A study of a curious arithmetic function

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Abstract

In this note, we study the arithmetic function \( f : \mathbb{Z}_+^* \to \mathbb{Q}_+^* \) defined by \( f(2^k \ell) = \ell^{1-k} \) (\( \forall k, \ell \in \mathbb{N}, \ell \text{ odd} \)). We show several important properties about that function and then we use them to obtain some curious results involving the 2-adic valuation.

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1 Introduction and notations

The purpose of this paper is to study the arithmetic function \( f : \mathbb{Z}_+^* \to \mathbb{Q}_+^* \) defined by:

\[
f(2^k \ell) = \ell^{1-k} \quad (\forall k, \ell \in \mathbb{N}, \ell \text{ odd}).
\]

We have for example \( f(1) = 1, f(2) = 1, f(3) = 3, f(12) = \frac{1}{3}, f(40) = \frac{1}{25}, \ldots \).

So it is clear that \( f(n) \) is not always an integer. However, we will show in what follows that \( f \) satisfies among others the property that the product of the \( f(r) \)'s \( (1 \leq r \leq n) \) is always an integer and it is a multiple of all odd prime number not exceeding \( n \). Further, we exploit the properties of \( f \) to establish some curious properties concerning the 2-adic valuation.

The study of \( f \) requires to introduce the two auxiliary arithmetic functions \( g : \mathbb{Q}_+^* \to \mathbb{Z}_+^* \) and \( h : \mathbb{Z}_+^* \to \mathbb{Q}_+^* \), defined by:

\[
g(x) := \begin{cases} x & \text{if } x \in \mathbb{N} \\ 1 & \text{else} \end{cases} \quad (\forall x \in \mathbb{Q}_+^*) \quad (1)
\]

\[
h(r) := \frac{r}{g(\frac{r}{2})g(\frac{r}{4})g(\frac{r}{8}) \cdots} \quad (\forall r \in \mathbb{Z}_+^*) \quad (2)
\]
Remark that the product in the denominator of the right-hand side of \( (2) \) is actually finite because \( g(\frac{r}{2^i}) = 1 \) for any sufficiently large \( i \); so \( h \) is well-defined.

**Some notations and terminologies.** Throughout this paper, we let \( N^* \) denote the set \( N \setminus \{0\} \) of positive integers. For a given prime number \( p \), we let \( v_p \) denote the usual \( p \)-adic valuation. We define the *odd part* of a positive rational number \( \alpha \) as the positive rational number, denoted \( \text{Odd}(\alpha) \), so that we have \( \alpha = 2^{v_2(\alpha)} \cdot \text{Odd}(\alpha) \). Finally, we denote by \( \lfloor . \rfloor \) the integer-part function and we often use in this paper the following elementary well-known property of that function:

\[
\forall a, b \in N^*, \forall x \in \mathbb{R} : \lfloor \frac{x}{a} \rfloor = \lfloor \frac{x}{ab} \rfloor.
\]

### 2 Results and proofs

**Theorem 2.1** Let \( n \) be a positive integer. Then the product \( \prod_{r=1}^{n} f(r) \) is an integer.

**Proof.** For a given \( r \in N^* \), let us write \( f(r) \) in terms of \( h(r) \). By writing \( r \) in the form \( r = 2^k \ell \) (\( k, \ell \in N, \ell \text{ odd} \)), we have by the definition of \( g \):

\[ g\left(\frac{r}{7}\right) g\left(\frac{r}{17}\right) g\left(\frac{r}{8}\right) \ldots = (2^{k-1} \ell) (2^{k-2} \ell) \times \ldots \times (2^0 \ell) = 2^{\frac{k(k-1)}{2}} \ell^k. \]

So, it follows that:

\[ h(r) := \frac{r}{g\left(\frac{r}{7}\right) g\left(\frac{r}{17}\right) g\left(\frac{r}{8}\right) \ldots} = \frac{2^k \ell}{2^{\frac{k(k-1)}{2}} \ell^k} = 2^{\frac{k(k-1)}{2}} \ell^{1-k} = 2^{\frac{k(k-1)}{2}} f(r). \]

Hence

\[ f(r) = 2^{\frac{v_2(r)(v_2(r)-3)}{2}} h(r). \tag{3} \]

Using (3), we get for all \( n \in N^* \):

\[ \prod_{r=1}^{n} f(r) = 2^{\sum_{r=1}^{n} \frac{v_2(r)(v_2(r)-3)}{2}} \prod_{r=1}^{n} h(r). \tag{4} \]

By taking the odd part of each of the two hand-sides of this last identity, we obtain:

\[ \prod_{r=1}^{n} f(r) = \text{Odd} \left( \prod_{r=1}^{n} h(r) \right) \quad (\forall n \in N^*). \tag{5} \]

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So, to confirm the statement of the theorem, it suffices to prove that the product \( \prod_{r=1}^{n} h(r) \) is an integer for any \( n \in \mathbb{N}^* \). To do so, we lean on the following sample property of \( g \):

\[
g \left( \frac{1}{a} \right) g \left( \frac{2}{a} \right) \cdots g \left( \frac{r}{a} \right) = \left\lfloor \frac{r}{a} \right\rfloor! \quad (\forall r, a \in \mathbb{N}^*).
\]

Using this, we have:

\[
\prod_{r=1}^{n} h(r) = \prod_{r=1}^{n} g \left( \frac{r}{2} \right) g \left( \frac{r}{4} \right) g \left( \frac{r}{8} \right) \cdots \frac{n!}{\left\lfloor \frac{n}{2} \right\rfloor! \left\lfloor \frac{n}{4} \right\rfloor! \left\lfloor \frac{n}{8} \right\rfloor! \cdots}.
\]

Hence

\[
\prod_{r=1}^{n} h(r) = \frac{n!}{\left\lfloor \frac{n}{2} \right\rfloor! \left\lfloor \frac{n}{4} \right\rfloor! \left\lfloor \frac{n}{8} \right\rfloor! \cdots}.
\]

(Remark that the product in the denominator of the right-hand side of (6) is actually finite because \( \left\lfloor \frac{n}{2^i} \right\rfloor = 0 \) for any sufficiently large \( i \).

Now, since \( \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + \cdots \leq \frac{n}{2} + \frac{n}{4} + \frac{n}{8} + \cdots = n \) then \( \frac{n!}{\left\lfloor \frac{n}{2} \right\rfloor! \left\lfloor \frac{n}{4} \right\rfloor! \left\lfloor \frac{n}{8} \right\rfloor! \cdots} \) is a multiple of the multinomial coefficient \( \left( \frac{n}{2}! \frac{n}{4}! \frac{n}{8}! \cdots \right) \) which is an integer. Consequently \( \frac{n!}{\left\lfloor \frac{n}{2} \right\rfloor! \left\lfloor \frac{n}{4} \right\rfloor! \left\lfloor \frac{n}{8} \right\rfloor! \cdots} \) is an integer, which completes this proof. □

**Theorem 2.2** Let \( n \) be a positive integer. Then \( \prod_{r=1}^{n} f(r) \) is a multiple of \( \text{Odd}(\text{lcm}(1, 2, \ldots, n)) \).

In particular, \( \prod_{r=1}^{n} f(r) \) is a multiple of all odd prime number not exceeding \( n \).

**Proof.** According to the relations (5) and (6) obtained during the proof of Theorem 2.1, it suffices to show that \( \frac{n!}{\left\lfloor \frac{n}{2} \right\rfloor! \left\lfloor \frac{n}{4} \right\rfloor! \left\lfloor \frac{n}{8} \right\rfloor! \cdots} \) is a multiple of \( \text{lcm}(1, 2, \ldots, n) \).

Equivalently, it suffices to prove that for all prime number \( p \), we have:

\[
v_p \left( \frac{n!}{\left\lfloor \frac{n}{2} \right\rfloor! \left\lfloor \frac{n}{4} \right\rfloor! \left\lfloor \frac{n}{8} \right\rfloor! \cdots} \right) \geq \alpha_p,
\]

where \( \alpha_p \) is the highest power of \( p \) dividing \( n! \).
where $\alpha_p$ is the $p$-adic valuation of $\text{lcm}(1, 2, \ldots, n)$, that is the greatest power of $p$ not exceeding $n$. Let us show (7) for a given arbitrary prime number $p$.

Using Legendre's formula (see e.g., [1]), we have:

$$v_p \left( \frac{n!}{\left( \frac{n}{2} \right)! \left( \frac{n}{4} \right)! \left( \frac{n}{8} \right)! \cdots} \right) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor - \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \left\lfloor \frac{n}{2^jp^i} \right\rfloor$$

$$= \sum_{i=1}^{\alpha_p} \left( \left\lfloor \frac{n}{p^i} \right\rfloor - \sum_{j=1}^{\alpha_p} \left\lfloor \frac{n}{2^jp^i} \right\rfloor \right) \quad (8)$$

Next, for all $i \in \{1, 2, \ldots, \alpha_p\}$, we have:

$$\sum_{j=1}^{\alpha_p} \left\lfloor \frac{n}{2^jp^i} \right\rfloor = \sum_{j=1}^{\alpha_p} \left\lfloor \frac{\alpha_p}{2^j} \right\rfloor \leq \sum_{j=1}^{\alpha_p} \left\lfloor \frac{\alpha_p}{2^j} \right\rfloor < \left\lfloor \frac{n}{p^i} \right\rfloor .$$

But since $\left( \left\lfloor \frac{n}{p^i} \right\rfloor - \sum_{j=1}^{\alpha_p} \left\lfloor \frac{n}{2^jp^i} \right\rfloor \right) (i \in \{1, 2, \ldots, \alpha_p\})$ is an integer, it follows that:

$$\left\lfloor \frac{n}{p^i} \right\rfloor - \sum_{j=1}^{\alpha_p} \left\lfloor \frac{n}{2^jp^i} \right\rfloor \geq 1 \quad (\forall i \in \{1, 2, \ldots, \alpha_p\}) .$$

By inserting those last inequalities in (8), we finally obtain:

$$v_p \left( \frac{n!}{\left( \frac{n}{2} \right)! \left( \frac{n}{4} \right)! \left( \frac{n}{8} \right)! \cdots} \right) \geq \alpha_p ,$$

which confirms (7) and completes this proof. $\blacksquare$

**Theorem 2.3** For all positive integer $n$, we have:

$$\prod_{r=1}^{n} h(r) \leq c^n ,$$

where $c = 4.01055487 \ldots .$

In addition, the inequality becomes an equality for $n = 1023 = 2^{10} - 1$.

**Proof.** First, we use the relation (9) to prove by induction on $n$ that:

$$\prod_{r=1}^{n} h(r) \leq n^{\log_2 n} \cdot 4^n \quad (9)$$

- For $n = 1$, (9) is clearly true.
- For a given $n \geq 2$, suppose that (9) is true for all positive integer $< n$.
and let us show that (9) is also true for \( n \). To do so, we distinguish the two following cases:

1\(^{st} \) case: (if \( n \) is even, that is \( n = 2m \) for some \( m \in \mathbb{N}^* \)).

In this case, by using (6) and the induction hypothesis, we have:

\[
\prod_{r=1}^{n} h(r) = \left(\frac{2m}{m}\right)^m \prod_{r=1}^{m} h(r) \\
\leq \left(\frac{2m}{m}\right)^m m^{\log_2 m 4^m} \\
\leq m^{\log_2 m 4^{2m}} \quad \text{(since} \left(\frac{2m}{m}\right) \leq 4^m) \\
\leq n^{\log_2 n 4^n},
\]

as claimed.

2\(^{nd} \) case: (if \( n \) is odd, that is \( n = 2m + 1 \) for some \( m \in \mathbb{N}^* \)).

By using (6) and the induction hypothesis, we have:

\[
\prod_{r=1}^{n} h(r) = (2m + 1) \left(\frac{2m}{m}\right)^m \prod_{r=1}^{m} h(r) \\
\leq (2m + 1) \left(\frac{2m}{m}\right)^m m^{\log_2 m 4^m} \\
\leq m^{\log_2 m + 1} 4^{2m+1} \quad \text{(since} 2m + 1 \leq 4m \text{ and} \left(\frac{2m}{m}\right) \leq 4^m) \\
\leq n^{\log_2 n 4^n},
\]

as claimed.

The inequality (9) thus holds for all positive integer \( n \). Now, to establish the inequality of the theorem, we proceed as follows:

— For \( n \leq 70000 \), we simply verify the truth of the inequality in question (by using the Visual Basic language for example).

— For \( n > 70000 \), it is easy to see that \( n^{\log_2 n} \leq (c/4)^n \) and by inserting this in (9), the inequality of the theorem follows.

The proof is complete.

Now, since any positive integer \( n \) satisfies \( \prod_{r=1}^{n} f(r) \leq \prod_{r=1}^{n} h(r) \) (according to (5) and the fact that \( \prod_{r=1}^{n} h(r) \) is an integer), then we immediately derive from Theorem 2.3 the following:

**Corollary 2.4** For all positive integer \( n \), we have:

\[
\prod_{r=1}^{n} f(r) \leq c^n,
\]
where \( c \) is the constant given in Theorem 2.3.

To improve Corollary 2.4, we propose the following optimal conjecture which is very probably true but it seems difficult to prove or disprove it!

**Conjecture 2.5** For all positive integer \( n \), we have:

\[
\prod_{r=1}^{n} f(r) < 4^n.
\]

Using the Visual Basic language, we have checked the validity of Conjecture 2.5 up to \( n = 100000 \). Further, by using elementary estimations similar to those used in the proof of Theorem 2.3, we can easily show that:

\[
\lim_{n \to +\infty} \left( \prod_{r=1}^{n} f(r) \right)^{1/n} = \lim_{n \to +\infty} \left( \prod_{r=1}^{n} h(r) \right)^{1/n} = 4,
\]

which shows in particular that the upper bound of Conjecture 2.5 is optimal.

Now, by exploiting the properties obtained above for the arithmetic function \( f \), we are going to establish some curious properties concerning the 2-adic valuation.

**Theorem 2.6** For all positive integer \( n \) and all odd prime number \( p \), we have:

\[
\sum_{r=1}^{n} v_2(r)v_p(r) \leq \sum_{r=1}^{n} v_p(r) - \left\lfloor \frac{\log n}{\log p} \right\rfloor.
\]

**Proof.** Let \( n \) be a positive integer and \( p \) be an odd prime number. Since (according to Theorem 2.2), the product \( \prod_{r=1}^{n} f(r) \) is a multiple of the positive integer \( \text{Odd}(\text{lcm}(1, 2, \ldots, n)) \) whose the \( p \)-adic valuation is equal to \( \left\lfloor \frac{\log n}{\log p} \right\rfloor \), then we have:

\[
v_p \left( \prod_{r=1}^{n} f(r) \right) = \sum_{r=1}^{n} v_p(f(r)) \geq \left\lfloor \frac{\log n}{\log p} \right\rfloor.
\]

But by the definition of \( f \), we have for all \( r \geq 1 \):

\[
v_p(f(r)) = (1 - v_2(r))v_p(r).
\]

So, it follows that:

\[
\sum_{r=1}^{n} (1 - v_2(r))v_p(r) \geq \left\lfloor \frac{\log n}{\log p} \right\rfloor,
\]

which gives the inequality of the theorem. \( \blacksquare \)
Theorem 2.7 Let $n$ be a positive integer and let $a_0 + a_12^1 + a_22^2 + \cdots + a_s2^s$ be the representation of $n$ in the binary system. Then we have:

$$\sum_{r=1}^{n} \frac{v_2(r)(3 - v_2(r))}{2} = \sum_{i=1}^{s} ia_i.$$ 

In particular, we have for all $m \in \mathbb{N}$:

$$\sum_{r=1}^{2^m} \frac{v_2(r)(3 - v_2(r))}{2} = m.$$

Proof. By taking the $2$-adic valuation in the two hand-sides of the identity (4) and then using (6), we obtain:

$$\sum_{r=1}^{n} \frac{v_2(r)(3 - v_2(r))}{2} = v_2\left(\prod_{r=1}^{n} h(r)\right) = v_2\left(\frac{n!}{\left\lfloor \frac{n}{2} \right\rfloor ! \left\lfloor \frac{n}{4} \right\rfloor ! \left\lfloor \frac{n}{8} \right\rfloor ! \cdots}\right).$$

It follows by using Legendre’s formula (see e.g., [1]) that:

$$\sum_{r=1}^{n} \frac{v_2(r)(3 - v_2(r))}{2} = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{2^i} \right\rfloor - \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \left\lfloor \frac{n}{2^{i+j}} \right\rfloor$$

$$= \sum_{i=1}^{\infty} \left\lfloor \frac{n}{2^i} \right\rfloor - \sum_{u=2}^{\infty} (u-1) \left\lfloor \frac{n}{2^u} \right\rfloor$$

$$= \sum_{i=1}^{\infty} \left\lfloor \frac{n}{2^i} \right\rfloor - \sum_{i=1}^{\infty} i \left\lfloor \frac{n}{2^{i+1}} \right\rfloor.$$

By adding to the last series the telescopic series $\sum_{i=1}^{\infty} ((i-1) \left\lfloor \frac{n}{2^i} \right\rfloor - i \left\lfloor \frac{n}{2^{i+1}} \right\rfloor)$ which is convergent with sum zero, we derive that:

$$\sum_{r=1}^{n} \frac{v_2(r)(3 - v_2(r))}{2} = \sum_{i=1}^{\infty} i \left( \left\lfloor \frac{n}{2^i} \right\rfloor - 2 \left\lfloor \frac{n}{2^{i+1}} \right\rfloor \right).$$

But according to the representation of $n$ in the binary system, we have:

$$\left\lfloor \frac{n}{2^i} \right\rfloor - 2 \left\lfloor \frac{n}{2^{i+1}} \right\rfloor =\begin{cases} a_i & \text{for } i = 1, 2, \ldots, s \\ 0 & \text{for } i > s \end{cases}.$$ 

Hence

$$\sum_{r=1}^{n} \frac{v_2(r)(3 - v_2(r))}{2} = \sum_{i=1}^{s} i a_i,$$ 

as required.

The second part of the theorem is nothing else an immediate application of its first part with $n = 2^m$. The proof is finished. ■
References

[1] G.H. Hardy and E.M. Wright. The Theory of Numbers, fifth ed., Oxford Univ. Press, London, 1979.