Coefficient Estimates and the Fekete–Szegö Problem for New Classes of $m$-Fold Symmetric Bi-Univalent Functions

Georgia Irina Oros 1,*,† and Luminița-Ioana Cotîrlă 2,‡

1 Department of Mathematics and Computer Science, University of Oradea, 410087 Oradea, Romania
2 Department of Mathematics, Technical University of Cluj-Napoca, 400114 Cluj-Napoca, Romania
* Correspondence: georgia_oros_ro@yahoo.co.uk
† These authors contributed equally to this work.

Abstract: The results presented in this paper deal with the classical but still prevalent problem of introducing new classes of $m$-fold symmetric bi-univalent functions and studying properties related to coefficient estimates. Quantum calculus aspects are also considered in this study in order to enhance its novelty and to obtain more interesting results. We present three new classes of bi-univalent functions, generalizing certain previously studied classes. The relation between the known results and the new ones presented here is highlighted. Estimates on the Taylor–Maclaurin coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ are obtained and, furthermore, the much investigated aspect of Fekete–Szegö functional is also considered for each of the new classes.

Keywords: $m$-fold symmetric; bi-univalent functions; analytic functions; Fekete–Szegö functional; coefficient bounds; coefficient estimates

1. Introduction and Preliminary Results

The study of bi-univalent functions has its origins in a 1967 paper published by Lewin [1], where he introduced and first investigated the class of bi-univalent functions. It was then proved that $|a_2| < 1.51$, with the estimation being further investigated only a few years later [2,3]. The definition of this class involves the well-known class of functions $A$ consisting of the functions having the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the open unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$ and normalized by the conditions $f(0) = 0$, $f'(0) = 1$. The subclass $S \subset A$ is formed of functions in class $A$ which are univalent in $U$.

In [4], the Koebe One-Quarter Theorem stated guarantees that a disk of radius $1/4$ is contained in $f(U)$ for every univalent function $f$. Hence, every function $f \in S$ admits an inverse function $f^{-1}$, defined as follows:

$$f^{-1}(f(z)) = z, z \in U$$

and

$$f(f^{-1}(w)) = w, |w| < r_0(f), r_0(f) \geq 1/4,$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (3a_2^3 - 5a_2 a_3 + a_4)w^4 + \ldots$$

A function $f \in A$ is said to be bi-univalent in $U$ if both $f$ and $f^{-1}$ are univalent in $U$. 

---

Citation: Oros, G.I.; Cotîrlă, L.-I. Coefficient Estimates and the Fekete–Szegö Problem for New Classes of $m$-Fold Symmetric Bi-Univalent Functions. Mathematics 2022, 10, 129. https://doi.org/10.3390/math10010129

Publisher’s Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Copyright: © 2022 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).
Let \( \Sigma \) denote the class of all bi-univalent functions in \( U \) given by (1).

The interest in bi-univalent functions resurfaced in 2010 when a paper authored by H. M. Srivastava et al. in [5] was published. It opened the door for many interesting developments on the topic. Soon, other new subclasses of bi-univalent functions were introduced [6,7], and special classes of bi-univalent functions were investigated such as Ma-Minda starlike and convex functions [8], analytic bi-Bazilević functions [9] and recently a family of bi-univalent functions associated with Bazilević functions and the \( \lambda - \) pseudo-starlike functions [10]. Brannan and Clunie’s conjecture [3] was further investigated [11] and subordination properties were also obtained for certain subclasses of bi-univalent functions [12]. New results continued to emerge in the recent years, such as coefficient estimates for some general subclasses of analytic and bi-univalent functions [13–15]. Horadam polynomials were used for applications on Bazilević bi-univalent functions satisfying subordination conditions [16] and for introducing certain classes of bi-univalent functions [17]. Operators were also included in the study as it can be seen in earlier publications [18] and in very recent ones [19]. In 2014, Srivastava et al. [20] defined \( m \)-fold symmetric bi-univalent functions following the concept of \( m \)-fold symmetric univalent functions. In this paper, some important results were proved, such as the fact that each bi-univalent function generates an \( m \)-fold symmetric bi-univalent function for each \( m \in \mathbb{N} \).

A domain \( D \) is said to be \( m \)-fold symmetric if a rotation of \( D \) about the origin through an angle \( 2\pi/m \) carries \( D \) on itself.

A function \( f \) holomorphic in \( D \) is said to be \( m \)-fold symmetric if \( f(e^{2\pi i/m} z) = e^{2\pi i/m} f(z) \).

A function is said to be \( m \)-fold symmetric if it has the following normalized form

\[
f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}, \quad z \in U, \quad m \in \mathbb{N} \cup \{0\}.
\]

(3)

The normalized form of \( f \) is given as in (3) and the series expansion for \( f^{-1}(z) \) is given below [20]:

\[
g(w) = f^{-1}(w) = w - a_{m+1} w^{m+1} + \\
+ [(m + 1)a_{m-1}^2 - a_{2m+1}][w^{2m+1} - \\
- \frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}]w^{3m+1} + \ldots
\]

(4)

Let \( \xi_m \) the class of \( m \)-fold symmetric univalent functions in \( U \) that are normalized by (3).

The functions in the class \( \xi \) are one-fold symmetric.

Examples of \( m \)-fold symmetric bi-univalent functions are

\[
\{ \left[ \frac{z^m}{1-z^m} \right]^{1/2}; \quad -\log(1-z^m) \}^{1/2}; \quad \frac{1}{3} \log(1-z^m) \}^{1/3};
\]

Interesting results regarding \( m \)-fold symmetric bi-univalent functions were published in the same year when this notion was introduced [21]; this continued to appear in the following years [22–25] and is still researched today [26,27], proving that the topic remains in development.

The Fekete–Szegö problem is the problem of maximizing the absolute value of the functional \( |a_3 - \mu a_2^2| \).

The Fekete–Szegö inequalities introduced in 1933, see [28], preoccupied researchers regarding different classes of univalent functions [29,30]; hence, it is obvious that such inequalities were obtained regarding bi-univalent functions too and very recently published papers can be cited to support the assertion that the topic still provides interesting results [31–33]. Inspiring new results emerged when quantum calculus was involved in the studies, as can be seen in many papers [34,35] and in studies published very recently [36–40]. Some elements of the (p; q)-calculus must be used for obtaining the original results contained in this paper. Further information can be found in [34,35].
Definition 1 ([34]). Let \( f \in \mathcal{A} \) given by (1) and \( 0 < q < p \leq 1 \). Then, the \((p, q)\)-derivative operator or \( p, q \)-difference operator for the function \( f \) of the form (1) is defined by

\[
D_{p,q}f(z) = \frac{f(pz) - f(qz)}{(p-q)z}, z \in U^* = U - \{0\}
\]  

(5)

and

\[
(D_{p,q}f)(0) = f'(0)
\]  

(6)

provided that the function \( f \) is differentiable at 0.

From relation (2), we deduce that

\[
D_{p,q}f(z) = 1 + \sum_{k=2}^{\infty} [k]_{p,q} a_k z^{k-1}
\]  

(7)

where the \((p, q)\)-bracket number or twin-basic is given by

\[
[k]_{p,q} = \frac{p^k - q^k}{p-q} = p^{k-1} + p^{k-2}q + p^{k-3}q^2 + \ldots + pq^{k-2} + q^{k-1}, p \neq q
\]

which is a natural generalization of the \( q \)-number.

Additionally, \( \lim_{p \to 1^-} [k]_{p,q} = [k]_q = \frac{1-q^k}{1-q} \).

Definition 2 ([41]). Let \( f \in \mathcal{A}, 0 \leq d < 1 \) and \( s \geq 1 \) is real. Then, \( f \in L_{s}(d) \) of \( s \)-pseudostarlike function of order \( d \) in \( U \) if and only if

\[
\Re\left(\frac{z|f'(z)|^s}{f(z)}\right) > d.
\]

Lemma 1 ([4,42]). Let the function \( w \in \mathcal{P} \) be given by the following series \( w(z) = 1 + w_1 z + w_2 z^2 + \ldots, z \in U \), where we denote by \( \mathcal{P} \) the class of Carathéodory functions analytic in the open disk \( U \),

\[
\mathcal{P} = \{ w \in \mathcal{A} | w(0) = 1, \Re(w(z)) > 0, z \in U \}.
\]

The sharp estimate given by \( |w_n| \leq 2, n \in \mathbb{N}^* \) holds true.

The tremendous impact quantum calculus has had when associated to univalent functions theory is nicely highlighted in the recent review paper [43].

In the next section of the paper, the original results obtained by the authors are presented in three definitions of new subclasses of bi-univalent functions and theorems concerning coefficient estimates and Fekete–Szego functional for the newly defined classes defined by \((p, q)\)-derivative operator given in relations (5)–(7). The connection with previously known results is revealed in some remarks following each result presented.

2. Main Results

Definition 3. The class \( \mathcal{N} - FS_{\Sigma_m}^{p,q,s,n}(d) \) \((m \in \mathbb{N}, 0 < q < p \leq 1, s \geq 1, 0 < d \leq 1, (z, w) \in U, 0 \leq n \leq 1) \) contains all the functions \( f \) given by (3) if the following conditions are satisfied:

\[
\left\{ f \in \Sigma_n \right\}
\]

\[
|\arg\{(1-n)(D_{p,q}f(z)) + n[z(D_{p,q}f(z))' + D_{p,q}f(z)](D_{p,q}f(z))^{-1}\}| < \frac{d\pi}{2}, (z \in U)
\]

and

\[
|\arg\{(1-n)(D_{p,q}g(w)) + n[w(D_{p,q}g(w))' + D_{p,q}g(w)](D_{p,q}g(w))^{-1}\}| < \frac{d\pi}{2}
\]

(8)

(9)
where the function \(g\) is given by (4).

**Remark 1.** When \(n = 0\) and \(s = 1\), we obtain the class \(H_{\Sigma}^{p,q,n}\) introduced in [15].

**Remark 2.** In the case when \(p = 1, n = 0, s = 1, m = 1\) (one fold – case) we have \(\lim_{q \to 1} \mathcal{N} - FS_{\Sigma,1}^{p,q,n}(d) = FS_{\Sigma}(d)\) and we obtain the class which was introduced by Srivastava et al. in [5].

The next theorem gives coefficient bounds for the functions belonging to the class \(\mathcal{N} - FS_{\Sigma,m}^{p,q,n}(d)\).

**Theorem 1.** Let \(f\) be a function in the class \(\mathcal{N} - FS_{\Sigma,m}^{p,q,n}(d), (m \in \mathbb{N}, 0 < q < p \leq 1, s \geq 1, 0 < d \leq 1, (z, w) \in U, 0 \leq n \leq 1)\), which has the form (3). Then,

\[
|a_{n+1}| \leq \frac{2d}{\sqrt{(m+1)d[2m+1]_{p,q}(s+2mn) + (s+2mn)(s-1)[m+1]_{p,q}^2d - (d-1)(s+mn)^2|m+1|_{p,q}^2}}
\]

and

\[
|a_{2m+1}| \leq \frac{2d}{(s+2mn)[2m+1]_{p,q}} + \frac{2(m+1)d^2}{(s+mn)^2|m+1|_{p,q}^2}.
\]

**Proof.** If we use relations (8) and (9), we obtain

\[
(1 - n)(D_{p,q}f(z))^s + n[z(D_{p,q}f(z))'] + D_{p,q}f(z)](D_{p,q}f(z))^{s-1} = [\alpha(z)]^d, z \in U
\]

and

\[
(1 - n)(D_{p,q}g(w))^s + n[w(D_{p,q}g(w))'] + D_{p,q}g(w)](D_{p,q}g(w))^{s-1} = [\beta(w)]^d, w \in U
\]

where \(\alpha(z)\) and \(\beta(w)\) in \(\mathcal{P}\) are given by

\[
\alpha(z) = 1 + \alpha_{m}z^m + \alpha_{2m}z^{2m} + \alpha_{3m}z^{3m} + \ldots
\]

and

\[
\beta(w) = 1 + \beta_{m}w^m + \beta_{2m}w^{2m} + \beta_{3m}w^{3m} + \ldots
\]

If we compare the coefficients in relations (12) and (13), we have

\[
(s + mn)[m + 1]_{p,q} a_{m+1} = d\alpha_{m},
\]

\[
(s + 2mn)[2m + 1]_{p,q} a_{2m+1} + \frac{(s + 2mn)(s-1)[m+1]_{p,q}^2 a_{m+1}^2}{2} = d\alpha_{2m} + \frac{d(d-1)}{2} \alpha_{m}^2,
\]

\[
- (s + mn)[m + 1]_{p,q} a_{m+1} = d\beta_{m},
\]

\[
(s + 2mn)[2m + 1]_{p,q}([m+1]_{p,q}^2 a_{m+1}^2 - a_{2m+1}) + \frac{(s + 2mn)(s-1)[m+1]_{p,q}^2 a_{m+1}^2}{2} = d\beta_{2m} + \frac{d(d-1)}{2} \beta_{m}^2.
\]

From relations (16) and (18), we obtain

\[
\alpha_{m} = -\beta_{m}
\]
and
\[ 2(s + mn)^2[m + 1]_p q^2 \alpha^2_{m+1} = d^2(\alpha^2_m + \beta^2_m) \] (21)

Now, from relations (17), (19) and (21), we obtain the next relation
\[ (s + 2mn)(s - 1)[m + 1]_p q^2 \alpha^2_{m+1} + (m + 1)(s + 2mn)[2m + 1]_p q^2 \alpha^2_{m+1} \]
\[ = d(\alpha_{2m} + \beta_{2m}) + \frac{d(d - 1)}{2} \left[ \frac{2(s + mn)^2[m + 1]_p q^2 \alpha^2_{m+1}}{2} \right]. \]
Therefore, we obtain
\[ \alpha^2_{m+1} = \frac{d(\alpha_{2m} + \beta_{2m})}{(m + 1)(s + 2mn)[2m + 1]_p q^2 + (s + 2mn)(s - 1)[m + 1]_p q^2 - (d - 1)(s + mn)^2[m + 1]_p q^2}. \]

Now, for the coefficients \( \alpha_{2m} \) and \( \beta_{2m} \), if we apply Lemma 1, we obtain relation (10):
\[ |a_{m+1}| \leq \frac{2d}{\sqrt{d(m + 1)(s + 2mn)[2m + 1]_p q + (s + 2mn)(s - 1)[m + 1]_p q^2 - (d - 1)(s + mn)^2[m + 1]_p q^2}}. \]

If we use relations (17) and (19), we obtain
\[ 2(s + 2mn)[2m + 1]_p q^2 (m + 1)\alpha_{2m+1} - (m + 1)(s + 2mn)[2m + 1]_p q^2 \alpha_{m+1} = \]
\[ = d(\alpha_{2m} - \beta_{2m}) + \frac{d(d - 1)}{2} (\alpha^2_m - \beta^2_m). \] (22)

From relations (20)–(22), we obtain
\[ \alpha_{2m+1} = \frac{d(\alpha_{2m} - \beta_{2m})}{2(s + 2mn)[2m + 1]_p q} + \frac{d^2(\alpha^2_m + \beta^2_m)(m + 1)}{4(s + mn)^2[m + 1]_p q^2}. \] (23)

If we apply Lemma 1 for the coefficients \( \alpha_m, \alpha_{2m}, \beta_m, \beta_{2m} \) and relation (23), we obtain relation (11):
\[ |a_{2m+1}| \leq \frac{2d}{(s + 2mn)[2m + 1]_p q} + \frac{2(m + 1)d^2}{(s + mn)^2[m + 1]_p q^2}. \]

\[ \square \]

**Definition 4.** The class \( \mathcal{N} = FS^{P,q,s,n}_\Sigma_m(e) \) \((0 < q < p \leq 1, 0 \leq e < 1, m \in \mathbb{N}, s \geq 1, (z, w) \in U, 0 \leq n \leq 1) \) contains all the functions \( f \) given by (3) if the following conditions are satisfied:

\[
\left\{ f \in \Sigma_m \right\} \begin{array}{l}
\Re\{(1 - n)(D_{p,q}f(z))^s + n[z(D_{p,q}f(z))' + D_{p,q}f(z)](D_{p,q}f(z))^{s-1}\} > e, z \in U \\
\Re\{(1 - n)(D_{p,q}g(w))^s + n[w(D_{p,q}g(w))' + D_{p,q}g(w)](D_{p,q}g(w))^{s-1}\} > e, w \in U,
\end{array}
\] (24)

where the function \( g \) is defined by relation (4).

**Remark 3.** (a) When \( n = 0 \) and \( s = 1 \), we obtain the class \( H^{\alpha,\beta}_\Sigma_m \), which was introduced in [15].

(b) When \( p = 1, m = 1 \) (one fold case), \( n = 0, s = 1 \) and \( \lim_{q \to 1} \mathcal{N} = FS^{P,q,s,n}_\Sigma_m(e) = \mathcal{N} = FS^E_\Sigma_m(e) \), we obtain the class which was introduced by Srinivasava et al. in [5]

The next theorem gives the coefficient bounds for the functions class \( \mathcal{N} = FS^{P,q,s,n}_\Sigma_m(e) \).
Theorem 2. Let $f$ be a function in the class $\mathcal{N} = FS_{\Sigma_m}^{p,q,n}(e)$, $(m \in \mathbb{N}, 0 < q < p \leq 1, s \geq 1, 0 \leq e < 1, (z, w) \in U, 0 \leq n \leq 1)$, which has the form (3). Then,

$$|a_{m+1}| \leq \min\left\{ \frac{2(1-e)}{(s + mn)^2[m + 1]_{p,q}} \sqrt{s + mn(m - 1)[m + 1]_{p,q}^2 + (s + 2mn)[m + 1]_{p,q}^2} \right\}$$  \hspace{1cm} (26)

and

$$|a_{2m+1}| \leq \frac{2(1-e)}{(s + 2mn)[m + 1]_{p,q}}.$$  \hspace{1cm} (27)

Proof. We use relations (24) and (25) to obtain

$$(1 - n)(D_{p,q}f(z))^s + n[z(D_{p,q}f(z))' + D_{p,q}f(z)](D_{p,q}f(z))^{s-1} = e + (1-e)a(z), \quad z \in U$$  \hspace{1cm} (28)

and

$$(1 - n)(D_{p,q}g(w))^s + n[w(D_{p,q}g(w))' + D_{p,q}g(w)](D_{p,q}g(w))^{s-1} = e + (1-e)b(w), \quad w \in U$$  \hspace{1cm} (29)

respectively, where $a(z)$ and $b(w)$ in $\mathcal{P}$ are given by relations (14) and (15).

We compare the coefficients from (28) and (29) and we obtain the following relations:

$$(s + mn)[m + 1]_{p,q}a_{m+1} = (1-e)a_m,$$  \hspace{1cm} (30)

$$(s + 2mn)[m + 1]_{p,q}a_{2m+1} + \frac{(s + 2mn)(m - 1)[m + 1]_{p,q}^2}{2} a_{m+1}^2 = (1-e)a_m,$$  \hspace{1cm} (31)

$$(s + mn)[m + 1]_{p,q}a_{m+1} = (1-e)b_m,$$  \hspace{1cm} (32)

$$(s + 2mn)[m + 1]_{p,q}[(m + 1)a_{m+1}^2 - a_{2m+1}] +$$  \hspace{1cm} (33)

$$+ \frac{(s + 2mn)(m - 1)[m + 1]_{p,q}^2}{2} a_{m+1}^2 = (1-e)b_{2m}.$$

Now, we obtain, from relations (32) and (30),

$$a_m = -b_m$$  \hspace{1cm} (34)

and

$$2(s + mn)^2[m + 1]_{p,q}a_{m+1}^2 = (1-e)^2(a_m^2 + b_m^2).$$  \hspace{1cm} (35)

We obtain, from relations (33) and (31), the next relation

$$(s + 2mn)[m + 1]_{p,q}(m + 1)a_{m+1}^2 + (s + 2mn)(m - 1)[m + 1]_{p,q}a_{m+1}^2 =$$  \hspace{1cm} (36)

$$= (1-e)(a_{2m} + b_{2m}).$$

We apply Lemma 1 for the coefficients $a_m, a_{2m}, b_m, b_{2m}$ and obtain

$$|a_{m+1}| \leq 2 \sqrt{\frac{1-e}{(s + 2mn)[m + 1]_{p,q}(m + 1)(s + 2mn) + (s - 1)(s + 2mn)[m + 1]_{p,q}^2}}$$

and then relation (26) hold. We use relations (33) and (31) to find the bound on $|a_{2m+1}|$, we obtain

$$-(m + 1)(s + 2mn)[m + 1]_{p,q}a_{m+1}^2 + 2(s + 2mn)[m + 1]_{p,q}(m + 1)a_{2m+1} = (1-e)(a_{2m} - b_{2m}).$$  \hspace{1cm} (37)

From relation (37), we obtain

$$a_{2m+1} = \frac{(1-e)(a_{2m} - b_{2m})}{2(s + 2mn)[m + 1]_{p,q}} + \frac{(m + 1)}{2} a_{m+1}^2.$$  \hspace{1cm} (38)
From relation (35), if we substitute the value of $a_{m+1}^2$, we obtain

$$a_{2m+1} = \frac{(1 - c)(\alpha_{2m} - \beta_{2m})}{2(s + 2mn)[2m + 1]_{p,q}^2} + \frac{(m + 1)(1 - c)^2(\alpha_{m}^2 + \beta_{m}^2)}{4(s + mn)^2[m + 1]_{p,q}^2}. \quad (39)$$

Now, we apply Lemma 1 for the coefficients $\alpha_m, \alpha_{2m}, \beta_m, \beta_{2m}$ and relation (39), and we obtain

$$|a_{2m+1}| \leq \frac{2(1 - c)}{(s + 2mn)[2m + 1]_{p,q}^2} + \frac{2(m + 1)(1 - c)^2}{(s + mn)^2[m + 1]_{p,q}^2}.$$

From relations (36) and (38), if we apply Lemma 1, we find that relation (27) holds:

$$|a_{2m+1}| \leq \frac{2(1 - c)}{(s + 2mn)[2m + 1]_{p,q}^2}.$$

In the next theorem, we compute the Fekete–Szegő functional for the class $N^\alpha_m - FS_{\Sigma m}^{\alpha, \beta, \gamma}(d)$.}

**Theorem 3.** Let $f$ be a function of the form (3) be in the class $N^\alpha_m - FS_{\Sigma m}^{\alpha, \beta, \gamma}(d)$. Then,

$$|a_{2m+1} - \rho a_{m+1}^2| \leq \begin{cases} \frac{2d}{(s + 2mn)[2m + 1]_{p,q}^2}, & |r(\rho)| \leq \frac{1}{(s + 2mn)[2m + 1]_{p,q}^2} \\ \frac{4d(s + 2mn)[2m + 1]_{p,q}^2 |r(\rho)|, |r(\rho)| \geq \frac{1}{(s + 2mn)[2m + 1]_{p,q}^2}, \end{cases} \quad (40)$$

where we denote by

$$r(\rho) = \frac{d((m + 1)[m + 1]_{p,q}^2 - 2\rho[m + 1]_{p,q}^2)}{2[2m + 1]_{p,q}^2(s + mn)(m + 1)d[2m + 1]_{p,q}^2 + [m + 1]_{p,q}^2(d(s - 2) + 1)}.$$

**Proof.** From the proof of Theorem 1, we know the values of the coefficients $a_{m+1}^2$ and $a_{2m+1}^2$.

$$a_{2m+1} = \frac{d(a_{2m} - \beta_{2m})}{2(s + 2mn)[2m + 1]_{p,q}^2} + \frac{(m + 1)d^2(\alpha_{m}^2 + \beta_{m}^2)}{4(s + mn)^2[2m + 1]_{p,q}^2}.$$

$$a_{m+1}^2 = \frac{d^2(\alpha_{2m} + \beta_{2m})}{(m + 1)d(s + 2mn)[2m + 1]_{p,q}^2 + d(s + 2mn)(s - 1)[m + 1]_{p,q}^2 - (d - 1)(s + mn)^2[m + 1]_{p,q}^2}.$$

We will start to compute $a_{2m+1} - \rho a_{m+1}^2$.

It follows that

$$a_{2m+1} - \rho a_{m+1}^2 =$$

$$= d\{a_{2m}2(s + 2mn)[2m + 1]_{p,q}^2 +$$

$$+ \frac{d((m + 1)[m + 1]_{p,q}^2 - 2\rho[m + 1]_{p,q}^2)}{2[2m + 1]_{p,q}^2(s + 2mn)(m + 1)d[2m + 1]_{p,q}^2 + [m + 1]_{p,q}^2(d(s - 2) + 1)}$$

$$+ \beta_{2m}\left[\frac{d((m + 1)[m + 1]_{p,q}^2 - 2\rho[m + 1]_{p,q}^2)}{2[2m + 1]_{p,q}^2(s + 2mn)(m + 1)d[2m + 1]_{p,q}^2 + [m + 1]_{p,q}^2(d(s - 2) + 1)} - \frac{1}{2(s + 2mn)[2m + 1]_{p,q}^2}\right]\}.$$
According to Lemma 1 and after some computations, we obtain relation (40):

\[
|a_{2m+1} - \rho a_{m+1}^2| \leq \begin{cases}
\frac{2d}{(s+2mn)[2m+1]_{p,q}} |r(\rho)| & \leq \frac{1}{(s+2mn)[2m+1]_{p,q}} \\
4d(s + 2mn)[2m+1]_{p,q} |r(\rho)|, |r(\rho)| \geq \frac{1}{(s+2mn)[2m+1]_{p,q}}
\end{cases},
\]

\(\square\)

In the next theorem, we compute the Fekete–Szegő functional for the class \(N - FS^{n,\alpha,\beta}_{\Sigma,m}(e)\).

**Theorem 4.** Let \(f\) be a function in the class \(N - FS^{n,\alpha,\beta}_{\Sigma,m}(e)\) which has the form (3). Then

\[
|a_{2m+1} - \rho a_{m+1}^2| \leq \begin{cases}
\frac{2(1+e)}{(s+2mn)[2m+1]_{p,q}} |r(\rho)| & \leq \frac{1}{2(s+2mn)[2m+1]_{p,q}} \\
4(s + 2mn)(1 - e)[2m+1]_{p,q}^2 |r(\rho)|, |r(\rho)| \geq \frac{1}{2(s+2mn)[2m+1]_{p,q}}
\end{cases},
\]

(41)

where we denote by

\[
r(\rho) = \frac{m + 1 - 2\rho}{2(s + 2mn)(2m+1)_{p,q}(m + 1) + (s - 1)[m+1]_{p,q}^2},
\]

**Proof.** We will compute \(a_{2m+1} - \rho a_{m+1}^2\) using the values of the coefficients \(a_{2m+1}^2\) and \(a_{2m+1}\) from the proof of Theorem 2:

\[
a_{2m+1} = \frac{(1-e)(a_{2m} - \beta_{2m})}{2[1 + 2m]_{p,q}(s + 2mn)} +
\]

\[
\frac{(m + 1)(1-e)(a_{2m} + \beta_{2m})}{2(s + 2mn)[2m+1]_{p,q}(m + 1) + (s - 1)[m+1]_{p,q}^2},
\]

\[
a_{m+1}^2 = \frac{(1-e)(a_{2m} + \beta_{2m})}{(s + 2mn)[2m+1]_{p,q}(m + 1) + (s - 1)[m+1]_{p,q}^2}
\]

It follows that

\[
a_{2m+1} - \rho a_{m+1}^2 =
\]

\[
= (1-e)\left\{ a_{2m}\left[ \frac{1}{2[1 + 2m]_{p,q}(s + 2mn)} \right] + \frac{m + 1 - 2\rho}{2(s + 2mn)(m + 1)_{p,q} + (s - 1)[m+1]_{p,q}^2} \right\} +
\]

\[
+ \beta_{2m}\left\{ \frac{m + 1 - 2\rho}{2(s + 2mn)[2m+1]_{p,q}(m + 1) + (s - 1)[m+1]_{p,q}^2} - \frac{1}{2[1 + 2m]_{p,q}(s + 2mn)} \right\}.
\]

According to Lemma 1 and after some computations, we obtain relation (41):

\[
|a_{2m+1} - \rho a_{m+1}^2| \leq \begin{cases}
\frac{2(1+e)}{(s+2mn)[2m+1]_{p,q}} |r(\rho)| & \leq \frac{1}{2(s+2mn)[2m+1]_{p,q}} \\
4(s + 2mn)(1 - e)[2m+1]_{p,q}^2 |r(\rho)|, |r(\rho)| \geq \frac{1}{2(s+2mn)[2m+1]_{p,q}}
\end{cases}.
\]

\(\square\)

**Definition 5.** Let \(l, t : U \rightarrow \mathbb{C}\) be analytic functions with the properties \(\min\{Re(l(z)), Re(t(z))\} > 0\), where \(z \in U, l(0) = t(0) = 1\).
The class \( N - FS_{\Sigma,m}^{d} \) contains all the functions \( f \) given by (3) if the following conditions are satisfied:

\[
(1 - n)(D_{p,q}f(z))^s + n[z(D_{p,q}f(z))' + D_{p,q}f(z)](D_{p,q}f(z))^{s-1} \in l(U), \ z \in U
\] (42)

and

\[
(1 - n)(D_{p,q}g(w))^s + n[w(D_{p,q}g(w))' + D_{p,q}g(w)](D_{p,q}g(w))^{s-1} \in t(U), \ w \in U,
\] (43)

where the function \( g \) is given by (4).

In the next theorem, we obtain coefficient bounds for the functions class \( N - FS_{\Sigma,m}^{d} \).

**Theorem 5.** Given a function \( f \) in the class \( N - FS_{\Sigma,m}^{d} \), we have

\[
|a_{m+1}| \leq \min \left\{ \sqrt{\frac{|l'(0)|^2 + |l'(0)|^2}{2(s + mn)^2[m + 1]_{p,q}^2}}, \sqrt{\frac{|l''(0)| + |l''(0)|}{(s + 2mn)(s - 1)[m + 1]_{p,q} + (s + 2mn)(m + 1)[2m + 1]_{p,q}}}, \right\};
\] (44)

\[
|a_{2m+1}| \leq \min \left\{ \frac{(|l'(0)|^2 + |l'(0)|^2)[(m + 1)[2m + 1]_{p,q} + (s - 1)[m + 1]_{p,q}^2]}{8(s + mn)^2[2m + 1]_{p,q}^2}, \frac{|l''(0)|^2 + |l''(0)|^2}{2(s + 2mn)[2m + 1]_{p,q}^2}, \right\} + \frac{(m + 1)[2m + 1]_{p,q} + (s - 1)[m + 1]_{p,q}^2}{4(s + 2mn)(2m + 1)[p,q]}(1 - s)[m + 1]_{p,q} + (m + 1)[2m + 1]_{p,q} \right\}. \] (45)

**Proof.** Using relations (42) and (43), we obtain the following relations:

\[
(1 - n)(D_{p,q}f(z))^s + n[z(D_{p,q}f(z))' + D_{p,q}f(z)](D_{p,q}f(z))^{s-1} = l(z),
\] (46)

and

\[
(1 - n)(D_{p,q}g(w))^s + n[w(D_{p,q}g(w))' + D_{p,q}g(w)](D_{p,q}g(w))^{s-1} = t(w),
\] (47)

where functions \( l(z) \) and \( t(w) \) having the following forms and satisfy the conditions from Definition 5:

\[
l(z) = 1 + l_1 z + l_2 z^2 + ... \] (48)

\[
t(w) = 1 + t_1 w + t_2 w^2 + ... \] (49)

Substituting relations (48) and (49) into (46) and (47), respectively, and equating the coefficients, we obtain

\[
(s + mn)[m + 1]_{p,q}a_{m+1} = l_1;
\] (50)

\[
(s + 2mn)[2m + 1]_{p,q}a_{2m+1} + \frac{(s + 2mn)(s - 1)[m + 1]_{p,q}^2}{2} a_{m+1}^2 = l_2;
\] (51)

\[
- (s + mn)[m + 1]_{p,q}a_{m+1} = t_1;
\] (52)

\[
(s + 2mn)[2m + 1]_{p,q}((m + 1)a_{m+1}^2 - a_{2m+1}) + \frac{(s + 2mn)(s - 1)[m + 1]_{p,q}^2}{2} a_{m+1}^2 = t_2. \] (53)

We obtain

\[
l_1 = -t_1
\] (54)

and

\[
l_1^2 + t_1^2 = 2(s + mn)^2[m + 1]_{p,q}^2 a_{m+1}^2
\] (55)

from relations (50) and (52).

Adding relations (51) and (53), we obtain

\[
a_{m+1}^2 \{ (s + 2mn)(s - 1)[m + 1]_{p,q}^2 + (s + 2mn)(m + 1)[2m + 1]_{p,q} \} = l_2 + t_2.
\] (56)
Now, from (55) and (56), we obtain

\[ a_{m+1}^2 = \frac{l_1^2 + l_2^2}{2(s + mn)^2[m + 1]_{P,q}} \]  

(57)

and

\[ a_{m+1}^2 = \frac{l_2 + t_2}{(s + 2mn)(s - 1)[m + 1]_{P,q} + (s + 2mn)(m + 1)[2m + 1]_{P,q}}. \]  

(58)

We find, from relations (57) and (58), that

\[ |a_{m+1}|^2 \leq \frac{|l_1'(0)|^2 + |l_1''(0)|^2}{2(s + mn)^2[m + 1]_{P,q}} \]

and

\[ |a_{m+1}|^2 \leq \frac{|l_2'(0)| + |l_2''(0)|}{(s + 2mn)(s - 1)[m + 1]_{P,q} + (s + 2mn)(m + 1)[2m + 1]_{P,q}}. \]

Hence, the coefficient \(|a_{m+1}|\) has the form given in (44).

Next, by subtracting relation (53) from (51), we obtain

\[ 2(s + 2mn)(2m + 1)_{P,q}a_{2m+1} + \]

\[ + \frac{(s + 2mn)(s - 1)[m + 1]_{P,q} - (s + 2mn)[2m + 1]_{P,q}(m + 1)}{2} \]

\[ a_{m+1} = l_2 - t_2. \]

Substituting the value of \(a_{m+1}^2\) from (57) into (59), it follows that

\[ a_{2m+1} = \]

\[ \frac{l_2 - t_2}{2(s + 2mn)(2m + 1)_{P,q}} + \frac{[(m + 1)[2m + 1]_{P,q} - (s - 1)[m + 1]_{P,q}](l_1^2 + l_2^2)}{8(s + mn)^2[m + 1]_{P,q}[2m + 1]_{P,q}}. \]

So,

\[ |a_{2m+1}| \leq \frac{\{(l_1'(0))^2 + (l_1''(0))^2\}(2m + 1)_{P,q}^2(m + 1) - (s - 1)[m + 1]_{P,q}^2}{8(s + mn)^2[m + 1]_{P,q}[2m + 1]_{P,q}} + \frac{|l_2'(0)|^2 + |l_2''(0)|^2}{2(s + 2mn)[2m + 1]_{P,q}}. \]

Using in relation (59), \(a_{2m+1}\) given by (58), we have

\[ a_{2m+1} = \]

\[ \frac{l_2 - t_2}{2(s + 2mn)[2m + 1]_{P,q}} + \frac{(l_2 + t_2)(2m + 1)_{P,q} + (s - 1)[m + 1]_{P,q}}{4(s + 2mn)[2m + 1]_{P,q}^2(m + 1)_{P,q}} \]

\[ \frac{(m + 1)[2m + 1]_{P,q} + (s + 1)[m + 1]_{P,q}[m + 1]_{P,q} + (m + 1)[2m + 1]_{P,q}}{4(s + 2mn)[2m + 1]_{P,q}^2(m + 1)_{P,q}}. \]

It follows that

\[ |a_{2m+1}| \leq \frac{|l_2'(0)| + |l_2''(0)|}{2(s + 2mn)[2m + 1]_{P,q}} + \frac{[(m + 1)[2m + 1]_{P,q} + (s - 1)[m + 1]_{P,q}][l_2'(0)]^2 + |l_2''(0)|}{4(s + 2mn)[2m + 1]_{P,q}^2(m + 1)_{P,q}}. \]

\[ \square \]

3. Conclusions

Following the line of research initiated by Srivastava et al. [20], three new classes of \(m\)-fold bi-univalent functions are introduced in Definitions 3–5. The classes introduced here have previously introduced and studied classes of bi-univalent functions as special cases. For these new classes, coefficient estimates are given regarding the Taylor–Maclaurin
coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ and the Fekete-Szegő problem is investigated for each class in the theorems from the Main Results section. The bounds of coefficient estimates obtained here are not sharp, and thus further investigation is required in order to improve these estimates.

The results are particularly interesting as a result of adding quantum calculus aspects in the research, an approach often seen in recent published and cited studies.

**Author Contributions:** Conceptualization, G.I.O. and L.-I.C.; methodology, G.I.O.; software, L.-I.C.; validation, G.I.O. and L.-I.C.; formal analysis, L.-I.C. and G.I.O.; investigation, L.-I.C.; resources, G.I.O.; data curation, G.I.O.; writing—original draft preparation, L.-I.C.; writing—review and editing, G.I.O.; visualization, L.-I.C.; supervision, G.I.O.; project administration, G.I.O.; funding acquisition, L.-I.C. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** Not applicable.

**Conflicts of Interest:** The authors declare no conflicts of interest.

**References**

1. Lewin, M. On a coefficient problem for bi-univalent functions. *Proc. Am. Math. Soc.* 1967, 18, 63–68. [CrossRef]
2. Netanyahu, E. The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$. *Arch. Rational Mech. Anal.* 1969, 32, 100–112.
3. Brannan, D.A.; Clunie, J.; Kirwan, W.E. Coefficient estimates for a class of starlike functions. *Can. J. Math.* 1970, 22, 476–485. [CrossRef]
4. Duren, P.L. *Univalent Functions, Grundlehren der Mathematischen Wissenschaften*; Springer: New York, NY, USA; Berlin/Hiedelberg, Germany; Tokyo, Japan, 1983.
5. Srivastava, H.M.; Mishra, A.K.; Gochhayat, P. Certain subclasses of analytic and bi-univalent functions. *Appl. Math. Lett.* 2010, 23, 1188–1192. [CrossRef]
6. Frasin, B.A.; Aouf, M.K. New subclasses of bi-univalent functions. *Appl. Math. Lett.* 2011, 24, 1569–1573. [CrossRef]
7. Frasin, B.A.; Aouf, M.K. Coefficient bounds for certain classes of bi-univalent functions. *Hacet. J. Math. Stat.* 2014, 43, 383–389. [CrossRef]
8. Ali, R.M.; Lee, S.K.; Ravichandran, V.; Supramaniam, S. Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions. *Appl. Math. Lett.* 2012, 25, 344–351. [CrossRef]
9. Jahangiri, J.M.; Hamidi, S.G. Faber polynomial coefficient estimates for analytic bi-Bazilevic functions. *Mat. Vesnik* 2015, 67, 123–129.
10. Srivastava, H.M.; Wanas, A.K.; Güney, H.O. New Families of Bi-univalent Functions Associated with the Bazilevič functions and the $\lambda$ – pseudo-starlike functions. *Iran. J. Sci. Technol. Trans. Sci.* 2021, 45, 1799–1804. [CrossRef]
11. Sivasubramanian, S.; Sivakumar, R.; Kanas, S.; Kim, S.-A. Verification of Brannan and Clunie’s conjecture for certain subclasses of bi-univalent functions. *Ann. Polon. Math.* 2015, 113, 295–304. [CrossRef]
12. Çağlar, M.; Aslan, S. Fekete-Szegő inequalities for subclasses of bi-univalent functions satisfying subordinate conditions. *AIP Conf. Proc.* 2016, 1726, 020078. [CrossRef]
13. Srivastava, H.M.; Gaboury, S.; Ghanim, F. Coefficient estimates for some general subclasses of analytic and bi-univalent functions. *Afr. Mat.* 2017, 28, 693–706. [CrossRef]
14. Pál-Szabó, A.O.; Oros, G.I. Coefficient Related Studies for New Classes of Bi-Univalent Functions. *Mathematics* 2020, 8, 1110. [CrossRef]
15. Cotrălă, L.I. New classes of analytic and bi-univalent functions. *AIMS Math.* 2021, 6, 10642–10651. [CrossRef]
16. Wanas, A.K.; Alb Lupas, A. Applications of Horadam Polynomials on Bazilevič Bi-Univalent Function Satisfying Subordinate Conditions. *IOP Conf. Ser. J. Phys. Conf. Ser.* 2019, 1294, 032003. [CrossRef]
17. Abirami, C.; Magesh, N.; Yamini, J. Initial bounds for certain classes of bi-univalent functions defined by Horadam Polynomials. *Abstr. Appl. Anal.* 2020, 2020, 7391058. [CrossRef]
18. Bucur, R.; Andrei, L.; Breaz, D. Coefficient bounds and Fekete-Szegő problem for a class of analytic functions defined by using a new differential operator. *Appl. Math. Sci.* 2015, 9, 1355–1368.
19. Patila, A.B.; Naik, U.H. On Coefficient Inequalities of Certain Subclasses of Bi-Univalent Functions Involving the Sălăgean Operator. *Filomat* 2021, 35, 1305–1313. [CrossRef]
20. Srivastava, H.M.; Sivasubramanian, S.; Sivakumar, R. Initial coefficient bounds for a subclass of m-fold symmetric bi-univalent functions. *Tbilisi Math. J.* 2014, 7, 1–10. [CrossRef]
21. Hamidi, S.G.; Jahangiri, J.M. Unpredictability of the coefficients of m-fold symmetric bi-starlike functions. *Int. J. Math.* 2014, 25, 1450064. [CrossRef]

22. Eker, S.S. Coefficient bounds for subclasses of m-fold symmetric bi-univalent functions. *Turkish J. Math.* 2016, 40, 641–646. [CrossRef]

23. Srivastava, H.M.; Gaboury, S.; Ghanim, F. Initial coefficient estimates for some subclasses of m-fold symmetric bi-univalent functions. *Acta Math. Sci.* 2016, 36, 863–871. [CrossRef]

24. Altinkaya, S.; Yalçın, S. On some subclasses of m-fold symmetric bi-univalent functions. *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.* 2018, 67, 29–36.

25. Sakar, F.M.; Güney, M.O. Coefficient estimates for certain subclasses of m-fold symmetric bi-univalent functions defined by the q-derivative operator. *Konuralp J. Math.* 2018, 6, 279–285.

26. Srivastava, H.M.; Wanas, A.K. Initial Maclaurin coefficient bounds for new subclasses of analytic and m-fold symmetric bi-univalent functions defined by a linear combination. *Kyungpook Math. J.* 2019, 59, 493–503.

27. Bulut, S.; Salehian, S.; Motamednezhad, A. Comprehensive subclass of m-fold symmetric bi-univalent functions defined by subordination. *Afr. Mat.* 2021, 32, 531–541. [CrossRef]

28. Fekete, M.; Szegö, G. Eine bemerkung über ungerade schlichte funktionen. *J. Lond. Math. Soc.* 1933, 8, 85–89. [CrossRef]

29. Kanas, S. An unified approach to the Fekete-Szegö problem. *Appl. Math. Comput.* 2012, 218, 8453–8461. [CrossRef]

30. Dziok, J. A general solution of the Fekete-Szegö problem. *Bound Value Probl.* 2013, 98, 13. [CrossRef]

31. Zaprawa, P. On the Fekete-Szegö problem for classes of bi-univalent functions. *Bull. Belg. Math. Soc. Simon Stevin* 2014, 21, 169–178. [CrossRef]

32. Al-Hawary, T.; Amourah, A.; Frasin, B.A. Fekete–Szegö inequality for bi-univalent functions by means of Horadam polynomials. *Bol. Soc. Mat. Mex.* 2021, 27, 79. [CrossRef]

33. Amourah, A.; Frasin, B.A.; Abdeljaward, T. Fekete-Szegö inequality for analytic and bi-univalent functions subordinate to Gegenbauer polynomials. *J. Funct. Spaces* 2021, 2021, 5574673. [CrossRef]

34. Corcino, R.B. On p; q-binomial coefficients. *Integers* 2008, 8, A29.

35. Sadjang, P.N. On the fundamental theorem of (p,q)-calculus and some (p,q)-Taylor formulas. *Results Math.* 2018, 73. [CrossRef]

36. El-Deeb, S.M.; Bulboaca, T.; El-Matary, B.M. Maclaurin Coefficient Estimates of Bi-Univalent Functions Connected with the q-Derivative. *Mathematics* 2020, 8, 418. [CrossRef]

37. Catas, A. On the Fekete-Szegö problem for certain classes of meromorphic functions using p,q-derivative operator and a p,q-wright type hypergeometric function. *Symmetry* 2021, 13, 2143. [CrossRef]

38. Amourah, A. Fekete-Szegö inequalities for analytic and bi-univalent functions subordinate to (p, q)-Lucas Polynomials. *arXiv* 2020, arXiv:2004.00409.

39. Wanas, A.K; Cotîrl˘ a, L.I. Initial coefficient estimates and Fekete-Szegö inequalities for new families of bi-univalent functions governed by (p − q) – Wanas operator. *Symmetry* 2021, 13, 2118. [CrossRef]

40. Srivastava, H.M.; Altinkaya, S.; Yalçın, S. Hankel Determinant for a Subclass of Bi-Univalent Functions Defined by Using a Symmetric q-Derivative Operator. *Filomat* 2018, 32, 503–516. [CrossRef]

41. Babalola, K.O. On λ –pseudo-starlike function. *J. Class. Anal.* 2013, 3, 137–147. [CrossRef]

42. Pommerenke, C. *Univalent Functions*; Vanderhoeck and Ruprecht: Gottingen, Germany, 1975; 376p.

43. Srivastava, H.M. Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis. *Iran. J. Sci. Technol. Trans. A Sci.* 2020, 44, 327–344. [CrossRef]