A FAMILY OF FLAT CONNECTIONS ON THE PROJECTIVE SPACE HAVING DIHEDRAL MONODROMY AND ALGEBRAIC GARNIER SOLUTIONS

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Abstract. A. Girand constructs an explicit two-parameter family of flat connections over the complex projective plane \( \mathbb{P}^2 \). These connections have dihedral monodromy and their polar locus is a prescribed quintic composed of a conic and three tangent lines. In this paper, we give a generalization of this construction. That is, we construct an explicit \( n \)-parameter family of flat connections over the complex projective space \( \mathbb{P}^n \). Moreover, we discuss the relation between these connections and the Garnier system.

1. Introduction

A meromorphic rank 2 connection \((E, \nabla)\) on a projective manifold \( X \) is the datum of a rank 2 vector bundle \( E \) equipped with a \( \mathbb{C} \)-linear morphism \( \nabla : E \rightarrow E \otimes \Omega^1_X(D) \) satisfying Leibniz rule

\[ \nabla(f \cdot s) = f \cdot \nabla(s) + df \otimes s \]

for any section \( s \) and function \( f \). Here \( D \) is the polar divisor of the connection \( \nabla \). The connection \( \nabla \) is flat when the curvature vanishes, that is \( \nabla \cdot \nabla = 0 \). For a flat meromorphic rank 2 connection, we can define its monodromy representation. When \( \det(E) = \mathcal{O}_X \) and the trace connection \( \text{tr}(\nabla) \) is the trivial connection on \( \mathcal{O}_X \), we say that \((E, \nabla)\) is an \( \mathfrak{sl}_2 \)-connection.

The main purpose of this paper is to construct flat meromorphic \( \mathfrak{sl}_2 \)-connections on the projective space \( \mathbb{P}^n \) explicitly. Now we recall the structure theorem of flat meromorphic \( \mathfrak{sl}_2 \)-connections on projective manifolds due to Loray–Pereira–Touzet. (For in detail, see [1] and [9]). We say that any two connection \((E, \nabla)\) and \((E', \nabla')\) are birationally equivalent when there is a birational bundle transformation \( \phi : E \rightarrow E' \) that conjugates the two operators \( \nabla \) and \( \nabla' \). The connection \((E, \nabla)\) is called regular if local \( \nabla \)-horizontal sections have moderate growth near the polar divisor \( \nabla_\infty \) (see [3, Chap. II, Definition 4.2] for details). We say that any two connection \((E, \nabla)\) and \((E', \nabla')\) are projectively equivalent if the induced \( \mathbb{P}^1 \)-bundles coincide \( \mathbb{P}(E) = \mathbb{P}(E') \), and if moreover \( \nabla \) and \( \nabla' \) induce the same projective connection \( \mathbb{P}(\nabla) = \mathbb{P}(\nabla') \).

Theorem 1.1 ([9 Theorem E]). Let \((E, \nabla)\) be a flat meromorphic \( \mathfrak{sl}_2 \)-connection on a projective manifold \( X \). Then at least one of the following assertions holds true.

1. There exists a generically finite Galois morphism \( f : Y \rightarrow X \) such that \( f^*(E, \nabla) \) is projectively birationally equivalent to one of the following connections defined on the trivial bundle:

\[
\nabla = d + \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix} \quad \text{or} \quad d + \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix}
\]

with \( \omega \) a rational closed 1-form on \( X \).

2. There exists a rational map \( f : X \dashrightarrow C \) to a curve and a meromorphic connection \((E_0, \nabla_0)\) on \( C \) such that \((E, \nabla)\) is projectively birationally equivalent to \( f^*(E_0, \nabla_0) \).

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(iii) The \( \mathfrak{sl}_2 \)-connection \((E, \nabla)\) has at worst regular singularities and there exists a rational map \( f : X \rightarrow \mathcal{S} \) which projectively factors the monodromy through one of the tautological representations of a polydisk Shimura modular orbifold \( \mathcal{S} \). In particular, the monodromy representation of \((E, \nabla)\) is quasi-unipotent at infinity, rigid, and Zariski dense.

We recall the assertion (i) for regular meromorphic connections in more detail. (See [4, Section 7.2]). Let \((E, \nabla)\) be a flat regular meromorphic \( \mathfrak{sl}_2 \)-connection on \( X \). The Riemann–Hilbert correspondence establishes a one-to-one correspondence between representations up to conjugacy and regular connections up to birational bundle transformations. For instance, if its monodromy representation is virtually abelian, i.e. abelian after a finite cover, then it is either diagonal or unipotent after the finite cover. These monodromy representations are realized by the connections as in [1]. From the view point of the assertion (i) in this theorem, we construct explicit flat regular meromorphic \( \mathfrak{sl}_2 \)-connections on the projective space \( \mathbb{P}^n \).

Girand constructs explicit flat regular meromorphic \( \mathfrak{sl}_2 \)-connections over \( \mathbb{P}^2 \) ([6]). The connections have dihedral monodromy representations, which are virtually abelian, and their polar locus is a prescribed quintic composed of a conic and three tangent lines. Our construction is based on his idea. When \( n = 2 \), our explicit connections over \( \mathbb{P}^n \) are coincide with Girand’s explicit connections over \( \mathbb{P}^2 \). Moreover, in [6], the explicit flat connections over \( \mathbb{P}^2 \) give algebraic solutions of the Garnier system. In fact, if we consider the restrictions of the explicit connections over \( \mathbb{P}^2 \) to generic lines on \( \mathbb{P}^2 \), then we have the Fuchsian systems with 5 regular singularities. If we vary parameters of generic lines, then we have isomonodromic families of the Fuchsian systems with 5 regular singularities. The families are parametrized by the space which parametrizes generic lines on \( \mathbb{P}^2 \). The dimension of this space is 2, which is coincide with the dimension of the space of time variables of the Garnier system associated to the isomonodromic deformation of the Fuchsian systems with 5 regular singularities. If we take a finite cover of the space which parametrizes generic lines, then we have a generically finite morphism from the finite cover to the space of time variables of the Garnier system. Then we obtain algebraic solutions of the Garnier system. If we restrict the explicit connections over \( \mathbb{P}^2 \) to special lines on \( \mathbb{P}^2 \), then we can produce algebraic solutions of the Painlevé VI equation, which contain a part of algebraic solutions in [8]. (See [6, Section 3.1]).

1.1. Main result. Let \([x : y : z_1 : \ldots : z_{n-2} : t]\) be the homogeneous coordinates of \( \mathbb{P}^n \). Set \( f(x, y, t) := x^2 + y^2 + t^2 - 2(xy + yt + tx) \). Let \( \mathcal{Q}_0 \) and \( \mathcal{Q}_i \) be the divisors on \( \mathbb{P}^n \) defined by \( \mathcal{Q}_0 := \{ f(x, y, t) = 0 \} \) and \( \mathcal{Q}_i := \{ f(x, y, t) - z_i^2 = 0 \}, \, i = 1, \ldots, n-2 \), respectively. Let \( D_n \) be the divisor on \( \mathbb{P}^n \) defined by \( D_n := \{ x = 0 \} + \{ y = 0 \} + \{ t = 0 \} + \mathcal{Q}_0 + \mathcal{Q}_1 + \cdots + \mathcal{Q}_{n-2} \).

For \( \lambda = (\lambda_0, \ldots, \lambda_{n-1}) \in \mathbb{C}^n \), we define rational 1-forms on \( \mathbb{P}^n \) as follows:

\[
\alpha_0(x, y) := \left( \frac{2\lambda_0 + \lambda_1}{2} \frac{dx}{x} - \frac{2\lambda_0 - \lambda_1}{2} \frac{dy}{y} \right), \\
\alpha_1(x, y) := 4 \frac{df(x, y, 1)}{f(x, y, 1)}, \\
\alpha_0^i(x, y, z_i) := \lambda_{i+1} \left( \frac{dz_i}{2f(x, y, 1)} - \frac{z_i df(x, y, 1) - z_i^2}{2f(x, y, 1) - z_i^2} \right), \\
\alpha_2(x, y, z_i) := \left( \frac{\alpha_0^i(x, y, z_i)}{f(x, y, 1)} \right),
\]

which are described by the affine coordinates \([x : y : z_1 : \ldots : z_{n-2} : 1]\).

Let \( D_\infty \) be the infinite dihedral group:

\[
D_\infty := \left\{ \begin{pmatrix} 0 & \alpha \\ -\alpha^{-1} & 0 \end{pmatrix}, \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix} \mid \alpha, \beta \in \mathbb{C}^* \right\} \leq \text{SL}_2(\mathbb{C}).
\]

In Proposition 3.3 below, for generic \( \lambda = (\lambda_0, \ldots, \lambda_{n-1}) \in \mathbb{C}^n \), we define a dihedral representation \( \rho_\lambda : \pi_1(\mathbb{P}^n \setminus D_n, *) \rightarrow D_\infty \) of the fundamental group \( \pi_1(\mathbb{P}^n \setminus D_n, *) \). This representation \( \rho_\lambda \) is virtually abelian.

**Theorem 1.2.** For generic \( \lambda = (\lambda_0, \ldots, \lambda_{n-1}) \in \mathbb{C}^n \), there exists an explicit flat meromorphic \( \mathfrak{sl}_2 \)-connection \( \nabla_\lambda \) over the trivial vector bundle \( \mathbb{P}^n \times \mathbb{C}^2 \rightarrow \mathbb{P}^n \) with the following properties:
(1) $\nabla_\lambda$ has at worst regular singularities. The polar divisor of $\nabla_\lambda$ is equal to $D_n$;
(2) The monodromy representation of $\nabla_\lambda$ is conjugated to $\rho_\lambda$.

The connection $\nabla_\lambda$ is given by

$$\nabla_\lambda = d + \left( \begin{array}{cc} \mathcal{A}_{11} & \mathcal{A}_{12} \\ -\mathcal{A}_{21} & -\mathcal{A}_{11} \end{array} \right) + \sum_{i=1}^{n-2} \left( \begin{array}{cc} \mathcal{A}_{i1}^j & \mathcal{A}_{i2}^j \\ -\mathcal{A}_{21}^j & -\mathcal{A}_{11}^j \end{array} \right),$$

where

$$\mathcal{A}_{11} := (x-1)\alpha_2(x,y) + \alpha_1(x,y) + \frac{1}{2} \frac{dy}{y}, \quad \mathcal{A}_{11}^j := (x-1)\alpha_2^j(x,y,z_i),$$
$$\mathcal{A}_{12} := \frac{dx + (x-1)^2\alpha_2(x,y) + 2(x-1)\alpha_1(x,y) + \alpha_0(x,y)}{y}, \quad \mathcal{A}_{12}^j := \frac{(x-1)^2\alpha_2^j(x,y,z_i) + \alpha_0^j(x,y,z_i)}{y},$$
$$\mathcal{A}_{21} := y\alpha_2(x,y), \quad \mathcal{A}_{21}^j := y\alpha_2^j(x,y,z_i),$$
in the affine coordinates $[x : y : z_1 : \ldots : z_{n-2} : 1]$.

1.2. Algebraic Garnier solution. We consider the Fuchsian system with $2n + 1$ regular singularities at $0, 1, t_1, \ldots, t_{2n-2}, \infty$:

$$d + \hat{H}_{2n-1} \frac{d\hat{x}}{\hat{x}} + \hat{H}_{2n} \frac{d\hat{x}}{\hat{x} - 1} + \sum_{i=1}^{2n-2} \hat{H}_i \frac{d\hat{x}}{\hat{x} - t_i},$$

where $\hat{H}_i$ ($i = 1, \ldots, 2n$) are $2 \times 2$ matrices independent to $\hat{x}$ and $t_i \neq t_j$ ($i \neq j$). We assume that $\hat{H}_{2n+1} := -\sum_{i=1}^{2n} \hat{H}_i$ is a diagonal matrix and the eigenvalues of $\hat{H}_i$ ($i = 1, \ldots, 2n + 1$) are as in Table 1. We fix a basis $\gamma_{\hat{x}} (\hat{x} = 0, 1, t_1, \ldots, t_{2n-2}, \infty)$ in the fundamental group $\pi_1(\mathbb{P}^1 \setminus \{0, 1, t_1, \ldots, t_{2n-2}, \infty\}, \ast)$. Here the loop $\gamma_{\hat{x}}$ on $\mathbb{P}^1$ is oriented counter-clockwise, $\hat{x}$ lies inside, while the other singular points lie outside.

**Table 1.** The eigenvalues of the residue matrices ($i = 1, \ldots, n - 2$).

| Resid matrices | $\hat{H}_1$ | $\hat{H}_2$ | $\hat{H}_{2i+1}$ | $\hat{H}_{2i+2}$ | $\hat{H}_{2n-1}$ | $\hat{H}_{2n}$ | $\hat{H}_{2n+1}$ |
|----------------|-------------|-------------|-----------------|-----------------|-----------------|----------------|-----------------|
| Eigenvalues    | $\pm \frac{1}{4}$ | $\pm \frac{1}{4}$ | $\pm \lambda_{i+1}$ | $\pm \lambda_i$ | $\pm \lambda_i$ | $\pm \lambda_{i+1}$ | $\pm \lambda_{i+1} + \lambda_i$ |

We restrict $\nabla_\lambda$ to a generic line on $\mathbb{P}^n$. The divisor $D_n$ consists of 3-lines and $(n - 1)$-conics. Then the restriction of $\nabla_\lambda$ is the Fuchsian system [2]. If we vary parameters of generic lines on $\mathbb{P}^n$, then we have an isomonodromic family of these Fuchsian systems parametrized by the space which parametrizes generic lines. The preserved monodromy representation of the fundamental group $\pi_1(\mathbb{P}^1 \setminus \{0, 1, t_1, \ldots, t_{2n-2}, \infty\}, \ast)$ of this isomonodromic family is conjugated to the representation given by Table 2. The dimension of the space which parametrizes generic lines on $\mathbb{P}^n$ is $2n - 2$, which is coincide with the dimension of the space of time variables of the Garnier system associated to the isomonodromic deformation of the Fuchsian systems with $(2n - 2) + 3$ regular singularities. If we take a finite cover of the space which parametrizes generic lines, then we have a generically finite morphism from the finite cover to the space of time variables of the Garnier system. Then our isomonodromic family is related to an algebraic solution of the Garnier system.

The organization of this paper is as follows. In Section 2 we construct an explicit $n$-parameter family $\nabla_\lambda$ of flat connections over the complex projective space $\mathbb{P}^n$. The explicit $n$-parameter family $\nabla_\lambda$ is parametrized by $n$-tuples of complex numbers $\lambda = (\lambda_0, \ldots, \lambda_{n-1})$. In Section 3 we compute the monodromy representation of $\nabla_\lambda$ for generic $\lambda$. We show that this monodromy representation is conjugated to $\rho_\lambda$. In Section 4 we discuss the relation between $\nabla_\lambda$ and the Garnier system ([4], [5], [11]).
Table 2. The representation of the fundamental group; here \( a_j = \exp(-\pi \sqrt{-1} \lambda_j) \) \( j = 0, 1, \ldots, n - 1 \).

| \( \gamma_0 \) | \( \gamma_1 \) | \( \gamma_{t_1} \) | \( \gamma_{t_2} \) |
|----------------|----------------|----------------|----------------|
| \((\begin{array}{cc} a_1 & 0 \\ 0 & a_1^{-1} \end{array})\) | \((\begin{array}{cc} -a_0 & 0 \\ 0 & -a_0^{-1} \end{array})\) | \((\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array})\) | \((\begin{array}{cc} 0 & a_0^{-2} \\ -a_0^2 & 0 \end{array})\) |

2. Construction of flat connections on projective spaces

2.1. Flat connection \((\nabla_0)_\lambda\) defined by rational closed 1-forms. Let \( \lambda_0, \ldots, \lambda_{n-1} \) be complex numbers. Set \( Y := \text{Spec} \mathbb{C}[u_0, u_1, z_1, \ldots, z_{n-2}] \). Let \( \omega_0 \) and \( \psi_n \) be the closed rational 1-forms on \( Y \) defined by

\[
\omega_0 := \lambda_0 \left( \frac{du_0}{u_0} - \frac{du_1}{u_1} \right) + \lambda_1 \left( \frac{du_0}{u_0 - 1} - \frac{du_1}{u_1 - 1} \right)
\]

\[
\psi_n := \left\{
\begin{array}{ll}
\sum_{i=1}^{n-2} 2\lambda_{i+1} \left( \frac{(u_0 - u_1)du_z - z_2 du_1}{(u_0 - u_1)^2 - z_1^2} \right) & n > 2 \\
0 & n = 2.
\end{array}
\right.
\]

We have a family of flat connections

\[
(\nabla_0)_\lambda := d + \frac{1}{2} \left( \begin{array}{cc} \omega_0 + \psi_n & 0 \\ 0 & -\omega_0 - \psi_n \end{array} \right)
\]

on the trivial rank 2 vector bundle \( E_0 \to Y \). The family \((\nabla_0)_\lambda\) is parametrized by \( \lambda = (\lambda_0, \ldots, \lambda_{n-1}) \).

On the associated projective bundle \( \mathbb{P}(E_0) \), we have the associated projective connection \( \mathbb{P}((\nabla_0)_\lambda) = dw_0 + (\omega_0 + \psi_n)w_0 \), where \( w_0 \) is a projective coordinate on the fibers.

2.2. Descent of the connection \((\nabla_0)_\lambda\). We consider the birational transformation of the projective connection \( \mathbb{P}((\nabla_0)_\lambda) \) defined by \( \Phi : \mathbb{P}(E_0) \dashrightarrow \mathbb{P}(E_0) \);

\[
(u_0, u_1, z_1, \ldots, z_{n-2}, [w_0^0 : w_1^0]) \mapsto (u_0, u_1, z_1, \ldots, z_{n-2}, [\tilde{w}_0^1 : \tilde{w}_1^1]),
\]

where

\[
\tilde{w}_0^1 = (u_0 - u_1)w_0^1 + w_1^0, \quad \tilde{w}_1^1 = w_0^1 - w_0^0.
\]

The rational function \( \tilde{w}_0^1/\tilde{w}_0^0 \) is an invariant of the involution \( \mathbb{P}(E_0) \to \mathbb{P}(E_0) \);

\[
(u_0, u_1, z_1, \ldots, z_{n-2}, [w_0^0 : w_1^0]) \mapsto (u_1, u_0, z_1, \ldots, z_{n-2}, [w_0^0 : w_1^0]).
\]

Put \( w_0 = w_0^1/w_0^0 \) and \( \tilde{w}_0 = \tilde{w}_0^1/\tilde{w}_0^0 \). We can check the following proposition by direct computation.

**Proposition 2.1.** We define a map \( f : Y \to \mathbb{P}^n \) by

\[
(u_0, u_1, z_1, \ldots, z_{n-2}) \mapsto [s_1 : s_2 : z_1 : \ldots : z_{n-2} : 1],
\]

where \( s_1 = u_0 + u_1 \) and \( s_2 = u_0u_1 \). The birational transformation \( (\Phi^{-1})^*(\mathbb{P}((\nabla_0)_\lambda)) \) on \( \mathbb{P}(E_0) \) descends to a projective connection on \( f(Y) \times \mathbb{P}^1 \to f(Y) \):

\[
(\Phi^{-1})^*\mathbb{P}((\nabla_0)_\lambda) = \frac{d\tilde{w}_0}{dw_0} (dw_0 + (\omega_0 + \psi_n)w_0)
\]

\[
= d\tilde{w}_0 + \left( \alpha_2(s_1, s_2) + \sum_{i=1}^{n-2} \alpha_2^i(s_1, s_2, z_i) \right) \tilde{w}_0^2
\]

\[
+ 2\alpha_1(s_1, s_2)\tilde{w}_0 + \left( \alpha_0(s_1, s_2) + \sum_{i=1}^{n-2} \alpha_0^i(s_1, s_2, z_i) \right),
\]

where \( \alpha_0(s_1, s_2, z_1) = \alpha_1(s_1, s_2, z_1) = 0 \).
\[ \begin{aligned}
\alpha_0(s_1, s_2) &:= \frac{2\lambda_0(1 - s_1 + s_2) + \lambda_1(-s_1 + 2s_2)}{2(1 - s_1 + s_2)} ds_1 - \frac{\lambda_0 s_1(1 - s_1 + s_2) + \lambda_1 s_2(s_1 - 2)}{2s_2(1 - s_1 + s_2)} ds_2, \\
\alpha_i(s_1, s_2, z_i) &:= \frac{\lambda_{i+1}}{4} \left( dz_i - \frac{z_i d(s_1^2 - 4s_2 - z_i^2)}{2(s_1^2 - 4s_2 - z_i^2)} \right), \\
\alpha_1(s_1, s_2) &:= -\frac{1}{4} \frac{d(s_1^2 - 4s_2)}{s_1^2 - 4s_2},
\end{aligned} \]

The corresponding connection \( (\nabla_1)_\lambda \) on \( f(Y) \times \mathbb{C}^2 \to f(Y) \) is

\[
(\nabla_1)_\lambda = d + \left( \begin{array}{cc}
\alpha_0(s_1, s_2) & \alpha_0(s_1, s_2) \\
-\alpha_0(s_1, s_2) & -\alpha_1(s_1, s_2)
\end{array} \right) + \sum_{i=1}^{n-1} \left( \begin{array}{cc}
-\alpha_0(s_1, s_2, z_i) & 0 \\
0 & \alpha_0(s_1, s_2, z_i)
\end{array} \right).
\]

We consider a relation between this connection and the connection \( (\nabla_0)_\lambda \). Let \( \nabla'_0 \) be the meromorphic connection on \( Y \times \mathbb{C} \to Y \) defined by \( \nabla'_0 := d - \frac{1}{2} \frac{d(u_0 - u_1)}{u_0 - u_1} \). We define a matrix \( M_1(u_0, u_1) \) on \( Y \) by

\[
M_1(u_0, u_1) := \left( \begin{array}{cc}
-1 & -u_0 + u_1 \\
1 & -u_0 + u_1
\end{array} \right).
\]

Let \( \nabla''_0 \) be the meromorphic connection on \( Y \times \mathbb{C}^2 \to Y \) defined by

\[
\nabla''_0 := d + M_1(u_0, u_1)^{-1} dM_1(u_0, u_1)
\]

\[
+ M_1(u_0, u_1)^{-1} \frac{1}{2} \left( \begin{array}{cc}
\omega_0 + \psi_n & 0 \\
0 & -\omega_0 - \psi_n
\end{array} \right) M_1(u_0, u_1).
\]

Then we have

\[
f^*(\nabla_1)_\lambda = \nabla''_0 \otimes \nabla'_0.
\]

Moreover, we consider the map \( \mathbb{P}^n \to \mathbb{P}^n \): \( [s_1 : s_2 : z_1 : \ldots : z_{n-2} : t] \mapsto [x : y : z_1 : \ldots : z_{n-2} : t] \), where \( x := t - s_1 + s_2 \) and \( y := s_2 \). Set

\[ f(x, y) := x^2 + y^2 + 1 - 2(xy + x + y). \]

Then the rational 1-forms \( \alpha_0 \) are transformed into

\[
\alpha_0(x, y) = \alpha_1(x, y, z_1) = \lambda_{i+1} \left( dz_i - \frac{z_i d(f(x, y) - z_i^2)}{2(f(x, y) - z_i^2)} \right),
\]

\[
\alpha_1(x, y) = \frac{1}{4} \frac{df(x, y)}{f(x, y)},
\]

which are described by the affine coordinates \([x : y : z_1 : \ldots : z_{n-2} : 1]\).

### 2.3. Birational transformations of the connection \((\nabla_1)_\lambda\)

From the connection \((\nabla_1)_\lambda\) on \( f(Y) \times \mathbb{C}^2 \to f(Y) \), we construct a connection on the trivial bundle \( \mathbb{P}^n \times \mathbb{C}^2 \to \mathbb{P}^n \) whose pole divisor is \( D_n \). If we extend the rational 1-forms \( \alpha_0 \) to rational 1-forms on \( \mathbb{P}^n \), then \( \alpha_0(x, y) \), \( \alpha_0(x, y, z_1) \) and \( \alpha_1(x, y) \) have poles of order 2, 2 and 1 along the divisor \( (t = 0) \), respectively. On the other hand, the rational 1-forms \( \alpha_2(x, y) \) and \( \alpha_2(x, y, z_1) \) have no pole along the divisor \( (t = 0) \). So we consider a birational transformation of the projective connection \((\nabla_1)\) as follows. The \( dy/y \) part of the projective connection \((\nabla_1)\) is

\[
d\tilde{w}_0 - \frac{\lambda_0(\tilde{w}_0 - x + 1)(\tilde{w}_0 + x - 1)}{x + 1} \frac{dy}{y}
\]

\[ + \text{ terms whose pole divisors do not contain the divisor } (y = 0) \].

Then we consider the following birational map

\[
\mathbb{P}^n \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^n \times \mathbb{P}^1
\]

\[
([x : y : z_1 : \ldots : z_{n-2} : 1], [1 : \tilde{w}_0]) \mapsto ([x : y : z_1 : \ldots : z_{n-2} : 1], [1 : w]),
\]
where $\tilde{w}_0 - x + 1 = wy$. By this birational transformation \[9\text{], the projective connection \[5\text{ is transformed into}
\begin{align*}
dw + \left( A_{21}(x, y) + \sum_{i=1}^{n-2} A_{21}^i(x, y, z_i) \right) w^2 \\
+ 2 \left( A_{11}^i(x, y) + \sum_{i=1}^{n-2} A_{11}^i(x, y, z_i) \right) w + A_{12}^i(x, y) + \sum_{i=1}^{n-2} A_{12}^i(x, y, z_i),
\end{align*}
(10)
where \begin{align*}
A_{21}(x, y) &:= y\alpha_2(x, y), \\
A_{11}(x, y) &:= (x - 1)\alpha_2(x, y) + \alpha_1(x, y) + \frac{1}{2} \frac{dy}{y}, \\
A_{12}(x, y) &:= \frac{dx + (x - 1)^2 \alpha_2(x, y) + 2(x - 1)\alpha_1(x, y) + \alpha_0(x, y)}{y}, \\
A_{12}^i(x, y, z_i) &:= \frac{(x - 1)^2 \alpha_2^i(x, y, z_i) + \alpha_0^i(x, y, z_i)}{y}.
\end{align*}
\]
The corresponding connection $\nabla_\lambda$ on $\mathbb{P}^n \times \mathbb{C}^2 \to \mathbb{P}^n$ is
\begin{equation}
\nabla_\lambda = d + \left( \begin{array}{cc}
A_{11}(x, y) & A_{12}(x, y) \\
-A_{21}(x, y) & -A_{11}(x, y)
\end{array} \right) + \sum_{i=1}^{n-2} \left( \begin{array}{cc}
A_{11}^i(x, y, z_i) & A_{12}^i(x, y, z_i) \\
-A_{21}^i(x, y, z_i) & -A_{11}^i(x, y, z_i)
\end{array} \right),
\end{equation}
whose polar divisor is $D_n$. This connection $\nabla_\lambda$ is the connection in Theorem 1.2. We consider a relation between $\nabla_\lambda$ and $(\nabla_1)_\lambda$. Let $\nabla_1'$ be the meromorphic connection on $\mathbb{P}^n \times \mathbb{C} \to \mathbb{P}^n$ defined by $\nabla_1' := d - \frac{1}{2} \frac{dy}{y}$. We define a matrix $M_2(x, y)$ on $Y$ by
\begin{equation}
M_2(x, y) := \begin{pmatrix} y & x - 1 \\
0 & 1 \end{pmatrix}.
\end{equation}
Let $\nabla''_1$ be the meromorphic connection on $\mathbb{P}^n \times \mathbb{C}^2 \to \mathbb{P}^n$ defined by
\begin{equation}
\nabla''_1 = d + M_2(x, y)^{-1} dM_2(x, y) + M_2(x, y)^{-1} \left( \begin{array}{cc}
\alpha_1(x, y) & \alpha_0(x, y) \\
-\alpha_2(x, y) & -\alpha_1(x, y)
\end{array} \right) M_2(x, y)
+ \sum_{i=1}^{n-2} M_2(x, y)^{-1} \left( \begin{array}{cc}
0 & \alpha_0^i(x, y, z_i) \\
-\alpha_2^i(x, y, z_i) & 0
\end{array} \right) M_2(x, y).
\end{equation}
We can check that
\begin{equation}
\nabla_\lambda = \nabla''_1 \otimes \nabla_1'.
\end{equation}
By a combination of the equalities \[7\text{ and \[13\text{, we have the following proposition:}
\begin{proposition}
The pull-back $f^* \nabla_\lambda$ is birationally equivalent to $(\nabla_0)_\lambda \otimes \nabla'_0 \otimes f^* \nabla_1'$.
\end{proposition}
\]
3. Monodromy representation
In this section, we consider the monodromy representation $\pi_1(\mathbb{P}^n \setminus D_n, \star) \to \text{SL}_2(\mathbb{C})$ of $\nabla_\lambda$. We show that the monodromy representation is conjugated to a certain dihedral representation of the fundamental group $\pi_1(\mathbb{P}^n \setminus D_n, \star)$.

3.1. Zariski’s hyperplane section theorem. Let $H_i (i = 1, \ldots, n - 2)$ be the hyperplanes in $\mathbb{P}^n$ defined by
\begin{equation}
H_i := (z_i - a_i x - b_i y - c_i t = 0) \quad i = 1, \ldots, n - 2.
\end{equation}
Here $a_i, b_i,$ and $c_i$ $(i = 1, \ldots, n - 2)$ are generic complex numbers. For simplicity, we assume that $0 < \lvert a_i \rvert \ll 1$ and $0 < \lvert b_i \rvert \ll 1$. Let $f(x, y, t)$ be the following quadratic polynomial

$$
f(x, y, t) := x^2 + y^2 + t^2 - 2(xy + yt + tx)$$

$$(14)$$

Let $\tilde{Q}_0$, $\tilde{Q}_i$, and $\tilde{D}_n$, be the divisors on $\mathbb{P}^2 = \mathbb{P}^n \cap (\cap_{i=1}^n H_i)$ defined by

$$\tilde{Q}_0 := (f(x, y, t) = 0),$$

$$\tilde{Q}_i := (f(x, y, t) - (a_i x + b_i y + c_i t)^2 = 0) \quad (i = 1, \ldots, n - 2),$$

$$\tilde{D}_n := (x = 0) + (y = 0) + (t = 0) + \tilde{Q}_0 + \tilde{Q}_1 + \cdots + \tilde{Q}_{n-2},$$

respectively. By Zariski’s hyperplane section theorem (for example see [I]), we have the natural isomorphism

$$\pi_1(\mathbb{P}^n \setminus D_n, \ast) \cong \pi_1(\mathbb{P}^2 \setminus \tilde{D}_n, \ast). \quad (15)$$

### 3.2. Zariski–Van-Kampen method

Let $\pi : \mathbb{P}^2 \setminus \tilde{D}_n \to \mathbb{P}^1$ be the projection defined by

$$\pi : \mathbb{P}^2 \setminus \tilde{D}_n \to \mathbb{P}^1$$

$$[x : y : t] \mapsto [x : t].$$

Let $\{[x^+ : 1], [x^- : 1]\} \subset \mathbb{P}^1$ be the roots of the discriminant of $f(x, y, t) - (a_i x + b_i y + c_i t)^2$ with respect to $y$. We denote $[x^+ : 1]$ and $[x^- : 1]$ by $x_i^+$ and $x_i^-$, respectively. Since $0 < \lvert a_i \rvert \ll 1$ and $0 < \lvert b_i \rvert \ll 1$, there exists an element of $\{x_i^+, x_i^-\}$ in a neighborhood of $\infty = [0 : 1]$. We assume that $x_i^-$ is a point in a neighborhood of $\infty$. Set $a = [a : 1]$ where $0 < \lvert a \rvert \ll 0$. For $i = 0, 1, \ldots, n - 2$, let $y_i^+$ and $y_i^-$ be the intersection of $\tilde{Q}_i$ and $\pi^{-1}(a)$: $\tilde{Q}_i \cap \pi^{-1}(a) = \{y_i^+, y_i^-\}$. Here we assume that

$$0 < \text{Arg} \left( \frac{y_i^+ - (a + 1)}{y_i^- - (a + 1)} \right) < \cdots < \text{Arg} \left( \frac{y_{n-2}^+ - (a + 1)}{y_{n-2}^- - (a + 1)} \right) < \pi.$$

**Figure 1.** Fibers of $\pi$.

We define natural numbers $i_k$ and $j_k$ $(k = 1, \ldots, n - 2)$ so that $\{i_1, \ldots, i_{n-2}\} = \{1, \ldots, n - 2\}$, $\{j_1, \ldots, j_{n-2}\} = \{1, \ldots, n - 2\}$,

$$0 < \text{Arg} \left( \frac{x_{i_k}^+}{x_{j_k}^-} \right) < \cdots < \text{Arg} \left( \frac{x_{i_{n-2}}^+}{x_{j_{n-2}}^-} \right) < 2\pi \quad \text{and} \quad 0 < \text{Arg} \left( \frac{1}{x_{j_1}} \right) < \cdots < \text{Arg} \left( \frac{1}{x_{j_{n-2}}} \right) < 2\pi.$$

Here we define the range of the principal value of arguments Arg by the closed-open interval $[0, 2\pi)$. Let $\Gamma_1$ and $\Gamma_2$ be the groups defined by

$$\Gamma_1 := \left\{ \begin{array}{c|c} \alpha_0, \alpha_0^+, \ldots, \alpha_{y_{n-2}^+}, & \alpha_0 \alpha_{y_1^+} \cdots \alpha_{y_{n-2}^+} \alpha_0 \alpha_{y_1^-} \cdots \alpha_{y_{n-2}^-} \alpha_0^+ \alpha_\infty = 1 \\ \alpha_{y_1^+}, \ldots, \alpha_{y_{n-2}^+}, \alpha_\infty & \end{array} \right\}$$

and

$$\Gamma_2 := \left\{ \begin{array}{c|c} \gamma_0, \gamma_1, \gamma_2^+, \ldots, \gamma_{x_{n-2}^+}, & \gamma_0 \gamma_1 \gamma_2^+ \cdots \gamma_{x_{n-2}^+} \gamma_0 = 1 \\ \gamma_{x_1^+}, \ldots, \gamma_{x_{n-2}^+}, \gamma_\infty & \end{array} \right\}.$$
Then we have \( \pi_1(\pi^{-1}(a) \setminus \tilde{D}_n \cap \pi^{-1}(a)), *) \cong \Gamma_1 \) and \( \pi_1(\mathbb{P}^1 \setminus \{0, 1, x_1^\pm, \ldots, x_{n-1}^\pm, \infty\}, a) \cong \Gamma_2 \). We can define the monodromy action of \( \Gamma_2 \) on \( \Gamma_1 \) naturally. Remark that the monodromy actions around the points on \( \mathbb{P}^1 \) which are the projections of the intersection of \( \tilde{Q}_i \cap \tilde{Q}_j \) are trivial for \( i \neq j \) (\( i, j = 0, 1, \ldots, n-2 \)), since \( \tilde{Q}_i \) and \( \tilde{Q}_j \) intersects transversally.

**Figure 2.** Loops on \( \pi^{-1}(a) \) and \( \mathbb{P}^1 = (\mathbb{C}_x)_0 \cup (\mathbb{C}_x)_\infty \).

Let \( \Gamma \) be the group defined by

\[
\Gamma := \langle \alpha_0, \alpha_{y_0}^+, \ldots, \alpha_{y_{n-2}}^+, \alpha_{y_0}^-, \ldots, \alpha_{y_{n-2}}^-; \gamma_0 \mid \gamma_x^\pm(\alpha_y) = \alpha_y (i = 1, \ldots, n-2), \gamma_1(\alpha_y) = \alpha_y, \gamma_0(\alpha_y) = \gamma_0^{-1} \alpha_y \gamma_0, y \in \{0, y_0, \ldots, y_{n-2}\} \rangle.
\]

By the Zariski–Van-Kampen method (for example see [2]), we have the isomorphism

\[
\pi_1(\mathbb{P}^2 \setminus \tilde{D}_n, *) \cong \Gamma.
\]

**Proposition 3.1.** The group \( \Gamma \) is generated by \( \alpha_0, \alpha_{y_0}^+, \ldots, \alpha_{y_{n-2}}^+ \), and \( \gamma_0 \).

**Proof.** The actions of \( \gamma_0 \) and \( \gamma_x^+ \) on \( \alpha_{y_0} \) and \( \alpha_{y_{i}} \) are respectively as follows:

\[
\gamma_0(\alpha_{y_0}^+) = \alpha_{y_0}^- \quad \text{and} \quad \gamma_x^+(\alpha_{y_{i}}^+) = \alpha_{y_{i}}^+ \alpha_{y_{i}}^- \alpha_{y_{i}}^{-1} \quad (i = 1, \ldots, n-2).
\]

Then \( \alpha_{y_i}^- \) (\( i = 0, \ldots, n-2 \)) are generated by \( \alpha_{y_0}^+, \ldots, \alpha_{y_{n-2}}^+ \), and \( \gamma_0 \). \( \square \)

**Proposition 3.2.** Set \( \tilde{\alpha} = \alpha_{y_1}^+ \cdots \alpha_{y_{n-2}}^+ \). For the elements \( \alpha_{y_0}^+, \alpha_0, \gamma_0, \) and \( \tilde{\alpha} \) of \( \Gamma \), we have the following equalities:

\[
\begin{align*}
(18) \quad \left[ \alpha_0, \gamma_0 \right] & = \left[ \alpha_0, \tilde{\alpha} \right] = 1, \\
(19) \quad \left( (\gamma_0 \tilde{\alpha}) \alpha_{y_0}^+ \right)^2 & = (\alpha_{y_0}^+(\gamma_0 \tilde{\alpha}))^2, \\
(20) \quad (\alpha_{y_0}^+ \alpha_0)^2 & = (\alpha_0 \alpha_{y_0}^+)^2.
\end{align*}
\]

**Proof.** The actions of \( \gamma_0 \) on \( \alpha_0 \) and \( \alpha_{y_i}^+ \) (\( i = 1, \ldots, n-2 \)) are trivial:

\[
\gamma_0(\alpha_0) = \alpha_0, \quad \gamma_0(\alpha_{y_i}^+) = \alpha_{y_i}^+ \quad (i = 1, \ldots, n-2).
\]

Then we have the equality (18).

Second, we show the equality (19). Put \( \tilde{\gamma}_\infty = \gamma_{2,1}^- \cdots \gamma_{2,n-2}^- \gamma_\infty \). The action of \( \tilde{\gamma}_\infty \) is as follows:

\[
(21) \quad \tilde{\gamma}_\infty(\alpha_{y_i}^+) = \alpha_0 \alpha_{y_0}^+ \alpha_0^{-1}.
\]
By the equalities (17), (21), \( a_\infty = \gamma_\infty \gamma_0 \), and (18), we have

\[
a_{y_0} = \gamma_\infty a_{y_0} \gamma_0^{-1} a_{y_0} \gamma_0 a_0^{-1} \gamma_0^{-1} \\
= \gamma_\infty a_{y_0} \gamma_0^{-1} a_{y_0} \gamma_0 a_0^{-1} \gamma_0^{-1} \\
= (a_{y_1} \cdots a_{y_{n-2}} a_{y_0} a_{y_1} \cdots a_{y_{n-2}})^{-1} a_{y_0} a_0 \gamma_0 (a_{y_1} \cdots a_{y_{n-2}} a_{y_0} a_{y_1} \cdots a_{y_{n-2}})^{-1} \\
= (\tilde{a}_0 a_{y_0} \gamma_0 a_{y_0}^{-1})^{-1} a_{y_0} a_0 \gamma_0 (\tilde{a}_0 a_{y_0} \gamma_0 a_{y_0}^{-1}).
\]

Then we have the equality (19).

The action of \( \gamma_1 \) on \( a_{y_0} \) is as follows:

\[
\gamma_1(a_{y_0}) = (a_0 a_{y_0} a_0^{-1} a_{y_0}^{-1})^{-1}.
\]

Then we have the equality (20). \( \square \)

### 3.3. Monodromy representation of \( \nabla_\lambda \)

Let \( D_\infty \) be the infinite dihedral group:

\[
D_\infty := \left\{ \left( \begin{array}{cc} \alpha & 0 \\ 0 & \beta^{-1} \end{array} \right) \in \mathbb{C}^* \right\} \leq \mathrm{SL}_2(\mathbb{C}).
\]

**Proposition 3.3.** For generic \( \lambda \), the monodromy representation of \( \nabla_\lambda \) is conjugated to the dihedral representation \( \rho_\lambda : \pi_1(\mathbb{P}^n \backslash D_n, *) \to D_\infty \) of the fundamental group \( \pi_1(\mathbb{P}^n \backslash D_n, *) \cong \Gamma \) defined by

\[
\rho_\lambda(a_0) = \begin{pmatrix} -\exp(-\pi \lambda_0) & 0 \\ 0 & -\exp(\pi \lambda_0) \end{pmatrix}, \quad \rho_\lambda(\gamma_0) = \begin{pmatrix} \exp(-\pi \lambda_1) & 0 \\ 0 & \exp(\pi \lambda_1) \end{pmatrix}, \\
\rho_\lambda(a_{y_0}^{+i}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho_\lambda(a_{y_0}^{-i}) = \begin{pmatrix} \exp(-\pi \lambda_{i+1}) & 0 \\ 0 & \exp(\pi \lambda_{i+1}) \end{pmatrix},
\]

where \( i = 1, \ldots, n - 2 \).

**Proof.** Let \( \rho_\nabla_\lambda : \pi_1(\mathbb{P}^n \backslash D_n, *) \to \mathrm{SL}_2(\mathbb{C}) \) be a monodromy representation of \( \nabla_\lambda \). Put \( A_0 := \rho_\nabla_\lambda(a_0) \), \( A_{y_0}^{+i} := \rho_\nabla_\lambda(a_{y_0}^{+i}) (i = 0, \ldots, n - 1) \) and \( C_0 := \rho_\nabla_\lambda(\gamma_0) \). Let \( U \) be some analytic open subset of \( \mathbb{P}^2(\tilde{Q}_0 \cup (y = 0) \cup (t = 0)) \) such that \( U \) is simply connected and \( U \) contains the loops \( a_{y_0}^{+i} (i = 1, \ldots, n - 1) \) and \( \gamma_0 \). On the open subset \( U \), the connection \( \nabla_\lambda \) is isomorphic to \( (\nabla_0)_\lambda \). Then by some conjugation, we may put

\[
C_0 = \begin{pmatrix} \exp(-\pi \lambda_1) & 0 \\ 0 & \exp(\pi \lambda_1) \end{pmatrix}, \quad A_{y_0}^{+i} = \begin{pmatrix} \exp(-\pi \lambda_{i+1}) & 0 \\ 0 & \exp(\pi \lambda_{i+1}) \end{pmatrix} (i = 1, \ldots, n - 2).
\]

Assume that \( \exp(-\pi \lambda_1) \neq \exp(\pi \lambda_1) \). By Proposition 3.2, we have the equality \( A_0 C_0 = C_0 A_0 \). Then we have

\[
A_0 = \begin{pmatrix} -\exp(-\pi \lambda_0) & 0 \\ 0 & -\exp(\pi \lambda_0) \end{pmatrix}.
\]

Note that the image \( \mathrm{Im}(\rho_\nabla_\lambda) \) is non-abelian. Since \( C_0, A_0, \) and \( A_{y_0}^{+i} (i = 1, \ldots, n - 2) \) are diagonal matrices, we may put

\[
A_{y_0}^{+i} = \begin{pmatrix} a_{11} & a_{12} \\ -1 & a_{22} \end{pmatrix}.
\]

Put \( \tilde{A} := A_{y_0}^{+1} \cdots A_{y_0}^{+n-2} \). By Proposition 3.2, we have the equalities \( (A_{y_0}^{+i}(C_0 \tilde{A}))^2 = ((C_0 \tilde{A}) A_{y_0}^{+i})^2 \) and \( (A_{y_0}^{+i} A_0)^2 = (A_0 A_{y_0}^{+i})^2 \). Assume that \( \exp(-\pi \lambda_1 - \pi \sum_{i=1}^{n-2} \lambda_{i+1})^2 \neq 1 \) and \( -\exp(-\pi \lambda_0)^2 \neq 1 \). Since \( A_{y_0}^{+i}(C_0 \tilde{A}) \neq (C_0 \tilde{A}) A_{y_0}^{+i} \) and \( A_{y_0}^{+i} A_0 \neq A_0 A_{y_0}^{+i} \), we have the equalities \( (A_{y_0}^{+i}(C_0 \tilde{A}))^2 = -I_2 \) and
Then we have the following equalities:

\[
\begin{align*}
    a_{11}a_{22} + a_{12} &= 1 \\
    a_{11}(\exp(-\pi\lambda_1 - \pi \sum_{i=1}^{n-2} \lambda_{i+1}))^2 &= a_{22} \\
    a_{11}(-\exp(-\pi\lambda_0))^2 &= a_{22}.
\end{align*}
\]

We assume that \((\exp(-\pi\lambda_1 - \pi \sum_{i=1}^{n-2} \lambda_{i+1}))^2 \neq (-\exp(-\pi\lambda_0))^2\). Then we have \(a_{11} = 0, a_{12} = 1,\) and \(a_{22} = 0.\)

\[\square\]

4. Algebraic Garnier solution

Assume that an \(n\)-tuple of complex numbers \(\lambda = (\lambda_0, \ldots, \lambda_{n-1})\) is sufficiently generic. In this section, we restrict the flat connection \(\nabla_\lambda\) to a generic line \(\mathbb{P}^n \cap (\bigcap_{i=1}^{n-2} H'_i)\), where

\[
\begin{align*}
H'_0 &= (y - ax - bt = 0) \\
H'_i &= (z_i - c_i x - d_i t = 0) \quad (i = 1, 2, \ldots, n - 2).
\end{align*}
\]

Here \(a, b, c_i,\) and \(d_i (i = 1, 2, \ldots, n - 2)\) are generic complex numbers. We consider the transformation \(\hat{x} = -\frac{x}{x}\). Let \(T\) be a Zariski open subset of \(\text{Spec} \mathbb{C}[a, b, c_i, d_i]_{i=1,\ldots,n-2}\). We consider the map \(\mathbb{P}^1 \times T \to \mathbb{P}^n\) defined by \(\eqref{22}\). Let \((\nabla_{\mathbb{P}^1 \times T})_\lambda\) be the flat connection on the trivial rank \(2\) vector bundle \(F_0\) over \(\mathbb{P}^1 \times T\) induced by the flat connection \(\nabla_\lambda\) over \(\mathbb{P}^n\). Let

\[
(\nabla_{\mathbb{P}^1 \times T/T})_\lambda : F_0 \to F_0 \otimes \Omega^1_{\mathbb{P}^1 \times T/T}(D_n)
\]

be the relative connection on \(F_0\) over \(\mathbb{P}^1 \times T\) associated to \((\nabla_{\mathbb{P}^1 \times T})_\lambda\). We discuss a relation between the relative connection \((\nabla_{\mathbb{P}^1 \times T/T})_\lambda\) and the Garnier system \([3], [5], [11]\).

4.1. Regular singular points of \((\nabla_{\mathbb{P}^1 \times T/T})_\lambda\).

By the pull-back of \(x^2 + y^2 + t^2 - 2(xy + yt + tx)\) and \(x^2 + y^2 + t^2 - 2(xy + yt + tx) - z_i^2\) under \(\mathbb{P}^1 \times T \to \mathbb{P}^n\), we have the following polynomials over \(T\):

\[
f(a, b, \hat{x}) := \frac{(a - 1)^2 b^2}{a^2} \hat{x}^2 + \frac{2b(1 + a + b - ab)}{a} \hat{x} + (b - 1)^2,
\]

\[
f_i(a, b, c_i, d_i, \hat{x}) := f(a, b, \hat{x}) - \frac{(ad_i - bc_i \hat{x})^2}{a^2},
\]

which are described on the affine coordinate \([\hat{x} : 1]\). Let \(I\) be the ideal of \(\mathbb{C}[a, b, \hat{b}, c_i, d_i, \hat{d}_i]_{i=1,\ldots,n-2}\) defined by \(I := (\hat{b}^2 - 4(a + b - ab), \hat{d}_i^2 - \Delta_i)_{i=1,\ldots,n-2}\), where \(\Delta_i\) is the discriminant of \(f_i(a, b, c_i, d_i, \hat{x})\) with respect to \(\hat{x}\). We have the natural morphism

\[
\text{Spec} \mathbb{C}[a, b, \hat{b}, c_i, d_i, \hat{d}_i]_{i=1,\ldots,n-2}/I \to \text{Spec} \mathbb{C}[a, b, c_i, d_i]_{i=1,\ldots,n-2}.
\]

Let \(\tilde{T}\) be the inverse image of \(T\) under this morphism: \(\tilde{T} \to T\). Let \(t_1\) and \(t_2\) be the rational functions on \(\tilde{T}\) defined by

\[
t_1 := \frac{a(\hat{b} - 2)^2}{4(a - 1)(b^2 - a)} \quad \text{and} \quad t_2 := \frac{a(\hat{b} + 2)^2}{4(a - 1)(b^2 - a)}.
\]

Then \(f(a, b, t_1) = f(a, b, t_2) = 0.\) Moreover, let \(t_{2i+1}\) and \(t_{2i+2}\) be the rational functions on \(\tilde{T}\) defined by

\[
t_{2i+1} := \frac{2ab(1 + a + b - ab - c_id_i)}{2b^2((a - 1)^2 - c_i^2)} \quad \text{and} \quad t_{2i+2} := \frac{2ab(1 + a + b - ab - c_id_i)}{2b^2((a - 1)^2 - c_i^2)}.
\]

Then \(f_i(a, b, c_i, d_i, t_{2i+1}) = f_i(a, b, c_i, d_i, t_{2i+2}) = 0.\) By these rational functions, we have a generically finite morphism

\[
(\tilde{T} \to \text{Spec} \mathbb{C}[t_1, t_2, \ldots, t_{2n-2}],
\]

if the Zariski open subset \(T\) shrinks. We take the pull-back \((\nabla_{\mathbb{P}^1 \times \tilde{T}})_{\lambda}\) of \((\nabla_{\mathbb{P}^1 \times T/T})_{\lambda}\) under the morphism \(\mathbb{P}^1 \times \tilde{T} \to \mathbb{P}^1 \times T\). Then \((\nabla_{\mathbb{P}^1 \times \tilde{T}/T})_{\lambda}\) is a family of the Fuchsian systems with \(2n + 1\) regular singularities at \(\hat{x} = 0, 1, t_1, \ldots, t_{2n-2}, \infty\) parametrized by \(\tilde{T}\).
4.2. Residue matrices of \((\nabla_{\mathbb{P}^1 \times \tilde{T}/\tilde{T}})^\lambda\). We describe the residue matrices of \((\nabla_{\mathbb{P}^1 \times \tilde{T}/\tilde{T}})^\lambda\) at the regular singular points. Put

\[
M_2(\tilde{x}) := \begin{pmatrix}
-ab(\tilde{x} - 1) & -b\tilde{x} - a \\
0 & 0
\end{pmatrix}
\quad \text{and}
\]

\[
\alpha_0^i(\tilde{x}) := \lambda_{i+1} \left( \begin{array}{c}
-bc_i \\
-a(\tilde{x} - t_{2i+1}) - \frac{2a}{2(\tilde{x} - t_{2i+1}) - \frac{8a(\tilde{x} - t_{2i+1})}{a}}
\end{array} \right)
\]

Let \(H_{2n-1}^\tilde{T}\) be the residue matrix at \(\tilde{x} = 0\). We have the following equality

\[
H_{2n-1}^\tilde{T} = M_2(0)^{-1} \begin{pmatrix}
0 & \frac{\lambda_{n}(\tilde{x} - t_{2i+1})}{8x(a-1)} \\
0 & 0
\end{pmatrix} M_2(0).
\]

Let \(H_{2n}^\tilde{T}\) be the residue matrix at \(\tilde{x} = 1\). We have the following equality

\[
H_{2n}^\tilde{T} = \begin{pmatrix}
1 - \frac{\lambda_{n}}{2} \left( 1 + \frac{4(a^2 - 8a)}{a^2 - 4a^2 - 1} \right) \sum_{i=1}^{n-2} \alpha_0^i(1) & 0 \\
0 & 1 + \frac{1}{b^2(a-1)(a-2)} \end{pmatrix} M_2(1).
\]

Let \(H_{2i+1}^\tilde{T}\) and \(H_{2i+2}^\tilde{T}\) be the residue matrices at \(\tilde{x} = t_{2i+1}\) and \(\tilde{x} = t_{2i+2}\), respectively. We have the following equalities

\[
H_{2i+1}^\tilde{T} = M_2(t_{2i+1})^{-1} \begin{pmatrix}
\frac{\lambda_{n}(a-1)}{2(b-2a)} & \sum_{i=1}^{n-2} \frac{a^2 \alpha_0^i(t_{2})}{b^2(a-1)^2 (t_{2i+1} - 1)} \left( \begin{array}{c}
0 \\
\frac{1}{4}
\end{array} \right)
\end{pmatrix} M_2(t_{2i+1})
\]

\[
H_{2i+2}^\tilde{T} = M_2(t_{2i+2})^{-1} \begin{pmatrix}
\frac{\lambda_{n}(a-1)}{2(b+2a)} & \sum_{i=1}^{n-2} \frac{a^2 \alpha_0^i(t_{2})}{b^2(a-1)^2 (t_{2i+2} - 1)} \left( \begin{array}{c}
0 \\
\frac{1}{4}
\end{array} \right)
\end{pmatrix} M_2(t_{2i+2})
\]

Let \(H_{2n+1}^\tilde{T}\) be the residue matrix at \(\tilde{x} = \infty\). Let \(A_{ij}^\tilde{T}(\tilde{x})\) be the relative rational 1-forms over \(\tilde{T}\) which are the relativization of the pull-backs of the rational 1-forms \((\nabla_{\mathbb{P}^1 \times \tilde{T}/\tilde{T}})^\lambda\) on \(\mathbb{P}^1 \times \tilde{T}\). We have the following equality

\[
\text{res}_{\tilde{x}=\infty} \begin{pmatrix}
A_{11}^\tilde{T}(\tilde{x}) \\
A_{21}^\tilde{T}(\tilde{x})
\end{pmatrix} = 0 \quad (i = 1, \ldots, n - 2).
\]

Then we have

\[
H_{2n+1}^\tilde{T} = \begin{pmatrix}
-1 & a - 2 \\
1 & a
\end{pmatrix} \begin{pmatrix}
\frac{\lambda_{n} + \lambda_1}{2} & 0 \\
\frac{\lambda_{n} + \lambda_1}{2} & 1
\end{pmatrix} \begin{pmatrix}
-1 & a - 2 \\
1 & a
\end{pmatrix}^{-1}
\]

4.3. Garnier system. Following [10], we recall the Garnier systems. Let \(A(\tilde{x})\) be the Fuchsian system with \(2n + 1\) regular singularities at \(t_1, \ldots, t_{2n}, \infty\):
We consider the isomonodromic deformation of the Fuchsian system
\( (31) \) (see \([10, \text{Theorem 2.7}]\)).

Let \(\tilde{\rho}_A\) be the representation of the fundamental group defined by Table 3.

\[\tilde{\rho}_A: \pi_1(\mathbb{P}^1 \setminus \{t_1, \ldots, t_{2n}, \infty\}, \ast) \to \text{SL}_2(\mathbb{C})\]

be the representation of the fundamental group defined by Table 3. We consider the isomonodromic deformation of the Fuchsian system \(A(\tilde{x})\) whose preserved monodromy representation is conjugated to \(\rho_A^0\). Let \(d + \sum_{i=1}^{2n} \bar{H}_i(t) \frac{dx}{x} \in t_i\) be the Fuchsian system with \(2n + 1\) regular singularities at \(t_i\) whose monodromy representation is conjugated to \(\rho_A^0\). There exists an open neighbourhood \(U_0 \subset \mathbb{C}^n\) of the point \(t^0 = (t_1^0, \ldots, t_{2n}^0)\) such that for any \(t \in U_0\), there exists a unique tuple \((\bar{H}_i(t))_{i=1,\ldots,2n}\) of analytic matrix valued functions such that \(\bar{H}_i(t^0) = \bar{H}_i^0, i = 1, \ldots, 2n\), and the monodromy representation of \(d + \sum_{i=1}^{2n} \bar{H}_i(t) \frac{dx}{x} \in t_i\) is conjugated to \(\rho_A^0\). The matrices \(\bar{H}_i(t) i = 1, \ldots, 2n\) are the solutions of the Cauchy problem with the initial data \((\bar{H}_i(t^0))_{i=1,\ldots,2n}\) for the Schlesinger equations (see \([10, \text{Theorem 2.7}]\)).

The \((2n-2)\)-variable Garnier system \(G_{2n-2}\) is the completely integrable Hamiltonian system
\[
G_{2n-2}: \left\{ \begin{array}{l}
\frac{\partial \rho_j}{\partial t_i} = -\frac{\partial K_i}{\partial \nu_j} \quad i, j = 1, \ldots, 2n-2 \\
\frac{\partial \nu_j}{\partial t_i} = \frac{\partial K_i}{\partial \nu_j} \quad i, j = 1, \ldots, 2n-2,
\end{array} \right.
\]
where
\[
K_i = -\frac{\Lambda(t_i)}{T'(t_i)} \sum_{k=1}^{2n-2} \frac{T(v_k)}{(v_k - t_i)X(v_k)} \left\{ \tilde{\rho}_A^0 - \sum_{m=1}^{2n} \frac{\theta_m - \delta_{im}}{v_k - t_m} \rho_k + \frac{\kappa}{v_k(v_k - 1)} \right\}
\]
with \(t_{2n-1} = 0, t_{2n} = 1, \kappa := \frac{1}{3} \left\{ \sum_{m=1}^{2n} \theta_m - 1 \right\} \), \(\Lambda(t) := \prod_{k=1}^{2n-1} (t - \nu_k) \) and \(T(t) := \prod_{k=1}^{2n} (t - t_k)\). Here \(\theta_m (m = 1, \ldots, 2n, \infty)\) is the constant parameters defined by
\[
\theta_1 = \frac{1}{2}, \quad \theta_2 = \frac{1}{2}, \quad \theta_{2i+1} = \lambda_i + 1, \quad \theta_{2i+2} = \lambda_i + 1 (i = 1, \ldots, n - 2)
\]
\[
\theta_{2n-1} = \lambda_1, \quad \theta_{2n} = \lambda_0 - 1, \quad \theta_{\infty} = \lambda_0 + \lambda_1.
\]
Let \(A(\tilde{x})\) be the Fuchsian system with \(2n + 1\) regular singularities at \(t_1, \ldots, t_{2n}, \infty\) as above. We fix the poles \(t_{2n-1}\) and \(t_{2n}\) at \(0\) and \(1\), respectively. Let \(\{\nu_1, \ldots, \nu_{2n-2}\}\) be the roots of the following equation of degree \(2n - 2\):
\[
\sum_{k=1}^{2n} \frac{\bar{H}_k}{\tilde{x} - t_k} = 0.
\]
For each \(\nu_i\), we define \(\rho_i\) by
\[
\rho_i := \sum_{k=1}^{2n} \frac{\bar{H}_k}{\nu_i - t_k}.
\]
If a tuple \((\tilde{H}_i(t))_{i=1,\ldots,2n}\) is a solution of the Schlesinger equations, then the corresponding functions \(\nu_j(t_1,\ldots,t_{2n-2})\) and \(\rho_j(t_1,\ldots,t_{2n-2})\) \((j = 1,\ldots,2n-2)\) satisfy the Garnier system \(G_{2n-2}\) (see [10 Theorem 2.1]).

4.4. **Algebraic solution.** By the morphism (23), we have a generically finite morphism

\[ \text{Spec } \mathbb{C}[\rho_i, \nu_i]_{1 \leq i \leq 2n-2} \times \tilde{T} \longrightarrow \text{Spec } \mathbb{C}[\rho_i, \nu_i]_{1 \leq i \leq 2n-2} \times \text{Spec } \mathbb{C}[t_1,\ldots,t_{2n-2}]. \]

We consider the algebraic solution of \(G_{2n-2}\) associated to the representation \(\rho'_\lambda\).

For the residue matrices \(H_\lambda^T\) of \((\nabla_{\mathcal{F}1 \times \mathcal{F}/\tilde{T}})\), we put

\[ H_\lambda^T := \begin{pmatrix} -1 & a-2 \\ 1 & a \end{pmatrix} H_\lambda \begin{pmatrix} -1 & a-2 \\ 1 & a \end{pmatrix}^{-1} \]

for \(i = 1,\ldots,2n\). Let \(\mathcal{A}_T(\tilde{x})\) be the family of the Fuchsian systems with \(2n+1\) regular singularities at \(0,1,t_1,\ldots,t_{2n-2},\infty\) parametrized by \(\tilde{T}\) defined by

\[ \mathcal{A}_T(\tilde{x}) := d + \tilde{H}_2^{T-1} \frac{d\tilde{x}}{\tilde{x}} + \tilde{H}_2^{T-1} \frac{d\tilde{x}}{\tilde{x}-1} + \sum_{i=1}^{2n-2} \tilde{H}_i^{T-1} \frac{d\tilde{x}}{\tilde{x}-t_i}. \]

Note that \(\tilde{H}_2^{T+1} := -\sum_{i=1}^{2n} \tilde{H}_i^{T}\) is a diagonal matrix. By Proposition 3.3 for each \(\tilde{t} \in \tilde{T}\), the Fuchsian system \(\mathcal{A}_T(\tilde{x})\) has the monodromy representation which is conjugated to \(\rho'_\lambda\), which is independent of \(\tilde{t} \in \tilde{T}\). That is, the family \(\mathcal{A}_T(\tilde{x})\) of the Fuchsian systems parametrized by \(\tilde{T}\) preserves their monodromy representations. By (31), (32), and (34), we have algebraic functions \(\nu_i, \rho_i (i = 1,\ldots,2n-2)\) on \(\tilde{T}\).

These algebraic functions give the solution of \(G_{2n-2}\) associated to the representation \(\rho'_\lambda\).

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