On the pinch technique beyond one loop

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A brief account is given of the problems involved in extended the pinch technique beyond the one-loop level, and of investigations into one possible approach to solving them.

1. INTRODUCTION

The pinch technique (PT) is a well-defined algorithm for the rearrangement of contributions to conventional one-loop n-point functions in gauge theories to construct one-loop “effective” n-point functions with improved theoretical properties. Most notably, the PT one-loop n-point functions are entirely gauge-independent, and satisfy simple QED-like Ward identities. This rearrangement of one-loop perturbation theory is based on the systematic use of the tree level Ward identities to cancel among Feynman integrands all factors of longitudinal four-momentum associated with gauge fields propagating in loops. As a result of these improved properties, the PT has been advocated as the appropriate theoretical framework for a wide range of applications in which one is forced to go beyond the strictly order-by-order computation of S-matrix elements, or to consider amplitudes for explicitly off-shell processes.

A fundamental criticism of the PT, however, is that it has yet to be shown how the PT algorithm may be consistently extended beyond the one-loop level to construct 1PI multi-loop n-point functions with the same desirable properties as at one loop. This extension requires the solution of two problems: (i) How to reorganize consistently contributions to multi-loop integrands so as to isolate explicitly the PT n-point functions as internal loop corrections? (ii) How to deal consistently with the factors of longitudinal internal gauge field four-momentum which themselves originate from such internal loop corrections?

Here we briefly describe investigations into one possible approach to solving these problems.

2. ON THE PT AT TWO LOOPS

We consider the construction of the PT two-loop i.e. $O(\alpha_s^2)$ gluon self-energy in massless QCD, starting from the four-fermion process $q\bar{q} \to q\bar{q}$. In the class of linear covariant gauges, the required Feynman rules are (cf. Fig. 1)

Fig. 1a: $Z^2_i k + \mathcal{i}\epsilon$  
Fig. 1b: $Z_1 \gamma^\mu T^m_{ji}$  
Fig. 1c: $-i\delta^{mn} \left( \frac{1}{Z_3} \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) + \xi k_\mu k_\nu \right)$  
Fig. 1d: $Z_{1YMG} f^{rst} T^{(0)}_{\rho \sigma \tau}(k_1, k_2, k_3)$  
Fig. 1e: $\frac{1}{Z_{2FP}} \mathcal{i}\delta^{rs} k^2 + \mathcal{i}\epsilon$  
Fig. 1f: $-Z_{1FP} g f^{rst} k_{2\tau}$

The renormalization constants $Z_i$ have perturbative expansions $Z_i = 1 + \sum_{n=1}^{\infty} (Z_i - 1)^{(n)} \alpha_s^n$. For order-by-order book keeping, the expressions are then split into lowest order $(Z_i = 1)$ and associated counterterm contributions. Here, however, it will be convenient to leave the Feynman rules as given, with the expansion implicit.

Fig. 1. The diagrams for the Feynman rules.
In the PT at one loop, the renormalization constants satisfy the QED-like relation
\[ \frac{Z_1}{Z_2} = \frac{Z_{1,YM}}{Z_{3,FP}} = 1 \] (7)
to \( \mathcal{O}(\alpha_s) \), and are gauge-independent.

We consider first the \( \mathcal{O}(\alpha_s^2) \) corrections to \( q\bar{q} \rightarrow q\bar{q} \) consisting of one-loop diagrams with one-loop counterterm insertions (cf. Fig. 2). Given that each loop contributes a correction of \( \mathcal{O}(\alpha_s) \), the \( Z_i \) must be expanded to \( \mathcal{O}(\alpha_s^2) \).

For the diagram 2a, using (7), the Feynman rules for the quarks are

\[ \Gamma^{(0)}_{\rho \sigma} (k_3; k_1, k_2) \] (11)
where (12) is the now-familiar decomposition (13)
\[ \begin{align*}
\Gamma^{(0)F}_{\rho \sigma} (k_3; k_1, k_2) &= (k_1 - k_2)_{\rho} g_{\rho \sigma} \\
&\quad -2k_{3 \rho} g_{\sigma \mu} + 2k_{3 \sigma} g_{\rho \mu} , \\
\Gamma^{(0)P}_{\rho \sigma} (k_3; k_1, k_2) &= -k_{1 \rho} g_{\sigma \mu} + k_{2 \sigma} g_{\rho \mu} .
\end{align*} \] (12)

for \( A^f_\rho (k_3) \) the external gluon and \( A^c_\rho (k_1), A^c_\rho (k_2) \) the internal gluons. The implementation of the PT is then precisely as usual at one loop, except for (i) \( \xi \rightarrow Z_3 \xi \) and (ii) overall factors of \( Z_i \) for each diagram, which from (7) are the same to \( \mathcal{O}(\alpha_s) \). But the PT one-loop \( n \)-point functions are individually \( \xi \)-independent, so (i) is irrelevant.

Thus, implementing the PT as usual, and using (7), the resulting PT rearrangement of the \( \mathcal{O}(\alpha_s^2) \) contributions to \( q\bar{q} \rightarrow q\bar{q} \) from the diagrams 2b–2f corresponds to the Feynman rules

\[ \frac{1}{Z_{1,YM} k^2 + i\epsilon g_{\mu \nu}} \] (14)
Fig. 1d: \[ Z_{1,YM} g f^{rst} \Gamma^{(0)F}_{\rho \sigma} (k_3; k_1, k_2) \] (15)
Fig. 1e: \[ \frac{1}{Z_{1,FP} k^2 + i\epsilon} \] (16)
Fig. 1f: \[ -Z_{1,FP} g f^{rst} (k_1 + k_2)_\tau \] (17)

The above rules hold for \( Z_{1,YM}, Z_{1,FP} \) to \( \mathcal{O}(\alpha_s) \). Using the above PT Feynman rules for the self-energy diagrams 2b and 2c, the renormalization constants \( Z_{1,YM} \) and \( Z_{1,FP} \) associated with the gluon and ghost, respectively, propagating in the loop cancel (alternatively, the four \( \mathcal{O}(\alpha_s) \) counterterm insertions in both the gluon and the ghost loop sum to zero). This is exactly analogous to the fermion case above, and to QED.

In essence, using the gluon Feynman rules directly in the form (7), (8), (10), (11), (12), (13), we have used not only the lowest order \( (Z_i = 1) \) components but also the corresponding \( \mathcal{O}(\alpha_s) \) counterterms to trigger the PT rearrangement. This results in the above property for the spin 0, \( \frac{1}{4} \) and 1 contributions to the \( \mathcal{O}(\alpha_s^2) \) gluon self-energy.

We note that in the background field method (BFM), the \( \mathcal{O}(\alpha_s) \) counterterm insertions in the gluon loop in Fig. 2b do not vanish [5] except in the Landau quantum gauge \( \xi_Q = 0 \). Thus, in the above approach, the correspondence between the PT \( n \)-point functions and those of the BFM at \( \xi_Q = 1 \) (8) does not persist beyond one loop.

We now turn to the \( \mathcal{O}(\alpha_s^2) \) corrections to \( q\bar{q} \rightarrow q\bar{q} \) consisting of two-loop diagrams. In general, in order to avoid problems with renormalizability, we are now obliged to make decompositions analogous to (9), (10) for the one-loop corrections to the gauge field propagator and triple gauge ver-
This requires the solution of the first problem in Sec. 1. For the restricted case of the subset of two-loop diagrams for \( q\bar{q} \rightarrow q\bar{q} \) involving always one fermion loop, this is straightforward, since the one-loop corrections to the gluon propagator and vertex are then due only to the quark loops, and so are trivially isolated. Furthermore, the tree level triple gauge vertices which occur then always have one external leg and two internal legs, and so can be dealt with as in the PT at one loop.

We thus have to isolate the factors of longitudinal gluon four-momentum due to the invariant tensor structure of these internal fermion loops.

For the diagrams involving internal quark loop two-point corrections \( \Pi^{(1,f)}(k) \), e.g. Fig. 3a, we use

\[
i \Pi^{(1,f)}(k) = i(k^2 g_{\mu\nu} - k_k k_\nu) \Pi^{(1,f)}(k^2). \quad (18)
\]

For the diagrams involving internal quark loop three-point corrections \( \Gamma^{(1,f)}(k_1, k_2, k_3) \), e.g. Fig. 3b, we use the general decomposition

\[
\Gamma^{(1,f)}(k_1, k_2, k_3) = A(k_1^2, k_2^2, k_3^2) g_{\rho\sigma}(k_1 - k_2)_\rho \nonumber
\]
\[
- B(k_1^2, k_2^2, k_3^2) g_{\rho\sigma} k_3^\rho \nonumber
\]
\[
- C(k_1^2, k_2^2, k_3^2) [(k_1 k_2)_{\rho\sigma} - k_1 k_2]_{\rho\sigma} (k_1 - k_2)_\tau \nonumber
\]
\[
+ \frac{1}{2} S(k_1^2, k_2^2, k_3^2) [(k_1 k_2)_{\rho\sigma} - k_1 k_2, k_3^\rho (k_1 - k_2)] \nonumber
\]
\[
+ F(k_1^2, k_2^2, k_3^2) [(k_1 k_2)_{\rho\sigma} - k_1 k_2, k_3^\rho (k_1 - k_2)] \nonumber
\]
\[
\times [k_1(k_1 - k_3) - k_2(k_2 - k_3)] \nonumber
\]
\[
+ H(k_1^2, k_2^2, k_3^2) [g_{\rho\sigma} (k_1(k_2 - k_3) - k_2(k_1 - k_3))] \nonumber
\]
\[
+ \frac{1}{2} (k_1 k_2 (k_3^\rho - k_1 k_2 k_3^\rho) \text{ cycl. prms}.) \quad (19)
\]

Writing \( k_{2\rho} = -(k_1 + k_3)_\rho \) and \( k_{1\sigma} = -(k_2 + k_3)_\sigma \), in (19), the extension to arbitrary order of the decomposition (11) is

\[
\Gamma^{(1,f)}(k_1, k_2, k_3) = \Gamma^{(1,f)}(k_1, k_2, k_3) \quad (20)
\]

with

\[
P_{\tau}(k_1, k_2) = -[A(k_1^2, k_2^2, k_3^2)]_{\tau\rho\sigma} (k_1 - k_2)_\rho (k_1 - k_2)_\sigma 
\]
\[
+ C(k_1^2, k_2^2, k_3^2) (k_1 - k_2)_\rho (k_1 - k_2)_\sigma + [C(k_1^2, k_2^2, k_3^2)]_{\tau\rho\sigma} 
\]
\[
- (k_1 k_3) F(k_1^2, k_2^2, k_3^2) [(k_1 k_2)_{\rho\sigma} - k_1 k_2, k_3^\rho (k_1 - k_2)] 
\]
\[
- S(k_1^2, k_2^2, k_3^2) [(k_1 k_2)_{\rho\sigma} - k_1 k_2, k_3^\rho (k_1 - k_2)] 
\]
\[
- F(k_1^2, k_2^2, k_3^2) [(k_1 k_2)_{\rho\sigma} - k_1 k_2, k_3^\rho (k_1 - k_2)] 
\]
\[
+ H(k_1^2, k_2^2, k_3^2) [(k_1 k_2)_{\rho\sigma} - k_1 k_2, k_3^\rho (k_1 - k_2)] \quad (22)
\]
\[
Q_{\tau}(k_1, k_2) = C(k_1^2, k_2^2, k_3^2) (k_1 - k_2), 
\]
\[
- F(k_1^2, k_2^2, k_3^2) [(k_1 k_2)_{\rho\sigma} - k_1 k_2, k_3^\rho (k_1 - k_2)] \quad (23)
\]

For the one-loop fermionic contribution to \( \Gamma_{\rho\sigma}(k_1, k_2, k_3) \), the Ward identity

\[
k_3^2 \Gamma^{(1,f)}(k_1, k_2, k_3) = \Pi^{(1,f)}(k_1^2) - \Pi^{(1,f)}(k_2^2) \quad (24)
\]

uniquely determines the one-loop fermionic contributions to the functions \( A, B, C, S \):

\[
A^{(1,f)}(k_1^2, k_2^2, k_3^2) = -\frac{1}{2} \left( \Pi^{(1,f)}(k_1^2) + \Pi^{(1,f)}(k_2^2) \right) 
\]
\[
B^{(1,f)}(k_1^2, k_2^2, k_3^2) = -\frac{1}{2} \left( \Pi^{(1,f)}(k_1^2) - \Pi^{(1,f)}(k_2^2) \right) 
\]
\[
C^{(1,f)}(k_1^2, k_2^2, k_3^2) = \frac{2}{k_1^2 - k_2^2} g^{(1,f)}(k_1^2, k_2^2, k_3^2) 
\]
\[
S^{(1,f)}(k_1^2, k_2^2, k_3^2) = 0 \quad (25)
\]

The one-loop massless quark contributions to the functions \( F \) and \( H \) may be found in [3].

Substituting the expressions (25) into (21) -- (23), the Ward identity (24) decomposes as

\[
k_3^2 \Gamma^{(1,f)}(k_1, k_2) = \quad (26)
\]
\[
\left( k_3^2 \Pi^{(1,f)}(k_2^2) - k_2^2 \Pi^{(1,f)}(k_3^2) \right) g_{\rho\sigma}, \quad (27)
\]
\[
k_2^2 \Gamma^{(1,f)}(k_1, k_2) = \quad (28)
\]
\[
k_2 k_2 \Pi^{(1,f)}(k_3^2) - k_1 k_1 \Pi^{(1,f)}(k_1^2) \quad (28)
\]

It is important to point out that, obviously, the integrands for the scalar functions \( \Pi^{(1,f)} \) in (26) and \( A^{(1,f)} - H^{(1,f)} \) in (25) may be projected out from the integrands for \( \Pi^{(1,f)} \) and \( \Gamma^{(1,f)} \), respectively: all rearrangements in the PT take place under the loop momentum integral sign(s).

Using the above decompositions, the PT can be implemented in a similar way to the one loop case. One finds [3] that the resulting mixed bosonic-fermionic contribution to the two-loop gluon self-energy is (i) gauge-independent, (ii) multiplicatively renormalizable by a local counterterm and
(iii) has ultra-violet divergence as specified by the corresponding coefficient \((2\pi C_A + 4C_F)T_F n_f\) of the two-loop QCD \(\beta\)-function. These properties are identical to those of the two-loop vacuum polarization in QED. Furthermore, as a result of the Ward identity \((26)\), these contributions to the two-loop gluon self-energy and gluon-fermion vertex obey the QED-like Ward identity as in the PT at one loop.

3. DISCUSSION AND CONCLUSIONS

In general, the extension of the PT beyond one loop requires the solution of the two problems in Sec. 1. For the case of two-loop QCD \(n\)-point functions which involve always one fermion loop, the first problem is easily dealt with, so that one can then investigate the second problem.

Here we have described the approach to this second problem in which all factors of longitudinal four-momentum associated with gauge fields propagating in loops are used to trigger the PT rearrangement. Such factors arise not only from the lowest order gauge field propagators and triple gauge vertices, but also from the invariant tensor structure of internal loop corrections, as well as the gauge field propagator and triple gauge vertex counterterms. It was described how, for the two-loop gluon self-energy constructed in this approach, the mixed bosonic-fermionic contribution to this function displays a set of properties precisely analogous to those of the two-loop vacuum polarization in QED.

For the purely bosonic i.e. gluon and ghost contributions to the two-loop gluon self-energy in the PT framework, the first problem in Sec. 1 has yet to be solved. However, this problem has been solved for the case of the PT two-loop quark self-energy \([1]\). A key element of this solution is the decomposition of the tree level triple gauge vertex

\[
\Gamma^{(0)}_{\rho\sigma\tau}(k_1, k_2, k_3) = \Gamma^{(0)F}_{\rho\sigma}(k_1; k_2, k_3) + \Gamma^{(0)F}_{\tau\rho}(k_2; k_3, k_1) \\
- \Gamma^{(0)F}_{\tau\sigma}(k_3; k_1, k_2) - 2\Gamma^{(0)F}_{\tau\rho}(k_3; k_1, k_2).
\]  

In general, it is as yet unclear whether the approach described here is that which is required to yield PT multi-loop “effective” \(n\)-point functions with all of the desirable properties displayed at one loop. It may well be that the way to resolve this question will be to extend the analysis of \([1]\) to the two-loop level to relate the PT two-loop gauge boson self-energy in a massive theory, e.g. the electroweak Standard Model, directly to experimental observables via dispersive techniques.

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