Internal DLA for Cylinders

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Dedicated to E. M. Stein

1 Introduction

Internal Diffusion-Limited Aggregation (internal DLA) is a random lattice growth model. Consider the two-dimensional lattice, $\mathbb{Z} \times \mathbb{Z}$. In the case of a single source at the origin, the random occupied set $A(T)$ of $T$ lattice sites is defined inductively as follows. Let $A(1)$ be the singleton set containing the origin. Given $A(T-1)$, start a random walk in $\mathbb{Z} \times \mathbb{Z}$ at the origin. Then

$$A(T) := \{n\} \cup A(T-1)$$

where $n \in \mathbb{Z} \times \mathbb{Z}$ is the first site reached by the random walk that is not in $A(T-1)$.

In this paper, we will discuss the continuum limit of internal DLA, which is governed by a deterministic fluid flow equation known as Hele-Shaw flow. Our main focus will be on fluctuations. In [JLS11] we characterized the average fluctuations of the model just described in terms of a close relative of the Gaussian Free Field, defined below. In this article we will prove the analogous results for the lattice cylinder. In the case of the cylinder, the fluctuations are described in terms of the Gaussian Free Field exactly. We will also state without proof an almost sure bound on the maximum fluctuation in the case of the cylinder analogous to the case of the planar

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lattice proved in [JLS12a]. The main tools used in the proofs are martingales. As we shall see, the martingale property in this context is the counterpart in probability theory of well-known conservation laws for Hele-Shaw flow.

The internal DLA model was introduced in 1986 by Meakin and Deutch [MD86] to describe chemical processes such as electropolishing, etching, and corrosion. Think of the occupied region as a blob of fluid. Figure 1 depicts a simulation of a cluster (blob) of size one million in dimension 2. At each step a corrosive molecule is introduced at a source, which, in this simulation is a single point at the origin. The corrosive particle wanders at random through the fluid until it reaches the fluid-metal boundary, where it eats away a tiny portion of metal and enlarges slightly the fluid region. The question that concerned Meakin and Deutch was the smoothness of the surface that is being polished, that is, how irregular the boundary is. Figure 2 is a close-up picture of the boundary fluctuations.

Figure 1 suggests that the limit shape from a point source is a disk.
Indeed, in 1992, Lawler, Bramson and Griffeath \cite{LBG92} proved that the rescaled limit shape of internal DLA from a point source is a ball in any dimension. In 1995, Lawler \cite{Law95} proved almost sure bounds on the cluster of the form

\[ B(r - Cr^{1/3}) \cap \mathbb{Z}^d \subset A(T) \subset B(r + Cr^{1/3}), \]

where \( T \) is the volume of the ball of radius \( r \) and \( C \) is a dimensional constant. On the other hand, the numerical simulations of Meakin and Deutch predicted fluctuations, on average, of size \( O(\sqrt{\log r}) \) in dimension 2 and \( O(1) \) in dimension 3. They made their predictions based on small values of \( T \), but much larger simulations are now possible and give the same results.

The theorems we will describe are consistent with the size of fluctuations predicted by Meakin and Deutch and reveal deeper structure, namely that the fluctuations obey a central limit theorem. The Fourier coefficients of the fluctuations tend to independent gaussians, whose variance we can compute. This gives a heuristic explanation of numerical results on average fluctuations and many other predictions such as what should be the best possible bound on maximum fluctuations. In 2010, Asselah and Gaudillièrè \cite{AG10} improved the power in Lawler’s bound in dimensions greater than 2. Later in 2010, Assellah and Gaudillièrè \cite{AG10a, AG10b} and the present authors \cite{JLS12a, JLS12b} independently proved logarithmic bounds on the maximum fluctuation.

**Theorem 1.** (Maximum Fluctuations) There is a dimensional constant \( C_d \), such that almost surely for sufficiently large \( r \),

\[ B(r - C_2 \log r) \cap \mathbb{Z}^2 \subset A(T) \subset B(r + C_2 \log r) \]

with \( T = \pi r^2 \). Moreover, for \( d \geq 3 \),

\[ B(r - C_d \sqrt{\log r}) \cap \mathbb{Z}^d \subset A(T) \subset B(r + C_d \sqrt{\log r}) \]

where \( T \) is the volume of the ball of radius \( r \).

The maximum fluctuations represent the worst case along the entire circumference as opposed to the average fluctuations observed by Meakin and Deutch. Whether one considers the average or the worst case, the model produces remarkably smooth surfaces — even more smooth in dimension 3 than in dimension 2.

Before going any further, we should add a disclaimer. Despite their superficial similarity, the internal DLA model and the Diffusion-Limited
Aggregation (DLA) model introduced by Witten and Sander [WS81] are very different. DLA is a model of particle deposition, in which a seed particle is placed at the origin in a lattice. Particles follow a random walk starting at infinity and attach to the existing cluster the first time they are adjacent to it. The particles form a cluster of fractal character and the continuum limit is very far from deterministic. In their 1986 article, Meakin and Deutch refer to the work of Witten and Sander and explain that the internal DLA model is better behaved than DLA and intended to describe quite different physical phenomena, ones that do not exhibit chaos. The Hele-Shaw model is also highly relevant to DLA, but it is the complement of the cluster that is interpreted as the fluid region. Thus the fluid region shrinks. When fluid is sucked away, the Hele-Shaw equation is ill-posed, and the methods of partial differential equations no longer apply except at very short time scales. Instead, algebraic methods are used. The subject is of great interest in statistical physics and has a direct connection with random matrices, but it is not the subject of this paper.

This paper discusses various aspects of several works of the authors [JLS12a, JLS12b, JLS11]. Rather than prove any of the theorems in those papers, which concern $\mathbb{Z}^d$, we prove two central limit theorems (Theorems 3 and 4) in which the set $\mathbb{Z}^2$ of [JLS11] is replaced by the lattice cylinder $(\mathbb{Z}/N\mathbb{Z}) \times \mathbb{Z}$. In the next section, we state our theorems in this new geometric setting. In the third section we explain the relationship between internal DLA and Hele-Shaw flow. Sections 4 and 5 give complete proofs of two central limit theorems for fluctuations of internal DLA on cylinders. We discuss the work of Levine and Peres concerning the relationship of internal DLA with the obstacle problem in Section 6. In the last section we make a few further remarks about the theorems of [JLS12a, JLS12b, JLS11], the effects of geometry on the problem, higher-dimensional questions, and questions related to more general random walks.

2 Main results for the cylinder.

In this section we state our main results in the case of the two-dimensional cylinder rather than the single source model in the plane which is carried out in [JLS12a]. We will make a comparison at the end of the paper.

Consider the cyclic group $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$, whose elements will typically be denoted $n_1 = 1, 2, \ldots, N$. In the lattice cylinder $\mathbb{Z}_N \times \mathbb{Z}$, define the set

$$A(0) = \{n = (n_1, n_2) \in \mathbb{Z}_N \times \mathbb{Z} : n_2 \leq 0\}$$
Figure 3: The symmetric difference of $A_N(T)$ and $\{y \leq T/N^2\}$ is the thin, ragged band at the top with early points in red above the line $y = T/N^2$ and late points in blue below. The bar at the bottom is the region $y \leq 0$.

For integers $T > 0$, the set $A(T)$ of lattice points is defined inductively, with source at $n_2 = -\infty$. Equivalently, given the set $A(T - 1)$, start a random walk in $\mathbb{Z}_N \times \mathbb{Z}$ at one of the sites $(n_1, 0)$, $n_1 = 1, \ldots, N$, with equal probability. $A(T) \setminus A(T - 1)$ consists of the site at which the random walk exits $A(T - 1)$ for the first time. Denote

$$A^+(T) = A(T) \setminus A(0)$$

A theorem analogous to Theorem 1, stated in a slightly more precise form, is

**Theorem 2.** Given $0 < y_1$ and $a < \infty$, there is a constant $C$ depending only on $y_1$, and $a$ such that with probability $1 - N^{-a}$, for all $y$, $0 \leq y \leq y_1$,

$$\{n : n_2 \leq yN - C \log N\} \cap (\mathbb{Z}_N \times \mathbb{Z}) \subset A(T) \subset \{n : n_2 \leq yN + C \log N\}$$

with $T = \lfloor yN^2 \rfloor$.

Next, we scale $A(T)$ by the factor $1/N$ to obtain a subset $A_N(T)$ of $\mathbb{T} \times \mathbb{R}$ with $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. For $n = (n_1, n_2)$, $n_1 = 1, \ldots, N$, $n_2 \in \mathbb{Z}$, and $0 < x \leq 1$ representing $x \in \mathbb{T}$, let

$$Q_N(n) = \{(x, y) \in \mathbb{T} \times \mathbb{R} : n_1 - 1 < Nx \leq n_1, \ n_2 - 1 < Ny \leq n_2\}$$

(1)

$Q_N(n)$ is the square of sidelength $1/N$ with $n/N$ at its upper right corner. Define

$$A_N(T) = \bigcup_{n \in A(T)} Q_N(n); \quad A_N^+(T) = \bigcup_{n \in A^+(T)} Q_N(n)$$

(2)
Thus $A^+_N(T)$ is the occupied subset of $\mathbb{T} \times \mathbb{R}_+$ consisting of $T$ squares of area $1/N^2$. We define a discrepancy function $D_{N,T}$ by

$$D_{N,T}(x,y) = N(A_N(T) - 1_{\{y \leq T/N^2\}}). \quad (3)$$

Figure 3 gives a closer look at the discrepancy between $A(T)$ and the expected strip by distinguishing early and late sites relative to the time $T = yN^2$. The figure depicts the sign of $D_{N,T}$ in different colors. $D_{N,T}$ takes on the values $\pm N$ and 0. $D_{N,T} > 0$ means that $Q$ is early relative to the time $T$. $D_{N,T} < 0$ means that $Q$ is late.

The factor $N$ in the definition of $D_{N,T}$ is the appropriate normalization so that the limit exists in the sense of distributions as $N \to \infty$. Informally, our next theorem says that

$$D_{N,T}(x,y) \to D(x)\delta(y - y_0) \quad (4)$$

in the sense of distributions with

$$D(x) \sim \sum_{k=1}^{\infty} \frac{a_k}{\sqrt{k}} \cos(2\pi kx) + \frac{b_k}{\sqrt{k}} \sin(2\pi kx)$$

and $a_k$ and $b_k$ independent, normally distributed random variables with mean zero and variance 1. The random variable $D(x)$ is not defined for individual values of $x$. For each $x$, the variance,

$$\sum_{k=1}^{\infty} \frac{1}{k} (\cos^2(2\pi x) + \sin^2(2\pi x)) = \infty$$

The precise statement of the theorem uses duality and involves weight factors that are merely asymptotic to $c/\sqrt{K}$.

Let $H_0$ be the Sobolev space of functions $\eta$ on $\mathbb{T} \times \mathbb{R}_+$ satisfying $\eta(x,0) = 0$ and square norm equal to the Dirichlet integral,

$$\|\eta\|^2_{H_0} = \int_0^\infty \int_0^1 |\nabla \eta(x,y)|^2 \, dx \, dy$$

The restriction of $H_0$ to the circle $y = y_0$ is the Sobolev space $H^{1/2}$. Its dual is the space of $H^{-1/2}$ distributions on $\mathbb{T}$ with dual norm given by

$$\|f\|_{(y_0)} = \sup \{ \int_0^1 f(x)\eta(x,y_0) \, dx : \|\eta\|_{H_0} \leq 1 \}$$
Fix an integer $K$, and consider test functions $\varphi \in C^\infty(T \times \mathbb{R})$ of the form

$$\varphi(x, y) = \sum_{|k| \leq K} \alpha_k(y) e^{2\pi i k x}$$

Assume that for each $k$, $\alpha_k$ is supported in the annulus $0 < c_1 \leq |y| \leq c_2$, and the $\varphi$ is real-valued, i.e., $\alpha_{-k} = \overline{\alpha_k}$.

**Theorem 3.** Let $T = [y_0, N^2]$. Then as $N \to \infty$,

$$D_{N,T}(\varphi) := \int_{T \times \mathbb{R}} D_{N,T}(x, y) \varphi(x, y) \, dx \, dy$$

tends in law to a normally distributed random variable with mean zero and variance

$$S^2_{y_0}(\varphi) := \|\varphi(\cdot, y_0)\|_2^2 = \sum_{0 < |k| \leq K} m_k |\alpha_k(y_0)|^2$$

with

$$m_k = \frac{1}{4\pi|k|} \left(1 - e^{-4\pi|k|y_0}\right)$$

The messy term $e^{-4\pi|k|y_0}$ in the coefficients $m_k$ comes from starting the growth process at $y = 0$. If we started at $y = -\infty$ it would disappear.

In general, a gaussian random variable relative to a Sobolev space has the form

$$X = \sum_j a_j \varphi_j$$

where $\varphi_j$ form an orthonormal basis for the Hilbert space and $a_j$ are mean zero, variance 1 independent random variables. Thus Theorem 3 asserts that $D_{N,T}$ tends to $D(x) \delta(y - y_0)$ in which $D$ is a (real-valued) gaussian random variable with mean value zero associated to the Hilbert space of functions $g \in H^{1/2}_{y_0}(T)$ with

$$\hat{g}(k) = \int_0^1 g(x) e^{-2\pi i k x} \, dx$$

$$\hat{g}(0) = 0, \hat{g}(-k) = \overline{\hat{g}(k)}$$

$$\|g\|_{H^{1/2}_{y_0}}^2 = \sum_{k \neq 0} |\hat{g}(k)|^2 / m_k$$

(5)

Roughly speaking, a discrepancy $|D(x)|$ of size one means that the particles arrive late or early by a unit distance in the original lattice or distance
1/N in the continuum cylinder. To illustrate this we consider the example, with $T = N^2$, in which $A_N(T)$ occupies all the squares of $y \leq 1 - 1/N$, exactly half of the $N$ squares in $1 - 1/N \leq y \leq 1$ and exactly half of the $N$ squares in $1 \leq y \leq 1 + 1/N$. Then $D_{N,T}(x,y) = N$ on each of the occupied squares of $1 < y < 1+1/N$ and $D_{N,T}(x,y) = -N$ on each of the unoccupied squares of $1 - 1/N < y < 1$. In both cases the integral of $|D_{N,T}|$ over the square is $N/N^2 = 1/N$ and there are $N$ such squares so the total is

$$\int_{T \times R} |D_{N,T}(x,y)| dxdy = 1$$

Thus, in this example, the limit satisfies $|D(x)| = 1$ (half positive and half negative),

With the appropriate interpretation of the size of $D$ in mind, we can confirm heuristically the predictions of Meakin and Deutch as follows. At scale $N$, it’s natural to truncate the series to $k \leq N$, and say

$$D(x) \sim \sum_{k=1}^{N} \frac{a_k}{\sqrt{k}} \cos(2\pi kx) + \frac{b_k}{\sqrt{k}} \sin(2\pi kx)$$

with independent unit variance gaussians as coefficients. For each fixed $x$, the variance of the right side is

$$\sum_{k=1}^{N} \frac{1}{k} \approx \log N$$

Thus the standard deviation of $D(x)$ is expected to be on the order of $\sqrt{\log N}$. On the other hand, we can also predict the maximum fluctuation over all $x$. At scale $1/N$, we have $N$ different values of $x$ at which the discrepancy is represented by a random variable of standard deviation $\sqrt{\log N}$. While these are not independent, they are less and less correlated as the separation gets larger. Thus we expect the largest of $D(x)$ and the largest $-D(x)$ to be on the order of a factor $\sqrt{\log N}$ larger than a single standard deviation, or $(\sqrt{\log N})^2 = \log N$. This is the maximum bound demonstrated in Theorems 1 and 2. The same heuristic reasoning applies in higher dimensions. The central limit theorems of [JLS11] in dimensions $d \geq 3$ yield a truncated variance of size $O(1)$ at typical boundary sites consistent with the higher-dimensional numerical evidence of Meakin and Deutch. Moreover, the same reasoning as above predicts that the maximum fluctuation in dimensions $d \geq 3$ is $O(\sqrt{\log T})$, where $T = N^d$ or $T = r^d$ is the number of particles. This higher dimensional estimate is proved in [JLS12b] and
Figure 4: Simulation of internal DLA on the cylinder with $N = 500$. In (a) late points ($L_N > 0$) and early points ($L_N < 0$) are indicated in blue and red, respectively. (b) The intensity of the colors indicates the size of $L_N$.

Very recently in [AG11], Asselah and Gaudilliére have confirmed that size $\sqrt{\log T}$ fluctuations do occur.

We will also analyze the fluctuations of the entire process as opposed to what happens at a single time $T$. In analogy with the discrepancy function $D_{N,T}(x,y)$, we will define a rescaled lateness function $L_N(x,y)$ that measures how early or late the point $(x,y)$ is reached by the cluster.

Before doing so, we will introduce a continuous time parameter $t$. Let $T(t)$ be a standard Poisson random variable, with $T(0) = 0$. $T(t)$ is an integer-valued process that produces in expectation $t$ particles at time $t$. For every $0 \leq t_0 < t_1 < \cdots < t_m$, the variables $T(t_j) - T(t_{j-1})$ are independent nonnegative integer-valued with expectation $t_j - t_{j-1}$, respectively. We also
assume that $T(t)$ is independent of the internal DLA growth process, and consider the process $A(T(t))$ depending on continuous time $t$.

For $n = (n_1, n_2) \in \mathbb{Z}_N \times \mathbb{Z}$, define

$$F(n) = \inf\{t : n \in A(T(t))\}$$

and

$$L_N(x, y) = F(n)/N - yN, \quad (x, y) \in Q_N(n)$$

For example, if $F(n) = (n_2 + 1)N$, then $n = (n_1, n_2)$ joins the cluster exactly one row late, in other words, by $N$ units of time $t$, which corresponds to a single row of width $N$ and height 1 in $\mathbb{Z}_N \times \mathbb{Z}$ or a single row of width 1 and height $1/N$ in $\mathbf{T} \times \mathbb{R}$. In that case, $L_N(n/N) = 1$. Figure 4 depicts simulations of $L_N$.

Informally, we say that $L_N$ tends as $N \to \infty$ to the gaussian free field with Dirichlet boundary conditions on $\mathbf{T} \times \mathbb{R}^+$, that is a gaussian random variable with respect to the Hilbert space $H_0$. The rigorous statement in dual form is as follows. Let $\varphi(x, y)$ be defined as above.

**Theorem 4.** As $N \to \infty$,

$$L_N(\varphi) := \int_{\mathbf{T} \times \mathbb{R}} L_N(x, y) \varphi(x, y) \, dx \, dy$$

tends in law to a mean zero gaussian random variable with variance

$$S^2(\varphi) := \|\varphi\|^2_{H_0} = \sum_{|k| \leq K} \int_0^\infty \left| \alpha_k(y') e^{2\pi k(y-y')} dy' \right|^2 \, dy'$$

We will establish an estimate on the error in the central limit theorem of order $O(N^{-2/15})$, depending on the size of $\varphi$ and the magnitude of $K$.

### 3 IDLA and Hele-Shaw flow

In this section, we give a heuristic description of the relationship between internal DLA and the Hele-Shaw model. This section contains no proofs, only formal derivations. The proof that the deterministic limit of internal DLA is Hele-Shaw flow, given in 2009 by Levine and Peres [LP10], proceeds via a discrete version of a classical obstacle problem. We will discuss their work in slightly more detail in Section 6.

Recall that in internal DLA from a single source in $\mathbb{Z}^2$ a particle takes a random walk from the origin in $A(T)$ until the first time it exits. Then it
stops and augments the cluster to form $A(T + 1)$. It stops at sites $y$ at unit distance from $A(T)$ with probability $p_T(y)$, and this first exit probability is the discrete harmonic measure. In other words, it satisfies

$$v(0) = \sum_{y \in \mathbb{Z}^2} v(y)p_T(y)$$

for every function $v : \mathbb{Z}^2 \to \mathbb{R}$ satisfying

$$\mathcal{L}v(x) = 0, \quad \text{for all } x \in A(T),$$

where $\mathcal{L}$ is the discrete Laplacian defined by

$$\mathcal{L}f(x) = \frac{1}{4}[f(x + e_1) + f(x - e_1) + f(x + e_2) + f(x - e_2)] - f(x)$$

This suggests that the deterministic continuum limit of the growth process is governed by harmonic measure. Indeed, the continuum limit of the random walk is Brownian motion, and, according to Kakutani’s theorem, the hitting probability of Brownian motion starting from a point of a domain is the harmonic measure relative to that point. The continuum process in which a region grows proportionally to its harmonic measure is known as Hele-Shaw flow.

Hele-Shaw flow describes the flow of fluid between two nearby parallel plates. The occupied region is essentially two-dimensional, so it is modeled by an open set $\Omega_t \subset \mathbb{R}^2$ at time $t$. Given a domain $\Omega_0$ at time $t = 0$, fluid is pumped in at the origin so that the area grows at a uniform speed, $|\Omega_t| = t + |\Omega_0|$. The pressure $p(x,t)$ satisfies $\Delta p(x,t) = -\delta$ in $\Omega_t$ and $p(x,t) = 0$ on $\partial\Omega_t$. The Hele-Shaw equation governing the growth says that the normal velocity of the boundary of $\Omega_t$ is $|\nabla p|$. Since $p$ is Green’s function for $\Omega_t$ with pole at the origin, Hele-Shaw’s equation can also be expressed as saying that the growth of the domain is proportional to its harmonic measure. The correspondence with the discrete case is $\Omega_0 \leftrightarrow A(T_0), \Omega_t \leftrightarrow A(T_1), |\Omega_0| = T_0/N^2, |\Omega_t| = T_1/N^2$, and $t = (T_1 - T_0)/N^2$.

One way to solve the Hele-Shaw equation is to solve instead for

$$u(x,t) = \int_0^t p(x,s)ds.$$  \hspace{1cm} (9)

It’s well known (c. f. [GV06]) that for each fixed $t$, $u$ solves an obstacle problem as follows. Choose $\gamma(x,t)$ to be a function on $\mathbb{R}^2$ solving $\Delta \gamma = t\delta + 1_{\Omega_0} - 1$. Let $w$ solve the obstacle problem

$$w(x,t) = \inf\{f : \Delta f \leq 0, \ f \geq \gamma\}$$
Although \( w \) depends on the choice of \( \gamma \), the set
\[
\Omega_t = \{ x \in \mathbb{R}^2 : w(x,t) > \gamma(x,t) \}
\]
and the function
\[
u(x,t) = w(x,t) - \gamma(x,t) \geq 0
\]
are independent of the choice of \( \gamma \). On \( \Omega_t \), \( \Delta u = -\Delta \gamma = 1 - t\delta - 1_{\Omega_0} \), and on \( \Omega_t^c \), \( u = 0 \). In fact,
\[
\Delta u = 1_{\Omega_t} - 1_{\Omega_0} - t\delta
\]
in all of \( \mathbb{R}^2 \).

Conversely, starting from \( u \), differentiate \((10)\) with respect to \( t \), to obtain
\[
\frac{\partial}{\partial t} \Delta u(x,t) = V\sigma_t - \delta
\]
where \( V \) is the normal velocity of the boundary of \( \Omega_t \) and \( \sigma_t \) is the arc length measure of \( \partial \Omega_t \). Define
\[
p(x,t) = \frac{\partial}{\partial t} u(x,t) \quad (11)
\]
Then
\[
\frac{\partial}{\partial t} \Delta u(x,t) = \Delta p(x,t) = -\delta + |\nabla p|\sigma_t,
\]
and hence \( p = (\partial/\partial t)u \) is the pressure for a Hele-Shaw fluid cell with normal velocity \( V = |\nabla p| \).

The formulas above yield conservation laws,
\[
v(0) = \frac{\partial}{\partial t} \int_{\Omega_t} v(x) \, dx \quad (12)
\]
for every harmonic function \( v \). We derive \((12)\) in integrated form by multiplying \((10)\) by \( v \) and integrating to obtain
\[
0 = \int_{\mathbb{R}^2} (\Delta v)u \, dx = \int_{\mathbb{R}^2} v(\Delta u) \, dx = \int_{\Omega_t} v \, dx - \int_{\Omega_0} v \, dx - tv(0)
\]
(One sees formally that the integration by parts has no boundary terms because \( u \) vanishes to second order on \( \partial \Omega_t \) and is identically zero outside.) These formulas are also known as quadrature formulas \([GV06] \).

We have now come nearly full circle. Let \( \omega_t \) be the harmonic measure of \( \Omega_t \) with respect to the origin, defined by the property
\[
v(0) = \int_{\partial \Omega_t} v(x)\omega_t(dx)
\]
for every harmonic function in $\Omega_t$ with, say, continuous boundary values. Then $\omega_t = |\nabla p|\sigma_t = V\sigma_t$, where $V$ is the normal velocity of $\partial\Omega_t$, and

$$v(0) = \frac{\partial}{\partial t} \int_{\Omega_t} v(x) \, dx = \int_{\partial\Omega_t} v(x)V\sigma_t(\partial x) = \int_{\partial\Omega_t} v(x)\omega_t(\partial x)$$

The discrete analogue is the equation we started with, (8).

For any fixed discrete harmonic function $v$, define

$$M(T) = \sum_{n \in A(T)} v(n)$$

If $v(0) = 0$, then (8) implies that the conditional expectation of $M(T + 1)$ given $A(T)$ is

$$\mathbb{E}(M(T + 1) | A(T)) = \sum_{y \in \mathbb{Z}^2} v(y)p_T(y) = v(0) = 0 \quad (13)$$

In other words, $M$ is a martingale. Martingales of this type for various choices of $v$ are the main tools in the proofs of theorems about fluctuations. The martingale property is an immediate consequence of the discrete version of Kakutani’s theorem. The continuum theorems won’t be necessary to us; they just help us to gain intuition.

Finally, we carry out a heuristic derivation that suggests the form of the central limit theorems concerning fluctuations. Suppose that the boundary is given by a perturbation of the disk, in polar coordinates,

$$r < R + \epsilon f(\theta), \quad f(\theta) = \sum_{k \in \mathbb{Z}} \alpha_k e^{ik\theta}$$

with $\overline{\alpha_{-k}} = \overline{\alpha_k}$. We calculate the linearization of Hele-Shaw flow for perturbations of the disk. The Hadamard variational formula says that the (first order in $\epsilon$) change in the gradient of Green’s function is minus the radial derivative of the harmonic extension of $f$,

$$-\frac{\partial}{\partial r} \sum_{k \in \mathbb{Z}} \alpha_k (r/R)^k e^{ik\theta} \bigg|_{r=R} = -\sum_{k \in \mathbb{Z}} \frac{k\alpha_k}{R} e^{ik\theta}$$

The minus sign is very important. When $f(\theta) > 0$, the perturbation is farther from the origin than the location $Re^{i\theta}$ on the circle and the harmonic measure is smaller than average, and fewer particles than average accumulate near $Re^{i\theta}$. This deterministic aspect of the process that keeps the shape close to circular.
Next, we guess as to the stochastic ingredients of the evolution. We propose that the modes vary independently. We expect that for some constant $c > 0$, $k > 0$, $t = \pi r^2$,

$$d\alpha_k = -k\alpha_k \frac{dr}{r} + c\, dB_k(\rho) = -k\alpha_k \, d\rho + c\, dB_k, \quad (\rho = \log r) \quad (14)$$

with independent white noise (derivative $dB_k$ of a Brownian motion $B_k$) of equal amplitude in each mode. The term $-k\alpha_k \, d\rho$ represents the deterministic drift back towards the disk coming from the calculation above.

With $c = 1$, this is the stochastic differential equation that yields the Gaussian Free Field. In [JLS12a] we find instead that the stochastic differential equation turns out to be

$$d\alpha_k = -(k + 1)\alpha_k \, d\rho + dB_k, \quad \rho = \log r$$

The fact that $k$ is replaced by $k + 1$ is related to the curvature of the boundary. The circumference circle of the circle increases with $r$, so there there is room for more particles at the larger radius, and the modes decrease slightly more than given in the rough calculation above. On the other hand, in the case of the cylinder, the circumference of the boundary circle of reference remains constant, and we show in this paper that we get exactly the Gaussian Free Field.

4 Proof of Theorem 3

Note first that if $g(x) = e^{2\pi ikx}$, $k > 0$ and

$$u(x, y) = \begin{cases} g(x) \frac{\sinh(2\pi ky)}{\sinh(2\pi k y_0)}, & 0 \leq y \leq y_0 \\ g(x) e^{-2\pi k(y - y_0)}, & y_0 \leq y < \infty \end{cases}$$

Then the restriction norm

$$\|g\|_{H^{1/2}} = \inf \{\|v\|_{H^0} : v(x, y_0) = g(x)\}$$

is achieved by the harmonic extension $u$. This is proved by computing

$$\int_0^\infty \int_0^1 |\nabla u|^2 \, dx \, dy = (4\pi k)/(1 - e^{-4\pi k y_0}) = 1/m_k$$

so that formula (5) holds.
Divide the outcomes of the cluster growth $A(T)$ into the three events. Event 1, with probability at least $1 - N^{-100}$ is the event that the conclusion of Theorem 2 holds, or, put another way, $D_{N,T}$ is supported in the set

$$F = \{(x,y) : |y - T/N^2| \leq C(\log N)/N\}$$

for all $T \leq C_1 N^2$. Event 2, is the event that $D_{N,T}$ is supported in $y \leq C_2$ for all $T \leq C_1 N^2$, but Event 1 does not hold. Thus Event 2 has probability at most $N^{-100}$.

Event 3 is the complement of Events 1 and 2.

To estimate the probability of Event 3, we recall from [JLS12a] that thin tentacles are rare events. Lemma A of [JLS12a] can be stated in a nearly equivalent form as follows.

Denote $B(n) = \{m \in \mathbb{Z}^N \times \mathbb{Z} : n_2 - N/2 \leq m_2 < n_2 + N/2\}$. This is a cylinder with about $N^2$ lattice sites.

**Lemma 5.** (Thin tentacles) There are positive absolute constants $C_0, b > 0$, and $c > 0$ such that for all $n \in \mathbb{Z}^N \times \mathbb{Z}$ with $n_2 \geq N$,

$$P\{n \in A(T) \text{ and } \#(A(T) \cap B(n)) \leq bN^2\} \leq C_0 e^{-cN^2/\log N}. \quad (15)$$

This lemma implies that that for $C_2$ sufficiently large relative to $C_1$, Event 3 has probability at most $O(e^{-cN^2/\log N})$. Indeed, suppose there is $n \in A(T)$ such that $n_2 > C_2 N$ for some $T \leq C_1 N^2$. Then since $\#A(T) = T$, and $A(T)$ is connected, for at least one $n' \in A(T)$ with $n'_2 \geq N$, $\#B(n') \cap A(T) \leq (2/C_2)C_1 N^2$. Thus if $C_1/C_2 < b$, Lemma 5 applies to $B(n')$ and Event 3 has probability at most $C_0 C_2 e^{-cN^2/\log N}$.

On Event 1 we will replace $\phi$ by a harmonic function. For $|k| \leq K < N$, define $q(k,N) \geq 0$ by

$$1 - \cos(2\pi k/N) = \cosh(q/N) - 1 \quad (16)$$

It follows that

$$q(k,N) = 2\pi|k| + O(1/N^2) \quad (17)$$

Define for $n \in \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}$

$$\psi_0(n, T, N) = \sum_{0 < |k| \leq K} a_k(T/N^2) e^{2\pi i n_k/N} e^{(q/N)(n_2 - T/N)} \quad (18)$$

The function $\psi_0$ is discrete harmonic on the grid of lattice points with spacing $1/N$ that equals an approximation to $\phi - \alpha_0$ on the circle $\{(x,T/N^2) : x \in T\}$. 

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We claim that on Event 1,
\[
\int_{T \times \mathbb{R}} D_{N,T}(x,y) \varphi(x,y) \, dx \, dy = \frac{1}{N} \sum_{n \in A^+(T)} \psi_0(n, T, N) + O(\log N/N) \quad (19)
\]
To prove this first note that
\[
\int_{T \times \mathbb{R}} D_{N,T}(x,y) \varphi(x,y) \, dx \, dy = \int_{T \times \mathbb{R}} D_{N,T}(x,y)(\varphi(x,y) - \alpha_0) \, dx \, dy
\]
\[
= \int_F D_{N,T}(x,y)(\varphi(x,y) - \alpha_0) \, dx \, dy
\]
\[
= N \int_F 1_{A_N(T)}(\varphi(x,y) - \alpha_0) \, dx \, dy
\]
Let \( F_N = \{ n \in \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z} : |n_2 - T/N| \leq C \log N \} \). Next, for \( n \in F_N \) and \((x,y) \in Q_n(n)\),
\[
|\varphi(x,y) - \psi_0(n, T, N)| \leq C_3(\log N)/N
\]
where \( C_3 = 10C \max |\nabla \varphi| \). Without loss of generality, \( C \log N \) is an integer. Therefore \( F \cap A_N(T) \) is a union of squares of side \( 1/N \) and we can match every such square with its corner lattice point and replace replace \( \varphi - \alpha_0 \) by \( \psi_0 \). Thus we obtain
\[
N \int_F 1_{A_N(T)}(\varphi(x,y) - \alpha_0) \, dx \, dy = \frac{1}{N} \sum_{n \in F_N} 1_{A(T)} \psi_0(n, T, N) + O((\log N)/N)
\]
Moreover,
\[
\frac{1}{N} \sum_{n \in F_N} 1_{A(T)} \psi_0(n, T, N) = \frac{1}{N} \sum_{n \in A^+(T)} \psi_0(n, T, N)
\]
This concludes the proof of (19).
Define
\[
M(s) = \frac{1}{N} \sum_{n \in A(T \wedge s)} \psi_0(n, T, N)
\]
Then \( M \) is a martingale. Denote by
\[
Q = \sum_{s=1}^T \mathbb{E}(|M(s) - M(s-1)|^2 | A(s-1))
\]
the quadratic variation of the martingale, and denote

\[ S^2 = \mathbb{E}(Q), \quad B = \mathbb{E}(|Q - S^2|^2), \quad A = \sum_1^T \mathbb{E}|M(s) - M(s - 1)|^4 \]

A theorem of Heyde and Brown \[\text{[HB70]}\] gives a bound on the rate of convergence in the martingale central limit theorem as follows. There is an absolute constant \( C \) such that

\[
\sup_{\lambda \in \mathbb{R}} |\mathbb{P}(M(T)/S \leq \lambda) - \Phi(\lambda)| \leq C \left( \frac{(A + B)/{S^4}}{1} \right)^{1/5}
\]

Note that

\[ Q = \frac{1}{N^2} \sum_{n \in A^+(T)} |\psi_0(n, T, N)|^2 \]

Define

\[ H(x, y) = \sum_{0 < |k| \leq K} \alpha_k(y_0)e^{2\pi i kx}e^{2\pi |k|(y - y_0)} \]

On Event 1, \( A^+(T) \) is up to a strip of unit width and height \( C(\log N)/N \), equal to the set \( 0 \leq y \leq y_0 \). Moreover, because \( q = 2\pi |k| + O(1/N^2) \), for \((x, y) \in Q_N(n)\),

\[ |H(x, y) - \psi_0(n, T, N)| = O(1/N) \]

Thus on Event 1,

\[ Q = \int_0^1 \int_0^{y_0} |H(x, y)|^2 \, dx \, dy + O(1/N) \]

Furthermore,

\[ \int_0^1 \int_0^{y_0} |H(x, y)|^2 \, dx \, dy = \sum_{0 < |k| \leq K} m_k|\alpha_k(y_0)|^2 \]

with

\[ m_k = \int_0^{y_0} e^{4\pi |k|(y - y_0)} \, dy = \frac{1}{4\pi |k|} (1 - e^{-4\pi |k| y_0}) \]

Hence \( |Q - S^2_{\{y_0\}}(\varphi)| \leq 1/\sqrt{N} \) with probability \( 1 - N^{-100} \). On Event 2, \( A(T) \subset \{n_2 \leq CT/N\} \), so that the factor \( e^{(q/N)(200N^{-T}/N)} \leq e^{200K} \) is bounded, and \( Q = O(N^2) \). Thus the expectation from Event 2 is at most \( O(N^2N^{-100}) = O(N^{-98}) \). Finally, on Event 3, the worst case, we still have
the trivial estimate $n \in A(T) \implies n_2 \leq T$. Hence $e^{(q/N)(n_2 - T/N)} \leq e^{CN}$ for constant $C$ depending only on $K$, and $Q = O(e^{CN})$. But Event 3 has probability of order $e^{-cN^2/\log N}$, which is much smaller than exponential. All together we have $\mathbb{E}|Q - S_{\{y_0\}}^2|^2 \leq 1/N$.

On Event 1 or 2, $|M(s) - M(s - 1)|^4 \leq C/N^4$, This contributes to $B$ a sum of size $O(N^2/N^4) = O(1/N^2)$. On Event 3, the worst size case is size $e^{CN}$ which is much smaller than $e^{-cN^2/\log N}$, and hence negligible in the sum representing $B$.

In all, $A + B = O(1/N)$ and we get the bound $N^{-1/5}$ for the discrepancy of the distribution with the one for the standard normal variable.

5 Proof of Theorem 4

We consider separately Events 1, 2, and 3 as in the proof of Theorem 3.

Lemma 6. Denote

$$L_N(\varphi) = \int_{T \times \mathbb{R}} L_N(x,y)\varphi(x,y) \, dx \, dy$$

On Event 1, with probability at least $1 - N^{-100}$,

$$L_N(\varphi) = \frac{1}{N^3} \int_0^\infty (t - T(t))\alpha_0(t/N^2) \, dt$$

$$+ \frac{1}{N^3} \int_0^\infty \sum_{n \in A^+(T(t))} \psi_0(n,t,N) \, dt + O((\log N)^2/N)$$

Proof of Lemma 6 Rewrite $L_N$ as

$$L_N(x,y) = \int_0^\infty (1_{y \leq T(t)/N^2} - 1_{A_N(T(t))}) \, dt + \int_0^\infty (1_{y \leq t/N^2} - 1_{y \leq T(t)/N^2}) \, dt$$

It follows that

$$\int_{T \times \mathbb{R}} L_N(x,y)\varphi(x,y) \, dx \, dy = U_1 + U_2 + U_3$$

with

$$U_1 = \frac{1}{N} \int_0^\infty \int_0^1 \int_0^\infty (1_{y \leq T(t)/N^2} - 1_{A_N(T(t))}) (\varphi(x,y) - \alpha_0(y)) \, dt \, dx \, dy$$

$$U_2 = \frac{1}{N} \int_0^\infty \int_0^1 \int_0^\infty (1_{y \leq T(t)/N^2} - 1_{A_N(T(t))}) \alpha_0(y) \, dt \, dx \, dy$$

$$U_3 = \frac{1}{N} \int_0^\infty \int_0^\infty (1_{y \leq t/N^2} - 1_{y \leq T(t)/N^2}) \alpha_0(y) \, dt \, dy$$
Almost surely, \( |T(t) - t| = O(t^{1/2} \log t) \). Furthermore, on Event 1, the integrand of \( U_1 \) is supported within distance \( O((\log N)/N \) of \( y = t/N^2 \). Therefore, on the support,

\[
|\varphi(x, y) - \alpha_0(y) - \psi_0(xN, yN, t, N)| = O((\log N)/N)
\]

Hence, when we replace \( \varphi(x, y) - \alpha_0(y) \) by \( \psi_0(xN, yN, t, N) \), the difference is dominated by

\[
\frac{1}{N} \int_0^{2y_1N^2} \int_0^{\infty} \int_0^1 (1_{\{y \leq T(t)/N^2\}} - 1_{A_N(T(t))}) \frac{(\log N)/N}{dx} dy dt
\]

\[
\leq \frac{1}{N} \int_0^{2y_1N^2} ((\log N)/N)^2 dt = O((\log N)^2/N)
\]

Next, we claim that

\[
\int_0^{\infty} \int_0^{\infty} \int_0^1 (1_{\{y \leq T(t)/N^2\}} - 1_{A_N(T(t))}) \psi_0(xN, yN, t, N) dx dy dt
\]

\[
= \frac{1}{N} \int_0^{\infty} \int_0^{\infty} \int_0^1 (1_{\{y \leq h(t)\}} - 1_{A_N(T(t))}) \psi_0(xN, yN, t, N) dx dy dt
\]

\[
= -\frac{1}{N^3} \sum_{n \in \mathbb{Z}_N \times \mathbb{Z}} (1_{\{n_2 \leq h(t)N\}} - 1_{A^+(T(t))}) \psi_0(n, t, N) dx dy dt
\]

\[
+ O((\log N)/N)
\]

\[
= -\frac{1}{N^3} \sum_{n \in A^+(T(t))} \psi_0(n, t, N) dx dy dt + O((\log N)/N)
\]

Indeed, the first equation is valid because

\[
\int_0^1 \psi_0(xN, yN, t, N) dx = 0
\]

In other words, we may replace \( 1_{y \leq T(t)/N^2} \) with \( 1_{y \leq h(t)} \) for any function \( h \). In order to justify the error bound in the second equation, choose \( h(t) = \lfloor t/N \rfloor/N \). With probability \( 1 - N^{-100} \) this, once again, confines the integration to a strip of width \( O((\log N)/N \). Replace \( \psi_0(xN, yN, t, N) \) on \( Q_N(n) \) with the value at the corner \( \psi_0(n, t, N) \) to obtain a discrete sum. (For convenience, we chose \( h(t) \) so that the corresponding discrete upper bound \( n_2 \leq Nh(t) = \lfloor t/N \rfloor \) is an integer.) Replacing \( (xN, yN) \) with \( n \) moves the point by a distance at most 1 in each variable, which changes \( \psi_0 \) by \( O(1/N) \). In the previous substitution, the difference in \[20\] was \( O((\log N)/N) \), so the
integrated error here is smaller by the factor $1/\log N$. Finally, in the last equation, replacing $h(t)$ by 0 does not change the sum because for each $n_2$,

$$\sum_{n_1=1}^{N} \psi_0(n_1, n_2, t, N) = 0$$

Next we confirm that with probability $1 - N^{-100}$,

$$U_2 = O((\log N)^2/N) \quad (21)$$

Let

$$f_N(t, y) = \int_{0}^{1} (1\{y \leq T(t)/N^2\} - 1_{A_N(T(t))}) dx$$

Since $A^+(T(t))$ consists of $T(t)$ squares of side length $1/N$,

$$\int_{0}^{\infty} f_N(t, y) dy = 0 \quad (22)$$

On Event 1, and using the almost sure estimate $|T(t) - t| = O(t^{1/2} \log t)$, we have

$$f_N(t, y) = 0 \quad \text{for} \quad |y - t/N^2| \geq C(\log N)/N \quad (23)$$

Consequently, since $|f_N(t, y)| \leq 2$,

$$\int_{0}^{\infty} |f(t, y)| dy = O((\log N)/N) \quad (24)$$

Hence, on Event 1,

$$U_2 = \int_{0}^{\infty} \int_{0}^{\infty} \frac{1}{N} f_N(t, y) \alpha_0(y) dy dt$$

$$= \int_{y_0 N^2/2}^{2 y_1 N^2} \int_{0}^{\infty} \frac{1}{N} f_N(t, y) \alpha_0(y) dy dt$$

$$= \int_{y_0 N^2/2}^{2 y_1 N^2} \int_{0}^{\infty} \frac{1}{N} f_N(t, y)(\alpha_0(y) - \alpha_0(t/N^2)) dy dt$$

$$\leq \int_{y_0 N^2/2}^{2 y_1 N^2} ((\log N)/N^2)((\log N)/N) dt = O((\log N)^2/N)$$

The second equation the fact that $\alpha_0$ is supported in $y_0 \leq y \leq y_1$ and (23). The third equation uses (22). The final inequality uses (24) and (23) to bound the difference in values of $\alpha_0$. This concludes the proof of (21).
Finally, we show that
\[
U_3 = \frac{1}{N^3} \int_0^\infty (t - T(t)) \alpha_0(t/N^2) \, dt + O((\log N)/N) \tag{25}
\]
Indeed, change variables to \( r = N^2 y \), to and define \( R_1 \) by
\[
U_3 = \frac{1}{N^3} \int_0^\infty \int_0^\infty \left( 1_{\{r \leq t/N^2\}} - 1_{\{r \leq T(t)/N^2\}} \right) \alpha_0(r/N^2) \, dr \, dt
\]
\[
= \frac{1}{N^3} \int_0^\infty \int_0^\infty \left( 1_{\{r \leq t/N^2\}} - 1_{\{r \leq T(t)/N^2\}} \right) \alpha_0(t/N^2) \, dr \, dt + R_1
\]
Using \(|T(t) - t| = O(t^{1/2} \log t)\), we see that
\[
|\alpha_0(r/N^2) - \alpha_0(t/N^2)| = O((\log N)/N)
\]
on the support of the integral representing the remainder term \( R_1 \). This and the support properties of \( \alpha_0 \) yield
\[
|R_1| \leq \frac{1}{N^3} \int_{y_0 N^2/2}^{y_0 N^2} \int_{t-C N \log N}^{t+C N \log N} ((\log N)/N) \, dr \, dt = O((\log N)^2/N)
\]
This concludes the proof of Lemma 6 with an error of \( O((\log N)^2/N) \).

For \( 0 \leq s < \infty \) define \( M(s) = M_1(s) + M_2(s) \) by
\[
M_1(s) = \frac{1}{N^3} \int_0^\infty (t \wedge s - T(t \wedge s)) \alpha_0(t/N^2) dt
\]
\[
M_2(s) = \frac{1}{N^3} \int_0^\infty \sum_{n \in A^+(T(t \wedge s))} \psi_0(n, t, N) dt
\]
The total quadratic variation of \( M \) is
\[
Q = \int_0^\infty Q(s) ds \quad \text{with} \quad Q(s) = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \mathbb{E}((M(s + \epsilon) - M(s))^2|A(T(s)))
\]
Since \( T(t) \) is independent of the process defining \( A(T) \), \( Q(s) = Q_1(s) + Q_2(s) \) the sum of the quadratic variation of \( M_1 \) and \( M_2 \) separately. Because the \( \alpha_k(y) \) are supported in \( y \leq y_1 \), \( M(s) \) is constant and \( Q(s) = 0 \) for \( s \geq y_1 N^2 \).

Let \( M^* = \lim_{s \to \infty} M(s) = M(y_1 N^2) \). Then we have shown that with probability \( 1 - N^{-100} \),
\[
|L_N(\varphi) - M^*| \leq C(\log N)^2/N
\]
We will use the continuous analogue of the quantitative form of the martingale central limit theorem mentioned above, proved by Haeuser [H88].
Lemma 7. Let \( S^2 = EQ \). Then
\[
\sup_{\lambda \in \mathbb{R}} |P(M^* / S \leq \lambda) - \Phi(\lambda)| \leq \left[ E(|Q - S^2|^2 / S^4) + E \sum_{0 \leq s \leq y_1 N^2} |\Delta M(s)|^4 / S^4 \right]^{1/5}
\]
in which we define
\[
\sum_s |\Delta M(s)|^4 = \sum_j |M(s_j^+) - M(s_j^-)|^4
\]
This last sum is over the (almost surely finite number) of times \( s_j \) in \( 0 \leq s \leq y_1 N^2 \) at which \( M(s) \) is discontinuous. In fact, we will replace \( S^2 \) by \( S^2(\varphi) \).
\[
S^2(\varphi) = \sum_{|k| \leq K} \int_0^\infty \left| \alpha_k(y') e^{2\pi k(y-y')} dy' \right|^2 dy
\]
In fact, these \( s_j \) are the times at which \( T(s_j^+) - T(s_j^-) = 1 \).
\[
M_1(s_j^+) - M_1(s_j^-) = -\frac{1}{N^3} \int_{s_j}^\infty \alpha_0(t/N^2) dt = -\frac{1}{N^3} \int_{s_j}^{y_1 N^2} \alpha_0(t/N^2) dt
\]
\[
M_2(s_j^+) - M_2(s_j^-) = \frac{1}{N^3} \int_{s_j}^\infty \psi_0(n_j, t, N) dt = \frac{1}{N^3} \int_{s_j}^{y_1 N^2} \psi_0(n_j, t, N) dt
\]
with \( \{n_j\} = A(T(s_j^+) \setminus A(T(s_j^-)) \). On \( \alpha_0 = O(1) \) and on Events 1 or 2, \( p_0(n_j, t, N) \) is bounded so that \( |\Delta M(s_j)| = O(1/N) \). On Event 3, \( |\Delta M(s_j)| = O(e^{CN}) \), which is much smaller than the probability \( e^{-cN^2 / \log N} \) of Event 3. There are are almost surely \( O(N^2) \) jumps \( s_j \). Therefore,
\[
E \sum_{0 \leq s \leq y_1 N^2} |\Delta M(s)|^4 / S^4 = O(1/N^2)
\]
We will show below that
\[
E(|Q - S^2(\varphi)|^2) = O(N^{-2/3})
\]
(26)
Once we have proved this, the proof of Theorem 4 is nearly complete. Lemma 7 says that \( M^* \) has the same distribution as a gaussian with variance \( S^2 = EQ \) up to \( O(N^{-2/15}) \). Moreover, (26) implies that \( |S^2 - S^2(\varphi)| = O(N^{-1/3}) \). According to Lemma 6, \( M^* \) differs from \( L_N(\varphi) \) by at most \( O((\log N)^2 / N) \) with probability \( O(1 - N^{-100}) \).
It remains to prove (26). Since $E((\epsilon - (T(s + \epsilon) - T(s))^2) = \epsilon$,

$$Q_1(s) = \left( \frac{1}{N^3} \int_s^\infty \alpha_0(t/N^2) dt \right)^2$$

and

$$Q_1 = \int_0^\infty Q_1(s) ds = \int_0^\infty \left( \int_y^\infty \alpha_0(y') dy' \right)^2 dy$$

**Lemma 8.** With probability $1 - N^{-100}$,

$$Q_2 = \sum_{0 < |k| \leq K} \int_0^\infty \left| \alpha_k(y') e^{2\pi k(y-s)} dy' \right|^2 dy + O(N^{-1/4})$$

Proof.

$$Q_2(s) = \sum_{n \in \mathbb{Z} \times \mathbb{Z}} \left| \frac{1}{N^3} \int_s^\infty \psi_0(n,t,N) dt \right|^2 p_s(n) \tag{27}$$

where $p_s(n)$ is the probability that the random walk starting at $(n_1,0)$, $n_1 = 1, \ldots, N$ with equal probability, exits $A(T(s))$ for the first time at the site $n$. Thus $p_s(n)$ is nonzero only on the boundary of $A(T(s))$, that is at sites at distance exactly 1 from $A(T(s))$. With probability $1 - N^{-100}$, $p_s(n) > 0$ implies $|n_2 - s/N| \leq C \log N$. In other words, the boundary of $A(T(s))$ is nearly a horizontal line. One can therefore deduce from barrier estimates (discrete harmonic majorants, not written down explicitly here) that the distribution of $p_s(n)$ is approximately uniform in the $n_1$ variable in the following sense. For any $a > 0$, define

$$R(a) = \{ n : 1 \leq n_1 \leq aN, \ |n_2 - s/N| \leq C \log N \}$$

Then

$$\sum_{n \in R(a)} p_s(n) = a + O(N^{-1/3})$$

(Sharper bounds are also valid; we have not attempted to optimize the power.)

Put the sites for which $p_s(n(j)) > 0$ in order according to their position horizontally, $1 \leq n_1(1) \leq n_1(2) \leq \cdots$ and consider disjoint intervals $I_j$ of $0 \leq x \leq 1$, so that the right endpoint of $I_j$ is the left endpoint of $I_{j+1}$ and the length $|I_j| = p_s(n(j))$. Then for all $x \in I_j$,

$$|\psi_0(n(j),t,N) - \rho_0(xN,s/N,t,N)| = O(N^{-1/3})$$

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Thus, if we replace the sum of \( \psi_0 \) on the lattice with the weighting \( p_s(n) \) by the integral \( dx \), we find that

\[
Q_2(s) = \int_0^1 \frac{1}{N^3} \int_s^{y_1 N^2} \psi_0(xN, t, N) \, dt \, dx + O(N^{-2}N^{-1/3})
\]

The worst error comes from cross terms in the integrand of the square with one factor of size \( O(N^{-1/3}) \) and the other of unit size. This yields errors which are a factor \( O(N^{-1/3}) \) smaller than the main term. The main term \( Q_2(s) \) is of size \( O(N^{-2}) \) as one can see from the fact that the expression inside the \( \cdot \) sign is an integral in \( t \) over an interval of length of order \( N^2 \) of a (roughly) unit sized integrand divided by \( N^3 \), thus of size \( 1/N \). This is squared and summed over the probability measure \( p_s(n) \).

Finally, integrating \( Q_2(s) \) over \( 0 \leq s \leq y_1 N^2 \), and changing variables, one finds the expression in Lemma 8, with an error that is a factor \( O(N^{-1/3}) \) smaller.

6 Obstacle problems

Levine and Peres had an entirely different motivation. In 1991 Diaconis and Fulton [DF91] defined the notion of smash sum. Consider two open subsets \( A \) and \( B \) of the plane and define a function \( \mu_N \) on the lattice \( \mathbb{Z}^2 \) by

\[
\mu_N(n) = 1_A(n/N) + 1_B(n/N)
\]

The function \( \mu_N \) represents an initial collection of particles, two at each site of \( A \cap B \) in a rescaled lattice of mesh size \( 1/N \) and one particle at each of the rest of the sites in \( A \cup B \). Choose any site with more than one particle, and move one of them to any of the four adjacent sites with equal probability. The order in which the particles move is irrelevant because they are interchangeable. A site can be occupied by many particles at the same time, and the process is said to stop the first time each occupied site has exactly one particle. Denote the final distribution by \( \nu_N \). The theorems of Levine and Peres characterized the deterministic limit of \( \nu_N \), the set \( C \) such that

\[
\lim_{N \to \infty} \frac{1}{N^2} \sum_{n \in \mathbb{Z}^2} \varphi(n/N) \nu_N(n) = \int_{\mathbb{R}^2} \varphi(x) 1_C(x) \, dx
\]

Figure 3 gives an instance of the smash sum of two disks.

We will now give a few more of the ideas behind the work of Levine and Peres [LP10]. The idea is to show that the continuum limit solves the obstacle problem.
The first step of their proof is to study a deterministic process on the lattice, known as the divisible sandpile. (The analogous continuum process is balayage or sweeping out \cite{GV06}.) Consider any nonnegative function $\mu : \mathbb{Z}^2 \to \mathbb{R}_+$, representing possibly fractional quantities of particles at each lattice site or else the height of a sandpile. Pick a site $x$ at which $\mu(x) > 1$, after a single step in the process, a toppling of the site $x$, the new heights are 1 at $x$ and $(\mu(x) - 1)/4$ extra at each of the four adjacent sites. This process continues until every site has at most height 1. One can show that the process stops in a finite number of steps for finitely supported $\mu$. Moreover, the final height function, denoted $\nu(x)$, is independent of the order in which the toppling occurs. The divisible sandpile reflects the average behavior of the random walk. Figure 6 depicts starting from the same $\mu_N$ as in the preceding picture, Figure 5. The resemblance is already striking for $N \approx 200$.

Levine and Peres show that the random process is well approximated by the deterministic sandpile process by means of an auxiliary function they call the odometer function $u(x)$. The function $u(x)$ records the (fractional) number of particles donated by the site $x$ in the course of the deterministic process. The word odometer reminds us that $u(x)$ does not represent a net loss of particles, but rather the total quantity donated without subtracting the number received. It is not hard to check that $u$ solves the discrete Laplace equation

$$\mathcal{L}u(x) = \nu(x) - \mu(x)$$
Moreover, $u$ can be obtained from the solution to the discrete obstacle problem as follows.

**Lemma 9.** (Levine-Peres) Fix any $\gamma(x)$ satisfying $L\gamma = \mu - 1$. Let $w$ solve the obstacle problem

$$w(x) = \inf\{f(x) : Lf \leq 0, \ f \geq \gamma\}.$$  

Let $u$ be the odometer function starting from $\mu$. Then

$$u = w - \gamma.$$

The fact that the discrete function $u$ tends to its continuum counterpart depends ultimately on estimating the difference between the fundamental solution of $L$ and $\Delta$.

## 7 Further remarks and questions

Theorem 1 is proved in [JLS12a] in dimension 2 and in [JLS12b] in dimensions $d \geq 3$. The proof uses martingales associated to the discrete analogue of Green’s function with a pole at a point near the putative boundary, either inside or outside. These martingales take values larger than the expected value if there are extra points in the cluster near the pole, and the martingale is smaller than its expected value if there are fewer than the typical number.
of occupied sites in the cluster near the pole. The central limit theorem is inadequate to the task of estimating large deviations of the martingale. Instead one uses the parametrization of the martingale by Brownian motion. The lemma concerning thin tentacles, Lemma 5, is crucial as well. The proof also involves an iteration of successively better estimates on the inner and outer deviations of the shape. The proof of Theorem 2 is analogous to that of Theorem 1, just as Theorems 3 and 4 are similar to the corresponding theorems for $\mathbb{Z}^2$.

In this paper, we chose to treat the case of the cylinder so as to identify a case in which the fluctuations are described exactly by the Gaussian Free Field. In [JLS11] we carried out the case of the disk. One difference with the case of the cylinder is that it’s somewhat harder to construct suitable discrete harmonic functions approximating $z^k$. Furthermore, the estimates require variants for averages with respect to discrete harmonic functions of van der Corput’s theorem counting lattice points in disks. We have not yet carried out the case $d = 3$, although we believe it follows from very similar methods. The technical difficulty is that it requires variants of theorems stronger theorems than van der Corput’s concerning the number of lattice points in a ball in 3-space, along the lines of improvements due to Vinogradov (see [IKKN04]).

Whereas the square Dirichlet norm is

$$\int_{\mathbb{R}^2} |\nabla f(x, y)|^2 \, dx \, dy = \int_0^{2\pi} \int_0^\infty (|r \partial_r f|^2 + |\partial_\theta f|^2) \frac{dr \, d\theta}{r}$$

$$= 2\pi \sum_{k \in \mathbb{Z}} \int_0^\infty (|r \partial_r f_k|^2 + |k f_k|^2) \frac{dr}{r}$$

in which

$$f(x, y) = \sum_{k \in \mathbb{Z}} f_k(r) e^{ik\theta}$$

The square of the norm of the gaussian random field representing fluctuations from a source at the origin in $\mathbb{Z}^2$ is

$$2\pi \sum_{k \in \mathbb{Z}} \int_0^\infty (|r \partial_r f_k|^2 + |(|k| + 1) f_k|^2) \frac{dr}{r}$$

In general, we expect that the random field will reflect the curvature of the deterministic region. But even in this simple case, the norm is expressed in terms of non-local (pseudo-differential) operators.

The expression for the norm in Theorem 3 at distance $y_0$, starting from the (exactly straight) boundary of $y \leq 0$ involves the factor $(1 - e^{-4\pi |k|y_0})$. 

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Thus in some average sense, the influence of deterministic behavior at \( y = 0 \) attenuates at an exponential rate at \( y = y_0 \). It would be nice to understand this better. At the same time one can ask about the mixing time, that is, given a known boundary at one time, how long do we need to wait before that configuration is mostly forgotten?

One can also ask questions about random walks other than the standard one. The first author\(^1\) supervised work on this subject in the summer of 2008 by a high school student, Max Rabinovich \([R08]\). He adapted the methods of Levine and Peres to the hexagonal lattice. His key observation is that the same methods work, provided one can approximate the discrete fundamental solution by the analogue of the Newtonian potential. At first it appears that the estimates need to be good to second order at infinity which they are not for the hexagonal lattice. But on closer inspection, what is required are error estimates for the difference of fundamental solutions, as compared to the gradient of the Newtonian potential. The gradient is of order \( 1/r^{d-1} \) as \( r \to \infty \) and the error term is one order better, \( 1/r^d \), which is two orders better than \( 1/r^{d-2} \) as required. Rabinovich’s theorem applies to all random walks on \( \mathbb{Z}^d \) given by a finitely supported probability measure \( p \) on \( \mathbb{Z}^d \) such that the random walk moves from \( x \) to \( x + \alpha \) with probability \( p(\alpha) \) and

\[
\sum_{\alpha \in \mathbb{Z}^d} p(\alpha)\alpha = 0
\]

This condition means that the random walk has no drift.

It remained to consider walks with drift. James Propp proposed the specific example of a walk that moves East or North, each with probability \( 1/2 \). If the source is the origin, this fills a cluster in the first quadrant. If there are \( T \) particles, it is natural to rescale by parabolic scaling \( u = x + y \), and \( v = x - y \) are replaced by \( U = (x + y)/T^{2/3}, V = (x - y)/T^{1/3} \). Then in parallel with the work of Levine and Peres, one expects the cluster to be associated with an obstacle problem based on parabolic operators in the \((U, V)\) variables as treated by Caffarelli, Petrosyan and Shahgholian \([CPS04]\).

After this lecture, Cyrille Lucas \([Lu12]\), carried out this program and in the process established the existence of a so-called heat ball. Take the limiting (and indeed simplest case) of the Hele-Shaw flow in which \( \Omega_0 \) shrinks to a point. Then \( \Omega_t \) is the Euclidean ball of volume \( t \). The conservation law

\[^1\text{The author thanks Pavel Etingof for suggesting this problem.}\]
in integrated form can be written
\[ v(0) = \frac{1}{\text{vol} B} \int_B v(x) \, dx \]
for any harmonic function \( v \). Of course, this is just the well known mean value property for harmonic functions. The domain analogous to the ball for the heat operator is a set \( H \subset \{(x,t) : t \geq 0\} \) of area 1 such that
\[ v(0,0) = \int_H u(x,t) \, dx \, dt \]
where \( v \) satisfies the adjoint or backwards heat equation
\[ \left[ (\partial/\partial x)^2 + (\partial/\partial t) \right] v(x,t) = 0 \]
Evidently, any parabolic dilation \( H_R = \{(Rx,R^2 t) : (x,t) \in H\} \) satisfies of
\[ v(0,0) = \frac{1}{R^3} \int_{H_R} v(x,t) \, dx \, dt \]
Many weighted averages of \( v \) produce \( v(0,0) \). This one is interesting because
the weight is constant, proportional to Lebesgue measure.

As mentioned in the lecture, the question that remains open is the regularity of the boundary of \( H \). The discrete construction of the divisible sandpile gives an approximation to the continuum set \( H \). The theorems of \cite{CPS04} give a criterion involving approximations to \( H \). Their criterion would imply that the boundary of \( H \) is smooth if there were a practical bound on the constants involved. It would also be interesting to show that \( H \) is convex, which looks rather obvious from the sandpile approximation. A typical approach would be to realize \( H \) as the level set of a log concave function. However, the odometer function \( u \) associated with \( H \) is not log concave. On the other hand, we have numerical evidence that \( u/h \) is log concave, in which \( h \) is the standard fundamental solution, \( t^{-1/2} e^{-x^2/4t} \). This would imply that \( H = \{u/h > 0\} \) is convex. It is not hard to show in the Propp example that the discrete analogue of \( u/h \) is log concave in the \( x \) direction. Unfortunately, one does find numerically a very few sites near the boundary at which log concavity fails slightly in the \( t \) direction. So at least in the Propp example, it’s hard to see how a combinatorial proof could succeed.
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