Eulerian Posets

**Definition**

A graded poset $P$ with a minimal and maximal element, denoted $\hat{0}$ and $\hat{1}$ respectively, is *Eulerian* if for all $x < y \in P$ we have

$$\sum_{x \leq z \leq y} (-1)^{\text{rk}(z)} = 0$$

Equivalently if $\mu(x, y) = (-1)^{\text{rk}(y) - \text{rk}(x)}$.

Examples include

- face lattices of polytopes
- face posets of regular CW spheres
- intervals in the Bruhat orders of Coxeter systems
- the lattices of regions of oriented matroids
The uncrossing poset

- rank \( \binom{n}{2} + 1 \)
- \((2n - 1)!! + 1\) elements
- \(n\)th Catalan number many atoms

**Theorem (Lam)**

*The uncrossing poset is Eulerian.*

**Theorem (Hersh-Kenyon)**

*The uncrossing poset is shellable, moreover it is a CW poset.*
A lattice is a poset $\mathcal{L}$ in which for all $x, y \in \mathcal{L}$ there exists a least upper bound or join $(x \vee y)$ and greatest lower bound or meet $(x \wedge y)$. In other words the join and meet satisfy:

\[
x \vee y \geq x, y \text{ and } z \geq x, y \Rightarrow z \geq x \vee y
\]
\[
x \wedge y \leq x, y \text{ and } z \leq x, y \Rightarrow z \leq x \wedge y
\]

A *join irreducible* of a lattice is an element which covers exactly one element.

Let $\text{irr}(\mathcal{L})$ denote the set of join irreducibles of $\mathcal{L}$. 
Definition

A set $X$ is a generating set for a lattice $\mathcal{L}$ if

- $\hat{0}_{\mathcal{L}} \in X$
- for all $\ell \in \mathcal{L}$, $\ell = x_1 \vee \ldots \vee x_k$ for some $x_i \in X$

Denote the lattice generated by $X$ as $\langle X \rangle$.

Definition

Let $\mathcal{L}$ be a lattice with generating set $\text{gen}(\mathcal{L})$, and $I \subseteq \text{gen}(\mathcal{L}) \setminus \{\hat{0}_{\mathcal{L}}\}$.

- The contraction of $\mathcal{L}$ by $I$ is $\mathcal{L}/I = \langle j \vee \bigvee_{i \in I} i : j \in \text{gen}(\mathcal{L}) \rangle$.
- The deletion of $\mathcal{L}$ by $I$ is $\mathcal{L} \setminus I = \langle \text{gen}(\mathcal{L}) \setminus I \rangle$.

A minor of $\mathcal{L}$ is a lattice with generating set which is obtained from $\mathcal{L}$ by some sequence of deletions and contractions.

For a minor $M$ denote the generating set by $\text{gen}(M)$. 
\[ \mathcal{L} = \langle \hat{0}, 1, 2, 3, 4 \rangle \]

\[ 1234 \]

\[ \mathcal{L} \setminus 1 = \langle \hat{0}, 2, 3, 4 \rangle \]

\[ 1234 \]

\[ \hat{0} \]

\[ \hat{0} \]

\[ \mathcal{L} / 1 = \langle 1, 12, 13, 1234 \rangle \]

\[ 1234 \]

\[ \mathcal{L} / 1 \setminus (1234) = \langle 1, 12, 13 \rangle \]

\[ 1234 \]

\[ 12 \]

\[ 13 \]

\[ 1 \]

\[ 24 \]

\[ 34 \]

\[ 2 \]

\[ 3 \]

\[ 4 \]

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Proposition

Let $G$ be a vertex labelled graph and $\mathcal{L}(G)$ its lattice of flats.

simple vertex labelled minors of $G \leftrightarrow$ minors of $\mathcal{L}(G)$

The bijection is $H \mapsto \mathcal{L}(H)$. 
The poset of minors

Definition
Let $\mathcal{L}$ be a lattice with generating set. For minors of $\mathcal{L}$ define a partial order by

\[ M_1 \leq M_2 \text{ if and only if } M_1 \text{ is a minor of } M_2 \]

Let $M_\mathcal{L}$ be the poset of minors of $\mathcal{L}$ together with a minimal element denoted $\emptyset$. 
Some basic results

Proposition

- The atoms of $M_L$ are the elements of $L$.
- $M_L$ is graded by $\text{rk}(M) = \# \text{gen}(M)$.
- The rank 2 elements of $M_L$ are minors $\langle x, x \lor i \rangle$ where $x \in L$ and $i \in \text{gen}(L)$ such that $i \not\leq x$.
- $M_L$ is thin (every rank 2 interval has 4 elements)

Proposition

Let $L$ be a lattice and $\text{gen}(L) = \{\widehat{0}_L, \ell_1, \ldots, \ell_n\}$. Let $\theta : B_n \to L$ be defined by $\theta(X) = \bigvee_{x \in X} \ell_x$.

The minors of $L$ are given by $\langle \theta(X) : X \in \text{irr}(I) \cup \{\widehat{0}_I\} \rangle$ where $I$ is an interval of $B_n$.

In particular the minors of $B_n$ are the intervals.
Some basic results

Proposition

Let $\mathcal{L}$ be a lattice with $n + 1$ generators. The interval $[\langle 0 \rangle, \mathcal{L}]$ in $M_{\mathcal{L}}$ is isomorphic to $B_n$. The subposet of $M_{\mathcal{L}}$ consisting of contractions of $\mathcal{L}$ is isomorphic to $\mathcal{L}^*$.

Proposition

1. The minor poset $M_{B_n}$ is isomorphic to the face lattice $Q_n$ of the $n$-dimensional cube.
2. Let $C_n$ be the length $n$ chain. The minor poset $M_{C_n}$ is isomorphic to $B_{n+1}$.

$C_3 = \langle \hat{0}, 1, 2, 3 \rangle \quad C_3 \setminus 1 = \langle \hat{0}, 2, 3 \rangle \quad C_3 / 1 = \langle 1, 2, 3 \rangle$

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PL spheres

Definition

Given a poset $P$ the order complex $\Delta(P)$ is the simplicial complex consisting of all chains in $P$.

A poset $P$ is said to be a PL sphere if there is a piecewise linear homeomorphism from $\Delta(P \setminus \{\hat{0}, \hat{1}\})$ to the boundary of a simplex.

Examples:

- Face lattices of polytopes
- Intervals of Bruhat orders [Reading]

Theorem (?)

If $P$ is a PL sphere then every interval of $P$ is a PL sphere.
The zip operation

An element \( z \in P \) is said to be a zipper if
- \( z \) only covers two elements \( x \) and \( y \)
- \( \{ p \in P : p < x \} = \{ p \in P : p < y \} \)
- \( z = x \lor y \)

\( \text{zip}(P, z) \) is the poset obtained from \( P \) by identifying \( x, y \) and \( z \).
The zip operation preserves Eulerianness and PL sphericity.
The main theorem

**Theorem**

Let \( \mathcal{L} \) and \( \mathcal{K} \) be lattices with generating sets such that there is a join preserving surjection from \( \mathcal{L} \) onto \( \mathcal{K} \) which descends to a surjection from \( \text{gen}(\mathcal{L}) \) onto \( \text{gen}(\mathcal{K}) \).

The poset of minors \( M_\mathcal{K} \) can be obtained from \( M_\mathcal{L} \) by a sequence of zip operations.

**Corollary**

For any lattice \( \mathcal{L} \) with generating set the poset of minors \( M_\mathcal{L} \) is Eulerian and a PL sphere.
Proof.

- $f : \mathcal{L} \to \mathcal{K}$ join preserving,
  
  $f(\text{gen}(\mathcal{L})) = \text{gen}(\mathcal{K})$

- View $f$ as a congruence via $x \equiv y(f) \iff f(x) = f(y)$

- Define a partial order on edges $(x_1 < y_1) \leq (x_2 < y_2) \iff x_1 = x_2 \lor z$ and $y_1 = y_2 \lor z$.

- $\equiv$ preserves joins if and only if the edges $x < y$ with $x \equiv y$ form a lower order ideal
Proof.

- \( F : M_\mathcal{L} \to M_\mathcal{K} \) \( F(\langle X \rangle) = \langle f(X) \rangle \)
- Nontrivial fibers of \( F \) are \( M, M_x, M_y \) where
  - \( x, y \in \text{gen}(M) \)
  - \( M_y = M \setminus y \)
  - \( M_x = \begin{cases} M/y & x = \hat{0}_M \\ M \setminus x & x \neq \hat{0}_M \end{cases} \)
- Each \( M \) is a zipper when these are zipped rank by rank
The cd-index

Let $P$ be a rank $n + 1$ poset with $\widehat{0}$ and $\widehat{1}$. Let $a$ and $b$ be noncommutative variables.

**Definition**

Let $C$ be a chain in $P$ which contains $\widehat{0}$ and $\widehat{1}$. Define $w(C) = w_1 \cdot \ldots \cdot w_n$ by

$$w_i(C) = \begin{cases} b & C \text{ goes through rank } i \\ (a - b) & C \text{ does not go through rank } i \end{cases}$$

**Definition**

The *ab-index* of $P$ is the polynomial

$$\Psi(P) = \sum_C w(C)$$

If $\Psi(P)$ is a polynomial in $c = a + b$ and $d = ab + ba$ then this polynomial is the *cd-index* of $P$ (also denoted $\Psi(P)$).
The cd-index continued

**Theorem (Bayer-Billera)**

If \( P \) is an Eulerian poset then it has a cd-index.

**Example**

Let \( P = B_3 \).

| \( \emptyset < 123 \) | \( (a - b)^2 \) |
|---------------------|----------------|
| \( \emptyset < 1 < 123, \emptyset < 2 < 123, \emptyset < 3 < 123 \) | \( b(a - b) \) |
| \( \emptyset < 12 < 123, \emptyset < 13 < 123, \emptyset < 23 < 123 \) | \( (a - b)b \) |
| \( \emptyset < 1 < 12 < 123, \emptyset < 1 < 13 < 123, \emptyset < 2 < 12 < 123 \) | \( b^2 \) |
| \( \emptyset < 2 < 23 < 123, \emptyset < 3 < 13 < 123, \emptyset < 3 < 23 < 123 \) | \( b^2 \) |

\[
\Psi(B_3) = (a - b)^2 + 3b(a - b) + 3(a - b)b + 6b^2
\]

\[
= a^2 + 2ba + 2ab + b^2 = c^2 + d
\]
The cd-index continued

**Theorem (Karu)**

The cd-index of a Gorenstein* poset has nonnegative coefficients. In particular the cd-index of an Eulerian spherical poset is nonnegative.

**Theorem (Reading)**

Let $z \succ x, y$ be a zipper in $P$, with $z \neq \hat{1}$. If $P$ is Eulerian then

$$
\Psi(\text{zip}(P, z)) = \Psi(P) - \Psi([\hat{0}, x]) \cdot d \cdot \Psi([z, \hat{1}])
$$

**Remark**

If $z = \hat{1}$ then $\Psi(P) = \Psi(\text{zip}(P, z)) \cdot c$
A corollary to the main theorem

**Corollary**

Let $\mathcal{L}$ and $\mathcal{K}$ be lattices with generating sets such that there is a join preserving surjection from $\mathcal{L}$ onto $\mathcal{K}$ which descends to a surjection from $\text{gen}(\mathcal{L})$ onto $\text{gen}(\mathcal{K})$.

The following inequality on cd-indices is satisfied coefficientwise.

$$\Psi(M_K) \cdot c^{\# \text{gen}(\mathcal{L}) - \# \text{gen}(\mathcal{K})} \leq \Psi(M_\mathcal{L}) \leq \Psi(Q_n)$$

**Example**

Let $\mathcal{L} = \langle 0, a, b, c \rangle$ as before.

$$\Psi(M_\mathcal{L}) = c^3 + 2cd + 3dc$$

$$\Psi(M_{B_3}) = c^3 + 4cd + 6dc$$