Chiral Anomaly for a New Class of Lattice Dirac Operators

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Abstract
A new class of lattice Dirac operators which satisfy the index theorem have been recently proposed on the basis of the algebraic relation
\[ \gamma_5(\gamma_5 D) + (\gamma_5 D)\gamma_5 = 2a^{2k+1}(\gamma_5 D)^{2k+2} \]
Here \( k \) stands for a non-negative integer and \( k = 0 \) corresponds to the ordinary Ginsparg-Wilson relation. We analyze the chiral anomaly and index theorem for all these Dirac operators in an explicit elementary manner. We show that the coefficient of anomaly is independent of a small variation in the parameters \( r \) and \( m_0 \), which characterize these Dirac operators, and the correct chiral anomaly is obtained in the (naive) continuum limit \( a \to 0 \).

1 Introduction
A new class of lattice Dirac operators \( D \) have been recently proposed on the basis of the algebraic relation[1]
\[ \gamma_5(\gamma_5 D) + (\gamma_5 D)\gamma_5 = 2a^{2k+1}(\gamma_5 D)^{2k+2} \]  
(1.1)
where \( k \) stands for a non-negative integer, and \( k = 0 \) corresponds to the ordinary Ginsparg-Wilson relation[2] for which an explicit example of the operator free of species doubling has been given by Neuberger[3]. It has been shown in [1] that we can in fact construct the lattice Dirac operator, which is free of species doublers, for all values of \( k \). Here \( \gamma_5 \) is a hermitian chiral Dirac matrix and \( \gamma_5 D \) is also hermitian.

When one defines
\[ \Gamma_5 \equiv \gamma_5 - (a\gamma_5 D)^{2k+1} \]  
(1.2)
the relation (1.1) is written as
\[ \Gamma_5(\gamma_5 D) + (\gamma_5 D)\Gamma_5 = 0. \]  
(1.3)
The index relation[4][5] on the lattice is generally written as
\[ Tr\Gamma_5 = n_+ - n_- , \]  
(1.4)
which is confirmed by[1]
\[ Tr\Gamma_5 \equiv \sum_{\lambda_n} \phi_n^\dagger \Gamma_5 \phi_n \]
\[
\sum_{\lambda_n=0} \phi_n^\dagger \Gamma_5 \phi_n + \sum_{\lambda_n \neq 0} \phi_n^\dagger \Gamma_5 \phi_n \\
= \sum_{\lambda_n=0} \phi_n^\dagger \Gamma_5 \phi_n \\
= \sum_{\lambda_n=0} \phi_n^\dagger [\gamma_5 - (a \gamma_5 D)^{2k+1}] \phi_n \\
= n_+ - n_- = \text{index} \quad (1.5)
\]

where \( n_\pm \) stand for the number of normalizable zero modes in

\[\gamma_5 D \phi_n = 0 \quad (1.6)\]

for the hermitian operator \( \gamma_5 D \) with simultaneous eigenvalues \( \gamma_5 \phi_n = \pm \phi_n \). We also used the relation following from (1.3)

\[\gamma_5 D \Gamma_5 \phi_n = -\lambda_n \Gamma_5 \phi_n \quad (1.7)\]

if

\[\gamma_5 D \phi_n = \lambda_n \phi_n \quad (1.8)\]

which suggests that either \( \Gamma_5 \phi_n \) for \( \lambda_n \neq 0 \) is orthogonal to \( \phi_n \) or else \( \Gamma_5 \phi_n = 0 \). The positive definite inner product is defined by summing over all the lattice points

\[\phi_n^\dagger \phi_n = (\phi_n, \phi_n) \equiv \sum_x a^4 \phi_n^*(x) \phi_n(x) \quad (1.9)\]

but the coordinate \( x \) is often omitted in writing \( \phi_n \).

The Euclidean path integral for a fermion is defined by

\[\int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp[\int \bar{\psi} D \psi] \quad (1.10)\]

where

\[\int \bar{\psi} D \psi \equiv \sum_{x,y} \bar{\psi}(x) D(x,y) \psi(y) \quad (1.11)\]

and the summation runs over all the points on the lattice. The relation (1.3) is re-written as

\[\gamma_5 \Gamma_5 \gamma_5 D + D \Gamma_5 = 0 \quad (1.12)\]

and thus the Euclidean action is invariant under the global “chiral” transformation[5]

\[\bar{\psi}(x) \to \bar{\psi}'(x) = \bar{\psi}(x) + i \sum_z \bar{\psi}(z) \epsilon \gamma_5 \Gamma_5(z, x) \gamma_5 \]

\[\psi(y) \to \psi'(y) = \psi(y) + i \sum_w \epsilon \Gamma_5(y, w) \psi(w) \quad (1.13)\]

with an infinitesimal constant parameter \( \epsilon \). Under this transformation, one obtains a Jacobian factor

\[\mathcal{D}\bar{\psi}' \mathcal{D}\psi' = J \mathcal{D}\bar{\psi} \mathcal{D}\psi \quad (1.14)\]
with
\[
J = \exp[-2iT\text{Re}\Gamma_5] = \exp[-2i\epsilon(n_+ - n_-)]
\]  
(1.15)

where we used the index relation (1.5). This derivation may be regarded as a lattice counter part of the continuum path integral[6].

In Ref.[1] it was shown by using the method in [7], which is a lattice extension of the method in [6], that the index \(n_+ - n_-\) appearing in the Jacobian factor is related to the Pontryagin number for any operator in (1.1) if the operator \(\gamma_5D\) satisfies suitable conditions. In this paper, we evaluate \(\text{Tr}\Gamma_5\) in a more explicit and elementary manner on the basis of explicit formulas for \(\gamma_5D\) in the continuum limit[1]. We show that these operators \(\gamma_5D\) for all \(k\) in fact reproduce the correct chiral anomaly and consequently correct Pontryagin number.

2 A brief summary of the model and notation

The operator \(\Gamma_5\) appearing in the index relation (1.5) has an explicit expression[1]
\[
\Gamma_5 = \gamma_5 - H^{(2k+1)}
\]  
(2.1)

with
\[
H^{(2k+1)} \equiv (\gamma_5aD)^{2k+1} = \frac{1}{2}\gamma_5[1 + D_W^{(2k+1)} - \frac{1}{\sqrt{(D_W^{(2k+1)})^\dagger D_W^{(2k+1)}}}].
\]  
(2.2)

The operator \(D_W^{(2k+1)}\) is in turn expressed as a generalization of the ordinary Wilson Dirac operator as
\[
D_W^{(2k+1)} = i(\mathcal{C})^{2k+1} + (B)^{2k+1} - (\frac{m_0}{a})^{2k+1}.
\]  
(2.3)

See Appendix for further details of the general solution to (1.1).

The ordinary Wilson Dirac operator \(D_W\), which corresponds to \(D_W^{(1)}\), is given by
\[
D_W(x, y) \equiv i\gamma^\mu C_\mu(x, y) + B(x, y) - \frac{1}{a}m_0\delta_{x,y},
\]
\[
C_\mu(x, y) = \frac{1}{2a}[\delta_{x,\hat{\mu}a,y}U_\mu(y) - \delta_{x,y+\hat{\mu}a}U_\mu^\dagger(x)],
\]
\[
B(x, y) = \frac{r}{2a}\sum_\mu[2\delta_{x,y} - \delta_{y,\hat{\mu}a}U_\mu^\dagger(x) - \delta_{y,x+\hat{\mu}a}U_\mu(y)],
\]
\[
U_\mu(y) = \exp[iagA_\mu(y)],
\]  
(2.4)

where we added a constant mass term to \(D_W\). Our matrix convention is that \(\gamma^\mu\) are anti-hermitian, \((\gamma^\mu)^\dagger = -\gamma^\mu\), and thus \(\mathcal{C} \equiv \gamma^\mu C_\mu(n, m)\) is hermitian
\[
\mathcal{C}^\dagger = \mathcal{C}.
\]  
(2.5)

Since the operators \(\mathcal{C}\) and \(B\) form the basis for any fermion operator on the lattice, we summarize the basic properties of \(\mathcal{C}\) and \(B\).

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1 The continuum limit in this paper stands for the so-called “naive” continuum limit with \(a \to 0\), and the lattice size is gradually extended to infinity for any finite \(a\) in the process of taking the limit \(a \to 0\).
2.1 Operators $\mathcal{C}$ and $B$ and Brillouin zone

For a square lattice, for which we work in this paper, one can explicitly show that the simplest lattice fermion action

$$S = \int \bar{\psi} i \mathcal{C} \psi$$  (2.6)

is invariant under the transformation\[8\]

$$\psi' = T \psi, \quad \bar{\psi}' = \bar{\psi} T^{-1}$$  (2.7)

where $T$ stands for any one of the following 16 operators

$$1, T_1T_2, T_1T_3, T_1T_4, T_2T_3, T_2T_4, T_3T_4, T_1T_2T_3T_4,$$  (2.8)

and

$$T_1, T_2, T_3, T_4, T_1T_2T_3, T_2T_3T_4, T_3T_4T_1, T_4T_1T_2.$$  (2.9)

The operators $T_\mu$ are defined by

$$T_\mu \equiv \gamma_\mu \gamma_5 \exp(i\pi x^\mu/a)$$  (2.10)

and satisfy the relation

$$T_\mu T_\nu + T_\nu T_\mu = 2\delta_{\mu\nu}$$  (2.11)

with $T_\mu^\dagger = T_\mu = T_\mu^{-1}$ for anti-hermitian $\gamma_\mu$. We denote the 16 operators by $T_n$, $n = 0 \sim 15$, in the following with $T_0 = 1$. By recalling that the operator $T_\mu$ adds the momentum $\pi/a$ to the fermion momentum $k_\mu$, we cover the entire Brillouin zone

$$-\frac{\pi}{2a} \leq k_\mu < \frac{3\pi}{2a}$$  (2.12)

by the operation (2.7) starting with the free fermion defined in

$$-\frac{\pi}{2a} \leq k_\mu < \frac{\pi}{2a}.$$  (2.13)

The operators in (2.8) commute with $\gamma_5$, whereas those in (2.9) anti-commute with $\gamma_5$ and thus change the sign of chiral charge, reproducing the 15 species doublers for (2.6) with correct chiral charge assignment; $\sum_{n=0}^{15} (-1)^n \gamma_5 = 0$.

One may define the near continuum configurations by the momentum $k_\mu$ carried by the fermion

$$-\frac{\pi}{2a} \leq k_\mu \leq \frac{\pi}{2a}$$  (2.14)

or

$$-\frac{\pi}{2} \leq ak_\mu \leq \frac{\pi}{2}$$  (2.15)

for sufficiently small $a$ and $\epsilon$ combined with the operation $T_n$ in (2.8) and (2.9). To identify each species doubler clearly in the near continuum configurations, we also keep $r/a$ and
$m_0/a$ finite for $a \to$ small [8], and the gauge fields are assumed to be sufficiently smooth. For these configurations, we can approximate the operator $D_W$ by

$$D_W = i\mathcal{D} + M_n + O(\epsilon^2) + O(a(gA_\mu)^2)$$

(2.16)

for each species doubler, where the mass parameters $M_n$ stand for $M_0 = -\frac{m_0}{a}$ and one of

$$\frac{2r}{a} - \frac{m_0}{a}, \ (4, -1); \quad \frac{4r}{a} - \frac{m_0}{a}, \ (6, 1)$$

$$\frac{6r}{a} - \frac{m_0}{a}, \ (4, -1); \quad \frac{8r}{a} - \frac{m_0}{a}, \ (1, 1)$$

(2.17)

for $n = 1 \sim 15$. Here we denoted (multiplicity, chiral charge) in the bracket for species double. In (2.16) we used the relation valid for the configurations (2.15), for example,

$$D_W e^{ikx} = \sum_y D_W(x, y)e^{iky}$$

$$= [\sum_\mu \gamma^\mu \sin \frac{ak_\mu}{a} + \frac{r}{a} \sum_\mu (1 - \cos \frac{ak_\mu}{a}) - \frac{m_0}{a}]e^{ikx}$$

$$= [\gamma^\mu k_\mu (1 + O(\epsilon^2)) + \frac{r}{a} O(\epsilon^2) - \frac{m_0}{a}]e^{ikx}$$

(2.18)

for vanishing gauge fields.

For the near continuum configurations, we thus have from (2.3)

$$D_W^{(2k+1)} = i(\mathcal{D})^{2k+1} + M_n^{(2k+1)} + O(\epsilon^2)$$

(2.19)

where the mass parameters $M_n^{(2k+1)}$ stand for

$$M_0^{(2k+1)} \equiv -\left(\frac{m_0}{a}\right)^{2k+1}$$

(2.20)

and one of

$$\left(\frac{2r}{a}\right)^{2k+1} - \left(\frac{m_0}{a}\right)^{2k+1}, \ (4, -1); \quad \left(\frac{4r}{a}\right)^{2k+1} - \left(\frac{m_0}{a}\right)^{2k+1}, \ (6, 1)$$

$$\left(\frac{6r}{a}\right)^{2k+1} - \left(\frac{m_0}{a}\right)^{2k+1}, \ (4, -1); \quad \left(\frac{8r}{a}\right)^{2k+1} - \left(\frac{m_0}{a}\right)^{2k+1}, \ (1, 1)$$

(2.21)

for $n = 1 \sim 15$, in the same notation as in (2.17).

To avoid the appearance of species doublers in $\gamma_5 D$, we choose $M_0^{(2k+1)} < 0$ and all other mass parameters $M_n^{(2k+1)} > 0$, $n \neq 0$, namely

$$0 < m_0 < 2r.$$ 

(2.22)

The choice

$$2m_0^{2k+1} = 1$$

(2.23)

normalizes properly the Dirac operator $H_{(2k+1)}$ in (2.2)

$$H_{(2k+1)} \simeq (i\gamma_5 a \mathcal{D})^{2k+1} + \gamma_5 (i\gamma_5 a \mathcal{D})^{2(2k+1)}$$

(2.24)

in the near continuum configurations for all $|M_n| \to$ large [1].
3 Evaluation of the lattice Jacobian

For an operator \( O(x, y) \) defined on the lattice, one may define

\[
O_{mn} \equiv \sum_{x,y} \phi_m(x) O(x, y) \phi_n(y),
\]

and the trace

\[
TrO = \sum_n O_{nn} = \sum_n \sum_{x,y} \phi_n(x) O(x, y) \phi_n(y) = \sum_x (\sum_{n,y} \phi_n(x) O(x, y) \phi_n(y)).
\]

The local version of the trace (or anomaly) is then defined by

\[
trO(x, x) \equiv \sum_{n,y} \phi_n(x) O(x, y) \phi_n(y).
\]

For the operator of our interest, we have

\[
tr \Gamma_5(x) = tr[\gamma_5 - (\gamma_5 aD)^{2k+1}] = -tr(\gamma_5 aD)^{2k+1}
\]

\[
= -\frac{1}{2} \gamma_5 \left[ 1 + D_W^{(2k+1)} \frac{1}{\sqrt{(D_W^{(2k+1)})^\dagger D_W^{(2k+1)}}} \right] = -\frac{1}{2} \gamma_5 D_W^{(2k+1)} \frac{1}{\sqrt{(D_W^{(2k+1)})^\dagger D_W^{(2k+1)}}}
\]

\[
= -\frac{1}{2} \sum_{n=0}^{15} \int \frac{d^4k}{(2\pi)^4} e^{-ikx} T_n e^{ikx}
\]

where we used the plane wave basis defined in the domain (2.13) combined with the operation \( T_n \). In this calculation, we repeatedly used the relation

\[
tr \gamma_5 = 0
\]

which is expected to be valid in lattice theory. We also used a short hand notation

\[
Oe^{ikx} = \sum_y O(x, y)e^{iky}.
\]

There are various ways to evaluate the above trace (3.4). We evaluate the trace (3.4) by following the procedure used for the overlap Dirac operator in Refs.[9][10]. Some of the basic papers of the lattice anomaly calculation are found in [11]-[14]. In this section we simplify the expression of the Jacobian, and its explicit evaluation is presented in the next section.
3.1 General analysis of the trace

Our starting formula is (by using the momentum domain (2.12))

\[-\frac{1}{2} (\frac{1}{a})^4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d^4 p}{(2\pi)^4} tr\gamma_5 \tilde{D}_W^{(2k+1)}(p) \frac{1}{\sqrt{(\tilde{D}_W^{(2k+1)}(p))^4 \tilde{D}_W^{(2k+1)}(p)}} \]

\( (3.7) \)

with

\[ \tilde{D}_W^{(2k+1)}(p) \equiv (a^{2k+1}) D_W^{(2k+1)}(p) \]

\[ = i [i \sum \gamma^\mu \sin p_\mu + a\tilde{\mathcal{Q}}]^{2k+1} + [r \sum \mu (1 - \cos p_\mu) + a\tilde{B}]^{2k+1} \]

\[ - (m_0)^{2k+1} \]

\( (3.8) \)

and we defined the integration variable

\[ p_\mu = ak_\mu. \]

\( (3.9) \)

We used the definitions

\[ e^{-ikx} a \mathcal{Q} e^{ikx} h(x) \equiv [i \sum \gamma^\mu \sin ak_\mu + a\tilde{\mathcal{Q}}] h(x) \]

\( (3.10) \)

and

\[ e^{-ikx} aB e^{ikx} h(x) \equiv [r \sum \mu (1 - \cos ak_\mu) + a\tilde{B}] h(x). \]

\( (3.11) \)

for a sufficiently smooth function \( h(x) \). In the following we often omit writing \( h(x) \).

Consequently,

\[ D_W^{(2k+1)}(k_\mu) \equiv e^{-ikx} D_W^{(2k+1)}(p) e^{ikx} \]

\[ = i [i \sum \gamma^\mu \sin ak_\mu / a + \tilde{\mathcal{Q}}]^{2k+1} + [r \sum \mu (1 - \cos ak_\mu) + \tilde{B}]^{2k+1} \]

\[ - (m_0 / a)^{2k+1} \]

\( (3.12) \)

and \( D_W^{(2k+1)}(p) \) is defined by setting \( k_\mu = p_\mu / a \) in \( D_W^{(2k+1)}(k_\mu) \).

In the continuum limit \( a \rightarrow 0 \) with \( p_\mu = ak_\mu \) kept fixed, the operator \( \tilde{\mathcal{Q}} \) approaches

\[ \tilde{\mathcal{Q}} = \sum \mu \gamma^\mu (\cos ak_\mu \partial_\mu + ig \cos ak_\mu A_\mu) + O(a) = \sum \gamma^\mu \cos p_\mu D_\mu + O(a) \]

\( (3.13) \)

and the leading term of \( \tilde{B} \) is known to be\[11\]

\[ \tilde{B} = -ir \sum \mu \sin ak_\mu D_\mu + O(a) = -ir \sum \mu \sin p_\mu D_\mu + O(a) \]

\( (3.14) \)

with the covariant derivative defined by

\[ D_\mu \equiv \partial_\mu + igA_\mu. \]

\( (3.15) \)
Note that the “conventional naive continuum limit” is defined by $a \to 0$ with $k_{\mu}$ kept fixed, instead of $p_{\mu} = ak_{\mu}$ being kept fixed as in the above limit.

In the denominator of (3.7), one has a factor
\[
(D_W^{(2k+1)}(p))^\dagger D_W^{(2k+1)}(p)
\]
\[
= [(i \sum \gamma^\mu \sin p_{\mu} + a C)]^{2k+1}
\]
\[
+ \{[r \sum (1 - \cos p_{\mu}) + a \tilde{B}]^{2k+1} - (m_0)^{2k+1}\}^2
\]
\[
- i[(i \sum \gamma^\mu \sin p_{\mu} + a C)^{2k+1}, [r \sum (1 - \cos p_{\mu}) + a \tilde{B}]^{2k+1}]
\]
\[
= \{\sum (\sin p_{\mu} - ai \tilde{C}_{\mu})^2 + \frac{a^2}{4}[\gamma^\mu, \gamma^\nu][\tilde{C}_{\mu}, \tilde{C}_{\nu}]\}^{2k+1}
\]
\[
+ \{[r \sum (1 - \cos p_{\mu}) + a \tilde{B}]^{2k+1} - (m_0)^{2k+1}\}^2
\]
\[
- i[(i \sum \gamma^\mu \sin p_{\mu} + a C)^{2k+1}, [r \sum (1 - \cos p_{\mu}) + a \tilde{B}]^{2k+1}].
\] (3.16)

Note that the first two terms in this last expression commute with $\gamma^5$, while the last term anti-commutes with $\gamma^5$. The last term of (3.16) is the interference term: From the structure of the commutator, one can confirm that it consists of terms with a factor
\[
- ia^2\gamma^\mu[\tilde{C}_{\mu}, \tilde{B}] = -ia^2 g r \sum_{\mu,\nu} \gamma^\mu \cos p_{\mu} F_{\mu\nu} + O(a^3)
\] (3.17)

and the $2k$ factors of $[i \sum \gamma^\mu \sin p_{\mu} + a \tilde{C}]$ and the $2k$ factors of $[r \sum (1 - \cos p_{\mu}) + a \tilde{B}]$.

We also note that
\[
[\tilde{C}_{\mu}, \tilde{C}_{\nu}] = ig \cos p_{\mu} \cos p_{\nu} F_{\mu\nu} + O(a).
\] (3.18)

To simplify various expressions in the following, we define the variables
\[
c_{\mu} = \cos ak_{\mu} = \cos p_{\mu}, \quad s_{\mu} = \sin ak_{\mu} = \sin p_{\mu}
\] (3.19)

and
\[
s' = \sum \gamma^\mu \sin p_{\mu}, \quad s^2 = \sum (s_{\mu})^2.
\] (3.20)

### 3.2 Contribution of mass terms

We now examine the integrand of (3.7) with only the “mass terms” in the numerator retained
\[
\frac{1}{a^4} tr\gamma_5 \{[r \sum (1 - c_{\mu}) + a \tilde{B}]^{2k+1} - (m_0)^{2k+1}\} \frac{1}{\sqrt{(D_W^{(2k+1)}(p))^\dagger D_W^{(2k+1)}(p)}}.
\] (3.21)

The numerator contains no $\gamma^\mu$’s. We expand the denominator in powers of the interference term, which contains an odd number of $\gamma^\mu$’s. By remembering that the rest of the
denominator factor contains an even number of $\gamma^\mu$’s and thus commute with $\gamma_5$, the odd powers in the interference term in this expansion anti-commute with $\gamma_5$ and thus vanish after taking the trace. Only the even powers in the interference term could survive the trace operation with $\gamma_5$. Since the interference term is of order $O(a^2)$, only the zeroth order term and the second order term in the interference term are important in the limit $a \to 0$.

We can see that the second order term in the interference vanishes. Since the second order term is already of order $O(a^4)$ because of (3.17), one can set $a = 0$ in all the remaining $2k$ powers of $[i \not{s} + a \not{C}]$ and the $2k$ powers of $[r \sum_\mu (1 - c_\mu) + a \not{B}]$. Namely these terms are replaced by $i \not{s}$ and $r \sum_\mu (1 - c_\mu)$, respectively. Since these factors commute with each other, the interference term consists of a sum of terms of the structure

$$
(i \not{s})^l ia^2 r \sum_{\mu,\nu} \gamma^\mu c_\mu s_\nu (i \not{s})^m [r \sum_\mu (1 - c_\mu)]^{2k}
$$

with $l + m = 2k$, if one uses (3.17): These terms are linear in $\gamma^\mu$ if one uses the relations

$$
(i \not{s})^2 = s^2,
$$

$$
(i \not{s})\gamma^\nu (i \not{s}) = -2(s^2) s_\nu - s^2 \gamma^\nu.
$$

We thus have only two $\gamma^\mu$’s with order $O(a^4)$ in the numerator of (3.21), which vanishes after the trace with $\gamma_5$. Note that the terms which contain $\gamma^\mu$’s in the denominator is of order $O(a^2)$ as in (3.16), and thus these terms cannot be used to supply extra $\gamma^\mu$’s.

We have thus established that only the zeroth order term in the interference survives in (3.21), namely, one can set the interference term to 0 in the denominator. In this case, from the expression in (3.16) we see that the $\gamma^\mu$ factors appear only in the combination

$$
\frac{a^2}{4} [\gamma^\mu, \gamma^\nu] \tilde{C}_\mu \tilde{C}_\nu.
$$

The second power in this factor is just sufficient to survive the trace operation and cancel $1/a^4$ in front of the integral. This means that we can set $a = 0$ everywhere except in the prefactor in (3.24). The expression (3.21) is then replaced by

$$
\frac{1}{a^4} tr \gamma_5 \left\{ [r \sum_\mu (1 - c_\mu)]^{2k+1} - (m_0)^{2k+1} \right\} \frac{1}{\sqrt{F(k)}}
$$

where

$$
F(k) \equiv \left( s^2 + i g \frac{a^2}{4} [\gamma^\mu, \gamma^\nu] c_\mu c_\nu F_{\mu\nu} \right)^{2k+1} + \left\{ [r \sum_\mu (1 - c_\mu)]^{2k+1} - (m_0)^{2k+1} \right\}^2.
$$

### 3.3 Contribution of kinetic term

Similarly, we analyze the integrand of (3.7) with the “kinetic term” in the numerator

$$
\frac{1}{a^4} tr \gamma_5 \left\{ i [i \not{s} + a \not{C}] \right\}^{2k+1} \frac{1}{\sqrt{(D_W^{(2k+1)}(p)) (D_W^{(2k+1)}(p))}}
$$

(3.27)
Since the numerator is now odd in powers of $\gamma^\mu$, only the odd powers of the interference term could survive. The third power of the interference is $O(a^6)$, and only the first power in the interference need to be analyzed.

We first rewrite the numerator factor as

$$[i s' + a\tilde{C}]^{2k+1} = \left\{ \sum_{\mu}(s_\mu - ai\tilde{C}_\mu)^2 + \frac{a^2}{4}[\gamma^\mu, \gamma^\nu][\tilde{C}_\mu, \tilde{C}_\nu] \right\} [i s' + a\tilde{C}]$$

which shows that we have only one $\gamma^\mu$ which is not multiplied by $a$. As we have already explained, the interference term in the denominator

$$-i[i s' + a\tilde{C}]^{2k+1}, [r \sum_{\mu}(1 - c_{\mu}) + a\tilde{B}]^{2k+1}$$

is written as a sum of terms with a single commutator

$$-ia^2\gamma^\mu[\tilde{C}_\mu, \tilde{B}] = -ia^2 gr \sum_{\mu,\nu} \gamma^\mu c_{\mu}s_{\nu} F_{\mu\nu}[i s' + a\tilde{C}]$$

multiplied by the $2k$ factors of $[i s' + a\tilde{C}]$ and the $2k$ factors of $[r \sum_{\mu}(1 - c_{\mu}) + a\tilde{B}]$. In such a term, if one exchanges the order of $[i s' + a\tilde{C}]$ and $[r \sum_{\mu}(1 - c_{\mu}) + a\tilde{B}]$, one generates another commutator as in (3.30). We then have a factor $a^4$ and thus we can set $a = 0$ in all other terms in the integrand. We recognize that such a term contains $\gamma^\mu$ only in the combination with $s'$ and the two factors of the above commutator (3.30). From this combination together with $s'$ in the numerator (3.28), we cannot form a non-vanishing contraction with the antisymmetric $\epsilon^{\mu\nu\alpha\beta}$ tensor.

This means that we can write the interference term as a sum of terms of the structure

$$[i s' + a\tilde{C}]^l(-i)a^2 gr \sum_{\mu,\nu} \gamma^\mu c_{\mu}s_{\nu} F_{\mu\nu}[i s' + a\tilde{C}]^m \times[r \sum_{\mu}(1 - c_{\mu}) + a\tilde{B}]^{2k}$$

where $l + m = 2k$. If both of $l$ and $m$ are even, we can use

$$[i s' + a\tilde{C}]^2 = \sum_{\mu}(s_\mu - ai\tilde{C}_\mu)^2 + \frac{a^2}{4}[\gamma^\mu, \gamma^\nu][\tilde{C}_\mu, \tilde{C}_\nu]$$

and $\gamma^\mu$ always appears in the combination $a\gamma^\mu$ except for the numerator term (3.28) which contains $s'$, and the commutator term (3.30), which contains $a^2\gamma^\mu$. Such a combination could give rise to a non-vanishing result. On the other hand, if both of $l$ and $m$ are odd, one has to deal with a left-over term

$$[i s' + a\tilde{C}](-i)a^2 gr \sum_{\mu,\nu} \gamma^\mu c_{\mu}s_{\nu} F_{\mu\nu}[i s' + a\tilde{C}]$$

$$= [i s' + a\tilde{C}](-2)(-i)a^2 gr \sum_{\mu,\nu} c_{\mu}s_{\nu} F_{\mu\nu}[is_\mu + a\tilde{C}_\mu]$$
\[ + [i \not{s} + a \not{C}]( -i )a^3 gr \sum_{\mu, \nu, \alpha} \gamma^\alpha \gamma^\mu c_\alpha c_\mu s_\nu D_\alpha F_{\mu \nu} \]

\[- \{ \sum_\mu (s_\mu - ai \bar{\gamma}_\mu)^2 + \frac{a^2}{4} [\gamma^\mu, \gamma^\nu][\bar{C}_\mu, \bar{C}_\nu] \} ( -i )a^2 gr \sum_{\mu, \nu} \gamma^\mu c_\mu s_\nu F_{\mu \nu}. \]

(3.33)

where we used \( \gamma^\mu \gamma^\alpha + \gamma^\alpha \gamma^\mu = -2 \eta^{\mu \alpha} \). The first term of this equation contains the factor \( a^2 [i \not{s} + a \not{C}] \), which should be replaced by \( a^2 i \not{s} \) and should be combined with the factor \( i \not{s} \) in the numerator factor (together with \( a^3 [\gamma^\mu, \gamma^\nu][\bar{C}_\mu, \bar{C}_\nu] \) from other factors) to obtain a possible non-zero result. But such a term cannot make a non-vanishing contraction with \( \epsilon^{\mu \nu \alpha \beta} \). The second term is of order \( O(a^3) \) and contains 3 \( \gamma^\mu \)’s. If this term is combined with \( [i \not{s} + a \not{C}] \) in the numerator, it becomes of order \( O(a^5) \) due to (3.32). It is combined with the derivative operator such as \( a \bar{C}_\mu \) in (3.16), when commuting with other denominator factors, it becomes \( O(a^4) \); such a term contains \( \not{s}^2 = - s^2 \) if combined with \( \not{s} \) in the numerator and vanishes after trace with \( \gamma_5 \). Thus we can set the first and second terms to 0 in the above equation (3.33).

Only the last term in (3.33) can survive: Among those surviving terms, the even \( l \) terms and odd \( l \) terms cancel pairwise except one term in the interference term.

By this way, we can write the total interference term as

\[-(2k + 1)ia^2 gr \sum_{\mu, \nu} \gamma^\mu c_\mu s_\nu F_{\mu \nu} \{ s^2 + i g \frac{a^2}{4} [\gamma^\mu, \gamma^\nu] c_\mu c_\nu F_{\mu \nu} \}^k \times [r \sum_\mu (1 - c_\mu)]^{2k} \]

(3.34)

where the factor \( 2k + 1 \) comes from the \( 2k + 1 \) powers of \( [r \sum_\mu (1 - c_\mu) + a \bar{B}] \). We also set \( a = 0 \) in all the terms without \( \gamma^\mu \) since this does not influence the surviving terms. We can also set \( a = 0 \) in the numerator factor except for the combination \( a^3 [\gamma^\mu, \gamma^\nu] \). Note that the order of \( \gamma^\mu \) and \( [\gamma^\mu, \gamma^\nu] \) can be changed freely in the expansion of the denominator in powers of \( a^2 \), since the surviving terms are contracted with \( \gamma_5 \) to give rise to \( \epsilon^{\mu \nu \alpha \beta} \).

To summarize this tedious analysis, we can write the integrand with the “kinetic” term (3.27) as

\[ \frac{1}{a^4} tr \gamma_5 \{ (2k + 1) r (i g \frac{a^2}{4}) [\gamma^\mu, \gamma^\nu] c_\mu c_\nu F_{\mu \nu} ( \sum_\alpha \frac{s^2}{4 c_\alpha} ) \times \{ s^2 + i g \frac{a^2}{4} [\gamma^\mu, \gamma^\nu] c_\mu c_\nu F_{\mu \nu} \}^{2k} [r \sum_\mu (1 - c_\mu)]^{2k} \} \frac{1}{F_{(k)}^{3/2}} \]

(3.35)

where \( F_{(k)} \) is defined in (3.26). In writing this final expression we used the following sequence of rewriting

\[ s \sum_{\mu, \nu} \gamma^\mu c_\mu s_\nu F_{\mu \nu} = \sum_\alpha \sum_{\mu, \nu} \gamma^\alpha \gamma^\mu c_\mu s_\nu s_\alpha F_{\mu \nu} \]

\[ = \frac{1}{2} \sum_\alpha \sum_{\mu, \nu} [\gamma^\alpha, \gamma^\mu] c_\mu s_\nu s_\nu \delta_{\nu, \alpha} F_{\mu \nu} \]

11
\[ \sum_{\mu,\nu} \gamma_{\nu} \gamma_{\mu} c_{\mu} c_{\nu} F_{\mu\nu} = -\frac{1}{2} \sum_{\mu,\nu} [\gamma_{\mu}, \gamma_{\nu}] c_{\mu} c_{\nu} F_{\mu\nu} \left( \sum_{\alpha} s_{\alpha}^2 / 4c_{\alpha} \right) \]  

(3.36)

Namely, only the term with two $\gamma_{\mu}$'s contributes to the final result, and the odd term in $s_{\nu}$ after integration over the momentum vanishes. In the last step, we used the lattice hypercubic symmetry by taking into account the contraction with the $\epsilon^{\mu\nu\alpha\beta}$ symbol later.

4 Formula for the chiral anomaly and parameter independence

The basic formula for the chiral anomaly is given by (3.7), (3.25) and (3.35). The next step is to expand the integrand in the powers of 

\[ (ig_4^2)_{\mu\nu} \gamma_{\mu}, \gamma_{\nu} c_{\mu} c_{\nu} F_{\mu\nu} \]

and retain only the terms which contain the second power of this factor. We then combine the expansion with the formula

\[ \text{tr}\gamma_5 \left\{ (ig_4^2)_{\mu\nu} \gamma_{\mu}, \gamma_{\nu} c_{\mu} c_{\nu} F_{\mu\nu} \right\}^2 = \left( \prod_{\alpha=1}^{4} c_{\alpha} \right) g^2 \text{tr} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \]  

(4.1)

where $\epsilon^{1234} = 1$. We thus write only the coefficients of the factor

\[ g^2 \text{tr} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \]

(4.2)

in the following.

The contribution from the “mass terms” (3.25) is given by

\[ -\frac{1}{16} \int_{-\pi/2}^{\pi/2} \frac{d^4p}{(2\pi)^4} \left( \prod_{\alpha=1}^{4} c_{\alpha} \right) (2k + 1) M_{(2k+1)} \frac{[3(2k + 1)(s^2)^{4k} - 4k(s^2)^{2k-1}H]}{H^{5/2}} \]  

(4.3)

where

\[ H \equiv (s^2)^{2k+1} + M_{(2k+1)}^2 \]

\[ M_{(2k+1)} \equiv [r \sum_{\mu} (1 - c_{\mu})]^{2k+1} - m_{0}^{2k+1}. \]  

(4.4)

The contribution from the “kinetic term” (3.35) is written as

\[ -\frac{1}{16} \int_{-\pi/2}^{\pi/2} \frac{d^4p}{(2\pi)^4} \left( \prod_{\alpha=1}^{4} c_{\alpha} \right) (2k + 1) \{ r(\sum_{\beta} s_{\beta}^2 / c_{\beta}) [r \sum_{\mu} (1 - c_{\mu})]^{2k} \} \times \{ 4k(s^2)^{2k-1}H - 3(2k + 1)(s^2)^{4k} \} \frac{1}{H^{5/2}}. \]  

(4.5)

Thus the total contribution is given by

\[ I_{2k+1} = -\frac{2k + 1}{16} \int_{-\pi/2}^{\pi/2} \frac{d^4p}{(2\pi)^4} \left( \prod_{\alpha=1}^{4} c_{\alpha} \right) \{ M_{(2k+1)} - r(\sum_{\beta} s_{\beta}^2 / c_{\beta}) [r \sum_{\mu} (1 - c_{\mu})]^{2k} \} \times \{ 3(2k + 1)(s^2)^{4k} - 4k(s^2)^{2k-1}H \} \frac{1}{H^{5/2}}. \]  

(4.6)
4.1 Parameter independence

To analyze the parameter independence of the coefficient of the chiral anomaly, we follow the procedure in Refs. [9][10]. We first rewrite the integral in the domain $-\frac{\pi}{2} \leq p_\mu \leq \frac{3\pi}{2}$ to the integral in the domain $-\frac{\pi}{2} \leq p_\mu \leq \frac{\pi}{2}$ by using the variables $s_\mu = \sin p_\mu$ as

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d^4p}{(2\pi)^4} \left( \prod_{a=1}^{4} c_\alpha \right) = \sum_{\epsilon_\mu = \pm} \left( \prod_{\mu} \epsilon_\mu \right) \int_{-1}^{1} \frac{d^4s}{(2\pi)^4} \left( \prod_{\alpha=1}^{4} c_\alpha \right)$$

(4.7)

where the symbol $\epsilon_\mu$ takes care of the 16 (would-be) species doublers

$$\epsilon_\mu = (\pm, \pm, \pm, \pm).$$

(4.8)

The following formula is also valid

$$c_\mu = \epsilon_\mu (1 - s_\mu^2)^{1/2}. \quad \text{(4.9)}$$

In this new notation, we have

$$H(s) \equiv (s^2)^{2k+1} + M_{(2k+1)}^2$$

$$M_{(2k+1)} \equiv [r \sum_\mu (1 - \epsilon_\mu (1 - s_\mu^2)^{1/2})]^{2k+1} - m_0^{2k+1}$$

(4.10)

and we evaluate

$$(2k+1)H(s) + \frac{1}{5} H(s)^{7/2} \sum_\mu s_\mu \frac{\partial}{\partial s_\mu} H(s)^{-5/2}$$

$$= (2k+1)H(s) + \frac{1}{3} H(s)^{5/2} \sum_\mu s_\mu \frac{\partial}{\partial s_\mu} H(s)^{-3/2}$$

$$= -(2k+1)M_{(2k+1)} \left\{ -[r \sum_\mu (1 - \epsilon_\mu (1 - s_\mu^2)^{1/2})]^{2k+1} + m_0^{2k+1} \right\}$$

$$+ [r \sum_\mu (1 - \epsilon_\mu (1 - s_\mu^2)^{1/2})]^{2k+1} r \sum_\mu s_\mu^2 \epsilon_\mu (1 - s_\mu^2)^{-1/2} \right\}$$

$$= (2k+1)M_{(2k+1)}$$

$$\times \{ M_{(2k+1)} - [r \sum_\mu (1 - \epsilon_\mu (1 - s_\mu^2)^{1/2})]^{2k+1} r \sum_\mu s_\mu^2 \epsilon_\mu (1 - s_\mu^2)^{-1/2} \}$$

(4.11)

which is a generalization of the identity discussed in [10]. By using these relations one can prove

$$\frac{1}{2k+1} \left[ \sum_\mu s_\mu \frac{\partial}{\partial s_\mu} + 4 \right] \left( \frac{(s^2)^{4k}}{H(s)^{5/2}} \right)$$

$$= -\left( \frac{(s^2)^{4k}}{H(s)^{5/2}} \right)$$

$$+ \{ M_{(2k+1)} - [r \sum_\mu (1 - \epsilon_\mu (1 - s_\mu^2)^{1/2})]^{2k+1} r \sum_\mu s_\mu^2 \epsilon_\mu (1 - s_\mu^2)^{-1/2} \}$$

$$\times 5M_{(2k+1)} \left( \frac{(s^2)^{4k}}{H(s)^{7/2}} \right)$$

(4.12)
and

\[
\frac{1}{2k+1} \left[ \sum_{\mu} s_\mu \frac{\partial}{\partial s_\mu} + 4 \right] \left( \frac{(s^2)^{2k}}{H(s)^{3/2}} \right) H(s)^{3/2} = - \frac{(s^2)^{2k-1}}{H(s)^{3/2}}
\]

\[
+ \{ M_{(2k+1)} - \left[ r \sum_{\mu} (1 - c_\mu) (1 - s_\mu^2)^{1/2} \right]^{2k} r \sum_{\mu} s_\mu^2 c_\mu (1 - s_\mu^2)^{-1/2} \}
\]

\[
\times 3 M_{(2k+1)} \left( \frac{(s^2)^{2k-1}}{H(s)^{5/2}} \right).
\quad (4.13)
\]

One can then show that

\[
\frac{\partial I_{(2k+1)}}{\partial m^0_{2k+1}} = \frac{2k+1}{16} \sum_{\epsilon_{\mu} = \pm} (\prod_{\mu} \epsilon_\mu) \int_{-1}^{1} \frac{d^4s}{(2\pi)^4}
\]

\[
\times \left\{ 3(2k+1) \left( \frac{(s^2)^{4k}}{H^{5/2}} \right) - 5 [M_{(2k+1)} - r \sum_{\beta} s_{\beta}^{2} \left( \sum_{\mu} (1 - c_\mu) \right)^{2k} M_{(2k+1)} \left( \frac{(s^2)^{4k}}{H^{7/2}} \right) \right] \right.
\]

\[
- 4k \left( \frac{(s^2)^{2k-1}}{H^{3/2}} \right) - 3 [M_{(2k+1)} - r \sum_{\beta} s_{\beta}^{2} \left( \sum_{\mu} (1 - c_\mu) \right)^{2k} M_{(2k+1)} \left( \frac{(s^2)^{2k-1}}{H^{5/2}} \right) \} \right)
\]

\[
= \frac{2k+1}{16} \sum_{\epsilon_{\mu} = \pm} (\prod_{\mu} \epsilon_\mu) \int_{-1}^{1} \frac{d^4s}{(2\pi)^4}
\]

\[
\times \left\{ \left[ -3 \left( \sum_{\mu} s_\mu \frac{\partial}{\partial s_\mu} + 4 \right) \left( \frac{(s^2)^{4k}}{H(s)^{5/2}} \right) \right] \right.
\]

\[
+ \frac{4k}{2k+1} \left\{ \left( \sum_{\mu} s_\mu \frac{\partial}{\partial s_\mu} + 4 \right) \left( \frac{(s^2)^{2k-1}}{H(s)^{3/2}} \right) \right\}
\quad (4.14)
\]

Similarly, we can show the relations by using (4.11)

\[
\frac{1}{2k+1} \left[ \sum_{\mu} s_\mu \frac{\partial}{\partial s_\mu} + 4 \right] \left( \frac{r \sum_{\mu} (1 - c_\mu)}{H(s)^{5/2}} \right) \left( \frac{(s^2)^{2k+1}}{H(s)^{5/2}} \right)
\]

\[
= - \left\{ \left[ r \sum_{\mu} (1 - c_\mu) \right]^{2k+1} - \left[ r \sum_{\mu} (1 - c_\mu) \right]^{2k} r \sum_{\mu} s_\mu^2 c_\mu (1 - s_\mu^2)^{-1/2} \right\} \left( \frac{(s^2)^{4k}}{H(s)^{5/2}} \right)
\]

\[
+ \{ M_{(2k+1)} - \left[ r \sum_{\mu} (1 - c_\mu) \right]^{2k} r \sum_{\mu} s_\mu^2 c_\mu (1 - s_\mu^2)^{-1/2} \}
\]

\[
\times 5 M_{(2k+1)} \left( \frac{r \sum_{\mu} (1 - c_\mu)}{H(s)^{7/2}} \right) \left( \frac{(s^2)^{4k}}{H(s)^{7/2}} \right)
\quad (4.15)
\]

and

\[
\frac{1}{2k+1} \left[ \sum_{\mu} s_\mu \frac{\partial}{\partial s_\mu} + 4 \right] \left( \frac{r \sum_{\mu} (1 - c_\mu)}{H(s)^{3/2}} \right) \left( \frac{(s^2)^{2k+1}}{H(s)^{3/2}} \right)
\]

\[
= - \left\{ \left[ r \sum_{\mu} (1 - c_\mu) \right]^{2k+1} - \left[ r \sum_{\mu} (1 - c_\mu) \right]^{2k} r \sum_{\mu} s_\mu^2 c_\mu (1 - s_\mu^2)^{-1/2} \right\} \left( \frac{(s^2)^{2k+1}}{H(s)^{3/2}} \right)
\]

\[
+ \{ M_{(2k+1)} - \left[ r \sum_{\mu} (1 - c_\mu) \right]^{2k} r \sum_{\mu} s_\mu^2 c_\mu (1 - s_\mu^2)^{-1/2} \}
\]

\[
\times 5 M_{(2k+1)} \left( \frac{r \sum_{\mu} (1 - c_\mu)}{H(s)^{5/2}} \right)
\quad (4.16)
\]
\[ + \{ M_{(2k+1)} - [r \sum_{\mu} (1-c_{\mu})]^{2k} r \sum_{\mu} s_{\mu}^2 \epsilon_{\mu} (1-s_{\mu}^2)^{-1/2} \} \]
\[ \times 3M_{(2k+1)} \left( \frac{[r \sum_{\mu} (1-c_{\mu})]^{2k+1} (s^2)^{2k-1}}{H(s)^{5/2}} \right) \]  
\[ (4.16) \]

where \( c_{\mu} = \epsilon_{\mu} (1-s_{\mu}^2)^{1/2} \) is understood. Using these relations, we have
\[ r^{2k+1} \frac{\partial I_{2k+1}}{\partial r^{2k+1}} = - \frac{2k+1}{16} \sum_{\epsilon_{\mu} = \pm} (\prod_{\mu} \epsilon_{\mu}) \int_{-1}^{1} \frac{d^4 s}{(2\pi)^4} \]
\[ \times \{ (2k+1) \left\{ \frac{[r \sum_{\mu} (1-c_{\mu})]^{2k+1} - [r \sum_{\mu} (1-c_{\mu})]^{2k} r \sum_{\mu} \epsilon_{\mu} s_{\mu}^2 (1-s_{\mu}^2)^{-1/2}}{H(s)^{5/2}} \right\} \]
\[ - [M_{(2k+1)} - [r \sum_{\mu} (1-c_{\mu})]^{2k} r \sum_{\mu} s_{\mu}^2 \epsilon_{\mu} (1-s_{\mu}^2)^{-1/2}] \]
\[ \times 5M_{(2k+1)} \left( \frac{[r \sum_{\mu} (1-c_{\mu})]^{2k+1} (s^2)^{4k}}{H(s)^{7/2}} \right) \]
\[ - 4k \left\{ \frac{[r \sum_{\mu} (1-c_{\mu})]^{2k+1} - [r \sum_{\mu} (1-c_{\mu})]^{2k} r \sum_{\mu} \epsilon_{\mu} s_{\mu}^2 (1-s_{\mu}^2)^{-1/2}}{H(s)^{3/2}} \right\} \]
\[ - [M_{(2k+1)} - [r \sum_{\mu} (1-c_{\mu})]^{2k} r \sum_{\mu} s_{\mu}^2 \epsilon_{\mu} (1-s_{\mu}^2)^{-1/2}] \]
\[ \times 3M_{(2k+1)} \left( \frac{[r \sum_{\mu} (1-c_{\mu})]^{2k+1} (s^2)^{2k-1}}{H(s)^{5/2}} \right) \}
\[ = \frac{2k+1}{16} \sum_{\epsilon_{\mu} = \pm} (\prod_{\mu} \epsilon_{\mu}) \int_{-1}^{1} \frac{d^4 s}{(2\pi)^4} \]
\[ \times \left\{ - 3 \sum_{\mu} s_{\mu} \frac{\partial}{\partial s_{\mu}} + 4 \left( \frac{[r \sum_{\mu} (1-c_{\mu})]^{2k+1} (s^2)^{4k}}{H(s)^{5/2}} \right) \right\} \]
\[ + \frac{4k}{2k+1} \left\{ \sum_{\mu} s_{\mu} \frac{\partial}{\partial s_{\mu}} + 4 \left( \frac{[r \sum_{\mu} (1-c_{\mu})]^{2k+1} (s^2)^{2k-1}}{H(s)^{3/2}} \right) \right\}. \]  
\[ (4.17) \]

The integrand becomes singular only if the following two relations simultaneously hold
\[ s^2 = 0, \quad [r \sum_{\mu} (1-\epsilon_{\mu} (1-s_{\mu}^2)^{1/2})]^{2k+1} - m_0^{2k+1} = 0 \]  
\[ (4.18) \]

namely, only when \( m_0/r = 0, \quad 2, \quad 4, \quad 6, \quad 8 \). We are working in the physical region
\[ 0 < m_0 < 2r \]  
\[ (4.19) \]

and thus the above integrals are regular, and we have from (4.14) and (4.17)
\[ \frac{\partial I_{2k+1}}{\partial m_0^{2k+1}} = \frac{\partial I_{2k+1}}{\partial r^{2k+1}} = 0 \]  
\[ (4.20) \]

after partial integration. It can be confirmed that boundary terms at \( s_{\mu} = \pm 1 \) give vanishing contributions after a summation over \( \sum_{\epsilon_{\mu} = \pm} \). This shows that the coefficient of the anomaly is stable under a smooth variation of the parameters \( r \) and \( m_0 \), which is expected for a topological quantity such as the chiral anomaly.
4.2 Explicit evaluation of the chiral anomaly

Since the coefficient of the anomaly is independent of the parameters $r$ and $m_0$, we evaluate the anomaly in the limit where both of $r$ and $m_0$ go to 0. To be precise we introduce an auxiliary parameter $a$, and take a limit $a \to 0$ with both of

$$\frac{r}{a}, \frac{m_0}{a}$$

kept fixed in the physical region (4.19). The parameter $a$ plays the role of an effective lattice spacing, though our formulas (4.3) and (4.5) are derived in the limit of the lattice spacing $a = 0$.

There are various ways to evaluate the coefficient of the anomaly, and one of these methods is given in [10] in the analysis of the overlap operator with $k = 0$. We present a calculation which reveals a close connection with the naive continuum limit. We first observe that the contribution of the “kinetic” term (4.5), which arises from the interference term in the denominator, vanishes in the above limit (4.21) \(^{2}\). To show this we examine

$$-\frac{1}{16} \sum_{\epsilon_{\mu}=\pm} \left( \prod_{\mu} \epsilon_{\mu} \right) \int_{-1}^{1} \frac{d^4s}{(2\pi)^4} \left\{ r \left( \sum_{\beta} \frac{s_\beta^2}{c_\beta} \right) \left[ r \sum_{\mu} (1 - c_{\mu}) \right]^{2k} \right\}$$

$$\times \left \{ 4k (s^2)^{2k-1} H - 3(2k + 1)(s^2)^{4k} \right \} \frac{1}{H^{5/2}}. \quad (4.22)$$

where $c_{\mu} = \epsilon_{\mu} (1 - s_{\mu}^2)^{1/2}$ is understood. We define the integration domain

$$-\epsilon \leq s_{\mu} \leq \epsilon \quad (4.23)$$

for all $\mu$ with a sufficiently small but finite $\epsilon$. Since $s^2 > 0$ and the denominator of the integrand is regular for the integration domain outside the above domain, the integral outside the domain (4.23) vanishes in the limit $a \to 0$. Note that the denominator of $\sum_{\beta} \frac{s_\beta^2}{c_\beta}$ does not cause any divergence in the integral (4.22). In fact one can even take $\epsilon \to 0$ in such a manner that

$$a/\epsilon^l \to 0 \quad (4.24)$$

for a suitable fixed positive integer $l$. This is because the integral outside the domain (4.23) vanishes at least linearly in $a$, and thus one can let $\epsilon \to 0$ simultaneously with the above constraint, where the denominator $\epsilon^l$ stands for the possible infrared singularity in this calculational procedure.

We thus examine the remaining integral

$$-\frac{1}{16} \sum_{\epsilon_{\mu}=\pm} \left( \prod_{\mu} \epsilon_{\mu} \right) \int_{-\epsilon}^{\epsilon} \frac{d^4s}{(2\pi)^4} \left\{ r \left( \sum_{\beta} \frac{s_\beta^2}{c_\beta} \right) \left[ r \sum_{\mu} (1 - c_{\mu}) \right]^{2k} \right\} \times \left \{ 4k (s^2)^{2k-1} H - 3(2k + 1)(s^2)^{4k} \right \} \frac{1}{H^{5/2}}. \quad (4.25)$$

\(^{2}\) This property has been used in the treatment of the overlap operator in Refs.[7] and [15].
Since \(|c_\mu| \simeq 1\) in the above integral, we can ignore the variation of \(c_\mu\). If one rescales the integration variable \(s_\mu = as'_\mu\) and defines

\[
    r' = r/a, \quad m'_0 = m_0/a \quad (4.26)
\]

the above integral is written as

\[
    -\frac{1}{16} \sum_{\mu=\pm} (\Pi_\epsilon \epsilon_\mu) \int_{-\epsilon/a}^{\epsilon/a} \frac{d^4s'}{(2\pi)^4} \{r'(\sum_{\beta} \frac{s_{\beta}^2}{c_\beta})[r'(\sum_{\mu}(1-c_\mu)]^{2k} \}
    \times \{4k((s')^2)^{2k-1}H - 3(2k+1)((s')^2)^{4k}\} \frac{1}{H^{5/2}} \quad (4.27)
\]

where \(H(s')\) is parametrized by \(r'\) and \(m'_0\), which are kept fixed in the limit \(a \to 0\). Note that the factor \((\sum_{\beta} \frac{s_{\beta}^2}{c_\beta})\) in the numerator, which is written in terms of the original variables, is \(O(\epsilon^2)\). The above integral is convergent in this limit \(a \to 0\) and of order \(O(\epsilon^2)\), and thus it can be made arbitrarily small. We can even make it vanish precisely by taking the limit (4.24). We can thus ignore the contribution of the “kinetic” term, which arises from the interference term in the denominator, in the above limit (4.21).

We now come to the main contribution of the “mass terms” in (4.3). It turns out to be more convenient to go back to (3.25), which gives rise to (4.3). If one uses the notation of (3.4) instead of (3.7), (3.25) is written as

\[
    \sum_{n=0}^{15} \left( \frac{-1}{2} \right)^n \frac{1}{a^4} \int_{-\pi/2}^{\pi/2} \frac{d^4p}{(2\pi)^4} tr\gamma_5 \{[r(\sum_{\mu}(1 \pm c_\mu)]^{2k+1} - (m_0)^{2k+1}\} \frac{1}{\sqrt{F(k)(n, p_\mu)}} \quad (4.28)
\]

where

\[
    F(k)(n, p_\mu) \equiv \{(i\not{s'} + \sum_{\mu} \frac{a\gamma_\mu c_\mu D_\mu)^2}{}}^{2k+1}
    +\{(r(\sum_{\mu}(1 \pm c_\mu)]^{2k+1} - (m_0)^{2k+1}\}^2. \quad (4.29)
\]

The summation runs over the 16 would-be species doublers and the factor \(r(\sum_{\mu}(1 \pm c_\mu)]\) arises from each momentum domain. For later convenience, we modified the denominator factor in (3.25) by replacing \(s^2 + ig\frac{a^2}{4}[\gamma^\mu, \gamma^\nu]c_\mu c_\nu F_{\mu\nu}\) with \((i\not{s'} + \sum_{\mu} a\gamma_\mu c_\mu D_\mu)^2\), but this does not change the result as was explained in detail in the passage from (3.21) to (3.25) in Section 3.2.

In the present integral, we can also define the domain

\[
    -\epsilon \leq p_\mu \leq \epsilon \quad (4.30)
\]

for arbitrarily small but finite \(\epsilon\). One can again confirm that the integral outside this domain vanishes at least linearly in \(a\) for \(a \to 0\). This is because we retain the second order term in \(a^2[\gamma^\mu, \gamma^\nu]F_{\mu\nu}\) to survive the trace with \(\gamma_5\), and this cancels the factor \(1/a^4\) in front of the integral. The resulting integral is finite outside the above domain, and it vanishes at least linearly in \(a\) in the limit (4.21). We can also let \(\epsilon \to 0\) as in (4.24).
We thus examine (4.28) inside the domain (4.30)

\[
\sum_{n=0}^{15} \frac{(-1)^n}{2} \frac{1}{a^4} \int_{-\epsilon}^{\epsilon} \frac{d^4p}{(2\pi)^4} \frac{[r \sum_{\mu}(1 \pm c_\mu)]^{2k+1} - (m_0)^{2k+1}}{\sqrt{F(k)(n, p_\mu)}}
\]

\[
= \sum_{n=0}^{15} \frac{(-1)^n}{2} \frac{1}{(2\pi)^4} \int_{-\epsilon/a}^{\epsilon/a} \frac{d^4k}{\sqrt{F(k)(n, ak_\mu)}}
\]

For sufficiently small \( \epsilon \), we have

\[
[r \sum_{\mu}(1 \pm \cos ak_\mu)]^{2k+1} - (m_0)^{2k+1} = M_n^{(2k+1)} + O(\epsilon^2)
\]

\[
F(k)(n, ak_\mu)/a^{2(2k+1)} = (i k' + D)^{2(2k+1)} + (M_n^{(2k+1)})^2 + O(\epsilon^2)
\]

(4.32)

where the mass parameters \( M_n^{(2k+1)} \) are defined in (2.20) and (2.21). The above integral in the limit \( a \to 0 \) with (4.24) approaches

\[
-\frac{1}{2} \sum_{n=0}^{15} \frac{(-1)^n}{2} \frac{1}{x^{2k+1} + 1} \frac{M_n^{(2k+1)}}{\sqrt{((i k' + D)^2/M_0^2)^{2k+1} + 1}}
\]

(4.33)

by recalling \( \hat{M}_0 < 0 \), and \( \hat{M}_n^2 = [(M_n^{(2k+1)})^2]^{1/(2k+1)} \). The sum of integrals in (4.33) gives rise to the anomaly for all \( M_n^2 \to \infty \) in the final stage[15]

\[
\lim_{M_n^2 \to \infty} tr \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} e^{-ikx} \gamma_5 f \left( \frac{D^2}{M_n^2} \right) e^{ikx} = \frac{g^2}{32\pi^2} tr e^{\mu\nu} F_{\mu\nu} F_{\alpha\beta}.
\]

(4.34)

Here we defined

\[
f(x) = \frac{1}{\sqrt{x^{2k+1} + 1}}
\]

(4.35)

which satisfies

\[
f(0) = 1, \quad f(\infty) = 0,
\]

\[
f'(x)x|_{x=0} = f'(x)x|_{x=\infty} = 0.
\]

(4.36)

The left-hand side of (4.34) is known to be independent of the choice of \( f(x) \) which satisfies the mild condition (4.36)[6].

By combining (1.15), (3.4) and (4.34), we recover the Atiyah-Singer index theorem (in continuum \( R^4 \) space)[16][17]

\[
n_+ - n_- = \int dx \frac{g^2}{32\pi^2} tr e^{\mu\nu} F_{\mu\nu} F_{\alpha\beta}
\]

(4.37)

in the (naive) continuum limit for any operator in (1.1), if one follows the construction of \( \gamma_5 D \) in [1].
5 Discussion

We have shown in an explicit and elementary manner that a new class of lattice Dirac operators proposed in [1] satisfy the correct anomaly relation. We have in particular shown that the anomaly coefficient is independent of a small variation in the parameters $r$ and $m_0$, which characterize these Dirac operators. This is in agreement with the more general but formal analysis in [1].

The Dirac operator in [1] is defined in a somewhat indirect manner as

$$\det H = (\det (H(2k+1)))^{1/(2k+1)}$$

as in (A.9) in Appendix. This definition is sufficient for the non-perturbative analysis, and as we have shown in this paper, it is also sufficient to evaluate the chiral anomaly explicitly. However, the perturbative treatment of these general class of Dirac operators is not well understood yet. It would therefore be interesting to extend the analyses in [14][18], for example, to the present general class of operators.

As for the locality issue of the present class of operators, the fact that the anomaly calculation in a naive continuum limit makes sense suggests that there is a certain range of gauge field configurations which make these operators local. The fact that we take a $2k+1$th root of $H(2k+1)$ in (5.1) by itself may not spoil much of the locality, since the eigenfunctions in (A.17) are defined in terms of $H(2k+1)$ and thus they may reflect the locality properties of $H(2k+1)$, which may not differ qualitatively from those of the overlap operator[19][20]. In any case, a direct analysis of this locality issue is left as an important problem.

Our construction of the Dirac operator (5.1) on the basis of the defining algebraic relation (1.1) suggests that we obtain better chiral properties if one increases the parameter $k$. An intuitive argument for this expectation is that the right-hand side $2a^{2k+1}(\gamma_5 D)^{2k+2}$ of the algebra (1.1), which breaks chiral symmetry, becomes more irrelevant for larger $k$ in the sense of renormalization group. At the same time, however, our construction requires a larger lattice for larger $k$ since the basic operator appearing in our construction spreads over far apart lattice points for large $k$. To maintain the locality of the Dirac operator, we need to take a smaller lattice spacing for larger $k$.

As for the chiral fermions[21][22], the present calculation of anomaly is readily extended to the evaluation of the fermion number anomaly of chiral theory and also to the so-called covariant form of non-Abelian anomalies[6] in the continuum limit. But the construction of chiral fermion theory at finite lattice spacing is a challenging unsolved problem not only in our general Dirac operators but also in the original overlap operator[3].

T-W. Chiu has recently informed us that a numerical study of some of basic properties of the operator $\gamma_5 D$ with $k = 1$, such as index theorem, chiral anomaly and the propagator, is in progress[24].

One of us (KF) thanks Ting-Wai Chiu for numerous helpful discussions from the very beginning of the present investigation.
A Basic construction of general Dirac operators

We start with (1.1) written in the form
\[ H \gamma_5 + \gamma_5 H = 2H^{2k+2} \] (A.1)
or equivalently
\[ \Gamma_5 H + H \Gamma_5 = 0 \] (A.2)
where \( H = a \gamma_5 D \) and \( \Gamma_5 = \gamma_5 - H^{2k+1} \). This algebraic relation implies that
\[ \gamma_5 H^2 = [\gamma_5 H + H \gamma_5] H - H[\gamma_5 H + H \gamma_5] + H^2 \gamma_5 = H^2 \gamma_5. \] (A.3)
Namely, the algebraic relation (A.1) is equivalent to the two relations
\[ \gamma_5 H^2 - H^2 \gamma_5 = 0 \] (A.4)
If one defines \( H_{(2k+1)} \equiv H^{2k+1} \), the first relation of (A.4) becomes
\[ H_{(2k+1)} \gamma_5 + \gamma_5 H_{(2k+1)} = 2H^2_{(2k+1)} \] (A.5)
with \( \Gamma_5 = \gamma_5 - H_{(2k+1)} \), which is identical to the conventional Ginsparg-Wilson relation with \( k = 0 \) in (1.1). We utilize this property to construct a solution to (A.1).

The physical condition for the operator \( H \) in (A.1) in the near continuum configuration is
\[ H \simeq \gamma_5 a \delta + \gamma_5 (\gamma_5 a \delta)^{2k+2} \] (A.6)
where the first term stands for the leading term in chiral symmetric terms, and the second term stands for the leading term in chiral symmetry breaking terms. Thus \( H_{(2k+1)} \) should satisfy
\[ H_{(2k+1)} \simeq [\gamma_5 a \delta + \gamma_5 (\gamma_5 a \delta)^{2k+2}]^{2k+1} \]
\[ \simeq (\gamma_5 a \delta)^{2k+1} + \gamma_5 (\gamma_5 a \delta)^{2(2k+1)} \] (A.7)
as can be confirmed by noting \( \gamma_5 \delta + \delta \gamma_5 = 0 \). Here only the leading terms in chiral symmetric and chiral symmetry breaking terms, respectively, are written.

One can thus construct a solution for \( H_{(2k+1)} \) by
\[ H_{(2k+1)} = \frac{1}{2} \gamma_5 [1 + \gamma_5 H_W^{(2k+1)} \sqrt{H_W^{(2k+1)} / H_W^{(2k+1)}}^1] \] (A.8)
in terms of the hermitian \( H_W^{(2k+1)} \equiv \gamma_5 D_W^{(2k+1)} = (H_W^{(2k+1)})^\dagger \). The operator \( D_W^{(2k+1)} \) is defined in (2.3). The physical condition (A.7) is satisfied by (2.19), as was noted in the body of the text.

We now discuss how to reconstruct \( H \), which satisfies (A.1), from \( H_{(2k+1)} \) defined above. The basic idea is to define in the representation where \( H_{(2k+1)} \) is diagonal
\[ H = (H_{(2k+1)})^{1/(2k+1)} \] (A.9)
in such a manner that $H$ thus obtained satisfies the second constraint in (A.4). For this purpose, we first recall the essence of the general representation of the algebra (A.1)[1].

If one defines the eigenvalue problem

$$H_{(2k+1)}\phi_n = (a\lambda_n)^{2k+1}\phi_n, \quad (\phi_n, \phi_n) = 1$$  \hspace{1cm} (A.10)

one can classify the eigenstates into the 3 classes:

(i) $n_\pm$ ("zero modes"),

$$H_{(2k+1)}\phi_n = 0, \quad \gamma_5\phi_n = \pm\phi_n,$$  \hspace{1cm} (A.11)

(ii) $N_\pm$ ("highest states"),

$$H_{(2k+1)}\phi_n = \pm\phi_n, \quad \gamma_5\phi_n = \pm\phi_n, \quad \text{respectively},$$  \hspace{1cm} (A.12)

(iii) "paired states" with $0 < |(a\lambda_n)^{2k+1}| < 1$,

$$H_{(2k+1)}\phi_n = (a\lambda_n)^{2k+1}\phi_n, \quad H_{(2k+1)}(\Gamma_5\phi_n) = -(a\lambda_n)^{2k+1}(\Gamma_5\phi_n),$$  \hspace{1cm} (A.13)

where

$$\Gamma_5 = \gamma_5 - H_{(2k+1)}.$$  \hspace{1cm} (A.14)

Note that $\Gamma_5(\Gamma_5\phi_n) \propto \phi_n$ for $0 < |(a\lambda_n)^{2k+1}| < 1$.

We have a chirality sum rule[23]

$$n_+ + N_+ = n_- + N_-$$  \hspace{1cm} (A.15)

where $N_\pm$ stand for the number of "highest states" in the classification (ii) above.

All the states $\phi_n$ with $0 < |(a\lambda_n)^{2k+1}| < 1$, which appear pairwise with $(a\lambda_n)^{2k+1} = \pm|(a\lambda_n)^{2k+1}|$, can be normalized to satisfy the relations

$$\Gamma_5\phi_n = [1 - (a\lambda_n)^{2(2k+1)}]^{1/2}\phi_{-n},$$
$$\gamma_5\phi_n = (a\lambda_n)^{2k+1}\phi_n + [1 - (a\lambda_n)^{2(2k+1)}]^{1/2}\phi_{-n},$$  \hspace{1cm} (A.16)

where $\phi_{-n}$ stands for the eigenstate with an eigenvalue opposite to that of $\phi_n$.

We can define the solution $H$ of (A.1) operationally by

$$H\phi_n \equiv a\lambda_n\phi_n$$  \hspace{1cm} (A.17)

by using the same set of eigenfunctions and (the real $2k + 1$th roots of) eigenvalues

$$\{\phi_n\}, \quad \{a\lambda_n\}$$  \hspace{1cm} (A.18)

as for $H_{(2k+1)}$ in (A.10). Note that the operator $\Gamma_5 = \gamma_5 - H_{(2k+1)} = \gamma_5 - H^{2k+1}$, which reverses the signature of eigenvalues of "paired states" and defines the index, is identical to (A.2) and (A.5).
We can confirm the second constraint in (A.4) and the defining algebraic relation (A.2) for any $\phi_n$ in (A.17) by using (A.16),

$$[H^2\gamma_5 - \gamma_5 H^2]\phi_n = 0$$

$$[\Gamma_5 H + H\Gamma_5]\phi_n = 0.$$  \hspace{1cm} (A.19)

The general representation of the algebra (A.1) is obtained from the standard representation, which is defined by $H$ and $\{\phi_n\}$ in (A.17), and $\gamma_5$ in (A.16), by applying a suitable unitary transformation.

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