Differential equations: Solving the oscillation system

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Abstract. The purpose of this research is to perform modeling and simulation to solve differential equations in cases of physical phenomena. The physical system studied is simple harmonic motion. The method used is the method of problem solving using computer software. The simulation uses the Matlab program by applying the Euler method. The result obtained is to distinguish the oscillator graph without any external forces with any forces. This solution helps students solve problems using computer software in giving physical meaning. Therefore, students must be trained and guided to have the ability to think to make it easier to solve problems with the help of computers.

1. Introduction
Differential equations are used to model a problem with various independent variables [1]. This field of study is an interesting and important issue from mathematics for modeling various phenomena in physics, chemistry, biology, engineering, or economics [2]. For example, the problem of transport phenomena, the mass-spring systems, capacitor inductance circuits, expansion, chemical reactions, pendulums, and so on. All of these cases can be modeled in the form of differential equations which arise because physical phenomena in scientific studies can be expressed by the rate of change [3,4]. Example,

\[ \frac{dq}{dt} = -0.25(q - 30)^{2/3} \]  

Equation (1) above is an equation to model the rate of change in temperature \( q \) of the body which loses heat due to the influence of environmental temperature. Therefore, differential equations are very important in mathematics because many laws and physical relationships emerge mathematically to engineer a model of solution [5]. The solution to an exact differential equation is a function that satisfies the equation and also satisfies the limit conditions for the initial value of the function. The solution can be done using analytical and numerical approaches. The analytic approach is also called the true approach because it provides true solutions or real solutions [6,7]. However, the analytical approach will encounter difficulties and a long time if we encounter complex and complicated phenomena. This approach is limited to problems that have a simple geometric interpretation and are of a low size.
Meanwhile, the physical phenomenon is a complex case, and it is complicated [8]. Therefore, a numerical approach was developed to solve problems that could not be solved analytically. Various methods are used to solve numerically, including Euler, Runge-Kutta, Runge-Kutta order 4 (RK 4), Trapezoid, and others [6,9]. In this article, we did discuss the Euler method. Solving differential equations using numerical methods produces a table of function values for several independent variables, but these are not explicitly stated in the form of a function formula.

The purpose of this research is to make modeling and simulation of cases of physical phenomena using the Matlab 2019 application tool. This solution is to help students solve problems. In this article, the physical phenomenon being studied is simple harmonics oscillation. This is because we often encounter them in everyday life. The phenomenon of this oscillatory motion is found in physics, including the motion of electrons in atoms, the behavior of currents and voltages in electric circuits, and planetary orbits.

1.1. Case: mass on spring
In this article, we will discuss mass on a spring [10]. The problem is “A block with mass m is attached to the end of a light horizontal spring and the other end is fixed to a non-movable wall. The block glides on a smooth surface”. See Figure 1.

![Figure 1. Mass on spring.](image)

Figure 1. at time t there is an extension of the spring x(t), which is the difference between the actual length of the spring x (0) and the length of the spring x(t). x(t) can also be used as a coordinate to determine the instantaneous horizontal displacement of the mass. The system equilibrium occurs when the mass is at rest, and the spring does not increase in length. In this state, the horizontal force acting on the mass is zero, so there is no reason to start moving. However, if the system is disturbed from an equilibrium position, then the mass experiences a horizontal force exerted by Hooke's law (if the block is moved so that the spring increases in length).

\[
\vec{F} = -k \cdot \vec{x}
\]  

Here, k > 0, this is called force constant of the spring. The negative sign indicates that f (x) is indeed a recovery force. Note that if a system oscillates around its equilibrium position, the restoration force acts on the system. The restoration force is directly proportional to the displacement of the block from the equilibrium position (|f | ∝ x).

This shows that Hooke's law applies to springs that experience relatively small extension. The beam displacement cannot be done too large. Therefore, this dynamic system motion represents the motion of various mechanical systems when it is slightly disturbed from a steady state of equilibrium. If the spring force is the only external force acting on the object, then Newton’s second law of motion gives following time evolution equation for the system:

\[
\vec{F}(x) = m. \ddot{a}
\]
There are two types of forces acting on the current system, namely the recovery force (eq. (2)) and the newton force (eq. (3)). From Equations (2) and (3) can be derived:

\[ m \cdot \ddot{a} = -k \cdot x \]
\[ m \cdot \frac{d^2x}{dt^2} = -k \cdot x \]

or

\[ \frac{d^2x}{dt^2} = -\frac{k}{m} \cdot x \]

(4)

Where \( \omega^2 = \frac{k}{m} \), so

\[ \frac{d^2x}{dt^2} = -\omega^2 \cdot x \]

(5)

Equation (5) is a ordinary differential equation and it is known as the simple harmonic oscillation. Suppose that, as it slides over the horizontal surface, the mass is subject to a frictional damping force that opposes its motion is directly proportional to its instantaneous velocity. When a spring system is given a force, the response that occurs depends on the external force exerted on the system and the damping experienced by the system. The total force acting on the mass in the damped system is

\[ F_y = -l \cdot \frac{dy}{dt} \]

So, expression to the law of second Newton

\[ m \cdot \frac{d^2y}{dt^2} = -k \cdot y - l \cdot \frac{dy}{dt} \]

Where \( m > 0 \) is the mass, \( k > 0 \) the spring force constant, which parameterizes the strength of the damping. The time evolution equation of the system thus becomes

\[ \frac{d^2y}{dt^2} + \frac{k}{m} \cdot y + \frac{l}{m} \cdot \frac{dy}{dt} = 0 \]

(6)

with,

\[ \omega^2 = \frac{k}{m} \text{ and } 2b = \frac{l}{m} \]

So,

\[ D^2 + 2bD + \omega^2 = 0 \]

(7)

Equation (7) above is ordinary differential equation, and it can solved by square root.

\[ D_{12} = -b \pm \sqrt{b^2 - \omega^2} \]

General solution:

- For \( b^2 > \omega^2 \), overdamped
- For \( b^2 = \omega^2 \), critical damped
- For \( b^2 < \omega^2 \), damped

2. Methods

Simulation programs using Matlab software R2019a 64 bit (win64). The Matlab program in this case uses data input of time and deviation to provide output values for variables. The following are the steps to determine the equation for oscillating motion. 1) Constructing a Matlab script is a harmonic oscillation. 2) Determine the input. 3) The data is plotted on a graph, where the x-axis is time (t) and the y-axis is the deviation (x(t)). 4) Read data from charts. The data required for equations (5) and (6) are:

- Amplitude. The data required is the hilltop point and the valley peak point.
Omega. The data to determine the angular frequency $\omega$ is to determine the start time $t_1$ and end time $t_2$. The distance requirement between $t_1$ and $t_2$ is 1 wave. The formula used to determine angular frequency and frequency is $\omega = 2\pi f$ and $f = \frac{1}{t_2-t_1}$.

Damping Coefficient. The amplitude of this damped oscillation motion is a decrease in the harmonic oscillation motion and the amplitude decreases based on $e^{-\gamma t}$. Therefore, the initial amplitude of the damped oscillation motion is the same as the amplitude of the harmonic motion. The damping coefficient will affect the amplitude value.

2.1. Numerical methods
From equation (5) we can derive an equation for the acceleration of the spring, which is the second derivative of motion to time. If the above differential equations are decomposed into Euler's method, then

$$x_{i+1} = x_i + hv$$

$$v_{i+1} = v_i + ha$$

with

$$t_i = t + h$$

3. Results and discussion
If there is no friction, the spring will continue to oscillate without stopping [10]. Equation (5) is a differential equation with solutions

$$x(t) = A\sin(\omega t + \theta)$$

or

$$x(t) = A\cos(\omega t + \theta)$$

(8)

The amplitude of the oscillations will decrease over time and eventually, the oscillations will stop [11-13]. It is said that the oscillatory motion is damped by friction so that this oscillatory motion is called damped harmonic motion. The extent of the friction force is proportional to the object's velocity and has a direction opposite to the object's velocity. In an oscillation system, mechanical energy is dissipated due to the frictional force. If mechanical energy decreases, it means that the motion in the system is damped.

Differential equations such as equations (5) and (7) can be solved using Matlab. The equation is composed of 4 variables, namely amplitude, angular frequency, horizontal shift and vertical shift. The matlab script for solving equations using the Euler method is

```
for i=1:n
xp(i)=t;
t=t+h;
yp(i)=x;
x=x+h*v;
zp(i)=v;
v=v+h*a;
a=-k/m*x-b/m*v;
end
```

The script above is writing in the Matlab program for the section process. Using this script, the completion chart for ideal and damped oscillations is illustrated in Figure 2. Figure 2 shows that (a) because there is no force affecting the load, the load will continue to oscillate. There is no friction. It appears that the graph is fixed. In (b), the influence of the frictional force affects the graph. The longer the oscillation will stop. It can be seen that the graph gets smaller the deviations that occur.

Solving differential equations using the Matlab program as the case above aims to make it easier for students to get solutions quickly. Another goal is to help students make graphics to facilitate interpretation and interpretation so that students have a good understanding and skills in physical phenomena [14,15]. Emphasized skills are the ability to represent mathematical equations in graphical form and the ability to give physical meaning to the results of graphic visualization. These aspects
include focusing on problems, connecting problems with physics concepts, planning strategies for finding solutions, executing plans, and interpreting and evaluating solutions [16–18].

**Figure 2.** Simple plot for: (a) oscillation no friction, (b) damped oscillations.

The computer is used to visualize various natural events that are difficult to observe directly. Computers are used in learning to simulate a system and students can interact with it [19]. Learning by multiplerepresentations was able to provide new insights to construct new knowledge and allow deep understanding and coherent knowledge [20]. Students can think systematically, try, and evaluate interpretations based on principles. The computer will provide information to students about the problem. Thus, concepts that are difficult to explain become easily understood by students with the help of computer visualization [7,9,21,22].

4. Conclusion

Skills in using mathematical language are the ability to do mathematical modeling and provide physical meaning. These skills can be developed in students with the help of computer software. Generic ability in the discussion of differential equations to understand physical phenomena and interpret physical phenomena. Therefore, students must be trained and guided to have the ability to think correctly, and easily solve problems with the help of computers.

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