An inverse problem for the heat equation

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Abstract

Let \( u_t = u_{xx} - q(x)u, 0 \leq x \leq 1, \ t > 0, \ u(0,t) = 0, u(1,t) = a(t), u(x,0) = 0, \)
where \( a(t) \) is a given function vanishing for \( t > T, \ a(t) \neq 0, \ \int_0^T a(t)dt < \infty. \)
Suppose one measures the flux \( u_x(0,t) := b_0(t) \) for all \( t > 0. \) Does this information
determine \( q(x) \) uniquely? Do the measurements of the flux \( u_x(1,t) := b(t) \) give
more information about \( q(x) \) than \( b_0(t) \) does?

The above questions are answered in this paper.

1 Introduction

Consider the heat transfer problem described by the equation

\[
\begin{align*}
  u_t &= u_{xx} - q(x)u, \quad 0 \leq x \leq 1, \ t > 0, \\
  u(x,0) &= 0, \\
  u(0,t) &= 0, \quad u(1,t) = a(t),
\end{align*}
\]

where \( a(t) \) is the prescribed temperature, and \( q(x) \) is a real-valued integrable function.
Assume that \( a(t) \) is a pulse-type function, that is,

\[
a(t) = 0 \text{ for } t > T, \quad \int_0^T a(t)dt < \infty, \ a(t) \neq 0.
\]

In particular, one can choose \( a(t) \) to be the delta-function \( a(t) = \delta(t). \) Suppose one
measures the flux at one of the ends of the rod, either measuring

\[
  u_x(1,t) := b(t),
\]

or

\[
  u_x(0,t) := b_0(t).
\]
The questions that are answered in this paper are:
1) Does the knowledge of \( a(t) \) and \( b_0(t) \) for all \( t > 0 \) determine \( q(x) \) uniquely?
2) Does the knowledge of \( a(t) \) and \( b(t) \) for all \( t > 0 \) determine \( q(x) \) uniquely?
3) How does one calculate \( q(x) \) given \( a(t) \) and \( b(t) \)?

The answers we give are:
1) The knowledge of \( a(t) \) and \( b_0(t) \) for all \( t > 0 \) does not determine \( q(x) \) uniquely, in general. It does, if \( q(x) \) is symmetric with respect to the point \( x = \frac{1}{2} \), that is, if \( q(x + \frac{1}{2}) = q(\frac{1}{2} - x) \), or if \( q(x) \) is known on the interval \([\frac{1}{2}, 1]\).
2) The knowledge of \( a(t) \) and \( b(t) \) for all \( t > 0 \) determines \( q(x) \) uniquely.
3) An algorithm for computing \( q(x) \) given \( a(t) \) and \( b(t) \) is given.

The answer to question 2) was given in [4] and earlier, under the additional assumption \( q(x) \geq 0 \), in [1]. The answer to question 1) is new, as far as the author knows. An algorithm for computing \( q(x) \) is similar to the one described in [4].

In section 2 the answers to questions 1) and 2) are given, and an answer to question 3) is given in section 3.

2 Answer to questions 1) and 2).

Let us Laplace-transform (1.1)-(1.3), (1.5) and (1.6). If \( v := v(x, \lambda) := \int_0^\infty u(x, t)e^{-\lambda t}dt \), then

\[
\begin{align*}
v'' - q(x)v - \lambda v &= 0, \quad 0 \leq x \leq 1 \\
v(0, \lambda) &= 0, \quad v(1, \lambda) = A(\lambda), \\
v_x(1, \lambda) &= B(\lambda), \\
v_x(0, \lambda) &= B_0(\lambda),
\end{align*}
\]

where \( A(\lambda) \), \( B(\lambda) \) and \( B_0(\lambda) \) are the Laplace transforms of \( a(t) \), \( b(t) \) and \( b_0(t) \), respectively.

**Proposition 2.1.** The data \( \{A(\lambda), B(\lambda)\} \) known for a set of \( \lambda > 0 \), which has a finite limiting point, determines \( q(x) \) uniquely.

**Proposition 2.2.** The data \( \{A(\lambda), B_0(\lambda)\} \), known for all \( \lambda > 0 \), does not determine \( q(x) \) uniquely, in general.

If \( q(x) \) is known on the interval \([\frac{1}{2}, 1]\) then \( q(x) \) on the interval \([0, \frac{1}{2}]\) is uniquely determined by the above data.

Also, if \( q(\frac{1}{2} - x) = q(\frac{1}{2} + x) \), then \( q(x) \) is uniquely determined on the interval \([0, 1]\) by the above data.

**Proof of Proposition 2.1.** Let \( \varphi(x, \nu) \) solve equation (2.1) with \( \lambda = -\nu \) and satisfy the condition

\[
\varphi(0, \nu) = 0, \quad \varphi'(0, \nu) = 1.
\]
The solution $\varphi(x, \nu)$ is an entire function of $\nu$ and of $k = \nu^{1/2} = i\lambda^{1/2}$ (see [3], [5]).

Since $\varphi$ and $v$ satisfy the first condition (2.2), one has:

$$v(x, \lambda) = c(\lambda)\varphi(x, -\lambda),$$

where $c(\lambda)$ does not depend on $x$. Thus

$$c(\lambda)\varphi(1, -\lambda) = A(\lambda), \quad c(\lambda)\varphi'(1, -\lambda) = B(\lambda).$$

Note that $v(x, \lambda)$ may be not defined for some $\lambda$, namely for some $\lambda$, namely for $-\lambda = \lambda_j$, where $\lambda_j$ are the eigenvalues of the problem

$$\ell\psi_j := \psi_j'' + q(x)\psi_j = \lambda_j\psi_j, \quad \psi_j(0) = \psi_j(1) = 0.$$  \hspace{1cm} (2.8)

Since $\lambda > 0$, the condition $\lambda = -\lambda_j$ can be satisfies only if $\lambda_j < 0$. There are at most finitely many negative eigenvalues of the selfadjoint Dirichlet operator $\ell = -\frac{d^2}{dx^2} + q(x)$ in $H := L^2[0, 1]$. For the problem (2.1)-(2.2) to be solvable, when $\lambda = -\lambda_j$ it is necessary and sufficient that the appropriate orthogonality conditions are satisfied. Namely one finds

$$v(x, \lambda) = -\sum_{j=1}^{\infty} \frac{A(\lambda)\psi_j'(1)}{\lambda + \lambda_j}\psi_j(x).$$  \hspace{1cm} (2.9)

For this series to be defined at $\lambda = -\lambda_j > 0$ it is necessary and sufficient that $A(-\lambda_j) = 0$. Note that $\psi_j'(1) \neq 0$ by the uniqueness of the solution to the Cauchy problem (see (2.8)).

Since we have assumed $a(t) = 0$ for $t > T$, the function $A(\lambda)$ is an entire function of $\lambda$ on the complex $\lambda$-plane. Therefore $v(\lambda)$ is well-defined as a meromorphic function of the parameter $\lambda$ with values in $H$. Note that problem (1.1)-(1.3) is always solvable, but if the operator $\ell$ has negative eigenvalues, then the solution to (1.1)-(1.3) may grow exponentially as $t \to +\infty$.

From (2.7) one concludes

$$\frac{B(\lambda)}{A(\lambda)} = \frac{\varphi'(1 - \lambda)}{\varphi(1, -\lambda)},$$

since $c(\lambda) \neq 0$. The zeros of the function

$$\varphi(1, \nu) = 0$$

are precisely the Dirichlet eigenvalues $\lambda_j$ of $\ell$, while the zeros of the function

$$\varphi'(1, \nu) = 0$$

are precisely the eigenvalues of the problem

$$\ell w_j = \mu_j w_j, \quad w_j(0) = 0, \quad w_j'(1) = 0.$$  \hspace{1cm} (2.13)
It is well known (see e.g. [3]) that the knowledge of \( \{\lambda_j\} \) and \( \{\mu_j\} \) for all \( j \) determines \( q(x) \) uniquely because two spectra of \( \ell \) with the same homogeneous boundary condition at \( x = 0 \) and two different homogeneous boundary condition at \( x = 1 \), determine \( q(x) \) uniquely.

The zeros of \( \frac{B(\lambda)}{A(\lambda)} = \frac{\varphi'(1,\lambda)}{\varphi(1,\lambda)} \) are the numbers \( \mu_j \) and only these numbers, while its poles are the numbers \( \lambda_j \) and only these numbers.

Proposition 2.1 is proved.

Remark 2.1. A different proof of Proposition 2.1, based on Property C for ODE, is given in [4].

Proof of Proposition 2.2. From (2.6) and (2.4) it follows that
\[
c(\lambda)\varphi(1,-\lambda) = A(\lambda), \quad c(\lambda)\varphi'(0,-\lambda) = B_0(\lambda).
\]

Thus
\[
\frac{B_0(\lambda)}{A(\lambda)} = \frac{\varphi'(0,-\lambda)}{\varphi(1,-\lambda)} = \frac{1}{\varphi(1,-\lambda)} = \frac{1}{\varphi(1,\nu)}, \quad \nu := -\lambda.
\]

The poles of the function (2.15) are the eigenvalues \( \lambda_j \), and this is the only information one can get from (2.15).

The knowledge of one spectrum \( \{\lambda_j\} \) of \( \ell \) determines, roughly speaking, “half of the potential”: namely, if \( q(x) \) is known on the interval \( [\frac{1}{2}, \ell] \), then the data \( \{\lambda_j\} \) known for all \( j \) determine \( q(x) \) uniquely (see [2], [4], [6]). By the same reason if \( q(x + \frac{1}{2}) = q(\frac{1}{2} - x) \) then \( q(x) \) is uniquely determined on \( [0,1] \) by the set \( \{\lambda_j\} \) known for all \( j \).

Proposition 2.2 is proved.

Remark 2.2. In [4] and [6] a general uniqueness result is obtained which says that if \( q(x) \) is known on \( [b,1] \), \( 0 < b < 1 \), where \( b \) is an arbitrary fixed number, then the set \( \{\lambda_{m(j)}\} \) determines \( q(x) \) on \( [0,b] \) uniquely provided that \( \sigma \geq 2b \). Here \( \lambda_{m(j)} \) is an arbitrary subset of \( \{\lambda_j\} \) such that \( m(j) = \frac{1}{\sigma}(1 + \varepsilon_j), \sum_{j=1}^{\infty} |\varepsilon_j| < \infty \). So, if \( m(j) = j \), then \( \sigma = 1, \varepsilon_j = 0, b \leq \frac{1}{2} \). For \( b = \frac{1}{2} \) one gets the uniqueness result used in the proof of Proposition 2.2 and obtained in [3].

Remark 2.3. From our arguments it follows that extra data (1.6) yields, roughly speaking, half of the information that data (1.5) yields, and therefore does not allow one to recover \( q(x) \) uniquely.


3 An algorithm for computing \(q(x)\)

If \(\{a(t), b(t)\}\) are our data, one takes the Laplace transform and gets \(\frac{B(\lambda)}{A(\lambda)}\). One calculates the zeros and poles of this function and gets the numbers \(\{\lambda_j\}\) and \(\{\mu_j\}\). In the literature (see [3]) there is an algorithm for calculating the spectral function \(\rho(\lambda)\) of the operator \(\ell\) from the knowledge of \(\{\lambda_j\} \cup \{\mu_j\}\). If \(\rho(\lambda)\) is found, then the Gelfand-Levitan algorithm allows one to calculate \(q(x)\) from \(\rho(\lambda)\). This algorithm is described in [3], [4], [5].

In this section we describe an algorithm which is a version of the one described in [5], pp.297-299 (see [4], p. 57), which is quite different from the Gelfand-Levitan one and may be numerically more stable.

Recall that

\[
\varphi(x, \nu) = \varphi_0(x, \nu) + \int_0^x K(x, y) \varphi_0(y, \nu) dy, \quad \varphi_0(x, \nu) := \frac{\sin(kx)}{k}, \quad k = \sqrt{\nu}, \quad (3.1)
\]

where \(K(x, y)\) is the transformation kernel, and

\[
q(x) = 2 \frac{dK(x, x)}{dx}. \quad (3.2)
\]

Since \(\varphi(y, \lambda_j) = 0\), one gets:

\[
\int_0^1 K(1, y) \varphi_0(y, \lambda_j) dy = -\varphi_0(1, \lambda_j), \quad j = 1, 2, \ldots \quad (3.3)
\]

Since the set \(\{\varphi_{0j}\} := \{\varphi_0(y, \lambda_j)\}_y\), forms a Riesz basis of \(H = L^2[0, 1]\), relations (3.3) allow one to find \(K(1, y)\).

Recall that a basis \(\{h_j\}\) of a Hilbert space \(H\) is called a Riesz basis if there is a linear bounded map \(A\) and \(A^{-1}\) is a linear bounded operator on \(H\), such that \(h_j = Af_j\), where \(\{f_j\}\) is an orthonormal basis of \(H\) (see [8], p. 148).

Numerically one may look for \(K(1, y)\) of the form

\[
K(1, y) = \sum_{j=1}^J c_j \varphi_0(y, \lambda_j), \quad (3.4)
\]

substitute (3.4) into (3.3) and get a linear system for \(c_j, 1 \leq j \leq J\). Here \(J\) is an arbitrary large positive integer. The matrix of the linear system is the Gram matrix

\[
(\varphi_{0j}, \varphi_{0m}) := \int_0^1 \varphi_{0j}(x) \varphi_{0m}(x) dx, \quad \varphi_{0j}(x) := \frac{\sin(k_jx)}{k_j}, k_j = \sqrt{\lambda_j},
\]

which is not ill-conditioned since \(\{\varphi_{0j}\}\) forms a Riesz basis.

Differentiate (3.1) with respect to \(x\) and set \(\nu = \mu_j\) \(x = 1\) to get

\[
0 = \varphi_0'(1, \mu_j) + K(1, 1) \varphi_0(1, \mu_j) + \int_0^1 K_x(1, y) \varphi_{0j}(y) dy, \quad j = 1, 2, \ldots \quad (3.5)
\]
These equations determine uniquely $K_x(1, y)$, since $\varphi'_0(1, \mu_j), K(1, 1)$ and $\varphi_0(1, \mu_j)$ are known numbers. Thus we can compute $K(1, y)$ and $K_x(1, y)$, $0 \leq y \leq 1$, from the data \{a(t), b(t)\}.

If $K(1, t)$ and $K_x(1, t)$ are known, then one can derive a Volterra integral equation for the unknown $U := \{q(x), K(x, y)\}$ (see [4], p. 56, and [7]).

In [4] it is proved that this equation can be solved by iterations, and therefore $q(x)$ can be computed by an iterative process.

For convenience of the reader we write down the above integral equation for $U := \{q(x), K(x, y)\}$ and an iterative process for the solution of this equation:

$$U = W(U) + h,$$  \hfill (3.6)

where

$$W(U) := \begin{pmatrix} -2 \int_1^1 q(s)K(s, 2x - s)ds \\ \frac{1}{2} \int_{D_{xy}} q(s)K(s, t)dsdt \end{pmatrix}.$$  \hfill (3.7)

$D_{xy}$ is the region bounded by the straight lines $s = 1, t - y = s - x$, and $t - y = x - s$ on the $(s,t)$ plane,

$$h = \begin{pmatrix} f \\ g \end{pmatrix},$$  \hfill (3.8)

$$f(x) := 2[K_y(1, 2x - 1) + K_x(1, 2x - 1)],$$  \hfill (3.9)

$$g(x, y) = \frac{K(1, y + x - 1) + K(1, y - x + 1)}{2} - \frac{1}{2} \int_{y+x-1}^{y-x+1} K_s(1, t)dt.$$  \hfill (3.10)

Note that $f$ and $g$ are computable from the data $K(1, x)$ and $K_x(1, x)$, and (3.6) is a nonlinear Volterra-type equation for the unknown $q(x)$ and $K(x, y)$.

It is proved in [4], p 57 (and in [7]) that the iterative process

$$U_{n+1} = W(U_n) + h, \quad U_0 = h$$  \hfill (3.11)

converges (at a rate of geometric series) to $U(x) = \begin{pmatrix} q(x) \\ K(x, y) \end{pmatrix}$. The details concerning the functional space in the norm of which the convergence hold are given in [4].

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