Magnetohydrodynamics from gravity

Cheng-Yong Zhang 1, Yi Ling 2, Chao Niu 1, Yu Tian 1 and Xiao-Ning Wu 3

1 College of Physical Sciences, Graduate University of Chinese Academy of Sciences, Beijing 100049, China
2 Institute of High Energy Physics, Chinese Academy of Sciences, Beijing 100049, China
3 Institute of Mathematics, Academy of Mathematics and System Science, CAS, Beijing 100190, China and Hua Loo-Keng Key Laboratory of Mathematics, CAS, Beijing 100190, China

Abstract

Imposing the Petrov-like boundary condition on the hypersurface at finite cutoff, we derive the hydrodynamic equation on the hypersurface from the bulk Einstein equation with electromagnetic field in the near horizon limit. We first get the general framework for spacetime with matter field, and then derive the incompressible Navier-Stokes equations for black holes with electric charge and magnetic charge respectively. Especially, in the magnetic case, the standard magnetohydrodynamic equations will arise due to the existence of the background electromagnetic field on the hypersurface.

*Electronic address: zhangchengy10@mails.gucas.ac.cn
†Electronic address: linyi@ihep.ac.cn
‡Electronic address: niuchao09@mails.gucas.ac.cn
§Electronic address: ytian@gucas.ac.cn
¶Electronic address: wuxn@amss.ac.cn
I. INTRODUCTION

Under the long wavelength limit, the correspondence between the bulk gravity and the fluid living on the horizon was firstly discovered by Damour\cite{1}. He investigated the surface effects in black hole physics and disclosed the similarity of the behavior of the excitations of a black hole horizon to those of a fluid. After a series of studies, the fluid/gravity correspondence was extended to the boundary at infinity in the context of AdS/CFT correspondence\cite{2–17}. Traditionally, this correspondence was constructed by directly disturbing the bulk metric under the regularity condition of the horizon. It was shown that in a long wavelength limit, the remaining dynamics on the boundary is governed by the incompressible Navier-Stokes equation\cite{18–21}.

In \cite{24}, Lysov and Strominger firstly studied the fluid/gravity correspondence by imposing the Petrov-like condition on the cutoff surface in the near horizon limit and found that the constraint equations for gravity can give rise to the incompressible Navier-Stokes equation for a fluid living on a flat spacetime with less one dimensions. In this sense the Petrov-like condition is of a holographic nature. Later, this framework was generalized to describe a dual fluid living on a cutoff surface that is spatially curved in \cite{26}. In this case the Navier-Stokes equation receives contributions from the curvature of the hypersurface. In \cite{27}, the holographic nature of the Petrov-like condition was further disclosed in a spacetime with a cosmological constant. Based on all the work above, it is reasonable to conjecture that in the approach of fluid/gravity duality, imposing the Petrov-like condition on the cutoff surface is equivalent to imposing the regularity condition on the future horizon at least in the near horizon limit. However, technically, imposing Petrov-like condition is much simpler than imposing the regularity condition.

In this paper, we will generalize this framework further to a spacetime with matter fields. In general, the presence of matter fields in a system involves more degrees of freedom. Therefore, the key issue that we need to solve is what kind of boundary condition should be imposed for matter fields on the cutoff surface such that the remaining degrees of freedom on the surface can lead to a right correspondence between the fluid and gravity. We will take the electromagnetic field as an example and investigate how to constrain its degrees of freedom so that the remaining degrees of freedom of the Einstein-Maxwell fields exactly matches those of a charged fluid\cite{22, 23}. Under this framework, we explicitly show that
the (magneto)hydrodynamics of the dual charged fluid is governed by the incompressible Navier-Stokes equations, where an external force will arise if a background electromagnetic field exists on the cutoff surface.

We organize the paper as follows. In section II, we firstly construct the general framework for imposing Petrov-like condition on the cutoff surface with matter fields, and then propose a boundary condition for the electromagnetic field on the cutoff in detail. In section III, we apply the framework to the charged $AdS_{p+2}$ black brane and then derive the incompressible Navier-Stokes equation on a flat hypersurface in the near horizon limit. In section IV, we consider a spacetime with Reissner-Nordstrom-AdS (RN-AdS) background in which the embedded hypersurface is intrinsically curved. In section V, the magnetic black hole is considered. We find an external force will arise due to the existence of the background electromagnetic field on the hypersurface. Our main results and conclusions are summarized in Section VI. In appendix A, we present a detailed calculation on the perturbations of the electromagnetic field. Appendix B is on the specific forms of the Hamiltonian constraint in the background of charged AdS black brane and RN-AdS spacetime, respectively.

II. THE GENERAL FRAMEWORK FOR A SPACETIME WITH MATTER FIELDS

We construct the general framework for imposing the Petrov-like condition on the cutoff surface in the presence of matter. Our general setup is following. Firstly, we require that the dynamics of the $p+2$ dimensional bulk spacetime is subject to the standard Einstein equation with matter. Secondly, the background contains a Killing horizon. Then given an embedded hypersurface which has a finite distance from the horizon, we consider the perturbations of the extrinsic curvature as well as the matter fields on the cutoff surface under appropriate boundary conditions. In this section we will consider the boundary condition for gravity at first, and then for Maxwell field later.

For the gravity part such a boundary condition is proposed to be the Petrov-like condition. It should be noticed that the Petrov conditions in its original form are proposed to classify the geometry of the whole spacetime such that it can be defined at each point in the bulk spacetime. For our purpose we only specify this condition on the cutoff surface, therefore the first thing we need to do is decomposing the $p + 2$ dimensional Weyl tensor appearing
in Petrov-like conditions in terms of the \( p + 1 \)-dimensional quantities, which includes the intrinsic curvature and extrinsic curvature of the hypersurface as well as the induced metric on it.

Consider a \( p + 2 \) dimensional spacetime with bulk metric \( g_{\mu \nu} \) which satisfies the Einstein equation with matter

\[
G_{\mu \nu} = R_{\mu \nu} - \frac{R}{2} g_{\mu \nu} = -\Lambda g_{\mu \nu} - T_{\mu \nu}, \quad (\mu, \nu = 0, \ldots, p + 1).
\]

Here \( \Lambda \) is a cosmological constant, and \( T_{\mu \nu} \) is the energy-momentum tensor of matter fields.

Now we embed a timelike hypersurface \( \Sigma_c \) with induced metric \( h_{ab} \) into this spacetime. The momentum constraints on \( \Sigma_c \) then could be written as

\[
D_a K_b^a - D_b K^a = -T_{\mu b} n^\mu, \quad (a, b = 0, \ldots, p),
\]

where \( K_{ab} \) is the extrinsic curvature of \( \Sigma_c \) and the covariant derivative operator \( D \) is compatible with the induced metric, namely \( D_a h_{bc} = 0 \). Besides, the Hamiltonian constraint on \( \Sigma_c \) has the form

\[
p + 1 R + K_{\mu \nu} K^{\mu \nu} - K^2 = 2\Lambda + 2T_{\mu \nu}n^\mu n^\nu.
\]

These constraints give \( p + 2 \) equations of \( K_{ab} \). Next we decompose the \( p + 2 \) dimensional Weyl tensor in terms of the intrinsic curvature and extrinsic curvature of the hypersurface. It turns out that the results are

\[
C_{\mu \nu \sigma \rho} = R_{\mu \nu \sigma \rho} - \frac{4(\Lambda + T)}{p(p + 1)} g_{\mu[\sigma} g_{\rho]\nu} + \frac{2}{p}(g_{\mu[\sigma} T_{\rho]\nu} - g_{\nu[\sigma} T_{\rho]\mu}),
\]

\[
C_{abcd} = p + 1 R_{abcd} - K_{ac} K_{bd} + K_{ad} K_{bc} - \frac{4(\Lambda + T)}{p(p + 1)} h_{a[c} h_{d]b}
\]

\[
+ \frac{2}{p} h_a^\alpha h_b^\beta h_c^\gamma h_d^\delta \left( g_{\alpha[\gamma} T_{\beta]_\beta} - g_{\beta[\gamma} T_{\alpha]_\beta} \right),
\]

\[
C_{abc(\mu)} = D_a K_{bc} - D_b K_{ac} + \frac{2}{p} h_a^\alpha h_b^\beta h_c^\gamma \left( g_{\alpha[\gamma} T_{\beta]_\beta} - g_{\beta[\gamma} T_{\alpha]_\beta} \right) n^\delta,
\]

\[
C_{a(\nu)b(\mu)} = -p + 1 R_{ab} + K K_{ab} - K_{ac} K_{b}^c + h_a^\alpha h_b^\gamma R_{\alpha \gamma} - \frac{2(\Lambda + T)}{p(p + 1)} h_{ab}
\]

\[
+ \frac{2}{p} h_a^\alpha h_b^\gamma \left( g_{\alpha[\gamma} T_{\beta]_\beta} - g_{\beta[\gamma} T_{\alpha]_\beta} \right) n^\delta n^\alpha.
\]

Here \( n^\mu \) is the unit normal vector of \( \Sigma_c \) and \( C_{abc(\mu)} = C_{abc\mu} n^\mu, C_{a(\nu)b(\mu)} = C_{\mu b\nu} n^\mu n^\nu, T = T_{\mu}^\nu \).

The Petrov-like conditions is defined as

\[
C_{(l)(l)} = l^\mu m_i^\nu l_{\mu}^\nu m_j^\rho C_{\mu \nu \sigma \rho} = 0
\]
on the hypersurface $\Sigma_c$. Here the $p + 2$ Newman-Penrose-like vector fields should satisfy the relations

$$l^2 = k^2 = 0, \quad (k, l) = 1, \quad (k, m_i) = (l, m_i) = 0, \quad (m_i, m_j) = \delta^i_j. \quad (8)$$

Furthermore, if we introduce the Brown-York tensor on the hypersurface which is defined as

$$t_{ab} = Kh_{ab} - K_{ab}, \quad (9)$$

then with equations (4)-(6), the Petrov-like conditions (7) can be expressed in terms of the Brown-York tensor. When the induce metric and the intrinsic curvature are fixed for the cutoff surface, we find that the only dynamical variables for gravity are the Brown-York tensor.

In the absence of matter fields, the Petrov-like condition on $\Sigma_c$ gives $\frac{p(p+1)}{2} - 1$ equations of the Weyl tensor. It reduces the $\frac{(p+2)(p+1)}{2}$ components of the extrinsic curvature $K_{ab}$ to $p + 2$ unconstrained variables, and so is the Brown-York tensor, which may be interpreted as the energy density, velocity field $v^i$ and pressure $P$ of a fluid living on the hypersurface. The $p + 2$ momentum and Hamiltonian constraints on $\Sigma_c$ then become an equation of state and evolution for the fluid variables. However, without any further expansion in the fluid described here has rather exotic dynamical equations\cite{24}. In the following section, we will do this expansion around a large mean curvature limit and get the familiar hydrodynamical equation in the near horizon limit.

In the presence of matter fields, obviously we need further impose appropriate boundary conditions for matter field to constrain the degrees of freedom of matter on the surface such that the total remaining degrees of freedom is what we need for a fluid living on the surface. In general, such boundary conditions depends on the content of matter fields as well as the property of the dual fluid. Obviously, a better understanding for Strominger’s boundary condition should be of benefit to find boundary condition for matter field. Now let’s reconsider Strominger’s Petrov like boundary condition. For the gravity/fluid correspondence, the bulk is a p+2 dim manifold which contains horizon and a time-like boundary\cite{17}. For vacuum case, such system can be controlled by the initial-boundary value problem(IBVP) method\cite{25}. Based on the work by Friedrich and Nagy, it is easy to see that Strominger’s boundary condition is closed related with free boundary data of IBVP. In other words, we can say Strominger’s boundary condition is coming from the free boundary data of IBVP.
of vacuum Einstein system. This observation gives us a guideline for searching a suitable
boundary condition for matter field. In this paper we intend to take the electromagnetic field
as an example and propose a boundary condition which is very analogous to the Petrov-like
condition for gravity. Specifically, the energy-momentum tensor of the electromagnetic field
is given by

$$T_{\mu\nu} = \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} - F_{\mu\rho} F_{\nu}^{\rho}. \quad (10)$$

Based on the result of IBVP of Maxwell field[25], we propose the boundary condition
for the Maxwell field to be

$$F_{(l)i} = F_{\mu\nu} l^\mu m_i^\nu = 0. \quad (11)$$

Physically, such a boundary condition can be intuitively understood as there is no outgoing
electromagnetic waves passing through the cutoff surface. Similarly as we fix the induced
metric $h_{ab}$ on the cutoff surface, we also fix $F_{ab}|_{\Sigma_i}$, which could be regarded as the Dirichlet-
like boundary condition. Here $a, b$ stand the components on hypersurface. Mathematically,
these equations give $p + \frac{p(p+1)}{2}$ constraints for Maxwell field and only one component is
remaining now. While we have the current conservation law $D_a J^a = 0$. With identification
$J^a = -n_\mu F^{\mu a}$ on boundary[3], it gives

$$- D_a J^a = D_a (n_\mu F^{\mu a}) = h_\beta^\alpha \nabla_\alpha (n_\mu F^{\mu \beta})$$

$$= F^{\mu \beta} h_\beta^\alpha \nabla_\alpha n_\mu + n_\mu h_\beta^\alpha \nabla_\alpha F^{\mu \beta}$$

$$= F^{\mu \beta} K_{\mu \beta} + n_\mu n_\beta \nabla_\alpha F^{\mu \beta} + n_\mu \nabla_\alpha F^{\mu \alpha} = 0. \quad (12)$$

The perturbation of this conservation law will be presented in the appendix. It is clear that
it gives a constraint to the Maxwell field.

Based on the above analysis, we find the total degrees of freedom for gravity reduce to
those for a charged fluid. This match makes it possible to construct a duality between
the gravity and fluid. To prove this, the key procedure is to obtain the hydrodynamical
equations for a fluid from the bulk Einstein equation. This is what we intend to do in the
following sections.
III. FROM PETROV-EINSTEIN TO NAVIER-STOKES IN CHARGED ADS BLACK BRANE

In this section, we investigate the dynamic behavior of the geometry on a flat embedding. Specifically, taking a timelike hypersurface in a charged $AdS_{p+2}$ black brane, we impose boundary conditions for both gravity and Maxwell fields in terms of Brown-York tensor and electromagnetic field tensor. Then we expand the perturbations of the extrinsic curvature as well as the Maxwell field around a large mean curvature, which in our cases indicates that the cutoff surface approaches to the horizon of the black hole. We find that the incompressible Navier-Stokes equation can be derived with the help of the Einstein momentum constraints.

A. Petrov-like conditions of Charged $AdS_{p+2}$ black brane

The spacetime we consider here is a charged $AdS_{p+2}$ black brane. Its metric reads as

$$ds^2_{p+2} = -f(r)dt^2 + 2dtdr + r^2\delta_{ij}dx^idx^j, \ (i, j = 1, ... p),$$

$$f(r) = \frac{r^2}{l^2} - \frac{2\mu}{r^{p-1}} + \frac{Q^2}{r^{2p-2}}. \quad (13)$$

Here $Q$ is the electric charge. Take an embedded hypersurface $\Sigma_c$ by setting $r = r_c$ such that the induced metric can be written as

$$ds^2_{p+1} = -f(r_c)dt^2 + r_c^2\delta_{ij}dx^idx^j. \quad (14)$$

It is obvious that $\Sigma_c$ is an intrinsically flat $p+1$ dimensional spacetime. In order to discuss the dynamical behavior of the geometry in the non-relativistic limit, we introduce a parameter $\lambda$ which rescales the time coordinate by $\tau = \lambda x^0 = \lambda \sqrt{f_c}t$. The induced metric on $\Sigma_c$ then could be written as

$$ds^2_{p+1} = -(dx^0)^2 + r_c^2\delta_{ij}dx^idx^j$$
$$= -\frac{1}{\lambda}d\tau^2 + r_c^2\delta_{ij}dx^idx^j. \quad (15)$$

Moreover, we identify $r_c - r_h = \alpha^2 \lambda^2$ so that the limit $\lambda \to 0$ can also be thought of as a kind of near horizon limit [24]. Here $r_h$ is the horizon of bulk spacetime, namely, $f(r_h) = 0$. $\alpha$ is a parameter which will be fixed later. It turns out that $\alpha$ depends on the specific form of the spacetime metric.
Then we turn to the Petrov-like condition. The base vector fields taken here are
\[ m_i = \frac{1}{r} \partial_i, \quad \sqrt{2l} = \frac{1}{\sqrt{f}} \partial_t - n = \partial_0 - n, \quad \sqrt{2k} = -\frac{1}{\sqrt{f}} \partial_t - n = -\partial_0 - n. \] (16)
in (7), so the Petrov-like condition becomes
\[ C_{0ij0} + C_{0ijn} + C_{0jin} + C_{ninj} = 0. \] (17)

This equation may be expressed in terms of the Brown-York tensor on \( \Sigma_c \). Its components in the coordinate system (15) are
\[ t = pK, \quad t^i = -K^i, \quad t^i_j = K^{ij} - K^i_j, \quad t^i = K - K^i. \] (18)

In terms of the above variables, equation (17) turns out to be
\[
0 = \frac{2}{\lambda^2} t^i_t t^j + \frac{p^2}{\lambda^2} h_{ij} - \frac{t}{p} t^i_j h_{ij} + t^i_j t_{ij} + 2\lambda \partial_r (\frac{t}{p} h_{ij} - t_{ij}) - \frac{2}{\lambda} D(t^i_j) - t_{ik} t^k_j \\
+ \frac{1}{p} (T_{\delta\beta} n^\beta n^\delta + 2\Lambda + T + T_{00} - 2T_{00} n^\delta) g_{ij} - T_{ij}. \] (19)

Now we need to express the energy-momentum tensor \( T_{\mu\nu} \) as the electromagnetic field tensor \( F_{\mu\nu} \). This will involve the rise of lower index of the \( p + 2 \) dimensional variables \( F_{\mu\nu} \). Since the Maxwell field is \( p + 2 \) dimensional, and the bulk metric is disturbed, it should be more cautious to deal with the Maxwell field. To express the energy-momentum tensor, we also need the behavior of the metric. Based on the method in [24], what we need is only the near horizon metric because the boundary will approach to the horizon under the large mean curvature limit. To control the near horizon metric, Bondi-like coordinates are always a convenient choice. This means that we could fix \( g_{rt} = 1 \) and \( g_{ri} = 0 \) on \( \Sigma_c \), which is equivalent to \( g^{rt} = 1, \ g^{ri} = 0 \) [28]. Here \( i \) is the pure space component index. With metric (13) and (16), the boundary condition for electromagnetic field could be written as \( F_{ti}|_{\Sigma_c} = 0 \) and \( F_{ij}|_{\Sigma_c} = 0 \). Together with constraints (11), it leads to \( F_{ri} = 0 \) and \( F^{ri} = 0 \). With these constraints for Maxwell field and the specific coordinate we have chosen, \( F_{\mu\nu} \) and \( F^{\mu\nu} \) could be written as in terms of \( F_{\mu\nu} \) on \( \Sigma_c \).
\[
F_{\mu\nu}|_{\Sigma_c} = F_{\mu\nu}, \quad\quad F^{\mu\nu}|_{\Sigma_c} = F^{\mu\nu} g^{rt}, \quad\quad F^{ij}|_{\Sigma_c} = 0, \quad\quad F^{ri}|_{\Sigma_c} = 0, \quad F_{t}^i|_{\Sigma_c} = F_t^i g^{ri}. \] (20)
After some straightforward calculations (see appendix A for details), the Petrov-like condition \([19]\) can be written as

\[
0 = \frac{2}{\lambda^2} t^r_i t^r_j + \frac{t^2}{p^2} h_{ij} - \frac{t}{p} t^r_i h_{ij} + t^r_i t_{ij} + 2\lambda \partial_r \left( \frac{t}{p} h_{ij} - t_{ij} \right) - \frac{2}{\lambda} D_i t^r_j - t_{ik} t^k_j \\
+ \frac{1}{p} \left( \lambda^2 f F_{rr} F_{rr} + 2\Lambda \right) \delta_j.
\]  

(21)

So far, all the calculations are exact. Next we will disturb the Einstein-Maxwell field in order to get the hydrodynamic equation.

**B. Perturbation of Einstein-Maxwell field**

Through this paper the induced metric of background is fixed such that the dynamical variable is the extrinsic curvature. We consider the perturbations of the extrinsic curvature on the cutoff surface. For the charged AdS black brane, it is straightforward to obtain the components of the extrinsic curvature of \(\Sigma_c\) as

\[
K^i_j = \frac{\sqrt{T}}{r} \delta^i_j, \quad K^r_r = \frac{1}{2\sqrt{T}} \partial_r f, \\
K^r_i = 0, \quad K = \frac{1}{2\sqrt{T}} \partial_r f + \frac{p\sqrt{T}}{r}.
\]  

(22)

The spacetime is disturbed by adding a corresponding term which has a higher order than the background. In terms of Brown-York tensor, these are

\[
t^r_i \to 0 + \lambda t^r_i^{(1)} + \cdots, \\
t^r_r \to \frac{p\sqrt{T}}{r} + \lambda t^r_r^{(1)} + \cdots, \\
t^i_j \to \left( \frac{1}{2\sqrt{T}} \partial_r f + \frac{(p - 1)\sqrt{T}}{r} \right) \delta^i_j + \lambda t^i_j^{(1)} + \cdots, \\
t \to \frac{p}{2\sqrt{T}} \partial_r f + \frac{p^2 \sqrt{T}}{r} + \lambda t^{(1)} + \cdots.
\]  

(23)

All the perturbation terms are at order \(O(\lambda^1)\). The next key step in our formalism is to consider the behavior of these perturbations as the cutoff surface approaches the black brane horizon. As mentioned above, our strategy is identifying the perturbation parameter \(\lambda\) with the location of the hypersurface \(r_c - r_h = x\) by \(x = \alpha^2 \lambda^2\), such that the near horizon limit can be achieved simultaneously with the non-relativistic limit. As a result, the following
The quantities should also be expanded as

\[
\frac{1}{r_c} = \frac{1}{r_h} \left( 1 - \frac{x}{r_h} \right) + \cdots = \frac{1}{r_h} - \frac{1}{r_h^2} \alpha^2 \lambda^2 + \cdots,
\]

\[
f_c = \frac{(r_h + x)^2}{l^2} - \frac{2\mu}{(r_h + x)^{p-1}} + \frac{Q^2}{(r_h + x)^{2p-2}}
\]

\[
= bx + cx^2 + \cdots = b\alpha^2 \lambda^2 + c\alpha^4 \lambda^4 + \cdots,
\]

\[
h^{ij}|_{r_c} = \frac{1}{r_h^2} \left( 1 - \frac{2x}{r_h} \right) \delta^{ij} + \cdots = \left( \frac{1}{r_h^2} - \frac{2\alpha^2 \lambda^2}{r_h^3} \right) \delta^{ij} + \cdots
\]

(24)

Here the brief notes \( b = \frac{2r_h^2 - 2\mu(1-p)}{r_h^3} + \frac{(2-2p)Q^2}{r_h^3} \), \( c = \frac{1}{r^2} + \frac{\mu p(1-p)}{r_h^3} + \frac{(1-p)(1-2p)Q^2}{r_h^{2p}} \) and \( h^{ij(0)} = \frac{1}{r_h^2} \delta^{ij} \), \( h^{ij(1)} = \frac{2\alpha^2 \lambda^2}{r_h^3} \delta^{ij} \) have been used.

Now we consider the perturbations of the Maxwell field. The background field of \( AdS_{p+2} \) black brane is

\[
F = \sqrt{p(p-1)} Q_r \frac{dt}{r^p} \wedge dr = C dt \wedge dr,
\]

(25)

where \( Q \) is the electric charge and \( C = \sqrt{p(p-1) \frac{Q}{r}} \). The perturbation of Maxwell field can be written as

\[
F_{\tau\tau} = \frac{1}{\lambda \sqrt{f}} C + F_{\tau\tau}^{(0)}.
\]

(26)

There is a need to explain why we take this perturbation. The order of \( F_{\tau\tau} \) is \( O(\lambda^{-1}) \) before we identify the non-relativistic limit and the near horizon limit. So the perturbation has just a higher order than the background. This is a natural way. In order to calculate the order of the electromagnetic field with upper index, we need to work out the order of the inverse metric. Here we point out that under appropriate coordinates, there are \( g^{\tau\tau} \sim O(\lambda^2), \ g^{ri} \sim O(\lambda^2), \ g^{ij} \sim O(\lambda^0), \ (i = 1, \ldots, p) \) [28]. With these preparations, the Petrov-like conditions [24] on \( \Sigma_c \) turn out to be order by order

\[
\lambda^{-2} : -\frac{b}{4\alpha^2} \delta^i_j + \frac{b}{4\alpha^2} \delta^i_j = 0,
\]

(27)

\[
\lambda^0 : \sqrt{\frac{b}{\alpha}} \tau^{(1)}_{ij} = 2h^{ik(0)}t^{\tau(1)}_{ik} - 2h^{ik(0)}t^{\tau(1)}_{(ik)} + \sqrt{\frac{b}{\alpha}} \frac{t^{(1)}}{p} \delta^i_j + \frac{1}{\alpha} \frac{C_h^2}{2\Lambda} \delta^i_j.
\]

(28)

Here \( C_h = \sqrt{p(p-1) \frac{Q}{r_h}} \) is a constant which is relevant to the background electromagnetic field. It’s obvious that the background (\( \lambda^{-2} \) order ) satisfies this condition automatically.
C. Navier-Stokes equation in charged AdS black brane

At last we come to the momentum constraints on $\Sigma_c$. Since $\Sigma_c$ is intrinsically flat, the momentum constraints can be written as

$$\partial_a t_a^n = T_{\mu b} n^\mu, \quad (a, b = 0, \ldots p). \tag{29}$$

The time component on the left-hand side could be expanded as

$$\partial_\mu t^\mu = \partial_\tau t^\tau + \partial_i t^i. \tag{30}$$

At leading order, there are

$$\begin{align*}
\partial_\tau t^\tau &= \lambda \partial_\tau t^{\tau(1)} + O(\lambda^2) = O(\lambda), \\
\partial_i t^i &= \partial_i (t^{i\mu} h^{\mu\tau}) = -\frac{1}{\lambda^2} \partial_i t^{\tau} = -\frac{1}{r^2 \lambda^2} \partial_i t^\tau = O(\lambda^{-1}). \tag{31}
\end{align*}$$

The time component on the right-hand side of (29) is

$$T^\mu_{\tau r} n_\mu = \frac{1}{\sqrt f} T^r_{\tau r} = -\frac{1}{\lambda f} F^{r\rho} F_{\tau \rho} = 0. \tag{32}$$

So the time component of momentum constraints at leading order turns to

$$\partial_i t^{\tau(1)} = 0. \tag{33}$$

Identifying

$$t_{i}^{\tau(1)} = \frac{v_i}{2}, \quad \frac{t^{(1)}}{p} = \frac{P}{2}. \tag{34}$$

We get the incompressible condition.

$$\partial_i v^i = 0. \tag{35}$$

Then consider the space components of the momentum constraints. The left hand side could be expanded as

$$\partial_\mu t^\mu_i = \partial_\tau t^\tau_i + \partial_k t^k_i. \tag{36}$$

At leading order

$$\begin{align*}
\partial_\tau t^\tau_i &= \lambda \partial_\tau t^{\tau(1)}_i + O(\lambda^2), \\
\partial_k t^k_i &= \lambda \partial_k t^{k(1)}_i + O(\lambda^2). \tag{37}
\end{align*}$$
The right hand side is

\[ T^\mu_\nu n_\mu = \frac{1}{\sqrt{f}} T^r_r = -\frac{1}{\sqrt{f}} \left( F^r t F_{ir} + F^r j F_{ij} \right) = 0. \quad (38) \]

So the space components of the momentum constraints turn to

\[ \partial_r t^r_t + \partial_k t^k_t = 0, \quad (39) \]

Combining with the identifying equation (34) and (28), we get

\[ \partial_r v_i + \partial_t P + \frac{\alpha}{\sqrt{b}} \left( v^k \partial_k v_i - \partial^k \partial_k v_i \right) = 0. \quad (40) \]

So we see that \( \alpha \) should be fixed to be \( \sqrt{b} \), which leads to

\[ \partial_r v_i + \partial_t P + v^k \partial_k v_i - \partial^k \partial_k v_i = 0. \quad (41) \]

This is exact the incompressible Navier-Stokes equation in \( p \) space dimensions.

In general, the hydrodynamic equation for a charged fluid is

\[ \partial_r v_i + \partial_t P + v^k \partial_k v_i - \partial^k \partial_k v_i = F^a_{ia} J^a. \quad (42) \]

Here \( a \) contains \( i \) and \( \tau \). In this equation there is a term caused by the external electromagnetic force. However, since \( F_{ab} |_{r_c} = 0 \) on the cutoff surface for the charged AdS black brane, the right-hand side of the Navier-Stokes equation vanishes. In section V, we will demonstrate that an external force does arise when the background electromagnetic field \( F_{ab} |_{r_c} \neq 0 \), which corresponds to hydrodynamics of a magnetofluid.

IV. FROM PETROV-EINSTEIN TO NAVIER-STOKES IN RN-ADS SPACETIME

The hypersurface in charged AdS black brane is intrinsic flat. Now let us consider an intrinsically curved hypersurface in Reissner-Nordstrom-AdS (RN-AdS) spacetime and study the influence of the space curvature. The bulk metric is

\[ ds^2_{p+2} = -f(r) dt^2 + 2 dt dr + r^2 d\Omega_p^2, \]

\[ f(r) = 1 - \frac{2 \mu}{r^{p-1}} + \frac{Q^2}{r^{2p-2}} + \frac{r^2}{l^2}. \quad (43) \]

Here \( d\Omega_p^2 \) is \( p \)-dimensional spherical metric which could be written as

\[ d\Omega^2 = d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \cdots + \sin^2 \theta_1 \cdots \sin^2 \theta_{p-1} d\theta_p^2. \quad (44) \]
The metric of the embedding hypersurface $\Sigma_c$ at $r = r_c$ is

$$ds_{p+1}^2 = -f(r_c)dt^2 + r_c^2dΩ_p^2. \quad (45)$$

Introducing a coordinate transformation $\tau = \lambda x^0 = \lambda \sqrt{f}t$, the induced metric could be written as

$$ds_{p+1}^2 = -(dx^0)^2 + r_c^2dΩ_p^2 = -\frac{1}{\lambda^2}d\tau^2 + r_c^2dΩ_p^2. \quad (46)$$

The form of the extrinsic curvature is just the same as that of charged AdS black brane. While the Newman-Penrose-like vector fields for RN-AdS take the following form

$$m_i = \frac{1}{r \sin \theta_1 \ldots \sin \theta_{i-1}} \partial_i, \quad \sqrt{2l} = \frac{1}{\sqrt{f}} \partial_t - n = \partial_0 - n, \quad \sqrt{2k} = \frac{-1}{\sqrt{f}} \partial_t - n = -\partial_0 - n. \quad (47)$$

The explicit expression of the Petrov-like condition is the same as (17).

Introducing Brown-York tensor $t_{ab} = Kh_{ab} - K_{ab}$, the Petrov-like condition in intrinsically curved spacetime with matter field could be written as.

$$0 = \frac{2}{\lambda^2}(t^*_it^*_j + \frac{t^2}{p^2}h_{ij} - \frac{t}{p}t^*_ih_{ij} + t^*_it^*_ij + 2\lambda \partial_t(-h_{ij} - t_{ij}) - \frac{2}{\lambda}D_{(i}t^*_{j)} - t_{ik}t^*_k$$

$$+ \frac{1}{p}(T_{\delta\beta}n^\nu n^\delta + 2\Lambda + T + T_{00} - 2T_{0n}n^\delta)g_{ij} - T_{ij} - p^{+1}R_{ij}. \quad (48)$$

Here $p^{+1}R_{00ij} = 0$ has been considered.

Now it is needed to express $T_{\delta\beta}$ in terms of the electromagnetic field $F_{\delta\beta}$. As have done after (19), we choose the same coordinate and constraints for electromagnetic field. After some similar calculations, (48) turns to

$$0 = \frac{2}{\lambda^2}(t^*_it^*_j + \frac{t^2}{p^2}h_{ij} - \frac{t}{p}t^*_ih_{ij} + t^*_it^*_ij + 2\lambda \partial_t(-h_{ij} - t_{ij}) - \frac{2}{\lambda}D_{(i}t^*_{j)} - t_{ik}t^*_k$$

$$+ \frac{1}{p}\left(\lambda^2 fF_{rr}F_{rr} + 2\Lambda\right)\delta^i_j - p^{+1}R_{ij}. \quad (49)$$

Here $p^{+1}R_{ij}$ is only relevant to the pure space components $\theta_i$ and has the order $O(\lambda^0)$.

Now we consider the perturbations of the Einstein-Maxwell field. Let us think about gravity firstly. Taking Brown-York tensor as the fundamental variables, the perturbation of gravity has just the form of (23).

Identifying $r_c - r_h = x$ with $\alpha^2\lambda^2$, we get the following expansion

$$f(r_c) = 1 - \frac{2\mu}{(r_h + x)^{p-1}} + \frac{Q^2}{(r_h + x)^{2p-2}} + \frac{(r_h + x)^2}{l^2}$$

$$= bx + cx^2 + \cdots = b\alpha^2 x^2 + a\alpha^4 x^4 + \cdots. \quad (50)$$
Here $b = -\frac{2\mu(1-p)}{r_h^3} + \frac{Q^2(2-2p)}{r_h^4} + \frac{2\mu}{r_h^2}$ and $c = \frac{P(1-p)}{r_h^3} + \frac{Q^2(1-p)(1-2p)}{r_h^4} + \frac{1}{r_h^2}$.

The perturbation of electromagnetic field takes the same form as equation (26). At the first non-trivial order, it gives

$$
\lambda^0 : \frac{\sqrt{b}}{\alpha} t^{(1)}_j = 2h^{ik(0)} t^{(1)}_k t^{(1)}_j - 2h^{ik(0)} t^{(1)}_{(j,k)} + \frac{\sqrt{b}}{\alpha} t^{(1)}_j \frac{1}{p} \delta^i_j + \frac{1}{p} \left(C_h^2 + 2\Lambda\right) \delta^i_j - \frac{p+1}{p} R^i_j. \quad (51)
$$

Here $C_h = \sqrt{p(p-1)} \frac{Q}{r_h}$ is a constant which is relevant to the background electromagnetic field.

Now considering the momentum constraints on $\Sigma_c$,

$$
D_a t^a_b = T_{\mu b} n^\mu, \quad (a,b = 0, \ldots p). \quad (52)
$$

The time component gives at leading order

$$
D_i t^{\tau i(1)} = 0. \quad (53)
$$

Identifying

$$
t^{(1)}_i = \frac{v_i}{2}, \quad \frac{t^{(1)}_i}{p} = \frac{P}{2}. \quad (54)
$$

the incompressible equation is derived

$$
D_i v^i = 0. \quad (55)
$$

The space components of the right hand side turn out to be zero which is similar to (38).

With equations (51)-(55) we get

$$
\partial_r v_i + D_i P + \frac{\sqrt{b}}{\alpha} \left(v^j D_j v_i - D^j D_j v_i - R^i_j v_j\right) = 0. \quad (56)
$$

from the momentum constraints. Here $R^i_j \propto \gamma^i_j$ has been considered since the pure space has a spherical metric. Again let $\alpha = \sqrt{b}$, we get the standard incompressible Navier-Stokes equation in spatially curved spacetime.

$$
\partial_r v_i + D_i P + v^j D_j v_i - D^j D_j v_i - R^i_j v_j = 0. \quad (57)
$$

Compare to the situation of intrinsic flat hypersurface, the hydrodynamic equation in intrinsic curved hypersurface has an extra term which is relevant to the Ricci tensor of the boundary. Similar to the case of charged AdS black brane, the electromagnetic has no influence on the hydrodynamic equation. This is a natural result because of the boundary conditions we have taken. $F_{ab}|_{r_c} = 0$ isolate the influence of the electromagnetic field and lead to the incompressible Navier-Stokes equation.
V. THE SITUATION OF MAGNETIC REISSNER-NORDSTROM BLACK HOLE

In previous sections, the background electromagnetic field on the cutoff surface vanishes. This leads to the incompressible Navier-Stokes equation in the first non-trivial leading order. Now we consider a 4-dimensional magnetic black hole with metric

\[ ds^2_{p+2} = -f(r)dt^2 + 2dtdr + r^2d\Omega^2_\mu, \]

\[ f(r) = 1 - \frac{2\mu}{r^{p-1}} + \frac{P^2}{r^2}. \]  

(58)

Here \( P \) is the magnetic charge and the electromagnetic potential is

\[ A = P \cos d\phi. \]  

(59)

So the electromagnetic field tensor \( F = -P \sin \theta d\theta \wedge d\phi \). The space component is no longer zero. It is expected that this term will lead to the external electromagnetic force in the Navier-Stokes equation at last. \( F^{\mu\nu} \) and \( F^\mu_\nu \) could be written in terms of \( F_{\mu\nu} \) on \( \Sigma_c \) as

\[ F^r_t|_{r_c} = F_{tr}, \quad F^{ri}|_{r_c} = F_{tr}g^{ri} + F_{kj}g^{ji}g^{kr}, \quad F^{ij}|_{r_c} = F_{lk}g^{kj}g^{li}, \]

\[ F^r_t|_{r_c} = F_{rt}g^{rr}, \quad F^{r}_i|_{r_c} = F_{ki}g^{kr}, \quad F^{i}_t|_{r_c} = F_{tr}g^{ri}. \]  

(60)

With the method in section IV, the Petrov condition takes the form

\[ 0 = \frac{2}{\lambda^2}t^i_t^j t^j_i + \frac{t^2}{p^2}h_{ij} - \frac{t^i_t h_{ij} + t^j_t t_{ij}}{p} + 2\lambda\partial_\tau \left( \frac{t^i}{p}h_{ij} - t_{ij} \right) - \frac{2}{\lambda}D(i_t^j) - t_{ik}t^k_j - p^{+1}R_{ij} \]

\[ + \frac{1}{p} \left( \frac{\lambda}{\sqrt{f}}F_{rt}F_{ki}g^{kr}g^{ri} - \frac{1}{f}F_{ij}F_{ki}g^{kr}g^{ri}g^{ji} + \lambda^2 fF_{rt}F_{rt} - \frac{1}{2}F_{ij}F_{lk}g^{il}g^{jk} + 2\Lambda \right) \delta^i_j \]

\[ + F_{im}F_{jk}g^{il}g^{mk}. \]  

(61)

Next we should think about the perturbation of Einstein-Maxwell field. The perturbation of gravity has just the form of \([23]\), but the perturbation of Maxwell field now should take the following form

\[ F_{rt} = F^{(0)}_{rt} + O(\lambda) \]  

(62)

since the background value of \( F_{rt} \) is zero. Similar to the situation of RN black hole, we have the expansion near horizon

\[ f(r_c) = 1 - \frac{2\mu}{r_h + x} + \frac{P^2}{(r_h + x)^2} \]

\[ = bx + cx^2 + \cdots = b\alpha^2 \lambda^2 + c\alpha^4 \lambda^4 + \cdots, \]  

(63)
where \( b = \frac{2a}{r_h} - \frac{2P^2}{r_h} \) and \( c = \frac{2a}{r_h} + \frac{3P^2}{r_h} \). At the first non-trivial leading order \( O(\lambda^0) \), the Petrov-like condition gives

\[
\frac{\sqrt{b}}{\alpha} t^{(1)}_{ij} = 2h^{ik(0)} t^{(1)}_k t^{(1)}_j - 2h^{ik(0)} t^{(1)}_{(j,k)} + \frac{\sqrt{b} t^{(1)}_i}{\alpha} \delta^i_j - \frac{p+1}{p} R^i_j + \frac{1}{p} \left( \frac{1}{2} F_{ij} F_{ik} g^{(0)}_{jl} g^{(0)}_{jk} + 2\Lambda \right) \delta^i_j + F_{im} F_{jk} g^{(0)}_{il} g^{mk(0)}.
\]

(64)

Now we come to the momentum constraints. It is easy to show that the time component still gives the incompressible equation \( D_i v^i = 0 \) under identification (34).

However, the space components of the momentum constraints turn out to be different to the previous results. The left hand side of the momentum constraints still gives the form the same as previous section. The right hand side gives

\[
T^\mu_i n_\mu = \frac{1}{\sqrt{f}} T^r_i = -\frac{1}{\sqrt{f}} F^r_{ij} F_{ij}.
\]

(65)

Since \( F^{ri} r_c = \lambda \sqrt{f} F_{rr} g^{ri} + F_{kj} g^{ji} g^{kr} \sim O(\lambda^2) \), the order of the right hand is \( O(\lambda) \) in the near horizon limit. With identification \( -n_\mu F^{\mu a} = J^a \), we get the external electromagnetic force term \( F_{ij} J^j \). Again let \( \alpha = \sqrt{b} \), the first order of the space components turns to be the standard incompressible magnetofluid equation

\[
\partial_\tau v_i + D_i P + v^j D_j v_i - D_j D_j v_i - R^i_j v_j = F_{ij} J^j.
\]

(66)

This is the expected form as equation (42), where the external force term comes from the background electromagnetic field on the cutoff surface.

VI. SUMMARY AND DISCUSSIONS

In this paper we have generalized the framework presented in [24, 26, 27] to a spacetime with matter field. We have demonstrated that with the help of Petrov-like condition and Einstein-Maxwell constraints, the incompressible Navier-Stokes equation can be derived for a charged fluid living on the cutoff surface which is embedded into a charged AdS black brane, RN-AdS black hole and magnetic black hole respectively. During the derivation imposing appropriate boundary condition for the Maxwell field is crucial. In order to find a suitable boundary condition, we use the result of initial boundary value problem of Einstein system and Maxwell system. This also give us a guideline for searching boundary condition
of general matter field. This also gives us another way to understand the gravity/fluid correspondence in terms of the evolution of partial differential equations. Since the background electromagnetic field on the cutoff surface is fixed, the matter field does not generate extra contributions to the incompressible Navier-Stokes equation in the charged AdS black brane and RN-AdS spacetime due to the fact that $F_{ab}|_{r_c} = 0$. However, for a black hole with magnetic charge, the background electromagnetic field on the cutoff surface is not zero, its coupling with charges of the fluid contributes an external force term to the hydrodynamic equation.

Now the holographic character of Petrov-like condition has been further disclosed in the spacetime with electromagnetic field. We conjecture that it should be a universal method to reduce the Einstein equation to the Navier-Stokes equation for a general spacetime in the presence of a horizon.

In this paper, the correspondence between gravity and fluid is feasible at the near horizon limit, while with the method of directly disturbing the bulk metric, the Navier-Stokes equation can be derived on arbitrary finite cutoff surface. So we wonder whether the method in this paper can be applied to the non-near horizon limit. This might be taken as further investigation.

**Acknowledgments**

YL was partly supported by NSFC (10875057, 11178002), Fok Ying Tung Education Foundation (No.111008), the key project of Chinese Ministry of Education (No.208072), Jiangxi young scientists (JingGang Star) program and 555 talent project of Jiangxi Province. YT and XW was partly supported by NSFC (11075206, 11175245).

**Appendix A: Calculations of the energy-momentum tensor of electromagnetic field**

Now we will show the concrete details that how to get equation (21) from (19). By definition

$$T_{\mu\nu} = \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} - F_{\mu\rho} F^{\rho}_{\nu},$$
we have
\[
T_{\mu\nu}n^\mu n^\nu = \frac{1}{4} F_{\rho\sigma} F^\rho{}^\sigma - F^\mu{}^\nu \nabla_\mu n_\nu = \frac{1}{4} F_{\rho\sigma} F^\rho{}^\sigma - \frac{1}{f} F^\rho F^{\nu\rho},
\]
\[
T = \frac{p - 2}{4} F_{\rho\sigma} F^\rho{}^\sigma,
\]
\[
T_{00} = \frac{1}{f} T_{tt} = -\frac{1}{4} F_{\rho\sigma} F^\rho{}^\sigma - \frac{1}{f} F_{tp} F^p_t,
\]
\[
T_{\delta\beta}n^\delta n^\beta = T_0^\delta n_\delta = \frac{1}{f} T^r_t = -\frac{1}{f} F^{\nu\rho} F_{t\rho},
\]
\[
T^i_j = \frac{1}{4} \delta^i_j F_{\rho\sigma} F^\rho{}^\sigma - F^{i\rho} F_{j\rho}.
\]

The second line of equation (19) then turns to
\[
\frac{1}{p} (T_{\delta\beta}n^\beta n^\delta + 2\Lambda + T + T_{00} - 2T_{\delta\beta}n^\delta) \delta^i_j - T^i_j
\]
\[
= \frac{1}{p} (-\frac{1}{2} F_{\rho\sigma} F^{\rho\sigma} - \frac{1}{f} F^{\nu\rho} F^{\nu\rho} + 2\Lambda - \frac{1}{f} F_{tp} F^p_t + \frac{2}{f} F^{\nu\rho} F_{t\rho}) \delta^i_j + F^{i\rho} F_{j\rho}.
\]

With equation (20), it is easy to work out
\[
-\frac{1}{2} F_{\rho\sigma} F^{\rho\sigma} = -\frac{1}{2} (F_{rt} F^{rt} + F_{tr} F^{tr} + F_{ij} F^{ij}) = F_{rt} F_{rt},
\]
\[
-\frac{1}{f} F^{\nu\rho} F^{\nu\rho} = -\frac{1}{f} (F^r_t F^{rt} + F^r_i F^{ri}) = \frac{1}{f} F_{rt} g^{rr},
\]
\[
-\frac{1}{f} F_{tp} F^p_t = -\frac{1}{f} (F_{tr} F^r_t + F_{ti} F^i_t) = -\frac{1}{f} F_{tp} g_{rr},
\]
\[
\frac{2}{f} F^{\nu\rho} F_{t\rho} = \frac{2}{f} F^{ri} F_{ti} = 0,
\]
\[
F^{i\rho} F_{j\rho} = F^{ik} F_{jk} = 0.
\]

So we have
\[
\frac{1}{p} (T_{\delta\beta}n^\beta n^\delta + 2\Lambda + T + T_{00} - 2T_{\delta\beta}n^\delta) \delta^i_j - T^i_j
\]
\[
= \frac{1}{p} [2\Lambda + F_{rt} F_{rt}] \delta^i_j
\]
\[
= \frac{1}{p} [\lambda^2 f F_{rr} F_{rr} + 2\Lambda] \delta^i_j.
\]

Considering the perturbations of electromagnetic field
\[
F_{rt} = \frac{1}{\lambda \sqrt{f}} C_h + F_{rr}^{(0)}.
\]
We finally get
\[
\frac{1}{p} (T_{\delta\beta} n^\beta n^\delta + 2\Lambda + T + T_{00} - 2T_{0\delta} n^\delta) \delta^i_j - T^i_j
\]
\[
= \frac{1}{p} (C_h^2 + 2\Lambda) \delta^i_j + O(\lambda).
\]

The situation of RN black hole and magnetic black hole are similar.

**Appendix B: Hamiltonian constraint of charged AdS black brane and RN-AdS spacetime**

The Hamiltonian constraint is
\[
p^{+1} R + K_{\mu\nu} K^{\mu\nu} - K^2 = 2\Lambda + 2T_{\mu\nu} n^\mu n^\nu.
\]

In terms of \( t_{ab} = K h_{ab} - K_{ab} \), it turns to
\[
p^{+1} R + (t_{\tau})^2 - \frac{2h_{ij} t_{i\tau} t_{j\tau}}{\lambda^2} t_{i\tau} t_{j\tau} - \frac{t^2}{p} + t_j^i t^j_i = 2\Lambda + 2T_{\mu\nu} n^\mu n^\nu.
\]

The second term on the right hand side
\[
2T_{\mu\nu} n^\mu n^\nu = \frac{1}{2} F_{\rho\sigma} F^{\rho\sigma} - \frac{2}{f} F^r_p F^{r_p}
\]
\[
= 2 \frac{F_{rt} F_{rt} g^{rr} - F_{rt} F_{rt}}{f}
\]
\[
= 2 \frac{\lambda^2 f F_{rr} F_{rr} g^{rr} - \lambda^2 f F_{rr} F_{rr}}{f}
\]
\[
= (\frac{2}{f} g^{rr} - 1) C_h^2 + O(\lambda)
\]
\[
= C_h^2 + O(\lambda).
\]

Hamiltonian constraint at order \( \lambda^{-2} \) is automatically satisfied:
\[
\frac{p}{4\lambda^2} - \frac{p}{4\lambda^2} = 0.
\]

In charged AdS black brane, the subheading order \( \lambda^0 \) gives
\[
- t_{i}^{\tau(1)} - 2h_{ij} t_{i}^{\tau(1)} t_{j}^{\tau(1)} - \frac{p_b}{r_h} = 2\Lambda + C_h^2.
\]

In RN-AdS, the subleading order \( \lambda^0 \) gives
\[
p^{+1} R - t_{r}^{\tau(1)} - 2h_{ij} t_{i}^{\tau(1)} t_{j}^{\tau(1)} - \frac{p_b}{r_h} = 2\Lambda + C_h^2.
\]

Here \( p^{+1} R \) is the scalar curvature of the hypersurface.
Appendix C: Current conservation law

\[ D_{a}J^{a} = 0 \iff \partial_{a} \left( \sqrt{-h}J^{a} \right) = 0. \]

Since \( g^{ri} \sim O(\lambda^2) \), and

\[ J^{r} = n_{r}F^{rr} = \lambda^2 \sqrt{f}F_{rr} = \lambda C + \lambda^2 \sqrt{f}F^{(0)}_{rr}, \]

\[ J^{i} = n_{r}F^{ri} = \lambda \sqrt{f}g^{ri}F_{rr} = g^{ri}C + \lambda \sqrt{f}g^{ri}F^{(0)}_{rr}. \]

The first non-trivial leading order gives

\[ \partial_{r}F^{(0)}_{rr} = 0. \]

---

[1] T. Damour, (1979), Quelques proprietes mecaniques, electromagnetiques, thermodynamiques et quantiques des trous noirs, Thèse de doctorat dEtat, Universite Paris 6. (available at [http://www.ihes.fr/damour/Articles/](http://www.ihes.fr/damour/Articles/)). T. Damour, (1982), Surface effects in black hole physics, in Proceedings of the Second Marcel Grossmann Meeting on General Relativity, Ed. R. Ruffini, North Holland, p. 587.

[2] G. Policastro, D.T. Son and A.O. Starinets, Phys. Rev. Lett. 87, 081601 (2001) [arXiv:hep-th/0104066]; JHEP 0209, 043 (2002) [arXiv:hep-th/0205052].

[3] P. Kovtun, D.T. Son and A.O. Starinets, “Holography and hydrodynamics: diffusion on stretched horizons”, JHEP 0310, 064 (2003) [arXiv:hep-th/0309213].

[4] A. Buchel and J.T. Liu, “Universality of the shear viscosity in supergravity”, Phys. Rev. Lett. 93, 090602 (2004) [arXiv:hep-th/0311175].

[5] N. Iqbal and H. Liu, “Universality of the hydrodynamic limit in AdS/CFT and the membrane paradigm”, Phys. Rev. D 79, 025023 (2009) [arXiv:0809.3808].

[6] S. Bhattacharyya, S. Minwalla and S. R. Wadia, “The Incompressible Non-Relativistic Navier-Stokes Equation from Gravity”, JHEP 0908, 059 (2009), arXiv:0810.1545 [hep-th].

[7] C. Eling, I. Fouxon and Y. Oz, “The Incompressible Navier-Stokes Equations From Black Hole Membrane Dynamics”, Phys. Lett. B 680, 496 (2009) [arXiv:0905.3638].

[8] S. Bhattacharyya, V. E. Hubeny, S. Minwalla, M. Rangamani, “Nonlinear Fluid Dynamics from Gravity”, JHEP 0802, 045 (2008), arXiv: 0712.2456 [hep-th].
[9] S. Bhattacharyya, V. E. Hubeny, R. Loganayagam, G. Mandal, S. Minwalla, T. Marita, M. Rangamani and H.S. Reall, “Local Fluid Dynamical Entropy from Gravity”, JHEP 0806, 055 (2008), arXiv:0803.2526 [hep-th]

[10] S. Bhattacharyya, R. Loganayagam, S. Minwalla, S. Nampuri, S.P. Trivedi, S.R. Wadia, “Forced Fluid Dynamics from Gravity”, JHEP 0902, 018 (2009), arXiv:0806.0006 [hep-th]

[11] S. Bhattacharyya, R. Loganayagam, I. Mandal, S. Minwalla, A. Sharma, “Conformal Nonlinear Fluid Dynamics from Gravity in Arbitrary Dimensions”, JHEP 0812, 116 (2008), arXiv:0809.4272 [hep-th]

[12] Marco M. Caldarelli, Oscar J.C. Dias, Dietmar Klemm, “Dyonic AdS black holes from magnetohydrodynamics”, JHEP 0903, 025(2009), arXiv: 0812.0801 [hep-th]

[13] James Hansen, Per Kraus, “Nonlinear Magnetohydrodynamics from Gravity”, JHEP 0904, 048 (2009), arXiv: 0811.3468 [hep-th]

[14] Geoffrey Combre, Paul McFadden, Kostas Skenderis, Marika Taylor, “The relativistic fluid dual to vacuum Einstein gravity”, JHEP 03(2012)076, arXiv: 1201.2678 [hep-th]

[15] Christopher Eling, Adiel Meyer, Yaron Oz, “The Relativistic Rindler Hydrodynamics”, JHEP 1205 (2012) 116, arXiv: 1201.2705 [hep-th]

[16] S. Bhattacharyya, R. Loganayagam, S.R. Wadia, “The Incompressible Non-Relativistic Navier-Stokes Equation from Gravity”, JHEP 0908, 059 (2009), arXiv:0810.1545 [hep-th]

[17] I. Bredberg, C. Keeler, V. Lysov and A. Strominger, “Wilsonian Approach to Fluid/Gravity Duality”, JHEP 1103, 141 (2011), arXiv:1006.1902 [hep-th]

[18] I. Bredberg, C. Keeler, V. Lysov and A. Strominger “FROM NAVIER-STOKES TO EINSTEIN”, arXiv:1101.2451 [hep-th]

[19] R.-G. Cai, L. Li and Y.-L. Zhang, “Non-Relativistic Fluid Dual to Asymptotically AdS Gravity at Finite Cutoff Surface”, JHEP 1107, 027 (2011), arXiv:1104.3281.

[20] Chao Niu, Yu Tian, and X.N. Wu, Yi Ling, “Incompressible Navier-Stokes Equation from Einstein-Maxwell and Gauss-Bonnet-Maxwell Theories”, Phys.Lett.B 711, 411 (2012), arXiv:1107.1430 [hep-th]

[21] R.-G. Cai, L. Li, Z.-Y. Nie and Y.-L. Zhang, “Holographic Forced Fluid Dynamics in Non-relativistic Limit”, arXiv:1202.4091 [hep-th].

[22] N. Banerjee, J. Bhattacharya, S. Bhattacharya, S. Dutta, R. Loganayagam, and P. Surowka, “Hydrodynamics from charged black branes”, JHEP 1101, 094 (2011), arXiv:0809.2596 [hep-
[23] Y.P. Hu, P. Sun, and J.H. Zhang, “Hydrodynamics with conserved current via AdS/CFT correspondence in the Maxwell-Gauss-Bonnet gravity”, Phys.Rev.D 83, 126003 (2011) arXiv:1103.3773 [hep-th]

[24] V. Lysov, and A. Strominger, “FROM PETROV-EINSTEIN TO NAVIER-STOKES”, arXiv:1104.5502 [hep-th]

[25] H. Friedrich and G. Nagy, “The Initial Boundary Value Problem for Einstein’s Vacuum Field Equation”, Commun. Math. Phys. 201 (1999) 619.

[26] T.Z. Huang, Yi Ling, W.J. Pan, Yu Tian, and X.N. Wu, “From Petrov-Einstein to Navier-Stokes in Spatially Curved Spacetime”, JHEP 1110, 079 (2011), arXiv:1107.1464 [gr-qc]

[27] T.Z. Huang, Yi Ling, W.J. Pan, Yu Tian, and X.N. Wu, “Fluid/Gravity duality with Petrov boundary condition in a spacetime with a cosmological constant”, Phys. Rev. D 85, 123531 (2012), arXiv:1111.1576 [hep-th]

[28] Xiaoming Wu, Chao-Guang Huang and Jia-Rui Sun, Phys. Rev. D 77 (2008) 124023.

[29] R.Moreau, Magnetohydrodynamics, KLUWER ACADEMIC PUBLISHERS, 1990.