THE BOUNDARY ANALOG OF THE CARATHÉODORY-SCHUR
INTERPOLATION PROBLEM

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Abstract. Characterization of Schur-class functions (analytic and bounded by one
in modulus on the open unit disk) in terms of their Taylor coefficients at the origin is
due to I. Schur. We present a boundary analog of this result: necessary and sufficient
conditions are given for the existence of a Schur-class function with the prescribed
nontangential boundary expansion
\[f(z) = s_0 + s_1(z - t_0) + \ldots + s_N(z - t_0)^N + o(|z - t_0|^N)\]
at a given point \(t_0\) on the unit circle.

1. Introduction

Let \(S\) denote the Schur class of analytic functions mapping the open unit disk \(D\) into
its closure (i.e., the closed unit ball of \(H^\infty\)). Characterization of Schur class functions in
terms of their Taylor coefficients goes back to I. Schur [18] (and to C. Carathéodory [14]
for a related class of functions).

Theorem 1.1. There is a function \(f(z) = s_0 + s_1 z + \ldots + s_{n-1} z^{n-1} + \ldots \in S\) if and
only if the lower triangular Toeplitz matrix \(U_n\) (see formula (2.4) below) is a contraction,
i.e., if and only if the matrix \(P = I_n - U_n U_n^*\) is positive semidefinite.

By a conformal change in variable, a similar result is established for an arbitrary point
\(\zeta \in D\) at which the Taylor coefficients are prescribed: there exists a function
\(f \in S\) of the form
\[f(z) = s_0 + s_1(z - \zeta) + \ldots + s_{n-1}(z - \zeta)^{n-1} + \ldots \tag{1.1}\]
if and only if a certain matrix \(P\) (explicitly constructed in terms of \(\zeta\) and \(s_0, \ldots, s_{n-1}\)) is
positive semidefinite. Furthermore, if \(P\) is positive definite, then there are infinitely many
functions \(f \in S\) of the form (1.1). If \(P \succeq 0\) is singular, then there is a unique
\(f \in S\) of the form (1.1) and this unique function is a finite Blaschke product of degree equal to the
rank of \(P\).

In this paper, we examine a similar question in the “boundary” setting where Taylor
expansion (1.1) at \(\zeta \in D\) is replaced by the asymptotic expansion at some point \(t_0\) on the
unit circle \(T\).

Question: Given a point \(t_0 \in T\) and given numbers \(s_0, \ldots, s_N \in \mathbb{C}\), does there exist a
function \(f \in S\) which admits the asymptotic expansion
\[f(z) = s_0 + s_1(z - t_0) + \ldots + s_N(z - t_0)^N + o(|z - t_0|^N) \tag{1.2}\]
as \(z\) tends to \(t_0\) nontangentially?

The complete answer to this question is given in Theorem 2.3 below which is the main
result of the paper. The necessary and sufficient conditions for the existence of a function
\(f \in S\) subject to (1.2) are given in terms of a certain positive semidefinite matrix (as in the
classical “interior” case) constructed explicitly in terms of the data set and (in contrast to
the classical case) of two additional numbers also constructed from \(\{t_0, s_0, \ldots, s_N\}\). This
Theorem 2.2. Given \( s_0, s_1 \in \mathbb{C} \), there exists a function \( f \in S \) such that

\[
f(z) = s_0 + s_1(z - t_0) + o(|z - t_0|) \quad \text{as} \quad z \to t_0
\]

if and only if either

1. \(|s_0| < 1\) or
2. \(|s_0| = 1\) and \( t_0 s_1 \mathbf{a}_0 \geq 0 \).

Such a function is unique and is equal identically to \( s_0 \) if and only if \(|s_0| = 1\) and \( s_1 = 0\).

Due to Theorem 2.2, we may focus in what follows on the case \( N \geq 2 \). Moreover, due to Lemma 2.1, it suffices to assume that \(|s_0| = 1\) and to characterize all tuples \( \{ s_1, \ldots, s_N \} \) for which the problem \( \text{BP}_N \) has a solution under the latter assumption. To present the result, we first introduce some needed definitions. In what follows, \( S^{(n)}(t_0) \) will stand for the class of Schur functions satisfying a Carathéodory-Julia type condition:

\[
f \in S^{(n)}(t_0) \quad \text{def} \quad f \in S \quad \& \quad \liminf_{z \to t_0} \frac{\partial^{2n-2}}{\partial z^{n-1} \partial \bar{z}^{n-1}} \frac{1 - |f(z)|^2}{1 - |z|^2} < \infty.
\]

We will identify \( S^{(0)}(t_0) \) with \( S \). The higher order Carathéodory-Julia condition (2.3) was introduced in [8] and studied later in [11] and [10]. This condition can be equivalently
reformulated in terms of the de Branges-Rovnyak space $\mathcal{H}(f)$ (we refer to [12] for the definition) associated with the function $f \in \mathcal{S}$ as follows: a Schur-class function $f$ belongs to $\mathcal{S}^{(n)}(t_0)$ if and only if for every $f \in \mathcal{H}(f)$, the boundary limits $f_j(t_0)$ exist for $j = 0, \ldots, n - 1$. As was shown in [13] (and earlier in [11] for inner functions), the latter de Branges-Rovnyak space property (and therefore, the membership in $\mathcal{S}^{(n)}(t_0)$) is equivalent to relation

$$\sum_k \frac{1 - |a_k|^2}{|t_0 - a_k|^{2n+2}} + \int_0^{2\pi} \frac{d\mu(\theta)}{|t_0 - e^{i\theta}|^{2n+2}} < \infty,$$

where the numbers $a_k$ come from the Blaschke product of the inner-outer factorization of $f$:

$$f(z) = \prod \frac{\bar{a}_k}{a_k} \cdot \frac{z - a_k}{1 - za_k} \cdot \exp \left\{-\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta) \right\}.$$

Several other equivalent characterizations of the class $\mathcal{S}^{(n)}(t_0)$ will be recalled in Theorem 3.1 below. Given a tuple $s = \{s_0, s_1, \ldots, s_N\}$, we define the lower triangular Toeplitz matrix $U^s_n$ and the Hankel matrix $H^s_n$ by

$$U^s_n = \begin{bmatrix} s_0 & 0 & \cdots & 0 \\ s_1 & s_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ s_{n-1} & \cdots & s_1 & s_0 \end{bmatrix}, \quad H^s_n = \begin{bmatrix} s_1 & s_2 & \cdots & s_n \\ s_2 & s_3 & \cdots & s_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_n & s_{n+1} & \cdots & s_{2n-1} \end{bmatrix} \quad (2.4)$$

for every appropriate integer $n \geq 1$ (i.e., for every $n \leq N + 1$ in the first formula and for every $n \leq (N + 1)/2$ in the second). Given a point $t_0 \in \mathbb{T}$, we introduce the upper triangular matrix

$$\Psi_n(t_0) = \begin{bmatrix} t_0 & -t_0^2 & \cdots & (-1)^{n-1}(n-1)_{0,0} \\ 0 & -t_0^3 & \cdots & (-1)^n(n-1)_{1,0} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & (-1)^n(n-1)_{n-1,0} \end{bmatrix} \quad (2.5)$$

with the entries

$$\Psi_{j\ell} = \begin{cases} 0, & \text{if } j > \ell, \\ (-1)^{\ell-1} \left( \begin{array}{c} \ell - 1 \\ j - 1 \end{array} \right) t_0^{j+\ell-j-1}, & \text{if } j \leq \ell, \quad (j, \ell = 1, \ldots, n), \end{cases} \quad (2.6)$$

and finally, for every $n \leq (N + 1)/2$, we introduce the structured matrix

$$P^s_n = [p^s_{ij}]_{i,j=1}^n = H^s_n \Psi_n(t_0) U^s_n \quad (2.7)$$

with the entries (as it follows from (2.4)–(2.7))

$$p^s_{ij} = \sum_{r=1}^{j} \left( \sum_{k=1}^{r} s_{i+k-1} \Psi_{kr} \right) \overline{\psi}_{j-r} \quad (2.8)$$

Although the matrix $P^s_n$ depends on $t_0$, we drop this dependence from notation. However, in the case that the parameters $s_j$ in (2.7) are equal to the angular boundary limits $f_j(t_0)$
Theorem 2.3. Let \( p \) be the set of the problem \( P \) that satisfies BP conditions for the problem \( P \) and let \( s \) be defined by (2.8) and \( s \) be also the principal submatrix of \( P_n^s \) for every \( k < n \). We also observe that formula (2.7) defines the numbers \( p^s_{ij} \) in terms of \( s = \{s_0, \ldots, s_N\} \) for every pair of indices \( (i, j) \) subject to \( i + j \leq N + 1 \). In particular, if \( n \leq N/2 \), one can define via this formula the column
\[
B_n := \begin{bmatrix}
\begin{array}{c}
p_{1,n+1}^s \\
\vdots \\
p_{n,n+1}^s \\
\end{array}
\end{bmatrix} = \begin{bmatrix}
\begin{array}{cccc}
\sigma_1 & s_2 & \cdots & s_{n+1} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_n & s_{n+1} & \cdots & s_{2n} \\
\end{array}
\end{bmatrix} \begin{bmatrix}
\Psi_{n+1}(t_0) \\
\vdots \\
\Psi_0 \\
\end{bmatrix},
\]

(2.10)

where the second equality follows from representation of type (2.7) for the matrix \( P_{n+1}^s \) which is determined from \( s = \{s_0, \ldots, s_N\} \) completely (if \( n < N/2 \)) or except for the entry \( p_{n+1,n+1}^s \) (if \( n = 2n \)).

The next theorem is the main result of the paper; it gives necessary and sufficient conditions for the problem \( BP_N \) to have a solution and also for this solution to be unique.

Theorem 2.3. Let \( t_0 \in \mathbb{T} \) and \( s = \{s_0, s_1, \ldots, s_N\} \) \((N \geq 2)\) be given. In case the matrix \( P_k^s \) is positive semidefinite for some \( k \geq 0 \), we let \( n \) \((0 \leq n \leq (N + 1)/2)\) to be the greatest integer such that \( P_n^s \geq 0 \). In case \( n \leq N/2 \), let \( p_{n+1,n}^s \) and \( p_{n,n+1}^s \) be defined by (2.8) and let \( B_n \) be as in (2.10). Then

1. The problem \( BP_N \) has a unique solution if and only if \(|s_0| = 1\), \( P_n^s \) is singular and either
   a. \( n = (N + 1)/2 \) and \( \text{rank } P_n^s = \text{rank } P_{n-1}^s \) or
   b. \( n = N/2 \),
   \[
   p_{n+1,n}^s = \overline{P}_{n,n+1}^s \quad \text{and} \quad \text{rank } P_n^s = \text{rank } [P_n^s B_n].
   \]
   The unique solution is a finite Blaschke product of degree equal \( \text{rank } P_n^s \).

2. The problem \( BP_N \) has infinitely many solutions if and only if either
   a. \(|s_0| < 1\) or
   b. \(|s_0| = 1\), \( P_n^s > 0 \) and one of the following holds:
      i. \( n = (N + 1)/2 \);
      ii. \( n = N/2 \) and \( t_0 \cdot (P_{n+1,n}^s - \overline{P}_{n,n+1}^s) \geq 0 \);
      iii. \( 0 < n < N/2 \) and \( t_0 \cdot (P_{n+1,n}^s - \overline{P}_{n,n+1}^s) > 0 \).

   In any of these three cases, every solution of the problem belongs to \( \mathcal{S}(n)(t_0) \).

3. Otherwise the problem has no solutions.

Part (1) in Theorem 2.3 can be formulated in the following more unified way (see Corollary 3.6 below for the proof):

Lemma 2.4. The uniqueness occurs if and only if the matrix \( P_n^s \) of the maximal possible size (i.e., \( n = \lceil \frac{N+1}{2} \rceil \)) is positive semidefinite (and singular) and admits a positive semidefinite extension \( P_{n+1}^s \) for an appropriate choice of \( s_{2n+1} \) (in case \( N = 2n \)) or of \( s_{2n+1} \) and \( s_{2n} \) (in case \( N = 2n - 1 \)).
Additional symmetry and rank conditions in part (1) of Theorem 2.3 guarantee that the above extension exists. Observe that the \( n \times (n+1) \) matrix \( \mathbb{P}_n^s \) in (2.11) is formed by the \( n \) top rows of the matrix \( \mathbb{P}_{n+1}^s \) which are completely specified by \( s = \{ s_0, \ldots, s_N \} \) whenever \( n \leq N/2 \).

If \( N = 1 \) or \( N = 2 \), the integer \( n \) (defined as in Theorem 2.3) is at most one and it follows from formula (2.7) that \( \mathbb{P}_n^s = P_{11}^s = \mathbb{H}_1^s \psi_1(t_0)\mathbb{U}_1^s = s_1 t_0 s_0 \). Furthermore, for \( N = 2 \), formula (2.8) gives

\[
p_{21}^s = t_0 s_2 s_0 \quad \text{and} \quad p_{12}^s = |s_1|^2 t_0 - s_1 s_0 t_0^2 - s_2 s_0 s_1^3.
\]

Letting \( N = 1 \) in Theorem 2.3 leads us to Theorem 2.2 while letting \( N = 1 \) gives the following result: \( \text{given } s_0, s_1, s_2 \in \mathbb{C}, \text{ there exists a function } f \in \mathcal{S} \text{ such that } f(z) = s_0 + s_1(\zbar{z} - t_0) + s_2(z - t_0)^2 + o(|z - t_0|^2) \quad \text{as } z \to t_0, \quad (2.12)
\]

if and only if either \(|s_0| < 1 \) or

\[
|s_0| = 1, \quad s_1 t_0 s_0 \geq 0 \quad \text{and} \quad 2\text{Re}(t_0^2 s_0 s_2) \geq |s_1|^2 - t_0 s_0 s_1. \quad (2.13)
\]

The uniqueness occurs if and only if \(|s_0| = 1 \) and \( s_1 = s_2 = 0 \) and the unique function of the required form is equal to \( s_0 \) identically.

In general, the algorithm determining whether or not there exists a Schur-class function with prescribed boundary derivatives can be designed as follows. If \(|s_0| \neq 1 \), then the definitive answer comes up. If \(|s_0| = 1 \), we do not have to check positivity of all the matrices \( \mathbb{P}_k^s \) for \( k = 1, 2, \ldots \) to find the greatest integer \( n \) such that \( \mathbb{P}_n^s \geq 0 \). It suffices to get the greatest \( n \) such that \( \mathbb{P}_n^s \) is Hermitian. If this Hermitian \( \mathbb{P}_n^s \) is not positive semidefinite, then the problem \( \mathbf{BP}_N \) has no solutions (see Remark 1 below). If \( \mathbb{P}_n^s \) is positive semidefinite (singular), then we check one of the two possibilities indicated in part (1) of Theorem 2.3 depending on the parity of \( N \). If \( \mathbb{P}_n^s > 0 \), then we verify exactly one of the three possibilities in part (2(b)). We illustrate this strategy by a numerical example.

**Example 1.** Let \( N = 3 \), \( t_0 = 0 \), \( s_0 = s_1 = 1 \) and \( s_2 = s_3 = 0 \). Then formula (2.7) gives \( \mathbb{P}_1 = 1 \) and \( \mathbb{P}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \). Thus, the greatest \( n \leq (N+1)/2 = 2 \) such that \( \mathbb{P}_n \) is Hermitian, is \( n = 2 \). Since \( \mathbb{P}_2 \) is positive semidefinite (singular) and since rank \( \mathbb{P}_2 = \text{rank} \mathbb{P}_1 = 1 \), it follows from part (i(a)) in Theorem 3.1 that there is a unique function \( f \in \mathcal{S} \) such that \( f(z) = 1 + (z - 1) + o((z - 1)^3) \) as \( z \to t_0 \). This unique function is clearly \( f(z) \equiv z \) which thus gives yet another proof of Theorem 2.1 in [13]: If \( f \in \mathcal{S} \) and if \( f(z) = z + o((z - 1)^3) \) as \( z \to 1 \), then \( f(z) \equiv z \).

In Section 3 we consider the case when the matrix \( \mathbb{P}_n^s \) chosen as in Theorem 2.3 is singular. The nondegenerate case is handled in Section 4 at the end of which we summarize all possible cases completing the proof of Theorem 2.3.

**3. The determinate case**

In this section we will consider the case when for some \( n \leq (N + 1)/2 \), the matrix \( \mathbb{P}_n^s \) constructed from the data set via formula (2.7) is positive semidefinite and singular. It is well known that for any Schur-class function \( f \), the Schwarz-Pick matrix

\[
\mathbb{P}_n^f(z) := \left[ \frac{1}{|z|} \frac{\partial^{i+j}}{\partial z^i \partial \bar{z}^j} \frac{1 - |f(z)|^2}{1 - |z|^2} \right]_{i,j=0}^{n-1}
\]
is positive semidefinite for every $n \geq 1$ and $z \in \mathbb{D}$; in fact it is positive definite unless $f$ is a finite Blaschke product in which case $\text{rank}(P_n^f(z)) = \min\{n, \deg f\}$. Given a point $t_0 \in \mathbb{T}$, the boundary Schwarz-Pick matrix is defined by

$$P_n^f(t_0) := \lim_{z \to t_0} P_n^f(z), \quad (3.1)$$

provided the nontangential limit in (3.1) exists. Thus, once the boundary Schwarz-Pick matrix $P_n^f(t_0)$ exists, it is positive semidefinite. It is readily seen from definition (2.3) that the membership $f \in \mathcal{S}^{(n)}(t_0)$ is necessary for the limit (3.1) to exist (it is necessary for the nontangential convergence of the rightmost diagonal entry in $P_n^f(z)$). In fact, it is also sufficient due the following theorem established in [8].

**Theorem 3.1.** Let $f \in \mathcal{S}$, $t_0 \in \mathbb{T}$ and $n \in \mathbb{N}$. The following are equivalent:

1. $f \in \mathcal{S}^{(n)}(t_0)$.
2. The boundary Schwarz-Pick matrix $P_n^f(t_0)$ exists.
3. The nontangential boundary limits $f_j(t_0)$ exist for $j = 0, \ldots, 2n - 1$ and satisfy

$$|f_0(t_0)| = 1 \quad \text{and} \quad P_n^f(t_0) \geq 0,$$

where $P_n^f(t_0)$ is the matrix defined in (2.7).

Moreover, if this is the case, then $P_n^f(t_0) = P_n^f(t_0)^\ast$.

We remark that in contrast to the boundary Schwarz-Pick matrix $P_n^f(t_0)$ which is positive semidefinite whenever it exists, the structured matrix $P_n^f(t_0)$ defined in terms of the angular limits $f_j(t_0)$ by formula (2.9) does not have to be positive semidefinite and even Hermitian. Theorem 3.1 states in particular that positivity of this structured matrix is an exclusive property of $\mathcal{S}^{(n)}(t_0)$-class functions. The following stronger version of the implication (3) $\Rightarrow$ (1) in Theorem 3.1 appears in Theorem 1.7 [11].

**Theorem 3.2.** Let $f \in \mathcal{S}$, $t_0 \in \mathbb{T}$ and let us assume that the nontangential boundary limits $f_j(t_0)$ exist for $j = 0, \ldots, 2n - 1$ and are such that $|f_0(t_0)| = 1$ and $P_n^f(t_0) = P_n^f(t_0)^\ast$. Then $f \in \mathcal{S}^{(n)}(t_0)$.

**Remark 1.** Theorems 3.1 and 3.2 show that for $f \in \mathcal{S}$ such that the boundary limits $f_j(t_0)$ exist for $j = 0, \ldots, 2n - 1$ and $|f_0| = 1$, the matrix $P_n^f(t_0)$ defined in (2.9) is Hermitian if and only if it is positive semidefinite and moreover, that this is the case if and only if $f \in \mathcal{S}^{(n)}(t_0)$.

In the rest of the section we prove the “if” part of statement (1) in Theorem 3.1. We first recall the following result (see Theorem 6.2 in [9] for the proof).

**Theorem 3.3.** Let $t_0 \in \mathbb{T}$ and $s = \{s_0, \ldots, s_{2n-1}\}$ be such that

$$|s_0| = 1, \quad P_n^s \geq 0 \quad \text{and} \quad \det P_n^s = 0. \quad (3.2)$$

Then there exists a unique $f \in \mathcal{S}$ such that

$$f_j(t_0) = s_j \quad (j = 0, \ldots, 2n - 2) \quad \text{and} \quad (-1)^{n}t_0^{2n-1}s_0(f_{2n-1}(t_0) - s_{2n-1}) \geq 0. \quad (3.3)$$

This unique $f$ is a finite Blaschke product of degree equal to the rank of $P_n^s$.

**Lemma 3.4.** Let $g \in \mathcal{S}^{(n)}(t_0)$. If $g$ is a finite Blaschke product, then

$$\text{rank } P_n^g(t_0) = \min\{n, \deg g\}. \quad (3.4)$$

Otherwise, $P_n^g(t_0) > 0,$
Proof. Since \( g \in S^{(n)}(t_0) \), from Theorem 3.1 we have \(|g_0(t_0)| = 1\) and \(P_n^g(t_0) \geq 0\). Let us assume that \(P_n^g(t_0)\) is singular and that \(\text{rank}(P_{n}^g(t_0)) = d\). Letting \(s_j := g_j(t_0)\) for \(j = 0, \ldots, 2n-1\), we conclude from Theorem 3.3 that there exists a unique function \(f \in S\) satisfying conditions (3.3) and that \(f\) is a Blaschke product of degree \(d\). Since \(g\) obviously satisfies the same conditions, we have \(f \equiv g\). Thus, if \(P_n^g(t_0) \geq 0\) is singular, then \(g\) is a finite Blaschke product and \(\text{rank}P_n^g(t_0) = \text{deg}f < n\). To complete the proof it remains to show that if \(g\) is a finite Blaschke product and \(\text{rank}P_n^g(t_0) = n\), then \(\text{deg}g \geq n\). To this end, observe that since \(P_n^g(t_0) = \lim_{z \to t_0} P_n^g(z)\) and since \(\text{rank}P_n^g(z) = \min\{n, \text{deg}g\}\) for every \(z \in \mathbb{D}\), we have

\[
n = \text{rank}P_n^g(t_0) \leq \text{rank}P_n^g(z) = \min\{n, \text{deg}g\}.
\]

Therefore, \(\text{deg}g \geq n\) which completes the proof. \(\square\)

Corollary 3.5. Let \(N \geq 2n + 1\), let \(t_0 \in \mathbb{T}\) and \(s = \{s_0, \ldots, s_N\}\) be such that (3.2) holds and let us assume that \(P_{n+1}^s \not\equiv 0\). Then the problem \(\text{BP}_N\) has no solutions.

Proof. Assume that \(f\) is a solution to the \(\text{BP}_N\). Then \(f\) satisfies conditions (3.3) and therefore, it is a finite Blaschke product of degree \(d = \text{rank}P_n^s < n\). Since \(f\) solves the problem \(\text{BP}_N\) and since \(N \geq 2n + 1\), it follows that \(f_{2n}(t_0) = s_{2n}\) and \(f_{2n+1}(t_0) = s_{2n+1}\). Therefore \(P_{n+1}^f(t_0) = P_{n+1}^s\). Since \(f \in S\), the matrix \(P_{n+1}^f(t_0)\) is positive semidefinite, and so is \(P_{n+1}^s\), which contradicts the assumption. \(\square\)

Corollary 3.6. Let \(N = 2n - 1\) or \(N = 2n\) and let \(t_0 \in \mathbb{T}\) and \(s = \{s_0, \ldots, s_N\}\) be such that (3.3) holds. Then the problem \(\text{BP}_N\) has a (unique) solution if and only if the matrix \(P_n^s\) admits a positive semidefinite structured extension \(P_{n+1}^s\).

Proof. Uniqueness follows from Theorem 3.3. If \(f\) solves the \(\text{BP}_N\), then it is a finite Blaschke product (by Theorem 3.3) and therefore \(f_j(t_0)\) exist for every \(j \geq 0\). Letting \(s_{2n+1} := f_{2n+1}(t_0)\) and also \(s_{2n} := f_{2n}(t_0)\) (in case \(N = 2n - 1\) where \(s_{2n}\) is not prescribed) we have \(P_{n+1}^f = P_{n+1}^s\). Letting \(n = 0\) which proves the “only if” part. Conversely, if \(P_{n+1}^s \geq 0\) for some choice of \(s_{2n}\) and \(s_{2n+1}\) (in case \(N = 2n - 1\)) or for some choice of \(s_{2n+1}\) (if \(N = 2n\) and hence \(s_{2n}\) is prescribed), then we conclude by virtue of Theorem 3.3 that there is an \(f \in S\) such that

\[
f_j(t_0) = s_j \quad (j = 0, \ldots, 2n) \quad \text{and} \quad (-1)^n t_0^{2n+1} s_0(f_{2n+1}(t_0) - s_{2n+1}) \geq 0.
\]

This \(f\) clearly is a solution to the problem \(\text{BP}_N\) for either \(N = 2n - 1\) or \(N = 2n\). \(\square\)

Lemma 3.7. Let us assume that \(t_0 \in \mathbb{T}\) and \(s = \{s_0, \ldots, s_{2n-1}\}\) meet conditions (3.3). Then the problem \(\text{BP}_{2n-1}\) has a (unique) solution if and only if \(\text{rank}P_n^s = \text{rank}P_{n-1}^s\).

Proof. By Theorem 3.3 there exists a unique \(f \in S\) satisfying conditions (3.3), which is a finite Blaschke product of degree \(d = \text{rank}P_n^s < n\). This \(f\) may or may not be a solution of the problem \(\text{BP}_{2n-1}\), i.e., it does or does not satisfy equality \(f_{2n}(t_0) = s_{2n}\) rather than inequality in (3.3). If it does, then \(P_{n}^f(t_0) = P_n^s\) and therefore, we have from (3.4)

\[
\text{rank}P_{n-1}^s = \text{rank}(P_{n-1}^f(t_0)) = \min\{n-1, d\} = d = \text{rank}P_n^s
\]

which proves the “only if” part. To verify the reverse direction, let us assume that the only function \(f\) satisfying conditions (3.3) is not a solution to the problem \(\text{BP}_{2n-1}\), i.e., that the strict inequality prevails in (3.3). Then it follows from the definitions (2.7) and (2.9) that all the corresponding entries in \(P_{n}^f(t_0)\) and \(P_n^s\) are equal, except for the rightmost
diagonal entries $p_{nn}^f$ and $p_{nn}^s$ which are subject to $p_{nn}^f < p_{nn}^s$. Write $\mathbb{P}_n$ and $\mathbb{P}_n^f(t_0)$ in the block form as

$$\mathbb{P}_n = \begin{bmatrix} \mathbb{P}_n^{-1} & B \\ B^* & p_{nn}^s \end{bmatrix}, \quad \mathbb{P}_n^f(t_0) = \begin{bmatrix} \mathbb{P}_n^{-1} & B \\ B^* & p_{nn}^f \end{bmatrix}.$$ 

Since the latter matrices are positive semidefinite, we have by the standard Schur complement argument,

$$\begin{align*}
\text{rank } \mathbb{P}_n^s & = \text{rank } \mathbb{P}_n^{-1} + \text{rank } (p_{nn}^s - X^* \mathbb{P}_n^{-1} X), \\
\text{rank } \mathbb{P}_n^f(t_0) & = \text{rank } \mathbb{P}_n^{-1} + \text{rank } (p_{nn}^f - X^* \mathbb{P}_n^{-1} X),
\end{align*}$$

where $X \in \mathbb{C}^{n-1}$ is any solution of the equation $\mathbb{P}_n^{-1} X = B$. Since $\text{rank } \mathbb{P}_n^f(t_0) = \text{rank } \mathbb{P}_n^{-1}(t_0) = \text{rank } \mathbb{P}_n^{-1}$, it follows from (3.6) that $p_{nn}^f = X^* \mathbb{P}_n^{-1} X$. Since $p_{nn}^f < p_{nn}^s$, we conclude from (3.5) that

$$\text{rank } \mathbb{P}_n^s = \text{rank } \mathbb{P}_n^{-1} + 1.$$

Thus, $\text{rank } \mathbb{P}_n^s \neq \text{rank } \mathbb{P}_n^{-1}$ which completes the proof. \hfill \Box

To proceed, we need the following “symmetry” result.

**Lemma 3.8.** Let us assume that $t_0 \in \mathbb{T}$ and $s = \{s_0, \ldots, s_{2n-1}\}$ are such that

$$|s(t_0)| = 1 \quad \text{and} \quad \mathbb{P}_n = \mathbb{P}_n^s.$$ \hspace{1cm} (3.7)

Let $p_{ij}^s$ be the numbers defined via formula (2.8) for

$$i, j \in \{1, \ldots, 2n-2\}, \quad \text{subject to} \quad 2 \leq i + j \leq 2n - 2.$$ \hspace{1cm} (3.8)

Then $p_{ij}^s = p_{ji}^s$ for all $i, j$ as in (3.8).

Observe that the positive definiteness of the associated matrix $\mathbb{P}_n^s$ is not required. Note also that since the numbers $\Psi_{j\ell}$ in (2.4) are defined for all $j, \ell \geq 1$, the data set $\{t_0, s_0, s_1, \ldots, s_{2n-1}\}$ is exactly what we need to define the numbers $p_{ij}^s$ in the index $(i, j)$ as in (3.8). The statement follows by combining some results from [11] and [7]. We will give the exact references below.

**Proof.** By [11] Theorem 1.9], conditions (3.7) are equivalent to the following matrix equality

$$\mathbb{U}_2^n \Psi_{2n}(t_0) \mathbb{U}_2^n = \Psi_{2n}(t_0),$$ \hspace{1cm} (3.9)

where the $2n \times 2n$ upper triangular matrices $\mathbb{U}_2^n$ and $\Psi_{2n}(t_0)$ are defined via formulas (2.4) and (2.5) and where $\mathbb{U}_2^n$ denotes the complex conjugate of $\mathbb{U}_2^n$. Let us define the matrices $T_{2n} \in \mathbb{C}^{2n \times 2n}$ and $E_{2n}, M_{2n} \in \mathbb{C}^{2n}$ by the formulas

$$T_{2n} = \begin{bmatrix} t_0 & 0 & \cdots & 0 \\ 1 & t_0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & t_0 \end{bmatrix}, \quad E_{2n} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad M_{2n} = \begin{bmatrix} s_0 \\ s_1 \\ \vdots \\ s_{2n-1} \end{bmatrix}.$$ \hspace{1cm} (3.10)

By [7] Theorem 10.5], condition (3.9) is necessary and sufficient for the Stein equation

$$Q - T_{2n} Q T_{2n}^* = E_{2n} E_{2n}^* - M_{2n} M_{2n}^*,$$ \hspace{1cm} (3.11)

to have a solution $Q = [q_{ij}]_{i,j=1}^{2n}$. It is not hard to see (see [7] Lemma 11.1] that the entries $q_{ij}$ are uniquely recovered from (3.11) for all $(i, j)$ as in (3.8); the explicit formula for each such $q_{ij}$ coincides with that in (2.8) for the corresponding $p_{ij}^s$. Thus, $q_{ij} = p_{ij}^s$ for all $(i, j)$ subject to (3.8). On the other hand, by taking adjoints in (3.11) we conclude that $Q^*$
solves (3.11) whenever \( Q \) does. By the above uniqueness, the \((i,j)\)-th entry of \( Q^* \) (which is \( q_{ij} \)) equals \( p_{ij}^* \) for every \((i,j)\) as in (3.8). Therefore, \( p_{ij}^* = q_{ji} \) for every \((i,j)\) subject to (3.8), which completes the proof. □

**Lemma 3.9.** Let us assume that \( t_0 \in \mathbb{T} \) and \( s = \{s_0, \ldots, s_{2n-1}, s_{2n}\} \) meet conditions (3.3). Then the problem \( BP_{2n} \) has a (unique) solution if and only if (2.11) hold.

**Proof.** Due to Corollary 3.6, it suffices to show that if conditions (3.2) are satisfied, then conditions (2.11) are necessary and sufficient for the existence of an \( s_{2n+1} \in \mathbb{C} \) such that the matrix \( P_{n+1}^s \) defined via formula (2.4) is positive semidefinite. Write \( P_{n+1}^s \) in the form

\[
P_{n+1}^s = \begin{bmatrix} P_n^s & B_n \\ C_n & p_{n+1,n+1}^s \end{bmatrix}, \quad \text{where} \quad C_n = [p_{n+1,1}^s, p_{n+1,2}^s, \ldots, p_{n+1,n}^s],
\]

where \( B_n \) is given in (2.10) and where accordingly to (2.8),

\[
p_{n+1,n+1}^s = \sum_{\ell=1}^{n-1} \sum_{r=1}^\ell s_{n+\ell}^s \Psi_{\ell r} s_{n+1-r}^s + \sum_{\ell=1}^n s_{n+\ell}^s \Psi_{\ell,n+1}^s s_{n+1-r}^s + (-1)^{n+1} t_0^2 s_{2n+1}^s s_{n+1}^s.
\]

(3.12)

Recall that the entry \( p_{n+1,n+1}^s \) in \( P_{n+1}^s \) is the only one which depends on \( s_{2n+1} \). Formula (3.12) shows that one can get any \( p_{n+1,n+1}^s \in \mathbb{C} \) by an appropriate choice of \( s_{2n+1} \). Since \( P_n^s \) is Hermitian and \( \|s_0\| = 1 \), it follows from Lemma 3.8 that \( p_{ij}^s = \overline{p_{ji}}^s \) for every \((i,j)\) subject to (3.8) in particular, \( p_{n+1,j}^s = \overline{p_{j,n+1}^s} \) for every \( j = 1, \ldots, n - 2 \). Therefore, the first condition in (2.11) is equivalent to \( C_n = B_n^* \) so that

\[
P_{n+1}^s = \begin{bmatrix} P_n^s & B_n \\ B_n^* & p_{n+1,n+1}^s \end{bmatrix},
\]

(3.13)

where \( B_n \) is given in (3.2). A well known result on positive semidefinite block matrices asserts that the matrix (3.13) is positive semidefinite if and only if the equation

\[
P_n^s X = B_n
\]

(3.14)

is consistent and \( p_{n+1,n+1}^s \geq X^* P_n^s X \) for any solution \( X \) to (3.14). Thus, the matrix (3.13) is positive semidefinite for some \( p_{n+1,n+1}^s \) (or equivalently, for some \( s_{2n+1} \)) if and only if equation (3.14) is consistent. The latter is equivalent to the second condition in (2.11). □

4. THE INDETERMINATE CASE

In this section we consider the cases listed in the second part of Theorem 2.3. Since the case where \( \|s_0\| < 1 \) is covered by Lemma 2.1, we can (and will) assume that \( t_0 \in \mathbb{T} \) and \( s = \{s_0, \ldots, s_N\} \) are such that

\[
\|s_0\| = 1 \quad \text{and} \quad P_n^s > 0
\]

(4.1)

where \( P_n^s \) is defined by formulas (2.4)–(2.7). For the maximal case where \( N = 2n - 1 \), the complete parametrization of all solutions of the \( BP_{2n-1} \) is known and will be recalled in
Theorem 4.1. Let $T \in \mathbb{C}^{n \times n}$ and $E, M \in \mathbb{C}^n$ be the matrices given by

\[
T = \begin{bmatrix}
t_0 & 0 & \ldots & 0 \\
1 & t_0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ldots & 1 & t_0
\end{bmatrix}, \quad E = \begin{bmatrix}
1 \\
0 \\
\vdots \\
0
\end{bmatrix}, \quad M = \begin{bmatrix}
s_0 \\
s_1 \\
\vdots \\
s_{n-1}
\end{bmatrix}
\]  

(4.2)

(\text{these matrices are of the same structure as those in (3.10) but twice smaller}) and let $\tilde{P}$ be the positive definite matrix defined as

\[
\tilde{P} := \mathbb{P}_n^* + MM^*.
\]  

(4.3)

It is not hard to show that the numbers $M^*\tilde{P}^{-1}M$ and $E^*\tilde{P}^{-1}E$ are less than one. We let

\[
\alpha = \sqrt{1 - M^*\tilde{P}^{-1}M}, \quad \beta = \sqrt{1 - E^*\tilde{P}^{-1}E}.
\]

Now we introduce the $2 \times 2$ matrix-function

\[
S = \begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\]  

(4.4)

with the entries

\[
a(z) = E^*(\tilde{P} - z\mathbb{P}_n^*T^*)^{-1}M, \\
b(z) = \beta \left(1 - zE^*(\tilde{P} - z\mathbb{P}_n^*T^*)^{-1}T^{-1}E \right), \\
c(z) = \alpha \left(1 - zM^*T^*(\tilde{P} - z\mathbb{P}_n^*T^*)^{-1}M \right), \\
d(z) = z\alpha\beta M^*(\mathbb{P}_n^*)^{-1}\tilde{P}(\tilde{P} - z\mathbb{P}_n^*T^*)^{-1}T^{-1}E.
\]  

(4.5) - (4.8)

It was shown in Theorem 6.4 \[9\] that $S$ is a rational function of McMillan degree $n$ which is inner in $\mathbb{D}$. Therefore, its entries (4.5), (4.6), (4.8) are rational Schur class functions analytic at $t_0$. Some properties of their Taylor coefficients at $t_0$ are recalled below (see Lemma 6.5 in \[9\] for the proof).

**Theorem 4.1.** Let

\[
a(z) = \sum_{j \geq 0} a_j(t_0)(z - t_0)^j, \quad b(z) = \sum_{j \geq 0} b_j(t_0)(z - t_0)^j, \quad c(z) = \sum_{j \geq 0} c_j(t_0)(z - t_0)^j
\]

(4.9)

be the Taylor expansions of the functions (4.5) - (4.7) at $t_0$. Then

1. $a_j(t_0) = s_j$ for $j = 0, \ldots, 2n - 1$ and $|d(t_0)| = 1$.
2. $b_j(t_0) = c_j(t_0) = 0$ for $j = 0, \ldots, n - 1$.
3. $b_n(t_0) \neq 0$, $c_n(t_0) \neq 0$ and moreover,

\[
t_0^{2n} b_n(t_0) = (-1)^{n-1} c_n(t_0) d(t_0) s_0.
\]

(4.10)

The next theorem (see Theorem 1.6 in \[9\]) describes the solution set of the problem $\text{BP}_{2n-1}$, that is, all functions $f \in S$ such that

\[
f(z) = s_0 + s_1(z - t_0) + \ldots + s_{2n-1}(z - t_0)^{2n-1} + o(|z - t_0|^{2n-1})
\]

(4.11)

and also the solution set of its slight modification $\text{BP}_{2n-1}$ which consists of finding $f \in S$ subject to the stronger nontangential asymptotic

\[
f(z) = s_0 + s_1(z - t_0) + \ldots + s_{2n-1}(z - t_0)^{2n-1} + O(|z - t_0|^{2n}) \quad \text{at} \quad t_0.
\]

(4.12)

**Theorem 4.2.** Let us assume that conditions (4.1) are in force.
1. A function \( f \) is a solution to the problem \( \text{BP}_{2n-1} \) if and only if it is of the form
\[
f(z) = T_s[E](z) := a(z) + \frac{b(z)c(z)E(z)}{1 - d(z)E(z)}
\]  
where the coefficient matrix \( S \) is given in (4.4)–(4.8) and where \( E \) is a Schur-class function such that either
\[
E(t_0) := \lim_{z \to t_0} E(z) \neq \overline{d(t_0)}
\]  
or the nontangential boundary limit \( E(t_0) \) does not exist.

2. A function \( f \) solves the problem \( \text{BP}_{2n-1} \), i.e., \( f \in S \) and satisfies (4.11) if and only if \( f \) is of the form (4.13) for an \( E \in S \) which is either as in (1) or is subject to equalities
\[
E(t_0) = \overline{d(t_0)} \quad \text{and} \quad \lim_{z \to t_0} \frac{1 - |E(z)|^2}{1 - |z|^2} = \infty.
\]  

Remark 2. The correspondence \( E \to f \) established by formula (4.13) is one-to-one and the inverse transformation is given by
\[
E(z) = T_s^{-1}[f](z) = \frac{f(z) - a(z)}{b(z)c(z) + d(z)(f(z) - a(z))}.
\]  
Therefore condition (4.15) explicitly describes the dichotomy between condition (4.12) and a weaker condition (4.11). Although condition (4.12) does not have a clear interpolation interpretation in general, it gets one while being restricted to rational Schur functions. In this case, (4.12) is equivalent to (4.11) and therefore, to conditions (2.1); we refer to [3] for rational boundary interpolation.

Remark 3. Substituting all Schur class functions \( E \) into (4.13) produces all functions \( f \in S \) satisfying conditions (3.3). This relaxed interpolation problem was studied in [7], [9]. Theorem 4.2 also describes the gap between the problem \( \text{BP}_{2n-1} \) and its relaxed version: the strict inequality holds in the last condition in (3.3) for a function \( f \) of the form (4.13) if and only if the corresponding parameter \( E \) is subject to \( E(t_0) = \overline{d(t_0)} \) and \( \liminf_{z \to t_0} \frac{1 - |E(z)|^2}{1 - |z|^2} < \infty \).

Theorem 4.2 shows that conditions (4.11) guarantee that the problem \( \text{BP}_{2n-1} \) has infinitely many solutions which covers therefore the case (b1) in the second part of Theorem 2.3. In case \( N \geq 2n \) we will use representation (4.13) to reduce the original problem \( \text{BP}_N \) to a similar problem with fewer number of interpolation conditions. Still assuming that conditions (4.11) are satisfied we use Taylor expansions (4.9) of rational functions (4.5)–(4.7) at \( t_0 \) and the Taylor expansion \( d(z) = \sum_{j=0}^{\infty} d_j(t_0)(z - t_0)^j \) of the function \( d \) from (4.8) to define the polynomials
\[
F(z) = \sum_{j=0}^{N-2n} (s_{2n+j} - a_{2n+j}(t_0))(z - t_0)^j, \quad D(z) = \sum_{j=0}^{N-2n} d_j(t_0)(z - t_0)^j, \quad B(z) = \sum_{j=0}^{N-2n} b_{n+j}(t_0)(z - t_0)^j, \quad C(z) = \sum_{j=0}^{N-2n} c_{n+j}(t_0)(z - t_0)^j
\]  

Theorem 4.2 also describes the gap between the problem \( \text{BP}_{2n-1} \) and its relaxed version: the strict inequality holds in the last condition in (3.3) for a function \( f \) of the form (4.13) if and only if the corresponding parameter \( E \) is subject to \( E(t_0) = \overline{d(t_0)} \) and \( \liminf_{z \to t_0} \frac{1 - |E(z)|^2}{1 - |z|^2} < \infty \).

Theorem 4.2 shows that conditions (4.11) guarantee that the problem \( \text{BP}_{2n-1} \) has infinitely many solutions which covers therefore the case (b1) in the second part of Theorem 2.3. In case \( N \geq 2n \) we will use representation (4.13) to reduce the original problem \( \text{BP}_N \) to a similar problem with fewer number of interpolation conditions. Still assuming that conditions (4.11) are satisfied we use Taylor expansions (4.9) of rational functions (4.5)–(4.7) at \( t_0 \) and the Taylor expansion \( d(z) = \sum_{j=0}^{\infty} d_j(t_0)(z - t_0)^j \) of the function \( d \) from (4.8) to define the polynomials
\[
F(z) = \sum_{j=0}^{N-2n} (s_{2n+j} - a_{2n+j}(t_0))(z - t_0)^j, \quad D(z) = \sum_{j=0}^{N-2n} d_j(t_0)(z - t_0)^j, \quad B(z) = \sum_{j=0}^{N-2n} b_{n+j}(t_0)(z - t_0)^j, \quad C(z) = \sum_{j=0}^{N-2n} c_{n+j}(t_0)(z - t_0)^j
\]  

Therefore condition (4.15) explicitly describes the dichotomy between condition (4.12) and a weaker condition (4.11). Although condition (4.12) does not have a clear interpolation interpretation in general, it gets one while being restricted to rational Schur functions. In this case, (4.12) is equivalent to (4.11) and therefore, to conditions (2.1); we refer to [3] for rational boundary interpolation.
and the rational function
\[
R(z) = \frac{F(z)}{B(z)C(z) + D(z)F(z)}.
\] (4.19)

Observe that since \(B(t_0)C(t_0) = b_n(t_0)c_n(t_0) \neq 0\) (by part (3) in Theorem 4.1), the numerator and the denominator in (4.19) cannot have a common zero at \(t_0\). Thus, \(R(z)\) is analytic at \(t_0\) if and only if
\[
B(t_0)C(t_0) + D(t_0)F(t_0) = b_n(t_0)c_n(t_0) + d(t_0)(s_{2n} - a_{2n}(t_0)) \neq 0.
\] (4.20)

**Remark 4.** If condition (4.20) is satisfied, then
\[
R_0 := R(t_0) = \frac{d(t_0)(s_{2n} - a_{2n}(t_0))}{(-1)^{n-1}t_0^n|c_n(t_0)|^2s_0 + s_{2n} - a_{2n}(t_0)} \neq d(t_0).
\] (4.21)

**Proof.** Evaluating (4.19) at \(z = t_0\) gives, on account of (4.17), (4.18),
\[
R(t_0) = \frac{s_{2n} - a_{2n}(t_0)}{b_n(t_0)c_n(t_0) + d(t_0)(s_{2n} - a_{2n}(t_0))}
\]
and substituting (4.11) into the right-hand side part of the latter equality gives
\[
R(t_0) = \frac{s_{2n} - a_{2n}(t_0)}{(-1)^{n-1}t_0^n|d(t_0)s_0|c_n(t_0)|^2 + d(t_0)(s_{2n} - a_{2n}(t_0))}
\]
which is equivalent to the second equality in (4.21) since \(|d(t_0)| = 1\) (by part (1) in Theorem 4.1). Since \(s_0 \neq 0\) (by assumption (4.1)) and \(c_n(t_0) \neq 0\) (by part (3) in Theorem 4.1), the inequality in (4.21) follows. \(\square\)

**Theorem 4.3.** Let \(t_0 \in \mathbb{T}\) and \(s = \{s_0, \ldots, s_N\}\) be such that conditions (4.4) hold for some \(n \leq N/2\) and let \(a, b, c, d\) be the rational functions defined in (4.5)–(4.8) with Taylor expansions (4.9) at \(t_0\).

1. If the problem \(\text{BP}_N\) admits a solution, then (4.20) holds, so that the function \(R\) defined in (4.19) is analytic at \(t_0\).
2. If condition (4.20) is satisfied, then a function \(f\) is a solution of the problem \(\text{BP}_N\) if and only if it is of the form (4.13) for some \(E \in \mathcal{S}\) such that
\[
E(z) = R(z) + o(|z - t_0|^{N-2n}) \quad \text{as} \quad z \to t_0.
\] (4.22)

**Proof.** By statement (1) in Theorem 4.1, \(a_j(t_0) = s_j\) for \(j = 0, \ldots, 2n - 1\) which together with definition (4.17) of \(F\) implies that
\[
a(z) + (z - t_0)^{2n}F(z) = \sum_{j=0}^{N} s_j(z - t_0)^j + O(|z - t_0|^{N+1}).
\]
Therefore, asymptotic equality (4.2) can be equivalently written as
\[
f(z) = a(z) + (z - t_0)^{2n}F(z) + o(|z - t_0|^N) \quad (z \to t_0).
\] (4.23)
Let \(f\) be a solution to the \(\text{BP}_N\), i.e., \(f \in \mathcal{S}\) and (4.23) holds. Since \(N \geq 2n\), \(f\) also satisfies (4.12) and therefore it is of the form (4.13) for some \(E \in \mathcal{S}\) (by Theorem 4.2). Observe the equalities
\[
d(z) = D(z) + o(|z - t_0|^{N-2n}) \quad (z \to t_0),
\] (4.24)
\[
b(z)c(z) = (z - t_0)^{2n}B(z)C(z) + o(|z - t_0|^N) \quad (z \to t_0)
\] (4.25)
which follow from definitions 4.17, 4.18 by statement (2) in Theorem 4.1. Substituting (4.23)–(4.25) into (4.16) (which is equivalent to (4.13)) gives

$$E(z) = \frac{F(z) + o(|z - t_0|^{N-2n})}{B(z)C(z) + D(z)F(z) + o(|z - t_0|^{N-2n})} \quad (z \to t_0). \quad (4.26)$$

Since $F$, $B$, $C$, $D$ are polynomials, the limit (as $z \to t_0$) of the expression on the right hand side of (4.26) exists (finite of infinite) and therefore the limit $E(t_0)$ exists too. Since $E$ is a Schur-class function, this limit is finite and therefore, (4.20) holds. Asymptotic equality (4.22) follows from (4.19) and (4.26) due to (4.20).

It remains to prove the “if” part in statement (2) of the theorem. To this end, let us assume that condition (4.20) is met so that $R$ is analytic at $t_0$. Let us assume that $E$ is a Schur-class function subject to asymptotic equality (4.22) and let $f$ be defined by the formula (4.13). Then $f \in S$ since $E \in S$ and the coefficient matrix (4.4) is inner. Substituting (4.22), (4.24) and (4.25) into (4.13) we obtain

$$f(z) = a(z) + \frac{(z - t_0)^{2n}B(z)C(z)R(z) + o(|z - t_0|^N)}{1 - D(z)R(z) + o(|z - t_0|^{N-2n})}, \quad (4.27)$$

By Remark 3.1 $R(t_0) \neq d(t_0)$ and since $|d(t_0)| = 1$ (by part (1) in Theorem 4.1), it follows that $1 - R(t_0)D(t_0) = 1 - R(t_0)d(t_0) \neq 0$. Then we can write (4.27) as

$$f(z) = a(z) + \frac{(z - t_0)^{2n}B(z)C(z)R(z) + o(|z - t_0|^N)}{1 - D(z)R(z)}. \quad (4.28)$$

Now we substitute formula (4.19) for $R$ into the latter equality and arrive at (4.23) which is equivalent to (4.22). Thus, $f$ solves $BP_N$ which completes the proof of the theorem.

**Corollary 4.4.** Let $t_0 \in T$ and $s = \{s_0, \ldots, s_N\}$ meet conditions (4.7) for some $n \leq N/2$ and let $f$ be of the form (4.13) for some function $E \in S$ subject to (4.14). Then the boundary limit $f_{2n}(t_0)$ exists if and only if the limit $E(t_0)$ exists. In this case,

$$f_{2n}(t_0) = a_{2n}(t_0) + \frac{(-1)^{n-1-t_0}|c_n(t_0)|^2s_0E(t_0)}{d(t_0) - E(t_0)} \quad (4.28)$$

**Proof.** Since conditions 4.21 are met, representation (4.13) for $f$ follows from Theorem 4.2 (part 2). Simultaneous existence of the limits follows from Theorem 4.3 (part 2) applied to the problem $BP_{2n}$ with data $s_0, \ldots, s_{2n-1}$ and $s_{2n} := f_{2n}(t_0)$. Since $E(t_0) = R(t_0)$ by (4.22), we have from (4.21)

$$E(t_0) = \frac{d(t_0)(f_{2n}(t_0) - a_{2n}(t_0))}{(-1)^{n-1-t_0}|c_n(t_0)|^2s_0 + f_{2n}(t_0) - a_{2n}(t_0)}. \quad (4.28)$$

Solving the latter equality for $f_{2n}(t_0)$ gives (4.28). \hfill $\square$

**Corollary 4.5.** The problem $BP_N$ has a solution if and only if there exists a function $E \in S$ satisfying asymptotic equality (4.22) which in turn is equivalent to boundary interpolation conditions

$$E_j(t_0) = R_j(t_0) \quad \text{for} \quad j = 0, \ldots, N - 2n. \quad (4.29)$$
The first statement follows directly from part (2) of Theorem 1.3. Since the function $R$ is analytic at $t_0$, the equivalence (4.22) $\leftrightarrow (\text{ref 3.27b})$ follows (see e.g., [7, Corollary 7.9] for the proof). Explicit formula for $R_0 = R(t_0)$ in terms of original data is given in (4.21).

Similar formulas for $j \geq 1$ can be written explicitly but as we will see below, they do not play any essential role in the subsequent analysis.

Now we take another look at formula (4.21). If we will think of $s_{0}, \ldots, s_{2n-1}$ as of given numbers satisfying conditions (4.1), then formula (4.21) establishes a linear fractional map $F: s_{2n} \mapsto R_0$ on the Riemann sphere (recall that the entries $d(t_0), c_n(t_0)$ and $a_2(t_0)$ in (4.21) are uniquely determined by $t_0$ and $s_0, \ldots, s_{2n-1}$). The only value of the argument $s_{2n}$ which does not meet condition (4.20) is $s_{2n}^0 = a_{2n}(t_0) - b_n(t_0)c_n(t_0)d(t_0)$. It is not hard to see from (4.20) that $F(s_{2n}^0) = \infty$ and $F(\infty) = \overline{d(t_0)}$. Thus, if we consider $F$ as a map from $\mathbb{C} \setminus \{s_{2n}^0\}$ into $\mathbb{C}$, then condition (4.20) and inequality in (4.21) will be satisfied automatically.

Still assuming that $t_0, s_0, \ldots, s_{2n-1}$ are fixed and varying $s_{2n}$, we can define two linear functions $s_{2n} \mapsto p_{n+1,n}^s$ and $s_{2n} \mapsto p_{n,n+1}^s$ by formula (2.8). Indeed, letting $(i,j) = (n+1,n)$ and $(i,j) = (n+1,n+1)$ in (2.8) and taking into account that $\Psi_{nn} = (1)^{n-1/2}a_{2n-1}$ and $\Psi_{n+1,n+1} = (-1)^{n}a_{2n+1}$ by (2.9), we have

\[ p_{n+1,n}^s = (-1)^{n-1/2}s_{2n}s_0 + \Phi, \quad p_{n,n+1}^s = (-1)^{n+1/2}s_{2n}s_0 + \Upsilon \quad (4.30) \]

where the terms

\[ \Phi = \sum_{r=1}^{n} \sum_{\ell=1}^{r} s_{n+\ell}^r \Psi_{\ell r} s_{n-r} + \sum_{\ell=1}^{n} s_{n+\ell}^r \Psi_{\ell n} s_0, \]

\[ \Upsilon = \sum_{r=1}^{n} \sum_{\ell=1}^{r} s_{n+\ell-1}^r \Psi_{\ell r} s_{n-1-r} + \sum_{\ell=1}^{n} s_{n+r}^{n-1} \Psi_{\ell n+1} s_0 \quad (4.31) \]

are completely determined from $t_0$ and $s_0, \ldots, s_{2n-1}$.

**Lemma 4.6.** Let $p_{n+1,n}^s$ and $p_{n,n+1}^s$ be defined by formulas (4.21) and (4.30) for some fixed $s_{2n}$. Then

\[ t_0(p_{n+1,n}^s - \overline{p_{n,n+1}^s}) = \frac{|c_n(t_0)|^2(1 - |R_0|^2)}{|d(t_0) - R_0|^2}. \quad (4.33) \]

**Proof.** Let us substitute the constant function $\mathcal{E}(z) \equiv -\overline{d(t_0)}$ into (4.13):

\[ h(z) := T_S[-\overline{d(t_0)}](z) = a(z) - \frac{b(z)c(z)d(t_0)}{1 + d(z)d(t_0)} \]

Since $\mathcal{E}$ is a unimodular constant function and the matrix $S$ of coefficients in (4.13) is inner, it follows that $h$ is a rational inner function, i.e., a finite Blaschke product. Since $\mathcal{E}(z) \equiv -\overline{d(t_0)}$ meets condition (4.14), the function $h$ solves the problem $BP_{2n-1}$ by Theorem 1.2. Thus,

\[ h_j(t_0) = s_j \quad \text{for} \quad j = 0, \ldots, 2n - 1 \quad (4.34) \]

and therefore $P_{n}^h(t_0) = P_{n}^s$ where the matrix $P_{n}^h(t_0)$ is defined via formula (2.9). The extended matrix $P_{n+1}^h(t_0)$ is positive semidefinite, since $h$ is a finite Blaschke product. In particular, the $(n + 1, n)$ and $(n, n + 1)$ entries in this matrix are complex conjugates of each other:

\[ P_{n+1,n}^h = \overline{P_{n,n+1}^h}. \quad (4.35) \]
These entries are defined via formula (2.3) but with \( h_j(t_0) \) replacing \( s_j \). Due to (4.34),
\[
p_{n+1,n}^h = (-1)^{n-1}t_0^{n-1}h_{2n}(t_0)s_0 + \Phi, \quad p_{n,n+1}^h = (-1)^nh_0^{2n+1}h_{2n}(t_0)s_0 + \Upsilon
\]
where \( \Phi \) and \( \Upsilon \) are the same as in (4.31), (4.32). Substituting the two latter equalities into (4.28):
\[
\Phi - \Upsilon = (-1)^{n-1}t_0^{2n-1}h_{2n}(t_0)s_0 + (-1)^nh_0^{2n+1}h_{2n}(t_0)s_0
\]
(4.36)
The formula for \( h_{2n}(t_0) \) can be obtained from Corollary 4.4 by plugging in \( E(t_0) = -d(t_0) \) into (4.28):
\[
h_{2n}(t_0) = a_{2n}(t_0) + \frac{(-1)^n}{2}\left| c_n(t_0) \right|^2 s_0.
\]
On the other hand, we have from (4.21)
\[
s_{2n} = a_{2n}(t_0) + \frac{(-1)^n - 1}{2}\left| c_n(t_0) \right|^2 s_0 R_0
\]
and we conclude from the two last equalities that
\[
t_0^{2n}s_0(s_{2n} - h_{2n}(t_0)) = (-1)^{n-1}\left| c_n(t_0) \right|^2 \left[ \frac{R_0}{d(t_0) - R_0} + \frac{1}{2} \right]
\]
\[
= (-1)^{n-1}\left| c_n(t_0) \right|^2 \left[ \frac{R_0}{d(t_0) + R_0} \right]
\]
(4.37)
Now we make subsequent use of (4.30), (4.36) and (4.37) to get
\[
t_0(p_{n+1,n}^s - P_{n,n+1}^s) = (-1)^{n-1}\left[ t_0^{2n}s_{2n} + t_0^{2n}s_{2n} + t_0\left( \Phi - \Upsilon \right) \right]
\]
\[
= (-1)^{n-1}\left[ t_0^{2n}s_0(s_{2n} - h_{2n}(t_0)) + t_0^{2n}s_0(s_{2n} - h_{2n}(t_0)) \right]
\]
\[
= \left| c_n(t_0) \right|^2 \cdot \text{Re} \left[ \frac{d(t_0) + R_0}{d(t_0) - R_0} \right] = \left| c_n(t_0) \right|^2 \left[ 1 - \left| R_0 \right|^2 \right]
\]
and thus, to complete the proof.

Proof of Theorem 2.3: We will check all possible cases for given data \( t_0, s_0, \ldots, s_N \). Recall that the integer \( n \) is chosen so that the matrix \( P_n \) is positive semidefinite and the larger matrix \( P_{n+1} \) (in case \( N > 2n \)) is not.

Case 1: If \( |s_0| < 1 \), the problem has infinitely many solutions by Lemma 2.1.

Case 2: Let \( |s_0| = 1 \) and \( n = 0 \). Then the problem has no solutions. Indeed, equality \( n = 0 \) means (by the very definition of \( n \)) that \( P_n = t_0s_1s_0^* \neq 0 \). Then it follows from Theorem 2.2 that there are no Schur functions of the form (2.2). Therefore, there are no Schur functions satisfying (1.2), that is solving the problem \( \text{BP}_N \).

Case 3: Let \( |s_0| = 1 \) and \( P_n \) is singular. By Corollary 3.5 Lemma 3.7 and Lemma 3.9 the problem has a unique solution if \( N = 2n - 1 \) or \( N = 2n \) with additional conditions indicated in the formulation of part (1) in Theorem 2.1 and it does not have a solution otherwise.

Case 4: If \( |s_0| = 1, P_n > 0 \) and \( N = 2n - 1 \), then the problem has infinitely many solutions by Theorem 4.7.

Case 5: Let \( N \geq 2n, |s_0| = 1, P_n > 0, \) and \( p_{n+1,n} = P_{n,n+1}^s \). Then the problem has infinitely many solutions if \( N = 2n \) and it has no solutions if \( N > 2n \).
Proof. Let $N = 2n$ so that $s_0, \ldots, s_{2n}$ are given and $s_{2n}$ is such that $p_{n+1,n}^s = \overline{p}_{n,n+1}^s$. By the arguments from the proof of Lemma 3.9 there exists an $s_{2n+1}$ such that the structured extension $P_{n+1}^s$ of $P_n^s$ is positive definite. Since $P_{n+1}^s > 0$, it follows by virtue of Theorem 3.2 that there are infinitely many solutions to the problem $BP_{2n+1}$ each one of which solves the $BP_{2n}$.

To complete the proof we recall a result from [11] (see Theorem 1.8 there):

Let $f \in S$ admit the nontangential boundary limits $f_j(t_0)$ for $j = 0, \ldots, 2n$ which are such that

$$|f_0(t_0)| = 1, \quad P_n^f(t_0) \geq 0 \quad \text{and} \quad p_{n+1,n}^f = \overline{p}_{n,n+1}^f. \quad (4.38)$$

If the nontangential boundary limit $f_{2n+1}(t_0)$ exists then necessarily $P_{n+1}^f \geq 0$.

Let $N > 2n$ and let us assume that $f$ is a solution to the problem $BP_N$. Since $N > 2n$, we have enough data to construct $P_n^s$ which must be equal to $P_{n+1}^f$ by the assumptions of the current case, conditions (4.33) are met and the limit $f_{2n+1}(t_0)$ exists. Therefore, the matrix $P_{n+1}^f = P_n^s$ is positive semidefinite which contradicts the choice of $n$. □

Case 6: Let $N \geq 2n$, $|s_0| = 1$, $P_n^s > 0$, and $p_{n+1,n}^s \neq \overline{p}_{n,n+1}^s$. Then the problem has infinitely many solutions if $t_0 \left(p_{n+1,n}^s - \overline{p}_{n,n+1}^s\right) > 0$ and it has no solutions if $t_0 \left(p_{n+1,n}^s - \overline{p}_{n,n+1}^s\right) < 0$.

Proof. By Corollary 1.5 the problem $BP_N$ has a solution if and only if there is a function $E \in S$ satisfying conditions (4.29) with $R_0(t_0)$ given by (4.21). By (4.33), the number $u := t_0 \left(p_{n+1,n}^s - \overline{p}_{n,n+1}^s\right)$ is real. Since $|c_n(t_0)| \neq 0$ (by statement (3) in Theorem 1.1) and $R_0 \neq \overline{d}(t_0)$ (by (4.21)), it follows from (4.33) that $|R_0| < 1$ if $u > 0$ and $|R_0| < 1$ if $u < 0$. In the first case, there are infinitely many functions $E \in S$ satisfying conditions (4.29) (by Lemma 2.1). Each such function lead to a solution $f$ of the problem $BP_N$. In the second case there is no $E \in S$ satisfying $E(t_0) = R_0$ and therefore, there are no solutions to the $BP_N$. □

All possible cases have been verified. They prove statement (3) and the “if” parts in statements (1) and (2). Since these cases are disjoint, the “only if” parts in statements (1) and (2) now follow. The fact that the unique solution (in part (1)) is a finite Blaschke product follows from Theorem 8.4. In the indeterminate case (2), any solution of the problem belongs to $S^{(n)}(t_0)$, by Theorem 8.2. This completes the proof of Theorem 2.5.

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