On supremum of bounded quantum observable*

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Abstract. In this paper, we present a new necessary and sufficient condition for which the supremum $A \vee B$ exists with respect to the logic order $\preceq$. Moreover, we give out a new and much simpler representation of $A \vee B$ with respect to $\preceq$, our results have nice physical meanings.

Keywords: Quantum observable, logic order, supremum.

PACS numbers: 02.10-v, 02.30.Tb, 03.65.Ta.

1 Introduction

There some basic notations: $H$ is a complex Hilbert space, $S(H)$ is the set of all bounded linear self-adjoint operators on $H$, $S^+(H)$ is the set of all positive operators in $S(H)$, $P(H)$ is the set of all orthogonal projection operators on $H$, $\mathcal{B}(\mathbb{R})$ is the set of all Borel subsets of real number set $\mathbb{R}$. Each element in $P(H)$ is said to be a quantum event on $H$. Each element in $S(H)$ is said to be a bounded quantum observable on $H$. For $A \in S(H)$, let $R(A)$ be the range of $A$, $\overline{R(A)}$ be the closure of $R(A)$, $P_A$ be the orthogonal projection on $\overline{R(A)}$, $P^A$ be the spectral measure of $A$, $\text{null}(A)$ be the null space of $A$, and $N_A$ be the orthogonal projection on $\text{null}(A)$.

Let $A, B \in S(H)$. If for each $x \in H$, $[Ax, x] \leq [Bx, x]$, then we say that $A \preceq B$. Equivalently, there exists a $C \in S^+(H)$ such that $A + C = B$. $\preceq$ is a partial order on $S(H)$. The physical meaning of $A \preceq B$ is that the expectation of $A$ is not greater than the expectation of $B$ for each state of the system. So the order $\preceq$ is said to be a numerical order of $S(H)$. But $(S(H), \preceq)$ is not a lattice. Nevertheless, as a well known theorem due to Kadison, $(S(\mathbb{H}), \preceq)$ is an anti-lattice, that is, for any two elements $A$ and $B$ in

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*This project is supported by Natural Science Found of China (10771191 and 10471124).
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In 2006, Gudder introduced a new order \( \preceq \) on \( S(H) \): if there exists a \( C \in S(H) \) such that \( AC = 0 \) and \( A + C = B \), then we say that \( A \preceq B \) (\([2]\)).

Equivalently, \( A \preceq B \) iff for each \( \Delta \in B(\mathbb{R}) \) with \( 0 \not\in \Delta \), \( P_A(\Delta) \leq P_B(\Delta) \) (\([2]\)). The physical meaning of \( A \preceq B \) is that for each \( \Delta \in B(\mathbb{R}) \) with \( 0 \not\in \Delta \), the quantum event \( P_A(\Delta) \) implies the quantum event \( P_B(\Delta) \). Thus, the order \( \preceq \) is said to be a logic order of \( S(H) \) (\([2]\)). In [2], it is proved that \( (S(H), \preceq) \) is not a lattice since the supremum of arbitrary \( A \) and \( B \) may not exist in general. In [3], it is proved that the infimum \( A \land B \) of \( A \) and \( B \) with respect to \( \preceq \) always exists. In [4, 5], the representation theorems of the infimum \( A \land B \) of \( A \) and \( B \) with respect to \( \preceq \) were obtained. In more recent, Xu and Du and Fang in [6] discussed the existence of the supremum \( A \lor B \) of \( A \) and \( B \) with respect to \( \preceq \) by the technique of operator block. Moreover, they gave out a sufficient and necessary conditions for the existence of \( A \lor B \) with respect to \( \preceq \). Nevertheless, their conditions are difficult to be checked since the conditions depend on an operator \( W \), but \( W \) is not easy to get. Moreover, their proof is so much algebraic that we can not understand its physical meaning.

In this paper, we present a new necessary and sufficient condition for which \( A \lor B \) exists with respect to \( \preceq \) in a totally different form. Furthermore, we give out a new and much simpler representation of \( A \lor B \) with respect to \( \preceq \), our results have nice physical meanings.

**Lemma 1.1 [2]**. Let \( A, B \in S(H) \). If \( A \preceq B \), then \( A = BP_A \).

**Lemma 1.2 [2]**. If \( P, Q \in P(H) \), then \( P \preceq Q \) iff \( P \preceq Q \), and \( P \) and \( Q \) have the same infimum \( P \land Q \) and the supremum \( P \lor Q \) with respect to the orders \( \preceq \) and \( \succeq \), we denote them by \( P \land Q \) and \( P \lor Q \), respectively.

**Lemma 1.3 [7]**. Let \( A, B \in S(H) \). Then \( P^A(\{0\}) = N(A) \), \( P_A = P^A(R\setminus\{0\}) \), \( P_A + N(A) = I \), \( P_A \lor P_B = I - N(A) \land N(B) \).

## 2 Some elementary lemmas

Let \( A, B \in S(H) \) and they have the following forms:

\[
A = \int_{-M}^{M} \lambda dA_\lambda
\]
and 
\[ B = \int_{-M}^{M} \lambda dB_{\lambda}, \]
where \( \{A_{\lambda}\}_{\lambda \in \mathbb{R}} \) and \( \{B_{\lambda}\}_{\lambda \in \mathbb{R}} \) be the identity resolutions of \( A \) and \( B \) ([7]), respectively, and \( M = \max(\|A\|, \|B\|) \).

If \( A \) has an upper bound \( F \) in \( S(H) \) with respect to \( \preceq \), then it follows from Lemma 1.1 that \( A = FP_{A} \). Note that \( A \in S(H) \), so \( FP_{A} = PAF \) and thus \( AF = FA \). Let \( F \) have the following form:
\[ F = \int_{-G}^{G} \lambda dF_{\lambda}, \]
where \( \{F_{\lambda}\}_{\lambda \in \mathbb{R}} \) is the identity resolution of \( F \) and \( G = \max(\|F\|, M) \). Then we have
\[ A = FP_{A} = (\int_{-G}^{G} \lambda dF_{\lambda})PA = \int_{-G}^{G} \lambda d(F_{\lambda}PA). \]

**Lemma 2.1.** Let \( A \in S(H) \) and \( F \in S(H) \) be an upper bound of \( A \) with respect to \( \preceq \). Then for each \( \Delta \in B(\mathbb{R}) \), we have
\[
P^{A}(\Delta) = \begin{cases} 
P^{F}(\Delta)PA, & 0 \not\in \Delta \\
N(A), & \Delta = \{0\} \\
P^{F}(\Delta \setminus \{0\})PA + N(A), & 0 \in \Delta \end{cases}
\]

**Proof.** We just need to check \( P^{A}(\Delta) = P^{F}(\Delta)PA \) when \( 0 \not\in \Delta \), the rest is trivial. Note that if we restrict on the subspace \( P_{A}(H) = \overline{R(A)} \), since \( AF = FA \), then \( \{F_{\lambda}PA\}_{\lambda \in \mathbb{R}} \) is the identity resolution of \( F|_{P_{A}(H)} \) ([7]). Let \( f \) be the characteristic function of \( \Delta \). Then the following equality proves the conclusion:
\[
P^{A}(\Delta) = f(A) = f(FPA) = \int_{-G}^{G} f(\lambda)d(F_{\lambda}PA) = \int_{\lambda \in \Delta} d(F_{\lambda}PA) = P^{F}(\Delta)PA.
\]

It follows from Lemma 2.1 immediately:

**Lemma 2.2.** Let \( A, B \in S(H) \) and \( F \in S(H) \) be an upper bound of \( A \) and \( B \) with respect to \( \preceq \). Then for any two Borel subsets \( \Delta_{1} \) and \( \Delta_{2} \) of \( \mathbb{R} \), if \( \Delta_{1} \cap \Delta_{2} = \emptyset \), \( 0 \not\in \Delta_{1} \), \( 0 \not\in \Delta_{2} \), we have
\[
P^{A}(\Delta_{1})P^{B}(\Delta_{2}) = P^{F}(\Delta_{1})PA P^{F}(\Delta_{2})PB = PA P^{F}(\Delta_{1}) P^{F}(\Delta_{2})PB = \theta.
\]
Lemma 2.3. Let $A, B \in S(H)$ and have the following property: For each pair $\Delta_1, \Delta_2 \in \mathcal{B}(\mathbb{R})$, whenever $\Delta_1 \cap \Delta_2 = \emptyset$ and $0 \notin \Delta_1$, $0 \notin \Delta_2$, we have $P^A(\Delta_1)P^B(\Delta_2) = \emptyset$, then the following mapping $E : \mathcal{B}(\mathbb{R}) \to P(H)$ defines a spectral measure:

$$E(\Delta) = \begin{cases} 
P^A(\Delta) \lor P^B(\Delta), & 0 \notin \Delta \\
N(A) \land N(B) = I - P_A \lor P_B, & \Delta = \{0\} \\
P^A(\Delta \setminus \{0\}) \lor P^B(\Delta \setminus \{0\}) + N(A) \land N(B), & 0 \in \Delta
\end{cases}$$

Proof. First, we show that for each $\Delta \in \mathcal{B}(\mathbb{R})$, $E(\Delta) \in P(H)$. It is sufficient to check the case of $0 \in \Delta$. Since $P^A(\Delta \setminus \{0\}) \lor P^B(\Delta \setminus \{0\}) \leq P^A(\Delta \setminus \{0\}) \lor P^B(\Delta \setminus \{0\}) = P_A \lor P_B$, so it follows from Lemma 1.3 that $P^A(\Delta \setminus \{0\}) \lor P^B(\Delta \setminus \{0\}) + N(A) \land N(B) \in P(H)$ and the conclusion is hold.

Second, we have

$$E(\emptyset) = P^A(\emptyset) \lor P^B(\emptyset) = \emptyset \lor \emptyset = \emptyset,$$

$$E(R) = P^A(R \setminus \{0\}) \lor P^B(R \setminus \{0\}) + N(A) \land N(B)$$

$$= P_A \lor P_B + N(A) \land N(B) = I.$$

Third, if $\Delta_1 \cap \Delta_2 = \emptyset$, there are two cases:

(i). $0$ doesn’t belong to any one of $\Delta_1$ and $\Delta_2$. It follows from the definition of $E$ that $E(\Delta_1)E(\Delta_2) = (P^A(\Delta_1) \lor P^B(\Delta_1))(P^A(\Delta_2) \lor P^B(\Delta_2))$. Note that $P^B(\Delta_1)P^A(\Delta_2) = \emptyset$ by the conditions of the lemma and $P^B(\Delta_1)P^B(\Delta_2) = \emptyset$, we have $P^B(\Delta_1)(P^A(\Delta_2) \lor P^B(\Delta_2)) = \emptyset$, similarly, we have also $P^A(\Delta_1)(P^A(\Delta_2) \lor P^B(\Delta_2)) = \emptyset$, thus,

$$E(\Delta_1)E(\Delta_2) = \emptyset.$$

Furthermore, we have

$$E(\Delta_1 \cup \Delta_2) = P^A(\Delta_1 \cup \Delta_2) \lor P^B(\Delta_1 \cup \Delta_2)$$

$$= P^A(\Delta_1) \lor P^A(\Delta_2) \lor P^B(\Delta_1) \lor P^B(\Delta_2)$$

$$= (P^A(\Delta_1) \lor P^B(\Delta_1)) \lor (P^A(\Delta_2) \lor P^B(\Delta_2))$$

$$= E(\Delta_1) + E(\Delta_2).$$

That is, in this case, we proved that

$$E(\Delta_1)E(\Delta_2) = \emptyset,$$

$$E(\Delta_1 \cup \Delta_2) = E(\Delta_1) + E(\Delta_2).$$
(ii). 0 belongs to one of $\Delta_1$ and $\Delta_2$. Without of losing generality, we suppose that $0 \in \Delta_1$, since $\Delta_1 \cap \Delta_2 = \emptyset$, so $0 \notin \Delta_2$, thus we have

$$E(\Delta_1)E(\Delta_2) = (P^A(\Delta_1 \{0\}) \lor P^B(\Delta_1 \{0\}) + N(B) \land N(A))(P^A(\Delta_2) \lor P^B(\Delta_2))$$

$$= (P^A(\Delta_1 \{0\}) \lor P^B(\Delta_1 \{0\}))(P^A(\Delta_2) \lor P^B(\Delta_2)) = \theta,$$

$$E(\Delta_1 \cup \Delta_2) = P^A(\Delta_1 \{0\} \cup \Delta_2) \lor P^B(\Delta_1 \{0\} \cup \Delta_2) + (N(B) \land N(A))$$

$$= (P^A(\Delta_1 \{0\}) \lor P^B(\Delta_1 \{0\}) + (N(B) \land N(A))) + (P^A(\Delta_2) \lor P^B(\Delta_2))$$

$$= (P^A(\Delta_1 \{0\}) \lor P^B(\Delta_1 \{0\}) + (N(A) \land N(B))) + (P^A(\Delta_2) \lor P^B(\Delta_2))$$

$$= E(\Delta_1) + E(\Delta_2).$$

Thus, it follows from (i) and (ii) that whenever $\Delta_1 \cap \Delta_2 = \emptyset$, we have

$$E(\Delta_1)E(\Delta_2) = \theta,$$

$$E(\Delta_1 \cup \Delta_2) = E(\Delta_1) + E(\Delta_2).$$

Final, if $(\Delta_n)_{n=1}^{\infty}$ is a sequence of pairwise disjoint Borel sets in $B(\mathbb{R})$, then it is easy to prove that

$$E(\bigcup_{n=1}^{\infty} \Delta_n) = \sum_{n=1}^{\infty} E(\Delta_n).$$

Thus, the lemma is proved.

3 Main results and proofs

**Theorem 3.1.** Let $A, B \in S(H)$ and have the following property: For each pair $\Delta_1, \Delta_2 \in B(\mathbb{R})$, whenever $\Delta_1 \cap \Delta_2 = \emptyset$ and $0 \notin \Delta_1, 0 \notin \Delta_2$, we have $P^A(\Delta_1)P^B(\Delta_2) = \theta$. Then the supremum $A \lor B$ of $A$ and $B$ exists with respect to the logic order $\leq$.

**Proof.** By Lemma 2.3, $E(\cdot)$ is a spectral measure and so it can generate a bounded quantum observable $K$ and $K$ can be represented by $K = \int_{-M}^{M} \lambda dE_\lambda$, where $\{E_\lambda\} = E(-\infty, \lambda], \lambda \in \mathbb{R}$ and $M = \max(\|A\|, \|B\|)$. Moreover, for each $\Delta \in B(\mathbb{R})$, $P^K(\Delta) = E(\Delta)$ ([7]). We confirm that $K$ is the supremum $A \lor B$ of $A$ and $B$ with respect to $\leq$. In fact, for each $\Delta \in B(\mathbb{R})$ with $0 \notin \Delta$, by the definition of $E$ we knew that $P^K(\Delta) = E(\Delta) = P^A(\Delta) \lor P^B(\Delta) \geq P^A(\Delta), P^K(\Delta) = E(\Delta) = P^A(\Delta) \lor P^B(\Delta) \geq P^B(\Delta)$. So it following from the equivalent properties of $\leq$ that $A \preceq K, B \preceq K$ ([2]). If $K'$ is another upper bound of $A$ and $B$ with respect to $\preceq$, then for each $\Delta \in B(\mathbb{R})$ with $0 \notin \Delta$, we
have $P^A(\Delta) \leq P^{K'}(\Delta)$, $P^B(\Delta) \leq P^{K'}(\Delta)$ ([2]), so $P^A(\Delta) \lor P^B(\Delta) = E(\Delta) = P^K(\Delta) \leq P^{K'}(\Delta)$, thus we have $K \preceq K'$ and $K$ is the supremum of $A$ and $B$ with respect to $\preceq$ is proved.

It follows from Lemma 2.2 and theorem 3.1 that we have the following theorem immediately:

**Theorem 3.2.** Let $A, B \in S(H)$. Then the supremum $A \lor B$ of $A$ and $B$ exists with respect to the logic order $\preceq$ iff for each pair $\Delta_1, \Delta_2 \in B(\mathbb{R})$, whenever $\Delta_1 \cap \Delta_2 = \emptyset$ and $0 \notin \Delta_1$, $0 \notin \Delta_2$, we have $P^A(\Delta_1)P^B(\Delta_2) = \theta$. Moreover, in this case, we have the following nice representation:

$$A \lor B = \int_{-M}^{M} \lambda dE_{\lambda},$$

where $\{E_{\lambda}\} = E(-\infty, \lambda], \lambda \in \mathbb{R}$ and $M = \max(\|A\|, \|B\|)$.

**Remark 3.3.** Let $A, B \in S(H)$. Note that for each $\Delta \in B(\mathbb{R})$, $P^A(\Delta)$ is interpreted as the quantum event that the quantum observable $A$ has a value in $\Delta$ ([2]), and the conditions: $\Delta_1 \cap \Delta_2 = \emptyset$, $0 \notin \Delta_1$, $0 \notin \Delta_2$ must have $P^A(\Delta_1)P^B(\Delta_2) = \theta$ told us that the quantum events $P^A(\Delta_1)$ and $P^B(\Delta_2)$ can not happened at the same time, so, the physical meanings of the supremum $A \lor B$ exists with respect to $\preceq$ iff for each pair $\Delta_1, \Delta_2 \in B(\mathbb{R})$, whenever $\Delta_1 \cap \Delta_2 = \emptyset$ and $0 \notin \Delta_1$, $0 \notin \Delta_2$, the quantum observable $A$ takes value in $\Delta_1$ and the quantum observable $B$ takes value in $\Delta_2$ can not happen at the same time.

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