Optimization-based Motion Planning for Multirotor Aerial Vehicles: a Review

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\section*{Abstract}

In general, optimal motion planning can be performed both as local and as global. In such a planning, the choice in favor of either local or global planning technique mainly depends on whether the environmental conditions are dynamic or static. Hence, the most adequate choice is to use local planning or local planning alongside global planning. When designing optimal motion planning both as local or as global, the key metrics to bear in mind are execution time, asymptotic optimality, quick reaction on dynamic obstacles. Such planning approaches can address the aforesaid target metrics more efficiently compared to other approaches such as path planning followed by smoothing. Thus, the foremost objective of this study is to analyse related literature in order to understand how the motion planning, specially trajectory planning, problem is formulated, when being applied for generating optimal trajectories in real-time for Multirotor Aerial Vehicles (MAVs), impact the listed metrics. As result of the research, trajectory planning problem was broken down into a set of subproblems, the lists of methods for addressing each of the problems were identified and described in detail. Subsequently, the most prominent results from 2010 to 2022 were summarized and presented in the form of a timeline.

\textit{Keywords:} MAVs, B-Spline, Minimum-snap, Model Predictive Control (MPC), Nonlinear Model Predictive Control (NMPC), Linear Quadratic Regulator (LQR), Differential Dynamic Programming (DDP), Optimal Control Problem (OCP), Quadratic Programming (QP), Safe Flight Corridor (SFC), Gradient-based Trajectory Optimization (GTO), Truncated Signed
Distance Field (TSDF)

List of Abbreviations

| Abbreviation | Description |
|--------------|-------------|
| BFGS         | Broyden—Fletcher—Goldfarb—Shanno. |
| CBFs         | Control Barrier Functions. |
| CHOMP        | Covariant Hamiltonian Optimization for Motion Planning. |
| CMPCC        | Corridor-based Model Predictive Contouring Control. |
| DDP          | Differential Dynamic Programming. |
| EO           | Elastic Optimization. |
| ESDF         | Euclidean Signed Distance Field. |
| GTC          | Geometric Tracking Control. |
| GTO          | Gradient-based Trajectory Optimization. |
| iLQR         | Iterative Linear Quadratic Regulator. |
| IRIS         | Iterative Regional Inflation by Semi-definite Programming. |
| JPS          | Jump Point Search. |
| KF           | Kalman Filter. |
| LQG          | Linear Quadratic Gaussian. |
| LQR          | Linear Quadratic Regulator. |
| LTI          | Linear Time Invariant. |
| MAV          | Multirotor Aerial Vehicle. |
| MAVs         | Multirotor Aerial Vehicles. |
| MHE          | Model Horizon Estimation. |
| MIQP         | Mixed Integer Quadratic Programming. |
| MPC          | Model Predictive Control. |
| MPCC         | Mathematical Program with Complementarity Constraints. |
| NLP          | Nonlinear Programming. |
| NMPC         | Nonlinear Model Predictive Control. |
| OCP          | Optimal Control Problem. |
| OCPs         | Optimal Control Problems. |
1. Introduction

Adroit motion planning of flying little creatures, such as birds and butterflies, is an extraordinarily demanding task for several reasons, including aggressive maneuver. An example of such high-speed maneuver need is one in particularly tight spots where the environment is obstacle-rich. Researchers have been trying to replicate similar maneuvers using two different types of aerial vehicles: conventional and unconventional. In this research we deal with conventional areal vehicles, for instance Unmanned Aerial Vehicles (UAVs), Multirotor Aerial Vehicles (MAVs), etc. Recent progression in computation capabilities and embedded sensing has been boosting the procedure of mimicking natural flying animals; this advancement has enabled plenty of new opportunities in diverse fields: inspection, autonomous transportation, logistic, delivery, areal photography, post-disaster and medical services. Yet optimal motion planning remains a crucial task in all the fields listed above. In optimal motion planning, the environment reasoning can not be predictable since environment conditions change.
rapidly. Hence, there are various challenges to be addressed to obtain highly efficient and optimal motion planning. In this paper, we mainly focus on how researchers have been addressing these challenges in optimal motion planning to obtain robust navigation in various domains for MAVs.

In most of the foregoing applications, the environment is entirely or partially unexplored. Furthermore, unpredictable events can occur anytime due to numerous reasons. Thus, to tackle those unexpected problems in real-time, a fast and accurate optimal motion planning technique is required. In general, optimal motion planning problem is divided into a few sub categories: path planning followed by smoothing, kinodynamic search-based trajectory generation, and motion primitive-based approaches. Among them, plan-based control approaches are the most widely used and efficient way to address the considered problem compared to the other two approaches. Plenty of plan-based control strategies have been proposed throughout the recent decade, showing promising results; this is one of the key motivation factors for reviewing plan-based control, especially for industrial MAVs. Most of the industrial MAVs such as quadrotors have their low-level controllers, for example PX4 [1], DJI [2], that can be operated independently irrespective of high-level execution commands. Moreover, such controllers reduce the overhead and complexity for developing high-level planning algorithms due to the independency. In other words, the same planner can be deployed on different firmware by implementing an interface between a high-level planner and low-level controller. Thus, we narrowed down our study to considering only plan-based control approaches (Fig. 1), particularly in application to industrial MAVs.

The main limitation of MAVs is low flight-time. Hence, a MAV should be capable of executing robust, agile, aggressive maneuver while ensuring dynamic feasibility and guaranteeing smoothness of the trajectory in low flight-time. Furthermore, trajectory plotting should be performed within an obstacle-free zone at high-speed to handle a given mission effectively. Such behaviour is imposed by adhering to a set of constraints. If and only if the constraints are incorporated appropriately, desired needs can be fulfilled. Obtaining the right
Figure 1: The overview of plan-based control paradigm in the context of trajectory planning problem formulation. There are various ways to formulate the trajectory planning problem, each of which consists of a set of sub-modules (green color boxes) depends on the problem behaviour.

Constraints at the right moment and applying appropriate control sequences to improve motion quality is the key objective of any plan-based control approach. Yet the procedure of obtaining such right constraints is an open research problem due to its complexity and numerous other challenges that should be handled simultaneously. For example, Multirotor Aerial Vehicle (MAV) has been widely employed in video making related fields in recent years, cinematographic aerial shooting being one of the popular areas interests during last five years. In such shooting, generating smooth, obstacle-free trajectories is the main challenge. Besides, various other challenges exist, most of them are application-specific. In this work, we examine the most common problems related to trajectory planning applications in the paradigm of plan-based control, and how researchers have been alleviating those problems by proposing compelling solutions.

In optimal trajectory planning, trajectory generation and controlling the MAV are strongly interconnected. For MAVs, trajectory generation process is relatively easy due to the dynamic properties of the MAVs. When dynamic ob-
Figure 2: The basic building blocks that encounter in trajectory planning problem. In general, a considered trajectory planning problem can be comprised of one or more blocks sequentially or in parallel to fulfill the desired needs.
Sampling-based method for time-optimal paths generation for a point-mass model [33], a continuous reference trajectory refinement technique for slow-speed maneuvering [34], trajectory planning approach considering geometrical configuration constraints and user-defined dynamic constraints for unconstrained control effort minimization [35], Logistic curve-based trajectory generation technique [36], Gaussian process-based residual dynamic learning [37], nonuniform kinodynamic search-based trajectory generation [38], a standard form of a two-point boundary-value problem using Pontryagin’s minimum principle-based approach is proposed [39].

Online teach and repeat planning technique was proposed [40], in which a geometric controller [41] was utilized for trajectory tracking. Moreover, an iterative trajectory refinement strategy was proposed to relieve the local minima problem where the free space was represented as a convex cluster, i.e., a set of convex polytopes [40], a faster approach for segmenting free space as a set of polytopes using point cloud [10], receding horizon trajectory generation was proposed in [42], whereas trajectory generation for moving target was proposed in [43].

Trajectory planning technique was proposed based on non-uniform B-splines ensuring kinodynamic feasibility [20] where Geometric Tracking Control (GTC) is used for controlling, incremental ESDF method for constructing the environment [44], B-spline based kinodynamic search algorithm followed by elastic-based optimization [23], perception-aware optimal trajectory generation with limited field of view [45], direct collocation method for trajectory generation [46], Minimum-time B-spline trajectory generation [24].

B-spline based kinodynamic search followed by refining the trajectory by using Elastic Optimization (EO) [57], fast marching method alone side with Bernstein basis polynomial trajectory generation [13], Topomap: three-dimensional topological map in which the sparse point cloud was directly utilized to construct the environment [48], continuous-time trajectory optimization technique was applied for generating the trajectory in which initial waypoints were generated using RRG. Furthermore, monocular visual-inertial fusion was used for constructing the environment [49].

Informed RRG method for finding an initial obstacle free path [19], uniform B-spline based trajectory generation [25], using visual features to construct dense map and utilized for extracting obstacle-free space [8], SFC for extracting obstacle-free regions as a convex set [16], free space was constructed as a set of convex polytopes based on stereographic projection [9], topologically distinctive online trajectory planning [50], proposing 3D Jump Point Search (JPS) [16].

Extending Minimum-snap as an unconstrained quadratic program in which path segments were jointly optimized [31], Mixed Integer Quadratic Programming (MIQP) based trajectory generation technique in which free space was segmented convexly by IRIS [52], generating safe avoidance trajectories [53] which was inspired by Covariant Hamiltonian Optimization for Motion Planning (CHOMP) and Minimum-snap. Moreover, it introduces random restart technique to avoid local minima, kinodynamic FMT* followed by Minimum-snap trajectory smoother [29], sophisticated octree-based partitioning tree-based obstacles representation [54].

Proposing IRIS for free space segmentation [17], Minimum-snap trajectory generation using MIQP in which IRIS used for free space segmentation [55], motion primitive based approach for polynomial trajectory generation [50]. Long range navigation based on teach and repeat where iterative closest point matching (ICP) was utilized [57], coordinate descent optimization in which objective was to minimize the along the coordinate hyperplanes [56].

Trajectory generation based on pre-computed convex regions, which were used to build the map [59], trajectory was generated seeking the Time-Optimal Parameterization of a given Path (TOPP) [60].

Local replanning for exploring in which motion primitives were used for ensure the dynamic feasibility [51], path planning by using A* for searching the optimal path in lattice space (x,y,z,heading) followed by motion primitive-based trajectory generator [53], asymptotically optimal kinodynamic RRT* trajectory planner [21], CHOMP trajectory generation and continuous improvement of the initial trajectory considering obstacles and smoothness of the trajectory [63].

Proposing MIQP based approach for trajectory generation [54], seeking different homology classes of trajectories and generating an optimal trajectory subject to those homology classes [55].

Minimum-snap trajectory generation [3], Gradient free optimization technique, STOMP [68], proposing quite faster search algorithm JPS in uniform grid [67].

Free space was extracted by discretizing the space via the 3D Delaunay triangulation [68].

Figure 3: The most prominent related research outcomes which led the success of the trajectory planning for MAVs in the last decade.
stacles are incorporated, the trajectory has to be refined at a high rate in order to keep smooth maneuver despite increased computational demands. Moreover, understanding of close-in obstacles’ positions relative to the MAV is crucial for making decisions in real-time; this arises a new challenge: the one of the rapidity and accuracy of relative environment reconstruction, which essentially is how obstacles constraints are added to the problem formulation. Yet another challenge is the one of the impact of the obstacles constraints on the smoothness and dynamic feasibility of the generated trajectory. After conducting an extensive literature review on the topic of trajectory planning for MAVs, we were able to isolate basic building blocks that are essential for the optimal motion planning as shown in Fig. 2. Each of the primary components plays a key role in the process of trajectory generation. The rest of the paper focuses on understanding of how those building blocks are interconnected in solving trajectory planning problem.

The rest of the paper is organized as follows: section 2 explains what type of motion model is likely to be suitable for defining the dynamics of MAV based on the chosen trajectory generation technique. Then, state-of-the-art techniques on how to find initial tentative waypoints for trajectory generation is explained in section 3. Section 4 presents an extensive review on initial trajectory generation techniques. Section 5 explains how free space is extracted and incorporated into the trajectory planning. Trajectory refinement process is explained in section 6. Horizon-based trajectory planning techniques are described in section 7. Various solvers which can be used to solve the optimization problem are detailed under section 8.

2. Motion Model Selection

Exact model, empirical model and differential flatness are the main techniques that can be employed for selecting the most appropriate motion model for a specified application. The appropriate motion model selection procedure varies depending on problem formulation. For example, planning followed by
controlling approaches does not necessarily have an exact motion model mainly due to high computational demands. In such scenarios, an empirical motion model is sufficient for planning, since a dedicated controller is utilized for controlling the quadrotor.

2.1. Exact Model

In general, MAV dynamics is described by 12-DOF. However, in planning followed by high-level controlling approaches it is not required to define an actual motion model for planning, since high-level controller consists of a fully-fledged quadrotor motion model. In most of the circumstances, the planner is comprised of an approximated quadrotor dynamics; this is due to computational complexity, which is not adequate for real-time on-board processing. Hence, the motion model selection process depends on the approach that formulates needs. In [6], the researchers proposed 12-DOF motion model whose state vector is defined by

\[ x = [p^\top, v^\top, \psi^\top, \omega^\top], \]

where \( \psi, p, v \) and \( \omega \) stand for orientation (rad), position (m), velocity (m/s) and angular velocity (rad/s) in \( \mathbb{R}^3 \), respectively.

The system input or total trust that is applied for each of the motors is given by

\[ f = [f_1, f_2, f_3, f_4]^T \text{ (N)}. \]

State estimation can be formulated as follows, given the current state:

\[ \dot{p} = v \]

\[ \dot{v} = -g \cdot e_z + \frac{(f \cdot \exp[\psi] \cdot e_z - k_v \cdot v)}{m} \]

\[ \dot{\psi} = \omega + \frac{1}{2} [\psi] \cdot \omega + \left(1 - \frac{1}{2 \tan \left(\frac{1}{2} \|\psi\|\right)}\right) [\psi]^2 \cdot \omega / \|\psi\|^2 \]

\[ \dot{\omega} = J^{-1}(\rho(f_2 - f_4)e_x) + \rho(f_3 - f_1)e_y + k_m(f_1 - f_2 + f_3 - f_4)e_z - [\omega] \cdot J \cdot w, \]

where \( g = 9.8 \text{ms}^{-2} \) and \( e_i \), \( i = x, y, z \) stand for standard basis vectors in \( \mathbb{R}^3 \), \( k_v, m, J, \rho \) and \( k_m \) are robot specific constants.

2.2. Empirical Model

Other than the exact model, a 6-DOF motion model was proposed for governing quadrotor in a distributed setup [69]. Later, it was reduced to 4-DOF
motion model [5]. Furthermore, in [70], 4-DOF motion was used for controlling several quadrotors in a distributed setup in which NMPC and Model Horizon Estimation (MHE) are incorporated for relative tracking where the relative motion model was defined as:

\[
\dot{x} = f_c(x, u, \psi_z) = \begin{bmatrix}
\dot{p}_x \\
\dot{p}_y \\
\dot{p}_z \\
\dot{\psi}_z
\end{bmatrix} = \begin{bmatrix}
v_x \cos(\psi_z) - v_y \sin(\psi_z) - \bar{v}_x + \bar{p}_y \bar{\psi}_z \\
v_x \sin(\psi_z) + v_y \cos(\psi_z) - \bar{v}_y - \bar{p}_x \bar{\psi}_z \\
\bar{v}_z - \bar{\psi}_z \\
\bar{\psi}_z - \bar{\psi}_z
\end{bmatrix},
\]

(2)

where the function \(f_c(\cdot) : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_{ru}} \rightarrow \mathbb{R}^{n_x}\) and \(n_x = n_u = n_{ru} = 4\). The current control input is given by \(u = [v_x, v_y, v_z, \dot{\psi}_z]\), whereas relative control input \(u_{ru}\) is denoted by \([\bar{v}_x, \bar{v}_y, \bar{v}_z, \bar{\psi}_z]\). \(x = [p_x, p_y, p_z, \psi_z]\) is the state of the motion model, where \(p_i, i \in \{x, y, z\}\) is the position of the MAV in the world frame. \(\psi_z\) and \(\bar{\psi}_z\) denote yaw angle or heading angle around z axis and relative yaw angle, respectively. Derivative of \(\psi_z\) and \(\bar{\psi}_z\) are denoted by \(\dot{\psi}_z\) and \(\bar{\dot{\psi}}_z\), respectively. \(v_i, i \in \{x, y, z\}\) denote the velocities on each direction, whereas \(\dot{p}_i, i \in \{x, y, z\}\) gives the derivatives of \(p_i\). Since discrete space was chosen for controlling the system, Euler discrete model (2) was formulated as follows:

\[
x^+ = f_d(x, u, \psi_z) = \begin{bmatrix}
p_x \\
p_y \\
p_z \\
\bar{\psi}_z
\end{bmatrix} + \delta \begin{bmatrix}
v_x \cos(\psi_z) - v_y \sin(\psi_z) - \bar{v}_x + \bar{p}_y \bar{\psi}_z \\
v_x \sin(\psi_z) + v_y \cos(\psi_z) - \bar{v}_y - \bar{p}_x \bar{\psi}_z \\
v_z - \bar{\psi}_z \\
\bar{\psi}_z - \bar{\psi}_z
\end{bmatrix},
\]

(3)

where \(\delta\) is the sampling period and \(f_d(\cdot) : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_{ru}} \rightarrow \mathbb{R}^{n_x}\). \(f_c\) and \(f_d\) denote continuous and discrete dynamics, respectively. \(x^+\) depicts the next state given the current state \(x\). Subsequently, the motion model was simplified to 4-DOF for trajectory tracking for a quadrotor [71, eq.(1)]. In this trajectory tracking approach, planning followed by the high-level controlling paradigm was applied. Such an approach was introduced because simplified motion model is a reasonable choice for achieving real-time performance. Quadrotor state was defined as \(x = [p_x, p_y, p_z, \psi_z]^T \in \mathbb{R}^{n_x}\), whereas input to the system was given...
by \( u = [v_x, v_y, v_z, \dot{\psi}_z]^T \in \mathbb{R}^{n_u} \). The simplified motion model was given by

\[
\dot{x} = f_c(x, u) = \begin{bmatrix}
\dot{p}_x \\
\dot{p}_y \\
\dot{p}_z \\
\dot{\psi}_z
\end{bmatrix} = \begin{bmatrix}
v_x \cos(\psi_z) - v_y \sin(\psi_z) \\
v_x \sin(\psi_z) + v_y \cos(\psi_z) \\
v_z \\
\dot{\psi}_z
\end{bmatrix}, \tag{4}
\]

where \( f_c(\cdot) : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \rightarrow \mathbb{R}^{n_x} \) and \( n_x = n_u = 4 \). The discretization of (4) was given by:

\[
x^+ = f_d(x, u) = \begin{bmatrix}
p_x \\
p_y \\
p_z \\
\dot{\psi}_z
\end{bmatrix} + \delta \begin{bmatrix}
v_x \cos(\psi_z) - v_y \sin(\psi_z) \\
v_x \sin(\psi_z) + v_y \cos(\psi_z) \\
v_z \\
\dot{\psi}_z
\end{bmatrix}, \tag{5}
\]

where \( f_d(\cdot) : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \rightarrow \mathbb{R}^{n_x} \).

### 2.3. Differential Flatness

Here differential flatness \cite{72} provides algebraic functions (e.g., polynomials) which analytically map the trajectory and whose higher-order derivatives map to system states and inputs. Since Nth order polynomial can be differentiated up to N-1 times, differential flatness property ensures the feasibility of the trajectory and generates appropriate control commands. More precisely, let

\[
\dot{x} = f_c(x, u) \quad x \in \mathbb{R}^{n_x}, u \in \mathbb{R}^{n_u}. \tag{6}
\]

be a nonlinear system. According to \cite{73}, if the system is differentially flat, there always exists a flat output, namely \( z \in \mathbb{R}^{n_z} \), where the dimension of the output is given by \( n_z \). In such a system, states and control inputs can also be formulated from the system flat outputs whose derivatives are mapped through functions, namely \( \varrho \) and \( \tau \). Let \( z = \Im(x, u, \dot{u}, ..., u^{(q)}) \) be the flat output, holding \( x = \varrho(z, \dot{z}, ..., z^{(r)}) \) and \( u = \tau(z, \dot{z}, ..., z^{(r)}) \), where apices \( ^{(i)} \) stipulates the \( i \)th derivative. Along with that, explicit trajectory generation process can benefit when it uses differentially flat systems, for example, \( \varrho \) and \( \tau \) can be a dth order polynomial \( p(t) \). Then, \( x^T(t) = [p^T(t) \; \dot{p}^T(t) \; \ddot{p}^T(t)] \) be the state of the system at
time $t$ in which $\dot{p}^T$ and $\ddot{p}^T$ indicate the velocity and acceleration of the system, respectively. Control inputs can be determined by jerk \cite{74}, namely $\ddot{p}^T(t)$ where $p(t) = \lambda_0 + \lambda_1 t + \lambda_2 t^2 + \ldots + \lambda_d t^d, \ t \in [0, dt]$, where $\lambda_i, i = 0, ..., d$ are the polynomial coefficients. There are various ways to construct these kinds of polynomials, including Minimum-snap, B-spline, etc.

3. Initial Waypoints Identification

Generally speaking, robots have a limited sensing range. So, planning a trajectory out of such sensing range would be counterproductive. Hence, local trajectory planning and refinement when a robot moves is the optimal choice. With the help of sensing capabilities within robots’ sensing range, the robot’s surrounded environment can be constructed as intersection of three separate disjoint sets: free-known ($C_{free}$), occupied ($C_{obs}$) and unknown ($C_{unknown}$). Once $C_{free} \cup C_{unknown}$ is identified, a set of intermediate waypoints is needed to navigate the robot along the trajectory from the start position to the desired position. There are various techniques for finding a set of intermediate waypoints: sampling-based techniques (e.g., RRT*, Probabilistic Road Map (PRM)), path searching techniques (e.g., A*, D*, JPS) and so forth. Moreover, kinodynamic properties are incorporated into preceding intermediate waypoints finding techniques to ensure the dynamic feasibility of the robot. One of the first kinodynamic-based path planning approaches was proposed in \cite{75} in which a variant of A* method alongside with kinodynamic properties was applied to ensure the dynamic feasibility. Subsequently, several different methods were proposed for enhancing the path planning, ensuring the dynamic feasibility by kinodynamic properties, including motion primitive-based approaches.

Motion primitive based approaches\cite{56,76,77} can be utilized for finding intermediate waypoints and for trajectory generation. Gordon et al. \cite{78} proposed a set of motion primitives for connecting edges of the graph that was constructed from A*. In this method, motion primitives were used to define state vector $\mathbf{x}(t)$ and control input $\mathbf{u}(t)$ as a Linear Time Invariant (LTI) sys-
where $p_\mu(t) = \sum_{d=0}^{d} \lambda_d t^d$, $\mu \in \{x,y,z\}$, which is formulated similar to (17), while $k_\tau$ and $d$ are the order of the derivative and the order of the polynomial, respectively.

Hence, given control policy $u_i(t)$ and initial state $x(0)$, a sequence of succeeding states for a given time duration is determined by

$$x_i(t) = e^{At}x(0) + \int_0^t e^{A(t-\gamma)}Bu(\gamma)d\gamma,$$  

where $\gamma$ is the time duration that control policy is applied to. In [78], to define the actual and heuristic cost of A*, the researchers used motion primitives, which are defined (as shown) in (9), and calculated initial waypoints set.

Another interesting approach to finding a set of initial intermediate waypoints is by using fast marching methods. In general, fast marching methods [79] are applied to track the propagation of a convoluted interface such as wavefront, especially in image processing. Let $\varphi$ be a close curve in a plane $\in \mathbb{R}^3$ that propagates orthogonal to the plane with a speed $v(p)$, assume $v > 0$. Given $\nabla T$ time period, propagation of the plane can be described by $|\nabla T(x)| = \frac{1}{v(p)}$ based on Eikonal partial differential equation [80], where $p$ is the position in $\mathbb{R}^3$ and the arrival time is formulated by $T(x)$. Fast marching concept was applied for path searching in [13] by proposing a method for calculating velocity map. In this method, the arrival time was determined by assessing the desired velocity at the considered position. Hence, arrival time was calculated by backtracking
from the goal pose to the start pose along the minimum cost path, which can be estimated from gradient descendant. Though gradient descendant may trap in a local minimum, when smart marching is applied, gradient descendant does not trap in local minimum due to fast marching nature; this property was proved in [81]. To define the velocity map, ESDF was utilized to get the closest obstacle poses from the given pose. A quadrotor should move faster when there are no close-in obstacles and should be slower when it is moving through a cluttered environment. Such a behaviour was mimicked by incorporating a hyperbolic tangential function, i.e., tanh. With such an assumption, the corresponding velocity was calculated based on (10)

\[
v(l) = \begin{cases} 
  v_{\text{max}}(\text{tanh}(l - e) + 1)/2, & 0 \leq l < l_0 \\
  0, & l < 0 
\end{cases}
\]

where \(v_{\text{max}}\) is the maximum velocity a quadrotor can fly, \(l\) is the distance to the closest obstacle from the considered pose \(p\) and \(e\) is Euler’s constant.

4. Initial Trajectory Generation

Let us consider a non-linear system in the form of \(\dot{x}(t) = f_c(x(t), u(t))\) with initial state \(x(t_0) = x_0\), where state vector and control inputs are denoted by \(x \in \mathbb{R}^n_x\) and \(u \in \mathbb{R}^n_u\), respectively. When generating an initial trajectory (\(\Gamma\)), ensuring dynamic feasibility is a must. In other words, \(x\) and \(u\) satisfy the following constraints:

\[
x \in X \subseteq \mathbb{R}^n_x, \quad u \in U \subseteq \mathbb{R}^n_u
\]

In addition to these constraints, safely constraints should also be imposed after reasoning the environment, to guarantee safety. The environment or configuration space \(C\) can be decomposed into \(C_{\text{obs}}\) and \(C_{\text{free}}\). Hence, a set of constraints should be introduced for the quadrotor to always be within free space \(x \in C_{\text{free}} = C / C_{\text{obs}}\). Hence, initial trajectory generation process has to consider both said types of constraints simultaneously so that the quadrotor would have smooth flying experience.
4.1. Define Trajectory

Let $\Gamma \leftarrow C \subset \mathbb{R}^d$ be an initial trajectory, which is parameterized as a function of time where $d$ denotes the $C$’s dimension. Since $\Gamma$ is a function, the objective of the trajectory generator is to determine the precise objective, which will eventually provide the optimal trajectory in timely manner satisfying constraints and hypotheses that are imposed. Hence, optimal trajectory, namely $\Gamma^*$, can be posed as a discrete or continuous OCP [82]:

$$\Gamma^* = \min_{u(\cdot)} J(x(0), u(\cdot))$$

s.t. $x(0) = x_0, \ x(t_n) = x_n$

$$\dot{x}(t) = f_c(x(t), u(t))$$

$$x(t) \in C_{free}, \ u(t) \in U, \ t \in [t_0, t_n],$$

where $t_0$ and $t_n$ denote the start and terminal time, respectively. Yet another challenging problem is to formulate the objective function, namely $J$. In the following subsections, we discuss several approaches to address this problem.

4.2. Minimum-snap based Trajectory Generation

Minimum-snap trajectory generation [3] uses the differential flatness property (section 2.3) to automate trajectory generation process. Let quadrotor trajectory be $\Gamma_T(t) = [r_T(t), \psi_T(t)]^T$ for flat output $[x, y, z, \psi_z]^T$ where $r = [x, y, z]$ is the center position of the MAV with respect to world coordinate system and $\psi_z$ is the yaw angle of the MAV. The continuous trajectory can be expressed as follows:

$$\Gamma(t) : [t_0, t_n] \leftarrow \mathbb{R}^d,$$

where $d$ is the dimension of the space, e.g., 3. As we defined in section 2.3, system states and inputs can be determined in terms of $\Gamma$ and its derivatives. $\Gamma$, $\dot{\Gamma}$ and $\ddot{\Gamma}$ will correspond to position, velocity and acceleration, respectively. Flat output and its derivatives estimation in Minimum-snap, refer to the original work [3] eqs.(1-35)].
In Minimum-snap trajectory parameterization, total time duration of the trajectory is divided into a set of sub-intervals, i.e., keyframes. Each keyframe consists of a desired position and a yaw angle. A safe corridor is constructed between consecutive keyframes as a set of piecewise polynomial functions to estimate smooth transitions through the keyframes. Let $m_d$ and $d$ be the number of keyframes and the order of the piecewise polynomial functions, respectively. Hence, $\Gamma_T(t)$ can be formulated as

\[
\Gamma_T(t) = \begin{cases} 
\sum_{i=0}^{d} \Gamma_i,1(t-t_0)^i & t_0 \leq t < t_1 \\
\sum_{i=0}^{d} \Gamma_i,2(t-t_1)^i & t_1 \leq t < t_2 \\
\vdots \\
\sum_{i=0}^{d} \Gamma_i,m_d(t-t_{m_d-1})^i & t_{m_d-1} \leq t < t_{m_d}
\end{cases} 
\]  \quad (14)

To generate an optimal trajectory, the following objective is utilized:

\[
J(r_T, \psi_T) = \int_{t_0}^{t_{m_d}} \xi_r \left\| \frac{d^k r_T}{dt^k} \right\|^2 dt + \xi_\psi \frac{d^k \psi_T}{dt^k} \right\|^2 dt 
\]

\[
\min_w J(r_T, \psi_T) \quad \text{s.t.} \quad \Gamma_T(t_i) = \Gamma_i, \quad i = 1, ..., m_d \\
\frac{d^p x_T}{dt^p}|_{t=t_j} \leq 0 \quad j = 0, m_d; \quad p = 1, ..., k_r \\
\frac{d^p y_T}{dt^p}|_{t=t_j} \leq 0 \quad j = 0, m_d; \quad p = 1, ..., k_r \\
\frac{d^p z_T}{dt^p}|_{t=t_j} \leq 0 \quad j = 0, m_d; \quad p = 1, ..., k_r \\
\frac{d^p \psi_T}{dt^p}|_{t=t_j} \leq 0 \quad j = 0, m_d; \quad p = 1, ..., k_\psi,
\]  \quad (15)

where $\xi_r$ and $\xi_\psi$ are regulation parameters, $k_r$ and $k_\psi$ are the order of derivation at each keyframe and $\Gamma_T(t_i) = [x_i, y_i, z_i, \psi_z]^T, i = 0, ..., T$. Time intervals, $t_1, t_2, ..., t_{m_d}$ can be kept constant or varying when deriving the Minimum-snap trajectory generation. In most of the cases, having varying time intervals between keyframes is necessary. Mellinger et al. [3] proposed a gradient descent-based approach for finding optimal time intervals between keyframes. Further, Chen et al. [12] utilized A* to find the intermediate waypoints. Based on these estimations, time segments or keyframes are calculated incorporating both ve-
locity and acceleration limits. In the latter approach, the steps listed below were used to obtain intermediate waypoints. Initially, environment was constructed as a map using OctoMap. Afterwards, the formed map was split into two subsets: allocated and non-allocated (a set of free spaces). Then, the discrete graph was constructed connecting consecutive free spaces, which were represented as cubes. Afterwards, A* was applied for finding the optimal path segment within each cube. Similar to (15), the researchers set $k_r = 3$ and minimized only total jerk (16) to minimize the angular velocity. As an aside, minimizing the angular velocity helps to avoid fast rotation.

$$J = \int_{t_0}^{t_{md}} \xi_r \left\| \frac{d^{k_r} \Gamma_r(t)}{dt^{k_r}} \right\|^2 dt. \quad (16)$$

4.3. Polynomial Trajectory Generation as QP

In Minimum-snap trajectory generation, total trust force, i.e., attitude acceleration, is proportional to fourth derivative (snap) of the trajectory [3]. The gracefulness of such behavior helps to avoid generating excessive control commands. Subsequently, a slight variation of Minimum-snap trajectory generation was proposed in [51], where segment times or keyframes were fixed initially. Once start and goal position were provided, RRT* [21] was utilized for finding obstacle-free path between the start and the goal poses as a sequence of optimal waypoints. Initial segment times ($m_d$), which were estimated using optimal waypoints, were calculated according to the maximum velocities that quadrotor is allowed to fly due to set technical limits. Let $p_i(t)$ be the $d$th order polynomial in the $i$th segment that describes as follows:

$$p_i(t) = \lambda_0 t^0 + \lambda_1 t^1 + \lambda_2 t^2 + \lambda_3 t^3 + ... + \lambda_d t^d. \quad (17)$$

Each $p_i(t)$ provides a flat output for a given time index $t$. $\lambda_j, j = 0, ..., d$ denotes the polynomial coefficients. The objective or cost function $J(\Gamma_i)$ can be fully determined by penalizing the derivatives of squares [51]:

$$J(\Gamma_i) = \int_{t_i}^{t_{i+1}} \xi_0 p_i(t)^2 + \xi_1 \dot{p}_i(t)^2 + \xi_2 \ddot{p}_i(t)^2 + ... + \xi_{k_r} \Gamma^{(k_r)}(t)^2 = P_i^T Q(T_i) P_i, \quad (18)$$
where $P_i$ is a vector whose elements contain polynomial coefficients: $\xi_0, \xi_1, \ldots, \xi_{k_i}$, $k_i^r$ is the highest order of derivative and $Q(T_i)$ is Hassin matrix, which contains the ith segment squares of derivatives. Since there are $m_d$ number of segments, total cost $J(\Gamma)$ can be expressed by

$$J(\Gamma) = \begin{bmatrix} P_1 \\ \vdots \\ P_{m_d} \end{bmatrix}^T \begin{bmatrix} Q(T_1) & \ddots & \vdots \\ \vdots & Q(T_{m_d}) \\ \vdots \end{bmatrix} \begin{bmatrix} P_1 \\ \vdots \\ P_{m_d} \end{bmatrix}.$$ (19)

For a smooth flight experience, ensuring the continuity of derivatives between segments is necessary. Hence, imposing constraints between segments, e.g., velocity, acceleration, jerk and snap is needed, which can be formulated as follows:

$$C_i p_i = d_i, \quad C_i = \begin{bmatrix} \xi_0 \\ \vdots \\ \xi_{k_i} \end{bmatrix}, \quad d_i = \begin{bmatrix} d_0 \\ \vdots \\ d_{k_i} \end{bmatrix},$$ (20)

where $C_i$ contains a mapping matrix whose entries contain the start and end coefficients of ith segment, whereas $d_i$ contains derivative values, i.e., start and end of ith segment. Taking all constraints of $m_n$ segments,

$$C \begin{bmatrix} p_1 \\ \vdots \\ p_{m_d} \end{bmatrix} = \begin{bmatrix} d_1 \\ \vdots \\ d_{m_d} \end{bmatrix}.$$ (21)

Now this can be solved as a constrained QP problem.

### 4.4. Unconstrained Polynomial Trajectory Generation

The techniques that are used for unconstrained trajectory optimization are faster than constraints optimization. In [51], the researchers extended Minimum-snap trajectory generation as an unconstrained QP. According to their findings, Minimum-snap works well for small segments size. For higher-order polynomials with varying segment sizes, Minimum-snap becomes ill-conditioned. Thus, an unconstrained QP was proposed. After substituting (20) and (21) into (19),
$J(\Gamma)$ can be reformulated as

$$J(\Gamma) = \begin{bmatrix} d_1 & \cdots & d_{m_d} \end{bmatrix}^T \begin{bmatrix} C(T_1) & \cdots & C(T_{m_d}) \end{bmatrix} \begin{bmatrix} Q(T_1) & \cdots & Q(T_{m_d}) \end{bmatrix}^{-1} \begin{bmatrix} d_1 \\ \vdots \\ d_{m_d} \end{bmatrix}$$

(22)

where $\mathbf{d}$ contains fixed derivatives ($d_f$) and free derivatives ($d_p$), $\mathbf{S}$ is a permutation matrix (ones and zeros), which is used to correct the order. Then, $\frac{\partial J(\Gamma)}{\partial d_p} = 0$ yields the optimal value for $d_p$:

$$d_p^* = -R_{pp}^{-1}R_{fp}^T d_f.$$  
(23)

Once $d_p$ is determined, a polynomial that corresponds to each segment can be recovered.

4.5. Unconstrained Polynomial Trajectory Generation with Collision Avoidance

Oleynikova et al. [53] extended what Richter [51] proposed for adding support for collision avoidance capabilities. They added additional term for calculating the collision cost,

$$J(\Gamma) = \xi_{\text{obs}} J_{\text{obs}}(\Gamma) + \xi_{\text{smooth}} J_{\text{smooth}}(\Gamma),$$

(24)

where $J_{\text{smooth}}$ exactly equals (22). To estimate $J_{\text{obs}}(\Gamma)$, it is required to initially calculate position $\mathbf{p}_i(t)$ [17] and velocity $\mathbf{v}_i(t)$ for each axis at time $t$ after
selecting the corresponding segment \((i, i = 1, ..., m_d)\)

\[
p_i(t) = T p_i, \quad p_i = [\lambda_0, \lambda_1, ..., \lambda_d]^T, \quad T = [t^0, t^1, t^2, ..., t^d],
\]

\[
v_i(t) = \dot{p}_i(t) = TV p_i,
\]

\[
\frac{\partial J_{\text{obs}}(\Gamma_i)}{\partial d p_i(t)} = \sum_{t=0}^{t^d} \|v_i(t)\| \nabla_i c(T(C^{-1}S)_{pp}) \Delta t + c(p_i(t)) \frac{v_i(t)}{\|v_i(t)\|} TV(C^{-1}S)_{pp} \Delta t,
\]

where \((C^{-1}S)_{pp}\) is the right-side matrix which corresponds to \(d_p\). For representing the collision cost \(c(p_i(t))\), a line integral of a potential function, i.e., (45), was used. As total cost is given (22), \(J_{\text{obs}}(\Gamma)\) can be calculated for all the segments provided that \(d^*_p\) can be estimated. In a cluttered environment, optimization problem is most likely to be non-linear as well as non-convex. Thus, Broyden—Fletcher—Goldfarb—Shanno (BFGS) [83] was used to solve the optimization problem. Yet the solver failed to obtain the global minimum most of the time. Hence, several random restarts were needed to find the optimal solution. A thorough discussion of how random restarts were invoked into the optimization problem was detailed in [59].

4.6. Covariant Gradients for Trajectory Generation

The significance of covariant gradients technique is that both \(J_{\text{obs}}(\Gamma)\) and \(J_{\text{smooth}}(\Gamma)\) depend solely on physical characteristic of the desired trajectory. In other words, the trajectory generation is invariant to its parameterization. If gradient descent is applied, it depends on the way trajectory is parameterized. The covariant gradients technique removes this dependency. Hence, covariant gradient technique depends solely on physical representation or dynamic quantities of the trajectory with respect to an operator, \(\Theta\).

\[
\|\Gamma\|^2_\Theta = \int \sum_{n=1}^{k} \xi(\Gamma(t)^{(n)})^2 dt,
\]

20
where $\xi$ is a constant and apices $(^n)$ determine the $n$th order derivative. The correlation of derivatives between two trajectories: $\Gamma_1$ and $\Gamma_2$, is defined by assuming inner product as given (28).

$$< \Gamma_1, \Gamma_2 > = \int \sum_{n=1}^{k} \xi \Gamma_1(t)^{(n)} \Gamma_2(t)^{(n)} dt.$$  \hspace{1cm} \text{(28)}

The primary objective of $\Theta$ is to distinguish the norm (27) and the inner product (28) from the L2 norm [63].

### 4.7. B-spline based Trajectory Generation

$d^{th}$ order B-spline can be defined for a given knot sequence $p_k = \{t_0, t_1, ..., t_{n_k}\}$ and control points $p_c = \{p_0, p_1, ..., p_{n_p}\}$, where $t_\ast$ $\in$ $\mathbb{R}$, $p_\ast$ $\in$ $\mathbb{R}^d$ and $n_k = n_p + d + 1$. If $d$ is set to 3, each $p_i$ represents position in $\mathbb{R}^3$, where $i = 0, ..., n_p$.

For a given time index $t$, the corresponding position $p(t)$ can be fully determined by using De-Boor-Cox formula [84].

$$p(t) = \text{DeBoorCox}(t, p_c).$$  \hspace{1cm} \text{(29)}

Estimation is not limited to the position; velocity, acceleration or any high order derivative of $p_c$ can be estimated using $\text{DeBoorCox}(t, p_c^{(\ast)})$ as given in Algorithm. 1, where $(\ast)$ depicts the order of the derivative of $p_c$ such that $(\ast) < d$.

Later, the B-spline matrix representation was proposed by Qin [85]. B-spline can be formulated as uniform or non-uniform. J. Hu et al. [86] detailed the uniform B-spline matrix representation. In uniform B-spline, knot span is the same for any considered consecutive time interval, i.e., $\Delta t = t_{i+1} - t_i$, $i \in [0, n_k)$.

Any position of the trajectory can be parameterized by considering only $d+1$ consecutive control points: $[p_i, p_{i+1}, ..., p_{i+d}]$. Hence, corresponding normalized time $q(t)$ can be calculated as follows:

$$q(t) = \frac{t - t_i}{t_{i+1} - t_i} = \frac{t - t_i}{\Delta t}, \quad t \in [t_i, t_{i+1}].$$  \hspace{1cm} \text{(30)}

In the matrix representation, $c(q(t))$, which is given in [29], can be determined
Algorithm 1 The B-spline trajectory \( (p) \) and its derivative estimation for a given time index \( t \), where \( p \) equals \( p^c(\ast) \)

1: procedure DeBoorCox\((t, p)\)
2: \[ t = \begin{cases} 
  p_k[d], & \text{if } t < p_k[d] \\
  p_k[n_k], & \text{if } t > p_k[n_k] \\
  t, & \text{otherwise} 
\end{cases} \]
3: \( k = d \)
4: while true do
5: \quad if \( p_k[k + 1] \geq t \) then
6: \quad \quad break
7: \quad k++
8: \quad \text{p}_e[d]
9: \quad for \( i \leftarrow 0 \) to \( d \) do
10: \quad \quad \text{p}_e[i] \leftarrow p[k - d + i]
11: \quad for \( r \leftarrow 1 \) to \( d \) do
12: \quad \quad for \( i \leftarrow d \) to \( r \) do
13: \quad \quad \quad \beta \leftarrow \frac{t - p_k[i + k - d]}{p_k[i + k + r] - p_k[i + k - d]}
14: \quad \quad \quad \text{p}_e[i] \leftarrow (1 - \beta) \times \text{p}_e[i - 1] + \beta \times \text{p}_e[i]
15: \quad return \text{p}_e[d]
by:

\[ c(q(t)) = q(t)M_d p_i, \quad q(t) = [1, q(t), q^2(t), ..., q^d(t)]^T, \quad p_i = [p_i, p_{i+1}, ..., p_{i+d}]^T, \]

\[ M_d \in \mathbb{R}^{d+1 \times d+1}, \quad M_{r,c} = \frac{1}{d!} \left( \frac{d}{d-r} \right) \sum_{s=c}^{d} (-1)^{s-c} \times \left( \frac{d}{s-c} \right) (d-s)^{d+1-r-s}. \]

(31)

Since each control point \( p_i \) belongs to \( d+1 \) of successive spans, B-spline can be controlled locally. Due to such controllability, b-spline is suitable for local trajectory planning \[25\]. Moreover, the derivatives of a given B-spline are also B-spline \[34\]. Hence, B-spline’s derivatives (e.g., velocity, acceleration, jerk) can be calculated considering corresponding span \([t_i, t_{i+1}]\) for a given \( d+1 \) consecutive control points \( p_i = [p_i, p_{i+1}, ..., p_{i+d}]^T \in \mathbb{R}^{d \times 3} \) and corresponding knot vector.

\[ \frac{dc(q(t))}{du} = \frac{1}{(\Delta t)} b_1 M_d v_i^T, \quad b_1 = [0, 1, u, ..., u^{d-1}] \in \mathbb{R}^{d+1}, \]

\[ \frac{d^2c(q(t))}{d^2u} = \frac{1}{(\Delta t^2)} b_2 M_d v_i^T, \quad b_2 = [0, 0, 1, u, ..., u^{d-2}] \in \mathbb{R}^{d+1}. \]

(32)

The explicit form of estimation of velocity and acceleration of a given time index is calculated as follows:

\[ \frac{dc(q(t))}{du} = d \cdot \frac{p_c(i+1) - p_c(i)}{p_k(i+d+1) - p_k(i+1)}, \]

\[ \frac{d^2c(q(t))}{d^2u} = (d^2 - d) \cdot \left( \frac{p_c(i+2) - p_c(i+1)}{p_k(i+d+2) - p_k(i+2)} - \frac{p_c(i+1) - p_c(i)}{p_k(i+d+1) - p_k(i+1)} \right). \]

(33)

In most of the situations, initial control points are generated as explained in section \[3\]. Such methods may be or may be not smooth enough for initial trajectory generation. There are various ways to generate intermediate waypoints to improve the quality of the trajectory using B-splines. For example, initial trajectory was constructed using cubic B-Spline in \[71\]. Such a capability is mainly due to B-spline’s properties.

It is particularly continuity and convex-hall properties that make B-spline trajectory generation such a robust technique.

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4.7.1. Convex Hull Property

Among the properties of B-spline, convex hull property is the most significant property due to its capabilities for checking the dynamical feasibility and the collision. How convex hull property is incorporated for calculating dynamical feasibility is given in (33). As shown in Fig. 4, \( d_h > 0 \) and \( d_h > d_c - r_h \) should be held for a considered point in the trajectory to ensure a collision-free trajectory, where \( d_c \) is the distance between a given control point and its closest obstacle position. In \( d \)th order B-spline, convex hull is formed by connecting any successive \( d + 1 \) control points, e.g., \( p_i, p_{i+1}, p_{i+2}, \ldots, p_{i+d} \) or union of all consecutive control points that lie on the corresponding b-spline curve [78]. Moreover, \( r_h \) can be substituted with \( d_{i,i+1} + d_{i+1,i+2} + d_{i+2,i+3} \) since \( r_h \leq d_{i,i+1} + d_{i+1,i+2} + d_{i+2,i+3} \), where \( d_{i,i+1} = \|p_{i+1} - p_i\|, \; d_{i+1,i+2} = \|p_{i+2} - p_{i+1}\| \) and \( d_{i+2,i+3} = \|p_{i+4} - p_{i+3}\| \). As mentioned in [20], the following condition should hold for collision free tra-
trajectory planning:

\[ d_{i,i+1} < \frac{d_c}{3}, \quad d_c > 0, \quad i \in \{1, 2, 3\}. \]  (34)

4.7.2. Continuity

B-spline based trajectory generation has several advantages over the piece-wise based trajectory generation [51, 53]. The boundary constraints are to be satisfied explicitly to guarantee the continuity of a piece-wise trajectory. In such a trajectory, the smoothness of the trajectory solely depends on the way control points are formed. On the other hand, boundary constraints can be neglected since the whole trajectory can be treated as one segment in B-spline based trajectory generation. Moreover, B-spline based trajectory can be controlled locally, as explained in section 4.7.1 without affecting the rest of the trajectory.

Figure 5: Trajectory generation using uniform B-spline. The smoothness of the curve is dependent on the degree of B-spline. The trajectory passes precisely through the given control points at the degree equal to 1, as depicted in light blue color. Smoothness of the trajectory increases with the order of the B-spline.
4.8. Bernstein Piecewise Trajectory Generation

Bernstein polynomial is a specific form of B-spline, which is similar to the Bezier curve [87, 88]. Bernstein polynomial can be described as follows:

$$\Gamma_j(t) = \lambda_0^j P_d^0(t) + \lambda_1^j P_d^1(t) + \ldots + \lambda_d^j P_d^d(t) = \sum_{i=0}^{d} \lambda_i^j P_d^i(t),$$

where $d$ is the degree of the polynomial, $\lambda_0^j, \lambda_1^j, \ldots, \lambda_d^j$ are the control points of $j$th polynomial segment and $t \in [0,1]$. Since Bezier is a particular form of B-spline curve, such curves hold convex hull property. Hence, given a sequence of control points, a constrained convex hull can be defined using the control points that are considered. Both the beginning and end of the curve are determined by the first and the last control points, respectively. Further, derivative of Bezier is also a Bezier curve.

$$\Gamma_{\mu}(t) = \begin{cases} 
    s_1 \cdot \sum_{i=0}^{d} \lambda_i^1 \mu P_d^i \left( \frac{t-t_0}{s_1} \right) & t_0 \leq t < t_1 \\
    s_2 \cdot \sum_{i=0}^{d} \lambda_i^2 \mu P_d^i \left( \frac{t-t_1}{s_2} \right) & t_1 \leq t < t_2 \\
    \vdots \\
    s_m \cdot \sum_{i=0}^{d} \lambda_i^m \mu P_d^i \left( \frac{t-t_{m-1}}{s_{m_d}} \right) & t_{m_d-1} \leq t < t_{m_d} 
\end{cases},$$

where $i, j$ refer to $i^{th}$ control point in $j^{th}$ segment, i.e., $\lambda_i^j, s_j$ is a scaling factor of $j^{th}$ segment for mapping time duration from $[0,1]$ to $[t_{j-1}, t_j]$ and $\mu \in \{x, y, z\}$. Once $\Gamma_{\mu}(t)$ is obtained, the objective is to minimize the total cost, which can be determined by taking integral of square error up to $k_r$ order as given in (16). Such a problem can be formulated as a QP constraint problem. For instance, Gao and Wu [13] proposed Bernstein-based trajectory optimization approach in which three types of constraints piecewise trajectory continuity, safety constraints which are based on convex hull property, and dynamical feasibility constraints enforced [13].

4.9. Comparison of several trajectory techniques

In the preceding subsections, several types of trajectory parameterization techniques were considered. We have selected three different types of trajectory
parameterization techniques for this comparison: piecewise-polynomials technique, fitting based on a sequence of points, and the third is uniform B-spline based technique. The objective of piecewise-polynomials is to find optimal polynomial coefficients [8] or end-derivatives [51] of consecutive segments, whereas the objective of the third technique is to find a set of points satisfying the provided constraints [26]. A comparison of how velocity, acceleration, jerk, and snap are varied for selected techniques in terms of mean, standard deviation (std), min and max for the same a set of control points and knot vector is present bellow. Considered control points and knot vector are

\[
p_{ctrl} = [[0.011, -0.0329, 2.017], [1.867, 3.408, 1.6], [7.514, 5.715, 3.735],
[8.410, 0.911, 1.600], [6.902, -5.531, 4.306], [1.899, -6.680, 3.082],
[-2.302, -0.611, 5.375]]
\]

\[
p_{knot} = [0.0, 5.0, 12.0, 18.0, 26.0, 31.0, 40]
\]

Each approach has its own set of parameters to fine-tune for obtaining an optimal trajectory. The generated trajectories are shown in Fig. 5 with different configuration setup (with different parameter set). Fig. 6 shows how the derivatives up to the 4th change over time on each direction, i.e., x, y, z, separately for each technique. When looking at the derivatives of each method, it is clear that smoothness, which is the main point to be considered for motion planning, is higher in both B-spline and Minimum-snap compared to CHOMP. Since uniform B-spline is used in this comparison, smoothness changes of each derivative between B-spline and Minimum-snap can not be compared directly due to time allocation when generating the trajectories. Hence, Minimum-snap trajectory smoothness can be changed, optimizing the time allocation process [51]. On the contrary, such a time allocation process is not necessary for uniform B-spline.

Yet control points are interpolated appropriately to generate a continuous and smooth trajectory.

We varied the parameter set of each approach appropriately and estimated mean, std, max, min of velocity, acceleration, jerk, and snap profile; the result is given in Table. 1. The results clearly indicate that the consistence of trajectory
depends on the parameters that are used to parameterize the trajectory. Hence, appropriate parameter set selection for a given task is of utmost importance, which can be seen by looking at the statistical properties (mean, std, min, and max) of higher order derivatives, e.g., velocity, acceleration, jerk, and snap. As described in the previous paragraph, time allocation process directly affects the parameter selection of Minimum-snap. Further, optimal polynomial coefficients process depends on time allocation as given in (14). On the other hand, Polytraj generation process has fewer parameters to be optimized since it uses free end-derivatives of each segment. Hence, the latter technique is faster that Minimum-snap.

Figure 6: Changes of position, velocity, acceleration, jerk, and snap profiles over time for the provided control points sequence and knot vector.

5. Free Space Extraction

Obstacles regions identification is utmost essential for optimal trajectory planning in real-time. In a cluttered environment, the way trajectory planning problem formulated is matters for fast reaction. Such trajectory planning ap-
proaches can be designed as QP mainly due to less computation power required for such tasks. Hence, forming obstacles-free regions in the form of convex has more advantages in terms of reducing the computation power, simplicity and fast convergence. Chen [12] attempted to define free space as a series of cubes between start and goal pose. Thenceforth, OctoMap [89] was used for constructing the map surrounding the quadrotor, where regions with no obstacles are considered as free spaces. After obtaining the free space information, obstacle constraints are enforced into (16) to generate optimal trajectory.

Let $C = [c_1^m, c_2^m, ...]$ be a set of consecutive grids within the OctoMap and corresponding free space regions be $C_{free} = [c_1^f, c_2^f, ...]$. Both $c_i^m$ and $c_i^f$ were defined as cubes, each of which is described by

$$c_i^m = \left[ c_{x_0}^m, c_{y_0}^m, c_{z_0}^m, c_{x_1}^m, c_{y_1}^m, c_{z_1}^m \right], \quad c_i^f = \left[ c_{x_0}^f, c_{y_0}^f, c_{z_0}^f, c_{x_1}^f, c_{y_1}^f, c_{z_1}^f \right], \quad (38)$$

Once $C_{free}$ was obtained, free space regions can be considered as a set of inequality constraints that can be added into the piece-wise polynomials trajectory
Table 1: Velocity, acceleration, jerk, and snap profile for generating an optimal trajectory for a given set of knot vector and control points (Fig. 6) using three different techniques: Minimum-snap [3], Poly-traj [51], and CHOMP [26].

| Type                           | Velocity | Acceleration |
|--------------------------------|----------|--------------|
|                                | mean     | std          | min     | max     | mean     | std          | min     | max     |
| Poly-traj, d: 8, mc: 2         | 0.0058   | 1.0154       | -1.4545 | 3.9179  | 0.0056   | 0.9051       | -2.835  | 3.6449  
| Poly-traj, d: 6, mc: 4         | 0.006    | 1.0708       | -1.7176 | 3.7864  | 0.0043   | 0.9307       | -2.7887 | 3.6032  
| Poly-traj, d: 8, mc: 4         | 0.0059   | 1.0299       | -1.4728 | 3.934   | 0.0053   | 0.9131       | -2.9157 | 3.5214  
| Poly-traj, d: 10, mc: 4        | 0.0058   | 1.0057       | -1.4428 | 3.9213  | 0.0052   | 0.8918       | -2.7541 | 3.631   
| Minimum-snap, d: 8, mc: 2      | 0.1258   | 1.2154       | -4.4345 | 3.1259  | 0.1258   | 1.2154       | -3.2874 | 3.3278  
| Minimum-snap, d: 8, mc: 6      | 0.045    | 0.0094       | -0.07   | 0.019   | 0.09     | 0.0007       | -0.0098 | 0.0104  
| Minimum-snap, d: 6, mc: 4      | 0.0689   | 1.0009       | -1.3416 | 3.2388  | 0.0012   | 0.4584       | -2.3189 | 3.2185  
| Minimum-snap, d: 8, mc: 4      | 0.0615   | 1.0412       | -1.3215 | 3.7543  | 0.0075   | 0.8763       | -2.5487 | 3.3215  
| Minimum-snap, d: 10, mc: 4     | 0.0036   | 1.0006       | -1.3428 | 3.7832  | 0.0099   | 0.4378       | -2.4548 | 3.4983  
| CHOMP, pd: 3                   | 0.0068   | 0.6421       | -0.9522 | 1.7255  | 0.0045   | 0.3786       | -1.131  | 1.476   
| CHOMP, pd: 5                   | 0.0065   | 0.644        | -0.634  | 1.7161  | 0.0044   | 0.3909       | -1.1082 | 1.4418  
| CHOMP, pd: 7                   | 0.0064   | 0.6443       | -0.966  | 1.7105  | 0.0043   | 0.3916       | -1.0551 | 1.4205  

| Type                           | Jerk     | Snap         |
|--------------------------------|----------|--------------|
|                                | mean     | std          | min     | max     | mean     | std          | min     | max     |
| Poly-traj, d: 8, mc: 2         | 0.007    | 1.2544       | -4.8056 | 3.9318  | -0.0151  | 2.3178       | -8.8029 | 6.9483  
| Poly-traj, d: 8, mc: 6         | 0.0117   | 1.568        | -5.5746 | 5.7423  | -0.1288  | 3.5271       | -13.4562 | 10.2578 |
| Poly-traj, d: 6, mc: 4         | -0.021   | 1.2562       | -4.1192 | 3.7562  | 0.0131   | 1.9593       | -7.7134 | 6.6049  
| Poly-traj, d: 10, mc: 4        | 0.0074   | 1.3399       | -5.5769 | 4.499   | 0.0054   | 1.0773       | -12.3429 | 9.9033  
| Minimum-snap, d: 8, mc: 2      | 0.0006   | 1.125        | -4.3413 | 3.5153  | -0.0042  | 2.1383       | -8.0056 | 6.3148  
| Minimum-snap, d: 8, mc: 6      | 0.0005   | 0.0004       | -0.0007 | 0.0049  | 0.0005   | 0.0064       | -0.0008 | 0.0009  
| Minimum-snap, d: 6, mc: 4      | 0.117    | 1.3456       | -5.2167 | 5.321   | -0.0693  | 4.214        | -12.5124 | 9.2134  
| Minimum-snap, d: 8, mc: 4      | -0.001   | 1.1321       | -3.7192 | 3.3217  | 0.0093   | 1.2145       | -5.6527 | 4.7854  
| Minimum-snap, d: 10, mc: 4     | 0.0009   | 1.2145       | -3.9987 | 3.9983  | -0.0067  | 2.8731       | -10.7653 | 8.8416  
| CHOMP, pd: 3                   | 0.0021   | 0.3643       | -1.2594 | 1.1584  | -0.0014  | 0.4239       | -1.8326 | 1.5425  
| CHOMP, pd: 5                   | 0.0023   | 0.3628       | -1.2553 | 1.1639  | 0.0005   | 0.4241       | -1.8526 | 1.6243  
| CHOMP, pd: 7                   | 0.0022   | 0.3614       | -1.2732 | 1.1769  | 0.0021   | 0.4247       | -1.7462 | 1.5996  

| d: order of polynomial, mc: maximum continuity or maximum continuity order in between consecutive segments, pd: number of proposed points or point density per defined time duration of the trajectory |

In such a trajectory, additional boundary constraints should be introduced if the extrema of $d$th order polynomial violate the boundary constraints corresponding to each axis, i.e., x, y and z in each segment [12, eq. 10]. Similar to
the preceding approach, Gao and Shen [14] proposed a sequence of spheres to represent free space from the initial position to the final position. To construct the environment, a map was not built; instead, they bypassed map building by constructing a KD-tree [11] based place holder to store raw point cloud for the LiDAR. Afterwards, a relative map to the current pose of the MAV was retrieved using nearest neighbour search; RRG [90] combined with A* was used to find a flight corridor or intermediate waypoints. Such intermediate waypoints were connected by overlapping spheres.

IRIS [7] is one of the first successful ideas in which obstacle-free spaces are extracted using a convex optimization technique. In this proposed approach, initially it is required to provide a seeking point and an area with a boundary box where an obstacle-free region to be searched. Seeking point is defined as a unit ball: \( \varepsilon(C, p_0) = \{ p = C\hat{p} + p_0 \mid \|\hat{p}\|_2 \leq 1 \} \), where \( p_0 \) is the center point. The linear constraints, which separate the boundary box into obstacle-free and obstacle-rich regions, are defined as a set of hyper-planes: \( P = \{ p \mid A p \leq b \} \).

Subsequently, finding the optimal representation of \( \varepsilon(C, p_0) \) and \( P \) with respect to given obstacles, \( \varepsilon_j, j = 1, ..., N \) is solved as an iterative process (39).

\[
\begin{align*}
\min_{C,p_0,A,b} & \quad -\log(\det C) \\
\text{s.t.} & \quad A_j^T p_k \geq b_j \quad \forall p_k \in \varepsilon_j, \quad j = 1, ..., N \\
& \quad \sup_{\|\hat{p}\|} A_i^T (C\hat{p} + p_0) \leq b_i \quad \forall i = 1, ..., N,
\end{align*}
\]

(39)

where \( A_i \) and \( b_i \) correspond to ith row of \( A \) and \( b \). The first constraint, i.e., \( A_j^T p_k \geq b_j \), is imposed to move obstacle into one side of the plane, \( A_j^T p = b_j \), whereas the second constraint, i.e., \( \sup_{\|\hat{p}\|} A_i^T (C\hat{p} + p_0) \leq b_i \), ensures the ellipsoid lies on the other side of the plane. The researches proposed to solve the (39) as a two-step process: searching, first, for proper constraints (i.e., \( A_i \) and \( b_i \)) and then the maximum volume that satisfies ellipsoid, ensuring preceding constraints. In other words, they attempted to find hyperplanes that separate obstacle regions and free region. Conceptually, hyperplanes separation is done by finding planes that intersect with obstacle boundaries. Afterwards,
the ellipsoid is uniformly expanded until it intersects with obstacle boundaries. Let $\alpha$ be the scaling factor which defines the expansion. Let $\varepsilon_\alpha = \{ C\tilde{p} + p_0 | \|\tilde{p}\|_2 \leq \alpha \}$ for $\alpha \geq 1$ be the expanded ellipsoid. Hence, the optimal $\alpha^*$ can be determined by

$$\alpha^* = \arg \min_\alpha \varepsilon_\alpha \cap t_j \neq \emptyset$$

(40)

After finding $\alpha^*$, it is possible to define the optimal inscribed ellipsoid ($\varepsilon^*$), which is the obstacle-free region [7, sec.3.3].

Sikang et al. [16] proposed a new, quite different from the aforesaid IRIS, approach for extracting obstacle-free regions as a convex set SFC. SFC searches a set of overlapping polyhedra from start pose to goal pose. To get intermediate obstacle-free positions, the researchers utilized a graph search technique, namely JPS [67]. The main reason for selecting JPS over sampling-based algorithms (e.g., RRT* and PRM) or search-based techniques such as A* or Dijkstra is due to the nature of JPS; it uses uniform-cost grid map with uniform voxels. In general, sampling-based techniques are not deterministic though probabilistically complete. Thus, there is no guarantee about the duration of searching time. On the other hand, computational time for search-based methods is pretty high if the environment is cluttered. However, JPS has lower searching time compared to A* [16]. Let $p_c = p_0, p_1, ..., p_n$ be the intermediate waypoints from start to goal pose and $l_i = \langle p_i, p_{i+1} \rangle$ be the $i^{th}$ line segment, where $i = 1, ..., n - 1$.

Each line segment constitutes a convex polyhedra, namely, $E_i$. Along with that, SFC can be expressed as $SFC(P) = \{ E_i | i = 0, ..., n - 1 \}$. SFC has two steps: finding $E_i$ that fits the $l_i$ and seeking a set of linear inequalities that are tangent to $E_i$. Let $E_i = \varepsilon_i(C_i, p_i^0) = \{ p = C_i\tilde{p} + p_i^0 | \\|\tilde{p}\|_2 \leq 1 \}$. In $\mathbb{R}^3$, $C_i$ can be decomposed as $R^T SR$, where $R$ gives the axis of rotation between considered line segment in between $p_i$ and $p_{i+1})$. Semi-axis of $E_i$ is given by $S = \text{diag}(a, b, c)$ as a diagonal matrix. $p_i^0$ is the center of $l_i$. The objective of SFC is to find each pair $E_i$ and $p_i^0$, given the $l_i$ and obstacles set ($\text{Obs}_i$), which touches the $E_i$.

Initially, ellipsoids are spheres whose center poses are located as the mid
Figure 8: Free space extraction using SFC. Once intermediate initial waypoints are defined, SFC calculates free space along the path, which is constructed form the initial waypoints points of \(l_i, \ i = 1, \ldots, n - 1\). Afterwards, semi-axes except for the axis along \(p_{i+1} - p_i\), are shrunk until the corresponding ellipsoid contains no obstacles. Let \(\varepsilon_i^{*}(C_i, p_{i}^0)\) be the ith ellipsoid after applying the shrinking process. \(p_j\) denotes the closest point that touches the \(\varepsilon_i^{*}(C_i, p_{i}^0)\), where \(j = 1, \ldots, m\) and \(m\) is the number of obstacles. Hence, corresponding half-space \(H_j = \{p_j \mid a_j^T p_j < b_j\}\) is defined as a plane that is tangential to \(\varepsilon_i^{*}(C, p_0)\), where \(a_j\) and \(b_j\) are determined by:

\[
a_j = \frac{d\varepsilon_i}{dp_{p=p_j}} = 2C_i^{-1}C_i^{-T}(p_j - p_{i}^0), \quad b_j = a_j^T p_j.
\]

Hence, the intersections of these \(m\) half spaces create a convex polyhedron \(C = \bigcup_{j=0}^{m} H_j = \{p \mid A^T p < b\}\). The same approach is applied to each line-segment, \(l_i\) in which we can generate each \(C_i\). All in all, \(SFC(P) = \{C \mid i = 0, \ldots, n - 1\}\) can be constructed. More descriptive formulation is in [16, Algorithm 1].

6. Continuous Trajectory Refinement

The objective function consists of several sub-objective functions: for improving the smoothness, for avoiding the obstacles and so forth. In this section,
a precise explanation is given on how to construct sub-objective functions for each of the various occasions. First, we examine the simplest case where only dynamic feasibility and obstacle avoidance constraints are taken into consideration. Let $J$ be the objective function or performance index

$$J(\Gamma) = \xi_{\text{smooth}} J_{\text{smooth}}(\Gamma) + \xi_{\text{obs}} J_{\text{obs}}(\Gamma).$$  \hspace{1cm} (42)

There are various formulations of how $J_{\text{obs}}$ and $J_{\text{smooth}}$ are determined. In general, $J_{\text{smooth}}$ can be expressed as:

$$J_{\text{smooth}}(\Gamma) = \frac{1}{2} \int_0^1 \left\| \frac{d\Gamma(t)}{dt} \right\|^2 dt. \hspace{1cm} (43)$$

Eliminating unnecessary higher-order motion is the main objective of the $J_{\text{smooth}}$. On the other hand, $J_{\text{obs}}$ encourages to generate or modify collision-free trajectory by trying to push control points away from the obstacle zone if the trajectory is already in collisions or penalizing parts of the trajectory that is close to the obstacles. Let $B \subset \mathbb{R}^d$ be the exterior boundary of the MAV and $c$ is the cost function of penalizing close-in obstacles with respect to $B$. Along with that, $J_{\text{obs}}$ can be formulated as follows:

$$J_{\text{obs}}(\Gamma) = \int_0^1 \int_{u \in B} c(f_c(\Gamma(t), p)) \left\| \frac{df_c(\Gamma(t), p)}{dt} \right\|^2 dp dt, \hspace{1cm} (44)$$

where the function $f_c(\Gamma(t), p)$, which was proposed by Ratliff at al. \cite{26}, can be defined as follows:

$$f_c(\Gamma(t), p) = \begin{cases} 
-\text{dis}(\Gamma(t), p) + \frac{1}{2} \delta_{\text{dis}} & \text{if } \text{dis}(\Gamma(t), p) < 0 \\
\frac{1}{\delta_{\text{dis}}^2}(\text{dis}(\Gamma(t), p) - \delta_{\text{dis}})^2 & \text{if } 0 < \text{dis}(\Gamma(t), p) \leq \delta_{\text{dis}} \ 
0 & \text{otherwise}
\end{cases} \hspace{1cm} (45)$$

where $\delta_{\text{dis}}$ denotes the distance from the boundary (B) of the quadrotor to a given obstacle position. Before taking gradient at i, $J(\Gamma)$ is linearized around i, $J_i(\Gamma) \approx J(\Gamma_i) + (\Gamma - \Gamma_i)^T \nabla J(\Gamma_i)$. Defining c and d is detailed in \cite{26} eqs.\cite{22-28}.

In \cite{20}, the cost of the trajectory was estimated based on the following formulation:

$$J(\Gamma) = \xi_{\text{obs}} J_{\text{obs}}(\Gamma) + \xi_{\text{smooth}} J_{\text{smooth}}(\Gamma) + \xi_{\text{soft}} J_{\text{soft}}(\Gamma), \hspace{1cm} J_{\text{soft}}(\Gamma) = J_v(\Gamma) + J_a(\Gamma), \hspace{1cm} (46)$$
where $J_{soft}(\Gamma)$ is determined by soft limits on acceleration and velocity. $J_{smooth}(\Gamma)$ is defined by considering only geometric information without minimizing snap and/or jerk \[3\]. Such minimization is required because of the following stages of the trajectory optimization. In such trajectory optimization, time reallocation has less impact on the objective function. Hence, $J_{smooth}(\Gamma)$ is defined as follows:

$$J_{smooth}(\Gamma) = \sum_{i=d-1}^{n+d} \left\| \frac{p_{i+1} - p_i + p_{i-1} - p_i}{f_{i+1,i}} \right\|^2,$$

where number of control points, denoted $n$, and $f_{i+1,i}$ and $f_{i-1,i}$ can be interpreted as connecting joint force of two springs between control points pairs: $(p_{i+1}, p_i)$ and $(p_i, p_{i-1})$, for example, control points lie on a straight line if the sum of all terms equals zero. As an aside, similar approaches were proposed in \[91, 92\].

The value of $J_{obs}(\Gamma)$ is determined by calculating the distance to the closest object pose from each control point in which the distance to the obstacle, i.e., $f_c(p_i)$, is given by

$$f_c(p_i) = \begin{cases} (\text{dis}(p_i) - \delta)^2 & \text{dis}(p_i) \leq \delta_{\text{dis}} \\ 0 & \text{dis}(p_i) > \delta_{\text{dis}} \end{cases},$$

where $\delta_{\text{dis}}$ is the free distance between MAV’s center and the pose of the closest obstacle. Hence, $J_{obs}(\Gamma) = \sum_{i=d}^{n-d} f_c(p_i)$ can be estimated based on a given trajectory in the form of control points. Soft constraints are defined by not exceeding both acceleration and velocity within the those max limits.

$$J_v(\Gamma) = \sum_{i=d-1}^{n-d} f_v(v_{i,\mu}), \quad J_a(\Gamma) = \sum_{i=d-2}^{k-d} f_a(a_{i,\mu})$$

$$f(v) = \begin{cases} (v_{\mu}^2 - v_{\text{max}}^2)^2 & v_{\mu} > v_{\text{max}} \\ 0 & v_{\mu} \leq v_{\text{max}}^2 \end{cases}, \quad f(a) = \begin{cases} (a_{\mu}^2 - a_{\text{max}}^2)^2 & a_{\mu} > a_{\text{max}} \\ 0 & a_{\mu} \leq a_{\text{max}}^2 \end{cases}.$$  

To calculate acceleration and velocity at each control point and when both acceleration and velocity exceed their maximum limits, convex hull property \[34\] of b-spline is utilized to penalize those control points. Based on the previous
method. [25] proposed an endpoint cost \( J_{\text{endpoint}}(\Gamma) \), into the objective function as an additional term. The key intuition behind adding \( J_{\text{endpoint}}(\Gamma) \) is to reduce the error between local trajectory and global trajectory, since \( J_{\text{endpoint}}(\Gamma) \) penalizes error of both velocity and position with respect to the desired global trajectory. \( J_{\text{endpoint}}(\Gamma) \) is determined as follows:

\[
J_{\text{endpoint}}(\Gamma) = J_{\text{end}}(\Gamma) = \xi_p p_{\text{end}}(t_{\text{end}}) - p_{\text{end}})^2 + \xi_v \dot{p}(t_{\text{end}}) - \dot{p}_{\text{end}})^2,
\]

(50)

where \( \xi_p \) and \( \xi_v \) are regularization parameters, whereas \( p_{\text{end}} \) and \( \dot{p}_{\text{end}} \) are desired end position and velocity of the trajectory.

7. Receding Horizon Trajectory Planning

On most occasions, paths which are obtained by planning techniques are sub-optimal. Hence, the initial trajectory that is generated based on the initial path is to be refined, ensuring dynamic feasibility for controlling the MAV. Various approaches can be applied for trajectory refinement. However, enabling recursive feasibility, incorporating terminal constraints and convergence to the desired state are the utmost importance considerations to be contemplated throughout the process. LQR and MPC are two most popular approaches that are being used for receding horizon planning. LQR is applied for linear systems, whereas iLQR and DDP are applied for non-linear system. Both in LQR or iLQR, OCP is defined as an open-loop control problem. On the other hand, MPC is designed as close-loop OCP. In other words, OCP is seeking actions knowing the behaviour of the surround environment.

7.1. LQR based trajectory generation

DDP [93, 94] is one of the first techniques proposed for solving optimal control problems. Let \( x_{k+1} = f_d(x_k, u_k) \) be the discrete-time system dynamics; the total cost of the trajectory can be formulated for a given control policy, i.e., \( \pi_{k+i} \), for all \( i = \{0, 1, ..., N - 1\} \).

\[
\sum_{i=0}^{N-1} c(x_{k+i}, u_{k+i}) + c_{\text{goal}}(x_{k+N}).
\]

(51)
The optimal control input, i.e., \( u_{k+i} = \pi_{k+i}(x_{k+i}) \), for a given time index, i.e., \( i+k \), can be obtained by minimizing the (51). Thus, cost (cost-to-go) which was proposed in [95] is fully determined by

\[
V_{k+i}(x_{k+i}) = \min_{u_{k+i}} \left( c(x_{k+i}, u_{k+i}) + V_{k+i+1}(f_d(x_{k+i}, u_{k+i})) \right).
\]  

The same procedure can be applied recursively in backward direction for seeking the optimal \( \pi_{k+i}(x_{k+i}) = \arg\min_{u_{k+i}} \left( c(x_{k+i}, u_{k+i}) + V_{k+i+1}(f_d(x_{k+i}, u_{k+i})) \right) \). DDP yields almost the same behaviour: first estimate optimal control and then apply a forward pass to determine the updated nominal trajectory. Consequently, LQR is a simplified version of DDP. LQR is one of the fundamental ways to obtain a closed-form solution for a given optimal control problem under which system dynamics is assumed to be linear. Let us assume the system dynamics is defined as in (5). The intuition of LQR is to estimate optimal control sequence for maneuvering the quadrotor from an initial position to the desired pose. Let \( N \) be the receding horizon whose optimal trajectory is to be determined. The total cost, i.e., \( J_k(x_k, \pi_N) \), consists of three parts: initial, intermediate and final costs, where \( \pi_N = \{ \pi_k, \pi_{k+1}, ..., \pi_{k+i}, ..., \pi_{N-1} \} \)

\[
J_k(x_k, \pi_N) = c_{\text{start}}(x_k) + \sum_{i=0}^{N-1} c(x_{k+i}, u_{k+i}) dt + c_{\text{end}}(x_{k+N}),
\]

where

\[
\frac{\partial^2 C_{\text{start}}(x_k)}{\partial x \partial x} \leq 0, \quad \frac{\partial^2 C_{\text{goal}}(x_{k+N})}{\partial x \partial x} \leq 0, \quad \frac{\partial^2 C}{\partial x \partial u} \leq 0, \quad \text{and} \quad \frac{\partial^2 C}{\partial u \partial u} \leq 0
\]

are positive semidefinite Hessians to guarantee the minimizing of the total cost. The total cost can be formulated in various ways. In LQR, the total cost is
defined as Quadratic costs as follows:

\[ c_{\text{start}}(x_k) = \frac{1}{2} x_k^T Q_{\text{start}} x_k + x_k^T q_{\text{start}}, \]

\[ c_{\text{goal}}(x_{k+N}) = \frac{1}{2} x_{k+N}^T Q_{\text{goal}} x_{k+N} + x_{k+N}^T q_{\text{goal}}, \]

\[ c(x_{k+i}, u_{k+i}) = \frac{1}{2} x_{k+i}^T Q x_{k+i} + \frac{1}{2} u_{k+i}^T R u_{k+i} + u_{k+i}^T P x_{k+i} + x_{k+i}^T p + \xi, \]

where \( i \in \{0, 1, ..., N - 1\} \), \( Q_{\text{start}} \in \mathbb{R}^{n_x \times n_x}, Q_{\text{goal}} \in \mathbb{R}^{n_x \times n_x}, Q \in \mathbb{R}^{n_x \times n_x}, R \in \mathbb{R}^{n_u \times n_u}, P \in \mathbb{R}^{n_u \times n_u}, q_{\text{start}} \in \mathbb{R}^{n_u}, q_{\text{goal}} \in \mathbb{R}^{n_u}, p \in \mathbb{R}^{n_u}, \) and \( \xi \in \mathbb{R} \) are predefined in which \( Q_{\text{start}}, Q_{\text{goal}}, Q, \) and \( R \) are positive definite, whereas \( J_k \geq 0 \) and \( j_k \geq 0 \) assumed to be positive semi-definite. LQR problem (53) and (54) provides an optimal \( \pi_N \) in close form solution as expressed in (52) the cost-to-go function, i.e., (52) can be reformulated as an explicit quadratic formulation as follows:

\[ V_{k+i}(x_{k+i}) = \frac{1}{2} x_{k+i}^T J_{k+i} x_{k+i} + x_{k+i}^T j_{k+i} + \xi, \]

Estimation of both \( J_{k+i} \) and \( j_{k+i} \) can be obtained in a recursive way starting from the goal position \( x_{k+N} \) to the initial position \( x_k \), using Riccati differential equation for all \( i \in \{0, ..., N - 1\} \).

\[ J_k = Q + A_k^T J_{k+1} A_k - \]

\[ (P + B_k^T J_{k+1} A_k) (R + B_k^T J_{k+1} B_k)^{-1} (P + B_k^T J_{k+1} A_k) \]

\[ j_k = p + A_k^T j_{k+1} + A_k^T J_{k+1} c_k \]

\[ -(P + B_k^T J_{k+1} A_k) (R + B_k^T J_{k+1} B_k)^{-1} (r + B_k^T j_{k+1} + B_k^T J_{k+1} c_k). \]

In general, system dynamics is described by:

\[ x_{k+1} = f_d(x_k, u_k) = A_k x_k + B_k u_k, \]

If the system dynamics is non-linear, \( A_k \) and \( B_k \) are recalculated by linearizing the \( f_d \) at each time index. Since linearization has to be carried out in each
iteration, it is called the iLQR \[27\].

\[
A_k = \frac{\partial f_c}{\partial x}(x_k, u_k), \quad B_k = \frac{\partial f_c}{\partial u_k}(x_k, u_k).
\] (58)

Boundary or goal position conditions are given by \( S_{k+N} = Q_{goal}, \quad j_{k+N} = q_{goal} \). The feedback control policy in LQR is fully determined as follows:

\[
\pi_k(x_k) = -(R + B_k^T J_{k+1} B_k)^{-1} \cdot (P + B_k^T J_{k+1} A_t) x_k
\]

\[
-(R + B_k^T J_{k+1} B_k)^{-1} \cdot (r + B_k^T j_{k+1} + B_k^T J_{k+1} B_k).
\] (59)

As given in \[56\], system stability depends on system dynamics. When quadrotor dynamics is non-linear, the stability of iLQR is not guaranteed. Jur and Berg \[28\] attempted to address the stability issue by proposing a novel method called LQR smoothing; this method can be applied for linear or non-linear systems to acquire the minimum-cost trajectory. The main difference in LQR smoothing compared to LQR is that LQR minimizes the cost of not only backward direction, i.e., cost-to-go, but also applies forward direction, i.e., cost-to-come \[28, 6, 96\]. However, the output of LQR, iLQR or LQR smoothing does not address the system noise. Both linear or nonlinear state estimator may eliminate the system noise. LQG \[29, 97\] is one of the ways to solve this problem. LQG consists of a state estimator ,i.e., Kalman Filter (KF), and state feedback, i.e., iLQR or LQR.

### 7.2. MPC based trajectory generation

Figure 9: The basic idea of MPC-based receding horizon planning

\[
J(x, u) = \sum_{i=0}^{N} \left( x_{k+i} - x_{k+i}^{ref} \right)^2 + \left( u_{k+i} - u_{k+i}^{ref} \right)^2_R
\]

\[
\min_w J(x, u)
\]

s.t. \( g_1(w) = 0, \quad g_2(w) \leq 0 \)

\( x_{\min} \leq x_{k+i} \leq x_{\max}, \quad \forall 0 \leq i \leq N \)

\( -v_{\max} \leq u_{k+i} \leq v_{\max}, \quad \forall 0 \leq i \leq N - 1 \)

\( u_k = u_0^* \)

Output

\( x_k \)
As detailed in section 7.1, unaccountability of addressing sudden disturbances is the main limitation of OCP techniques (e.g., LQR, DDP); this is due to its nature. LQR calculates fixed receding control policy and applies to the system; there is no intervention during the control policy execution. MPC is one of the ways to address the preceding problem, which is characteristic of both LQR and DDP. The difference between MPC and LQR is that only the first portion of the control policy is applied to system in MPC through the calculation of full control policy rather than employing full control policy as in LQR.

Let us assume the system dynamics as given in \( \text{(3)} \). In general, MPC can be formed as follows:

\[
\begin{align*}
\min_w & \ J_{\text{end}}(x_{k+N}, x^{ref}_{k+N}) + J_k(x, u, x^{ref}, u^{ref}) \\
\text{s.t.} & \ x_{k+1} = f_d(x_k, u_k) \\
& \ x_{\text{min}} \leq x_{k+i} \leq x_{\text{max}} \quad \forall 0 \leq i \leq N \\
& \ u_{\text{min}} \leq u_{k+i} \leq u_{\text{max}} \quad \forall 0 \leq i < N - 1 \\
& \ g_1(w) = 0 \\
& \ g_2(w) \leq 0,
\end{align*}
\]

where \( w = u_k, ..., u_{k+N-1} \) is the optimal control sequence to be estimated in each iteration. Variable \( J_{\text{end}}(x_{k+N}) \) plays a significant role in terms of the stability of the system locally and globally. Presenting local stability is relatively easy, e.g., Lyapunov’s analysis compared to global stability. In addition to terminal cost, terminal constraints for states should be enforced, which is quite computationally challenging for real-time applications. Moreover, enforcing terminal constraints is even difficult for non-linear dynamics. Thus, in most of the practical applications, terminal constraints are not enforced into the optimization procedure. Furthermore, classical MPC lacks recursive feasibility. Several varieties of MPC have been proposed to address processing issues to a certain extent. For a linear system, the performance index, i.e., \( J_k(x, u, z^{ref}, u^{ref}) \), can
be defined as follows:

\[
J_k(\mathbf{x}, \mathbf{u}, \mathbf{z}^{ref}, \mathbf{u}^{ref}) = \sum_{i=0}^{N-1} ((\mathbf{x}_{k+i} - \mathbf{x}^{ref}_{k+i})^T Q_x (\mathbf{x}_{k+i} - \mathbf{x}^{ref}_{k+i}) + (\mathbf{u}_{k+i} - \mathbf{u}^{ref}_{k+i})^T R_u (\mathbf{u}_{k+i} - \mathbf{u}^{ref}_{k+i})) \\
+ (\mathbf{x}_{k+N} - \mathbf{x}^{ref}_{k+N})^T P (\mathbf{x}_{k+N} - \mathbf{x}^{ref}_{k+N})
\]

(61)

where \( Q_x \), which is a positive semi-definite matrix, consists of the state error penalty coefficients, whereas \( R_u \) should be positive definite and \( P \) is state error on the terminal cost. In principle, stability and feasibility are not assured implicitly. Consequently, stability and feasibility tend to improve for the longer receding horizon, which is quite challenging due to computational demands.

Quadrotor dynamics is usually expressed in a non-linear fashion. Therefore LQR or linear MPC can not be applied without linear approximation. Hence, Nonlinear Programming (NLP)-based approach has to be applied. Direct multiple shooting and direct collocation are the main two techniques that are used to transform OCP into NLP. In both direct multiple shooting and direct collocation, the state is minimized in addition to controlling inputs. Direct multiple shooting differs from direct collocation due to the way of the problem formulation. In multiple shooting, the problem is quantized into \( N \) subintervals, i.e., receding horizon length. In direct collocation, however, those subintervals are further described by a set of polynomials such as B-spline or Lagrangian; this will increase the problem sparsity. On the contrary, the number of optimization parameters to be optimized has dramatically increased in direct collocation compared to multiple shooting. This, collocation is better when it is accuracy-wise, but direct multiple shooting is better when it is performance-wise. In [71], the trajectory tracking problem is formulated based on the direct collocation and multiple shooting. Further, the researchers have proved that multiple shooting has lower computational footprint compared to direct collocation.
7.3. Disturbance Estimation

In the context of optimal trajectory planning, simultaneously computing optimal control policy, which is required to respond to unknown, sudden disturbances, and handling kinematics (i.e., obstacle avoidance) as well as dynamics (i.e., satisfying velocity and acceleration constraints) yields a challenging problem, especially for quadrotors. While geometry-based path planning techniques [98, 99] ensure the asymptotical optimality of a path, they however do not consider quadrotor dynamics. But, it is essential that generation of an optimal control policy ensures the dynamic feasibility. So, in [100, 101], LQR was incorporated into path planning, by which both dynamic feasibility and local optimality were guaranteed. However, local optimality does not necessarily yield global optimality [102]. In [101, 13], a set of motion primitives was used to find feasible trajectories ensuring both global and local optimality. When dealing with unknown disturbances, MPC is a more robust technique than LQR. In [16], MPC-based trajectory planning approach was proposed, ensuring both the local and global optimality. However, none of the aforesaid approaches formally guarantees stability and safety. Lyapunov’s analysis can be applied to confirm the local stability. Moreover, the terminal constraints set [103] can be incorporated. However, those measures are time-consuming, which directly affects the real-time performance [104]. A set of CBFs was proposed for improving real-time performance without affecting the system stability in [105, 106, 32].

Recently, reference governors-based techniques were proposed in [107, 108], enforcing safety constraints. It is natural that designing a path planer is followed by the actual controller to maneuver MAV. In such approaches, a reference governor can be used to handle the stability and constraint satisfaction separately to ensure system stability [109].

The above approaches are employed to estimate optimal control policy for safe navigation while imposing stability either using Lyapunov-functions or reference governors. On the other hand, Li et al. [110] proposed to obtain an optimal control policy using a State-dependant Distance Metric (SDDM). They
have modelled the system dynamics as a linear, time-invariant as follows:

\[ \dot{x} = Ax + Bu, \quad (62) \]

where \( u \) indicates the control input. System state, i.e., \( x := (p(t), y(t)) \), consists of two parts: \( p \) and \( y \), where \( p(t) \) denotes the quadrotor position at a given time \( t \) and \( y(t) \) describes the higher-order terms, e.g., velocity, acceleration, etc. In the latter work, quadratic norm was utilized to represent the error between robot position and close-in obstacles positions. The quadratic norm is defined as \( \|p\|_R := \sqrt{p^T R p} \), where \( R \) is a symmetric positive matrix. \( R[\psi_z] \) is fully determined by the MAV heading direction \( \psi_z \) at a given time instance as follows:

\[
R[\psi_z] = \begin{cases} 
  c_2 I + (c_1 - c_2) \frac{\psi_z \psi_z^T}{\|\psi_z\|^2}, & \text{if } \psi_z \neq 0 \\
  c_1 I, & \text{otherwise}
\end{cases}, \quad (63)
\]

where both \( c_1 \) and \( c_2 \) are predefined scales such that \( c_2 > c_1 > 0 \); this process is called the SDDM, trajectory will be bounded incorporating SDDM information. Since quadrotor dynamics is linear, a reference governor \[108\] is introduced to maintain safety and stability. Other than LQR and MPC, there exist several receding horizon-based techniques for optimal trajectory planning as given in Table 2.

8. Solving Trajectory Planning Problem

As explained in the preceding sections, several constraints (e.g., soft and hard) are imposed to ensure dynamic feasibility, smooth navigation, handling disturbances, etc. Hence, optimal trajectory planning is posed as a constraints optimization problem in most of the situations. Constraint-based optimization problems are solved in two different ways: adding hard constraints or introducing soft constraints. In general, constraint-based optimization problem can be formulated as a quadruple, i.e., \( P_{\text{constraint}} = (c, g_1, g_2, J) \), where \( c \) stands for performance index or cost function, whereas equality and inequality constraints are given by \( g_1 \) and \( g_2 \), respectively. The objective function is given by \( J \).
| Algorithm                                           | Motion Model | Gradient Estimator |
|----------------------------------------------------|--------------|--------------------|
|                                                   | Linear       | Nonlinear          | Hamiltonian | Gradient |
| Differential Dynamic Programming (DDP) [111]        | ✗            | ✓                  | ✗           | ✓        |
| Linear Quadratic Regulator (LQR) [112]             | ✓            | ✗                  | ✗           | ✗        |
| Iterative LQR (iLQR) [113]                         | ✗            | ✓                  | ✗           | ✓        |
| Linear Model Predictive Control (MPC) [114]        | ✓            | ✗                  | ✗           | ✗        |
| Nonlinear Model Predictive Control (NMPC) [71]     | ✓            | ✓                  | ✓           | ✗        |
| Constrained Nonlinear Model Predictive Control CGMRES (NMPC-CGMRES) [115] | ✗            | ✓                  | ✓           | ✗        |
| Corridor-based Model Predictive Contouring Control (CMPCC) [31] | ✓            | ✗                  | ✗           | ✗        |
| Constrained Nonlinear Model Predictive Control Newton (NMPC-Newton) [116] | ✗            | ✓                  | ✗           | ✗        |
| Model Predictive Path Integral Control (MPPI) [117] | ✓            | ✓                  | ✗           | ✗        |
| Cross Entropy Method (CEM) [118]                   | ✓            | ✓                  | ✗           | ✗        |
hard constraint-based formulation, optimal solution, i.e., $w$, for $P_{\text{constraint}}$ is calculated, ensuring all the constraints. In soft constraints formulation, objective function does not need to satisfy all the constraints, but satisfying those constraints will improve the final $w$. D. Mellinger and V.Kumar [3] took the lead for proposing a successful approach for trajectory generation as a hard constraint-based optimization approach, i.e., Minimum-snap. Subsequently, in [51], the researchers extended the Minimum-snap trajectory generation as an unconstrained or soft constraint-based optimization problem.

When generating trajectories, ensuring a collision-free path is essential. Hence, representing free-space in a structured way and imposing obstacle constraints for trajectory generation is a must for safety. Free space can be represented in different ways such as cubes ([12, 13]), spheres ([14, 15]) and polyhedrons ([16]). The intuition of these approaches is to apply path planning through the free space to obtain the intermediate waypoints. Once intermediate waypoints are extracted, trajectory generation procedure is utilised for retrieving smooth, feasible, collision-free trajectory. On the other hand, in [47, 23, 13], kinodynamic path planning followed by B-spline based trajectory generation is considered.

Most of the works that were proposed for soft constraint-based trajectory generation formulated optimal trajectory planning as nonlinear optimization problems in which smoothness and safety were introduced as soft constraints. Most of the time gradient-descent based [63] or gradient-free approaches [66, 53] were applied for minimizing the cost of smoothness and safety.

The constrains optimization problem can be designed in either QP or NLP form. In QP, the procedure is to minimize or maximize the objective subject to a set of linear constraints in most of the situations. On the other hand, non-quadratic programming is used to handle the non-linear constraints each of which has a unique nature to solve the problem. In general, QP objective can
be described as:

\[
\min_x \frac{1}{2}x^TQx + c^T x \\
\text{s.t. } Ax \succeq b,
\]

where \(Ax \succeq b\) stands for the set of linear inequalities. \(Q\) is a positive symmetric matrix. There are various ways to solve QP, including interior point, active set and gradient projection. In some situations, multiple variables that are to be optimized are integer values; those are solved as MIQP. For example, FASTER [119] used MIQP for safe trajectory planning with aggressive controls [52].

Most of the recent optimal trajectory planning techniques [20, 25, 19, 53] were formulated as GTO in which optimization problem was designed as a non-linear form. The gradient descent is performed with respect to each parametrization index of \(\Gamma\) to minimize the different, i.e., \(\Gamma_{i+1} - \Gamma_i\). Hence, \(\Gamma_{i+1}\) can be determined by solving the following optimization problem as given in [120, 63].

\[
\Gamma_{i+1} = \arg \min_{\Gamma} J(\Gamma) + (J(\Gamma) - J(\Gamma_i))^T \nabla J(\Gamma_i) + \frac{\eta}{2} \|\Gamma - \Gamma_i\|_M^2,
\]

where \(M\) is a weighting matrix and \(\eta\) is a regularization parameter. GTO is rather popular due to its ability to deform ineffability trajectory segments, low memory requirement and high throughput. Despite having the listed advantages, GTO can not avoid the problem of a local minimum. STOMP [66] is one of the early techniques proposed to address the local minima problem. STOMP is based on the gradient-free technique. However, STOMP is unable to obtain real-time performance. Besides STOMP, the local minima problem has been addressed by various recent works. Yet, this remains an open problem to be solved. Zhou [121] proposed a method, i.e., Path-guided Optimization (PGO), for overcoming local minima problem by generating topologically distinct paths and doing parallel optimization. Furthermore, various solvers can be utilized for solving optimization problems, including BOBYQA [122], L-BFGS [123, 34], ACADO [124], SLSQP [125], Proximal Operator Graph Solver
Shravan et al. [74] proposed a trajectory optimization technique in a distributed setup in which the researchers evaluated their formulation with several solvers. According to their observations, BOBYQA is faster compared to BFGS and SLSQP, while MMA yielded similar performance to that of BOBYQA. In [129], L-BFGS was proposed for finding the shortest path in real time; in this research effort however L-BFGS does not guarantee the optimality, only feasibility is enforced. Mathematical Program with Complementarity Constraints (MPCC) [130] yet another proposed method for fast trajectory optimization in real time. Moreover, Mathieu and Nicolas [131] proposed a SQP-based trajectory generation approach for carrying augmented loads. The intuition behind selecting SQP over other solvers is due to its superlinear convergency and ability to handle non-linear constraints within milliseconds.

9. Conclusion

All in all, we have thoroughly reviewed trajectory planning problem in the paradigm of plan-based control for MAVs. Such trajectory planning problem was broken down into the set of subproblems: free-space segmentation, motion model selection, initial waypoints identification, initial trajectory generation, continuous trajectory refinement, and receding horizon trajectory planning. Afterwards, for each subproblem we examined how previous research have addressed those by presenting and evaluating various approaches to the considered subproblem. Finally, several selected recent approaches were listed (Table 3) according to the listed subproblems we have identified. With that, we concluded that trajectory planning problem can be designed by addressing those subproblems carefully for MAVs.
| Approach | Dynamics Model | Intermediate Waypoint Selection | Initial Trajectory Generation | Continuous trajectory refinement and solver | Free space extraction | Receding horizon planning or controlling |
|----------|----------------|---------------------------------|------------------------------|---------------------------------------------|----------------------|------------------------------------------|
| [121] DP | Sampling-based topological search | PGO based B-splines | GTO | ESDF | - |
| [27] DP | Kinodynamic-based search | B-splines | EO using QCQP | ESDF | - |
| [104] DP | Kinodynamic-based search | Linear Quadratic Minimum Time | unconstrained QP | ESDF | - |
| [53] DP | Informed-RRT* | Continuous time polynomial | BFGS | ESDF | - |
| [25] DP | - | Piecewise Bézier-based curve with minimum-jerk | Elastic band optimization | Convex Cluster | - |
| [28] DP | A* kinodynamic search | B-spline | Nlopt [133] | ESDF | GTC |
| [23] DP | B-spline kinodynamic search | EO | "QCQP" | TSDF | - |
| [13] DP | - | CHOMP | Functional gradient [120] | ESDF | CHOMP |
| [104] DP | A* | uniform B-spline | BFGS, L-BFGS, T-NEWTON [122] | ESDF | - |
| [12] DP | fast marching-based search | Bernstein polynomial | Moser [112] | TSDP | - |
| [24] DP | BRG combined with A* | piecewise polynomials | QCQP | KD-tree | GTC |
| Exact | line search | Iterated LQR Smoothing | Iterated LQR Smoothing | - | - |
| Exact | A* | Visual-Inertial Navigation System (VINS) | Gradient-based | TSDP | GTC |
| [86] DP | RRT* | Uniform-B-spline | MMA and BFGS | OctoMap and Circular Buffer | GTC |
| [12] DP | JPS | Minimum-span | Constrained QP | SFC | RHC |
| [10] DP | piecewise-linear path | SDDM | SDDM | Constrained QP | - |
| [19] DP | JPS | Cubic Bézier curve | MIQP using Gurobi [137] | SFC | - |
| [11] DP | - | CMPC | OSQP [125] | SFC | RHC |
| [10] DP | - | NMPC | ACADO [124] | - | MHE |
| [13] DP | A* | Multi-segment polynomial | OOQP [128] | OctoMap | GTC |
| Exact | RRT* | Minimum-span | Unconstrained QP | OctoMap | GTC |
| Exact | MINVO basis | Uniform B-spline | Augmented Lagrangian [144] | Outer polyhedral | - |
| Exact | Piecewise linear path | Piecewise polynomial | MIQP using Moser | IRIS | - |
| [174] DP | Non-uniform kinodynamic search | Uniform B-spline | Constrained QP | ESDF | RHC |
| [21] DP | Uniform B-spline | NMPC | CasADi [145] with Ipopt [146] | ESDF | PID |
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