Topology Change and the Propagation of Massless Fields

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We analyse the massless wave equation on a class of two dimensional manifolds consisting of an arbitrary number of topological cylinders connected to one or more topological spheres. Such manifolds are endowed with a degenerate (non-globally hyperbolic) metric. Attention is drawn to the topological constraints on solutions describing monochromatic modes on both compact and non-compact manifolds. Energy and momentum currents are constructed and a new global sum rule discussed. The results offer a rigorous background for the formulation of a field theory of topologically induced particle production.
1. Introduction

It is well known that the global structure of a manifold is fundamental in constructing regular solutions to tensor equations. Furthermore field quantisation is sensitive to the topology of the underlying base space [1], [2], [3]. However little attention has been given to the elucidation of classical solutions of field equations on manifolds with degenerate geometries that can accommodate non-trivial topology change in general relativity. Such solutions arise from equations that are not globally hyperbolic.

Some of the earliest mathematical work on the study of partial differential equations that change from being hyperbolic to elliptic was done by Tricomi [4]. This early treatise involved considerable technicalities that have not been extensively pursued in the mathematical literature. Even in two dimensions the analysis of second order partial differential equations with indefinite characteristics is often non-trivial and the general theory using modern techniques has only recently been considered in topologically trivial manifolds. [5], [6], [7]. Such techniques are relevant for the general study of (non-linear) equations that can arise on manifolds with a degenerate geometry but need to be supplemented by further data to provide well posed or interesting problems.

Kundt first [8] discussed the non-existence of certain topologically non-trivial spacetimes assuming that every geodesic is complete. Geroch [9] exploited the notion of global hyperbolicity to reach a similar conclusion.

In this paper we consider two dimensional manifolds with smooth (degenerate) metrics. For the applications that we have in mind we require the existence of asymptotically flat Lorentzian domains foliated by compact space-like hypersurfaces.
A non-trivial two dimensional example is the trouser space. It may be realized as a pair of trousers embedded in a Minkowskian spacetime of 3 dimensions such that spacelike circles, disconnected at some time, become connected at another. Such a manifold cannot sustain a global metric with a Lorentzian signature. The domain where the metric becomes degenerate depends on the embedding but cannot be eliminated.

One of the fundamental issues that arises in describing fields on such manifolds is the dependence of the field equation on the regularity of the metric tensor field. We adopt a pragmatic approach in this paper and impose natural conditions that enable us to construct non-singular scalar fields that are globally $C^1$ in the presence of a degenerate $C^\infty$ metric field.

An example of a two-dimensional trouser-type manifold with a metric that is singular at a single point may be found in [10]. In our approach we consider manifolds endowed with a smooth everywhere regular covariant metric tensor that is however degenerate. Such manifolds are therefore not causally connected [11], [12], [13].

Since the trousers embedding is only for ease of visualisation one may equivalently consider two (or more) cylinders surgically attached to a punctured 2-sphere. It is then possible with the aid of smooth bump functions to endow such a topological space with a metric that has Euclidean signature on the punctured sphere and has Lorentzian signature on a domain of the cylinders. The transition between Euclidean and Lorentzian signature is handled smoothly by the bump functions.

We describe below regular global solutions to the massless scalar field equation

$$d \star d\psi = 0$$

(1)
on some interesting spaces modeled on the above. Here $d$ denotes the exterior derivative, $\star$ the Hodge map and $\psi$ a complex scalar field. Since the punctured sphere is conformally flat one may use stereographic projections as charts to map (the real part of) suitable complex analytic functions that solve (1) on the punctured 2-plane, to the punctured sphere. Since one can also solve (1) on the Lorentzian cylinders it is possible to match them across the degeneracy curves to construct a global solution.

2. Differential Structure

We first establish an atlas to describe a 2-dimensional manifold, an example of which may be visualised as the embedding sketched in Figure 1. We shall then endow this manifold with a $C^\infty$ metric that has Euclidean signature where the cylinders are connected to the sphere and Lorentzian signature elsewhere.

The manifold $M$ is constructed by first removing $n$ non-intersecting caps $\{\text{Cap}^\alpha\}_{\alpha=1\ldots n}$ from a topological sphere $S^2$. Let $S = S^2 - \bigcup_\alpha \text{Cap}^\alpha$. Then from a set of $n$ cylinders, $\{\text{cylinder}^\alpha\}_{\alpha=1\ldots n}$, smoothly attach each cylinder onto each of the holes made by removing the caps successively.

For each $\alpha$ construct the coordinate chart $(U^\alpha, \Phi^\alpha)$ where

\[ U^\alpha = M - \{\text{all the cylinders except cylinder } \alpha\} \]
\[ = S \cup \text{cylinder}^\alpha \]

and

\[ \Phi^\alpha: U^\alpha \to \mathbb{R}^2 : x \mapsto (\tau, \phi) \]

where $0 \leq \phi < 2\pi$ and $-\infty < \tau \leq \pi$, (see Figure 2). For $(\tau, \phi) \in \Phi^\alpha(S)$ we adopt the
standard $S^2$ metric for a sphere of radius $R_1$:

$$g^\alpha|_S = R_1^2 \left( d\tau \otimes d\tau + \sin^2(\tau) d\phi \otimes d\phi \right). \quad (2)$$

For each cap let $\tau_s$ be the angle subtended between the edge of the cap, the center of the sphere, and the center of the cap $s_\alpha \in S^2$. For $\tau > \tau_s$, $(\tau, \phi) \in \Phi^\alpha(S)$ are standard spherical coordinates. Thus $\Phi^\alpha(U^\alpha)$ has holes in it corresponding to the other cylinders, but all the holes are in the region $\Phi^\alpha(S)$. When it is necessary to distinguish coordinates belonging to charts adapted to different cylinders we shall append a chart label as a superscript to the corresponding coordinates. A change of coordinates from $(\tau^\alpha, \phi^\beta) \in \Phi^\alpha(S)$ to $(\tau^\beta, \phi^\beta) \in \Phi^\beta(S)$ corresponds to an $SO(3)$ isometry of this metric. (This will be exploited with the aid of Mobius transformations below).

For each $\alpha$, choose a value of $\tau_M$ such that $\tau_M < \tau_s$. Define the region $M^\alpha$ as follows:

$$\Phi^\alpha(M^\alpha) = \{ (\tau, \phi) : -\infty < \tau \leq \tau_M, 0 \leq \phi < 2\pi \} \quad (3)$$

On region $M^\alpha$ we adopt the standard flat Lorentzian metric for a cylinder of radius $R_2$ given by

$$g^\alpha|_{M^\alpha} = -d\tau \otimes d\tau + R_2^2 d\phi \otimes d\phi \quad (4)$$

Although $\frac{\partial}{\partial \tau}$ is directed towards the Euclidean region for each $\alpha$ we are free to choose an independent time-orientation for each Lorentzian domain. With this in mind we are free to label cylinders as either incoming or outgoing. We now smoothly connect the metric in regions $S$ and $M^\alpha$ with the aid of bump functions. Consider the function $B_p : \mathbb{R} \to \mathbb{R}$ with $B_p \in C^\infty$ and $B_p(x) = 0, x \leq 0$ and $B_p(x) = 1, x \geq 1$ with $B_p$ increasing. There are standard techniques for constructing such a function [14]. Introduce

$$f(\tau) = (R_1^2 + 1) B_p \left( \frac{\tau - \tau_M}{\tau_s - \tau_M} \right) - 1 \quad (5)$$
so that \( f(\tau) = -1 \) for \( \tau < \tau_M \) and \( f(\tau) = R_1^2 \) for \( \tau > \tau_S \).

Define \( \tau_S \) with \( \tau_M < \tau_S < \tau_S \) so that \( f(\tau_S) = 0 \) and hence \( f(\tau) < 0 \) for \( \tau < \tau_S \) and \( f(\tau) > 1 \) for \( \tau > \tau_S \).

Further define

\[
h(\tau) = (R_1^2 \sin^2(\tau) - R_2^2) \text{Bo} \left( \frac{\tau - \tau_M}{\tau_S - \tau_M} \right) + R_2^2
\]

so that in the region \( U^\alpha \) the metric is given by:

\[
g^\alpha = f(\tau)d\tau \otimes d\tau + h(\tau)d\phi \otimes d\phi.
\]

Such a metric smoothly interpolates between Lorentzian and Euclidean regions. Typical metric components are sketched in Figure 3.

We have adopted a metric with an axial Killing symmetry in order to expedite our discussion of (angular) momentum conservation below.

It is convenient to introduce the regions \( E^\alpha, L^\alpha \) and the rings \( \Sigma^\alpha \) defined in the table below.

| \( \tau \) | \( \phi^\alpha(M^\alpha) \) | \( \phi^\alpha(L^\alpha) \) | \( \phi^\alpha(\Sigma^\alpha) \) | \( \phi^\alpha(E^\alpha) \) | \( \phi^\alpha(S) \) |
|---|---|---|---|---|---|
| \(-\infty < \tau \leq \tau_M \) | \( \tau_M < \tau < \tau_S \) | \( \tau_S < \tau \leq \pi \) | \( \tau_S \leq \tau < \pi \) | \( \tau_M < \tau \leq \tau_S \) | \( \tau_S < \tau < \pi \) |
| \( 0 \leq \phi < \frac{2\pi}{3} \) | \( 0 \leq \phi < \frac{2\pi}{3} \) | \( 0 \leq \phi < \frac{2\pi}{3} \) | \( 0 \leq \phi < \frac{2\pi}{3} \) | \( 0 \leq \phi < \frac{2\pi}{3} \) | \( 0 \leq \phi < \frac{2\pi}{3} \) |
| \( f(\tau) = -1 \) | \( f(\tau) < 0 \) | \( f(\tau) = 0 \) | \( f(\tau) > 0 \) | \( f(\tau) = R_1^2 \) | \( f(\tau) = R_1^2 \) |
| \( h(\tau) = R_2^2 \) | \( h(\tau) > 0 \) | \( h(\tau) > 0 \) | \( h(\tau) > 0 \) | \( h(\tau) = R_1^2 \) sin^2(\tau) | \( h(\tau) = R_1^2 \) sin^2(\tau) |

so \( U^\alpha = S \cup E^\alpha \cup L^\alpha \cup M^\alpha \). Note: The constants \( \tau_S, \tau_S, \tau_M, R_2 \) and the functions \( f(\tau), h(\tau) \) may be different for each cylinder \( \alpha \) and will be written \( \tau_S^\alpha, \tau_S^\alpha, \tau_M^\alpha, R_2^\alpha, f^\alpha(\tau), h^\alpha(\tau) \) respectively, when we wish to distinguish them.
It is worth noting that metrics of this type can readily be found that give rise to bounded curvature scalars where the signature changes. For example if
\[ f(\tau_\Sigma + t) = a_1 t + a_3 t^3 + \cdots \]
and
\[ h(\tau_\Sigma + t) = b_0 + b_3 t^3 + b_4 t^4 + \cdots \]
in the vicinity of \( \tau = \tau_\Sigma + t \) then the magnitude of the curvature scalar is \( \frac{9b_3}{2a_1b_0} \) at \( \tau = \tau_\Sigma \), where the metric becomes degenerate.

3. Matching Conditions

We wish to construct functions \( \psi: M \mapsto \mathbb{C} \) from solutions to the equation (1). Since the Hodge map is singular where the metric tensor is degenerate we restrict to solutions that are \( C^1 \) across the rings \( \Sigma^\alpha \). By deriving (1) as a local extremum of the action
\[ \Lambda = \int_M d\psi \wedge \star d\psi \]  
(8)
one recognises a hyperbolic wave equation in regions \( \bigcup \alpha M^\alpha \cup L^\alpha \) with Lorentzian signature and an elliptic (Laplace) Equation in the Euclidean region \( \bigcup \alpha S \cup M^\alpha \). The closure of these regions intersect on the 1 dimensional rings \( \Sigma^\alpha \). We assume that solutions to these local equations are continuous at these rings:
\[ [\psi]_{\Sigma^\alpha} = 0 \quad \forall \alpha \]  
(9)
where, for any ring \( \Sigma \), \( [\omega]_{\Sigma} \) represents the discontinuity
\[ [\omega]_{\Sigma} = \lim_{x \to x_0 \in \Sigma, x \in E^\alpha} \omega_x - \lim_{x \to x_0 \in \Sigma, x \in L^\alpha} \omega_x. \]  
(10)
By demanding that the contributions to the variations of the action cancel on \( \Sigma^\alpha \) one derives [15] the natural junction conditions
\[ [\Sigma^\alpha \ast (d\psi)]_{\Sigma^\alpha} = 0. \]  
(11)
Such conditions also arises naturally in a distributional description [15]. It is the purpose of this paper to find \( C^1 \) regular solutions on \( M \) that satisfy (1), (9) and (11).
4. Mapping the Euclidean Domain into the Complex Plane

Since the manifold is two dimensional and (1) is conformally covariant under scalings of the Euclidean metric it is natural to use the complex plane. However although one may map most of a sphere to the complex plane using the canonical stereographic mapping, we note that the entire Euclidean region $M^E$ of our problem comprises the compact set

$$M^E = S \cup \left( \bigcup_\alpha E^\alpha \cup \Sigma^\alpha \right) \subset M.$$  

It is useful therefore to first consider an injection of the Euclidean region into the sphere

$$\mathcal{P} : M^E \mapsto S^2$$

given by

$$\mathcal{P}|_S : S \mapsto S^2 \text{ is the identity} \quad (12)$$

and

$$\mathcal{P}|_{E^\alpha} : E^\alpha \mapsto S^2$$

$$\left( \theta = 2 \arctan \left( \frac{\rho e^{G(\tau)}}{2R_1} \right), \phi = \phi \right) \quad (13)$$

where $(\theta, \phi)$ are the usual spherical coordinates for $S^2$ about the point $s_\alpha$ labeling the centre of cylinder $\alpha$.

The pair $(\tau, \phi)$ denote the coordinates of the chart $(U^\alpha, \Phi^\alpha)$ restricted to the Euclidean region and

$$G(\tau) = \int_{\tau}^{\tau} \left| f \left( \tau' \right) \right|^2 \frac{1}{h \left( \tau' \right)} d\tau', \quad \rho = 2R_1 \tan \left( \frac{\tau_0}{2} \right) e^{-G(\tau_S)}$$

The constant $\rho$ and the function $G(\tau)$ may be different for each cylinder $\alpha$ and will be written $\rho_\alpha, G^\alpha(\tau)$ respectively when we wish to distinguish them.

The definition (13) of $\mathcal{P}|_{E^\alpha}$ extends naturally to $\mathcal{P}|_{E^\alpha \cup S}$ and agrees with $\mathcal{P}|_S$ of definition
We may now map the entire Euclidean domain $M^E$ into $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$, the one point compactified complex plane with an $S^2$ topology. Recall that for each $\alpha$, $s_\alpha \in S^2$ is the center of the Cap$^\alpha$ (removed in the construction of the manifold $M$). Denote by $s_\alpha^\infty \in S^2$ the point antipodal to $s_\alpha$. For each cylinder $\alpha$ there exists a stereographic projection $\hat{p}^\alpha : S^2 - \{s_\alpha^\infty\} \mapsto \mathbb{C}$ such that $\hat{p}^\alpha(s_\alpha) = 0 \in \mathbb{C}$. We extend this to $\hat{p}^\alpha : S^2 \mapsto \mathbb{C}_\infty$
\[
\hat{p}^\alpha : S^2 \mapsto \mathbb{C}_\infty
\]
\[
: (\theta, \phi) \mapsto 2R_1 e^{i\phi} \tan(\frac{\theta}{2})
\]
\[
: s_\alpha^\infty \mapsto \infty
\]
We now combine these maps $p^\alpha : M^E \mapsto \mathbb{C}_\infty$
\[
p^\alpha = \hat{p}^\alpha \circ \mathcal{P}
\]
to construct the required projection of $M^E$ into $\mathbb{C}_\infty$. If $x \in S \cup E^\alpha = M^E \cap U^\alpha$ then in the $(\tau, \phi)$ chart we have $\Phi^\alpha(x) = (\tau, \phi)$ with $p^\alpha(x) = \rho e^{G(\tau) + i\phi}$. Examples of the maps $p^\alpha$, $\hat{p}^\alpha$ and $\mathcal{P}$ are sketched in Figure 4.

Since $p^\beta p^{\alpha^{-1}} : p^\beta(M^E) \mapsto p^\alpha(M^E)$ is the same as $\hat{p}^\beta \hat{p}^{\alpha^{-1}} : \mathbb{C}_\infty \mapsto \mathbb{C}_\infty$ where both are defined we have a mapping between two different projections of a sphere. Such a mapping may be represented by a special Mobius Transformation representing an $SO(3)$ rotation of
the sphere:

\[
\hat{p}^\beta \hat{p}^{-1}(z) = e^{ia_3^{(\alpha \beta)}} \left(\frac{-2R_1 \tan(\frac{1}{2}a_2^{(\alpha \beta)}) + ze^{ia_1^{(\alpha \beta)}}}{1 + \frac{\tan(\frac{1}{2}a_2^{(\alpha \beta)})ze^{ia_1^{(\alpha \beta)}}}{2R_1}}\right)
\]  

(15)

where \(\{a_1^{(\alpha \beta)}, a_2^{(\alpha \beta)}, a_3^{(\alpha \beta)}\} \in \mathbb{R}^3\),

\[
2R_1 \tan(\frac{1}{2}a_2^{(\alpha \beta)})e^{-ia_1^{(\alpha \beta)}} = \hat{p}^\alpha(s_\beta) \quad \text{and} \quad -2R_1 \tan(\frac{1}{2}a_2^{(\alpha \beta)})e^{ia_3^{(\alpha \beta)}} = \hat{p}^\beta(s_\alpha).
\]

The parameters \(\{a_1^{(\alpha \beta)}, a_2^{(\alpha \beta)}, a_3^{(\alpha \beta)}\}\) represent the Euler angles of the SO(3) rotation. In terms of the coordinates on the sphere, the pair \((a_2^{(\alpha \beta)}, -a_1^{(\alpha \beta)})\) denote the \((\theta, \phi)\) coordinates of the point \(s_\beta\) with respect to the spherical coordinate system about \(s_\alpha\). Similarly \((-a_2^{(\alpha \beta)}, a_3^{(\alpha \beta)})\) are the \((\theta, \phi)\) coordinates of \(s_\alpha\) with respect to the spherical coordinate system about \(s_\beta\).

By looking at the inverse of \(\hat{p}^\alpha \hat{p}^{-1}\beta\) we note that \(a_1^{(\alpha \beta)} = -a_3^{(\alpha \beta)}\), \(a_2^{(\alpha \beta)} = a_2^{(\beta \alpha)}\), and \(a_3^{(\alpha \beta)} = -a_1^{(\beta \alpha)}\). Since we are free to choose the origin of the \(\phi\) coordinate for each \(\alpha\) we may use this freedom to make some of the \(a_1^{(\alpha \beta)}, a_3^{(\alpha \beta)}\) to vanish. \((a_2^{(\alpha \beta)}\) is fixed by the location of holes on the manifold.) For example if there are just two cylinders we may choose \(a_1^{(12)} = 0\) and \(a_3^{(12)} = 0\) by the choice of the origins of \(\phi(1)\) and \(\phi(2)\). For a third cylinder, we may take \(a_3^{(13)} = 0\) by the choice of origin of \(\phi(3)\), but \(a_1^{(23)}, a_3^{(23)}, a_1^{(13)}\) must now be calculated using \(\hat{p}^{(2)} \hat{p}^{(3)-1} = (\hat{p}^{(3)} \hat{p}^{(1)-1}) \circ (\hat{p}^{(1)} \hat{p}^{(2)-1})\) and in general this will be non zero.

5. Construction of Global Solutions

**Theorem (1.1)**

Given the analytic functions \(\psi^\alpha_\pm : p^\alpha(M^E) \mapsto \mathbb{C}\) and a set of constants \(A^\delta, B^\delta \in \mathbb{C}\) for each cylinder \(\delta\) satisfying \(\sum_\delta A^\delta = 0\), then \(\psi|_{M^E} : M^E \mapsto \mathbb{C}\) is a solution to (1) given by

\[
\psi|_{M^E}(x) = \psi_+^\alpha(p^\alpha(x)) + \psi_-^\alpha(p^\alpha(x)) + \sum_{s_{\delta}, s_{\alpha} \neq s_{\delta}} A^\delta \log |p^\alpha(x) - \hat{p}^\alpha(s_{\delta})| + B^\alpha.
\]  

(16)
The sum $\sum_{s_\delta \neq s_\infty^\alpha}$ here is over all caps $\delta$ excluding the cap whose centre is $s_\infty^\alpha$, should it exist.

(This is because if $s_\delta = s_\infty^\alpha$ then $\log |p^\alpha(x) - \hat{p}^\alpha(s_\delta)| = \log |p^\alpha(x) - \hat{p}^\alpha(s_\infty^\alpha)| = \log(\infty).$)

**Theorem (1.2)**

Given periodic functions $\psi_{L\alpha}^\pm : S^1 \mapsto \mathbb{C}$ then in the Lorentzian region of the cylinder $\alpha$, a solution $\psi|_{L^\alpha \cup M^\alpha} : L^\alpha \cup M^\alpha \mapsto \mathbb{C}$ is given in the chart $\Phi^\alpha(x) = (\tau, \phi)$ by

$$\psi|_{L^\alpha \cup M^\alpha}(x) = \psi_+^L(\phi + G^\alpha(\tau)) + \psi_-^L(\phi - G^\alpha(\tau)) - A^\alpha G^\alpha(\tau) + A^\alpha \log(\rho^\alpha) + B^\alpha.$$  \hspace{1cm} (17)

**Theorem (1.3)**

If $\psi^\alpha_\pm, \psi^L_\pm$ satisfy the conditions

$$\psi_+^L(\phi) = \frac{1}{2}(1 + i)(\hat{\psi}_+^\alpha(\rho e^{i\phi}) - i\hat{\psi}_-^\alpha(\rho e^{i\phi}))$$

$$\psi_-^L(\phi) = \frac{1}{2}(1 - i)(\hat{\psi}_+^\alpha(\rho e^{i\phi}) + i\hat{\psi}_-^\alpha(\rho e^{i\phi}))$$  \hspace{1cm} (18)

where

$$\hat{\psi}_\pm^\alpha : p^\alpha(E^\alpha \cup \Sigma^\alpha) \mapsto \mathbb{C}$$

$$\hat{\psi}_\pm^\alpha(\hat{\psi}_\pm^\alpha + \sum_{s_\delta \neq s_\alpha^\infty, s_\infty^\alpha} A^\delta \log |z - \hat{p}^\alpha(s_\delta)| \quad \forall z \in p^\alpha(E^\alpha \cup \Sigma^\alpha)$$  \hspace{1cm} (19)

then $\psi|_{U^\alpha}$ satisfies the conditions (9) and (11). The sum $\sum_{s_\delta \neq s_\alpha^\infty, s_\infty^\alpha}$ here is over all caps $\delta$ excluding the cap $\alpha$ but also excluding the cap whose centre is $s_\infty^\alpha$, should it exist.

**Theorem (1.4)**

Any solution to (1), (9) and (11) can be written in this form.

**Theorem (1.5)**

Under change of coordinates $\Phi^\alpha$ to $\Phi^\beta$

$$\psi_+^\beta \circ p^\beta = \psi_+^\alpha \circ p^\alpha$$

$$\psi_-^\beta \circ p^\beta = \psi_-^\alpha \circ p^\alpha$$  \hspace{1cm} (20)

$$B^\beta = B^\alpha + \sum_{s_\delta \neq s_\alpha^\infty, s_\infty^\beta} A^\delta \log |\hat{p}^\alpha(s_\delta) - \hat{p}^\alpha(s_\infty^\beta)| + A^\beta \log |\hat{p}^\beta(s_\infty^\alpha)| + A^\beta^\infty \log |p^\beta(s_\alpha) - \hat{p}^\beta(s_\infty^\alpha)|.$$
The sum $\sum_{s_\delta \neq s_\alpha, s_\beta}^\infty \delta$ here is over all caps $\delta$ excluding the caps whose centres are either $s_\alpha^\infty$ or $s_\beta^\infty$ should they exist. In the case when $s_\beta^\infty$ is the centre of a cap (say $s_\beta^\infty$ is $s_\delta$) let $A_\beta^\infty$ be the corresponding constant $A_\delta$.

**Proof of Theorem (1.1)**

If $s_\alpha^\infty \in p^\alpha(M^E)$ then since $\sum_\delta A_\delta = 0$, $\psi|_{M^E}$ is a well defined function. Also since $M^E$ is compact, so is $p^\alpha(M^E)$, hence $\psi(p^\alpha(M^E))$ is closed and bounded. Thus $\psi : M \mapsto \mathbb{C}$ is non-singular. In the Euclidean region let $\Phi^\alpha(x) = (\tau, \phi)$ where $x \in M^E \cap U^\alpha = S \cup E^\alpha$. Then $p^\alpha(x) = \rho e^{i\phi + G(\tau)}$. It is straightforward to verify that this satisfies (1) in $M^E \cap U^\alpha$.

**Proof of Theorem (1.2)**

Trivial. ■

**Proof of Theorem (1.3)**

For this part of the proof we drop the $\alpha$ and write $\psi_\pm^\alpha = \psi_\pm$, $\psi_L^\alpha = \psi_L$, $\rho_\alpha = \rho$, $G^\alpha(\tau) = G(\tau)$. In the Euclidean region let $\Phi^\alpha(x) = (\tau, \phi)$ where $x \in M^E \cap U^\alpha = S \cup E^\alpha$. Then

$$\psi|_{M^E \cap U^\alpha}(x) = \psi_+(\rho e^{i\phi + G(\tau)}) + \psi_-(\rho e^{i\phi + G(\tau)}) + \sum_\delta A_\delta \log|\rho e^{i\phi + G(\tau)} - \hat{p}^\alpha(s_\delta)| + B^\alpha.$$

Putting $\tau = \tau_\Sigma$, since $G(\tau_\Sigma) = 0$, the continuity condition (9) is satisfied.

We next pull back this Euclidean solution to $\Sigma^\alpha$. Since $p^\alpha(E^\alpha \cup \Sigma^\alpha)$ is an annulus about $0 \in \mathbb{C}$, $\log(z - \hat{p}^\alpha(s_\delta))$ is a well defined function for all $\delta \neq \alpha$. Thus $\hat{\psi}_\pm$ is well defined and

$$\psi|_{E^\alpha}(x) = \hat{\psi}_+(\rho e^{i\phi + G(\tau)}) + \hat{\psi}_-(\rho e^{i\phi + G(\tau)}) + A^\alpha G(\tau) + A^\alpha \log(\rho) + B^\alpha.$$

The Hodge dulls are

$$\star d\tau = \frac{1}{G'(\tau)} d\phi \quad \text{and} \quad \star d\phi = -G'(\tau) d\tau.$$
In order to prove Theorem (1.4) we need the following lemmas:

Lemma (1.4.1)

Let \( D \subset \mathbb{C}_\infty \) be a closed pathwise connected subset and \( \psi: D \mapsto \mathbb{R} \) a solution of Laplace’s
equation $d \star d\psi = 0$, where $\mathbb{C}$ has the standard $\mathbb{R}^2$ Euclidean metric. Each hole in $D$ is a connected open subset $C^\alpha$, so that $\bigcup_\alpha C^\alpha = \mathbb{C}_\infty - D$ is a disjoint union. For each hole $C^\alpha$ choose $a^\alpha \in C^\alpha$. Then there exists an analytic function $f: D \rightarrow \mathbb{C}$ and constants $A^\alpha \in \mathbb{R}$ such that we can write
\[
\psi(z) = \text{Re}(f(z)) + \sum_\alpha A^\alpha \log(|z - a^\alpha|).
\]

**Proof of Lemma (1.4.1)**

If $D$ were simply connected then writing $z = x + iy$ and $f(x + iy) = \psi(x + iy) + i\varphi(x + iy)$ the Cauchy Riemann equations imply
\[
\frac{\partial \psi}{\partial x} = \frac{\partial \varphi}{\partial y} \quad \text{and} \quad \frac{\partial \psi}{\partial y} = -\frac{\partial \varphi}{\partial x}.
\]
Given $\psi$ we can use these to determine $\varphi$ up to a constant.

Since $D$ is not simply connected, $\varphi$ need not be single valued but can at worst be a function of the winding numbers $n^\alpha \in \mathbb{Z}$ around each hole $C^\alpha$. Suppose $\varphi$ increase by $2\pi n^\alpha A^\alpha$ as it winds round $C^\alpha$ once. By subtracting $A^\alpha \log(z - a^\alpha)$ from $f(z)$ we are left with a single valued function in a neighbourhood of the hole $\alpha$. Hence
\[
\hat{f}(z) \overset{\text{def}}{=} f(z) - \sum_\alpha A^\alpha \log(z - a^\alpha) = \psi(z) + i\varphi(z) - \sum_\alpha A^\alpha \log(z - a^\alpha)
\]
is now single valued. Taking the real part we prove the lemma. ■

**Lemma (1.4.2)**

\[
\sum_\delta A^\delta = 0.
\]

**Proof of Lemma (1.4.2)**

If $\infty \notin D \subset \mathbb{C}_\infty$, then $\infty \in \mathbb{C}_\infty - D$ and $\infty \in C^\delta$ for one of the $\delta$. By considering the
variation of $\varphi$ along a contour $\Gamma$ just inside $C - \delta$, so that $\Gamma$ encloses all the other holes \( \{C^\alpha\}_{\alpha \neq \delta} \) we see that $\sum_{\alpha \neq \delta} A^\alpha = -A^\delta$.

If $\infty \in D \subset \mathbb{C}_\infty$, then $\psi: D \mapsto \mathbb{R}$ is bounded as $z \to \infty$ so again considering a contour that $\Gamma$ encloses all the holes, we obtain the same condition. ■

Lemma (1.4.3)

If $\psi: \mathbb{R} \times S^1 \mapsto \mathbb{R}$ satisfies the hyperbolic equation $d \ast d\psi = 0$, where $\mathbb{R} \times S^1$ has the standard Lorentzian metric, then there exist functions $\psi_\pm: S^1 \mapsto \mathbb{R}$ and a constant $A \in \mathbb{R}$ such that

$$
\psi(\tau, \phi) = \psi_+(\phi + \tau) + \psi_-(\phi - \tau) + A\tau
$$

where $(\tau, \phi)$ are coordinates for $\mathbb{R} \times S^1$.

Proof of Lemma (1.4.3)

Trivial. ■

With these lemmas we return to the proof of Theorem (1.4).

Suppose $\psi: M \mapsto \mathbb{C}$ satisfies equation (1). Since $p^\alpha$ is a conformal mapping, $\psi \circ p^{\alpha^{-1}}: p^\alpha(M^E) \mapsto \mathbb{C}$ satisfies Laplace’s equation. Then $\text{Re}(\psi \circ p^{\alpha^{-1}}), \text{Im}(\psi \circ p^{\alpha^{-1}}): p^\alpha(M^E) \mapsto \mathbb{R}$ each satisfy Laplace’s equation. Also $p^\alpha(M^E)$ is a pathwise connected closed subset of $\mathbb{C}_\infty$.

First assume that $\bar{p}^\alpha(s_\delta) \neq \infty$ for all $s_\delta$. Choose $a^\delta = \bar{p}^\alpha(s_\delta)$. Therefore from lemma (1.4.1) there exist analytic functions $f_1, f_2: p^\alpha(M^E) \mapsto \mathbb{C}$ and constants $A_2^\alpha, A_1^\alpha \in \mathbb{R}$ such that we can write (with $x \in M^E$ and $z = p^\alpha(x)$):

$$
\text{Re}(\psi(x)) = \text{Re}(f_1(z)) + \sum_{s_\delta} A_1^\delta \log(|z - \bar{p}^\alpha(s_\delta)|)
$$

$$
\text{Im}(\psi(x)) = \text{Re}(f_2(z)) + \sum_{s_\delta} A_2^\delta \log(|z - \bar{p}^\alpha(s_\delta)|).
$$
Let
\[ \psi^\alpha_+(z) = f_1(z) + if_2(z) \]
\[ \psi^\alpha_-(z) = f_1(z) - if_2(z) \]
\[ A^\delta = A^\delta_1 + A^\delta_2. \]

Then
\[ \psi|_{\text{M}E}(x) = \psi^\alpha_+(p^\alpha(x)) + \psi^\alpha_-(\hat{p}^\alpha(x)) + \sum_{s^\delta} A^\delta \log |p^\alpha(x) - \hat{p}^\alpha(s^\delta)| + B^\alpha. \]  

(22)

If on the other hand \( s^\alpha = s_\beta \) for some \( \beta \) then we are left with an extra term \( A^\alpha \log |z - a^\beta| \)
where \( z = \hat{p}^\alpha(x) \). However \( \log |z - a^\beta| = \log |1 - a^\beta/z| + \log |z| \) and \( \log(1 - a^\beta/z) \) is analytic
on \( p^\alpha(M^E) \). So (22) becomes
\[ \psi|_{\text{M}E}(x) = \left( \psi^\alpha_+(p^\alpha(x)) + A^\alpha \log(p^\alpha(x)) \right) + \left( \psi^\alpha_-(p^\alpha(x)) + A^\alpha \log(p^\alpha(x)) \right) \]
\[ + \sum_{s^\delta \neq s^\alpha, s^\alpha} A^\delta \log |p^\alpha(x) - \hat{p}^\alpha(s^\delta)| + (A^\alpha + A^\alpha) \log |\hat{p}^\alpha(x)| + B^\alpha \]

and **Theorem (1.4)** is true in the Euclidean region. Furthermore the Lorentzian region on
one cylinder \( L^\alpha \cup M^\alpha \) is conformal to the cylinder \( \mathbb{R} \times S^1 \), so proving **Theorem (1.4)** for
such a region is equivalent to **Lemma (1.4.3)**

Finally we wish to show that the joining conditions (9) and (11), imply the relationship (18).
Let \( A^L, B^L \) be the \( A^\alpha, B^\alpha \) of equation (17) and \( A^E, A^E \) be the \( A^\alpha, B^\alpha \) of equation (16). If
\( \tau = \tau_\Sigma \) then (9) implies
\[ \psi^L_+(\phi) + \psi^L_-(\phi) + B^L = \psi_+(\rho e^{i\phi}) + \psi_+(\rho e^{i\phi}) + B^E. \]  

(23)

Also from (11) we have
\[ \left( \psi^L_+(\phi) + \psi^L_-(\phi) + A^L \right) d\phi = \left( \rho e^{i\phi} \psi_+(\rho e^{i\phi}) + \rho e^{i\phi} \psi_+(\rho e^{i\phi}) + A^E \right) d\phi. \]

Integrating this gives
\[ \psi^L_+(\phi) + \psi^L_-(\phi) + A^L \phi = -i \psi_+(\rho e^{i\phi}) + i \psi_+(\rho e^{i\phi}) + A^E \phi + C \]

and solving these equations we get (18) up to the choice of \( B \).
Proof of Theorem (1.5)

By substituting $z' = p^\beta p^{\alpha-1}(z)$ where $z = p^\alpha(x)$ in (16), the first two of (20) are automatically satified.

If $\mathcal{M} : \mathbb{C}_\infty \mapsto \mathbb{C}_\infty$ is a Mobius transformation then for $a, b \in \mathbb{C}$

$$\frac{a - b}{a - \mathcal{M}(\infty)} = \frac{\mathcal{M}^{-1}(b) - \mathcal{M}^{-1}(a)}{\mathcal{M}^{-1}(b) - \mathcal{M}^{-1}(\infty)}. \quad (24)$$

Substituting $\mathcal{M} = p^\alpha p^{\beta-1}$, $a = p^\alpha(y)$, $b = p^\alpha(x)$ then $\mathcal{M}(\infty) = \hat{p}^\alpha(s_\beta^\infty)$ and $\mathcal{M}^{-1}(\infty) = \hat{p}^\beta(s_\alpha^\infty)$ in this gives

$$\frac{p^\alpha(y) - p^\alpha(x)}{p^\alpha(y) - \hat{p}^\alpha(s_\beta^\infty)} = \frac{p^\beta(x) - p^\beta(y)}{p^\beta(x) - \hat{p}^\beta(s_\alpha^\infty)}.$$

Writing $y = s_\delta$ and taking logs gives the relation:

$$\log |p^\alpha(x) - \hat{p}^\alpha(s_\delta)| = \log |p^\beta(x) - \hat{p}^\beta(s_\delta)| - \log |p^\beta(x) - \hat{p}^\beta(s_\alpha^\infty)| + \log |\hat{p}^\alpha(s_\delta) - \hat{p}^\alpha(s_\beta^\infty)|.$$

This is valid except for the case when $s_\delta \neq s_\alpha^\infty$ and $s_\delta \neq s_\beta^\infty \forall \delta$ since then $\hat{p}^\alpha(s_\alpha^\infty) = \infty$. For this case we first note that

$$(\mathcal{M}(\infty) - b)(\mathcal{M}^{-1}(b) - \mathcal{M}^{-1}(\infty)) = K \quad (25)$$

where $K$ is independent of $b$. Subtracting (25) with $b = p^\alpha(x)$ from (25) with $b = 0$ gives

$$(\hat{p}^\alpha(s_\beta^\infty) - p^\alpha(x))(p^\beta(x) - \hat{p}^\beta(s_\alpha^\infty)) = \hat{p}^\alpha(s_\beta^\infty)(p^\beta(s_\alpha) - \hat{p}^\beta(s_\alpha^\infty))$$

and hence

$$\log |\hat{p}^\alpha(s_\beta^\infty) - p^\alpha(x)| = -\log |p^\beta(x) - \hat{p}^\beta(s_\alpha^\infty)| + \log |\hat{p}^\alpha(s_\beta^\infty)| + \log |p^\beta(s_\alpha^\infty)|.$$
Thus
\[
\sum_{s_{\delta} \neq s_{\parallel}^{\infty}} A^{\delta} \log |p^{\alpha}(x) - \tilde{p}^{\alpha}(s_{\delta})| \\
= \sum_{s_{\delta} \neq s_{\alpha}^{\infty}, s_{\parallel}^{\infty}} A^{\delta} \log |p^{\alpha}(x) - \tilde{p}^{\alpha}(s_{\delta})| + A^{\beta^{\infty}} \log |p^{\alpha}(x) - \tilde{p}^{\alpha}(s_{\parallel}^{\infty})| \\
= \sum_{s_{\delta} \neq s_{\alpha}^{\infty}, s_{\parallel}^{\infty}} A^{\delta} \log |p^{\beta}(x) - \tilde{p}^{\beta}(s_{\delta})| - \sum_{s_{\delta} \neq s_{\alpha}^{\infty}, s_{\parallel}^{\infty}} A^{\delta} \log |p^{\beta}(x) - \tilde{p}^{\beta}(s_{\alpha}^{\infty})| \\
+ \sum_{s_{\delta} \neq s_{\alpha}^{\infty}, s_{\parallel}^{\infty}} A^{\delta} \log |p^{\beta}(x) - \tilde{p}^{\beta}(s_{\parallel}^{\infty})| - A^{\beta^{\infty}} \log |p^{\beta}(x) - \tilde{p}^{\beta}(s_{\alpha}^{\infty})| + A^{\beta^{\infty}} \log |p^{\alpha}(s_{\parallel}^{\infty})| + \\
A^{\beta^{\infty}} \log |p^{\beta}(s_{\alpha}) - \tilde{p}^{\beta}(s_{\alpha}^{\infty})|.
\]

Now \(\psi\) and \(p^{\alpha}(M^{E})\) satify one or other of the conditions for Lemma (1.4.2) \((\sum_{\delta} A^{\delta} = 0)\) so
\[
- \sum_{s_{\delta} \neq s_{\alpha}^{\infty}, s_{\parallel}^{\infty}} A^{\delta} \log |p^{\beta}(x) - \tilde{p}^{\beta}(s_{\alpha}^{\infty})| = (A^{\alpha^{\infty}} + A^{\beta^{\infty}}) \log |p^{\beta}(x) - \tilde{p}^{\beta}(s_{\alpha}^{\infty})| \\
- \sum_{s_{\delta} \neq s_{\alpha}^{\infty}, s_{\parallel}^{\infty}} A^{\delta} \log |p^{\beta}(x) - \tilde{p}^{\beta}(s_{\parallel}^{\infty})| + A^{\beta^{\infty}} \log |p^{\beta}(x) - \tilde{p}^{\beta}(s_{\alpha}^{\infty})| = \sum_{s_{\delta} \neq s_{\alpha}^{\infty}} A^{\delta} \log |p^{\beta}(x) - \tilde{p}^{\beta}(s_{\delta})| \\
\sum_{s_{\delta} \neq s_{\alpha}^{\infty}} A^{\delta} \log |p^{\alpha}(x) - \tilde{p}^{\alpha}(s_{\delta})| = \sum_{s_{\delta} \neq s_{\alpha}^{\infty}} A^{\delta} \log |p^{\beta}(x) - \tilde{p}^{\beta}(s_{\delta})| + \sum_{s_{\delta} \neq s_{\alpha}^{\infty}, s_{\parallel}^{\infty}} A^{\delta} \log |p^{\alpha}(s_{\delta}) - \tilde{p}^{\alpha}(s_{\parallel}^{\infty})| \\
+ A^{\beta^{\infty}} \log |p^{\alpha}(s_{\parallel}^{\infty})| + A^{\beta^{\infty}} \log |p^{\beta}(s_{\alpha}) - \tilde{p}^{\beta}(s_{\alpha}^{\infty})|.
\]
Finally
\[
\sum_{s_{\delta} \neq s_{\alpha}^{\infty}} A^{\delta} \log |p^{\alpha}(x) - \tilde{p}^{\alpha}(s_{\delta})| + B^{\alpha} = \sum_{s_{\delta} \neq s_{\alpha}^{\infty}} A^{\delta} \log |p^{\beta}(x) - \tilde{p}^{\beta}(s_{\delta})| + B^{\beta}
\]
and Theorem (1.3) is satified. }

Observe that in the flat Minkowski regions \(M^{\alpha}\), with \(\Phi^{\alpha}(x) = (\tau, \phi), (17)\) becomes
\[
\psi|x|_{M}^{\alpha} = \psi_{+}^{\alpha} \left( \phi + \frac{\tau - \epsilon}{R_{2}} \right) + \psi_{-}^{\alpha} \left( \phi - \frac{\tau - \epsilon}{R_{2}} \right) - A^{\alpha} \frac{\tau - \epsilon}{R_{2}} + A^{\alpha} \log(\rho_{\alpha}) + B^{\alpha}
\]
where
\[
\epsilon = \epsilon^{\alpha} = G^{\alpha}(\tau_{M}^{\alpha}) - \tau_{M}^{\alpha}.
\]

We also note that the global solution is \(C^{1}\) and piecewise \(C^{\infty}\).
6. Field Energy and Momentum

For a field configuration $\psi$ and a local vector field $X$ on $M$ we define the forms

$$T_X = i_Xd\psi \wedge *d\psi + d\psi \wedge i_X *d\psi.$$  \hspace{1cm} (27)

In Lorentzian regions where $X$ is one of the Killing vectors $\frac{\partial}{\partial \tau}$ or $\frac{\partial}{\partial \phi}$ these identify energy and (angular) momentum density 1-forms [16]. Integrating these over a ring of constant $\tau$ in a Lorentzian region gives an energy and (angular) momentum appropriate to that hypersurface.

$$\mathcal{E}^\alpha(\tau) = \int_0^{2\pi} \left((i_{\frac{\partial}{\partial \tau}}d\psi) *d\psi + d\psi(i_{\frac{\partial}{\partial \tau}} *d\psi)\right)$$

$$\mathcal{L}^\alpha(\tau) = \int_0^{2\pi} \left((i_{\frac{\partial}{\partial \phi}}d\psi) *d\psi + d\psi(i_{\frac{\partial}{\partial \phi}} *d\psi)\right).$$  \hspace{1cm} (28)

We may define similar quantities by integrating over a ring of constant $\tau$ (where it exists) in the Euclidean region. Where the metric has axial symmetry, i.e. on $E^\alpha \cup \Sigma^\alpha \cup L^\alpha \cup M^\alpha$,

$$L \frac{\partial}{\partial \phi} g = 0$$  \hspace{1cm} (29)

where $L$ denotes the Lie derivative. It follows that the momentum $\mathcal{L}^\alpha(\tau)$ is a constant in all regions where the metric is nondegenerate. Furthermore $\frac{\partial}{\partial \tau}$ is also a Killing vector in the flat Lorentzian regions, $M^\alpha$, of the manifold:

$$L \frac{\partial}{\partial \tau} g = 0.$$  \hspace{1cm} (30)

In these regions we therefore have two constants of the motion $\mathcal{E}^{L\alpha}$ and $\mathcal{L}^{L\alpha}$ since (28) are independent of $\tau$. We now compute the forms (28) corresponding to the Lorentzian solutions (17):

$$\mathcal{E}^{L\alpha}(\tau) = 4 \left(\frac{-f(\tau)}{h(\tau)}\right)^\frac{1}{2} \left(\int_0^{2\pi} \psi_+(\phi + G(\tau)) \psi_+^{L\alpha}(\phi + G(\tau)) + \psi_-^{L\alpha}(\phi - G(\tau)) \psi_-^{L\alpha}(\phi - G(\tau)) d\phi + \pi A^\alpha \bar{A}^\alpha\right)$$  \hspace{1cm} (31)
and

\[ \mathcal{L}^{L\alpha} = 4 \int_0^{2\pi} \left( \psi_+ L\alpha^t(\phi - G(\tau)) \overline{\psi_+ L\alpha^t(\phi - G(\tau))} - \psi_- L\alpha^t(\phi + G(\tau)) \overline{\psi_- L\alpha^t(\phi + G(\tau))} \right) d\phi. \]  

(32)

Since each integrand is periodic in \( \phi \) these may be simplified by the substitutions \( \phi \to \phi \pm G(\tau) \):

\[ \mathcal{E}^{L\alpha}(\tau) = 4 \left( -\frac{f(\tau)}{h(\tau)} \right)^{\frac{1}{2}} \left( \int_0^{2\pi} \left( \psi_+ L\alpha^t(\phi) \overline{\psi_+ L\alpha^t(\phi)} + \psi_- L\alpha^t(\phi) \overline{\psi_- L\alpha^t(\phi)} \right) d\phi + \pi A^\alpha \overline{A^\alpha} \right) \]  

(33)

\[ \mathcal{L}^{L\alpha} = 4 \int_0^{2\pi} \left( \psi_- L\alpha^t(\phi) \overline{\psi_- L\alpha^t(\phi)} - \psi_+ L\alpha^t(\phi) \overline{\psi_+ L\alpha^t(\phi)} \right) d\phi. \]  

(34)

For the solution in the Euclidean region \( \mathcal{E}^\alpha \):

\[ \mathcal{E}^{E\alpha}(\tau) = -4 \left( \frac{f(\tau)}{h(\tau)} \right)^{\frac{1}{2}} \text{Re} \left( 2\rho^2 \int_0^{2\pi} \left( \hat{\psi}_+^{\alpha^t}(\rho e^{i\phi + G(\tau)}) \overline{\hat{\psi}_+^{\alpha^t}(\rho e^{i\phi + G(\tau)})} + \hat{\psi}_-^{\alpha^t}(\rho e^{i\phi + G(\tau)}) \overline{\hat{\psi}_-^{\alpha^t}(\rho e^{i\phi + G(\tau)})} e^{2(i\phi + G(\tau))} \right) d\phi + \pi A^\alpha \overline{A^\alpha} \right) \]  

(35)

and

\[ \mathcal{L}^{E\alpha} = 8\rho^2 \text{Im} \left( \int_0^{2\pi} \left( \hat{\psi}_+^{\alpha^t}(\rho e^{i\phi + G(\tau)}) \overline{\hat{\psi}_-^{\alpha^t}(\rho e^{i\phi + G(\tau)})} e^{2(i\phi + G(\tau))} \right) d\phi \right) \]  

(36)

where \( \hat{\psi}_\pm : p^\alpha M^E \mapsto \mathbb{C} \) is defined in (19). Since \( S \) has holes in it this integral cannot be extended to all of \( S \). However these expressions are proportional to the real and imaginary parts of an integral of an analytic function:

\[ \mathcal{L}^\alpha = 8\text{Re}(\mathcal{I}_1^\alpha) \quad \text{and} \quad \mathcal{E}^{E\alpha}(\tau) = \left( \frac{f(\tau)}{h(\tau)} \right)^{\frac{1}{2}} 8\text{Im}(\mathcal{I}_1^\alpha) \]  

(37)

where by substituting \( z = \rho e^{i\phi} \)

\[ \mathcal{I}_1^\alpha = \oint_{|z| = \rho e^{G(\tau)}} \hat{\psi}_+^{\alpha^t}(z) \overline{\hat{\psi}_-^{\alpha^t}(z)} z \, dz + \frac{1}{2} \pi i A^\alpha \overline{A^\alpha}. \]  

(38)

By Cauchy’s theorem this integral is invariant under continuous deformations of the contour and is therefore independent of \( \tau \) in the region \( \mathcal{E}^\alpha \). Thus we may write in this region

\[ \mathcal{I}_1^\alpha = \oint_{|z| = \rho} \hat{\psi}_+^{\alpha^t}(z) \overline{\hat{\psi}_-^{\alpha^t}(z)} z \, dz + \frac{1}{2} \pi i A^\alpha \overline{A^\alpha} \]
so
\[ E^α(τ) = -4 \left( \frac{f(τ)}{h(τ)} \right)^{\frac{1}{2}} \text{Re} \left( 2ρ^2 \int_0^{2\pi} \hat{ψ}_+^{αI}(ρe^{iφ}) \hat{ψ}_-^{αI}(ρe^{iφ}) e^{2iφ} dφ + πA^αA^α \right) \]  
(39)

and
\[ L^α = 8ρ^2 \text{Im} \left( \int_0^{2\pi} \hat{ψ}_+^{αI}(ρe^{iφ}) \hat{ψ}_-^{αI}(ρe^{iφ}) e^{2iφ} dφ \right). \]  
(40)

By substituting (18) into (40) we get (34) and note that
\[ L^α = L^Lα \overset{\text{def}}{=} L^α. \]

Although \( E^α \) has little immediate physical significance we shall refer to it as a pseudo energy in the following.

7. A Conservation Identity

Given any solution in the asymptotic Lorentzian domain of any cylinder it is now possible to use the previous theorem to construct a global solution compatible with our continuity and junction conditions. This is the analogue of solving a Cauchy problem for the field on \( M \). One can then compute the energy and momentum currents in the asymptotic Lorentzian domain of any other cylinder. In general the energy and momentum are not globally conserved. In itself this is not surprising since the field \( ψ \) propagates through gravitational fields in addition to being diffracted through Euclidean domains. By axial symmetry the momentum is always conserved for a topology with a pair of collinear cylinders. We shall also demonstrate below that for topologies containing any number of cylinders, two of which are collinear, solutions, monochromatic in one of these collinear cylinders, give rise to momentum conservation. No similar statements are possible for the energy. However the canonical expressions for the energy and momentum forms are inspired by symmetry considerations that ensure the forms constructed with the aid of Killing vector fields are locally closed and hence locally
conserved. In the presence of metric degeneracy such arguments break down and it is of interest to search for alternative quantities that may be conserved in the global topology under consideration.

Motivated by the pseudo-energy and momentum integrals above define

\[ I_\alpha = \oint_{p(\Sigma^\alpha)} \tilde{\psi}_+^{\alpha'}(z) \tilde{\psi}_-^{\alpha'}(z) z^n \, dz \]

where

\[ \tilde{\psi}_\pm^{\alpha'} : p^\alpha(M^E) \mapsto \mathbb{C} \]

\[ \tilde{\psi}_+^{\alpha'}(z) = \psi_+^{\alpha}(z) + \sum_{s \delta \neq s 0} \frac{1}{2} A^\delta(z - \hat{p}^\alpha(s \delta))^{-1} \]

\[ \tilde{\psi}_-^{\alpha'}(z) = \psi_+^{\alpha}(z) + \sum_{s \delta \neq s 0} \frac{1}{2} A^\delta(z - \hat{p}^\alpha(s \delta))^{-1} \]

are well defined functions.

Also let

\[ L^\alpha = (-R_1 I_0^\alpha + \frac{1}{4R_1} I_2^\alpha) e_1^\alpha - i(R_1 I_0^\alpha + \frac{1}{4R_1} I_2^\alpha) e_2^\alpha + I_1^\alpha e_3^\alpha \in \mathbb{C}^3 \]

where \( \{e_1^\alpha, e_2^\alpha, e_3^\alpha\} \) is a natural orthonormal basis for \( \mathbb{R}^3 \). We embed \( M \) in \( \mathbb{R}^3 \) with the axis of cylinder \( \alpha \) pointing in the direction of \( e_3^\alpha \). The projection of \( L^\alpha \) along \( e_3^\alpha \) is \( I_1^\alpha \) which agrees with definition (38) since

\[ \oint_{p(\Sigma^\alpha)} \tilde{\psi}_+^{\alpha'}(z) \tilde{\psi}_-^{\alpha'}(z) z \, dz = \oint_{p(\Sigma^\alpha)} \left( \tilde{\psi}_+^{\alpha'}(z) + \frac{1}{2} A^\alpha z^{-1} \right) \left( \tilde{\psi}_-^{\alpha'}(z) + \frac{1}{2} \bar{A}^\alpha z^{-1} \right) z \, dz \]

\[ = \oint_{p(\Sigma^\alpha)} \tilde{\psi}_+^{\alpha'}(z) \tilde{\psi}_-^{\alpha'}(z) z \, dz + \frac{1}{2} \pi i A^\alpha \bar{A}^\alpha + \frac{1}{2} \oint_{p(\Sigma^\alpha)} \left( \bar{A}^\alpha \tilde{\psi}_+^{\alpha'}(z) + A^\alpha \tilde{\psi}_-^{\alpha'}(z) \right) \, dz \]

and the last term is zero because \( \psi_\pm^{\alpha'} \) has no term in \( z^{-1} \). The real and imaginary parts of \( I_1^\alpha \) determine the momentum and pseudo-energy by (37).

We assert:

**Theorem 2**

\[ \sum_\alpha L^\alpha = 0 . \]

(41)
**Proof**

A change of chart \( \alpha \to \beta \) induces \( z \to y = M^{-1}(z) = p^\beta p^{\alpha -1}(z) \) hence

\[
\tilde{\psi}^\beta_+ (z) = \frac{d\tilde{\psi}^\beta_+ (z)}{dz} = \frac{d}{dz} (\tilde{\psi}^\alpha_+(y)) = \tilde{\psi}^\alpha_+(y) \frac{dy}{dz} = \tilde{\psi}^\alpha_+(y) \frac{M'(y)}{M'(y)}.
\]

If \( C : \mathbb{R} \to \mathbb{C} \) is any closed curve then

\[
\oint_{p^\beta(C)} \tilde{\psi}^\beta_+(z) \tilde{\psi}^\beta_-(z) dz = \oint_{p^\alpha(C)} \tilde{\psi}^\alpha_+(y) \tilde{\psi}^\alpha_-(y) \frac{M(y)}{M'(y)} dy.
\]

Let \( \{a_1^{(\alpha\beta)}, a_2^{(\alpha\beta)}, a_3^{(\alpha\beta)}\} \) be the Euler angles introduced above so that using (15)

\[
\frac{M(y)}{M'(y)} = y \cos(a_2^{(\alpha\beta)}) + \sin(a_2^{(\alpha\beta)}) \left(-R_1 e^{-ia_1^{(\alpha\beta)}} + \frac{1}{4 R_1} \gamma^2 e^{ia_1^{(\alpha\beta)}}\right).
\]

By a change of basis from \( \{e^\alpha_1, e^\alpha_2, e^\alpha_3\} \) to \( \{e^\beta_1, e^\beta_2, e^\beta_3\} \)

\[
(e^\alpha_1, e^\alpha_2, e^\alpha_3) = (e^\beta_1, e^\beta_2, e^\beta_3) \begin{pmatrix} \cos(a_3^{(\alpha\beta)}) & \sin(a_3^{(\alpha\beta)}) & 0 \\ \sin(a_3^{(\alpha\beta)}) & \cos(a_3^{(\alpha\beta)}) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(a_2^{(\alpha\beta)}) & 0 & \sin(a_2^{(\alpha\beta)}) \\ 0 & 1 & 0 \\ \sin(a_2^{(\alpha\beta)}) & 0 & \cos(a_2^{(\alpha\beta)}) \end{pmatrix} \begin{pmatrix} \cos(a_1^{(\alpha\beta)}) & \sin(a_1^{(\alpha\beta)}) & 0 \\ -\sin(a_1^{(\alpha\beta)}) & \cos(a_1^{(\alpha\beta)}) & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

giving

\[
e^\beta_3 \cdot (e^\beta_1, e^\beta_2, e^\beta_3) = \left( \sin(a_2^{(\alpha\beta)}) \cos(a_1^{(\alpha\beta)}) \, \sin(a_2^{(\alpha\beta)}) \sin(a_1^{(\alpha\beta)}) \, \cos(a_2^{(\alpha\beta)}) \right).
\]

Now if \( f : p^\alpha(M^E) \to \mathbb{C} \) is any analytic function then

\[
\sum_{\alpha} \oint_{p^\beta(\Sigma^\alpha)} f(z) \, dz = 0.
\]

So with \( f(z) = \tilde{\psi}^\beta_+(z) \tilde{\psi}^\beta_-(z) z \)

\[
0 = \sum_{\alpha} \oint_{p^\beta(\Sigma^\alpha)} \tilde{\psi}^\beta_+(z) \tilde{\psi}^\beta_-(z) z \, dz
= \sum_{\alpha} \oint_{p^\alpha(\Sigma^\alpha)} \tilde{\psi}^\alpha_+(z) \tilde{\psi}^\alpha_-(z) \frac{M(z)}{M'(z)} \, dz
= \sum_{\alpha} \oint_{p^\alpha(\Sigma^\alpha)} \tilde{\psi}^\alpha_+(z) \tilde{\psi}^\alpha_-(z) \left( z \cos(a_2^{(\alpha\beta)}) + \sin(a_2^{(\alpha\beta)}) \left(-R_1 e^{-ia_1^{(\alpha\beta)}} + \frac{1}{4 R_1} \gamma^2 e^{ia_1^{(\alpha\beta)}}\right) \right) \, dz
= \sum_{\alpha} T^\alpha_1 \cos(a_2^{(\alpha\beta)}) + \sin(a_2^{(\alpha\beta)}) \left(-R_1 e^{-ia_1^{(\alpha\beta)}} T^\alpha_0 + \frac{1}{4 R_1} \gamma^2 e^{ia_1^{(\alpha\beta)}} T^\alpha_2 \right)
= \sum_{\alpha} T^\alpha_1 \cos(a_2^{(\alpha\beta)}) + \sin(a_2^{(\alpha\beta)}) \cos(a_1^{(\alpha\beta)}) \left(-R_1 T^\alpha_0 + \frac{1}{4 R_1} T^\alpha_2 \right)
\]
\[- \sin(a_2^{(a\beta)}) \sin(a_1^{(a\beta)}) i (R_1 T_0^\alpha + \frac{1}{4m_1} T_2^\alpha) \]

\[= \sum_\alpha \sin(a_2^{(a\beta)}) \cos(a_1^{(a\beta)}) (L^\alpha \cdot e_1^\alpha) + \sin(a_2^{(a\beta)}) (L^\alpha \cdot e_2^\alpha) + \cos(a_2^{(a\beta)}) (L^\alpha \cdot e_3^\alpha) \]

\[= \sum_\alpha (L^\alpha \cdot e_1^\alpha) (e_1^\alpha \cdot e_3^\beta) + (L^\alpha \cdot e_2^\alpha) (e_2^\alpha \cdot e_3^\beta) + (L^\alpha \cdot e_3^\alpha) (e_3^\alpha \cdot e_3^\beta) \]

\[= \left( \sum_\alpha L^\alpha \right) \cdot e_3^\beta \quad \forall \beta. \]

If the set \(\{e_3^\beta\}\) span \(\mathbb{R}^3\) then this implies \(\sum_\alpha L^\alpha = 0\). In the case where the \(\{e_3^\beta\}\) span \(\mathbb{R}^2\) or \(\mathbb{R}^1\) we compare with a topology containing an additional cylinder \(\gamma\) transverse to the existing set. By Cauchy’s theorem \(T_n^\gamma = 0\) for \(n = 0, 1, 2\) so \(L^\gamma = 0\). The expression \((\sum_\alpha L^\alpha) \cdot e_3^\beta = 0\) above becomes \((\sum_\alpha L^\alpha) \cdot e_3^\gamma + L^\gamma \cdot e_3^\gamma = 0\) where now \(\{e_3\}\) span \(\mathbb{R}^3\).

An immediate corollary is that if \(M\) has one cylinder only it cannot admit a regular solution with non-zero momentum.

**Proof:** \(L^\alpha = 0\) hence \(T_1^\alpha = 0\) and \(L^\alpha = 0\).

We can also calculate \(T_\alpha^n\) in the chart \(\Phi^\alpha(x) = (\tau, \phi)\) using the solutions \(\psi_{L^\alpha}^\pm\) on the Lorentzian region \(L^\alpha \cup M^\alpha\) restricted to \(\Sigma^\alpha:\)

\[T_\alpha^n = \frac{1}{4} (\rho_\alpha)^{n-1} \int_0^{2\pi} e^{i\phi(n-1)} (2\psi_{L^\alpha}(\phi) \overline{\psi_{L^\alpha}(\phi)} - 2\psi_{L^\alpha}(\phi) \overline{\psi_{L^\alpha}(\phi)}) \]

\[- 2i\psi_{L^\alpha}(\phi) \overline{\psi_{L^\alpha}(\phi)} - 2i\psi_{L^\alpha}(\phi) \overline{\psi_{L^\alpha}(\phi)}) \]

\[+ \overline{A^\alpha} \left((1 - i) \psi_{L^\alpha}(\phi) + (1 + i) \psi_{L^\alpha}(\phi)\right) + \]

\[A^\alpha \left((1 - i) \psi_{L^\alpha}(\phi) + (1 + i) \psi_{L^\alpha}(\phi)\right) + iA^\alpha \overline{A^\alpha} d\phi. \]

Thus the sum-rule (41) can be seen to correlate properties of the Lorentzian solutions.

8. Constraints on the Manifold for the Propagation of Monochromatic Modes

In the introduction we alluded to the general fact that proving the existence of regular solutions to tensor equations on a manifold is a problem in global analysis. In this section
we construct a particular global solution corresponding to a monochromatic, positive energy, propagating mode in a preferred cylinder. The nature of this construction will clearly indicate what topological constraints must be imposed for it to exist on $M$. A space of solutions of this type is relevant to the construction of the Fock spaces that feature in a field quantisation.

The classical solutions below may be used to construct the Bogolubov coefficients necessary to estimate particle production induced by a degenerate gravitational field.

Let the cylinder $\beta$ be oriented at an angle $a_{\alpha\beta}^2 = \omega$ with the cylinder $\alpha$. In constructing the map between charts adapted to these cylinders we choose the Euler angles \( \{a_{\alpha\beta}^1, a_{\alpha\beta}^2, a_{\alpha\beta}^3\} \) with $a_{\alpha\beta}^1 = 0$ and $a_{\alpha\beta}^3 = 0$. The required Mobius transformation is then given by:

\[
p^{\beta}_{\alpha} p^{\alpha^{-1}}(z) = \frac{-2R_1 \tan\left(\frac{\omega}{2}\right) + z}{1 + \frac{\tan\left(\frac{\omega}{2}\right)z}{2R_1}}. \tag{43}
\]

Introduce the metric constants $\xi_{\alpha}$ and $\xi_{\beta}$ by the equations $\rho_{\alpha} = 2R_1 \tan(\xi_{\alpha}/2)$, $\rho_{\beta} = 2R_1 \tan(\xi_{\beta}/2)$ where $\rho_{\alpha}$ and $\rho_{\beta}e$ are defined by (14). Also the constants $\epsilon_{\alpha}$, $\epsilon_{\beta}$ follow from (26). On $(\tau, \phi) \in \Phi^\alpha(L^\alpha \cup M^\alpha)$ define the null coordinates

\[
\gamma_{\pm}^\alpha = \phi \pm G^\alpha(\tau) \quad \text{such that} \quad \gamma_{\pm}^\alpha = \phi \pm \frac{\tau - \epsilon_{\alpha}}{R_2^\alpha} \text{ on } M^\alpha \tag{44}
\]

and similarly for $\beta$. For $x \in L^\alpha \cup M^\alpha$ with $(\tau, \phi) = \Phi^\alpha(x)$ let the solution to the wave equation be the left-moving single mode with frequency $k \in \mathbb{Z}^+$:

\[
\psi^\alpha_{L} |_{L^\alpha \cup M^\alpha}(x) = e^{ik\gamma_{+}^\alpha}.
\]

Therefore from (18), $\psi_{+}^{L\alpha}(\gamma_{+}^\alpha) = e^{ik\gamma_{+}^\alpha}$ and $\psi_{+}^{L\alpha}(\gamma_{-}^\alpha) = 0$ so

\[
\psi_{+}^{L\alpha}(\gamma) = \frac{1}{2}(1 + i) \left( \psi_{+}^{\alpha}(\rho_{\alpha} e^{i\gamma}) - i \psi_{-}^{\alpha}(\rho_{\alpha} e^{i\gamma}) \right) = e^{ik\gamma} \\
\psi_{-}^{L\alpha}(\gamma) = \frac{1}{2}(1 - i) \left( \psi_{+}^{\alpha}(\rho_{\alpha} e^{i\gamma}) + i \psi_{-}^{\alpha}(\rho_{\alpha} e^{i\gamma}) \right) = 0. \tag{45}
\]
Thus from (18), with $z$

Solving these we get
\[ \psi^\alpha_+(\rho \alpha e^{i\gamma}) = \frac{1}{2} (1 - i) e^{i k \gamma} \]
\[ \psi^\alpha_-(\rho \alpha e^{i\gamma}) = \frac{1}{2} (1 - i) e^{-i k \gamma}. \]

Analytically continuing these to the domain $\mathcal{P}(M^E)$ yields
\[ \psi^\alpha_+(z) = \frac{1}{2} (1 - i) \rho^k \alpha z^k \]
\[ \psi^\alpha_-(z) = \frac{1}{2} (1 - i) \rho^k \alpha z^{-k}. \]

Hence there are no log terms for this particular solution. Furthermore since $\psi^\alpha_+ \sim z^k$ then $s^\alpha_\infty \notin \mathcal{P}(M^E)$. In order that this solution be regular we have excised the point $z = \infty$. This is equivalent to requiring that a cylinder $\beta$ exist such that $\mathcal{P}(\Sigma^\beta)$ enclose $s^\alpha_\infty$.

In order to describe the above solution on any $\beta$ cylinder we must transform it to $(U^\beta, \phi^\beta)$ coordinates. From (20) we have
\[ \psi_{L^\beta \cup M^\beta}(x) = \psi_{+}^{L^\beta}(\gamma^\beta_+) + \psi_{-}^{L^\beta}(\gamma^\beta_-) \]
\[ = \frac{1}{2} (1 + i) \left( \psi^\beta_+(\rho \beta e^{i\gamma^\beta_+}) - i \psi^\beta_-(\rho \beta e^{i\gamma^\beta_+}) \right) + \frac{1}{2} (1 - i) \left( \psi^\beta_+(\rho \beta e^{i\gamma^\beta_-}) + i \psi^\beta_-(\rho \beta e^{i\gamma^\beta_-}) \right) \]
\[ = \frac{1}{2} \left( \frac{-\tan(\frac{1}{2} \omega) + \tan(\frac{1}{2} \xi^\beta) e^{i \gamma^\beta_+}}{\tan(\frac{1}{2} \xi^\beta)(1 + \tan(\frac{1}{2} \omega) \tan(\frac{1}{2} \xi^\beta) e^{i \gamma^\beta_+})} \right)^k + \frac{1}{2} \left( \frac{\tan(\frac{1}{2} \xi^\beta)(1 + \tan(\frac{1}{2} \omega) \tan(\frac{1}{2} \xi^\beta) e^{-i \gamma^\beta_+})}{-\tan(\frac{1}{2} \omega) + \tan(\frac{1}{2} \xi^\beta) e^{-i \gamma^\beta_+}} \right)^k \]
\[ - \frac{i}{2} \left( \frac{-\tan(\frac{1}{2} \omega) + \tan(\frac{1}{2} \xi^\beta) e^{i \gamma^\beta_-}}{\tan(\frac{1}{2} \xi^\beta)(1 + \tan(\frac{1}{2} \omega) \tan(\frac{1}{2} \xi^\beta) e^{i \gamma^\beta_-})} \right)^k + \frac{i}{2} \left( \frac{\tan(\frac{1}{2} \xi^\beta)(1 + \tan(\frac{1}{2} \omega) \tan(\frac{1}{2} \xi^\beta) e^{-i \gamma^\beta_-})}{-\tan(\frac{1}{2} \omega) + \tan(\frac{1}{2} \xi^\beta) e^{-i \gamma^\beta_-}} \right)^k. \]

We have seen that for this solution to exist the topology must contain at least one cylinder attached to the cap containing the point $s^\alpha_\infty$. (It is interesting to note that this constraint
can be relaxed for standing wave solutions in an asymptotically flat Lorentzian domain.)

We may use theorem 2 to calculate the momentum associated with the solution on this
cylinder. Since \( T_1^\alpha = -8k^2\pi \), \( T_0^\alpha = 0 \) and \( T_2^\alpha = 0 \) then \( L^\alpha = -8k^2\pi e_3^\alpha \). Hence \( T_1^\beta = L^\beta \cdot e_3^\beta = (-L^\alpha) \cdot e_3^\beta = 8k^2\pi e_3^\alpha \cdot e_3^\beta = -8k^2\pi \cos(\omega) \) and the momentum in cylinder \( \beta \) is \( L^\beta = 8k^2\pi \cos(\omega) \). As \( L^\alpha = -8k^2\pi \) the momentum is conserved for \( \omega = \pi \), i.e. collinear cylinders.

The momentum associated with this solution in any cylinder other than these two is zero by
Cauchy’s theorem.

For collinear cylinders, \( \omega = \pi \), the solution becomes, with \((\tau, \phi) = \Phi^\beta(x)\):

\[
\psi_{L^\beta M^\beta}(x) = \frac{(-1)^k}{2} e^{-ik\gamma^\beta} \left( (\tan(\frac{1}{2}\xi_\alpha) \tan(\frac{1}{2}\xi_\beta))^k + (\tan(\frac{1}{2}\xi_\alpha) \tan(\frac{1}{2}\xi_\beta))^{-k} \right)
+ i \frac{(-1)^k}{2} e^{-ik\gamma^\beta} \left( (\tan(\frac{1}{2}\xi_\alpha) \tan(\frac{1}{2}\xi_\beta))^k - (\tan(\frac{1}{2}\xi_\alpha) \tan(\frac{1}{2}\xi_\beta))^{-k} \right). \tag{48}
\]

We can only remove the right-moving wave term, \( e^{-ik\gamma^\beta} \), if \( \tan(\frac{1}{2}\xi_\alpha) \tan(\frac{1}{2}\xi_\beta) = \pm 1 \), i.e. \( \xi_\alpha + \xi_\beta = \pi \), which means that the \( \Sigma \)-rings coincide and there is no Euclidean region.

The solution \( \psi_{L^\beta} \) diverges when either \( \omega = \xi_\beta \) or \( \omega + \xi_\beta = \pi \). The first of these implies that \( p^\beta \Sigma^\beta \) contains \( s_\alpha \) and the second implies that \( p^\beta \Sigma^\beta \) contains \( s_\alpha^\infty \). Since the cylinders have
non-zero radii these cannot occur.

The energy of the solution (47) can be computed using (33). For \( k = 1 \) one finds:

\[
\mathcal{E}^{L^\beta}(\tau) = 4\pi \sin^2(\xi_\beta) \left( \frac{-f(\tau)}{h(\tau)} \right)^{\frac{1}{2}} \left( \frac{(1 + \cos(\omega) \cos(\xi_\beta))(1 + \cos(\xi_\alpha))}{\cos(\omega) + \cos(\xi_\beta)^3 (1 - \cos(\xi_\alpha))} + \frac{(1 - \cos(\omega) \cos(\xi_\beta))(1 - \cos(\xi_\alpha))}{\cos(\omega) - \cos(\xi_\beta)^3 (1 + \cos(\xi_\alpha))} \right) \tag{49}
\]
which is manifestly positive definite.
9. Constraints for the Existence of Solutions on a Compact Manifold

Here we construct regular solutions in the simplest compact manifold with an axially symmetric degenerate metric. It may be visualised as a pair of Euclidean spheres connected by a single cylinder. See the bone Figure 5. The metric becomes degenerate on a pair of rings in the cylinder and partitions its geometry so that the middle section is Lorentzian. Such a manifold may be viewed as a simple model of a signature changing closed cosmology.

**Theorem 3**

The Lorentzian region of $M$ will admit monochromatic standing wave solutions to (1) of frequency $k \in \mathbb{Z}$ if and only if $\epsilon$ satisfies the condition:

$$\cos(k\epsilon) = 0.$$ (50)

where $\epsilon$ is a real parameter determined by the metric (equation (51) below).

**Proof**

Let $M$ have the $S^2$ topology above where $\Sigma^1$, $\Sigma^2$ denote two non-intersecting rings where the metric changes signature. Thus $M$ is $E^1 \cup \Sigma^1 \cup L \cup \Sigma^2 \cup E^2$ where $E^\alpha$ are the Euclidean domains and $L$ is the Lorentzian domain. Let $(U^\alpha, \Phi^\alpha)$ be charts for $U^\alpha = E^\alpha \cup \Sigma^\alpha \cup L$ where

$$\Phi_\alpha : U^\alpha \mapsto \mathbb{R}^2 \quad \alpha = 1, 2.$$ $x \mapsto (\tau_\alpha, \phi_\alpha).$

We note that the $\tau$ coordinates induce opposite coordinate time orientations on $L$. The differential structure $(\Phi_1) \circ (\Phi_2)^{-1} : (\tau_2, \phi_2) \mapsto (\tau_1, \phi_1)$ is given by the equations:

$$\tau_1 + \tau_2 = \tau_0$$ $$\phi_1 + \phi_2 = \phi_0.$$

This differential structure is compatible with $M$ being orientable.
It is now convenient to write

\[ G^\alpha : \tau_\alpha (\Phi_\alpha (U^\alpha)) \mapsto \mathbb{R} \]

\[ : \tau_\alpha \mapsto \int_{\tau_\alpha (\Sigma^\alpha)} \left| \frac{-f^1(\tau')}{h^1(\tau')} \right|^2 d\tau' \]

where \( \{(\tau_\alpha, \phi_\alpha) : \tau_\alpha = \tau_\alpha (\Sigma^\beta)\} = \Phi^\alpha (\Sigma^\beta) \). For \( x \in U^1 \cap U^2 \)

\[ G^1(\tau_1(x)) + G^2(\tau_2(x)) = \int_{\tau_1 (\Sigma^1)} \left( \frac{-f(t_1)}{h(t_1)} \right)^2 dt_1 + \int_{\tau_2 (\Sigma^2)} \left( \frac{-f(t_2)}{h(t_2)} \right)^2 dt_2 \]

\[ = \int_{\tau_1 (\Sigma^1)} \left( \frac{-f(t_1)}{h(t_1)} \right)^2 dt_1 - \int_{\tau_1 (\Sigma^1)} \left( \frac{-f(t_1)}{h(t_1)} \right)^2 dt_1 \]

\[ = \epsilon < 0 \]

where \( t_2 = \tau_0 - t_1 \). From theorem (1.2) the solution \( \psi|_L \) in the Lorentzian region is given as

\[ \psi|_L (x) = \psi^L_+ (\phi_1 + G^1(\tau_1)) + \psi^L_1 (\phi_1 - G^1(\tau_1)) = \psi^L_+ (\phi_2 + G^2(\tau_2)) + \psi^L_- (\phi_2 - G^2(\tau_2)) \]

\[ = \psi^L_+ (\phi_0 + \epsilon - \phi_1 - G^2(\tau_1)) + \psi^L_- (\phi_0 - \epsilon - \phi_1 + G^2(\tau_1)). \]

The absence of any log terms is demanded by regularity.

Since \( \phi_1 + G^1(\tau_1) \) and \( \phi_1 - G^1(\tau_1) \) are independent

\[ \psi^L_+ (\phi) = \psi^L_+ (\phi_0 + \epsilon - \phi) \quad \forall \phi, \]

\[ \psi^-_L (\phi) = \psi^-_- (\phi_0 - \epsilon - \phi) \]

The Euclidean solutions in \( E^1 \) and \( E^2 \) will be constrained to match this Lorentzian solution.

From theorem 1.3 the Euclidean solutions on \( \Sigma^1 \) and \( \Sigma^2 \) satisfy:

\[ \psi^2_+ (\rho_2 e^{i\phi}) = \frac{i}{2} \psi^1_+ (\rho_1 e^{i(\phi_0 + \epsilon - \phi)}) + \frac{i}{2} \psi^1_+ (\rho_1 e^{i(\phi_0 - \epsilon - \phi)}) + \frac{i}{2} \psi^1_+ (\rho_1 e^{i(\phi_0 + \epsilon - \phi)}) - \frac{i}{2} \psi^1_+ (\rho_1 e^{i(\phi_0 - \epsilon - \phi)}) \]

\[ \psi^2_- (\rho_2 e^{i\phi}) = \frac{i}{2} \psi^1_+ (\rho_1 e^{i(\phi_0 + \epsilon - \phi)}) - \frac{i}{2} \psi^1_+ (\rho_1 e^{i(\phi_0 - \epsilon - \phi)}) + \frac{i}{2} \psi^1_+ (\rho_1 e^{i(\phi_0 + \epsilon - \phi)}) + \frac{i}{2} \psi^1_+ (\rho_1 e^{i(\phi_0 - \epsilon - \phi)}). \]

It is convenient to write \( \overline{\psi}_\pm (z) \) as \( \tilde{\psi}_\pm (z) \). (Observe \( \tilde{\psi}_\pm \) are analytic in \( z \).)
The analytic continuation of (52) may be written:

\[
\psi_+^2(z) = \frac{1}{2} \psi_+^1 \left( \frac{\rho_1 \rho_2 e^{i(\phi_0 - \epsilon)}}{z} \right) + \frac{i}{2} \psi_+^1 \left( \frac{\rho_1 \rho_2 e^{i(\phi_0 + \epsilon)}}{z} \right) + \frac{i}{2} \psi_+^1 \left( \frac{\rho_1 z}{\rho_2 e^{i(\phi_0 - \epsilon)}} \right) - \frac{i}{2} \psi_+^1 \left( \frac{\rho_1 z}{\rho_2 e^{i(\phi_0 + \epsilon)}} \right)
\]

\[
\psi_-^2(z) = \frac{1}{2} \psi_-^1 \left( \frac{\rho_1 \rho_2 e^{i(\phi_0 - \epsilon)}}{z} \right) + \frac{i}{2} \psi_-^1 \left( \frac{\rho_1 \rho_2 e^{i(\phi_0 + \epsilon)}}{z} \right) - \frac{i}{2} \psi_-^1 \left( \frac{\rho_1 z}{\rho_2 e^{i(\phi_0 - \epsilon)}} \right) - \frac{i}{2} \psi_-^1 \left( \frac{\rho_1 z}{\rho_2 e^{i(\phi_0 + \epsilon)}} \right).
\]

(53)

We now express the Euclidean solutions as the general Laurent expansions

\[
\psi_+^1(z) = \sum_{k \leq 0} A_k^1 z^k \quad \psi_+(z) = \sum_{k \leq 0} B_k^1 z^k \quad \psi_+^2(z) = \sum_{k \leq 0} A_k^2 z^k \quad \psi_-^2(z) = \sum_{k \leq 0} B_k^2 z^k.
\]

(54)

The integers \( k \leq 0 \) since \( \psi_+^a \) must be bounded as \( z \to \infty \). The functions \( \psi_+^a \) are not required to be bounded as \( z \to 0 \), since \( p^a S^a \subset \mathbb{C} = \mathbb{C} - D \) where \( D \) is a disc at the origin.

Substituting these expansions into (53) yields

\[
\sum_{k \leq 0} A_k^2 z^k = -\sum_{k \leq 0} B_k^1 \frac{\rho_1^k}{\rho_2^k} z^k e^{-ik\phi_0} \sin(k\epsilon) + \sum_{k \leq 0} A_k^1 \frac{\rho_1^k \rho_2^k}{z^k} e^{ik\phi_0} \cos(k\epsilon)
\]

\[
\sum_{k \leq 0} B_k^2 z^k = \sum_{k \leq 0} A_k^1 \frac{\rho_1^k}{\rho_2^k} z^k e^{-ik\phi_0} \sin(k\epsilon) + \sum_{k \leq 0} B_k^1 \frac{\rho_1^k \rho_2^k}{z^k} e^{ik\phi_0} \cos(k\epsilon).
\]

Hence comparing coefficients of \( z \) we deduce \( \cos(k\epsilon) = 0 \) and

\[
A_k^2 = -\frac{B_k^1 \rho_1^k}{\rho_2^k} e^{-ik\phi_0} \sin(k\epsilon) \quad \text{and} \quad B_k^2 = \frac{A_k^1 \rho_2^k}{\rho_1^k} e^{-ik\phi_0} \sin(k\epsilon).
\]

For a given \( \epsilon \) we may now write the solution in the Lorentzian region as

\[
\psi|_L((\mathcal{G}_1)^{-1}(\tau, \phi)) = \sum_{k \leq 0} \rho_1^k \left( A_k^1 e^{ik\phi} + B_k^1 e^{-ik\phi} \right) \left( \cos(kG_1(\tau)) - \sin(kG_1(\tau)) \right)
\]

(55)

where for convergence

\[
\sum_{k \leq 0} \rho_\alpha^k \left( |A_k^\alpha| + |B_k^\alpha| \right) < \infty \quad \alpha = 1, 2,
\]

and it is understood that only those modes satisfying (50) are to be included in the sums above. We note that (50) admits no (non-constant) solution for \( \epsilon = 0 \). This is consistent with Louville’s theorem.
Also for (50) to admit solutions $\epsilon/\pi$ must be rational and
\[
\frac{\epsilon}{\pi} = \frac{2m + 1}{2k}
\]
where $m, k \in \mathbb{Z}$.

Furthermore, given $\epsilon$, if $k$ is a solution to (50) so is $(2m + 1)k$ for all $m \in \mathbb{Z}$. From the corollary to theorem 2 it follows that all solutions for this manifold have zero momentum, $\mathcal{L} = 0$. By direct computation, the energy associated with (55) is:
\[
\mathcal{E}^L(\tau) = \left(\frac{-f^1(\tau)}{h^1(\tau)}\right)^{\frac{1}{2}} \sum_{k \leq 0} 4\pi k^2 \rho_1 2^k (|A_k^1|^2 + |B_k^1|^2)
\]
where the summation is restricted as above.

10. Discussion

We have analysed the massless wave equation on a class of two dimensional manifolds with smooth degenerate metrics. We have drawn particular attention to the interplay between the topological structure of the manifold and the existence of regular solutions. Such solutions are piecewise smooth but globally $C^1$. The difference between solutions on compact and non-compact manifolds has been stressed and the effects of the topology on the currents induced by the Killing symmetries of the metric have been explicitly calculated. We have also introduced a sum rule that may be interpreted as a conservation law for a new type of momentum in the presence of signature change.

Topology change in two dimensional field theories is relevant in a number of contemporary problems. String field theory interactions proceed by processes in which the classical two dimensional world sheet exhibits a change in topology. Indeed Figure 1 may be regarded as a non-compact immersion in a spacetime describing the interaction of a set of closed strings. In such a description the metric on the world sheet is dynamically induced by an extremal immersion. However if the immersion is in a spacetime of Lorentzian signature then the
induced metric cannot be non-degenerate. In order to accommodate the change in signature of the induced metric it is necessary to confront the problems discussed in this paper.

The above results pertain to two dimensional manifold with genus zero. However it is straightforward in principle to extend these techniques to the case of non-zero genus. One may also expect that the general features discussed above will have analogues in higher dimension. Such results can then be interpreted as the effects of spacetime topology on the propagation of matter in the presence of signature change. [17] [18] [19] [20] [21] [22] [23] [24] [25]

If changes of signature can occur on a Planckian scale then it is imperative to understand how to extend standard quantum field theory to such topologically non-trivial backgrounds. We have surmised elsewhere [26], [15] that asymptotically flat Lorentzian domains that are connected via a Euclidean domain may induce matter interactions that can be interpreted as particle creation by analogy with particle creation by localised gravitational curvature. The results in this paper are an attempt to provide a rigorous background for the formulation of such a field theory of topologically induced particle interactions.

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REFERENCES

1. S J Avis, C J Isham, Recent Developments in Gravitation, Ed. M Lèvy, S Deser, Cargèse, Plenum Press, 1978
2. A Ashketar, Lectures on Non-Perturbative Canonical Gravity, Advanced series in Astrophysics and Cosmology, Vol 6, World Scientific, 1991
3. J S Dowker, J Phys. A 10 (1967) 115, 5 (1971) 1375
4. M M Smirnov, Trans. Amer. Math. Monographs (Amer. Math. Soc) 51 1957
5. R Gardner, N Kamran, Hyperbolic Equations in the plane.
6. D H Hartley, E D Fackerell R W Tucker, An Obstruction to the Integrability of a Class of Non-Linear Wave Equations by 1-Stable Cartan Characteristics, J Diff Equations, To Appear.
7. D H Hartley, P Tuckey, R W Tucker, Equivalence of Darboux and Gardner methods for Integrating hyperbolic equations in the Plane, Duke Math. J. To Appear
8. W Kundt Comm. Math. Phys. 4 (1967) 143
9. R Geroch, J. Math. Phys. 11 (1970) 437
10. H Ishikawa, Prog. Theor. Phys. 57 (1977) 339
11. H Ishikawa, J. Math. Phys. Phys. 18 (1977) 2375
12. F J Tipler Ann. Phys. 108 (1977) 1
13. S Hawking, R K Sachs, Comm. Math. Phys. 35 (1974) 287
14. Th Bröcker, K Jänich, Introduction to Differential Topology, CUP (1982)
15. T D Dray, C A Manogue, R W Tucker, Phys. Rev. D48 (1993) 2587
16. I M Benn, R W Tucker, An Introduction to Geometry and Spinors with Applications in Physics. Adam Hilger (1988)
17. G Ellis, A Sumeruk, D Coule, C Hellaby Class. Q. Grav. 1535 9 1992
18. T Dereli, Önder, R W Tucker, Phys. Lett. B324 (1994) 143
19. T Dereli, R W Tucker, Class. Quantum Grav. 10 (1993) 365
20. T Dereli, Önder, R W Tucker, Class. Quantum Grav. 10 (1993) 1425
21. S S Hayward, Class. Q. Grav. L7 10 1993
22. M Kossowski, M Kriele Class. Q. Grav. 1157 10 1993
23. M Kossowski, M Kriele Class. Q. Grav. 2363 10 1993
24. M Kossowski, M Kriele Proc. Roy. Sc. Lond. 297 A444 1994
25. R Kerner, J Martin Class. Q. Grav. 2111 10 1993
26. T Dray, C A Manogue, R W Tucker, Gen. Rel. Grav. 23 (1991) 967
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Figure 1. The Manifold.
Figure 2. The Chart $(U^\alpha, \Phi^\alpha)$
Figure 3. The Metric Functions $f$ and $h$. 
Figure 4. The Effects of $P$, $\hat{p}^1$ and $p^1$
Figure 5. The Bone