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Complete convergence for arrays of ratios of order statistics

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Abstract: Let \( \{ X_{n,k}, 1 \leq k \leq m_n, n \geq 1 \} \) be an array of independent random variables from the Pareto distribution. Let \( X_{n,k}^{(n)} \) be the \( k \)th largest order statistic from the \( n \)th row of the array and set \( R_{n,i_n,j_n} = X_{n,j_n}^{(n)}/X_{n,i_n}^{(n)} \) where \( j_n < i_n \). The aim of this paper is to study the complete convergence of the ratios \( \{ R_{n,i_n,j_n}, n \geq 1 \} \).

Keywords: Ratios; Pareto distribution; complete convergence.

MSC: 60F15, 62G30

1 Introduction

Let \( \{ X_n, n \geq 1 \} \) be a sequence of independent and identically distributed random variables and let \( X_n^{(n)} \leq X_n^{(n-1)} \leq \cdots \leq X_n^{(1)} \) be the order statistics. During the past many years, the influence of order statistics has attracted considerable attention. This topic is relevant in various practical situations like in (re)insurance applications when a significant proportion of the sum of claims is consumed by a small number of claims (or even by a single claim) due to earthquakes, floods, hurricanes, terrorism, etc. Numerous authors gave necessary and/or sufficient conditions for the convergence of certain ratios of extreme terms (order statistics and terms of maximum modulus) and sums. Smid and Stam [1] proved that the successive quotients of the order statistics in decreasing order are asymptotically independent with some distribution functions, under certain conditions. O’Brien [2] studied the ratio variate \( X_{n(j)}/S_n \), where \( S_n = X_1 + \cdots + X_n \), which is a quantity arising in the analysis of process speedup and the performance of scheduling. Balakrishnan and Stepanov [3] discussed the asymptotic properties of ratios \( X_{n(j)}/X_{n(i)} \) for \( i \neq j \). Furthermore, there are many scholars who have studied the more-refined properties for some specific distributions. For example, Malik and Trudel [4] found the distribution of the quotient of and two order statistics from the Pareto, Power and Weibull distributions, by using the Mellin transform technique.

In the present paper, we want to study some limit theorems for the order statistics from the Pareto distribution. Pareto distribution describes the income of individuals, and since its introduction, it has found wide applications in many fields of studies such as economics, insurance premium, population distributions, stock market analysis, and queuing theory. Johnson et al. [5] discussed some potential applications of this distribution in different subject fields. Mann et al. [6] studied different estimation procedures for the unknown

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parameters of the Pareto distribution. Miao et al. [7] establish two large deviations for the Pareto distribution, and discussed the maxima of sums of the two-tailed Pareto random variables.

Let \( X \) be a random variable with the Pareto distribution, i.e., the probability density function of \( X \) is

\[
  f(x) = \frac{k x_m^k}{x^{k+1}} I(x \geq x_m)
\]

where \( x_m \) is the (necessarily positive) minimum possible value of \( X \), and \( k \) is a positive parameter. In the present paper, we consider an array of independent random variables \( \{X_{n,k}, 1 \leq k \leq m_n, n \geq 1\} \) with density

\[
  f(x) = p_n x^{-p_n - 1} I(x \geq 1)
\]

where \( p_n > 0 \). Let \( X_{n(m_n)} \) be the \( m_n \)th largest order statistic from each row of the array. Hence we have

\[
  X_{n(m_n)} \leq X_{n(m_n-1)} \leq \cdots \leq X_{n(2)} \leq X_{n(1)}.
\]

Next let us define the ratios of the order statistics

\[
  R_n := R_{n,i_n,j_n} = \frac{X_{n(i_n)}}{X_{n(j_n)}}, \quad j_n < i_n
\]

which implies that \( R_n \geq 1 \). It is not difficult to check that the density of \( R_n \) is

\[
  f_{R_n}(r) = \frac{p_n(i_n - 1)!}{(i_n - j_n - 1)!(j_n - 1)!} r^{p_n - 1} (1 - r^{p_n})^{j_n - 1} I(r \geq 1).
\]

It is obvious that the density of \( R_n \) is free of \( m_n \). Adler [8–10] studied the limit behaviors for the weighted sums of the ratios. In [9], Adler assumed that the parameters \( i_n, j_n \) were fixed. In [8], Adler was allowed to let \( m_n \) and \( i_n \) grow, but \( j_n \) was fixed. The paper [10] was a natural extension of Adler [8, 9] and allow all subscripts to grow, but the distance between \( i_n \) and \( j_n \) was fixed, i.e., \( \Delta := i_n - j_n \) was fixed. Adler [10] obtained the following results. The first theorem establishes an unusual strong law where \( \Delta = 1 \).

**Theorem 1.** [10] If \( p_n j_n = 1, \Delta = 1 \) and \( \alpha > -2 \), then

\[
  \lim_{N \to \infty} \frac{\sum_{n=1}^{N} ((\log n)^\alpha / n) R_n}{(\log N)^{\alpha+2}} = \frac{1}{\alpha + 2}, \quad \text{a.s.}
\]

The following results are to consider the case \( \Delta > 1 \).

**Theorem 2.** [10] If \( p_n j_n = 1, j_n = o(\log n), \Delta > 2 \) and \( \alpha > -2 \), then

\[
  \lim_{N \to \infty} \frac{\sum_{n=1}^{N} ((\log n)^\alpha / (n j_n^{\Delta - 1})) R_n}{(\log N)^{\alpha+2}} = \frac{1}{(\alpha + 2)(\Delta - 1)!}, \quad \text{a.s.}
\]

**Theorem 3.** [10] If \( p_n j_n = 1, j_n \sim \log n, \Delta > 2 \) and \( \alpha > -2 \), then

\[
  \lim_{N \to \infty} \frac{\sum_{n=1}^{N} ((\log n)^\alpha / (n j_n^{\Delta - 1})) R_n}{(\log N)^{\alpha+2}} = \frac{\gamma_{\Delta}}{(\alpha + 2)(\Delta - 1)!}, \quad \text{a.s.}
\]

where

\[
  \gamma_{\Delta} = \sum_{k=1}^{\Delta-1} \frac{(\Delta-1)!(-1)^{k+1} e^{-k}}{k} - \sum_{k=2}^{\Delta-1} \frac{1}{k},
\]

or, if one wishes

\[
  \gamma_{\Delta} = 1 + \sum_{k=1}^{\Delta-1} \frac{(\Delta-1)!(-1)^{k}(1 - e^{-k})}{k},
\]

where naturally, if \( \Delta = 2 \) we have \( \sum_{k=2}^{\Delta-1} \frac{1}{k} = 0 \).
Theorem 1.4. [10] If \( p_{n,j,n} = 1, j_n \sim (\log n)\alpha \), where \( 1 < \alpha < 2, \Delta = 2 \) and \( \alpha > -3 \), then
\[
\lim_{N \to \infty} \frac{\sum_{n=1}^{N} (\log n)^{\alpha} / n) R_{n}}{(\log N)^{\alpha+3}} = \frac{1}{2(\alpha + 3)}, \text{ a.s.}
\]

There is some literature concerning the limit theorems, for example, Miao et al. [11, 12] established some limit theorems for the ratios of order statistics from exponentials and uniform distribution. In the present paper, we are interested in the complete convergence for the ratios \( R_{n} \). In order to prove the complete convergence, since the expectation of \( R_{n} \) does not exist, we need to deal with the tailed term by using the results in Sung et al. [13]. Throughout the paper, we use the constant \( C \) to denote a generic real number that is not necessarily the same in each appearance, and we define \( \log x = \log(\max(\epsilon, x)) \).

2 Complete convergence theorems

In this section, we consider the complete convergence theorems for the weighted sums of the ratios \( R_{n} \). The concept of complete convergence of a sequence of random variables was introduced by Hsu and Robbins [14] as follows. A sequence \( \{ U_{n}, n \geq 1 \} \) is said to converge completely to a constant \( C \) if
\[
\sum_{n=1}^{\infty} P(|U_{n} - C| > \epsilon) < \infty, \text{ for all } \epsilon > 0.
\]

By using Borel-Cantelli lemma, this implies that \( U_{n} \to C \) almost surely.

2.1 Several lemmas

Before giving the main results, we need to recall the following some lemmas.

Lemma 2.1. [13] Let \( \{ X_{n,k}, 1 \leq k \leq m_{n}, n \geq 1 \} \) be an array of rowwise independent random variables and \( \{ a_{n}, n \geq 1 \} \) a sequence of positive constants such that
\[
\sum_{n=1}^{\infty} a_{n} = \infty.
\]

Suppose that for every \( r > 0 \) and some \( \epsilon > 0 \):
(i) \( \sum_{n=1}^{\infty} a_{n} \sum_{i=1}^{m_{n}} P(|X_{n,i}| > r) < \infty \),
(ii) there exists \( J \geq 2 \) such that
\[
\sum_{n=1}^{\infty} a_{n} \left( \sum_{i=1}^{m_{n}} E X_{n,i}^{2} I(|X_{n,i}| \leq \epsilon) \right) < \infty,
\]
(iii) \( \sum_{i=1}^{m_{n}} E X_{n,i} I(|X_{n,i}| \leq \epsilon) \to 0 \) as \( n \to \infty \).

Then we have
\[
\sum_{n=1}^{\infty} a_{n} P \left( \left| \sum_{i=1}^{m_{n}} X_{n,i} \right| > r \right) < \infty \text{ for all } r > 0.
\]

Lemma 2.2. [10] For \( \Delta > 1 \) and \( p_{n,j,n} = 1 \), we have
\[
p_{n}^{j_{n}-1} \frac{j_{n}^{\Delta-1}}{(j_{n} - 1)! (j_{n} - 1)!} \sim \frac{1}{(\Delta - 1)!}
\]
where “\( a_{n} \sim b_{n} \)” denotes \( \lim_{n \to \infty} a_{n} / b_{n} = 1 \).

Lemma 2.3. [10] If \( 1 < \alpha < 2 \), then
\[
\lim_{x \to \infty} \frac{1 + 3 x^{-1} \log x + x^{\alpha-1} [(e^{x} x^{3})^{-1} x^{\alpha-1} - 1]}{x^{1-a}} = 1/2.
\]
2.2 Main results

In the subsection, we give the complete convergence of the weighted sums with different forms. The distance between \( n \) and \( j_n \) is fixed, i.e., \( \Delta \) is fixed. We consider the most interesting case and assume that \( p_nj_n = 1 \). The growth of \( j_n \) can’t be very fast, so we need a logarithmic growth rate for \( j_n \). The conclusion breaks into three cases for different forms.

Case 1: \( \Delta = 1 \) and \( a > -2 \);
Case 2: \( j_n = o(\log n) \), \( \Delta = 2 \) and \( a > -2 \);
Case 3: \( j_n \sim (\log n)^a \), \( \Delta = 2 \) and \( a > -3 \), where \( 1 < a < 2 \).

**Theorem 2.1.** Let \( \{b_N, N \geq 1\} \) and \( \{r_N, N \geq 1\} \) be sequences of positive numbers satisfying

\[
\sum_{N=1}^{\infty} b_N = \infty, \quad \frac{(\log \log N)(\log \log N + \log r_N)}{r_N \log N} \to 0 \quad (2.1)
\]

and

\[
\sum_{N=1}^{\infty} b_N \log N \max \left\{ 1, \frac{1}{r_N} \right\} < \infty. \quad (2.2)
\]

If \( p_nj_n = 1, \Delta = 1 \) and \( a > -2 \), then

\[
\sum_{N=1}^{\infty} b_N P \left( \max_{1 \leq k \leq N} \left| \sum_{n=1}^{k} \frac{(\log n)^a}{n} (R_n - ER_n I(1 \leq R_n \leq c_n)) \right| \geq r_N (\log N)^{a+2} \right) < \infty \quad (2.3)
\]

where \( c_n = n(\log n)^2 \).

**Remark 2.1.** The convergence of the series in (2.3) holds trivially for any \( \sum_{N=1}^{\infty} b_N < \infty \).

**Theorem 2.2.** Let \( \{b_N, N \geq 1\} \) and \( \{r_N, N \geq 1\} \) be sequences of positive numbers satisfying (2.1) and (2.2). If \( p_nj_n = 1, j_n = o(\log n), \Delta = 2 \) and \( a > -2 \), then

\[
\sum_{N=1}^{\infty} b_N P \left( \max_{1 \leq k \leq N} \left| \sum_{n=1}^{k} \frac{(\log n)^a}{n} (R_n - ER_n I(1 \leq R_n \leq c_n)) \right| \geq r_N (\log N)^{a+2} \right) < \infty \quad (2.4)
\]

where \( c_n = n(\log n)^{a+1} \).

**Remark 2.2.** The conclusion of Theorem 2.2 can also be obtained when \( j_n \sim \log n \) and \( c_n = n(\log n)^{a+1} \).

**Theorem 2.3.** Let \( \{b_N, N \geq 1\} \) and \( \{r_N, N \geq 1\} \) be sequences of positive numbers satisfying

\[
\sum_{N=1}^{\infty} b_N = \infty, \quad \frac{(\log \log N)(\log \log N + \log r_N)}{r_N (\log N)^{2-a}} \to 0 \quad (2.5)
\]

and

\[
\sum_{N=1}^{\infty} b_N \log N \max \left\{ 1, \frac{1}{r_N} \right\} < \infty \quad (2.6)
\]

where \( 1 < a < 2 \). If \( p_nj_n = 1, j_n \sim (\log n)^a, \Delta = 2 \) and \( a > -3 \), then

\[
\sum_{N=1}^{\infty} b_N P \left( \max_{1 \leq k \leq N} \left| \sum_{n=1}^{k} \frac{(\log n)^a}{n} (R_n - ER_n I(1 \leq R_n \leq c_n)) \right| \geq r_N (\log N)^{a+3} \right) < \infty
\]

where \( c_n = n(\log n)^3 \).
2.3 Proofs of main results

Before giving the proofs, we want to show the outlines of the proofs. For any \( k \), we partition \( t_n \left( R_n - ER_nI(1 \leq R_n \leq c_n) \right) \) into the following two terms:

\[
\sum_{n=1}^{k} t_n \left( R_n - ER_nI(1 \leq R_n \leq c_n) \right) = \sum_{n=1}^{k} t_n [R_nI(1 \leq R_n \leq c_n) - ER_nI(1 \leq R_n \leq c_n)]
+ \sum_{n=1}^{k} t_n R_nI(R_n > c_n) =: I_k + II_k,
\]

(2.7)

where \( t_n \) are our weights. Next we discuss separately the complete convergence of the above terms. The complete convergence of the first term can be verified by the different density of \( R_n \), Lemma 2.2 and Kolmogorov’s inequality. Since the expectation of \( R_n \) dose not exists, we will deal with the tailed term by using the results in Sung et al. [13]. Therefore, the complete convergence of the second term can be obtained by Lemma 2.1.

Proof of Theorem 2.1. Firstly we partition \( \sum_{n=1}^{k} ((log n)^{\alpha}/n) \left( R_n - ER_nI(1 \leq R_n \leq c_n) \right) \) into the following two terms:

\[
\sum_{n=1}^{k} ((log n)^{\alpha}/n) \left( R_n - ER_nI(1 \leq R_n \leq c_n) \right) = \sum_{n=1}^{k} ((log n)^{\alpha}/n)[R_nI(1 \leq R_n \leq c_n) - ER_nI(1 \leq R_n \leq c_n)]
+ \sum_{n=1}^{k} ((log n)^{\alpha}/n)R_nI(R_n > c_n)
= :I_k + II_k.
\]

(2.8)

Note that the density for the ratio of our adjacent order statistics, that is, \( \Delta = 1 \), is

\[ f_{R_n}(x) = x^{-2}I(x \geq 1), \]

then it is not difficult to see that

\[ ER_n^2I(1 \leq R_n \leq c_n) = c_n - 1 = n(log n)^2 - 1. \]

By denoting

\[ \hat{R}_n := R_nI(1 \leq R_n \leq c_n) - ER_nI(1 \leq R_n \leq c_n), \]

and by the Kolmogorov’s inequality, we have

\[
P\left( \max_{1 \leq k \leq N} \left| \sum_{n=1}^{k} ((log n)^{\alpha}/n)\hat{R}_n \right| > r_N(log N)^{\alpha+2} \right) \leq \frac{1}{r_N^2(log N)^{2(\alpha+2)}} Var \left[ \sum_{n=1}^{N} ((log n)^{\alpha}/n)\hat{R}_n \right]
\leq \frac{N}{r_N^2(log N)^{2(\alpha+2)}} Var[R_nI(1 \leq R_n \leq c_n)]
\leq \frac{N}{r_N^2(log N)^{2(\alpha+2)}} (log n)^2 \leq \frac{1}{r_N^2} \frac{\log \log N}{log N}. \]

(2.9)

Here we use the fact

\[
\sum_{n=1}^{N} \frac{(log n)^{2(\alpha+2)}}{n(log N)^{2(\alpha+2)}} \leq \begin{cases} \frac{C}{\log N}, & \text{for } \alpha > -2, \alpha \neq -\frac{3}{2}, \\ \frac{C}{\log N}, & \text{for } \alpha = -\frac{3}{2} \end{cases},
\]

(2.10)

for all \( N \) large enough. From the condition (2.2), we can get

\[
\sum_{N=1}^{\infty} b_N P \left( \max_{1 \leq k \leq N} |I_k| \geq r_N(log N)^{\alpha+2} \right) < \infty.
\]

(2.11)
Next, we consider the complete convergence of the term \( II_k \), from using Lemma 2.1. By denoting

\[ X_{N,n} := \frac{(\log n)^a/n)R_nI(R_n > c_n)}{r_n(\log N)^{a+2}}, \]

then by the similar discussions in (2.10), for any \( r > 0 \), we have

\[
\sum_{n=1}^{N} P \left( X_{N,n} \geq r \right) \leq \sum_{n=1}^{N} P \left( R_n \geq \max \left\{ c_n, nr \cdot r_n(\log N)^{a+2}/(\log n)^a \right\} \right)
\]

\[
\leq C \sum_{n=1}^{N} \max \left\{ \frac{1}{n(\log n)^2}, \frac{(\log n)^a}{nr_n(\log N)^{a+2}} \right\} \leq C \log \log N \frac{\log N}{r_n \log N}
\]

which implies

\[
\sum_{N=1}^{\infty} b_N \sum_{n=1}^{N} P \left( X_{N,n} \geq r \right) < \infty.
\]

Furthermore, for any \( \varepsilon > 0 \), we have

\[
\sum_{n=1}^{N} EX_{N,n}I \left( X_{N,n} \leq \varepsilon \right) \leq C \frac{\log \log N + \log r_n}{r_n(\log N)^{a+2}} \sum_{n=1}^{N} \frac{(\log n)^a}{n}
\]

\[
\leq C \frac{(\log \log N)(\log \log N + \log r_n)}{r_n \log N} \to 0
\]

as \( N \to \infty \), since the condition (2.1). At last

\[
\sum_{n=1}^{N} EX_{N,n}I \left( X_{N,n} \leq \varepsilon \right) \leq C \sum_{n=1}^{N} \frac{(\log n)^a}{r_n(\log N)^{a+2}} \leq \frac{C \log \log N}{r_n \log N},
\]

then we get

\[
\sum_{N=1}^{\infty} b_N \left( \sum_{n=1}^{N} EX_{N,n}I \left( X_{N,n} \leq \varepsilon \right) \right)^2 < \infty.
\]

By Lemma 2.1 and (2.12)-(2.14), we have

\[
\sum_{N=1}^{\infty} b_N P \left( \max_{1 \leq k \leq N} |I_{Ik}| \geq r \cdot r_n(\log N)^{a+2} \right) = \sum_{N=1}^{\infty} b_N P \left( \frac{((\log n)^a/n)R_nI(R_n > c_n)}{r_n(\log N)^{a+2}} \geq r \right) < \infty.
\]

Therefore, the desired result can be obtained by (2.11) and (2.15).

\( \Box \)

**Proof of Theorem 2.2.** As the proof of Theorem 2.1, we partition \( \sum_{n=1}^{\infty} ((\log n)^a/(n^{\alpha-1}))(R_n - ER_nI(1 \leq R_n \leq c_n)) \) into the following two terms:

\[
\sum_{n=1}^{k} ((\log n)^a/(n^{\alpha-1}))(R_n - ER_nI(1 \leq R_n \leq c_n))
\]

\[
= \sum_{n=1}^{k} ((\log n)^a/(n^{\alpha-1}))[R_nI(1 \leq R_n \leq c_n) - ER_nI(1 \leq R_n \leq c_n)]
\]

\[
+ \sum_{n=1}^{k} ((\log n)^a/(n^{\alpha-1}))R_nI(R_n > c_n)
\]

\[= I_k + II_k.\]

For the case \( \Delta \geq 2, p_n \Delta n = 1 \) and by Lemma 2.2, the density for the ratio of our adjacent order statistics is

\[
f_{R_n}(x) = \frac{p_n(\Delta n - 1)!}{(\Delta n - 1)!} x^{-p_n(\Delta n - 1)} (1 - x^{-p_n})^{\Delta n - 1} I(x \geq 1)
\]

\[\sim \frac{\Delta n - 1}{(\Delta - 1)!} x^{-2} (1 - x^{-1})^{\Delta n - 1} I(x \geq 1).\]
Then it follows that

$$ER_n I(1 \leq R_n \leq c_n) \sim \frac{j_n^{\Delta-1}}{(\Delta - 1)!} \int_1^{c_n} x^{-1}(1 - x^{-1/j_n})^{\Delta-1} \, dx$$

$$= \frac{j_n^{\Delta-1}}{(\Delta - 1)!} \int_1^{c_n} x^{-1} \sum_{k=0}^{\Delta-1} \binom{\Delta-1}{k} (-1)^k x^{-k/j_n} \, dx$$

$$= \frac{j_n^{\Delta-1}}{(\Delta - 1)!} \left[ \log c_n + \sum_{k=1}^{\Delta-1} \binom{\Delta-1}{k} (-1)^k (k/j_n)^{-1}(1 - c_n^{-k/j_n}) \right]$$

$$\sim \frac{j_n^{\Delta-1} \log n}{(\Delta - 1)!}$$

and

$$ER_n^2 I(1 \leq R_n \leq c_n) \leq C j_n^{\Delta-1} c_n = C n^{-2(\Delta-1)}(\log n)^2.$$

By the Kolmogorov’s inequality, we have

$$P \left( \max_{1 \leq k \leq N} \left| \sum_{n=1}^{N} \left( (\log n)^a/(n j_n^{\Delta-1}) \right) \tilde{R}_n \right| > r_N(\log N)^{a+2} \right)$$

$$\leq \frac{\sum_{n=1}^{N} \left( (\log n)^a/(n j_n^{\Delta-1}) \right)^2 \Var[R_n I(1 \leq R_n \leq c_n)]}{r_N^2(\log N)^{2(a+2)}} \leq \frac{\log \log N}{r_N^2 \log N}$$

where

$$\tilde{R}_n := R_n I(1 \leq R_n \leq c_n) - ER_n I(1 \leq R_n \leq c_n).$$

From the condition (2.2), we can get

$$\sum_{N=1}^{\infty} b_N P \left( \max_{1 \leq k \leq N} |I_k| > r_N(\log N)^{a+2} \right) < \infty.$$  

(2.18)

Next, we consider the complete convergence of the term $I_k$, from using Lemma 2.1. By denoting

$$X_{N,n} := \frac{(\log n)^a/(n j_n^{\Delta-1}) R_n I(R_n > c_n)}{r_N(\log N)^{a+2}},$$

then for any $r > 0$, we have

$$\sum_{n=1}^{N} P \left( X_{N,n} \geq r \right) \leq \sum_{n=1}^{N} P \left( R_n \geq \max \left\{ c_n, nr \cdot r_N j_n^{\Delta-1}(\log N)^{a+2}/(\log n)^a \right\} \right)$$

$$\leq C \sum_{n=1}^{N} \max \left\{ \frac{1}{n(\log n)^2}, \frac{(\log n)^a}{nr_N(\log N)^{a+2}} \right\} \leq C \log \log N \frac{\log \log N}{r_N^2 \log N},$$

which implies

$$\sum_{N=1}^{\infty} b_N \sum_{n=1}^{N} P \left( X_{N,n} \geq r \right) < \infty.$$  

(2.19)

Furthermore, for any $\varepsilon > 0$, we have

$$\sum_{n=1}^{N} E X_{N,n} I(X_{N,n} \leq \varepsilon) \leq \frac{C}{r_N(\log N)^{a+2}} \sum_{n=1}^{N} \frac{(\log n)^a j_n^{\Delta-1}(\log \log N + \log r_N)}{(\Delta - 1)!}$$

$$\leq C \frac{\log \log N + \log r_N}{r_N(\log N)^{a+2}} \sum_{n=1}^{N} \frac{(\log n)^a}{n}$$

$$\leq C \frac{(\log \log N)(\log \log N + \log r_N)}{r_N \log N} \to 0$$  

(2.20)
as \( N \to \infty \). At last, since
\[
\sum_{n=1}^{N} E X_{N,n}^2 I \left( X_{N,n}^2 \leq \varepsilon \right) \leq C \sum_{n=1}^{N} \frac{\varepsilon (\log n)^a}{(n-j)^{a-1} r_N (\log N)^{a+2}}
\]
\[
\leq C \sum_{n=1}^{N} \frac{(\log n)^a}{r_N (\log N)^{a+2}}
\]
\[
\leq C \log \log N \frac{1}{r_N \log N},
\]
we get
\[
\sum_{N=1}^{\infty} b_N \left( \sum_{n=1}^{N} E X_{N,n}^2 I \left( X_{N,n} \leq \varepsilon \right) \right)^2 < \infty. \tag{2.21}
\]

By Lemma 2.1 and (2.19)-(2.21), we have
\[
\sum_{N=1}^{\infty} b_N P \left( \max_{1 \leq k \leq N} |I_k| \geq r \cdot r_N (\log N)^{a+2} \right) = \sum_{N=1}^{\infty} b_N P \left( \frac{((\log n)^a / (n-j)^{a-1}) R_n I \left( R_n > c_n \right) \geq r}{r_N (\log N)^{a+2}} \right) < \infty. \tag{2.22}
\]

Therefore, the desired result can be obtained by (2.18) and (2.22).

**Proof of Theorem 2.3.** As the proof of Theorem 2.1, we partition \( \sum_{n=1}^{k} ((\log n)^a / n)(R_n - E R_n I(1 \leq R_n \leq c_n)) \) into the following two terms:
\[
\sum_{n=1}^{k} ((\log n)^a / n)(R_n - E R_n I(1 \leq R_n \leq c_n))
\]
\[
= \sum_{n=1}^{k} ((\log n)^a / n)[R_n I(1 \leq R_n \leq c_n) - E R_n I(1 \leq R_n \leq c_n)]
\]
\[
+ \sum_{n=1}^{k} ((\log n)^a / n) R_n I(R_n > c_n)
\]
\[
=: I_k + II_k. \tag{2.23}
\]

For the case \( \Delta = 2, p_n j_n = 1 \) and by Lemma 2.2, the density for the ratio of our order statistics is
\[
f_{R_n}(x) = \frac{p_n (i_n - 1)!}{(i_n - j_n - 1)! (j_n - 1)!} x^{p_n i_n - 1} (1 - x^{-p_n})^{j_n - 1} I(x \geq 1)
\]
\[
\sim j_n x^{-2} (1 - x^{-1/j_n}) I(x \geq 1).
\]

Then it follows that
\[
E R_n^2 I(1 \leq R_n \leq c_n) \sim j_n \int_{1}^{C_n} (1 - x^{-1/j_n}) dx \leq C j_n c_n \sim C n (\log n)^{a+3}.
\]

and
\[
P \left( \max_{1 \leq k \leq N} \sum_{n=1}^{k} ((\log n)^a / n) R_n \bigg| > r_N (\log N)^{a+3} \right)
\]
\[
\leq \sum_{n=1}^{N} \frac{(\log n)^a / n}{r_N^2 (\log N)^{2(a+3)}} \text{Var}[R_n I(1 \leq R_n \leq c_n)] \tag{2.24}
\]
\[
\leq \sum_{n=1}^{N} \frac{(\log n)^a / n}{r_N^2 (\log N)^{2(a+3)}} n (\log n)^{a+3} \leq C \frac{\log \log N}{r_N^2 (\log N)^{2-a}}
\]
where
\[
\hat{R}_n := R_n I(1 \leq R_n \leq c_n) - E R_n I(1 \leq R_n \leq c_n).
\]
From the condition (2.6), we can get
\[ \sum_{N=1}^{\infty} b_N P \left( \max_{1 \leq k \leq N} |I_k| \geq r_N (\log N)^{a+3} \right) < \infty. \] (2.25)

Next, we consider the complete convergence of the term \( II_k \), from using Lemma 2.1. By denoting
\[ X_{N,n} := \frac{((\log n)^a/n) R_n I(R_n > c_n)}{r_N (\log N)^{a+3}}, \]
then for any \( r > 0 \), we have
\[
\sum_{n=1}^{N} P \left( X_{N,n} \geq r \right) \leq \sum_{n=1}^{N} P \left( R_n \geq \max \left\{ c_n, nr \frac{(\log n)^a}{r_N (\log N)^{a+3}} \right\} \right)
\leq C \sum_{n=1}^{N} \max \left\{ \frac{j_n}{c_n}, \frac{(\log n)^a}{nr_N (\log N)^{a+3}} \right\}
\leq C \sum_{n=1}^{N} \max \left\{ \frac{1}{n (\log n)^{3-a}}, \frac{(\log n)^a}{nr_N (\log N)^{a+3}} \right\}
\leq C \log \log N \frac{1}{r_N (\log N)^{2-a}}
\]
which implies
\[ \sum_{n=1}^{\infty} b_N \sum_{n=1}^{N} P \left( X_{N,n} \geq r \right) < \infty. \] (2.26)
Furthermore, for any \( \varepsilon > 0 \), we have
\[
\sum_{n=1}^{N} EX_{N,n} I \left( X_{N,n} \leq \varepsilon \right) \leq C \frac{\log \log N + \log r_N}{r_N (\log N)^{a+3}} \sum_{n=1}^{N} \frac{(\log n)^a}{n}
\leq C \left( \log \log N \frac{(\log N + \log r_N)}{r_N (\log N)^2} \right) \rightarrow 0
\]
as \( N \rightarrow \infty \), since the condition (2.5). Then
\[
\sum_{n=1}^{N} EX_{N,n}^2 I \left( X_{N,n} \leq \varepsilon \right) \leq C \sum_{n=1}^{N} \frac{(\log n)^a}{r_N (\log N)^{a+3}} \leq C \log \log N \frac{1}{r_N (\log N)^{2-a}},
\]
and
\[ \sum_{N=1}^{\infty} b_N \left( \sum_{n=1}^{N} EX_{N,n}^2 I \left( X_{N,n} \leq \varepsilon \right) \right)^2 < \infty. \] (2.28)

By Lemma 2.1 and (2.26)-(2.28), we get
\[ \sum_{N=1}^{\infty} b_N P \left( \max_{1 \leq k \leq N} |II_k| \geq r \cdot r_N (\log N)^{a+3} \right)
\leq \sum_{N=1}^{\infty} b_N P \left( \frac{((\log n)^a/n) R_n I(R_n > c_n)}{r_N (\log N)^{a+3}} \geq r \right) < \infty. \] (2.29)

Therefore, the desired result can be obtained by (2.25) and (2.29).

\section{Corollaries and examples}

In this section, we investigate some corollaries and examples for the complete convergence theorems in Section 2. Adler [10] examined strong laws involving weighted sums of \( \{ R_n, n \geq 1 \} \), and get some unusual limit theorems. Our corollaries show the complete convergence about them. And then, we search some examples to enhance the persuasive of the conclusion for specific \( r_N \) and \( b_N \), \( N \geq 1 \).
Corollary 3.1. Let \( \{b_N, N \geq 1\} \) and \( \{r_N, N \geq 1\} \) satisfy the condition (2.1) and (2.2). If \( p_{n,j_n} = 1, \Delta = 1 \) and \( \alpha > -2 \), then
\[
\sum_{N=1}^{\infty} b_N P \left( \sum_{n=1}^{N} \left( \frac{(\log n)^{\alpha}}{(\log N)^{\alpha+2}} \frac{1}{n} \right) R_n - 1 \right) \geq r_N < \infty, \quad (3.1)
\]

Proof. We can check directly the conditions in the proof of Theorem 2.1 to get the corollary, since
\[
ER_n I(1 \leq R_n \leq c_n) = \log c_n \sim \log n
\]
and
\[
\sum_{N=1}^{N} \left( \frac{(\log n)^{\alpha}}{(\log N)^{\alpha+2}} \right) \frac{1}{n} \sim \frac{1}{\alpha + 2}.
\]

Example 3.1. For \( p_{n,j_n} = 1, \Delta = 1 \) and \( \alpha > -2 \), let \( r_N = N \) and \( b_N = \frac{1}{(\log N)^{\delta}} \) for any \( \delta > 0 \). Then we can get
\[
\sum_{N=1}^{\infty} \frac{1}{(\log N)^{\delta}} \left( \sum_{n=1}^{N} \left( \frac{(\log n)^{\alpha}}{(\log N)^{\alpha+2}} \frac{1}{n} \right) R_n - 1 \right) \geq N < \infty, \quad (3.2)
\]

Example 3.2. For \( p_{n,j_n} = 1, \Delta = 1 \) and \( \alpha > -2 \), let \( r_N = \frac{1}{\log \log N} \) and \( b_N = \frac{1}{N(\log \log N)^{\delta}} \) for any \( \delta > 0 \). Then we can get
\[
\sum_{N=1}^{\infty} \frac{1}{N(\log \log N)^{\delta}} \left( \sum_{n=1}^{N} \left( \frac{(\log n)^{\alpha}}{(\log N)^{\alpha+2}} \frac{1}{n} \right) R_n - 1 \right) \geq \frac{1}{\log \log N} < \infty, \quad (3.3)
\]

Example 3.3. If \( p_{n,j_n} = 1, \Delta = 1 \) and \( \alpha > -2 \), let \( r_N = r \) be a constant and \( b_N = \frac{1}{N(\log \log N)^{\delta}} \), where \( 0 < \delta \leq 1 \). Then for \( r > 0 \), we have
\[
\sum_{N=1}^{\infty} \frac{1}{N(\log \log N)^{\delta}} \left( \sum_{n=1}^{N} \left( \frac{(\log n)^{\alpha}}{(\log N)^{\alpha+2}} \frac{1}{n} \right) R_n - 1 \right) \geq r < \infty, \quad (3.4)
\]

Corollary 3.2. For \( p_{n,j_n} = 1, \Delta = 2 \) and \( \alpha > -2 \), let \( \{b_N, N \geq 1\} \) and \( \{r_N, N \geq 1\} \) satisfy the condition (2.1) and (2.2). If \( j_n = o(\log n) \), then
\[
\sum_{N=1}^{\infty} b_N P \left( \sum_{n=1}^{N} \left( \frac{(\log n)^{\alpha}/(nj_n^{\alpha-1})}{(\log N)^{\alpha+2}} \frac{1}{n} \right) R_n - 1 \right) \geq r_N < \infty, \quad (3.5)
\]

If \( j_n \sim \log n \), then
\[
\sum_{N=1}^{\infty} b_N P \left( \sum_{n=1}^{N} \left( \frac{(\log n)^{\alpha}/(nj_n^{\alpha-1})}{(\log N)^{\alpha+2}} \frac{1}{n} \right) R_n - 1 \right) \geq r_N < \infty, \quad (3.6)
\]

where
\[
\gamma_{\Delta} = 1 + \sum_{k=1}^{\Delta-1} \frac{(-1)^{k} (1 - e^{-k})}{k},
\]

Proof. For the case \( j_n = o(\log n) \) and \( c_n = nj_n^{\alpha-1}(\log n)^2 \), we have
\[
ER_n I(1 \leq R_n \leq c_n) \sim \frac{\log n}{(\Delta - 1)!}
\]
and
\[
\sum_{n=1}^{N} \left( \frac{(\log n)^{\alpha}/(nj_n^{\alpha-1})}{(\log N)^{\alpha+2}} \frac{1}{n} \right) R_n \sim \frac{1}{(\alpha + 2)(\Delta - 1)!}.
\]

By checking directly the conditions in the proof of Theorem 2.2, the conclusion (3.5) can be obtained.
For the case \( j_n \sim \log n \), let \( c_n = n(\log n)^{\Delta+1} \), then it is not difficult to check that \( c_n^{-1/j_n} \to e^{-1} \) as \( n \to \infty \). Hence we get

\[
ER_n I(1 \leq R_n \leq c_n) \sim \frac{j_n^{\Delta-1}}{(\Delta-1)!} \int_1^{c_n} x^{-1} (1 - x^{-1/j_n})^{\Delta-1} dx
\]

\[
= \frac{j_n^{\Delta-1}}{(\Delta-1)!} \int_1^{c_n} x^{-1} \sum_{k=0}^{\Delta-1} \binom{\Delta-1}{k} (-1)^k x^{-k/j_n} dx
\]

\[
= \frac{j_n^{\Delta-1}}{(\Delta-1)!} \left[ \log c_n + \sum_{k=1}^{\Delta-1} \binom{\Delta-1}{k} (-1)^k (k/j_n)^{-1} (1 - c_n^{-k/j_n}) \right]
\]

\[
\sim \frac{\log n}{(\Delta-1)!} \left[ 1 + \sum_{k=1}^{\Delta-1} \frac{1}{k} \binom{-1}{k} (-1)^k (1 - e^{-k}) \right]
\]

\[
= \frac{\gamma_\Delta (\log n)^\Delta}{(\Delta-1)!}
\]

and

\[
\sum_{n=1}^{N}(\log n)^a/(n(j_n^{\Delta-1}))ER_n I(1 \leq R_n \leq c_n) \sim \frac{\sum_{n=1}^{N}(\log n)^a/(n(j_n^{\Delta-1}))\gamma_\Delta (\log n)^\Delta}{(\log N)^{a+2}(\Delta-1)!}
\]

\[
\sim \frac{\gamma_\Delta}{(\alpha+2)(\Delta-1)!}.
\]

By checking the conditions in the proof of Theorem 2.2 where \( j_n \sim \log n \), the conclusion (3.6) can be obtained.

\( \square \)

**Example 3.4.** For \( p_nj_n = 1, \Delta \geq 2 \) and \( \alpha > -2 \), let \( r_N = N \) and \( b_N = \frac{1}{(\log N)^\delta} \) for any \( \delta > 0 \). If \( j_n = o(\log n) \), then

\[
\sum_{N=1}^{\infty} \frac{1}{(\log N)^\alpha} P \left( \frac{\sum_{n=1}^{N} \frac{(\log n)^a}{(n(j_n^{\Delta-1}))R_n}{(\log N)^{a+2}}}{(\alpha+2)(\Delta-1)!} \geq N \right) < \infty.
\]

If \( j_n \sim \log n \), then

\[
\sum_{N=1}^{\infty} \frac{1}{(\log N)^\alpha} P \left( \frac{\sum_{n=1}^{N} \frac{(\log n)^a}{(n(j_n^{\Delta-1}))R_n}{(\log N)^{a+2}}}{(\alpha+2)(\Delta-1)!} \geq N \right) \frac{\gamma_\Delta}{(\alpha+2)(\Delta-1)!} < \infty.
\]

where

\[
\gamma_\Delta = 1 + \sum_{k=1}^{\Delta-1} \frac{(-1)^k (1 - e^{-k})}{k}.
\]

**Example 3.5.** For \( p_nj_n = 1, \Delta \geq 2 \) and \( \alpha > -2 \), let \( r_N = \frac{1}{\log \log N} \) and \( b_N = \frac{1}{N(\log \log N)^\delta} \) for any \( \delta > 0 \). If \( j_n = o(\log n) \), then

\[
\sum_{N=1}^{\infty} \frac{1}{N(\log N)^{\delta+\bar{2}}} P \left( \frac{\sum_{n=1}^{N} \frac{(\log n)^a}{(n(j_n^{\Delta-1}))R_n}{(\log N)^{a+2}}}{(\alpha+2)(\Delta-1)!} \geq \frac{1}{(\log N)} \right) < \infty.
\]

If \( j_n \sim \log n \), then

\[
\sum_{N=1}^{\infty} \frac{1}{N(\log N)^{\delta+\bar{2}}} P \left( \frac{\sum_{n=1}^{N} \frac{(\log n)^a}{(n(j_n^{\Delta-1}))R_n}{(\log N)^{a+2}}}{(\alpha+2)(\Delta-1)!} \geq \frac{1}{(\log N)} \frac{\gamma_\Delta}{(\alpha+2)(\Delta-1)!} \right) < \infty.
\]
Example 3.6. Let \( p_n j_n = 1, \Delta > 2 \) and \( \alpha > -2 \). Let \( r_N = r \) be a constant and \( b_N = \frac{1}{N \log N^\delta} \), where \( 0 < \delta \leq 1 \). If \( j_n = o(\log n) \), then for \( r > 0 \),
\[
\sum_{N=1}^{\infty} \frac{1}{N \log N^\delta} P \left( \left| \sum_{n=1}^{N} \frac{(\log n)^a / (n j_n^{-1}) R_n}{(\log N)^a+2} - \frac{1}{(\alpha + 2)N^{\Delta - 1}} \right| \geq r \right) < \infty.
\]
If \( j_n \sim \log n \), then
\[
\sum_{N=1}^{\infty} \frac{1}{N \log N^\delta} P \left( \left| \sum_{n=1}^{N} \frac{(\log n)^a / (n j_n^{-1}) R_n}{(\log N)^a+2} - \frac{1}{(\alpha + 2)N^{\Delta - 1}} \right| \geq r \right) < \infty.
\]

Corollary 3.3. Let \( \{b_N, N \geq 1\} \) and \( \{r_N, N \geq 1\} \) satisfy the condition (2.5) and (2.6). If \( p_n j_n = 1, j_n \sim (\log n)^a \), where \( 1 < a < 2, \Delta = 2 \) and \( \alpha > -3 \), then
\[
\sum_{N=1}^{\infty} b_N P \left( \left| \sum_{n=1}^{N} \frac{(\log n)^a / n R_n}{(\log N)^a+3} - \frac{1}{2(\alpha + 3)} \right| \geq r_N \right) < \infty.
\]

Proof. We can check directly the conditions in the proof of Theorem 2.3 to get the corollary. Since
\[
ER_n I(1 \leq R_n \leq c_n) \sim j_n \int_{1}^{c_n} x^{-1}(1 - x^{-1/\Delta}) dx
\]
\[
= j_n \int_{1}^{c_n} (x^{-1} - x^{-1/\Delta}) dx
\]
\[
= j_n \left[ \log c_n + j_n(c_n^{-1/\Delta} - 1) \right]
\]
\[
\sim (\log n)^a (\log n + 3 \log \log n) + (\log n)^{2a}((\log(\log n)^3)^{-1/(\log n)^a} - 1),
\]
by Lemma 2.3 with \( t = \log n \), we have
\[
ER_n I(1 \leq R_n \leq c_n) \sim t^a(t + 3 \log t) + t^{2a}[(e^{t^{3}})^{-1/\alpha} - 1]
\]
\[
= t^{2a}[1 + \log \log n] + t^{2a}[(e^{t^{3}})^{-1/\alpha} - 1]
\]
\[
\sim t^2/2 = (\log n)^2/2.
\]
Thus we have
\[
\frac{\sum_{n=1}^{N} ((\log n)^a / n) ER_n I(1 \leq R_n \leq c_n)}{(\log N)^{a+3}} \sim \frac{1}{2(\alpha + 3)}.
\]
Then the conclusion can be obtained.

Example 3.7. For \( p_n j_n = 1, j_n \sim (\log n)^a \), where \( 1 < a < 2, \Delta = 2 \) and \( \alpha > -3 \), let \( r_N = N \) and \( b_N = \frac{1}{N \log N^\delta} \) for any \( \delta > a - 1 \). Then we have
\[
\sum_{N=1}^{\infty} \frac{1}{N \log N^\delta} P \left( \left| \sum_{n=1}^{N} \frac{(\log n)^a / n R_n}{(\log N)^{a+3}} - \frac{1}{2(\alpha + 3)} \right| \geq N \right) < \infty.
\]

Example 3.8. For \( p_n j_n = 1, j_n \sim (\log n)^a \), where \( 1 < a < 2, \Delta = 2 \) and \( \alpha > -3 \), let \( r_N = \frac{1}{N \log \log N} \) and \( b_N = \frac{1}{N \log N^\delta} \) for any \( \delta > 0 \). Then we have
\[
\sum_{N=1}^{\infty} \frac{1}{N \log N^\delta} P \left( \left| \sum_{n=1}^{N} \frac{(\log n)^a / n R_n}{(\log N)^{a+3}} - \frac{1}{2(\alpha + 3)} \right| \geq \frac{1}{\log \log N} \right) < \infty.
\]

Example 3.9. If \( p_n j_n = 1, j_n \sim (\log n)^a \), where \( 1 < a < 2, \Delta = 2 \) and \( \alpha > -3 \), let \( r_N = r \) be a constant and \( b_N = \frac{1}{N \log N^\delta} \), where \( a - 1 < \delta \leq 1 \). Then for \( r > 0 \), we have
\[
\sum_{N=1}^{\infty} \frac{1}{N \log N^\delta} P \left( \left| \sum_{n=1}^{N} \frac{(\log n)^a / n R_n}{(\log N)^{a+3}} - \frac{1}{2(\alpha + 3)} \right| \geq r \right) < \infty.
\]
4 Conclusion

The work examines some limit theorems for the order statistics from the Pareto distribution proposed in current work, and investigates the complete convergence for the ratios of the order statistics. In this paper, we firstly get the complete convergence of the weighted sums with different forms by discussing three different cases. Then we obtain some corollaries. Some examples are also given to support the conclusion. There are more relevant properties regarding order statistics that will be investigated by us in the future.

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References

[1] Smid B., Stam A.J., Convergence in distribution of quotients of order statistics, Stochastic Processes Appl., 1975, 3, 287-292
[2] O’Brien G.L., A limit theorem for sample maxima and heavy branches in Galton-Watson trees, J. Appl. Probab., 1980, 17, 539-545
[3] Balakrishnan N., Stepanov A., Asymptotic properties of the ratio of order statistics, Statist. Probab. Lett., 2008, 78, 301-310
[4] Malik H., Trudel R., Probability density function of quotient of order statistics from the Pareto, power and Weibull distributions, Comm. Statist. A-Theory Methods, 1982, 11, 801-814
[5] Johnson N. L., Kotz S., Balakrishnan N., Continuous univariate distributions, 1994, John Wiley & Sons, Inc., New York.
[6] Mann N.R., Schafer R.E., Singpurwalla N. D., Methods for statistical analysis of reliability and life data, 1974, John Wiley & Sons, New York-London-Sydney.
[7] Miao Y., Wang Y.L., Ma X.W., Some limit results for Pareto random variables, Comm. Statist. Theory Methods, 2013, 42, 4384-4391
[8] Adler A., Limit theorems for arrays of ratios of order statistics, Bull. Inst. Math. Acad. Sinica., 2005, 33, 327-344
[9] Adler A., Exact laws for sums of ratios of order statistics from the Pareto distribution, Cent. Eur. J. Math., 2006, 4, 1-4
[10] Adler A., Unusual strong laws for arrays of ratios of order statistics, Braz. J. Probab. Stat., 2011, 25, 34-43
[11] Miao Y., Sun Y., Wang R.J., Dong M.R., Various limit theorems for ratios from the uniform distribution, Open Math., 2016, 4, 393-403
[12] Miao Y., Wang R.J., Adler A., Limit theorems for order statistics from exponentials, Statist. Probab. Lett., 2016, 110, 51-57
[13] Sung S.H., Volodin A.I., Hu T.C., More on complete convergence for arrays, Statist. Probab. Lett., 2005, 71, 303-311
[14] Hsu P.L., Robbins H., Complete convergence and the law of large numbers, Proc. Nat. Acad. Sci. U. S. A., 1947, 33, 25-31