Morse theory methods for a class of quasi-linear elliptic systems of higher order

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Abstract
We develop the local Morse theory for a class of non-twice continuously differentiable functionals on Hilbert spaces, including a new generalization of the Gromoll–Meyer’s splitting theorem and a weaker Marino–Prodi perturbation type result. They are applicable to a wide range of multiple integrals with quasi-linear elliptic Euler equations and systems of higher order.

Mathematics Subject Classification Primary 58E05 · 49J52 · 49J45

1 Introduction
Since Palais and Smale [48,50,55] generalized finite-dimensional Morse theory [43,45] to nondegenerate $C^2$ functionals on infinite dimensional Hilbert manifolds and used it to study multiplicity of solutions for semilinear elliptic boundary value problems, via many people’s effort, such a direction has very successful developments, see a few of nice books [2,12,13,42,47,51,52,67] and references therein for details. The Morse theory for functionals on an infinite dimensional Hilbert manifold has two main aspects: Morse relations related critical groups to Betti numbers of underlying spaces (global), computation of critical groups (local). Combining use of both is the most effective in applications. The global aspect is well-developed, for example, $C^1$-smoothness for functionals are sufficient. The basic tools for the local aspect mainly consist of Gromoll–Meyer’s generalized Morse lemma (or splitting theorem) in [28] and the perturbation theorem of Marino and Prodi [41], which are stated for $C^2$ functionals on Hilbert spaces (cf. [12,42]). It is for such reasons that most of applications of the Morse theory to differential equations are restricted to semi-linear elliptic equations and Hamiltonian systems [12,42,47]. Applications to quasi-linear elliptic equations and systems

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require a suitable local Morse theory for either non-twice continuously differentiable functionals on Hilbert spaces or twice continuously differentiable functionals on Banach spaces. There exists significant progress for some special versions of quasi-linear elliptic equations and systems, e.g. \([10,11,17,18,21,52,58,59]\), though no satisfactory local Morse theory in these two cases is developed.

This work is motivated by studies of quasi-linear elliptic equations and systems of higher order given by the following multi-dimensional variational problem (1.3) under Hypothesis \(\mathcal{H}_{p,N,m,n}\) on the integrand \(F\). Since in this situation the functional \(\mathcal{F}(1.3)\) cannot, in general, be of class \(C^2\) on its natural domain space \(W^{m,p}(\Omega, \mathbb{R}^N)\) for \(p = 2\), the known local Morse theory is helpless. This requires us to develop the local Morse theory for this class of non-twice continuously differentiable functionals on Hilbert spaces, for example, some generalization of the Gromoll–Meyer’s splitting theorem and some weaker Marino–Prodi perturbation type result.

Throughout this paper, unless stated otherwise, we will use the following notations: For normed linear spaces \(X, Y\) we denote by \(X^*\) the dual space of \(X\), and by \(\mathcal{L}(X, Y)\) the space of linear bounded operators from \(X\) to \(Y\). We also abbreviate \(\mathcal{L}(X) := \mathcal{L}(X, X)\). Denote by

\[
B_X(y, r) := \{x \in X \mid \|x - y\|_X < r\} \quad \text{and} \quad \bar{B}_X(y, r) := \{x \in X \mid \|x - y\|_X \leq r\}
\]

the open and closed balls in \(X\) with radius \(r\) and centred at \(y\), respectively. The corresponding closed ball. The (norm)-closure of a set \(S \subset X\) will be denoted by \(\overline{S}\) or \(Cl(S)\). Let \(m, n \geq 1\) be two integers, \(\Omega \subset \mathbb{R}^n\) a bounded domain with boundary \(\partial \Omega\). Denote the general point of \(\Omega\) by \(x = (x_1, \ldots, x_n) \in \mathbb{R}^n\) and the element of Lebesgue \(n\)-measure on \(\Omega\) by \(dx\). A multi-index is an \(n\)-tuple \(\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N}_0)^n\), where \(\mathbb{N}_0 = \mathbb{N} \cup \{0\}\). \(|\alpha| := \alpha_1 + \cdots + \alpha_n\) is called the length of \(\alpha\). Denote by \(M(k)\) the number of such \(\alpha\) of length \(|\alpha| \leq k\), \(M_0(k) = M(k) - M(k - 1), k = 0, \ldots, m\), where \(M(-1) = 0\). Then \(M(0) = M_0(0)\) only consists of \(0 = (0, \ldots, 0) \in (\mathbb{N}_0)^n\).

Let \(p \in [2, \infty)\) be a real number, and let \(N \geq 1, n > 1\) be integers. We make Hypothesis \(\mathcal{H}_{p,N,m,n}\). For each multi-index \(\gamma\) as above, let

\[
p_{\gamma} \in (2, \infty) \text{ if } |\gamma| = m - n/p, \quad p_{\gamma} = \frac{np}{n - (m - |\gamma|)p}\quad \text{if } m - n/p < |\gamma| \leq m, \\
q_{\gamma} = 1 \text{ if } |\gamma| < m - n/p, \quad q_{\gamma} = \frac{p_{\gamma}}{p_{\gamma} - 1}\quad \text{if } m - n/p \leq |\gamma| \leq m;
\]

and for each two multiindexes \(\alpha, \beta\) as above, let \(p_{\alpha\beta} = p_{\beta\alpha}\) be defined by the conditions

\[
p_{\alpha\beta} = 1 - \frac{1}{p_{\alpha}} - \frac{1}{p_{\beta}} \quad \text{if } |\alpha| = |\beta| = m, \\
p_{\alpha\beta} = 1 - \frac{1}{p_{\alpha}} \quad \text{if } m - n/p \leq |\alpha| \leq m, \quad |\beta| < m - n/p, \\
p_{\alpha\beta} = 1 \quad \text{if } |\alpha|, |\beta| < m - n/p, \\
0 < p_{\alpha\beta} < 1 - \frac{1}{p_{\alpha}} - \frac{1}{p_{\beta}} \quad \text{if } |\alpha|, |\beta| \geq m - n/p, \quad |\alpha| + |\beta| < 2m.
\]

For \(M_0(k) = M(k) - M(k - 1), k = 0, \ldots, m\) as above, we write \(\xi \in \prod_{k=0}^{m} \mathbb{R}^{N \times M_0(k)}\) as \(\xi = (\xi_0, \ldots, \xi^m)\), where \(\xi_0 = (\xi_0^1, \ldots, \xi_0^N)^T \in \mathbb{R}^N\) and for \(k = 1, \ldots, m\), \(\xi^k = (\xi_{\alpha}^k) \in \mathbb{R}^{N \times M_0(k)}\), where \(1 \leq i \leq N\) and \(|\alpha| = k\). Denote by

\[
\xi_{\alpha}^k = [\xi_{\alpha}^k : |\alpha| < m - n/p], \quad k = 1, \ldots, N.
\]
Let \( \Omega \times \prod_{k=0}^{m} \mathbb{R}^{N \times M_0(k)} \ni (x, \xi) \mapsto F(x, \xi) \in \mathbb{R} \) be twice continuously differentiable in \( \xi \) for almost all \( x \), measurable in \( x \) for all values of \( \xi \), and \( F(\cdot, \xi) \in L^1(\Omega) \) for \( \xi = 0 \). Suppose that derivatives of \( F \) fulfill the following properties:

(i) For \( i = 1, \ldots, N \) and \( |\alpha| \leq m \), functions \( F_{\alpha}^{i}(x, \xi) := F_{\alpha^i_{a}}(x, \xi) \) for \( \xi = 0 \) belong to \( L^1(\Omega) \) if \( |\alpha| < m - n/p \), and to \( L^{q_{\alpha}}(\Omega) \) if \( m - n/p \leq |\alpha| \leq m \).

(ii) There exists a continuous, positive, nondecreasing functions \( g_{1} \) such that for \( i, j = 1, \ldots, N \) and \( |\alpha|, |\beta| \leq m \) functions \( \Omega \times \mathbb{R}^{M(m)} \rightarrow \mathbb{R} \), \( (x, \xi) \mapsto F_{\alpha\beta}^{ij}(x, \xi) := F_{\alpha^i_{a}\beta^j_{b}} \) satisfy:

\[
|F_{\alpha\beta}^{ij}(x, \xi)| \leq g_{1}\left(\sum_{k=1}^{N} |\xi_{\alpha}^{k}|\right)\left(1 + \sum_{k=1}^{N} \sum_{m-n/p \leq |\gamma| \leq m} |\xi_{\gamma}^{k}|p_{\gamma}\right)^{p_{\alpha\beta}}. \tag{1.1}
\]

(iii) There exists a continuous, positive, nondecreasing functions \( g_{2} \) such that:

\[
\sum_{i=1}^{N} \sum_{|\alpha|=|\beta|=m} F_{\alpha\beta}^{ij}(x, \xi)\eta_{\alpha}^{i}\eta_{\beta}^{j} \geq g_{2}\left(\sum_{k=1}^{N} |\xi_{\alpha}^{k}|\right)\left(1 + \sum_{k=1}^{N} \sum_{|\gamma|=m} |\xi_{\gamma}^{k}|p_{\gamma}\right)^{2} \sum_{i=1}^{N} \sum_{|\alpha|=m} (\eta_{\alpha}^{i})^{2}. \tag{1.2}
\]

for any \( \eta = (\eta_{\alpha}^{i}) \in \mathbb{R}^{N \times M_0(m)} \).

Note: (a) If \( m \leq n/p \) the functions \( g_{1} \) and \( g_{2} \) should be understand as positive constants.

(b) For \( N = 1 \) the conditions in Hypothesis \( H_{p,N,m,n} \) were introduced in [52, §3.1] (also see [53, §1.2] and [54, pp. 110,118]); but it was only required that \( p_{\gamma} \in (0, \infty) \) if \( |\gamma| = m - n/p \) there. We modify it as “\( p_{\gamma} \in (2, \infty) \) if \( |\gamma| = m - n/p \)” so as to coincide with the condition “\( 0 < p_{\alpha\beta} < 1 - \frac{1}{p_{\alpha}} - \frac{1}{p_{\beta}} \) if \( |\alpha| = |\beta| = m - n/p \).” This is only needed in case \( mp \geq n \).

(c) The controllable growth condition [27, p. 40] (also called ‘common condition of Morrey’ or ‘the natural assumption of Ladyzhenskaya and Ural’tseva’ [27, p. 38, (I)]) is stronger than Hypothesis \( H_{2,N,1,n} \), see Proposition 4.22; the Lagrangian function in De Giorgi’s example (cf. [27, p. 54]) satisfies Hypothesis \( H_{2,n,1,n} \), but does not fulfill the controllable growth condition on \( \Omega = B_{1}^{n}(0) = \{|x| \in \mathbb{R}^{n} : |x| < 1\}, n \geq 3 \).

Let \( \Omega \subset \mathbb{R}^{n} \) be a bounded domain such that the Sobolev embeddings theorems for the spaces \( W^{m,p}(\Omega) \) hold. For an element of \( W^{m,p}(\Omega, \mathbb{R}^{N}), \tilde{u} = (u^{1}, \ldots, u^{N}) : \Omega \rightarrow \mathbb{R}^{N} \), we denote by \( D^{k}\tilde{u} \) the set \( \{D^{k}u^{i} : |\alpha| = k, i = 1, \ldots, N\} \) for \( k = 1, \ldots, m \), and form the expression \( F(x, \tilde{u})(x), D^{m}\tilde{u}(x), \) in which \( \tilde{u}(x) \) and \( D^{m}u^{i}(x) \) take the place of \( \xi^{0} \) and \( \xi_{\alpha}^{i} \), respectively. Let \( V = \tilde{V} + V_{0} \subset W^{m,p}(\Omega, \mathbb{R}^{N}), \) where \( V_{0} \) is a closed subspace containing \( W_{0}^{m,p}(\Omega, \mathbb{R}^{N}) \). Consider the variational integral

\[
\mathcal{F}(\tilde{u}) = \int_{\Omega} F(x, \tilde{u}, \ldots, D^{m}\tilde{u})dx, \quad \tilde{u} \in V. \tag{1.3}
\]

Call critical points of \( \mathcal{F} \) generalized solutions of the boundary value problem corresponding to \( V \):

\[
\sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^{\alpha}F_{\alpha}^{i}(x, \tilde{u}, \ldots, D^{m}\tilde{u}) = 0, \quad i = 1, \ldots, N. \tag{1.4}
\]

When \( N = 1 \), we write \( \xi \in \mathbb{R}^{M(m)} \) as \( \xi = \{\xi_{\alpha} : |\alpha| \leq m\} \), \( \xi_{0} = \{\xi_{\alpha} : |\alpha| < m - n/p\} \) (this is empty if \( mp \leq n \)), and \( F_{\alpha\beta}^{ij}(x, \xi) =: F_{\alpha^i_{a}\beta^j_{b}}(x, \xi) \). As stated in [52, §3.4, Lemma 16, §5.2] and [54, pp. 118–119], under Hypothesis \( H_{p,1,m,n} \) the functional \( \mathcal{F} \) in (1.3) is of class \( C^{1} \), and the (derivative) mapping \( \mathcal{F} : W_{0}^{m,p}(\Omega) \rightarrow [W_{0}^{m,p}(\Omega)]^{*} \) is Fréchet differentiable if \( p > 2 \), but only Gâteaux-differentiable if \( p = 2 \). The latter is best possible. In fact, it was
shown on [54, Chap.5, Sec. 5.1, Theorem 1]: If $p = 2$, $m = 1$ and $F \in C^2(\overline{\Omega} \times \mathbb{R}^1 \times \mathbb{R}^n)$ has uniformly bounded mixed partial derivatives $F_{\xi_1 \xi_j}, F_{\xi_1 u}$ and $F_{uu}$ (therefore $F$ satisfies Hypothesis $\mathcal{Z}_{2.1,m,n}$), then the functional $\mathcal{Z}$ on $W^{1,2}_0(\Omega)$ has Fréchet second derivative at zero if and only if $F(x, 0, \xi) = \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j + \sum_{i=1}^n b_i(x)\xi_i + c(x)$. So, generally speaking, under Hypothesis $\mathcal{Z}_{2.1,m,n}$ the known Morse–Palais lemma cannot be used for $\mathcal{Z}$. Even so, by improving Smale’s method in [55], Skrypnik [52, Chapter 5] obtained Morse inequalities for $\mathcal{Z}$ on $W^{m,2}_0(\Omega)$ provided that $\mathcal{Z}$ is coercive and that each critical point $u$ of $\mathcal{Z}$ is nondegenerate in the sense that the Gâteaux derivative of $\mathcal{Z}$ at $u$ is an invertible bounded linear self-adjoint operator on $W^{m,2}_0(\Omega)$. (If $p = \dim \Omega = 2$ and $F \in C^{k,a}$ for some $a \in (0, 1)$ and an integer $k \geq 3$, it was proved in [54, Chapter 7, Th. 4.4] that every critical point $u$ of $\mathcal{Z}$ on $W^{m,2}_0(\Omega)$ sits in $C^{k+m-1,a}(\overline{\Omega})$; in fact $u$ is also analytic in $\Omega$ provided that $F$ is analytic in its arguments.)

For effectively using Morse theory methods to study critical points of $\mathcal{Z}$ on $W^{m,2}(\Omega, \mathbb{R}^N)$, it is expected that there exists a corresponding Gromoll–Meyer’s splitting theorem for this functional. Recently, the author in [32, Theorem 1.1] proved a generalization of Gromoll–Meyer’s splitting theorem in [28] and used it to study periodic solutions of Lagrangian systems on compact manifolds which are strongly convex and has quadratic growth on the fibers. It includes the case of $\dim \Omega = 1$ (and similar one appeared in some optimal control problems [61]). Lu [32, Theorem 1.1] was also generalized to a class of continuously directional differentiable functions on Hilbert spaces in [33, Theorem 2.1]. Our design of these splitting theorems is based on a key fact that the involved solutions have higher smoothness, which is usually satisfied for many one-dimensional variational problems. Such an assumption of regularity ensured that the implicit function theorem can be used in the proofs of [32, Theorem 1.1] and [33, Theorem 2.1]. If $N = 1$, $\dim \Omega = 2$ and $F$ is smooth enough, we may prove under Hypothesis $\mathcal{Z}_{2.1,m,2}$ that [33, Theorem 2.1] is applicable for the functional $\mathcal{Z}$ on $W^{m,2}_0(\Omega)$. However, if $\dim \Omega > 2$, for the variational problem (1.3), it seems helpless because of lack of the priori regularity of critical points; see Sect. 4.4 for details. Thus new ideas and methods are needed. We need establish an implicit function theorem for only Gâteaux differentiable map $\mathcal{F}'$. After carefully analyzing this map, we propose the following fundamental assumption and arrive at the expected goal.

**Hypothesis 1.1** Let $H$ be a Hilbert space with inner product $(\cdot, \cdot)_H$ and the induced norm $\| \cdot \|_H$, and let $X$ be a dense linear subspace in $H$. Let $V$ be an open neighborhood of the origin $\theta \in H$, and let $L \in C^1(V, \mathbb{R})$ satisfy $L'(\theta) = 0$. Assume that the gradient $\nabla L$ has a Gâteaux derivative $B(u) \in L_s'(H)$ at every point $u \in V \cap X$, and that the map $B : V \cap X \to L_s'(H)$ has a decomposition $B = P + Q$, where for each $x \in V \cap X$, $P(x) \in L_s'(H)$ is positive definite and $Q(x) \in L_s'(H)$ is compact, and they also satisfy the following properties:

(D1) All eigenfunctions of the operator $B(\theta)$ that correspond to non-positive eigenvalues belong to $X$.

(D2) For any sequence $(x_k) \subset V \cap X$ with $\|x_k\| \to 0$, $\|P(x_k)u - P(\theta)u\| \to 0$ for any $u \in H$.

(D3) The map $Q : V \cap X \to L_s(H)$ is continuous at $\theta$ with respect to the topology on $H$.

(D4) For any sequence $(x_k) \subset V \cap X$ with $\|x_k\| \to 0$, there exist constants $C_0 > 0$ and $k_0 \in \mathbb{N}$ such that $(P(x_k)u, u)_H \geq C_0\|u\|^2$ for all $u \in H$ and for all $k \geq k_0$.

The condition (D4) is equivalent to (D4*) in [33] by Lemma 2.7. Lemma 2.8 shows that Hypothesis 1.1 with $X = H$ is hereditary on closed subspaces.

Under Hypothesis 1.1, if $\theta$ is nondegenerate, i.e., $\mathrm{Ker}(B(\theta)) = \{\theta\}$, we prove a new generalization of Morse–Palais Lemma, Theorem 2.1. If Hypothesis 1.1 holds with $X = H$
we establish a new splitting lemma, Theorem 2.2. Strategies of their proofs will be given at the end of Sect. 2. Actually, we prove a more general parameterized splitting theorem, Theorem 2.16, which will be used to generalize many bifurcation theorems for potential operators in [39]. Comparing with splitting lemmas in [32,33], the new ones may largely simplify the arguments for Lagrangian systems in [32]. However, the former may, sometime, provide more elaborate results, for example, as we have done modifying the proof ideas of them may yield the desired splitting lemma for the Finsler energy functional on the space of $H^1$-curves in [36]. It is not clear how to complete this with the present one. In accord with Hypothesis 1.1, a weaker Marino–Prodi perturbation type result, Theorem 3.2, is also presented in Sect. 3.

In Sect. 4, we first list some fundamental analytic properties of the functional $F$ under Hypothesis $F_{p,N,m,n}$. In particular, Corollary 4.4 shows that Hypothesis $F_{2,N,m,n}$ assures $F$ to satisfy Hypothesis 1.1 on any closed subspace of $W^{m,2}(\Omega, \mathbb{R}^N)$ for a bounded Sobolev domain $\Omega \subset \mathbb{R}^n$. Their proofs are not difficult, but cumbersome, and may be completed by non-essentially changing that of [38, Theorem 3.1]. Then we are only satisfied to give Morse inequalities and some corollaries. Finally, we also make compares with previous work and explore applicability of them in Sect. 4.4.

Further essential applications may be found in the sequel papers [39,40]. We showed in [39] that Theorem 2.9 can be effectively used to generalize some famous bifurcation theorems for potential operators, which led to many bifurcation results for quasi-linear elliptic Euler equations and systems of higher order. Using the theory developed in this paper we can also generalize the results in [32] to a class of Lagrangian systems of higher order with lower smoothness conditions for Lagrangians.

2 The splitting lemmas for a class of non-$C^2$ functionals

2.1 Statements of main results

We always assume that Hypothesis 1.1 holds without special statements. Then it implies that $\nabla L$ is of class $(S)_+$ near $\theta$ as proved in [33, p.2966-2967]. In particular, $L$ satisfies the (PS) condition near $\theta$.

Let $H = H^+ \oplus H^0 \oplus H^-$ be the orthogonal decomposition according to the positive definite, null and negative definite spaces of $B(\theta)$. Denote by $P^*$ the orthogonal projections onto $H^*$, $* = +, 0, -$. By [33, Proposition B.2] Hypothesis 1.1 implies that there exists a constant $C_0 > 0$ such that each $\lambda \in (-\infty, C_0)$ is either not in the spectrum $\sigma(B(\theta))$ or is an isolated point of $\sigma(B(\theta))$ which is also an eigenvalue of finite multiplicity. It follows that both $H^0$ and $H^-$ are finitely dimensional, and that there exists a small $a_0 > 0$ such that $[-2a_0, 2a_0] \subset \sigma(B(\theta))$ at most contains a point 0, and hence

$$ (B(\theta)u, u)_H \geq 2a_0 ||u||^2 \quad \forall u \in H^+, \quad (B(\theta)u, u)_H \leq -2a_0 ||u||^2 \quad \forall u \in H^-.$$  \hfill (2.1)

Note that (D1) implies $H^- \oplus H^0 \subset X, v := \dim H^0$ and $\mu := \dim H^-$ are called the Morse index and nullity of the critical point $\theta$. In particular, if $v = 0$ the critical point $\theta$ is said to be nondegenerate. Without special statements, all nondegenerate critical points in this paper are in the sense of this definition. Moreover, such a critical point must be isolated by (2.4).

Our first result is the following Morse–Palais Lemma, a special case of Theorem 2.9. Comparing with that of [33, Remark 2.2(i)], the smoothness of $L$ is strengthened, but other conditions are suitably weakened.
Theorem 2.1 Under Hypothesis 1.1, if \( \theta \) is nondegenerate, then it is an isolated critical point, and there exist a small \( \epsilon > 0 \), an open neighborhood \( W \) of \( \theta \) in \( H \) and an origin-preserving homeomorphism, \( \phi : B_{H^+}(\theta, \epsilon) + B_{H^-}(\theta, \epsilon) \rightarrow W \), such that

\[
\mathcal{L} \circ \phi(u^+ + u^-) = \|u^+\|^2 - \|u^-\|^2, \quad \forall (u^+, u^-) \in B_{H^+}(\theta, \epsilon) \times B_{H^-}(\theta, \epsilon).
\]

Moreover, if \( \hat{H} \) is a closed subspace containing \( H^- \), and \( \hat{H}^+ \) is the orthogonal complement of \( H^- \) in \( \hat{H} \), i.e., \( \hat{H}^+ = \hat{H} \cap H^+ \), then \( \phi \) restricts to a homeomorphism \( \hat{\phi} : (B_{\hat{H}^+}(\theta, \epsilon) + B_{\hat{H}^-}(\theta, \epsilon)) \rightarrow \hat{W} := W \cap \hat{H} \), and \( \mathcal{L} \circ \hat{\phi}(u^+ + u^-) = \|u^+\|^2 - \|u^-\|^2 \) for all \((u^+, u^-) \in B_{\hat{H}^+}(\theta, \epsilon) \times B_{\hat{H}^-}(\theta, \epsilon)\).

Under the assumptions of this theorem, if \( X = H \) we can prove that \( \nabla \mathcal{L} \) is locally invertible near \( \theta \) in Theorem 2.13. Theorem 2.1 is also key for us to prove Theorem 2.16, whose special case is:

Theorem 2.2 (Splitting Theorem) Let Hypothesis 1.1 hold with \( X = H \). Suppose \( v \neq 0 \). Then there exist small positive numbers \( \epsilon, r, s \), a unique continuous map \( \phi : B_{H^0}(\theta, \epsilon) \rightarrow H^+ \oplus H^- \) satisfying

\[
\phi(\theta) = \theta \quad \text{and} \quad (I - P^0)\nabla \mathcal{L}(z + \phi(z)) = 0 \quad \forall z \in B_{H^0}(\theta, \epsilon),
\]

for all \( \epsilon \) and \( \epsilon \). Moreover, if \( \hat{\phi} \) is of class \( C^{1-0} \), and we have also:

(a) For each \( z \in B_{H^0}(\theta, \epsilon) \), \( \Phi(z, \theta) = z + \phi(z) \), \( \phi_z(u^+ + u^-) \in H^- \) if and only if \( u^+ = \theta \);

(b) The functional \( B_{H^0}(\theta, \epsilon) \ni z \mapsto \mathcal{L}(z + \phi(z)) \) is of class \( C^1 \) and \( D\mathcal{L}(z + \phi(z))[v] = D\mathcal{L}(z + \phi(z))[v] \) for all \( v \in H^0 \).

Since the map \( \phi \) satisfying (2.2) is unique, as [32,33] it is possible to prove in some cases that \( \phi \) and \( \mathcal{L}^0 \) are of class \( C^1 \) and \( C^2 \), respectively.

Theorems 2.1 and 2.2 cannot be derived from those of [23]. In fact, according to the conditions (c) and (d) in [23, Theorem 1.3] the functional \( \mathcal{L} \) in Theorem 2.1 should satisfy:

\[(c') \exists \eta > 0, \delta > 0 \text{ such that } \|(B(u)(u + z) - B(\theta)(u + z), h)\| < \eta \|u + z\| \cdot \|h\| \text{ for all } u \in B_H(\theta, \delta), z \in H^0 \text{ and } h \in H \setminus \{\theta\};
\]

\[(d') \exists \delta > 0 \text{ such that } (\nabla \mathcal{L}(z + u_1^+ + u_1^-) - \nabla \mathcal{L}(z + u_2^+ + u_2^-), (u_1^+ - u_2^+) + (u_1^- - u_2^-)) > 0 \text{ for all } (u_1^+, u_1^-), (u_2^+, u_2^-) \in B_{H^+}(\theta, \delta) \times B_{H^-}(\theta, \delta) \text{ with } u_1^+ + u_1^- \neq u_2^+ + u_2^-.
\]

The former implies \( \|B(u)(u + z) - B(\theta)(u + z)\| \leq \eta \|u + z\| \) for all \( u \in B_H(\theta, \delta), z \in H^0 \); and the latter implies, for some \( t \in (0, 1), (B(z + u_2^+ + u_2^- + tu^+ + tu^-)(u^+ + u^-), u^+ + u^-) > 0 \) with \( u^+ = u_1^+ - u_2^+ \) and \( u^- = u_1^- - u_2^- \). From these it is not hard to see that under our assumptions the conditions \( (c') \) and \( (d') \) cannot be satisfied in general.

Let \( K \) always denote an Abel group (without special statements), and let \( H_q(A, B; K) \) denote the \( q \)th relative singular homology group of a pair \((A, B)\) of topological spaces with
coefficients in $K$. For each $q \in \mathbb{N} \cup \{0\}$ the $q$th critical group (with coefficients in $K$) of $L$ at $\theta$ is defined by $C_q(L, \theta; K) := H_q(L_c \cap U, L_c \cap U \setminus \{\theta\}; K)$, where $c = L(\theta)$, $L_c = \{L \leq c\}$ and $U$ is a neighborhood of $\theta$ in $H$. Under the assumptions of Theorem 2.1 we have $C_q(L, \theta; K) = \delta_{q\mu} K$ as usual. For the degenerate case, though our $L^0$ is only class $C^1$, the proofs in [42, Theorem 8.4] and [13, Theorem 5.1.17] (or [12, Theorem 1.5.4]) may be slightly modify to get the following shifting theorem, a special case of Theorem 2.18.

**Theorem 2.23** (Shifting Theorem) Under the assumptions of Theorem 2.2, if $\theta$ is an isolated critical point of $L$, then $C_q(L, \theta; K) \cong C_q(-\mu(L^0, \theta; K)$ for all $q \in \mathbb{N}_0$. Consequently, $\text{rank} C_q(L, \theta; K) = 0$ if $q < \mu$ or $q > \mu + \nu$.

As done for $C^2$ functionals in [12,13,42,47] some critical point theorems can be derived from Theorem 2.3. For example, $C_q(L, \theta; K)$ is equal to $\delta_{q\mu} K$ (resp. $\delta_{(\mu+\nu)\mu}$) if $\theta$ is a local minimizer (resp. maximizer) of $L^0$, and $C_q(L, \theta; K) = 0$ for $q \leq \mu$ and $q \geq \mu + \nu$ if $\theta$ is neither a local minimizer nor local maximizer of $L^0$. Similarly, the corresponding generalizations of Theorems 2.1, 2.1', 2.2, 2.3 and Corollary 1.3 in [12, Chapter II] can be obtained with Theorems 2.1 and 2.2 and their equivariant versions in Sect. 2.5. In particular, as a generalization of [12, Theorem II.1.6] (or [13, Theorem 5.1.20]) we have

**Theorem 2.24** Let Hypothesis 1.1 hold with $X = H$, and let $\theta$ be an isolated critical point of mountain pass type, i.e., $C_1(L, \theta; K) \neq 0$. Suppose that $\nu > 0$ and $\mu = 0$ imply $\nu = 1$. Then $C_q(L, \theta; K) = \delta_{q\mu} K$.

When $\nu > 0$ and $\mu = 1$, $C_0(L^0, \theta; K) \neq 0$ by Theorem 2.3. We can change $L^0$ outside a very small neighborhood $\theta \in B_{\mu^0}(\theta, \epsilon)$ to get a $C^1$ functional on $H^0$ which is coercive (and so satisfies the (PS)-condition). Then it follows from $C_0(L^0, \theta; K) \neq 0$ and [47, Proposition 6.95] that $\theta$ is a local minimizer of $L^0$. As a generalization of Corollary 3.1 in [12, page 102] we have also: Under the assumptions of Theorem 2.4, if the smallest eigenvalue $\lambda_1$ of $B(\theta) = d^2 L(\theta)$ is simple whenever $\lambda_1 = 0$, then $\lambda_1 \leq 0$, and index($\nabla L, \theta$) = $-1$. Theorem 5.1 and Corollary 5.1 in [12, page 121] are also true if “$f \in C^2(M, \mathbb{R})$” and “Fredholm operators $d^2 f(x)$” are replaced by “$f \in C^1(M, \mathbb{R})$ and $\nabla f$ is Gâteaux differentiable” and “under some chart around $p_1$ the functional $f$ has a representation that satisfies Hypothesis 1.1”, respectively. We can also generalize many critical point theorems in [33,37] to the setting above, for example, combing with [29] a corresponding result to [33, Theorem 2.10] may be proved under suitable assumptions. They will be given in other places.

Strategies of the proof of Theorem 2.2 and arrangements in this section Under the assumptions of Theorem 2.2, no known implicit function theorems or contraction mapping principles can be used to get $\varphi$ in (2.2), which is rather different from the case in [32,33]. The methods in [23] provide a possible way to construct such a $\varphi$. However, as shown below Theorem 2.2, our assumptions cannot guarantee the conditions ($c'$) and ($d'$) above. Fortunately, it is with Lemma 2.10 and Theorem 2.1 that we can complete this construction.

In Sect. 2.2 we list some lemmas, and prove a more general parameterized version of Theorem 2.1. It is necessary for a key implicit function theorem for a family of potential operators, Theorem 2.12, which is proved in Sect. 2.3; we also give an inverse function theorem, Theorem 2.13, there. In Sect. 2.4 we shall prove a parameterized splitting theorem, Theorem 2.16, and a parameterized shifting theorem, Theorem 2.18; Theorems 2.2 and 2.3 are special cases of them, respectively. The equivariant case is considered in Sect. 2.5.

**2.2 Lemmas and a parameterized version of Theorem 2.1**

Under Hypothesis 1.1 we have the following two lemmas as proved in [32,33].
Lemma 2.5 There exist a function \( \omega : V \cap X \to [0, \infty) \) such that \( \omega(x) \to 0 \) as \( x \in V \cap X \) and \( \|x\| \to 0 \), and that for any \( x \in V \cap X \), \( u \in H^0 \oplus H^* \) and \( v \in H \),

\[
(B(x)u, v)_H - (B(\theta)u, v)_H \leq \omega(x)\|u\| \cdot \|v\|.
\]

Lemma 2.6 There exist a small neighborhood \( U \subset V \) of \( \theta \) in \( H \) and a number \( c_0 < 0 \) such that for any \( x \in U \cap X \),

- \( (B(x)u, u)_H \geq c_0\|u\|^2 \forall u \in H^+ \);
- \( |(B(x)u, v)_H| \leq \omega(x)\|u\| \cdot \|v\| \forall v \in H^+ \cap H^0 \);
- \( (B(x)u, u)_H \leq -c_0\|u\|^2 \forall u \in H^- \).

Lemma 2.7 (D4) is equivalent to the condition (D4*) in [33]:

\( (D4*) \) There exist positive constants \( \eta_0 > 0 \) and \( c_0' > 0 \) such that

\[
(P(x)u, u) \geq c_0'\|u\|^2 \quad \forall u \in H, \quad \forall x \in B_H(\theta, \eta_0) \cap X.
\]

Indeed, since each \( P(x) \) is a positive definite bounded linear operator, its spectral set is a bounded closed subset in \( (0, \infty) \), and \( \sigma(\sqrt{P(x)}) = \{ \sqrt{\lambda} | \lambda \in \sigma(P(x)) \} \). It follows that (D4) implies (D4*).

The following result is easily verified, see [38].

Lemma 2.8 Suppose that Hypothesis 1.1 with \( X = H \) is satisfied. Then for any closed subspace \( \tilde{H} \subset H \), \((\hat{H}, \hat{V}, \hat{L})\) satisfies Hypothesis 1.1 with \( X = H \), where \( \hat{V} := V \cap \hat{H} \) and \( \hat{L} := L|_{\hat{V}} \).

For later applications in [39], we shall prove the following more general version of Theorem 2.1.

Theorem 2.9 Under Hypothesis 1.1, let \( G \in C^1(V, \mathbb{R}) \) satisfy:

- \( G'(\theta) = \theta \),
- \( G''(u) \in \mathcal{L}_1(H) \) at any \( u \in V \), and \( G''(u) : V \to \mathcal{L}_1(H) \) are continuous at \( \theta \).

Suppose that the critical point \( \theta \) of \( L \) is nondegenerate. Then there exist \( \rho > 0 \), \( \epsilon > 0 \), a family of open neighborhoods of \( \theta \) in \( H \), \( \{ W_\lambda | |\lambda| \leq \rho \} \) and a family of origin-preserving homeomorphisms, \( \phi_\lambda : B_{H^+}(\theta, \epsilon) + B_{H^-}(\theta, \epsilon) \to W_\lambda, \lambda \leq \rho \), such that

\[
(L + \lambda G) \circ \phi_\lambda(u^+ + u^-) = \|u^+\|^2 - \|u^-\|^2, \quad (u^+, u^-) \in B_{H^+}(\theta, \epsilon) \times B_{H^-}(\theta, \epsilon).
\]

Moreover, \([-\rho, \rho] \times (B_{H^+}(\theta, \epsilon) + B_{H^-}(\theta, \epsilon)) \ni (\lambda, u) \mapsto \phi_\lambda(u) \in H \) is continuous, and \( \theta \) is an isolated critical point of each \( L + \lambda G \). Finally, if \( \hat{H} \) is a closed subspace containing \( H^- \), and \( \hat{H}^+ \) is the orthogonal complement of \( H^- \) in \( \hat{H} \), i.e., \( \hat{H}^+ = \hat{H} \cap H^+ \), then each \( \phi_\lambda \) restricts to a homeomorphism \( \hat{\phi}_\lambda : B_{\hat{H}^+}(\theta, \epsilon) + B_{\hat{H}^-}(\theta, \epsilon) \to \hat{W}_\lambda := W_\lambda \cap \hat{H}, \) and \( (L + \lambda G) \circ \hat{\phi}_\lambda(u^+ + u^-) = \|u^+\|^2 - \|u^-\|^2 \) for all \( (u^+, u^-) \in B_{\hat{H}^+}(\theta, \epsilon) \times B_{\hat{H}^-}(\theta, \epsilon) \).

Proof Take a small \( \epsilon > 0 \) so that \( B_{\hat{H}^+}(\theta, \epsilon) + B_{\hat{H}^-}(\theta, \epsilon) \) is contained in the open neighborhood \( U \) in Lemma 2.6. As in Step 3 of the proof of [32, Theorem 1.1] (or the proof of [33, Lemma 3.5]), it follows from the mean value theorem and Lemma 2.6 that

\[
\begin{align*}
D_L(u^+ + u^-)(u^+_2 - u^-_1) - D_L(u^+ + u^-)(u^+_1 - u^-_2) &\leq -a_0\|u^+_2 - u^-_1\|^2, \\
D_L(u^+ + u^-)(u^+ - u^-) &\geq a_1\|u^+\|^2 + a_0\|u^-\|^2
\end{align*}
\]

for all \( u^+ \in B_{\hat{H}^+}(\theta, \epsilon) \) and \( u^- \in B_{\hat{H}^-}(\theta, \epsilon), i = 1, 2 \). (See [38] for details).

Since \( G'' : V \to \mathcal{L}_1(H) \) are continuous at \( \theta \), as in the proofs of (2.3) and (2.4) in [38] we may shrink \( \epsilon > 0 \) and find \( \rho > 0 \) such that
In particular, this implies that
\[ p(u^+ + u^-) \leq \frac{a_0}{2} \| u^+ - u^- \|^2, \]
\[ |\lambda| D\tilde{G}(u^+ + u^-)[u_2^- - u_1^-] - D\tilde{G}(u^+ + u_1^-)[u_2^- - u_1^-] \leq \frac{a_1}{2} \| u_2^- - u_1^- \|^2, \]
\[ |\lambda| D\tilde{G}(u^+ + u^-)[u^+ - u^-] \leq \frac{a_1}{2} \| u^+ \|^2 + \frac{a_0}{2} \| u^- \|^2 \]
for all \( \lambda \in [-\rho, \rho], u^+ \in \tilde{B}_{H^+}(\theta, \epsilon) \) and \( u^-, u_i^- \in \tilde{B}_{H^-}(\theta, \epsilon), i = 1, 2 \). The first inequality and (2.3) lead to
\[ D(\mathcal{L} + \lambda \tilde{G})(u^+ + u^-)[u_2^- - u_1^-] - D(\mathcal{L} + \lambda \tilde{G})(u^+ + u_1^-)[u_2^- - u_1^-] \leq -\frac{a_0}{2} \| u_2^- - u_1^- \|^2, \quad \forall (\lambda, u^+, u^-) \in [-\rho, \rho] \times \tilde{B}_{H^+}(\theta, \epsilon) \times \tilde{B}_{H^-}(\theta, \epsilon). \]
(2.5)
The latter and (2.4) yield for all \( (\lambda, u^+, u^-) \in [-\rho, \rho] \times \tilde{B}_{H^+}(\theta, \epsilon) \times \tilde{B}_{H^-}(\theta, \epsilon), \)
\[ D(\mathcal{L} + \lambda \tilde{G})(u^+ + u^-)[u^+ - u^-] \geq \frac{a_1}{2} \| u^+ \|^2 + \frac{a_0}{2} \| u^- \|^2. \]
(2.6)
In particular, this implies that \( \theta \) is an isolated critical point of each \( \mathcal{L} + \lambda \tilde{G} \) and that
\[ D(\mathcal{L} + \lambda \tilde{G})(u^+)[u^+] \geq \frac{a_1}{2} \| u^+ \|^2 > p(\| u^+ \|), \quad \forall (\lambda, u^+) \in [-\rho, \rho] \times \tilde{B}_{H^+}(\theta, \epsilon) \setminus \{\theta\}, \]
where \( p : (0, \epsilon) \to (0, \infty) \) is a non-decreasing function given by \( p(t) = \frac{a_1}{a_0} t^2 \). This, (2.5) and (2.6) show that the conditions of [33, Theorem A.1] are satisfied. The first two conclusions follow immediately.

For the final claim, note that (2.5) and (2.6) naturally hold for all \( u^+ \in \tilde{B}_{H^+}(\theta, \epsilon) \) and \( u^-, u_i^- \in \tilde{B}_{H^-}(\theta, \epsilon), i = 1, 2 \). Carefully checking the proof of [33, Theorem A.1] the conclusion is easily obtained. (Note that this claim seems unable to be directly derived from Lemma 2.8.)

\[ \square \]

2.3 An implicit function theorem for a family of potential operators

Under Hypothesis 1.1, we shall prove an implicit function theorem, Theorem 2.12, which implies the first claim in Theorem 2.2, and an inverse function theorem, Theorem 2.13.

Take \( \epsilon > 0, r > 0 \) and \( s > 0 \) so small that the closures of both
\[ Q_{r,s} := B_{H^+}(\theta, r) \oplus B_{H^-}(\theta, s) \quad \text{and} \quad B_{H^0}(\theta, \epsilon) \oplus Q_{r,s} \]
are contained in the neighborhood \( U \) in Lemma 2.6. Since \( H^0 \subset X, X \cap Q_{r,s} \) is also dense in \( Q_{r,s} \). Let \( P^\perp = I - P^0 = P^+ + P^- \). By Lemma 2.6 we obtain \( a_0' > 0, a_1' > 0 \) such that
\[ (P^\perp \nabla \mathcal{L}(z + u), u^+) = (\nabla \mathcal{L}(u), u^+) \geq a_1' \| u^+ \|^2 - a_0' [\omega(z + u)]^2 \| u^- \|^2, \]
(2.7)
\[ (P^\perp \nabla \mathcal{L}(z + u), u^-) = (\nabla \mathcal{L}(u), u^-) \leq -a_1' \| u^- \|^2 + a_0' [\omega(z + u)]^2 \| u^+ \|^2 \]
(2.8)
for all \( u \in Q_{r,s} \) and \( z \in \tilde{B}_{H^0}(\theta, \epsilon) \). Since \( \omega(z + u) \to 0 \) as \( \| z + u \| \to 0 \), by shrinking \( r > 0, s > 0 \) and \( \epsilon > 0 \) we can require that \( [\omega(z + u)]^2 < \frac{a_1'}{2a_0'} \) for all \( (z, u) \in \tilde{B}_{H^0}(\theta, \epsilon) \times Q_{r,s} \).
This, (2.7) and (2.8) lead to, respectively,
\[ (P^\perp \nabla \mathcal{L}(z + u), u^+) \geq a_1' \| u^+ \|^2 - \frac{a_1'}{2} \| u^- \|^2 \quad \forall (u, z) \in Q_{r,s} \times \tilde{B}_{H^0}(\theta, \epsilon), \]
\[ (P^\perp \nabla \mathcal{L}(z + u), u^-) \leq -a_1' \| u^- \|^2 + \frac{a_1'}{2} \| u^+ \|^2 \quad \forall (u, z) \in Q_{r,s} \times \tilde{B}_{H^0}(\theta, \epsilon), \]
and hence for all \( u \in \overline{Q_{r,s}}, z_j \in \bar{B}_H^0(\theta, \epsilon) \), \( j = 1, 2, \) and \( t \in [0, 1] \),

\[
(tP^\perp \nabla L(z_1 + u) + (1 - t)P^\perp \nabla L(z_2 + u), u^+)_H \geq a_1' \|u^+\|^2 - \frac{a_1'}{2} \|u^+\|^2, \quad (2.9)
\]

\[
(tP^\perp \nabla L(z_1 + u) + (1 - t)P^\perp \nabla L(z_2 + u), u^-)_H \leq -a_1' \|u^-\|^2 + \frac{a_1'}{2} \|u^+\|^2.
\]

(2.10)

**Lemma 2.10** Let \( \Omega = [0, 1] \times \bar{B}_H^0(\theta, \epsilon) \times \bar{B}_H^0(\theta, \epsilon) \). Then

\[
\inf\{|t P^\perp \nabla L(z_1 + u) + (1 - t) P^\perp \nabla L(z_2 + u)| |(t, z_1, z_2, u) \in \Omega| > 0.
\]

**Proof** Since \( \partial \overline{Q_{r,s}} = [(\partial B^+_H(\theta, r)) \oplus \bar{B}_H^-(\theta, s)] \cup [\bar{B}_H^+(\theta, r) \oplus (\partial \bar{B}_H^+(\theta, s))] \), we have \( \Omega = \Lambda_1 \cup \Lambda_2 \), where \( \Lambda_1 = [0, 1] \times \bar{B}_H^0(\theta, \epsilon) \times \bar{B}_H^0(\theta, \epsilon) \) and \( \Lambda_2 = [0, 1] \times \bar{B}_H^0(\theta, \epsilon) \times \bar{B}_H^0(\theta, \epsilon) \times \bar{B}_H^+(\theta, r) \oplus (\partial \bar{B}_H^-(\theta, s)) \). Firstly, let us prove

\[
\inf\{|t P^\perp \nabla L(z_1 + u) + (1 - t) P^\perp \nabla L(z_2 + u)| |(t, z_1, z_2, u) \in \Lambda_1| > 0. \quad (2.11)
\]

By a contradiction, suppose that there exist sequences \( (t_n) \subset [0, 1] \) and

\[
(z_n), (z'_n) \subset \bar{B}_H^0(\theta, \epsilon), \quad (u_n) \subset (\partial B^+_H(\theta, r)) \oplus \bar{B}_H^-(\theta, s)
\]

such that \( |t_n P^\perp \nabla L(z_n + u_n) + (1 - t_n) P^\perp \nabla L(z'_n + u_n)| \rightarrow 0 \). We can assume

\[
(t_n P^\perp \nabla L(z_n + u_n) + (1 - t_n) P^\perp \nabla L(z'_n + u_n), u^+_n)_H \leq \frac{a_1'r^2}{4}, \quad \forall n \in \mathbb{N},
\]

(2.12)

\[
(t_n P^\perp \nabla L(z_n + u_n) + (1 - t_n) P^\perp \nabla L(z'_n + u_n), u^-_n)_H \geq -\frac{a_1'r^2}{4}, \quad \forall n \in \mathbb{N}. \quad (2.13)
\]

Note that \( u^+_n \in \partial B^+_H(\theta, r) \) and \( u^-_n \in \bar{B}_H^-(\theta, s) \). So (2.12) and (2.9) lead to

\[
\frac{a_1'r^2}{4} \geq (t_n P^\perp \nabla L(z_n + u_n) + (1 - t_n) P^\perp \nabla L(z'_n + u_n), u^+_n)_H \geq \frac{a_1'r^2}{4} - \frac{a_1'}{2} \|u^-_n\|^2
\]

and therefore

\[
\frac{r^2}{\|u^-_n\|^2} \leq \frac{2}{3}, \quad \forall n \in \mathbb{N}. \quad (2.14)
\]

Moreover, from (2.10) and (2.13) we conclude that

\[
-\frac{a_1'r^2}{4} \leq (t_n P^\perp \nabla L(z_n + u_n) + (1 - t_n) P^\perp \nabla L(z'_n + u_n), u^-_n)_H \leq -\frac{a_1'}{2} \|u^-_n\|^2 + \frac{a_1'r^2}{2}
\]

and hence \( \frac{r^2}{\|u^-_n\|^2} \geq \frac{4}{3}, \quad \forall n \in \mathbb{N} \), which contradicts (2.14). Equation (2.11) is proved.

Next, we only need to prove

\[
\inf\{|t P^\perp \nabla L(z_1 + u) + (1 - t) P^\perp \nabla L(z_2 + u)| |(t, z_1, z_2, u) \in \Lambda_2| > 0
\]

again. As above, suppose that there exist sequences \( (t_n) \subset [0, 1] \) and

\[
(z_n), (z'_n) \subset \bar{B}_H^0(\theta, \epsilon), \quad (v_n) \subset B^+_H(\theta, r) \oplus (\partial B^+_H(\theta, s))
\]

such that \( |t_n P^\perp \nabla L(z_n + v_n) + (1 - t_n) P^\perp \nabla L(z'_n + v_n)| \rightarrow 0 \). As above we can assume
Lemma 2.11

Note that \( v_n^+ \in B_{H^+}(\theta, r) \) and \( v_n^- \in \partial B_{H^-}(\theta, s) \) for all \( n \in \mathbb{N} \). Then (2.10) and (2.16) imply

\[
-\frac{a_1's^2}{4} \leq (t_n P^\perp \nabla L(z_n + v_n) + (1 - t_n) P^\perp \nabla L(z_n' + v_n), v_n^+)_H \leq \frac{a_1's^2}{4}, \quad \forall n \in \mathbb{N},
\]

and so

\[
\frac{s^2}{\|v_n^+\|^2} \leq \frac{2}{3}, \quad \forall n \in \mathbb{N}.
\]

With the same methods, (2.9) and (2.15) yield

\[
\frac{a_1's^2}{4} \geq (t_n P^\perp \nabla L(z_n + v_n) + (1 - t_n) P^\perp \nabla L(z_n' + v_n), v_n^-)_H \geq \frac{a_1'}{2} \|v_n^-\|^2 - \frac{a_1's^2}{4}
\]

and so

\[
\frac{s^2}{\|v_n^-\|^2} \geq \frac{4}{3}, \quad \forall n \in \mathbb{N}.
\]

This contradicts (2.17). The desired claim is proved. \( \square \)

Since (D4) is equivalent to (D4*) by Lemma 2.7, it was proved in [33, p. 2966–2967] that \( \nabla L \) is of class \((S)_+\) under the conditions \((S), (F), (C)\) and \((D)\) in [33]. In particular, this is also true under the assumptions of Theorem 2.1 (without requirement \( H^0 = \{\theta\} \)).

In the following we always assume that \( r > 0, s > 0 \) and \( \epsilon > 0 \) are as in Lemma 2.10.

Lemma 2.11

For each \( z \in B_{H^0}(\theta, \epsilon) \), the map

\[
f_z : \overline{Q_{r,s}} \rightarrow H^+ \oplus H^-, \quad u \mapsto P^\perp \nabla L(z + u),
\]

is of class \((S)_+\). Moreover, for any two points \( z_0, z_1 \in B_{H^0}(\theta, \epsilon) \) the map

\[
\mathcal{H} : [0, 1] \times \overline{Q_{r,s}} \rightarrow H^+ \oplus H^-, \quad (t, u) \mapsto (1 - t) P^\perp \nabla L(z_0 + u) + t P^\perp \nabla L(z_1 + u)
\]

is a homotopy of class \((S)_+\) (cf. [47, Definition 4.40]).

Proof

By [47, Proposition 4.41] we only need to prove the first claim. Let a sequence \( (u_j) \subset \overline{Q_{r,s}} \) weakly converge to \( u \in H^+ \oplus H^- \), and satisfy \( \lim (P^\perp \nabla L(z + u_j), u_j - u)_H \leq 0 \). It suffices to prove \( u_j \rightarrow u \) in \( H^+ \oplus H^- \). Note that \( u_j \rightarrow u \) in \( H \) because \( \overline{Q_{r,s}} \subset H^+ \oplus H^- \).

So is \( z + u_j \rightarrow z + u \) in \( H \). Moreover, \( u_j - u \in H^+ \oplus H^- \) implies

\[
(P^\perp \nabla L(z + u_j), u_j - u)_H = (\nabla L(z + u_j), u_j - u)_H = (\nabla L(z + u_j), (z + u_j) - (z + u))_H.
\]

It follows that \( \lim (\nabla L(z + u_j), (z + u_j) - (z + u))_H \leq 0 \). But \( \nabla L \) is of class \((S)_+\) near \( \theta \in H \), we have \( z + u_j \rightarrow z + u \) and so \( u_j \rightarrow u \). \( \square \)

Let \( \operatorname{deg} \) denote the Browder–Skrュrnik degree for demicontinuous \((S)_+\)-maps ([7,8,52–54]), see [47, §4.3] for a nice exposition. By Lemma 2.10 \( \deg(f_0, Q_{r,s}, \theta) \) is well-defined and using the Poincaré–Hopf theorem (cf. [16, Theorem 1.2]) we have

\[
\deg(f_0, Q_{r,s}, \theta) = \sum_{q=0}^{\infty} (-1)^q \text{rank} C_q(f_0, \theta; G).
\]
Note that $\mathcal{L}|_{Q_{r,s}}$ satisfies the conditions of Theorem 2.1 at $\theta \in H^+ \oplus H^-$. It follows that $C_q(f_0, \theta; G) = \delta_{\mu q} G$, where $\mu = \dim H^-$. Hence (2.18) becomes
\[
\deg(f_0, Q_{r,s}, \theta) = (-1)^{\mu}.
\] (2.19)

For each $z \in B_{H^0}(\theta, \epsilon)$, we derive from Lemma 2.10 that
\[
\inf \{\|f_z(u)\| : u \in \partial Q_{r,s}\} > 0, \quad \inf \{\|f_z(u) + (1 - t)f_0(u)\| : t \in [0, 1], u \in \partial Q_{r,s}\} > 0.
\]
The former implies that $\deg(f_z, Q_{r,s}, \theta)$ is well-defined, the latter and Lemma 2.11 lead to
\[
\deg(f_z, Q_{r,s}, \theta) = \deg(f_0, Q_{r,s}, \theta) = (-1)^{\mu}.
\] (2.20)

So there exists a point $u_z \in Q_{r,s}$ such that
\[
P_\perp \nabla L(z + u_z) = f_z(u_z) = \theta.
\] (2.21)

**Theorem 2.12** (Parameterized Implicit Function Theorem) Under the assumptions of Theorem 2.2, suppose further that $G_1, \ldots, G_n \in C^1(V, \mathbb{R})$ satisfy

(i) $G_j'(|\theta|) = \theta, \quad j = 1, \ldots, n$;

(ii) for each $j = 1, \ldots, n$, the gradient $\nabla G_j$ has Gâteaux derivative $G_j''(u) \in \mathcal{L}_s(H)$ at any $u \in V$, and $G_j' : V \to \mathcal{L}_s(H)$ is continuous at $\theta$.

Then by shrinking $r > 0, s > 0$ and $\epsilon > 0$ in Lemma 2.10 (if necessary) we have $\delta > 0$ and a unique continuous map
\[
\psi : [-\delta, \delta]^n \times B_{H}(\theta, \epsilon) \cap H^0 \to Q_{r,s} \subset (H^0)^\perp
\] (2.22)
such that for all $(\tilde{\lambda}, z) \in [-\delta, \delta]^n \times B_{H}(\theta, \epsilon) \cap H^0$ with $\tilde{\lambda} = (\lambda_1, \ldots, \lambda_n)$, $\psi(\tilde{\lambda}, \theta) = \theta$ and
\[
P_\perp \nabla L(z + \psi(\tilde{\lambda}, z)) + \sum_{j=1}^{n} \lambda_j P_\perp \nabla G_j(z + \psi(\tilde{\lambda}, z)) = \theta,
\] (2.23)
where $P_\perp$ is as in (2.21). This $\psi$ also satisfies
\[
\|\psi(\tilde{\lambda}, z_1) - \psi(\tilde{\lambda}, z_2)\| \leq 3\|z_1 - z_2\|. \quad \forall (\tilde{\lambda}, z) \in [-\delta, \delta]^n \times B_{H}(\theta, \epsilon) \cap H^0.
\] (2.24)

Moreover, if $G$ is a compact Lie group acting on $H$ orthogonally, $V, \mathcal{L}$ and all $G_j$ are $G$-invariant (and hence $H^0, (H^0)^\perp$ are $G$-invariant subspaces, and $\nabla L, \nabla G_j$ are $G$-equivariant), then $\psi$ is equivariant with respect to $z$, i.e., $\psi(\tilde{\lambda}, g \cdot z) = g \cdot \psi(\tilde{\lambda}, z)$ for $(\tilde{\lambda}, z) \in [-\delta, \delta]^n \times B_{H}(\theta, \epsilon) \cap H^0$ and $g \in G$.

**Proof** Step 1 There exist numbers $\rho_1, \delta \in (0, 1)$ satisfying: (i) $B_{H}(\theta, 2\rho_1) \subset V$, (ii) if $\tilde{\lambda}_k = (\lambda_{k,1}, \ldots, \lambda_{k,n}) \in [-\delta, \delta]^n$ converges to $\tilde{\lambda}_0 = (\lambda_{0,1}, \ldots, \lambda_{0,n}) \in [-\delta, \delta]^n$, $u_k \in B_{H}(\theta, 2\rho_1)$ weakly converges to $u_0 \in B_{H}(\theta, 2\rho_1)$, and it also holds that
\[
\lim \nabla L(u_k) + \sum_{j=1}^{n} \lambda_j \nabla G_j(u_k), u_k - u_0)_H \leq 0.
\] (2.25)

Then $u_k \to u_0$. In particular, for each $\tilde{\lambda} \in [-\delta, \delta]^n$, the map
\[
B_{H}(\theta, 2\rho_1) \times [0, 1] \to H^+ \oplus H^-, \quad (t, u) \mapsto P_\perp \nabla L(u) + \sum_{j=1}^{n} t \lambda_j P_\perp \nabla G_j(u)
\]
is a homotopy of class $(S)_+$ (cf. [47, Definition 4.40]).

In fact, by [33, (5.8)] we have $\rho_1 > 0$ and $C'_0 > 0$ such that $B_{H}(\theta, 2\rho_1) \subset V$ and...
\[
(\nabla L(u), u - u')_H \geq \frac{C'_0}{2} \|u - u'\|^2 + (\nabla L(u'), u - u')_H \\
+ (Q(\theta)(u - u'), u - u')_H, \quad \forall u, u' \in B_H(\theta, 2\rho_1). \quad (2.26)
\]

Similarly, for each fixed \(j \in \{1, \ldots, n\}\), we have \(\tau = \tau(u, u') \in (0, 1)\) such that
\[
(\nabla G_j(u), u - u')_H = (\nabla G_j(u) - \nabla G_j(u'), u - u')_H + (\nabla G_j(u'), u - u')_H
\]

\[
= (\nabla G'_j(\tau u + (1 - \tau)u') - \nabla G''_j(\theta))(u - u')_H + (\nabla G_j(u'), u - u')_H \\
+ (\nabla G'_j(\theta)(u - u'), u - u')_H, \quad \forall u, u' \in B_H(\theta, 2\rho_1).
\]

Since \(V \ni v \mapsto G''_j(v) \in \mathcal{L}_z(H)\) is continuous at \(\theta\), we may shrink \(\rho_1 > 0\) so that
\[
\|G''_j(v) - G''_j(\theta)\| \leq \frac{C'_0}{8n}, \quad \forall v \in B_H(\theta, 2\rho_1), \quad j = 1, \ldots, n. \quad (2.27)
\]

It follows that for all \(u, u' \in B_H(\theta, 2\rho_1)\) and \(j = 1, \ldots, n,\)
\[
|(\nabla G_j(u), u - u')_H| \leq \frac{C'_0}{8n} \|u - u'\|^2 + |(\nabla G_j(u'), u - u')_H| + |(G''_j(\theta)(u - u'), u - u')_H|.
\]

Take \(\delta \in (0, 1)\) so that \(\delta \sum_{j=1}^n \|G''_j(\theta)\| < C'_0/8\). These and (2.26) imply that
\[
(\nabla L(u), u - u')_H + \sum_{j=1}^n \lambda_j (\nabla G_j(u), u - u')_H \\
\geq \frac{C'_0}{4} \|u - u'\|^2 + (\nabla L(u'), u - u')_H + (Q(\theta)(u - u'), u - u')_H \\
- \sum_{j=1}^n |(\nabla G_j(u'), u - u')_H|, \quad \forall \tilde{\lambda} = (\lambda_1, \ldots, \lambda_n) \in [-\delta, \delta]^n.
\]

Replacing \(u, u'\) and \(\lambda_j\) by \(\bar{u}_k, u_0\) and \(\lambda_{k,j}\) in the inequality, we derive from (2.25) that \(u_k \to u_0\) since (D3) implies that \((\nabla L(u_0), u_k - u_0)_H \to 0, (Q(\theta)(u_k - u_0), u_k - u_0)_H \to 0\) and \((\nabla G_j(u_0), u_k - u_0)_H \to 0\).

Note: The above proof shows that the family \(\{\nabla L_{\tilde{\lambda}} := \nabla L + \sum_{j=1}^n \lambda_j \nabla G_j | \tilde{\lambda} \in [-\delta, \delta]^n\}\) satisfies the (PS) condition on \(B_H(\theta, \epsilon)\) for any \(\epsilon < 2\rho_1\), that is, if \(\tilde{\lambda}_k \in [-\delta, \delta]^n\) converges to \(\tilde{\lambda}_0 \in [-\delta, \delta]^n\), and \(u_k \in B_H(\theta, \epsilon)\) satisfies \(\nabla L_{\tilde{\lambda}_k}(u_k) \to \theta\) and \(\sup_k |\nabla L_{\tilde{\lambda}_k}(u_k)| < \infty\), then \((u_k)\) has a converging subsequence \(u_{k_i} \to u_0 \in B_H(\theta, \epsilon)\) with \(\nabla L_{\tilde{\lambda}_0}(u_0) = \theta\).

Step 2 Since \(\nabla L\) and \(\nabla G_1, \ldots, \nabla G_n\) are all locally bounded, for \(r > 0, s > 0\) and \(\epsilon > 0\) in Lemma 2.10, by shrinking them we can assume that \(\overline{B}_{\rho_1}(\theta, \epsilon) \times \overline{Q}_{r,s} \subset B_H(\theta, 2\rho_1)\) and
\[
\sup\{|\nabla L_{\tilde{\lambda}}(z, u)| | \tilde{\lambda}, z, u \in [-1, 1]^n \times \overline{B}_{\rho_1}(\theta, \epsilon) \oplus \overline{Q}_{r,s}\} < \infty. \quad (2.28)
\]

Then by Lemma 2.10 we may shrink \(\delta \in (0, 1)\) so that
\[
\inf t P^\perp \left(\nabla L + \sum_{j=1}^n \lambda_j \nabla G_j\right)(z_1 + u) + (1 - t) P^\perp \left(\nabla L + \sum_{j=1}^n \lambda_j \nabla G_j\right)(z_2 + u) > 0.
\]
where the infimum is taken for all \((t, z_1, z_2, u) \in [0, 1] \times \tilde{B}_{H^0}(\theta, \epsilon) \times \tilde{B}_{H^0}(\theta, \epsilon) \times \partial \Omega_{r,s}\) and \((\lambda_1, \ldots, \lambda_n) \in [-\delta, \delta]^n\). This implies that for each \((\tilde{\lambda}, \tilde{z}) \in [-\delta, \delta]^n \times B_{H^0}(\theta, \epsilon)\), the map

\[
f_{\tilde{\lambda}, \tilde{z}}: \Omega_{r,s} \ni u \mapsto P_\perp \nabla \mathcal{L}(z + u) + \sum_{j=1}^n \lambda_j P_\perp \nabla \mathcal{G}_j(z + u) \in H^+ \oplus H^-
\]

has a well-defined Browder–Skrypnik degree \(\deg(f_{\tilde{\lambda}, \tilde{z}}, \Omega_{r,s}, \theta)\) and

\[
\deg(f_{\tilde{\lambda}, \tilde{z}}, \Omega_{r,s}, \theta) = \deg(f_{0,0}, \Omega_{r,s}, \theta) = \deg(f_{0,0}, \Omega_{r,s}, \theta) = (-1)^\mu, \tag{2.29}
\]

where \(f_{0,0}\) is as in (2.19). Hence for each \((\tilde{\lambda}, \tilde{z}) \in [-\delta, \delta]^n \times B_{H^0}(\theta, \epsilon)\) there exists a point \(u_{\tilde{\lambda}, \tilde{z}} \in \Omega_{r,s}\) such that

\[
P_\perp \nabla \mathcal{L}(z + u_{\tilde{\lambda}, \tilde{z}}) + \sum_{j=1}^n \lambda_j P_\perp \nabla \mathcal{G}_j(z + u_{\tilde{\lambda}, \tilde{z}}) = f_{\tilde{\lambda}, \tilde{z}}(u_{\lambda, \zeta}) = 0. \tag{2.30}
\]

By shrinking the above \(\epsilon > 0, r > 0\) and \(s > 0\) (if necessary), \(\omega\) and \(a_0, a_1\) in Lemma 2.6 can satisfy

\[
\omega(z + u) < \min\{a_0, a_1\}/2, \quad \forall (z, u) \in \tilde{B}_{H^0}(\theta, \epsilon) \times \Omega_{r,s}. \tag{2.31}
\]

**Step 3** If \(\delta \in (0, 1)\) is sufficiently small, then \(u_{\tilde{\lambda}, \tilde{z}}\) is a unique zero point of \(f_{\tilde{\lambda}, \tilde{z}}\) in \(\Omega_{r,s}\). In fact, suppose that there exists another different \(u'_{\tilde{\lambda}, \tilde{z}} \in \Omega_{r,s}\) satisfying (2.30). Consider the decomposition \(u_{\tilde{\lambda}, \tilde{z}} - u'_{\tilde{\lambda}, \tilde{z}} = (u_{\tilde{\lambda}, \tilde{z}} - u'_{\tilde{\lambda}, \tilde{z}})^+ + (u_{\tilde{\lambda}, \tilde{z}} - u'_{\tilde{\lambda}, \tilde{z}})^-\). We may prove the conclusion in three cases:

- \(\|(u_{\tilde{\lambda}, \tilde{z}} - u'_{\tilde{\lambda}, \tilde{z}})^+\| > \|(u_{\tilde{\lambda}, \tilde{z}} - u'_{\tilde{\lambda}, \tilde{z}})^-\|\),
- \(\|(u_{\tilde{\lambda}, \tilde{z}} - u'_{\tilde{\lambda}, \tilde{z}})^+\| = \|(u_{\tilde{\lambda}, \tilde{z}} - u'_{\tilde{\lambda}, \tilde{z}})^-\|\),
- \(\|(u_{\tilde{\lambda}, \tilde{z}} - u'_{\tilde{\lambda}, \tilde{z}})^+\| < \|(u_{\tilde{\lambda}, \tilde{z}} - u'_{\tilde{\lambda}, \tilde{z}})^-\|\).

Let us write \(\mathcal{L}' = \mathcal{L} + \sum_{j=1}^n \lambda_j \mathcal{G}_j\) for conveniences. Then (2.30) implies

\[
0 = (P_\perp \nabla \mathcal{L}'(z + u_{\tilde{\lambda}, \tilde{z}}) - P_\perp \nabla \mathcal{L}'(z + u_{\lambda, \zeta}), (u_{\tilde{\lambda}, \tilde{z}} - u'_{\tilde{\lambda}, \tilde{z}})^+)_H
= (P_\perp \nabla \mathcal{L}(z + u_{\tilde{\lambda}, \tilde{z}}) - P_\perp \nabla \mathcal{L}(z + u'_{\lambda, \zeta}), (u_{\tilde{\lambda}, \tilde{z}} - u'_{\tilde{\lambda}, \tilde{z}})^+)_H
+ \sum_{j=1}^n \lambda_j (P_\perp \nabla \mathcal{G}_j(z + u_{\tilde{\lambda}, \tilde{z}}) - P_\perp \nabla \mathcal{G}_j(z + u'_{\lambda, \zeta}), (u_{\tilde{\lambda}, \tilde{z}} - u'_{\tilde{\lambda}, \tilde{z}})^+)_H. \tag{2.32}
\]

For simplicity we write \(u_{\tilde{\lambda}, \tilde{z}}\) and \(u'_{\tilde{\lambda}, \tilde{z}}\) as \(u_z\) and \(u'_{z}\), respectively. In the first two cases, we may use the mean value theorem to get \(\tau \in (0, 1)\) such that

\[
(P_\perp \nabla \mathcal{L}(z + u_z) - P_\perp \nabla \mathcal{L}(z + u'_{z}), (u_z - u'_{z})^+)_H
= (\nabla \mathcal{L}(z + u_z) - \nabla \mathcal{L}(z + u'_{z}), (u_z - u'_{z})^+)_H
= (B(z + \tau u_z + (1 - \tau)u_z')(u_z - u'_{z}), (u_z - u'_{z})^+)_H
= (B(z + \tau u_z + (1 - \tau)u_z')(u_z - u'_{z})^+, (u_z - u'_{z})^+)_H
+ (B(z + \tau u_z + (1 - \tau)u_z')(u_z - u'_{z})^-, (u_z - u'_{z})^+)_H
\geq a_1\|(u_z - u'_{z})^+\|^2 - \omega(z + \tau u_z + (1 - \tau)u_z')\|(u_z - u'_{z})^+\| \cdot \|(u_z - u'_{z})^+\|
\geq a_1\|(u_z - u'_{z})^+\|^2 - \frac{a_1}{4}\|(u_z - u'_{z})^+\|^2 + \|(u_z - u'_{z})^+\|^2
\]

\[
\|u_{\tilde{\lambda}, \tilde{z}} - u'_{\tilde{\lambda}, \tilde{z}}\|^2
\]

\[
\|u_{\tilde{\lambda}, \tilde{z}} - u'_{\tilde{\lambda}, \tilde{z}}\|^2
\]
\[
\geq a_1 \|(u_z - u'_z)^+\|^2 - \frac{a_1}{2} \|(u_z - u'_z)^+\|^2 \\
= \frac{a_1}{2} \|(u_z - u'_z)^+\|^2, \tag{2.33}
\]

where the first inequality comes from Lemma 2.6(i)–(ii), the second is derived from (2.31) and the inequality \(2|ab| \leq |a|^2 + |b|^2\), and the third is because \(\|(u_z - u'_z)^-\|^2 \leq \|(u_z - u'_z)^+\|^2\). It follows from (2.32)–(2.33) that

\[
0 = (P^\perp \nabla \mathcal{L}_\lambda(z + u_{\lambda, z}^+)) - P^\perp \nabla \mathcal{L}_\lambda(z + u_{\lambda, z}^+), (u_{\lambda, z}^+ - u_{\lambda, z}^+)^+)_H \\
\geq \sum_{j=1}^n \lambda_j (G_j''(z + \tau u_{\lambda, z}^+ + (1 - \tau)u_{\lambda, z}^+)(u_{\lambda, z}^+ - u_{\lambda, z}^+), (u_{\lambda, z}^+ - u_{\lambda, z}^+)^+)_H \\
+ \frac{a_1}{2} \|(u_{\lambda, z}^+ - u_{\lambda, z}^+)^+\|^2. \tag{2.34}
\]

By (2.27) we have a constant \(M > 0\) such that

\[
\sup\{\|G_j'(z + w)\| \mid (z, w) \in \overline{B}_{|M|}(\theta, \epsilon) \times \overline{Q}_r, j = 1, \ldots, n\} < M. \tag{2.35}
\]

From this and the inequality \(\|(u_z - u'_z)^-\|^2 \leq \|(u_z - u'_z)^+\|^2\) we deduce

\[
\sum_{j=1}^n |\lambda_j (G_j''(z + \tau u_{\lambda, z}^+ + (1 - \tau)u_{\lambda, z}^+)(u_{\lambda, z}^+ - u_{\lambda, z}^+), (u_{\lambda, z}^+ - u_{\lambda, z}^+)^+)_H| \\
\leq nM \|(u_{\lambda, z}^+ - u_{\lambda, z}^+)^-\|^2, \tag{2.36}
\]

Let us shrink \(\delta > 0\) in Step 2 so that \(\delta < \frac{a_1}{8nM}\). Then (2.34) and (2.36) lead to

\[
0 = (P^\perp \nabla \mathcal{L}_\lambda(z + u_{\lambda, z}^+)) - P^\perp \nabla \mathcal{L}_\lambda(z + u_{\lambda, z}^+), (u_{\lambda, z}^+ - u_{\lambda, z}^+)^+)_H \geq \frac{a_1}{4} \|(u_{\lambda, z}^+ - u_{\lambda, z}^+)^+\|^2.
\]

This contradicts \((u_{\lambda, z}^+ - u_{\lambda, z}^+)^+ \neq \theta\).

Similarly, for the third case, as in (2.33) we may use Lemma 2.6(ii)–(iii) to obtain

\[
0 = (P^\perp \nabla \mathcal{L}(z + u_z^+)) - P^\perp \nabla \mathcal{L}(z + u_z^+), (u_z^+ - u_z^+)^-)_H \\
= (\nabla \mathcal{L}(z + u_z^+) - \nabla \mathcal{L}(z + u_z^+), (u_z^+ - u_z^+)^-)_H \\
= (B(z + tu_z + (1 - t)u_z^+)(u_z - u_z^+), (u_z - u_z^+)^-)_H \\
= (B(z + tu_z + (1 - t)u_z^+)(u_z - u_z^+)^-, (u_z - u_z^+)^-)_H \\
+ (B(z + tu_z + (1 - t)u_z^+)(u_z - u_z^+)^+, (u_z - u_z^+)^-)_H \\
\leq -a_0 \|(u_z - u_z^+)^-\|^2 + \omega(z + tu_z + (1 - t)u_z^+)(u_z - u_z^+)^- \cdot \|(u_z - u_z^+)^+\| \\
\leq -a_0 \|(u_z - u_z^+)^-\|^2 + \frac{a_0}{4} \|(u_z - u_z^+)^-\|^2 + \|(u_z - u_z^+)^+\|^2 \\
\leq -a_0 \|(u_z - u_z^+)^-\|^2 + \frac{a_0}{2} \|(u_z - u_z^+)^-\|^2 \\
= -\frac{a_0}{2} \|(u_z - u_z^+)^-\|^2,
\]
and hence
\[
0 = (P^\perp \nabla \mathcal{L}_{\hat{\lambda}}(z + u_{\hat{\lambda}, z}^\perp) - P^\perp \nabla \mathcal{L}_{\hat{\lambda}}(z + u_{\hat{\lambda}, z}^\perp), (u_{\hat{\lambda}, z}^\perp - u_{\hat{\lambda}, z}^\perp)^-)_H \\
\leq \sum_{j=1}^{n} \lambda_j \langle G_j^\ast(z + \tau u_{\hat{\lambda}, z} + (1 - \tau)u_{\hat{\lambda}, z}^\perp), (u_{\hat{\lambda}, z}^\perp - u_{\hat{\lambda}, z}^\perp)^- \rangle_H \\
- \frac{a_0}{2} \| (u_{\hat{\lambda}, z}^\perp - u_{\hat{\lambda}, z}^\perp)^+ \|^2.
\]
(2.37)
As in (2.36) we may deduce
\[
\sum_{j=1}^{n} |\lambda_j \langle G_j^\ast(z + \tau u_{\hat{\lambda}, z} + (1 - \tau)u_{\hat{\lambda}, z}^\perp), (u_{\hat{\lambda}, z}^\perp - u_{\hat{\lambda}, z}^\perp)^- \rangle_H |
\leq n\delta M \| (u_{\hat{\lambda}, z}^\perp - u_{\hat{\lambda}, z}^\perp)^- \| \cdot \| (u_{\hat{\lambda}, z}^\perp - u_{\hat{\lambda}, z}^\perp)^- \|
\leq n\delta M \| (u_{\hat{\lambda}, z}^\perp - u_{\hat{\lambda}, z}^\perp)^- \|^2 + \| (u_{\hat{\lambda}, z}^\perp - u_{\hat{\lambda}, z}^\perp)^- \| \cdot \| (u_{\hat{\lambda}, z}^\perp - u_{\hat{\lambda}, z}^\perp)^+ \|
\leq 2n\delta M \| (u_{\hat{\lambda}, z}^\perp - u_{\hat{\lambda}, z}^\perp)^- \|^2.
\]
So if the above $\delta > 0$ is also shrunk so that $\delta < \frac{a_0}{8nM}$, we may derive from this and (2.37) that
\[
0 = (P^\perp \nabla \mathcal{L}_{\hat{\lambda}}(z + u_{\hat{\lambda}, z}^\perp) - P^\perp \nabla \mathcal{L}_{\hat{\lambda}}(z + u_{\hat{\lambda}, z}^\perp), (u_{\hat{\lambda}, z}^\perp - u_{\hat{\lambda}, z}^\perp)^-)_H \leq -\frac{a_0}{4} \| (u_{\hat{\lambda}, z}^\perp - u_{\hat{\lambda}, z}^\perp)^- \|^2,
\]
which also leads to a contradiction. As a consequence, we have a well-defined map
\[
\psi : [-\delta, \delta]^n \times B_{H^0}(\theta, \epsilon) \to Q_{r,s}, \quad (\lambda, z) \mapsto u_{\hat{\lambda}, z}^\perp.
\]
(2.38)
**Step 4** $\psi$ is continuous. Let sequences $\tilde{\lambda}_k \in [-\delta, \delta]^n$ and $(z_k) \subset B_{H^0}(\theta, \epsilon)$ converge to $\tilde{\lambda}_0 \in [-\delta, \delta]^n$ and $z_0 \in B_{H^0}(\theta, \epsilon)$, respectively. We want to prove that $\psi(\tilde{\lambda}_k, z_k) \to \psi(\tilde{\lambda}_0, z_0)$. Since $\psi(\tilde{\lambda}_k, z_k) \in Q_{r,s}$, $k = 1, 2, \ldots$, we can suppose $\psi(\tilde{\lambda}_k, z_k) \to u_0 \in Q_{r,s}$ in $H$. Noting $\psi(\tilde{\lambda}_k, z_k) - u_0 \in H^+ \oplus H^-$, by (2.30) we have
\[
(\nabla \mathcal{L}_{\hat{\lambda}_k}(z_k + \psi(\tilde{\lambda}_k, z_k)), \psi(\tilde{\lambda}_k, z_k) - u_0) = (P^\perp \nabla \mathcal{L}_{\hat{\lambda}_k}(z_k + \psi(\tilde{\lambda}_k, z_k)), \psi(\tilde{\lambda}_k, z_k) - u_0) = 0.
\]
It follows from this and (2.28) that
\[
\| (\nabla \mathcal{L}_{\hat{\lambda}_k}(z_k + \psi(\tilde{\lambda}_k, z_k)), (z_k + \psi(\tilde{\lambda}_k, z_k)) - (z_0 + u_0) \|
= \| (\nabla \mathcal{L}_{\hat{\lambda}_k}(z_n + \psi(\tilde{\lambda}_k, z_k)), z_k - z_0) \| \leq \| \nabla \mathcal{L}_{\hat{\lambda}_k}(z_k + \psi(\tilde{\lambda}_k, z_k)) \| \cdot \| z_k - z_0 \| \to 0.
\]
As in the proof of Step 1, we may derive from this that $z_k + \psi(\tilde{\lambda}_k, z_k) \to z_0 + u_0$ and so $\psi(\tilde{\lambda}_k, z_k) \to u_0$. Observe that (2.30) implies $P^\perp \nabla \mathcal{L}_{\hat{\lambda}_k}(z_k + \psi(\tilde{\lambda}_k, z_k)) = 0$, $k = 1, 2, \ldots$. The $C^1$-smoothness of $\mathcal{L}$ and all $G_j$ leads to $P^\perp \nabla \mathcal{L}_{\hat{\lambda}_0}(z_0 + u_0) = 0$. By Step 3 we arrive at $\psi(\tilde{\lambda}_0, z_0) = u_0$ and hence $\psi$ is continuous at $(\tilde{\lambda}_0, z_0)$.

**Step 5** For any $(\tilde{\lambda}, z_i) \in [-\delta, \delta]^n \times B_H(\theta, \epsilon) \cap H^0, i = 1, 2$, by the definition of $\psi$, we have
\[
P^\perp \nabla \mathcal{L}(z_i + \psi(\tilde{\lambda}, z_i)) + \sum_{j=1}^{n} \lambda_j \langle P^\perp \nabla G_j(z_i + \psi(\tilde{\lambda}, z_i)), \psi(\tilde{\lambda}, z_i) \rangle = \theta, \quad i = 1, 2,
\]
and hence for $\Xi = z_1 - z_2 + \psi(\tilde{\lambda}, z_1) - \psi(\tilde{\lambda}, z_2) = \Xi^0 + \Xi^+ + \Xi^-$ we derive

$$0 = (P^\perp \nabla \mathcal{L}(z_1 + \psi(\tilde{\lambda}, z_1)) - P^\perp \nabla \mathcal{L}(z_2 + \psi(\tilde{\lambda}, z_2)), \Xi^+)_H$$

$$+ \sum_{j=1}^n \lambda_j (P^\perp \nabla G_j(z_1 + \psi(\tilde{\lambda}, z_1)) - P^\perp \nabla G_j(z_2 + \psi(\tilde{\lambda}, z_2)), \Xi^+)_H. \quad (2.39)$$

As in the proof of (2.33) we obtain $\tau \in (0, 1)$ such that

$$(P^\perp \nabla \mathcal{L}(z_1 + \psi(\tilde{\lambda}, z_1)) - P^\perp \nabla \mathcal{L}(z_2 + \psi(\tilde{\lambda}, z_2)), \Xi^+)_H$$

$$= (B(\tau z_1 + \tau \psi(\tilde{\lambda}, z_1) + (1 - \tau)z_2 + (1 - \tau)\psi(\tilde{\lambda}, z_2)) \Xi^+, \Xi^+)_H$$

$$+ (B(\tau z_1 + \tau \psi(\tilde{\lambda}, z_1) + (1 - \tau)z_2 + (1 - \tau)\psi(\tilde{\lambda}, z_2)) (\Xi^0 + \Xi^-), \Xi^+)_H$$

$$\geq a_1 \| \Xi^+ \|^2 - \frac{a_1}{4} \|[\Xi^- + \Xi^0]^2 + \| \Xi^+ \|^2 \|$$

$$= \frac{3a_1}{4} \|[\Xi^- \|^2 - \frac{a_1}{4} \| \Xi^0 \|^2 - \frac{a_1}{4} \| \Xi^- \|^2. \quad (2.40)$$

Let us further shrink $\delta > 0$ in Step 3 so that $\delta < \frac{\min[a_0, a_1]}{16nM}$. As in (2.36) we may deduce

$$\sum_{j=1}^n \lambda_j (P^\perp \nabla G_j(z_1 + \psi(\tilde{\lambda}, z_1)) - P^\perp \nabla G_j(z_2 + \psi(\tilde{\lambda}, z_2)), \Xi^+)_H \quad (2.41)$$

$$\leq \sum_{j=1}^n |\lambda_j (G_j^\prime(\tau z_1 + (1 - \tau)z_2 + \tau \psi(\tilde{\lambda}, z_1) + (1 - \tau)\psi(\tilde{\lambda}, z_2)) \Xi, \Xi^+)_H|$$

$$\leq n\delta M \| \Xi \| \cdot \| \Xi^+ \| \leq 2n\delta M \|[\Xi \|^2 + \| \Xi^+ \|^2 \|$$

$$\leq \frac{a_1}{8} \|[\Xi^- \|^2 + \| \Xi^0 \|^2 + 2\| \Xi^+ \|^2 \|. \quad (2.42)$$

This, (2.39) and (2.40) lead to

$$0 \geq \frac{3a_1}{4} \|[\Xi^- \|^2 - \frac{a_1}{4} \| \Xi^0 \|^2 - \frac{a_1}{4} \| \Xi^- \|^2 - \frac{a_1}{8} \|[\Xi^- \|^2 + \| \Xi^0 \|^2 + 2\| \Xi^+ \|^2 \|$$

and so

$$0 \geq 4\|[\Xi^- \|^2 - 3\| \Xi^0 \|^2 - 3\| \Xi^- \|^2. \quad (2.43)$$

Similarly, replacing $\Xi^+$ by $\Xi^-$ in (2.40) and (2.41) we derive

$$(P^\perp \nabla \mathcal{L}(z_1 + \psi(\tilde{\lambda}, z_1)) - P^\perp \nabla \mathcal{L}(z_2 + \psi(\tilde{\lambda}, z_2)), \Xi^-)_H$$

$$\leq -\frac{3a_0}{4} \|[\Xi^- \|^2 + \frac{a_0}{4} \| \Xi^0 \|^2 + \frac{a_0}{4} \| \Xi^- \|^2, \quad \sum_{j=1}^n \lambda_j (P^\perp \nabla G_j(z_1 + \psi(\tilde{\lambda}, z_1)) - P^\perp \nabla G_j(z_2 + \psi(\tilde{\lambda}, z_2)), \Xi^-)_H$$

$$\leq \frac{a_0}{8} \|[\Xi^+ \|^2 + \| \Xi^0 \|^2 + 2\| \Xi^- \|^2 \|.$$

As above these two inequalities and the equality

$$0 = (P^\perp \nabla \mathcal{L}(z_1 + \psi(\tilde{\lambda}, z_1)) - P^\perp \nabla \mathcal{L}(z_2 + \psi(\tilde{\lambda}, z_2)), \Xi^-)_H$$

$$+ \sum_{j=1}^n \lambda_j (P^\perp \nabla G_j(z_1 + \psi(\tilde{\lambda}, z_1)) - P^\perp \nabla G_j(z_2 + \psi(\tilde{\lambda}, z_2)), \Xi^-)_H$$
yield: $0 \geq 4\|\Xi^-\|^2 - 3\|\Xi^0\|^2 - 3\|\Xi^+\|^2$. Combing with (2.43) we obtain

$$\|\Xi^+ + \Xi^-\|^2 = \|\Xi^+\|^2 + \|\Xi^-\|^2 \leq 6\|\Xi^0\|^2.$$  

Note that $\Xi^0 = z_1 - z_2$ and $\Xi^+ + \Xi^- = \psi(\lambda, z_1) - \psi(\lambda, z_2)$. The desired claim is proved.

**Step 6** The uniqueness of $\psi$ implies that it is equivariant with respect to $z$.  

As a by-product we have also the following result though it is not used in this paper.

**Theorem 2.13** (Inverse Function Theorem) If the assumptions of Theorem 2.1 hold with $X = H$, then $\nabla L$ is a homeomorphism near $\theta$.

**Proof** We can assume that $\nabla L$ is of class $(S)_+$ in $Q_{r,s}$. Since $H^0 = \{\theta\}$ and $\nabla L = f_0$,

$$\deg(\nabla L, Q_{r,s}, \theta) = (-1)^\mu$$

by (2.19). Moreover, $\varrho := \inf\{\|\nabla L(u)\| \mid u \in \partial Q_{r,s}\} > 0$ by Lemma 2.10. For any given $v \in B_H(\theta, \varrho)$, let us define $\mathcal{H} : [0, 1] \times Q_{r,s} \to H$, $(t, u) \mapsto \nabla L(u) - tv$. Then

$$\|\mathcal{H}(t, u)\| = \|\nabla L(u) - tv\| \geq \|\nabla L(u)\| - \|v\| \geq \varrho - \|v\| > 0, \forall (t, u) \in [0, 1] \times \partial Q_{r,s}.$$  

Let $t_n \to t$ in $[0, 1], (u_n) \subset Q_{r,s}$ converge weakly to $u$ in $H$, and $\limsup_{n \to \infty} \|\mathcal{H}(t_n, u_n) - u\|_H \leq 0$. Then $(\nabla L(u_n), u_n - u)_H = (\mathcal{H}(t_n, u_n), u_n - u)_H + t_n(v, u_n - u)_H$ leads to

$$\limsup_{n \to \infty} (\nabla L(u_n), u_n - u)_H \leq 0.$$  

It follows that $u_n \to u$ in $H$ because $\nabla L$ is of class $(S)_+$ in $Q_{r,s}$. Hence $\mathcal{H}$ is a homotopy of class $(S)_+$, and thus (2.44) gives $\deg(\nabla L - v, Q_{r,s}, \theta) = \deg(\nabla L, Q_{r,s}, \theta) = (-1)^\mu$. This implies $\nabla L(\xi_v) = v$ for some $\xi_v \in Q_{r,s}$. By Step 3 in the proof of Theorem 2.12 (taking $\lambda = 0$) it is easily seen that the equation $\nabla L(u) = v$ has a unique solution in $Q_{r,s}$, and hence $\xi_v$ is unique. Then we get a map $B_H(\theta, \varrho) \ni v \mapsto \xi_v \in Q_{r,s}$ to satisfy $\nabla L(\xi_v) = v$ for all $v \in B_H(\theta, \varrho)$. We claim that this map is continuous. Arguing by contradiction, assume that there exists a sequence $v_n \to v$ in $B_H(\theta, \varrho)$, such that $\xi_{v_n} \to \xi^*$ in $H$ and $\|\xi_{v_n} - \xi^*\| \geq \epsilon_0$ for some $\epsilon_0 > 0$ and all $n = 1, 2, \ldots$. Note that

$$(\nabla L(\xi_{v_n}), \xi_{v_n} - \xi^*)_H = (v_n, \xi_{v_n} - \xi^*)_H = (v_n - v, \xi_{v_n} - \xi^*)_H + (v, \xi_{v_n} - \xi^*)_H \to 0.$$  

We derive that $\xi_{v_n} \to \xi^*$ in $H$, and so $\nabla L(\xi_{v_n}) = v_n$ can lead to $\nabla L(\xi^*) = v$. The uniqueness of solutions implies $\xi^* = \xi_v$. This prove the claim. Hence $\nabla L$ is a homeomorphism from an open neighborhood $\{\xi_v \mid v \in B_H(\theta, \varrho)\}$ of $\theta$ in $Q_{r,s}$ onto $B_H(\theta, \varrho)$.

**Theorem 2.13** cannot be derived from the invariance of domain theorem (5.4.1) of Berger [3] or [25, Theorem 2.5]. Recently, Ekeland proved an weaker inverse function theorem, [24, Theorem 2]. Since we cannot insure that $B(u)$ has a right-inverse $L(u)$ which is uniformly bounded in a neighborhood of $\theta$, Theorem 2.13 cannot be derived from [24, Theorem 2] either.

### 2.4 Parameterized splitting and shifting theorems

To shorten the proof of the main theorem, we shall write parts of it into two propositions.
Proposition 2.14 Under the assumptions of Theorem 2.12, for each $(\tilde{\lambda}, z) \in [-\delta, \delta] \times B_{H^0}(\theta, e)$, let $\psi_H(z) = \psi(\tilde{\lambda}, z)$ be given by (2.22). Then it satisfies

$$L_{\tilde{\lambda}}(z + \psi_H(z)) = \min\{L_{\tilde{\lambda}}(z + u) \mid u \in B_H(\theta, r) \cap H^+\} \text{ if } H^- = \{\theta\},$$

$$L_{\tilde{\lambda}}(z + \psi_H(z)) = \min\{L_{\tilde{\lambda}}(z + u + P^-\psi_H(z)) \mid u \in B_H(\theta, r) \cap H^+\} \text{ if } H^- \neq \{\theta\}.$$ 

Proof Case $H^- = \{\theta\}$. Then (2.23) becomes $P^+\nabla L_{\tilde{\lambda}}(z + \psi_H(z)) = 0 \forall z \in B_{H^0}(\theta, e)$ since $Q_{r,s} = B_H(\theta, r) \cap H^+$. This and the integral mean value theorem give for each $u \in Q_{r,s}$,

$$L_{\tilde{\lambda}}(z + u) - L_{\tilde{\lambda}}(z + \psi_H(z))$$

$$= \int_0^1 (\nabla L_{\tilde{\lambda}}(z + \psi_H(z) + \tau(u - \psi_H(z))), u - \psi_H(z))_H d\tau$$

$$= \int_0^1 (P^+\nabla L_{\tilde{\lambda}}(z + \psi_H(z) + \tau(u - \psi_H(z))), u - \psi_H(z))_H d\tau$$

$$= \int_0^1 (P^+\nabla L_{\tilde{\lambda}}(z + \psi_H(z) + \tau(u - \psi_H(z))) - P^+\nabla L_{\tilde{\lambda}}(z + \psi_H(z)), u - \psi_H(z))_H d\tau$$

$$= \int_0^1 (\nabla L_{\tilde{\lambda}}(z + \psi_H(z) + \tau(u - \psi_H(z))) - \nabla L_{\tilde{\lambda}}(z + \psi_H(z)), u - \psi_H(z))_H d\tau$$

$$= \int_0^1 \tau d\tau \int_0^1 (B(z + \psi_H(z) + \rho\tau(u - \psi_H(z)))(u - \psi_H(z)), u - \psi_H(z))_H d\rho$$

$$+ \sum_{j=1}^n \lambda_j \int_0^1 \tau d\tau \int_0^1 (G''_j(z + \psi_H(z) + \rho\tau(u - \psi_H(z)))(u - \psi_H(z)), u - \psi_H(z))_H d\rho$$

$$\geq \frac{d_1}{2} \|u - \psi_H(z)\|^2$$

$$\sum_{j=1}^n \lambda_j \int_0^1 \tau d\tau \int_0^1 (G''_j(z + \psi_H(z) + \rho\tau(u - \psi_H(z)))(u - \psi_H(z)), u - \psi_H(z))_H d\rho$$

$$\leq 2n\delta M \|u - \psi_H(z)\|^2 \leq \frac{d_1}{4} \|u - \psi_H(z)\|^2.$$

These lead to

$$L_{\tilde{\lambda}}(z + u) - L_{\tilde{\lambda}}(z + \psi_H(z)) \geq \frac{d_1}{4} \|u - \psi_H(z)\|^2,$$ (2.45)

which implies the desired conclusion.

Case $H^- \neq \{\theta\}$. For each $u \in B_H(\theta, r) \cap H^+$ we have $u + P^-\psi_H(z) \in Q_{r,s}$. As above we can use (2.23) to derive

$$L_{\tilde{\lambda}}(z + u + P^-\psi_H(z)) - L_{\tilde{\lambda}}(z + \psi_H(z)) \geq \frac{d_1}{4} \|u - P^+\psi_H(z)\|^2,$$ (2.46)
and therefore the second equality. Similarly, for each \( v \in B_{H}(\theta, r) \cap H^{-} \) we have \( P^{+}\psi_{\lambda}(z) + v \in Q_{r,s} \), and use (2.23) and Lemma 2.6(ii)–(iii) to deduce

\[
\mathcal{L}_{\lambda}(z + P^{+}\psi_{\lambda}(z) + v) - \mathcal{L}_{\lambda}(z + \psi_{\lambda}(z))
= \int_{0}^{1} (\nabla \mathcal{L}_{\lambda}(z + \psi_{\lambda}(z) + t(u - P^{+}\psi_{\lambda}(z))), v - P^{-}\psi_{\lambda}(z))_{H} dt
\]

\[
= \int_{0}^{1} (\nabla \mathcal{L}_{\lambda}(z + \psi_{\lambda}(z) + t(v - P^{-}\psi_{\lambda}(z))) - \nabla \mathcal{L}_{\lambda}(z + \psi_{\lambda}(z)), v - P^{-}\psi_{\lambda}(z))_{H} dt
\]

\[
= \int_{0}^{1} t \int_{0}^{1} (B(z + \psi_{\lambda}(z) + \tau t(v - P^{-}\psi_{\lambda}(z)))(v - P^{-}\psi_{\lambda}(z)), v - P^{-}\psi_{\lambda}(z))_{H} dt d\tau
\]

\[
+ \sum_{j=1}^{n} \lambda_{j} \int_{0}^{1} t dt \int_{0}^{1} (\mathcal{G}_{j}(z + \psi_{\lambda}(z) + \tau t(v - P^{-}\psi_{\lambda}(z)))(v - P^{-}\psi_{\lambda}(z)), v - P^{-}\psi_{\lambda}(z))_{H} dt
\]

\[
\leq -\frac{a_{0}}{2} \|v - P^{-}\psi_{\lambda}(z)\|^{2}
\]

\[
+ \sum_{j=1}^{n} \lambda_{j} \int_{0}^{1} t dt \int_{0}^{1} (\mathcal{G}_{j}(z + \psi_{\lambda}(z) + \tau t(v - P^{-}\psi_{\lambda}(z)))(v - P^{-}\psi_{\lambda}(z)), v - P^{-}\psi_{\lambda}(z))_{H} dt
\]

\[
\leq -\frac{a_{0}}{4} \|v - P^{-}\psi_{\lambda}(z)\|^{2},
\]

and hence the third equality.

Proposition 2.15 Under the assumptions of Theorem 2.12, for each \( (\tilde{\lambda}, z) \in [-\delta, \delta]^{n} \times B_{H^{0}}(\theta, \epsilon) \), let \( \psi_{\lambda}(z) = \psi(\tilde{\lambda}, z) \) be given by (2.22). Then

\[
\mathcal{L}_{\lambda}^{0}(z) := \mathcal{L}_{\lambda}(z + \psi_{\lambda}(z)) = \mathcal{L}(z + \psi(\tilde{\lambda}, z)) + \sum_{j=1}^{n} \lambda_{j} \mathcal{G}_{j}(z + \psi(\tilde{\lambda}, z))
\]

(2.47)

defines a \( C^{1} \) functional on \( B_{H}(\theta, \epsilon) \cap H^{0} \), and its differential is given by

\[
D \mathcal{L}_{\lambda}^{0}(z)[h] = D \mathcal{L}(z + \psi(\tilde{\lambda}, z))[h] + \sum_{j=1}^{n} \lambda_{j} D \mathcal{G}_{j}(z + \psi(\tilde{\lambda}, z))[h], \quad \forall h \in H^{0}.
\]

(2.48)

(Clearly, this implies that \( [-\delta, \delta]^{n} \ni \tilde{\lambda} \mapsto \mathcal{L}_{\lambda}^{0} \in C^{1}(\tilde{B}_{H}(\theta, \epsilon) \cap H^{0}) \) is continuous by shrinking \( \epsilon > 0 \) since \( \dim H^{0} < \infty \).

Proof Case \( H^{-} \neq \{\theta\} \). For fixed \( z \in B_{H}(\theta, \epsilon) \cap H^{0}, h \in H^{0} \), and \( t \in \mathbb{R} \) with sufficiently small \( |t| \), the last two equalities in Proposition 2.14 imply

\[
\mathcal{L}_{\lambda}(z + th + P^{+}\psi_{\lambda}(z + th) + P^{-}\psi_{\lambda}(z)) - \mathcal{L}_{\lambda}(z + P^{+}\psi_{\lambda}(z + th) + P^{-}\psi_{\lambda}(z))
\]

\[
\leq \mathcal{L}_{\lambda}(z + th + \psi_{\lambda}(z + th)) - \mathcal{L}_{\lambda}(z + \psi_{\lambda}(z))
\]

\[
\leq \mathcal{L}_{\lambda}(z + th + P^{+}\psi_{\lambda}(z) + P^{-}\psi_{\lambda}(z + th)) - \mathcal{L}_{\lambda}(z + P^{+}\psi_{\lambda}(z) + P^{-}\psi_{\lambda}(z + th)).
\]

(2.49)
Since $\mathcal{L}_\lambda$ is $C^1$ and $\psi_\lambda$ is continuous we deduce,

\[
\lim_{t \to 0} \frac{\mathcal{L}_\lambda(z + th + P^+\psi_\lambda(z + th) + P^-\psi_\lambda(z) - \mathcal{L}_\lambda(z + P^+\psi_\lambda(z + th) + P^-\psi_\lambda(z))}{t} = \lim_{t \to 0} \int_0^1 D\mathcal{L}_\lambda(z + sth + P^+\psi_\lambda(z + th) + P^-\psi_\lambda(z))[h]ds = D\mathcal{L}_\lambda(z + \psi_\lambda(z))[h].
\]

(2.50)

Here the last equality follows from the Lebesgue’s Dominated Convergence Theorem since \{D\mathcal{L}_\lambda(z + sth + P^+\psi_\lambda(z + th) + P^-\psi_\lambda(z))[h] | 0 \leq s \leq 1, |t| \leq 1\} is bounded by the compactness of \{z + sth + P^+\psi_\lambda(z + th) + P^-\psi_\lambda(z) | 0 \leq s \leq 1, |t| \leq 1\}. Similarly, we have

\[
\lim_{t \to 0} \frac{\mathcal{L}_\lambda(z + th + P^+\psi_\lambda(z) + P^-\psi_\lambda(z) + th) - \mathcal{L}_\lambda(z + P^+\psi_\lambda(z) + P^-\psi_\lambda(z) + th))}{t} = D\mathcal{L}_\lambda(z + \psi_\lambda(z))[h].
\]

(2.51)

Using the Sandwich Theorem we conclude from (2.49), (2.50) and (2.51) that

\[
\lim_{t \to 0} \frac{\mathcal{L}_\lambda(z + th + \psi_\lambda(z + th)) - \mathcal{L}_\lambda(z + \psi_\lambda(z))}{t} = D\mathcal{L}_\lambda(z + \psi_\lambda(z))[h], \quad \forall h \in H^0.
\]

That is, $\mathcal{L}^0_\lambda$ is Gâteaux differentiable and $D\mathcal{L}^0_\lambda(z) = D\mathcal{L}_\lambda(z + \psi_\lambda(z))[H^0]$. The latter implies that $\mathcal{L}^0_\lambda$ is of class $C^1$ because both $D\mathcal{L}_\lambda$ and $\psi_\lambda$ are continuous.

**Case $H^{-} = \{\theta\}$.** For fixed $z \in B_H(\theta, \epsilon) \cap H^0$ and $h \in H^0$, and $t \in \mathbb{R}$ with sufficiently small $|t|$, the first equality in Proposition 2.14 implies

\[
\mathcal{L}_\lambda(z + th + \psi_\lambda(z + th)) - \mathcal{L}_\lambda(z + \psi_\lambda(z + th)) \\
\leq \mathcal{L}_\lambda(z + th + \psi_\lambda(z + th)) - \mathcal{L}_\lambda(z + \psi_\lambda(z)) \\
\leq \mathcal{L}_\lambda(z + th + \psi_\lambda(z)) - \mathcal{L}_\lambda(z + \psi_\lambda(z)).
\]

(2.52)

By the continuity of $\nabla \mathcal{L}_\lambda$ and $\psi_\lambda$ we obtain

\[
\lim_{t \to 0} \frac{\mathcal{L}_\lambda(z + th + \psi_\lambda(z + th)) - \mathcal{L}_\lambda(z + \psi_\lambda(z + th))}{t} = \lim_{t \to 0} \int_0^1 D\mathcal{L}_\lambda(z + sth + \psi_\lambda(z + th))[h]ds = D\mathcal{L}_\lambda(z + \psi_\lambda(z))[h].
\]

(2.53)

(As above this follows from the Lebesgue’s Dominated Convergence Theorem because $[z + sth + \psi_\lambda(z + th)] | 0 \leq s \leq 1, 0 \leq t \leq 1$ is compact and thus \{D\mathcal{L}_\lambda(z + sth + \psi_\lambda(z + th))[h] | 0 \leq s \leq 1, |t| \leq 1\} is bounded). Similarly, we may prove

\[
\lim_{t \to 0} \frac{\mathcal{L}_\lambda(z + th + \psi_\lambda(z)) - \mathcal{L}_\lambda(z + \psi_\lambda(z))}{t} = D\mathcal{L}_\lambda(z + \psi_\lambda(z))[h],
\]

(2.54)

and thus

\[
\lim_{t \to 0} \frac{\mathcal{L}_\lambda(z + th + \psi_\lambda(z + th)) - \mathcal{L}_\lambda(z + \psi_\lambda(z))}{t} = D\mathcal{L}_\lambda(z + \psi_\lambda(z))[h]
\]

by (2.52), (2.52) and (2.54). The desired claim follows immediately.
Theorem 2.16 (Parameterized Splitting Theorem) Under the assumptions of Theorem 2.12, by shrinking $\delta > 0$, $\epsilon > 0$ and $r > 0$, $s > 0$, we obtain an open neighborhood $W$ of $\theta$ in $H$ and an origin-preserving homeomorphism

$$[-\delta, \delta]^n \times B_{H^0}(\theta, \epsilon) \times (B_{H^+}(\theta, r) + B_{H^-}(\theta, s)) \to [-\delta, \delta]^n \times W,$$

such that

$$L_\lambda \circ \Phi_\lambda(z, u^+ + u^-) = ||u^+||^2 - ||u^-||^2 + L_\lambda(z + \psi(\lambda, z))$$

for all $(\lambda, z, u^+ + u^-) \in [-\delta, \delta]^n \times B_{H^0}(\theta, \epsilon) \times (B_{H^+}(\theta, r) + B_{H^-}(\theta, s))$, where $\psi$ is given by (2.22). The functional $L_\lambda^0 : B_H(\theta, \epsilon) \cap H^0 \to \mathbb{R}$ given by (2.47) is of class $C^1$, and its differential is given by (2.48). Moreover, (i) if $L$ and $G_j$, $j = 1, \ldots, n$, are of class $C^{2,-}$, then so is $L_\lambda^0$ for each $\lambda \in [-\delta, \delta]^n$; (ii) if a compact Lie group $G$ acts on $H$ orthogonally, and $V$, $L$ and $G$ are $G$-invariant (and hence $H^0$, $(H^0)_\bot$ are $G$-invariant subspaces), then for each $\lambda \in [-\delta, \delta]^n$, $\psi(\lambda, \cdot)$ and $\Phi_\lambda(\cdot, \cdot)$ are $G$-equivariant, and $L_\lambda^0(z) = L_\lambda(z + \psi(\lambda, z))$ is $G$-invariant.

If the corresponding conditions with [32, Theorem 1.1] or [33, Remark 3.2] are also satisfied, we can prove: $\psi(\lambda, \cdot)$ is of class $C^1$, $L_\lambda^0$ is of class $C^2$, and

$$D L_\lambda^0(z)[u] = (\nabla L_\lambda^0(z + \psi(\lambda, z)), u)_H, \quad (2.57)$$

$$d^2 L_\lambda^0(z)[u, v] = (L''_\lambda(z + \psi(\lambda, z))(u + D_z\psi(\lambda, z)[u]), v)_H \quad (2.58)$$

for all $z \in B_H(\theta, \epsilon) \cap H^0$ and $u, v \in H^0$. Note that $\psi(\lambda, \theta) = \theta$. We have

$$d^2 L_\lambda^0(\theta)[z_1, z_2] = (L''_\lambda(\theta)(z_1 + D_z\psi(\lambda, \theta)[z_1]), z_2)_H$$

$$= -\sum_{j=1}^n \lambda_j(G''_j(\theta)(z_1 + D_z\psi(\lambda, \theta)[z_1]), z_2)_H, \quad \forall z_1, z_2 \in H^0,$$

and $d^2 L_0^0(\theta) = 0$ by $D_z\psi(\theta, \theta) = \theta$. Moreover, if $L''(\theta)G''_j(\theta) = G''_j(\theta)L''(\theta)$ for $j = 1, \ldots, n$, then

$$d^2 L_\lambda^0(\theta)[z_1, z_2] = -\sum_{j=1}^n \lambda_j(G''_j(\theta)(z_1, z_2)_H, \quad \forall z_1, z_2 \in H^0. \quad (2.59)$$

Claim 2.17 In this situation, if $\theta \in H$ is a nondegenerate critical point of $L_\lambda^0$ then $\theta \in H^0$ is such a critical point of $L_\lambda^0$ too.

In fact, suppose that $z_1 \in H^0$ satisfies $d^2 L_\lambda^0(\theta)[z_1, z_2] = 0 \forall z_2 \in H^0$. Then (2.59) implies

$$(P^0 L''_\lambda(\theta)(z_1 + D_z\psi(\lambda, \theta)[z_1]), u)_H = (P^0 L''_\lambda(\theta)(z_1 + D_z\psi(\lambda, \theta)[z_1]), P^0 u)_H = 0 \forall u \in H.$$ 

Hence $P^0 L''_\lambda(\theta)(z_1 + D_z\psi(\lambda, \theta)[z_1]) = 0$. Moreover, since $(I - P^0)\nabla L_\lambda(z + \psi(\lambda, z)) = \theta$ for all $z \in B_H(\theta, \epsilon) \cap H^0$, Differentiating this equality at $z = \theta$ we get $(I - P^0)\nabla L_\lambda(z_1 + D_z\psi(\lambda, \theta)[z_1]) = \theta$ for all $z_1 \in H^0$. It follows that $L''_\lambda(\theta)(z_1 + D_z\psi(\lambda, \theta)[z_1]) = 0$ and hence $z_1 + D_z\psi(\lambda, \theta)[z_1] = \theta$. Note that $z_1 \in H^0$ and $D_z\psi(\lambda, \theta)[z_1] \in (H^0)_\bot$. We arrive at $z_1 = \theta$. 

\(\blacksquare\) Springer
Proof of Theorem 2.16  Let \( N = H^0 \), and for each \( \tilde{\lambda} \in [-\delta, \delta]^n \) we define a map
\[
F_{\tilde{\lambda}} : B_N(\theta, \epsilon) \times Q_{r,s} \rightarrow \mathbb{R}, \ (z,u) \mapsto L_{\tilde{\lambda}}(z + \psi(\tilde{\lambda},z) + u) - L_{\tilde{\lambda}}(z + \psi(\tilde{\lambda},z)).
\]
(2.61)

Then \( D_2 F_{\tilde{\lambda}}(z,u)[v] = (P_{\perp} \nabla L_{\tilde{\lambda}}(z + \psi(\tilde{\lambda},z) + u), v)_{H} \) for \( z \in B_N(\theta, \epsilon), u \in Q_{r,s} \) and \( v \in N^\perp \). Moreover it holds that
\[
F_{\tilde{\lambda}}(z, \theta) = 0 \quad \text{and} \quad D_2 F_{\tilde{\lambda}}(z, \theta)[v] = 0 \quad \forall v \in N^\perp.
\]
(2.62)

Since \( B_N(\theta, \epsilon) \oplus Q_{r,s} \) has the closure contained in the neighborhood \( U \) in Lemma 2.6, and \( \psi(\tilde{\lambda}, \theta) = \theta \), we can shrink \( \nu > 0, \epsilon > 0, r > 0 \) and \( s > 0 \) so small that
\[
z + \psi(\tilde{\lambda},z) + u^+ + u^- \in U, \quad \forall (\tilde{\lambda},z,u^+ + u^-) \in [-\delta, \delta]^n \times B_N(\theta, \epsilon) \times Q_{r,s}.
\]
(2.63)

Let us verify that each \( F_{\tilde{\lambda}} \) satisfies conditions (ii)–(iv) in [33, Theorem A.1].

**Step 1**

For \( \tilde{\lambda} \in [-\delta, \delta]^n, z \in B_N(\theta, \epsilon), u^+ \in B_{H^+}(\theta, r) \) and \( u^-, u^- \in B_{H^-}(\theta, \epsilon) \), we have
\[
D_2 F_{\tilde{\lambda}}(z,u^+ + u^-)[u^- - u^+] - D_2 F_{\tilde{\lambda}}(z,u^+ + u^-)[u^- - u^+] = \langle \nabla L_{\tilde{\lambda}}(z + \psi(\tilde{\lambda},z) + u^+ + u^-), u^- - u^- \rangle_H
\]
\[\quad - \langle \nabla L_{\tilde{\lambda}}(z + \psi(\tilde{\lambda},z) + u^+ + u^-), u^- - u^- \rangle_H.
\]
(2.64)

Since the function \( u \mapsto \langle \nabla L_{\tilde{\lambda}}(z + \psi(\tilde{\lambda},z) + u^+ + u^-), u^- - u^- \rangle_H \) is Gâteaux differentiable, the mean value theorem yields \( t \in (0, 1) \) such that
\[
\langle \nabla L_{\tilde{\lambda}}(z + \psi(\tilde{\lambda},z) + u^+ + u^-), u^- - u^- \rangle_H = \langle B(z + \psi(\tilde{\lambda},z) + u^+ + u^- + t(u^- - u^-)), u^- - u^- \rangle_H
\]
\[\quad + \sum_{j=1}^{n} \lambda_j \langle G_j''(z + \psi(\tilde{\lambda},z) + u^+ + u^- + t(u^- - u^-)), u^- - u^- \rangle_H
\]
\[\quad \leq \sum_{j=1}^{n} \lambda_j \langle G_j''(z + \psi(\tilde{\lambda},z) + u^+ + u^- + t(u^- - u^-)), u^- - u^- \rangle_H
\]
\[\quad - a_0 \| u^- - u^- \|^2.
\]
(2.65)

because of Lemma 2.6(iii). Recall that we have assumed \( \delta < \frac{\min(a_0, a_1)}{8nM} \) in Step 3 of the proof of Theorem 2.12. From this and (2.35) it follows that
\[
\sum_{j=1}^{n} |\lambda_j \langle G_j''(z + \psi(\tilde{\lambda},z) + u^+ + u^- + t(u^- - u^-)), u^- - u^- \rangle_H|
\]
\[\quad \leq n \delta M \| u^- - u^- \|^2 \leq \frac{a_0}{8} \| u^- - u^- \|^2.
\]

This, (2.64) and (2.65) lead to
\[
(D_2 F_{\tilde{\lambda}}(z, u^+ + u^-) - D_2 F_{\tilde{\lambda}}(z, u^+ + u^-))[u^- - u^-] \leq -\frac{a_0}{2} \| u^- - u^- \|^2.
\]

This implies the condition (ii) of [33, theorem A.1].

**Step 2**

For \( \tilde{\lambda} \in [-\delta, \delta]^n, z \in B_N(\theta, \epsilon), u^+ \in B_{H^+}(\theta, r) \) and \( u^- \in B_{H^-}(\theta, s) \), by (2.62) and the mean value theorem, for some \( t \in (0, 1) \) we have
\[
D_2 F_{\tilde{\lambda}}(z, u^+ + u^-)[u^+ - u^-]
\]
Thus the condition (iii) of \([33, \text{Theorem A.1}]\) is satisfied. In particular, (2.67) also implies because of Lemma 2.6(i) and (iii). As above we have

\[
\sum_{j=1}^{n} |\lambda_j \left( G_j''(z + \psi(z)) + t(u^+ + u^-))(u^+ + u^-), u^+ - u^- \right) H | \\
\leq n\delta M \|u^+ + u^-\| \cdot \|u^+ - u^-\| \\
\leq \min\{a_0, a_1\} (\|u^+\|^2 + \|u^-\|^2) \\
\leq \frac{a_1}{4} \|u^+\|^2 + \frac{a_0}{4} \|u^-\|^2.
\]

This and (2.66) give

\[
D_2 F_\lambda(z, u^+ + u^-)[u^+ - u^-] \geq \frac{a_1}{2} \|u^+\|^2 + \frac{a_0}{2} \|u^-\|^2. \tag{2.67}
\]

Thus the condition (iii) of \([33, \text{Theorem A.1}]\) is satisfied. In particular, (2.67) also implies

\[
D_2 F_\lambda(z, u^+)[u^+ - u^-] \geq \frac{a_1}{2} \|u^+\|^2 > p(\|u^+\|), \quad \forall u^+ \in \bar{B}_{H^+}(\theta, s) \setminus \{\theta\},
\]

where \(p : (0, \varepsilon) \to (0, \infty)\) is a non-decreasing function given by \(p(t) = \frac{a_1}{4} t^2\). Namely, \(F_\lambda\) satisfies the condition (iv) of \([33, \text{Theorem A.1}]\) (the parameterized version of \([22, \text{Theorem 1.1}]\)).

The other arguments are as before.

**Step 3** The claim (i) in the part of “Moreover” follows from (2.24) directly. For the second one, since \(\psi(\lambda, \cdot)\) is \(G\)-equivariant, and \(L_\lambda\) is \(G\)-invariant, we derive from (2.61) that \(F_\lambda\) is \(G\)-invariant. By the construction of \(\Phi_\lambda(\cdot, \cdot)\) (cf. \([22]\) and \([32, \text{Theorem A.1}]\)), it is expressed by \(F_\lambda(z, \cdot, \cdot)\), one easily sees that \(\Phi_\lambda(\cdot, \cdot)\) is \(G\)-equivariant. \(\square\)

**Theorem 2.18** (Parameterized Shifting Theorem) Suppose for some \(\tilde{\lambda} \in [\delta, \delta]'\) that \(\theta \in H\) is an isolated critical point of \(L_\lambda\) (thus \(\theta \in H^0\) is that of \(L_\lambda^0\)). Then

\[
C_q(L_\lambda^0, \theta; K) = C_{q-\mu}(L_\lambda^0, \theta; K) \quad \forall q \in \mathbb{N} \cup \{0\}, \tag{2.68}
\]

where \(L_\lambda^0(z) = L_\lambda(z + \psi(\tilde{\lambda}, z)) = L(z + \psi(\tilde{\lambda}, z)) + \sum_{j=1}^{n} \lambda_j G_j(z + \psi(\tilde{\lambda}, z))\) is as in (2.47).
Proof} Though $\mathcal{L}_\mathcal{H}$ and $\mathcal{L}_\mathcal{O}$ are only of class $C^1$, the construction of the Gromoll–Meyer pair on the pages 49–51 of [13] is also effective for them (see [14]). Hence the result can be obtained by repeating the proof of [13, Theorem I.5.4]. Of course, with a stability theorem of critical groups the present case can also be reduced to that of [13, Theorem I.5.4]. See [38] for a detailed proof.

2.5 Splitting and shifting theorems around critical orbits

We shall list main results and related corollaries for convenience of later applications as in Sect. 4 and [39]. Outlines for their proofs are also given because our methods are completely different from those in the literature. Let $H$ be a Hilbert space with inner product $(\cdot, \cdot)_H$ and let $(\mathcal{H}, (\cdot, \cdot))$ be a $C^2$ Hilbert-Riemannian manifold modeled on $H$. Let $O \subset \mathcal{H}$ be a compact $C^3$ submanifold without boundary, and let $\pi : NO \to O$ denote the normal bundle of it. The bundle is a $C^2$-Hilbert vector bundle over $O$, and can be considered as a subbundle of $T_O \mathcal{H}$ via the Riemannian metric $(\cdot, \cdot)_H$.

are self-adjoint with respect to the inner product $(\cdot, \cdot)$.

Let $\mathcal{L} : \mathcal{H} \to \mathbb{R}$ be a $C^1$ functional. A connected $C^3$ submanifold $O \subset \mathcal{H}$ is called a critical manifold of $\mathcal{L}$ if $\mathcal{L}|_O$ is constant and $D\mathcal{L}(x)[v] = 0$ for any $x \in O$ and $v \in T_x \mathcal{H}$. If there exists a neighborhood $\mathcal{V}$ of $O$ such that $\mathcal{V} \setminus O$ contains no critical points of $\mathcal{L}$ we say $O$ to be isolated. We make:

Hypothesis 2.19

The gradient field $\nabla \mathcal{L} : \mathcal{H} \to T\mathcal{H}$ is Gâteaux differentiable and thus there exists an operator $(d^2 \mathcal{L}(x)) \in \mathcal{L}_x(T\mathcal{H})$ for each $x \in O$; moreover, $O \ni x \mapsto d^2 \mathcal{L}(x)$ is a continuous section of $\mathcal{L}_x(T\mathcal{H}) \to O$, $\dim \ker(d^2 \mathcal{L}(x)) = 0$ for any $x \in O$, and there exists $a_0 > 0$ such that $(\sigma(d^2 \mathcal{L}(x))) \cap \{-2a_0, 2a_0\} = 0$, $\forall x \in O$.

This implies that $O \ni x \mapsto B_x(\theta_x) := \Pi_x \circ d^2 \mathcal{L}(x)|_{NO_x} = d^2(\mathcal{L} \circ \exp_{NO_x})(\theta_x)$ is a continuous section of $\mathcal{L}_x(NO) \to O$, $\dim \ker(B_x(\theta_x)) = 0$ for all $x \in O$, and $\sigma(B_x(\theta_x)) \cap \{-2a_0, 2a_0\} = 0$. Denote by $N^O \mathcal{H} = P^* \mathcal{H}, \ast = +, -, 0$. Clearly, $B_x(\theta_x)(N^O \mathcal{H}) \subset N^O \mathcal{H}$ for any $x \in O$ and $\ast = +, -, 0$. By [12, Lem.7.4], we have $N^O \mathcal{H} = N^+ \mathcal{H} \oplus N^- \mathcal{H} \oplus N^0 \mathcal{H}$. If $\text{rank} N^0 \mathcal{H} = 0$, the critical orbit $O$ is called nondegenerate.

In the following we only consider the case $O$ is a critical orbit of a compact Lie group. The general case can be treated as in [35]. The following assumption implies naturally Hypothesis 2.19 in this case.

Hypothesis 2.20

(i) Let $G$ be a compact Lie group, and let $H$ be a $C^3$ Hilbert-Riemannian $G$-space (that is, $H$ is a $C^3$ $G$-Hilbert manifold with a Riemannian metric $(\cdot, \cdot)$) such that $T\mathcal{H}$ is a $C^2$ Riemannian $G$-vector bundle, see [65]).

(ii) The $C^1$ functional $\mathcal{L} : \mathcal{H} \to \mathbb{R}$ is $G$-invariant, $\nabla \mathcal{L} : \mathcal{H} \to T\mathcal{H}$ is Gâteaux differentiable (i.e., under any $C^3$ local chart the functional $\mathcal{L}$ has a Gâteaux differentiable gradient map), and $O$ is an isolated critical orbit which is a $C^3$ critical submanifold with Morse index $\mu_O$. 

\[ \square \]
Since \( \exp_{g \cdot x}(g \cdot v) = g \cdot \exp_x(v) \) for any \( g \in G \) and \((x, v) \in T \mathcal{H}\), we have \( \mathcal{L} \circ \exp(g \cdot x, g \cdot v) = \mathcal{L}(\exp(g \cdot x, g \cdot v)) = \mathcal{L}(g \cdot \exp(x, v)) = \mathcal{L}(\exp(x, v)) \). It follows that

\[
\nabla \left( \mathcal{L} \circ \exp \right)_{g \cdot x}(g \cdot v) = g \cdot \nabla \left( \mathcal{L} \circ \exp \right)_{x}(v)
\]

for any \( g \in G \) and \((x, v) \in \mathcal{O} \), which leads to

\[
d^2 \left( \mathcal{L} \circ \exp \right)_{g \cdot x}(g \cdot v) \cdot g = g \cdot d^2 \left( \mathcal{L} \circ \exp \right)_{x}(v)
\]

as bounded linear operators from \( \mathcal{O} \) onto \( \mathcal{O} \).

Corresponding to Theorems 2.9 and 2.16 we have the following two theorems.

**Theorem 2.21** (Parameterized Morse–Palais lemma around critical orbits) Under Hypothesis 2.20, let for some \( x_0 \in \mathcal{O} \) the pair \( (\mathcal{L} \circ \exp_{x_0}, B_{T_{x_0}H}(\theta, \epsilon)) \) and so the pair \( (\mathcal{L} \circ \exp |_{\mathcal{O}(x_0)}, \mathcal{O}(x_0)) \) by Lemma 2.8) satisfies the corresponding conditions in Hypothesis 1.1 with \( X = H \). Let \( G \)-invariant functionals \( G_j \in C^1(H, \mathbb{R}) \), \( j = 1, \ldots, n \), have value zero and vanishing derivative at each point of \( \mathcal{O} \), and also fulfill:

(i) gradients \( \nabla G_j \) have Gâteaux derivatives \( G_j''(u) \) at each point \( u \) near \( \mathcal{O} \),

(ii) \( G_j'(u) \) are continuous at each point \( u \in \mathcal{O} \) (and hence each \( \mathcal{G}_j \) is of class \( C^{2-0} \) near \( \mathcal{O} \)).

Suppose that the critical orbit \( \mathcal{O} \) is nondegenerate. Then there exist \( \delta > 0 \), \( \epsilon > 0 \) and a continuous map \( \Phi : [-\delta, \delta]^n \times N^0 \mathcal{O}(\epsilon) \oplus N^+ \mathcal{O}(\epsilon) \oplus N^- \mathcal{O}(\epsilon) \rightarrow \mathcal{O} \) such that each \( \Phi(\lambda, \cdot) : N^+ \mathcal{O}(\epsilon) \oplus N^- \mathcal{O}(\epsilon) \rightarrow \mathcal{O} \) is a \( G \)-equivariant homeomorphism onto an open neighborhood of the zero section preserving fibers, and that \( \mathcal{L}_\lambda := \mathcal{L} + \sum_{j=1}^n g_j \) satisfies

\[
\mathcal{L}_\lambda \circ \exp \circ \Phi(\lambda, x, u^+ + u^-) = \|u^+\|^2_\lambda - \|u^-\|^2_\lambda + L^{(2)}_\lambda |_{\mathcal{O}}
\]

for any \( \lambda \in [-\delta, \delta]^n \), \( x \in \mathcal{O} \) and \((u^+, u^-) \in N^0 \mathcal{O}(\epsilon)_x \times N^- \mathcal{O}(\epsilon)_x \).

This theorem will be proved after the proof of the following theorem.

**Theorem 2.22** (Parameterized Splitting Theorem around critical orbits) Suppose that the critical orbit \( \mathcal{O} \) in Theorem 2.21 is degenerate, i.e., \( \text{rank} N^0 \mathcal{O} > 0 \). Then for sufficiently small \( \epsilon > 0 \), \( \delta > 0 \), the following hold:

(I) There exists a unique continuous map

\[
\mathcal{H} : [-\delta, \delta]^n \times N^0 \mathcal{O}(3\epsilon) \rightarrow N^+ \mathcal{O} \oplus N^- \mathcal{O}, \quad (\lambda, x, v) \mapsto h_{\lambda}(\lambda, v),
\]

such that for each \( \lambda \in [-\delta, \delta]^n \), \( h(\lambda, \cdot) : N^0 \mathcal{O}(3\epsilon) \rightarrow N^+ \mathcal{O} \oplus N^- \mathcal{O} \) is a \( G \)-equivariant topological bundle morphism that preserves the zero section and satisfies

\[
(P^+_x + P^-_x) \circ \Pi_x \nabla(\mathcal{L}_\lambda \circ \exp_x)(v + h_{\lambda}(\lambda, v)) = 0 \quad \forall (x, v^0) \in N^0 \mathcal{O}(\epsilon).
\]

(II) There exists a continuous map \( \Phi : [-\delta, \delta]^n \times N^0 \mathcal{O}(\epsilon) \oplus N^+ \mathcal{O}(\epsilon) \oplus N^- \mathcal{O}(\epsilon) \rightarrow \mathcal{O} \) such that for each \( \lambda \in [-\delta, \delta]^n \), \( \Phi(\lambda, \cdot) : N^0 \mathcal{O}(\epsilon) \oplus N^+ \mathcal{O}(\epsilon) \oplus N^- \mathcal{O}(\epsilon) \rightarrow \mathcal{O} \) is a \( G \)-equivariant homeomorphism onto an open neighborhood of the zero section preserving fibers, and such that

\[
\mathcal{L}_\lambda \circ \exp \circ \Phi(\lambda, x, v, u^+ + u^-) = \|u^+\|^2_\lambda - \|u^-\|^2_\lambda + L^{(2)}_\lambda |_{\mathcal{O}}(v + h_{\lambda}(\lambda, v))
\]

for any \( \lambda \in [-\delta, \delta]^n \), \( x \in \mathcal{O} \) and \((v, u^+, u^-) \in N^0 \mathcal{O}(\epsilon)_x \times N^+ \mathcal{O}(\epsilon)_x \times N^- \mathcal{O}(\epsilon)_x \).

(III) For each \((\lambda, x) \in [-\delta, \delta]^n \times \mathcal{O} \) the functional

\[
N^0 \mathcal{O}(\epsilon)_x \rightarrow \mathbb{R}, \quad v \mapsto L^{(2)}_{\lambda,x}(v) := \mathcal{L}_\lambda \circ \exp_x(v + h_{\lambda}(\lambda, v))
\]
is $G_x$-invariant, of class $C^1$, and has differential given by
\[ DC^0_{\lambda,x}(v)[v'] = D(L_{\lambda} \circ \exp_x)(v + h_x(\lambda, v))[v'], \quad \forall v' \in N^0\mathcal{O}_x. \]

Moreover, each $h_x(\lambda, \cdot)$ is of class $C^{1-0}$, and if $\mathcal{L}$ is of class $C^{2-0}$ so is $LC^0_{\lambda,x}$.

**Proof** We only outline main procedures in case $\lambda = 0$, i.e., $LC^0 = \mathcal{L}$. By the assumption and (2.70) we deduce that each pair $(\mathcal{L} \circ \exp|_{N\mathcal{O}(\epsilon)x}, N\mathcal{O}(\epsilon)x)$ satisfies the corresponding conditions with Hypothesis 1.1 with $X = H$ too, and that there exists $a_0 > 0$ such that
\[ \sigma \left( d^2 \left( \mathcal{L} \circ \exp|_{N\mathcal{O}(\epsilon)x} \right)(\theta_x) \right) \cap \left( [-2a_0, 2a_0] \setminus \{0\} \right) = \emptyset, \quad \forall x \in \mathcal{O}. \quad (2.72) \]

By Theorem 2.2 we have $\epsilon \in (0, \epsilon/3)$ and a continuous map $h_{x_0} : N^0\mathcal{O}(3\epsilon)x_0 \to N^\pm\mathcal{O}(\epsilon/2)x_0$, such that $h_{x_0}(g \cdot v) = g \cdot h_{x_0}(v)$, $h_{x_0}(\theta_{x_0}) = \theta_{x_0}$ and
\[ (P_{x_0}^+ + P_{x_0}^-) \nabla \left( \mathcal{L} \circ \exp|_{N\mathcal{O}(\epsilon)x_0} \right)(v + h_{x_0}(v)) = 0, \quad \forall v \in N^0\mathcal{O}(3\epsilon)x_0. \]

Furthermore, the function
\[ \mathcal{L}^e_{x_0} : N^0\mathcal{O}(\epsilon)x_0 \to \mathbb{R}, \quad v \mapsto \mathcal{L} \circ \exp_{x_0}(v + h_{x_0}(v)) \]

is of class $C^1$, and $D\mathcal{L}^e_{x_0}(v)[u] = D(\mathcal{L} \circ \exp|_{N\mathcal{O}(\epsilon)x_0})(v + h_{x_0}(v))[u]$. Define
\[ h : N^0\mathcal{O}(3\epsilon) \to TH, \quad (x, v) \mapsto g \cdot h_{x_0}(g^{-1} \cdot v), \]

where $g \cdot x_0 = x$. We claim: $h$ is continuous. Otherwise, there exists a sequence $(x_j, v_j) \subset N^0\mathcal{O}(3\epsilon)$ converging to a point $(\tilde{x}, \tilde{v}) \in N^0\mathcal{O}(3\epsilon)$, such that $(h(x_j, v_j))$ has no intersection with an open neighborhood $U$ of $h(\tilde{x}, \tilde{v})$ in $TH$. Let $\tilde{g}, g_j \in G$ be such that $\tilde{g} \cdot x_0 = \tilde{x}$ and $g_j \cdot x_0 = x_j$, $j = 1, 2, \ldots$. Then $h(\tilde{x}, \tilde{v}) = \tilde{g} \cdot h_{x_0}(\tilde{g}^{-1} \cdot \tilde{v})$ and $h(x_j, v_j) = g_j \cdot h_{x_0}(g_j^{-1} \cdot v_j)$ for each $j \in \mathbb{N}$. Note that $\tilde{g}^{-1} \cdot U$ is an open neighborhood of $h_{x_0}(\tilde{g}^{-1} \cdot \tilde{v}) = \tilde{g}^{-1} \cdot h(\tilde{x}, \tilde{v})$ and that the sequences $\tilde{g}^{-1} \cdot h(x_j, v_j) = \tilde{g}^{-1} \cdot g_j \cdot h_{x_0}(g_j^{-1} \cdot v_j)$ have no intersection with $\tilde{g}^{-1} \cdot U$. Since $G$ is compact, we may assume $\tilde{g}^{-1} \cdot g_j \to \hat{g} \in G$ and so $g_j^{-1} \to (\tilde{g} \hat{g})^{-1} \in G$ after passing to a subsequence (if necessary). Then
\[ \tilde{g}^{-1} \cdot h(x_j, v_j) = \tilde{g}^{-1} \cdot g_j \cdot h_{x_0}(g_j^{-1} \cdot v_j) \to \hat{g} \cdot h_{x_0}((\tilde{g} \hat{g})^{-1} \cdot \tilde{v}) = h_{x_0}(\tilde{g}^{-1} \cdot \tilde{v}). \]

It follows that $h_{x_0}(\tilde{g}^{-1} \cdot \tilde{v})$ does not belong to $\tilde{g}^{-1} \cdot U$. This contradicts the fact that $\tilde{g}^{-1} \cdot U$ is an open neighborhood of $h_{x_0}(\tilde{g}^{-1} \cdot \tilde{v})$.

By the definition of $h$, it is clearly $G$-equivariant and satisfies
\[ (P_{x}^+ + P_{x}^-) \nabla \left( \mathcal{L} \circ \exp|_{N\mathcal{O}(\epsilon)x} \right)(v + h_x(v)) = 0, \quad \forall (x, v) \in N^0\mathcal{O}(3\epsilon). \quad (2.73) \]

Moreover, the map $F : N^0\mathcal{O}(\epsilon) \oplus N^+\mathcal{O}(\epsilon) \oplus N^-\mathcal{O}(\epsilon) \to \mathbb{R}$ defined by
\[ F(x, v, u^+ + u^-) = F_x(v, u^+ + u^-) = \mathcal{L} \circ \exp_x(v + h_x(v) + u^+ + u^-) - \mathcal{L} \circ \exp_x(v + h_x(v)), \quad (2.74) \]

is $G$-invariant, and satisfies for any $(x, v) \in N^0\mathcal{O}(\epsilon)$ and $u \in N^+\mathcal{O}_x \oplus N^-\mathcal{O}_x$,
\[ F_x(v, \theta_x) = 0 \quad \text{and} \quad D_2F_x(v, \theta_x)[u] = 0. \quad (2.75) \]

By (2.69), (2.70) and Lemmas 2.5 and 2.6 we can immediately obtain:
Lemma 2.23 There exist positive numbers \( \varepsilon_1 \in (0, \varepsilon) \) and \( a_1 \in (0, 2a_0) \), and a function \( \Omega : N(\varepsilon_1) \to [0, \infty) \) with the property that \( \Omega(x, v) \to 0 \) as \( \|v\|_x \to 0 \), such that for any \((x, v) \in N(\varepsilon_1)\) the following conclusions hold with \( B_x = d^2 \left( L \circ \exp_{N(\varepsilon_1)} \right) \):

(i) \( \|\langle B_x(v)u, w \rangle_x \| \leq \Omega(x, v) \|u\|_x \cdot \|w\|_x \) for any \( u \in N^0O_x \oplus N^-O_x \) and \( w \in N^0\lambda \); 

(ii) \( \langle B_x(v)u, w \rangle_x \geq a_1 \|u\|_x^2 \) for all \( u \in N^+O_x \); 

(iii) \( \|\langle B_x(v)u, w \rangle_x \| \leq \Omega(x, v) \|u\|_x \cdot \|w\|_x \) for all \( u \in N^+O_x, w \in N^-O_x \); 

(iv) \( \langle B_x(v)u, w \rangle_x \leq -a_0 \|u\|_x^2 \) for all \( u \in N^0O_x \).

Let us choose \( \varepsilon_2 \in (0, \varepsilon/2) \) so small that \((x, v^0 + \delta_x(v^0) + u^+ + u^-) \in N(\varepsilon_1)\) for \((x, v^0) \in N^0(\varepsilon_2)\) and \((x, u^*) \in N^*O(\varepsilon_2)\). As in the proof of [33, Lemma 3.5], we may use [33, Lemma 2.4] to derive

**Lemma 2.24** Let the constants \( a_1 \) and \( a_0 \) be given by Lemma 2.23(ii), (iv). For the above \( \varepsilon_2 > 0 \) and each \( x \in O \) the restriction of the functional \( F_x \) to \( N^0(\varepsilon_2), \oplus \{N^+O(\varepsilon_2), \ominus \} \) satisfies:

(i) \( D_2F_x(v^0, u^+ + u^-)[u^\pm - u^\pm_j] - D_2F_x(v^0, u^+ + u^-)[u^\pm - u^\pm_j] \leq -a_1 \|u^\pm - u^\pm_j\|_x^2 \) for any \((x, v^0) \in N^0O(\varepsilon_2), (x, +u^+) \in N^+O(\varepsilon_2)\) and \((x, u^*) \in N^0O(\varepsilon_2), j = 1, 2; \)

(ii) \( D_2F_x(v^0, u^+ + u^-)[u^+ - u^-] \geq a_1 \|u^+\|_x^2 + a_0 \|u^-\|_x^2 \) for any \((x, v^0) \in N^0O(\varepsilon_2)\) and \((x, u^*) \in N^*O(\varepsilon_2), \ast = +, -; \)

(iii) \( D_2F_x(v^0, u^+)[u^+] \geq a_1 \|u^+\|_x^2 \) for any \((x, v^0) \in N^0O(\varepsilon_2)\) and \((x, u^+) \in N^+O(\varepsilon_2). \)

Denote by bundle projections \( \Pi_0 : \frac{N^0O(\varepsilon_2)}{N^0O(\varepsilon_2)} \to O \) and \( \Pi_{\pm} : \frac{N^+O \oplus N^-O}{N^0O(\varepsilon_2)} \to O, \Pi_\ast : \frac{N^*O}{N^0O(\varepsilon_2)} \to O, \ast = +, - \). Let \( \Lambda = \frac{N^0O(\varepsilon_2)}{N^0O(\varepsilon_2)} \), \( p : \mathcal{E} \to \Lambda \) and \( p_{\ast} : \mathcal{E} \to \Lambda \) be the pullbacks of \( N^+O \oplus N^-O \) and \( N^*O \) via \( \Pi_0, \ast = +, - \). Then \( \mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^- \), and for \( \lambda = (x, v^0) \in \Lambda \) we have \( E_\lambda = N^+O_x \oplus N^-O_x \) and \( E^\lambda = N^*O_x, \ast = +, - \). Moreover, for each \( \eta > 0 \) we write 

\[
B_\eta(\mathcal{E}) = \left\{ (\lambda, w) \mid \lambda = (x, v^0) \in \Lambda \land w \in (N^0O \oplus N^0O)_x(\eta) \right\},
\]

\[
\tilde{B}_\eta(\mathcal{E}) = \left\{ (\lambda, w) \mid \lambda = (x, v^0) \in \Lambda \land w \in (N^0O \oplus N^0O)_x(\eta) \right\}.
\]

Similarly, \( B_\eta(\mathcal{E}) \) and \( \tilde{B}_\eta(\mathcal{E}) \) are defined. Let \( J : B_{2\varepsilon_2}(\mathcal{E}) \to \mathbb{R} \) be given by 

\[
J(\lambda, \varepsilon^0) = J_\lambda(\varepsilon^0) = \mathcal{F}(x, v^0, \varepsilon^0), \quad \forall \lambda = (x, v^0) \in \Lambda \land \forall \varepsilon^0 \in B_{2\varepsilon_2}(\mathcal{E})_\lambda.
\]

(2.76)

It is continuous, and of class \( C^1 \) with respect to \( \varepsilon^0 \). From (2.75) and Lemma 2.24 we directly obtain:

**Lemma 2.25** The functional \( J_\lambda \) satisfies the conditions in Theorem A.2 of [33] (the bundle parameterized version of [22, Theorem 1.1]), that is,

(i) \( J_\lambda(\varepsilon^0) = 0 \) and \( D J_\lambda(\varepsilon^0) = 0; \)

(ii) \( D J_\lambda(\lambda, u^+ + u^-)[u^\pm - u^\pm_j] - D J_\lambda(\lambda, u^+ + u^-)[u^\pm - u^\pm_j] \leq -a_1 \|u^\pm - u^\pm_j\|_x^2 \) for any \( \lambda = (x, v^0) \in \Lambda, u^\pm \in B_{2\varepsilon_2}(\mathcal{E})_\lambda \) and \( u^\pm_j \in B_{2\varepsilon_2}(\mathcal{E}^-)_\lambda, j = 1, 2; \)

(iii) \( D J_\lambda(\lambda, u^+ + u^-)[u^+ - u^-] \geq a_1 \|u^+\|_x^2 + a_0 \|u^-\|_x^2 \) for any \( \lambda = (x, v^0) \in \Lambda \) and \( u^+ \in B_{2\varepsilon_2}(\mathcal{E}^+)_\lambda, \ast = +, -; \)

(iv) \( D J_\lambda(\lambda, u^+)[u^+] \geq a_1 \|u^+\|_x^2 \) for any \( \lambda = (x, v^0) \in \Lambda \) and \( u^+ \in B_{2\varepsilon_2}(\mathcal{E}^+)_\lambda. \)
By this we can use Theorem A.2 of [33] to get \( \epsilon \in (0, \varepsilon_2) \), an open neighborhood \( U \) of the zero section \( 0_\mathcal{E} \) of \( \mathcal{E} \) in \( B_{2\varepsilon_2}(\mathcal{E}) \) and a homeomorphism
\[
\phi : B_\varepsilon(\mathcal{E}^+) \oplus B_\varepsilon(\mathcal{E}^-) \to U, \quad (\lambda, u^+ + u^-) \mapsto (\lambda, \phi_\lambda(u^+ + u^-))
\]
(2.77) such that for all \((\lambda, u^+ + u^-) \in B_\varepsilon(\mathcal{E}^+) \oplus B_\varepsilon(\mathcal{E}^-)\) with \( \lambda = (x, v^0) \in \Lambda, \)
\[
J(\phi(\lambda, u^+ + u^-)) = \|u^+\|^2_\mathcal{X} - \|u^-\|^2_\mathcal{X}.
\]
(2.78) Moreover, for each \( \lambda \in \Lambda, \phi_\lambda(\theta_\lambda) = \theta_\lambda, \phi_\lambda(x + y) \in \mathcal{E}_\mathcal{X}^- \) if and only if \( x = \theta_\lambda, \) and \( \phi \) is a homeomorphism from \( B_\varepsilon(\mathcal{E}^-) \) onto \( U \cap \mathcal{E}^- \).

Note that \( B_\varepsilon(\mathcal{E}^+) \oplus B_\varepsilon(\mathcal{E}^-) = N^0(O(2\varepsilon_2) \oplus N^+ O(\varepsilon) \oplus N^- O(\varepsilon)) \) and \( U = N^0(O(2\varepsilon_2) \oplus \hat{U}, \) where \( \hat{U} \) is an open neighborhood of the zero section of \( N^+ O \oplus N^- O \) in \( N^0 O(2\varepsilon_2) \oplus N^- O(\varepsilon_2) \).

Let \( \mathcal{V} = N^0 O(\varepsilon) \oplus \hat{U}, \) which is an open neighborhood of the zero section of \( N O \) in \( N^0 O(2\varepsilon_2) \oplus N^+ O(2\varepsilon_2) \oplus N^- O(\varepsilon_2). \) By (2.77) we get a homeomorphism
\[
\phi : N^0 O(\varepsilon) \oplus N^+ O(\varepsilon) \oplus N^- O(\varepsilon) \to \mathcal{V}, \quad (x, v, u^+ + u^-) \mapsto (x, v, \phi(x,v)(u^+ + u^-)),
\]
and therefore a topological embedding bundle morphism that preserves the zero section,
\[
\Phi : N^0 O(\varepsilon) \oplus N^+ O(\varepsilon) \oplus N^- O(\varepsilon) \to NO,
\]
\[
(x, v, u^+ + u^-) \mapsto (x, v + h_\mathcal{X}(v), \phi(x,v)(u^+ + u^-)).
\]
From (2.74), (2.76) and (2.78) it follows that \( \Phi \) and \( \phi \) satisfy
\[
\mathcal{L} \circ \exp \circ \Phi(x, v, u^+ + u^-) = \mathcal{L} \circ \exp_\mathcal{X}(v + h_\mathcal{X}(v) + \phi(x,v)(u^+ + u^-)) = \|u^+\|^2_\mathcal{X} - \|u^-\|^2_\mathcal{X} + \mathcal{L} \circ \exp_\mathcal{X}(v + h_\mathcal{X}(v))
\]
for all \((x, v + u^+, u^-) \in N^0 O(\varepsilon) \oplus N^+ O(\varepsilon) \oplus N^- O(\varepsilon)\). The other conclusions easily follow from the above arguments. Theorem 2.22 is proved.

**Proof of Theorem 2.21** We also consider the case \( \lambda = 0 \) merely. In the present case Lemma 2.23 also holds with \( N^0 O_\mathcal{X} = \{\theta_\lambda\} \forall x \in \mathcal{O}. \) But we need to replace the map \( \mathcal{F} \) in (2.74) by
\[
\mathcal{F}(x, u^+ + u^-) : N^0 O(\varepsilon) \oplus N^- O(\varepsilon) \to \mathbb{R}, \quad (x, u^+ + u^-) \mapsto \mathcal{L} \circ \exp_\mathcal{X}(u^+ + u^-).
\]
For any \( x \in O_\mathcal{X}, \) let \( \mathcal{F}_x \) be the restriction of \( \mathcal{F} \) to \( N^+ O(\varepsilon)_x \oplus N^- O(\varepsilon)_x. \) As in the proof of Theorem 2.1, Lemma 2.24 is still true with \( \overline{N^0 O(2\varepsilon_2)}_x = \{\theta_\lambda\}. \) Then the desired conclusions can be obtained by applying [33, Theorem A.2] to \( \Lambda = O \) and \( J_\mathcal{X} = \mathcal{F}_x \) with \( \lambda = x \in O. \) \( \square \)

As in [5,12,42,64,65], from Theorems 2.21, 2.22 we may, respectively, deduce

**Corollary 2.26** Under the assumptions of Theorem 2.21, let \( \theta^- \) be the orientation bundle (or sheaf) of \( N^- O \) and \( \mathbf{K} \) any commutative ring. Then it holds that
\[
C_*(L_\mathcal{X}, O; K) \cong H_{* - \mu O}(O; \theta^- \otimes K) \quad \text{and} \quad C^*_G(L_\mathcal{X}, O; K) \cong H^*_{G(-\mu O}(O; \theta^- \otimes K),
\]
(2.79) where for \( q \in \mathbb{N}_0, C^*_G(L_\mathcal{X}, O; K) = H^q(E \times G ((L_\mathcal{X})_c \cap U), E \times G ((L_\mathcal{X})_c \setminus O) \cap U);K) \) is the so-called the \( q^{th} \) \( G \) critical group of \( O \) defined with a universal smooth principal \( G \)-bundle \( E \to B_G, \) a \( G \)-invariant neighborhood \( U \) of \( O \) and \( c = L_\mathcal{X}(O). \) In particular, for \( \mathbf{K} = \mathbb{Z}_2 \) there hold
\[
C_*(L_\mathcal{X}, O; \mathbb{Z}_2) \cong H_{* - \mu O}(O; \mathbb{Z}_2) \quad \text{and} \quad C^*_G(L_\mathcal{X}, O; \mathbb{Z}_2) \cong H^*_{G(-\mu O}(O; \mathbb{Z}_2).
\]
(2.80)
Corollary 2.27 (Shifting Theorem) Under the assumptions of Theorem 2.22, if \( \mathfrak{O} \) has trivial normal bundle then \( C_q(\mathcal{L}_s, \mathfrak{O}; \mathbf{K}) \cong \bigoplus_{j=0}^{\infty} C_{q-j-\mu}(\mathcal{L}_s)^{\mathfrak{O}}; \theta_{\mathcal{L}}; \mathbf{K}) \otimes H_j(\mathfrak{O}; \mathbf{K}) \) for any commutative group \( \mathbf{K} \) and \( x \in \mathfrak{O} \).

3 A generalization of Marino–Prodi’s perturbation theorem

Marino and Prodi [41] studied local Morse function approximations for \( C^2 \) functionals on Hilbert spaces. We shall generalize their result to a class of functionals satisfying the following stronger assumption than Hypothesis 1.1.

Hypothesis 3.1 Let \( V \) be an open set of a Hilbert space \( H \) with inner product \((\cdot, \cdot)_H\), and \( \mathcal{L} \in C^1(V, \mathbb{R}) \). Assume that the gradient \( \nabla \mathcal{L} \) has a Gâteaux derivative \( B(u) \in \mathcal{L}(H) \) at every point \( u \in V \), and that the map \( B: V \to \mathcal{L}(H) \) has a decomposition \( B = P + Q \), where for each \( u \in V \), \( P(u) \in \mathcal{L}(H) \) is positive definite, \( Q(u) \in \mathcal{L}(H) \) is compact, and they also satisfy the following properties:

(i) For any \( u \in H \), the map \( V \ni x \mapsto P(x)u \in H \) is continuous;
(ii) The map \( Q: V \to \mathcal{L}(H) \) is continuous;
(iii) \( P \) is a positive definite uniformly, i.e., each \( u_0 \in V \) has a neighborhood \( \mathcal{U} \) such that for some constants \( C_0 > 0 \), \( (P(u)v, v)_H \geq C_0 \|v\|^2 \), \( \forall v \in H, \forall u \in \mathcal{U}(u_0) \).

As in the proofs of Theorems 4.1 and 4.2 under Hypothesis \( \mathfrak{F}_{2,n,m,n} \), we can check that the functional \( \mathfrak{F} \) in (1.3) satisfies this hypothesis. By improving methods in [12,19,41] we may prove

Theorem 3.2 Under Hypothesis 3.1, suppose: (a) \( u_0 \in V \) is a unique critical point of \( \mathcal{L} \), (b) the corresponding maps \( \varphi \) and \( \mathcal{L}^0 \) as in Theorem 2.22 near \( u_0 \) are of classes \( C^1 \) and \( C^2 \), respectively, (c) \( \mathcal{L} \) satisfies the (PS) condition. Then for any \( \epsilon > 0 \) and \( r > 0 \) such that \( \mathcal{B}_H(u_0, r) \subset V \) and \( \sup \{\|\mathcal{L}(u)\| \mid u \in \mathcal{B}_H(u_0, r)\} \leq \infty \), there exists a functional \( \mathcal{L} \in C^1(V, \mathbb{R}) \) with the following properties:

(i) \( \mathcal{L} \) satisfies Hypothesis 3.1 and the (PS) condition;
(ii) \( \sup_{u \in V} \|\mathcal{L}(u) - \mathcal{L}(\tilde{u})\| < \epsilon, \sup_{u \in V} \|\mathcal{L}'(u) - \mathcal{L}'(\tilde{u})\| < \epsilon \) and \( \sup_{u \in V} \|\mathcal{L}''(u) - \mathcal{L}''(\tilde{u})\| < \epsilon \), where \( \mathcal{L}'(u) \) and \( \mathcal{L}''(u) \) are Gâteaux derivatives of \( \mathcal{L}'(u) \) and \( \mathcal{L}'(u) \), respectively;
(iii) \( \mathcal{L}(x) = \mathcal{L}(\tilde{x}) \) if \( x \in V \) and \( \|u - u_0\| \geq r \);
(iv) the critical points of \( \mathcal{L} \), if any, are in \( B_H(u_0, r) \) and nondegenerate (so finitely many by the arguments below 2.1); moreover the Morse indexes of these critical points sit in \([m^-, m^- + n^0]\), where \( m^- \) and \( n^0 \) are the Morse index and nullity of \( u_0 \), respectively.

As shown, the functionals in [32,40] satisfy the conditions of this theorem. If \( N = 1 \), \( \dim \Omega = 2 \) and \( F \) is smooth enough, we may also prove under Hypothesis \( \mathfrak{F}_{2,1,m,2} \) that Theorem 3.2 is applicable for the functional \( \mathfrak{F} \) on \( W_m^{1,2}(\Omega) \). In general, under Hypothesis \( \mathfrak{F}_{2,2,m,n} \), for a critical point \( \tilde{u} \) of the functional \( \mathfrak{F}_H \) on \( H := W_m^{1,2}(\Omega, \mathbb{R}^N) \) defined by the right side of (1.3), if there exist a real \( p \geq 2 \) and an integer \( k > m + \frac{n}{p} \) such that \( \tilde{u} \in C^k(\Omega, \mathbb{R}^N) \), and \( F \) is of classes \( C^{k-m+2} \) and \( C^{k-1,1} \), respectively, then Theorem 4.16 (or Theorem 4.19) shows that (b) of Theorem 3.2 can be satisfied for \( \mathfrak{F}_H \) near \( \tilde{u} \).

Marino–Prodi’s result has many important applications in the critical point theory, see [12,19,26,31] and literature therein. With Theorem 3.2 they may be given in our framework. Moreover, it is very possible to give a corresponding result with Theorem 3.2 in the setting of [32–34].
Marino–Prodi’s perturbation theorem in [41] was also generalized to the equivariant case under the finite (resp. compact Lie) group action by Wasserman [65] (resp. Viterbo [63]), see the proof of Theorem 7.8 in [12, Chapter I] for full details. Similarly, we can present an equivariant version of Theorem 3.2 for compact Lie group action, but it is omitted here.

**Proof of Theorem 3.2** Without loss of generality we may assume \( \theta \in V \) and \( u_0 = \theta \). By the assumption (b) we have a \( C^2 \) reduction functional \( L^0 : B_H(\theta, \delta) \cap H^0 \to \mathbb{R} \) such that \( \theta \) is the unique critical point of it. In this case, from (2.2) and (2.58) with \( C \)

then

\[
\tilde{\omega}(z) = \omega(z + \varphi(z)) < \frac{1}{2} \min\{a_0, a_1\}, \quad \forall z \in B_H(\theta, \delta) \cap H^0.
\]  

(3.1)

By the uniqueness of solutions we can also require that if \( v \in B_H(\theta, \delta) \) satisfies \((I - P^0)\nabla L(v) = 0\) then \( v = z + \varphi(z) \) for some \( z \in B_H(\theta, \delta) \cap H^0 \).

Take a smooth function \( \rho : [0, \infty) \to \mathbb{R} \) satisfying: \( 0 \leq \rho \leq 1, \rho(t) = 1 \) for \( t \leq \delta/2, \rho(t) = 0 \) for \( t \geq \delta, \text{and} |\rho'(t)| < 4/\delta \). For \( b \in H^0 \) we set

\[
L^0_b(z) = L^0(z) + \rho(\|z\|)(b, z)_H.
\]

Then

\[
DL^0_b(z)[\xi] = DL(z + \varphi(z))[\xi + \varphi'(z)\xi] + \rho(\|z\|)(b, \xi)_H
+ \rho'(\|z\|)(b, z)_H(z/\|z\|, \xi)_H, \quad \forall \xi \in H^0.
\]

(3.2)

Note that \( v := \inf \{\|DL^0(z)\| : z \in \tilde{B}_{H^0}(\theta, \delta) \setminus B_{H^0}(\theta, \delta/2)\} > 0 \). Suppose \( \|b\| < v/5 \). Then

\[
\|DL^0_b(z)\| = \|DL^0(z) + \rho(\|z\|)b + (b, z)_H\rho'(\|z\|)z/\|z\|\| \geq v - 5\|b\| > 0,
\]

(3.3)

and therefore \( L^0_b \) has no critical point in \( \tilde{B}_{H^0}(\theta, \delta) \setminus B_{H^0}(\theta, \delta/2) \). By Sard’s theorem we may take arbitrary small \( b \neq 0 \) such that the critical points of \( L^0_b \), if any, are nondegenerate. Choose a \( C^2 \) function \( \beta : H \to \mathbb{R} \) such that \( \beta(u) = 0 \) for \( u \in H \setminus B_H(\theta, r) \), and \( \beta(u) = 1 \) for \( u \in B_H(\theta, \delta) \). Clearly, we can require \( \sup\{\|\beta(u)\|, \|\beta'(u)\|, \|\beta''(u)\| : u \in H\} \leq M \) for some \( M > 0 \). Define

\[
\tilde{L}_b(u) = L(u) + \beta(u)\rho(\|P^0u\|)(b, P^0u)_H
\]

(3.4)

We shall prove that \( \tilde{L}_b \) satisfies the expected requirements for sufficiently small \( b \neq 0 \) produced by Sard’s theorem above.

**Step 1** Prove that \( \tilde{L}_b \) satisfies (iv) if \( b \) is small enough. Since \( L \) satisfies the (PS) condition, \( c := \inf \{\|DL(u)\| : u \in B_H(\theta, r) \setminus B_H(\theta, \delta)\} > 0 \). Hence all critical points of \( \tilde{L}_b \) belong to \( B_H(\theta, \delta) \) as long as \( b \) is small enough.

Let us prove that each critical point \( v \) of \( \tilde{L}_b \) in \( B_H(\theta, \delta) \) is nondegenerate. Observe that

\[
0 = \tilde{L}_b'(v)[\xi] = (\nabla L(v), \xi)_H + \rho(\|P^0v\|)(b, P^0\xi)_H
+ \rho'(\|P^0v\|)(b, P^0v)_H(P^0v, P^0\xi)_H/\|P^0v\|, \quad \forall \xi \in H.
\]

(3.5)

Since \( \rho(\|P^0v\|) = 1 \) for \( \|P^0v\| \leq \|v\| < \delta \), this implies \( (\nabla L(v), \xi)_H = 0 \) for any \( \xi \in H^+ \oplus H^- \), i.e., \((I - P^0)\nabla L(v) = 0\). It follows that \( v = z + \varphi(z) \) for some \( z \in B_H(\theta, \delta) \cap H^0 \).

[This \( z \) is nonzero. Otherwise, \( v = \theta \). But \( \theta \) is not a critical point of \( \tilde{L}_b \) if \( b \neq \theta \).] Note that \((\nabla L(z + \varphi(z)), \varphi'(z)\xi)_H = 0 \forall \xi \in H^0 \) because \( \varphi'(z)\xi \in H^+ \oplus H^- \). (3.5) leads to
0 = (∇L(z + ϕ(z)), ξ)_H + ρ(∥z∥)(b, ξ)_H + ρ'(∥z∥)(b, z)_H(z, ξ)_H/∥z∥
= (∇L(z + ϕ(z)), ξ)_H + (∇L(z + ϕ(z)), ϕ'(z)ξ)_H
+ ρ(∥z∥)(b, ξ)_H + ρ'(∥z∥)(b, z)_H(z, ξ)_H/∥z∥ ∀ξ ∈ H^0,
and therefore D_L^0(z) = 0 by (3.2). That is, z is a critical point of L_b^0, and so z ∈ B_{H^0}(θ, δ/2) by (3.3). It follows from (3.4) that

Let ξ ∈ Ker(Ż''_b(v)). By (3.6), we have

L''(v)[ξ, η] = (L''(z + ϕ(z))[ξ], η)_H = 0, ∀η ∈ H. (3.7)

Decompose ξ into ξ_0 + ξ_⊥, where ξ_0 ∈ H^0 and ξ_⊥ ∈ H^+ ⊕ H^−. A direct computation yields

(L''(z + ϕ(z))[ξ_0], η + ϕ'(z)[η])_H + (L''(z + ϕ(z))[ξ_⊥], η + ϕ'(z)[η])_H = 0, ∀η ∈ H^0. (3.8)

Note that (I − P^0)∇L(w + ϕ(w)) = 0 ∀w ∈ B_{H^0}(θ, δ) by (2.2). Hence

(∇L(w + ϕ(w)), ζ)_H = 0 ∀ζ ∈ H^+ ⊕ H^−.

Differentiating this equality with respect to w yields

(L''(w + ϕ(w))[τ + ϕ'(w)[τ]], ζ)_H = 0, ∀τ ∈ H^0, ∀w ∈ B_{H^0}(θ, δ), ∀ζ ∈ H^+ ⊕ H^−.

In particular, we have (L''(z + ϕ(z))[ξ_⊥], η + ϕ'(z)[η])_H = 0 ∀η ∈ H^0. This and (3.8) yield

d^2L^0(z)[ξ_0, η] = (L''(z + ϕ(z))[ξ_0], η + ϕ'(z)[η])_H = 0, ∀η ∈ H^0. (3.9)

Moreover, d^2L^0_b(z') = d^2L^0(z') ∀z' ∈ B_{H^0}(θ, δ/2) by the construction of L_b^0. We obtain that d^2L^0_b(z)[ξ_0, η] = 0 ∀η ∈ H^0. Since z is a nondegenerate critical point of L_b^0 by the choice of b, ξ_0 = θ and thus ξ = ξ_⊥. By (3.6), (3.7) and (L''_b(v)[ξ], η)_H = 0 ∀η ∈ H, we get

(L''(z + ϕ(z))[ξ_⊥], η)_H = (L''(z + ϕ(z))[ξ], η)_H = 0, ∀η ∈ H. (3.10)

Hence L''(z + ϕ(z))[ξ_⊥] = 0. Decompose ξ_⊥ into ξ_+ + ξ_−, where ξ_+ ∈ H^+ and ξ_− ∈ H^−.

Then L''(z + ϕ(z))[ξ_+] = −L''(z + ϕ(z))[ξ_−]. By Lemma 2.6 and (3.1) we derive

a_1∥ξ_+∥^2 ≤ L''(z + ϕ(z))[ξ_+], ξ_+)_H = (L''(z + ϕ(z))[ξ_−], ξ_+)_H ≤ a_1∥ξ_+∥ ⋅ ∥ξ_−∥,

−a_0∥ξ_−∥^2 ≥ L''(z + ϕ(z))[ξ_−], ξ_−)_H = (L''(z + ϕ(z))[ξ_+], ξ_−)_H ≥ −a_0∥ξ_−∥ ⋅ ∥ξ_+∥.

These imply that ξ_+ = ξ_− = θ and so ξ = θ. Hence v is a nondegenerate critical point of L_b.

Note that Lemma 2.6 and (3.6) give rise to

L''_b(v)[ξ, ξ] = (L''(v)[ξ], ξ)_H ≥ a_1∥ξ∥^2, ∀ξ ∈ H^+, L''_b(v)[ξ, ξ] = (L''(v)[ξ], ξ)_H ≤ −a_0∥ξ∥^2, ∀ξ ∈ H^−.
But $H = H^+ \oplus H^0 \oplus H^-$, dim $H^+ = m^+$ and dim $H^0 = n^0$. These show that the Morse index of $\tilde{L}_b'(u)$ must sit in $[m^-, m^- + n^0]$. This is proved.

**Step 2** Prove that $\tilde{L}_b$ satisfies Hypothesis 3.1 on $V$ if $b \not= 0$ is small enough. By (3.4) we have for all $\xi, \eta \in H$,

$$
\tilde{L}_b'(u)[\xi] = \mathcal{L}'(u)[\xi] + (\beta'(u)[\xi])\rho(\|P^0 u\|)(b, P^0 u)_H + \beta(u)\rho(\|P^0 u\|)(b, P^0 u)_H \\
+ \beta(u)\rho'(\|P^0 u\|)(b, P^0 u)_H \left(\frac{1}{\|P^0 u\|}\right)
$$

and

$$(\tilde{L}_b^\prime(u)[\eta], \xi)_H = (\mathcal{L}''(u)[\eta], \xi)_H + (\beta''(u)[\eta], \xi)_H \rho(\|P^0 u\|)(b, P^0 u)_H \\
+ (\beta''(u)[\eta])\rho(\|P^0 u\|)(b, P^0 u)_H + \beta(u)\rho(\|P^0 u\|)(b, P^0 u)_H \left(\frac{1}{\|P^0 u\|}\right)^2 \\
+ \beta(u)\rho'(\|P^0 u\|)(b, P^0 u)_H \left(\frac{1}{\|P^0 u\|}\right) \\
+ \beta(u)\rho''(\|P^0 u\|)(b, P^0 u)_H \left(\frac{1}{\|P^0 u\|}\right)^2 \\
- \beta(u)\rho''(\|P^0 u\|)(b, P^0 u)_H \left(\frac{1}{\|P^0 u\|}\right)^3
$$

By the constructions of $\beta$ and $\rho$, after the tedious estimate we get a constant $M_2 > 0$ such that

$$
|\mathcal{Y}(u, b, \xi, \eta)| \leq M_2\|b\| \cdot \|\xi\| \cdot \|\eta\|, \quad \forall u \in V, \forall \xi, \eta \in H.
$$

Since we may require that the support of $\beta$ can be contained a neighborhood of $\theta$ on which (iii) of Hypothesis 3.1 holds, for sufficiently small $b \not= 0$ the positive definite part $\tilde{P}$ of $\mathcal{L}_b''$ given by $\tilde{P}(u, \xi, \eta)_H = (P(u, \xi, \eta)_H + \mathcal{Y}(u, b, \xi, \eta)$, is also uniformly positive definite on this neighborhood. Hence $\tilde{L}_b$ satisfies Hypothesis 3.1.

**Step 3** Prove that (ii) and (iii) can be satisfied if $b \not= 0$ is small. Indeed, (3.12) implies that $\|\tilde{L}_b''(u) - \mathcal{L}_b''(u)\| \leq M_2\|b\|$ for all $u \in V$. Moreover, by (3.4) and (3.11) we have positive numbers $M_i$, $i = 0, 1, 2$, such that $\|\tilde{L}_b(u) - \mathcal{L}(u)\| \leq M_0\|b\|$ and $\|\tilde{L}_b'(u) - \mathcal{L}'(u)\| \leq M_1\|b\|$ for all $u \in V$. Hence it suffices to require that $\|b\| < \epsilon/M_i$ for $i = 0, 1, 2$.

**Step 4** Prove that $\tilde{L}_b$ satisfies the $(PS)$ condition for small $b$. By (ii) and (iii) in Hypothesis 3.1, there exists $\epsilon \in \left(0, \delta/2\right)$ such that for all $u \in B_H(\theta, \epsilon)$ and $\xi \in H$,

$$
(P(u, \xi, \eta)_H \geq C_0\|\xi\|^2 \quad \text{and} \quad \|Q(u) - Q(\theta)\| < C_0/2.
$$

Recall that $\mathcal{L}$ is bounded in $\tilde{B}_H(u_0, r)$ and that $\theta$ is a unique critical point of $\mathcal{L}$ in $V$. Since $\mathcal{L}$ satisfies the $(PS)$ condition, we have $v_0 > 0$ such that $\|\mathcal{L}'(u)\| \geq v_0$ for all $u \in \tilde{B}_H(u_0, r) \setminus B_H(\theta, \epsilon)$. Choose $b$ so small that

$$
\|\tilde{L}_b'(u)\| \geq v_0/2, \quad \forall u \in \tilde{B}_H(u_0, r) \setminus B_H(\theta, \epsilon).
$$
Let \( (u_n) \subset V \) satisfy \( \hat{L}_b(u_n) \to 0 \) and \( \sup_n |\hat{L}_b(u_n)| < \infty \). Assume that \( (u_n) \) has a subsequence \( (u_{n_k}) \) sitting in \( V \setminus B_H(u_0, r) \). Since \( L \) satisfies the (PS) condition, by (iii) we deduce that \( (u_{n_k}) \) has a converging subsequence. Thus after removing finitely many terms we may assume that \( (u_n) \subset B_H(u_0, r) \), and by (3.14) we may further assume that \( (u_n) \subset B_H(\theta, \epsilon) \).

It follows from (3.11) that \( \nabla \hat{L}_b(u_n) = \nabla L(u_n) + P^0b \) for all \( n \). For any two natural numbers \( n \) and \( m \), using the mean value theorem we have \( \tau \in (0, 1) \) such that

\[
(\nabla L(u_n) - \nabla L(u_m), u_n - u_m)_H = (B(\tau u_n + (1 - \tau)u_m)(u_n - u_m), u_n - u_m)_H
\]

\[
= (P(\tau u_n + (1 - \tau)u_m)(u_n - u_m), u_n - u_m)_H + (Q(\tau u_n - (1 - \tau)u_m)(u_n - u_m), u_n - u_m)_H
\]

\[
\geq C_0 \|u_n - u_m\|^2 - \frac{C_0}{m} \|u_n - u_m\|^2 + (Q(\tau u_n - (1 - \tau)u_m)(u_n - u_m), u_n - u_m)_H,
\]

where the last inequality comes from (3.13). Passing to a subsequence we may assume \( u_n \to u_0 \). Since \( Q(\theta) \) is compact, \( Q(\theta)u_n \to Q(\theta)u_0 \) and so \( (Q(\theta)(u_n - u_m), u_n - u_m)_H \to 0 \) as \( n, m \to \infty \). Note that \( \nabla L(u_n) - \nabla L(u_m) = (\nabla L(u_n) + P^0b) - (\nabla L(u_m) + P^0b) \to 0 \) as \( n, m \to \infty \). From the above inequality we conclude that \( \|u_n - u_m\| \to 0 \) as \( n, m \to \infty \). This implies \( u_n \to u_0 \). Theorem 3.2 is proved. \( \Box \)

### 4 Applications to quasi-linear elliptic systems of higher order

#### 4.1 Fundamental analytic properties for functionals \( \mathcal{F} \)

For \( p \in [2, \infty) \) and integers \( m \geq 1 \), \( n \geq 2 \), a bounded domain \( \Omega \in \mathbb{R}^n \) is said to be a Sobolev domain for \( (p, m, n) \) if the Sobolev embeddings theorems for the spaces \( W^{m,p}(\Omega) \) hold. The following two theorems summarize fundamental analytic properties of the functional \( \mathcal{F} \).

**Theorem 4.1** Given \( p \in [2, \infty) \) and integers \( m, N \geq 1 \), \( n \geq 2 \), let \( \Omega \subset \mathbb{R}^n \) be a Sobolev domain for \( (p, m, n) \), and let \( V \) be a closed subspace of \( W^{m,p}(\Omega, \mathbb{R}^N) \) and \( V = \bar{\vec{u}} + V_0 \) for some \( \bar{\vec{u}} \in W^{m,p}(\Omega, \mathbb{R}^N) \). Suppose that (i)--(ii) in Hypothesis \( \mathcal{F}_{p,N,m,n} \) hold. Then we have

**A.** The restriction of the functional \( \mathcal{F} \) in (1.3) to \( V, \mathcal{F}_V \), is bounded on any bounded subset, of class \( C^1 \), and the derivative \( D\mathcal{F}_V(\bar{\vec{u}}) \) of it at \( \bar{\vec{u}} \) is given by

\[
\langle D\mathcal{F}_V(\bar{\vec{u}}), \bar{\vec{v}} \rangle = \sum_{i=1}^N \sum_{|\alpha| \leq m} \int_\Omega F_{ij}^\alpha(x, \bar{\vec{u}}(x), \ldots, D^m \bar{\vec{u}}(x)) D^\alpha v_j^i dx, \quad \forall \bar{\vec{v}} \in V_0. \tag{4.1}
\]

Moreover, the map \( V \ni \bar{\vec{u}} \to \mathcal{F}_V(\bar{\vec{u}}) \in V_0^* \) also maps bounded subset into bounded ones.

**B.** The map \( \mathcal{F}_V \) is of class \( C^1 \) on \( V \) if \( p > 2 \), Gâteaux differentiable on \( V \) if \( p = 2 \), and for each \( \bar{\vec{u}} \in V \) the derivative \( D\mathcal{F}_V(\bar{\vec{u}}) \in \mathcal{L}(V_0, V_0^*) \) is given by

\[
\langle D\mathcal{F}_V(\bar{\vec{u}})[\bar{\vec{v}}], \bar{\vec{\phi}} \rangle = \sum_{i,j=1}^N \sum_{|\alpha| \leq m, |\beta| \leq m} \int_\Omega F_{ij}^{\alpha \beta}(x, \bar{\vec{u}}(x), \ldots, D^m \bar{\vec{u}}(x)) D_{ij}^\alpha \bar{\vec{v}}_j^\beta \cdot D^\alpha \bar{\vec{\phi}}^i dx. \tag{4.2}
\]

(In the case \( p = 2 \), equivalently, the gradient map of \( \mathcal{F}_V, V \ni \bar{\vec{u}} \mapsto \nabla \mathcal{F}_V(\bar{\vec{u}}) \in V_0 \), given by

\[
\langle \nabla \mathcal{F}_V(\bar{\vec{u}}), \bar{\vec{v}} \rangle_{m,2} = \langle \mathcal{F}_V'(\bar{\vec{u}}), \bar{\vec{v}} \rangle \quad \forall \bar{\vec{v}} \in V_0. \tag{4.3}
\]

has a Gâteaux derivative \( D(\nabla \mathcal{F}_V)(\bar{\vec{u}}) \in \mathcal{L}_2(V_0) \) at every \( \bar{\vec{u}} \in V \).) Moreover, \( D\mathcal{F}_V \) also satisfies the following properties:
(i) For every given $R > 0$, $\{D^2\alpha_{\nu}(\vec{u}) \mid \|\vec{u}\|_{m,p} \leq R\}$ is bounded in $L_2(V_0)$. Consequently, when $p = 2$, $\bar{\delta}_V$ is of class $C^{2,0}$.

(ii) For any $\vec{v} \in V_0$, $\vec{u}_k \to \vec{u}_0$ implies $D^2\alpha_{\nu}(\vec{u}_k)[\vec{v}] \to D^2\alpha_{\nu}(\vec{u}_0)[\vec{v}]$ in $V_0^*$. When $\|\vec{v}\|_{2} \leq \tilde{C}$, the final terms in the definitions of $P$ and $Q$ may be deleted. Consequently, $\bar{\delta}_V$ is completely continuous for each $\vec{u} \in V$.

Theorem 4.2 Under assumptions of Theorem 4.1, suppose that (iii) in Hypothesis $\bar{\delta}_p,N,m,n$ is also satisfied. Then

C). $\bar{\delta}' : W^{m,p}(\Omega, \mathbb{R}^N) \to (W^{m,p}(\Omega, \mathbb{R}^N))^*$ is of class $(S)_+$. 

D). Suppose $p = 2$. For $u \in V$, let $D(\nabla \bar{\delta}_V)(\vec{u})$, $P(\vec{u})$ and $Q(\vec{u})$ be operators in $L(V_0)$ defined by

$$(D(\nabla \bar{\delta}_V)(\vec{u})(\vec{v}), \vec{\phi})_{m,2} = \sum_{i,j=1}^{N} \sum_{|\alpha| \leq m, |\beta| \leq m} \int_{\Omega} F_{ij}^{\alpha\beta}(x, \vec{u}(x), \ldots, D^m\vec{u}(x)) D^\beta v^j \cdot D^\alpha \vec{\phi} dx,$$

$$(P(\vec{u})\vec{v}, \vec{\phi})_{m,2} = \sum_{i,j=1}^{N} \sum_{|\alpha| = |\beta| = m} \int_{\Omega} F_{ij}^{\alpha\beta}(x, \vec{u}(x), \ldots, D^m\vec{u}(x)) D^\beta v^j \cdot D^\alpha \vec{\phi} dx + \sum_{i=1}^{N} \sum_{|\alpha| \leq m-1} \int_{\Omega} D^\alpha v^i \cdot D^\alpha \vec{\phi} dx,$$

$$(Q(\vec{u})\vec{v}, \vec{\phi})_{m,2} = \sum_{i,j=1}^{N} \sum_{|\alpha| \leq m, |\beta| \leq m} \int_{\Omega} F_{ij}^{\alpha\beta}(x, \vec{u}(x), \ldots, D^m\vec{u}(x)) D^\beta v^j \cdot D^\alpha \vec{\phi} dx - \sum_{i=1}^{N} \sum_{|\alpha| \leq m-1} \int_{\Omega} D^\alpha v^i \cdot D^\alpha \vec{\phi} dx,$$

respectively. (If $V \subset W^{m,p}_0(\Omega, \mathbb{R}^N)$, the final terms in the definitions of $P$ and $Q$ may be deleted.) Then $D(V \bar{\delta}_V) = P + Q$, and

(i) for any $\vec{v} \in V_0$, the map $V \ni \vec{u} \mapsto P(\vec{u})\vec{v} \in V_0$ is continuous;

(ii) for every given $R > 0$ there exist positive constants $C(R, n, m, \Omega)$ such that

$$(P(\vec{u})\vec{v}, \vec{v})_{m,2} \geq C\|\vec{v}\|^2_{m,2}, \quad \forall \vec{v} \in V_0, \forall \vec{u} \in V with \|\vec{u}\|_{m,2} \leq R;$$

(iii) $V \ni \vec{u} \mapsto Q(\vec{u}) \in L(V_0)$ is continuous, and $Q(\vec{u})$ is completely continuous for each $\vec{u}$;

(iv) for every given $R > 0$ there exist positive constants $C_j(R, n, m, \Omega)$, $j = 1, 2$ such that

$$(D(\nabla \bar{\delta}_V)(\vec{u})(\vec{v}), \vec{v})_{m,2} \geq C_1\|\vec{v}\|^2_{m,2} - C_2\|\vec{v}\|^2_{m-1,2}, \quad \forall \vec{v} \in V_0, \forall \vec{u} \in V with \|\vec{u}\|_{m,2} \leq R.$$
general case. For the case \( N > 1 \), proofs of Theorems 4.1 and 4.2 can be completed by non-essentially changing that of [38, Theorem 3.1], i.e., only using the following proposition (easily verified as in the proof of [38, Prop.4.3]) adding or estimating more terms in each step. We omit them.

**Proposition 4.3** For the function \( g_1 \) in Hypothesis \( \mathfrak{R}_{p,N,m,n} \), let continuous positive nondecreasing functions \( g_k : [0, \infty) \to \mathbb{R}, \ k = 3, 4, 5 \), be given by

\[
g_3(t) := 1 + g_1(t)[t^2 M(m)N + t(M(m)N + 1)^2] + g_1(t)(M(m)N + 1)^2, \]

\[
g_4(t) := g_1(t) + g_1(t) \quad \text{and} \quad g_5(t) := (M(m)N + 1)g_1(t)(t + 1). \]

Then (ii) in Hypothesis \( \mathfrak{R}_{p,N,m,n} \) implies that for all \((x, \xi)\),

\[
|F(x, \xi)| \leq |F(x, 0)| + \left( \sum_{k=1}^{N} |\xi_k^i|^p \right) \left( 1 + \sum_{i=1}^{N} \sum_{|\alpha| < m - n/p} |\xi_i^{\alpha}|^p \right),
\]

\[
|F^k_{\alpha}(x, \xi)| \leq |F^k_{\alpha}(x, 0)| + g_4 \left( \sum_{i=1}^{N} |\xi_i^{\beta}| \right) \left( 1 + \sum_{i=1}^{N} \sum_{m-n/p < |\gamma| \leq m} |\xi_i^\gamma|^p \right) \sum_{j=1}^{N} |\xi_j^\gamma|; \tag{4.4}
\]

for the latter we further have

\[
|F^k_{\alpha}(x, \xi)| \leq |F^k_{\alpha}(x, 0)| + g_5 \left( \sum_{i=1}^{N} |\xi_i^{\beta}| \right) \left( 1 + \sum_{i=1}^{N} \sum_{m-n/p < |\gamma| \leq m} |\xi_i^\gamma|^p \right), \tag{4.5}
\]

if \(|\alpha| < m - n/p\), and

\[
|F^k_{\alpha}(x, \xi)| \leq |F^k_{\alpha}(x, 0)| + g_5 \left( \sum_{i=1}^{N} |\xi_i^{\beta}| \right) \left( 1 + \sum_{i=1}^{N} \sum_{m-n/p < |\gamma| \leq m} |\xi_i^\gamma|^p \right)^{1/q_\alpha} \tag{4.6}
\]

if \(m - n/p \leq |\alpha| \leq m\).

As a direct consequence of Theorems 4.1 and 4.2 we have:

**Corollary 4.4** Let \( N, m \geq 1, n \geq 2 \) be integers and let \( \Omega \subset \mathbb{R}^n \) a bounded Sobolev domain. Under Hypothesis \( \mathfrak{R}_{2,N,m,n} \), the restriction of the functional \( \mathfrak{R} \) in (1.3) with \( V = W^{m,2}(\Omega, \mathbb{R}^N) \) to any closed subspace \( H \) of \( W^{m,2}(\Omega, \mathbb{R}^N) \) satisfies Hypothesis 1.1 with \( X = H \).

**Remark 4.5** Theorems 4.1 and 4.2 have also more general versions in the setting of [49, 50, 55]. Let \( M \) be a \( n \)-dimensional compact \( C^\infty \) manifold with a strictly positive smooth measure \( \mu \), and possibly with boundary, and \( \pi : E \to M \) a real finite dimensional \( C^\infty \) vector space bundle over \( M \) of rank \( N \). A \( m \)-th order Lagrangian \( L \) on \( E \) is said to satisfy Hypothesis \( \mathfrak{R}_{p,N,m,n} \) if it has a representation satisfying Hypothesis \( \mathfrak{R}_{p,N,m,n} \) under any
local trivialization of $E$. (As usual, $W^{m,p}(M, E)$ is identified with $W^{m,p}(M, \mathbb{R}^N)$ if $E$ is a trivial bundle $M \times \mathbb{R}^N \to M$.) The integral functional of such a Lagrangian on $W^{m,p}(M, E)$ possess corresponding conclusions as in Theorems 4.1 and 4.2; the full detail will be given at other places. In particular, if $M$ is the torus $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$, a $m$th order Lagrangian on $M \times \mathbb{R}^N$ fulfilling Hypothesis $\mathfrak{F}_{p,N,m,n}$ is understood as a function $F: \mathbb{R}^n \times \prod_{k=0}^m \mathbb{R}^N \times M_0(k) \to \mathbb{R}$, which is not only 1-periodic in each variable $x_i$, $i = 1, \ldots, n$, but also satisfies Hypothesis $\mathfrak{F}_{p,N,m,n}$ with $\Omega = [0, 1]^n$. Then Theorems 4.1 and 4.2 also hold if $W^{m,p}(\Omega, \mathbb{R}^N)$ is replaced by $W^{m,p}(\mathbb{T}^n, \mathbb{R}^N)$.

4.2 (PS)- and (C)-conditions

A $C^1$ functional $\varphi$ on a Banach $X$ is said to satisfy $(PS)_c$-condition (resp. $(C)_c$-condition) at the level $c \in \mathbb{R}$ if every sequence $(x_j) \subset X$ such that $\varphi(x_j) \to c \in \mathbb{R}$ and $\varphi'(x_j) \to 0$ (resp. $(1 + \|x_j\|)\varphi'(x_j) \to 0$) in $X^*$ has a convergent subsequence in $X$. When $\varphi$ satisfies the $(PS)_c$-condition (resp. $(C)_c$-condition) at every level $c \in \mathbb{R}$ we say that it satisfies the $(PS)$-condition (resp. $(C)$-condition). For a $C^1$ functional $\varphi$ on a Banach space $X$, which is bounded below, it was further proved in [47, Proposition 5.23] that $\varphi$ satisfies the $(PS)$-condition if and only if it does the $(C)$-condition. If $\varphi \in C^1(X, \mathbb{R})$ is bounded below and satisfies the $(PS)$-condition, then it is coercive [9]. Conversely, Proposition 3 in [1, Chap.4, §5] claimed that any Gâteaux differentiable, convex, lower semicontinuous coercive functional $\varphi$ on a reflexive Banach space $X$ satisfies condition (weak $C$), that is, for any sequence $(x_n) \subset X$ such that $\sup |\varphi(x_n)| < \infty$ and $(\varphi'(x_n)) \subset X^* \setminus \{0\}$ and $\varphi'(x_n) \to 0$ in $X^*$, where $X^*$ is the dual space of $E$, there is some point $\bar{x} \in X$ such that $\varphi'(\bar{x}) = 0$ and $\liminf \varphi(x_n) \leq \varphi(\bar{x}) \leq \limsup \varphi(x_n)$. For $\mathfrak{F}_V$ we have a similar result.

Theorem 4.6 Let $\Omega \subset \mathbb{R}^n$, $N \in \mathbb{N}$, $p \in [2, \infty)$ and $V \subset W^{m,p}(\Omega, \mathbb{R}^N)$ be as in Theorem 4.1. Suppose that Hypothesis $\mathfrak{F}_{p,N,m,n}$ hold and that $\mathfrak{F}_V$ is coercive. Then $\mathfrak{F}_V$ satisfies the $(PS)$- and (C)-conditions on $V$. In particular, the controllable growth conditions (see “Appendix A”) imply that $\mathfrak{F}$ is coercive on any closed affine subspace of $W^{1,2}(\Omega, \mathbb{R}^N)$.

Proof Since the coercivity of $\mathfrak{F}$ implies that it is bounded below, by [47, Proposition 5.23] it suffices to prove that $\mathfrak{F}$ satisfies the $(PS)$-condition. Let $(\bar{u}_j) \subset V$ satisfy $\mathfrak{F}'(\bar{u}_j) \to c \in \mathbb{R}$ and $\mathfrak{F}(\bar{u}_j) \to 0$. Since $\mathfrak{F}$ is coercive, $(\bar{u}_j)$ is bounded. Note that $\bar{u}_j = \bar{v} + \bar{v}_j$, $\bar{v}_j \in V_0$ and that $V_0$ is a Hilbert subspace. After passing to a subsequence we may assume $\bar{v}_j \to \bar{v}$ in $V_0$. Moreover, $\mathfrak{F}'(\bar{u}_j) \to 0$ implies

$$\lim_{j \to \infty} \langle \mathfrak{F}'(\bar{u}_j), \bar{u}_j - \bar{u} \rangle = \lim_{j \to \infty} \langle \mathfrak{F}'(\bar{u}_j), \bar{v}_j - \bar{v} \rangle = \lim_{j \to \infty} \langle \mathfrak{F}'(\bar{u}_j), \bar{v}_j - \bar{v} \rangle = 0.$$

By (C) of Theorem 4.2, $\mathfrak{F}'$ is of class $(S)_+$, and hence $\bar{u}_j \to \bar{u}$ in $V$. The final claim is obvious.

There exist some explicit conditions on $F$ under which $\mathfrak{F}$ is coercive on $W^{m,p}_0(\Omega, \mathbb{R}^N)$, for example, there exist two positive constants $c_0, c_1$ such that

$$F(x, \xi) \geq c_0 \sum_{i=1}^N \sum_{|\alpha|=m} |\xi^{\alpha}_i|^p - c_1, \quad \forall (x, \xi).$$

The coercivity requirement is too strong. In fact, the proof of Theorem 4.6 shows that under Hypothesis $\mathfrak{F}_{p,N,m,n}$ we only need to add some conditions so that
\[
\sup_j |\tilde{y}(\tilde{u}_j)| < \infty \quad \text{and} \quad \tilde{y}'(\tilde{u}_j) \to 0 \implies \sup_j \|\tilde{u}_j\|_{m,p} < \infty.
\]

For example, the following two results are easily verified, see [38] for full proofs.

**Theorem 4.7** Let \( N \in \mathbb{N} \), \( p \in (2, \infty) \) and \( \Omega \subset \mathbb{R}^n \) be a Sobolev domain for \((p, m, n)\). Then \( \mathcal{F} \) satisfies the (PS)- and (C)-conditions on \( W^{m,p}_0(\Omega, \mathbb{R}^N) \) provided that Hypothesis \( \mathcal{H}_{p,n,m} \) is satisfied and that there exist \( \kappa \in \mathbb{R} \) and \( \gamma \in L^1(\Omega) \) such that

\[
F(x, \xi) = \kappa \sum_{i=1}^N \sum_{|\alpha| \leq m} F^i(x, \xi)\xi^i_\alpha \geq c_0 \sum_{i=1}^N \sum_{|\alpha| = m} |\xi^i_\alpha|^p - c_1 \sum_{i=1}^N |\xi^i_0|^p - \gamma(x) \quad \forall (x, \xi),
\]

where \( c_0 > 0 \) and \( c_0 - c_1 S_{m,p} > 0 \) with

\[
\int_\Omega |u|^p dx \leq S_{m,p} \int_\Omega |D^m u|^p dx = S_{m,p} \sum_{|\alpha| = m} \int_\Omega |D^\alpha u|^p \quad \forall u \in W^{m,p}_0(\Omega, \mathbb{R}^N).
\]

**Theorem 4.8** Let \( \Omega \subset \mathbb{R}^n \) be a Sobolev domain for \((2, m, n)\). Suppose that Hypothesis \( \mathcal{H}_{2,n,m,n} \) is satisfied with the constant function \( c_2 = 0 \), and that

\[
F(x, \hat{\xi}, 0) \leq \varphi(x) + C \sum_{i=1}^N \sum_{|\alpha| \leq m-1} |\xi^i_\alpha|^r, \quad \forall (x, \hat{\xi}, 0) \in \omega_0 \times \prod_{k=0}^{m-1} \mathbb{R}^N 
\]

where \( \varphi \in L^1(\Omega) \) and \( 1 \leq r < 2 \). Then \( \mathcal{F} \) satisfies the (PS)- and (C)-conditions on \( W^{m,2}_0(\Omega, \mathbb{R}^N) \).

When \( m = 1 \) and \( F \) does not depend on \( x, \hat{\xi} \), more characterizations of coercivity for \( \mathcal{F} \) can be found in [15] and references therein.

### 4.3 Morse inequalities and corollaries

Firstly, we show that Corollary 4.4 and Theorems 2.21 and 2.22 (taking \( \lambda = 0 \)) imply:

**Theorem 4.9** Let \( \Omega \subset \mathbb{R}^n \) be a Sobolev domain for \((2, m, n)\), \( N \in \mathbb{N} \), and \( H \) a closed subspace of \( W^{m,2}(\Omega, \mathbb{R}^N) \), \( \mathcal{H} = \tilde{\omega} + H \) for some \( \tilde{\omega} \in W^{m,2}(\Omega, \mathbb{R}^N) \). Let \( G \) be a compact Lie group which acts on \( \mathcal{H} \) in a \( C^3 \)-smooth isometric way. Suppose that Hypothesis \( \mathcal{F}_{2,n,m,n} \) is satisfied and that the restriction functional \( \mathcal{F}_{\mathcal{H}} := \mathcal{F}|_{\mathcal{H}} \) is \( G \)-invariant, where \( \mathcal{F} \) is given by (1.3). Let \( \mathcal{O} \) be an isolated critical orbit of \( \mathcal{F}_{\mathcal{H}} \) and also a compact \( C^3 \) submanifold. Its normal bundle \( N\mathcal{O} \) has fiber at \( \tilde{u} \in \mathcal{O} \), \( N\mathcal{O}_{\tilde{u}} = \{ \tilde{v} \in H \mid (\tilde{v}, \tilde{w})_{m,2} = 0 \ \forall \tilde{w} \in T_{\tilde{u}}O \subset H \} \). Let \( N^+\mathcal{O}_{\tilde{u}}, N^0\mathcal{O}_{\tilde{u}} \) and \( N^-\mathcal{O}_{\tilde{u}} \) be the positive definite, null and negative definite spaces of the bounded linear self-adjoint operator associated with the bilinear form

\[
N\mathcal{O}_{\tilde{u}} \times N\mathcal{O}_{\tilde{u}} \ni (\tilde{v}, \tilde{w}) \mapsto \sum_{i=1}^N \sum_{|\alpha| \leq m, |\beta| \leq m} \int_\Omega F_{ij}^{ab}(x, \tilde{u}(x), \ldots, D^m \tilde{u}(x)) D^\beta v^i \cdot D^\alpha w^i dx.
\]

Then \( \dim N^0\mathcal{O}_{\tilde{u}} \) and \( \dim N^-\mathcal{O}_{\tilde{u}} \) are finite and independent of choice of \( \tilde{u} \in \mathcal{O} \). They are called nullity and Morse index of \( \mathcal{O} \), denoted by \( v_\mathcal{O} \) and \( m_\mathcal{O} \), respectively. Moreover, the following holds.
(i) If \( v_\mathcal{O} = 0 \) (i.e., the critical orbit \( \mathcal{O} \) is nondegenerate), there exist \( \epsilon > 0 \) and a \( G \)-equivariant homeomorphism onto an open neighborhood of the zero section preserving fibers \( \Phi : N^+\mathcal{O}(\epsilon) \oplus N^-\mathcal{O}(\epsilon) \to N\mathcal{O} \) such that for any \( \bar{u} \in \mathcal{O} \) and \((\bar{v}_+, \bar{v}_-) \in N^+\mathcal{O}(\epsilon)_{\bar{u}} \times N^-\mathcal{O}(\epsilon)_{\bar{u}},
\[
\mathcal{H} \circ E \circ \Phi(\bar{u}, \bar{v}_+, \bar{v}_-) = \|\bar{v}_+\|_{m,2}^2 - \|\bar{v}_-\|_{m,2}^2 + \mathcal{F}|_{\mathcal{O}},
\]
where \( E : N\mathcal{O} \to \mathcal{H} \) is given by \( E(\bar{u}, \bar{v}) = \bar{u} + \bar{v} \).

(ii) If \( v_\mathcal{O} \neq 0 \) there exist \( \epsilon > 0 \), a \( G \)-equivariant topological bundle morphism that preserves the zero section,
\[
\mathcal{H} : N^0\mathcal{O}(3\epsilon) \to N^+\mathcal{O} \oplus N^-\mathcal{O} \subset \mathcal{H} \times \mathcal{H}, \quad (\bar{u}, \bar{v}) \mapsto \mathcal{H}(\bar{v}),
\]
and a \( G \)-equivariant homeomorphism onto an open neighborhood of the zero section preserving fibers, \( \Phi : N^0\mathcal{O}(\epsilon) \oplus N^+\mathcal{O}(\epsilon) \oplus N^-\mathcal{O}(\epsilon) \to N\mathcal{O} \), such that the following properties hold:

(ii.1) for any \( \bar{u} \in \mathcal{O} \) and \((\bar{v}_0, \bar{v}_+, \bar{v}_-) \in N^0\mathcal{O}(\epsilon)_{\bar{u}} \times N^+\mathcal{O}(\epsilon)_{\bar{u}} \times N^-\mathcal{O}(\epsilon)_{\bar{u}},
\[
\mathcal{H} \circ E \circ \Phi(\bar{u}, \bar{v}_0, \bar{v}_+, \bar{v}_-) = \|\bar{v}_+\|_{m,2}^2 - \|\bar{v}_-\|_{m,2}^2 + \mathcal{G}(\bar{u} + \bar{v}_0 + \mathcal{H}(\bar{v}));
\]

(ii.2) for each \( \bar{u} \in \mathcal{O} \) the function
\[
N^0\mathcal{O}(\epsilon)_{\bar{u}} \to \mathbb{R}, \quad \bar{v} \mapsto \mathcal{G}_{\bar{u}}^c(\bar{v}) := \mathcal{G}(\bar{u} + \bar{v} + \mathcal{H}(\bar{v}))
\]
is \( G_{\bar{u}} \)-invariant, of class \( C^1 \), and satisfies:
\[
D\mathcal{G}_{\bar{u}}^c(\bar{v})[\bar{w}] = D\mathcal{G}(\bar{u} + \bar{v} + \mathcal{H}(\bar{v}))[\bar{w}], \quad \forall \bar{w} \in N^0\mathcal{O}_{\bar{u}}.
\]

**Proof** Since \( T\mathcal{H} = \mathcal{H} \times \mathcal{H} \), the exponential map \( \exp : T\mathcal{H} \to \mathcal{H} \) (with respect to the Riemannian-Hilbert structure on \( \mathcal{H} \) induced by the inner product \( \langle \cdot, \cdot \rangle_{m,2} \)) is given by \( \exp(\bar{u}, \bar{v}) = \bar{u} + \bar{v} \) for \( \bar{u}, \bar{v} \in \mathcal{H} \times \mathcal{H} \). Let \( \mathcal{F}_{N\mathcal{O}_{\bar{u}}} \) be the restriction of \( \mathcal{F}_{\mathcal{H}} \) to the fiber of \( N\mathcal{O} \) at \( \bar{u} \in \mathcal{O} \). Then \( \mathcal{F}_{N\mathcal{O}_{\bar{u}}}(\bar{v}) = \mathcal{F}(\bar{u} + \bar{v}) \) for \( \bar{v} \in N\mathcal{O}_{\bar{u}} \). It follows from Corollary 4.4 that \( \mathcal{F}_{N\mathcal{O}_{\bar{u}}} \) satisfies Hypothesis 1.1 with \( X = N\mathcal{O}_{\bar{u}} \) around the origin of \( N\mathcal{O}_{\bar{u}} \). Theorems 2.21 and 2.22 lead to the desired conclusions immediately. \( \square \)

If \( n = 2 \), \( \partial \Omega \) is smooth, and either \( F \) is analytic, or \( m = 1 \) and \( F \) is suitable smooth, then the Morse indexes of critical points of \( \mathcal{F} \) on \( W^{m,2}_{0}(\Omega, \mathbb{R}^N) \) can be computed by Uhlenbeck’s generalizations [60, Theorem 3.5] for Smale’s Morse index theorem [56]. The case of Neumann type boundary conditions may still be considered by Dalbono and Portaluri [20].

Write \( \mathcal{F}_{\mathcal{H},d} = \{ x \in \mathcal{H} \mid \mathcal{F}(x) \leq d \} \) for \( d \in \mathbb{R} \). Using Corollary 2.26, the standard arguments (see [12, Chapter I, Theorem 7.6], [42, Chapter 10] and [3, Corollary 6.5.10]) yield

**Theorem 4.10** Under the assumptions of Theorem 4.9, let \( a < b \) be two regular values of \( \mathcal{F}_{\mathcal{H}} \) \( \mathcal{F}_{\mathcal{H}}^{-1}([a, b]) \) contains only nondegenerate critical orbits \( \mathcal{O}_j \) with Morse indexes \( \mu_j \), \( j = 1, \ldots, k \). Suppose that \( \mathcal{F}_{\mathcal{H}} \) satisfies the \((PS)_c \) condition for each \( c \in [a, b] \). (This is true if either \( \mathcal{F}_{\mathcal{H}} \) is coercive or one of Theorems 4.6 and 4.7 holds in case \( \mathcal{H} = W^{m,2}_{0}(\Omega, \mathbb{R}^N) \).

Then
\[
\sum_{j=1}^{k} \dim H_{q-\mu_{\mathcal{O}_j}}(\mathcal{O}_j; \mathbb{Z}_2) = \dim H_q(\mathcal{F}_{\mathcal{H},b}, \mathcal{F}_{\mathcal{H},a}; \mathbb{Z}_2), \quad \forall q \in \mathbb{N}_0,
\]

\[\square\] Springer
and there exists a polynomial with nonnegative integral coefficients \( Q(t) \) such that
\[
\sum_{i=0}^{\infty} \sum_{j=1}^{k} \text{rank} H_G^i (\mathcal{O}_j, \theta_j^{-} \otimes K) t^{i+j} = \sum_{i=0}^{\infty} \text{rank} H_G^i (\mathcal{F}_{\mathcal{T}_k}, \mathcal{F}_{\mathcal{T}_k}, K) t^i + (1 + t) Q(t),
\]
(4.10)
where \( \theta_j^{-} \) is the orientation bundle of \( N - \mathcal{O}_j, j = 1, \ldots, k \). In particular, if \( G \) is a trivial group and each \( \mathcal{O}_j \) becomes a nondegenerate critical point \( \tilde{u}_j \), we have the Morse inequalities:
\[
\sum_{j=0}^{l} (-1)^{l-j} N_j(a, b) \geq \sum_{j=0}^{l} (-1)^{l-j} \beta_j(a, b), \quad \forall l \in \mathbb{N}_0,
\]
(4.11)
where for each \( q \in \mathbb{N}_0, N_q(a, b) = \sharp \{ 1 \leq i \leq k \mid \mu_i = q \} \) (the number of points in \( \{ \tilde{u}_j \}_{j=1}^k \) with Morse index \( q \) and \( \beta_q(a, b) = \sum_{i=1}^{k} \text{rank} H_{\mathcal{T}_k} (\mathcal{F}_{\mathcal{T}_k}, \mathcal{F}_{\mathcal{T}_k}, K) \). Furthermore, if \( \mathcal{F}_{\mathcal{T}_k} \) is coercive, has only nondegenerate critical points, and for each \( q \in \mathbb{N}_0 \) there exist only finitely many critical points with Morse index \( q \), then the following relations hold:
\[
\sum_{i=0}^{q} (-1)^{q-i} N_i \geq (-1)^q, \quad \forall q \in \mathbb{N}_0, \quad \text{and} \quad \sum_{i=0}^{\infty} (-1)^{i} N_i = 1,
\]
(4.12)
where \( N_i \) is the number of critical points of \( \mathcal{F}_{\mathcal{T}_k} \) with Morse index \( i \).

**Remark 4.11**
(i) When \( N = 1, \mathcal{T} = W_{0}^{m,2}(\Omega) \) and \( \mathcal{F}_{\mathcal{T}_k} \) is coercive, (4.11) was first obtained by Skrypnik in [52, §5.2] and [53, Theorem 4.7, Chap.1]. Instead of using Morse–Palais lemma, his ideas are similar to Smale’s [55], but some new techniques are employed, which motivated our current work.

(ii) If \( m = N = 1, \partial \Omega \) is of class \( C^{2+\alpha} \) for some \( \alpha \in (0, 1) \), and \( p > n = \dim \Omega \) such that \( W_{p,2} \subset C^{1} \), under some conditions on \( F \) Ströhmer [57] proved a handle body theorem for \( \mathcal{F} \) on \( Z_{\varphi} = \{ u \in W^{2,p}(\Omega) \mid |u|_{\partial \Omega} = \varphi |_{\partial \Omega} \} \) for \( \varphi \in C^{2+\alpha} \). His conditions and those of Theorem 4.10 cannot be contained each other.

(iii) As in Remark 4.5, we may give the corresponding versions of Theorems 4.9 and 4.10 in the setting of [49,50,55]. In particular, replace \( \Omega \subset \mathbb{R}^n \) by \( \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n \) and assume that a \( C^2 \) function \( F : \mathbb{T}^n \times \prod_{k=0}^{m} \mathbb{R}^{N \times M_0(k)} \to \mathbb{R} \) satisfies Hypothesis \( \mathcal{F}_{2,N,m,n} \) when restricted to \( [0, 1]^n \times \prod_{k=0}^{m} \mathbb{R}^{N \times M_0(k)} \), then for the action of \( G = \mathbb{T}^n \) or \( \mathbb{T}^1 \) on \( W_{m,2}(\mathbb{T}^n, \mathbb{R}^N) \) given by the isometric linear representation
\[
\{ (t_1, \ldots, t_n) \cdot \tilde{u} \}(x_1, \ldots, x_n) = \tilde{u}(x_1 + t_1, \ldots, x_n + t_n), \quad \{ t \cdot \tilde{u} \}(x_1, \ldots, x_n) = \tilde{u}(x_1 + t, \ldots, x_n + t), \quad \{ t \} \in \mathbb{T}^n,
\]
\[
\{ (t) \cdot \tilde{u} \}(x_1, \ldots, x_n) = \tilde{u}(x_1 + t, \ldots, x_n + t), \quad \{ t \} \in \mathbb{T}^1
\]
Theorems 4.9 and 4.10 hold true. These provide necessary tools for generalizing works in [32,62]. If the function \( F \) is defined on \( \mathbb{T}^n \times \mathbb{T}^n \times \prod_{k=1}^{m} \mathbb{R}^{N \times M_0(k)} \) the corresponding variational problem on \( W_{m,2}(\mathbb{T}^n, \mathbb{T}^N) \) is related to [46] and may be also considered with our theory.

**Corollary 4.12**
Given integers \( m, N \geq 1, n \geq 2, \) let \( \Omega \subset \mathbb{R}^n \) be a Sobolev domain for \( (2, m, n) \), and let \( V_0 \) be a closed subspace of \( W_{m,2}(\Omega, \mathbb{R}^N) \) and \( V = \tilde{w} + V_0 \) for some \( \tilde{w} \in W_{m,2}(\Omega, \mathbb{R}^N) \). Suppose that Hypothesis \( \mathcal{F}_{2,N,m,n} \) holds. Then each critical point of \( \mathcal{F}_V \) has finite Morse index \( \mu \) and nullity \( v \); moreover, if \( \tilde{u} \in V \) is an isolated critical point of \( \mathcal{F}_V \), for any Abelian group \( K \), rank\( C_j(\mathcal{F}_V, \tilde{u}; K) < \infty \) \( \forall j \in \mathbb{N}_0 \), and \( C_j(\mathcal{F}_V, \tilde{u}; K) = 0 \) for \( j < \mu \) or \( j > \mu + v \).
This is a direct consequence of Theorem 2.3 and Corollary 4.4. From Corollary 4.4, Theorems 2.1 and 4.6 and [47, Proposition 6.93] we immediately deduce

**Corollary 4.13** Let $V \subset W^{m,2}(\Omega, \mathbb{R}^N)$ be as in Corollary 4.12. If $\mathcal{F}_V$ is bounded below, satisfies the (PS)-condition, and has a nondegenerate critical point which is not a global minimizer, then it has at least three critical points.

The last two corollaries also hold if $\Omega$ is replaced by $\mathbb{T}^n$. For a $C^2$ Lagrangian satisfying the controllable growth conditions the corresponding integral functional is bounded below and coercive. From the above results we immediately get

**Theorem 4.14** Let $\Omega \subset \mathbb{R}^n$ be a Sobolev domain for $(2,1,n)$, $N \in \mathbb{N}$, and $H$ a closed subspace of $W^{1,2}(\Omega, \mathbb{R}^N)$, $\mathcal{H} = \tilde{\omega} + H$ for some $\tilde{\omega} \in W^{1,2}(\Omega, \mathbb{R}^N)$. Assume that $\Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \ni (x,z,p) \mapsto F(x,z,p) \in \mathbb{R}$ is a $C^2$ function fulfilling the controllable growth conditions (see “Appendix A”). Let $G$ be a compact Lie group which acts on $V$ in a $C^3$-smooth isometric way. Suppose that the restriction functional $\mathcal{F}|_H := \mathcal{F}|_H$ is $G$-invariant. Then

(i) If $a < b$ are two regular values of $\mathcal{F}|_H$ and $\mathcal{F}|_H^{-1}((a,b))$ contains only nondegenerate critical orbits $O_j$ with Morse indexes $\mu_j$, $j = 1, \ldots, k$, then (4.9) and (4.10) hold; in particular, if $G$ is trivial and $\mathcal{F}|_H^{-1}((a,b))$ contains only nondegenerate critical points, then (4.11) holds.

(ii) If $\mathcal{F}|_H$ has only nondegenerate critical points, and for each $q \in \mathbb{N}_0$ there exist only finitely many critical points with Morse index $q$, then (4.12) holds.

(iii) If $\tilde{u}$ is a critical point of $\mathcal{F}$ on $V := \tilde{w} + W^{1,2}_0(\Omega, \mathbb{R}^N) \subset W^{1,2}(\Omega, \mathbb{R}^N)$ which is not a global minimizer, then it has at least three critical points on $V$ provided that the bilinear form

$$W^{1,2}_0(\Omega, \mathbb{R}^N) \times W^{1,2}_0(\Omega, \mathbb{R}^N) \ni (\tilde{v}, \tilde{w}) \mapsto \sum_{i=1}^N \sum_{|\alpha|,|\beta| \leq 1} \int_{\Omega} F_{ij}^{ij}(x, \tilde{u}(x), D\tilde{u}(x))D^\alpha\beta v^i \cdot D^\alpha\beta w^j \, dx$$

is nondegenerate.

(iv) If $\Omega$ is replaced by $\mathbb{T}^n$ in (i) and (ii) the corresponding conclusions also holds.

4.4 Applicability of related previous work

In this section we study under what conditions on $F$ splitting theorems in [4,29,32,33] are applicable. As consequences, under Hypothesis $\mathcal{F}_H$ of $H$ on $H := W^{m,2}_0(\Omega, \mathbb{R}^N)$ defined by the right side of (1.3) is smooth enough then the critical groups of $\mathcal{F}_H$ at $\tilde{u}$ are equal to those of the restriction of $\mathcal{F}_H$ to a smaller appropriate space containing it. For these, the following special case of [44, Theorem 6.4.8] is very key.

**Proposition 4.15** For a real $p \geq 2$ and an integer $k \geq m + \frac{n}{p}$, let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class $C^{k-1,1}$, $N \in \mathbb{N}$, and let bounded and measurable functions on $\Omega$, $A^{ij}_{ab}$, $i, j = 1, \ldots, N$, $|\alpha|, |\beta| \leq m$, fulfill the following conditions:

(i) $A^{ij}_{ab} \in C^{k-|\alpha|-2m-1,1}(\overline{\Omega})$ if $2m - k < |\alpha| \leq m$,

(ii) there exists $c_0 > 0$ such that

$$\sum_{i,j=1}^N \sum_{|\alpha| = |\beta| = m} \int_{\Omega} A^{ij}_{ab} \eta^i_{\alpha} \eta^j_{\beta} \geq c_0 \sum_{i=1}^N \sum_{|\alpha| = m} |\eta^i_{\alpha}|^2, \quad \forall \eta \in \mathbb{R}^N \times M_0(m).$$
Suppose that \( \tilde{u} = (u^1, \ldots, u^N) \in W^{m,2}_0(\Omega, \mathbb{R}^N) \) and \( \lambda \in (-\infty, 0] \) satisfy

\[
\sum_{i,j=1}^{N} \sum_{|\alpha|,|\beta| \leq m} \oint_{\Omega} (A^{ij}_{\alpha\beta} - \lambda \delta_{ij} \delta_{\alpha\beta}) D^\beta u^i \cdot D^\alpha v^j \, dx = 0 \quad \forall v \in W^{m,2}_0(\Omega, \mathbb{R}^N).
\]

Then \( \tilde{u} \in W^{k,p}(\Omega, \mathbb{R}^N) \). Moreover, for \( f_j = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha f^j_{\alpha} \), where \( f^j_{\alpha} \in W^{k-2m+|\alpha|,p}(\Omega) \) if \( |\alpha| > 2m - k \), and \( f^j_{\alpha} \in L^p(\Omega) \) if \( |\alpha| \leq 2m - k \), suppose that \( \tilde{u} \in W^{m,2}_0(\Omega, \mathbb{R}^N) \) satisfy

\[
\oint_{\Omega} \sum_{j=1}^{N} \sum_{|\alpha| \leq m} \left[ \sum_{i=1}^{N} \sum_{|\beta| \leq m} A^{ij}_{\alpha\beta} D^\beta u^i - f^j_{\alpha} \right] D^\alpha v^j \, dx = 0, \quad \forall v \in W^{m,2}_0(\Omega, \mathbb{R}^N),
\]

we have also \( \tilde{u} \in W^{k,p}(\Omega, \mathbb{R}^N) \).

Without special statements, the Hilbert space \( H = W^{m,2}_0(\Omega, \mathbb{R}^N) \) with the usual inner product

\[
(u, v)_{H} = \sum_{i=1}^{N} \sum_{|\alpha| = m} \oint_{\Omega} D^\alpha u^i D^\alpha v^j \, dx.
\]

**Theorem 4.16** Under Hypothesis \( \mathfrak{F}_{2,N, m,n} \), let \( \mathfrak{F}_H \) denote the functional on \( H \) defined by the right side of (1.3), and let \( \tilde{u}^* \in H \) be a critical point of \( \mathfrak{F}_H \). Suppose that there exist a real \( p > 1 \) and an integer \( k > m + \frac{n}{p} \) such that \( \tilde{u}^* \) is contained in the Banach subspace \( X_{k,p} := W^{k,p}(\Omega, \mathbb{R}^N) \cap W^{m,2}_0(\Omega, \mathbb{R}^N) \) of \( W^{k,p}(\Omega, \mathbb{R}^N) \). Moreover, if \( F \) is of class \( C^{k-1,1} \), then (2.1) in [33, page 2944] is also fulfilled, and the negative definite space of \( D(\nabla \mathfrak{F}_H)(\tilde{u}^*) \) is contained in \( X_{k,p} \).

**Proof** Denote by \( \nabla \mathfrak{F}_H \) the gradient of \( \mathfrak{F}_H \). For \( s < 0 \) let \( W^{s,p}(\Omega) = [W^{-s,p'}_0(\Omega)]^* \) as usual, where \( p' = \frac{p}{p-1} \). Note that the \( m \)-th power of the Laplace operator, \( \Delta^m \), is an isomorphism from a Banach subspace \( W^{k,p}(\Omega) \cap W^{m,2}_0(\Omega) \) of \( W^{k,p}(\Omega) \) to \( W^{k-2m,p}(\Omega) \), and thus that its inverse, denoted by \( \Delta^{-m} \), is from \( W^{k-2m,p}(\Omega) \) to \( W^{k,p}(\Omega) \cap W^{m,2}_0(\Omega) \). By (4.1) and (4.13), it is easily computed that for \( i = 1, \ldots, N \),

\[
(\Delta^{-m} \mathfrak{F}_H(\tilde{u}))^i = \Delta^{-m} \sum_{|\alpha| \leq m} (-1)^{m+|\alpha|} D^\alpha (F^i_{\alpha}(\cdot, \tilde{u}(\cdot), \ldots, D^m \tilde{u}(\cdot))), \quad \forall \tilde{u} \in H.
\]

As in Theorem 4.1, \( \nabla \mathfrak{F}_H \) has the Gâteaux derivative \( D(\nabla \mathfrak{F}_H)(\tilde{u}) \in L^2(H) \) at \( \tilde{u} \in H \) such that for any \( \tilde{v}, \tilde{\varphi} \in H \), \( D(\nabla \mathfrak{F}_H)(\tilde{u})(\tilde{v}), \tilde{\varphi}) \) is given by the right side of (4.2). Denote by \( \mathcal{B} \) the restriction of \( D(\nabla \mathfrak{F}_H) \) to \( X_{k,p} \). For \( \tilde{u} \in X_{k,p} \) and \( \tilde{v} \in H \) let \( \mathcal{B}(\tilde{u}) \tilde{v} = (\mathcal{B}(\tilde{u}) \tilde{v})^1, \ldots, (\mathcal{B}(\tilde{u}) \tilde{v})^N \). It is easily verified that

\[
(\mathcal{B}(\tilde{u}) \tilde{v})^i = \Delta^{-m} \sum_{j=1}^{N} \sum_{|\alpha| \leq m, |\beta| \leq m} (-1)^{m+|\alpha|} D^\alpha (F^{ij}_{\alpha\beta}(\cdot, \tilde{u}(\cdot), \ldots, D^m \tilde{u}(\cdot)) D^\beta v^j).
\]

Let \( \Delta \) denote the restriction of \( \nabla \mathfrak{F}_H \) to \( X_{k,p} \). We have
Claim 4.17 \( \mathbb{A} \) is a \( C^1 \) map from \( X_{k,p} \) to itself, and satisfies \( d\mathbb{A}(\vec{u})[\vec{v}] = \mathbb{B}(\vec{u})\vec{v} \) for all \( \vec{u}, \vec{v} \in X_{k,p} \).

We first admit this claim and postpone its proof.

For each \( \vec{u} \in X_{k,p} \), we may write \( \mathbb{P}(\vec{u}) = \mathbb{P}(\vec{u}) + \mathbb{Q}(\vec{u}) \), where for \( i = 1, \ldots, N \),

\[
(\mathbb{P}(\vec{u})\vec{v})^j = \Delta^{-m} \sum_{j=1}^{N} \sum_{|\alpha|=|\beta|=m} (-1)^{m+|\alpha|} D^\alpha(F^i_{\alpha\beta}(. , \vec{u}(.), \ldots, D^m\vec{u}(.)) D^\beta \vec{v}^j),
\]

(4.16)

\[
(\mathbb{Q}(\vec{u})\vec{v})^j = \Delta^{-m} \sum_{j=1}^{N} \sum_{|\alpha| \leq m, \beta \leq m, \alpha + \beta < 2m} D^\alpha(F^i_{\alpha\beta}(., \vec{u}(.), \ldots, D^m\vec{u}(.)) D^\beta \vec{v}^j).
\]

(4.17)

As in the proofs of Theorems 4.1 and 4.2 (cf. [38]) we can derive that \( \mathbb{P}(\vec{u}) \) and \( \mathbb{Q}(\vec{u}) \) are positive definite and completely continuous, respectively, and they also satisfy the condition (D) in [33, page 2944]. By [33, Proposition B.2] we also see that (C1) in [33, page 2944] holds for the operator \( \mathbb{B}(\vec{u}^*) \).

Next, we prove the second claim. Let \( \Lambda_{\alpha\beta}^i := F_{\alpha\beta}(., \vec{u}^*(), \ldots, D^m\vec{u}^*(.)). \) They sit in \( C^{k-m}(\Omega, \mathbb{R}^N) \) because \( \vec{u}^* \in C^k(\Omega, \mathbb{R}^N) \). Let \( \vec{u} \in H \) be such that \( \vec{w} := \mathbb{B}(\vec{u}^*) \vec{u} \) sits in \( X_{k,p} \). Then (4.15) implies that

\[
\int_{\Omega} \sum_{j=1}^{N} \sum_{|\alpha| \leq m} \left[ \sum_{i=1}^{N} \Lambda_{\alpha\beta}^i D^\beta \vec{u}^i - f_{\alpha}^j \right] D^\alpha \vec{v}^j \ dx = 0, \forall \vec{v} \in W_0^{m,2}(\Omega, \mathbb{R}^N),
\]

where \( f_{\alpha}^j = 0 \) for \( |\alpha| < m \), and for \( f_{\alpha}^j = (-1)^{|\alpha|} D^\alpha \vec{u}^j \) for \( |\alpha| = m \). Note that \( \vec{u} \in W^{k,p}(\Omega) \) and \( k > m \) lead to \( f_{\alpha}^j \in W^{k-m,p}(\Omega) = W^{k-2m+|\alpha|,p}(\Omega) \) for \( |\alpha| = m > 2m - k \). Since \( p \geq 2, \beta \Omega \) is of classes \( C^{k-1,1} \) and (1.2) implies that Proposition 4.15(ii) is satisfied, we may use the second claim of Proposition 4.15 to deduce that \( \vec{u} \in W^{k,p}(\Omega, \mathbb{R}^N) \). That is, (C2) in [33, page 2944] is fulfilled.

The final conclusion may be derived from the first claim of Proposition 4.15 as above. \( \square \)

When \( n = 2 \), we see from [54, Chapter 7, Th. 4.4] that the conditions of this theorem can be satisfied if \( F \) is smooth enough.

Proof of Claim 4.17 Let \( r = \dim(\Omega) = \prod_{k=0}^{m} M_0^{N \times M_0(k)} = n + (m+1) N + \sum_{k=0}^{m} M_0(k) \). For \( \vec{u} \in X_{k,p} \) and \( x \in \Omega \) put \( u(x) = (x, \vec{u}(x), \ldots, D^m\vec{u}(x)) \). Then \( \Upsilon(\vec{u}) = u \) defines an affine (and thus smooth) map \( \Upsilon \) from \( X_{k,p} \) to \( W^{k-m,p}(\Omega, \mathbb{R}^r) \). Since \( (k-m)p > n \) and \( F_{\alpha}^i \) is of class \( C^{k-m+1} \), by [66, Lemma 2.96] (with \( X = W^{k-m,p} \)) the map

\[
\Phi_{F_{\alpha}^i} : W^{k-m,p}(\Omega, \mathbb{R}^r) \to W^{k-m,p}(\Omega), \ u \mapsto F_{\alpha}^i \circ u
\]

is of class \( C^1 \), and \( d\Phi_{F_{\alpha}^i}(u)v = (dF_{\alpha}^i \circ u)v \) for any \( u, v \in W^{k-m,p}(\Omega, \mathbb{R}^r) \). Hence

\[
\Lambda_{\alpha,i} = F_{\alpha}^i \circ \Upsilon : X_{k,p} \to W^{k-m,p}(\Omega), \vec{u} \mapsto F_{\alpha}^i(., \vec{u}(.), \ldots, D^m\vec{u}(.))
\]

is of class \( C^1 \) and

\[
d\Lambda_{\alpha,i}(\vec{u})(\vec{v}) = d(\Phi_{F_{\alpha}^i} \circ \Upsilon)(\vec{u})(\vec{v}) = d(\Phi_{F_{\alpha}^i})(\Upsilon(\vec{u}))(d\Upsilon(\vec{u})(\vec{v})) = (dF_{\alpha}^i \circ u)(v - \psi),
\]

\( \diamond \)
where \( \psi : \Omega \to \mathbb{R}^r \) is given by \( \psi(x) = (x, 0, \ldots, 0) \) for \( x \in \Omega \). Clearly, (4.14) implies
\[
(\mathbb{A}(\vec{u}))^i = \Delta^{-m} \sum_{|\alpha| \leq m} (-1)^{m+|\alpha|} D^\alpha (\mathbb{A}_i \alpha(\vec{u})).
\]

Since \( \Delta^{-m} : W^{k-2m,p}(\Omega) \to W^{k,p}(\Omega) \cap W^{m,2}(\Omega) \) is a Banach space isomorphism and \( D^\alpha : W^{k-m,p}(\Omega) \to W^{k-m-|\alpha|,p}(\Omega) \) is a continuous linear operator, we deduce that \( \mathbb{A} \) is a \( C^1 \) map from \( X_{k,p} \) to itself. The second conclusion easily follows from the above arguments. We may also obtain it as follows. Denote by the inclusion \( \iota : X_{k,p} \hookrightarrow H \). Since \( \iota \circ \mathbb{A} = \nabla \mathcal{H}|_{X_{k,p}} \) and \( (D(\nabla \mathcal{H})(\vec{u})[\vec{v}], \vec{v})_H \) is equal to the right side of (4.2), it follows that \( d(\mathbb{A}(\vec{u}))[\vec{v}] = \mathbb{B}(\vec{u}) \vec{v} \).

Let \( Y \) be the Banach subspace \( C^m(\overline{\Omega}, \mathbb{R}^N) \cap W^{m,2}_0(\Omega, \mathbb{R}^N) \) of \( C^m(\overline{\Omega}, \mathbb{R}^N) \). (Since \( k \geq m+1 \) and \( \partial \Omega \) is of class \( C^{k-1,1} \), we have
\[
Y = \{ \vec{u} \in C^m(\overline{\Omega}, \mathbb{R}^N) \mid D^s \vec{u} |_{\partial \Omega} = 0, \ s = 0, \ldots, m-1, \ D^{m-1} \vec{u} \in W^{1,2}_0 \}
\]

by [6, Theorem 9.17].) Let \( \mathcal{H} \) denote the restriction of \( \nabla \mathcal{H} \) to \( Y \). Since \( F \) is of class \( C^{k-2+m} \), it follows from \( \omega \)-lemma (cf. [66, Lemma 2.96]) that \( \mathcal{H} \) is of class \( C^{k-2+m} \). Define \( \mathcal{B} : Y \to \mathcal{C}_y(H) \) by \( \mathcal{B}(\vec{u}) = D(\nabla \mathcal{H})(\vec{u}) \). Then \( \mathcal{D}(\mathcal{H})(\vec{u})[\vec{v}, \vec{w}] = (\mathcal{B}(\vec{u}) \vec{v}, \vec{w})_H \) for any \( \vec{u}, \vec{v}, \vec{w} \in Y \).

For \( \vec{u}, \vec{v} \in X_{k,p} \) (resp. \( \vec{u}, \vec{v} \in Y \)) and \( \psi, \vec{v}, \vec{w} \in H \), (4.2) yields
\[
((\mathbb{B}(\vec{u}) - \mathbb{B}(\vec{v}))(\vec{v}, \vec{w}))_H \quad (\text{resp. } ((\mathbb{B}(\vec{u}) - \mathbb{B}(\vec{v}))(\vec{v}, \vec{w}))_H)
\]
\[
= \sum_{i,j=1}^{N} \sum_{|\alpha| \leq m, |\beta| \leq m} \int_{\Omega} F^{ij}_{\alpha\beta}(x, \vec{u}(x), \ldots, D^m \vec{u}(x)) \psi^i \cdot D^\alpha \psi^j \, dx.
\]

Since \( X_{k,p} \) may be continuously embedded into the space \( Y \), the above equality implies

**Claim 4.18** The map \( \mathbb{B} : X_{k,p} \to \mathcal{C}_y(H) \) (resp. \( \mathcal{B} : Y \to \mathcal{C}_y(H) \)) is uniformly continuous on any bounded subset of \( X_{k,p} \) (resp. \( Y \)).

**Theorem 4.19** For a real \( p \geq 2 \) and an integer \( k \geq m + \frac{n}{p} \), let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with boundary of class \( C^{k-1,1} \), \( N \in \mathbb{N} \), and let \( \overline{\Omega} \times \prod_{k=0}^{m} \mathbb{R}^{N \times M_0(k)} \ni (x, \xi) \mapsto F(x, \xi) \in \mathbb{R} \) be of class \( C^{k-2+m} \). Suppose that a critical point \( \vec{u}^* \) of \( \mathcal{H}_{X_{k,p}} \) on \( X_{k,p} = W^{k,p}(\Omega, \mathbb{R}^N) \cap W^{m,2}_0(\Omega, \mathbb{R}^N) \) belongs to \( C^k(\overline{\Omega}, \mathbb{R}^N) \) and that there exists \( c > 0 \) such that for any \( \eta = (\eta^i_\alpha) \in \mathbb{R}^{N \times M_0(m)} \) and for any \( x \in \overline{\Omega} 
\]
\[
N \sum_{i,j=1}^{N} \sum_{|\alpha|=|\beta|=m} F^{ij}_{\alpha\beta}(x, \vec{u}^*(x), D\vec{u}^*(x), \ldots, D^m \vec{u}^*(x)) \eta^i_\alpha \eta^j_\beta \geq c \sum_{i=1}^{N} \sum_{|\alpha|=m} (\eta^i_\alpha)^2.
\]

(4.18)

For \( \vec{u} \in X_{k,p} \) (resp. \( \vec{u} \in Y \)) and \( \vec{v} \in H \), let \( (\mathbb{A}(\vec{u}))^i \) and \( (\mathbb{B}(\vec{u}) \vec{v})^i \) be still defined by the right side of (4.14) and (4.15), respectively. Then maps \( \mathbb{A} : X_{k,p} \to X_{k,p} \) and \( \mathcal{B} : Y \to \mathcal{C}_y(H) \) satisfy Claims 4.17 and 4.18, and so \( (\mathcal{H}_{X_{k,p}}, \mathbb{A}, X_{k,p}, Y) \) fulfills the conditions of [29, Theorem 2.5] near \( \vec{u}^* \in X_{k,p} \). Moreover the negative definite space of \( \mathbb{B}(\vec{u}^*) \in \mathcal{C}_y(H) \) is contained in \( X_{k,p} \).

**Corollary 4.20** For a real \( p \geq 2 \) and an integer \( k \geq m + \frac{n}{p} \), let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with boundary of class \( C^{k-1,1} \), \( N \in \mathbb{N} \), and let \( \overline{\Omega} \times \prod_{k=0}^{m} \mathbb{R}^{N \times M_0(k)} \ni (x, \xi) \mapsto F(x, \xi) \in \mathbb{R} \) be of class \( C^{k-2,m} \). Then
(i) if a critical point $\tilde{u}^*$ of $\delta_X_{k,p}$ on $X_{k,p} = W^{k,p}(\Omega, \mathbb{R}^N) \cap W^{m,2}_0(\Omega, \mathbb{R}^N)$ belongs to $C^k(\overline{\Omega}, \mathbb{R}^N)$ and (4.18) also holds for some $c > 0$ and for all $\eta = (\eta^i_\alpha) \in \mathbb{R}^{N \times M_0(m)}$ and $x \in \overline{\Omega}$, we have $C_*(\delta_X_{k,p}, \tilde{u}^*; K) = C_*(\delta_Y, \tilde{u}^*; K)$ provided that $\tilde{u}^*$ is an isolated critical point for $\delta_Y$ (and so for $\delta_X_{k,p}$);

(ii) if Hypothesis $\delta_{2,N,m,n}$ is also satisfied and a critical point $\tilde{u}^*$ of $\delta_H$ (as in Theorem 4.16) belongs to $C^k(\overline{\Omega}, \mathbb{R}^N)$, we have $C_*(\delta_H, \tilde{u}^*; K) = C_*(\delta_X_{k,p}, \tilde{u}^*; K)$ provided that $\tilde{u}^*$ is an isolated critical point for $\delta_H$ (and so for $\delta_X_{k,p}$).

By Theorem 4.19, (i) follows from [29, Corollary 2.8]. Using Theorem 4.16 we may obtain (ii) from Remark 2.2(i) and Corollary 2.6 in [33].

Finally, we state the following more general version of [4, Theorems 2.1.2.2].

**Theorem 4.21** For a real $p \geq 2$ and an integer $k > m + \frac{n}{p}$, let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class $C^{k-1,1}$, $N \in \mathbb{N}$, and let $\overline{\Omega} \times \prod_{k=0}^m \mathbb{R}^{N \times M_0(k)} \ni (x, \xi) \mapsto F(x, \xi) \in \mathbb{R}$ be of class $C^{k-m+3}$. As above, we have the functional $\delta_X_{k,p}$ on $X_{k,p} = W^{k,p}(\Omega, \mathbb{R}^N) \cap W^{m,2}_0(\Omega, \mathbb{R}^N)$, which is of class $C^{k-m+3}$ by [66, Lemma 2.96] (with $X = C^0$) as in the proof of Claim 4.17. Suppose that a critical point of $\delta_X_{k,p}, \tilde{u}^*$, belongs to $C^k(\overline{\Omega}, \mathbb{R}^N)$ and that there exists $c > 0$ such that (4.18) holds for any $\eta = (\eta^i_\alpha) \in \mathbb{R}^{N \times M_0(m)}$ and for any $x \in \overline{\Omega}$. Let $\mathbb{B}(\tilde{u}^*)$ be given by (4.15). Then it has finite dimensional kernel and negative definite space, $H^0$ and $H^-$, which are contained in $X_{k,p}$. Denote by $X_{k,p}^+$ the intersection of $X_{k,p}$ with the positive definite space of $\mathbb{B}(\tilde{u}^*)$, and by $P^0$, $P^-$, $P^+$ the projections onto $H^0$, $H^-$, $X_{k,p}^+$ yielded by the Banach space direct sum decomposition $X_{k,p} = H^0 \oplus H^- \oplus X_{k,p}^+$. Then we have:

(i) if $H^0 = \{0\}$, there exists a $C^1$ diffeomorphism $\varphi : U \to X_{k,p}$ in some neighborhood $U \subset X_{k,p}$ of zero such that $\varphi(0) = \theta$ and

$$\delta_X_{k,p}(\varphi(\tilde{u}) + \tilde{u}^*) = \| P^+ \tilde{u} \|^2_H - \| P^- \tilde{u} \|^2_H + \delta_X_{k,p}(\tilde{u}^*) \quad \forall \tilde{u} \in U;$$

(ii) if $H^0 \neq \{0\}$, there exist $\epsilon > 0$, a (unique) $C^1$ map $\eta : B_{H^0}(\theta, \epsilon) \to X_{k,p}^+ \oplus H^-$ satisfying $\eta(\theta) = \theta$ and $(P^+ + P^-)\nabla \delta_X_{k,p}(\tilde{u}^* + z + \eta(z)) = 0 \forall z \in B_{H^0}(\theta, \epsilon)$, and a $C^1$ diffeomorphism $\varphi : U \to X_{k,p}$ in some neighborhood $U \subset X_{k,p}$ of zero such that $\varphi(0) = \theta$ and

$$\delta_X_{k,p}(\varphi(\tilde{u}) + \tilde{u}^*) = \| P^+ \tilde{u} \|^2_H - \| P^- \tilde{u} \|^2_H + \delta_X_{k,p}(P^0 \tilde{u}) \quad \forall \tilde{u} \in U,$$

where $\nabla \delta_X_{k,p}$ is the gradient of $\delta_X_{k,p}$ with respect to the inner product in (4.13), and $\delta_X_{k,p}^o$ is a $C^2$ map on $B_{H^0}(\theta, \epsilon)$ defined by $\delta_X_{k,p}^o(z) = \delta_X_{k,p}(z + \eta(z) + \tilde{u}^*)$, which has zero as a critical point.

In the present case, for $\tilde{u} \in X_{k,p}$ and $\tilde{v} \in H$, we still assume that $(A(\tilde{u}))^i_j$ and $(\mathbb{B}(\tilde{u}^*) \tilde{v})^i_j$ are defined by the right side of (4.14) and (4.15), respectively. Then $d\delta_X_{k,p}(\tilde{u})[\tilde{w}] = (A(\tilde{u}), \tilde{w})_H \forall \tilde{w} \in X_{k,p}$, and Claims 4.17 and 4.18 also hold for maps $A : X_{k,p} \to X_{k,p}$ and $B : X_{k,p} \to \mathcal{L}(H)$, respectively. Claim 4.17 implies that $\mathbb{B}(\tilde{u})$ restricts to an element in $\mathcal{L}(X_{k,p})$, still denoted by $\mathbb{B}(\tilde{u})$, and that $B : X_{k,p} \to \mathcal{L}(X_{k,p})$ is $C^0$. Moreover, if $F$ is of class $C^{k-m+3}$, the map $\Phi_{F^0} : W^{k-m,p}(\Omega, \mathbb{R}^r) \to W^{k-m,p}(\Omega)$ in the proof of Claim 4.17 will be of class $C^2$, and so is $A$. This implies that $\mathbb{B} : X_{k,p} \to \mathcal{L}(X_{k,p})$ is $C^1$. For a $C^2$ map $A$ from Banach spaces to $X$ and any fixed $x_0 \in X$ it easily follows from the Hahn-Banach theorem and the mean value theorem that there exists a ball $B(x_0, r) \subset X$ centred at $x_0$ such that $A(x)$ is uniformly continuously differentiable on $B(x_0, r)$. These and
Claim 4.18 show that \( \mathfrak{F}_{X,\nu,\mu,\lambda} \) is \((B(\bar{u}^*, r), H)\)-regular for some ball \( B(\bar{u}^*, r) \subset X_{\nu,\mu,\lambda} \). Using Proposition 4.15 we can also prove that for the spectrum \( \sigma(\mathfrak{B}(\bar{u})) \) of the complexification of \( \mathfrak{B}(\bar{u}) \) in \( \mathcal{L}_0(X,\nu,\mu,\lambda,0) \) other \( \sigma(\mathfrak{B}(\bar{u})) \) or \( \sigma(\mathfrak{B}(\bar{u})) \setminus \{0\} \) is bounded away from the imaginary axis, see [39, Theorem 7.17] for some related proof details. Hence Theorems 1.1 and 1.2 in [4] lead to Theorem 4.21.

In applications, we may use the regularity results for solutions of the Euler–Lagrangian equations or systems to modify \( F \) suitably so that useful information can be obtained by combing the theories developed in this paper with results in this subsection. We expect that they can be used in studies of geometric variational problems such as minimal surfaces and harmonic maps.

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### A Appendix: Comparing Hypothesis \( \mathfrak{F}_{2,\nu,\mu,\lambda,\kappa} \) with controllable growth conditions

It is easily checked that Hypothesis \( \mathfrak{F}_{2,\nu,\mu,\lambda,\kappa} \) for \( n \geq 2 \) may be equivalently formulated as Hypothesis \( \mathfrak{F}_{2,\nu,\mu,\lambda,\kappa} \). Let \( z = (z_1, \ldots, z_N) \in \mathbb{R}^N, p = (p^i_\alpha) \in \mathbb{R}^{N \times n} \), where \( 1 \leq i \leq N \) and \( \alpha \in \mathbb{N}^N \) with \( |\alpha| = 1 \). Let \( \mathfrak{O} \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \ni (x, z, p) \mapsto F(x, z, p) \in \mathbb{R} \) be twice continuously differentiable in \( (x, p) \) for almost all \( x \), measurable in \( x \) for all values of \( (z, p) \), and \( F(\cdot, z, p) \in L^1(\Omega) \) for \( (z, p) = 0 \). Let \( \kappa_n = 2n/(n-2) \) for \( n > 2 \), and \( \kappa_n \in (2, \infty) \) for \( n = 2 \). The derivatives of \( F \) fulfill the following properties:

(i) \( F_{z_i}(\cdot, 0) \in L^{\kappa_n/(\kappa_n-1)} \) and \( F_{p^j_\alpha}(\cdot, 0) \in L^{2} \) for \( i = 1, \ldots, N \) and \( |\alpha| = 1 \).

(ii) There exist positive constants \( g_1, g_2 \) and \( s \in (0, \frac{\kappa_n-2}{\kappa_n}), r_\alpha \in (0, \frac{\kappa_n-2}{2\kappa_n}) \) for each \( \alpha \in \mathbb{N}^N \) with \( |\alpha| = 1 \), such that for \( i, j = 1, \ldots, N \), \( |\alpha| = |\beta| = 1 \),

\[
|F_{p^j_\alpha}(x, z, p)| \leq g_1 \left( 1 + \sum_{l=1}^N |z|^{|\alpha|} + \sum_{k=1}^N |p^k_\alpha|^2 \right)^{r_\alpha},
\]

\[
|F_{z_i j}(x, z, p)| \leq g_1 \left( 1 + \sum_{l=1}^N |z|^{|\alpha|} + \sum_{k=1}^N |p^k_\alpha|^2 \right)^{s},
\]

\[
\sum_{i,j=1}^N \sum_{|\alpha|=|\beta|=1} g_2 \sum_{i=1}^N \sum_{|\alpha|=1} (n_\alpha^i \bar{n}_\beta^j)^2, \quad \forall \eta = (n_\alpha^i) \in \mathbb{R}^{N \times n}.
\]

The **controllable growth conditions** (abbreviated to CGC below) [27, page 40] (that is, the so-called ‘common condition of Morrey’ or ‘the natural assumptions of Ladyzhenskaya and Ural’tseva’ [27, page 38,(1)]) may be, in our notation, expressed as:

**CGC:** \( \mathfrak{O} \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \ni (x, z, p) \mapsto F(x, z, p) \in \mathbb{R} \) is of class \( C^2 \), and there exist positive constants \( v, \mu, \lambda, M_1, M_2 \), such that with \( |z|^2 := \sum_{i=1}^N |z|^2 \) and \( |p|^2 := \sum_{|\alpha|=1} \sum_{k=1}^N |p^k_\alpha|^2 \),

\[
v \left( 1 + |z|^2 + |p|^2 \right) - \lambda \leq F(x, z, p) \leq \mu \left( 1 + |z|^2 + |p|^2 \right),
\]

\[
|F_{p^i_\alpha}(x, z, p)|, |F_{p^i_\alpha x_l}(x, z, p)|, |F_{z_i x_l}(x, z, p)|, |F_{z_i j}(x, z, p)| \leq \mu \left( 1 + |z|^2 + |p|^2 \right)^{1/2},
\]

\[
|F_{p^i_\alpha j}(x, z, p)|, |F_{z_i j}(x, z, p)| \leq \mu,
\]

\[
M_1 \sum_{i=1}^N \sum_{|\alpha|=1} \sum_{|\beta|=1} (n_\alpha^i \bar{n}_\beta^j)^2 \leq M_2 \sum_{i=1}^N \sum_{|\alpha|=1} (n_\alpha^i)^2, \quad \forall \eta = (n_\alpha^i) \in \mathbb{R}^{N \times n}.
\]
Moreover, if $F = F(x, p)$ does not depend explicitly on $z$, the first three lines are replaced by

$$
\nu \left(1 + \left| p \right|^2 \right) - \lambda \leq F(x, p) \leq \mu \left(1 + \left| p \right|^2 \right)
$$

and

$$
\left| F_{p_\alpha}^i (x, p) \right|, \left| F_{p_\alpha, x_j}^i (x, p) \right| \leq \mu \left(1 + \left| p \right|^2 \right)^{1/2}.
$$

From these it is not hard to see

**Proposition 4.22.** CGC implies Hypothesis $\mathfrak{F}_{2,N,1,n}$.

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