Wasserstein GAN Can Perform PCA

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Abstract—Generative Adversarial Networks (GANs) have become a powerful framework to learn generative models that arise across a wide variety of domains. While there has been a recent surge in the development of numerous GAN architectures with distinct optimization metrics, we are still lacking in our understanding on how far away such GANs are from optimality.

In this paper, we make progress on a theoretical understanding of the GANs under a simple linear-generator Gaussian-data setting where the optimal maximum-likelihood generator is known to perform Principal Component Analysis (PCA). We find that the original GAN by Goodfellow et. al. fails to recover the optimal PCA solution. On the other hand, we show that Wasserstein GAN can perform PCA, and hence it may serve as a basis for an optimal GAN architecture that yields the optimal generator for a wide range of data settings.

I. INTRODUCTION

The problem of learning the probability distribution of data is one of the most fundamental problems in statistics and machine learning. A generative model plays a crucial role as an underlying framework by providing a functional block (called the generator in the literature) which can create fake data which resembles the distribution of real data. Generative Adversarial Networks (GANs) have provided very powerful and efficient solutions for learning the generative model. The GAN framework includes two major components: Generator and Discriminator. This is inspired by a two-player game in which one player, Generator, wishes to generate fake samples that are close to real data, while the other player, Discriminator, wants to discriminate real samples against fake ones. Since the firstly introduced GAN (that we call vanilla GAN), there has been a proliferation of GAN architectures with distinct optimization metrics including f-GAN, MMD-GAN, WGAN, Least-Squares GAN, Boundary equilibrium GAN, etc. One natural question that arises in this context is: Is there an optimal GAN architecture among such GANs that provides optimal solutions for learning true distributions? More specifically, is there a proper optimization metric among those employed in such GANs that yields a generated distribution which maximizes the likelihood function?

In an effort to make progress towards answering this question, we take into consideration a simple canonical setting in which the optimal solution for learning distributions is well-known and hence one can readily figure out an optimization metric (if any) that yields such a solution. The simple setting represents the case in which the data has a high-dimensional Gaussian distribution and a generator is subject to a linear operation with a Gaussian input. It has been shown in this benchmark setting that the maximum-likelihood optimal solution performs Principal Component Analysis (PCA), i.e., the covariance matrix of the generated distribution takes principal components of the true covariance matrix.

The first finding of this work is that vanilla GAN does not recover the PCA solution under the Gaussian setting. In the population limit where there is an infinite number of data samples, vanilla GAN can be translated into an optimization that minimizes Jensen-Shannon (JS) divergence between the true and generated distributions. Here we find that whenever the rank of the covariance matrix of the generated Gaussian distribution is smaller than that of the true distribution (which is a typical scenario), the JS divergence attains the same value of log 2 regardless of the generator. Hence, vanilla GAN fails to achieve the PCA solution.

Recently such issue on the JS divergence motivated Arjovsky et. al. to propose Wasserstein GAN (WGAN) which replaces the JS divergence with the first-order Wasserstein distance that does not pose the issue. Then, one may wonder if WGAN can recover the PCA solution in the linear Gaussian setting at hand? The main contribution of this paper is to show that this is the case, i.e., WGAN indeed performs PCA. To prove this, we first translate WGAN optimization into another equivalent optimization. Exploiting the key condition that the optimal solution of the translated optimization should satisfy, we show that the optimal solution enables PCA. Our finding suggests that WGAN may be a good candidate for an optimal GAN architecture that yields an optimal generator for a wide variety of settings beyond the Gaussian case.

We also investigate stability issues for two prominent neural-net-based algorithms that intend to achieve the Nash equilibrium promised by the WGAN optimization. The WGAN optimization can be formulated as a minimax optimization which is actually a challenging problem as the convergence to a bad local optima may often occur. We show via empirical results that the two algorithms are actually practically appealing at least for the Gaussian setting. Our empirical simulation reveals that the deep-learning-based algorithms ensure the fast convergence to the Nash equilibrium with a small gap to the optimality.

Related work: Recently, Feizi et. al. explored a natural way of specifying a loss function that leads to a unified GAN architecture. The authors have investigated a quadratic loss based on the second-order Wasserstein distance to show that the corresponding GAN architecture (which they call Quadratic GAN) can perform PCA under the linear Gaussian...
setting. This suggests that Quadratic GAN may also be a good candidate for an optimal architecture. However, their algorithm for implementing Quadratic GAN is tailored for the Gaussian setting, so the development of generic algorithms that span a wide spectrum of data settings has been out of reach. On the other hand, we focus on WGAN for which generic neural-net-based algorithms have been well established, and promise that WGAN may be more practically appealing towards an optimal GAN architecture.

II. GAN ARCHITECTURE

Fig. 1 illustrates an architecture for Generative Adversarial Networks (GANs). Suppose we are given the number $n$ of real data samples. We denote those by $(Y_1, Y_2, \ldots, Y_n)$ where $Y_i \in \mathbb{R}^d$ and $d$ indicates a dimension of each sample. Let $P_Y$ denote the probability distribution of $Y_i$. Let $(X_1, X_2, \ldots, X_n)$ be latent-space signals: inputs to the generator $G(\cdot)$ where $X_i \in \mathbb{R}^r$, and $r$ indicates a dimension of the latent signal, usually $r \leq d$. Let $\tilde{Y}_i := G(X_i) \in \mathbb{R}^d$ be the $i$th fake sample and $P_{\tilde{Y}}$ be the probability distribution of $\tilde{Y}_i$.

We focus on a linear-generator Gaussian-data setting in which $Y_i$’s are i.i.d. each according to $P_Y = \mathcal{N}(0, \mathbf{K}_Y)$; $X_i$’s are i.i.d. $\sim \mathcal{N}(0, \mathbf{I})$; and $G(\cdot)$ is a linear operator implemented by a matrix $\mathbf{G} \in \mathbb{R}^{d \times r}$.

The GAN architecture is inspired by a two-player game in which one player is the discriminator $D(\cdot)$ who wishes to discriminate real samples against fake ones; and the other is the generator $G(\cdot)$ who wants to fool the discriminator. Many GAN approaches have been developed as an effort to achieve such goals, and corresponding optimization problems are formulated with different optimization metrics [11–17].

III. VANILLA GAN

In an effort to achieve such goals, Goodfellow et. al. [11] came up with an insightful interpretation: Viewing $D(\cdot)$ as the probability that the input (taking either a real sample $Y_i$ or a fake one $\tilde{Y}_i$) is a real sample. This viewpoint motivated them to introduce the following minimax optimization. Given $(X_1, Y_1), \ldots, (X_n, Y_n)$:

$$\min_{G(\cdot)} \max_{D(\cdot)} \frac{1}{n} \sum_{i=1}^{n} \log D(Y_i) + \frac{1}{n} \sum_{i=1}^{n} \log(1 - D(\tilde{Y}_i)).$$

Notice that the discriminator intends to maximize the logarithm of “1 – such probability” $1 - D(\cdot)$ for a fake sample.

For simplicity, we consider a regime in which the sample size $n$ is large enough that the objective function in the above can be well approximated as the population limit. So the optimization can be approximated as:

$$\min_{G(\cdot)} \max_{D(\cdot)} \mathbb{E}_Y [\log D(Y)] + \mathbb{E}_{\tilde{Y}} [\log(1 - D(\tilde{Y}))]. \quad (1)$$

In this work, we find that vanilla GAN does not recover the optimal PCA solution even for the simple Gaussian setting. To see this, we first rewrite the objective function as:

$$\int_{z \in \mathcal{Y} \cup \mathcal{\tilde{Y}}} [P_Y(z) \log D(z) + P_{\tilde{Y}}(z) \log(1 - D(z))] dz, \quad (2)$$

where $\mathcal{Y}$ and $\mathcal{\tilde{Y}}$ indicate the ranges of $Y$ and $\tilde{Y}$, respectively. Here we set:

$$P_Y(z)dz := 0 \text{ if } z \in \mathcal{\tilde{Y}} \setminus \mathcal{Y};$$
$$P_{\tilde{Y}}(z)dz := 0 \text{ if } z \in \mathcal{Y} \setminus \mathcal{\tilde{Y}}.$$ 

Observe that for any $(a, b) \in \mathbb{R}^2 \setminus \{0, 0\}$ and $t \in (0, 1)$, the function $\log t + b \log(1 - t)$ is maximized at $t^* = \frac{a}{a+b}$. Hence, for a fixed $G(\cdot)$, the optimal discriminator reads:

$$D^*(z) = \frac{P_Y(z)}{P_Y(z) + P_{\tilde{Y}}(z)}.$$ 

Plugging this into (1), we get:

$$\min_{G} \mathbb{E}_Y [\log D^*(Y)] + \mathbb{E}_{\tilde{Y}} [\log(1 - D^*(\tilde{Y}))]$$

where $KL$ and $JS$ indicate the KL and JS divergences, respectively [12], [13]: For distributions $p$ and $q$,

$$JSD(p||q) := \frac{1}{2} \left( \text{KL} \left( p \left\| \frac{p + q}{2} \right\| \right) + \text{KL} \left( q \left\| \frac{p + q}{2} \right\| \right) \right).$$

In the linear Gaussian setting, $\tilde{Y}$ is also Gaussian $\mathcal{N}(0, \mathbf{K}_{\tilde{Y}})$ where $\mathbf{K}_{\tilde{Y}} = \mathbb{E}[GX(GX)^T] = \mathbf{G}\mathbf{G}^T$. Typically the dimension of the latent signals is smaller than than of data. So in this case, the support of $P_{\tilde{Y}}$ has a strictly lower dimension than that of $P_Y$, yielding:

$$\frac{P_Y(z) + P_{\tilde{Y}}(z)}{2} dz = \begin{cases} \frac{P_Y(z)}{2} dz & \text{if } z \in \mathcal{Y} \setminus \mathcal{\tilde{Y}}; \\ \frac{P_{\tilde{Y}}(z)}{2} dz & \text{if } z \in \mathcal{\tilde{Y}} \setminus \mathcal{Y}. \end{cases} \quad (3)$$

This then gives $\text{KL} \left( P_Y \left\| \frac{P_Y + P_{\tilde{Y}}}{2} \right\| \right) = \text{KL} \left( P_Y \left\| \frac{P_Y + P_{\tilde{Y}}}{2} \right\| \right) = \log 2$, which in turn yields:

$$JSD(P_Y || P_{\tilde{Y}}) = \log 2, \quad (4)$$

regardless of how we design $G$. This implies that an optimal $G^*$ does not necessarily perform PCA.
IV. WASSERSTEIN GAN

The main reason that vanilla GAN fails to recover the optimal solution is that it is based on the JS divergence which poses the critical issue reflected in (4). As an effort to avoid such an issue, one may consider another prominent GAN architecture developed by Arjovsky et al. [3]: Wasserstein GAN (WGAN) which employs the first-order Wasserstein distance instead of the JS divergence. The first-order Wasserstein distance is defined as: Given two distributions $P_Y$ and $P_{\tilde{Y}}$,

$$W(P_Y, P_{\tilde{Y}}) := \min_{P_{\tilde{Y}}} \mathbb{E}_{P_Y} \left[ \|Y - \tilde{Y}\| \right],$$

where the minimization is over all joint distributions which respect the marginals $P_Y$ and $P_{\tilde{Y}}$. Here $\| \cdot \|$ indicates the $\ell_2$ norm. Unlike the JS divergence, it provides a meaningful non-saturating value even when dimensions of supports of two distributions are distinct. WGAN intends to solve the following optimization:

$$\min_G W(P_Y, P_{\tilde{Y}}).$$

The main contribution of this paper is that unlike vanilla GAN, WGAN recovers the optimal PCA solution under the linear Gaussian setting, formally stated below.

**Theorem 1.** Let $Y \sim \mathcal{N}(0, K_Y)$ where $K_Y$ has a full rank, i.e., $\text{rank}(K_Y) = d$. Let $X \sim \mathcal{N}(0, L_Y)$ where $r \leq d$. The optimal solution of the Wasserstein GAN optimization (6) under a linear generator is the $r$-PCA solution.

**Proof.** The proof consists of two parts:
(a) The WGAN optimization (6) is translated into another equivalent optimization which was investigated in depth in prior works [9];
(b) We show that the $r$-PCA solution respects the unique condition (derived in [9]) that the optimal solution of the translated optimization satisfies.

By the definition of the Wasserstein distance (5) and the assumption of a linear generator,

$$\min_G W(P_Y, P_{GX}) = \min_G \min_{P_{\tilde{Y}}} \mathbb{E} [\|Y - GX\|].$$

Lemma 1 below casts the WGAN optimization (6) into another equivalent optimization.

**Lemma 1.** Under a linear generator,

$$\min_G \min_{P_{\tilde{Y}}} \mathbb{E} [\|Y - GX\|] = \min_{U^T U = I} \mathbb{E} [\|Y - UU^T Y\|].$$

**Proof.** See Section IV-A

Remark: Lemma 1 suggests that for the optimal $G^*$ and $U^*$, we have $G^* X = U^* U^{*T} Y$. Considering the covariances of these quantities, we see that

$$G^* G^{*T} = U^* U^{*T} K_Y U^* U^{*T}.$$  \hfill (9)

For the translated optimization (8), [9] derived the key condition that the optimal solution $U^*$ should satisfy. This condition turns to play a crucial role to prove the theorem. We first introduce a notation which serves describing the condition. Let

$$M = \mathbb{E} \left[ \frac{YY^T}{\|Y - U^* U^{*T} Y\|} \right].$$

Now the key condition w.r.t. $U^*$ says:

$$U^* = r\text{-PrincipalEigenvectors}(M),$$

which means a concatenation of the $r$ principal eigenvectors of $M$. It was also shown in [9] that $U^*$ satisfying the key condition (11) is unique. Here what we prove is that such $U^*$ satisfies (see below for the proof):

$$U^* = r\text{-PrincipalEigenvectors}(K_Y).$$

Applying (12) into (9), we can find the optimal $G^*$ satisfies:

$$G^* G^{*T} = V \text{diag}(\sigma_1^2, \ldots, \sigma_r^2, 0, \ldots, 0) V^T$$

where $V := [v_1, \ldots, v_d] \in \mathbb{R}^{d \times d}$ and $\sigma_i^2$ are eigen-components of $K_Y$:

$$K_Y = V \text{diag}(\sigma_1^2, \ldots, \sigma_r^2) V^T =: \Sigma \Sigma V^T.$$  \hfill (14)

Here $v_i$’s are orthonormal vectors and $\sigma_1 \geq \cdots \geq \sigma_d$. This implies that the optimal generator $G^*$ is formed by taking the $r$ principal components of $K_Y$. This completes the proof. From below, we will prove the main claim (12).

**Proof of (12):** Since $V$ is an orthonormal matrix, we get:

$$Y = V V^T Y = \sum_{i=1}^d Z_i v_i,$$

where $Z_i := v_i^T Y$. Since $Y \sim \mathcal{N}(0, \Sigma V \Sigma V^T)$ (14),

$$Z := V^T Y = [Z_1; \cdots; Z_d] \sim \mathcal{N}(0, \Sigma).$$

Let $V_r := [v_1, \ldots, v_r]$ ($r$-principal eigenvectors of $K_Y$). Now we intend to show that $V_r = U^*$. To this end, it suffices to show that $V_r$ satisfies the key condition of (10) due to the uniqueness of $U^*$. So, we plug $V_r$ into (10) (by replacing $U^*$), thus obtaining:

$$M = \mathbb{E} \left[ \frac{YY^T}{\|Y - V_r V_r^T Y\|} \right] \overset{(a)}{=} \mathbb{E} \left[ \frac{YY^T}{\sqrt{\sum_{i=r+1}^d Z_i^2}} \right] \overset{(b)}{=} \sum_{j=1}^d \mathbb{E} \left[ \frac{Z_j^2}{\sqrt{\sum_{i=r+1}^d Z_i^2}} \right] v_j v_j^T + 2 \mathbb{E} \left[ \frac{Z_j Z_k}{\sqrt{\sum_{i=r+1}^d Z_i^2}} \right] v_j v_k^T,$$
where \((a)\) follows from the orthonormality of \(v_i\)'s and \((b)\) follows from the fact that
\[
\mathbb{E} \left[ \frac{Z_j Z_k}{\sqrt{\sum_{i=r+1}^{d} Z_i^2}} \right] = 0, \quad (17)
\]
which we will show in the sequel.

**Proof of (17).** Let \(Z^r := [Z_{r+1}, \ldots, Z_d] \). Using the tower property and the fact that \(Z = [Z_1; \ldots; Z_d] \sim \mathcal{N}(0, \Sigma)\) (see (15)), we get
\[
\mathbb{E} \left[ \frac{Z_j Z_k}{\sqrt{\sum_{i=r+1}^{d} Z_i^2}} \right] = \mathbb{E}_{Z^r, Z_k} \left[ \mathbb{E}_{Z_j} \left[ \frac{Z_j Z_k}{\sqrt{\sum_{i=r+1}^{d} Z_i^2}} \right] Z^r, Z_k \right]
\]
\[
= \mathbb{E}_{Z^r, Z_k} \left[ \mathbb{E}_{Z_j} \left[ \frac{-Z_j Z_k}{\sqrt{\sum_{i=r+1}^{d} Z_i^2}} \right] Z^r, Z_k \right]
\]
\[
= - \mathbb{E} \left[ \frac{Z_j Z_k}{\sqrt{\sum_{i=r+1}^{d} Z_i^2}} \right],
\]
where \((a)\) follows from the fact that \(Z_j \sim \mathcal{N}(0, \sigma_j^2)\) has a symmetric pdf and \(\frac{Z_j Z_k}{\sqrt{\sum_{i=r+1}^{d} Z_i^2}}\) is an odd function of \(Z_j\).

Therefore,
\[
\mathbb{E} \left[ \frac{Z_j Z_k}{\sqrt{\sum_{i=r+1}^{d} Z_i^2}} \right] = 0.
\]

Computing \(\mathbf{M} v_k\) with the help of (16), we get:
\[
\mathbf{M} v_k = \mathbb{E}\left[ \frac{Z_k^2}{\sqrt{\sum_{i=r+1}^{d} Z_i^2}} \right] v_k.
\]
This implies that \(v_k\) is an eigenvector of \(\mathbf{M}\) and the corresponding eigenvalue is \(\mathbb{E}\left[ \frac{Z_k^2}{\sqrt{\sum_{i=r+1}^{d} Z_i^2}} \right]\).

Now it suffices to show that \((v_1, \ldots, v_r)\) are \(r\)-principal eigenvectors of \(\mathbf{M}\). To show this, we will demonstrate below that for \(j \leq r\) and \(k > r\),
\[
\mathbb{E} \left[ \frac{Z_j^2}{\sqrt{\sum_{i=r+1}^{d} Z_i^2}} \right] \geq \mathbb{E} \left[ \frac{Z_k^2}{\sqrt{\sum_{i=r+1}^{d} Z_i^2}} \right]. \quad (18)
\]
Since \(j \leq r\) and \(Z_i\)'s are independent, we have:
\[
\mathbb{E} \left[ \frac{Z_j^2}{\sqrt{\sum_{i=r+1}^{d} Z_i^2}} \right] = \mathbb{E} \left[ Z_j^2 \right] \mathbb{E} \left[ \frac{1}{\sqrt{\sum_{i=r+1}^{d} Z_i^2}} \right]
\]
\[
\geq \mathbb{E} \left[ Z_j^2 \right] \mathbb{E} \left[ \frac{1}{\sqrt{Z^2 + \sum_{i>r, i \neq k} Z_i^2}} \right]
\]
\[
\geq \mathbb{E}_{Z_k} \left[ Z_k^2 \right] \mathbb{E}_{Z^r} \left[ \frac{1}{\sqrt{Z_k^2 + \sum_{i>r, i \neq k} Z_i^2}} \right].
\]

**A. Proof of Lemma 7**

Let \(\tilde{Y} = \mathbf{G} \mathbf{X}\) be a random vector whose support lies in \(S\) \((r\)-dimensional subspace in \(\mathbb{R}^d\)). Then we can write \(Y\) as:
\[
Y = Y_{S'} + Y_S, \quad (19)
\]
where \(Y_S\) indicates the projection of \(Y\) onto \(S\). Using this,
\[
\min_{\mathbf{G}} W(\mathbb{P}_Y, \mathbb{P}_{\tilde{Y}}) = \min_{\mathbf{G}} \min_{\mathbb{P}_{\tilde{Y}}} \mathbb{E}[||Y - \tilde{Y}||]
\]
\[
= \min_{\mathbf{G}} \min_{\mathbb{P}_{\tilde{Y}}} \mathbb{E}\left[ ||Y_{S'} + Y_S - \tilde{Y}||^2 \right]
\]
\[
= \min_{\mathbf{G}} \min_{\mathbb{P}_{\tilde{Y}}} \mathbb{E}\left[ ||Y_{S'}||^2 + ||Y_S - \tilde{Y}||^2 \right]
\]
\[
\geq \min_{\mathbf{G}} \mathbb{E}\left[ ||Y_{S'}||^2 \right] = \min_{\mathbf{G}} \mathbb{E}[||Y - Y_S||]
\]
\[
= \min_{\mathbf{U}} \mathbb{E}\left[ ||Y - \mathbf{U} \mathbf{U}^T Y|| \right]
\]
where \((a)\) follows from the orthogonality between \(Y_{S'}\) and \((Y_S - \tilde{Y})\); and \((b)\) is because the choice of \(\tilde{Y} = Y_S\) minimizes \(\mathbb{E}[||Y_{S'}||^2 + ||Y_S - \tilde{Y}||^2]\).

**V. NEURAL-NET-BASED WGAN ALGORITHMS**

So far we have shown that WGAN recovers the optimal PCA solution for the linear Gaussian setting. This suggests that WGAN can be a good candidate for the optimal GAN architecture.

In practice, however, it is early to arrive at this conclusion. To implement the WGAN solution, usually we hinge upon the Kantorovich dual [14] of the primal optimization, which includes two function optimizations that can be efficiently solved via neural networks with high accuracy:
\[
\min_{\mathbf{G}} W(\mathbb{P}_Y, \mathbb{P}_{\mathbf{G} \mathbf{X}}) = \min_{\mathbf{G}} \max_{\mathbb{P}_D} \mathbb{E}_Y[D(Y)] - \mathbb{E}_{\tilde{Y}}[D(\tilde{Y})],
\]
where \( \|D\|_{L} \leq 1 \) means 1-Lipschitz functions \( D \): For all \( y_{1} \) and \( y_{2} \), \( \|D(y_{1}) - D(y_{2})\| \leq \|y_{1} - y_{2}\| \). But here an issue arises: The Kantorovich dual is of minimax optimization where the convergence to a local optimum frequently occurs. Hence, we also need to investigate the stability of neural-net-based algorithms that implement the minimax solution.

In this work, we consider such two prominent algorithms: (1) WGAN with weight clipping (WGAN-VC); (2) WGAN with gradient penalty (WGAN-GP).

Another contribution of this work is to empirically show that the two algorithms yield good stability for the Gaussian setting, i.e., ensure the fast convergence to the Nash equilibrium with a small gap to the optimality.

An empirical verification is done for the following setting. We generate \( n \in \{1000, 10000, 100000, 1000000\} \) i.i.d. samples from a \( d \)-dimensional Gaussian distribution \( Y \sim \mathcal{N}(0, \Sigma) \). We set \( d = 32 \). A generation for \( \Sigma \) takes the following procedure. First, we generate \( V \Sigma V^{T} \) such that \( |V|_{ij} \sim \mathcal{N}(0, 1) \) and \( \sigma^{2} \sim \text{Uniform}(0, 10) \). Next, we normalize \( V \Sigma V^{T} \) so as to have the unit Frobenius norm. The generator is constructed with a single matrix \( G \). The discriminator is implemented with a fully connected neural network. As a performance metric, we employ the Frobenius norm between the ground-truth covariance and the generated covariance. As a setting that incorporates a more complex network, we consider the case which \( r = 8 \), \( n = 100,000 \), and the number of hidden units for each layer \( n_{h} \in \{10, 50, 100, 200, 300\} \); see Fig. 3(a). Here we employ WGAN-GP. The discriminator is implemented with a neural network composed of five layers in which hidden layers include the same number of hidden units: (a) the performance of WGAN-GP with the sample size \( n = 100,000 \) in different sizes of hidden units \( n_{h} \in \{10, 50, 100, 200, 300\} \); (b) the performance of WGAN-GP with different sample sizes \( n \in \{1000, 10000, 100000, 1000000\} \) for the case of \( n_{h} = 300 \). We found that the gap reduces when a more complex neural network is employed together with a larger \( n \). Here are details - also see Figs. 3 and 4.

As a setting that incorporates a more complex network, we consider the case which \( r = 8 \), \( n = 100,000 \), and the number of hidden units for each layer \( n_{h} \in \{10, 50, 100, 200, 300\} \); see Fig. 3(a). Here we employ WGAN-GP. The discriminator is implemented with a neural network composed of five layers in which hidden layers include the same number of hidden units and we use Adam optimizer with the learning rate \( 10^{-3} \). We see from Fig. 3(a) that the gap reduces with an increasing complexity, converging almost to 0. Fig. 3(b) demonstrates the performance of WGAN-GP with different sample sizes \( n \in \{1000, 10000, 100000, 1000000\} \) for the case of \( n_{h} = 300 \). We also see a very small gap for a large sample size. Fig. 4 shows the performance as a function of iterations when \( n_{h} = 300 \) and the sample size \( n = 1,000,000 \). We can also see a small gap in the convergence.

VI. CONCLUSION

We showed that WGAN can recover the optimal PCA solution under the linear-generator Gaussian-data setting. This promises that WGAN may form the basis of an optimal GAN architecture. Our future work includes: (1) Discovering a class of GAN architectures that performs PCA under the Gaussian setting (if any); (2) Exploring other settings in which other GAN architectures provide optimal solutions (if any); (3)...

Training of WGAN takes less than 60 seconds on a TESLA_P40 GPU.
Fig. 4. The performance of WGAN-GP for the linear Gaussian setting in which the data dimension $d = 32$ and latent signal dimension $r = 8$ with the sample size $n = 1,000,000$. The discriminator is implemented with a neural network composed of five layers, each with 300 neurons and ReLU activation functions. We can observe a small gap in the convergence.

Developing novel algorithms which can resolve stability issues (if required).

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