DERIVED EQUIVALENCES FOR RATIONAL CHEREDNIK ALGEBRAS

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Abstract. Let $W$ be a complex reflection group and $H_c(W)$ the Rational Cherednik algebra for $W$ depending on a parameter $c$. One can consider the category $\mathcal{O}$ for $H_c(W)$. We prove a conjecture of Rouquier that the categories $\mathcal{O}$ for $H_c(W)$ and $H_{c'}(W)$ are derived equivalent provided the parameters $c, c'$ have integral difference. Two main ingredients of the proof are a connection between the Ringel duality and Harish-Chandra bimodules and an analog of a deformation technique developed by the author and Bezrukavnikov. We also show that some of the derived equivalences we construct are perverse.

1. Introduction

1.1. Hecke algebras. Let $W$ be the Weyl group of some connected reductive group $G$. By $S$ we denote the set of reflections in $W$. The group algebra $CW$ admits a classical deformation, the Hecke algebra $H_q(W)$, where $q \in (\mathbb{C}^\times)^{S/W}$. The representation theory of $H_q(W)$ is most interesting when $q$ has finite (and sufficiently small) order. In this case this representation theory is similar to (but easier than) the modular representation theory of the group $W$.

The Hecke algebras still make sense when $W$ is a complex reflection group, see [BMR], we will recall the definition below. Those are still algebras $H_q(W)$ with $q \in (\mathbb{C}^\times)^{S/W}$. Their structure is more complicated than in the Weyl group case, for example, it is not known in the full generality whether $\dim H_q(W) = |W|$. However, the algebra $H_q(W)$ always has the maximal finite dimensional quotient $H_q(W)$ and the dimension of this quotient is $|W|$, [L6]. When $q$ is Zariski generic, we have $H_q(W) \cong CW$.

Let us point out that the algebra $H_q(W)$, in general, has infinite homological dimension so is “singular”. So one can ask about a “resolution of singularities”. Such resolutions are provided by categories $\mathcal{O}$ for Rational Cherednik algebras to be described briefly in the next subsection.

1.2. Cherednik algebras and their categories $\mathcal{O}$. Let $\mathfrak{h}$ denote the reflection representation of a complex reflection group $W$. A Rational Cherednik algebra $H_c(W)$ is a flat deformation of the skew-group ring $S(\mathfrak{h} \oplus \mathfrak{h}^\ast)\#W$ depending on a parameter $c \in p := \mathbb{C}^{S/W}$. We write $H_c$ instead of $H_c(W)$ if this does not create ambiguity.

This algebra admits a triangular decomposition $H_c = S(\mathfrak{h}^\ast) \otimes CW \otimes S(\mathfrak{h})$ (as a vector space), where $S(\mathfrak{h}^\ast), CW, S(\mathfrak{h})$ are embedded as subalgebras. So it makes sense to consider a category $\mathcal{O}$. This is a full subcategory in $H_c$-mod consisting of all modules that are finitely generated over $S(\mathfrak{h}^\ast)$ and have locally nilpotent action of $\mathfrak{h}$. Let us denote the category $\mathcal{O}$ by $\mathcal{O}_c$ or by $\mathcal{O}_c(W)$. It has analogs of Verma modules, $\Delta_c(\lambda)$, parameterized by the irreducible representations $\lambda$ of $W$. Each $\Delta_c(\lambda)$ has a unique irreducible quotient, $\Delta_c(\lambda)$.
Let \( L_c(\lambda) \), and the assignment \( \lambda \mapsto L_c(\lambda) \) is a bijection between the sets of the irreducible \( W \)-modules and the set of the irreducible objects in \( \mathcal{O}_c \).

The category \( \mathcal{O}_c \) has a so called highest weight structure that axiomatizes certain upper triangularity properties similar to those of the BGG categories \( \mathcal{O} \). We will recall a precise definition later. One consequence of being highest weight is that \( \mathcal{O}_c \) has finite homological dimension.

Moreover, there is a quotient functor \( KZ_c : \mathcal{O}_c \to \mathcal{H}_q \)-mod introduced in [GGG] that is fully faithful on the projective objects (a highest weight cover in the terminology of Rouquier, [R1]). So we can view \( \mathcal{O}_c \) as a “resolution of singularities” for \( \mathcal{H}_q \)-mod. Here \( q \) is recovered from \( c \) by some kind of exponentiation: there is a \( \mathbb{Z} \)-lattice \( p, q \subset \mathfrak{p} \) such that the set of Hecke parameters is identified with \( p/p \mathfrak{q} \) and \( q = c + p \mathfrak{q} \).

1.3. Derived equivalences. Now let \( c, c' \) be two Cherednik parameters with \( c - c' \in p \mathfrak{q} \) so that \( \mathcal{O}_c, \mathcal{O}_{c'} \) are two resolutions of singularities for \( \mathcal{H}_q \)-mod. A natural question to ask is whether these two resolutions are derived equivalent. Rouquier conjectured that this is so in [R2, Conjecture 5.6]. The main goal of this paper is to prove this conjecture.

**Theorem 1.1.** Let \( c, c' \in p \) and \( c - c' \in p \mathfrak{q} \). Then there is a derived equivalence \( D^b(\mathcal{O}_c) \cong D^b(\mathcal{O}_{c'}) \) intertwining the functors \( KZ_c, KZ_{c'} \).

Theorem 1.1 was known for \( W = G(\ell, 1, n) \). Recall that this group is realized as \( \mathfrak{S}_n \ltimes \mu^n_\ell \), where \( \mu_\ell \) denotes the group of \( \ell \)th roots of 1, and its reflection representation is \( \mathbb{C}^n \). In this case, Theorem 1.1 was proved in [GL, Section 5] and is a consequence of the quantized derived McKay equivalence. Peculiarly, the proof is based on the study of actual algebro-geometric resolutions of \( (\mathfrak{h} \oplus \mathfrak{h}^*)/W \).

1.4. Perverse equivalences. We will also prove in Section 9 that some of the equivalences in Theorem 1.1 are perverse in the sense of [R2, 2.6]. Some special cases of this were established in [BL, Section 7].

Let us recall the definition of a perverse equivalence. Let \( C^1, C^2 \) be two abelian categories equivalent to categories of modules over some finite dimensional associative algebras. Suppose \( C^i, i = 1, 2 \), is equipped with a filtration \( C^i = C^i_0 \supset C^i_1 \supset C^i_2 \supset \ldots \supset C^i_q \supset C^i_{q+1} = \{0\} \) by Serre subcategories. We say that an equivalence \( \varphi : D^b(C^1) \to D^b(C^2) \) of triangulated categories is perverse with respect to the filtrations if the following holds:

(I) \( \varphi \) restricts to an equivalence of \( D^b(C^1_j) \) and \( D^b(C^2_j) \) for all \( j = 1, \ldots, q \). Here we write \( D^b(C^i) \) for the full subcategory of \( D^b(C^i) \) consisting of all complexes with homology in \( C^i_j \).

(II) For \( M \in C^1_j \) we have \( H_k(\varphi M) = 0 \) for \( k < j \).

(III) The functor \( M \mapsto H_j(\varphi M) \) induces an equivalence \( C^1_j/C^1_{j+1} \cong C^2_j/C^2_{j+1} \). Moreover, \( H_k(\varphi M) \in C^1_{j+1} \) for \( k > j \).

The definition of filtrations on \( C^1 := \mathcal{O}_c, C^2 := \mathcal{O}_{c'} \) making some of equivalences in Theorem 1.1 is technical. Roughly speaking these filtrations are obtained by degenerating the filtration by dimension of support.

1.5. Ideas of the proof and the content. Our key idea is the same as in the proof of [BL, Theorem 7.2]: we want to prove Theorem 1.1 at a generic point of a hyperplane (if \( \mathcal{O}_c \) is not semisimple, then \( c \) lies in a countable union of hyperplanes) and then to degenerate to a special point. At a Weil generic point our derived equivalence will be the
Let us consider two important families of examples. First, assume \( W = C \) for the set of complex reflections in \( S \). We denote the collection of parameters \( q_{H,i} \) by \( q \). By definition, \( \mathcal{H}_q(= \mathcal{H}_q(W)) \) is the quotient of \( \mathbb{C}W \) by the relations \( \prod_{i=1}^{\ell_H} (T_H - q_{H,i}) \), one for each reflection hyperplane \( H \). For example, if we put \( q_{H,i} = \exp(2\pi\sqrt{-1}i/\ell_H) \), then we get \( \mathbb{C}W \). We remark that rescaling the parameters \( q_{H,i} \) is actually isomorphic to the Hecke algebra is actually \( \mathbb{C}W/w \), where we write \( S \) for the set of complex reflections in \( W \).

Let us consider two important families of examples. First, assume \( W \) is a real reflection group so that all \( \ell_H \) are equal to 2. For simplicity, we assume that \( W \) is irreducible. In the corresponding Coxeter diagram \( I \), let \( m_{ij} \) be the multiplicity of the edge between vertices \( i, j \). The braid group \( B_W \) is generated by elements \( T_i, i \in I \), subject to the relation \( T_i T_j T_i \cdots = T_j T_i T_j T_i \cdots, \) where on both sides we have \( m_{ij} \) factors. The number of parameters \( q \) is either one or two and the additional relations for the Hecke algebra are \( (T_i - q_i)(T_i + 1) = 0 \) (we have \( q_i = q_j \) if the reflections \( s_i, s_j \) in \( W \) are conjugate).

The second family is for the groups \( W = G(\ell, 1, n) \) (for notational simplicity, let us restrict to \( \ell > 1 \)). In this case, the braid group \( B_W \) is the affine braid group of type \( A \), it is generated by elements \( T_0, \ldots, T_{n-1} \) subject to the relations \( T_0 T_1 T_0 = T_1 T_0 T_0, T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \) for \( 1 \leq i \leq n - 2 \) and \( T_i T_j = T_j T_i \) for \( |i - j| > 1 \). The Hecke algebra is the quotient of \( \mathbb{C}W \) by the relations \( (T_i + 1)(T_i - q) = 0 \) for \( i > 0 \) and...
\[ \prod_{j=0}^{\ell-1}(T_0 - Q_j) = 0. \] Here \( Q_0, \ldots, Q_{\ell-1} \) are nonzero complex numbers, we can take \( Q_0 = 1 \) without changing the algebra.

Let us point out that we can define the Hecke algebra \( \mathcal{H}_{R,q} \) over any commutative ring \( R \) (the entries of \( q \) are supposed to be invertible elements of \( R \)).

It was shown in [L6] that the algebra \( \mathcal{H}_{q}(W) \) admits a maximal finite dimensional quotient to be denoted by \( \mathcal{H}_{q}(W) \) whose dimension equals \( |W| \). These algebras form a flat family over \( (\mathbb{C}^*)^{S/W} \). So for a \( \mathbb{C}[(\mathbb{C}^*)^{S/W}] \)-algebra \( R \), we have an algebra \( \mathcal{H}_{R,q}(W) \) that is a projective \( R \)-module of rank \( |W| \).

2.2. Rational Cherednik algebras. Recall that we have chosen a \( W_H \)-eigenvector \( \alpha_H \in \mathfrak{h} \) with nontrivial eigencharacter. Pick a \( W_H \)-eigenvector \( \alpha_H \in \mathfrak{h}^* \) with \( \langle \alpha_H, \alpha_H^\vee \rangle = 2 \). For a complex reflection \( s \) we write \( \alpha_s, \alpha_s^\vee \) for \( \alpha_H, \alpha_H^\vee \) where \( H = \mathfrak{h}^s \). Let \( c : S \to \mathbb{C} \) be a function constant on the conjugacy classes. The space of such functions is denoted by \( \mathfrak{p} \), it is a vector space of dimension \( |S/W| \).

By definition, [EG, 1.4], [GGOR, 3.1], the Rational Cherednik algebra \( H_c(= H_c(W)) \) is the quotient of \( T(\mathfrak{h} \oplus \mathfrak{h}^*)\#W \) by the following relations:

\[ [x, x'] = [y, y'] = 0, [y, x] = \langle y, x \rangle - \sum_{s \in S} c(s) \langle x, \alpha_s^\vee \rangle \langle y, \alpha_s \rangle s, \quad x, x' \in \mathfrak{h}^*, y, y' \in \mathfrak{h}. \]

We would like to point out that \( H_c \) is the specialization to \( c \) of a \( \mathbb{C}[\mathfrak{p}] \)-algebra \( H_p \) defined as follows. The space \( \mathfrak{p}^* \) has basis \( c_s \) naturally numbered by the conjugacy classes of reflections. Then \( H_p \) is the quotient of \( T(\mathfrak{h} \oplus \mathfrak{h}^*)\#W \otimes \mathbb{C}[\mathfrak{p}] \) by the relations similar to the above but with \( c(s) \in \mathbb{C} \) replaced with \( c_s \in \mathfrak{p}^* \). For a commutative algebra \( R \) with a \( W \)-invariant map \( c : S \to R \) we can consider the algebra \( H_{R,c} = R \otimes \mathbb{C}[\mathfrak{p}] \). If \( R = \mathbb{C}[\mathfrak{p}'] \) for an affine subspace \( \mathfrak{p}' \subset \mathfrak{p} \), then we write \( H_{R,c} \) instead of \( H_{R,c} \).

Let us recall some structural results about \( H_c \). The algebra \( H_c \) is filtered with \( \deg \mathfrak{h}^* = 0, \deg W = 0, \deg \mathfrak{h} = 1 \). The associated graded is \( S(\mathfrak{h} \oplus \mathfrak{h}^*)\#W \), [EG 1.2]. This yields the triangular decomposition \( H_c = S(\mathfrak{h}^*) \otimes CW \otimes S(\mathfrak{h}) \), [GGOR 3.2]. Similarly, the algebra \( H_p \) is filtered. We can either set \( \deg \mathfrak{p} = 1 \) (this is our usual choice) or \( \deg \mathfrak{p} = 0 \). The latter choice shows that \( H_p = S(\mathfrak{h}^*) \otimes \mathbb{C}[\mathfrak{p}]W \otimes S(\mathfrak{h}) \) as a \( \mathbb{C}[\mathfrak{p}] \)-module.

Consider the element \( \delta := \prod_s \alpha_s^\ell_s \in S(\mathfrak{h}^*)^W \), where \( \ell_s := \ell_{\mathfrak{h}^s} \). Since \( ad \delta \) is locally nilpotent, the quotient \( H_c[\delta^{-1}] \) is well-defined. There is a natural isomorphism \( H_c[\delta^{-1}] \cong D(\mathfrak{h}^\text{reg})\#W \), [EG 1.4], [GGOR 5.1].

Consider the averaging idempotent \( e := |W|^{-1} \sum_{w \in W} w \in CW \subset H_c \). The spherical subalgebra by definition is \( e H_c e \), it is a deformation of \( S(\mathfrak{h} \oplus \mathfrak{h}^*)^W \). When the algebras \( e H_c e \) and \( H_c \) are Morita equivalent (automatically, via the bimodule \( H_c e \)), we say that the parameter \( c \) is spherical.

There is an Euler element \( h \in H_c \) satisfying \( [h, x] = x, [h, y] = -y, [h, w] = 0 \). It is constructed as follows. Pick a basis \( y_1, \ldots, y_n \in \mathfrak{h} \) and let \( x_1, \ldots, x_n \in \mathfrak{h}^* \) be the dual basis. For \( s \in S \), let \( \lambda_s \) denote the eigenvalue of \( s \) in \( \mathfrak{h}^* \) different from 1. Then

\[
(2.1) \quad h = \sum_{i=1}^n x_i y_i + \frac{n}{2} - \sum_{s \in S} \frac{2c(s)}{1 - \lambda_s} s = \sum_{i=1}^n y_i x_i - \frac{n}{2} + \sum_{s \in S} \frac{2c(s)}{1 - \lambda_s^{-1}} s.
\]

2.3. Category \( O \). Following [GGOR 3.2], we consider the full subcategory \( O_c(W) \) of \( H_c \)-mod consisting of all modules \( M \) that are finitely generated over \( S(\mathfrak{h}^*) \) and such that \( \mathfrak{h} \) acts on \( M \) locally nilpotently. Equivalently, a module \( M \) lies in \( O_c(W) \) if it is finitely generated over \( S(\mathfrak{h}^*) \) and is graded in such a way that the grading is compatible with that.
on $H_c$ induced by $\text{ad}(h)$. For example, pick an irreducible representation $\lambda$ of $W$. Then the Verma module $\Delta_c(\lambda):= H_c \otimes_{S(h)} W \lambda$ (here $h$ acts by 0 on $\lambda$) is in $\mathcal{O}_c(W)$. Often we drop $W$ from the notation and just write $\mathcal{O}_c$.

To a module $M \in \mathcal{O}_c$ we can assign its associated variety $V(M)$ that, by definition, is the support of $M$ (as a coherent sheaf) in $h$. Clearly, $V(M)$ is a closed $W$-stable subvariety.

A basic result about $\mathcal{O}_c$ is that it is a highest weight category. Let us recall the general definition. Let $\mathcal{C}$ be a $\mathbb{C}$-linear abelian category equivalent to the category of modules over some finite dimensional associative $\mathbb{C}$-algebra. Let $\Lambda$ be an indexing set for the simples in $\mathcal{C}$, we write $L(\lambda)$ for the simple object indexed by $\lambda$ and $P(\lambda)$ for its projective cover. By a highest weight category we mean a triple $(\mathcal{C}, \leq, \{\Delta(\lambda)\}_{\lambda \in \Lambda})$, where $\leq$ is a partial order on $\Lambda$ and $\Delta(\lambda), \lambda \in \Lambda$, is a collection of standard objects in $\mathcal{C}$ satisfying the following conditions:

(i) $\text{Hom}_\mathcal{C}(\Delta(\lambda), \Delta(\mu)) \neq 0$ implies $\lambda \leq \mu$.

(ii) $\text{End}_\mathcal{C}(\Delta(\lambda)) = \mathbb{C}$.

(iii) There is an epimorphism $P(\lambda) \twoheadrightarrow \Delta(\lambda)$ whose kernel admits a filtration with successive quotients of the form $\Delta(\mu)$ with $\mu > \lambda$.

Axiom (iii) allows to recover projective objects from standard objects as follows. Take a linear ordering on $\Lambda$ refining the order $\leq$ above: $\lambda_1 > \lambda_2 > \ldots > \lambda_n$. We construct the object $P(\lambda_k)$ inductively. Set $P_k(\lambda_k) := \Delta(\lambda_k)$. If $P_i(\lambda_k)$ with $i \leq k$ is already constructed, for $P_{i-1}(\lambda_k)$ we take the universal extension of $P_i(\lambda_k)$ by $\text{Ext}^1(P_i(\lambda_k), \Delta(\lambda_{k-1})) \otimes \Delta(\lambda_{k-1})$. Then $P(\lambda_k) = P_i(\lambda_k)$.

Let us describe a highest weight structure on $\mathcal{O}_c$, [GGOR Theorem 2.19]. For the standard objects we take the Verma modules. A partial order on $\Lambda = \text{Irr}(W)$ is introduced as follows. The element $\sum_{s \in S} 2c(s) \lambda_s - 1 \in \mathbb{C}W$ is central so acts by a scalar, denoted by $c_\lambda$ (and called the c-function), on $\lambda$. We set $\lambda < \mu$ if $c_\lambda - c_\mu \in \mathbb{Q}_{\succ 0}$ (we could take the coarser order by requiring the difference to lie in $\mathbb{Z}_{\succ 0}$ but we do not need this). We write $\lambda <^c \mu$ if we want to indicate the dependence on the parameter $c$.

Since $\mathcal{O}_c$ is a highest weight category, we see that the classes $[\Delta_c(\lambda)]$ form a basis in $K_0(\mathcal{O}_c)$. So we can identify $K_0(\mathcal{O}_c)$ with $K_0(W\text{-mod})$ by sending $[\Delta_c(\lambda)]$ to $[\lambda]$.

We will need a construction of a projective object containing $P(\lambda)$ as a summand, [GGOR 2.4]. Namely, consider the object $\nabla_n(\lambda) := H_c \otimes_{S(h)} W (\lambda \otimes S(h)/(h^n))$ so that $\Delta(\lambda) = \Delta_1(\lambda)$. The module $\nabla_n(\lambda)$ is graded, $\nabla_n(\lambda) = \sum_{k \in \mathbb{Z}} \nabla_n(\lambda)_k$, the grading is induced from that on $H_c$ by the eigenvalues of $\text{ad}(h)$. The graded components $\nabla_n(\lambda)_k$ are finite dimensional and are preserved by the action of $h$. Let $\nabla_n(\lambda)_k$ denote the generalized eigenspace for $h$ in $\nabla_n(\lambda)_k$ with eigenvalue $k + c_\lambda$. Then $\nabla_n(\lambda) := \bigoplus_k \nabla_n(\lambda)_k$ is a submodule of $\nabla_n(\lambda)$. It is not difficult to see that a natural surjection $\nabla_{n+1}(\lambda) \twoheadrightarrow \nabla_n(\lambda)$ is an isomorphism for $n$ large enough. Denote the stable module $\nabla_n(\lambda)$ by $\nabla(\lambda)$. It is easy to see that this module is projective and admits a surjection onto $\Delta(\lambda)$. As a corollary of this construction we get the following.

**Lemma 2.1.** There is a direct summand in $\varprojlim_{n \to \infty} H_c/H_c h^n$ that is a projective generator of $\mathcal{O}_c$.

Recall that in any highest weight category one has costandard objects $\nabla(\lambda), \lambda \in \Lambda$, with $\text{dim Ext}^i(\Delta(\lambda), \nabla(\mu)) = \delta_{i,0} \delta_{\lambda,\mu}$. In the case of the category $\mathcal{O}_c$ one can construct
the costandard objects $\nabla_c(\lambda)$ as follows. Consider the parameter $c^*$ defined by $c^*(s) := -c(s^{-1})$. There is an isomorphism $H_c(W, \mathfrak{h}) \cong H_{c^*}(W, \mathfrak{h}^{\star})$ that is the identity on $\mathfrak{h}^*, \mathfrak{h}$ and is the inversion on $W$. Take the Verma module $\Delta_c(\lambda) \in O_c(W, \mathfrak{h}^*)$. Its restricted dual $\Delta_c(\lambda^*):= \bigoplus_k \Delta_c(\lambda^*)_k$ is a left $H_c(W, \mathfrak{h})$-module and it lies in the category $O$. This module is $\nabla_c(\lambda)$. The category $O^{opp}_c$ is highest weight with the same order and with standard objects $\nabla_c(\lambda)$.

Here are some basic properties of the standard and the costandard objects.

**Lemma 2.2.** The following is true:

1. We have $[\nabla_c(\lambda)] = [\Delta_c(\lambda)]$ for all $\lambda$.
2. If $L_c(\mu)$ lies in the socle of $\Delta_c(\lambda)$, then $V(L_c(\mu)) = \mathfrak{h}$.
3. If $L_c(\mu)$ lies in the head of $\nabla_c(\lambda)$, then $V(L_c(\mu)) = \mathfrak{h}$.

**Proof.** (1) is a part of [GGOR, Proposition 3.3]. (2) follows from the observation that $\Delta_c(\lambda)$ is torsion free over $S(\mathfrak{h}^*)$. (3) is a corollary of the fact that taking the restricted dual is a category equivalence $O_c(W, \mathfrak{h}) \cong O_{c^*}(W, \mathfrak{h}^*)$ preserving the Gelfand-Kirillov dimensions.

As with an arbitrary highest weight category, we have tilting objects in $O_c$. Recall that an object is tilting if it is both standardly filtered and costandardly filtered. The indecomposable tilting objects are again labeled by $\lambda$. More precisely, we have an indecomposable tilting $T(\lambda)$ such that $\Delta(\lambda) \subset T(\lambda)$ and $T(\lambda)/\Delta(\lambda)$ has a filtration with successive quotients $\Delta(\mu), \mu < \lambda$.

Here are some basic properties of the tilting objects.

**Lemma 2.3.** Let $T_c$ stand for a tilting generator – the direct sum of all indecomposable tilting objects. Then the following is true:

1. If $\text{Hom}(T_c, L_c(\mu)) \neq 0$ or $\text{Hom}(L_c(\mu), T_c) \neq 0$, then $V(L(\mu)) = \mathfrak{h}$.
2. If $\text{Ext}^1(T_c, L_c(\mu)) \neq 0$ or $\text{Ext}^1(L_c(\mu), T_c) \neq 0$, then $\text{codim}_V(L_c(\mu)) \leq 1$.

**Proof.** (1) is a direct consequence of Lemma 2.2. The implication $\text{Ext}^1(T_c, L_c(\mu)) \neq 0 \Rightarrow \text{codim}_V(L_c(\mu)) \leq 1$ follows from the categorical description of the GK dimension given in [GGOR, 4.3]. The remaining implication in (2) now follows from the claim that taking the restricted dual is a category equivalence preserving the GK dimensions and, moreover, it maps tiltings to tiltings.

Let us now discuss the right handed analog of the category $O_c$. By definition, it consists of the finitely generated (over $S(\mathfrak{h}^*)$) right $H_c$-modules with locally nilpotent action of $\mathfrak{h}$. We denote this category by $O^{r}_c$. It also has Verma modules $\Delta'_c(\lambda) := (\bigwedge^n \mathfrak{h}^* \otimes \lambda^*) \otimes_{S(\mathfrak{h}) \neq W} H_c$, where $\bigwedge^n \mathfrak{h}^* \otimes \lambda$ is viewed as a left $W$-module so that $\bigwedge^n \mathfrak{h}^* \otimes \lambda^*$ is a right $W$-module. The category $O^{r}_c$ is highest weight with order $\leq^{c_r}$ given as follows.

The scalar by which $\sum_{s \in S} \frac{2c(s)}{1 - \lambda_s^{-1}} s$ acts on the right $W$-module $\bigwedge^n \mathfrak{h}^* \otimes \lambda^*$ coincides with $c_\lambda$. Then we set $\lambda <^{c_r} \mu$ if $c_\mu - c_\lambda \in \mathbb{Q}_{>0}$ (we choose the sign in this way because we are dealing with right modules so the multiplication by $x$ decreases the eigenvalue for $h$ by 1). Note that this order is opposite to the $c$-order for $O_c$. We will be mostly considering the highest weight category $O^{r,opp}_c$.

To finish this subsection, let us note that one can also define the category $O_{R,c}$ for a commutative algebra $R$: it consists of all $H_{R,c}$-modules that are finitely generated over
2.4. KZ functor. Recall that $H_c[\delta^{-1}] \cong D(\mathfrak{h}^{reg})\# W$. So we have the localization functor $\text{loc} : \mathcal{O}_c \to \text{LS}^W(\mathfrak{h}^{reg})$, where on the right hand we have the category of $W$-equivariant local systems on $\mathfrak{h}^{reg}$. The functor is given by $\text{loc}(M) := M[\delta^{-1}]$. There is a standard equivalence $\text{LS}^W(\mathfrak{h}^{reg}) \cong \text{LS}(\mathfrak{h}^{reg}/W), N \mapsto eN$. One can show that the image of $\mathcal{O}_c$ lies in the subcategory $\text{LS}_{\text{reg}}(\mathfrak{h}^{reg}/W)$ of the local systems with regular singularities. The latter is equivalent to the category $\mathbb{C}B_W$-mod of the finite dimensional $B_W$-modules, the equivalence sends a local section to its fiber at a point equipped with the monodromy action.

It follows from [GGOR, Section 5] that the essential image of the functor $\mathcal{O}_c \to \mathbb{C}B_W$-mod coincides with $\mathbb{H}_q$-mod. The parameter $q$ is computed as follows. We can find elements $h_{H,j} \in \mathbb{C}$ with $j = 0, \ldots, \ell_H - 1$ and $h_{H,j} = h_{H',j}$ for $H' \in WH$ such that

$$c(s) = \sum_{j=1}^{\ell_H-1} \frac{1 - \lambda^j_s}{2}(h_{j,s} - h_{j,s-1})$$

Clearly, for fixed $H$, the numbers $h_{H,0}, \ldots, h_{H,\ell_H-1}$ are defined up to a common summand. We can recover the elements $h_{H,i}$ by the formula

$$h_{H,i} = \frac{1}{\ell_H} \sum_{s \in WH \setminus \{1\}} \frac{2c(s)}{\lambda_s - 1} \lambda_s^{-i}$$

Note that $\sum_{i=0}^{\ell_H-1} h_{H,i} = 0$. We will view $h_{H,i}$ as an element of $\mathfrak{p}^*$ whose value on $c : S \to \mathbb{C}$ is given by $\{23\}$.

We set

$$q_{H,j} := \exp(2\pi \sqrt{-1}(h_{H,j} + j/\ell_H)).$$

So we get the functor $KZ : \mathcal{O}_c \to \mathbb{H}_q$-mod. Let us list properties of this functor obtained in [GGOR, Section 5].

**Proposition 2.4.** The functor $KZ$ has the following properties:

1. $KZ$ is a quotient functor, its kernel consists of all modules $M \in \mathcal{O}_c$ that are torsion over $S(\mathfrak{h}^*) \leftrightarrow V(M) \neq \mathfrak{h}$.
2. $KZ$ is fully faithful on the projective objects. Also it is fully faithful on the tilting objects.
3. Suppose that we have $q_{H,i} \neq q_{H,j}$ for any reflection hyperplane $H$ and $i \neq j$. Then $KZ$ is fully faithful on the standardly filtered objects (=the objects admitting a filtration with standard successive quotients).

Let $P_{KZ}$ denote the projective object in $\mathcal{O}_c(W)$ defining the functor $KZ$ so that there is a distinguished isomorphism $\mathbb{H}_q(W) \cong \text{End}(P_{KZ})^{\text{opp}}$. The object $P_{KZ}$ is the sum of all objects in $\mathcal{O}_c$ that are simultaneously projective and injective (hence tilting) with suitable multiplicities.

We also have a version of $KZ$ over rings. Namely, let $R$ be a regular complete local ring with residue field $\mathbb{C}$. Then the exponential map still makes sense and we get a quotient functor $KZ : \mathcal{O}_{R,c}(W) \to \mathbb{H}_{R,q}(W)$-mod.
2.5. **Ringel duality.** Let \( C_1, C_2 \) be two highest weight categories. Let \( C_2^L, C_1^\vee \) denote the full subcategories of standardly and costandardly filtered objects in \( C_2, C_1 \), respectively. Let \( R \) be an equivalence \( C_1^\vee \xrightarrow{\sim} C_2^L \). Let \( T \) denote the tilting generator of \( C_1 \), i.e., the sum of all indecomposable tilting objects. Then \( C_2 \) gets identified with \( \text{End}(T)^{\text{opp}} \text{-mod} \) and the equivalence \( R \) above becomes \( \text{Hom}(T, \bullet) \). We also have a derived equivalence \( R \text{Hom}(T, \bullet) : D^b(C_1) \to D^b(C_2) \). This equivalence maps injectives to tiltings and, obviously, tiltings to projectives. We write \( C_1^* \) for \( C_2 \). The functor \( R \) is called the (covariant) Ringel duality, and the category \( C_1^\vee \) is called the Ringel dual of \( C_1 \).

In the case when \( C_1 = \mathcal{O}_c \) the Ringel duality was realized explicitly in [GGQR, 4.1]. Namely, set \( n := \dim \mathfrak{h} \) and consider the functor \( D := R \text{Hom}_{\mathcal{O}}(\bullet, H_c)[n] \). It defines a derived equivalence between \( D^b(\mathcal{O}_c) \) and \( D^b(\mathcal{O}_c^{\text{opp}}) \), that maps \( \Delta_c(\lambda) \) to \( \Delta_c^\vee(\lambda) \). Hence the functor \( D \) realizes \( R^{-1} \) and \( \mathcal{O}_c = \mathcal{O}_c^{\text{opp}, \nabla} \).

An important property of the functor \( D \) is that it is a perverse equivalence with respect to the support filtrations. Namely, set \( C_1 := \mathcal{O}_c, C_2 := \mathcal{O}_c^{\text{opp}} \). Consider a filtration \( C^1 := C_0^1 \supset C_1^1 \supset C_2^1 \supset \ldots \supset C_j^1 \supset \ldots \supset C_{n+1}^1 := \{0\} \), where, by definition, \( C_j^1 \) consists of all modules \( M \in C^1 \) with \( \dim V(M) \leq n - j \). Define the filtration \( \mathcal{O}_c^{\text{opp}, \nabla} = C_0^2 \supset C_1^2 \supset \ldots \supset C_{n+1}^2 = \{0\} \) similarly. The definition of a perverse equivalence was given in Subsection 2.3.

**Lemma 2.5.** The equivalence \( D : D^b(\mathcal{O}_c) \to D^b(\mathcal{O}_c^{\text{opp}}) \) is perverse with respect to the filtrations introduced above.

**Proof.** Pick \( M \in \mathcal{O}_c \) and equip it with a good filtration. Let \( H_{\mathfrak{h}, c} \) denote the Rees algebra, and \( M_{\mathfrak{h}} \) be the Rees module over \( H_{\mathfrak{h}, c} \) constructed from the filtration on \( M \). The right \( H_{\mathfrak{h}, c} \)-module \( \text{Ext}_{H_{\mathfrak{h}, c}}^i(M_{\mathfrak{h}}, H_{\mathfrak{h}, c}) \) is graded and \( \text{Ext}_{H_{\mathfrak{h}, c}}^i(M_{\mathfrak{h}}, H_{\mathfrak{h}, c}/(h - 1) = \text{Ext}_{H_{\mathfrak{h}, c}}^i(M, H_c) \).

So \( \text{Ext}_{H_{\mathfrak{h}, c}}^i(M, H_c) \) is equipped with a filtration. The module \( \text{Ext}_{H_{\mathfrak{h}, c}}^i(M, H_{\mathfrak{h}, c}) \) is finitely generated so the filtration is good. Further, we have a standard short exact sequence

\[
\text{Ext}_{H_{\mathfrak{h}, c}}^i(M, H_{\mathfrak{h}, c}) \xrightarrow{h} \text{Ext}_{H_{\mathfrak{h}, c}}^i(M, H_{\mathfrak{h}, c}) \to \text{Ext}_{H_{\mathfrak{h}, c}}^i(M, H_{\mathfrak{h}, c}/(h))
\]

Note that the last term is naturally identified with \( \text{Ext}_{S(\mathfrak{h} \oplus \mathfrak{h}^*) \# W}(gr M, S(\mathfrak{h} \oplus \mathfrak{h}^*) \# W) \).

We conclude that \( gr \text{Ext}_{H_{\mathfrak{h}, c}}^i(M, H_c) \subset \text{Ext}_{gr H_{\mathfrak{h}, c}}^i(gr M, gr H_c) \).

If the support of \( gr M \) has codimension \( n+s \), then \( \text{Ext}_{gr H_{\mathfrak{h}, c}}^k(gr M, gr H_c) = 0 \) for \( k < n+s \), and \( \text{Ext}_{gr H_{\mathfrak{h}, c}}^k(gr M, gr H_c) \) has support of codimension larger than \( n+s \) for \( k > n+s \). Both claims follow from the observations that \( \text{Ext}_{gr H_{\mathfrak{h}, c}}^k(gr M, gr H_c) \to \text{Ext}_{S(\mathfrak{h} \oplus \mathfrak{h}^*) \# W}(gr M, S(\mathfrak{h} \oplus \mathfrak{h}^*) \# W) \).

So (I) and (II) in the definition of a perverse equivalence hold for \( D \). (III) follows from \( D^2 = \text{id} \) and a standard spectral sequence for the composition of derived functors.

We see that the inverse Ringel duality \( R^{-1} \) is perverse.

2.6. **Remarks on orderings and parameterizations.** We consider the \( \mathbb{Z} \)-lattice and the \( \mathbb{Q} \)-lattice \( p \subset p_{\mathfrak{h}} \subset p^* \) spanned by the elements \( h_{\mathfrak{h}, i} - h_{\mathfrak{h}, j} \) and the dual lattices \( p \subset p_{\mathfrak{h}} \subset p \). So \( c - d \in p_{\mathfrak{h}} \) if and only if there are nonzero scalars \( \alpha_H, H \in \mathfrak{h}/W \) such that \( q_{\mathfrak{h}, i}^H = \alpha_H q_{\mathfrak{h}, i} \) for all \( H \) and \( i \).

We will need a certain sublattice in \( p_{\mathfrak{h}} \). In [BC, 7.2], Berest and Chalykh established a group homomorphism \( \text{tw} : p_{\mathfrak{h}} \to \text{Bij} (\text{Irr } W) \) called the KZ twist. Set \( p_{\mathfrak{h}} := \ker \text{tw} \).

We will use another spanning set for \( p_{\mathfrak{h}} \). We can assign an element in \( p_{\mathfrak{h}} \) to a one-dimensional character of \( W \) as follows. There is a homomorphism \( \text{Hom}(W, \mathbb{C}^*) \to \)
\[\prod_{H\in\Delta/W} \text{Irr}(W_H)\] given by the restriction. It turns out that this map is an isomorphism, see [R1, 3.3.1]. So to an arbitrary collection of elements \((a_H)\) with \(0 \leq a_H \leq \ell_H - 1\) we can assign the character of \(W\) that sends \(s\) to \(\lambda_s^{-a_H}\). To a character \(\chi\) given in this form we assign the element \(\tilde{\chi} \in \mathfrak{p}\) given by \(h_H, i(\tilde{\chi}) = 1 - \frac{a_i}{\ell_H}\) if \(i \geq \ell - a_H\) and \(-\frac{a_i}{\ell_H}\) if \(i < \ell - a_H\). The motivation behind this definition will be explained in Subsection 3.1. Clearly, the elements of the form \(\tilde{\chi}\) span \(\mathfrak{p}_\mathbb{Q}\).

Let us proceed to orders.

**Lemma 2.6.** The function \(c \mapsto c_\lambda\) is rational on \(\mathfrak{p}_\mathbb{Q}\).

**Proof.** The action of the element \(\varphi_H = \sum_{s \in W_H \setminus \{1\}} \frac{2c(s)}{\lambda_s - 1} s\) on the \(W_H\)-isotypic component corresponding to the character \(s \mapsto \lambda_s^j\) is by the scalar \(\ell_H h_{H,-j}\). The claim follows. \(\square\)

Define an equivalence relation \(\sim\) on \(\text{Irr}(W)\) by setting \(\lambda \sim \lambda'\) if \(c_\lambda = c_{\lambda'}\) for every parameter \(c\). Note that different one-dimensional representations cannot be equivalent. Now if \(\lambda \not\sim \mu\), then we have the hyperplane \(\Pi_{\lambda,\mu}\) in \(\mathfrak{p}\) given by \(c_\lambda = c_\mu\). All the hyperplanes \(\Pi_{\lambda,\mu}\) are rational.

Fix a coset \(c + \mathfrak{p}_\mathbb{Z}\) and consider \(c'\) in this coset. We write \(c \sim c'\) if \(\lambda \leq c' \lambda'\) implies \(\lambda \leq c' \lambda'\). We write \(c \prec c'\) if \(c \sim c'\) and \(c' \prec c\). The equivalence classes are relative interiors in the cones defined by the hyperplane arrangement \(\{\Pi_{\lambda,\mu}, \lambda \not\sim \mu, c_\lambda - c_\mu \in \mathbb{Q}\\}\) on \(c + \mathfrak{p}_\mathbb{Z}\).

We are mostly interested in the open cones. Below the open cones in this stratification will be called open chambers. For each open chamber we have its opposite chamber, where the order is opposite. Note that if \(c\) is Weil generic in \(\mathfrak{p}\), we have just one open chamber, while for a Weil generic \(c\) on a rational hyperplane parallel to \(\Pi_{\lambda,\mu}\) we have exactly two open cones that are opposite to each other.

### 2.7 Rouquier equivalence theorem

In [R1, 4.2], Rouquier established some tools to prove an equivalence of categories \(\mathcal{O}_c, \mathcal{O}_{c'}\) for different parameters \(c\).

We start with a general setting. Let \(A_h\) be a \(\mathbb{C}[[h]]\)-algebra that is free of finite rank as a module over \(\mathbb{C}[[h]]\). Assume that \(A_h[h^{-1}]\) is split semisimple. Let \(B_h\) be another \(\mathbb{C}[[h]]\)-algebra (free of finite rank) and let \(P_h\) be a projective \(B_h\)-module with a fixed isomorphism \(\text{End}_{B_h}(P_h)^{\text{opp}} \cong A_h\). Assume that \(P_h[h^{-1}]\) is a projective generator of \(B_h[h^{-1}]-\text{mod}\). So we have an exact functor \(\pi_h = \text{Hom}_{B_h}(P_h, \cdot) : B_h-\text{mod} \to A_h-\text{mod}\) that is an equivalence after inverting \(h\). Next, suppose that \(B_h\)-mod is a highest weight category over \(\mathbb{C}[[h]]\) with \(\Lambda\) being an indexing set of simples.

Let \(B, A, \pi\) be the specializations of \(B_h, A_h, \pi_h\) to \(h = 0\). Note that functor \(\pi_h\) defines a bijection between \(\Lambda\) in \(A_h[h^{-1}]-\text{mod}\) given by \(\lambda \mapsto \Delta_h(\lambda)[h^{-1}]\).

We say that \(\pi\) is 0-faithful if it is fully faithful on standardly filtered objects.

The following result is due to Rouquier, [R1, Proposition 4.42, Theorem 4.49].

**Proposition 2.7.** Let \((B_h, P_h, \pi_h)\) and \((B'_h, P'_h, \pi'_h)\) be two triples as above. Suppose that the following hold:

1. The functors \(\pi, \pi'\) are 0-faithful.
2. There is an order on \(\text{Irr}(B) \cong \text{Irr}(A_h[h^{-1}]) \cong \text{Irr}(B')\) that is highest weight for both \(B-\text{mod}\) and \(B'-\text{mod}\).

Then there is an equivalence \(B_h-\text{mod} \cong B'_h-\text{mod}\) that intertwines the functors \(\pi_h, \pi'_h\).

The proof goes as follows. First, one notices that \(\pi_h(\Delta_h(\lambda)) \cong \pi'_h(\Delta'_h(\lambda))\), this follows from [R1, Lemma 4.48] and uses only (2). Then one shows that the functors \(\pi_h, \pi'_h\)
are 1-faithful, i.e., preserve both Hom’s and Ext’s between standardly filtered objects, Proposition 4.42, this follows from (1). Finally, one uses a construction of the indecomposable projectives in $B_h$-mod recalled in Subsection 2.3 and gets $\pi_h(P_h(\lambda)) \cong \pi'_h(P'_h(\lambda))$. This completes the proof.

Rouquier applied this result to Cherednik categories $O$.

**Proposition 2.8.** Suppose that $c,c' \in p$ satisfy the following conditions:

(i) $c - c' \in p_Z$.

(ii) $\text{tw}(c' - c) = \text{id}$ and the ordering $\leq^c$ refines $\leq^{c'}$.

(iii) For each hyperplane $H$ and different $i,j$, we have $q_{H,i} \neq q_{H,j}$.

Then there is an equivalence $O_c \cong O_{c'}$ of highest weight categories mapping $\Delta_c(\lambda)$ to $\Delta_{c'}(\lambda)$ that intertwines the $KZ$ functors $O_c, O_{c'} \to H_q$-mod.

Below we will see that assumption (iii) is not necessary.

The scheme of the proof is as follows. Choose a generic line through $c$. Consider the algebra $H_c(W) \coloneqq \mathbb{C}[h] \otimes_{\mathbb{C}[p]} H_p$, where the homomorphism $\mathbb{C}[p] \to \mathbb{C}[h]$ is given by restricting to the formal neighborhood of 0 in $\ell$. Form an analogous algebra $\tilde{H}_{c'}(W)$ (for the line $\ell + c' - c$). Let $\tilde{P}, \tilde{P}'$ be projective generators in the corresponding categories $\mathcal{O}$. Let $B_h, B'_h$ be the endomorphism algebras (with opposite multiplications) for these projectives. For $P_h, P'_h$ we will take the objects $P_{KZ}$ in these categories (or more precisely the deformations of $P_{KZ} \in O_c, P_{KZ} \in O_{c'}$). The endomorphism algebras of $P_h, P'_h$ is the same algebra $H_q(W)$ by (i). So we can apply Proposition 2.7 (1) there holds because of (iii). The identification $\text{Irr}(W) \cong \text{Irr}(H_q) \cong \text{Irr}(W)$ is the identity because of $\text{tw}(c - c') = \text{id}$ and (2) now holds because of (ii).

### 2.8. Induction and restriction functors for category $O$.

Let $W' \subset W$ be a parabolic subgroup. We have a natural homomorphism $H_q(W') \to H_q(W)$ that gives rise to the restriction functor $H^\text{Res}_{W'} : H_q(W')$-mod $\to H_q(W)$-mod. When we write $q$ in $H_q(W')$ we mean the parameter $q'$ given by $q'_{H,i} = q_{H,i}$ for every reflection hyperplane of $W'$. The functor $H^\text{Res}_{W'}$ has left (induction) $H^\text{Ind}_{W'}$ and right (coinduction) $H^\text{Coind}_{W'}$, functors.

On the other hand, in [BE, 3.5], Bezrukavnikov and Etingof defined induction and restriction functors for the Cherednik categories $\mathcal{O}$. Namely, we have the restriction functor $O^\text{Res}_{W'} : O_c(W') \to O_c(W)$ and its left adjoint, the induction functor $O^\text{Ind}_{W'} : O_c(W) \to O_c(W')$.

Let us recall the construction of $O^\text{Res}_{W'}$ from [BE]. Pick $b \in \mathfrak{h}$ such that $W_b = W'$. We can consider the completion $H^{\wedge}_c := \mathbb{C}[b/W]^{\wedge} \otimes_{\mathbb{C}[b/W]} H_c$, where $\mathbb{C}[b/W]^{\wedge}$ is the completion of $\mathbb{C}[b/W]$ with respect to the maximal ideal defined by $b$. The completion $H^{\wedge}_c$ is a filtered algebra. Similarly, we can consider the completion $H_{c}(W', \mathfrak{h})^{\wedge}_c := \mathbb{C}[b/W]^{\wedge} \otimes_{\mathbb{C}[b/W]} H_c(W', \mathfrak{h})$ (where we write $H_c(W', \mathfrak{h})$ for $H_c(W') \otimes D(\mathfrak{h}^{W'})$) and form the centralizer algebra $Z(W,W', H_c(W', h)^{\wedge}_c)$ as in [BE, 3.2], as an algebra, this is just $\text{Mat}_{W/W'}(H_c(W', \mathfrak{h})^{\wedge}_c)$, Bezrukavnikov and Etingof, [BE, 3.3], produced an explicit filtration preserving isomorphism $\theta_b : H_c(W, \mathfrak{h})^{\wedge}_c \cong Z(W,W', H_c(W', h)^{\wedge}_c)$. Form the completion $M^{\wedge} : \mathbb{C}[b/W]^{\wedge} \otimes_{\mathbb{C}[b/W]} M$. This gives rise to a functor from $O_c$ to the category $O(H^{\wedge}_c)$ of all $H^{\wedge}_c$-modules finitely generated over $S(\mathfrak{h}^{*})^{\wedge}_c$. Then we take elements in $M' := e(W') \theta_b(M^{\wedge})$ that are finite for the action of the Euler element of $H_c(W', h)$ (here $e(W')$ is a primitive idempotent in $Z(W,W', CW')$ defining a Morita equivalence between $Z(W,W', CW')$ and $CW'$). Let $M'_{\text{fin}}$ be the resulting $H_c(W', h)$-module. Then
Proof. (1) is a part of [BEG, Lemma 3.3]. (2) is straightforward. In (3) notice that \( \bar{\chi} \) is also locally nilpotent.

It was shown in [L3], see also [S] for the same result but under additional restrictions, that the functors \( \mathcal{O} \text{Res}_W^W \) and \( \mathcal{O} \text{Ind}_W^W \) are biadjoint. Moreover, Shan has checked in [S, Theorem 2.1] that the restriction functors intertwine the KZ functors: \( \text{KZ} \circ \mathcal{O} \text{Res}_W^W = \mathcal{H} \text{Res}_W^W \circ \text{KZ} \). Here \( \text{KZ} \) stands for the KZ functor \( \mathcal{O}_c(W) \to \mathcal{H}_q(W') - \text{mod} \).

It is clear from the construction in [BE, 3.5] that the functor \( \mathcal{H}_q(W') - \text{mod} \to \mathcal{H}_q(W) - \text{mod} \). It follows that \( \mathcal{H} \text{Res}_W^W \) admits a biadjoint functor, so \( \mathcal{H} \text{Ind}_W^W \cong \mathcal{H} \text{Coind}_W^W \). We also note that the induction functors intertwine the KZ functors as well.

For \( M \in \mathcal{O}_c(W) \), the associated variety \( V(M) \) is the union of the strata of the form \( Wb^W \), where \( W' \) is a parabolic subgroup. A stratum \( Wb^W \) is the union of irreducible components of \( V(M) \) if and only if \( \mathcal{O} \text{Res}_W^W(M) \) is finite dimensional and nonzero.

3. Harish-Chandra bimodules

3.1. Harish-Chandra bimodules. In this subsection, we recall the definition and some basic results about Harish-Chandra (HC) bimodules over the algebras \( H_{R,c} \) and \( eH_{R,c}e \) (we write \( H_c \) for \( H_{R,c}(W) \), etc.).

A definition of a HC \( H_c \)-\( H_c \)-bimodule was introduced in [BEG, Section 3]. A HC bimodule, by definition, is a finitely generated \( H_c \)-\( H_c \)-bimodule, where the adjoint actions of \( S(\mathfrak{h}^*)^W \), \( S(\mathfrak{h})^W \) are locally nilpotent.

Here are some basic properties of HC bimodules.

**Proposition 3.1.** Let \( M \) be a HC \( H_c \)-\( H_c \)-bimodule. Then the following is true:

1. \( M \) is finitely generated as a left \( H_c \)-module, as a right \( H_c \)-module and as a \( S(\mathfrak{h}^*)^W \otimes S(\mathfrak{h})^W \)-module (with \( S(\mathfrak{h}^*)^W \subset H_c^c \), \( S(\mathfrak{h})^W \subset H_c^c \)).
2. If \( N \) is a HC \( H_c \)-\( H_c \)-bimodule, then \( N \otimes_{H_c} M \) is also HC.
3. If \( N \in \mathcal{O}_c \), then \( M \otimes_{H_c} N \in \mathcal{O}_c \).

**Proof.** (1) is a part of [BEG, Lemma 3.3]. (2) is straightforward. In (3) notice that \( M \otimes_{H_c} N \) is finitely generated over \( S(\mathfrak{h}^*)^W \) thanks to (1) and has locally nilpotent action of the augmentation ideal \( S(\mathfrak{h}^*)_+^W \subset S(\mathfrak{h})^W \). The latter easily implies that the action of \( \mathfrak{h} \) is also locally nilpotent. \( \square \)

We can give an analogous definition for \( H_p \)-bimodules. Namely, we say that an \( H_p \)-bimodule \( M \) is HC if there is \( \psi \in p \) such that \( pm = m(p - \langle \psi, p \rangle) \) for any \( p \in p^* \) and the adjoint actions of \( S(\mathfrak{h}^*)^W, S(\mathfrak{h})^W \) are locally nilpotent. Let \( \mathcal{HC}(H_p, \psi) \) denote the category of such HC bimodules. We could also relax the condition on the compatibility between the left and the right \( \mathbb{C}[p] \)-actions but this is technical. Also we can speak about HC bimodules over the spherical subalgebras.

Let us provide an important example. Let \( \chi \) be a character of \( W \), \( e_{\chi} \in \mathbb{C}W \) be the corresponding idempotent, \( \bar{\chi} \) be the element in \( p_z \) constructed in Subsection 2.6. According to [BC, 5.4], there is an isomorphism \( \varphi : eH_p e \sim e_{\chi} H_p e_{\chi} \) that maps \( p \in p^* \) to \( p + \langle \bar{\chi}, p \rangle \).

**Lemma 3.2.** \( e_{H_{c+\bar{\chi}}} e_{\chi} \) is a HC \( e_{H_{c+\bar{\chi}}} e_{H_{c+\bar{\chi}}} \)-bimodule.
Proof. According to the construction of the isomorphism $\varphi$ in [BC 5.4], this isomorphism preserves the filtrations given by $\deg \mathfrak{h}, \deg W = 0, \deg \mathfrak{h}^* = 1$ and the gradings induced by ad $h$. Moreover, the associated graded isomorphism $S(\mathfrak{h} \oplus \mathfrak{h}^*)^W = \text{gr} eHe \xrightarrow{\sim} \text{gr} e_{\chi}H_{e+\chi}e_{\chi} = S(\mathfrak{h} \oplus \mathfrak{h}^*)^W$ is the identity. The associated graded of $eH_{e+\chi}e_{\chi}$ is the $S(\mathfrak{h} \oplus \mathfrak{h}^*)^W$-bimodule $S(\mathfrak{h} \oplus \mathfrak{h}^*)^W, x^{-1}$. Pick a homogeneous element $a \in S(\mathfrak{h})^W$. The operator induced by $[a, \cdot]$ on $\text{gr} eH_{e+\chi}e_{\chi} = S(\mathfrak{h} \oplus \mathfrak{h}^*)^W, x^{-1}$ is zero hence $[a, \cdot]$ is locally nilpotent. Now let us pick a homogeneous element $b \in S(\mathfrak{h}^*)^W$ and prove that $[b, \cdot]$ is locally nilpotent on $eH_{e+\chi}e_{\chi}$. The bimodule $eH_{e+\chi}e_{\chi}$ is graded and the grading is compatible with the filtration. Then we can twist the filtration using the grading, compare to Remark 3.3 below, so that the multiplication by $\mathfrak{h}^*$ preserves the filtration, and this does not change the associated graded. This shows that $[b, \cdot]$ is locally nilpotent.

So we get an $H_{e+\chi}H_c$ bimodule

$$B_{e+\chi} := H_{e+\chi}e \otimes_{eH_{e+\chi}e} eH_{e+\chi}e_{\chi} \otimes_{eHe} eH_c.$$  

Similarly, we get the $H_cH_{e+\chi}$ bimodule $B_{c+\chi}$. These bimodules are HC. We also can define the objects $B_{p, \pm \chi} \in \text{HC}(H_p, \pm \chi), B_{p, \mp \chi} \in \text{HC}(H_p, \mp \chi)$ that these bimodules are HC follows from Lemma 3.2 if the adjoint actions of $S(\mathfrak{h})^W, S(\mathfrak{h}^*)^W$ are locally nilpotent on all fibers over $p$, then they are locally nilpotent on the whole bimodule.

There is an alternative definition of HC bimodules given in [L2]. Equip the algebra $H_p$ with a filtration, $H_p = \bigcup_{i\geq 0} F_i H_p$, by setting $\deg \mathfrak{h} = \deg \mathfrak{p} = 1, \deg CW = \deg \mathfrak{h}^* = 0$. The algebra $\text{gr} H_p$ is finite over its center denoted by $Z_p$ (recall the Satake isomorphism from [EG] Theorem 3.1], $Z_p \cong e(\text{gr} H_p)e$, given by $z \mapsto ze$). By a Harish-Chandra $H_p$-bimodule we mean a bimodule $M$ that can be equipped with an increasing filtration $M = \bigcup_{i \in \mathbb{Z}} M_i$, such that $\text{gr} M$ is finitely generated over $\text{gr} H_p$ and, moreover, the left and the right actions of $Z_p$ commute. Such a filtration is called good. One can give a definition of a HC $eH_p$-bimodule in a similar fashion.

Remark 3.3. Let us remark that we used a different filtration, denote it here by $F'$, in [L2 3.4]. The filtrations are related as follows:

$$F'_i H_p = \bigoplus_k (F_k H_p) \cap \{ a \in H_p | [h, a] = (i - 2k)a \}.$$  

It follows that a less technical definition we use now is equivalent to what we have used in [L2 3.4].

We have checked in [L2 5.4] that any HC bimodule in the sense of [BEG] is also HC in the sense of [L2]. Conversely, let $M$ be a HC bimodule in the sense of [L2] such that $pm = m(p - (\psi, p))$. Then $M \in \text{HC}(H_p, \psi)$.

To $M \in \text{HC}(H_p, \psi)$ we can assign its associated variety, $V(M) \subset (\mathfrak{h} \oplus \mathfrak{h}^*)/W$. By definition, this is the support of $\text{gr} M/\mathfrak{p} \text{gr} M$, where the associated graded is taken with respect to a good filtration.

There is one important property of HC bimodules that is easy to see from the definition in [L2]. Namely, for a HC $H_p$-bimodule $M$ we can consider its specialization $M_c := M \otimes_{\mathfrak{C}_p} \mathbb{C}_c$. By the right support of $M$ we mean $\text{Supp}'(M) := \{ c \in \mathfrak{p} | M_c \neq 0 \}$. The following lemma is proved completely analogously to [BL] Lemma 5.5, Corollary 5.6.

Lemma 3.4. Let $\mathfrak{p}' \subset \mathfrak{p}$ be an affine subspace, $\psi \in \mathfrak{p}$, and let $M \in \text{HC}(H_{\mathfrak{p}'}, \psi)$. Then the following is true:
(1) There is \( f \in \mathbb{C}[\mathfrak{p}'] \) such that \( M \otimes_{\mathbb{C}[\mathfrak{p}']} \mathbb{C}[\mathfrak{p}'_f] \) is a free module over \( \mathbb{C}[\mathfrak{p}'_f] \). Here \( \mathfrak{p}'_f \) is the principal open subset in \( \mathfrak{p}' \) defined by \( f \).

(2) The support \( \text{Supp}^r(M) \) is a constructible subset of \( \mathfrak{p}' \).

Let us deduce some corollaries from Lemma 3.4.

**Corollary 3.5.** The following is true:

1. There is a Zariski open subset of \( \mathfrak{p} \) consisting of spherical parameters.
2. Let \( \chi \) be a character of \( W \). Then there is a Zariski open subset of parameters \( c \) such that \( \mathcal{B}_{c,\chi} \) and \( \mathcal{B}_{c+\chi,-\chi} \) are mutually inverse Morita equivalences.

**Proof.** The algebra \( H_c \) is simple for a Weil generic \( c \), see, e.g., \([L2, 4.2]\), so such \( c \) is spherical. (1) follows from (2) of Lemma 3.4 applied to the bimodule \( H_{\delta}/H_{\delta}eH_{\delta} \).

Let us proceed to (2). Note that we have natural homomorphisms

\[
\mathcal{B}_{\mathfrak{p},-\chi} \otimes_{H_{\delta}} \mathcal{B}_{\mathfrak{p},\chi} \to H_{\delta}, \quad \mathcal{B}_{\mathfrak{p},\chi} \otimes_{H_{\delta}} \mathcal{B}_{\mathfrak{p},-\chi} \to H_{\delta}.
\]

Their specializations to Weil generic \( c \) are isomorphisms because they are always isomorphisms after inverting \( \delta \) and \( H_c \) is simple. So they are also isomorphisms for a Zariski generic \( c \).

Also a direct analog of \([BL, \text{Corollary 5.7}]\) holds.

**Remark 3.6.** It is easy to see that a direct analog of Lemma 3.4 holds for the category \( \mathcal{O}_{\mathfrak{p}'} \).

3.2. **Restriction functors for HC bimodules: construction.** Pick a parabolic subgroup \( W' \subset W \). Set \( \Xi := \mathcal{N}_W(W')/W' \). Let \( \mathfrak{h}_{W'} \) denote a unique \( W' \)-stable complement to \( \mathfrak{h}^{W'} \), the spaces \( \mathfrak{h}_{W'}, \mathfrak{h}^{W'} \) are \( \mathcal{N}_W(W') \)-stable. Form the algebra \( H_{\mathfrak{p}}(W') \) for the \( W' \)-action on \( \mathfrak{h}_{W'} \). By definition \( H_{\mathfrak{p}}(W') := \mathbb{C}[\mathfrak{p}] \otimes_{\mathbb{C}[\mathfrak{p}']} H_{\mathfrak{p}'}(W') \), where \( \mathfrak{p}' \) is the parameter space for \( W' \), it comes with a natural linear map \( \mathfrak{p} \to \mathfrak{p}' \) induced by the restriction from \( S \) to \( S \cap W' \). We consider the category of \( \Xi \)-equivariant \( H_{\mathfrak{p}}(W') \)-modules, by definition, it consists of the HC \( H_{\mathfrak{p}}(W') \)-bimodules \( N \) equipped with a \( \mathcal{N}_W(W') \)-action that

- restricts to the adjoint \( W' \)-action,
- and makes the structure map \( H_{\mathfrak{p}}(W') \otimes N \otimes H_{\mathfrak{p}}(W') \to N \) equivariant for the \( \mathcal{N}_W(W') \)-action.

We denote the category of \( \Xi \)-equivariant HC \( H_{\mathfrak{p}}(W') \)-bimodules by \( \mathcal{HC}_{\Xi}(H_{\mathfrak{p}}(W')) \).

In \([L2]\), we have introduced a functor \( \bullet_{\mathfrak{p},W'} : \mathcal{HC}(H_{\mathfrak{p}}(W), \psi) \to \mathcal{HC}_{\Xi}(H_{\mathfrak{p}}(W'), \psi) \). Here we are going to explain a construction of this functor that is equivalent to but simpler than the one given in \([L2]\).

Set \( Y := \{ b \in \mathfrak{h} | W_b = W' \} \), \( \mathfrak{h}^{\text{reg}}_{W'} = \{ b \in \mathfrak{h} | W_b \subset W' \} \) so that \( Y \subset \mathfrak{h}^{\text{reg}}_{W'} \) is closed and \( \mathfrak{h}^{\text{reg}}_{W'} \subset \mathfrak{h} \) is a principal open subset. Set

\[
H_{\mathfrak{p},Y} := \mathbb{C}[\mathfrak{h}^{\text{reg}}_{W'}/W'^{\text{reg}}]^{\mathfrak{p}Y} \otimes_{\mathbb{C}[\mathfrak{h}/W]} H_{\mathfrak{p}},
\]

where \( \mathbb{C}[\mathfrak{h}^{\text{reg}}_{W'}/W'^{\text{reg}}]^{\mathfrak{p}Y} \) is the usual completion along a closed subvariety, note that this algebra is etale over \( \mathbb{C}[\mathfrak{h}/W] \). The space \( H_{\mathfrak{p},Y} \) is easily seen to be an algebra and this algebra is filtered. Moreover, the group \( \Xi \) acts on \( H_{\mathfrak{p},Y} \) by filtration preserving algebra isomorphisms. So we can introduce a notion of a \( \Xi \)-equivariant HC \( H_{\mathfrak{p},Y} \)-bimodule: it is a \( \Xi \)-equivariant \( H_{\mathfrak{p},Y} \)-bimodule \( M' \) equipped with a filtration such that \( \text{gr} M' \) is a finitely generated \( \mathbb{C}[\mathfrak{h}^{\text{reg}}_{W'}/W'^{\text{reg}}]^{\mathfrak{p}Y} \otimes_{\mathbb{C}[\mathfrak{h}/W]} Z_{\mathfrak{p}}(W) \)-module. We have a functor \( \mathcal{F}' : \mathcal{HC}(H_{\mathfrak{p}}, \psi) \to \mathcal{HC}_{\Xi}(H_{\mathfrak{p},Y}, \psi) \) given by \( M \mapsto \mathbb{C}[\mathfrak{h}^{\text{reg}}_{W'}/W'^{\text{reg}}]^{\mathfrak{p}Y} \otimes_{\mathbb{C}[\mathfrak{h}/W]} M \).
On the other hand, we can form the algebra $H_p(W', \mathfrak{h})^{\wedge_Y} := \mathbb{C}[\mathfrak{h}^{reg-W'}/W']^{\wedge_Y} \otimes_{\mathbb{C}[\mathfrak{h}/W]} H_p(W', \mathfrak{h})$. The algebra $H_p(W', \mathfrak{h})^{\wedge_Y}$ is filtered. Further, form the centralizer algebra $Z(W, W', H_p(W', \mathfrak{h})^{\wedge_Y})$ from [BE 3.2]. The group $\Xi$ acts on $Z(W, W', H_p(W', \mathfrak{h})^{\wedge_Y})$. Then, similarly to [BE 3.2], there is a filtration preserving isomorphism

$$H_p^{\wedge_Y} \cong Z(W, W', H_p(W', \mathfrak{h})^{\wedge_Y})$$

that coincides with a natural isomorphism

$$\mathbb{C}[\mathfrak{h}^{reg-W'}/W']^{\wedge_Y} \otimes_{\mathbb{C}[\mathfrak{h}/W]} (\mathbb{C}[\mathfrak{h}] \# W) \cong Z(W, W', \mathbb{C}[\mathfrak{h}^{reg-W'}]^{\wedge_Y})$$
on the filtration zero components, similarly to [BE 3.3]. The isomorphism (3.1) is $\mathbb{C}[\mathfrak{p}]$-linear and $\Xi$-equivariant. It is induced by an isomorphism from [L2 2.13] by passing to $\mathbb{C}^{\times}$-finite elements and then taking the quotient by $\hbar - 1$, compare with [L3 2.3]. Note that the isomorphism we use does not need to be given by formulas in [BE 3.3].

The isomorphism (3.1) gives rise to an equivalence

$$\text{HC}^{\Xi}(H_p^{\wedge_Y}, \psi) \cong \text{HC}^{\Xi}(H_p(W', \mathfrak{h})^{\wedge_Y}, \psi)$$
given by the push-forward under (3.1) followed by the multiplication by a primitive idempotent $e(W') \in Z(W, W', CW')$, compare with Subsection 2.8. Let $\mathcal{F}: \text{HC}(H_p, \psi) \rightarrow \text{HC}^{\Xi}(H_p(W', \mathfrak{h})^{\wedge_Y}, \psi)$ be the resulting functor, it is exact.

On the other hand, we have a functor $\mathcal{G}: \text{HC}^{\Xi}(H_p(W'), \psi) \rightarrow \text{HC}^{\Xi}(H_p(W', \mathfrak{h})^{\wedge_Y}, \psi)$ given by $N \mapsto \mathbb{C}[Y \times \mathfrak{h}_{W'}/W']^{\wedge_Y} \otimes_{\mathbb{C}[Y \times \mathfrak{h}_{W'}/W']} (D(Y) \otimes N)$.

**Lemma 3.7.** The functor $\mathcal{G}$ is a full embedding whose image contains that of $\mathcal{F}$. The functor $\mathcal{G}^{-1} \circ \mathcal{F}: \text{HC}(H_p, \psi) \rightarrow \text{HC}^{\Xi}(H_p(W'), \psi)$ coincides with the functor $\bullet_\mathfrak{h}$ from [L2].

**Proof.** Let us show that the functor $\mathcal{G}$ is a full embedding by producing a left inverse functor. First, take the centralizer of $D(Y)$ in $\mathcal{G}(N)$. The result is $\mathbb{C}[\mathfrak{h}_{W'}/W']^{\wedge_Y} \otimes_{\mathbb{C}[\mathfrak{h}_{W'}/W']} N$. Then take the elements that are locally finite for the Euler element $h' \in H_p(W')$. The resulting bimodule is $N$. So we have constructed a left inverse functor for $\mathcal{G}$.

The remaining two claims are proved similarly. In [L2 3.6] the functor $\bullet_\mathfrak{h}$ was constructed as follows. Pick $M \in \text{HC}(H_p, \psi)$, equip it with a good filtration and form the Rees bimodule $M_h$. Then we complete the bimodule $M_h$ with respect to the symplectic leaf $L_{W'}$ corresponding to $W'$, this leaf is given by $\{(x, y) \in \mathfrak{h} \oplus \mathfrak{h}^*|W(x, y) = W'\}/\Xi$. The corresponding completion $R_h(H_p)^{\wedge_{\xi_{W'}}}$ was shown in [L2 Section 2] to be isomorphic to a twist of

$$Z \left(W, W', R_h(H_p(W', \mathfrak{h}))^{\wedge_{\xi_{W'}}}\right)^{\Xi},$$

where we write $\tilde{L}_{W'}$ for $\{(x, y) \in \mathfrak{h} \oplus \mathfrak{h}^*|W(x, y) = W'\}$. By a twist we mean a sheaf whose sections on open affine subsets are the same but gluing maps are different. In order to construct $\bullet_{\mathfrak{h}, W'}$, we first lift $M_h^{\wedge_{\xi_{W'}}}$ (viewed as a sheaf on $L_{W'}$) to a sheaf on $\tilde{L}_{W'}$, the resulting sheaf is given by

$$Z \left(W, W', R_h(H_p(W', \mathfrak{h}))^{\wedge_{\xi_{W'}}}\right)^{tw} \otimes_{Z(W, W', R_h(H_p(W', \mathfrak{h}))^{\wedge_{\xi_{W'}}})^{tw, \Xi}} M_h^{\wedge_{\xi_{W'}}}.$$
Let \( \mathcal{N}_h \), take the finite vectors (for the \( \mathbb{C}^\times \)-action induced by the dilations on \( \mathfrak{h} \oplus \mathfrak{h}^* \)) in \( \mathcal{N}_h \) and mod out \( h - 1 \). The resulting bimodule is \( \mathcal{M}_{i,W'} \).

Thanks to that construction, what we need to show is that \( R_h(\mathcal{F}(M)) \) coincides with the \( \mathbb{C}^\times \)-finite part of the global sections of the lift of \( \mathcal{M}_h^{\aleph W'} \) to the open subset \( Y \times \mathfrak{h}^{\aleph W'} \subset Y \) (here we consider the \( \mathbb{C}^\times \)-action that is trivial on \( \mathfrak{h}^* \) and by the dilations on \( \mathfrak{h} \)). The sections of interest is nothing else but \( \mathbb{C}[\mathfrak{h}^{\aleph W'}/W'\wedge Y \otimes_{\mathbb{C}[h/W]} M_h^{\aleph h} \) and we need to show that the \( \mathbb{C}^\times \)-finite part coincides with \( \mathbb{C}[\mathfrak{h}^{\aleph W'}/W/\wedge Y \otimes_{\mathbb{C}[h/W]} M_h \). Recall that \( \text{gr} M \) is finitely generated as a \( \mathbb{C}[\mathfrak{h}^{W'}] \otimes S(\mathfrak{h})^{W'} \otimes \mathbb{C}[\mathfrak{p}] \)-module. Since both \( \mathfrak{h} \) and \( \mathfrak{p} \) have degree 1, it follows that the degree \( n \) part in \( M_h^{\aleph h} \) is finitely generated over \( \mathbb{C}[\mathfrak{h}^{W'}] \). The coincidence we need easily follows.

\[ \square \]

3.3. Restriction functors for HC bimodules: properties. Let us quote some properties of the functor \( \bullet_{i,W'} \) established mostly in \([L2]\).

1) The functor \( \bullet_{i,W'} \) is exact and \( \mathbb{C}[\mathfrak{p}] \)-linear. This follows directly from the definition.

2) The functor \( \bullet_{i,W'} \) intertwines the tensor product functors.

3) It is known (and easy to show) that the associated variety of a HC bimodule is the union of the leaves \( \mathcal{L}_W \) of \( \mathcal{L}_W \). On the level of associated varieties the functor \( \bullet_{i,W'} \) behaves as follows. Let the associated variety of \( M \in \text{HC}(H_p,\psi) \) be the union of the leaves \( \mathcal{L}_W \) corresponding to the conjugacy classes of the parabolic subgroups \( W_1,\ldots,W_k \). Then the associated variety of \( M_{i,W'} \) is the union of the leaves corresponding to all parabolic subgroups in \( W' \) conjugate to one of \( W_i \). This is established in \([L2\text{ Prop. 3.6.5}]\).

4) For a HC bimodule \( M \in \text{HC}(H_\psi,\psi) \) we can define its generic rank to be the generic rank of \( e \text{ gr} M \), where the associated graded is taken with respect to any good filtration. The functors \( \mathcal{F}, \mathcal{G} \) preserve the generic ranks and so does the functor \( \bullet_{i,W'} \).

5) We can define the functor \( \bullet_{i,W'} \) for the HC bimodules over the spherical algebras because \([3.1]\) induces an isomorphism \( eH_p^{\wedge Y} \cong eW'\psi(H_p(\mathfrak{h}))^{\wedge Y} \). It can easily be seen from the definition that \( (e\mathcal{F}\mathcal{G})_{i,W'} \cong eW'(\mathcal{B}_{i,W'})eW' \), where \( eW' \) denotes the averaging idempotent in \( \mathcal{W}' \).

6) Consider the categories \( \text{HC}_{\partial\mathcal{L}_{W'}}(H_p,\psi) \subset \text{HC}_{\mathcal{L}_{W'}}(H_p,\psi) \) consisting of all HC bimodules \( B \) with \( V(B) \) contained in the boundary of \( \mathcal{L}_{W'} \) and in the closure of \( \mathcal{L}_{W'} \), respectively. Let \( \text{HC}_{\mathcal{L}_{W'}}(H_p,\psi) \) denote the quotient category. Then \( \bullet_{i,W'} \) gives rise to a well-defined functor \( \bullet_{i,W'} : \text{HC}_{\mathcal{L}_{W'}}(H_p,\psi) \to \text{HC}_{\mathcal{L}_0}(H_p(\mathfrak{w}'),\psi) \) that is a full embedding with image closed under taking subquotients. This is a part of \([L2\text{ Thm. 3.4.5}]\).

7) The functor

\[ \bullet_{i,W'} : \text{HC}_{\mathcal{L}_{W'}}(H_p,\psi) \to \text{HC}_{\mathcal{L}_0}(H_p(\mathfrak{w}'),\psi) \]

admits a right adjoint to be denoted by \( \bullet_{i,W'}^\dagger \). Both kernel and cokernel of the adjunction functor morphisms \( M \to (M_{i,W'})_{i,W'}^\dagger \) are supported on \( \partial\mathcal{L}_{W'} \). This is a part of \([L2\text{ Thm. 3.4.5}]\).

8) We have a natural isomorphism \( \mathcal{B}_{i,W'} \otimes_{H_p(\mathfrak{w}')} \text{Res}^W_W(M) \cong \text{Res}^W_W(\mathcal{B} \otimes_{H_p} M) \) for \( \mathcal{B} \in \text{HC}(H_p,\psi) \), \( M \in \mathcal{O}_p \). This is an easy consequence of \([L2\text{ Thm. 5.5}]\).

3.4. Tor’s and Ext’s. Here we will investigate various Tor’s and Ext’s involving HC bimodules.

**Proposition 3.8.** Let \( \mathcal{B}_1, \mathcal{B}_2 \) be HC \( H_p \)-bimodules and \( N \in \mathcal{O}_p \). Then \( \text{Ext}^{i}_{H_p}(\mathcal{B}_1, \mathcal{B}_2) \) and \( \text{Tor}^H_i(\mathcal{B}_1, \mathcal{B}_2) \) are HC bimodules, while \( \text{Ext}^{i}_{H_p}(\mathcal{B}_1, N), \text{Tor}^H_i(\mathcal{B}_1, N) \) are in \( \mathcal{O}_p \).
Proof. The proofs of the claims involving HC bimodules are similar, we will do the case of $\text{Ext}^1_{H_p}(B_1, B_2)$.

Let us equip $B_1, B_2$ with good filtrations. Let $H_{p,h}$ denote the Rees algebra, and let $B_{1,h}, B_{2,h}$ be the Rees bimodules. Then $\text{Ext}^1_{H_p,h}(B_{1,h}, B_{2,h})$ is a finitely generated graded $H_{p,h}$-bimodule. Moreover, $\text{Ext}^1_{H_p,h}(B_{1,h}, B_{2,h})/(h - 1) = \text{Ext}^1_{H_p}(B_1, B_2)$. So it remains to prove that, for an element $a \in H_{p,h}$ that lies in $Z_p$ modulo $h$, the operator $[a, \bullet]$ maps $\text{Ext}^1_{H_p,h}(B_{1,h}, B_{2,h})$ to $h \text{Ext}^1_{H_p,h}(B_{1,h}, B_{2,h})$. We have an exact sequence

$$\text{Ext}^1_{H_p,h}(B_{1,h}, B_{2,h}) \rightarrow \text{Ext}^1_{H_p,h}(B_{1,h}, B_{2,h}) \rightarrow \text{Ext}^1_{H_p,h}(B_{1,h}, B_{2,h})/(h)$$

The last term coincides with $\text{Ext}^1_{H_p,h}(B_{1,h}/(h), B_{2,h}/(h))$ so the operator $[a, \bullet]$ acts trivially on that term. This implies our claim.

The proofs of the claims involving category $O$ are similar and are based on the observation that the objects of the category $O_p$ are precisely the graded modules $M$ whose associated varieties (in $(\mathfrak{h} + \mathfrak{h}^*)/W$) lie in $\mathfrak{h}/W$.

Proposition 3.8 immediately extends to HC bimodules in $\text{HC}(H_{p', \psi})$, and Tor’s/Ext’s taken over $H_{p'}$ or $H_{p'+\psi}$.

Now let us investigate derived tensor products of Harish-Chandra bimodules with a projective generator $P_c$ of $O_c$.

Lemma 3.9. We have $\text{Tor}^{H_c}_i(B, P_c) = 0$ for $i > 0$.

Proof. Recall that $P_c$ is a direct summand in $H^{\mathfrak{h}} := \varprojlim_{n \to \infty} H_c/H_c h^n$. So it is enough to show that $\text{Tor}^{H_c}_i(B, H^{\mathfrak{h}}) = 0$ for $i > 0$. We will prove that this holds for every finitely generated $S(\mathfrak{h}^*)^W-\text{HC}$-bimodule $B$ such that the adjoint action of $S(\mathfrak{h}^*)^W$ is locally nilpotent. Let us note that every such bimodule is finitely generated over $S(\mathfrak{h}^*)^W \otimes S(\mathfrak{h})$, the proof of this repeats that of [BEG, Lemma 3.3,(ii)].

Assume that we already know that $\text{Tor}^{H_c}_j(B, H^{\mathfrak{h}}) = 0$ for $j = 1, \ldots, i - 1$ for any bimodule $B$ as above. Since $S(\mathfrak{h}^*)^W \otimes H_c$ is Noetherian, we see that there is a finite filtration on $B$ such that the successive quotients are generated by elements commuting with $S(\mathfrak{h}^*)^W$. So it is enough to prove that $\text{Tor}^{H_c}_i(B, H^{\mathfrak{h}}) = 0$ for a bimodule $B$ generated by elements commuting with $S(\mathfrak{h}^*)^W$. We have an epimorphism $H^{\mathfrak{h}}_{c, k} \rightarrow B$ of $S(\mathfrak{h}^*)^W-\text{HC}$-bimodules, let $K$ denote the kernel. Of course, $K$ is still a bimodule of the required form. Then we have an exact sequence $\text{Tor}^{H_c}_i(H^{\mathfrak{h}}_{c, k}, H^{\mathfrak{h}}) \rightarrow \text{Tor}^{H_c}_i(B, H^{\mathfrak{h}}) \rightarrow \text{Tor}^{H_c}_i(K, H^{\mathfrak{h}})$. If $i > 1$, we are done by the inductive assumption.

Let us consider the case $i = 1$. It is enough to show that the functor $\bullet^{\mathfrak{h}} := \bullet \otimes_{H_c} H^{\mathfrak{h}}$ is exact on the category of bimodules in consideration. This functor coincides with the $\mathfrak{h}$-adic completion on the right that is exact on the category of finitely generated $S(\mathfrak{h}^*)^W \otimes S(\mathfrak{h})$-modules by the standard Commutative algebra.

Lemma 3.10. Let $M$ be a HC $H_{c'}-\text{HC}$-bimodule. If $M \otimes_{H_c} P_c = 0$, then $M = 0$.

Proof. Assume $M \neq 0$. Recall that there is a parabolic subgroup $W' \subset W$ such that $M_{t,W'}$ is a nonzero finite dimensional bimodule. So there is a finite dimensional $H_c(W')$-module $L'$ such that $M_{t,W'} \otimes_{H_c(W')} L' \neq \{0\}$. Note that $\mathcal{O} \text{Res}^{W'}_W P_c$ is a projective generator of $\mathcal{O}_c(W')$. That the module is projective is a consequence of the existence of a biadjoint functor to $\mathcal{O} \text{Res}^{W'}_W$. As was checked in [39, Proposition 2.7], the induction functor $\mathcal{O} \text{Ind}^{W'}_W$, does not annihilate any nonzero module. So $\mathcal{O} \text{Res}^{W'}_W P_c$ is a generator. We deduce that
$M_{t,W} \otimes_{H_c(W')}^C \text{Res}_W^W P_t \neq \{0\}$. But the left hand side is $\text{Res}_W^W (M \otimes_{H_c} P_t) \neq \{0\}$. This contradiction finishes the proof. □

Finally, let us investigate the compatibility of Tor’s with the restriction functors.

**Lemma 3.11.** We have a natural isomorphism $\text{Tor}_i^{H_c} (\mathcal{B}_1, \mathcal{B}_2)_{t,W'} \simeq \text{Tor}_i^{H_c(W')} (\mathcal{B}_1, \mathcal{B}_2)_{t,W'}$.

**Proof.** In the notation of Lemma 3.7 it suffices to show that the functors $\mathcal{F}, \mathcal{G}$ intertwine Tor’s. For $\mathcal{F}$, this follows from the observation that $\mathbb{C}[\mathfrak{h}^{\text{reg}} - W'/W']^\mathcal{W}$ is a flat $\mathbb{C}[\mathfrak{h}/W]$-module. The proof for $\mathcal{G}$ is similar. □

Similarly, we see that $\text{Res}_W^W (\text{Tor}_i^{H_c} (\mathcal{B}, M)) \simeq \text{Res}_W^W (\mathcal{B}_t,W', \text{Res}_W^W (M))$.

### 3.5. Relation to quantized quiver varieties

Here we deal with the case when $W$ is a cyclic group. We will need an interpretation of the spherical subalgebras $eH,e$ as quantized quiver varieties due to Holland, [H], and some constructions and results from [BL, L]. The results of this subsection will be used in Subsection 3.5.

Let $W = \mathbb{Z} / \ell \mathbb{Z}$. Consider the space $R := \mathbb{C}^\ell$ and the group $G := (\mathbb{C}^*)^\ell$ acting on $R$ via $(t_1, \ldots, t_\ell) (x_1, \ldots, x_\ell) = (t_2x_1t_1^{-1}, t_3x_2t_2^{-1}, \ldots, t_\ell x_\ell t_\ell^{-1})$. The induced action of $G$ on $T^*R = R \oplus R^*$ is Hamiltonian with moment map $\mu((x_i, y_i)_{i=1}^\ell) = (\sum x_2y_2 - x_1y_1, \ldots, x_\ell y_\ell - x_\ell y_\ell)$. It is easy to see that $\mu^{-1}(0)/G$ is identified with $\mathbb{C}^2/W$.

There is a quantum analog of this isomorphism originally due to Holland. Consider the Weyl algebra $A(R \oplus R^*)$. We have a (symmetrized) quantum comoment map $\Phi : g \to A(R \oplus R^*)$ given by $\epsilon_i \mapsto \frac{1}{\ell} (x_{i+1}y_{i+1} + y_{i+1}x_{i+1} - x_iy_i - y_ix_i)$, where $\epsilon_i, i = 1, \ldots, \ell$, is an element of the tautological basis in $g = \mathbb{C}^\ell$. Then, for $\lambda \in g^*$, we can form the quantum Hamiltonian reduction $A_\lambda := [A(R \oplus R^*)/A(R \oplus R^*) \{x - \langle \lambda, x \rangle\}]^G$. This is a filtered algebra (we consider the filtration by the degree of a differential operator) with $\text{gr} A_\lambda = \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]_W$. Here $\lambda$ is recovered from $\epsilon$ by the following formulas:

$$\sum_{i=1}^\ell \lambda_i = 0, \lambda_i = \frac{1}{\ell} (1 + 2 \sum_{j=1}^{\ell-1} c_j \exp(2\pi \sqrt{-1} ij/\ell)), i = 1, \ldots, \ell - 1.$$

We will also need resolutions of singularities of $\mu^{-1}(0)/G$ and their quantizations. Pick $\theta \in \mathbb{Z}^\ell \cong \text{Hom}(G, \mathbb{C}^*)$ that satisfies $\sum_{i=1}^\ell \theta_i = 0, \theta_i \neq \theta_j$ for $i \neq j$. Then we can form the $\theta$-semistable locus $(T^*R)^{\theta-ss}$ and the corresponding GIT reduction $X^\theta := \mu^{-1}(0)^{\theta-ss}/G$. The variety $X^\theta$ is a smooth symplectic variety (in fact, independent of $\theta$ up to an isomorphism) with a resolution of singularities morphism $\rho : X^\theta \to \mathbb{C}^2/W$. This variety can be quantized by the microlocal sheaf of algebras, $A_\lambda^\theta$, that is also constructed by quantum Hamiltonian reduction. We microlocalize $A(T^*R)$ to a sheaf in conical topology on $T^*R$ so that the restriction $A(T^*R)|_{(T^*R)^{\theta-ss}}$ makes sense. Then we set $A_\lambda^\theta := [A(T^*R)|_{(T^*R)^{\theta-ss}}/A(T^*R)|_{(T^*R)^{\theta-ss}} \{x - \langle \lambda, x \rangle\}]^G$; this is a sheaf of filtered algebras on $X^\theta$ in conical topology with $\text{gr} A_\lambda^\theta = \mathcal{O}_{X^\theta}$. We have $\Gamma(A_\lambda^\theta) = A_\lambda$, while the higher cohomology groups of $A_\lambda^\theta$ vanish.

We can consider the category $A_\lambda^\theta$-mod of the quasi-coherent $A_\lambda^\theta$-modules. Then we have the global section functor $\Gamma : A_\lambda^\theta$-Mod $\to A_\lambda$-Mod. There is a criterion for this functor to be an equivalence, see [Bo] (the formalism of $\mathbb{Z}$-algebras used in Boyarchenko’s paper is equivalent to the formalism we use by [BPW, 5.3]). Namely, let us consider the permutation $\sigma$ of $\{1, 2, \ldots, \ell\}$ such that $\theta_{\sigma(1)} > \theta_{\sigma(2)} > \ldots > \theta_{\sigma(\ell)}$. Then the functor $\Gamma$ is an equivalence if and only if $\lambda_{\sigma(i)} - \lambda_{\sigma(j)} \not\in \mathbb{Z}_{\leq 0}$ for $i < j$. If $\Gamma : A_\lambda^\theta$-Mod $\to A_\lambda$-Mod is an equivalence, then we say that $(\lambda, \theta)$ satisfies the abelian localization.
Let us now construct some HC bimodules. Let \( \varphi \) be a character of \( G \). We consider the \( \mathcal{A}_{\lambda+\varphi}^{\theta} \)-\( \mathcal{A}_{\lambda}^{\theta} \)-bimodule

\[
\mathcal{A}_{\lambda+\varphi}^{\theta} := \left[ A(T^* R) \big|_{(T^* R)^{\theta-ss}} / A(T^* R) \big|_{(T^* R)^{\theta-ss}} \{ x - \langle \lambda, x \rangle \} \right]^{G,\varphi},
\]

where the superscript \( G,\varphi \) indicates that we take \( (G,\varphi) \)-semiinvariants, and the \( \mathcal{A}_{\lambda+\varphi}^{\theta} \)-\( \mathcal{A}_{\lambda}^{\theta} \)-bimodule \( \mathcal{A}_{\lambda+\varphi}^{\theta} := \Gamma(\mathcal{A}_{\lambda+\varphi}^{\theta}) \).

We need to realize the inverse Ringel duality functor \( R^{-1} \) as \( \mathcal{A}_{\lambda+\varphi}^{\theta} \otimes_{\mathcal{A}_{\lambda}} L \cdot \). Namely, assume that \( (\lambda, -\theta), (\lambda + \varphi, \theta) \) satisfy the abelian localization. This implies, in particular, that the algebras \( \mathcal{A}_{\lambda}, \mathcal{A}_{\lambda+\varphi} \) have finite homological dimension, hence the corresponding Cherednik parameters are spherical, see [E2, Theorem 5.5]. So it makes sense to speak about the categories \( \mathcal{O} \) for \( \mathcal{A}_{\lambda}, \mathcal{A}_{\lambda+\varphi} \); those categories are highest weight.

**Lemma 3.12.** There is an equivalence \( \mathcal{O}(\mathcal{A}_{\lambda+\varphi})^\vee \cong \mathcal{O}(\mathcal{A}_{\lambda}) \) such that \( R^{-1} : D^b(\mathcal{O}(\mathcal{A}_{\lambda})) \xrightarrow{\sim} D^b(\mathcal{O}(\mathcal{A}_{\lambda+\varphi})) \) gets identified with \( \mathcal{A}_{\lambda+\varphi}^{\theta} \otimes_{\mathcal{A}_{\lambda}} L \cdot \).

**Proof.** The functors \( \Gamma \) identify the categories \( \mathcal{O} \) for \( \mathcal{A}_{\lambda}, \mathcal{A}_{\lambda+\varphi} \) with those for \( \mathcal{A}_{\lambda}^{\theta}, \mathcal{A}_{\lambda+\varphi}^{\theta} \), see [BLPW] 3.3 for definitions of the latter categories. The functor \( \mathcal{A}_{\lambda+\varphi}^{\theta} \otimes_{\mathcal{A}_{\lambda}} L \cdot \) is the long wall-crossing functor \( \mathfrak{WC}_{\theta \rightarrow \theta} \) from [BL] 2.5,4.1. Now our claim follows from [L5, 4.1], see Remark 4.3 there, in particular. \( \square \)

4. **Ringel duality via HC bimodules**

4.1. **Main result.** The goal of this section is to prove that the Ringel duality functor is realized as the derived tensor product with a HC bimodule. More precisely, we are going to prove the following.

**Theorem 4.1.** Let \( c \in \mathfrak{p}, \psi \in \mathfrak{p}_\mathbb{Z} \) be such that the parameters \( c, c - \psi \) lie in opposite open chambers and are spherical. Then there is a labeling preserving equivalence \( \mathcal{O}_c \cong \mathcal{O}_{c-\psi} \) and a HC \( H_{c-\psi} \)-\( H_{c} \)-bimodule \( \mathcal{B}_c(\psi) \) such that \( R^{-1} \cong \mathcal{B}_c(\psi) \otimes_{H_c} L \cdot \).

4.2. **Improved equivalence theorem.** Our goal here is to prove an improved version of Proposition [L8] with condition (iii) omitted. The proof in the case of \( G(\ell, 1, n) \) basically appeared in [L4, Section 8].

**Proposition 4.2.** Suppose that \( c, c' \in \mathfrak{p} \) satisfy the following conditions:

(i) \( c - c' \in \mathfrak{p}_\mathbb{Z} \).

(ii) \( \text{tw}(c' - c) = \text{id} \) and the ordering \( \leq \) refines \( \leq' \).

Then there is an equivalence \( \mathcal{O}_c \xrightarrow{\sim} \mathcal{O}_{c'} \) of highest weight categories mapping \( \Delta_c(\lambda) \) to \( \Delta_{c'}(\lambda) \) that intertwines the KZ functors \( \mathcal{O}_c, \mathcal{O}_{c'} \rightarrow \mathcal{H}_q \text{-mod} \).

Let \( B_h \) have the meaning as in the discussion after Proposition [L8] so that \( B_h \)-mod is naturally equivalent to the category \( \mathcal{O} \) over the deformation \( H_\ell(W) \). Define a projective \( B_h \)-module \( P_h \) as follows. Take all \( \lambda_1, \ldots, \lambda_k \in \text{Irr}(W) \) such that \( \text{codim}_h V(L_\ell(\lambda_i)) \leq 1 \). Set \( \hat{P}_h := \bigoplus_{i=1}^k P_h(\lambda_i) \), where \( P_h(\lambda_i) \) stands for the deformation of the projective \( B \)-module \( P(\lambda_i) \) to an automatically projective \( B_h \)-module. Let us point out that if (iii) of Proposition [L8] holds, then the indecomposable summands of \( \hat{P} \) are precisely those of \( P_{KZ} \).

Set \( \hat{A}_h := \text{End}_{B_h}(\hat{P}_h) \) and let \( \hat{\pi}_h \) be the natural quotient functor \( B_h \)-mod \rightarrow \hat{A}_h \)-mod. We write \( \hat{A}, \hat{\pi} \) for the specializations to \( h = 0 \).
The proof of the following lemma repeats that of [La, 8.7].

**Lemma 4.3.** The functor $\hat{\pi}$ is 0-faithful.

Set $P_h := \hat{\pi}_h(P_{KZ,h})$ and let $\hat{\pi}_h$ denote the functor $\text{Hom}_{\hat{A}_h}(P_h, \bullet) : \hat{A}_h\text{-mod} \to A_h\text{-mod}$. It follows that $\pi_h = \hat{\pi}_h \circ \hat{\pi}_h$.

Now let $B'_h$ have the same meaning as in the discussion after Proposition 2.8. We construct the algebra $\hat{A}'_h$ and the functors $\hat{\pi}'_h, \hat{\pi}'_h$ similarly to the above. It remains to show that there are progenitors in $\hat{A}_h\text{-mod}, \hat{A}'_h\text{-mod}$ whose images under $\pi_h, \pi'_h$ are isomorphic. Indeed, the functors $\pi_h, \pi'_h$ are fully faithful on the projectives because the functors $\pi_h, \pi'_h$ are. So the claim on the coincidence of the images gives an equivalence $\hat{A}_h\text{-mod} \sim \hat{A}'_h\text{-mod}$ that intertwines the functors

$$\pi_h : \hat{A}_h\text{-mod} \to A_h\text{-mod}, \pi'_h : \hat{A}'_h\text{-mod} \to A_h\text{-mod}.$$  

This equivalence induces the identity identification

$$\text{Irr}(W) \cong \text{Irr}(\hat{A}_h[h^{-1}]) \cong \text{Irr}(A_h[h^{-1}]) \cong \text{Irr}(\hat{A}'_h[h^{-1}]) \cong \text{Irr}(W)$$

(because the functors $\pi_h, \pi'_h$ factor through $\pi_h, \pi'_h$). Now the existence of an equivalence $B_h\text{-mod} \sim B'_h\text{-mod}$ that intertwine the functors $\pi_h, \pi'_h$ and preserves the labels follows from Proposition 2.7.

So let us show that there are projective generators $\hat{P}_h$ of $\hat{A}_h\text{-mod}$ and $\hat{P}'_h$ of $\hat{A}'_h\text{-mod}$ such that $\pi_h(\hat{P}_h) \cong \pi'_h(\hat{P}'_h)$.

First, let us deal with the case $\dim \mathfrak{h} = 1$. For $i, j \in \{1, \ldots, \ell_H\}$ we write $i \sim j$ if $q_i = q_j$ equivalently, $h_i - h_j + \frac{i-j}{\ell} \in \mathbb{Z}$. The category $\mathcal{H}_q(W)\text{-mod}$ splits into the sum of blocks, one per each equivalence class in $\{1, \ldots, \ell\}$. The labels $\lambda_i$ (we write $\lambda_i$ for the label corresponding to the character $z \mapsto z^{-i}$) belonging to the same block have pairwise different values of the $c$-function with integral pairwise differences (recall that the values of the $c$-function is $\ell h_i$). So they are ordered linearly in a highest weight order. This characterizes the images of the projectives in $A_h\text{-mod}$ uniquely, see [La, 8.3].

Now let us deal with the general case. Let us decorate the objects related to $W_H$ with the superscript “$H$”, e.g., for a character $\lambda$ of $W_H$ let $c^H_\lambda$ denote the value of the $c$-function for $W_H$. For $\hat{P}_h$ we take the sum of all objects of the form $\text{Ind}^W_W P^H_h$ (or more precisely the image of this object under $\hat{\pi}_h$), where $H$ runs over the conjugacy classes of the reflection hyperplanes and $P^H_h$ is the sum of the indecomposable projectives in the deformed category $\mathcal{O}_c(W_H)$. The object $P_h$ is a projective generator in $A_h\text{-mod}$. Moreover, $\pi_h(\text{Ind}^W_W P^H_h) = \pi_h^H(\pi_h^H(P^H_h))$ by what was recalled in Subsection 2.8. Note that, for two characters $\lambda, \mu$ of $\hat{W}_H$ (that extend to characters of $W$ as was recalled in Subsection 2.8), we have $c^\lambda_\lambda - c^\mu_\mu = \frac{1}{N} (c_\lambda - c_\mu)$, where $N$ is the number of hyperplanes in the conjugacy class of $H$. From the previous paragraph, it follows that $\pi^H_h(P^H_h) = \pi'^H_h(P'^H_h)$. This completes the proof.

4.3. **KZ vs Ringel duality.** Pick a parameter $c$ in an open chamber. Let $\psi \in \mathbb{Z}_c$ be such that $c - \psi$ lies in the opposite chamber. The main result of this subsection is the following proposition.

**Proposition 4.4.** There is an identification $\mathcal{O}_{c-\psi} \cong \mathcal{O}_c$ such that the Ringel duality $R^{-1} : D^b(\mathcal{O}_c) \to D^b(\mathcal{O}_{c-\psi})$ maps $\Delta_c(\lambda)$ to $\nabla_{c-\psi}(\lambda)$ and intertwines the KZ functors.
We will establish a labeling preserving equivalence of \( \mathcal{O}_{c-\psi} \) with \( \mathcal{O}_{c}^{\text{opp}} \) preserving the associated varieties. For this, we need to establish an analog of the KZ functor \( \text{KZ}^r: \mathcal{O}_{c}^{\text{opp}} \to \mathbb{H}_q(W) \text{-mod} \) (together with its deformed version, meaning deformations over \( \mathbb{C}[\mathfrak{h}] \)) and establish its compatibility with the Ringel duality. Then we will use an argument that is completely analogous to the previous subsection.

We set \( \text{KZ}^r := \text{KZ} \circ D^{-1} \).

**Lemma 4.5.** For \( M \in \mathcal{O}_{c}^{\text{opp}} \), the object \( \text{KZ}^r(M) \) is in homological degree 0. The functor \( \text{KZ}^r \) defines an equivalence of \( \mathcal{O}_{c}^{\text{opp}}/(\mathcal{O}_{c}^{\text{opp}})_{\text{tor}} \) and \( \mathbb{H}_q(W) \text{-mod} \).

**Proof.** This is a direct consequence of Lemma 2.5 and the claim that KZ induces an equivalence \( \mathcal{O}_c/\mathcal{O}_{c}^{\text{tor}} \cong \mathbb{H}_q(W) \text{-mod} \).

We still can consider the restriction functors \( \mathcal{O}_r \mathcal{R}_{W}^W \) for \( \mathcal{O}_{c}(W) \). They are compatible with the homological dualities in the following sense. Let \( D_W \) denote the homological duality \( D^b(\mathcal{O}_c(W')) \to D^b(\mathcal{O}_c(W')) \).

**Lemma 4.6.** We have a functor isomorphism \( D_W \circ \mathcal{O}_r \mathcal{R}_{W}^W \cong \mathcal{H}_{\psi} \mathcal{R}_{W}^W \circ \text{KZ}^r \).

Here is an immediate corollary of Lemma 4.6, the definition of \( \text{KZ}^r \) and the claim that the usual restriction functors intertwine KZ. Here \( \text{KZ}^r \) is the KZ functor \( \mathcal{O}_c(W')^{\text{opp}} \to \mathbb{H}_q(W') \text{-mod} \).

**Corollary 4.7.** We have \( \text{KZ}^r \circ \mathcal{O}_r \mathcal{R}_{W}^W \cong \mathcal{H}_{\psi} \mathcal{R}_{W}^W \circ \text{KZ}^r \).

**Proof of Lemma 4.6.** It is easy to see that the homological duality commutes with the completion functor \( M \mapsto M^{\text{tor}} \). Also it obviously commutes with \( \theta_{\mathfrak{b}} \) and with multiplying by \( e(W') \). So the functor \( M \mapsto e(W')\theta_{\mathfrak{b}}(M^{\text{tor}}) \) commutes with the homological duality. On the other hand, the functor \( N \mapsto (\mathbb{C}[\mathfrak{h}]^W \otimes N)^{\text{tor}} \) also commutes with the homological duality. The latter functor is an equivalence of suitably defined categories \( \mathcal{O} \) and this completes the proof.

Let us make two remarks about the functors above. First of all, the functor \( \mathcal{O}_r \mathcal{R}_{W}^W \) admits a left and, simultaneously, a right adjoint functor, \( \mathcal{O}_r \mathcal{I}_{W}^W \), the proof is the same as for the left-handed analogs. Second of all, we can define straightforward analogs of the functors above for the deformed categories, for example, \( \tilde{\text{KZ}}^r : \tilde{\mathcal{O}}_{c}^{\text{opp}} \to \tilde{\mathbb{H}}_q(W) \text{-mod} \) defined as \( \tilde{\text{KZ}}^r := \text{KZ} \circ D^{-1} \). Straightforward analogs of Lemmas 4.5,4.6 and also of Corollary 4.7 still hold.

**Proof of Proposition 4.4.** Now we can adopt the argument of Subsection 4.2 to prove that there is a label preserving highest weight equivalence \( \mathcal{O}_{c}^{\text{opp}} \cong \mathcal{O}_{c-\psi} \). For this we note that the \( c \)-orders for both categories are opposite to the \( c \)-order for \( \mathcal{O}_c \). Also the identification of the labels in \( \mathcal{O}_{c-\psi}(W) \) and \( \mathcal{O}_c(W)^{\text{opp}} \) is the identity. This is because \( \text{KZ}(\Delta_{c-\psi}(\lambda))[h^{-1}] \cong \text{KZ}(\Delta_{c}(\lambda))[h^{-1}] \) and \( \text{KZ}^r(\tilde{\nabla}_{c-\psi}(\lambda)) = \text{KZ}^r(\tilde{\nabla}_{c}(\lambda)) \) (we use the notation like \( \Delta, \nabla \) for the standard and costandard modules in the deformed categories \( \mathcal{O} \)). The former isomorphism holds because \( \text{tw}(\psi) = \text{id} \), the latter is a consequence of the definition of \( \text{KZ}^r \). So we see that the direct analogs of (i) and (ii) in Proposition 2.8 hold. Thanks to (the deformed version of) Corollary 4.7, the deformed \( \text{KZ}^r \)-functors intertwine (again deformed) induction functors. We can now get rid of (iii) as in the previous subsection. A labeling preserving equivalence \( \mathcal{O}_{c}^{\text{opp}} \cong \mathcal{O}_{c-\psi} \) is established, it intertwines the functors \( \text{KZ}^r \) and KZ. \( \square \)
Let us also point out that the equivalence $\mathcal{O}_c^{\text{opp}} \cong \mathcal{O}_{c-\psi}$ preserves the supports. This follows from Corollary 4.7 combined with [GL, 6.4.9].

4.4. Bimodule $\mathcal{B}_c(\psi)$. Let $c$ be a parameter and $\psi \in \mathfrak{p}_\mathbb{Z}$.

**Lemma 4.8.** There is a unique simple $HC H_{c-\psi}$-$H_c$-bimodule $\mathcal{B}_c(\psi)$ with the following properties:

1. $\mathcal{B}_c(\psi)$ is simple.
2. $\mathcal{B}_c(\psi)[\delta^{-1}]$ is the regular $D(\mathfrak{h}^{\text{reg}})^\#W$-bimodule.

**Proof.** Let us recall that, for a character $\chi$ of $W$, we have the HC $H_{c'+\chi}$-$H_{c'}$-bimodule $\mathcal{B}_{c',\chi}$ and the HC $H_{c'}$-$H_{c+\chi}$-bimodule $\mathcal{B}_{c,\chi}$. Taking an appropriate tensor product of the bimodules of the form $\mathcal{B}_{c',\chi}, \mathcal{B}_{c+\chi,-\chi}$ we get a $H_{c-\psi}$-$H_c$-bimodule that we denote by $\mathcal{B}_{c,\psi}$ (this notation is ambiguous as the bimodule depends on the choice of tensor factors but this is not important for us). By the construction, $e\mathcal{B}_{c,\chi}e[\delta^{-1}] = D(\mathfrak{h}^{\text{reg}})^W\chi, e\mathcal{B}_{c+\chi,-\chi}e[\delta^{-1}] = D(\mathfrak{h}^{\text{reg}})^W\chi^{-1}$. These bimodules are isomorphic to $D(\mathfrak{h}^{\text{reg}})^W$, isomorphisms are given by multiplying by suitable products of elements $\alpha_s$. Therefore $e\mathcal{B}_{c,\psi}e[\delta^{-1}] = D(\mathfrak{h}^{\text{reg}})^W$. So $\mathcal{B}_{c,\psi}[\delta^{-1}] = D(\mathfrak{h}^{\text{reg}})^\#W$. The algebra $D(\mathfrak{h}^{\text{reg}})^\#W$ is simple. So there is a unique simple composition factor of $\mathcal{B}_{c,\psi}$ that does not vanish under inverting $\delta$. We take this composition factor for $\mathcal{B}_c(\psi)$.

Let us prove the uniqueness of $\mathcal{B}_c(\psi)$. From (2) and the construction of $\bullet_{\{1\}}$ recalled in Subsection 3.2 it follows that $\mathcal{B}_c(\psi)_{\{1\}}$ is the trivial $W$-module, let us write $1_{c,\psi}$ for this bimodule. So we get a homomorphism $\mathcal{B}_c(\psi) \rightarrow 1_{c,\psi}$ whose kernel and cokernel have proper associated varieties by 7) of Subsection 3.3. Now (1) and (2) determine $\mathcal{B}_c(\psi)$ uniquely.

We will need an equivalent formulation of (2).

**Lemma 4.9.** Let $\mathcal{B}$ be a $HC H_{c-\psi}$-$H_c$-bimodule. Then the following two conditions are equivalent:

1. $\mathcal{B}[\delta^{-1}]$ is a regular $D(\mathfrak{h}^{\text{reg}})^\#W$-bimodule.
2. The functor $\mathcal{B} \otimes_{H_c} \bullet : \mathcal{O}_c \rightarrow \mathcal{O}_{c-\psi}$ intertwines the KZ functors.

**Proof.** In (2) we can replace the KZ functors with the localization functors. Now (1) obviously implies (2). Let us prove the implication (2) $\Rightarrow$ (1). We have

$$e\mathcal{B}c[\delta^{-1}] \otimes_{D(\mathfrak{h}^{\text{reg}})^W} e\text{loc}(\Delta_c(\text{triv})) \cong e\text{loc}(\Delta_c(\text{triv})).$$

Since the adjoint action of $\mathbb{C}[\mathfrak{h}^{\text{reg}}]^W$ on $e\mathcal{B}c[\delta^{-1}]$ is locally nilpotent, we see that the previous isomorphism gives rise to a $D(\mathfrak{h}^{\text{reg}})^W$-bimodule homomorphism

$$e\mathcal{B}c[\delta^{-1}] \rightarrow \text{Diff}(e\text{loc}(\Delta_c(\text{triv}))), e\text{loc}(\Delta_c(\text{triv}))),$$

where on the right hand side we have the space of differential maps. But $e\text{loc}(\Delta_c(\text{triv})))$ = $\mathcal{O}_{\mathfrak{h}^{\text{reg}}/W}$ and so the space of the differential maps we need is just $D(\mathfrak{h}^{\text{reg}})^W$. Since this bimodule is simple, we conclude that $e\mathcal{B}c[\delta^{-1}] \rightarrow D(\mathfrak{h}^{\text{reg}})^W$. If $K$ is the kernel of this map, then $K \otimes_{D(\mathfrak{h}^{\text{reg}})^W} \mathcal{O}_{\mathfrak{h}^{\text{reg}}/W} = 0$. By passing to the completion at a point of $\mathfrak{h}^{\text{reg}}/W$, we see that $K = \{0\}$.

Now let us describe $\mathcal{B}_c(\psi)_{\{1\}}$, where $W'$ be a parabolic subgroup of $W$.

**Proposition 4.10.** We have $\mathcal{B}_c(\psi)_{\{1\}} = \mathcal{B}_c'(\psi)$, where the right hand side is the analog of $\mathcal{B}_c(\psi)$ for $W'$. 
Proof. The proof is in three steps.

(1) The socle of the bimodule $B_c(\psi)_{1,W'}$ is a simple HC bimodule whose associated variety coincides with $(\mathfrak{h}_{W'} \oplus \mathfrak{h}_{W'})/W'$.

(2) The head of the bimodule $B_c(\psi)_{1,W'}$ is a simple HC bimodule whose associated variety coincides with $(\mathfrak{h}_{W'} \oplus \mathfrak{h}_{W'})/W'$.

(3) The bimodule $B_c(\psi)_{1,W'}$ satisfies the analogs of (1) and (2) in Lemma 4.8

Let us show (1). It is enough to show that there are no HC bimodules with proper associated variety in the socle of $B_c(\psi)_{1,W'}$. Indeed, then the socle is simple because the generic rank of $B_c(\psi)_{1,W'}$ is 1.

So assume the converse: there is a subbimodule in $B_c(\psi)_{1,W'}$ with proper associated variety. So there is a parabolic subgroup $W'' \subset W'$ such that the ideal $J' \subset \mathbb{C}[\mathfrak{h}]^W$ of the stratum corresponding to $W''$ has nonzero annihilator in $B_c(\psi)_{1,W'}$. From the construction of $\bullet_{1,W'}$, it follows that the ideal $J \subset \mathbb{C}[\mathfrak{h}]^W$ corresponding to $W''$ has nonzero annihilator in $\mathbb{C}[\mathfrak{h}]_{W''/W'}^{\psi} \otimes_{\mathbb{C}[\mathfrak{h}/W]} \mathbb{C}_c(\psi)$. But since $\mathbb{C}[\mathfrak{h}]_{W''/W'}^{\psi}$ is a flat $\mathbb{C}[\mathfrak{h}/W]$-algebra, we see that

$$\text{Hom}_{\mathbb{C}[\mathfrak{h}/W]}(\mathbb{C}[\mathfrak{h}/W]/J, \mathbb{C}[\mathfrak{h}]_{W''/W'}^{\psi} \otimes_{\mathbb{C}[\mathfrak{h}/W]} \mathbb{C}_c(\psi)) = \mathbb{C}[\mathfrak{h}]_{W''/W'}^{\psi} \otimes_{\mathbb{C}[\mathfrak{h}/W]} \text{Hom}_{\mathbb{C}[\mathfrak{h}/W]}(\mathbb{C}[\mathfrak{h}/W]/J, \mathbb{C}_c(\psi)).$$

So $J$ has nonzero annihilator in $B_c(\psi)$ as well. The union of the annihilators of the ideals $J^m, m \in \mathbb{Z}_{>0}$, in $B_c(\psi)$ is a subbimodule in $B_c(\psi)$ that needs to coincide with $B_c(\psi)$ because the latter is simple. The associated variety of this subbimodule is contained in $\mathcal{O}_{W''}$ because we have an epimorphism $(H_{c,\psi}/J^k)^{\oplus r} \twoheadrightarrow B_c(\psi)$ of left $H_{c,\psi}$-modules. Contradiction with the fact that the associated variety of $B_c(\psi)$ is $(\mathfrak{h} \oplus \mathfrak{h}^*)/W$.

Let us proceed to (2). Assume the contrary: there is an epimorphism $B_c(\psi)_{1,W'} \twoheadrightarrow \mathcal{B}'$, where $\mathcal{B}'$ is a $H_{c,\psi}(W')/H_{c}(W')$-bimodule with proper associated variety. So, in the notation of Lemma 3.7, $\mathcal{F}'(B_c(\psi)) \twoheadrightarrow \iota \circ \mathcal{G}(\mathcal{B}')$ (an epimorphism of $H_{c,\psi}(\mathfrak{h})^{\psi}$-bimodules), where $\iota$ stands for the Morita equivalence between the categories of $H_{c,\psi}(\mathfrak{h})^{\psi}$-bimodules. Note that the composition $B_c(\psi) \twoheadrightarrow \iota \circ \mathcal{G}(\mathcal{B}')$ cannot be zero because $B_c(\psi)$ generates $\mathcal{F}(B_c(\psi))$. On the other hand, $\iota \circ \mathcal{G}(\mathcal{B}')$ is annihilated by some nontrivial ideal in $\mathbb{C}[\mathfrak{h}]^W$. Since $B_c(\psi)$ is simple, it also needs to be annihilated by that ideal. This contradiction proves (2).

It remains to show that $B_c(\psi)_{1,W'}$ satisfies (1) and (2) of Lemma 4.8. (1) follows from the above. (2) follows from Lemma 4.9 and the functor isomorphisms $KZ' \circ \mathcal{O} \mathcal{R}_{W'} \cong \mathcal{H} \mathcal{R}_{W'} \circ KZ$ and $B_{1,W'} \otimes_{H_{c}(W') \mathcal{O} \mathcal{R}_{W'}} \mathcal{O} \mathcal{R}_{W'}(\bullet) \cong \mathcal{O} \mathcal{R}_{W'}(B_c(\psi) \otimes_{H_{c} \mathcal{O}} \bullet)$. More precisely, the functor isomorphisms imply that the functor $B_c(\psi)_{1,W'} \otimes_{H_{c}(W') \mathcal{O}} \bullet$ intertwines $KZ'$ on the essential images of $\mathcal{R}_{W'}$. But, as we have seen in the proof of Lemma 3.10, the essential image contains a projective generator. So $B_c(\psi)_{1,W'} \otimes_{H_{c}(W')} \bullet$ intertwines the functors $KZ'$.

Now let us describe the bimodule $B_c(\psi)_{1,W'}$ under the following assumptions

- The parameters $c, c - \psi$ (or, more precisely, their restrictions to $W_H \cap S$) lie in opposite open chambers for $W_H$.
- The parameters $c, c - \psi$ are spherical.

Recall an isomorphism $\epsilon_{W_H} H_c e_{W_H} \cong \mathcal{A}_\lambda$ from Subsection 3.5. Let $\lambda$ be the parameter corresponding to $c - \psi$ and $\theta$ be a stability condition such that $(\lambda^-, \theta)$ satisfies the abelian localization.
Proposition 4.11. We have an isomorphism $e_{W_H}\mathcal{B}_c(\psi)_{\dagger; W_H}e_{W_H} \cong A_{\lambda, \lambda - \lambda}^{(\theta)}$.

Proof. Let $\mathcal{B}_c^H(\psi)$ be the analog of $\mathcal{B}_c(\psi)$ for $W_H$ so that $\mathcal{B}_c(\psi)_{\dagger; W_H} = \mathcal{B}_c^H(\psi)$. It remains to show that $e_{W_H}\mathcal{B}_c^H(\psi)e_{W_H} = A_{\lambda, \lambda - \lambda}^{(\theta)}$. Let us notice that both sides are simple bimodules. For the right hand side, this is [BL, Lemma 5.1] and for the left hand side this follows from the definition. Also when we localize $\delta_H$, we will get the regular $D(\mathbb{C}^x)^{W_H}$-bimodules. For the left hand side, this again follows from the definition. For the right hand side, this follows from the construction: $A_{\lambda, \lambda - \lambda}^{(\theta)}$ quantizes a line bundle on $X^\theta$ and the restriction of any line bundle to $\mathbb{C}^x \times \mathbb{C}$/$W_H$ is trivial because the latter variety is factorial. The isomorphism of interest now follows from the uniqueness part of Lemma 4.8. □

4.5. Proof of Theorem 4.1. First, we will need a corollary of Proposition 4.1. Assume that $c, c - \psi$ lie in the opposite chambers. Let $T_{c - \psi}$ denote the sum of all indecomposable tiltings in $\mathcal{O}_{c - \psi}$ and $P_c$ be the sum of all indecomposable projectives in $\mathcal{O}_c(W)$. Recall the functor $\text{loc} : \mathcal{O}_c \to \text{LS}_{rs}(\mathfrak{h}^{req}/W)$.

Corollary 4.12. We have $\text{loc}(P_c(\lambda)) \cong \text{loc}(T_{c - \psi}(\lambda))$.

Proof. Let us remark that the images of $\mathcal{O}_c, \mathcal{O}_{c - \psi}$ under $\text{loc}$ coincide, for example, because the Hecke algebras are the same. Further, by Proposition 4.1, the images of $\text{loc}(P_c(\lambda)), \text{loc}(T_{c - \psi}(\lambda))$ under the equivalence $\text{im} \text{loc} \cong \mathcal{H}_{c - \psi}$-mod are isomorphic. □

Proof of Theorem 4.1. Let us show that $\mathcal{B}_c(\psi) \otimes_{H_c} P_c \cong T_{c - \psi}$.

First of all, since $\mathcal{B}_c(\psi)[\delta^{-1}] = D(\mathfrak{h}^{req})\# W$ and $\text{loc}(P_c) \cong \text{loc}(T_{c - \psi})$, we get an isomorphism $\text{loc}(\mathcal{B}_c(\psi) \otimes_{H_c} P_c) \cong \text{loc}(T_{c - \psi})$. This gives rise to a homomorphism $\iota : \mathcal{B}_c(\psi) \otimes_{H_c} P_c \to \text{KZ}^* \circ \text{KZ}(T_{c - \psi})$, where we write $\text{KZ}^*$ for the right adjoint functor of $\text{KZ}$. Note that $T_{c - \psi} \subset \text{im} \iota$. Indeed, by Lemma 2.3, either $\text{KZ}^* \circ \text{KZ}(T_{c - \psi}) = T_{c - \psi}$ or

$$\text{codim } \text{V}(\text{KZ}^* \circ \text{KZ}(T_{c - \psi})) / T_{c - \psi} = 1$$

and, furthermore, the head of $T_{c - \psi}$ consists of simples with associated variety $\mathfrak{h}$. Now $T_{c - \psi} \subset \text{im} \iota$ follows from

$$\text{loc}(\text{im} \iota) = \text{loc}(T_{c - \psi})$$

(4.1)

All simples in the head of $\mathcal{B}_c(\psi) \otimes_{H_c} P_c$ have associated variety $\mathfrak{h}$. Indeed, an epimorphism $\mathcal{B}_c(\psi) \otimes_{H_c} P_c \twoheadrightarrow L$, where $L$ is a simple, gives rise to a nonzero homomorphism $\mathcal{B}_c(\psi) \to L(P_c, L)$, where the target bimodule is the Harish-Chandra part of $\text{Hom}_c(P_c, L)$. If $V(L) \neq \mathfrak{h}$, then $V(L(P_c, L)) \neq (\mathfrak{h} \oplus \mathfrak{h}^*)/W$. This is impossible because $\mathcal{B}_c(\psi)$ is simple and $V(\mathcal{B}_c(\psi)) = (\mathfrak{h} \oplus \mathfrak{h}^*)/W$. This proves the claim in the beginning of the paragraph. Together with (4.1) this implies $\iota : \mathcal{B}_c(\psi) \otimes_{H_c} P_c \twoheadrightarrow T_{c - \psi}$.

By Lemma 3.12 the derived tensor product with the bimodule $A_{\lambda, \lambda - \lambda}^{(-\theta)}$ is the inverse Ringel duality. It follows that the object $^o \text{Res}_W^{W_H}(\mathcal{B}_c(\psi) \otimes_{H_c} P_c) \cong \mathcal{B}_c(\psi)_{\dagger; W_H} \otimes_{H_c(W_H)}^o \text{Res}_W^{W_H}(P_c)$ is tilting so there are no finite dimensional modules in the socle of $^o \text{Res}_W^{W_H}(\mathcal{B}_c(\psi) \otimes_{H_c} P_c)$. Together with (4.1) this implies

$$^o \text{Res}_W^{W_H}(\ker \iota) = 0, \forall H.$$  

(4.2)

The object $T_{c - \psi}$ has no extensions by simples with associated variety of codimension more than 1 in either direction, Lemma 2.3. So (4.2) implies that $\mathcal{B}_c(\psi) \otimes_{H_c} P_c = T_{c - \psi} \oplus \ker \iota$. But we have already seen in this proof that $\text{Hom}_{\mathcal{O}_{c - \psi}}(\mathcal{B}_c(\psi) \otimes_{H_c} P_c, \ker \iota) = 0$. This finally implies that $\iota$ is an isomorphism.
Note that under this isomorphism, the summand $\mathcal{B}_c(\psi) \otimes_{H_c} P_c(\lambda)$ coincides with $T_{c-\psi}(\lambda)$. This is because of Corollary 4.12.

By the above, $\mathcal{B}_c(\psi) \otimes_{H_c} \ast = \mathcal{B}_c(\psi) \otimes_{H_c} \ast$ gives a functor $\mathcal{O}_c(W) \dashrightarrow \mathcal{O}_{c-\psi}(W) \dashrightarrow$. Both this functor and $R^{-1}$ make the following diagram commutative.

\[
\begin{array}{ccc}
\mathcal{O}_c\text{-proj} & \xrightarrow{=} & \mathcal{O}_{c-\psi}\text{-tilt} \\
\downarrow \text{KZ(}\mathcal{O}_c\text{-proj)} & & \downarrow \text{KZ(}\mathcal{O}_{c-\psi}\text{-tilt)} \\
\mathcal{KZ(}\mathcal{O}_c\text{-proj)} & \xrightarrow{=} & \mathcal{KZ(}\mathcal{O}_{c-\psi}\text{-tilt)}
\end{array}
\]

Since KZ is fully faithful on both $\mathcal{O}_c$-proj, $\mathcal{O}_{c-\psi}$-tilt, we conclude that $\mathcal{B}_c(\psi) \otimes_{H_c} \ast \cong R^{-1}$.

**Corollary 4.13.** Under the assumptions of Theorem 4.1, we have $\mathcal{B}_c(\psi) \otimes_{H_c} \Delta_c(\lambda) = \nabla_{c-\psi}(\lambda)$ and $\text{Tor}_i^{\mathcal{H}_c}(\mathcal{B}_c(\psi), \Delta_c(\lambda)) = 0$ for $i > 0$.

### 5. Derived equivalences

#### 5.1. Scheme of proof

In this section, we prove Theorem 1.1. The proof is basically in three steps. The first one is Theorem 4.1.

Second, let $p'$ be a hyperplane in $p$ and $\psi \in \mathbb{p}_c^\mathbb{Z}$ be such that $c \in p', c - \psi$ lie in opposite open chambers provided $c$ is Weil generic (let us note that Weil generic points of a hyperplane define equal $c$-orders). We will produce a HC $H_{p-\psi} - H_p$ bimodule $B_{p'}(\psi)$ whose Weil generic fiber coincides with $B_c(\psi)$. This will be done in Subsection 5.2.

Third, Subsection 5.3, we will prove that, for a Zariski generic $c \in p'$, the functor $\mathcal{B}_c(\psi) \otimes_{H_c} \ast$ is a derived equivalence $D'(\mathcal{O}_c) \rightarrow D'(\mathcal{O}_{c-\psi})$.

After these three steps, Theorem 1.1 follows from Proposition 4.2 and several easy observations. We will prove the theorem carefully in Subsection 5.4. In the last subsection of this section we will provide an application of Theorem 1.1 to counting the simple objects in $\mathcal{O}$ with given associated variety.

#### 5.2. Family of HC bimodules

Let $p'$ be an affine subspace in $p$ and $\psi \in \mathbb{p}_c^\mathbb{Z}$ be such that $c \in p', c - \psi$ lie in opposite open chambers provided $c$ is Weil generic in $p'$. Our goal is to produce a HC bimodule $B_{p'}(\psi) \in \text{HC}(H_{p'}, -\psi)$ with $B_{p'}(\psi)_c = B_c(\psi)$ for a Weil generic $c \in p'$.

The idea is as follows. The bimodule $B_{p', \psi}$ still makes sense and its specialization to $c \in p'$ is $B_{c, \psi}$. We need to “cut” $B_{p', \psi}$ removing everything with proper associated variety (for Weil generic $c$) from the head and from the socle. We will see that there is an ideal $I$ in $H_{p'}$ such that $H_c/I_c$ is the maximal quotient with proper associated variety for Weil generic $c$. Then we cut “small” bimodules from the socle by using the induction and restriction functors and from the head by multiplying by $I$. To produce $I$ we first produce $I \subset H_{p'}$ such that $I_c \subset H_c$ is the minimal ideal of finite codimension in $H_c$ for Weil generic $c$.

**Lemma 5.1.** Let $p' \subset p$ be an affine subspace. There is a two-sided ideal $I \subset H_{p'}$ with the following two properties:

(i) $H_{p'}/I$ is a finitely generated $\mathbb{C}[p']$-module.

(ii) For a Weil generic $c \in p'$, the specialization $I_c := \mathbb{C}_c \otimes_{\mathbb{C}[p']} I$ is the maximal ideal of finite codimension in $H_c$. 
Proof. Consider the ideal \( I(k) \subset H_{p'} \) generated by elements of the form
\[
\sum_{\sigma \in S_{2k}} \text{sgn}(\sigma) a_{\sigma(1)} \cdots a_{\sigma(2k)}
\]
for arbitrary \( a_1, \ldots, a_{2k} \in H_{p'} \). By the Amitsur-Levitski theorem, any \( k \)-dimensional representation of \( H_{p'} \) factors through \( H_{p'}/I(k) \). Also it is clear from the definition that \( I(1) \supset I(2) \supset \ldots \). The quotients \( I(k-1)/I(k) \) are HC \( H_{p'} \)-bimodules and so their supports in \( p' \) are constructible subsets, Lemma 5.1. Also note that \( V(H_{p'}/I(k)) = \{0\} \). This is proved by analogy with the proof of [L1, Theorem 7.2.1] using the decomposition of completions of \( H_{p'} \), see [L2, 3.10]. In particular, for any \( c \), the ideal \( I_c(m) \subset H_c \) has finite codimension for any \( m \).

We claim that only finitely many of the supports of \( I(k-1)/I(k) \) are dense in \( p' \). Indeed, assume the contrary: there is an infinite sequence \( k_1 < k_2 < \ldots \) such that the support of \( I(k_i-1)/I(k_i) \) is dense in \( p' \). Let \( p'_i \) be a Zariski open subset of \( p' \) with the property that \( H_{p'}/I(k_i) \) is flat over \( p'_i \). Take \( c \in \cap p'_i \). Then we have a sequence of ideals, \( H_c \supset I_c(k_1) \supset I_c(k_2) \supset \ldots \). Being a Serre subcategory in the category \( \mathcal{O} \), the category of finite dimensional \( H_c \)-modules has enough projectives (and there are finitely many of those). So \( H_c/I_c(k_i) \) is the quotient of the direct sum of certain projectives. In particular, \( I_c(k_i) \) contains the annihilator of the direct sum of all projectives that is an ideal of finite codimension. So the claim in the beginning of this paragraph is proved. So, for some \( m \), the support of \( I(k_i-1)/I(k_i) \) for \( k > m \) is contained in some proper Zariski closed subset of \( p' \) (depending on \( k \)).

We set \( I := I(m) \). Condition (ii) follows from the choice of \( m \). (i) now follows from the first paragraph of the proof. \( \Box \)

Now let us define a two-sided ideal \( \tilde{I} \subset H_{p'} \). Let \( W' \) be a parabolic subgroup of \( W \). For a two-sided ideal \( J \subset H_{p'}(W') \) such that \( H_{p'}(W')/J \) is finitely generated over \( \mathbb{C}[p'] \), define the ideal \( J^1,H,W \) as the kernel of \( H_{p'} \rightarrow (H_{p'}(W')/J)^{1,H,W} \).

Let \( I(W') \) stand for the ideal in \( H_{p'}(W') \) defined similarly to \( I \subset H_{p'} \). We set
\[
\tilde{I} := \left( \bigcap_{W' \neq \{1\}} I(W')^{1,H,W} \right)^n.
\]

Lemma 5.2. For a Weil generic \( c \) and any HC \( H_{c-\psi} \)-\( H_c \)-bimodule \( M \) with proper associated variety, we have \( M\tilde{I}_c = 0 \) and \( \tilde{I}^2_c = \tilde{I}_c \).

Proof. Take \( c \) so that (ii) of Lemma 5.1 is satisfied for all possible \( W' \) and such that \( H_{p'}(W')/\tilde{I}, \tilde{I} \) are flat over some Zariski open neighborhood of \( c \) in \( p' \).

Let \( W_1, \ldots, W_\ell \) be the parabolic subgroups corresponding to the irreducible components of \( V(M) \). Consider the morphism \( M \rightarrow \bigoplus_{i=1}^\ell (M_{i,W_i})^{1,H,W_i} \) and let \( K \) denote its kernel. The \( H_c(W_i) \)-bimodule \( M_{i,W_i} \) is finite dimensional and so is annihilated by \( I_c(W_i) \). So \( (M_{i,W_i})^{1,H,W_i} \) is annihilated by \( I_c(W_i) \). It follows that \( M\bigcap_{i=1}^\ell (I_c(W_i))^{1,H,W_i} \subset K \). Note that \( V(K) \subset V(M) \setminus \bigcup_{i=1}^\ell Z_{W_i} \). An easy induction shows that \( M\tilde{I}_c = 0 \).

The equality \( \tilde{I}^2_c = \tilde{I}_c \) (for a Weil generic \( c \)) follows from the observation that \( H_c/\tilde{I}^2_c \) has proper support. \( \Box \)

Now let \( p', \psi \) be as in the beginning of the subsection. Our goal is to produce a bimodule \( B_{\psi} \in HC(H_{p'}, -\psi) \). We start with the \( H_{p'-\psi} \)-\( H_{p'} \)-bimodule \( B_{\psi} \). Consider a natural
homomorphism

\[
(5.1) \quad \mathcal{B}_{p',\psi} \to (\mathcal{B}_{p',\psi}, \{1\})^{\dagger,\{1\}}.
\]

Let \(\hat{\mathcal{B}}_{p',\psi}\) denote the image. Let us take \(c\) in a Zariski open subset \(U\), where the kernel, the image and the cokernel of \(5.1\) are flat over \(\mathbb{C}[p']\). We claim that for \(c \in U\), the bimodule \(\hat{\mathcal{B}}_{c,\psi}\) has no subbimodules with proper associated variety. Indeed, thanks to the left exactness of \(\bullet^{\dagger,\{1\}}\), we see that \(\mathcal{B}_{c,\psi} \to (\mathcal{B}_{c,\psi}, \{1\})^{\dagger,\{1\}}\) factors through \(\mathcal{B}_{c,\psi} \to \hat{\mathcal{B}}_{c,\psi}\). Since \((\mathcal{B}_{c,\psi}, \{1\})^{\dagger,\{1\}}\) has no submodules with proper associated variety, our claim is proved.

We set \(\mathcal{B}_{p'}(\psi) := \hat{\mathcal{B}}_{p',\psi} \tilde{I}\). We claim that for \(c\) as in Lemma 5.2 \(\mathcal{B}_c(\psi)\) has no quotients with proper associated variety. Indeed, by Lemma 5.2 such a quotient of \(\mathcal{B}_c(\psi)\) has to be annihilated by \(\tilde{I}_c\) that is impossible because \(\tilde{I}_c^2 = \tilde{I}_c\). So we conclude that, for a Weil generic \(c\) the following holds:

- The specialization \(\mathcal{B}_{p'}(\psi)_c\) has no submodules and quotients with proper associated variety.
- \(\mathcal{B}_{p'}(\psi)_c\) is a subquotient of \(\mathcal{B}_{c,\psi}\).

But \(\mathcal{B}_{c,\psi}\) has a unique composition factor with full associated variety and this factor is \(\mathcal{B}_c(\psi)\). The equality \(\mathcal{B}_{p'}(\psi)_c = \mathcal{B}_c(\psi)\) follows.

Below we write \(\mathcal{B}_c(\psi)\) for \(\mathcal{B}_{p'}(\psi)_c\) when \(c\) is Zariski generic.

### 5.3. Degeneration.

Let an affine subspace \(p'\) in \(p\) and \(\psi \in \mathcal{L}_c\) be such that

- For a Weil generic \(c \in p'\), the parameters \(c\) and \(c - \psi\) lie in opposite open chambers.
- For a Zariski generic \(c \in p'\), the parameters \(c\) and \(c - \psi\) are spherical.

Our goal in this subsection is to prove the following result.

**Proposition 5.3.** There is a non-empty Zariski open subset \(U \subset p'\) with the following properties:

1. \(\mathcal{B}_c(\psi) \otimes_{H_c} \bullet : \mathcal{B}^C(O_c) \to \mathcal{B}^C(O_{c - \psi})\) is an equivalence of triangulated categories.
2. \(\text{Tor}_i^{H_c}(\mathcal{B}_c(\psi), \Delta_c(\lambda)) = 0\) for \(i > 0\) and \([\mathcal{B}_c(\psi) \otimes_{H_c} \Delta_c(\lambda)] = [\nabla_{c - \psi}(\lambda)]\).

The scheme of the proof is as follows.

(i) We prove that \(H_c \cong R\text{End}_{H_c}(\mathcal{B}_c(\psi))\) (meaning, in particular, that all higher Ext’s vanish) for a Weil generic \(c \in p'\).

(ii) We extend (i) to the case when \(c\) is Zariski generic.

(iii) We prove (2) of Proposition 5.3.

(iv) Then we deduce (1) of Proposition 5.3 from (ii) and (iii).

**Step (i).** We start by proving a general result.

**Lemma 5.4.** Let \(\mathcal{B} \in \text{HC}(H_c, c' - c)\) be such that \(\mathcal{B} \otimes_{H_c} \bullet : \mathcal{B}^C(O_c) \to \mathcal{B}^C(O_{c'})\) is an equivalence of triangulated categories. Then \(\text{Ext}_i^{H_c}(\mathcal{B}, \mathcal{B}) = 0\) for \(i > 0\), and \(\text{End}_{H_c}(\mathcal{B}) = H_c\).

**Proof.** Let \(D^b_0(H_c)\) denote the full subcategory in \(D^b(H_c\text{-mod})\) consisting of all complexes with homology in \(O_c\); this category is naturally identified with \(D^b(O_c)\) by [E2] Proposition 4.4. Also consider the category \(D^b_{HC}(H_c' - H_c)\) of all complexes of \(H_c' - H_c\)-bimodules with HC homology. Then \(\bullet \otimes_{HC} \bullet : D^b(H_c' - H_c) \times D^b(H_c) \to D^b(H_c')\) restricts to

\[
D^b_{HC}(H_c' - H_c) \times D^b(O_c) \to D^b(O_c).
\]
and a similar statement holds for $R\text{Hom}_{H_c}(\bullet, \bullet)$ thanks to Proposition 3.8.

Since $B \otimes_{H_c} \bullet$ is an equivalence, we see that a quasi-inverse equivalence is given by $R\text{Hom}_{H_c}(B, \bullet)$ and we have an isomorphism of functors $\text{Id} \to R\text{Hom}_{H_c}(B, B \otimes_{H_c} \bullet)$. But the latter functor is nothing else but $R\text{End}_{H_c}(B, B) \otimes^L \bullet$. Consider $R\text{End}_{H_c}(B, B) \otimes^L P_c$. Thanks to Lemma 3.9, we have $H^i(R\text{End}_{H_c}(B, B) \otimes^L P_c) = \text{Ext}^i_{H_c}(B, B) \otimes_{H_c} P_c$. But the Ext is a HC bimodule and since its tensor product with the projective generator is zero, by Lemma 3.10, we see that the bimodule itself is zero. Also we see that, for a natural homomorphism $H_c \to \text{End}_{H_c}(B, B)$, we have $H_c \otimes_{H_c} P_c \sim \text{End}_{H_c}(B, B) \otimes_{H_c} P_c$. It follows that the tensor products of the kernel and of the cokernel of $H_c \to \text{End}_{H_c}(B, B)$ with $P_c$ are zero. This implies that the homomorphism $H_c \to \text{End}_{H_c}(B, B)$ is an isomorphism. □

Since $B_c(\psi) \otimes_{H_c} \bullet$ is an equivalence for a Weil generic $c$, the claim of Step (i) follows.

Step (ii). Recall that all $\text{Ext}^i_{H_c}(B_c(\psi), B_c(\psi))$ are HC $H_c$-bimodules and hence are generically flat over $\mathbb{C}[p']$. It follows that, for a Zariski generic $c$, the HC bimodule $\text{Ext}^i_{H_c}(B_c(\psi), B_c(\psi))$ is the specialization of $\text{Ext}^i_{H_{p'}}(B_{p'}(\psi))$ at $c$. From Step (i), we now deduce that $\text{Ext}^i_{H_c}(B_c(\psi), B_c(\psi)) = 0$ for a Zariski generic parameter $c \in p'$. Also we see that the natural homomorphism $H_{p'} \to \text{End}_{H_{p'}}(B_{p'}(\psi))$ is an isomorphism after specializing to a Weil generic $c$. So $H_c \sim \text{End}_{H_c}(B_c(\psi), B_c(\psi))$ for Zariski generic $c$, as well. This completes Step (ii).

Step (iii). Recall, Proposition 3.8 that $\text{Tor}^i_{H_{p'}}(B_{p'}(\psi), \Delta_{p'}(\lambda))$ is in $O_{p' - \psi}$ for all $i$. In particular, $\text{Tor}^1_{H_{p'}}(B_{p'}(\psi), \Delta_{p'}(\lambda))$ is generically flat over $\mathbb{C}[p']$, similarly to (1) of Lemma 3.4. By Corollary 4.13 we see that $\text{Tor}^i_{H_c}(B_c(\psi), \Delta_c(\lambda)) = 0$ for $i > 0$ and Weil generic $c$. It follows that $\text{Tor}^i_{H_c}(B_c(\psi), \Delta_c(\lambda)) = 0$ for Zariski generic $c$. Also we see that $B_{p'}(\psi) \otimes_{H_{p'}} \Delta_{p'}(\lambda)$ is generically flat over $\mathbb{C}[p']$ and the specialization to a Weil generic $c$ coincides with $\nabla_{c - \psi}(\lambda)$ by Corollary 4.13. It follows that the class of $B_c(\psi) \otimes_{H_c} \Delta_c(\lambda)$ in $K_0$ for a Zariski generic $c$ coincides with that of $\nabla_{c - \psi}(\lambda)$. This can be seen by looking at the formal characters of modules in the Cherednik categories $O$ defined, for example, in [31] Definition 11.10.

This completes Step (iii).

Step (iv). Now suppose that we have a HC $H_{c'}$-$H_c$-bimodule $B$ with the following properties:

(a) $\text{Ext}^i_{H_c}(B, B) = 0$ for $i > 0$ and $H_c \sim \text{End}_{H_c}(B, B)$.

(b) $B \otimes_{H_c} \Delta_c(\lambda)$ is concentrated in homological degree 0 and its class in $K_0$ coincides with $[\nabla_{c}(\lambda)]$.

We are going to show that $R\text{Hom}_{H_{c'}}(B, \bullet)$ is an equivalence $D^b(O_{c'}) \to D^b(O_c)$. The proof is inspired by that of [BK], Theorem 1.1] and also the argument of [LA, Section 10].

Set $F := R\text{Hom}_{H_{c'}}(B, \bullet), G := B \otimes_{H_{c'}} \bullet$. It follows from (a) that $F \circ G \cong \text{id}$. So to prove that $F$ (and hence $G$) is an equivalence, it is enough to show that $\text{im}G$ coincides with $D^b(O_c)$.

Let us order the labels $\lambda_1, \ldots, \lambda_N$ in some linear order making $O_{c - \psi}$ into a highest weight category. It follows from (b) that $L_{c - \psi}(\lambda_i)$ occurs in the composition series of $G(\Delta_c(\lambda_i))$ once and all other simple constituents are of the form $L_{c - \psi}(\lambda_j)$ with $j < i$. Using an ascending induction on $i$, we check that $L_{c - \psi}(\lambda_i)$ lies in the image of $G$. From here we deduce that $G$ is essentially surjective.

The proof of Proposition 5.3 is now complete.
5.4. Proof of the main result. Now we are ready to prove Theorem 1.1. Pick a parameter $c$. The order $c^E$ is refined by $\leq$ for $\tilde{c} \in c + p_{\mathbb{Z}}$ lying in an open chamber. Thanks to Proposition 4.2, we may replace $c$ with $\tilde{c}$ without changing the abelian category and assume that $c$ lies in an open chamber $\mathcal{C}$. Also it is enough to establish a derived equivalence in the case when $c'$ lies in a chamber $\mathcal{C}'$ that shares a wall $\Pi_0$ with $\mathcal{C}$ (in the general case, we take the composition of a sequence of equivalences, each crossing a single wall). Next, we may assume that $c' - c \in p_{\mathbb{Z}}$. Indeed, otherwise we can modify $c$ by subtracting an element $\psi' \in p_{\mathbb{Z}}$ such that a bimodule $B_{c,\psi'}$ with $c' - c + \psi' \in p_{\mathbb{Z}}$ is a Morita equivalence and $c - \psi' \in \mathcal{C}$.

Then replacing both $c, c'$ with points of $(c + p_{\mathbb{Z}}) \cap \mathcal{C}, (c' + p_{\mathbb{Z}}) \cap \mathcal{C}'$ (so that the categories $\mathcal{O}_c, \mathcal{O}_{c'}$ stay the same) we may assume that $c, c'$ lie on hyperplanes $\Pi, \Pi'$ parallel to the wall $\Pi_0$ separating $\mathcal{C}, \mathcal{C}'$ such that Zariski generic parameters in $\Pi, \Pi'$ are spherical. Now we can modify $c, c'$ by the same element of $\Pi_0 \cap p_{\mathbb{Z}}$ (this intersection is a lattice in $\Pi_0$ because $\Pi_0$ is defined over $\mathbb{Q}$) staying in the same chambers so that $B_{c}(\psi) \otimes_{H_c}^L \bullet$ is an equivalence.

Finally, we need to show that the equivalence $B_{c}(\psi) \otimes_{H_c}^L \bullet$ intertwines the KZ functors. It follows from the construction of $B_{c}(\psi)$ that $B_{c}(\psi)[\delta^{-1}] = B_{c,\psi}[\delta^{-1}] = D(h^{reg})\#W$. So $B_{c}(\psi) \otimes_{H_c}^L \bullet$ intertwines the functors $\text{loc}$ and hence KZ. This completes the proof of Theorem 1.1.

5.5. Application to counting. The associated variety of a simple in $\mathcal{O}_c$ coincides with $Wb^W$ for some parabolic subgroup $W' \subset W$, see [BL] 3.8. Let $n_{W'}(c)$ denote the number of associated simples with Weyl group $Wb^W$.

Proposition 5.5. Let $\psi \in p_{\mathbb{Z}}$. Then $n_{W'}(c) = n_{W'}(c - \psi)$ for all $c$.

Proof. If $\psi = -\tilde{\chi}$ and $B_{c,\tilde{\chi}}$ is a Morita equivalence, the claim is clear. When $c, c' := c - \psi$ satisfy the assumptions of Proposition 4.2, the equivalence of that proposition preserves the supports, see [GL] 6.4.9. So we can assume that $B_{c}(\psi) \otimes_{H_c}^L \bullet : D^{b}(\mathcal{O}_c) \rightarrow D^{b}(\mathcal{O}_{c-\psi})$ is an equivalence. Let $D^b_W(\mathcal{O}_c)$ denote the full subcategory of $D^{b}(\mathcal{O}_c)$ consisting of all complexes with homology having associated variety inside $Wb^W$. From the compatibility of the restriction functors and Tor’s (see the end of Subsection 3.4) it follows that $B_{c}(\psi) \otimes_{H_c}^L \bullet$ maps $D^{b}_W(\mathcal{O}_c)$ to $D^{b}_W(O_{c-\psi})$. This implies $\sum_{W' \subset W} n_{W'}(c) \leq \sum_{W' \subset W} n_{W'}(c - \psi)$, which in turn implies the equality of the proposition. $\square$

6. Perverse equivalences

6.1. Main result. In this section we are going to prove that there are perverse equivalences between some categories $D^{b}(\mathcal{O}_c), D^{b}(\mathcal{O}_{c'})$. Namely, suppose an affine subspace $p' \subset p$ and $\psi \in p_{\mathbb{Z}}$ are such that

(1) Both $p', p' - \psi$ contain Zariski open subsets of spherical elements.

(2) For a Weil generic $c \in p'$, the parameters $c, c - \psi$ lie in opposite open chambers.

Then, for a Zariski generic $c$, the functor $\varphi_c := B_{c}(\psi) \otimes_{H_c}^L \bullet : D^{b}(\mathcal{O}_c) \rightarrow D^{b}(\mathcal{O}_{c'})$ is an equivalence of triangulated categories. We are going to show that (possibly after restricting to a smaller Zariski open subset) the equivalence $\varphi_c$ is perverse. This is an analog of [BL] Theorem 7.2.

The corresponding filtrations are produced similarly to [BL] Section 7. Namely, recall the ideals $I(W')^{1,H,W'} \subset H_{p'}(W)$ defined before Lemma 5.2. Define the ideal $J_{k} \subset H_{p'}$ as
follows:

\[ \mathcal{J}_k := \left( \bigcap_{W'} I(W')^{1,H,W'} \right)^k, \]

where the intersection is taken over all parabolic subgroups \( W' \) with \( \dim h^{W'} \leq k - 1 \). The ideal \( \mathcal{J}_k \) has the following important property. Let \( c \) be a Weil generic element of \( p' \), then the specialization \( \mathcal{J}_{k,c} \) coincides with the minimal ideal \( J \subset H_c \) such that \( \dim V(H_c/J) < 2k \). Note that, in this case, \( \mathcal{J}_{k,c}^2 = \mathcal{J}_{k,c} \), in particular, the subcategory in \( \mathcal{O}_c \) of all modules annihilated by \( \mathcal{J}_{k,c} \) is closed under extensions and hence is a Serre subcategory. Also note that \( H_c = \mathcal{J}_{0,c} \supset \mathcal{J}_{1,c} \supset \mathcal{J}_{2,c} \supset \ldots \supset \mathcal{J}_{n,c} \supset \mathcal{J}_{n+1,c} = \{0\} \) for all parameters \( c \). Let \( C^1_c = \mathcal{O}_c \) and \( C^1_j \) be the full subcategory in \( \mathcal{O}_c \) consisting of all modules annihilated by \( \mathcal{J}_{n+1-j,c} \). So we get a filtration of \( C^1_c \) by Serre subcategories.

Define ideals \( \mathcal{J}'_i \subset H_{p'-\psi} \) and subcategories \( C^2_c \subset C^2 := \mathcal{O}_{c-\psi} \) in a similar way.

**Theorem 6.1.** There is a non-empty Zariski open subset \( U \subset p' \) such that \( \mathcal{J}_{k,c} = \mathcal{J}_{k,c} \) for all \( k \) and the equivalence \( B_c(\psi) \otimes_{H_c} L \) is perverse with respect to the filtrations introduced above.

In case \( c, c' \) lie in the opposite chambers, Theorem 6.1 follows from Lemma 2.5. Indeed, the filtrations on \( \mathcal{O}_c, \mathcal{O}_{c'} \) are the filtrations by dimensions of support as in Subsection 2.5. Recall that the equivalence \( \mathcal{O}_{\text{app-r}} \cong \mathcal{O}_{c'} \) preserves supports, see the end of Subsection 4.3. This implies our claim.

In general, we will, roughly speaking, show that in our situation the perversity is preserved under degeneration.

### 6.2. Double wall-crossing bimodule

As in [BL, Section 7], we consider the complexes \( D_{p'} := B_{p'-\psi}(-\psi) \otimes_{H_{p'-\psi}} B_{p'}(\psi) \) and \( D_c := B_{c-\psi}(-\psi) \otimes_{H_c} B_c(\psi) \). Clearly, over the Zariski open locus where all homology of \( D_{p'} \) are flat over \( \mathbb{C}[^p] \), the homology of \( D_c \) coincide with the specialization of that of \( D_{p'} \) to \( c \).

**Lemma 6.2.** The following is true:

1. For a Weil generic \( c \in p' \), the functor \( D_c \otimes_{H_c} L : Db(\mathcal{O}_c) \to Db(\mathcal{O}_c) \) is the Serre duality, it maps the projective \( P_c(\lambda) \) to the injective \( I_c(\lambda) \).
2. The higher homology of \( D_{p'} \) are torsion \( \mathbb{C}[^p]\)-modules.

**Proof.** The first claim follows from the observation that the functor is the square of the inverse Ringel duality. To prove the second one, it is enough to show that \( H_i(D_c) = 0 \) for a Weil generic \( c \) and \( i > 0 \). But \( H_i(D_c) \otimes_{H_c} P_c = H_i(D_c \otimes_{H_c} H_c P_c) = 0 \). So \( H_i(D_c) = 0 \). □

Now let us investigate the structure of the complex \( D_c \otimes_{H_c} H_c / \mathcal{J}_{1,c} \).

**Lemma 6.3.** For a Zariski generic \( c \in p' \), we have an isomorphism \( D_c \otimes_{H_c} H_c / \mathcal{J}_{1,c} \cong H_c / \mathcal{J}_{1,c}[-2n] \).

**Proof.** The proof is in two steps.

**Step 1.** First, let us establish the claim in the case when \( c \) is Weil generic. The modules in \( \mathcal{O}_c \) annihilated by \( \mathcal{J}_{1,c} \) are precisely the finite dimensional ones. For a finite dimensional \( H_c \)-module \( M \), we have \( \text{Tor}_i^{H_c}(D_c, M) = 0 \) unless \( i = 2n \) because \( D_c \otimes_{H_c} L \) is the square of the Ringel duality. For the same reason, \( M \mapsto \text{Tor}_2^{H_c}(D_c, M) \) is an auto-equivalence of the category of the finite dimensional modules in \( \mathcal{O}_c \). It follows that \( D_c \otimes_{H_c} M / \mathcal{J}_{1,c} \cong \mathcal{O}_{c-\psi}[-2n] \), where \( \mathcal{J}_{c} \) is an invertible \( H_c / \mathcal{J}_{1,c} \)-bimodule. On the other hand, if we identify \( \mathcal{O}_{c-\psi} \) with
Proposition 6.4. For a Zariski generic element $\phi$, the equivalence $M \mapsto \mathcal{C}_c(\psi) \otimes_{\mathcal{H}_c} M[-n]$ becomes $R\text{Hom}(M, H_c)[n]$. Since $D^2 = \text{id}$, we see that $\mathcal{D}_c \otimes_{\mathcal{H}_c} \bullet [-2n]$ is the identity functor. This implies that $\mathcal{J}_c = H_c/\mathcal{J}_{1,c}$.

Step 2. Step 1 implies that, for a Zariski generic $c$, we have $\text{Tor}^H_c(\mathcal{D}_c, H_c/\mathcal{J}_{1,c}) = 0$ for $i \neq 2n$. Set $\mathcal{J}_c := \text{Tor}^H_{2n}(\mathcal{D}_c, H_c/\mathcal{J}_{1,c})$. Then $\mathcal{J}_c$ form a flat (over some Zariski open subset $U \subset p'$) family of $H_c/\mathcal{J}_{1,c}$-bimodules. We claim that if $H_c/\mathcal{J}_{1,c} \cong \mathcal{J}_c$ for a Weil generic point $c$, then the same is true for a Zariski generic point. The proof goes as follows. Let $A_U$ be an associative $\mathbb{C}[U]$-algebra that is a free module of finite rank over $\mathbb{C}[U]$. Let $M_U, N_U$ be two $A_U$-modules that are free of equal finite ranks, say $r$, over $\mathbb{C}[U]$. Then, for a Zariski generic $c \in U$, we have $\text{Hom}_{A_U}(M_U, N_U)_c = \text{Hom}_{A_c}(M_c, N_c)$. Then we consider the determinant function on $\text{Hom}_{A_U}(M_U, N_U) \subset \text{Mat}_r(\mathbb{C}[U])$. The determinant is nonzero and so we are done.

Now we want to study the tensor products $\mathcal{D}_c \otimes_{\mathcal{H}_c} H_c/\mathcal{J}_{1,c}$ for $i > 1$. The following proposition is an analog of [BL, Theorem 7.1(2)] and is a crucial step of the proof of Theorem 6.1. Let $\tilde{\mathcal{J}}_{\ell,c}$ stand for $\{a \in \mathcal{J}_{\ell-1,c} | \mathcal{J}_{\ell-1,c} a \subset \mathcal{J}_{\ell,c}\}$.

Proposition 6.4. For a Zariski generic element $c \in p'$, the following is true:

(i) $\text{Tor}^H_c(\mathcal{D}_c, H_c/\mathcal{J}_{\ell,c}) = 0$ for $i < 2(n + 1 - \ell)$.
(ii) $\text{Tor}^H_c(\mathcal{D}_c, H_c/\mathcal{J}_{\ell,c})$ is annihilated by $\mathcal{J}_{\ell-1,c}$ from the left and from the right when $i > 2(n + 1 - \ell)$.
(iii) $\text{Tor}^H_c(\mathcal{D}_c, H_c/\mathcal{J}_{\ell,c}) \to \mathcal{J}_{\ell-1,c}/\mathcal{J}_{\ell,c}$ and the kernel is annihilated by $\mathcal{J}_{\ell-1,c}$ from the left and from the right.

Proof. Let us start by establishing (i) and (ii) for Weil generic $c$, these claims for a Zariski generic $c$ will then follow. Since $\mathcal{D}_c \otimes_{\mathcal{H}_c} \bullet$ is the squared Ringel duality and since the Ringel duality is perverse with respect to the support filtration, we see that $\text{Tor}^H(\mathcal{D}_c, P_c/\mathcal{J}_{\ell,c} P_c) = 0$ for $i < 2(n + 1 - \ell)$, while $\dim V(\text{Tor}^H(\mathcal{D}_c, P_c/\mathcal{J}_{\ell,c} P_c)) < \ell - 1$ for $i > 2(n + 1 - \ell)$. But

(6.1) $\text{Tor}^H(\mathcal{D}_c, P_c/\mathcal{J}_{\ell,c} P_c) = H_i(\mathcal{D}_c \otimes_{\mathcal{H}_c} H_c/\mathcal{J}_{\ell,c} \otimes_{\mathcal{H}_c} P_c) = \text{Tor}^H(\mathcal{D}_c, H_c/\mathcal{J}_{\ell,c}) \otimes_{\mathcal{H}_c} P_c$.

(i) follows from Lemma 3.10. To prove (ii), note that, under the assumptions in (ii), $\dim V(\mathcal{B} \otimes_{\mathcal{H}_c} P_c) < \ell - 1$, where $\mathcal{B} := \text{Tor}^H_c(\mathcal{D}_c, H_c/\mathcal{J}_{\ell,c})$. So if $W'_c \subset W$ is a parabolic subgroup with $\dim \mathfrak{h}_{W_c} \geq \ell - 1$, then $\mathfrak{c} \text{Res}^W_c(\mathcal{B} \otimes_{\mathcal{H}_c} P_c) = 0$. Equivalently, $\mathcal{B}_c \otimes_{\mathcal{H}_c(W')} = 0$. But $\mathfrak{c} \text{Res}^W_c(P_c)$ is a projective generator of $\mathcal{O}_c(W')$ as was noted in the proof of Lemma 3.10. So, from Lemma 3.10 it follows that $\mathcal{B}_c \otimes_{\mathcal{H}_c(W')} = 0$. Since $c$ is Weil generic in $p'$ it follows that $\mathcal{J}_{\ell-1,c} \mathcal{B} = \mathcal{B} \mathcal{J}_{\ell-1,c} = 0$.

Let us proceed to the proof of (iii). As in the proof of the corresponding part of [BL, Theorem 7.1], it is enough to establish $\tilde{\mathcal{B}} \in \text{HC}(H_{p'})$ with homomorphisms $\varphi^1 : \text{Tor}^{H_{p'}}_{2(n+1-\ell)}(\mathcal{B}_c(\mathcal{D}_{p'}), H_{p'}/\mathcal{J}_{\ell}) \to \tilde{\mathcal{B}}, \varphi^2 : \mathcal{J}_{\ell-1}/\mathcal{J}_{\ell} \to \tilde{\mathcal{B}}$ such that, for a Weil generic $c \in p'$, the following holds:

(a) $\text{im} \varphi^1_c = \text{im} \varphi^2_c$.
(b) $\ker \varphi^2_c = \mathcal{J}_{\ell,c}/\mathcal{J}_{\ell,c}$, while $\dim V(\ker \varphi^1_c) < 2(\ell - 1)$.

Let $W'_1, \ldots, W'_k$ be all parabolic subgroups (viewed up to conjugacy) with $\dim \mathfrak{h}_{W'_c} = \ell - 1$. Recall the ideals $I(W'_i) \subset H_{p'}(W'_i)$. We set

$\tilde{\mathcal{B}} := \bigoplus_{i=1}^k (H_{p'}(W'_i)/I(W'_i))^{\dagger, W'_i}$. 

Let us construct $\varphi^1, \varphi^2$. These homomorphisms will be induced from

$$
\varphi^1_{1, W'_i} : \text{Tor}_{2(n+1-\ell)}^H(D_{p'}, H_c/J_{\ell,c}, W'_i) \rightarrow H_{p'}(W'_i)/I(W'_i);
$$

$$
\varphi^2_{1, W'_i} : (J_{\ell-1}/J_{\ell})_{1, W'_i} \rightarrow H_{p'}(W'_i)/I(W'_i).
$$

Let us construct $\varphi^1_{W'_i}$. Note that, by Lemma [BL, Theorem 7.2] (here $D_{p'}(W'_i)$ stands for an analog of $D_{p'}$ for $W'_i$), where the equality is of $N_W(W'_i)/W'_i$-equivariant HC bimodules. Further, from the construction of $J_{\ell}$ we get $(J_{\ell})_{1, W'_i} \subset I(W'_i)$ and the support of $I(W'_i)/(J_{\ell})_{1, W'_i}$ is not dense in $p'$ (indeed, for Weil generic $c$, the ideal $(J_{\ell,c})_{1, W'_i}$ has finite codimension and hence contains $I_c(W'_i)$). By Lemma 3.11 we have

$$
(6.2) \quad \text{Tor}_{2(n+1-\ell)}(H_0(D_{p'}), H_{p'}/J_{\ell})_{1, W'_i} \cong \text{Tor}_{2(n+1-\ell)}(H_0(D_{p'}), H_{p'}/(J_{\ell})_{1, W'_i}).
$$

Recall, Lemma 6.3, that over some nonempty Zariski open subset $U \subset p'$ we have an isomorphism of the right hand side of (6.2) with $H_{p'}(W'_i)/I(W'_i)$ as well as, Lemma 6.2 an isomorphism of $D_{p'}$ with $H_0(D_{p'})$. Multiplying the former isomorphism with an appropriate polynomial $f \in \mathbb{C}[p']$ (to extend it from $U$ to $p'$), we get a required homomorphism $\varphi^1_{W'_i}$. We construct $\varphi^2_{W'_i}$ in a similar fashion.

For $\varphi^j$, we take the homomorphisms induced by $\bigoplus_{i=1}^{r} \varphi^j_{W'_i}$. Let us check properties (a),(b). We start with (a). Since the bimodules $[H_c(W'_i)/I_c(W'_i)]_{1, W'_i}$ are supported on the closures of the pairwise different leaves, it is enough to show that

$$(a') \quad \text{the images of } \text{Tor}_{2(n+1-\ell)}(H_0(D_{c}), H_c/J_{\ell,c}, J_{\ell-1,c}/J_{\ell,c}) \text{ in } [H_c/I_c(W'_i)]_{1, W'_i} \text{ coincide.}
$$

We claim that both bimodules $\text{Tor}_{2(n+1-\ell)}(H_0(D_{c}), H_c/J_{\ell,c}, J_{\ell-1,c}/J_{\ell,c})$ do not have quotients annihilated by $J_{\ell-1,c}$. Indeed, for $J_{\ell-1,c}/J_{\ell,c}$, this follows from $J_{\ell-1,c} = J_{\ell-1,c}$. Suppose

$$
\text{Tor}_{2(n+1-\ell)}(D_{c}, H_c/J_{\ell,c})J_{\ell-1,c} \neq \text{Tor}_{2(n+1-\ell)}(D_{c}, H_c/J_{\ell,c}).
$$

This is equivalent to $H_{2(n+1-\ell)}(D_{c} \otimes L P_c/J_{\ell-1,c}P_c) \neq 0$. But we already know that $H_i(D_{c} \otimes L P_c/J_{\ell-1,c}P_c) = 0$ for $i < 2(n+2-\ell)$, a contradiction. The claim in the beginning of the paragraph is established.

Recall that the cokernel of the homomorphism

$$
J_{\ell-1,c}/J_{\ell,c} \rightarrow ((J_{\ell-1,c}/J_{\ell,c})_{1, W'_i})_{1, W'_i} = [H_c(W'_i)/I_c(W'_i)]_{1, W'_i}
$$

is annihilated by $J_{\ell-1,c}$. Since $\text{Tor}_{2(n+1-\ell)}(D_{c}, H_c/J_{\ell,c})_{1, W'_i} \cong H_c(W'_i)/I_c(W'_i)$, we deduce (a') and hence (a).

Let us prove (b). Since the homomorphisms $\varphi^j_{W'_i}, j = 1, 2$ are isomorphisms for a Weil generic $c$, we see that the kernels of $\varphi^j$ are precisely the maximal sub-bimodules with support of dimension less than $2(\ell - 1)$. That the maximal sub-bimodule of $J_{\ell-1,c}/J_{\ell,c}$ with this property coincides with $J_{\ell,c}/J_{\ell,c}$ follows from Lemma 5.2. \qed

6.3. Proof of Theorem [6.1] With Proposition 6.4 in hand, the proof of Theorem 6.1 basically repeats that of [BL, Theorem 7.2]. As we have mentioned before, the equality $J_{\ell,c} = J_{\ell,c}$ holds for a Zariski generic $c \in p'$ so the filtrations are well-defined.

Observe that, for $M \in \mathcal{C}_1$, we have

$$
(6.3) \quad B_c(\psi) \otimes_{H_c}^L M = (B_c(\psi) \otimes_{H_c}^L H_c/J_{n+1-j,c}) \otimes_{H_c/J_{n+1-j,c}}^L M.
$$
and, for $M' \in \mathcal{C}_j^2$,

$$R \text{Hom}_{H_c-\psi}(\mathcal{B}_c(\psi), M') = R \text{Hom}_{H_c-\psi/\mathcal{J}_{n+1-j,c}}(H_{c-\psi}/\mathcal{J}_{n+1-j,c} \otimes_h^L H_{c-\psi} \mathcal{B}_c(\psi), M').$$

Let us prove (I) in the definition of a perverse equivalence. By the definition of $\mathcal{J}_{\ell,c}$, for a Weil generic $c$, we see that a bimodule annihilated by this ideal on the right is annihilated by $\mathcal{J}_{\ell,c}$ on the left. So $\mathcal{J}_{n+1-j,c} H_k(\mathcal{B}_c(\psi) \otimes_h^L H_{c/\mathcal{J}_{n+1-j,c}}) = 0$ for a Weil generic $c$ and hence also for a Zariski generic $c$. From (6.3) we deduce that $\varphi_c(C_j^1) \subset C_j^2$ for a Zariski generic $c$. Similarly, using (6.4), we see that $\varphi_c^{-1}(C_j^2) \subset C_j^1$ for a Zariski generic $c$. (I) is proved.

Let us prove (II). We know that $\text{Tor}_{i}^H(\mathcal{B}_c(\psi), P_c/\mathcal{J}_{n+1-j,c} P_c) = 0$ for $i < j$ and Weil generic $c$. As in (6.1), this is equivalent to $\text{Tor}_{i}^H(\mathcal{B}_c(\psi), H_{c/\mathcal{J}_{n+1-j,c}}) = 0$ for a Weil generic $c$. So the Tor above vanishes for a Zariski generic $c$. We use (6.3) and see that (II) holds.

Let us check (III). We set

$$\mathcal{B}_c(\psi)_j := \text{Tor}_{j}^H(\mathcal{B}_c(\psi), H_{c/\mathcal{J}_{n+1-j,c}}),$$

$$\mathcal{B}_{c-\psi}(-\psi)_j := \text{Tor}_{j}^{H_{c-\psi}}(\mathcal{B}_{c-\psi}(-\psi), H_{c-\psi/\mathcal{J}_{n+1-j,c}}).$$

As in [BL, Theorem 7.1(4)], we have

$$\text{Tor}_{2j}^H(\mathcal{D}_c, H_{c/\mathcal{J}_{n+1-j,c}}) = \mathcal{B}_{c-\psi}(-\psi)_j \otimes_{H_{c-\psi}} \mathcal{B}_c(\psi)_j,$$

$$\text{Tor}_{2j}^{H_{c-\psi}}(\mathcal{D}_c, H_{c-\psi/\mathcal{J}_{n+1-j,c}}) = \mathcal{B}_c(\psi)_j \otimes_{H_{c-\psi}} \mathcal{B}_{c-\psi}(-\psi)_j,$$

where we write $\mathcal{D}_c'$ for $\mathcal{B}_c(\psi) \otimes_{H_{c-\psi}} \mathcal{B}_{c-\psi}(-\psi)$. Similarly to [BL] 7.7, (III) follows from (iii) of Proposition 6.4 (and its direct analog for $\mathcal{D}_c'$).

7. Open problems

7.1. Extension to general SRA. We have constructed derived equivalences between categories $\mathcal{O}$. One can ask to construct such equivalences between the categories of all modules. We conjecture that $D^b(H_{c-\psi} \text{-mod}) \simto D^b(H_{c-\psi} \text{-mod})$ for any $\psi \in \mathfrak{p}_Z$. This conjecture is true when $W = G(\ell, 1, n)$, this was shown in [GL, Section 5]. Moreover, in [GL, Section 5] the conjecture was also proved for the Symplectic reflection algebras corresponding to the groups $\Gamma_n := \mathfrak{S}_n \ltimes \Gamma_1^n$, where $\Gamma_1$ is a finite subgroup of $\text{SL}_2(\mathbb{C})$. Now let $\Gamma \subset \text{Sp}(V)$ be an arbitrary symplectic reflection group. We conjecture that $D^b(H_{c-\psi} \text{-mod})$ for any $c \in \mathfrak{p}$ and any $\psi$ in a lattice $\mathfrak{p}_Z$ defined in this case as follows. The parameter space $\mathfrak{p}$ is the direct sum $\bigoplus_{i=1}^r \mathfrak{p}_i$, where $r$ is the number of the conjugacy classes of subspaces of the form $V^s \subset V$ of codimension 2 and $\mathfrak{p}_i^s$ is the parameter space for the pointwise stabilizer $\Gamma_i^s$ of $V^s$ (acting on $V/V^s \cong \mathbb{C}^2$). Since $\Gamma_i^s$ is a Kleinian group, we have a lattice $\mathfrak{p}_Z^i \subset \mathfrak{p}_i^s$ used in [GL, Section 5]. We set $\mathfrak{p}_Z := \bigoplus_{i=1}^r \mathfrak{p}_Z^i$. This definition is compatible both with the Cherednik case and with the case of $\Gamma_n$.

7.2. Wall-crossing bijections. Let $\varphi_c : D^b(\mathcal{O}_c) \simto D^b(\mathcal{O}_{c-\psi})$ be a perverse equivalence constructed above. As any perverse equivalence, it induces a bijection $\text{ Irr}(\mathcal{O}_c) \rightarrow \text{ Irr}(\mathcal{O}_{c-\psi})$. The question is to compute this bijection (a generalization of the Mullineux involution). The case of the groups $G(\ell, 1, n)$ will be addressed in a subsequent paper.
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