On homomorphism spaces of metrizable groups

Gábor Lukács†

March 29, 2022

Abstract

For two not necessarily commutative topological groups \( G \) and \( K \), let \( \mathcal{H}(G, K) \) denote the space of all continuous homomorphisms from \( G \) to \( K \) with the compact-open topology. We prove that if \( G \) is metrizable and \( K \) is compact then \( \mathcal{H}(G, K) \) is a \( k \)-space. As a consequence we obtain that if \( D \) is a dense subgroup of \( G \) then \( \mathcal{H}(D, K) \) is homeomorphic to \( \mathcal{H}(G, K) \), and if \( G \) is separable \( h \)-complete, then the natural map \( G \to \mathcal{C}(\mathcal{H}(G, K), K) \) is open onto its image.

The aim of the present paper is to generalize the result of Chasco [2] that for every abelian metrizable group \( G \), its dual group \( \hat{G} \) (i.e. the group of homomorphisms into the unit circle, \( \mathbb{T} \)) is a \( k \)-space under the compact-open topology. We prove that the space of homomorphisms \( \mathcal{H}(G, K) \) is a \( k \)-space whenever \( G \) is a (not necessarily commutative) metrizable topological group and \( K \) is a compact topological group which satisfies assumptions that we call “radical-based” below.

Definition 1. A topological group \( K \) is \textit{radical-based}, if it has a countable base \( \{\Lambda_n\} \) at \( e \), such that each \( \Lambda_n \) is symmetric, and for all \( n \in \mathbb{N} \):

1. \( (\Lambda_n)^n \subset \Lambda_1 \);
2. \( a^1, a^2 \ldots, a^n \in \Lambda_1 \) implies \( a \in \Lambda_n \).

Any topological subgroup \( K \) of the unitary group of a \( C^* \)-algebra is radical-based: one can define \( \Lambda_n = \{u \in K \mid \|u - e\| < \varepsilon_n\} \) for a suitably chosen sequence \( \{\varepsilon_n\} \).

Recall that a Hausdorff topological space \( X \) is called a \textit{k-space} if \( F \subset X \) is closed if and only if \( F \cap C \) is closed for every closed compact subset \( C \) of \( X \).

Theorem 1. For a metrizable topological group \( G \) and a radical-based compact group \( K \), \( \mathcal{H}(G, K) \) is a \( k \)-space.

\footnote{2000 Mathematics Subject Classification 22A05}
\footnote{I gratefully acknowledge the financial support received from York University and Ontario Graduate Scholarship Program that enabled me to do this research.}
In order to prove Theorem \ref{thm:main} we will need the two results below. To shorten notations, for \( \alpha \in \mathcal{H}(G, K) \) we put \( S_\alpha(A, B) = S(A, B) \cap \mathcal{H}(G, K) \) where \( S(A, B) = \{ \gamma \mid \gamma(A) \subset B \} \) \( (A \subset G \text{ and } B \subset K) \).

**Lemma 1.** If \( K \) is radical-based then its base \( \{ \Lambda_n \} \) at \( e \) satisfies:

(a) \( \Lambda_{2k}\Lambda_{2k} \subset \Lambda_k \) for all \( k \in \mathbb{N} \);

(b) \( \Lambda_{2k} \subset \Lambda_k \) for all \( k \in \mathbb{N} \). \( \Box \)

**Lemma 2.** Suppose that \( G \) is metrizable and \( K \) is compact and radical-based.

Let \( \alpha \in \mathcal{H}(G, K) \) and \( U \) be a neighborhood of \( e \) in \( G \). Then \( S_\alpha(U, \Lambda_2) \) is precompact, i.e. \( \overline{S_\alpha(U, \Lambda_2)} \) is compact.

**Proof of Theorem \ref{thm:main}**. Let \( \Phi \subset \mathcal{H}(G, K) \) be a set such that for any compact subset \( \Xi \) of \( \mathcal{H}(G, K) \), \( \Phi \cap \Xi \) is closed. We have to prove that \( \Phi \) is closed. To that end let \( \zeta \in \mathcal{H}(G, K) \) such that \( \zeta \notin \Phi \). It suffices to find a compact subset \( C \) of \( G \) and \( l \geq 2 \) such that \( S_\zeta(C, \Lambda_{2l}) \cap \Phi = \emptyset \).

The group \( G \) is first countable, so let \( \{ U_n \}_{n=1}^\infty \) be a base at \( e \). We may assume that \( U_n \) is decreasing. Set \( U_0 = G \). We are going to find \( l \geq 2 \) and construct inductively a family \( \{ F_n \}_{n=0}^\infty \) of finite subsets of \( G \) such that for all \( n \geq 0 \)

\[
\begin{align*}
(1) & \quad F_n \subset U_n, \\
(2) & \quad \bigcap_{k=1}^n S_\zeta(F_k, \Lambda_{2l}) \cap \overline{S_\zeta(U_{n+1}, \Lambda_2)} \cap \Phi = \emptyset.
\end{align*}
\]

First we have to construct \( F_0 \). By Lemma \ref{lem:compact-open} \( \overline{S_\zeta(U_1, \Lambda_2)} \) is compact, thus by the assumption \( \overline{S_\zeta(U_1, \Lambda_2)} \cap \Phi \) is closed. On compact subsets of \( \mathcal{H}(G, K) \) the compact-open topology coincides with the topology of pointwise convergence. But \( \zeta \notin \overline{S_\zeta(U_1, \Lambda_2)} \) \( \cap \Phi \), so there exists a neighborhood of \( \zeta \) in the pointwise topology which is disjoint from \( \overline{S_\zeta(U_1, \Lambda_2)} \cap \Phi \). It is clear that sets of the form \( S_\zeta(F, \Lambda_1) \) where \( F \subset G \) is finite form a base at \( \zeta \) for the pointwise topology on \( \mathcal{H}(G, K) \). So there exists \( F_0 \) such that

\[
S_\zeta(F_0, \Lambda_1) \cap \overline{S_\zeta(U_1, \Lambda_2)} \cap \Phi = \emptyset. \tag{1}
\]

(Without loss of generality we may assume \( l \geq 2 \).) By Lemma \ref{lem:compact-open} \( \Lambda_{2l} \subseteq \Lambda_l \), thus \( S_\zeta(F_0, \Lambda_{2l}) \subset S_\zeta(F_0, \Lambda_l) \). In particular:

\[
S_\zeta(F_0, \Lambda_{2l}) \cap \overline{S_\zeta(U_1, \Lambda_2)} \cap \Phi = \emptyset. \tag{2}
\]

Suppose that we have already constructed \( F_0, \ldots, F_{n-1} \) such that (1) and (2) hold. For all \( x \in U_n \) we define

\[
\Delta_x = \bigcap_{k=0}^{n-1} S_\zeta(F_k, \Lambda_{2l}) \cap S_\zeta(\{ x \}, \Lambda_{2l}) \cap \overline{S_\zeta(U_{n+1}, \Lambda_2)} \cap \Phi. \tag{3}
\]
Notice, that the sets $\Delta_x$ are closed, because each $S_\xi(F_k, \bar{\Lambda}_{2l})$ is closed even in the pointwise topology. But then
\[
\bigcap_{x \in U_n} \Delta_x = \bigcap_{k=0}^{n-1} S_\xi(F_k, \bar{\Lambda}_{2l}) \cap S_\xi(U_n, \bar{\Lambda}_{2l}) \cap S_\xi(U_{n+1}, \Lambda_2) \cap \Phi. \tag{4}
\]
Since $S_\xi(U_n, \bar{\Lambda}_{2l}) \subset S_\xi(U_n, \Lambda_2)$, this means (using assumption (2)) that
\[
\bigcap_{x \in U_n} \Delta_x \subset \bigcap_{k=0}^{n-1} S_\xi(F_k, \bar{\Lambda}_{2l}) \cap S_\xi(U_n, \Lambda_2) \cap \Phi = \emptyset. \tag{5}
\]
$\Delta_x$ are closed subsets of $\overline{S_\xi(U_{n+1}, \Lambda_2)}$, which is compact by Lemma 2. Therefore, there must be a finite set $F_n \subset U_n$ such that $\bigcap_{x \in F_n} \Delta_x = \emptyset$, in other words:
\[
\bigcap_{k=0}^{n-1} S_\xi(F_k, \bar{\Lambda}_{2l}) \cap S_\xi(F_n, \bar{\Lambda}_{2l}) \cap S_\xi(U_{n+1}, \Lambda_2) \cap \Phi = \emptyset, \tag{6}
\]
as desired.

Let $C = \bigcup_{n=0}^{\infty} F_n \cup \{e\}$. We have $F_n \subset U_n$, so $C$ is a set of elements converging to $e$. Thus $C$ is sequentially compact, but since $G$ is metrizable, it means that $C$ is compact. It is clear that $S_\xi(C, \bar{\Lambda}_{2l}) \cap S_\xi(U_n, \Lambda_2) \cap \Phi = \emptyset$. Since $H(G, K) = \bigcup_{n=0}^{\infty} S_\xi(U_n, \Lambda_2)$, this means that $S_\xi(C, \bar{\Lambda}_{2l}) \cap \Phi = \emptyset$. Therefore $S_\xi(C, \Lambda_2) \cap \Phi = \emptyset$. $\square$

A topological space $X$ is hemicompact if $X$ is the countable union of compact subspaces $X_n$, such that every compact subset of $X$ is contained in a finite union of the sets $X_n$.

**Corollary 1.** For a metrizable topological group $G$ and a radical-based compact group $K$, $\mathcal{E}(H(G, K), K)$ is completely metrizable.

**Proof.** Once metrizability has been shown the completeness is obvious, because $H(G, K)$ is a $k$-space, and $K$ is complete (because it is compact). Since in [11] it was shown that if $X$ is hemicompact then $\mathcal{E}(X, K)$ is metrizable, it suffices to show that $H(G, K)$ is hemicompact.

Let $\Xi$ be a compact subset of $H(G, K)$. By the Ascoli Theorem $\Xi$ is equicontinuous, in particular there exists a neighborhood $U$ of $e$ such that $\xi(U) \subset \Lambda_2$ for all $\xi \in \Xi$. In other words, $\Xi \subset S_\xi(U, \Lambda_2)$. Let $\{U_n\}$ be a base at $e \in G$. For some $n \in \mathbb{N}$, $U_n \subset U$, thus $\Xi \subset S_\xi(U_n, \Lambda_2) \subset S_\xi(U_n, \Lambda_2)$. By Lemma 2 $\overline{S_\xi(U_n, \Lambda_2)}$ is compact, and clearly $H(G, K) = \bigcup_{n=1}^{\infty} \overline{S_\xi(U_n, \Lambda_2)}$, hence $H(G, K)$ is hemicompact, as desired. $\square$

**Definition 2.** $G$ is $h$-complete if for any continuous homomorphism $f : G \to H$ the subgroup $f(G)$ is closed in $H$. 

3
The Corollary below generalizes a theorem by Chasco [2] stating that for a separable metrizable complete abelian group $G$, the natural map $G \to \hat{G}$ is an isomorphism of topological groups if and only if it is bijective.

**Corollary 2.** Suppose that $G$ is separable metrizable and $h$-complete, and suppose further that $K$ is compact and radical-based. Then, the natural map $N : G \to C(\mathcal{H}(G, K), K)$ is continuous and open onto the image. (In particular it is an embedding if and only if it is one-to-one.)

**Proof.** Since $G$ is metrizable it is clear that $N$ is continuous, because the natural map $G \to C(\mathcal{H}(G, K), K)$ is continuous. To see that it is open onto its image, we notice that since $G$ is $h$-complete, $N(G)$ is closed in $C(\mathcal{H}(G, K), K)$, and therefore it is complete metric. Hence $N(G)$ is a Baire space. Applying an open map type of theorem ([3 Cor. 32.4]) one obtains that $N$ is open onto the image. □

**Theorem 2.** Let $K$ be a compact radical-based group. If $D$ is a dense subgroup of the metrizable group $G$ then $\mathcal{H}(D, K) \cong \mathcal{H}(G, K)$.

**Proof.** Clearly, $\mathcal{H}(D, K) = \mathcal{H}(G, K)$ as sets, and we have an induced map $\iota : \mathcal{H}(G, K) \to \mathcal{H}(D, K)$ by restriction. Since $\iota$ is continuous, and bijective, we only have to show that $\iota$ is open. To that end we will show that the inverse image of a compact set is compact. Since $\mathcal{H}(D, K)$ is a $k$-space it will imply that $\iota$ is open.

Take a compact subset $\Phi$ of $\mathcal{H}(D, K)$. Then, by the Ascoli Theorem $\Phi$ is equicontinuous; in particular there exists a neighborhood $U$ of $e$ in $G$, such that $\zeta(U \cap D) \subset \Lambda_1$ for all $\zeta \in \Phi$, and so $\zeta(U \cap D) \subset \Lambda_2$ for all $\zeta \in \Phi$. Thus $\Phi \subset S_e(U \cap D, \Lambda_2)$.

Let $V$ be a symmetric neighborhood of $e$ in $G$ such that $V^2 \subset U$, and let $x \in V$. There exists a sequence $\{x_n\} \subset D$ such that $x_n \to x$, and thus for $n \geq n_0$, $x_n \in xV \subset VV \subset U$. Since $x_n \in D$, $x_n \in U \cap D$ for $n \geq n_0$, thus $x \in U \cap D$, and hence $V \subset U \cap D$. Therefore $\Phi \subset S_e(U \cap D, \Lambda_2) \subset S_e(V, \Lambda_2)$. By Lemma 2, $S_e(V, \Lambda_2)$ is precompact, hence $\Phi$ is compact in $\mathcal{H}(G, K)$ as it is closed there. □

We note that Theorem 2 is a generalization to the non-abelian case of a similar result by Chasco [2].

**Acknowledgements**

The author is deeply indebted to his PhD thesis supervisor, Prof. Walter Tholen, for his dedicated mentorship that made this research possible.

He thanks Dr. Gavin Seal, for his assistance in simplifying some proofs and preparing the paper for submission, and he also appreciates the helpful comments of the anonymous referee.
Last, but not least, the author offers special thanks to his students for their enormous encouragement.

References

[1] R. Arens, *A topology for spaces of transformations*, Ann. of Math. **47** (1946), 480-495.

[2] M. J. Chasco, *Pontryagin duality for metrizable groups*, Arch. Math. **70** (1998), 22-28.

[3] Ryszard Engelking, *General Topology*. Polish Scientific Publishers, 1977.

[4] T. Husain, *Introduction to Topological Groups*. Saunders, Philadelphia, 1966.

[5] J. R. Isbell, *Uniform Spaces*, Mathematical Surveys, *Number 12*. American Mathematical Society, 1964.

[6] J. L. Kelley, *General Topology*, Van Nostrand, Princeton, 1955

[7] L. S. Pontryagin, *Topological Groups*, volume 2 of *L. S. Pontryagin Selected Works*. Gordon and Breach Science Publishers, New York, 1986.

Department of Mathematics & Statistics
York University, 4700 Keele Street
Toronto, Ontario, M3J 1P3
Canada

e-mail: lukacs@mathstat.yorku.ca