One particle distribution function and shear viscosity in magnetic field: a relaxation approach

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We calculate the $\delta f$ correction to the one particle distribution function in presence of magnetic field and non-zero shear viscosity within the relaxation time approximation. The $\delta f$ correction is found to be electric charge dependent. Subsequently, we also calculate one longitudinal and four transverse shear viscous coefficients as a function of dimensionless Hall parameter $\chi_H$ in presence of the magnetic field. Calculation of invariant yield of $\pi^-$ in a simple Bjorken expansion with cylindrical symmetry shows no noticeable change in spectra due to the $\delta f$ correction for realistic values of the magnetic field and relaxation time. However, when transverse expansion is taken into account using a blast wave type flow field we found noticeable change in spectra and elliptic flow coefficients due to the $\delta f$ correction. The $\delta f$ is also found to be very sensitive on the magnitude of magnetic field. Hence we think it is important to take into account the $\delta f$ correction in more realistic numerical magnetohydrodynamics simulations.

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I. INTRODUCTION

Ultra-intense transient electromagnetic fields are generated in the initial stages of high energy heavy ion collisions [1–7]. The possibility of the existence of such an intense magnetic field has encouraged theorician to study QCD under the intense field. This has so far resulted in a number of a conjectured new phenomenon which is believed to exist in presence of an intense electromagnetic field. For example, the phenomena of chiral magnetic effect (CME), chiral magnetic wave, and change in the photon and dilepton productions to name a few [4, 8–12]. Some other theoretical developments related to the magnetic field in heavy ion collisions includes magnetovortical evolution [13–15], calculation of Wigner functions for fermions in strong magnetic fields [17], shear viscosity in an anisotropic unitary Fermi gas [16], the shear and bulk viscosity of Quark-Gluon-Plasma (QGP) in strong magnetic fields [18–20] etc. On the other hand, several model studies in the last decade show that the QGP created in high energy heavy ion collisions possesses a very small value of shear viscosity to entropy density ratio. Among several of these model studies, relativistic hydrodynamics played one of the most important roles to extract the value of shear viscosity to entropy density ratios from the available experimental data [21–29]. However, almost all of these hydrodynamics model studies have so far ignored the effect of a large magnetic field on the fluid evolution, hence, the extracted values of shear viscosity are probably not as precise as is usually claimed. Only in the recent year’s people finally start investigating the effect of magnetic field on QGP evolution [30–40]. In addition to that theoretical efforts are going on to better understand the relativistic magnetohydrodynamics from first principle calculations, for example the formulation of relativistic non-resistive dissipative magnetohydrodynamics from a kinetic theory approach can be found in [41], and some application to astrophysical problems is described in [42], the calculation of transport coefficients in magnetic field using Kubo formula was investigated in [43]. Here we would like to point out that almost all of the recent numerical hydrodynamic model studies with a non-zero magnetic field have concentrated only on the effect of the field on the fluid evolution. The effect of the magnetic field on the freezeout distribution function and hence on the corresponding correction to the invariant yield has so far been neglected in all of these magneto hydrodynamical model studies. As it is well known, the freezeout distribution function is used in the Cooper-Frye prescription to convert the fluid elements to particles (hadrons) during the kinetic freezeout in order to get the invariant yields of particle spectra. In the present study, we use the relaxation time approximation to calculate the $\delta f$ correction to the one particle equilibrium distribution function $f_0$ and the corresponding shear viscosity coefficients in presence of a magnetic field. It is also known that under the influence of an external magnetic field locally equilibrated thermal system becomes anisotropic and corresponding shear viscosity has five different coefficients. We would like to reiterate that the motivation for calculating the $\delta f$ correction arises to complete the study of the transverse momentum spectra and the corresponding flow harmonics calculated in hydrodynamics simulation with non-zero magnetic field. However, in this study we shall use the calculated $\delta f$ in a simplified Bjorken model with and without transverse flow to investigate the effect on invariant yield coming only from the $\delta f$. Before proceeding further, we note that the local equilibrium distribution function in presence of the electromagnetic field is known to have the following close form [44, 45]

$$f_0^m(p) = \frac{1}{(2\pi)^3} \exp\left(-\beta \left[ (p^\mu + qA^\mu)u_\mu - \mu \right] \right), \quad (1)$$

where $q$ is the electric charge, $A^\mu$ is the four potential corresponding to an electro-magnetic field, and $\mu$ is the chemical potential. We also note, that $A^\mu$ is not uniquely defined for an arbitrary given magnetic field (or in other
word using a different gauge a new $A'_{\mu}$ can also give
the same magnetic field as before) and this ambiguity in
defining $A'$ makes it difficult to use $f_0^{\text{rec}}(p)$ in the
Cooper-Frye formula
\[
E^{0N} = \int f_0^{\text{rec}} p^\mu d\Sigma_{\mu}.
\]
(2)

Where $d\Sigma_{\mu}$ is the differential freezeout hypersurface, and
$p^\mu \to p^\mu + q A^\mu$ is the canonical momentum.

Thus we cannot use Eq.(1) in the Cooper-Frye freeze-
out formula Eq.(2) in order to study the effect of mag-
netic field on the freezeout distribution function. There-
fore we choose a different approach and calculate the cor-
correction $\delta f$ to the local equilibrium distribution function
$f_0(p)$ in presence of an external magnetic field by consider-
ing the $\delta f$ to be small in comparison to the $f_0(p)$.

The paper is organized as follows. In the next section, we
discuss the formulation where $\delta f$ is calculated from the
linearised Boltzmann transport equation in presence of a
non-zero velocity gradient and magnetic field. This
section also contains some results on particle spectra in
Bjorken expansion with and without transverse flow and we
discuss the shear viscosity for Bose and Fermi gas in
presence of the magnetic field. Finally, in section III we
discuss conclusion and outlook. We use the natural unit
where $h = c = k_B = 1$ and four vectors are denoted by
Greek indices and three vectors by Latin indices.

II. FORMALISM

For the sake of completeness, let us start with the rel-
ativistic Boltzmann transport equation:
\[
p^\mu \frac{\partial f}{\partial x^\mu} + m_0 F^\mu \frac{\partial f}{\partial p^\mu} = C[f],
\]
(3)

where $m_0$ is rest mass, $p^\mu = \gamma (m_0, m_0 \vec{v}) = (p^0, \vec{p})$ is
the four momentum and $F^\mu = \gamma \vec{F} \cdot \vec{v}, \vec{F} = (F^0, \vec{F})$
is the four force. $C[f]$ on the right hand side of Eq.(3) is the
collision integral which is given by (with the assumption of
hypothesis of molecular chaos and binary collisions)
\[
C[f] = \frac{1}{2} \int \frac{d^3 p_1}{p_1} \frac{d^3 p}{p} \frac{d^3 p_1}{p_1} \frac{d^3 p_1}{p_1}
\left[ f(x, p, p_1 | p, p_1) - f(x_1, p, p_1 | p_1, p_1) \right],
\]
(4)

where $f, f_1, f_\mu$ and $f_\mu_1$ stands for one particle distribu-
tion $f(x^i, p^i), f(x^i, p_1^i), f(x^i, p^i)$ and $f(x^i, p_1^i)$ respect-
ively, $w(p, p_1 | p_1, p_1)$ is the transition rate. In equilib-
rium, the collision integral $C[f]$ vanishes for the equi-
librium distribution function $f_0(x^i, p^i)$. In principle one
can evaluate corresponding collision integral and solve
the Boltzmann transport equation. However, the gen-
eral form of collision integral is not easy to handle and
usually, it is non-trivial to solve the transport equation
analytically. A great deal of simplification can be made if
we replace the collision integral by $C[f] = -\frac{\tau_c}{\tau_s}$, where
$\tau_c$ is the relaxation time, during which the system slightly
away from equilibrium, relaxes back to the nearest equi-
librium state. In other words, for systems slightly away
from the equilibrium the distribution function can be ap-
proximated as $f = f_0 + \delta f$, where $\delta f$ denotes the devia-
tion from the equilibrium distribution function with the
additional assumption of $\frac{\tau_c}{\tau_s} \ll 1$. The calculation of $\delta f$
and hence kinetic coefficients (viscosity) can be simplified
by noting that they don’t depend on the fluid velocity $V$
explicitly. It is sufficient to consider at any point in the
fluid where $V$ is zero but has non-zero spatial derivative
i.e., we consider the fluid rest frame. In this case one
can express the Boltzmann equation in relaxation time
approximation by considering only magnetic force and
shear viscosity as [46, 47]
\[
\left( v_i p_j \frac{\partial V_i}{\partial x_j} - \frac{1}{3} v_i p_j \nabla \cdot \vec{V} \right) \left( \frac{\partial f_0}{\partial \vec{p}} \right) = -\frac{\delta f}{\tau_c} + q \varepsilon_{ijk} v_j B_k \frac{\partial f}{\partial p_i},
\]
(5)

where $q$ is the electric charge, and $\varepsilon_{ijk}$ is the totally
antisymmetric Levi-civita tensor. The last term in Eq.(5)
corresponds to the velocity dependent magnetic force on
charged particles. As mentioned earlier, here we consider
the $\delta f$ correction for the non-zero velocity gradient and
magnetic field, the corresponding viscous stress tensor
$\sigma^{ij}$ is proportional to the symmetric stress tensor of fluid
velocity as
\[
\sigma^{ij} = \eta^{ijkl} V_{kl}
\]
(6)

where $\eta^{ijkl}$ is the viscosity tensor,
\[
V^{kl} = \frac{1}{2} \left( \frac{\partial V^k}{\partial x^j} + \frac{\partial V^l}{\partial x^k} \right)
\]
(7)

and the fluid velocity $\vec{V}$ is assumed to be non-relativistic.
If we consider only shear viscosity (zero bulk viscosity)
in a magnetic field $\vec{B}$ with the unit magnetic field vector
$\vec{b} = \frac{\vec{B}}{B_0}$ then Eq.(6) can be written as
\[
\sigma^{ij} = \frac{4}{n_0} \eta^{ijkl} S^{ij}_{(n)}
\]
(8)

The above expression is constructed in such a way that
each of the second rank tensor $S^{ij}_{(n)}$ (for $n = 0 \rightarrow 4$
gives zero on contraction with respect to the indices i,j.
$\eta^{00}, \eta^{11}, \eta^{22}, \eta^{33}, \eta^{44}$ are shear viscosity co-efficient.
The second rank symmetric trace zero tensor $S^{ij}_{(n)}$‘s are
constructed out of $\delta^{ij}, V^{ij}, b^i, b^i$, and $b^{ij}$, where $b^{ij} = \varepsilon^{ij} b_k$.
\[
S^{ij}_{(0)} = (3b^i b^j - \delta^{ij}) \left( b^l b^k V_{lk} - \frac{1}{3} \nabla \cdot \vec{V} \right),
\]
(9)
\[
S^{ij}_{(1)} = 2 \delta^{ij} \nabla \cdot \vec{V} - 2 V^{ik} b_k b^j - 2 b^i V^{jk} b_k +
\]
(10)
\[
S^{ij}_{(2)} = 2 \left( V^{ik} b_k b^j + b^i V^{jk} b_k - 2 b^j V^{ik} b_k b_l \right),
\]
(11)
\[
S^{ij}_{(3)} = -b^k V_{lk} b^j - V^{ik} b_k b^j + b^i b^j V_{kl} b_k,
\]
(12)
\[
S^{ij}_{(4)} = -2 \left( b^{ik} V_{kl} b^j + b^i b^j V_{kl} b_k \right).
\]
(13)
Viscous coefficient \( \eta_{(0)} \) in Eq. (8) is called longitudinal shear viscosity since \( S_{ij}^{(0)} b_i b_j \neq 0 \), for similar reasons rest of the \( \eta_{(n)} \)'s are called transverse viscosity since they are transverse to \( b_i b_j \).

In order to find the five shear viscosity coefficients, we first note that the definition of shear stress from kinetic theory is

\[
\sigma^{ij} = -\frac{g}{(2\pi)^3} \int \delta f(p) v^i v^j d^3p
\]  

where \( g \) is degeneracy, \( v \) is the particle velocity, and \( p \) is the corresponding momentum. The correction to the equilibrium distribution function \( \delta f(p) \) can be written as

\[
\delta f(p) = -\left( \frac{\partial f_0}{\partial \epsilon} \right) C_{ijkl}(\epsilon)v^i v^j V^{kl},
\]

where \( \epsilon \) is the energy. Hitherto unknown fourth-rank tensor \( C_{ijkl}(\epsilon) \) needs to be evaluated in order to determine \( \delta f \). We note that \( C_{ijkl}(\epsilon) \) is contracted with symmetric tensors \( V^{kl} \) and \( v^i v^j \) to make \( \delta f \) scalar, and this imply the following symmetry properties

\[
C_{ijkl} = C_{jikl} = C_{ijlk}.
\]

The fourth rank tensor \( C_{ijkl} \) can be constructed from the linear combinations of \( b_i, \delta_{ij}, \) and \( \epsilon_{ijkl} \). We can construct following eight linearly independent fourth rank tensor [46]

\[
\begin{align*}
\epsilon^{(1)}_{ijkl} &= \delta_{ik} \delta_{jl} + \delta_{ij} \delta_{jk}, \\
\epsilon^{(2)}_{ijkl} &= \delta_{ij} \delta_{kl}, \\
\epsilon^{(3)}_{ijkl} &= \delta_{ik} b_j b_l + \delta_{jk} b_i b_l + \delta_{il} b_j b_k + \delta_{jl} b_i b_k, \\
\epsilon^{(4)}_{ijkl} &= \delta_{ij} b_k b_l, \\
\epsilon^{(5)}_{ijkl} &= b_i b_j \delta_{kl}, \\
\epsilon^{(6)}_{ijkl} &= b_i b_k b_l, \\
\epsilon^{(7)}_{ijkl} &= b_i b_j \delta_{kl} + b_j b_i \delta_{kl} + b_i \delta_{jk} + b_j \delta_{ik} \\
\epsilon^{(8)}_{ijkl} &= b_i b_k b_l + b_j b_i b_l + b_k b_j b_l + b_l b_j b_k \ 
\end{align*}
\]

The \( C_{ijkl} \) is linear combination of the above fourth rank tensors

\[
C_{ijkl}(\epsilon) = \tau_c \sum_{n=1}^{8} c^{(n)}(\epsilon) \epsilon^{(n)}_{ijkl},
\]

Now let us substitute \( \delta f \) from Eq.(15) into Eq.(14) which yields

\[
\sigma^{ij} = -\frac{g}{(2\pi)^3} \int \left( \frac{\partial f_0}{\partial \epsilon} \right) C^{klmn}(\epsilon) v^k p^i v^l p^m d^3p V^{mn}.
\]

Comparing Eq.(26) with Eq.(6) we have

\[
\eta^{ijkl} = -\frac{g}{(2\pi)^3} \int \left( \frac{\partial f_0}{\partial \epsilon} \right) C^{klmn}(\epsilon) v_m p_n v^i p^j d^3p.
\]

Since \( v^i = \xi^i_\rho \), the above equation can be re-written as

\[
\eta^{ijkl} = -\frac{g}{(2\pi)^3} \int \left( \frac{\partial f_0}{\partial \epsilon} \right) C^{klmn}(\epsilon) \frac{p^i p^j p^m p^n}{\epsilon^2} d^3p.
\]

The product \( p^i p^j p^m p^n \) on the right hand side of Eq.(28) can be written as

\[
p^i p^j p^m p^n = \frac{1}{15} (\delta_{ij}\delta_{mn} + \delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) p^4.
\]

The normalization constant \( \frac{1}{15} \) is obtained by contracting the index \( i \) with \( j \) and \( m \) with \( n \) (note the superscript and subscript have the same meaning since we are dealing with three dimensional vectors). Using Eq.(29) in Eq.(28) we have

\[
\eta^{ijkl} = -\frac{g}{15(2\pi)^3} \int \left( \frac{\partial f_0}{\partial \epsilon} \right) \frac{p^i}{\epsilon^2} D^{ijkl} d^3p,
\]

where

\[
D^{ijkl} = \left( \xi^{ijmn}_1 + \xi^{ijmn}_2 \right) C^{mnkl}(\epsilon).
\]

In order to find \( C^{ijkl} \) and hence \( \delta f \) let us substitute Eq.(15) and Eq.(17-24) in the Boltzmann equation Eq.(5)

\[
\left( \xi^{(1)}_{ijkl} + \chi_H \xi^{(7)}_{ijkl} \right) C_{klmn} = \tau_c \left( \xi^{(1)}_{ijmn} - \frac{2}{3} \xi^{(2)}_{ijmn} \right).
\]

Here \( \chi_H = \frac{gB}{\tau_c} \) is a dimensionless quantity also known as Hall parameter. Using the expansion Eq.(25) in Eq.(32) and taking appropriate tensor contraction one can evaluate \( c^{(n)}(\epsilon) \)'s given in Eq.(25). It is a straightforward but tedious calculation part of which we discuss in the next section and in appendix A. The shear viscosity coefficients can be calculated once we know the \( D^{ijkl} \) given in Eq.(31). This can be done in a similar way like the evaluation of \( c^{(n)} \)'s and also discussed later in the text and in Appendix A.

### A. \( \delta f \) correction due to magnetic field

As mentioned earlier, in order to evaluate the \( \delta f \) we need to find the unknown coefficient \( c^{(n)} \)'s in Eq.(25). This can be achieved by using Eq.(25) in Eq.(32) and taking the appropriate inner product on both sides. Alternatively one can also evaluate \( c^{(n)} \)'s from Eq.(31) by first evaluating \( D^{ijkl} \) (discussed in the next section). Here we solve Eq.(32) and obtain the following values of \( c^{(n)} \)'s (for details see Appendix A)

\[
\begin{align*}
c^{(1)} &= \frac{1}{2(1 + \chi_H^2)}, \\
c^{(2)} &= -\frac{(1 - \chi_H^2)}{3(1 + \chi_H^2)}, \\
c^{(3)} &= \frac{3\chi_H^2}{2(1 + \chi_H^2)^2}.
\end{align*}
\]
dependence of $c$ or even power of $c$ and $c$. Here we note that the correction to distribution function $\delta f$ is now readily obtained as

$$\delta f(p) = -\tau_c \sum_{n=1}^{8} c_n \xi(\chi_H) \left( \frac{\partial f_0}{\partial c} \right) \psi^p \psi^{ijkl}. \quad (34)$$

Here we note that $c^{(4)} = c^{(5)}$, also the coefficients $c^{(7)}$ and $c^{(8)}$ depend linearly and third power on the Hall coefficient $\chi_H = \frac{2H}{e} \tau_c$, as all other $c^{(n)}$’s contain quadratic or even power of $\chi_H$. The implication is that $c^{(7)}$ and $c^{(8)}$ are electric charge dependent and hence the $\delta f$. The dependence of $c^{(n)}$ on $\chi_H$ is shown in figure 1. It will be interesting to investigate in future the effect of the magnetic field at the freezeout on the charge dependent elliptic flow when the $\delta f$ correction (Eq.(34)) is used in the Cooper-Frye prescription for calculating the invariant yield. We left this detail investigation for a possible future work and consider (in a later section) a simple Bjorken expansion of fluid to study the effect of $\delta f$ on transverse momentum spectra of pion.

B. shear viscosity in magnetic field

Here we discuss calculation of $\eta(\alpha)$’s, for that we shall first determine the equation for $D^{ijkl}$. Using the definition of $D^{ijkl}$ from Eq. (31) in Eq.(32) and using table I of Appendix A we have

$$\left( \xi^{ijkl} - \frac{2}{5} \xi^{ijkl} + \chi_H \xi^{ijkl} \right) D_{klmn} = 2\tau_c \left( \xi^{ijmn} - \frac{2}{3} \xi^{ijkl} \right). \quad (35)$$

Let us express $D^{ijkl}$ in terms of unknown coefficients $d^{(1)}(\epsilon), ..., d^{(8)}(\epsilon)$ and $\tau_c$ as

$$D_{ijkl}(\epsilon) = \tau_c \sum_{n=1}^{8} d^{(n)}(\epsilon) \xi(\epsilon). \quad (36)$$

Solving Eq.(35) we obtain the values of $d^{(n)}(\epsilon)$ (the details of which is given in the Appendix A)

$$d^{(1)} = \frac{1}{1 + 4 \chi_H^2}, \quad d^{(2)} = -\frac{2}{3} \frac{2 - 2 \chi_H^2}{1 + 4 \chi_H^2}, \quad d^{(3)} = \frac{3}{(1 + 4 \chi_H^2)(1 + \chi_H^2)},$$

$$d^{(4)} = \frac{-4 \chi_H^2}{1 + 4 \chi_H^2}, \quad d^{(5)} = \frac{-3}{1 + 4 \chi_H^2},$$

$$d^{(6)} = \frac{-12 \chi_H^4}{1 + 4 \chi_H^2}, \quad d^{(7)} = \frac{-\chi_H^4}{1 + 4 \chi_H^2},$$

$$d^{(8)} = \frac{-3 \chi_H^6}{(1 + 4 \chi_H^2)(1 + \chi_H^2)}. \quad (37)$$

Similar to $c^{(n)}$ here we also observe $d^{(4)} = d^{(5)}$. Now we can calculate the $\eta_1 ... \eta_5$ from Eq.(28) by using these $d^{(n)}$’s. In order to do that we use the definition of traceless shear tensor $\sigma^{ijkl}$ given in Eq.(8), and also use Eq.(9-13) and Eq.(17-24) to express $\eta^{ijkl}$ in terms of $\xi(\epsilon)$ as

$$\eta^{ijkl} = \eta^{(0)} \left( 3 \xi(ij) - \xi(5) - \xi(4) + \xi(2)/3 \right)^{ijkl} + \eta^{(1)} \left( \xi(1) - \xi(2) - \xi(3) + \xi(4) + \xi(5) + \xi(6) \right)^{ijkl} + \eta^{(2)} \left( \xi(3) - 4 \xi(6) \right)^{ijkl} - \eta^{(3)} \left( \xi(7)/2 - \xi(8)/2 \right)^{ijkl} - \eta^{(4)} \left( \xi(8) \right)^{ijkl}. \quad (38)$$

Using the above expansion of $\eta^{ijkl}$ on the left hand side of Eq.(30) and contracting both side with the appropriate tensors we obtain $\eta(\alpha)$. The straightforward but tedious calculation of $\eta(\alpha)$’s are given in Appendix B, here we only note the values of $\eta(\alpha)$

$$\begin{align*}
\eta^{(1)} & = \frac{1}{1 + 4 \chi_H^2} \eta^{(0)}, \\
\eta^{(2)} & = \frac{1}{1 + \chi_H^2} \eta^{(0)}, \\
\eta^{(3)} & = \frac{2 \chi_H}{1 + 4 \chi_H^2} \eta^{(0)}, \\
\eta^{(4)} & = \frac{\chi_H}{1 + \chi_H^2} \eta^{(0)}. \quad (39)
\end{align*}$$
Here we observe that $\eta(3) = 2\chi_H\eta(1)$ and $\eta(4) = \chi_H\eta(2)$. The variation of $\eta(j)/\eta(0)$ (where $j = 1 - 4$) as a function of $\chi_H$ is shown in figure 2. One still need to evaluate the longitudinal shear viscosity $\eta(0)$. Once $\eta(0)$ is known for a given system, the rest of the $\eta(n)$’s are obtained from Eq.(39) for a given value of $\chi_H$. We shall evaluate $\eta(0)$ for a few cases.

C. calculation of $\eta(0)$ for Bose and Fermi gas

The expression for longitudinal shear viscosity as evaluated in Appendix B (Eq.(B8)) is

$$\eta(0) = \frac{1}{15} \frac{g_T e}{2(2\pi)^3} \int \frac{d^3 p}{e^2} \frac{p^4}{T^3} d^3 p. \quad (40)$$

For massless Bose and Fermi gas $f_0 = \frac{1}{\exp[(e - \mu)/T] + 1}$, where $e = p$ and $\mu$ is the chemical potential. We note that for this case $\frac{\partial f_0}{\partial \mu} = \frac{-\delta f_0}{\delta \mu}$ and Eq.(40) can be expressed as

$$\eta(0) = \frac{1}{15} \frac{g_T e}{2(2\pi)^3} \frac{\partial}{\partial \mu} \int \frac{d^4 p}{T^3} L^0 d^3 p. \quad (41)$$

Eq.(41) is a special case of the following general form

$$B_n = \int_0^\infty \frac{x^n dx}{\exp(x - a) + 1} = \pm \Gamma(n + 1)L_i_{1+n}(\pm e^a), \quad (42)$$

where $a$ is a constant, $L_i_x(x)$ is the poly-logarithmic function of order $n$, and $\Gamma(n)$ is the gamma function. Using the general result Eq.(42) in Eq.(41) yields

$$\eta(0) = \frac{4g_T e L_i(x(\pm e^\mu/T))}{5\pi^2} T^4, \quad (43)$$

where $\pm$ corresponds to Boson and Fermi gas respectively.

The $\eta_0$ at $\mu = 0$ for bosons and fermions are $\eta(0) = \frac{2}{27} g_T e \pi^2 T^4$, and $\eta(0) = \frac{7}{900} g_T e \pi^2 T^4$ respectively.

Now let us discuss the case for non-zero mass. It is not possible to evaluate the integral Eq.(40) in a closed form for an arbitrary mass of the particles, however, for $m \ll T$, the integral in Eq.(40) can be expanded in terms of $m/T$, and we have

$$\eta(0) = \frac{8}{150} \frac{g_T e T^4}{\pi^2} \left( 24L_i(x(\pm e^\mu/T)) - 5L_i(x(\pm e^\mu/T)) \left( \frac{m}{T} \right)^2 + \cdots \right) \quad (44)$$

where again the symbol $\pm$ corresponds to bosons or fermions respectively.

III. EFFECT OF $\delta f$ TO THE TRANSVERSE MOMENTUM SPECTRA

A. In a boost invariant one dimensional Bjorken expansion

One needs to implement the $\delta f$ correction in the Cooper-Frye freezeout formula in a numerical magnetohydrodynamics code to investigate the actual effect of $\delta f$ on invariant yield and flow coefficients. Such a involved study is out of the scope of the present work. For the sake of simplicity, here we consider a Bjorken expansion of fluid and calculate the corresponding invariant yield of pion in a magnetic field by using the distribution function obtained in Eq.(34). The fluid four velocity in this case takes the following form ($u^\tau = 1$, $u^\eta = u^\phi = 0$), where we define the longitudinal proper time $\tau = \sqrt{x^2 - z^2}$, space-time rapidity $\eta_s = \frac{1}{T} \ln \left( \frac{1 + z}{1 - z} \right)$, $r = \sqrt{x^2 + y^2}$ and $\phi = \arctan(y/x)$. The only non vanishing component of the shear stress for this velocity profile is $V_{k\ell} = V_{\eta_s} = -\frac{1}{T}$ [48]. When the above
value of $V_{n_1,n_2}$ is used in Eq.(34) the terms which are non-vanishing, contains magnetic field only along the $\eta_v$ direction. Hence we are forced to choose a hypothetical external magnetic field which is strongest along the $\eta_v$ direction $(0,0,0,B_{n_v})$. In reality we know that the magnetic field is strongest in the transverse plane and vanishing along longitudinal direction.

The invariant yield of pions are obtained from the Cooper-Frye freezeout formula which for the present case for a Boltzmann gas is

$$\frac{d^2 N(0)}{d^2 p_T dy} + \sum_{i=1}^{8} \frac{d^2 N(i)}{d^2 p_T dy} = \frac{1}{(2\pi)^2} \int p^\mu d\Sigma^\mu \left( f_0 + \sum_{i=1}^{8} \delta f^{(i)} \right)$$

FIG. 5: Top panel: Invariant yield of $\pi^-$ as a function of $p_T$ or various values of relaxation time when magnetic field kept constant $qB = 0.1 m_\pi^2$. Bottom panel: same as top panel but for elliptic flow $v_2$ of $\pi^-$. 

where $x = m_T/T$, modified Bessel function of second kind $K_0(x)$ the differential freezeout hypersurface $d\Sigma^\mu = (\tau d\eta, r dr d\phi, 0, 0, 0)$, and the four momentum $p^\mu = (m_T \cos y, p_T \cos \phi, p_T \sin \phi, m_T \sinh y)$. The superscript 0 in Eq.(45) indicates the equilibrium case. The first integral on the right hand side is $\frac{2 m_T \pi R_0^2}{(2\pi)^3} K_1(x)$ and the second integral is the correction to the invariant yield. We evaluate the correction by using the appropriate form of $\delta f^{(i)}$’s (given in appendix C) in Eq.(45) and evaluating the integral numerically. For the calculation we have also used $c_H(c) = (qB \tau_c)/(m_T \cos y)$, $\tau_c = 0.5$ fm, and we take midrapidity i.e., $y = 0$. We found that the $p_T$ spectra hardly changes in magnetic field compared to without magnetic field case even for a ambitious value of $qB \sim 100 m_\pi^2$. For a more realistic fluid expansion which contains transverse as well as azimuthal variation of fluid velocity, the correction to the $p_T$ spectra and flow coefficients might be quite different than what is obtained here, this case is discussed in the next section. It is also worthwhile to mention that $\delta f$ contains two terms $\delta f^{(7)}$ and $\delta f^{(8)}$ which are charge dependent ($c^{(7)}, c^{(8)}$ in Eq.(33)) which might be important for studying CME. For the present case of one dimensional Bjorken expansion these charge dependent terms vanishes.

FIG. 4: Elliptic flow $v_2$ of $\pi^-$ as a function of $p_T$. Solid red line corresponds to without magnetic field and zero shear stress $(V_{\perp} = 0)$ and other lines correspond to different values of magnetic field.

in Eq.(33))

B. Transverse expansion along with the longitudinal Bjorken expansion

In order to study a more realistic fluid expansion with non-zero transverse flow, following Ref. [48] we use a generalise version of the blast wave model. We would like to mention that the blast wave model is a simple model of the flow fields and a full dissipative magnetohydrodynamics simulation is needed to estimate actual effects. By assuming a linear rise of transverse velocity as a function of radius of the fireball in the transverse plane and a velocity field with a small elliptic flow component we use a generalise version of the blast wave model. We would like to generalise version of the blast wave model. We would like to mention that the blast wave model is a simple model of the flow fields and a full dissipative magnetohydrodynamics simulation is needed to estimate actual effects. By assuming a linear rise of transverse velocity as a function of radius of the fireball in the transverse plane and a velocity field with a small elliptic flow component we have the following hydrodynamical fields :
For a head-on collision the elliptic flow component \( u_2 = 0 \). Here we have taken, \( u_0 = 0.56 \) and \( u_2 = 0.1 \). These values approximately corresponds to a mid central heavy ion collisions at top RHIC energies. It is useful to realise that \( \tau u^\eta \) and \( r u^\phi \) are velocities in \( \eta \) and \( \phi \) directions respectively. For boost invariant flow \( u^\eta = 0 \) and for rotationally invariant flow \( u^\phi = 0 \). Here we also assume longitudinal boost-invariance, the invariant yield is calculated by using Eq.(45) for the given hydrodynamic fields Eq.(46) . Calculation of corresponding \( V_{kl} \) are given in Appendix D. The invariant yield of \( \pi^- \) for various values of magnetic fields are shown in figure 3. For comparison we also show the zero magnetic field and zero shear stress case in the same figure by the solid red line. Unlike one dimensional expansion we found visible correction to the invariant yield when non-zero transverse expansion is taken into account (note that \( qB = 0 \) does not imply ideal case, one also needs to set zero shear viscosity which is achieved by setting \( V_{kl} = 0 \)). In order to calculate the \( v_2 \) we use the following formula which is obtained by considering the correction to be small (see Ref. [48])

\[
v_2(p_T) = v_2^0(p_T) \left( 1 - \frac{\int d\phi \frac{d^2N^{(1)}}{dp_T dp^2_T}}{\int d\phi \frac{d^2N^{(0)}}{p_T dp_T dp^2_T}} + \frac{\int d\phi cos(2\phi) \frac{d^2N^{(2)}}{dp_T dp^2_T}}{\int d\phi \frac{d^2N^{(0)}}{p_T dp_T dp^2_T}} \right)
\]

The \( v_2 \) for different values of magnetic field are shown in figure 4. One can clearly see that \( v_2 \) changes due to the \( \delta f \) correction in presence of magnetic field compared to the without magnetic field case (shown by red line in figure 4). The above mentioned results are obtained for a constant relaxation time \( \tau_c = 0.5 \) fm. The dependence of invariant yield and \( v_2 \) on \( \tau_c \) for a fixed value of magnetic field \( qB \) are shown in figure 5. Finally we would like to comment on the non-monotonic behaviour of \( \delta f \) correction as observed in invariant yield and \( v_2 \) (see figure 3 and 4 ). We note that the contribution from \( \delta f_1 \) and \( \delta f_2 \) shows non-monotonic behaviour as a function of magnetic field. This is shown in figure 6 where we plot corresponding corrections in invariant yields \( \frac{d^2N^{(1)}}{dp_T dp^2_T} \) and \( \frac{d^2N^{(2)}}{dp_T dp^2_T} \) respectively as function of \( p_T \) for two different values of magnetic field \( qB = 0.1m^2_\pi \) (orange lines) and \( qB = 10m^2_\pi \) (blue lines). One can clearly see that the corrections due to two terms cancel each other for smaller values of magnetic field, whereas for a larger magnetic field they act coherently. This behaviour is reflected in the correction to invariant yield and \( v_2 \).

\[
T(\tau_0, \eta_s, r, \phi) = T_0 \Theta(R_0 - r),
\]

\[
u^\prime(\tau_0, \eta_s, r, \phi) = \frac{u_0}{r_0} \left( 1 + u_2 \cos(2\phi) \right) \Theta(R_0 - r),
\]

\[
u^\phi = 0,
\]

\[
u^\eta = 0,
\]

\[
u^\tau = \sqrt{1 + (\nu^\phi)^2}.
\]

\[
(46)
\]

IV. CONCLUSION

We calculated the \( \delta f \) correction for a fluid with finite velocity gradient and under the influence of magnetic field in the relaxation time approximation. The \( \delta f \) is found to be composed of eight different terms some of which are electric charge dependent. When used in the Cooper-Feynman freezeout formula for a simple fluid evolution in one dimensional Bjorken expansion, the change in invariant yield of negative pion is found to be negligible for realistic values of the magnetic field. Interestingly, however, when the transverse expansion of the fluid is taken into account we found that both spectra and \( v_2 \) shows noticeable changes. We also evaluated the shear viscous coefficients in magnetic field. One longitudinal \( \eta(0) \) and four transverse \( (\eta_{(1)}, \eta_{(2)}, \eta_{(3)}, \eta_{(4)}) \) viscous coefficients have been calculated as a function of Hall parameter \( \chi_H \). It is observed that the transverse viscosities are smaller in comparison to the \( \eta(0) \) for \( 0 \leq \chi_H \leq 5 \). It has also been shown that \( \eta_{(2)} - \eta_{(4)} \) can be expressed in terms of \( \eta(0) \) with coefficients which are functions of \( \chi_H \) only. All the transverse shear viscous coefficients are found to decrease with respect to the \( \eta_0 \) for \( \chi_H > 1 \). At smaller \( \chi_H \leq 1 \), the \( \eta_3 \) and \( \eta_4 \) shows non-monotonic behavior. We would like to mention that it is important to calculate the same \( \delta f \) in other methods such as Chapman-Enskog or Grad’s moment method in order to corroborate our findings. For example, it is
interesting to compare results from other methods to see whether one also get similar electric charge dependent terms in the $\delta f$ corrections which might be important for studying CME. These open problems are out of the scope of the present study and we leave it for a possible future investigation.

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Appendix A: calculation of $c^{(n)}$ and $d^{(n)}$’s

Here we discuss in details the calculation of coefficients $c^{(n)}$’s and $d^{(n)}$’s appeared in Eq.33) and Eq.37). First let us calculate $c_n$’s from Eq. 32

$$
\left(\xi_{ijkl}^{(1)} + \chi H \xi_{ijkl}^{(7)}\right) C_{klmn} = \tau_c \left(\xi_{ijkl}^{(1)} - \frac{2}{3} \xi_{ijkl}^{(2)}\right)
$$

(A1)

where $C_{ijkl} = \tau_c \sum_{n=1}^{8} e^{(n)}(e)\xi_{ijkl}^{(n)}$

\begin{equation}
\left(\xi_{ijkl}^{(1)} + \chi H \xi_{ijkl}^{(7)}\right) \tau_c \sum_{n=1}^{8} c^{(n)}(e)\xi_{klmn}^{(n)} = \tau_c \left(\xi_{ijkl}^{(1)} - \frac{2}{3} \xi_{ijkl}^{(2)}\right)
\end{equation}

(A2)

For convenience let us write $c^{(n)}(e)$ as $c^{(n)}$, swap the right and left hand side of the above equation and writing the tensor index as a common term we have

$$
\left(\xi_{ijkl}^{(1)} - \frac{2}{3} \xi_{ijkl}^{(2)}\right)_{ijmn} = \left(\xi_{ijkl}^{(1)} + \chi H \xi_{ijkl}^{(7)}\right)_{ijkl} \xi_{klmn}^{(1)}
+ \left(\xi_{ijkl}^{(1)} + \chi H \xi_{ijkl}^{(7)}\right)_{ijkl} \xi_{klmn}^{(2)}
+ \left(\xi_{ijkl}^{(1)} + \chi H \xi_{ijkl}^{(7)}\right)_{ijkl} \xi_{klmn}^{(3)}
+ \left(\xi_{ijkl}^{(1)} + \chi H \xi_{ijkl}^{(7)}\right)_{ijkl} \xi_{klmn}^{(4)}
+ \left(\xi_{ijkl}^{(1)} + \chi H \xi_{ijkl}^{(7)}\right)_{ijkl} \xi_{klmn}^{(5)}
+ \left(\xi_{ijkl}^{(1)} + \chi H \xi_{ijkl}^{(7)}\right)_{ijkl} \xi_{klmn}^{(6)}
+ \left(\xi_{ijkl}^{(1)} + \chi H \xi_{ijkl}^{(7)}\right)_{ijkl} \xi_{klmn}^{(7)}
+ \left(\xi_{ijkl}^{(1)} + \chi H \xi_{ijkl}^{(7)}\right)_{ijkl} \xi_{klmn}^{(8)}
\end{equation}

(A3)

Now we need to evaluate the tensor products like

$$
\left(1 - \frac{1}{2} \xi_{ijkl}^{(1)} - \frac{1}{3} \xi_{ijkl}^{(2)}\right)_{ijmn} = \left(\xi_{ijkl}^{(1)} - \frac{2}{3} \xi_{ijkl}^{(2)}\right)_{ijmn}
+ \left(\xi_{ijkl}^{(1)} + \chi H \xi_{ijkl}^{(7)}\right)_{ijkl} \xi_{klmn}^{(1)}
+ \left(\xi_{ijkl}^{(1)} + \chi H \xi_{ijkl}^{(7)}\right)_{ijkl} \xi_{klmn}^{(2)}
+ \left(\xi_{ijkl}^{(1)} + 3 \chi H \xi_{ijkl}^{(7)} - \chi H \xi_{ijkl}^{(8)}\right)_{ijkl} \xi_{klmn}^{(3)}
+ \left(\xi_{ijkl}^{(1)} + 4 \chi H \xi_{ijkl}^{(7)}\right)_{ijkl} \xi_{klmn}^{(4)}
+ \left(\xi_{ijkl}^{(1)} + 4 \chi H \xi_{ijkl}^{(7)}\right)_{ijkl} \xi_{klmn}^{(5)}
+ \left(\xi_{ijkl}^{(1)} + \chi H \xi_{ijkl}^{(7)}\right)_{ijkl} \xi_{klmn}^{(6)}
+ \left(\xi_{ijkl}^{(1)} + \chi H \xi_{ijkl}^{(7)}\right)_{ijkl} \xi_{klmn}^{(7)}
+ \left(\xi_{ijkl}^{(1)} + \chi H \xi_{ijkl}^{(7)}\right)_{ijkl} \xi_{klmn}^{(8)}
$$

(A4)

Equating the coefficients of $\xi_{ijkl}^{(n)}$ on both side of Eq.(A4) we have,

$$
c^{(1)} - \frac{4}{3} \chi H c^{(7)} = \frac{1}{2}
$$

$$
c^{(2)} + 4 \chi H c^{(7)} = -\frac{1}{3}
$$

$$
c^{(3)} + 3 \chi H c^{(7)} - \chi H c^{(8)} = 0
$$

$$
c^{(4)} - 4 \chi H c^{(7)} = 0
$$

$$
c^{(5)} - 4 \chi H c^{(7)} = 0
$$

$$
c^{(6)} + 4 \chi H c^{(8)} = 0
$$

$$
c^{(7)} + \chi H c^{(1)} = 0
$$

$$
c^{(8)} + \chi H c^{(3)} = 0
$$

(A5)

By solving these eight equations, we obtain the values of $c^{(n)}$’s given in Eq.33).

The calculation for $d^{(n)}$’s proceed in a similar way. From Eq.(35) we have

$$
\left(\xi_{ijkl}^{(1)} - \frac{2}{5} \xi_{ijkl}^{(2)} + \chi H \xi_{ijkl}^{(7)}\right) D_{klmn} = 2 \tau_c \left(\xi_{ijkl}^{(1)} - \frac{2}{3} \xi_{ijkl}^{(2)}\right)
\end{equation}

(A6)

where

$$
D_{ijkl}(e) = \tau_c \sum_{n=1}^{8} d^{(n)}(e)\xi_{ijkl}^{(n)}
\end{equation}

(A7)

Thus we have
Comparing the coefficients of $\xi^{(n)}_{ijkl}$ on both sides of the above equations we have

$$d^{(1)} - 4\chi H d^{(7)} = 1$$
$$d^{(2)} - \frac{2}{5}d^{(2)} + \frac{1}{5}d^{(5)} - 4\chi H d^{(7)} = \frac{2}{3}$$
$$d^{(3)} + 3\chi H d^{(7)} - \chi_0 d^{(8)} = 0$$
$$-4\chi d^{(3)} - \frac{2}{5}d^{(4)} - \frac{1}{5}d^{(6)} - 4\chi H d^{(7)} = 0$$
$$d^{(5)} - 4\chi H d^{(7)} = 0$$
$$d^{(6)} + 4\chi H d^{(8)} = 0$$
$$d^{(7)} + \chi_0 d^{(4)} = 0$$
$$d^{(8)} + \chi_0 d^{(3)} = 0.$$ (A10)

The values of $d^{(n)}$’s (see Eq.37) are obtained by solving these eight equations.

**Appendix B: calculation of $\eta_{(n)}$**

Here we shall discuss the evaluation of longitudinal viscosity $\eta_{(0)}$ for which the final expression is given in Eq.(40).

Using the tabulated values (table 1) for the tensor product and using the shorthand notation for tensor indices we have

$$\begin{pmatrix} \xi^{(1)} - \frac{2}{3} \xi^{(2)} \end{pmatrix}_{ijmn} = d^{(1)} \begin{pmatrix} 2\xi^{(1)} - \frac{4}{5} \xi^{(2)} + 2\chi H \xi^{(7)} \end{pmatrix}_{ijmn}$$

$$+ d^{(2)} \begin{pmatrix} \frac{4}{5} \xi^{(2)} \end{pmatrix}_{ijmn}$$

$$+ d^{(3)} \begin{pmatrix} 2\xi^{(3)} - \frac{8}{5} \xi^{(4)} + 2\chi H \xi^{(8)} \end{pmatrix}_{ijmn}$$

$$+ d^{(4)} \begin{pmatrix} \frac{4}{5} \xi^{(4)} \end{pmatrix}_{ijmn}$$

$$+ d^{(5)} \begin{pmatrix} 2\xi^{(5)} - \frac{2}{5} \xi^{(2)} \end{pmatrix}_{ijmn}$$

$$+ d^{(6)} \begin{pmatrix} 2\xi^{(6)} - \frac{2}{5} \xi^{(4)} \end{pmatrix}_{ijmn}$$

$$+ d^{(7)} \begin{pmatrix} 2\xi^{(7)} - 8\chi H \end{pmatrix}_{ijmn}$$

$$+ d^{(8)} \begin{pmatrix} 2\xi^{(8)} + 2\chi H \left(4\xi^{(6)} - \xi^{(3)} \right) \end{pmatrix}_{ijmn}.$$ (A8)

As given in Eq.(38) the $\eta_{ijkl}$ can be decomposed as

$$\eta_{ijkl} = \sum_{i=0}^{4} \eta_{(n)} I_{(n)}^{ijkl},$$ (B1)

where $I_{(n)}^{ijkl}$’s are defined as

$$I_{(0)}^{ijkl} = (3\xi^{(6)} - \xi^{(4)} - \xi^{(5)} + \xi^{(2)}/3)^{ijkl},$$ (A9)
In order to find $\eta$ Eq. (B1) on the left hand side of the above equation
and multiply both side by $f_{ijkl}^{(0)}$ yield

$$\eta_{(0)} f_{ijkl}^{(0)} = - \frac{1}{15} \frac{g}{(2\pi)^4} \int d^3 p \left( \frac{\partial f_0}{\partial \epsilon} \right) \frac{p_i p_j}{\epsilon^2} f_{ijkl}^{(0)}.$$  

(B4)

In order to evaluate the above integral we need to find the product $D^{ijkl} f_{ijkl}^{(0)}$. By using the expression for $D^{ijkl}$ Eq. (36) we have

$$D^{ijkl} f_{ijkl}^{(0)} = \tau_c \sum_{n=1}^{8} d_n \xi_n^{ijkl} f_{ijkl}^{(0)}.$$  

(B5)

For simplicity and to avoid writing repetitively the tensor index $ijkl$ we remove the indices in the following equation

\[ \sum_{n=1}^{8} d_n \xi_n I^{(0)} = d_{(1)} \xi_{(1)} \left( 3\xi^{(0)} - \xi^{(4)} - \xi^{(5)} + \frac{\xi^{(2)}}{3} \right) + d_{(2)} \xi_{(2)} \left( 3\xi^{(6)} - \xi^{(4)} - \xi^{(5)} + \frac{\xi^{(2)}}{3} \right) + d_{(3)} \xi_{(3)} \left( 3\xi^{(6)} - \xi^{(4)} - \xi^{(5)} + \frac{\xi^{(2)}}{3} \right) + d_{(4)} \xi_{(4)} \left( 3\xi^{(6)} - \xi^{(4)} - \xi^{(5)} + \frac{\xi^{(2)}}{3} \right) + d_{(5)} \xi_{(5)} \left( 3\xi^{(6)} - \xi^{(4)} - \xi^{(5)} + \frac{\xi^{(2)}}{3} \right) + d_{(6)} \xi_{(6)} \left( 3\xi^{(6)} - \xi^{(4)} - \xi^{(5)} + \frac{\xi^{(2)}}{3} \right) + d_{(7)} \xi_{(7)} \left( 3\xi^{(6)} - \xi^{(4)} - \xi^{(5)} + \frac{\xi^{(2)}}{3} \right) + d_{(8)} \xi_{(8)} \left( 3\xi^{(6)} - \xi^{(4)} - \xi^{(5)} + \frac{\xi^{(2)}}{3} \right) \]

(B6)

From kinetic theory the $\eta_{ijkl}$ can be expressed in terms of $\delta f$ as

$$\eta_{ijkl} = - \frac{1}{15} \frac{g}{(2\pi)^4} \int d^3 p \left( \frac{\partial f_0}{\partial \epsilon} \right) \frac{p_i p_j}{\epsilon^2} D^{ijkl}$$  

(B3)

In order to find $\eta_{(0)}$ we use the decomposition given in Eq. (B1) on the left hand side of the above equation and

\[ \sum_{n=1}^{8} d_n \xi_n I^{(0)} = d_{(1)} \xi_{(1)} \left( 3\xi^{(0)} - \xi^{(4)} - \xi^{(5)} + \frac{\xi^{(2)}}{3} \right) + d_{(2)} \xi_{(2)} \left( 3\xi^{(6)} - \xi^{(4)} - \xi^{(5)} + \frac{\xi^{(2)}}{3} \right) + d_{(3)} \xi_{(3)} \left( 3\xi^{(6)} - \xi^{(4)} - \xi^{(5)} + \frac{\xi^{(2)}}{3} \right) + d_{(4)} \xi_{(4)} \left( 3\xi^{(6)} - \xi^{(4)} - \xi^{(5)} + \frac{\xi^{(2)}}{3} \right) + d_{(5)} \xi_{(5)} \left( 3\xi^{(6)} - \xi^{(4)} - \xi^{(5)} + \frac{\xi^{(2)}}{3} \right) + d_{(6)} \xi_{(6)} \left( 3\xi^{(6)} - \xi^{(4)} - \xi^{(5)} + \frac{\xi^{(2)}}{3} \right) + d_{(7)} \xi_{(7)} \left( 3\xi^{(6)} - \xi^{(4)} - \xi^{(5)} + \frac{\xi^{(2)}}{3} \right) + d_{(8)} \xi_{(8)} \left( 3\xi^{(6)} - \xi^{(4)} - \xi^{(5)} + \frac{\xi^{(2)}}{3} \right) \]

(B7)

\[ \sum_{n=1}^{8} d_n \xi_n I^{(0)} = \frac{8}{3} d_{(3)}^{(3)} + d_{(5)}^{(5)} + \frac{2}{3} d_{(6)}^{(6)} \]

(B8)

\[ \sum_{n=1}^{8} d_n \xi_n I^{(0)} = \frac{8}{3} \left( \frac{1}{1 + 4\chi_H^2} \right) + \frac{3\chi_H^2}{(1 + 4\chi_H^2)(1 + \chi_H^2)} + \left( \frac{-4\chi_H^2}{1 + 4\chi_H^2} \right) + \frac{2}{3} \left( \frac{12\chi_H^2}{(1 + 4\chi_H^2)(1 + \chi_H^2)} \right) \]

(B9)

\[ \sum_{n=1}^{8} d_n \xi_n I^{(0)} = \left( \frac{1 + 4\chi_H^2}{1 + \chi_H^2} \right) \left( \frac{1 + \chi_H^2}{1 + \chi_H^2} \right) \]

(B10)
In evaluating the tensor contraction in the above calculation we have used values from table (II).

Now let us calculate the product \( I^{ijkl}_{(0)} I^{(0)ijkl} \) which appeared on the left hand side of Eq. (B4) Again we shall use values given in table (II) for the following calculation.

\[
I^{ijkl}_{(0)} I^{(0)ijkl} = \left( 3\xi(6) - \xi(4) - \xi(5) + \frac{\xi(2)}{3} \right)_{ijkl} \left( 3\xi(6) - \xi(4) - \xi(5) + \frac{\xi(2)}{3} \right)_{ijkl}
= 9 - 3 - 3 + 1 - 3 + 1 - 3 + 1 + 3 - 1 + 3 - 1 + 1 - 1 + 1 + 1
= 4.
\]

Finally using these values in Eq. (B4) we have the expression for \( \eta_{(0)} \) as

\[
\eta_{(0)} = -\frac{1}{15} \frac{g_T}{(2\pi)^3} \int d^4p \frac{\partial f_0}{\partial \xi} \frac{p^4 e^4}{c^2}.
\]

Calculation of \( \eta_{(1)} \), \( \eta_{(4)} \) are similar and we don’t include them here.

Appendix C: \( \delta f \) for Bjorken expansion

Here we discuss the \( \delta f^{(i)} \) used in Eq. (45) for the longitudinal boost invariant case. The shear stress for this case is. For the boost invariant expansion without transverse flow, \( u_r = 1 \) and all other components of velocity are zero \( (u_\theta = u_\phi = 0) \). Thus the only non vanishing component of the shear stress \( V_{kk} = V_{\eta \eta} = -\frac{1}{2} \). The corresponding \( \delta f^{(3)} \) for the magnetic field \( 0,0,0, B_\eta \) turns out to be of the following form

\[
\delta f^{(1)} = -2c(1)\tau_{c} \left( \frac{f_0}{T \tau_{c}} \right) m^2_{T} \sin^2 \eta_{s},
\]
\[
\delta f^{(2)} = -c(2)\tau_{c} \left( \frac{f_0}{T \tau_{c}} \right) m^2_{T} \sin^2 \eta_{s},
\]
\[
\delta f^{(3)} = -4c(3)\tau_{c} \left( \frac{f_0}{T \tau_{c}} \right) b^2_{0} m^2_{T} \sin^2 \eta_{s},
\]
\[
\delta f^{(4)} = -c(4)\tau_{c} \left( \frac{f_0}{T \tau_{c}} \right) b^2_{0} m^2_{T} \sin^2 \eta_{s},
\]
\[
\delta f^{(5)} = -c(5)\tau_{c} \left( \frac{f_0}{T \tau_{c}} \right) b^2_{0} m^2_{T} \sin^2 \eta_{s},
\]
\[
\delta f^{(6)} = -c(6)\tau_{c} \left( \frac{f_0}{T \tau_{c}} \right) b^2_{0} m^2_{T} \sin^2 \eta_{s},
\]
\[
\delta f^{(7)} = 0, 
\]
\[
\delta f^{(8)} = 0.
\]

Appendix D: \( \delta f \) for Bjorken expansion along with transverse expansion

Here we calculate the \( \delta f \) correction for the case of transverse expansion as discussed in section III.B. Assuming boost invariance, the spatial components of viscous tensor \( V^{kl} \) are given by (in order to be consistent with our definition of \( V^{kl} \) we neglect the term which makes \( \nabla^\mu u^\nu \) traceless in Ref.[48]),

\[
V^{rr} = -2\partial_r u^r - 2u^r Du^r,
\]
\[
V^{\phi \phi} = -2\partial_\phi u^\phi - 2u^\phi Du^\phi,
\]
\[
V^{\eta \eta} = -\frac{u^\tau}{\tau},
\]
\[
V^{rr} = -r \partial_r u^r - ru^r Du^r - ru^r Du^\phi,
\]
\[
V^{r \phi} = V^{r \phi} = 0,
\]

and the derivatives in the rest frame \( Du^\mu \) are given by,

\[
Du^r = u^r \partial_r u^r + u^r \partial_r u^r + u^\phi \partial_\phi u^\phi - r(u^\phi)^2,
\]
\[
r Du^\phi = u^r \partial_r (ru^\phi) + u^\phi \partial_\phi (ru^\phi) + u^\phi \partial_\phi (ru^\phi) + u^\phi u^r.
\]

To fix the values of the time derivatives \( (\partial_r u^\phi, \partial_\phi u^r, \partial_r u^r) \) appear in the above equations, it is sufficient to consider the ideal equation of motion. Thus the time derivatives can be determined as following,

\[
\partial_r u^\phi = 0 
\]
\[
\partial_\tau u^r = \frac{c_\eta^2 v}{1 - c_\eta^2 v} \left( \frac{u^r + u^r}{\tau} + (1 + u^2) \partial_\tau u^r \right) - v \partial_\tau u^r 
\]
\[
\partial_\tau u^r = v \partial_\tau u^r
\]

Here \( v = u^\phi / u^r \) is the radial velocity and \( c_\eta \) denotes the velocity of sound. We use \( c_\eta = 0.1 \) which is taken.
from lattice QCD results at temperature $T_f = 130$ MeV [49].

The corresponding $\delta f^{(i)}$ for the magnetic field

$$\delta f^{(1)} = c^{(1)}/T_e \left( f_0/T_e \right) 2 [p^2 r V_{rr} + 2 p_r \phi b_r V_{rr} + p^2_b V_{\phi \phi} + p^2_\eta V_{\eta \eta}],$$

$$\delta f^{(2)} = c^{(2)}/T_e \left( f_0/T_e \right) [(p^2 r^2 + p^2_\eta)(V_{rr} + V_{\phi \phi} + V_{\eta \eta})],$$

$$\delta f^{(3)} = c^{(3)}/T_e \left( f_0/T_e \right) [p^2 b^2_r V_{rr} + p^2 b^2_\phi V_{\phi \phi} + (p^2_r + p^2_\eta)b_r b_r V_{rr} + 2 p_r \phi b_r b_\phi (V_{rr} + V_{\phi \phi}) + p_r \phi (b^2_r + b^2_\phi) V_{rr}],$$

$$\delta f^{(4)} = c^{(4)}/T_e \left( f_0/T_e \right) [(p^2_r + p^2_\eta)(b^2_r V_{rr} + b^2_\phi V_{\phi \phi}) + 2 b_r b_\phi V_{rr}],$$

and $\delta f^{(5)}$ for the electromagnetic field ($0, b_r, b_\phi, 0$) turns out to be of the following form

$$\delta f^{(5)} = c^{(5)}/T_e \left( f_0/T_e \right) [(p^2 r^2 + p^2_\eta b^2_r + 2 p_r \phi b_r b_\phi) (V_{rr} + V_{\phi \phi} + V_{\eta \eta})],$$

$$\delta f^{(6)} = c^{(6)}/T_e \left( f_0/T_e \right) [(p^2 r^2 b^2_r + p^2_\eta b^2_\phi + 2 p_r \phi b_r b_\phi),$$

$$\delta f^{(7)} = c^{(7)}/T_e \left( f_0/T_e \right) 4 p_r [p_r b_\phi (V_{rr} - V_{\phi \phi})$$

$$+ (p_r b_\phi - p_\phi b_r) V_{rr} + p_\phi b_r (V_{\phi \phi} - V_{\eta \eta})],$$

$$\delta f^{(8)} = c^{(8)}/T_e \left( f_0/T_e \right) 4 p_r [(p_r b^2_\phi + p_\phi b_r b_\phi) (V_{rr} - V_{\phi \phi})$$

$$+ (p_r b_\phi^2 - p_\phi b_r b_\phi + p_\phi b_\phi^2) V_{rr}].$$

(D4)

[1] A. Bzdak and V. Skokov, “Event-by-event fluctuations of magnetic and electric fields in heavy ion collisions,” Phys. Lett. B 710, 171 (2012).

[2] W. T. Deng and X. G. Huang, “Event-by-event generation of electromagnetic fields in heavy-ion collisions,” Phys. Rev. C 85, 044907 (2012).

[3] V. Roy and S. Pu, Phys. Rev. C 92, 064902 (2015) doi:10.1103/PhysRevC.92.064902 [arXiv:1508.03761 [nucl-th]].

[4] D. E. Kharzeev, L. D. McLerran and H. J. Warringa, “The Effects of topical charge change in heavy ion collisions: ‘Event by event P and CP violation.” Nucl. Phys. A 803, 227 (2008).

[5] Rafeelski, Johann and Müller, Berndt. Magnetic Splitting of Quasimolecular Electronic States in Strong Fields, Phys. Rev. Lett., 1630.10.517-520 (1976)

[6] H. Li, X. I. Sheng and Q. Wang, “Electromagnetic fields with electric and chiral magnetic conductivities in heavy ion collisions,” Phys. Rev. C 94, no. 4, 044903 (2016).

[7] K. Tuchin, “Time and space dependence of the electromagnetic field in relativistic heavy-ion collisions,” Phys. Rev. C 88, no. 2, 024911 (2013) doi:10.1103/PhysRevC.88.024911 [arXiv:1305.5806 [hep-ph]].

[8] K. Tuchin, “Electromagnetic field and the chiral magnetic effect in the quark-gluon plasma,” Phys. Rev. C 91, no. 6, 064902 (2015) doi:10.1103/PhysRevC.91.064902 [arXiv:1411.1363 [hep-ph]].

[9] K. Tuchin, “Synchrotron radiation by fast fermions in heavy-ion collisions,” Phys. Rev. C 82, 034904 (2010) [Phys. Rev. C 83, 039903 (2011)].

[10] K. Tuchin, “Photon decay in strong magnetic field in heavy-ion collisions,” Phys. Rev. C 83, 017901 (2011) doi:10.1103/PhysRevC.83.017901 [arXiv:1008.1604 [nucl-th]].

[11] G. Basar, D. Kharzeev and V. Skokov, “Conformal anomaly as a source of soft photons in heavy ion collisions,” Phys. Rev. Lett. 109, 202303 (2012) doi:10.1103/PhysRevLett.109.202303 [arXiv:1206.1334 [hep-ph]].

[12] Y. Hirono, T. Hirano and D. E. Kharzeev, “The chiral magnetic effect in heavy-ion collisions from event-by-event anomalous hydrodynamics,” Phys. Rev. C 91, 054915 (2015).

[13] A. Dash, V. Roy and B. Mohanty, “Magnetovortical evolution of QGP in heavy ion collisions,” arXiv:1705.05657 [nucl-th].

[14] W. Florkowski, A. Kumar and R. Ryblewski, “Vortex-like solutions and internal structures of covariant ideal magnetohydrodynamics,” arXiv:1803.06695 [nucl-th].

[15] B. Mcmnnes, “A Bound on Vorticity,” arXiv:1803.02528 [hep-ph].

[16] R. Samanta, R. Sharma and S. P. Trivedi, “Shear viscosity in an anisotropic unitary Fermi gas,” Phys. Rev. A 96, no. 5, 053601 (2017) doi:10.1103/PhysRevA.96.053601 [arXiv:1607.04799 [cond-mat.stat-mech]].

[17] X. I. Sheng, D. H. Rischke, D. Vasak and Q. Wang, “Wigner functions for fermions in strong magnetic fields,” Eur. Phys. J. A 54, no. 2, 21 (2018) doi:10.1140/epja/i2018-12414-9 [arXiv:1707.01388 [hep-ph]].

[18] S. Li and H. U. Yee, “Shear Viscosity of Quark-Gluon Plasma in Weak Magnetic Field in Perturbative QCD: Leading Log,” Phys. Rev. D 97, 056024 (2018) doi:10.1103/PhysRevD.97.056024 [arXiv:1707.00795 [hep-ph]].

[19] K. Hatorti, X. G. Huang, D. H. Rischke and D. Satow, “Bulk Viscosity of Quark-Gluon Plasma in Strong Magnetic Fields,” Phys. Rev. D 96, no. 9, 094009 (2017) doi:10.1103/PhysRevD.96.094009 [arXiv:1708.00515 [hep-ph]].

[20] S. Ghosh, B. Chatterjee, P. Mohanty, A. Mukharjee and H. Mishra, “Perfect fluid nature in weakly-interacting magnetized quark matter,” arXiv:1804.00812 [hep-ph].

[21] C. Shen, S. A. Bass, T. Hirano, P. Huovinen, Z. Qiu, H. Song and U. Heinz, “The QGP shear viscosity: Elusive goal or just around the corner?,” J. Phys. G 38, 124045 (2011).

[22] P. Romatschke and U. Romatschke, Phys. Rev. Lett. 99, 172301 (2007); M. Luzum and P. Romatschke, Phys. Rev. C 78, 044915 (2008).

[23] H. Song and U. W. Heinz, Phys. Lett. B 658, 279 (2008).
