Large-System Analysis of Joint User Selection and Vector Precoding for Multiuser MIMO Downlink

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Abstract

Joint user selection (US) and vector precoding (US-VP) is proposed for multiuser multiple-input multiple-output (MU-MIMO) downlink. The main difference between joint US-VP and conventional US is that US depends on data symbols for joint US-VP, whereas conventional US is independent of data symbols. The replica method is used to analyze the performance of joint US-VP in the large-system limit, where the numbers of transmit antennas, users, and selected users tend to infinity while their ratios are kept constant. The analysis under the assumptions of replica symmetry (RS) and 1-step replica symmetry breaking (1RSB) implies that optimal data-independent US provides nothing but the same performance as random US in the large-system limit, whereas data-independent US is capacity-achieving as only the number of users tends to infinity. It is shown that joint US-VP can provide a substantial reduction of the energy penalty in the large-system limit. Consequently, joint US-VP outperforms separate US-VP in terms of the achievable sum rate, which consists of a combination of vector precoding (VP) and data-independent US. In particular, data-dependent US can be applied to general modulation, and implemented with a greedy algorithm.

Index Terms

Multiuser multiple-input multiple-output (MU-MIMO) downlink, Multiple-input multiple-output broadcast channel (MIMO-BC), zero-forcing transmit beamforming, user selection, vector precoding, energy penalty, achievable rate, large-system analysis, order statistics, statistical physics, replica method, replica symmetry breaking (RSB).

I. INTRODUCTION

MULTIPLE-input multiple-output (MIMO) systems use multiple transmit and receive antennas to increase the spectral efficiency [1], [2]. In early work, point-to-point MIMO or multiuser MIMO (MU-MIMO) uplink was investigated [3]–[8]. In these MIMO systems, the receiver can utilize all received signals to detect the transmitted data. Recent research activities have been shifted to MU-MIMO downlink, in which one base station (BS) communicates with non-cooperative users. In the MU-MIMO downlink the main part of signal processing is at the transmitter side, whereas it is at the receiver side for the MU-MIMO uplink.

Transmit strategies used for the MU-MIMO downlink depend on duplexing. For the MU-MIMO downlink with frequency-division duplexing (FDD), channel state information (CSI) is not available at the transmitter side. Instead, the BS may utilize limited feedback information about channel quality, transmitted through the uplink channels [9], [10]. For the MU-MIMO downlink with time-division duplexing (TDD), on the other hand, channel state information (CSI) is used to pre-cancel inter-user interference (IUI) at the transmitter side. The CSI may be estimated by utilizing the fact that fading coefficients in both links are identical for TDD [11], [12]. In particular, it is possible for the BS to attain accurate CSI when the coherence time is sufficiently long. In this paper, the MU-MIMO downlink with TDD is considered under the assumption that the coherence time is sufficiently long. For simplicity, we assume that perfect CSI is available at the transmitter and that the number of receive antenna for each user is one.

The MU-MIMO downlink we consider is mathematically modeled as the MIMO broadcast channel (MIMO-BC) with perfect CSI at the transmitter. Recent excellent papers [13]–[16] have proved that the capacity region of the MIMO-BC with perfect CSI at the transmitter is achieved by dirty-paper coding (DPC) [17], which is a sophisticated scheme that pre-cancels IUI at the transmitter side. Since DPC is infeasible in terms of the computational complexity, however, it is an active research area and the target in this paper to construct a suboptimal scheme that achieves an acceptable tradeoff between performance and complexity.

Zero-forcing transmit beamforming (ZFBF) [18]–[20] is a simple approach for pre-cancelling IUI at the transmitter side. The ZFBF decomposes the MIMO-BC into per-user interference-free channels. A drawback of the ZFBF is that energy penalty, which is the energy required for the pre-cancellation of IUI, increases rapidly as the number of (supported) users gets closer...
to the number of transmit antennas. An increase of the energy penalty results in a degradation of the receive signal-to-noise ratio (SNR).

The number of users is commonly larger than the number of transmit antennas for MU-MIMO downlink. In order to reduce the energy penalty, user selection (US) has been proposed [21]–[24]. A subset of users is selected to mitigate the increase of the energy penalty. Interestingly, it has been shown that a greedy algorithm of US can achieve the sum capacity of the MIMO-BC when only the number of users tends to infinity [23], [25]. This result can be understood as follows: If the channel vectors for all users were orthogonal to each other, the ZFBF would be optimal. However, there are dependencies between the channel vectors in general. In US, the BS attempts to select a subset of users with almost orthogonal channel vectors. It is possible to pick a finite number of almost orthogonal channel vectors from an infinite number of channel vectors under proper conditions. Thus, the ZFBF with US can achieve the sum capacity of the MIMO-BC when the number of users tends to infinity. Since the number of selected users should be comparable to the number of transmit antennas, the interpretation above implies that the performance of US degrades significantly as the number of transmit antennas gets closer to the number of users.

The situation in which the number of transmit antennas is comparable to the number of users is becoming practical [12]. As an alternative limit representing this situation, we consider the large-system limit in which the number of transmit antennas and the number of users tend to infinity while their ratio is kept constant.

Vector perturbation or vector precoding (VP) is an effective pre-coding scheme that works well in the large-system limit [26]–[28]. In VP, the data symbols are modified to take values in relaxed alphabets to reduce the energy penalty. As relaxed alphabets, lattice-type alphabets [26] and a continuous alphabet [27] have been proposed. In this paper, VP schemes with lattice-type and continuous alphabets are referred to as “lattice VP (LVP)” and “continuous VP (CVP),” respectively. The search for a modified data symbol vector to minimize the energy penalty is NP-hard for LVP, so that LVP is infeasible for large alphabets or a large number of users. On the other hand, the search for CVP reduces to a quadratic optimization problem [29], which may be solved by using an efficient algorithm. The large-system analysis in [27], [30] has been shown that the performance of CVP is comparable to that of LVP in the large-system limit. In this paper, we only focus on CVP.

A drawback of CVP is that the modulation is restricted to quadrature phase shift keying (QPSK). This restriction results in poor performance especially for the high SNR regime. In this paper, we propose a novel precoding scheme that is applicable for any modulation. The basic idea is to combine US and VP (US-VP). Joint US-VP we propose should not be confused with separate US-VP [28], in which a subset of users is first selected on the basis of CSI and subsequently VP is performed for the selected users. The crucial difference between the two schemes is that US depends on the data symbols for joint US-VP, whereas it is independent of the data symbols for separate US-VP. In this paper, joint US-VP is simply referred to as US-VP.

Data-dependent US (DD-US) proposed in our previous work [31] can be regarded as a special example of US-VP: It is equivalent to US-VP with the original alphabet as the relaxed alphabet. DD-US allows us to use any modulation, as conventional US does. Furthermore, DD-US can be easily implemented with a suboptimal greedy algorithm for DD-US [31].

The goal of this paper is to assess the performance of US-VP. For that purpose, we consider the large-system limit in which the number of transmit antennas, the number of users, and the number of selected users tend to infinity while their ratios are kept constant. The replica method is used to analyze the performance of US-VP in the large-system limit. The replica method was originally developed in statistical physics [32]–[34], and has been used to analyze the performance of MIMO systems [27], [30], [35]–[41] since Tanaka’s pioneering work [42].

The weakness of the replica method is that the method is based on several non-rigorous assumptions, such as the commutativity between the large-system limit and the other limits, replica continuity, and replica symmetry (RS). The commutativity was justified for a spin glass model [43], called Sherrington-Kirkpatrick (SK) model. The validity of replica continuity is open. The RS assumption may be broken for several models. In this case, the assumption of replica symmetry breaking (RSB) should be considered [44]. The RS assumption corresponds to the situation under which an energy function, called free energy, is unimodal. On the other hand, the RSB assumption corresponds to the situation under which the free energy has many local minima [33]. The simplest (strongest) assumption for RSB is referred to as 1-step RSB (1RSB). The most complex (weakest) assumption for RSB is called full-step RSB (full-RSB), which includes the RS assumption and the other lower-step RSB assumptions. In this paper, only the RS and 1RSB assumptions are considered, since the assumption of higher-step RSB yields numerically unsolvable results for our problem. Thus, the results presented in this paper should be regarded as an approximation for the true ones.

Recently, the validity of several results obtained from the replica method has been investigated. Korada and Montanari [45] proved Tanaka’s formula based on the RS assumption. Guerra and Talagrand’s excellent works [46], [47] proved that the replica analysis under the full-RSB assumption provides the correct result for the SK model. The latter methodology might be applicable for our problem.

This paper is organized as follows: After summarizing the notation used in this paper, in Section II we introduce the MIMO-BC and US-VP. Section III summarizes the main results of this paper. In Section IV we present numerical results based on the main results. Section V concludes this paper. The derivations of the main results are summarized in the appendices.
A. Notation

For a complex number \( z \in \mathbb{C} \), the real and imaginary parts of \( z \) are denoted by \( \Re[z] \) and \( \Im[z] \), respectively. Furthermore, \( z^* \) stands for the complex conjugate of \( z \). For a matrix \( A \), the transpose, conjugate transpose, trace, and the determinant of \( A \) are denoted by \( A^T \), \( A^H \), \( \text{Tr}(A) \), and \( \det(A) \), respectively. \( I_N \) stands for the \( N \times N \) identity matrix. \( 1_N \) represents the \( N \)-dimensional vector whose elements are all one. \( \text{diag}\{a_1, \ldots, a_N\} \) stands for the \( N \times N \) diagonal matrix with \( a_n \) as the \( n \)-th diagonal element. The Kronecker product operator between two matrices is denoted by \( \otimes \).

For a set \( A = \{a_i : i = 1, \ldots, N\} \), \( \setminus a_i \) stands for the set \( \{a_{i'} : \text{for all } i' \neq i\} \) obtained by eliminating \( a_i \) from \( A \). Similarly, \( \setminus A_i \) denotes the union \( \cup_{i' \neq i} A_i \) for sets \( \{A_i\}_{i=1}^N \). The direct product \( A_1 \times \cdots \times A_N \) is denoted by \( \prod_{i=1}^N A_i \).

For a random variable \( X \), \( \mathbb{E}[X] \) and \( \mathbb{V}[X] \) stand for the mean and variance of \( X \), respectively. For the sequence of real random variables \( \{X_i\}_{i=1}^N \), \( X_i \) denotes the \( i \)-th order statistic of \( \{X_i\} \), i.e. \( X_1 \leq \cdots \leq X_N \) \([48]\). \( N(m, \Sigma) \) represents a real Gaussian distribution with mean \( m \) and covariance matrix \( \Sigma \). Similarly, \( \mathcal{CN}(m, \Sigma) \) stands for a proper complex Gaussian distribution with mean \( m \) and covariance matrix \( \Sigma \) \([49]\).

For a discrete random variable \( X \), the entropy of \( X \) is denoted by \( H(X) \). If \( X \) is a continuous random variable, \( h(X) \) represents the differential entropy. For two random variables \( X \) and \( Y \), the mutual information between \( X \) and \( Y \) is denoted by \( I(X;Y) \). Throughout this paper, all logarithms are taken to base 2, while the natural logarithm is denoted by \( \ln \).

Finally, we summarize several functions used in this paper. The function \( \delta(\cdot) \) represents the Dirac delta function. For a proposition \( P \), the indicator function \( 1(P) \) is defined as

\[
1(P) = \begin{cases} 
1 & P \text{ is true} \\
0 & P \text{ is false}. 
\end{cases}
\]

(1)

The probability density function (pdf) of a circularly symmetric complex Gaussian random variable with variance \( \sigma^2 \) is denoted by

\[
p_{\mathcal{CG}}(z; \sigma^2) = \frac{1}{\pi \sigma^2} e^{-\frac{|z|^2}{\sigma^2}} \quad \text{for } z \in \mathbb{C}.
\]

(2)

Similarly, the pdf of a zero-mean real Gaussian random variable with variance \( \sigma^2 \) is written as

\[
p_{\mathcal{G}}(x; \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} \quad \text{for } x \in \mathbb{R}.
\]

(3)

The standard Gaussian measure \( Dx \) is defined as

\[
Dx = p_{\mathcal{G}}(x;1)dx.
\]

(4)

Furthermore, the function \( Q(x) \) is given by

\[
Q(x) = \int_x^\infty Dy.
\]

(5)

II. SYSTEM MODELS

A. MIMO Broadcast Channel

We consider the MIMO-BC which consists of one BS with \( N \) transmit antennas and \( K \) users with one receive antenna. For simplicity, Rayleigh block-fading channels are assumed: The channel gains between the BS and each user are fixed during \( T_c \) time slots, and at the beginning of the next fading block the channel gains are independently sampled from a circularly symmetric complex Gaussian distribution. Let \( y_{k,t} \in \mathbb{C} \) denote the received signal for user \( k \) in time slot \( t \). The receive vector \( \mathbf{y}_t = (y_{1,t}, \ldots, y_{K,t})^T \in \mathbb{C}^K \) consisting of all received signals in time slot \( t \) is given by

\[
\mathbf{y}_t = \mathbf{H}\mathbf{u}_t + \mathbf{n}_t, \quad t = 0, \ldots, T_c - 1.
\]

(6)

In (6), \( \mathbf{n}_t = (n_{1,t}, \ldots, n_{K,t})^T \sim \mathcal{CN}(0, \mathbf{N}_0\mathbf{I}_K) \) denotes the additive white Gaussian noise (AWGN) vector. The vector \( \mathbf{u}_t \in \mathbb{C}^N \) is the transmit vector in time slot \( t \), which will be defined shortly. Each row vector of the channel matrix \( \mathbf{H} \in \mathbb{C}^{K \times N} \) corresponds to the channel gains between the BS and each user. It is natural for the MIMO-BC to assume that each element of the channel matrix is \( O(1) \) and that the time-average transmit power is constrained to below \( P \). For convenience in analysis, we make an equivalent assumption: We assume that \( \mathbf{H} \) has independent circularly symmetric complex Gaussian elements with variance \( 1/N \), and that the time-average transmit power is constrained to below \( NP \), i.e.

\[
\frac{1}{T_c} \sum_{t=0}^{T_c-1} ||\mathbf{u}_t||^2 \leq NP.
\]

(7)

Under these assumptions, the transmit SNR is defined as \( P/N_0 \).

Slow fading is considered in this paper, i.e. \( T_c \to \infty \). Note that the channel matrix in (6) is fixed during \( T_c \) time slots. In this situation, we can assume that the channel matrix \( \mathbf{H} \) is known to the transmitter, since the transmitter can estimate the channel matrix on the basis of pilot signals transmitted from each user in a negligibly small portion of one fading block.
B. Zero-Forcing Transmit Beamforming

Let $x_t = (x_{1,t}, \ldots, x_{K,t})^T \in \mathbb{C}^K$ denote the data symbol vector in time slot $t$. The set $\mathcal{X}_k^{(all)} = \{x_{k,t} : t = 0, \ldots, T_c - 1\}$ corresponds to the data symbols sent to user $k$, and is assumed to be independent for different $k$. Throughout this paper, power allocation is not considered: The data symbols $\{x_{k,t}\}$ are assumed to be independent and identically distributed (i.i.d.) zero-mean complex random variables with unit variance.

The BS uses the information about the channel matrix to pre-cancel IUI. For $K \leq N$, the simplest method for pre-cancellation is ZFBF \cite{20}, in which the transmit vector $u_t$ is given by

$$u_t = \sqrt{\frac{NP}{\mathcal{E}(H, \{x_t\})}} u_t^{ZF}(H, x_t),$$

with

$$u_t^{ZF}(H, x_t) = H^H (HH^H)^{-1} x_t.$$  \hspace{1cm} (8)

In (8), the energy penalty $\mathcal{E}(H, \{x_t\})$ is defined as the time-average power of the ZFBF vectors (9).

$$\mathcal{E}(H, \{x_t\}) = \frac{1}{T_c} \sum_{t=0}^{T_c-1} \|u_t^{ZF}(H, x_t)\|^2.$$  \hspace{1cm} (10)

It is straightforward to confirm that the transmit vector (8) satisfies the power constraint (7).

The ZFBF (8) decomposes the MIMO-BC (6) into per-user channels

$$y_{k,t} = \sqrt{\frac{NP}{\mathcal{E}(H, \{x_t\})}} x_{k,t} + n_{k,t},$$

for all $k$. This implies that the receive SNR is given by $NP/(\mathcal{E}(H, \{x_t\})N_0)$. The drawback of the ZFBF is an increase of the energy penalty (10). Substituting the ZFBF vector (9) into (10) yields

$$\mathcal{E}(H, \{x_t\}) = \frac{1}{T_c} \sum_{t=0}^{T_c-1} x_t^H (HH^H)^{-1} x_t$$

$$\rightarrow \text{Tr} \left\{ \left( HH^H \right)^{-1} \right\},$$

in $T_c \to \infty$. The Marčenko-Pastur law \cite{50} implies that the energy penalty (13) per user converges almost surely to

$$\frac{1}{K} \mathcal{E}(H, \{x_t\}) \to \frac{1}{1 - \alpha},$$

in the large-system limit, where both $K$ and $N$ tend to infinity with their ratio $\alpha = K/N$ kept constant. The asymptotic energy penalty (14) diverges as $\alpha$ gets closer to 1 from below. Since the receive SNR $NP/(\mathcal{E}(H, \{x_t\})N_0)$ is inversely proportional to the energy penalty, this divergence results in a fatal degradation of the receive SNR.

C. Vector Precoding

As a method for improving the drawback of ZFBF, VP with ZFBF was proposed \cite{26, 27}. In VP, each data symbol $x_{k,t}$ is modified to take values in a relaxed alphabet $\mathcal{M}_{x_{k,t}} \subset \mathbb{C}$, depending on the original data symbol $x_{k,t}$, to reduce the energy penalty. The modified data symbol vector $\tilde{x}_t \in \prod_{k=1}^{K} \mathcal{M}_{x_{k,t}}$ based on the minimization of the energy penalty (12) is given by

$$\tilde{x}_t = \arg\min_{\tilde{x}_t \in \mathcal{M}_{x_{1,t}} \times \cdots \times \mathcal{M}_{x_{K,t}}} \tilde{x}_t^H (HH^H)^{-1} \tilde{x}_t.$$  \hspace{1cm} (15)

Note that the modified vector (15) to minimize each instantaneous power $\|u_t^{ZF}(H, x_t)\|^2$ for the ZFBF (9) minimizes the energy penalty (12) for the ZFBF.

**Example 1** (CVP). Suppose that QPSK is used. For CVP \cite{27}, the relaxed alphabet $\mathcal{M}_x$ for a QPSK data symbol $x$ is given by

$$\mathcal{M}_x = \tilde{\mathcal{M}}_{\Re[x]} + i \tilde{\mathcal{M}}_{\Im[x]},$$

with

$$\tilde{\mathcal{M}}_b = \begin{cases} [b, \infty) & \text{for } b = 1/\sqrt{2} \\ (-\infty, b] & \text{for } b = -1/\sqrt{2}. \end{cases}$$  \hspace{1cm} (17)
Miller et al. [27] showed that the CVP results in a significant reduction of the energy penalty, compared to the conventional ZFBF. The minimization problem (15) reduces to a quadratic optimization problem [29], so that one can use an efficient algorithm to solve (15).

The point of the CVP is that the modified data symbol vector $\tilde{x}_t$ depends on the channel matrix $H$. Consequently, the energy penalty $E(H, \{\tilde{x}_t\})$, given by (10), for the CVP never tends to (13) in $T_c \to \infty$. In fact, the energy penalty for the CVP was shown to be bounded in the limit $\alpha \to 1$ after taking the large-system limit [30].

D. Joint User Selection and Vector Precoding

We propose US-VP based on the combination of US and VP. US-VP is performed every $T$ ($\ll T_c$) time slots. Let $\mathcal{K}_i \subset \mathcal{K}_{all} = \{1, \ldots, K\}$, with size $\tilde{K} = |\mathcal{K}_i| (\leq N)$, denote the set of selected users in the $i$th block of US ($i = 0, \ldots, T_c/T - 1$). The corresponding modified data symbol vectors are denoted by $\tilde{x}_{\mathcal{K}_i, t} \in \prod_{k \in \mathcal{K}_i} \mathcal{M}_{x_k, t}$ for $t = iT, \ldots, (i + 1)T - 1$. The set of selected users $\mathcal{K}_i$ and the corresponding modified vectors $\{\tilde{x}_{\mathcal{K}_i, t} : t = iT, \ldots, (i + 1)T - 1\}$ are selected to minimize the energy penalty (10):

$$
(\mathcal{K}_i; \{\tilde{x}_{\mathcal{K}_i, t}\}) = \arg \min_{\mathcal{K}_i, \{\tilde{x}_{\mathcal{K}_i, t}\}} E(H_{\mathcal{K}_i}, \{\tilde{x}_{\mathcal{K}_i, t}\}),
$$

where the minimization is taken over $\{\mathcal{K}_i \subset \mathcal{K}_{all} : |\mathcal{K}_i| = \tilde{K}\}$ and $\{\tilde{x}_{\mathcal{K}_i, t} \in \prod_{k \in \mathcal{K}_i} \mathcal{M}_{x_k, t} : t = iT, \ldots, (i + 1)T - 1\}$, with

$$
E(H_{\mathcal{K}_i}, \{\tilde{x}_{\mathcal{K}_i, t}\}) = \frac{1}{T} \sum_{t = iT}^{(i+1)T-1} \left\| u^{(ZF)}_i(H_{\mathcal{K}_i}, \tilde{x}_{\mathcal{K}_i, t}) \right\|^2.
$$

In (19), the ZFBF vector $u^{(ZF)}_i(H_{\mathcal{K}_i}, \tilde{x}_{\mathcal{K}_i, t})$ is given by (9). Furthermore, $H_{\mathcal{K}_i} \in \mathbb{C}^{K \times N}$ denotes the channel matrix corresponding to the selected users $\mathcal{K}_i$, which is obtained by collecting the row vectors for the selected users $\mathcal{K}_i$ from the channel matrix $H$.

**Example 2** (DD-US). DD-US is defined as the US-VP (18) with the original alphabet as the relaxed alphabet, i.e. $\mathcal{M}_{x_k, t} = \{x_k, t\}$. Thus, the modified data symbol vector $\tilde{x}_{\mathcal{K}_i, t}$ is equal to the original data symbol vector $x_{\mathcal{K}_i, t} \in \mathbb{C}^{\tilde{K}}$, obtained by stacking the data symbols $\{x_k, t\}$ for the selected users $\mathcal{K}_i$. The minimization problem (18) for the DD-US can be approximately solved by using a greedy algorithm proposed in [37].

**Example 3** (US-CVP). Suppose that QPSK is used. Joint US and CVP (US-CVP) is defined as the US-VP (18) with the CVP (16). Unfortunately, the minimization problem (18) is not convex. It may be possible to extend the greedy algorithm for the DD-US [37] to the US-CVP. Obviously, the obtained algorithm should be more complex than the greedy algorithm for the DD-US.

The transmit vector $u_t$ for US-VP (18) in time slot $t$ is given by

$$
u_t = \sqrt{\frac{NP}{E(H_{\mathcal{K}_i}, \{\tilde{x}_{\mathcal{K}_i, t}\})}} u^{(ZF)}_i(H_{\mathcal{K}_i}, \tilde{x}_{\mathcal{K}_i, t}),
$$

where the energy penalty $E(H_{\mathcal{K}_i}, \{\tilde{x}_{\mathcal{K}_i, t}\})$ for the US-VP is defined as

$$
E(H_{\mathcal{K}_i}, \{\tilde{x}_{\mathcal{K}_i, t}\}) = \frac{1}{T_c/T} \sum_{i=0}^{T_c/T-1} E_i(H_{\mathcal{K}_i}, \{\tilde{x}_{\mathcal{K}_i, t}\}),
$$

with (19). In order to simplify detection in each user, the data symbols for non-selected users are discarded at the transmitter side [38]. This implies that the ZFBF (20) with US-VP (18) decomposes the MIMO-BC (6) into per-user channels

$$
\begin{align*}
y_{k, t} &= \sqrt{\frac{NP}{E(H_{\mathcal{K}_i}, \{\tilde{x}_{\mathcal{K}_i, t}\})}} \left\{ s_{k, i} \tilde{x}_{k, t} \right\} + n_{k, t},
\end{align*}
$$

for all $k$. In (22), $\tilde{x}_{k, t} \in \mathcal{M}_{x_k, t}$ denotes the modified data symbol corresponding to the original data symbol $x_{k, t}$. The variable $s_{k, i} \in \{0, 1\}$ indicating whether user $k$ has been selected in block $i$ is defined as

$$
\begin{align*}
s_{k, i} &= \begin{cases} 1 & k \in \mathcal{K}_i \ 0 & k \notin \mathcal{K}_i. \end{cases}
\end{align*}
$$

Furthermore, $I_{k, t} \in \mathbb{C}$ denotes the interference to the non-selected user $k \notin \mathcal{K}_i$, given by

$$
I_{k, t} = \bar{h}_{k} u^{(ZF)}_i(H_{\mathcal{K}_i}, \tilde{x}_{\mathcal{K}_i, t}),
$$

If there are multiple solutions, one solution is selected randomly and uniformly.
where \( \tilde{h}_k \in \mathbb{C}^{1 \times N} \) denotes the \( k \)th row vector of the channel matrix \( H \). Note that the indices \( t \) of \( y_{k,t} \) and \( x_{k,t} \) in (22) are identical to each other, since the data symbols for the non-selected users \( k \notin K_i \) have been discarded. This simplifies the detection of (23).

It is easy for user \( k \) to blind-detect one variable \( s_{k,i} \) from the \( T \) observations \( \{y_{k,t}\} \) in each block. Using the decision-feedback of \( \hat{x}_{k,t} \) from the decoder improve the accuracy of detection (21). In order to reduce the energy penalty, small \( T \) should be used. On the other hand, too small \( T \) makes it difficult to detect. As one option, dozens of time slots should be used as the block length \( T \). For example, the energy loss due to detection errors is at most 0.2–0.5 dB for \( T = 16 \) (31).

**Remark 1.** Let us discuss the relationship between the DD-US and conventional US. The set of selected users \( K_0 \) in the first block for the DD-US is given by

\[
K_0 = \arg \min_{K \subseteq K_{\text{all}} : |K| = \hat{K}} E_0 (H_{K_0}, \{x_{K_0,t}\}),
\]

(25)

with (19). On the other hand, when the minimization of the energy penalty (23) for the ZFBF or equivalently of (22) in \( T_c \to \infty \) is used as the US criterion, the set of selected users \( K \) for conventional US is given by

\[
K = \arg \min_{K \subseteq K_{\text{all}} : |K| = \hat{K}} \lim_{T \to \infty} E_0 (H_K, \{x_{K,t}\}),
\]

(26)

where we have re-written the coherence time \( T_c \) as \( T \). Note that (26) is independent of the data symbols, since the object function tends to \( \text{Tr}\{(H_K H_H)^{-1}\} \).

The minimization and the limit in (26) is not commutative. It is straightforward to prove the inequality

\[
\lim_{T \to \infty} \min_{K_0 \subseteq K_{\text{all}} : |K_0| = \hat{K}} E_0 (H_{K_0}, \{x_{K_0,t}\}) \leq \min_{K \subseteq K_{\text{all}} : |K| = \hat{K}} \lim_{T \to \infty} E_0 (H_K, \{x_{K,t}\}).
\]

(27)

Comparing (25) and (26), we find that the energy penalty of the DD-US in \( T \to \infty \) provides a lower bound on that of the conventional US. Let us prove the inequality (27). We start with a trivial inequality

\[
\min_{K_0 \subseteq K_{\text{all}} : |K_0| = \hat{K}} E_0 (H_{K_0}, \{x_{K_0,t}\}) \leq E_0 (H_K, \{x_{K,t}\}),
\]

(28)

where \( K \subseteq K_{\text{all}} \) with \( |K| = \hat{K} \) denotes the set of selected users (26) for the conventional US. We next take the limit \( T \to \infty \). Since \( K \) is independent of the data symbols, we can use the weak law of large numbers for the right-hand side (RHS) of (28) to find that \( E_0 (H_K, \{x_{K,t}\}) \) converges in probability to \( \text{Tr}\{(H_K H_H)^{-1}\} \) or equivalently the RHS of (27) in \( T \to \infty \). Thus, we obtain the inequality (27).

### III. Main Results

#### A. Large-System Analysis

We use the replica method to analyze the performance of US-VP in the large-system limit where the number of transmit antennas \( N \), the number of users \( K \), and the number of selected users \( \hat{K} \) tend to infinity while their ratios \( \alpha = K/N \) and \( \kappa = \hat{K}/K \) are kept constant. Without loss of generality, we focus on the first block \( i = 0 \) of US-VP and drop the subscripts \( i \) from \( E_i (H_{K_i}, \{x_{K_i,t}\}) \), \( K_i \), and \( s_{k,i} \).

The asymptotic performance of US-VP is characterized via a solvable US-VP problem. We first define the solvable problem. For a positive parameter \( q \), let us define a random variable \( \tilde{E}_k(\{\tilde{x}_{k,t}\}, q) \) as

\[
\tilde{E}_k(\{\tilde{x}_{k,t}\}, q) = \frac{1}{T} \sum_{t=0}^{T-1} \left| \tilde{x}_{k,t} - \sqrt{q} x_{k,t} \right|^2.
\]

(29)

In (29), \( \{\tilde{x}_{k,t}\} \) are independent circularly symmetric complex Gaussian random variables with unit variance. The normalized parameter \( q/\alpha K \) will be shown to be equal to the average energy penalty per selected user in the large-system limit. The solvable US-VP problem is the following minimization problem:

\[
E_K = \min_{K \subseteq K_{\text{all}} : |K| = K} \frac{1}{K} \sum_{k \in K} \tilde{E}_k(\{\tilde{x}_{k,t}\}, q).
\]

(30)

The asymptotic performance of US-VP is characterized via three quantities for (30) in the large-system limit. The minimization in (30) with respect to \( \{\tilde{x}_{k,t}\} \) is straightforwardly solved to obtain

\[
E_K = \min_{K \subseteq K_{\text{all}} : |K| = K} \frac{1}{K} \sum_{k \in K} E_k(q),
\]

(31)
with
\[ E_k(q) = \frac{1}{T} \sum_{t=0}^{T-1} |\tilde{x}_{k,t}^{(opt)}(q) - \sqrt{q}z_{k,t}|^2, \] (32)

where \( \tilde{x}_{k,t}^{(opt)}(q) \) denotes the optimal modified data symbol, given by
\[ \tilde{x}_{k,t}^{(opt)}(q) = \arg\min_{\tilde{x}_{k,t} \in \mathcal{X}_{k,t}} |\tilde{x}_{k,t} - \sqrt{q}z_{k,t}|^2. \] (33)

In order to solve the minimization (31) analytically, we write the order statistics for the random variables \( \{E_k(q)\} \) as \( \{E_{(k)}(q)\} \), i.e. \( E_{(1)}(q) \leq \cdots \leq E_{(K)}(q) \). The minimization (31) reduces to
\[ E_K = \frac{1}{K} \sum_{k=1}^{K} \left( \frac{k}{K} \leq \kappa \right) E_{(k)}(q). \] (34)

The three quantities that characterize the asymptotic performance of US-VP is the mean and variance for (34) in the large-system limit, and the \( K \)th order statistic \( E_{(K)}(q) \) for (32) in the large-system limit. The three quantities are given via the cumulative distribution function (cdf) of (32),
\[ F_T(x; q) = \Pr(E_k(q) \leq x). \] (35)

Note that the cdf (35) is monotonically increasing, because of \( z_{k,t} \sim \mathcal{CN}(0, 1) \). Thus, there exists the inverse function of (35), denoted by \( F_T^{-1}(x; q) \).

**Lemma 1.** Let \( \xi_{\kappa,T}(q) \) denote the \( \kappa \)-quantile for the cdf (35),
\[ \xi_{\kappa,T}(q) = F_T^{-1}(\kappa; q). \] (36)

Then, the \( K \)th order statistic \( E_{(K)}(q) \) for (32) converges in probability to the \( \kappa \)-quantile \( \xi_{\kappa,T}(q) \) in the large-system limit.

**Proof of Lemma 1.** Since \( E_{(K)}(q) \) is a sample \( \kappa \)-quantile for the independent random variables (32) with the common cdf (35), Bahadur’s theorem [51] or its modification [52] implies that
\[ E_{(K)}(q) = \xi_{\kappa,T}(q) - \frac{\hat{F}_T(\xi_{\kappa,T}(q); q) - \kappa}{\hat{F}_T(\xi_{\kappa,T}(q); q)} + o(K^{-1/2}), \] (37)
in the large-system limit, where \( \hat{F}_T(x; q) \) is the empirical cdf for (32), given by
\[ \hat{F}_T(x; q) = \frac{1}{K} \sum_{k=1}^{K} 1(E_k(q) \leq x). \] (38)

The mean and variance of the empirical cdf (38) at \( x = \xi_{\kappa,T}(q) \) are given by
\[ \mathbb{E}[\hat{F}_T(\xi_{\kappa,T}(q); q)] = \kappa, \] (39)
\[ \text{Var}[\hat{F}_T(\xi_{\kappa,T}(q); q)] = \frac{\kappa(1 - \kappa)}{K}, \] (40)
respectively. Thus, the second term on the RHS of (37) is a quantity of \( O(K^{-1/2}) \). This observation implies that (37) converges in probability to the \( \kappa \)-quantile (36) in the large-system limit.

**Lemma 2** (Stigler 1974). Let \( \mu_{\kappa,T}(q) \) and \( \sigma^2_{\kappa,T}(q) \) denote the mean of \( E_K \) and the variance of \( \sqrt{K}E_K \), given by (34), in the large-system limit. Then,
\[ \mu_{\kappa,T}(q) = \int_0^\infty F_T^{-1}(x; q)dx, \] (41)
\[ \sigma^2_{\kappa,T}(q) = \int_0^{\xi_{\kappa,T}(q)} \int_0^{\xi_{\kappa,T}(q)} [F_T(\min(x, y); q) - F_T(x; q)F_T(y; q)]dxdy, \] (42)
where the \( \kappa \)-quantile \( \xi_{\kappa,T}(q) \) is given by (36).

**Proof of Lemma 2.** The function \( 1(k/K \leq \kappa) \) in (34) is bounded and continuous almost everywhere (a.e.) \( F_T^{-1} \), since the cdf (35) is monotonically increasing. Thus, we can use Stigler’s theorems [53] Theorems 1 and 3] to obtain (41) and (42).

We need calculate the cdf (35) to evaluate the three quantities (36), (41), and (42). See Appendix A for how to calculate the cdf (35).
B. Average Energy Penalty

The average of the energy penalty (21) for US-VP [18], denoted by \( \bar{\mathcal{E}} = \mathbb{E}[\mathcal{E}(\{H_{K_i}\}, \{\hat{x}_{K_i}, \ell\})] \), is analyzed in the large-system limit. We use the replica method under the RS and 1RSB assumptions [23], [34]. Roughly speaking, the RS assumption corresponds to postulating that there are no local minimizers to the minimization (18) in the large-system limit. On the other hand, the 1RSB assumption is the simplest assumption for the case where there are many local minimizers in the large-system limit.

**Proposition 1.** Suppose that \( q_0 \) is the solution to the fixed-point equation

\[
q_0 = \alpha \mu_{\kappa, \chi}(q_0),
\]

where \( \mu_{\kappa, \chi}(q) \) is given by (41). If (43) has multiple solutions, the smallest solution \( q_0 \) is selected. Under the RS assumption, the average energy penalty per selected user \( \bar{\mathcal{E}}/K \) converges to \( q_0/(\alpha \kappa) \) in the large-system limit.

**Derivation of Proposition 1.** See Appendix C.

**Proposition 2.** Suppose that \( q_1 \) satisfies the coupled fixed-point equations

\[
\ln \left(1 + \frac{q_1}{\chi}\right) = \frac{\alpha}{\chi} \left(\mu_{\kappa, \chi}(q_1) - \frac{T \sigma_{\kappa, \chi}^2(q_1)}{2 \chi}\right),
\]

\[
\frac{q_1}{\chi + q_1} = \frac{\alpha}{\chi} \left(\mu_{\kappa, \chi}(q_1) - \frac{T \sigma_{\kappa, \chi}^2(q_1)}{\chi}\right),
\]

for some \( 0 < \chi < \infty \), in which \( \mu_{\kappa, \chi}(q) \) and \( \sigma_{\kappa, \chi}^2(q) \) are given by (41) and (42), respectively. If there are multiple solutions, the smallest solution \( q_1 \) is selected. Under the 1RSB assumption, the average energy penalty per selected user \( \bar{\mathcal{E}}/K \) converges to \( q_1/(\alpha \kappa) \) in the large-system limit.

**Derivation of Proposition 2.** See Appendix D.

The asymptotic energy penalty for VP was calculated with the R-transform for the empirical eigenvalue distribution of \( hh^{H}i^{-1} \) [27], [30]. Since it is difficult to apply this method to our case, another method is used in the calculation of the energy penalty, as presented in Appendix C. Note that the meanings of RSB are different for the two methods. The two methods should yield the same result under the full-RSB assumption, since the full-RSB assumption is expected to provide the correct solution [46], [47]. However, they may yield different results under the RS and 1RSB assumptions, since these assumptions are approximations. In fact, the two methods seem to yield different results under the 1RSB assumption, whereas the same result is obtained under the RS assumption.

It is straightforward to show that the RS assumption provides a smaller prediction of the energy penalty than the 1RSB assumption.

**Property 1.** Let \( q_0 \) and \( q_1 \) denote the solutions defined in Proposition 1 and Proposition 2, respectively. Then,

\[
q_0 < q_1.
\]

**Proof of Property 1.** Eliminating \( \sigma_{\kappa, \chi}^2(q_1) \) with (44) and (45) yields

\[
2\chi \ln \left(1 + \frac{q_1}{\chi}\right) - \frac{\chi q_1}{\chi + q_1} = \alpha \mu_{\kappa, \chi}(q_1).
\]

We write the left-hand side (LHS) of (47) as \( f(q_1, \chi) \). It is straightforward to prove \( f(q, \chi) < q \) for any \( q > 0 \) and \( \chi > 0 \). Calculating the first and second derivatives of \( f(q, \chi) \) with respect to \( \chi \), we obtain

\[
\frac{\partial f}{\partial \chi} = 2 \ln \left(1 + \frac{q}{\chi}\right) - \frac{q(2\chi + 3q)}{(\chi + q)^2},
\]

\[
\frac{\partial^2 f}{\partial \chi^2} = -\frac{2q^3}{\chi(\chi + q)^3} < 0.
\]

Since \( \lim_{\chi \to \infty} \partial f/\partial \chi = 0 \), (48) and (49) imply \( \partial f/\partial \chi > 0 \) for any \( q > 0 \). Furthermore, \( \lim_{\chi \to \infty} f(q, \chi) = q \) indicates \( f(q, \chi) < q \) for any \( q > 0 \) and \( \chi > 0 \).

Let us prove \( q_0 < q_1 \). Since (41) is positive in \( q \to 0 \), from (45) we find \( q \leq \alpha \mu_{\kappa, \chi}(q) \) for any \( q \in (0, q_0) \). Then,

\[
f(q, \chi) < q \leq \alpha \mu_{\kappa, \chi}(q),
\]

for any \( q \in (0, q_0] \) and \( \chi > 0 \). This inequality implies that (47) has no solutions for any \( q_1 \in (0, q_0] \). Thus, we obtain \( q_0 < q_1 \).

We next calculate the average energy penalty in \( T \to \infty \) to derive a performance bound for separate US-VP.
Corollary 1. Suppose that \( q_0 \) is the solution to the fixed-point equation
\[
q_0 = \alpha \kappa \mathbb{E} \left[ \min_{\tilde{x}_{k,t} \in \mathcal{M}_{q_{k,t}}} |\tilde{x}_{k,t} - \sqrt{q_0 z_{k,t}}|^2 \right].
\] (51)

If (51) has multiple solutions, the smallest solution \( q_0 \) is selected. Under the RS assumption, the average energy penalty per selected user \( \tilde{E}_K / K \) converges to \( q_0 / (\alpha \kappa) \) in \( T \to \infty \) after taking the large-system limit.

Proof of Corollary 1. See Appendix \[.]

An informal derivation of (51) is as follows: First of all, one should recall that \( \mu_{\kappa, T}(q) \) have been defined as the mean of (31) in the large-system limit. The weak law of large numbers implies that each term (32) in (31) converges in probability to the expectation \( \mathbb{E}[E_k(q)] \) in \( T \to \infty \), which is equal to the expectation on the RHS of (51) with \( q_0 = q \). Thus, the minimization in (31) should make no sense in \( T \to \infty \), i.e., (51) should tend to \( \kappa \mathbb{E}[E_k(q)] \) in \( T \to \infty \). This implies that the fixed-point equation (43) reduces to (51) in \( T \to \infty \).

As noted in Remark 1, the energy penalty for the DD-US in \( T \to \infty \) provides a lower bound on that for the conventional (data-independent) US (26). For the DD-US, it is possible to solve (51).

\[
\frac{q_0}{\alpha \kappa} = \frac{1}{1 - \alpha \kappa}.
\] (52)

Comparing (14) and (52), we find that the energy penalty for the DD-US in \( T \to \infty \) is achievable by the ZFBF with random US (RUS), referred to as ZFBF-RUS in this paper, in which a subset of users with size \( \tilde{K} \) is selected randomly and uniformly. This observation under the RS assumption implies that the performance of the ZFBF with the optimal US is equal to that of the ZFBF-RUS in the large-system limit. Furthermore, from Property 1 we can conclude that the 1RSB assumption yields a wrong result in \( T \to \infty \). The same statements also hold for separate US-CVP: Under the RS assumption, the performance of separate US-CVP is equal to that of the CVP with RUS (CVP-RUS) in the large-system limit. Furthermore, the 1RSB assumption yields a wrong result in \( T \to \infty \).

One cannot conclude from the results under the RS and 1RSB assumptions that conventional (data-independent) US makes no sense in the large-system limit, since there is a possibility that the full-RSB assumption provides a smaller energy penalty than the RS assumption. Unfortunately, it is difficult to calculate the energy penalty under higher-step RSB assumptions, so that whether this statement is correct should be checked by using another methodology. We leave this issue as future work, since it is beyond the scope of this paper.

C. Sum Rate

Before investigating the achievable sum rate of US-VP, the joint distribution of the indicator variable (23) and the modified data symbols \( \tilde{X}_k = \{ \tilde{x}_{k,t} : t = 0, \ldots, T - 1 \} \) given the data symbols \( X_k = \{ x_{k,t} : t = 0, \ldots, T - 1 \} \) is analyzed in the large-system limit. This joint distribution is used to calculate the achievable sum rate.

Let \( \{ A_t \} \) denote measurable subsets of \( \mathbb{C} \) for \( t = 0, \ldots, T - 1 \). The joint distribution is shown to be characterized via the conditional probability
\[
\Pr \left( E_k(q) \leq \xi_{\kappa, T}(q), \{ \tilde{x}_{k,t}^{(\text{opt})}(q) \in A_t \} \big| X_k \right),
\] (53)

where \( E_k(q), \tilde{x}_{k,t}^{(\text{opt})}(q) \), and \( \xi_{\kappa, T}(q) \) are given by (32), (35), and (36), respectively.

Proposition 3. Suppose that \( q_0 \) is the same solution as in Proposition 1. Under the RS assumption, the conditional joint probability \( \Pr(s_k = 1, \tilde{X}_k \in \prod_{t=0}^{T-1} A_t | X_k) \) converges to (52) with \( q = q_0 \) in the large-system limit.

Derivation of Proposition 3. See Appendix \[.

Proposition 4. Suppose that \( q_1 \) is the same solution as in Proposition 2. Under the 1RSB assumption, the conditional joint probability \( \Pr(s_k = 1, \tilde{X}_k \in \prod_{t=0}^{T-1} A_t | X_k) \) converges to (52) with \( q = q_1 \) in the large-system limit.

Derivation of Proposition 4. See Appendix \[.

It is straightforward to find \( \Pr(s_k = 1) = \kappa \): Marginalizing (53) yields
\[
\Pr(s_k = 1) = \Pr \left( E_k(q) \leq \xi_{\kappa, T}(q) \right),
\] (54)

which is equal to the cdf \( F_T(\xi_{\kappa, T}(q); q) \), given by (35). The definition of the \( \kappa \)-quantile (36) implies \( \Pr(s_k = 1) = \kappa \) for any \( q \).

We next investigate the achievable sum rate of US-VP in the large-system limit. The achievable sum rate \( R \) is given by
\[
R = \sum_{k=1}^{K} R_k,
\] (55)
where the achievable rate $R_k$ for user $k$ is defined as the mutual information per time slot between all data symbols $X^{(all)}_k$ for user $k$ and all received signals $Y^{(all)}_k = \{y_{k,t} : t = 0, \ldots, T - 1\}$ for user $k$ [34], [55],

$$R_k = \lim_{T \to \infty} \frac{1}{T} I \left( X^{(all)}_k; Y^{(all)}_k \right). \quad (56)$$

In (56), the received signal $y_{k,t}$ is transmitted through the equivalent channel (22), in which $y_{k,t}$ depends on the data symbol $x_{k,t}$ through the modified data symbol $\tilde{x}_{k,t} \in M_{x_{k,t}}$.

A crucial assumption in evaluating the achievable rate (56) is the self-averaging property of the energy penalty (21) in the large-system limit.

**Assumption 1** (Self-Averaging Property). The energy penalty per selected user for US-VP converges in probability to the expectation $\bar{E} = \mathbb{E}[\{(H_{K_k}), \{\tilde{x}_{K_k,t}\})]$ in the large-system limit.

The energy penalty corresponds to free energy in the low-temperature limit or equivalently ground state energy in statistical physics [33], [34] (See Appendix C). Normalized ground state energy is believed to be self-averaging for many disordered systems. In fact, the self-averaging property of ground state energy was proved for a generalized spin glass model [56]–[58] and for MIMO systems [59]. Since the proof of Assumption 1 is beyond the scope of this paper, we postulate the self-averaging property of the energy penalty in the large-system limit.

The following lemma provides a genie-aided upper bound on the achievable sum rate (55) in the large-system limit.

**Lemma 3.** Suppose that Assumption 1 holds. Then, the achievable sum rate (55) per transmit antenna $R/N$ is bounded from above by

$$\bar{C} = \frac{\alpha}{T} \left( H(\kappa) + \kappa I(X^{(all)}_k; Y_k|s_k = 1) \right) \quad (57)$$

in the large-system limit, with $X_k = \{x_{k,t} : t = 0, \ldots, T - 1\}$ and $Y_k = \{y_{k,t} : t = 0, \ldots, T - 1\}$ denoting the data symbols and the received signals in the first block, respectively. In (57), $H(\kappa)$ denotes the binary entropy function

$$H(\kappa) = -\kappa \log \kappa - (1 - \kappa) \log(1 - \kappa). \quad (58)$$

**Proof of Lemma 3.** The received signal $y_{k,t}$ given by (22) depends on all data symbols for user $k$ through the energy penalty (21). Under Assumption 1, this dependencies disappear in the large-system limit: The equivalent channel (22) reduces to

$$y_{k,t} = \sqrt{\frac{P}{q}} \left\{ s_{k,i} \tilde{x}_{k,t} + (1 - s_{k,i}) I_{k,t} \right\} + n_{k,t} \quad (59)$$

in the large-system limit, where $q$ is equal to $q_0$ or $q_1$ in Propositions 1 and 2. Since the US-VP (18) is performed block by block, the received signals (59) are i.i.d. block by block. As a result, the achievable rate (56) reduces to

$$R_k = \frac{1}{T} I(X_k; Y_k) \quad (60)$$

in the large-system limit.

We next consider a genie-aided upper bound on (60), in which a genie informs each user about whether he/she has been selected in the first block,

$$R_k \leq \frac{1}{T} I(X_k; Y_k, s_k), \quad (61)$$

where $s_k$ is the indicator variable (23) to represent whether user $k$ has been selected. In (61), we have dropped the subscript $i$ from $s_{k,i}$. The upper bound (61) is formally obtained from the chain rule for mutual information [54],

$$I(X_k; Y_k, s_k) = I(X_k; Y_k) + I(X_k; s_k|Y_k) \quad > I(X_k; Y_k). \quad (62)$$

Applying the chain rule for mutual information to the RHS of (61) yields

$$I(X_k; Y_k, s_k) = I(X_k; s_k) + I(X_k; Y_k|s_k). \quad (63)$$

Since $s_k$ is a binary variable, the conditional entropy $H(s_k|X_k)$ is non-negative [54], so that the first term on the RHS of (63) is bounded from above by the entropy of $s_k$,

$$I(X_k; s_k) = H(s_k) - H(s_k|X_k) < H(s_k). \quad (64)$$

Since the prior probability $\Pr(s_k = 1)$ is equal to $\kappa$, the entropy $H(s_k)$ is equal to the binary entropy function $H(\kappa)$. On the other hand, the second term should be equal to

$$I(X_k; Y_k|s_k = 1) = \kappa I(X_k; Y_k|s_k = 1), \quad (65)$$

where

$$I(X_k; Y_k|s_k = 1) = \kappa I(X_k; Y_k|s_k = 1). \quad (65)$$
where $\kappa$ is the probability with which $s_k$ takes 1. This can be understood as follows: The equivalent channel (59) implies that user $k$ receives the interference (54) when $s_k = 0$. In this case, the interference $I_{k,t}$ does not contain the desired data symbols $\{\tilde{x}_{k,t}\}$ for user $k$. Strictly speaking, there may be dependencies between the received signals for $s_k = 0$ and the desired data symbols, since the set of selected users depends on the desired data symbols for user $k$. However, these dependencies should be negligible in the large-system limit, so that we obtain (65). Combining (55), (61), (63), (64) with $H(s_k) = H(\kappa)$, and (65), we arrive at the upper bound (67).

In the derivation of the upper bound (67), we have used the two upper bounds (61) and (64). The looseness of (67) due to the latter bound is negligible as $T \to \infty$, since $T^{-1}H(\kappa)$ tends to zero. On the other hand, the genie-aided bound (61) also becomes tight as $T \to \infty$, since the detection of $s_k$ becomes easy as $T$ increases. See [31] for an iterative algorithm to detect $s_k$. For example, the SNR loss required for detecting $s_k$ is at most 0.5 dB for a sum rate per transmit antenna of 0.5 bps/Hz when $T = 16$ and QPSK are used. Furthermore, the SNR loss is at most 0.2 dB for 1 bps/Hz. These observations may imply that the upper bound (67) is reasonably tight for a few dozen $T$.

As shown in Appendix B, $s_k$ is independent of $X_k$ for the DD-US with QPSK. As a result, the mutual information $I(X_k;Y_k|s_k)$ in (67) reduces to

$$
I(X_k;Y_k|s_k = 1) = \kappa TI(x_{k,t};y_{k,t}|s_k = 1)
$$

for the DD-US with QPSK. However, it is still hard to calculate the upper bound (67) for general modulation, since $s_k$ depends on $X_k$. From Lemma 3, we obtain an upper bound for the DD-US that is possible to calculate for large $T$.

**Corollary 2** (Bound for DD-US). Suppose that Assumption 1 holds. Then, the achievable sum rate (55) per transmit antenna $R/N$ for the DD-US is bounded from above by

$$
\mathcal{C}_{DD-US} = \alpha \left\{ \frac{H(\kappa)}{T} + \kappa I(x_{k,t};y_{k,t}|s_k = 1) \right\},
$$

where $H(\kappa)$ denotes the binary entropy function (58).

**Proof of Corollary 2**. It is sufficient from Lemma 3 to prove the following upper bound:

$$
I(X_k;Y_k|s_k = 1) \leq TI(x_{k,t};y_{k,t}|s_k = 1).
$$

By definition,

$$
I(X_k;Y_k|s_k = 1) = h(Y_k|s_k = 1) - h(Y_k|X_k, s_k = 1).
$$

Since $\tilde{x}_{k,t} = x_{k,t}$ holds for the DD-US, the second term $h(Y_k|X_k, s_k = 1)$ is equal to the differential entropy $TH(n_{k,t})$ of the noise $\{n_{k,t}\}$ in (59). On the other hand, the conditional differential entropy $h(Y_k|s_k = 1)$ is bounded from above by the sum of the conditional differential entropy for each received signal (54).

$$
h(Y_k|s_k = 1) \leq \sum_{t=0}^{T-1} h(y_{k,t}|s_k = 1).
$$

These observations imply that the RHS of (69) is equal to the upper bound (68).

We shall explain how to calculate the conditional mutual information $I(x_{k,t};y_{k,t}|s_k = 1)$, which is given by

$$
I(x_{k,t};y_{k,t}|s_k = 1) = \mathbb{E} \left[ \log \frac{p(y_{k,t}|x_{k,t},s_k = 1)}{p(y_{k,t}|s_k = 1)} \right],
$$

with

$$
p(y_{k,t}|s_k = 1) = \mathbb{E}_{x_{k,t}} [p(y_{k,t}|x_{k,t},s_k = 1)|s_k = 1],
$$

where the conditional pdf $p(y_{k,t}|x_{k,t},s_k = 1)$ is given by

$$
p(y_{k,t}|x_{k,t},s_k = 1) = \mathbb{E}_{\tilde{x}_{k,t}} [p(y_{k,t}|\tilde{x}_{k,t},s_k = 1)|x_{k,t},s_k = 1].
$$

In (73), the pdf $p(y_{k,t}|\tilde{x}_{k,t},s_k = 1)$ characterizes the equivalent channel (59),

$$
p(y_{k,t}|\tilde{x}_{k,t},s_k = 1) = PCG \left( y_{k,t} - \sqrt{\frac{P}{q}} \tilde{x}_{k,t}; N_0 \right),
$$

with (2), where $q$ is equal to $q_0$ or $q_1$ in Propositions 1 and 2.

2 This statement does not hold for the US-CVP. Consequently, we need to derive a tight lower bound of the second term to obtain an upper bound on (59). Unfortunately, we are unable to find such a tight lower bound that can be calculated easily.
In order to calculate the expectations in (72) and (73), we need the joint posterior probability of $\tilde{x}_{k,t}$ and $x_{k,t}$ given $s_k = 1$, given by

$$
\Pr(\tilde{x}_{k,t} \in \tilde{A}, x_{k,t} \in A | s_k = 1) = \frac{\Pr(s_k = 1, \tilde{x}_{k,t} \in \tilde{A}|x_{k,t})\Pr(x_{k,t} \in A)}{\Pr(s_k = 1)},
$$

with measurable sets $\tilde{A} \subset \mathbb{C}$ and $A \in \mathbb{C}$. In (75), $\Pr(x_{k,t} \in A)$ denotes the prior probability of $x_{k,t}$. Furthermore, the conditional probability $\Pr(s_k = 1, \tilde{x}_{k,t} \in \tilde{A}|x_{k,t})$ is characterized via Proposition 3 or Proposition 4:

$$
\Pr(s_k = 1, \tilde{x}_{k,t} \in \tilde{A}|x_{k,t}) = \Pr \left( E_k(q) \leq \xi_{\kappa,T}(q), \tilde{x}_{k,t}^{(\text{opt})}(q) \in \tilde{A} | x_{k,t} \right),
$$

with $q = q_0$ and $q = q_1$ under the RS and 1RSB assumptions, respectively. In summary, it is possible to calculate the mutual information (71) from (72)–(76). See Appendix B for the details.

IV. NUMERICAL RESULTS

A. Energy Penalty

US-VP is compared to ZFBF-RUS and CVP-RUS in terms of the average energy penalty. As noted in Section III-B, ZFBF-RUS and CVP-RUS provide lower bounds on the energy penalties of conventional US with ZFBF and separate US-CVP, respectively. The block length $T$ is kept finite, while the coherence time $T_c$ is implicitly assumed to tend to infinity.

We first focus on the DD-US with Gaussian signaling $x_{k,t} \sim \mathcal{CN}(0,1)$. Figure 1 shows the average energy penalty per selected user $\bar{E}/\tilde{K}$ based on Propositions 1 and 2. For comparison, the energy penalties of ZFBF-RUS and CVP-RUS are also shown on the basis of Corollary 1. The energy penalty of CVP-RUS was originally evaluated in [27]. Furthermore, we plot the energy penalty of a greedy algorithm for the DD-US with Gaussian signaling proposed in [31]. We obtain three observations: First, the RS solution is indistinguishable from the 1RSB solution for low-to-moderate $\alpha\kappa = \tilde{K}/N$, whereas there is a gap between the two solutions for large $\alpha\kappa$. Secondly, the energy penalty of the greedy algorithm is close to the RS and 1RSB solutions for low-to-moderate $\alpha\kappa$. This observation implies that the two solutions can provide acceptable approximations for the actual energy penalty in the same region. Finally, the DD-US outperforms ZFBF-RUS and CVP-RUS for low-to-moderate $\alpha\kappa$. Note that the energy penalty of CVP-RUS for finite-sized systems gets closer from above to the asymptotic one [30], whereas the energy penalty of the DD-US gets closer from below to the asymptotic one, as shown in Fig. 1. This implies that the performance gap between the DD-US and CVP-RUS should be larger for finite-sized systems.

We next focus on the average energy penalty of the DD-US with QPSK, shown in Fig. 2. For comparison, the energy penalty of the US-CVP is also shown on the basis of Propositions 1 and 2. Furthermore, we plot the energy penalty of the greedy algorithm for the DD-US with QPSK [31]. Three observations are obtained: First, the RS and 1RSB solutions for the DD-US are indistinguishable from the respective solutions for the US-CVP in the low-to-moderate regime of $\alpha\kappa$. Second, the RS and 1RSB solutions for the US-CVP are close to each other for moderate-to-large $\alpha\kappa$, whereas there is a gap between the two solutions for small $\alpha\kappa$. Finally, the 1RSB solution provides an acceptable approximation for moderate-to-large $\alpha\kappa$, while the RS solution does for small $\alpha\kappa$. 

![Fig. 1. $\bar{E}/\tilde{K}$ versus $\alpha\kappa = \tilde{K}/N$ for $\alpha = 4$, $T = 64$, and Gaussian signaling.](image)
Finally, we investigate the impacts of $T$ and $\alpha$ on the average energy penalty. Figure 3 shows the energy penalty of the DD-US with Gaussian signaling versus $T$. For small $T$, the energy penalty of the DD-US increases quickly as $T$ grows. For moderate-to-large $T$, on the other hand, it increases slowly toward that for ZFBF-RUS. Figure 4 shows the energy penalty of the DD-US with Gaussian signaling versus $\alpha$. The RS solution is indistinguishable from the 1RSB solution, except for large $\alpha$. We find that the gap between the analytical predictions and the energy penalty of the greedy algorithm [31] becomes large as $\alpha$ increases. This may be due to the suboptimality of the greedy algorithm [31].

**B. Sum Rate**

The DD-US is compared to ZFBF-RUS, CVP-RUS, and the DPC without power allocation in terms of the achievable sum rate. For the DD-US, we use the upper bound (67) on the achievable sum rate of the DD-US. The achievable sum rate of CVP-RUS was evaluated in [30]. The achievable sum rate of the DPC without power allocation is equal to the sum capacity of a dual MIMO uplink [14] with no power allocation. The sum capacity of the dual MIMO uplink is possible to calculate in the large-system limit [7].

Before presenting the achievable rates, the distribution of the power of the modified symbol $\tilde{x}_{k,t}$ given $s_k = 1$ is investigated for the DD-US with Gaussian signaling, which can be calculated via (75). Figure 5 shows the pdf of $|\tilde{x}_{k,t}|^2$ given $s_k = 1$. For comparison, we also plot the prior pdf of the original data symbol $x_{k,t}$. The DD-US selects the data symbols with smaller power to reduce the energy penalty. Consequently, the pdf of the power of the modified symbol $\tilde{x}_{k,t}$ has lighter tail than the prior pdf. This non-Gaussianity of the modified symbol results in a rate loss.

Figure 6 shows the upper bound (67) on the achievable sum rate per transmit antenna of the DD-US. There is optimal $\alpha_k$ or equivalently the optimal number of selected users to maximize the sum rate for all schemes. This can be understood as follows: Increasing the number of selected users results in a degradation of the energy penalty and in an increase of spatial
The latter effect is dominant for small $\alpha K$, whereas the former is for large $\alpha K$. Consequently, the sum rates are maximized at an optimal number of selected users.

Figure 7 shows the upper bound (67) with the optimal number of selected users. The RS and 1RSB solutions for the DD-US with Gaussian signaling are close to each other. Furthermore, the upper bounds for the DD-US with Gaussian signaling are larger than the achievable sum of CVP-RUS [30] for all transmit SNRs, while the DD-US with QPSK is comparable to CVP-RUS. Unfortunately, the upper bounds for the DD-US with Gaussian signaling are far from the achievable sum rate of DPC. For sum rates per transmit antenna of 0.5 bps/Hz and 1 bps/Hz, the DD-US with Gaussian signaling provides performance gains of 1.2 dB and 1.4 dB, respectively, compared to CVP-RUS. Note that the SNR loss required for detecting whether each user has been selected is ignored for the upper bound (67). The upper bound becomes tight as $T$ grows. For example, the SNR loss for an iterative detection algorithm proposed in [31] is 0.5 dB for a sum rate per transmit antenna of 0.5 bps/Hz when $T = 16$ and QPSK are used. Furthermore, the SNR loss is 0.2 dB for 1 bps/Hz. These results may imply that the DD-US with Gaussian signaling outperforms CVP-RUS in terms of the actual achievable sum rate.

V. CONCLUSIONS

Joint US-VP has been compared to separate US-VP in the large-system limit, where the numbers of transmit antennas, users, and selected users tend to infinity while their ratios are kept constant. The analyses under the RS and 1RSB assumptions have shown that conventional (data-independent) US may make no sense in the large-system limit: Under the RS and 1RSB assumptions, RUS achieves the same performance as optimal data-independent US in the large-system limit. Since conventional US is capacity-achieving as only the number of users tends to infinity, this implies that whether conventional US works well depends on how to take asymptotic limits. Joint US-VP can provide a substantial reduction of the energy penalty in the
large-system limit. Consequently, joint US-VP outperforms separate US-VP in terms of the achievable sum rate. In particular, DD-US can be applied to general modulation, and implemented easily with a greedy algorithm.

**APPENDIX A**

**CALCULATION OF (35)**

A. Fourier Representation

The cdf (35) can be calculated via the characteristic function of (32). Let $G_T(\omega)$ denote the characteristic function of (32),

$$G_T(\omega) = \mathbb{E}\left[e^{i\omega E_k(q)}\right].$$  

(77)

Since (32) is the sum of i.i.d. random variables, (77) is decomposed into

$$G_T(\omega) = G\left(\frac{\omega}{2T}\right)^{2T},$$  

(78)

with

$$G(\omega) = \mathbb{E}\left[e^{i\omega\min_{|\hat{y}_k|\in M_k,|z_k|<1}(\sqrt{2}\Re[\hat{y}_k] - \sqrt{2}\Re[z_k])^2}\right].$$  

(79)

In (79), $M_k$ is given by $\{x\}$ for the DD-US and (17) for the US-CVP, respectively.

It is well-known that the pdf of (32) is given by the inverse Fourier transform

$$p(E_k(q) = E) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_T(\omega)e^{-i\omega E}d\omega.$$  

(80)
Integrating the pdf (80) from 0 to \( x \), we obtain

\[
F_T(x; q) = \int_{-\infty}^{\infty} \frac{1 - e^{-i\omega x}}{2\pi i\omega} G_T(\omega) d\omega. \tag{81}
\]

Since

\[
\int_{-\infty}^{\infty} \frac{e^{i\omega E_k(q)}}{2\pi i\omega} d\omega = \int_{-\infty}^{\infty} \frac{\sin(\omega E_k(q))}{2\pi\omega} d\omega = \frac{1}{2}, \tag{82}
\]

(81) reduces to

\[
F_T(x; q) = \frac{1}{2} - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{\omega} G \left( \frac{\omega}{2\pi} \right)^{2T} e^{-i\omega x} d\omega, \tag{83}
\]

where we have used (78). It is possible to calculate (83) numerically when the characteristic function (79) is given.

**B. DD-US**

Let us calculate the characteristic function (79) for the DD-US. Since \( \sqrt{2q} \Re[x_{k,t}] \sim \mathcal{N}(0, q) \) and \( \mathcal{M}_x = \{ x \} \) for the DD-US, we obtain

\[
G(\omega) = \mathbb{E} \left[ \frac{1}{\sqrt{1-2i\omega q}} \exp \left( \frac{2i\omega \Re[x_{k,t}]^2}{1-2i\omega q} \right) \right]. \tag{84}
\]

For QPSK \( \Re[x_{k,t}] = \pm 1/\sqrt{2} \),

\[
G(\omega) = \frac{1}{\sqrt{1-2i\omega q}} \exp \left( \frac{i\omega}{1-2i\omega q} \right). \tag{85}
\]

For Gaussian signaling \( \Re[x_{k,t}] \sim \mathcal{N}(0, 1/2) \), (84) reduces to

\[
G(\omega) = \frac{1}{\sqrt{1-2(1+q)\omega}}, \tag{86}
\]

which is associated with the characteristic function for the chi-square distribution with one degree of freedom. In this case, the cdf (85) is associated with that for the chi-square distribution with \( 2T \) degrees of freedom:

\[
F_T(x; q) = \gamma \left( T, \frac{T \mathcal{M}_x}{1+q} \right), \tag{87}
\]

where \( \gamma(a, x) \) is the incomplete gamma function

\[
\gamma(a, x) = \frac{1}{\Gamma(a)} \int_0^x y^{a-1} e^{-y} dy, \tag{88}
\]

with \( \Gamma(x) \) denoting the gamma function.

**C. US-CVP**

Let us calculate the characteristic function (79) for the US-CVP. From (17), we obtain

\[
G(\omega) = \mathbb{E} \left[ \exp \left( i\omega \min_{\tilde{x} \in [1, \infty]} (\tilde{x} - z)^2 \right) \right], \tag{89}
\]

where the expectations are taken with respect to \( z \sim \mathcal{N}(0, q) \). Calculating the expectation yields

\[
G(\omega) = \int_{-\infty}^{1/\sqrt{q}} e^{i\omega(1-\sqrt{q})u} Du + Q \left( \frac{1}{\sqrt{q}} \right). \tag{90}
\]

In (90), \( Du \) denotes the standard Gaussian measure (4). Furthermore, \( Q(x) \) is given by (5).
APPENDIX B
SUM RATE FOR DD-US

A. Calculation of \((76)\)

The conditional probability \((76)\) for the DD-US can be calculated in the same manner as in Appendix A. Let \(G_T(\omega, \{\omega_i\}; X_k)\) denote the conditional characteristic function of \((32)\),

\[
G_T(\omega, \{\omega_i\}; X_k) = \mathbb{E} \left[ e^{i\omega T \sum_{t=0}^{T-1} \Re[\omega_i z_{k,t}^{(\text{opt})}(q)]} \right] X_k,
\]

where \(z_{k,t}^{(\text{opt})}(q)\) is given by \((33)\). From \((32)\), the characteristic function \((91)\) is decomposed into

\[
G_T(\omega, \{\omega_i\}; X_k) = \prod_{t=0}^{T-1} G \left( \frac{\omega}{2T}, \omega_t; x_{k,t} \right),
\]

where \(G(\omega, \omega_t; x_{k,t})\) is given by

\[
G(\omega, \omega_t; x_{k,t}) = \mathbb{E} \left[ \exp \left\{ 2i\omega \left| z_{k,t}^{(\text{opt})}(q) - \sqrt{q} \bar{z}_{k,t} \right|^2 
                      + i\Re[\omega^* \bar{z}_{k,t}^{(\text{opt})}(q)] \right\} x_{k,t} \right].
\]

Then, the conditional probability \((76)\) is given by

\[
\Pr(s_k = 1, \bar{x}_{k,t} \in \hat{A}|x_{k,t}) = \mathbb{E} \left[ \int_{\hat{A} \times \mathbb{C}^{T-1}} p(s_k = 1, \bar{x}_k|X_k) d\bar{x}_k \left| x_{k,t} \right. \right],
\]

with

\[
p(s_k = 1, \bar{x}_k|X_k) = \int_{-\infty}^{\infty} G(\omega, \omega_t; x_{k,t}) e^{-i\Re[\omega^* \bar{x}_{k,t}]} d\omega_t.
\]

In \((85)\), \(f(\omega, \bar{x}_{k,t}; x_{k,t})\) is defined as

\[
f(\omega, \bar{x}_{k,t}; x_{k,t}) = \frac{1}{(2\pi)^2} \int_{\mathbb{C}} G(\omega, \omega_t; x_{k,t}) e^{-i\Re[\omega^* \bar{x}_{k,t}]} d\omega_t,
\]

with \((93)\).

Let us calculate \((93)\) for the DD-US to calculate \((94)\). In the same manner as in the derivation of \((84)\), we obtain

\[
G(\omega, \omega_t; x_{k,t}) = G(\omega; x_{k,t}) e^{i\Re[\omega^* \bar{x}_{k,t}]}\]

with

\[
G(\omega; x_{k,t}) = \frac{1}{1 - 2i\eta \omega} \exp \left( \frac{2i\omega |x_{k,t}|^2}{1 - 2i\eta \omega} \right).
\]

Substituting \((97)\) into \((96)\) yields

\[
f(\omega, \bar{x}_{k,t}; x_{k,t}) = G(\omega; x_{k,t}) \delta(\bar{x}_{k,t} - x_{k,t}),
\]

which implies that \((95)\) reduces to

\[
p(s_k = 1, \bar{x}_k|X_k) = \prod_{t=0}^{T-1} \delta(\bar{x}_{k,t} - x_{k,t})
\]

\[
\cdot \left[ \frac{1}{2} - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \mathbb{P} \left( \frac{\omega}{2T}; x_{k,t} \right) e^{-i\Re[\omega^* \bar{x}_{k,t}]} d\omega \right],
\]

with \((98)\). The expressions \((98)\) and \((100)\) imply that \(s_k\) is independent of \(X_k\) for the DD-US when QPSK \(|x_{k,t}|^2 = 1\) is used. Substituting \((100)\) into \((94)\), we find that \((94)\) for the DD-US is given by

\[
\Pr(s_k = 1, \bar{x}_{k,t} \in \hat{A}|x_{k,t}) = 1(x_{k,t} \in \hat{A}) \left[ \frac{1}{2} - \frac{1}{2\pi i} \int_{-\infty}^{\infty} G \left( \frac{\omega}{2T}; x_{k,t} \right) \right.
\]

\[
\cdot \left. \left[ 2(T-1) \frac{\exp \left( -i\Re[\omega^* \bar{x}_{k,t}] \right)}{\omega} \right] d\omega \right],
\]

where \(G(\omega)\) and \(G(\omega; x_{k,t})\) are given by \((84)\) and \((98)\), respectively.
B. Calculation of (71)

The conditional pdf (73) for the DD-US reduces to

\[ p(y_{k,t}|x_{k,t}, s_k = 1) = p(y_{k,t}|\tilde{x}_{k,t} = x_{k,t}, s_k = 1) \]

(102)

with (74). We shall evaluate the conditional pdf (72) for Gaussian signaling \( x_{k,t} \sim \mathcal{CN}(0, 1) \). Substituting (75) with (101) into (72) and then calculating the integration with respect to \( x_{k,t} \), we obtain

\[ p(y_{k,t}|s_k = 1) = \frac{1}{\kappa} \frac{1}{2} p_{CG} \left( y_{k,t}; \frac{P}{q} + N_0 \right) - \frac{1}{2\pi i} \int_{-\infty}^{\infty} G \left( \frac{\omega}{2T} \right)^{2T} \]

\[ p_{CG} \left( y_{k,t}; \frac{P}{q} \sigma^2 \left( \frac{\omega}{T} \right) + N_0 \right) e^{-i\omega \xi(s)} \frac{d\omega}{\omega} \]

(103)

with

\[ \sigma^2(\omega) = \frac{1 - i q \omega}{1 - i(1 + q)\omega}. \]

(104)

In (103), \( p_{CG}(z; \sigma^2) \) and \( G(\omega) \) are given by (2) and (86), respectively. It is possible to calculate the mutual information (71) with (102) and (103).

APPENDIX C
DERIVATION OF PROPOSITION 1

A. Statistical Physics

Before deriving Proposition 1, we shall present a brief introduction on statistical physics. Statistical physics elucidates macroscopic properties of many-body systems that consist of many microscopic elements with interaction. Let \( s_i \) denote a variable that represents the state of the \( i \)th microscopic element for \( i = 1, \ldots, N \). Suppose that the interactions between the microscopic elements are characterized by Hamiltonian \( H(s) \), which is a real-valued function of the configuration \( s = (s_1, \ldots, s_N)^T \). It is known that the distribution of \( s \) is given by the so-called Gibbs-Boltzmann distribution with a positive parameter \( \beta > 0 \),

\[ \Pr(s; \beta) = \frac{1}{Z(\beta)} e^{-\beta H(s)}, \]

with

\[ Z(\beta) = \sum_{\{s\}} e^{-\beta H(s)}. \]

(106)

The parameter \( \beta \) is called “inverse temperature.” Let \( S_\beta \) denote the set of ground states \( \{s\} \) to minimize the Hamiltonian \( H(s) \). Only the ground states contribute to the Gibbs-Boltzmann distribution in the low-temperature limit \( \beta \to \infty \): The Gibbs-Boltzmann distribution (105) converges to

\[ \Pr(s; \beta) \to \frac{1}{|S_\beta|} 1(s \in S_\beta), \]

(107)

in the low-temperature limit \( \beta \to \infty \). The normalization constant (106) is called “partition function,” and is utilized to calculate several macroscopic quantities. As an example, let us calculate the internal energy \( \langle H(s) \rangle_\beta \), with \( \langle \cdots \rangle_\beta \) denoting the expectation with respect to the Gibbs-Boltzmann distribution (105). We define the free energy as

\[ f(\beta) = -\frac{1}{\beta} \ln Z(\beta), \]

with (106). It is straightforward to find that the internal energy is given by

\[ \langle H(s) \rangle_\beta = \frac{\partial}{\partial \beta} (\beta f(\beta)). \]

(109)

This implies that calculating the internal energy reduces to calculating the free energy.

Since the Gibbs-Boltzmann distribution (105) converges to (107) in the low-temperature limit, the internal energy tends to the ground state energy, which is the minimum of the Hamiltonian \( H(s) \), in the low-temperature limit. The ground state energy is possible to calculate from (108) directly:

\[ \langle H(s) \rangle_\infty = \lim_{\beta \to \infty} f(\beta). \]

(110)

We will use the formula (110) to calculate the energy penalty.
B. Formulation

The average energy penalty \( \bar{E} = \mathbb{E} \{ \mathcal{E} \{ \{ H_{K_i} \}, \{ \hat{x}_{K_i,t} \} \} \} \) for US-VP (18), given via (21), is equal to the average \( \bar{E}_i \) of the energy penalty (19) for any block \( i \). Without loss of generality, we focus on the first block \( i = 0 \) and drop the subscripts \( i \) from \( \bar{E}_i, K_i, \) and \( s_{k,i} \).

The asymptotic energy penalty for VP was analyzed with the R-transform for the empirical eigenvalue distribution of \( (HH^{\dagger})^{-1} \) (27), (30). Unfortunately, it is difficult to apply this method to our case, since the empirical eigenvalue distribution of \( (HH^{\dagger})^{-1} \) is hard to calculate. Instead, we use the following lemma to calculate the average energy penalty \( \bar{E} \) without using the R-transform.

**Lemma 4.** Let us define \( S = \text{diag} \{ s_1, \ldots, s_K \} \) and \( \bar{x}_t = (\hat{x}_{1,t}, \ldots, \hat{x}_{K,t})^T \in \prod_{k=1}^K M_{s_{k,t}}, \) with \( s_k \) given by (23). The energy penalty (19) for the first block \( i = 0 \) is equal to

\[
\mathcal{E}_\text{min} = \lim_{\lambda \to \infty} \min_{\{ s_k \}} \min_{\{ \bar{x}_t \}} \min_{\{ u_t \}} \frac{1}{T} \mathcal{H}_\lambda(S, \{ \bar{x}_t \}, \{ u_t \}),
\]

where the minimizations with respect to \( \{ s_k \}, \{ \bar{x}_t \}, \) and \( \{ u_t \} \) are over \( \{ 0, 1 \}^K, \prod_{t=0}^{T-1} \prod_{k=1}^K M_{s_{k,t}} \), and \( C^{NT} \), respectively. In (117), \( \mathcal{H}_\lambda(S, \{ \bar{x}_t \}, \{ u_t \}) \), is given by

\[
\mathcal{H}_\lambda(S, \{ \bar{x}_t \}, \{ u_t \}) = \sum_{t=0}^{T-1} \| u_t \|^2 + \lambda g(S, \{ \bar{x}_t \}, \{ u_t \}),
\]

with

\[
g(S, \{ \bar{x}_t \}, \{ u_t \}) = \sum_{t=0}^{T-1} \| S (H u_t - \bar{x}_t) \|^2 + \left( \text{Tr} S - \bar{K} \right)^2.
\]

**Proof of Lemma 4** Since the function (113) is non-negative, the function (112) is bounded in \( \lambda \to \infty \) only when \( g(S, \{ \bar{x}_t \}, \{ u_t \}) = 0 \). This implies

\[
\sum_{k=1}^K s_k = \bar{K},
\]

\[
u_t = H_K^{\dagger} \left( H_K H_K^{\dagger} \right)^{-1} \bar{x}_{K,t} = u_t^{(ZF)}(H_K, \bar{x}_{K,t}),
\]

where we have used \( \mathcal{K} = \{ k \in \mathcal{K}_{\text{all}} : s_k = 1 \} \), obtained from (23). Thus, (111) reduces to

\[
\mathcal{E}_\text{min} = \min_{\mathcal{K} \subset \mathcal{K}_{\text{all}}} \min_{|\mathcal{K}| = \bar{K}} \min_{\{ s_k \}_{k \in \mathcal{K}}} \lim_{\lambda \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} \left\| u_t^{(ZF)}(H_K, \bar{x}_{K,t}) \right\|^2,
\]

which is equal to the energy penalty (19) with US-VP (18).

We start with defining the free energy as

\[
f = -\frac{1}{\beta KT} \mathbb{E} \left[ \ln Z(\beta, \lambda) \right],
\]

where the so-called partition function \( Z(\beta, \lambda) \) is given by

\[
Z(\beta, \lambda) = \sum_{\{ s_k \}_{k \in \{ 0, 1 \}}} \prod_{t=0}^{T-1} \left( \int_{\prod_{k=1}^K M_{s_{k,t}}} \mathcal{H}_\lambda(S, \{ \bar{x}_t \}, \{ u_t \}) \right)
\]

\[
\int_{C^{NT}} e^{-\beta \mathcal{H}_\lambda(S, \{ \bar{x}_t \}, \{ u_t \})} \prod_{t=0}^{T-1} d\bar{x}_t \prod_{t=0}^{T-1} du_t,
\]

with (112). Only the minimums of (112) contribute to the free energy (117) in \( \beta \to \infty \), so that taking the limit \( \beta \to \infty \) in (117) before \( \lambda \to \infty \) yields

\[
\lim_{\lambda \to \infty} \lim_{\beta \to \infty} f = \frac{1}{K} \mathbb{E} \left[ \mathcal{E}_\text{min} \right],
\]

which is equal to the average energy penalty per selected user \( \bar{E}/\bar{K} \) for US-VP (18) from Lemma 4. Thus, calculating the average energy penalty is equivalent to evaluating the free energy (117).

We use the replica method to calculate the free energy (117) in the large-system limit. The replica method is based on the identity

\[
f = -\lim_{u \to 0} \frac{1}{\beta u KT} \ln \mathbb{E} \left[ Z(\beta, \lambda)^u \right].
\]
Since the RHS is difficult to calculate for real \( u > 0 \), we regard \( u \) as a natural number to obtain a special expression for (120) with (118).

\[
f = -\lim_{u \to 0} \frac{1}{\beta u KT} \ln Z_u(\beta, \lambda),
\]

with

\[
Z_u(\beta, \lambda) = \mathbb{E} \left[ \prod_{a=0}^{u-1} \left\{ \sum_{\{s_{k,a} \in \{0,1\}\}} \prod_{t=0}^{T-1} \left( \int_{\prod_{k=1}^{K} \mathcal{M}_{s_{k,t}}} \right) \right. \right.
\]
\[
\left. \int_{\mathbb{C}^{NT}} e^{-\beta \mathcal{H}_\lambda(S_u, \{\tilde{x}_{t,a}\}, \{u_{t,a}\})} \prod_{t=0}^{T-1} (d\tilde{x}_{t,a} d u_{t,a}) \right] \bigg|_{u=0}
\]

In (122), \( u_{t,a} \in \mathbb{C}^N, \tilde{x}_{t,a} \in \prod_{k=1}^{K} \mathcal{M}_{s_{k,t}} \), and \( s_{k,a} \in \{0,1\} \) denote replicas of the transmit vector \( u_t \), the modified data symbol vector \( \tilde{x}_t \), and the indicator variable \( s_{k} \), respectively. Furthermore, the diagonal matrix \( S_u \) is given by \( S_u = \text{diag}\{s_{1,a}, \ldots, s_{K,a}\} \).

\[\text{C. Average over Quenched Randomness}\]

We first evaluate the expectation in (122) with respect to the channel matrix \( H \). Using (112) yields

\[
Z_u(\beta, \lambda) = \int_{\mathbb{C}^{NT}} \Xi_{\beta \lambda}^u(\{u_{t,a}\}) \prod_{a=0}^{u-1} \prod_{t=0}^{T-1} \left\{ e^{-\beta\|u_{t,a}\|^2} d u_{t,a} \right\},
\]

with

\[
\Xi_{\beta \lambda}^u(\{u_{t,a}\}) = \mathbb{E} \left[ \prod_{a=0}^{u-1} \left\{ \sum_{\{s_{k,a} \in \{0,1\}\}} \prod_{t=0}^{T-1} \left( \int_{\prod_{k=1}^{K} \mathcal{M}_{s_{k,t}}} \right) \right. \right.
\]
\[
\left. \left. \int_{\mathbb{C}^{NT}} e^{-\beta \lambda g(S_u, \{\tilde{x}_{t,a}\}, \{u_{t,a}\})} \prod_{t=0}^{T-1} (d\tilde{x}_{t,a} d u_{t,a}) \right|_{u=0} \right]\]

where \( g(\{s_{k,a}\}, \{\tilde{x}_{t,a}\}, \{u_{t,a}\}) \) is given by (113). Let us define a random vector \( v_u(k) \in \mathbb{C}^T \) as

\[
v_u(k) = \sum_{n=1}^{N} (H)_{k,n} u_u(n),
\]

with \( u_u(n) = (u_{n,0,a}, \ldots, u_{n,T-1,a})^T \), in which \( u_{n,t,a} = (u_{t,a})_{n} \) denotes the \( n \)th element of \( u_{t,a} \). Since we have assumed that \( H \) has independent circularly symmetric complex Gaussian elements with variance \( 1/N \), \( v(k) = (v_0(k)^T, \ldots, v_{u-1}(k)^T)^T \) conditioned on \( \{u_{t,a}\} \) is a circularly symmetric complex Gaussian random vector with the covariance matrix

\[
Q = \frac{1}{N} \sum_{n=1}^{N} u(n) u(n)^H,
\]

with \( u(n) = (u_0(n)^T, \ldots, u_{u-1}(n)^T)^T \). The function (113) in (124) depends on \( \{u_{t,a}\} \) only through the covariance matrix (126), so that we can re-write (124) as \( \Xi_{\beta \lambda}^u(Q) \) to find that (123) reduces to

\[
Z_u(\beta, \lambda) = \left( \frac{\pi}{\beta} \right)^{uNT} \int_{\mathbb{C}^{NT}} \Xi_{\beta \lambda}^u(Q) \prod_{n=1}^{N} \left\{ \left( \frac{\beta}{\pi} \right)^{uT} e^{-\beta\|u(n)\|^2} d u(n) \right\}.
\]

In (127), \( \Xi_{\beta \lambda}^u(Q) \) is given by

\[
\Xi_{\beta \lambda}^u(Q) = \mathbb{E} \left[ \prod_{a=0}^{u-1} \left\{ \sum_{\{s_{k,a} \in \{0,1\}\}} \prod_{k=1}^{K} \left( \int_{\prod_{t=0}^{T-1} \mathcal{M}_{s_{k,t}}} \right) \right. \right.
\]
\[
\left. \left. e^{-\beta \lambda g(s_{k,a}, \{\tilde{x}_{a}(k)\}, \{u_{a}(k)\})} \prod_{k=1}^{K} (d\tilde{x}_{a}(k)) \right] \right],
\]
\[
\hat{g}(\{s_{k,a}\}, \{\tilde{x}_a(k)\}, \{u_a(k)\}) = \sum_{k=1}^{K} s_{k,a} \|v_a(k) - \tilde{x}_a(k)\|^2 + \left( \sum_{k=1}^{K} s_{k,a} - \tilde{K} \right)^2,
\]
with \(\tilde{x}_a(k) = ((\tilde{x}_0,a)_{T}, \ldots, (\tilde{x}_{T-1,a})_{T})^T\).

D. Average over Spin Variables

We next calculate the integration in (127) with respect to \(\{u(n)\}\). The expression (127) implies that \(\{u(n)\}\) can be regarded as independent circularly symmetric complex Gaussian random vectors with the covariance matrix \(\beta^{-1}I_u\). Thus, the covariance matrix (126) is regarded as a complex Wishart matrix \([50]\) with \(N\) degrees of freedom, so that the pdf of (126) is given by
\[
p(Q) = C_u e^{-\beta N \text{Tr} Q} \det Q^{N-u_T},
\]
with
\[
C_u = \frac{(\beta N)^{uT}}{\pi^{uT(uT-1)/2} \prod_{i=1}^{uT}(N-i)!}.
\]
Replacing the integration in (127) with respect to \(\{u(n)\}\) by the average over \(Q\) after substituting (127) into the free energy (121), we obtain
\[
f = -\lim_{u \to 0} \frac{1}{\beta uK} \ln \int C_u \det Q^{-uT} \cdot \exp \left\{ \beta N \left( \frac{1}{\beta N} \ln \Xi^{(u)}_{\beta}(Q) - I_u(Q) \right) \right\} dQ
\]
\[-\frac{1}{\beta \alpha K} \ln \left( \frac{\pi}{\beta} \right),
\]
with
\[
I_u(Q) = \text{Tr} Q - \frac{1}{\beta} \ln \det Q.
\]
Assuming that the large-system limit and the limit \(u \to 0\) are commutative, we use the saddle-point method to arrive at
\[
\lim_{K \to \infty} f = \lim_{u \to 0} \frac{1}{u \alpha K} \Phi_u(Q_s) - \frac{1}{\beta \alpha K} \ln(\pi e),
\]
with
\[
\Phi_u(Q) = I_u(Q) - \lim_{K \to \infty} \frac{\alpha}{\beta K} \ln \Xi^{(u)}_{\beta}(Q),
\]
where we have used the asymptotic formula for (131)
\[
\lim_{K \to \infty} \frac{1}{\beta uK} \ln C_u = \frac{1}{\beta \alpha K} \ln(\pi e) + o(1)
\]
in the large-system limit. In (134), the limit \(\lim_{K \to \infty}\) denotes the large-system limit. Furthermore, \(Q_s\) is the solution to minimize (135):
\[
Q_s = \arg\min_Q \Phi_u(Q).
\]
E. Replica Symmetry Solution

Let us assume RS for the solution (137).

Assumption 2 (Replica Symmetry).
\[
Q_s = (\chi I_u + q_0 1_u 1_u^T) \otimes I_T.
\]
We first calculate (133) to obtain
\[
\frac{1}{u^T} I_u(Q_s)
= \frac{1}{u} \left[ u(\chi + q_0) - \frac{1}{\beta} \ln(\chi + uq_0) - \frac{u - 1}{\beta} \ln \chi \right]
\rightarrow \chi + q_0 - \frac{q_0}{\beta \chi} - \frac{1}{\beta} \ln \chi,
\]
in \(u \to 0\).
We next evaluate (128). The RS assumption (138) implies that (125) is represented as
\[ v_a(k) = \sqrt{\chi} w_a(k) + \sqrt{q_0} z(k), \]  \tag{141} \]
where \( \{w_a(k) \in \mathbb{C}^T\} \) and \( \{z(k) \in \mathbb{C}^T\} \) are independent circularly symmetric complex Gaussian random vectors with covariance \( I_T \), respectively. We calculate the expectation with respect to \( \{w_a(k)\} \) to obtain
\[ \Xi_{\beta \lambda}^{(u)}(Q_s) = \mathbb{E} \left[ \left( \Xi_{\beta \lambda}^{(RS)}(\{z(k)\}) \right)^u \right], \]  \tag{142} \]
with
\[ \Xi_{\beta \lambda}^{(RS)}(\{z(k)\}) = \sum_{\{s_k \in \{0,1\}\}} \prod_{k=1}^{K} \left( \frac{\left( \int_{\mathcal{M}_{x_k,t}} \right)^{\frac{1}{\beta \lambda}}}{(1 + \beta \lambda)^T \sum_{k=1}^{K} d\hat{x}(k)} \right) e^{-H_{\beta \lambda}^{(RS)}(\{s_k, \{\hat{x}(k)\}, \{z(k)\})}, \]  \tag{143} \]
where \( H_{\beta \lambda}^{(RS)}(\{s_k, \{\hat{x}(k)\}, \{z(k)\}) \) is given by
\[ H_{\beta \lambda}^{(RS)}(\{s_k, \{\hat{x}(k)\}, \{z(k)\}) = \beta \lambda \left( \sum_{k=1}^{K} s_k - \tilde{K} \right)^2 + \sum_{k=1}^{K} \frac{\beta \lambda s_k}{1 + \beta \lambda s_k} || \hat{x}(k) - \sqrt{q_0} z(k) ||^2. \]  \tag{144} \]
Taking \( u \to 0 \) yields
\[ \lim_{u \to 0} \lim_{K \to \infty} \frac{1}{\beta u KT} \ln \Xi_{\beta \lambda}^{(u)}(Q_s) = \lim_{K \to \infty} \frac{1}{\beta KT} \mathbb{E} \left[ \ln \Xi_{\beta \lambda}^{(RS)}(\{z(k)\}) \right], \]  \tag{145} \]
with (143). Since (144) should be \( O(\beta) \) in \( \beta \to \infty, \) \( \chi \) must be \( O(\beta^{-1}) \) in \( \beta \to \infty. \) Taking \( \beta \to \infty \) with \( \hat{\chi} = \beta \chi \) fixed before \( \lambda \to \infty \) yields
\[ \lim_{\lambda \to \infty} \lim_{\beta \to \infty} \lim_{u \to 0} \lim_{K \to \infty} \frac{1}{\beta u KT} \ln \Xi_{\beta \lambda}^{(u)}(Q_s) = -\frac{1}{\hat{\chi}} \lim_{K \to \infty} \mathbb{E}[E_{RS}(q_0)], \]  \tag{146} \]
where \( E_{RS}(q_0) \) is given by
\[ E_{RS}(q_0) = \frac{1}{K} \min_{\{s_k \in \{0,1\}\} \sum_{k=1}^{K} s_k = K} \sum_{k=1}^{K} s_k E_{k}^{(RS)}(q_0), \]  \tag{147} \]
with
\[ E_{k}^{(RS)}(q_0) = \frac{1}{T} \sum_{t=0}^{T-1} \min_{\hat{x}_{k,t} \in \mathcal{M}_{x_k,t}} || \hat{x}_{k,t} - \sqrt{q_0} (z(k))_t ||^2. \]  \tag{148} \]
In order to evaluate the expectation of (147), we write the order statistics of \( \{E_{k}^{(RS)}(q_0)\} \) as \( \{E_{(k)}^{(RS)}(q_0)\} \), i.e. \( E_{1}^{(RS)}(q_0) \leq E_{2}^{(RS)}(q_0) \leq \cdots \leq E_{(k)}^{(RS)}(q_0) \) \[48\]. Since (147) can be represented as
\[ E_{RS}(q_0) = \frac{1}{K} \sum_{k=1}^{K} E_{(k)}^{(RS)}(q_0), \]  \tag{149} \]
Lemma 2 implies
\[ \lim_{K \to \infty} \mathbb{E}[E_{RS}(q_0)] = \mu_{n,T}(q_0), \]  \tag{150} \]
with (41).
Finally, we substitute (140) and (146) with (150) into the free energy (134) to arrive at
\[ \lim_{\lambda \to \infty} \lim_{\beta \to \infty} \lim_{K \to \infty} f = \frac{1}{\alpha \hat{\chi}} \left( q_0 - \frac{q_0 - \alpha \mu_{n,T}(q_0)}{\hat{\chi}} \right), \]  \tag{151} \]
where \( \lim_{\beta \to \infty} \) denotes the limit in which \( \beta \to \infty \) and \( \chi \to 0 \) with \( \hat{\chi} = \beta \chi \) fixed. In (151), \( \hat{\chi} \) and \( q_0 \) are chosen so as to extremize the free energy (151). The stationarity condition for \( \hat{\chi} \) implies that \( q_0 \) is the solution to the fixed-point equation

\[
q_0 = \alpha \mu_{\nu, \tau}(q_0). \tag{152}
\]

Substituting (152) into the free energy (151) yields \( f = q_0/(\alpha \kappa) \).

If the fixed-point equation (152) has multiple solutions, the solution \( q_0 \) to minimize (135) or equivalently the free energy is selected. Since the free energy is given by \( q_0/(\alpha \kappa) \), this criterion is equivalent to selecting the smallest solution to the fixed-point equation (152).

**APPENDIX D**

**DERIVATION OF PROPOSITION 2**

We start with (134). Let us assume 1RSB for the solution (137).

**Assumption 3** (1-step Replica Symmetry Breaking).

\[
Q_s = \left[ \chi I_u + q_0 1_u 1_u^T + q_1 I_{u/m_1} \otimes (1_{m_1} 1_{m_1}^T) \right] \otimes I_T, \tag{153}
\]

for a positive integer \( m_1 \) satisfying \( u/m_1 \in \mathbb{N} \).

We first calculate (133) to obtain

\[
\frac{1}{u} T I_u(Q_s) = \frac{1}{\beta u} \left[ \beta u (\chi + q_0 + q_1) - \frac{u(m_1 - 1)}{m_1} \ln \chi \right.
\]

\[
- \left( \frac{u}{m_1} - 1 \right) \ln ( \chi + m_1 q_1 ) - \ln ( \chi + u q_0 + m_1 q_1 ) \right]
\]

\[
\to \chi + q_0 + q_1 - \frac{q_0}{\beta (\chi + m_1 q_1)} - \frac{1}{\beta m_1} \ln \left( 1 + \frac{m_1 q_1}{\chi} \right)
\]

\[
- \frac{1}{\beta} \ln \chi, \tag{154}
\]

in \( u \to 0 \).

We next evaluate (128). The 1RSB assumption (153) implies that (125) is represented as

\[
v_a(k) = \sqrt{\chi} w_a(k) + \sqrt{q_0} z_{\nu/a/m_1}(k), \tag{156}
\]

where \( \{ w_a(k) \in \mathbb{C}^T \} \), \( \{ z(k) \in \mathbb{C}^T \} \), and \( \{ z_r(k) \in \mathbb{C}^T \} \) are independent circularly symmetric complex Gaussian random vectors with covariance \( I_T \), respectively. We calculate the expectation with respect to \( \{ w_a(k) \} \) to obtain

\[
\Xi_{\beta \lambda}^{(u)}(Q_s)
\]

\[
= \mathbb{E} \left[ E_{(x_0(k))} \left\{ \Xi_{\beta \lambda}^{(1RSB)}(\{ z(k) \}, \{ z_0(k) \}) \right\}^{u/m_1} \right], \tag{157}
\]

with

\[
\Xi_{\beta \lambda}^{(1RSB)}(\{ z(k) \}, \{ z_0(k) \}) = \sum_{\{ s_k \in \{0, 1\} \}} \prod_{k \in K} \left( \int_{\prod_{t=1}^{T-k} \mathcal{M}_{x_k,t}} d\hat{x}(k) \right)
\]

\[
e^{-H_{\beta \lambda}^{(1RSB)}(\{ s_k \}, \{ \hat{x}(k) \}, \{ z(k) \}, \{ z_0(k) \})} \prod_{k=1}^{K} \int_{\prod_{t=1}^{T-k} \mathcal{M}_{s_k,t}} d\hat{x}(k), \tag{158}
\]

where \( H_{\beta \lambda}^{(1RSB)}(\{ s_k \}, \{ \hat{x}(k) \}, \{ z(k) \}, \{ z_0(k) \}) \) is given by

\[
H_{\beta \lambda}^{(1RSB)}(\{ s_k \}, \{ \hat{x}(k) \}, \{ z(k) \}, \{ z_0(k) \})
\]

\[
= \sum_{k=1}^{K} \frac{\beta \lambda s_k}{1 + \beta \lambda s_k \chi} ||\hat{x}(k) - \sqrt{q_0} z(k) - \sqrt{q_0} z_0(k)||^2
\]

\[
+ \frac{\beta \lambda}{\chi} \left( \sum_{k=1}^{K} s_k - \hat{K} \right)^2. \tag{159}
\]
Taking $u \to 0$ yields
\[
\lim_{u \to 0} \lim_{K \to \infty} \frac{1}{\beta u KT} \ln \mathbb{E} \left[ \ln E(z_0(k)) \right] = \lim_{K \to \infty} \frac{1}{\beta m_1 K T} \mathcal{E} \left[ \ln \mathbb{E} \left[ z_0(k) \right] \right],
\]
(160)
with (158). The function (160) converges in the limit $\beta \to \infty$, $m_1 \to 0$, and $\chi \to 0$ with $\mu_1 = \beta m_1$ and $\tilde{\chi} = \beta \chi$ fixed. Taking this limit before $\lambda \to \infty$ yields
\[
\lim_{\lambda \to \infty} \lim_{\beta \to \infty} \lim_{u \to 0} \lim_{K \to \infty} \frac{1}{\beta u KT} \ln \mathbb{E} \left[ \ln E(z_0(k)) \right] = \lim_{K \to \infty} \frac{1}{\mu_1 K T} \mathcal{E} \left[ \ln \mathbb{E} \left[ z_0(k) \right] \right],
\]
(161)
where $E_{1\text{RSB}}(q_0, q_1)$ is given by
\[
E_{1\text{RSB}}(q_0, q_1) = \frac{1}{K} \min_{\{s_k \in \{0, 1\}\} : \Sigma_{k=1}^K s_k = 1} \sum_{k=1}^K s_k E_k^{(1\text{RSB})}(q_0, q_1),
\]
(162)
with
\[
E_k^{(1\text{RSB})}(q_0, q_1) = \frac{1}{T} \sum_{t=0}^{T-1} \left\| \hat{x}_{k,t} - \sqrt{q_0(z(k))}_t - \sqrt{q_1(z_0(k))}_t \right\|^2.
\]
(163)
In order to evaluate the distribution of (162), we write the order statistics of $\{E_k^{(1\text{RSB})}(q_0, q_1)\}$ as $\{E_k^{(1\text{RSB})}(q_0, q_1)\}$, i.e.
\[
E_1^{(1\text{RSB})}(q_0, q_1) \leq E_2^{(1\text{RSB})}(q_0, q_1) \leq \cdots \leq E_K^{(1\text{RSB})}(q_0, q_1)
\]
(164). Expression (162) can be represented as
\[
E_{1\text{RSB}}(q_0, q_1) = \frac{1}{K} \sum_{k=1}^K E_k^{(1\text{RSB})}(q_0, q_1).
\]
(165)
Since $E_{1\text{RSB}}(q_0, q_1)$ conditioned on $\{\hat{x}(k)\}$ and $\{z(k)\}$ converges in law to a Gaussian random variable in the large-system limit [53, Theorem 6], (161) reduces to
\[
\lim_{\lambda \to \infty} \lim_{\beta \to \infty} \lim_{u \to 0} \lim_{K \to \infty} \frac{1}{\beta u KT} \ln \mathbb{E} \left[ \ln E(z_0(k)) \right] = \lim_{K \to \infty} \left\{ \frac{\mathbb{E}[E_{1\text{RSB}}(q_0, q_1)]}{\chi} - \frac{\mu_1 T K V E_{1\text{RSB}}(q_0, q_1)}{2 \chi^2} \right\}.
\]
(166)
Lemma 2 implies
\[
\lim_{K \to \infty} \mathbb{E}[E_{1\text{RSB}}(q_0, q_1)] = \mu_{\kappa,T}(q_0 + q_1),
\]
(167)
\[
\lim_{K \to \infty} K V E_{1\text{RSB}}(q_0, q_1) = \sigma_{\kappa,T}^2(q_0 + q_1),
\]
(168)
with (41) and (42).

Finally, we substitute (155) and (165) with (166) and (167) into the free energy (134) to arrive at
\[
\lim_{\lambda \to \infty} \lim_{\beta \to \infty} \lim_{u \to 0} \lim_{K \to \infty} f
\]
\[
= \frac{1}{\alpha K} \left[ q_0 + q_1 + 1 \left\{ \mu_1 q_0 \frac{\mu_1 q_1}{\hat{\chi} + \mu_1 q_1} + \ln \left( \frac{\hat{\chi} + \mu_1 q_1}{\hat{\chi}} \right) \right\} - \alpha \left\{ \frac{\mu_1 \mu_{\kappa,T}(q_0 + q_1)}{\hat{\chi}} - \frac{T \mu_1^2 \sigma_{\kappa,T}^2(q_0 + q_1)}{2 \chi^2} \right\} \right],
\]
(169)
where $\lim_{\beta \to \infty}$ denotes the limit $\beta \to \infty$, $m_1 \to 0$, and $\chi \to 0$ with $\mu_1 = \beta m_1$ and $\tilde{\chi} = \beta \chi$ fixed. In (168), $\mu_1$, $\tilde{\chi}$, $q_0$, and $q_1$ or equivalently $\mu_1$, $\tilde{\chi} = \hat{\chi}/\mu_1$, $q_0$, and $q_1$ are chosen so as to extremize the free energy (168). The stationarity conditions for $\mu_1$ and $\tilde{\chi}$ are given by
\[
\frac{q_0}{\hat{\chi} + q_1} + \ln \left( \frac{1 + q_1}{\hat{\chi}} \right) = \alpha \left[ \frac{\mu_{\kappa,T}(q_0 + q_1)}{\hat{\chi}} - \frac{T \sigma_{\kappa,T}^2(q_0 + q_1)}{2 \chi^2} \right],
\]
(169)
\[
\frac{q_0}{(\bar{x} + q_1)^2} - \frac{q_1}{\bar{x}(\bar{x} + q_1)} = \alpha \left[ -\frac{1}{\bar{x}^2} \mu_{\kappa,T}(q_0 + q_1) + \frac{T}{\bar{x}^2} \sigma^2_{\kappa,T}(q_0 + q_1) \right],
\]
respectively. From the stationarity conditions for \(q_0\) and \(q_1\), we obtain
\[
\frac{q_0}{(\bar{x} + q_1)^2} = 0,
\]
which implies \(q_0 \to 0\), \(\bar{x} \to \infty\), or \(q_1 \to \infty\). The free energy \(\psi(q,\ln y)\) diverges in \(q_1 \to \infty\). Furthermore, the limit \(\bar{x} \to \infty\) corresponds to the RS solution. Taking \(q_0 \to 0\) yields
\[
\lim_{\lambda \to \infty} \lim_{\beta \to \infty} \lim_{K \to \infty} f = \frac{q_1}{\alpha \kappa},
\]
where \(q_1\) satisfies the coupled fixed-point equations,
\[
\ln \left( 1 + \frac{q_1}{\bar{x}} \right) = \alpha \left[ \frac{1}{\bar{x}} \mu_{\kappa,T}(q_1) - \frac{T}{2\bar{x}^2} \sigma^2_{\kappa,T}(q_1) \right],
\]
\[
\frac{q_1}{\bar{x} + q_1} = \alpha \left[ \frac{1}{\bar{x}} \mu_{\kappa,T}(q_1) - \frac{T}{\bar{x}^2} \sigma^2_{\kappa,T}(q_1) \right].
\]
If the coupled fixed-point equations \(173\) and \(174\) have multiple solutions, the solution \(q_1\) to minimize the free energy or equivalently the smallest solution \(q_1\) to the coupled fixed-point equations is selected.

**APPENDIX E**

**Proof of Corollary 1**

We prove that the fixed-point equation \(43\) reduces to \(51\) in \(T \to \infty\). We first show that the \(\kappa\)-quantile \(36\) converges to the expectation
\[
E[E_k(q)] = E \left[ \min_{\tilde{z}_{k,t} \in M_{k,t}} |\tilde{z}_{k,t} - \sqrt{q}z_{k,t}|^2 \right]
\]
in \(T \to \infty\). Let \(\xi\) denote a variable that satisfies
\[
1 = F_T(\xi; q)
\]
in \(T \to \infty\). Since the cdf \(35\) is monotonically increasing, we find \(\xi \geq \xi_{\kappa,T}(q)\), with \(36\). The weak law of large numbers implies that the random variable \(32\) converges in probability to \(175\) in \(T \to \infty\), so that the cdf \(35\) converges to
\[
\lim_{T \to \infty} F_T(x; q) = 1(x \geq E[E_k(q)])
\]
in \(T \to \infty\). Thus, \(176\) reduces to
\[
1 = 1(\xi \geq E[E_k(q)])
\]
in \(T \to \infty\). The smallest variable \(\xi\) satisfying \(178\) is given by \(175\). This implies that \(36\) is bounded from above by \(175\) in \(T \to \infty\):
\[
\limsup_{T \to \infty} \xi_{\kappa,T}(q) \leq E[E_k(q)].
\]
Similarly, considering a variable \(\underline{\xi}\) that satisfies
\[
0 = F_T(\underline{\xi}; q)
\]
in \(T \to \infty\), we obtain the lower bound on \(36\)
\[
\liminf_{T \to \infty} \xi_{\kappa,T}(q) \geq E[E_k(q)].
\]
Combining the two bounds \(179\) and \(181\) yields
\[
\lim_{T \to \infty} \xi_{\kappa,T}(q) = E[E_k(q)],
\]
with \(175\).
We next calculate \(41\) in \(T \to \infty\). Integrating \(41\) by parts after the transformation \(y = F_T^{-1}(x; q)\), we obtain
\[
\mu_{\kappa,T}(q) = \int_{0}^{\xi_{\kappa,T}(q)} gF_T'(y; q)dy
= \kappa \xi_{\kappa,T}(q) - \int_{0}^{\xi_{\kappa,T}(q)} F_T(x; q)dx,
\]
with \(175\).
with (36). Applying (177) and (182) to (183) yields

$$\lim_{T \to \infty} \mu_{k,T}(q) = \kappa \mathbb{E}[E_k(q)],$$

(184)

with (175). This implies that (43) reduces to (51) in $T \to \infty$.

**APPENDIX F**

**DERIVATION OF PROPOSITION 3**

**A. Replica Method**

As shown in Appendix C, the Gibbs-Boltzmann distribution (105) converges to (107) in the low-temperature limit $\beta \to \infty$. This implies that the marginal distribution $\Pr(s; \beta) = \sum_i \Pr(s_i; \beta)$ tends to

$$\Pr(s_i; \beta) \rightarrow \frac{1}{|\mathcal{S}_g(i)|} \mathbb{1}(s_i \in \mathcal{S}_g(i))$$

(185)

in the low-temperature limit, where $\mathcal{S}_g(i)$ is the set of the $i$th element $s_i$ included in the ground states $\mathcal{S}_g$. Thus, evaluating the conditional joint distribution $\Pr(s_k = 1, \tilde{X}_k \in \prod_{t=0}^{T-1} \mathcal{A}_t | X_k)$ reduces to calculating a marginal distribution of the Gibbs-Boltzmann distribution associated with the Hamiltonian (112).

We start with the identity

$$\Pr\left(s_k = 1, \tilde{X}_k \in \prod_{t=0}^{T-1} \mathcal{A}_t \middle| X_k \right) = \int_{\mathcal{A}_0 \times \cdots \times \mathcal{A}_{T-1}} p(s_k = 1, \tilde{X}_k| X_k) d\tilde{X}_k,$$

(186)

with

$$p(s_k, \tilde{X}_k| X_k) = \lim_{\lambda \to \infty} \lim_{\beta \to \infty} \lim_{u \to 0} Z(\beta, \lambda)^{u-1} \mathbb{E} \left[ \sum_{s_k} \int e^{-\beta H_{\lambda}(\mathcal{S}_k, \{\tilde{x}_t\}, \{u_t\})} d\tilde{x}_k \prod_{t=0}^{T-1} du_t \middle| X_k \right],$$

(187)

where $H_{\lambda}(\mathcal{S}, \{\tilde{x}_t\}, \{u_t\})$ and $Z(\beta, \lambda)$ are given by (112) and (118), respectively. In (187), $\sum_{s_k}$ denotes the marginalization over $\{s_k \in \{0, 1\} : k' \neq k\}$. Furthermore, $\int d\tilde{x}_k$ represents the marginalization over $\{\tilde{x}_{k',t} \in \mathcal{M}_{x_{k',t}} : \text{for all } t \text{ and } k' \neq k\}$. Regarding $u$ in (187) as a non-negative integer gives

$$p(s_k, \tilde{X}_k| X_k) = \lim_{\lambda \to \infty} \lim_{\beta \to \infty} \lim_{u \to 0} Z_u(s_k, \tilde{X}_k, X_k; \beta, \lambda),$$

(188)

with

$$Z_u(s_k, \tilde{X}_k, X_k; \beta, \lambda) = \left(\frac{\pi}{\beta}\right)^{uN_T} \int \Xi_{\beta\lambda}^{(u)}(Q, s_k, \tilde{X}_k, X_k)p(Q)dQ.$$

(189)

In (189), the pdf $p(Q)$ is given by (130). Furthermore, $\Xi_{\beta\lambda}^{(u)}(Q, s_k, \tilde{X}_k, X_k)$ is defined as

$$\Xi_{\beta\lambda}^{(u)}(Q, s_k, 0, \tilde{X}_k, 0, X_k) = \mathbb{E} \left[ \sum_{\{s_{k,0}, \tilde{x}_{k,0}\}} \prod_{a=0}^{u-1} \exp\left\{ -\beta \lambda \frac{k}{a} \{s_{k,a}\}, \{\tilde{x}_{k,a}\} \} \right\} d\tilde{x}_{k,0} \right],$$

(190)

with (129), where $\tilde{x}_{k,0}$ is given by $\tilde{x}_{k,0} = \{\tilde{x}_{k,t} : t = 0, \ldots, T-1\}$. Substituting (130) into (189) yields

$$Z_u(s_k, \tilde{X}_k, X_k; \beta, \lambda) = \left(\frac{\pi}{\beta}\right)^{uN_T} C_u \int dQ \det Q^{-uT} \exp \left\{ \beta N \left( \frac{1}{\beta N} \ln \Xi_{\beta\lambda}^{(u)}(Q, s_k, \tilde{X}_k, X_k) - I_u(Q) \right) \right\},$$

(191)
with (133). Assuming that the large-system limit and the limits in (188) are commutative, we use the saddle-point method to obtain

\[
p(s_k, \tilde{X}_k | \mathcal{X}_k) = \lim_{\lambda \to \infty} \lim_{\beta \to \infty} \lim_{u \to K} \lim_{K \to \infty} \left\{ e^{-\beta N I_u(Q_s)} \Xi^{(u)}_{\beta \lambda}(Q_s, s_k, \tilde{X}_k, \mathcal{X}_k) \right\},
\]

where \(Q_s\) is the solution to minimize (135), given by (137). In the derivation of (192), we have used the fact that the difference between \(\ln \Xi^{(u)}_{\beta \lambda}(Q_s, s_k, \tilde{X}_k, \mathcal{X}_k)\) and \(\ln \Xi^{(u)}_{\beta \lambda}(Q)\) should be \(O(1)\).

### B. Replica Symmetry Solution

We evaluate (192) under the RB assumption (138). The order parameter \(q_0\) satisfies the fixed-point equation (152). Furthermore, from (139), it is straightforward to find that \(I_u(Q_s)\) tends to zero in \(u \to 0\).

We next calculate (190) with (141) to obtain

\[
\lim_{u \to 0} \lim_{K \to \infty} \Xi^{(u)}_{\beta \lambda}(Q_s, s_k, 0, \tilde{X}_k, 0, \mathcal{X}_k) = \lim_{K \to \infty} E \left[ \Xi^{(RS)}_{\beta \lambda}(\{k\}) \right]^{-1} \cdot \sum_{\{s_k\}, \{\tilde{s}_k\}} e^{-H^{(RS)}_{\beta \lambda}(\{s_k\}, \{\tilde{s}_k\})} (1 + \beta \chi) \sum_{s_{k'}=1}^{\infty} q_{k',0} d \tilde{X}_k, 0 | \mathcal{X}_k,
\]

where \(\Xi^{(RS)}_{\beta \lambda}(\{k\})\) and \(H^{(RS)}_{\beta \lambda}(\{s_k\}, \{\tilde{s}_k\}, \{z(k)\})\) are given by (143) and (144), respectively. Substituting (193) into (192) and then taking \(\beta \to \infty\) with \(\tilde{\chi} = \beta \chi\) fixed before \(\lambda \to \infty\), we have

\[
p(s_k, \tilde{X}_k | \mathcal{X}_k) = \lim_{K \to \infty} E \left[ \exp \left\{ -\frac{\beta}{\tilde{\chi}} H^{(RS)}(s_k, \tilde{X}_k) \right\} \right] \mathcal{X}_k,
\]

with

\[
H^{(RS)}(s_k, \tilde{X}_k) = \frac{s_k}{T} \sum_{t=0}^{T-1} |\tilde{x}_{k,t} - \sqrt{q_0} z(k)|^2
+ \min_{\{s_k\} \in A} \sum_{k'=1}^{K} s_{k'} E_k^{(RS)}(q_0)
- \min_{\{s_k\} \in A} \sum_{k'=1}^{K} s_{k'} E_k^{(RS)}(q_0),
\]

where \(E_k^{(RS)}(q_0)\) is given by (148). The quantity (195) is non-negative for any \(s_k\) and \(\tilde{X}_k\), and zero if and only if \((s_k, \tilde{X}_k)\) is equal to the optimal solution \((s_k^{(opt)}(q_0), \{\tilde{x}_{k,t}^{(opt)}(q_0)\})\), given by

\[
\tilde{x}_{k,t}^{(opt)}(q_0) = \min_{\tilde{x}_{k,t} \in A} |\tilde{x}_{k,t} - \sqrt{q_0} z(k)|^2, \quad \text{and}

s_k^{(opt)}(q_0) = \arg\min_{s_k \in (0,1)} \left\{ s_k E_k^{(RS)}(q_0) \right\}
+ \min_{\{s_k\} \in A} \sum_{k'=1}^{K} s_{k'} E_k^{(RS)}(q_0), \quad \text{and}
\]

with (148). Substituting (193) into (186), we arrive at

\[
\Pr \left( s_k = 1, \tilde{X}_k \in \prod_{t=0}^{T-1} A_t \middle| \mathcal{X}_k \right) = \lim_{K \to \infty} E \left[ \prod_{t=0}^{T-1} 1 \left( s_k^{(opt)}(q_0) = 1 \right) 1 \left( \tilde{x}_{k,t}^{(opt)}(q_0) \in A_t \right) \right] \mathcal{X}_k,
\]

where \(\tilde{x}_{k,t}^{(opt)}(q_0)\) and \(s_k^{(opt)}(q_0)\) are given by (196) and (197), respectively.
In order to complete the derivation of Proposition 3 we prove that (198) reduces to (53) with \( q = q_0 \). The solution (197) takes 1 if and only if \( E_k^{(RS)}(q_0) \) is smaller than the \( K \)th order statistic, i.e. \( E_k^{(RS)}(q_0) \leq E_{(K)}(q_0) \). Lemma 1 implies that the \( K \)th order statistic \( E_{(K)}(q_0) \) converges in probability to the \( \kappa \)-quantile \( \xi_{\kappa,T}(q_0) \), given by (36), in the large-system limit. This observation implies that (198) reduces to (53) with \( q = q_0 \).

APPENDIX G
DERIVATION OF PROPOSITION 4

We start with (192). Let us calculate (192) under the 1RSB assumption (153). As shown in Appendix D, \( q_0 \) tends to zero, and \( q_1 \) satisfies the coupled fixed-point equations (173) and (174). Furthermore, from (154), it is straightforward to find that \( I_q(Q_z) \) tends to zero in \( u \to 0 \).

We next calculate (190) with (156) to obtain

\[
\lim_{u \to 0} \lim_{K \to \infty} \frac{\Xi(s_k, s_{k,0}, \tilde{X}_k, X_k)}{H^{(RS)}(s_k, \tilde{X}_k)} = \lim_{K \to \infty} \mathbb{E} \left[ \exp \left\{ -\frac{\beta}{\chi} H^{(1RSB)}(s_k, \tilde{X}_k) \right\} \right] = \lim_{\beta \to \infty} \lim_{\chi \to 0} \mathbb{E} \left[ \exp \left\{ -\frac{\beta}{\chi} H^{(1RSB)}(s_k, \tilde{X}_k) \right\} \right],
\]

with

\[
H^{(1RSB)}(s_k, \tilde{X}_k) = \frac{s_k}{T} \sum_{t=0}^{T-1} |\tilde{x}_{k,t} - \sqrt{q_1}(z_0(k))_t|^2 + \min_{\{s_{k,0} \in \{0,1\}\} : \sum_{k'=1}^K s_{k'} = K} \sum_{k'=1}^K s_{k'} E^{(1RSB)}_{k'}(0, q_1),
\]

(201)

where \( E^{(1RSB)}_{k'}(q_0, q_1) \) is given by (163). The quantity (201) is non-negative for any \( s_k \) and \( \tilde{X}_k \), and zero if and only if \((s_k, \tilde{X}_k)\) is equal to the optimal solution \((s_k^{(opt)}(0, q_1), \{\tilde{x}_{k,t}^{(opt)}(0, q_1)\}_t)\), given by

\[
\tilde{x}_{k,t}^{(opt)}(0, q_1) = \arg \min_{\tilde{x}_{k,t} \in A_{s_k,t}} |\tilde{x}_{k,t} - \sqrt{q_1}(z_0(k))_t|^2,
\]

(202)

\[
s_k^{(opt)}(0, q_1) = \arg \min_{s_k \in \{0,1\}} \left\{ s_k E^{(1RSB)}_{K}(0, q_1) \right\}.
\]

(203)

with (163). Substituting (200) into (186), we arrive at

\[
\Pr \left( s_k = 1, \tilde{X}_k \in \prod_{t=0}^{T-1} A_t \mid X_k \right) = \lim_{K \to \infty} \mathbb{E} \left[ 1 \left( s_k^{(opt)}(0, q_1) = 1 \right) \prod_{t=0}^{T-1} \left( \tilde{x}_{k,t}^{(opt)}(0, q_1) \in A_t \right) \mid X_k \right],
\]

(204)

where \( \tilde{x}_{k,t}^{(opt)}(0, q_1) \) and \( s_k^{(opt)}(0, q_1) \) are given by (202) and (203), respectively. Repeating the argument in the end of Appendix F-B we find that (204) reduces to (53) with \( q = q_1 \).
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