THE FOX-HATCHER CYCLE AND A VASSILIEV INVARIANT OF ORDER THREE

SAKI KANOU AND KEIICHI SAKAI

Abstract. We show that the integration of a 1-cocycle $I(X)$ of the space of long knots in $\mathbb{R}^3$ over the Fox-Hatcher 1-cycles gives rise to a Vassiliev invariant of order exactly three. This result can be seen as a continuation of the previous work of the second named author [14], proving that the integration of $I(X)$ over the Gramain 1-cycles is the Casson invariant, the unique nontrivial Vassiliev invariant of order two (up to scalar multiplications). The result in the present paper is also analogous to part of Mortier’s result [11]. Our result differs from, but is motivated by, Mortier’s one in that the 1-cocycle $I(X)$ is given by the configuration space integrals associated with graphs while Mortier’s cocycle is obtained in a combinatorial way.

1. Introduction

The spaces of smooth embeddings of manifolds are receiving a lot of attention in topology, on the ground that various important methods in algebraic and geometric topology are being applied to the spaces. In this paper we study the space of (framed) long knots in $\mathbb{R}^3$.

Definition 1.1. A long knot is an embedding $f: \mathbb{R}^1 \hookrightarrow \mathbb{R}^3$ satisfying $f(x) = (x, 0, 0)$ for any $x \in \mathbb{R}^1$ with $|x| \geq 1$. A framed long knot is a smooth map $\tilde{f} = (f, w): \mathbb{R}^1 \to \mathbb{R}^3 \times SO(3)$ such that $f$ is a long knot, the first column of $w(x) \in SO(3)$ is equal to $f'(x)/|f'(x)|$ and $w(x)$ is the identity matrix for any $x \in \mathbb{R}^1$ with $|x| \geq 1$. The space of all long knots (respectively framed long knots) is denoted by $\mathcal{K}$ (respectively $\tilde{\mathcal{K}}$).

The recent studies of $\mathcal{K}$ (and its high dimensional analogues) are revealing relations between the topological nature of $\mathcal{K}$ and the Vassiliev invariants (see for example [9]) for knots and links. In [14] the second named author has constructed a de Rham 1-cocycle $I(X)$ of $\mathcal{K}$ (see [3]), by means of the integrations over configuration spaces associated with graphs, and has shown that the integration of $I(X)$ over the Gramain cycles of $\mathcal{K}$ gives rise to the Casson invariant $v_2$, the Vassiliev invariant of order two uniquely characterized by $v_2(\text{trivial knot}) = 0$ and $v_2(\text{trefoil knot}) = 1$. This may be seen as a real valued version of [16, Theorem 2]. After that Mortier has given another 1-cocyle $\alpha^1_1$ of $\mathcal{K}$ in a combinatorial way and has shown that its evaluations over the Gramain cycles and the Fox-Hatcher cycles $\mathcal{F}H$ are Vassiliev invariants of orders respectively two and three [11, Theorem 4.1]. In [6] 1-cocycles on $\mathcal{K}$ are also studied in detail from a combinatorial viewpoint.

The main result in the present paper is analogous to the order three part of Mortier’s result.

Theorem 1.2. The integration of $I(X)$ over the Fox-Hatcher cycles gives rise to a Vassiliev invariant of order three for framed long knots. More precisely we have

$$\int_{\mathcal{F}H} I(X) = 6v_3(f) - \text{lk}(\tilde{f})v_2(f),$$

where

- $p: \tilde{\mathcal{K}} \to \mathcal{K}$ is the first projection and $f = p(\tilde{f})$,
- $v_2$ is the Casson invariant, and $v_3$ is the Vassiliev invariant of order three characterized by the conditions

$$v_3(\text{trivial knot}) = 0, \quad v_3(3^+_1) = 1, \quad v_3(3^-_1) = -1$$

($3^+_1$ and $3^-_1$ are respectively the right-handed and the left-handed trefoil knots), and
- $\text{lk}(\tilde{f}) \in \mathbb{Z}$ is the framing number of $\tilde{f}$ (see Remark 1.3 below).

Remark 1.3. The framing number $\text{lk}(f)$ is the linking number of $f = p(\tilde{f})$ and $f'$, where $f'$ is the long knot obtained by moving $f$ slightly into the direction of the second column of $w$. In fact the map $p \times \text{lk}: \tilde{\mathcal{K}} \to \mathcal{K} \times \mathbb{Z}$ is a homotopy...
The key ingredient is Theorem 4.1 and is proved in §4.2. The formula (1.1) is verified in §4.3.

Vassiliev invariants (0-cocycles of $K$) are obtained from trivalent graphs, while our 1-cocycle $I_c$ is constructed by means of the configuration space integral associated with graphs, that was developed in [1, 3, 10] to describe Vassiliev invariants and was generalized in [5] to obtain a cochain map from a graph complex to $\Omega_{DR}^*(\mathcal{K})$ (up to some correction terms, that vanish in the cases of the spaces of long knots in high dimensional spaces). Vassiliev invariants (0-cocycles of $\mathcal{K}$) are obtained from trivalent graphs, while our 1-cocycle $I_c$ comes from non-trivalent graphs (see Figure 3.3). It is very interesting, although not strange, that non-trivalent graphs may also have information of Vassiliev invariants.

We note that the right hand side of (1.1) coincides with the formula for Mortier’s invariant of order three. We thus expect that the 1-cocycle $I_c$ is cohomologous to Mortier’s $\alpha_3$ (this is true on the connected components of torus and hyperbolic knots; see [8, p. 2]).

This paper is organized as follows. In §2 the Fox-Hatcher cycle is introduced, and in §3 the construction of the 1-cocycle $I_c$ is reviewed. Our invariant $\nu$, the left hand side of (1.1), is shown to be of order three in Corollary 4.2. The key ingredient is Theorem 4.1 and is proved in §4.2. The formula (1.1) is verified in §4.3.

2. The Fox-Hatcher cycle

2.1. The Fox-Hatcher cycle. The Fox-Hatcher cycle was introduced in [7], and was later studied in [8] from the viewpoint of the space of knots. If $f = p(\tilde{f})$ is not trivial, it then gives a non-zero element of $\pi_1(\mathcal{K}_f)$, where $\mathcal{K}_f$ is the path component of $\mathcal{K}$ containing $\tilde{f}$.

The Fox-Hatcher cycle is defined as follows. A framed long knot can be seen as a based embedding $f : S^1 \hookrightarrow S^3$ (we see $S^3$ as in $\mathbb{R}^4$) together with a framing $w$, with a prescribed behavior near the basepoint. For $t \in S^1$, $w(t)$ is an orthonormal basis of $T_{f(t)}S^3$ whose first vector is $f'(t)/|f'(t)|$. There exists an $S^1$-action on the space of such embeddings defined by $(\theta \cdot (f, w))(t) = (A(\theta)^{-1}f(t - \theta), A(\theta)^{-1}w(t - \theta))$, where $A(\theta) \in SO(4)$ is the matrix given by $A(\theta) = (w(\theta), f(\theta))$. For any $\tilde{f} \in \mathcal{K}$, this action determines a 1-cycle $FH_{\tilde{f}} : S^1 \to \mathcal{K}_f$ and we call it the Fox-Hatcher cycle. We notice that the $S^1$-action looks very similar to the natural $S^1$-action on free loop spaces by the reparametrization, and in fact this action defines a BV-operation on $H_*(\mathcal{K})$ [15].

Practically it is convenient to describe $FH$ on knot diagrams. In this paper a framed long knot is drawn in a usual knot diagram with so-called blackboard framing.

Definition 2.1. Let $D$ be a knot diagram of $\tilde{f}$ with blackboard framing and $c$ the “left-most” crossing, namely the crossing that we meet first when traveling from $f(-1)$ along the natural orientation of $f$. We call the transformation shown in Figure 2.1 the Fox-Hatcher move (FH-move for short) on $c$. The left-most crossing $c$ disappears after the FH-move on $c$ and the right-most crossing $c'$ is created. If the arc that moves in the FH-move is the over-arc (resp. under-arc) at $c$, then after the FH-move it becomes the over-arc (resp. under-arc) at $c'$. We arrive the original diagram $D$ after performing the FH-moves for all the other crossings $c$ of $D$ and the newborn crossings $c'$. The sequence of these FH-moves realizes $FH_{\tilde{f}}$.

2.2. FH moves and Gauss diagrams. The configuration of crossings of a knot diagram is encoded by (linear) Gauss diagrams. Here we see how the FH-move on the left-most crossing changes the Gauss diagram.

Definition 2.2. A (linear) Gauss diagram is a subdivision of $\{1, 2, \ldots, 2n\}$ for some natural number $n$ into a union $\bigcup_{1 \leq k \leq n} [i_k, j_k]$ of $n$ subsets of cardinality 2.
A Gauss diagram can be seen as a graph on $\mathbb{R}^1$ with even number of vertices all of which are on $\mathbb{R}^1$ and each vertex is joined by exactly one edge with another vertex. Here segments in $\mathbb{R}^1$ interposed between two vertices are not regarded as edges. See Figure 2.2 for example.

**Definition 2.3** ([14] Definition 3.3). Let $c_1, \ldots, c_n$ be (part of the) crossings of a knot diagram of $f \in \mathcal{K}$ such that each $c_i$ corresponds to $f(p_i)$ and $f(q_i)$, with $-1 < p_1 < \cdots < p_n < 1$ and $p_i < q_i$ for any $i = 1, \ldots, n$. We say that the crossings $c_1, \ldots, c_n$ respect a Gauss diagram $G$ if $G$ is isomorphic to the Gauss diagram $G_{c_1, \ldots, c_n}$ obtained by joining $p_i$ and $q_i$ for $i = 1, \ldots, n$. See Figure 2.2.

Under the setting of Definition 2.3, the left-most crossing is $c_1$. Let $G$ be the Gauss diagram that $c_1, \ldots, c_n$ respect. Then the new knot diagram obtained by performing the FH-move on $c_1$ has crossings $c_2, \ldots, c_n, c'_1$ that respect the Gauss diagram $G'$ obtained by moving the left-most vertex (corresponding to $c_1$) to the right-most one. See Figure 2.3.

We eventually arrive the original Gauss diagram after performing the FH-moves on all the crossings $c$ of the original diagram and the newborn crossings $c'$. This sequence produces a cycle of Gauss diagrams (see Figures 2.4, 2.4, 2.5). In this way the set of all the Gauss diagrams is decomposed into the disjoint cycles.

### 3. The cocycle $I(X)$

In this section we give a quick review of the construction of differential forms on $\mathcal{K}$ associated with graphs. See also [13][5][10][17] for details.

By a graph we mean the oriented real line $\mathbb{R}^1$ together with two kind of vertices, one is called interval and the other free, and oriented edges connecting them (see Figure 3.1). The interval vertices (or i-vertices for short) are placed on the oriented line while the free vertices (or f-vertices for short) are not on the line. The i-vertices and the f-vertices of a graph $X$ are labeled by respectively the numbers $1, \ldots, n$, where $v_1$ and $v_\ell$ are respectively the numbers of the i-vertices and the f-vertices of $X$, so that the labels of the i-vertices respect the orientation of the real line. We allow graphs to have a loop, an edge that has exactly one i-vertex as its endpoint (see Figure 3.1).

For a graph $X$, let $E_X$ be the configuration space

$$
E_X := \{(f, (y_1, \ldots, y_{v_1+v_\ell})) \in \mathcal{K} \times \text{Conf}_{v_1+v_\ell}(\mathbb{R}^3) \mid y_i = f(x_i) \text{ for some } x_i \in \mathbb{R}^1 \text{ for } i = 1, \ldots, v_1, \}
$$

where

$$
\text{Conf}_k(M) := \{(x_1, \ldots, x_k) \in M^{\times k} \mid x_i \neq x_j \text{ if } i \neq j \}
$$

is the space of $k$-point configurations on a space $M$. 

### References

[13][5][10][17]
To an oriented edge $\alpha$ of $X$ from the $i$-th vertex to the $j$-th vertex ($i \neq j$), we assign a map

\begin{equation}
\varphi_\alpha : E_X \to S^2, \quad \varphi_\alpha(f, (y_1, \ldots, y_\ell)) := \frac{y_j - y_i}{|y_j - y_i|}.
\end{equation}

To a loop $\alpha$ at $k$-th i-vertex ($1 \leq i \leq \nu_i$) we assign

\begin{equation}
\varphi_\alpha : E_X \to S^2, \quad \varphi_\alpha(f, (y_1, \ldots, y_\ell)) := \frac{f'(x_k)}{|f'(x_k)|},
\end{equation}

where $x_k \in \mathbb{R}^1$ satisfies $y_k = f(x_k)$.

Let $\omega \in \Omega^2_{DR}(S^2)$ be a unit volume form of $S^2$ that is anti-symmetric, meaning that $i^*\omega = -\omega$ for the antipodal map $i : S^2 \to S^2$. Define $\omega_X \in \Omega^2_{DR}(E_X)$ by

\begin{equation}
\omega_X := \bigwedge_{\text{edges } \alpha \text{ of } X} \varphi_\alpha^*(\omega),
\end{equation}

where $e$ is the number of edges of $X$. The order of the edges is not important because $\deg \omega = 2$ is even.

Let $\pi_X : E_X \to K$ be the first projection. This is a fiber bundle with fiber

\begin{equation}
\pi_X^{-1}(f) = \{y \in \text{Conf}_{\nu_i+3}(\mathbb{R}^3) | y_i = f(x_i) \text{ for some } x_i \in \mathbb{R}^1 \text{ for } i = 1, \ldots, \nu_i\}
\end{equation}

of dimension $\nu_i + 3v$. Integrating $\omega_X$ along the fiber, we get

\begin{equation}
I(X) := \pi_X^*(\omega_X) \in \Omega^{2\nu_i-3v}(K).
\end{equation}

**Remark 3.1.** The integration (3.7) converges since we can compactify all the fibers of $\pi_X$ by adding the boundary faces to (3.5) so that the maps $\varphi_\alpha$ are smoothly extended to the compactification. See [2, 3, 5, 10].

**Example 3.2.** Let $X$ be the graph that has only one edge $\alpha$ joining two i-vertices (Figure 3.2, the left). Then $E_X \simeq K \times \text{Conf}_2(\mathbb{R}^1)$ and $I(X) \in \Omega^0_{DR}(K)$ is a function on $K$.

In this paper we use an anti-symmetric unit volume form $\omega$ whose support is contained in (small) neighborhoods $U_x$ of the poles $(0, 0, \pm 1) \in S^2$. Suppose $f \in K$ is “almost planer,” meaning that

- the image of $f$ coincides with a knot diagram $D$ on $\mathbb{R}^2 \times [0]$ except for neighborhoods of crossings of $D$,
- near the crossings the image of $f$ is contained in $\mathbb{R}^2 \times (-\epsilon, \epsilon)$, and
- the unit tangent vectors $f'(x)/|f'(x)|$ are not contained in $U_x$.

Then $\varphi_\alpha : \{f\} \times \text{Conf}_2(\mathbb{R}^1) \to S^2$ has its image in $U_x$ only on the subspace of $(x_1, x_2)$ such that $f(x_1)$ and $f(x_2)$ are on the over- and under-arcs of a crossing of $D$, one on each arc (Figure 3.2, the center). Each crossing contributes to the value $I(X)(f)$ by the half of its sign; because this contribution is the half of the linking number of the “Hopf link” (Figure 3.2, the right), which is equal to the sign of the crossing.

By the generalized Stokes’ theorem for fiber integrations, we have

\begin{equation}
\left.dI(X) = \pi_X^*(d\omega_X) \pm \pi_X^2(\omega) = \pm \pi_X^2(\omega), \right.
\end{equation}

where $\pi_X^2$ is the restriction of $\pi_X$ to the fiberwise boundary. There exists “almost” 1-1 correspondence between

- the codimension 1 faces of the boundary that nontrivially contribute to $dI(X)$, and
- the graphs obtained from $X$ by contracting one of its edges and arcs (segments in $\mathbb{R}^1$ interposed between two i-vertices).
Here we in fact need the anti-symmetry of vol. We thus have

\[ dI(X) = I(\partial X) + \text{(correction terms)}, \]

where \( \partial X \) is a formal sum of graphs obtained from \( X \) by contracting one of its edges and arcs. The above correspondence is not rigorously 1-1 and we need “correction terms,” that are conjectured to vanish. We can therefore get a closed form of \( K \) if we have a graph cocycle, a formal sum \( X \) of graphs with \( \partial X = 0 \) (and if we have appropriate correction terms). It is known that any \( \mathbb{R} \)-valued Vassiliev invariant can be produced from a trivalent graph cocycle.

In [13, 14] the second named author has given an example of non-trivalent graph cocycle

\[ X = \sum_{1 \leq k \leq 9} a_k X_k, \quad (a_1, \ldots, a_9) = (-2, 1, 2, -2, -1, 1, -1, 1) \]

(see Figure 3.3), and has proved that \( I(X) \in H^1_{DR}(\mathcal{K}) \) is not zero. This follows from Theorem 3.3 ([14]).

The differential form \( I(X) \in \Omega^1_{DR}(\mathcal{K}) \) is closed, and its integration over the Gramain cycle \( \Gamma_f \) (see Remark 3.4 below) is equal to the Casson invariant \( v_2(f) \).

**Remark 3.4.** The Gramain 1-cycle \( \Gamma_f : S^1 \to \mathcal{K} \) for \( f \in \mathcal{K} \) is a cycle that rotates \( f \) around the “long axis” \( \mathbb{R}^1 \times \{(0,0)\} \). Explicitly \( \Gamma_f \) is given by

\[ \Gamma_f(\theta)(x) := \begin{pmatrix} 1 & \cos \theta & \sin \theta \end{pmatrix} f(x) \quad \text{for} \quad \theta \in S^1, \ x \in \mathbb{R}^1. \]

Mortier [11, Theorem 4.1] has given a 1-cocycle \( \alpha_3^1 \) of \( \mathcal{K} \) in a combinatorial way and has proved that

\[ \langle \alpha_3^1, \Gamma_f \rangle = v_2(f) \quad \text{and} \quad \langle \alpha_3^1, p^*FH_{(f,w)} \rangle = 6v_3(f) - w \cdot v_2(f) \]

for \( (f, w) \in \mathcal{K} \times \mathbb{Z} \approx \tilde{\mathcal{K}} \). This result motivates us to compute the integration of \( I(X) \) over the FH-cycles.

**4. Integration of \( I(X) \) over the Fox-Hatcher cycle**

Recall that \( p : \tilde{\mathcal{K}} \to \mathcal{K} \) is the map forgetting the framing of \( \tilde{f} \). For any \( \tilde{f} \in \tilde{\mathcal{K}} \) we define

\[ v(\tilde{f}) := \int_{p^*FH_{\tilde{f}}} I(X) = \sum_{1 \leq k \leq 9} a_k \int_{p^*FH_{\tilde{f}}} I(X_k). \]

This gives an isotopy invariant \( v \) for framed long knots. Our goal is to describe \( v \) as a linear combination of the Vassiliev invariants of order less or equal to three.
4.1. The invariant $v$ is of order three. For any $\tilde{f} \in \tilde{K}$ and crossings $c_1, \ldots, c_n$ of its diagram, define

$$D^4 v(\tilde{f}) := \sum_{e_1, \ldots, e_n \in \{+1, -1\}} e_1 \cdots e_n v(\tilde{f}_{e_1, \ldots, e_n}),$$

where $\tilde{f}_{e_1, \ldots, e_n}$ is a framed long knot obtained by changing, if necessary, the crossings $c_i$ so that its sign is equal to $e_i$. What we want to show is $D^4 v(\tilde{f}) = 0$ for any choice of $\tilde{f}$ and $c_1, \ldots, c_4$.

Let $c_1, c_2, c_3$ be (part of the) crossings of a diagram $D$ of $\tilde{f} \in \tilde{K}$ respecting the Gauss diagram $G$ (Definition 2.3). Let us perform the FH-moves (described in §2) on all the crossings $c$ of $D$ and the corresponding newborn crossings $c'$. The Gauss diagram that the three crossings under consideration respect changes as in Figure 2.3 when the FH-move is performed on one of $c_i$ and $c'_i$ ($i = 1, 2, 3$), and in the sequence of the FH-moves realizing the FH-cycle, six Gauss diagrams (some of which may be equal to each other) respected by the three crossings under consideration form a cycle. Figures 4.1, 4.2 and 4.3 show three such cycles. There are 15 Gauss diagrams with three edges, and only 10 of them are included in these three cycles. The remaining five Gauss diagrams form the other two cycles, that we omit since in fact they do not contribute to our computation in §4.2.

Figure 4.1. Type I cycle of the Gauss diagrams respecting three crossings under consideration; $\{x, y, z\} = \{c_1, c_2, c_3\}$

Figure 4.2. Type II cycle of the Gauss diagrams respecting three crossings under consideration; $\{x, y, z\} = \{c_1, c_2, c_3\}$

Figure 4.3. Type III cycle of the Gauss diagrams respecting three crossings under consideration; $\{x, y, z\} = \{c_1, c_2, c_3\}$
Theorem 4.1. $D^3 v(\tilde{f})$ is given by
\[
D^3 v(\tilde{f}) = \begin{cases} 
-2 & \text{if } c_1, c_2 \text{ and } c_3 \text{ respect one of the Gauss diagrams in type I cycle}, \\
2 & \text{if } c_1, c_2 \text{ and } c_3 \text{ respect one of the Gauss diagrams in type II cycle}, \\
6 & \text{if } c_1, c_2 \text{ and } c_3 \text{ respect the unique Gauss diagram in type III cycle}, \\
0 & \text{otherwise.} 
\end{cases}
\]

Corollary 4.2. The invariant $v$ is a Vassiliev invariant for framed long knots of order exactly three.

Proof. Let $c_1, \ldots, c_4$ be crossings of a diagram of $\tilde{f} \in \mathcal{K}$. Let $\tilde{f}_\pm$ be knots obtained by changing $c_4$ so that its sign is respectively $\pm 1$. Then by definition
\[
D^4 v(f) = D^3 v(f_+) - D^3 v(f_-).
\]
Moreover $c_1, c_2$ and $c_3$ of $f_+$ and $f_-$ respect the same Gauss diagram. Thus we have $D^3 v(\tilde{f}_+) = D^3 v(\tilde{f}_-)$ by Theorem 4.1, concluding $D^4 v(\tilde{f}) = 0$.

Theorem 4.1 also says that $D^3 v(\tilde{f})$ can be nonzero, and $v$ is not of order two nor less. \qed

The next subsection is devoted to the proof of Theorem 4.1.

4.2. Computation of $D^3 v$. As in Example 3.2, we assume that

- $\text{vol} \in \Omega_{2p}(S^2)$ is an anti-symmetric unit volume form of $S^2$ whose support is contained in small neighborhoods of poles $(0, 0, \pm 1) \in S^2$, and
- we compute $D^3 v(\tilde{f})$ after transforming $\tilde{f}$ to be “almost planar.”

For $k = 1, \ldots, 9$, consider the pullback square
\[
\begin{array}{ccc}
(p \circ FH_\tilde{f})^* E_{X_k} & \xrightarrow{p \circ FH_\tilde{f}} & E_{X_k} \\
\pi_{X_k} & \downarrow & \pi_{X_k} \\
S^1 & \xrightarrow{FH_\tilde{f}} & \mathcal{K} \\
\end{array}
\]

Then
\[
\int_{p \circ FH_\tilde{f}} I(X_k) = \int_{S^1} (p \circ FH_\tilde{f})^* \pi_{X_k} \omega_{X_k} = \int_{S^1} \pi_{X_k} (p \circ FH_\tilde{f})^* \omega_{X_k} = \int_{(p \circ FH_\tilde{f})^* E_k} (p \circ FH_\tilde{f})^* \omega_{X_k}.
\]

Note that $(p \circ FH_\tilde{f})^* E_{X_k}$ is explicitly given by
\[
(p \circ FH_\tilde{f})^* E_{X_k} = \left\{ (p(FH_\tilde{f}(\theta)), y) \in \mathcal{K} \times \text{Conf}_{V_1, V_1} (\mathbb{R}^3) \mid \theta \in S^1, \ y_i = p(FH_\tilde{f}(\theta))(x_i) \right\} \subset E_{X_k}.
\]

Suppose a diagram $D$ of $\tilde{f}$ has $n$ crossings. Then $FH_\tilde{f}$ can be realized on knot diagram by the sequence of $2n$ FH-moves on $c$ or $c'$, where $c$ is one of the crossings of $D$ and $c'$ is a newly created crossing after the FH-move on $c$. We can decompose $S^1$ into $2n$ intervals
\[
S^1 = \bigcup_c (I_c \cup I_c')
\]
such that $FH_\tilde{f}$ restricted on $I_c$ (resp. $I_c'$) realizes the FH-move on $c$ (resp. $c'$).

Definition 4.3. Under the above setting, define
\[
E_{k,c,c'} := \{ (p(FH_\tilde{f}(\theta)), y) \in (p \circ H_\tilde{f})^* E_{X_k} \mid \theta \in I_c \cup I_c' \}.
\]

By definition we have
\[
(p \circ FH_\tilde{f})^* E_{X_k} = \bigcup_{c,c'} E_{k,c,c'} 
\]
and hence
\[
\int_{(p \circ FH_\tilde{f})^* E_k} \omega_{X_k} = \sum_c \int_{E_{k,c,c'}} (p \circ FH_\tilde{f})^* \omega_{X_k}.
\]

Combining \eqref{eq:4.1}, \eqref{eq:4.2}, \eqref{eq:4.6} and \eqref{eq:4.10}, we have
\[
D^3 v(\tilde{f}) = \sum_{1 \leq k \leq 9} a_k \sum_{c,c'} \sum_{e_1, e_2, e_3 \in \{+1, -1\}} e_1 e_2 e_3 \int_{E_{k,c,c'}} (p \circ FH_\tilde{f})^* \omega_{X_k}.
\]
4.2.1. Eliminating \(X_3, \ldots, X_9\). Let \(h_i \ (i = 1, 2, 3)\) be the “distance” between two arcs at \(c_i\), \(i = 1, 2, 3\) (Figure 4.4). We may compute \(D^3v(f)\) in the limit \(h_i \to 0 \ (i = 1, 2, 3)\) since \(v\) is an invariant. In this limit, only the graphs \(X_1\) and \(X_2\) essentially contribute to \(D^3v(f)\);

**Proposition 4.4.** (1) For \(k = 1, \ldots, 9\) and any crossing \(c\) other than \(c_1, c_2, c_3\), we have

\[
\lim_{h_1, h_2, h_3 \to 0} \sum_{c \in \{c_1, c_2, c_3\}} \sum_{1 \leq i, j, l \leq 1} \epsilon_i \epsilon_j \epsilon_l \int_{E_{k,c'}} p \circ FH_{\tilde{\varphi}_{i,j,l}} \omega_{X_{k,l}} = 0.
\]

(2) If \(k = 3, \ldots, 9\), then the equation (4.12) also holds for \(c \in \{c_1, c_2, c_3\}\).

Consequently

\[
D^3v(f) = \lim_{h_1, h_2, h_3 \to 0} \sum_{k=1}^{3} a_k \sum_{c \in \{c_1, c_2, c_3\}} \sum_{1 \leq i, j, l \leq 1} \epsilon_i \epsilon_j \epsilon_l \int_{E_{k,c'}} p \circ FH_{\tilde{\varphi}_{i,j,l}} \omega_{X_{k,l}}.
\]

**Proof of Proposition 4.4(1).** Let \(-1 < p_i < q_i < 1 \ (i = 1, 2, 3)\) with \(p_1 < p_2 < p_3\) be the real numbers such that \(f(p_i)\) and \(f(q_i)\) correspond to \(c_i\), and let \(A_i, \ B_i\) be small open intervals that include respectively \(p_i\) and \(q_i\) (see Figure 4.5). Let \(E_{k,c',c''} \subset E_{k,c} \) be the subspace consisting of \((\theta, y)\) with no \(y_j\) \((1 \leq j \leq 9)\) being in \(A_1\). Then even if we set \(h_1 = 0\), any two points \(y_{j1}\) and \(y_{j2}\) corresponding to endpoints of a single edge of \(X_k\) do not collide in \(E_{k,c',c''}\), and the maps \(\varphi_{i,j}\) and hence the integrand \(\omega_{X_k}\) can be defined on \(E_{k,c',c''}\). This implies

\[
\lim_{h_1 \to 0} \left( \int_{E_{k,c',c''}} p \circ FH_{\tilde{\varphi}_{i,j,l}} \omega_{X_k} - \int_{E_{k,c',c''}} p \circ FH_{\tilde{\varphi}_{i,j,l}} \omega_{X_k} \right) = 0.
\]

If we analogously define \(E_{k,c',c''}A_m\) and \(E_{k,c',c''}B_m\), then similar cancellation to (4.12) occurs for them. Moreover we have

\[
\bigcup_{m=1,2,3} (E_{k,c',c''}A_m \cup E_{k,c',c''}B_m) = E_{k,c',c''}
\]

because no \(X_k\) has six or more \(i\)-vertices. Although \(A_1, \ldots, B_3\) are not disjoint, we can arrange them to be disjoint by considering their difference sets and intersections (on which the same argument is valid). Thus we have (4.12).

**Proof of Proposition 4.4(2) for \(k = 7, 8, 9\).** It is enough to consider the case \(c = c_1\); the cases \(c = c_2, c_3\) can be proved similarly.

The similar argument in the proof of (1) also implies (4.14) with \(h_1\) replaced respectively by \(A_m\) (or \(B_m\)) and \(h_m, m = 2, 3\). We thus complete the proof, because \(X_k (k = 7, 8, 9)\) has three or less \(i\)-vertices and we have

\[
E_{k,c_1,c'_1} = \bigcup_{m=2,3} (E_{k,c_1,c'_1}A_m \cup E_{k,c_1,c'_1}B_m).
\]

**Proof of Proposition 4.4(2) for \(k = 5, 6\).** It is enough to consider the case \(c = c_1\).

Let \(E \subset E_{k,c'_1}\) be the subspace of \(E_{k,c'_1}\) consisting of \((\theta, y)\) with each of \(A_2, B_2, A_3\) and \(B_3\) containing at least one \(y_j\) corresponding to an \(i\)-vertex \(j\) of \(X_k\). Then the integrations in (4.12) with \(E_{k,c'_1}\) replaced by \(E_{k,c'_1}\setminus E\) are defined even if we set \(h_m = 0\) for at least one \(m \in \{2, 3\}\), and the cancellation similar to (4.14) occurs, similarly as the above.
Type II; the subspace Figure 4.8 shows the configurations in $E$

Proof of Proposition 4.4 (2) for $k$

Then it suffices to show (4.12) with

$$E$$

Proof of Proposition 4.4 (2) for $k$

A $i$-vertices, each of

$\subset$

$\theta, y$

$\alpha$

$\bar{\varphi}$

$\varphi_a$ is not included in $\text{supp}(\text{vol})$ and hence $\varphi_a^\ast \text{vol} = 0$, because $\text{supp}(\text{vol})$ is assumed to be in neighborhoods of $(0, 0, \pm 1) \in S^2$ and our $\bar{f}$ is almost planar. The integrand $\omega_{X_4}$ is therefore zero on $E_{II}$. □

Proof of Proposition 4.4 (2) for $k = 4$. Consider the case $c = c_1$ (the same arguments are valid for $c = c_2, c_3$). Let $E \subset E_{k_3, k'_3}$ be the subspace consisting of $(\theta, y)$ with each of $A_2, B_2, A_3$ and $B_3$ contains at least one point. It is then enough to show (4.12) with $E_{k_3, k'_3}$ replaced by $E$, as in the above proofs.

As $X_4$ has four $i$-vertices, each of $A_2, B_2, A_3$ and $B_3$ contains exactly one point on $E$. In particular $y_4 \in A_2$, and the map $\varphi_4$ for the loop $\alpha$ at the vertex 1 has the image outside $\text{supp}(\text{vol})$ by our assumption on $\bar{f}$ and vol, and hence $\omega_{X_4}$ vanishes on $E$.

Proof of Proposition 4.4 (2) for $k = 3$. Again consider the case $c = c_1$. Let $E \subset E_{3, k'_3}$ be the subspace consisting of $(\theta, y)$ satisfying both (i) and (ii):

(i) $y_1$ is on the arc $C$ that moves in the FH-moves on $c_1$, (ii) each of $A_2, B_2, A_3$ and $B_3$ contains exactly one of $y_2, \ldots, y_5$.

Then it suffices to show (4.12) with $E_{3, k'_3}$ replaced by $E$. This is because:

- If $E'$ denotes the subspace of $E_{k_3, k'_3}$ consisting of $(\theta, y)$ that does not satisfy (ii), then the integrations in (4.12) with $E_{k_3, k'_3}$ replaced by $E'$ are defined even if we set $h_m = 0$ for at least one $m \in \{2, 3\}$ and the cancellation similar to (4.14) occurs, by the same reason as in the above proofs.
- If $E''$ denotes the subspace of $E_{3, k'_3}$ consisting of $(\theta, y)$ that satisfies (ii) but does not satisfy (i), then the map $\varphi_3$ (with $\alpha$ is the loop of $X_3$ at the $i$-vertex labeled by 1) has its image outside $\text{supp}(\text{vol})$ since $\bar{f}$ is supposed to be almost planar, and hence $\omega_{X_3}$ vanishes on $E''$.

Figure 4.8 shows the configurations in $E$ that may non-trivially contribute to the integration of $I(X_3)$. Let $J_s$ ($s = 1, 2$) be the unit intervals identified with those on $C$ drawn with thick curves in Figure 4.8. We write $p, FH_j(\theta)$ as $f_\theta$ for
short. Define \( \phi_1 : I_{r_1} \times J \to S^2 \) (s = 1, 2), \( \phi_{24} : A_2 \times B_2 \to S^2 \) and \( \phi_{35} : A_3 \times B_3 \to S^2 \) by

\[
\phi_1(t, t) := \frac{f_i'(t)}{f_j'(t)}, \quad \phi_{ij}(t, u) := \frac{f(u) - f(t)}{|f(u) - f(t)|}, \quad (i, j) = (2, 4), (3, 5).
\]

Then

\[
\int_E P \circ FH_J \omega_{X_i} = \int_{I_{r_1} \times (J \cup J_2)} \phi_1^* \text{vol} \int_{A_1 \times B_1} \phi_{24}^* \text{vol} \int_{A_2 \times B_2} \phi_{35}^* \text{vol}.
\]

Define the diffeomorphisms \( \xi : J_1 \to J_2 \) and \( \eta : \mathbb{R}^3 \to \mathbb{R}^3 \) by

\[
\xi(t) = 1 - t, \quad \eta(x, y, z) := (-x, y, z).
\]

Then, with respect to the coordinates of \( \mathbb{R}^3 \) shown in Figure 4.8, the following diagram commutes;

\[
\begin{array}{ccc}
I_{r_1} \times J_1 & \xrightarrow{\phi_1} & S^2 \\
\downarrow \text{id} \times \xi & & \downarrow \eta \\
I_{r_1} \times J_2 & \xrightarrow{\phi_1} & S^2
\end{array}
\]

and since \( \xi \) reverses the orientation and \( \eta \) preserves the orientation, we have

\[
\int_{I_{r_1} \times J_2} \phi_1^* \text{vol} = -\int_{I_{r_1} \times J_1} \phi_1^* \text{vol} \quad \text{and hence} \quad \int_{I_{r_1} \times (J_1 \cup J_2)} \phi_1^* \text{vol} = \sum_{s=1,2} \int_{I_{r_1} \times J_s} \phi_1^* \text{vol} = 0.
\]

Thus (4.18) is zero. \( \square \)

Thus we only need to compute the alternating sums of the integrations of \( I(X_i) \) and \( I(X_j) \) in the limit \( h_1, h_2, h_3 \to 0 \).

4.2.2. Computation of \( I(X_i) \). The following two subspaces of \( E_{X, \epsilon_j} \) (j = 1, 2, 3) do not essentially contribute to the alternating sum of \( I(X_i) \).

- The subspace where the arc near the left-most crossing moving in the FH-move contains no point; because the integrals on the subspace are the same for \( \epsilon_j = +1 \) and \( \epsilon_j = -1 \) and they cancel in the alternating sum.
- The subspace where no edge joins points on the arc moving in the FH-move contains no point; because all the maps \( \varphi_{ij} \) and hence the integrand \( \omega_{X_i} \) can be defined even if \( h_m = 0 \) and thus the cancellation similar to (4.14) occurs.

Thus only the subspaces of types (1-a) and (1-b) consisting \((\theta, y)\) as shown in Figure 4.9 can essentially contribute to the integrations of \( I(X_i) \). In both cases, the arc near the left-most crossing contains \( y_2 \) (case (1-a)) or \( y_4 \) (case (1-b)) moves right in the FH-move, and when the arc comes over or under the middle crossing, the map \( \varphi_{12} \) or \( \varphi_{14} \) has its image in \( \text{supp}(\text{vol}) \) and the integrand is not zero at that moment.

If three crossings \( c_1, c_2, c_3 \) under consideration respect one of the Gauss diagrams in Type I cycle (Figure 4.1), then in the FH-cycle we meet the situation (1-a) in Figure 4.9 once, because the Gauss diagram \( G_{1(a)} \) appears once in Type I cycle. If \( c_1, c_2, c_3 \) respect one of the Gauss diagrams in Type II cycle (Figure 4.2), then in the FH-cycle we meet the situation (1-b) in Figure 4.9 twice, because the Gauss diagram \( G_{1(b)} \) appears twice in Type II cycle. Otherwise we do not meet the situations (1-a) nor (1-b) and the integration vanishes.
Then changing the valuables suitably, the left hand side of (4.22) is equal to
\[ \epsilon \text{(4.25)} \]
where \( \text{vol} \times \text{(4.26)} \)\( y \text{(4.23)} \)\( y \text{and define} \ \phi \text{(4.24)} \)

\[ \text{Proposition 4.5.} \text{ We have} \]

\[
(4.22) \quad \epsilon_1 \epsilon_2 \epsilon_3 \sum_{x \in \{1, 2, 3\}} \int_{E_{x \epsilon_1 x_2 x_3}} p \circ \text{FH}_{\tilde{f}_{x_1 x_2 x_3}} \omega_{X_i}
\]

\[
= \begin{cases} 
1/8 & \text{if } c_1, c_2, c_3 \text{ respect one of the Gauss diagrams in Type I cycle (Figure 4.1)}, \\
-1/4 & \text{if } c_1, c_2, c_3 \text{ respect one of the Gauss diagrams in Type II cycle (Figure 4.2)}, \\
0 & \text{otherwise.}
\end{cases}
\]

**Proof.** Consider the first case; we may assume that \( c_1, c_2, c_3 \) respect the Gauss diagram \( G_{(1-a)} \). Then only \( E_{123}c_1 \) can contain the configurations of type (1-a) and non-trivially contribute to the alternating sum of the integrations of \( \tilde{I}(X_1) \).

Let \( b: \mathbb{R}^1 \to \mathbb{R}^1 \) be an even function whose graph looks as in Figure 4.10. For \( (\theta, x_1, \ldots, x_5) \in \mathbb{R}^5 \), consider \( y_1, \ldots, y_5 \in \mathbb{R}^3 \) given by

\[
(4.23) \quad y_1 = (x_1, 0, 0), \ y_2 = (0, -\epsilon_2 x_2, b(\epsilon_2 x_2)), \ y_3 = (x_3, 0, 0), \ y_4 = (\theta, -\epsilon_1 x_4, 2b(\epsilon_1 x_4/2)), \ y_5 = (0, -\epsilon_3 x_5, b(x_3))
\]

and define \( \varphi: \mathbb{R}^5 \to (S^2)^3 \) by

\[
(4.24) \quad \varphi(\theta, x_1, \ldots, x_5) := \left( \frac{y_2 - y_1}{|y_2 - y_1|}, \frac{y_3 - y_2}{|y_3 - y_2|}, \frac{y_4 - y_3}{|y_4 - y_3|} \right).
\]

Then changing the valuables suitably, the left hand side of (4.22) is equal to

\[
(4.25) \quad \epsilon_1 \epsilon_2 \epsilon_3 \int_{\mathbb{R}^5} \varphi^t(\text{vol}^{3 \times 3}),
\]

where \( \text{vol}^{3 \times 3} = \text{pr}_1^t \text{vol} \wedge \text{pr}_2^t \text{vol} \wedge \text{pr}_3^t \text{vol} \in \Omega_0^{3 \times 3}((S^2)^3). \)

Define \( \Phi: \mathbb{R}^3 \to (\mathbb{R}^3)^3 \) and \( \psi_1: \mathbb{R}^2 \to S^2 (s = 1, 2) \) by respectively

\[
(4.26) \quad \Phi(\theta, x_1, \ldots, x_5) := ((x_1, \epsilon_2 x_2), (x_1 - \theta, \epsilon_1 x_4), (x_3, \epsilon_3 x_5)),
\]

\[
(4.27) \quad \psi_1(x, x') := \frac{y' - y}{|y' - y|}, \ \psi_2(x, x') := \frac{y'' - y}{|y'' - y|}, \ \text{where} \ y := (x, 0, 0), \ y' := (0, -x, b(x)), \ y'' := (0, -x', 2b(x'/2)).
\]
Then $\Phi$ is a linear diffeomorphism whose determinant is $\epsilon_1 \epsilon_2 \epsilon_3$, and the following diagram is commutative;

\[
\mathbb{R}^6 \xrightarrow{\psi} (S^2)^{\times 3} \xrightarrow{\Phi} (\mathbb{R}^2)^{\times 3} \xrightarrow{\phi^* \times \phi^*} (\mathbb{R}^2)^{\times 3}
\]

Thus (4.25) is equal to

\[
(\epsilon_1 \epsilon_2 \epsilon_3)^2 \left( \int_{\mathbb{R}^2} \psi^*_1 \text{vol} \right)^2 \int_{\mathbb{R}^2} \psi^*_2 \text{vol} = \left( \frac{1}{2} \right)^3 = \frac{1}{8},
\]

here $1/2$ appears by exactly the same reason as in Example 3.2.

The second case that $c_1, c_2, c_3$ respect the Gauss diagram $G_{(1, b)}$ can be similarly computed, replacing

- (4.28) and (4.24) respectively with

\[
y_1 = (x_1, 0, 0), \ y_2 = (\theta, -\epsilon_2 x_2, b(\epsilon_2 x_2/2)), \ y_3 = (x_3, 0, 0), \ y_4 = (0, -\epsilon_1 x_4, b(x_4)), \ y_5 = (0, -\epsilon_1 x_5, b(x_5)),
\]

(4.30)

\[
\varphi(\theta, x_1, \ldots, x_5) := \left( \frac{y_4 - y_1}{|y_4 - y_1|}, \frac{y_5 - y_3}{|y_5 - y_3|}, \frac{y_2 - y_1}{|y_2 - y_1|} \right)
\]

(namely $y_2$ and $y_4$ are swapped), and

- (4.26) with

\[
\Phi(\theta, x_1, \ldots, x_5) := ((x_1, \epsilon_2 x_2), (x_1 - \theta, \epsilon_1 x_4), (x_3, \epsilon_3 x_5)).
\]

Then the determinant of $\Phi$ is $-\epsilon_1 \epsilon_2 \epsilon_3$, and because we meet the situation (1-b) twice in the FH-cycle, the left hand side of (4.22) in this case is equal to

\[
-2(\epsilon_1 \epsilon_2 \epsilon_3)^2 \left( \int_{\mathbb{R}^2} \psi^*_1 \text{vol} \right)^2 \int_{\mathbb{R}^2} \psi^*_2 \text{vol} = -\frac{1}{4}. \tag{4.33}
\]

4.2.3. Computation of $I(X_2)$. The computation of $I(X_2)$ goes similarly to that of $I(X_1)$. Only the subspaces of types (2-a) and (2-b) consisting of $(\theta, y)$ as shown in Figure 4.11 can essentially contribute to the alternating sum of the integrations of $I(X_2)$. If three crossings $c_1, c_2, c_3$ under consideration respect one of the Gauss diagrams in Type II cycle (Figure 4.2), then in the FH-cycle we meet the situation (2-a) in Figure 4.9 twice, because the Gauss diagram $G_{(2, a)}$ appears twice in Type II cycle. If $c_1, c_2, c_3$ respect one of the Gauss diagrams in Type III cycle (Figure 4.3), then in the FH-cycle we meet the situation (2-b) in Figure 4.9 six times, because the Gauss diagram $G_{(2, b)}$ appears six times in Type III cycle.

Figure 4.11. Configurations essentially contributing to $I(X_2)$; they can exist only if the three crossings under consideration respect the Gauss diagrams $G_{(2-a)}$ or $G_{(2-b)}$. 

\[\begin{array}{c}
G_{(2-a)} \\
\end{array}\]

\[\begin{array}{c}
G_{(2-b)} \\
\end{array}\]

\[\begin{array}{c}
\text{the FH-move} \\
\end{array}\]

\[\begin{array}{c}
\text{the FH-move} \\
\end{array}\]
Proposition 4.6. We have

\[(4.34) \quad e_1 e_2 e_3 \sum_{c \in \{e_1, e_2, e_3\}} \int_{\mathbb{R}^2} p \circ \tilde{F} \circ \tilde{H} \cdot f_{c_1, c_2, c_3} \omega_{x_2} \]

\[= \begin{cases} 
-1/4 & \text{if } c_1, c_2, c_3 \text{ respect one of the Gauss diagrams in Type II cycle (Figure 4.2)}, \\
3/4 & \text{if } c_1, c_2, c_3 \text{ respect one of the Gauss diagrams in Type III cycle (Figure 4.3)}, \\
0 & \text{otherwise.}
\end{cases} \]

Proof. Consider the first case that \(c_1, c_2, c_3\) respect the Gauss diagram \(G_{(2-a)}\). Then only \(E_{2x_1, c_{1}'}\) can contain the configurations of type (2-a) and non-trivially contribute to the integral. The proof of this case goes very similarly to the above one; we just need to replace

- \[4.23\] and \[4.24\] respectively with

\[(4.35) \quad y_1 = (x_1, 0, 0), \quad y_2 = (x_2, 0, 0), \quad y_3 = (0, -e_2 x_3, b(e_2 x_3)), \quad y_4 = (0, e_1 x_4, 2b(e_1 x_4/2)), \quad y_5 = (0, e_3 x_5, b(e_3 x_3)), \]

\[\varphi(\theta, x_1, \ldots, x_5) := \left( \frac{y_3 - y_1}{y_3 - y_1}, \frac{y_5 - y_2}{y_5 - y_2}, \frac{y_4 - y_1}{y_4 - y_1} \right),\]

- \[4.26\] with

\[(4.37) \quad \Phi(\theta, x_1, \ldots, x_5) := ((x_1, e_2 x_3), (x_1 - \theta, e_1 x_4), (x_2, e_3 x_5)).\]

Then \(\Phi\) is a linear diffeomorphism with the determinant \(-e_1 e_2 e_3\), and because we meet the situation (2-a) twice in the FH-cycle, the left hand side of \[(4.34)\] in this case is equal to

\[(4.38) \quad -2(e_1 e_2 e_3)^2 \left( \int_{\mathbb{R}^2} \psi_1^1 \text{vol} \right)^2 \int_{\mathbb{R}^2} \psi_2^2 \text{vol} = -\frac{1}{4}.\]

Consider the second case that \(c_1, c_2, c_3\) respect the Gauss diagram \(G_{(2-b)}\). The proof of this case goes very similarly to that of the case (1-b) in Proposition \[4.5\]; we just need to replace

- \[4.23\] and \[4.24\] respectively with

\[(4.39) \quad y_1 = (x_1, 0, 0), \quad y_2 = (x_2, 0, 0), \quad y_3 = (0, -e_1 x_3, 2b(e_1 x_3/2)), \quad y_4 = (0, e_2 x_4, b(e_2 x_4)), \quad y_5 = (0, e_3 x_5, b(e_3 x_3)), \]

\[\varphi(\theta, x_1, \ldots, x_5) := \left( \frac{y_4 - y_1}{y_4 - y_1}, \frac{y_5 - y_2}{y_5 - y_2}, \frac{y_3 - y_1}{y_3 - y_1} \right),\]

- \[4.26\] with

\[(4.41) \quad \Phi(\theta, x_1, \ldots, x_5) := ((x_1 - \theta, e_2 x_3), (x_1, e_1 x_4), (x_2, e_3 x_5)).\]

Then \(\Phi\) is a linear diffeomorphism with the determinant \(e_1 e_2 e_3\), and because we meet the situation (2-b) six times in the FH-cycle, the left hand side of \[(4.34)\] in this case is equal to

\[(4.42) \quad 6(e_1 e_2 e_3)^2 \left( \int_{\mathbb{R}^2} \psi_1^1 \text{vol} \right)^2 \int_{\mathbb{R}^2} \psi_2^2 \text{vol} = \frac{3}{4}.\]
Proof of Theorem 4.1. Let $c_1$, $c_2$ and $c_3$ respect one of the Gauss diagrams in type I cycle (Figure 4.1). Then by (4.13) and Propositions 4.5, 4.6 we have

\begin{equation}
D^3 v(f) = \sum_{k=1,2} a_k \sum_{\epsilon \in \{c_1, c_2, c_3\}} \sum_{e_1, e_2, e_3 \in \{+1, -1\}} \epsilon_1 \epsilon_2 \epsilon_3 \int_{E_{k,e_1,e_2,e_3}} p \circ FH_{f_{e_1,e_2,e_3}} \omega_{k,e_1,e_2,e_3}.
\end{equation}

\begin{equation}
= (-2) \cdot \sum_{\epsilon_1, \epsilon_2, \epsilon_3 \in \{+1, -1\}} \left( \frac{1}{8} \right) + 1 \cdot \sum_{\epsilon_1, \epsilon_2, \epsilon_3 \in \{+1, -1\}} \left( \frac{1}{4} \right) = 2.
\end{equation}

Lastly suppose that $c_1$, $c_2$ and $c_3$ respect one of the Gauss diagrams in type III cycle (Figure 4.1). Then

\begin{equation}
D^3 v(f) = \sum_{k=1,2} a_k \sum_{\epsilon \in \{c_1, c_2, c_3\}} \sum_{e_1, e_2, e_3 \in \{+1, -1\}} \epsilon_1 \epsilon_2 \epsilon_3 \int_{E_{k,e_1,e_2,e_3}} p \circ FH_{f_{e_1,e_2,e_3}} \omega_{k,e_1,e_2,e_3} = (-2) \cdot 0 + 1 \cdot \sum_{\epsilon_1, \epsilon_2, \epsilon_3 \in \{+1, -1\}} \frac{3}{4} = 6.
\end{equation}

If $c_1$, $c_2$ and $c_3$ respect no Gauss diagram in three cycles, then $D^3 v(f) = 0$. \hfill \Box

4.3. An explicit description of $v$. It is known (see [9, p. 215] for example) that the space of the Vassiliev invariants for framed knots of order less than or equal to three are multiplicatively generated by the framing number $lk$, the Casson invariant $v_2$ and the order three invariant $v_3$ (characterized by the conditions in Theorem 1.2). Thus all the Vassiliev invariants of order less than or equal to three are linear combinations of

$$lk, \quad v_2, \quad lk^2, \quad v_3, \quad lk \cdot v_2, \quad lk^3.$$

Lemma 4.7. Our invariant $v$ is of the form $v = av_3 + b lk \cdot v_2 + cv_2$ for some constants $a, b, c \in \mathbb{R}$.

Proof. The value of $v$ on the trivial long knot $f_0(x) = (x, 0, 0)$ together with a framing number $w \in \mathbb{Z}$ is a linear combination of $w$, $w^2$ and $w^3$ because $v_2(f_0) = v_3(f_0) = 0$. But by the definition $p_v H_{f_0, w}$ is a constant loop of $K$ for any $w \in \mathbb{Z}$. Thus $v(f_0, w) = 0$ for any $w \in \mathbb{Z}$, and the coefficients of $lk$, $lk^2$ and $lk^3$ must be zero. \hfill \Box

Below we compute the constants $a, b, c$ in Lemma 4.7. We denote by $3^+_1$ and $3^-_1$ respectively the right-handed and the left-handed trefoil knots, by $4_1$ the figure eight knot. By the formulas for $v_2$ and $v_3$ in [12, Theorems 1, 2] we have

\begin{equation}
v_2(3^+_1) = v_2(3^-_1) = 1, \quad v_2(4_1) = -1, \quad v_3(4_1) = 0.
\end{equation}

Proposition 4.8. We have $a = 6, b = -1$.

Proof. Consider the “standard” diagram of $3^+_1$ in Figure 2.2 and write it as $f = f_{4, ++}$. This can be seen as a framed long knot with framing number $+3$. The diagram $f_{4, --}$ is $3^-_1$ with framing number $-3$ and all the other $f_{e_1, e_2, e_3}$ are trivial. The Gauss diagram in Figure 2.2 appears in the Type III cycle in Figure 4.3 and $D^3 v(f) = 6$ by Theorem 4.1. Thus we have

\begin{equation}
6 = D^3 v(f) = (av_3(3^+_1) + b \cdot 3 + cv_2(3^+_1)) - (av_3(3^-_1) + b \cdot (-3) + cv_2(3^-_1)) = 2a + 6b,
\end{equation}

here the last equality holds by (1.3) and (4.47).

Next consider the diagram of $4_1$ in Figure 4.13. We write it as $g = g_{4, --}$, focusing on $c_1, c_2, c_3$. This can be seen as
a framed long knot with framing number 0. Then \(g_{\ldots,-}\) is the 3-1 cycle appearing in the Type II cycle in Figure 4.2 and \(D^3 v(g) = 2\) by Theorem 4.1. Thus we have

\[
2 = D^3 v(g) = -(av_3(4_1) + b \cdot 0 + cv_2(4_1)) - (av_3(3_1^1) + b \cdot (-4) + cv_2(3_1^1)) = a + 4b,
\]

here the last equality holds again by (1.2) and (4.47). Therefore \(a = 6, b = -1\) by (4.48) and (4.49).

**Proposition 4.9.** We have \(c = 0\).

**Proof.** As explained in [3], for the “standard” diagram of \(\tilde{f} = 3_1^1\) (with framing number +3, see Figure 2.2), the FH-cycle \(p,\tilde{f}\) is homologous to 3 times the Gramain cycle \(G_f\) (see Remark 3.4), where \(f = p(\tilde{f}) \in \mathcal{K}\). This is because, as we can see in the Figure in [8] p. 3, each of the FH-moves on the crossings of \(\tilde{f}\) is the rotation around the long axis by degree \(\pi\), and in the FH-cycle we perform six times of the FH-moves. Thus

\[
6 v_3(3_1^1) - 3 v_2(3_1^1) + cv_2(3_1^1) = v(\tilde{f}) = \int_{p,\tilde{f}} I(X) = 3 \int_{G_f} I(X).
\]

In [14] we have proved that the integration of \(I(X)\) over \(G_f\) is equal to \(v_2(f)\). Therefore (4.50) can be rewritten as

\[
6 \cdot 1 - 3 \cdot 1 + c \cdot 1 = 3 \cdot 1,
\]

and we have \(c = 0\). \(\square\)

**Acknowledgments**

The authors are deeply grateful to Arnaud Mortier for their invaluable comments and discussions. They also express their appreciation to Thomas Fiedler for sharing the information about his 1-cocycles in his book.

**References**

[1] D. Altschuler and L. Freidel, Vassiliev knot invariants and Chern-Simons perturbation theory to all orders, Comm. Math. Phys. 187 (1997), no. 2, 261–287.

[2] S. Axelrod and I. M. Singer, Chern-Simons perturbation theory II, J. Diff. Geom. 39 (1994), 173–213.

[3] R. Bott and C. Taubes, On the self-linking of knots, J. Math. Phys. 35 (1994), no. 10, 5247–5287.

[4] R. Budney, Little cubes and long knots, Topology 46 (2007), 1–27.

[5] A. Cattaneo, P. Cotta-Ramusino, and R. Longoni, Configuration spaces and Vassiliev classes in any dimensions, Algebr. Geom. Topol. 2 (2002), 949–1000.

[6] T. Fiedler, Polynomial one-cocycles for knots and closed braids, Series on Knots and Everything 64, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2020.

[7] R. H. Fox, Rolling, Bull. Amer. Math. Soc. 72 (1966), 162–164.

[8] A. Hatcher, Topological moduli space of knots, [http://www.math.cornell.edu/~hatcher/Papers/knotspaces.pdf](http://www.math.cornell.edu/~hatcher/Papers/knotspaces.pdf)

[9] D. M. Jackson and I. Moffatt, An introduction to quantum and Vassiliev knot invariants, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, Springer, Cham, 2019.

[10] T. Kohno, Vassiliev invariants and de Rham complex on the space of knots, Symplectic geometry and quantization (Sanda and Yokohama, 1993), Contemp. Math., vol. 179, pp. 123–138.

[11] A. Mortier, Combinatorial cohomology of the space of long knots, Algebr. Geom. Topol. 15 (2015), no. 6, 3435–3465.

[12] M. Polyak and O. Viro, Gauss diagram formulas for Vassiliev invariants, Internat. Math. Res. Notices (1994), no. 11, 445–453.

[13] K. Sakai, Nontrivalent graph cocycle and cohomology of the long knot space, Algebr. Geom. Topol. 8 (2008), 1499–1522.

[14] _______. An integral expression of the first non-trivial one-cocycle of the space of long knots in \(\mathbb{R}^3\), Pac. J. Math. 250 (2011), no. 2, 407–419.

[15] _______. BV-structures on the homology of the framed long knot space, J. Homotopy Relat. Struct. 11 (2016), no. 3, 425–441.

[16] V. Turchin, Calculating the First Nontrivial 1-Cocycle in the Space of Long Knots, Math. Notes 80 (2006), no. 1, 101–108.

[17] I. Vodić, A survey of Bott-Taubes integration, J. Knot Theory Ramifications 16 (2007), no. 1, 1–42.

Faculty of Mathematics, Shinshu University, 3-1-1 Asahi, Matsumoto, Nagano 390-8621, Japan

Email address: 20ss104f@gmail.com

Email address: ksakai@math.shinshu-u.ac.jp