CENTRAL LIMIT THEOREM FOR ARTIN L-FUNCTIONS

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Abstract. We show that the sum of the traces of Frobenius elements of Artin L-functions in a family of G-fields satisfies the Gaussian distribution under certain counting conjectures. We prove the counting conjectures for $S_4$ and $S_5$-fields. We also show central limit theorem for modular form L-functions with the trivial central character with respect to congruence subgroups as the level goes to infinity.

1. Introduction

Let $G$ be a finite group which is a transitive subgroup of a certain symmetry group $S_{d+1}$. A number field $K$ of degree $d + 1$ is called a $G$-field if its Galois closure $\bar{K}$ over $\mathbb{Q}$ is a $G$-Galois extension. For a $G$-field $K$, we attach the Artin L-function

$$L(s, \rho, K) = \frac{\zeta_K(s)}{\zeta(s)} = \prod_{n=1}^{\infty} a_\rho(n)n^{-s},$$

where $\rho$ is $d$-dimensional representation of $G$. Note that $-1 \leq a_\rho(p) \leq d$. If $G = S_{d+1}$, $\rho$ is the $d$-dimensional standard representation of $S_{d+1}$. Let $L(X)^{r_2}$ be the set of $G$-fields $K$ with $|d_K| < X$ and signature $(r_1, r_2)$. In this paper, we restrict to the case $G = S_{d+1}$, and consider the sum

$$\sum_{p \leq x} a_\rho(p)$$

in the family of $S_{d+1}$-fields, $L(X)^{r_2}$ and show that under the counting conjectures (2.1) and (2.2), it has the Gaussian distribution, namely, for a continuous real function $h$ on $\mathbb{R}$, if

$$\frac{\log X}{\log x} \to \infty \quad \text{as} \quad x \to \infty,$$

then

$$\frac{1}{\#L(X)^{r_2}} \sum_{L(s, \rho, K) \in L(X)^{r_2}} h \left( \frac{\sum_{p \leq x} a_\rho(p)}{\pi(x)} \right) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t)e^{-\frac{t^2}{2}} dt.$$

When $L(X)^{r_2}$ is replaced by $\mathcal{F}_k$, the family of all normalized holomorphic Hecke eigen cusp forms of weight $k$ with respect to $SL_2(\mathbb{Z})$, Nagoshi [13] showed that if

$$\frac{\log k}{\log x} \to \infty \quad \text{as} \quad x \to \infty,$$

then

$$\frac{1}{\#\mathcal{F}_k} \sum_{f \in \mathcal{F}_k} h \left( \frac{\sum_{n \leq x} \lambda_f(n)}{\pi(x)} \right) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t)e^{-\frac{t^2}{2}} dt.$$
and called it central limit theorem.

Here we note that we do not need the Artin conjecture nor the strong Artin conjecture in the proof of (1.1). The estimates (2.1) and (2.2) are proved by Taniguchi and Thorne [17] for $S_3$ fields. For $G = S_4, S_5$, the estimate (2.1) was proved in [3] and [16], resp. We prove (2.2) in Sections 6 and 7. Hence (1.1) is unconditional for $S_3, S_4$ and $S_5$-fields. These estimates will be used in computing the $n$-level densities of Artin $L$-functions [7], [8].

We also study the distribution of the prime sums $\sum_{p \leq x} a_\rho(p)^r$ for a positive integer $r$. The effective Chebotarev density theorem implies the analogue of Sato-Tate distribution. Namely, 

$$\frac{1}{\pi(x)} \sum_{p \leq x} a_\rho(p)^r \to n_r \text{ as } x \to \infty,$$

where $n_r$ is the multiplicity of the trivial representation in $\rho^r$.

We also study distribution of $\sum_{p \leq x} a_f(p)$ for $f \in S_k(N)$, the set of normalized Hecke eigen cusp forms of weight $k$ with respect to $\Gamma_0(N)$ with the trivial central character. We prove the Gaussian distribution as $N \to \infty$.

2. Counting $S_{d+1}$-fields with local conditions

Let $S = (LC_p)$ be a finite set of local conditions. $LC_p = S_{p,C}$ means that $p$ is unramified in $K$ and the conjugacy class of $\text{Frob}_p$ is $C$. Let $|S_{p,C}| = \frac{|C|}{|G(1+f(p))|}$ for some positive function $f(p)$ which satisfies $f(p) = O(\frac{1}{p})$. There are also several splitting types of ramified primes, which are denoted by $r_1, r_2, \ldots, r_w$. If $LC_p = S_{p,r_i}$, then it means that $p$ is ramified and its splitting type is $r_i$. Assume that we can choose explicit positive functions $c_1(p), c_2(p), \ldots, c_w(p)$ with $\sum_{i=1}^w c_i(p) = f(p)$. Define $|S_{p,r_i}| = \frac{c_i(p)}{1+f(p)}$ and $|S| = \prod_p |LC_p|$.

Let $L(X;S)^{r_2}$ be the set of $S_{d+1}$-fields $K$ with $|d_K| < X$ and the local condition $S$.

Then assume that

\begin{align*}
(2.1) \quad |L(X)^{r_2}| &= A(r_2)X + O(X^{\delta}), \\
(2.2) \quad |L(X;S)^{r_2}| &= |S|A(r_2)X + O \left( \prod_{p \in S} \left( \frac{1}{\prod_p} \right)^{X^{\gamma}} \right),
\end{align*}

for some positive constant $\delta$ and $\gamma$, and the implied constant is uniformly bounded for $p$ and local conditions at $p$. 

\[\sqrt{\frac{\sum_{p \leq x} a_f(p)}{x}} \to \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} h(t)e^{-t^2} dt},\]
This assumption is satisfied when $G = S_3, S_4$ and $S_5$. When $G = S_3$, Taniguchi and Thorne \cite{17} obtained more precise results: Let $L(X)^\pm$ be the set of cubic fields $K$ with $\pm d_K < X$. Then

$$|L(X)^\pm| = \frac{C^\pm}{12\zeta(3)}X + \frac{4K^\pm}{5\Gamma(2/3)^3}X^{5/6} + O(X^{7/9+\epsilon}),$$

where $C^- = 3$, $C^+ = 1$, $K^- = \sqrt{3}$, and $K^+ = 1$. Here, we count only one cubic field from three conjugate fields. Let $TS_p, PS_p,$ and $IN_p$ be the local conditions of $p$ which means that $p$ is totally split, partially split and inert respectively. Let $S = \{LC_p| i = 1, 2, \cdots, u\}$ be a set of local conditions at $p_i$. Then

$$|LC_p| = \begin{cases} \frac{1}{6} & \text{if } LC_p = TS_p, \\ \frac{3}{6} & \text{if } LC_p = PS_p, \\ \frac{2}{6} & \text{if } LC_p = IN_p, \\ \frac{1}{p} & \text{if } p \text{ is partially ramified,} \\ \frac{1}{p^2} & \text{if } p \text{ is totally ramified.} \end{cases}$$

and $|L(X; S)^\pm| = |S|A^\pm X + O(E_S(X))$, where $A^\pm = C_1\frac{C^\pm}{12\zeta(3)}$, and

$$E_S(X) = \begin{cases} X^{5/6} & \text{if } (\prod_{p \in S} p^{\frac{e_p}{p}})^{\frac{1}{1+e_p}} < X^{\frac{1}{1+e_p}}, \\ (\prod_{p \in S} p^{\frac{e_p}{p}})^{\frac{\epsilon}{\frac{1}{1+e_p}}} & \text{if } (\prod_{p \in S} p^{\frac{e_p}{p}})^{\frac{\epsilon}{\frac{1}{1+e_p}}} \geq X^{\frac{1}{1+e_p}}, \end{cases}$$

where $e_p = 1$ if $p$ is unramified and otherwise, $e_p = 2$.

For $G = S_4, S_5$, the estimate (2.1) was proved in \cite{3} and \cite{16}. We prove (2.2) in Sections 6 and 7.

We identify $L(X)^{r_2}$ with the set of Artin $L$-functions $L(s, \rho, K)$ where $K$ is a $G$-field with $|d_K| < X$ with signature $(r_1, r_2)$. Here we count only one $G$-field for each $d + 1$ conjugate fields. Throughout the article, we implicitly assume that the size of $L(X)^{r_2}$ is same with the number of number fields which satisfy the requirements of $L(X)^{r_2}$. This claim deals with arithmetic equivalence of number fields. Two number fields $K_1$ and $K_2$ are arithmetically equivalent if $\zeta_{K_1}(s) = \zeta_{K_2}(s)$. We say that a number field $K$ is arithmetically solitary if $\zeta_K(s) = \zeta_F(s)$ implies that $K$ and $F$ are conjugate. It is known that $S_d$-fields and $A_d$ fields are arithmetically solitary. See Chapter II in \cite{12}.

For simplicity, we denote $L(s, \rho, K) \in L(X)^{r_2}$ by $\rho \in L(X)$.

3. Central limit theorem of Artin L-functions

Consider, for a continuous real function $h$ on $\mathbb{R}$,
\[ \frac{1}{\# L(X)} \sum_{\rho \in L(X)} h \left( \frac{\sum_{p \leq x} a_{\rho}(p)}{\sqrt{\pi(x)}} \right). \]

We assume that \( x \) grows more slowly than \( X \); namely, \( \frac{\log X}{\log x} \to \infty \) as \( x \to \infty \). So for an arbitrary positive real number \( a \), we have \( X > x^a \).

By Theorem 25.8 and Theorem 30.2 (the method of moments) in [1], it is enough to consider \( h(x) = x^r \). Consider

\[ \sum_{\rho \in L(X)} \left( \frac{\sum_{p \leq x} a_{\rho}(p)}{\sqrt{\pi(x)}} \right)^r. \]

By multinomial formula,

\[ \left( \sum_{p \leq x} a_{\rho}(p) \right)^r = \sum_{u=1}^{r} \binom{r}{u} \frac{1}{u!} \prod_{u} \left( \sum_{\rho \in L(X)} a_{\rho}(p_1)^{r_1} \cdots a_{\rho}(p_u)^{r_u} \right), \]

where \( \sum_{(r_1, \ldots, r_u)} \) means the sum over the \( u \)-tuples \((r_1, \ldots, r_u)\) of positive integers such that \( r_1 + \cdots + r_u = r \), and \( \sum_{(p_1, \ldots, p_u)} \) means the sum over the \( u \)-tuples \((p_1, \ldots, p_u)\) of distinct primes such that \( p_i \leq x \) for each \( i \). Then

\[ \begin{aligned} \sum_{(r_1, \ldots, r_u)} &= \pi(x)^{-\frac{r}{2}} \sum_{u=1}^{r} \frac{1}{u!} \sum_{(r_1, \ldots, r_u)} \frac{1}{r_1! \cdots r_u!} \sum_{(p_1, \ldots, p_u)} \left( \sum_{\rho \in L(X)} a_{\rho}(p_1)^{r_1} \cdots a_{\rho}(p_u)^{r_u} \right). \end{aligned} \]

Now we claim that except when \( r \) is even, \( u = \frac{r}{2} \), and \( r_1 = \cdots = r_u = 2 \), it gives rise to the error term.

Now suppose \( r_i \geq 2 \) for all \( i \), and \( r_j > 2 \) for some \( j \). Then since \( r_1 + \cdots + r_u = r \), \( u \leq \frac{r-1}{2} \).

Hence by the trivial estimate, such term is majorized by

\[ \pi(x)^{-\frac{r}{2}} \sum_{u=1}^{r} \frac{1}{u!} \sum_{(r_1, \ldots, r_u)} \frac{1}{r_1! \cdots r_u!} \sum_{d^{r_1+\cdots+r_u}} |L(X)| \sum_{u} \frac{1}{u!} \sum_{d^{r}} X \pi(x)^{-\frac{r}{2}}. \]

This gives rise to the error term.

Suppose \( r_i \leq 2 \) for all \( i \). Suppose \( r_1 = 1 \) for some \( i \). We may assume that \( r_1 = 1 \).

Let \( N \) be the number of conjugacy classes of \( G \), and partition the sum \( \sum_{\rho \in L(X)} \) into \( (N + w)^u \) sums, namely, given \((S_1, \ldots, S_u)\), where \( S_i \) is either \( S_{\rho_i, C} \) or \( S_{\rho_i, r_j} \), we consider the set of \( \rho \in L(X) \) with the local conditions \( S_i \) for each \( i \). Note that in each such partition, \( a_{\rho}(p_1)^{r_1} \cdots a_{\rho}(p_u)^{r_u} \) remains a constant.
Suppose $p_1$ is unramified, and fix the splitting types of $p_2, \cdots, p_u$, and let $\text{Frob}_{p_i}$ runs through the conjugacy classes of $G$. Then by (2.2), the sum of such $N$ partitions is

$$
\sum_C \left( \frac{|C|a_\rho(p_1)}{|G|(1 + f(p_1))} A(S_2, \ldots, S_u)X + O((p_1 \cdots p_u)^\gamma X^\delta) \right),
$$

for a constant $A(S_2, \ldots, S_u)$. Let $\chi_\rho$ be the character of $\rho$. Then $a_\rho(p) = \chi_\rho(g)$, where $g = \text{Frob}_p$. By orthogonality of characters, $\sum_C |C|a_\rho(p_1) = \sum_{g \in G} \chi_\rho(g) = 0$. Hence the above sum is $O((p_1 \cdots p_u)^\gamma X^\delta)$, and it is majorized by $\pi(x)^{-\frac{\gamma}{2} + \mu} x^\gamma X^\delta$.

Hence we can assume that $r_i \leq 2$ for each $i$, and $p_j$ is ramified when $r_j = 1$. Suppose $r_1 + \cdots + r_v + r_{v+1} + \cdots + r_u = r$, $r_1 = \cdots = r_v = 1$ and $r_{v+1} = \cdots = r_u = 2$. Then $u - v \leq \frac{r}{2}$, and $p_1, \ldots, p_v$ are ramified. The partition of fixed splitting types of $p_{v+1}, \ldots, p_u$ is majorized by

$$
\prod_{i=1}^u \frac{f(p_i)}{1 + f(p_i)} B(S_{v+1}, \ldots, S_u)X + O((p_1 \cdots p_u)^\gamma X^\delta),
$$

for some constant $B(S_{v+1}, \ldots, S_u)$. Since $\frac{f(p)}{1 + f(p)} \ll \frac{1}{p}$, it contributes to

$$
\pi(x)^{u-v-\frac{r}{2}} (\log \log x)^v X + \pi(x)^{-\frac{r}{2} + u} x^\gamma X^\delta \ll X (\log \log x)^v \pi(x)^{-\frac{r}{2}} + \pi(x)^{-\frac{r}{2} + u} x^\gamma X^\delta.
$$

Now let $r$ be even, $u = \frac{r}{2}$, and $r_1 = \cdots = r_u = 2$. If one of $p_1, p_2, \cdots, p_u$ is ramified, their contribution is majorized by $X \pi(x)^{-1} \log \log x$. Now we assume that all primes are unramified. Then the corresponding term is

$$
(3.3) \quad \pi(x)^{-\frac{r}{2}} \frac{1}{u!} \frac{r!}{2^u} \sum_{(p_1, \ldots, p_u)} \left( \sum_{(L(s, \rho) \in L(X))} a_\rho(p_1)^2 \cdots a_\rho(p_u)^2 \right).
$$

Let $N$ be the number of conjugacy classes of $G$, and partition the sum $\sum_{\rho \in L(X)}$ into $N^u$ sums where $(C_1, \ldots, C_u)$ is the set of $\rho \in L(X)$ such that $\text{Frob}_{p_i} \in C_i$ for each $i$. Then,

$$
\sum_{\rho \in L(X)} a_\rho(p_1)^2 \cdots a_\rho(p_u)^2 = \sum_{(C_1, \ldots, C_u)} \chi_\rho(p_1)^2 \cdots \chi_\rho(p_u)^2 \left( \sum_{\rho \in L(X)} 1 \right)
$$

$$
= \sum_{(C_1, \ldots, C_u)} \chi_\rho(p_1)^2 \cdots \chi_\rho(p_u)^2 \left( \prod_{i=1}^u \frac{|C_i|}{|G|(1 + f(p_i))} |L(X)| + O((p_1 \cdots p_u)^\gamma X^\delta) \right).
$$

Now

$$
\sum_{(C_1, \ldots, C_u)} \chi_\rho(p_1)^2 \cdots \chi_\rho(p_u)^2 \prod_{i=1}^u \frac{|C_i|}{|G|(1 + f(p_i))} = \prod_{i=1}^u \left( \sum_{C_i} \frac{\chi_\rho(p_i)^2 |C_i|}{|G|(1 + f(p_i))} \right).
$$
Here \( \chi_\rho(p)^2 = \chi_{\rho^2}(p) = \chi_{\text{Sym}^2\rho}(p) + \chi_{\wedge^2\rho}(p) \). We observed in [7] that since \( \rho \) is an irreducible real self-dual representation, \( \text{Sym}^2\rho \) contains the trivial representation and \( \wedge^2\rho \) does not contain the trivial representation ([11], page 274). Hence \( \chi_\rho(p)^2 = 1 + \sum_{j=1}^{l} \eta_j(p) \), where \( \eta_j \)'s are non-trivial irreducible characters of \( G \). By the orthogonality of characters, for each \( j \),

\[
\sum_{\rho \in L(X)} \eta_j(p) = 0.
\]

Hence, \( \sum_{\rho \in L(X)} \chi_\rho(p)^2 = 1 + \sum_{r \in \mathbb{C}} \eta_j(p) \). By the orthogonality of characters, for each \( j \),

\[
\sum_{\rho \in L(X)} \eta_j(p) = 0.
\]

Therefore,

\[
\sum_{(p_1, \ldots, p_u)} \left( \sum_{\rho \in L(X)} a_\rho(p_1) \cdots a_\rho(p_u) \right) = \pi(x)^u |L(X)| + O(\pi(x)^u |L(X)| \log \log x) + O(\pi(x)^u x^{\gamma u} X^\delta).
\]

Note

\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^r e^{-\frac{t^2}{2}} dt = \begin{cases} 
\frac{r!}{(r/2)^{r/2}}, & \text{if } r \text{ is even}, \\
0, & \text{if } r \text{ is odd}.
\end{cases}
\]

Hence we have proved

**Theorem 3.1.** Suppose \( \frac{\log X}{\log x} \to \infty \) as \( x \to \infty \). Then

\[
\frac{1}{|L(X)|} \sum_{\rho \in L(X)} \left( \sum_{p \leq x} a_\rho(p) \right)^r = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^r e^{-\frac{t^2}{2}} dt + O \left( \frac{(\log \log x)^r}{\pi(x)^{\frac{r}{2}}} \right).
\]

This proves ([11]).

## 4. Central Limit Theorem for Hecke eigenforms; Level aspect

In this section, in analogy to ([12]), we consider central limit theorem for modular form \( L \)-functions with the trivial central character with respect to congruence subgroups as the level goes to infinity. We follow [11] closely. For \( k \geq 2 \), let \( S_k(N) \) be the set of normalized Hecke eigen cusp forms of weight \( k \) with respect to \( \Gamma_0(N) \) with the trivial central character. Let

\[
f(z) = \sum_{n=1}^{\infty} a_f(n) n^{k-1} e^{2\pi i n z}; \quad a_f(mn) = a_f(m)a_f(n), \quad \text{if } (m,n) = 1; \quad a_f(1) = 1; \quad a_f(p^j) = a_f(p)a_f(p^{j-1}) - a_f(p^{j-2}).
\]

We show

**Theorem 4.1.** For a continuous real function \( h \) on \( \mathbb{R} \), (assume that \( \frac{\log N}{\log x} \to \infty \) as \( x \to \infty \))

\[
\frac{1}{\#S_k(N)} \sum_{f \in S_k(N)} h \left( \frac{\sum_{p \leq x} a_f(p)}{\sqrt{\pi(x)}} \right) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) e^{-\frac{t^2}{2}} dt \quad \text{as } x \to \infty.
\]
We have, from [14],

**Lemma 4.2.** Suppose $k \geq 2$. Let $S_k(N, \chi)$ be the set of normalized Hecke eigen cusp forms of weight $k$ with respect to $\Gamma_0(N)$ with a character $\chi \pmod{N}$. Then

$$
\sum_{f \in S_k(N, \chi)} a_f(n) = \frac{k-1}{12} \chi(\sqrt{n}) n^{-\frac{1}{2}} \psi(N) + O(n^{\epsilon} N^{\frac{1}{2}} d(N)),
$$

for some constant $c$, independent of $n, N$.

Here $\psi(N) = N \prod_{l \mid N} (1 + \frac{1}{l})$, and $d(N)$ is the number of positive divisors of $N$. Note that $\psi(N) = |\mathbb{SL}_2(\mathbb{Z}) : \Gamma_0(N)|$. Here $\chi(x) = 0$ if $x$ is not a positive integer prime to $N$. In particular, if $n$ is not a square, $\sum_{f \in S_k(N, \chi)} a_f(n) = O(n^{\epsilon} N^\frac{1}{2} d(N))$. Taking $n = 1$ and $\chi = 1$, we have

$$
\#S_k(N) = \frac{k-1}{12} \psi(N) + O(N^\frac{1}{2} d(N)).
$$

We need to compute, for a positive integer $r$,

$$
(4.1) \quad \sum_{f \in S_k(N)} \left( \frac{\sum_{p \leq x} a_f(p)}{\sqrt{\pi(x)}} \right)^r.
$$

By multinomial formula,

$$
(4.1) = \pi(x)^{-\frac{r}{2}} \sum_{u=1}^{r} \frac{1}{u!} \sum_{(r_1, \ldots, r_u)} \frac{r!}{r_1! \cdots r_u!} \sum_{(p_1, \ldots, p_u)} \left( \sum_{f \in S_k(N)} a_f(p_1)^{r_1} \cdots a_f(p_u)^{r_u} \right).
$$

Now we claim that except when $r$ is even, $u = \frac{r}{2}$, and $r_1 = \cdots = r_u = 2$, it gives rise to the error term.

By [13], Lemma 2, we can show that $a_f(p)^n = \sum_{n=0}^{\infty} h_n(j) a_f(p^j)$, where $h_n(j) = 0$ if $n$ is odd and $j$ is even, or if $n$ is even and $j$ is odd. For $u$-tuples $(r_1, \ldots, r_u)$ and $(p_1, \ldots, p_u)$, we define

$$
A(r_1, \ldots, r_u) = \sum_{(p_1, \ldots, p_u)} B(r_1, \ldots, r_u; p_1, \ldots, p_u),
$$

$$
B(r_1, \ldots, r_u; p_1, \ldots, p_u) = \sum_{f \in S_k(N)} a_f(p_1)^{r_1} \cdots a_f(p_u)^{r_u}.
$$

Then

$$
B(r_1, \ldots, r_u; p_1, \ldots, p_u) = \sum_{0 \leq j_1 \leq r_1, \ldots, 0 \leq j_u \leq r_u} h_{r_1}(j_1) \cdots h_{r_u}(j_u) \sum_{f \in S_k(N)} a_f(p_1^{j_1} \cdots p_u^{j_u}).
$$

As in [13], if $r_l$ is odd for some $l$, $A(r_1, \ldots, r_u) \ll N^\frac{1}{2} d(N) \pi(x)^u x^{cur}$. Now let $r_1 = \cdots = r_u = 2$. Then $r$ is even, and $u = \frac{r}{2}$.
\[ A(r_1,\ldots,r_u) = \pi(x)^{\frac{u}{2}} \# S_k(N) + O(\pi(x)^{\frac{u}{2}-1}(\log \log x)^{\frac{u}{2}} \# S_k(N)). \]

Now suppose that all \( r_i \)'s are even, and \( r_i > 2 \) for some \( i \). Then \( u \leq \frac{r}{2} - 1 \). Then
\[ A(r_1,\ldots,r_u) \ll \pi(x)^{-1} \# S_k(N). \]

Hence, as in Theorem 3.1, we have

**Proposition 4.3.** Assume that \( \frac{\log N}{\log x} \to \infty \) as \( x \to \infty \). Then
\[
\frac{1}{\# S_k(N)} \sum_{f \in S_k(N)} \left( \frac{\sum_{p \leq x} a_f(p)}{\sqrt{\pi(x)}} \right)^r = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^r e^{-\frac{t^2}{4}} dt + O\left( \frac{\log \log x^{\frac{u}{2}}}{\pi(x)} \right).
\]
This proves Theorem 4.1.

### 5. Analogues of Sato-Tate distribution

For a Hecke eigenform \( f \in \mathcal{F}_k \), Sato-Tate conjecture says that for a continuous real function \( h \) on \([-2,2]\),
\[
\frac{1}{\pi(x)} \sum_{p \leq x} h(a_f(p)) \to \frac{1}{2\pi} \int_{-2}^{2} h(t) \sqrt{4 - t^2} dt, \quad \text{as} \quad x \to \infty.
\]

Let \( a_f(p) = 2 \cos \theta_f(p) \) for \( \theta_f(p) \in [0, \pi] \). Then \( \{\theta_f(p)\} \) is uniformly distributed with respect to the measure \( \frac{2}{\pi} \sin^2 \theta d\theta \) on \([0,\pi]\). This is proved in [2].

For a vertical Sato-Tate distribution, one can consider, for a fixed prime \( p \),
\[
(5.1) \quad \sum_{f \in \mathcal{F}_k} a_f(p)^n.
\]

Conrey-Duke-Farmer [9] proved, for a holomorphic form of weight \( k \),
\[
\sum_{f \in \mathcal{F}_k} a_f(p)^n = \frac{k}{6\pi} \left( 1 + \frac{1}{p} \right) \int_0^\pi 2^n \cos^n \theta \frac{\sin^2 \theta}{(1 - \frac{1}{p})^2 + \frac{4}{p} \sin^2 \theta} d\theta + O(p^{\frac{n}{2}+\epsilon}).
\]

This implies that \( \{\theta_f(p), f \in \mathcal{F}_k\} \) is uniformly distributed with respect to the measure
\[
\frac{2}{\pi} \left( 1 + \frac{1}{p} \right) \frac{\sin^2 \theta}{(1 - \frac{1}{p})^2 + \frac{4}{p} \sin^2 \theta} d\theta.
\]

For Artin \( L \)-function analogue of Sato-Tate distribution, we consider, for \( r \geq 1 \),
\[
(5.2) \quad \frac{1}{\pi(x)} \sum_{p \leq x} a_{\rho}(p)^r.
\]
In our case, note that $-1 \leq a_p(p) \leq d$. By effective Chebotarev density theorem (cf. [15], page 132), for $\log x \gg |G|(|d_R|)^{1/2}$,

$$
\sum_{\substack{p \leq x \\
\text{Frob}_p \in C}} a_p(p) = \frac{|C|}{|G|} \pi(x) + O\left(\pi(x^{1/2}) + xe^{-c|G|^{-1/2}(\log x)^{1/2}}\right),
$$

where $\beta$ is an exceptional zero of $\zeta_K(s)$ such that $1 - \beta \leq \frac{1}{4} \log d_K$, if it exists. Hence

$$
\sum_{\substack{p \leq x \\
\text{Frob}_p \in C}} a_p(p) = \sum_{C} a_p(p) \left(\sum_{\substack{p \leq x \\
\text{Frob}_p \in C}} 1\right) = \sum_{C} a_p(p) \left|\frac{C}{|G|}\pi(x) + O\left(\pi(x^{3/2}) + xe^{-c|G|^{-1/2}(\log x)^{1/2}}\right)\right|.
$$

Now $\sum_{C} |C|a_p(p) = \sum_{g \in G} \chi_p(g)^r$ and $\chi_p(g) = \chi_{\rho^r}(g)$. Note that

$$
\frac{1}{|G|} \sum_{g \in G} \chi_{\rho^r}(g) = n_r,
$$

which is the multiplicity of the trivial representation in $\rho^r$. Hence

$$
\sum_{\substack{p \leq x}} a_p(p) = n_r \pi(x) + O\left(\pi(x^{3/2}) + xe^{-c|G|^{-1/2}(\log x)^{1/2}}\right).
$$

Therefore,

$$
\frac{1}{\pi(x)} \sum_{\substack{p \leq x}} a_p(p) \rightarrow n_r, \quad \text{as } x \to \infty.
$$

For vertical Sato-Tate distribution, for a fixed prime $p$, consider

$$
\frac{1}{|L(X)|} \sum_{\rho \in L(X)} a_p(p)^r.
$$

Then by (2.2),

$$
\sum_{\rho \in L(X)} a_p(p)^r = \sum_{C} a_p(p)^r \left(\sum_{\substack{\rho \in L(X) \\
\text{Frob}_p \in C}} 1\right) + a_p(p)^r \left(\sum_{\substack{\rho \in L(X) \\
\text{Frob}_p \text{ is ramified}} \right) + O\left(|L(X)|^{1/2+\gamma}\right) + O\left(\frac{X}{p}\right)
$$

$$
= \frac{|L(X)|}{|G|(1 + f(p))} \sum_{C} |C|a_p(p) + O(p^\gamma X^\delta) + O\left(\frac{X}{p}\right) = \frac{|L(X)|n_r}{1 + f(p)} + O(p^\gamma X^\delta) + O\left(\frac{X}{p}\right).
$$

So if $X > p^{1+\delta}$,

$$
\frac{1}{|L(X)|} \sum_{\rho \in L(X)} a_p(p)^r = \frac{n_r}{1 + f(p)} + O(p^{-1}).
$$
6. Counting $S_5$ quintic fields with local conditions

Shankar and Tsimerman [16] recently counted $S_5$ quintic fields with a power saving error terms. For $i = 0, 1, 2$, let $N_5^{(i)}(X)$ be the number of $S_5$ quintic fields of signature $(5 - 2i,i)$ with $|d_K| < X$. Then they showed

$$N_5^{(i)}(X) = D_i X + O_{\varepsilon} \left( X^{\frac{399}{240} + \varepsilon} \right),$$

where $D_i = d_i \prod_p (1 + p^{-2} - p^{-4} - p^{-5})$ and $d_0,d_1,d_2$ are $\frac{1}{240}, \frac{1}{24}$ and $\frac{1}{16}$, respectively.

We can count quintic fields with finitely many local conditions. Let $C$ be a conjugacy class of $S_5$ and $f(p) = p^{-1} + 2p^{-2} + 2p^{-3} + p^{-4}$. Let $\mathcal{S} = \{LC_p\}$ be a finite set of local conditions. Define $|S_{p,C}| = \frac{|C|}{|\mathcal{C}|^{(1+f(p))}}$, $|S_{p,r_i}| = \frac{c_i(p)}{(1+ f(p))}$, and $|\mathcal{S}| = \prod_p |LC_p|$, where $c_i(p)$’s are given explicitly at the end of this section.

**Theorem 6.1.** Let $N_5^{(i)}(X,\mathcal{S})$ be the number of $S_5$ quintic fields of signature $(5 - 2i,i)$ with $|d_K| < X$, and with the local condition $\mathcal{S}$. Then

$$N_5^{(i)}(X,\mathcal{S}) = |\mathcal{S}| D_i X + O_{\varepsilon} \left( \prod_{p \in \mathcal{S}} p \right)^{2-\varepsilon} X^{\frac{399}{240} + \varepsilon}.$$ 

We follow the notations in [16]. Let $V_\mathbb{Z}$ be the space of 4-tuples of $5 \times 5$ alternating matrices with integer coefficients. The group $G_\mathbb{Z} = GL_4(\mathbb{Z}) \times SL_5(\mathbb{Z})$ acts on $V_\mathbb{Z}$ via

$$(g_4,g_5) \cdot (A,B,C,D)^t = g_4(5A5_g,t)g_5B5_g,tg_5C5_g,tg_5D5_g,t)^t.$$ 

Here $g_4 \cdot (A,B,C,D)^t$ means $(a_1(A,B,C,D)^t, a_2(A,B,C,D)^t, a_3(A,B,C,D)^t, a_4(A,B,C,D)^t)$, where $a_i$ is the $i$th row of $g_4$.

There is a canonical bijection between the set of $G_\mathbb{Z}$-equivalence classes of elements $(A,B,C,D) \in V_\mathbb{Z}$, and the set of isomorphism classes of pairs of $(R,R')$, where $R$ is a quintic ring and $R'$ is a sextic resolvent ring of $R$. (See [5].) Let $V$ be an element of $V_\mathbb{Z}$. Over the residue field $\mathbb{F}_p$, the element $V$ determines a quintic $\mathbb{F}_p$-algebra $R(V)/(p)$. Let us define the splitting symbol $(V,p)$ by

$$(V,p) = (f_1^{e_1} f_2^{e_2} \cdots),$$

whenever $R(V)/(p) \cong \mathbb{F}_{p^{e_1}}[t_1]/(t_1^{e_1}) \oplus \mathbb{F}_{p^{e_2}}[t_2]/(t_2^{e_2}) \oplus \cdots$. Then there are 17 possible splitting types for $(V,p)$: $(11111), (11112), (11122), (11222), (12222), (11113), (11123), (11223), (11232), (12223), (12232), (12322), (13322), (13332), (13333), (13334)$, and $(13335)$. Let $\sigma$ be one of 17 splitting types. Then define $T_p(\sigma)$ to be the set of $V \in V_\mathbb{Z}$ such that $(V,p) = \sigma$ and $U_p(\sigma)$ to be the set of elements in $T_p(\sigma)$ corresponding
to quintic rings that are maximal at \( p \). The set \( U_p(\sigma) \) is defined by congruence conditions on coefficients of \( V \) modulo \( p^2 \). Let \( \mu(U_p(\sigma)) \) be the \( p \)-adic density of \( S \) in \( V_{Z_p} \). They are computed in Lemma 4 in [5]. Let \( U_p \) denote the union of the 17 \( U_p(\sigma) \). Then Lemma 20 of [5] implies that

\[
\mu(U_p) = (p - 1)^8 p^{12}(p + 1)^4(p^2 + 1)^2(p^4 + p^3 + p^2 + p + 1)(p^4 + p^3 + 2p^2 + 2p + 1)/p^{40}.
\]

Note that

\[
d_i\zeta(2)^2\zeta(3)^2\zeta(4)^2\zeta(5)\prod_p \mu(U_p) = d_i\prod_p (1 + p^{-2} - p^{-4} - p^{-5}),
\]

which is the coefficient of the main term in counting quintic fields. Here we need to interpret \( \mu(U_p) \) in the following way: \( U_p \) can be considered as a subset of \((\mathbb{Z}/q^2\mathbb{Z})^{40}\), or the union of \( k \) translates of \( p^2V_Z \), where \( k \) is the size of the set. Here \( k = \mu(U_p)q^{80} \). Let \( W_p \) be the complement of \( U_p \) in \( V_Z \), then \( \mu(W_p) = 1 - \mu(U_p) \). Then \( W_p \) is the union of \( \mu(W_p)q^{80} \) translates of \( q^2V_Z \).

For \( q \) square-free, let \( W_q \subset V_Z \) be the set of elements corresponding to quintic rings that are not maximal at each prime dividing \( q \). Then \( W_q \) is the union of \( \prod_{p|q} \mu(W_p) \cdot q^{80} \) translates of \( q^2V_Z \) by the Chinese Remainder Theorem.

Let \( V_Z^{\text{ndeg}} \) denote the set of elements in \( V_Z \) that correspond to orders in \( S_5 \)-fields, and let \( V_Z^{\text{deg}} \) be the complement of \( V_Z^{\text{ndeg}} \). A point in \( V_Z \) corresponds to a maximal order in an \( S_5 \) quintic fields precisely if it is in \( \cap_p U_p \cap V_Z^{\text{ndeg}} \). For a \( G_Z \)-invariant subset \( S \) of \( V_Z \), let \( N(S,X) \) denote the number of irreducible \( G_Z \)-orbits in \( S^{\text{ndeg}} \) having discriminant bounded by \( X \). For a set \( S \) which is not \( G_Z \)-invariant, we can define \( N^*(S,X) \) which also counts the orbits of degenerate points in \( S \).

Now we choose a finite set of primes \( \{p_1, p_2, \ldots, p_n\} \). And choose a splitting type \( \sigma_{p_k} \) for each \( p_k \), \( k = 1, 2, \ldots, n \). Define \( U'_p \) to be \( U_p \) if \( p \neq p_k \), \( k = 1, 2, \ldots, n \). If \( p = p_k \) for some \( k \), then \( U'_p = U_p(\sigma_{p_k}) \). Let \( W'_p \) be the complement of \( U'_p \). Then \( W_q' \) is the union of \( \prod_{p|q} \mu(W'_p) \cdot q^{80} \) translates of \( q^2V_Z \).

Let \( N(i)(X,\{\sigma_{p_k}\}_{k=1,2,\ldots,n}) \) be the number of \( S_5 \) quintic fields of signature \((5 - 2i,i)\) with \(|d_K| < X \) and the splitting types of \( p_k \)'s are \( \sigma_{p_k} \) for \( k = 1, 2, \ldots, n \). Then by inclusion-exclusion method,

\[
N(i)(X,\{\sigma_{p_k}\}_{k=1,2,\ldots,n}) = \sum_q \mu(q)N(W_q' \cap V_Z^{(i)}, X).
\]

We use two estimates of \( N(W_q' \cap V_Z^{(i)}, X) \) for small \( q \)'s and large \( q \)'s separately. Lemma 3 in [16] says that \( N(W_q', X) = O(X/q^{2-\epsilon}) \) and it implies that \( N(W_q', X) \ll \epsilon (p_1p_2\cdots p_n)^{2-\epsilon} X/q^{2-\epsilon} \) for
q square-free. If \( L \) is a translate of \( mV_Z \), then we have

\[
N^*(\{ x \in L \cap V^{(i)}_Z : a_{12} \neq 0 \}, X) = c_i \frac{X}{m^{\frac{1}{10}}} + O \left( m^{-39} X^{\frac{39}{40}} \right),
\]

where \( c_i = d_i \zeta(2)^2 \zeta(3)^2 \zeta(4)^2 \zeta(5) \). (See Equation 28 in [6].) Since \( W'_q \) is an union of \( \prod_{p | q} \mu(W'_q) \cdot q^80 \) translates of \( q^2V_Z \),

\[
N^*(\{ x \in W'_q \cap V^{(i)}_Z : a_{12} \neq 0 \}, X) = \prod_{p | q} \mu(W'_q) \cdot c_i X + O \left( q^2 X^{\frac{39}{40}} \right) = \mu(W'_q) \cdot c_i X + O \left( q^2 X^{\frac{39}{40}} \right).
\]

Then

\[
N^{(i)}_5(X, \{ \sigma_p \}_{i=1,2,\ldots,n}) = \sum_q \mu(q)N(W'_q \cap V^{(i)}_Z, X)
\]

\[
= \sum_{q \leq Q} \left( \mu(W'_q) \cdot c_i X + O \left( q^2 X^{\frac{39}{40}} \right) - \mu(q) N_{12}^*(W_q \cap V^{deg.(i)}_Z, X) \right) + \sum_{q > Q} O_{\epsilon} \left( (p_1 p_2 \cdots p_n)^{2-\epsilon} X^{\frac{39}{q^2}} \right)
\]

\[
= \sum_{q \leq Q} \left( \mu(W'_q) \cdot c_i X - \mu(q) N_{12}^*(W_q \cap V^{deg.(i)}_Z, X) \right) + O_{\epsilon} \left( (p_1 p_2 \cdots p_n)^{2-\epsilon} X/Q^{1-\epsilon} + X^{39Q^{1+\epsilon}} \right)
\]

\[
= \sum_{q} c_i \mu(W'_q) X + (p_1 p_2 \cdots p_n)^{2-\epsilon} O_{\epsilon} \left( X/Q^{1-\epsilon} + X^{39Q^{3+\epsilon}} + X^{\frac{399}{380}Q^{1+\epsilon}} \right)
\]

\[
= \sum_{q} c_i \mu(U'_q) X + (p_1 p_2 \cdots p_n)^{2-\epsilon} O_{\epsilon} \left( X/Q^{1-\epsilon} + X^{39Q^{3+\epsilon}} + X^{\frac{399}{380}Q^{1+\epsilon}} \right).
\]

Putting \( Q = X^{\frac{1}{400}} \), we have

\[
(p_1 p_2 \cdots p_n)^{2-\epsilon} O_{\epsilon} \left( X/Q^{1-\epsilon} + X^{39Q^{3+\epsilon}} + X^{\frac{399}{380}Q^{1+\epsilon}} \right) \ll_{\epsilon} (p_1 p_2 \cdots p_n)^{2-\epsilon} X^{\frac{399}{400}+\epsilon}.
\]

Note that

\[
c_i \prod_p (\mu(U'_q)) = \prod_{k=1}^n \mu(U_p(\sigma_{p_k})) c_i \prod_p \mu(U_p) = \prod_{k=1}^n \mu(U_p(\sigma_{p_k})) \frac{d_i \prod_p (1 + p^{-2} - p^{-4} - p^{-5})}{\mu(U_p)}.
\]

From Lemma 20 in [5], we can see that, for \( f(p) = p^{-1} + 2p^{-2} + 2p^{-3} + p^{-4} \),

\[
\frac{\mu(U_p(\sigma))}{\mu(U_p)} = \frac{1/120}{1 + f(p)}, \quad \frac{1/12}{1 + f(p)}, \quad \frac{1/8}{1 + f(p)}, \quad \frac{1/6}{1 + f(p)}, \quad \frac{1/6}{1 + f(p)}, \quad \frac{1/4}{1 + f(p)}, \quad \frac{1}{1 + f(p)}.
\]
for $\sigma = (11111), (1112), (122), (113), (23), (14), (15)$, respectively, and

$$\frac{\mu(U_p(\sigma))}{\mu(U_p)} = \frac{1/6 \cdot 1/p}{1 + f(p)} \cdot \frac{1/2 \cdot 1/p}{1 + f(p)} \cdot \frac{1/3 \cdot 1/p}{1 + f(p)} \cdot \frac{1/2 \cdot 1/p^2}{1 + f(p)} \cdot \frac{1/2 \cdot 1/p^2}{1 + f(p)},$$

for $\sigma = (1^2111), (1^212), (1^23), (1^21^21), (2^21), (1^311), (1^32), (1^31^2), (1^41), (1^5)$, respectively. Hence we have proved the theorem.

7. Counting $S_4$ quartic fields with local conditions

In [3], Belabas, Bhargava and Pomerance obtained a power saving error term for counting $S_4$ quartic fields. For $i = 0, 1$, let $N^{(i)}_4(X)$ be the number of isomorphism classes of $S_4$-quartic fields of signature $(4 - 2i, i)$ with $|d_K| < X$. Then

$$N^{(i)}_4(X) = D_iX + O(X^{23/24+\epsilon}),$$

where $D_i = d_i \prod_p (1 + p^{-2} - p^{-3} - p^{-4})$, and $d_0 = \frac{1}{45}, d_1 = \frac{1}{8},$ and $d_0 = \frac{1}{16}$.

Let $V_{\mathbb{Z}}$ be the space of pairs of $(A, B)$ of integral ternary quadratic forms. The group $G_{\mathbb{Z}} = GL_2(\mathbb{Z}) \times SL_3(\mathbb{Z})$ acts on $V_{\mathbb{Z}}$ in the following way. For $g_2 \in GL_2(\mathbb{Z})$ and $g_3 \in SL_3(\mathbb{Z})$,

$$(g_2, g_3) \cdot (A, B)^t = g_2 (g_3 A g_3^t, g_3 B g_3^t)^t.$$  

Here $g_2 \cdot (A, B)$ means that $(a_1(A, B)^t, a_2(A, B)^t)$, where $a_i$ is the $i$th row of $g_2$.

There is a canonical bijection between the set of $G_{\mathbb{Z}}$-equivalence classes of elements $(A, B) \in V_{\mathbb{Z}}$, and the set of isomorphism classes of pairs of $(Q, R)$, where $Q$ is a quartic ring and $R'$ is a cubic resolvent ring of $Q$. (See [4].)

A prime $p$ has 11 possible splitting type in an $S_4$ quartic field $K$. They are $(1111), (112), (13), (4), (22), (1^211), (1^21^2), (2^2), (1^31),$ and $(1^4).$ As we did in the previous section, we can define $U_p(\sigma)$ and $U_p$, resp. and $\mu(U_p(\sigma))$’s are computed in Lemma 23, [4].

Using their result, Yang [18] was able to count $S_4$ quartic fields with one local condition with a power saving error term. He showed that

$$N^{(i)}_4(X, \sigma_p) = \frac{\mu(U_p(\sigma_p))}{\mu(U_p)} D_iX + O\left(p^2 X^{\frac{143}{144} + \epsilon}\right).$$
By the same argument in the previous section, we can extend Yang’s result to the case of finitely many local conditions:

\[
N^{(i)}(X, \{\sigma_{p_k}\}_{k=1,2,\ldots,n}) = \left( \prod_{k=1}^{n} \frac{\mu(U_p(\sigma_{p_k}))}{\mu(U_{p_k})} \right) D_i X + O \left( (p_1 \cdots p_k)^2 X^{\frac{1}{144} + \varepsilon} \right).
\]

From Lemma 23 in [4], we can see that, for \( f(p) = p^{-1} + 2p^{-2} + p^{-3} \),

\[
\frac{\mu(U_p(\sigma))}{\mu(U_p)} = \frac{1/24}{1 + f(p)}, \frac{1/4}{1 + f(p)}, \frac{1/3}{1 + f(p)}, \frac{1/8}{1 + f(p)}, \text{ and } \frac{1/4}{1 + f(p)}
\]

for \( \sigma = (1111), (112), (13), (22), (4) \), respectively, and

\[
\frac{\mu(U_p(\sigma))}{\mu(U_p)} = \frac{1/2 \cdot 1/p}{1 + f(p)}, \frac{1/2 \cdot 1/p}{1 + f(p)}, \frac{1/2 \cdot 1/p^2}{1 + f(p)}, \frac{1/p^2}{1 + f(p)}, \frac{1/p^2}{1 + f(p)}, \text{ and } \frac{1/p^3}{1 + f(p)}
\]

for \( \sigma = (1^211), (1^22), (1^21^2), (2^2), (1^31), (1^4) \), respectively. Hence we have proved

**Theorem 7.1.** Let \( N^{(i)}(X, S) \) be the number of \( S_4 \) quartic fields of signature \( (4 - 2i, i) \) with \(|d_K| < X\), and with the local condition \( S \). Then

\[
N^{(i)}(X, S) = |S| D_i X + O\left( \left( \prod_{p \in S} p \right)^{2 X^{\frac{1}{144} + \varepsilon}} \right).
\]

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