On the position representation of mechanical power, force and torque operators

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Quantizing the transfer of energy and momentum between interacting particles, we obtain a quantum impulse equation and relations that the corresponding mechanical power, force and torque satisfy. In addition to the energy-frequency and momentum-wavelength relations, we introduce the power-frequency and force-wavelength analogs, respectively. Further, we obtain an operator representation for the mechanical power and impact force in the position space and discuss their correspondence with the relevant momentum operator. The position representation of the torque operator and its relation to the orbital angular momentum operator is also considered. The results are grounded by the presence of a constant that appears as fundamental as the Planck's constant to all obtained relations.

I. INTRODUCTION

Quantum mechanics is considered as one of the greatest achievements in theoretical physics [1–3]. Describing phenomena governed by principles beyond those defining the macroscopic world, it would always raise discussions on a possible interdependence with classical mechanics [4–13]. The foundations of quantum mechanics are well established and verified in time [14–19]. In that respect, the quantum theory appears as an invaluable tool for interpreting experimental findings from the nano-scale world and thus being a ground for vast number of researches accounting for different atomic and subatomic processes. Some prominent examples in the field of condensed matter physics are the photon-electron interactions [20–22], the electrons' interactions that underpin the magnetic properties of matter [23–28] and different transport phenomena [29–34]. Furthermore, determining the time response of the electrons [35] and the time that it takes for quantum jumps to occur [36] are really fascinating and promising achievements of quantum physics nowadays. From the perspective of present technological progress, the possibility of observing and controlling quantum effects such as tunneling [37–39] and entanglement [40–44] promises a number of future applications in the field of logical devices [45–50].

Whether one studies problems in the field of fundamental or applied quantum mechanics the definition of the Hamiltonian and momentum of all constituent units of a quantum system is a key point in understanding the corresponding dynamics. On the other hand, the knowledge about all processes of energy and momentum transfer and the rate at which these processes take place over time may strongly enrich the study. Understanding the dynamics of energy and momentum transfer requires additional investigation about the observables such as the mechanical power that a certain interaction generate and the applied forces. In that respect, it is worth mention that the existence of Hermitian linear operators representing these dynamical observables in the coordinate space is an unexplored question. Respectively, the impact of such representations in all fields of quantum physics is still obscure.

The present article discuss the quantization of energy and momentum transfer processes between interacting particles within the standard quantum mechanics. Accordingly, we report a possible lower limit for the rate at which the processes take place over time and thus fundamental relations that the relevant dynamical quantities such as the mechanical power, force and torque satisfy. In particular, we represent the Planck–Einstein and de Broglie relations in terms of the mechanical power and force, respectively. We demonstrate that the impulse-momentum theorem for quantum-scale objects is grounded by an impulse equation that has no classical analogue. Further, we introduce the position representation of the operators associated to the respective power, force and torque and establish the commutation relations these observables obey. The uncertainty in mechanical power and force is also studied. A key role within all results plays a constant that has a unit measure of energy. It emerges naturally in the theory and has as fundamental part in all equations as the Planck's constant. The Planck's constant, however, remains unique in terms of the study discussed in Refs. [16, 51, 52].

The rest of this paper is organized as follows: In Section II we briefly review the fundamental approaches and related equations used as a ground in the present research; In Section III we introduce the main equation and all originating relations; The results are further discussed in Section IV within an example of a system in which there is an absence of non-contact forces; Section V includes summary of the obtained results.

II. GENERALITIES

A. Classical observables

The Lagrangian and Hamiltonian mechanics are among the most successfully applied approaches when studying the dynamics of an arbitrary mechanical system. Although their consideration usually includes generalized variables, henceforth, with respect to the objective to be followed, all functions and relations are given in terms of Cartesian coordinates.

For an isolated system consisting of a collection of point
particles, the Lagrangian \( L = L(r_i, \dot{r}_i) \), where \( r_i = (r_{1i}, r_{2i}, \ldots) \) are the position vectors of the particles and \( \dot{r}_i = (\dot{r}_{1i}, \dot{r}_{2i}, \ldots) \) are the respective velocities, collects all the knowledge needed to determine how the considered system evolves over time. Respectively, predicting the particles configuration at any given moment of time, one has to solve the Euler–Lagrange equations. On the other hand, knowing the relevant momenta \( p_i = (p_{1i}, p_{2i}, \ldots) \), the evolution in time can be studied with the aid of the Hamiltonian mechanics by using the Hamiltonian \( H = H(r_i, p_i) \) and solving the Hamiltonian equations. Both, the Lagrangian and Hamiltonian approaches are related such that one can determine the total energy of the system by accounting for the relation

\[
H = \sum_i \dot{r}_i \cdot p_i - L. \tag{2.1}
\]

Although knowing the total energy is usually sufficient in determining the dynamical state of a mechanical system, studying the rate at which the transfer of energy and momentum between the interacting particles takes place with time may strongly enrich the study. To this end, in addition to (2.1), one has to account for the mechanical power \( P \) and the respective net forces \( F_i = \partial_i H, L \) satisfying the relation

\[
P = \sum_i F_i \cdot \dot{r}_i - W, \tag{2.2}
\]

where \( W \equiv \dot{H} \) gives the rate at which the system's energy changes with time. For closed systems \( W \) is a zero function and it is included in equation (2.2) for the sake of clarity. Further, the impulse associated to the \( i \)-th particle reads

\[
J_i = \int_{T_i} F_i \, dt, \tag{2.3}
\]

where the domain \( T_i \) defines the time interval within which the \( i \)-th force acts. In the processes of elastic collision, for all \( i \), the impulse is distributed over an infinitesimal short time-interval \( \lambda_i \in T_i \). Accordingly, \( F_i \) represents an impact force given by \( F_i = J_i \delta(t - \lambda_i) \).

### B. Quantum observables

In quantum mechanics, the information about the dynamics of a multi particle system, with position vectors \( r_i = (r_{1i}, r_{2i}, \ldots) \), is embodied in a state function \( \Psi = \Psi(r_i, t) \), where \( ||\Psi||^2 = 1 \). Each state function is related to the probability of observing the system in a particular dynamical state characterized by a number of physical quantities with a given uncertainty in their values. Satisfying the Schrodinger equation

\[
i\hbar \partial_t \Psi(r_i, t) = \hat{H} \Psi(r_i, t), \tag{2.4}
\]

the explicit representation of \( \Psi(r_i, t) \) is related to the Hamiltonian's representation, which in turn depends on the introduced interactions and hence the way the system is being manipulated, i.e. observed. Respectively, depending on how one interacts with the system's constituents, the latter may demonstrate either their wave-like or point-like character.

The wave-like character of the collective point-like particles' dynamics described by (2.1) and (2.2) become apparent on the very frontier between the classical and quantum theoretical approaches of representing observables. One prominent example that represents the wave-particle duality concept is a system composed of free particles. In this case, the representation of the Hamiltonian in (2.4) maps its classical counterpart in (2.1) accordingly. Thus, we have

\[
\hat{H} \Psi_p(r_i, t) = \sum_i \dot{r}_i \cdot \hat{p}_i \Psi_p(r_i, t) - \hat{L} \Psi_p(r_i, t), \tag{2.5}
\]

where in the absence of spin degrees of freedom, the state function \( \Psi_p(r_i, t) \) is represented as a product of plane waves each with phase given by

\[
\phi_i = \hbar^{-1}(\hat{p}_i \cdot \hat{r}_i - E_i t), \tag{2.6}
\]

\( E_i \)-denotes the \( i \)-th particle's kinetic energy. Respectively, we have

\[
\hat{H} \Psi_p(r_i, t) = \sum_i E_i \Psi_p(r_i, t), \quad \hat{p}_i \Psi_p(r_i, t) = \hat{p}_i \Psi_p(r_i, t), \quad \hat{L} \Psi_p(r_i, t) = \hat{L} \Psi_p(r_i, t), \tag{2.7}
\]

where Lagrangian \( L \) and the associated operator are a functions only of a kinetic terms and take into account both relativistic and non-relativistic cases. As an example, for a system composed of massless particles one has \( \hat{L} \Psi_p(r_i, t) = 0 \) and \( |\Psi_p| = c \) for all \( i \), where \( c \) is the light speed in vacuum.

The phase associated to the \( i \)-th particle is further given by

\[
\phi_i = k_i \cdot \hat{r}_i - \omega_i t, \tag{2.8}
\]

where \( \omega_i \) and \( k_i \) are the corresponding angular frequency and wave vector, respectively. Both (2.6) and (2.8) satisfy equation (2.5) simultaneously and stand as a ground for the Planck–Einstein and de Broglie relations given by

\[
E_i = \hbar \omega_i, \quad p_i = \hbar k_i, \tag{2.9}
\]

respectively, that marked the origin of quantum theory. It is essential to emphasize that since \( \Psi_p(r_i, t) \) represents a plane wave, the transformation of equation (2.5) into (2.1) holds under no semi-classical approximation.

### III. Mechanical power, force and torque as quantum observables

In addition to the representation of energy, momentum and orbital angular momentum in quantum mechanics, equation (2.2) can be used as a ground for deriving the relevant representations of mechanical power, force and torque. Since the power-force relation in (2.2) doesn't depend on the explicit representation of the Lagrangian in (2.1), its quantum mechanical counterpart will be adequate to any type
of interactions. Therefore, for an arbitrary Hamiltonian representation in (2.4), the quantum mechanical analogue of (2.2) reads
\[ \hat{P}_i \Psi_F(r_i, t) = \sum_l r_l \cdot \hat{F}_l \Psi_F(r_i, t) - \hat{W} \Psi_F(r_i, t), \tag{3.1} \]
where the state function satisfies the following equations
\[ \hat{P}_i \Psi_F(r_i, t) = \sum_l P_l \Psi_F(r_i, t), \quad \hat{F}_l \Psi_F(r_i, t) = F_l \Psi_F(r_i, t), \]
\[ \hat{W} \Psi_F(r_i, t) = W \Psi_F(r_i, t). \tag{3.2} \]

Hence, \( \Psi_F(r_i, t) \) is given as a direct product of plane waves, such that the \( i \)-th one is characterized by the phase
\[ \varphi_i = e^{-1}(F_i \cdot r_i - P_i t), \tag{3.3} \]
where \( e \) is a constant that has the unit measure of energy. From (3.1) follows that, for any system, in addition to the eigenstates in (2.4), there exists a state function \( \Psi_F(r_i, t) \) associating to each individual process of energy and momentum transfer a quanta described by a plane wave. Accordingly, the energy and momentum of the \( i \)-th quanta should be equal to the change in kinetic energy \( \delta E_i \) and momentum \( \delta p_i \) of the \( i \)-th particle within the time-interval that the applied forces act. Therefore, for all \( i \), the corresponding quanta has energy \( \delta E_i \equiv \delta \varphi_i = \hbar \partial \varphi_i \) and momentum \( \delta p_i \equiv p_i = \hbar \kappa_i \), where \( \varphi_i = \delta \varphi_i \) and \( \kappa_i = \delta \kappa_i \) represent the change in associated to the \( i \)-th particle angular frequency and wave vector, respectively. Therefore, the phase in (3.3) is further given by
\[ \varphi_i = \hbar^{-1}(p_i \cdot r_i - \varphi_i t) \tag{3.4} \]
and hence,
\[ \varphi_i = (\kappa_i \cdot r_i - \varphi_i t). \tag{3.5} \]
As the total energy of the system is conserved the change in total momentum is zero \( \sum_i \delta p_i = 0 \) and the sum \( \sum_i \delta \varphi_i \) equals the change in total kinetic energy.

A. Power-frequency and force-momentum relations

Representing the de Broglie hypothesis, equation (3.1) leads to an additional Planck–Einstein relation. Taking into account (3.3) and (3.5), for all \( i \), we obtain two fundamental relations representing the mechanical power \( P_i \) and force \( F_i \) as a function of the frequency \( \varphi_i \) and the vector \( \kappa_i \), respectively. Thus, we have
\[ P_i = \varepsilon \varphi_i, \quad F_i = \varepsilon \kappa_i. \tag{3.6} \]
The last relations show that the greater the frequency \( \varphi_i \) associated to the \( i \)-th quanta the greater the mechanical power generated in the process of energy transfer. Accordingly, the greater the change in momentum \( \delta p_i = \hbar \kappa_i \) the greater the force acting on the \( i \)-th particle.

Equations (3.6) lead to alternative with respect to the classical dynamics force-momentum relation and formulation of the mechanical power. Taking into account relations (3.4) and (3.6), we get
\[ P_i = \frac{\varepsilon}{\hbar} \delta E_i, \quad F_i = \frac{\varepsilon}{\hbar} \delta p_i, \tag{3.7} \]
where the constant \( \hbar/\varepsilon \) has a unit measure of time. In contrast to the classical dynamics and the Ehrenfest’s theorem, relations (3.7) give a fundamental limit for the rate at which the processes of energy and momentum transfer may happened under the discussed conditions. In other words, on a quantum level the time-interval for transfer energy and momentum when particles interact, appears as a fundamental constant equal to \( \hbar/\varepsilon \).

We would like to point out that, in the case the \( i \)-th particle’s momentum changes \( N \gg N_A \) times, where \( N_A \) is the Avogadro constant, the force-momentum relation can be rewritten according to the substitutions \( \rho_i \to \hbar/\varepsilon \) and \((\hbar/\varepsilon)\,dN \to df\).

B. Impulse-momentum theorem and torque

The power and force given in (3.7) depend only on the change in particle’s energy and momentum, respectively. As a result, for all \( i \), the impulse equation associated to the relation on the right hand side in (3.7) reads
\[ J_i = \frac{\hbar}{\varepsilon} F_i. \tag{3.8} \]
The last equation represents the Impulse-momentum theorem within the concept of wave-particle duality. In contrast to (2.3), the impulse in (3.8) is defined for a discrete time-interval \( \hbar/\varepsilon \).

In addition to relations (3.7) we can further obtain an expression for the torque applied to the \( i \)-th particle. Denoting its orbital angular momentum by \( I_i \), and the applied torque by \( \tau_i \), from the right hand side of equation (3.7) we have
\[ \tau_i = \frac{\varepsilon}{\hbar} \delta I_i. \tag{3.9} \]
As the transfer of a momentum between particles is restricted in time, the time that it takes for the relevant orbital angular momentum to change is also a constant equals \( \hbar/\varepsilon \). We would like to point out that the total orbital angular momentum is conserved and hence with respect to (3.4) the orbital angular momentum of the associated quanta reads \( I_i = \delta I_i \).

IV. IMPACT FORCE: ZERO INITIAL OR FINAL MOMENTUM

In reference to the system described by (2.5), for all \( i \), the observables \( F_i \) in (3.6) represent impact forces occurring due to incident collisions between the constituent particles. Therefore, interacting once the \( i \)-th particle’s initial energy
and momentum change. In that respect, suggesting a zero initial or final momentum, depending on whether the particle starts form rest or transfer all of its momentum, respectively, equations (3.6) reduce to

\[ P_i = \varepsilon \omega_i, \quad F_i = \varepsilon k_i. \] (4.1)

Moreover, equations (3.7) and (3.9) read

\[ P_i = \frac{\varepsilon}{\hbar} E_i, \quad F_i = \frac{\varepsilon}{\hbar} p_i, \quad \tau_i = \frac{\varepsilon}{\hbar} I_i. \] (4.2)

One then may ask is there any relation that for all \( i \), \( P_i \) satisfies in the case of zero momentum. Answering that question, we rewrite the representations (2.8), (2.8) and (3.3) in terms of the four-momentum, four-wavevector and four-force, respectively. Accordingly, the force-momentum relation in (4.2) can be rewritten in Minkowski space, where the contravariant four-momentum is given by \( p_i^\mu = (E_i/c, p_i) \). The corresponding relativistic relation then reads

\[ \mathcal{F}_i = \frac{\varepsilon}{\hbar} p_i^\mu, \] (4.3)

where \( \mathcal{F}_i = (P_i/c, F_i) \) is the respective contravariant four-force. With the aid of equation (4.3) we obtain expression for the mechanical power that the \( i \)-th particle with rest mass \( m_i \) may apply interacting with other constituents form the considered system,

\[ P_i = \sqrt{p_i^2 c^2 + \frac{\varepsilon^2}{\hbar^2} m_i^2 c^4}, \] (4.4)

where \( F_i = |F_i| \). Hence, form (4.4) follows that for the \( i \)-th particle being at rest, \( P_i \) will depend only on the particle’s rest mass \( m_i \). For example, when that condition is satisfied, we have

\[ P_i = \frac{\varepsilon c^2}{\hbar} m_i. \]

The last relation gives the exact amount of mechanical power that will be generated if in the process of interaction the \( i \)-th particle loses all of its rest mass within the time-interval \( \hbar/\varepsilon \). The existence of such fundamental limit is due to the presence of the constant \( \varepsilon \) according to which relations (3.7) and hence (4.2) are defined.

### A. Time evolution and position representation

The energy and momentum transfer in a physical system is the main indication for its evolution. Since the power is an observable that accounts for such a processes in addition to the Hamiltonian of a quantum mechanical system, one can rely on the respective power operator given on the left hand side in equation (3.2). Therefore, in the framework of the considered case described by equations (4.2), in complement to the Schrodinger equation (2.4), we have

\[ i \varepsilon \partial_t \psi(r_i, t) = \hat{P} \psi(r_i, t). \] (4.5)

The operator \( \hat{P} \) commute with the respective Hamiltonian \( \hat{H} \) such that for \( \hat{H} \psi(r_i, t) = \sum_i E_i \psi(r_i, t) \) and all \( i \), one obtains the relation on the left hand side in (4.2). Nevertheless, \( P \) has to be treated in accordance to the problem one is solving. For example, let the \( i \)-th particle is at rest and represents an independent quantum harmonic oscillator with frequency \( \omega_i \). Further, consider a hypothetical collision in which the \( i \)-th particle transfers all of its energy. Then, the maximum mechanical power that can be generated within the considered interaction is given by

\[ P_{n_i} = \varepsilon \omega_i \left( n_i + \frac{1}{2} \right), \quad n_i \in \mathbb{N}_0. \] (4.6)

On the other hand, in the case of (4.2), the position representation of \( \hat{P}_i \) reads

\[ \hat{P}_i \equiv -\frac{i \varepsilon}{\hbar} \nabla_i. \] (4.7)

The components of the operator in (4.7) obey the commutation relation

\[ [\hat{P}_i, \hat{P}_j^\nu] = i \varepsilon \delta_{\beta \nu}, \quad \beta, \nu \in \mathbb{K} \]

and commute with each one component of the relevant momentum operator. We would like to point out that in the considered case the kinetic energy of the system is conserved. Thus, one can includes only potential energy operators that commute with each particle’s momentum operator. In that respect, it is worth giving the time derivative of the force operator,

\[ d_t F_i = \frac{\varepsilon}{\hbar} d_t \hat{P}_i, \] (4.8)

where taking into account the Ehrenfest’s theorem we get

\[ d_t \langle \hat{F}_i^\nu \rangle = \frac{i \varepsilon}{\hbar} \langle [\hat{P}_i^\nu, \hat{H}] \rangle + \langle [\hat{P}_i^\nu, \hat{H}] \rangle, \quad \nu \in \mathbb{K} \]

and

\[ d_t \langle \hat{F}_i^\nu \rangle = -i \frac{\varepsilon}{\hbar} \langle [\hat{F}_i^\nu, \hat{H}] \rangle + \langle [\hat{F}_i^\nu, \hat{H}] \rangle. \]

Hence, \( d_t \langle \hat{F}_i \rangle = 0 \), since the system exhibits only a random elastic collisions of non-potential interactions.

The observable \( \tau_i \) gives the acquired orbital angular momentum of the \( i \)-th particle due to the applied impact force \( F_i \). It is represented by a three component operator \( \hat{F}_i = (\hat{F}_i^\nu)_{\nu \in \mathbb{K}} \) and similar to the orbital and spin angular momenta, its components satisfy the algebra

\[ [\hat{F}_i^\nu, \hat{F}_j^\beta] = i \varepsilon \epsilon_{\gamma \beta \nu} \hat{F}_i^\gamma, \quad \gamma, \beta, \nu \in \mathbb{K}. \] (4.9)
In coordinate space the torque has the following representation
\[ \hat{\tau}_i = -i\hbar \langle \hat{r}_i \times \nabla \rangle, \]
where \( \hat{r}_i \) is the position operator associated with the \( i \)-th particle space coordinates. As the operator in (4.7) commute with the relevant momentum operator, the following commutation relations hold
\[ [\hat{P}_i, \hat{P}^\dagger_j] = i\hbar \delta_{ij}, \]
where \( \hat{P}_i = (\hat{P}_i)_{\nu \varepsilon} \) is the corresponding orbital angular momentum operator. Therefore, the magnitude of the torque applied to the \( i \)-th particle, which acquires an orbital angular momentum with quantum number \( l_i \), reads \( |\tau_i| = \varepsilon \sqrt{\tau_i(\tau_i + 1)} \), where \( \tau_i \equiv l_i \). The equation giving the rate at which the corresponding orbital angular momentum changes with time is analogous to the equation (4.8). It is written as
\[ d_t \hat{\tau}_i = \frac{\varepsilon}{\hbar} d_\lambda \hat{I}_i \]
and shows us that any variation in the orbital angular momentum corresponds to a change of the relevant torque by a constant of proportionality \( \hbar / \varepsilon \).

In the relativistic case, the expression for the four-force operator follows from equation (4.3) such that one has
\[ \hat{F}^\mu_i = i\epsilon \partial^\mu_i \]
where the derivative \( \partial^\mu = (\partial_\nu, -\nabla) \).

\section*{B. Uncertainty relations}

With respect to the operator representations of the mechanical power and impact force discussed in Sec. IV A, the Heisenberg uncertainty principle applies. For all \( i \) and \( \nu \in \mathbb{N} \), one obtains the following inequalities
\[ \Delta v_i \Delta F_i^\nu \geq \frac{\varepsilon}{2} \]
and
\[ \Delta t \Delta p_i \geq \frac{\varepsilon}{2}. \]

Therefore, gaining a knowledge for the position of a particle leads to a lack of information for the applied force and hence the change in its momentum. Furthermore, the greater the energy that a particle may transfer the greater the uncertainty in power.

\section*{C. Electromagnetic field example}

Expressing the free-space electromagnetic field operators in terms of (3.3), in addition to the Hamiltonian and total momentum operators of the electromagnetic field, we obtain the power operator
\[ \hat{P} = \sum_{k, \sigma} \varepsilon \omega_k \hat{n}_{k, \sigma}, \]
where the number operator \( \hat{n}_{k, \sigma} \) gives the number of photons in a mode determined by the wave vector \( k \) and the polarization index \( \sigma \). The operator in (4.11) gives the maximum mechanical power generated when creating the photons \( |(k, \sigma)^n, . . . , (k', \sigma')^\eta \rangle \) form the vacuum |0\rangle. Respectively, the operator
\[ \hat{F} = \sum_{k, \sigma} \varepsilon k \hat{n}_{k, \sigma}, \]
describes the same processes in terms of applied force. In other words, it gives the applied force during the creation of the aforementioned number of photons within the time-interval \( \hbar / \varepsilon \).

\section*{V. CONCLUSION}

Studying the processes of energy and momentum transfer between interacting particles in quantum mechanics, we obtain a lower limit for the duration of these processes. Accordingly, the change in kinetic energy and momentum of a particle is quantized. Introducing a quanta for the transfer of energy and momentum leads to a unique representation of the impulse-momentum theorem and hence relations for the respective mechanical power, force and torque. A key role in all derived relations plays a constant, denoted accordingly by \( \varepsilon \), that has a unit measure of energy and appears as fundamental as the Planck’s constant in the theory. In particular, \( \varepsilon \) is the proportionality constant between the mechanical power of a quanta of interaction and the frequency associated to that quanta, see equation (3.6). The presence of the named constant points out to the aforementioned lower limit for the rate at which the relevant energy and momentum transfer takes place over time, see equations (3.7) and (3.8).

In conclusion, studying the role of \( \varepsilon \) in physics more closer, we definitely have to test its uniqueness and make efforts in determining its value.

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