The Analytical Structure of Acoustic and Elastic Material Properties

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In this paper, we take an in-depth look at the analytical structure of the material transfer functions which govern acoustic and elastic response. These include wavenumber (κ), in such media and refractive index (n), density (ρ) and its inverse, stiffness (C) and compliance (D) tensors as well as the Bulk modulus (B), and finally the broader generalization of these properties which is now known as the Willis tensor (L). Our goal is to clarify the appropriate dispersion relations applicable to these properties from the perspective of passivity. Under some mild assumptions, causality ensures that these properties are analytical in the upper half but deriving dispersion relations for them requires one to know how they behave in the limit |ω| → ∞. Unlike electromagnetism, such a determination cannot be made on physical grounds since in that limit the continuum approximation breaks down. Instead, we can exploit the properties of the Herglotz-Nevanlinna (H-N) functions along with their tensorial counterparts which characterize the transfer functions of certain passive systems and for which the appropriate dispersion relation is known. Our aim, therefore, is to clarify the relationship that these transfer functions have with Herglotz functions, which in turn determines the appropriate dispersion relation for them. Our analysis shows that based upon passivity alone, dispersion relations of minimum order 1 apply to the Fourier transforms of D, ρ, n′, and the inverse of B, order 3 apply to C, B, and the inverse of ρ, and order 2 applies to κ.

I. INTRODUCTION

If a cause-effect relation adopts a convolution form, then the assumption that the effect cannot exist before its cause – the colloquial statement of causality – has strong implications for the transfer function of the relationship. Such transfer functions are ubiquitous in physics and engineering. For the purpose of the current study, we will be concerned with the transfer functions which represent dynamic acoustic and elastic material responses. These transfer functions include the compliance, D, and stiffness, C, of solid materials, bulk modulus, B, of a liquid or air, density, ρ, of materials, and finally, wavenumber (κ) and refractive index (n′) – quantities which characterize wave propagation in such materials. They also include the vectorial or tensorial forms of these quantities as well as the general Willis tensor, L [1,2], which has come to characterize metamaterial response. Considering any one of these quantities as a time dependent function, m(t), causality states that m(t) = 0∀t < 0. Under some conditions, causality can give rise to relations between the real and imaginary parts of the Fourier transform of m(t). Denoting by ˜m(ω), the Fourier transform of m(t), causality can result in the following relations [3]:

\[ \Re ˜m(\omega) = \frac{\omega}{\pi} \Re \int_{-\infty}^{\infty} \frac{3\Re ˆm(\omega')}{\omega' - \omega} d\omega' \]
\[ \Im ˜m(\omega) = -\frac{\omega}{\pi} \Im \int_{-\infty}^{\infty} \frac{3\Re ˆm(\omega')}{\omega' - \omega} d\omega' \]

In the above, \( \Re \) is the Cauchy Principal value and the relations are called the generalized Kramers Kronig relations [4,5]. Here, n is any integer greater than or equal to some integer l and the value of l depends upon the behavior of ˆm(ω) in the limit |ω| → ∞. The main aim of this paper is to establish the correct value of l for the transfer functions alluded to above. A certain value of l has been used in a recent paper [6] in the context of Willis tensors but without the supporting arguments necessary.

For electromagnetic waves, the situation is markedly simpler due to the limiting speed of light – a benefit which does not exist in acoustics or elastodynamics [7]. For electromagnetics, the refractive index, n′(ω), is proportionally related to \( \sqrt{\epsilon(\omega)} \) (assuming that the magnetic permeability, μ, is equal to unity). \( \epsilon(\omega) \) is in turn related to the susceptibility of the medium χ(ω), which relates the physical quantities, electric polarization and electric field, through a convolution relation. χ(ω) is automatically causal from physical considerations and, therefore, n′(ω), κ(ω)/ω are causal as well. Furthermore, since the dielectric is underlined by a vacuum and the high frequency behavior of a wave approaches that of vacuum propagation [8,9], physics dictates that in the high frequency limit, ε(ω) tends to 1 (since χ(ω) goes down as 1/ω² and ε = 1 + 4πχ). Since ε(ω) tends to 1 in the high frequency limit, so does n′(ω). For electromagnetic

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wave propagation, \( n'(\omega) = n_r + i(n_0\beta/2\omega) \), where the real part of \( n' \), \( n_r \), is called the real refractive index, and the factor \( \beta \) is called the extinction coefficient which governs the attenuation of the medium. \( c_0 \) is the speed of light which is a constant. In the high frequency limit, since \( n'(\omega) \to 1 \), we have \( n'(\omega) - 1 \) tending to zero. Hence, we see that for electromagnetics, there are fundamental limits on the behavior of material properties such as \( \epsilon(\omega) \) and wave properties such as \( n'(\omega) \) in the limit \( |\omega| \to \infty \). Determining how these quantities behave in the high frequency limit allows us to determine the correct value of \( l \) for these properties.

The same arguments cannot be made for acoustics or elastodynamics. Unlike dielectrics which behave as vacuum in the high frequency limit, acoustic or elastic mediums are not underlined by a reference response in such limits. In fact, it does not make sense to talk about material properties such as compliance or stiffness in such limits because the continuum approximation breaks down. To navigate this problem, researchers in acoustics and elastodynamics have resorted to empirical arguments. For instance, Ginzberg [10] essentially assumed that \( \kappa(\omega)/\omega \) exists as \( |\omega| \to \infty \), and that it approaches some limiting value independent of \( \arg \omega \), which allowed him to derive a value of \( l \) for \( \kappa(\omega) \). The same approach was followed by Futterman [11] in his application of dispersion relation to seismic wave propagation (see also [12–16] for further discussions on dispersion in seismic waves and connections to Kramers-Kronig relationships). His essential argument is that it is difficult to envision that the structure of the Earth would resonate to a disturbance at infinite frequency. This allows him to say that the imaginary part of \( n'(\omega) \), which is proportional to attenuation, must be 0 in that limit and the real part must equal some constant \( n_r(\infty) \). For acoustic wave propagation, the derivation of the dispersion relations is often based upon assuming a functional form for attenuation [17–19]. Consider \( \kappa(\omega) = \omega/c(\omega) + io(\omega) \), where \( c(\omega) \) is the phase velocity of the wave, and \( o(\omega) \) is the attenuation constant. For media in which the attenuation satisfies a frequency power law, \( o(\omega) = o_0|\omega|^\gamma \), the correct value of \( l \) depends upon the power coefficient \( \gamma \) [5, 20–22]. Thus we see that the determination of \( l \) in acoustics and elastodynamics is generally either based upon empirical assumptions on the functional form of the property under consideration or on the high frequency behavior of these properties – the latter especially being based on a set of assumptions which may be difficult to defend on physical grounds.

There is one especially notable work which attempts to deduce the correct value of \( l \) for passive acoustic and elastodynamic media through the use of Herglotz functions [23], sometimes also called Nevalninna or Pick functions. These are functions which are analytic in the upper half of the complex plane where they have non-negative imaginary part. Herglotz functions have well known behavior in the high frequency limit and the corresponding correct value of \( l \) which may be derived for them is well known [8, 21–25]. Weaver and Pao [9] showed that for passive media, \( \kappa(\omega) \) is a Herglotz function and derived dispersion relations for it. Our work picks up from Weaver and Pao’s work and fills in some missing details. Weaver and Pao restricted their discussions to \( \kappa(\omega) \) and did not consider other properties of interest described above. This is understandable in part because the ideas of frequency dependent density, for example, have only become popular with the advent of metamaterials [26]. Here, we extend the Herglotz function based analysis that Weaver and Pao pursued to the other properties which control the dynamic behavior of acoustic and elastodynamic media including the Willis tensor. Furthermore, instead of following the Cauchy integral based proofs presented by Weaver and Pao, we present proofs which are based upon distribution theory. These are more succinct and applicable to generalized functions as well. Furthermore, our analysis applies to tensorial properties which were not considered by Weaver and Pao. Our approach in the rest of the paper will be the following: for the properties under consideration, we will show that they are related to Herglotz functions and then we will use the theory of Herglotz functions to derive the correct value of \( l \) for these properties thus determining the correct dispersion relations for them. To show that these properties are related to Herglotz functions, we will extensively invoke the principle of passivity. In other words, the main conclusions in this paper only apply to materials and mediums which do not have sources of energy.

II. GENERALIZED KRAMERS-KRONIG RELATIONSHIPS

Here we are concerned with the space of temperate distributions and we summarize some relevant results for the same. For more details on distribution theory and the space of temperate distributions, we refer the reader to exhaustive references on the topic [27]. In this section, we only present the immediately useful results without providing any proofs. We define \( S \) as the space of rapidly decreasing test functions characterized by \( \phi(t) \in C^\infty \) which, together with all their derivatives, decrease faster than any inverse power of \( t \) as \( |t| \to \infty \):

\[
\lim_{|t| \to \infty} |t^p \partial^m \phi(t)| = 0; \quad p, m = 0, 1, \ldots
\]  

(2)

We define the class of temperate distributions \( S' \) as the set of distributions which are linear functionals on \( S \) and the Fourier transform of a distribution \( g \in S' \) through the relation \( \langle \mathcal{F}g, \phi \rangle = \langle g, \mathcal{F}\phi \rangle \). \((x, y)\) represents the inner-product
\[ \mathcal{F}\phi(t) = \tilde{\phi}(\omega) = \int_{-\infty}^{\infty} \phi(t)e^{i\omega t}dt \]  

Fourier transform of a distribution \(g(t) \in \mathcal{S}'\), denoted by \(G(\omega), \mathcal{F}g, \hat{g}\), exists and belongs to \(\mathcal{S}'\). With \(k = \omega + is\), the Laplace transform is defined through the Fourier transform using \(Lg = G(k) = \mathcal{F}(g(t)e^{-st})\). A causal distribution in \(\mathcal{S}'\) is one which is zero for \(t < 0\) and belongs to a subspace of \(\mathcal{S}'\) denoted by \(\mathcal{S}'_+\). If \(g(t) \in \mathcal{S}'_+\) then its Laplace transform has a region of convergence \(s > 0\) and its boundary value is the Fourier transform \(G(\omega)\). Furthermore, the following generalized Hilbert transform applies to \(G(\omega)\):

\[ G(\omega) = -\frac{\omega^n}{\pi i} \left[ \frac{G(\omega)}{\omega^n} \ast \mathcal{P} \left( \frac{1}{\omega} \right) \right]. \tag{4} \]

\(\ast\) denotes the convolution operation, \(\mathcal{P}\) denotes the principal value distribution and \(n\) is any integer greater than or equal to a specific non-negative integer \(l\). Equating the real and imaginary parts of (4), we arrive at the generalized dispersion relationships (generalized Kramers-Kronig relationships) that were also mentioned in (1). The value of \(l\) depends upon the growth properties of \(G(\omega)\) and, equivalently, the discontinuity properties of \(g(t)\). To be more specific, we note the result that every distribution is a finite order derivative of a continuous function. \(l\) is the order of the derivative which connects \(g(t)\) to some continuous function. Once \(l\) is determined, one can derive a set of valid dispersion relations by taking \(n = l\) in (1) and separating the real and imaginary parts. Higher order dispersion relations are also valid if one takes \(n > l\), however, dispersion relations with \(n < l\) are invalid. Our next step is to collate the set of proofs which establish the correct value of \(l\) for Herglotz functions. In what follows, we will use letters (either in lower or higher case) such as \(x, f, h\) to represent quantities in the time domain and by hat such as \(\hat{x}, \hat{f}, \hat{h}\) to represent their Laplace or Fourier transforms. Sometimes, we will refer to classical results from electrical networks and control theory where the convention is to refer to the region \(s > 0\) as the right half plane. This convention emerges from the definition of the complex frequency, \(p\) which parametrizes the Laplace transform, as \(p = s + i\omega\). In this convention, we will refer to the Laplace transform as \(X(p)\), as an example. At other times we will refer to the convention more common in physics where the region \(s > 0\) signifies the upper half plane. This emerges from defining the complex frequency, \(k = \omega + is\). In this convention, we will refer to the Laplace transform as \(\tilde{X}(k)\), as an example. Fourier transform \((\tilde{X}(\omega)\), for example\) will be indicated by using the dependence on the frequency \(\omega\). The mentioned complex frequency domains are illustrated in Fig. 1.

**FIG. 1.** Two representation of the complex frequency plane (a) \(p = s + i\omega\) and (b) \(k = \omega + is\).

### III. CORRECT VALUE OF \(l\) FOR HERGLOTZ FUNCTIONS

Here, we introduce the Herglotz integral representation [28–30] for functions which are holomorphic in the upper half plane of the complex frequency with positive imaginary part and show [31 32] that their inverse Fourier transforms are a second order derivative of continuous functions. There is a related concept in areas such as control theory where the right half of the complex plane is used more commonly. There, functions which are holomorphic in the right half and which posses a non-negative real part there, are termed positive functions [33 34]. We emphasize that a Herglotz function is just a mapped version of a positive function. In the rest of this section, we show that a Herglotz function represents the transform of a time domain function which is a second order derivative of a continuous function – in other words, \(l = 2\) applies to all Herglotz functions.
Consider the Herglotz representation of any function which is holomorphic in the upper half plane of \( k \) (complex frequency; \( k = \omega + is \)) and possesses a non-negative imaginary part there\[28\]:

\[
\hat{h}(k) = a + bk + \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{1 + rk}{r - k} d\nu(r)
\]  

(5)

The reader is referred to section\[VIII\] subsection A, for further information about the derivation of this expression. In Eq. (5), \( a, b \in \mathbb{R} \), \( b \geq 0 \) and \( \nu \) is a positive bounded measure. It should be noted that this representation is equivalent to the Cauer representation for positive functions\[36\]. The above represents the Laplace transform of the time domain function \( h(t) \) whose Fourier transform can be found simply by taking the limit \( s \to 0^+ \) of \( \hat{h}(k) \). This is done by considering the identity

\[
\lim_{s \to 0^+} \frac{1}{\omega + is} = \mathcal{P}(1/\omega) - i\pi \delta(\omega),
\]

which allows us to write,

\[
\hat{h}(\omega) = a + b\omega + \frac{1}{\pi} \int_{-\infty}^{+\infty} (1 + r\omega) \left[P \left(\frac{1}{\omega - r}\right) - i\pi \delta(\omega - r)\right] d\nu(r)
\]  

(6)

\( \hat{h}(\omega) \) is understood as a distribution and in distributional analysis applied to passivity, it is customary to assume that \( \hat{h} \in \mathcal{S}' \). We can now write the equation above using distributional notation:

\[
(\hat{h}(\omega), \phi(\omega)) = (a, \phi(\omega)) + (b, \phi(\omega)) - 2i \left(\int_{-\infty}^{+\infty} (1 + r\omega) \left[\frac{1}{2} \delta(\omega - r) - \frac{1}{2\pi i} P \left(\frac{1}{\omega - r}\right)\right] d\nu(r), \phi(\omega)\right)
\]  

(7)

where \( \phi \in \mathcal{S} \). \((.,.)\) is the inner product, linear with respect to both terms. We’d like to derive the inverse Fourier transform of the above. The inverse Fourier transform of the first term is simply \( a\delta(t) \) whereas for the second term, it is \( b\delta^{(1)}(t) \) with \( \delta^{(1)}(t) \) being the first derivative of \( \delta(t) \). To find the inverse Fourier transform of the last term, we recall the following relevant identities for distributional Fourier transforms\[8\]

\[
(\mathcal{F}\{f(t)\}, \phi(\omega)) = (\hat{f}(\omega), \tilde{\phi}(t)); \quad \mathcal{F}\left\{\frac{e^{-i\xi t}\theta(t)}{2\pi}\right\} = \frac{1}{2} \delta(\omega - \xi) - \frac{1}{2\pi i} P \left(\frac{1}{\omega - \xi}\right)
\]  

(8)

This allows us to transform the integral term in Eq. (7) as

\[
-2i \left(\int_{-\infty}^{+\infty} (1 + r\omega) \left[\frac{\theta(t)e^{-irt}}{2\pi}\right] e^{i\omega t} dtd\nu(r), \phi(\omega)\right)
\]  

(9)

which can be written in the following form after noting that \( \mathcal{F}\{(i\omega)^n f(\omega)\} = \tilde{f}^{(n)}(t)\):

\[
\frac{1}{\pi} \left[i \left(\int_{-\infty}^{+\infty} \theta(t)e^{-irt} d\nu(r), \tilde{\phi}(t)\right) + \left(\int_{-\infty}^{\infty} r\theta(t)e^{-irt} d\nu(r), \tilde{\phi}^{(1)}(t)\right)\right]
\]  

(10)

To proceed further, we recall the classical result that in

\[
P(t) = \int_{-\infty}^{+\infty} e^{-i\xi t} dF(\xi),
\]  

(11)

if the function \( F \) is a function with bounded variations on \((-\infty, +\infty)\), then \( P(t) \) – called the after-effect function – will be bounded and continuous\[32\][33][37\]. Integration by parts in Eq. (10) gives:

\[
-i \int_{-\infty}^{+\infty} r d\nu(r) \int_{-\infty}^{\infty} \frac{\theta(t)}{r} e^{-i\eta t} \tilde{\phi}^{(2)}(t) dt + i \int_{-\infty}^{+\infty} r d\nu(r) \int_{0}^{\infty} \tilde{\phi}^{(2)}(t) \frac{d^2}{dt^2} \left(\frac{d^2}{dt^2} (P(t)\theta(0))\right)
\]  

(12)

Substituting Eq. (12) into Eq. (10) and then substituting the new form of Eq. (10) into Eq. (7) and by using identity in Eq. (8), we can write the inverse Fourier transform of \( h \) as:

\[
h(t) = a\delta(t) + b\delta^{(1)}(t) - \frac{i}{\pi} \left[P(t)\theta(t) - \frac{d^2}{dt^2} (P(t)\theta(t)) + \frac{d^2}{dt^2} (\theta(t)P(0))\right]
\]  

(13)

thus showing that \( h(t) \) is a second order derivative of a continuous function, or that \( l = 2 \) applies for \( \hat{h}(\omega) \).
A. Symmetric Herglotz representation

For physical applications, \( h(t) \) generally represents a real transfer function. If we insist that \( h(t) \) is real then we have the following symmetry relations on its Laplace transform:

\[
\hat{h}^*(k) = \hat{h}(-k^*) 
\]

where complex conjugation is implied by *. Immediately one can write:

\[
\hat{h}(k) = \frac{1}{2} \left[ \hat{h}(k) + \hat{h}^*(-k^*) \right] 
\]

Now the integral representation of \( \hat{h}(k) \) can be modified by applying the above identity to Eq. [5]:

\[
\hat{h}(k) = a + \int_{-\infty}^{+\infty} \frac{1 + k^2}{r^2 - k^2} r d\nu(r) = \hat{h}(k) = a + \frac{1}{2} \int_{-\infty}^{+\infty} (1 + k^2) \left( \frac{1}{r + k} + \frac{1}{r - k} \right) d\nu(r) 
\]

The above is called a symmetric Herglotz representation[29]. It can be shown that its inverse transform is given by:

\[
h(t) = a\delta(t) - P_2(t)\theta(t) + \frac{d^2}{dt^2} [P_2(t)\theta(t)] 
\]

where \( P_2(t) = \int_{-\infty}^{\infty} \sin(\xi t) dF(\xi) \). It clear that in the above, \( h(t) \) is a fully real function and is, again, a second order derivative of a continuous function. Therefore, by keeping in mind the generalized form of Hilbert pairs in Eq. [1] and noting the findings of this section one can write the dispersion relation for any Herglotz function as:

\[
\hat{h}(\omega) = \frac{\omega^2}{\pi i} \left[ \frac{\hat{h}(\omega)}{\omega^2} \star P \left( \frac{1}{\omega} \right) \right]. 
\]

IV. PASSIVITY CONSIDERATIONS

Transfer functions of passive systems are Herglotz functions under certain considerations, which we discuss in this section. Here, we consider all relevant quantities in tensorial and distributional forms. A tensor of distributions \( f(t) \) is defined through its actions on a test function \( \phi(t) \), both in appropriate spaces. Specifically, \( (f(t), \phi(t)) \) is the matrix of complex numbers obtained by replacing each element of \( f(t) \) by the number that this element assigns to the testing function \( \phi(t) \) through the inner product operation. Zemanian introduced tensorial distribution spaces to admit tensors of distributions of appropriate ranks. For example, \( \mathcal{S}'_{n \times n \times n \times n} \) is the space of all fourth order tensors whose elements are distributions in \( \mathcal{S}' \) etc. Zemanian showed that a single-valued, linear, time-invariant, and continuous input output relation can be written in the convolution form, \( v = z \ast j \), where \( v, z, j \) are tensors of distributions in appropriate spaces, and \( \ast \) denotes a convolution in time as well as appropriate tensorial contraction:

\[
v = z \ast j : v_l(t) = z_{lm}(t) \ast j_m(t); \quad l, m = 1, 2...n 
\]

A. Immittance and scattering forms of passivity

For a physical system with an input output relation in the convolution form, the requirement that the system be passive (output energy cannot exceed input energy) automatically implies that the system is causal as well [3]. The statement of passivity can be framed in two equivalent forms – scattering and immittance. Consider, for example, an input-output relationship \( x = g \ast f \), whose passivity condition is given by the following scattering form:

\[
\int_{-\infty}^{t} (f^\dagger f - x^\dagger x) \, dt' \geq 0, \, \forall \, t 
\]

where \( \dagger \) represents a conjugate transpose operation and \( f^\dagger f \) is indicative of the \( L_2 \) energy in the input at time \( t \). The above statement says that the total energy consumed in generating the output at any time \( t \) can never exceed the total energy in the input to the system up to that time.
The passivity relations can be framed in another form, called the immittance form, which emerges naturally in certain problems. The introduction of new variables \( v(t) = f(t) + x(t) \) and \( j(t) = f(t) - x(t) \) allows us to write the passivity conditions as:

\[
\Re \int_{-\infty}^{t} v^j j^t dt^t \geq 0, \forall \ t
\]

The interpretation of both forms of passivity is that in a passive system, the net absorbed energy of the system is non-negative. If the net absorbed energy is zero, then the system under consideration is conservative but still admissible as a passive system. Using the statement of passivity in the immittance form and assuming that \( v = z * j \) applies, we reiterate important results from Zemanian [31] which are relevant here. For real transfer functions \( z(t) \), Zemanian showed that the following are true in the right half plane – these are Herglotz functions. Functions possessing similar properties in the right half are called positive functions and when such functions represent real transfer functions in the time domain, then they are called positive real functions. Positive real functions are the scalar analogues of positive real matrices which were discussed in this section under the context of passivity in the immittance form. As mentioned earlier, the two concepts – Herglotz functions/matrices and positive functions/matrices – are connected to each other through a simple mapping. The former is defined in terms of \( k = \omega + is \) whereas the latter is defined in terms of \( p = s + i\omega \). Thus, it becomes evident that for an input-output relationship whose passivity statement may be written in the immittance form, the Laplace transform of its transfer is closely related to a Herglotz matrix. The connection is the following:

1. \( \hat{g}(p) \) is analytic/holomorphic.
2. \( \hat{g}(p^*) = \hat{g}^*(p) \)
3. \( I - \hat{g}^{(1)}(p)\hat{g}(p) \) is non-negative definite.

For the scattering formalism, assuming that the input-output relationship is \( x = g * f \), the following results are true in the region \( s > 0 \) [25]:

- \( \hat{g}(p) \) is holomorphic
- \( \hat{g}(p^*) = \hat{g}^*(p) \)
- \( I - \hat{g}^{(1)}(p)\hat{g}(p) \) is non-negative definite.

where \( I \) is the identity operator in the appropriate dimension. In the above, \( \hat{z} \) is called a positive real matrix whereas \( \hat{g} \) is called a bounded-real matrix [25], and the two concepts are closely related to each other. In the preceding section, an integral representation was introduced for functions which have non-negative imaginary parts and are holomorphic in the upper half plane – these are Herglotz functions. Functions possessing similar properties in the right half are called positive functions and when such functions represent real transfer functions in the time domain, then they are called positive real functions. Positive real functions are the scalar analogues of positive real matrices which were discussed in this section under the context of passivity in the immittance form. As mentioned earlier, the two concepts – Herglotz functions/matrices and positive functions/matrices – are connected to each other through a simple mapping. The former is defined in terms of \( k = \omega + is \) whereas the latter is defined in terms of \( p = s + i\omega \). Thus, it becomes evident that for an input-output relationship whose passivity statement may be written in the immittance form, the Laplace transform of its transfer is closely related to a Herglotz matrix. The connection is the following:

A positive real matrix with properties mentioned in the last sub-section has an integral representation [31]:

\[
\hat{W}(p) = -iC + Dp + \int_{-\infty}^{+\infty} \frac{p}{p^2 + r^2} (1 + r^2) d\nu(r) + \int_{-\infty}^{+\infty} \frac{1 - p^2}{p^2 + r^2} (1 + r^2) dL(r)
\]

In this representation, \( C \) is an \( n \times n \) skew-symmetric matrix with pure imaginary elements, \( D \) is an \( n \times n \) non-negative definite symmetric matrix with real elements, \( \nu \) is a symmetric matrix with real elements with bounded variations which are odd functions of \( r \), and \( L \) is a skew-symmetric matrix with real elements with bounded variations which are even functions of \( r \). The above representation for positive real matrices can be transformed into a representation for Herglotz matrices through a simple mapping. To be more specific, following the way used in Beltrami [33] to derive the integral representation for positive real matrices, one can apply the introduced mappings through the steps and find an integral representation for matrices which possess the properties of Herglotz like:

\[
\hat{W}(k) = A - iBk + \int_{-\infty}^{+\infty} \frac{1 + k^2}{r^2 - k^2} r d\nu(r) + i \int_{-\infty}^{+\infty} \frac{1 + r^2}{k^2 - r^2} kdL(r)
\]
Here, $A$ is a symmetric $n \times n$ matrix with real elements and $B$ is a non-negative definite and skew-symmetric matrix with real members. $\nu$ ($L$) is a symmetric (skew-symmetric) matrix whose elements are real with bounded variations and even (odd) functions of $r$.

We can now take a similar set of steps as we took in section III and check whether the inverse transform of Eq. (23) is a finite order derivative of a continuous matrix. Here, we do not reiterate the steps and just mention the inverse Fourier transform of the boundary value of Eq. (23) in the $S'$ topology:

$$\left( w(t), \tilde{\phi}(t) \right) = \left( A\delta(t), \tilde{\phi}(t) \right) - \left( iB\delta(t) + 1, \tilde{\phi}(t) \right) - \left( \int \sin(rt) \theta(t) d\nu(r), \tilde{\phi}(2)(t) \right)$$

$$+ \left( \int \sin(rt) \theta(t) d\nu(r), \tilde{\phi}(t) \right) + \left( \int \cos(rt) \theta(t) dL(r), \tilde{\phi}(t) \right)$$

$$- \left( \int \cos(rt) \theta(t) dL(r), \tilde{\phi}(2)(t) \right) + \left( \theta(t) dL(r), \tilde{\phi}(2)(t) \right)$$

Thus we get:

$$w(t) = A\delta(t) - iB\delta(t) + \frac{d^2}{dt^2} \left[ P_2(t)\theta(t) - P_1(t)\theta(t) + P_1(0)\theta(t) \right] + P_2(t)\theta(t) + P_1(t)\theta(t)$$

Where, $P_2(t) = \int_{-\infty}^{\infty} \sin(\xi t) dF(\xi)$ and $P_1(t) = \int_{-\infty}^{\infty} \cos(\xi t) dF(\xi)$. It can be seen that all the non-zero terms in this equation are real and it is a second order derivative of two matrices with whose elements are continuous functions. Thus $l = 2$ applies here as well which allows us to write the appropriate dispersion relation for $\tilde{W}(\omega)$ as:

$$\tilde{W}(\omega) = -\frac{\omega^2}{\pi i} \left[ \tilde{W}(\omega) + P \left( \frac{1}{\omega} \right) \right] .$$

V. RELATION BETWEEN $C, D, L, B, \rho$ AND Herglotz FUNCTIONS - AN APPEAL TO PASSIVITY.

The results in sections III and IV allow us to derive the correct order generalized Hilbert pairs for the material and metamaterial properties in acoustics and elastodynamics. Consider, for example, the case of elastodynamics whose equation of motion is given by:

$$\sigma_{ij,j} + f_i = \ddot{p}_i,$$

where $\sigma$, $f$ and $p$ are space and time dependant representations of stress, external force and momentum respectively. This equation of motion is augmented with constitutive relations which, in the linear regime, are given by [39]:

$$\epsilon(x, t) = D \ast \sigma = \int D(x, t - \tau) : \sigma(x, \tau) d\tau,$$

$$p(x, t) = \rho \ast \dot{u} = \int \rho(x, t - \tau) \dot{u}(x, \tau) d\tau,$$

where $\epsilon$, $\dot{u}$ are strain and velocity respectively. In the above, $D$ is the time domain compliance tensor and $\rho$ is the time domain density tensor. The field variables and the constitutive tensors are all assumed to belong to appropriate distribution spaces (see [39] for details). The total absorbed energy at time $t$ in a system characterized by Eqs. (27,28) and occupying a region $\Omega$ can be derived as:

$$E(t) = \int_{-\infty}^{t} \frac{dE(\tau)}{dt} d\tau = \Re \frac{1}{2} \int_{-\infty}^{t} d\tau \int_{\Omega} dx \frac{\partial}{\partial \tau} \left[ \sigma(x, \tau) : \epsilon^*(x, \tau) + p(x, \tau) \dot{u}^*(x, \tau) \right].$$

Passivity implies that $E(t) \geq 0 \forall t$. Since the absorbed energy $E(t)$ is related to the power $P(t)$ through $P(t) = dE/dt$ and since the power $P(t)$ may be expressed in terms of the work done by the body forces $f$ and surface tractions $t$, one can eventually arrive at the following relation implied by passivity [39]:

$$\Re \int_{-\infty}^{t} ds \left[ \sigma_{ij}^*(\tau) \dot{t}_{ij}^* + \ddot{p}_i(\tau) \dot{u}_i^* \right] \geq 0$$

(30)
One can view the above relation as an example of the immittance form of passivity. By employing the constitutive relations (28) and keeping in mind that the real part of the equation above is being taken, one can show that the passivity equation implies the following conditions:

\[ \hat{D}^h(p), \hat{\rho}^h(p) \geq 0 \]  
(31)

In the above, the hat represents the Laplace transform, the superscript \( h \) (\( nh \)) represents the hermitian (non-hermitian) part of the tensor, and the inequality is understood in terms of positive semi-definiteness. Thus, passivity implies that \( \hat{D}, \hat{\rho} \) are positive tensors if the Laplace transform is defined with respect to the parameter \( p \). Consequently, \( \hat{iD}(k), \hat{i\rho}(k) \) are Herglotz tensors. This has further consequences due to the general Laplace transform relation \( L( f^{(n)}(t) ) = (-ik)^nf(k) \). For example, from \( \hat{D} = -ik\hat{D} \), we get \( k\hat{D} = \hat{iD} \). Since \( \hat{iD}(k) \) has a positive semi-definite non-hermitian part on account of it being Herglotz, it implies that \( k\hat{D} \) also has a positive semi definite non hermitian part. As a corollary, for real frequency \( \omega \) and a scalar compliance \( D \), this result means that the imaginary part of \( \omega \hat{D} \) must be a non negative quantity due to passivity. Similar results hold for \( \hat{\rho} \) as well. Since \( \hat{iD}(k), \hat{i\rho}(k) \) are Herglotz tensors, it automatically means that the lowest order dispersion relations that one can write on \( \hat{iD}(k), \hat{i\rho}(k) \) is order 2 \( (l = 2) \). Therefore, we have the following dispersion relations:

\[
\begin{align*}
\omega \hat{D}(\omega) &= -\frac{\omega^2}{\pi i} \mathcal{P} \int_{-\infty}^{+\infty} \frac{\omega' \hat{D}(\omega')}{\omega^2(\omega - \omega')^2} d\omega' \\
\omega \hat{\rho}(\omega) &= -\frac{\omega^2}{\pi i} \mathcal{P} \int_{-\infty}^{+\infty} \frac{\omega' \hat{\rho}(\omega')}{\omega^2(\omega - \omega')^2} d\omega'
\end{align*}
\]
(32)

After some algebraic manipulations and using the symmetry relations \( \hat{D}(-\omega) = \hat{D}^*(\omega) \) and \( \hat{\rho}(-\omega) = \hat{\rho}^*(\omega) \), the dispersion relations in Eq. (32) can be re-written only for positive frequencies:

\[
\Re \hat{D}(\omega) = -\frac{\omega}{\pi} \mathcal{P} \int_{0}^{\infty} \frac{\omega^2 \Re \hat{D}(\omega')}{\omega^2(\omega^2 - \omega'^2)} d\omega' ; \quad \Im \hat{D}(\omega) = \frac{\omega}{\pi} \mathcal{P} \int_{0}^{\infty} \frac{\omega^3 \Im \hat{D}(\omega')}{\omega^2(\omega^2 - \omega'^2)} d\omega'
\]
(33)

for the real and imaginary parts of \( \hat{D} \), and:

\[
\Re \hat{\rho}(\omega) = -\frac{\omega}{\pi} \mathcal{P} \int_{0}^{\infty} \frac{\omega^2 \Re \hat{\rho}(\omega')}{\omega^2(\omega^2 - \omega'^2)} d\omega' ; \quad \Im \hat{\rho}(\omega) = \frac{\omega}{\pi} \mathcal{P} \int_{0}^{\infty} \frac{\omega^3 \Im \hat{\rho}(\omega')}{\omega^2(\omega^2 - \omega'^2)} d\omega'
\]
(34)

for the real and imaginary parts of \( \hat{\rho} \). In addition to the above, there are some further interesting conclusions on the analytic structure of the inverse of the compliance tensor – the stiffness tensor \( \hat{C} = \hat{D}^{-1} \) – due to the fact that the inverse of a positive real matrix is also positive real[10].

Since \( \hat{p}\hat{D}(p) \) is a positive real tensor, it follows that \( 1/p\hat{D}(p) \text{ or } \hat{C}(p)/p \) is also a positive real tensor. As a corollary, it also follows that \( \hat{-C}(k)/k \) is Herglotz. At this point, the positive realness of \( \hat{C}(p)/p \) immediately means that the lowest order dispersion relation for \( \hat{C} \) is 3 \( (l = 3) \). This result is also provable by employing the passivity statement in the immittance form, this time by assuming the strain tensor, \( \epsilon \), as the input (see Appendix II). Thus, the lowest order dispersion relations applicable to \( \hat{C}(\omega) \), based purely upon passivity, are:

\[
\Re \hat{C}(\omega) = -\frac{\omega^3}{\pi} \mathcal{P} \int_{0}^{\infty} \frac{\omega^3 \Re \hat{C}(\omega')}{\omega^2(\omega^2 - \omega'^2)} d\omega' ; \quad \Im \hat{C}(\omega) = \frac{\omega^3}{\pi} \mathcal{P} \int_{0}^{\infty} \frac{\omega^3 \Im \hat{C}(\omega')}{\omega^2(\omega^2 - \omega'^2)} d\omega'
\]
(35)

Some important points need to be made here. Our analysis shows that passivity in elastodynamics implies that \( \omega \hat{D}(\omega) \) is Herglotz and in this conclusion, the compliance tensor is analogous to the electrical permittivity tensor \( \epsilon \) of electromagnetism which also has the same Herglotz behavior[29]. Bernland et al. [29] have shown that \( \omega \epsilon(\omega) \) is Herglotz but then they go on to derive a zero order dispersion relation on \( \epsilon(\omega) - \epsilon(\infty) \). However, it must be noted that in doing so, they assume further restrictions on the behavior of \( \epsilon(\omega) \) than asserted merely by passivity. These include not only continuity and boundedness but also that \( \epsilon(\omega) - \epsilon(\infty) \) goes down as \( \mathcal{O}(1/\omega) \) as \( |\omega| \to \infty \). Similarly, Muheleinstein et al. [6] assert a zero order dispersion relation on the Willis tensor. It is likely that similar constraints are implied in their results as well but they are not explicitly clarified. However, it should be noted that these constraints,
while reasonable, follow from neither passivity nor causality. Passivity implies causality and, by itself, only implies that order 1 dispersion relations apply to the compliance tensor of elastodynamics and the electrical permittivity tensor of electromagnetism.

Similarly to the treatments applied to a general elastodynamic case, a slightly simpler example can be considered which allows one to evaluate the effect of passivity on the bulk modulus $B$. We first express Eq. (30) in an alternate form involving the deviatoric and volumetric parts of stress and strain:

$$
\sigma_{ij} \dot{\epsilon}_{ij} = (s_{ij} - M \delta_{ij}) \left( \dot{\epsilon}_{ij} + \frac{\dot{d}}{3} \delta_{ij} \right) ^* \tag{36}
$$

Here, $s_{ij}$ is the deviatoric part of the stress tensor and $M = -\frac{\sigma_{kk}}{3}$ is the volumetric part. Similarly, $\dot{\epsilon}_{ij}$ is the first derivative of the deviatoric part of the strain tensor and $\dot{d} = \dot{\epsilon}_{kk}$ is the time derivative of the volumetric part. At this point, the passivity statement (30) can be employed only keeping the volumetric effects. By employing the constitutive relation between $d$ and $M$, $M(t, x) = -B(t, x) \ast d(t, x)$, and considering velocity and the volumetric strain separately as the inputs to the system, we arrive at the result that the Bulk modulus behaves in a similar fashion as the stiffness tensor and the inverse of the Bulk modulus has the same behavior as the compliance tensor. More precisely, $l = 3$ for $\hat{B}$ and $l = 1$ for $1/\hat{B}$.

The above analysis can be extended to materials characterized by passive linear Willis tensors – a class of constitutive relations which encompasses both the elastodynamic and acoustic cases \[6, 42–54\]. The details of the following analysis are given in Srivastava [39] and here we outline only the most pertinent points. Linear Willis materials are those for which the constitutive relations exhibit a coupled form. Defining the stress and the velocity as the inputs (represented in the vector $\mathbf{w}(t)$) and the strain and momentum as the outputs (represented in the vector $\mathbf{v}(t)$) of the system, and noting that the elements of the input vectors are assumed to be in $D$, one can write a single convolution relation for a linear, real, time-invariant and causal Willis material:

$$
\mathbf{v}(t) = \int_{-\infty}^{\infty} L(t-s) \mathbf{w}(s) ds \tag{37}
$$

In Eq. (37), $\mathbf{v} = \begin{pmatrix} \epsilon \\ p \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} \sigma \\ u \end{pmatrix}$. Moreover, the kernel of the integral in Eq. (37), is a $n \times n$ matrix whose elements are distributions of slow growth. At this point, the requirement enforced by the passivity can be represented like:

$$
E(t) = \Re \int_{-\infty}^{t} \mathbf{w}^\dagger(s) \dot{\mathbf{v}}(s) ds \geq 0 \tag{38}
$$

By applying the convolution relation (37), one can write:

$$
E(t) = \Re \int_{-\infty}^{t} ds \mathbf{w}^\dagger(s) \int_{-\infty}^{\infty} \hat{L}(\nu) \mathbf{w}(s-\nu) d\nu \geq 0 \tag{39}
$$

where,

$$
\hat{L} = \begin{pmatrix} D & S_1 \\ S_2 & \rho \end{pmatrix}
$$

is the coupled Willis constitutive tensor. In the above, $D$ and $\rho$ are compliance and density tensors and $S_1$ and $S_2$ are coupling tensors. Employing a similar method which was used for materials which are not of Willis type to take the integrands to the Laplace domain, one can conclude that passivity implies:

$$
y^\dagger \hat{L}^h(p) y \geq 0 \tag{40}
$$

The rest of the analysis follows from the previous concerns. Specifically, we arrive at the conclusion that $p\hat{L}(p)$ is a positive real tensor and that $k\hat{L}(k)$ is Herglotz. Thus, it follows that $l = 1$ applies to $\hat{L}$ and $l = 3$ applies to $L^{-1}$.

**VI. DISPERSION RELATIONS FOR $\kappa, n'$**

Appropriate dispersion relations may also be derived for the wavenumber $\kappa$ and the refractive index $n'$ which emerge in wave propagation problems in acoustics and elastodynamics, however, the treatment does not parallel the
one which we employed for constitutive tensors. Analyticity and non-negative imaginary part in the upper half-plane for \( \kappa(k) \) were shown indirectly by Weaver and Pao [9] by relying on the analyticity of the Green’s function in a simple wave propagation problem (Fig. 2). In this problem, an input plane wave given by \( \hat{f}(k) = e^{-ikt} \) is incident on a slab at \( z = 0 \). As it travels a distance \( z \) in the slab, it is transformed into a form \( \hat{u}(k) = A(k)e^{i(\kappa z - kt)} \) where \( A(k) \) encapsulates both the phase and amplitude modifications of the wave as it travels in the slab. The relation between \( \hat{u}(k) \) and \( \hat{f}(k) \) is, therefore, \( \hat{u}(k) = A(k)\hat{f}(k) \) where \( A(k)e^{i\kappa z} = \hat{g}(k, z) \) is the Green’s function of the problem.

Weaver and Pao used two slabs made of the same material (Fig. 2) but of thicknesses \( z_0, z_1 = z_0 + d \) to arrive at the relation \( \kappa = -\frac{i}{d} \ln \frac{\hat{g}(k, z_1)}{\hat{g}(k, z_0)} \). They showed that, given the analyticity of the Green’s function, the analyticity of the wavenumber follows. In addition, passivity requires that the energy contained in the wave, as it travels through the slab, must be monotonically non-decreasing. This results in a passivity statement in the scattering form:

\[
\int_{-\infty}^{t} (f^*f - u^*u) \, dt' \geq 0, \forall \, t
\]  

(41)

From this passivity statement, Weaver and Pao concluded that the wavenumber has a non-negative imaginary part in the upper half. Analyticity combined with the non-negative imaginary part in the upper half ensure that \( \kappa \) is a Herglotz function but not necessarily a symmetric one, which allowed Weaver and Pao to derive a dispersion relation for it.

![FIG. 2. Schematic of the propagation of a plane wave through a thin slab.](image)

Knowing that \( \kappa \) is a Herglotz function and it is not necessarily symmetric immediately implies that the lowest dispersion relation that one can write for \( \kappa \) is 2 \((l = 2)\):

\[
\kappa(\omega) = -\frac{\omega^2}{\pi i} \mathcal{P} \int_{-\infty}^{+\infty} \frac{\kappa(\omega')}{\omega'^2(\omega - \omega')} \, d\omega'
\]  

(42)

As a final point, we can consider the dispersion relations on the refractive index \( n' \). For acoustic waves, for instance, the phase velocity \( c_p \) is related to the Bulk modulus and density through the relation \( c_p = \sqrt{B/\rho} \) and the wavenumber is related to the phase velocity as \( \kappa = k/c_p \). Referring back to the analysis associated with Fig. (2), we now have the Green’s function relationship \( G(z, k) = A(k)e^{i\kappa n'(k)z} \). Since the Green’s function satisfies the passivity relationship in the scattering form, it follows that \( kn'(k) \) is a Herglotz function\([9]\). Therefore, we can conclude that the lowest dispersion relation order for \( n' \) coming from passivity is 1, \((l = 1)\):

\[
n'(\omega) = -\frac{\omega}{\pi i} \mathcal{P} \int_{-\infty}^{+\infty} \frac{n'(\omega')}{\omega'(\omega - \omega')} \, d\omega'
\]  

(43)

VII. CONCLUSIONS

In this paper, we have derived dispersion relations for the material tensors and transfer functions of acoustic and elastodynamic materials and metamaterials. Generally this requires one to know the behavior of these properties in the
limit $|\omega| \to \infty$, however, this information is not easy to ascertain. In fact, it may not even make sense to talk about this asymptotic limit as the continuum approximation breaks down in the high frequency regime. To sidestep this issue, we resort to the principle of passivity and its connection to Herglotz functions and positive functions. Our analysis in this paper concerns only passive systems – systems which encompass no sources of energy. Since Herglotz functions have well known dispersion relations, our overarching aim in this paper is to relate the various transfer functions and material properties of passive acoustic and elastodynamic systems to Herglotz functions. However, for completeness sake, we also include some relevant and classical results. We first clarify that the inverse Fourier transform of any Herglotz function is a second order derivative of a continuous function. This immediately establishes the classical result that dispersion relations of order 2 apply to Herglotz functions. Then we describe the immittance and scattering forms of passivity, especially clarifying the classical result which connects the transfer functions appearing in immittance forms of passivity to positive functions. We subsequently clarify the connection between positive functions and Herglotz functions. Thus, we clarify the connections between positive functions, Herglotz functions, passivity, and dispersion relations. These developments then allow us to derive the appropriate dispersion relations on wavenumber ($\kappa$), refractive index ($n$), density ($\rho$) and its inverse, stiffness ($C$) and compliance ($D$) tensors, the Bulk modulus ($B$), and finally the broader generalization of these properties which is now known as the Willis tensor ($L$). Our analysis shows that based upon passivity alone, dispersion relations of minimum order 1 apply to the Fourier transforms of $D$, $\rho$, $n'$, and the inverse of $B$, order 3 apply to $C$, $B$, and the inverse of $\rho$, and order 2 applies to $\kappa$.

VIII. APPENDIX

A. Appendix I

In this appendix, we present an introduction about an integral representation called "Herglotz" for two related functions – those which are holomorphic in the upper half of the complex domain and have positive imaginary parts and those which are holomorphic in the right half plane and have positive real parts. Cauchy’s integral formula is an integral representation which describes the value of a holomorphic function at a certain point in terms of an integral over the boundary of a closed contour containing that point. Assuming a function, $f(z)$, analytic in the unit circle in the complex plane, Cauchy’s integral gives:

$$f(z) = \frac{1}{2\pi i} \int_{|\xi|=1} \frac{f(\xi)}{\xi - z} d\xi$$ (44)

If $f$ has a positive real part inside the unit circle, then it is possible to represent the integral formula in the following representation [28]:

$$f(z) = i \Im(\{f(0)\}) + \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{is} + z}{e^{is} - z} d\nu(s)$$ (45)

where where $\nu(s)$ on $U$ is called a Borel measure and $U = [0, 2\pi]$. The measure is real and uniformly bounded. Eq. (45) can be mapped to any of the half planes under discussion in this paper using conformal mappings. For instance, to obtain the so called Herglotz representation [29, 35], we first map the function to the upper half plane and then define another transformation $h = if$. Since $f$ has a positive real part inside the unit circle, $h$ will have a positive imaginary part in the upper half. The composite mapping function which accomplishes this is:

$$k = \frac{1}{2} \left( 1 + \frac{1}{1 - z} \right)$$ (46)

where $k$ is a complex number, and the function, after the transformations is:

$$h(k) = \Im\{f(0)\} + \frac{1}{2\pi} \nu(0) k + \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{1}{r - k} - \frac{r}{1 + r^2} \, d\mu(r)$$ (47)

where, $d\nu = \frac{2}{1 + r^2} d\mu(r)$, $e^{is} = \frac{r^2}{r^2 + 1}$, and $-\infty < r < \infty$ is real. Equivalently, Eq. (47) can be re-written as:

$$h(k) = a + bk + \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{1 + rk}{r - k} \, d\nu(r)$$ (48)
which is the usual Herglotz representation where $a$ is real, $b$ is non-negative and $\nu$ is a positive bounded measure. To get to the positive representation ($p-$representation), one can simply map the unit circle to the right half with:

$$p \equiv \frac{1 + z}{1 - z}$$  \hspace{1cm} (49)

which yields [39]:

$$h(p) = c + dk + \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{irp - 1}{ir - p} dv(r)$$  \hspace{1cm} (50)

where $d$ is non-negative and $c$ is a pure imaginary constant. Similar to the symmetric Herglotz representation, by considering the symmetry relation $h(\bar{p}) = \bar{h}(p)$ (if $h(p)$ is real in its time domain), one can write:

$$h(p) = dp + \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{p(1 + r^2)}{p^2 + r^2} dv(r)$$  \hspace{1cm} (51)

which is the so called positive-real function representation, equivalent to the symmetric Herglotz representation.

**B. Appendix II**

In this appendix, following the discussion on the dispersion relation for the stiffness matrix, we review the application of the passivity statement in its immittance form when the strain tensor is the input instead of the stress tensor. To avoid mentioning unnecessary equations here, we immediately start from Eq. (30). Using the constitutive relation in the time domain, the first term of Eq. (30), can be re-written:

$$\Re \int_{-\infty}^t ds \left[ \dot{\epsilon}_{ij}(s) \int_{-\infty}^{+\infty} C_{ijkl} \epsilon_{kl}(s - \nu) d\nu \right] \geq 0$$  \hspace{1cm} (52)

Since we are not considering a restrictive problem in which the inputs are members of $L^2$, instead of using the Plancherel’s theorem to transform the stiffness matrix to the Laplace domain, we go through the process which Zemanian[31] and Srivastava[39] employed in their derivations. Recalling that the stiffness matrix here is considered to be a tempered distribution, to make sure that the Laplace transform is definable, the input $\epsilon(s)$ is expressed as $\epsilon_{ij}(s) = \epsilon_{ij} \phi(s)$, where $\epsilon_{ij}$ is a constant matrix and $\phi(s) \in \mathcal{S}$. Since the members of $\mathcal{S}$ are infinitely differentiable and continuous, one can also define, $\dot{\epsilon}_{ij}(s) = \epsilon_{ij} \dot{\phi}(s)$. Substituting the introduced expressions for $\epsilon_{ij}$ and $\dot{\epsilon}_{ij}$ into Eq. (52), and by assuming $\phi(s) = e^{ps}$, we arrive at:

$$\Re \left[ p^* \hat{C}_{ijkl}(p) \right] \geq 0$$

Or, equivalently:

$$\Re \left[ \frac{1}{p} \hat{C}_{ijkl}(p) \right] \geq 0$$  \hspace{1cm} (53)

Instead of setting $\phi(s) = e^{ps}$ one can also use $\phi(s) = e^{-iks}$ to have the Laplace transform of the stiffness matrix in the $k$-notation:

$$\Re \left[ \frac{1}{-ik} \hat{C}_{ijkl}(k) \right] \geq 0$$  \hspace{1cm} (54)

In essence, we have shown that $\hat{C}(p)/p$ is positive.

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