

\textbf{C^{0}}-\text{STABILITY OF TOPOLOGICAL ENTROPY FOR CONTACTOMORPHISMS}

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Abstract. Topological entropy is not lower semi-continuous: small perturbation of the dynamical system can lead to a collapse of entropy. In this note we show that for some special classes of dynamical systems (geodesic flows, Reeb flows, positive contactomorphisms) topological entropy at least is stable in the sense that there exists a nontrivial continuous lower bound, given that a certain homological invariant grows exponentially.

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1. Introduction and results

Topological entropy. Topological entropy \( h_{\text{top}}(\varphi) \) is a good numerical measure for the complexity of a self-map \( \varphi : M \to M \) of a compact metrizable space \( M \). We use the following definition of Bowen [4] and Dinaburg [6]: Fix a metric \( d \) generating the topology of \( M \). A \((K, \delta)\)-separated set is a subset \( N \subseteq M \) such that for all \( n \neq n' \in N \) there is a \( k \in [0, K] \) such that \( d(\varphi^k(n), \varphi^k(n')) \geq \delta \). Topological entropy is defined as the growth rate of maximal cardinality of \((T, \delta)\)-separated sets:

\[
h_{\text{top}}(\varphi) = \sup_{\delta > 0} \Gamma(\text{maximal cardinality of a } (T, \delta)\text{-separated set}),
\]

where for a sequence of non-negative numbers \( a_k \), \( \Gamma(a_k) := \limsup_{\delta \searrow 0} \frac{1}{\delta} \log a_k \) is the exponential growth rate.

Unfortunately, topological entropy is very hard to compute explicitly, see [15, 20, 12]. One reason for this is the lack of lower semi-continuity of the function \( \varphi \mapsto h_{\text{top}}(\varphi) \) that associates to a map its entropy. The examples in Section 2 show that even for smooth diffeomorphisms of 3-dimensional closed manifolds, \( h_{\text{top}} \) can collapse to 0 under perturbation, i.e. can jump up from 0 when passing to a limit. Only in very special situations the topological entropy is continuous. For example, in the class of diffeomorphisms of closed 2-dimensional surfaces, topological entropy

\textit{Date:} January 17, 2020.

This work is supported by SFB/TRR 191 ‘Symplectic Structures in Geometry, Algebra and Dynamics’ funded by the DFG.
is continuous due to the existence and stability of hyperbolic sets, see e.g. [14, S5.6]. For further stability results see [16] or [9] and the references therein.

Remark 1.1 (Upper semi-continuity). On the class of smooth maps, $h_{\text{top}}$ is upper semi-continuous in the $C^\infty$ topology, see [21]. This fails on the class of $C^r$ maps.

Stability of topological entropy. Since there is no hope for lower semi-continuity of topological entropy, we aim at the next best thing: we look for an interesting class of spaces and maps for which the topological entropy is “stable”, i.e. there is a nontrivial continuous lower bound for $h_{\text{top}}$. Given stability, it is still hard to explicitly compute entropy, but one can estimate it from below, since total collapse of entropy under perturbation is prevented.

To formalize this, let $\mathcal{P}$ be a parameter space which continuously parametrizes a subset of the space of smooth self-maps $\mathcal{P} \to \mathcal{D} \subseteq C^\infty(M, M)$ equipped with the $C^0$-topology.

Definition 1.2. Topological entropy is stable on $\mathcal{P}$ if there exists a not identically vanishing continuous function $\gamma: \mathcal{P} \to \mathbb{R}_{\geq 0}$ such that $\gamma(p) \leq h_{\text{top}}(\varphi_p)$.

Note that since topological entropy is upper semi-continuous in $C^\infty(M, M)$ for smooth maps, points where $\gamma(p) = h_{\text{top}}(\varphi_p)$ are points of continuity of topological entropy on $\mathcal{D}$.

Remark 1.3. Given stability of topological entropy, one still can collapse topological entropy by large perturbations. See e.g. [1], where it is proven that every contact structure on a closed manifold admits a sequence of contact forms with fixed volume whose Reeb flows have topological entropy converging to 0. This is complementary to the present article, where we show stability under small perturbations.

A criterion for stability. We work in the setting of maps $\varphi$ which are time-1 maps of twisted periodic flows, a class of flows containing autonomous flows: let $\varphi^t$ be a smooth 1-parameter family of maps such that $\varphi^0 = \text{id}$ and $\dot{\varphi}^t = X^t$, where $X^t$ is a 1-periodic vector field. Then, iterations $(\varphi)^k$ of $\varphi := \varphi^1$ coincide with $\varphi^t$ at integer times $t = k$. We formulate a criterion that implies stability of topological entropy of $\varphi$. It is used in all the examples given at the end of the introduction.

Let a class of twisted periodic flows $\varphi^t$ of a closed manifold $M$ be parametrized by $\mathcal{P}$. Our main assumption is the abundance of chords from a fixed submanifold to a generic fiber of some fibered region in the manifold:

Criterion 1.4. There is a submanifold $A^k \subseteq M$ of dimension $k$ and a subset $\mathcal{N} \subseteq \mathcal{M}$, where $\pi: \mathcal{N} \to \mathcal{B}$ is a fibration over a manifold $B^{k+1}$ of dimension $k + 1$ such that the following holds. There is a continuous positive function $\alpha: \mathcal{P} \to \mathbb{R}_{>0}$ such that for almost all $b \in B$ the number $n(A, b, T, \varphi^t)$ of $\varphi^t$-chords from $A$ to the fiber over $b$ of length $\leq T$ is finite and grows uniformly $\alpha$-exponentially in $T$, i.e. for some $T_0 = T_0(\varphi^t)$ we have for $T \geq T_0$:

$$n(A, b, T, \varphi^t) \geq e^{\alpha T}.$$ 

Note that then the exponential growth rate of $n(A, b, T, \varphi^t)$ is at least $\alpha$. We assume the number $T_0$ to be uniform in $b$, but not necessarily continuous in $\mathcal{P}$.

The following result has been used in specific situations multiple times before. We give a proof for the following general formulation in Section 3.
Theorem 1. If a class of twisted periodic flows $\varphi^t$ of smooth self-maps of a compact manifold $M$ satisfies Criterion 1.4 then $\alpha$ constitutes a stability function for the topological entropy of $\varphi = \varphi^1$.

Applications of Theorem 1. The following is to my knowledge the most general class of smooth 1-parameter families of smooth self-maps for which volume growth is stable. It is an extension of the result in [7].

Let $(M, \Lambda, \alpha)$ be a closed contact manifold with closed Legendrian $\Lambda$ which is fillable by the triple $(W, L, \lambda)$ consisting of a Liouville domain with asymptotically conical exact Lagrangian $L$. This means that $M = \partial W, \Lambda = \partial L = L \cap M, \alpha = \lambda|_T M$. We parametrize the class of positive paths of contactomorphisms by the set of positive contact Hamiltonians $P = \{ h \in C^\infty(M \times [0, 1], \mathbb{R}) \mid 0 < h \}$.

The functions $h$ define the contact Hamiltonian vector fields by $\alpha(X_h) = h; \quad \iota_{X_h} d\alpha = -dh + dh(R)\alpha,$ and these vector fields generate the paths $\varphi^t_h$ by

$\varphi^0_h = id; \quad \varphi^t_h = X_h(\varphi^t_h).$

A positive contactomorphism is the endpoint of a positive path of contactomorphisms. We can extend $\varphi^t$ to a twisted periodic flow by convex combination with a Reeb flow. Note that the function $h = 1$ generates the Reeb flow of $\alpha$. The stability then comes from the growth of the positive part of the filtered Lagrangian Rabinowitz–Floer homology $\text{RFH}^T(W, L, h)$, which is defined for the subset $P_{\text{reg}}$ of Hamiltonians for which $\bigcup_{t \neq 0} \phi^t_h(\Lambda) \neq \Lambda$ (see Section 4 for more on this homology).

Theorem 2. If for some $h \in P_{\text{reg}}$ the positive Lagrangian Rabinowitz–Floer homology $\text{RFH}^T(W, L, h)$ has positive dimensional growth $\Gamma_{\text{RFH}}(h) := \Gamma(\dim \iota_a \text{RFH}^{<T}(h))$, then for $h \in P_{\text{reg}}$ we have $h_{\text{top}}(\varphi^t_h) \geq \Gamma_{\text{RFH}}(h)$.

This lower bound extends continuously to all of $P$, is positive and stable in the $C^0$-topology on $P$: $\Gamma_{\text{RFH}}(h) \geq \min_{x,t} h(x, t) \cdot \Gamma_{\text{RFH}}(1) > 0$.

Remark 1.5. Our proof holds for the much more general class of (not necessarily twisted periodic) positive paths of contactomorphisms, as long as we impose uniform bounds $0 < c \leq h \leq C$. However, the conclusions only hold for volume growth and we do not know the connection to topological entropy. For more on this, see Remark 3.1.

We conclude this introduction by a list of examples where the Criterion 1.4 is satisfied.

Example 1.6. As a first example let $Q$ be a closed manifold with exponentially growing fundamental group. Let $P = \{\text{Riemannian metrics on } Q\}$ and $D = \{\varphi^t \mid \varphi^t \text{ is a geodesic flow on } SQ\}$. The sets in Criterion 1.4 are then $A = S_Q, B \subseteq Q$ is a neighbourhood of $p$ and $N = \pi^{-1}(B)$. The number of geodesic chords from $A$ to $\pi^{-1}(q)$ of length at most $R$ is then at least the number of elements of $\pi_1(Q, p, q)$ that are represented by paths that lift into the ball $\tilde{B}_R(p) \subseteq \tilde{Q}$ in the universal cover, since the minimizer of length is a geodesic. The function $\gamma$ is then given

...
by the growth of this number, which is independent of \( q \) and positive if the group growth of \( \pi_1(Q) \) is positive. Continuity of \( \gamma \) with respect to \( g \) in \( C^0 \)-norm comes from the fact that the balls \( \tilde{B}_R(p) \) vary continuously with \( g \). This is a classic discussion: We define the ball volume growth\(^1\) as the growth of volume of a ball in the universal cover of \( Q \): \( \Gamma_{\text{ball}}(M, g) = \Gamma(\text{vol}(\tilde{B}_R(p))) \). For the relation between \( \Gamma_{\text{ball}} \) and the group growth of \( \pi_1 \) and for the fact that \( \Gamma_{\text{ball}} \) is a lower bound for topological entropy, see [18].

A bit more intricate is the situation if the fundamental group is finite and the homology of the based loop space grows exponentially. The function \( \gamma \) is then given by the growth \( \Gamma_{\text{dim}}(\dim \mathcal{H}M_{\leq T}(\mathcal{E})) \) of the Morse homology of the set of loops in \( M \) of energy at most \( T \). As Gromov [13] showed, positivity of \( \gamma \) is a topological invariant of \( Q \).

These examples can be generalized to \( P \) to boundaries of Liouville domains with exponentially growing wrapped Floer homology (the open string analog to symplectic homology). The role of \( A \) is then taken by some Legendrian sphere that is fillable by an exact Lagrangian, and \( N \) is a neighbourhood of \( A \) which is fibered by Legendrian spheres. In this framework Alves and Meiwes managed to construct many examples of contact manifolds different from cosphere bundles that admit a nonvanishing function \( \gamma \). Among the examples are (non-standard) spheres of dimension \( \geq 7 \), \( S^3 \times S^2 \), and non-standard contact structures on any plumbing of cosphere bundles of base dimension \( \geq 4 \).

All the above examples were generalized by the first author to the class of positive contactomorphisms [7]. As a subclass we find lightlike geodesic flows on globally hyperbolic Lorentz manifolds, which might be of independent interest, see [5].

As a final example we mention Reeb flows on 3-manifolds with a Legendrian knot whose cylindrical contact homology has exponential growth, studied by Alves [2]. This situation arises in hyper-tight contact manifolds with pseudo-anosov fundamental group.

**Acknowledgments.** I wish to thank Leonid Polterovich and Felix Schlenk for inputs and advice. This work is supported by SFB/TRR 191 ‘Symplectic Structures in Geometry, Algebra and Dynamics’ funded by the DFG.

2. **Examples for total collapse of topological entropy**

The following example by Milnor [20] shows that topological entropy is not lower semi-continuous, even in very simple settings: it fails for smooth maps from the closed unit disk to itself.

**Example 2.1.** The family \( f_t : D \subseteq \mathbb{C} \rightarrow D ; \ z \mapsto tz^2 \) of maps is smooth. Looking at the restriction \( f_1|_{\partial D} : \partial D \rightarrow \partial D \) one sees that \( h_{\text{top}}(f_1) = \log 2 \). However, \( f_t \) has zero entropy for all \( t < 1 \) since the chain recurrent set of \( f_t \) is the origin.

\(^1\)In Riemannian geometry this is called volume growth. We use the different term to avoid confusion with \( \Gamma_{\text{vol}} \).
If one imposes that the surface be closed and the map be a \( C^r, r \geq 1 \), diffeomorphism, we have lower semi-continuity because of the existence and local stability of hyperbolic sets, as described in [13, S5.6]. However, this result is 2-dimensional in essence and ceases to hold in higher dimensions. The following example shows the lack of lower semi-continuity of topological entropy in the category of \( C^r \) diffeomorphisms (\( r \geq 1 \) or \( r = \infty \)) of closed manifolds in three dimensions:

**Example 2.2.** Let \( \Sigma \) be a closed surface and \( f : \Sigma \to \Sigma \) be a \( C^r \) diffeomorphism with \( h_{\text{top}}(f) > 0 \) that is isotopic to the identity through a smooth path \( f_s \) of \( C^r \) diffeomorphisms. We can assume that \( f_s = f \) for \( s \) near 0 and \( f_s = \text{id} \) for \( s \) near 1. Let \( \tau : T^1 \to \mathbb{R} \) be a bump function on the circle \( T^1 = \mathbb{R}/\mathbb{Z} \) supported in \((0, \frac{1}{2})\) with \( \tau(\frac{1}{2}) = 1 \). Further, let \( g_t \) be the negative gradient flow of a Morse function on \( T^1 \) with critical points only in 0 and \( \frac{1}{2} \). Then \( F_t : \Sigma \times T^1 \to \Sigma \times T^1; (x, \theta) \mapsto (f_{\tau(\theta)}(x), g_t(\theta)) \) is a smooth family of \( C^r \) diffeomorphisms such that \( F_0 = (f_\tau(\theta), \text{id}) \) has positive topological entropy in the fiber \( \theta = \frac{1}{2} \) and such that \( F_t \) has zero topological entropy for every \( t > 0 \) since the restriction to the chain recurrent set \( \Sigma \times \{0, \frac{1}{2}\} \) of \( F_t \) is the identity.

### 3. Proof of Theorem 1

Here, we give a proof of the Stability Theorem 1 under the assumption of Criterion 1. The proof is based on the proofs in the special situations, see e.g. [10, 7].

In this paper, we always work with smooth maps \( \varphi \) on compact manifolds \( M \), so by a combined theorem of Yomdin [23] and Newhouse [21], topological entropy coincides with volume growth:

\[
h_{\text{top}}(\varphi^t) = \Gamma_{\text{vol}}(\varphi^t) = \sup_{S \subseteq M} \Gamma(\text{vol}_S \varphi^t(S)),
\]

where the supremum is taken over all compact submanifolds \( S \) of arbitrary codimension and \( \text{vol} \) is taken with respect to any Riemannian metric. Since volume growth is independent of the choice of Riemannian metric, we can choose a nice one. Let \( g_B \) be a Riemannian metric on the base \( B^{k+1} \) and \( \text{vol}_{B^{k+1}} \) its induced volume. Then we choose \( g \) on \( M \) such that the induced \( k + 1 \)-dimensional volume of \( g \) in \( \mathcal{N} \) is larger than its shadow on \( B \), i.e.

\[
(3.1) \quad \text{vol}_{g}^{k+1} \geq \pi \text{vol}_{B}^{k+1}.
\]

We denote the set of \( b \in B \) for which the growth condition holds by \( B_{\text{reg}} \), and we denote the trace left behind by \( \varphi^t A \) by \( A^T := \bigcup_{t \in [0,T]} \varphi^t A \).

The proof is completed in 6 steps:

- **Step 1** \( \Gamma_{\text{vol}}(\varphi^t) \geq \Gamma(\text{vol}^k(\varphi^t A)) \);
- **Step 2** \( \Gamma(\text{vol}^k(\varphi^t A)) \geq \Gamma(\text{vol}^{k+1}(A^T)) \);
- **Step 3** \( \Gamma(\text{vol}^{k+1}(A^T)) \geq \Gamma(\text{vol}^{k+1}(A^T) \cap \mathcal{N}) \);
- **Step 4** \( \Gamma(\text{vol}^{k+1}(A^T) \cap \mathcal{N}) \geq \Gamma(\int_{A^T} \pi \text{d} \text{vol}_{B}^{k+1}) \);
- **Step 5** \( \Gamma(\int_{A^T} \pi \text{d} \text{vol}_{B}^{k+1}) \geq \Gamma(\int_{B_{\text{reg}}} n(A, b, T, \varphi^t) \text{d} \text{vol}_{B}^{k+1}) \);
- **Step 6** \( \Gamma(\int_{B_{\text{reg}}} n(A, b, T, \varphi^t) \text{d} \text{vol}_{B}^{k+1}) \geq \alpha \).

**Step 1** holds since the supremum is an upper bound.
Step 2 holds since the vector field $X$ generating $\phi^t$ is bounded above since it is periodic on a compact space:

$$\text{vol}^{k+1}(A^T) = \int_0^T \int_{\phi^t(A)} \|pr_{\phi^t(A)} \cdot X\| \, d\text{vol}^k \, dt \leq \sup \|X\| \int_0^T \text{vol}^k(\phi^t(A)) \, dt.$$ 

Then, we note that for $f$ with $\limsup_{t \to -\infty} f(t) \geq 1$ we have that $\Gamma(\int_0^T f(t) \, dt) \leq \Gamma(f(T))$.

Step 3 holds since volume is monotone under inclusion and since $f \leq g$ implies $\Gamma(f) \leq \Gamma(g)$. Step 4 is an immediate consequence of (3.1). Step 5 is just a reformulation.

Step 6 is concluded since we assumed Criterion 1.4 and thus for some $T_0$ we have $n(A, b, T, \phi^t) \geq e^{\alpha T}$ for all $b \in B$ and $T \geq T_0$. This completes the proof.

Remark 3.1 (Non-autonomous dynamical systems). For volume growth $\Gamma_{\text{vol}}$ it makes sense to work with the much more general class of non-autonomous dynamical systems, i.e. for smooth families of smooth maps $\phi^t$ with $\phi^0 = \text{id}$, which are not necessarily twisted periodic. The above proof goes through as long as we impose that $X^t := \phi^t$ is bounded from above. Also, the proof of Theorem 3 in Section 4 does not rely on twisted periodicity, but only on the Hamiltonian $h$ being uniformly bounded for all time $0 < c \leq h \leq C$.

One can also define topological entropy in this more general setting: A $(T, \delta)$-separated set is a subset $N \subseteq M$ such that for all $n \neq n' \in N$ there is a $t \in [0, T]$ such that $d(\phi^t(n), \phi^t(n')) \geq \delta$. Topological entropy is then defined as the exponential growth rate of maximal cardinality of $(T, \delta)$-separated sets:

$$h_{\text{top}}(\phi^t) = \sup_{\delta > 0} \Gamma(\text{maximal cardinality of a } (T, \delta)-\text{separated set}).$$

This generalized notion of topological entropy for non-autonomous systems is studied only in a few papers. For example, in [17] it is shown that in the discrete time setting this definition of generalized topological entropy (which is analogous to the definition of Bowen and Dinaburg) coincides with the definition that generalizes the definition of topological entropy by Adler, Konheim and McAndrew.

For this article it is of interest under what conditions the theorems of Yomdin [23] and Newhouse [21] generalize to non-autonomous dynamical systems.

Question 3.2. How are volume growth and topological entropy related for non-autonomous dynamical systems?

4. Proof of Theorem 2

In this section we show how to obtain Criterion 1.4 in the setup of [7], which proves Theorem 2. Since this should serve as a model for similar situations, we refer for details to [7] and focus on the structure of the proof. The main point is the setup of a persistence module for each element of $P$ and proving that a perturbation of the parameter changes the persistence module Lipschitz-continuously with respect to (logarithmized) interleaving distance. We refer to [22] for basic notions on persistence modules.
Liouville domains and conical Lagrangians. Let \((M, \Lambda, \alpha)\) be a contact manifold with Legendrian \(\Lambda\) which is fillable by the triple \((W, L, \lambda)\) consisting of a Liouville domain \((W, \lambda)\) with asymptotically conical exact Lagrangian \(L\). This means that \(M = \partial W, \Lambda = \partial L = L \cap M, \alpha = \lambda|_{TM}\).

**Example 4.1.** The stereotypical example for this setup is the cosphere bundle of a compact manifold \((S^*Q, S^*_\alpha Q, \lambda)\), where \(\lambda = pdq\) is the tautological 1-form, together with the Legendrian being a fiber over a point. We can realize the explicitly as the level set of a fiberwise starshaped Hamiltonian function \(H\). We can then take the sublevel set as Liouville domain (i.e. the codisk bundle \(D^*_\alpha Q\)), where a codisk fiber fills \(S^*_\alpha Q\) asymptotically conically.

Positive contactomorphisms. In the introduction we parametrize the class of twisted periodic positive paths of contactomorphisms by positive periodic contact Hamiltonians. However, in view of Remark 3.1 it makes sense to directly parametrize the set of positive paths of contactomorphisms by the set of bounded positive contact Hamiltonians.

The Hamiltonian vector field of such a function generates the smooth family of contactomorphisms \(\varphi^t_h\), which are positively transverse to the contact structure \(\ker \alpha\) (since \(h \geq 0\)). If \(h = 1\), then \(X_1\) is the Reeb vector field of \(\alpha\). If \(h\) is autonomous, then \(X_h\) is the Reeb vector field of \(\frac{1}{h}\alpha\). Nonautonomous Hamiltonians generate all paths of contactomorphisms. This is a contrast to symplectic dynamics, where there is an obstruction in \(H^1(M; \mathbb{R})\) for a path of symplectomorphisms to be Hamiltonian, called flux. The set \(Cont_+(M, \ker \alpha)\) of positive contactomorphisms consists of all contactomorphisms that are reached through positive paths of contactomorphisms.

**Example 4.2.** Continuing the example of the cosphere bundle \(S^*Q\) of a compact manifold, we observe that the Hamiltonian flow of the defining starshaped Hamiltonian is a reparametrization of the Reeb flow on the spherization. If the Hamiltonian is \(\frac{1}{2}p^2_g\) with respect to some Riemannian metric \(g\), then the induced flow on the \(\frac{1}{2}\)-level \(\Sigma_g\) set is the co-geodesic flow on \(Q\). Any other fiberwise starshaped hypersurface \(\Sigma\) is graphical over \(\Sigma_g\) by radial dilation: \(\Sigma = f\Sigma_g\), where \(f: \Sigma_g \to \mathbb{R}_{>0}\) and \(\alpha\Sigma = f\alpha_{\Sigma_g}\). Thus, studying characteristic flows of fiberwise starshaped hypersurfaces amounts to the same as studying the set of autonomous contact Hamiltonian flows.

**Persistent Rabinowitz–Floer homology.** Given a Liouville domain and asymptotically conical exact Lagrangian \((W, L, \lambda)\) that is bounded by a contact manifold with a Legendrian \((M, \Lambda, \alpha)\), we say a Hamiltonian \(h \in \mathcal{P}\) is regular if \(\bigcup_{t \neq 0} \varphi^t_h(\Lambda) \cap \Lambda\). We denote the set of regular Hamiltonians by \(\mathcal{P}_{\text{reg}}\). Note that \(\mathcal{P}_{\text{reg}} \subseteq \mathcal{P}\) is comeager. For a regular Hamiltonian we can define the action filtered positive Lagrangian Rabinowitz–Floer chain complex \(\text{RFC}_{Z}^*(W, L; h)\). The induced homology is the action filtered positive Lagrangian Rabinowitz–Floer homology

\[ \text{RFH}_{Z}^*(W, L; h). \]

We drop \(W, L\) from the notation if there is no possibility of confusion. For the analytical details of this homology, especially the discussion of the differential, we refer to [7]. Note that the results below on interleavings require that \(h \in \mathcal{P}_{\text{reg}}\).
whereas the results on topological entropy do not. The reason will become clear in the last paragraph, where perturbations of the Legendrian are discussed.

Positive Lagrangian Rabinowitz–Floer homology has the following properties, cf. [7] and the references therein.

1. The chain complex is a $\mathbb{Z}_2$-vector space generated by chords of the contact Hamiltonian vector field $X_h$ that start and end at $\Lambda$ and have length in $(0, T)$. Therefore,
   \[
   \dim \text{RFH}^T_+(h) \leq \# \{X_h\text{-chords from } \Lambda \text{ to } \Lambda \text{ of length } \leq T \}.
   \]

2. The family of vector spaces $\{\text{RFH}^T_+(h)\}_{T \in \mathbb{R}_{>0} \cup \{+\infty\}}$ is a persistence module: it is a direct system directed by morphisms induced by inclusion of generators for $T \leq T'$,
   \[
   \text{RFH}^T_+(h) \xrightarrow{\iota_{T, T'}} \text{RFH}^{T'}_+(h),
   \]
   which satisfy $\iota_{T, T''} \circ \iota_{T', T} = \iota_{T, T''}$, and for each finite $T$ the chain complex $\text{RFC}^T_+$ is finite dimensional. We define the total homology as the direct limit
   \[
   \text{RFH}_+(h) := \lim T \downarrow \text{RFH}^T_+(h).
   \]

3. For $h, k \in \mathcal{P}$ with $h \leq k$ pointwise, we have continuation morphisms
   \[
   \text{RFC}^T_+(h) \xrightarrow{\phi_{h,k}^T} \text{RFC}^T_+(k).
   \]
   Given $h \leq k \leq g$ the morphisms satisfy $\phi_{k,g}^T \circ \phi_{h,k}^T = \phi_{h,g}^T$. These morphisms commute with the morphisms induced by inclusion and induce an isomorphism in the total homology. In other words, the following diagram commutes.

   \[
   \text{RFH}^T_+(h) \xrightarrow{\iota_{T, T'}} \text{RFH}^{T'}_+(h) \xrightarrow{\iota_{\infty}} \text{RFH}_+(h) \cong \text{RFH}_+(k) \xrightarrow{\iota_{\infty}} \text{RFH}^T_+(k)
   \]

4. There is a cofinal subfamily of $\mathcal{C} \subseteq \mathcal{P}_{\text{reg}}$ that is closed under scaling, i.e. $c \in \mathcal{C}, \lambda > 0 \Rightarrow \lambda c \in \mathcal{C}$, such that continuation morphisms with respect to scaling coincide with persistent morphisms, i.e. for $\lambda > 0$
   \[
   \text{RFH}^T_+(\lambda c) \cong \text{RFH}^{\lambda T}_+(c).
   \]

Cofinal means that $\forall h, k \in \mathcal{P}$ there exists $c \in \mathcal{C}$ such that $c \geq h$ and $c \geq k$.

**Remark 4.3.** In the following we assume that $1 \in \mathcal{P}_{\text{reg}}$, so that every positive constant is in $\mathcal{P}_{\text{reg}}$ and we can choose $\mathcal{C} = \mathbb{R}_{>0}$. If $1 \notin \mathcal{P}_{\text{reg}}$, then we choose any regular autonomous Hamiltonian and scalings thereof, for they also admit the scaling property. Alternatively we can choose $\mathcal{C}$ to be the set of regular autonomous Hamiltonians. In dynamical terms, autonomous Hamiltonians parametrize positive paths of contactomorphisms that are flows.
Interleaving stability of persistence modules. We can combine the diagram in (3) with the property in (4) to obtain for $0 < c < h < C$ and for every $T$

\[ RFH^+_T(c) \xrightarrow{\phi^+_T,h} RFH^+_T(h) \xrightarrow{\phi^+_T,C} RFH^+_T(C) \cong RFH^+_T(c) \]

If we logarithmize the persistence parameter $T = e^\tau$, concatenating the above diagram for $T = T$ and $T = \frac{C}{e} T$ yields an interleaving of the two persistence modules $\iota_X RFH^+_T(h)$ and $\iota_X RFH^+_T(c)$ with interleaving distance $\log \frac{C}{e}$:

\[ RFH^+_T(c) \xrightarrow{\phi^+_T,C} RFH^+_T(c) \xrightarrow{\phi^+_T,H} RFH^+_T(h) \xrightarrow{\phi^+_T,C} RFH^+_T(h). \]

Thus, our choice of $P$ implies that every induced persistence module is interleaved with the module of a Reeb flow, with interleaving distance given by the logarithmized oscillation of the Hamiltonian.

Infinite persistence module. From the persistence module $RFH^+_T(h)$ we can define a new one by taking the image in the total homology $\iota_X RFH^+_T(h) \subseteq RFH_+$, where the persistence morphisms are given by $\iota^\tau_{T,T'} \circ \iota_X = \iota_X \circ \iota^T_{T,T'}$ and where continuation morphisms for different $h, k$ are given by $\phi^\tau_{h,k} \circ \iota_X = \iota_X \circ \phi^T_{h,k}$. This amounts to deleting all finite bars from the associated barcodes. Since all morphisms commute with $\iota_X$, the properties above also hold for $\iota_X RFH^+_T(h)$. The interleaving diagram from before becomes

\[ \iota_X RFH^+_T(c) \xrightarrow{\subseteq} \iota_X RFH^+_T(c) \xrightarrow{\subseteq} \iota_X RFH^+_T(h) \xrightarrow{\subseteq} \iota_X RFH^+_T(h). \]

All persistence morphisms $\iota^\tau_{T,T'}$ are inclusions as subspaces of $RFH_+$ and thus for any $h \in P$ the dimension of $\iota_X RFH^+_T(h)$ is monotone in $T$. Since the above interleaving diagram commutes and since the horizontal arrows are inclusions, the vertical and diagonal arrows must be injections. We conclude that

\[ \dim \iota_X RFH^+_T(h) \in \left[ \dim \iota_X RFH^+_T(c), \dim \iota_X RFH^+_T(C) \right], \]

which implies that 

\[ \Gamma(\dim \iota_X RFH^+_T(h)) \geq \Gamma(\dim \iota_X RFH^+_T(c)) = e \Gamma(\dim \iota_X RFH^+_T(1)) \]

as claimed.

Remark 4.4. Note that the bound from above is not interesting since there might be homologically invisible chords, and since the volume growth of the Legendrian submanifold might not realize the supremum in the definition of volume growth. In rare cases this is not the case, as for geodesic flows of hyperbolic manifolds.

On the way we have reproved the following classical corollary for the logarithmized persistence parameter:

\[ RFH^+_T(c) \xrightarrow{\phi^+_T,C} RFH^+_T(c) \xrightarrow{\phi^+_T,H} RFH^+_T(h) \xrightarrow{\phi^+_T,C} RFH^+_T(h). \]
Corollary 3. The persistence modules $RFH^e_T(h)$ and $RFH^e_T(\min h)$ are $\log \frac{C}{e}$-interleaved. Likewise, the persistence modules $\iota_x RFH^e_T(h)$ and $\iota_x RFH^e_T(\min h)$ are $\log \frac{C}{e}$-interleaved.

Connection with wrapped homology. If $1 \in \mathcal{P}_{\text{reg}}$, then the positive Lagrangian Rabinowitz–Floer homology of the constant Hamiltonian $1$ (which induces the Reeb flow of $\alpha$) is isomorphic to the positive part of wrapped Floer homology, cf. [7]. That is, if $0 \leq a$, then

$$WH^{(0,a)}(W, L) \cong RFH^+_a(1).$$

This isomorphism is analogous to the isomorphism between Rabinowitz–Floer homology and symplectic homology [5].

This is of much interest to our situation, since in some cases wrapped Floer homology is more computable than Lagrangian Rabinowitz–Floer homology. In particular, wrapped Floer homology admits a Pontrjagin product, which can be used to study the dimensional growth $\Gamma^{\text{symp}}(W, L)$ of $WH^{(0,a)}(W, L)$. This is used in [3] to find examples of contact manifolds different from cotangent bundles such that every Reeb flow has positive topological entropy. By the above isomorphism, all positive contactomorphisms on these spaces have positive topological entropy.

Stability of chord counting under perturbations of the Legendrian. As the last step in the construction, we want to make a statement about a generic nearby Lagrangian of $\Lambda$. For this we take the following result from [7] Proposition 1.7.

Proposition 4.5. Let $\Lambda'$ be a Legendrian that is isotopic through Legendrians to $\Lambda$. Let $\psi$ be a contactomorphism that takes $\Lambda$ to $\Lambda'$ so that $(\psi^{-1})^*\alpha = f \alpha$. Then the exponential growth of the number of $\varphi^t$-chords from $\Lambda$ to $\Lambda'$ of length $\leq T$ is at least $\min f \cdot \min h \cdot \Gamma^{\text{symp}}(W, L)$.

We sketch the proof. Suppose that a Legendrian $\Lambda$ is perturbed through an isotopy of Legendrians to a nearby Legendrian $\Lambda'$. The Legendrian isotopy can be extended to a path of contactomorphisms $\{\psi_t\}_{t \in [0,1]}$. The $X_h$ chords from $\Lambda$ to $\Lambda'$ are in 1-1 correspondence with $D(\psi_t)^{-1}X_h$ chords from $(\psi_t)^{-1}\Lambda$ to $\Lambda$.

We can define a new Hamiltonian $g$ which generates $\psi_t$ for time $t \in [0, 1]$ and thus takes $\Lambda$ to $(\psi_t)^{-1}\Lambda$ in time 1, and then generates $D(\psi_t)^{-1}X_h$. The transformation formula of the contact Hamiltonian tells us that

$$g(x, t)|_{t \geq 1} = \alpha_x (D(\psi_t)^{-1}X_h(\psi(x), t-1)) = f(x)h(\psi(x), t-1)$$

and therefore $g|_{t \geq 1} \geq \min f \cdot \min h$. Unfortunately, the Hamiltonian defined this way is not necessarily smooth at 1 and is not necessarily positive in $[0, 1]$.

Both problems can be solved by convex combination with a Reeb flow, which strongly flows forward in time $[0, 1]$ and then is reverted in time $[1, T_0]$ for $T_0$ large enough such that combination with $g$ affects the lower bounds only a little. The chords from $\Lambda$ to $\Lambda'$ of the resulting path of contactomorphisms of length $\geq T_0$ are in 1-1 correspondence with the chords of length $\geq T_0 - 1$ from $\Lambda$ to $\Lambda'$, where the correspondence shifts the period by 1, which proves the proposition.

If the deformation of the Legendrian is small, then $\min f$ is close to 1. A convenient consequence of this perturbative stability is that for almost all deformations $\Lambda'$ of $\Lambda$ the Hamiltonian $g$ produced in the proof of Proposition 4.5 is regular, $g \in \mathcal{P}_{\text{reg}}$. Thus, the conclusion of Theorem 2 holds for all Hamiltonians $h \in \mathcal{P}$.
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