Shift Harnack Inequality and Integration by Part Formula for Semilinear SPDE*

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Abstract

Shift Harnack and integration by part formula are establish for semilinear spde with delay and a class of stochastic semilinear evolution equation which cover the hyperdissipative Naiver-Stokes/Burges equation. For the case of stochastic equation with delay, an extension to path space is given.

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1 Introduction

Recently, using a new coupling argument, [17] provides a new type Harnack inequality, called shift Harnack inequality, and derive Driver’s integration by part formula, see [6]. The main idea is that construct two processes which start from the same point and at the expected time T they separate at a fixed vector almost surely. In [17] there, for the case of semilinear stochastic partial differential equations(SPDE), two problems remains, the first one is that how to establish shift Harnack inequality and integration by part formula for semilinear SPDE with delay, the second is that whether the two processes can separate at arbitrarily vector. In this paper, we try to find the answer to the two problems. We construct coupling in the spirit of [17], for the case of stochastic functional equation it even dates back to [7]. With a little knowledge of control

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theory and regularity theory of semigroups, explicit coupling is constructed, then we derive the shift Harnack and integration by part formula.

In the second part of the paper, we deal with semilinear SPDE with delay and generalized to non-Lipschitz case. In the third part, we extend the integration by part formula to the path space of the solution of the stochastic functional equation, some application are given. The last part, we establish the shift Harnack inequality and integration by part formula for a class of stochastic evolution equation, which covers the hyperdissipative Navier-Stokes/ Burgers equation.

Before our main results, we need some preparation. For any $T > 0$, assume that $U, H$ are Hilbert spaces, $B \in \mathcal{L}(U, H)$, and $-A$ generates an analytic semigroup, define two operators as follow, which are well known in control theory,

\begin{equation}
L^B_T : L^2([0, T], U) \to H, \quad L^B_T f = \int_0^T e^{-(T-t)A} B f(t) dt,
\end{equation}

\begin{equation}
R^B_T : H \to H, \quad R^B_T h = \int_0^T e^{-tA} B^* B B h dt,
\end{equation}

and let

\begin{equation}
D_A(1/2, 2) = \{ x \in H \mid ||x||^2_2 := \int_0^\infty ||A e^{-tA} x||^2 dt < \infty \}. \tag{1.3}
\end{equation}

The following proposition are well known in semigroup theory and control theory, see Theorem 3.1 in page 143 in [3] and Appendix B in [5] for details,

**Proposition 1.1.** (1) Assume that $-A$ generate an analytic semigroup, then for each $T > 0$, the map

\begin{equation}
(1.4) \quad u \to (u' + Au, u(0)) : W^{1,2}([0, T], H) \cap L^2([0, T], \mathcal{D}(A)) \to L^2([0, T], H) \times D_A(1/2, 2),
\end{equation}

is an isomorphism, and

\begin{equation}
(1.5) \quad W^{1,2}([0, T], H) \cap L^2([0, T], \mathcal{D}(A)) \subset C([0, T], D_A(1/2, 2)).
\end{equation}

(2) For the two operator $L^B_T$ and $R^B_T$, if $B^{-1} \in \mathcal{L}(H, U)$, then

\begin{equation}
(1.6) \quad \text{Im}(L^B_T) = \text{Im}((R^B_T)^{1/2}) = D_A(1/2, 2),
\end{equation}

\begin{equation}
(1.7) \quad ||(R^B_T)^{-1/2} x||^2 = \min \left\{ \int_0^T ||f(s)||^2_U ds \mid L_T f = x \right\}, \quad x \in D_A(1/2, 2).
\end{equation}

Here $(R^B_T)^{-1/2}$ means the pseudo-inverse of $(R^B_T)^{1/2}$.

**Remark 1.2.** By Proposition 1.1, for any $T > 0$, $x \in D_A(1/2, 2)$, there exists $f \in L^2([0, T], U)$ such that

\begin{equation}
(1.8) \quad L^B_T f = x, \quad ||f||^2_{L^2([0, T], U)} = \min \left\{ \int_0^T ||f(s)||^2_U ds \mid L^B_T f = x \right\}.
\end{equation}
The following lemma will give us the time behavior

**Lemma 1.3.** If \( B^{-1} \in \mathcal{L}(H,U) \) and \( Ae^{-tA}(D_A(1/2,2)) \subset U \) for all \( t > 0 \), then exists \( C > 0 \) which independent of \( T \), such that

\[
||(R^B_T)^{-1/2}x||^2 \leq 2||B^{-1}||^2 \left( \frac{||x||^2}{T} + 2||x||^2 \right)
\]

**Proof.** For all \( x \in D_A(1/2,2) \), let \( f(t) = B^{-1} \left( e^{-(T-t)A}x + \frac{2T}{T}Ae^{-(T-t)A}x \right) \). Then \( L_Tf = x \)

\[
||(R^B_T)^{-1/2}x||^2 \leq ||f||^2_{L^2([0,T],U)} \leq ||B^{-1}||^2 \left( \frac{2||x||^2}{T} + 4 \int_0^T ||Ae^{-(T-t)A}x||^2 dt \right)
\]

\[
\leq 2||B^{-1}||^2 \left( \frac{||x||^2}{T} + 2||x||^2 \right).
\]

\[ \square \]

## 2 Semilinear SPDE with Delay

\( H \) is Hilbert space with norm \( || \cdot || \), \( \mathcal{C} = C([-\tau,0],H) \), consider the following equation

\[
dx(t) = -Ax(t)dt + F(x(t))dt + \sigma(t)dW(t),
\]

satisfies the following conditions

(H1) \(-A\) generates analytic semigroup, there exists \( a \in \mathbb{R} \) such that \(-A - a\) is dissipative, and there exists \( \alpha \in (0,\frac{1}{2}) \) so that \( \int_0^T t^{-2\alpha}||e^{-tA}||^2_{HS} dt < \infty \),

(H2) \( F : \mathcal{C} \to H \) is Lipschitz with Lipschitz constant \( L \),

(H3) \( \sigma : [0,T] \to \mathcal{L}(H) \) measurable and bounded, and there is \( M > 0 \), such that \( ||\sigma(\cdot)^{-1}||_{\infty} \leq M \).

Denote the solution of the equation with initial value \( \xi \) by \( x(t,\xi) \), related segment process \( x_t(\xi) \), and \( P_Tf(\xi) = \mathbb{E} f(x_T(\xi)) \). In this section, we choose that \( B = I \) in Proposition 1.1. Now, we state our result

**Theorem 2.1.** Assume that (H1) to (H3) hold and \( T > \tau \). Let

\[
\eta \in W^{1,2}([-\tau,0],H) \bigcap L^2([-\tau,0],\mathcal{D}(A)), \psi(t) = \eta'(t - T) + A\eta(t - T), t \in [T - \tau, T].
\]

Then for any \( \xi \in \mathcal{C}, f \in \mathcal{B}(\mathcal{C}) \),

\[
(P_Tf(\xi))^p \leq P_Tf^p(\cdot + \eta)(\xi) \exp \left\{ \frac{pM^2}{p - 1} \left[ 2 + \frac{L^2(T - \tau)^2}{2} ||(R^I_{T-\tau})^{-1/2} \eta(\tau)||^2 \right] \right\}
\]

\[
+ \int_{T-\tau}^T ||\psi(t)||^2 dt + \tau \left\{ \left[ (T - \tau) ||(R^I_{T-\tau})^{-1/2} \eta(\tau)||^2 \right] \vee ||\eta||^2_{\infty} \right\}
\]
Moreover, if we assume that $F : \mathcal{C} \to H$ is Gâteaux derivable with $\|\nabla F(\cdot)\|_{\infty} \leq L < \infty$ in addition, then for any $\phi \in L^2([0, T - \tau], H)$ such that $L^1_{T-\tau} = \eta(-\tau)$, and $\psi$ defined as above, we have

\begin{equation}
(P_T \nabla \eta f)(\xi) = \mathbb{E} \left\{ f(x_T(\xi)) \int_0^T (\sigma(t))^{-1} \left[ \phi(t)1_{[0, T-\tau]}(t) + \psi(t)1_{[T-\tau, T]}(t) - \nabla \Gamma_i F(x_i(\xi)) \right], dW(t) \right\}, \ f \in C^1_b(\mathcal{C}),
\end{equation}

where

\begin{equation}
\Gamma(t) = \begin{cases} \eta(t - T), & t \geq T - \tau, \\ \int_0^t e^{-(t-s)A} \phi(s) ds, & t < T - \tau. \end{cases}
\end{equation}

**Proof of Theorem 2.1** By Proposition 1.1 for the case that $B = I$ and $U = H$, $\psi$ are well defined. By Remark 1.2, firstly we choose $\phi \in L^2([0, T - \tau], H)$ such that

\begin{equation}
L^1_{T-\tau} \phi = \eta(-\tau), \quad \|\phi\|_{L^2([0, T - \tau], H)} = \|R^1_{T-\tau}\eta(-\tau)\|.
\end{equation}

We construct another process as follow

\[
\begin{cases} 
\mathrm{d}y(t) = -Ay(t)\mathrm{d}t + F(x(t))\mathrm{d}t + \sigma(t)\mathrm{d}W(t) + \epsilon(\phi(t)1_{[0, T-\tau]}(t)\mathrm{d}t + \psi(t)1_{[T-\tau, T]}(t))\mathrm{d}t, \\
y_0 = \xi,
\end{cases}
\]

then

\begin{equation}
y(t) = x(t) + \epsilon \int_0^t e^{-(t-s)A} \phi(s)1_{[0, T-\tau]}(s)ds + \epsilon \int_0^t e^{-(t-s)A} \psi(s)1_{[T-\tau, T]}(s)ds.
\end{equation}

For $t \geq T - \tau$,

\begin{equation}
\int_0^t e^{-(t-s)A} \phi(s)1_{[0, T-\tau]}(s)ds = e^{-(t-T+\tau)A} \int_0^{T-\tau} e^{-(T-\tau-s)A} \phi(s)ds = e^{-(t-T+\tau)A} \eta(-\tau).
\end{equation}

Since $\eta \in W^{1,2}([-\tau, 0], H) \bigcap L^2([-\tau, 0], \mathcal{F}(A))$, and

\begin{equation}
\psi(t) = \eta'(t - T) + A\eta(t - T), \quad t \geq T - \tau,
\end{equation}

that means $\eta(\cdot - T)$ is the solution of the following equation

\begin{equation}
\frac{d\eta(t - T)}{dt} = -A\eta(t - T) + \psi(t), \quad t \geq T - \tau,
\end{equation}

with initial value $\eta(-\tau)$ at $T - \tau$, or in the integration form

\begin{equation}
\eta(t - T) = e^{-(t-T+\tau)}\eta(-\tau) + \int_{T-\tau}^t e^{-(t-s)A} \psi(s) ds, \quad t \geq T - \tau,
\end{equation}

4
thus, for $t \geq T - \tau$,
\begin{align}
y(t) &= x(t) + e^{-(t-T+\tau)A} \eta(-\tau) + \epsilon \eta(t - T) - e^{-(t-T+\tau)} \eta(-\tau) \\
&= x(t) + \epsilon \eta(t - T),
\end{align}
that means
\begin{equation}
y_T = x_T + \epsilon \eta,
\end{equation}
therefore, for all $t \in [0, T]$,
\begin{equation}
y(t) - x(t) = \epsilon \Gamma(t), \ y_t - x_t = \epsilon \Gamma_t.
\end{equation}
Let
\begin{equation}
h^t(s) = \epsilon \sigma(t)^{-1} \left( \phi(t) 1_{[0, T-\tau]}(t) + \psi(t) 1_{[T-\tau, T]}(t) \right) + \sigma(t)^{-1}(F(x_t) - F(y_t)),
\end{equation}
\begin{equation}
d\tilde{W}(t) = dW(t) + h^t(t)dt, \ R_T^\epsilon = \exp \left\{ - \int_0^T \langle h^t(s), dW(s) \rangle - \frac{1}{2} \int_0^T ||h^t(s)||^2 ds \right\}.
\end{equation}
Then we can rewrite the equation of $y$ as
\begin{equation}
dy(t) = -Ay(t)dt + F(y_t)dt + \sigma(t)d\tilde{W}(t), \ y_0 = \xi.
\end{equation}
By (H1) to (H3), as in [17], and noting that
\begin{align}
\int_0^T ||\Gamma||_\infty^2 dt &\leq \int_0^{T-\tau} \left( \int_0^T ||\phi(s)||ds \right)^2 dt + \int_{T-\tau}^T ||\Gamma||^2 dt \\
&\leq \int_0^{T-\tau} t \int_0^{T-\tau} ||\phi(s)||^2 ds dt + \tau \left\{ \left[(T-\tau) \int_0^{T-\tau} ||\phi(s)||^2 ds \right] \vee ||\eta||^2_{\infty} \right\} \\
&\leq \frac{(T - \tau)^2}{2} ||(R_T^{T-\tau})^{-1/2} \eta(-\tau)||^2 + \tau \left\{ \left[ \frac{(T-\tau)||(R_T^{T-\tau})^{-1/2} \eta(-\tau)||^2}{2} \right] \vee ||\eta||^2_{\infty} \right\},
\end{align}
we can prove that $\{\tilde{W}(t)\}_{t \in [0, T]}$ is Brownian motion by Girsanov theorem and get the shift Harnack inequality
\begin{align}
(P_T f(\xi))^p &\leq P_T f^p(\cdot + \eta)(\xi)(\mathbb{E}(R_T^1)^{\frac{p}{p-1}})^{p-1} \\
&\leq P_T f^p(\cdot + \eta)(\xi) \exp \left[ \frac{p}{2(p-1)} \int_0^T ||h(t)||^2 dt \right] \\
&\leq P_T f^p(\cdot + \eta)(\xi) \exp \left[ \frac{pM^2}{p-1} \left( 2 + \frac{L_T^2(T-\tau)^2}{2} \right) ||(R_T^{T-\tau})^{-1/2} \eta(-\tau)||^2 \right] \\
&\quad + \int_{T-\tau}^T ||\psi||^2 dt + \tau \left\{ \left[ \frac{2 + \frac{L_T^2(T-\tau)^2}{2} ||(R_T^{T-\tau})^{-1/2} \eta(-\tau)||^2}{2} \right] \vee ||\eta||^2_{\infty} \right\} \right\}
\end{align}
in the last inequality, we have used that (2.5). Further more since $F$ has bounded Gâteaux derivative, choosing $\phi \in L^2([0, T-\tau], H)$ such that $L_T^{T-\tau}\phi = \eta(-\tau)$, then
\begin{align}
\frac{d}{d\epsilon} |_{\epsilon=0} R_T^\epsilon &= - \int_0^T \langle \sigma(t)^{-1} \left[ \phi(t) 1_{[0, T-\tau]}(t) + \psi(t) 1_{[T-\tau, T]}(t) \right], dW(t) \rangle \\
&\quad + \int_0^T \langle \sigma(t)^{-1} \nabla \Gamma_t F(x_t), dW(t) \rangle.
\end{align}
hold in \( L^1(\mathbb{P}) \). Therefore

\[
(P_T \nabla_{\eta} f)(\xi) = \lim_{\epsilon \to 0^+} \frac{P_T f(\cdot - \epsilon \eta)(\xi) - \mathbb{E} f(x_T(\xi))}{-\epsilon}
\]

(2.18)

\[
= \lim_{\epsilon \to 0^+} \frac{\mathbb{E} R^\epsilon f(y_T(\xi) - \epsilon \eta) - \mathbb{E} f(x_T(\xi))}{-\epsilon}
\]

\[= \mathbb{E} \left\{ f(x_T(\xi)) \int_0^T \langle \sigma(t)^{-1} \left[ \phi(t)1_{[0,T-\tau]}(t) + \psi(t)1_{[T-\tau,T]}(t) - \nabla_{T_*} F(x_t(\xi)) \right], dW(t) \right\}.
\]

\[\square\]

**Remark 2.2.** The second condition in (2.3) is only used to shift Harnack inequality, to get explicit integration by part formula, one can choose “\( \phi \)”, here we provided a procedure to get it and give an example. Fix any \( T > 0 \). For any \( x \in D_A(1/2) \), \( h \in L^2([0,T], H) \), let

\[
\phi_1(t) = e^{-tA}x + \int_0^t e^{-(t-s)A} h(s)ds,
\]

Then \( \phi_1 \in W^{1,2}([0,T], H) \cap L^2([0,T], \mathcal{G}(A)) \) by Proposition 1.4. Let \( u \in C^1([0,T], \mathbb{R}) \), \( u(0) = 0 \), \( u(T) = 1 \). Then

\[
\phi(t) = \frac{d}{dt} (u(t)\phi_1(T-t)) + u(t)A\phi_1(T-t).
\]

It's clear that \( \phi \in L^2([0,T], H) \) and

\[
\int_0^T e^{-(T-t)A} \phi(t) dt = u(T)\phi_1(0) - u(0)\phi_1(T) = x.
\]

For example, one can choose \( h = 0 \), \( u(t) = \frac{t}{T} \), then \( \phi(t) = \frac{e^{-\left(T-t\right)A}x}{T} + \frac{2t}{T} Ae^{-\left(T-t\right)A}x \).

For general case we can use Lemma 1.3 to get the following inequality, for more sharp estimate we expect more better inequality.

**Corollary 2.3.** Assume that \((H1)\) to \((H3)\) hold, \( T > \tau \) and \( \psi \) as in Theorem 2.1, then

\[
(P_T f(\xi))^p \leq P_T f^p(\cdot + \eta)(\xi) \exp \left\{ \frac{2M^2}{p - 1} \left[ \left( 2 + L^2(T - \tau) \right)^2 \left( \frac{||x||^2}{T - \tau} + 2||x||^2 \right) + \int_{T-\tau}^T ||\psi(t)||^2 dt + \tau \left\{ 2(T - \tau) \left( \frac{||x||^2}{T - \tau} + 2||x||^2 \right) \right\} \right\}
\]

(2.19)

When \( A \) is self adjoint, we have

**Corollary 2.4.** When \( A \) is self adjoint operator and \( A \geq \lambda_0 > 0 \), under the assumption in Theorem 2.1, we have the shift Harnack inequality holds with \( ||(R^T_{T-\tau})^{-1/2}||^2 \) replaced by \( \frac{2||A^{1/2} \eta(\tau - T)||^2}{1 - e^{-2(T-\tau)\lambda_0}} \).

**Proof.** In this situation,

\[R^T_{T-\tau} = \int_0^{T-\tau} e^{-2tA} dt = \frac{A^{-1}}{2}(I - e^{-2(T-\tau)A}),
\]

(2.20)
and $D_A(1/2, 2) = \mathcal{D}(A^{1/2})$, then

\begin{equation}
(2.21) \quad ||(R^f_{T-\tau})^{-1/2}h(-\tau)||^2 = 2||(I - e^{-2(T-\tau)A})^{-1/2}A^{1/2}h(-\tau)||^2 \leq \frac{2||A^{1/2}h(-\tau)||^2}{1 - e^{-2(T-\tau)\lambda_0}}.
\end{equation}

**Corollary 2.5.** Assume that $(H1), (H3)$ hold and there is an increasing continuous function $\gamma : [0, \infty) \rightarrow [0, \infty)$ such that

\begin{equation}
(2.22) \quad ||F(x) - F(y)|| \leq \gamma(||x - y||_\infty), \forall x, y \in \mathcal{C},
\end{equation}

and equation (2.7) has pathwise unique mild solution, then for any $p > 1$, $f \in \mathcal{B}^+_p(H)$,

\begin{equation}
(2.23) \quad (P_Tf)^p \leq P_Tf^p(e + \cdot) \exp \left[ \frac{M^2p}{p-1} \left( ||(R^f_{T-\tau})^{-1/2}h(-\tau)||^2 + T\gamma^2(\sqrt{T-\tau})||R^f_{T-\tau})^{-1/2}h(-\tau)||^2 \right) + \int^T_{T-\tau} ||\psi(t)||^2dt \right].
\end{equation}

**Proof.** We use the notation in Theorem 2.1. In this case, the shift Harnack inequality follows from the following estimate

\begin{equation}
(2.24) \quad \frac{1}{2M^2} \int^T_0 ||h(t)||^2dt \leq \left( \int^T_0 ||\phi(t)||^2dt + \int^T_{T-\tau} ||\psi(t)||^2dt + \int^T_0 \gamma^2(||\Gamma||)dt \right)
\leq \left( ||(R^f_{T-\tau})^{-1/2}h(-\tau)||^2 + \int^T_{T-\tau} ||\psi(t)||^2dt \right)
\end{equation}

\begin{equation}
+ T\gamma^2 \left( \int^T_0 ||\phi(s)||ds \vee ||\eta||_\infty \right) dt
\leq \left( ||(R^f_{T-\tau})^{-1/2}h(-\tau)||^2 + \int^T_{T-\tau} ||\psi(t)||^2dt \right)
\end{equation}

\begin{equation}
+ T\gamma^2 \left( ||\eta||_\infty \vee (T-\tau)||G^f_{T-\tau})^{-1/2}h(-\tau)||^2 \right).
\end{equation}

**Remark 2.6.** For the existence and uniqueness of stochastic functional differential equations with non-Lipschitz coefficients and nontrivial examples, one can see [4, 14, 20] and references there in.

## 3 Extend to Path Space

Firstly, we shall give an integration by part formula on path space of solution of stochastic functional differential equation, it follows from [9, 15]. For simplicity, we assume that $\nabla F$ is bounded, $\sigma \equiv I$. Denote $\nabla^j$ the partial derivative of the j-th component, $W^x_T$ the path space of segment process $x^\xi$ on $[0, T]$. Denote all the bounded smooth cylindrical function on $\mathcal{C}$ by $\mathcal{F}C^\infty_b(W^x(T))$, i.e.

$$\mathcal{F}C^\infty_b(\mathcal{C}) = \{ G(\gamma) = g(\gamma_{t_1}, \cdots, \gamma_{t_n}), g \text{ is smooth on } \mathcal{C}^n \text{ with bounded all derivative }, \gamma \in W^x(T) \}$$
Definition 1. Let $G$ be a function on $W^\xi(T)$, its directive along the direction $\eta$ at $\gamma \in W^\xi(T)$, if the following limit exists
\[ \nabla_\eta G(\gamma) := \lim_{\epsilon \to 0^+} \frac{G(\gamma + \epsilon \eta) - G(\gamma)}{\epsilon}. \]
If $\nabla G(\gamma)$ provides a continuous linear functional on $W^{1,2}_0([0, T], H) := \left\{ f \in W([0, T], H) \mid f(0) = 0 \right\}$, the gradient $\nabla G(\gamma)$ is defined as its Riesz representation, i.e., $\nabla G(\gamma)$ is an element in $W^{1,2}_0([0, T], H)$ such that $\langle \nabla G(\gamma), \eta \rangle_{W^{1,2}_0} = \nabla_\eta G(\gamma)$.

By definition, for $G(\gamma) = g(\gamma(t_1), \ldots, \gamma(t_n)), g \in C^1_b(H^n)$, it’s easy to see that $\frac{d}{dt}(\nabla G(\gamma))(t) = \sum_{i=1}^n 1_{[t<t_i]} \nabla^i g(\gamma)$.

Proposition 3.1. If $F$ is Fréchet differentiable on $\mathcal{C}$ with $\|\nabla F\| \leq L$ and $\nabla F$ is uniformly continuous on bounded set of $\mathcal{C} \times \mathcal{C}$. Then for all $\eta \in W^{1,2}_0([0, T], H) \cap L^2([0, T], \mathcal{D}(A))$, extending it to $[-\tau, 0]$ by zero, we have
\[ \mathbb{E}\nabla_\eta G(x^\xi([0, T])) = \mathbb{E}G(x^\xi([0, T])) \int_0^T \langle \frac{d\eta(t)}{dt} + A\eta(t) - \nabla_\eta F(x_t), dW(t) \rangle, \ G \in C^1_b(W^\xi(T)). \]

Proof. By the definition of $\eta$, we can define
\[ h(t) = \eta(t) + \int_0^t A\eta(s)ds - \int_0^t \nabla_{\eta_s} F(x_s)ds. \]
Then $h \in W^{1,2}_0([0, T], H)$ $\mathbb{P}$-a.s. and it’s adapted. Since $\nabla F$ is Fréchet differentiable on $\mathcal{C}$ with $\|\nabla F\| \leq L$ and $\nabla F$ is uniformly continuous on bounded set of $\mathcal{C} \times \mathcal{C}$, Let $D_hx$ be the Malliavin derivative of $x$ along $h$, then is the mild solution of the following equation
\[ dD_hx(t) = -AD_hx(t)dt + \nabla_{D_hx} F(x_t)dt + h'(t)dt, \ D_hx_0 = 0, \]
by [2] Theorem A.2. Noting that $\eta$ and $D_hx$ satisfy the same differential equation with the same initial value, then $D_hx = \eta \mathbb{P}$-a.s. Therefore
\[ \mathbb{E}\nabla_\eta G(x^\xi([0, T])) = \mathbb{E}\nabla_{D_hx} G(x^\xi([0, T])) = \mathbb{E}D_h G(x^\xi([0, T])) \]
\[ = \mathbb{E}G(x^\xi([0, T])) \int_0^T \langle h(t), dW(t) \rangle \]
\[ = \mathbb{E}G(x^\xi([0, T])) \int_0^T \langle \frac{d\eta}{dt} + A\eta(t) - \nabla_\eta F(x_t), dW(t) \rangle. \]

Let $\xi(\Phi, \Psi) = \mathbb{E}\langle \Phi, \Psi \rangle_{W^{1,2}_0(x^\xi([0, T]))}, \ \Phi, \Psi \in \mathcal{F}C^\infty_b(W^\xi(T)).$ Then we have

Corollary 3.2. $\left( \xi(\cdot, \mathcal{F}C^\infty_b(W^\xi(T))) \right)$ is closable in $L^2(W^\xi(T), \Pi^\xi(T))$. 

8
Proof. Let \( \phi \in C^2_b(\mathbb{R}^{n \times m}) \) and
\[
\Phi(\gamma) = \phi(\langle \gamma(t_1), e_1 \rangle, \ldots, \langle \gamma(t_n), e_n \rangle, \ldots, \langle \gamma(t_m), e_n \rangle),
\]
for any \( \Psi \in \mathcal{F} C^\infty_0(W^\xi(T)) \), by integration by part formula,
\[
\mathbb{E}[\nabla \Phi, \nabla \Psi]_{W^{1,2}_0} = \sum_{i,j=1}^{m,n} \mathbb{E}[\partial \phi]_{ij} \langle \Psi, \cdot \wedge t_i e_j \rangle_{W^{1,2}_0}
\]
(3.2)
\[
= \sum_{i,j=1}^{m,n} \mathbb{E}[\nabla \cdot \nabla t e_j] [\Psi(\partial \phi)_{ij} - \Psi \nabla \cdot \nabla t e_j (\partial \phi)_{ij}] 
= \mathbb{E} \Psi \sum_{i,j=1}^{m,n} \left[ (\partial \phi)_{ij} \delta(\cdot \wedge t_i e_j) - \nabla \cdot \nabla t e_j (\partial \phi)_{ij} \right].
\]
By this formula and that this type of \( \nabla \Phi \) is dense in \( L^2(W^\xi(T) \rightarrow W^{1,2}_0([0, T], H), \Pi^\xi(T)) \), we can prove the corollary. \( \square \)

Next we shall prove the log Sobolev inequality for this Dirichlet form. Let \( y(\cdot, h) \) be the the mild solution of the following equation in pathwise sense
\[
dy(t) = -Ay(t)dt + \nabla_y F(x_t)dt + h'(t)dt, \quad y_0 = 0,
\]
where \( h \in L^2(\Omega \rightarrow W^{1,2}_0([0, T], H)) \), then \( y \in C([0, T], H) \) \( \mathbb{P} \)-a.s., if we assume that \( \nabla F \) is bounded, then it’s clear that \( \nabla_y F(x_t) + h'(\cdot) \in L^2([0, T], H) \), \( \mathbb{P} \)-a.s., by Proposition [1], we find that \( y(\cdot, h) \in W^{1,2}_0([0, T], H) \cap L^2([0, T], \mathcal{D}(A)) \), that means it’s a continuous operator on \( W^{1,2}_0([0, T], H) \). Before the estimate of the operator norm, we recall a priori estimate in [3, Lemma 3.3 in page 141].

**Lemma 3.3.** Assume that \( J \) generates an analytic semigroup with negative type and the resolvent satisfies \( ||(\lambda - J)^{-1}|| \leq M/|\lambda| \), \( Re \lambda > 0 \). For all \( f \in W^{1,2}([0, \infty), H) \) and \( T > 0 \). Let \( v \) be the solution of the following equation
\[
dv(t) = Jv(t)dt + f(t)dt, \quad v(0) = 0.
\]
Then
\[
\int_0^T ||Jv(t)||^2dt \leq (M + 1)^2 \int_0^T ||f(t)||^2dt,
\]
\[
\int_0^T ||v'(t)||^2dt \leq M^2 \int_0^T ||f(t)||^2dt.
\]
**Proof.** The first inequality is the result of [3, Lemma 3.3 in page 141]. The second one was missing there, but it can be proved follow [3, Lemma 3.3 in page 141] completely, a proof is given here just for convenient. Let \( \bar{v}(t) \) be the solution of
\[
d\bar{v}(t) = J\bar{v}(t)dt + \bar{f}(t)dt, \quad \bar{v}(0) = 0, \quad t \in \mathbb{R},
\]
where \( \bar{f}(t) = f(t)1_{(0, T)}(t) \). Then
\[
\bar{v}(t) = \begin{cases} 
v(t) & t \in [0, T] \\
0 & t \leq 0 \\
e^{(t-T)J}v(T) & t \geq T. \end{cases}
\]
Apply Fourier transform to equation (3.4), letting
\[ \hat{v}(k) = \int_0^\infty e^{-ikt}v(t)dt, \quad \hat{f}(k) = \int_0^\infty e^{-ikt}f(t)dt, \]
we arrive at
\[ ik\hat{v}(k) = A\hat{v}(k) + \hat{f}(k), \]
then
\[ \hat{v}(k) = (ik - J)^{-1}\hat{f}(k), \]
this implies that
\[ \|ik\hat{v}(k)\| = \|(ik - J)^{-1}\hat{f}(k)\| = \|ik\hat{f}(k)\| \leq M\|\hat{f}(k)\|, \]
by Parseval’s inequality, we get the second inequality.

**Lemma 3.4.** Assume that \( \sigma \equiv I, \|\nabla F\| \leq L \). Then \( y \) is a continuous operator on \( W^{1,2}([0, T], H) \) and
\[
\|y(\cdot, h)\|_{W^{1,2}}^2 \leq e^{2Ta + T^2(a^2 + 2Ta)}(1 + LT e^{T(L+a^2)})\|h\|_{W^{1,2}}^2, \quad \forall h \in W^{1,2}([0, T], H).
\]

**Proof.** It’s no harm to assume that \( a \geq 0 \). Since \( y(\cdot, h) \in W^{1,2}_0([0, T], H) \cap L^2([0, T], \mathcal{D}(A)) \), we view \( \nabla_y F(x, \cdot) + h'(\cdot) \) as a inhomogeneous term in equation (3.3). Replacing \( y(t) \) by \( e^{at}(t) \) and \( -A \) by \( -A - a \), we get
\[
\int_0^t \|(e^{ar}y(r))'\|^2dr \leq \int_0^t e^{2ra}\|\nabla_y F(x, r) + h'(r)\|^2dr, \quad t \leq T,
\]
by Lemma 3.3. Then
\[
\int_0^t e^{2ra}\|y'(r)\|^2dr - \int_0^t e^{2ra}\|\nabla_y F(x, r) + h'(r)\|^2dr
\]
\[ \leq -\int_0^t |ae^{ra}y(r)|^2dr - 2\int_0^t \langle ae^{ra}y(r), e^{ra}y'(r)\rangle dr
\]
\[ \leq \int_0^t a^2e^{2ra}\|y(r)\|^2dr \leq Ta^2e^{2Ta} \int_0^r \|y'(s)\|^2dsdr.
\]
By Gronwall’s inequality
\[
\int_0^T \|y'(t)\|^2dt \leq e^{T^2a^2e^{2Ta} + 2Ta} \int_0^T (L||y_t||_\infty + ||h'||(t))^2dt.
\]
On the other hand
\[
d\|y(t)\|^2 \leq 2a\|y(t)\|^2dt + 2L||y_t||_\infty\|y(t)||dt + 2||h'(t)|| \cdot \|y(t)\|dt,
\]
then
\[
d\|y(t)\| \leq a\|y(t)\|^2dt + L||y_t||_\infty dt + ||h'(t)||dt.
\]
This implies that
\[ \|y_t\|_{\infty} \leq (L + a) \int_0^t \|y_r\|_{\infty} dr + \int_0^t \|h'(r)\| dr, \quad t \geq 0. \]

By Gronwall’s lemma,
\[ \|y_t\|_{\infty} \leq e^{t(L+a)} \int_0^t \|h'(r)\| dr. \]

By this estimate, one can find that
\[ L^2 \int_0^T \|y_t\|_2^2 \leq L^2 \int_0^T e^{t(L+a)} \left( \int_0^t \|h'(s)\| ds \right)^2 dt, \]
and
\[ 2L \int_0^T \|y_t\| \|h'\| dt = 2L \int_0^T e^{t(L+a)} \int_0^t \|h'(r)\| dr \|h'(t)\| dt \]
\[ \leq LTe^{L(t+a)} \int_0^T \|h'(r)\|^2 dr - L(L+a) \int_0^T e^{t(L+a)} \left( \int_0^t \|h'(s)\| ds \right)^2 dt \]
(3.7)

Considering we have assume \( a \geq 0 \), therefore
\[ \int_0^T \|y'(t)\|^2 dt \leq e^{2Ta^2+T^2(a^2)^2} 2^{2Ta^-} \left( 1 + LTe^{T(L+a)} \right) \int_0^T \|h'(r)\|^2 dr. \]
\[ \square \]

**Proposition 3.5.** Assume that \( \sigma \equiv I \), \( \|\nabla F\| \leq L \) and \( \nabla F \) is uniformly continuous on bounded set of \( \mathcal{C} \times \mathcal{C} \), then we have the following log Sobolev inequality holds
\[ \mathbb{E}G^2 \log G^2 - \mathbb{E}G^2 \log \mathbb{E}G^2 \leq e^{2(Ta^2+T^2(a^2)^2)2} \left( 1 + LTe^{T(L+a)} \right) \mathbb{E}^x \left( G, \log G \right), \]
for all \( G \in \mathcal{F} \mathcal{C}_{b}^{\infty}(W^{\xi}(T)) \). In particularly, for \( G(\gamma) = g(\gamma(T)) \), \( g \in \mathcal{C}_{b}^{1}(H) \), we have
\[ \mathbb{E}(g^2 \log g^2) - \mathbb{E}g^2 \log \mathbb{E}g^2 \leq e^{2(Ta^2+T^2(a^2)^2)2} \left( 1 + LTe^{T(L+a)} \right) \mathbb{E}^x \left( \nabla g \right)^2 (x^{\xi}(T)). \]

**Proof.** Consider the gradient \((Dx)^* \nabla G\). Then for all adapted \( h \in L^2(\Omega \to W^{1,2}([0,T],H) ; \mathbb{P})\) with \( \|h\|_{W^{1,2}} \) bounded \( \mathbb{P}\)-a.s., we have
\[ \mathbb{E}\langle (Dx)^* \nabla G,h \rangle = \mathbb{E}\langle \nabla G, D_h x \rangle = \mathbb{E}D_h G(x^{\xi}([0,T])) - \mathbb{E}G(x^{\xi}([0,T])) \int_0^T \langle h'(t), dW(t) \rangle. \]

By martingale representation theorem, It’s standard that
\[ \mathbb{E} \left[ G(x^{\xi}([0,T])) \big| \mathcal{F}_t \right] = \mathbb{E}G(x^{\xi}([0,T])) + \int_0^t \mathbb{E} \left[ \left( \frac{d}{ds} (Dx)^* \nabla G \big| \mathcal{F}_s \right) dW(s). \]

Let \( m_t = \mathbb{E} \left[ G^2(x^{\xi}([0,T])) \big| \mathcal{F}_t \right] \). By Itô’s formula, we have
\[ \mathbb{E} m_T \log m_T - m_0 \log m_0 = \int_0^T \mathbb{E} \left( \frac{d}{dt} (Dx)^* \nabla G^2 \big| \mathcal{F}_t \right)^2 \frac{2}{m_t} dt \]
\[ \leq 2 \int_0^T \mathbb{E} \left( \frac{d}{dt} (Dx)^* \nabla G \big| \mathcal{F}_t \right)^2 \frac{2}{m_t} dt \leq 2 \mathbb{E} \| (Dx)^* \nabla G \|^2 \leq 2 \left( 1 + LTe^{LT} \right) \mathbb{E} \| \nabla G \|^2. \]
\[ \square \]
Next, we shall extend the above result to the path space of segment processes. Let \( W^\xi(T) \) the path space of \( x^\xi(\cdot) \) on \([0, T]\), \( \mathcal{H} \) consist of all the segment functions of \( W^1_0([0, T], H) \), i.e.

\[
\mathcal{H} = \{ \psi \ | \ \psi(\cdot) \in W^1_0([0, T], H) \text{ extended to } [-\tau, 0] \text{ by zero} \}.
\]

\( S \) be the natural embedding from \( W^\xi(T) \) to \( W^\xi_T \), i.e. \( (S\gamma_\tau) = \gamma_\tau \). Then we can introduce an inner product structure on \( \mathcal{H} \) as follow such that it becomes a Hilbert space

\[
\langle \phi, \psi \rangle_{\mathcal{H}} := \langle S^{-1}\phi, S^{-1}\psi \rangle_{W^1_0}, \ \phi, \psi \in \mathcal{H}.
\]

Let \( G \in C^1_b(W^\xi_T) \) and \( \nabla G(\gamma) \) be the derivative of \( G \) at \( \gamma \in W^\xi_T \) along the direction \( \eta \in \mathcal{H} \). If \( \nabla G(\gamma) \) gives a continuous linear functional on \( \mathcal{H} \), we define the gradient \( \nabla G(\gamma) \) as its Riesz representation, i.e. \( \langle \nabla G(\gamma), \eta \rangle_{\mathcal{H}} = \nabla G(\gamma) \). Let \( W^{1,2}_T = W^{1,2}([-\tau, 0], H) \). On \( W^{1,2}([-\tau, 0], H) \) we rig the inner product \( \langle \phi, \psi \rangle_{W^{1,2}_T} := \langle \phi(0), \psi(0) \rangle + \int_{-\tau}^0 \langle \phi(s), \psi(s) \rangle \mathrm{d}s \), \( \phi, \psi \in W^{1,2}([-\tau, 0], H) \). Let \( G(\gamma) = g(\gamma_t, \cdots, \gamma_t_n) \), \( g \in C^1_b(\mathbb{R}^n) \), \( \nabla^i g \) be the partial derivative of the \( i \)-th component, as an element of \( W^{1,2}_T \) just as above. Then for any \( \eta \in \mathcal{H} \)

\[
\nabla G(\gamma) = \sum_{i=1}^n \langle \nabla^i g(\gamma), \eta_i \rangle_{W^{1,2}_T} = \sum_{i=1}^n \left[ \int_{-\tau}^0 \langle \nabla^i g(\gamma)(s), \dot{\eta}(t_i + s) \rangle \mathrm{d}s + \langle \nabla^i g(\gamma)(0), \eta(t_i) \rangle \right] = \sum_{i=1}^n \left[ \int_{-\tau}^T \langle 1_{[t_i, t_{i+1}]}(s) \nabla^i g(\gamma)(s - t_i) \rangle \mathrm{d}s + \int_{-\tau}^T \langle \eta \rangle \nabla^i g(\gamma)(0), \dot{\eta}(s) \rangle \mathrm{d}s \right]
\]

\[= \int_0^T \sum_{i=1}^n \left[ 1_{[t_i, t_{i+1}]}(s) \nabla^i g(\gamma)(s - t_i) + 1_{[s, t_i]} \nabla^i g(\gamma)(0), \dot{\eta}(s) \right] \mathrm{d}s,
\]

we have

\[
\left( \frac{\mathrm{d}}{\mathrm{d}t} S^{-1} \nabla G(\gamma) \right)(s) = \sum_{i=1}^n \left[ 1_{[t_i, t_{i+1}]}(s) \nabla^i g(\gamma)(s - t_i) + 1_{[s, t_i]} \nabla^i g(\gamma)(0) \right].
\]

A counterpart of \((Dx)^* \nabla G(\gamma)\) as in Proposition 3.5 is \( S(Dx)^* S^{-1} \nabla G(\gamma) \). By these definition, just as in the previous discussion, we have the results on \( W^\xi_T \).

**Proposition 3.6.** Under the same assumption of Proposition 3.5, For all \( \eta \in W^{1,2}_0([0, T], H) \bigcap L^2([0, T], \mathcal{D}(A)) \),

we have the integration by part formula

\[
\mathbb{E} \nabla S_T \gamma \mathbb{E} = \int_0^T G(x^\xi_{[0,T]}) \langle \dot{\eta}(t) + A\eta(t) - \nabla \eta F(x_t), \mathrm{d}W(t) \rangle.
\]

Let \( \mathcal{E}^\xi_T(\Phi, \Psi) = \mathbb{E} \langle \nabla \Phi, \nabla \Psi \rangle_{\mathcal{H}}(x^\xi_{[0,T]}) \). Then \( \left( \mathcal{E}^\xi_T, \mathcal{F} C^\infty_b(W^\xi_T) \right) \) is closable in \( L^2(W^\xi_T, \mathbb{P}^\xi_T) \). Log Sobolev inequality holds

\[
\mathbb{E} G^2 \log G^2 - \mathbb{E} G^2 \log \mathbb{E} G^2 \leq 2e^{2T\alpha + T^2(\alpha^2 + 2T\alpha^3)} \left( 1 + LT e^{T(L + \alpha^2)} \right) \mathbb{E} ||\nabla G||^2_{\mathcal{H}}.
\]
In particular, for $G(\gamma) = g(\gamma T)$, $g \in C_b^1(\mathcal{C})$,

$$P_T g^2 \log g^2(\xi) - P_T g^2(\xi) \log P_T g(\xi) \leq 2(T + 1)e^{2Ta^+ + T^2(a^+)^2}e^{2Ta^+} \left(1 + LT e^{T(L+a^+)}\right) \mathbb{E}\|\nabla g\|_{W^{1,2}}^2(\xi).$$

4 Stochastic evolution equation with non-Lipschitz coefficients

Here we consider the following equation in Hilbert space $H$

\begin{equation}
(4.1) \quad dx(t) = -Ax(t)dt + B(x(t))dt + QdW(t).
\end{equation}

We shall use the notation following

\begin{equation}
(4.2) \quad (V_\theta, \| \cdot \|_{V_\theta}) = (\mathcal{D}(A^\theta), \| A^\theta \cdot \|), \quad \| \cdot \|_Q = \| Q^{-1} \cdot \|.
\end{equation}

The coefficients of the equation may satisfy some of the following conditions

(A1) $A$ is positive self adjoint operator with $A \geq \lambda_0 > 0$, $Q \in \mathcal{L}_{HS}(H)$ is non-degenerated,

(A2) $B$ is hemicontinuous, i.e. the map $s \to \langle B(v_1 + sv_2), v \rangle$ is continuous on $\mathbb{R}$, and there exists $\gamma \in [0, 2)$, $\alpha \in [0, 1]$ and $K_1, K_2 \geq 0$ such that

\begin{align}
(4.3) & \quad \langle B(u) - B(v), u - v \rangle \leq (\rho(v) + K_1)\|u - v\|_{V_\alpha}^\gamma \|u - v\|^{2-\gamma} \\
(4.4) & \quad \langle B(u - v), v \rangle \leq K_2\|v\|_{V_\alpha} \|u - v\|_{V_\alpha}^\gamma \|u - v\|^{2-\gamma}
\end{align}

where $\rho : V \to \mathbb{R}^+$ is measurable, locally bounded function, $\rho(0) = 0$,

(A3) There exists $\theta \in (0, 1]$ and $K_3 > 0$ such that

\begin{equation}
(4.5) \quad \|u\|_Q^2 \leq K_3\|u\|_{V_\theta}^2,
\end{equation}

(A4) There exists a constant $K_4 > 0$ such that

\begin{equation}
(4.6) \quad \|B(u) - B(v)\|_Q^2 \leq \beta(u - v)(1 + \|v\|_V + \|u\|_V)^2,
\end{equation}

where $\beta : V \to \mathbb{R}^+$ is locally bounded measurable function.

(A5) $B$ is Gâteaux differentiable from $V$ to $Q(H)$ and there exists $K_4 \geq 0$ such that

\begin{equation}
(4.7) \quad \|\nabla B(v)\|_Q \leq K_4(1 + \|v\|_V),
\end{equation}

here we endow $Q(H)$ with the norm $\| \cdot \|_Q$ such that it becomes a Banach space,

(A6) There is $K_5 \geq 0$ such that

\begin{equation}
(4.8) \quad \langle B(w), w \rangle \leq K_5(1 + \|w\|^2), \quad \forall w \in V.
\end{equation}
Remark 4.1. (1) By (4.4), we have
(4.9) \[ \langle B(w), v \rangle = \langle B(w + v - v), v \rangle \leq K_2 \|v\|_{V_\alpha} \|\nabla_v w\|^2 \gamma, \forall v, w \in V, \]
thus
(4.10) \[ \|B(w)\|_{V^*} \leq C \|\nabla_v w\|^2 \gamma, \forall w \in V, \]
and from (4.3),
(4.11) \[ \langle B(w), w \rangle \leq \|B(0)\| \cdot \|w\| + K_1 \|\nabla_v w\|^2 \gamma, \forall w \in V. \]
Therefore, by [13] and directly calculus we can prove that under the conditions (A1) and (A2),
equation (4.1) has uniqueness strong solution.
(2) It’s easy to see that (A5) implies that (A4) holds in the following form
(4.12) \[ \|B(u) - B(v)\|_Q \leq 2K_4 \|u - v\|_V (1 + \|u\|_V + \|v\|_V). \]
(3) Though it’s easy to see that Navier-Stokes operator satisfies (A2), but unfortunately, it does
not satisfies (A3) to (A5).

Theorem 4.2. Assume that (A1) to (A4) hold and \( e \in \mathcal{D}(A_1 + \theta_2) \), then the shift log-Harnack
inequality holds
(4.13) \[ P_T \log f(x) \leq \log P_T f(e + \cdot)(x) + \Psi_2(T, e), \forall f \in \mathcal{B}_b(H), \]
here
(4.14) \[ \Psi(x, T, e) = C \{ \frac{(T + 1)}{T} \|A^{(1+\theta)/2}e\|^2 + b_e \beta \left( \|Q\|^2_{HS} + \|B(0)\|^2 + \|x\| \right)^T + \frac{\|e\|^2}{2} + \frac{\sqrt{2 - \gamma}}{4} \|A^{1/4}e\|^2 \|A^{(2\alpha+\gamma-2)/4}e\|^2 \|2(2-\gamma) \}
\times \exp \left\{ CT \left[ 1 + \frac{2 - \gamma}{4} \|A^{(2\alpha+\gamma-2)/4}e\|^4 \right] \right\}, \]
where \( C \) is constant depending on \( \gamma, K_1, K_2, K_3 \) and
(4.15) \[ b_e := \sup_{\|v\|_V \leq \|e\|_V} \beta(v). \]
If we assume that (A1) to (A5) and (A4) hold, then for all \( \phi \in L^2([0, T], V_\theta), L_T^\phi = e \), the
integration by part formula holds
(4.16) \[ P_T \nabla_e f(x) = \mathbb{E} f(x(T)) \int_0^T \langle Q^{-1}(\phi(t) - \nabla_{\Gamma(t)}B(x(t))), dW(t) \rangle, \forall f \in C_b^1(H), \]
where
(4.17) \[ \Gamma(t) = \int_0^t e^{-(t-s)A_1} \phi(s) ds. \]
Proof. Consider the operator $L_T^{\theta/2}$ which maps from $L^2([0, T], V_\theta)$ to $H$. Since $A^{-\theta/2} : H \to V_\theta$ is isometric,

\[(4.18) \quad L_T^{\theta/2} : L^2([0, T], V_\theta) \to V_1 \]

is surjective, by proposition 1. Note that $A$ is self adjoint, thus $A^{-\theta/2}$ is adjoint of $A^{\theta/2}$ as an operator from $V_\theta$ to $H$, then $R_T^{\theta/2} = \int_0^T e^{-2tA}dt$. Firstly we shall choose special $\phi$ to get log Haranck inequality. Since $A^{\theta/2}e \in V_1$, as in Lemma 1.3 replacing $x$ by $A^{\theta/2}e$ and $B^{-1}$ by $A^{-\theta/2}e$, we have $\phi(t) = \frac{1}{T}e^{-(T-t)A}e + \frac{T}{2} A e^{-(T-t)A}e$. Then

\[(4.19) \quad \Gamma(t) = \frac{t}{T} e^{-(T-t)A}e, \quad L_T \phi = e, \]
\[(4.20) \quad \int_0^T ||A^{\theta/2} \phi(s)||^2 ds = \int_0^T ||\phi(s)||^2_{V_\theta} ds \leq \frac{2(1+T)}{T} ||A^{\theta/2}\||^2, \]

and by (A3)

\[(4.21) \quad \int_0^T ||Q^{-1} \phi(s)||^2 ds \leq K_3 \int_0^T ||A^{\theta/2} \phi(s)||^2 ds \leq \frac{2(1+T)K_3}{T} ||A^{\theta/2}\||^2. \]

We construct another process

\[(4.22) \quad dy(t) = -Ay(t)dt + B(x(t))dt + QdW(t) + \phi(t)dt, \quad y(0) = x, \]

then $y(t) = x(t) + \Gamma(t)$, in particular, $y(T) = x(T) + e$. Let

\[(4.23) \quad d\tilde{W}(t) = dW(t) + Q^{-1} \phi(t)dt + Q^{-1}(B(x(t)) - B(x(t) + \Gamma(t)))dt, \]

and

\[(4.24) \quad R_t = \exp \left[ -\int_0^t \langle Q^{-1}(\phi(s) + B(x(s)) - B(x(s) + \Gamma(s))), dW(s) \rangle \right. \]
\[\left. - \frac{1}{2} \int_0^t ||Q^{-1}(\phi(s) + B(x(s)) - B(x(s) + \Gamma(s)))||^2 ds \right], \]

we can rewrite $y$ as

\[(4.25) \quad dy(t) = -Ay(t)dt + B(y(t))dt + Qd\tilde{W}(t). \]

Next we shall prove that $\{\tilde{W}(t)\}_{t \in [0, T]}$ is $R_T^P$-Brownian Motion, then $y$ is a weak solution of equation (4.1), and since equation (4.1) has pathwise unique solution, $y$ and $x$ has the same law under the probability measures respectively, then

\[(4.26) \quad \mathbb{E}f(x(T)) = P_T f(x) = \mathbb{E}R_T f(y(T)) = \mathbb{E}R_T f(x(T) + e), \]

therefore the argument in [17] can be applied. To this end, we shall adapt the argument in [16] [17] to estimate $\mathbb{E}R_t \log R_t$. Note that

\[(4.27) \quad \sup_{t \in [0, T]}||\Gamma(t)||_V \leq ||e||_V, \quad \sup_{t \in [0, T]} \beta(\Gamma(t)) \leq \sup_{||v||_V \leq ||e||_V} \beta(v). \]
Then

\[ ||Q^{-1}(B(x(t)) - B(x(t) + \Gamma(t)))||^2 \leq \beta(\Gamma(t))(1 + ||x(t)||_V + ||x(t) + \Gamma(t)||_V)^2 \]

\[ \leq 3b_{\epsilon}(1 + 3||x(t)||_V^2 + 2||\Gamma(t)||_V^2) \]

Let

\[ (4.29) \tau_n = \inf\{t \in [0, T] \mid \int_0^t ||x(s)||_V^2 ds + ||x(t)||^2 \geq n\}. \]

Then by Girsanov theorem, for \( s \leq T \), \( \{\bar{W}(t)\}_{t \leq s \wedge \tau_n} \) is Brownian Motion under the probability \( R_{s \wedge \tau_n}P \). Rewrite the equation of \( x \), we have

\[ dx(t) = -Ax(t)dt + B(x(t))dt + Qd\bar{W}(t) \]

\[ = -Ax(t)dt + B(x(t))dt + Qd\bar{W}(t) - \phi(t)dt - (B(x(t)) - B(x(t) + \Gamma(t)))dt \]

\[ = -Ax(t)dt + B(x(t) + \Gamma(t))dt + Qd\bar{W}(t) - \phi(t)dt, \ t \leq s \wedge \tau_n, \]

by Itô’s formula and \( (A^2) \), as what we do to equation \((4.30)\), we rewrite it in the form of \( \bar{W} \), then get that, for any \( t \leq s \wedge \tau_n \)

\[ d||x(t)||^2 + 2||x(t)||_V^2 dt - ||Q||_{HS} dt + 2\langle \phi(t), x(t) \rangle dt \]

\[ = 2\langle B(x(t) + \Gamma(t)), x(t) \rangle dt + 2\langle Qd\bar{W}(t), x(t) \rangle \]

\[ \leq (2||B(0)|| \cdot ||x(t) + \Gamma(t)|| + 2K_1||x(t) + \Gamma(t)||^2 \gamma ||x(t) + \Gamma(t)||^{2-\gamma}) dt \]

\[ + 2K_2||\Gamma(t)||_V^2 ||x(t) + \Gamma(t)||^2 \gamma ||x(t) + \Gamma(t)||^{2-\gamma} dt + 2\langle Qd\bar{W}(t), x(t) \rangle. \]

In following \( C \) is constant depend on \( \gamma, K_1, K_2 \) may change from line to line. By B-D-G inequality and Hölder inequality, we have

\[ \mathbb{E}R_{s \wedge \tau_n} \sup\limits_{r \in [0, t \wedge \tau_n]} ||x(r)||^2 + \mathbb{E}R_{s \wedge \tau_n} \int_0^{t \wedge \tau_n} ||x(r)||_V^2 dr \]

\[ \leq (C + ||B(0)||^2 + ||x|| + ||Q||_{HS}^2)t + 2\mathbb{E}R_{s \wedge \tau_n} \sup\limits_{r \in [0, t \wedge \tau_n]} \left| \int_0^r \langle Qd\bar{W}(u), x(u) \rangle \right| \]

\[ + \int_0^t ||\phi(r)||^2 dr + C \int_0^T ||\Gamma(t)||_V^2 (1 + ||\Gamma(t)||_{V_\alpha}^\gamma) dt \]

\[ + C\mathbb{E}R_{s \wedge \tau_n} \left( \int_0^{t \wedge \tau_n} ||x(r)||^2 dr + \int_0^{t \wedge \tau_n} ||x(r)||_V^2 \Gamma(r)||_{V_\alpha}^\gamma dr \right), \]

\[ \leq (C + ||Q||_{HS}^2 + ||B(0)||^2 + ||x||)t + \int_0^t ||\phi(r)||^2 dr \]

\[ + C \int_0^T ||\Gamma(t)||_{V_\alpha}^\gamma (1 + ||\Gamma(t)||_{V_\alpha}^\gamma) dt \]

\[ + C\mathbb{E}R_{s \wedge \tau_n} \left( \int_0^{t \wedge \tau_n} ||x(r)||^2 dr + \int_0^{t \wedge \tau_n} ||x(r)||_V^2 ||\Gamma(r)||_{V_\alpha}^\gamma dr \right), \]
In order to use the Gronwall’s lemma, we need more calculate. Note that for the last term, we have

\[ \mathbb{E} R_{s \wedge \tau_n} \int_0^{t \wedge \tau_n} \| x(r) \|^2 \| \Gamma(r) \| \frac{2}{V_\alpha} dr \]

\[ \leq C \left( \int_0^t \| \Gamma(r) \| \frac{2}{V_\alpha} dr \right)^\frac{1}{2} \mathbb{E} R_{s \wedge \tau_n} \left( \int_0^{t \wedge \tau_n} \| x(r) \|^4 dr \right)^\frac{1}{2} \]

\[ \leq C \left( \int_0^t \| \Gamma(r) \| \frac{2}{V_\alpha} dr \right)^\frac{1}{2} \mathbb{E} R_{s \wedge \tau_n} \left( \sup_{r \in [0, t \wedge \tau_n]} \| x(r) \| \right) \left( \int_0^{t \wedge \tau_n} \| x(r) \|^2 dr \right)^\frac{1}{2} \]

\[ \leq \frac{1}{2} \mathbb{E} R_{s \wedge \tau_n} \sup_{r \in [0, t \wedge \tau_n]} \| x(r) \|^2 + C \left( \int_0^t \| \Gamma(r) \| \frac{2}{V_\alpha} dr \right) \mathbb{E} R_{s \wedge \tau_n} \int_0^{t \wedge \tau_n} \| x(r) \|^2 dr \]

In order clear relation with \( e \), we shall calculate the integration term relate to \( \Gamma \). By Minkowski inequality

\[ \int_0^T || \Gamma(r) || \frac{2}{V_\alpha} dr = T^{-4/(2-\gamma)} \int_0^T || r A^{\alpha/2} e^{-(T-r)\lambda} ||^{2/(2-\gamma)} e^{(2-\gamma)dr} \]

\[ = T^{-4/(2-\gamma)} \left( \int_0^T \left( \int_{\lambda_0}^{\infty} r^{2} \lambda^{\alpha} e^{-2(T-r)\lambda} d|| E_\lambda e ||^2 \right)^{2/(2-\gamma)} \right)^{1/2} \gamma \frac{2}{2} \lambda \]

\[ \leq T^{-4/(2-\gamma)} \left( \int_{\lambda_0}^{\infty} \lambda^\alpha \left( \int_0^T \gamma^{1/(2-\gamma)} e^{-(T-r)\lambda} d|| E_\lambda e ||^2 \right)^{2/(2-\gamma)} \right) \frac{2}{2} \gamma \]

\[ \leq \frac{2}{4} \gamma || A^{(2\alpha+\gamma-2)/4} e ||^{4/(2-\gamma)}. \]

In particular, for \( \alpha = 1 \) and \( \gamma = 0 \), we have

\[ \int_0^T || \Gamma(r) ||^2 \frac{2}{V_\alpha} dr \leq \frac{1}{2} || e ||^2. \]

For \( \alpha = 1 \), \( \gamma = 1 \),

\[ \int_0^T || \Gamma(r) ||^{2/2} || \Gamma(r) ||^{2/2} dr \leq \left( \int_0^T || \Gamma(r) ||^4 \frac{2}{V_\alpha} dr \right)^{1/2} \left( \int_0^T || \Gamma(r) ||^4 \frac{2}{V_\alpha} dr \right)^{1/2} \]

\[ \leq \frac{\sqrt{2 - 2}}{4} || A^{1/4} e ||^2 || A^{(2\alpha+\gamma-2)/4} e ||^{2/(2-\gamma)}. \]

At last

\[ \int_0^T || \phi(r) ||^{2} dr \leq \frac{2(T + 1)}{T} || A^{1/2} e ||^2. \]

Therefore

\[ \mathbb{E} R_{s \wedge \tau_n} \sup_{r \in [0, t \wedge \tau_n]} || x(r) ||^2 + \mathbb{E} R_{s \wedge \tau_n} \int_0^{t \wedge \tau_n} || x(r) ||^2 \frac{2}{V_\alpha} dr \]

\[ \leq C \left\{ || Q ||_H^2 + || B(0) ||^2 + || x || \right\} T + \frac{1}{2} \frac{2 - \gamma}{4} || A^{1/4} e ||^2 || A^{(2\alpha+\gamma-2)/4} e ||^{2/(2-\gamma)} \}

\[ + C \left( 1 + \frac{2 - \gamma}{4} || A^{(2\alpha+\gamma-2)/4} e ||^{4/(2-\gamma)} \right) \int_0^{t \wedge \tau_n} || x(r) ||^2 dr \]
By Gronwall’s inequality

\begin{equation}
\mathbb{E} \sup_n \left[ R_{s \wedge \tau_n} \sup_{r \in [0, T \wedge \tau_n]} \| x(r) \|^2 + \mathbb{E} R_{s \wedge \tau_n} \int_0^{t \wedge \tau_n} \| x(r) \|^2 \, dr \right] \leq \infty.
\end{equation}

By these estimate and (A4), we have

\begin{equation}
\mathbb{E} R_{s \wedge \tau_n} \log R_{s \wedge \tau_n} = \frac{1}{2} \mathbb{E} R_{s \wedge \tau_n} \int_0^{s \wedge \tau_n} \| Q^{-1}(\phi(t) + B(x(t)) - B(x(t) - \Gamma(t))) \|^2 \, dt
\end{equation}

\begin{equation}
\leq K_3 \int_0^T \| \phi(t) \|^2 \, dt + 3b \mathbb{E} R_{s \wedge \tau_n} \int_0^{s \wedge \tau_n} (1 + 4\| x(t) \|_V^2 + \| \Gamma(t) \|_V^2) \, dt
\end{equation}

\begin{equation}
= \Psi(x, T, e).
\end{equation}

therefore, as in [17], we can prove that \{\hat{W}(t)\}_{t \in [0,T]} is B.M. and

\begin{equation}
\mathbb{E} R_T \log R_T \leq \Psi(x, T, e).
\end{equation}

By this estimate and Young’s inequality, we have the shift log-Harnack inequality,

\begin{equation}
P_T \log f(x) = \mathbb{E} Q \log f(y_T) = \mathbb{E} R_T \log f(x_T + e)
\leq \mathbb{E} R_T \log R_T + \log \mathbb{E} f(x_T + e) \leq \log P_T f(e + \cdot)(x) + \Psi(x, T, e).
\end{equation}

For Integration by part formula, one can choose any \( \phi \in L^2([0,T], V_\theta) \) such that \( L_T^\epsilon \phi = e \). Replacing \( e \) by \( \epsilon e \), \( \phi \) by \( \epsilon \phi \) and \( \Gamma \) by \( \epsilon \Gamma \), just as in the case \( \epsilon = 1 \) above, and by Lemma 4.3 we have

\begin{equation}
\frac{d}{d\epsilon} |_{\epsilon=0} R_T^\epsilon = - \int_0^T \langle Q^{-1} \left( \phi(t) - \nabla \Gamma(t) B(x(t)) \right), dW(t) \rangle,
\end{equation}

holds in \( L^1(\mathbb{P}) \), then

\begin{equation}
P_T \nabla e f(x) = \mathbb{E} f(x(T)) \int_0^T \langle Q^{-1} \left( \phi(t) - \nabla \Gamma(t) B(x(t)) \right), dW(t) \rangle, \quad f \in C^1_b(H).
\end{equation}

We adapted the argument in [8] to prove that

\textbf{Lemma 4.3.} Under conditions (A1) to (A3) and (A5), then \( \left\{ \frac{R_T - 1}{\epsilon} \right\}_{\epsilon \in (0,1)} \) is uniformly integrable w.r.t \( \mathbb{P} \), consequently

\begin{equation}
\frac{d}{d\epsilon} |_{\epsilon=0} R_T^\epsilon = - \int_0^T \langle Q^{-1} \left( \phi(t) - \nabla \Gamma(t) B(x(t)) \right), dW(t) \rangle,
\end{equation}

holds in \( L^1(\mathbb{P}) \).

\textbf{Proof.} Denote

\begin{equation}
\Theta_1(s) = Q^{-1}(\epsilon \phi(s) + B(x(s)) - B(x(s) + \epsilon \Gamma(s)))
\end{equation}

\begin{equation}
\Theta_2(s) = Q^{-1}(\phi(s) + \nabla \Gamma(s) B(x(s) + \epsilon \Gamma(s)))
\end{equation}

\begin{equation}
\Theta_3(s) = \epsilon Q^{-1}(\phi(s) - B(x(s)) + B(x(s) + \epsilon \Gamma(s))).
\end{equation}
Since
\[
\mathbb{E} \int_0^T \sup_{\epsilon \in [0,1]} ||\nabla_{\Gamma(r)} B(x(r) + \epsilon \Gamma(r))||_Q^2 dr \\
\leq \mathbb{E} \int_0^T ||\Gamma(s)||^2_V (||x(s)||^2_V + ||\Gamma(s)||^2_V) ds \\
\leq \sup_{s \in [0,T]} ||\Gamma(s)||^2_V \mathbb{E} \int_0^T ||x(s)||^2_V ds + \int_0^T ||\Gamma(s)||^4_V ds < \infty,
\]
we have, for any $\epsilon \in [0,1)$,
\[
\frac{d}{d\epsilon} R_T^\epsilon = - R_T^\epsilon \int_0^T \langle \Theta^\epsilon_2(s), dW(s) \rangle - R_T^\epsilon \int_0^T \langle \Theta^\epsilon_3(s), \Theta^\epsilon_2(s) \rangle ds, \ a.s.,
\]
and then
\[
\frac{|R_T^\epsilon - 1|}{\epsilon} = \frac{1}{\epsilon} \int_0^\epsilon R_T^r \left( \int_0^T \langle \Theta^r_2(s), dW(s) \rangle + \langle \Theta^r_1(s), \Theta^r_2(s) \rangle ds \right) dr \\
\leq \frac{1}{\epsilon} \int_0^\epsilon R_T^r \int_0^T \langle \Theta^r_2(s), dW(s) \rangle dr + \frac{1}{\epsilon} \int_0^\epsilon R_T^r \int_0^T \langle \Theta^r_1(s), \Theta^r_2(s) \rangle ds dr.
\]
Note that
\[
\frac{1}{\epsilon} \int_0^\epsilon R_T^r \int_0^T \langle \Theta^r_1(s), \Theta^r_2(s) \rangle ds dr \\
\leq \frac{1}{\epsilon} \int_0^T R_T^r \int_0^T r [||\phi(s)||_Q + ||\Gamma(s)||_V(||\Gamma(s)||_V + ||x(s)||_V)]^2 ds dr \\
\leq \int_0^T R_T^r \int_0^T [||\phi(s)||_Q + ||\Gamma(s)||_V(||\Gamma(s)||_V + ||x(s)||_V)]^2 ds dr
\]
and just as in the case of $\epsilon = 1$, we can prove that
\[
\mathbb{E} \int_0^1 R_T^r \int_0^T [||\phi(s)||_Q + ||\Gamma(s)||_V(||\Gamma(s)||_V + ||x(s)||_V)]^2 ds dr \\
\leq \int_0^T ||\phi(s)||^2_Q ds + \int_0^T ||\Gamma(s)||^4_V ds + \sup_{s \in [0,T]} ||\Gamma(s)||^2_V \int_0^1 \mathbb{E} R_T^r \int_0^T ||x(s)||^2_V ds dr
\]
By these estimate, follow the line of [Lemma 2.4.] completely, one can prove the lemma. □

**Corollary 4.4.** Assume that (A1) to (A3) and (A4) hold, further more (A4) or (A4) hold in the following form
\[
||B(u) - B(v)||_Q \leq K_4 ||u - v|| (1 + ||u||_V + ||v||_V),
\]
let
\[
\delta_e = \frac{e^{(\lambda_0 - 2K_3)T}}{18K_4||Q||^2||e||^2_V T}
\]

then for \( r \in (0, \sqrt{\delta_e}) \) and \( p > \frac{\sqrt{8\delta_e r^2 + r^4} + 2\Delta_e + r^2}{2\Delta_e - r^2} \),

\[
(P_T f(x))^p \leq P_T f^p (r e + \cdot) (x) \exp \left\{ \frac{p - 1}{4} \frac{||x||^2}{T} + \left( \frac{||Q||^2_{HS} + 2K_5}{2} \right) ((\lambda_0 - 2K_5)^+ T \lor 1) \right\}
\]

(4.55)

\[+ \frac{(p + 1)p}{2(p - 1)} \left( \frac{2K_3(T + 1)}{T} \right) ||A^{(1+\theta)/2}||^2 + \frac{3}{2} ||A^{1/4} e||^4 + \frac{3K_4}{2} ||e||^2 \].

If strengthen \((A7)\) to be

(4.56)

\[||B(u) - B(v)||_Q \leq \beta (u - v),\]

then, for any \( p > 1 \), the following shift Harnack inequality holds

(4.57)

\[(P_T f)^p \leq P_T f^p (e + \cdot) \exp \left\{ \frac{p}{p - 1} \left( \frac{2||A^{1/2} e||}{1 - e^{-2\lambda_0 T}} + \int_0^T \beta (\Gamma(s)) ds \right) \right\} \]

**Proof.** By Remark 4.1, we assume \((4.53)\) holds. We adapted the technology used in [19, Lemma 3.1]. Let \( \lambda = \frac{c_0}{4||Q||^2} \) and

\[\beta(t) = \left[ \frac{c_0 (1 - e^{-(\lambda_0 - 2K_5)t})}{\lambda_0 - 2K_5} + c_0 T e^{-(\lambda_0 - 2K_5)t} \right]^{-1},\]

for \( \lambda_0 - 2K_5 = 0 \) we define it as \( \frac{1}{c_0 T} \). By Itô’s formula for \( ||x(t)||^2 \beta (t) \) and Hölder inequality, we can prove that

\[
\mathbb{E} \exp \left\{ 2\lambda \int_0^{T \wedge \tau_n} \beta (t) ||x(t)||_V^2 dt - \lambda \beta (0) ||x||_Q^2 - \lambda ||Q||^2_{HS} + 2K_5 \int_0^T \beta (t) dt \right\}
\]

(4.58)

\[\leq \left[ \mathbb{E} \exp \left\{ 2\lambda \int_0^{T \wedge \tau_n} [(2K_5 - \lambda_0) \beta (t) + \beta' (t)] \cdot ||x(t)||_V^2 dt + 4\lambda \int_0^{T \wedge \tau_n} \beta (t) ||x(t)||^2 dt \right\} \right]^{1/2}
\]

\[\times \left[ \mathbb{E} \exp [2\lambda \int_0^{T \wedge \tau_n} \beta (t) ||x(t)||_V^2 dt] \right]^{1/2},\]

then by the definition of \( \lambda \) and \( \beta \), we have

(4.59)

\[
\mathbb{E} \exp \left\{ \frac{e^{(\lambda_0 - 2K_5)^- T}}{2T ||Q||^2} \int_0^T ||x(t)||^2_V dt \right\}
\]

\[\leq \exp \left\{ \frac{||x||^2}{2T ||Q||^2} + \frac{||Q||^2_{HS} + 2K_5}{2 ||Q||^2} ((\lambda_0 - 2K_5)^+ T \lor 1) \right\}.
\]

For \( r \in (0, \sqrt{\delta_e}) \), just replacing \( e \) by \( re \) in Theorem 4.2, for and \( p > \frac{\sqrt{8\delta_e r^2 + r^4} + 2\Delta_e + r^2}{2\Delta_e - r^2} \), we can prove that

(4.60)

\[
(\mathbb{E} R_{T^p}^{\frac{p}{p-1}})^{p-1} \exp \left\{ - \frac{p(p + 1)}{2(p - 1)} \left( K_3 \int_0^T ||\phi(t)||_V dt + 6K_4 \int_0^T ||\Gamma(t)||^2_V dt + 3K_4 \int_0^T ||\Gamma(t)||^2_V dt \right) \right\}
\]

\[\leq \left( \mathbb{E} \exp \left[ \frac{18K_4 p(p + 1)}{(p - 1)^2} \int_0^T ||\Gamma(t)||^2_V ||x(t)||^2_V dt \right] \right)^{\frac{p-1}{2}}
\]

\[\leq \left( \mathbb{E} \exp \left[ \frac{9K_4 p(p + 1)}{(p - 1)^2} ||re||^2_V \int_0^T ||x(t)||^2_V dt \right] \right)^{\frac{p-1}{2}},\]

20
by the definition of $\delta_e$, we have
\begin{equation}
(4.61) \quad \frac{9K_4p(p+1)||e||^2_v}{(p-1)^2} \leq \frac{e^{(\lambda_0-2K_5)^- T}}{2T||Q||^2},
\end{equation}
then
\begin{equation}
(4.62) \quad (\mathbb{E}R_T^{\frac{p}{p-1}})^{p-1} \exp \left\{ -\frac{p(p+1)}{2(p-1)} \left( K_3 \int_0^T ||\phi(t)||^2_{V_o} dt + 6K_4 \int_0^T ||\Gamma(t)||^2_{H} dt + 3K_4 \int_0^T ||\Gamma(t)||^2_{v} dt \right) \right\}
\leq \exp \left\{ \frac{p-1}{4||Q||^2} \left[ \frac{||e||^2}{T} + \left( ||Q||^2_{H^S} + 2K_5 \right) [(\lambda_0 - 2K_5)^+ T \vee 1] \right] \right\}.
\end{equation}
Combine this with (4.34), (4.20) and (4.35), we prove the first inequality. The second inequality is similar to corollary 2.5.

\[ \square \]

**Corollary 4.5.** For hyperdissipative stochastic Navier-Stokes/Burgers equation in [19], (A1) to (A6) hold.

**Proof.** We just have to verify the conditions (A1) to (A6). (A3) is the (A0) there, and by the bilinear, it’s Fréchet differentiable form $V$ to $Q(H)$, and by (A3) in [19],
\begin{equation}
(4.63) \quad ||\nabla_u B(v)||_Q = ||B(u, v) + B(v, u)||_Q \leq C||u||_{V_o}||v||_{V_o} \leq C||u||_V||v||_V,
\end{equation}
then (A5) holds. By (A3) in [19],
\begin{equation}
(4.64) \quad ||B(u) - B(v)||_Q = ||B(u - v + v) - B(v)||_Q \\
\leq ||B(u - v)||_Q + ||B(u - v, v)||_Q + ||B(v, u - v)||_Q \\
\leq C \left( ||u - v||^2_{V_o} + 2||u - v||_{V_o}||v||_{V_o} \right) \\
\leq C||u - v||_V (||u||_V + ||v||_V),
\end{equation}
then (A4) holds with $\beta(\cdot) = C||\cdot||_V$ and $K_4 = 0$. By (A2) in [19], and
\begin{equation}
(4.65) \quad \langle B(u) - B(v), u - v \rangle = \langle B(u - v + v) - B(v), u - v \rangle \\
= \langle B(u - v, v) + B(v, u - v) + B(u - v), u - v \rangle \\
= \langle B(v, u - v), u - v \rangle + \langle B(u - v, v), u - v \rangle \\
\leq C(||v|| \cdot ||u - v||_V ||u - v|| + ||u - v|| \cdot ||v||_V ||u - v||) \\
\leq C||v||_V ||u - v||_V ||u - v||.
\end{equation}
(4.4) and (4.3) holds for $\gamma = 1$, $\rho = ||\cdot||_V$, the hemicontinuous follows from bilinear. Therefore, we prove the corollary.

\[ \square \]

At last, we give a simple corollary to discuss the density of solution of equation (1.1).
**Corollary 4.6.** Under the condition of Theorem 4.2. Let \( \pi_n \) be the orthogonal projection from \( H \) to some \( n \) dimension subspace \( H_n \). Then the distribution of \( \pi_n x(T) \) has density \( \rho_n \) with respective to Lebesgue measure on \( H_n \) and

\[
\nabla \log \rho_n(x) = -\mathbb{E}(N|\pi_n x(T) = x), \quad \mathbb{P}_{\pi_n x(T)} - \text{a.s.},
\]

where \( N \) such that \( P_T \nabla_{e} f = \mathbb{E}f(x(T))N(e) \).

**Proof.** For all \( f \in C^1_b H_n \), let \( f_n(x) = f(\pi_n x) \forall x \in H \). Then for all \( h \in H_n \),

\[
\mathbb{E}\nabla_h f(\pi_n x(T)) = \mathbb{E}\nabla_h f_n(x(T)) = \mathbb{E}f_n(x(T))N(h) = \mathbb{E}f(\pi_n x(T))N(h),
\]

by Theorem 2.4 in [17], we prove the corollary.

**Remark 4.7.** The more interesting case for the density of the projection of the solution is SPDE driven by degenerate noise, see [1] and reference there in.

For more applications of shift Harnack inequality and integration by part formula, one can see [17].

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