SIGN-CHANGING SOLUTIONS FOR A CLASS OF FRACTIONAL SCHRÖDINGER EQUATIONS WITH VANISHING POTENTIALS

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Abstract. In this paper we consider a class of fractional Schrödinger equations with potentials vanishing at infinity. By using a minimization argument and a quantitative deformation Lemma, we prove the existence of a sign-changing solution.

1. INTRODUCTION

In the past years there has been a considerable amount of research related to the existence of nontrivial solutions for Schrödinger-type equations

\[
\begin{aligned}
-\Delta u + V(x)u &= K(x)f(u) \quad \text{in } \mathbb{R}^N \\
u &\in D^{1,2}(\mathbb{R}^N)
\end{aligned}
\]

(1.1)

where \( V : \mathbb{R}^N \to \mathbb{R} \) and \( K : \mathbb{R}^N \to \mathbb{R} \) are positive and continuous functions, and \( f : \mathbb{R} \to \mathbb{R} \) is a continuous function satisfying suitable growth assumptions. An important class of problems associated to (1.1) is the so called zero mass case, which occurs when the potential \( V(x) \) vanishes at infinity. Such class of problems has been investigated by many authors by using several variational methods; see for instance [1, 3, 4, 13, 14, 15].

Recently, the study of nonlinear equations involving the fractional Laplacian has gained tremendous popularity due to their intriguing analytic structure and in view of several applications in different subjects, such as, optimization, finance, anomalous diffusion, phase transition, flame propagation, minimal surface. The literature on fractional and non-local operators of elliptic type and their applications is quite large, for example, we refer the interested reader to [5, 9, 16, 17, 19, 28, 32, 33, 36, 37] and references therein. For the basic properties of fractional Sobolev spaces with applications to partial differential equations, we refer the reader to [22, 31] and references therein.

Motivated by the interest shared by the mathematical community in this topic, the purpose of this paper is to study sign-changing (or nodal) solutions for the following class of fractional equations

\[
(-\Delta)\alpha u + V(x)u = K(x)f(u) \quad \text{in } \mathbb{R}^N,
\]

(1.2)

where \( \alpha \in (0, 1) \) and \( N > 2\alpha \), \( (-\Delta)^\alpha \) is the fractional Laplacian which can be defined for a function \( u \) belonging to the Schwartz space \( \mathcal{S}(\mathbb{R}^N) \) of rapidly decaying functions as

\[
(-\Delta)^\alpha u(x) = C_{N,\alpha} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2\alpha}}dy, \quad x \in \mathbb{R}^N,
\]

where P.V. stands for the Cauchy principal value and \( C_{N,\alpha} \) is a normalizing constant [22]. Here, we assume that \( V, K : \mathbb{R}^N \to \mathbb{R} \) are continuous on \( \mathbb{R}^N \), and that satisfy the following conditions. Indeed, we say \( (V, K) \in \mathcal{K} \) if

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(VK₁) \( V(x), K(x) > 0 \) for all \( x \in \mathbb{R}^N \) and \( K \in \mathcal{L}^\infty(\mathbb{R}^N) \); 

(VK₂) If \( \{A_n\}_n \subset \mathbb{R}^N \) is a sequence of Borel sets such that the Lebesgue measure \( m(A_n) \leq R \), for all \( n \in \mathbb{N} \) and some \( R > 0 \), then

\[
\lim_{r \to +\infty} \int_{A_n \cap B_r(0)} K(x) \, dx = 0, \quad \text{uniformly in } n \in \mathbb{N}.
\]

Furthermore, one of the below conditions occurs

(VK₃) \( \frac{K}{V} \in \mathcal{L}^\infty(\mathbb{R}^N) \)

or

(VK₄) there exists \( m \in (2, 2^*_\alpha) \) such that

\[
\frac{K(x)}{V(x)^{\frac{2^*_\alpha - m}{2^*_\alpha}}} \to 0 \quad \text{as } |x| \to +\infty,
\]

where \( 2^*_\alpha = \frac{2N}{N - 2\alpha} \).

Concerning the nonlinearity \( f : \mathbb{R} \to \mathbb{R} \), we assume that \( f \) is a \( C^1 \)-function and fulfills the following growth conditions in the origin and at infinity:

(\( f_1 \)) \( \lim_{|t| \to 0^+} \frac{f(t)}{|t|} = 0 \) if (VK₃) holds 

or

(\( \tilde{f}_1 \)) \( \lim_{|t| \to 0^+} \frac{f(t)}{|t|^{m - 1}} = 0 \) if (VK₄) holds with \( m \in (2, 2^*_\alpha) \) defined in (VK₄).

(\( f_2 \)) \( f \) has a quasicritical growth at infinity, namely

\[
\lim_{|t| \to +\infty} \frac{f(t)}{|t|^{2^*_\alpha - 1}} = 0,
\]

(\( f_3 \)) There exists \( \theta \in (2, 2^*_\alpha) \) so that

\[
0 < \theta F(t) = \theta \int_0^t f(\tau) \, d\tau \leq f(t) \, t \quad \text{for all } |t| > 0,
\]

(\( f_4 \)) The map \( f \) and its derivative \( f' \) satisfy

\[
f'(t) < \frac{f(t)}{t} \quad \text{for all } t \neq 0.
\]

Let us observe that by (\( f_4 \)) follows that \( t \mapsto \frac{f(t)}{|t|} \) is strictly increasing for all \( |t| > 0 \). Moreover,

\[
t \mapsto \frac{1}{2} f(t) t - F(t) \quad \text{is strictly increasing for every } t > 0
\]

strictly decreasing for every \( t < 0 \) \hspace{1cm} (1.3)

and, in particular

\[
\frac{t^2}{2} f'(t) - \frac{t}{2} f(t) > 0 \quad \text{for all } t \neq 0.
\] \hspace{1cm} (1.4)

Problem (1.2) appears in a lot of studies, for instance, the existence of standing wave solutions \( \psi(x, t) = u(x) e^{-i\omega t} \) for the fractional Schrödinger equation

\[
th \frac{\partial \psi}{\partial t} = h^2 (-\Delta)\alpha \psi + W(x) \psi - f(x, \psi) \quad \text{in } \mathbb{R}^N
\]
where ℏ is the Planck’s constant, $W : \mathbb{R}^N \to \mathbb{R}$ is an external potential and $f$ a suitable nonlinearity. This equation plays an important role in the fractional quantum mechanic, and was introduced by Laskin [26, 27] through expanding the Feynman path integral from the Brownian-like to the Lévy-like quantum mechanical paths. Lately the study of fractional Schrödinger equations has attracted the attention of many mathematicians; see for instance [6, 7, 20, 21, 23, 24, 25, 30, 34, 35] and references therein. In spite of the fact that there are many papers dealing with existence and multiplicity of solutions of fractional Schrödinger equations in $\mathbb{R}^N$, to our knowledge there are no papers investigating the existence of sign-changing solutions for fractional Schrödinger equations with potentials vanishing at infinity, and here we would like to go further in this direction.

The main result of this paper is the following:

**Theorem 1.1.** Suppose that $(V, K) \in \mathcal{K}$ and $f \in C^1(\mathbb{R}, \mathbb{R})$ verifies $(f_1)$ or $(\tilde{f}_1)$, and $(f_2) - (f_4)$. Then, problem (1.2) admits a least energy sign-changing weak solution.

For weak solution to (1.2), we mean a function $u \in \mathbb{X}$ such that

$$\int\int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2\alpha}} \, dx\, dy + \int_{\mathbb{R}^N} V(x)u(x)\varphi(x) \, dx = \int_{\mathbb{R}^N} K(x)f(u)\varphi(x) \, dx$$

for all $\varphi \in \mathbb{X}$, where

$$\mathbb{X} = \left\{ u \in \mathcal{D}^{\alpha,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^2 \, dx < +\infty \right\}.$$

The proof of Theorem 1.1 is obtained by adapting some arguments developed in [2, 8]. More precisely, we minimize the following Euler-Lagrange functional

$$J(u) := \frac{1}{2} \int\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\alpha}} \, dx\, dy + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|u|^2 \, dx - \int_{\mathbb{R}^N} K(x)F(u) \, dx$$

on the nodal set

$$\mathcal{M} := \{ w \in \mathcal{N} : w^+ \neq 0, w^- \neq 0, \langle J'(w), w^+ \rangle = 0 = \langle J'(w), w^- \rangle \}.$$

Then we prove that the minimum is achieved and, by using a suitable variant of the quantitative deformation Lemma, we show that it is a critical point of $J$. Clearly, due to the presence of the nonlocal term $\int\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\alpha}} \, dx\, dy$, the Euler-Lagrange functional $J$ does no longer satisfy the decompositions

$$J(u) = J(u^+) + J(u^-)$$

$$\langle J'(u), u^\pm \rangle = \langle J'(u^\pm), u^\pm \rangle,$$

which were very useful to apply the variational methods exploited in the classical literature; see for instance [8, 10, 11, 12, 18]. Therefore, in order to prove the existence of a sign-changing solution to (1.2), a more accurate investigation is needed in our setting.

The paper is organized as follows. In Section 2 we present the variational framework of the problem and compactness results which will be useful for the next sections. In Section 3 we give some technical lemmas used in the proof of the main result. Finally, the proof of Theorem 1.1 is given in Section 4.
2. Preliminary results

Firstly we recall some basic notation and facts which will be used in the sequel of the paper. We denote by $\mathcal{D}^{\alpha,2}(\mathbb{R}^N)$ the closure of functions $C_c^\infty(\mathbb{R}^N)$ with respect to the so called Gagliardo seminorm

$$[u]^2 = \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\alpha}} \, dx \, dy.$$ 

In order to prove that problem (1.2) has a variational structure, let us introduce the Hilbert space

$$X = \left\{ u \in \mathcal{D}^{\alpha,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^2 \, dx < +\infty \right\}$$

equipped with the norm

$$\|u\|^2 = [u]^2 + \int_{\mathbb{R}^N} V(x)|u|^2 \, dx.$$ 

Let $q \in \mathbb{R}$ such that $q \geq 1$, and let us define the weighted Lebesgue space

$$L^q_K(\mathbb{R}^N) = \left\{ u : \mathbb{R}^N \rightarrow \mathbb{R} \text{ measurable and } \int_{\mathbb{R}^N} K(x)|u|^q \, dx < \infty \right\}$$

equipped with the norm

$$\| \cdot \|_{L^q_K(\mathbb{R}^N)} = \left( \int_{\mathbb{R}^N} K(x)|u|^q \, dx \right)^{\frac{1}{q}}.$$ 

Now we prove the following continuous and compactness results, whose proofs can be obtained adapting the arguments in [1]. For the reader's convenience we give the proofs.

**Lemma 2.1.** Assume that $(V, K) \in \mathcal{K}$. Then $X$ is continuously embedded in $L^q_K(\mathbb{R}^N)$ for all $q \in [2, 2\alpha]$ if $(VK_3)$ holds. Moreover, $X$ is continuously embedded in $L^\infty_K(\mathbb{R}^N)$ if $(VK_4)$ holds.

**Proof.** We distinguish two cases.

Assume that $(VK_3)$ is true. The proof is trivial if $q = 2$ or $q = 2\alpha$.

Fix $q \in (2, 2\alpha)$ and let $\lambda = \frac{2\alpha - q}{2\alpha - 2}$. We can observe that $q$ can be written as $q = 2\lambda + (1 - \lambda)2\alpha$.

Then we have

$$\int_{\mathbb{R}^N} K(x)|u|^q \, dx = \int_{\mathbb{R}^N} K(x)|u|^{2\lambda}|u|^{(1-\lambda)2\alpha} \, dx$$

$$\leq \left( \int_{\mathbb{R}^N} |K(x)|^{\frac{\lambda}{2}} |u|^2 \, dx \right)^{\lambda} \left( \int_{\mathbb{R}^N} |u|^{2\alpha} \, dx \right)^{1-\lambda}$$

$$\leq \left( \sup_{x \in \mathbb{R}^N} \frac{|K(x)|}{|V(x)|^{\lambda}} \right) \left( \int_{\mathbb{R}^N} V(x)|u|^2 \, dx \right)^{\lambda} \left( \int_{\mathbb{R}^N} |u|^{2\alpha} \, dx \right)^{1-\lambda}$$

$$\leq C \left( \sup_{x \in \mathbb{R}^N} \frac{|K(x)|}{|V(x)|^{\lambda}} \right) \left( \int_{\mathbb{R}^N} V(x)|u|^2 \, dx \right)^{\lambda} \left( \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\alpha}} \, dx \, dy \right)^{(1-\lambda)\frac{2\alpha}{2}}$$

$$\leq C \left( \sup_{x \in \mathbb{R}^N} \frac{|K(x)|}{|V(x)|^{\lambda}} \right) \|u\|^{2\lambda + (1-\lambda)2\alpha}$$

$$= C \left( \sup_{x \in \mathbb{R}^N} \frac{|K(x)|}{|V(x)|^{\lambda}} \right) \|u\|^q.$$
Taking into account that $K \in L^\infty(\mathbb{R}^N)$ and $(VK_3)$ holds true, we conclude that

$$\|u\|_{L^2_k(\mathbb{R}^N)} \leq C\|u\|.$$ 

Now, we will suppose that $(VK_4)$ is true. Denoting $\lambda_0 = \frac{2^*_\alpha - m}{2^*_\alpha - 2}$, $m$ can be written as $m = 2\lambda_0 + (1 - \lambda_0)2^*_\alpha$. As above, we have

$$\int_{\mathbb{R}^N} K(x)|u|^m \, dx = \int_{\mathbb{R}^N} K(x)|u|^{2\lambda_0}|u|^{(1-\lambda_0)2^*_\alpha} \, dx$$

$$\leq \left( \int_{\mathbb{R}^N} |K(x)| \frac{1}{2^*_\alpha} |u|^2 \, dx \right)^{\lambda_0} \left( \int_{\mathbb{R}^N} |u|^{2^*_\alpha} \, dx \right)^{1-\lambda_0}$$

$$\leq \left( \sup_{x \in \mathbb{R}^N} \frac{|K(x)|}{|V(x)|^{\lambda_0}} \right) \left( \int_{\mathbb{R}^N} V(x)|u|^2 \, dx \right)^{\lambda_0} \left( \int_{\mathbb{R}^N} |u|^{2^*_\alpha} \, dx \right)^{1-\lambda_0}$$

$$\leq C \left( \sup_{x \in \mathbb{R}^N} \frac{|K(x)|}{|V(x)|^{\lambda_0}} \right) \|u\|^m.$$ 

Since $(VK_4)$ holds true, we have that $\frac{K(x)}{V(x)^{2^*_\alpha - 2}} \in L^\infty(\mathbb{R}^N)$, and hence

$$\|u\|_{L^\infty(\mathbb{R}^N)} \leq C\|u\|.$$ 

This complete the proof of our lemma. \hfill \Box

**Proposition 2.1.** Assume $(V, K) \in K$. The following facts hold:

1. $\mathcal{X}$ is compactly embedded into $L^q(\mathbb{R}^N)$ for all $q \in (2, 2^*_\alpha)$ if $(VK_3)$ holds;
2. $\mathcal{X}$ is compactly embedded into $L^m(\mathbb{R}^N)$ if $(VK_4)$ holds.

**Proof.** (1) Assume that $(VK_3)$ holds. Fix $q \in (2, 2^*_\alpha)$ and let $\varepsilon > 0$. Then, there exist $0 < t_0 < t_1$ and a positive constant $C$ such that

$$K(x)|t|^q \leq \varepsilon C \left[ V(x) |t|^2 + |t|^{2^*_\alpha} \right] + C K(x) \chi_{[t_0, t_1]}(|t|)|t|^{2^*_\alpha}, \text{ for all } t \in \mathbb{R}.$$ 

Integrating over $\mathcal{B}^r_0(0)$ we have, for all $u \in \mathcal{X}$ and $r > 0$,

$$\int_{\mathcal{B}^r_0(0)} K(x)|u|^q \, dx \leq \varepsilon C \int_{\mathcal{B}^r_0(0)} \left[ V(x)|u|^2 + |u|^{2^*_\alpha} \right] \, dx + C t_1^{2^*_\alpha} \int_{A \cap \mathcal{B}^r_0(0)} K(x) \, dx$$

$$=: \varepsilon C \mathcal{Q}(u) + C t_1^{2^*_\alpha} \int_{A \cap \mathcal{B}^r_0(0)} K(x) \, dx \tag{2.1}$$

where we set

$$\mathcal{Q}(u) := \int_{\mathcal{B}^r_0(0)} \left[ V(x)|u|^2 + |u|^{2^*_\alpha} \right] \, dx \text{ and } A = \{x \in \mathbb{R}^N : t_0 \leq |u(x)| \leq t_1\}.$$ 

Now, if $\{u_n\}_n \subset \mathcal{X}$ is a sequence such that $u_n \rightharpoonup u$ in $\mathcal{X}$, then there is $M > 0$ such that

$$\|u_n\|^2 \leq M \text{ and } \int_{\mathbb{R}^N} |u_n|^{2^*_\alpha} \, dx \leq M \forall n \in \mathbb{N}. \tag{2.2}$$
This implies that \( \{ Q(u_n) \} \) is bounded from above by a positive constant. Let us denote by 
\[ A_n = \{ x \in \mathbb{R}^N : t_0 \leq |u_n| \leq t_1 \} \]. By (2.2) we deduce 
\[ t_0^{2^*_m} m(A_n) \leq \int_{A_n} |u_n|^{2^*_m} \, dx \leq M \quad \forall n \in \mathbb{N}, \]
from which \( \sup_{n \in \mathbb{N}} |m(A_n)| < +\infty \). Therefore, from \((VK_2)\) there exists a positive radius \( r \) large enough such that 
\[ \int_{A_n \cap B_r^c(0)} K(x) \, dx < \frac{\varepsilon}{t_1^{2^*_m}} \text{ for all } n \in \mathbb{N}. \quad (2.3) \]
Therefore (2.1) and (2.3) lead to 
\[ \int_{B_r^c(0)} K(x)|u_n|^{q} \, dx \leq C M + C t_1^{2^*_m} \int_{A_n \cap B_r^c(0)} K(x) \, dx \leq (C M + C)\varepsilon \text{ for all } n \in \mathbb{N}. \quad (2.4) \]
Recalling that \( q \in (2, 2^{*}_m) \) and that \( K \) is a continuous function, by Sobolev embedding follows that 
\[ \lim_{n \to \infty} \int_{B_r(0)} K(x)|u_n|^{q} \, dx = \int_{B_r(0)} K(x)|u|^{q} \, dx. \quad (2.5) \]
From (2.4) for \( \varepsilon > 0 \) small enough and (2.5) it holds 
\[ \lim_{n \to \infty} \int_{\mathbb{R}^N} K(x)|u_n|^{q} \, dx = \int_{\mathbb{R}^N} K(x)|u|^{q} \, dx \]
from which we conclude that 
\[ u_n \to u \text{ in } L^q_K(\mathbb{R}^N), \text{ for every } q \in (2, 2^{*}_m). \]

(2) Let us suppose that \((VK_4)\) is true. Moreover, we can see that for each \( x \in \mathbb{R}^N \) fixed, the function 
\[ g(t) = V(x)t^{2^*_m - m} + t^{2^*_m - m}, \text{ for every } t > 0, \]
has \( C_m V(x)^{\frac{2^*_m - m}{2^*_m - 2}} \) as its minimum value, where \( C_m = \left( \frac{2^*_m - 2}{2^*_m - m} \right)^{\frac{m - 2}{2^*_m - 2}}. \) Hence 
\[ C_m V(x)^{\frac{2^*_m - m}{2^*_m - 2}} \leq V(x)t^{2^*_m - m} + t^{2^*_m - m}, \text{ for every } x \in \mathbb{R}^N \text{ and } t > 0. \]
Then, in combination with \((VK_4)\), for any \( \varepsilon > 0 \), in correspondence of a positive radius \( r > 0 \) sufficiently large it follows 
\[ K(x)|t|^{m} \leq \varepsilon C_m' \left[ V(x)|t|^2 + |t|^{2^*_m} \right], \text{ for every } t \in \mathbb{R} \text{ and } |x| \geq r, \]
where \( C_m' \) is the inverse of \( C_m \), and integrating over \( B_r^c(0) \) we get 
\[ \int_{B_r^c(0)} K(x)|u|^m \, dx \leq \varepsilon C_m' \left[ \|u\|^2 + \|u\|^{2^*_m}_{L^{2^*_m}(\mathbb{R}^N)} \right] \text{ for all } u \in \mathbb{X}. \quad (2.6) \]
If \( \{u_n\}_n \subset \mathbb{X} \) is a sequence such that \( u_n \to u \) in \( \mathbb{X} \), by (2.6) we deduce that 
\[ \int_{B_r^c(0)} K(x)|u|^m \, dx \leq \varepsilon C_m'' \text{ for all } n \in \mathbb{N}. \quad (2.7) \]
Recalling that \( m \in (2, 2_\alpha^*) \) and that \( K \) is a continuous function, it follows from Sobolev embedding that

\[
\lim_{n \to \infty} \int_{B_r(0)} K(x)|u_n|^m dx = \int_{B_r(0)} K(x)|u|^m dx.
\] (2.8)

Then, from (2.7) for \( \varepsilon > 0 \) small enough and (2.8) it holds

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} K(x)|u_n|^m dx = \int_{\mathbb{R}^N} K(x)|u|^m dx.
\]

from which we conclude that

\( u_n \to u \) in \( L^m_K(\mathbb{R}^N) \), for every \( m \in (2, 2_\alpha^*) \).

Then, the proof of our proposition is complete. \( \square \)

The following lemma is a compactness result related to the nonlinear term.

Lemma 2.2. Assume \((V,K) \in \mathcal{K} \) and \( f \) satisfies \((f_1) - (f_2)\) or \((f_1') - (f_2')\). Let \( u_n \) be a sequence such that \( u_n \rightharpoonup u \) in \( \mathbb{X} \), then, up to a subsequence,

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} K(x)F(u_n) \, dx = \int_{\mathbb{R}^N} K(x)F(u) \, dx
\]

and

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} K(x)f(u_n)u_n \, dx = \int_{\mathbb{R}^N} K(x)f(u)u \, dx.
\]

Proof. Assume that \((VK_3)\) holds. From \((f_1) - (f_2)\), fixed \( q \in (2, 2_\alpha^*) \) and given \( \varepsilon > 0 \), there exists \( C > 0 \) such that

\[
|K(x)f(t)t| \leq \varepsilon C \left[ V(x)|t|^2 + |t|^{2_\alpha} \right] + CK(x)|t|^q, \text{ for all } t \in \mathbb{R}. \tag{2.9}
\]

From Proposition 2.1 since

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} K(x)|u_n|^q \, dx = \int_{\mathbb{R}^N} K(x)|u|^q \, dx,
\]

there exists \( r > 0 \) such that

\[
\int_{B_r(0)} K(x)|u_n|^q \, dx < \varepsilon, \text{ for all } n \in \mathbb{N}. \tag{2.10}
\]

Since \( \{u_n\}_n \subset \mathbb{X} \) is bounded, there exists a positive constant \( C' \) such that

\[
\int_{\mathbb{R}^N} V(x)|u_n|^2 \, dx \leq C' \text{ and } \int_{\mathbb{R}^N} |u_n|^{2_\alpha} \, dx \leq C', \text{ for all } n \in \mathbb{N}. \tag{2.11}
\]

Combining (2.9), (2.10), and (2.11) we have

\[
\int_{B_r(0)} K(x)|u_n|^q \, dx < (2CC' + 1)\varepsilon, \text{ for all } n \in \mathbb{N}.
\]

Assume that \((VK_4)\) holds. Similarly to the second part of Proposition 2.1, given \( \varepsilon > 0 \) sufficiently small, there exists \( r > 0 \) large enough such that

\[
K(x) \leq \varepsilon C'_m \left[ V(x)|t|^{2-m} + |t|^{2_\alpha-m} \right], \text{ for every } |t| > 0 \text{ and } |x| > r.
\]

Consequently, for all \( |t| > 0 \) and \( |x| > r \)

\[
K(x)|f(t)t| \leq \varepsilon C'_m \left[ V(x)|f(t)t||t|^{2-m} + |f(t)t||t|^{2_\alpha-m} \right].
\]
From \((f_1)\) and \((f_2)\), there exist \(C, t_0, t_1 > 0\) satisfying
\[
K(x)|f(t)t| \leq \varepsilon C \left[ V(x)|t|^2 + |t|^{2\alpha} \right], \text{ for every } t \in I \text{ and } |x| > r
\]
where \(I = \{ t \in \mathbb{R} : |t| < t_0 \text{ or } |t| > t_1 \}\). Therefore, for every \(u \in X\), setting
\[
Q(u) = \int_{\mathbb{R}^N} V(x)|u|^2 \, dx + \int_{\mathbb{R}^N} |u|^{2\alpha} \, dx
\]
and \(A = \{ x \in \mathbb{R}^N : t_0 \leq |u(x)| \leq t_1 \}\), the following estimate holds
\[
\int_{B_r(0)} K(x)f(u)u \, dx \leq \varepsilon C Q(u) + C \int_{A \cap B_r(0)} K(x) \, dx.
\]
Since \(\{u_n\}_n \subset X\) is bounded, there exists \(C' > 0\) such that
\[
\int_{\mathbb{R}^N} V(x)|u_n|^2 \, dx \leq C' \text{ and } \int_{\mathbb{R}^N} |u_n|^{2\alpha} \, dx \leq C', \text{ for all } n \in \mathbb{N}.
\]
Therefore
\[
\int_{B_r(0)} K(x)f(u_n)u_n \, dx \leq \varepsilon C'' + C \int_{A_n \cap B_r(0)} K(x) \, dx,
\]
where \(A_n = \{ x \in \mathbb{R}^N : r_0 \leq |u_n(x)| \leq r_1 \}\). Following the same arguments in the proof of Proposition 2.1 and by \((VK_2)\) we deduce that
\[
\int_{A_n \cap B_r(0)} K(x) \, dx \to 0 \text{ as } r \to +\infty
\]
uniformly in \(n \in \mathbb{N}\) and, for \(\varepsilon > 0\) small enough
\[
\left| \int_{B_r(0)} K(x)f(u_n)u_n \, dx \right| < (C'' + 1)\varepsilon.
\]
In order to complete the proof, we have to prove that
\[
\lim_{n \to +\infty} \int_{B_r(0)} K(x)f(u_n)u_n \, dx = \int_{B_r(0)} K(x)f(u)u \, dx
\]
which follows by the compactness Lemma of Strauss [14]. \(\square\)

3. Technical Lemmas

In the following we look for sign-changing weak solutions of problem \((1.2)\), that is a function \(u \in X\) such that \(u^+ := \max\{u, 0\}, \ u^- := \min\{u, 0\}\) in \(\mathbb{R}^N\) and
\[
\iint_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2\alpha}} \, dx \, dy + \int_{\mathbb{R}^N} V(x)u(x)\varphi(x) \, dx = \int_{\mathbb{R}^N} K(x)f(u)\varphi(x) \, dx
\]
for all \(\varphi \in X\).

In view of assumptions on \(V, K\) and \(f\), we can see that the functional \(J : X \to \mathbb{R}\) defined by
\[
J(u) := \frac{1}{2}[u]^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|u|^2 \, dx - \int_{\mathbb{R}^N} K(x)F(u) \, dx
\]
is Fréchet differentiable and that its differential \(J'\) is given by
\[
\langle J'(u), \varphi \rangle = \iint_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2\alpha}} \, dx \, dy + \int_{\mathbb{R}^N} V(x)u(x)\varphi(x) \, dx - \int_{\mathbb{R}^N} K(x)f(u)\varphi(x) \, dx
\]
for all \( u, \varphi \in X \). Then, the weak solutions of problem (1.2) are the critical points of \( J \). Associated to \( J \), we introduce the following Nehari manifold

\[
\mathcal{N} := \{ u \in X \setminus \{0\} : \langle J'(u), u \rangle = 0 \}.
\]

Since we look for sign-changing solution to (1.2), it is natural to seek functions \( w \in M \) such that

\[
J(w) = \inf_{v \in M} J(v),
\]

where

\[
M := \{ w \in \mathcal{N} : w^+ \neq 0, w^- \neq 0, \langle J'(w), w^+ \rangle = 0 = \langle J'(w), w^- \rangle \}.
\]

Let us point out that for all \( u \in X \)

\[
[u]^2 = [u^+]^2 + [u^-]^2 - \int_{\mathbb{R}^{2N}} \frac{(u^+(x)u^-(y) + u^-(x)u^+(y))}{|x - y|^{N+2\alpha}} \, dx \, dy \geq [u^+]^2 + [u^-]^2
\]

so, we can deduce that

\[
J(u) = J(u^+) + J(u^-) - \int_{\mathbb{R}^{2N}} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x - y|^{N+2\alpha}} \, dx \, dy,
\]

and

\[
\langle J'(u), u^+ \rangle = \langle J'(u^+), u^+ \rangle - \int_{\mathbb{R}^{2N}} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x - y|^{N+2\alpha}} \, dx \, dy.
\]

In particular, for all \( w \in M \) it results

\[
\langle J'(w^\pm), w^\pm \rangle \leq 0.
\]

As in [2, 8], we are able to prove the existence of a minimizer of \( J \) on \( M \) and that it is a weak solution to (1.2) by using a suitable deformation argument.

Now, we collect some preliminary lemmas which will be used in the last section to prove our main result.

**Lemma 3.1.**

(i) For all \( u \in \mathcal{N} \) such that \( \|u\| \to +\infty \), then \( J(u) \to +\infty \);

(ii) There exists \( \varrho > 0 \) such that \( \|u\| \geq \varrho \) for all \( u \in \mathcal{N} \) and \( \|w^\pm\| \geq \varrho \) for all \( w \in M \).

**Proof.** (i) By using the definition of \( \mathcal{N} \) and taking into account the assumption \((f_3)\) we get

\[
J(u) = J(u) - \frac{1}{\theta} \langle J'(u), u \rangle
\]

\[
= \left( \frac{1}{2} - \frac{1}{\theta} \right) \|u\|^2 - \int_{\mathbb{R}^{N}} K(x) \left[ F(u) - \frac{1}{\theta} f(u)u \right] \, dx
\]

\[
= \left( \frac{1}{2} - \frac{1}{\theta} \right) \|u\|^2 + \frac{1}{\theta} \int_{\mathbb{R}^{N}} K(x) \left[ f(u)u - \theta F(u) \right] \, dx
\]

\[
\geq \left( \frac{1}{2} - \frac{1}{\theta} \right) \|u\|^2.
\]

So, when \( \|u\| \to +\infty \), the last inequality implies that \( J(u) \to +\infty \).
(ii) By assumptions \((f_1) - (f_2)\) or \((f_1) - (f_2)\) we have that, for any \(\varepsilon > 0\) there exists a positive constant \(C_\varepsilon\) such that
\[
|f(t)| \leq \varepsilon |t|^2 + C_\varepsilon |t|^{2n}, \quad \text{for all } t \in \mathbb{R} \\
|f(t)| \leq \varepsilon |t|^m + C_\varepsilon |t|^{2n}, \quad \text{for all } t \in \mathbb{R}.
\]
Since \(u \in \mathcal{N}\) we have \(\langle J'(u), u \rangle = 0\), that is
\[
\|u\|^2 = \int_{\mathbb{R}^N} K(x) f(u) \, dx.
\]
Now we distinguish two cases.

Assume that \((VK_3)\) holds, then, by applying \((3.1)\) and Proposition 2.1 we get
\[
\|u\|^2 \leq \frac{\|K\|_{L^\infty(\mathbb{R}^N)}}{V} \varepsilon \int_{\mathbb{R}^N} V(x) |u|^2 \, dx + C_\varepsilon \int_{\mathbb{R}^N} K(x) |u|^{2n} \, dx \\
\leq \frac{\|K\|_{L^\infty(\mathbb{R}^N)}}{V} \|u\|^2 + C_\varepsilon S \|K\|_{L^\infty(\mathbb{R}^N)} \|u\|^{2n} \tag{3.4}
\]
where \(S\) is the Sobolev embedding constant.

Let us suppose that \((VK_4)\) holds true. By \((3.2)\) and Proposition 2.1 we have
\[
\|u\|^2 \leq \varepsilon \int_{\mathbb{R}^N} K(x) |u|^m \, dx + C_\varepsilon \int_{\mathbb{R}^N} K(x) |u|^{2n} \, dx \\
\leq \varepsilon \|u\|^m + C_\varepsilon S \|K\|_{L^\infty(\mathbb{R}^N)} \|u\|^{2n} \tag{3.5}
\]
Since \(m \in (2, 2n)\), we can choose \(\varepsilon\) small enough in order to find \(\varrho > 0\) such that \(\|u\| \geq \varrho\).

Now, for \(w \in \mathcal{M}\), we have that \(\langle J'(w), w^\pm \rangle = 0\), so
\[
\langle J'(w^\pm), w^\pm \rangle = \iint_{\mathbb{R}^{2N}} \frac{w^+(x) w^-(y) + w^-(x) w^+(y)}{|x-y|^{N+2\alpha}} \, dx\, dy \leq 0
\]
which gives
\[
\|w^\pm\|^2 \leq \int_{\mathbb{R}^N} K(x) w^\pm f(w^\pm) \, dx.
\]
Then we can proceed as in the proof of \((i)\). \qed

**Lemma 3.2.** Let \(\{w_n\}_n \subset \mathcal{M}\) such that \(w_n \rightharpoonup w\) in \(\mathbb{X}\). Then \(w^\pm \neq 0\).

**Proof.** Firstly we observe that by Lemma 3.1 there exists \(\varrho > 0\) such that
\[
\|w_n^\pm\| \geq \varrho \quad \text{for all } n \in \mathbb{N}. \tag{3.6}
\]
Since \(w_n \in \mathcal{M}\), we have \(\langle J'(w_n), w_n^\pm \rangle = 0\), that is
\[
\|w_n^\pm\|^2 - \iint_{\mathbb{R}^{2N}} \frac{w_n^+(x) w_n^-(y) + w_n^-(x) w_n^+(y)}{|x-y|^{N+2\alpha}} \, dx\, dy = \int_{\mathbb{R}^N} K(x) f(w_n^\pm) w_n^\pm \, dx. \tag{3.7}
\]
At this point, recalling that
\[
- \iint_{\mathbb{R}^{2N}} \frac{w_n^+(x) w_n^-(y) + w_n^-(x) w_n^+(y)}{|x-y|^{N+2\alpha}} \, dx\, dy \geq 0,
\]
by (3.6) and (3.7) we deduce that
\[ g^2 \leq \|w_n^\pm\|^2 \leq \int_{\mathbb{R}^N} K(x)f(w_n^\pm)w_n^\pm \, dx. \] (3.8)

Now, by \( w_n \to w \) in \( X \) and Proposition 2.1, we know that \( w_n \to w \) in \( L_{K}^r(\mathbb{R}^N) \). Moreover, by using \( |t^\pm - s^\pm| \leq |t-s| \) for all \( t, s \in \mathbb{R} \), we can deduce that \( w_n^\pm \to w^\pm \) in \( L_{K}^r(\mathbb{R}^N) \), and being \( K(x) > 0 \) for all \( x \in \mathbb{R}^N \), we also have \( w_n^\pm \to w^\pm \) a.e. in \( \mathbb{R}^N \). Then, we can argue as in Lemma 2.2 to deduce that
\[ \int_{\mathbb{R}^N} K(x)f(w_n^\pm)w_n^\pm \, dx \to \int_{\mathbb{R}^N} K(x)f(w^\pm)w^\pm \, dx. \] (3.9)

Putting together (3.8) and (3.9) we have
\[ 0 < g^2 \leq \int_{\mathbb{R}^N} K(x)f(w^\pm)w^\pm \, dx \]
showing that \( w^\pm \neq 0 \). \( \square \)

**Lemma 3.3.** If \( v \in X : v^\pm \neq 0 \), then there exist \( s,t > 0 \) such that
\[ \langle J'(tv^+ + sv^-), v^+ \rangle = 0 \quad \text{and} \quad \langle J'(tv^+ + sv^-), v^- \rangle = 0. \]

As a consequence \( tv^+ + sv^- \in M \).

**Proof.** Let \( G : (0, +\infty) \times (0, +\infty) \to \mathbb{R}^2 \) be a continuous vector field given by
\[ G(t,s) = (\langle J'(tv^+ + sv^-), tv^+ \rangle, \langle J'(tv^+ + sv^-), sv^- \rangle) \]
for every \( t, s \in (0, +\infty) \times (0, +\infty) \). We distinguish two cases.

Assume that \( (VK_3) \) holds. By using (3.1) and Proposition 2.1 we have
\[ \langle J'(tv^+ + sv^-), tv^+ \rangle = t^2\|v^+\|^2 - st \int_{\mathbb{R}^{2N}} \frac{v^+(x)v^-(y) + v^-(x)v^+(y)}{|x-y|^{N+2\alpha}} \, dxdy - \int_{\mathbb{R}^N} K(x)tv^+ f(tv^+) \, dx \]
\[ \geq t^2\|v^+\|^2 - \int_{\mathbb{R}^N} K(x)tv^+ f(tv^+) \, dx \]
\[ \geq t^2\|v^+\|^2 - \|K\|_{L^\infty(\mathbb{R}^N)} t^2\|v^+\|^2 - C_\epsilon t^{2\alpha} \|K\|_{L^\infty(\mathbb{R}^N)} \|v^+\|^{2\alpha} \]
\[ = \left( 1 - \epsilon \|K\|_{L^\infty(\mathbb{R}^N)} \right) t^2\|v^+\|^2 - C_\epsilon t^{2\alpha} \|K\|_{L^\infty(\mathbb{R}^N)} \|v^+\|^{2\alpha}. \]

Suppose that \( (VK_4) \) holds. Then by (3.1) and Proposition 2.1 we deduce
\[ \langle J'(tv^+ + sv^-), tv^+ \rangle = t^2\|v^+\|^2 - st \int_{\mathbb{R}^{2N}} \frac{v^+(x)v^-(y) + v^-(x)v^+(y)}{|x-y|^{N+2\alpha}} \, dxdy - \int_{\mathbb{R}^N} K(x)tv^+ f(tv^+) \, dx \]
\[ \geq t^2\|v^+\|^2 - \epsilon t^m\|v^+\|^m - C_\epsilon t^{2\alpha} \|K\|_{L^\infty(\mathbb{R}^N)} \|v^+\|^{2\alpha}. \]

Then there exists \( r > 0 \) small enough such that \( \langle J'(rv^+ + sv^-), rv^+ \rangle > 0 \) for all \( s > 0 \), and similarly there exists \( \tilde{r} > 0 \) small enough such that \( \langle J'(tv^+ + \tilde{r}v^-), \tilde{r}v^- \rangle > 0 \) for all \( t > 0 \).

By assumption \((f_3)\) there exists a positive constant \( C_1 \) such that
\[ F(t) \geq C_1 t^{\theta}, \quad \text{for every } t \text{ sufficiently large}. \] (3.10)
Hence, taking into account that $\theta \in (2, 2_\alpha)$, we get
\[
\langle J'(tv^+ + sv^-), tv^+ \rangle = t^2\|v^+\|^2 - st \int_{\mathbb{R}^{2N}} \frac{v^+(x)v^-(y) + v^-(x)v^+(y)}{|x-y|^{N+2\alpha}} \, dx \, dy - \int_{\mathbb{R}^N} K(x)tv^+f(tv^+) \, dx \\
\leq t^2\|v^+\|^2 - st \int_{\mathbb{R}^{2N}} \frac{v^+(x)v^-(y) + v^-(x)v^+(y)}{|x-y|^{N+2\alpha}} \, dx \, dy - t^\theta C_1 \int_{\mathbb{R}^N} K(x)|v^+|^\theta \, dx \\
\to -\infty \text{ as } t \to +\infty.
\]
Then, there exists $R > 0$ sufficiently large such that $\langle J'(Rv^+ + sv^-), Rv^+ \rangle < 0$ for all $s > 0$ and similarly we can find $\tilde{R} > 0$ such that $\langle J'(tv^+ + \tilde{R}v^-), \tilde{R}v^- \rangle < 0$ for all $t > 0$. As a consequence, we have proved the existence of suitable $0 < r < R$ such that, for all $t, s \in [r, R]$ it holds
\[
\langle J'(rv^+ + sv^-), rv^+ \rangle > 0 \text{ and } \langle J'(tv^+ + \tilde{r}v^-), \tilde{r}v^- \rangle > 0,
\]
and
\[
\langle J'(Rv^+ + sv^-), Rv^+ \rangle < 0 \text{ and } \langle J'(tv^+ + \tilde{R}v^-), \tilde{R}v^- \rangle < 0.
\]
By applying Miranda’s theorem [29] we can conclude. □

For each $v \in X$ with $v^\pm \neq 0$, let us consider the function $h^v : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ given by
\[
h^v(t, s) = J(tv^+ + sv^-)
\]
and its gradient $\Phi^v : [0, +\infty) \times [0, +\infty) \to \mathbb{R}^2$ defined by
\[
\Phi^v(t, s) = (\Phi^v_1(t, s), \Phi^v_2(t, s))
\]
\[= \left( \frac{\partial h^v}{\partial t}(t, s), \frac{\partial h^v}{\partial s}(t, s) \right)
\]
\[= \left( \langle J'(tv^+ + sv^-), v^+ \rangle, \langle J'(tv^+ + sv^-), v^- \rangle \right). \quad (3.12)
\]
Furthermore, we consider the Jacobian matrix of $\Phi^v$:
\[
(\Phi^v)'(t, s) = \begin{pmatrix}
\frac{\partial \Phi^v_1}{\partial t}(t, s) & \frac{\partial \Phi^v_1}{\partial s}(t, s) \\
\frac{\partial \Phi^v_2}{\partial t}(t, s) & \frac{\partial \Phi^v_2}{\partial s}(t, s)
\end{pmatrix}
\]
In the following we aim to prove that, if $w \in \mathcal{M}$, the function $h^w$ has a critical point and in particular a global minimum in $(t, s) = (1, 1)$.

**Lemma 3.4.** If $w \in \mathcal{M}$, then
(a) $h^w(t, s) < h^w(1, 1) = J(w)$, for all $t, s \geq 0$ such that $(t, s) \neq (1, 1)$;
(b) $\det(\Phi^w)'(1, 1) > 0$.

**Proof.** (a) Since $w \in \mathcal{M}$, then $\langle J'(w), w^\pm \rangle = 0$, that is
\[
\|w^+\|^2 - \int_{\mathbb{R}^{2N}} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{N+2\alpha}} \, dx \, dy = \int_{\mathbb{R}^N} K(x)w^+f(w^+) \, dx \\
\|w^-\|^2 - \int_{\mathbb{R}^{2N}} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{N+2\alpha}} \, dx \, dy = \int_{\mathbb{R}^N} K(x)w^-f(w^-) \, dx.
\]
From this and by the definition of $\Phi^w$, it follows that $(1, 1)$ is a critical point of $h^w$.

Let $t, s \geq 0$. Then, by $(f_3)$ we get

$$h^w(t, s) = J(tw^+ + sw^-)$$
$$\leq \frac{1}{2} \|tw^+ + sw^-\|^2 - C_0 \int_{\mathbb{R}^N} K(x)|tw^+ + sw^-|^\theta \, dx.$$  

Let us suppose that $|t| \geq |s| > 0$. Thus, by using $F(t) \geq 0$ for all $t \in \mathbb{R}$, we have

$$h^w(t, s) \leq \frac{(t^2 + s^2)}{2} \left[ \|w^+\|^2 + \|w^-\|^2 - \int_{\mathbb{R}^{2N}} \frac{w^+(x)w^-(y) + w^+(y)w^-(x)}{|x - y|^{N+2\alpha}} \, dxdy \right]$$
$$- C_0 t^\theta \int_{A^+} K(x)|w^+|^\theta \, dx$$

where $A^+ \subset \text{supp}(w^+)$ has positive and finite measure. By using $t^2 + s^2 \leq 2t^2$ we can see that

$$\frac{h^w(t, s)}{t^2 + s^2} \leq \frac{1}{2} \left[ \|w^+\|^2 + \|w^-\|^2 - \int_{\mathbb{R}^{2N}} \frac{w^+(x)w^-(y) + w^+(y)w^-(x)}{|x - y|^{N+2\alpha}} \, dxdy \right] - \frac{t^{\theta-2}}{2} \int_{A^+} K(x)|w^+|^\theta \, dx$$

and taking into account that $\theta > 2$, we can infer that

$$\lim_{|(t, s)| \to +\infty} h^w(t, s) = -\infty.$$ 

By using the continuity of $h^w$ we can deduce the existence of $(\bar{t}, \bar{s}) \in [0, +\infty) \times [0, +\infty)$ that is a global maximum point of $h^w$.

Now we prove that $\bar{t}, \bar{s} > 0$. Suppose by contradiction that $\bar{s} = 0$. Then $\langle J'(\bar{t}w^+), \bar{t}w^+ \rangle = 0$, that is

$$\|w^+\|^2 = \int_{\mathbb{R}^N} K(x)(w^+)^2 \frac{f(w^+)}{tw^+} \, dx. \quad (3.13)$$

Since

$$\langle J'(w^+), w^+ \rangle = \langle J'(w), w^+ \rangle + \int_{\mathbb{R}^{2N}} \frac{w^+(x)w^-(y) + w^-(y)w^+(x)}{|x - y|^{N+2\alpha}} \, dxdy < 0$$

we get

$$\|w^+\|^2 < \int_{\mathbb{R}^N} K(x)(w^+)^2 \frac{f(w^+)}{w^+} \, dx. \quad (3.14)$$

Then, combining (3.13) and (3.14) we obtain

$$0 \leq \int_{\mathbb{R}^N} K(x) \left[ \frac{f(w^+)}{w^+} - \frac{f(\bar{t}w^+)}{\bar{t}w^+} \right] (w^+)^2 \, dx$$

so, by \((f_4)\), we deduce that \(\bar{t} \leq 1\). Taking into account \((1.3)\) we can infer
\[
\tilde{h}^w(\bar{t}, 0) = J(\bar{t} w^+)
= J(\bar{t} w^+) - \frac{1}{2} \langle J'(\bar{t} w^+), \bar{t} w^+ \rangle \\
= \int_{\mathbb{R}^N} K(x) \left[ \frac{1}{2} \bar{t} w^+ f(\bar{t} w^+) - F(\bar{t} w^+) \right] \, dx \\
\leq \int_{\mathbb{R}^N} K(x) \left[ \frac{1}{2} \bar{t} w^+ f(\bar{t} w^+) - F(\bar{t} w^+) \right] \, dx + \int_{\mathbb{R}^N} K(x) \left[ \frac{1}{2} \bar{t} w^- f(\bar{t} w^-) - F(\bar{t} w^-) \right] \, dx \\
= J(w) - \frac{1}{2} \langle J'(w), w \rangle \\
= J(w) = h^w(1, 1).
\]
Then \(\tilde{h}^w(\bar{t}, 0) < h^w(1, 1)\), and this gives a contradiction because \((\bar{t}, 0)\) is a global maximum point. Similarly we can prove that \(\bar{t} > 0\).

Now we show that \(\bar{s}, \bar{t} \leq 1\). Since \((\tilde{h}^w)'(\bar{t}, \bar{s}) = 0\), we get
\[
\bar{t}^2 \|w^+\|^2 - \bar{s} \int \int_{\mathbb{R}^{2N}} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{N+2\alpha}} \, dx \, dy = \int_{\mathbb{R}^N} K(x)\bar{t} w^+ f(\bar{t} w^+) \, dx \\
\bar{s}^2 \|w^-\|^2 - \bar{s} \int \int_{\mathbb{R}^{2N}} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{N+2\alpha}} \, dx \, dy = \int_{\mathbb{R}^N} K(x)\bar{s} w^- f(\bar{s} w^-) \, dx.
\]
Assume that \(\bar{t} \geq \bar{s}\). Since
\[
\int \int_{\mathbb{R}^{2N}} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{N+2\alpha}} \, dx \, dy \leq 0,
\]
we have
\[
\bar{t}^2 \|w^+\|^2 - \bar{s}^2 \|w^-\|^2 - \bar{s} \int \int_{\mathbb{R}^{2N}} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{N+2\alpha}} \, dx \, dy \geq \int_{\mathbb{R}^N} K(x)\bar{t} w^+ f(\bar{t} w^+) \, dx. \quad (3.15)
\]
Since \(\langle J'(w), w^+ = 0 \ (w \in \mathcal{M})\), we deduce that
\[
\|w^+\|^2 - \int \int_{\mathbb{R}^{2N}} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{N+2\alpha}} \, dx \, dy = \int_{\mathbb{R}^N} K(x)w^+ f(w^+) \, dx
\]
which together with \((3.15)\) gives
\[
0 \geq \int_{\mathbb{R}^N} K(x) \left[ \frac{f(\bar{t} w^+)}{\bar{t} w^+} - \frac{f(w^+)}{w^+} \right] \, dx.
\]
By \((f_4)\) we can infer that \(\bar{t} \leq 1\).

Now we aim to prove that \(h^w\) does not assume a global maximum in \([0, 1] \times [0, 1] \setminus \{(1, 1)\}\), namely
\[
h^w(\bar{t}, \bar{s}) < h^w(1, 1) \text{ for every } (\bar{t}, \bar{s}) \in [0, 1] \times [0, 1] \setminus \{(1, 1)\}.
\]
Let us observe that by the linearity of \(F\) and the positivity of \(K\) it follows that
\[
\int_{\mathbb{R}^N} K(x)F(w) \, dx = \int_{\mathbb{R}^N} K(x)(F(w^+) + F(w^-)) \, dx.
\]
Then, by the definition of $h^w$ and (1.3) we get
\begin{align*}
h^w(\bar{t}, \bar{s}) &= J(\bar{t}w^+ + \bar{s}w^-) - \frac{1}{2} \langle J'(\bar{t}w^+ + \bar{s}w^-), \bar{t}w^+ \rangle - \frac{1}{2} \langle J'(\bar{t}w^+ + \bar{s}w^-), \bar{s}w^- \rangle \\
&= \frac{\bar{t}^2}{2} \|w^+\|^2 + \frac{\bar{s}^2}{2} \|w^-\|^2 - \bar{t} \int_{\mathbb{R}^N} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{N+2\alpha}} \, dx dy - \int_{\mathbb{R}^N} K(x)F'(\bar{t}w^+) \, dx \\
&\quad - \int_{\mathbb{R}^N} K(x)F'(\bar{s}w^-) \, dx - \frac{\bar{t}^2}{2} \|w^+\|^2 - \frac{\bar{s}^2}{2} \|w^-\|^2 + \bar{t} \int_{\mathbb{R}^N} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{N+2\alpha}} \, dx dy \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^N} K(x)\bar{t}w^+ f(\bar{t}w^+) \, dx + \frac{1}{2} \int_{\mathbb{R}^N} K(x)\bar{s}w^- f(\bar{s}w^-) \, dx \\
&= \frac{1}{2} \int_{\mathbb{R}^N} K(x)[\bar{t}w^+ f(\bar{t}w^+) - F(\bar{t}w^+)] \, dx + \frac{1}{2} \int_{\mathbb{R}^N} K(x)[\bar{s}w^- f(\bar{s}w^-) - F(\bar{s}w^-)] \, dx \\
&< \frac{1}{2} \int_{\mathbb{R}^N} K(x)[w^+ f(w^+) - F(w^+)] \, dx + \frac{1}{2} \int_{\mathbb{R}^N} K(x)[w^- f(w^-) - F(w^-)] \, dx \\
&= \frac{1}{2} \int_{\mathbb{R}^N} K(x)[wf(w) - F(w)] \, dx = h^w(1, 1).
\end{align*}

(b) Firstly, let us observe that
\begin{align}
\frac{\partial \Phi^w(t, s)}{\partial t}(t, s) &= \|w^+\|^2 - \int_{\mathbb{R}^N} K(x)f'(\bar{t}w^+)(w^+)^2 \, dx \\
\frac{\partial \Phi^w(t, s)}{\partial s}(t, s) &= \|w^-\|^2 - \int_{\mathbb{R}^N} K(x)f'(\bar{s}w^-)(w^-)^2 \, dx \\
\frac{\partial \Phi^w(t, s)}{\partial s}(t, s) &= -\int_{\mathbb{R}^N} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{N+2\alpha}} \, dx. \tag{3.16}
\end{align}

Then, by using the fact that $w \in \mathcal{M}$, (3.16) and (1.4) we have
\begin{align*}
det(\Phi^w)'(1, 1) &= \left[ \|w^+\|^2 - \int_{\mathbb{R}^N} f'(w^+)(w^+)^2 \, dx \right] \left[ \|w^-\|^2 - \int_{\mathbb{R}^N} f'(w^-)(w^-)^2 \, dx \right] \\
&= \left( \int_{\mathbb{R}^N} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{N+2\alpha}} \, dx \right)^2 \\
&= \left( \int_{\mathbb{R}^N} (w^+ f(w^+) - f'(w^+)(w^+)^2) \, dx + \int_{\mathbb{R}^N} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{N+2\alpha}} \, dx \right) \\
&\quad \times \left( \int_{\mathbb{R}^N} (w^- f(w^-) - f'(w^-)(w^-)^2) \, dx + \int_{\mathbb{R}^N} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{N+2\alpha}} \, dx \right) \\
&= \left( \int_{\mathbb{R}^N} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{N+2\alpha}} \, dx \right)^2 > 0.
\end{align*}

\[\square\]

4. Proof of Theorem 1.1

In this section we will prove the existence of $w \in \mathcal{M}$ in which the infimum of $J$ is achieved on $\mathcal{M}$. Then, by using a quantitative deformation lemma, we show that $w$ is a critical point of $J$, so a sign-changing solution of (1.2).
By using Lemma 3.1 we know that there exists a minimizing sequence \( \{w_n\} \subset \mathcal{M} \), bounded in \( X \), such that

\[
J(w_n) \to \inf_{v \in \mathcal{M}} J(v) =: c_0 > 0. 
\] (4.1)

By using Proposition 2.1 we can assume that

\[
w_n^+ \rightharpoonup w^+ \text{ in } X, \\
w_n^+ \to w^+ \text{ in } L^m_K(\mathbb{R}^N), \\
w_n^+ \to w^+ \text{ a.e. in } \mathbb{R}^N.
\]

From Lemma 3.2 we deduce that \( w^\pm \neq 0 \), so \( w = w^+ + w^- \) is sign-changing. By Lemma 3.3, there exist \( s,t > 0 \) such that

\[
\langle J'(tw^+ + sw^-), w^+ \rangle = 0, \quad \langle J'(tw^+ + sw^-), w^- \rangle = 0 \quad (4.2)
\]

and \( tw^+ + sw^- \in \mathcal{M} \). Now, we prove that \( s,t \leq 1 \). Since \( w_n \in \mathcal{M} \), we have \( \langle J'(w_n), w_n^\pm \rangle = 0 \) or equivalently

\[
\|w_n^+\|^2 - \int_{\mathbb{R}^N} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{N+2\alpha}} dxdy = \int_{\mathbb{R}^N} K(x)w_n^+ f(w_n^+) dx 
\] (4.3)

\[
\|w_n^-\|^2 - \int_{\mathbb{R}^N} \frac{w^+(x)w^-(y) + w^-(x)w^+(y)}{|x-y|^{N+2\alpha}} dxdy = \int_{\mathbb{R}^N} K(x)w_n^- f(w_n^-) dx. \quad (4.4)
\]

The weak lower semicontinuity of the norm \( \| \cdot \| \) in \( X \) yields

\[
\|w^\pm\|^2 \leq \liminf_{n \to \infty} \|w_n^\pm\|^2, \quad (4.5)
\]

and by using similar argument as in Lemma 2.2, we have

\[
\int_{\mathbb{R}^N} K(x) f(w_n^\pm) w_n^\pm dx \to \int_{\mathbb{R}^N} K(x) f(w^\pm) w^\pm dx. \quad (4.6)
\]

Taking into account (4.3), (4.4), (4.5), (4.6), and by applying Fatou’s lemma we deduce

\[
\langle J'(w), w^+ \rangle \leq 0 \text{ and } \langle J'(w), w^- \rangle \leq 0. \quad (4.7)
\]

Then, putting together (4.2) and (4.7), and arguing as in the proof of Lemma 3.4-(a) we deduce that \( s,t \leq 1 \). Next, we show that \( J(tw^+ + sw^-) = c_0 \) and \( t = s = 1 \).
By using $tw^+ + sw^- \in \mathcal{M}$, $w_n \in \mathcal{M}$, (1.3), (4.1) and $s,t \in (0,1]$ we can see $c_0 \leq J(tw^+ + sw^-)$

$$= J(tw^+ + sw^-) - \frac{1}{2}(J'(tw^+ + sw^-), tw^+ + sw^-)$$

$$= \int_{\mathbb{R}^N} K(x) \left[ \frac{1}{2} f(tw^+ + sw^-)(tw^+ + sw^-) - F(tw^+ + sw^-) \right] dx$$

$$= \int_{\mathbb{R}^N} K(x) \left[ \frac{1}{2} f(tw^+)(tw^+) - F(tw^+) \right] dx + \int_{\mathbb{R}^N} K(x) \left[ \frac{1}{2} f(sw^-)(sw^-) - F(sw^-) \right] dx$$

$$\leq \int_{\mathbb{R}^N} K(x) \left[ \frac{1}{2} f(w^+)(w^+) - F(w^+) \right] dx + \int_{\mathbb{R}^N} K(x) \left[ \frac{1}{2} f(w^-)(w^-) - F(w^-) \right] dx$$

$$= \lim_{n \to \infty} \left\{ \int_{\mathbb{R}^N} K(x) \left[ \frac{1}{2} f(w_n^+)(w_n^+) - F(w_n^+) \right] dx + \int_{\mathbb{R}^N} K(x) \left[ \frac{1}{2} f(w_n^-)(w_n^-) - F(w_n^-) \right] dx \right\}$$

$$= \lim_{n \to \infty} \left[ J(w_n) - \frac{1}{2}\langle J'(w_n), w_n \rangle \right]$$

$$= \lim_{n \to \infty} J(w_n) = c_0.$$

Thus, we have proved that there exist $t,s \in (0,1]$ such that $tw^+ + sw^- \in \mathcal{M}$ and $J(tw^+ + sw^-) = c_0$. Let us observe that by the above calculation we can infer that $t = s = 1$, so $w = w^+ + w^- \in \mathcal{M}$ and $J(w^+ + w^-) = c_0$.

Finally we prove that $w$ is a critical point of $J$, that is $J'(w) = 0$. We argue by contradiction. Then, we can find a positive constant $\beta > 0$ and $v_0 \in X$ with $\|v_0\| = 1$, such that $\langle J'(w), v_0 \rangle = 2\beta > 0$. By the continuity of $J'$, we can choose a radius $R$ so that $\langle J'(v), v_0 \rangle = \beta > 0$ for every $v \in B_R(w) \subset X$ with $v^\pm \neq 0$.

Let $\mathcal{A} = (\xi, \chi) \times (\bar{\xi}, \bar{\chi}) \subset \mathbb{R}^2$ with $0 < \xi < 1 < \chi$ such that

(i) $(1, 1) \in \mathcal{A}$ and $\Phi^w(t,s) = (0,0)$ in $\bar{\mathcal{A}}$ if, and only if, $(t,s) = (1,1)$,

(ii) $c_0 \notin h^w(\partial \mathcal{A})$,

(iii) $\{tw^+ + sw^- : (t,s) \in \overline{\mathcal{A}}\} \subset B_R(w)$

where $h^w$ and $\Phi^w$ are defined by (3.11) and (3.12), and satisfy Lemma 3.4. Then we can choose a radius $0 < r < R$ such that

$$B = \overline{B_r(w)} \subset B_R(w) \text{ and } B \cap \{tw^+ + sw^- : (t,s) \in \partial \mathcal{A}\} = \emptyset. \quad (4.8)$$

Now, let us define a continuous mapping $\rho : X \to [0, +\infty)$ such that

$$\rho(u) := \text{dist}(u, B^c) \text{ for all } u \in X,$$

and we consider a bounded Lipschitz vector field $V : X \to X$ given by $V(u) = -\rho(u)v_0$. For every $u \in X$, denoting by $\eta(\tau) = \eta(\tau, u)$, we consider the following Cauchy problem

$$\left\{ \begin{array}{l}
\eta'(\tau) = V(\eta(\tau)) \text{ for all } \tau > 0, \\
\eta(0) = u.
\end{array} \right.$$}

We can observe that there exist a continuous deformation $\eta(\tau, u)$ and $\tau_0 > 0$ such that for all $\tau \in [0, \tau_0]$ the following properties hold:

(a) $\eta(\tau, u) = u$ for all $u \notin B$,
(b) $\tau \to J(\eta(\tau, u))$ is decreasing for all $\eta(\tau, u) \in \mathcal{B}$,
(c) $J(\eta(\tau, w)) \leq J(w) - \frac{r\beta}{2} \tau$.

Indeed, (a) follows by the definition of $\rho$. Regarding (b), we can observe that $\langle J'(\eta(\tau)), v_0 \rangle = \beta > 0$ for $\eta(\tau) \in \mathcal{B} \subset \mathcal{B}_R(w)$, and, by the definition of $\rho$, we can infer $\rho(\eta(\tau)) > 0$. Then
\[
\frac{d}{d\tau}(J(\eta(\tau))) = \langle J'(\eta(\tau)), \eta'(\tau) \rangle = -\rho(\eta(\tau)) \langle J'(\eta(\tau)), v_0 \rangle = -\rho(\eta(\tau)) \beta < 0, \forall \eta(\tau) \in \mathcal{B},
\]
that is $J(\eta(\tau, u))$ is decreasing with respect to $\tau$.

Now we prove (c). Being $\tau_0 > 0$ such that $\eta(\tau, u) \in \mathcal{B}$ for every $\tau \in [0, \tau_0]$, we can assume that
\[
\|\eta(\tau, w) - w\| \leq \frac{r}{2} \text{ for any } \tau \in [0, \tau_0].
\]

Since $\rho(\eta(\tau, w)) = \text{dist}(\eta(\tau, w), \mathcal{B}^c) \geq \frac{r}{2}$, we can deduce that
\[
\frac{d}{d\tau} J(\eta(\tau, w)) \leq -\rho(\eta(\tau, w)) \beta \leq -\frac{r\beta}{2}
\]
and, integrating on $[0, \tau_0]$, we get
\[
J(\eta(\tau_0, w)) - J(w) \leq -\frac{r\beta}{2} \tau_0.
\]

Now, we consider a suitable deformed path $\tilde{\eta}_0 : \overline{\mathcal{A}} \to \mathbb{X}$ defined by
\[
\tilde{\eta}_0(t, s) := \eta(\tau_0, tw^+ + sw^-), \text{ for all } (t, s) \in \overline{\mathcal{A}}.
\]

We can note that
\[
\max_{(t,s) \in \overline{\mathcal{A}}} J(\tilde{\eta}_0(t,s)) < c_0.
\]

Indeed, by (b) and the fact that $\eta(0, u) = u$, we have
\[
J(\tilde{\eta}_0(t, s)) = J(\eta(\tau_0, tw^+ + sw^-)) \leq J(\eta(0, tw^+ + sw^-))
= J(tw^+ + sw^-) = h^w(t, s) < c_0 \forall (t, s) \in \overline{\mathcal{A}} \setminus \{(1, 1)\},
\]
and for $(t, s) = (1, 1)$, in virtue of (c) we have
\[
J(\tilde{\eta}_0(1, 1)) = J(\eta(\tau_0, w^+ + w^-)) = J(\eta(\tau_0, w))
\leq J(w) - \frac{r\beta}{2} \tau_0 < J(w) = c_0.
\]

Then, $\tilde{\eta}_0(\overline{\mathcal{A}}) \cap \mathcal{M} = \emptyset$, that is
\[
\tilde{\eta}_0(t,s) \notin \mathcal{M} \text{ for all } (t,s) \in \overline{\mathcal{A}}. \tag{4.9}
\]

On the other hand, defining $\Psi_{\tau_0} : \mathcal{A} \to \mathbb{R}^2$ such that
\[
\Psi_{\tau_0} := \begin{pmatrix}
\langle J'(\tilde{\eta}_0(t,s)), (\tilde{\eta}_0(t,s))^+ \rangle \\
\langle J'(\tilde{\eta}_0(t,s)), (\tilde{\eta}_0(t,s))^− \rangle
\end{pmatrix}
t, s
\]
we can see that, for all $(t, s) \in \partial \mathcal{A}$, by (4.8) and (a) for $\tau = \tau_0$, it holds
\[
\Psi_{\tau_0}(t,s) = \langle J'(tw^+ + sw^-), w^+ \rangle, \langle J'(tw^+ + sw^-), w^- \rangle = \Phi^w(t,s).
\]

Then, by using Brouwer’s topological degree, we have
\[
\text{deg}(\Psi_{\tau_0}, \mathcal{A}, (0,0)) = \text{deg}(\Phi^w, \mathcal{A}, (0,0)) = \text{sgn}(\det(\Phi^w)'(1,1)) = 1,
\]
so we deduce that $\Psi_{\tau_0}$ has a zero $(\bar{t}, \bar{s}) \in A$, that is
$$
\langle J'(\bar{\eta}_{\tau_0}(\bar{t}, \bar{s})), (\bar{\eta}_{\tau_0}(\bar{t}, \bar{s}))^{\pm} \rangle = 0.
$$
Therefore, there exists $(\bar{t}, \bar{s}) \in A$ such that $\bar{\eta}_{\tau_0}(\bar{t}, \bar{s}) \in M$ and this contradicts (4.9).

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