Revisiting evaluation of the integral module from the product of two real functions

V P Hranilov 1, L P Kogan 2, P V Gorev 3, P N Kuzmichev 4, P A Egorova 5

1 Nizhny Novgorod State Technical University, Nizhny Novgorod, 24, Minin Str., 603950, Russia
2 Nizhny Novgorod State University of Architecture and Civil Engineering, Nizhny Novgorod, 65, Ilyinskaya Str., 603950, Russia,
3 Giprogascenter JSC, 26, Nizhny Novgorod, Alekseevskaya Str., 603950, Russia,
4 Giprogascenter JSC, 26, Nizhny Novgorod, Alekseevskaya Str., 603950, Russia,
5 Nizhny Novgorod State Technical University, Nizhny Novgorod, 24, Minin Str., 603950, Russia

E-mail: L_kog@list.ru

Abstract. The article envisages the question of estimating the integral module from the product of two real functions. To solve this problem, it is required to transit to the integral by the range values of one of the multipliers that allows transiting to the probabilistic notation of the assumed integral. As a result, it is possible to obtain the required estimation in the form of the composite function product from one of multipliers by the integral of another factor.

1. Introduction
This work raises the question on evaluating the integral module from the product of two multipliers of type

$$\xi = \int_a^b s(x)g(x)dx,$$

(1)

where $s(x)$ and $g(x)$ are functions of real ones at $x \in [a, b]$ that can alternate within this interval. The similar integrals often occur in the problems concerning the mathematical modeling of the wave propagation in the heterogeneous media and directing systems. It is considered that dependence $s(x)$ is continuous throughout boundaries $x \in [a, b]$, as well as with the limited number of possible points of discontinuity of the first-order derivative, whereas $g(x)$ is regarded as the continuously differentiable function. Additionally, the nonoccurrence of the finite extent segments within the $[a, b]$ interval with the fulfilled condition $g(x) = \text{const}$ is required. It is considered that values $g(x)$ at $x \in [a, b]$ are changed within $g_{\min} \leq g(x) \leq g_{\max}$.

The authors set a goal to obtain the relevant estimations in the form that is close to that one occurring in the course of extended mean-value theorem application [1] – [6]. This is due to the advantage of the given theorem (in comparison with other integral inequalities (see [7] – [12]), which relates to the fact that the resulting equation is in the form of the product coefficient associated with one of the subintegral functions by the assumed integral of the second multiplier. Such property is significant, especially, when determining the degree of convergence of Neumann series in the defined
problems. Moreover, in this paper some other relations and estimations, which are also important for the investigation of the properties of certain integrals, have been obtained. As a consequence of calculations presented below, the integral of equation (1) leads to the kind, which is associated with the integration by the range of values of one of the multipliers (see also [7]), that is equal to the integration in a probability sense. This allows, in particular, using the probability theory for giving the estimation.

2. Materials and methods

It is necessary to write equation (1) as the sum of the integrals by the sections of function monotonicity.

\[ \xi = \sum_{i=1}^{N} \xi_i; \quad \xi_i = \int_{X_{i-1}}^{X_i} s(x) g(x) dx. \]  

(2)

Here, \( X_0 = a, \ X_N = b \) and \( x = X_i, \ 1 \leq i \leq N - 1 \) are values of inner points of \( g(x) \) function extrema within the interval of integration into equation (1); function \( g(x) \) possesses values \( Z_i = g(X_i) \) at \( x = X_i, \ 0 \leq i \leq N \). If changing variable \( z = g(x) \), it is required to transform \( \xi_i \) from equation (2) to the integral of the \( \xi_i \) type of equation (3):

\[ \xi_i = \int_{Z_{i-1}}^{Z_i} s(x_i(z)) A_i(z) dz, \]  

(3)

where \( x_i(z) \) is an argument of equation (3), which is the solution of equation \( g(x) = z \) belonging to interval \([X_{i-1}, X_i]\). We consider that the number of such solutions is \( M(z) \). It is obvious that the \( x_i = x_i(z) \) dependence involves the \( z \in [\min \{Z_{i-1}, Z_i\}, \max \{Z_{i-1}, Z_i\}] \) definition domain being a branch of the function that is opposite to \( g(x) \). By condition, function \( g(x) \) is differentiable and thus continuous at \( x \in [a, b] \), as well as strictly monotone at any interval \([X_{i-1}, X_i]\). As a consequence, function \( x_i(z) \) is available and continuous on the specified sections of the \( z \) axis according to the continuity theorem of the inverse function [1].

There is a derivative of the relevant inverse function branch in \( A_i(z) = \frac{dx_i(z)}{dz} \) (3). Consideration of the differentiability of \( g(x) \), the availability and boundedness of function \( A_i(z) \) at the specified interval of axis \( z \) (except for infinitesimal neighborhood of values \( z = \tilde{z} \) corresponding to the stationary points of function \( g(x) \)) is resulted from the theorem on the derivative of the inverse function.

Values \( A_i(z) \) at \( \tilde{z} \) points go to the infinity, that is why integral \( \xi_i \) of equation (3) (by values of \( z \)) is an improper integral of the second kind, which convergence is resulted from the convergence of equation (2). It is simply, if \( Z_{i-1} < Z_i \), then \( A_i(z) \geq 0 \), and when \( Z_{i-1} > Z_i \), then \( A_i(z) \leq 0 \). Thus, it is convenient to rewrite it as \( \xi_i = \sum_{i=1}^{M(z)} s[x_i(z)] |A_i(z)| \) dz .

As a result we obtain:

\[ \xi = \int_{\xi_{\min}}^{\xi_{\max}} \left( \sum_{i=1}^{M(z)} s[x_i(z)] |A_i(z)| \right) zdz. \]  

(4)

It is necessary to admit that index \( i \) of equation (3) and in any sum of the \( \sum_{i=1}^{M(z)} \) kind means the branch number of inverse function \( x_i(z) \) responding to reaching value \( z \) by function \( g(x) \) while this index is equal to the number of the monotonicity interval of function \( g(x) \) in equation (1).

Let us multiply and divide the right-hand side of equation (3) by function \( f(z) = \frac{\sum_{i=1}^{M(z)} |A_i(z)|}{b-a} \). Then, the following equation is obtained:

\[ \xi = (b-a) \int_{\xi_{\min}}^{\xi_{\max}} f(z) dz, \]  

(5)
where \( T(z) = \sum_{i=1}^{M(z)} s[x_i(z)] |A_i(z)| \) is a multiplier available in equation (4). From the definition of \( f(z) \), it follows that \( f(z)dz = \frac{\sum_{i=1}^{M(z)} |A_i(z)|}{b-a} |s[x_i(z)]|dz \). Here, the right member is equal to the ratio of sums of \( M(z) \) paths of infinitely small intervals of the X-axis where \( z \leq g(x) \leq z + d \) is a relation valid for every interval \((b-a)\) content function of the whole integration domain of equation (1). Such relation is equal to elementary probability \( P(z \leq g(x) \leq z + d) = P(z, z + d) \) calculated by the enormous number of tests for the event consisting in entering function values \( g(x) \) into interval \([z, z + d]\) of its range of values under condition of the proportional and random distribution of argument \( x \) within interval \([a, b]\). Thus, dependence \( f(z) \) is identically equal to the probability density of random variable \( g(x) \) relating to the defined distribution of argument \( x \).

It is to be admitted that if \( g(x) \) is a sinusoid function (i.e. when the module of \( g'(x) \) is the same in all points, where \( g(x) \) possesses the same values, which are peculiar for function \( \sin(x) \)), so equation \( T(z) \) is too simplified. Whereas all values \( |A_i(z)| \) with this provision in \( T(z) \) become the same at all \( i \) indexes (the same for every \( z \), in general), the following equation is obtained:

\[
T(z) = \frac{S(z)}{M(z)} = \sum_{i=1}^{M(z)} s[x_i(z)] |A_i(z)|
\]

(6)

where designation \( S(z) = \sum_{i=1}^{M(z)} s[x_i(z)] \) is introduced and symbol \( \bar{s}[g(x)=z] \) means the averaging of function \( s(x) \) by the set of points of the X-axis that belong to interval \( x \in [a, b] \) with \( g(x) = z \).

The relation of equation (7) allows obtaining the number of estimations. Firstly, equation (7) is obtained when applying the first law of mean \([1]\) to the integral on the right-hand side of the equation and changing the ‘mean value’ that is nearly always unknown into the largest value of the subintegral function:

\[
|\xi| \leq (b-a) \max \left\{ \bar{s}[g(x)=z] \right\} \max \{ |g_{\min}|, |g_{\max}| \}.
\]

(8)

It is considered here that \( f(z)dz = 1 \).

Let us assume that \( M_z \) is a number of points that belong to interval \( x \in [a, b] \) and in which \( |s(x)| \) reaches the largest value within this interval. Thus, if \( \min \{ M(z) \} x \in [g_{\min}, g_{\max}] > M_z \), then \( \max \{ \bar{s}[g(x)=z] \} x \in [g_{\min}, g_{\max}] < \max \{ |s(x)| \} x \in [a, b] \). As a result, the estimation of equation (8) for \( |\xi| \) becomes more precise than that oriented immediately to the integral of equation (1) due to the first law of mean (as well as changing the ‘mean’ value of the subintegral function into the largest one according to the modulo).

It is also admitted that if \( g_{\min} \) and \( g_{\max} \) are variables with the same sign, then the equation will be obtained in accordance with the extended mean value theorem:

\[
|\xi| \leq (b-a) \max \left\{ \bar{s}[g(x)=z] \right\} \max \{ |g_{\min}|, |g_{\max}| \} \int_{g_{\min}}^{g_{\max}} z f(z) dz = \max \left\{ \bar{s}[g(x)=z] \right\} \max \{ |g_{\min}|, |g_{\max}| \} \int_{g_{\min}}^{g_{\max}} z f(z) dz.
\]

(9)

It is considered here that \( f(z)dz = \frac{b-g(x)dx}{b-a} \). It must be emphasized that it will be necessary to use multiplier \( \max \{ s(x) \} x \in [a, b] \) in most cases of evaluating \( |\xi| \) if directly applying the extended mean value theorem to integral of equation (1) instead of coefficient \( \max \{ s(x) \} x \in [a, b] \) proceeding the integral sign in equation (9). This will practically always result in the increasing of the upper bound of values \( |\xi| \) compared to the results obtained in this paper. Additionally, it is to be admitted that relation \( (z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} L(p) \exp(-ipz) dp \), where \( L(p) \) is a characteristic function
corresponding to random variable \( g = g(x) \) at the random distribution of argument \( x \) within interval \([a, b]\) specified above, is valid for probability density \( f(z) \). Thus, considering equation (5), integral \( \xi \) may be written as

\[
\xi = \frac{1}{2\pi} (b-a) \int_{\theta_{\min}}^{\theta_{\max}} L(p) \exp(-ipz) \, dp \, T(z) \, dz = (b-a) \int_{-\infty}^{+\infty} L(p) \Phi(p) \, dp = \\
= i(b-a) \int_{-\infty}^{+\infty} L(p) d\Phi_1(p),
\]

where \( \Phi(p) = \frac{1}{2\pi} \int_{\theta_{\min}}^{\theta_{\max}} T(z)z \exp(-ipz) \, dz = i\Phi'(p) \); here \( \Phi_1(p) = \frac{1}{2\pi} \int_{\theta_{\min}}^{\theta_{\max}} T(z) \exp(-ipz) \, dz \).

3. Conclusion
The relations of equations (5) and (9) are the major results of this paper. Using equation (5), it is possible to write the assumed integral of equation (1) as that allows using the machinery of probability theory for its estimating. Particularly, it allows \( |\xi| \) to reach inequation (9). Upon conditions specified above and frequently realized for sinusoid functions, this estimation is more precise than that which would be valid when directly applying the extended mean value theorem. Besides, inequation (8), which affords to define more precise results of the first law of mean at set conditions, as well as equations (10) are actual for evaluating the integrals of equation of the (1) kind.

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