ON THE ANTICANONICAL SHEAF OF A SPHERICAL HOMOGENEOUS SPACE

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Abstract. Let $G/H$ be a spherical homogeneous space and $U \subseteq G/H$ the open orbit of a Borel subgroup. We consider the $G$-linearized anticanonical sheaf $\omega_{G/H}$. Up to a constant factor, there is a natural choice of a global section $s \in \Gamma(G/H, \omega_{G/H})$ whose restriction generates $\Gamma(U, \omega_{G/H})$. Our purpose is to give two different characterizations of this section. Furthermore, this is exactly the section considered by Brion, and computing its divisor is a way to establish the equivalence of the definitions of the types of colors due to Luna and Knop.

Introduction

Throughout the paper, we work with algebraic varieties and algebraic groups over the field of complex numbers $\mathbb{C}$.

We first recall some facts about spherical varieties. Let $G$ be a connected reductive group and $B \subseteq G$ a Borel subgroup. A closed subgroup $H \subseteq G$ is called spherical if $B$ acts on $G/H$ with an open orbit $U$. In this case $G/H$ is called a spherical homogeneous space. A $G$-equivariant open embedding $G/H \hookrightarrow X$ into a normal irreducible $G$-variety $X$ is called a spherical embedding and $X$ is called a spherical variety.

An equivalent characterization for a homogeneous space to be spherical is that the $G$-module $\Gamma(G/H, \mathcal{L})$ is multiplicity-free for every $G$-linearized invertible sheaf $\mathcal{L}$, i.e. the multiplicity of any simple $G$-module in the decomposition of the module of global sections is at most 1. It is known that $U$ is isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$. Since $U$ is affine, its complement is of pure codimension 1, and the irreducible components of the complement to the open $B$-orbit $U$ in $G/H$ are $B$-invariant prime divisors $D_1, \ldots, D_k$, which are called the colors of $G/H$.

In this paper, the main object of interest is the anticanonical sheaf $\omega_{G/H}$, i.e. the top exterior power of the tangent sheaf $\mathcal{L}$, i.e. the $G$-module $\Gamma(G/H, \mathcal{L})$ is multiplicity-free for every $G$-linearized invertible sheaf $\mathcal{L}$, i.e. the multiplicity of any simple $G$-module in the decomposition of the module of global sections is at most 1. It is known that $U$ is isomorphic to $(\mathbb{C}^*)^r \times \mathbb{C}^*$. Since $U$ is affine, its complement is of pure codimension 1, and the irreducible components of the complement to the open $B$-orbit $U$ in $G/H$ are $B$-invariant prime divisors $D_1, \ldots, D_k$, which are called the colors of $G/H$.

In this paper, the main object of interest is the anticanonical sheaf $\omega_{G/H}$, i.e. the top exterior power of the tangent sheaf, of a spherical homogeneous space $G/H$. It is equipped with a natural $G$-linearization (see Section 1).

Since $U$ has trivial divisor class group, the invertible sheaf $\omega_{G/H}$ is trivial on $U$, and we might ask whether there is a natural choice of a generator $s \in \Gamma(U, \omega_{G/H})$. Uniqueness will always be understood to be up to a constant factor.

We first recall the well-known case where $G = B = T$ is an algebraic torus and $H$ is trivial. A spherical embedding $G/H \hookrightarrow X$ is then simply a toric variety $X$ with embedded torus $U = G/H \cong T$, and it is possible to show that there is a unique generator $s \in \Gamma(U, \omega_{G/H})$ which is $T$-invariant. This generator can be explicitly written as

$$s = x_1 \frac{\partial}{\partial x_1} \wedge \ldots \wedge x_n \frac{\partial}{\partial x_n}$$

where $x_1, \ldots, x_n$ is a choice of coordinates for the algebraic torus $T$. In addition to being $T$-invariant, this section has the important property that it has a zero of order 1 along every $T$-invariant divisor in $X$.

In general, however, there need not be any $B$-invariant section $s \in \Gamma(U, \omega_{G/H})$ as the following example shows.
Example. Let $G = SL_2(\mathbb{C})$ and $H = B$ a Borel subgroup. Then the open $B$-orbit $U$ in $G/H$ is isomorphic to $\mathbb{C}$. It is not difficult to see that there is only one $B$-semi-invariant section in $\Gamma(U, \omega_{G/B})$ and that it is not $B$-invariant.

As we can not expect a $B$-invariant section, we have to look for something else. By the multiplicity-freeness of spherical homogeneous spaces, we obtain

$$\Gamma(G/H, \omega_{G/H}) \cong \bigoplus V_\chi,$$

where $\chi$ runs over pairwise different dominant weights of $B$ and $V_\chi$ is the simple $G$-module of highest weight $\chi$.

This decomposition is very simple when $Q \subseteq G$ is a parabolic subgroup since then $\Gamma(G/Q, \omega_{G/Q})$ is a simple $G$-module by the Borel-Weil-Bott theorem (see [Dem68] or [Dem73]). In particular, exactly one dominant weight occurs (compare this with the example above), and there is a unique choice of a $B$-semi-invariant section $s \in \Gamma(G/Q, \omega_{G/Q})$ which restricts to a generator $s \in \Gamma(U, \omega_{G/H})$.

In general, one associates a natural parabolic subgroup $P \subseteq G$ to any spherical homogeneous space, namely the stabilizer of the open $B$-orbit $U$.

When $H$ contains a maximal unipotent subgroup of $G$, the homogeneous space $G/H$ is called horospherical, and the normalizer of $H$ in $G$ is a parabolic subgroup conjugated to the opposite parabolic of $P$, which we denote by $P^-$. Hence there is a natural morphism $\pi : G/P \to G/P^-$, which is known to be a torus fibration. Therefore $\pi^*(\omega_{G/P^-}) = \omega_{G/H}$, the simple $G$-module $\Gamma(G/P^-, \omega_{G/P^-})$ is a direct summand of $\Gamma(G/H, \omega_{G/H})$, and a unique $B$-semi-invariant section $s \in \Gamma(G/P^-, \omega_{G/P^-}) \subseteq \Gamma(G/H, \omega_{G/H})$ exists, whose weight we denote by $\kappa_P$.

In general, there is no natural morphism $\pi : G/H \to G/P^-$, but the following statement is nevertheless valid.

Theorem A. Let $G/H$ be a spherical homogeneous space, $U \subseteq G/H$ the open $B$-orbit, and $P$ the stabilizer of $U$. Then the simple $G$-module $\Gamma(G/P^-, \omega_{G/P^-})$ is a direct summand of $\Gamma(G/H, \omega_{G/H})$. Equivalently, there exists a $B$-semi-invariant section

$$s \in \Gamma(G/H, \omega_{G/H})$$

of weight $\kappa_P$, which restricts to a generator $s \in \Gamma(U, \omega_{G/H})$.

Another characterization of $s$ is the following.

Theorem B. Let $G/H$ be a spherical homogeneous space, and $U \subseteq G/H$ the open $B$-orbit. Then the restriction of the $B$-semi-invariant section $s$ of Theorem A to $U$ generates $\Gamma(U, \omega_{G/H})$, and this generator is uniquely determined by the following property: For any spherical embedding $G/H \hookrightarrow X$ it has a zero of order 1 along any $G$-invariant prime divisor. Note that we do not assume that the generator can be extended to a global section on $G/H$.

We will see that $s$ is the section considered by Brion in [Bri97, Proposition 4.1]. To complete the picture, we compute $\text{div} \ s$ on an arbitrary spherical embedding $G/H \hookrightarrow X$. Since $s$ is a generator of the anticanonical sheaf on the open $B$-orbit, its divisor is a linear combination of divisors in the boundary of $U$, i.e. of the colors and the $G$-invariant divisors $X_1, \ldots, X_n$. By Theorem B, we have

$$\text{div} \ s = \sum_{i=1}^k m_i D_i + \sum_{j=1}^n X_j.$$

In order to compute the coefficients $m_i$, we need to divide the colors into types. This is done, e.g. in [Lan01], but we propose a different approach depending on [Kno95], and which can sometimes be easier to apply in practice (see the examples in the
The pullback of differential forms with respect to the action morphism \( \varphi \) yields a morphism \( G \), i.e. the diagram in Figure 1 commutes.

**Definition.** We say that a color is

- of type \( a \) if \( \Phi_{\alpha}(H_\alpha) \) is a maximal torus,
- of type \( 2a \) if \( \Phi_{\alpha}(H_\alpha) \) is the normalizer of a maximal torus,
- of type \( b \) if \( \Phi_{\alpha}(H_\alpha) \) contains a maximal unipotent subgroup.

By computing \( \text{div} s \), we will show that this definition is in agreement with the definition of Luna. This implies the following explicit formulae for the coefficients \( m_i \) in the expression for \( \text{div} s \) due to Brion and Luna ([Br97] [Lun97]).

**Proposition.** We have

\[
    m_i = \begin{cases} 
        \frac{1}{2} \langle \alpha^\vee, \kappa_P \rangle = 1 & \text{for } D_i \text{ of type } a \text{ or } 2a, \\
        \langle \alpha^\vee, \kappa_P \rangle & \text{for } D_i \text{ of type } b.
    \end{cases}
\]

As a byproduct, we obtain that Brion’s description of the anticanonical sheaf

\[
    \mathcal{Z}_X = \mathcal{O}(D_1)^{\otimes m_1} \otimes \ldots \otimes \mathcal{O}(D_k)^{\otimes m_k} \otimes \mathcal{O}(X_1) \otimes \ldots \otimes \mathcal{O}(X_l)
\]

of an arbitrary spherical variety \( X \) is not only valid inside \( \text{Pic}(X) \), but even inside the group of isomorphism classes of linearized invertible sheaves \( \text{Pic}^G(X) \), when the invertible sheaves \( \mathcal{O}_X(X_i) \) and \( \mathcal{O}_X(D_i) \) are equipped with a canonical \( G \)-linearization (see Section 4).

This work is organized in 5 sections. In Section 1, we recall the notion of a \( G \)-linearized quasicoherent sheaf on a \( G \)-variety and the natural \( G \)-linearization of its cotangent and tangent sheaves. In Section 2, we prove Theorems A and B. In Section 3, we recall the different definitions for the types of colors and state some related observations. In Section 4, we then determine the coefficients \( m_i \) and show that the two definitions are equivalent. Finally, we explicitly compute the types of colors and the coefficients \( m_i \) for some examples in Section 5.

## 1. The co- and tangent sheaves of a \( G \)-variety

In this section, let \( X \) be a \( G \)-variety for an algebraic group \( G \). Then \( G \) acts on \( X \) by an action morphism \( \alpha : G \times X \to X \). We denote by \( \mu : G \times G \to G \) the multiplication morphism of the algebraic group \( G \). Let us repeat the definition of a \( G \)-linearization of a quasicoherent sheaf (see [Tim11] Definition C.2] or [MFK94] Definition 1.6)).

**Definition 1.1.** A \( G \)-linearization of a quasicoherent sheaf \( \mathcal{F} \) on \( X \) is an isomorphism of quasicoherent sheaves \( \tilde{\alpha} : \pi_X^* \mathcal{F} \to \alpha^* \mathcal{F} \) satisfying the cocycle condition, i.e. the diagram in Figure 1 commutes.

Recall that the cotangent sheaf \( \Omega_X \) of \( X \) is, locally on affine open neighbourhoods \( U \), given as the sheaf associated to the module of Kähler differentials \( \Omega_{\mathcal{O}_X(U)/\mathcal{O}_U} \). The pullback of differential forms with respect to the action morphism \( \alpha \) yields a \( G \)-linearization of the cotangent sheaf, namely \( \tilde{\alpha} : \alpha^* \Omega_X \to \pi_X^* \Omega_X \). Making the
The restriction of \( \lambda_4 \) is given by 

\[ (\pi_X \circ \pi_{G \times X})^* \delta \xrightarrow{\pi_{G \times X}^* \hat{\alpha}} (\alpha \circ \pi_{G \times X})^* \delta \]

\[ (\pi_X \circ (\text{id}_G \times \alpha))^* \delta \xrightarrow{(\text{id}_{G} \times \alpha)^* \hat{\alpha}} (\alpha \circ (\text{id}_G \times \alpha))^* \delta \]

\[ (\pi_X \circ (\mu \times \text{id}_X))^* \delta \xrightarrow{(\mu \times \text{id}_X)^* \hat{\alpha}} (\alpha \circ (\mu \times \text{id}_X))^* \delta \]

**Figure 1.** The cocycle condition.

Further assumption that \( X \) is smooth, we may dualize \( \hat{\alpha} \) and obtain a \( G \)-linearization of the tangent sheaf \( \mathcal{T}_X := \text{Hom}_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X) \), namely \( \hat{\alpha}^\vee : \pi_X^* \mathcal{T}_X \to \alpha^* \mathcal{T}_X \).

For an affine open subset \( U \subseteq X \) and \( g \in G \), let us denote the coordinate rings of \( U \) and \( g \cdot U \) by \( A \) and \( B \) respectively. The element \( g \in G \) acts on a local section \( \delta \in \text{Der}_{C}(A, A) = \Gamma(U, \mathcal{T}_X) \) by restricting the \( G \)-linearization to \( \{ g \} \times X \), i.e. \( g \cdot \delta := \hat{\alpha}^\vee |_{\{ g \} \times X}(\delta) \). It is straightforward to check that

\[ g \cdot \delta = \lambda_g^\# \circ \delta \circ \lambda_g^{-1} \in \text{Der}_{C}(B, B) = \Gamma(g \cdot U, \mathcal{T}_X), \]

where \( \lambda_g : X \to X \) is given by \( x \mapsto g^{-1} \cdot x \).

Let \( G = (C, +) \) or \( G = (C^*, \cdot) \) be a one-dimensional connected algebraic group with neutral element \( e \in G \). We will recall how one can associate a global vector field \( u \in \Gamma(X, \mathcal{T}_X) \) to a one-parameter subgroup \( u : G \to G \). The coordinate ring of \( G \) is either the polynomial ring \( C[t] \) or the Laurent polynomial ring \( C[t^{\pm 1}] \). In both cases, we have a natural choice of a basis of the tangent space of \( G \) over the point \( e \), namely \( \frac{\partial}{\partial t^\ell} |_e \in \mathcal{T}_G|_e \). Let \( U \) be an affine open subset of \( X \) with coordinate ring \( A \). The restriction of \( u \) to \( U \) lies in \( \text{Der}_{C}(A, A) \) and is given by

\[ (u|_U f)(x) := \frac{\partial}{\partial t^\ell} |_e f(u(t) \cdot x) \]

for \( f \in A \), \( x \in U \). It is straightforward to check that these local sections are well-defined and glue to a global section \( u \in \Gamma(X, \mathcal{T}_X) \).

**Remark 1.2.** Assume that \( X = G \) and \( G \) acts on itself by left translation. It is then straightforward to check that \( u \) is an invariant vector field, i.e. \( \rho^g \circ u \circ \rho_{g^{-1}}^g = u \) for all \( g \in G \), where \( \rho_g : G \to G \) is given by \( h \mapsto hg. \) Moreover, \( u|_e \in \text{Lie}_u(G) \).

**2. Proof of Theorems A and B**

Let \( G/H \hookrightarrow X \) be a spherical embedding. We denote the \( G \)-invariant prime divisors in \( X \) by \( X_1, \ldots, X_n \). We may assume a basepoint \( x_0 \) with stabilizer \( H \) inside the open \( B \)-orbit \( U \). As we are interested in the anticanonical sheaf of \( X \), we may assume that \( X \) does not contain \( G \)-orbits of codimension two or greater. In particular, \( X \) is smooth and \( \text{Pic}(X) = \text{Cl}(X) \).

Let \( T \subseteq B \) a maximal torus, \( R \subseteq X(T) \) the associated root system, and \( S \subseteq R \) the set of simple roots corresponding to \( B \). We denote by \( S^p \) the set of simple roots such that the corresponding parabolic subgroup \( P_{S^p} = P \subseteq G \) is the stabilizer of the open \( B \)-orbit.

Since \( X \) is a smooth toroidal variety, it has the following local structure (see [Tim11: Theorem 29.1]): The set \( X^0 := X \setminus \bigcup_{i=1}^k D_i \) is stable by \( P \). There exists a
Levi subgroup $L$ of $P$ and a closed $L$-stable subvariety $Z$ of $X^0$ such that

$$R_u(P) \times Z \to X^0$$

$$(u, z) \mapsto u \cdot z$$

is a $P$-equivariant isomorphism. The kernel of the $L$-action on $Z$, which we denote by $L_0$, contains $(L, L)$ and $Z$ is a toric embedding of $L/L_0$. Every $G$-orbit intersects $Z$ in a unique $L/L_0$-orbit.

Under the isomorphism above, we let $x_0$ correspond to $(u_0, z_0) \in R_u(P) \times Z$. By appropriately choosing $x_0$, we may assume that $u_0$ is the neutral element of $R_u(P)$.

We denote by $\tilde{\omega}_X := \bigwedge \dim X \mathcal{F}_X$, i.e. the top exterior power of the tangent sheaf, the anticanonical sheaf of $X$. It is an invertible sheaf and carries a natural $G$-linearization induced from the $G$-linearization of $\mathcal{F}_X$.

We recapitulate some observations of Brion (see [Bri97, Section 4]). The variety $Z$ is a toric variety for a quotient torus of $T$. Let $T_0$ be the kernel of the $T$-action on $Z$ and let $T_1$ be a subtorus of $T$ with $T = T_0 T_1$ such that $T_0 \cap T_1$ is finite. We have a commutative diagram of equivariant morphisms with respect to the action of $B_0 := R_u(P) T_1$:

$$\begin{array}{ccc}
B_0 & \to & B \cdot x_0 \\
\downarrow & & \downarrow \\
B_0 / (T_0 \cap T_1) & \to & B_0 / (T_0 \cap T_1)
\end{array}$$

The arrows are finite coverings. In particular, the tangent space of $X$ at $(u_0, z_0)$ is isomorphic to the direct sum of the tangent spaces of $R_u(P)$ and $T_1$ at the corresponding neutral elements. This is the Lie algebra of $B_0$ which decomposes as

$$\text{Lie } B_0 = \text{Lie } T_1 \oplus \bigoplus_{\alpha \in R^+ \backslash \{S^P\}} \mathfrak{g}_\alpha$$

where $(S^P)$ denotes the root system generated by $S^P$ and $\mathfrak{g}_\alpha$ denotes the subspace of $T$-semi-invariant vectors of weight $\alpha$ in $\mathfrak{g} := \text{Lie } G$. Choose a realization $(u_\alpha)_{\alpha \in R}$ of the root system $R$ (see [Spr09, §8.1]) and choose a basis $\lambda_1, \ldots, \lambda_r$ of the lattice of one-parameter multiplicative subgroups of $T_1$. In Section 1 we have seen how to associate a global vector field $u_\alpha$ (resp. $l_1, \ldots, l_r$) to the one parameter subgroup $u_\alpha$ (resp. $\lambda_1, \ldots, \lambda_r$). We obtain a global section

$$s := \left( \bigwedge_{\alpha \in R^+ \backslash \{S^P\}} u_\alpha \right) \wedge l_1 \wedge \ldots \wedge l_r \in \Gamma(X, \tilde{\omega}_X).$$

**Proposition 2.1** (see [Bri97, Proposition 4.1]). The zero set of $s$ is exactly the union of the colors $D_i$ and the boundary divisors $X_j$.

**Proof.** Let $u$ be a one-parameter subgroup used in the definition of $s$, i.e. $u = u_\alpha$ or $u \in \{\lambda_1, \ldots, \lambda_r\}$. Let $Y$ be a $B$-stable divisor, i.e. $Y = D_i$ or $Y = X_j$. Since $u$ maps into $B$, by construction, $u|_y \in \mathcal{F}_Y|_y$ for all $y \in Y$. Since $Y$ has codimension 1 in $X$ and the number of vector fields which have been wedged to obtain $s$ is $\dim X$, the global section $s$ vanishes on $Y$.

We now show that $s$ vanishes nowhere on the open $B$-orbit. Since $u$ maps into $B_0$, we may define a global vector field $u'$ on $B_0$. By the local structure theorem, $B_0$ is a finite covering of $B \cdot x_0$. In particular, we have a well-defined pushforward of vector fields, and $u|_{B \cdot x_0}$ is the pushforward of $u'$. Since $u'$ is $T_0 \cap T_1$-invariant (see Remark 1.2), it suffices to show that $s' \in \Gamma(B_0, \tilde{\omega}_{B_0})$, i.e. the global section of the anticanonical sheaf of $B_0$ which arises by wedging all the global vector fields $u'$, vanishes nowhere.
Since \( w'|_e \in \text{Lie } u(G) \), it follows that \( s'|_e \) arises by wedging a basis of \( \text{Lie } B_0 \). In particular, \( s' \) does not vanish at \( e \). Since \( s' \) is invariant, it follows that it vanishes nowhere. \( \square \)

Recall from the introduction that, by the Borel-Weil-Bott theorem, the space \( \Gamma(G/P^-, \tilde{\omega}_{G/P^-}) \) is a simple \( G \)-module, whose highest weight we denote by \( \kappa_p \).

**Proposition 2.2.** The global section \( s \in \Gamma(X, \tilde{\omega}_X) \) is \( B \)-semi-invariant of weight \( \kappa_p \).

**Proof.** Since \( U \) has trivial Picard group, every \( B \)-linearized invertible sheaf on \( U \) is \( B \)-equivariantly isomorphic to \( \mathcal{O}_U(\chi) \) for some \( \chi \in \mathcal{X}(B) \). We have maps \( \tilde{\omega}_U \to \mathcal{O}_U(\chi) \to \mathcal{O}_U \), where the first map is chosen to be a \( B \)-equivariant isomorphism, and the second map is canonical, but not necessarily \( B \)-equivariant. Let \( f \) be the image of \( s \) under the composed map. By [KKV89, Proposition 1.3, (ii)], the regular function \( f \) is \( B \)-semi-invariant. It follows that \( s \) is \( B \)-semi-invariant as well, where the weight has to be corrected by the twist \( \chi \).

Next, we determine the weight of \( s \). Let \( w \in T \), \( f \in \Gamma(U, \mathcal{O}_U) \), and \( x \in U \). Then we have

\[
\begin{align*}
((w \cdot u_\alpha)(f))(x) &= \left( \left( \lambda^*_w \circ u_\alpha \circ \lambda^*_{w^{-1}} \right)(f) \right)(x) \\
&= \left. \frac{\partial}{\partial t} \right|_{t=0} f(wu(t)w^{-1} \cdot x) \\
&= \left. \frac{\partial}{\partial t} \right|_{t=0} f(u(\alpha(w)t) \cdot x) \\
&= \left( \left( \alpha(w)u_\alpha \right)(f) \right)(x).
\end{align*}
\]

Hence \( u_\alpha \) is \( T \)-semi-invariant of weight \( \alpha \). Analogously, one can show that \( l_i \) is \( T \)-invariant. In particular, the weight of \( s \) depends only on the set \( S^p \). As \( G/P^+ \) and \( X \) have the same stabilizer of the open \( B \)-orbit, it follows that \( s \) is \( B \)-semi-invariant of weight \( \kappa_p \). \( \square \)

**Remark 2.3.** Observe that Proposition 2.2 implies Theorem A, and additionally shows that the section \( s \in \Gamma(G/H, \tilde{\omega}_{G/H}) \) from Theorem A can be extended to a global section on \( X \) for every spherical embedding \( G/H \hookrightarrow X \).

**Remark 2.4.** A nonzero \( B \)-semi-invariant rational section of \( \tilde{\omega}_X \) is uniquely determined by its weight \( \chi \in \mathcal{X}(B) \) up to a constant factor, and such a rational section exists if and only if \( \chi \in \kappa_p + \mathcal{M} \).

**Corollary 2.5** (see [Bri97 Proposition 4.1]). We have

\[
\text{div } s = \sum_{i=1}^k m_i D_i + \sum_{j=1}^n X_j
\]

with \( m_i \in \mathbb{Z}_{>0} \).

**Proof.** Since \( s \) is a \( B \)-semi-invariant section, it follows that its divisor is a linear combination of the \( B \)-invariant divisors of \( X \), i.e.

\[
\text{div } s = \sum_{i=1}^k m_i D_i + \sum_{j=1}^n r_j X_j
\]

for integers \( m_i \) and \( r_j \). Since \( s \) vanishes on every \( B \)-invariant divisor, it follows that the \( m_i \) and \( r_j \) are positive. To show that \( r_j = 1 \), we consider the restriction \( s' := s|_{X^0} \). Above, we have seen that the open subset \( X^0 \) is isomorphic to the product variety \( R_\alpha(P) \times Z \). In particular, \( \tilde{\omega}_{X^0} \cong \pi_1^* \tilde{\omega}_{R_\alpha(P)} \otimes \pi_2^* \tilde{\omega}_Z \) where \( \pi_1 \) denotes the
projection onto the \(i\)-th factor. The section \(s'\) behaves well under this product decomposition. Indeed, set \(s_1 := \bigwedge_{j \in R^+ \setminus \{S^p\}} u_j | R_u(P)\) and \(s_2 := t_1 | Z \wedge \ldots \wedge t_1 | Z\). Then under the isomorphism above the section \(s'\) corresponds to the section \(s_1 \otimes s_2\). Since \(s_1\) does not vanish on \(R_u(P)\), we obtain

\[
\sum_{j=1}^n r_j X_j = (\text{div} s)|_{\mathfrak{X}_0} = \text{div} s' = \text{div}(s_1 \otimes s_2) = \pi_2^* \text{div} s_2.
\]

Now, \(Z\) is a toric variety with respect to the quotient torus \(T_1/(T_0 \cap T_1)\), and the pullback under \(\pi_2\) of its torus-invariant divisors are exactly the \(G\)-invariant divisors \(X_j\) of \(X\). Since \(\sigma_2\) is invariant under the action of the torus \(T_1/(T_0 \cap T_1)\), it follows, by toric geometry, that \(r_j = 1\).

We denote by \(\mathcal{M} \subseteq \mathfrak{X}(B)\) the weight lattice of \(B\)-semi-invariants in the function field \(\mathbb{C}(G/H)\) and by \(\mathcal{N} := \text{Hom}(\mathcal{M}, Z)\) the dual lattice. We define a map \(\iota: \mathcal{V} \to \mathcal{N}\) from the set \(\mathcal{V}\) of \(G\)-invariant discrete valuations on \(\mathbb{C}(G/H)\) to \(\mathcal{N}\) by \(\langle \iota(\nu), \chi \rangle := \nu(f_\chi)\) where \(f_\chi \in \mathbb{C}(G/H)\) is \(B\)-semi-invariant of weight \(\chi \in \mathcal{M}\). This is possible since \(f_\chi\) is unique up to a constant factor. The map \(\iota\) is actually injective, so we may consider \(\mathcal{V}\) as a subset of the vector space \(\mathcal{N} \otimes \mathbb{Z} Q\). It is known that \(\mathcal{V}\) is a cosimplicial cone (see [Bri90]), called the valuation cone of \(G/H\). In particular, the valuation cone is full-dimensional.

**Proof of Theorem B.** By Corollary [2.5], it follows that the \(B\)-semi-invariant section \(s\) of Theorem A has a zero of order 1 along any \(G\)-invariant prime divisor of any spherical embedding \(G/H \hookrightarrow X\).

Now assume that \(s'\) is a generator of \(\Gamma(U, \omega_{G/H})\) which has a zero of order 1 along any \(G\)-invariant prime divisor of any spherical embedding \(G/H \hookrightarrow X\). Since the open \(B\)-orbit \(U\) of \(G/H\) has trivial divisor class group, we have maps \(\omega_U \to O_U(\chi) \to O_U\), where the first map is chosen to be a \(B\)-equivariant isomorphism, and the second map is canonical, but not necessarily \(B\)-equivariant. Under this isomorphism \(s\) and \(s'\) correspond to functions \(f, f' \in \Gamma(U, O_{G/H})\). Since \(s\) and \(s'\) are generators, the functions \(f\) and \(f'\) are invertible. In particular, \(f/f' \in \Gamma(U, O_{G/H}^\ast)\). By [KKV89] Proposition 1.3, \(\langle f/f', \nu_1 \rangle = 0\). It follows that the valuation cone \(\mathcal{V}\) of \(G/H\) is contained in the subspace \(\{w \in \mathcal{N}_Q : \langle f/f', w \rangle = 0\}\). Since \(\mathcal{V}\) is a full-dimensional cone, this is only possible if the \(B\)-weight of \(f/f'\) is 0, i.e. \(f\) and \(f'\) coincide up to a scalar multiple.

3. **Types of colors**

We first recall the usual definition of the types of colors of a spherical variety. We denote by \(\Delta := \{D_1, \ldots, D_k\}\) the set of colors and define the map \(\rho: \Delta \to \mathcal{N}\) by \(\langle \rho(D), \chi \rangle := \nu_D(f_\chi)\) where \(f_\chi \in \mathbb{C}(G/H)\) is \(B\)-semi-invariant of weight \(\chi \in \mathcal{M}\). For \(\alpha \in S\) we denote by \(P_\alpha \subseteq G\) the corresponding minimal standard parabolic subgroup containing \(B\), and define

\[
\Delta(\alpha) := \{D_1 \in \Delta : P_\alpha \cdot D_1 \neq D_1\}.
\]

This means that \(S^p\) is the set of simple roots \(\alpha \in S\) such that \(\Delta(\alpha) = \emptyset\).

The primitive generators in \(\mathcal{M}\) of the extremal rays of the negative of the dual of the valuation cone \(\mathcal{V}\) are called the spherical roots of \(G/H\).
The usual way (first introduced in [Lum01], see also [Tim11 Section 30.10]) to define the type of a color \( \Delta_\alpha \) is now as follows: If \( \alpha \) is a spherical root, we say that \( D_\alpha \) is of type \( a \). If \( 2\alpha \) is a spherical root, we say that \( D_\alpha \) is of type \( 2a \). Otherwise, we say that \( D_\alpha \) is of type \( b \).

**Example 3.1.** As an illustration, let \( T \subseteq SL(2) \) be a maximal torus, \( N \subseteq SL(2) \) its normalizer, and \( U \subseteq SL(2) \) a maximal unipotent subgroup. Then the homogeneous space \( SL(2)/T \) contains two colors of type \( a \), the homogeneous space \( SL(2)/N \) contains one color of type \( 2a \), and the homogeneous space \( SL(2)/U \) contains one color of type \( b \).

We will, however, give and use another definition of the types of colors. We first repeat some observations of Knop (see [Kno95 Section 3]). We denote the stabilizer of the basepoint \( x_0 \) inside \( P_\alpha \) by \( H_\alpha \). For \( \alpha \in S \) we have \( P_\alpha /B \cong \mathbb{P}^1 \). The action of \( P_\alpha \) induces a surjective morphism of algebraic groups

\[
\Phi_\alpha : P_\alpha \to Aut(P_\alpha /B ) \cong PGL(2).
\]

Since \( P_\alpha \cdot x_0 \) contains the open \( B \)-orbit, the minimum of the codimensions of the \( B \)-orbits which are contained in \( P_\alpha \cdot x_0 \) is 0. By [Kno95 Lemma 3.1], we have \( \dim \Phi_\alpha (H_\alpha) \geq 1 \), i.e. \( \Phi_\alpha (H_\alpha) \) is a spherical subgroup of \( PGL(2) \).

**Remark 3.2.** The following results are given in [Kno95 Lemma 3.2]: If \( \Phi_\alpha (H_\alpha) = PGL(2) \), then \( \alpha' (M) = 0 \) and \( P_\alpha \cdot x_0 = B \cdot x_0 \). The other possibilities are that \( \Phi_\alpha (H_\alpha) \) is a maximal torus (in which case there are three \( B \)-orbits in \( P_\alpha \cdot x_0 \)), that \( \Phi_\alpha (H_\alpha) \) is the normalizer of a maximal torus (in which case there are two \( B \)-orbits in \( P_\alpha \cdot x_0 \)), and that \( \Phi_\alpha (H_\alpha) \neq PGL(2) \) contains a maximal unipotent subgroup (in which case there are again two \( B \)-orbits in \( P_\alpha \cdot x_0 \)).

**Proposition 3.3.** The orbit \( P_\alpha \cdot x_0 \) is open and meets exactly the colors in \( \Delta(\alpha) \). In particular, \( |\Delta(\alpha)| \leq 2 \).

**Proof.** As orbits are open in their closures, it follows that \( P_\alpha \cdot x_0 \) is open. It is clear that \( P_\alpha \cdot x_0 \) can only meet colors in \( \Delta(\alpha) \). On the other hand, if \( D_\alpha \in \Delta \) does not meet \( P_\alpha \cdot x_0 \), then it is a connected component of the complement to \( P_\alpha \cdot x_0 \) in \( G/H \) and therefore \( P_\alpha \cdot x_0 \), since \( P_\alpha \) is connected.

By Remark 3.2 there are at most three \( B \)-orbits in \( P_\alpha \cdot x_0 \). Since the open \( B \)-orbit is contained in \( P_\alpha \cdot x_0 \), it follows that \( |\Delta(\alpha)| \leq 2 \). \( \square \)

**Definition 3.4.** We say that a color \( D_\alpha \in \Delta(\alpha) \) is

- of type \( a \) if \( \Phi_\alpha (H_\alpha) \) is a maximal torus,
- of type \( 2a \) if \( \Phi_\alpha (H_\alpha) \) is the normalizer of a maximal torus,
- of type \( b \) if \( \Phi_\alpha (H_\alpha) \) contains a maximal unipotent subgroup.

**Remark 3.5.** We denote the sets containing the colors of type \( a \), \( 2a \) and \( b \) by \( \Delta^a \), \( \Delta^{2a} \) and \( \Delta^b \). We have \( \Delta = \Delta^a \cup \Delta^{2a} \cup \Delta^b \). Indeed, if there were a color \( D_\alpha \) invariant by every \( P_\alpha \) \( (\alpha \in S) \), then it would be \( G \)-invariant. It is not yet clear that this union is disjoint and that Definition 3.4 is equivalent to the usual one. We will prove this in Theorem 4.1.

**Proposition 3.6.** The pullback of a color of \( PGL(2)/\Phi_\alpha (H_\alpha) \) with respect to \( \Phi_\alpha (B) \) under the natural map

\[
\Phi_\alpha : P_\alpha \cdot x_0 \cong P_\alpha /H_\alpha \to PGL(2)/\Phi_\alpha (H_\alpha)
\]

is a prime divisor whose closure is an element of \( \Delta(\alpha) \), and pulling back induces a bijection between the colors in \( \Delta(\alpha) \) and the colors of \( PGL(2)/\Phi(H_\alpha) \).
Proposition 3.9. Let $D \in \Delta(\alpha)$ be of type $a$ or $2a$. Then we have $\langle \alpha^\vee, \kappa_P \rangle = 2$.

Proof. Since $\rho_S$ is equal to the sum of the fundamental dominant weights, we have $\langle \alpha^\vee, \kappa_P \rangle = \langle \alpha^\vee, 2\rho_S - 2\rho_{SP} \rangle = 2 - \langle \alpha^\vee, 2\rho_{SP} \rangle$. It remains to verify $\langle \alpha^\vee, \rho_{SP} \rangle = 0$. 

Remark 3.8. For $I \subseteq S$ we denote by $\rho_I$ the half-sum of the positive roots in the root system generated by $I$. By Proposition 2.2 we have $\kappa_P = 2\rho_S - 2\rho_{SP}$. It is also known that $\rho_I$ is the sum of the fundamental dominant weights of that root system (see [Hum78] Lemma 13.3A).
It suffices to show that $\langle \alpha', \beta \rangle = 0$ for all simple roots $\beta$ in $S^p$. By Remark 3.2, every simple root in $S^p$ is orthogonal to $M$. Thus the result follows by verifying that $\alpha \in M$. The map $\Phi_\alpha$ from Proposition 3.6 induces an inclusion of lattices $M' \hookrightarrow M$, where $M'$ is the weight lattice of $\Phi_\alpha(B)$-semi-invariants in $C(\mathrm{PGL}(2)/\Phi_\alpha(H_\alpha))$. In the case of type $a$ or $2a$, i.e. when $\Phi_\alpha(H_\alpha)$ is a maximal torus or its normalizer, it is not difficult to check that a multiple of the simple root $\alpha'$ of $\mathrm{PGL}(2)$ with respect to $\Phi_\alpha(T)$ and $\Phi_\alpha(B)$ lies in $M'$. Since $\Phi_\alpha^*(\alpha') = \alpha$, it follows that a multiple of $\alpha$ lies in $M$. □

4. Computation of the coefficients

The following construction has already been introduced in [Gag12, Section 3]. We refer the reader to this paper for details.

The $f_i \in C[G]$ are right $H$-semi-invariant. We denote by $\eta_i \in \chi(H)$ the right $H$-weight of $f_i$. We define

$$G = G \times (C^*)^k, \quad B = B \times (C^*)^k, \quad C = C \times (C^*)^k,$$

and denote by $(\epsilon_i)_{1 \leq i \leq k}$ the standard basis of the character lattice of $(C^*)^k$. We also define the subgroup

$$\mathcal{H} := \left\{ (h, \eta_1(h), \ldots, \eta_k(h)) : h \in H \right\} \subseteq G.$$

For the spherical homogeneous space $G/\mathcal{H}$ we define $\Xi, \Xi' \subseteq \chi(B)$, $\varphi : \Xi \to \Xi'$, and $\kappa_p$ as in the previous sections. The characters $\kappa_p$ and $\kappa_p^\prime$ coincide as characters of the semisimple part $G_1$. We will hence identify these two characters. We will denote by $\overline{s}$ the global section of $\omega_{G/\mathcal{H}}$ as constructed in Section 2.

Remark 4.1 (See [Gag12, Section 3]). We have $\Pic(G/\mathcal{H}) = 0$ and $\overline{\Xi} = \overline{\Xi}' \oplus \chi(\overline{\mathcal{B}})^\mathcal{H}$, where $\overline{\Xi}'$ is the sublattice of $\chi(\overline{\mathcal{B}})$ with basis $(\chi_1 + \epsilon_i)_{1 \leq i \leq k}$. The group $\overline{G}$ also acts on $G/H$ via the natural projection $\pi : \overline{G} \to G$. Dividing $\overline{G}/\mathcal{H}$ by $(C^*)^k$ yields the natural projection map

$$\pi : \overline{G}/\mathcal{H} \to G/H,$$

which is $\overline{G}$-equivariant and a good geometric quotient. It can also be interpreted as associated fiber bundle

$$\pi : G \times_H \prod_{i=1}^k C^*_{\eta_i} \to G/H,$$

which is the direct sum of $C^*$-bundles. The colors of $\overline{G}/\mathcal{H}$ are given by the equations $f_i \otimes \epsilon_i^{-1} \in C[\overline{G}/\mathcal{H}]$ for $1 \leq i \leq k$. In particular, the pullback of Cartier divisors under $\pi$ induces a bijection between $\Delta$ and $\overline{\Xi}$. We will denote the pullback of $D_1 \in \Delta$ by $\overline{D}_i$.

Proposition 4.2. We have $\pi^*(\omega_{G/H}) = \omega_{G/\mathcal{H}} = \Pic(\overline{G}/\mathcal{H})$ and $\pi^*(s) = s$, where we keep in mind that the two sections are only determined up to a constant factor.

Proof. By Remark 4.1 $\pi$ is $\overline{G}$-equivariant, and it is easy to see that $\overline{\pi}$ is a smooth morphism of smooth varieties. We obtain an equivariant version of the relative cotangent exact sequence

$$0 \to \pi^*(\Omega_{G/H}) \to \Omega_{G/\mathcal{H}} \to \Omega_{\overline{\pi}} \to 0.$$

It follows that $\det\pi^*(\Omega_{G/H}) \otimes \det\Omega_{\overline{\pi}} = \det\Omega_{G/\mathcal{H}}$, i.e. $\pi^*(\omega_{G/H}) \otimes \det\Omega_{\overline{\pi}} = \omega_{G/\mathcal{H}}$. Therefore it remains to show that $\det\Omega_{\overline{\pi}} = 0 \in \Pic(\overline{G}/\mathcal{H})$. 

We can cover $G/H$ by open orbits $U$ of different Borel subgroups of $G$. Since orbits of a Borel subgroup have trivial Picard group, the restriction
\[ \pi|_{U} : \pi^{-1}(U) \cong U \times (C^*)^k \to U \]
is trivial. As the central torus $C$ is contained in every Borel subgroup of $G$, the open subvarieties $U$ are $C$-invariant.

On any $U \times (C^*)^k$ we denote by $(t_1, \ldots, t_k)$ the natural coordinates of $(C^*)^k$ and consider the section $\bigwedge_{i=1}^{k} \frac{dt_i}{t_i}$ of $\det \Omega_{\pi}$. By Remark 4.1 we have transition functions $g_i \in C[U \cap U']^*$ of $\pi$ to any other open orbit $U'$ of a Borel subgroup. We obtain
\[ \bigwedge_{i=1}^{k} \frac{dt_i}{t_i} = \bigwedge_{i=1}^{k} \frac{dg_i t'_i}{g_i t'_i} = \bigwedge_{i=1}^{k} \frac{dt'_i}{t'_i} \]
where $(t'_1, \ldots, t'_k)$ are the natural coordinates of $(C^*)^k$ over $U$. It follows that $\text{div} \bigwedge_{i=1}^{k} \frac{dt_i}{t_i} = 0$. As $\overline{G}$ acts trivially on this section, we obtain $\det \Omega_{\pi} = 0 \in \text{Pic}(\overline{G}/H)$. Since $\text{Pic}(\overline{G}/H) \to \text{Pic}(\overline{G}/H)$ is injective, this concludes the proof of the first statement.

Finally, it follows from the $G$-equivariance of $\pi$ that $\pi^*(s)$ is $B$-semi-invariant of weight $\kappa_P$.

As in Corollary 2.5 we denote by $m_i \in \mathbb{Z}_{>0}$ the order of $D_i$ in $\text{div } s$ and by $\overline{m}_i \in \mathbb{Z}_{>0}$ the order of $\overline{D}_i$ in $\text{div } \overline{s}$. By Proposition 4.2,
\[ \bigwedge_{i=1}^{k} m_i D_i = \text{div } \overline{s} = \text{div}(\pi^*(s)) = \pi^*(\text{div } s) = \sum_{i=1}^{k} m_i \pi^*(D_i), \]
where the divisors are restricted to the corresponding homogeneous spaces. Hence it suffices to determine the coefficients $\overline{m}_i$.

**Proposition 4.3.** There exists a character $\vartheta \in \mathcal{X}(\overline{G})$ such that $\xi - \vartheta \in \overline{M}$, and we have $\overline{m}_i = \langle \overline{\pi}(D_i), \kappa_P - \vartheta \rangle$ for $1 \leq i \leq k$.

**Proof.** Since $\text{Pic}(\overline{G}/H) = 0$, we have maps $\overline{\pi}_*: \mathcal{O}(\overline{G}/H) \to \mathcal{O}(\overline{G}/H)$ for a character $\vartheta \in \mathcal{X}(\overline{G})$, where the first map is chosen to be a $\overline{G}$-equivariant isomorphism, and the second map is canonical, but not necessarily $\overline{G}$-equivariant. The image of the global section $\pi$ under the above map corresponds to a $B$-semi-invariant regular function $f_{\kappa_P - \vartheta}$ of weight $\kappa_P - \vartheta$. Hence we have $\overline{m}_i = \nu_{\overline{\pi}}(f_{\kappa_P - \vartheta}) = \langle \overline{\pi}(D_i), \kappa_P - \vartheta \rangle$. \hfill \Box

**Corollary 4.4.** We have
\[ \kappa_P - \vartheta = \chi + \sum_{i=1}^{k} \overline{m}_i (\chi_i + \varepsilon_i) \]
for some $\chi \in \mathcal{X}(\overline{G})$. In particular,
\[ \kappa_P = \sum_{i=1}^{k} \overline{m}_i \chi_i \quad \text{and} \quad -\vartheta = \chi + \sum_{i=1}^{k} \overline{m}_i \varepsilon_i. \]

**Proof.** Since $\overline{M} = \overline{M'} \oplus \mathcal{X}(\overline{G})^+$ where $\overline{M'}$ has basis $(\chi_i + \varepsilon_i)_{1 \leq i \leq k}$, we can write $\kappa_P - \vartheta = \chi + \sum_{i=1}^{k} a_i (\chi_i + \varepsilon_i)$ for some $\chi \in \mathcal{X}(\overline{G})$ and $a_i \in \mathbb{Z}$. Since, by Remark 4.1 the colors of $\overline{G}/H$ are given by the equations $f_i \otimes \varepsilon^{-1}_i$, we have
\[ \langle \overline{\pi}(D_i), \chi_{i_1} + \varepsilon_{i_2} \rangle = \begin{cases} 1 & \text{if } i_1 = i_2, \\ 0 & \text{otherwise}. \end{cases} \]
Since $\chi \in \mathcal{X}(\overline{G}/\overline{H})$ is a non-vanishing regular function on $\overline{G}/\overline{H}$, we have $\langle \pi(T_i), \chi \rangle = 0$ for $1 \leq i \leq k$. By Proposition $4.3$ the result follows.

**Proposition 4.5.** The equation

$$\kappa_P = \sum_{i=1}^{k} m_i \chi_i$$

has a unique solution provided that the integers $m_i$ are positive. It is given by $m_1 = \langle \alpha^\vee, \kappa_P \rangle$ for $D_1$ of type $b$ and $m_1 = \frac{1}{2} \langle \alpha^\vee, \kappa_P \rangle = 1$ for $D_1$ of type $a$ or $2a$.

**Proof.** Let $\alpha \in S\setminus S^p$. We apply Proposition $3.7$. If $D_1 \in \Delta(\alpha)$ is of type $b$, we obtain $\langle \alpha^\vee, \kappa_P \rangle = m_i$. If $D \in \Delta(\alpha)$ is of type $2a$, we obtain, by Proposition $3.9$, $2 = \langle \alpha^\vee, \kappa_P \rangle = 2m_i$. Finally, if $D_1, D_2 \in \Delta(\alpha)$ are of type $a$, we obtain $2 = \langle \alpha^\vee, \kappa_P \rangle = m_{i_1} + m_{i_2}$ by Proposition $3.9$. Therefore the only possible solution in positive integers is $m_{i_1} = m_{i_2} = 1$.

**Theorem 4.6.** The union $\Delta = \Delta^a \cup \Delta^{2a} \cup \Delta^b$ is disjoint, and Definition $3.4$ is equivalent to the usual definition for the types of colors.

**Proof.** By Proposition $3.7$, $\Delta^{2a}$ is disjoint from the union $\Delta^a \cup \Delta^b$. By Proposition $4.5$, the two sets $\Delta^a$ and $\Delta^b$ are disjoint. It follows that every color is of exactly one type.

The equivalent of Proposition $3.7$ for the usual definition of the types of colors (see [Bos98, Tim11, Lemma 30.24]) proves the equivalence for colors of type $2a$. The proof is concluded by observing that we have $|\Delta(\alpha)| = 2$ exactly for colors of type $a$ under both definitions.

We conclude by explaining the last remark from the introduction. There exist (unique) $G$-linearizations of the invertible sheaves $\mathcal{O}_X(D)$ and $\mathcal{O}_X(X_i)$ such that their canonical sections are $C$-invariant (see [Bri07, 4.1]). The equality

$$\mathcal{O}_X = \mathcal{O}(D_1)^{\otimes m_1} \otimes \ldots \otimes \mathcal{O}(D_k)^{\otimes m_k} \otimes \mathcal{O}(X_1) \otimes \ldots \otimes \mathcal{O}(X_t)$$

inside $\text{Pic}^G(X)$ then follows from the fact that both $G$-sheaves admit a $C$-invariant rational (in our case even global) section with the same divisor of zeroes and poles.

### 5. Examples

In this section, we compute the coefficients $m_i$ for several well-known spherical homogeneous spaces. In 5.1, 5.2, and 5.3 we give examples for every type of color. Example 5.4 illustrates $S^p \neq \emptyset$ and $m_i > 2$. Observe that in 5.1, 5.3, and 5.4 the computation is simplified by applying Definition $3.4$ for the types of colors.

**Example 5.1.** Consider $G := \text{SL}(n) \times \text{SL}(n)$ and $H := \text{SL}(n) \subseteq G$ embedded diagonally. Then $G/H$ is isomorphic to $\text{SL}(n)$ where $G$ acts with the first factor from the left and with the second factor from the right after inverting. We denote by $D(n) \subseteq \text{SL}(n)$ the subgroup of diagonal matrices, by $T(n) \subseteq \text{SL}(n)$ the subgroup of upper triangular matrices, and by $T(n)^- \subseteq \text{SL}(n)$ the subgroup of lower triangular matrices. We define $B := T(n)^- \times T(n)$ and $T := D(n) \times D(n)$, and obtain the set of simple roots $S = \{\alpha_1, \ldots, \alpha_{n-1}, \beta_1, \ldots, \beta_{n-1}\}$.

The homogeneous space $G/H$ is spherical, and there are $n-1$ colors of $G/H$, i.e. $\Delta = \{D_1, \ldots, D_{n-1}\}$, given by $D_i = V(f_i)$ where $f_i \in \mathcal{C}[\text{SL}(n)]$ is the upper-left principal minor of size $i \times i$. It is not difficult to see that $\Delta(\alpha_i) = \Delta(\beta_i) = \{D_i\}$.

Furthermore, all colors are of type $b$ since for every $\alpha_i$ the image of $H \cap P_{\alpha_i}$ under $\Phi_{\alpha_i} : P_{\alpha_i} \rightarrow \text{PGL}(2)$ is a Borel subgroup of PGL(2) and therefore contains a maximal unipotent subgroup.
The stabilizer of the open $B$-orbit in $G/H$ is $B$ itself. Therefore we have $S^p = \emptyset$ and $\kappa_p = 2\rho_S$. Since $\rho_S$ is equal to the sum of the fundamental dominant weights, we obtain $m_i = \langle \alpha_i^\vee, 2\rho_S \rangle = 2$ for $1 \leq i \leq n - 1$.

**Example 5.2.** Consider $G := \text{SL}(2) \times \text{SL}(2) \times \text{SL}(2)$ and $H := \text{SL}(2) \subseteq G$ embedded diagonally. We denote by $B \subseteq G$ the Borel subgroup of lower triangular matrices and by $T \subseteq B$ the maximal torus of diagonal matrices. We obtain the set of simple roots $\{\alpha, \beta, \gamma\}$ corresponding to the three factors in $G$.

The homogeneous space $G/H$ is spherical, and there are 3 colors of $G/H$, i.e. $\Delta = \{D_{12}, D_{13}, D_{23}\}$. To be more precise, let $A_{ij}$ for $1 \leq i < j \leq 3$ be the $2 \times 2$ matrix whose rows are given by the first rows of the $i$-th and $j$-th factor of $G$. Then $\det A_{ij}$ is an equation for $D_{ij}$. It is not difficult to see that $\Delta(\alpha) = \{D_{12}, D_{13}\}$, $\Delta(\beta) = \{D_{12}, D_{23}\}$, and $\Delta(\gamma) = \{D_{13}, D_{23}\}$. In particular, it follows that all colors are of type $\alpha$, and therefore $m_{ij} = 1$ for $1 \leq i < j \leq 3$.

By the usual definition for the types of colors (see Section 3), it follows that $\alpha, \beta, \gamma$ are contained in $\mathcal{M}$, and the valuation cone is given by

$$\mathcal{V} = \{v \in \mathcal{N}_Q : \langle v, \alpha \rangle \leq 0, \langle v, \beta \rangle \leq 0, \langle v, \gamma \rangle \leq 0\}.$$

**Example 5.3.** Consider $G := \text{SL}(n)$ for $n \geq 3$ and $H := \text{SO}(n)$, the subgroup of orthogonal matrices. Let $G$ act on the space $\text{Sym}(n)$ of symmetric $n \times n$ matrices via $A \cdot M = AMAT^T$. Then the stabilizer of the identity matrix is $H$. We denote by $B \subseteq G$ the Borel subgroup of lower triangular matrices and by $T \subseteq B$ the maximal torus of diagonal matrices. We obtain the set of simple roots $\{\alpha_1, \ldots, \alpha_{n-1}\}$.

The homogeneous space $G/H$ is spherical, and there are $n - 1$ colors of $G/H$, i.e. $\Delta = \{D_{1i}, \ldots, D_{n-1}\}$, given by $D_i = \mathcal{V}(f_i)$ where $f_i \in \mathbb{C}[\text{SL}(n)]$ is again the upper-left principal minor of size $i \times i$. It is not difficult to see that $\Delta(\alpha_i) = \{D_i\}$. Furthermore, all colors are of type $2\alpha$ since for every $\alpha_i$ the image of $H \cap P_{\alpha_i}$ under $\Phi_{\alpha_i} : P_{\alpha_i} \rightarrow \text{PGL}(2)$ is the normalizer of a maximal torus. It follows that $m_i = 1$ for $1 \leq i \leq n - 1$.

As in the previous example, it follows that $2\alpha_i$ is contained in $\mathcal{M}$ for $1 \leq i \leq n - 1$, and the valuation cone is given by

$$\mathcal{V} = \{v \in \mathcal{N}_Q : \langle v, 2\alpha_i \rangle \leq 0 \text{ for every } 1 \leq i \leq n - 1\}.$$

**Example 5.4.** Consider $G := \text{SL}(n)$ for $n \geq 3$ and $H := \text{SL}(n - 1)$ embedded as the block diagonal matrices with entries on the lower-right of $\text{SL}(n)$. Let $B \subseteq G$ be the Borel subgroup of upper triangular matrices and $T \subseteq G$ the subgroup of diagonal matrices. We obtain the set of simple roots $S = \{\alpha_1, \ldots, \alpha_{n-1}\}$.

Let $G$ act on $\mathbb{C}^n \times \mathbb{C}^n$ by acting naturally on the first factor and with the contragredient action on the second factor. Denoting the coordinates of the first factor by $X_1, \ldots, X_n$ and the coordinates of the second factor by $Y_1, \ldots, Y_n$, we obtain

$$G/H \cong \mathcal{V}(X_1Y_1 + \ldots + X_nY_n - 1) \subseteq \mathbb{C}^n \times \mathbb{C}^n.$$
Acknowledgments

We would like to thank our teacher Victor Batyrev for encouragement and highly useful advice, as well as Jürgen Hausen for several useful discussions. We are also grateful to Dmitry Timashev for elaborating on Section 30.4 of his book.

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