The $su(2)_{-1/2}$ WZW model and the $\beta\gamma$ system

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Abstract

The bosonic $\beta\gamma$ ghost system has long been used in formal constructions of conformal field theory. It has become important in its own right in the last few years, as a building block of field theory approaches to disordered systems, and as a simple representative – due in part to its underlying $su(2)_{-1/2}$ structure – of non-unitary conformal field theories. We provide in this paper the first complete, physical, analysis of this $\beta\gamma$ system, and uncover a number of striking features. We show in particular that the spectrum involves an infinite number of fields with arbitrarily large negative dimensions. These fields have their origin in a twisted sector of the theory, and have a direct relationship with spectrally flowed representations in the underlying $su(2)_{-1/2}$ theory. We discuss the spectral flow in the context of the operator algebra and fusion rules, and provide a re-interpretation of the modular invariant consistent with the spectrum.

1 Introduction

This is the first of two papers dedicated to the study of various facets of $c = -1$ conformal field theory (CFT). Our general goal is to explore in physical terms certain non-unitary CFTs of physical importance. Belonging to this family of theories are the various supergroup Wess-Zumino-Witten (WZW) and sigma models used in the description of phase transitions in disordered electronic materials [1, 2, 3, 4, 5, 6], the Liouville theory which describes the conformal mode of 2D gravity [7, 8], and the $sl(2, \mathbb{R})$ WZW model which plays a crucial role in the study of strings on $AdS_3$ [9].

A common manifestation of non-unitarity in a CFT is the presence of operators with a negative dimension. The simplest example of a non-unitary CFT is the minimal $M_{2,5}$ or Yang-Lee model, which differs little from minimal unitary CFTs. In particular, the theory is rational and has a spectrum of dimensions bounded from below. The models we are interested in have richer structures. They may involve logarithms [10, 11], for example, and/or exhibit a large, possibly infinite, set of operators with negative dimensions.

A great deal of effort has been made to characterize and classify abstractly such theories [12, 13]. It is fair to say, however, that very few explicit examples are well understood. The case of $c = -2$ has given rise to surprisingly complicated results (see [12, 13] for a review), while for potentially more interesting physical theories (such as sigma models on superprojective spaces), partial results reveal a truly baffling complexity [14].

Our goal in this paper is to discuss another example that shares many of the features of the more interesting models, but can still be studied in depth. It is the $\beta\gamma$ system, which plays a crucial role in the free-field representation of supergroup WZW models [3, 15], for example. The $\beta\gamma$ system is a deceptively
simple, “free” theory, with action

\[ S = \frac{1}{2\pi} \int d^2 z \left( \gamma_L \partial \beta_L - \beta_L \partial \gamma_L + (L \to R) \right) \]

and central charge \( c = -1 \). It bears a striking formal resemblance to a non-compactified complex boson.

It turns out, however, that this model has a lot of structure, including strong non-unitary problems due to the ill-defined nature of the functional integral (1). Note that the action (1) has an obvious \( Sp(2)_R \otimes Sp(2)_L \) global symmetry.

Here we will rather exploit the underlying \( su(2) \) symmetry discovered in [16]. The \( \beta \gamma \) system can actually be understood as a \( \hat{su}(2)_{-1/2} \) WZW model, as we show here. As is well-known, the action of a WZW model based on a compact Lie group is well-defined only when the level is integer. An option for bypassing this obstruction for non-integer level is to consider a non-unitary model as being defined purely algebraically, in terms of an affine Lie algebra at fractional level and its representation theory.

The cornerstone of this idea is an observation by Kac and Wakimoto [17] on \( \hat{su}(2)_k \) for fractional level \( k = t/u \) with \( t \in \mathbb{Z} \) and \( u \in \mathbb{N} \) co-prime, and \( t + 2u - 2 \geq 0 \). They found that there is a finite number of primary fields associated to highest-weight representations that transform linearly among themselves under the modular group. They are called admissible representations. An example is provided when \( k = -1/2 \) (in which case \( c = 3k/(k + 2) = -1 \)), inviting one to have the reasonable expectation that the \( \hat{su}(2)_{-1/2} \) model is a rational CFT.

Although the \( \beta \gamma \) ghosts and twists are described naturally in terms of the admissible representations, we show in this paper that neither the \( \beta \gamma \) system nor the \( \hat{su}(2)_{-1/2} \) WZW model is a rational CFT in the conventional sense. In brief, this is established as follows. Using a free-field representation of the \( \beta \gamma \) system, we can show that multiple fusions of twist fields with themselves can generate fields with arbitrarily large negative dimensions. These are interpreted physically in terms of deeper twists. Within the context of the WZW model, the presence of an unbounded spectrum is explained naturally in terms of the concept of spectral flow. A posteriori, it is then quite understandable that the non-rationality of the WZW model at fractional level reveals itself in the context of the hitherto puzzling issue of fusion rules. Proper interpretations of the known fusion rules and their limitations are also provided.

The paper is organized as follows. In Section 2, we review basic facts on the \( \beta \gamma \) system. We discuss thoroughly the twist fields, and show that their \( u(1) \) charge is a free parameter. Their conformal dimension, on the other hand, is determined completely by monodromy considerations, and is found to be \( h = -1/8 \). We also discuss the free-field representation of the \( \beta \gamma \) system, which is based on a \( c = -2 \) fermionic \( \mathfrak{q} \mathfrak{g} \) system, and a free boson, \( \phi \), with negative metric.

Section 3 collects some results on the \( \hat{su}(2)_{-1/2} \) model, such as a characterization of the admissible representations and a description of the associated primary fields. An alternative derivation of the \( \hat{su}(2)_{-1/2} \) spectrum using the vacuum null-vector is also presented. For later reference, we list the fusion rules computed by enforcing the decoupling of the singular vectors [18], as well as those obtained by the Verlinde formula.

In Section 4, we study the \( \hat{su}(2) \) structure of the \( \beta \gamma \) system, and show how the twist fields can be organized in representations of the base \( su(2) \) algebra. We show that a particular set of twist fields (which corresponds to choosing a particular normal ordering for the \( \beta \gamma \) system in the Ramond sector) fit into lowest- and highest-weight representations of spin \( j = -1/4 \) or \( j = -3/4 \). Thus, an infinite number of twists are re-organized in terms of infinite-dimensional representations of \( su(2) \). We argue that this gives rise to a free-field realization of the \( \hat{su}(2)_{-1/2} \) model.

In the subsequent section, we confirm this identification by comparing with the correlators of the \( \hat{su}(2)_{-1/2} \) model determined using the Knizhnik-Zamolodchikov (KZ) equation.

In the conceptually most important Section 6, we study the fusion rules in the \( \beta \gamma \) system, and hence, in the \( \hat{su}(2)_{-1/2} \) model. We show that fusions of the basic twists generate fields of arbitrarily large negative dimensions. This can be interpreted in terms of the deeper twists in the \( \beta \gamma \) system, or in

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1Another option to bypass this difficulty would be to look for a non-compact symmetry group, in which case the constraint \( k \in \mathbb{N} \) is no longer needed. Here, we would then consider a model with global \( SL(2, \mathbb{R}) \) invariance, whose spectrum generating algebra, when \( c = -1 \), is \( \hat{s}(2, \mathbb{R})_{1/2} \).
terms of the action of the spectral flow in the $\hat{su}(2)_{-1/2}$ WZW model. This is in sharp contrast to most expectations in the literature, where the existence of the modular invariant in $[17, 19]$ has been interpreted (at least implicitly) as a “proof” that the $\hat{su}(2)_{-1/2}$ model is a rational CFT. These observations have some overlap with the results obtained recently by Maldacena and Oguri $[3]$ on the $\hat{sl}(2, \mathbb{R})$ WZW model, and by Gaberdiel on the $\hat{su}(2)_{-4/3}$ WZW model $[20]$. In particular, the addition of extra fields under the spectral flow is reminiscent of the solution to the $\hat{sl}(2, \mathbb{R})$ WZW model $[9]$. However, the two models are different, and the appearance of continuous representations in $[9]$ is not replicated here. Finally, we show that the modular invariant partition function written in $[17]$ is compatible with the infinite operator content that we encounter, provided the characters are properly interpreted.

Section 7 contains some concluding remarks. The issue of logarithms alluded to above is the subject of our forthcoming paper $[21]$.

2 The $\beta\gamma$ system

2.1 Generalities

The bosonic $\beta\gamma$ system or $c = -1$ ghost theory is defined by the (first order) action $[1]$. Here we focus on the left-moving sector and omit the subscript $L$ for simplicity. The functional integral is clearly not well-defined. How formal manipulations of the model can give rise to a meaningful physical theory, is to a large extent the essence of this paper. At first sight, a formal treatment of the $\beta\gamma$ system produces very simple results. The elementary correlators, obtained through analytic continuation of the Gaussian integrals, read

$$\langle \beta(z)\gamma(w) \rangle = -\langle \gamma(z)\beta(w) \rangle = \frac{1}{z-w}$$

The stress-energy tensor is

$$T = \frac{1}{2} (\beta \partial \gamma : - \partial \beta \gamma : )$$

and leads to the central charge $c = -1$, while $\beta$ and $\gamma$ both have weight $h = \frac{1}{2}$. There is also an obvious $u(1)$ charge with current

$$J^3 = -\frac{1}{2} : \gamma \beta :$$

with respect to which $\beta$ has charge $1/2$ and $\gamma$ charge $-1/2$.

The ghost fields can be periodic (NS sector, $p = 0$) or anti-periodic (R sector, $p = 1$), and have the mode expansions:

$$\beta(z) = \sum_{n \in \mathbb{Z}} z^{-n-1-p/2} \beta_{n+(1+p)/2} \quad \gamma(z) = \sum_{n \in \mathbb{Z}} z^{-n-1+p/2} \gamma_{n+(1-p)/2}$$

The associated commutators read

$$[\beta_s, \gamma_t] = \delta_{s+t,0}$$

In the NS sector ($p = 0$) the ground state is defined by $\beta_r |\phi_0\rangle = \gamma_r |\phi_0\rangle = 0$ for $r > 0$ and $r \in \mathbb{N} + 1/2$. The normal-ordered Hamiltonian is then

$$L_0 = \sum_{n \geq 0} (n + 1/2) \left( \gamma_{-n-1/2} \beta_{n+1/2} - \beta_{-n-1/2} \gamma_{n+1/2} \right)$$

In the R sector ($p = 1$), the Hamiltonian reads

$$L_0 = -\frac{1}{8} + \sum_{n > 0} n (\gamma_{-n} \beta_n - \beta_{-n} \gamma_n)$$
Because of the existence of zero modes, \( \beta_0 \) and \( \gamma_0 \), different choices can be made for the ground state. These choices have deep implications in terms of the \( su(2) \) content of the theory. This will be addressed in the context of the spectral flow in Section 6. One of the simplest choices is to demand that

\[
\beta_{n+1} |\varphi_1\rangle = \gamma_n |\varphi_1\rangle = 0, \quad n \geq 0
\]

(9)
in which case \( \beta_0 \langle \varphi_1 | \beta_1 \) \neq 0. The vacuum state is infinitely degenerate since \( \beta_0^N |\varphi_1\rangle \) is a vacuum state for any \( N \). The states are naturally organized in terms of the parity of the number of acting \( \beta \) modes. This infinite number of states will thus split into two sequences of definite parity, and each sequence will eventually be associated to an infinite \( su(2) \) representation.

The \( u(1) \) charge in the R sector reads

\[
J_0^3 = -\frac{1}{4} (\beta_0 \gamma_0 + \gamma_0 \beta_0) = \frac{1}{4} - \frac{1}{2} \beta_0 \gamma_0
\]

(10)
The \( u(1) \) charge of the state \( |\varphi_1\rangle \) is \( J_0^3 = \frac{1}{4} \), while the \( u(1) \) charge of \( \beta_0^N |\varphi_1\rangle \) is \( J_0^3 = \frac{1}{4} + \frac{N}{2} \). One could as well exchange the roles of \( \beta \) and \( \gamma \), and define the ground state in the R sector through

\[
\gamma_{n+1} |\varphi_1\rangle = \beta_n |\varphi_1\rangle = 0, \quad n \geq 0
\]

(11)
In this case, \( |\varphi_1\rangle \) has charge \( J_0^3 = -\frac{1}{4} \), and the states \( \beta_0^N |\varphi_1\rangle \) charge \( J_0^3 = -\frac{1}{4} - \frac{N}{2} \). Finally, it is also possible that the ground state is annihilated neither by \( \beta_0 \) nor \( \gamma_0 \). This would lead to a tower of states with values of \( J_0^3 \) extending infinitely in both directions.

In the rest of this section, we choose the ground state of the NS and R sector to obey the highest-weight conditions

\[
\beta_{n+(1+p)/2} |\varphi_p\rangle = \gamma_{n-(1-p)/2} |\varphi_p\rangle = 0, \quad n \geq 0
\]

(12)
The character, \( \chi \), of the Verma module is obtained by counting all possible applications of the lowering operators \( \beta_{n+(1+p)/2} \) and \( \gamma_{n-(1-p)/2} \) with \( n < 0 \). Keeping track of the powers of \( \beta \) with a factor \( y \) and those of \( \gamma \) with a factor \( y^{-1} \), we thus have

\[
\chi_p(q,y) = q^{1/24} q^{-\frac{1}{2}(1-(1-p)^2)} \prod_{n=0}^{\infty} \frac{1}{1 + y q^{n+(1-p)/2}} \frac{1}{1 + y^{-1} q^{n+(1+p)/2}}
\]

(13)
In the expansion of the infinite product, the coefficient of \( q^0 \) (which can occur only for \( p = 1 \)) is \( \sum_{m \geq 0} y^m \). This reflects the infinite degeneracy of the vacuum.

Up to this point, the \( c = -1 \) theory defined with the above highest-weight conditions resembles the \( c = 2 \) theory of a complex boson \[16\]. This is not surprising. For instance, the formal partition function of the \( \beta \gamma \) system in the RR sector on the torus would read \( Z \propto 1/\det \Delta \), where \( \Delta \) is the Laplacian. This extends to the partition function when NS and R sectors are combined. Recall that the determinant of the Laplacian acting on functions with boundary conditions (the complex variable on the torus is denoted \( \zeta \) to avoid confusion)

\[
f(\zeta + \omega_1) = -e^{2\pi i \alpha} f(\zeta), \quad f(\zeta + \omega_2) = -e^{2\pi i \beta} f(\zeta), \quad \tau = \omega_2/\omega_1
\]

(14)
takes the form (cf. the character \[13\] with \( \alpha = p/2 \) and \( e^{2\pi i \beta} = y \)):

\[
\frac{1}{\det \Delta_{\alpha+1/2,\beta+1/2}} = q^{1/24} q^{-\alpha^2/2} \prod_{n=0}^{\infty} \frac{1}{1 + e^{2\pi i \beta} q^{n+1/2-\alpha}} \frac{1}{1 + e^{-2\pi i \beta} q^{n+1/2+\alpha}} \times (c.c.)
\]

(15)
The associated partition function reads (\( \beta \) is denoted \( z \) for future convenience)

\[
Z = \left( \frac{1}{\det \Delta_{1/2,z+1/2}} + \frac{1}{\det \Delta_{1/2,z}} + \frac{1}{\det \Delta_{0,z+1/2}} + \frac{1}{\det \Delta_{0,z}} \right)
\]

(16)
The “natural case” is \( z = 0 \), and corresponds to what one would like the partition function to be, based on formal functional integral calculations with the \( \beta \gamma \) action. This coincides, of course, with the partition function of a non-compactified complex boson. The infinite degeneracy of the vacuum in the operator formulation can then be interpreted as a manifestation of the zero mode of the Laplacian. Partition functions (characters) in different sectors can be matched, and it is tempting to argue that the \( \beta \gamma \) system and the non-compactified complex boson are essentially equivalent, the former being a sort of twisted version of the latter [16].

We shall see in the following that this is not true. In particular, the \( \beta \gamma \) system defined through the highest-weight conditions (12) is a complicated CFT where the divergences present in the functional integral give rise to a spectrum unbounded from below, and the partition function (16) will not be the final answer. The formal similarity with the non-compactified complex boson is indeed only formal and hides a subtle resummation of this spectrum.

To uncover the physical content of the \( \beta \gamma \) system requires a deeper understanding of the fields in the \( R \) sector, to which we now turn.

2.2 Twist fields and monodromy considerations

The \( R \) sector corresponds to anti-periodic boundary conditions for the \( \beta \) and \( \gamma \) fields on the plane. These boundary conditions are generated by twist fields, for which we demand the local monodromy conditions

\[
\beta(z)\tau(1) \propto (z - z_1)^{-1/2}, \quad \gamma(z)\tau(1) \propto (z - z_1)^{-1/2}
\]

The square root singularities multiply “excited” twist fields. Here and below we use the abbreviation \( \tau(i) = \tau(z_i) \).

Let us calculate the conformal weight of the twist fields. For this, we consider the ratio of correlators

\[
g_2(z, w) = \frac{\langle \beta(z)\gamma(w)\tau(2)\tau(1) \rangle}{\langle \tau(2)\tau(1) \rangle}
\]

in the limit where \( z_1 = 0 \) and \( z_2 = \infty \). It follows that

\[
g_2(z, w) = z^{-1/2}w^{-1/2} \frac{Az + (1 - A)w}{z - w}
\]

where the square root part is due to (17), while the meromorphic function follows from the constraint that \( g_2(z, w) \propto (z - w)^{-1} \) as \( z \to w \), cf. (2). \( A \) is unconstrained and is left as a free parameter. Its meaning is immediately elucidated when considering the \( u(1) \) current (4). Namely, from \( g_2(z, w) \) we find the OPE

\[
\frac{\langle \beta \gamma : (w)\tau(2)\tau(1) \rangle}{\langle \tau(2)\tau(1) \rangle} = \frac{A - 1/2}{w} + \ldots
\]

as \( w \to z_1 \), and thus, the \( u(1) \) charge of the twist field at \( z_1 \) is \( J_0^3 = -\frac{1}{2}(A - 1/2) \). It must be opposite to the \( u(1) \) charge of the conjugate twist field at \( z_2 \) in order for the correlator not to vanish. Thus, despite the notation used above, the two twist fields within the correlator may be different – the twist field is not unique.

The \( u(1) \) charge does not affect the conformal weight, in contrast to what would happen in the fermionic \( \eta \xi \) system, for example. To see this, we use the stress-energy tensor (3). Evaluating the leading singularity as \( w \to z_1 \) along the same lines as above, one finds

\[
\frac{\langle \partial\beta \gamma : (w)\tau(2)\tau(1) \rangle}{\langle \tau(2)\tau(1) \rangle} = \left( \frac{3}{8} \frac{A}{2} \right) \frac{1}{w^2} + \ldots
\]

and

\[
\frac{\langle \partial\gamma \beta : (w)\tau(2)\tau(1) \rangle}{\langle \tau(2)\tau(1) \rangle} = -\left( \frac{3}{8} \frac{(1 - A)}{2} \right) \frac{1}{w^2} + \ldots
\]
It follows that $h = -\frac{1}{3}$, independently of the value of $A$.

In theories of free Majorana and Dirac fermions, or of free bosons, monodromy conditions like (13) define the twist fields uniquely. That is not so in the $\beta\gamma$ system: although the conformal weight of $\tau$ is fixed uniquely by the local monodromy ($h = -\frac{1}{3}$), an ambiguity related to the $u(1)$ charge remains.

### 2.3 Free-field representation: $\beta\gamma$ twists vs $\eta\xi$ twists

To understand better the role of the $u(1)$ charge, it is convenient to introduce the free-field representation

$$\beta = e^{-i\phi}\eta, \quad \gamma = e^{i\phi}\partial\xi$$

where $\phi$ is a free boson with negative metric:

$$\langle \phi(z)\phi(w)\rangle = +\ln(z - w)$$

We use an implicit notation where vertex operators like $e^{i\phi}$ are normal ordered. $\eta$ and $\xi$ are fermions of weight $h = 1$ and $h = 0$, respectively, obeying

$$\langle \eta(z)\xi(w)\rangle = \langle \xi(z)\eta(w)\rangle = \frac{1}{z - w}$$

Since the exponentials in the free-field representation have weight $h = -\frac{1}{2}$, $\beta$ and $\gamma$ have weight $h = \frac{1}{2}$, as required. Recall that the $\eta\xi$ system has stress-energy tensor $T = :\partial\xi\eta:$; and central charge $c = -2$.

It is important to notice that the $\beta\gamma$ system has a single $u(1)$ charge, while this representation allows for two of them. The free-field representation of $J^3$ involves the $u(1)$ charge of the boson only, $J^3 = \frac{1}{2}i\partial\phi$, but there is also the $u(1)$ charge of the $\eta\xi$ system, $j = :\xi\eta:$.

The free-field representation makes it clear that to create a branch for $\beta$ and $\gamma$, we have a continuum of possibilities. Indeed, introduce a twist\footnote{The difference in the two OPEs stems from the fact that $\xi$ and $\eta$ have different dimensions. Twists are more easily defined in the symplectic fermion theory, using two fermionic currents.} for the fermionic fields:

$$\partial\xi(z)\tilde{\sigma}_\lambda(w) \propto (z - w)^{\lambda - 1}, \quad \eta(z)\tilde{\sigma}_\lambda(w) \propto (z - w)^{-\lambda}$$

or more precisely

$$\partial\xi(z)e^{2\pi i}\tilde{\sigma}_\lambda(w) = e^{2\pi i(\lambda - 1)}\partial\xi(z)\tilde{\sigma}_\lambda(w), \quad \eta(z)e^{2\pi i}\tilde{\sigma}_\lambda(w) = e^{-2\pi i\lambda}\eta(z)\tilde{\sigma}_\lambda(w)$$

Such twists have been studied before\cite{22,23}. Their conformal dimension and $\eta\xi$ charge (i.e., $j_0$ eigenvalue) are given by

$$h_{\tilde{\sigma}_\lambda} = -\frac{\lambda(1 - \lambda)}{2}, \quad q_{\tilde{\sigma}_\lambda} = \lambda$$

In particular, $\xi$ and $\eta$ have $\eta\xi$ charges $+1$ and $-1$, respectively. To complement this, we can introduce a “magnetic” charge operator, $e^{i\alpha(\phi - \tilde{\phi})}$, in the free-boson theory, and select $\alpha$ in order for $\beta$ and $\gamma$ to be anti-periodic when acting on $\tilde{\sigma}_\lambda e^{i\alpha(\phi - \tilde{\phi})}$. For instance,

$$\beta(z)e^{2\pi i} e^{i\alpha\phi(w)}\tilde{\sigma}_\lambda(w) = e^{2\pi i(\alpha - \lambda)}\beta(z) e^{i\alpha\phi(w)}\tilde{\sigma}_\lambda(w)$$

and we want the phase to be $e^{-i\pi}$, so that

$$2\alpha - 2\lambda = -1$$

One then has

$$\gamma(z)e^{2\pi i} e^{i\alpha\phi(w)}\tilde{\sigma}_\lambda(w) = e^{2\pi i(-\alpha + \lambda - 1)}\gamma(z) e^{i\alpha\phi(w)}\tilde{\sigma}_\lambda(w)$$
with a corresponding phase $e^{-i\tau}$ as well. In the following, $\alpha$ and $\lambda$ are always assumed to be related by (30). The expression for the full twist field follows:

$$\tilde{\tau}_\lambda = \tilde{\sigma}_\lambda e^{i(\lambda - \frac{1}{2})(\phi - \tilde{\phi})}$$ (32)

Its dimension is

$$h = -\lambda(1 - \lambda) - \frac{(1 - 2\lambda)^2}{8} = -\frac{1}{8}$$ (33)

independent of $\lambda$, as desired. As already mentioned, in the free-field representation the $u(1)$ current of the $\beta\gamma$ system reads $J^3 = \frac{i}{\pi} \partial \phi$. Thus, the charge of the twist field is $J^3_{\tilde{\tau}} = -\frac{1}{2} (\lambda - \frac{3}{2})$, and $\lambda$ coincides with the parameter $A$ introduced above. Conjugate twist fields are obtained by replacing $\lambda$ by $1 - \lambda$.

Let us now try to identify particular $c = -2$ twist fields. Given its dimension and charge, we can set $\tilde{\sigma}_0 = I$. Note that $\tilde{\sigma}_1$ also has dimension 0 but it has $\eta \xi$ charge $+1$, hence it cannot be equivalent to the identity field. But since it has the same dimension and charge as $\xi$, it is natural to set $\tilde{\sigma}_1 = \xi$. There are two other values of $\lambda$ that suggest simple identifications: $\tilde{\sigma}_{-1}$ and $\tilde{\sigma}_2$ both have dimension 1, and their charges are $-1$ and $+2$, respectively. It is thus natural to set $\tilde{\sigma}_{-1} = \eta$ and $\tilde{\sigma}_2 = \xi \partial \xi$. These identifications are supported by the following, slightly different OPEs, compatible with charge conservation:

$$\eta(z)\tilde{\sigma}_{-1}(w) \sim (z - w)^{-\lambda} \tilde{\sigma}_{-1}(w), \quad \partial \xi(z)\tilde{\sigma}_2(w) \sim (z - w)^{\lambda - 1} \tilde{\sigma}_{-1}(w)$$ (34)

These OPEs in turn (as well as the charge and dimension assignments) are compatible with the following more general identifications ($n > 0$):

$$\tilde{\sigma}_n = \xi \partial \xi \cdots \partial^{n-1} \xi \quad \text{with} \quad h_{\tilde{\sigma}_n} = \frac{n(n - 1)}{2}, \quad q_{\tilde{\sigma}_n} = n$$

$$\tilde{\sigma}_{-n} = \eta \partial \eta \cdots \partial^{n-1} \eta \quad \text{with} \quad h_{\tilde{\sigma}_{-n}} = \frac{n(n + 1)}{2}, \quad q_{\tilde{\sigma}_{-n}} = -n$$ (35)

It is now crucial to recall that there are different choices of $c = -2$ theories, which will lead to different realizations of “the” $\beta\gamma$ system. Here, we will choose to use not the $\eta \xi$ system but rather the $\eta \partial \xi$ system (though, for simplicity refer to it as the $\eta \xi$ system). In the terminology of [24, 23], we thus work in the small algebra, generated by the modes of $\eta$ and $\partial \xi$. We denote the corresponding twist fields by $\sigma_\lambda$. Instead of (35), for instance, we then have

$$\sigma_n = \partial \xi \cdots \partial^{n-1} \xi \quad \text{with} \quad h_{\sigma_n} = \frac{n(n - 1)}{2}, \quad q_{\sigma_n} = n - 1$$

$$\sigma_{-n} = \eta \partial \eta \cdots \partial^{n-1} \eta \quad \text{with} \quad h_{\sigma_{-n}} = \frac{n(n + 1)}{2}, \quad q_{\sigma_{-n}} = -n$$ (36)

Particular examples are $\sigma_1 = I$ and $\sigma_2 = \partial \xi$. These identifications are compatible with the OPEs (24) for $n \neq 0, 1$. For $n = 0, 1$, since $\sigma_0 = \sigma_1 = I$, the OPEs have to be changed by the addition of integer powers of $z - w$.

The definition of the twist fields in the $\beta\gamma$ system is now

$$\tau_\lambda \equiv \sigma_\lambda e^{i(\lambda - \frac{1}{2})(\phi - \tilde{\phi})}$$ (37)

($\tilde{\phi}$ is omitted in the following). With this definition, the OPEs of twists in the $\beta\gamma$ system are generically satisfied, except for some special values of $\lambda$. For instance, one has $\beta(z)\tau_0(w) \sim (z - w)^{-1/2}$ and $\gamma(z)\tau_0(w) \sim (z - w)^{1/2}$. Similarly, we have $\beta(z)\tau_1(w) \sim (z - w)^{1/2}$ and $\gamma(z)\tau_1(w) \sim (z - w)^{-1/2}$. Otherwise $\beta(z)\tau_{-1}(w) \sim (z - w)^{-1/2}$ and $\gamma(z)\tau_{-1}(w) \sim (z - w)^{1/2}$. The marginal cases can be obtained within our formalism as limits of the generic cases. For instance, the four-point function $(\beta\gamma\tau_1\tau_0)/(\tau_1\tau_0)$ goes as $(z/w)^{1/2}(z - w)^{-1}$, and corresponds to the particular case $A = 1$.

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Footnote: This choice is justified by the observation that the $\widehat{su}(2)_{-1/2}$ symmetry constructed in the $\beta\gamma$ system commutes with a BRST charge $Q_\eta = \tilde{\phi} \eta$ in the $\eta \xi \phi$ free-field representation. How logarithms may appear when going beyond this choice, will be the subject of our forthcoming paper [21].
2.4 Twist correlators

We now turn to the calculation of the four-point functions of twist fields. There are several ways to tackle this question. A possibility is to follow the lines of \([23,22]\), that is, to implement local as well as global monodromy constraints in the \(\beta\gamma\) system directly. The starting point would be the ratio of correlators

\[
g_4(u, w) = \frac{\langle \beta(u)\gamma(w)\tau(4)\tau(3)\tau(2)\tau(1) \rangle}{\langle \tau(4)\tau(3)\tau(2)\tau(1) \rangle}\tag{38}
\]

(the charge labels being introduced below). Using monodromy arguments based on \([17]\) and the result \([3]\), would, however, not lead to a complete determination of \(g_4\). There would, once again, remain ambiguities because of the \(u(1)\) charge of the twist fields. If we were to assume, for instance, that \(\tau(1)\) and \(\tau(3)\) on the one hand, and \(\tau(2)\) and \(\tau(4)\) on the other, had the same charge, it would follow that

\[
g_4(u, w) = \prod_{i}(u - z_i)^{-1/2}(w - z_i)^{-1/2}
\]

\[
\times \left[ \frac{A(u - z_1)(w - z_2)(w - z_4)(w - z_3)}{(u - w)} + \frac{B(u - z_2)(u - z_1)(w - z_4)(w - z_3)}{(u - w)} + C(z_i, \bar{z}_i) \right] \tag{39}
\]

with the only constraint \(A + B = 1\) (\(A, B\) are constants, while \(C\) is a constant with respect to \(z\) and \(w\) but depends on the other arguments). The choice of \(A\) (and hence of \(B\)) would correspond to a choice of \(u(1)\) charge for each of the twist fields, exactly as in the case of the two-point function.

In fact, to evaluate this correlator, it is easier to use the free-field representation. The bosonic part of the twist correlator is then evaluated straightforwardly using Wick’s theorem. As for the fermionic part, we can use the method of \([23,22]\). Let us indicate how. The starting point is

\[
g_4(u, w) = \frac{\langle \eta(u)\partial\xi(w)\sigma_{1+\rho}(4)\sigma_\nu(3)\sigma_{1+\mu}(2)\sigma_\lambda(1) \rangle}{\langle \sigma_{1+\rho}(4)\sigma_\nu(3)\sigma_{1+\mu}(2)\sigma_\lambda(1) \rangle}\tag{40}
\]

with \(\lambda + \mu + \nu + \rho = 0\). Local monodromy constraints lead to the introduction of the two forms

\[
\omega_1(u) = (u - z_1)^{-\lambda}(u - z_2)^{-1-\mu}(u - z_3)^{-\nu}(u - z_4)^{-1-\rho}
\]

\[
\omega_2(w) = (w - z_1)^{\lambda-1}(w - z_2)^{\mu}(w - z_3)^{\nu-1}(w - z_4)^{\rho}
\] \tag{41}

\(g_4(u, w)\) can then be written as the product \(\omega_1\omega_2\) times an analytic function of \(u, w, z_i\), which can be determined uniquely. Sending \(z \to w\), extracting the stress-energy tensor from the \(\eta\partial\xi\) OPE, and using the OPE of \(T\) with primary fields, leads (after setting \(z_1 = 0, z_2 = z, z_3 = 1\) and \(z_4 = \infty\)) to a differential equation in \(z\) for the object

\[
G(z, \bar{z}) = \lim_{z_\infty \to \infty} |z_\infty|^\lambda |\bar{z}_\infty|^\bar{\lambda}\sigma(z, \bar{z})\sigma_\nu(1, 1)\sigma_{1+\mu}(z, \bar{z})\sigma_\lambda(0, 0)\] \tag{42}

In the related free-boson problem, it is referred to as the quantum correlator. The second part of the calculation involves the function

\[
h_4(u, \bar{w}) = \frac{\langle \eta(u)\partial\xi(\bar{w})\sigma_{1+\rho}(4)\sigma_\nu(3)\sigma_{1+\mu}(2)\sigma_\lambda(1) \rangle}{\langle \sigma_{1+\rho}(4)\sigma_\nu(3)\sigma_{1+\mu}(2)\sigma_\lambda(1) \rangle}\tag{43}
\]

While we had the OPE \(\xi(u)\sigma_\lambda(w, \bar{w}) \propto (u - \bar{w})^{-\lambda}\), we now need \(\xi(\bar{u})\sigma_\lambda(w, \bar{w}) \propto (\bar{u} - \bar{w})^{-\lambda+1}\). It follows that \(h_4(u, \bar{w}) = \omega_1(u)\omega_1(\bar{w})D(z_i, \bar{z}_i)\), where \(D\) is a constant (with respect to \(z\) and \(w\)). The final step is to demand that \(\xi\) is uniquely defined when going around contours that encircle the four twist fields. This involves the integral of \(\omega_1\) along various cycles. For the benefit of our presentation, we focus on the ones that are related to the fundamental integral

\[
\int_0^1 dy y^{-\lambda}(1 - y)^{-1-\mu}(1 - zy)^{-\nu} = \frac{\Gamma(1 - \lambda)\Gamma(-\mu)}{\Gamma(1 - \lambda - \mu)} F(\nu, 1 - \lambda; 1 - \lambda - \mu; z) \tag{44}
\]
This integral formula is valid for $\text{Re}(1 - \lambda)$, $\text{Re}(-\mu) > 0$ and $|z| < 1$. Here $F$ denotes the hypergeometric function $_2F_1$. The final result is

$$G \propto |z|^{\mu(\lambda-1)}[1 - z]^{\nu(1+\mu)} [F(\nu, 1 - \lambda; 1 - \lambda - \mu; z)F(\nu, 1 - \lambda; 1 - \lambda - \mu; 1 - \bar{z}) + (c.c.)]$$

Going back to the physical correlator, one has

$$\langle \sigma_{+\rho}(4)\sigma_{-\nu}(3)\sigma_{+\mu}(2)\sigma_{-\lambda}(1) \rangle \propto |z_{12}|^{-h_1-h_2+h_3+h_4}|z_{13}|^{-h_1+h_2+h_3-h_4}|z_{24}|^{-h_1-h_2-h_3-h_4}$$

$$\times |z|^{8(1-\lambda-\mu-\nu)}[1 - z]^{8(1+\mu)} [F(\nu, 1 - \lambda; 1 - \lambda - \mu; z)F(\nu, 1 - \lambda; 1 - \lambda - \mu; 1 - \bar{z}) + (c.c.)]$$

with $z_{ij} = z_i - z_j$. Note again that we are interested in values of $\lambda, \mu, \nu, \rho$ for which the OPEs initially written may not hold exactly, and differ from the ones used in this derivation by integer powers of $z - w$. Our practical definition of the twist fields will be through the four-point function, which we will demand to always be given by (46). If it so happens that the hypergeometric function is then ill-defined, the pole may formally be factored out, and a new hypergeometric function, this time well-defined, is substituted. This simple procedure corresponds to supplementing (14) with an integral formula valid for other values of the parameters $\lambda, \mu, \nu, \rho$, and implementing it in the expression for the four-point function (46). See also the expansion discussed in Section 5.1.

The above correlator is compatible with all the identifications proposed previously, in particular with setting $\sigma_n = :\partial \xi \cdots \partial^{n-1} \xi :$ and $\sigma_{-n} = :\eta \partial \eta \cdots \partial^{n-1} \eta :$. To illustrate this, we consider

$$\sigma_3 = :\partial \xi \partial^2 \xi : \quad \sigma_{-2} = :\eta \partial \eta :$$

and compute the four-point function

$$G = \langle :\partial \xi \partial^2 \xi : (4) :\partial \xi \partial^2 \xi : (3) :\eta \partial \eta : (2) :\eta \partial \eta : (1) \times (c.c.) \rangle$$

We should then check that it coincides with the corresponding case of our general result:

$$G \propto |z_{13}z_{24}|^{-12}|z(1 - z)|^{-12} [F(3, -2; 1; z)F(3, -2; 1; 1 - \bar{z}) + (c.c.)]$$

Since $F(3, -2; 1; z) = 1 - 6z + 6z^2$, we end up with

$$G \propto |z_{13}z_{24}|^{-12}|z(1 - z)|^{-12}(1 - 6z + 6z^2)(1 - 6\bar{z} + 6\bar{z}^2)$$

Finally, we can go back to the free-field representation of the $\beta \gamma$ system and obtain

$$\langle \tau_{+\rho}(4)\tau_{-\nu}(3)\tau_{+\mu}(2)\tau_{-\lambda}(1) \rangle \propto |z_{13}z_{24}|^{1/2}|z(1 - z)|^{1/2}$$

$$\times |z|^{-\lambda-\nu}[1 - z]^{-\lambda-\mu} [F(1 - \lambda, \lambda; 1; z)F(1 - \lambda, \lambda; 1; 1 - \bar{z}) + (c.c.)]$$

A particularly simple case is

$$\langle \tau_{-\lambda}(4)\tau_{-\lambda}(3)\tau_{-\lambda}(2)\tau_{-\lambda}(1) \rangle \propto |z_{13}z_{24}|^{1/2}|z(1 - z)|^{1/2} [F(1 - \lambda, \lambda; 1; z)F(1 - \lambda, \lambda; 1; 1 - \bar{z}) + (c.c.)]$$

This computation demonstrates the equivalence of the twist-field correlators based on the monodromy definition, and the explicit representation in terms of fermions obtained in the previous subsection.

### 3 The $\widehat{su}(2)_{-1/2}$ WZW model

In this section we review some aspects of the $\widehat{su}(2)_{-1/2}$ WZW model. Its relation to the $\beta \gamma$ system is discussed in the subsequent sections.
3.1 Admissible representations of the $\hat{su}(2)_{-1/2}$ model

A fractional level $k$, for the $\hat{su}(2)_k$ algebra, is said to be admissible if $k = t/u$, with $t \in \mathbb{Z}$ and $u \in \mathbb{N}$ relative prime, and $t + 2u - 2 \geq 0$ [17] (see also [26], Section 18.6). The admissible $\hat{su}(2)_k$ representations are then characterized by those spins $j$ which can be parameterized by two non-negative integers $r$ and $s$ as

$$2j + 1 = r - (k + 2)s, \quad 1 \leq r \leq t + 2u - 1, \quad 0 \leq s \leq u - 1 \quad (53)$$

In the present case $t + 2u - 1 = u = 2$, so that $r = 1, 2$ and $s = 0, 1$. Thus, there are four admissible representations, with $(r, s) = (1, 0), (2, 0), (1, 1), (2, 1)$. With this ordering, they correspond to

$$j = 0, \frac{1}{2}, \frac{3}{4}, \frac{1}{4} \quad (54)$$

Their highest-weight states have conformal dimension $h_j$ given by

$$h_j = \frac{j(j + 1)}{k + 2} \quad (55)$$

The admissible conformal dimensions are thus

$$h_0 = 0, h_{1/2} = \frac{1}{2}, h_{-3/4} = h_{-1/4} = -\frac{1}{8} \quad (56)$$

The characters of the admissible representations are given by

$$\chi_j^\epsilon(q, y) = \text{Tr}_{D_j^\epsilon} q^{L_0 - c/24} y^{J_3} \quad (57)$$

$D_j^\epsilon$ denotes the representation of spin $j$ with $\epsilon = \pm$ for highest or lowest weight, while

$$q = e^{2\pi i \tau}, \quad y = e^{2\pi iz} \quad (58)$$

This trace can be summed and for the highest-weight representations of Kac and Wakimoto, we get [17] (see e.g. [26], Eq. 18.185)

$$\chi_j^\epsilon(\tau, z) = \frac{\Theta_{b^\epsilon}^{(d)}(z/u; \tau) - \Theta_{b^\epsilon}^{(d)}(z/u; \tau)}{\Theta_{-1}^{(d)}(z; \tau) - \Theta_{2}^{(d)}(z; \tau)} \quad (59)$$

Here we have used the notation

$$\Theta_{b^\epsilon}^{(d)} = \sum_{l \in \mathbb{Z} + b^\epsilon} q^{dl^2} y^{-dl} \quad (60)$$

with $k = -1/2 = t/u$, $u = 2$, $d = u^2(k + 2) = 6$, and

$$b^\pm = u[\pm(r) - (k + 2)s] \quad (61)$$

For the identity, for example, we have $b^\pm = \pm 2$, while for the field with $j = -1/4$, we have $b_+ = 1$ and $b_- = -7$.

The $\hat{su}(2)_{-1/2}$ admissible characters turn out to have simple expressions in terms of the usual theta functions:

$$\chi_0(\tau, z) = \frac{1}{2} \left[ \frac{\eta}{\vartheta_4(\tau, z)} + \frac{\eta}{\vartheta_3(\tau, z)} \right]$$

$$\chi_{1/2}(\tau, z) = \frac{1}{2} \left[ \frac{\eta}{\vartheta_4(\tau, z)} - \frac{\eta}{\vartheta_3(\tau, z)} \right]$$

$$\chi_{-1/4}(\tau, z) = \frac{1}{2} \left[ \frac{\eta}{i \vartheta_1(\tau, z)} + \frac{\eta}{\vartheta_2(\tau, z)} \right]$$

$$\chi_{-3/4}(\tau, z) = \frac{1}{2} \left[ \frac{\eta}{i \vartheta_1(\tau, z)} - \frac{\eta}{\vartheta_2(\tau, z)} \right]$$

(62)
These admissible character functions close under the modular group \([17]\). For this reason, the \(\hat{su}(2)_{-1/2}\) WZW model is often said to be a rational CFT (and likewise for more general admissible level \(\hat{su}(2)_{k}\) theories). This statement will be re-evaluated in Section 6 when we are more careful about the convergence regions of the traces \([53]\) when summed to the character functions \([59]\) and \([62]\).

The associated diagonal modular invariant \([4]\) is given by \([3]\)

\[
Z(\tau, z) = |\chi_0(\tau, z)|^2 + |\chi^+_{1/4}(\tau, z)|^2 + |\chi_{1/2}(\tau, z)|^2 + |\chi^+_{3/4}(\tau, z)|^2 = \sum_{i=1}^4 |\eta_i(\tau, z)|^2 \quad (63)
\]

This can be shown to coincide with \([16]\).

We stress that in the present work, when we refer to highest- or lowest-weight representations, the qualitative “highest” or “lowest” refers to the grade-zero \(su(2)\) representation: a highest-weight state is annihilated by \(J_0^+\) while a lowest-weight state is annihilated by \(J_0^-\). The superscript, + or −, on the character indicates that the representation is a highest-weight or lowest-weight representation, respectively. Note that for \(j = 0, 1/2\), the representation is both a highest- and a lowest-weight representation. On the other hand, both types of representations are affine highest-weight representations, that is, they are annihilated by the action of \(J_n^+\), \(J_n^−\) and \(J_0^\pm\) for \(n > 0\). The functional form above is not a property which relies upon choosing the representations to be \(su(2)\) highest- or lowest-weight representations. For example, \(\chi^+_{3/4} = -\chi_{-1/4}\) and \(\chi^+_{1/4} = -\chi_{-3/4}\) (see below). In other words, the admissible representations could be regarded either as highest- or lowest-weight representations.

### 3.2 Generating the \(\hat{su}(2)_{-1/2}\) spectrum from the vacuum singular vector

Another way of generating the spectrum of the model is to use the constraints induced by the presence of the non-trivial vacuum singular vector, cf. \([20]\). The \(\hat{su}(2)_{-1/2}\) vacuum representation has a null state at level 4. It can be written explicitly as

\[
\left( J^3_{-4} - \frac{14}{127} J^1_{-3} J^1_{+1} + \frac{184}{127} J^3_{-3} J^3_{1} - \frac{62}{127} J^5_{-3} J^5_{1} + \frac{27}{127} J^5_{-2} J^5_{2} \right.
\]

\[
- \frac{38}{127} J^3_{-2} J^3_{2} + \frac{100}{127} J^3_{-1} J^3_{1} + \frac{64}{127} J^5_{-2} J^5_{1} - \frac{16}{127} J^3_{-1} J^3_{1} J^3_{1} J^3_{1} - \frac{32}{127} J^3_{-1} J^3_{1} J^3_{1} J^3_{1} \left) |0\rangle \quad (64)
\]

\[^4\text{This is not the sole invariant we can write, however. Given that charge conjugation (defined from the square of the modular S matrix) relates } j \text{ to } -j - 1, \text{ we also have the charge conjugate version of the diagonal invariant:}
\]

\[
Z^C = |\chi_0|^2 + |\chi_{1/2}|^2 + |\chi^+_{1/4}|^2 + |\chi^+_{3/4}|^2
\]

\[^5\text{If the variable } z \text{ transforms as } z \tau/(a\tau + b) \text{ as } \tau \text{ is changed to } (a\tau + b)/(c\tau + d), \text{ there is no prefactor in this partition function. If, on the other hand, } z \text{ is changed to } z/(c\tau + d), \text{ there is the prefactor } \exp(2c\Im(z)^2/\Im(\tau)), \text{ cf. appendix B of } [4].
\]

\[^6\text{In } [4], \text{ this technique is presented as a simplified implementation of the Zhu’s algebra. It can be traced back to early works in CFT. For instance, in } [27], \text{ the field associated to the non-trivial vacuum singular vector is called the model’s equation of motion, and the way the full spectrum can be extracted from it is illustrated for the minimal model describing the Yang-Lee singularity. This approach is also used extensively in the construction of new } W \text{ algebras in } [28], \text{ for example.}
\]

\[^7\text{The singular vectors for the admissible representations were obtained long ago by Malikov, Feigin and Fuchs } [52], \text{ and are referred to as MFF singular vectors. Their theorem 3.2 } [13], \text{ expresses a singular vector as a monomial involving fractional powers: for the vacuum representation of the } k = -1/2 \text{ model, the MFF singular vector reads}
\]

\[
(J^{3}_{+1})^{3/2}(J^{0}_{-})^{0}(J^{+1}_{-1})^{1/2}|0\rangle
\]

Using the commutation relations, one can re-express this result as a sum of terms where each generator has an integer power. The point here is the following. The vector \([40]\) and the above MFF singular vector both have \(L_0\) eigenvalue 4. However, the MFF singular vector has \(J^3_0\) eigenvalue 2, while \([40]\) has \(J^3_0\) eigenvalue 0. Thus, \(J^3_0\) must be applied twice to get the latter from the former.
The zero mode of this null vector provides a constraint that fixes the allowed representations of the theory. We get the following conditions

\[(3 + 16C) [3(J_0^3)^2 - C] = 0\]  

(65)

where the Casimir, \(C\), is given by

\[C = \frac{1}{2}(J_0^- J_0^+ - J_0^+ J_0^-) + J_0^3 J_0^3\]  

(66)

Let us look at these conditions on the highest-weight state of the highest-weight representation of spin \(j\), denoted \(D_j^+\). We will write the state associated to the field in the admissible representation of spin \(j\) as \(|j, m\rangle_n\), where \(m\) and \(n\) are the \(J_0^3\) and \(L_0\) eigenvalues, respectively:

\[
\begin{align*}
C|j, m\rangle_n &= j(j + 1)|j, m\rangle_n \\
J_0^3|j, m\rangle_n &= m|j, m\rangle_n \\
L_0|j, m\rangle_n &= (h_j + n)|j, m\rangle_n
\end{align*}
\]  

(67)

The highest-weight state of \(D_j^+\) satisfies \(J_0^+|j, j\rangle_0 = 0\).

From the condition (63), we find the following set of highest-weight representations \(D_j^+\):

\[
\begin{align*}
j &= 0, & C &= 0 \\
j &= 1/2, & C &= 3/4 \\
j &= -1/4, & C &= -3/16 \\
j &= -3/4, & C &= -3/16
\end{align*}
\]  

(68)

These are in one-to-one correspondence with the admissible representations for the \(\widehat{su}(2)_{-1/2}\) model.

Also, to each highest-weight representation, there corresponds a lowest-weight representation, with the same Casimir eigenvalue. The lowest-weight states are \(|j, -j\rangle_0\), and the corresponding representations are denoted by \(D_j^-\). Note that the states in \(D_j^+\) are different from those in \(D_j^-\) unless \(j\) is integer or half-integer (in which case the corresponding \(su(2)\) representation – the grade-zero content of the affine representation – is finite-dimensional). For \(j = 1/2\) (or the trivial \(j = 0\)) the representation is both a lowest- and a highest-weight representation. The two notions can then be identified.

As pointed out in [2], in a related context, another infinite-dimensional representation is allowed. It is neither a highest- nor a lowest-weight representation, and for that reason it cannot be assigned a \(j\) value. Nevertheless, the representation does have a well-defined Casimir eigenvalue, \(C = -3/16\), and is defined by

\[
\begin{align*}
J_0^3|m\rangle_0 &= |m + 1\rangle_0 \\
J_0^+|m\rangle_0 &= |m + 1\rangle_0 \\
J_0^-|m\rangle_0 &= (-3/16 - m(m - 1))|m - 1\rangle_0
\end{align*}
\]  

(69)

It is denoted \(E\). In fact, this representation extends to a continuous set of representations by setting \(m \in \mathbb{Z} + t\) with \(t \in [0, 1]\), excluding the values of \(t\) used for the admissible representations obtained above (i.e., \(t \neq -1/4, -3/4\)). These are denoted \(E_t\).

In this approach to the determination of the spectrum, it appears that both highest- and lowest-weight representations are present. Again, the distinction is meaningful when \(j \notin \mathbb{N}/2\), i.e., when \(s \neq 0\) with \(s\) defined in (63). Two such representations always appear pairwise \((j, -1 - j)\), and they have the same conformal dimension. Moreover, the associated character function \(\chi_j^+\), viewed as a highest-weight representation (hence the superscript +), is related to the conjugate character function \(\chi_{-1-j}^-\), viewed as a lowest-weight representations [30, 31]:

\[
\chi_j^+(\tau, z) = -\chi_{-1-j}^-(\tau, -z) \quad (s \neq 0)
\]  

(70)
One observes that all fusions combine a highest- with a lowest-weight representation. The rules obtained by decoupling the singular vectors \[ \text{(18)} \] are invariant \[ \text{(63)} \], it is not difficult to realize that it could just as well be written either as

\begin{equation}
\phi^+_j (w,x) = \sum_{m \in (j + \mathbb{Z}_\leq)} \phi^{(m)}_j (w) x^{j + m}
\end{equation}

while the field associated to a lowest-weight representation reads

\begin{equation}
\phi^-_j (w,x) = \sum_{m \in (-j + \mathbb{Z}_\geq)} \phi^{(m)}_j (w) x^{j + m}
\end{equation}

Having this set of highest- and lowest-weight representations, the question arises: which representations are actually present in the Kac-Wakimoto diagonal modular invariant? When revisiting the diagonal set of results are rather different. The latter fusion rules were recovered in \[ \text{(35)} \] using cohomology theory. On the other hand, fusion rules can also be computed by enforcing the decoupling of singular vectors \[ \text{(18)} \]. The two methods of BRST cohomology \[ \text{(33)} \] and the vertex-operator \[ \text{(34)} \] methods yield identical results. In the context of non-unitary WZW models, there are contradicting results concerning fusion rules. The latter fusion rules were recovered in \[ \text{(35)} \] using cohomology theory of infinite-dimensional Lie algebras, and in \[ \text{(32)} \] using a Coulomb gas method based on the Wakimoto free-field realization to compute correlators. Notice also that the Verlinde formula incorporates the results of \[ \text{(32)} \] but in those cases where the latter are ill-defined, it yields negative signs \[ \text{(33)} \].

As a preparation for our discussions in subsequent sections, we here review both sets of results for \( k = -1/2 \). In both cases, we omit discussing the trivial fusions with the identity field. First, the fusion rules obtained by decoupling the singular vectors \[ \text{(18)} \] are

\begin{align}
D^+_{-3/4} \times D^-_{-3/4} &= D_0 \\
D^+_{-1/4} \times D^-_{-1/4} &= D_0 \\
D^\pm_{-3/4} \times D^\mp_{1/4} &= D_{1/2} \\
D^\pm_{-3/4} \times D_{1/2} &= D^\pm_{-1/4} \\
D^\pm_{-1/4} \times D_{1/2} &= D^\pm_{-3/4} \\
D_{1/2} \times D_{1/2} &= D_0
\end{align}

(75)

One observes that all fusions combine a highest- with a lowest-weight representation: \( D^+ \times D^- \) or \( D^- \times D^+ \). Recall that \( D_0 \) and \( D_{1/2} \) are both highest- and lowest-weight representations.
Let us explain the reason for which we insist on interpreting these fusion rules in terms of fields with specified representations. The derivation presented in [13] considers the fusion of three fields \( \psi_j(z, x) \). Two of the associated representations (highest, lowest or even continuous) are initially not specified, while one field carries a highest-weight representation (or equivalently a lowest-weight representation). Assuming it is the field \( \psi_{j_1}(z_1, x_1) \), one then considers the decoupling of the singular vector of the highest-weight representation \( j_1 \) from the three-point function

\[
\langle j_1 | \psi_{j_2}(z_2, x_2) | j_1 \rangle \quad (76)
\]

Applying the null vector to this correlator then provides a set of products depending on \( j_1, j_2, j_3 \) which must be set to zero in order to have decoupling. This leads to a set of conditions on the three spins, thus providing the fusion rules. At first sight, it then seems that only the representation of the field \( \psi_{j_1}(z_1, x_1) \) has been specified. However that is not true. From projective and global SU(2) invariance, one obtains the well-known form of the generating-function three-point correlator:

\[
\langle \psi_{j_3}(z_1, x_3) \psi_{j_2}(z_2, x_2) \psi_{j_1}(z_1, x_1) \rangle \propto \frac{(z_2 - x_1)^{2j_2+j_1-j_3}(z_3 - x_2)^{2j_3+j_2-j_1}(x_3 - x_1)^{2j_3+j_1-j_2}}{(z_2 - z_1)^{h_2+h_1-h_3}(z_3 - z_2)^{h_3+h_2-h_1}(z_3 - z_1)^{h_3+h_1-h_2}} \quad (77)
\]

One observes that for the triplets of spins appearing in the fusion rules (77), at most one of the combinations \( j_2 + j_1 - j_3 \), \( j_3 + j_2 - j_1 \) and \( j_3 + j_1 - j_2 \) is not a non-negative integer. This means that at most one of the monomials \((z_2 - x_1)^{2j_2+j_1-j_3}, \) say) requires an infinite expansion. As a result, one of the involved fields will correspond to an infinite-dimensional highest-weight representation, while the other will correspond to an infinite-dimensional lowest-weight representation. This is reflected in the fusion rules. At most two of the representations are infinite-dimensional, and in the cases where they appear on each side of the fusion identity, they are both highest or lowest weight. That is due to the fact that three-point functions correspond to couplings of three representations to the singlet. Thus, extracting fusion rules from three-point functions requires considering the conjugate representation to one of the fields, interchanging a highest- with a lowest-weight representations.

In a rational CFT, the fusion algebra of the finitely many primary fields must close. However, the decoupling method does not predict the outcome of the fusion \( D^+_{-1/4} \times D^+_{-1/4} \), for example. One may also worry about the associated modular invariant. In that vein, it is noted that the fusion rules (74) seem to superficially contradict the various forms of the modular invariant written before, none of which, at first sight, contains all the fields (associated to \( D_0, D_{1/2}, D^+_{-1/4} \) and \( D^+_{-3/4} \)) at the same time.

Note finally that the fusions (73) are invariant under the action of the \( \hat{su}(2) \) outer automorphism \( a \) which acts on the spin labels as \( a : j \to a(j) = k/2 - j \). The invariance property of the fusion rules is

\[
D_a(j_1) \times D_{a'}(j_2) = \sum_{j_3} D_{a a'}(j_3) \quad (78)
\]

with the +/- specification omitted. A more refined version of this symmetry relation will be considered later.

Another set of fusion rules are computed from a direct application of the Verlinde formula. In the case of the diagonal modular invariant containing all highest-weight representations, the rules are (cf. [17]):

\[
\begin{align*}
D^+_{-3/4} \times D^+_{-3/4} & = -D_{1/2} \\
D^+_{-1/4} \times D^+_{-1/4} & = -D_{1/2} \\
D^+_{-3/4} \times D^+_{-1/4} & = -D_0 \\
D^+_{-3/4} \times D_{1/2} & = D^+ \\
D^+_{-1/4} \times D_{1/2} & = D^+ \\
D_{1/2} \times D_{1/2} & = D_0
\end{align*}
\quad (79)
\]
(here the representations $D_0$ and $D_{1/2}$ are viewed as highest-weight representations). The Bernard-Felder rules \cite{30,31} correspond to considering only the last three fusions.

The two sets of fusion rules are obviously different, though they do have common features. In particular, all fusions with the identity (not written) as well as the last three fusions in each set, are identical (when restricting to highest-weight representations). However, the first three are rather different. These are precisely the ones that involve a negative sign in the Verlinde case. An immediate consequence is that the Verlinde fusions are not invariant under (78) (with all highest weights). However, the relation (70) suggests a simple way to re-conciliate these different results \cite{30,31}, and to interpret the negative signs, e.g.,

$$D^{+}_{-3/4} \times D^{+}_{-3/4} = -D_{1/2} \quad \rightarrow \quad D^{+}_{-3/4} \times (-D^{-}_{-1/4}) = -D_{1/2}$$

Although appealing at first sight, we stress that this prescription cannot make the two sets of fusion rules identical in the general case (that is, when $u \geq 3$, where $u$ is the denominator of $k$) since the decoupling method yields more terms than predicted by the Verlinde formula. Roughly, the fusion rules in both methods split into separated fusions in the integral and fractional sector, that is, in terms of the $r$ and $s$ labels \cite{33}. The decoupling fusions are isomorphic to those of $\widehat{su}(2)_{u(k+2)-2} \otimes \widehat{osp}(1,2)_{u-1}$, while the fusions obtained by the Verlinde formula are of the type $\widehat{su}(2)_{u(k+2)-2} \otimes \widehat{u}(1)_{u-1}$ (see, e.g., \cite{38}).

The apparent contradiction between the two sets of fusion rules is a further signal that the CFT interpretation of the WZW model with only four primary fields may be wrong. In order to address in a concrete way the issues raised here, we first describe the connection between the $\widehat{su}(2)_{-1/2}$ WZW model and the $\beta\gamma$ system. That will provide us with a free-field representation of the WZW model. It is established in the next section, and confirmed in Section 5 at the level of correlation functions. Thus armed, we revisit in Section 6 the various puzzling issues raised in this section.

4 The $\widehat{su}(2)_{-1/2}$ model vs the $\beta\gamma$ system

4.1 The $\beta\gamma$ representation of the current algebra

As pointed out in \cite{10}, the $\widehat{su}(2)$ currents live in the universal covering of the $\beta\gamma$ algebra. More precisely, each current can be represented as a bilinear in these ghosts as

$$J^+ = \frac{1}{2}: \beta^2 :$$

$$J^- = \frac{1}{2}: \gamma^2 :$$

$$J^3 = \frac{1}{2}: \gamma \beta :$$

The OPEs read

$$J^+(z)J^-(w) \sim -\frac{1}{2} \frac{\beta(w)}{(z-w)^2} + \frac{2J^3(w)}{(z-w)}$$

$$J^3(z)J^\pm(w) \sim \pm J^\pm(w) \frac{(z-w)}{(z-w)}$$

$$J^3(z)J^3(w) \sim -\frac{1}{4} \frac{(z-w)^2}{(z-w)^2}$$

Having found the representation of the currents, the next step is to understand the structure of the WZW primary fields in terms of the $\beta\gamma$ system. It is rather easy to verify that the $(\beta, \gamma)$ pair forms a spin-1/2 multiplet:

$$J^3(z)\beta(w) \sim \frac{1}{2} \frac{\beta(w)}{(z-w)}, \quad J^+(z)\beta(w) \sim 0$$
In terms of the twist fields and representations of \( \hat{\mathfrak{su}}(2) \), we must be able to describe the infinite number of states at grade zero in the \( \beta\gamma \) system. We then need to understand how the twist fields are organized to form irreducible representations. In particular, we must be able to describe the infinite number of states at grade zero in the \( j = -1/4 \) and \( j = -3/4 \) admissible representations. To proceed further along these lines, we turn to the \( \eta\xi\phi \) free-field representation.

4.2 Twist fields and representations of \( \hat{\mathfrak{su}}(2)_{-1/2} \)

In terms of the \( \eta\xi\phi \) fields, the \( \hat{\mathfrak{su}}(2)_{-1/2} \) currents are represented by

\[
\begin{align*}
J^+ &= \frac{1}{2} e^{-2i\phi} : \partial \eta \eta : \\
J^- &= \frac{1}{2} e^{2i\phi} : \partial^2 \xi \xi : \\
J^3 &= \frac{1}{2} i \partial \phi
\end{align*}
\]

We now study their action on the twist fields, noting that

\[
J^3 \tau_\lambda = \frac{(2\lambda - 1)}{4} \tau_\lambda
\]

Let us first identify those twist fields that correspond to highest- and lowest-weight states (characterized by the vanishing of the single pole in the OPE with \( J^\pm \), respectively). There are two highest weights (hw) and two lowest weights (lw):

\[
\begin{align*}
J^3_0 \tau_0 &= 0, & J^3_0 \tau_0 &= \frac{1}{4} \tau_0 & \Rightarrow \text{lw} : j = -\frac{1}{4}, m = \frac{1}{4} \\
J^3_0 \tau_{-1} &= 0, & J^3_0 \tau_{-1} &= \frac{3}{4} \tau_{-1} & \Rightarrow \text{lw} : j = -\frac{3}{4}, m = -\frac{3}{4} \\
J^3_0 \tau_1 &= 0, & J^3_0 \tau_1 &= -\frac{1}{4} \tau_1 & \Rightarrow \text{hw} : j = -\frac{1}{4}, m = -\frac{1}{4} \\
J^3_0 \tau_2 &= 0, & J^3_0 \tau_2 &= -\frac{3}{4} \tau_2 & \Rightarrow \text{hw} : j = -\frac{3}{4}, m = -\frac{3}{4}
\end{align*}
\]

More generally, the twist fields with \( \lambda \in \mathbb{Z} \) can be organized to form the admissible representations:

\[
\begin{align*}
\tau_{-2n}, & \quad n \in \mathbb{Z}_\geq : \quad D^-_{-1/4} \\
\tau_{-2n-1}, & \quad n \in \mathbb{Z}_\geq : \quad D^-_{-3/4} \\
\tau_{2n+1}, & \quad n \in \mathbb{Z}_\geq : \quad D^+_{-1/4} \\
\tau_{2n+2}, & \quad n \in \mathbb{Z}_\geq : \quad D^+_{-3/4}
\end{align*}
\]

These twist fields are all expressed in terms of the free boson and differential monomials in \( \eta \) or \( \xi \). Moreover, the above result indicates that generic twist fields are in representations which are neither highest- nor lowest-weight representations! Indeed, for \( \lambda \notin \mathbb{Z} \), the twist fields \( \tau_{\lambda+2n} \ (n \in \mathbb{Z}) \) form infinite-dimensional representations which are neither highest- nor lowest-weight representations.

We consider the \( \beta\gamma \) free-field representation of the \( \hat{\mathfrak{su}}(2)_{-1/2} \) model further by examining four-point functions in the next section.
5 The KZ equation and correlators

In this section we want to construct the four-point functions for the \( \widehat{su}(2)_{-1/2} \) model using the KZ equation. Let us introduce the differential operator realization of the \( su(2) \) generators

\[
J_0^+ = -x^2 \partial_x + 2jx, \quad J_0^- = \partial_x, \quad J_0^3 = x\partial_x - j
\]  

(88)

The Casimir operator is given by

\[
C = \eta_{ab} J_0^a J_0^b = j(j+1)
\]

(89)

with \( \eta_{+-} = \eta_{-+} = \eta_{33}/2 = 1/2 \). Note that the WZW primary fields \( \phi_j(w, x) \) satisfy

\[
J^a(z)\phi_j(w, x) \sim \frac{1}{z-w} J^a_0 \phi_j(w, x)
\]

(90)

We recall that the KZ equation captures the constraint that follows from the insertion of the null vector

\[
\left( L_{-1} - \frac{1}{k+2} \eta_{ab} J_0^a J_0^b \right) |\phi\rangle
\]

(91)

into a four-point function, for example. Using the realization (88), the four-point KZ equation reads

\[
(k+2)\partial_z + \sum_{j \neq i} \frac{\eta_{ab}(J_0^a)_i \otimes (J_0^b)_j}{z_i - z_j} \langle \phi_{j_4}(z_4, x_4)\phi_{j_3}(z_3, x_3)\phi_{j_2}(z_2, x_2)\phi_{j_1}(z_1, x_1) \rangle = 0
\]

(92)

Here \( (J_0^a)_j \) means that \( J_0^a \) acts on the field labeled by \( j \) and positioned at \( z_j \). The generic solution to these equations was computed in [23], and contains logarithms.

From the similarity with the \( c = 2 \) CFT, for example, it is natural to expect that a solution to the KZ equation may involve an infinite series in \( x_1 \), and not just be polynomial. This is indeed what happens. Introduce the differential operators

\[
P = -x^2(1-x) \partial_x^2 + [- (j_1 + j_2 + j_3 - j_4 + 1)x^2 + 2j_1 x + 2j_2 x(1-x)] \partial_x
\]

\[
+2j_2(j_1 + j_2 + j_3 - j_4)x - 2j_1j_2
\]

(93)

and

\[
Q = -(1-x)^2 x \partial_x^2 + [(j_1 + j_2 + j_3 - j_4 + 1)(1-x)^2 + 2j_3(1-x)
\]

\[
+2j_2(1-x)] \partial_x + 2j_2(j_1 + j_2 + j_3 - j_4)(1-x) - 2j_2j_3
\]

(94)

The anharmonic ratios, \( z \) and \( x \), are defined by

\[
z = \frac{(z_2 - z_1)(z_4 - z_3)}{(z_3 - z_1)(z_4 - z_2)}, \quad x = \frac{(x_2 - x_1)(x_4 - x_3)}{(x_3 - x_1)(x_4 - x_2)}
\]

(95)

The KZ equation may then be written

\[
(k+2)\partial_z F_{j_1,j_2,j_3,j_4}(z, x) = \left[ \frac{P}{z} + \frac{Q}{z-1} \right] F_{j_1,j_2,j_3,j_4}(z, x)
\]

(96)

where the function \( F_{j_1,j_2,j_3,j_4} \) is defined by

\[
F(z, x)_{j_1,j_2,j_3,j_4} = \lim_{w \to \infty} z_4^{2h_{j_4}} w^{-2h_{j_4}} \langle \phi_{j_4}(z_4, x_4)\phi_{j_3}(1,1)\phi_{j_2}(z, x)\phi_{j_1}(0,0) \rangle
\]

(97)

Some solutions to the KZ equation are related to each other under the action of the outer automorphism \( a \). Indeed, under the map \( j_i \to \frac{\alpha}{2} - j_i \), \( F_{j_1,j_2,j_3,j_4} \) may be shown to obey the same KZ equation as

\[
(x - z)^{-j_{1} - j_{2} - j_{3} - j_{4}} z^\alpha (1-z)^\beta F_{\frac{\alpha}{2} - j_{1}, \frac{\alpha}{2} - j_{2}, \frac{\alpha}{2} - j_{3}, \frac{\alpha}{2} - j_{4}}
\]

(98)
\[\alpha \text{ and } \beta \text{ are determined through the commutation of (103) with the differential operators } P \text{ and } Q. \]

The correlators of interest here are the ones for \( k = -1/2 \) involving fields with dimension \( h = -1/8 \). Let us focus on the case \( j_1 = j_2 = j_3 = j_4 = -1/4 \). Relating the solution (103)

\[
F(z,x)_{0,0,0,0} = A' \left[ \frac{2 \log z}{3} + \log x \right] + B' \left[ \frac{2 \log(1-z)}{3} + \log(1-x) \right] + C' 
\]

(99)

(where \( A', B' \) and \( C' \) are constants) to \( F(z,x)_{-1/4,-1/4,-1/4,-1/4} \), we find \( \alpha = \beta = 1/4 \), and hence

\[
F(z,x)_{-1/4,-1/4,-1/4,-1/4} = \frac{[z(1-z)]^{1/4}}{\sqrt{(x-z)}} 
\]

\[
\times \left\{ A' \left[ \frac{2 \log z}{3} + \log x \right] + B' \left[ \frac{2 \log(1-z)}{3} + \log(1-x) \right] + C' \right\} 
\]

(100)

Other solutions may be obtained using a symmetry under \( j \rightarrow -1 - j \). The symmetry is expressed as

\[
\frac{\partial^{1+2j}}{\partial x^{1+2j}} F_{j_1,j_2,j_3,j_4}(z,x) = F_{j_1,-1-j_2,j_3,j_4}(z,x) 
\]

(101)

As discussed in (39), this equation only makes sense when \( 1 + 2j \in \mathbb{Z}_+ \). However, if we consider the change of more than one field, we can formally combine fractional derivatives to obtain a well-defined operation. This is useful here since the spins are \( j = -1/4 \), so that \( 1 + 2j = 1/2 \). Thus, the application of this operation on two spins relates, by a simple derivative, the solution (103) to one for which two of the spins are now \( j = -3/4 \).

### 5.1 Generating function for twist correlators

We will postpone the discussion of logarithms to our subsequent paper (21), and here concentrate on the simplest solution deduced from (103):

\[
z^{1/4}(1-z)^{1/4}(x_3-x_1)^{-1/2}(x_4-x_2)^{-1/2} \left( \frac{(x_1-x_2)(x_3-x_4)}{(x_3-x_1)(x_4-x_2)} - z \right)^{-1/2} 
\]

(102)

All the prefactors left over in the study of the KZ equation have been re-installed. Our goal is to prove that this expression is a generating function for the four-point functions of twist correlators determined in Section 2, hence providing further support to our interpretation of the \( \beta \gamma \) system as an \( \tilde{su}(2)_{-1/2} \) theory.

In order to be able to expand (102), we choose to consider the region defined by

\[
z < x, \quad x_1 < x_2 < x_3 < x_4 
\]

(103)

This is also the “natural region” to consider from the point of view of using loop projective invariance to fix three of the \( x_i \)'s to the standard values, cf. (17). We may now expand the \( x \)-dependent part of (102):

\[
(x_3-x_1)^{-1/2}(x_4-x_2)^{-1/2} \left( \frac{(x_2-x_1)(x_4-x_3)}{(x_3-x_1)(x_4-x_2)} - z \right)^{-1/2} 
\]

\[
= (x_2-x_1)^{-1/2}(x_4-x_3)^{-1/2} \sum_{n \geq 0} \left( -\frac{1}{n} \right) \left( -z \frac{(x_3-x_1)(x_4-x_2)}{(x_2-x_1)(x_4-x_3)} \right)^n 
\]

\[
= (x_2x_4)^{-1/2} \sum_{n,m_1,m_2,m_3,m_4 \geq 0} (-1)^{n+m_1+m_2+m_3+m_4} \left( -\frac{1}{n} \right) z^n 
\]

\[
\times \left( -\frac{1}{m_1} \right) \left( -\frac{1}{m_2} \right) \left( \frac{n}{m_3} \right) \left( \frac{n}{m_4} \right) x_1^{N_1} x_2^{N_2} x_3^{N_3} x_4^{N_4} 
\]

(104)
where we have introduced

\[
\begin{align*}
N_1 &= m_1 + m_3 \\
N_2 &= -m_1 + m_4 - n \\
N_3 &= m_2 - m_3 + n \\
N_4 &= -m_2 - m_4
\end{align*}
\] (105)

A first and almost trivial observation is that

\[
N_1 + N_2 + N_3 + N_4 = 0
\] (106)

which simply corresponds to the conservation of momentum (and follows from global SU(2) invariance). The coefficient to \(x_1^{N_1}x_2^{N_2-1/2}x_3^{N_3}x_4^{N_4-1/2}\) may now be evaluated, and using (105) we find

\[
X(N_1, N_2, N_3, N_4; z) = \sum_{n \geq 0} (-1)^{N_2+N_3+n} \left(\frac{-1/2}{n}\right) z^n \sum_{m \geq 0} \left(\frac{-1/2 - n}{m}\right)
\times \left(\frac{n}{N_1 + N_3 - n - m}\right) \left(\frac{n}{N_1 - m}\right) \left(\frac{n}{N_2 + n + m}\right)
\] (107)

One has to be careful when analyzing this double summation. The summation ranges of \(n\) and \(m\) must be cut, and one is left with a sum of several double summations. In each of them, one may express the summation over \(m\) as a terminating \(4F_3\) hypergeometric function with argument 1. Unfortunately, the hypergeometric functions are not balanced, while most known results pertain to such functions. In particular, the 6-j symbols are associated to balanced hypergeometric functions.

Nevertheless, according to the twist-field approach, we should expect to be able to express (107) in terms of an ordinary hypergeometric function \(\Hyper{2}{1}{a, b; c; z}\) with \(a, b, c\) depending on \(N_1, N_2, N_3\). Indeed, we find

\[
\begin{align*}
X(N_1, N_2, N_3, N_4; z) &= (-1)^{N_1+N_3} \left(\frac{-1/2}{N_1}\right) \left(\frac{-1/2 + N_3 + N_4}{N_3}\right) \left(\frac{N_1}{N_1 + N_2}\right) \\
&\times z^{N_1+N_2}(1-z)^{-(N_2+N_3)} \Hyper{2}{1}{-2N_3, 1 + 2N_1; 1 + 2|N_1 + N_2|; z} \\
&+ (-1)^{N_1+N_3} \left(\frac{-1/2}{N_3}\right) \left(\frac{-1/2 + N_1 + N_2}{N_1}\right) \left(\frac{N_3}{N_3 + N_4}\right) \\
&\times z^{N_3+N_4}(1-z)^{-(N_1+N_4)} \Hyper{2}{1}{-2N_1, 1 + 2N_3; 1 + 2|N_3 + N_4|; z} \\
&- \delta_{N_1+N_2, 0} (-1)^{N_1+N_3} \left(\frac{-1/2}{N_1}\right) \left(\frac{-1/2}{N_3}\right) \\
&\times (1-z)^{N_1-N_4} \Hyper{2}{1}{-2N_3, 1 + 2N_1; 1; z}
\end{align*}
\] (108)

\(N_4\) has been included to emphasize the symmetry

\[
X(N_1, N_2, N_3, N_4; z) = X(N_3, N_4, N_1, N_2; z)
\] (109)

Note that

\[
(1-z)^{N_1-N_3} \Hyper{2}{1}{-2N_3, 1 + 2N_1; 1; z} = (1-z)^{N_1+N_3} \Hyper{2}{1}{-2N_1, 1 + 2N_3; 1; z}
\] (110)

showing the symmetry of the final term [108]. The absolute values ensure that the hypergeometric functions are well-defined, while the binomials split the result according to \(N_1 + N_2 \geq 0\) or \(N_1 + N_2 \leq 0\). The overlap \(N_1 + N_2 = 0\) is taken care of by the final subtraction.

We have verified (108) in many explicit examples, and been able to prove it analytically in several cases. One of those cases is the particularly interesting situation when \(N_1 = -N_2 = N_3 = -N_4\):

\[
X(N, -N, N, -N; z) = \left(\frac{-1/2}{N}\right) \left(\frac{-1/2}{N}\right) \Hyper{2}{1}{2N + 1, -2N; 1; z}
\] (111)
Let us indicate how one may prove this result. We see that \([107]\) reduces to

\[
X(N, -N, N, -N; z) = \left(\sum_{n=0}^{N} \sum_{m=-n}^{N} + \sum_{n=N+1}^{2N} \sum_{m=0}^{2N-n}\right) (-1)^n z^n
\]

\[
\times \left(\frac{1}{n} \right) \left(\frac{1}{m} \right) \left(\frac{1}{2N-n-m} \right) \left(\frac{n}{N-m} \right) \left(\frac{n}{N-m} \right)
\]

\[
= \sum_{n=0}^{N} (-1)^n z^n \left(\frac{1}{n} \right) \left(\frac{1}{2N-n} \right) \left(\frac{1}{N-n} \right) \left(\frac{1}{N-n} \right)
\]

\[
\times 4F3 \left[ \begin{array}{l}
N+1/2, -N, -n, -n \\
N-n+1, -N-n+1/2, 1; 1
\end{array} \right]
\]

\[
+ \sum_{n=N+1}^{2N} (-1)^n z^n \left(\frac{1}{n} \right) \left(\frac{1}{2N-n} \right) \left(\frac{n}{N} \right) \left(\frac{n}{N} \right)
\]

\[
\times 4F3 \left[ \begin{array}{l}
n+1/2, -N, -N, -(2N-n) \\
2N+1/2, n-N+1, n-N+1; 1
\end{array} \right]
\]

(112)

Now we use Eq. (2.4.2.3) of \([10]\):

\[
4F3 \left[ \begin{array}{l}
f, 1+f-h, h-a, d \\
h, 1+f+a-h, g; 1
\end{array} \right]
\]

\[
= \frac{\Gamma(g)\Gamma(g-f-d)}{\Gamma(g-f)\Gamma(g-d)} 4F3 \left[ \begin{array}{l}
a, d, 1+f-g, \frac{1}{2}f, \frac{1}{2}(1+f) \\
a, h, 1+f+a-h, \frac{1}{2}(1+f+d-g), \frac{1}{2}(2+f+d-g); 1
\end{array} \right]
\]

(113)

which is valid when \(f\) or \(d\) is a negative integer. Applying (113) to the two hypergeometric functions of our interest (112), the resulting \(4F3\) hypergeometric functions reduce to Saalschützian \(3F2\) hypergeometric functions. They may be summed using Saalschütz’s theorem, and (111) follows straightforwardly.

Other interesting situations appear when one of the \(N_i\) vanishes. It is straightforward to verify analytically that \([107]\) then sums to

\[
X(0, N_2, N_3, N_4; z) = (-1)^{N_3} \left(\frac{1/2}{-N_2} \right) \left(\frac{-1/2+N_2}{-N_4} \right) z^{-N_2(1-z)-N_4}
\]

\[
X(N_1, 0, N_3, N_4; z) = (-1)^{N_4} \left(\frac{-1/2}{-N_1} \right) \left(\frac{-1/2+N_1}{N_3} \right) z^{N_1(1-z)-N_3}
\]

\[
X(N_1, N_2, 0, N_4; z) = (-1)^{N_4} \left(\frac{-1/2}{-N_4} \right) \left(\frac{-1/2+N_4}{-N_2} \right) z^{-N_4(1-z)-N_2}
\]

\[
X(N_1, N_2, N_3, 0; z) = (-1)^{N_2} \left(\frac{-1/2}{-N_3} \right) \left(\frac{-1/2+N_3}{N_1} \right) z^{N_3(1-z)-N_1}
\]

(114)

with the arguments still subject to \([106]\).

In conclusion, we have found that the general four-point chiral block is given by

\[
(\phi_{-1/4}(z_1, x_4)\phi_{-1/4}(z_3, x_3)\phi_{-1/4}(z_2, x_2)\phi_{-1/4}(z_1, x_1))
\]

\[
= (z_3-z_1)^{1/4}(z_4-z_2)^{1/4}2^{1/4}(1-z)^{1/4}
\]

\[
\times \sum_{n_1, n_2, n_3, n_4 \geq 0} x_1^{n_1} x_2^{-1/2-n_2} x_3^{-1/2-n_4} (1)^{n_1+n_3} \delta_{n_1-n_2+n_3-n_4,0}
\]

\[
\times \left(\frac{-1/2}{n_1} \right) \left(\frac{-1/2+n_3-n_4}{n_3} \right) \left(\frac{n_1}{n_1-n_2} \right)
\]

\[
\times z^{n_1-n_2}(1-z)^{n_2-n_3} 2F1 [-2n_3, 1+2n_1; 1+2|n_1-n_2|; z]
\]
A comparison is now straightforward. For one ghost field. For instance, we have those involving the twist fields. At first, let us look at the simple fusion involving one twist field and

We first return to the 6.1 Twist fusion rules from the free-field representation content, and the modular invariant.

In this section, we probe the darker corners of our models in order to address the puzzles raised in the previous sections. Our key tool is the free-field representation. We revisit the fusion rules, the operator intertwining operators. It was referred to as adjustable monodromy, since the primary fields would ap-

primary fields. That merely reflects that the generating-function correlator expands on different pairings of representations depending on the order of its arguments. An interpretation of the expansion of the primary fields themselves was addressed in [36] in the context of a Wakimoto free-field realization. There it was argued that the expansions depend on the numbers of screening operators associated to the various intertwining operators. It was referred to as adjustable monodromy, since the primary fields would appear with different monodromy properties depending on the context. Our results here on the generating functions are in accordance with [36] [11]. The first systematic work on correlators in fractional-level WZW models appeared in [42].

6 Fusion rules, characters and the modular invariant

In this section, we probe the darker corners of our models in order to address the puzzles raised in the previous sections. Our key tool is the free-field representation. We revisit the fusion rules, the operator content, and the modular invariant.

6.1 Twist fusion rules from the free-field representation

We first return to the \( \beta \gamma \) system and use its \( \eta \xi \phi \) representation to reconsider the fusion rules, in particular those involving the twist fields. At first, let us look at the simple fusions involving one twist field and one ghost field. For instance, we have

A hitherto implicit \( z \)-dependent prefactor has been included, and the notation has been changed according to \( N_1, N_2, N_3, N_4 \to n_1, -n_2, n_3, -n_4 \).

We can now compare this result with the correlators for twist fields in the \( \beta \gamma \) system, recalling the expansions (71) and (72). Now, suppose we insert at \( z_1 \) and \( z_3 \) lowest-weight representations \( D_{-1/4}^+ \), and at \( z_2 \) and \( z_4 \) highest-weight representations \( D_{-1/4}^+ \). The expansion of the correlator should then read

A comparison is now straightforward. For \( n_1 - n_2 \geq 0 \), the matching of the first term in (115) with the corresponding twist correlator (51) is obvious. For \( n_1 - n_2 < 0 \), we recall that our simplified prescription for the twist correlators involved extracting the (possible) pole from the hypergeometric function by a formal manipulation. This may be achieved using

and a match with the second term in (115) is obtained.

We observe that the choice of order used here (103) dictates the representations carried by the primary fields. That merely reflects that the generating-function correlator expands on different pairings of representations depending on the order of its arguments. An interpretation of the expansion of the primary fields themselves was addressed in [50] in the context of a Wakimoto free-field realization. There it was argued that the expansions depend on the numbers of screening operators associated to the various intertwining operators. It was referred to as adjustable monodromy, since the primary fields would appear with different monodromy properties depending on the context. Our results here on the generating functions are in accordance with [36] [11]. The first systematic work on correlators in fractional-level WZW models appeared in [42].

\[
\begin{align*}
\beta(z)\tau_\lambda(w) & \sim \frac{\tau_{\lambda-1}(w)}{(z-w)^{1/2}} \\
\beta(z)\tau_0(w) & \sim \frac{\tau_{-1}(w)}{(z-w)^{1/2}}, \quad \beta(z)\tau_2(w) \sim \frac{\tau_1(w)}{(z-w)^{1/2}}
\end{align*}
\]
They are of the types \(D^{-1/4} \times D_{1/2} = D^{-3/4}\) and \(D^{-3/4} \times D_{1/2} = D^{+1/4}\), respectively. This result is confirmed by computations with generic descendants. Other examples of this type lead to the rules

\[
\begin{align*}
D^{-3/4} \times D_{1/2} &= D^{-1/4} \\
D^{-1/4} \times D_{1/2} &= D^{+3/4}.
\end{align*}
\]

(120)
(121)

The \(D_{1/2}\) representation intertwines the two infinite-dimensional representations with \(j = -1/4\) and \(j = -3/4\). The remaining fusion involving the \(j = 1/2\) representation, namely \(D_{1/2} \times D_{1/2} = D_{0}\), is easily verified. These fusion rules agree with the last three in (73) and (74).

Let us now consider fusions involving two twist fields. Take for instance fusions of the type \(D^{+1/4} \times D^{-1/4}\). The simplest case is

\[
\tau_1(z)\tau_0(w) = e^{i\phi(z)/2}e^{-i\phi(w)/2} \sim (z-w)^{1/4}I
\]

(where as usual \(I\) stands for the identity field). A somewhat more complicated case would be

\[
\tau_3(z)\tau_{-2}(w) = (\partial\xi\partial^2\xi : e^{3i\phi/2})(z)(\eta\partial\eta : e^{-5i\phi/2})(w) \sim (z-w)^{1/4}I
\]

(123)

More generally, we get \(\tau_{2m+1}\tau_{-2n} \sim (J^\pm)^{n-m}|I\). These computations confirm the following fusion rule

\[
D^{+1/4} \times D^{-1/4} = D_{0}
\]

(124)

This result agrees with the second fusion in (73). Similar calculations confirm the first and the third cases in (74). Notice that if the Verlinde fusions (79) were blind to the specification of the representations, the corresponding fusions would be invalidated.

All fusions computed so far have been of the type \(D^+ \times D^-\) (since \(D_{1/2}\) can be viewed either as a highest- or a lowest-weight representation). Consider now fusions of the form \(D^\pm \times D^\pm\). These are precisely those for which the Verlinde formula is supposed to apply. The simplest case of a twist product of the type \(D^- \times D^-\) is

\[
\tau_0(z)\tau_0(w) = e^{-i\phi(z)/2}e^{-i\phi(w)/2} \sim (z-w)^{-1/4}e^{-i\phi(w)}
\]

(125)

It corresponds to the fusion of the “bottom” fields in \(D^{-1/4} \times D^{-1/4}\). The result, however, is not a twist field. In other words, there is no value of \(\lambda\) for which \(e^{-i\phi}\) can be written in the form \(\sigma_\lambda e^{i\eta(\lambda-1/2)\phi}\). This is clear since \(e^{-i\phi}\) has dimension \(-1/2\) while twist fields have dimension \(-1/8\). Moreover, its products with the ghost fields

\[
\beta(z)e^{-i\phi(w)} \sim (z-w)^{-1}(\eta e^{-2i\phi})(w), \quad \gamma(z)e^{-i\phi(w)} \sim (z-w)\partial\xi(w)
\]

(126)
do not have the monodromy properties that characterize the twist fields. Products of generic descendants in \(D^{-1/4} \times D^{-1/4}\), namely \(\tau_{2n} \times \tau_{-2m}\), yield \(J^+\) descendant of \(e^{-i\phi}\). Therefore, the set of fields appearing in \(D^{-1/4} \times D^{-1/4}\) is \(\{(J^+)^n e^{-i\phi}| n \in \mathbb{Z}_2\}\). They all have dimension \(-1/2\).

Consider a sample product associated to \(D^{-1/4} \times D^{-1/4}\):

\[
\tau_1(z)\tau_1(w) = e^{i\phi(z)/2}e^{i\phi(w)/2} \sim (z-w)^{-1/4}e^{i\phi(w)}
\]

(127)

Again, \(e^{i\phi}\) is not a twist field. It is another new field, also with dimension \(-1/2\). The various fields occurring in \(D^{-1/4} \times D^{+1/4}\) are \(\{(J^-)^n e^{i\phi}| n \in \mathbb{Z}_2\}\).

Next, consider the simplest product within \(D^{-3/4} \times D^{-3/4}\):

\[
\tau_{-1}(z)\tau_{-1}(w) = (\eta e^{-3i\phi/2})(z)(\eta e^{-3i\phi/2})(w) \sim (z-w)^{-5/4}(\partial\eta\eta : e^{-3i\phi})(w)
\]

(128)
The field appearing on the right hand side has dimension $-3/2$. It is not a new field, however, since it can be expressed as a $J^+$ descendant of one of the new fields already found. It is easily recognized as $J^+_1 e^{-i\phi}$:

\[
J^+_1 e^{-i\phi(w)} = \frac{1}{2\pi i} \oint dz(z - w) J^+_1(z) e^{-i\phi(w)}
= \frac{1}{2\pi i} \oint dz \frac{1}{2} \frac{1}{2(z - w)} (e^{-3i\phi} : \partial \eta \eta :)(w)
= \frac{1}{2} (e^{-3i\phi} : \partial \eta \eta :)(w)
\]

This also shows that $e^{-i\phi}$ is not associated to an affine highest-weight state, even though it is a Virasoro highest weight. The $su(2)$ descendants in $D_{-3/4}^+ \times D_{-3/4}^-$ are $\{(J^+_0)^n(J^+_1 e^{-i\phi})| n \in \mathbb{Z}_+\}$.

Finally, the structure of the product $D_{-3/4}^+ \times D_{-3/4}^-$ is fixed by that of its top fields:

\[
\tau_2(z) \tau_2(w) = (\partial \xi e^{3i\phi/2})(z)(\partial \xi e^{3i\phi/2})(w) \sim (z - w)^{-5/4}(\partial^2 \xi \partial \xi : e^{3i\phi})(w) \sim (z - w)^{-5/4}(J^+_1 e^{i\phi})(w)
\]

with the descendants comprising the set $\{(J^+_0)^n(J^+_1 e^{i\phi})| n \in \mathbb{Z}_+\}$.

Next we consider the two remaining products $D_{-1/4}^+ \times D_{-3/4}^-$ and $D_{-1/4}^+ \times D_{-3/4}^-$. Of the first kind, we have the fusion

\[
\tau_0(z) \tau_{-1}(w) = e^{-i\phi/2}(z) \eta e^{-3i\phi/2}(w) \sim (z - w)^{-3/4}(\partial \xi \partial \xi : e^{3i\phi})(w) \sim (z - w)^{-3/4}(\beta^\pm_2 e^{-i\phi})(w)
\]

with descendants $\{(J^+_0)^n(\beta^\pm_2 e^{-i\phi})| n \in \mathbb{Z}_+\}$. Associated to the second product, we have

\[
\tau_1(z) \tau_2(w) = e^{i\phi(z)/2}(\partial \xi e^{3i\phi/2})(w) \sim (z - w)^{-3/4}(\partial \xi e^{2i\phi})(w) \sim (z - w)^{-3/4}(\gamma^\pm_2 e^{i\phi})(w)
\]

with descendants $\{(J^+_0)^n(\gamma^\pm_2 e^{i\phi})| n \in \mathbb{Z}_+\}$.

In summary, by considering products of the form $D_{-l/4}^+ \times D_{-l'/4}^-$, $l, l' = 1, 3$, we have found the following set of new fields:

\[
e^{-i\phi}, \quad (\beta^\pm_2 e^{-i\phi}), \quad (J^+_1 e^{-i\phi}), \quad \text{and their } J^+_0 \text{ descendants} \\
e^{i\phi}, \quad (\gamma^\pm_2 e^{i\phi}), \quad (J^+_1 e^{i\phi}), \quad \text{and their } J^-_0 \text{ descendants}
\]

This provides an explicit construction of all the fields in the products $D_{-l/4}^+ \times D_{-l'/4}^-$. Now, by taking products of twist fields with these new fields, we produce still new fields. The simplest case is $\tau_0 \times e^{-i\phi}$ which generates $e^{-3i\phi/2}$, with dimension $-9/8$. Its product with $\tau_0$ in turn generates $e^{-2i\phi}$, with dimension $-2$, and so on. Proceeding in this way, we obviously get new fields at every step, with conformal dimensions that become more and more negative. A sample new field occurring at the $n$-th step is $e^{-n_i\phi/2}$, of dimension $-n^2/8$.

These computations call for a description of these new fields in terms of the $\beta\gamma$ system and the $\hat{su}(2)_{-1/2}$ admissible representations. Both points are addressed in turn in the following two subsections. We stress that these results immediately invalidate the Verlinde fusion rules in the form (74), and demonstrate the incompleteness of the Awata-Yamada fusion rules (74).

---

8We note that a possible way to get rid of this problem would be to show that the $\beta\gamma$ system is a reduction of the $\eta\xi$ system, and that there are fields in the latter that are not present in the former. This applies, in particular, to the new fields generated in the fusion of $D_{-1/4}^+ \times D_{-1/4}^-$ as represented in the $\eta\xi$ system. We have, however, found no indication that such a reduction is possible nor necessary.
6.2 Deeper-twist fields

The usual twist fields \( \tau \) are primary fields of the \( \beta\gamma \) chiral algebra in the R sector. Apart from the vacuum, these are the only affine primary fields in the NS sector. However, the field structure of the \( \beta\gamma \) system is much richer than that. It contains composites of the twist fields, which will be called deeper twists. There is an infinite number of them, parameterized by a positive integer \( n \). The deeper twists will be denoted \( \tau^{(n)} \). They can be defined from their monodromy property with respect to the ghost fields, by demanding, for instance, that the leading terms in the OPEs be:

\[
\beta(z)\tau^{(n)}(w) \propto (z - w)^{-n/2}, \quad \gamma(z)\tau^{(n)}(w) \propto (z - w)^{-n/2}
\]

In this notation, the \( \tau \) introduced previously would be \( \tau^{(1)} \).

To illustrate, let us discuss \( \tau^{(2)} \) in more details. We can define the Green function \( g_2^{(2)}(z, w) \) as before, i.e.,

\[
g_2^{(2)}(z, w) = \frac{\langle \beta(z)\gamma(w)\tau^{(2)}(z)\tau^{(2)}(w) \rangle}{\langle \tau^{(2)}(z)\tau^{(2)}(w) \rangle}
\]

and find that

\[
g_2^{(2)}(z, w) = z^{-1}w^{-1} \frac{A z^2 + (1 - A)w^2}{z - w}
\]

From this it follows that the dimension of \( \tau^{(2)} \) is \( h = -\frac{1}{2} \), while its charge is \( J_0 = -\frac{1}{2}(2A - 1) \). In terms of the \( \eta\xi \) system, one can represent it as

\[
\tau^{(2)} = \sigma^{(2)}_\lambda e^{i(\lambda - 1)\phi}
\]

(so that \( A = \frac{1}{2} \)), with the monodromies

\[
\eta(z)\sigma^{(2)}_\lambda(w) \sim (z - w)^{-\lambda}, \quad \partial\xi(z)\sigma^{(2)}_\lambda(w) \sim (z - w)^{\lambda - 2}
\]

These fields \( \sigma^{(2)}_\lambda \) have weight \( h = -\frac{\lambda(2 - \lambda)}{2} \). Note that \( \sigma^{(2)} \) is not what we so far have called a twist field in the \( \eta\xi \) system because of the difference between its expansions with \( \eta \) and \( \xi \). As before, the above relations will not always hold when \( A = 0 \) or \( A = 1 \).

It is important to realize that the deeper-twist fields are not new objects. In fact, \( \tau^{(2)}_{2\lambda} \) appears naturally in the OPE of \( \tau^{(1)}_\lambda \) with itself. This can be seen most clearly if one considers

\[
g_4(z, w) = \frac{\langle \eta(z)\partial\xi(w)\tau_{1-\lambda}(4)\tau_\lambda(3)\tau_{1-\lambda}(2)\tau_\lambda(1) \rangle}{\langle \tau_{1-\lambda}(4)\tau_\lambda(3)\tau_{1-\lambda}(2)\tau_\lambda(1) \rangle}
\]

Introducing the forms

\[
\omega_1(z) = [(z - z_1)(z - z_3)]^{-\lambda}[(z - z_2)(z - z_4)]^{-1+\lambda}
\]

\[
\omega_2(w) = [(w - z_1)(w - z_3)]^{\lambda}[(w - z_2)(w - z_4)]^{1-\lambda}
\]

one finds that

\[
g_4(z, w) = \omega_1(z)\omega_2(w) \left[ \frac{\lambda(z - z_1)(z - z_3)(w - z_2)(w - z_4)}{(z - w)} + (1 - \lambda)\frac{(z - z_2)(z - z_4)(w - z_1)(w - z_3)}{(z - w)} + C \right]
\]

Letting \( z_1 \to z_3 \) and \( z_2 \to z_4 \) shows the appearance of singularities \( (z - z_1)^{-2\lambda} \) and \( (w - z_2)^{2\lambda - 2} \), characteristics of the \( \sigma^{(2)}_{2\lambda} \) core of the field \( \tau^{(2)}_{2\lambda} \). The reason why the core field \( \sigma^{(2)}_{2\lambda} \) (in the \( c = -2 \) \( \eta\xi \) system) was not noticed before, is that the twist fields that were considered had \( \lambda = \frac{1}{4} \), i.e. \( \sigma^{(1)}_{1/2} \). The
point is that the case \( \lambda = \frac{1}{2} \) is special, and the coupling to the field \( \sigma^{(2)}_1 \) vanishes. (It vanishes both in the numerator and the denominator of \( g \), which is why it still appears formally in the previous equations. But one can check that the OPE coefficient vanishes at that point.) In the \( \beta \gamma \) system, twist fields generically involve \( \tau^{(2)}_\lambda \) with \( \lambda \) arbitrary, and therefore no truncation occurs. This can also be seen directly at the level of the four-point twist correlator. From that perspective, it is the \( \eta \xi \) system that appears special, at least as long as one restricts to rational twists, i.e., \( \lambda \in \mathbb{Q} \). The significance of an \( \eta \xi \) system with irrational twists remains to be explored.

The representation (137) generalizes to deeper twists with \( n > 2 \):

\[
\tau^{(n)}_\lambda = \sigma^{(n)}_\lambda e^{i(\lambda - \frac{n}{2})\phi}
\]  

(142)

The dimension of \( \sigma^{(n)}_\lambda \) is \( h = -\frac{\lambda(n-\lambda)}{2} \), while that of \( \tau^{(n)}_\lambda \) reads

\[
h_{\tau^{(n)}_\lambda} = -\frac{n^2}{8}
\]

(143)

Thus, the spectrum of the deeper twists is unbounded from below.

Now, as one can naturally expect, the deeper twists \( \tau^{(2)} \) are exactly the fields identified in the products \( D^\pm_{-l/l'} \times D^\pm_{-l'/l} \) (with \( l,l' = 1,3 \)). In particular, we have \( \tau_0^{(2)} = e^{-i\phi} \), and this identification is further supported by the OPEs

\[
\beta(z)e^{-i\phi}(w) \propto (z-w)^{-1}, \quad \gamma(z)e^{-i\phi}(w) \propto (z-w)
\]

(144)

Similarly, we have \( \tau_2^{(2)} = e^{i\phi} \). Both expressions are indeed of the form (142) with \( \sigma^{(2)}_0 = \sigma^{(2)}_2 = I \). It should be noticed that no deeper twists \( \tau^{(2)}_\lambda \) for \( \lambda \) odd are needed to close the operator algebra in the \( \beta \gamma \) system. If they can actually be constructed, the simple presence of \( \sigma^{(1)}_{1/2} \) in the \( c = -2 \) sector is not sufficient because of the zero occurring in the OPE, cf. the comment made above. To settle whether there is a consistent theory with a closed operator algebra containing these twists, is beyond the scope of this paper.

The images of the deeper-twist fields under the zero-mode algebra are also deeper-twist fields. For instance,

\[
J_0^+ e^{-i\phi} \propto (-2i\partial\phi \partial\eta + \partial^2 \eta \eta) e^{-3i\phi}
\]

(145)

The term in bracket conspires to exactly produce the same singularities in the expansion with \( \beta \) and \( \gamma \) as the initial deeper twist \( e^{-i\phi} \).

Twist fields deeper than \( \tau^{(2)} \) will appear in products \( \tau_\lambda(z)\tau^{(2)}_{\lambda'}(w) \) and more generally, in products \( \tau^{(n)}_{\lambda}(z)\tau^{(m)}_{\lambda'}(w) \), as long as \( \text{sign}(\lambda) = \text{sign}(\lambda') \). We have already identified an explicit and simple example of arbitrary deeper twist appearing in the repeated product of \( \tau_0 \) with itself, namely \( \tau^{(n)}_0 = e^{-in\phi/2} \).

Another simple example is \( \tau^{(n)}_1 = e^{in\phi/2} \), which is similarly obtained from the repeated product of \( \tau_1 \) with itself.

That settles the question of these new fields appearing in the fusions of the type \( D^\pm \times D^\pm \) from the point of view of the \( \beta \gamma \) system. We now have to see how these deeper twists can be described from the \( \hat{su}(2) \) point of view.

### 6.3 Spectral flow of the \( \hat{su}(2)_{-1/2} \) admissible representations

At first sight, the identification of the deeper twists in terms of representations of the \( \hat{su}(2)_{-1/2} \) WZW model appears to be rather problematic. Indeed, the irreducible representations that are modular covariant are precisely the admissible representations: there are only four of them and their spectrum is bounded from below by the value \(-1/8\). The question of how to generate an unbounded spectrum from these admissible representations still has to be addressed.
The resolution to this problem lies in a remarkable feature of the admissible representations. Namely, they are not mapped onto themselves under twisting, or in the $\hat{su}(2)$ terminology, under the spectral flow. To see this, we first discuss the spectral flow which is a symmetry transformation of the $\hat{su}(2)$ algebra:

$$J^3_n = J^3_n - \frac{k}{2} w \delta_{n,0}, \quad \hat{J}^\pm_n = J^\pm_{n\mp w}$$

In terms of the currents themselves, these transformations read

$$\hat{J}^3 = J^3 - \frac{k}{2} w, \quad \hat{J}^\pm = J^\pm e^{\pm i w}$$

The new Sugawara stress-energy tensor is

$$\hat{T} = T - wJ^3 + \frac{k}{4} w^2$$

or in terms of modes

$$\hat{L}_n = L_n - wJ^3_n + \frac{k}{4} w^2 \delta_{n,0}$$

In the following, we mainly need the zero-mode transformations, which (for $k = -1/2$) read

$$\hat{L}_0 = L_0 - wJ^3_0 - \frac{w^2}{8}, \quad \hat{J}^3_0 = J^3_0 + \frac{w}{4}$$

The spectral flow (with $w \in \mathbb{Z}$) is nothing but the action of the automorphisms $\pi_w$, acting on operators as

$$\pi_w(J^\pm_m)^{n^{-1}}_w = J^\pm_{m\mp w}, \quad \pi_w(J^3_m)^{n^{-1}}_w = J^3_m - \frac{k}{2} w \delta_{m,0}$$

With these relations, the action of $\pi_w$ can be studied at the level of the representations.\footnote{This result is easily obtained starting from the classical version of the Sugawara stress-energy tensor, $T = (1/2k)(2J^3J^3 + J^+J^- + J^-J^+) + \frac{k^2}{4} w^2$ which becomes, under the transformation of the currents, $\hat{T} = (1/2k)(2\hat{J}^3\hat{J}^3 + \hat{J}^+\hat{J}^- + \hat{J}^-\hat{J}^+) - wJ^3 + \frac{k^2}{4} w^2$. Upon quantization, the prefactor in front of the bilinear term gets renormalized in the usual way: $k \rightarrow k + 2$.\footnote{From these transformations, we can easily check that in spite of the quadratic character of the $L_0$ transformation, flowing successively with $w$ and $w'$ is equivalent to flowing with $w + w'$.\footnote{In terms of WZW models, the spectral flow has a simple interpretation in the classical limit. Indeed, recall that solutions to the equations of motion have the form $g = g(z)\hat{g}(\bar{z})$. If we put the theory on a cylinder with space period $\omega_1$, periodicity simply requires that $g(z + \omega_1) = g(z)M$ and $\hat{g}(z + \omega_1) = M^{-1}g(z), M$ an element of $SU(2) \ (or \ SL(2, \mathbb{R}))$. The spectral flow affects such a solution by multiplying it by group elements whose Euler angles are linear in $z$. This corresponds to the action of an element of the loop group which is not continuously connected to the identity. Since $\Pi_1(SU(2))$ is trivial, this interpretation requires the model to be defined in terms of the $SL(2, \mathbb{R})$ group structure. But recall that this interpretation is action dependent, and the existence of an action is a feature that is lost in the $\hat{su}(2)$ case at fractional level.}}$}
Let us consider in detail the action of \( \pi_{-1} \) on the different admissible representations viewed as highest-weight representations. Let us start with the vacuum representation \( D_0 \), whose highest-weight state \( |0\rangle \) satisfies
\[
J_0^3 |0\rangle = 0, \quad J_1^- |0\rangle = 0, \quad J_0^- |0\rangle = 0, \quad J_0^+ |0\rangle = 0
\]
(153)
These relations are transformed into
\[
(J_0^3 - 1/4) |\pi_{-1}(0)\rangle = 0, \quad J_0^- |\pi_{-1}(0)\rangle = 0
\]
(154)
This fixes \( |\pi_{-1}(0)\rangle \) to be a lowest-weight state with \( m = -j = 1/4 \), so that
\[
\pi_{-1}(D_0) = D_{-1/4}
\]
(155)
Proceeding in a similar way, still focusing on the highest-weight state, we find that
\[
\pi_{-1}(D_{1/2}) = D_{-3/4}, \quad \pi_{-1}(D_{1/4}^-) = D_0, \quad \pi_{-1}(D_{3/4}^-) = D_{1/2}
\]
(156)
where the last relation indicates that the highest-weight state \( | -3/4, -3/4 \rangle \) is mapped to the lowest-weight state of \( D_{1/2} \): \( |1/2, -1/2\rangle \). Understanding the labeling \( \pm \) for \( j = 0, 1/2 \) in that sense, we thus have
\[
\pi_{-1}(D_j^+) = D_{k/2-j}^-
\]
(157)
Similarly, we find
\[
\pi_1(D_0^-) = D_{-1/4}^+, \quad \pi_1(D_{1/2}^-) = D_{-3/4}^+, \quad \pi_1(D_{1/4}^-) = D_{1/2}^+, \quad \pi_1(D_{-1/4}^-) = D_0^-
\]
(158)
that is,
\[
\pi_1(D_j^-) = D_{k/2-j}^+
\]
(159)
All actions of \( \pi_w \) not listed above lead to representations that are not affine highest weights. An example is given in Figure 1 with the filled dots indicating the extremal points of the representations.

Let us make this somewhat more concrete by considering the transformation of the admissible characters. Since the characters are defined as
\[
\chi = \text{Tr} e^{2i\pi \tau(L_0-c/24)} e^{4i\pi \tau J_0^3}
\]
(160)
\footnote{To further illustrate these computations, note that in the analysis of \( \pi_1(D_{1/2}^-) = D_{-3/4}^+ \), we find in particular that the simplest singular vector of \( D_{1/2} \), namely \( (J_0^3)^2 |1/2, -1/2\rangle \), is sent to \( (J_0^3)^2 | -3/4, -3/4 \rangle \), as it should.}
they transform as follows under the spectral flow:

\[ \chi_j(\tau, \theta) \rightarrow \tilde{\chi}_j = \text{Tr}_{D_j} e^{2i\pi(r(L_0 - c/24) + 3)} e^{-i\pi w^2/4} e^{i\pi z} \chi_j(\tau, z - w\tau/2) \]  

(161)

Take the case \( w = -1 \). Using the transformation formulas

\[ \begin{align*}
\theta_1(z + \tau/2, \tau) &= e^{-i\pi z} e^{-i\pi \tau/4} \theta_4(z, \tau) \\
\theta_2(z + \tau/2, \tau) &= e^{-i\pi z} e^{-i\pi \tau/4} \theta_3(z, \tau) \\
\theta_3(z + \tau/2, \tau) &= e^{-i\pi z} e^{-i\pi \tau/4} \theta_2(z, \tau) \\
\theta_4(z + \tau/2, \tau) &= e^{-i\pi z} e^{-i\pi \tau/4} \theta_1(z, \tau)
\end{align*} \]  

(162)

and the theta-function form (62) of the admissible characters, we find the following transformations:

\[ \begin{align*}
\chi_0 &\rightarrow \chi_{-1/4}^+ \\
\chi_{1/2} &\rightarrow \chi_{-3/4}^+ \\
\chi_{-1/4}^+ &\rightarrow -\chi_{1/2} \\
\chi_{-3/4}^+ &\rightarrow \chi_0
\end{align*} \]  

(163)

In view of (73), they are compatible with the results just obtained. Notice the required period 4 in these maps. Since the spectral flow shifts conformal weights by \( w \) In view of (70), they are compatible with the results just obtained. Notice the required period 4 in these maps. Since the spectral flow shifts conformal weights by \( w^2/8 \), an increase of \( w \) by a multiple of 4 ensures that the weights differ by integers.

Let us now turn to fusion rules and see how the flowed representations could appear. The key step is the assumption [20] that the fusion rules should be invariant under the full action of \( \pi \):

\[ \pi_w(\phi) \times \pi_w(\phi') = \pi_{w+w'}(\phi \times \phi') \]  

(164)

and not just with respect to the action of the outer automorphism \( a \) on spins (cf. (78)). Applying this to a simple example, we find that the product \( D_{-1/4}^{-1} \times D_{-1/4}^{-1} \) with itself, for example, amounts to

\[ D_{-1/4}^{-1} \times D_{-1/4}^{-1} = \pi_{-1}(D_0) \times \pi_{-1}(D_0) = \pi_{-2}(D_0 \times D_0) = \pi_{-2}(D_0) \]  

(165)

The \( \pi_{-2} \) flowed dimension of the vacuum state being \( -1/2 \), we recover our deeper twist \( \tau_0^{(2)} \). It is now viewed as the “lowest weight” of a representation that is not itself affine lowest-weight (cf. Fig 1).

If we consider the list of novel fields that appear in products \( D^\pm \times D^\pm \), we see that we only need to account for the presence of \( e^{\pm i\phi} \), the other “new fields” being natural composites with either the ghosts or the mode currents. The two fields \( e^{\pm i\phi} \) have the same conformal dimension but they differ by their \( J_0^3 \) eigenvalues. It is thus clear that if one of the fields corresponds to \( \pi_{-2}(I) \), the other one is \( \pi_{2}(I) \). More generally, the twist fields \( \tau_0^{(n)} \) and \( \tau_n^{(n)} \) are the flowed versions \( \pi_{\pm n}(I) \) [20].

We thus conclude that those operators generated under fusions computed by the \( \eta \xi \phi \) representation and having increasingly negative dimensions, correspond to the spectrally flowed representations. In other words, our free-field computations corroborates the assumption (164).

In light of these observations, it is natural to expect that the operator algebra only closes if all flowed representations are included. Since both highest- and lowest-weight representations are generated for \( j = -1/4, -3/4 \) by acting with \( \pi_{\pm j} \) on \( j = 0, 1/2 \), none of the previous interpretations of the diagonal modular invariant can be correct. Some fields were missing in each case. We revisit the invariant in the next section and show how a consistent theory can be constructed. We find that the modular invariant is obtained essentially by summing over the orbits of the spectral flow.

\[^{14}\text{Notice that the action of the automorphism on representations is opposite to that on the operators and hence on their eigenvalues.}\]
6.4 The $\hat{su}(2)_{-1/2}$ partition function revisited

Let us look again at the character of an admissible highest-weight representation:

$$\chi^+_j(q, y) = \text{Tr}_{D^+_j} q^{L_0 + 1/24} y^{J_3} \quad (q = e^{2\pi i \tau}, \ y = e^{2\pi i z})$$  \hfill (166)

We suppose that $|q| < 1$ and focus on the expansion of the character in terms of the variable $y$. As a function of $y$, this character has poles, which means that the summed expression (166) is only defined in a particular region in the complex plane. In other words, the character of the spin-$j$ representation is given by the function and the specification of a region of convergence. In this case, the region is given by the annulus $1 < |y| < 1/|q|^{1/2}$.

Once we flow the representation, the new character converges in a region determined by the flow. Since the character transforms under the flow as

$$\chi^+_{j,w}(q, y) = q^{-w^2/8} y^{w/2} \chi^+_j(q, y^{w/2})$$  \hfill (167)

the new region of convergence becomes (cf. (168))

$$|q|^{w/2} < |y| < |q|^{(w-1)/2}$$  \hfill (168)

Despite the notation (167) may result in a character associated to a lowest-weight representation (cf. (70)).

Now we know that the functional form above falls back on one of our original functions, but we should be careful. Since the original region of convergence is mapped to a new region of convergence for the “flowed character” (when interpreting the character in terms of an operator or representation), we get the following dimension for the flowed operator:

$$h^+_{j,w} = \frac{j(j+1)}{3/2} - wj - \frac{1}{8} w^2$$  \hfill (169)

(where the upper index + indicates that we use $m = j$ here). For instance, for $j = -1/4$, we see that $w = 1$ falls back on the identity, while $w = -1$ gives $h = -1/2$ which corresponds to $\pi_{-2}(f)$. In this way we see all the fields, but they are linked to particular convergence regions. In particular, the highest- and lowest-weight representations are all there. It is only that, say, $D^-_{-1/4}$ and $D^+_{-3/4}$ have different regions of convergence. Stated differently, flowing among representations amounts to perform analytic continuations.

The fact that we have to define convergence regions is not obvious from the original “trace point of view”. But since we know the result of the sum (cf. (53)), we see that the reason is due to the singularities in the summed function. The presence of these singularities is a feature particular to the fractional-level case. For $k$ integer, the sum is a holomorphic character function, converging everywhere in the plane. As a result, performing a spectral flow does not change the region of convergence. As already mentioned, it is a characteristic of the integer-level case that the flowed integrable representations are all mapped back to integrable representations.

Let us stress the following. Even though the set of admissible character functions (when combined to form the Kac-Wakimoto invariant) is invariant under the modular transformations, the physical states are associated with particular expansions, in well-defined regions of convergence. Hence, a character function is not mapped to a unique field or $\hat{su}(2)$ module. Rather, it is the decomposition on states which is required in the determination of the spectrum.

Keeping this in mind, we can revisit the modular invariance. When a transformation of the form $\tau \to \tau + 1$ is performed, character functions are indeed mapped into each other. The associated regions of convergence, on the other hand, are not mapped into each other. They are mapped into the regions of convergence of the spectrally flowed operators. This means that to have a full-fledged modular invariant, one has to include the infinite set of flowed representations. The invariant should really be interpreted
as a sum over all regions of convergence of all the expansion series. Thus, the partition function that includes all the twisted modules reads

$$Z = \sum_{w \in \mathbb{Z}} \sum_{j=0,1/2} |q^{-w^2/8} y^{w/2}\chi_j(q, yq^{-w/2})|^2$$

(170)

Since the partition functions have period four under the flow, we finally arrive at the usual functional form (here the superscript is included to distinguish the functions, not the representations)

$$Z = \sum_{D} \left( |\chi_0^+(q, y)|^2 + |\chi_{1/2}^+(q, y)|^2 + |\chi_{1/4}^+(q, y)|^2 + |\chi_{3/4}^+(q, y)|^2 \right)$$

(171)

This sum is understood to be over all domains of convergences over which we expand the functions to get the characters. Thus functionally, there is an infinite constant multiplying the usual partition function.

Now, with these comments, one should be careful when interpreting the Verlinde formula. The potential problem linked to the fact that the modular transformations relate different regions of convergences, is not taken into account in the derivation of the Verlinde formula. Therefore, it is well established only for integrable representations, or equivalently, for holomorphic character functions. As we have seen in our case, twisted modules appear under the modular transformations, meaning that we do not flow back onto the original set of fields when going around cycles on the torus. This clearly indicates that the Verlinde formula does not apply to the $\tilde{s}u(2)_{-1/2}$ WZW model (and nor to more general fractional-level WZW models). The previous belief that the four admissible fields close under fusion is incorrect, even though it is naively (but only naively) supported by the Verlinde formula.

### 7 Conclusion

The main conclusion of our study is that the $\tilde{s}u(2)_{-1/2}$ model defined algebraically à la Kac and Wakimoto, and the $\beta\gamma$ system with a standard choice of normal ordering in the R sector (i.e., the ground state is annihilated by one of the ghost zero modes), are not rational CFTs in the conventional sense. In both formulations of this $c = -1$ model, the spectra contain operators of arbitrarily large negative dimensions which are not primary fields with respect to the chiral algebra (either $\beta\gamma$ or $\tilde{s}u(2)_{-1/2}$). On the WZW side, a formal, yet meaningful theory, can be obtained by extending the basic set of admissible fields to include their orbits under the spectral flow. Viewed from the $\beta\gamma$ perspective, this amounts to taking into account an infinite number of deeper twists. We stress that without these extensions, the theories are not consistent. This can be seen either at the level of the fusion rules, which otherwise do not close, or at the level of the modular invariant, since modular transformations map characters to their flowed versions.

Let us rephrase this conclusion somewhat in order to settle some loose points in our initial discussion of the $\beta\gamma$ system in Section 2.1. As pointed out there, the $\beta\gamma$ system with the usual highest-weight conditions is plagued with divergences because the functional integral cannot be properly defined. Naive analytic continuation leads to a partition function which is essentially the inverse of $\text{det} \, \Delta$. A better procedure is to define the model in terms of the associated $\tilde{s}u(2)$ CFT. We have seen, however, that this WZW model has a rich operator content. Its spectrum extends infinitely beyond the simple set of admissible representations, by including all their images under the spectral flow. Nevertheless, there is a way to define characters of the flowed representations using analytic continuation such that the partition function of the complicated theory with no ground state, coincides formally with $1/\text{det} \, \Delta$ in the vicinity of $z = 0$. The naive result is thereby put into context. Ultimately, the original singularity at $z = 0$ appears as one copy of an infinite number of singularities, and the positions of these singularities determine regions of convergence in the complex plane that are associated to the characters of the various deeper twists. We thus see that, roughly, the original singularity hides two types of interrelated infinities: the infinite degeneracy of the twist fields (which are distinguished by their $u(1)$ charges), and the infinite number of deeper twists (each of which being degenerate). The spectrum is unbounded from below.

14In the terminology of [44], these are quasi-rational CFTs in that there is an infinite but countable number of characters in the partition function, while only a finite number of fields appear in any given fusion.
It should be clear that the choice of the highest-weight conditions (12) is not essential, since the alternative conditions (11) are recovered by flowing (cf. the analysis of Section 6.3 where $D^\pm$ representations are mapped into each others by the action of $\pi_{\pm1}$). Note, however, that the flow cannot generate a $\beta\gamma$ vacuum that is not annihilated by any of the ghost zero modes. Such a vacuum would lead to representations that are neither highest nor lowest weight. They would extend infinitely in both directions (and are called continuous representations).

A particularity of those $\beta\gamma$ twist fields that have been found to be organized in infinite-dimensional $\hat{su}(2)_{-1/2}$ representations, is that they live entirely in the free-fermionic sector of the $c = -2$ component. In other words, they do not involve those twists of the $\eta\xi$ sector which have no fermionic description. This allows the theory to avoid, at least in this version, the logarithmic extensions found in $c = -2$ CFT. Clearly, it is possible to introduce operators going beyond the model we have studied. The symplectic fermions provide an example, where the extension amounts to including two fermionic zero modes. Preliminary results show the presence of logarithms in that case. This will be the subject of our forthcoming paper [21].

We stress that the $c = -1$ model, independently of its physical applications, is a particularly good laboratory because it admits a faithful free-field representation [16]. It allows us to study general properties of non-unitary WZW models in a very controlled way. Note, however, that if we measure the complexity of a non-unitary WZW model by the number of its admissible representations, the $\hat{su}(2)_{-1/2}$ model is not the simplest one. Indeed, it has four admissible representations, while the model with the least possible number of admissible representations is $\hat{su}(2)_{-4/3}$, with three (the vacuum and two fields of dimension $-1/3$) [15]. This latter model has central charge $c = -6$, and has been studied extensively in [20]. There it is indicated that logarithms (originating from indecomposable $su(2)$ representations) may be unavoidable. This may not be a generic feature of fractional-level models, as our analysis has shown. It is still an open question, though, to understand the origin of the logarithms in the $\hat{su}(2)_{-4/3}$ WZW model, as it is presented in [20], and to understand the differences from the $\hat{su}(2)_{-1/2}$ model described here and in [21] (for its logarithmic lift).

It should be clear that our main conclusion concerning the non-rational character of the WZW models at fractional level, is independent of the specific value of $k$ studied. Although the presence of an unbounded negative spectrum can be traced back to the sign of $k$ (cf. the expression for the flowed dimension of the vacuum), the non-analytic property of the admissible characters is a generic feature.

The presence of a spectrum of dimensions which is unbounded from below casts doubts about proposals to use $\beta\gamma$ systems in the CFT description of fixed points in disordered electronic systems [17]. On the other hand, we have seen that some objects like the partition function of the $\beta\gamma$ system are, in a formal sense, insensitive to the unboundedness of the spectrum. It is thus plausible that, at the same level of formality, useful physical quantities can be calculated using such CFT methods. A somewhat related situation occurred in the study of the $U(1, 1)$ WZW model and the Alexander polynomial [17].

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15 In that vein, notice that the representation [16] can be viewed as a non-unitary parafermionic description of the $\hat{su}(2)_{1/2}/\hat{u}(1)$ coset, where the negative sign of the level is absorbed in a re-definition of the boson metric. Here the dimension of the basic parafermions (represented by $\partial\eta$ and $\partial^2\xi\phi$) is thus 3, and the parafermionic algebra is bound to reduce to the triplet algebra of $su(2)$.

16 Note that both models are associated to a Virasoro $c(1, p)$ model up to a $u(1)$ factor (with $p = 1, 2$, respectively) – one of the lowest-dimensional fields in each case being associated to $\phi_{1,p}$. 31
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