Nearly incompressible turbulence for different 2D and slab energy ratios

L Adhikari¹, G P Zank¹,², P Hunana¹,², R Bruno³, D Telloni⁴, Q Hu¹,² and D Shiota⁵

¹Center for Space Plasma and Aeronomic Research (CSPAR), University of Alabama in Huntsville, Huntsville, AL 35899, USA
²Department of Space Science, University of Alabama in Huntsville, Huntsville, AL 35899, USA
³INAF-IAPS Instituto di Astrofisica e Planetologia Spaziali, Via del Fosso del Cavaliere 100, 00133 Roma, Italy
⁴INAF - Astrophysical Observatory of Torino, Via Osservatorio 20, 10025 Pino Torinese, Italy
⁵Institute for Space-Earth Environmental Research, Nagoya University, Nagoya, Aichi 464-8601, Japan

E-mail: la0004@uah.edu

Abstract. Zank et al 2017 [1] developed equations describing the transport of 2D and slab turbulence based on nearly incompressible magnetohydrodynamic (NI MHD) theory. Zank et al [1] discussed preliminary solutions of the model equations, and Adhikari et al [2] provided a detailed analysis of the model equations. Their analyses of the 2D and slab turbulence transport model equations were based on the ratio of the inner boundary values of the 2D and slab energies being of the order 80:20. Here, we investigate the effect of choosing the 2D and slab energy ratios to be 80:20, 70:30, 60:40, and 55:45 for the boundary energy values, and solve the 2D and slab turbulence transport model equations. We compare our theoretical results with observations made by Voyager 2 and Ulysses. We find that the 2D and slab turbulent quantities such as the energy in forward and backward propagating modes, the fluctuating kinetic and magnetic energy, for example, are different for different boundary values, but show similar trends with increasing heliocentric distance. In addition, we also investigate the model equations by including a shear source of turbulence in the slab turbulence transport equations. This was not considered by [1] and [2]. We find that the slab turbulent quantities behave more reasonably in the presence of both shear and pickup ion sources of turbulence.
1. Introduction
The plasma beta $\beta$ is defined as the ratio of the plasma pressure and the magnetic pressure. The plasma beta of the solar wind and the solar corona is of the order of $\beta \sim 1$ and $\beta \ll 1$ when away from the heliospheric current sheet region. Therefore, a turbulence transport model that reflects plasma beta values of order $\beta \sim 1$ or $\beta \ll 1$ is required to describe the evolution of fluctuations in the solar wind, in contrast to previous 3D incompressible turbulence transport models [3–17], which are appropriate to a plasma beta $\beta \gg 1$ [18]. The nearly incompressible magnetohydrodynamic (NI MHD) theory allows for the formulation of turbulence transport model equations in the $\beta \sim 1$ or $\beta \ll 1$ regime [18, 19]. In such a description, the turbulence can be decomposed into a dominant 2D and minority slab component so that the total turbulence field is a superposition of 2D and slab turbulence. Zank & Mattaheus [19] and Bieber et al [20] showed theoretically and observationally that the ratio of the 2D and slab turbulence energy is 80:20. This ratio of the energies can have different values in the slow and fast solar wind [21]. Smith et al [22] showed that the ratio of the energy between the two populations of slab (1D) and 2D wavevectors is roughly equally distributed at high latitudes during solar minimum.

Initially, a form of NI MHD was developed to describe the observed $k^{-5/3}$, (where, $k$ is a wave number), spectrum of the interstellar electron density fluctuations [23, 24]. The NI theory was mostly developed in the early 1990s for homogeneous flows [18, 19, 25–29] before being extended to inhomogeneous flows [30–33]. Zank et al 2017 [1] derived the turbulence transport model equations using an Elsässer variables formulation for both homogeneous and inhomogeneous flows. Zank et al 2017 [1] introduced a further higher-order coupling of the 2D and slab turbulence model equations for inhomogeneous flows, which is discussed further by Zank et al 2017 (this issue) for homogeneous flows. Zank et al 2017 [1] show that the Alfvén velocity does not appear in the leading order 2D turbulence transport model equations in contrast to models that start from 3D incompressible MHD (for which $\beta \gg 1$) [34–36], that have the Alfvén velocity in their 2D turbulence transport equations. The 2D turbulence transport equations of Zank et al 2017 [1] based on NI MHD [18, 19, 30], shows that the fluctuations lie in a plane orthogonal to a direction of the large scale magnetic field, and are convected by the large scale solar wind speed. Further, the 2D transport equations for an inhomogeneous flow do not introduce any type of Alfvén critical point as do other turbulence transport model equations.

In this manuscript, we solve the Zank et al [1] 2D and slab turbulence transport model equations for a variety of energy ratios 80:20, 70:30, 60:40, and 55:45, and compare the numerical solutions with Voyager 2 and Ulysses measurements. The Voyager 2 observed values are obtained from Adhikari et al [2], and the Ulysses observed values are obtained from Bavassano et al [37] and Yang et al [38]. Zank et al [1], and Adhikari et al [2] presented numerical solutions of the 2D and slab model equations. Both papers used a boundary condition in which the 2D and slab turbulence energy has the ratio 80:20. In Zank et al [1] (and also [2]), the coupled turbulence transport equations included a shear source of turbulence in the 2D turbulence transport equations only, and a pickup ion source of turbulence in the slab turbulence transport equations. Here, along with solving the [1] turbulence transport equations for different energy ratios, we also solve the
coupled equations when both sources of turbulence are included in the slab turbulence transport equations. Section 2 describes the 2D and slab turbulence transport model equations. Section 3 presents a comparison of the numerical solutions of the 2D and slab turbulence transport equations with Voyager 2 and Ulysses observations, where a shear source is included in the 2D turbulence equations and a pickup ion source in the slab turbulence equations. Section 4 presents a comparison of the numerical solutions with observations, when a shear source is included in the 2D turbulence model, and shear and pickup ion sources are included in the slab turbulence model. Finally, Section 5 presents some discussion and conclusions.

2. 2D and Slab Model Equations

In this section, we present the 1D steady-state coupled 2D and slab turbulence transport model equations. The details of the derivation of the 2D and slab model equations are given in [1] (see also [2]). The 1D steady state 2D turbulence transport model equations in a spherical coordinate system $r$ are [1]

$$
U \frac{d}{dr} \langle z^\pm \rangle + \frac{U}{r} \langle z^\pm \rangle + \frac{U}{r} E_D^\pm + \frac{1}{r} \langle z^\pm \rangle \langle z^{-2} \rangle / \lambda^\pm = -2 \langle z^\pm \rangle \langle z^{-2} \rangle / \lambda^\pm + 2C_{sh}^D r_0 |\Delta U| V_4^2; 
$$

$$
= -2 \langle z^\pm \rangle \langle z^{-2} \rangle / \lambda^\pm + 2C_{sh}^D r_0 |\Delta U| V_4^2; 
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$$

where $\langle z^\pm \rangle$ is the energy in forward propagating modes, $\langle z^{-2} \rangle$ is the energy in backward propagating modes, $E_D^\pm$ is the residual energy, $L^\pm$ are correlation functions corresponding to $\langle z^\pm \rangle$, and $L_D^\pm$ a correlation function for $E_D^\pm$. The first terms on the right-hand side (RHS) of Equations (1) and (2) are the non-linear dissipation terms, the second terms are the higher order correction terms associated with the dissipation of slab turbulence and the corresponding injection of 2D energy through the 3-wave resonance condition [39]. The third terms are stream shear sources of turbulence [1, 2]. The parameters $C_{sh}^\pm$ and $C_{sh}^{E_D}$ are parametrized strengths of the shear source of energy in forward (+) and backward modes (-) and the residual energy ($E_D$), respectively. Here, $r_0 (= 1 \text{AU})$ is a reference location, $\Delta U$ is the difference between fast and slow solar wind.
speed, and \( V_{A0} \) is the Alfvén velocity at 1 AU. The normalized cross-helicity is defined as \( \sigma_{sh}^C \equiv E_{sh}^C / E_{sh}^{2F} \neq 0 \), where \( E_{sh}^C = (z^{+2} - z^{-2})/2 \) is the cross-helicity and \( E_{sh}^{2F} = (z^{+2} + z^{-2})/2 \) is the total turbulent energy.

Similarly, the 1D steady-state slab turbulence transport model equations are [1]

\[
(U + V_{A0} \frac{r_0}{r} \frac{d(z^{+2})}{dr} - (2b - 1) \frac{U}{r} (z^{+2}) + 2(3b - \frac{1}{2}) \frac{U}{r} E^*_D \pm (4b - 1) \frac{V_{A0}}{r_0} \frac{r_0}{r} E^*_D \pm \frac{V_{A0}}{r_0} \frac{r_0}{r} )^2 + 2 \left( \frac{z^{+2}}{\lambda_{\infty}^2} - \frac{z^{+2}}{\lambda_{\infty}^2} \right) \frac{1}{2} \right]
\]

\[
-2 \frac{(z^{+2}) (z^{+2})}{\lambda_{\infty}^2} + 2 \left( C^{+}_{sh} \frac{r_0}{r} |\Delta U| V_{A0}^2 \right) \frac{f_{D} n_{H}^\infty U V_{A0}}{n_{sw} r_{ion}} \exp \left( -\frac{L}{r} \right)
\]

\[
(U + V_{A0} \frac{r_0}{r} \frac{d(z^{+2})}{dr} - (2b - 1) \frac{U}{r} (z^{+2}) + 2(3b - \frac{1}{2}) \frac{U}{r} E^*_D - (4b - 1) \frac{V_{A0}}{r_0} \frac{r_0}{r} )^2
\]

\[
+ \frac{1}{2} \left( E^*_D (z^{+2}) (z^{+2}) / \lambda_{\infty}^2 - \frac{(z^{+2}) (z^{+2})}{\lambda_{\infty}^2} \right) - \frac{(z^{+2}) (z^{+2})}{\lambda_{\infty}^2} + 2 C^{+}_{sh} \frac{r_0}{r} |\Delta U| V_{A0}^2
\]

\[
(U + V_{A0} \frac{r_0}{r} \frac{dL_{+}}{dr} - (2b - 1) \frac{U}{r} L_{+} + 2 \left( \frac{3b}{2} - \frac{1}{4} \right) \frac{U}{r} L_{+} \pm \frac{V_{A0}}{r_0} \frac{r_0}{r} )^2 L_{+}\]

\[
\pm \frac{V_{A0}}{r_0} \frac{r_0}{r} L_{+}^2 + \frac{1}{r} \left( L_{+} - L_{-}^* \right) \frac{1}{2} \frac{(z^{+2}) (z^{+2})}{2} = 0;
\]

\[
U \frac{dL_{-}^2}{dr} - 2 \left( b - \frac{1}{2} \right) \frac{U}{r} L_{-} + 2 \left( 3b - \frac{1}{2} \right) \frac{U}{r} (L_{+}^2 - L_{-}^2) - (4b - 1) \left( L_{+}^2 - L_{-}^2 \right) \frac{V_{A0}}{r_0} \frac{r_0}{r} )^2
\]

\[
- \frac{1}{r} \left( L_{+}^2 - L_{-}^2 \right) (z^{+2}) (z^{+2}) / 2 + (L_{+}^2 - L_{-}^2) \frac{1}{2} \left( \frac{(z^{+2}) (z^{+2})}{2} = 0,
\]

where \( C^{+}_{sh} \) and \( C^{+}_{sh} \) are the strengths of the shear source of turbulence. Equations (5) and (6) are different from the Zank et al [1] and Adhikari et al [2] in that these equations have an additional shear source of turbulence which were not included in [1] and [2]. We solve the coupled equations for two cases; i) \( C^{+}_{sh} = 0 \) and \( C^{+}_{sh} = 0 \), and ii) \( C^{+}_{sh} \neq 0 \) and \( C^{+}_{sh} \neq 0 \). The parameter \( \langle z^{+2} \rangle \) is the slab energy in forward propagating modes, \( \langle z^{-2} \rangle \) is the slab energy in backward propagating modes, \( E^*_D \) is the slab residual energy, \( L_{+}^2 \) are the correlation functions corresponding to \( \langle z^{+2} \rangle \), and \( L_{D}^2 \) the correlation function for \( E^*_D \). \( E^*_D \) is the total slab turbulent energy, and \( E^*_C \) is the slab cross-helicity. The parameter \( L(\sim 8 \text{ AU}) \) is the ionization cavity length scale, and \( f_{D} \) is the fraction of pickup ion energy transferred into excited waves [40]. The parameter
\[ n_{SW}^\infty = 0.1 \text{ cm}^{-3} \] is the number density of interstellar neutrals entering the heliosphere, 
\[ \tau_{\text{ion}}^j = 10^6 \text{ s} \] is the neutral ionization time at 1 AU, 
\[ n_0^\infty = 5 \text{ cm}^{-3} \] is the solar wind density at 1 AU, and 
\[ 0 < b < 1 \] is a constant. The third term on the RHS of Equation (5) is a pickup ion source of slab turbulence [8].

The 1D steady-state variance of density fluctuations is [1]

\[ U \frac{d}{dr} \langle \rho^{\infty 2} \rangle + \frac{4U}{r} \langle \rho^{\infty 2} \rangle + \frac{4}{r} \langle u^{\infty 2} \rangle^{1/2} \langle \rho^{\infty 2} \rangle = -\frac{\langle u^{\infty 2} \rangle^{1/2} \langle \rho^{\infty 2} \rangle}{\lambda_u^\infty} + \eta_1 \langle \rho^{\infty 2} \rangle_0 \frac{r^2 |\Delta U|}{r^3} \]

\[ + \eta_2 \langle \rho^{\infty 2} \rangle_0 \frac{U}{V_A0 \tau_{\text{ion}}^n n_{SW}^0} \exp \left( -\frac{L}{r} \right), \]  

(9)

where \( \eta_1 \) and \( \eta_2 \) are constant, and \( \langle u^{\infty 2} \rangle \) and \( \lambda_u^\infty \) are the variance of the velocity fluctuations and the corresponding correlation length, i.e.,

\[ \langle u^{\infty 2} \rangle = \frac{\langle z^{\infty 2} \rangle + \langle z^{\infty -2} \rangle + 2E_\infty^D}{4}, \]  

and

\[ \lambda_u^\infty = \frac{(E_T^\infty + E_C^\infty) \lambda_u^+ + (E_T^\infty - E_C^\infty) \lambda_u^- + E_D^\infty \lambda_D^\infty}{2(E_T^\infty + E_C^\infty)}. \]  

(10)

In Equation (9), the first term on the right-hand side is the non-linear dissipation term. It shows that the 2D incompressible velocity fluctuations \( \langle u^{\infty 2} \rangle \) are responsible for the dissipation through mixing of the density fluctuations. In Equation (9), if \( \langle u^{\infty 2} \rangle \sim 0 \), the non-linear dissipation term (first term on rhs of (9)) and mixing term (third term on the left-hand side) become zero. In such a case, along with the absence of a source of turbulence (second and third terms on the rhs), the variance of density fluctuations decays as \( r^{-4} \), which is due to the expansion of the solar wind. However, \( \langle u^{\infty 2} \rangle \neq 0 \) ensures that the non-linear dissipation and mixing terms are non-zero, in which case the rate of dissipation of \( \langle \rho^{\infty 2} \rangle \) is faster than \( r^{-4} \) [1, 2]. The 1D steady-state solar wind temperature equation is given by

\[ U \frac{dT}{dr} + (\gamma - 1) \frac{2UT}{r} = \frac{1}{3} \frac{m_p}{k_B} \alpha \left[ \frac{2 \langle z^{\infty +2} \rangle \langle z^{\infty -2} \rangle^{1/2}}{\lambda_u^+} + \frac{2 \langle z^{\infty -2} \rangle \langle z^{\infty +2} \rangle^{1/2}}{\lambda_u^-} \right] + E_D^* \left( \frac{\langle z^{\infty -2} \rangle^{1/2}}{\lambda_u^+} + \frac{\langle z^{\infty +2} \rangle^{1/2}}{\lambda_u^-} \right) \]

\[ + \frac{2 \langle z^{\infty -2} \rangle \langle z^{\infty +2} \rangle^{1/2}}{\lambda_u^-} + E_D^* \left( \frac{\langle z^{\infty -2} \rangle^{1/2}}{\lambda_u^+} + \frac{\langle z^{\infty +2} \rangle^{1/2}}{\lambda_u^-} \right), \]  

(11)

where \( \gamma = 5/3 \) is the adiabatic index, \( m_p \) is the proton mass, \( k_B \) is the Boltzmann constant, and \( \alpha \) is the von Kármán-Taylor constant. The RHS of (11) describes the heating of the thermal background plasma by the dissipation of 2D turbulence, i.e., the loss terms on the RHS of the 2D equations (1)–(4). Only the correlation lengths of 2D turbulence are associated with the solar wind heating. We solve the coupled turbulence transport equations (1)–(11) using a 4th-order Runge-kutta method from 1 to 75 au.
3. Results: \( C_{sh}^{h,0} = 0 \) & \( C_{sh}^{E,D} = 0 \)

In this section, we present numerical solutions of the coupled turbulence transport equations (1)–(11), setting \( C_{sh}^{h,0} = 0 \) & \( C_{sh}^{E,D} = 0 \), and compare with Voyager 2 observations from 1–75 au. We also compare the energy ratio of the inward and outward propagating modes with Ulysses measurements from 1–5 au. We use the boundary conditions shown in Tables 1 and 3 to solve the coupled equations (1)–(11) and then compute several turbulent quantities such as the energy in forward and backward propagating modes, the total turbulent energy, the normalized residual energy and cross-helicity, the correlation functions and correlation lengths corresponding to forward and backward propagating modes and the residual energy, the fluctuating kinetic and magnetic energy, the Alfvén ratio, the variance of density fluctuations and the solar wind temperature. In Tables 1 and 3, the first column identifies the turbulent quantities. The second, third, fourth, and fifth columns represent the boundary conditions corresponding to the 80:20, 70:30, 60:40 and 55:45 energy ratios between 2D and slab turbulence, respectively. Table 2 shows the parameter values that are used in the model equations [2]. Figures 1–4 are solutions of the turbulence transport model equations corresponding to the inner boundary values shown in Table 1, and Figures 5–8 are solutions corresponding to the boundary conditions shown in Table 3.

| Parameters | 80:20 | 70:30 | 60:40 | 55:45 |
|------------|-------|-------|-------|-------|
| \( \langle z^{+2} \rangle \) (km\(^2\)s\(^{-2}\)) | 4000  | 3500  | 3000  | 2750  |
| \( \langle z^{-2} \rangle \) (km\(^2\)s\(^{-2}\)) | 800   | 700   | 600   | 550   |
| \( E_D^\infty \) (km\(^3\)s\(^{-2}\)) | -100  | -87.5 | -75   | -68.75 |
| \( L^+_\infty \) (km\(^3\)s\(^{-2}\)) | 8.25 \times 10^9 | 7.22 \times 10^9 | 6.19 \times 10^9 | 5.67 \times 10^9 |
| \( L^-_\infty \) (km\(^3\)s\(^{-2}\)) | 1.33 \times 10^9 | 1.16 \times 10^9 | 9.93 \times 10^8 | 9.09 \times 10^8 |
| \( L_D^\infty \) (km\(^3\)s\(^{-2}\)) | -2.11 \times 10^8 | -1.85 \times 10^8 | -1.58 \times 10^8 | -1.45 \times 10^8 |
| \( \langle z^{+2} \rangle \) (km\(^2\)s\(^{-2}\)) | 1000  | 1500  | 2000  | 2250  |
| \( \langle z^{-2} \rangle \) (km\(^2\)s\(^{-2}\)) | 200   | 300   | 400   | 450   |
| \( E_D^s \) (km\(^2\)s\(^{-2}\)) | -25   | -37.5 | -50   | -56.25 |
| \( L^+_s \) (km\(^3\)s\(^{-2}\)) | 7.4 \times 10^8 | 1.1 \times 10^9 | 1.47 \times 10^9 | 1.66 \times 10^9 |
| \( L^-_s \) (km\(^3\)s\(^{-2}\)) | 5.96 \times 10^8 | 8.93 \times 10^8 | 1.19 \times 10^9 | 1.33 \times 10^9 |
| \( L_D^s \) (km\(^3\)s\(^{-2}\)) | -5.27 \times 10^7 | -7.92 \times 10^7 | -1.06 \times 10^8 | -1.19 \times 10^8 |
| \( \langle \rho^{\infty,2} \rangle \) (cm\(^{-6}\)) | 1.4   | 1.4   | 1.4   | 1.4   |
| \( T \) (K) | 8 \times 10^4 | 8 \times 10^4 | 8 \times 10^4 | 8 \times 10^4 |

Table 1. Boundary values for 80:20, 70:30, 60:40, and 55:45 energy ratios between 2D and slab turbulence.

Figure 1 shows the comparison between the theoretical 2D and slab turbulent energies with observed values as a function of heliocentric distance. The observed turbulence energy corresponding to Voyager 2 observations is considered as being a superposition of 2D and slab energy [2]. The top-left plot of Figure 1 shows the energy in forward propagating modes \( \langle z^{+2} \rangle \), the top-right plot the energy in backward propagating modes \( \langle z^{-2} \rangle \), the middle-left the total turbulent energy \( E_T \), the middle-right the fluctuating magnetic energy \( \langle B^2 \rangle \), and the bottom-left the fluctuating kinetic
Table 2. Model parameters [2].

| Parameters | Values | Parameters | Values |
|------------|--------|------------|--------|
| $b$        | 0.26   | $f_D$      | 0.25   |
| $C^+$      | 0.9    | $\tau_{\text{ion}}^0$ | $10^6$ s |
| $C^-$      | 0.9    | $n_{\text{sw}}^0$    | 5 cm$^{-3}$ |
| $C_{sh}^+$ | 0.5    | $n_{\infty}^H$      | 0.1 cm$^{-3}$ |
| $U$        | 400 km s$^{-1}$ | $\eta_1$  | $1.8 \times 10^{-2}$ |
| $\Delta U$ | 200 km s$^{-1}$ | $\eta_2$  | $5 \times 10^{-5}$ |
| $V_{A0}$   | 40 km s$^{-1}$ | $\alpha$ | 0.2 |
| $r_0$      | 1 AU   |             |        |

energy $\langle u^2 \rangle$. The bottom-right plot shows a comparison of the theoretical energy ratio between the backward and forward propagating modes, i.e., $\langle z^{-2} \rangle/\langle z^{+2} \rangle$ (also represented as $R_{vA}^2$ (cal)) and the corresponding Ulysses observations $R_{vA}^2$ (obs) [38]. $R_{vA}^2$ (cal) is obtained from the observed Walén slope through an analytical expression, and the observed $R_{vA}^2$ is based on a direct decomposition of original Alfvenic fluctuations into outward- and inward-propagating Alfven waves [38, 41]. Walén slope describes to what extent the fluctuations are purely Alfvenic and can be related to the amplitude ratio of inward (sunward) to outward (antisunward) Alfven waves. In Figure 1, the solid curves illustrate 2D quantities, the dashed curves the slab quantities, the dashed-dotted-dashed the superposition of 2D and slab quantities, and the scatter plots the observed values. Furthermore, the red curves are the numerical solutions corresponding to the 80:20 energy ratio, the blue curves the 70:30, the green curves the 60:40, and the black curves the 55:45. Figure 1 shows that the 2D and slab energies are slightly different from each other for different boundary values, but nonetheless show similar trends with increasing heliocentric distance. As we change the energy ratios from 80:20 to 55:45, the 2D energies decrease slightly (solid red to black curves in Figure 1) and the slab energies increase slightly (dashed red to black curves in Figure 1) in the heliosphere. However, the total energy, i.e., the sum of 2D and slab energy (dashed-dotted-dashed curves), remains approximately the same in each case with increasing heliocentric distance.

Figure 1 also shows that the slab energies $\langle z^{+2} \rangle$, $E_T^s$, $\langle b^{+2} \rangle$, and $\langle u^{+2} \rangle$ decay faster than the 2D energies $\langle z^{\infty+2} \rangle$, $E_T^\infty$, $\langle B^{\infty+2} \rangle$, and $\langle u^{\infty+2} \rangle$ below $\sim 2 - 4$ au, similar to [1] and [2]. The slab energies start to increase beyond $\sim 4 - 5$ au, due to the presence of pickup ions driving turbulence in the outer heliosphere [1, 2]. The 2D and slab turbulence exhibit different levels of energy throughout the heliosphere, indicating that an anisotropy is present in the 2D and slab turbulence [1]. Furthermore, it shows that the backward propagating modes $\langle z^{+2} \rangle$ decreases initially unlike the forward propagating modes $\langle z^{+2} \rangle$. This result indicates that backward propagating modes are created by the reflection of forward modes in the presence of magnetic field, solar wind velocity, and solar wind density gradients.

In the bottom-right plot of Figure 1, the black triangles are taken from [37], the blue squares and red circles are taken from [38]. The comparison between the theoretical energy ratio and the observed energy ratio is quite reasonable. The energy ratio increases
towards \sim 1 \text{ with increasing heliocentric distance, most rapidly within } \sim 10 \text{ au, and more slowly beyond } \sim 10 \text{ au. Here, } \langle z^{-2} \rangle / \langle z^{+2} \rangle < 1 \text{ below } \sim 10 \text{ au, which indicates that the turbulence is imbalanced, whereas } \langle z^{-2} \rangle / \langle z^{+2} \rangle \sim 1 \text{ beyond } \sim 10 \text{ au, indicating that the turbulence is approximately balanced. We re-emphasize the point made in [1], which is that the balanced turbulence state is a consequence of turbulence driving by the creation of PU1s and not an intrinsic property of MHD turbulence.}

The top-left, top-right, and middle-left plots of Figure 2 show the comparison between the theoretical and observed normalized residual energy \( \sigma_D \), the normalized cross-helicity \( \sigma_c \), and the Alfvén ratio \( r_A \) as a function of heliocentric distance. Similarly, in these plots the solid red, blue, green, and black curves illustrate 2D quantities, and the corresponding dashed curves the slab quantities. As in Figure 1, the 2D and slab \( \sigma_D \), \( \sigma_c \) and \( r_A \) for each 2D:slab ratio evolve in a way that is almost similar with increasing heliocentric distance.

The middle-right plot of Figure 2 shows the comparison between the theoretical and observed variance of density fluctuations \( \langle \rho^2 \rangle \) with increasing heliocentric distance. The solid curves are the theoretical \( \langle \rho^2 \rangle \), the scatter “+” symbols are the observed \( \langle \rho^2 \rangle \), and the dashed red line illustrates the \( r^{-4} \) curve, where \( r \) is the heliocentric distance. Similarly, it also shows that the theoretical \( \langle \rho^2 \rangle \) evolves similarly throughout the heliosphere for each choice of 2D:slab ratio.

The bottom plot of Figure 2 shows the comparison between the theoretical and observed solar wind temperature \( T \) as a function of heliocentric distance. In the plot, the dashed red line is an \( r^{-4/3} \) curve, i.e. depicting an adiabatic profile of the solar wind temperature. Again, the temperature profiles in each case are approximately similar throughout the heliosphere, and are in good agreement with the observed solar wind temperature.

The top-left, top-right, and bottom plots of Figure 3 show the normalized 2D and slab correlation functions corresponding to forward and backward propagating modes and the residual energy as a function of heliocentric distance. In Figure 3, the solid and dashed curves illustrate the 2D and slab correlation functions, respectively. Similarly, the red, blue, green, and black curves are the numerical solutions corresponding to the 80:20, 70:30, 60:40, and 55:45 energy ratios, respectively.

Figure 4 shows the comparison between the theoretical and observed correlation length with increasing heliocentric distance. The top left and right plots of Figure 4 illustrate the correlation length corresponding to forward and backward propagating modes. The middle left and right plots show the correlation length for the residual energy and the variance of the velocity fluctuations. The bottom left and right plots depict the correlation length for the variance of the magnetic field fluctuations and the cross-correlation of the covariance of the velocity and magnetic field fluctuations. The blue triangles are the observed correlation lengths, and are scattered widely.

In this manuscript, we also solve the coupled equations (1)–(11) for a different set of boundary conditions as shown in Table 3. Here, the inner boundary values of the 2D and slab energies in the Table 1 and Table 3 are same, but the boundary values of the correlation functions are different. Use of different boundary values allows us to explore their effect on the evolution of turbulence in the heliosphere.

Figure 5 illustrates the comparison between the theoretical and observed turbulent
energies corresponding to the boundary values shown in Table 3. Figure 5 shows that the trends of the turbulence energies are similar to the observed values, and also these energies are approximately similar to the results shown in Figure 1. Figure 5 shows that the theoretical 2D energies are larger than the theoretical slab energies within \( \sim 10 \text{ au} \). However, beyond \( \sim 10 \text{ au} \), the 2D energy is larger than the slab energy only for the 80:20 ratio, while in the other cases the slab energies are larger than the 2D energies. In contrast, the theoretical 2D energies are larger than the theoretical slab energies throughout the heliosphere for the boundary conditions shown in Table 1. Moreover, in Figure 5 (bottom-right) the ratio of the backward and forward slab energies is closer to the \( R_{vA}^2 \) (cal) (red circles), whereas the ratio of the 2D backward and forward propagating modes energies (and also ratio of total backward and forward propagating energy (dashed-dotted-dashed curves)) is larger than the observed values. In Figure 1 (bottom-right), the ratio of 2D backward and forward propagating modes energies (and also the ratio of total backward and forward propagating energy) is closer to the [37] and [38] \( R_{vA}^2(\text{obs}) \) (blue circles) observed values.

Figure 6 shows the comparison between the theoretical and observed normalized residual energy and cross-helicity, the Alfvén ratio, the variance of the density fluctuations, and the solar wind temperature as a function of heliocentric distance. The theoretical results are approximately similar to the results shown in Figure 2. Figure 7 illustrates the normalized correlation functions as a function of heliocentric distance. Figure 8 shows the comparison between the theoretical and observed correlation lengths as a function of heliocentric distance. The correlation lengths shown in Figure 8 and Figure 4 have similar trends, although with different values.

### Table 3. Boundary values for 80:20, 70:30, 60:40, and 55:45 energy ratios between 2D and slab turbulence.

| Parameters | 80:20 | 70:30 | 60:40 | 55:45 |
|------------|-------|-------|-------|-------|
| \( \langle z^+ \rangle \) (km²s⁻²) | 4000 | 3500 | 3000 | 2750 |
| \( \langle z^- \rangle \) (km²s⁻²) | 800 | 700 | 600 | 550 |
| \( E_D^+ \) (km²s⁻²) | -100 | -87.5 | -75 | -68.75 |
| \( L^+ \) (km³s⁻²) | \( 2.95 \times 10^9 \) | \( 2.59 \times 10^9 \) | \( 2.22 \times 10^9 \) | \( 2.03 \times 10^9 \) |
| \( L^- \) (km³s⁻²) | \( 1.33 \times 10^9 \) | \( 1.16 \times 10^9 \) | \( 9.99 \times 10^8 \) | \( 9.15 \times 10^8 \) |
| \( L_D^+ \) (km³s⁻²) | \( -2.11 \times 10^8 \) | \( -1.85 \times 10^8 \) | \( -1.59 \times 10^8 \) | \( -1.46 \times 10^8 \) |
| \( E_D^- \) (km²s⁻²) | 200 | 300 | 400 | 450 |
| \( L_D^- \) (km³s⁻²) | -25 | -37.5 | -50 | -56.25 |
| \( L^0 \) (km³s⁻²) | \( 1.03 \times 10^9 \) | \( 1.54 \times 10^9 \) | \( 2.07 \times 10^9 \) | \( 2.34 \times 10^9 \) |
| \( L_D^0 \) (km³s⁻²) | \( 9.93 \times 10^8 \) | \( 1.49 \times 10^9 \) | \( 1.99 \times 10^9 \) | \( 2.25 \times 10^9 \) |
| \( \langle \rho^2 \rangle \) (cm⁻⁶) | 1.4 | 1.4 | 1.4 | 1.4 |
| \( T \) (K) | \( 8 \times 10^4 \) | \( 8 \times 10^4 \) | \( 8 \times 10^4 \) | \( 8 \times 10^4 \) |
4. Results: \( C_{s\parallel}^{\pm} \neq 0 \) \& \( C_{s\parallel}^{\pm} \neq 0 \)

In this section, we show numerical solutions of the coupled turbulence transport equations (1)–(11) setting \( C_{s\parallel}^{\pm} \neq 0 \) \& \( C_{s\parallel}^{\pm} \neq 0 \). Here, we use the boundary conditions shown in Table 4. We use the same parameter values shown in Table 2 except \( \eta_1 = 4.5 \times 10^{-2} \) and \( \eta_2 = 10^{-4} \). Furthermore, we use \( C_{s\parallel}^{\pm} = C_{s\parallel}^{\pm}/2.9 \) and \( C_{sh}^{\pm} = C_{sh}^{\pm}/2.9 \), where the values of \( C_{s\parallel}^{\pm} \) and \( C_{sh}^{\pm} \) are given in Table 2 i.e., we assume that the slab shear source terms are weaker than those for 2D turbulence.

The solutions of the coupled turbulence transport model equations are shown in Figures (9) to (12). The descriptions of Figures (9), (10), (11), and (12) are similar to Figures 1, 2, 3, and 4, respectively. Figures (9)–(12) show that by including the shear source, the slab forward and backward propagating energy \( \langle \sigma_{s\parallel}^{\pm} \rangle \) (top left & right of Figure 9), the total turbulent energy \( E_{T}^{\pm} \) (middle left of Figure 9), the fluctuating kinetic energy \( \langle u^{\pm} \rangle \) (bottom left of Figure 9), and the fluctuating magnetic energy \( \langle B^{\pm} \rangle \) (middle right of Figure 9) decay gradually with increasing heliocentric distance.

| Parameters | 80:20 | 70:30 | 60:40 | 55:45 |
|------------|-------|-------|-------|-------|
| \( \langle \sigma_{s\parallel}^{\pm} \rangle \) (km\(^2\)s\(^{-2}\)) | 1600 | 1400 | 1200 | 1100 |
| \( \langle \sigma_{s\parallel}^{-} \rangle \) (km\(^2\)s\(^{-2}\)) | 160 | 140 | 120 | 110 |
| \( E_{T}^{\pm} \) (km\(^3\)s\(^{-2}\)) | -80 | -70 | -60 | -55 |
| \( L_{+}^{\pm} \) (km\(^3\)s\(^{-2}\)) | 2.95 \times 10^9 | 2.58 \times 10^9 | 2.21 \times 10^9 | 2.03 \times 10^9 |
| \( L_{-}^{\pm} \) (km\(^3\)s\(^{-2}\)) | 2.65 \times 10^8 | 2.32 \times 10^8 | 1.99 \times 10^8 | 1.82 \times 10^8 |
| \( L_{D}^{\pm} \) (km\(^3\)s\(^{-2}\)) | -1.7 \times 10^8 | -1.49 \times 10^8 | -1.27 \times 10^8 | -1.16 \times 10^8 |
| \( \langle \sigma_{s\parallel}^{\pm} \rangle \) (km\(^3\)s\(^{-2}\)) | 400 | 600 | 800 | 900 |
| \( \langle \sigma_{s\parallel}^{-} \rangle \) (km\(^3\)s\(^{-2}\)) | 40 | 60 | 80 | 90 |
| \( E_{T}^{\pm} \) (km\(^3\)s\(^{-2}\)) | -20 | -30 | -40 | -45 |
| \( L_{+}^{\pm} \) (km\(^3\)s\(^{-2}\)) | 2.96 \times 10^8 | 4.44 \times 10^8 | 5.93 \times 10^8 | 6.67 \times 10^8 |
| \( L_{-}^{\pm} \) (km\(^3\)s\(^{-2}\)) | 1.66 \times 10^8 | 2.49 \times 10^8 | 3.33 \times 10^8 | 3.75 \times 10^8 |
| \( L_{D}^{\pm} \) (km\(^3\)s\(^{-2}\)) | -4.25 \times 10^7 | -6.38 \times 10^7 | -8.5 \times 10^7 | -9.57 \times 10^7 |
| \( \langle \rho_{\infty} \rangle \) (cm\(^{-3}\)) | 1.4 | 1.4 | 1.4 | 1.4 |
| \( T \) (K) | 8 \times 10^4 | 8 \times 10^4 | 8 \times 10^4 | 8 \times 10^4 |

Table 4. Boundary values for 80:20, 70:30, 60:40, and 55:45 energy ratios between 2D and slab turbulence.

The slab normalized residual energy \( \sigma_{\lambda}^{\pm} \) plot (top left of Figure 10) shows the slab fluctuating magnetic energy dominates within the ionization cavity and beyond (< 25 au) due to the presence of a shear source. Also, a shear source of slab turbulence is responsible for the normalized cross-helicity \( \sigma_{\lambda}^{\pm} \) (top right of Figure 10) decreasing from \( \sim 1 \) au, unlike in Section 3 where it decreases from beyond 2 au. The Alfven ratio \( r_{a}^{\pm} \) (middle left of Figure 10) is also different from Section 3 (middle left of figures 2 & 6) in that \( r_{a}^{\pm} \) now decreases towards smaller values initially i.e., the variance of the
slab magnetic field fluctuations dominates the variance of the slab solar wind velocity fluctuations.

The 2D correlation functions and the 2D correlation lengths (Figures 11 & 12, respectively) show similar trends to that of the previous results. Also, the slab correlation functions shown in Figure 11 look approximately similar to the previous results. However, the slab correlation lengths (dashed curves in Figure 12) are different from the previous results. A reason for this is that the shear sources appear only in the transport equations of the forward and backward propagating modes, and the residual energy, and not in the correlation functions. Figure 12 shows that the slab correlation lengths corresponding to different energy modes increase gradually as a function of heliocentric distance. In Figure 12, there are no peaks in the slab correlation lengths as in the previous results (Figures 4 & 8).

5. Discussion and Conclusions

We solved the coupled 1D steady-state turbulence transport equations (1)–(11) from 1 to 75 au for two cases; i) \( C^{*\pm}_{sh} = 0 \) \& \( C^{*E}_{sh} = 0 \), and ii) \( C^{*\pm}_{sh} \neq 0 \) \& \( C^{*E}_{sh} \neq 0 \), and compared the solutions to Voyager 2 and Ulysses measurements. In the first case, we solve the model equations (1)–(11) by including a shear source of turbulence in the 2D turbulence transport equations, and a pickup ion source of turbulence in the slab turbulence transport equations. In second case, we solve the equations (1)–(11) by including a shear source of 2D turbulence, and a shear and pickup ion source of slab turbulence. Here, the first case is similar to that solved by Zank et al [1] and Adhikari et al [2]. We computed several 2D and slab turbulent quantities: the energy in forward and backward propagating modes, the total turbulent energy, the normalized residual energy and cross-helicity, the correlation lengths and normalized correlation functions corresponding to forward and backward propagating modes and the residual energy, the fluctuating kinetic and magnetic energies, the solar wind temperature, and the variance of the density fluctuations for the boundary conditions shown in Table 1 and Table 3 for the first case, and Table 4 for the second case.

Zank et al 2017 [1] developed the coupled 2D and slab turbulence transport model equations, and presented preliminary results. Later, Adhikari et al [2] presented a detailed analysis of these model equations. Both papers used boundary values for the 2D and slab energies of the order of 80:20 [19, 20]. However, these ratios of the 2D and slab energies can be different in the fast and slow solar wind, at higher and lower latitudes, and in the ecliptic plane. Solar maximum and minimum conditions can also affect the value of the ratio. In this manuscript, we solved the model equations for four 2D to slab energy ratios, 80:20, 70:30, 60:40, and 55:45, and compared the solutions with Voyager 2 and Ulysses measurements. We found that the turbulence parameters in each case are different, however, the trends with heliocentric distance are similar. We found that the boundary values can change the turbulence energies in the outer heliosphere - in certain cases, the 2D energy is larger than the slab energy, and in others the slab energy can be larger than the 2D energy. In addition, the slab turbulent quantities in the presence of both sources of turbulence behave more reasonably than when only a pickup ion source of slab turbulence is included.

In summary, we find that the choice of energies in the 2D and slab components,
together with the correlation functions, at the inner boundary and the sources of turbulence are important in determining the evolution of turbulence throughout the heliosphere.

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Figure 1. Comparison between the theoretical results and the values observed by Voyager 2 (V2) and Ulysses (bottom-right) as a function of heliocentric distance. Top left and right: the energy in forward and backward propagating modes. Middle left and right: the total turbulent energy and the fluctuating magnetic energy. Bottom left and right: the fluctuating kinetic energy and the energy ratio between inward and outward propagating modes. The solid curves describe 2D quantities, the dashed curves the slab quantities, and the dashed-dotted-dashed the sum of the 2D and slab quantities. Red curves correspond to boundary 2D and slab energy ratios of 80:20, the blue curves 70:30, the green curves 60:40, and the black curves 55:45. The blue scatter “+” symbols are observed values taken from [2]. The black triangles are obtained from [37] and the blue squares and red circles are obtained from [38]. $R_{v,A}$ is the energy ratio of the backward and forward Alfvén waves.
Figure 2. Comparison between the theoretical results and observations as a function of heliocentric distance. Top left and right: the normalized residual energy and cross-helicity. Middle left and right: the Alfvén ratio and the variance of the density fluctuations. Bottom: the solar wind temperature. The red dashed curve of the middle right panel indicates a $r^{-4}$ curve, and the bottom plot $r^{-4/3}$, where $r$ is the heliocentric distance. The description of the curves in the figure is similar to Figure 1.
Figure 3. Normalized correlation functions as a function of heliocentric distance. Top-left, top-right and bottom plots are the correlation functions corresponding to forward and backward propagating modes, and the residual energy. $L_0$ is the normalization constant. Description of the curves in the figure is similar to Figure 1.
Figure 4. Comparison of the theoretical and observed correlation lengths as a function of heliocentric distance. Top left and right: correlation length corresponding to forward and backward propagating modes. Middle left and right: correlation length corresponding to the residual energy and the variance of velocity fluctuations. Bottom left and right: correlation length corresponding to the variance of magnetic field fluctuations and the cross-correlation between the co-variance between the velocity and the magnetic field fluctuations. The blue triangles are the observed correlation lengths. Description of curves in the figure is similar to Figure 1.
Figure 5. Comparison between the theoretical results derived from Table 3 and the observed values as a function of heliocentric distance. Description of the curves of the figure is similar to Figure 1.
Figure 6. Comparison between the theoretical results derived from Table 3 and the observed values as a function of heliocentric distance. Description of the curves of the figure is similar to Figure 2.
**Figure 7.** Normalized correlation functions derived from Table 3 as a function of heliocentric distance. Description of the figure is similar to Figure 3.
Figure 8. Comparison of the theoretical and observed correlation lengths derived from Table 3 with increasing heliocentric distance. Description of the figure is similar to Figure 4.
Figure 9. Comparison between the theoretical results derived from Table 4 and the observed values as a function of heliocentric distance. Description of the curves of the figure is similar to Figure 1.
Figure 10. Comparison between the theoretical results derived from Table 4 and the observed values as a function of heliocentric distance. Description of the curves of the figure is similar to Figure 2.
Figure 11. Normalized correlation functions derived from Table 4 as a function of heliocentric distance. Description of the figure is similar to Figure 3.
Figure 12. Comparison of the theoretical and observed correlation lengths derived from Table 4 with increasing heliocentric distance. Description of the figure is similar to Figure 4.