A new version of distributional chaos and the relations between
distributional chaos in a sequence and other concepts of chaos

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Abstract
In this paper we consider relations between distributional chaos in a sequence
with distributional chaos, (ω-chaos, R-T chaos, DC3, respectively). We give a sufficient
condition and prove that the distributional chaos is equivalent to the distributional
chaos in a sequence under this condition. Besides, we get that distributional chaos in a
sequence and ω-chaos(R-T chaos, DC3, respectively) don’t imply each other. Finally, we
give a new definition of chaos, named DC2’, which is similar to DC2, and show that for
any integer N > 0, f is DC2’ if and only if \( f^N \) is also DC2’.

Keywords: distributional chaos in a sequence; ω-chaos; R-T chaos; DC2’

1. Introduction

The existence of chaotic behavior in deterministic systems has attracted researchers for many
years. Since Li and Yorke first gave the definition of chaos with strict mathematical terms in 1975
(see [1]), the research on chaos has had a great influence on modern science including natural
science and many humanities, such as economics, sociology and philosophy. We can say that the
influence has covered almost all disciplines. In almost all fields relating to dynamical progress,
there exists a chaotic phenomenon. The theory of chaos convinces scientists that a simple
definite system can produce complicated properties and a complex system possibly follow a
simple law. As the ultimate aim of the scientists is to clarify the essence of the complexity, the
chaotic systems with irregularly complex dynamical behaviors naturally become one common
subject. However, depending on different perspective and understanding (people from different
fields try to describe the behaviors by providing definitions of chaos according to their
understanding of the subject.), the various concepts of chaos have been given, such as Li–Yorke
chaos, Schweizer–Smital chaos (also called distributional chaos; see [2]), Devaney chaos [3],
ω-chaos[4], R-T chaos[5], etc. Among them, distributional chaos has some actual significance.
Later, three mutually nonequivalent versions of distributional chaos of type 1-3(DC1–DC3) were
considered[6]. And more and more researchers give their attention to the properties of
distributional chaos. Each definition tries to describe some kind of unpredictability of the system.
Therefore, there exists much ambiguity in academic intercourse of different fields. At the same
time, this situation cannot be tolerated in mathematical field which is based on strict
mathematical definitions. Therefore, it is very significant to further explore the essence of chaos,
unify the definition of chaos, and discuss the inner relations between the different definitions of
chaos. Hence, in order to establish a satisfactory definitional and terminological framework for
chaotic system, we need to reveal the inner link between the various concepts which characterize the complexity. Since then, surely, it is an important question to understand the relation among the various definitions. And there are many results about that [6, 8-19]. In order to reveal the links between distributional chaos and chaos in the sense of Li and Yorke, the concept of distributional chaos in a sequence was introduced in [7]. In particular, The more related results about the distributional chaos in a sequence is referred in[7-11,17].

In this paper we study some different definitions of chaos and relations among them. We have known that the distributional chaos implies the distributional chaos in a sequence but the verse is not sure[11], in a word, the distributional chaos is stronger than the distributional chaos in a sequence. However, in this paper, we will give a sufficient condition to show that the distributional chaos is equivalent to the distributional chaos in a sequence under this condition.

Besides, we give a new definition of chaos named distributional chaos of type 2'(DC2'),which contrast to DC2, and now the four versions of distributional chaos are mutually nonequivalent. Then we get a theorem that f is DC2' if and only if f\(^N\) is too. Finally, we research the relationship about the other notions of chaos chaos(\(\omega\)-chaos, R_T chaos, DC3 ) with distributional chaos in a sequence.

This paper is organized as follows. In Section 2, we will first give some preliminaries and definitions. The main conclusions will be given in Section 3.

2. Preliminaries and definitions

We denote the \(N\)-fold iterates of f by \(f^N\).

**Definition 1.** Let \((X,d)\) be a metric space. A continuous map \(f:X \to X\) is called Li-Yorke chaotic if there exists an uncountable subset \(S \subseteq X\) such that for every pair \(x, y \in S\) of distinct points we have
\[
\liminf_{n \to \infty} d\left(f^n(x), f^n(y)\right) = 0 \quad \text{and} \quad \limsup_{n \to \infty} d\left(f^n(x), f^n(y)\right) > 0.
\]

**Definition 2.** Let \((X,d)\) be a metric space, and \(f:X \to X\) be a continuous map. Note
\[
F(f,x,y,t) = \liminf_{n \to \infty} \frac{1}{n} \# \left\{ i \left| d(f^i(x), f^i(y)) < t, 0 \leq i < n \right. \right\},
\]
\[
F^*(f,x,y,t) = \limsup_{n \to \infty} \frac{1}{n} \# \left\{ i \left| d(f^i(x), f^i(y)) < t, 0 \leq i < n \right. \right\}.
\]
f is called distributional chaos if there exists an uncountable subset \(S \subseteq X\) such that for every pair \(x, y \in S\) of distinct points we have \(F(f,x,y,\epsilon) = 0\) for some \(\epsilon > 0\) and \(F^*(f,x,y,\epsilon) = 1\) for all \(t > 0\).

**Remark 1.** Distributional chaos was generalized in [6], so the above f is also called distributional chaos of type 1, briefly, DC1. If there exists an uncountable subset \(S \subseteq X\) such that for every pair \(x, y \in S\) of distinct points, we have
\[
F(f,x,y,\epsilon) < 1\ for \ some \ \epsilon > 0 \ and \ F^*(f,x,y,\epsilon) = 1 \ for \ all \ t > 0,
\]
or
\[
F(f,x,y,t) < F^*(f,x,y,t) \ for \ all \ t \in J, \ where \ J \ is \ some \ nondegenerate \ interval,
\]
then we say that f exhibits distributional chaos of type 2-3, briefly, DC2 or DC3, respectively.
In the following definition, we will generalize further the distributional chaos. It is similar to DC2, so we call it DC2'.

**Definition 3.** Let $(X, d)$ be a metric space, and $f: X \to X$ be a continuous map. Note

$$F(f, x, y, t) = \liminf_{n \to \infty} \frac{1}{n} \# \{ i \mid d(f^i(x), f^i(y)) < t, 0 \leq i < n \},$$

$$F^*(f, x, y, t) = \limsup_{n \to \infty} \frac{1}{n} \# \{ i \mid d(f^i(x), f^i(y)) < t, 0 \leq i < n \},$$

$f$ is called a distributional chaos of type 2', briefly DC2', if there exists an uncountable subset $S \subseteq X$ such that for every pair $x, y \in S$ of distinct points we have $F(f, x, y, \varepsilon) = 0$ for some $\varepsilon > 0$ and $F^*(f, x, y, t) > 0$ for all $t > 0$.

**Definition 4.** Let $(X, d)$ be a metric space, $f: X \to X$ be a continuous map and $(q_i)$ be a strictly infinitely increasing sequence of positive integers. Note

$$F(f, x, y, t, q_i) = \liminf_{n \to \infty} \frac{1}{n} \# \{ i \mid d(f^{q_i}(x), f^{q_i}(y)) < t, 0 \leq i < n \},$$

$$F^*(f, x, y, t, q_i) = \limsup_{n \to \infty} \frac{1}{n} \# \{ i \mid d(f^{q_i}(x), f^{q_i}(y)) < t, 0 \leq i < n \},$$

$f$ is called a distributional chaos in a sequence briefly SDC if there exists an uncountable subset $S \subseteq X$ such that for every pair $x, y \in S$ of distinct points we have $F(f, x, y, \varepsilon, q_i) = 0$ for some $\varepsilon > 0$ and $F^*(f, x, y, t, q_i) = 1$ for all $t > 0$.

**Remark 2.** We can also generalize the distributional chaos in a sequence named distributional chaos in a sequence of type 1-3 which is similar to the distributional chaos, briefly SDC1-3. That is if

$$F(f, x, y, \varepsilon, q_i) < 1 \text{ for some } \varepsilon > 0 \text{ and } F^*(f, x, y, t, q_i) = 1 \text{ for all } t > 0,$$

or

$$F(f, x, y, t, q_i) < F^*(f, x, y, t, q_i) \text{ for all } t \in j, \text{ where } j \text{ is some nondegenerate interval},$$

then we say that $f$ exhibits distributional chaos in a sequence of type 2-3, briefly, SDC2 or SDC3, respectively.

From the above definitions it is easy to see that the following statements hold:

(a) DC1 implies DC2, SDC and DC2';
(b) both DC2 and DC2' respectively implies DC3,
(c) all DC1, DC2, DC2' and SDC respectively implies chaos in the sense of Li and Yorke,
(d) SDC1 implies SDC2, and SDC2 implies SDC3.

It has been proved that three versions of distributional chaos of type 1-3 (DC1–DC3) are mutually nonequivalent[12] and DC1 is not equivalent to SDC1[11]. In the following example, we will show that DC2' is also neither equivalent to DC1 nor DC2. DC2' is not equivalent to DC3 by (c) and (13). So the four versions of distributional chaos (DC1, DC2, DC2', DC3) are mutually nonequivalent.

**Example 1**

Let $X = [0, +\infty)$ and define the metric $d: X \times X \to [0, 1]$ be
\[
d(x, y) = \begin{cases} 
0, & x = y \\
\frac{1}{2^k}, & [x] = [y] \equiv 0 \text{ (mod 2)}, x \neq y \\
1, & \text{else}
\end{cases}
\]

where \( b_1 = 1, b_i = 2^{b_{i-1}} + b_{i-1} \), it is easily to see that \((X, d)\) is a discrete metric space. Let \( f: ([0, +\infty), d) \rightarrow ([0, +\infty), d)\) be \( f(x) = x + 1 \). We will claim that \( f \) is DC2'.

(i) Take \( D_0 = (0, 1), \forall 0 < t < \frac{1}{2}, \exists k_0 \in N, \text{ such that } \frac{1}{2k_0} < t, \text{ note } \sum_{j=1}^{k} b_j = L_k. \forall x, y \in D_0(x \neq y), \forall k \geq k_0, \forall 2i \in (L_{2k}, L_{2k+1} - 1), \text{ notices that } f^{2i}(x) = 2i + x \in (L_{2k}, L_{2k+1}), f^{2i}(y) = 2i + y \in (L_{2k}, L_{2k+1}) \text{ and } [f^{2i}(x)] = [f^{2i}(y)] = 2i, \text{ so } d(f^{2i}(x), f^{2i}(y)) = d(2i + x, 2i + y) = \frac{1}{2^k} \leq \frac{1}{2k_0} < t, \text{ t, 0 \leq i < L_{2k+1}} \geq \frac{1}{L_{2k+1}} \left( \frac{L_{2k+1} - 1}{L_{2k+1} - 1} \right) \]

So that \( \limsup_{n \to \infty} \frac{1}{n} \# \{ i \mid d(f^i(x), f^i(y)) < t, 0 \leq i < n \} \geq \limsup_{k \to \infty} \frac{1}{L_{2k+1}} \left( \frac{L_{2k+1} - 1}{L_{2k+1} - 1} \right) = \limsup_{k \to \infty} \frac{2^{b_1 + b_2 + \cdots + b_{2k}}}{2^{b_1 + b_2 + \cdots + b_{2k+1}} - 1} = \frac{1}{2} \).

(1)

Since \([x] = [y] \equiv 0 \text{ (mod 2)}\) where \( x, y \in D_0(x \neq y), \text{ then } \forall i \in N, [f^{2i+1}(x)] = [x] + 2i + 1 = [y] + 2i + 1 = [f^{2i+1}(y)] \equiv 1 \text{ (mod 2)}, \text{ so } d(f^{2i+1}(x), f^{2i+1}(y)) = 1.

So that \( \limsup_{n \to \infty} \frac{1}{n} \# \{ i \mid d(f^i(x), f^i(y)) < t, 0 \leq i < n \} \leq \limsup_{n \to \infty} \frac{n+1}{n} = \frac{1}{2}. \) (2)

Therefore, \( \limsup_{n \to \infty} \frac{1}{n} \# \{ i \mid d(f^i(x), f^i(y)) < t, 0 \leq i < n \} = \frac{1}{2} \) by (1) and (2).

(ii) \( \forall x, y \in D_0(x \neq y), \forall k \in N, \forall i \in (L_{2k+1}, L_{2k+2} - 1), \text{ notices that } f^i(x) \neq f^i(y) \in (L_{2k+1}, L_{2k+2}), \text{ so } d(f^i(x), f^i(y)) = 1.

\[
\frac{1}{L_{2k+2}} \# \{ i \mid d(f^i(x), f^i(y)) < \frac{1}{2}, 0 \leq i < L_{2k+2} \} \leq \frac{1}{L_{2k+2}} (L_{2k+1} + 1)
\]

\[
\liminf_{n \to \infty} \frac{1}{n} \# \{ i \mid d(f^i(x), f^i(y)) < \frac{1}{2}, 0 \leq i < n \} \leq \liminf_{k \to \infty} \frac{1}{L_{2k+2}} \# \{ i \mid d(f^i(x), f^i(y)) < \frac{1}{2}, 0 \leq i < L_{2k+2} \} \leq \liminf_{k \to \infty} \frac{L_{2k+1} + 1}{L_{2k+2}} = \frac{b_1 + b_2 + \cdots + b_{2k+1} + 1}{b_1 + b_2 + \cdots + b_{2k+1} + 2} = 0.
\]

Therefore \( f \) is DC2' by (i) and (ii).

**Definition 5.** Let \( \omega(x, f) \) denote the set of \( \omega \)-limit points of \( f \), and \( S \subseteq X \). We say that \( S \) is an \( \omega \)-scrambled set if, for any \( x, y \in S \) with \( x \neq y \),

(1) \( \omega(x, f) \setminus \omega(x, f) \) is uncountable;
(2) \( \omega(x, f) \cap \omega(x, f) \) is nonempty; and
(3) \( \omega(x, f) \) is not contained in the set of periodic points.

We say that \( f \) is \( \omega \)-chaotic, if there exists an uncountable \( \omega \)-scrambled set.
**Definition 6.** We say that a semiflow \((T, X)\) is Ruelle–Takens chaotic (R-T chaos) if it is point-transitive and sensitive.

**Lemma 1**[1] Let \(f: I \to I\) be a continuous map, then \(f\) is Li-Yorke chaos if and only if \(f\) is distributional chaos in a sequence.

**Lemma 2**[1] Let \(f: I \to I\) be a continuous map, then \(f\) is \(\omega\) – chaos if and only if \(f\) has positive topological entropy.

**Lemma 3**[1] Let \(X\) be a compact metric space, \(f: X \to X\) be continuous, \(x, y \in X, \forall N > 0\), we have

(i) If for \(t > 0\), \(F(f(x, y), t) = 0\), then \(F(f^n(x, y), t) = 0\).

(ii) If for \(t > 0\), \(F(f^n(x, y), t) = 0\), then \(F(f(x, y), t) = 0\).

3. **Main results**

**Theorem 1.** Let \((X, f)\) be compact dynamical system and \(Q = \{q_i\}\) be a strictly infinitely increasing sequence of positive integers. If \(\exists M\) such that \(\forall i, we have q_{i+1} - q_i \leq M\), then that \(f\) is distributional chaos if and only if that \(f\) is distributional chaos in a sequence \(Q\).

**Proof.** Necessity. (i) Since \(f\) is distributional chaos, \(F(f, x, y, \varepsilon) = 0\), then there is an increasing sequence \(\{n_k\}\) of positive integers such that

\[
\text{for } k \to \infty, \frac{1}{n_k} \left| \left\{ i \mid d(f^{q_i}(x), f^{q_i}(y)) < \varepsilon, 0 \leq i < n_k \right\} \right| < 0.
\]

Put \(m_k = \left\lfloor \frac{n_k}{M} \right\rfloor\).

Then for each \(k\),

\[
\left| \left\{ i \mid d(f^{q_i}(x), f^{q_i}(y)) < \varepsilon, 0 \leq i < m_k \right\} \right| \leq \left| \left\{ i \mid d(f^i(x), f^i(y)) < \varepsilon, 0 \leq i < n_k \right\} \right|,
\]

and further

\[
\frac{1}{n_k} \left| \left\{ i \mid d(f^{q_i}(x), f^{q_i}(y)) < \varepsilon, 0 \leq i < m_k \right\} \right| \leq \frac{1}{n_k} \left| \left\{ i \mid d(f^i(x), f^i(y)) < \varepsilon, 0 \leq i < n_k \right\} \right|.
\]

This gives for \(k \to \infty, \frac{F(f, x, y, \varepsilon, q_i)}{M} = 0\) by (3). Therefore \(F(f, x, y, \varepsilon, q_i) = 0\).

(ii) Since \(f\) is distributional chaos, \(F^*(f, x, y, t) = 1\), for all \(t > 0\), then there is an increasing sequence \(\{n_k\}\) of positive integers such that

\[
\text{for } k \to \infty, \frac{1}{n_k} \left| \left\{ i \mid d(f^i(x), f^i(y)) < t, 0 \leq i < n_k \right\} \right| \to 1,
\]

and so

\[
\frac{1}{n_k} \left| \left\{ i \mid d(f^i(x), f^i(y)) \geq t, 0 \leq i < n_k \right\} \right| \to 0.
\]

Put \(m_k = \left\lfloor \frac{n_k}{M} \right\rfloor\).

Then for each \(k\),
\[ \#\{i \mid d(f^{q_i}(x), f^{q_i}(y)) \geq t, 0 \leq i < m_k\} \leq \#\{i \mid d(f^i(x), f^i(y)) \geq t, 0 \leq i < n_k\} \]

And further
\[ \frac{1}{n_k} \#\{i \mid d(f^{q_i}(x), f^{q_i}(y)) \geq t, 0 \leq i < m_k\} \leq \frac{1}{n_k} \#\{i \mid d(f^i(x), f^i(y)) \geq t, 0 \leq i < n_k\}. \]

This gives for \( k \to \infty, \) \( \frac{1}{M} \sum_{i=1}^{n_k} \#\{i \mid d(f^{q_i}(x), f^{q_i}(y)) \geq t, 0 \leq i < m_k\} \to 0 \) by (4).

Therefore
\[ \frac{1}{m_k} \sum_{i=1}^{n_k} \#\{i \mid d(f^{q_i}(x), f^{q_i}(y)) \geq t, 0 \leq i < m_k\} \to 0 \text{ for } k \to \infty, \]

\( F^*(x, y, t, q_i) = 1. \)

Therefore f is distributional chaos in a sequence \( Q. \)

Sufficiency. (iii) Since f is distributional chaos in a sequence \( Q, \) \( F(f, x, y, \epsilon, q_i) = 0, \) then there is an increasing sequence \( \{n_k\} \) of positive integers such that

\[ \text{for } k \to \infty, \frac{1}{n_k} \sum_{i=1}^{n_k} \#\{i \mid d(f^{q_i}(x), f^{q_i}(y)) < \epsilon, 0 \leq i < n_k\} \to 0 \] (5)

Since \( f \) is continuous and \( X \) is compact, \( f^j \) is uniformly continuous for each \( j = 1, 2, \ldots, M. \)

Consequently, for fixed \( \epsilon > 0, \) there exist \( s > 0 \) such that

\[ d(f^{q_i}(x), f^{q_i}(y)) < \epsilon \text{ when } d(f^{q_{i-1}}(x), f^{q_{i-1}}(y)) < s, \text{ for each } j = 1, 2, \ldots, M, \]

so

\[ \sum_{j=1}^{M} \#\{i \mid d(f^{q_{i-1}}(x), f^{q_{i-1}}(y)) < s, 0 \leq i < n_k\} \leq M \cdot \#\{i \mid d(f^{q_i}(x), f^{q_i}(y)) < \epsilon, 0 \leq i < n_k\} \]

for \( k \to \infty, \) \( \frac{1}{n_k} \sum_{j=1}^{M} \#\{i \mid d(f^{q_{i-1}}(x), f^{q_{i-1}}(y)) < s, 0 \leq i < n_k\} \to 0 \) by (5) and (6).

(iv) Since f is distributional chaos in a sequence \( Q, \) \( F^*(f, x, y, p, q_i) = 1 \) for all \( p > 0, \) then there is an increasing sequence \( \{n_k\} \) of positive integers such that

\[ \text{for } k \to \infty, \frac{1}{n_k} \sum_{i=1}^{n_k} \#\{i \mid d(f^{q_i}(x), f^{q_i}(y)) < p, 0 \leq i < n_k\} \to 1, \]

and so

\[ \frac{1}{n_k} \#\{i \mid d(f^i(x), f^i(y)) \geq p, 0 \leq i < n_k\} \to 0. \] (7)

Since \( f \) is continuous and \( X \) is compact, \( f^j \) is uniformly continuous for each \( j = 1, 2, \ldots, M. \)

Consequently, for \( \forall t > 0, \) there exist \( s > 0 \) such that

\[ d(f^{q_{i+j-1}}(x), f^{q_{i+j-1}}(y)) < t \text{ when } d(f^{q_i}(x), f^{q_i}(y)) < s, \text{ for each } j = 1, 2, \ldots, M. \]

So we have \( d(f^{q_i}(x), f^{q_i}(y)) \geq s \text{ when } d(f^{q_{i+j-1}}(x), f^{q_{i+j-1}}(y)) \geq t \text{ for each } j = 1, 2, \ldots, M. \)

So
Since $X$ is compact, there exists an increasing sequence of positive integers such that

$$
\limsup_{k \to \infty} q_{i+1} - q_i = \infty.
$$

Therefore $f$ is distributional chaos.

**Remark.** The theorem 1 is the generalization of [14], where $N=M$, i.e., $Q = \{q_i\} = \{Ni\}$. If $f$ is distributional chaos in a sequence $Q = \{q_i\}$ but is not distributional chaos, then we have $\limsup_{k \to \infty} q_{i+1} - q_i = \infty$.

**Theorem 2.** Let $X$ be a compact metric space. Then $f$ is DC2 if and only if $f^N$ is too.

**Proof.** Necessity. Since $f$ is DC2, so $F^n(f, x, y, t) > 0$ for all $t > 0$, then there exists an increasing sequence $\{m_k\}$ of positive integers such that

$$
\text{for all } t > 0, \frac{1}{m_k} \# \{i \mid d \left( f^i(x), f^i(y) \right) < t, 0 \leq i < m_k \} > 0 \ (k \to \infty)
$$

(9)

Since $X$ is compact, $f^i$ is uniformly continuous for each $i = 1, 2, ..., N$.

Consequently, for $s > 0$, there exists $p > 0$ such that for all $u, v$ and each $i = 1, 2, ..., N, d \left( f^i(u), f^i(v) \right) \geq p$ whenever $d \left( f^N(u), f^N(v) \right) \geq s$. This implies

$$
N \left( \# \left\{ i \mid d \left( f^N(x), f^N(y) \right) \geq s, 0 \leq i < n_k \right\} - 1 \right) 
\leq \# \left\{ i \mid d \left( f^i(x), f^i(y) \right) \geq p, 0 \leq i < N \cdot n_k \right\}
$$

(10)

Take $n_k = \left\lfloor \frac{m_k}{N} \right\rfloor$ for all $k \geq 1$, by a simple calculation, we may derive from (10) that

$$
\frac{1}{m_k} \# \{ i \mid d \left( f^i(x), f^i(y) \right) < p, 0 \leq i < m_k \} \leq \frac{1}{n_k} \# \{ i \mid d \left( f^N(x), f^N(y) \right) < s, 0 \leq i < n_k \}
\leq \frac{1}{n_k} \# \{ i \mid d \left( f^i(x), f^i(y) \right) \geq p, 0 \leq i < N \cdot n_k \}
$$

(11)

By (9) and (11), we have

$$
\text{for all } s > 0, \frac{1}{n_k} \# \{ i \mid d \left( f^N(x), f^N(y) \right) < s, 0 \leq i < n_k \} > 0 \ (k \to \infty).
$$

(12)

(12) and Lemma 3 show that $f^N$ is DC2.
Sufficiency. Since $f^N$ is DC2', so $F^*(f^N, x, y, t) > 0$ for all $t > 0$, then there exists an increasing sequence $\{m_k\}$ of positive integers such that

$\text{for all } t > 0, \frac{1}{m_k} \# \{ i \bigg| d \left( f^N_i(x), f^N_i(y) \right) < t, 0 \leq i < m_k \} > 0 \ (k \to \infty)$

Take $n_k = m_k \cdot N$, then for all $k \geq 1$ and all $t > 0$,

$\frac{1}{m_k} \# \left\{ i \bigg| d \left( f^i(x), f^i(y) \right) < t, 0 \leq i < m_k \right\} \geq \frac{1}{m_k} \# \left\{ i \bigg| d \left( f^N_i(x), f^N_i(y) \right) < t, 0 \leq i < m_k \right\} > 0 \ (k \to \infty)$

so $\frac{1}{n_k} \# \left\{ i \bigg| d \left( f^i(x), f^i(y) \right) < t, 0 \leq i < n_k \right\} > 0 \ (k \to \infty)$. \hspace{1cm} (13)

(13) and lemma 3 show that $f$ is DC2'.

**Theorem 3.** SDC and $\omega -$chaos don't imply each other.

**Proof.** On one hand, let $f: I \to I$ be a continuous map, and be Li-Yorke chaos with zero topological entropy, then $f$ is distributional chaos in a sequence by lemma 1, but is not $\omega -$chaos by lemma 2.

On the other hand, there is a dynamical system $(X, g)$ that $g$ is $\omega -$chaos but is not Li-Yorke chaos in [15]. Because distributional chaos in a sequence implies Li-Yorke chaos, $g$ is not distributional chaos in a sequence.

**Theorem 4.** SDC and R-T chaos don't imply each other.

**Proof.** On one hand, let $(X, f)$ be a minimal distal non equicontinuity dynamical system, then $(X, f)$ is R-T chaos but not Li-Yorke chaos, so $(X, f)$ is not distributional chaos in a sequence like the proof of theorem 3.

On the other hand, Professor Wang and Li give a dynamical system $(X, f)$, which is distributional chaos in a sequence but not R-T chaos in[16].

**Theorem 5.** SDC and DC3 don't imply each other.

**Proof.** On one hand, there is a continuous map $f: I \to I$, which is Li-Yorke chaos but not DC3 in [3], then $f$ is distributional chaos in a sequence by lemma 1.

On the other hand, there is a continuous map $f$ which is DC3 but not Li-Yorke chaos in [6], then $f$ is not distributional chaos in a sequence like the proof of theorem 3.
References

[1] T. Y. Li and J. A. Yorke, Period three implies chaos, Amer. Math. Monthly 82(1975), 985-992.
[2] Schweizer B., Smítal, J. Measures of chaos and a spectral decomposition of dynamical systems on the interval[J]. Transactions of the American Mathematical Society, 1994, 344(2):737-754.
[3] Devaney R L. An Introduction to Chaotic Dynamical Systems[M]. The Benjamin/Cummings Publishing Co. Inc, 1986.
[4] Li S. w-chaos and topological entropy[J]. Transactions of the American Mathematical Society, 1993, 339(1):243-249.
[5] Auslander J, Yorke J A. Interval maps, factors of maps, and chaos[J]. Tohoku Mathematical Journal First, 1980, 32(2):177-188.
[6] Balibrea F, Smital J, Stefankova M. The three versions of distributional chaos[J]. Chaos Solitons & Fractals, 2005, 23(5):1581-1583.
[7] Wang L, Huang G, Huan S. Distributional chaos in a sequence[J]. Nonlinear Analysis Theory Methods & Applications, 2007, 67(7):2131-2136.
[8] Liu H, Wang L, Chu Z. Devaney's chaos implies distributional chaos in a sequence[J]. Nonlinear Analysis, 2009, 71(12):6144-6147.
[9] Li J, Oprocha P. On n-scrambled tuples and distributional chaos in a sequence[J]. Journal of Difference Equations and Applications, 2013.
[10] Piotr O. Families, filters and chaos[J]. Bulletin of the London Mathematical Society, 2010(4):713-725.
[11] Song W, Wang L, Zhang A. Sequential distributional chaos nonequivalent to distributional chaos[J]. Acta entiariwm Naturalium Universitatis Jilinensis, 2002.
[12] Wang H, Liao G, Fan Q. Substitution systems and the three versions of distributional chaos[J]. Topology and its Applications, 2008, 156(2):262-267.
[13] Wang H, Lei F, Wang L. DC3 and Li–Yorke chaos[J]. Applied Mathematics Letters, 2014, 31:29-33.
[14] Wang L, Huan S, Huang G. A note on Schweizer–Smital chaos[J]. Nonlinear Analysis Theory Methods & Applications, 2008, 68(6):1682-1686.
[15] Pikula, Rafał. On some notions of chaos in dimension zero[J]. Colloquium Mathematicum, 2007, 107(2):167-177.
[16] Li-Dong W. Chaos in a Sequence being not Equivalent to R—T Chaos[J]. Journal of Dalian Nationalities University, 2007.
[17] Tang YJ, Yin JD. Distributional chaos and distributional chaos in a sequence occurring on a subset of the one-sided symbolic system[J] BULLETIN of THE KOREAN MATHEMATICAL SOCIETY 95-108.
[18] Kolyada S F. Li-Yorke sensitivity and other concepts of chaos[J]. Ukrainian Mathematical Journal, 2004, 56(8):1242-1257.
[19] Smítal, J. Various notions of chaos, recent results, open problems[J]. Real Analysis Exchange, 2002, 26(1):81-86.