AN OPTIMAL CONTROL PROBLEM FOR SOME NONLINEAR ELLIPTIC EQUATIONS WITH UNBOUNDED COEFFICIENTS

GABRIELLA ZECCA

Dipartimento di Matematica e Applicazioni “R. Caccioppoli”
Università degli Studi di Napoli Federico II, Complesso Universitario Monte Sant’Angelo
Via Cintia - 80126 Napoli, Italy

Abstract. We study an optimal control problem associated to a Dirichlet boundary value problem of the type

\[(BVP) \quad \text{div} \left[ \beta(x) \nabla u(x) + \left( A \frac{x}{|x|^2} + g(x) \right) u(x) \right] = \text{div} F, \quad u \in W^{1,p}_0(\Omega), \]

where \( \Omega \) is a bounded regular domain of \( \mathbb{R}^N \), \( 0 \in \Omega \), \( \beta : \Omega \to \mathbb{R} \) is an unbounded function satisfying \( \beta(x) \geq \lambda_0 > 0 \) a.e., \( A \) is a suitably small constant, and \( g \in L^\infty(\Omega; \mathbb{R}^N) \).

We consider the vector field \( F \) as the control and the corresponding weak solution \( u \) to (BVP) as the state. Our aim is to find the optimal vector field \( F \in L^p(\Omega) \) so that the corresponding state \( u \in W^{1,p}_0(\Omega) \) is close to the desired profile in \( L^p(\Omega) \) while the norm of \( u \) in \( W^{1,p}(\Omega) \) is not too large.

We prove that, for every \( p \) less than 2 and suitably close to 2, (BVP) admits an unique weak solution and for such values of \( p \), we prove the existence of optimal pairs.

1. Introduction. In this paper we consider the Dirichlet problem for a class of elliptic operators with unbounded coefficients including as a particular case the following:

\[
\begin{aligned}
&\text{div} \left[ \beta(x) \nabla u(x) + \left( A \frac{x}{|x|^2} + g(x) \right) u(x) \right] = \text{div} F(x), \\
&u \in W^{1,p}_0(\Omega),
\end{aligned}
\]

where \( \Omega \) is a bounded regular domain in \( \mathbb{R}^N \), \( 0 \in \Omega \), \( \beta : \Omega \to \mathbb{R} \) is unbounded in \( \Omega \), \( \beta(x) \geq \lambda_0 > 0 \) a.e. \( x \in \Omega \), \( A \) is a suitable small positive constant and \( g \in L^\infty(\Omega; \mathbb{R}^N) \).

We regard \( F \in L^p(\Omega, \mathbb{R}^N) \) as a control variable and the corresponding solution \( u \in W^{1,p}_0(\Omega) \) to (1.1) as the corresponding state variable.

For a given profile \( z \in L^p(\Omega) \), we try to find a vector field \( F \in L^p(\Omega) \) so that the corresponding solution \( u \in W^{1,p}_0(\Omega) \) of Problem (1.1) is close to \( z \) in \( L^p(\Omega) \) and \( u \) itself is not too large in \( W^{1,p}_0(\Omega) \).

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Precisely, our aim in this paper is to study the following optimal control problem:

**Problem (OCP).** For a given $z \in L^p(\Omega)$,

$$\text{Minimize } \left\{ J(F) = \int_{\Omega} (|\Phi(F) - z|^p + (|\nabla \Phi(F)||\beta(x)|)^p) \, dx, \ F \in L^p(\Omega, \mathbb{R}^N) \right\}$$

(1.2)

where for every $F \in L^p(\Omega)$, $\Phi(F) = u$ denotes the distributional solution to

$$\text{div} \left[ \beta(x) \nabla u(x) + \left( A \frac{x}{|x|^2} + g(x) \right) u(x) \right] = \text{div} \, F,$$

$u \in W^{1,p}_0(\Omega),$

Optimal control problems for partial differential equations with unbounded coefficients is a classical problem initiated by Lions [31]. The range of such problems is very wide, including as well optimal shape design problems, some problems originating in mechanics and others.

Our model (1.1) also appears for example in the stationary diffusion-convection equation describing flows, or alternatively, describing a stochastically-changing system. The same or similar equation arises also in many contexts unrelated to flows through space. It is formally identical to the Fokker-Planck equation for the velocity of a particle and it is closely related to the Black-Scholes equation and other equations in financial mathematics.

Note that in our case we have a singular drift term of potential type. Typically, see for example [43, 39] such a term has an opposite sign.

Here in this paper we will consider optimal control problems where the corresponding Dirichlet (BVP) is well posed, referring to a forthcoming paper for cases where for some control $F$ the corresponding (BVP) admits many solutions or is unsolvable.

It is known that in general, for problems of type (1.1), the uniqueness of solutions fails in $W^{1,p}_0(\Omega)$ when $p$ is far from 2 (see the pathological example contained in [36]). Anyway, when $\beta$ is bounded, existence and uniqueness of such solutions have been obtained for $p$ sufficiently close to 2 in [37, 5, 12, 3, 33, 40, 14, 4, 19, 25]. Other extension of the optimal control problem for the elliptic system (1.1) can be found also in [41, 11].

In the first part of the paper we shall prove the following result, that could be of independent interest:

**Theorem 1.1.** Let $\Omega$ be a bounded regular domain in $\mathbb{R}^N$ containing the origin, $N > 2$. Let $\beta : \Omega \to \mathbb{R}$ be a bounded mean oscillation (BMO) function in $\Omega$ with $\beta(x) \geq \lambda_0 > 0$ a.e. $x \in \Omega$, and assume $g \in L^\infty(\Omega)$. There exist two positive constants $d = d(\|\beta\|_{\text{BMO}}, \lambda_0)$ and $\varepsilon_0 = \varepsilon_0(\|\beta\|_{\text{BMO}}, \lambda_0)$ such that if

$$A < d$$

then, for every $2 - \varepsilon_0 < p \leq 2$, and for every $F \in L^p(\Omega, \mathbb{R}^N)$, Problem (1.1) admits a unique distributional solution $u \in W^{1,p}_0(\Omega)$.

We point out that Theorem 1.1 also applies to more general nonlinear Leray-Lions type operators (See Theorem 2.1 below).

Note that in Theorem 1.1 we are dealing with problems that are in general not coercive. When the problem is coercive, distributional solutions of (1.1) when $\beta$ is a BMO function have been obtained in [8, 38, 16].
Several results for optimal control problems related to elliptic PDE’s with unbounded coefficients have been also obtained in [26, 21, 22, 27, 28, 9, 10] and the recent papers [20], [30] with the reference therein.

We observe that, by Theorem 1.1, the cost functional $J(F) \geq 0$ defined in (1.2) is finite for every $F \in L^r(\Omega)$, $p < r \leq 2$.

We establish the existence of an optimal pair for (OCP), proving the following main result:

**Theorem 1.2.** Under the same assumptions of Theorem 1.1, there exist two positive constants $d = d(\|\beta\|_{BMO}, \lambda_0)$ and $\varepsilon_0 = \varepsilon_0(\|\beta\|_{BMO}, \lambda_0)$ such that if $A < d$ then, for every $2 - \varepsilon_0 < p < 2$, Problem (OCP) admits a solution $F_0 \in L^p(\Omega)$.

The paper is organized as follows: the definitions and some basic known property of the involved function spaces are contained in Section 3. Sections 2, 4 and 5 are devoted to the elliptic problem. In particular, in Section 2 we address the assumptions and the statement of the existence and uniqueness result for the state equations. In Section 4 we obtain some preliminary lemmas and Section 5 contains the proof of Theorem 1.1.

Finally, Section 6 contains the proof of Theorem 1.2

2. **The state equation.** In this section we establish existence and uniqueness of distributional solutions of the following Dirichlet problem:

$$\begin{cases}
    \text{div} \ (A(x, \nabla u) + B(x, u)) = \text{div} \ F & \text{in } \Omega \\
    u \in W_0^{1,p}(\Omega)
\end{cases}$$

(2.1)

when $1 < p \leq 2$ and $\Omega \subset \mathbb{R}^N$ is a bounded regular domain, $N > 2$. We refer the reader to Section 3 for definitions of the involved functional spaces.

We assume that $A : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function i.e.

$$x \to A(x, \xi) \text{ is measurable for any } \xi \in \mathbb{R}^N;$$

(2.2)

$$\xi \to A(x, \xi) \text{ is continuous for almost every } x \in \Omega.$$  

(2.3)

We also assume that there exists a real function $\beta(x) \geq \lambda_0 > 0$ belonging to the space of functions with bounded mean oscillation, $\beta \in BMO(\Omega)$, such that for almost every $x \in \Omega$,

$$|A(x, \xi) - A(x, \eta)| \leq \alpha(\beta(x))|\xi - \eta|$$

(2.4)

$$\beta(x)|\xi - \eta|^2 \leq \langle A(x, \xi) - A(x, \eta), \xi - \eta \rangle$$

(2.5)

$$A(x, 0) = 0$$

(2.6)

for any vectors $\xi$ and $\eta$ in $\mathbb{R}^N$, where $\alpha \geq 1$ is a constant. Moreover, we assume that $B : \Omega \times \mathbb{R} \to \mathbb{R}^N$ is a Carathéodory function verifying the following two properties:

i) There exists a non negative function $E : \Omega \to \mathbb{R}_+$ in the *Weak-L^N space* (or *Marcinkiewicz space*) $E \in L^{N, \infty}(\Omega)$ such that

$$|B(x, s) - B(x, t)| \leq E(x)|s - t|,$$

(2.7)

for a.e. $x \in \Omega$ and for every $s, t \in \mathbb{R}$.

ii) $B(x, 0) = 0$, for almost all $x \in \Omega$.  

(2.8)
For every $F \in L^p(\Omega, \mathbb{R}^N)$, $p > 1$, we consider distributional solutions to Problem (2.1), i.e. Sobolev functions $u \in W^{1,p}_0(\Omega)$ verifying the equality

\[ \int_\Omega \langle A(x, \nabla u) + B(x, u), \nabla \varphi \rangle \, dx = \int_\Omega \langle F, \nabla \varphi \rangle \, dx \quad (2.9) \]

for every test functions $\varphi \in C_0^\infty(\Omega)$.

Note that our assumption $\beta \in BMO(\Omega)$ ensure, using (2.4), (2.6) and Lemma 3.1 below, that for $u \in W^{1,p}_0(\Omega)$ we have

\[ \int_\Omega \langle A(x, \nabla u), \nabla \varphi \rangle \, dx \leq \|\beta\|_{L^p(\Omega)} \|\nabla u\|_{L^p(\Omega)} \|\nabla \varphi\|_{L^\infty(\Omega)} < \infty \quad \forall \varphi \in C_0^\infty(\Omega), \]

$p' = \frac{p}{p-1}$.

On the lower order term, we point out that our assumption $E \in L^{N,\infty}(\Omega)$ is natural and essentially optimal in order to have that the left hand side of (2.9) be absolutely convergent. In fact, in our assumptions, if $u \in W^{1,p}_0(\Omega)$, using (3.1), (3.2) and the Sobolev embedding in Lorenz spaces $W^{1,p}_0(\Omega) \subset L^{p^*}(\Omega), N > p$ and $p^* = \frac{Np}{N-p}$ (see Theorem 3.1 below) we have:

\[ \|B(x, u)\|_{L^p} \leq \|E\|_{N,\infty} \|u\|_{L^p} \leq \|E\|_{N,\infty} \|u\|_{p^*, s} \leq S_p \|E\|_{N,\infty} \|\nabla u\|_p. \]

When $E \equiv 0$, in [42] a detailed consideration of the question about the boundedness of bilinear forms corresponding to second order divergence elliptic problems with BMO coefficients is provided.

We note that the only assumption $E \in L^{N,\infty}(\Omega)$ does not guarantee the existence of a solution to (2.1) also when $\beta$ is bounded (see the example contained in [19]). Since $L^{\infty}(\Omega)$ is not dense in $L^{N,\infty}(\Omega)$, it is meaningful to consider the distance with respect to the $L^{N,\infty}$ norm of a function $E \in L^{N,\infty}(\Omega)$ to $L^{\infty}(\Omega)$.

We obtain the following:

**Theorem 2.1.** Let assumptions (2.2)–(2.8) be verified. There exist two positive constants $d = d(\alpha, \|\beta\|_{BMO}, N, \lambda_0)$ and $\varepsilon_0 = \varepsilon_0(\alpha, \|\beta\|_{BMO}, N, \lambda_0)$ such that if

\[ dist_{L^{N,\infty}(\Omega)}(E, L^\infty) < d \quad (2.10) \]

then, for every $2 - \varepsilon_0 < p \leq 2$, and every $F \in L^p(\Omega, \mathbb{R}^N)$, Problem (2.1) admits a solution $u \in W^{1,p}_0(\Omega)$, and such solution is unique.

The proof of Theorem 2.1 is addressed in Section 5.

The distance to $L^\infty$ in some function spaces has been studied in [7]. When $\beta \in L^\infty$ assumption (2.10) has been considered in [13] for the linear case $A(x, \nabla u) = A(x) \nabla u$, and in [17, 18, 19] for the nonlinear case.

Note that our existence result applies for example in case where $E(x) = (\frac{A(x)}{\|A\|_1} + |g(x)|$ (and $0 \in \Omega$) when $A$ is positive and sufficiently small and $g \in L^{\infty}(\Omega; \mathbb{R}^N)$; in this case in fact (see Example 1 for details), by computation one has $dist_{L^{N,\infty}(\Omega)}(E, L^\infty) = A\omega_N^{\frac{1}{N}}$, where $\omega_N$ denotes the volume of the unit ball of $\mathbb{R}^N$. Then we recover problems considered in [4] when $\beta \in L^\infty(\Omega)$. Our result also applies when $E$ belongs to the Lorentz space $L^{N,q}(\Omega)$ with $1 \leq q < +\infty$, without any assumption on the smallness of the norm of $E$. In this case in fact, since $L^{\infty}(\Omega)$ is dense in $L^{N,q}(\Omega)$ then $dist_{L^{N,\infty}(\Omega)}(E, L^\infty) = 0$. We are also related to [35] where the case $p \geq 2$ is treated.
3. Some functional space. In this section we recall the definitions of some known function spaces. We refer the reader to [2] for more details.

Assume $1 < p, q < +\infty$ and let $\Omega \subset \mathbb{R}^N$ be a bounded regular domain. The Lorentz space $L^{p,q}(\Omega)$ consists of all measurable function $g$ defined on $\Omega$ for which

$$\|g\|_{p,q}^q = p \int_0^{+\infty} |\Omega_t|^q t^{q-1} dt < \infty$$

where $\Omega_t = \{ x \in \Omega : |g(x)| > t \}$ and $|\Omega_t|$ is the Lebesgue measure of $\Omega_t$. Note that $\| \cdot \|_{p,q}$ is equivalent to a norm and $L^{p,q}$ becomes a Banach space when endowed with it (see [34]). For $p = q$, the Lorentz space $L^{p,p}$ reduces to the standard Lebesgue space $L^p$. For $q = \infty$, the class $L^{p,\infty}$ consists of all measurable functions $g$ defined on $\Omega$ such that

$$\|g\|_{p,\infty}^p = \sup_{t > 0} t^p |\Omega_t| < +\infty,$$

and it coincides with the Marcinkiewicz class, weak-$L^p$.

Here below we just recall some properties that will be useful in the sequel:

i) Whenever $1 < q < p < r \leq \infty$, the following inclusions hold:

$$L^r(\Omega) \subset L^{p,q}(\Omega) \subset L^{p,p}(\Omega) = L^p(\Omega) \subset L^{p,r}(\Omega) \subset L^{p,\infty}(\Omega) \subset L^q(\Omega).$$

ii) for $1 < p, s < +\infty$ and $1 < q < s < \infty$ it holds

$$\|u||_{p,q}^q = \|u||_{sp,sq}^s.$$

iii) (Hölder inequality in Lorentz spaces) Assume $1 < p < \infty$, $1 \leq q < \infty$ and $\frac{1}{p} + \frac{1}{r} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$ (for $q = \infty$ we assume $q' = 1$). If $f \in L^{p,q}(\Omega)$ and $g \in L^{p',q'}(\Omega)$, then

$$\int_{\Omega} |f(x)g(x)| dx \leq \|f||_{p,q}\|g||_{p',q'}. \quad (3.2)$$

The following Sobolev embedding theorem in Lorentz spaces holds true (see [34], [1], [15])

**Theorem 3.1.** Let $1 < p < N$ and $1 \leq q \leq p$. Then, every function $g \in W^{1,1}_0(\Omega)$ verifying $|\nabla g| \in L^{p,q}$ actually belongs to $L^{p',q'}$, where $p' = \frac{Np}{N-p}$ and

$$\|g||_{p',q'} \leq S_p\|\nabla g||_{p,q}$$

where $S_p = C(N)\frac{p}{N-p}$.

We remark that $L^\infty(\Omega)$ is not dense in $L^{p,\infty}(\Omega)$, $p \in [1, \infty]$. We define the distance of a given function $f \in L^{p,\infty}(\Omega)$ to $L^\infty(\Omega)$ as usual assuming

$$\text{dist}_{L^{p,\infty}}(f, L^\infty) = \inf_{g \in L^\infty} \|f - g||_{p,\infty}.$$

Note that, since $\| \cdot \|_{p,\infty}$ is not a norm, $\text{dist}_{L^{p,\infty}}$ is just equivalent to a metric.

For $\rho > 0$, we consider the truncation operator

$$T_\rho(y) = \frac{y}{|y|} \min\{|y|, \rho\}.$$

To have a formula for the distance we note that by [7] we have

$$\text{dist}_{L^{p,\infty}}(f, L^\infty) = \lim_{\rho \to \infty} \|f - T_\rho f||_{p,\infty}. \quad (3.3)$$

Indeed, $\forall g \in L^\infty, \forall \rho \geq \|g\|_{\infty}$, we have for a.e. $x \in \Omega$,

$$|f(x) - g(x)| \geq |f(x) - T_\rho f(x)|$$
Example 1. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain containing the origin and let $q \in ]1, \infty[$. The function

$$E(x) = A|x|^{-N/q} \in L^{q, \infty}(\Omega) \quad (\text{and } E \notin L^{q}(\Omega))$$

verifies

$$\text{dist}_{L^{q, \infty}}(E, L^{\infty}) = A\omega N^{1/q}$$

where $\omega N$ denotes the measure of the unit ball of $\mathbb{R}^N$. In fact, by easy computations one has

$$(E - T_\rho E)(x) = \begin{cases} A|x|^{-N/q} - \rho, & |x| \leq \left(\frac{A}{\rho}\right)^{\frac{q}{q-1}} \\ 0, & |x| > \left(\frac{A}{\rho}\right)^{\frac{q}{q-1}} \end{cases}$$

and then, for $\lambda > 0$

$$|\Omega_\lambda| = |\{x \in \Omega : (E - T_\rho E)(x) > \lambda\}| = \omega N A^q(\lambda + \rho)^{-q},$$

$\rho$ sufficiently large. Hence

$$\text{dist}_{L^{q, \infty}}(E, L^{\infty}) = \lim_{\rho \to +\infty} \sup_{\lambda > 0} \lambda |\Omega_\lambda|^\frac{1}{q} = A\omega N^{1/q}$$

For other comments on the distance to $L^{\infty}$ and some applications we refer to [7].

A function $\beta \in L^1(\Omega)$ is a function with bounded mean oscillation on $\Omega$, (and we will write $\beta \in BMO(\Omega)$), whenever

$$\|\beta\|_{BMO(\Omega)} = \sup_{Q} \frac{1}{|Q|} \int_Q |\beta(x) - \beta_Q| \, dx < \infty$$

where the supremum is taken over all cubes $Q \subset \Omega$ with sides which are parallel to the coordinate axis.

Lemma 3.1. Let $\Omega$ be a bounded domain of $\mathbb{R}^N$ and let $\beta \geq 0$ be a function in $BMO(\Omega)$. Then, for every $1 < q < \infty$,

$$\int_\Omega |\beta(x)|^q \, dx \leq q|\Omega|^\frac{q}{q-1} C(\beta, \Omega).$$

Proof. The proof is an easy consequence of the elementary inequality $\beta^p e^{-\beta} \leq \frac{\beta^p}{p} e^{-p}$, $\forall \beta \geq 0$. $\square$

For basic properties of the classical $BMO$ space we refer to [24].

Assume now $\beta : \Omega \to [0, \infty)$ be a locally integrable function such that $\beta(x) > 0$ for a.e. $x \in \Omega$, i.e. a weight. We define the weighted Lebesgue Space $L^p(\beta, \Omega)$, $1 \leq p < \infty$, as the set

$$L^p(\beta, \Omega) = \{f : \Omega \to \mathbb{R} \text{ measurable : } \int_\Omega |f(x)|^p \beta(x) \, dx < \infty\}.$$

$L^p(\beta, \Omega)$ becomes a Banach space when equipped with the following norm:

$$\|f\|_{p, \beta} = \left(\int_\Omega |f(x)|^p \beta(x) \, dx\right)^\frac{1}{p}.$$

We denote by $W^{1,p}(\beta, \Omega)$, $1 \leq p < \infty$ the weighted Sobolev space consisting of all functions $f \in L^p(\beta, \Omega)$ with weak derivatives, and satisfying

$$\int_\Omega |\nabla f(x)|^p \beta(x) \, dx < \infty.$$
The weighted space $W^{1,p}(\beta, \Omega)$, $1 \leq p < \infty$ is a Banach space if equipped with the norm
\[ \|f\|_{W^{1,p}(\beta, \Omega)} = \|f\|_{p,\beta} + \|\nabla f\|_{\beta,\Omega} \] (3.4)
If $1 < p < \infty$ and $\beta = \frac{1}{p} - \frac{1}{2}$, then $C_0^\infty(\Omega)$ is a subset of $W^{1,p}(\beta, \Omega)$, and we can introduce the space $W^{1,p}_0(\beta, \Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm (3.4). For more details see [29].

4. Preliminaries. This section contains some preliminary result that will be useful in the sequel.

It is well known that using the Hodge decomposition, any vector field $F \in L^p(\Omega, \mathbb{R}^N)$ can be written uniquely as
\[ F = \nabla \psi + H \]
where $\psi \in W^{1,p}(\Omega)$ and $H$ is a divergence free vector in $L^p(\Omega)$.

The following stability property of the Hodge decomposition holds (see [8], Lemma 2). It is a refined version of a result obtained in [23].

Theorem 4.1 (Stability of the Hodge decomposition [8, 23]). Let $\Omega$ be a bounded regular domain of $\mathbb{R}^N$ and consider $\beta \in BMO$ such that $\frac{1}{p} \in BMO$. If $w \in W^{1,2-\varepsilon}(\beta, \Omega)$, $1 < q < \infty$ and $-1 < 2\varepsilon < q - 1$, then there exists $\psi \in W^{1,\frac{q}{q-\varepsilon}}(\beta, \Omega)$ and a divergence free vector field $H \in L^{\frac{q}{q-\varepsilon}}(\beta, \Omega)$ such that
\[ |\nabla w|^{-\frac{\varepsilon}{q}} \nabla w = \nabla \psi + H \]
and
\[ \|\nabla \psi\|_{L^{\frac{q}{q-\varepsilon}}(\beta, \Omega)} \leq K \|\nabla w\|_{L^{\frac{q}{q-\varepsilon}}(\beta, \Omega)}^{1-\frac{\varepsilon}{q}} \]
\[ \|H\|_{L^{\frac{q}{q-\varepsilon}}(\beta, \Omega)} \leq K \varepsilon \|\nabla w\|_{L^{\frac{q}{q-\varepsilon}}(\beta, \Omega)}^{1-\frac{\varepsilon}{q}} \]
where
\[ K = K(N, q, \|\beta\|_{BMO}) = c(N, q)(1 + \|\beta\|_{BMO})^{\gamma(q)} \]
Here $\gamma(q)$ is an exponent depending only on $q$.

Proposition 4.1. Assume (2.2) — (2.8) and $\beta \in L^\infty(\Omega)$. Then, there exist three positive constants $\varepsilon_0$, $d$, and $C$, depending only on $N, \alpha, \|\beta\|_{BMO}$ and $\lambda_0$, with $\varepsilon_0 < \frac{\lambda_0}{\pi + \Omega}$, $\varepsilon_0$ and $d$ decreasing and $c$ increasing with respect to the BMO-norm of $\beta$, satisfying the following property:

If
\[ \|E\|_{L^{N,\infty}} < \frac{d}{q} \]
then for every $2 - \varepsilon_0 < p \leq 2$ and for $F, \mathcal{G} \in L^p(\Omega; \mathbb{R}^N)$, each of the two problems
\[ \left\{ \begin{array}{l} \text{div} \ [A(x, \nabla u) + B(x, u)] = \text{div} \ F \quad \text{in } \Omega \\ u \in W^{1,p}_0(\Omega) \end{array} \right. \]
(4.1)
and
\[ \left\{ \begin{array}{l} \text{div} \ [A(x, \nabla v) + B(x, v)] = \text{div} \ \mathcal{G} \\ v \in W^{1,p}_0(\Omega) \end{array} \right. \]
(4.2)
admits a unique solution, and the following continuity estimate holds true
\[ \|\nabla u - \nabla v\|_p \leq \varepsilon \|\mathcal{F} - \mathcal{G}\|_p . \]
(4.3)
Moreover, the following regularity result for the difference of such solutions $(u - v)$ holds:
\[ \forall q \in (p, 2], \quad (\mathcal{F} - \mathcal{G}) \in L^q(\Omega) \implies (\nabla u - \nabla v) \in L^q(\Omega). \]
(4.4)
Proof. The proof is an easy consequence of [8], Proposition 4.1 due to M. Carozza, G. Moscariello and A. Passarelli di Napoli (see also [12] and [5]). It can be obtained following line by line Proposition 2.1 of [19]. For the convenience of the reader we give here some details.

Assume
\[ \varepsilon_0 = \min \left\{ \frac{4}{N + 2} \frac{1}{\alpha c(N)(1 + 2\|B_{BMO}\|)\gamma} \right\} \] (4.5)
with \( c(N) \) and \( \gamma \) appearing in Theorem 4.1 with \( q = 2 \). Let \( 2 - \varepsilon_0 < p \leq 2 \) and \( \mathcal{F}, \mathcal{G} \in L^p(\Omega) \).

Let \( u, v \) be solutions to Problems (4.1) and (4.2) respectively. To prove (4.3), we note that by (2.7), (2.8) and Theorem 3.1 we have that \( [\mathcal{F} - B(x, u)] \) and \( [\mathcal{G} - B(x, v)] \) are in \( L^p(\Omega; \mathbb{R}^N) \). Hence, by Proposition 4.1 in [8], rewriting our problems as
\[
\begin{align*}
\begin{cases}
\text{div } A(x, \nabla u) = \text{div } [\mathcal{F} - B(x, u)] & \text{in } \Omega \\
u \in W^{1, p}_0(\Omega)
\end{cases}
\end{align*}
\]
and
\[
\begin{align*}
\begin{cases}
\text{div } A(x, \nabla v) = \text{div } [\mathcal{G} - B(x, v)] & \text{in } \Omega \\
v \in W^{1, p}_0(\Omega)
\end{cases}
\end{align*}
\]
we have that
\[ \|\nabla u - \nabla v\|_p \leq c(\|\mathcal{F} - \mathcal{G}\|_p + \|E\|_p u - v\|_p) \leq c(\|\mathcal{F} - \mathcal{G}\|_p + \|E\|_{N, \infty} u - v\|_{p^*, p}) \]
with \( c = c(N, \alpha, \|\beta\|_{BMO}, \lambda_0) \), and \( c \) increasing with respect to the BMO-norm of \( \beta \).

Using again Theorem 3.1, for \( d = (cS_{2,N})^{-1} \) we obtain Inequality (4.3) and the uniqueness of the solutions of Problems (4.1) and (4.2).

The existence of solutions for \( 2 - \varepsilon_0 < p \leq 2 \) easily follows by standard approximation argument, considering a sequence of \( \mathcal{F}_j \in L^2(\Omega) \) converging to \( \mathcal{F} \) in \( L^p \) and using the continuity estimate (4.3). The proof of (4.4) is contained in [19] Lemma 2.1 and we omit it.

5. Proof of Theorem 2.1. This section is devoted to the proof of Theorem 2.1. We shall divide it into three principal steps. In the first step we obtain existence of solutions to Problem (2.1) under the additional assumption \( \beta \in L^\infty(\Omega) \). Here a crucial result for our aims is an estimate of the gradient of solutions in terms of quantities depending on the \( BMO \)-norm of \( \beta \) and not on its \( L^\infty \)-norm (see estimate (5.3)). In the second step, we remove the assumption \( \beta \in L^\infty(\Omega) \) using an approximation argument: precisely, we construct a sequence of problems considering “suitable truncated operators” \( A_n \) of \( A \) (see (5.11) and (5.16)) and then we use the a priori estimate obtained in the first step of the proof. Last step 3 is devoted to prove the uniqueness of solutions.

Proof of Theorem 2.1. Let \( d = d(\|\beta\|_{BMO}, \lambda_0, \alpha, N) \) and \( \varepsilon_0 = \varepsilon_0(\|\beta\|_{BMO}, \lambda_0, \alpha, N) \) be the positive constants obtained in Proposition 4.1. Assume that
\[ \text{dist}_{L^{N, \infty}(\Omega)}(E, L^\infty(\Omega)) < d. \]

By formula (3.3) we can fix a constant \( \rho = \rho(\alpha, \|\beta\|_{BMO}, N, \lambda_0) \geq 1 \) such that the truncated function \( T_\rho E \) at level \( \rho \) satisfies
\[ \|E - T_\rho E\|_{N, \infty} < d \] (5.1)
Here and below, for such fixed value of $\rho$ we denote
\[
\vartheta(x) = \begin{cases} \frac{T_\rho E(x)}{E(x)} & \text{if } E(x) \neq 0 \\ 1 & \text{if } E(x) = 0. \end{cases} (5.2)
\]

Let now assume $2 - \varepsilon_0 < p \leq 2$, and $F \in L^p(\Omega, \mathbb{R}^N)$.

**Step 1.** Assume $\beta \in L^\infty(\Omega)$.

We prove that, in this case, Problem (2.1) admits a solution $u \in W^{1,p}_0(\Omega)$ and the estimate
\[
\|\nabla u\|_p \leq C(\|F\|_p + k_0 |\Omega|^{\frac{1}{2}}) \tag{5.3}
\]
holds with $C$ and $k_0$ positive constants depending only on $\alpha, N, \|\beta\|_{BMO}, \|F\|_p, \lambda_0$, and increasing with respect to $\|\beta\|_{BMO}$.

To prove this, we shall use a fixed point theorem, exactly as in Theorem 1.1 of [19], where the case $\beta \in L^\infty(\Omega)$ is treated. We sketch here the proof, just to convince the reader on the independence of estimate (5.3) on the $L^\infty$-norm of $\beta$.

We refer the reader to [19] for the details.

By Proposition 4.1, for every given $\bar{u} \in W^{1,p}_0(\Omega)$, the following auxiliary problem:
\[
\begin{align*}
\text{div} \left[ A(x, \nabla u) + (1 - \theta(x))B(x, u) \right] &= \text{div} \left[ F - \theta(x)B(x, \bar{u}) \right] \\
u \in W^{1,p}_0(\Omega) \tag{5.4}
\end{align*}
\]
admits a unique solution $u \in W^{1,p}_0(\Omega)$. In fact, by (5.1) and (5.2)
\[
(1 - \theta)E_{N,\infty} < d \quad \text{and} \quad [F - \theta(\cdot)B(\cdot, \bar{u})] \in L^p(\Omega; \mathbb{R}^N).
\]

Then, we can define the operator
\[
\mathcal{F}: W^{1,p}_0(\Omega) \to W^{1,p}_0(\Omega)
\]
assuming that, $\forall \bar{u} \in W^{1,p}_0(\Omega)$, $u := \mathcal{F}(\bar{u})$ is the unique solution to Problem (5.4).

A fixed point of $\mathcal{F}$ is a solution to Problem (2.1). Note that $\mathcal{F}$ is a continuous and compact operator on $W^{1,p}_0(\Omega)$: this can be easily seen by using Estimate (4.3) of Proposition 4.1.

To prove the existence of a fixed point of $\mathcal{F}$ we use the following:

**Leray-Shauder Fixed Point Theorem:** $\mathcal{F}$ has a fixed point if there exists a constant $K > 1$ such that the a priori estimate $\|\nabla u\|_p < K$ holds for every $u \in W^{1,p}_0(\Omega)$ and $t \in [0, 1]$ satisfying $u - t\mathcal{F}(u) = 0$.

To find such constant $K$, we consider couple of constants $s, t \in (0, 1]$, and the corresponding couple of solutions $u, v \in W^{1,p}_0(\Omega)$ of the equations $v = s\mathcal{F}(v)$ and $u = t\mathcal{F}(u)$, i.e. we consider $u, v \in W^{1,p}_0(\Omega)$ solutions in $\Omega$ of
\[
\begin{align*}
\text{div} \left( A(x, \frac{\nabla u}{t}) + (1 - \theta(x))B(x, \frac{u}{t}) \right) &= \text{div} \left[ F - \theta(x)B(x, \bar{u}) \right] \\
u \in W^{1,p}_0(\Omega)
\end{align*}
\]
and
\[
\begin{align*}
\text{div} \left( A(x, \frac{\nabla v}{s}) + (1 - \theta(x))B(x, \frac{v}{s}) \right) &= \text{div} \left[ F - \theta(x)B(x, \bar{u}) \right] \\
u \in W^{1,p}_0(\Omega)
\end{align*}
\]
respectively. Our strategy is to consider the obvious inequality
\[
\|\nabla u\|_p \leq \left\| \frac{\nabla v}{s} \right\|_p + \left\| \frac{\nabla u}{t} - \frac{\nabla v}{s} \right\|_p
\]
We shall obtain estimates for the difference \( \frac{u}{t} - \frac{v}{s} \). Then, we will fix the constant \( s \) and the corresponding solution \( v \) in a suitable way.

**Claim 1.** For every \( s, t \in (0, 1] \), and \( u, v \in W^{1,p}_0(\Omega) \) solutions of the equations \( v = s\mathcal{F}(v) \) and \( u = t\mathcal{F}(u) \) we have

\[
\left\| \frac{\nabla v}{s} - \frac{\nabla u}{t} \right\|_2 \leq cp \left( |\Omega_k|^\frac{1}{2} \left[ \left\| \frac{\nabla v}{s} - \frac{\nabla u}{t} \right\|_2 + \left\| \frac{v}{s} \right\|_2 + k|\Omega|^\frac{1}{2} \right] \right) \tag{5.5}
\]

with \( c = c(N, \lambda_0, \Omega) \), where for every \( k > 0 \), by \( \Omega_k \) we denote the superlevel set

\[
\Omega_k = \left\{ x \in \Omega : \frac{u}{t} - \frac{v}{s} > k \right\}. \tag{5.6}
\]

**Proof of Claim 1.** Let \( s, t, u, v \) be as before. Using (2.7) and the Sobolev embedding theorem, we observe that \( \theta(x) [B(x, u) - B(x, v)] \) belongs to \( L^{p^*} (\Omega) \), with \( p^* = \frac{Np}{N-p} \geq 2 \). Then by (4.4) of Proposition 4.1, \( \left( \frac{u}{t} - \frac{v}{s} \right) \in W^{1,2}_0(\Omega) \) and so, from (2.4) and (2.7) we deduce that

\[
\left\{ A \left( x, \frac{\nabla u}{t} \right) - A \left( x, \frac{\nabla v}{s} \right) + (1 - \theta(x)) \left[ B(x, \frac{u}{t}) - B(x, \frac{v}{s}) \right], \nabla \varphi \right\} \in L^2(\Omega, \mathbb{R}^N)
\]

This allows us to use test functions in \( W^{1,2}_0(\Omega) \) when we consider the difference of previous equations, i.e.

\[
\int_{\Omega} \left( A \left( x, \frac{\nabla u}{t} \right) - A \left( x, \frac{\nabla v}{s} \right) + (1 - \theta(x)) \left[ B(x, \frac{u}{t}) - B(x, \frac{v}{s}) \right], \nabla \varphi \right) dx
\]

\[
= \int_{\Omega} \theta(x) [B(x, v) - B(x, u)], \nabla \varphi \right) dx \quad \forall \varphi \in W^{1,2}_0(\Omega)
\]

We then use \( \varphi = \left( \frac{u}{t} - \frac{v}{s} \right) \in W^{1,2}_0(\Omega) \) as test function in (5.19), and we arrive to Estimate (5.5).

**Claim 2.** For every \( k \in \mathbb{N} \) the following estimate for the superlevel set \( |\Omega_k| \) defined in (5.6) holds true:

\[
|\Omega_k|^\frac{1}{2} \leq \frac{cp}{\log(1 + k)} \left( \left\| \frac{v}{s} \right\|_2 + |\Omega|^\frac{1}{2} \right). \tag{5.7}
\]

with \( c \) depending only on \( \Omega, \lambda_0 \) and \( N \).

**Proof of Claim 2.** Following the technique introduced in [3], we use

\[
\varphi := \frac{\frac{u}{t} - \frac{v}{s}}{1 + \frac{u}{t} - \frac{v}{s}} \in W^{1,2}_0(\Omega)
\]
as test function in (5.19), to have:

\[
\left\| \frac{\nabla \left( \frac{u}{t} - \frac{v}{s} \right)}{1 + \frac{u}{t} - \frac{v}{s}} \right\|_2 \leq cp \left( |\Omega|^\frac{1}{2} + \left\| \frac{v}{s} \right\|_2 \right),
\]

with \( c = c(N, \lambda_0, \Omega) \). This and the Sobolev embedding theorem imply

\[
\left\| \log \left( 1 + \frac{u}{t} - \frac{v}{s} \right) \right\|_2 \leq cp \left( \left\| \frac{v}{s} \right\|_2 + |\Omega|^\frac{1}{2} \right),
\]

from which we deduce the desired estimate (5.7).

At this point we fix the value of \( s \in (0, 1) \) and the function \( v \) as follows: we remark that, by Proposition 4.1, assumption (2.8) and Poincaré inequality, for \( 0 <
s \ll 1$, the operator $sF: W^{1,p}_0(\Omega) \to W^{1,p}_0(\Omega)$ is a contraction. Indeed, for every $\bar{u}, \bar{v} \in W^{1,p}_0(\Omega)$,

$$\|\nabla(sF(\bar{u}) - sF(\bar{v}))\|_p \leq sC\rho\|\bar{u} - \bar{v}\|_p \leq sC\rho\|\nabla\bar{u} - \nabla\bar{v}\|_p.$$  

Hence, for $0 < s \ll 1$ there exists a unique $v \in W^{1,p}_0(\Omega)$ solution to the equation $v = sF(v)$. Moreover, using again Proposition 4.1, (2.6) and (2.8), for $0 < s \ll 1$ sufficiently small and $s$ decreasing with respect to $\|\beta\|_{BMO}$,

$$\left\|\frac{\nabla u}{s}\right\|_p \leq C\|F\|_p$$  

(5.8)

with $C$ increasing with respect to $\|\beta\|_{BMO}$ and independent on $s$.

So, let us fix such a value of $s$ and this unique solution $v$ to the equation $v = sF(v)$. Combining (5.7) and (5.8) and using the Sobolev inequality we then have

$$|\Omega_k|^\frac{1}{p} \leq \frac{C}{\log(1 + k)} \left(\|F\|_p + |\Omega|^\frac{1}{2}\right).$$  

(5.9)

with $C = C(\lambda_0, N, \Omega, \|\beta\|_{BMO})$ positive constant increasing with respect to $\|\beta\|_{BMO}$.

By last estimate and (5.5), one can find $k_0 = k_0(\lambda_0, N, \Omega, \|F\|_p, \|\beta\|_{BMO})$ increasing with respect to $\|\beta\|_{BMO}$ s.t.

$$\left\|\frac{\nabla u}{s} - \frac{\nabla \bar{u}}{t}\right\|_2 \leq C \left(\|F\|_p + k_0|\Omega|^\frac{1}{2}\right).$$  

(5.10)

Hence, we can conclude that for every $t \in [0, 1]$ and $u \in W^{1,p}_0(\Omega)$ solution of the equation $u = tF(u)$, it holds

$$\|\nabla u\|_p \leq \left\|\frac{\nabla v}{s}\right\|_p + \left\|\frac{\nabla u}{t} - \frac{\nabla \bar{u}}{s}\right\|_p \leq C \left(\|F\|_p + k_0|\Omega|^\frac{1}{2}\right)$$

Hence, $F$ has a fixed point by applying the Leray-Shauder Fixed Point Theorem with

$$K = C \left(\|F\|_p + k_0|\Omega|^\frac{1}{2}\right)$$

and this completes our proof in case $\beta \in L^\infty(\Omega)$.

**Step 2.** Remove the additional assumption $\beta \in L^\infty$.

Now we prove the existence of solutions to (2.1) assuming $\beta \in BMO \setminus L^\infty(\Omega)$. For every $n \in \mathbb{N}$ and for almost every $x \in \Omega$ define

$$A_n(x, \xi) = \begin{cases} A(x, \xi) & \text{if } \beta(x) \leq n \\ n\frac{A(x, \xi)}{\beta(x)} & \text{if } \beta(x) > n \end{cases}$$  

(5.11)

It is not hard to see that for any $n \in \mathbb{N}$ and for almost every $x \in \Omega$, $A_n$ satisfies

$$|A_n(x, \xi) - A_n(x, \eta)| \leq \alpha \beta_n(x)|\xi - \eta|$$  

(5.12)

$$\beta_n(x)|\xi - \eta|^2 \leq (A_n(x, \xi) - A_n(x, \eta), \xi - \eta)$$  

(5.13)

$$A_n(x, 0) = 0$$  

(5.14)

for any vectors $\xi$ and $\eta$ in $\mathbb{R}^N$, where $\beta_n$ denotes as usual the truncated of $\beta$ at level $n$ so that $\beta_n \in L^\infty(\Omega)$. Note also that $\beta_n(x) \geq \lambda_0 > 0$ for $n$ sufficiently large. Moreover,

$$\|\beta_n\|_{BMO} \leq 2\|\beta\|_{BMO}$$  

(5.15)
(see [6]). By Step 1 we have that there exists a solution \( u_n \) to the problem

\[
\begin{cases}
\text{div}(A_n(x, \nabla u_n) + B(x, u_n)) = \text{div} F & \text{in } \Omega \\
u_n \in W^{1,p}_0(\Omega)
\end{cases}
\]  

(5.16)

and

\[
||\nabla u_n||_{L_p(\Omega)} \leq C(||F||_p + k_0|\Omega|^{\frac{1}{2}})
\]  

(5.17)

with \( C \) and \( k_0 \) increasing with respect to \( ||\beta_n||_{BMO} \) (see (5.9) and (5.10)). Then, by (5.15), we can assume \( C \) and \( k_0 \) independent on \( n \).

By (5.17) we can extract a subsequence \( \{u_n\} \) that converges weakly in \( W^{1,p}_0(\Omega) \) and strongly in \( L^p(\Omega) \) to a function \( u \in W^{1,p}_0(\Omega) \). To prove that the limit \( u \) solves Problem (2.1) we will prove that \( \{u_n\} \) is a Cauchy sequence in \( W^{1,r}_0(\Omega) \) for any \( 2 - \varepsilon_0 < r < p \).

To this aim for any \( n, m \in \mathbb{N} \) let us note that \( (u_n - u_m) \in W^{1,r}_0(\beta_n, \Omega) \) so that by Theorem 4.1 there exist \( \varphi \in W^{1,r}_0(\beta_n, \Omega) \) and a divergence free vector field \( h \in L^{r-1}(\beta_n, \Omega) \) such that

\[
|\nabla u_n - \nabla u_m|^{r-2}(\nabla u_n - \nabla u_m) = \nabla \varphi + h,
\]

\[
||\nabla \varphi||_{L^{\infty}(\beta_n, \Omega)} \leq c(N)(1 + ||\beta_n||_{BMO})^{\gamma}||\nabla u_n - \nabla u_m||^{r-1}_{L^r(\beta_n, \Omega)}
\]

(5.18)

\[
||h||_{L^{r-1}(\beta_n, \Omega)} \leq c(N)(1 + ||\beta_n||_{BMO})^{\gamma}(2-r)||\nabla u_n - \nabla u_m||^{r-1}_{L^r(\beta_n, \Omega)}.
\]

Without loss of generality we assume \( m < n \). By using (5.13), (5.18), (5.12), (5.11) and (5.15) we have

\[
\begin{align*}
&\int_{\Omega} |\nabla u_n - \nabla u_m|^{r\beta_n}(x)dx \\
&\leq \int_{\Omega} \langle A_n(x, \nabla u_n) - A_n(x, \nabla u_m), \nabla u_n - \nabla u_m \rangle |\nabla u_n - \nabla u_m|^{r-2}dx \\
&= \int_{\Omega} \langle A_n(x, \nabla u_n) - A_n(x, \nabla u_m), h \rangle dx \\
&+ \int_{\Omega} \langle A_n(x, \nabla u_n) - A_n(x, \nabla u_m), \nabla \varphi \rangle dx \\
&+ \int_{\Omega} \langle A_n(x, \nabla u_m) - A_n(x, \nabla u_m), \nabla \varphi \rangle dx \\
&\leq \alpha \int_{\Omega} |\nabla u_n - \nabla u_m||h|\beta_n dx + \int_{\Omega} |B(x, u_n) - B(x, u_m)||\nabla \varphi|dx \\
&+ \int_{m \leq \beta < n} \left\langle \frac{m}{\beta(x)} A(x, \nabla u_m) - A(x, \nabla u_m), \nabla \varphi \right\rangle dx \\
&+ \int_{\beta > n} \left\langle \left( \frac{m}{\beta} - \frac{n}{\beta} \right) A(x, \nabla u_m), \nabla \varphi \right\rangle dx \\
&\leq \alpha ||\nabla u_n - \nabla u_m||_{L^r(\beta_n)}||h||_{L^{r-1}(\beta_n)} + ||B(x, u_n) - B(x, u_m)||_{L^r(\Omega)}||\nabla \varphi||_{L^{r-1}(\Omega)} \\
&+ \alpha \int_{\Omega} |\beta_m - \beta_n| |\nabla u_m| |\nabla \varphi|dx \\
&\leq \alpha c(N)(1 + 2||\beta||_{BMO})^{\gamma}(2-r)||\nabla u_n - \nabla u_m||_{L^r(\beta_n, \Omega)}
\end{align*}
\]
β
\[ \text{since the proof of Proposition 4.1 can be repeated line by line using the continuity}\]
\[ \text{statement of Proposition 4.1 can be obtained assuming}\]
\[ \text{Step 3.}\]
\[ \text{The uniqueness.}\]
\[ \text{where}\]
\[ \text{C}\]
\[ \text{is a Cauchy sequence in}\]
\[ \text{W}\]
\[ \text{−}\]
\[ \text{Theorem 3.1 we have as before,}\]
\[ \text{L}\]
\[ \text{(2.1) verifies}\]
\[ \text{in Ω}\]
\[ \|\nabla\|\text{ test function}\]
\[ \text{ϕ}\]
\[ \text{Hence, every function}\]
\[ \text{Let}\]
\[ \text{To prove the uniqueness of solutions to Problem (2.1) we start by observing}\]
\[ \text{We conclude by observing that by (5.17) the solution}\]
\[ \text{u} \in W^{1,p}_0(Ω) \text{ to problem}\]
\[ \text{(2.1) verifies}\]
\[ \|\nabla u\|_{L^p(Ω)} \leqslant \liminf_{n \to \infty} \|\nabla u_n\|_{L^p(Ω)} \leqslant C[\|\mathcal{F}\|_p + k_0|Ω|^{\frac{1}{p}}]\]
\[ \text{where}\]
\[ C\]
\[ k_0\]
\[ \text{are positive constants depending on}\]
\[ N,\|\beta\|_{BMO},\lambda_0,\alpha,\|\mathcal{F}\|_p.\]
\[ \text{Step 3. The uniqueness.}\]
\[ \text{To prove the uniqueness of solutions to Problem (2.1) we start by observing that}\]
\[ \text{using the existence result obtained in the previous Step 2, exactly the same}\]
\[ \text{statement of Proposition 4.1 can be obtained assuming}\]
\[ \text{β} \in BMO(Ω) \setminus L^∞(Ω),\]
\[ \text{since the proof of Proposition 4.1 can be repeated line by line using the continuity}\]
\[ \text{estimate contained in}[8],\pg 927\text{ obtained for}\]
\[ \text{β} \in BMO(Ω).\]
\[ \text{Let} u, v \in W^{1,p}_0(Ω), 2 − ε_0 < p ≤ 2 \text{ be solutions to Problem (2.1). This means that}\]
\[ \int_{Ω} \langle A(x, ∇u) − A(x, ∇v) + (1 − \theta(x))[B(x, u) − B(x, v)], ∇φ⟩dx\]
\[ = \int_{Ω} (θ(x)[B(x, v) − B(x, u)], ∇φ)x, \quad ∀φ ∈ C^∞_0(Ω) \quad (5.19)\]
\[ \text{As observed in Step 1, using assumptions (2.7), (2.8), Theorem 3.1 and (4.5),}\]
\[ \theta(x)[B(x, u) − B(x, v)] \in L^p(Ω), \text{ with } p^* > 2. \text{ By (4.4), this implies (}\]
\[ \text{∇u − ∇v) ∈ L^{p^*}(Ω, \mathbb{R}^N)\text{ and so, by Lemma 3.1, (2.4) and (2.7) we deduce that}\]
\[ \{A(x, ∇u) − A(x, ∇v) + (1 − \theta(x))[B(x, u) − B(x, v)]\} \in L^2(Ω, \mathbb{R}^N)\]
\[ \text{Hence, every function}\]
\[ φ \in W^{1,2}_0(Ω)\text{ is an admissible test function in (5.19). At this}\]
\[ \text{point, using similar argument as in}[3],[19]\text{ we can conclude our proof. We give}\]
\[ \text{here some details for the convenience of the reader. For every} \varepsilon > 0 \text{ we consider as}\]
\[ \text{test function} φ \text{ in (5.19), the truncated function}\]
\[ φ := T_\varepsilon(u − v) \in W^{1,2}_0(Ω).\]
By using (2.4) – (2.7) we obtain
\[
\begin{align*}
&\langle \lambda_0 - S_{2,N}\|E - T_\rho E\|_{N,\infty}\|\nabla T_\varepsilon (u - v)\|_2^2 \\
&\leq \lambda_0 \|\nabla T_\varepsilon (u - v)\|_2^2 - \int \Omega |E - T_\rho E||u - v|\nabla T_\varepsilon (u - v)|\,dx \\
&\leq \int \Omega (A(x, \nabla u) - A(x, \nabla v) + (1 - \theta(x))[B(x, u) - B(x, v), \nabla T_\varepsilon (u - v)])\,dx \\
&\leq \int_{0 < |u-v| \leq \varepsilon} T_\rho E(x) |u - v|\nabla T_\varepsilon (u - v)|\,dx \\
&\leq \rho \varepsilon \{x \in \Omega : 0 < |u - v| \leq \varepsilon\}|\frac{2}{2}||\nabla T_\varepsilon (u - v)||_2.
\end{align*}
\]

Then, assuming \(d < \frac{\lambda_0}{S_{2,N}}\),
\[
||\nabla T_\varepsilon (u - v)||_2^2 \leq \varepsilon^2 \left(\frac{\rho}{\lambda_0 - dS_{2,N}}\right)^2 \{x \in \Omega : 0 < |(u - v)(x)| \leq \varepsilon\}
\tag {5.20}
\]

Now, let \(0 < \varepsilon < \eta\), so that
\[
\varepsilon^2 |\Omega_\eta| = \int_{|u-v| > \eta} |T_\varepsilon (u - v)|^2 \,dx \leq \varepsilon \int \Omega |\nabla T_\varepsilon (u - v)|^2 \,dx,
\]

where \(c = c(N)\). Combining the above inequality with (5.20), letting \(\varepsilon \to 0^+\) we obtain that \(|\Omega_\eta| = 0\). Being \(\eta > 0\) arbitrary, we conclude that \(u = v\) a.e in \(\Omega\). \(\square\)

6. Proof of Theorem 1.2. In this section we prove the existence of an optimal control for Problem (OCP). Here and below we assume \(E(x) := A\frac{x}{|x|^2} + g(x), g \in L^\infty(\Omega; \mathbb{R}^N)\).

Proof of Theorem 1.2. Let \((\mathcal{F}_j)_{j \in \mathbb{N}} \subseteq L^p(\Omega)\) be a minimizing sequence for Problem (OCP), i.e. let \(\mathcal{F}_j \in L^p(\Omega)\) be such that
\[
\mathcal{J}(\mathcal{F}_j) = \int \Omega (|\Phi(\mathcal{F}_j)|^2 + (\nabla \Phi(\mathcal{F}_j))\beta(x))^p \,dx \to \inf_{\mathcal{F} \in L^p(\Omega)} \mathcal{J}(\mathcal{F}) = m \in [0, +\infty)
\]

By Theorem 1.1, for every \(j\) there exists a unique state \(u_j = \Phi(\mathcal{F}_j) \in W^{1,p}_0(\Omega)\) such that
\[
\int \Omega (\beta(x)\nabla u_j + E(x)u_j, \nabla \varphi) \,dx = \int \Omega \mathcal{F}_j, \nabla \varphi \,dx
\tag {6.1}
\]

for every test function \(\varphi \in C^\infty_0(\Omega)\).

Since \(\mathcal{J}(\mathcal{F}_j) \to m \in [0, +\infty)\) as \(j \to \infty\), then for \(j\) sufficiently large,
\[
||\beta \nabla u_j||_p^p \leq \mathcal{J}(\mathcal{F}_j) < m + 1.
\]

Then, there exists \(V \in L^p(\Omega)\) such that \(\beta \nabla u_j \rightharpoonup V\) weakly in \(L^p(\Omega)\).

On the other hand, since \(\beta(x) \geq \lambda_0\) a.e in \(\Omega\), for \(j\) sufficiently large
\[
\lambda_0 \left(\int \Omega |\nabla u_j|^p \,dx\right)^{1/p} \leq \left(\int \Omega (|\nabla u_j|\beta(x))^p \,dx\right)^{1/p} < (m + 1)^{1/p}.
\]

Hence, also the sequence \((u_j)_{j \in \mathbb{N}}\) is bounded in \(W^{1,p}_0(\Omega)\) and then there exists \(u_0 \in W^{1,p}_0(\Omega)\) such that \(u_j \to u_0\) weakly in \(W^{1,p}_0(\Omega)\), and a.e. in \(\Omega\) and
\[
u_j \to u_0\] strongly in \(L^p(\Omega)\). \(\tag {6.2}\)
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This gives, using Lemma 3.1,\( \beta \nabla u_j \rightharpoonup \beta \nabla u_0 \) weakly in \( L^r(\Omega) \) for every \( r < p \) and so
\[
\beta \nabla u_0 = V \in L^p(\Omega) \quad \text{and} \quad \beta \nabla u_j \rightharpoonup \beta \nabla u_0 \text{ weakly in } L^p(\Omega). \tag{6.3}
\]

Consider now the sequence \( E(x)u_j \). Since \( u_j \to u_0 \) a.e. on \( \Omega \), also \( E(x)u_j \to E(x)u_0 \) a.e. in \( \Omega \). Moreover, since \( (u_j) \) is bounded in \( W^{1,p}_0(\Omega) \), using (3.1), (3.2) and the Sobolev imbedding Theorem 3.1,
\[
\|E|u_j|\|_p \leq \|E\|_{N,\infty}\|u_j\|_p, \leq C.
\]
This gives \( Eu_j \to Eu_0 \) strongly in \( L^s(\Omega; \mathbb{R}^N) \) for every \( s < p \) and then
\[
Eu_j \rightharpoonup Eu_0 \text{ weakly in } L^p(\Omega). \tag{6.4}
\]
as \( j \to \infty \), with
\[
Eu_0 \in L^p(\Omega)
\]
Note that, combining (6.3) and (6.4) we can pass to the limit in (6.1) as \( j \to \infty \), having in particular
\[
\int_{\Omega} \langle \beta(x) \nabla u_j + E(x)u_j, \nabla \varphi \rangle \, dx \to \int_{\Omega} \langle \beta(x) \nabla u_0 + E(x)u_0, \nabla \varphi \rangle \, dx, \quad \forall \varphi \in W^{1,p}_0(\Omega)
\]
and then for the minimizing sequence \( \{F_j\}_{j \in \mathbb{N}} \), by (6.1) we have, unless to pass to a subsequence,
\[
\int_{\Omega} \langle F_j, \nabla \varphi \rangle \, dx \to \int_{\Omega} \langle F_0, \nabla \varphi \rangle \, dx \quad \forall \varphi \in W^{1,p'}_0(\Omega)
\]
as \( j \to \infty \), with
\[
F_0 = (\beta \nabla u_0 + Eu_0) \in L^p(\Omega; \mathbb{R}^N), \quad u_0 = \Phi(F_0).
\]

Now it remains to prove that \( F_0 \) is an optimal control i.e,
\[
J(F_0) = \inf_{F \in L^p(\Omega)} J(F)
\]
To this aim, by the lower semicontinuity of the norm \( \| \cdot \|_{L^p(\Omega)} \) with respect to the weak topology of \( L^p(\Omega) \), by (6.2) and (6.3) we get
\[
\liminf_{j \to \infty} J(F_j) = \liminf_{j \to \infty} \left[ \int_{\Omega} (\|\Phi(F_j) - z\|_p + (|\nabla \Phi(F_j)|\beta(x))^p) \, dx \right]
\]
\[
= \liminf_{j \to \infty} \left[ \|u_j - z\|_p + \|\beta \nabla u_j\|_p \right]
\]
\[
\geq \|u_0 - z\|_p + \|\beta \nabla u_0\|_p
\]
\[
= \int_{\Omega} (|\Phi(F_0) - z|_p + (|\nabla \Phi(F_0)|\beta(x))^p) \, dx.
\]
Then,
\[
J(F_0) \geq \inf_{F \in L^p(\Omega)} J(F) = \lim_{j \to \infty} J(F_j) = \liminf_{j \to \infty} J(F_j) \geq J(F_0)
\]
and hence the pair \( (u_0, F_0) \) is optimal for Problem (OCP). This completes our proof. \( \square \)
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E-mail address: g.zecca@unina.it