BLOWUP RATE FOR MASS CRITICAL ROTATIONAL NONLINEAR SCHRÖDINGER EQUATIONS

NYLA BASHARAT, YI HU, AND SHIJUN ZHENG

Abstract. We consider the blowup rate for blowup solutions to $L^2$-critical, focusing NLS with a harmonic potential and a rotation term. Under a suitable spectral condition we prove that there holds the “log-log law” when the initial data is slightly above the ground state. We also construct minimal mass blowup solutions near the ground state level with distinct blowup rates.

1. Introduction

Consider the focusing nonlinear Schrödinger equation (NLS) with an angular momentum term on $\mathbb{R}^{1+n}$:

$$
\begin{aligned}
\begin{cases}
    iu_t = -\Delta u + Vu - |u|^{p-1}u + i A \cdot \nabla u \\
u(0,x) = u_0 \in H^1
\end{cases}
\end{aligned}
$$

where $u = u(t,x) \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$ denotes the wave function, $V(x) = \gamma^2 |x|^2$, $\gamma > 0$ is a trapping harmonic potential that confines the movement of particles, and $A(x) = Mx$, with $M = -M^T$ being an $n \times n$ real-valued skew-symmetric matrix. The linear hamiltonian $H_{A,V} := -\Delta + V + i A \cdot \nabla$ is essentially self-adjoint in $L^2$, whose eigenvalues are associated to the Landau levels as quantum numbers. The angular momentum operator $L_{Au} := i A \cdot \nabla u$ generates the rotation $e^{tA \cdot \nabla} u = u(e^{tM} x)$ in $\mathbb{R}^n$. The space $\mathcal{H}^1 = \mathcal{H}^{1,2}$ denotes the weighted Sobolev space given by: for $r \in (1, \infty)$,

$$
\mathcal{H}^{1,r}(\mathbb{R}^n) := \{ f \in L^r : \nabla f, x f \in L^r \},
$$

which is endowed with the norm $|f|_{\mathcal{H}^{1,r}} = |\nabla f|_r + |xf|_r + |f|_r$, here $| \cdot |_r := | \cdot |_{L^r}$ denoting the $L^r$-norm.

When $n = 3$, equation (1) is also known as Gross-Pitaevskii equation which models rotating Bose-Einstein condensation (BEC) with attractive particle interactions in a dilute gaseous ultra-cold superfluid. The...
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operator $L_A$ are usually denoted by $-\Omega \cdot L$, where $\Omega = (\Omega_1, \Omega_2, \Omega_3) \in \mathbb{R}^3$ is a given angular velocity vector and $L := -ix \wedge \nabla$, in which case $M = \begin{pmatrix}
0 & -\Omega_3 & \Omega_2 \\
\Omega_3 & 0 & -\Omega_1 \\
-\Omega_2 & \Omega_1 & 0
\end{pmatrix}$.

Such a system describing rotating particles in a harmonic trap has acquired significance in connection with optics and atomic physics in theoretical and experimental physics [13, 18, 25, 4, 1, 3]. Meanwhile, it demands rigorous mathematical analysis on the evolution and dynamics of the quantized flow. For $p \in (1, 1 + 4/(n - 2))$, the local in time existence and uniqueness of (1) has been established in [14, 15, 2, 6], see also [8, 11, 28] for the treatment in a general magnetic setting. The purpose of this article is to study how the rotation affects the wave collapse as well as energy concentration under a trapping potential. In particular, we will address the blowup rate for the blowup solution of the $L^2$-critical focusing equation in (1) where $p = 1 + 4/n$.

Let $H^1 = \{ u \in L^2 : \nabla u \in L^2 \}$ be the usual Sobolev space. Recall that the standard NLS reads

\[
\begin{cases}
  i\varphi_t = -\Delta \varphi - |\varphi|^{p-1}\varphi, \\
  \varphi(0, x) = \varphi_0 \in H^1,
\end{cases}
\]

and the well-posedness problem for $p \in (1, 1 + 4/(n - 2)]$ has been studied for a few decades and is quite well understood in the euclidean space. Let $Q \in H^1(\mathbb{R}^n)$ be the unique positive radial function that satisfies (17, 26)

\[
-Q = -\Delta Q - Q^{1+\frac{4}{n}}.
\]

When $p = 1 + \frac{4}{n}$ and $|\varphi_0|^2 = |Q|^2$, Merle [19] was able to determine the profile for all blowup solutions with minimal mass at the ground state level, which are obtained from pseudo-conformal transform. Hence all blowup solutions have blowup rate $(T - t)^{-1}$. In the mass critical and supercritical case $p \in [1 + 4/n, \infty)$, the wave collapse dynamics appears very subtle issue for $|u_0|^2 = |Q|^2$. Within an arbitrarily small neighborhood of $Q$, there always exist $\varphi_0$ and $\psi_0$ such that the flow $\varphi_0 \mapsto \varphi$ blowups in finite time, and, $\psi_0 \mapsto \psi$ exists global in time and scatters as $t \to \infty$ in $H^1$.

When $p = 1 + \frac{4}{n}$ and $|\varphi_0|^2 > |Q|^2$, Bourgain and Wang [5] constructed solutions of positive energy having blowup rate $(T - t)^{-1}$ in dimensions $n = 1, 2$, which was later shown unstable however. Perelman [23] gave the first rigorous demonstration of the existence and stability for the log-log speed for generic blowup solutions in 1d. More recently, under
certain spectral condition in the Spectral Property (Section 3), Merle and Raphaël [21] proved the sharp blowup rate of the solutions for (2), i.e., there exists a universal constant $\alpha^*$ such that if $\varphi_0 \in B_{\alpha^*}$ with negative energy, then $\varphi(t, x)$ is a blowup solution to (2) with maximal interval of existence $[0, T)$ satisfying the log-log blowup rate as $t \to T$

$$|\nabla \varphi(t)|_2 \approx \sqrt{\frac{\log \log(T-t)}{T-t}},$$

where

$$B_{\alpha} := \left\{ \phi \in H^1 : \int |Q|^2 dx < \int |\phi|^2 dx < \int |Q|^2 dx + \alpha \right\},$$

see Theorem 3.1. Such a blowup rate is also known to be stable in $H^1$.

In the presence of a rotational term, we will show how to prove such a “log-log law” for the NLS (1). Our proof is based on a virial identity for (1), the $\mathcal{R}$-transform (7) that maps solutions of (2) to solutions of (1), and an application of the above result of Merle and Raphaël’s. This treatment is motivated by [29], where the analogous result is obtained for the case $A = 0$ and $V$ being a harmonic potential. Our main result is stated as follows. Let $\alpha^*$ be the above-mentioned universal constant.

**Theorem 1.1.** Let $p = 1 + \frac{4}{n}$ in (1), $1 \leq n \leq 5$. Suppose $u_0 \in B_{\alpha^*}$ satisfies

$$\int |\nabla u_0|^2 - \frac{n}{n+2} \int |u_0|^{2+\frac{4}{n}} < 0.$$  \hspace{1cm} (4)

Then $u \in C([0, T); H^1)$ is a blowup solution of (1) in finite time $T < \infty$, with the log-log blowup rate

$$\lim_{t \to T} \frac{\left| \nabla u(t) \right|_2}{\left| \nabla Q \right|_2} \sqrt{\frac{T-t}{\log \log(T-t)}} = \frac{1}{\sqrt{2\pi}},$$

where $Q$ is the unique solution of (3).

2. **Local wellposedness for (1)**

For $\varphi_0 \in H^1$, the local well-posedness of (1) was proved for $1 \leq p < 1 + \frac{2}{n-2}$, see e.g. in [11, 28]. The proof for the local well-posedness relies on local in time dispersive estimates for $U(t) = e^{-itH_{A,V}}$, the fundamental solution on $[0, \delta)$ (for some small $\delta > 0$) constructed in [27]. The vectorial function $A$ represents a magnetic potential that induces the Coriolis effect or centrifugal force for the spinor particles. Alternatively, this can also be done by means of the explicit formula in (12) for $U(t)$ defined in $(0, \pi/2\gamma)$. This formula is obtained from the so-called $\mathcal{R}$-transform in (17), a type of pseudo-conformal transform.
Proposition 2.1. Let $1 + \frac{4}{n} \leq p < 1 + \frac{4}{n-2}$. Suppose $u_0 \in \mathcal{H}^1$.

(a) [Blowup Alternative]
(i) If $1 \leq p < 1 + \frac{4}{n-2}$, then there exists $T^* > 0$ such that (1) has a unique maximal solution $u \in C([0, T^*), \mathcal{H}^1) \cap L^q_{loc}([0, T^*), \mathcal{H}^{1, r})$, where $r = p + 1$ and $q = \frac{4(p+1)}{n(p-1)}$.
(ii) If $T^*$ is finite, then $|\nabla u|_2 \to \infty$ as $t \to T^*$ with a lower bound:
$$|\nabla u(t)|_2 \geq \frac{C}{\sqrt{T^* - t}}.$$ 

(b) [Conservation Laws] The following are conserved on the maximal lifespan $[0, T^*)$.

- (mass) $M(u) = \int |u|^2$
- (energy) $E(u) = \int \left( |\nabla u|^2 + V|u|^2 - \frac{2}{p+1}|u|^{p+1} + i\bar{u}A \cdot \nabla u \right)$
- (angular momentum) $\ell_A(u) = \int i\bar{u}A \cdot \nabla u$.

In the critical case $p = 1 + 4/n$, from [6] we know that $|Q|_2$ is the sharp threshold such that:

(a) If $|u_0| < |Q|_2$, then (1) has a unique global in time solution.
(b) For all $c \geq |Q|_2$, there exists $u_0$ with $|u_0|_2 = c$ so that $u$ is a finite time blowup solution of (1).

As we mentioned in the introduction section, if $|u_0| = |Q|_2$, from Merle’s characterization for the blowup profile of (2), all such blowup solutions have the blowup rate $|\nabla u(t)|_2 \approx \frac{C}{T^*-t}$ as $t \to T^*$, see Proposition [4.5].

3. A SPECTRAL PROPERTY AND THE log-log LAW

To show the blowup rate for initial data above the ground state $Q$ as stated in Theorem [1.1], we need the following Spectral Property. Let $y$ denote the spatial variable in $\mathbb{R}^n$.

**Spectral Property.** Consider the two Schrödinger operators

$$L_1 := -\Delta + \frac{2}{n} \left( \frac{4}{n} + 1 \right) Q^{\frac{1}{n}-1} y \cdot \nabla Q, \quad L_2 := -\Delta + \frac{2}{n} Q^{\frac{1}{n}-1} y \cdot \nabla Q,$$

and the real-valued quadratic form for $\varepsilon = \varepsilon_1 + i\varepsilon_2 \in H^1$

$$H(\varepsilon, \varepsilon) := (L_1\varepsilon_1, \varepsilon_1) + (L_2\varepsilon_2, \varepsilon_2).$$
Let
\[ Q_1 := \frac{n}{2} Q + y \cdot \nabla Q, \quad Q_2 := \frac{n}{2} Q_1 + y \cdot \nabla Q_1. \]

Then there exists a universal constant \( \delta_0 > 0 \) such that for every \( \varepsilon \in H^1 \), if
\[ (\varepsilon_1, Q) = (\varepsilon_1, Q_1) = (\varepsilon_1, y_j Q)_{1 \leq j \leq n} = (\varepsilon_2, Q_1) = (\varepsilon_2, Q_2) = (\varepsilon_2, \partial_y Q)_{1 \leq j \leq n} = 0, \]
then
\[ H(\varepsilon, \varepsilon) \geq \delta_0 \left( \int |\nabla \varepsilon|^2 dy + \int |\varepsilon|^2 e^{-|y|} dy \right). \]

The proof of the Spectral Property in any dimension is not complete. It has been proved in [20] for dimension \( n = 1 \) by using the explicit solution \( Q(x) = \left( \frac{3}{\cosh^2(2x)} \right)^{\frac{1}{4}} \) to (3). One can also find a computer assisted proof of the Spectral Property in dimensions \( n = 2, 3, 4 \) in [12]. For dimension 5 and higher see Remark 3.2.

The Spectral Property is equivalent to the coercivity for \( L_1 \) and \( L_2 \) on quadratic forms, the study of which involving the ground state solution \( Q \) naturally appears in a perturbation setting when dealing with stability problem. These two operators are related to the Lyapunov functionals \( L_\pm \), where
\[ L_+ = -\Delta + 1 - (1 + \frac{4}{n}) Q_+^2 \quad \text{and} \quad L_- = -\Delta + 1 - Q_-^2, \]
see [12].

Using the Spectral Property, Merle and Raphaël obtained the following blowup rate for (2) in the absence of potentials, see [21, 12, 22].

**Theorem 3.1.** Let \( p = 1 + \frac{4}{n} \) in (2). Let \( 1 \leq n \leq 5 \). There exists a universal constant \( \alpha^* > 0 \) such that the following is true. Suppose \( \varphi_0 \in B_{\alpha^*} \) satisfies
\[ \int |\nabla \varphi_0|^2 - \frac{n}{n+2} \int |\varphi_0|^{2+\frac{4}{n}} < 0. \]
Then \( \varphi \in C([0, T); H^1) \) is a blowup solution of (2) in finite time \( T < \infty \), which admits the log-log blowup rate
\[ \lim_{t \to T} \frac{|\nabla \varphi(t)|}{|\nabla Q|} \sqrt{\frac{T - t}{\log |\log(T - t)|}} = \frac{1}{\sqrt{2\pi}}. \]

**Remark 3.2.** From [12] we know that the Spectral Property is true in dimensions \( n = 1, 2, 3, 4 \). If \( n = 5 \), Theorem 3.1 continues to hold true as soon as the Spectral Property verifies the orthogonality condition with \( (\varepsilon_1, Q_1) = 0 \) replaced by \( (\varepsilon_1, |y|^2 Q) = 0 \), which is numerically verified in [12]. For the above-mentioned reason, Theorem 3.1 remains open in dimensions \( n \geq 6 \).
Remark 3.3. It is well-known that the log-log law is a generic behavior for those blowup solutions in the theorem, whose proof relies on algebraic cancellations related to the topological degeneracy of the linear operators $L_1$ and $L_2$ around $Q$. Such blowup rate is stable in the sense that the set $U_0$ in the log-log regime is open in $H^1$, where $U_0$ denotes the set of all initial data $\varphi_0$ in $B_\alpha^*$ so that the flow $\varphi_0 \mapsto \varphi(t)$ of (2) collapses in finite time $T^* < \infty$ with the log-log speed given in (5), see [24, 12].

4. The blowup rate for the rotational NLS

In this section we prove Theorem 1.1. We will always assume $p = 1 + \frac{4}{n}$ in both (1) and (2). We will need a virial identity for (2) and the $\mathcal{R}$-transform introduced in Proposition 4.3. This transform gives a relation between the two solutions of (1) and (2), which is coined as a combination of the lens transform and the rotation $e^{tA}$. One can view it as certain pseudo-conformal symmetry in the rotational case, see [7, 29] in the presence of a quadratic potential only, i.e., $M = 0$ and $\gamma \neq 0$.

The following is a standard virial identity for (2) in the weighted Sobolev space $\mathcal{H}^1$, which can be proved by a direct calculation.

**Lemma 4.1.** Let $\varphi$ be a solution to the problem (2) in $C([0,T), \mathcal{H}^1)$. Define $J(t) := \int |x|^2 |\varphi|^2 dx$. Then

$$J'(t) = 4 \mathfrak{Im} \int x \bar{\varphi} \cdot \nabla \varphi dx, \quad J''(t) = 8 \mathcal{E}(\varphi_0),$$

where

$$\mathcal{E}(\varphi) = \int \left( |\nabla \varphi|^2 - \frac{n}{n+2} |\varphi|^{\frac{4}{n+2}} \right) dx.$$

**Lemma 4.2.**

(a) Given any real $n$ by $n$ matrix $M$ and any function $f$ in $C_0^\infty \cap L^2(\mathbb{R}^n)$, we have

$$e^{t(Mx) \cdot \nabla} f(x) = f(e^{tM} x).$$

(b) If $M$ is a real skew-symmetric matrix, then $e^{tM} \in SO(n)$ for all $t$, where $SO(n)$ is the group of $n$ by $n$ orthogonal matrices with determinant 1. Moreover, $(Mx) \cdot \nabla$ and $\Delta$ commute, i.e.,

$$[(Mx) \cdot \nabla, \Delta] = 0.$$

Part (a) can be proven by showing that both sides of (6) obey the ODE

$$\partial_t F(t,x) = (Mx) \cdot \nabla F(t,x), \quad F(0,x) = f(x).$$
Part (b) follows from a straightforward calculation.

**Proposition 4.3.** Let \( \varphi(t, x) \in C([0, T), H^1) \) be a solution to (2) where \( T > 0 \). Define the \( \mathcal{R} \) transform \( \varphi \mapsto \mathcal{R}(\varphi) \) to be

\[
\mathcal{R}(\varphi)(t, x) := \frac{1}{\cos^2(2\gamma t)} e^{-i \frac{\gamma^2}{2} |x|^2} \tan(2\gamma t) \left( \frac{\tan(2\gamma t)}{2\gamma}, \frac{e^{tM} x}{\cos(2\gamma t)} \right). \tag{7}
\]

Then \( u = \mathcal{R} \varphi \) is a solution to (1) in \( C([0, \frac{\arctan(2\gamma T)}{2\gamma}), \mathcal{H}^1) \).

Conversely, let \( u(t, x) \in C([0, T^*), \mathcal{H}^1) \) be a solution to (1) where \( T^* \in (0, \frac{\pi}{4\gamma}] \). Then \( \varphi = \mathcal{R}^{-1}u \), given by

\[
\varphi(t, x) := \frac{1}{(1 + (2\gamma t)^2)^{\frac{1}{2}}} e^{i \frac{\gamma^2}{2} |x|^2} \left( \frac{\arctan(2\gamma t)}{2\gamma}, \frac{e^{-tM} x}{\sqrt{1 + (2\gamma t)^2}} \right), \tag{8}
\]

is a solution to (2) in \( C([0, \frac{\tan(2\gamma T^*)}{2\gamma}), H^1) \), where \( \mathcal{R}^{-1} \) is the inverse of \( \mathcal{R} \).

**Proof.** We will only briefly check (7) for \( u = \mathcal{R} \varphi \). The other one \( u \mapsto \varphi \) is the inverse transform. By direct computation, we have

\[
\begin{align*}
u_t(t, x) &= n\gamma \sin(2\gamma t) \frac{\sin(2\gamma t)}{\cos^2(2\gamma t)} e^{-i \frac{\gamma^2}{2} |x|^2} \tan(2\gamma t) \varphi \left( \frac{\tan(2\gamma t)}{2\gamma}, \frac{e^{tM} x}{\cos(2\gamma t)} \right) \\
&\quad - i\gamma^2 |x|^2 \frac{1}{\cos^2(2\gamma t)} e^{-i \frac{\gamma^2}{2} |x|^2} \tan(2\gamma t) \varphi \left( \frac{\tan(2\gamma t)}{2\gamma}, \frac{e^{tM} x}{\cos(2\gamma t)} \right) \\
&\quad + \frac{1}{\cos^2(2\gamma t)} e^{-i \frac{\gamma^2}{2} |x|^2} \tan(2\gamma t) \varphi_t \left( \frac{\tan(2\gamma t)}{2\gamma}, \frac{e^{tM} x}{\cos(2\gamma t)} \right) \\
&\quad + \frac{1}{\cos^2(2\gamma t)} e^{-i \frac{\gamma^2}{2} |x|^2} \tan(2\gamma t) (e^{tM} M x) \cdot \nabla \varphi \left( \frac{\tan(2\gamma t)}{2\gamma}, \frac{e^{tM} x}{\cos(2\gamma t)} \right) \\
&\quad + 2\gamma \sin(2\gamma t) \frac{\sin(2\gamma t)}{\cos^2(2\gamma t)} e^{-i \frac{\gamma^2}{2} |x|^2} \tan(2\gamma t) (e^{tM} x) \cdot \nabla \varphi \left( \frac{\tan(2\gamma t)}{2\gamma}, \frac{e^{tM} x}{\cos(2\gamma t)} \right). \tag{9}
\end{align*}
\]

To compute \( \Delta u \) and \( i(M x) \cdot \nabla u \), first note that

\[
\begin{align*}
\nabla \left( \varphi \left( \frac{\tan(2\gamma t)}{2\gamma}, \frac{e^{tM} x}{\cos(2\gamma t)} \right) \right) &= \frac{1}{\cos(2\gamma t)} (e^{tM})^T \nabla \varphi \left( \frac{\tan(2\gamma t)}{2\gamma}, \frac{e^{tM} x}{\cos(2\gamma t)} \right), \\
\Delta \left( \varphi \left( \frac{\tan(2\gamma t)}{2\gamma}, \frac{e^{tM} x}{\cos(2\gamma t)} \right) \right) &= \frac{1}{\cos^2(2\gamma t)} \Delta \varphi \left( \frac{\tan(2\gamma t)}{2\gamma}, \frac{e^{tM} x}{\cos(2\gamma t)} \right),
\end{align*}
\]

where we used

(a) If \( B \in M_{n \times n} \) is a constant matrix, then \( \nabla (\varphi(Bx)) = B^T (\nabla \varphi)(Bx) \);

(b) \( \Delta = \text{div}(\nabla), \) \( \text{div } F = \text{tr} \left( \frac{\partial F_i}{\partial x_j} \right) = \text{trace of the Jacobian of } F; \)
(c) If \( C \) is a constant square matrix, \( W = [w_1, \ldots, w_n]^T \) is a vector-valued function of \( x \in \mathbb{R}^n \), then
\[
\text{div } (C W) = \text{tr } ([\nabla w_1, \ldots, \nabla w_n]C^T).
\]

(d) \( \text{tr } (U^* \Lambda U) = \text{tr } (\Lambda) \) if \( U \) is a unitary matrix and \( \Lambda \in M_{n \times n} \).

Thus we obtain
\[
\Delta u(t, x) = -in\gamma \sin(2\gamma t) e^{-\frac{i\gamma}{2} |x|^2 \tan(2\gamma t)} \varphi \left( \frac{\tan(2\gamma t)}{2\gamma}, \frac{e^{tM}x}{\cos(2\gamma t)} \right)
\]

\[
- \gamma^2 |x|^2 \frac{\sin^2(2\gamma t)}{\cos^2(\frac{\pi}{2} + z)} e^{-\frac{i\gamma}{2} |x|^2 \tan(2\gamma t)} \varphi \left( \frac{\tan(2\gamma t)}{2\gamma}, \frac{e^{tM}x}{\cos(2\gamma t)} \right)
\]

\[
- i2\gamma \frac{\sin(2\gamma t)}{\cos^2(\frac{\pi}{2} + z)} e^{-\frac{i\gamma}{2} |x|^2 \tan(2\gamma t)} \left( e^{tM}x \right) \cdot \nabla \varphi \left( \frac{\tan(2\gamma t)}{2\gamma}, \frac{e^{tM}x}{\cos(2\gamma t)} \right)
\]

\[
+ \frac{1}{\cos^2(\frac{\pi}{2} + z)} e^{-\frac{i\gamma}{2} |x|^2 \tan(2\gamma t)} \Delta \varphi \left( \frac{\tan(2\gamma t)}{2\gamma}, \frac{e^{tM}x}{\cos(2\gamma t)} \right),
\]

and, noting that \((Mx) \cdot x = 0\),
\[
i(Mx) \cdot \nabla u = \frac{i}{\cos^2(\frac{\pi}{2} + z)} e^{-\frac{i\gamma}{2} |x|^2 \tan(2\gamma t)} (Me^{tM}x) \cdot \nabla \varphi \left( \frac{\tan(2\gamma t)}{2\gamma}, \frac{e^{tM}x}{\cos(2\gamma t)} \right).
\]

Bring (7), (9), (10), and (11) into Cauchy problem (1), and recall that \( \varphi \) satisfies (2) with \( p = 1 + 4/n \), hence, we see that \( u \) is a solution to (1) with \( u(0, x) = \varphi(0, x) \).

The above virtually shows that under the relation \( u = R \varphi \iff \varphi = R^{-1}u \), \( u \) satisfies (1) if and only if \( \varphi \) satisfies (2). Therefore the second part of the proposition is also true.

\[\square\]

**Remark 4.4.** The \( R \) transform also allows us to solve the linear equation for (1). The equation \( i\partial_t \varphi = -\Delta \varphi \) has the fundamental solution
\[
e^{it\Delta}(x, y) = \frac{1}{(4\pi it)^n} e^{i\frac{|x-y|^2}{2t}}.
\]

Applying (7) we then obtain the fundamental solution to \( i\partial_t u = H_{AV}u \):
\[
e^{-itH_{AV}}(x, y) = \left( \frac{\gamma}{2\pi i \sin(2\gamma t)} \right)^{\frac{n}{2}} e^{i\frac{\gamma}{2} [(|x|^2 + |y|^2) \cot(2\gamma t)]} e^{-i\frac{\gamma}{2} \frac{(e^{tM}x) \cdot y}{\sin(2\gamma t)}}.
\]

This expression is significantly simpler than the one in (10) per Mehler’s formula. To our best knowledge, (12) might be the first simply unified explicit formula compared with [16] [14] [15] [2].

Now we are ready to prove Theorem 1.1.
Proof of Theorem 1.1. Let \( u \in C([0, T), \mathcal{H}^1) \) be the blowup solution to the problem (11), where \([0, T)\) is the maximal interval of existence. Then by (8), there is a \( \varphi(t, x) \in C([\frac{\tan(2\gamma T)}{2\gamma}, \mathcal{H}^1]) \) that solves (2), where \([\frac{\tan(2\gamma T)}{2\gamma}, \mathcal{H}^1]\) is the maximal interval of existence. Note that \( u_0 = \varphi_0 \), and according to Theorem 3.1, the condition (11) suggests that \( \varphi \) is a blowup solution. Recall from (7), for \( T \in (0, \frac{\pi}{4\gamma}] \),

\[
\begin{align*}
    u(t, x) &= \frac{1}{\cos^\frac{\gamma}{2}(2\gamma t)} e^{-i\frac{\pi}{2}|x|^2 \tan(2\gamma t)} \varphi \left( \frac{\tan(2\gamma t)}{2\gamma}, \frac{e^{iMx}}{\cos(2\gamma t)} \right).
\end{align*}
\]

Then for all \( t \in [0, T) \), we have

\[
\begin{align*}
    \nabla_x u(t, x) &= -i\gamma x \frac{\sin(2\gamma t)}{\cos^\frac{\gamma}{2+1}(2\gamma t)} e^{-i\frac{\pi}{2}|x|^2 \tan(2\gamma t)} \varphi \left( \frac{\tan(2\gamma t)}{2\gamma}, \frac{e^{iMx}}{\cos(2\gamma t)} \right) \\
    &\quad + \frac{1}{\cos^\frac{\gamma}{2+1}(2\gamma t)} e^{-i\frac{\pi}{2}|x|^2 \tan(2\gamma t)} e^{iM} T \nabla \varphi \left( \frac{\tan(2\gamma t)}{2\gamma}, \frac{e^{iMx}}{\cos(2\gamma t)} \right) \\
    &:= I_1 + I_2.
\end{align*}
\]

For \( I_1 \), a change of variable gives

\[
|I_1|_2 = \gamma \sin(2\gamma t) \left| x \varphi \left( \frac{\tan(2\gamma t)}{2\gamma}, x \right) \right|_2.
\]

Let \( J(t) := |x \varphi(t, x)|_2^2 \). Then

\[
J(t) = J(0) + J'(0) t + \int_0^t J''(\tau) (t - \tau) d\tau.
\]

Note that

\[
|J'(0)| = \left| x \varphi_0 \right|_2^2 \leq |\varphi_0|^2_{\mathcal{H}^1},
\]

and by Lemma 4.1, we have

\[
|J''(0)| = \left| 4 \Im \int x \varphi_0 \cdot \nabla \varphi_0 dx \right| \leq 4||x \varphi_0||_2 ||\nabla \varphi_0||_2 \leq ||\varphi_0||^2_{\mathcal{H}^1},
\]

\[
J''(t) = 8 E(\varphi_0).
\]

Thus

\[
\begin{align*}
    \left| x \varphi \left( \frac{\tan(2\gamma t)}{2\gamma}, x \right) \right|_2^2 &= J \left( \frac{\tan(2\gamma t)}{2\gamma} \right) \\
    &\leq |J(0)| + |J'(0)| \frac{\tan(2\gamma t)}{2\gamma} + 4 E(\varphi_0) \left( \frac{\tan(2\gamma t)}{2\gamma} \right)^2 \\
    &\leq |\varphi_0|^2_{\mathcal{H}^1} + |\varphi_0|^2_{\mathcal{H}^1} \frac{\tan(2\gamma T)}{2\gamma} + 4 E(\varphi_0) \left( \frac{\tan(2\gamma T)}{2\gamma} \right)^2,
\end{align*}
\]
and so

$$|I_1|_2 \leq C(\varphi_0, T).$$  \hspace{1cm}  \text{(14)}$$

For $I_2$, in view of Lemma 4.2, $e^{tM} \in SO(n)$, a change of variable gives for $t \in [0, T)$, $(T \leq \pi/4\gamma)$

$$|I_2|_2 = \frac{1}{\cos(2\gamma t)} \left| \nabla \varphi \left( \frac{\tan(2\gamma t)}{2\gamma}, x \right) \right|_2.$$  \hspace{1cm} \text{As } t \to T,

$$\frac{\tan(2\gamma t)}{2\gamma} \to \frac{\tan(2\gamma T)}{2\gamma},$$

so by Theorem 3.1

$$\lim_{t \to T} \frac{\left| \nabla \varphi \left( \frac{\tan(2\gamma t)}{2\gamma}, x \right) \right|_2}{\left| \nabla Q \right|_2} \sqrt{\frac{\tan(2\gamma T) - \tan(2\gamma t)}{2\gamma} - \tan(2\gamma t)} = \frac{1}{\sqrt{2\pi}}.$$  \hspace{1cm} \text{Note that as } t \to T,

$$\frac{\sin(2\gamma(T-t))}{2\gamma(T-t)} \to 1 \quad \text{and} \quad \frac{\log \sin(2\gamma(T-t))}{\log(T-t)} \to 1,$$

so the above blowup rate can be simplified as

$$\lim_{t \to T} \frac{\left| \nabla \varphi \left( \frac{\tan(2\gamma t)}{2\gamma}, x \right) \right|_2}{\left| \nabla Q \right|_2} \sqrt{\frac{T-t}{\log |\log(T-t)|}} = \frac{\cos(2\gamma T)}{\sqrt{2\pi}}$$

to yield

$$\lim_{t \to T} \frac{|I_2|_2}{\left| \nabla Q \right|_2} \sqrt{\frac{T-t}{\log |\log(T-t)|}} = \frac{1}{\sqrt{2\pi}}.$$  \hspace{1cm}  \text{(15)}$$

Therefore, combining (14) and (15) we obtain

$$\lim_{t \to T} \frac{\left| \nabla u \right|_2}{\left| \nabla Q \right|_2} \sqrt{\frac{T-t}{\log |\log(T-t)|}} = \frac{1}{\sqrt{2\pi}}.$$  \hspace{1cm} \text{4.1. Blowup rate at the ground state } Q. \text{ We conclude with some discussions on the wave collapse rates for (1) when the initial data is near } Q, \text{ which could be a subtle issue. Notice that this } Q \text{ is not the ground state for (1), instead, it is the one for (2). If } |u_0|_2 = |Q|_2, \text{ then the wave collapse for (1) is different than in the case } |u_0|_2 > |Q|_2. \text{ Applying the transform (7) to the solitary wave } \varphi = e^{tQ} \text{ we can construct a blowup solution with blowup rate } (T-t)^{-1}:

$$u(t, x) = \frac{1}{\cos^2(2\gamma t)} e^{-\frac{1}{2}|x|^2 \tan(2\gamma t)} e^{\frac{\tan(2\gamma t)}{2\gamma}} \frac{e^{tM} Q}{\cos(2\gamma t)}.$$  \hspace{1cm}  \text{(16)}$$
One easily checks that $u$ blows up at $T = \frac{\pi}{4\gamma}$ satisfying

$$|\nabla u|_2 \approx \frac{1}{(\frac{n}{2} - 2\gamma t)} |\nabla Q|_2$$

as $t \to T = \frac{\pi}{4\gamma}$.

Note that solutions of the form (16) with such blowup time and singularity can also be obtained with other nonpositive, non-radial bound states $Q_b$ as the profile in place of $Q$, where $|Q_b|_2 > |Q|_2$ and $n \geq 2$.

Suppose that $u$ is a blowup solution to (1) on $[0, T^*)$ with $T^* < \frac{\pi}{4\gamma}$ and $|u_0|_2 = |Q|_2$. Then by means of (8) we may define a blowup solution to (2) with the same initial data which blows up at $T_0 = \frac{\tan(2\gamma T)}{2\gamma}$. Merle [19] showed that up to the scaling and phase invariances of (2), the only minimal mass blowup solutions are of the form

$$\varphi(t, x) = \frac{1}{(T_0 - t)^{\frac{n}{4}}} e^{-\frac{i|x|^2}{4(T_0 - t)}} e^{\frac{i}{2}\gamma(T - t) - x_0}$$

for some $x_0 \in \mathbb{R}^n$. By the $R$-transform (7) and the uniqueness of (1) (Proposition 2.1), we then establish a characterization for all minimal mass blowup solutions of (1).

**Proposition 4.5.** Let $|u_0|_2 = |Q|_2$. Let $u$ be a blowup solution of (1) on $[0, T^*)$ with $T^* < \frac{\pi}{4\gamma}$. Then $u$ must assume the following form (up to scaling and phase invariance): There exists $x_1 \in \mathbb{R}^n$ such that

$$u(t, x) = \left( \frac{2\gamma \cos(2\gamma T)}{\sin(2\gamma(T - t))} \right)^\frac{n}{4} e^{-\frac{i|x|^2}{4}\cot(2\gamma(T - t))} e^{i\gamma \cos(2\gamma T) \tan(2\gamma T) \cos(2\gamma T) \sin(2\gamma(T - t))}$$

$$\times Q \left( \frac{2\gamma \cos(2\gamma T) e^{i\gamma x} - x_1}{\sin(2\gamma(T - t))} \right).$$

Moreover, $|\nabla u|_2 \approx (T - t)^{-1}$ as $t \to T$.

Note that the blowup solution given in (16) is not covered by (17). For example, if $T^* = \pi/8\gamma$, $x_1 = 0$, then $u_0(x) = (2\gamma)^{n/4} e^{-\frac{i|x|^2}{4}} e^{i\gamma x} Q(2\gamma x)$. Rather, (16) can be viewed as a bordering case of the assertion in Proposition 4.5 corresponding to $T^* = \pi/4\gamma$.

**Remark 4.6.** If the initial value is of the form $u_0 = (1 + \varepsilon)Q$ with $0 < \varepsilon < \sqrt{1 + \frac{\alpha^*}{|Q|_2^2}} - 1$, then the corresponding solution to (1) will blowup at the rate stated in Theorem 1.1. Indeed, by the range for $\varepsilon$ and the Pohozaev identity $\int |\nabla Q|^2 = \frac{n}{n+2} \int |Q|^{2+\frac{4}{n}}$, it is easy to verify $u_0 \in B_{\alpha^*}$ and condition (4).

**Remark 4.7.** For large initial data one can also derive a general lower bound for the collapse rate. If the solution of (1) satisfies $\lim_{t \to T^*} |\nabla u|_2 = 0$, then
∞, then there exists $C = C_{p,n} > 0$ such that
\[ |\nabla u(t)|_2 \geq C(T^* - t)^{-\left(\frac{1}{p-1} - \frac{n-2}{4}\right)}. \]
This follows from quite standard argument as in [9] that is used to show the l.w.p and blowup alternative for the Cauchy problem (1) on $[0, T^*)$. In the $L^2$-critical case $p = 1 + \frac{4}{n}$, the lower bound becomes
\[ |\nabla u(t)|_2 \geq C(T^* - t)^{-\frac{3}{2}}. \]

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(Nyla Basharat) DPHS, AUGUSTA UNIVERSITY, AUGUSTA, GA 30912
*E-mail address:* nbasharat@augusta.edu

(Yi Hu) DEPARTMENT OF MATHEMATICAL SCIENCES, GEORGIA SOUTHERN UNIVERSITY, STATESBORO, GA 30460
*E-mail address:* yihu@georgiasouthern.edu

(Shijun Zheng) DMS, GEORGIA SOUTHERN UNIVERSITY, STATESBORO, GA 30460
*E-mail address:* szheng@georgiasouthern.edu