Large-scale velocity fluctuations of turbulence

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Abstract. Even at scales much larger than the scale of the energy supply, turbulence exhibits significant fluctuations. These large-scale fluctuations are described in a formalism that has the same mathematical structure as used for a canonical ensemble in the statistical mechanics. The formalism yields a universal law for the energy distribution of the fluctuations, which is confirmed with experimental data of a variety of turbulent flows. We thereby demonstrate that turbulence is related through its large-scale fluctuations to the statistical mechanics.

1. Introduction

Turbulence is induced by supplying kinetic energy at some scale \( L \). This energy is transferred to both the larger and the smaller scales. However, as shown in Fig. 1, it is on average transferred to smaller and smaller scales because it is eventually dissipated into heat at the smallest scale, i.e., the Kolmogorov length \( \eta \). The energy transfer from \( L \) to \( \eta \) consists of many random steps, each of which occurs between neighboring scales. Hence, although motions at the scale \( L \) depend on the flow configuration, such dependence is lost in the energy transfer. The resultant small-scale motions exhibit universal features that have been studied in detail (see Sreenivasan & Antonia, 1997).

The energy is transferred to scales much larger than \( L \) and could cause velocity fluctuations there (Fig. 1). We expect some universality for these large-scale fluctuations. To lose dependence on the flow configuration, the energy transfer has a sufficient number of random steps. This is because any step prefers to occur between neighboring scales.

For stationary and homogeneous turbulence, we also expect that the large-scale fluctuations are analogous to thermal fluctuations in the equilibrium statistical mechanics (see Callen, 1985).

Figure 1. Sketch of large- and small-scale motions in three-dimensional stationary turbulence.
The stationarity means that no net energy is transferred across the large scales, while the homogeneity means that no net energy is transferred in space. This is close to the case of the thermal equilibrium, where no mean heat transfer occurs. On the other hand, both the fluctuations have many degrees of freedom. At the large scales of the turbulence, spatial correlations are expected to be negligible. Then, the fluctuation energy is additive. Its value in a large-scale region is the sum of its values in the yet large-scale subregions that are not correlated with one another. This is just the case of the thermal fluctuations.

The large-scale fluctuations are known to be significant, regardless of the flow configuration (Mouri et al., 2006, 2009, and references therein), as considered by Landau & Lifshitz (1959) and Obukhov (1962). However, their details are still not known. Experimentally or numerically, any detailed study requires long data for many samples of the large scales. Such data have not been available. The situation is nevertheless improving (Mouri et al., 2006, 2009, and references therein), as considered by Landau & Lifshitz (1959) and Onsager (1949; Sinai, 1972). The formalism is confirmed with long experimental data of a variety of turbulent flows.

2. Basic configuration and coarse graining

Let us consider a velocity component \( v(x) \) along a one-dimensional cut \( x \) of three-dimensional stationary turbulence. The turbulence is assumed to be homogeneous at least in the \( x \) direction. The average \( \langle v \rangle \) is subtracted so as to have \( \langle v \rangle = 0 \). As a characteristic scale \( L \) of the energy supply, we use the correlation length of the local energy \( \delta E \):

\[
L = \frac{\langle (\delta E - \langle \delta E \rangle)^2 \rangle}{2\langle \delta E \rangle^2} \quad \text{with} \quad \tilde{L} = \int_0^\infty \frac{\langle [v^2(x + r) - \langle v^2 \rangle][v^2(x) - \langle v^2 \rangle] \rangle}{\langle (\delta E - \langle \delta E \rangle)^2 \rangle} \, dr.
\]

The usual definition is \( \tilde{L} \), but more convenient to the present study is \( L \). We have \( L = \tilde{L} \) if the distribution of \( v \) is Gaussian, \( \langle v^4 \rangle = 3\langle v^2 \rangle^2 \).

The one-dimensional cut \( x \) is divided into segments with length \( R \). For each segment, the center of which is tentatively defined as \( x_c \), the energy \( v^2 \) is averaged as

\[
v^2(x_c) = \frac{1}{R} \int_{-R/2}^{+R/2} v^2(x_c + x) \, dx.
\]

We focus on this coarse-grained energy. The mean square of \( v^2_R \) around its average \( \langle v^2_R \rangle = \langle v^2 \rangle \) is

\[
\langle (v^2_R - \langle v^2_R \rangle)^2 \rangle = \frac{2}{R^2} \int_0^R (R - r) \frac{\langle [v^2(x + r) - \langle v^2 \rangle][v^2(x) - \langle v^2 \rangle] \rangle \, dr}{\langle (\delta E - \langle \delta E \rangle)^2 \rangle}
\]

(see Rice, 1954). Since the energy is supplied at around the scale \( L \), we assume that any \( n \)-point spatial correlation of \( v^2 \) is negligible at \( r \gg L \). In other words, we assume additivity of \( Rv^2_R \) at \( R \gg L \). Then, Eqs. (1) and (3a) yield

\[
\langle (v^2_R - \langle v^2_R \rangle)^2 \rangle = \frac{4L}{R} \langle v^2 \rangle^2 \quad \text{at} \quad R \gg L.
\]

The assumption also implies \( \langle (v^2_R - \langle v^2_R \rangle)^n \rangle \propto \langle v^2 \rangle^n \), for \( n = 3, 4, 5, \ldots \), which could depend on the spatial correlations of \( v^2 \) among up to \( n \) points. We specify the coefficient of such a relation [Eq. (8)] by utilizing a thermostatistical formalism.
3. Thermostatistical formalism

There is an analogue of Eq. (3b) in the equilibrium statistical mechanics, i.e., formula for thermal fluctuations of the energy $E$ in a canonical ensemble that has the size $R$ and is in contact with a heat bath at temperature $T$ (see Callen, 1985):

$$\langle (E - \langle E \rangle)^2 \rangle = C_R T^2 \quad \text{with} \quad C_R = \left( \frac{\partial \langle E \rangle}{\partial T} \right)_R.$$  \hspace{1cm} (4)

We have assumed that $R v_R^2$ at $R \gg L$ is additive. Since $E$ is also additive, Eq. (4) is equivalent to Eq. (3b) through the correspondences

$$T = \frac{\langle v^2 \rangle}{\sqrt{\zeta}} \quad \text{and} \quad E = N \left[ v_R^2 - (1 - \sqrt{\zeta}) \langle v^2 \rangle \right] \quad \text{at} \quad N = \frac{R}{4L} \gg 1,$$  \hspace{1cm} (5a)

and hence

$$\langle E \rangle = \zeta N T \quad \text{and} \quad C_R = \zeta N.$$  \hspace{1cm} (5b)

Here $\zeta > 0$ is an arbitrary constant that is determined later [Eq. (7)]. Each of the segments with length $R$ is composed of $N$ subsegments with length $4L$. These are the units of the segments and correspond to the energy-containing eddies. The segments are in equilibrium with surrounding turbulence that serves as the heat bath at $T = \langle v^2 \rangle / \sqrt{\zeta}$. Although this is not a true temperature, the analogy is significant.

The energy distribution $P(E)$ in any canonical ensemble is determined by the heat capacity $C_R$ (see Callen, 1985), through basic relations of the statistical mechanics. Since $C_R$ is related to the entropy $\langle S \rangle$ as $C_R = T(\partial T/\partial S)_R$, we integrate $C_R = \zeta N$ in Eq. (5b) to derive

$$\langle S \rangle = \zeta N \left[ \ln \left( \frac{T}{T_0} \right) + 1 \right],$$  \hspace{1cm} (6a)

with a constant of integration $\zeta N (1 - \ln T_0)$. The Helmholtz free energy $\langle F \rangle = \langle E \rangle - T \langle S \rangle$ and subsequently the partition function $Z = \exp(-\langle F \rangle/T)$ are derived as

$$\langle F \rangle = -\zeta N T \ln \left( \frac{T}{T_0} \right) \quad \text{and} \quad Z = \left( \frac{T}{T_0} \right)^{\zeta N}.$$  \hspace{1cm} (6b)

From the inverse of the Laplace transformation $Z(T) = \int_0^\infty \Omega(E) \exp(-E/T) dE$, we derive the density of states $\Omega(E) = E^{\zeta N - 1} / \Gamma(\zeta N) T_0^{\zeta N}$, where $\Gamma$ is the Gamma function. Finally, $P(E) = \Omega(E) \exp(-E/T)/Z(T)$ is derived as

$$P(E) = \frac{E^{\zeta N - 1} \exp(-E/T)}{\Gamma(\zeta N) T_0^{\zeta N}}.$$  \hspace{1cm} (6c)

This is independent of $T_0$. With an increase in $N$, the distribution becomes narrower, and also it approaches to a Gaussian distribution in accordance with the central limit theorem (see Kendall & Stuart, 1977; Callen, 1985).

To determine the value of $\zeta$, we assume universality of $P(E)$ at $N \gg 1$. Let us first consider a special case where the $N$ subsegments are not correlated but has the same energy distribution that corresponds to the square of a Gaussian random variable. The resultant $P(E)$ at any $N$ is the $\chi^2$ distribution with $N$ degrees of freedom (see Kendall & Stuart, 1977), which is reproduced from Eq. (6c) with

$$\zeta = \frac{1}{2}.$$  \hspace{1cm} (7)

Then, also at $N \gg 1$ in other general cases, the universality ensures the same value for $\zeta$. It should be noted that $\zeta = 1/2$ yields $\langle E \rangle = NT/2$ [Eq. (5b)], i.e., the law of energy equipartition among the $N$ degrees of freedom (see Callen, 1985).
Table 1. Parameters of grid turbulence (G1 and G2), boundary layer (B1 and B2), and jet (J1 and J2) from Mouri et al. (2009).

| Quantity                          | Units | G1     | G2     | B1     | B2     | J1     | J2     |
|----------------------------------|-------|--------|--------|--------|--------|--------|--------|
| Kinematic viscosity              | $\nu$ | cm$^2$s$^{-1}$ | 0.143  | 0.142  | 0.138  | 0.143  | 0.139  | 0.139  |
| rms velocity fluctuation         | $(v^2)^{1/2}$ | m/s$^{-1}$ | 0.683  | 1.06   | 0.464  | 1.96   | 1.36   | 2.06   |
| Kolmogorov length                | $\eta$ | cm     | 0.0180 | 0.0138 | 0.0322 | 0.0123 | 0.0179 | 0.0137 |
| Taylor microscale                | $\lambda$ | cm     | 0.597  | 0.548  | 1.35   | 0.806  | 1.21   | 1.08   |
| Correlation length [Eq. (1)]     | $L$    | cm     | 2.35   | 2.65   | 8.33   | 7.34   | 12.9   | 13.1   |
| Microscale Reynolds number       | $Re_\lambda$ |      | 285    | 409    | 454    | 1103   | 1183   | 1603   |

4. Confirmation by experimental data

The theoretical distribution of $E$ [Eq. (6c)] yields the distribution of $v^2_R$, which is confirmed with six sets of experimental data of Mouri et al. (2009) for grid turbulence (G1 and G2), boundary layer (B1 and B2), and jet (J1 and J2). Their parameters are summarized in Table 1.

The experiments were conducted under stationary conditions in a wind tunnel that has a test section of $18 \times 3 \times 2$ m$^3$. At a position where the turbulence was fully developed, Mouri et al. (2009) measured temporal fluctuations of the spanwise velocity $v(t)$. They were converted into the spatial fluctuations $v(x)$, by utilizing Taylor’s hypothesis $x = \sqrt{\langle v^2 \rangle} \cdot U t$, where $U$ is the mean streamwise velocity. Since the data length is as long as 30 km in B1, 80 km in B2, and 100–130 km in the others, the statistics are reliable.

Figure 2 shows the two-point correlation of the local energy $v^2$, which is used to calculate the subsegment length $4L$ [Eq. (1)]. The correlation appears to be negligible above the scale of $4L$ (arrows) as assumed in our formalism. For a range of $N = R/4L$, we calculate the coarse-grained energy $v^2_R$ in each segment with length $R$ [Eq. (2)].

Having the length $4L \approx 0.1–0.5$ m (Table 1), the subsegments are local enough to represent local regions of the turbulence that were actually existent in the wind tunnel. They are connected to make up homogeneous segments with any length $R$. Although the turbulence in the wind tunnel was not homogeneous over scales $\gg 4L$, these segments are still of use. The statistical mechanics allows us to make up a canonical system by connecting systems that might be isolated from one another, i.e., our subsegments, if they are in contact with the same heat bath, i.e., turbulence at $T = \langle v^2 \rangle / \zeta$ [Eq. (5a)] surrounding the subsegments (see Callen, 1985).

Figure 3 shows the probability density distribution of $v^2_R / \langle v^2 \rangle$ at $N = R/4L = 10$ and 30. The solid and the dotted lines are the theoretical predictions of Eq. (6c) via Eq. (5a) for $\zeta = 1/2$ and 1, which depend on $N$ alone. The experiments are in agreement with one another and with the theory for $\zeta = 1/2$ [Eq. (7)].

Figure 2. Two-point correlation $\langle \langle v^2(x+r) \rangle - \langle v^2 \rangle | \langle v^2(x) \rangle - \langle v^2 \rangle \rangle$ normalized by its value at $r = 0$ as a function of $r/\eta$ in grid turbulence G1 (○), boundary layer B2 (△), and jet J1 (□). The arrows indicate $r = 4L$. 
Figure 3. Probability density distribution of $\frac{v_R^2}{\langle v_R^2 \rangle}$ at $N = R/4L = 10$ and 30 in grid turbulence G1 and G2 ($\bigcirc$), boundary layer B1 and B2 ($\triangle$), and jet J1 and J2 ($\square$). The solid and the dotted lines are the theoretical predictions for $\zeta = 1/2$ and 1.

Figure 4. Moments $\langle (v_R^2 - \langle v_R^2 \rangle)^n \rangle$ normalized by $\langle v_R^2 \rangle^n$ for $n = 2$ and 3 as well as skewness $\langle (v_R^2 - \langle v_R^2 \rangle)^3 \rangle / \langle (v_R^2 - \langle v_R^2 \rangle)^2 \rangle^{3/2}$ as a function of $N = R/4L$. The symbols are the same as in Fig. 3. In the upper panel, the solid line matches the dotted line.

5. Concluding remarks

For stationary and homogeneous turbulence, we have described the large-scale fluctuations in a thermostatistical formalism. By utilizing an analogy between the fluctuations of the coarse-grained energy $v_R^2$ [Eq. (3b)] and the thermal fluctuations of the energy $E$ in a canonical ensemble [Eq. (4)], we have related $v_R^2$ to $E$ [Eq. (5)]. The resultant formalism [Eqs. (6) and (7)] reproduces the distribution of $v_R^2$ at $R/4L = N \gtrsim 10^4$ in Figs. 3 and 4. Therefore, through the large-scale fluctuations, turbulence is related to the statistical mechanics.

The thermostatistical formalism of Onsager (1949) for a class of two-dimensional turbulence is well known. We have demonstrated that such a formalism also exists at large scales of usual three-dimensional turbulence.

Our formalism assumes that $Rv_R^2$ is additive and $v_R^2$ obeys a universal distribution at $R \gg L$. This assumption has been confirmed along with the formalism. Especially in Figs. 3 and 4, we have observed the universal distribution of $v_R^2$.

However, the additivity and the universality might not be exact. Some spatial correlation of $v^2$ might not be exactly negligible in flows such as those known to have large-scale spatial structures,
where the additivity might be lost in some manner specific to the flow configuration. Nevertheless, our formalism is at least a good approximation because we have found no disagreement with the experiments. Similar discussions exist about universality at the small scales (Landau & Lifshitz, 1959; Obukhov, 1962; Mouri et al., 2006, and references therein).

To complete our formalism, we infer the functional form of $T/T_0$ in Eq. (6). This quantity is dimensionless and characterizes the turbulence in the subsegments. Judging from $T = \langle v^2 \rangle / \sqrt{z}$ [Eq. (5a)], it is natural to equate $T/T_0$ with the square of the Reynolds number $Re^2$ for those subsegments with length $4L$:

$$Re^2 = \frac{16L^2 \langle v^2 \rangle}{\nu^2} = \frac{\sqrt{z} R^2 T}{N^2 \nu^2}. \quad (9)$$

Here $\nu$ is the kinematic viscosity. The functions such as $Z$ in Eq. (6) are now described as $Z = Re^{2CN}$ and so on (for details, see Mouri et al., 2011).

Finally, we note that our formulation is applicable to large-scale fluctuations other than those of $v^2_R$ if they possess additivity, etc., and are thereby analogous to thermal fluctuations in the equilibrium statistical mechanics. We only have to write the mean square in the form of Eq. (3b).

By comparing this with Eq. (4), we obtain the correspondences as in Eq. (5) and a formalism as in Eq. (6). The formalism could be constrained by some additional feature, e.g., universality in Eq. (7). Examples are expected to be found in a variety of fluctuations, far beyond those of turbulence.

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