More examples of pseudosymmetric braided categories

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Abstract
We study some examples of braided categories and quasitriangular Hopf algebras and decide which of them is pseudosymmetric, respectively pseudotriangular. We show also that there exists a universal pseudosymmetric braided category.

Introduction

Braided categories have been introduced by Joyal and Street in [4] as natural generalizations of symmetric categories. Roughly speaking, a braided category is a category that has a tensor product with a nice commutation rule. More precisely, for every two objects $U$ and $V$ we have an isomorphism $c_{U,V} : U \otimes V \to V \otimes U$ that satisfies certain conditions. These conditions are chosen in such a way that for every object $V$ in the category there exists a natural way to construct a representation for the braid group $B_n$ on $V^\otimes n$, therefore the name braided categories. If we impose the extra condition $c_{V,U}c_{U,V} = id_{U \otimes V}$ for all objects $U, V$ in the category, we recover the definition of symmetric categories. It is well known that symmetric categories can be used to construct representations for the symmetric group $\Sigma_n$.

Pseudosymmetric categories are a special class of braided categories and have been introduced in [9]. The motivation was the study of certain categorical structures called twines, strong twines and pure-braided structures (introduced in [1], [8] and [13]). A braiding on a strict monoidal category is called pseudosymmetric if it satisfies a sort of modified braid relation; any symmetric braiding is pseudosymmetric. One of the most intriguing results obtained in [9] was that the category of Yetter-Drinfeld modules over a Hopf algebra $H$ is pseudosymmetric if and only if $H$ is commutative and cocommutative. We proved in [10] that pseudosymmetric categories can be used to construct representations for the group $P\Sigma_n = [\beta_n/P_n]$; the quotient of the braid group by the commutator subgroup of the pure braid group. There exists also a Hopf algebraic analogue of pseudosymmetric braidings: a quasitriangular structure on a Hopf algebra is called pseudotriangular if it satisfies a sort of modified quantum Yang-Baxter equation.

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In this paper we tie some lose ends from \cite{9} and \cite{10}. We study more examples of braided categories and quasitriangular Hopf algebras and decide when they are pseudosymmetric, respectively pseudotriangular. Namely, we prove that the canonical braiding of the category $\mathcal{LR}(H)$ of Yetter-Drinfeld-Long bimodules over a Hopf algebra $H$ (introduced in \cite{11}) is pseudosymmetric if and only if $H$ is commutative and cocommutative. We show that any quasitriangular structure on the $4\nu$-dimensional Radford’s Hopf algebra $H_\nu$ (introduced in \cite{12}) is pseudotriangular. We analyze the positive quasitriangular structures $R(\xi, \eta)$ on a Hopf algebra with positive bases $H(G; G_+, G_-)$ (as defined in \cite{6}, \cite{7}), where $\xi, \eta$ are group homomorphisms from $G_+$ to $G_-$, and we present a list of necessary and sufficient conditions for $R(\xi, \eta)$ to be pseudotriangular. If $R(\xi, \eta)$ is normal (i.e. if $\xi$ is trivial) these conditions reduce to the single relation $\eta(uv) = \eta(vu)$ for all $u, v \in G_+$.

In the last section we recall the pseudosymmetric braided category $\mathcal{PS}$ introduced in \cite{10} and we show that it is a universal pseudosymmetric category. More precisely, we prove that it satisfies two universality properties similar to the ones satisfied by the universal braid category $\mathcal{B}$ (see \cite{5}).

1 Preliminaries

We work over a base field $k$. All algebras, linear spaces, etc, will be over $k$; unadorned $\otimes$ means $\otimes_k$. For a Hopf algebra $H$ with comultiplication $\Delta$ we denote $\Delta(h) = h_1 \otimes h_2$, for $h \in H$. For terminology concerning Hopf algebras and monoidal categories we refer to \cite{5}.

Definition 1.1 (\cite{9}) Let $C$ be a strict monoidal category and $c$ a braiding on $C$. We say that $c$ is pseudosymmetric if the following condition holds, for all $X, Y, Z \in C$:

$$(c_{Y,Z} \otimes id_X)(id_Y \otimes c_{-1,X,Y}^{-1})(c_{X,Y} \otimes id_Z) = (id_Z \otimes c_{X,Y})(c_{-1,X,Y}^{-1} \otimes id_Y)(id_X \otimes c_{Y,Z}).$$

In this case we say that $C$ is a pseudosymmetric braided category.

Proposition 1.2 (\cite{9}) Let $C$ be a strict monoidal category and $c$ a braiding on $C$. Then $c$ is pseudosymmetric if and only if the family $T_{X,Y} := c_{Y,X,Y} c_{X,Y} : X \otimes Y \to X \otimes Y$ satisfies the condition $(T_{X,Y} \otimes id_Z)(id_X \otimes T_{Y,Z}) = (id_X \otimes T_{Y,Z})(T_{X,Y} \otimes id_Z)$ for all $X, Y, Z \in C$.

Definition 1.3 (\cite{9}) Let $H$ be a Hopf algebra and $R \in H \otimes H$ a quasitriangular structure. Then $R$ is called pseudotriangular if $R_{12} R_{31}^{-1} R_{23} = R_{23} R_{31}^{-1} R_{12}$.

Proposition 1.4 (\cite{9}) Let $H$ be a Hopf algebra and let $R$ be a quasitriangular structure on $H$. Then $R$ is pseudotriangular if and only if the element $F = R_{21} R \in H \otimes H$ satisfies the relation $F_{12} F_{23} = F_{23} F_{12}$.

2 Yetter-Drinfeld-Long bimodules

For a braided monoidal category $C$ with braiding $c$, let $C^{in}$ be equal to $C$ as a monoidal category, with the mirror-reversed braiding $c_{M,N} := c_{N,M}^{-1}$, for all objects $M, N \in C$. Directly from the definition of a pseudosymmetric braiding, we immediately obtain:

Proposition 2.1 Let $C$ be a strict braided monoidal category. Then $C$ is pseudosymmetric if and only if $C^{in}$ is pseudosymmetric.
Let $H$ be a Hopf algebra with bijective antipode $S$. Consider the category $H\mathcal{YD}^H$ of left-right Yetter-Drinfeld modules over $H$, whose objects are vector spaces $M$ that are left $H$-modules (denote the action by $h \otimes m \mapsto h \cdot m$) and right $H$-comodules (denote the coaction by $m \mapsto m(0) \otimes m(1) \in M \otimes H$) satisfying the compatibility condition

$$(h \cdot m)(0) \otimes (h \cdot m)(1) = h_2 \cdot m(0) \otimes h_3 m(1) S^{-1}(h_1), \quad \forall \ h \in H, \ m \in M.$$ 

It is a monoidal category, with tensor product given by

$$h \cdot (m \otimes n) = h_1 \cdot m \otimes h_2 \cdot n, \quad (m \otimes n)(0) \otimes (m \otimes n)(1) = m(0) \otimes n(0) \otimes n(1)m(1).$$

Moreover, it has a (canonical) braiding given by

$$c_{M,N} : M \otimes N \to N \otimes M, \quad c_{M,N}(m \otimes n) = n(0) \otimes n(1) \cdot m,$$

$$c_{M,N}^{-1} : N \otimes M \to M \otimes N, \quad c_{M,N}^{-1}(n \otimes m) = S(n(1)) \cdot m \otimes n(0).$$

Consider also the category $H\mathcal{YD}$ of left-left Yetter-Drinfeld modules over $H$, whose objects are vector spaces $M$ that are left $H$-modules (denote the action by $h \otimes m \mapsto h \cdot m$) and left $H$-comodules (denote the coaction by $m \mapsto m^{(-1)} \otimes m^{(0)} \in H \otimes M$) with compatibility condition

$$(h_1 \cdot m)^{(-1)} h_2 \otimes (h_1 \cdot m)^{(0)} = h_1 m^{(-1)} \otimes h_2 \cdot m^{(0)}, \quad \forall \ h \in H, \ m \in M.$$ 

It is a monoidal category, with tensor product given by

$$h \cdot (m \otimes n) = h_1 \cdot m \otimes h_2 \cdot n, \quad (m \otimes n)^{(-1)} \otimes (m \otimes n)^{(0)} = m^{(-1)} n^{(-1)} \otimes m^{(0)} \otimes n^{(0)}.$$ 

Moreover, it has a (canonical) braiding given by

$$c_{M,N} : M \otimes N \to N \otimes M, \quad c_{M,N}(m \otimes n) = m^{(-1)} \cdot n \otimes m^{(0)},$$

$$c_{M,N}^{-1} : N \otimes M \to M \otimes N, \quad c_{M,N}^{-1}(n \otimes m) = m^{(0)} \otimes S^{-1}(m^{(-1)}) \cdot n.$$ 

**Proposition 2.2** ([2]) For the categories $H\mathcal{YD}^H$ and $H\mathcal{YD}$ with braidings as above, we have an isomorphism of braided monoidal categories $(H\mathcal{YD}^H)^{\text{in}} \simeq H\mathcal{YD}$. 

**Proposition 2.3** ([2]) The canonical braiding of $H\mathcal{YD}^H$ is pseudosymmetric if and only if $H$ is commutative and cocommutative.

As a consequence of Propositions 2.1, 2.2 and 2.3 we obtain:

**Proposition 2.4** The canonical braiding of $H\mathcal{YD}^H$ is pseudosymmetric if and only if $H$ is commutative and cocommutative.

We recall now the braided monoidal category $\mathcal{LR}(H)$ defined in [11]. The objects of $\mathcal{LR}(H)$ are vector spaces $M$ endowed with $H$-bimodule and $H$-bicomodule structures (denoted by $h \otimes m \mapsto h \cdot m, m \otimes h \mapsto m \cdot h, m \mapsto m^{(-1)} \otimes m^{(0)}, m \mapsto m^{<0>} \otimes m^{<1>}$, for all $h \in H, m \in M$), such that $M$ is a left-left Yetter-Drinfeld module, a left-right Long module, a right-right Yetter-Drinfeld module and a right-left Long module, i.e. (for all $h \in H, m \in M$):

$$(h_1 \cdot m)^{(-1)} h_2 \otimes (h_1 \cdot m)^{(0)} = h_1 m^{(-1)} \otimes h_2 \cdot m^{(0)}, \quad (2.1)$$
Moreover, Proof.

LR Yetter-Drinfeld conditions appearing in the definition of \( X, Y, Z \) is (2.1) and (2.3) become respectively

\[
(h \cdot m)^{<0>} \otimes (h \cdot m)^{<1>} = h \cdot m^{<0>} \otimes m^{<1>},
\]

\[
(m \cdot h_2)^{<0>} \otimes h_1 (m \cdot h_2)^{<1>} = m^{<0>} \cdot h_1 \otimes m^{<1>} h_2,
\]

\[
(m \cdot h)^{(\!-\!1)} \otimes (m \cdot h)^{(0)} = m^{(\!-\!1)} \otimes m^{(0)} \cdot h.
\]

Conversely, assume that \( H \) is \( H \)-bilinear \( H \)-bicomodule and \( H \)-bimodule maps, if \( k \) with unit \( \text{id} \) then \( M \otimes N \in \mathcal{LR}(H) \) as follows (for all \( m \in M, n \in N, h \in H \)):

\[
h \cdot (m \otimes n) = h_1 \cdot m \otimes h_2 \cdot n, \quad (m \otimes n) \cdot h = m \cdot h_1 \otimes n \cdot h_2,
\]

\[
(m \otimes n)^{(\!-\!1)} \otimes (m \otimes n)^{(0)} = m^{(\!-\!1)} n^{(\!-\!1)} \otimes (m^{(0)} \otimes n^{(0)}),
\]

\[
(m \otimes n)^{<0>} \otimes (m \otimes n)^{<1>} = (m^{<0>} \otimes n^{<0>}) \otimes m^{<1>} n^{<1>}.
\]

Moreover, \( \mathcal{LR}(H) \) has a (canonical) braiding defined, for \( M, N \in \mathcal{LR}(H), m \in M, n \in N, \) by

\[
c_{M,N} : M \otimes N \to N \otimes M, \quad c_{M,N}(m \otimes n) = m^{(\!-\!1)} \cdot n^{<0>} \otimes m^{(0)} \cdot n^{<1>},
\]

\[
c_{M,N}^{-1} : N \otimes M \to M \otimes N, \quad c_{M,N}^{-1}(n \otimes m) = m^{(0)} \cdot S^{-1}(n^{<1>}) \otimes S^{-1}(m^{(\!-\!1)}) \cdot n^{<0>}.
\]

**Proposition 2.5** The canonical braiding of \( \mathcal{LR}(H) \) is pseudosymmetric if and only if \( H \) is commutative and cocommutative.

Proof. Assume that the canonical braiding of \( \mathcal{LR}(H) \) is pseudosymmetric. As noted in \( HYD \) with its canonical braiding is a braided subcategory of \( \mathcal{LR}(H) \), so the canonical braiding of \( HYD \) is pseudosymmetric; by Proposition 2.4 it follows that \( H \) is commutative and cocommutative. Conversely, assume that \( H \) is commutative and cocommutative. Then one can see that the two Yetter-Drinfeld conditions appearing in the definition of \( \mathcal{LR}(H) \) become Long conditions, that is (2.1) and (2.3) become respectively

\[
(h \cdot m)^{(\!-\!1)} \otimes (h \cdot m)^{(0)} = m^{(\!-\!1)} \otimes h \cdot m^{(0)},
\]

\[
(m \cdot h)^{<0>} \otimes (m \cdot h)^{<1>} = m^{<0>} \cdot h \otimes m^{<1>}.
\]

Let now \( X, Y, Z \in \mathcal{LR}(H) \); we compute, for \( x \in X, y \in Y, z \in Z \):

\[
(c_{Y,Z} \otimes \text{id}_X)(\text{id}_X \otimes c_{Z,X}^{-1})(c_{X,Y} \otimes \text{id}_Z)(x \otimes y \otimes z)
\]

\[=
(c_{Y,Z} \otimes \text{id}_X)(\text{id}_Y \otimes c_{Z,X}^{-1})(x^{(\!-\!1)} \cdot y^{<0>} \otimes x^{(0)} \cdot y^{<1>} \otimes z)
\]

\[=
(c_{Y,Z} \otimes \text{id}_X)(x^{(\!-\!1)} \cdot y^{<0>} \otimes z^{(0)} \cdot S^{-1}(x^{(0)} \cdot y^{<1>})^{<1>})
\]

\[\otimes S^{-1}(z^{(\!-\!1)}) \cdot (x^{(0)} \cdot y^{<1>}^{<0>})
\]

\[\overset{2.6}{=}\]

\[=
(c_{Y,Z} \otimes \text{id}_X)(x^{(\!-\!1)} \cdot y^{<0>} \otimes z^{(0)} \cdot S^{-1}(x^{(0)} \otimes y^{<0>}) \otimes S^{-1}(z^{(\!-\!1)} \cdot x^{(0)} \cdot y^{<0>}) \cdot y^{<1>})
\]

\[\overset{2.5, 2.6}{=}\]

\[=
y^{<0>}(z^{(0)} \cdot x^{(0)} \cdot y^{<0>}) \cdot S^{-1}(x^{(0)} \otimes y^{<0>}) \otimes x^{(\!-\!1)} \cdot y^{<0> \!} \cdot z^{(\!-\!1)} \cdot x^{(\!-\!0)} \cdot y^{<1>},
\]

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(id_Z \otimes c_{X,Y})(c_{Z,X}^{-1} \otimes id_Y)(id_X \otimes c_{Y,Z})(x \otimes y \otimes z)

= (id_Z \otimes c_{X,Y})(c_{Z,X}^{-1} \otimes id_Y)(x \otimes y^{(-1)} \cdot z^{<0>} \otimes y^{(0)} \cdot z^{<1>})

= (id_Z \otimes c_{X,Y})(y^{(-1)} \cdot z^{<0>} \cdot S^{-1}(x^{<1>}) \otimes x^{<0>} \otimes y^{(0)} \cdot z^{<1>})

\leq (id_Z \otimes c_{X,Y})(y^{(-1)} \cdot z^{<0>} \cdot y^{-1}(x^{<1>}) \otimes S^{-1}(z^{<0>} \cdot x^{<1>}) \cdot x^{<0>} \otimes y^{(0)} \cdot z^{<1>})

\leq (id_Z \otimes c_{X,Y})(y^{(-1)} \cdot z^{<0>} \cdot y^{-1}(x^{<1>}) \otimes S^{-1}(z^{<0>} \cdot x^{<1>}) \cdot x^{<0>} \otimes y^{(0)} \cdot z^{<1>})

= \sum_{i,l=0}^{2\nu-1} \omega^{-il} g^i \otimes g^{sl} + \beta \sum_{i,l=0}^{2\nu-1} \omega^{-il} g^i x \otimes g^{sl+\nu} x.

It was also proved in [12] that \( R_{s,\beta} \) is triangular if and only if \( s = \nu \).

Following [12], we introduce an alternative description of \( R_{s,\beta} \), more appropriate for our purpose. For every natural number \( 0 \leq l \leq 2\nu - 1 \), we define

\[ e_l = \frac{1}{2\nu} \sum_{i=0}^{2\nu-1} \omega^{-il} g^i, \]

regarded as an element in the group algebra of the cyclic group of order \( 2\nu \) generated by the element \( g \) (which in turn may be regarded as a Hopf subalgebra of \( H_\nu \) in the obvious way). Then, by [12], the following relations hold:

\[ 1 = e_0 + e_1 + \ldots + e_{2\nu-1}, \]
\[ e_i e_j = \delta_{ij} e_i, \]
\[ g^i e_j = \omega^{ij} e_j, \]

for all \( 0 \leq i, j \leq 2\nu - 1 \). Also, a straightforward computation shows that we have
\[
\sum_{i=0}^{2\nu-1} (-1)^i e_i = g^\nu.
\]

Note also that, since \( \omega \) is a primitive \( 2\nu^{th} \) root of unity, we have
\[ \omega^\nu = -1. \]

With this notation, the quasitriangular structure \( R_{s,\beta} \) may be expressed (cf. [12]) as
\[
R_{s,\beta} = \sum_{l=0}^{2\nu-1} e_l \otimes g^{sl} + \beta \left( \sum_{l=0}^{2\nu-1} e_l x \otimes g^{s\nu+l} x \right).
\]

We are interested to see for what \( s, \beta \) is \( R_{s,\beta} \) pseudotriangular. We note first that for \( \beta = 0 \), \( R_{s,0} \) is actually a quasitriangular structure on the group algebra of the cyclic group of order \( 2\nu \), which is a commutative Hopf algebra, so \( R_{s,0} \) is pseudotriangular.

Consider now \( R_{s,\beta} \) an arbitrary quasitriangular structure on \( H_\nu \). We need to compute first \( (R_{s,\beta})_{21} R_{s,\beta} \). By using the defining relations \( x^2 = 0 \) and \( gx + xg = 0 \), the properties of the elements \( e_l \) listed above and the fact that \( s \) and \( \nu \) are odd numbers, a straightforward computation yields:
\[
(R_{s,\beta})_{21} R_{s,\beta} = \sum_{l,t=0}^{2\nu-1} \omega^{2sl+2st} e_l \otimes e_t + \beta \left( \sum_{l,t=0}^{2\nu-1} \omega^{2sl+2s\nu+t} e_l x \otimes e_t x \right)
\]
\[- \beta \left( \sum_{l,t=0}^{2\nu-1} (-1)^{l+t} \omega^{2sl+2\nu+l} x e_l \otimes e_t x \right).
\]

Let us denote this element by \( T \). We need to compare \( T_{12} T_{23} \) and \( T_{23} T_{12} \), so we first compute them, using repeatedly the defining relations of \( H_\nu \) and the properties of the elements \( e_l \):
\[
T_{12} T_{23} = \sum_{l,t,j=0}^{2\nu-1} \omega^{2sl+2stj} e_l \otimes e_t \otimes e_j + \beta \left( \sum_{l,t,i,j=0}^{2\nu-1} \omega^{2sl+2sij+\nu j} e_l \otimes e_t e_i \otimes e_j x \right)
\]
\[- \sum_{l,t,i,j=0}^{2\nu-1} (-1)^{l+j} \omega^{2sl+2sij+\nu i} e_l \otimes e_t xe_i \otimes e_j x + \sum_{l,t,i,j=0}^{2\nu-1} \omega^{2sl+2sij+\nu t} e_l x \otimes e_t xe_i \otimes e_j
\]
\[- \sum_{l,t,i,j=0}^{2\nu-1} (-1)^{l+t} \omega^{2sl+2sij+\nu l} x e_l \otimes e_t xe_i \otimes e_j \]
\[
= \sum_{l,t,j=0}^{2\nu-1} \omega^{2sl+2stj} e_l \otimes e_t \otimes e_j + \beta \left( \sum_{l,t,j=0}^{2\nu-1} (-1)^j \omega^{2sl+2stj} e_l \otimes e_t x \otimes e_j x \right)
\]
\[- \sum_{l,t,i,j=0}^{2\nu-1} (-1)^j \omega^{2sl+2sij} e_l \otimes e_t xe_i \otimes e_j x + \sum_{l,t,i,j=0}^{2\nu-1} (-1)^t \omega^{2sl+2sij} e_l x \otimes e_t xe_i \otimes e_j \]
\[
\begin{align*}
&\quad - \sum_{l,t,i,j=0}^{2\nu-1} (-1)^l \omega^{2slt+2sij} x_{e_l} \otimes e_t x_{e_i} \otimes e_j \\
&= \sum_{l,t,j=0}^{2\nu-1} \omega^{2stlj} e_l \otimes e_t \otimes e_j + \beta \left( \sum_{l,t=0}^{2\nu-1} (-1)^l \omega^{2stlt} e_l \otimes e_t x \otimes g^{2st} e_j x \right) \\
&\quad - \sum_{l,t,i,j=0}^{2\nu-1} (-1)^l g^{2st} e_l \otimes e_t x e_i \otimes g^{2si} e_j x + \sum_{l,t,i=0}^{2\nu-1} \omega^{2sij} e_l x \otimes g^{2stl} e_t x e_i \otimes e_j \\
&\quad - \sum_{l,t,i,j=0}^{2\nu-1} (-1)^l \omega^{2sij} x e_l \otimes g^{2stl} e_t x e_i \otimes e_j \\
&= \sum_{l,t,j=0}^{2\nu-1} \omega^{2stlj} e_l \otimes e_t \otimes e_j + \beta \left( \sum_{l,t=0}^{2\nu-1} g^{2stl} e_l \otimes e_t x \otimes g^{2stl+\nu} x \right) \\
&\quad - \sum_{l,t,i=0}^{2\nu-1} g^{2stl} e_t x e_i \otimes g^{2si+\nu} x + \sum_{l,i,j=0}^{2\nu-1} e_l x \otimes g^{2stl+2sij+\nu} x e_i \otimes e_j \\
&\quad - \sum_{l,i,j=0}^{2\nu-1} x e_l \otimes g^{2stl+2sij+\nu} x e_i \otimes e_j \\
&= \sum_{l,t,j=0}^{2\nu-1} \omega^{2stlj} e_l \otimes e_t \otimes e_j + \beta \left( \sum_{l,t=0}^{2\nu-1} g^{2stl} e_t x \otimes g^{2stl+\nu} x \right) \\
&\quad - \sum_{l,i=0}^{2\nu-1} g^{2stl} e_t x e_i \otimes g^{2si+\nu} x + \sum_{l,j=0}^{2\nu-1} e_l x \otimes g^{2stl+2sij+\nu} x \otimes e_j \\
&\quad - \sum_{l,j=0}^{2\nu-1} x e_l \otimes g^{2stl+2sij+\nu} x \otimes e_j, \\

T_{23} T_{12} &= \sum_{l,t,j=0}^{2\nu-1} \omega^{2stlt+2stlj} e_l \otimes e_t \otimes e_j + \beta \left( \sum_{l,t,j=0}^{2\nu-1} (-1)^l \omega^{2stlt+2stlj} e_l x \otimes e_t x \otimes e_j \right) \\
&\quad - \sum_{l,t,j=0}^{2\nu-1} (-1)^l \omega^{2stlt+2stlj} e_l x e_t \otimes e_j x + \sum_{l,t,i,j=0}^{2\nu-1} (-1)^i \omega^{2stlt+2stlj} e_l \otimes e_t x e_i \otimes e_j x \\
&\quad - \sum_{l,t,j=0}^{2\nu-1} (-1)^j \omega^{2stlt+2stlj} e_l \otimes x e_t \otimes e_j x \end{align*}
\]
\[ T_{12}T_{23} - T_{23}T_{12} = \beta \left( \sum_{t=0}^{2\nu-1} g^{2st} \otimes e_t x \otimes g^{2st+\nu} x - \sum_{t,i=0}^{2\nu-1} g^{2st} \otimes e_t x e_i \otimes g^{2si+\nu} x \right. \\
\left. - \sum_{t,i=0}^{2\nu-1} g^{2si} \otimes e_t x e_i \otimes g^{2st+\nu} x + \sum_{t=0}^{2\nu-1} g^{2st} \otimes x e_t \otimes g^{2st+\nu} x \right) \]

Thus, we can see that we have

\[ x e_l = e_{l-\nu} x, \]

for all \( 0 \leq l \leq 2\nu - 1 \), where the subscripts are taken mod \( 2\nu \). We use the following facts:

\[ \omega^\nu = -1, \quad x g^i = (-1)^i g^i x = \omega^{\nu i} g^i x. \]

We have:

\[ x e_l = \frac{1}{2\nu} \sum_{i=0}^{2\nu-1} \omega^{-il} g^i \]
Now we compute:

\[ \sum_{t,i=0}^{2\nu-1} g^{2st} \otimes e_t x e_i \otimes g^{2^{i+\nu}} x = \sum_{t,i=0}^{2\nu-1} g^{2st} \otimes e_t e_{i-\nu} x \otimes g^{2^{i+\nu}} x \]

\[ = \sum_{t,i=0}^{2\nu-1} g^{2st} \otimes \delta_{t,i-\nu} e_t x \otimes g^{2^{i+\nu}} x \]

\[ = \sum_{t=0}^{2\nu-1} g^{2st} \otimes e_t x \otimes g^{2s(t+\nu)+\nu} x \]

\[ = \sum_{t=0}^{2\nu-1} g^{2st} \otimes e_t x \otimes g^{2^{s+\nu}} g^{2^{t+\nu}} x \]

\[ = \sum_{t=0}^{2\nu-1} g^{2st} \otimes e_t x \otimes g^{2^{t+\nu}} x, \]

so we have \( \sum_{t=0}^{2\nu-1} g^{2st} \otimes e_t x \otimes g^{2^{s+\nu}} x - \sum_{t,i=0}^{2\nu-1} g^{2st} \otimes e_t x e_i \otimes g^{2^{i+\nu}} x = 0 \). Similarly, we have:

\[ \sum_{t,i=0}^{2\nu-1} g^{2si} \otimes e_t x e_i \otimes g^{2^{s+\nu}} x = \sum_{t,i=0}^{2\nu-1} g^{2si} \otimes e_{t+\nu} e_i \otimes g^{2^{s+\nu}} x \]

\[ = \sum_{t,i=0}^{2\nu-1} g^{2si} \otimes x \delta_{t+\nu,i} e_i \otimes g^{2^{s+\nu}} x \]

\[ = \sum_{i=0}^{2\nu-1} g^{2si} \otimes x e_i \otimes g^{2s(i-\nu)+\nu} x \]

\[ = \sum_{i=0}^{2\nu-1} g^{2si} \otimes x e_i \otimes g^{2^{s+\nu}} x, \]

so we have \( \sum_{t=0}^{2\nu-1} g^{2si} \otimes x e_t \otimes g^{2^{s+\nu}} x - \sum_{t,i=0}^{2\nu-1} g^{2si} \otimes e_t x e_i \otimes g^{2^{s+\nu}} x = 0 \). Consequently, we have \( T_{12} T_{23} - T_{23} T_{12} = 0 \), and so we obtained:
The relations between these actions and the decompositions of two subgroups $G = G_+ G_-$ of $G$ have been classified in [6], [7] as follows.

A basis of a Hopf algebra over $\mathbb{C}$ is called positive if all the structure constants (for the unit, counit, multiplication, comultiplication and antipode) with respect to this basis are nonnegative real numbers. The finite dimensional Hopf algebras having a positive basis and the positive quasitriangular structures on them have been classified in [6], [7] as follows:

Let $G$ be a group (we denote by $e$ its unit). A unique factorization $G = G_+ G_-$ of $G$ consists of two subgroups $G_+$ and $G_-$ of $G$ such that any $g \in G$ can be written uniquely as $g = g_+ g_-$, with $g_+ \in G_+$ and $g_- \in G_-$. By considering the inverse map, we can also write uniquely $g = \overline{g}_- \overline{g}_+$, with $\overline{g}_- \in G_-$ and $\overline{g}_+ \in G_+$.

Let $u \in G_+$, $x \in G_-$; then we can write uniquely

$$ xu = (x u)(x^u), \text{ with } x u \in G_+ \text{ and } x^u \in G_-, $$
$$ ux = (u x)(u^x), \text{ with } u x \in G_- \text{ and } u^x \in G_+. $$

So, we have the following actions of $G_+$ and $G_-$ on each other (from left and right):

$$ G_- \times G_+ \rightarrow G_+ \text{, } (x, u) \mapsto x u, $$
$$ G_- \times G_+ \rightarrow G_- \text{, } (x, u) \mapsto x^u, $$
$$ G_+ \times G_- \rightarrow G_- \text{, } (u, x) \mapsto u x, $$
$$ G_+ \times G_- \rightarrow G_+ \text{, } (u, x) \mapsto u^x. $$

The relations between these actions and the decompositions $g = g_+ g_- = \overline{g}_- \overline{g}_+$ are:

$$ \overline{g}_- \overline{g}_+ = g_+; \overline{g}_+ = g_-; g_+^g g_- = \overline{g}_+; (g^g_+ g_-)(g^g_-) = g_+ g_-; (g_+^g g_-)(g_+^g) = g_- g_+. $$

Given a unique factorization $G = G_+ G_-$ of a finite group $G$, one can construct a finite dimensional Hopf algebra $H(G; G_+, G_-)$, which is the vector space spanned by the set $G$ (we denote by $\{g\}$ an element $g \in G$ when it is regarded as an element in $H(G; G_+, G_-)$) with the following Hopf algebra structure:

multiplication: $\{g\} \{h\} = \delta_{g_+ g_-} \{gh\}$

unit: $1 = \sum_{g_+ \in G_+} \{g_+\}$

comultiplication: $\Delta(\{g\}) = \sum_{h_+ \in G_+} \{g_+ h_+^{-1} (h_+ g_-)\} \otimes \{h_+ g_-\}$

unit: $\varepsilon(\{g\}) = \delta_{g_+ e}$

antipode: $S(\{g\}) = \{g^{-1}\}$

The Hopf algebra $H(G; G_+, G_-)$ has $G$ as the obvious positive basis. Conversely, it was proved in [3] that all finite dimensional Hopf algebras with positive bases are of the form $H(G; G_+, G_-)$.

The positive quasitriangular and triangular structures on $H(G; G_+, G_-)$ have been described in [7] as follows:

**Theorem 3.1** ([7]) Any quasitriangular structure $R_{s, \beta}$ on Radford’s Hopf algebra $H_\nu$ is pseudotriangular.

**Theorem 4.1** ([7]) Let $G = G_+ G_-$ be a unique factorization of a finite group $G$. Let $\xi, \eta : G_+ \rightarrow G_-$ be two group homomorphisms satisfying the following conditions:

$$ \xi(u)^v = \xi(u^{\eta(v)}), \quad (4.1) $$
be a positive quasitriangular structure on $H$.

Moreover, $R(\xi, \eta) := \sum_{u, v \in G_+} \{u(\eta(v)^u)^{-1}\} \otimes \{v(\eta(u)^v)^{-1}\}$ is a positive quasitriangular structure on $H(G; G_+, G_-)$. Conversely, every positive quasitriangular structure on $H(G; G_+, G_-)$ is given by the above construction.

Moreover, each of the conditions \((4.1) - (4.3)\) is equivalent to the corresponding property below:

\begin{align}
\xi(u) &= \xi(\eta(v)^u), \\
\eta(v)^u &= \eta(\xi(u)^v), \\
uv &= \eta(\xi(u)^v), \\
xv &= \xi(u)x, \\
ux &= \eta(u)x.
\end{align}

Moreover, $R(\xi, \eta)$ is triangular if and only if $\xi = \eta$.

Our aim now is to characterize those $R(\xi, \eta)$ that are pseudotriangular. So, let $R = R(\xi, \eta)$ be a positive quasitriangular structure on $H(G; G_+, G_-)$. We have (see \[7\]):

\[ R_{21}R = \sum_{u, v \in G_+} \{v(\eta(u)^v)^{-1}\} \otimes \{u(\eta(v)^u)^{-1}\} \otimes \{v(\eta(v)^v)^{-1}\} \otimes \{u(\eta(u)^u)^{-1}\}, \]

where we denoted $\overline{v} = v^\xi(u)$ and $\overline{u} = \eta(v)^u$.

We denote $T = R_{21}R$ and we compute (by using the formula for the multiplication of $H(G; G_+, G_-)$):

\[ T_{12}T_{23} = \left( \sum_{u, v \in G_+} \{v(\eta(u)^v)^{-1}\} \otimes \{u(\eta(v)^u)^{-1}\} \otimes \{v(\eta(v)^v)^{-1}\} \otimes \{u(\eta(u)^u)^{-1}\} \otimes 1 \right) \]

\[ \left( \sum_{s, t \in G_+} \{s(\eta(t)^s)^{-1}\} \otimes \{t(\eta(s)^t)^{-1}\} \otimes \{s(\eta(t)^s)^{-1}\} \otimes \{t(\eta(s)^t)^{-1}\} \right) \]

\[ = \sum_{u, v, s, t \in G_+} \{v(\eta(u)^v)(\eta(\overline{v})^u)^{-1}\} \otimes \{u(\eta(v)^u)^{-1}\} \otimes \{v(\eta(v)^v)^{-1}\} \otimes \{u(\eta(u)^u)^{-1}\} \]

\[ \otimes \{s(\eta(t)^s)^{-1}\} \]

\[ = \sum_{u, v, t \in G_+} \{v(\eta(u)^v)(\eta(\overline{v})^u)^{-1}\} \otimes \{u(\eta(v)^u)^{-1}\} \otimes \{v(\eta(v)^v)^{-1}\} \otimes \{u(\eta(u)^u)^{-1}\} \]

\[ \otimes \{s(\eta(t)^s)^{-1}\} \]

\[ \otimes \{t(\eta(s)^t)^{-1}\}, \]

where $t = u(\eta(v)^u)^{-1}\xi(v^\xi(u))$, and

\[ T_{23}T_{12} = \left( \sum_{a, b \in G_+} 1 \otimes \{b(\eta(b)^a)^{-1}\} \otimes \{a(\eta(b)^a)^{-1}\} \otimes \{b(\eta(b)^a)^{-1}\} \right) \]

\[ \otimes \{a(\eta(b)^a)^{-1}\} \]

\[ \otimes \{b(\eta(b)^a)^{-1}\} \]

\[ \otimes \{a(\eta(b)^a)^{-1}\} \]
The three conditions in the above Proposition may be simplified, so we obtain that
\( \text{is pseudotriangular if and only if we have:} \)
\[
\text{triangular if and only if} \quad \text{for all} \quad a, b, c \in G_+.
\]
Proof. If \( \text{for all} \quad \) normal positive quasitriangular structure
\[ G \]
be simplified, in particular we have
\[
(\eta(v)u)^{-1} \xi(u) = (\eta(v)u)^{-1} \xi(u),
\]
\[
(\eta(t)^{-1} \xi(t^{-1})c = (\eta(t)^{-1} \xi(t^{-1}),
\]
for all \( u, v, s \in G_+ \), where \( t = u(\eta(v)u)^{-1} \xi(uv) \) and \( c = u(\eta(s)u)^{-1} \xi(uv) \).

A better description may be obtained for a certain class of positive quasitriangular structures.

**Definition 4.3** \( [7] \) A positive quasitriangular structure \( R(\xi, \eta) \) on \( H(G; G_+, G_-) \) is called normal if \( \xi(u) = e \) for all \( u \in G_+ \).

**Theorem 4.4** A normal positive quasitriangular structure \( R(\xi, \eta) \) on \( H(G; G_+, G_-) \) is pseudotriangular if and only if \( \eta(vu) = \eta(vu) \) for all \( u, v \in G_+ \).

**Proof.** We note first that, since \( \xi(u) = e \) for all \( u \in G_+ \), some of the relations (4.1)-(4.10) may be simplified, in particular we have \( \eta(v) = \eta(v) \), \( uv = v(u^v) \), \( \eta(v)u = \eta(v)u \), \( uv = (\eta(u)^{\eta(u)}uv \)
for all \( u, v \in G_+ \). By using these relations, together with the fact that \( \xi(u) = e \) for all \( u \in G_+ \), the three conditions in the above Proposition may be also simplified, so we obtain that \( R(\xi, \eta) \) is pseudotriangular if and only if we have:
\[
\eta(vuv^{-1}) = \eta(vuv^{-1}),
\]
\[
\eta(v)^{-1} \eta(tst^{-1})^{-1} = \eta(usu^{-1})^{-1} \eta(v)^{-1},
\]
\[
\eta(t)^{-1} = \eta(u)^{-1},
\]
for all \( u, v, s \in G_+ \), where \( t = uuv^{-1} \) and \( c = usus^{-1}u^{-1} \), and one can easily see that each of these three conditions is equivalent to the condition \( \eta(vu) = \eta(vu) \), for all \( u, v \in G_+ \).

We recall from \([9]\) that the canonical quasitriangular structure on the Drinfeld double of a finite dimensional Hopf algebra \( H \) is pseudotriangular if and only if \( H \) is commutative and cocommutative. In particular, if \( G \) is a finite group, the canonical quasitriangular structure on the Drinfeld double of the dual \( k[G]^* \) of the group algebra \( k[G] \) is pseudotriangular if and only if \( G \) is abelian. We want to reobtain this result (over \( \mathbb{C} \)) as an application of Theorem 4.4.
We consider the unique factorization $G = G_+ G_-$, where $G_+ = G$ and $G_- = \{ e \}$ (so the Hopf algebra $H(G; G_+, G_-)$ is exactly $k[G]^*$). As in [7], we consider the group $\tilde{G} = G \times G$, with the unique factorization $\tilde{G} = \tilde{G}_+ \tilde{G}_-$, where $\tilde{G}_+ = G \times \{ e \}$ and $\tilde{G}_- = \{ (g, g) : g \in G \}$. By [7], the group homomorphisms $\xi, \eta : \tilde{G}_+ \to \tilde{G}_-$ defined by $\xi(g, e) = (e, e)$ and $\eta(g, e) = (g, g)$ induce a positive quasitriangular structure $R(\xi, \eta)$ on $H(\tilde{G}; \tilde{G}_+, \tilde{G}_-)$ and moreover $H(\tilde{G}; \tilde{G}_+, \tilde{G}_-)$ is the Drinfeld double of $H(G; G_+, G_-) = k[G]^*$ and $R(\xi, \eta)$ is its canonical quasitriangular structure. Obviously $R(\xi, \eta)$ is normal, so we may apply Theorem 4.4 and we obtain that $R(\xi, \eta)$ is pseudotriangular if and only if $(gh, gh) = (hg, hg)$ for all $g, h \in G$, i.e. if and only if $G$ is abelian.

5 Universality of the pseudosymmetric category $\mathcal{PS}$

In this section we use terminology, notation and some results from [5] (but we use the term "monoidal" instead of "tensor" when we speak about tensor categories and tensor functors).

Our aim is to show that the pseudosymmetric category $\mathcal{PS}$ introduced in [10] has two universality properties similar to the ones of the braid category $\mathcal{B}$, the universal braided monoidal category (see [5]). First, we recall from [10] the definition of $\mathcal{PS}$. The objects of $\mathcal{PS}$ are natural numbers $n \in \mathbb{N}$. The set of morphisms from $m$ to $n$ is empty if $m \neq n$ and is $\mathcal{PS}_n := \frac{B_n}{\{P_n, P_n\}}$ if $m = n$, where $B_n$ (respectively $P_n$) is the braid group (respectively pure braid group) on $n$ strands. The monoidal structure of $\mathcal{PS}$ is defined as the one for $\mathcal{B}$, and so is the braiding, namely (we denote as usual by $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ the standard generators of $B_n$ and by $\pi_n$ the natural morphism from $B_n$ to $P_n$):

\[
c_{n,m} : n \otimes m \to m \otimes n, \quad c_{0,n} = id_n = c_{n,0}, \\
c_{n,m} = \pi_{n+m}((\sigma_m \sigma_{m-1} \cdots \sigma_1)(\sigma_{m+1} \sigma_m \cdots \sigma_2) \cdots (\sigma_{m+n-1} \sigma_{m+n-2} \cdots \sigma_n)) \quad \text{if} \quad m, n > 0.
\]

In order to introduce the first universality property for $\mathcal{PS}$, we need the following definition, motivated by results in [10] and by the definition of Yang-Baxter operators from [5].

**Definition 5.1** If $V$ is an object in a monoidal category $(\mathcal{C}, \otimes, I, a, l, r)$, an automorphism $\sigma$ of $V \otimes V$ is called a pseudosymmetric Yang-Baxter operator on $V$ if the following two dodecagons
(for $\sigma$ and $\sigma^{-1}$) commute:

Note that a pseudosymmetric Yang-Baxter operator is a special type of Yang-Baxter operator as defined in [5], p. 323. Moreover, just like Yang-Baxter operators, they can be transferred by using functors between monoidal categories:

**Lemma 5.2** Let $(F, \varphi_0, \varphi_2) : \mathcal{C} \to \mathcal{D}$ be a monoidal functor between two monoidal categories. If $\sigma \in \text{Aut}(V \otimes V)$ is a pseudosymmetric Yang-Baxter operator on the object $V \in \mathcal{C}$, then

$$\sigma' = \varphi_2(V, V)^{-1} \circ F(\sigma) \circ \varphi_2(V, V)$$

is a pseudosymmetric Yang-Baxter operator on $F(V)$.

**Proof.** The proof follows exactly as in [5], Lemma XIII.3.2, by using also the identity

$$(\sigma')^{-1} = \varphi_2(V, V)^{-1} \circ F(\sigma^{-1}) \circ \varphi_2(V, V)$$

in order to prove the pseudosymmetry of $\sigma'$.

We define the category $\text{PSYB}(\mathcal{C})$ of pseudosymmetric Yang-Baxter operators to be a full subcategory of $\text{YB}(\mathcal{C})$, the category of Yang-Baxter operators defined in [5]. An object in $\text{PSYB}(\mathcal{C})$ is a pair $(V, \sigma)$ where $V$ is an object in $\mathcal{C}$ and $\sigma$ is a pseudosymmetric Yang-Baxter operator.

Recall the following construction from [5]. Suppose that $(F, \varphi_0, \varphi_2) : \mathcal{B} \to \mathcal{C}$ is a monoidal functor from the universal braid category $\mathcal{B}$ to a given monoidal category $\mathcal{C}$. Since $c_{1,1} = \sigma_1$ is a Yang-Baxter operator on the object $1 \in \mathcal{B}$, it follows that $\sigma = \varphi_2^{-1}(1,1)F(c_{1,1})\varphi_2(1,1)$ is a Yang-Baxter operator on $F(1) \in \mathcal{C}$. In this way we get a functor $\Theta : \text{Tens}(\mathcal{B}, \mathcal{C}) \to \text{YB}(\mathcal{C})$, where $\text{Tens}(\mathcal{B}, \mathcal{C})$ is the category of monoidal functors from $\mathcal{B}$ to $\mathcal{C}$. It was proved in [5] that:

**Theorem 5.3** ([5]) For any monoidal category $\mathcal{C}$, the functor $\Theta : \text{Tens}(\mathcal{B}, \mathcal{C}) \to \text{YB}(\mathcal{C})$ is an equivalence of categories.
One can note that we have a natural monoidal functor \( \pi : B \to P \mathcal{S} \) induced by the group epimorphism \( \pi_n : B_n \to P S_n \). This allows us to identify the category \( \text{Tens}(P \mathcal{S}, C) \) with a subcategory of \( \text{Tens}(B, C) \). More precisely, we identify it with the full subcategory of all monoidal functors \( F : B \to C \) with the property that there exists a monoidal functor \( G : P \mathcal{S} \to C \) such that \( F = G \circ \pi \).

We can state now the first universality property of \( P \mathcal{S} \):

**Theorem 5.4** For any monoidal category \( C \), the functor \( \tilde{\Theta} : \text{Tens}(P \mathcal{S}, C) \to \text{PSYB}(C) \) defined as \( \tilde{\Theta}(G) = \Theta(G \circ \pi) \) is an equivalence of categories.

**Proof.** First we note that \( \pi(c_1, 1) \) is a pseudosymmetric Yang-Baxter operator in \( P \mathcal{S} \) and so by Lemma 5.2 we have \( \varphi_2^{-1}(1, 1)G(\pi(c_1, 1))\varphi_2(1, 1) \in \text{PSYB}(C) \). This means that \( \tilde{\Theta} \) is well defined.

Since \( \Theta \) is fully faithful and \( \tilde{\Theta} \) is its restriction to a full subcategory, it is enough to show that \( \tilde{\Theta} \) is essentially surjective. This follows from the next lemma. \( \square \)

**Lemma 5.5** Let \( C \) be a strict monoidal category and \((V, \sigma)\) an object in \( \text{PSYB}(C) \). Then there exists a unique strict monoidal functor \( G : P \mathcal{S} \to C \) such that \( G(1) = V \) and \( G(\pi(c_1, 1)) = \sigma \).

**Proof.** From [5], Lemma XIII.3.5 we know that for all \((V, \sigma) \in YB(C)\) there exists a unique strict monoidal functor \( F : B \to C \) such that \( F(1) = V \) and \( F(c_1, 1) = \sigma \). It is enough to show that when \((V, \sigma) \in \text{PSYB}(C)\) the functor \( F \) factors through \( \pi \). But this follows immediately from the fact (see [10]) that

\[
PS_n = \frac{B_n}{< \sigma_i \sigma_{i+1}^{-1} \sigma_i = \sigma_{i+1} \sigma_i^{-1} \sigma_{i+1} \mid 1 \leq i \leq n-2 >}
\]

and the definition of a pseudosymmetric Yang-Baxter operator. \( \square \)

**Definition 5.6** ([5]) A monoidal functor \((F, \varphi_0, \varphi_2)\) from a braided monoidal category \( C \) to a braided monoidal category \( D \) is braided if for every pair \((U, V)\) of objects in \( C \) the square

\[
\begin{array}{ccc}
F(U) \otimes F(V) & \xrightarrow{\varphi_2} & F(U \otimes V) \\
\varepsilon_{F(U), F(V)} & & \varepsilon_{F(U,V)} \\
F(V) \otimes F(U) & \xrightarrow{\varphi_2} & F(V \otimes U)
\end{array}
\]

commutes. Denote by \( \text{Br}(C, D) \) the category whose objects are braided monoidal functors and morphisms are natural monoidal transformations.

**Theorem 5.7** ([5]) For a braided monoidal category \( C \), the functor \( \Theta' : \text{Br}(B, C) \to C \) defined by \( \Theta'(F) = F(1) \) is an equivalence of categories.

In the definition of a pseudosymmetric braided category \( C \) introduced in [9] was assumed that \( C \) was a strict monoidal category. The next proposition is the analogue of Theorem 3.7 from [9] for monoidal categories with nontrivial associativity constraints. Note that the proof that we present here is very direct and is inspired by the results in [10].
Proposition 5.8  Let \((\mathcal{C}, \otimes, I, a, l, r, c)\) be a braided monoidal category. The following conditions are equivalent:

(i) For every \(U, V, W \in \mathcal{C}\) the following diagram is commutative:

(ii) For every \(U, V, W \in \mathcal{C}\) the following diagram is commutative:

Proof. Take \(U, V, W \in \mathcal{C}\). Using only the fact that \(\mathcal{C}\) is a braided category we have

\[
((c_{U,V}c_{U,V}) \otimes id_W)a_{U,V,W}^{-1}(id_U \otimes (c_Wc_{V,W}))a_{U,V,W}
\]

\[
= (c_{V,U} \otimes id_W)a_{U,V,W}^{-1}(id_U \otimes c_{U,V})a_{U,V,W}(c_{V,U} \otimes id_W)
\]

\[
a_{U,V,W}^{-1}(id_U \otimes c_Wc_{V,W})a_{U,V,W}
\]

\[
= (c_{V,U} \otimes id_W)a_{U,V,W}^{-1}(id_U \otimes c_{U,V})a_{U,V,W}(c_{U,V} \otimes id_W)
\]

\[
a_{U,V,W}^{-1}(id_U \otimes c_Wc_{V,W})a_{U,V,W}
\]
Theorem 5.12

For a pseudosymmetric braided category $\mathcal{C}$, the functor $\Theta : \text{Br}(\mathcal{C}) \to \mathcal{C}$ defined by $\Theta(G) = G(1)$ is an equivalence of categories.
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