Emergence of a bicritical end point in the random crystal field Blume-Capel model

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We obtain the phase diagram for the Blume-Capel model with bimodal distribution for random crystal fields, in the space of three fields: temperature ($T$), crystal field ($\Delta$) and magnetic field ($H$). We find that three critical lines meet at a tricritical point, but only for weak disorder. As disorder strength increases there is no tricritical point in the phase diagram. We instead find a bicritical end point, where only two of the critical lines meet on a first order surface in the $H = 0$ plane. For intermediate strengths of disorder, the phase diagram has critical end points along with the bicritical end point. One needs to look at the phase diagram in the space of three fields to identify various such multicritical points.

Multicritical points typically occur in systems described by three or more thermodynamic fields. In these systems, there can be critical points that can be reached only by fixing three or more thermodynamic parameters. Hence the full phase diagram of such systems is multidimensional \[1-3\]. Such critical points are ubiquitous in nature, in systems like binary fluids \[4, 5\], metamagnets \[6\], alloys of magnetic and non-magnetic materials \[7\], $He^3 - He^4$ mixtures \[8\], quantum metals \[9\], polymer collapse \[10\] and quantum chromodynamics \[11\]. Tricritical point (TCP) is one of the most widely studied and well understood multicritical points \[12\]. Solvable models which display higher order critical points are useful in outlining the topology of phase diagrams \[13\]. In this context, mean field Blume-Capel models \[14, 15\] has been very useful and is one of the most well studied model. It is the simplest model to exhibit a TCP. TCP is an example of a multicritical point, which is a point of confluence of three critical lines in the space of three fields ($T, \Delta, H$). Here $T$ and $H$ are the temperature and external field respectively and $\Delta$ is a non-ordering field, known as the crystal field \[12, 16\]. In the ($\Delta$) plane (with $H = 0$), TCP shows itself as a critical line ending in a first order line.

Introducing randomness in bond strength or field strength is known to affect the phase diagram. In two dimensions it was shown that even an infinitesimal amount of random field disorder can change a first order transition to continuous transition or can destroy it altogether \[17, 18\]. Most of the previous studies of random crystal field Blume-Capel model have focussed on ($T, \Delta$) plane \[19-27\]. In this paper, we revisit the problem and look at the phase diagram in the three fields space. We find that the phase diagram changes nontrivially with the strength of disorder and other multicritical points show up in the phase diagram. Identifying these higher order critical points requires the study of the full phase diagram, in the space of three fields, which we present in this paper.

We find that the tricritical point persists for only very weak disorder strengths. As the disorder strength increases, the tricritical point vanishes and a different multicritical point, bicritical end point (BEP) emerges where only two of the three critical lines end on a first order surface \[28, 30\].

BEP has been comparatively less observed and studied in literature. Two well known examples where BEP has been observed are: anisotropic continuous spin systems as a spin flop \[28\] and in spin 3/2 systems with crystal
The emergence of BEP as a consequence of disorder is a new result being presented in this paper. We also find that the model exhibits critical end points (CEP) for intermediate strengths of disorder. Critical end point is a critical point where a line of second order transitions terminates at a line of first order transitions. Alternately, it can also be defined as a point where two phases become critical in the presence of one or more ordered phases, known as the spectator phases, in systems with multiple phases. We find three different phase diagrams depending on the strength of disorder. For weak disorder, the three critical lines meet at a tricritical point (see Fig 1(a)). For intermediate disorder strengths, the two critical lines with $H \neq 0$ meet at a BEP and the line of continuous transition in $(T, \Delta)$ plane (known as $\lambda$ line) meets a line of first order transition at a CEP. CEP and BEP are connected via a quadruple line, along which the four phases co-exist (see Fig 1(b)). For strong disorder, the BEP persists but CEP vanishes and the $\lambda$ line continues to $\Delta \to \infty$ (see Fig. 1(c)). We will study these three topologies in this paper. We expect the change in topology of the phase diagram as a function of strength of disorder to be robust and not restricted to mean-field solutions.

The plan of the paper is as follows: In Section II we discuss the Blume-Capel model in the presence of external field and derive the equations for critical lines in $(T, \Delta, H)$ space. In Section III we show the presence of BEP for strong disorder. In Section IV we briefly discuss the case of weak disorder and in Section V we try to get a Landau description for BEP and CEP. We conclude with a short discussion in Section VI.

I. MODEL

We study the Blume-Capel model with random crystal field disorder in the presence of external field on a fully connected graph. The Hamiltonian can be written as

$$H(C_N) = -\frac{1}{2N}(\sum_i s_i)^2 - \sum_i \Delta_i s_i^2 - H \sum_i s_i \tag{1}$$

where $\Delta_i$ represent quenched random crystal field at each site, $H$ is the external field and $s_i$ are spin $-1$, $0$, and $1$ random variables which can take $\pm 1, 0$ values. There are two order parameters: magnetisation, $m = \bar{s}$ and density of $\pm 1$ spins, $q = \bar{s^2}$. These are obtained by taking a quenched average of the random variables $s$ and $s^2$ respectively. We draw random crystal fields from bimodal distribution of the kind:

$$P(\Delta_i) = p\delta(\Delta_i - \Delta) + (1-p)\delta(\Delta_i + \Delta) \tag{2}$$

Since $p = 0$ or $1$ will imply no disorder and $p = 1/2$ would be the most random case, hence it is enough to look for $0 \leq p \leq 0.5$.

It can be shown that the probability of a configuration $C_N$ satisfies large deviation principle (LDP) in the presence of random crystal field disorder, i.e.

$$P(C_N : \sum_i s_i/N = x_1; \sum_i s_i^2/N = x_2) \sim \exp(-NI(x_1, x_2)) \tag{3}$$

The rate function $I(x_1, x_2)$ for bimodal random crystal field disorder in the absence of external field was calculated using tilted LDP recently. Using the same method, the rate function in the presence of external field is:

$$I(x_1, x_2) = x_1 \tanh^{-1}(\frac{x_1}{x_2}) + x_2 \left[ \ln \left( \frac{z}{2cosh^{-1}(\frac{x_1}{x_2})} \right) - p\ln(1 + ze^{p\Delta}) - (1-p) \ln(1 + ze^{-p\Delta}) \right] + p\ln(1 + 2e^{p\Delta}) + (1-p) \ln(1 + 2e^{-p\Delta}) - \frac{\beta x_1^2}{2} - \beta H x_1 \tag{4}$$

The rate function is like a Landau free energy functional, whose minima in $(x_1, x_2)$ plane gives the free energy for a given $\beta(= 1/T), \Delta$ and $H$. Hence the values of $x_1$ and $x_2$ which minimise $I(x_1, x_2)$ are the value of $m$ and $q$ respectively for a given set of thermodynamic variables. Minimising $I(x_1, x_2)$ with respect to $x_1$ and $x_2$ results in the following equations for $m$ and $q$:

$$\tanh(\beta(m + H)) = \frac{q}{m} \tag{5}$$

$$z = \frac{2}{\sqrt{1 - m^2/q^2}} \tag{6}$$

where $z$ is related to $q$ via the following equation:

$$\frac{q}{z} = \frac{pe^{\beta\Delta}}{1 + ze^{\beta\Delta}} + \frac{(1-p)e^{-\beta\Delta}}{1 + ze^{-\beta\Delta}} \tag{7}$$

At fixed points the rate function can be replaced by a one parameter functional $\hat{f}(m)$, which comes out to be:

$$\hat{f}(m) = \frac{\beta m^2}{2} - p \log(1 + 2e^{\beta\Delta} \cosh \beta(m + H)) - (1-p) \log(1 + 2e^{-\beta\Delta} \cosh \beta(m + H)) + p \log(1 + 2e^{\beta\Delta}) + (1-p) \log(1 + 2e^{-\beta\Delta}) \tag{8}$$

From this we get the following self-consistent equation for $m$:

$$m = 2 \sinh \beta(m + H) \left[ \frac{pe^{\beta\Delta}}{1 + 2e^{\beta\Delta} \cosh \beta(m + H)} + \frac{(1-p)e^{-\beta\Delta}}{1 + 2e^{-\beta\Delta} \cosh \beta(m + H)} \right] \tag{9}$$

For $H = 0$, the phase diagram has been studied before. Expanding Eq. 5 in powers of $m$, assuming $m$ to be small gives the equation for a line of continuous
transition in the $H = 0$ plane. The line of continuous transition in $H = 0$ plane is known as the $\lambda$-line and satisfies the following equation

$$5 - 4\beta = 2(\beta p - 1)e^{\beta \Delta} + 2(\beta - \beta p - 1)e^{-\beta \Delta}$$  \hspace{1cm} (10)

This is valid only when the higher order terms in the expansion can be ignored and breaks down surely when the $\beta$ and $\Delta$ which satisfy Eq. (10) also satisfy the following condition:

$$\cosh(\beta \Delta) = \frac{12\beta - 19}{8}$$  \hspace{1cm} (11)

The value of $(\beta, \Delta)$ (or equivalently $(T, \Delta)$) which satisfy Eqs. (10) and (11) simultaneously gives the location of TCP for a given $p$. It was found [27] that beyond $p_c = 0.0454$ the two equations cannot be satisfied simultaneously and hence there is no TCP, and the $\lambda$ line in the $(T, \Delta)$ plane extends to $\Delta \to \infty$. This raises a natural question: What is the effect of disorder on the two critical lines $\lambda_+$ and $\lambda_-$? We will focus on the effect of disorder on these two critical lines in this paper.

Since $m \neq 0$ along the $\lambda_+$ and $\lambda_-$ lines, expanding $f(m)$ in powers of $m$ is not useful. In general at the critical point first, second and third derivative of the free energy functional with respect to $m$ should be zero [12] (and fourth derivative should be greater than zero). We use this condition to locate all the critical points in the $(T, \Delta, H)$ phase diagram. For $H = 0$ and $m = 0$ the third derivative is trivially zero and second derivative gives the same condition as Eq. (10).

In general, equating second and third derivative of $f(m)$ to zero we get the following two conditions respectively:

$$p(2x^2 + xy) + (1-p)(2 + xy)/(1 + 2xy)^2 = \frac{1}{2\beta}$$  \hspace{1cm} (12)

$$p(x - 8x^3 - 2x^2y)/(1 + 2xy)^3 + (1-p)(x^2 - 8 + 2xy)/(1 + 2xy)^3 = 0$$  \hspace{1cm} (13)

here $x = \exp(\beta \Delta)$ and $y = \cosh \beta(m + H)$. For $p \neq 0$, the two equations are quartic and sextic in $x$.

For $p = 0$, they reduce to the following simpler equations:

$$\frac{2 + xy}{[x + 2y]^2} = \frac{1}{2\beta}$$  \hspace{1cm} (14)

$$\frac{x^2 - 8 - 2xy}{[x + 2y]^3} = 0$$  \hspace{1cm} (15)

Solving these equations we get

$$y = \cosh \beta(m + H) = \frac{\beta - 2}{\sqrt{4 - \beta}}$$  \hspace{1cm} (16)

$$x = e^{\beta \Delta} = \frac{4}{\sqrt{4 - \beta}}$$  \hspace{1cm} (17)

Hence we reproduce the classic results of Blume,Emery and Griffiths [57]. There is a line of critical points for $4 \geq \beta \geq 3$ for $H > 0$ and another for $H < 0$. Both critical lines extend to $\Delta \to \infty$. These two lines enclose two first order surfaces which meet in the $H = 0$ plane along a triple line (line with three phase co-existence).

Above $\beta = 4$ there is no value of $x$ and $y$ that can satisfy Eqs. (12) and (13) simultaneously. The magnetisation along these two critical lines is not zero and is equal to

$$m = \pm \sqrt{\frac{\beta - 3}{\beta}}$$  \hspace{1cm} (18)

This can be used to get the value of $H$ along the critical lines, which comes out to be

$$H = \pm \frac{1}{\beta} \log \left( \frac{\beta - 2 + \sqrt{\beta^2 - 3\beta}}{\sqrt{4 - \beta}} \right) - m$$  \hspace{1cm} (19)

These two critical lines meet in the $H = 0$ plane at a point with $T_{TCP} = 1/\beta$ and $\Delta_{TCP} = 0.462098$. This is the well known TCP in $(T, \Delta)$ plane for $p = 0$ (can be obtained by solving Eq. (10) and (11) simultaneously for $p = 0$).

For $p \neq 0$, we use Mathematica [58] to solve Eq. (12) and Eq. (13) simultaneously to get the two critical lines numerically. To solve the equations for any arbitrary $p$, we scan different values of $\beta$ and $\Delta$ and hence $x$ and solve the Eq. (12) (corresponding to $f''(m) = 0$) exactly to get the corresponding value of $y$. Then we substitute the value of $x$ and $y$ in Eq. (13) to check if $(x, y)$ satisfy the condition, $f'''(m) = 0$.

For each set of $(x, y)$ that satisfy Eq. (12) and Eq. (13) simultaneously, we can calculate $m$ using the equation:

$$m = \pm 2\sqrt{y^2 - 1} \left[ \frac{px}{1 + 2xy} + \frac{(1 - p)}{y + 2x} \right]$$  \hspace{1cm} (20)

The corresponding value of $H$ along the critical lines can then be calculated by inverting, $y = \cosh \beta(m + H)$.

For a TCP to exist the two critical lines in the $H \neq 0$ plane should meet in $H = 0$ plane at the point where second order line ends in a first order transition line in the $(T, \Delta)$ plane. We can put $H = 0$ and $m = 0$ in Eqs. (12) and (13) to directly look for this point. Hence, we separately solve the two equations for $y = 1$. Interestingly, we find that for $y = 1$, the two equations can be solved simultaneously only for $p \leq p_c (= 0.0454)$.

This also indicates the value of $p$ beyond which linear stability analysis breaks down and Eq. (10) is not valid anymore. More interestingly even though the two equations can be solved for $H = 0$ till $p \leq 0.0454$, we find that for $p > 0.022$, one more solution shows up, with $m \neq 0$ and $H = 0$. For $p > 0.0454$, all possible solutions have $m \neq 0$.

Hence we find that the two critical lines, $\lambda_+$ and $\lambda_-$ meet $\lambda$ line at a TCP for $p < 0.022$. For $p > 0.022$, the two critical lines, $\lambda_+$ and $\lambda_-$ meet inside the first order surface, i.e at a point where $m \neq 0$. This point hence is not a TCP, but a BEP. Furthermore, we find that for $p > 0.022$ there are different kinds of phase diagrams possible: For $0.022 < p \leq 0.1078$ the phase
diagram is as shown in Fig. 1(b): In \( H = 0 \) plane there is a four phase coexistence line starting from the BEP which separates the two ordered phases. This line meets the \( \lambda \)-line defined via Eq. 10 giving rise to a CEP. From CEP there is a three phase coexistence line which ends in another CEP. For 0.1078 < \( p \) ≤ 0.5 the phase diagram is as shown in Fig. 1(c): There is a four phase coexistence line from BEP which never crosses the \( \lambda \) line defined via Eq 10 and goes all the way to \( T = 0 \). Moreover we find that \( \lambda^+ \) and \( \lambda^- \) critical lines exist for all strengths of disorder (i.e for all values of \( p \)). We give more details of these multicritical points in the next few sections.

II. STRONG DISORDER AND BEP

For \( p > 0.022 \), the two critical lines for \( H \neq 0 \) do not meet at the potential TCP point as given by simultaneous solution of Eq. 10 and 11. Instead they meet inside the ordered plane. We find that the two wings are separated by a first order line in \( H = 0 \) plane, which behaves differently for 0.022 < \( p \) ≤ 0.1078 and for 0.1078 < \( p \) ≤ 0.5. Hence we will look at these two regimes separately.

A. 0.1078 < \( p \) ≤ 0.5

For this range of \( p \), along the first order line in \( H = 0 \) plane there is a four phase coexistence, which ends in a bicritical end point (see Fig 1(c)). This line is a first order transition line between two ordered states with different values of magnetisations. These two different ordered states are a result of disorder and are not present in the pure system. At low temperatures, the system prefers \( \pm 1 \) spin states when the \( \Delta \) is small. As \( \Delta \) increases, due to disorder, states with finite fraction of zero spins compete with the states with only \( \pm 1 \) spins. This can be seen by looking at the order parameter \( q \) as a function of \( \Delta \), as shown in Fig. 2.

One can see all the transitions clearly by plotting \( \tilde{f}(m) \) in different regions of the phase diagram as shown in Fig. 3 for \( p = 0.2 \). From the plots we can see that the \( H = 0 \) line separates the two ordered phases. Along \( H \neq 0 \) critical lines, two of these phases become critical and at BEP the two critical phases coexist.

To understand the nature of transition especially at BEP, we looked at the magnetisation \( m \) and magnetic susceptibility \( \chi = \frac{\partial m}{\partial H} |_{H \to 0} \). Let us first look at the magnetisation as a function of \( T \) in the \( H = 0 \) plane.
for different fixed values of $\Delta$ (see Fig. 4). We find that for $\Delta < \Delta_{BEP}$, the magnetisation changes its slope near $T = T_{BEP}$, the change becomes sharper as one approaches $\Delta = \Delta_{BEP}$. For $\Delta > \Delta_{BEP}$ (but close to $\Delta_{BEP}$), the magnetisation undergoes a first order transition as it crosses the quadruple line and then changes slope near $T = T_{BEP}$. For $\Delta$ much larger than $\Delta_{BEP}$, as we increase $T$ there is no first order jump or change of slope around $T = T_{BEP}$. We also looked at $m$ as a function of $\Delta$ for three different values of $T$ (see fig. 5). First order jump as one crosses the quadruple line is clear for $T < T_{BEP}$. For $T > T_{BEP}$ there is no signature of any transition.

It is hard to deduce the nature of transition at BEP by looking at the magnetisation alone. Hence we studied the magnetic susceptibility near BEP. First we look at it for fixed value of $\Delta$. As we fix $\Delta = \Delta_{BEP}$ and vary $T$, we find that there is an infinite peak at the $T$ of $\lambda$ transition. There is another peak at $T = T_{BEP}$, but this peak seems to be finite (see Fig. 6). This behaviour can be contrasted with the behaviour at $\Delta > \Delta_{BEP}$ as shown in Fig 7. We find a discontinuity where it crosses the first order line and a finite peak near $T = T_{BEP}$.

We also study magnetic susceptibility as we vary $\Delta$ at $T_{BEP}$ (Fig 8). Again around $\Delta = \Delta_{BEP}$, we see only a finite peak. This can be contrasted with the clear first order jump in magnetic susceptibility for fixed $T < T_{BEP}$ (see Fig. 9).

We scanned a large region in ($T, \Delta$) plane near BEP. The affect of the presence of BEP is felt even far away from the point. It was shown via scaling arguments [29] that if the two critical lines meeting at BEP are in the same universality class and are symmetric, then the singular behaviour contribution to the phase boundary cancels out [29, 30]. In our case the two critical lines $\lambda_{+}$ and $\lambda_{-}$ lie in the same universality class. Looking at the three dimensional phase diagram it is clear that there is only one phase in the system in the sense that there exist a path between any two non-singular points in the phase diagram which does not have to encounter a singularity.

At BEP the first three derivatives of $\tilde{f}(m)$ w.r.t $m$ are zero and free energy is not analytic at this point. BEP is a point of two phase co-existence and there is no critical transition from one phase to another at BEP.
FIG. 9: Magnetic susceptibility(χ) vs Δ plot for p = 0.2 at T < T_{BEP}

As p increases we find that the critical lines enclosing the wings become flatter and the temperature at which they meet in H = 0 plane decreases. We have tabulated the range of T for different p in Table 1.

B. 0.022 < p ≤ 0.1078

In this region the wings meet at BEP as before, but the first order quadruple line now intersects the λ-line at a critical end point (we will call this critical end point as CEP1 to distinguish it from the other critical end point in the phase diagram at a lower temperature, which we will call as CEP2). After that it becomes a line of triple point (see Fig 1(b)). In Fig 10(c) we plot the free energy functional along this line. Along the first order line there is a line of four phase coexistence between BEP and CEP1 and then there is a usual triple line between CEP1 and CEP2. As shown in Fig 10(c), CEP1 itself is neither a quadruple or a triple point. It is instead a point where a critical state coexists with two ordinary stable phases. Between CEP2 and 0 temperature there is again a quadruple line as shown in Fig 10(e).

The CEP is a point where two phases become critical in the presence of one or more non critical spectator phase. At CEP, f(m) for m = 0 and for m ≠ 0 should be equal (i.e. f(m = 0) = f(m ≠ 0)) along with their derivative with respect to m (f'(m = 0) = f'(m ≠ 0)). If this point lies on the λ line, then we get the condition for CEP. Hence to find CEP, we explore the λ-line for a point where f(m = 0) = f(m ≠ 0) along with f'(m = 0) = f'(m ≠ 0). We find that for p > 0.107875 the condition cannot be satisfied.

In Table 2 we tabulate the location of BEP, CEP1 and CEP2 for different values of p. The CEP2 is an artifact of the fact that we are working on a complete graph and we do not expect it to be present in finite dimensions.

We plot the magnetic susceptibility as a function of Δ for T = T_{BEP} and for T = T_{CEP} in H = 0 plane. As expected first plot shows two peaks: a finite peak at BEP and an infinite peak at intersection with the λ-line (see Fig 11), while the second plot shows one peak only at CEP1 (see Fig. 12).
### TABLE I: Width of the wing lines for different $p$. $T_{lc}$ and $\Delta_{lc}$ represent the values of $T$ and $\Delta$ for $H = 0$ where the $\lambda_+$ and $\lambda_-$ lines meet and $T_{uc}$ is the value along the critical line as $\Delta \to \infty$ and $H \to \infty$.

| $p$   | $T_{lc}$ | $\Delta_{lc}$ | $T_{uc}$ | $\delta T$ |
|------|----------|---------------|----------|------------|
| 0.0453 | 0.28043  | 0.501175      | 0.23866  | 0.0417665  |
| 0.05  | 0.276396 | 0.50468       | 0.237473 | 0.038923   |
| 0.07  | 0.26185  | 0.518896      | 0.23245  | 0.029399   |
| 0.1   | 0.2451   | 0.538417      | 0.224972 | 0.020128   |
| 0.2   | 0.2058   | 0.596376      | 0.2058   | 0.0058     |
| 0.3   | 0.17643  | 0.6490843     | 0.174978 | 0.001452   |
| 0.4   | 0.15024  | 0.69968       | 0.1499   | 0.000248   |
| 0.5   | 0.125016 | 0.7499884     | 0.12498  | 0.000036   |

### TABLE II: Co-ordinates of the BEP and CEP’s for $0.022 < p < 0.107$.

| $p$   | $T_{BEP}$ | $\Delta_{BEP}$ | $T_{CEP1}$ | $\Delta_{CEP1}$ | $T_{CEP2}$ | $\Delta_{CEP2}$ |
|------|-----------|---------------|-----------|----------------|-----------|----------------|
| 0.03  | 0.2961208 | 0.489187      | 0.295197  | 0.489166       | 0.03      | 0.497729       |
| 0.044 | 0.28153  | 0.500195      | 0.27585   | 0.500186       | 0.0401    | 0.521585       |
| 0.07  | 0.26185  | 0.518896      | 0.24036   | 0.519398       | 0.07099   | 0.533953       |
| 0.107 | 0.24166  | 0.542         | 0.15972   | 0.547514       | 0.13975   | 0.549068       |

**FIG. 11: Magnetic susceptibility($\chi$) vs $\Delta$ plot for $p = 0.044$ at $T = T_{BEP}$**

**FIG. 12: Magnetic susceptibility($\chi$) vs $\Delta$ plot for $p = 0.044$ at $T = T_{CEP}$**

### III. WEAK DISORDER AND TCP

Along the region $0 \leq p \leq 0.022$ the wings meet the $\lambda$-line at the TCP and phase diagram is similar to the pure case. Along the first order line there is three phase coexistence. As $p$ increases, the TCP shifts towards smaller $T$ and larger $\Delta$. Again at very low temperature there is a CEP, similar to the case discussed in the Section II B for all $p > 0$, which is an artifact of working on a complete graph.

### IV. LANDAU THEORY

In the previous sections we studied the phase diagram by looking at the full free energy functional and its derivatives. In general there is no theory to get the higher order critical points via pure thermodynamic considerations. Usually Landau theory is a very useful tool to classify different kind of transitions and even though it might not be accurate quantitatively, it helps in understanding different possible topologies of the phase diagram. But while very successful in explaining ordinary critical point, it is not always possible to find a Landau description for
higher order critical point, i.e. it is perhaps possible to define a free functional always, but it might not always be Taylor expansible [39]. In this section we expand the free energy functional to check if we can explain the phase diagrams based on the coefficients of different powers of the order parameter. For example, the Ising universality class critical point can be determined easily by expanding up to fourth power in $m$, provided that the higher order coefficients are positive. For TCP one need to expand till sixth order. A sixth order Landau theory hence allows only for ordinary critical points and TCPs. Hence we expect that we need to keep more terms in the expansion, if we expect to find higher order critical points like CEP and BEP [40]. Hence we expanded the free energy functional till eighth power of $m$. We get

$$\tilde{f}(m) = a_2 m^2 + a_4 m^4 + a_6 m^6 + a_8 m^8$$  \hspace{1cm} (21)$$

where $a_i$'s are Landau coefficients, which come out to be:

$$a_2 = \frac{\beta}{2} \left( 1 + \frac{2\beta(p-1)}{2 + e^{2\beta \Delta}} - \frac{2\beta e^{\beta \Delta}}{1 + 2e^{\beta \Delta}} \right)$$

$$a_4 = \frac{\beta^4}{12} \left( \frac{(-4 + e^{\beta \Delta})(p-1)}{(2 + e^{\beta \Delta})^2} + \frac{pe^{\beta \Delta}(-1 + 4e^{\beta \Delta})}{(1 + 2e^{\beta \Delta})^2} \right)$$

$$a_6 = \frac{\beta^6}{360} \left( \frac{(64 - 26e^{3\beta \Delta} + e^{6\beta \Delta})(p-1)}{(2 + e^{\beta \Delta})^3} - \frac{pe^{3\beta \Delta}(1 - 26e^{2\beta \Delta} + 64e^{3\beta \Delta})}{(1 + 2e^{\beta \Delta})^3} \right)$$

$$a_8 = \frac{\beta^8}{20160} \left( \frac{(1188e^{3\beta \Delta} - 2176 - 120e^{3\beta \Delta} + e^{6\beta \Delta})(p-1)}{(2 + e^{\beta \Delta})^4} + \frac{pe^{3\beta \Delta}(-1 + 120e^{\beta \Delta} - 1188e^{2\beta \Delta} + 2176e^{3\beta \Delta})}{(1 + 2e^{\beta \Delta})^4} \right)$$  \hspace{1cm} (22)$$

The second order transition is given by $a_2 = 0$, provided $a_4 > 0$. Equating $a_2 = 0$ gives us:

$$1 + \frac{2\beta(p-1)}{2 + e^{\beta \Delta}} = \frac{2\beta e^{\beta \Delta}}{1 + 2e^{\beta \Delta}}$$  \hspace{1cm} (23)$$

This equation is same as Eq. (10) obtained by linear expansion around $m = 0$. According to the Landau theory, a new universality class, namely the TCP occurs when $a_4$ becomes equal to 0, provided $a_6 > 0$. We find that the condition for $a_4 = 0$ along the $\lambda$-line is the same as given by substituting Eq. (11) into Eq. (10). For $p > p_c = 0.0454$, $a_4$ is never 0 and hence beyond $p_c$, the condition for occurrence of TCP cannot be satisfied. For $p > 0.022$, $a_6 < 0$ at the point where $a_4 = 0$. Hence sixth order Landau theory while sufficient for $p < 0.022$, is not enough for $p > 0.022$.

Hence for $a_6 < 0$, we consider the expansion till eighth order, since $a_8 > 0$ for all ranges of the parameters. CEP will be a point along the $\lambda$-line (given by Eq. (23)) where the $\tilde{f}(T_c, m_c) = 0$ and $\tilde{f}'(T_c, m_c) = 0$ and $m_c \neq 0$. Solving these, we get the condition for the existence of CEP to be

$$\frac{a_2^2}{4a_4a_8} = 1$$  \hspace{1cm} (24)$$

We find that Eq. (24) can be satisfied only for $0.022 < p \leq 0.0454$, and that too at a point very close to the point where $a_4 = 0$. This is way off the actual position of CEP as determined in Section [11]. In fact in Section [11] we had found numerically that CEP is present for a much larger range of $p$: $0.022 < p \leq 0.1078$.

To estimate BEP using truncated $\tilde{f}(m)$, we equate the first three derivative of truncated $\tilde{f}(m)$ in Eq. (22) w.r.t $m$ to 0. For $m \neq 0$, this gives the condition for BEP to be: $a_6 = -\sqrt{\frac{8a_4a_8}{3}}$. Again this condition gets satisfied only for $0.022 < p \leq 0.0454$. This gives a BEP very close to CEP and the actual location does not match with the numerical estimates of Section [11]. We tried including more terms in the expansion of $\tilde{f}(m)$, but we could not locate BEP using a truncated $\tilde{f}(m)$, suggesting that full $\tilde{f}(m)$ is needed for locating the BEP.

V. DISCUSSION

Blume-Capel model is a very useful model due to its simplicity and rich phase diagram. While its phase diagram in the presence of disorder in $(T, \Delta)$ plane has been studied using many different techniques like renormalisation group techniques [21], field theories [20], Bethe lattice [22], replica [26] and numerical methods [11], the three field phase diagram has not been studied. In this paper, we found a BEP in the phase diagram of Blume-Capel model with bimodal random crystal field on a fully connected graph by looking at the three field $(T, \Delta, H)$ phase diagram. It would be interesting to see if a similar phase diagram is realized in finite dimensions using numerical simulations [11, 43]. The origin of BEP in this case is different from the usual pure anisotropic continuous spin systems, where the BEP was seen as a point of spin flop [28]. A two parameter Landau theory description exist for spin flop [32]. It would be interesting to see if a one parameter Landau theory can be built, which has a BEP as seen in the strong disorder case in our work. A study of three fields diagram for random field Blume Capel model would also be useful to understand the nature of TCPs reported [14] in the $(T, \Delta)$ phase diagram [15].

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