Sharp Bounds Between Two Rényi Entropies of Distinct Positive Orders

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Abstract

Many axiomatic definitions of entropy, such as the Rényi entropy, of a random variable are closely related to the $\ell_\alpha$-norm of its probability distribution. This study considers probability distributions on finite sets, and examines the sharp bounds of the $\ell_\beta$-norm with a fixed $\ell_\alpha$-norm, $\alpha \neq \beta$, for $n$-dimensional probability vectors with an integer $n \geq 2$. From the results, we derive the sharp bounds of the Rényi entropy of positive order $\beta$ with a fixed Rényi entropy of another positive order $\alpha$. As applications, we investigate sharp bounds of Ariomoto’s mutual information of order $\alpha$ and Gallager’s random coding exponents for uniformly focusing channels under the uniform input distribution.

I. INTRODUCTION

Information measures for random variables play important roles in many fields of engineering, such as coding theorems. Let the $n$-dimensional probability simplex be denoted by

$$\Delta_n := \left\{ (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n \mid p_i \geq 0 \text{ and } \sum_{i=1}^{n} p_i = 1 \right\}$$

for an integer $n \geq 2$. For a random variable $X \sim p \in \Delta_n$, the Shannon entropy [23] is defined by

$$H(X) = H(p) := -\sum_{i=1}^{n} p_i \ln p_i,$$

which is one of famous information measures of uncertainty of $X$. Previously, the Shannon entropy was axiomatically generalized to several forms [4], [5], [7], [15], [20], [26]. One of famous extensions of the Shannon entropy is the Rényi entropy [20] of order $\alpha$, which is defined by

$$H_\alpha(X) = H_\alpha(p) := \frac{\alpha}{1-\alpha} \ln \|p\|_\alpha$$

for $X \sim p \in \Delta_n$ and $\alpha \in (0, 1) \cup (1, \infty)$, where

$$\|p\|_\alpha := \left( \sum_{i=1}^{n} p_i^\alpha \right)^{1/\alpha}$$
denotes the $\ell_\alpha$-norm of $p$ for $\alpha \in (0, \infty)$. Thus, we see from (3) that the Rényi entropy is closely related to the
$\ell_\alpha$-norm. Moreover, for $\alpha \in \{1, \infty\}$, the Rényi entropy is also defined by
\[
H_1(p) := \lim_{\alpha \to 1} H_\alpha(p) = H(p),
\]
\[
H_\infty(p) := \lim_{\alpha \to \infty} H_\alpha(p) = -\ln \|p\|_\infty,
\]
where the last equality of (5) follows by L'Hôpital’s rule and the $\ell_\infty$-norm used in (6) is given by
\[
\|p\|_\infty := \lim_{\alpha \to \infty} \|p\|_\alpha = \max \{p_1, p_2, \ldots, p_n\}.
\]
Note that the minimum error probability $P_e(X)$ of guessing the value of $X \sim p$ is calculated by $P_e(X) = 1 - \|p\|_\infty$ (cf. [10, Eq. (2)]). The Rényi entropy of order 0 is called the Hartley entropy or the max-entropy; however, it is
omitted and we only consider the Rényi entropy of positive order in this study.

In the previous study, the sharp bounds of the Shannon entropy $H(X)$ with a fixed error probability $P_e(X)$, i.e.,
\[
\min_{p \in \Delta_n; P_e(X) = p_e} H(p) \leq H(X) \leq \max_{p \in \Delta_n; P_e(X) = p_e} H(p)
\]
for $p_e \in [0, (n-1)/n]$, were independently derived by Kovalevsky [17], Tebbe and Dwyer [25], and Feder and
Merhav [10]. Note that the upper bound of (8) is the unconditional version of Fano’s inequality [8] (cf. the remark
of [6, p. 40]). We now define the following two characteristic $n$-dimensional probability vectors: (i) the distribution $v_n(\cdot)$ is defined by
\[
v_n(p) := (v_1, v_2, \ldots, v_n) \in \Delta_n,
\]
\[
v_i = \begin{cases} 
1 - (n-1)p & \text{if } i = 1, \\
p & \text{otherwise}
\end{cases}
\]
for $p \in [0, 1/(n-1)]$, and (ii) the distribution $w_n(\cdot)$ is defined by
\[
w_n(p) := (w_1, w_2, \ldots, w_n) \in \Delta_n,
\]
\[
w_i = \begin{cases} 
1 - \lfloor 1/p \rfloor p & \text{if } i = \lfloor 1/p \rfloor + 1, \\
p & \text{if } i \leq \lfloor 1/p \rfloor, \\
0 & \text{otherwise}
\end{cases}
\]
for $p \in [1/n, 1]$, where $\lfloor x \rfloor := \max\{z \in \mathbb{Z} \mid z \leq x\}$ denotes the floor function of $x \in \mathbb{R}$. Then, the results of [10],
[17], [25] show that the upper and lower bounds of (8) are attained by the distributions $v_n(\cdot)$ and $w_n(\cdot)$, respectively.
In addition, by using topological method, Harremoës and Topsøe [14] derived that the upper and lower bounds of the Shannon entropy with a fixed index of coincidence $IC(p) := \|p\|_2^2$ are also attained by the distributions $v_n(\cdot)$ and $w_n(\cdot)$, respectively. In the above previous works, we note that the error probability $P_e(X)$ and the index of coincidence $IC(p)$ are closely related to the $\ell_\infty$-norm and the $\ell_2$-norm, respectively. Namely, these results are related to the min-entropy $H_\infty(X)$ and the collision entropy $H_2(X)$. As a generalization of the above results, we derived in [21] that the extremal Shannon entropies with a fixed $\ell_\alpha$-norm, $\alpha \in (0, 1) \cup (1, \infty)$, are attained by the
distributions \(v_n(\cdot)\) and \(w_n(\cdot)\). Therefore, the sharp bounds of the Rényi entropy of order \(\alpha \in (0, 1) \cup (1, \infty)\) with a fixed Shannon entropy, and vice versa, were derived in [21]. Furthermore, in [22], we extended the result of [21] to the relations between the conditional Shannon entropy and the expectation of \(\ell_{\alpha}\)-norm.

In this study, we investigate the sharp bounds of \(\ell_{\beta}\)-norm with a fixed \(\ell_{\alpha}\)-norm for \(n\)-dimensional probability vectors, as shown in Theorem 1 of Section II. Note that the case \(\alpha = \beta\) is omitted in the study since it is obvious. Since the Rényi entropy is closely related to the \(\ell_{\alpha}\)-norm, Theorem 1 implies the sharp bounds between two Rényi entropies of distinct orders, which is described in Theorem 2 of Section II. On the other hand, in Theorem 3 of Section III, we show the exact feasible regions between expectations of \(\ell_{\alpha}\)- and \(\ell_{\beta}\)-norms for probability distributions on a finite set. Since Arimoto’s conditional Rényi entropy [3] is closely related to the expectations of \(\ell_{\alpha}\)-norm, we can apply Theorem 3 to it as Fig. 3 of Section IV. As applications of the above results, Section IV investigates the sharp bounds on reliability functions, such as the mutual information of order \(\alpha\) [3] and the \(E_0\) function [12], for uniformly focusing channels of Definition 2 of Section IV under the uniform input distribution.

II. Extremality of \(\ell_{\alpha}\)-norm and Rényi entropy

In this section, the sharp bounds between the \(\ell_{\alpha}\)-norm and the \(\ell_{\beta}\)-norm, \(\alpha \neq \beta\), for \(n\)-dimensional probability vectors are examined. As the result, Theorem 1 shows that the distributions \(v_n(\cdot)\) and \(w_n(\cdot)\) take extremal values of norms for \(n\)-dimensional probability vectors. The following lemma shows the monotonicity of the \(\ell_{\alpha}\)-norms of \(v_n(p)\) and \(w_n(p)\) with respect to \(p\).

**Lemma 1.** For a fixed \(\alpha \in (0, 1)\), \(\|v_n(p_v)\|_{\alpha}\) (resp. \(\|w_n(p_w)\|_{\alpha}\)) is strictly increasing (resp. strictly decreasing) for \(p_v \in [0, 1/n]\) (resp. \(p_w \in [1/n, 1]\)). Conversely, for a fixed \(\alpha \in (1, \infty)\), \(\|v_n(p_v)\|_{\alpha}\) (resp. \(\|w_n(p_w)\|_{\alpha}\)) is strictly decreasing (resp. strictly increasing) for \(p_v \in [0, 1/n]\) (resp. \(p_w \in [1/n, 1]\)).

**Proof of Lemma 1:** We first prove Lemma 1 for \(\|v_n(p)\|_{\alpha}\). If \(\alpha = 1\), then Lemma 1 is reduced to [21, Lemma 1]. In this proof, we omit the case of \(\alpha = 1\) and we only consider \(\|v_n(p)\|_{\alpha}\) for \(\alpha \in (0, 1) \cup (1, \infty)\). A direct calculation shows

\[
\frac{\partial \|v_n(p)\|_{\alpha}}{\partial p} = \frac{\partial}{\partial p} \left( (1 - (n - 1)p)^{\alpha} + (n - 1)p^{\alpha} \right)^{1/\alpha}
\]

\[
= \frac{1}{\alpha} \left( (1 - (n - 1)p)^{\alpha} + (n - 1)p^{\alpha} \right)^{(1/\alpha) - 1} \left( \frac{\partial}{\partial p} \left( (1 - (n - 1)p)^{\alpha} + (n - 1)p^{\alpha} \right) \right)
\]

\[
= \frac{1}{\alpha} \left( (1 - (n - 1)p)^{\alpha} + (n - 1)p^{\alpha} \right)^{(1/\alpha) - 1} \left( -\alpha(n - 1)(1 - (n - 1)p)^{\alpha - 1} + \alpha(n - 1)p^{\alpha - 1} \right)
\]

\[
= (n - 1) \left( (1 - (n - 1)p)^{\alpha} + (n - 1)p^{\alpha} \right)^{(1/\alpha) - 1} \left( p^{\alpha - 1} - (1 - (n - 1)p)^{\alpha - 1} \right)
\]
for $p \in (0, 1/(n - 1))$. Let
\[
\text{sgn}(x) := \begin{cases} 
1 & \text{if } x > 0, \\
0 & \text{if } x = 0, \\
-1 & \text{if } x < 0
\end{cases}
\] (17)
denote the sign function of $x \in \mathbb{R}$. Since
\[
\text{sgn} \left( p^{\alpha - 1} - (1 - (n - 1)p)^{\alpha - 1} \right) = \begin{cases} 
1 & \text{if } \alpha < 1, \\
0 & \text{if } \alpha = 1, \\
-1 & \text{if } \alpha > 1
\end{cases}
\] (18)
for $p \in (0, 1/n)$, we obtain
\[
\text{sgn} \left( \frac{\partial \|v_n(p)\|_\alpha}{\partial p} \right) = \text{sgn} \left( (n - 1) \left( (1 - (n - 1)p) + (n - 1)p^{\alpha} \right)^{(1/\alpha) - 1} \left( p^{\alpha - 1} - (1 - (n - 1)p)^{\alpha - 1} \right) \right)
\] 
\[
= \text{sgn}(n - 1) \cdot \text{sgn} \left( (1 - (n - 1)p)^{\alpha} + (n - 1)p^{\alpha} \right)^{(1/\alpha) - 1} \cdot \text{sgn} \left( p^{\alpha - 1} - (1 - (n - 1)p)^{\alpha - 1} \right)
\] (19)
\[
= \begin{cases} 
1 & \text{if } \alpha < 1, \\
0 & \text{if } \alpha = 1, \\
-1 & \text{if } \alpha > 1
\end{cases}
\] (20)
for $p \in (0, 1/n)$, which implies that

- if $\alpha \in (0, 1)$, then $\|v_n(p)\|_\alpha$ is strictly increasing for $p \in [0, 1/n]$, and
- if $\alpha \in (1, \infty)$, then $\|v_n(p)\|_\alpha$ is strictly decreasing for $p \in [0, 1/n]$.

Moreover, since $\|v_n(p)\|_\infty = 1 - (n - 1)p$, the $\ell_\infty$-norm $\|v_n(p)\|_\infty$ is also strictly decreasing for $p \in [0, 1/n]$.

We next prove Lemma 1 for $\|w_n(p)\|_\alpha$. In this part, we also omit the case of $\alpha = 1$ and we only consider $\|w_n(p)\|_\alpha$ for $\alpha \in (0, 1) \cup (1, \infty)$. For an integer $m \in [2, n]$, we readily see that
\[
\|w_n(p)\|_\alpha = \|v_m(p)\|_\alpha
\] (22)
for $p \in [1/m, 1/(m - 1)]$; hence, we get
\[
\frac{\partial \|w_n(p)\|_\alpha}{\partial p} \equiv (m - 1) \left( (1 - (m - 1)p)^{\alpha} + (m - 1)p^{\alpha} \right)^{(1/\alpha) - 1} \left( p^{\alpha - 1} - (1 - (m - 1)p)^{\alpha - 1} \right)
\] (23)
for $p \in (1/m, 1/(m - 1))$. Since
\[
\text{sgn} \left( p^{\alpha - 1} - (1 - (m - 1)p)^{\alpha - 1} \right) = \begin{cases} 
1 & \text{if } \alpha > 1, \\
0 & \text{if } \alpha = 1, \\
-1 & \text{if } \alpha < 1
\end{cases}
\] (24)
for $p \in (1/m, 1/(m - 1))$, we obtain
\[
\text{sgn} \left( \frac{\partial \|w_n(p)\|_\alpha}{\partial p} \right) = \text{sgn} \left( (m - 1) \left( (1 - (m - 1)p)^\alpha + (m - 1)p^\alpha \right)^{(1/\alpha) - 1} \right)
\] (25)
\[
= \text{sgn}(m - 1) \cdot \text{sgn} \left( (1 - (m - 1)p)^\alpha + (m - 1)p^\alpha \right)^{(1/\alpha) - 1} \cdot \text{sgn} \left( p^{\alpha - 1} - (1 - (m - 1)p)^{\alpha - 1} \right)
\] (26)
\[
= \begin{cases} 
1 & \text{if } \alpha > 1, \\
0 & \text{if } \alpha = 1, \\
-1 & \text{if } \alpha < 1
\end{cases}
\] (27)
for $p \in (1/m, 1/(m - 1))$. Since
\[
\|w_n(1/n)\|_\alpha = \left( \left( \frac{1}{p} \right) p^\alpha + \left( 1 - \frac{1}{p} \right) \right)^{1/\alpha} \bigg|_{p=1/n}
\] (28)
\[
= (n^{1-\alpha} + 0^\alpha)^{1/\alpha}
\] (29)
\[
= n^{(1/\alpha) - 1},
\] (30)
\[
\lim_{p \to (1/m)^-} \|w_n(p)\|_\alpha = \lim_{p \to (1/m)^-} \left( \left( \frac{1}{p} \right) p^\alpha + \left( 1 - \frac{1}{p} \right) \right)^{1/\alpha}
\] (31)
\[
= \lim_{p \to (1/m)^-} \left( m p^\alpha + (1 - m) p^\alpha \right)^{1/\alpha}
\] (32)
\[
= (m^{1-\alpha} + 0^\alpha)^{1/\alpha}
\] (33)
\[
= m^{(1/\alpha) - 1},
\] (34)
for $\alpha \in (0, \infty)$ and an integer $m \in [2, n - 1]$, the $\ell_\alpha$-norm $\|w_n(p)\|_\alpha$ is continuous for $p \in [1/n, 1]$. Thus, it follows from (27) that

- if $\alpha \in (0, 1)$, then $\|w_n(p)\|_\alpha$ is strictly decreasing for $p \in [1/n, 1]$, and
- if $\alpha \in (1, \infty)$, then $\|w_n(p)\|_\alpha$ is strictly increasing for $p \in [1/n, 1]$.

Moreover, since $\|w_n(p)\|_\infty = p$, the $\ell_\infty$-norm $\|w_n(p)\|_\infty$ is also strictly increasing for $p \in [1/n, 1]$.

Since the strictly monotonic function has an inverse function, Lemma 1 implies that the distributions $v_n(\cdot)$ and $w_n(\cdot)$ are uniquely determined by a given $\ell_\alpha$-norm. By using distributions $v_n(\cdot)$ and $w_n(\cdot)$, we show extremal properties of $\ell_\alpha$-norm of $n$-dimensional probability vectors, as shown in Lemmas 2 and 3.

**Lemma 2.** For any $n \geq 2$, $p \in \Delta_n$, and $\alpha \in (0, 1) \cup (1, \infty)$, there exists $p \in [0, 1/n]$ such that

\[
\|v_n(p)\|_\alpha = \|p\|_\alpha, \tag{43}
\]

\[
\|v_n(p)\|_\beta \leq \|p\|_\beta \quad \text{for all } \beta \in (\min\{1, \alpha\}, \max\{1, \alpha\}), \tag{44}
\]

\[
\|v_n(p)\|_\beta \geq \|p\|_\beta \quad \text{for all } \beta \in (0, \min\{1, \alpha\}) \cup (\max\{1, \alpha\}, \infty). \tag{45}
\]

**Proof of Lemma 2:** Following the notation used in [18], for a probability vector $p = (p_1, p_2, \ldots, p_n) \in \Delta_n$, let

\[
p_1 \geq p_2 \geq \cdots \geq p_n\tag{46}
\]
denote the components of $p$ in decreasing order; and let

\[
p_{\downarrow} \coloneqq (p_1, p_2, \ldots, p_n)\tag{47}
\]
denote the decreasing rearrangement of $p$. Note that

\[
1 \leq \|p\|_\alpha \leq n^{(1/\alpha) - 1} \quad \text{for } 0 < \alpha \leq 1,
\]

\[
n^{(1/\alpha) - 1} \leq \|p\|_\alpha \leq 1 \quad \text{for } 1 \leq \alpha \leq \infty \tag{49}
\]

for $p \in \Delta_n$, where it is assumed in (49) that $n^{(1/\alpha) - 1} = 1/n$ when $\alpha = \infty$.

We first consider the obvious case of Lemma 2 such that $n = 2$, $\|p\|_\alpha = 1$, or $\|p\|_\alpha = n^{(1/\alpha) - 1}$. It can be easily seen that, for any $p = (p_1, p_2) \in \Delta_2$, there exists $p \in [0, 1/2]$ such that $p_{\downarrow} = v_2(p)$; therefore, the lemma obviously holds when $n = 2$.

It follows from Minkowski’s inequality (see, e.g., the proof of [5, Theorem 3]) that

- $\|p\|_\alpha$ is strictly concave in $p \in \Delta_n$ for a fixed $\alpha \in (0, 1)$ and
- $\|p\|_\alpha$ is strictly convex in $p \in \Delta_n$ for a fixed $\alpha \in (1, \infty)$.
In addition, since
\[
\|\lambda p + (1 - \lambda)q\|_{\infty} = \max_{1 \leq i \leq n} \{\lambda p_i + (1 - \lambda)q_i\} \tag{50}
\]
\[
\leq \lambda \left( \max_{1 \leq i \leq n} p_i \right) + (1 - \lambda) \left( \max_{1 \leq i \leq n} q_i \right) \tag{51}
\]
\[
= \lambda \|p\|_{\infty} + (1 - \lambda)\|q\|_{\infty} \tag{52}
\]
for \( p, q \in \Delta_n \) and \( \lambda \in [0, 1] \), we also get that \( \|p\|_{\infty} \) is convex in \( p \in \Delta_n \). Note that the convexity of \( \|p\|_{\infty} \) is not strict since the inequality of (51) holds with equality if
\[
\arg\max_{1 \leq i \leq n} p_i = \arg\max_{1 \leq i \leq n} q_i. \tag{53}
\]

The above convexity and concavity imply that
\[
\forall \alpha \in (0, \infty], \quad \|p\|_{\alpha} = n^{(1/\alpha) - 1} \quad \iff \quad p = \left( \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n} \right) = v_n \left( \frac{1}{n} \right), \tag{54}
\]
\[
\forall \alpha \in (0, \infty], \quad \|p\|_{\alpha} = 1 \quad \iff \quad p = (1, 0, \ldots, 0) = v_n(0). \tag{55}
\]

Therefore, the lemma obviously holds when either \( \|p\|_{\alpha} = 1 \) or \( \|p\|_{\alpha} = n^{(1/\alpha) - 1} \).

Thus, Lemma 2 is proved for the cases \( n = 2 \) and \( \|p\|_{\alpha} \in \{1, n^{(1/\alpha) - 1}\} \) for \( \alpha \in (0, 1) \cup (1, \infty] \). Hence, it is enough to prove the lemma for \( p \in \Delta_n \) such that \( n \geq 3 \) and \( \|p\|_{\alpha} \in (\min\{1, n^{(1/\alpha) - 1}\}, \max\{1, n^{(1/\alpha) - 1}\}) \) for \( \alpha \in (0, 1) \cup (1, \infty] \) in the later analyses.

We now begin to prove Lemma 2 for \( \alpha \neq \infty \) and \( \beta \neq \infty \). For fixed \( n \geq 3 \), \( \alpha \in (0, 1) \cup (1, \infty) \), and \( A \in (\min\{1, n^{(1/\alpha) - 1}\}, \max\{1, n^{(1/\alpha) - 1}\}) \), we assume for \( p \in \Delta_n \) that
\[
\|p\|_{\alpha} = A. \tag{56}
\]

For that \( p \) satisfying (56), let \( k \in \{2, 3, \ldots, n - 1\} \) be an index such that \( p_{[k-1]} > p_{[k+1]} = p_{[n]} \); namely, the index \( k \) is chosen to satisfy the following inequalities:
\[
p_{[1]} \geq p_{[2]} \geq \cdots \geq p_{[k-1]} \geq p_{[k]} \geq p_{[k+1]} = p_{[n]} = \cdots = p_{[n]} \quad (p_{[k-1]} > p_{[k+1]}). \tag{57}
\]

Since \( p_1 + p_2 + \cdots + p_n = 1 \), we observe that
\[
\sum_{i=1}^{n} p_i = 1 \tag{58}
\]
\[
\Rightarrow \quad \frac{d}{dp_{[k]}} \left( \sum_{i=1}^{n} p_i \right) = \frac{d}{dp_{[k]}} (1) \tag{59}
\]
\[
\iff \quad \frac{d}{dp_{[k]}} \left( \sum_{i=1}^{n} p_i \right) = 0 \tag{60}
\]
\[
\iff \quad \frac{d}{dp_{[k]}} p_{[k]} + \sum_{i=1; i \neq k}^{n} \frac{dp_{[i]}}{dp_{[k]}} = 0 \tag{61}
\]
\[
\iff \quad \sum_{i=1; i \neq k}^{n} \frac{dp_{[i]}}{dp_{[k]}} = -1. \tag{62}
\]
In this proof, we assume that
\[
\frac{dp_{[i]}}{dp_{[k]}} = 0
\]  
(63)
for \(i \in \{2, 3, \ldots, k - 1\}\) and
\[
\frac{dp_{[j]}}{dp_{[k]}} = \frac{dp_{[n]}}{dp_{[k]}}
\]  
(64)
for \(j \in \{k + 1, k + 2, \ldots, n\}\). By constraints (63) and (64), we get
\[
\sum_{i=1}^{n} p_i = 1
\]  
(65)
(62)
\[
\sum_{i=1; i \neq k}^{n} \frac{dp_{[i]}}{dp_{[k]}} = -1
\]  
(66)
\[
\sum_{i=1}^{k-1} \frac{dp_{[i]}}{dp_{[k]}} + \sum_{j=k+1}^{n} \frac{dp_{[j]}}{dp_{[k]}} = -1
\]  
(67)
(63)
\[
\frac{dp_{[1]}}{dp_{[k]}} + \sum_{j=k+1}^{n} \frac{dp_{[j]}}{dp_{[k]}} = -1
\]  
(68)
(64)
\[
\frac{dp_{[1]}}{dp_{[k]}} + (n - k) \frac{dp_{[n]}}{dp_{[k]}} = -1
\]  
(69)
\[
\frac{dp_{[1]}}{dp_{[k]}} = -1 - (n - k) \frac{dp_{[n]}}{dp_{[k]}}
\]  
(70)
Defining the \(\alpha\)-logarithm function \([27]\) as
\[
\ln_{\alpha} x := \begin{cases} 
\ln x & \text{if } \alpha = 1, \\
\frac{x^{1-\alpha} - 1}{1-\alpha} & \text{if } \alpha \neq 1
\end{cases}
\]  
(71)
for \(x > 0\), where note that L’Hôpital’s rule shows
\[
\lim_{\alpha \to 1} \frac{x^{1-\alpha} - 1}{1-\alpha} = \ln x,
\]  
(72)
we observe from the constraint (56) that
\[
\|p\|_{\alpha} = A
\]  
(73)
\[
\sum_{i=1}^{n} p_i^\alpha = A^\alpha
\]  
(74)
\[
\frac{d}{dp_{[k]}} \left( \sum_{i=1}^{n} p_i^\alpha \right) = \frac{d}{dp_{[k]}} (A^\alpha)
\]  
(75)
\[
\frac{d}{dp_{[k]}} \left( \sum_{i=1}^{n} p_i^\alpha \right) = 0
\]  
(76)
\[
\sum_{i=1}^{n} \frac{d}{dp_{[k]}} (p_i^\alpha) = 0
\]  
(77)
\[
\frac{d}{dp_{[k]}} (p_{[k]}^\alpha) + \sum_{i=1; i \neq k}^{n} \frac{d}{dp_{[k]}} (p_i^\alpha) = 0
\]  
(78)
\[ \alpha p_{[k]}^{\alpha - 1} + \sum_{i=1,i \neq k}^{n} \frac{d}{dp_{[i]}}(p_{[i]}^\alpha) = 0 \] (79)

\[ \sum_{i=1,i \neq k}^{n} \left( \frac{dp_{[i]}}{dp_{[k]}} \left( \frac{d}{dp_{[i]}}(p_{[i]}^\alpha) \right) \right) = -\alpha p_{[k]}^{\alpha - 1} \] (80)

\[ \sum_{i=1,i \neq k}^{n} \left( \frac{dp_{[i]}}{dp_{[k]}} \right) (\alpha p_{[i]}^{\alpha - 1}) = -\alpha p_{[k]}^{\alpha - 1} \] (81)

\[ \sum_{i=1,i \neq k}^{n} \left( \frac{dp_{[i]}}{dp_{[k]}} \right) p_{[i]}^{\alpha - 1} = -p_{[k]}^{\alpha - 1} \] (82)

\[ \sum_{i=1}^{k-1} \left( \frac{dp_{[i]}}{dp_{[k]}} \right) p_{[i]}^{\alpha - 1} + \sum_{j=k+1}^{n} \left( \frac{dp_{[j]}}{dp_{[k]}} \right) p_{[j]}^{\alpha - 1} = -p_{[k]}^{\alpha - 1} \] (83)

\[ \sum_{i=1}^{k-1} \left( \frac{dp_{[i]}}{dp_{[k]}} \right) p_{[i]}^{\alpha - 1} + \sum_{j=k+1}^{n} \frac{dp_{[j]}}{dp_{[k]}} p_{[j]}^{\alpha - 1} = -p_{[k]}^{\alpha - 1} \] (84)

\[ \sum_{i=1}^{k-1} \left( \frac{dp_{[i]}}{dp_{[k]}} \right) p_{[i]}^{\alpha - 1} + \sum_{j=k+1}^{n} \frac{dp_{[j]}}{dp_{[k]}} p_{[j]}^{\alpha - 1} = -p_{[k]}^{\alpha - 1} \] (85)

\[ \left( \frac{dp_{[1]}}{dp_{[k]}} \right) p_{[1]}^{\alpha - 1} + \frac{dp_{[n]}}{dp_{[k]}} p_{[n]}^{\alpha - 1} = -p_{[k]}^{\alpha - 1} \] (86)

\[ \left( \frac{dp_{[1]}}{dp_{[k]}} \right) p_{[1]}^{\alpha - 1} + (n-k) \left( \frac{dp_{[n]}}{dp_{[k]}} \right) p_{[n]}^{\alpha - 1} = -p_{[k]}^{\alpha - 1} \] (87)

\[ (n-k) \left( \frac{dp_{[n]}}{dp_{[k]}} \right) (p_{[n]}^{\alpha - 1} - p_{[1]}^{\alpha - 1}) = - \left( p_{[n]}^{\alpha - 1} - p_{[1]}^{\alpha - 1} \right) \] (88)

\[ \left( \frac{p_{[k]}}{p_{[1]}} \right) \alpha^{-1} - 1 = \left( \frac{p_{[1]}}{p_{[k]}} \right) \alpha^{-1} - 1 \] (89)

\[ \left( \frac{p_{[k]}}{p_{[1]}} \right) 1^{-\alpha} - 1 = \left( \frac{p_{[1]}}{p_{[k]}} \right) 1^{-\alpha} - 1 \] (90)

\[ \left( \frac{p_{[k]}}{p_{[1]}} \right) \alpha^{-1} - 1 = \left( \frac{p_{[1]}}{p_{[k]}} \right) \alpha^{-1} - 1 \] (91)

\[ \left( \frac{p_{[k]}}{p_{[1]}} \right) 1^{-\alpha} - 1 = \left( \frac{p_{[1]}}{p_{[k]}} \right) 1^{-\alpha} - 1 \] (92)

\[ \left( \frac{p_{[k]}}{p_{[1]}} \right) \alpha^{-1} - 1 = \left( \frac{p_{[1]}}{p_{[k]}} \right) \alpha^{-1} - 1 \] (93)

\[ \left( \frac{p_{[k]}}{p_{[1]}} \right) 1^{-\alpha} - 1 = \left( \frac{p_{[1]}}{p_{[k]}} \right) 1^{-\alpha} - 1 \] (94)

\[ \left( \frac{p_{[k]}}{p_{[1]}} \right) \alpha^{-1} - 1 = \left( \frac{p_{[1]}}{p_{[k]}} \right) \alpha^{-1} - 1 \] (95)

where (a) follows by the chain rule of the derivative.
We now check the sign of the right-hand side of (95). Since $\ln_\alpha x$ is a strictly increasing function of $x > 0$ for every $\alpha \in \mathbb{R}$, we get

$$0 \leq \ln_\alpha \left( \frac{p[1]}{p[k]} \right) < \ln_\alpha \left( \frac{p[1]}{p[n]} \right)$$

for $1 > p[1] \geq p[k] \geq p[n] > 0$ ($p[1] > p[n]$), where the left-hand inequality holds with equality if and only if $p[1] = p[k]$. Note that the monotonicity of $\alpha$-logarithm function can be verified as follows: For $\alpha \in \mathbb{R} \setminus \{1\}$, a simple calculation yields

$$\frac{\partial}{\partial x} \ln_\alpha x = \frac{\partial}{\partial x} \left( \frac{x^{1-\alpha} - 1}{1-\alpha} \right)$$

$$= \frac{1}{1-\alpha} \left( \frac{\partial}{\partial x} (x^{1-\alpha}) \right)$$

$$= \frac{1}{1-\alpha} \left( (1-\alpha)x^{-\alpha} \right)$$

$$= x^{-\alpha}$$

$$> 0$$

for $x > 0$, which implies that $\ln_\alpha x$ is a strictly increasing function of $x > 0$ for every $\alpha \in \mathbb{R} \setminus \{1\}$; on the other hand, if $\alpha = 1$, then $\ln_1 x = \ln x$ is clearly a strictly increasing function of $x > 0$. It follows from (96) that

$$0 \leq \frac{\ln_\alpha \left( \frac{p[1]}{p[k]} \right)}{\ln_\alpha \left( \frac{p[1]}{p[n]} \right)} < 1;$$

and therefore, we obtain

$$\text{sgn} \left( \frac{dp[n]}{dp[k]} \right) = \begin{cases} 0 & \text{if } p[1] = p[k], \\ -1 & \text{otherwise}, \end{cases}$$

which implies that $p[n]$ is strictly decreasing for $p[k]$ under the constraints (56), (57), (63), and (64).

Similarly, we check the sign of the right-hand side of (70):

$$\frac{dp[1]}{dp[k]} = -1 - (n-k) \frac{dp[n]}{dp[k]}$$

$$= -1 + \ln_\alpha \left( \frac{p[1]}{p[n]} \right),$$

It follows from (102) that

$$-1 \leq \frac{dp[1]}{dp[k]} < 0,$$

which implies that $p[1]$ is also strictly decreasing for $p[k]$ under the constraints (56), (57), (63), and (64).
We next consider \( \|p\|_\beta \) for a fixed \( \beta \in (0, 1) \cup (1, \infty) \). A direct calculation yields

\[
\frac{d\|p\|_\beta}{dp_{[k]}} = \frac{d}{dp_{[k]}} \left( \sum_{i=1}^{n} p_i^\beta \right)
\]

(107)

\[
= \beta \left( p_{[k]}^{\beta-1} + \sum_{i=1, i \neq k}^{n} \left( \frac{dp_{[i]}}{dp_{[k]}} \right) p_{[i]}^{\beta-1} \right)
\]

(108)

\[
= \beta \left( p_{[k]}^{\beta-1} + \sum_{i=1}^{k-1} \left( \frac{dp_{[i]}}{dp_{[k]}} \right) p_{[i]}^{\beta-1} + \sum_{j=k+1}^{n} \left( \frac{dp_{[j]}}{dp_{[k]}} \right) p_{[j]}^{\beta-1} \right)
\]

(109)

\[
= \beta \left( p_{[k]}^{\beta-1} + \sum_{i=1}^{k-1} \left( \frac{dp_{[i]}}{dp_{[k]}} \right) p_{[i]}^{\beta-1} + \sum_{j=k+1}^{n} \left( \frac{dp_{[j]}}{dp_{[k]}} \right) p_{[j]}^{\beta-1} \right)
\]

(110)

\[
= \beta \left( p_{[k]}^{\beta-1} + \sum_{i=1}^{k-1} \left( \frac{dp_{[i]}}{dp_{[k]}} \right) p_{[i]}^{\beta-1} + \sum_{j=k+1}^{n} \left( \frac{dp_{[j]}}{dp_{[k]}} \right) p_{[j]}^{\beta-1} \right)
\]

(111)

\[
= \beta \left( p_{[k]}^{\beta-1} + \sum_{i=1}^{k-1} \left( \frac{dp_{[i]}}{dp_{[k]}} \right) p_{[i]}^{\beta-1} + \sum_{j=k+1}^{n} \left( \frac{dp_{[j]}}{dp_{[k]}} \right) p_{[j]}^{\beta-1} \right)
\]

(112)

\[
= \beta \left( p_{[k]}^{\beta-1} + \left( \frac{dp_{[1]}}{dp_{[k]}} \right) p_{[1]}^{\beta-1} + \left( \frac{dp_{[n]}}{dp_{[k]}} \right) p_{[n]}^{\beta-1} \right)
\]

(113)

\[
= \beta \left( p_{[k]}^{\beta-1} - p_{[1]}^{\beta-1} + (n-k) \left( \frac{dp_{[1]}}{dp_{[k]}} \right) p_{[1]}^{\beta-1} + (n-k) \left( \frac{dp_{[n]}}{dp_{[k]}} \right) p_{[n]}^{\beta-1} \right)
\]

(114)

\[
= \beta \left( p_{[k]}^{\beta-1} - p_{[1]}^{\beta-1} + (n-k) \left( \frac{dp_{[1]}}{dp_{[k]}} \right) p_{[1]}^{\beta-1} \right)
\]

(115)

\[
= \beta \left( p_{[k]}^{\beta-1} - p_{[1]}^{\beta-1} + (n-k) \left( \frac{\ln \alpha \left( \frac{p_{[1]}}{p_{[k]}} \right)}{\ln \alpha \left( \frac{p_{[1]}}{p_{[n]}} \right)} \right) \right)
\]

(116)

\[
= \beta \left( p_{[k]}^{\beta-1} - p_{[1]}^{\beta-1} - \frac{\ln \alpha \left( \frac{p_{[1]}}{p_{[k]}} \right)}{\ln \alpha \left( \frac{p_{[1]}}{p_{[n]}} \right)} \right)
\]

(117)

\[
= \beta \left( p_{[n]}^{\beta-1} - p_{[1]}^{\beta-1} \right) \left( p_{[1]}^{\beta-1} - p_{[n]}^{\beta-1} \right)
\]

(118)

\[
= \beta \left( \frac{p_{[n]}^{\beta-1} - p_{[1]}^{\beta-1}}{p_{[n]}^{\beta-1} - p_{[1]}^{\beta-1}} \right) \left( \frac{p_{[1]}^{\beta-1} - p_{[n]}^{\beta-1}}{p_{[1]}^{\beta-1} - p_{[n]}^{\beta-1}} \right)
\]

(119)

Since \( \|p\|_\beta = (\|p\|_\beta)^{1/\beta} \), it follows by the chain rule that

\[
\frac{d\|p\|_\beta}{dp_{[k]}} = \left( \frac{1}{\beta} \right) \frac{d\|p\|_\beta^{1/\beta}}{dp_{[k]}} \cdot \left( \frac{d\|p\|_\beta^{1/\beta}}{dp_{[k]}} \right)
\]

(120)

\[
= \left( \frac{n}{1/\beta - 1} \right) \left( p_{[n]}^{\beta-1} - p_{[1]}^{\beta-1} \right) \left( \frac{\ln \alpha \left( \frac{p_{[1]}}{p_{[k]}} \right)}{\ln \alpha \left( \frac{p_{[1]}}{p_{[n]}} \right)} \right) \right)
\]

(121)
Thus, we get
\[
\text{sgn} \left( \frac{d\left\| p \right\|_\beta}{dp_{[k]}} \right) = \text{sgn} \left( \sum_{i=1}^{n} p_i^{\beta-1} \left( p_{[n]}^{\beta-1} - p_{[1]}^{\beta-1} \right) \frac{\ln \beta \left( \frac{p_{[1]}}{p_{[k]}} \right) - \ln \alpha \left( \frac{p_{[1]}}{p_{[n]}} \right)}{\ln \beta \left( \frac{p_{[1]}}{p_{[k]}} \right) - \ln \alpha \left( \frac{p_{[1]}}{p_{[n]}} \right)} \right) \text{sgn} \left( p_{[n]}^{\beta-1} - p_{[1]}^{\beta-1} \right) \frac{\ln \beta \left( \frac{p_{[1]}}{p_{[k]}} \right) - \ln \alpha \left( \frac{p_{[1]}}{p_{[n]}} \right)}{\ln \beta \left( \frac{p_{[1]}}{p_{[k]}} \right) - \ln \alpha \left( \frac{p_{[1]}}{p_{[n]}} \right)} \right) \right) \tag{122}
\]

\[
= \text{sgn} \left( \sum_{i=1}^{n} p_i^{\beta-1} \right) \cdot \text{sgn} \left( p_{[n]}^{\beta-1} - p_{[1]}^{\beta-1} \right) \cdot \text{sgn} \left( \ln \beta \left( \frac{p_{[1]}}{p_{[k]}} \right) - \ln \alpha \left( \frac{p_{[1]}}{p_{[n]}} \right) \right) \tag{123}
\]

\[
= \text{sgn} \left( p_{[n]}^{\beta-1} - p_{[1]}^{\beta-1} \right) \cdot \text{sgn} \left( \ln \beta \left( \frac{p_{[1]}}{p_{[k]}} \right) - \ln \alpha \left( \frac{p_{[1]}}{p_{[n]}} \right) \right) \tag{124}
\]

For \( p_{[1]} > p_{[n]} > 0 \), we readily see that
\[
\text{sgn} \left( p_{[n]}^{\beta-1} - p_{[1]}^{\beta-1} \right) = \begin{cases} 
1 & \text{if } \beta < 1, \\
0 & \text{if } \beta = 1, \\
-1 & \text{if } \beta > 1.
\end{cases} \tag{125}
\]

It was derived in [21, Lemma 5] that, for \( \alpha < \beta \) and \( 1 \leq x \leq y \) \((y \neq 1)\),
\[
\frac{\ln \alpha x}{\ln \alpha y} \leq \frac{\ln \beta x}{\ln \beta y}
\]

with equality if and only if \( x \in \{1, y\} \). Therefore, since
\[
1 \leq \frac{P_{[1]}}{P_{[k]}} \leq \frac{P_{[1]}}{P_{[n]}} \quad \left( 1 \neq \frac{P_{[1]}}{P_{[n]}} \right)
\]

for \( 1 > P_{[1]} \geq P_{[k]} \geq P_{[n]} > 0 \) \((P_{[1]} > P_{[n]})\), it follows from (126) that
\[
\text{sgn} \left( \frac{\ln \beta \left( \frac{P_{[1]}}{P_{[k]}} \right)}{\ln \alpha \left( \frac{P_{[1]}}{P_{[n]}} \right)} \right) = \begin{cases} 
1 & \text{if } \beta > \alpha \text{ and } P_{[1]} > P_{[k]} > P_{[n]}, \\
0 & \text{if } \beta = \alpha \text{ or } P_{[1]} = P_{[k]} \text{ or } P_{[k]} = P_{[n]}, \\
-1 & \text{if } \beta < \alpha \text{ and } P_{[1]} > P_{[k]} > P_{[n]}.
\end{cases} \tag{128}
\]

Combining (125) and (128), if \( P_{[1]} > P_{[k]} > P_{[n]} \), we obtain
\[
\text{sgn} \left( \frac{d\left\| p \right\|_\beta}{dp_{[k]}} \right) = \text{sgn} \left( \ln \beta \left( \frac{P_{[1]}}{P_{[k]}} \right) - \ln \alpha \left( \frac{P_{[1]}}{P_{[n]}} \right) \right) \cdot \text{sgn} \left( p_{[n]}^{\beta-1} - p_{[1]}^{\beta-1} \right) \cdot \text{sgn} \left( \ln \beta \left( \frac{P_{[1]}}{P_{[k]}} \right) - \ln \alpha \left( \frac{P_{[1]}}{P_{[n]}} \right) \right)
\]

\[
= \begin{cases} 
1 & \text{if } \beta \in \{\min\{1, \alpha\}, \max\{1, \alpha\}\}, \\
0 & \text{if } \beta \in \{1, \alpha\}, \\
-1 & \text{if } \beta \in (0, \min\{1, \alpha\}) \cup (\max\{1, \alpha\}, \infty),
\end{cases} \tag{129}
\]

which implies that
- if \( \alpha \in (0, 1) \), then
  - \( \left\| p \right\|_\beta \) with a fixed \( \beta \in (\alpha, 1) \) is strictly increasing for \( p_{[k]} \), and
  - \( \left\| p \right\|_\beta \) with a fixed \( \beta \in (0, \alpha) \cup (1, \infty) \) is strictly decreasing for \( p_{[k]} \),
• if $\alpha \in (1, \infty)$, then
  
  - $\|p\|_\beta$ with a fixed $\beta \in (1, \alpha)$ is strictly increasing for $p[k]$, and
  
  - $\|p\|_\beta$ with a fixed $\beta \in (0, 1) \cup (\alpha, \infty)$ is strictly decreasing for $p[k]$. 

Note that the above monotonicity hold under the constraints (56), (57), (63), and (64).

To accomplish the proof of Lemma 2 for $\alpha \neq \infty$ and $\beta \neq \infty$ by using the above relations, we repeat the following operation until the vector $p = (p_1, p_2, \ldots, p_n)$ satisfies $p[2] = p[n]$, i.e.,

$$p[1] > p[2] = p[3] = \cdots = p[n] > 0,$$

which is equivalent to the vector $v_n(\cdot)$. If the index $k \in \{3, 4, \ldots, n-1\}$ of constraint (57) satisfies $p[k] > p[k+1]$, then we reset the index $k$ to $k-1$; namely, we choose the index $k \in \{2, 3, \ldots, n-1\}$ to satisfy the following inequalities:

$$p[1] \geq p[2] \geq \cdots \geq p[k-1] \geq p[k] > p[k+1] = p[k+2] = \cdots = p[n] \geq 0. \tag{132}$$

For that index $k$, we consider to decrease the value $p[k]$ under the constraints (56), (57), (63), and (64). It follows from (103) that the value $p[1]$ is strictly increased by according to decreasing $p[k]$. Hence, if $p[k]$ is decreased, then the strict inequality $p[1] > p[2]$ must be held. Similarly, it follows from (64) and (106) that, for all indices $j \in \{k+1, k+2, \ldots, n\}$, the value $p[j]$ is also strictly increased by according to decreasing $p[k]$. Hence, if $p[k]$ is decreased, then $p[k+1] = p[k+2] = \cdots = p[n] > 0$ also must be held. Let $q = (q_1, q_2, \ldots, q_n)$ denote the probability vector such that made from $p$ by decreasing $p[k]$ until the equality $p[k] = p[k+1]$ holds under the conditions of (56), (57), (63), (64), and (132). Namely, the vector $q$ satisfies the following inequalities:

$$q[1] > q[2] \geq q[3] \geq \cdots \geq q[k-1] > q[k] = q[k+1] = \cdots = q[n] > 0. \tag{133}$$

Since $q$ is made from $p$ under the constraint (56), note that

$$\|q\|_\alpha = \|p\|_\alpha \tag{134}$$

for a fixed $\alpha \in (0, 1) \cup (1, \infty)$. Moreover, it follows from (130) that

$$\|q\|_\beta \leq \|p\|_\beta \quad \text{for } \beta \in (\min\{1, \alpha\}, \max\{1, \alpha\}), \tag{135}$$

$$\|q\|_\beta \geq \|p\|_\beta \quad \text{for } \beta \in (0, \min\{1, \alpha\}) \cup (\max\{1, \alpha\}, \infty). \tag{136}$$

Repeating these operation until (131) holds, we have

$$\|v_n(p)\|_\alpha = \|p\|_\alpha, \tag{137}$$

$$\|v_n(p)\|_\beta \leq \|p\|_\beta \quad \text{for } \beta \in (\min\{1, \alpha\}, \max\{1, \alpha\}), \tag{138}$$

$$\|v_n(p)\|_\beta \geq \|p\|_\beta \quad \text{for } \beta \in (0, \min\{1, \alpha\}) \cup (\max\{1, \alpha\}, \infty) \tag{139}$$

for any $p \in \Delta_n$, any fixed $\alpha \in (0, 1) \cup (1, \infty)$, and some $p \in [0, 1/n]$.

Finally, we consider the $\ell_\beta$-norm with a fixed $\ell_\infty$-norm for $\beta \in (0, 1) \cup (1, \infty)$, i.e., $\alpha = \infty$. In a similar way to [16, p. 1214], we now prove the Schur-convexity of the $\ell_\beta$-norm for $\beta \in (0, 1) \cup (1, \infty)$ by introducing
the method of majorization [18]. A probability vector \( p = (p_1, p_2, \ldots, p_n) \in \Delta_n \) is said to be majorized by \( q = (q_1, q_2, \ldots, q_n) \in \Delta_n \) if
\[
\forall k = 1, 2, \ldots, n - 1, \quad \sum_{i=1}^{k} p[i] \leq \sum_{i=1}^{k} q[i],
\]
and we write it as \( p \prec q \). Then, a function \( \phi : \Delta_n \to \mathbb{R} \) is said to be Schur-convex if
\[
p \prec q \implies \phi(p) \leq \phi(q)
\]
for \( p, q \in \Delta_n \), where \( \phi(\cdot) \) is also said to be strictly Schur-convex if the inequality of (141) is strict. Similarly, a function \( \phi : \Delta_n \to \mathbb{R} \) is said to be Schur-concave if
\[
p \prec q \implies \phi(p) \geq \phi(q)
\]
for \( p, q \in \Delta_n \), where \( \phi(\cdot) \) is also said to be strictly Schur-concave if the inequality of (142) is strict. It is known that if a function \( g : [0, 1] \to \mathbb{R} \) is concave, then the function
\[
\phi(p) = \sum_{i=1}^{n} g(p_i)
\]
is Schur-convex in \( p \in \Delta_n \) (cf. [18, p. 64]). Since \( -\phi(\cdot) \) is Schur-concave when \( \phi(\cdot) \) is Schur-convex, it also follows that if a function \( g : [0, 1] \to \mathbb{R} \) is concave, then the function
\[
\phi(p) = \sum_{i=1}^{n} g(p_i)
\]
is Schur-concave in \( p \in \Delta_n \). Thus, since \( x^\beta \) is strictly concave in \( x \geq 0 \) for every \( \beta \in (0, 1) \), we see that
\[
\|p\|_\beta^\alpha = \sum_{i=1}^{n} p_i^\alpha
\]
is Schur-concave in \( p \in \Delta_n \) for every \( \alpha \in (0, 1) \); similarly, since \( x^\beta \) is strictly convex in \( x \geq 0 \) for every \( \beta \in (1, \infty) \), we also see that \( \|p\|_\beta^\alpha \) is strictly Schur-convex in \( p \in \Delta_n \) for every \( \alpha \in (1, \infty) \). Therefore, since \( x \mapsto x^{1/\beta} \) is a strictly increasing function of \( x \geq 0 \) for every \( \beta \in (0, \infty) \), we have that \( \|p\|_\beta = (\|p\|_\beta^\beta)^{1/\beta} \) is strictly Schur-concave in \( p \in \Delta_n \) for every \( \beta \in (0, 1) \) and \( \|p\|_\beta \) is strictly Schur-convex in \( p \in \Delta_n \) for every \( \beta \in (1, \infty) \).

To consider probability vectors \( p \in \Delta_n \) with a fixed \( \ell_\infty \)-norm, we assume in (56) that \( \alpha = \infty \), i.e.,
\[
\|p\|_\infty = p[1] = A
\]
for a constant \( A \in (1/n, 1) \). Then, since
\[
v_n\left(\frac{1 - A}{(n - 1)}\right) = \left(A, \frac{1 - A}{(n - 1)}, \frac{1 - A}{(n - 1)}, \ldots, \frac{1 - A}{(n - 1)}\right) \prec p
\]
for all \( p \in \Delta_n \) under the constrain (146), it follows from the Schur-convexity of the \( \ell_\beta \)-norm that
\[
\|v_n\left(\frac{1 - A}{(n - 1)}\right)\|_\beta \leq \|p\|_\beta \quad \text{for } \beta \in (1, \infty),
\]
\[
\|v_n\left(\frac{1 - A}{(n - 1)}\right)\|_\beta \geq \|p\|_\beta \quad \text{for } \beta \in (0, 1)
\]
for all \( p \in \Delta_n \) under the constraint (146), where note that
\[
0 < \frac{1 - A}{(n - 1)} < \frac{1}{n} \tag{150}
\]
for \( A \in (1/n, 1) \). Combining (137)–(139), (148), and (149), for any \( n \geq 2, p \in \Delta_n, \) and \( \alpha \in (0, 1) \cup (1, \infty) \), we have that there exists \( p \in [0, 1/n] \) such that
\[
\|w_n(p)\|_\alpha = \|p\|_\alpha, \tag{151}
\]
\[
\|w_n(p)\|_\beta \leq \|p\|_\beta \quad \text{for} \quad \beta \in (\min\{1, \alpha\}, \max\{1, \alpha\}), \tag{152}
\]
\[
\|w_n(p)\|_\beta \geq \|p\|_\beta \quad \text{for} \quad \beta \in (0, \min\{1, \alpha\}) \cup (\max\{1, \alpha\}, \infty], \tag{153}
\]
where the inequality (153) for \( \beta = \infty \) follows from (148), (149), and the monotonicity of \( \|v_n(p)\|_\infty \) for \( p \in [0, 1/n] \) (cf. Lemma 1). This completes the proof of Lemma 2.

**Lemma 3.** For any \( n \geq 2, p \in \Delta_n, \) and \( \alpha \in (0, 1) \cup (1, \infty) \), there exists \( p \in [1/n, 1] \) such that
\[
\|w_n(p)\|_\alpha = \|p\|_\alpha, \tag{154}
\]
\[
\|w_n(p)\|_\beta \geq \|p\|_\beta \quad \text{for all} \quad \beta \in (\min\{1, \alpha\}, \max\{1, \alpha\}), \tag{155}
\]
\[
\|w_n(p)\|_\beta \leq \|p\|_\beta \quad \text{for all} \quad \beta \in (0, \min\{1, \alpha\}) \cup (\max\{1, \alpha\}, \infty]. \tag{156}
\]

**Proof of Lemma 3:** It can be easily seen that, for any \( p \in \Delta_2 \), there exists \( p \in [1/2, 1] \) such that \( p_\Delta = w_2(p) \); therefore, the lemma obviously holds when \( n = 2 \). As with (54) and (55), we readily see that
\[
\forall \alpha \in (0, \infty), \quad \|p\|_\alpha = n^{(1/\alpha) - 1} \iff p = \left( \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n} \right) = w_n\left( \frac{1}{n} \right), \tag{157}
\]
\[
\forall \alpha \in (0, \infty), \quad \|p\|_\alpha = 1 \iff p_\Delta = (1, 0, \ldots, 0) = w_n(1) \tag{158}
\]
Therefore, the lemma obviously holds when \( \|p\|_\alpha \in \{1, n^{(1/\alpha) - 1}\} \); thus, Lemma 3 is proved for the case \( n = 2 \) and \( \|p\|_\alpha \in \{1, n^{(1/\alpha) - 1}\} \) for \( \alpha \in (0, 1) \cup (1, \infty) \). Hence, it is enough to prove the lemma for \( p \in \Delta_n \) such that \( n \geq 3 \) and \( \|p\|_\alpha \in (\min\{1, n^{(1/\alpha) - 1}\}, \max\{1, n^{(1/\alpha) - 1}\}) \) for \( \alpha \in (0, 1) \cup (1, \infty) \) in the late analyses.

We now begin to prove Lemma 3 for \( \alpha \neq \infty \) and \( \beta \neq \infty \). For fixed \( n \geq 3, \alpha \in (0, 1) \cup (1, \infty) \), and \( A \in (\min\{1, n^{(1/\alpha) - 1}\}, \max\{1, n^{(1/\alpha) - 1}\}) \), we assume for \( p \in \Delta_n \) that
\[
\|p\|_\alpha = A. \tag{159}
\]
For that \( p \) satisfying (159), let \( k, l \in \{2, 3, \ldots, n\} \) \((k < l)\) be indices such that \( p_{[k]} = p_{[k-1]} > p_{[k+1]} \) and \( p_{[l]} > p_{[l+1]} = 0 \); namely, the indices \( k, l \) are chosen to satisfy the following inequalities:
\[
p_{[1]} = \cdots = p_{[k-1]} \geq p_{[k]} \geq p_{[k+1]} \geq \cdots \geq p_{[l-1]} \geq p_{[l]} > p_{[l+1]} = \cdots = p_{[n]} = 0 \quad (p_{[k-1]} > p_{[k+1]}). \tag{160}
\]
Since \( p_1 + p_2 + \cdots + p_n = 1 \), we observe as with (62) that
\[
\sum_{i=1}^{n} p_i = 1 \quad \implies \quad \sum_{i=1; i \neq k}^{n} \frac{dp_i}{dp_{[k]}} = -1. \tag{161}
\]
In this proof, we assume that

\[
\frac{dp_{[i]}}{dp_{[k]}} = \frac{dp_{[j]}}{dp_{[k]}} \quad (162)
\]

for \( i \in \{2, 3, \ldots, k - 1\} \),

\[
\frac{dp_{[j]}}{dp_{[k]}} = 1 \quad (163)
\]

for \( j \in \{k + 1, k + 2, \ldots, l - 1\} \), and

\[
\frac{dp_{[m]}}{dp_{[k]}} = 0 \quad (164)
\]

for \( m \in \{l + 1, l + 2, \ldots, n\} \). By constraints (162), (163), and (164), we get

\[
\sum_{i=1}^{n} p_i = 1 \quad (165)
\]

\[
\sum_{i=1, i \neq k}^{n} \frac{dp_{[i]}}{dp_{[k]}} = -1 \quad (166)
\]

\[
\iff \quad \sum_{i=1}^{k-1} \frac{dp_{[i]}}{dp_{[k]}} + \sum_{j=k+1}^{l-1} \frac{dp_{[j]}}{dp_{[k]}} + \frac{dp_{[k]}}{dp_{[k]}} + \sum_{m=l+1}^{n} \frac{dp_{[m]}}{dp_{[k]}} = -1 \quad (167)
\]

\[
(1 - k) \frac{dp_{[1]}}{dp_{[k]}} + \sum_{j=k+1}^{l-1} \frac{dp_{[j]}}{dp_{[k]}} + \frac{dp_{[k]}}{dp_{[k]}} + \sum_{m=l+1}^{n} \frac{dp_{[m]}}{dp_{[k]}} = -1 \quad (168)
\]

\[
(1 - k) \frac{dp_{[1]}}{dp_{[k]}} + (l - k - 1) \frac{dp_{[l]}}{dp_{[k]}} + \sum_{m=l+1}^{n} \frac{dp_{[m]}}{dp_{[k]}} = -1 \quad (169)
\]

\[
(1 - k) \frac{dp_{[1]}}{dp_{[k]}} + (l - k - 1) \frac{dp_{[l]}}{dp_{[k]}} = -1 \quad (170)
\]

\[
(1 - k) \frac{dp_{[1]}}{dp_{[k]}} = -(l - k) \quad (171)
\]

\[
(1 - k) \frac{dp_{[1]}}{dp_{[k]}} = -(l - k) - \frac{dp_{[l]}}{dp_{[k]}} \quad (172)
\]

\[
\frac{dp_{[1]}}{dp_{[k]}} = \frac{1}{k - 1} \left( (l - k) + \frac{dp_{[l]}}{dp_{[k]}} \right) \quad (173)
\]

where note in (173) that \( k \geq 2 \). Moreover, we observe from the constraint (159) that

\[
\|p\|_\alpha = A \quad (174)
\]

\[
\sum_{i=1}^{n} p_i^\alpha = A^\alpha \quad (175)
\]

\[
\sum_{i=1}^{l} p_i^\alpha = A^\alpha \quad (176)
\]

\[
\sum_{i=1, i \neq k}^{l} \left( \frac{dp_{[i]}}{dp_{[k]}} \right) p_{[i]}^{\alpha-1} = -p_{[k]}^{\alpha-1} \quad (177)
\]

\[
\sum_{i=1}^{k-1} \left( \frac{dp_{[i]}}{dp_{[k]}} \right) p_{[i]}^{\alpha-1} + \sum_{j=k+1}^{l-1} \left( \frac{dp_{[j]}}{dp_{[k]}} \right) p_{[j]}^{\alpha-1} + \left( \frac{dp_{[l]}}{dp_{[k]}} \right) p_{[l]}^{\alpha-1} = -p_{[k]}^{\alpha-1} \quad (178)
\]
\( \sum_{i=1}^{k-1} \frac{dp_i}{dp_k} p_i^{\alpha - 1} + \sum_{j=k+1}^{l-1} \frac{dp_j}{dp_k} p_j^{\alpha - 1} + \left( \frac{dp_l}{dp_k} \right) p_l^{\alpha - 1} = -p_k^{\alpha - 1} \) (179)

\( (k - 1) p_{[1]}^{\alpha - 1} \left( \frac{dp_{[1]}}{dp_k} \right) + \sum_{j=k+1}^{l-1} p_j^{\alpha - 1} + \left( \frac{dp_l}{dp_k} \right) p_l^{\alpha - 1} = -p_k^{\alpha - 1} \) (180)

\( (k - 1) p_{[1]}^{\alpha - 1} \left( \frac{dp_{[1]}}{dp_k} \right) + \sum_{j=k+1}^{l-1} p_j^{\alpha - 1} + \left( \frac{dp_l}{dp_k} \right) p_l^{\alpha - 1} = -p_k^{\alpha - 1} \) (181)

\( (k - 1) p_{[1]}^{\alpha - 1} \left( \frac{dp_{[1]}}{dp_k} \right) + \sum_{j=k+1}^{l-1} p_j^{\alpha - 1} + \left( \frac{dp_l}{dp_k} \right) p_l^{\alpha - 1} = -p_k^{\alpha - 1} \) (182)

\( (k - 1) p_{[1]}^{\alpha - 1} \left( \frac{dp_{[1]}}{dp_k} \right) + \sum_{j=k+1}^{l-1} p_j^{\alpha - 1} + \left( \frac{dp_l}{dp_k} \right) p_l^{\alpha - 1} = -p_k^{\alpha - 1} \) (183)

\( -(l - k) p_{[1]}^{\alpha - 1} - \left( \frac{dp_{[1]}}{dp_k} \right) p_{[1]}^{\alpha - 1} + \left( \frac{dp_{[1]}}{dp_k} \right) p_{[1]}^{\alpha - 1} = -p_k^{\alpha - 1} \) (184)

\( -(l - k) p_{[1]}^{\alpha - 1} + \left( \frac{dp_{[1]}}{dp_k} \right) p_{[1]}^{\alpha - 1} - p_{[1]}^{\alpha - 1} = -p_k^{\alpha - 1} \) (185)

\( \left( \frac{dp_{[1]}}{dp_k} \right) \left( p_{[1]}^{\alpha - 1} - p_{[1]}^{\alpha - 1} \right) = -p_k^{\alpha - 1} \) (186)

\[ \frac{dp_{[1]}}{dp_k} = -\sum_{j=k}^{l-1} \left( p_{[j]}^{\alpha - 1} - p_{[1]}^{\alpha - 1} \right) \] (187)

\[ \frac{dp_{[1]}}{dp_k} = -\sum_{j=k}^{l-1} \left( \frac{p_{[1]}}{p_{[j]}} \right)^{\alpha - 1} - 1 \] (188)

\[ \frac{dp_{[1]}}{dp_k} = -\sum_{j=k}^{l-1} \left( \frac{p_{[1]}}{p_{[j]}} \right)^{1-\alpha} - 1 \] (189)

\[ \frac{dp_{[1]}}{dp_k} = -\sum_{j=k}^{l-1} \left( \frac{p_{[1]}}{p_{[j]}} \right) \ln \alpha \left( \frac{p_{[1]}}{p_{[j]}} \right) \] (190)

We now check the sign of the right-hand side of (190). Since \( p_{[1]} \geq p_{[k]} \geq p_{[j]} \) for \( j \geq k \geq 2 \), we get

\[ \frac{dp_{[1]}}{dp_k} \overset{(190)}{=} -\sum_{j=k}^{l-1} \left( \frac{p_{[1]}}{p_{[j]}} \right) \ln \alpha \left( \frac{p_{[1]}}{p_{[j]}} \right) \]

\[ \overset{(a)}{=} -\sum_{j=k}^{l-1} \left( \frac{p_{[1]}}{p_{[j]}} \right) \ln \alpha \left( \frac{p_{[1]}}{p_{[j]}} \right) \]

\[ = -(l - k) \left( \frac{p_{[1]}}{p_{[j]}} \right) \ln \alpha \left( \frac{p_{[1]}}{p_{[j]}} \right) \] (193)
where (a) follows by the monotonicity of the $\alpha$-logarithm function (cf. (101)), the inequality (a) holds with equality if and only if $p[k] = p[l-1]$, and the inequality (b) holds with equality if and only if $p[1] = p[k]$ for $k \leq l - 1$. Thus, it follows from (194) that $p[l]$ is strictly decreasing for $p[k]$ under the constraints (159), (160), (162), (163), and (164).

Similarly, we check the sign of the right-hand side of (173) as follows:

$$\frac{dp[1]}{dp[k]} = -\frac{1}{k-1} \left( l - k + \frac{dp[l]}{dp[k]} \right)$$ (195)

$$\leq - \frac{l - k}{k-1} + \frac{1}{k-1} \sum_{j=k}^{l-1} \frac{\ln_{\alpha} (p[i])}{\ln_{\alpha} (p[l])}$$ (196)

$$\leq - \frac{l - k}{k-1} + \frac{1}{k-1} \sum_{j=k}^{l-1} \frac{\ln_{\alpha} (p[i])}{\ln_{\alpha} (p[l])}$$ (197)

$$= - \frac{l - k}{k-1} + \frac{1}{k-1} \sum_{j=k}^{l-1} 1$$ (198)

$$= - \frac{l - k}{k-1} + \frac{l - k}{k-1}$$ (199)

$$= 0,$$ (200)

where (a) follows by $p[1] > p[l]$ (cf. the constraint (160)) and the monotonicity of the $\alpha$-logarithm function (cf. (101)). Hence, it follows from (200) that $p[l]$ is also strictly decreasing for $p[k]$ under the constraints (159), (160), (162), (163), and (164).

We next consider $\|p\|_{\beta}$ for a fixed $\beta \in (0, 1) \cup (1, \infty)$. A direct calculation yields

$$\frac{d\|p\|_{\beta}}{dp[k]} = \frac{d}{dp[k]} \left( \sum_{i=1}^{n} p[i]^\beta \right)$$ (201)

$$= \frac{d}{dp[k]} \left( \sum_{i=1}^{n} p[i]^\beta \right)$$ (202)

$$= \beta \left( p[k]^{\beta-1} + \sum_{i=1; i\neq k}^{l} \left( \frac{dp[i]}{dp[k]} \right) p[i]^{\beta-1} \right)$$ (203)

$$= \beta \left( p[k]^{\beta-1} + \sum_{i=1}^{k-1} \left( \frac{dp[i]}{dp[k]} \right) p[i]^{\beta-1} + \sum_{j=k+1}^{l} \left( \frac{dp[j]}{dp[k]} \right) p[j]^{\beta-1} + \left( \frac{dp[l]}{dp[k]} \right) p[l]^{\beta-1} \right)$$ (204)

$$= \beta \left( p[k]^{\beta-1} + \sum_{i=1}^{k-1} \left( \frac{dp[i]}{dp[k]} \right) p[i]^{\beta-1} + \sum_{j=k+1}^{l} \left( \frac{dp[j]}{dp[k]} \right) p[j]^{\beta-1} + \left( \frac{dp[l]}{dp[k]} \right) p[l]^{\beta-1} \right)$$ (205)

$$= \beta \left( p[k]^{\beta-1} + (k-1) p[l]^{\beta-1} \left( \frac{dp[l]}{dp[k]} \right) + \sum_{j=k+1}^{l} \left( \frac{dp[j]}{dp[k]} \right) p[j]^{\beta-1} + \left( \frac{dp[l]}{dp[k]} \right) p[l]^{\beta-1} \right)$$ (206)

$$= \beta \left( p[k]^{\beta-1} + (k-1) p[l]^{\beta-1} \left( \frac{dp[l]}{dp[k]} \right) + \sum_{j=k+1}^{l} \left( \frac{dp[j]}{dp[k]} \right) p[j]^{\beta-1} \right)$$ (207)
\[ \begin{align*} &\beta \left( (k - 1) p_{[1]}^{\beta - 1} \left( \frac{dp_{[1]}}{dp_{[k]}} \right) + \sum_{j=k}^{l-1} p_{[j]}^{\beta - 1} \left( \frac{dp_{[j]}}{dp_{[k]}} \right) \right) \\
&\overset{(173)}{=} \beta \left( (k - 1) p_{[1]}^{\beta - 1} \left( \frac{1}{k - 1} \left( (l - k) + \frac{dp_{[1]}}{dp_{[k]}} \right) \right) + \sum_{j=k}^{l-1} p_{[j]}^{\beta - 1} \left( \frac{dp_{[j]}}{dp_{[k]}} \right) \right) \\
&\overset{(170)}{=} \beta \left( (l - k) p_{[1]}^{\beta - 1} - \sum_{j=k}^{l-1} p_{[j]}^{\beta - 1} \right) \left( \frac{dp_{[1]}}{dp_{[k]}} \right) \left( \frac{dp_{[j]}}{dp_{[k]}} \right) p_{[1]}^{\beta - 1} \right) \\
&\overset{(190)}{=} \beta \left( \sum_{j=k}^{l-1} \left( p_{[j]}^{\beta - 1} - p_{[1]}^{\beta - 1} \right) \right) - \sum_{j=k}^{l-1} \left( \ln_{\alpha} \left( \frac{p_{[j]}}{p_{[1]}} \right) \right) \left( p_{[j]}^{\beta - 1} - p_{[1]}^{\beta - 1} \right) \\
&\overset{(71)}{=} \beta \left( p_{[1]}^{\beta - 1} - p_{[1]}^{\beta - 1} \right) \sum_{j=k}^{l-1} \left( \ln_{\beta} \left( \frac{p_{[j]}}{p_{[1]}} \right) - \ln_{\alpha} \left( \frac{p_{[j]}}{p_{[1]}} \right) \right) . \\
\end{align*} \]

Since \( \|p\|_\beta = (\|p\|_\beta^\beta)^{1/\beta} \), it follows by the chain rule that

\[ \frac{d\|p\|_\beta}{dp_{[k]}} = \left( \frac{1}{\beta} \frac{\|p\|_\beta}{\|p\|_\beta} \right) \left( \frac{d\|p\|_\beta^\beta}{dp_{[k]}} \right) \]

\[ \overset{(214)}{=} \sum_{i=1}^{n} p_{i}^{(1/\beta) - 1} \left( p_{[i]}^{\beta - 1} - p_{[1]}^{\beta - 1} \right) \sum_{j=k}^{l-1} \left( \ln_{\beta} \left( \frac{p_{[j]}}{p_{[1]}} \right) - \ln_{\alpha} \left( \frac{p_{[j]}}{p_{[1]}} \right) \right) . \]

Thus, we get

\[ \text{sgn} \left( \frac{d\|p\|_\beta}{dp_{[k]}} \right) = \text{sgn} \left( \sum_{i=1}^{n} p_{i}^{(1/\beta) - 1} \left( p_{[i]}^{\beta - 1} - p_{[1]}^{\beta - 1} \right) \sum_{j=k}^{l-1} \left( \ln_{\beta} \left( \frac{p_{[j]}}{p_{[1]}} \right) - \ln_{\alpha} \left( \frac{p_{[j]}}{p_{1}} \right) \right) \right) \]

\[ \overset{(217)}{=} \text{sgn} \left( \sum_{i=1}^{n} p_{i}^{(1/\beta) - 1} \right) \cdot \text{sgn} \left( p_{[1]}^{\beta - 1} - p_{[1]}^{\beta - 1} \right) \cdot \text{sgn} \left( \sum_{j=k}^{l-1} \left( \ln_{\beta} \left( \frac{p_{[j]}}{p_{[1]}} \right) - \ln_{\alpha} \left( \frac{p_{[j]}}{p_{[1]}} \right) \right) \right) \]

\[ \overset{(218)}{=} \text{sgn} \left( p_{[1]}^{\beta - 1} - p_{[1]}^{\beta - 1} \right) \cdot \text{sgn} \left( \sum_{j=k}^{l-1} \left( \ln_{\beta} \left( \frac{p_{[j]}}{p_{[1]}} \right) - \ln_{\alpha} \left( \frac{p_{[j]}}{p_{[1]}} \right) \right) \right) \right) . \]

Since \( p_{[1]} > p_{[l]} > 0 \) by the constraint (160), we readily see that

\[ \text{sgn} \left( p_{[1]}^{\beta - 1} - p_{[1]}^{\beta - 1} \right) = \begin{cases} 1 & \text{if } \beta < 1, \\
0 & \text{if } \beta = 1, \\
-1 & \text{if } \beta > 1. \end{cases} \]

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Moreover, since
\[
1 \leq \frac{p_1}{p_j} < \frac{p_1}{p_l} \quad (1 \neq \frac{p_1}{p_l})
\]
for \( j \in \{k, k+1, \ldots, l-1\} \), it follows from (126) (cf. [21, Eq. (20)]) that
\[
\text{sgn} \left( \ln_\beta \left( \frac{p_1}{p_l} \right) - \ln_\alpha \left( \frac{p_1}{p_l} \right) \right) = \begin{cases} 
1 & \text{if } \beta > \alpha \text{ and } p_1 > p_j > p_l, \\
0 & \text{if } \beta = \alpha \text{ or } p_1 = p_j \text{ or } p_j = p_l, \\
-1 & \text{if } \beta < \alpha \text{ and } p_1 > p_j > p_l
\end{cases}
\]
for \( j \in \{k, k+1, \ldots, l-1\} \); and therefore, we have
\[
\text{sgn} \left( \sum_{j=k}^{l-1} \left( \ln_\beta \left( \frac{p_1}{p_l} \right) - \ln_\alpha \left( \frac{p_1}{p_l} \right) \right) \right) = \begin{cases} 
1 & \text{if } \beta > \alpha \text{ and } (p_1 > p_k \text{ or } p_k > p_l), \\
0 & \text{if } \beta = \alpha \text{ or } (p_1 = p_k \text{ and } p_{k+1} = p_l) \text{ or } p_k = p_l, \\
-1 & \text{if } \beta < \alpha \text{ and } (p_1 > p_k \text{ or } p_k > p_l)
\end{cases}
\]
Combining (220) and (223), if \( p_1 > p_k \) or \( p_k > p_l \), we obtain
\[
\text{sgn} \left( \frac{d||p||_\beta}{dp_k} \right) = \text{sgn} \left( \frac{p_1^\beta - p_k^\beta}{p_l^\beta - p_1^\beta} \right) \begin{cases} 
1 & \text{if } \beta \in (\min\{1, \alpha\}, \max\{1, \alpha\}), \\
0 & \text{if } \beta \in \{1, \alpha\}, \\
-1 & \text{if } \beta \in (0, \min\{1, \alpha\}) \cup (\max\{1, \alpha\}, \infty),
\end{cases}
\]
which implies that
- if \( \alpha \in (0, 1) \), then
  - \( ||p||_\beta \) with a fixed \( \beta \in (\alpha, 1) \) is strictly increasing for \( p_k \), and
  - \( ||p||_\beta \) with a fixed \( \beta \in (0, \alpha) \cup (1, \infty) \) is strictly decreasing for \( p_k \),
- if \( \alpha \in (1, \infty) \), then
  - \( ||p||_\beta \) with a fixed \( \beta \in (1, \alpha) \) is strictly increasing for \( p_k \), and
  - \( ||p||_\beta \) with a fixed \( \beta \in (0, 1) \cup (\alpha, \infty) \) is strictly decreasing for \( p_k \).
Note that the above monotonicity hold under the constraints (159), (160), (162), (163), and (164).

To accomplish the proof of Lemma 3 for \( \alpha \neq \infty \) and \( \beta \neq \infty \) by using the above relations, we repeat the following operation until the vector \( p = (p_1, p_2, \ldots, p_n) \) satisfies \( p_{k-1} = p_k \) and \( l = k + 1 \), i.e.,
\[
P_1 = p_2 = \cdots = p_{k-1} = p_k = p_{k+1} \geq p_{k+2} = p_{k+3} \cdots = p_n = 0,
\]
which is equivalent to the vector \( w_n(\cdot) \). If \( p_{k-1} = p_k \) and \( k < l - 1 \), then we reset the index \( k \in \{2, 3, \ldots, n-2\} \) to \( k+1 \); namely, we now choose the indices \( k, l \in \{2, 3, \ldots, n\} \) \( (k < l) \) to satisfy the following inequalities:
\[
P_1 = p_2 = \cdots = p_{k-1} > p_k \geq p_{k+1} \geq \cdots \geq p_{l-1} \geq p_l > p_{l+1} = p_{l+2} = \cdots = p_n = 0.
\]
For that index $k$, we consider to increase $p[k]$ under the constraints (159), (160), (162), (163), and (164). Note that the constraint (163) implies that, for $j \in \{k + 1, k + 2, \ldots, l - 1\}$, the value $p[j]$ is increased at the same speed as $p[k]$. It follows from (162) and (200) that, for $i \in \{1, 2, \ldots, k - 1\}$, the value $p[i]$ is strictly decreased by according to increasing $p[k]$. Similarly, it follows from (194) that $p[i]$ is also strictly decreased by according to increasing $p[k]$.

Let $q = (q_1, q_2, \ldots, q_n)$ denote the probability vector such that made from $p$ by increasing $p[k]$ until the equality $p[k - 1] = p[k]$ or $p[l] = 0$ holds under the conditions of (159), (162), (163), (164), and (227). Namely, the vector $q$ satisfies either

$$q[1] = q[2] = \cdots = q[k - 1] = q[k] \geq q[k + 1] \geq \cdots \geq q[l - 1] > q[l] \geq q[l + 1] = q[l + 2] = \cdots = q[n] = 0$$

or

$$q[1] = q[2] = \cdots = q[k - 1] \geq q[k] \geq q[k + 1] \geq \cdots \geq q[l - 1] > q[l] = q[l + 1] = q[l + 2] = \cdots = q[n] = 0.$$  

Note that there is a possibility that both of (228) and (229) hold as follows:

$$q[1] = q[2] = \cdots = q[k - 1] = q[k] \geq q[k + 1] \geq \cdots \geq q[l - 1] > q[l] = q[l + 1] = q[l + 2] = \cdots = q[n] = 0.$$  

Since $q$ is made from $p$ under the constraint (159), note that

$$\|q\|_\alpha = \|p\|_\alpha$$

for a fixed $\alpha \in (0, 1) \cup (1, \infty)$. Moreover, it follows from (225) that

$$\|q\|_\beta \geq \|p\|_\beta \quad \text{for } \beta \in (\min\{1, \alpha\}, \max\{1, \alpha\}),$$

$$\|q\|_\beta \leq \|p\|_\beta \quad \text{for } \beta \in (0, \min\{1, \alpha\}) \cup (\max\{1, \alpha\}, \infty).$$

Repeating these operation until (226) holds, we have

$$\|w_n(p)\|_\alpha = \|p\|_\alpha,$$

$$\|w_n(p)\|_\beta \geq \|p\|_\beta \quad \text{for } \beta \in (\min\{1, \alpha\}, \max\{1, \alpha\}),$$

$$\|w_n(p)\|_\beta \leq \|p\|_\beta \quad \text{for } \beta \in (0, \min\{1, \alpha\}) \cup (\max\{1, \alpha\}, \infty)$$

for any $p \in \Delta_n$, any fixed $\alpha \in (0, 1) \cup (1, \infty)$, and some $p \in [1/n, 1]$.

Finally, we consider the $\ell_\beta$-norm with a fixed $\ell_\infty$-norm for $\beta \in (0, 1) \cup (1, \infty)$. To consider probability vectors $p \in \Delta_n$ with a fixed $\ell_\infty$-norm, we assume in (159) that $\alpha = \infty$, i.e,

$$\|p\|_\infty = p[1] = A$$

for a constant $A \in (1/n, 1)$. Recall from the proof of Lemma 2 that

- $\|p\|_\beta$ is strictly Schur-concave in $p \in \Delta_n$ for every $\beta \in (0, 1)$ and
- $\|p\|_\beta$ is strictly Schur-convex in $p \in \Delta_n$ for every $\beta \in (1, \infty)$.  

Thus, since

\[ w_n(A) = \left( A, A, \ldots, A, 1 - \left[ \frac{1}{A} \right] A, 0, 0, \ldots, 0 \right) \times p \]

(238)

for all \( p \in \Delta_n \) under the constrain (237), it follows from the Schur-convexity of the \( \ell_\beta \)-norm that

\[ \|w_n(A)\|_\beta \geq \|p\|_\beta \quad \text{for } \beta \in (1, \infty), \]

(239)

\[ \|w_n(A)\|_\beta \leq \|p\|_\beta \quad \text{for } \beta \in (0, 1) \]

(240)

for all \( p \in \Delta_n \) under the constraint (146). Combining (234)–(236), (239), and (240), for any \( n \geq 2 \), \( p \in \Delta_n \), and \( \alpha \in (0, 1) \cup (1, \infty) \), we have that there exists \( p \in [1/n, 1] \) such that

\[ \|w_n(p)\|_\alpha = \|p\|_\alpha, \]

(241)

\[ \|w_n(p)\|_\beta \leq \|p\|_\beta \quad \text{for } \beta \in (\min\{1, \alpha\}, \max\{1, \alpha\}), \]

(242)

\[ \|w_n(p)\|_\beta \geq \|p\|_\beta \quad \text{for } \beta \in (0, \min\{1, \alpha\}) \cup (\max\{1, \alpha\}, \infty), \]

(243)

where the inequality (243) for \( \beta = \infty \) follows from (239), (240), and the monotonicity of \( \|w_n(p)\|_\infty \) for \( p \in [1/n, 1] \) (cf. Lemma 1). This completes the proof of Lemma 3.

Lemmas 2 and 3 show that, among all \( n \)-dimensional probability vectors with a fixed \( \ell_\alpha \)-norm for a given \( \alpha \in (0, 1) \cup (1, \infty) \), the distributions \( v_n(\cdot) \) and \( w_n(\cdot) \) take extremal \( \ell_\beta \)-norm for all positive \( \beta \) (\( \neq \alpha \)). Combining Lemmas 2 and 3, we have the following theorem.

**Theorem 1.** For any \( n \geq 2 \), \( p \in \Delta_n \), and \( \alpha \in (0, 1) \cup (1, \infty) \), there exist unique numbers \( p_v \in [0, 1/n] \) and \( p_w \in [1/n, 1] \) such that

\[ \|p\|_\alpha = \|v_n(p_v)\|_\alpha = \|w_n(p_w)\|_\alpha, \]

(244)

\[ \|v_n(p_v)\|_\beta \leq \|p\|_\beta \leq \|w_n(p_w)\|_\beta \quad \text{for all } \beta \in (\min\{1, \alpha\}, \max\{1, \alpha\}), \]

(245)

\[ \|v_n(p_v)\|_\beta \leq \|p\|_\beta \leq \|w_n(p_w)\|_\beta \quad \text{for all } \beta \in (0, \min\{1, \alpha\}) \cup (\max\{1, \alpha\}, \infty). \]

(246)

**Proof of Theorem 1:** For \( n \geq 2 \), \( p \in \Delta_n \), and \( \alpha \in (0, 1) \cup (1, \infty) \), it follows from Lemmas 2 and 3 that there exist \( p_v \in [0, 1/n] \) and \( p_w \in [1/n, 1] \) such that satisfy the following:

\[ \|p\|_\alpha = \|v_n(p_v)\|_\alpha = \|w_n(p_w)\|_\alpha, \]

(247)

\[ \|v_n(p_v)\|_\beta \leq \|p\|_\beta \leq \|v_n(p_v)\|_\beta \quad \text{for all } \beta \in (\min\{1, \alpha\}, \max\{1, \alpha\}), \]

(248)

\[ \|v_n(p_v)\|_\beta \leq \|p\|_\beta \leq \|w_n(p_w)\|_\beta \quad \text{for all } \beta \in (0, \min\{1, \alpha\}) \cup (\max\{1, \alpha\}, \infty), \]

(249)

which are equivalent to (244)–(246). The uniqueness of the values \( p_v \) and \( p_w \) follow by Lemma 1.

Theorem 1 shows that, among all \( n \)-dimensional probability vectors with a fixed \( \ell_\alpha \)-norm for a given \( \alpha \in (0, 1) \cup (1, \infty) \), the distributions \( v_n(\cdot) \) and \( w_n(\cdot) \) take the extremal \( \ell_\beta \)-norm for \( \beta \in (0, 1) \cup (1, \infty) \), \( \beta \neq \alpha \). Hence,
Theorem 1 identifies the boundaries of the region
\[
R_n(\alpha, \beta) := \{ (\|p\|_\alpha, \|p\|_\beta) \mid p \in \Delta_n \}
\]
for \(n \geq 2\) and \(\alpha, \beta \in (0, 1) \cup (1, \infty], \alpha \neq \beta\). We illustrate boundaries of \(R_n(\alpha, \beta)\) in Fig. 1.

By using Theorem 1, we derive the sharp bounds of the Rényi entropy of order \(\beta\) with a fixed Rényi entropy of another order \(\alpha\), as shown in Theorem 2.

**Theorem 2.** For any \(n \geq 2\), \(p \in \Delta_n\), and \(\alpha \in (0, \infty]\), there exist unique numbers \(p_v \in [0, 1/n]\) and \(p_w \in [1/n, 1]\) such that
\[
H_\alpha(p) = H_\alpha(v_n(p_v)) = H_\alpha(w_n(p_w)),
\]
\[
H_\beta(w_n(p_w)) \leq H_\beta(p) \leq H_\beta(v_n(p_v)) \quad \text{for all } \beta \in (0, \alpha),
\]
\[
H_\beta(v_n(p_v)) \leq H_\beta(p) \leq H_\beta(w_n(p_w)) \quad \text{for all } \beta \in (\alpha, \infty].
\]

**Proof of Theorem 2:** If \(\alpha = 1\), then Theorem 2 is reduced to [21, Corollary 1]. Similarly, if \(\beta = 1\), then Theorem 2 is also reduced to [21, Theorem 2]. Therefore, in this proof, it is enough to prove Theorem 2 for \(\alpha \in (0, 1) \cup (1, \infty]\) and \(\beta \in (0, 1) \cup (1, \infty]\).

Consider the function
\[
f_\alpha(x) = \begin{cases} 
\frac{\alpha}{1-\alpha} \ln x & \text{if } \alpha \in (0, 1) \cup (1, \infty), \\
-\ln x & \text{if } \alpha = \infty
\end{cases}
\]
for \(x > 0\). Then, we readily see that
\[
H_\alpha(p) = f_\alpha(\|p\|_\alpha)
\]
for \(\alpha \in (0, 1) \cup (1, \infty]\) and \(p \in \Delta_n\). It is clear that \(f_\alpha(x)\) is a strictly monotonic function of \(x > 0\) for every \(\alpha \in (0, 1) \cup (1, \infty]\); hence, it follows from (255) that (245) of Theorem 1 implies the equalities:
\[
H_\alpha(p) = H_\alpha(v_n(p_v)) = H_\alpha(w_n(p_w)),
\]
which is equivalent to (251) of Theorem 2.

We now suppose that \( \alpha \in (0, 1) \). Then, Eq. (245) of Theorem 1 can be written as

\[
\|v_n(p_v)\|_\beta \leq \|p\|_\beta \leq \|w_n(p_w)\|_\beta \quad \text{for all } \beta \in (\alpha, 1).
\]  

(257)

Since \( f_\beta(x) \) of (254) is a strictly increasing function of \( x > 0 \) for every \( \beta \in (0, 1) \), it follows from (255) that (257) implies the inequalities:

\[
H_\beta(v_n(p_v)) \leq H_\beta(p) \leq H_\beta(w_n(p_w)) \quad \text{for all } \beta \in (\alpha, 1).
\]  

(258)

In addition, Eq. (246) of Theorem 1 can be written as

\[
\|w_n(p_w)\|_\beta \leq \|p\|_\beta \leq \|v_n(p_v)\|_\beta \quad \text{for all } \beta \in (0, \alpha) \cup (1, \infty).
\]  

(259)

Since \( f_\beta(x) \) is a strictly increasing function of \( x > 0 \) for every \( \beta \in (0, 1) \), it follows from (255) that (259) implies the inequalities:

\[
H_\beta(w_n(p_w)) \leq H_\beta(p) \leq H_\beta(v_n(p_v)) \quad \text{for all } \beta \in (0, \alpha);\]

(260)

similarly, since \( f_\beta(x) \) is a strictly decreasing function of \( x > 0 \) for every \( \beta \in (1, \infty) \), it follows from (255) that (259) implies the inequalities:

\[
H_\beta(v_n(p_v)) \leq H_\beta(p) \leq H_\beta(w_n(p_w)) \quad \text{for all } \beta \in (1, \infty].
\]  

(261)

Combining (258), (260), and (261), we have

\[
H_\beta(w_n(p_w)) \leq H_\beta(p) \leq H_\beta(v_n(p_v)) \quad \text{for all } \beta \in (0, \alpha),
\]

(262)

\[
H_\beta(v_n(p_v)) \leq H_\beta(p) \leq H_\beta(w_n(p_w)) \quad \text{for all } \beta \in (\alpha, 1) \cup (1, \infty],
\]

(263)

which are (252) and (253) of Theorem 2, respectively, when \( \alpha \in (0, 1) \).

On the other hand, we now suppose that \( \alpha \in (1, \infty] \). Then, Eq. (245) of Theorem 1 can be written as

\[
\|v_n(p_v)\|_\beta \leq \|p\|_\beta \leq \|w_n(p_w)\|_\beta \quad \text{for all } \beta \in (1, \alpha).
\]  

(264)

Since \( f_\beta(x) \) of (254) is a strictly decreasing function of \( x > 0 \) for every \( \beta \in (1, \infty] \), it follows from (255) that (264) implies the inequalities:

\[
H_\beta(w_n(p_w)) \leq H_\beta(p) \leq H_\beta(v_n(p_v)) \quad \text{for all } \beta \in (1, \alpha).
\]  

(265)

In addition, Eq. (246) of Theorem 1 can be written as

\[
\|w_n(p_w)\|_\beta \leq \|p\|_\beta \leq \|v_n(p_v)\|_\beta \quad \text{for all } \beta \in (0, \alpha) \cup (\alpha, \infty).
\]  

(266)

Since \( f_\beta(x) \) is a strictly increasing function of \( x > 0 \) for every \( \beta \in (0, 1) \), it follows from (255) that (266) implies the inequalities:

\[
H_\beta(w_n(p_w)) \leq H_\beta(p) \leq H_\beta(v_n(p_v)) \quad \text{for all } \beta \in (0, 1);
\]  

(267)
similarly, since $f_\beta(x)$ is a strictly decreasing function of $x > 0$ for every $\beta \in (1, \infty)$, it follows from (255) that (266) implies the inequalities:

$$H_\beta(v_n(p)) \leq H_\beta(p) \leq H_\beta(w_n(p)) \quad \text{for all } \beta \in (\alpha, \infty].$$

(268)

Combining (265), (267), and (268), we have

$$H_\beta(w_n(p_w)) \leq H_\beta(p) \leq H_\beta(v_n(p_v)) \quad \text{for all } \beta \in (0, 1),$$

(269)

and

$$H_\beta(v_n(p_v)) \leq H_\beta(p) \leq H_\beta(w_n(p_w)) \quad \text{for all } \beta \in (0, 1) \cup (\alpha, \infty],$$

(270)

which are (252) and (253) of Theorem 2, respectively, when $\alpha \in (1, \infty]$.

Theorem 2 shows that, for a fixed Rényi entropy of order $\alpha \in (0, \infty]$, the distributions $v_n(\cdot)$ and $w_n(\cdot)$ take the maximum and minimum Rényi entropy of order $\beta \in (0, \alpha)$, respectively, and the distributions $v_n(\cdot)$ and $w_n(\cdot)$ take the minimum and maximum Rényi entropy of order $\beta \in (\alpha, \infty]$, respectively. Therefore, the distributions $v_n(\cdot)$ and $w_n(\cdot)$ have extremal properties in the sense between two Rényi entropies of distinct positive orders. We plot the bounds of Theorem 2 in Fig. 2.

**Remark 1.** We remark that Theorem 2 is a generalization of [21, Corollary 1] since the Rényi entropy of order 1 is the Shannon entropy. Many axiomatic definitions of entropy are closely related to the $\ell_\alpha$-norm of the probability distribution (cf. [21, Table 1]). Therefore, in a similar way to Theorem 2, we can obtain the sharp bounds on several entropies [4], [5], [7], [15], [26] with two distinct orders.

We now consider to extend Theorem 2 from the Rényi entropy to the Rényi divergence [20]. For $p, q \in \Delta_n$ such
that $p \ll q$, i.e., $p$ is absolutely continuous with respect to $q$, the Rényi divergence of order $\alpha$ is defined by

$$D_\alpha(p \parallel q) := \frac{1}{\alpha - 1} \ln \sum_{i=1}^{\infty} p_i^\alpha q_i^{1-\alpha}$$

(271)

for $\alpha \in (0, 1) \cup (1, \infty)$. Moreover, it is also defined that

$$D_1(p \parallel q) := \lim_{\alpha \to 1} D_\alpha(p \parallel q) = \sum_{i=1}^{\infty} p_i \ln \frac{p_i}{q_i},$$

(272)

$$D_\infty(p \parallel q) := \lim_{\alpha \to \infty} D_\alpha(p \parallel q) = \ln \max_{1 \leq i \leq n, q_i > 0} \left(\frac{p_i}{q_i}\right),$$

(273)

where the most right-hand side of (272) is called the relative entropy or the Kullback-Leibler divergence. For the $n$-ary uniform distribution $u_n := (1/n, 1/n, \ldots, 1/n) \in \Delta_n$, since

$$D_\alpha(p \parallel u_n) = \ln n - H_\alpha(p)$$

(274)

for $\alpha \in (0, \infty]$, the Rényi divergence from the uniform distribution $u_n$ is a strictly decreasing function of the Rényi entropy; thus, the following corollary holds from Theorem 2.

**Corollary 1.** For any $n \geq 2$, $p \in \Delta_n$, and $\alpha \in (0, \infty]$, there exist unique values $p_v \in [0, 1/n]$ and $p_w \in [1/n, 1]$ such that

$$D_\alpha(p \parallel u_n) = D_\alpha(v_n(p_v) \parallel u_n) = D_\alpha(w_n(p_w) \parallel u_n),$$

(275)

$$D_\beta(v_n(p_v) \parallel u_n) \leq D_\beta(p \parallel u_n) \leq D_\beta(w_n(p_w) \parallel u_n) \quad \text{for all } \beta \in (0, \alpha),$$

(276)

$$D_\beta(w_n(p_w) \parallel u_n) \leq D_\beta(p \parallel u_n) \leq D_\beta(v_n(p_v) \parallel u_n) \quad \text{for all } \beta \in (\alpha, \infty].$$

(277)

As with Theorem 2, Corollary 1 also shows the sharp bounds of Rényi divergence from the uniform distribution of order $\beta$ with a fixed Rényi divergence from the uniform distribution of another order $\alpha$.

### III. Feasible Regions of Arimoto’s Conditional Rényi Entropies for Two Distinct Orders

In this section, we extend the results of Section II from unconditional settings to conditional settings. Consider a pair of discrete random variables $(X, Y) \sim P_{X|Y} P_Y$ such that $P_Y(y) > 0$ for all $y \in Y$. The conditional Shannon entropy [23] of $X$ given $Y$ is defined by

$$H(X \mid Y) := \sum_{y \in Y} P_Y(y) H(X \mid Y = y)$$

(278)

$$= \sum_{y \in Y} P_Y(y) \left( - \sum_{x \in X} P_{X|Y}(x \mid y) \ln P_{X|Y}(x \mid y) \right),$$

(279)

which is also called the equivocation. When we denote by

$$N_\alpha(X \mid Y) := \sum_{y \in Y} P_Y(y) \left( \sum_{x \in X} P_{X|Y}(x \mid y)^\alpha \right)^{1/\alpha}$$

(280)
the expectation of $\ell_\alpha$-norm for $\alpha \in (0, \infty]$ and $(X,Y) \sim P_X | Y \cdot P_Y$, Arimoto [3] defined the conditional Rényi entropy of order $\alpha$ as

$$H_\alpha(X | Y) := \frac{\alpha}{1-\alpha} \ln N_\alpha(X | Y)$$  \hspace{1cm} (281)

for $\alpha \in (0, 1) \cup (1, \infty)$. As with (5) and (6), it is defined that

$$H_1(X | Y) := \lim_{\alpha \to 1} H_\alpha(X | Y) = H(X | Y),$$ \hspace{1cm} (282)

$$H_\infty(X | Y) := \lim_{\alpha \to \infty} H_\alpha(X | Y) = -\ln N_\infty(X | Y),$$ \hspace{1cm} (283)

where the last equality of (282) is shown in [3, Theorem 2] and [11, Proposition 1], and the last equality of (283) is shown in [11, Proposition 1]. If the cardinality of the finite set is denoted by $|\cdot|$, for the region

$$\mathcal{R}_n^{\text{cond}}(\alpha, \beta) := \left\{ (N_\alpha(X | Y), N_\beta(X | Y)) \mid (X,Y) \in \mathcal{X} \times \mathcal{Y}, |\mathcal{X}| = n, \text{ and } |\mathcal{Y}| \geq 2 \right\},$$ \hspace{1cm} (284)

we present the following theorem.

**Theorem 3.** $\mathcal{R}_n^{\text{cond}}(\alpha, \beta) = \text{Conv} (\mathcal{R}_n(\alpha, \beta))$, where Conv($\mathcal{R}$) denotes the convex hull of the set $\mathcal{R} \subset \mathbb{R}^k$ ($k \in \mathbb{N}$).

*Proof of Theorem 3:* We prove Theorem 3 in a similar way to [22, Theorem 3]. It is clear that if $|\mathcal{X}| = n$, then the point $(N_\alpha(X | Y), N_\beta(X | Y))$ is a convex combination of the points $(\|p_i\|_\alpha, \|p_i\|_\beta)$ for $p_1, p_2, \cdots \in \Delta_n$ (cf. (280)). Therefore, Theorem 3 holds.

It is worth noting that $\mathcal{R}_n(\alpha, \beta)$ is derived from Theorem 1 (cf. (250)). Since the points $(N_\alpha(X | Y), N_\beta(X | Y))$ and $(H_\alpha(X | Y), H_\beta(X | Y))$ are in one-to-one correspondence (cf. (281)) for every distinct $\alpha, \beta \in (0, 1) \cup (1, \infty)$, we obtain the exact feasible regions of two conditional Rényi entropies of distinct orders from Theorem 3, where the exact feasible regions when the order $\alpha$ or $\beta$ is 1 are derived from [22, Theorem 3]. We illustrate an application of Theorem 3 in Fig. 3. In Fig. 3-(b), note that the dotted lines are identical to the boundaries shown in Fig. 2.
IV. UNIFORMLY FOCUSING CHANNELS

We consider a discrete memoryless channel (DMC) as an application of the previous sections in the rest of paper. Let finite sets $\mathcal{X}$ and $\mathcal{Y}$ be the input and output alphabets of a DMC, respectively. Let random variables $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$ be the input and output of a DMC, respectively. The input distribution of a DMC is denoted by $P_X(x)$ for $x \in \mathcal{X}$. In particular, let $U_X$ be the uniform input distribution on $\mathcal{X}$, i.e., $U_X(x) = 1/|\mathcal{X}|$ for all $x \in \mathcal{X}$. Then, the DMC consists of a transition probability distribution $\{P_{Y|X}(y | x) \mid (x, y) \in \mathcal{X} \times \mathcal{Y}\}$. We now define the following three classes of DMCs.

**Definition 1.** A DMC $P_{Y|X}$ is said to be uniformly dispersive [19] or uniform from the input [9] if there exists a permutation $\pi_x : \mathcal{Y} \to \mathcal{Y}$ for each $x \in \mathcal{X}$ such that

$$P_{Y|X}(\pi_x(y) \mid x) = P_{Y|X}(\pi_{x'}(y) \mid x')$$

for all $(x, x', y) \in \mathcal{X}^2 \times \mathcal{Y}$.

**Definition 2.** A DMC $P_{Y|X}$ is said to be uniformly focusing [19] or uniform from the output [9] if there exists a permutation $\pi_y : \mathcal{X} \to \mathcal{X}$ for each $y \in \mathcal{Y}$ such that

$$P_{Y|X}(y \mid \pi_y(x)) = P_{Y|X}(y' \mid \pi_{y'}(x))$$

for all $(x, y, y') \in \mathcal{X} \times \mathcal{Y}^2$.

**Definition 3.** A DMC is said to be strongly symmetric [19] or doubly uniform [9] if it is both uniformly dispersive and uniformly focusing.

If suppose that $\mathcal{X} = \mathcal{Y} = \{0, 1, \ldots, n-1\}$, we define the following two $n$-ary input and output strongly symmetric channels: (i) the strongly symmetric channel $V_{Y|X} : \mathcal{X} \to \mathcal{Y}$ is defined by

$$V_{Y|X}(y \mid x) := \begin{cases} 1 - (n-1)p & \text{if } y = x, \\ p & \text{if } y \neq x \end{cases}$$

for $(x, y) \in \mathcal{X} \times \mathcal{Y}$ and some $p \in [0, 1/n]$, and (ii) the strongly symmetric channel $W_{Y|X} : \mathcal{X} \to \mathcal{Y}$ is defined by

$$W_{Y|X}(y \mid x) := \begin{cases} p & \text{if } y \equiv x + i \pmod{n} \text{ for } 0 \leq i < \lfloor 1/p \rfloor, \\ 1 - \lfloor 1/p \rfloor p & \text{if } y \equiv x + \lfloor 1/p \rfloor \pmod{n}, \\ 0 & \text{otherwise} \end{cases}$$

for $(x, y) \in \mathcal{X} \times \mathcal{Y}$ and some $p \in [1/n, 1]$. It is clear that, for all $x \in \mathcal{X}$, the decreasing orders of the conditional distributions $\{V_{Y|X}(y \mid x) \mid y \in \mathcal{Y}\}$ and $\{W_{Y|X}(y \mid x) \mid y \in \mathcal{Y}\}$ are identical to the distributions $v_n(\cdot)$ and $w_n(\cdot)$, respectively. Note that the channel $V_{Y|X}$ is sometimes called the $n$-ary symmetric channel.

For a DMC $(X, Y) \sim P_X P_{Y|X}$, Arimoto [3] proposed the mutual information of order $\alpha$ as

$$I_\alpha(P_X; P_{Y|X}) := H_\alpha(X) - H_\alpha(X \mid Y)$$

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for \( \alpha \in (0, \infty] \), where note that \( I(P_X; P_{Y|X}) := I_1(P_X; P_{Y|X}) \) is the (ordinary) mutual information. Since
\[
I_\alpha(U_X; P_{Y|X}) = D_\alpha(P_X|Y(\cdot | y) \parallel U_X)
\]
(290) for any uniformly focusing channel \( P_{Y|X} : X \to Y \) and all \( y \in Y \) (cf. [21, Eq. (325)]), we apply Corollary 1 to uniformly focusing channels in Theorem 4.

**Theorem 4.** For any uniformly focusing channel \( P_{Y|X} \) and \( \alpha \in (0, \infty] \), there exist unique channels \( V_{Y|X} \) and \( W_{Y|X} \) such that
\[
I_\alpha(U_X; P_{Y|X}) = I_\alpha(U_X; V_{Y|X}) = I_\alpha(U_X; W_{Y|X}),
\]
(291)
\[
I_\beta(U_X; V_{Y|X}) \leq I_\beta(U_X; P_{Y|X}) \leq I_\beta(U_X; W_{Y|X}) \quad \text{for all } \beta \in (0, \alpha),
\]
(292)
\[
I_\beta(U_X; W_{Y|X}) \leq I_\beta(U_X; P_{Y|X}) \leq I_\beta(U_X; V_{Y|X}) \quad \text{for all } \beta \in (\alpha, \infty].
\]
(293)

**Proof of Theorem 4:** It is easy to see that
\[
\sum_{x \in X} V_{Y|X}(y | x) = \sum_{x \in X} W_{Y|X}(y | x) = 1
\]
(294) for all \( y \in Y \). For a DMC \( (X, Y) \sim U_X V_{Y|X} \), defined in (287), we readily see that
\[
P_{X|Y}(\cdot | y) = \frac{(1/|X|) V_{Y|X}(y | \cdot)}{\sum_{x' \in X} (1/|X|) V_{Y|X}(y | x')} = \frac{V_{Y|X}(y | \cdot)}{\sum_{x' \in X} V_{Y|X}(y | x')}
\]
(295)
\[
= V_{Y|X}(y | \cdot)
\]
(296)
\[
= V_{Y|X}(y | x')
\]
(297)
\[
= V_{Y|X}(y | \cdot)
\]
(298)
for all \( y \in Y \) and some \( p_v \in [0, 1/n] \). Similarly, for a DMC \( (X, Y) \sim U_X W_{Y|X} \), defined in (288), it also follows that
\[
P_{X|Y}(\cdot | y) = W_{Y|X}(y | \cdot)
\]
(299)
\[
= w_n(p_w)
\]
(300)
for all \( y \in Y \) and some \( p_w \in [1/n, 1] \). Hence, Theorem 4 directly follows from Corollary 1 and (290).

In Theorem 4, we note that the input alphabets of channels \( P_{Y|X}, V_{Y|X}, \) and \( W_{Y|X} \) are identical. Therefore, as with \( v_n(\cdot) \) and \( w_n(\cdot) \) for Theorem 2 and Corollary 1, the strongly symmetric channels \( V_{Y|X} \) and \( W_{Y|X} \) have extremal properties in the sense of the mutual information of order \( \alpha \) for uniformly focusing channels.

For a DMC \( P_{Y|X} \), we now consider the \( E_0 \) function
\[
E_0(\rho, P_X, P_{Y|X}) := -\ln \sum_{y \in Y} \left( \sum_{x \in X} P_X(x) P_{Y|X}(y | x) \right)^{1/(1+\rho)}
\]
for a DMC \( P_{Y|X} \). We now consider the \( E_0 \) function
\[
E_0(\rho, P_X, P_{Y|X}) := -\ln \sum_{y \in Y} \left( \sum_{x \in X} P_X(x) P_{Y|X}(y | x) \right)^{1/(1+\rho)}
\]
(301)
for $\rho \in (-1, \infty)$, which is defined by Gallager [12]. The $E_0$ function is used in the random coding exponent [12]  
\[ E_r(R, P_X, P_{Y|X}) := \max_{\rho \in [0,1]} \left\{ E_0(\rho, P_X, P_{Y|X}) - \rho R \right\} \tag{301} \]
for a rate $R \geq 0$, and other error exponents [2], [24]. It is known, e.g., [1, Eq. (6)], that  
\[ E_0(\rho, U_X, P_{Y|X}) = I_{1/(1+\rho)}(U_X; P_{Y|X}) \tag{302} \]
for $\rho \in (-1, 0) \cup (0, \infty)$. Note that the limiting value of the left-hand side of (302) as $\rho \to 0$ is the (ordinary) mutual information $I(U_X; P_{Y|X})$. Namely, the $E_0$ function is closely related to the mutual information of order $\alpha$, and sharp bounds of two distinct $E_0$ functions for uniformly focusing channels can be obtained in a similar manner to Theorem 4. Therefore, we present the sharp bounds of the error exponent with a fixed mutual information of order $\alpha$ in Theorem 5.

**Theorem 5.** For any uniformly focusing channel $P_{Y|X}$ and $\alpha \in (0, 1/2] \cup [1, \infty]$, there exist unique channels $V_{Y|X}$ and $W_{Y|X}$ such that satisfy  
\[ I_\alpha(U_X; P_{Y|X}) = I_\alpha(U_X; V_{Y|X}) = I_\alpha(U_X; W_{Y|X}) \tag{303} \]
and the following: (i) if $\alpha \in (0, 1/2]$, then  
\[ E_r(R, U_X, W_{Y|X}) \leq E_r(R, U_X, P_{Y|X}) \leq E_r(R, U_X, V_{Y|X}) \tag{304} \]
for all $R \geq 0$, and (ii) if $\alpha \in [1, \infty]$, then  
\[ E_r(R, U_X, V_{Y|X}) \leq E_r(R, U_X, P_{Y|X}) \leq E_r(R, U_X, W_{Y|X}) \tag{305} \]
for all $R \geq 0$.

**Proof of Theorem 5:** Let $P_{Y|X}$ be a uniformly focusing channel. For a fixed $\alpha \in (0, \infty)$, assume that  
\[ I_\alpha(U_X; P_{Y|X}) = I_\alpha(U_X; V_{Y|X}) = I_\alpha(U_X; W_{Y|X}). \tag{306} \]
Then, it follows from Theorem 4 that  
\[ I_\beta(U_X; V_{Y|X}) - R \leq I_\beta(U_X; P_{Y|X}) - R \leq I_\beta(U_X; W_{Y|X}) - R \quad \text{for } \beta \in (0, \alpha), \tag{307} \]
\[ I_\beta(U_X; W_{Y|X}) - R \leq I_\beta(U_X; P_{Y|X}) - R \leq I_\beta(U_X; V_{Y|X}) - R \quad \text{for } \beta \in (\alpha, \infty] \tag{308} \]
for all $R \geq 0$. By change the variable as  
\[ \beta = \frac{1}{1 + \rho} \iff \rho = \frac{1 - \beta}{\beta}, \tag{309} \]
\[1\]In the paper, we omit the maximizing over the input distribution $P_X$.  

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we rewrite (307) and (308) as

\[
I_{1/(1+\rho)}(U_X; V_Y|X) - R \leq I_{1/(1+\rho)}(U_X; P_Y|X) - R \leq I_{1/(1+\rho)}(U_X; W_Y|X) - R \quad \text{for } \rho \in \left(\frac{1-\alpha}{\alpha}, \infty\right),
\]

(310)

\[
I_{1/(1+\rho)}(U_X; W_Y|X) - R \leq I_{1/(1+\rho)}(U_X; P_Y|X) - R \leq I_{1/(1+\rho)}(U_X; V_Y|X) - R \quad \text{for } \rho \in \left(-1, \frac{1-\alpha}{\alpha}\right)
\]

(311)

for all \( R \geq 0 \). Then, since

\[
E_0(\rho, U_X, P_Y|X) - \rho R = \rho I_{1/(1+\rho)}(U_X; P_Y|X) - \rho R
\]

(312)

\[
= \rho \left(I_{1/(1+\rho)}(U_X; P_Y|X) - R\right),
\]

(313)

it follows from (310) and (311) that

\[
E_0(\rho, U_X, V_Y|X) - \rho R \leq E_0(\rho, U_X, P_Y|X) - \rho R \leq E_0(\rho, U_X, W_Y|X) - \rho R
\]

(314)

for \( \rho \in (-1, \min\{0, (1-\alpha)/\alpha\}\} \cup \{\max\{0, (1-\alpha)/\alpha\}, \infty\} \) and \( R \geq 0 \), and

\[
E_0(\rho, U_X, W_Y|X) - \rho R \leq E_0(\rho, U_X, P_Y|X) - \rho R \leq E_0(\rho, U_X, V_Y|X) - \rho R
\]

(315)

for \( \rho \in \{\min\{0, (1-\alpha)/\alpha\}\}, \max\{0, (1-\alpha)/\alpha\}\} \) and \( R \geq 0 \). Note that

\[
E_\epsilon(U_X, P_Y|X, R) = \max_{\rho \in [0,1]} \left\{ E_0(\rho, U_X, P_Y|X) - \rho R \right\}.
\]

(316)

By dividing the range of the order \( \alpha \in (0, \infty) \), we consider (314) and (315) as follows:

**Case (i):** \( 0 < \alpha \leq 1/2 \)

If \( \alpha \in (0, 1/2] \), then \( (1-\alpha)/\alpha \geq 1 \); hence, it follows from (315) that

\[
E_0(\rho, U_X, W_Y|X) - \rho R \leq E_0(\rho, U_X, P_Y|X) - \rho R \leq E_0(\rho, U_X, V_Y|X) - \rho R
\]

(317)

for all \( \rho \in [0, 1] \) and \( R \geq 0 \). Therefore, we get

\[
E_\epsilon(U_X, W_Y|X, R) \leq E_\epsilon(U_X, P_Y|X, R) \leq E_\epsilon(U_X, V_Y|X, R)
\]

(318)

for all \( R \geq 0 \).

**Case (ii):** \( 1 \leq \alpha \leq \infty \)

If \( \alpha \in [1, \infty) \), then \( \lim_{x \to \alpha}(1-x)/x \leq 0 \); hence, it follows from (314) that

\[
E_0(\rho, U_X, V_Y|X) - \rho R \leq E_0(\rho, U_X, P_Y|X) - \rho R \leq E_0(\rho, U_X, W_Y|X) - \rho R
\]

(319)

for all \( \rho \in [0, 1] \) and \( R \geq 0 \). Therefore, we get

\[
E_\epsilon(U_X, V_Y|X, R) \leq E_\epsilon(U_X, P_Y|X, R) \leq E_\epsilon(U_X, W_Y|X, R)
\]

(320)

for all \( R \geq 0 \).
Theorem 5 shows that, among all n-ary input uniformly focusing channels with a fixed mutual information of order $\alpha \in (0, 1/2] \cup [1, \infty)$ under the uniform input distribution, the strongly symmetric channels $V_{Y|X}$ and $W_{Y|X}$ take the extremal random coding exponents. We illustrate the sharp bounds of $E_r(U_X, P_{Y|X})$ for uniformly focusing channels $P_{Y|X}$ in Fig. 4.

Finally, note that the uniform input distribution maximizes the mutual information of order $\alpha$ if a channel is strongly symmetric (cf. [3, Eq. (20)] and [12, p. 145]).

**V. Conclusion**

In this paper, we examined the sharp bounds between the $\ell_\alpha$-norm and the $\ell_\beta$-norm of n-dimensional probability vectors for distinct $\alpha, \beta \in (0, 1) \cup (1, \infty)$, as shown in Theorem 1. By using the result, Theorem 2 established the sharp bounds between two Rényi entropies of distinct positive orders. In Remark 1, we mentioned that sharp bounds on other axiomatic definitions of entropy [4], [5], [7], [15], [26] can be obtained by using Theorem 1, as with Theorem 2. In Section III, we considered to extend the above results from unconditional settings to conditional settings. Then, Theorem 3 identified the exact feasible regions between two expectations of $\ell_\alpha$- and $\ell_\beta$-norms, which implies the exact feasible regions between two conditional Rényi entropies of distinct orders (cf. Fig. 3). Finally, Section IV examined the sharp bounds on channel reliability functions, such as the mutual information of order $\alpha$ and the $E_0$ function, for uniformly focusing channels under the uniform input distribution. Then, Theorem 5 provided the sharp bounds on the random coding exponent of uniformly focusing channels under the uniform input distribution with a fixed mutual information of order $\alpha$. Finally, we remark that the sharp bounds of error exponents, such as the strong converse [2] and the sphere packing [24] exponents, of uniformly focusing channels under the uniform input distribution with a fixed mutual information of order $\alpha$ are also obtained from Theorem 4, as with Theorem 5.
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