = |\varphi(s, t_2, z, u) - \varphi(s, t_1, z, u)| \leq \int_{t_1}^{t_2} |\varphi_t(s, \theta, z, u)| \, d\theta, \]

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where

\[ \phi_t(s,\theta,z,u) = \frac{\partial}{\partial t} \phi(s,t,z,u) = \frac{\sigma_2(u,t)}{2 \sqrt{2\pi \sigma_1^2(u,t)}} - \frac{KK'}{\sqrt{2\pi\sigma_1}} e^{-\frac{K^2}{2}}, \]

the prime ' is the derivative with respect to \( t \) and \( \sigma_2 = (\sigma_1)' \). Denoting by \( \mu_1' = \mu' \) we obtain that

\[ K' = \left( \frac{z - s\mu}{\sigma_1} \right)' = \frac{s\mu_1}{\sigma_1} - \frac{z - s\mu}{\sigma_1} (\sigma_1)' = -\frac{s\mu_1}{\sigma_1} - \frac{1}{2} K \frac{\sigma_2}{2\sigma_1^2}. \]

Taking into account that, \( \mu_1 \) is bounded, we obtain that for some \( c^* > 0 \)

\[ \left| \frac{\partial}{\partial t} \phi(s,t,z,u) \right| \leq c^* (1 + |s|) e^{-\frac{K^2}{2}(K^2 + |K| + 1)} \frac{1}{\sigma_1^3}. \]

Therefore, for some \( c^* > 0 \) and \( u > t \)

\[ \int_\mathbb{R} \left| \frac{\partial}{\partial t} \phi(s,t,z,u) \right| dz \leq \frac{c^* (1 + |s|)}{u - t}, \]

and we get

\[ |I_2(\delta)| \leq c^* (1 + |s|) \int_{t_1}^{t_2} \left( \int_{t_2 + \delta_1}^{T} \frac{1}{u - \theta} du \right) d\theta \leq c^* (1 + |s|) \delta \int_{t_2 + \delta_1}^{T} du \leq c^* (1 + |s|) \delta \ln \delta. \]

Therefore, for some \( c^* > 0 \)

\[ \limsup_{\delta \to 0} \frac{1}{\delta^2} |h(s,t_2) - h(s,t_1)| \leq c^* (1 + |s|). \]

Now to prove the second part we firstly take the partial derivative of the function \( h \) which may represented by

\[ \frac{\partial}{\partial s} h(s,t) = \frac{1}{\sqrt{2\pi}} \int_t^T \mu(u,t) \left( \int_R \Psi_h(u,z) K e^{-\frac{K^2}{2}} dz \right) du. \tag{7.2} \]

Then,

\[ \frac{\partial}{\partial s} h(s,t) = \int_t^T \mu(u,t) \left( \int_R \Psi_h(s\mu + \sigma_1 K, u) \frac{e^{-\frac{K^2}{2}}}{\sqrt{2\pi}} dK \right) du = \int_t^T \frac{\mu(u,t)}{\sigma_1(u,t)} \left( \mathbf{E} \Psi_h(s\mu(u,t) + \sigma_1(u,t) \xi, u) \xi \right) du, \]

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where $\xi \sim \mathcal{N}(0, 1)$. So, we can represent the derivative Eq. (7.2) as

$$\frac{\partial}{\partial s} h(s, t) = \int_t^T q(t, u) du \quad \text{and} \quad q(t, u) = q_1(t, u)q_2(t, u),$$

where $q_1(t, u) = E \xi \Psi h(s \mu(u, t) + \sigma_1(u, t) \xi, u)$ and $q_2(t, u) = \mu(u, t)/\sigma_1(u, t)$. Setting now $q_3(u) = q(t_2, u) - q(t_1, u)$, we obtain that

$$\frac{\partial}{\partial s} h(s, t_2) - \frac{\partial}{\partial s} h(s, t_1) = \int_{t_2}^T q_3(u) du - \int_{t_1}^{t_2} q(t_1, u) du.$$

Now we recall, that the function $\Psi_h$ is bounded, i.e. $|q(t, u)| \leq c^*/\sqrt{u - t}$ for some $c^* > 0$. Therefore,

$$\left| \frac{\partial}{\partial s} h(s, t_2) - \frac{\partial}{\partial s} h(s, t_1) \right| \leq \int_{t_2}^T |q_3(u)| du + \int_{t_1}^{t_2} \frac{c^*}{\sqrt{u - t_1}} du \leq I_1^*(\delta) + I_2^*(\delta) + 2c^*\sqrt{\delta},$$

where

$$I_1^*(\delta) = \int_{t_2}^{t_2 + \delta_1} |q_3(u)| du \quad \text{and} \quad I_2^*(\delta) = \int_{t_2 + \delta_1}^T |q_3(u)| du.$$

Similarly, for $0 < t_1 < t_2$

$$I_1^*(\delta) \leq c^* \int_{t_2}^{t_2 + \delta_1} \left( \frac{1}{\sqrt{u - t_2}} + \frac{1}{\sqrt{u - t_1}} \right) du \leq 4c^* \sqrt{\delta_1}.$$

To estimate $I_2^*(\delta)$, note that

$$|q_3(u)| = |q_1(u, t_2)q_2(u, t_2) - q_1(u, t_1)q_2(u, t_1)|$$

$$\leq \left| q_2(u, t_2) \left( q_1(u, t_2) - q_1(u, t_1) \right) \right| + \left| q_1(u, t_1) \left( q_2(u, t_2) - q_2(u, t_1) \right) \right|.$$

Moreover, noting that

$$q_2(u, t) = \frac{\mu(u, t)}{\sigma_1(u, t)} \leq \frac{1}{\sigma_1(u, t)} \leq \frac{c^*}{\sqrt{u - t}},$$

we obtain that for $u > t$,

$$|q_3(u)| \leq c \left( |q_2(u, t_2) - q_2(u, t_1)| + \frac{1}{\sqrt{u - t_1}} |q_1(u, t_2) - q_1(u, t_1)| \right).$$
From the definition of $q_1$, we can obtain that for some $c^* > 0$

$$|q_2(u, t_2) - q_2(u, t_1)| \leq \int_{t_1}^{t_2} \frac{1}{(u - \theta)^\frac{1}{2}} d\theta \leq \frac{\delta}{(u - t_2)^\frac{1}{2}}.$$  

and

$$|q_3(u)| \leq \frac{\delta}{(u - t_2)^\frac{1}{2}} + \frac{c^*}{\sqrt{u - t_1}} |q_1(u, t_2) - q_1(u, t_1)|.$$  

It should be noted that Proposition 7.1 implies that for any $0 < \beta < 1$ and for some $c^* > 0$,  

$$|\Psi_h(s_2, t) - \Psi_h(s_1, t)| \leq c^* |s_2 - s_1|^\beta. \quad (7.3)$$

We recall that $|\sigma_1(u, t_2) - \sigma_1(u, t_1)| \leq \delta/\sqrt{u - t_2}$. Therefore,  

$$|q_1(u, t_2) - q_1(u, t_1)| \leq (1 + |s|^\beta) \left( |\mu(u, t_2) - \mu(u, t_1)|^\beta + |\sigma_1(u, t_2) - \sigma_1(u, t_1)|^\beta \right).$$

We recall that $|\sigma_1(u, t_2) - \sigma_1(u, t_1)| \leq \delta/\sqrt{u - t_2}$. Therefore,

$$|q_1(u, t_2) - q_1(u, t_1)| \leq (1 + |s|^\beta) \left( \frac{\delta^\beta}{(u - t_2)^\beta} \right) \leq (1 + |s|^\beta) \left( \frac{\delta^\beta}{(u - t_2)^\beta} \right).$$

Thus,

$$|q_3(u)| \leq \frac{\delta}{(u - t_2)^\frac{1}{2}} + \frac{(1 + |s|^\beta)}{\sqrt{u - t_2}} \frac{\delta^\beta}{(u - t_2)^\beta}.$$  

Therefore,

$$I_2^*(\delta) \leq (1 + |s|^\beta) \int_{t_2 + \delta_1}^T \left( \frac{\delta}{(u - t_2)^\frac{1}{2}} + \frac{\delta^\beta}{(u - t_2)^\frac{3}{2} + \frac{1}{2}} \right) du \leq \left( \frac{\delta}{\sqrt{\delta_1}} + \delta^\beta (\delta_1)^\frac{1 - \beta}{2} \right) (1 + |s|^\beta).$$

Therefore, for any $0 < \beta < 1/2$

$$\lim_{\delta \to 0} \frac{I_1^*(\delta) + I_2^*(\delta)}{\delta^\beta} < +\infty.$$  

Hence, Proposition 7.2. \qed
8 Proofs

8.1 Proof of Theorem 5.1

Let \( h \in X \) be the fixed point for the mapping \( L \), i.e. \( h = L_h \). Consider now the following equation

\[
Y_t(s,t) + \frac{\sigma^2 Y_{ss}(s,t)}{2} + s g_1(t) Y_s(s,t) + \Psi_h(s,t) = 0, \quad Y(s,T) = 0, \tag{8.1}
\]

where \( g_1(t) = \gamma_1 g(t) - \gamma_2 > 0 \) and \( \Psi_h(s,t) \) is given in Eq. (3.8). Then we change the variables as \( u(s,t) = Y(s,T - t) \), so we get

\[
u_t(s,t) - \frac{\sigma^2 u_{ss}(s,t)}{2} - sg_1(t) u_s(s,t) - \Psi_h(s,t) = 0, \quad u(s,0) = 0. \tag{8.2}
\]

We can rewrite the previous equation as

\[
u_t(s,t) - \frac{\sigma^2 u_{ss}(s,t)}{2} + a(s,t,u,u_s) = 0, \quad u(s,0) = 0,
\]

where

\[
a(s,t,u,p) = -sg_1(t)p - \Psi_h(s,t).
\]

Taking into account that

\[
\Psi_{\text{max}} = \sup_{s \in \mathbb{R}} \sup_{t} \Psi_h(s,t) < \infty,
\]

we obtain that \( a(s,t,u,0)u = -\Psi_{\text{max}}|u| \), i.e. the condition in Eq. (A.7) holds with \( \Phi(r) = \Psi_{\text{max}} \) and \( b = 0 \). In view of Propositions 7.1 and 7.2, the function \( \Psi_h \) satisfies the Hölder condition \( C_5 \) for any \( 0 < \beta < 1/2 \). By using Theorem A.2 we obtain that Eq. (8.2) has a bounded solution. Therefore, there exists a solution of Eq. (8.1). In order to prove this proposition we use the probabilistic representation. Now, we define a stopping time \( \tau_n \)

\[
\tau_n = \inf \left\{ n \geq t : |\eta_u^{s,t}| \geq n \right\} \wedge T,
\]

where the process \( (\eta_u^{s,t})_{u \geq t} \) is defined in Eq. (3.6). By the Itô formula we obtain that

\[
Y(s,t) = -\int_t^{\tau_n} \left( (Y_t(\eta_u^{s,t}, u) + g_1(u)\eta_u^{s,t}Y_s(\eta_u^{s,t}, u) + \frac{\sigma^2}{2} Y_{ss}(\eta_u^{s,t}, u) \right) du
\]

\[
- \int_t^{\tau_n} Y_s(\eta_u^{s,t}, u) d\tilde{W}_u + Y(\eta_u^{s,t}, \tau_n).
\]

taking into account the equation Eq. (4.7) we obtain that

\[
Y(s,t) = \int_t^{\tau_n} \Psi_h(\eta_u^{s,t}, u) \, du - \int_t^{\tau_n} Y_s(\eta_u^{s,t}, u) \, d\tilde{W}_u + Y(\tau_n, \eta_u^{s,t}).
\]
Taking into account that $\mathbb{E} \int_{t}^{\tau_n} Y_s(\eta_{u}^{s,t}, u) \, d\tilde{W}_u = 0$, we obtain

$$Y(s, t) = \mathbb{E} \int_{t}^{\tau_n} \Psi_h(\eta_{u}^{s,t}, u) \, du + \mathbb{E} Y(\tau_n, \eta_{\tau_n}^{s,t}).$$

Note here that the solution of the equation Eq. (8.1) is bounded. So, by Dominated Convergence theorem and in view of the boundary condition in Eq. (8.1) we obtain that

$$\lim_{n \to \infty} \mathbb{E} Y(\eta_{\tau_n}^{s,t}, \tau_n) = \mathbb{E} \lim_{n \to \infty} Y(\eta_{\tau_n}^{s,t}, T) = 0.$$

Moreover, taking into account that $\Psi_h \geq 0$, by the Monotone Convergence theorem we obtain

$$Y(s, t) = \mathbb{E} \lim_{n \to \infty} \int_{t}^{\tau_n} \Psi_h(\eta_{u}^{s,t}, u) \, du = \mathbb{E} \int_{t}^{T} \Psi_h(\eta_{u}^{s,t}, u) du,$$

i.e. $Y(s, t) = \mathcal{L}_h(s, t) = h$. Hence Theorem 5.1. \(\square\)

### 8.2 Proof of Theorem 5.2

To proof this theorem we use the verification theorem (A.1) and find the solution to the HJB equation using FK mapping with $h$ a fixed point for the mapping $\mathcal{L}$. Therefore, the function

$$z(\varsigma, t) = \varpi \gamma U(s, t) \quad \text{and} \quad U(s, t) = \exp \left\{ \frac{s^2}{2} g(t) + h(s, t) \right\}, \quad (8.3)$$

is the solution of the HJB equation Eq. (4.5). By using this function we calculate the optimal control variables in Eq. (4.9) and we obtain the strategies Eq. (4.10) - Eq. (4.11). Hence $H_3).$ Now we want to check condition $H_4).$ First note that the equation

$$d\varsigma^*_t = a^*(\varsigma^*_t, t)dt + b^*(\varsigma^*_t, t)dW_t, \quad t \geq 0, \quad \varsigma^*_0 = x$$

is identical to the equation (4.1). By the assumptions on the market parameters, all the coefficients of (4.1) are continuous and bounded. so the usual integrability and Lipschitz conditions are satisfied, this implies $H_4).$

**Lemma 8.1.** There exists some $\delta > 1$ such that for any $0 \leq t < T < T_0$

$$\sup_{\tau \in \mathcal{M}_t} \mathbb{E} \left( Z^\delta(\varsigma_t, \tau)|\varsigma_t = \varsigma \right) < +\infty,$$

where $\mathcal{M}_t$ is the set of all stopping times with $t \leq \tau \leq T$ and the function $z$ is given in Eq. (8.3).
Lemma 8.1 yields condition $H_5$), where $z(s, t) = \varphi x^{\gamma} \exp \{s^2g(t)/2+Y(s, t)\}$. Now, the Verification Theorem A.1 implies Theorem 5.2. □

8.3 Proof of Theorem 5.3

We set $\Delta_n(y, t) = h(y, t) - h_n(y, t)$. So,

$$\Upsilon^*_n(y, t) = \sup_{(y, t) \in K} (|\Delta_n(y, t)| + |D_y \Delta_n(y, t)|) \leq e^{\gamma T} \rho(h, h_n)$$

$$\leq (B_0 + B_1) \frac{\lambda^n}{1 - \lambda} e^{\gamma T} = (B_0 + B_1) \exp\{H(\lambda)\},$$

where $B_0$ and $B_1$ are defined in Eq. (3.3) and $H(\lambda) = B_2 T/\lambda^2 + n \ln n - \ln(1 - \lambda)$.

If we take $\lambda = 1/\sqrt{n}$ and $\kappa = n(B_2)^2$ then we obtain

$$\Upsilon^*_n(y, t) = O(n^{-\delta n}),$$

for any $0 < \delta < 1/2$. Hence Theorem 5.3. □

9 Appendix

A.1 Verification theorem

Now we give the verification theorem from [1]. Consider on the interval $[0, T]$. The stochastic control process given by the $N$-dimensional Itô process

$$\begin{align*}
\dot{\varsigma}^v_t &= a(\varsigma^v_t, t, v)dt + b(t, \varsigma^v_t, v)dW_t, \quad t \geq 0, \\
\varsigma^v_0 &= x \in \mathbb{R}^N, \\
\end{align*}$$

(A.1)

where $(W_{0 \leq t \leq T})$ is a standard $k-$ dimensional Brownian motion. We assume that the control process $v$ takes values in some set $\Theta$. Moreover, we assume that the coefficients $a$ and $b$ satisfy the following conditions:

$V_1$) for all $t \in [0, T]$ the functions $a(., ., .)$ and $b(., ., .)$ are continuous on $\mathbb{R}^N \times \Theta$; where $\Theta \in \mathbb{R} \times \mathbb{R}_+$.

$V_2$) for every deterministic vector $v \in \Theta$ the stochastic differential equation

$$d\varsigma^v_t = a(\varsigma^v_t, t, v)dt + b(\varsigma^v_t, t, v)dW_t$$

has a unique strong solution.
Now we introduce admissible control process for the equation (A.1). We set
\[ F_t = \sigma\{W_u, 0 \leq u \leq t\}, \quad \text{for any } 0 < t \leq T, \]
where a stochastic control process \( \nu = (\nu_t)_{t \geq 0} = (\alpha_t, c_t)_{t \geq 0} \) is called admissible on \([0, T]\) with respect to equation (A.1) if it is \((F_t)_{0 \leq t \leq T}\) progressively measurable with values in \(\Theta\), and equation (A.1) has a unique strong a.s. continuous solution \((\varsigma^\nu_t)_{0 \leq t \leq T}\) such that
\[
\int_0^T (|a(\varsigma^\nu_t, t, \nu_t)| + |b(\varsigma^\nu_t, t, \nu_t)|^2) dt < \infty \quad \text{a.s.}
\]
We denote by \(V\) the set of all admissible control processes with respect to the equation (A.1).

Moreover, let \(f : \mathbb{R}^m \times [0, T] \times \Theta \to [0, \infty)\) and \(h : \mathbb{R}^m \to [0, \infty)\) be continuous utility functions. We define the cost function by
\[
J(x, t, \nu) = \mathbb{E}_{x,t} \left( \int_t^T f(\varsigma, u, \nu_u) du + h(\varsigma^\nu_T) \right), \quad 0 \leq t \leq T,
\]
where \(\mathbb{E}_{x,t}\) is the expectation operator conditional on \(\varsigma^\nu_T = x\). Our goal is to solve the optimization problem
\[
J^*(x, t) := \sup_{\nu \in V} J(x, t, \nu). \tag{A.2}
\]

To this end we introduce the Hamilton function, i.e. for any \(\varsigma\) and \(0 \leq t \leq T\), with \(q \in \mathbb{R}^N\) and symmetric \(N \times N\) matrix \(M\) we set
\[
H(\varsigma, t, q, M) := \sup_{\theta \in \Theta} H_0(\varsigma, t, q, M, \theta),
\]
where
\[
H_0(\varsigma, t, q, M, \theta) := a'(\varsigma, t, \theta)q + \frac{1}{2} tr[bb'(\varsigma, t, \theta)M] + f(\varsigma, t, \theta).
\]
In order to find the solution to (A.2) we investigate the HJB equation
\[
\begin{cases}
z_t(\varsigma, t) + H(\varsigma, t, z(\varsigma, t), z_{\varsigma}(\varsigma, t)) = 0, \quad t \in [0, T], \\
z(\varsigma, T) = h(\varsigma), \quad \varsigma \in \mathbb{R}^N.
\end{cases} \tag{A.3}
\]
Here \(z_t\) denote the partial derivatives of \(z\) with respect to \(t\), \(z_{\varsigma}(\varsigma, t)\) the gradient vector with respect to \(\varsigma\) in \(\mathbb{R}^N\) and \(z_{\varsigma\varsigma}(x, t)\) denotes the symmetric hessian matrix, that is the matrix of the second order partial derivatives with respect to \(\varsigma\).

We assume the following conditions hold:
The functions \( f \) and \( h \) are non negative.

There exists a function \( z(\varsigma) \) from \( C^{2,1}(\mathbb{R}^N \times [0,T]), t \) from \( \mathbb{R}^N \times [0,T] \to (0, \infty) \) which satisfies the HJB equation.

There exists a measurable function \( \theta^* : \mathbb{R}^N \times [0,T] \to \Theta \) such that for all \( \varsigma \in \mathbb{R}^N \) and \( 0 \leq t \leq T \)

\[
H(\varsigma, t, z(\varsigma, t), z_{\varsigma\varsigma}(\varsigma, t)) = H_0(\varsigma, t, z(\varsigma, t), z_{\varsigma\varsigma}(\varsigma, t), \theta^*(\varsigma^0, t)).
\]

There exists a unique strong solution to the Itô equation

\[
d\varsigma^*_t = a(\varsigma^*_t, t)dt + b(\varsigma^*_t, t)dW_t, \quad \varsigma^*_0 = x, \quad t \geq 0,
\]

where \( a(., t) = a(., t, \theta^*(., t)) \) and \( b(., t) = b(., t, \theta^*(., t)) \). Moreover, the optimal control process \( \upsilon^*_t = \theta^*(\upsilon^*_t, t) \) for \( 0 \leq t \leq T \) belongs to \( \mathcal{V} \).

\[
E \left( \int_0^T f(\upsilon)du + h(\varsigma^*_T) \right) < +\infty.
\]

**Theorem A.1.** Assume that conditions \( H_1 \)–\( H_5 \) holds

\[
\Rightarrow \upsilon^*_t = (\upsilon^*_t)_{0 \leq t \leq T}
\]

is a solution for this problem.

**A.2 Cauchy Problem**

Suppose \( u(x, t) \) is the classical solution of the following a nonlinear problem

\[
\begin{cases}
  \mathcal{L}u \equiv u_t - \sum_{1 \leq i,j \leq n} a_{ij}(x,t,u,u_x)u_{x,x_j} + a(x,t,u,u_x) = 0 \\
  u|_{t=0} = u(x,0) = \psi_0(x).
\end{cases}
\]  

(A.4)

We assume that there exist some functions \((a_1, a_2, \ldots, a_n)\), such that

\[
a_{ij}(x,t,u,p) \equiv \frac{\partial a_i(x,t,u,p)}{\partial p_j} \quad \text{and} \quad A(x,t,u,p) \equiv a(x,t,u,p) - \sum_{i=1}^n \frac{\partial a_i}{\partial u} p_i - \sum_{i=1}^n \frac{\partial a_i}{\partial x_i}.
\]

(A.5)

(A.6)
Now for any \( N \geq 1 \),
\[
\Gamma_N = \{(x, t) : |x| \leq N, \ 0 \leq t \leq T\}
\]
We introduce the following conditions for ensuring the existence of the solution \( u(x, t) \) of Cauchy problem.

Suppose that the following conditions hold.

**C_1**) For all \( N \geq 1 \),
\[
\psi_0(x) \in H^{2+\beta}(\Gamma_N) \quad \text{and} \quad \max_{E_n} |\psi_0(x)| < \infty.
\]

**C_2**) There exist \( h \geq 0 \) and some \( \mathbb{R}_+ \to \mathbb{R}_+ \) function \( \Phi \), such that for all \( x \in \mathbb{R}^n, u \in \mathbb{R} \) and for all \( 0 \leq t \leq T \)
\[
A(x, t, u, 0)u \geq -\Phi(|u|)|u| - b,
\]
and
\[
\int_0^\infty \frac{d\tau}{\Phi(\tau)} = \infty.
\]

**C_3**) For \( t \in (0, T] \) and arbitrary \( x, \ u, \ p \in \mathbb{R}^n \), and any \( \xi = (\xi_1, \xi_2, ..., \xi_n) \in \mathbb{R}^n \), there exists \( 0 < \nu < \mu \) such that
\[
\sum_{1 \leq i,j \leq n} a_{ij}(x, t, u, p)\xi_i\xi_j \geq 0 \quad \text{and} \quad \nu|\xi|^2 \leq a_{ij}(x, t, u, p)\xi_i\xi_j \leq \mu|\xi|^2.
\]

**C_4**) The functions \( a_i(x, t, u, p) \) and \( a(x, t, u, p) \) are continuous, and the functions \( (a_i)_{1 \leq i \leq n} \) are differentiable with respect to \( x, u \) and \( p \in \mathbb{R}^n \), and for any \( N \geq 1 \) there exists \( \mu_1 = \mu_1(N) \) such that
\[
\sup_{(x, t) \leq \Gamma_N} \frac{\sum_{i=1}^n (|a_i| + |\partial a_i/\partial u|)(1 + |p|) + \sum_{i,j=1}^n |\partial a_i/\partial x_j| + |a|}{(1 + |p|)^2} \leq \mu_1(N).
\]

**C_5**) For all \( N \geq 1 \), and for all \( |x| \leq N, \ 0 \leq t \leq T, \ |u| \leq N \) and \( |p| \leq N \), the functions \( a_i, \ a, \ \partial a_i/\partial p_j, \ \partial a_i/\partial u, \) and \( \partial a_i/\partial x_i \) are continuous functions satisfying a Hölder condition in \( x, \ t, \ u \) and \( p \) with exponents \( \beta, \ \beta/2, \ \beta \) and \( \beta \) respectively for some \( \beta > 0 \).

**Theorem A.2** (See Theorem 8.1, p. 495 of [13]). Assume that the conditions \( C_1) - C_5 \) hold. Then there exists at least one solution \( u(x, t) \) of Cauchy problem (A.4) that is bounded in \( \mathbb{R}^N \times [0, T] \) which belongs to \( H^{2+\beta, 1+\beta/2}(\Gamma_N) \) for any \( N \geq 1 \).
A.3 Proof of Proposition 6.1

One can check directly that the set $\mathcal{X}$ is closed in the $C^{1,0} (\mathbb{R} \times [0, T])$ which is complete. So, the space $(\mathcal{X}, \rho)$ is complete also. Hence Proposition 6.1. □

A.4 Proof of Proposition 6.3

Firstly, note that from the definition of the mapping in Eq. (3.8) we get for any $\delta > 0$

$$\frac{\mathcal{L}_h(s + \delta, t) - \mathcal{L}_h(s, t)}{\delta} = \int_t^T \int_{\mathbb{R}} \left( \Psi_h(z, u) \left( \frac{\varphi(s + \delta, t, z, u) - \varphi(s, t, z, u)}{\delta} \right) \right) dz \, du.$$  

Taking into account that the function $\rho$ is continuously differentiable, we can rewrite

$$\frac{\varphi(s + \delta, t, z, u) - \varphi(s, t, z, u)}{\delta} = \frac{1}{\delta} \int_s^{s+\delta} \varrho(\nu, t, z, u) d\nu = \varrho(s, t, z, u) + D_\delta(s, t, z, u),$$  

where $\varrho(s, t, z, u) = \partial \varphi(s, t, z, u)/\partial s$ and

$$D_\delta(s, t, z, u) = \frac{1}{\delta} \int_s^{s+\delta} \left( \varrho(\nu, t, z, u) - \varrho(s, t, z, u) \right) d\nu.$$  

So,

$$\frac{\mathcal{L}_h(s + \delta, t) - \mathcal{L}_h(s, t)}{\delta} = \int_t^T \left( \int_{\mathbb{R}} \Psi_h(z, u) \varrho(s, t, z, u) dz \right) du + G_\delta,$$

where $G_\delta = \int_t^T \left( \int_{\mathbb{R}} \Psi_h(z, u) D_\delta(s, t, z, u) dz \right) du$. Now we have to prove that the term $G_\delta$ goes to zero as $\delta \to 0$.

As $\Psi_h(s, t)$ is a bounded function as for $h \in \mathcal{X}$, therefore,

$$|G_\delta| \leq \Psi^* \int_t^T \frac{1}{\delta} \left( \int_s^{s+\delta} L(\nu, u) d\nu \right) du \leq \Psi^* \int_t^T L^*_\delta(u) du$$

where $\Psi^* = \sup_{z \in \mathbb{R}, 0 \leq t \leq T} |\Psi_h(z, u)|$, $L^*_\delta(u) = \max_{s \leq \nu \leq s + \delta} L(\nu, u)$ and

$$L(\nu, u) = \int_{\mathbb{R}} \left| \varrho(\nu, t, z, u) - \varrho(s, t, z, u) \right| dz.$$  

We can check directly that for some $c^* > 0$

$$\sup_{0 < \delta < 1} L^*_\delta(u) \leq \frac{c^*}{\sqrt{u - t}}.$$
Moreover, note that for some $N > 1$

\[
L(\nu, u) \leq \int_{|z| \leq N} |g(\nu, t, z, u) - g(s, t, z, u)|dz + \int_{|z| > N} |g(\nu, t, z, u) - g(s, t, z, u)|dz.
\]

The first part approach zero when $N \to 0$.

\[
\int_{|z| > N} |g(\nu, t, z, u)|dz = \frac{\mu(u, t)}{\sqrt{2\pi\sigma_1(u, t)}} \int_{|\sigma_1 y + \nu u| > N} |y| e^{-\frac{y^2}{2}} dy 
\leq \frac{\mu(u, t)}{\sqrt{2\pi\sigma_1(u, t)}} \int_{|y| > N_1} |y| e^{-\frac{y^2}{2}} dy \to 0 \quad \text{as} \quad N \to \infty.
\]

where $N_1 = \left( N - (|s| + \delta)|\mu| \right)/\sigma_1$, and $s, \mu, \sigma_1$ are fixed.

Thus, for any $s, t, \text{and} \ u$,

\[
\lim_{\delta \to 0} L^*_\delta(u) = 0.
\]

(A.8)

So, by the Lebesgue dominated theorem $\int_T L^*_\delta(u) du \to 0$.

Hence Proposition 6.3. □

A.5 Proof of Lemma 8.1

From the optimal wealth process given in Eq. (4.10) through Itô formula we have that

\[
X^*_t = x \exp \left\{ \int_0^t a^*(u) du \right\} E_{0,t}(b^*),
\]

where the function $a^*$ and $b^*$ are defined in Eq. (4.10) and

\[
E_{0,t}(b^*) = \exp \left\{ \int_0^t b^*(u) dW_u - \frac{1}{2} \int_0^t (b^*(u))^2 du \right\}.
\]

We will show Lemma 8.1 for $\delta = 1 + (1 - \gamma)/2\gamma$, taking into account that

\[
z(s, t) \leq c^* x^\gamma \exp\{s^2 g(0)/2\}.
\]

To this end it is sufficient to show that

\[
\sup_{\tau \in \mathcal{M}_t} \mathbb{E} \left( (X^*_\tau)^{\delta_t} \exp \left\{ \frac{\delta_1}{2}\sigma_{\tau}^2 \right\} \left| X_t = x, S_t = s \right\} \right) < +\infty,
\]

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where $\delta_1 = \gamma \tilde{\delta} = (1 + \gamma)/2 < 1$ and $\tilde{\delta}_1 = g(0)\tilde{\delta}$. Note that $(\mathcal{E}_{t,u})_{u \geq t}$ is supermartingale and $\mathbb{E}\mathcal{E}_{t,\tau}(b^*) \leq 1$ for any stopping time $\tau \in \mathcal{M}$. Moreover, note that

$$S_\tau = e^{-\kappa(T-t)}s + \xi_{t,\tau} \quad \text{and} \quad \xi_{t,\tau} = \sigma e^{-\kappa \tau} \int_t^\tau e^{\kappa u} dW_u.$$  

Since $|S_\tau| \leq |s| + |\xi_{t,\tau}|$, one needs to check that

$$\sup_{\tau \in \mathcal{M}} \mathbb{E} \left( (X_\tau^*)^{\delta_1} \exp \left\{ \delta_1 \xi_{t,\tau}^2 \right\} \right) < +\infty.$$  

By Hölder inequality we obtain that for $p = (1+\delta_1)/2\delta_1$ and $q = (1+\delta_1)/(1-\delta_1)$

$$\mathbb{E}_{t,t}(X_\tau^*)^{\delta_1} \exp \left\{ \delta_1 \xi_{t,\tau}^2 \right\} \leq \left( \mathbb{E}_{t,t}(X_\tau^*)^{\delta_2} \right)^{\frac{p}{q}} \left( \mathbb{E}_{t,t} \exp \left\{ g \delta_1 \xi_{t,\tau}^2 \right\} \right)^{\frac{q}{p}},$$  

where $\delta_2 = p\delta_1 = (1 + \delta_1)/2 < 1$. Note that

$$\mathbb{E}_{t,t}(X_\tau^*)^{\delta_2} = x^{\delta_2} \mathbb{E} \left\{ \delta_2 \int_t^\tau a^*(u) du \right\} \left( \mathbb{E}_{t,\tau}(b^*) \right)^{\delta_2}. $$  

By Hölder inequality, for $r = 1/\delta_2$ and $q_1 = 1/(1-\delta_2)$

$$\mathbb{E}_{t,t}(X_\tau^*)^{\delta_2} \leq x^{\delta_2} \left( \mathbb{E} \left\{ \delta_2 \int_t^\tau a^*(u) du \right\} \right)^{1-\delta_2} \left( \mathbb{E}_{t,\tau}\mathcal{E}_{t,\tau}(b^*) \right)^{\delta_2} \leq x^{\delta_2} \left( \mathbb{E} \left\{ \frac{\delta_2}{1-\delta_2} \int_t^T |a^*(u)| du \right\} \right)^{1-\delta_2}. $$  

Moreover, note that

$$|a^*(t)| \leq \left( g(0) + \frac{\kappa_1}{\sigma^2} \right) \kappa_1 s^2 + \kappa_1 |s| B_1 + 1 + r \leq \kappa_2 s^2 + c^*,$$

where $c^*$ is some constant and $\kappa_2 = \kappa_1^2/2 + 1/2 + g(0)\kappa_1$. So, for some $c^* > 0$

$$\int_t^T |a^*(u)| du \leq 2\kappa_2 \int_t^T \xi_{t,u}^2 du + c^*.$$  

Let us show now that

$$\mathbb{E} \exp \left\{ \kappa_2 \int_t^T \xi_{t,u}^2 du \right\} < +\infty,$$

...
where \( \tilde{\kappa}_2 = 2\delta_2\kappa_2/(1 - \delta_2) = \left(2(3 + \gamma)\kappa_1^2\left(1/\sigma^2 + 1/2 + g(0)/\kappa_1\right)/(1 - \gamma)\right). 

\[
E \exp \left\{ \tilde{\kappa}_2 \left( \int_t^T \xi_{t,u}^2 \, du \right) \right\} = \sum_{m=0}^{\infty} \frac{\tilde{\kappa}_2^m}{m!} E \left( \int_t^T \xi_{t,u}^2 \, du \right)^m < +\infty.
\]

Moreover, note that in view of the Hölder inequality

\[
E \left( \int_t^T \xi_{t,u}^2 \, du \right)^m \leq (T - t)^{m-1} \int_t^T E \xi_{t,u}^{2m} \, du.
\]

Taking into account that \( \xi_{t,u} \sim \mathcal{N}(0, \int_t^u e^{-2\kappa(u-v)} \, dv) \), we obtain that

\[
E \xi_{t,u}^{2m} = (2m - 1)!! \left( \int_t^u e^{-2\kappa(u-v)} \, dv \right)^m \leq \frac{m!}{\kappa^m}
\]

and

\[
E \left( \int_t^T \xi_{t,u}^2 \, du \right)^m \leq m! \frac{T^m}{\kappa^m}.
\]

Therefore,

\[
E \exp \left\{ \tilde{\kappa}_2 \int_t^T \xi_{t,u}^2 \, du \right\} = \sum_{m=0}^{\infty} \frac{\tilde{\kappa}_2^m}{m!} E \left( \int_t^T \xi_{t,u}^2 \, du \right)^m \leq \sum_{m=0}^{\infty} \left( \frac{\tilde{\kappa}_2 T}{\kappa} \right)^m.
\]

In view of the definition of \( T_0 \) in Eq. (3.11) we obtain that the condition \( T < T_0 \) implies that \( T < \kappa/\tilde{\kappa}_2 \), i.e. this series is finite. Moreover, by Proposition A.4, we have that \( E \xi_{t,\tau} \leq m!(2\sigma^2 T)^m \) for all \( m \geq 1 \), and for any \( \tau \in \mathcal{M}_t \), So

\[
E_{\xi,t} \exp \{ q \delta_1 \xi_{t,\tau}^2 \} = 1 + \sum_{m=1}^{\infty} \frac{(q \delta_1)^m}{m!} E \xi_{t,\tau}^m \leq 1 + \sum_{m=1}^{\infty} \frac{(q \delta_1)^m}{m!} m!(2\sigma^2 T)^m \leq 1 + \sum_{m=1}^{\infty} (2q \delta_1 \sigma^2 T)^m < +\infty,
\]

for \( T < 1/2q^2 \delta_1 = \gamma(1 - \gamma)/(3 + \gamma)(1 + \gamma)\sigma^2 g(0) \), which is true due to the condition \( T < T_0 \). Hence Lemma 8.1. □
A.6  The smoothness properties for the function $E_Q(\eta_{u,t}^s, u)$

Lemma A.3. For any bounded function $Q$ in $\mathcal{X}$ and for $u > t$

$$\left| \frac{\partial}{\partial s} \int_t^T E_Q(\eta_{u,t}^s, u) du \right| \leq Q_t^* \frac{2\sqrt{2(T-t)}}{\pi}$$

where

$$Q_t^* = \sup_{s \in \mathbb{R}, \ t < u \leq T} |Q(s, u)|.$$

Proof. By Fubini theorem if a function $Q > 0$ then

$$\frac{\partial}{\partial s} E \int_t^T Q(\eta_{u,t}^s, u) du = \frac{\partial}{\partial s} \int_t^T E_Q(\eta_{u,t}^s, u) du.$$

As

$$E_Q(\eta_{u,t}^s, u) = \frac{1}{\sigma_1(u, t)} \int_{\mathbb{R}} Q(y, u) \varphi \left( \frac{y - s\mu(u, t)}{\sigma_1(u, t)} \right) dy,$$

where

$$\varphi(\theta) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\theta^2}{2}} \quad \text{and} \quad \theta = \frac{z - x\mu(u, t)}{\sigma_1(u, t)}.$$

Then we have that,

$$E_Q(\eta_{u,t}^s, u) = \frac{1}{\sigma_1(u, t)} \int_{\mathbb{R}} Q(y, u) \exp \left\{ - \frac{(y - s\mu(u, t))^2}{2\sigma_1^2(u, t)} \right\} dy.$$

Thus by deriving the last expression with respect to $s$,

$$\frac{\partial}{\partial s} E_Q(\eta_{u,t}^s, u) = \frac{\mu(u, t)}{\sqrt{2\pi\sigma_1^2(u, t)}} \int_{\mathbb{R}} Q(s\mu(u, t) + v\sigma_1(u, t), u) ve^{-v^2/2} dv.$$

Then by letting

$$v = \frac{y - s\mu(u, t)}{\sigma_1(u, t)}.$$

$$\frac{\partial}{\partial s} E_Q(\eta_{u,t}^s, u) = \frac{\mu(u, t)}{\sqrt{2\pi\sigma_1^2(u, t)}} \int_{\mathbb{R}} Q(s\mu(u, t) + v\sigma_1(u, t), u) ve^{-v^2/2} dv.$$

By taking the absolute value for both sides we get

$$\left| \frac{\partial}{\partial s} E_Q(\eta_{u,t}^s, u) \right| \leq \frac{\mu(u, t)}{\sqrt{2\pi\sigma_1^2(u, t)}} \int_{\mathbb{R}} |Q(s\mu(u, t) + v\sigma_1(u, t), u)||v|e^{-v^2/2} dv,$$

$$\leq Q_t^* \frac{\mu(u, t)}{\sqrt{2\pi\sigma(u, t)}} \int_{\mathbb{R}} |v|e^{-v^2/2} dv,$$

$$= Q_t^* \frac{\mu(u, t)}{\sqrt{2\pi\sigma(u, t)}} \left( \int_{\mathbb{R}} \mu^2(u, z) dz \right)^{1/2}.$$
where $Q^*_t = \sup_{y \in \mathbb{R}, u \geq t} |Q(y, u)|$. Therefore,
\[
\left| \frac{\partial}{\partial s} E Q(\eta^s_{u}, u) \right| \leq Q^*_t \sqrt{\frac{2}{\pi}} \frac{\mu(u, t)}{\sigma_1(u, t)}.
\]

Since the integral
\[
\int_t^T \frac{\mu(u, t)}{\sigma_1(u, t)} du = \int_t^T \frac{e^{-\int_t^u g_1(v) dv}}{\sigma_1 u \sqrt{2 \pi}} \frac{1}{\sqrt{u-t}} du = \frac{2}{\sigma} \sqrt{T-t}.
\]

Therefore,
\[
\left| \frac{\partial}{\partial s} E Q(\eta^s_{u}, u) \right| \leq Q^*_t \sqrt{\frac{2}{\pi}} \frac{1}{\sigma \sqrt{u-t}}. \tag{A.10}
\]

Therefore by taking the integral from $t$ to $T$,
\[
\left| \frac{\partial}{\partial s} \int_t^T E Q(\eta^s_{u}, u) du \right| \leq Q^*_t \sqrt{\frac{2}{\pi}} \int_t^T \frac{1}{\sigma \sqrt{u-t}} du \leq Q^*_t \frac{2}{\sigma} \sqrt{2(T-t)} \tag{A.11}.\]

Hence Lemma A.3. □

We need the following Proposition from [7].

**Proposition A.4.** Let $y$ be the solution to the following equation
\[
dy_t = f(y_t, t)dt + G_t dW_t, \quad y_0 = 0. \tag{A.12}
\]

where $yf(y, t) \leq -\kappa y^2$ for all $y$ and for $\kappa > 0$, and
\[
\sup_{0 \leq t \leq T} |G_t| \leq M, \quad \text{for all} \quad M > 0.
\]

Then for any stopping time $0 \leq \tau \leq T$
\[
E|y_\tau|^{2m} \leq m!(2M^2T)^m.
\]

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References

[1] B. Berdjane & S. M. Pergamenchtchikov (2013), *Optimal consumption and investment for markets with random coefficients*, Finance and stochastics, 17, (2), pp. 419-446

[2] M. Boguslavsky & E. Boguslavskaya (2004), *Arbitrage under power*, Risk.

[3] J. Caldeira & G. Moura (2013), *Selection of a portfolio of pairs based on cointegration: a statistical arbitrage strategy*, available at SSRN http://ssrn.com/abstract=2196391.

[4] Duffie, Filipovic & Schachermayer (2003), *Affine processes and applications in finance*, Annals of Applied Probability.

[5] R. J. Elliott, J. V. Der Hoek, and W. P. Malcolm (2005), *Pairs trading*, Quantitative Finance, 5, No. 3, pp. 271–276. ISSN 1469–7688 print/ ISSN 1469–7696 online

[6] P. B. Girma & A. S. Paulson (1999), *Risk arbitrage opportunities in petroleum futures spreads*. J. Futures Markets, 19(8), pp.931–955,

[7] Yu. M. Kabanov & S. M. Pergamenshchikov (2003), *Two-scale stochastic systems. Asymptotic Analysis and Control*. Applications of mathematics. stochastic modelling and applied probability. springer-Verlag Berlin Heidelberg New York.

[8] Kallsen & Muhle-Karbe (2010), *Utility maximization in affine stochastic volatility models*, International Journal of Theoretical and Applied Finance.

[9] I. Karatzas (1989), *Optimization problems in the theory of continuous trading*, SIAM, J. Control and Opt., 27, pp. 1221–1259.

[10] I. Karatzas & S. E. Shreve (1998), *Methods of Mathematical Finance*. springer, Berlin.

[11] Kraft, Seiferling & Seifried (2017), *Optimal consumption and investment with Epstein-Zin recursive utility*, Finance and Stochastics.

[12] Ch. Krauss (2017), *Statistical arbitrage pairs trading strategies: review and outlook*, J. Economic Surveys, 31, No. 2, pp. 513–545.
[13] O. A. Ladyženskaja, V. A. Solonnikov, & N. N. Ural’ceva (1988), Linear and quasilinear equations of parabolic type. (S. Smith, Trans.). Providence, R. I.: American Mathematical society. (Original work published 1967).

[14] R. Merton (1971), Optimal consumption and portfolio rules in a continuous time model, Journal of Economic Theory, 3, pp. 373–413.

[15] M. A. Monroe & R. A. Cohn (1986), The relative efficiency of the gold and treasury bill futures markets, J. Fut. Mark, 6, No 3, pp. 477–493. doi:10.1002/fut.3990060311

[16] A. A. Zebedee, & M. Kasch-Haroutounian (2009), A closer look at co-movements among stock returns, J. Economics and Business, Elsevier, 61(4), pp. 279-294, July.