Global calculus in BRST cohomology

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Abstract The iterated BRST cohomology is studied by computing cohomology of the variational complex on the infinite order jet space of a smooth fibre bundle. This computation also provides a solution of the global inverse problem of the calculus of variations in Lagrangian field theory.

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1 Introduction

We address the Lagrangian antifield BRST formalism of [3, 4, 6, 13], phrased in terms of exterior forms of finite jet order on the infinite order jet space of physical fields, ghosts and antifields on a base manifold $X$. Horizontal (semibasic) exterior forms constitute a bicomplex with respect to the BRST operator $s$ and the horizontal (total) differential $d_H$. It is graded by the ghost number $k$ and the form degree $p$. We aim to study the iterated $s$-cohomology $H^{k,p}(s|d_H)$ of the $d_H$-cohomology groups of this bicomplex (i.e., the term $E_2^{*,*}$ of its spectral sequence [16]). In terms of form degree $p = n = \dim X$, this cohomology coincides with the well-known local BRST cohomology (i.e., $s$-cohomology modulo $d_H$). If $p < n$, iterated BRST cohomology unlike local BRST cohomology is defined only for $d_H$-closed forms.

The above mentioned BRST formalism is usually formulated on a contractible base $X$ when the $d_H$-cohomology of form degree $0 < p < n$ are trivial in accordance with the algebraic Poincaré lemma (see Lemma 3 below). If $X = \mathbb{R}^n$, there is an isomorphism of the iterated (and local) BRST cohomology groups $H^{k,n}(s|d_H)$, $k \neq -n$, to the cohomology $H^{k+n}_{\text{tot}}$ groups of the total BRST operator $\tilde{s} = s + d_H$ on horizontal exterior forms of total ghost number $k + n \mathbb{R}$. We generalize this result to the case of iterated BRST cohomology of form degree $p < n$ and an arbitrary connected manifold $X$ (see Corollary 2 below). To construct the corresponding (global) descent equations, one needs the $d_H$-cohomology of exterior forms on the infinite order jet space. The study of this cohomology is the key point of our consideration.

Note that the descent equations for representatives of local BRST cohomology groups of form degree $p < n$ are also constructed, but this cohomology fails to be related to cohomology
of the total BRST operator \([1]\) (see also \([9]\) where BRST cohomology modulo the exterior differential \(d\) in the Yang–Mills gauge theory are considered).

To avoid the sophisticate techniques of ghosts and antifields \([17]\), we here study iterated cohomology of the graded differential algebra \(O^*_{\infty}\) of exterior forms of finite jet order on the infinite order jet space \(J^\infty Y\) of an affine bundle \(Y \to X\). Note that affine bundles provide a standard framework in quantum field theory. Let \(O^*_{\infty}\) be endowed with a nilpotent form degree preserving endomorphism \(s\) such that horizontal elements of \(O^*_{\infty}\) constitute a bicomplex with respect to \(s\) and \(d_H\), i.e., \(sd_H + d_Hs = 0\). We agree to call a gradation degree \(k\) with respect to \(s\) the ghost number. Suppose that \(s\) vanishes on exterior forms on \(X\) and that these forms are of zero ghost number and are not \(s\)-exact. The goal is the following.

**THEOREM 1.** If \(Y \to X\) is an affine bundle, the iterated \(s|d_H\)-cohomology is the following.

(i) The \(H^{\neq, p<n}(s|d_H)\) is trivial.

(ii) \(H^{0, p<n}(s|d_H) = H^p(X)\) where \(H^*(X)\) is de Rham cohomology of \(X\).

(iii) \(H^{k,n}(s|d_H) = H^{k+n}_{\text{tot}}, \ k < -n\ or \ k > -1\).

(iv) Let \(\gamma_p : H^p(X) \to H^p_{\text{tot}}, \ 0 \leq p < n, \) be a natural homomorphism corresponding to the monomorphism of the algebra \(O^*(X)\) of exterior forms on \(X\) to \(O^*_{\infty}\). Put \(\overline{H}^p = H^p_{\text{tot}}/\text{Im} \gamma_p\). If \(-n \leq k < -1\), there is a monomorphism \(\overline{H}^{k+n} \to H^{k,n}(s|d_H)\) such that \(H^{k,n}(s|d_H)/\overline{H}^{k+n} = \text{Ker} \gamma_{k+n+1}\). In particular, \(\text{Ker} \gamma_0 = 0\) and \(\overline{H}^0 = H^0_{\text{tot}}/H^0(X)\).

(v) \(H^{-1,n}(s|d_H) = \overline{H}^{n-1}\).

Note that the operator \(s\) in Theorem 1 may have different physical origins. The following corollary of Theorem 1 corresponds to the case of iterated BRST cohomology.

**COROLLARY 2.** Let \(Y \to X\) be a vector bundle and \(P^*_\infty \subset O^*_{\infty}\) a subalgebra of exterior forms which are polynomial in fibre coordinates of \(J^\infty Y \to X\). There is the decomposition of \(C^\infty(X)\)-modules \(P^*_\infty = O^*(X) \oplus (P^*_\infty)_{>0}\). Let \(s(P^*_\infty) \subset (P^*_\infty)^{>0}\). Then, \(\gamma_p, \ 0 \leq p < n, \) are monomorphisms, and the items (iv), (v) of Theorem 1 state isomorphisms \(H^{k,n}(s|d_H) = H^{k+n}_{\text{tot}}/H^{k+n}(X), \ -n \leq k \leq -1\).

In particular, if \(X = \mathbb{R}^n\), the iterated BRST cohomology of form degree \(p < n\) (except \(H^{0,0}(s|d_H) = \mathbb{R}\)) is always trivial in contrast with the local BRST cohomology.

In fact, to prove Theorem 1, we will obtain \(d_H\)– and \(\delta\)-cohomology of the variational complex of the graded differential algebra \(O^*_{\infty}\) in the case of an arbitrary smooth bundle \(Y \to X\).

The \(O^*_{\infty}\) is the direct limit of graded differential algebras of exterior forms on finite order jet manifolds. It consists of exterior forms on finite order jet manifolds modulo the pull-back identification. Passing to the direct limit of the de Rham complexes of exterior forms on finite order jet manifolds, de Rham cohomology of \(O^*_{\infty}\) has been found to coincide with de Rham cohomology of the bundle \(Y\) \(\mathbb{A}, \mathbb{F}\). However, this is not a way of studying other cohomology groups of \(O^*_{\infty}\). Therefore, we enlarge \(O^*_{\infty}\) to the structure algebra \(T^\infty\) of the
sheaf of germs of exterior forms on finite order jet manifolds. One can say that $T^*_\infty$ consists of exterior forms of locally finite jet order on $J^\infty Y$. The $d_H$- and $\delta$-cohomology of $T^*_\infty$ has been investigated in [19]. We simplify this investigation due to Lemma 4 below and prove that $O^*_\infty$ and $T^*_\infty$ have the same $d_H$- and $\delta$-cohomology (see Theorem 9 below). In particular, this provides a solution of the global inverse problem of the calculus of variations in the class of exterior forms of finite jet order.

For the proof of Theorem 1, it is quite important that, if $Y \to X$ is an affine bundle, $d_H$-cohomology $H^{<n}(d_H; O^*_\infty)$ of $O^*_\infty$ coincides with de Rham cohomology of the base $X$. It follows that every $d_H$-closed ($k < n$)-form $\phi \in O^*_\infty$ splits into the sum $\phi = \varphi + d_H \xi$ where $\xi \in O^*_\infty$ and $\varphi$ is a closed form on $X$. Since the operator $s$ annihilates these forms, the system of global descent equations can be constructed though its right-hand side is not zero. We come to the same result for the polynomial algebra $P^*_\infty$ and its subalgebra $P^*_\infty$ of $x$-independent forms.

2 The differential calculus on $J^\infty Y$

Smooth manifolds throughout are assumed to be real, finite-dimensional, Hausdorff, paracompact, and connected. Put further $\dim X = n$. The standard notation of jet formalism [10, 17] is utilized. Following the terminology of [14], by a sheaf $S$ on a topological space $Z$ is meant a sheaf bundle $S \to Z$. Accordingly, $\Gamma(S)$ denotes the canonical presheaf of sections of the sheaf $S$, and $\Gamma(Z, S)$ is the set of global sections of $S$.

Recall that the infinite order jet space $J^\infty Y$ of a smooth bundle $Y \to X$ is defined as a projective limit $(J^\infty Y, \pi_\infty)$ of the inverse system

$$
X \hookrightarrow^\pi Y \hookrightarrow^\pi_1 \cdots \hookrightarrow^\pi_{r-1} J^{r-1}Y \hookrightarrow^\pi_r J^r Y \hookrightarrow \cdots
$$

of finite order jet manifolds $J^r Y$ of $Y \to X$, where $\pi_{r-1}$ are affine bundles. Bearing in mind Borel’s theorem, one can say that $J^\infty Y$ consists of the equivalence classes of sections of $Y \to X$ identified by their Taylor series at points of $X$. Endowed with the projective limit topology, $J^\infty Y$ is a paracompact Fréchet manifold [19]. A bundle coordinate atlas \( \{U_Y, (x^\lambda, y^i)\} \) of $Y$ yields the manifold coordinate atlas \( \{(\pi_0^\infty)^{-1}(U_Y), (x^\lambda, y^i_\Lambda)\} \), $0 \leq |\Lambda|$, of $J^\infty Y$, together with the transition functions

$$
y^i_{\lambda+\Lambda} = \frac{\partial x^\mu}{\partial x'^\lambda} d_\mu y^i_\Lambda, \tag{2}
$$

where $d_\mu$ denotes the total derivative $d_\mu = \partial_\mu + \sum_{|\Lambda| \geq 0} y^i_{\mu+\Lambda} \partial^\Lambda_i$.

With the inverse system (1), one has the direct system

$$
O^*(X) \xrightarrow{\pi^*} O^*_0 \xrightarrow{\pi_0^*} O^*_1 \xrightarrow{\pi_1^*} \cdots \xrightarrow{\pi_{r-1}^*} O^*_r \to \cdots \tag{3}
$$
of \( \mathbb{R} \)-algebras \( \mathcal{O}_r^* \) of exterior forms on finite order jet manifolds, where \( \pi_{r-1}^* \) are pull-back monomorphisms. Its direct limit is the above mentioned graded differential \( \mathbb{R} \)-algebra \( \mathcal{O}_\infty^* \). It is a differential calculus over the \( \mathbb{R} \)-ring \( \mathcal{O}_\infty^0 \) of continuous real functions on \( J^\infty Y \) which are the pull-back of smooth functions on finite order jet manifolds. Let us enlarge the ring \( \mathcal{O}_\infty^0 \) to the \( \mathbb{R} \)-ring \( \mathcal{T}_\infty^0 \) of continuous real functions on \( J^\infty Y \) such that, given \( f \in \mathcal{T}_\infty^0 \) and any point \( q \in J^\infty Y \), there exists a neighborhood of \( q \) where \( f \) coincides with the pull-back of a smooth function on some finite order jet manifold. Let \( \mathcal{O}_r^* \) be a sheaf of germs of exterior forms on the \( r \)-order jet manifold \( J^r Y \) and \( \Gamma(\mathcal{O}_r^*) \) its canonical presheaf. There is the direct system of canonical presheaves

\[
\Gamma(\mathcal{O}_X^*) \xrightarrow{\pi^*} \Gamma(\mathcal{O}_0^*) \xrightarrow{\pi_1^*} \Gamma(\mathcal{O}_1^*) \xrightarrow{\pi_2^*} \cdots \xrightarrow{\pi_{r-1}^*} \Gamma(\mathcal{O}_r^*) \rightarrow \cdots.
\]

Its direct limit \( \mathcal{O}_\infty^* \) is a presheaf of graded differential \( \mathbb{R} \)-algebras on \( J^\infty Y \). Let \( \mathfrak{T}_\infty^* \) be the sheaf constructed from \( \mathcal{O}_\infty^* \) and \( \Gamma(\mathfrak{T}_\infty^*) \) its canonical presheaf. The structure algebra \( \mathfrak{T}_\infty^0 = \Gamma(J^\infty Y, \mathfrak{T}_\infty^*) \) of the sheaf \( \mathfrak{T}_\infty^* \) is a differential calculus over the \( \mathbb{R} \)-ring \( \mathcal{T}_\infty^0 \). There are the \( \mathbb{R} \)-algebra monomorphisms \( \mathcal{O}_r^* \rightarrow \Gamma(\mathfrak{T}_\infty^*) \) and \( \mathcal{O}_\infty^* \rightarrow \Gamma(\mathfrak{T}_\infty^*) \). Since the paracompact space \( J^\infty Y \) admits a partition of unity by elements of \( \mathfrak{T}_\infty^0 \), sheaves of \( \mathfrak{T}_\infty^0 \)-modules on \( J^\infty Y \) are fine and acyclic. Then, the abstract de Rham theorem on cohomology of a sheaf resolution \([14]\) can be called into play in order to obtain cohomology of the algebra \( \mathfrak{T}_\infty^* \).

For short, we agree to call elements of \( \mathfrak{T}_\infty^* \) the exterior forms on \( J^\infty Y \). Restricted to a coordinate chart \((\pi^0_\infty)^{-1}(U_Y)\) of \( J^\infty Y \), they can be written in a coordinate form, where horizontal and contact forms \( \{dx^\lambda; \theta^i_A = dy^i_A - y^{i+\lambda}_A dx^\lambda\} \) provide generators of the algebra \( \mathfrak{T}_\infty^* \). There is the canonical decomposition of \( \mathfrak{T}_\infty^* \) into \( \mathfrak{T}_\infty^0 \)-modules \( \mathfrak{T}_\infty^{k,s} \) of \( k \)-contact and \( s \)-horizontal forms:

\[
\mathfrak{T}_\infty^* = \bigoplus_{k,s} \mathfrak{T}_\infty^{k,s}, \quad h_k : \mathfrak{T}_\infty^* \rightarrow \mathfrak{T}_\infty^{k,*}, \quad h^s : \mathfrak{T}_\infty^* \rightarrow \mathfrak{T}_\infty^{*,s}; \quad 0 \leq k, \quad 0 \leq s \leq n.
\]

Accordingly, the exterior differential on \( \mathfrak{T}_\infty^* \) splits into the sum \( d = d_H + d_V \) of horizontal and vertical differentials such that

\[
\begin{align*}
d_H \circ h_k &= h_k \circ d \circ h_k, & d_H(\phi) = dx^\lambda \wedge d_A(\phi), & \phi \in \mathfrak{T}_\infty^*, \\
d_V \circ h^s &= h^s \circ d \circ h^s, & d_V(\phi) = \theta_A^i \wedge \partial^A_i \phi.
\end{align*}
\]

### 3 The horizontal complex

Being nilpotent, the differentials \( d_V \) and \( d_H \) provide the natural bicomplex \( \{\mathfrak{T}_\infty^{k,m}\} \) of the graded differential algebra \( \mathfrak{T}_\infty^* \). Let us consider its row

\[
0 \rightarrow \mathbb{R} \rightarrow \mathfrak{T}_\infty^0 \xrightarrow{d_H} \mathfrak{T}_\infty^0 \wedge \mathfrak{T}_\infty^0 \rightarrow \cdots \xrightarrow{d_H} \mathfrak{T}_\infty^{0,n} \wedge \mathfrak{T}_\infty^0 \rightarrow 0 \quad (4)
\]

\[\cdot\]
called the horizontal complex. The corresponding complex of sheaves

\[ 0 \rightarrow \mathbb{R} \rightarrow \mathbb{T}^0_\infty \xrightarrow{d_H} \mathbb{T}^{0,1}_\infty \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathbb{T}^{0,n}_\infty \xrightarrow{d_H} 0 \] (5)

except the last term is exact (see Lemma 3 below). Then, since the sheaves \( \mathbb{T}^{0,<n} \) of \( \mathcal{T}^0_\infty \) -modules on \( J^\infty Y \) are fine, we obtain from the abstract de Rham theorem and Lemma (5) below that \( d_H \)-cohomology \( H^{<n}(d_H; \mathcal{T}^\ast_\infty) \) of the horizontal complex (4) is equal to de Rham cohomology \( H^\ast(Y) \) of the bundle \( Y \). Theorem 9 below shows that \( d_H \)-cohomology \( H^{<n}(d_H; \mathcal{O}^\ast_\infty) \) of the horizontal complex

\[ 0 \rightarrow \mathbb{R} \rightarrow \mathcal{O}^0_\infty \xrightarrow{d_H} \mathcal{O}^{0,1}_\infty \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathcal{O}^{0,n}_\infty \xrightarrow{d_H} 0 \] (6)

of the algebra \( \mathcal{O}^\ast_\infty \) is the same. However, one should complete the horizontal complex (4) in the variational one in order to say something on the \( n \)th cohomology group of \( d_H \) (see the relation (19) below).

4 The variational complex

Let us consider the variational operator \( \delta = \tau \circ d \) on \( \mathcal{T}^\ast_\infty \) where

\[ \tau = \sum_{k>0} \frac{1}{k} \tau \circ h_k \circ h^n, \quad \tau(\phi) = \sum_{0 \leq |\Lambda|} (-1)^{|\Lambda|} \theta^i \wedge [d_\Lambda(\partial_i \phi)], \quad \phi \in \mathcal{T}^{0,n}_\infty, \]

is the projection \( \mathbb{R} \)-module endomorphism of \( \mathcal{T}^\ast_\infty \) such that \( \tau \circ d_H = 0 \) (see, e.g., [5, 10, 20]). The \( \delta \) is nilpotent, and obeys the relation

\[ \delta \circ \tau - \tau \circ d = 0. \] (7)

Put \( \mathcal{E}_k = \tau(\mathbb{T}^{k,n}_\infty) \), \( E_k = \tau(\mathcal{T}^{k,n}_\infty) \), \( k > 0 \). Since \( \tau \) is a projector, there are isomorphisms

\[ \Gamma(\mathcal{E}_k) = \tau(\Gamma(\mathbb{T}^{k,n}_\infty)), \quad E_k = \Gamma(J^\infty Y, \mathcal{E}_k). \]

With operators \( d_H \) and \( \delta \), we have the variational complex

\[ 0 \rightarrow \mathbb{R} \rightarrow \mathbb{T}^0_\infty \xrightarrow{d_H} \mathbb{T}^{0,1}_\infty \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathbb{T}^{0,n}_\infty \xrightarrow{\delta} \mathcal{E}_1 \xrightarrow{\delta} \mathcal{E}_2 \xrightarrow{\delta} \cdots \] (8)

of the sheaf \( \mathbb{T}^\ast_\infty \) and that

\[ 0 \rightarrow \mathbb{R} \rightarrow \mathcal{T}^0_\infty \xrightarrow{d_H} \mathcal{T}^{0,1}_\infty \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathcal{T}^{0,n}_\infty \xrightarrow{\delta} \mathcal{E}_1 \xrightarrow{\delta} \mathcal{E}_2 \xrightarrow{\delta} \cdots \] (9)

of its structure algebra \( \mathcal{T}^\ast_\infty \). The similar variational complex \( \{ \mathcal{O}^\ast_\infty, \mathcal{E}_k \} \) of the algebra \( \mathcal{O}^\ast_\infty \) takes place. There are the well-known statements summarized usually as the algebraic Poincaré lemma (see, e.g., [18, 20]).
LEMMA 3. If $Y$ is a contractible bundle $\mathbb{R}^{n+p} \to \mathbb{R}^n$, the variational complex $\{\mathcal{O}_\infty^*, \overline{E}_k\}$ of the graded differential algebra $\mathcal{O}_\infty^*$ is exact.

It follows that the variational complex of sheaves (8) is exact for any smooth bundle $Y \to X$. Moreover, the sheaves $\mathcal{T}^{*,n}_\infty$ in this complex are fine, and so are the sheaves $\mathcal{E}_k$ in accordance with Lemma 4 below. Hence, the variational complex (8) is a resolution of the constant sheaf $\mathbb{R}$ on $J^\infty Y$.

LEMMA 4. Sheaves $\mathcal{E}_k$, $k > 0$, are fine.

Proof. Though the $\mathbb{R}$-modules $E_{k>1}$ fail to be $\mathcal{T}_\infty^0$-modules [20], one can use the fact that the sheaves $E_{k>0}$ are projections $\tau(\mathcal{T}^{k,n}_\infty)$ of sheaves of $\mathcal{T}_\infty^0$-modules. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a locally finite open covering of $J^\infty Y$ and $\{f_i \in \mathcal{T}_\infty^0\}$ the associated partition of unity. For any open subset $U \subset J^\infty Y$ and any section $\varphi$ of the sheaf $\mathcal{T}^{k,n}_\infty$ over $U$, let us put $h_i(\varphi) = f_i \varphi$. The endomorphisms $h_i$ of $\mathcal{T}^{k,n}_\infty$ yield the $\mathbb{R}$-module endomorphisms

$$
\overline{h}_i = \tau \circ h_i : \mathcal{E}_k \xrightarrow{\text{inh}} \mathcal{T}^{k,n}_\infty \xrightarrow{h_i} \mathcal{T}^{k,n}_\infty \xrightarrow{\tau} \mathcal{E}_k
$$

of the sheaves $\mathcal{E}_k$. They possess the properties required for $\mathcal{E}_k$ to be a fine sheaf. Indeed, for each $i \in I$, $\text{supp} f_i \subset U_i$ provides a closed set such that $\overline{h}_i$ is zero outside this set, while the sum $\sum_{i \in I} \overline{h}_i$ is the identity morphism. $\square$

5 Cohomology of $\mathcal{T}_\infty^*$

LEMMA 5. There is an isomorphism

$$
H^*(J^\infty Y, \mathbb{R}) = H^*(Y, \mathbb{R}) = H^*(Y)
$$

between cohomology $H^*(J^\infty Y, \mathbb{R})$ of $J^\infty Y$ with coefficients in the constant sheaf $\mathbb{R}$, that $H^*(Y, \mathbb{R})$ of $Y$, and de Rham cohomology $H^*(Y)$ of $Y$.

Proof. Since $Y$ is a strong deformation retract of $J^\infty Y$ (see, e.g., [11]), the first isomorphism in (10) follows from the Vietoris–Begle theorem [7], while the second one results from the well-known de Rham theorem. $\square$

Let us consider the de Rham complex of sheaves

$$
0 \to \mathbb{R} \to \mathcal{T}^0_\infty \xrightarrow{d} \mathcal{T}^1_\infty \xrightarrow{d} \cdots
$$

on $J^\infty Y$ and the corresponding de Rham complex of their structure algebras

$$
0 \to \mathbb{R} \to \mathcal{T}^0_\infty \xrightarrow{d} \mathcal{T}^1_\infty \xrightarrow{d} \cdots
$$

(12)
The complex (11) is exact due to the Poincaré lemma, and is a resolution of the constant sheaf \( \mathbb{R} \) on \( J^\infty Y \) since sheaves \( \mathcal{T}^r \) are fine. Then, the abstract de Rham theorem and Lemma 6 lead to the following.

**Proposition 6.** De Rham cohomology \( H^*(\mathcal{T}^*_\infty) \) of the graded differential algebra \( \mathcal{T}^*_\infty \) is isomorphic to that \( H^*(Y) \) of the bundle \( Y \).

It follows that every closed form \( \phi \in \mathcal{T}^*_\infty \) splits into the sum
\[
\phi = \varphi + d\xi, \quad \xi \in \mathcal{T}^*_\infty,
\] where \( \varphi \) is a closed form on the bundle \( Y \).

Similarly, from the abstract de Rham theorem and Lemma 5, we obtain the following.

**Proposition 7.** There is an isomorphism between \( d_H \)- and \( \delta \)-cohomology of the variational complex (9) and de Rham cohomology of the bundle \( Y \), namely,
\[
H^{k<n}(d_H; \mathcal{T}^*_\infty) = H^{k<n}(Y), \quad H^{k-n}(\delta; \mathcal{T}^*_\infty) = H^{k\geq n}(Y).
\]

This isomorphism recovers the results of [1, 19], but note also the following. The relation (4) for \( \tau \) and the relation \( h_0 d = d_H h_0 \) for \( h_0 \) define a homomorphisms of the de Rham complex (12) of the algebra \( \mathcal{T}^*_\infty \) to its variational complex (3). The corresponding homomorphism of their cohomology groups is an isomorphism by virtue of Proposition 6 and Proposition 7. Then, the splitting (13) leads to the following decompositions.

**Proposition 8.** Any \( d_H \)-closed form \( \sigma \in \mathcal{T}^{0,m}, m < n, \) is represented by the sum
\[
\sigma = h_0 \varphi + d_H \xi, \quad \xi \in \mathcal{T}^{m-1}_\infty,
\] where \( \varphi \) is a closed \( m \)-form on \( Y \). Any \( \delta \)-closed form \( \sigma \in \mathcal{T}^{k,n}, k \geq 0, \) splits into
\[
\begin{align*}
\sigma &= h_0 \varphi + d_H \xi, & k &= 0, & \xi &\in \mathcal{T}^{0,n-1}_\infty, \\
\sigma &= \tau(\varphi) + \delta(\xi), & k &= 1, & \xi &\in \mathcal{T}^{0,n}_\infty, \\
\sigma &= \tau(\varphi) + \delta(\xi), & k &> 1, & \xi &\in E_{k-1},
\end{align*}
\] where \( \varphi \) is a closed \((n+k)\)-form on \( Y \).
6 Cohomology of $\mathcal{O}_\infty^*$

THEOREM 9. Graded differential algebra $\mathcal{O}_\infty^*$ has the same $d_H$- and $\delta$-cohomology of the variational complex as $\mathcal{T}_\infty^*$.

Proof. Let the common symbol $D$ stand for $d_H$ and $\delta$. Bearing in mind decompositions (14) – (17), it suffices to show that, if an element $\phi \in \mathcal{O}_\infty^*$ is $D$-exact in the algebra $\mathcal{T}_\infty^*$, then it is so in the algebra $\mathcal{O}_\infty^*$. Lemma 3 states that, if $Y$ is a contractible bundle and a $D$-exact form $\phi$ on $J^\infty Y$ is of finite jet order $[\phi]$ (i.e., $\phi \in \mathcal{O}_\infty^*$), there exists an exterior form $\varphi \in \mathcal{O}_\infty^*$ on $J^\infty Y$ such that $\phi = D\varphi$. Moreover, a glance at the homotopy operators for $d_H$ and $\delta$ [13] shows that the jet order $[\varphi]$ of $\varphi$ is bounded for all exterior forms $\phi$ of fixed jet order. Let us call this fact the finite exactness of the operator $D$. Given an arbitrary bundle $Y$, the finite exactness takes place on $J^\infty Y|_U$ over any open subset $U$ of $Y$ which is homeomorphic to a convex open subset of $\mathbb{R}^{\dim Y}$. Let us prove the following.

(i) Suppose that the finite exactness of the operator $D$ takes place on $J^\infty Y$ over open subsets $U$, $V$ of $Y$ and their non-empty overlap $U \cap V$. Then, it is also true on $J^\infty Y|_{U \cup V}$.

(ii) Given a family $\{U_\alpha\}$ of disjoint open subsets of $Y$, let us suppose that the finite exactness takes place on $J^\infty Y|_{U_\alpha}$ over every subset $U_\alpha$ from this family. Then, it is true on $J^\infty Y$ over the union $\bigcup_{\alpha} U_\alpha$ of these subsets.

If these assertions hold, the finite exactness of $D$ on $J^\infty Y$ takes place because one can construct the corresponding covering of the manifold $Y$ (9.5).

Proof of (i). Let $\phi = D\varphi \in \mathcal{O}_\infty^*$ be a $D$-exact form on $J^\infty Y$. By assumption, it can be brought into the form $D\varphi_U$ on $(\pi_0^\infty)^{-1}(U)$ and $D\varphi_V$ on $(\pi_0^\infty)^{-1}(V)$, where $\varphi_U$ and $\varphi_V$ are exterior forms of finite jet order. Let us consider their difference $\varphi_U - \varphi_V$ on $(\pi_0^\infty)^{-1}(U \cap V)$. It is a $D$-exact form of finite jet order which, by assumption, can be written as $\varphi_U - \varphi_V = D\sigma$ where $\sigma$ is also of finite jet order. Lemma 10 below shows that $\sigma = \sigma_U + \sigma_V$ where $\sigma_U$ and $\sigma_V$ are exterior forms of finite jet order on $(\pi_0^\infty)^{-1}(U)$ and $(\pi_0^\infty)^{-1}(V)$, respectively. Then, putting $\varphi'|_U = \varphi_U - D\sigma_U$, $\varphi'|_V = \varphi_V + D\sigma_V$, we have $\phi = D\varphi'$ on $(\pi_0^\infty)^{-1}(U \cap V)$ where $\varphi'$ is of finite jet order.

Proof of (ii). Let $\phi \in \mathcal{O}_\infty^*$ be a $D$-exact form on $J^\infty Y$. The finite exactness on $(\pi_0^\infty)^{-1}(\bigcup U_\alpha)$ holds since $\phi = D\varphi_\alpha$ on every $(\pi_0^\infty)^{-1}(U_\alpha)$ and, as was mentioned above, the jet order $[\varphi_\alpha]$ is bounded on the set of exterior forms $D\varphi_\alpha$ of fixed jet order $[\phi].$ \hfill $\square$

LEMMA 10. Let $U$ and $V$ be open subsets of a bundle $Y$ and $\sigma \in \mathcal{O}_\infty^*$ an exterior form of finite jet order on $(\pi_0^\infty)^{-1}(U \cap V) \subset J^\infty Y$. Then, $\sigma$ splits into a sum $\sigma_U + \sigma_V$ of exterior forms $\sigma_U$ and $\sigma_V$ of finite jet order on $(\pi_0^\infty)^{-1}(U)$ and $(\pi_0^\infty)^{-1}(V)$, respectively.

Proof. By taking a smooth partition of unity on $U \cup V$ subordinate to the cover $\{U, V\}$ and passing to the function with support in $V$, one gets a smooth real function $f$ on $U \cup V$ and taking the form $\sigma_U + \sigma_V$.

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which is 0 on a neighborhood of $U - V$ and 1 on a neighborhood of $V - U$ in $U \cup V$. Let $(\pi_0^\infty)^* f$ be the pull-back of $f$ onto $(\pi_0^\infty)^{-1}(U \cup V)$. The exterior form $((\pi_0^\infty)^* f) \sigma$ is zero on a neighborhood of $(\pi_0^\infty)^{-1}(U)$ and, therefore, can be extended by 0 to $(\pi_0^\infty)^{-1}(U)$. Let us denote it $\sigma_U$. Accordingly, the exterior form $(1 - (\pi_0^\infty)^* f) \sigma$ has an extension $\sigma_V$ by 0 to $(\pi_0^\infty)^{-1}(V)$. Then, $\sigma = \sigma_U + \sigma_V$ is a desired decomposition because $\sigma_U$ and $\sigma_V$ are of finite jet order which does not exceed that of $\sigma$. 

\[ \Box \]

7 The global inverse problem

The expressions (15) – (16) in Proposition 8 provide a solution of the global inverse problem of the calculus of variations on fibre bundles in the class of Lagrangians $L \in T_{0,n}^\infty$ of locally finite order [1, 19] (which is not so interesting for physical applications). These expressions together with Theorem 9 give a solution of the global inverse problem of the finite order calculus of variations.

PROPOSITION 11. (i) A finite order Lagrangian $L \in \mathcal{O}_{0,n}^0$ is variationally trivial, i.e.,

\[ \delta(L) = 0 \]

iff

\[ L = h_0 \varphi + d_H \xi, \quad \xi \in \mathcal{O}_{\infty}^{0,n-1}, \]

where $\varphi$ is a closed $n$-form on $Y$. (ii) A finite order Euler–Lagrange-type operator satisfies the Helmholtz condition $\delta(\mathcal{E}) = 0$ iff

\[ \mathcal{E} = \delta(L) + \tau(\phi), \quad L \in \mathcal{O}_{\infty}^{0,n}, \]

where $\phi$ is a closed $(n + 1)$-form on $Y$ (see also [21]).

A solution of the global inverse problem of the fixed order calculus of variations has been suggested in [1] by computing cohomology of the fixed order variational sequence. However, the proof of the local exactness of this variational sequence requires rather sophisticated \textit{ad hoc} techniques in order to be reproduced (see also [21]). The first thesis of [1] agrees with that of Proposition [11], but says that the jet order of the form $\xi$ in the expression (18) is $k - 1$ if $L$ is a $k$-order variationally trivial Lagrangian. The second one states that a $2k$-order Euler–Lagrange operator can be always associated with a $k$-order Lagrangian.

One obtains from Proposition [11(i)] that the cohomology group $H^n(d_H; \mathcal{O}_{\infty}^\infty)$ of the complex (3) obeys the relation

\[ H^n(d_H; \mathcal{O}_{\infty}^\infty)/H^n(Y) = \delta(\mathcal{O}_{\infty}^{0,n}), \]

where $\delta(\mathcal{O}_{\infty}^{0,n})$ is the $\mathbb{R}$-module of Euler–Lagrange forms on $J^\infty Y$. 

9
8 The case of an affine bundle

Let $Y \to X$ be an affine bundle. Since $X$ is a strong deformation retract of $Y$, de Rham cohomology of $Y$ is equal to that of $X$. It leads to the cohomology isomorphisms

$$H^{<n}(d_H; \mathcal{O}_\infty^*) = H^{<n}(X), \quad H^0(\delta; \mathcal{O}_\infty^*) = H^n(X), \quad H^k(\delta; \mathcal{O}_\infty^*) = 0.$$ 

Hence, every $d_H$-closed form $\phi \in \mathcal{O}_\infty^{0,m<n}$ splits into the sum

$$\phi = \varphi + d_H \xi, \quad \xi \in \mathcal{O}_\infty^{0,m-1}, \quad (20)$$

where $\varphi$ is a closed form on $X$.

In the case of an affine bundle $Y \to X$, horizontal complexes (4) – (5) induce similar complexes on the base $X$ as follows. It is quite important for the cohomology calculation of polynomial complexes in next Section.

Let us consider the open surjection $\pi^\infty : J^\infty Y \to X$ and the direct image $\{\pi^\infty_\ast \mathcal{F}_\infty^*\}$ on $X$ of the sheaf $\mathcal{F}_\infty^*$. Its stalk at a point $x \in X$ consists of the equivalence classes of sections of the sheaf $\mathcal{F}_\infty^*$ which coincide on the inverse images $(\pi^\infty)^{-1}(U_x)$ of neighbourhoods $U_x$ of $x$. Put further the notation $\mathcal{I}_x^\infty = \pi^\infty_\ast \mathcal{F}_x^*$. Since $\pi^\infty_\ast \mathbb{R} = \mathbb{R}$, we have the following complex of sheaves on $X$:

$$0 \to \mathbb{R} \to \mathcal{I}_x^\infty_0 \xrightarrow{d_H} \mathcal{I}_x^\infty_{0,1} \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathcal{I}_x^\infty_{0,n} \xrightarrow{d_H} 0. \quad (21)$$

Every point $x \in X$ has a base of open contractible neighbourhoods $\{U_x\}$ such that the sheaves $\mathcal{I}_x^\infty_0$ of $\mathcal{T}_\infty^*$-modules are acyclic on the inverse images $(\pi^\infty)^{-1}(U_x)$ of these neighbourhoods. Then, in accordance with the Leray theorem [12], cohomology of $J^\infty Y$ with coefficients in the sheaves $\mathcal{I}_x^\infty_0$ are isomorphic to that of $X$ with coefficients in their direct images $\mathcal{I}_x^\infty_0$; i.e., the sheaves $\mathcal{I}_x^\infty_0$ on $X$ are acyclic. Furthermore, Lemma 8 shows that the complexes of sections of sheaves $\mathcal{I}_x^\infty_{0<n}$ over $(\pi^\infty_0)^{-1}(U_x)$ are exact. It follows that the horizontal complex (21), except the last term, is also exact. Due to the $\mathbb{R}$-algebra isomorphism $\mathcal{T}_\infty^* = \Gamma(X, \mathcal{I}_x^\infty_0)$, one can think of the horizontal complex (4) as being the complex of the structure algebras of sheaves of the horizontal complex (21) on $X$.

9 Cohomology of polynomial complexes

Given the sheaf $\mathcal{I}_x^\infty_0$ on $X$, let us consider its subsheaf $\mathcal{P}_\infty^*$ of germs of exterior forms which are polynomials in the fiber coordinates $y^\Lambda_\infty$, $|\Lambda| \geq 0$, of the topological fiber bundle $J^\infty Y \to X$. This property is coordinate-independent due to the transition functions (2). The $\mathcal{P}_\infty^*$ is a sheaf of $C^\infty(X)$-modules. Its structure algebra $\mathcal{P}_\infty^*$ is a $C^\infty(X)$-subalgebra of $\mathcal{T}_\infty^*$. For short, one can say that $\mathcal{P}_\infty^*$ consists of exterior forms on $J^\infty Y$ which are locally polynomials in fiber coordinates $y^\Lambda_\infty$. 

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We have the subcomplex

\[ 0 \to \mathbb{R} \to \mathcal{P}_0^0 \overset{d_H}{\longrightarrow} \mathcal{P}_0^1 \overset{d_H}{\longrightarrow} \cdots \overset{d_H}{\longrightarrow} \mathcal{P}_0^n \]  

(22)

describing the horizontal complex (21) on \( X \). As a particular variant of the algebraic Poincaré lemma, the exactness of the complex (22) has been repeatedly proved (see, e.g., [4]). Since the sheaves \( \mathcal{P}_0^* \) of \( C^\infty(X) \)-modules on \( X \) are acyclic, the complex (22) is a resolution of the constant sheaf \( \mathbb{R} \) on \( X \). Hence, cohomology of the complex

\[ 0 \to \mathbb{R} \to \mathcal{P}_0^0 \overset{d_H}{\longrightarrow} \mathcal{P}_0^1 \overset{d_H}{\longrightarrow} \cdots \overset{d_H}{\longrightarrow} \mathcal{P}_0^n \]  

(23)

of the structure algebras \( \mathcal{P}_0^{*,n} \) of sheaves \( \mathcal{P}_0^{*,n} \) is equal to de Rham cohomology of \( X \). It follows that every \( d_H \)-closed polynomial form \( \phi \in \mathcal{P}_0^{*,n} \) splits into the sum

\[ \phi = \varphi + d_H \xi, \quad \xi \in \mathcal{P}_0^{0,n-1}, \]  

(24)

where \( \varphi \) is a closed form on \( X \). Let \( P_* \) be \( C^\infty(X) \)-subalgebra of the polynomial algebra \( \mathcal{P}_\infty^* \) which consists of exterior forms which are polynomials in the fiber coordinates \( y_i \). Obviously, \( P_*^\infty \) is a subalgebra of \( \mathcal{O}_*^\infty \). As a repetition of Theorem 9, one can show that \( P_*^\infty \) have the same cohomology as \( \mathcal{P}_*^\infty \), i.e., if \( \phi \) in the decomposition (24) is an element of \( \mathcal{P}_*^\infty \), then \( \xi \) is so.

Let us consider the subsheaf \( \mathcal{T}_*^\infty \) of the sheaf \( \mathcal{P}_*^\infty \) which consists of germs of \( x \)-independent polynomial forms. Its structure algebra \( \mathcal{T}_*^\infty \) is a subalgebra of the algebra \( \mathcal{P}_*^\infty \). We have the complex of sheaves

\[ 0 \to \mathbb{R} \to \mathcal{T}_\infty^0 \overset{d_H}{\longrightarrow} \mathcal{T}_\infty^1 \overset{d_H}{\longrightarrow} \cdots \overset{d_H}{\longrightarrow} \mathcal{T}_\infty^n \]  

which fails to be exact. The obstruction to its exactness at the term \( \mathcal{T}_\infty^k \) is provided by the germs of \( k \)-forms on \( X \) with constant coefficients [4]. Let us denote the sheaf of such germs by \( S_k^X \). For any \( 0 < k < n \), we have the short exact sequence of sheaves

\[ 0 \to \text{Im} \, d_H \to \text{Ker} \, d_H \to S_k^X \to 0 \]

and the sequence of their structure modules

\[ 0 \to \Gamma(X, \text{Im} \, d_H) \to \Gamma(X, \text{Ker} \, d_H) \to \Gamma(X, S_k^X) \to 0 \]

which is exact because \( S_k^X \) is a subsheaf of \( \mathbb{R} \)-modules of the sheaf \( \text{Ker} \, d_H \). Then, the \( k \)th cohomology group of the horizontal complex

\[ 0 \to \mathbb{R} \to \mathcal{T}_\infty^0 \overset{d_H}{\longrightarrow} \mathcal{T}_\infty^1 \overset{d_H}{\longrightarrow} \cdots \overset{d_H}{\longrightarrow} \mathcal{T}_\infty^n \]  

of the algebra \( \mathcal{T}_*^\infty \) is isomorphic to the \( \mathbb{R} \)-module \( \Gamma(X, S_k^X) \) of global constant \( k \)-forms on the manifold \( X \). If a manifold \( X \) does not admit an affine coordinate atlas, the module \( \Gamma(X, S_k^X) \) is empty and, consequently, the differential \( d_H \) is exact on the algebra \( \mathcal{T}_*^{0,n} \). Otherwise, any \( d_H \)-closed element \( \phi \in \mathcal{T}_*^{0,k}, \) \( 0 < k < n \), splits into the sum

\[ \phi = \varphi + d_H \xi, \quad \varphi \in \Gamma(X, S_k^X), \quad \xi \in \mathcal{T}_*^{0,k-1}. \]
10 Iterated cohomology

Turn to the proof of Theorem 1. By assumption, the horizontal complex \( (6) \) is a \( s|d_H \)-bicomplex. Its iterated cohomology group \( H^{k,m}(s|d_H) = E_2^{k,m}, \ k \in \mathbb{Z}, \ 0 \leq m \leq n, \) consists of \( d_H \)-closed horizontal \( m \)-forms \( \omega \in \mathcal{O}^{0,m} \) of ghost number \( k \) such that \( s\omega \) is \( d_H \)-exact, which are taken modulo exterior forms \( s\psi + d_H \xi \) where \( \psi \) is a \( d_H \)-closed form. Then, the assertions (i) and (ii) of Theorem 1 follows immediately from the decomposition (20) and the assumption that forms on \( X \) are of zero ghost number and are not \( s \)-exact. The proof of assertions (iii) – (v) is based on the analysis of descent equations. Since the operator \( s \) annihilates exterior forms on \( X \), descent equations can be constructed, but their right-hand side is not necessarily zero. The key point lies in the existence of closed forms on \( X \) which are exact with respect to the total operator \( \tilde{s} = s + d_H \).

Proof of (iii). Let \( \omega_n \) be a representative of the iterated cohomology group \( H^{k,n}(s|d_H), k < -n \) or \( k \geq -1 \). It is a horizontal \( n \)-form of ghost number \( k \) such that \( s\omega_n \) is \( d_H \)-exact, i.e.,

\[
 s\omega_n + d_H \omega_{n-1} = 0. \tag{25}
\]

Acting on this equality by \( s \), we observe that \( s\omega_{n-1} \) is a \( d_H \)-closed form of non-vanishing ghost number \( k + 2 \). Therefore, it is \( d_H \)-exact, i.e.,

\[
 s\omega_{n-1} + d_H \omega_{n-2} = 0.
\]

Iterating the arguments, one concludes the existence of a family \( \{\omega_{n-p}\}, 0 \leq p \leq n, \) of horizontal \( (n - p) \)-forms \( \omega_{n-p} \) of non-vanishing ghost numbers \( k + p \) which obey the descent equations

\[
 s\omega_{n-p} + d_H \omega_{n-p-1} = 0, \quad s\omega_0 = 0, \quad 0 \leq p < n. \tag{26}
\]

It may happen that \( \omega_{n-p} = 0, p \geq p_0, \) for some \( p_0 \). Put

\[
 \tilde{\omega}_n = \sum_{p=0}^{n} \omega_{n-p}. \tag{27}
\]

It is an \( \tilde{s} \)-closed form of total ghost number \( k + n \). Let \( \{\omega'_{n-p}\} \) be another solution of the descent equations (26) for given \( \omega_n \). It is readily observed that \( \tilde{\omega}_n - \tilde{\omega}'_n \) is an \( \tilde{s} \)-exact form. Let the iterated cohomology class of \( \omega_n \) be zero, i.e. \( \omega_n = s\xi_n + d_H \xi_{n-1} \) where \( \xi_n \) is \( d_H \)-closed. Then, \( \{\omega_n, s\xi_{n-1}, 0, \cdots, 0\} \) is a solution of the descent equations such that \( \tilde{\omega}_n \) (27) is an \( \tilde{s} \)-exact form. Conversely, let a horizontal exterior form \( \tilde{\omega} \) of total ghost number \( n + k \) be \( \tilde{s} \)-closed. It splits into the sum (27) whose summands obey the descent equations (26). The higher term \( \omega_n \) of this sum fulfills the relation (25), i.e., is a representative of iterated cohomology. Since all \( d_H \)-closed forms of non-vanishing ghost number are \( d_H \)-exact, the
descent equations \((29)\) show that: (i) if \(\omega_n = 0\), then \(\omega\) is \(\mathfrak{s}\)-exact, and (ii) if \(\omega = \mathfrak{s}\xi\) is \(\mathfrak{s}\)-exact, then \(\omega_n = \mathfrak{s}\xi_n + d_H\xi_{n-1}\) is of zero cohomology class. Thus, we come to a desired isomorphism.

**Proof of (iv).** In contrast with the previous case, \(d_H\)-closed forms now are not necessarily \(d_H\)-exact. Therefore, a representative \(\omega_n\) of iterated cohomology defines a system of descent equations where the descent equation of vanishing ghost number

\[
s\omega_{n+k+1} + d_H\omega_{n+k} = \varphi
\]

has a closed \((n + k + 1)\)-form \(\varphi\) on \(X\) in its right-hand side. Accordingly, the form \(\omega_n\) \((27)\) fulfills the equality \(\tilde{s}\omega_n = \varphi\). It follows that the system of descent equations

\[
\tilde{s}\omega^\varphi = \varphi,
\]

which contains the equation \((28)\), admits a solution \(\tilde{s}\omega^\varphi\) iff the cohomology class of \(\varphi\) belongs to \(\ker \gamma_{n+k+1}\). A higher term \(\omega^\varphi_n\) of every such solution obeys the relation \((23)\) and, consequently, is a representative of iterated cohomology. If \(\omega^\varphi_n = \omega_n^\varphi + s\xi_n + d_H\xi_{n-1}\) is another representative of the same cohomology class, then \(\tilde{\omega} + \mathfrak{s}\xi_{n-1}\) is also a solution of the same descent equations \((23)\). Consequently, any closed \((n + k + 1)\)-form \(\varphi\) on \(X\) whose cohomology class belongs to \(\ker \gamma_{n+k+1}\) defines a subset \(A_\varphi\) of the iterated cohomology group \(H^{k,n}(s|d_H)\), given by the higher terms of solutions of the descent equations \((23)\). In particular, let us consider \(A_{\varphi=0}\). The difference from the proof of item (iii) lies in the fact that, if the higher term \(\omega_n\) of an \(\mathfrak{s}\)-closed form \(\tilde{\omega}\) vanishes, then \(\omega = \mathfrak{s}\xi + \psi\) where \(\psi\) is a \((n + k)\)-closed form on \(X\). It follows that \(A_{\varphi=0} = \mathcal{T}^{k+n}\). One can justify easily that: (i) \(\omega^\varphi_n \in A_\varphi\) and \(\omega^0_n \in A_{\varphi=0}\) implies \(\omega^\varphi_n + \omega^0_n \in A_{\varphi}\), and (ii) \(\omega^\varphi_n, \omega^\varphi_n \in A_\varphi\) implies \(\omega^\varphi_n - \omega^\varphi_n \in A_{\varphi=0}\). It follows that \(A_\varphi\) is an affine space modelled over the linear space \(\mathcal{T}^{k+n}\). Let \(\varphi' = \varphi + d\sigma\). Then, a solution \(\tilde{s}\omega^\varphi\) of the descent equations \((29)\) defines a solution \(\tilde{s}\omega^\varphi + \sigma\) of the descent equations \((29)\) where \(\varphi\) is replaced with \(\varphi'\). These solutions have the same higher term \(\omega^\varphi_n = \omega^\varphi_n\) and, consequently define the same representative of iterated cohomology. It follows that \(A_\varphi = A_{\varphi+dH\sigma}\), i.e., \(A_\varphi\) is set by the cohomology class \([\varphi] \in \ker \gamma_{k+n+1}\) of \(\varphi\). It remains to show that \(H^{k,n}(s|d_H)\) is a disjoint union of sets \(A_{[\varphi]}\), \([\varphi] \in \ker \gamma_{k+n+1}\). Indeed, let a representative of iterated cohomology defines different systems of descent equations \((23)\) with \(\varphi\) and \(\varphi'\) in the right-hand side. Then, one can easily justify that \(\varphi' - \varphi\) is an exact form. Turn to the particular case \(k = -n\). If a constant function on \(X\) is \(\mathfrak{s}\)-exact, it is \(s\)-exact. Therefore, \(\ker \gamma = 0\).

**Proof of (v).** In this case, we have the descent equations \((23)\) with the zero right-hand side, but \(H^{-1,n}(s|d_H) = H_{\text{tot}}^{-1}/\im \gamma_{n-1}\).

Note that, in the case of BRST cohomology (see Corollary \(3)\), the right-hand side of the global descent equations for any total ghost number remains zero.
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