ON THE GRIFFITHS GROUP OF THE CUBIC SEVENFOLD

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2 August 1993

Abstract. In this paper we prove that the Griffiths group of a general cubic sevenfold is not finitely generated, even when tensored with \( \mathbb{Q} \). Using this result and a theorem of Nori, we provide examples of varieties which have some Griffiths group not finitely generated but whose corresponding intermediate Jacobian is trivial.

1. Introduction

For \( X \) a smooth complex projective variety, one of the important questions in algebraic geometry is the existence and the behavior of non trivial subvariety on \( X \), i.e., subvarieties that are not the intersection of \( X \) with a hypersurface. In general, the approach is as follows: we let \( Z^d(X) \) be the free abelian group generated by the irreducible subvarieties of codimension \( d \) in \( X \), and introduce various equivalence relations on \( Z^d(X) \). The problem is then to understand the quotients of \( Z^d(X) \) under these relations. The first, and most obvious, is homological equivalence, but there are other interesting relations, namely algebraic and rational equivalence. The so-called classical cases, when \( d = 1 \) (divisors) or \( d = \dim X \) (0–cycles) are well understood, but little is known in general. In these cases, homological and algebraic equivalence coincide, and the group of cycles algebraically equivalent to zero modulo rational equivalence is isomorphic to an abelian variety.

One of the first result for general \( d \) is due to Griffiths [8], who showed that homological and algebraic equivalence do not coincide if \( d \neq 1, \dim X \). The group of cycles on \( X \) of codimension \( d \) homologous to zero modulo algebraic equivalence is called the \( d \)-th Griffiths group of \( X \) and is denoted by \( G^d(X) \); we will mainly consider the \( \mathbb{Q} \)-vector space \( G^d_\mathbb{Q}(X) = G^d(X) \otimes \mathbb{Q} \), or equivalently, the Griffiths group modulo torsion. Clemens [5] then showed that \( G^d_\mathbb{Q}(X) \) can have infinite dimension over \( \mathbb{Q} \).

1991 Mathematics Subject Classification. 14C30, 14C10, 14J40, 14K30.

Partially supported by Science Project “Geometry of Algebraic Varieties”, n. 0-198-SC1, and by fundings from M.U.R.S.T. and G.N.S.A.G.A. (C.N.R.), Italy

Typeset by \( \textsf{AMS-T\TeX} \)
Clemens’ example is given by a general quintic hypersurface in $\mathbb{P}^4$, and this opened the way to the construction of other examples (see [1], [2], [13], [17]).

In each of these examples, the techniques of proof were quite ad hoc, and there seemed not to be a general pattern. Only recently, two fundamental results have been obtained: the first is Voisin’s proof of Clemens’ theorem on the quintic threefolds [21], and the other is Nori’s beautiful theorems on Hodge theoretic connectivity [14].

In showing that $G_d^Q(X)$, for some variety $X$, is infinite dimensional, there are basically two steps: first, one must find infinitely many cycles and then one has to show that they are independent. In all of the previous proofs, the cycles were found by geometrical considerations, which cannot generalize to more complicated situations. Voisin gives a very general method for finding cycles, based on an argument of Green concerning the Noether–Lefschetz locus of a suitably defined family of varieties; then, using Griffiths’ description of cohomology via residues of rational differential forms, she is able to compute explicitly (this is a highly non trivial computation) an infinitesimal invariant attached to an algebraic cycle, which permits to conclude that the cycles so found are independent. In [21], she works out the case of quintic threefolds, but her technique is quite general (Bardelli and Müller-Stach have worked out the case of the complete intersection of two cubics in $\mathbb{P}^5$).

A result in the opposite direction is Green’s theorem ([7] and Voisin, unpublished): other than for cubic hypersurfaces of dimension 3 and 5 and for quartic hypersurfaces of dimension 3, for which the intermediate Jacobian is an abelian variety and the Abel–Jacobi map is known to be surjective, the only cases in which the image of the Abel–Jacobi map for a general hypersurface may be non torsion, i.e., $G_d^Q(X)$ may be non zero, are quintic threefolds (where the image is non torsion [5]) and cubic sevenfolds, i.e., cubic hypersurfaces in $\mathbb{P}^8$. Moreover, the middle cohomology of a smooth cubic sevenfold carries a Hodge structure of level 3, similar to the one of a smooth quintic threefold. As Voisin’s proof is Hodge–theoretic, we were led to see if her techniques could be carried over to this case: in fact, following [21] we prove the following:

**Theorem 1.** For a general smooth cubic sevenfold $X$, the group of 3–cycles homologous to zero modulo algebraic equivalence is not finitely generated, even when tensored with $\mathbb{Q}$.

The main difference with the previously known cases of infinite generation is that the cycles are of codimension four and they are not curves. This makes it more difficult to treat the situation geometrically, and in fact we use in essential way Voisin’s method of doing things Hodge–theoretically to overcome this difficulty.

On the other hand, our example is not too surprising: Voisin has recently shown ([22]) that the Griffiths group is non torsion for any Calabi–Yau threefold, i.e., a Kähler threefold $X$ such that $K_X$ is trivial and $h^{1,0}(X) = h^{2,0}(X) = 0$. It is an open question whether the Griffiths group is always infinitely generated for such threefolds. Calabi–Yau threefolds have attracted much interest from physicist, in connection with problems in string theory. One of the main conjectures (based on
physics considerations) is the so-called “mirror symmetry”, that roughly states that
to each Calabi–Yau threefold \( X \) (plus some extra data) is attached another Calabi–
Yau threefold \( \tilde{X} \), the “mirror of \( X \)”, whose Hodge diamond is the mirror of the Hodge
diamond of \( X \). This conjecture is full of mathematical implications, some of which
have been verified (see [12] for further information, and for a precise statement of
the mirror conjecture). One potential counterexample to this conjecture is given by
rigid threefolds, i.e., with \( h^{2,1}(X) = 0 \), since then the mirror should have \( h^{1,1} = 0 \)
and so cannot be Kähler. In a recent preprint [3], Candelas, Derrick and Parkes show
how a cubic sevenfold can be considered the mirror (in an appropriate sense) of one
such rigid Calabi–Yau threefold. So, in some sense, the cubic sevenfold should have
properties similar to those of Calabi–Yau threefolds: Theorem 1 shows that this is so
with respect to the Griffiths group, which is not only non torsion but even infinitely
generated (modulo torsion).

But things can be even worse. The essential tool that has always been used to
show that two cycles are algebraically independent is the Abel–Jacobi map \( \theta_d \) from
the group of cycles of codimension \( d \) homologous to zero modulo rational equivalence
to the Griffiths intermediate Jacobian \( J^d(X) \). In [14], Nori introduces a deeper version
of Griffiths approach: his theorems allow to show independence even when the Abel–
Jacobi map is identically zero. He provides the first example of cycles homologically
equivalent to zero but not algebraically equivalent to zero on a particular variety
without the use of the Abel–Jacobi map on the variety, and in fact belonging to the
kernel of the Abel–Jacobi map. He shows that the algebraic part of the primitive
cohomology of a hypersurface \( X \) maps injectively under restriction to a subgroup
of the Griffiths group of a complete intersection of \( X \) with general hypersurfaces of
sufficiently high degree. However, in this way one cannot obtain infinitely generated
subgroups of the Griffiths group.

To get such examples, we use Theorem 1 and Nori’s theorem. Let \( X \) be a general
smooth cubic sevenfold, \( D_1, D_2 \) two general hypersurfaces of sufficiently high degree
and let \( Y = X \cap D_1 \cap D_2 \).

**Theorem 2.** The (rational) Griffiths group \( G^4_{\mathbb{Q}}(Y) \) of codimension 4 cycles in \( Y \) is
not finitely generated even if the intermediate Jacobian \( J^4(Y) \) is zero.

The same proof shows that also for \( Y' = X \cap D_1 \) the Griffiths group is not finitely
generated, so we have examples in both even and odd dimension.

We conclude this introduction explaining in details how Theorem 2 follows from
Nori’s theorem and Theorem 1. Let \( X \) be a smooth projective variety \( X \), and let
\( CH^d(X) \) be the Chow group of cycles of codimension \( d \) on \( X \): Nori defines an
increasing filtration \( A_r CH^d(X) \), whose main property is that if \( \eta \in A_r CH^d(X) \otimes \mathbb{Q} \)
then there is an algebraic subset \( Z \subset X \) of pure codimension \( (d - r - 1) \) so that \( Z \)
contains the support of \( \eta \), and the fundamental class of \( \eta \) vanishes in \( H_{2n-2d}(Z, \mathbb{Q}) \),
where \( n = \dim X \). In particular, \( A_0 CH^d(X) \) consists of cycles algebraically equivalent
to zero.

If \( \eta \in A_r CH^d(X) \), then not only \( \eta \) is homologous to zero, but \( \theta_d(\eta) \) has to satisfy
some restriction: if \( J^d_r(X) \) is the intermediate Jacobian of the largest integral Hodge structure contained in \( F^{d-r-1}H^{2d-1}(X) \), then \( \theta_d(\eta) \in J^d_r(X) \). Note that \( J^d_0(X) \) is the “largest abelian subvariety” of the intermediate Jacobian \( J^d(X) \). We can now state [14, Theorem 2]:

**Theorem.** (Nori) Let \( X \subset P^m \) be smooth, projective, and let \( Y \) be the intersection of \( X \) with \( h \) general hypersurfaces of sufficiently large degrees. Let \( \xi \in CH^{d}(X) \) and let \( \eta = \xi|Y \). Assume that \( r + d < \text{dimension of } Y \). If \( \eta \in A_r CH^{d}(Y) \otimes \mathbb{Q} \), then

1. The cohomology class of \( \xi \) vanishes in \( H^{2d}(X, \mathbb{Q}) \), and
2. the Abel–Jacobi image of a non-zero multiple of \( \xi \) belongs to \( J^d_r(X) \).

It is now clear how to prove Theorem 2: Let \( R : CH^{4}_{hom}(X) \rightarrow CH^{4}_{hom}(Y) \) be the restriction map and \( \pi : CH^{4}_{hom}(Y) \rightarrow G^{4}(Y) \) be the canonical projection. Let \( \xi \in CH^{4}_{hom}(X) \): if \( \pi \circ R(\xi) = 0 \) then, by Nori’s theorem (with \( r = 0 \)), we would have \( \theta_4(\xi) \in J^4_0(X) \). But for general \( X \), a monodromy argument shows that \( J^4_0(X) = 0 \) and hence \( \ker(\pi \circ R) \subseteq \ker \theta_4 \). Since for a general cubic sevenfold \( X \) we have, from the proof of Theorem 1, that the image \( \theta_4(CH^{4}_{hom}(X)) \) is not finitely generated, even when tensored with \( \mathbb{Q} \), the same is true for \( G^{4}(Y) \otimes \mathbb{Q} \).

2. Cycles on the cubic sevenfold

In this and the following sections we show how Voisin’s argument can be used to prove that the Griffiths group of a generic cubic sevenfold is infinitely generated. There are three parts in the proof: the first two (§2 and 3) are very similar to the corresponding ones in [21], and so we indicate only the changes needed in the present case, giving precise references for all the results we quote. The computation of the infinitesimal invariants in the last part however (§4), is quite different, and we will give full details. For ease of reference, we use the same notations as in [21].

Let \( (\lambda, \Sigma, X) \) be a triple, where \( X \subset P^8 \) is a smooth cubic hypersurface defined by an equation \( F \), \( \Sigma \subset X \) is a smooth hyperplane section, and \( \lambda \in H^{6}(\Sigma, \mathbb{Z})^{\text{prim}} \cap H^{3,3}(\Sigma) \) is the class of a primitive algebraic 3–cycle on \( \Sigma \).

We note the following two useful facts.

**Lemma 2.1.** Let \( X \) and \( \Sigma \) be as above. Then:

1. \( h^{7,0}(X) = h^{6,1}(X) = 0; h^{5,2}(X) = 1; h^{4,3}(X) = 84 \).
2. \( h^{6,0}(\Sigma) = h^{5,1}(\Sigma) = 0; h^{4,2}(\Sigma) = 8; h^{3,3}(\Sigma) = 36 \).
3. \( h^{0}(N_{\Sigma/X}) = 8 \).

**Proof.** The simplest way to compute these Hodge numbers is to use Griffiths’ description of the Hodge filtration of \( X \) and \( \Sigma \) with rational forms with poles along the hypersurface [8]. Since the Hodge numbers are invariant under deformations, one can compute them for the Fermat hypersurface, whose Jacobian ideal is particularly simple. Alternatively, one can use the formulas for the Hodge numbers of complete intersections given in [6, Théorème 2.3].

The last formula is clear. \( \square \)
Hence, the Hodge numbers of $X$ and $\Sigma$ “look like” the ones of a quintic threefold and its hyperplane section and moreover $h^{4,2}(\Sigma) = h^0(\Sigma, N_{\Sigma/X})$.

The second observation is crucial.

**Proposition 2.2.** The Hodge $(3,3)$–conjecture is true for any smooth cubic sixfold $\Sigma \subset \mathbb{P}^7$.

**Proof.** This is due to Steenbrink [19]. $\square$

Let $G$ be the smooth variety that parametrizes the pairs $(X, \Sigma)$ such that $\Sigma \subset X$, and let $\lambda \in H^6(\Sigma, \mathbb{Z})^{\text{prim}} \cap H^{3,3}(\Sigma)$. Since $\lambda$ is real, by Lemma 2.1, the condition “$\lambda$ remains of type $(3,3)$ in $H^6(\Sigma)$” is given by $h^{4,2}(\Sigma) = \dim H^2(\Sigma, \Omega^2_\Sigma) = 0$ equations on $G$, so that locally the family $F_\lambda$ of the triples $(\lambda, \Sigma, X)$ is a subvariety of $G$ of codimension at most $h^{4,2}$. In particular, $F_\lambda$ is smooth at any point in which its tangent space is of codimension $h^{4,2}$ in the tangent space of $G$.

The exact sequence

$$0 \longrightarrow T\Sigma \longrightarrow T_{X|\Sigma} \longrightarrow \mathcal{O}_\Sigma(1) \longrightarrow 0$$

induces a Kodaira–Spencer map $\rho : H^0(\mathcal{O}_\Sigma(1)) \rightarrow H^1(T\Sigma)$. We assume that $\rho$ is injective (this will be true in our situation).

Let $\alpha : T_X \rightarrow T_{X|\Sigma}$ be the restriction map, and $\beta : T\Sigma \rightarrow T_{X|\Sigma}$ the inclusion map. Also, we denote by $\bullet$ the cup–product $H^1(\Sigma, T\Sigma) \otimes H^3(\Omega^3_\Sigma) \rightarrow H^4(\Omega^2_\Sigma)$.

With these notations, we have:

**Lemma 2.3.** Assume that $\rho$ is injective. Then the infinitesimal deformations of the triple $(\lambda, \Sigma, X)$ are parametrized by the subspace $T\mathcal{F}_{(\Sigma, X)} \subset H^1(T\Sigma) \times H^1(T_X)$ given by:

$$(u, v) \in T\mathcal{F}_{(\Sigma, X)} \iff \alpha(v) = \beta(u) \text{ in } H^1(T_{X|\Sigma}) \text{ and } u \bullet \lambda^{3,3} = 0 \text{ in } H^4(\Omega^2_\Sigma)$$

**Proof.** [21, 1.1 and 1.3] $\square$

Let $\pi : T\mathcal{F}_{(\Sigma, X)} \rightarrow H^1(T_X)$ be the map induced by the second projection. Consider the diagram induced by (1):

$$
\begin{array}{c}
H^0(\mathcal{O}_\Sigma(1)) \\
\downarrow^\rho \hspace{1cm} \pi
\end{array}
\begin{array}{c}
\longrightarrow
\begin{array}{c}
H^1(T\Sigma) \hspace{1cm} \longrightarrow
\begin{array}{c}
H^1(T_{X|\Sigma}) \hspace{1cm} \longrightarrow
\begin{array}{c}
H^1(\mathcal{O}_\Sigma(1)) \hspace{1cm} \longrightarrow
\begin{array}{c}
H^4(\Omega^2_\Sigma)
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
$$

Since $H^1(\mathcal{O}_\Sigma(1)) = 0$, a simple diagram chase shows that $\pi$ is surjective if and only if the maps $\lambda^{3,3}$ and $\lambda^{3,3} \circ \rho$ have the same image. Moreover, if $\rho$ is injective, then $\pi$ is injective if and only if $\lambda^{3,3} \circ \rho$ is injective.
Now, \( \dim H^0(\mathcal{O}_\Sigma(1)) = \dim H^4(\Omega^2_\Sigma) \), and so \( \lambda^{3,3} \circ \rho \) is injective if and only if it is surjective, and in this case \( \pi \) is an isomorphism, i.e., the family \( F_\lambda \) project isomorphically onto the family \( \mathcal{Y} \) of deformations of \( X \).

The description of the map \( \lambda^{3,3} \circ \rho \) in terms of polynomial multiplication in the Jacobian ring of \( \Sigma \) is given in [21, 1.4]. Since of course the grading is different in our case, we repeat it here, mainly to fix notations.

Let \((x_0, \ldots, x_8)\) be homogeneous coordinates in \( \mathbb{P}^8 \) such that \( \Sigma = X \cap \{x_8 = 0\} \). Let \( F \) be the equation of \( X \) and let \( G = F|_{\mathbb{P}^7} \) be the equation of \( \Sigma \). The Jacobian ring \( R(\Sigma) \) is defined as:

\[
R(\Sigma) = \mathbb{C}[x_0, \ldots, x_7]/(\partial G/\partial x_0, \ldots, \partial G/\partial x_7) = S(\mathbb{P}^7)/J(G).
\]

\( R(\Sigma) \) is a graded ring, and by Griffiths’ residue theory there are well–known isomorphisms

\[
\begin{align*}
(1) & \quad H^1(T_\Sigma) \cong R^3(\Sigma), \\
(2) & \quad H^3(\Omega^3)_{\text{prim}} \cong R^4(\Sigma), \\
(3) & \quad H^4(\Omega^2) \cong R^7(\Sigma), \\
(4) & \quad H^1(T_\Sigma(-1)) \cong R^2(\Sigma).
\end{align*}
\]

These isomorphisms are compatible in the following sense: if a polynomial class \( P_\lambda \in R^4(\Sigma) \) corresponds under (2) to a primitive cohomology class \( \lambda^{3,3} \), then the map \( \lambda^{3,3} : H^1(T_\Sigma) \to H^4(\Omega^2) \) corresponds, under the isomorphisms (1) and (3), to multiplication by \( P_\lambda \).

Let now \( e \in R^2(\Sigma) \) be the image of the polynomial \( \partial F/\partial x_8|_{\mathbb{P}^7} \). Under the isomorphism (4), \( e \) corresponds to the extension class of the exact sequence

\[
0 \longrightarrow T_\Sigma \longrightarrow T_{X|\Sigma} \longrightarrow \mathcal{O}_\Sigma(1) \longrightarrow 0
\]

Since the isomorphisms (1) and (4) are compatible with polynomial multiplication, we obtain that \( \rho : H^0(\mathcal{O}_\Sigma(1)) \to H^1(T_\Sigma) \) is given by multiplication by \( e \) from \( R^1(\Sigma) \cong H^0(\mathcal{O}_\Sigma(1)) \) to \( R^3(\Sigma) \).

From the above discussion we obtain:

**Proposition 2.4.** Let \((\lambda, \Sigma, X)\) be a triple as above. If the multiplication by \( P_\lambda \cdot e \) induces an isomorphism \( R^1(\Sigma) \cong R^7(\Sigma) \), then the map \( \pi : T\mathcal{F}_\lambda \to H^1(T_X) \) is an isomorphism, and hence the family \( \mathcal{F}_\lambda \) of deformations of the triple \((\lambda, \Sigma, X)\) is smooth, of codimension \( h^{4,2}(\Sigma) \) in \( \mathcal{G} \), and projects isomorphically (locally) onto the family \( \mathcal{Y} \) of deformations of \( X \).

This is [21, Proposition 1.5]. We only note that the hypothesis of the proposition implies that the Kodaira–Spencer map is injective, and then we can use Lemma 2.3.

Let \( \mathcal{S}(X) \) be the Noether–Lefschetz locus for the hyperplane sections of \( X \), i.e., the set of those \( \Sigma \) such that \( H^6(\Sigma, \mathbb{Z})_{\text{prim}} \cap H^{3,3}(\Sigma) \neq 0 \). If \( \pi \) is an isomorphism at a point \((\lambda, \Sigma, X)\), then the component \( \mathcal{S}_\lambda \) of \( \mathcal{S}(X) \) given by \( \lambda \), in a neighborhood of \( \Sigma \) consists only of the isolated reduced point \( \Sigma \), and has the expected codimension equal to \( h^{4,2}(\Sigma) \). Hence, as in [21, 1.6], we can use Green’s argument ([4, §5]) to obtain:
Proposition 2.5. If there exists a triple \((\lambda, \Sigma, X)\) such that \(\pi : \mathcal{T}_\mathcal{F}_\lambda \to H^1(T_X)\) is an isomorphism, then the 0–dimensional components of the Noether–Lefschetz locus \(\mathcal{S}(X)\) form a countable dense subset of \(\mathbb{P}(H^0(\mathcal{O}_X(1)))\).

We will give later (Proposition 4.2) an explicit example of such a triple, using Proposition 2.4. This will show that there is a Zariski open set of the family \(\mathcal{Y}\) of deformations of \(X\) over which such triples exist.

3. The infinitesimal invariant of normal functions

To each triple \((\lambda, \Sigma, X)\) as above, we can associate an element \(Z_\lambda \in CH^3_{\text{Hom}}(X)\), the group of 3–cycles on \(X\) homologous to 0, modulo rational equivalence, defining \(Z_\lambda = j_* \lambda\), where \(j : \Sigma \to X\) is the inclusion map. In fact, by Proposition 2.2, \(\lambda\) gives a primitive algebraic 3–cycle on \(\Sigma\), and hence \(Z_\lambda\) is homologous to 0 in \(X\).

If \((\lambda, \Sigma, X)\) satisfies the hypothesis of Proposition 2.4, the local family \(\mathcal{F}_\lambda\) is isomorphic to \(\mathcal{Y}\) in a neighborhood of the point \(o \in \mathcal{Y}\) corresponding to \(X\). Hence there is a universal family \(p : \mathcal{X} \to \mathcal{Y}\) and a relative cycle \(Z_\lambda \in CH^3_{\text{Hom}}(\mathcal{X}/\mathcal{Y})\). From this one obtains a normal function \(\nu_\lambda\), with infinitesimal invariant \(\delta \nu_\lambda\) (we will recall later the definition of \(\delta \nu_\lambda\)).

Let \(q : \mathcal{J} \to \mathcal{Y}\) be the intermediate Jacobian bundle whose fibre in \(t \in \mathcal{Y}\) is \(q^{-1}(t) = J(\mathcal{X}_t)\), the intermediate Jacobian of \(\mathcal{X}_t\). The normal function \(\nu_\lambda\) is the holomorphic section of this bundle given by \(\nu_\lambda(t) = \Phi_{\mathcal{X}_t}(Z_\lambda(t))\), where \(\Phi_{\mathcal{X}_t}\) is the Abel–Jacobi map of \(\mathcal{X}_t\).

By definition, \(Z_\lambda = Z^1_\lambda - Z^2_\lambda\), where for \(t\) in a small disk around 0, \(Z^i_\lambda(t)\) is contained in a hyperplane section \(\Sigma_t \subset \mathcal{X}_t\), varying holomorphically with \(t\). It would be convenient to have \(Z^1_\lambda(t)\) smooth in \(\Sigma_t\), but this is not necessarily true. However, by a result of Kleiman [11], it is possible to find a smooth cycle rationally equivalent to \(2(Z_\lambda^1(t) + D(t))\), where \(D(t)\) is a sufficiently large multiple of a hyperplane section of codimension 3 in \(\Sigma_t\). This simply means that we have to multiply by 2 all the formulas in [21], and hence we might as well assume that the cycles \(Z_\lambda^1(t)\) are already smooth. We can also, up to rational equivalence inside \(\Sigma_t\), assume that the intersection of \(Z_\lambda^1(t)\) and \(Z_\lambda^2(t)\) is transversal and consists of finitely many points. Blowing up these points, we can assume that \(Z_\lambda^1(t)\) and \(Z_\lambda^2(t)\) do not meet inside \(\Sigma_t\) and are homologous in \(\mathcal{X}_t\). We now choose a differentiable chain \(\Gamma_t\), varying smoothly with \(t\), such that \(\partial \Gamma_t = Z^1_\lambda(t) - Z^2_\lambda(t)\).

Let \(\mathcal{H}^7 = R^i p_* \mathcal{C} \otimes \mathcal{O}_Y\), \(\mathcal{H}^{i,j} = R^j p_* (\Omega^i_{\mathcal{X}/\mathcal{Y}})\) and \(F^4 \mathcal{H}^7 = \oplus_{k \geq i} \mathcal{H}^{k,7-k}\). Then the normal function \(\nu_\lambda\) lifts to a holomorphic section \(\psi_\lambda\) of \((F^4 \mathcal{H}^7)^{\vee}\) by

\[
\psi_\lambda(t)(\omega_t) = \int_{\Gamma_t} \omega_t
\]

where \((\omega_t)_{t \in \mathcal{Y}}\) is a section of \((F^4 \mathcal{H}^7)^{\vee}\).

We briefly recall here Griffiths’ definition of the infinitesimal invariant \(\delta \nu_\lambda\); let

\[
\nabla : \mathcal{H}^{4,3} \otimes T_{\mathcal{Y}} \to \mathcal{H}^{3,4}
\]
be the variation of Hodge structure on $\mathcal{Y}$. In our case $\nabla$ is surjective in every point and so $\ker\nabla$ is a vector bundle over $\mathcal{Y}$. The infinitesimal invariant $\delta\nu_\lambda$ is the section of $(\ker\nabla)^\vee$ defined in the following way: let $\sum_i \omega_i \otimes \chi_i \in (\ker\nabla)_{t_0}$ and let $\tilde{\omega}_i(t)$ be a section of $F^4\mathcal{H}^7$ in a neighborhood of $t_0$, such that $\tilde{\omega}_i(t_0) = \omega_i$. By definition of $\ker\nabla$, $\sum_i \nabla_{\chi_i}(\tilde{\omega}_i) \in F^4\mathcal{H}^7(t_0)$, and hence we can define

$$\delta\nu_\lambda(\sum_i \omega_i \otimes \chi_i) = \sum_i \chi_i(\psi_\lambda(\tilde{\omega}_i)) - \psi_\lambda(t_0)(\sum_i \nabla_{\chi_i}(\tilde{\omega}_i)).$$

By quasi–horizontality of normal functions, this formula gives a well defined element of $(\ker\nabla)_{t_0}^\vee$, i.e., it does not depend on the choice of the $\tilde{\omega}_i$ [9].

We can now repeat the argument of [21, 2.4–2.12] to rewrite the infinitesimal invariant in terms of Jacobian rings. We explicitly note two facts: first of all, the proof of the crucial Proposition 2.4 of [21] holds in our case (with an obvious change in the grading and dimension) since the cycles $\Sigma_\lambda$ are smooth. Secondly, if $R(X)$ is the jacobian ring of $X$, let $\mu_X : R^3(X) \otimes R^3(X) \rightarrow R^6(X)$ be the multiplication map. Then, as in [21, 2.5], we can identify the map $\nabla(0)$ with $\mu_X$.

The formula [21, 2.4] holds only for tensors of rank one. To obtain a general formula we need:

**Lemma 3.1.** For $X$ generic, $\ker\mu_X$ is generated by tensors of rank one.

**Proof.** A simple proof can be given in the following way: let $X$ be the Fermat hypersurface of degree 3; then the Jacobian ideal of $X$ is given by the classes of the monomials $x_i x_j x_k$ with $i < j < k$.

Since $(x_i x_j x_k) x_a \otimes (x_a x_b x_c) - (x_i x_j x_a) x_a \otimes (x_b x_c) = x_i x_j (x_a + x_k) \otimes x_b x_c (x_a - x_k) - x_i x_j x_a \otimes x_a x_b x_c + x_i x_j x_k \otimes x_b x_c x_k$, we see that any monomial $P \otimes Q = x_i x_j x_k \otimes x_a x_b x_c$ can be rewritten, modulo tensors of rank 1 belonging to $\ker\mu_X$, with the indices $\{i, j, k, a, b, c\}$ in strictly increasing order (or else it already belonged to $\ker\mu_X$).

Let $\omega = \sum_i P_i \otimes Q_i \in \ker\mu_X$. The observation above allows to rewrite, modulo tensors of rank 1 belonging to $\ker\mu_X$, all the summands in a standard form that must then be zero.

To prove the claim for $X$ generic, one observes that $\mu_X$ induces a linear map $\mathbb{P}(R^3 \otimes R^3) \rightarrow \mathbb{P}(R^6)$ and the tensors of rank one are simply the image of $\mathbb{P}(R^3) \times \mathbb{P}(R^3)$ in $\mathbb{P}(R^3 \otimes R^3)$ via the Segre map. For $X$ the Fermat hypersurface, the intersection of this image with $\mathbb{P}(\ker\mu_X)$ generates $\mathbb{P}(\ker\mu_X)$ and so the same will be true generically. □

From the this discussion, one obtains the following formula for the infinitesimal invariant $\delta\nu_\lambda$ in terms of Jacobian ideals: Let $\mu_\Sigma : R^3(\Sigma) \otimes R^3(\Sigma) \rightarrow R^6(\Sigma)$ be the multiplication map. Then, as Voisin shows in an ingenious way in [21, 2.11 and 2.12], the infinitesimal invariant $\delta\nu_\lambda$ can be viewed as an element of $(\ker\mu_\Sigma)^\vee$, i.e., it can be computed using the hyperplane section $\Sigma$ of $X$. With the same argument we obtain

**Proposition 3.2.** If the multiplication by $P_\lambda e$ induces an isomorphism $f_\lambda : R^1(\Sigma) \rightarrow R^7(\Sigma)$, then

$$\delta\nu_\lambda(\sum_i P_i \otimes R_i) = \sum_i P_\lambda P_i (f_\lambda^{-1}(P_\lambda R_i)) \in R^8(\Sigma) \cong \mathbb{C}.$$
4. Infinite generation of the Griffiths group

We can now study the independence over \( \mathbb{Q} \) of the \( \delta \nu_\lambda \). For this, we apply the formula of proposition 3.2 to the cohomology classes \( \lambda \) of a cubic sixfold \( \Sigma \) for which the rank of \( H^3(\Omega^3_\Sigma)^{\text{prim}} \cap H^6(\Sigma, \mathbb{Z}) \) is at least 2. One such is the Fermat cubic sixfold \( \Sigma \), whose equation is \( F = x_0^3 + \cdots + x_7^3 \). The structure of the Hodge classes of \( \Sigma \) is well known, as it is for all Fermat hypersurfaces. We recall here the relevant facts for the case at hand. Complete proofs, in the general case, can be found in [15], [16], [18]. We also note again that since the Hodge conjecture is true for \( \Sigma \), it is enough to work with rational cohomology classes of type \((3,3)\), since these will correspond to algebraic 3–cycles on \( \Sigma \).

Let \( \mu_3 \) be the group of third roots of unity, let \( \zeta = e^{2\pi i/3} \) and let \( G = (\mu_3)^8/\Delta \), where \( \Delta \) is the diagonal. \( G \) acts on \( \Sigma \) by

\[
(\zeta_0, \ldots, \zeta_7) \cdot (x_0 : \cdots : x_7) = (\zeta_0 x_0 : \cdots : \zeta_7 x_7).
\]

The group of characters of \( G \) is

\[
\hat{G} = \left\{ (a_0, \ldots, a_7) \in \mathbb{Z}_3^8 \mid \sum_i a_i \equiv 0 \pmod{3} \right\},
\]

where, if \( g = (\zeta_0, \ldots, \zeta_7) \in G \) and \( \alpha = (a_0, \ldots, a_7) \in \hat{G} \), then \( \alpha(g) = \prod_i \zeta_i^{a_i} \). To describe the action of \( G \) on \( H^6(\Sigma, \mathbb{C}) \), we use Griffiths’ residue theory to represent cohomology classes with differential forms with poles along \( \Sigma \) [8].

Let \( \Omega = \sum_i (-1)^i x_i dx_0 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_7 \), where \( \hat{\cdot} \) means omitted, be the rational top form on \( \mathbb{P}^7 \). Then the isomorphism \( R^{3q-8}(\Sigma) \rightarrow H^{6-q+1,q-1}(\Sigma)^{\text{prim}} \) is given by

\[
P \mapsto \text{Res} \left( \frac{P \Omega}{F_q} \right). 
\]

Since the Jacobian ideal of \( F \) is generated by the monomials \( x_0^2, \ldots, x_7^2 \), the monomials of degree \( i \) which do not contain squares form a basis for \( R_i \); moreover \( F \) is invariant under \( G \) and the action of \( G \) commutes with \( \text{Res} \), and hence these monomials give a basis of eigenvectors. More precisely, the monomial \( P = \prod_i x_i^{b_i} \) gives an eigenvector of eigenvalue \((b_0 + 1, \ldots, b_7 + 1)\) (we are identifying the monomial \( P \) with the cohomology class it corresponds to). From what we have said, \( b_i \) can only be 0 or 1, and hence the characters with non–vanishing eigenspaces are \( \{(a_0, \ldots, a_7) \in \hat{G} \mid a_i \neq 0 \ \forall i\} \). It is also clear from this description that the eigenspaces are all one–dimensional. Moreover, since \( \text{Res} \frac{P \Omega}{F_{q+1}} \in H^{6-q,q}(\Sigma) \), we see that the characters occurring in \( H^{6-q,q}(\Sigma) \) are those for which \( \sum_i a_i = 3(q + 1) \), thinking of the \( a_i \) as integers between 1 and 2.

In fact, these eigenspaces are defined over \( \mathbb{Q}(\zeta) \), and the action of \( \text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) = \mathbb{Z}_3 \) is given by

\[
(3) \quad \sigma_a \cdot (a_0, \ldots, a_7) = (aa_0, \ldots, aa_7),
\]
where $\sigma_a(\zeta) = \zeta^a$. This description allows us to analyze the $\mathbb{Q}$-structure of $H^6(\Sigma, \mathbb{C})$.

The cohomology class given by a monomial is only in $H^6(\Sigma, \mathbb{C})$, but since the eigenspaces are one-dimensional, an appropriate multiple of an eigenvector (by a transcendental number, usually) will be in $H^6(\Sigma, \mathbb{Q}(\zeta))$. These numbers are of course related to the periods of the Fermat hypersurface, but since we don’t need them explicitly, we will ignore their precise values in what follows (also, we don’t know these values; the periods for Fermat hypersurfaces have been computed only for the cases of curves by Rohrlich [10, Appendix] and surfaces by Tretkoff [20].) We will use the following notation: if $\alpha \in \hat{G}$ is a character whose eigenspace is nonzero, we let $P_\alpha$ be a monomial such that $\text{Res} \frac{P_\alpha \Omega}{F^4} \in H^6(\Sigma, \mathbb{Q}(\zeta))$ is an eigenvector relative to $\alpha$.

We look now for rational cohomology classes in $H^{3,3}(\Sigma)^{\text{prim}}$. By the above discussion, a cohomology class in $H^6(\Sigma, \mathbb{Q}(\zeta))$ is rational if and only if it is invariant under the action of the Galois group. Let $\alpha \in \hat{G}$ be such that $\sum_i a_i = 12$, i.e., four of the $a_i$ are equal to 1 and the other four are equal to 2. Then $P_\alpha$ belongs to $H^{3,3}(\Sigma)^{\text{prim}}$ and the class corresponding to $P_\alpha + P_{2\alpha}$ will be a primitive rational class. In this way we can construct many rational primitive classes.

We now make explicit choices for the polynomials appearing in the formula of Proposition 3.2, and compute the infinitesimal invariant of the corresponding normal functions. Let $\alpha = (2, 2, 2, 1, 1, 1, 1)$ and $\beta = (2, 2, 2, 1, 2, 1, 1, 1)$, and let $a, b \in \mathbb{Z}$. Then $P_\alpha = Ax_0x_1x_2x_3$, $P_\beta = Bx_0x_1x_2x_4$, $P_{2\alpha} = Cx_4x_5x_6x_7$, $P_{2\beta} = Dx_3x_5x_6x_7$, where $A, B, C$ and $D$ are nonzero complex numbers. Let $\lambda_{a,b}$ be the primitive rational class corresponding to $P_{a} + P_{2a}$. Let $e = x_0x_1 + \frac{1}{A}x_2x_3 + \frac{1}{A}x_4x_5 + x_6x_7 + \frac{h}{D}x_3x_5$, where $h$ is a transcendental number. Let $Q = \frac{1}{A}x_4x_5x_6$ and $R = \frac{1}{B}x_3x_5x_7$ be in $R^3(\Sigma)$, so that $Q \otimes R \in \ker \mu_\Sigma$.

**Proposition 4.1.** Let the polynomials $P_\alpha$, $P_{2\alpha}$, $P_\beta$, $P_{2\beta}$, $e$, $Q$ and $R$ be chosen as above, and let $a, b \in \mathbb{Z}$. Then:

1. if $a \neq 0$, $a^2AC - b^2BD \neq 0$, the multiplication by $(P_{a,b} \cdot e)$ induces an isomorphism $f : R^1(\Sigma) \to R^3(\Sigma)$;
2. $\delta \nu_{\lambda_{a,b}}(Q \otimes R) = P_{a,b}Q(f^{-1}(P_{a,b}R)) = \frac{ab}{a + bh} \cdot x_0x_1x_2x_3x_4x_5x_6x_7$.

**Proof.** Let $\{f_0, \ldots, f_7\}$ be a basis of $R^1(\Sigma)$, where $f_i$ is the image of $x_i$, and let $\{g_0, \ldots, g_7\}$ be a basis of $R^3(\Sigma)$, where $g_i$ is the image of $\prod_{j \neq i} x_j$. It is then straightforward to check that, using these bases, the matrix of the multiplication by $(P_{a,b} \cdot e)$
is
\[
M = \begin{pmatrix}
0 & a & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & aC & bD & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & bD & 0 & 0 & aA & 0 \\
0 & 0 & 0 & bB & aA & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & a + bh \\
0 & 0 & 0 & 0 & 0 & 0 & a + bh
\end{pmatrix}
\]
Since \( \det M = a^2(a + bh)^2(a^2AC - b^2BD)^2 \), the multiplication by \((P_{a,b} \cdot e)\) is an isomorphism as claimed.

A simple computation now gives that \( P_{a,b}R = bg_6 \), and \( f^{-1}(bg_6) = \frac{b}{a + bh}f_7 \). The formula (2) follows. \( \square \)

We can now prove the theorem announced in the introduction:

**Theorem 4.2.** For the generic smooth cubic sevenfold, the Griffiths group is not finitely generated.

**Proof.** The argument in [21, 3.7] goes through word by word, provided we show that the numbers \( \frac{ab}{a + bh} \) span an infinite dimensional \( \mathbb{Q} \)-vector space. Let then \( \sum_{i=1}^{n} c_i \frac{a_i b_i}{a_i + b_i h} = 0 \) be a relation of linear dependence over \( \mathbb{Q} \). We assume that \( b_i = 1 \) for all \( i \), and that the \( a_i \) are all distinct and non-zero. Clearing denominators, we obtain an algebraic equation of degree \( n - 1 \) of which \( h \) is a root. Since \( h \) is transcendental, all coefficients must be zero. This gives a homogeneous linear system in the \( c_i \)'s, whose coefficient matrix has determinant a Van der Monde in the \( a_i \)'s, and hence is non-zero. This implies that \( c_i = 0 \) for all \( i \). \( \square \)

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