CONCORDANCES FROM DIFFERENCES OF TORUS KNOTS TO L–SPACE KNOTS

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Abstract. It is known that connected sums of positive torus knots are not concordant to L–space knots. Here we consider differences of torus knots. The main result states that the subgroup of the concordance group generated by two positive torus knots contains no nontrivial L–space knots other than the torus knots themselves. Generalizations to subgroups generated by more than two torus knots are also considered.

1. Introduction

In [4], Krcatovich showed that all L–space knots are prime. Thus no nontrivial connected sum of knots is an L–space knot. We consider instead concordances from knots to L–space knots. In [12], Zemke uses involutive knot Floer homology to obstruct a knot from being concordant to an L–space knot. He gives a few examples of this obstruction for sums and differences of torus knots. Expanding on this, Livingston [6], showed that a connected sum of positive torus knots is not concordant to an L–space knot using the Ozsváth–Szabó tau invariant, the Alexander polynomial, and the knot signature. In his paper, Livingston discusses a few examples in which his strategy does or does not apply to showing that differences of torus knots are not concordant to L–space knots. We show the following:

Theorem 1.1. If the connected sum of distinct positive torus knots \( mT(p, q) \# nT(r, s) \) is concordant to an L–space knot, then either \( m = 0 \) and \( n = 1 \) or \( m = 1 \) and \( n = 0 \).

The proof uses an approach similar to that of Livingston, with two additions:

- properties of the Ozsváth–Stipsicz–Szabó Upsilon invariant,
- and the result of Hedden and Watson [3] that the leading terms of the Alexander polynomial of an L–space knot of genus \( g \) must be \( t^{2g} - t^{2g-1} \).

More generally, we make the following conjecture:

Conjecture 1.2. If a connected sum of torus knots is concordant to an L–space knot, then it is concordant to a positive torus knot.

Note that Litherland [5] proved that torus knots are linearly independent in the concordance group. So a connected sum of torus knots is concordant to a positive torus knot \( T(p, q) \) only if it is of the form \( mT(p, q) \# (m-1)T(p, q) \) where \( m \geq 1 \).

As progress towards this conjecture, we give conditions under which a connected sum of torus knots is not concordant to an L–space knot. These conditions involve all of the aforementioned invariants, as well as the relations among Upsilon functions for torus knots discovered by Feller and Krcatovich [2].

Notation. Throughout this paper, all torus knots \( T(a, b) \) considered are such that \( 1 < a < b \) and \( \gcd(a, b) = 1 \).
2. Preliminaries

In this section, we gather some useful facts and references concerning $L$–space knots, torus knots, the Ozsváth–Szabó tau invariant, the Alexander polynomial, the Ozsváth–Stipsicz–Szabó Upsilon invariant, and the Levine–Tristram knot signature.

In [9], Ozsváth and Szabó introduced the Heegaard Floer invariant $\widehat{HF}(Y)$ which associates a graded abelian group to a closed 3–manifold $Y$. A rational homology 3–sphere $Y$ is called an $L$–space if $\text{rank}(\widehat{HF}(Y)) = |H_1(Y;\mathbb{Z})|$ (see [10]). A knot $K$ is called an $L$–space knot if it admits a positive $L$–space surgery. Since lens spaces are $L$–spaces, positive torus knots are $L$–space knots.

The Heegaard–Floer knot complex $\text{CFK}^\infty(K)$ was introduced in [8]. For $L$–space knots, the complex $\text{CFK}^\infty(K)$ is always a staircase complex where the height and width of each step is determined by the gaps in the exponents of the Alexander polynomial of $K$; the Alexander polynomial can be written as

$$\Delta_K(t) = \sum_{i=0}^{d} (-1)^i t^{a_i}$$

for some sequence of integers $\{a_i\}$ and $\text{CFK}^\infty(K)$ is a staircase of the form $[a_1-a_0, a_2-a_1, ..., a_d-a_{d-1}]$ where the indices alternate between horizontal and vertical steps. For more details, see [10] and [1].

In [11], Ozsváth, Stipsicz, and Szabó defined the Upsilon invariant, $\Upsilon_K(t)$, a piecewise–linear function with domain $[0, 2]$. See [11] for explicit computations for the family of knots $T(p, p+1)$ and for formulas for computing $\Upsilon$ for $L$–space knots from $\text{CFK}^\infty$. The derivative $\Upsilon'_K(t)$ is piecewise–constant with singularities at $t$–values where the slope changes in $\Upsilon_K(t)$. See Livingston [7] for results on computing $\Upsilon'_K(t)$ from $\text{CFK}^\infty(K)$.

Finally, consider the Levine–Tristram signature function $\sigma_K(t)$. The signature function is piecewise–constant and integer–valued with possible jumps occurring at zeroes of the Alexander polynomial of $K$. Livingston in [6] showed that if the cyclotomic polynomial $\left(\phi_c(t)\right)^k$ divides $\Delta_{T(p,q)}(t)$, then $\sigma_{T(p,q)}(t)$ jumps by $-2k$ at $t = 1/c$. We use this fact along with the following to prove Theorem 3.1.

Theorem 2.1 ([3, 7, 10]). If $K$ is an $L$–space knot, then

(a) $2\tau(K) = \deg(\Delta_K(t))$.
(b) $\Upsilon'_K(t)$ is increasing.
(c) $\Delta_K(t)$ has highest order terms $t^{2g} - t^{2g-1}$.

3. Proof of the main theorem

We break the proof into several smaller propositions. Note that if $J$ is concordant to $K$, then $\tau(K) = \tau(J)$, $\sigma_K(t) = \sigma_J(t)$, and $\Upsilon_K(t) = \Upsilon_J(t)$, as these are all concordance invariants.

Livingston’s result in [6] includes the cases where both $m$ and $n$ are nonnegative, so we need only show the result for at most one of $m, n$ positive. The first case is easy; we use the Ozsváth–Szabó tau invariant to rule out the case where $m, n \leq 0$.

Proposition 3.1. The knot $K = mT(p, q) \# nT(r, s)$ where $m, n \leq 0$ is not concordant to a nontrivial $L$–space knot.
Proof. If \( m = n = 0 \), then \( K \) is the unknot. So assume that \( m, n \leq 0 \) with at most one of \( m, n \) equal to \( 0 \). Suppose that \( J \) is a nontrivial \( L \)-space knot concordant to \( K \) and consider the Ozsváth–Szabó tau invariant, \( \tau(K) \). Recall that \( 0 < p < q \) and \( 0 < r < s \). Note that since the knots \( J, T(p, q), \) and \( T(r, s) \) are nontrivial \( L \)-space knots, \( \tau(J), \tau(T(p, q)), \) and \( \tau(T(r, s)) \) are positive by Theorem 2.1 (a). However, since \( J \) is concordant to \( K \), by the additivity of \( \tau \) under forming connected sums, we have that

\[
\tau(J) = \tau(K) = m\tau(T(p, q)) + n\tau(T(r, s)) < 0,
\]
a contradiction. Thus \( K \) is not concordant to an \( L \)-space knot. \( \square \)

The remaining cases are those where \( K = mT(p, q) \# nT(r, s) \) and \( m \cdot n < 0 \). Without loss of generality, we will assume that \( m > 0 \) and \( n < 0 \). For ease of notation, we write this as \( K = mT(p, q) \# -nT(r, s) \) with \( m, n > 0 \). Next, we use the Ozsváth–Stipsicz–Szabó Upsilon invariant to rule out the case where \( r > p \).

**Proposition 3.2.** The knot \( K = mT(p, q) \# -nT(r, s) \), where \( m, n > 0 \) and \( r > p \), is not concordant to an \( L \)-space knot.

**Proof.** Suppose that \( J \) is a \( L \)-space knot concordant to \( K \) and consider the Ozsváth–Stipsicz–Szabó Upsilon invariant, \( \Upsilon_K(t) \). Recall that \( 0 < p < q \) and \( 0 < r < s \). Since the knots \( J, T(p, q), \) and \( T(r, s) \) are all nontrivial \( L \)-space knots, \( \Upsilon'_J(t), \Upsilon'_T(p, q)(t), \) and \( \Upsilon'_T(r, s)(t) \) must be increasing. Analyzing \( \text{CFK}^\infty(T(p, q)) \) and \( \text{CFK}^\infty(T(r, s)) \), we see that \( \Upsilon_T(p, q)(t) \) has its first jump at \( t = 2/p \) and \( \Upsilon_T(r, s)(t) \) has its first jump at \( t = 2/r \). Since \( \Upsilon \) is additive under connected sum, \( \Upsilon_K(t) \) has its first jump at \( t = \min\{2/p, 2/r\} = 2/r \). Because \( -n < 0 \), this first jump is negative, implying that \( \Upsilon'_K(t) \) is not increasing. Since \( \Upsilon_J(t) = \Upsilon_K(t) \), we have that \( \Upsilon'_J(t) \) is not increasing, which is a contradiction. \( \square \)

For the remaining cases, we rely heavily on the work of Livingston in [6] for analyzing the relationship between the Alexander polynomial and the Levine–Tristram signature function of connected sums of torus knots.

**Proposition 3.3.** If the knot \( K = mT(p, q) \# -nT(r, s) \), where \( m, n \geq 1 \) and \( p \leq r \), is concordant to an \( L \)-space knot \( J \), then \( rs \) divides \( pq \) and

\[
\frac{(\Delta_{T(p, q)}(t))^m}{(\Delta_{T(r, s)}(t))^n} = \Delta_J(t).
\]

**Corollary 3.4.** If the knot \( K = mT(p, q) \# -nT(r, s) \), where \( m, n \geq 1 \) and \( p \leq r \), is concordant to an \( L \)-space knot \( J \), then \( m = n + 1 \).

**Proof of Proposition [6] Assume that \( J \) is an \( L \)-space knot concordant to \( K \). Recall that \( 0 < p < q \) and \( 0 < r < s \). Consider the Alexander polynomial of \( K \). It is a product of cyclotomic polynomials \( \phi_c(t) \):

\[
\Delta_K(t) = (\Delta_{T(p, q)}(t))^m (\Delta_{T(r, s)}(t))^n = \left( \prod_{a_j \neq b_j \neq 1} \phi_{a_j b_j}(t) \right)^m \left( \prod_{a_j \neq b_j \neq 1} \phi_{a_j b_j}(t) \right)^n.
\]

Suppose there exists \( c \) such that \( \phi_c(t) \mid \Delta_{T(r, s)}(t) \) and \( \phi_c(t) \mid \Delta_{T(p, q)}(t) \). By Livingston [6], this implies that the Levine–Tristram signature function of \( K \), \( \sigma_K(t) \), jumps by \(-2(m - n)\) at \( t = 1/c \).
Therefore, since \( J \) is concordant to \( K \), we would have that \((\phi_c(t))^{m-n}\) divides \( \Delta_J(t) \). Similarly, if \( c \) is such that \( \phi_c(t) \mid \Delta_{T(r,s)}(t) \) and \( \phi_c(t) \mid \Delta_{T(p,q)}(t) \), then \((\phi_c(t))^{n}\) divides \( \Delta_J(t) \), or if \( c \) is such that \( \phi_c(t) \not\mid \Delta_{T(r,s)}(t) \) and \( \phi_c(t) \mid \Delta_{T(p,q)}(t) \), then \((\phi_c(t))^{m}\) divides \( \Delta_J(t) \). Thus
\[
\text{deg}(\Delta_J(t)) \geq \text{deg}(\Delta_K(t)) - 2n \sum_{c \in C} \text{deg}(\phi_c(t))
\]
where \( C = \{c : \phi_c(t) \mid \Delta_{T(r,s)}(t) \text{ and } \phi_c(t) \not\mid \Delta_{T(p,q)}(t)\} \). On the one hand, we know that for \( L \)-space knots the degree of the Alexander polynomial is equal to twice the \( \tau \) invariant of the knot. So, since \( J \) is concordant to \( K \), we have that
\[
\text{deg}(\Delta_J(t)) = 2\tau(J) = 2\tau(K) = 2\left(m \cdot \frac{(p-1)(q-1)}{2} - n \cdot \frac{(r-1)(s-1)}{2}\right).
\]
On the other hand, we have
\[
\text{deg}(\Delta_K(t)) = m(p - 1)(q - 1) + n(r - 1)(s - 1).
\]
Therefore,
\[
\sum_{c \in C} \text{deg}(\phi_c(t)) \geq \frac{\text{deg}(\Delta_K(t)) - \text{deg}(\Delta_J(t))}{2n} = \frac{2n(r - 1)(s - 1)}{2n} = (r - 1)(s - 1) = \text{deg}(\Delta_{T(r,s)}(t)).
\]
So, if there exists \( c \) such that \( \phi_c(t) \mid \Delta_{T(r,s)}(t) \) but \( c \notin C \), then
\[
\sum_{c \in C} \text{deg}(\phi_c(t)) < \text{deg}(\Delta_{T(r,s)}(t)),
\]
which is a contradiction. Thus it must be that \( \Delta_{T(r,s)}(t) \mid \Delta_{T(p,q)}(t) \). Note that this implies that \( rs \) divides \( pq \). Also, \( \Delta_{T(r,s)}(t) \mid \Delta_{T(p,q)}(t) \) implies that \( \sigma_K(t) \) jumps by \(-2m\) at \( 1/c \) for each \( \phi_c(t) \) dividing \( \Delta_{T(p,q)}(t) \) and not dividing \( \Delta_{T(r,s)}(t) \) and jumps by \(-2(m-n)\) at \( 1/c \) for each \( \phi_c(t) \) dividing both. So it must be that
\[
\frac{(\Delta_{T(p,q)}(t))^{m}}{(\Delta_{T(r,s)}(t))^{n}} \text{ divides } \Delta_J(t).
\]
By the argument above involving the degrees of the polynomials, we find that
\[
\frac{(\Delta_{T(p,q)}(t))^{m}}{(\Delta_{T(r,s)}(t))^{n}} = \Delta_J(t).
\]

**Proof of Corollary 3.4.** Suppose that \( J \) is an \( L \)-space knot concordant to \( K \). Then by Proposition 3.3, we know that
\[
\frac{(\Delta_{T(p,q)}(t))^{m}}{(\Delta_{T(r,s)}(t))^{n}} = \Delta_J(t). \tag{3.1}
\]
In [8], Hedden and Watson show that the Alexander polynomial of an \( L \)-space knot of genus \( g \) must have highest degree terms \( t^{2g} - t^{2g-1} \). The symmetry of the Alexander polynomial then implies that the Alexander polynomial of an \( L \)-space knot must have lowest degree terms \( 1 - t \). Since \( T(p, q), T(r, s), \) and \( J \) are all \( L \)-space knots, Equation 3.1 implies that
\[
\frac{(1 - t + \cdots)^{m}}{(1 - t + \cdots)^{n}} = 1 - t + \cdots.
\]
Rearranging and expanding, we see that

\[(1 - t + \cdots)^m = (1 - t + \cdots)(1 - t + \cdots)^n\]

\[1 - mt + \cdots = (1 - t + \cdots)(1 - nt + \cdots)\]

\[1 - mt + \cdots = 1 - (n + 1)t + \cdots.\]

So it must be that \(m = n + 1\).

Finally, we prove Theorem 1.1.

**Proof of Theorem 1.1.** By Propositions 3.1, 3.2, and 3.3 and Corollary 3.3, we can consider only the case of \(K = mT(p, q) \neq (m - 1)T(r, s)\), where \(m \geq 2\), \(p \leq r\), and \(rs\) divides \(pq\). Suppose that \(J\) is an \(L\)-space knot concordant to \(K\). We apply Proposition 3.3 to have that

\[
J \Delta_k(t) = \frac{(t^{pq} - t^p - t^q + 1)^{m-1}}{(t^{pq} - t^p - t^q + 1)^{m-1}} \cdot \Delta_{T(p, q)}(t),
\]

where the last equality is due to the fact that \(rs|pq\). Rearranging, expanding, and focusing on lower degree terms, we get

\[
(t^{pq} - t^p - t^q + 1)^{m-1} J(t) = (t^{pq} - ts + \cdots + t^{rs} + 1)^{m-1}(t^{rs} - t^r - t^s + 1)^{m-1} \Delta_{T(p, q)}(t),
\]

\[
(t^{pq} - t^p - t^q + 1)^{m-1}(1 - t + \cdots) = (1 - t^r - t^s + \cdots)^{m-1}(1 - t + t^p - \cdots),
\]

\[
(1 - (m - 1)t^p + \cdots)(1 - t + \cdots) = (1 - (m - 1)t^r + \cdots)(1 - t + t^p - \cdots).
\]

Notice that on the left-hand side, the coefficients of \(t^2, t^3, \ldots, t^{p-1}\) are the same as those in \(J(t)\). The coefficient of \(t^p\) in \(J(t)\) must be either 0 or 1 since \(J\) is an \(L\)-space knot. So on the left-hand side, the coefficient of \(t^p\) is either \(-(m - 1)\) or \(-(m - 1) + 1\). On the right-hand side, the coefficients of \(t^2, t^3, \ldots, t^{r-1}\) are all 0 and the coefficient of \(t^r\) is \(-(m - 1)\). Therefore, equating coefficients, we see that if \(r < p\),

\[
J(t) = 1 - t - (m - 1)t^r + \cdots,
\]

and if \(r = p\), either

\[
J(t) = 1 - t - (m - 1)t^r + \cdots
\]

or

\[
J(t) = 1 - t + (-(m - 1) + 1)t^r + \cdots.
\]

Since \(L\)-space knots have Alexander polynomials with coefficients of alternating sign and \(-(m - 1) < 0\), we have reached a contradiction except when \(r = p\) and

\[
J(t) = 1 - t + (-(m - 1) + 1)t^r + \cdots.
\]

In this case, we must have that \(-(m - 1) + 1 = 0\) or 1. If \(-(m - 1) + 1 = 1\), then \(m = 1\), which is a contradiction to our assumption that \(m \geq 2\). If \(-(m - 1) + 1 = 0\), then \(m = 2\).
So we need only consider the case where \( K = 2T(p, q) \# - T(p, s) \). By Proposition 3.3, we know that \( rs_1pq \) and it follows in this case that \( s/q \). So \( s = pk + i \) for some \( 0 < i < p \) and \( k > 0 \) and \( q = sa = (pk + i)a \) for some \( a > 1 \). Applying Proposition 3.3 once more, we have that

\[
\Delta_J(t) = \frac{\Delta_{T(p, s)}(t)}{\Delta_{T(p, s)}(t)} = \frac{(tp^s - 1)(t - 1)}{(t^p - 1)(t^{sa} - 1)}.
\]

Rearranging, we see

\[
(1 - t^p - t^{ps} + \cdots)\Delta_J(t) = (1 - t^s + t^{s+1} + \cdots)
\]

and equating coefficients, the Alexander polynomial of \( J \) is determined to be

\[
\Delta_J(t) = 1 - t + t^p - t^{p+1} + t^{2p} - t^{2p+1} + \cdots + t^{kp} - t^{kp+1} - t^s + \text{higher degree terms}.
\]

Recalling that \( s = kp + i \), if \( i = 1 \) then the coefficient of \( t^s \) is \(-2\), and if \( 1 < i < p \) then the coefficients of \( t^{kp+1} \) and \( t^s \) are both \(-1\). In the former case, we have reached a contradiction since \( L \)-space knots have Alexander polynomials with coefficients 1, 0, or \(-1\). In the latter case, there are no terms of degree \( t^{kp+2}, \ldots, t^{s-1} \), thus \( \Delta_J(t) \) has two consecutive terms with coefficient \(-1\), contradicting the fact that \( L \)-space knots have Alexander polynomials with coefficients with alternating sign. Therefore \( K \) is not concordant to an \( L \)-space knot. \( \square \)

4. Towards the more general case

In this section, we state and prove some results which restrict concordances from connected sums of torus knots to \( L \)-space knots. First, we give generalizations of Proposition 3.3 and Corollary 3.4.

Proposition 4.1. If the knot

\( K = T(p_1, q_1) \# T(p_2, q_2) \# \cdots \# T(p_m, q_m) \# - T(p'_1, q'_1) \# - T(p'_2, q'_2) \# \cdots \# - T(p'_n, q'_n) \),

where \( m, n \geq 1 \), is concordant to an \( L \)-space knot \( J \), then

\[
\prod_{i=1}^m \Delta_{T(p_i, q_i)}(t) \cdot \prod_{i=1}^n \Delta_{T(p'_i, q'_i)}(t) = \Delta_J(t).
\]

Corollary 4.2. If the knot

\( K = T(p_1, q_1) \# T(p_2, q_2) \# \cdots \# T(p_m, q_m) \# - T(p'_1, q'_1) \# - T(p'_2, q'_2) \# \cdots \# - T(p'_n, q'_n) \),

where \( m, n \geq 1 \), is concordant to a nontrivial \( L \)-space knot, then \( m = n + 1 \).

Proof of Proposition 4.1. Suppose that \( J \) is an \( L \)-space knot concordant to \( K \). Consider the Alexander polynomial of \( K \). It is a product of cyclotomic polynomials \( \phi_c(t) \):

\[
\Delta_K(t) = \prod_{i=1}^m \Delta_{T(p_i, q_i)}(t) \cdot \prod_{i=1}^n \Delta_{T(p'_i, q'_i)}(t) = \left( \prod_{i=1}^m \prod_{a_j, b_j \neq 1} \phi_{a_j, b_j}(t) \right) \cdot \left( \prod_{i=1}^n \prod_{a_j, b_j \neq 1} \phi_{a_j, b_j}(t) \right).
\]
Let \( K^+ = T(p_1, q_1) \# T(p_2, q_2) \# \cdots \# T(p_m, q_m) \) and \( K^- = T(p'_1, q'_1) \# T(p'_2, q'_2) \# \cdots \# T(p'_n, q'_n) \). Note that \( K = K^+ \# - K^- \). Let
\[
C_+ = \{(c, k) : k > 0 \text{ and } (\phi_c(t))^k \mid \Delta_{K^+}(t) \text{ and } (\phi_c(t))^{k+1} \mid \Delta_{K^+}(t) \}\]
and
\[
C_- = \{(c, k) : k > 0 \text{ and } (\phi_c(t))^k \mid \Delta_{K^-}(t) \text{ and } (\phi_c(t))^{k+1} \mid \Delta_{K^-}(t) \}.
\]
Suppose there exists \( c \) such that \((c, k) \in C_+ \) and \((c, l) \in C_-\). By Livingston \cite{4}, this implies that the Levine–Tristram signature function of \( K \), \( \sigma_K(t) \), jumps by \(-2(k - l)\) at \( t = 1/c \). Therefore, since \( J \) is concordant to \( K \), we would have that \((\phi_c(t))^{k-l} \) divides \( \Delta_J(t) \). Similarly, if \( c \) is such that \((c, k) \in C_- \) but for all \( l \), \((c, l) \notin C_+ \), then \((\phi_c(t))^k \) divides \( \Delta_J(t) \), or if \( c \) is such that \((c, k) \in C_+ \) but for all \( l \), \((c, l) \notin C_- \), then \((\phi_c(t))^k \) divides \( \Delta_J(t) \). Thus
\[
\deg(\Delta_J(t)) \geq \deg(\Delta_K(t)) - \sum_{(c, k) \in C_+ \text{ and } (c, l) \in C_-} 2l \cdot \deg(\phi_c(t)).
\]
On the one hand, we know that for \( L \)-space knots the degree of the Alexander polynomial is equal to twice the \( \tau \) invariant of the knot. So, since \( J \) is concordant to \( K \), we have that
\[
\deg(\Delta_J(t)) = 2\tau(J) = 2\tau(K) = 2 \left( \sum_{i=1}^{m} \frac{(p_i - 1)(q_i - 1)}{2} - \sum_{i=1}^{n} \frac{(p'_i - 1)(q'_i - 1)}{2} \right).
\]
On the other hand, we have
\[
\deg(\Delta_K(t)) = \sum_{i=1}^{m} (p_i - 1)(q_i - 1) + \sum_{i=1}^{n} (p'_i - 1)(q'_i - 1).
\]
Therefore,
\[
\sum_{(c, k) \in C_+ \text{ and } (c, l) \in C_-} l \cdot \deg(\phi_c(t)) \geq \frac{1}{2} (\deg(\Delta_K(t)) - \deg(\Delta_J(t))) = \sum_{i=1}^{n} (p'_i - 1)(q'_i - 1) = \deg(\Delta_{K^-}(t)).
\]
If there exists \( c \) such that \((c, k) \in C_- \) but for all \( l \), \((c, l) \notin C_+ \), then
\[
\sum_{(c, k) \in C_+ \text{ and } (c, l) \in C_-} l \cdot \deg(\phi_c(t)) < \sum_{(c, l) \in C_-} l \cdot \deg(\phi_c(t)) = \deg(\Delta_{K^-}(t)),
\]
which is a contradiction. Thus it must be that \( \Delta_{K^-}(t) \mid \Delta_{K^+}(t) \). Finally, from the jumps of \( \sigma_K(t) = \sigma_J(t) \), we can conclude that
\[
\frac{\Delta_{K^+}(t)}{\Delta_{K^-}(t)} \text{ divides } \Delta_J(t).
\]
By the argument above involving the degrees of the polynomials, we find that
\[
\frac{\Delta_{K^+}(t)}{\Delta_{K^-}(t)} = \Delta_J(t),
\]
as asserted. \( \square \)
Proof of Corollary 4.2. Suppose that \( J \) is an \( L \)-space knot concordant to \( K \). Then by Proposition 4.1 we know that
\[
\prod_{i=1}^{m} \Delta_{T(p_i,q_i)}(t) = \Delta_{T(p,q)}(t).
\] (4.1)

In [3], Hedden and Watson show that the Alexander polynomial of an \( L \)-space knot must have leading terms \( 1 - t \). Since \( T(p_i,q_i) \), \( T(p_i',q_i') \), and \( J \) are all \( L \)-space knots, Equation 4.1 implies that
\[
\prod_{i=1}^{m} \frac{(1 - t + \cdots)}{(1 - t + \cdots)} = 1 - t + \cdots.
\]
Rearranging and expanding, we see that
\[
\prod_{i=1}^{m} (1 - t + \cdots) = (1 - t + \cdots) \prod_{i=1}^{n} (1 - t + \cdots)
\]
\[
1 - mt + \cdots = (1 - t + \cdots)(1 - nt + \cdots)
\]
\[
1 - mt + \cdots = 1 - (n + 1)t + \cdots.
\]
So it must be that \( m = n + 1 \).

Lastly, we use the following result of Feller and Krcatovich to give a condition on the Upsilon function of
\[ K = T(p_1,q_1) \# T(p_2,q_2) \# \cdots \# T(p_{m},q_m) \# -T(p_1',q_1') \# -T(p_2',q_2') \# \cdots \# -T(p_n',q_n'), \]
under which \( K \) cannot be an \( L \)-space knot.

**Theorem 4.3** (Feller–Krcatovich, [2]). Let \( p < q \) be coprime integers. Then
\[ \Upsilon_{T(p,q)}(t) = \Upsilon_{T(p,q-p)}(t) + \Upsilon_{T(p,p+1)}(t). \]

**Theorem 4.4.** Suppose that
\[ K = T(p_1,q_1) \# T(p_2,q_2) \# \cdots \# T(p_{m},q_m) \# -T(p_1',q_1') \# -T(p_2',q_2') \# \cdots \# -T(p_n',q_n') \]
with
\[ \Upsilon_K(t) = c_1 \Upsilon_{T(a_1,a_1+1)}(t) + c_2 \Upsilon_{T(a_2,a_2+1)}(t) + \cdots + c_s \Upsilon_{T(a_s,a_s+1)}(t) - c_1' \Upsilon_{T(a_1',a_1'+1)}(t) 
\]
\[ + c_2' \Upsilon_{T(a_2',a_2'+1)}(t) + \cdots + c_s' \Upsilon_{T(a_s',a_s'+1)}(t). \]
where \( c_i, c_i' > 0 \) for all \( i \), \( a_1 > a_2 > \cdots > a_r \), and \( a_1' > a_2' > \cdots > a_s' \). If there exists \( a_i' \) such that \( a_i' \) does not divide \( a_j \) for any \( j \), then \( K \) is not concordant to an \( L \)-space knot.

Before we prove the theorem, we offer an example.

**Example 4.5.** Consider the knot \( K = 3T(5,6) \# -T(2,5) \# -T(3,5) \). Here we have that
\[ \Upsilon_K(t) = 3 \Upsilon_{T(5,6)}(t) - \Upsilon_{T(3,4)}(t) - 3 \Upsilon_{T(2,3)}(t). \]
For an \( L \)-space knot \( J \), \( \Upsilon'_J(t) \) is an increasing, piecewise–constant function. The Upsilon function \( T(p,p+1) \) then has increasing, piecewise–constant derivative. From CFK∞(\( T(p,p+1) \)), we also know that \( \Upsilon'_J(t) \) has jumps in \([0, 1]\) only at \( t = 2i/p \) for \( i \) such that \( 0 < 2i/p \leq 1 \). Let \( K^+ = 3T(5,6) \) and \( K^- = T(2,5) \# T(3,5) \). Since Upsilon is additive under forming connected sums, we have that \( \Upsilon'_{K^+}(t) \) is increasing with jumps at \( 2/5 \) and \( 4/5 \). The function \( \Upsilon'_{K^-}(t) \) is decreasing with jumps at \( 2/3 \) and \( 1 \). Thus \( \Upsilon'_{K}(t) \) has a negative jump at \( t = 2/3 \) since \( \Upsilon'_{K^+}(t) \) is constant there. So \( K \) is not an \( L \)-space knot.
Proof of Theorem 4.4. Again, for an $L$–space knot $J$, $\Upsilon'_J(t)$ is an increasing, piecewise–constant function. The Upsilon function $T(p,p+1)$ then has increasing, piecewise–constant derivative. By analyzing $\text{CFK}_\infty^L(T(p,p+1))$, we also know that $\Upsilon'_{T(p,p+1)}(t)$ has jumps in $[0,1]$ only at $t = 2i/p$ for $i$ such that $0 < 2i/p \leq 1$. Let $K^+ = T(p_1,q_1) \# T(p_2,q_2) \# \cdots \# T(p_m,q_m)$ and $K^- = T(p'_1,q'_1) \# T(p'_2,q'_2) \# \cdots \# T(p'_n,q'_n)$.

If $\Upsilon_K(t)$ is as stated in the theorem, with $a'_i$ such that $a'_i$ does not divide $a_j$ for any $j$, we have that $\Upsilon'_{K^-}(t)$ has a negative jump at $2/a'_i$. Since no $a_j$ is divisible by $a'_i$, we have that $2k/a_j \neq 2/a'_i$ for any $j$. Thus $\Upsilon'_{K^+}(t)$ is constant at $2/a'_i$. This implies that $\Upsilon'_K(t) = \Upsilon'_{K^+} \# K^-(t)$ has a negative jump at $2/a'_i$ and so $\Upsilon_K(t)$ is not the Upsilon function of an $L$–space knot. □

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