Wave propagation in a strongly nonlinear locally resonant granular crystal

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\textbf{A B S T R A C T}
In this work, we study the wave propagation in a recently proposed acoustic structure, the locally resonant granular crystal. This structure is composed of a one-dimensional granular crystal of hollow spherical particles in contact, containing linear resonators. The relevant model is presented and examined through a combination of analytical approximations (based on ODE and nonlinear map analysis) and of numerical results. The generic dynamics of the system involves a degradation of the well-known traveling pulse of the standard Hertzian chain of elastic beads. Nevertheless, the present system is richer, in that as the primary pulse decays, secondary ones emerge and eventually interfere with it creating modulated wave trains. Remarkably, upon suitable choices of parameters, this interference “distills” a weakly nonlocal solitary wave (a “nanopteron”). This motivates the consideration of such nonlinear structures through a separate Fourier space technique, whose results suggest the existence of such entities not only with a single-side tail, but also with periodic tails on both ends. These tails are found to oscillate with the intrinsic oscillation frequency of the out-of-phase motion between the outer hollow bead and its internal linear attachment.

\section{Introduction}

Dynamics of one-dimensional granular chains has attracted substantial interest from the researchers of quite different scientific areas [1–24] due to their exciting dynamical properties. These chains support the formation of highly robust, strongly localized and genuinely traveling elastic stress waves. The existence of traveling waves was originally proved in [7] using the variational approach of [25], yet no information was given on their profile. Their single pulse character (in the strain variables) was rigorously shown in [26], following the approach of [27], and the doubly exponential character of their spatial decay in the absence of precompression was established. Earlier work on the basis of long wavelength approximations and numerical computations had conjectured that the waves were genuinely compact (spanning a finite number of elements) [1].

Recent studies [9–18] in the area have been mainly concerned with the effect of various types of structural inhomogeneities induced in the granular chain. The latter leads to a modulation of the solitary waves, as well as to new kinds of breathing modes produced either robustly [12–14,28,29] or transiently [30]. Wave propagation in tapered and decorated granular chains has been extensively studied in [9–11] both analytically and numerically. The approximations developed in these works for the estimation of the maximal pulse velocity recorded on each one of the granules along with its propagation through such inhomogeneous granular chains have demonstrated a good correspondence of the analytic predictions with the numerical simulations. Additional experimental, computational and analytical studies were devoted to the dynamics of the periodic granular chains (e.g. diatomic chains, granular containers, etc.) under various conditions of initial pre-compression [12–18]. Dynamics of primary pulses in the non-compressed granular chain perturbed by a weak dissipation has been considered in [19–24]. These, in turn, shed light on the evolution of the primary pulses in the dissipative 1D granular media and provide some qualitative theoretical (in some cases in connection with experiments [23]) estimations for modeling the dissipation in the chain as well as depicting the rate of decay of the primary pulse. A systematic theoretical attempt to capture the (decaying) evolution of a primary pulse in the granular chain subject to on-site perturbation has been provided in the extensive study of [31].

In the present paper we study a novel acoustic structure which has been recently considered in some experimental and theoretical studies [32–34], the locally resonant granular crystal. The fundamental unit cell of these periodic systems is made of an outer mass...
The governing equations of motion can then be written as follows,
\[ M_i \frac{d^2 U_i}{dt^2} = \left( \frac{4}{3} \right) E^* \sqrt{R_i} \left[ (U_{i-1} - U_i)^{3/2} - (U_i - U_{i+1})^{3/2} \right] \]
\[ + k (u_i - U_i), \quad \forall i, i \in N \]
\[ m_i \frac{d^2 u_i}{dt^2} = -k (u_i - U_i). \]  

Here \( U_i \) is the displacement of the \( i \)th sphere, while \( u_i \) is the displacement of the small mass, linearly coupled at the center of the \( i \)th sphere, \( r_i \) is the radius of the sphere, \( M_i \) is the mass of the sphere; and \( E^* = E/2 (1 - \mu^2) \); \( E \) is the elastic (Young’s) modulus and \( \mu \) is the Poisson’s ratio of the sphere. We note that the interaction force between the neighboring elements is given by \( F = (4/3) E^* \sqrt{R_i} \Delta^{3/2} \), where \( R_i \) is the radius (assumed implicitly to be uniform in the above expression i.e., independent of \( i \)) and \( \Delta \) is their relative displacement. Moreover, the \((+/-)\) subscripts in (1) and (3) indicate that only non-negative values of the expressions in parentheses are considered, i.e., the interaction is tensionless. We should also mention in passing that a mathematically similar system with (isolated) external resonators (a so-called mass–mass–mass setting) has recently been proposed in [38]. It is noteworthy that despite the mathematical equivalence of these two settings, their experimental realizations are quite different.

The system nondimensionalization is performed as follows
\[ X_i = \frac{U_i}{R_i}, \quad x_i = \frac{u_i}{R_i}, \quad \tau = \frac{E^*}{\sqrt{2\pi R_i^2 \rho}} \left[ t; \tilde{k} = \frac{3\sqrt{2k}}{4R_i E^*} \right] \]
\[ v = \frac{m_i}{M_i} \frac{d}{dt} = \frac{m}{M}. \]  

It is important to note that in the present study we assume (with one caveat to be explained below) that the outer and inner masses are uniform all through the chain (i.e. \( R_i = R, M_i = M, m_i = m \)).

Substituting (2) into (1) we end up with the following, non-dimensional set of equations governing the system dynamics
\[ X_{i,-\tau} \left[ (X_{i-1} - X_i)^{3/2} - (X_i - X_{i+1})^{3/2} \right] + \tilde{k} \left( x_i - X_i \right) \]
\[ x_{i,-\tau} = -\tilde{k} \left( x_i - X_i \right). \]  

To make the further analysis of (3) somewhat simpler it is convenient to introduce the coordinates of relative displacements (i.e., strains) for both outer and inner masses,
\[ \Delta_i = X_i - X_{i+1} \]
\[ d_i = x_i - x_{i+1}. \]  

Substituting (4) and (5) into (3) we obtain the following set of equations,
\[ \Delta_{i,-\tau} = \Delta_{i-1,\tau} - 2\Delta_{i-1,\tau}^3 + \Delta_{i+1,\tau} + \tilde{k} (d_i - \Delta_i) \]
\[ x_{i,-\tau} = -\tilde{k} (d_i - \Delta_i). \]  

The goal of the present study is to examine the wave propagation along the contacts of the outer masses (which contribute to highly nonlinear dynamics) in the presence of the internal mass attachments. We assume that the coupling between the internal mass and the outer sphere is linear and weak such that \( \tilde{k} \) is treated as a small system parameter \( k = \varepsilon \). We anticipate that as the primary pulse propagates down the chain, it is possible for it to “transfer” energy to the internal, linear attachments, storing it in the form of potential energy and thus depriving the original pulse from its initial kinetic energy. This, in turn, is expected to yield a decay of the amplitude of the primary pulse, which we now consider in three distinct asymptotic limits, namely: (1) \( \nu \ll 1 \), (2) \( \nu = O(1) \) and (3) \( \nu \gg 1 \).

2. Physical model

In the present study we consider the uncompressed, one-dimensional, locally resonant granular crystal composed of hollow elastic spheres in contact, containing linear resonators, as this is illustrated in Fig. 1. According to [37], the contact interaction of two hollow spheres depends strongly on the thickness of the spherical shells. However, for relatively thick spherical shells, the interaction contact follows the Hertzian contact law [1].

Fig. 1. Scheme of the model under consideration.
