Magneto-spin Hall conductivity of a two-dimensional electron gas

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Abstract – It is shown that the interplay of long-range disorder and in-plane magnetic field gives rise to an out-of-plane spin polarization and a finite spin Hall conductivity of the two-dimensional electron gas in the presence of Rashba spin-orbit coupling. A key aspect is provided by the electric-field-induced in-plane spin polarization. Our results are obtained first in the clean limit where the spin-orbit splitting is much larger than the disorder broadening of the energy levels via the diagrammatic evaluation of the Kubo formula. Then the results are shown to hold in the full range of the disorder parameter αpFτ by means of the quasiclassical Green function technique.

It is well established that the peculiar linear-in-momentum dependence of the Rashba (and of Dresdelsehaus) spin-orbit coupling leads in a two-dimensional electron gas (2DEG) to the vanishing of the spin Hall conductivity [1–4]. This can be directly recognized by considering the continuity-like equation for the in-plane spin polarization, where the spin-nonconserving terms can be written as the spin current associated to the out-of-plane spin polarization and to the spin Hall effect [5–7]. In this paper, we show that the interplay of an in-plane magnetic field, B, taken parallel to the electric field, E, (say along the e_x-axis) and long-range disorder changes this behavior providing then a potential handle on the spin Hall effect. In particular, we show that while the out-of-plane spin polarization is linear in the magnetic field, the spin Hall conductivity is quadratic. Our analysis is carried out in two steps. In the first step, we calculate the out-of-plane spin polarization using the diagrammatic approach of ref. [3], valid in the clean limit when the spin-orbit splitting is much larger than the disorder-induced broadening, 2αpF ≫ τ⁻¹. In order to make contact with the analysis of ref. [9] performed in the opposite dirty limit (αpF ≪ τ⁻¹), we present, in

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the second step, a derivation based on the Eilenberger equation for the quasiclassical Green function in the presence of spin-orbit coupling [11]. The advantage of doing so is that the analysis is valid for an arbitrary value of the parameter $2a_Fp_F^2/3$. We notice that the adoption of the above approximation for the vertex and corresponds to the inclusion of disorder effects to leading order in the parameter $1/(e_F^2a_F)$.

In order to consider the effect of long-range disorder, we expand the above scattering probability as

$$|V|^2 = V_0 + 2V_1 \cos(\varphi - \varphi') + 2V_2 \cos(2\varphi - 2\varphi') + \cdots,$$

where $\varphi - \varphi'$ is the angle between the two momenta $p$ and $p'$. The harmonics, $V_0, V_1, V_2$ are functions of $|p|$ and $|p'|$. In the following we will ignore this dependence and take both momenta at $p_F$. To first order in the magnetic field the diagrams to be evaluated are shown in fig. 1. Notice that all the vertices and propagators appearing in the diagrams must be evaluated at zero magnetic field. It is then convenient, following ref. [3], to use the disorder-free Hamiltonian eigenstates

$$|p\pm\rangle = \frac{1}{\sqrt{2}}(\pm\exp(-i\varphi)|p\uparrow\rangle + |p\downarrow\rangle)$$

corresponding to the eigenvalues $E_{1,2} = p^2/2m \pm \alpha p$ with $\tan(\varphi) = p_y/p_x$. In terms of the transformation matrix, $U$, defined by eq. (8), the Pauli matrices transform as

$$U\sigma_x U^\dagger = \hat{p}_y \sigma_x + \hat{p}_x \sigma_y, \quad \hat{p}_x \equiv \cos(\varphi),$$

$$U\sigma_y U^\dagger = -\hat{p}_x \sigma_x + \hat{p}_y \sigma_y, \quad \hat{p}_y \equiv \sin(\varphi),$$

As a consequence the spin density vertex, the magnetic field insertion and the charge current vertex become

$$U\sigma_x U^\dagger = -\frac{1}{2} \sigma_x,$$

$$U(-\omega_s \sigma_x) U^\dagger = -\omega_s (\hat{p}_y \sigma_x + \hat{p}_x \sigma_y),$$

Upon impurity averaging the spin and charge vertices get renormalized. In terms of the renormalized quantities, eq. (5) becomes to first order in the Zeeman field

$$s_z = -\frac{i}{4\pi} \int_\mathbb{R} \frac{d\xi}{4\pi} \sum_{\mu\nu} \hat{p}_x \mu (\Gamma_{\mu\nu} G_{\nu} - \Gamma_{\mu\nu} G_{\nu}^A) G_{\nu}^R G_{\mu}^A J_{\nu\mu},$$

$$(13)$$

$$\hat{p}_x \mu (\Gamma_{\mu\nu} G_{\nu} - \Gamma_{\mu\nu} G_{\nu}^A) G_{\nu}^R G_{\mu}^A J_{\nu\mu},$$

$$(12)$$

$$U j^z \sigma_z U^\dagger = \left( \frac{p}{m} \sigma_0 + \alpha \sigma_z \right) \hat{p}_x - \alpha \hat{p}_y \hat{p}_y. $$

$$\begin{align*}
\Sigma^{R,A}(p) &= \sum_{p'} |V(p - p')|^2 G^{R,A}(p'), \\
\end{align*}$$

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\end{align*}$$

$$(6)$$

$$(5)$$

$$(4)$$

$$(3)$$

$$(2)$$

$$(1)$$

$$(11)$$

$$(10)$$

$$(9)$$

$$(8)$$

$$s_z = -\frac{1}{2\pi} \text{Tr}_\sigma \left[ s_z G(p) j^z G^A(p) \right] |\xi| e_x,$$
where the first (second) term in the brackets refers to the magnetic field insertion in the top (bottom) Green function line and $\mu = \pm$ labels the eigenstates. The quantities $\Gamma_{\mu\nu}$ and $J_{\nu,\mu}$ are the dressed vertices corresponding to $s_z$ and $j_x^2$, respectively, and the Green functions are evaluated via the self-energy given in eq. (6). Apart from the spin vertex $\Gamma_{\nu\mu}$, all the other quantities have been evaluated in ref. [3], where it has been shown that the off-diagonal matrix element $J_{\nu,\mu}$ vanishes and the self-energy is diagonal in the eigenstate basis with

$$\Sigma^{(A)}_{\pm} = -\frac{(+) i}{2\tau_\pm}, \quad \tau_\pm = \tau \left(1 \mp \frac{V_1}{V_0} v_F \right), \quad \frac{1}{\tau} = 2\pi N_0 V_0.$$  

(14)

The spin vertex obeys the equation

$$\Gamma_{\nu\mu} = 1 + \sum_{\nu'} \Gamma_{\nu\nu'} G^{R}_{\nu'} G^{A}_{\nu} |V|^2 \frac{1 + \mu \cos(\varphi - \varphi')}{2},$$

(15)

which yields

$$\Gamma_{\nu\mu} = 1 + \frac{\mu_i}{2\alpha v_F} V_1.$$

(16)

By using $G^{R}_{\nu\nu'} G^{A}_{\nu} = i\tau_\mu (G^{R}_{\nu\nu'} - G^{A}_{\nu})$, integrating over the energy, $\xi = p^2/2m - \mu$, and keeping terms up to order $\alpha/v_F$, eq. (13) becomes

$$s_z = -\frac{|e|}{2} \frac{\omega_s}{\omega_F} \left(1 - \frac{V_1}{V_0}\right) \sum_{\mu = \pm} \frac{N_\mu J_{\mu} \tau_\mu}{p_\mu^2}.$$  

(17)

where $N_\pm = N_0 (1 \mp \alpha/v_F)$, $p_\pm = p_F (1 \mp \alpha/v_F)$ are the density of states and the Fermi momentum of the two spin subbands and $J_{\pm\pm} (p_\pm) = J_{\pm\pm} (p_\pm) = J_{\pm\pm} (p_\pm)$ are the Fermi-surface expressions of the charge vertices. Finally, borrowing from ref. [3] the expression for the vertices

$$J_{\pm} = v_F \left(\frac{V_0}{V_0 - V_1} \mp \frac{\alpha}{v_F} \frac{V_1 + V_2}{V_0 - V_2}\right),$$

(18)

one gets the out-of-plane spin polarization

$$s_z = -\frac{|e|}{2} |\epsilon| \frac{\omega_s}{\alpha v_F} \frac{N_0}{2} \frac{V_1 - V_2}{V_0 - V_2},$$

(19)

and, via eq. (4), the spin Hall conductivity

$$\sigma_{sH} = -\frac{|e|}{4\pi} \frac{\omega_s}{\alpha v_F} \left(\frac{V_1}{V_0 - V_2} - \frac{V_2}{V_0 - V_2}\right).$$  

(20)

Remarkably the above central result shows that the sign of the spin Hall conductivity depends on the relative strength of the harmonics of the scattering probability. This may explain the sign change in the numerical evaluation of ref. [8]. On the other hand, eq. (19) is inconsistent with ref. [9], where for the disorder model of eq. (7) one would expect a vanishing out-of-plane polarization. Notice that eqs. (19), (20) have been derived for $\omega_s$ going to zero, that is $\omega_s \ll \alpha v_F$.

Recall that we derived eqs. (19) and (20) in the clean limit. We now rederive the above results by means of the kinetic equations approach of ref. [11] and find that eqs. (19) and (20) are valid for all values of the disorder parameter $\alpha v_F$. Furthermore we will derive the effective Bloch equations in the dirty limit. We start with the Eilenberger equation

$$\partial_\tau \tilde{g} = -\frac{1}{2} \sum_{\mu = \pm} \left\{ \frac{p_\mu}{m} + \frac{\partial}{\partial p} (b_\mu \cdot \sigma), \frac{\partial}{\partial \sigma} g_\mu \right\},$$

(21)

for the quasiclassical Green function ($\tilde{g} \equiv g_{t1t2} (\tilde{p}; x)$)

$$\tilde{g} = \frac{1}{\pi} \int d\xi \tilde{G}_{t1t2} (p; x), \quad \tilde{G} = \begin{pmatrix} G^R & G^A \\ G^A & G^R \end{pmatrix},$$

(22)

where $\tilde{G}_{t1t2} (p; x)$ is the Wigner representation of the Green function, which has both matrix structure in the Keldysh (denoted by the check symbol) and spin spaces. $[,]$ and $\{,\}$ indicate commutator and anticommutator. As for the diagrammatic approach, the index $\mu = \pm$ labels the two spin subbands ($b_\pm = b (p_\pm)$). In integrations like in eq. (22) the corresponding poles in the Green functions yield the two-component decomposition of the quasiclassical Green function

$$\tilde{g}_{\pm} = \frac{1}{\pi} \begin{pmatrix} 1 \pm b_0 \cdot \sigma \end{pmatrix}. $$

(23)

The “0” subscript denotes evaluation at the Fermi surface in the absence of spin-orbit coupling. In the following we are going to use eq. (21) to first order in the parameter $|b_0|/\epsilon_F$. The connection to the physical observables is made by integrating over the energy $\epsilon$, which is the Fourier conjugated variable of the time difference $t_1 - t_2$. For instance, the out-of-plane spin density is given by the angular average of the Keldysh component

$$s_z = \frac{3}{8} \int d\epsilon \langle \sigma_{z} \rangle, \quad \langle \ldots \rangle \equiv \int_{0}^{2\pi} \frac{d\phi}{2\pi} \ldots .$$  

(24)

In order to solve, to linear order in the electric field, the Keldysh component of the Eilenberger equation (21), we use the minimal substitution $\partial_\xi g \rightarrow -\frac{e}{2} |\epsilon| \tilde{g}_z \tilde{g}_x \partial_\xi g_{eq}$ where $g_{eq} = \tanh(\epsilon/2T) (g_{eq}^R - g_{eq}^A)$ with $g_{eq}^R = g_{eq}^A = -|\tilde{g}_x| = = -\frac{1}{2} - \frac{1}{2} b_0 \cdot \sigma$ is the equilibrium quasiclassical Green function. As in the diagrammatic treatment previously developed, we find it convenient to transform the equations to the eigenstate basis via eq. (9). After expressing the quasiclassical Green function as a four-dimensional column vector

$$\tilde{g} = U g U^\dagger = \tilde{g}_0 \sigma_0 + \tilde{g}_x \cdot \sigma \rightarrow (\tilde{g}_0 \tilde{g}_x \tilde{g}_y \tilde{g}_z)^\dagger,$$

(25)

the Eilenberger equation (21) can be then written as a linear system of four equations for the components of $\tilde{g}$

$$\partial_\tau \tilde{g} = -\frac{1}{\tau} (M_0 + M_1) \tilde{g} + \frac{1}{\tau} (1 + N) (K \tilde{g}) + S_0 + S_1,$$  

(26)

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where
\[
N = -\frac{\alpha}{v_F} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]  
(27)
\[
M_0 = 1 + \frac{V_1}{V_0} N + 2\alpha p_F \tau \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},
\]  
(28)
\[
M_1 = 2\omega_s \tau \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\tilde{p}_x & 0 & -\tilde{p}_y \\ \frac{\alpha}{v_F} \tilde{p}_x & 0 & 0 & \frac{\alpha}{v_F} \tilde{p}_y \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]  
(29)
In eq. (26), \(\ldots\) denotes angle integration over \(\varphi'\) with the scattering kernel that can be expandend into angular harmonics as
\[
K(\varphi - \varphi') = K^{(0)} + \cos(\varphi - \varphi') K^{(a)} + \sin(\varphi - \varphi') K^{(b)} + \ldots,
\]  
(30)
each coefficient being itself a matrix
\[
K^{(0)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{V_1}{V_0} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{V_1}{V_0} \end{pmatrix},
\]  
\[
K^{(b)} = \frac{V_0 - V_2}{V_0} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix},
\]  
and
\[
K^{(a)} = \frac{1}{V_0} \begin{pmatrix} 2V_1 & 0 & 0 & 0 \\ 0 & V_0 + V_2 & 0 & 0 \\ 0 & 0 & 2V_1 & 0 \\ 0 & 0 & 0 & V_0 + V_2 \end{pmatrix}.
\]  
Finally the source electric-field dependent terms are
\[
S_0 = E \begin{pmatrix} \tilde{p}_x \\ \tilde{p}_x \\ -\frac{\alpha}{v_F} \tilde{p}_y \\ 0 \end{pmatrix},
\]  
and
\[
S_1 = \frac{\omega_s}{v_F p_F} E \begin{pmatrix} 0 \\ \tilde{p}_x \tilde{p}_y \\ 0 \\ \tilde{p}_x^2 \end{pmatrix},
\]  
where \(E = |e|E_{\text{ex}}v_F \partial_x (2 \tanh(\epsilon/2T))\). Notice that, consistently with the accuracy we are working, one may use for the charge density component the solution obtained in the absence of both spin-orbit coupling and magnetic field
\[
j_{c,x} \sim \langle \tilde{p}_x \tilde{g}^{(0)} \rangle = \frac{1}{2} \frac{V_0}{V_0 - V_1} \tau E, \tag{31}
\]  
where the characteristic transport time renormalization \(\tau_T = \tau_{V_0}/(V_0 - V_1)\) appears. We seek now a stationary solution of eq. (26) which is evaluated at first order in the magnetic field \(\tilde{g} = \tilde{g}^{(0)} + \tilde{g}^{(1)} + \ldots\). We then get
\[
\langle \tilde{g}^{(1)} \rangle = (M_0 - K^{(0)})^{-1} \tau \langle S_1 \rangle - \langle M_1 \tilde{g}^{(0)} \rangle.
\]  
(32)
According to the transformation of eq. (10), the out-of-plane spin polarization is related to
\[
s_z \sim \langle \tilde{g}^{(1)} \rangle = -\frac{\omega_s}{2\alpha p_F} \left( \frac{1}{v_F p_F} E + \frac{1}{\alpha p_F \tau} \frac{V_0 - V_1}{V_0} \langle \tilde{p}_x \tilde{g}^{(0)} \rangle \right),
\]  
(33)
which is expressed in terms of \(\langle \tilde{p}_x \tilde{g}_3 \rangle\) evaluated at zero magnetic field. This latter quantity is nothing but the in-plane spin polarization (cf. the second line of eq. (9)). By multiplying the second component of the system equation (26) by \(\tilde{p}_x\) and performing the angle average, one obtains the generalization, for long-range disorder, of the Edelstein result [10]
\[
s_y \sim \langle \tilde{p}_x \tilde{g}_3^{(0)} \rangle = -\frac{\alpha}{v_F} \frac{V_0}{V_0 - V_2} \tau E.
\]  
(34)
Finally, by using eqs. (33), (34) into eq. (24), one recovers the result (19) of the diagrammatic approach, which is now manifestly valid for any strength of the disorder. To understand the meaning of eq. (34), it is useful to recall the origin of the in-plane polarization [10]: In the presence of an electric field the Fermi surface is shifted by \(\partial \tilde{p}_x \sim |e|E_{\text{ex}} \tau T\). As a result the total spin of the electrons neither in the plus nor in the minus band adds up to zero. Although the contributions of both bands tend to cancel, a finite spin polarization remains due to the \(\alpha/v_F\) corrections in the density of states. For long-range disorder, the Fermi surface shift is proportional to the transport time \(\tau_T\), so one might expect the transport time also in the in-plane spin polarization. However due to the \(\alpha/v_F\) corrections each band has its own effective transport time, \(\tau_{T_{\pm}} \equiv J_{\pm} \tau_{\pm}/v_F\) and the explicit result reads \((s_0 = \alpha N_0 |e|E_{\text{ex}} \tau\)
\[
s_y = \frac{v_F}{4} (N_+ \tau_{T_{\tau,\tau}} - N_- \tau_{\tau,\tau}) |e|E_{\text{ex}} = -\frac{V_0}{V_0 - V_2} s_0,
\]  
(35)
where the last equality has been obtained by using the explicit expressions of \(N_{\pm}, \tau_{\pm}\) and \(J_{\pm}\) introduced in eqs. (14), (17), (18). At last we study the combined effect of magnetic and electric field in the diffusive regime, \(\omega_s \tau, \alpha p_F \tau \ll 1\). The effective Bloch equations for the spin density are
\[
\partial_t s_x = -\tau_s^{-1} (s_x - s_y \tau),
\]  
(36)
\[
\partial_t s_y = -\tau_s^{-1} [s_y + s_0 V_0/(V_0 - V_2)^{-1}] + 2\omega_s s_z,
\]  
(37)
\[
\partial_t s_z = -2\tau_s^{-1} s_z - 2\omega_s [s_y + s_0 \tau_{T_{\tau}}],
\]  
(38)
where \(\tau_s^{-1} = 2(\alpha p_F)^2 \tau_T\), and \(s_y \equiv N_0 \omega_s\). Equations (36) and (38) agree with what was found in ref. [9], the only difference is in the term proportional to the electric field (i.e. \(s_0\)) in eq. (37). The stationary spin polarization as a function of magnetic field is now determined as
\[
s_y = \frac{V_0 s_0}{V_0 - V_2} 1 + \frac{2\omega_s^2 \tau_s}{V_0 - V_2},
\]  
(39)
\[
s_z = \frac{V_0 s_0}{V_0 - V_2} \frac{V_1 - V_2}{V_0 - V_2} \omega_s \tau_s,
\]  
(40)
showing an out-out-plane contribution as observed experimentally in ref. [12]. Notice that eq. (40) agrees completely with the diagrammatic result of eq. (19) to linear order in the magnetic field, when $\omega_s \tau_s \ll 1$.

As stated previously, our analysis has been performed at the level of the ladder approximation so that weak localization effects have been neglected. The latter may be of comparable size, at least in the diffusive regime $\alpha p_F \tau \ll 1$, as has been discussed in ref. [7]. The investigation of the weak localization correction in the presence of an in-plane magnetic field is certainly a relevant open question, but beyond the scope of this work.

In conclusion, we have shown that the combined effect of an in-plane magnetic field, long-range disorder and spin-orbit coupling gives rise to an out-of-plane spin polarization and finite spin Hall conductivity, whose value does not depend on the concentration of defects as long as the 2DEG is in the metallic regime. To obtain the correct value of the electric-field–induced in-plane spin polarization it is essential to take into account the different transport times in the two spin-orbit splitted bands.

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