A comprehensive co-variant gauge theory of fracton phase of matter.

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Abstract

Basing on the recently proposed covariant action for the fracton model $^1$ where a paradigm shift of the existing research on fracton gauge theory is mooted. A holistic analysis of the fracton gauge theory is presented in this paper which connects the apparently bizarre properties of the fractons in the realm of quasi particles, like nearly vanishing mobility and also the nature of interaction exhibited by them which ranges from electromagnetic to gravitation-like effects. The twine principles of covariance and gauge invariances, the leitmotivs of modern theoretical physics form the basis of our formulation, thereby dispelling all the confusion and the contradiction of the existing fracton gauge theory. The implementation of the symmetry require the introduction of a novel scalar field theory based on the Galileon scalars. The ubiquity of the Galileon scalars as representation of fracton matter shown here is indeed remarkable.

1 Introduction

The fracton phase of matter is one of the marvels of strongly correlated electrons in soft condensed matter physics. Discovered only in the first decade of this century, it has already made its place in the challenge-list of theoretical physics. In spite of the voluminous work done in this field $^2$-$^3$ there are many riddles. As quasi-particles fractons are distinguished by the very low (often vanishing) mobility, in contrast with the other (quasi) particles, due to its immobility fractons are very slow to respond to an applied electric field. But composites of fractons can move freely. On the theoretical side this immobility of the fracton in isolated state and mobility in presence of other fractons is really an obscure phenomena. Note that the immobility of fractons can be explained by some special symmetry which does not act on any other particles in similar situations. The theorists introduced a phenomenology based on a tensor gauge theory. The tensor gauge theory is assumed to have dipole moment conservation, in addition to the well known electric
charge conservation. This points to the existence of another symmetry these symmetries were however not much explained [4].

An important break through was achieved in [4], where a reverse procedure was adopted by asking the question- which gauge field theory could give the dipole moment conservation along with the charge conservation? We refer the reader the original work [4] and the review [2][3] for the details of the algorithm and explanation of the phenomenology of matter in fracton phase. It will be seen that though it is the correct approach to apply two gauge symmetry corresponding to the two types of symmetries , but the method of formulating the gauge principle have several shortcomings. An array of scalar fields were assumed to represent the fracton phase of matter. A combination of these fields and their derivatives was defined so that the two types of symmetries are inbuilt with it. However, all these calculations were done by taking a particular coordinate system. The theory thus obtained was declared to have no symmetry under boosts , as stated in the paper [4]. It is clear that there is no guarantee that the particular choice of the “gauge invariant combination” will hold in another inertial frame. Again the whole exercise is clearly an effort to accommodate the two types of symmetries, without any attempt to answer the question where from they followed . So we still have the two types of symmetries as primitives in the theory of [4] , the origin of which is the experimental observation.

From the gauge principle we know that if there is a global symmetry of the matter fields then the gauge field appears in the localization process of that symmetry to compensate the variation of the transformation parameters from one point to the other. The matter field should carry a representation of the group that is to be gauged. Fracton matter must then carry the two types of symmetries which give the conserved quantities i.e. the charge and dipole moment. Again these conserved charges have a hierarchy in the sense that without the first one, the second quantity is not properly defined. In [4] there was no such symmetry of the un-gauged theory rather the un-gauged symmetry is not global according to them . The achievement of [4] is the pointer to the necessity of a gauge field theory. This trend is then pursued by many contributors . After [4] all the analysis done in this line have shared the same conceptions. There was no attempt to apply the gauge principle to the matter sector and construct a self consistent fracton matter coupled with the gauge field. Rather the gauge symmetry was commented not to hold for the matter action.

Very recently we proposed a new way of thinking in the research outlook in this field [1]. The focus in our paper was on the construction of an appropriate charged matter field action which has two different global symmetries. The paper [1] is the first study in the literature where a covariant gauge theory was presented. It was remarkable that all the results of fracton gauge theory, which were semi empirically discussed so far in the literature were cleanly derived from our model. Since then many more interesting results have been obtained . So the present paper came into being. Specifically, we will discuss the localization of the global symmetries of free fracton which was reported in [1] rather sketchily. The results of electromagnetic interaction, the self interaction of the fractons as well as gravitation type effects will be given here along with the connected concepts. In short the present paper gives a holistic account of the covariant fracton gauge theory.

After the introductory section we will briefly write the points of the present established “fracton gauge theory” which are difficult to understand physically , and sometimes lead to contradictory results. To show this we used the arguments of a representative pa-
per on this subject [4]. Specifically we will indicate where the algorithm is deviated from the gauging of space-time symmetries and internal symmetries [5][6]. The discussion in this section motivates a new algorithm true to the principles of gauge in variance and space time invariance. Shift from the empirical methods pursued in the present literature. In the next section we will pose our action mooted in the previous paper [1] and establish it comprehensively from the various symmetry requirements. Then we review the symmetries of the action and the corresponding conservation laws. Relation between the conserved charges reveal the simultaneous conservation of the charges and dipole moments, thereby show the existing empiricism in the analysis is actually deep rooted in. Note that it is a unique derivation of the two conserved charge. So far we have surveyed the literature, not many papers can be found where a derivation from the first principles is available.

From section IV, new results concerning the applications of the gauge principle to the phase rotation symmetry, shift symmetry and various other puzzling aspects will be discussed. The ‘gauging the symmetry approach’ will be our only tool. We start with enunciating the precise algorithm [6] in a short review. In the next subsection the algorithm is illustrated by applying it to the complex K-G field. The rest of the section will be devoted to the localization of our model i.e. the un-gauged fracton gauge theory. As a result of localization in addition to the well known maxwell’s theory with vector gauge field, we get another theory of a tensor gauge field $B_{\mu\nu}$. Look how methodically the tensor gauge field appears in our analysis as a consequence of the symmetries. The properties of this tensor gauge field has so far been obtained from the fracton phenomenology. In next section we give our analysis of the connection of fracton with the other field by comparison with the Maxwell’s field theory. Once again we see the efficacy of our algorithm.

In section VI, we discuss the localization of the space-time symmetry (Poincare symmetry). A surprising fallout is the emergence of another second rank tensor gauge field. Note carefully that in the existence theory the same tensor gauge field is used in both electromagnetic and gravitational sectors. On the face of it, it is a startling assumption which may indicate some sort of unification of the electromagnetic and gravitational excitation. In reality this two have been proved to be orthogonal. It is gratifying to observe that no such gastronomic assumption is required in our theory. In an extra dividend we derived the self interaction of the fracton, a puzzling issue in the present state of art. Note that though we are working in the backdrop of the classical fields theory, the results obtained in this paper are fur more general, because we base on such basic symmetries like Lie groups of symmetries and Poincare symmetries.

2 A critique of the existing ‘fracton gauge theory’

The existing fracton gauge theory suffers from a number of serious problems. The first question is the space-time in which the theory is formulated. It is nowhere stated. But the absence of the temporal part in their theory it is evident that a Galilean boost-violating non relativistic theory is considered. Such boost violating theory have occured in condensed matter physics, at the Lifitsitz point [7] as well as high energy physics and Horava gravity [9]. But there the required space time geometry, abandon boost symmetry[7][8].
In the present problem no such pressing need is observed. On the contrary in relativistic or non relativistic gauge theories, observing the phenomena in space-time is mandatory. Thus this formulation of the problem implies we don’t know whether the results reveal physical truth or it is merely an artifact of the gauges.

In the un-gauged theory [4] the phase rotation is given by

$$\Psi \rightarrow e^{i\lambda x}\Psi$$

(1)

But it is not globally constant, so one cannot identify the symmetry with a global gauge transformation. What happens to the gauge principle? - the question is not answered.

Following the arguments of [4] one can find that there are similar confusions throughout the paper. If one goes through the arguments of [4] it is apparent that the confusions are due to the complicated constructions for getting the two types of symmetry. Our letter [1] pointed out this confusions are due to the matter sector not being a proper representation of the symmetry group. Once this is rectified the confusions will disappear.

The discussion of this section is to highlight the nature of our differences with the existing theories which now, are known as the ‘fracton gauge theories’. The concept is observed to be introduced by [4] and followed till date [2] [11][12]. The unusually small mobility [12] [13] of fracton in the isolated state was the main characteristic of fracton physics. The explanation of the immobility of the fracton was provided [4] starting from the assumption of conserved dipole moment, in addition to the usual charge conservation. Mathematically,

$$Q = \int \rho d^Dx = \text{constant}; J^k = \int \rho x^k d^Dx$$

(2)

Since the idea of anything related to the existence of the fractons were dependent upon the basic interaction which was not available, so various groups groped the problem from different direction. As there were no proper suggestion for the un-gauge theory and its role in construction of the covariant derivatives, the application of the gauge principle became some what obscure. Thus they defined a gauge covariant operator to the lowest order [4] as,

$$\Psi \partial_i \partial_j \Psi - \partial_i \Psi \partial_j \Psi$$

(3)

. The implementation of the gauge principle does not end here. But let us accept it. Now at the first encounter one must be startled by the occurrence of second derivatives in the gauge covariant object under the gauge transformations. But the above form appears in the definition of the gauge - covariant second derivative $D_{ij}$ as

$$D_{ij}\Psi^2 = \Psi \partial_i \partial_j \Psi - \partial_i \Psi \partial_j \Psi - iA_{ij}\Psi^2$$

(4)

where $D_{ij}$ is a bi-linear operator acting on two functions, $\Phi$ and $\Psi$. After all these steps, one can write a Lagrangian in terms of $\Phi^2$ where the time part was suddenly introduced.

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1We saw that the author remarked “This symmetry transformation is (neither) a gauge ...” [4].

2It must be mentioned that in [11] we find a conscious attempt to understand the role of space-time symmetries, but that the matter part must carry the symmetries involve, escaped notice.
Probably the idea behind this was the complete independence of the temporal and spatial variables. But every body knows that to preserved the Galileo-Newton principle of relativity one has to assume the Galilean principle of relativity which corresponds to the Galilean transformation of the space time coordinates. So even in the non-relativistic scenario Galilean transformation connect the space and time coordinates.

But we discussed in [1] that if one chooses the zero mass complex Klein Gordon field to determine the dynamics of the charged matter then there is no scope for the 2nd gauge symmetry and thus we constructed our Lagrangian with the Galileon scalar fields $(\pi_1, \pi_2)$[14]. The Galileon field is a scalar under Poincare transformations and has a unique symmetry under the shift,

$$\pi \rightarrow \pi + a + b_\mu x^\mu$$

where $a$ and $b_\mu$ are real numbers. Note that we some times indicate the first term $a$ as a translation and second part $b_\mu x^\mu$ as shift, while some time we will refer the whole transformation as shift. It will understood from the context, which meaning is implied. The complex Galileon field defined as

$$\phi = \pi_1 + i\pi_2$$
$$\phi^* = \pi_1 - i\pi_2$$

We then constructed the Lagrangian of our theory in terms of $\phi$ and $\phi^*$, which is given by ,

$$\mathcal{L} = \left(\phi \Box \phi^* + \phi^* \Box \phi\right) \partial_\mu \phi \partial^\mu \phi^*$$

In the next section we give a short review of model (7). Before ending this short critique we must admit that the gauge theory with two symmetries suggested in [4] brings the fracton gauge theory at a turning point. But because of the lapse to relate the symmetry of the matter part dynamics, it failed to turn. This is possible to understand from the various comments in the same paper such as claiming an un-gauged theory to have a space time dependent phase rotation symmetry or the statements like ‘Note that the field $\Phi_i$ does not transform nicely under rotations’ or branding the not to well defined matter as ‘simply be a mathematical artifact’. As far as we know, this formalism is the base of the fracton gauge theories introduced till date . Thus if the reader apppreciates the research on the fracton gauge theories from our approach he/she will not hesitate to accept that the present paper is really a paradigm shift in the fracton gauge research, as we have remarked earlier.

3 Galileon based fracton model:

In our previous paper [1] we constructed our Lagrangian [7] with the help of Galileon scalar field $(\pi_1, \pi_2)$ and proved that the Lagrangian [7] has following properties :

a. The equation of motion satisfied by Lagrangian [7] is second order in time derivative of $\phi$ and $\phi^*$.

$$\Box \phi = \Box \phi^* = 0$$
b. The Lagrangian is invariant under full Poincare group i.e. the theory has both Lorentz and translation invariance.

c. The Lagrangian is symmetric under global phase rotation $\phi \rightarrow e^{i\alpha}\phi$ and $\phi^* \rightarrow e^{-i\alpha}\phi^*$. The corresponding conserved current is

$$\theta^\sigma{}^\mu = -i \left( \phi \partial^\sigma \phi^* \partial^\alpha \phi^* - \phi^* \partial^\sigma \phi \partial^\alpha \phi^* \right)$$

The charge density is computed by putting $\sigma = 0$. So electric charge is

$$Q = -\int i \left( \phi \dot{\phi} \partial^\alpha \phi \partial^\alpha \phi^* - \phi^* \dot{\phi} \partial^\alpha \phi \partial^\alpha \phi^* \right) d^3x$$

Naturally this is the charge of the fracton.

d. The Lagrangian is symmetric under shift symmetry with the parameters extended to complex values (using (5) and (6))

$$\phi \rightarrow \left( \phi + (1 + i)(a + b_\mu x^\mu) \right)$$

$$\phi^* \rightarrow \left( \phi^* + (1 - i)(a + b_\mu x^\mu) \right)$$

The charges corresponding to $b_\mu$ and $a$ respectively are

$$J^k_\mu = -\int x^k \left[ (1 + i)\dot{\phi}^* \partial_\alpha \phi \partial^\alpha \phi^* + (1 - i)\dot{\phi} \partial_\alpha \phi \partial^\alpha \phi^* \right] d^3x$$

$$Q' = -\int \left[ (1 + i)\dot{\phi}^* \partial_\alpha \phi \partial^\alpha \phi^* + (1 - i)\dot{\phi} \partial_\alpha \phi \partial^\alpha \phi^* \right] d^3x$$

We will now point out a very simple correspondence between the phase rotation symmetry and the translation part of shift symmetry. If we denote $\phi$ by a phasor in the complex phase space, the phase rotation shifts the phasor by a certain displacement. Now this can be exactly compensated by the translation of the tip of the phasor as shown in the figure.

It is also understandable that by varying the different points on the phasor, the initial one is exactly map to the final the position. It means that there is a correspondence between the two symmetries. This correspondence can be work out by equating the conserved charges obtained in both ways. Thus from (10) and (13) we can find a correspondence of the charges between the shift symmetry and the phase rotation parameter

Note that the symmetry transformations are a generalization of the well known shift symmetries. The complex scalar field defined here is a novel theory. Thus the analysis in this paper, adds new dimension to the results derived herein. Also the complex shift symmetries revealed here is a new symmetry. In fact on retrospection, we see that the complex scalar field introduced here, is a new scalar representation of the special unitary group $U(1)$, which has a conventional representation in the form of the massless
4 Interaction of fractons with electromagnetic field:

We will employ the celebrated gauge principle to localize the different symmetry of our model. It will be convenient to introduce the concept of “gauge principle” in the familiar context of maxwell’s electromagnetic theory. We start with a free theory namely the zero mass Klein-Gordon Lagrangian

\[ \mathcal{L}_{kg} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi^* \quad (14) \]

This is symmetric under the global gauge transformation \( \phi \rightarrow \phi e^{i \alpha} \), where \( \alpha \) is a constant.

The construction of the ungauged theory is most important for localization process. Note that here the particular transformations of \( \phi \) and \( \partial_\mu \phi \) are such that the Lagrangian (14) is invariant. Indeed, under the infinitesimal phase rotation \( \Lambda \),

\[
\begin{align*}
\delta \phi &= i \Lambda \phi \; \delta \partial_\mu \phi = -i \Lambda \partial_\mu \phi \\
\delta \phi^* &= -i \Lambda \phi^* \; \delta \partial_\mu \phi^* i = i \Lambda \partial_\mu 
\end{align*}
\quad (15)
\]

and this is the way the fields and their derivatives should transform so as to ensure invariance.

When the phase rotation symmetries are localized the parameters of the transformation are no longer constant but arbitrary functions of space time. So we consider \( \Lambda \) to be a function of \( x \) and \( t \). Then the transformation of the field \( \delta \phi \) retains its form but derivatives of it deviates from how they would do earlier. To compensate this change we have to replace the partial derivatives by

\[
\nabla_\mu \phi = (\partial_\mu + i A_\mu) \phi 
\quad (16)
\]

where \( A_\mu \) is called a gauge field. Its transformation under the local gauge transformation is assumed so that \( \nabla_\mu \) transform just as \( \partial_\mu \) transform in un-gauged theory. Invariance is thus retained. This is the gauge principle which we see is instrumental to obtain a Lagrangian invariant under local transformation. It is observed that the commutator of \( \nabla_\mu \) and \( \nabla_\nu \) satisfy

\[
[\nabla_\mu, \nabla_\nu] = F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu 
\quad (17)
\]

and the dynamics of the field \( A_\mu \) is completely given by \( F_{\mu\nu} \). Also it is anti-symmetric over the interchange of \( \mu \) and \( \nu \). It is then straightforward to identify the interaction generated by \( A_\mu \) as the electromagnetic force. This is nothing new but a well known fact. The gauge principle is thus a very useful tool in model building. In the following it will be seen that the gauge principle has similar ubiquity in constructing the fracton gauge theory.

In fracton physics, to understand the dynamics of the fracton and the electromagnetic field, we have to localize the shift symmetry and phase rotation symmetry by making

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3From what has been stated in the above, one should not get the impression that the whole electromagnetic theory can be thus constructed. It is rather the opposite. IT took more than 200 years of experimenting and theorizing to know \( U(1) \) is the appropriate symmetry!
the symmetry parameters \((\alpha, b_\mu)\) arbitrary function of space and time. The localization of \(\alpha\) and \(b_\mu\) introduced respectively a new vector gauge field and a new tensor gauge field. Thus we have to replace \(\partial_\mu\) by

\[
D_\mu \phi = \partial_\mu \phi + i A_\mu \phi + (1 + i) B_{\mu\nu} x^\nu \phi
\]  

(18)

Where \(A_\mu\) and \(B_{\mu\nu}\) are the new gauge fields. \(A_\mu\) is a vector under Lorentz transformations. It is the well known electromagnetic field, which we have encountered already, at the beginning of this section. But the appearance of the new field \(B\) given by the \(B_{\mu\nu}\) as matrix elements is an object of great interest. That the field is a tensor gauge field can easily be established. A little exercise proves that \(B_{\mu\nu}\) may be assumed symmetric, as is shown in the following.

Let it be a general second rank tensor.
Consider a virtual displacement \(\delta x^\mu\) of the system then from (18) the contraction

\[
\delta x^\mu D_\mu \phi = \delta x^\mu (\partial_\mu \phi) + i \delta x^\mu A_\mu \phi + (1 + i) \delta x^\mu B_{\mu\nu} x^\nu \phi
\]

(19)

Now any general second rank tensor can be split up in anti-symmetric and symmetric parts

\[
B_{\mu\nu} = B^{(A)}_{\mu\nu} + B^{(S)}_{\mu\nu}
\]

(20)

where

\[
B^{(A)}_{\mu\nu} = \frac{1}{2} \left( B_{\mu\nu} - B_{\nu\mu} \right)
\]

\[
B^{(S)}_{\mu\nu} = \frac{1}{2} \left( B_{\mu\nu} + B_{\nu\mu} \right)
\]

(21)

The left hand side of (22) is interpreted as the change of the field in the local translation \(\delta x^\mu\). One or two steps of algebra show that the anti-symmetric part contributes nothing to the factor \(\delta x^\mu D_\mu \phi\). So any anti-symmetric part of \(B_{\mu\nu}\) is irrelevant. It implies that \(B_{\mu\nu}\) may be assumed symmetric in (18).

Note that under the shift symmetry and phase rotation symmetry the fields \((\phi, \phi^*)\) and their derivative transforms as

\[
\delta \phi = \left( i \alpha \phi + (1 + i) b_\mu x^\mu \right)
\]

\[
\delta \phi^* = -i \alpha \phi^* + (1 - i) b_\mu x^\mu
\]

\[
\delta (\partial_\mu \phi) = \left( i \alpha \partial_\mu \phi + (1 + i) b_\mu \right)
\]

\[
\delta (\partial_\mu \phi^*) = -i \alpha \partial_\mu \phi^* + (1 - i) b_\mu
\]

\[
\delta (\partial^\mu \partial_\mu \phi) = \left( i \alpha \partial^\mu \partial_\mu \phi \right)
\]

\[
\delta (\partial^\mu \partial_\mu \phi^*) = -i \alpha \partial^\mu \partial_\mu \phi^*
\]

(22)

\(^4\)As there is some correspondence between the shift symmetry parameter \(a\) and the phase rotation parameter \(\alpha\), so we does not need to localize \(a\).

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The above analysis gives us the general methods of gauging the symmetry of we have listed the variations of the Fields and their derivatives under the global phase rotation symmetry, the complex shift symmetry and space time symmetries. under

Now the transformations of $A_\mu$ and $B_{\mu\nu}$ must ensure that the “new derivative ($D_\mu \phi$)” will transform in the same way as the usual derivatives (22) do in the global theory. Thus the variation of $D_\mu \phi$ must be

$$\delta(D_\mu \phi) = \left( i\alpha D_\mu \phi + (1 + i)b_\mu \right)$$  \hspace{1cm} (23)

After some calculation we found that in order to get the variation (23) the transformation of $A_\mu$ and $B_{\mu\nu}$ becomes

$$\phi \delta A_\mu = -\left( \phi \partial_\mu \alpha + 2b_\mu x^\nu A_\mu + 2B_{\mu\nu}b_\alpha x^\alpha x^\nu \right)$$

$$\phi \delta B_{\mu\nu} = \left( A_\mu b_\nu - \partial_\mu b_\nu \right)$$  \hspace{1cm} (24)

Our complex Galileon model (7) also contains second derivative of $\phi$ and $\phi^*$. The transformations of the second derivatives (22), however, do not have any shift symmetry contribution. Consequently, the second derivative $\partial_\mu \partial^\mu \phi$ should be replaced by $\bar{D}_\mu D^\mu \phi$.

Where $\bar{D}_\mu \phi$ is defined as

$$\bar{D}_\mu \phi = \partial_\mu \phi + iA_\mu \phi$$  \hspace{1cm} (25)

Explicit calculation shows that under local shift and phase rotation transformations $\bar{D}_\mu D^\mu \phi$ transform in the same way as $\partial_\mu \partial^\mu \phi$ i.e.

$$\delta(\bar{D}_\mu D^\mu \phi) = \left( i\alpha \bar{D}_\mu D^\mu \phi \right)$$  \hspace{1cm} (26)

In the above we have found out that both the phase rotations and the complex shift symmetries have appeared in global gauge theory. These symmetries together define fracton interaction with the applied electromagnetic fields.

5 Generalized Maxwell’s theory :

In the last section we have seen that the localization process applied to the shift symmetry produces two new gauge fields $A_\mu$ and $B_{\mu\nu}$. It is apparent that $A_\mu$ appearing from the localization of the translation part is nothing but the electromagnetic field obeying Maxwell’s equation. So it remains to determine the physical content of the new interaction. From the following analysis it will be apparent the smoothness of our formulation in contrast with the usual fracton gauge theory, as noted in section 1.

Proceeding the localization process as above we get the covariant derivative (15). Now the commutator of two covariant derivative give the field tensor from which the dynamics can be obtained. Now the commutator of $D_\mu$ and $D_\nu$ are given by
\[ [D_\mu, D_\nu] = i \left( \partial_\mu A_\nu - \partial_\nu A_\mu \right) + (1 + i)(\partial_\mu B_{\nu\rho} - \partial_\nu B_{\mu\rho})x^\rho + (B_{\mu\nu} - B_{\nu\mu}) \quad (27) \]

As \( B_{\mu\nu} \) is shown to be implementable as a symmetric tensor (see the discussing below) thus

\[ [D_\mu, D_\nu] = i \left( \partial_\mu A_\nu - \partial_\nu A_\mu \right) + (1 + i)(\partial_\mu B_{\nu\rho} - \partial_\nu B_{\mu\rho})x^\rho \quad (28) \]

The dynamics of the fields \( A_\mu \) and \( B_{\mu\nu} \) are completely given by the equation (28). So to construct the dynamics of the gauge field in our theory we will introduce \( F_{\mu\nu} \) where \( F_{\mu\nu} \) is given by

\[ F_{\mu\nu} = [D_\mu, D_\nu] \quad (29) \]

Using the equation (28), we get

\[ F_{\mu\nu} = F_{\mu\nu} + G_{\mu\nu\rho}x^\rho \quad (30) \]

Where

\[ F_{\mu\nu} = i(\partial_\mu A_\nu - \partial_\nu A_\mu) \]
\[ G_{\mu\nu\rho} = (1 + i)(\partial_\mu B_{\nu\rho} - \partial_\nu B_{\mu\rho}) \quad (31) \]

From (31) we see that \( F_{\mu\nu} \) is anti-symmetric and after some little calculation we can prove that \( G_{\mu\nu\rho} \) is also anti-symmetric. If we interchange \( \mu \) and \( \nu \) then we get

\[ G_{\nu\mu\rho} = (1 + i)(\partial_\nu B_{\mu\rho} - \partial_\mu B_{\nu\rho}) = -G_{\mu\nu\rho} \quad (32) \]

again from the definition of \( G_{\nu\mu\rho} \) we get

\[ G_{\rho\mu\nu} = (1 + i)(\partial_\rho B_{\mu\nu} - \partial_\mu B_{\rho\nu}) \]
\[ G_{\nu\rho\mu} = (1 + i)(\partial_\nu B_{\rho\mu} - \partial_\rho B_{\nu\mu}) \quad (33) \]

adding these two term we get

\[ G_{\rho\mu\nu} + G_{\nu\rho\mu} = (1 + i)\left[ (\partial_\rho B_{\mu\nu} - \partial_\mu B_{\rho\nu}) - (\partial_\mu B_{\rho\nu} - (\partial_\nu B_{\rho\mu}) \right] \]

as \( B_{\mu\nu} \) is a symmetric tensor thus we get

\[ G_{\rho\mu\nu} + G_{\nu\rho\mu} = G_{\nu\mu\rho} \quad (34) \]
\[ G_{\rho\mu\nu} + G_{\nu\rho\mu} + G_{\mu\nu\rho} = 0 \]  

(35)

Thus \( G_{\mu\nu\rho} \) satisfied Jacob’s identity, so \( G_{\mu\nu\rho} \) must be a anti-symmetric tensor. Now from (30) we can say that \( F_{\mu\nu} \) is also a anti-symmetric tensor. We now defined the dual of \( F_{\mu\nu} \), which is given by

\[ \tilde{F}_{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \]  

(36)

using (30) the partial derivative of \( \tilde{F}_{\mu\nu} \) can be written as

\[ \partial_{\mu} \tilde{F}_{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} \left[ \partial_{\mu} F_{\alpha\beta} + \partial_{\mu} (G_{\alpha\beta\rho} x^\rho) \right] = \epsilon^{\mu\nu\alpha\beta} \left[ \partial_{\mu} F_{\alpha\beta} + \partial_{\mu} (G_{\alpha\beta\rho}) x^\rho + G_{\alpha\beta\mu} \right] \]  

(37)

using the definition of \( F_{\alpha\beta} \) and \( G_{\alpha\beta\rho} \) we can easily show that first two term of (37) is zero and also the third term is nothing but Jacob’s identity, thus third term is also zero. So from (37)

\[ \partial_{\mu} \tilde{F}_{\mu\nu} = 0 \]  

(38)

Thus in analogy with Maxwell’s theory we can define electric field and magnetic field as

\[ E^l = F^{l0} = F^{l0} + G^{l0k} x_k = i(\partial^l A^0 - \partial^0 A^l) + (1 + i)(\partial^l B^{0k} - \partial^0 B^{lk}) x_k \]

\[ B^l = \tilde{F}^{l0} = \epsilon^{l0mn} \left[ F_{mn} + G_{mnk} x^k \right] = \epsilon^{l0mn} \left[ i(\partial_m A_n - \partial_n A_m) + (1 + i)(\partial_m B_{nk} - \partial_n B_{mk}) x^k \right] \]  

(39)

In terms of field tensor \( F^{\mu\nu} \) f two inhomogenous Maxwell’s equation can be written as

\[ \partial_{\mu} F^{\mu\nu} = J^\nu \]  

(40)

The last two Maxwell’s equation can be written as

\[ \partial_{\mu} \tilde{F}^{\mu\nu} = 0 \]  

(41)

again as \( F^{\mu\nu} \) is anti-symmetric, thus

\[ \partial_{\nu} \partial_{\mu} F^{\mu\nu} = 0 \]  

(42)
Thus using (10) we can write

$$\partial_\nu J^\nu = 0 \quad (43)$$

This is the equation of continuity. We can then write all the relevant equations of electromagnetism in terms of the matrix $F^{\mu\nu}$, as they are expressed by in the corresponding theory of $F^{\mu\nu}$. So the second part of the tensor $F^{\mu\nu}$ provides another representation of the $U(1)$ gauge group, where the basic interacting elements are the dipoles with a certain dipole moment. In the following section we will discuss this new symmetry elaborately.

6 Gravitational effect of fractons:

There is a conjecture that as fracton theory is formulated in terms of a second rank symmetric tensor $B_{\mu\nu}$, it may have a gravitational effect [2]. As there was no covariant fracton gauge theory, so they were forced to consider the same symmetric tensor $B_{\mu\nu}$ is responsible for the emergence of gravity. This is a singular assumption. However in combination with the other conclusions it should be looked upon as a very difficult paradox that whether it suggests a connection between electromagnetic theory and general theory of relativity or not? It is a paradox of the existing fracton gauge theory which can never be resolved in the confines in the existing picture.

All these questions and paradoxes associated with the fractons gravity interaction in the existing theory are absent in our formulation, based as it is on two general principles of covariance and gauge invariances. We know the formulation of the ‘seventh route to geometrodynamics’ [17], the essence of which is gauging of symmetries of a Poincare invariant matter theory, known as Poincare gauge theory (PGT) [9], [18],[]. From the localization of the shift symmetry of a Poincare invariant theory we get a geometrical manifold (Einstein-Cartan manifold) which goes to Riemannian geometry under the assumption of symmetric connections. From the commutator of the covariant derivatives we get the Riemann tensor. Also the Ricci tensor $R_{\mu\nu}$ is a product of the Riemann tensor, obtained as contraction of the indices as follows

$$R_{\mu\nu} = R^{\lambda}_{\quad \mu\lambda\nu} \quad (44)$$

This Ricci tensor vanish for the flat space time( Minkowski space-time). So a nonzero Riemann tensor is a guarantee that there exist curved space time which is a signature of gravity. How the matter interacts with the gravitational field is given by the Einstein’s equation(in natural units $c = \hbar = 1$),

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \quad (45)$$

where the symmetric energy-moment tensor is defined by $T_{\mu\nu}$

$$T_{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} \quad (46)$$
Where $S$ is the action of the matter theory. In our theory (7)

\[
T_{\mu\nu} = 2 \left( \phi^* \partial_\mu \partial_\nu \phi + \phi \partial_\mu \partial_\nu \phi^* \right) \partial_\alpha \phi \partial^\alpha \phi^* + 2 \left( \phi \Box \phi^* + \phi^* \Box \phi \right) \partial_\mu \phi \partial_\nu \phi^* - \delta_{\mu\nu} \left( \phi \Box \phi^* + \phi^* \Box \phi \right) \partial_\alpha \phi \partial^\alpha \phi^* \tag{47}
\]

We can easily show that the trace of this energy momentum tensor is

\[
T_\mu^\mu = 0 \tag{48}
\]

This is consistent with the Einstein’s equation. Note that we

Now in PGT corresponding to the basic fields $\Sigma_\mu^i$ and $M^ij_\mu$ the Lorentz field strength $R_{ij\mu\nu}^i$ and the translation field strength $T_{ij\mu\nu}$ are obtained following the usual procedure in gauge theory. The commutator of two covariant derivatives gives \[5\]

\[
[\nabla_k, \nabla_l] \phi = \frac{1}{2} \Sigma_k^\mu \Sigma_l^\nu R_{ij\mu\nu}^{ij} \phi - \Sigma_k^\mu \Sigma_l^\nu T_{ij\mu\nu} \nabla_i \phi \tag{49}
\]

These defining equations give the following expressions for the field-strengths

\[
T_{ij\mu\nu} = \partial_\mu \Sigma_i^\sigma \nu + M_i^{j\sigma \nu} \Sigma_k^\sigma - \partial_\nu \Sigma_i^\sigma \mu - M_i^{j\sigma \nu} \Sigma_k^\sigma
\]

\[
R_{ij\mu\nu} = \partial_\mu M_i^{j\nu} - \partial_\nu M_i^{j\mu} + M_i^{j\sigma \nu} M_k^{k\sigma \mu} - M_i^{j\sigma \nu} M_k^{k\sigma \mu} \tag{50}
\]

and also

\[
R_{\rho\sigma \mu\nu} = \Sigma_i^\rho \Sigma_j^\sigma T_{ij\mu\nu} \tag{51}
\]

So far the theory is in the Minkowski space and has been developed as a gauge theory. From the point of view of geometric interpretation, the Lorentz field strength $R_{ij\mu\nu}^i$ and the translation field strength $T_{ij\mu\nu}$, correspond to the Riemann tensor and the torsion. Using these basic structures, gravity can be formulated in the framework of PGT. Now for zero torsion (49) becomes

\[
[\nabla_k, \nabla_l] \phi = \frac{1}{2} \Sigma_k^\mu \Sigma_l^\nu R_{ij\mu\nu}^{ij} \phi \tag{52}
\]

But before localizing the space time symmetry we first show that our Lagrangian (7) is invariant under combined Poincare and Galileon transformation. Now under combined Poincare and Galileon transformation the fields and their derivative transforms as \[35\]

\[
\delta \phi = -\xi^\lambda \partial_\lambda \phi + (1 + i)b_\mu x_\mu \\
\delta \phi^* = -\xi^\lambda \partial_\lambda \phi^* + (1 - i)b_\mu x_\mu \\
\delta (\partial_\mu \phi) = -\xi^\lambda \partial_\mu \partial_\lambda \phi + \theta_\mu^\lambda \partial_\lambda \phi + (1 + i)b_\mu \\
\delta (\partial_\mu \phi^*) = -\xi^\lambda \partial_\mu \partial_\lambda \phi^* + \theta_\mu^\lambda \partial_\lambda \phi^* + (1 - i)b_\mu \\
\delta (\partial_\mu \partial_\nu \phi) = -\xi^\lambda \partial_\mu \partial_\nu \partial_\lambda \phi + \theta_\mu^\lambda \partial_\nu \partial_\lambda \phi + \theta_\nu^\lambda \partial_\mu \partial_\lambda \phi \\
\delta (\partial_\mu \partial_\nu \phi^*) = -\xi^\lambda \partial_\mu \partial_\nu \partial_\lambda \phi^* + \theta_\mu^\lambda \partial_\nu \partial_\lambda \phi^* + \theta_\nu^\lambda \partial_\mu \partial_\lambda \phi^* \tag{53}
\]
Here, $\xi^\lambda = \epsilon^\lambda + \theta^\lambda_\mu x^\mu$ are the infinitesimal Poincare transformation parameters. Under the transformation (53) the total change of Lagrangian (7) is

$$\Delta L = \delta L + \xi^\lambda \partial_\lambda L + \partial_\lambda \xi^\lambda L$$

(54)

where $\delta L$ is the form variation of $L$. Now using the variation (53) and equation of motion $\Box \phi = \Box \phi^* = 0$, we easily shows that

$$\Delta L = 0$$

(55)

Thus our Lagrangian (7) is invariant under combined Poincare and Galileon transformation.

Now we are ready to localized the space-time symmetry. In order to localized Poincare symmetry we have to replace $\partial_\mu$ by

$$\nabla_a \phi = \Sigma_a^\mu D_\mu \phi = \Sigma_a^\mu \left( \partial_\mu \phi + \frac{1}{2} M_{\mu bc} \sigma_{bc} \phi \right)$$

(56)

where $\sigma_{ab}$ is the Lorentz spin matrix and $\Sigma_a^\mu$ and $M_{\mu bc}$ are the new gauge fields corresponding to translation and Lorentz transformation. As The Galileon field has an additional shift symmetry, so we require to introduce further gauge fields $P_{\mu \nu}$. The new covariant derivatives are now defined as,

$$\nabla_a \phi = \Sigma_a^\mu \bar{D}_\mu \phi; \quad \bar{D}_\mu \phi = (D_\mu \phi + F_\mu \phi; F_\mu = (1 + i) P_{\mu \nu} x^\nu)$$

(57)

The transformations of the new fields are obtained by demanding that the covariant derivatives (57) transform as the ordinary one in (53),

$$\delta (\nabla_a \phi) = -\xi^\nu \partial_\nu (\nabla_a \phi) + \theta_a^b \nabla_b \phi + (1 + i) b_a$$

(58)

This yields,

$$\delta \Sigma_a^\mu = -\xi^\lambda \partial_\lambda \Sigma_a^\mu + \partial_\lambda \xi^\mu \Sigma_a^\lambda + \theta_a^b \Sigma_b^\mu$$

$$\delta M_{\mu ab} = -\xi^\lambda \partial_\lambda M_{\mu ab} - \partial_\mu \theta_{ab} - \partial_\mu \xi^\lambda M_{\mu ab} + \theta^a_{bc} M_{\mu cb} + \theta^b_{ac} M_{\mu ac}$$

$$\delta (x^\nu P_{\mu \nu} \phi) = -\xi^\lambda \partial_\lambda (x^\nu P_{\mu \nu} \phi) - \partial_\mu \xi^\lambda (x^\nu P_{\lambda \nu} \phi) - x^\nu \partial_\mu b_\nu$$

(59)

The Galileon symmetries manifested through these relations are compatible with analogous results in [20,33] where the algebra of Galileon generators has been defined. Also note that

$$\delta (\bar{D}_\mu \phi) = -\xi^\lambda \partial_\lambda (\bar{D}_\mu \phi) - \partial_\mu \xi^\lambda \bar{D}_\lambda \phi + (1 + i) b_\mu$$

$$\delta (F_\mu \phi) = -\xi^\lambda \partial_\lambda (F_\mu \phi) - \partial_\mu \xi^\lambda (F_\lambda \phi) - (1 + i) x^\nu \partial_\mu b_\nu$$

(60)

Thus both $F_\mu \phi$ and $\bar{D}_\mu \phi$ transform as four vectors under local Poincare transformations.

---

The gauge fields corresponding to the Lorentz rotation do not appear because $\phi$ is a Lorentz scalar.
The transformations of the second derivatives, however, do not have any Galileon contribution (see the last two eq. of (53)). Consequently, the second derivative \( \partial_\mu \partial_\nu \phi \) should be replaced by \( \nabla_a \nabla_b \phi \). Explicit calculation shows that under local Poincare plus Galileon transformations \( \nabla_a \nabla_b \phi \) transform in the same way as \( \partial_\mu \partial_\nu \phi \) i.e.

\[
\delta(\nabla_a \nabla_b \phi) = -\xi^d \partial_d(\nabla_a \nabla_b \phi) + \theta_a^d (\nabla_d \nabla_b \phi) + \theta_b^d (\nabla_a \nabla_d \phi)
\]

(61)

provided,

\[
\nabla_a b_c = 0
\]

(62)

This is a natural generalization of the condition \( \partial_a b_c = 0 \) in flat space. We can define \( F_a \) as,

\[
F_a = \Sigma_a^\mu F_\mu
\]

(63)

and,

\[
\nabla_a (F_b \phi) = \Sigma_a^\mu [\partial_\mu (\Sigma_b^\nu F_\nu \phi) + \frac{1}{2} M_{\mu cd} \Sigma_b^\nu F_\nu \phi]
\]

(64)

for zero torsion,

\[
\nabla_a \Sigma_c^\mu - \nabla_c \Sigma_a^\mu = 0
\]

(65)

thus we find,

\[
\nabla_a (F_b \phi) - \nabla_b (F_a \phi) = \Sigma_a^\mu \Sigma_b^\nu [\partial_\mu (F_\nu \phi) - \partial_\nu (F_\mu \phi)]
\]

(66)

The variation under Galileon transformations now yields,

\[
\delta[\nabla_a (F_b \phi) - \nabla_b (F_a \phi)] = (1 + i) \Sigma_a^\mu \Sigma_b^\nu [\partial_\mu b_\nu - \partial_\nu b_\mu]
\]

\[
= (1 + i)(\nabla_a b_b - \nabla_b b_a) = 0
\]

(67)

on account of (62). This implies that the choice,

\[
[\nabla_a (F_b \phi) - \nabla_b (F_a \phi)] = 0
\]

(68)

may be consistently implemented.

Now the commutator of \( \nabla_a \) and \( \nabla_b \) are given by

\[
[\nabla_a, \nabla_b] \phi = [\nabla_a, \nabla_b] \phi + [\nabla_a, F_b] \phi + [F_a, \nabla_b] \phi + [F_a, F_b] \phi
\]

(69)

After some calculation, using the symmetric property of \( P_{\mu\nu} \) and the condition (68) we get

\[
[\nabla_a, \nabla_b] \phi = \Sigma_a^\mu \Sigma_b^\nu \left[ \frac{1}{2} \sigma_{\alpha\beta} \left( \partial_\mu M_{\nu}^{\alpha\beta} - \partial_\nu M_{\mu}^{\alpha\beta} + M_{\rho\nu}^{\alpha} M_{\mu}^{\beta\rho} - M_{\rho\mu}^{\alpha} M_{\nu}^{\beta\rho} \right) \right]
\]

\[
(1 + i) x^\rho (\partial_\mu P_{\nu\rho} - \partial_\nu P_{\mu\rho})
\]

\[
+ \frac{1}{2} (1 + i) x^\rho \sigma_{cd} (P_{\mu\rho} M_{\nu}^{cd} - P_{\nu\rho} M_{\mu}^{cd}) \right] \phi
\]

(70)

\[\text{A general discussion of localising the Poincare sector of a higher derivative theory is given in [34].}\]
comparing this with (52) and using (51) we get

\[ R_{\mu \nu}^{\lambda \sigma} = \Sigma_\alpha \Sigma_\beta R_{\mu \nu}^{\alpha \beta} = \Sigma_\alpha \Sigma_\beta \left[ (\partial_{\mu} M_{\nu}^{\alpha \beta} - \partial_{\mu} M_{\mu}^{\alpha \beta} + M_\mu^{\alpha} M_\nu^{\beta \mu} - M_\mu^{\alpha} M_\nu^{\beta \mu}) \\
+ 2(1 + i) x^{\rho} \sigma^{\alpha \beta} (\partial_{\mu} P_{\rho \mu} - \partial_{\nu} P_{\rho \mu}) \\
+ (1 + i) x^{\rho} (P_{\rho \mu} M_{\nu}^{\alpha \beta} - P_{\rho \nu} M_{\mu}^{\alpha \beta}) \right] \]  

(71)

Now from (44) we get the Ricci tensor as,

\[ R_{\nu \sigma} = R_{\mu \nu}^{\mu \sigma} = \Sigma_\alpha \Sigma_\beta \left[ (\partial_{\mu} M_{\nu}^{\alpha \beta} - \partial_{\mu} M_{\mu}^{\alpha \beta} + M_\mu^{\alpha} M_\nu^{\beta \mu} - M_\mu^{\alpha} M_\nu^{\beta \mu}) \\
+ 2(1 + i) x^{\rho} \sigma^{\alpha \beta} (\partial_{\mu} P_{\rho \mu} - \partial_{\nu} P_{\rho \mu}) \\
+ (1 + i) x^{\rho} (P_{\rho \mu} M_{\nu}^{\alpha \beta} - P_{\rho \nu} M_{\mu}^{\alpha \beta}) \right] \]  

(72)

The equation (72) is, to say the least extraordinary. The first term of this on the 1st, contains a second rank tensor \( M_{\mu \nu} \). Clearly another second rank tensor \( P_{\mu \nu} \), which appears in the second and the last terms. Again we see that in the third term only the gauge fields may be identified with the external gravitation. Thus one can appreciate the importance of this formulation. Note that a fracton

7 Conclusion:

In this paper our purpose was to bring out essential novelties of the co variant fracton gauge theory [1] where the concepts of gauge invariance and covariance under Poincare symmetries (space-time translation and rotation symmetry) were interwoven in a unique way. The applications of these to the fracton gauge theory was remarked by us as a paradigm shift of the research methodology in the subject. Our endeavor was to give explicit expressions in support of our remark. The gauge principle has three cardinal concerns were the construction of un-gauge theory which will carry the symmetries globally, then comes the question how to extract the symmetries following a general algorithm and refraining from arbitrary assumption as much as possible. After this step the calculations of the various symmetry generators and the physical meaning of them falling the standard method of gauging the symmetries which have been formulated over last 50 years [?, ?], [6], [30], [31], [32], [33], [34], [35], [36], [37], [38], [39], [40], [41], [42].

We start with a brief review of the resistance works on 'fracton gauge theory'. Specifically it was pointed out that later paper on the subject though originally in many regards were follows of [4], but the algorithm which was suggested by [4] was neither space-time symmetric nor gauge symmetric. Hence the conclusions reach by them are only meaning full for a particular coordinate system for a particular gauge. Naturally such a theory has little predictive power. The input for which [4] really significant is the proposal for two different symmetries of the system. Only electromagnetic interaction is there so it is really puzzling that why should there be a peculiar symmetry, independent
of the U(1) phase rotation symmetry (independent but having a hierarchy relationship so that the second symmetry lose their meaning if the first symmetry does not hold). However the type of interaction is such that there is no gauge photon except the ordinarily known electromagnetic interaction carrier. So second symmetry gives a point like interaction without any propagating degree of freedom. Incidentally this point was not noticed the existing theory. Also we observed that no special consideration is given for the symmetry of the matter sector. Hence the situations were demanding a new way of thinking. We modestly claim that our works will provide an initiation in this line.

We have postulated a complex scalar field consisting of the Galileons scalar. It may be remembered that the Galileons are introduced in the process of obtaining a scalar field theory which originated from as a certain limit of DGP modified gravity theory in relation to the search for dark energy of cosmology. So a very abstract concept of cosmology appears in the fracton gauge theory which is related to such a mundane things such as glassy object. So fur we know this complex scalar field which has been used as a representation of the U(1) symmetry having two different gauge symmetries. Another point which goes in favor of Galileons is the second order nature of its equations of motion, notwithstanding the higher order theory. More over the action suggested by us has natural inbuilt U(1) symmetry. Also it shears CPT compatibility. So this theory is naturally suggested. Also the complex scalar field appearing here is a unique construction which may have significant role in theoretical physics.

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\footnote{Actually in four dimension there are five different members in the set of Hondesky system osfactions of which our Galileons constituents a particular class level $L_1$ and $L_3$. In a future communication this point will be very significant.}
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