Holographic multiverse

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Abstract

We explore the idea that the dynamics of the inflationary multiverse is encoded in its future boundary, where it is described by a lower dimensional theory which is conformally invariant in the UV. We propose that a measure for the multiverse, which is needed in order to extract quantitative probabilistic predictions, can be derived in terms of the boundary theory by imposing a UV cutoff. In the inflationary bulk, this is closely related (though not identical) to the so-called scale factor cutoff measure.
I. INTRODUCTION

In a theory with many low-energy vacua, the dynamics of eternal inflation generates a variety of environments where observers may develop and experiments may take place. One would then like to calculate probability distributions for such environments, including, for instance, probabilities for what we commonly call the cosmological parameters of the standard Big Bang model, or the parameters of the low energy standard model of particle physics (for recent reviews, see e.g. [1, 2]).

Suppose the universe starts in an inflationary false vacuum with a sufficiently low decay rate. Localized bubbles of neighboring vacua will occasionally nucleate and subsequently expand into the false vacuum. Each bubble takes a small fraction of the volume in the original inflating vacuum into a new phase, but since the parent vacuum is inflating, the daughter bubbles never deplete it completely. Some of the vacua within these daughter bubbles may also inflate, decaying into other vacua, and so on. This leads to a never ending cascade of bubbles within bubbles, where all possible phases in the theory are eventually realized.

Here, and for most of the paper, we assume that the “landscape” of vacua is irreducible. This means that we can access any given inflationary vacuum from any other one through a finite sequence of transitions. The transitions or “decays” can happen from high energy inflationary vacua to low energy inflationary vacua, but also in the opposite direction. If the landscape is not irreducible, then for present purposes we may think of the different irreducible sectors as different theories. In an irreducible landscape, the volume distribution of the different vacua quickly approaches an attractor solution as a function of time [3]. The explicit form of the volume distribution depends on which time coordinate we use in order to slice the inflationary “multiverse”, but the important point is that the attractor is independent of the initial state.¹

¹ Nonetheless, inflation is not eternal to the past [4]. It must start somehow, and a description of the beginning of inflation may be needed for logical completeness. For instance, in the case where the landscape of the theory is reducible, this may help addressing the question of which irreducible sector are we more likely to be in.
based on a statistical counting of occurrences of the different kinds of processes we may be interested in [5]. However, the results of this counting depend very much on how we regulate the infinities, and different regulators give vastly different results for the probabilities [6]. This is the so-called “measure problem” of inflationary cosmology.

There is by now a fairly extensive literature of proposals for the measure, some of which have already been superseded or simply abandoned because of inherent paradoxes or inconsistencies [7]. The fact remains, however, that in order to confront theory with observations the measure is needed, and the question must be addressed. Progress has been made in recent years, by way of identifying some generic problems which a measure should avoid. Amongst those we may highlight the following:

*The youngness paradox.* This corresponds to the prediction that we are exponentially more likely to have emerged much earlier in cosmic history, in strong conflict with observations [8, 9, 10]. This problem afflicts the so-called “global proper-time cutoff measure”, which is defined as follows. First, we choose an arbitrary spacelike surface Σ, which serves as the origin of proper time along a congruence of geodesics orthogonal to it. In the ensemble of all events, we count only those which happen before a fixed proper time. The probabilities are calculated in terms of this counting, in the limit when the proper time cutoff is sent to infinity. The same "youngness" effect arises if we use some other time variable in order to impose the global cutoff. Nevertheless, there is a narrow class of choices of the cutoff time variable where the “paradox” turns into a quite benign “youngness bias” [11]. The scale-factor cutoff measure, to be discussed below, belongs to this class.

*The Boltzmann brain paradox.* This corresponds to the prediction that we are most likely to be disembodied brains, created by large quantum fluctuations in an otherwise empty phase, dreaming of CMB multipoles and other exquisite data [12, 13, 14, 15, 16]. This paradox afflicts a broad class of measure proposals, and is due to the fact that thermalized regions like ours (which are thought to arise as a consequence of an earlier phase of slow roll inflation) are relatively rare in the multiverse. Most of the space-time volume is occupied by empty quasi-de Sitter phases. Because of that, “freak” observers, created by large quantum fluctuations in vacuum, may easily outnumber the “ordinary” observers who live in thermalized regions. Of course, this will depend on how we regulate the numbers of both types of observers, and it is conceivable that some measures may avoid the problem (provided that the landscape of vacua satisfies certain properties). This issue is currently
The Q-catastrophe. This afflicts proposals where probabilities are rewarded according to the amount slow-roll inflationary expansion which precedes thermalization, such as for instance, the “pocket-based” measure introduced in \[3, 19, 20\]. In this case, observables which are correlated with the number of e-foldings of slow-roll tend to suffer an exponential bias towards large or small values \[21\]. One such observable is the amplitude of density perturbations \(Q\) caused by quantum fluctuations of the inflaton field. The exponential bias would push the likely values of \(Q\) towards the boundaries of what is anthropically allowed, while the observed value happens to sit comfortably in the middle of the anthropic range. There may be ways out of this “catastrophe” (such as the possibility that density perturbations are seeded by a curvaton rather than the inflaton), but correlations of the parameters with the duration of slow roll inflation remains a potential nuisance.

These “paradoxes” and “catastrophes” can be thought of as phenomenological constraints, useful to narrow down the possible definitions of a measure. Additionally, we may take the point of view that \textit{initial conditions} at the beginning of inflation should be irrelevant for the purposes of making predictions \(^2\). The justification is that the dominant part of the spacetime volume in the eternally inflating “multiverse” is in the asymptotic future, where the volume distribution of different vacua (and physical processes therein) is well described by an attractor solution which is insensitive to initial conditions.

It has recently been emphasized \[11, 17, 18\] that the scale factor cutoff measure (which we shall discuss in Section IV) is free from the youngness paradox, does not suffer from the Q-catastrophe, and is independent of initial conditions. It also provides a good fit to the data when it is used in order to predict the likely values of the cosmological constant \[11\] and, with some relatively mild assumptions about the landscape, it avoids the Boltzmann brain paradox \[17, 18\]. It thus appears to be a promising candidate for the measure of the multiverse.

The purpose of this paper is to formulate a measure for the multiverse which may be connected to the underlying dynamical theory. Our discussion relies on the existence of the late time attractor for the volume distribution of vacua, and it is inspired by the holographic ideas which have been developed in the context of string theory. The idea is to formulate the

\(^2\) See, however, the caveat mentioned in the previous footnote.
calculation of probabilities directly at the “future boundary” of space-time, the place where everything has been said and done. Data at the future boundary contains information on everything that has happened in the multiverse, and this makes it a natural locus to do our counting.

The plan of the paper is the following. In Section II we review the causal structure of the inflationary multiverse, and formalize our definition of the future boundary. In Section III we propose that the dynamics of the multiverse can be mapped into a lower dimensional Euclidean field theory, which lives at the future boundary. Under this mapping, the late time (or “infrared”) self-similar behaviour of the attractor solution in the bulk, corresponds to scale invariance in the “ultraviolet” of the boundary theory. Conversely, the infrared properties of the boundary correspond to the initial stages of inflation, which depend on initial conditions. The measure is discussed in Section IV. By analogy with field theory, we propose to regulate infinities at the boundary by imposing a UV cutoff. Probabilities for different types of events are then defined as the ratios of occurrences in the limit when the cutoff is removed. As we shall see, this procedure is closely related to the scale-factor cutoff measure. Our conclusions are summarized and discussed in Section V.

II. THE FUTURE BOUNDARY

The future causal boundary, which we shall denote as $c^+$, can be defined as the set of endpoints of inextendible time-like curves. More precisely, “points” or elements of $c^+$ are defined as the chronological pasts of inextendible time-like curves. Two curves with the same past will therefore define the same “endpoint” at $c^+$. The future boundary contains points of various different types.

In the inflationary multiverse, most time-like curves will eventually exit the inflating region and fall into one of the non-inflating vacua, which we shall refer to as terminal vacua. Terminal vacua may have vanishing or negative vacuum energy, and will be denoted as Minkowski and anti-de Sitter (AdS) vacua respectively. Bubbles of AdS vacua develop a spacelike singularity or “big crunch” in their interior, and timelike curves hit this singularity in a finite proper time. The corresponding endpoints will be said to belong to the singular part of $c^+$. Time-like curves which enter a metastable Minkowski vacuum will also fall, eventually, into one of the AdS vacua, adding to the singular boundary. However, supersym-
metric Minkowski vacua are completely stable [23], and time-like curves entering them will span the future null and time-like infinities of the Minkowski conformal boundary, usually referred to as $\mathcal{I}^+$ and $i^+$ respectively. Fig. 1 represents the causal structure of an eternally inflating universe with terminal bubbles of different types. The conformal future boundary of Minkowski bubbles has the shape of a “hat” [25], whereas the spacelike singularity at the future boundary of AdS bubbles is represented by a broken line.

FIG. 1: Causal diagram of the inflationary multiverse. The vertical direction represents time, and the horizontal direction is space. Bubbles of different types nucleate and start expanding close to the speed of light. Bubbles with positive vacuum energy (dS bubbles) inflate eternally. Inflation stops in bubbles with vanishing or negative vacuum energy (Minkowski and AdS bubbles, respectively).

On the other hand, because inflation is eternal, some time-like curves will remain forever in inflating regions of space. The corresponding endpoints will be called eternal points, and
they can also be classified in different types. For instance, there are time-like geodesics which, after a finite sequence of transitions, remain forever in a given inflating vacuum. If we assign a “color” to each inflating vacuum, the corresponding endpoints would have a definite color. However, eternal time-like curves are more likely to jump back and forth between different vacua on their way to infinity, without ever settling into any of them. In this case, the corresponding endpoints would have, so to speak, a mixed color. Clearly, there are many different hues the eternal points may have.

The future boundary can be endowed with a topology, whose open sets are defined in terms of the future of points in the manifold. We say that a point \( p \in c^+ \) belongs to the open set \( U_{\text{int}}(q) \), if the time-like curves whose endpoint define \( p \) have some intersection with the chronological future of point \( q \) in the manifold \( M \). Intuitively, the intersection of the future lightcone of point \( q \) with \( c^+ \) draws the boundary of an open set \( U_{\text{int}}(q) \) in \( c^+ \). Likewise, we say that \( p \) belongs to the open set \( U_{\text{ext}}(q) \) if its defining time-like curves do not intersect the causal future of point \( q \). By definition, arbitrary unions and finite intersections of the \( U_{\text{int}}(q) \) and the \( U_{\text{ext}}(q') \) for all \( q, q' \in M \), are also open sets. It is unclear to us whether one can make \( c^+ \) into a differentiable manifold by using this topology. For this we would need an invertible map from the open sets \( U_{\text{int}}(q) \) and \( U_{\text{ext}}(q) \) onto “coordinate” open sets of \( \mathbb{R}^3 \), in such a way that the changes of coordinates are differentiable where open sets overlap. While this remains an interesting possibility, we shall not pursue it here. The reason is that we need not work with the full set \( c^+ \), but only with the set of eternal points (and its boundary), as we shall now describe.

A. The fiducial spacelike hypersurface \( \Sigma_3 \)

Consider a space-like hypersurface \( \Sigma_3 \) embedded in the multiverse and a congruence of time-like geodesics \( \gamma(x) \) orthogonal to it (see Fig. 2). Here \( x \) are coordinates on \( \Sigma_3 \). It is not necessary that \( \Sigma_3 \) be a Cauchy surface for the whole multiverse, but we do require that at least one of the geodesics in the congruence be eternal. This guarantees that the attractor solution is reached at late times in the spacetime region \( \mathcal{S} \) spanned by the congruence. Because of that, the portion of \( c^+ \) attached to this region will be a fair sample of the whole of \( c^+ \).

The congruence defines co-moving coordinates in \( \mathcal{S} \), and the metric \( g_{ij}(x) \) induced on \( \Sigma_3 \)
FIG. 2: We assign co-moving coordinates to the points in the eternal set $E$, by introducing a fiducial hypersurface $\Sigma_3$ and a congruence of geodesics orthogonal to it. The metric $g_{ij}(x)$ on $\Sigma_3$ can be used in order to assign co-moving distances amongst the points in $E$.

defines co-moving distances between different geodesics. Bubbles nucleating to the future of $\Sigma_3$ can be projected backwards along the congruence. The “image” of a bubble consists of all those points $x$ in $\Sigma_3$ such that $\gamma(x)$ intersects the given bubble. The congruence also allows us to define “scale factor time” $a$. At any spacetime point $x$ in $S$ this is given by

$$\ln a(x) = \int_{x(x)}^{x} H d\tau.$$  

(1)

Here $\tau$ is proper time and the integral is taken along the geodesic in the congruence that connects $x$ to some point $x(x)$ on $\Sigma_3$, while $H$ is one third of the expansion $\nabla_\mu u^\mu$ of the congruence. Here $u^\mu = dx^\mu/d\tau$ is the tangent vector. Note that with this definition,

$$a(x \in \Sigma_3) = 1.$$  

(2)

The co-moving volume of the image of a bubble of vacuum $i$ nucleating in inflating vacuum $j$ at scale factor time $a$ is given by

$$V_i(a) = \frac{4\pi}{3} H_j^{-3} a^{-3},$$  

(3)

where $H_j = (\Lambda_j/3)^{-1/2}$ is determined by the effective vacuum energy $\Lambda_j$. Hence, bubbles nucleating later in time will have smaller images on $\Sigma_3$. 

8
Here, and for the rest of the paper, we disregard geodesic crossing. Of course, due to gravitational instability, structure will form in some of the pocket universes and some geodesics will eventually cross each other. To avoid this effect, we shall define co-moving distances as is usually done in standard cosmology, by considering geodesics on a metric which is smoothed out on sufficiently large scales. In this case, the congruence will always be diverging, except in the collapsing AdS regions. As we shall see, we will not need to be specific about the details of the congruence inside of terminal bubbles.

FIG. 3: The future boundary of a Minkowski bubble is a ”hat”, consisting of the union of future null infinity $\mathcal{I}^+$ and time-like infinity $i^+$. The worldline of a ”census taker” ending at $i^+$ is also represented.

Each geodesic $\gamma(x)$ defines an “endpoint” $p(x)$ in $c^+$, and so the congruence maps the fiducial hypersurface $\Sigma_3$ onto the future boundary. However, this map is not one to one. Indeed, the image of a stable Minkowski bubble occupies a finite co-moving volume on $\Sigma_3$, 

9
given by Eq. (3), while the corresponding geodesics all end up at the same single point in $c^+$, namely, the point $i^+$ of the conformal boundary of Minkowski. Conversely, the future null infinity $\mathcal{I}^+$ of a bubble of the stable Minkowski vacuum is not reached by any of the time-like geodesics $\gamma(x)$, and so it has no anti-image on $\Sigma_3$. Fortunately, this will not necessarily be a problem.

The reason is that the map fails to be one to one only in the regions corresponding to the “hats” of stable Minkowski bubbles. However, it has been argued in [24, 25] that the bulk region in the interior of these bubbles is holographically described by a field theory which lives on the 2 dimensional surface $\Sigma_2$ which lies at the boundary of $\mathcal{I}^+$, where the hat meets the “horizontal” part of $c^+$ (see Fig. 3). In this sense, we need not worry about points on $c_+$ which are inside of $\Sigma_2$. This means that on the fiducial hypersurface $\Sigma_3$ we can excise the images of stable Minkowski bubbles: these will be accounted for by degrees of freedom which live at the boundary of the excised holes. As a matter of fact, we will also argue that the images of AdS bubbles, corresponding to big crunch singular points, should be excised in a similar way. In the case of AdS bubbles, the surface $\Sigma_2$ is the boundary of the corresponding set of “big crunch” singular points on $c_+$.

III. HOLOGRAPHY

In this Section we elaborate on the idea that the dynamics of the multiverse is dual to a boundary field theory. This is of course inspired by the holographic AdS/CFT correspondence, and the idea of applying it to inflationary cosmology is not new. The version we advocate here builds up from two earlier proposals.

In [26] it was proposed that the dynamics of de Sitter space is dual to a Euclidean CFT which lives at the conformal future boundary. That construction, however, did not allow for the possibility that a given de Sitter phase decays into neighboring vacua. Such decays drastically modify the structure of the future boundary, making it very different from the conformal boundary of de Sitter.

A somewhat related proposal was developed in Refs. [24, 25, 27]. There, it is argued that the bulk dynamics of a pocket universe corresponding to a stable Minkowski vacuum is dual to a 2-dimensional Euclidean field theory, which lives at the boundary $\Sigma_2$ of the Minkowski “hat” representing the future null infinity of that pocket. For a single bubble,
this boundary has the topology of a 2-sphere. The field theory on $\Sigma_2$ encodes the information corresponding to any observation that can be made by a hypothetical “census taker” who lives in the Minkowski bulk, and who is allowed to observe for an indefinite amount of time all the way to the tip of the hat, at $i^+$ (see Fig. 3). An important feature of this boundary theory is that it includes a Liouville field $L$ living in $\Sigma_2$. This field accounts for the time evolution in dual Minkowski bulk, and it has been argued that conformal invariance is recovered in the limit of large $L$, which corresponds to late times $[24, 25]$. Another interesting aspect of this picture is that the census taker would observe collisions between the reference Minkowski bubble and other bubbles that nucleate in its neighborhood. In this way, she would receive information about other vacua, different from the “parent” vacuum where the Minkowski bubble nucleated. Hence, the field theory on $\Sigma_2$ should encode a substantial amount of information about the dynamics in the landscape of vacua. It has been shown that, accounting for bubble collisions, the surface $\Sigma_2$ can have any genus, and so the boundary theory should include a sum over topologies $[27]$.

Based on the analogy with the black hole horizon complementarity $[28]$, it was proposed in $[24, 25]$ that the dynamics of the entire multiverse may be holographically dual to that of the causal patch of a single census taker. The Euclidean theory on $\Sigma_2$ would then provide a complete description of the multiverse. We note however that there may be many different stable Minkowski vacua in the landscape, and there seems to be no good reason to restrict attention to the field theory associated with a particular one of them. Also, it is not clear whether any such field theory on the surface $\Sigma_2$ would fully represent the underlying dynamics, since the census taker may not be able to see the full set of vacua.

In particular, there are some events in AdS whose future light cone is completely engulfed by the big crunch, and those cannot be seen by a Minkowski census taker. Hence, it appears that we need to enlarge our holographic screen in order to account for these regions. As mentioned above, we propose that the bulk dynamics of AdS bubbles is encoded in their boundary as well. Intuitively, this seems reasonable since we know that the bulk of AdS spacetime does have a holographic description. Here, with AdS bubbles, the situation is somewhat different because a future singularity develops. It is conceivable that this simply changes the boundary description $[29]$, and we shall tentatively assume that this is given in terms of degrees of freedom living in $\Sigma_2$ (as mentioned above, for AdS bubbles $\Sigma_2$ is defined as the boundary of the set of corresponding singular points at $c_+)$.
A conjecture

Here, we propose that the dynamics of the landscape is encoded in a Euclidean field theory defined on a set which includes all eternal points in $c^+$, and not just those belonging to $\Sigma_2$. The set $E$ of eternal points is known to be a fractal [30, 31], which we can represent on the fiducial hypersurface $\Sigma_3$. For any finite co-moving resolution $\xi$ this looks like a 3 dimensional "sponge", with holes on it which correspond to the images of AdS and Minkowski terminal bubbles. The images of bubbles of inflating vacua look like sponges, whose pores are occupied by other inflating vacua, or by holes.
bubbles, whose co-moving size is larger than $\xi$ (see Fig. 4). Note that the points in the eternal set are mapped onto $\Sigma_3$, where they have a single image. Hence, we have a notion of co-moving distance in the eternal set given by the metric $g_{ij}(x)$ of the fiducial surface $\Sigma_3$.

Our conjecture is that the bulk dynamics of the eternally inflating universe is represented by a Euclidean field theory living on that sponge, where the resolution scale $\xi$ plays the role of a renormalization scale. The degrees of freedom of the field theory live not only in the 3D "bulk" of the sponge, but also at the surfaces $\Sigma_2$ which are at the boundary of the holes corresponding to the images of all terminal bubbles. In this picture, we are invoking the dS/CFT correspondence of Ref. [26] for the inflating set, while we are borrowing from the ideas of Refs. [24, 25, 27] to incorporate the bulk dynamics of terminal bubbles. For the reasons explained in the previous Subsection, we are also including the boundaries of AdS bubbles, in contrast with Refs. [24, 25, 27] where only the boundary of a single Minkowski bubble was considered.

Inside the "sponge" representing the inflating region, we will have the images of bubbles of different inflating vacua, nested within each other. Each one of these images will be thought of as an instanton in the boundary theory [24]. Nested within the image $B_i$ of a bubble of vacuum of type $i$, we will have the images of bubbles that nucleate out of this vacuum later in time, and whose co-moving size is therefore smaller. In this way, $B_i$ will itself look like a sponge, whose “pores” contain vacua of different types. Some of the “pores” may correspond to other inflating vacua, in which case they will themselves look like sponges, and so forth. Of course, some of the “pores” may correspond to Minkowski or AdS vacua, in which case they are actual “holes” in the eternal set (these would correspond to bubbles of “nothing”). As the regulator is made finer and finer, we will see more and more of this structure.

As mentioned above, in the eternal set we have a notion of co-moving distance which is based on the metric $g_{ij}$ induced on an arbitrarily chosen fiducial hypersurface $\Sigma_3$. A notion of distance is in fact necessary in order to define the Wilsonian cut-off $\xi$. If we change the surface, the metric changes, and we shall now argue that this corresponds to Weyl rescalings in the boundary theory.

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3 The bubble walls of these instantons may also carry some degrees of freedom, such as zero modes, but these are unrelated to the holographic description of the bubble interiors.
Let us first discuss the case of de Sitter space. We can consider a pencil of geodesics of the standard congruence $C_0$ associated with the flat coordinates of de Sitter (dS), in which

$$ds^2 = dt^2 - e^{2Ht}d\mathbf{x}^2.$$  \hspace{1cm} (4)

Suppose now we choose another congruence, $C$. Asymptotically, $C$ becomes comoving with $C_0$, so the surfaces orthogonal to $C$ are nearly orthogonal to $C_0$ at late times. Consider one such (late-time) surface $\Sigma_t$. In terms of the standard coordinates, this can be represented as

$$\Sigma_t : \quad t = f(\mathbf{x}),$$  \hspace{1cm} (5)

where $f(\mathbf{x})$ is a very slowly varying function. The asymptotic metric in the coordinates defined by the congruence $C$ is

$$ds^2 \approx dt^2 - e^{2Ht}e^{-2Hf(\mathbf{x})}d\mathbf{x}^2.$$  \hspace{1cm} (6)

This shows that the change of congruence corresponds to a Weyl rescaling of the metric at the future boundary.

In the multiverse, each bubble is a part of dS space, but, as we described above, what remains of it asymptotically is just a "sponge". A change of congruence will induce a Weyl transformation in each sponge. In principle, from the argument above, it is not clear whether the transformation is the same or not in different sponges, because the Hubble rates $H$ are different in different vacua. However, this has to be the case because bubbles of one of the vacua have to fit neatly into the "pores" of the progenitor.\(^4\)

Weyl rescalings can be used in order to change the size of instantons. For a given congruence, by choosing the fiducial surface to be at a later time, the size of all instantons becomes bigger. However, because of the self-similar structure of the bulk attractor at late times, the result of this rescaling will not change the distribution of sizes and of bubbles of different types at the boundary. Thus, self-similarity at late times corresponds to UV scale invariance in the boundary theory. On the other hand, if we look at the boundary theory

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\(^4\) Suppose the parent and daughter vacua have expansion rates $H$ and $H'$, respectively. In flat de Sitter coordinates they expand respectively as $\exp(\mathcal{H}t)$ and $\exp(\mathcal{H}'t)$. If we chose the same origin for the time coordinate, then the continuation of the surface $t = \text{const}$ of the parent vacuum into a bubble is $t' = (H/H')t$. This matching of the exterior and interior regions gives a surface of constant scale factor; it is easy to see that the corresponding rescaling is indeed continuous across the bubble boundaries.
on the largest possible scales, the distribution of instantons of different vacua will reflect the
initial conditions of inflation, and will not have this invariance.\footnote{There is some evidence that the theory is invariant not just under rescalings, but under the full conformal
group. This will be discussed elsewhere \cite{32}.}

IV. THE BOUNDARY MEASURE

Here, we propose that the natural way to address the measure problem of inflationary
cosmology is to use the Wilsonian cutoff of the boundary theory.

Formally, we may consider the amplitude,

\[ Z[\bar{\phi}(x)] = \int D\phi \ e^{iS[\phi]} . \]  

Here \( S \) is the bulk action, and the integral is over bulk fields \( \phi \) approaching the prescribed
\( \phi = \bar{\phi}(x) \) at the future boundary. Bulk fields do not really approach constant values at late
times, since in particular their values keep changing due to bubble nucleation (as well as due quantum
fluctuations of light fields). In order to make sense of Eq. (7), bulk fields should be
smeared over a fixed co-moving scale \( \xi \), and likewise for the boundary values \( \bar{\phi} \). With this
course-graining, the values of the fields are frozen after the co-moving wavelength \( \xi \) crosses
the horizon, and the boundary condition can be implemented. We may use the amplitude
(7) in order to compute correlators. For instance, the two point function is given by

\[ \langle \bar{\phi}(x) \bar{\phi}(x') \rangle = \int D\bar{\phi} \ \bar{\phi}(x) \ \bar{\phi}(x') \ |Z[\bar{\phi}]|^2. \]  

In principle, we should also specify boundary conditions for \( \phi \) on some initial fiducial surface,
such as the \( \Sigma_3 \) we have discussed above. However, based on the arguments we presented in
the previous Section, we expect that the initial boundary condition will only determine the
infrared behaviour of correlators. On the other hand, we are interested in the UV fixed point
(corresponding to the attractor behaviour in the bulk description), and the initial boundary
condition will not play a role for our present purposes. Hence, we shall simply omit it in
the following discussion. By analogy with AdS/CFT, we now posit that the bulk dynamics
is dual to a Euclidean theory living at the boundary, where now the \( \bar{\phi}(x) \) play the role of
sources for operators in the boundary theory. The conjecture is that (7) is also given by

\[ Z[\bar{\phi}(x)] = e^{iW_{\text{CFT}}[\bar{\phi}]} , \]  

\( Z[\bar{\phi}(x)] = e^{iW_{\text{CFT}}[\bar{\phi}]} , \)
where $W_{CFT}$ is the effective action for a boundary field theory with appropriate couplings to the external sources $\bar{\phi}$. If the theory is regularized with a cutoff $\xi$, then we should think of the configurations $\bar{\phi}(x)$ as coarse-grained on the scale $\xi$.

We propose that in order to determine the probabilities of given semiclassical processes in the bulk, we should do the counting in the regularized boundary theory, where this counting will be finite. The idea is that any bulk process will also be represented in the boundary theory. In the coarse grained description, only a finite number of these processes will be resolved, and relative probabilities can be defined as the ratios of occurrences in the limit $\xi \to 0$.

Let us now argue that this definition of the measure is closely related to the so called scale factor cutoff measure, where we take into account only those processes which happen before a fixed scale factor time $a_c$, and then we take the limit where $a_c \to \infty$ in order to determine the probabilities for the processes to occur. Suppose we are interested in 4D bulk processes which require a resolution corresponding to the physical length scale $\lambda_{\text{min}}$. For example, if we have in mind some cosmological process at a given scale, we may think of $\lambda_{\text{min}}$ as a somewhat smaller scale, just so that the process can be properly identified. Now, the co-moving wavelength $\xi a$ corresponding to the boundary cut-off will be smaller than $\lambda_{\text{min}}$ provided that the process takes place at sufficiently early times,

$$a < a_c = \frac{\lambda_{\text{min}}}{\xi}.$$  

(10)

This equation relates the Wilsonian cutoff $\xi$ of the boundary theory to the scale factor cutoff $a_c$. Note that the infrared limit in the bulk theory, $a_c \to \infty$, corresponds to the UV limit in the boundary theory, $\xi \to 0$. This relation is of course familiar from the analogous context of AdS/CFT. There, the RG flow is associated with radial displacement in the bulk, whereas here it is associated with scale-factor time evolution.

The argument is somewhat more involved for terminal bubbles. The interior of these bubbles looks like an open FRW universe,

$$ds^2 = d\tau^2 - a_{\text{FRW}}^2(\tau)(d\zeta^2 + \sinh^2\zeta d\Omega^2),$$

(11)

6 Note that the resolution scale $\lambda_{\text{min}}$ can be thought of as a Wilsonian UV cutoff in the bulk theory.

7 As noted in the text, a connection between time evolution and RG flow is to be expected by analogy with AdS/CFT, and had already been observed e.g. in Refs. [24, 25, 26]. Here, we are making the connection more precise, by relating the RG flow to scale factor time evolution (as opposed to, say, proper time evolution).
and the corresponding boundary theory lives at space-like infinity of the 3-dimensional space-like hyperboloids that foliate this universe, $\zeta \rightarrow \infty$. The size of the images on the holographic screen is determined by the value of the scale factor $a_{FRW}$, as well as the radial distance $\zeta$ to the center of the hyperboloid. For a terminal bubble nucleated in parent vacuum $i$ at a scale factor $a_{nuc}$, the comoving radius of its future boundary is $R = (H_i a_{nuc})^{-1}$. The regulator scale $\xi$ applied to this boundary subtends an angle $\theta_\xi \approx \xi / R$ from the bubble center, and the corresponding physical distance on a hypersurface of constant $\tau$ at radial coordinate $\zeta$ is

$$d_\xi(\zeta, \tau) = \theta_\xi a_{FRW}(\tau) \sinh \zeta. \quad (12)$$

(We assume that $\xi \ll R$.) Requiring that $d_\xi$ is smaller than the resolution scale $\lambda_{min}$, we have

$$H_i a_{nuc} a_{FRW}(\tau) \sinh \zeta < \lambda_{min} / \xi. \quad (13)$$

Now, for $\zeta \gg 1$, the expression on the left-hand side is precisely the scale factor $a$ in the bubble interior $^{10, 17, 33}$. The factors $a_{nuc}$, $\sinh \zeta$ and $H_i a_{FRW}(\tau)$ account respectively for the expansion from the fiducial hypersurface to nucleation, from nucleation to the time when the geodesic at a given $\zeta$ crosses the bubble wall, and for the expansion inside the bubble. (Note that with the definitions we adopted, $a_{nuc}$ is dimensionless, while $a_{FRW}$ has the dimension of length.) Thus, we recover Eq. (11), that is, the scale factor cutoff.$^8$

The correspondence between the boundary and scale factor measures is nonetheless only approximate, and it can break down when we are interested in processes involving wavelengths much smaller than the Hubble radius (as is often the case). The physical reason is simple: while their physical size is smaller than the Hubble radius, these modes can be affected by all sorts of other subhorizon processes and need not simply evolve by conformal stretching with the expansion of the universe.$^9$

A related observation is that all finite co-moving wavelengths at the future boundary correspond to frozen modes (since the co-moving size of the horizon shrinks to zero asymptotically). This means that the coarse grained configurations $\tilde{\phi}(x)$, from which we must

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$^8$ Throughout the paper, we disregard focusing of geodesics at domain wall crossing and the ambiguities associated with continuing geodesic congruences into the “fuzzy” quantum regions in the vicinity of bubble nucleation events. These issues require further study.

$^9$ A similar situation is encountered in AdS/CFT, where wavelengths which are smaller than the AdS curvature radius are not trivially mapped into the boundary theory.
retrieve the information about bulk events, are configurations which are frozen in with the expansion. Physically, it is not surprising that the information gets to future infinity in the form of long wavelength modes. The events we are trying to reconstruct will give away, say, electromagnetic or gravitational radiation whose wavelength is conformally stretched. More generally, information is bound to leak from short to long wavelengths through interactions. Once the information is in the form of wavelengths bigger than the horizon, it becomes indestructible: no causal process can erase it. However, the precise way in which the information about bulk events is encoded at infinity can be more complicated than just conformal stretching from small scales, and so the correspondence with scale factor cutoff is not exact.

In fact, the scale factor measure itself is not uniquely defined on sub-horizon scales. In regions of structure formation, where geodesics converge and cross, the scale factor is not a good time variable, and this leads to ambiguities. The scale factor may start decreasing along some geodesics, until it vanishes at a caustic, and then start increasing again. A given value of the scale factor may thus be reached multiple times as we move along a geodesic, and it is not clear which of these occurrences should be used to implement the cutoff. In Ref. it was proposed that the first occurrence should be used. On the other hand, it was pointed out in that the resulting cutoff surface is strongly influenced by the local details of structure formation. It has a rather “spiky” appearance and is not generally spacelike. An alternative possibility, indicated in is to define the geodesic congruence using the spacetime metric smoothed over some characteristic scale. If this scale is chosen to be larger than the typical scale of structure formation, then the congruence will always be diverging, except in the collapsing AdS regions. In AdS regions, the scale factor will reach some maximum value \( a_{\text{max}} \) and then decrease down to zero, so additional prescriptions are needed to handle this case. If the cutoff value is \( a_c < a_{\text{max}} \) this value will occur twice on the geodesic, and hence it is not very clear how to implement the scale factor cutoff.

Information about an event in the 4-dimensional spacetime travels to the future infinity along null and timelike geodesics. In the case of events occurring in a dS vacuum, such as perhaps our own, this information is represented in a region within the comoving horizon of that event on the future boundary. For example, one can expect that the collapse of a protocloud resulting in galaxy formation will be encoded in the field values of the entire comoving horizon region, rather than being localized near the comoving location of the
galaxy. This seems to suggest that the boundary measure is not likely to be influenced by the local situation in the vicinity of the galaxy, as it would be with the version of the scale factor cutoff measure adopted in [11]. Instead, we may expect that in this case the boundary measure will be well approximated by the scale factor cutoff, with a geodesic congruence based on the metric smoothed on the scale of the horizon.

V. CONCLUSIONS AND DISCUSSION

We have argued that the dynamics of the inflationary multiverse may have a dual description in the form of a lower dimensional Euclidean field theory defined at the future infinity. The measure of the multiverse can then be defined by imposing a Wilsonian ultraviolet cutoff $\xi$ in that theory. In the limit of $\xi \to 0$, the boundary theory becomes conformally invariant, approaching a UV fixed point.

On super-horizon scales, the UV cutoff $\xi$ corresponds to an infrared (late time) scale factor cutoff in the bulk theory, and the renormalization group flow corresponds to the scale factor time evolution. The asymptotic scale invariance of the boundary theory is reflected in the late-time attractor behavior of eternal inflation. The correspondence between the boundary measure and scale factor cutoff is not precise on sub-horizon scales, but it is expected to hold approximately if the scale factor is defined using the metric suitably averaged over the horizon.

The proposal we have outlined in this paper is only a sketch of the boundary measure, with a number of open questions left for future research. One of these questions is related to the geodesic crossing. In order to avoid geodesic crossing, we assumed that our geodesic congruences are constructed from a metric averaged over the structure formation scale. We have also ignored the focusing of geodesics as they go through domain walls separating different phases. Such approximations appear to be out of place in a fundamental theory. In fact, it is not clear to what extent geodesic congruences are necessary for our construction. They are of course necessary to establish the correspondence with the scale factor measure, but one can hope that the duality between the bulk and boundary theories and the boundary measure can be formulated entirely in the framework of field theory, without reference to geodesic congruences.

The future infinity, where the boundary theory is defined, is a fractal set consisting of
infinitely fine “sponges” representing different inflating vacua. At any finite resolution $\xi$, each sponge is a 3D manifold, which is bounded by its borders with other sponges and by the boundaries of terminal bubbles. The latter boundaries, as well as the sponges themselves, are sites of (asymptotically) conformal field theories; the corresponding central charges have been estimated in [25, 26]. In the limit of $\xi \to 0$, the sponges become self-similar fractals of dimension $d_S < 3$ and their boundaries become fractals of $d_B > 2$, due to bubble collisions. (For low bubble nucleation rates, which is usually the case, $d_S$ and $d_B$ are very close to 3 and 2, respectively.) It would be very interesting to see if the boundary theory can be defined directly on this fractal set. If so, it will have to be a rather unconventional field theory.

We note, finally, that the detailed dynamics of the boundary theory may not be needed in order to apply the boundary measure. The advantage of duality is precisely that calculations can be done in the bulk, where the theory is weakly coupled. To make use of this procedure, it suffices to find out what are the asymptotic co-moving wavelengths carrying the information about the process of our interest. Technically, this may be more or less complicated depending on the process. But in principle, one should be able to determine it from standard bulk physics.

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