Reflected Generalized Beta Inverse Weibull Distribution: definition and properties

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Abstract

In this paper, we study a broad class of distribution functions which is defined by means of reflected generalized beta distribution. This class includes that of Beta-generated distribution as a special case. In particular, we use this class to extend the Inverse Weibull distribution in order to obtain the Reflected Generalized Beta Inverse Weibull Distribution. For this new distribution, moments, entropy and a reliability measure are derived. The link between the Inverse Weibull and the Dagum distribution is generalized. Then the maximum likelihood estimators of the parameters are examined and the observed Fisher information matrix provided. Finally, the usefulness of the model is illustrated by means of two applications to real data.

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1 Introduction

In the literature, various techniques are commonly used to extend a family of distribution function; some of these are based on simple transformations of the distribution function or of the survival function. In particular, let $X$ be a non-negative random variable (rv), representing the lifetime of an individual or unit, with cumulative distribution function (cdf), survival function (sf) and probability density function (pdf), respectively, denoted by $G(x; \tau)$, $\bar{G}(x; \tau) = 1 - G(x; \tau)$ and $g(x; \tau)$, where $\tau \in \Xi \subset \mathbb{R}^p$ with $p \geq 1$ and $p \in \mathbb{N}$. The transformations $F(x; \eta) = [G(x; \tau)]^a$ and $\bar{F}(x; \eta) = [\bar{G}(x; \tau)]^b$ define the proportional reversed hazard model (or Lehmann type I distribution) and proportional hazard model (or Lehmann type II distribution), respectively. Usually, the distribution $G(x; \tau)$ is denoted as a parent or
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In this context, Marshall and Olkin (2007) point out that the parameters $a$ and $b$ assume the meaning of resilience and frailty parameters, respectively. In this context, it is important to observe that once a resilience or frailty parameter has been introduced, the reintroduction of the same kind of parameter does not expand the family of distribution function; for example, in the case of the Weibull distribution, the introduction of the frailty parameter does not expand the distribution because the Weibull is already a proportional hazard family. Further extensions of the families of distribution functions may be obtained by a combination of the two methods outlined above; a remarkable example in this direction is the Beta generated methods which, in brief, define the following pdf

$$f_{B-G}(x; \eta) = \frac{1}{B(a,b)}[G(x; \tau)]^{a-1}[1 - G(x; \tau)])]^{b-1}g(x; \tau),$$

where $\eta = (a, b, \tau)$ and $B(\cdot, \cdot)$ is the Beta function. Thus, in a sense, the ideas of both frailty and resilience are embodied in this formulation of the pdf (Marshall and Olkin, 2007, pag.238).

The class of Beta-generated distributions has received considerable attention in recent years. In particular, after the studies of Eugene et al. (2002) and Jones (2004), many Beta-generated distributions have been proposed. Among these, we recall the Beta-Gumbel (Nadarajah and Kotz, 2004), the Beta-Exponential (Nadarajah and Kotz, 2006) and the Beta-Weibull (Famoye et al., 2005). The reader is referred to Barreto-Souza et al. (2010), Paranaíba et al. (2011) and Domma and Condino (2013) for the Beta-Generalized Exponential, the Beta-Burr XII and Beta-Dagum distribution, respectively, for more recent developments. A more complete list of Beta-generated distributions and an interesting review of this method for generating families of distributions can be found in Lee et al. (2013). Moreover, the Beta generated method can be obtained by means of a transformation of a random variable, i.e. if $U \sim Be(a, b)$ then the pdf of $X = G^{-1}(U; \eta)$ is $f_{B-G}(x; \eta)$. Recently, in order to increase the flexibility of the generator, Alexander et al. (2012) used the Generalized Beta of first type (GB1), first proposed by McDonald (1984), instead of the classic Beta function as generator. The pdf of a GB1 is

$$f(u; a, b, c) = \frac{c}{B(a, b)}u^{ac-1}(1 - u^c)^{b-1}$$

for $u \in (0, 1)$ and $a, b, c > 0$. With this generator, the random variable $X$ is said to have a generalized beta of first type (GB1) generated distribution, with pdf and cdf, respectively, given by
\[ f_{GB1-G}(x; \eta) = \frac{c}{B(a, b)} [G(x; \tau)]^{ac-1} (1 - [G(x; \tau)]^c)^{b-1} g(x; \tau) \]  (1.2)

and

\[ F_{GB1-G}(x; \eta) = \frac{1}{B(a, b)} \int_0^{[G(x; \tau)]^c} u^{a-1} (1 - u)^{b-1} du \]  (1.3)

where \( \eta = (a, b, c, \tau) \). In effect, as discussed in Alexander et al. (2012), (1.3) increases the flexibility of the generator for the presence of parameter \( c \). Nevertheless, there are cases where this does not occur; in particular, when \( G(x; \tau) \) is already a reversed proportional hazard distribution. In such cases, to give effect to the expansion of the distribution function it is possible to refer to the reflected version of (1.1) obtained by transformation \( v = 1 - u \), with pdf and cdf, respectively, given by

\[ f(v; a, b, c) = \frac{c}{B(a, b)} (1 - v)^{ac-1} (1 - (1 - v)^c)^{b-1} \]  (1.4)

\[ F(v; a, b, c) = 1 - \frac{c}{B(a, b)} \int_0^{1-v} w^{ac-1} (1 - w^c)^{b-1} dw = 1 - \frac{B((1-v)^c; a, b)}{B(a, b)} \]  (1.5)

for \( v \in (0, 1) \), \( a, b, c > 0 \), where \( B(z; a, b) \) is the incomplete beta function. Putting \( V = G(X; \tau) \), the random variable \( X \) is said to have a reflected generalized beta of first type (rGB1) generated distribution, with pdf and cdf, respectively, given by

\[ f_{rGB1-G}(x; \eta) = \frac{c}{B(a, b)} [1-G(x; \tau)]^{ac-1} (1 - [1-G(x; \tau)]^c)^{b-1} g(x; \tau) \]  (1.6)

and

\[ F_{rGB1-G}(x; \eta) = 1 - \frac{c}{B(a, b)} \int_0^{[1-G(x; \tau)]^c} v^{ac-1} (1 - v^c)^{b-1} dv = 1 - \frac{1}{B(a, b)} \int_0^{[1-G(x; \tau)]^c} w^{a-1} (1 - w)^{b-1} dw = \frac{B([1-G(x; \tau)]^c; a, b)}{B(a, b)}. \]  (1.7)

The class of generated distributions defined in (1.6) is called \textit{Beta-Exponential-X family} by Lee et al. (2013). Clearly, the baseline distribution \( G(x; \tau) \) is
a special case of (1.7) when \( a = b = c = 1 \). If \( c = 1 \), we obtain the Beta-generated distribution. Moreover, the exponentiated generalized class of distributions proposed by Cordeiro et al. (2013a) and Cordeiro et al. (2013b) turns out to be a special case of \( F_{rGB1-G}(x; \eta) \) setting \( a = 1 \). We thus observe that the latter class of distributions can be thought of as reflected Kumaraswamy-generated distributions. For \( b > 0 \) real non-integer, using the power series representation

\[
(1 - [1 - G(x; \tau)])^c b^{-1} = \sum_{j_1=0}^{\infty} p_{j_1, b} [1 - G(x; \tau)]^{c j_1} \tag{1.8}
\]

where \( p_{j_1, b} = \frac{(-1)^{j_1} \Gamma(b)}{\Gamma(b-j_1) \Gamma(j_1+1)} \), we can write the pdf (1.6) as

\[
f_{rGB1-G}(x; \eta) = \frac{c}{B(a,b)} \sum_{j_1=0}^{\infty} p_{j_1, b} [1 - G(x; \tau)]^{c(a+j_1)-1} g(x; \tau) \tag{1.9}
\]

with cdf

\[
F_{rGB1-G}(x; \eta) = \frac{1}{B(a,b)} \sum_{j_1=0}^{\infty} \frac{p_{j_1, b}}{(a+j_1)} \left\{ 1 - [1 - G(x; \tau)]^{c(a+j_1)} \right\} . \tag{1.10}
\]

The expansions of the pdf shown in (1.9) can be used to determine certain properties of \( rBG1-G \) distribution as, for example, the \( k \)-th moment,

\[
E(X^k) = \frac{c}{B(a,b)} \sum_{j_1=0}^{\infty} p_{j_1, b} \int_0^\infty x^k [1 - G(x; \tau)]^{c(a+j_1)-1} g(x; \tau) dx =
\frac{c}{B(a,b)} \sum_{j_1=0}^{\infty} p_{j_1, b} E_G \left\{ X^k [1 - G(X; \tau)]^{c(a+j_1)-1} \right\} \tag{1.11}
\]

where \( E_G(.) \) denotes the expectation with respect to the baseline distribution \( G(X; \tau) \). In this paper, we use the generator defined by (1.7) to extend the Inverse Weibull distribution. The rest of the paper is organized as follows. In Section 2, we define the Reflected Generalized Beta of first type Inverse Weibull distribution and derive the expansions for its distribution and probability density functions. The quantile, the moments, the entropy and a measure of reliability are derived in Section 3. In Section 4, the link between the Inverse Weibull distribution and the Dagum distribution is generalized. The maximum likelihood estimation is discussed in Section 5. Two applications on real data set are reported in Section 6. Finally, some of the mathematical results are reported in the Appendix.
2 Reflected Generalized Beta of first type Inverse Weibull distribution (rGB1-IWei)

Several authors have highlighted the fact that there are a considerable number of contexts in which the Inverse Weibull (IWei) distribution may be an appropriate model, mainly because the empirical hazard rate is unimodal (see, for example, Kundu and Howlader (2010); Singh et al. (2013)). Recently, Erto (2013) has shown certain peculiar properties of the IWei distribution; in particular, he formulated three real and typical degenerative mechanisms which lead exactly to the IWei random variable. The cumulative distribution function and probability density function of the IWei random variable, respectively, are:

\[ G_{\text{IWei}}(x; \tau) = e^{-\gamma x^{-\theta}} \]  

and

\[ g_{\text{IWei}}(x; \tau) = \theta \gamma x^{-\theta-1} e^{-\gamma x^{-\theta}}, \]

where \( \tau = (\gamma, \theta) \), with \( k \)-th moment about zero given by

\[ E(X^k) = \gamma^k \theta \Gamma \left( 1 - \frac{k}{\theta} \right) \]

where \( \Gamma(\cdot) \) is the Gamma function. Recently, Gusmão et al. (2011) have tried to generalize the IWei distribution by introducing the resilience parameter in (2.1), i.e. \( G(x) = \left[ e^{-\gamma x^{-\theta}} \right]^c \), but, as highlighted by Jones (2012), the proposed distribution is only a reparametrization of \( G_{\text{IWei}}(x; \tau) \). In other words, they do not expand the class of distributions because the authors introduce a resilience parameter in a reversed proportional hazard model. In order to extend the IWei distribution, given that this distribution is a reversed hazard rate model, we use the reflected Generalized Beta generator defined in (1.7). So, the rGB1-IWei cumulative distribution function is defined by inserting (2.1) in (1.7), i.e.

\[ F_{\text{rGB1-IWei}}(x; \eta) = 1 - \frac{1}{B(a, b)} \int_0^{[1-G_{\text{IWei}}(x; \tau)]^c} v^{a-1}(1-v)^{b-1} dv \]

\[ = 1 - I_{(1-G_{\text{IWei}}(x; \tau)^c)}(a, b) \]

where \( \eta = (\gamma, \theta, a, b, c) \) and \( I_{\text{G}}(a, b) \) is the incomplete beta function ratio. The corresponding pdf of the new distribution can be written as

\[ f_{\text{rGB1-IWei}}(x; \eta) = \frac{c \gamma \theta x^{-\theta-1} e^{-\gamma x^{-\theta}}}{B(a, b)} \left[ 1 - e^{-\gamma x^{-\theta}} \right]^{ac-1} \left( 1 - [1 - e^{-\gamma x^{-\theta}}]^c \right)^{b-1} \]
Figure 1: \( rGB1-IWei \) density for certain values of the parameters

Figure 1 shows various behaviours of the density \( f_{rGB1-IWei}(x; \eta) \), including the bimodal one, obtained for different values of parameters \( \gamma, \theta, a, b \) and \( c \).

The hazard rate for \( rGB1-IWei \) distribution, given by \( h_{rGB1-IWei}(x; \eta) = \frac{f_{rGB1-IWei}(x; \eta)}{[1 - G_{IWei}(x; \tau)]^{(a,b)} c} \), is a complex function of \( x \). However, from the plots reported in Fig. 2, we can state that \( h_{rGB1-IWei}(x; \eta) \) is much more flexible than the hazard rate of the \( IWei \) distribution. In particular, by denoting the strictly increasing (decreasing) failure rate as IFR (DFR) and the hazard rate with a minimum or a maximum as Bathtub (BT) and Upside-down Bathtub (UBT), respectively, we can observe that the \( rGB1-IWei \) model enables us to obtain IFR and DFR hazard rates or hazard rates with a
non-monotonic behaviour, such as UBT, BT-UBT or UBT-BT-UBT hazard rate. This wide range of different behaviours of the hazard rate makes the \( rGB1-IWei \) distribution a suitable model for reliability theory and survival analysis.

2.1. Expansions for the Cumulative Distribution Function and for the Probability Density Function. In this subsection, we present some representations of cdf and pdf of \( rGB1-IWei \) in terms of infinite sums. Using the relations (1.9), (2.1) and (2.2), the pdf of \( rGB1-IWei \) can be written as

\[
f_{rGB1-IWei}(x; \eta) = \frac{c}{B(a, b)} \sum_{j=0}^{\infty} p_{j,b} [1 - G_{IWei}(x; \tau)]^{c(a+j)-1} g_{IWei}(x; \tau) \quad (2.6)
\]
with cdf given by

\[ F_{rGB1-IWei}(x; \eta) = \frac{1}{B(a, b)} \sum_{j=0}^{\infty} \frac{p_{j,b}}{(a + j)} \left\{ 1 - [1 - G_{IWei}(x; \tau)]^{c(a+j)} \right\} \]

\[ = \frac{1}{B(a, b)} \sum_{j=0}^{\infty} \frac{p_{j,b}}{(a + j)} \left\{ 1 - [1 - e^{-\gamma x^{-\theta}}]^{c(a+j)} \right\} \] (2.7)

Another expansion of \( f_{rGB1-IWei}(x; \eta) \) may be derived from the following relation between pdf and cdf of a IWei distribution, i.e. \( g_{IWei}(x; \tau) = \gamma \theta x^{-\theta-1} G_{IWei}(x; \tau) \). After algebra, we can write \( f_{rGB1-IWei}(x; \eta) \) in term of an infinite sum of IWei density functions, i.e.

\[ f_{rGB1-IWei}(x; \eta) = \frac{c}{B(a, b)} \sum_{j_1=0}^{\infty} p_{j_1,b} \left( \sum_{j_2=0}^{\infty} \frac{p_{j_{2,c(a+j_1)}}}{(j_2 + 1)} g_{IWei}(x; \gamma(j_2 + 1), \theta) \right) \] (2.8)

The linear combination (2.8) enables us to obtain some mathematical properties of the rGB1-IWei distribution directly from those of the IWei distribution such as the moments, moment generating function and reliability.

### 3 Statistical Properties

This section is devoted to studying the statistical properties of the rGB1-IWei distribution, specifically quantile function, moments and moment generating function, entropy and reliability.

#### 3.1. Moments and Moment Generating Function.

In this subsection, we discuss the \( r \)th moment for the rGB1-IWei distribution. Moments are necessary in any statistical analysis, especially in applications. They can be used to study the most important features and characteristics of a distribution (e.g. tendency, dispersion, skewness and kurtosis).

**Proposition 1.** If \( X \) has rGB1-IWei distribution, then the \( r \)th moment of \( X \) is given by the following

\[ \mu_r(\eta) = \frac{c}{B(a, b)} \sum_{j_1=0}^{\infty} p_{j_1,b} \left( \sum_{j_2=0}^{\infty} \frac{p_{j_{2,c(a+j_1)}}}{(j_2 + 1)} [(j_2 + 1)\gamma]^{\frac{r}{\theta}} \cdot \Gamma\left(1 - \frac{r}{\theta}\right) \right) \] (3.1)

**Proof.** It is an immediate consequence of the expansions of the pdf defined in (2.8) and by equation (2.3).

Now, we shall derive the moment generating function of rGB1-IWei distribution.
Proposition 2. If $X$ has $rGB1-IWei$ distribution, then the moment generating function $M_X(t)$ has the following form

$$M_X(t) = \sum_{r=0}^{\infty} \left( \frac{t^r}{r!} \right) \frac{c}{B(a,b)} \sum_{j_1=0}^{\infty} P_{j_1,b} \left( \sum_{j_2=0}^{\infty} \frac{P_{j_2,c(a+j_1)}(j_2+1)}{(j_2+1)} \right) [\gamma(j_2 + 1)]^{\frac{r}{\theta}} \Gamma \left( 1 - \frac{r}{\theta} \right).$$  \hspace{1cm} (3.2)

Proof. Using the series expansion $e^{tx} = \sum_{r=0}^{\infty} \frac{(tx)^r}{r!}$, we can write

$$M_X(t) = \sum_{r=0}^{\infty} \frac{(t)^r}{r!} \int_0^{\infty} x^r f_{rGB1-IWei}(x; \eta) dx = \sum_{r=0}^{\infty} \frac{(t)^r}{r!} \mu_r(\eta).$$  \hspace{1cm} (3.3)

Substituting (3.1) into (3.3), we get the expression of $M_X(t)$.

3.2. Quantile of $rGB1-IWei$. The quantile $x_q$ of the $rGB1-IWei$ distribution can be easily obtained considering that

$$q = F_{rGB1-IWei}(x_q; \eta) = 1 - I_{[1-G_{IWei}(x_q; \tau)]^c}(a,b)$$

from which

$$[1 - G_{IWei}(x_q; \tau)]^c = [1 - e^{-\gamma(x_q)^\theta}]^c = I^{-1}(1 - q; a,b)$$

and

$$x_q(\eta) = \left\{ -\frac{1}{\gamma} \ln \left( 1 - [I^{-1}(1 - q; a,b)]^{1/c} \right) \right\}^{-1/\theta}.$$  \hspace{1cm} (3.4)

The median can be derived from (3.4) by setting $q = \frac{1}{2}$, that is

$$Med_{rGB1-IWei}(\eta) = \left\{ -\frac{1}{\gamma} \ln \left( 1 - [I^{-1}(0.5; a,b)]^{1/c} \right) \right\}^{-1/\theta}.$$  \hspace{1cm} (3.5)

We consider the expression (3.4) to compute the Galton’s skewness measure and the Moors’ kurtosis measure (Gilchrist, 2000, pag. 71), given respectively by:

$$G(\eta) = \frac{x_{0.75}(\eta) + x_{0.25}(\eta) - 2x_{0.5}(\eta)}{x_{0.75}(\eta) - x_{0.25}(\eta)}$$  \hspace{1cm} (3.6)

and

$$K(\eta) = \frac{x_{7/8}(\eta) - x_{5/8}(\eta) + x_{3/8}(\eta) - x_{1/8}(\eta)}{x_{0.75}(\eta) - x_{0.25}(\eta)}$$  \hspace{1cm} (3.7)

Plots of these skewness and kurtosis measures of the $rGB1-IWei$ distribution for selected values of parameters are reported in Figs. 3 and 4. Furthermore, in Table 1, we report the obtained values for Galton’s skewness and Moors’
Figure 3: Galton’s skewness measures of the $rGB1-IWei$ distribution for certain values of parameters

kurtosis measures for some choice of parameter values. Except the parameter $\gamma$, which is a scale parameter, all the other parameters influence the shape of the $rGB1-IWei$ distribution. From the obtained values and from the graphical analysis, we can state that the $rGB1-IWei$ model possess a positive skewness and it can be platykurtic or leptokurtic, according to the parameter values, given that the $K(\eta)$ measure can assume lower or higher values than the Moors’ kurtosis reference value of the normal distribution, equal to 1.233.

3.3. Entropy. The concept of entropy plays a vital role in information theory. The entropy of a random variable is defined in terms of its probability distribution and can be shown to be a valid measure of randomness or
uncertainty. In this section, we present Rényi entropy $I_R(\rho)$ and Shannon entropy $H(f_{rGB1-IWei}(x; \eta))$.

**Proposition 3.** If $X$ has $rGB1-IWei$ distribution, then the Rényi entropy is given by

$$I_R(\rho) = \frac{\rho}{1-\rho} \log\left(\frac{c}{B(a,b)}\right) + \frac{1}{1-\rho} \log\Gamma\left(\rho + \frac{(\rho-1)}{\theta}\right) - \log(\gamma\theta)$$

$$+ \frac{1}{1-\rho} \log \left( \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \frac{p_{j_1,\rho(b-1)+1}p_{j_2,\rho(ac-1)+j_1+1}}{[(j_2 + \rho)\gamma]^{(\rho\frac{(\rho-1)}{\theta})}} \right)$$

(3.8)
Proof. The Rényi entropy is defined as

\[ I_R(\rho) = \frac{1}{1 - \rho} \log \left[ \int_0^\infty f_{rGB1-IWei}(x; \eta)^\rho dx \right], \]

where \( \rho > 0, \) and \( \rho \neq 1. \) For the \( rGB1-IWei \) distribution, using the power series representation, the integral in \( I_R(\rho) \) can be reduced to

\[ \int_0^\infty f_{rGB1-IWei}(x; \eta)^\rho dx = \left( \frac{c\gamma \theta}{B(a, b)} \right)^\rho \sum_{j_1=0}^\infty \sum_{j_2=0}^\infty p_{j_1, \rho(b-1)+j_1c+1} \int_0^\infty x^{-\rho(x+1)} e^{-(j_2+\rho)\gamma x^{-\theta}} dx. \]

Setting \( \gamma(j_2 + \rho)x^{-\theta} = w, \) after algebra, we obtain

\[ \int_0^\infty f_{rGB1-IWei}(x; \eta)^\rho dx = \left( \frac{c\gamma \theta}{B(a, b)} \right)^\rho \frac{\Gamma \left( \frac{(\rho - 1)}{\theta} \right)}{\gamma \theta} \sum_{j_1=0}^\infty \sum_{j_2=0}^\infty p_{j_1, \rho(b-1)+1}p_{j_2, \rho(ac-1)+j_1c+1} \int_0^\infty x^{-\rho(x+1)} e^{-(j_2+\rho)\gamma x^{-\theta}} dx. \]

through simple algebra we obtain (3.8).

Proposition 4. If \( X \) has \( rGB1-IWei \) distribution, then the Shannon entropy is given by

\[ H(f_{rGB1-IWei}(x; \eta)) = -\log \left( \frac{c\theta \gamma}{B(a, b)} \right) + \frac{c}{B(a, b)} \sum_{j_1=0}^\infty \sum_{j_2=0}^\infty p_{j_1, bPj_2, c(a+j_1)} \frac{(\theta + 1)}{\theta} \left( \log \left( \frac{(j_2 + 1)\gamma}{(j_2 + 1)} \right) - \frac{(ac - 1)}{(j_2 + 1)} \left[ \psi(1) - \psi(j_2 + 2) \right] \right) + \frac{1}{(j_2 + 1)^2} - (b - 1) \sum_{i=0}^\infty \frac{p_{i,j_2+1}}{(i+1)^2} \left[ \psi(1) - \psi(i + 1) \right] (3.9) \]

Proof. The Shannon entropy is given by

\[ H(f_{rGB1-IWei}(x; \eta)) = -E \left[ \log f_{rGB1-IWei}(x; \eta) \right] = -\log(c\gamma \theta) + \log B(a, b) + (\theta + 1)E \left[ \log(X) \right] + E \left[ \gamma X^{-\theta} \right] -(ac - 1)E \left[ \log \left( 1 - e^{-\gamma X^{-\theta}} \right) \right] - (b - 1)E \left[ \log \left( 1 - \left[ 1 - e^{-\gamma X^{-\theta}} \right] \right) \right]. \]

Using the power series representation, we have

\[ E \left[ \log(X) \right] = \frac{c\theta \gamma}{B(a, b)} \sum_{j_1=0}^\infty \sum_{j_2=0}^\infty p_{j_1, bPj_2, c(a+j_1)} \int_0^\infty \log(x)x^{-(\theta+1)} e^{-(j_2+1)\gamma x^{-\theta}} dx. \]
Table 1: Galton’s skewness and Moors’ kurtosis values for certain parameter values, fixing $\gamma = 2$

| $\theta$ | $a$  | $b$  | $c$  | $\text{Med}(\eta)$ | $G(\eta)$  | $K(\eta)$ |
|---------|------|------|------|----------------------|-------------|-----------|
| 0.3     | 0.8  | 0.2  | 0.5  | 0.49591              | 0.978986    | 122.661   |
|         | 0.9  |      |      | 0.248557             | 0.94168     | 25.10809  |
|         | 2    |      |      | 0.115492             | 0.8637      | 7.851909  |
| 0.5     | 0.5  |      |      | 46.29557             | 0.991846    | 247.4652  |
|         | 0.9  |      |      | 8.072375             | 0.951087    | 27.9598   |
|         | 2    |      |      | 1.538344             | 0.829595    | 6.513944  |
| 1.5     | 0.5  |      |      | 27212.01             | 0.996317    | 376.6278  |
|         | 0.9  |      |      | 541.3055             | 0.955605    | 29.52281  |
|         | 2    |      |      | 22.03688             | 0.791734    | 5.60134   |
| 2.5     | 0.2  |      |      | 0.104087             | 0.854192    | 7.653882  |
|         | 0.9  |      |      | 0.065709             | 0.786764    | 4.709425  |
|         | 2    |      |      | 0.038303             | 0.695215    | 3.058152  |
| 0.5     | 0.5  |      |      | 1.705054             | 0.848704    | 7.789657  |
|         | 0.9  |      |      | 0.676817             | 0.741391    | 4.157021  |
|         | 2    |      |      | 0.254087             | 0.600894    | 2.521507  |
| 1.5     | 0.5  |      |      | 76.90645             | 0.868805    | 9.004874  |
|         | 0.9  |      |      | 11.55732             | 0.722108    | 3.980305  |
|         | 2    |      |      | 1.964581             | 0.534977    | 2.236769  |
| 3       | 0.8  | 0.2  | 0.5  | 0.932267             | 0.385066    | 1.890363  |
|         | 0.9  |      |      | 0.870047             | 0.296777    | 1.564662  |
|         | 2    |      |      | 0.805852             | 0.204397    | 1.354807  |
| 0.5     | 0.5  |      |      | 1.467418             | 0.415396    | 1.931137  |
|         | 0.9  |      |      | 1.232254             | 0.298808    | 1.556721  |
|         | 2    |      |      | 1.044012             | 0.188426    | 1.348938  |
| 1.5     | 0.5  |      |      | 2.776362             | 0.422986    | 1.925816  |
|         | 0.9  |      |      | 1.876481             | 0.298002    | 1.551785  |
|         | 2    |      |      | 1.362433             | 0.18593     | 1.356837  |
| 2.5     | 0.2  | 0.5  |      | 0.797517             | 0.198424    | 1.363006  |
|         | 0.9  |      |      | 0.761662             | 0.147846    | 1.285699  |
|         | 2    |      |      | 0.721645             | 0.093757    | 1.22673   |
| 0.5     | 0.5  |      |      | 1.054809             | 0.20527     | 1.38468   |
|         | 0.9  |      |      | 0.961717             | 0.144374    | 1.305912  |
|         | 2    |      |      | 0.871964             | 0.083534    | 1.254479  |
| 1.5     | 0.5  |      |      | 1.543819             | 0.219396    | 1.398329  |
|         | 0.9  |      |      | 1.277279             | 0.153343    | 1.315053  |
|         | 2    |      |      | 1.06986             | 0.092726    | 1.265593  |
Reflected generalized beta inverse Weibull distribution

\[
E \left[ \gamma X^{-\theta} \right] = \gamma E \left( X^{-\theta} \right) = \frac{c}{B(a, b)} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} p_{j_1, b} p_{j_2, c(a+j_1)} \int_0^{\infty} \{ \log [(j_2 + 1)\gamma] - \log(w) \} e^{-w} dw
\]

Moreover, we have

\[
E \left\{ \log \left[ 1 - e^{-\gamma X^{-\theta}} \right] \right\} = \frac{c \theta \gamma}{B(a, b)} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} p_{j_1, b} p_{j_2, c(a+j_1)} \int_0^{\infty} \log \left[ 1 - e^{-\gamma x^{-\theta}} \right] x^{-(\theta+1)} \left[ e^{-\gamma x^{-\theta}} \right]^{j_2+1} dx,
\]

setting \( w = 1 - e^{-\gamma x^{-\theta}} \), after algebra, we obtain

\[
E \left\{ \log \left[ 1 - e^{-\gamma x^{-\theta}} \right] \right\} = \frac{c}{B(a, b)} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} p_{j_1, b} p_{j_2, c(a+j_1)} \int_0^{1} \log(w)(1-w)^{j_2} dw = \frac{c}{B(a, b)} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \frac{p_{j_1, b} p_{j_2, c(a+j_1)}}{(j_2 + 1)} \{ \psi(1) - \psi(2 + j_2) \},
\]

where \( \psi(.) \) denote the digamma function. Finally, the last integral is given by

\[
E \left\{ \log \left[ 1 - e^{-\gamma x^{-\theta}} \right]^c \right\} = \frac{c \theta \gamma}{B(a, b)} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} p_{j_1, b} p_{j_2, c(a+j_1)} \int_0^{\infty} \log \left( 1 - e^{-\gamma x^{-\theta}} \right)^c x^{-(\theta+1)} \left[ e^{-\gamma x^{-\theta}} \right]^{j_2+1} dx = \frac{1}{B(a, b)} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} p_{j_1, b} p_{j_2, c(a+j_1)} \int_0^{1} \log(w)(1-w)^{\frac{j_2}{c}-1} \left[ 1 - (1-w)^{\frac{1}{c}} \right]^{j_2+1} dw = \frac{1}{B(a, b)} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} p_{j_1, b} p_{j_2, c(a+j_1)} \sum_{i=0}^{\infty} p_{i, j_2+1} \int_0^{1} \log(w)(1-w)^{\frac{i+1}{c}-1} dw = \frac{c}{B(a, b)} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} p_{j_1, b} p_{j_2, c(a+j_1)} \sum_{i=0}^{\infty} p_{i, j_2+1} \left\{ \psi(1) - \psi \left( \frac{i + 1}{c} + 1 \right) \right\}.}
\]
3.4. Reliability. In reliability studies, the stress-strength term is often used to describe the life of a component which has a random strength $X$ and is subject to a random stress $Y$. The component fails if the stress applied to it exceeds the strength, and the component will function satisfactorily whenever $Y < X$. Thus, $R = P(Y < X)$ is a measure of a component reliability which has many applications in physics, engineering, genetics, psychology and economics (Kotz et al. 2003). In this subsection, we derive the reliability $R$ when $X \sim rGB1-IWei(\eta_x)$ and $Y \sim rGB1-IWei(\eta_y)$ are independent random variables. Using expansion (2.8) of the pdf of the $rGB1-IWei$ random variable, after algebra, we can write

$$R = \frac{c_x c_y}{B(a_x, b_x) B(a_y, b_y)} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} \sum_{j_4=0}^{\infty} p_{j_1, b_y} p_{j_2, c_y (a_y + j_1)} p_{j_3, b_x} p_{j_4, c_x (a_x + j_3)} R_{j_2, j_4}$$

where

$$R_{j_2, j_4} = \int_0^{\infty} g_{IWei}(x; \gamma_x(j_4 + 1), \theta_x) G_{IWei}(x; \gamma_y(j_2 + 1), \theta_y) dx \quad (3.10)$$

is the reliability between the independent Inverse Weibull random variables. Hence, the reliability between $rGB1-IWei$ random variables is a linear combination of the reliabilities between $IWei$ random variables. Putting $\theta_x = \theta_y = \theta$, after algebra, the $R_{j_2, j_4}$ is given by

$$R_{j_2, j_4} = \gamma_x(j_4 + 1) \theta \int_0^{\infty} x^{-\theta-1} e^{-x}[\gamma_x(j_4 + 1) + \gamma_y(j_2 + 1)] dx = \frac{\gamma_x(j_4 + 1)}{[\gamma_x(j_4 + 1) + \gamma_y(j_2 + 1)]}.$$  

Finally, the reliability between independent $rGB1-IWei$ random variables, in the case $\theta_x = \theta_y = \theta$, is given by

$$R = \frac{c_x c_y}{B(a_x, b_x) B(a_y, b_y)} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} \sum_{j_4=0}^{\infty} p_{j_1, b_y} p_{j_2, c_y (a_y + j_1)} p_{j_3, b_x} p_{j_4, c_x (a_x + j_3)} \gamma_x(j_4 + 1)$$

$$\frac{1}{(j_2 + 1)(j_4 + 1) [\gamma_x(j_4 + 1) + \gamma_y(j_2 + 1)]}.$$  

4 Link between Generalizations of Inverse Weibull and Generalizations of Dagum distribution

In this section, we generalize the link between the Inverse Weibull distribution and Dagum distribution, via mixing Gamma density (see, for example,(Kleiber and Kotz, 2003, pag.192)). In the literature, it is recognized that a pdf, say $p_1(x)$, has a mixture representation if it can be written as
\[ p_1(x) = \int p_2(x|\theta)p_3(\theta)d\theta, \] where \( \theta \) is regarded as a random variable with pdf \( p_3(\cdot) \), called mixing density, and \( p_2(x|\theta) \) is the conditional pdf of \( rv \ X \) given \( \theta \). Mixtures are also called compound distributions. The following Proposition shows that the Burr III distribution (or special case of the Dagum distribution) has a mixture representation with mixing Gamma density.

**Proposition 5.** If the conditional \( rv \ X \) given \( \gamma \) has Inverse Weibull distribution, i.e. \( X|\gamma \sim IWei(\gamma, \theta) \), and the \( rv \ \gamma \) is Gamma distributed, i.e. \( \gamma \sim Ga(\beta) \), then the \( rv \ X \) is distributed as a Burr III distribution, i.e. \( X \sim BurrIII(\beta, \theta) \).

**Proof.** See, for example, Kotz and Kleiber (2003), pag. 192.

Next, by means of two propositions, we shall discuss the connection between certain generalizations of the \( IWei \) distribution and those of the Dagum distribution via mixing Gamma density.

**Proposition 6.** If \( X|\gamma \sim BeIWei(a, b, \gamma, \theta) \) and \( \gamma \sim Ga(\beta) \), then \( X \sim Mixture of Dag(\beta, \lambda_j, \theta) \), with \( \lambda_j = (a + j) \).

**Proof.** The conditional pdf of the \( rv \ X|\gamma \) and the marginal pdf of the \( rv \ \gamma \), respectively, are

\[
 f(x|\gamma; a, b, \theta) = \frac{\gamma \theta}{B(a, b)} x^{-\theta-1} e^{-\gamma x^{-\theta}} \left[ e^{-\gamma x^{-\theta}} \right]^{a-1} \left[ 1 - e^{-\gamma x^{-\theta}} \right]^{b-1}
\]

and

\[
 f(\gamma; \beta) = \frac{\gamma^{\beta-1} e^{-\gamma}}{\Gamma(\beta)}.
\]

Observe that \([1 - e^{-\gamma x^{-\theta}}]^{b-1} = \sum_{j=0}^{\infty} (-1)^j \Gamma(b) \Gamma(b-j) \Gamma(j) \Gamma(-\beta) \Gamma(\beta) \gamma^{j+1} e^{-\gamma j x^{-\theta}} \]

after algebra, the joint pdf between \( X \) and \( \gamma \) is

\[
 f(x, \gamma; a, b, \beta, \theta) = \frac{\theta x^{-\theta-1}}{B(a, b) \Gamma(\beta)} \sum_{j=0}^{\infty} (-1)^j \Gamma(b) \Gamma(b-j) \Gamma(j) \Gamma(-\beta) \Gamma(\beta) \gamma^{j+1} e^{-\gamma [1+(a+j) x^{-\theta}]}
\]

marginalizing with respect to \( \gamma \), we obtain the marginal pdf of \( rv \ X \), i.e.

\[
 f(x, a, b, \beta, \theta) = \frac{\theta x^{-\theta-1}}{B(a, b) \Gamma(\beta)} \sum_{j=0}^{\infty} (-1)^j \Gamma(b) \Gamma(b-j) \Gamma(j) \Gamma(-\beta) \Gamma(\beta) \gamma^{j+1} e^{-\gamma [1+(a+j) x^{-\theta}]}
\]

\[
 \int_0^\infty \gamma^\beta e^{-\gamma [1+(a+j) x^{-\theta}]} d\gamma
\]

\[
 = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(a+j) \Gamma(\beta)}{\Gamma(a) \Gamma(b-j) j! (a+j)} f_{Dag}(x; \beta, (a+j), \theta)
\]
where \( f_{\text{Dag}}(x; \beta, (a+j), \theta) = \beta(a+j) \theta x^{-\theta-1} [1 + (a+j)x^{-\theta}]^{-\beta-1} \) is the pdf of a Dagum rv. Moreover, given that \( \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(a+b)}{\Gamma(j+1)(a+b)} = 1 \), see Domma and Condino (2013), we can say that \( f(x, a, b, \beta, \theta) \) is a mixture of Dagum distribution.

**Proposition 7.** If \( X|\gamma \sim rGB1 - IWei(a, b, c, \gamma, \theta) \) and \( \gamma \sim Ga(\beta) \) then \( X \sim rGB1 - Da(a, b, c, \beta, \lambda_j, \theta) \), with \( \lambda_j = (a+j) \).

**Proof.** The conditional pdf of \( X|\gamma \) is

\[
 f_{rGB1-IWei}(x|\gamma; a,b,c,\theta) = \frac{c}{B(a,b)} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} p_{j_1,b} \left( \frac{p_{j_2,c(a+j_1)}}{(j_2+1)} \gamma^{(j_2+1)} \theta x^{-\theta-1} e^{-\gamma x^{-\theta}} \right).
\]

Assuming that the rv \( \gamma \) has pdf \( f(\gamma; \beta) = \frac{e^{-\gamma}}{\Gamma(\beta)} \), the marginal pdf of \( X \) is given by

\[
 f(x; \eta_1) = \int_0^{\infty} f_{rGB1-IWei}(x|\gamma; a,b,c,\alpha,\theta) f(\gamma; \beta) d\gamma
 = \frac{c}{B(a,b)} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} p_{j_1,b} \left( \frac{p_{j_2,c(a+j_1)}}{(j_2+1)} \gamma^{(j_2+1)} \theta x^{-\theta-1} e^{-\gamma x^{-\theta}} \right) f_Da(x; \beta, (j_2+1), \theta).
\]

\[ \triangleright \]

### 5 Estimation and Inference

In order to estimate the parameters \( \eta = (a, b, c, \gamma, \theta) \) of the \( rGB1-IWei \) distribution, we use the maximum likelihood (ML) method. Let \( \mathbf{x} = (x_1, x_2, ..., x_n) \) be a random sample of size \( n \) from the \( rGB1-IWei \) given by (2.5). The log-likelihood function for the vector of parameters \( \eta = (a, b, c, \gamma, \alpha, \theta) \) can be expressed as

\[
 \ell(\eta) = n \log(c) - n \log(B(a,b))+ (ac - 1) \sum_{i=1}^{n} \log \left[ 1 - G_{IWei}(x_i; \tau) \right]
\]
\[ + (b-1) \sum_{i=1}^{n} \log \left\{ 1 - [1 - G_{IWei}(x_i; \tau)]^c \right\} + \sum_{i=1}^{n} \log \left\{ g_{IWei}(x_i; \tau) \right\} \] (5.1)

where \( \tau = (\gamma, \theta) \). In what follows, we denote with \( \hat{h}_y(y, z) = \frac{\partial h(y, z)}{\partial y} \), \( \hat{h}_{yy}(y, z) = \frac{\partial^2 h(y, z)}{\partial y^2} \) and \( \hat{h}_{yz}(y, z) = \frac{\partial^2 h(y, z)}{\partial y \partial z} \) the partial derivatives of first order, of second order and mixed of a function, say \( h(y, z) \), respectively. Moreover, to simplify the notation, we use \( G(x_i; \tau) = G_{IWei}(x_i; \tau) \) and \( g(x_i; \tau) = g_{IWei}(x_i; \tau) \).

Differentiating the likelihood \( \ell(\eta) \) with respect to \( a, b, c, \gamma \) and \( \theta \), respectively, and setting the results equal to zero, we have

\[
\begin{align*}
\frac{\partial \ell(\eta)}{\partial a} &= -n \frac{\beta_x(a, b)}{B(a, b)} + c \sum_{i=1}^{n} \lg [1 - G(x_i; \tau)] = 0 \\
\frac{\partial \ell(\eta)}{\partial b} &= -n \frac{B_x(a, b)}{B(a, b)} + \sum_{i=1}^{n} \lg (1 - [1 - G(x_i; \tau)]^c) = 0 \\
\frac{\partial \ell(\eta)}{\partial c} &= \frac{n}{c} + a \sum_{i=1}^{n} \lg [1 - G(x_i; \tau)] - (b-1) \sum_{i=1}^{n} \frac{[1-G(x_i; \tau)]^c [1-G(x_i; \tau)]^c}{(1-[1-G(x_i; \tau)]^c)} = 0 \\
\frac{\partial \ell(\eta)}{\partial \tau_j} &= -(ac-1) \sum_{i=1}^{n} \frac{\beta_{x_j}(x_i; \tau)}{1-G(x_i; \tau)} + c(b-1) \sum_{i=1}^{n} \frac{[1-G(x_i; \tau)]^c [1-G(x_i; \tau)]^c}{(1-[1-G(x_i; \tau)]^c)} + \sum_{i=1}^{n} \frac{g_{x_j}(x_i; \tau)}{g(x_i; \tau)} = 0
\end{align*}
\]

for \( j = 1, 2, 3 \) and with \( \tau_1 = \gamma \) and \( \tau_2 = \theta \), where \( \hat{G}_{x_j}(x_i; \tau) \) and \( \hat{g}_{x_j}(x_i; \tau) \) are reported in the Appendix. The system does not admit any explicit solutions; therefore, the ML estimates \( \hat{\eta} = (\hat{a}, \hat{b}, \hat{c}, \hat{\gamma}, \hat{\theta}) \) can only be obtained by means of numerical procedures. Under regularity conditions, the ML estimator \( \hat{\eta} \) is consistent and asymptotically normally distributed. Moreover, the asymptotic variance-covariance matrix of \( \hat{\eta} \) can be approximated by the inverse of observed information matrix given by

\[
J(\eta) = 
\begin{bmatrix}
J_{aa} & J_{ab} & J_{ac} & J_{a\gamma} & J_{a\theta} \\
.. & J_{bb} & J_{bc} & J_{b\gamma} & J_{b\theta} \\
.. & .. & J_{cc} & J_{c\gamma} & J_{c\theta} \\
.. & .. & .. & J_{\gamma\theta} \\
.. & .. & .. & .. & J_{\theta\theta}
\end{bmatrix}
\]

whose elements are reported in the Appendix. In order to build the confidence interval and hypothesis tests, we use the fact that the asymptotic distribution of \( \hat{\eta} \) can be approximated by the multivariate normal distribution, \( N_6(\mathbf{0}, [J(\hat{\eta})]^{-1}) \), where \([J(\hat{\eta})]^{-1}\) is the inverse of observed information matrix evaluated at \( \hat{\eta} \).

6 Applications

In order to show the potentiality of the proposed model, in this section, we consider two real data sets and fit the proposed model to these data.
Example 1. The first data set concerning the survival times (in days) of guinea pigs injected with different doses of tubercle bacilli. These data, divided by 1000, have already been used in Kundu and Howlader (2010) in order to illustrate some results derived for the IWei distribution.

We compare the fit of the IWei model, resulting from the maximum likelihood estimates of the parameters reported by the authors for the uncensored data case, with that obtained by rGB1-IWei model.

For the rGB1-IWei model, we obtain the following ML estimates and the corresponding standard errors: \( \hat{\gamma} = 0.8176 \ (5.421) \), \( \hat{\theta} = 0.1284 \ (0.627) \), \( \hat{a} = 21.0134 \ (192.611) \), \( \hat{b} = 76.0581 \ (720.825) \), \( \hat{c} = 3.9858 \ (50.724) \). Both the Akaike Information Criterion (AIC) and the Kolmogorov-Smirnov (KS) statistics suggest that the rGB1-IWei model provides a better representation of the data than the IWei model. Indeed, considering the different number of parameters and the log-likelihood values reported for the IWei and rGB1-IWei models (\( \hat{l}_{IWei} = 101.644 \), \( \hat{l}_{rGB1-IWei} = 107.1106 \)), we obtain the AIC values, equal to \(-199.2888 \) and \(-204.2213 \), respectively. The values of the KS statistic for the two models are \( KS_{IWei} = 0.1364 \) and \( KS_{rGB1-IWei} = 0.0999 \). The associated \( p \) value for the rGB1-IWei, equal to 0.4692, supports the hypothesis that the data are drawn from the proposed distribution. Furthermore, since the IWei is a sub-model of the rGB1-IWei
when \( a = b = c = 1 \), we can consider the maximum values of the unrestricted and restricted log-likelihoods in order to obtain the LR statistics for testing the need of the extra parameters. Based on the LR statistic, equal to 10.932, and the corresponding \( p \) value, equal to 0.0121, at 5 % significance level, we reject the null hypothesis in favor of the new distribution. The plot of the empirical and the fitted densities are given in Fig. 5. Finally, we examine the empirical and the fitted hazard rate. In their paper, Kundu and Howlader (2010) affirm that the use of the \( IWei \) model seems reasonable since the plot of the empirical version of the scaled TTT transform for the data indicates that the hazard rate is unimodal. As suggested by Bo and Klefsjö (1984), we use this plotting procedure as a complementary technique for model identification. If we compare the fitted curves for the \( IWei \) and the \( rGB1-IWei \) models (see Fig. 6), the latter better illustrates the empirical behaviour of the hazard rate, again confirming the superiority of the proposed model in describing such data.

Example 2. The second data set, dealing with the relation among histopathological changes and cadmium concentration in the kidney cortex of horses, is taken from Ning et al. (2008). In order to show the ability of the proposed model to describe bimodal distributions, we fit the \( rGB1-IWei \) model to these data. The obtained ML estimates of the parameters with the related standard errors are: \( \hat{\gamma} = 442.6035 \ (464.49), \hat{\theta} = 1.1642 \ (0.31), \)
\hat{a} = 0.06967 \ (0.35), \ \hat{b} = 0.1994 \ (1.10) \ and \ \hat{c} = 309.4807 \ (0.33). \ From \ the 
graphical \ analysis \ of \ the \ empirical \ and \ fitted \ distributions \ reported \ in \ Fig. \ 7, 
we \ note \ that \ the \ model \ provides \ a \ good \ fit \ of \ these \ data, \ particularly \ as 
regards \ the \ bimodality \ that \ characterizes \ this \ distribution. \ These \ findings 
are \ also \ confirmed \ by \ the \ results \ obtained \ from \ the \ Kolmorov-Smirnov \ test 
(D = 0.0794; \ p \ value = 0.9296) \ and \ from \ the \ Anderson-Darling \ test \ (AD 
= 0.2525, \ p \ value = 0.9689), \ that \ strongly \ support \ the \ hypothesis \ that \ the 
data \ are \ drawn \ from \ the \ rGB1-IWei \ distribution.

7 \ Final \ Remarks

In \ this \ paper, \ we \ introduced \ the \ Reflected \ Generalized \ Beta \ of \ Inverse 
Weibull \ Distribution \ defined \ by \ using \ the \ Generalized \ Beta \ of \ first \ type 
distribution \ as \ a \ generator \ function, \ as \ proposed \ by \ Alexander \ et \ al. \ (2012), 
considering \ the \ reflected \ version \ for \ this \ generator. \ This \ method \ allows \ us 
to \ obtain \ a \ much \ more \ flexible \ distribution \ than \ the \ Inverse \ Weibull \ one. 
Indeed, \ both \ the \ density \ and \ the \ hazard \ function \ show \ additional \ behaviours 
compared \ to \ the \ Inverse \ Weibull \ distribution. \ This \ considerable \ flexibility 
makes \ the \ new \ distribution \ a \ suitable \ model \ for \ different \ applications, \ such 
as, \ for \ example, \ reliability \ and \ survival \ analysis. \ We \ provided \ full \ treatment 
of \ mathematical \ properties, \ obtaining \ moments, \ entropy \ and \ a \ reliability 
measure. \ We \ discussed \ maximum \ likelihood \ estimation \ and \ obtained \ the 
observed \ Fisher \ Information \ Matrix. \ Finally, \ two \ applications \ to \ a \ real \ data 
set \ are \ given \ to \ illustrate \ the \ usefulness \ of \ the \ proposed \ distribution.
Appendix

A

The elements of the observed information matrix $J(\eta)$ for the parameters $(a, b, c, \gamma, \theta)$ are given by

$$J_{aa} = -n \frac{\ddot{B}_{aa}(a, b)B(a, b) - \dot{B}_a(a, b)}{[B(a, b)]^2} ;$$

$$J_{ab} = -n \frac{\ddot{B}_{ab}(a, b)B(a, b) - \dot{B}_a(a, b)\dot{B}_b(a, b)}{[B(a, b)]^2} ;$$

$$J_{ac} = \sum_{i=1}^{n} \log [1 - G(x_i; \tau)] ;$$

$$J_{bb} = -n \frac{\ddot{B}_{bb}(a, b)B(a, b) - \dot{B}_b(a, b)}{[B(a, b)]^2} ;$$

$$J_{bc} = -\sum_{i=1}^{n} \frac{[1 - G(x_i; \tau)]^c \log [1 - G(x_i; \tau)]}{(1 - [1 - G(x_i; \tau)]^c)} ;$$

$$J_{cc} = -\frac{n}{c^2} - (b - 1) \sum_{i=1}^{n} \frac{[1 - G(x_i; \tau)]^c (\log [1 - G(x_i; \tau)])^2}{(1 - [1 - G(x_i; \tau)]^c)^2} ;$$

$$J_{a\tau_j} = -c \sum_{i=1}^{n} \frac{\dot{G}_{\tau_j}(x_i; \tau)}{[1 - G(x_i; \tau)]} ; \quad J_{b\tau_j} = c \sum_{i=1}^{n} \frac{[1 - G(x_i; \tau)]^{c-1} \dot{G}_{\tau_j}(x_i; \tau)}{[1 - G(x_i; \tau)]^c} ;$$

$$J_{c\tau_j} = -\sum_{i=1}^{n} \frac{\dot{G}_{\tau_j}(x_i; \tau)}{[1 - G(x_i; \tau)]} \left\{ 1 + (b - 1) \frac{[1 - G(x_i; \tau)]^c ([1 - G(x_i; \tau)]^c - c \log [1 - G(x_i; \tau)]) - 1)}{(1 - [1 - G(x_i; \tau)]^c)^2} \right\} ;$$

for $j = 1, 2$ and $\tau_1 = \gamma$ and $\tau_2 = \theta$;

$$J_{\tau_j \tau_h} = (1 - ac) \sum_{i=1}^{n} \left( \frac{\ddot{G}_{\tau_j \tau_h}(x_i; \tau) [1 - G(x_i; \tau)] + \dot{G}_{\tau_j}(x_i; \tau) \dot{G}_{\tau_h}(x_i; \tau)}{[1 - G(x_i; \tau)]^2} + \right.$$
\[
\begin{align*}
\frac{\ddot{g}_{r_j r_h}(x_i; \tau) g(x_i; \tau) - \ddot{g}_{r_j}(x_i; \tau) \dot{g}_{r_h}(x_i; \tau)}{[g(x_i; \tau)]^2} + c(b - 1) \sum_{i=1}^{n} \frac{[1 - G(x_i; \tau)]^{c-2}}{(1 - [1 - G(x_i; \tau)]^c)^2} \\
\{ (1 - [1 - G(x_i; \tau)]^c \} \left( \dot{G}_{r_j}(x_i; \tau) \dot{G}_{r_h}(x_i; \tau) + [1 - G(x_i; \tau)] \ddot{G}_{r_j r_h}(x_i; \tau) \right) \\
- c \ddot{G}_{r_j}(x_i; \tau) \dot{G}_{r_h}(x_i; \tau) \} 
\end{align*}
\]

for \( j = 1, 2 \) and \( h = 1, 2 \) with \( j \leq h \), and \( \tau_1 = \gamma \) and \( \tau_2 = \theta \).

Where

\[
\begin{align*}
\dot{G}_\gamma(x_i; \tau) &= -x_i^{-\theta} G(x_i; \tau) \quad \dot{G}_\theta(x_i; \tau) = \gamma x_i^{-\theta} \log(\gamma x_i) G(x_i; \tau) \quad \ddot{G}_\gamma(x_i; \tau) = -x_i^{-\theta} \dot{G}_\gamma(x_i; \tau) \\
\dot{G}_{\gamma\theta}(x_i; \tau) &= x_i^{-\theta} \left\{ \log(x_i) G(x_i; \tau) - \dot{G}_\theta(x_i; \tau) \right\} \\
\dot{G}_{\theta\theta}(x_i; \tau) &= \gamma x_i^{-\theta} \log(x_i) \left\{ \dot{G}_\theta(x_i; \tau) - \log(x_i) G(x_i; \tau) \right\} \\
\dot{g}_\gamma(x_i; \tau) &= \theta x_i^{-\theta - 1} \left\{ G(x_i; \tau) + \gamma \dot{G}_\gamma(x_i; \tau) \right\} \\
\dot{g}_\theta(x_i; \tau) &= \gamma \theta x_i^{-\theta - 1} \left\{ [1 - \theta \log(x_i)] G(x_i; \tau) + \theta \dot{G}_\theta(x_i; \tau) \right\} \\
\ddot{g}_\gamma(x_i; \tau) &= \theta x_i^{-\theta - 1} \left\{ 2 \dot{G}_\gamma(x_i; \tau) + \gamma \ddot{G}_\gamma(x_i; \tau) \right\} \\
\ddot{g}_{\gamma\theta}(x_i; \tau) &= x_i^{-\theta - 1} \left\{ [1 - \theta \log(x_i)] \left[ G(x_i; \tau) + \gamma \dot{G}_\gamma(x_i; \tau) \right] + \theta \left[ \dot{G}_\theta(x_i; \tau) + \gamma \ddot{G}_{\gamma\theta}(x_i; \tau) \right] \right\} \\
\ddot{g}_{\theta\theta}(x_i; \tau) &= \gamma x_i^{-\theta - 1} \left\{ \theta \ddot{G}_{\theta\theta}(x_i; \tau) - [2 - \theta \log(x_i)] \log(x_i) G(x_i; \tau) + 2 \left[ 1 - \theta \log(x_i) \right] \dot{G}_\theta(x_i; \tau) \right\}
\end{align*}
\]
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