Axisymmetric Solutions of the Euler Equations
for Sub-Square Polytropic Gases

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Abstract. We establish rigorously the existence of a three-parameter family of self-similar, globally bounded, and continuous weak solutions in two space dimensions to the compressible Euler equations with axisymmetry for $\gamma$-law polytropic gases with $1 \leq \gamma < 2$. The initial data of these solutions have constant densities and outward-swirling velocities. We use the axisymmetry and self-similarity assumptions to reduce the equations to a system of three ordinary differential equations, from which we obtain detailed structures of solutions besides their existence. These solutions exhibit familiar structures seen in hurricanes and tornadoes. They all have finite local energy and vorticity with well-defined initial and boundary values.

1. Introduction.

We are interested in finding some solutions to the initial value problem for the two-dimensional compressible Euler equations. Our approach is to try to generalize to two dimensions some of the results on Riemann problems for the one-dimensional compressible Euler equations. One natural generalization is to consider initial data which consist of four constant states, or any finite number of constant states [13–15]. The well-known configurations of regular and Mach reflections are special cases of such a generalization. However, no rigorous proof of existence of any nontrivial solutions to these problems has been established.

A more complete generalization is to consider initial data which depend only on the polar angle in the two-dimensional space of positions. This generalization looks certainly more formidable, if one hopes to solve it all. However, it now contains a special three-parameter family of data, namely, the axisymmetric initial data, which allows us to assume axisymmetry of the solutions and reduce the initial value problem of the partial differential equations to a boundary value problem of a system of ordinary differential equations. With this symmetry we were able to construct in [16] a two-parameter family of selfsimilar solutions corresponding to pure rotational initial data. In paper [17], we gave rigorous proofs for these solutions and constructed an additonal one-parameter family of solutions for the square polytropic cases. This additional parameter allows the initial flows to swirl outward. In the current paper we take on the sub-square cases: we construct rigorously a three-parameter family of solutions.

Our main task here is to study the resulting boundary value problem of the non-autonomous system of three ordinary differential equations in the cases $1 \leq \gamma < 2$. Singularity (stationary) points of the system consist of two-dimensional manifolds in the four-dimensional phase space. These singularity points correspond to surfaces of characteristics, where solutions may not be differentiable in the physical space and time. Linearizations at some stationary points yield no structure for the solutions due to the high order degeneracy of the system at these points. The existence of connecting orbits follows from high-order local analysis at stationary points and global analysis of various invariant regions of the system. A typical global solution may consist of as many as three nontrivial connecting orbits chained together continuously. The complete construction of the three-parameter family of solutions is described in the conclusions section, section 8, at the end of the paper.
We find that our solutions capture qualitatively some typical properties of swirling flows such as hurricanes and tornadoes. As an illustration, we plot a typical solution in Fig. 1.1–2. These figures are placed out of the main flow of the text due to their large sizes. See Fig. 8.1–2 for more precise description on the eye and wall regions. We refer the interested reader to paper [16] for explicit presentations of spiralling particle trajectories.

We point out that our initial data are restricted to the set of data with nonnegative radial velocities (swirling outward), in addition to axisymmetry and radial symmetry. Shock waves may arise if the initial radial velocities are allowed to be negative (swirling inward). Due to complications of the global dynamics of the ordinary differential equations, we have only studied the special case \( \gamma = 2 \) in a companion paper [17]. In this paper, we study the case \( 1 \leq \gamma < 2 \). The case \( \gamma > 2 \) will be considered in a forthcoming paper. For some partial results on the case \( \gamma > 2 \), we refer the reader to our paper [16]. Preliminary steps (Sect. 1–5) are reproduced here from [17] for readers’ convenience.

There is a great deal of related work, from which we mention only the most closely related. For general existence of weak solutions with axisymmetry for the 2-D compressible Euler equations outside a core region, we refer the reader to the recent work of Chen and Glimm [5]. For some explicit solutions of the compressible Euler equations with spherical symmetry but without swirls, see Courant and Friedrichs [4]. For viscous swirling motions, we refer the reader to Bellamy-Knights [1], Colonius, Lele, and Moin [3], Mack [9], Powell [11], and Serrin [12].

2. The problem.

We consider the two-dimensional compressible and polytropic Euler equations

\[
\begin{align*}
\rho_t + (\rho u)_x + (\rho v)_y &= 0, \\
(\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y &= 0, \\
(\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y &= 0,
\end{align*}
\]

where \( p = p(\rho) \) is a given increasing function of \( \rho \). Global existence of weak solutions to its initial value problem is open. Attempts have been made through considering special situations such as the diffraction of a planar shock at a wedge or generalized Riemann problems with four different initial constant states.

Here we consider a situation which involves swirling motions. We impose axisymmetry to the system. That is, we assume that our solutions \((u, v, \rho)\) have the property

\[
\rho(t, r, \theta) = \rho(t, r, 0)
\]

\[
\begin{pmatrix}
  u(t, r, \theta) \\
  v(t, r, \theta)
\end{pmatrix} = \begin{pmatrix}
  \cos \theta & -\sin \theta \\
  \sin \theta & \cos \theta
\end{pmatrix} \begin{pmatrix}
  u(t, r, 0) \\
  v(t, r, 0)
\end{pmatrix}
\]

for all \( t \geq 0, \theta \in \mathbb{R} \) and \( r > 0 \), where \((r, \theta)\) is the polar coordinates of the \((x, y)\) plane. With this symmetry, system (2.1) can be reduced for continuous solutions to

\[
\rho_t + (\rho u)_r + \frac{\rho u}{r} = 0,
\]
\[ u_t + uu_r + \frac{p_r}{\rho} - \frac{v^2}{r} = 0, \quad (2.3) \]
\[ v_t + uv_r + \frac{uv}{r} = 0, \]

where \( \rho = \rho(t, r, 0) \), etc. Note now that \( u \) and \( v \) in (2.3) represent the radial and pure rotational (a.k.a. tangential) velocities in the flow, respectively.

We limit ourselves to a Riemann-type of initial data; that is, we require the initial data to be independent of the radial variable \( r > 0 \):

\[ (\rho(0, r, \theta), u(0, r, \theta), v(0, r, \theta)) = (\rho_0(\theta), u_0(\theta), v_0(\theta)). \quad (2.4) \]

We remark in passing that the initial value problem of system (2.1) with the type of data in (2.4) seems to be more in the spirit to be called the two-dimensional Riemann problem for (2.1). It degenerates to the classical Riemann problem for the one-dimensional case, it is simple, and yet general enough to contain important waves such as swirling motions as well as shock and rarefaction waves and slip lines (surfaces).

When the axisymmetry condition (2.2) is imposed onto (2.4), we find that our data are limited to be

\[ u(0, r, \theta) = u_0 \cos \theta - v_0 \sin \theta, \]
\[ v(0, r, \theta) = u_0 \sin \theta + v_0 \cos \theta, \quad (2.5) \]
\[ \rho(0, r, \theta) = \rho_0, \]

where \( \rho_0 > 0, (u_0, v_0) \in \mathbb{R}^2 \) are arbitrary constants. Hence our data for system (2.3) are

\[ \rho(0, r, 0) = \rho_0, \quad u(0, r, 0) = u_0, \quad v(0, r, 0) = v_0. \quad (2.6) \]

Since the problem (2.3)(2.6) is invariant under self-similar transformations, we look for self-similar solutions \((\rho, u, v)\) which depend only on \( \xi = r/t \). We thus have the following boundary value problem of a system of ordinary differential equations:

\[ \rho_r = \frac{\rho \Theta}{r \Delta}, \quad u_r = \frac{1}{r \Delta}, \quad v_r = \frac{uv}{r(r - u)}, \quad (2.7) \]

\[ \lim_{r \to +\infty} (\rho, u, v) = (\rho_0, u_0, v_0) \quad (2.8) \]

where

\[ \Delta \equiv c^2 - (u - r)^2, \]
\[ \Theta \equiv v^2 - u(r - u), \]
\[ \Sigma \equiv (r - u)\Theta - u\Delta = v^2(r - u) - uc^2, \]
\[ c \equiv \sqrt{p'(\rho)}, \]
and \( r \) is used in place of \( \xi \) for notational convenience. We note that \( c \) is the sound speed.

We will construct global continuous solutions or establish their existence for problem (2.7–8) for any \( \rho_0 > 0, u_0 \geq 0, v_0 \in \mathbb{R} \) and \( p(\rho) = A_2 \rho^\gamma \) where \( 1 \leq \gamma < 2 \) and \( A_2 > 0 \) is a constant. We shall be mainly concerned with the case \( 1 < \gamma < 2 \) in Sections 4–7, but we will mention the differences for \( \gamma = 1 \) in Section 8. We will also assume \( v_0 \geq 0 \) because the case \( v_0 < 0 \) can be transformed by \( v \to -v \) to the case \( v_0 > 0 \). It can be verified that these solutions are also global continuous solutions to the original Euler equations.

3. Far-field solutions.

We show that problem (2.7-8) has a local solution near \( r = +\infty \) for any datum \((\rho_0, u_0, v_0)\) with \( \rho_0 > 0 \). Let \( s = \frac{1}{r} \). Then (2.7-8) can be written as

\[
\frac{d\rho}{ds} = \frac{\rho[u(1 - us) - sv^2]}{s^2c^2 - (1 - us)^2},
\]

\[
\frac{du}{ds} = \frac{suc^2 - v^2(1 - us)}{s^2c^2 - (1 - us)^2},
\]

\[
\frac{dv}{ds} = -\frac{uv}{1 - us},
\]

\[
(\rho, u, v)_{s=0} = (\rho_0, u_0, v_0).
\]

Problem (3.1-2) is a classically well-posed problem which has a unique local solution for any initial datum with \( \rho_0 > 0 \).

We find that \( v = 0 \) and

\[
sv^2 - u(1 - us) = 0
\]

are invariant surfaces in the four-dimensional \((\rho, u, v, s)\) space. It can be verified that

\[
\frac{d}{ds}[sv^2 - u(1 - us)] = \frac{(1 - 3su)s^2c^2 + u(1 - su)^2}{(1 - su)[s^2c^2 - (1 - su)^2]}[sv^2 - u(1 - us)].
\]

Explicit solutions. In the special case \( u_0 = 0 \), we find in [16] from the invariant surface (3.3) a set of explicit solutions near \( r = +\infty \):

\[
\rho = \rho_0, \quad u = \frac{v_0}{r}, \quad v = \frac{v_0}{r} \sqrt{r^2 - v_0^2}, \quad r \geq r^*
\]

(3.4)

where

\[
r^* \equiv \frac{1}{2} \left( \sqrt{v_0^2 + 4v_0^2 + c_0^2} \right),
\]

(3.5)

and

\[
c_0 \equiv \sqrt{p'(\rho_0)}.
\]
We comment that the functions in (3.4) are actually defined for all \( r \geq v_0 \), but we cannot use them up to \( v_0 \) with absolute certainty since (3.1) has a singularity at the point \( r = r^* \) on the curve (3.4). In fact, we find that the position \( r \), the radial velocity \( u \), and the sound speed \( c \) at \( r^* \) along (3.4) have the relation

\[
r^* = u(r^*) + c_0,
\]

which is to say that \( r^* \) is the radial characteristic speed; that is, distance that a small disturbance generated from the origin at time zero can travel radially in time \( t = 1 \). Thus, \( r^* \) in (3.5) is a characteristic position.

We point out that the Mach number along any solution in (3.4) is a constant:

\[
M \equiv \sqrt{u^2 + v^2/c} = v_0/c_0.
\]

For those who are interested in the pseudo Mach number defined in the self-similar coordinates \((\xi, \eta)\) as

\[
M_s \equiv \sqrt{(u-\xi)^2 + (v-\eta)^2/c},
\]

we find that

\[
M_s = \sqrt{r^2 - v_0^2/c_0}
\]

along a solution of (3.4). At the end point \( r = r^* \), we find

\[
M_s(r^*) = \left(\sqrt{M_0^2 + 1/4 + 1/4}\right)^{1/2} > 1,
\]

where

\[
M_0 \equiv v_0/c_0
\]

is the Mach number for the initial data in this case. Pseudo Mach number determines the type (hyperbolic or other) of the 2-D Euler equations in the self-similar plane \((\xi, \eta)\): hyperbolic if \( M_s > 1 \), other types if \( M \leq 1 \), see [15] for example. So the solutions (3.4) are pseudo supersonic and in hyperbolic regions.

4. Intermediate field equations.

We can simplify system (3.1) by assuming the polytropic relation

\[
p(\rho) = A_2\rho^\gamma
\]

for some \( A_2 > 0 \) and \( \gamma > 1 \), and introducing the variables

\[
I = su, \quad J = sv, \quad K = sc.
\]
Then system (3.1) can be written into the form
\[
\frac{dI}{ds} = \frac{2IK^2 - (1 - I)[J^2 + I(1 - I)]}{K^2 - (1 - I)^2}
\]
\[
\frac{dJ}{ds} = \frac{J}{1 - I} \left(1 - 2I\right)
\]
\[
\frac{dK}{ds} = \frac{K}{2} \left(2K^2 - 2(1 - I)^2 - (\gamma - 1)[J^2 - I(1 - I)]\right)
\]

(4.3)

Corresponding to the initial data (3.2), we shall look for solutions of (4.3) with the following initial condition:
\[
(I, J, K) \sim s(u_0, v_0, c_0)
\]

(4.4)
as \(s \to 0^+\). We note that (4.3) is now autonomous for \(I, J, K\) with respect to the new variable \(s' = \ln s\).

The invariant surfaces of (4.3) are the surface \(J = 0\), the surface \(K = 0\), and the surface
\[
H \equiv J^2 - I(1 - I) = 0
\]

(4.5)
since
\[
s \frac{dH}{ds} = \frac{(1 - 2I)[2K^2 - 3(1 - I)^2]}{(1 - I)[K^2 - (1 - I)^2]} H.
\]
The explicit solutions (3.4) are all in the invariant surface \(H = 0\). We postpone the description of these solutions in the variables \((I, J, K)\) to Sect. 6 when we have a more complete picture of all solutions.

**Scaling symmetry.** System (4.3) is invariant under the coordinate transformation \(s \to \alpha s\) for any constant \(\alpha > 0\). In particular, we can take \(\alpha = 1/c_0\). Thus we may assume that \(\rho_0 > 0\) is such that \(c_0 = \sqrt{p' (\rho_0)} = 1\). Hence the structure of any solution of problem (4.3-4) will depend only on the dimensionless quantities \(u_0/c_0\) and \(v_0/c_0\).

5. Solutions without swirls.

Let us first determine the distribution of integral curves on the invariant surface \(J = 0\).

Assume \(v_0 = 0\). We look for solutions to problem (4.3-4) with \(J = 0\). Hence we have a subsystem for \((I, K)\):
\[
\begin{cases}
\frac{dI}{ds} = I \frac{2K^2 - (1 - I)^2}{K^2 - (1 - I)^2} \\
\frac{dK}{ds} = K \frac{K^2 - (1 - I)^2 + \frac{\gamma - 1}{2} I(1 - I)}{K^2 - (1 - I)^2}
\end{cases}
\]

(5.1)

(5.2)

Introducing a new parameter \(\tau\), we can rewrite (5.1-2) as
\[
\begin{cases}
\frac{dI}{d\tau} = I \left[(1 - I)^2 - 2K^2\right] \\
\frac{dK}{d\tau} = K \left[(1 - I)^2 - K^2 - \frac{\gamma - 1}{2} I(1 - I)\right] \\
\frac{ds}{d\tau} = s \left[(1 - I)^2 - K^2\right].
\end{cases}
\]

(5.3)

(5.4)

(5.5)
Note that (5.3-4) form an autonomous subsystem. If \( u_0 \) also vanishes, then we have a trivial solution \( \rho = \rho_0, \ u = v = 0. \)

Suppose \( u_0 > 0. \) It can be verified that our far-field solutions starting at \( s = 0^+ \) enter the region \( \Omega \subset \mathbb{R}^2 \) in the \( (I, K) \) phase space given by

\[
\Omega : \begin{cases}
I > 0, \ K > 0,
\ a \equiv (1-I)^2 - K^2 - \frac{\gamma - 1}{2} I(1-I) > 0 \quad \text{in} \quad 0 < I \leq \frac{1}{\gamma} \\
\ b \equiv (1-I)^2 - 2K^2 > 0 \quad \text{in} \quad \frac{1}{\gamma} \leq I < 1
\end{cases}
\]

See Figure 5.1.

![Figure 5.1. Phase portrait of solutions without swirls.](image)

We show that solutions starting in the closure \( \overline{\Omega} \) will not leave \( \overline{\Omega} \) as \( s \) increases. Notice first that \( s > 0 \) is an increasing function of \( \tau \) in \( \Omega \) by equation (5.5) so we will show that solutions of the two equations (5.3-4) starting in the closure \( \overline{\Omega} \) will not leave \( \overline{\Omega} \) as \( \tau \) increases. The stationary points of (5.3-4) in \( \overline{\Omega} \) are the points \( (I, K) = (0, 0), (0, 1), (1, 0), \) and \( \left( \frac{1}{\gamma}, \frac{1}{\sqrt{2}} \left( 1 - \frac{1}{\gamma} \right) \right) \). The axis \( K = 0 \) in \( 0 < I < 1 \), and the axis \( I = 0 \) in \( 0 < K < 1 \) are trivial solutions. On the boundary \( b = 0 \), i.e.,

\[
K = \frac{1}{\sqrt{2}} (1 - I), \quad \frac{1}{\gamma} < I < 1, \tag{5.6}
\]

we find that \( \frac{dI}{d\tau} = 0, \ \frac{dK}{d\tau} < 0. \) So solutions enter \( \Omega \) on (5.6). Finally on the boundary \( a = 0 \), i.e.,

\[
K^2 = (1-I)^2 - \frac{\gamma - 1}{2} I(1-I), \quad 0 < I < \frac{1}{\gamma}, \tag{5.7}
\]
we find that $\frac{dL}{d\tau} < 0$, and $\frac{dK}{d\tau} = 0$. So solutions enter $\Omega$ on (5.7) also.

We conclude without showing further details that some solutions in $\Omega$ will go to the point $(1, 0)$, while others go to the point $(0, 1)$, with exactly one integral curve (the heteroclinic orbit) going to the point $\left(\frac{1}{\gamma}, \frac{1}{\sqrt{2}} \left(1 - \frac{1}{\gamma}\right)\right)$.

\[dL/dI = -L \frac{(\gamma - 1)(1 - I) - 2IL}{(1 - I)^2 - 2I^2L}, \quad 0 < I < 1\]  
(5.8)

\[L = M_0^{-2} \text{ at } I = 0\]  
(5.9)

where

\[M_0 \equiv u_0/c_0\]

denotes the Mach number of the initial states $(u_0, \rho_0)$. Problem (5.8-9) is well-posed for any $M_0 > 0$. The transitional solution goes from the point $(I, L) = (0, M_0^{-2})$ to $\left(\frac{1}{\gamma}, \frac{1}{\sqrt{2}}(\gamma - 1)^2\right)$. However, there does not seem to have an explicit formula for the value $M_0$ which yield the transitional solution. We give an estimate instead. For convenience we let $M_h(\gamma)$ denote the value of the initial Mach number for the transitional (critical, heteroclinic) solution. It can be seen that $M_0 = \frac{\sqrt{2}}{\gamma - 1}$ is an upper bound for the transitional Mach number $M_h(\gamma)$. In fact, any solution $L(I)$ of (5.8-9) with datum $M_0 \geq \frac{\sqrt{2}}{\gamma - 1}$ starts as a decreasing function of $I \geq 0$ till $I = \frac{1}{\gamma}$. In the interval $I \in \left[\frac{1}{\gamma}, 1\right]$, the solution remains below the two curves on which the numerator and denominator of the right-hand side
of (5.8) vanish respectively, therefore remains decreasing until the final stationary point \((I, L) = (1, 0)\), see Figure 5.2, where \(d = (\gamma - 1)^2/2, f = M_h^{-2}(\gamma)\), and \(N\) and \(D\) are where the numerator and denominator of the right-hand side of (5.8) vanish respectively.

This transitional solution yields a one-parameter family of smooth solutions in terms of \((r, u, v, \rho)\). We see from equation (5.5) that \(\ln s\) approaches infinity as the solutions approach the point \((I, K) = \left(\frac{1}{\gamma}, \frac{1}{\sqrt{2}} \left(1 - \frac{1}{\gamma}\right)\right)\), since \((1 - I)^2 - K^2 \neq 0\) at the point. So \(s \to +\infty\) and thus \(r \to 0^+\). Also the solutions have the asymptotics

\[
\begin{align*}
  u(r) &= \frac{1}{\gamma} r, \\
  c(r) &= \frac{1}{\sqrt{2}} \left(1 - \frac{1}{\gamma}\right) r
\end{align*}
\]
as \(r \to 0^+\). These global transitional solutions are similar and one is sketched in Figure 5.3.

\[
\begin{align*}
  &u(r) = \frac{1}{\gamma} r, \\
  &c(r) = \frac{1}{\sqrt{2}} \left(1 - \frac{1}{\gamma}\right) r
\end{align*}
\]

Fig. 5.3. A zero-swirl transitional solution.

The Mach number at \(r = 0\) of the transitional solution is \(M = \sqrt{2}/(\gamma - 1)\). The corresponding pseudo Mach number is \(M_s = \sqrt{2}\).

5.2. Finiteness of the parameter \(s\). We show next that the parameter \(s\) approaches finite numbers when solutions of (5.3-5) approach either the points \((I, K) = (1, 0)\) or \((0, 1)\).

We can linearize the two equations (5.3-4) at \((I, K) = (0, 1)\) to find

\[
\begin{align*}
  \frac{dI}{d\tau} &= -I \\
  \frac{d(K-1)}{d\tau} &= -\frac{\gamma + 3}{2} I - 2(K - 1).
\end{align*}
\]

The eigenvalues of (5.10) are \(\lambda_1 = -1, \lambda_2 = -2\). So solutions of (5.3-4) near \((0, 1)\) approach \((0, 1)\) exponentially as \(\tau \to +\infty\). From equation (5.5), we find

\[
\ln \frac{s}{s_0} = \int_{\tau_0}^{\tau} [(1 - I)^2 - K^2] \, d\tau
\]
for some constants $s_0 > 0$ and $\tau_0$. So $s$ approaches a finite number as $\tau \to \infty$ since $(1 - I)^2 - K^2$ approaches zero exponentially.

Linearization at the point $(I, K) = (1, 0)$ of the two equations (5.3-4) yield the trivial system of zero right-hand side. We need a different approach. We show that solutions near $(1, 0)$ will enter $(1, 0)$ in the sector bounded by $K = 0$ and the line

$$K = \alpha(1 - I)$$

(5.11)

for some $0 < \alpha < \frac{1}{\sqrt{2}}$, such that $\alpha^2 + \frac{3 - \gamma}{4} > 0$. See Figure 5.4.

![Figure 5.4. The parameter $s$ is finite near (1,0).](image)

In fact, a vector in the normal direction of (5.11) is $(\alpha, 1)$. We calculate the inner product of the vector field of (5.3-4) with the direction $(\alpha, 1)$ to find

$$\frac{d}{d\tau}(I, K) \cdot (\alpha, 1) = -\alpha \left[ 2I \left( \alpha^2 + \frac{\gamma - 3}{4} \right) - (1 - \alpha^2)(1 - I) \right] (1 - I)^2. \quad (5.12)$$

The expression (5.12) is negative if $I$ is close to 1 and $\alpha$ is such that $\alpha^2 + \frac{3 - \gamma}{4} > 0$. So we conclude that every solution that ends at $(1, 0)$ will be such that

$$K < \alpha(1 - I)$$

(5.13)

near $I = 1$ for some $\alpha < \frac{1}{\sqrt{2}}$, since $\gamma > 1$.

Now we look at the equation (5.1) and use (5.13) to find

$$s \frac{dI}{ds} > I \frac{1 - 2\alpha^2}{1 - \alpha^2}$$
when \( I \) is close to 1. Thus
\[
\frac{dI}{ds} > \text{a positive constant}
\]
when \( I \) is near 1. Therefore \( s \) is finite at the end \((1, 0)\) since \( I \) is bounded and the geometric integral \( \int_{1}^{+\infty} \frac{1}{s} ds \) diverges.

5.3. The global construction. We are now ready to construct global solutions for (5.1-2). For each solution ending at \((I, K) = (0, 1)\), we continue the solution by the constant state \((u, v, \rho) = (0, 0, \rho^{**})\) where \(\rho^{**}\) is the value of \(\rho\) at the ending point. These are continuous extensions. The relation between the density \(\rho^{**}\) and the ending value \(s^{**}\) is
\[
p'(\rho^{**}) = s^{**} - 2.
\]
Let
\[
c^{**} = \sqrt{p'(\rho^{**})},
\]
then
\[
c^{**} = r^{**}
\]
when \(r^{**}\) is the radius of the circle of the constant state in the physical plane \(t = 1\). So the core region has a constant density which expands at its sound speed. The Mach and pseudo Mach numbers at the edge of the core region are 0 and 1 respectively. In the core, the pseudo Mach number \(M_s = r/c^{**} < 1\). We therefore have constructed global solutions in this case.

For each solution ending at the point \((I, K) = (1, 0)\), we continue the solution by the vacuum state \(\rho = 0\). We do not need to specify the functions \(u\) or \(v\) in the vacuum since the Euler equations have \(\rho\) as a factor in every term. Each vacuum occupies a circular region of radius \(r^{**}\) determined by \(r^{**} = \frac{1}{\sqrt{c^{**}}} = u^{**}\) in the physical plane at \(t = 1\), where \(u^{**}\) is the terminal radial velocity of the fluid at the edge of the vacuum. Asymptotically, we find (see also [16])
\[
u - r = -\frac{2(\gamma - 1)}{\gamma + 1}(r - r^{**})(1 + o(1)),
\]
\[
c = \frac{(\gamma - 1)\sqrt{3 - \gamma}}{\gamma + 1}(r - r^{**})(1 + o(1))
\]
at the edge for \(\gamma < 3\). For \(\gamma = 3\), the asymptotics is given by
\[
u - r = -(r - r^{**})(1 + o(1)),
\]
\[
c = \frac{1}{2}(r - r^{**}) \ln^{-1/2} \frac{1}{r - r^{**}}(1 + o(1)).
\]
And for \(\gamma > 3\), it is given by
\[
u - r = -(r - r^{**})(1 + o(1)),
\]
\[
c = K_0(r - r^{**})^{\frac{2}{\gamma - 1}}(1 + o(1)), \quad K_0 > 0.
\]
From these we find that the Mach number $M = \infty$ for all $\gamma > 1$ and the pseudo Mach number $M_s = \frac{2}{\sqrt{3-\gamma}}$ for $\gamma < 3$, and $M_s = \infty$ for $\gamma \geq 3$ at the edge.

In all, we have constructed global solutions to the reduced system (5.1-2), the special case of (4.3-4) with zero swirl.

6. General solutions in the intermediate field.

Now consider the case $v_0 > 0$ and $u_0 \geq 0$ for system (4.3). Let $\Omega_3 \subset \mathbb{R}^3$ be the set of points $(I, J, K)$ satisfying $0 < I < 1$, $J > 0$, $K > 0$,

$$H = J^2 - I(1 - I) < 0,$$

$$B \equiv (1 - I)[J^2 + I(1 - I)] - 2IK^2 > 0 \text{ if } \frac{1}{\gamma} \leq I < 1,$$

$$A \equiv 2(1 - I)^2 + (\gamma - 1)[J^2 - I(1 - I)] - 2K^2 > 0 \text{ if } 0 < I < \frac{1}{\gamma}.$$

See Figure 6.1. Note that the surface $A = 0$ intersects the coordinate plane $K = 0$ on the line

$$J^2 = \frac{\gamma + 1}{\gamma - 1}(1 - I)(1 - \frac{2}{\gamma + 1})$$

which lies inside $H < 0$. It can be verified that all far-field solutions with $u_0 \neq 0$, $v_0 > 0, \rho_0 > 0$ enter the region $\Omega_3$ in $s > 0$. Far-field solutions with $u_0 = 0$ enter the side $H = 0$ of $\Omega_3$, see Sect. 4. We omit these tedious verifications.

Also let $\Omega_{33} \subset \mathbb{R}^3$ be the set of points $(I, J, K)$ satisfying $0 < I < 1$, $J > 0$, $0 < K < 1 - I$, and $H < 0$. See Fig.6.1.

We find that it is convenient to introduce a new variable $\tau$, as in section 5, to write the system (4.3) in the following form

$$\frac{dI}{d\tau} = (1 - I)B \quad (6.1)$$

$$\frac{dJ}{d\tau} = J(1 - 2I)[(1 - I)^2 - K^2] \quad (6.2)$$

$$\frac{dK}{d\tau} = \frac{1}{2}K(1 - I)A \quad (6.3)$$

$$\frac{ds}{d\tau} = s(1 - I)[(1 - I)^2 - K^2] \quad (6.4)$$

This is an autonomous system for $(I, J, K, s)$, and the first three equations (6.1-3) form an autonomous sub-system for $(I, J, K)$.

The stationary points of the system (6.1-3) contained in the closure $\overline{\Omega}_{33}$ are found to be given by the edge

$$K = 1 - I, \ J^2 = I(1 - I), \forall \ I \in [0, 1] \quad (6.5)$$

13
and

\[(I, J, K) = \left( \frac{1}{\gamma}, \ 0, \ \frac{1}{\sqrt{2}} \left( 1 - \frac{1}{\gamma} \right) \right) \quad (6.6)\]

in the case \(\gamma \neq 2\). For the case \(\gamma = 2\), the stationary points of (6.1-3) are given by edge (6.5) and the curve

\[
\begin{cases}
I = \frac{1}{2}, \\
K^2 = \frac{1}{2} \left( J^2 + \frac{1}{4} \right), \quad 0 \leq J < \frac{1}{2},
\end{cases} \quad (6.7)
\]

which is also the intersection of \(A = 0\) with \(B = 0\) in this case. Hence there is no stationary point in the open region \(\Omega_{33}\) when \(1 < \gamma < 2\); all the stationary points are on the boundary of \(\Omega_{33}\) in this case.

Fig. 6.1. The domains \(\Omega_3\) and \(\Omega_{33}\)

We claim that solutions inside \(\Omega_{33}\) do not leave \(\Omega_{33}\) from its sides (excluding possibly edges, corners, or stationary points) as \(s\) increases for all \(\gamma > 1\). Note that the sides of \(\Omega_{33}\) in the surfaces \(K = 0\), or \(J = 0\) or \(H = 0\) are invariant regions. We need only to prove that no solution leaves \(\Omega_{33}\) from the top side \(K = 1 - I\). The top side has an outward normal

\[\vec{n} = (1, 0, 1)\]
arranged in the coordinate order \((I, J, K)\). We calculate the inner product of the normal \(\mathbf{n}\) with the tangent vector of an integral curve of (6.1-3) on the top surface to yield

\[
\mathbf{n} \cdot \frac{d}{d\tau}(I, J, K) = (1 - I)(B + \frac{1}{2}KA) < 0
\]

which proves the claim, since \(\frac{ds}{d\tau} > 0\) in \(\Omega_{33}\).

We analyze the local structure of solutions at the stationary point (6.6). Let

\[
\mathcal{I} = I - \gamma, \quad \mathcal{J} = J, \quad \mathcal{K} = K - \frac{\gamma - 1}{\sqrt{2}\gamma}.
\]

To first order system 6.1-3 can be reduced to

\[
\frac{d\mathcal{I}}{d\tau} = -\frac{2(\gamma - 1)^2}{\gamma^3}(\mathcal{I} + \sqrt{2} \mathcal{K})
\]

\[
\frac{d\mathcal{J}}{d\tau} = -\frac{(2 - \gamma)(\gamma - 1)^2}{2\gamma^3} \mathcal{J}
\]

\[
\frac{d\mathcal{K}}{d\tau} = -\frac{(\gamma - 1)^3}{2\gamma^3}(\frac{\gamma + 2}{2} \mathcal{I} + \sqrt{2} \mathcal{K}).
\]

Its three eigenvalues are found to be

\[
\lambda_\pm = -\frac{(\gamma - 1)^2}{2\gamma^3} \left[ \gamma + 1 \pm \sqrt{(\gamma + 1)^2 + 4\gamma(\gamma - 1)} \right]
\]

\[
\lambda_2 = -\frac{(2 - \gamma)(\gamma - 1)^2}{2\gamma^3}.
\]

Corresponding eigenvectors are found to be

\[
\mathbf{v}_\pm = \left(4\sqrt{2}, 0, \gamma - 3 \pm \sqrt{(\gamma + 1)^2 + 4\gamma(\gamma - 1)}\right)
\]

\[
\mathbf{v}_2 = (0, 1, 0).
\]

It follows that \(\lambda_+ < 0, \lambda_2 < 0, \text{ and } \lambda_- > 0\) for \(1 < \gamma < 2\). Hence the stationary point (6.6) is hyperbolic. The local structure is depicted in Fig. 6.2.

To determine monotonicity of solutions, we need to deal with the surfaces \(A = 0\) and \(B = 0\) inside \(\Omega_{33}\). Consider first the surface

\[
B = I(1 - I)^2 + (1 - I)J^2 - 2IK^2 = 0 \quad \text{in} \quad 0 < I < 1. \quad (6.8)
\]

An outward normal is given by in the coordinate order \((I, J, K)\)

\[
(2K^2 - (1 - I)^2 + 2I(1 - I) + J^2, -2(1 - I)J, 4IK) \equiv \mathbf{n}_B.
\]
We calculate the inner product of the normal $\vec{n}_B$ with the tangent vector of an integral curve of (6.1-3) on the surface (6.8) to yield

$$\vec{n}_B \cdot \frac{d}{d\tau} (I, J, K) = (1 - I)^2 H J_b,$$

where

$$J_b \equiv (\gamma I - 1)(1 - I) - (2 - \gamma)J^2.$$

See figure 6.1 for the position of $J_b = 0$. So integral curves of (6.1–3) go from $B > 0$ to $B < 0$ on the surface $B = 0$ with $J_b < 0$, but reverse their directions on the surface $B = 0$ with $J_b > 0$ as $\tau$ increases.

![Diagram](image)

Fig. 6.2. Local structure of stationary point (6.6).
Now consider the surface
\[ A = 2(1 - I)^2 + (\gamma - 1)[J^2 - I(1 - I)] - 2K^2 = 0, \quad 0 < I < 1. \]

An outward normal is given by
\[
(4(1 - I) + (\gamma - 1)(1 - 2I), -2(\gamma - 1)J, 4K) \equiv \vec{n}_A
\]
in the order \((I, J, K)\). We similarly calculate the inner product of the normal \(\vec{n}_A\) with the tangent of the integral curves on this surface to yield
\[
\vec{n}_A \cdot \frac{d}{d\tau} (I, J, K) = H_{Ja},
\]
where
\[
J_a \equiv [\gamma + 3 - 2(\gamma + 1)I](1 - I)(1 - \gamma I) + (\gamma - 1)^2 J^2 (1 - 2I).
\]
See figure 6.1 for the position of \(J_a = 0\). In particular the cylindrical surface \(J_a = 0\) intersects the \((I, 0, K)\)-plane at \(I = 1/\gamma\) and the surface \(H = 0\) at \(I = \frac{\gamma + 3}{2(3\gamma - 1)}\). Therefore integral curves of (6.1–3) go from \(A > 0\) to \(A < 0\) on the surface \(A = 0\) with \(J_a < 0\), but reverse their directions on the surface with \(J_a > 0\) as \(\tau\) increases. The portion of the surface \(A = 0\) below \(K = 0\) will not be used.

![Diagram](image)

Fig. 6.3. Integral curves from the edge E for gamma between 1 and 2.

We need to study the local structure of solutions near stationary points given in formula (6.5). It is helpful to first describe solutions of (6.1-4) with data \(u_0 = 0, \quad v_0 > 0, \quad \rho_0 > 0\). Our far-field solutions (3.4-5) can be written in terms of \(I, J, K\) as follows:
\[
\begin{align*}
I &= s^2 v_0^2 \\
J &= s v_0 \sqrt{1 - s^2 v_0^2} \\
K &= s c_0
\end{align*}
\]  

(6.9)
valid for \( s \in [0, s^*] \) where \( s^* = 1/r^* \). These solutions are all in the surface \( H = 0 \) and they all start from the origin \((I, J, K) = (0, 0, 0)\) and end at points of the stationary edge given by (6.5). As the initial Mach number \( M_0 = |v_0|/c_0 \) varies in \((0, \infty)\), the ending points of the solutions cover all the interior points of the stationary edge (6.5) exactly once.

Now we study the structure of solutions near the stationary edge (6.5). More precisely, let us use \( E \) to denote the open set
\[
E \equiv \{ (I, J, K) = (\alpha, \sqrt{\alpha(1-\alpha)}, 1-\alpha) \mid \forall \alpha \in (0, 1) \}
\]
which contains the interior points of the curve (6.5). At any point of \( E \), the linear part of the right-hand side of (6.1–3) is found to be given by the matrix
\[
-(1-\alpha) \begin{bmatrix}
(1-\alpha)(1+2\alpha) & -2(1-\alpha)J_\alpha & 4\alpha(1-\alpha) \\
-2(2\alpha-1)J_\alpha & 0 & -2(2\alpha-1)J_\alpha \\
(1-\alpha) [\frac{1}{2} - \frac{1}{2} (1-2\alpha) + 2(1-\alpha)] & -(1-\alpha)(\gamma-1)J_\alpha & 2(1-\alpha)^2 \\
\end{bmatrix}
\]
where \( J_\alpha \equiv \sqrt{\alpha(1-\alpha)} \). The eigenvalues and eigenvectors of this matrix can be found as follows. Since the whole line \( E \) is stationary, we expect that \( \lambda_1 = 0 \) is an eigenvalue, and the tangent vectors of \( E \) are associated eigenvectors. The solutions of (6.9) offer another set of eigenvectors which can further simplify the calculation. We find the other two eigenvalues to be
\[
\lambda_2 = -2(1-\alpha)^2(1+\alpha) < 0, \quad \lambda_3 = -(1-\alpha)^2(1-2\alpha)
\]
with associated eigenvectors
\[
\vec{v}_2 = (2\sqrt{\alpha}, \frac{1-2\alpha}{\sqrt{1-\alpha}}, \frac{1-\alpha}{\sqrt{\alpha}}) \\
\vec{n}_3 \equiv (2(1-\alpha)[2(\gamma-1)\alpha - 1], -2(\gamma + 1)\sqrt{\alpha(1-\alpha)(2\alpha - 1)}, -(1-\alpha)[2(3\gamma-1)\alpha - (\gamma + 3)]).
\]
Along this direction (6.11), we calculate the following information:
\[
\vec{n}_3 \cdot \vec{v}_A \big|_E = 2(1-\alpha)(2\alpha - 1)[2(3\gamma - 1)\alpha - (\gamma + 3)]
\]
\[
\vec{n}_3 \cdot \vec{n}_B \big|_E = 2(1-\alpha)^2(1 - 2\alpha)[2(\gamma - 1)\alpha - 1]
\]
\[
\vec{n}_3 \cdot \vec{n}_H \big|_E = 2(1-\alpha)(1 - 2\alpha)(1 + 4\alpha)
\]
where \( \vec{n}_H \) is used to denote an outward normal to the surface \( J^2 = I(1-I) \).

From now on we restrict ourselves to the case \( 1 < \gamma < 2 \), since we have done the other cases \( \gamma \geq 2 \) to earlier papers. The integral curves along \( \vec{n}_3 \) starting from \( E \) in the direction of increasing \( s \) are depicted in Fig. 6.3.
It can be seen that $\vec{n}_3$ points into different directions relative to the positions of $A = 0$, $B = 0$, and $H = 0$ and the sign of the eigenvalue $\lambda_3$ changes when $\alpha$ varies. Our goal is to use the appropriate direction to construct solutions with monotone increasing $s$. Relative to the surface $B = 0$ we find that $\vec{n}_3$ points into $\Omega_3$ for $\alpha \in \left( \frac{1}{2}, \frac{1}{2(\gamma - 1)} \right)$, into $\Omega_{33}$ and $B < 0$ for $\alpha \in \left( \frac{1}{2}, \frac{1}{2(\gamma - 1)} \right)$, and $-\vec{n}_3$ points into $\Omega_3$ for $I \in \left( 0, \frac{1}{2} \right)$. Relative to the surface $A = 0$ we find that $\vec{n}_3$ points into $\Omega_{33}$ and $A < 0$ for $\alpha \in \left( \frac{\gamma + 3}{2(3\gamma - 1)}, 1 \right)$, into $A > 0$ for $\alpha \in \left( \frac{1}{2}, \frac{\gamma + 3}{2(3\gamma - 1)} \right)$, and $-\vec{n}_3$ points into $\Omega_3$ for $\alpha \in \left( 0, \frac{1}{2} \right)$.

Now we can depict the integral curves inside $\Omega_{33}$, see Figure 6.4.

![Figure 6.4](image)

Fig. 6.4. Integral curves in the intermediate field.

First we observe that $s$ is an increasing function of $\tau$ inside $\Omega_{33}$. $J$ is an increasing function of $\tau$ if $0 < I < \frac{1}{2}$, but changes to decreasing when $I \in \left( \frac{1}{2}, 1 \right)$. It is useful to observe the stable manifold of the system (6.1–3) at the point (6.6) which contains the transitional integral curve in the case $v_0 = 0$ considered in Sect. 5 and the transitional integral curve from a point of $E$ with a critical $I = I_h$. There are three kinds of integral curves depending on their positions relative to the stable manifold. The first kind consists of integral curves which are below the manifold and go to the stationary point $(1, 0, 0)$. Each of the second kind is right on the manifold and goes to the stationary point (6.6).
Each of the third kind is above the manifold and goes to a stationary point on the curve $E$ with $I \in (0, \frac{1}{2})$ given in (6.10). We mention in particular that no integral curve from inside $\Omega_{33}$ goes to a point of $E$ between $\frac{1}{2} < I < 1$ because $I$ is an increasing function of $\tau$ and $\vec{n}_3$ is pointing towards $(1, 0, 0)$ for $\frac{1}{2} < I < 1$. See Lemma A.1 of the Appendix for a complete proof. Also there is no integral curve from inside $\Omega_{33}$ which goes to the point $(0, 0, 1)$ because $J$ is an increasing function of $\tau$ for $I \in (0, \frac{1}{2})$.

We need to extend the intermediate field solutions to all $s$. Integral curves ending at point (6.6) are already defined for all $s > 0$ since $\tau \to +\infty$ and the right-hand side of equation (6.4) does not vanish at (6.6). For the first kind of integral curves which end at $(1, 0, 0)$, we show that $s$ approaches finite values although $\tau \to \infty$. Since the proof is long and tedious, we put it in the Appendix to avoid interruption of our construction. We use the natural continuation of vacuum $\rho = 0$ to extend our solutions till $s = \infty$. The values of $u, v$ are not needed to be specified in the vacuum. Near the edge of the vacuum we find

$$v = b(r - r^{**})^\beta, \quad \beta \equiv \frac{\gamma + 1}{2(\gamma - 1)}$$

where $b > 0$ is any constant. The asymptotics of $u$ and $c$ given in Sect. 5 are still valid. From these asymptotics we find that $M = \infty$ and $M_s = \frac{2}{\sqrt{2 - \gamma}}$, which are the same as in Sect. 5.

For the third kind of integral curves which end on the upper half of $E$ with $\alpha \in \left(0, \frac{1}{2}\right)$, we show that $s$ approaches finite values also. The proof is easy because we have

$$\frac{ds}{dJ} = \frac{s}{J(1 - 2I)}$$

from the second equation of system (4.3). The right-hand side of (6.15) is nonsingular for $(I, J, K) \in E$ with $0 < I < \frac{1}{2}$. Thus $s$ is finite around any point of $E$ with $0 < I < \frac{1}{2}$.

We comment that integral curves on the surface $J^2 = I(1 - I)$ starting from the origin and ending on a point of $E$ with $\alpha \in (I_h, 1)$, where $I_h \in (1/\gamma, 1/(2(\gamma - 1)))$, can be continued through the direction (6.11) into $\Omega_{33}$, and they go toward $(1, 0, 0)$ with finite ending $s$ values. The further extension by vacuum is also valid. But integral curves on the surface $J^2 = I(1 - I)$ starting from the origin and ending on a point of $E$ with $\alpha \in (1/2, I_h)$ are continued along the direction (6.11) into $\Omega_{33}$ and curl back to a point of $E$ with $0 < \alpha < 1/2$. Finally integral curves on the surface $J^2 = I(1 - I)$ starting from the origin and ending on a point of $E$ with $0 < \alpha < 1/2$ will be continued along direction (6.11) into $H > 0$ and discussed in the next section. It would be interesting to find explicitly the function $I_h = I_h(\gamma)$. We know that $I_h = 1/2$ for $\gamma = 2$, see [16, 17]. Numerically, we find that

$$I_h(1.1) = 0.93, \quad I_h(1.4) = 0.77, \quad I_h(1.7) = 0.63,$$

see paper [16] for numerical procedures used. For the special data $(u_0, v_0, \rho_0)$ with $u_0 = 0$, the critical value $I_h$ can be expressed through the solution formula (6.9) by the critical initial Mach number

$$M_h(\gamma) \equiv \sqrt{I_h/(1 - I_h)}.$$  

(6.16)
Since $I_h \in (1/\gamma, 1/(2(\gamma - 1)))$, we find that
\[
\frac{\sqrt{\gamma}}{\gamma - 1} < M_h(\gamma) < \frac{\sqrt{2(\gamma - 1)}}{2\gamma - 3}.
\] (6.17)

The lower bound is valid for all $\gamma \in (1, 2)$, but the upper bound is valid only for $\gamma \in (3/2, 2)$. So we have constructed all solutions globally except those which end on $E$ with $0 < I < \frac{1}{2}$.

7. Inner-field solutions.

We extend solutions that end on the set $E$ of (6.10) with $0 < I < \frac{1}{2}$ in this section. We find that these solutions go along the directions $\vec{n}_3$ given in (6.11) into the region between $A = 0$ and $B = 0$ in $0 < I < \frac{1}{2}$ and $J^2 > I(1 - I)$, and eventually go to infinity. The scaled variables $I, J,$ and $K$ are not suitable for this portion of the solutions.

We restart from system (2.7) with data $(u, v, \rho, r)$ satisfying the relations $E: K = 1 - I$ and $J^2 = I(1 - I)$ for $I \in (0, \frac{1}{2})$. In terms of $(u, v, c, r)$, these data are in the form
\[
\begin{cases}
u |_{r=\alpha} &= \beta \\
v |_{r=\alpha} &= \sqrt{\beta(\alpha - \beta)} \\
c |_{r=\alpha} &= \alpha - \beta
\end{cases}
\] (7.1)

where $\alpha > 0$ and $\beta \in (0, \frac{1}{2})$ are arbitrary. The directions $\vec{n}_3$ given in (6.11) point into the region $\Delta > 0$, $\Theta > 0$, $\Sigma > 0$.

Asymptotic analysis shows that problem (2.7) and (7.1) has solutions $(u(r), v(r), \rho(r))$ which vanish as $r \to 0^+$. The solutions are in the region $\Delta > 0$, $\Theta > 0$, $\Sigma > 0$. We perform an asymptotic analysis a priori to determine the orders at which $(u, v, \rho)$ vanish as $r \to 0^+$. For polytropic gases $p(\rho) = A_2 \rho^\gamma$, we find
\[
u = \frac{ar}{\ln \sigma}, \quad v = \frac{k}{(\ln \sigma)^a}, \quad \rho = \frac{d}{(\ln \sigma)^{2a}}
\] (7.2)

where
\[
a = \frac{1}{2(2 - \gamma)}, \quad k^2 = \frac{\gamma}{2 - \gamma}d^{\gamma - 1},
\]
and $\sigma > 0$ and $d > 0$ are arbitrary. We can easily verify that these directions are such that $\Delta > 0, \Theta > 0, \Sigma > 0$ for $r > 0$. In terms of the sound speed $c$, we find
\[
c = \frac{k \sqrt{A_2(2 - \gamma)}}{(\ln \sigma)^{a(\gamma - 1)a}}.
\]

We are motivated by this asymptotic analysis to use the scaled variables
\[
U = \frac{u}{c}, \quad V = \frac{v}{c}, \quad R = \frac{r}{c}.
\] (7.3)
We can rewrite system (2.7) into a new form

\[
\begin{align*}
\frac{du}{d\tau} &= \frac{r-u}{c^3} \Sigma \\
\frac{dv}{d\tau} &= \frac{xu}{c^3} \Delta \\
\frac{dr}{d\tau} &= \frac{r(r-u)}{c^2} \Delta \\
\frac{dc}{d\tau} &= \frac{\gamma-1}{2} \frac{r-u}{c^2} \Theta
\end{align*}
\]  

(7.4)

where \( \tau \) is a parameter. In terms of the variables in (7.3), we find

\[
\begin{align*}
\frac{dU}{d\tau} &= (R-U)\tilde{\Sigma} - \lambda U(R-U)\tilde{\Theta} \equiv A_1 \\
\frac{dV}{d\tau} &= UV \tilde{\Delta} - \lambda V(R-U)\tilde{\Theta} \equiv C_1 \\
\frac{dR}{d\tau} &= R(R-U) \tilde{\Delta} - \lambda R(R-U)\tilde{\Theta} \equiv B_1
\end{align*}
\]  

(7.5)

where \( \lambda \equiv \frac{\gamma - 1}{2} \) and \( \tilde{\Delta} \equiv 1 - (U - R)^2 \), \( \tilde{\Theta} \equiv V^2 - U(R-U) \) and \( \tilde{\Sigma} \equiv (R-U)\tilde{\Theta} - U \tilde{\Delta} \). System (7.5) is autonomous for \((U, V, R)\). We find that the last equation in (7.4) can be written as

\[
\frac{dc}{cd\tau} = \lambda(R-U)\tilde{\Theta}.
\]  

(7.6)

So \( c \) can be integrated from (7.6) once \((U, V, R)\) are obtained from (7.5). The corresponding data of (7.1) for (7.5) and (7.6) are any stationary point \((U^*, V^*, R^*, c^*)\) satisfying

\[
R^* - U^* = 1, \quad V^{*2} = U^*, \quad 0 < 2U^* < R^*, \quad c^* > 0.
\]  

(7.7)

(7.8)

The asymptotic analysis (7.2) shows that

\[
(U, \ V, \ R) \rightarrow (0, \ 0, \ 0), \quad r \rightarrow 0 +.
\]  

(7.9)

After (7.5-9) are solved, we use the third equation in (7.4) to show that \( r \) is an increasing function of \( \tau \in \mathbb{R} \) and \( r \rightarrow 0 \) or \( \alpha \in (0, +\infty) \) as \( \tau \) goes to \( \mp \infty \) respectively.

For more refined change of variables, we further find that our asymptotic analysis (7.2) implies that

\[
U = abr \left( \ln \frac{\sigma}{r} \right)^{(\gamma-1)a-1}
\]

\[
V = r \left( \ln \frac{\sigma}{r} \right)^{-\frac{1}{2}}
\]  

(7.10)

\[
R = br \left( \ln \frac{\sigma}{r} \right)^{(\gamma-1)a},
\]
where we set $b \equiv (k \sqrt{(2 - \gamma)A_2})^{-1}$. It follows that the above asymptotic behavior yields the function relations of $(R, U)$ to $V$

$$R = \frac{\sigma}{\sqrt{\gamma A_2 d^{\gamma - 1}}} \exp \left( -\frac{b^2 k^2}{V^2} \right) \left( \frac{bk}{V} \right)^{2(\gamma - 1)a}$$

$$U = \frac{1}{2} RV^2$$

which are not analytic at $V = 0$. It can be shown that the directions (7.10) enter the region $A_1 > 0$, $B_1 > 0$, and $C_1 > 0$. We omit the proof.

We transform the direction $\vec{n}_3$ of (6.11) to the new variables $(U, V, R)$ at a point of (7.7). It is possible to use linearization for the derivation, but we prefer a direct transformation. Let $\vec{n} = (n_1, n_2, n_3)$ be an arbitrary vector in the coordinate space $(I, J, K)$, which is a tangent direction of an integral curve of (6.1–3). So we have

$$\frac{dJ}{dI} = \frac{n_2}{n_1}, \quad \frac{dK}{dI} = \frac{n_3}{n_1}$$

along the integral curve. Because the variables $(I, J, K)$ are related to $(U, V, R)$ by

$$I = U/R, \quad J = V/R, \quad K = 1/R,$$

we find that the differentials $(dR, dU, dV)$ satisfy

$$\frac{RdV - V dR}{RdU - UdR} = \frac{n_2}{n_1}, \quad \frac{-dR}{RdU - UdR} = \frac{n_3}{n_1},$$

which can be solved to yield

$$\frac{dU}{dR} = \frac{Un_3 - n_1}{Rn_3}, \quad \frac{dV}{dR} = \frac{Vn_3 - n_2}{Rn_3}.$$ 

Thus the corresponding tangent vector in the new coordinate space $(U, V, R)$ of $\vec{n} = (n_1, n_2, n_3)$ in the coordinate space $(I, J, K)$ is a multiple of

$$(Un_3 - n_1, Vn_3 - n_2, Rn_3), \quad (7.11)$$

where $(n_1, n_2, n_3)$ as well as $(U, V, R)$ are evaluated at the same point $(u, v, c, r)$. For our particular problem $\vec{n}_3$ of (6.11), we use $\alpha = U/R$ for a point of (7.7) to find the corresponding direction to be a multiple of

$$\vec{n}_3^* = (2 + 3(3 - \gamma)U - 5(\gamma - 1)U^2, \sqrt{U}[7 - 3\gamma)U - (\gamma - 1)],$$

$$(1 + U)[\gamma + 3 - 5(\gamma - 1)U]). \quad (7.12)$$
We determine the relative positions of the regions where $A_1 > 0$, $B_1 > 0$, or $C_1 > 0$ in the region $R > U$, $U > 0$, $V > 0$. The common intersection $A_1 = B_1 = C_1 = 0$ is the curve (7.7) of stationary points for all $0 < U < R$. The intersection $A_1 = B_1 = 0$ is in the plane $R = \gamma U$. The intersection $B_1 = C_1 = 0$ is in the plane $R = 2U$. The intersection $A_1 = C_1 = 0$ is in the plane $R = \frac{2}{3-\gamma}U$. We consider two domains

$$\Omega_{41} = \{(U, V, R) \mid U > 0, V > 0, R > 2U, A_1 > 0, B_1 > 0\}$$

$$\Omega_{42} = \{(U, V, R) \mid U > 0, V > 0, 2U > R > \gamma U, A_1 > 0, C_1 > 0\}.$$ 

See Fig. 7.1.
We show that integral curves of (7.5) on the surface $A_1 = 0$ of the boundaries of both $\Omega_{41}$ and $\Omega_{42}$ are inward. In fact, the surface $A_1 = 0$ is represented by

$$V^2 = \frac{U[2 - (\gamma - 1)U(R - U)]}{2R - (\gamma + 1)U}$$

with an outward normal

$$\vec{n}_{A_1} = \left( n_1, -2V, \frac{-U[4 - (\gamma - 1)^2U^2]}{2R - (\gamma + 1)U^2} \right)$$

where $n_1$ is some expression that we do not need to compute. We calculate the inner product of this normal with the tangent direction of integral curves of (7.5)

$$\frac{d}{d\tau}(U, V, R) \cdot \vec{n}_{A_1} = -2VC_1 - \frac{U[4 - (\gamma - 1)^2U^2]}{[2R - (\gamma + 1)U^2]}B_1 < 0$$

since both $C_1 > 0$ and $B_1 > 0$ on $A_1 = 0$, in the range $R > \gamma U$.

We show next that integral curves of (7.5) are incoming on the surface $B_1 = 0$ of the domain $\Omega_{41}$ where $R > 2U$. In fact, the surface $B_1 = 0$ is represented by

$$\lambda V^2 = 1 + \lambda U(R - U) - (R - U)^2$$

with an inward normal

$$\vec{n}_{B_1} = (\lambda(R - 2U) + 2(R - U), -2\lambda V, \lambda U - 2(R - U)).$$

We find the inner product

$$\frac{d}{d\tau}(U, V, R) \cdot \vec{n}_{B_1} = A_1[\lambda(R - 2U) + 2(R - U)] - 2\lambda VC_1 > 0$$

since $A_1 > 0$ and $C_1 < 0$ in the region $R > 2U$ on the surface $B_1 = 0$.

We show next that integral curves of (7.5) are incoming on the surface $C_1 = 0$ of the domain $\Omega_{42}$ where $2U > R > \gamma U$. In fact, the surface $C_1 = 0$ is represented by

$$\lambda V^2 = \frac{U}{R - U} - (1 - \lambda)U(R - U)$$

with an inward normal

$$\vec{n}_{C_1} = \left( \frac{R}{(R - U)^2} + (1 - \lambda)(2U - R), -2\lambda V, -(1 - \lambda)U - \frac{U}{(R - U)^2} \right).$$

We find

$$\frac{d}{d\tau}(U, V, R) \cdot \vec{n}_{C_1} = \left[ \frac{R}{(R - U)^2} + (1 - \lambda)(2U - R) \right] A_1 - \left[ (1 - \lambda)U + \frac{U}{(R - U)^2} \right] B_1 > 0$$

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since $A_1 > 0$, $B_1 < 0$ on $C_1 = 0$ in the region $\frac{2}{3} \gamma U < R < 2U$ which contains $\gamma U < R < 2U$.

It is easy to see that integral curves of (7.5) are incoming on the surface $U = 0$ of the domain $\Omega_{41}$ since
\[
\frac{dU}{d\tau} = R^2 V^2 > 0
\]
on $U = 0$.

We show that integral curves of (7.5) go from $\Omega_{41}$ to $\Omega_{42}$ on the common interface $R = 2U$. A normal of $R = 2U$ pointing from $\Omega_{41}$ to $\Omega_{42}$ is $\vec{n} = (2, 0, -1)$ and
\[
\frac{d}{d\tau}(U, V, R) \cdot \vec{n} = 2U^2(1 - U^2 - V^2).
\]
on $R = 2U$. Using $A_1 \geq 0$ in $\Omega_{41}$ and $\Omega_{42}$ on $R = 2U$, we find
\[
V^2 + U^2 - 1 \geq \frac{\lambda + (1 - 2\lambda)U^2}{1 - \lambda} \geq 0,
\]
thus integral curves of (7.5) go from $\Omega_{41}$ to $\Omega_{42}$ on the interface $R = 2U$.

So integral curves in $\Omega_{41}$ exit $\Omega_{41}$ only through the side $R = 2U$ or the stationary edge (7.7).

We need to study the local structure of solutions of (7.5) at the origin $(U, V, R) = (0, 0, 0)$. In particular, we would like to prove the asymptotic behavior (7.2). To reduce the order of degeneracy, we introduce the variables
\[
X = \frac{U}{R}, \quad V_2 = V^2, \quad R_2 = R^2.
\]
We find that
\[
X = a \left( \ln \frac{\sigma}{r} \right)^{-1}, \quad V_2 = 2X
\]
as $r \to 0^+$ from the asymptotic analysis. So the origin $(X, V_2, R_2) = (0, 0, 0)$ will still be the place for us to look for outgoing solutions. Introducing $\tau'$ by
\[
d\tau' = R d\tau,
\]
we find the equations (7.5) in terms of $(X, V_2, R_2)$ to be
\[
\begin{align*}
\frac{dX}{d\tau'} &= (1 - X) \left[ V_2(1 - X) - 2X + X(1 - X)^2 R_2 \right] \equiv (1 - X) A_{11} \quad (7.15) \\
\frac{dV_2}{d\tau'} &= 2V_2 \left\{ X - X(1 - X)^2 R_2 - \lambda(1 - X) [V_2 - X(1 - X)R_2] \right\} \quad (7.16) \\
\frac{dR_2}{d\tau'} &= 2R_2(1 - X) \left\{ 1 - (1 - X)^2 R_2 - \lambda [V_2 - X(1 - X)R_2] \right\} \quad (7.17)
\end{align*}
\]
where we have set the expression in the brackets in (7.15) to be $A_{11}$. We need the local structure of (7.15–17) at $(0,0,0)$. Linearization at the point $(X, V_2, R_2) = (0,0,0)$ yields

$$
\begin{align*}
\frac{dX}{d\tau'} &= -2X + V_2 \\
\frac{dV_2}{d\tau'} &= 0 \\
\frac{dR_2}{d\tau'} &= 2R_2
\end{align*}
$$

which has eigenvalues

$$
\lambda_1 = -2, \quad \lambda_2 = 2, \quad \lambda_3 = 0,
$$

and associated eigenvectors

$$(X, V_2, R_2) = (1,0,0), (0,0,1), (1,2,0).$$

Fig. 7.2. The center-unstable manifold of the origin.

By the Center Manifold Theorem [8], system (7.15-17) has a $C^k$ center manifold which is tangent to the vector $(1,2,0)$ for any $k < \infty$. Suppose the center manifold takes the form

$$
X = g(V_2), \quad g(0) = 0, \quad g'(0) = 1/2 \quad (7.18)
$$

$$
R_2 = h(V_2), \quad h(0) = h'(0) = 0. \quad (7.19)
$$
Then the flow on the center manifold is described by

\[
\frac{dV_2}{d\tau'} = 2V_2 \left\{ g - g(1-g)^2h - \lambda(1-g) [V_2 - g(1-g)h] \right\}. \tag{7.20}
\]

Using (7.18-19), we can approximate (7.20) to find

\[
\frac{dV_2}{d\tau'} = (2 - \gamma)V_2^2 + O(V_2^3).
\]

So the center manifold is unstable since \(2 - \gamma > 0\). Now we consider the center-unstable manifold consisting of the unstable manifold and all center manifolds, see Kelley’s paper [8] for the existence and smoothness. This manifold is two-dimensional. It can be shown easily that they enter the region \(A_1 > 0, B_1 > 0, C_1 > 0\). Furthermore, we can use Henry’s approximation of center manifold ([7,8,2]) to find

\[
g = \frac{1}{2}V_2 - \frac{3 - \gamma}{4}V_2^2 + O(V_2^3)
\]

\[
h = O(V_2^3),
\]

which can be used to show that all center manifolds enter the region \(A_{11} > 0\).

The center-unstable manifold enter the region \(\Omega_{41}\). Note that the unstable manifold, i.e., the \(R_2\) axis, ends on one end of (7.7). We conclude that nearby integral curves in the center-unstable manifold end on (7.7) too, see Fig. 7.2.

By our earlier analysis, we know that integral curves in the center-unstable manifold can only exit \(\Omega_{41}\) through the divider between \(\Omega_{41}\) and \(\Omega_{42}\) or points on (7.7). From the continuity of the center-unstable manifold, we conclude that every point on (7.7) is covered by the center-unstable manifold.

Finally, we show that \(r \to 0^+\) at the origin \((X, V_2, R_2) = (0, 0, 0)\). It is easy to see that \(c\) is an increasing function of \(\tau\) from (7.6) since \(\tilde{\Theta}\) is positive along the center manifolds. Thus \(c\) is bounded as \(\tau \to -\infty\). From (7.3), we have \(r = Rc\). Hence \(r \to 0\) since \(R \to 0\) as \(\tau \to -\infty\).

We point out that both the Mach number \(M = \sqrt{u^2 + v^2}\) and the pseudo Mach number \(M_s = \sqrt{(U-R)^2 + V^2}\) go to 0 as \(r \to 0\). Also, both \(U\) and \(R - U\) go to 0 faster than \(V\), which explains the spiralling phenomena in both the physical space and in the self-similar coordinates.

Thus each point on (7.7) has a solution going to the origin through \(\Omega_{41}\) with \(r \to 0^+\).

8. Conclusions.

We summarize our results. By a weak solution to the \(2 - D\) Euler equations (2.1) we mean a bounded vector function \((u, v, \rho)\) satisfying the equations in the sense of distributions. Since we deal with only continuous weak solutions in this paper, we shall not mention any other requirements such as Rankine-Hugoniot relation or entropy conditions. Let \(c = \sqrt{p'(\rho)}\) be the sound speed. Let \(M = \sqrt{u^2 + v^2}/c\) be the Mach number. Let \(M_s = [((u-\xi)^2 + (v-\eta)^2)^{1/2}/c\) be the pseudo Mach number of the self-similar flow in the
self-similar coordinates \((\xi, \eta)\). In the notation of system (2.7), the pseudo Mach number \(M_s = [(u - r)^2 + v^2]^{1/2}/c\). Set
\[
M_0 \equiv \sqrt{u_0^2 + v_0^2}/c_0, \quad c_0 \equiv \sqrt{p'/(\rho_0)}
\]
which are in consistency with previous versions of \(M_0\) (Sect. 5 & 6). Also, let us introduce the vector parameter ("Mach vector")
\[
\vec{M}_0 \equiv \left(\frac{u_0}{c_0}, \frac{v_0}{c_0}\right).
\]
Recall that we can assume \(v_0 \geq 0\) without loss of generality. Also, the case \(v_0 = 0\) has been resolved in [17].

Recall that we can assume \(v_0 \geq 0\) without loss of generality. Also, the case \(v_0 = 0\) has been resolved in [17].

Fig. 8.1.

A solution \((u, v, \rho)\) vs. the \(x\)-axis at time \(t = 1\) with a datum \(u_0 = 0\) and \(M_0 < \sqrt{2}\).

**Theorem.** Assume \(p(\rho) = A_2 \rho^\gamma\) for any constants \(A_2 > 0\) and \(1 < \gamma < 2\). Then for any datum \((u_0, v_0, \rho_0)\) with \(u_0 \geq 0, v_0 > 0, \text{ and } \rho_0 > 0\), there exists a weak solution \((u, v, \rho)\) to the initial value problem (2.1) and (2.5). The solution is continuous, self-similar, and axisymmetric for \(t > 0\). It takes on initial datum almost everywhere and in \(L^q_{\text{loc}}\) for any \(q > 1\) as \(t \to 0^+\). Additionally, we have the following properties.

(I). If \(u_0 = 0\), the solution consists of two or three smooth pieces, depending on the initial Mach number \(M_0\). The first piece of the solution is always given explicitly by the formula (3.4) with constant Mach number near \(r = \infty\).

(I.a). If \(M_0 < \sqrt{2}\), the solution consists of two smooth pieces. The end point of the first piece of the solution is on the set of stationary points (6.5) with an \(I < 1/2\), which is transformed to be the stationary set (7.7) from which the second piece begins in
the direction given in (6.11) or equivalently (7.12). It is an integral curve on the center-unstable manifold of the ODE (7.15-17) at the origin. See Fig. 7.2 and 8.1. Both the Mach and the pseudo Mach numbers are zero at the center point of the solution.

(I.b). If \( M_0 > \sqrt{2} \), but \( M_0 < M_h(\gamma) \) (which is defined in (6.16) ), the solution consists of three smooth pieces. The second piece is given by the solution of the ODE(4.3) with direction (6.11) such that the solution goes to a stationary point on (6.5) with an \( I < 1/2 \). The third piece is identical to the second piece of a solution described in the previous case \( M_0 < \sqrt{2} \).

(I.c). If \( M_0 = M_h(\gamma) \), the solution consists of two pieces and the second piece is given by the solution of the ODE(4.3) with direction (6.11) such that the solution goes to the stationary point (6.6).

(I.d). If \( M_0 > M_h(\gamma) \), the solution consists of three smooth pieces and the second piece is given by the ODE (4.3) with direction (6.11), see Fig. 8.2. The radial and pure rotational components and the density function of the solution are increasing functions of the spatial radius \( r \) in the region \( u^{**}t < r < r^*t \) where \( r^* \) is given by (3.5) and \( u^{**} > 0 \) is the radial velocity at the inner end of the second piece. The third piece is the vacuum \( \rho = 0 \) with domain \( 0 \leq r < u^{**}t \). At the edge of the vacuum, there is no rotation, the Mach number is infinity, and the pseudo Mach number \( M_s = \frac{2}{\sqrt{3-\gamma}} \) (for \( \gamma < 3 \)).

(I.e). The critical Mach number \( M_h(\gamma) \) has a lower bound
\[
\sqrt{\gamma}/(\gamma - 1) < M_h(\gamma)
\]
for all \( \gamma \in (1, 2) \). It has an upper bound
\[
M_h(\gamma) < \sqrt{2(\gamma - 1)/(2\gamma - 3)}
\]
if \( \gamma \in (3/2, 2) \).

(II). Assume \( u_0 > 0, \rho_0 > 0, \) and \( v_0 > 0 \). Depending on the vector parameter \( \mathbf{M}_0 \), the solution can be globally smooth, contain a region of vacuum as in case (I.a), or contain an inner piece of a solution of (I.d). In terms of the selfsimilar coordinates and variables, the ODE (6.1-3) has at the point (6.6) a stable manifold, which contains the transitional solutions in the special cases \( u_0 = 0 \) of case (I) and \( v_0 = 0 \). If the vector parameter \( \mathbf{M}_0 \) is such that the solution is on the stable manifold, then the entire solution consists of one smooth piece and is governed by the system of ODE (3.1). If the vector parameter \( \mathbf{M}_0 \) is such that the solution is below the stable manifold, then the solution consists of a vacuum region at the center and the far-field solution given by the system of ODE (3.1) with data (3.2). If the vector parameter \( \mathbf{M}_0 \) is such that the solution is above the stable manifold, then the solution consists of two pieces, one is given by the system of ODE (3.1) with data (3.2), the other by an integral curve in the center-unstable manifold near the origin of (7.15-17), and the density \( \rho \) of the solution vanishes only at one point, the origin of space.

We omit the proof of the theorem with the remark that it can be verified that the pieces of integral curves when combined continuously in the order specified in the theorem form weak solutions to (2.1)(2.4).
The case $\gamma = 1$. We now point out necessary changes in Sections 4–7 to accommodate the case $\gamma = 1$. The equations (4.3) are still valid although the equation for $K$ is trivial since the sound speed $c$ is a constant. We need to supplement (4.3) with the equation for $\rho$ in (3.1), the equation for $\rho$ being decoupled from (4.3) now. Conclusions in Sect. 5 are modified as follows: all integral curves in $\Omega$ go to the point $(I, K) = (0, 1)$ except the trivial one $K = 0$. In Sect. 6 all integral curves in $\Omega_3 = \Omega_{33}$ from the origin $(I, J, K) = (0, 0, 0)$ enter the set $E$ with $\alpha < 1/2$. In Sect. 7 the conclusion is the same, but the proof, which we shall omit, needs nontrivial changes since the domains $\Omega_{41}$ and $\Omega_{42}$ are not bounded anymore. The overall conclusion for $\gamma = 1$ is the same as stated in the previous theorem, the main difference is that the threshold Mach number $M_h(1) = \infty$ and the threshold stable manifold mentioned in Step II is the bottom surface $K = 0$, thereby all solutions in the case $\gamma = 1$ have point cavities at the centers.

Fig. 8.2.

A solution $(u, v, \rho)$ vs. the $x$-axis at time $t = 1$ with a datum $u_0 = 0$ and $M_0 > M_h(\gamma)$. 

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Appendix. Finiteness of the parameter $s$ at point $(1,0,0)$

In this appendix we show mainly that the parameter $s$, the reciprocal of the radius $r$, approaches a finite value as an integral curve approaches the stationary point $(1,0,0)$ in $\Omega_3$ defined in section 6.

**Lemma A.1.** Assume $\gamma > 1$. Then for any integral curve of the system (6.1-4) that goes from $I < 1/2$ to $I > 1/2$ inside the domain $\Omega_{33}$, there exists an $\varepsilon \in (0,1)$ such that the integral curve is inside the cylinder

$$C_\varepsilon \equiv J^2 - \varepsilon I(1-I) < 0 \quad (A.1)$$

whenever its $I$ component is in $(\frac{1}{2}, 1)$.

**Proof.** We choose $\varepsilon \in (0,1)$ such that the integral curve is inside the cylinder $J^2 < \varepsilon I(1-I)$ at the special point $I = \frac{1}{2}$. For this $\varepsilon$, we show that integral curves on the surface of the cylinder $J^2 = \varepsilon I(1-I)$ inside $\Omega_{33}$ in the portion $I \in (\frac{1}{2}, 1)$ are all going into the cylinder $J^2 < \varepsilon I(1-I)$. We first calculate an outward normal to the cylinder $J^2 = \varepsilon I(1-I)$ to be

$$\vec{n}_\varepsilon = (-\varepsilon(1-2I), 2J, 0).$$

We then calculate the inner product of the normal $\vec{n}_\varepsilon$ with the tangent direction of any integral curve on the cylinder $J^2 = \varepsilon I(1-I)$

$$\vec{n}_\varepsilon \cdot \frac{d}{d\tau}(I, J, K)|_{J^2=\varepsilon I(1-I)} = -\varepsilon(1-2I)(1-I)[I(1-I)^2(1+\varepsilon) - 2IK^2]$$

$$+ 2\varepsilon I(1-I)(1-2I) \cdot [(1-I)^2 - K^2]$$

$$= \varepsilon I(1-2I)(1-I)\{-(1+\varepsilon)(1-I)^2 + 2K^2 + 2(1-I)^2 - 2K^2\}$$

$$= \varepsilon(1-\varepsilon)I(1-I)^3(1-2I) < 0 \quad \text{for } I \in \left(\frac{1}{2}, 1\right).$$

Hence the chosen integral curve will remain inside the cylinder $J^2 < \varepsilon I(1-I)$ in $I \in \left(\frac{1}{2}, 1\right)$. This completes the proof of Lemma A.1.

**Remark.** From the proof of Lemma A.1 it is clear that any integral curve entering $\Omega_{33}$ from the edge $E$ (6.10) will get into a cylinder $J^2 < \varepsilon I(1-I)$ for some $\varepsilon \in (0,1)$ for $I$ close to 1.

**Lemma A.2.** Assume $\gamma \geq 3/2$. Then for any integral curve that ends at $(1,0,0)$ of the system (6.1-4) in the domain $\Omega_3$, there are three numbers $\varepsilon \in (0,1)$, $\beta \in (2,\infty)$, and $\bar{I} \in (\frac{1}{2}, 1)$ such that the integral curve is inside the cylinder $C_\varepsilon < 0$ and below the surface

$$B_{\beta} \equiv (1-I)J^2 + I(1-I)^2 - \beta IK^2 = 0 \quad (A.2)$$

for all $I \in (\bar{I}, 1)$.
Proof. We first compute an outward normal \( \vec{n}_\beta \) to the surface \( B_\beta = 0 \).

\[
\vec{n}_\beta = (\beta K^2 - (1 - I)^2 + 2I(1 - I) + J^2, -2(1 - I)J, 2\beta IK).
\]

We calculate the inner product of this \( \vec{n}_\beta \) with the tangent vector of any integral curve on the surface \( B_\beta = 0 \):

\[
\vec{n}_\beta \cdot \frac{d}{d\tau} (I, J, K) \big|_{B_\beta = 0} = [\beta K^2 - (1 - I)^2 + 2I(1 - I) + J^2] \cdot (1 - I) \left[ (1 - I)^2 I + (1 - I)^2 - 2IK^2 \right]
\]

\[
- 2(1 - I)J^2(1 - 2I)[(1 - I)^2 - K^2] + \beta IK^2(1 - I) \left[ (\gamma - 1)J^2 - (\gamma - 1)I(1 - I) + 2(1 - I)^2 - 2K^2 \right]
\]

\[
= \left[ 2I(1 - I) + \frac{1}{I} J^2 \right] (1 - I)^2 \left( 1 - \frac{2}{\beta} \right) \left[ J^2 + I(1 - I) \right]
\]

\[
+ 2(1 - I)^2(1 - 2I)J^2 \frac{1}{\beta I} \left[ J^2 - (\beta - 1)I(1 - I) \right] + (1 - I)^2 \left[ J^2 + I(1 - I) \right] \frac{1}{I},
\]

\[
\left\{ (\gamma - 1)I - \frac{2}{\beta} (1 - I) \right\} J^2 - (\gamma - 1)I^2(1 - I) + 2 \left( 1 - \frac{1}{\beta} \right) I(1 - I)^2 \right\}.
\]

Rewriting the last expression in the form of a polynomial of \( J \), we obtain

\[
\vec{n}_\beta \cdot \frac{d}{d\tau} (I, J, K) \big|_{B_\beta = 0} = J^4 \left\{ \frac{1}{I} (1 - I)^2 \left( 1 - \frac{2}{\beta} \right) + \frac{2}{\beta I} (1 - I)^2(1 - 2I) + \frac{(1 - I)^2}{I} \left[ (\gamma - 1)I - \frac{2}{\beta} (1 - I) \right] \right\}
\]

\[
+ J^2 \left\{ 2I(1 - I)^3 \left( 1 - \frac{2}{\beta} \right) + (1 - I)^3 \left( 1 - \frac{2}{\beta} \right) - 2\frac{\beta - 1}{\beta} (1 - I)^3(1 - 2I) \right.
\]

\[
+ (1 - I)^3 \left[ (\gamma - 1)I - \frac{2}{\beta} (1 - I) \right] + 2 \left( 1 - \frac{1}{\beta} \right) (1 - I)^4 - (\gamma - 1)I(1 - I)^3 \right\}
\]

\[
+ 2I^2(1 - I)^4 \left( 1 - \frac{2}{\beta} \right) - (\gamma - 1)I^2(1 - I)^4 + 2 \left( 1 - \frac{1}{\beta} \right) I(1 - I)^5
\]

\[
= J^4 \frac{(1 - I)^2}{\beta I} \left\{ \beta - 2 + [\beta(\gamma - 1) - 2]I \right\} + J^2(1 - I)^3 \left[ 4I \left( 1 - \frac{1}{\beta} \right) + 1 - \frac{4}{\beta} \right]
\]

\[
+ I(1 - I)^4 \left\{ 2I \left( 1 - \frac{2}{\beta} \right) - (\gamma - 1)I + 2 \left( 1 - \frac{1}{\beta} \right) (1 - I) \right\}.
\]
Observing that both coefficients of $J^4$ and $J^2$ are positive when $\beta > 2$, $\gamma \geq 2$, and $I \in (\frac{1}{2}, 1)$, we find in this case in the region $C_\varepsilon < 0$ that

$$
\vec{n}_\beta \cdot \frac{d}{d\tau}(I, J, K)|_{B_{\beta=0}, C_\varepsilon < 0} < \frac{\varepsilon^2}{\beta} I(1 - I)^4 \{\beta - 2 + [\beta(\gamma - 1) - 2] I \}
$$

$$
+ \varepsilon I(1 - I)^4 \left[ 4I \left( 1 - \frac{1}{\beta} \right) + 1 - \frac{4}{\beta} \right]
$$

$$
+ I(1 - I)^4 \left[ -\frac{2I}{\beta} - (\gamma - 1)I + 2 \left( 1 - \frac{1}{\beta} \right) \right]
$$

$$
= I(1 - I)^4 \left\{ \varepsilon^2 \left[ 1 - \frac{2}{\beta} + (\gamma - 1 - \frac{2}{\beta}) I \right] + 4\varepsilon I \left( 1 - \frac{1}{\beta} \right) + \varepsilon - \frac{4\varepsilon}{\beta} - \left( \frac{2}{\beta} + \gamma - 1 \right) I + 2 \left( 1 - \frac{1}{\beta} \right) \right\}
$$

$$
\equiv I(1 - I)^4 F(\varepsilon, \beta, I, \gamma)
$$

where $F$ denotes the expression in the curly brackets.

We show that the inequality (A.3) still holds even when $\gamma \in [3/2, 2)$. The quadratic polynomial of $J^2$ attains its maximum at

$$
(J^2)_{\text{max}} = \frac{I(1 - I)[4I(\beta - 1) + \beta - 4]}{2I[2 - \beta(\gamma - 1)] + 4 - 2\beta}.
$$

(A.4)

At $\beta = 2$, we calculate

$$
(J^2)_{\text{max}} = \frac{1 - I}{2(2 - \gamma)} + O((1 - I)^2)
$$

(A.5)

for $I$ close to 1. Since $(J^2)_{\text{max}}$ is a continuous function of $\beta$ near 2, we see that $(J^2)_{\text{max}}$ is positive for all $\gamma \in (1, 2)$ when $\beta$ is close to 2 and $I < 1$ is close to 1. Hence the quadratic polynomial of $J^2$ is an increasing function of $J^2$ in the interval $[0, (J^2)_{\text{max}}]$. We show next that

$$
\varepsilon I(1 - I) \leq (J^2)_{\text{max}}
$$

(A.6)

for all $\gamma \in [3/2, 2)$, and $\varepsilon < 1$ when $I$ is close to 1 and $\beta$ is close to 2. In fact, (A.6) holds by continuity, the fact (A.5) and the inequality

$$
\varepsilon < \frac{1}{2(2 - \gamma)}
$$

which holds for all $\gamma \in [3/2, 2)$. So inequality (A.3) still holds in the region $C_\varepsilon < 0$ even when $\gamma \in [3/2, 2)$. 

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We need $F$ to be negative near $I = 1$ for some $\varepsilon > 0$, $\beta > 2$, and $\gamma \geq 3/2$. For any fixed integral curve that goes to the point $(1,0,0)$, we first choose $\varepsilon \in (0,1)$ so that the integral curve lies inside $C_\varepsilon < 0$ for $I \in (\tilde{I}, 1)$ for some $\tilde{I} < 1$. Fix this $\varepsilon$. We calculate the value of $F$ at the extreme point $\beta = 2, I = 1$:

$$
F(\varepsilon, 2, 1, \gamma) = \varepsilon^2(\gamma - 2) + \varepsilon - \gamma + 1
$$

$$
= (1 - \varepsilon)[1 + 2\varepsilon - \gamma(1 + \varepsilon)]
$$

$$
\leq (1 - \varepsilon)[1 + 2\varepsilon - \frac{3}{2}(1 + \varepsilon)]
$$

$$
= -\frac{1}{2}(1 - \varepsilon)^2 < 0.
$$

Since $F$ is a continuous function of $(\beta, I)$ near the point $(2, 1)$, we conclude that for each $\gamma \geq 3/2$ there exists an $\tilde{I} \in \left(\frac{1}{2}, 1\right)$ such that $F(\varepsilon, \beta, I, \gamma) < 0$ for all $I \in [\tilde{I}, 1)$ when $\beta$ is close to 2. Since $B_\beta = 0$ is close to $B = 0$, we can choose a $\beta > 2$ such that the integral curve lies below $B_\beta = 0$ at $I = \tilde{I}$. This integral curve will remain under the surface $B_\beta = 0$ for all $I > \tilde{I}$ because of the sign $F < 0$. This completes the proof of Lemma A.2.

Lemma A.2 does not hold for $\gamma < 3/2$. We need to deal with the case $\gamma < 3/2$ separately. Observe that the intersection point of the curve $J_b = 0$ with the stationary edge (6.5) is topologically different for $\gamma < 3/2$, see Fig.6.1.

**Lemma A.3.** Assume $1 < \gamma < 3/2$. Then integral curves in $\Omega_3$ of system (6.1–3) on the surface of the domain

$$(2 - \gamma)J^2 \leq \varepsilon(\gamma I - 1)(1 - I), \quad (A.7)$$

where $\varepsilon \in (0, 1)$, always enter the domain.

**Proof.** We calculate an outward normal to (A.7):

$$
\widehat{n} = (-\varepsilon(\gamma + 1 - 2\gamma I), \ 2(2 - \gamma)J, \ 0).
$$

We then calculate the inner product of this normal with the tangent vector of an integral curve of (6.1–3) on the surface of the domain (A.7):

$$
\widehat{n} \cdot \frac{d}{d\tau}(I, J, K) = \varepsilon(1 - I)(2\gamma I - \gamma - 1)B - 2(2 - \gamma)J^2(2I - 1) \left[(1 - I)^2 - K^2\right] \quad (A.8)
$$

$$
= \varepsilon(1 - I)^2(2\gamma I - \gamma - 1) \left[J^2 + I(1 - I)\right] - 2(2 - \gamma)J^2(2I - 1)(1 - I)^2 + K^2 \left[2(2 - \gamma)J^2(2I - 1) - 2\varepsilon I(1 - I)(2\gamma I - \gamma - 1)\right]
$$

$$
= \varepsilon(1 - I)^2(2\gamma I - \gamma - 1)[J^2 + I(1 - I)] - 2(2 - \gamma)J^2(2I - 1)(1 - I)^2 + 2\varepsilon(1 - I)^2K^2.
$$
We observe that this inner product is an increasing function of $K^2$. In the domain $\Omega_3$, the largest $K^2$ is achieved on the surface $B = 0$. From the step (A.8), we find that this inner product in negative for $I > 1/2$. This completes the proof of Lemma A.3.

**Lemma A.4.** Assume $1 < \gamma < 3/2$. Then for any integral curve that gets into (at least once) the domain

$$(2 - \gamma)J^2 \leq (\gamma I - 1)(1 - I) \quad (A.9)$$

and ends at $(1,0,0)$ of the system (6.1-4) in the domain $\Omega_3$, there are three numbers $\varepsilon \in (0,1)$, $\beta \in (2,\infty)$, and $I_0 \in \left(\frac{1}{2}, 1\right)$ such that the integral curve is inside the cylinder (A.7) and below the surface (A.2):

$$B_\beta \equiv (1 - I)J^2 + I(I - 1)^2 - \beta IK^2 = 0 \quad (A.2)$$

when $I \in (I_0, 1)$.

**Proof.** The proof parallels that of Lemma 2: use the surface of the domain (A.7) in places where $C_\varepsilon = 0$ is used.

**Theorem A.** For all $\gamma > 1$, the parameter $s$ is finite for any integral curve of (6.1-4) that goes to the point $(1,0,0)$ inside $\Omega_3$.

**Proof.** (i) Assume $\gamma \geq 3/2$ or the integral curve gets into the domain (A.7) at least once, hence the integral curve remains in $B_\beta > 0$ for $I$ close to 1. We find from the first equation of (4.3) that

$$s \frac{dI}{ds} = 2I - (1 - I) \frac{I(1 - I) - J^2}{(1 - I)^2 - K^2} \quad (A.10)$$

The right hand side of (A.10) is a decreasing function of $K^2$ in $\Omega_3$. So for $I$ close to 1 we find

$$s \frac{dI}{ds} \geq 2I - (1 - I)(1 - \frac{1}{\beta}) \frac{I(1 - I) - J^2}{(1 - I)^2 - \frac{1}{\beta}(1 - I)J^2}$$

$$= 2I - I\beta \frac{I(1 - I) - J^2}{\frac{1}{\beta}(1 - I)(1 - I) - J^2} = 2I - \beta I + \frac{\beta(\beta - 2)(1 - I)}{(\beta - 1)(1 - I) - J^2} \quad (A.11)$$

The very last expression of (A.11) is an increasing function of $J^2$, so we further find by using $J^2 \geq 0$ that

$$s \frac{dI}{ds} \geq (2 - \beta)I + \frac{\beta(\beta - 2)}{\beta - 1} I = \frac{\beta - 2}{\beta - 1} I.$$ 

So it can only take a finite amount of $s$ for $I$ to reach 1. (ii) If $1 < \gamma < 3/2$ and the integral curve in entirely outside the domain (A.9) when $I$ is close to 1, we have

$$(2 - \gamma)J^2 \geq (\gamma I - 1)(1 - I) \quad (A.12)$$
for $I$ close to 1. We can use equations (6.2) and (6.4) to find

$$\frac{dJ}{ds} = \frac{J(1-2I)}{s(1-I)}.$$ (A.13)

Using (A.12) in (A.13), we find

$$\frac{d\ln(\frac{1}{J})}{d\ln s} = \frac{2I - 1}{1 - I} > \alpha^2 \left(\frac{1}{J}\right)^2$$

where $\alpha^2$ is a positive constant. It is easy to derive that $s < \infty$ from this last differential inequality.

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Figure Captions.

Fig. 1.1. The velocity vector of a typical swirling solution. Here $\gamma = 1.4$, $u_0 = 0.0$, $v_0 = 1.0$, $\rho_0 = 1.0$, and $A_2 = 1.0$. The figure shows the solution at time $t = 1$ in the square $|x| \leq 1.7535$, and $|y| \leq 1.7535$.

Fig. 1.2. The fluid density of the typical swirling solution in Fig. 1.1. Outside the square the density is the constant 1.

Fig. 5.1. Phase portrait of solutions without swirls.

Fig. 5.2. Estimate of the transitional Mach number $M_h(\gamma) < \frac{\sqrt{2}}{\gamma - 1}$. In this figure $d = (\gamma - 1)^2/2$, $f = M_h^{-2}(\gamma)$, and $N$ and $D$ are where the numerator and denominator of the right-hand side of (5.8) vanish respectively.

Fig. 5.3. A zero-swirl transitional solution.

Fig. 5.4. The parameter $s$ is finite near $(1, 0)$.

Fig. 6.1. The region $\Omega_3$ and $\Omega_{33}$.

Fig. 6.2. Local structure of solutions at the stationary point (6.6).

Fig. 6.3. Solutions in the intermediate field for data with $u_0 = 0$.

Fig. 6.4. Integral curves in the intermediate field.

Fig. 7.1. The domains $\Omega_{41}$ and $\Omega_{42}$.

Fig. 7.2. The center-unstable manifold.

Fig. 8.1. A solution $(u, v, \rho)$ vs. the $x$-axis at time $t = 1$ with a datum $u_0 = 0$ and $M_0 < \sqrt{2}$.

Fig. 8.2. A solution $(u, v, \rho)$ vs. the $x$-axis at time $t = 1$ with a datum $u_0 = 0$ and $M_0 > M_h(\gamma)$. 

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