Abstract The formalism of Quantum Mechanics is based by definition on conserving probabilities and thus there is no room for the description of dissipative systems in Quantum Mechanics. The treatment of time-irreversible evolution (the arrow of time) is therefore ruled out by definition in Quantum Mechanics. In Quantum Field Theory it is, however, possible to describe time-irreversible evolution by resorting to the existence of infinitely many unitarily inequivalent representations of the canonical commutation relations (ccr). In this paper I review such a result by discussing the canonical quantization of the damped harmonic oscillator (dho), a prototype of dissipative systems. The irreversibility of time evolution is expressed as tunneling among the unitarily inequivalent representations. Canonical quantization is shown to lead to time dependent SU(1,1) coherent states. The exact action for the dho is derived in the path integral formalism of the quantum Brownian motion developed by Schwinger and by Feynman and Vernon. The doubling of the phase-space degrees of freedom for dissipative systems is related to quantum noise effects. Finally, the role of dissipation in the quantum model of the brain and the occurrence that the cosmological arrow of time, the thermodynamical one and the biological one point into the same direction are shortly mentioned.

1. Introduction

The formalism of Quantum Mechanics (QM) is based on conserving probabilities. In principle, therefore, there is no room for the description of time-irreversible evolution (the arrow of time) in QM. One has to introduce some sort of generalized quantum formalism in order to describe dissipative systems. The developments of the theory of unstable states going beyond the Breit-Wigner treatment and other phenomenological approaches have been frequently reported in the literature. See for example refs. [1]-[5].
Dissipative systems have been analyzed in the path integral formalism by Schwinger \[6\] and by Feynman and Vernon \[7\] from the point of view of the quantum theory for Brownian motion and are of course a major topic in non-equilibrium statistical mechanics and non-equilibrium Quantum Field Theory (QFT) at finite temperature \[8\]-\[11\].

In this paper I report on the results \[12\]-\[16\] on dissipative systems in quantum theory which show that QFT does allow a correct treatment of the arrow of time provided the full set of unitarily inequivalent (ui) representations of the canonical commutation relations (ccr) is used. I show \[17\] that the proper algebraic structure of QFT is the deformed Hopf algebra \[18, 19\] and that the doubling of the phase-space degrees of freedom implied by such a structure is related to quantum noise effects in the case of dissipative systems \[15\].

The microscopic theory for a dissipative system must include the details of the processes responsible for dissipation, including quantum effects. One may start since the beginning with a Hamiltonian that describes the system, the bath and the system-bath interaction. Subsequently, the description of the original dissipative system is recovered by the reduced density matrix obtained by eliminating the bath variables which originate the damping and the fluctuations. The problem with dissipative systems in QM is indeed that ccr are not preserved by time evolution due to damping terms. The rôle of fluctuating forces is in fact the one of preserving the canonical structure.

It is known since long time \[20\] that, at a classical level, the attempt to derive from a variational principle the equations of motion defining the dissipative system requires the introduction of additional complementary equations.

This latter approach has been pursued since several years also in context of quantum theory. In refs. \[21\] and \[12\]-\[16\] the quantization of the damped harmonic oscillator (dho) has been studied by doubling the phase-space degrees of freedom (see also \[22\] for the study of unstable particle in QFT). The doubled degrees of freedom play the rôle of the bath degrees of freedom. Let me observe that the canonical formalism is devised solely for closed systems, therefore in order to produce the canonical quantization of the damped oscillator, it is necessary and sufficient to close the system, namely to "balance" the energy flux, the momentum exchange, etc.. For that task, and only for that task, we do not really need to know the details of the environment and not even the details of the system-environment coupling: therefore, for such a limited task, we may "simulate" the environment as a collection of oscillators whose k-modes match the k-modes of our damped oscillator. With such a choice the environment, depicted as the system time-reversed double, is treated as the "effective" environment. Of course, in those cases in which such a crude simplification is not enough (we might be really interested in the details of the system-environment interface, for example) much more care is needed and the doubling picture is not enough. In Sec. 2 I present the approach based on the system doubling.

I would like to stress that the analysis for dissipative systems and the arrow of time presented in this paper should not be considered to be something just formal. It is a real problem the one of the description of open systems in a mathematically consistent formalism in QFT. QFT is in fact the only available theoretical scheme to describe high energy physics, as well as condensed matter physics, and quantum systems are always open systems interacting with their environment. It is true that in many cases the approximation of treating them as closed systems is very useful and successful for phenomenological computations, nevertheless there are many cases in which dissipative effects and breakdown of time reversal symmetry cannot be neglected. In these latter circumstances we do need a reliable, mathematically consistent QFT formalism.

The approach here presented has revealed to be useful in several applications of physical interest, ranging from unstable particles \[22\], to coherence in quantum Brownian motion \[23\], squeezed states in quantum optics \[12, 24, 25\], topologically massive theories in the infrared
region in 2+1 dimensions [26], the Chern-Simons-like dynamics of Bloch electrons in solids [26], and has features also common to two-dimensional gravity models [27], to the study of quantization arising from the loss of information [28, 29], to the quantization of matter in curved background [30]. Moreover, it has been applied [31, 32, 33] to the study of the memory capacity problem in the quantum model for the brain [34].

It has been known [21] that in QM time evolution of the dho leads out of the Hilbert space of states; in other words, the QM treatment of dho does not provide a unitary irreducible representation of SU(1,1) [35]. To cure these pathologies one must move to QFT, where infinitely many unitarily inequivalent representations of the ccr are allowed (in the infinite volume or thermodynamic limit). The reason for this is that the set of the states of the damped oscillator splits into ui representations (i.e. into disjoint folia, in the C*-algebra formalism) each one representing the states of the system at time \( t \): the time irreversible evolution is described as tunneling between ui representations. A remarkable feature of this description thus emerges: at microscopic level the irreversibility of time evolution (the arrow of time) is expressed by the non unitary evolution across the ui representations of the ccr.

I remark that the nature of the ground states of the ui representations is the one of the SU(1,1) generalized coherent states. Furthermore, the squeezed coherent states of light entering quantum optics [36, 24, 25] can be identified [12], up to elements of the group \( G \) of automorphisms of \( su(1,1) \), with the states of the quantum dho.

It has been also shown [14] that the dho states are time dependent thermal states, as expected due to the statistical nature of dissipation. This is reported in Sec. 3. The formalism for the dho turns out to be similar to the one of real time QFT at finite temperature, also called thermo-field dynamics (TFD) [1, 11, 15]. In refs. [37] and [38] such a connection with TFD has been further analyzed and the master equation has been discussed [37].

In ref. [15] the exact action for the dho in the path integral formalism of Schwinger and Feynman and Vernon has been obtained. The initial values of the doubled variables have been related to the probability of quantum fluctuations in the vacuum, a result which is interesting also in the more general case of thermal field theories. I report such results in Sec. 4.

In Sec. 5 I show that the proper algebraic structure of QFT is the Hopf algebra [24], which includes the usually considered Weyl-Heisenberg algebra (WH). I then show [17] that dissipative systems are properly described in the frame of the q-deformed Hopf algebra [18, 19, 39]. The q-deformation parameter turns out to be related with time parameter in the case of dho and with temperature in the case of thermal field theory. In both cases, the q-parameter acts as a label for the ui representations. Such a conclusion confirms a general analysis [10] which shows that the Weyl representations in QM and the ui representations in QFT are indeed labeled by the deformation parameter.

Sec. 6 is devoted to the conclusions. There I mention some recent developments which point to the rôle of dissipation in the quantization procedure [28, 29] and I also shortly recall the rôle of dissipation in the quantum model of the brain [31, 34] and on the occurrence that the cosmological arrow of time, the thermodynamical one and the biological one point into the same direction [32, 33].

2. The damped harmonic oscillator

In this section I want to perform the canonical quantization of the damped harmonic oscillator with classical equation

\[
 m\ddot{x} + \gamma \dot{x} + \kappa x = 0 \quad .
\] (1)
In the following I closely follow the approach of refs. [12]-[14] and [21]. The canonical quantization scheme can only deal with an isolated system. It is then necessary to double the phase-space dimensions [20, 21] in order to close the system (1). The closed system Lagrangian is then obtained as

\[ L = m\dot{x}\dot{y} + \frac{1}{2}\gamma(xy - \dot{x}\dot{y}) - \kappa xy \]  

Eq. (1) is obtained by varying eq. (2) with respect to \( y \), whereas variation with respect to \( x \) gives

\[ m\ddot{y} - \gamma \dot{y} + \kappa y = 0 \]

which appears to be the time reversed \((\gamma \rightarrow -\gamma)\) of eq. (1). \( y \) may be thought of as describing an effective degree of freedom for the heat bath to which the system (1) is coupled. The canonical momenta are then given by \( p_x \equiv \frac{\partial L}{\partial \dot{x}} = m\dot{y} - \frac{1}{2}\gamma y \); \( p_y \equiv \frac{\partial L}{\partial \dot{y}} = m\dot{x} + \frac{1}{2}\gamma x \). The Hamiltonian is

\[ H = p_x \dot{x} + p_y \dot{y} - \frac{1}{m} p_x p_y + \frac{1}{2m} \gamma (yp_y - xp_x) + \left( \kappa - \frac{\gamma^2}{4m} \right) xy \]  

For a discussion of Hamiltonian systems of this kind see also [41]. Canonical quantization is performed by introducing the commutators \([x, p_x] = i\hbar = [y, p_y], [x, y] = 0 = [p_x, p_y]\), and the corresponding sets of annihilation and creation operators

\[ \alpha \equiv \left( \frac{1}{2\hbar\Omega} \right)^{\frac{1}{2}} \left( \frac{p_x}{\sqrt{m}} - i\sqrt{m}\Omega x \right) \quad \alpha^\dagger \equiv \left( \frac{1}{2\hbar\Omega} \right)^{\frac{1}{2}} \left( \frac{p_x}{\sqrt{m}} + i\sqrt{m}\Omega x \right) \]  

\[ \beta \equiv \left( \frac{1}{2\hbar\Omega} \right)^{\frac{1}{2}} \left( \frac{p_y}{\sqrt{m}} - i\sqrt{m}\Omega y \right) \quad \beta^\dagger \equiv \left( \frac{1}{2\hbar\Omega} \right)^{\frac{1}{2}} \left( \frac{p_y}{\sqrt{m}} + i\sqrt{m}\Omega y \right) \]  

\[ [\alpha, \alpha^\dagger] = 1 = [\beta, \beta^\dagger] \quad [\alpha, \beta] = [\alpha, \beta^\dagger] = 0 \]

I have introduced \( \Omega \equiv \left[ \frac{1}{m} \left( \kappa - \frac{\gamma^2}{4m} \right) \right]^{\frac{1}{2}} \), the common frequency of the two oscillators eq. (1) and eq. (3), assuming \( \Omega \) real, hence \( \kappa > \frac{\gamma^2}{4m} \) (case of no overdamping). The Feshbach and Tikochinsky [21] quantum Hamiltonian is then obtained as

\[ H = \hbar\Omega(\alpha^\dagger \beta + \alpha \beta^\dagger) - \frac{i\hbar\gamma}{4m} \left[ (\alpha^2 - \alpha^\dagger \beta^\dagger) - (\beta^2 - \beta \alpha^\dagger) \right] \]  

In Sec. 4 I show that, at quantum level, the \( \beta \) modes allow quantum noise effects arising from the imaginary part of the action [15]. In Sec. 5 the doubling of the degrees of freedom will be shown to be a quite natural operation implied by the physically unavoidable requirement of the additivity of basic observables such as the energy, the angular momentum, etc..

By using the canonical linear transformations \( A \equiv \frac{1}{\sqrt{2}}(\alpha + \beta), B \equiv \frac{1}{\sqrt{2}}(\alpha - \beta) \), \( H \) is written as

\[ H = H_0 + H_I \]  

\[ H_0 = \hbar\Omega(A^\dagger A - B^\dagger B) \quad H_I = i\hbar\Gamma(A^\dagger B^\dagger - AB) \]

where the decay constant for the classical variable \( x(t) \) is denoted by \( \Gamma \equiv \frac{\gamma}{2m} \).

I observe that the states generated by \( B^\dagger \) represent the sink where the energy dissipated by the quantum damped oscillator flows: the \( B \)-oscillator represents the reservoir or heat bath coupled to the \( A \)-oscillator.
The dynamical group structure associated with the system of coupled quantum oscillators is that of SU(1,1). The two mode realization of the algebra su(1,1) is indeed generated by

\[ J_+ = A^\dagger B^\dagger, \quad J_- = J_+^\dagger = AB, \quad J_3 = \frac{1}{2}(A^\dagger A + B^\dagger B + 1), \quad [J_+, J_-] = -2J_3, \quad [J_3, J_\pm] = \pm J_\pm. \]

The Casimir operator \( C \) is \( C^2 \equiv \frac{1}{4} + J_3^2 - \frac{1}{2}(J_+ J_- + J_- J_+) = \frac{1}{4}(A^\dagger A - B^\dagger B)^2 \).

I also observe that \( [H_0, H_I] = 0 \). The time evolution of the vacuum \( |0(\cdot)\rangle \equiv |n_A = 0, n_B = 0 \rangle \), \( A|0\rangle = 0 = B|0\rangle \), is controlled by \( H_I \)

\[
|0(t)\rangle = \exp \left( -i\frac{H}{\hbar} \right) |0\rangle = \exp \left( -i\frac{H_I}{\hbar} \right) |0\rangle,
\]

\[
\lim_{t\to\infty} <0(t)|0 > \equiv \lim_{t\to\infty} \exp (-t\Gamma) = 0.
\]

Notice that once one sets the initial condition of positiveness for the eigenvalues of \( H_0 \), such a condition is preserved by the time evolution since \( H_0 \) is the Casimir operator (it commutes with \( H_I \)). In other words, there is no danger of dealing with energy spectrum unbounded from below. Time evolution for creation and annihilation operators is given by

\[
A \mapsto A(t) = e^{-i\frac{\Gamma}{\hbar}H_I} A e^{i\frac{\Gamma}{\hbar}H_I} = A \cosh (\Gamma t) - B^\dagger \sinh (\Gamma t),
\]

\[
B \mapsto B(t) = e^{-i\frac{\Gamma}{\hbar}H_I} B e^{i\frac{\Gamma}{\hbar}H_I} = B \cosh (\Gamma t) - A^\dagger \sinh (\Gamma t)
\]

and h.c., and the corresponding ones for \( A(t), B(t) \) and h.c. I note that eqs. (14) and (15) are Bogolubov transformations: they are canonical transformations preserving the ccr. Eq. (13) expresses the instability (decay) of the vacuum under the evolution operator \( \exp \left( -i\frac{H_I}{\hbar} \right) \). This means that the QM framework is not suitable for the canonical quantization of the dho. In other words time evolution leads out of the Hilbert space of the states and in ref. [14] it has been shown that the proper way to perform the canonical quantization of the dho is to work in the framework of QFT. In fact for many degrees of freedom the time evolution operator \( U(t) \) and the vacuum are formally (at finite volume) given by

\[
U(t) = \prod_{\kappa} \exp \left( -\frac{\Gamma_{\kappa} t}{2}(\alpha_\kappa^2 - \alpha_\kappa^{i2}) \right) \exp \left( \frac{\Gamma_{\kappa} t}{2}(\beta_\kappa^2 - \beta_\kappa^{i2}) \right)
\]

\[
\quad = \prod_{\kappa} \exp \left( \Gamma_{\kappa} t(A_{\kappa}^\dagger B_{\kappa}^\dagger - A_{\kappa} B_{\kappa}) \right),
\]

\[
|0(t)\rangle = \prod_{\kappa} \frac{1}{\cosh (\Gamma_{\kappa} t)} \exp \left( \tanh (\Gamma_{\kappa} t) A_{\kappa}^\dagger B_{\kappa}^\dagger \right) |0\rangle,
\]

with \( <0(t)|0(t) >= 1 \), \( \forall t \). Using the continuous limit relation \( \sum_{\kappa} \mapsto \frac{V}{(2\pi)^3} \int d^3 \kappa \), in the infinite-volume limit we have (for \( \int d^3 \kappa \Gamma_{\kappa} \) finite and positive)

\[
<0(t)|0 > \to 0 \quad \text{as} \quad V \to \infty \quad \forall t,
\]

and in general, \( <0(t)|0(t') > \to 0 \) as \( V \to \infty \) \( \forall t \) and \( t' \neq t \). At each time \( t \) a representation \( \{|0(t)\rangle \} \) of the ccr is defined and turns out to be unitary to any other representation \( \{|0(t')\rangle \},\forall t' \neq t \) in the infinite volume limit. In such a way the quantum dho evolves in
time through unitary representations of the ccr (tunneling). I remark that $|0(t)>$ is a two-mode time dependent generalized coherent state [22, 13].

One thus see that the Bogolubov transformations, eqs. (14) and (15) can be implemented for every $\kappa$ as inner automorphism for the algebra $su(1,1)_\kappa$. At each time $t$ we have a copy $\{A_\kappa(t), A^\dagger_\kappa(t), B_\kappa(t), B^\dagger_\kappa(t); |0(t)> | \forall \kappa \}$ of the original algebra induced by the time evolution operator which can thus be thought of as a generator of the group of automorphisms of $\bigoplus_\kappa su(1,1)_\kappa$ parameterized by time $t$ (we have a realization of the operator algebra at each time $t$, which can be implemented by Gel’fand-Naimark-Segal construction in the C*-algebra formalism [3]). Notice that the various copies become unitarily inequivalent in the infinite-volume limit, as shown by eqs. (19): the space of the states splits into unitary representations of the ccr each one labeled by time parameter $t$. As usual, one works at finite volume and only at the end of the computations the limit $V \to \infty$ is performed.

3. Thermal features of quantum dissipation

In refs. [13] and [14] it has been shown that the representation $\{|0(t)>\}$ is equivalent to the TFD representation $\{|0(\beta(t)>\}$, thus recognizing the relation between the dho states and the finite temperature states. In particular, one may introduce the free energy functional for the $A$-modes

$$F_A \equiv \langle 0(t)|\left( H_A - \frac{1}{\beta} S_A \right)|0(t)> ,$$

(20)

where $H_A$ is the part of $H_0$ relative to $A$-modes only, namely $H_A = \sum_\kappa \hbar \Omega_\kappa A^\dagger_\kappa A_\kappa$, and the entropy $S_A$ is given by

$$S_A \equiv -\sum_\kappa \left\{ A^\dagger_\kappa A_\kappa \ln \sinh^2(\Gamma_\kappa t) - A_\kappa A^\dagger_\kappa \ln \cosh^2(\Gamma_\kappa t) \right\} .$$

(21)

One then considers the stability condition $\frac{\partial F_A}{\partial \vartheta_\kappa} = 0 \ \forall \kappa$ \ , $\vartheta_\kappa \equiv \Gamma_\kappa t$ to be satisfied in each representation, and using the definition $E_\kappa \equiv \hbar \Omega_\kappa$, one finds

$$N_{A_\kappa}(t) = \sinh^2(\Gamma_\kappa t) = \frac{1}{e^{\beta(t)E_\kappa} - 1} ,$$

(22)

namely the Bose distribution for $A_\kappa$ at time $t$. $\{|0(t)>\}$ is thus recognized to be a representation of the ccr at finite temperature, equivalent to the TFD representation $\{|0(\beta)>\}$ [13, 14]. I also notice that $H_0$ and $H_I$ in eq. (10) are the free Hamiltonian and the generator of Bogolubov transformations, respectively, also in TFD (provided one sets $\Gamma t \equiv \theta(\beta)$ and $\Omega$ is given a proper expression). Use of eq. (21) shows that

$$\frac{\partial}{\partial t}|0(t)> = -\left( \frac{1}{2} \frac{\partial S}{\partial t} \right)|0(t)> .$$

(23)

One thus see that $i\left( \frac{1}{2} \hbar \frac{\partial S}{\partial t} \right)$ is the generator of time translations, namely time evolution is controlled by the entropy variations [22]. It is remarkable that the same dynamical variable $S$ whose expectation value is formally the entropy also controls time evolution: Damping (or, more generally, dissipation) implies indeed the choice of a privileged direction in time evolution (arrow of time) with a consequent breaking of time-reversal invariance. One may also show that $dF_A = dE_A - \frac{1}{\beta} dS_A = 0$, which expresses the first principle of thermodynamics for a system coupled with environment at constant temperature and in absence of mechanical work. One may
define as usual heat as \( dQ = \frac{1}{\gamma} dS \) and see that the change in time \( d\mathcal{N}_A \) of particles condensed in the vacuum turns out into heat dissipation \( dQ \).

It is interesting to observe that the thermodynamic arrow of time, whose direction is defined by the increasing entropy direction, points in the same direction of the cosmological arrow of time, namely the inflating time direction for the expanding Universe. This can be shown by considering indeed the quantization of inflationary models [44] (see also [30]). The concordance between the two arrows of time (also with the psychological arrow of time, cf. Sec. 6) is not at all granted and is a subject of an ongoing debate (see, e.g., [45]).

4. Quantum noise and the doubling of the degrees of freedom

Let me now ask the following question: Does the doubling of the degrees of freedom, namely the introduction of an “extra coordinate”, make any sense in the context of conventional QM? To answer to such a question I consider the special case of zero mechanical resistance. Let me begin with the Hamiltonian for an isolated particle and the corresponding density matrix equation

\[
H = -(\hbar^2/2m)(\partial/\partial Q)^2 + V(Q). \tag{24}
\]

\[
i\hbar(\partial \rho/\partial t) = [H, \rho], \tag{25}
\]

which indeed requires two coordinates (say \( Q_+ \) and \( Q_- \)). In the coordinate representation, we have [15]

\[
i\hbar(\partial \rho/\partial t) < Q_+ |\rho(t)|Q_- >= \tag{26}
\]

\[
\{- (\hbar^2/2m)[(\partial/\partial Q_+)^2 - (\partial/\partial Q_-)^2] + [V(Q_+) - V(Q_-)] \} < Q_+ |\rho(t)|Q_- >. \tag{27}
\]

In terms of the coordinates \( x \) and \( y \), it is \( Q_\pm = x \pm (1/2)y \), and the density matrix function \( W(x, y, t) = < x + (1/2)y |\rho(t)| x - (1/2)y > \). From eq. (27) the Hamiltonian now reads \( H_0 = (p_x p_y/m) + V(x + (1/2)y) - V(x - (1/2)y) \), with \( p_x = -i\hbar(\partial/\partial x), \ p_y = -i\hbar(\partial/\partial y) \), which, of course, may be constructed from the “Lagrangian”

\[
\mathcal{L}_0(\dot{x}, \dot{y}, x, y) = m\dot{x}\dot{y} - V(x + (1/2)y) + V(x - (1/2)y), \tag{28}
\]

One has then the justification for introducing eq. (3) at least for the case \( \gamma = 0 \). Notice indeed that for \( V(x \pm (1/2)y) = (1/2)k(x \pm (1/2)y) \) eq. (28) gives eq. (2) for the case \( \gamma = 0 \).

Next, my task is to explore the manner in which the Lagrangian model for quantum dissipation of refs. [12] - [16], [21] arises from the formulation of the quantum Brownian motion problem as described by Schwinger [3] and by Feynman and Vernon [4].

Let me suppose that the particle interacts with a thermal bath at temperature \( T \). The interaction Hamiltonian between the bath and the particle is taken as \( H_{int} = -fQ \), where \( Q \) is the particle coordinate and \( f \) is the random force on the particle due to the bath. In the Feynman-Vernon formalism the effective action for the particle has the form

\[
\mathcal{A}[x, y] = \int_{t_i}^{t_f} dt \mathcal{L}_0(\dot{x}, \dot{y}, x, y) + \mathcal{I}[x, y], \tag{29}
\]

where \( \mathcal{L}_0 \) is defined in eq. (28) and

\[
e^{i(\hbar)\mathcal{I}[x, y]} = < (e^{-i(\hbar)\int_{t_i}^{t_f} f(t)Q-(t)dt})_+ \rho(t) Q_+(t) dt (e^{i(\hbar)\int_{t_i}^{t_f} f(t)Q_-(t)dt})_+ > . \tag{30}
\]
In eq. (31) the average is with respect to the thermal bath; “(.)_+” denotes time ordering and “(.)_−” denotes anti-time ordering. If the interaction between the bath and the coordinate Q were turned off, then the operator f of the bath would develop in time according to 
\[ f(t) = e^{iH_R t/\hbar} f e^{-iH_R t/\hbar} \]
where \( H_R \) is the Hamiltonian of the isolated bath (decoupled from the coordinate Q). \( f(t) \) is the force operator of the bath to be used in eq. (31). Assuming that the particle makes contact with the bath at the initial time \( t_i \), the reduced density matrix function is at a final time

\[ W(x_f, y_f, t_f) = \int_{-\infty}^{\infty} dx_i \int_{-\infty}^{\infty} dy_i K(x_f, y_f, t_f; x_i, y_i, t_i) W(x_i, y_i, t_i), \quad (31) \]

\[ K(x_f, y_f, t_f; x_i, y_i, t_i) = \int_{x(t_i)=x_i}^{x(t_f)=x_f} D x(t) \int_{y(t_i)=y_i}^{y(t_f)=y_f} D y(t) e^{i(\hbar/\hbar) A[x,y]}, \quad (32) \]

The correlation function for the random force on the particle is given by \( G(t-s) = (i/\hbar) < f(t)f(s) > \). The retarded and advanced Greens functions are defined by \( G_{ret}(t-s) = \theta(t-s)[G(t-s) - G(s-t)] \) and \( G_{adv}(t-s) = \theta(s-t)[G(s-t)-G(t-s)] \). The mechanical resistance is defined \( R = \lim_{\omega \to 0} ReZ(\omega + i0^+) \) with the mechanical impedance \( Z(\zeta) \) (analytic in the upper half complex frequency plane \( Im \zeta > 0 \)) determined by the retarded Greens function \( -i\zeta Z(\zeta) = \int_0^\infty dt G_{ret}(t)e^{i\zeta t} \). The time domain quantum noise in the fluctuating random force is \( N(t-s) = (1/2) < f(t)f(s) + f(s)f(t) > \).

The time ordered and anti-time ordered Greens functions describe both the retarded and advanced Greens functions as well as the quantum noise,

\[ G_\pm(t-s) = \pm(1/2)[G_{ret}(t-s) + G_{adv}(t-s)] + (i/\hbar)N(t-s). \quad (33) \]

The interaction between the bath and the particle is evaluated by following Feynman and Vernon and we find (13) for the real and the imaginary part of the action

\[ ReA[x,y] = \int_{t_i}^{t_f} dt \mathcal{L}, \quad (34) \]

\[ \mathcal{L} = m \dot{x}\dot{y} - [V(x + (1/2)y) - V(x - (1/2)y)] + (1/2)[xF_{ret}^y + yF_{adv}^x], \quad (35) \]

\[ ImA[x,y] = (1/2\hbar) \int_{t_i}^{t_f} \int_{t_i}^{t_f} dt ds N(t-s)\dot{y}(t)y(s), \quad (36) \]

respectively, where the retarded force on \( y \) and the advanced force on \( x \) are defined as \( F_{ret}^y(t) = \int_{t_i}^{t_f} ds G_{ret}(t-s)y(s), \quad F_{adv}^x(t) = \int_{t_i}^{t_f} ds G_{adv}(t-s)x(s). \)

Eqs. (34) - (36) are rigorously exact for linear passive damping due to the bath when the path integral eq. (32) is employed for the time development of the density matrix.

I therefore conclude that the lagrangian eq. (2) can be viewed as the approximation to eq. (35) with \( F_{ret}^y = \gamma \dot{y} \) and \( F_{adv}^x = -\gamma \dot{x} \).

I also observe that at the classical level the “extra” coordinate \( y \), is usually constrained to vanish. Note that \( y(t) = 0 \) is a true solution to eqs. (3) so that the constraint is not in violation of the equations of motion. From eqs. (34) - (36) one sees that at quantum level nonzero \( y \) allows quantum noise effects arising from the imaginary part of the action. On the contrary, in the classical “\( \hbar \to 0 \)” limit nonzero \( y \) yields an “unlikely process” in view of the large imaginary part of the action implicit in eq. (34). Thus, the meaning of the constraint \( y = 0 \) at the classical level is the one of avoiding such “unlikely process”.
5. Hopf algebra, q-deformation and quantum dissipation

Quantum deformations [18, 39] of Lie algebras are well studied mathematical structures and therefore their properties need not to be presented again in this paper. I only recall that they are deformations of the enveloping algebras of Lie algebras and have Hopf algebra structure [19]. In this Section I will show [17] that dissipative systems (as well as the finite temperature non-equilibrium systems) are properly described in the frame of the q-deformed Hopf algebra. Moreover, I will argue that the proper algebraic structure of QFT is the deformed Hopf algebra. The q-deformation parameter turns out to be related with the time parameter in the case of dho (and with temperature in the case of thermal field theory). In both cases, the q-parameter acts as a label for the ui representations.

I observe that one central ingredient of Hopf algebras is the operator doubling implied by the coalgebra. The coproduct operation is indeed a map \( \Delta : A \rightarrow A \otimes A \) which duplicates the algebra. Lie-Hopf algebras are commonly used in the familiar addition of energy, momentum and angular momentum, e.g., for the "addition" of the angular momentum \( J^\alpha, \alpha = 1, 2, 3 \), of two particles one has: \( \Delta J^\alpha = J^\alpha \otimes 1 + 1 \otimes J^\alpha \equiv J^\alpha_1 + J^\alpha_2 \), \( J^\alpha \in su(2) \). Thus, the physical meaning of the coproduct is that it provides the prescription for operating on two modes.

In the following, for simplicity, let me focus on the case of bosons. The conclusions can also be extended to fermions [17].

The bosonic Hopf algebra, also called \( h(1) \), is generated by the set of operators \( \{ a, a^\dagger, H, N \} \) with commutation relations:

\[
[a, a^\dagger] = 2H, \quad [N, a] = -a, \quad [N, a^\dagger] = a^\dagger, \quad [H, \bullet] = 0.
\] (37)

Here \( a \) and \( a^\dagger \) denote generic annihilation and creation operators. For notational simplicity I omit the momentum suffix \( \kappa \) which will be restored later on. Later we will see how the present discussion relates to the dho operators introduced in the previous Sections. \( H \) is a central operator, constant in each representation. The Casimir operator is given by \( C = 2NH - a^\dagger a \). \( h(1) \) is equipped with the coproduct operation, defined by:

\[
\Delta a = a \otimes 1 + 1 \otimes a \equiv a_1 + a_2 , \quad \Delta a^\dagger = a^\dagger \otimes 1 + 1 \otimes a^\dagger \equiv a^\dagger_1 + a^\dagger_2 ,
\] (38)

\[
\Delta H = H \otimes 1 + 1 \otimes H \equiv H_1 + H_2 , \quad \Delta N = N \otimes 1 + 1 \otimes N \equiv N_1 + N_2 .
\] (39)

I remark that usually one introduces the operator algebra necessary to set up QFT by limiting himself to the introduction of the boson Weyl-Heisenberg (WH) algebra [37]. The assumption of the additivity of some observables such as the energy, the momentum and the angular momentum is so obvious that one does not even bother to spell it out. It is implicitly given as granted. However, if one is asked to express it explicitly and formally, then it becomes natural to introduce the coproduct map, as shown above, and thus to realize that the boson WH algebra [37] is only a part of the full algebraic structure. One needs the Hopf structure. The full algebraic structure which is needed, however, has to take into account one of the very special features of QFT, the one which characterizes it and makes it different from QM, namely the existence of infinitely many representations of theCCR (in QM all the representations of the CCR are unitary equivalent due to the von Neumann theorem). Then one is led to consider the quantum deformation of the Hopf algebra, as it appears from the following.

The \( q \)-deformation of \( h(1) \) is the Hopf algebra \( h_q(1) \):

\[
[a_q, a_q^\dagger] = [2H]_q , \quad [N, a_q] = -a_q , \quad [N, a_q^\dagger] = a_q^\dagger , \quad [H, \bullet] = 0 ,
\] (40)
where $N_q \equiv N$ and $H_q \equiv H$. The Casimir operator $C_q$ is given by $C_q = N[2H]_q - a_q^\dagger a_q$, where $[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}$. The coproduct is defined by
\[
\Delta a_q = a_q \otimes q^H + q^{-H} \otimes a_q, \quad \Delta a_q^\dagger = a_q^\dagger \otimes q^H + q^{-H} \otimes a_q^\dagger,
\]
whose algebra of course is isomorphic with $\{[39]\}$: $\{\Delta a_q, \Delta a_q^\dagger \} = \{2\Delta H\}_q$, etc. Note that $h_q(1)$ is a structure different from the commonly considered $q$-deformation of the harmonic oscillator that does not have a coproduct (and thus cannot allow for the duplication of space).

Let me denote by $\mathcal{F}_1$ the single mode Fock space, i.e. the fundamental representation $H = 1/2, C = 0$. In such a representation $h(1)$ and $h_q(1)$ coincide as it happens for $su(2)$ and $su_q(2)$ for the spin-$\frac{1}{2}$ representation. The differences appear in the coproduct and in the higher spin representations.

As customary, one requires that $a$ and $a^\dagger$, and $a_q$ and $a_q^\dagger$, are adjoint operators. This implies that $q$ can only be real or of modulus one. In the two mode Fock space $\mathcal{F}_2 = \mathcal{F}_1 \otimes \mathcal{F}_1$, for $|q| = 1$, the hermitian conjugation of the coproduct must be supplemented by the inversion of the two spaces for consistency with the coproduct isomorphism.

Summarizing, on $\mathcal{F}_2 = \mathcal{F}_1 \otimes \mathcal{F}_1$ it can be written:
\[
\Delta a = a_1 + a_2, \quad \Delta a^\dagger = a_1^\dagger + a_2^\dagger,
\]
\[
\Delta a_q = a_1 q^{1/2} + q^{-1/2} a_2, \quad \Delta a_q^\dagger = a_1^\dagger q^{1/2} + q^{-1/2} a_2^\dagger,
\]
\[
\Delta H = 1, \quad \Delta N = N_1 + N_2.
\]

Note that $[a_i, a_j] = [a_i, a_j^\dagger] = 0, \ i \neq j$.

It is now possible to show that the full set of infinitely many unitarily inequivalent representations of the ccr in QFT are classified by use of the deformed Hopf algebra. To do that it is sufficient to show that the Bogolubov transformations are directly obtained by use of the deformed coproduct operation. As well know, indeed, the Bogolubov transformations relate different (i.e. unitary inequivalent) representations. I consider therefore the following operators (cf. $\{11\}$ with $H = 1/2$):
\[
\alpha_q(\theta) \equiv \frac{\Delta a_q}{\sqrt{[2]_q}} = \frac{1}{\sqrt{[2]_q}} (e^{\theta} a_1 + e^{-\theta} a_2),
\]
\[
\beta_q(\theta) \equiv \frac{1}{\sqrt{[2]_q}} \frac{\delta}{\delta \theta} \Delta a_q = \frac{q}{\sqrt{[2]_q}} \frac{\delta}{\delta q} \Delta a_q = \frac{1}{\sqrt{[2]_q}} (e^{\theta} a_1 - e^{-\theta} a_2),
\]
and h.c., with $q(\theta) \equiv e^{2\theta}$. A set of commuting operators with canonical commutation relations is given by
\[
\alpha(\theta) \equiv \frac{\sqrt{[2]_q}}{2\sqrt{2}} [\alpha_q(\theta) + \alpha_q(-\theta) - \beta_q^\dagger(\theta) + \beta_q^\dagger(-\theta)],
\]
\[
\beta(\theta) \equiv \frac{\sqrt{[2]_q}}{2\sqrt{2}} [\beta_q(\theta) + \beta_q(-\theta) - \alpha_q^\dagger(\theta) + \alpha_q^\dagger(-\theta)].
\]
and h.c. One then introduces

\begin{align}
A(\theta) & \equiv \frac{1}{\sqrt{2}} (\alpha(\theta) + \beta(\theta)) = A \cosh \theta - B^\dagger \sinh \theta , \\
B(\theta) & \equiv \frac{1}{\sqrt{2}} (\alpha(\theta) - \beta(\theta)) = B \cosh \theta - A^\dagger \sinh \theta ,
\end{align}

with

\[ [A(\theta), A^\dagger(\theta)] = 1 , \quad [B(\theta), B^\dagger(\theta)] = 1 . \]

All other commutators are equal to zero and \( A(\theta) \) and \( B(\theta) \) commute among themselves.

Eqs. (50) and (51) are nothing but the Bogolubov transformations for the \((A, B)\) pair, to be compared with the corresponding transformations (14) and (15) in the case of the dho. In other words, eqs. (50), (51) show that the Bogolubov-transformed operators \( A(\theta) \) and \( B(\theta) \) are linear combinations of the coproduct operators defined in terms of the deformation parameter \( q(\theta) \) and of their \( \theta \)-derivatives.

From this point on one can re-obtain the results discussed in the previous Sections for the dho, provided one sets \( \theta \equiv \Gamma t \). Notice that

\[ -i \frac{\delta}{\delta \theta} A(\theta) = [G, A(\theta)] , \quad -i \frac{\delta}{\delta \theta} B(\theta) = [G, B(\theta)] , \]

and h.c., where \( G \equiv -i(A^\dagger B^\dagger - AB) \) denotes the generator of (52) and (53). For a fixed value \( \theta \), we have

\[ \exp(i\bar{\theta} p_\theta) A(\theta) = \exp(i\bar{\theta} G) A(\theta) \exp(-i\bar{\theta} G) = A(\theta + \bar{\theta}) , \]

and similar equations for \( B(\theta) \).

In eq. (54) the definition \( p_\theta = -i \frac{\delta}{\delta \theta} \) has been used. It can be regarded as the momentum operator "conjugate" to the "degree of freedom" \( \theta \), which thus acquires formal definiteness in the sense of the canonical formalism. In the infinite volume limit \( < 0(\theta)|0(\theta') > \equiv 0 \). In other words, the deformation parameter \( \theta = \frac{1}{2} \ln q \) acts as a label for the inequivalent representations, consistently with the results of Refs. [10, 24]. It is remarkable that the "conjugate momentum" \( p_\theta \) generates transitions among inequivalent (in the infinite volume limit) representations: \( \exp(i\bar{\theta} p_\theta) |0(\theta) > = |0(\theta + \bar{\theta}) > \).

In conclusion, one obtains, by use of the deformed Hopf algebra, the typical structure one deals with in QFT. In this connection, I observe that variation in time of the deformation parameter is related with the so-called heat-term in dissipative systems. In such a case, in fact, the Heisenberg equation for \( A(t, \theta(t)) \) is

\[ -i \dot{A}(t, \theta(t)) = -i \frac{\delta}{\delta t} A(t, \theta(t)) - i \frac{\delta \theta}{\delta t} \frac{\delta}{\delta \theta} A(t, \theta(t)) = \]

\[ [H, A(t, \theta(t))] + \frac{\delta \theta}{\delta t} [G, A(t, \theta(t))] = [H + Q, A(t, \theta(t))] , \]

where \( Q \equiv \frac{\delta \theta}{\delta t} G \) denotes the heat-term, and \( H \) is the Hamiltonian (responsible for the time variation in the explicit time dependence of \( A(t, \theta(t)) \)). \( H + Q \) is therefore to be identified with the free energy \([14, 24]\). In this way the results of Sec. 3 are also recovered. Thus, the conclusion is that variations in time of the deformation parameter actually involve dissipation.

When the proper field description is taken into account, \( A \) and \( B \) carry dependence on the momentum \( \kappa \) and, as customary in QFT, one deals with the algebras \( \bigoplus_\kappa h_\kappa(1) \) (cf. Sec. 2).
6. Concluding remarks

The dho total Hamiltonian is invariant under the transformations generated by \( J_2 = \bigoplus_\kappa J^{(\kappa)}_2 \). The vacuum however is not invariant under \( J_2 \) (see eq. (19)) in the infinite volume limit. Moreover, at each time \( t \), the representation \( \{ |0(t) \rangle \} \) may be characterized by the expectation value in the state \( |0(t)\rangle \), e.g., \( J^{(\kappa)}_3 - \frac{1}{2} \); thus the total number of particles \( n_A + n_B = 2n \) can be taken as an order parameter. Therefore, at each time \( t \) the symmetry under \( J_2 \) transformations is spontaneously broken. On the other hand, \( H_I \) is proportional to \( J_2 \). Thus, in addition to the breakdown of time-reversal (discrete) symmetry, already mentioned in Section 2, we also have spontaneous breakdown of time translation (continuous) symmetry. In other words, dissipation (i.e. energy non-conservation), has been described as an effect of the breakdown of time translation and time-reversal symmetry. It is an interesting question asking which is the zero-frequency mode, playing the rôle of the Goldstone mode, related with the breakdown of continuous time translation symmetry: I observe that since \( n_A - n_B \) is constant in time, the condensation (annihilation and/or creation) of AB-pairs does not contribute to the vacuum energy so that AB-pair may play the rôle of a zero-frequency mode.

In the discussion presented above a crucial rôle is played by the existence of infinitely many \( u_i \) representations of the ccr in QFT. In refs. \([24, 40]\) the q-WH algebra has been discussed in relation with the von Neumann theorem in QM and it has been shown on a general ground that the q-deformation parameter acts as a label for the Weyl systems in QM and for the \( u_i \) representations in QFT; the mapping between different (i.e. labeled by different values of \( q \)) representations (or Weyl systems) being performed by the Bogolubov transformations. Damped harmonic oscillator and finite temperature systems are explicit examples clarifying the physical meaning of such a labeling. Further examples are provided by unstable particles in QFT \([22]\), by quantization of the matter field in curved space-time \([30]\), by theories with spontaneous breakdown of symmetry where different values of the order parameter are associated to different \( u_i \) representations (different phases). In the case of damping, as well as in the case of time-dependent temperature, the system time-evolution is represented as tunneling through \( u_i \) representations: the non-unitary character of time-evolution (arrow of time) is thus expressed by the non-unitary equivalence of the representations in the infinite volume limit. It is remarkable that at the algebraic level this is made possible through the q-deformation mechanism which organizes the representations in an ordered set by means of the labeling.

In conclusion, from the point of view of boson condensation, time evolution in the presence of damping may be thus thought of as a sort of continuous transition among different phases, each phase corresponding, at time \( t \), to the coherent state representation \( \{ |0(t)\rangle \} \). The damped oscillator thus provides an archetype of system undergoing continuous phase transition.

As already mentioned in the introduction, dissipation in classical deterministic systems (such as the couple of classical oscillators described by eqs. \((1)\) and \((3)\) in Sec. 2) has been shown \([23]\) to lead under suitable conditions to a quantum behavior, as originally proposed by ‘t Hooft \([28]\). In particular, dissipation manifests as a geometric Berry-Anandan-like phase \([24]\) and it appears to be responsible for the zero point energy contribution in the oscillator energy spectrum \([29]\).

The features of the dissipative quantum dynamics discussed in this paper have been also used \([31, 32, 33]\) to implement an infinite memory capacity in the quantum model of brain \([34]\). The key point is in the fact that dissipative dynamics implies infinitely many degenerate vacua (i.e. the zero eigenvalue eigenstates of \( H_0 \), eqs. \((8)\) and \((18)\)), each of them describing a possible memory state according to the brain model of ref. \([34]\). Moreover, in view of the thermal character of such vacua illustrated in Sec. 3, the irreversible time evolution of the brain memory states, which is perceived as the arrow of time at a psychological experience level,
appears to proceed in the same direction of the thermodynamical and of the cosmological arrow of time mentioned in Sec. 3. It is interesting to note that in a somewhat unexpected way it emerges a possible answer to the questions raised by the ongoing debate [45] on the coincidence (or not) of the directions of the three arrows of times just mentioned. Finally, the dissipative quantum model of brain has revealed to be also interesting in the study of features related with consciousness mechanisms [33].

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