On the Ornstein-Zernike behaviour for the Bernoulli bond percolation on $\mathbb{Z}^d$, $d \geq 3$, in the supercritical regime

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Abstract

We prove Ornstein-Zernike behaviour in every direction for finite connection functions of bond percolation on $\mathbb{Z}^d$ for $d \geq 3$ when $p$, the probability of occupation of a bond, is sufficiently close to 1. Moreover, we prove that equi-decay surfaces are locally analytic, strictly convex, with positive Gaussian curvature.

1 Introduction and results

Ornstein-Zernike behaviour of correlation and connection functions has been rigorously proved for many models of statistical mechanics and percolation in the high temperature or low probability regime, first for extreme values of the parameter (see e.g. [BF]) and then up

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to the critical point (see [CCC], [CI], [CIV]). Above the critical probability in [CIL] it was proved that in two dimensions finite connection functions, i.e. the probabilities that two sites belong to the same finite cluster, exhibit a different asymptotic behaviour, which is related to the probability that two independent random walks in dimension 2 do not intersect. In higher dimensions, for $d \geq 3$, one expects that Ornstein-Zernike behaviour holds for finite connection probabilities above critical probability. This was proved in [BPS] when $p$ (the probability that a bond is open) is close to one for the connection probabilities in the direction of the axes. The proof is based on cluster expansion.

The problem of the asymptotic behaviour of finite connection functions in arbitrary directions presents an important difference with respect to that in the directions of coordinate axes. Indeed, in the limit of $p$ tending to 1 the probability distribution of the finite cluster containing two sites on a coordinate axis, conditioned to its existence, tends to a delta measure concentrated on the segment joining the two sites. The cluster expansion presented in [BPS] can be thought of as a perturbation about this configuration that plays the role of ground state. In the case of two sites that don’t lie on the same coordinate axis, the limiting distribution is not supported on a single configuration: one can say that the ground state is degenerate. This makes the extension of the method used in [BPS] problematic.

In this paper we prove Ornstein-Zernike behaviour in every direction for finite connection functions of bond percolation on $\mathbb{Z}^d$ for $d \geq 3$ when $p$, the probability of occupation of a bond, is sufficiently close to 1. Moreover, we prove that equi-decay surfaces are locally analytic, strictly convex with positive Gaussian curvature.

Our proofs rely in part on the methods developed in [CI], based on multi-dimensional renewal theory and local limit theorem, but we have to deal with new problems; in particular FKG inequality does not apply to finite connection functions, as they are probabilities of non-monotone events. By developing specific techniques we are able to treat the case when $p$ is sufficiently close to 1.

Here in the following are the main results of the paper and the notation that we will use. In Section 2 we introduce the relevant connectivity functions and their renewal structure. In Section 3 we prove the existence of the mass-gap for the direct connectivity function and prove the main results of the paper.

**Theorem 1** For any $d \geq 3$, there exists $p^* = p^*(d) \in (0, 1)$ such that, $\forall p \in (p^*, 1)$, uniformly in $x \in \mathbb{Z}^d$, $\|x\| \to \infty$,

$$
\Pr_p \left( 0 \leftrightarrow x, |C_{\{0,x\}}| < \infty \right) = \frac{\Phi_p(\hat{x})}{\sqrt{(2\pi \|x\|)^{d-1}}} e^{-\tau_p(x)} (1 + o(1)) ,
$$

where $\phi_p$ is a positive real analytic function on $\mathbb{S}^{d-1}$ and $\tau_p$ an equivalent norm in $\mathbb{R}^d$.

As a by-product of the proof of the previous theorem we also obtain the following result.
Theorem 2 For any $d \geq 3$, there exists $p^* = p^* (d) \in (0,1)$ such that, $\forall p \in (p^*,1)$, the equi-decay set is locally analytic and strictly convex. Moreover, the Gaussian curvature of the equi-decay set is uniformly positive.

1.1 Notation

For any $x \in \mathbb{R}^d$, $d \geq 1$, Let us denote by $|x| := \sum_{i=1}^{d} |x_i|$, by $\langle \cdot, \cdot \rangle$ the scalar product in $\mathbb{R}^d$ and by $\|\cdot\| := \sqrt{\langle \cdot, \cdot \rangle}$ the associated Euclidean norm. We then set $\mathbb{S}^{d-1} := \{ z \in \mathbb{R}^d : \|z\| = 1 \}$ and $\hat{x} := \frac{x}{\|x\|}$. Given a set $A \subset \mathbb{R}^d$, let us denote by $A^c$ its complement and by $\mathcal{P} (A)$ the collection of all subsets of $A$. We also set $\mathcal{P}_2 (A) := \{ A \in \mathcal{P} (A) : |A| = 2 \}$, where $|A|$ is the cardinality of $A$. Moreover, we denote by $\mathcal{A}, \overline{\mathcal{A}}$ respectively the interior of $A$ and the closure of $A$ and set $\partial \mathcal{A} := \overline{\mathcal{A}} \setminus \mathcal{A}$ the boundary of $\mathcal{A}$ in the Euclidean topology. Furthermore, if $x \in \mathbb{R}^d$, we set

\[ x + \mathcal{A} := \{ y \in \mathbb{R}^d : y - x \in \mathcal{A} \} \quad (2) \]

and, denoting by $B$ the closed unit ball in $\mathbb{R}^d$, for $r > 0$, let $rB := \{ x \in \mathbb{R}^d : \|x\| \leq r \}$, $B_r (x) := x + rB$.

To make the paper self-contained, we will now introduce those notions of graph theory which are going to be used in the sequel and refer the reader to [3] for an account on this subject.

Let $G = (V,E)$ be a graph whose set of vertices and set of edges are given respectively by a finite or denumerable set $V$ and $E \subset \mathcal{P}_2 (V)$. $G' = (V',E')$ such that $V' \subseteq V$ and $E' \subseteq \mathcal{P}_2 (V') \cap E$ is said to be a subgraph of $G$ and this property is denoted by $G' \subseteq G$. If $G' \subseteq G$, we denote by $V (G')$ and $E (G')$ respectively the set of vertices and the collection of the edges of $G'$. $|V (G')|$ is called the order of $G'$ while $|E (G')|$ is called its size.

Given $G_1, G_2 \subseteq G$, we denote by $G_1 \cup G_2 := (V (G_1) \cup V (G_2), E (G_1) \cup E (G_2)) \subset G$ the graph union of $G_1$ and $G_2$. A path in $G$ is a subgraph $\gamma$ of $G$ such that there is a bijection $\{0,..,|E (\gamma)|\} \ni i \rightarrow v (i) := x_i \in V (\gamma)$ with the property that any $e \in E (\gamma)$ can be represented as $\{x_{i-1},x_i\}$ for $i = 1,..,|E (\gamma)|$. A walk in $G$ of length $l \geq 1$ is an alternating sequence $x_0, e_1, x_1,..,e_l, x_l$ of vertices and edges of $G$ such that $e_i = \{x_{i-1},x_i\}$ $i = 1,..,l$.

Therefore, paths can be associated to walks having distinct vertices. Two distinct vertices $x, y$ of $G$ are said to be connected if there exists a path $\gamma \subseteq G$ such that $x_0 = x$, $x_{|E (\gamma)|} = y$. A graph $G$ is said to be connected if any two distinct elements of $V (G)$ are connected. The maximal connected subgraphs of $G$ are called components of $G$. Given $E' \subseteq E$, we denote by $G (E') := (V,E')$ the spanning graph of $E$. We also define

\[ V (E') := \left( \bigcup_{e \in E'} e \right) \subset V . \quad (3) \]

Given $V' \subseteq V$, we set

\[ E (V') := \{ e \in E : e \subset V' \} \quad (4) \]
and denote by $G[V'] := (V', E(V'))$ that is called the subgraph of $G$ induced or spanned by $V'$. Moreover, if $G' \subseteq G$, we denote by $G \setminus G'$ the graph $G[V \setminus V (G')] \subseteq G$ and define the boundary of $G'$ as the set

$$\partial G' := \{ e \in E \setminus E(G') : |e \cap V(G')| = 1 \} \subseteq E. \quad (5)$$

Let $\mathbb{L}^d$ be the $d$-dimensional cubic lattice, that is the geometric graph whose set of vertices is $\mathbb{Z}^d$ and whose set of edges is

$$E^d := \{ \{x, y\} \in \mathcal{P}^{(2)}(\mathbb{Z}^d) : |x - y| = 1 \}. \quad (6)$$

If $G$ is a subgraph of $\mathbb{L}^d$ of finite order, we denote by $\overline{G}$ the graph induced by the union of $V(G)$ with the sets of vertices of the connected components of the $\mathbb{L}^d \setminus G$ of finite size. We define the external boundary of $G$ to be $\partial G := \partial \overline{G}$. We remark that, given $G_i := (V_i, E_i), i = 1, 2$ two connected subgraphs of $\mathbb{L}^d$ of finite size, by (5), $\partial (G_1 \cup G_2) \subseteq \partial G_1 \cup \partial G_2$. Moreover,

$$\partial (G_1 \cup G_2) = \partial (\overline{G_1 \cup G_2}) \subseteq \partial \overline{G_1} \cup \partial \overline{G_2}. \quad (7)$$

Furthermore, looking at $\mathbb{L}^d$ as a cell complex, i.e. as the union of $\mathbb{Z}^d$ and $\mathbb{E}^d$ representing respectively the collection of 0-cells and of 1-cells, we denote by $(\mathbb{Z}^d)^*$ the collection of $d$-cells dual 0-cells in $\mathbb{L}^d$, that is the collection of unit $d$-cubes centered in the elements of $\mathbb{Z}^d$ (Voronoi cells of $\mathbb{L}^d$), and by $(\mathbb{E}^d)^*$ the collection of $(d - 1)$-cells dual 1-cells in $\mathbb{L}^d$, usually called plaquettes in the physics literature. We also define

$$\mathcal{E} := \left\{ \{e_1^*, e_2^*\} \in \mathcal{P}^{(2)}((\mathbb{E}^d)^*) : \text{codim}(\partial e_1^* \cap \partial e_2^*) = 2 \right\}. \quad (8)$$

and consider the graph $\mathcal{G} := ((\mathbb{E}^d)^*, \mathcal{E})$.

A bond percolation configuration on $\mathbb{L}^d$ is a map $\mathbb{E}^d \ni e \mapsto \omega_e \in \{0, 1\}$. Setting $\Omega := \{0, 1\}^{\mathbb{E}^d}$ we define

$$\Omega \ni \omega \mapsto E(\omega) := \{ e \in \mathbb{E}^d : \omega_e = 1 \} \in \mathcal{P}(\mathbb{E}^d). \quad (9)$$

Let $\mathcal{F}$ be the $\sigma$-algebra generated by the cylinder events of $\Omega$. Given $p \in [0, 1]$, we consider the independent Bernoulli probability measure $\mathbb{P}_p$ on $(\Omega, \mathcal{F})$ with parameter $p$.

Denoting by $\mathcal{G}^d := \{ G \subseteq \mathbb{L}^d : G = G(E), E \in \mathcal{P}(\mathbb{E}^d) \}$ the collection of spanning subgraphs of $\mathbb{L}^d$, we define the random graph

$$\Omega \ni \omega \mapsto G(\omega) := G(E(\omega)) \in \mathcal{G}^d. \quad (10)$$

Then, given $l \geq 1$, $x_1, \ldots, x_l \in \mathbb{Z}^d$, we denote by

$$\Omega \ni \omega \mapsto C_{\{x_1, \ldots, x_l\}}(\omega) \in \mathcal{P}(\mathbb{Z}^d) \quad (11)$$

the common open cluster of the points $x_1, \ldots, x_l \in \mathbb{Z}^d$, that is the set of vertices of the connected component of the random graph $G(\omega)$ to which these points belong, provided it exists, and define, in the case $C_{\{x_1, \ldots, x_l\}}$ is finite, the random set $\overline{\partial C}_{\{x_1, \ldots, x_l\}}$ to be equal to $\partial G$ if $G$ is the component of $G(\omega)$ whose set of vertices is $C_{\{x_1, \ldots, x_l\}}$ and the random set $S_{\{x_1, \ldots, x_l\}} := (\overline{\partial C}_{\{x_1, \ldots, x_l\}})^*$.  

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2 Analysis of connectivities

Given \( x, y \in \mathbb{Z}^d \), we set
\[
\varphi(x, y) := \begin{cases} 
\min \left\{ \left| \partial C_{(x,y)}(\omega) \right| : \omega \in \{0 < \left| C_{(x,y)}(\omega) \right| < \infty \} \right\} & \text{if } x \neq y \\
0 & \text{if } x = y
\end{cases}
\]
(12)

\( \varphi \) is symmetric and translation invariant, therefore in the sequel we will write
\[
\varphi(x, y) = \varphi(x - y).
\]
(13)

For any \( x \in \mathbb{Z}^d \) and \( k \geq \varphi(x) \), let us set
\[
A_k(x) := \{ \omega \in \Omega : \left| \partial C_{\{0,x\}}(\omega) \right| = k \} = \{ \omega \in \Omega : \left| S_{\{0,x\}}(\omega) \right| = k \}
\]
(14)

and \( A^k(x) := \bigvee_{l \geq k} A_l(x) \). We define
\[
\psi_k(x) := \min \left\{ \left| E(C_{\{0,x\}}(\omega)) \right| : \omega \in A_k(x) \right\},
\]
(15)
\[
\Psi_k(x) := \min \left\{ \left| E(C_{\{0,x\}}(\omega)) \right| : \omega \in A^k(x) \right\} = \min_{l \geq k} \psi_l(x)
\]
(16)

and set \( A(x) := A_{\varphi(x)}(x) \) and consequently \( \psi(x) := \psi_{\varphi(x)}, \Psi(x) := \Psi_{\varphi(x)}(x) \).

Remark 3 Given \( x \in \mathbb{Z}^d \), for \( i = 0, \ldots, d \), let \( \gamma_i \) be the path such that \( \gamma_0 = \emptyset \) and
\[
V(\gamma_i) = \{ \text{sign} \left( x_i \right) u_i, 2 \text{sign} \left( x_i \right) u_i, \ldots, |x_i| \text{sign} \left( x_i \right) u_i \} \quad i = 1, \ldots, d.
\]
(17)

Let also
\[
\gamma := \bigcup_{i=0}^{d-1} \left[ \left( \sum_{j=0}^{i} x_j u_j \right) + \gamma_{i+1} \right].
\]
(18)

By construction
\[
|\gamma| := |E(\gamma)| = |x|,
\]
(19)
\[
|\partial V(\gamma)| = |\partial V(\gamma)| = 2 (d-1) (|x| + 1) + 2.
\]
(20)

Hence, by (15), (16) and (14),
\[
\psi(x) \geq \Psi(x) = |x| = \min \left\{ |\omega| : \omega \in \{0 \leftrightarrow x\} \right\},
\]
(21)
\[
\varphi(x) \leq 2 (d-1) (|x| + 1) + 2 \leq 2 (d-1) (\psi(x) + 1) + 2.
\]
(22)
Lemma 4  For any $x, y \in \mathbb{Z}^d$,
\[ \varphi (x) \leq \varphi (y) + \varphi (x - y). \]  

\textbf{Proof.} Given $x, y \in \mathbb{Z}^d$, let
\begin{align*}
\omega_1 & \in \{ \omega \in \Omega : |\overline{\partial} C_{\{y,x\}}| = \varphi (x - y) \} , \\
\omega_2 & \in \{ \omega \in \Omega : |\overline{\partial} C_{\{y,y\}}| = \varphi (y) \} .
\end{align*}
there exists $\omega_3 \in \{ 0 < |C_{\{0,x\}} (\omega)| < \infty \}$ such that $C_{\{0,x\}} (\omega_3) = C_{\{0,y\}} (\omega_2) \cup C_{\{y,x\}} (\omega_1)$.
Moreover, $\overline{\partial} C_{\{0,x\}} \subseteq \overline{\partial} C_{\{0,y\}} \cup \overline{\partial} C_{\{y,x\}}$. Thus,
\[ \varphi (x) \leq |\overline{\partial} C_{\{0,x\}} (\omega_3)| \leq |\overline{\partial} C_{\{0,y\}} (\omega_2)| + |\overline{\partial} C_{\{y,x\}} (\omega_1)| = \varphi (x - y) + \varphi (y). \]  

\[ \square \]

Proposition 5  Let, for any $n \in \mathbb{N}$, $\mathbb{R}^d \ni x \mapsto \bar{\varphi}_n (x) := \frac{\varphi (nx)}{n} \in \mathbb{R}^+$. The sequence $\{ \bar{\varphi}_n \}_{n \in \mathbb{N}}$ converges pointwise to $\bar{\varphi}$ which is a convex, homogeneous-of-order-one function on $\mathbb{R}^d$. Moreover, $\{ \bar{\varphi}_n \}_{n \in \mathbb{N}}$ converges uniformly on $\mathbb{S}^{d-1}$.

\textbf{Proof.} Given $x \in \mathbb{R}^d$ and $m, n \in \mathbb{N}$, by the previous lemma,
\[ \varphi ([n + m] x) \leq \varphi ([nx]) + \varphi ([(n + m) x] - [nx]). \]  
Since $|[n + m] x - [nx]| - |nx| \leq d$, it is possible by adding a finite number of open bonds to construct from each $\omega \in \{ 0 < |C_{\{0,\lfloor \lfloor (n+m)x \rfloor \rfloor \rfloor} (\omega)| < \infty \}$ a
\[ \omega' \in \{ 0 < |C_{\{0,\lfloor \lfloor (n+m)x \rfloor \rfloor} (\omega)| < \infty \}. \]  
Hence, there exists a constant $c_1 = c_1 (d)$ such that
\[ |\overline{\partial} C_{\{0, \lfloor \lfloor (n+mx) \rfloor \rfloor \rfloor} (\omega')| \leq |\overline{\partial} C_{\{0, \lfloor nx \rfloor \rfloor} (\omega)| + c_1. \]  
Then,
\[ \varphi ([n + m] x) \leq \varphi ([nx]) + \varphi (\lfloor mx \rfloor) + c_1, \]
which imply the existence and the homogeneity of order one of $\bar{\varphi}$. The same argument shows that, for any $x, y \in \mathbb{R}^d$ and $n \in \mathbb{N}$,
\[ \varphi ([nx]) \leq \varphi ([ny]) + \varphi ([nx] - [ny]) \]
\[ \leq \varphi ([ny]) + \varphi ([n(x - y)]) + c_1. \]  
Setting $x = \lambda x_1 + (1 - \lambda) x_2, y = \lambda x_1$, with $x_1, x_2 \in \mathbb{R}^d$ and $\lambda \in (0, 1)$, dividing by $n$ and taking the limit $n \to \infty$ we obtain the convexity of $\bar{\varphi}$. Moreover, by (31) and (22), $\forall x, y \in \mathbb{R}^d, n \in \mathbb{N}$, there exists a constant $c'_1 = c'_1 (d)$ such that
\[ |\bar{\varphi}_n (x) - \bar{\varphi}_n (y)| \leq c'_1 \|x - y\|. \]  
Hence, the collection $\{ \bar{\varphi}_n \}_{n \in \mathbb{N}}$ is uniformly equicontinuous which, by the compactness of $\mathbb{S}^{d-1}$, implies that $\{ \bar{\varphi}_n \}_{n \in \mathbb{N}}$ converges uniformly. \[ \square \]
Lemma 6 There exists $c_2 = c_2(d) > 1$ such that, for any $x \in \mathbb{Z}^d$,

$$c_2^{-1} \leq \frac{\varphi(x)}{\psi(x)} \leq c_2,$$

(33)

Proof. For any $x \in \mathbb{Z}^d$, (22) implies $\varphi(x) \leq (2d + 1) \psi(x)$.

On the other hand, for any $x \in \mathbb{Z}^d$, let

$$\Omega_x := \{ \omega \in \{ 0 < |C_{0,x}(\omega)| < \infty \} : |\mathcal{C}_{0,x}(\omega)| = \varphi(x), \ |E(\mathcal{C}_{0,x}(\omega))| = \psi(x) \}.$$

(34)

Given $\omega \in \Omega_x$, $C_{0,x}(\omega)$ is contained, as a subset of $\mathbb{R}^d$, in the compact connected set $\bigcup_{x \in C_{0,x}(\omega)} x^*$. The function

$$\mathbb{Z}^d \ni x \mapsto v(x) := \min_{\omega \in \Omega_x} \left| \bigcup_{x \in C_{0,x}(\omega)} x^* \right| \in \mathbb{N}$$

(35)

is easily seen to satisfy subadditive type inequalities of the kind (23) and (30). Hence, arguing as in the previous proposition, the sequence $\{ \bar{v}_n \}_{n \in \mathbb{N}}$, where, $\forall n \in \mathbb{N}$,

$$\mathbb{R}^d \ni x \mapsto \bar{v}_n(x) := \frac{v\left(\left\lfloor \frac{nx}{n} \right\rfloor\right)}{n} \in \mathbb{R}^+,$$

(36)

converges to $\bar{v}$ which can be proved to be a convex, homogeneous-of-order-one function. Both $\bar{c}$ and $\bar{v}$ are equivalent norms in $\mathbb{R}^d$, therefore, there exists a positive constant $c'_2 = c_2(d)$ such that $\bar{v} \leq c'_2 \bar{c}$. Moreover,

$$\lim_{\|x\| \to \infty} \frac{v(x)}{\|x\|} = \bar{v}(\hat{x}) \ ; \ \lim_{\|x\| \to \infty} \frac{\bar{c}(x)}{\|x\|} = \bar{c}(\hat{x}).$$

(37)

Hence, for any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that, $\forall x \in (R_\varepsilon B)^c \cap \mathbb{Z}^d$, $v(x) \leq \bar{v}(\hat{x})(1 + \varepsilon)$.

Then,

$$\psi(x) \leq v(x) \leq (1 + \varepsilon) \|x\| \bar{v}(\hat{x}) \leq (1 + \varepsilon) \|x\| c'_2 \bar{c}(\hat{x}).$$

(38)

Proposition 7 There exists a constant $c_3 = c_3(d) > 1$ such that, for any $p \in \left( 1 - \frac{1}{c_3}, 1 \right)$ and any $\delta > \delta^*$, with

$$\delta^* = \delta^*(p, d) := \frac{\log \frac{c_3(d)}{p^{\alpha}(d)}}{\log \frac{1}{c_3(d)(1-p)}},$$

(39)

$$\mathbb{P}_p \left( \{|\mathcal{C}_{0,x}| \geq (1 + \delta) \psi(x)\} \cap \{|C_{0,x}| < \infty\} \right) \leq \frac{1}{1 - c_3(1-p)} \left[ \frac{c_3^{1+\delta} (1-p)^{\delta^*}}{p^{\alpha^*}} \right] v(x).$$

(40)
**Proof.** Let $L_d^0$ be the collection of subgraphs of $\mathbb{L}^d$ of finite order. Setting $G_d^0 := \{ G \in L_d^0 : \partial G = \overline{G} \}$ and denoting by $G_c^d$ the collection of connected elements of $G_d^0$, we define, for any $k \geq 2d$, the (possibly empty) collection of lattice’s subset

$$G_k := \{ G \subset \mathcal{G} : G = G \left[ (\partial G)^* \right] , \ G' \in G_c^d ; \ |V(G)| = k \} .$$

(41)

Since

$$\{0 < |C_{\{0,x\}}| < \infty \} = \bigvee_{k \geq \varphi(x)} A_k (x) ,$$

(42)

then,

$$P_p (A_k (x)) = (1 - p)^k \sum_{G \in G_k} \mathbb{P}_p \{ \omega \in \Omega : G \left[ S_{\{0,x\}}(\omega) \right] = G \}$$

and we get

$$P_p (A_k (x)) \leq |G_k| (1 - p)^k .$$

(43)

(44)

and we get

$$P_p (A_k (x)) \leq |G_k| (1 - p)^k .$$

(45)

We can choose for each $G \in G_k$ a minimal spanning tree $T_G$ and consider the collection of graphs

$$T_k := \{ T_G : G \in G_k \} .$$

(46)

Since given a connected tree there is a walk passing only twice through any edge of the graph, there exists a constant $c_3 = c_3 (d) > 1$ such that $|G_k| = c_3^k$. Therefore, by (43),

$$P_p \{ 0 < |C_{\{0,x\}}| < \infty \} = \sum_{k \geq \varphi(x)} \mathbb{P}_p (A_k (x)) \geq p^{\varphi(x)} (1 - p)^{\varphi(x)}$$

and we get

$$P_p (A_k (x)) \geq [p^{\varphi^2} (1 - p)]^{\varphi(x)} .$$

(47)

Therefore, $\forall \delta > \delta^*$, we have

$$P_p \{ |\partial C_{\{0,x\}}| \geq (1 + \delta) \varphi (x) \} \{ 0 < |C_{\{0,x\}}| < \infty \} \leq \frac{\sum_{k \geq (1 + \delta) \varphi (x)} p^k p^{\varphi(x)}}{p^{\varphi^2} (1 - p)^{\varphi(x)}} \leq \frac{c_3^{1 + \delta} (1 - p)^{\varphi(x)}}{p^{\varphi^2} (1 - p)} .$$

\[ \blacksquare \]

### 2.1 Renewal structure of connectivities

Given $t \in S^{d-1}$ we define

$$\mathcal{H}_y^t := \{ x \in \mathbb{R}^d : \langle t, x \rangle = \langle t, y \rangle \} \quad y \in \mathbb{R}^d$$

(48)
to be the \((d-1)\)-dimensional hyperplane in \(\mathbb{R}^d\) orthogonal to the vector \(t\) passing through a point \(y \in \mathbb{R}^d\) and the corresponding half-spaces
\[
\mathcal{H}_{y-}^t := \left\{ x \in \mathbb{R}^d : \langle t, x \rangle \leq \langle t, y \rangle \right\},
\]
\[
\mathcal{H}_{y+}^t := \left\{ x \in \mathbb{R}^d : \langle t, x \rangle \geq \langle t, y \rangle \right\}.
\]

Let \(t \in \mathbb{S}^d\). Given two points \(x, y \in \mathbb{Z}^d\) such that \(\langle x, t \rangle \leq \langle y, t \rangle\), we denote by \(C^t_{\{x,y\}}\) the cluster of \(x\) and \(y\) inside the strip \(S^t_{\{x,y\}} := \mathcal{H}_{y+}^t \cap \mathcal{H}_{y-}^t\) provided it exists.

Let \(u\) be the first of the unit vectors in the direction of the coordinate axis \(u_1, \ldots, u_d\) such that \(\langle t, u \rangle\) is maximal.

**Definition 8** Given \(t \in \mathbb{S}^{d-1}\), let \(x, y \in \mathbb{Z}^d\) such that \(\langle x, t \rangle \leq \langle y, t \rangle\), be connected in \(S^t_{\{x,y\}}\). The points \(b \in C^t_{\{x,y\}}\) such that:

1. \(\langle t, x + u \rangle \leq \langle t, b \rangle \leq \langle t, y - u \rangle\);
2. \(C^t_{\{x,y\}} \cap S^t_{\{b-u,b+u\}} = \{b - u, b, b + u\}\);

are said to be \(t\)-break points of \(C_{\{x,y\}}\). The collection of such points, which we remark is a totally ordered set with respect to the scalar product with \(t\), will be denoted by \(B^t(x,y)\).

**Definition 9** Given \(t \in \mathbb{S}^{d-1}\), let \(x, y \in \mathbb{Z}^d\) such that \(\langle x, t \rangle \leq \langle y, t \rangle\), be connected in \(S^t_{\{x,y\}}\). An edge \(\{b, b + u\}\) such that \(b, b + u \in B^t(x,y)\) is called \(t\)-bond of \(C_{\{x,y\}}\). The collection of such edges will be denoted by \(E^t(x,y)\), while \(B^t_{\{x,y\}}(x,y) \subset B^t(x,y)\) will denote the subcollection of \(t\)-break points \(b\) of \(C_{\{x,y\}}\) such that the edge \(\{b, b + u\} \in E^t(x,y)\).

**Notation 10** In the sequel we will omit the dependence on \(x\) in the notation of the random sets \(B^t(x,y)\), \(B^t_{\{x,y\}}(x,y)\) and \(E^t(x,y)\) if such point is taken to be the origin.

**Definition 11** Given \(t \in \mathbb{S}^{d-1}\), let \(x, y \in \mathbb{Z}^d\) such that \(\langle t, x \rangle \leq \langle t, y \rangle\) be connected. \(x, y \in \mathbb{Z}^d\) are said to be \(h_t\)-connected if

1. \(x\) and \(y\) are connected in \(S^t_{\{x,y\}}\) and \(|C^t_{\{x,y\}}| < \infty\);
2. \(x + u, y - u \in B^t(x,y)\).

Moreover, denoting by \(\{x \xleftarrow{h_t} y\}\) the event that \(x\) and \(y\) are \(h_t\)-connected, we set
\[
h_t^{(p)}(x,y) := \mathbb{P}_p \left\{ x \xleftarrow{h_t} y \right\}.
\]

Notice that, by translation invariance, \(h_t^{(p)}(x,y) = h_t^{(p)}(y-x,0)\) so in the sequel we will denote it simply by \(h_t^{(p)}(y-x)\). We also define by convention \(h_t^{(p)}(0) = 1\).
Definition 12 Let $t \in \mathbb{S}^{d-1}$ and $x, y \in \mathbb{Z}^d$ be $h_t$-connected. If $B^t (x + u, y - u)$ is empty, then $x$ and $y$ are said to be $f_t$-connected and the corresponding event is denoted by $\{x \leftrightarrow f_t y\}$. We then set

$$f_t^{(p)} (y - x) := \mathbb{P}_p \left\{ x \leftrightarrow f_t y \right\} .$$ (52)

We define by convention $f_t^{(p)} (0) = 0$.

Definition 13 Given $t \in \mathbb{S}^{d-1}$, let $x, y \in \mathbb{Z}^d$ such that $\langle t, x \rangle \leq \langle t, y \rangle$ be connected. Then:

1. $x, y$ are called $h_t$-connected and the corresponding event is denoted by $\{x \leftrightarrow h_t y\}$, if $C_{\{x,y\}} \cap S'_{\{y-u,y\}} = \{y - u, y\}$ and $|C_{\{x,y\}} \cap \mathcal{H}_y^t| < \infty$.

2. $x, y$ are called $f_t$-connected and the corresponding event is denoted by $\{x \leftrightarrow f_t y\}$, if they are $h_t$-connected and $B^t (x, y) = \emptyset$.

Definition 14 Given $t \in \mathbb{S}^{d-1}$, let $x, y \in \mathbb{Z}^d$ such that $\langle t, x \rangle \leq \langle t, y \rangle$ be connected. Then:

1. $x, y$ are called $h_t$-connected and the corresponding event is denoted by $\{x \leftrightarrow h_t y\}$, if:

   (a) $C_{\{x,y\}} \cap S'_{\{x,x+u\}} = \{x, x + u\}$;

   (b) $|C_{\{x,y\}} \cap \mathcal{H}_x^t| < \infty$.

2. $x, y$ are called $f_t$-connected and the corresponding event is denoted by $\{x \leftrightarrow f_t y\}$, if they are $h_t$-connected and $B^t (x, y) = \emptyset$.

The functions $\tilde{h}_t^{(p)} (x, y) := \mathbb{P}_p \left\{ x \leftrightarrow h_t y \right\}$ and $\tilde{f}_t^{(p)} (x, y) := \mathbb{P}_p \left\{ x \leftrightarrow f_t y \right\}$ are translation invariant.

Denoting by $g_t^{(p)} (x, y)$, for $t \in \mathbb{S}^{d-1}$, the probability of the event

$$\left\{ x \leftrightarrow^{g_t} y \right\} := \left\{ x \leftrightarrow y, |C_{\{x,y\}}| < \infty, |B^t (x, y)| \leq 1 \right\} ,$$ (53)

which is also translation invariant, we obtain

$$\mathbb{P}_p \{0 \leftrightarrow x, |C_{\{0,x\}}| < \infty\} = g_t^{(p)} (x) + \sum_{x_1, x_2 \in \mathbb{Z}^d} \tilde{f}_t^{(p)} (z_1) h_t^{(p)} (z_2 - z_1) \tilde{f}_t^{(p)} (x - z_2) ,$$ (54)

$$h_t^{(p)} (x) = \sum_{x \in \mathbb{Z}^d} f_t^{(p)} (z) h_t^{(p)} (x - z) .$$ (55)
Proposition 15 Given $t \in S^{d-1}$, for any $p \in (0, 1)$ and $x \in \mathbb{R}^d$ such that $\langle t, x \rangle > 0$,
\[
\tau_p^t (x) := - \lim_{n \to \infty} \frac{1}{n} \log h_t^{(p)} ([nx])
\]  
exists and is a convex and homogeneous-of-order-one function on $\mathbb{R}^d$. Moreover, for $p \in \left(1 - \frac{1}{c_3}, 1\right)$,
\[
\tau_p^t (x) \geq \varphi (x) \log \frac{1}{c_3 (1 - p)}.
\]

Proof. We proceed as in [ACC]. If $x, y \in \mathbb{Z}^d$ are $h_t$-connected and $S'_{\{x+u, y-u\}} \cap \mathbb{Z}^d \neq \emptyset$, by the definition of the $h_t$-connection, for any $z \in S'_{\{x+u, y-u\}}$ and $p \in (0, 1)$,
\[
h_t^{(p)} (y - x) \geq h_t^{(p)} (z - x) h_t^{(p)} (y - z).
\]
Then, for any $x \in \mathbb{Q}^d$ and $n, m \in \mathbb{N}$, if $k \in \mathbb{N}$ is such that $kx \in \mathbb{Z}^d$, we have
\[
\log h_t^{(p)} ((n + m)kx) \geq \log h_t^{(p)} (nkx) + \log h_t^{(p)} ((n + m)kx - nkx)
\]  
\[= \log h_t^{(p)} (nkx) + \log h_t^{(p)} (mkx),
\]
which imply that, for any $x \in \mathbb{Q}^d$, the limit $\tau_p^t$ in (56) exists and has the property
\[
\tau_p^t (\lambda x) = |\lambda| \tau_p^t (x), \lambda \in \mathbb{Q}.
\]
Moreover, given $x, y \in \mathbb{Q}^d, \lambda \in \mathbb{Q} \cap [0, 1]$ and $n \in \mathbb{N}$ such that $n\lambda x \in \mathbb{Z}^d$,
\[n (1 - \lambda) y \in \mathbb{Z}^d \cap S'_{\{u, n(n\lambda x + n(1-\lambda)y) - u\}}, \]
by (58),
\[
\log h_t^{(p)} (n (1 - \lambda) y + n\lambda x) \geq \log h_t^{(p)} (n (1 - \lambda) y) + \log h_t^{(p)} (n\lambda x),
\]
which, together with the previous property, implies
\[
\tau_p^t (n (1 - \lambda) y + n\lambda x) \leq \tau_p^t (n (1 - \lambda) y) + \tau_p^t (n\lambda x).\]
Therefore, $\tau_p^t$ is a convex, hence continuous, function on $\mathbb{Q}^d$ and can be extended to a convex and homogeneous-of-order-one function on $\mathbb{R}^d$. Furthermore, for $p \in \left(1 - \frac{1}{c_3}, 1\right)$, from the inequality
\[
h_t^{(p)} ([nx]) \leq P_p \{0 < |C_{\{0, |nx|\}}| < \infty\} \leq \frac{C_3 (1 - p)^{\varphi (|nx|)}}{1 - c_3 (1 - p)}
\]  
and Proposition 3 we get (57). □

Since for any $p \in (0, 1)$ and $d \geq 2$ (cf. [G] section 8.5) there exists $c_\perp = c_\perp (p, d) > 0$ such that
\[
P_p \{0 \leftrightarrow x, \ |C_{\{0, x\}}| < \infty\} \leq e^{-c_\perp \|x\|},
\]  
(64)

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while \(46\) implies that there exists \(c_+ = c_+ (p, d) > 0\) such that
\[
h_t^{(p)} (x) \geq e^{-c_+ \|x\|}.
\]
(65) it follows that \(\tau_p\) is finite and is an equivalent norm in \(\mathbb{R}^d\).

Furthermore, let
\[
\mathbb{R}^d \ni s \mapsto H_t^{(p)} (s) := \sum_{x \in \mathbb{Z}^d} h_t^{(p)} (x) e^{(s, x)} \in \overline{\mathbb{R}}.
\]
(66) implies that \(\forall p \in (p_c (d), 1),\) the effective domain of \(H_t (s),\)
\[
\mathcal{D}_t^p := \left\{ s \in \mathbb{R}^d : H_t^{(p)} (s) < \infty \right\},
\]
(67) is not empty since \(\mathcal{D}_t^p \supseteq \mathcal{K}_t^p \supseteq 0,\) where
\[
\mathcal{K}_t^p := \bigcap_{\hat{x} \in S^{d-1}} \left\{ s \in \mathbb{R}^d : \langle s, \hat{x} \rangle \leq \tau_t^p (\hat{x}) \right\}
\]
(68) is the convex body polar with respect to \(\mathcal{U}_t^p := \left\{ x \in \mathbb{R}^d : \tau_t^p (x) \leq 1 \right\}.

### 3 Renormalization

Let
\[
\mathcal{W} := \bigcap_{\hat{x} \in S^{d-1}} \left\{ s \in \mathbb{R}^d : \langle s, \hat{x} \rangle \leq \bar{\varphi} (\hat{x}) \right\}
\]
(69) and, for any \(x \in \mathbb{R}^d,\) let \(S_{x+1}^d := \left\{ s \in \mathcal{W}, \langle s, \hat{x} \rangle = \bar{\varphi} (\hat{x}) \right\}.

**Definition 16** Given \(t \in S^{d-1},\) for any \(x, y \in \mathbb{Z}^d\) such that \(\langle t, x \rangle \leq \langle t, y \rangle,\) let
\[
\left\{ \mathcal{H}_x \leftrightarrow \mathcal{H}_y \right\} := \left\{ \omega \in \Omega : 0 < |C_{x', y'}| < \infty ; x' \in \mathcal{H}_x^\circ \cap \mathbb{Z}^d, y' \in \mathcal{H}_y^\circ \cap \mathbb{Z}^d \right\}.
\]
(70) We define
\[
\varphi_t (x, y) := \min \left\{ \left| S_{\{x,y\}} (\omega) \right| : \omega \in \left\{ \mathcal{H}_x \leftrightarrow \mathcal{H}_y \right\} \right\}
\]
(71) and
\[
\phi_t (x, y) := \min_{\omega \in \left\{ \mathcal{H}_x \leftrightarrow \mathcal{H}_y \right\}} \left\{ \left| e^{\ast} \in S_{\{x,y\}} (\omega) : e^{\ast} \subset S_{\{x,y\}}^i \right| \right\}.
\]
(72)

Notice that, by translation invariance, \(\phi_t (x, y) = \phi_t (y - x, 0)\) therefore we are allowed to write \(\phi_t (x, y) = \phi_t (y - x)\). Moreover,
\[
\varphi_t (x, y) = \min_{(z', y') \in \mathcal{H}_x^\circ \cap \mathbb{Z}^d \times \mathcal{H}_y^\circ \cap \mathbb{Z}^d} \varphi (y' - z') = \min_{y' \in \mathcal{H}_y^\circ \cap \mathbb{Z}^d} \varphi (y' - x).
\]
(73) Hence we set \(\varphi_t (y - x) := \varphi_t (x, y)\). We also remark that Lemma 4 and Proposition 5 imply the existence of \(\bar{\varphi}_t (x) := \lim_{n \to \infty} \frac{\varphi_t (nx)}{n}\) for any \(x \in \mathbb{R}^d\) such that \(\langle t, x \rangle > 0.\)
Lemma 17  For any $x \in \mathbb{R}^d$ and $t \in S_x^{d-1}$, there exists $\lim_{n \to \infty} \frac{\phi_t([nx])}{n} = \bar{\varphi}(x)$.

Proof. For any $t \in S_x^{d-1}$, $\phi_t$ is superadditive and consequently $\bar{\varphi}_t(x) := \lim_{n \to \infty} \frac{\phi_t([nx])}{n}$ exists for any $x \in \mathbb{R}^d$ such that $\langle t, x \rangle > 0$. Furthermore, $\bar{\varphi}_t$ is a homogeneous-of-order-one function.

Given $t \in S_x^{d-1}$ and $x \in \mathbb{Z}^d$ such that $\langle t, x \rangle > 0$, by the definition of $\phi_t$ and $\bar{\varphi}_t$, it follows that $\phi_t(x) \leq \bar{\varphi}_t(x)$. Moreover, by the convexity of $\bar{\varphi}$, if $t$ and $x$ are chosen to be in polar relation with respect to $\bar{\varphi}$, we have $\bar{\varphi}_t(x) = \bar{\varphi}(x)$. Hence, $\forall x \in \mathbb{R}^d$ and $t \in S_x^{d-1}$, $\bar{\varphi}_t(x) \leq \bar{\varphi}(x)$.

On the other hand, for any $\hat{x} \in S_x^{d-1}$ and $t \in S_x^{d-1}$, let us consider the slab $S^t_{[0,[\hat{x}])]}$, with $n$ a large integer. Given $\delta \in \left(0, \frac{1}{2(d-1)} \right)$, $N \in \mathbb{N}$, and $K_n^N(\delta) := \left[\frac{1-\delta}{N}\right]$, we can decompose $S^t_{[0,[\hat{x}])]}$ as follows

$$S^t_{[0,[\hat{x}])] = S^t_{[0,[K_n^N(\delta)\hat{x}])} \cup S^t_{\{[K_n^N(\delta)\hat{x}] - [K_n^N(\delta)\hat{x}]\}} \cup S^t_{\{[\hat{x}] - [K_n^N(\delta)\hat{x}]\}}$$

(74)

where

$$S^t_{[0,[K_n^N(\delta)\hat{x}])} = \bigcup_{i=0,\ldots,K_n^N(\delta)-1} S^t_{\{i\hat{x}] - [(i+1)\hat{x}]\}}$$

(75)

and

$$S^t_{\{[\hat{x}] - [K_n^N(\delta)\hat{x}]\}} = \bigcup_{i=0,\ldots,K_n^N(\delta)-1} S^t_{\{[\hat{x}] - [(i+1)\hat{x}]\}}$$

(76)

For any configuration in $\omega \in \Omega : |\partial C_{[0,[\hat{x}])]| = \varphi_t([\hat{x}])]$, since there exists a constant $c_+$ such that $\varphi_t([\hat{x}])] \leq c_+ n$, there is at least one slab in the collection $\{S^t_{\{i\hat{x}] - [(i+1)\hat{x}]\}}\}_{i=0,\ldots,K_n^N(\delta)-1}$, which we denote by $S^t_{\{[j\hat{x}] - [(j+1)\hat{x}]\}}$ such that

$$|\{e^* \in S_{[0,[\hat{x}])] : e^* \subset S^t_{\{[j\hat{x}] - [(j+1)\hat{x}]\}}\}| \leq c_4 n^{2\delta} ,$$

(77)

with $c_4 = c'_4(N)$. The same argument also apply to the slabs in the collection $\{S^t_{\{[\hat{x}] - [(i+1)\hat{x}]\}}\}_{i=0,\ldots,K_n^N(\delta)-1}$, hence there exists $l \in \{0,\ldots,K_n^N(\delta)-1\}$ such that

$$|\{e^* \in S_{[0,[\hat{x}])] : e^* \subset S^t_{\{[\hat{x}] - [(l+1)\hat{x}]\}}\}| \leq c_4 n^{2\delta} .$$

(78)

Let $C_{j,l}(\omega)$ be a connected component of $C_{[0,[\hat{x}])] \cap S^t_{\{[j\hat{x}] - [(j+1)\hat{x}]\}}$ connecting $H_{[j\hat{x}]}^{\mu}$ with $H_{[\hat{x}] - [l\hat{x}]}^{\mu}$. Since the sum of the diameters of the components of the subgraph of $\mathcal{G}$ induced by $\{e^* \in S_{[0,[\hat{x}])] : e^* \cap H_{[j\hat{x}]}^{\mu} > 0\}$, with $y = [j\hat{x}]$, $[\hat{x}] - [l\hat{x}]$, is smaller than $c_4 n^{2\delta}$, there exists $c_4'' = c_4''(N)$ such that

$$|\{e^* \in (\partial C_{j,l}(\omega))^* : e^* \cap H_{[j\hat{x}]}^{\mu} > 0\}| \leq c_4'' n^{2\delta(d-1)} ,$$

(79)

and

$$|\{e^* \in (\partial C_{j,l}(\omega))^* : e^* \cap H_{[\hat{x}] - [l\hat{x}]}^{\mu} > 0\}| \leq c_4'' n^{2\delta(d-1)} .$$

(80)

Therefore,

$$\varphi_t([\hat{x}] - 2 [K_n^N(\delta)\hat{x}]) \leq \phi_t([\hat{x}]) + 2 c_4'' n^{2\delta(d-1)} .$$

(81)

Dividing by $n - 2n^{1-\delta}$ and taking the limit for $n \to \infty$, we obtain $\bar{\varphi}(x) \leq \bar{\varphi}_t(x)$. }
3.1 Proof of Theorem 1

For any \( x \in \mathbb{Z}^d \) and \( p \in \left( 1 - \frac{1}{c_3}, 1 \right) \), by Proposition 7 we are left with the estimate of the probability that there exists a finite clusters \( C_{(0,x)} \) such that \( |\mathcal{S} C_{(0,x)}| = |S_{(0,x)}| \leq (1 + \delta) \phi(x) \), for \( \delta \) larger than the value \( \delta^* \) given in (39).

Let \( t \in \mathbb{S}^d_{x} \). Moreover, given \( N \in \mathbb{N} \) larger than 1, let us set \( K^N(x) := \left\lfloor \frac{\|x\|}{N} \right\rfloor - 1 \) and

\[
\mathcal{H}^i := \mathcal{H}^i_{[iN\mathbb{Z}]} ; \mathcal{H}^i_{+} := \mathcal{H}^i_{[iN\mathbb{Z}]} ; i = 0, ..., K^N(x) ;
\]

\[
\mathcal{H}^i_{(K^N(x)+1)N\mathbb{Z}} := \mathcal{H}^i_{x} ; \mathcal{H}^i_{+} := \mathcal{H}^i_{[(K^N(x)+1)N\mathbb{Z}]};
\]

\[
S^i := \mathcal{H}^i_{+} \cap \mathcal{H}^i_{x+1}.
\]

With a slight notational abuse we still denote by \( S_{(0,x)} \) its representation as a hypersurface in \( \mathbb{R}^d \) and define

\[
C^i := C_{(0,x)} \cap S^i ; S^i := S_{(0,x)} \cap S^i.
\]

Hence, \( C_{(0,x)} = \bigcup_{i=0}^{\left\lfloor \frac{\|x\|}{N} \right\rfloor - 1} C^i \) and \( S_{(0,x)} \cap S^i_{(0,x)} \subseteq \bigcup_{i=0}^{\left\lfloor \frac{\|x\|}{N} \right\rfloor - 1} S^i \).

We say that a slab \( S^i \) is bad if \( S^i \) is not connected, otherwise we call it good, and call crossing any connected component \( s \) of \( S^i \) such that, denoting by \( \mathcal{T}(s) \) the compact subset of \( S^i \) whose boundary is \( s \), there is a path in \( L^d \cap \mathcal{T}(s) \) connecting \( \mathcal{H}^i \) with \( \mathcal{H}^i_{+} \).

We remark that since \( C_{(0,x)} \) is connected, the existence of two crossings in \( S^i \) implies the existence of two disjoint paths connecting \( \mathcal{H}^i \) and \( \mathcal{H}^i_{+} \) while the converse does not hold true in general.

Let \( \eta \) be the fraction of slabs where there are at least two crossings. Since any crossing is composed by at least \( \phi_t([N\mathbb{Z}]) \) plaquettes, we have

\[
\frac{\|x\|}{N} (\eta \phi_t([N\mathbb{Z}]) + (1 - \eta) \phi_t([N\mathbb{Z}])) = \frac{\|x\|}{N} (1 + \eta) \phi_t([N\mathbb{Z}]) < (1 + \delta) \phi(x).
\]

Moreover, given \( \varepsilon > 0 \), there exists \( R_{\varepsilon} > 0 \) such that, for any \( x \in \mathbb{Z}^d \cap (R_{\varepsilon}B)^c \), \( \phi(x) \leq \bar{\phi}(x)(1 + \varepsilon) \). Hence, choosing \( N \) sufficiently large such that \( \phi_t([N\mathbb{Z}]) \leq \bar{\phi}_t(N\mathbb{Z})(1 + \varepsilon) \), since \( t \in \mathbb{S}^d_{x} \), \( \phi_t(N\mathbb{Z}) = \bar{\phi}(N\mathbb{Z}) \) and, by the previous inequality, we get \( \eta < \delta \).

Then, there are at most \( \frac{\|x\|}{N} - \eta \frac{\|x\|}{N} \) 3-tuple of consecutive slabs containing a single crossing and therefore at most the same number of bad slabs containing a single crossing. Choosing, \( \eta \leq \frac{1}{6} \), it is possible to modify the configuration of at most \( c_5N^3 \) bonds, with \( c_5 = c_5(d, \delta) \), inside any 3-tuple of consecutive slabs containing a single crossing in such a way that the resulting cluster will have at least one \( t \)-bond inside each of these slabs. Since these modifications can be performed independently, this fact and the previous proposition imply that there exists a positive constant \( c_6 = c_6(p) \) such that

\[
\mathbb{P} \{ 0 < |C_{(0,x)}| < \infty \} \leq e^{-\|x\|c_6}
\]

(87)
and consequently the mass-gap condition $f_t^{(p)}(x) \leq e^{-c_7\|x\|} h_t^{(p)}(x)$, $c_7 = c_7(p) > 0$, uniformly in $t \in \mathbb{S}^{d-1}_x$.

Thus, from (54) we have

$$
\mathbb{P}_p \left\{ 0 \leftrightarrow x, \left| C_{\{0,x\}} \right| < \infty \right\} \leq c_8,
$$

(88)

with $c_8 = c_8(d) > 0$, for any $t \in \mathbb{S}^{d-1}_x$.

We now proceed as in Section 4.3 of [CI]. Given $t \in \mathbb{S}^{d-1}_x$, we extend $f_t^{(p)}$ to a function defined on the whole lattice by setting it equal to zero where it is non defined and set

$$
\mathbb{R}^d \ni s \mapsto -\nabla Q_t^{(p);s} := \sum_{x \in \mathbb{Z}^d} f_t^{(p)}(x) e^{\langle s,x \rangle} \in \mathbb{R}.
$$

(89)

The renewal equation (55) imply

$$
H_t^{(p)}(s) = \frac{1}{1 - F_t^{(p)}(s)}.
$$

(90)

For $s \in K_t^p$, since

$$
(s, x) \leq \max_{s \in K_t^p} (s, x) = \tau_t^p(x) \leq 1
$$

(91)

and $h_t^{(p)}(x) \leq e^{-\tau_t^p(x)}$, $F_t^{(p)}(s)$ is finite, moreover it is continuous, then, $\forall s \in \partial K_t^p$,

$$
\mathbb{Z}^d \ni x \mapsto q_{t,s}^{(p)}(x) := f_t^{(p)}(x) e^{\langle s,x \rangle} \in \mathbb{R}
$$

(92)

is the density of the probability measure $Q_t^{(p);s}$ on $(\mathbb{Z}^d, \mathcal{B}(\mathbb{Z}^d))$ which, has exponentially decaying tails:

$$
f_t^{(p)}(x) e^{\langle s,x \rangle} \leq e^{-c_7\|x\|} h_t^{(p)}(x) e^{\langle s,x \rangle} \leq e^{-c_7\|x\|}.
$$

(93)

If $X$ is a random vector with probability distribution $Q_t^{(p);s}$, denoting by $E_t^{(p);s}$ the expectation of a random variable under $Q_t^{(p);s}$, we set

$$
\mu_t^{(p)}(s) := E_t^{(p);s} [X] = \nabla \log F_t^{(p)}(s),
$$

(94)

while

$$
C_t^{(p)}(s) := \text{Hess} \log F_t^{(p)}(s)
$$

(95)

denotes the covariance matrix of $X$. Since $f_t^{(p)}(x) > 0$ on a whole half-space, $C_t^{(p)}(s)$ is non degenerate. Hence,

$$
\partial K_t^p = \left\{ s \in \mathbb{R}^d : F_t^{(p)}(s) = 1 \right\} \subseteq \mathbb{R}^d \setminus D_t^p
$$

(96)
is a real analytic strictly convex surface with Gaussian curvature uniformly bounded away from zero and therefore, because $Q_{t,s}^{(p)}$ is supported on $H^t_s \cap \mathbb{Z}^d$, $\mu^p_t (s) \neq 0$ and $(s, \mu^p_t (s)) > 0$ for any $s \in B_r (t) \cap \partial K^p_t$ with $r$ sufficiently small.

Let then $s \in B_r (t) \cap \partial K^p_t$, for any $\mu \in B (\mu^p_t (s)) \cap H^s_{\mu^p_t (s)}$, if $\{X_i\}_{i \geq 1}$ is a sequence of i.i.d. random vectors distributed according to $Q_{t,s}^{(p)}$, for $n \in \mathbb{N}$, we can rewrite (55) as

$$h_t^{(p)} ([n\mu]) = \delta_0 ([n\mu]) + e^{-\langle |n\mu|, s \rangle} \sum_{k \geq 1} \bigotimes_{i=1}^k Q_{t,s}^{(p)} \left\{ \sum_{i=1}^k X_i = |n\mu| \right\}.$$  \hspace{1cm} (97)

Hence, there exist two positive constant $c_9 = c_9 (p)$ and $c_{10} = c_{10} (p)$ such that

$$\left\| n\mu - \sum_{i=1}^k \mathbb{E}_p [X_i] \right\| = \| n\mu - k\mu^p_t (s) \| \geq nc_9 \| \mu - \mu^p_t (s) \| + |n-k| c_{10} \| \mu^p_t (s) \|. \hspace{1cm} (98)$$

Therefore, the standard large deviation upper bound for

$$\bigotimes_{i=1}^k Q_{t,s}^{(p)} \left\{ \sum_{i=1}^k X_i = |n\mu| \right\} \leq e^{-c_{11} \left( n - \frac{k}{n} \right)^2 - |n-k| - c_{12} \frac{n^2}{2} \| \mu - \mu^p_t (s) \|^2} \hspace{1cm} (99)$$

leads to

$$h_t^{(p)} ([n\mu]) \leq c_{13} \sqrt{n} e^{-\left( |\langle n\mu, s \rangle| + c_{14} n \right) \| \mu - \mu^p_t (s) \|}} \hspace{1cm} (100)$$

with $c_{13} = c_{13} (p) \hspace{.1cm}, \hspace{.1cm} c_{14} = c_{14} (p)$ positive constants. Then, (55) and (100) imply

$$\tau^{(p)}_t (\mu) \geq \left\langle \mu, s \right\rangle + c_{14} \| \mu - \mu^p_t (s) \|,$$  \hspace{1cm} (101)

that is the strict convexity of $\tau^{(p)}_t$. Moreover, because $\tau^{(p)}_t$, being an equivalent norm in $\mathbb{R}^d$, is lower semicontinuous, from (101) it follows that $\mu^p_t (s)$ and $s$ are in polar relation with respect to $\tau^{(p)}_t$, namely $\left\langle s, \mu^p_t (s) \right\rangle = \tau^{(p)}_t (\mu^p_t (s))$.

Furthermore, for any $x \in \mathbb{Z}^d$, let $t \in S^{d-1}$. There exist $r' > 0$, $s \in B_{r'} (t) \cap \partial K^p_t$, and $n_x \in \mathbb{N}$ such that

$$\| x - n_x \mu^p_t (s) \| \leq c_{15}, \hspace{1cm} (102)$$

where $c_{15} = c_{15} (p) > 0$. Hence, choosing $\alpha \in \left( 0, \frac{1}{2} \right)$, since $Q_{t,s}^{(p)}$ is a centered lattice distribu-
tion, we can apply the local central limit theorem for \( \bigotimes_{i=1}^{k} Q_{i:s}^{(p)} \left\{ \sum_{i=1}^{k} X_i = x \right\} \) to get,

\[
\sum_{k \in \mathbb{N} : |k-n_x| < n_x^{\frac{1}{2}+\alpha}} \bigotimes_{i=1}^{k} Q_{i:s}^{(p)} \left\{ \sum_{i=1}^{k} X_i = x \right\} = \sum_{k \in \mathbb{N} : |k-n_x| < n_x^{\frac{1}{2}+\alpha}} \frac{\exp \left\{ \frac{-(n_x-k)^2}{2k} \left( \langle (C_i^p)^{-1} (s) \mu_i^p (s) , \mu_i^p (s) \rangle \right) \right\}}{\sqrt{2\pi k \det C_i^p (s)}} \left( 1 + o (1) \right) \]

\[
= \sum_{k \in \mathbb{N} : |k-n_x| < n_x^{\frac{1}{2}+\alpha}} \frac{\exp \left\{ \frac{-(n_x-k)^2}{2n_x \left( 1 + O \left( n_x^{\alpha-\frac{1}{2}} \right) \right)} \left( \langle (C_i^p)^{-1} (s) \mu_i^p (s) , \mu_i^p (s) \rangle \right) \right\}}{\sqrt{2\pi n_x \left( 1 + O \left( n_x^{\alpha-\frac{1}{2}} \right) \right) \det C_i^p (s)}} \left( 1 + o (1) \right) 
\]

\[
= \frac{1}{\sqrt{(2\pi n_x)^{d-1} \det C_i^p (s) \left( \langle (C_i^p)^{-1} (s) \mu_i^p (s) , \mu_i^p (s) \rangle \right)}} \left( 1 + o (1) \right) .
\]

On the other hand, in the complementary range of \( k \)'s, proceeding as in (100) we obtain

\[
\sum_{k \in \mathbb{N} : |k-n_x| \geq n_x^{\frac{1}{2}+\alpha}} \bigotimes_{i=1}^{k} Q_{i:s}^{(p)} \left\{ \sum_{i=1}^{k} X_i = x \right\} \leq e^{-c_{15} n_x^{2\alpha} (104)}
\]

which, by (102), (97) and (SS), since \( s = s (\hat{x}) \), gives back

\[
\mathbb{P}_p \left\{ 0 \leftrightarrow x , \ |C_{[0,x]}| < \infty \right\} = \frac{\Phi_p (\hat{x})}{\sqrt{(2\pi \| x \|)^{d-1}}} e^{-\tau_p (x)} (1 + o (1)) ,
\]

where

\[
\mathbb{S}^{d-1} \ni \hat{x} \mapsto \Phi_p (\hat{x}) := \frac{\| \mu_i^p (s (\hat{x})) \|^{d-1}}{\sqrt{\det C_i^p (s (\hat{x})) \left( \langle (C_i^p)^{-1} (s (\hat{x})) \mu_i^p (s (\hat{x})) , \mu_i^p (s (\hat{x})) \rangle \right)}} \in \mathbb{R}^+ .
\]

is a real analytic function on \( \mathbb{S}^{d-1} \).

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On the Ornstein-Zernike behaviour for the Bernoulli bond percolation on $\mathbb{Z}^d$, $d \geq 3$, in the supercritical regime

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Abstract

We prove Ornstein-Zernike behaviour in every direction for finite connection functions of bond percolation on $\mathbb{Z}^d$ for $d \geq 3$ when $p$, the probability of occupation of a bond, is sufficiently close to 1. Moreover, we prove that equi-decay surfaces are locally analytic, strictly convex, with positive Gaussian curvature.

1 Introduction and results

Ornstein-Zernike behaviour of correlation and connection functions has been rigorously proved for many models of statistical mechanics and percolation in the high temperature or low probability regime, first for extreme values of the parameter (see e.g. [BF]) and then up

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to the critical point (see [CCC], [CI], [CIV]). Above the critical probability in [CIL] it was
proved that in two dimensions finite connection functions, i.e. the probabilities that two sites
belong to the same finite cluster, exhibit a different asymptotic behaviour, which is related
to the probability that two independent random walks in dimension 2 do not intersect. In
higher dimensions, for \( d \geq 3 \), one expects that Ornstein-Zernike behaviour holds for finite
connections probabilities above critical probability. This was proved in [BPS] when \( p \) (the
probability that a bond is open) is close to one for the connection probabilities in the direction
of the axes. The proof is based on cluster expansion.

The problem of the asymptotic behaviour of finite connection functions in arbitrary di-
rections presents an important difference with respect to that in the directions of coordinate
axes. Indeed, in the limit of \( p \) tending to 1 the probability distribution of the finite clus-
ter containing two sites on a coordinate axis, conditioned to its existence, tends to a delta
measure concentrated on the segment joining the two sites. The cluster expansion presented
in [BPS] can be thought of as a perturbation about this configuration that plays the role of
ground state. In the case of two sites that don’t lie on the same coordinate axis, the limiting
distribution is not supported on a single configuration: one can say that the ground state is
degenerate. This makes the extension of the method used in [BPS] problematic.

In this paper we prove Ornstein-Zernike behaviour in every direction for finite connection
functions of bond percolation on \( \mathbb{Z}^d \) for \( d \geq 3 \) when \( p \), the probability of occupation of
a bond, is sufficiently close to 1. Moreover, we prove that equi-decay surfaces are locally
analytic, strictly convex with positive Gaussian curvature.

Our proofs rely in part on the methods developed in [CI], based on multi-dimensional
renewal theory and local limit theorem, but we have to deal with new problems; in particular
FKG inequality does not apply to finite connection functions, as they are probabilities of
non-monotone events. By developing specific techniques we are able to treat the case when \( p \)
is sufficiently close to 1.

Here in the following are the main results of the paper and the notation that we will use.
In Section 2 we introduce the relevant connectivity functions and their renewal structure. In
Section 3 we prove the existence of the mass-gap for the direct connectivity function and
prove the main results of the paper.

**Theorem 1** For any \( d \geq 3 \), there exists \( p^* = p^* (d) \in (0, 1) \) such that, \( \forall p \in (p^*, 1) \), uniformly
in \( x \in \mathbb{Z}^d, \|x\| \to \infty \),

\[
\mathbb{P}_p \left( 0 \leftrightarrow x , \ |C_{\{0,x\}}| < \infty \right) = \frac{\Phi_p (\hat{x})}{\sqrt{(2\pi \|x\|)^{d-1}}} e^{-\tau_p (x)} (1 + o(1)) ,
\]

where \( \hat{x} := \frac{x}{\|x\|} \), \( \Phi_p \) is a positive real analytic function on \( \mathbb{S}^{d-1} \) and \( \tau_p \) an equivalent norm in
\( \mathbb{R}^d \).

As a by-product of the proof of the previous theorem we also obtain the following result.
Theorem 2 For any $d \geq 3$, there exists $p^* = p^*(d) \in (0,1)$ such that, $\forall p \in (p^*,1)$, the equi-decay set is locally analytic and strictly convex. Moreover, the Gaussian curvature of the equi-decay set is uniformly positive.

1.1 Notation

For any $x \in \mathbb{R}^d$, $d \geq 1$, Let us denote by $|x| := \sum_{i=1}^{d} |x_i|$, by $\langle \cdot, \cdot \rangle$ the scalar product in $\mathbb{R}^d$ and by $\| \cdot \| := \sqrt{\langle \cdot, \cdot \rangle}$ the associated Euclidean norm. We then set $\mathbb{S}^{d-1} := \{ z \in \mathbb{R}^d : \| z \| = 1 \}$ and $\hat{x} := \frac{x}{\|x\|}$. Given a set $A \subset \mathbb{R}^d$, let us denote by $A^c$ its complement and by $\mathcal{P}(A)$ the collection of all subsets of $A$. We also set $\mathcal{P}_2(A) := \{ A \in \mathcal{P}(A) : |A| = 2 \}$, where $|A|$ is the cardinality of $A$. Moreover, we denote by $\overline{A}$, $\mathring{A}$ respectively the interior of $A$ and the closure of $A$ and set $\partial A := \overline{A} \setminus \mathring{A}$ the boundary of $A$ in the Euclidean topology. Furthermore, if $x \in \mathbb{R}^d$, we set

$$x + A := \{ y \in \mathbb{R}^d : y - x \in A \}$$

and, denoting by $B$ the closed unit ball in $\mathbb{R}^d$, for $r > 0$, let $rB := \{ x \in \mathbb{R}^d : \| x \| \leq r \}$, $B_r(x) := x + rB$.

To make the paper self-contained, we will now introduce those notions of graph theory which are going to be used in the sequel and refer the reader to [13] for an account on this subject.

Let $G = (V,E)$ be a graph whose set of vertices and set of edges are given respectively by a finite or denumerable set $V$ and $E \subset \mathcal{P}_2(V)$. $G' = (V',E')$ such that $V' \subseteq V$ and $E' \subseteq \mathcal{P}_2(V') \cap E$ is said to be a subgraph of $G$ and this property is denoted by $G' \subseteq G$. If $G' \subseteq G$, we denote by $V(G')$ and $E(G')$ respectively the set of vertices and the collection of the edges of $G'$. $|V(G')|$ is called the order of $G'$ while $|E(G')|$ is called its size. Given $G_1, G_2 \subseteq G$, we denote by $G_1 \cup G_2 := (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2)) \subset G$ the graph union of $G_1$ and $G_2$. A path in $G$ is a subgraph $\gamma$ of $G$ such that there is a bijection $\{0,..,|E(\gamma)|\} \ni i \mapsto v(i) := x_i \in V(\gamma)$ with the property that any $e \in E(\gamma)$ can be represented as $\{x_{i-1},x_i\}$ for $i = 1,..,|E(\gamma)|$. A walk in $G$ of length $l \geq 1$ is an alternating sequence $x_0,e_1,x_1,..,e_l,x_l$ of vertices and edges of $G$ such that $e_i = \{x_{i-1},x_i\}$ $i = 1,..,l$. Therefore, paths can be associated to walks having distinct vertices. Two distinct vertices $x,y$ of $G$ are said to be connected if there exists a path $\gamma \subseteq G$ such that $x_0 = x$, $x_{|E(\gamma)|} = y$. A graph $G$ is said to be connected if any two distinct elements of $V(G)$ are connected. The maximal connected subgraphs of $G$ are called components of $G$. Given $E' \subseteq E$, we denote by $G(E') := (V,E')$ the spanning graph of $E$. We also define

$$V(E') := \left( \bigcup_{e \in E'} e \right) \subset V .$$

Given $V' \subseteq V$, we set

$$E(V') := \{ e \in E : e \subset V' \}$$

(4)
and denote by $G[V'] := (V', E(V'))$ that is called the subgraph of $G$ induced or spanned by $V'$. Moreover, if $G' \subset G$, we denote by $G \setminus G'$ the graph $G[V \setminus V(G')] \subseteq G$ and define the boundary of $G'$ as the set

$$\partial G' := \{ e \in E \setminus E(G') : |e \cap V(G')| = 1 \} \subset E .$$

(5)

Let $\mathbb{L}^d$ be the $d$-dimensional cubic lattice, that is the geometric graph whose set of vertices is $\mathbb{Z}^d$ and whose set of edges is

$$E^d := \{ \{ x, y \} \in \mathcal{P}^{(2)} (\mathbb{Z}^d) : |x - y| = 1 \} .$$

(6)

If $G$ is a subgraph of $\mathbb{L}^d$ of finite order, we denote by $\overline{G}$ the graph induced by the union of $V(G)$ with the sets of vertices of the connected components of the $\mathbb{L}^d \setminus G$ of finite size. We define the external boundary of $G$ to be $\partial G := \partial \overline{G}$. We remark that, given $G_i := (V_i, E_i), i = 1, 2$ two connected subgraphs of $\mathbb{L}^d$ of finite size, by (10), $\partial (G_1 \cup G_2) \subseteq \partial G_1 \cup \partial G_2$. Moreover,

$$\overline{\partial} (G_1 \cup G_2) = \partial (\overline{G_1} \cup \overline{G_2}) \subseteq \partial \overline{G_1} \cup \partial \overline{G_2} .$$

(7)

Furthermore, looking at $\mathbb{L}^d$ as a cell complex, i.e. as the union of $\mathbb{Z}^d$ and $\mathbb{E}^d$ representing respectively the collection of 0-cells and of 1-cells, we denote by $(\mathbb{Z}^d)^*$ the collection of $d$-cells dual 0-cells in $\mathbb{L}^d$, that is the collection of unit $d$-cubes centered in the elements of $\mathbb{Z}^d$ (Voronoi cells of $\mathbb{L}^d$), and by $(\mathbb{E}^d)^*$ the collection of $(d - 1)$-cells dual 1-cells in $\mathbb{L}^d$, usually called plaquettes in the physics literature. We also define

$$\mathcal{E} := \left\{ \{ e_1^*, e_2^* \} \in \mathcal{P}^{(2)} ((\mathbb{E}^d)^*) : \text{codim} (\partial e_1^* \cap \partial e_2^*) = 2 \right\}$$

(8)

and consider the graph $\mathcal{G} := ((\mathbb{E}^d)^*, \mathcal{E})$.

A bond percolation configuration on $\mathbb{L}^d$ is a map $\mathbb{E}^d \ni e \longmapsto \omega_e \in \{0,1\}$. Setting $\Omega := \{0,1\}^{\mathbb{E}^d}$ we define

$$\Omega \ni \omega \longmapsto E(\omega) := \{ e \in \mathbb{E}^d : \omega_e = 1 \} \in \mathcal{P}(\mathbb{E}^d) ,$$

(9)

Let $\mathcal{F}$ be the $\sigma$-algebra generated by the cylinder events of $\Omega$. Given $p \in [0,1]$, we consider the independent Bernoulli probability measure $\mathbb{P}_p$ on $(\Omega, \mathcal{F})$ with parameter $p$.

Denoting by $\mathcal{G}^d := \{ G \subseteq \mathbb{L}^d : G = G(E), E \in \mathcal{P}(\mathbb{E}^d) \}$ the collection of spanning subgraphs of $\mathbb{L}^d$, we define the random graph

$$\Omega \ni \omega \longmapsto G(\omega) := G(E(\omega)) \in \mathcal{G}^d .$$

(10)

Then, given $l \geq 1, x_1, ..., x_l \in \mathbb{Z}^d$, we denote by

$$\Omega \ni \omega \longmapsto C_{\{x_1, ..., x_l\}}(\omega) \in \mathcal{P}(\mathbb{Z}^d)$$

(11)

the common open cluster of the points $x_1, ..., x_l \in \mathbb{Z}^d$, that is the set of vertices of the connected component of the random graph $G(\omega)$ to which these points belong, provided it exists, and define, in the case $C_{\{x_1, ..., x_l\}}$ is finite, the random set $\overline{\partial} C_{\{x_1, ..., x_l\}}$ to be equal to $\partial G$ if $G$ is the component of $G(\omega)$ whose set of vertices is $C_{\{x_1, ..., x_l\}}$ and the random set $S_{\{x_1, ..., x_l\}} := (\overline{\partial} C_{\{x_1, ..., x_l\}})^*$.
2 Analysis of connectivities

Given $x, y \in \mathbb{Z}^d$, we set

$$
\varphi (x, y) := \begin{cases} 
0 & \text{if } x = y \\
\min \left\{ \left| \partial C_{\{x,y\}} (\omega) \right| : \omega \in \{ 0 < |C_{\{x,y\}} (\omega)| < \infty \} \right\} & \text{if } x \neq y
\end{cases}
$$

(12)

(13)

$$
\varphi (x, y) = \varphi (x - y)
$$

$\varphi$ is symmetric and translation invariant, therefore in the sequel we will write

$$
\varphi (x, y) = \varphi (x - y)
$$

For any $x \in \mathbb{Z}^d$ and $k \geq \varphi (x)$, let us set

$$
A_k (x) := \{ \omega \in \Omega : |\partial C_{\{0,x\}} (\omega)| = k \} = \{ \omega \in \Omega : |S_{\{0,x\}} (\omega)| = k \}
$$

and $A^k (x) := \bigvee_{l \geq k} A_l (x)$. We define

$$
\psi_k (x) := \min \{ |E (C_{\{0,x\}} (\omega))| : \omega \in A_k (x) \},
$$

(15)

$$
\Psi_k (x) := \min \{ |E (C_{\{0,x\}} (\omega))| : \omega \in A^k (x) \} = \min_{l \geq k} \psi_l (x)
$$

(16)

and set $A (x) := A_{\varphi(x)} (x)$ and consequently $\psi (x) := \psi_{\varphi(x)}$, $\Psi (x) := \Psi_{\varphi(x)} (x)$.

Remark 3 Given $x \in \mathbb{Z}^d$, for $i = 0, \ldots, d$, let $\gamma_i$ be the path such that $\gamma_0 = \varnothing$ and

$$
V (\gamma_i) = \{ \text{sign} (x_i) u_i, 2 \text{sign} (x_i) u_i, \ldots, |x_i| \text{sign} (x_i) u_i \} \quad i = 1, \ldots, d.
$$

(17)

Let also

$$
\gamma := \bigcup_{i=0}^{d-1} \left[ \left( \sum_{j=0}^{i} x_j u_j \right) + \gamma_{i+1} \right].
$$

(18)

By construction

$$
|\gamma| := |E (\gamma)| = |x|,
$$

(19)

$$
|\partial V (\gamma)| = |\partial V (\gamma)| = 2 (d-1) (|x| + 1) + 2.
$$

(20)

Hence, by (15), (16) and (14),

$$
\psi (x) \geq \Psi (x) = |x| = \min \{ |\omega| : \omega \in \{ 0 \leftrightarrow x \} \},
$$

(21)

$$
\varphi (x) \leq 2 (d-1) (|x| + 1) + 2 \leq 2 (d-1) (\psi (x) + 1) + 2.
$$

(22)
Lemma 4 For any \( x, y \in \mathbb{Z}^d \),
\[
\varphi (x) \leq \varphi (y) + \varphi (x - y) .
\] (23)

Proof. Given \( x, y \in \mathbb{Z}^d \), let
\[
\omega_1 \in \{ \omega \in \Omega : |\mathcal{C}_{\{y,x\}}(\omega)| = \varphi (x - y) \} ,
\] (24)
\[
\omega_2 \in \{ \omega \in \Omega : |\mathcal{C}_{\{y\}}(\omega)| = \varphi (y) \} .
\] (25)
there exists \( \omega_3 \in \{ 0 < |C_{\{y\}}(\omega)| < \infty \} \) such that \( C_{\{y\}}(\omega_3) = C_{\{y\}}(\omega_2) \cup C_{\{y\}}(\omega_1) \).

Moreover, \( \mathcal{C}_{\{y,x\}} \leq \mathcal{C}_{\{y\}} \cup C_{\{y\}} \). Thus,
\[
\varphi (x) \leq |\mathcal{C}_{\{y,x\}}(\omega_3)| \leq |\mathcal{C}_{\{y\}}(\omega_2)| + |\mathcal{C}_{\{y\}}(\omega_1)| = \varphi (x - y) + \varphi (y) .
\] (26)

Proposition 5 Let, for any \( n \in \mathbb{N} \), \( \mathbb{R}^d \ni x \mapsto \tilde{\varphi}_n (x) := \frac{\varphi (nx)}{n} \in \mathbb{R}^+ \). The sequence \( \{ \tilde{\varphi}_n \}_{n \in \mathbb{N}} \) converges pointwise to \( \tilde{\varphi} \) which is a convex, homogeneous-of-order-one function on \( \mathbb{R}^d \). Moreover, \( \{ \tilde{\varphi}_n \}_{n \in \mathbb{N}} \) converges uniformly on \( S^{d-1} \).

Proof. Given \( x \in \mathbb{R}^d \) and \( m, n \in \mathbb{N} \), by the previous lemma,
\[
\varphi ([(n + m)x]) \leq \varphi ([nx]) + \varphi ([n + m]x - [nx]) .
\] (27)

Since \( |[(n + m)x - [nx]) - [mx]| \leq d \), it is possible by adding a finite number of open bonds to construct from each \( \omega \in \{ 0 < |\mathcal{C}_{\{nx\}}(\omega)| < \infty \} \) a
\[
\omega' \in \{ 0 < |\mathcal{C}_{\{nx\}}(\omega')| < \infty \} .
\] (28)

Hence, there exists a constant \( c_1 = c_1 (d) \) such that
\[
|\mathcal{C}_{\{nx\}}(\omega')| \leq |\mathcal{C}_{\{nx\}}(\omega)| + c_1 .
\] (29)

Then,
\[
\varphi ([n + m]x) \leq \varphi ([nx]) + \varphi ([mx]) + c_1 ,
\] (30)
which imply the existence and the homogeneity of order one of \( \tilde{\varphi} \). The same argument shows that, for any \( x, y \in \mathbb{R}^d \) and \( n \in \mathbb{N} \),
\[
\varphi ([nx]) \leq \varphi ([ny]) + \varphi ([nx] - [ny]) 
\leq \varphi ([ny]) + \varphi ([n(x - y)]) + c_1 .
\] (31)

Setting \( x = \lambda x_1 + (1 - \lambda) x_2, y = \lambda x_1, \) with \( x_1, x_2 \in \mathbb{R}^d \) and \( \lambda \in (0, 1) \), dividing by \( n \) and taking the limit \( n \to \infty \) we obtain the convexity of \( \tilde{\varphi} \). Moreover, by (31) and (22), \( \forall x, y \in \mathbb{R}^d, n \in \mathbb{N} \),
there exists a constant \( c_1' = c_1' (d) \) such that
\[
|\tilde{\varphi}_n (x) - \tilde{\varphi}_n (y)| \leq c_1' \| x - y \| .
\] (32)

Hence, the collection \( \{ \tilde{\varphi}_n \}_{n \in \mathbb{N}} \) is uniformly equicontinuous which, by the compactness of \( S^{d-1} \), implies that \( \{ \tilde{\varphi}_n \}_{n \in \mathbb{N}} \) converges uniformly. \( \blacksquare \)
Lemma 6 There exists \( c_2 = c_2(d) > 1 \) such that, for any \( x \in \mathbb{Z}^d \),

\[
c_2^{-1} \leq \frac{\varphi(x)}{\psi(x)} \leq c_2,
\]

(33)

Proof. For any \( x \in \mathbb{Z}^d \), \([22]\) implies \( \varphi(x) \leq (2d + 1) \psi(x) \).

On the other hand, for any \( x \in \mathbb{Z}^d \), let

\[
\Omega_x := \{ \omega \in \{ 0 < |C_{(0,x)}(\omega)| < \infty \} : |\mathcal{T}C_{(0,x)}(\omega)| = \varphi(x), \ |E(C_{(0,x)}(\omega))| = \psi(x) \} .
\]

(34)

Given \( \omega \in \Omega_x \), \( C_{(0,x)}(\omega) \) is contained, as a subset of \( \mathbb{R}^d \), in the compact connected set \( \bigcup_{x \in C_{(0,x)}(\omega)} x^* \). The function

\[
\mathbb{Z}^d \ni x \mapsto \upsilon(x) := \min_{\omega \in \Omega_x} \left| \bigcup_{x \in C_{(0,x)}(\omega)} x^* \right| \in \mathbb{N}
\]

(35)

is easily seen to satisfy subadditive type inequalities of the kind \([23]\) and \([30]\). Hence, arguing as in the previous proposition, the sequence \( \{ \bar{\upsilon}_n \}_{n \in \mathbb{N}} \), where, \( \forall n \in \mathbb{N} \),

\[
\mathbb{R}^d \ni x \mapsto \bar{\upsilon}_n(x) := \frac{\upsilon\left( \left\lfloor nx \right\rfloor \right)}{n} \in \mathbb{R}^+ ,
\]

(36)

converges to \( \bar{\upsilon} \) which can be proved to be a convex, homogeneous-of-order-one function. Both \( \bar{\varphi} \) and \( \bar{\upsilon} \) are equivalent norms in \( \mathbb{R}^d \), therefore, there exists a positive constant \( c'_2 = c'_2(d) \) such that \( \bar{\upsilon} \leq c'_2 \bar{\varphi} \). Moreover,

\[
\lim_{||x|| \to \infty} \frac{\upsilon(x)}{||x||} = \bar{\upsilon}(\hat{x}) ; \lim_{||x|| \to \infty} \frac{\varphi(x)}{||x||} = \bar{\varphi}(\hat{x}) .
\]

(37)

Hence, for any \( \varepsilon > 0 \), there exists \( R_\varepsilon > 0 \) such that, \( \forall x \in (R_\varepsilon B)^c \cap \mathbb{Z}^d \),

\[
\frac{\upsilon(x)}{||x||} \leq \bar{\upsilon}(\hat{x})(1 + \varepsilon) . \text{ Then,}
\]

\[
\psi(x) \leq \upsilon(x) \leq (1 + \varepsilon) ||x|| \bar{\upsilon}(\hat{x}) \leq (1 + \varepsilon) ||x|| c'_2 \bar{\varphi}(\hat{x}) .
\]

(38)

\[
\square
\]

Proposition 7 There exists a constant \( c_3 = c_3(d) > 1 \) such that, for any \( p \in \left( 1 - \frac{1}{c_3}, 1 \right) \) and any \( \delta > \delta^* \), with

\[
\delta^* = \delta^*(p,d) := \frac{\log c_3(d)}{p(1-p)} ,
\]

(39)

\[
\mathbb{P}_p \left( \{|\mathcal{T}C_{(0,x)}| \geq (1 + \delta) \varphi(x) \} \middle| \{ |C_{(0,x)}| < \infty \} \right) \leq \frac{1}{1 - c_3(1-p)} \left[ \frac{c_3^{1+\delta}(1-p)^d}{p^2} \right]^{\psi(x)} .
\]

(40)
Proof.  Let $\mathcal{L}_0^d$ be the collection of subgraphs of $\mathbb{L}^d$ of finite order. Setting $\mathbb{C}_0^d := \{ G \in \mathcal{L}_0^d : \partial G = \mathbb{J}G \}$ and denoting by $\mathbb{C}_c^d$ the collection of connected elements of $\mathbb{C}_0^d$, we define, for any $k \geq 2d$, the (possibly empty) collection of lattice’s subset

$$
\mathcal{G}_k := \{ G \subset \mathbb{C} : G = G \left[ (\partial G')^* \right] , \ G' \in \mathbb{C}_c^d ; \ |V(G)| = k \} .
$$

(41)

Since

$$
\{ 0 < |C_{\{0,x\}} | < \infty \} = \bigvee_{k \geq \varphi(x)} A_k (x) ,
$$

(42)

then,

$$
P_p (A_k (x)) = (1 - p)^k \sum_{G \in \mathcal{G}_k} P_p \{ \omega \in \Omega : G \left[ S_{\{0,x\}} (\omega) \right] = G \}
$$

(43)

and we get

$$
P_p (A_k (x)) \leq |\mathcal{G}_k| (1 - p)^k .
$$

(44)

We can choose for each $G \in \mathcal{G}_k$ a minimal spanning tree $T_G$ and consider the collection of graphs

$$
T_k := \{ T_G : G \in \mathcal{G}_k \} .
$$

(45)

Since given a connected tree there is a walk passing only twice through any edge of the graph, there exists a constant $c_3 = c_3 (d) > 1$ such that $|G_k| = c_3^k$. Therefore, by (33),

$$
P_p \{ 0 < |C_{\{0,x\}} | < \infty \} = \sum_{k \geq \varphi(x)} P_p (A_k (x)) \geq p^{\psi(x)} (1 - p)^{\varphi(x)}
$$

(46)

$$
\geq \left[ p^{c_3^{\delta}} (1 - p)^{\delta} \right]^{\varphi(x)} .
$$

Therefore, $\forall p \in \left( 1 - \frac{1}{c_3}, 1 \right)$, choosing $\delta^*$ as in (39), $\forall \delta > \delta^*$, we have

$$
P_p \{ \{ \partial C_{\{0,x\}} \} \geq (1 + \delta) \varphi (x) \} \{ 0 < |C_{\{0,x\}} | < \infty \} \leq \frac{\sum_{k \geq (1+\delta)\varphi(x)} c_3^k (1 - p)^k}{[p^{c_3^{\delta}} (1 - p)]^{\varphi(x)}} \leq \frac{1}{1 - c_3 (1 - p)} \left[ \frac{c_3^{1+\delta} (1 - p)^{\gamma} \varphi(x)}{p^{c_3^{\delta}}} \right]^{\varphi(x)} .
$$

(47)

2.1 Renewal structure of connectivities

Given $t \in S^{d-1}$ we define

$$
\mathcal{H}_y^t := \{ x \in \mathbb{R}^d : \langle t, x \rangle = \langle t, y \rangle \} \quad y \in \mathbb{R}^d
$$

(48)
to be the \((d-1)\)-dimensional hyperplane in \(\mathbb{R}^d\) orthogonal to the vector \(t\) passing through a point \(y \in \mathbb{R}^d\) and the corresponding half-spaces

\[
\mathcal{H}_y^{t,-} := \{ x \in \mathbb{R}^d : \langle t, x \rangle \leq \langle t, y \rangle \}, \quad (49)
\]

\[
\mathcal{H}_y^{t,+} := \{ x \in \mathbb{R}^d : \langle t, x \rangle \geq \langle t, y \rangle \}. \quad (50)
\]

Let \(t \in \mathbb{S}^d\). Given two points \(x, y \in \mathbb{Z}^d\) such that \(\langle x, t \rangle \leq \langle y, t \rangle\), we denote by \(C^t_{\{x,y\}}\) the cluster of \(x\) and \(y\) inside the strip \(S^t_{\{x,y\}} := \mathcal{H}_x^{t,+} \cap \mathcal{H}_y^{t,-}\) provided it exists.

Let \(u\) be the first of the unit vectors in the direction of the coordinate axis \(u_1, \ldots, u_d\) such that \(\langle t, u \rangle\) is maximal

**Definition 8** Given \(t \in \mathbb{S}^{d-1}\), let \(x, y \in \mathbb{Z}^d\) such that \(\langle x, t \rangle \leq \langle y, t \rangle\), be connected in \(S^t_{\{x,y\}}\). The points \(b \in C^t_{\{x,y\}}\) such that:

1. \(\langle t, x + u \rangle \leq \langle t, b \rangle \leq \langle t, y - u \rangle\);  
2. \(C^t_{\{x,y\}} \cap S^t_{\{b-u,b+u\}} = \{ b - u, b, b + u \}\).

are said to be \(t\)-break points of \(C_{\{x,y\}}\). The collection of such points, which we remark is a totally ordered set with respect to the scalar product with \(t\), will be denoted by \(B^t(x, y)\).

**Definition 9** Given \(t \in \mathbb{S}^{d-1}\), let \(x, y \in \mathbb{Z}^d\) such that \(\langle x, t \rangle \leq \langle y, t \rangle\), be connected in \(S^t_{\{x,y\}}\). An edge \(\{b, b + u\}\) such that \(b, b + u \in B^t(x, y)\) is called \(t\)-bond of \(C_{\{x,y\}}\). The collection of such edges will be denoted by \(E^t(x, y)\), while \(B^t_e(x, y) \subset B^t(x, y)\) will denote the subcollection of \(t\)-break points \(b\) of \(C_{\{x,y\}}\) such that the edge \(\{b, b + u\} \in E^t(x, y)\).

**Notation 10** In the sequel we will omit the dependence on \(x\) in the notation of the random sets \(B^t(x, y)\), \(B^t_e(x, y)\) and \(E^t(x, y)\) if such point is taken to be the origin.

**Definition 11** Given \(t \in \mathbb{S}^{d-1}, x, y \in \mathbb{Z}^d\) such that \(\langle t, x \rangle \leq \langle t, y \rangle\) be connected. \(x, y \in \mathbb{Z}^d\) are said to be \(h_t\)-connected if

1. \(x\) and \(y\) are connected in \(S^t_{\{x,y\}}\) and \(|C^t_{\{x,y\}}| < \infty\);
2. \(x + u, y - u \in B^t(x, y)\).

Moreover, denoting by \(\{ x \overset{h_t}{\leftarrow} y \}\) the event that \(x\) and \(y\) are \(h_t\)-connected, we set

\[
h_t^{(p)} (x, y) := \mathbb{P}_p \left\{ x \overset{h_t}{\leftarrow} y \right\}. \quad (51)
\]

Notice that, by translation invariance, \(h_t^{(p)}(x,y) = h_t^{(p)}(y-x,0)\) so in the sequel we will denote it simply by \(h_t^{(p)}(y-x)\). We also define by convention \(h_t^{(p)}(0) = 1\).
Definition 12 Let \( t \in \mathbb{S}^{d-1} \) and \( x, y \in \mathbb{Z}^d \) be \( h_t \)-connected. If \( B^t (x + u, y - u) \) is empty, then \( x \) and \( y \) are said to be \( f_t \)-connected and the corresponding event is denoted by \( \{ x \leftrightarrow y \} \).

We then set

\[
 f_t^{(p)} (y - x) := \mathbb{P}_p \left\{ x \leftrightarrow y \right\} .
\]  

(52)

We define by convention \( f_t^{(p)} (0) = 0 \).

Definition 13 Given \( t \in \mathbb{S}^{d-1} \), let \( x, y \in \mathbb{Z}^d \) such that \( \langle t, x \rangle \leq \langle t, y \rangle \) be connected. Then:

1. \( x, y \) are called \( h_t \)-connected and the corresponding event is denoted by \( \{ x \leftrightarrow y \} \), if

\[
 C_{\{x,y\}} \cap S_t'_{\{y-u,y\}} = \{ y - u, y \} \quad \text{and} \quad |C_{\{x,y\}} \cap H_y^{t-}| < \infty.
\]

2. \( x, y \) are called \( f_t \)-connected and the corresponding event is denoted by \( \{ x \leftrightarrow y \} \), if they are \( h_t \)-connected and \( B^t (x, y) = \emptyset \).

Definition 14 Given \( t \in \mathbb{S}^{d-1} \), let \( x, y \in \mathbb{Z}^d \) such that \( \langle t, x \rangle \leq \langle t, y \rangle \) be connected. Then:

1. \( x, y \) are called \( h_t \)-connected and the corresponding event is denoted by \( \{ x \leftrightarrow y \} \), if:

   (a) \( C_{\{x,y\}} \cap S_t'_{\{x,x+u\}} = \{ x, x + u \} \);

   (b) \( |C_{\{x,y\}} \cap H_x^{t+}| < \infty \).

2. \( x, y \) are called \( f_t \)-connected and the corresponding event is denoted by \( \{ x \leftrightarrow y \} \), if they are \( h_t \)-connected and \( B^t (x, y) = \emptyset \).

The functions \( \tilde{h}_t^{(p)} (x, y) := \mathbb{P}_p \left\{ x \leftrightarrow y \right\} \) and \( \tilde{h}_t^{(p)} (x, y) := \mathbb{P}_p \left\{ x \leftrightarrow y \right\} \) are translation invariant.

Denoting by \( g_t^{(p)} (x, y) \), for \( t \in \mathbb{S}^{d-1} \), the probability of the event

\[
 \{ x \leftrightarrow y \} := \{ x \leftrightarrow y, |C_{\{x,y\}}| < \infty, |B^t_e (x, y)| \leq 1 \} ,
\]

(53)

which is also translation invariant, we obtain

\[
 \mathbb{P}_p \{ 0 \leftrightarrow x, |C_{\{0,x\}}| < \infty \} = g_t^{(p)} (x) + \sum_{z_1, z_2 \in \mathbb{Z}^d} \tilde{f}_t^{(p)} (z_1) h_t^{(p)} (z_2 - z_1) \tilde{f}_t^{(p)} (x - z_2) ,
\]

(54)

\[
 h_t^{(p)} (x) = \sum_{z \in \mathbb{Z}^d} f_t^{(p)} (z) h_t^{(p)} (x - z) .
\]

(55)
Proposition 15 Given $t \in S^{d-1}$, for any $p \in (0, 1)$ and $x \in \mathbb{R}^d$ such that $\langle t, x \rangle > 0$,

$$\tau_p^t (x) := - \lim_{n \to \infty} \frac{1}{n} \log h_t^{(p)} ([nx])$$

(56)

exists and is a convex and homogeneous-of-order-one function on $\mathbb{R}^d$. Moreover, for $p \in \left( 1 - \frac{1}{c_3}, 1 \right)$,

$$\tau_p^t (x) \geq \varphi (x) \log \frac{1}{c_3 (1 - p)} .$$

(57)

Proof. We proceed as in [ACC]. If $x, y \in \mathbb{Z}^d$ are $h_t$-connected and $S'_{\{x+u,y-u\}} \cap \mathbb{Z}^d \neq \emptyset$, by the definition of the $h_t$-connection, for any $z \in S'_{\{x+u,y-u\}}$ and $p \in (0, 1)$,

$$h_t^{(p)} (y - x) \geq h_t^{(p)} (z - x) h_t^{(p)} (y - z) .$$

(58)

Then, for any $x \in \mathbb{Q}^d$ and $n, m \in \mathbb{N}$, if $k \in \mathbb{N}$ is such that $kx \in \mathbb{Z}^d$, we have

$$\log h_t^{(p)} ((n + m) kx) \geq \log h_t^{(p)} (nkx) + \log h_t^{(p)} ((n + m) kx - nkx)$$

$$= \log h_t^{(p)} (nkx) + \log h_t^{(p)} (mkx) ,$$

(59)

which imply that, for any $x \in \mathbb{Q}^d$, the limit $\tau_p^t$ in (56) exists and has the property

$$\tau_p^t (\lambda x) = |\lambda| \tau_p^t (x) , \lambda \in \mathbb{Q} .$$

(60)

Moreover, given $x, y \in \mathbb{Q}^d, \lambda \in \mathbb{Q} \cap [0, 1]$ and $n \in \mathbb{N}$ such that $n\lambda x \in \mathbb{Z}^d$, $n (1 - \lambda) y \in \mathbb{Z}^d \cap S'_{\{u,n(\lambda x+(1-\lambda)y)-u\}}$, by (58),

$$\log h_t^{(p)} (n (1 - \lambda) y + n\lambda x) \geq \log h_t^{(p)} (n (1 - \lambda) y) + \log h_t^{(p)} (n\lambda x) ,$$

(61)

which, together with the previous property, implies

$$\tau_p^t (n (1 - \lambda) y + n\lambda x) \geq \tau_p^t (n (1 - \lambda) y) + \tau_p^t (n\lambda x) .$$

(62)

Therefore, $\tau_p^t$ is a convex, hence continuous, function on $\mathbb{Q}^d$ and can be extended to a convex and homogeneous-of-order-one function on $\mathbb{R}^d$. Furthermore, for $p \in \left( 1 - \frac{1}{c_3}, 1 \right)$, from the inequality

$$h_t^{(p)} ([nx]) \leq \mathbb{P}_p \{ 0 < |C_{\{0,[nx]\}}| < \infty \} \leq \frac{(c_3 (1 - p))^{\varphi ([nx])}}{1 - c_3 (1 - p)}$$

(63)

and Proposition 5 we get (57). \qed

Since for any $p \in (0, 1)$ and $d \geq 2$ (cf. [G] section 8.5) there exists $c_- = c_- (p, d) > 0$ such that

$$\mathbb{P}_p \{ 0 \leftrightarrow x , |C_{\{0,x\}}| < \infty \} \leq e^{-c_- \|x\|} ,$$

(64)
while \(46\) implies that there exists \(c_+ = c_+ (p, d) > 0\) such that
\[
h_t(p)(x) \geq e^{-c_+|x|}.
\]
(65)

it follows that \(\tau_p\) is finite and is an equivalent norm in \(\mathbb{R}^d\).

Furthermore, let
\[
\mathbb{R}^d \ni s \mapsto H_t(p)(s) := \sum_{x \in \mathbb{Z}^d} h_t(p)(x) e^{\langle s, x \rangle} \in \overline{\mathbb{R}}.
\]
(66)

(64) implies that \(\forall p \in (p_c (d), 1)\), the effective domain of \(H_t(s)\),
\[
D_t^p := \left\{ s \in \mathbb{R}^d : H_t(p)(s) < \infty \right\},
\]
(67)

is not empty since \(D_t^p \supseteq \hat{K}_t^p \supseteq 0\), where
\[
\hat{K}_t^p := \bigcap_{\hat{x} \in S^{d-1}} \left\{ s \in \mathbb{R}^d : \langle s, \hat{x} \rangle \leq \tau_t^p(\hat{x}) \right\}
\]
(68)
is the convex body polar with respect to \(U_t^p := \left\{ x \in \mathbb{R}^d : \tau_t^p(x) \leq 1 \right\}\).

\section{Renormalization}

Let
\[
\mathcal{W} := \bigcap_{\hat{x} \in S^{d-1}} \left\{ s \in \mathbb{R}^d : \langle s, \hat{x} \rangle \leq \tilde{\varphi}(\hat{x}) \right\}
\]
(69)

and, for any \(x \in \mathbb{R}^d\), let \(S_t^x := \left\{ \hat{s} \in S^{d-1} : \langle s, \hat{x} \rangle = \tilde{\varphi}(\hat{x}) \right\}\).

\textbf{Definition 16} Given \(t \in S^{d-1}\), for any \(x, y \in \mathbb{Z}^d\) such that \(\langle t, x \rangle \leq \langle t, y \rangle\), let
\[
\left\{ \mathcal{H}_x \leftrightarrow \mathcal{H}_y \right\} := \left\{ \omega \in \Omega : 0 < |C_{x,y'}| < \infty ; x' \in \mathcal{H}_x^- \cap \mathbb{Z}^d, y' \in \mathcal{H}_y^+ \cap \mathbb{Z}^d \right\}.
\]
(70)

We define
\[
\varphi_t(x, y) := \min \left\{ |S_{x,y}|(\omega) : \omega \in \left\{ \mathcal{H}_x \leftrightarrow \mathcal{H}_y \right\} \right\}
\]
(71)

and
\[
\phi_t(x, y) := \min_{\omega \in \left\{ \mathcal{H}_x \leftrightarrow \mathcal{H}_y \right\}} \left\{ e^* \in S_{x,y}(\omega) : e^* \subset S_{x,y}^t \right\}.
\]
(72)

Notice that, by translation invariance, \(\phi_t(x, y) = \phi_t(y - x, 0)\) therefore we are allowed to write \(\phi_t(x, y) = \phi_t(y - x)\). Moreover,
\[
\varphi_t(x, y) = \min_{(z', y') \in \mathcal{H}_x^- \cap \mathbb{Z}^d \times \mathcal{H}_y^+ \cap \mathbb{Z}^d} \varphi(y' - z') = \min_{y' \in \mathcal{H}_y^+ \cap \mathbb{Z}^d} \varphi(y' - x).
\]
(73)

Hence we set \(\varphi_t(x - y) := \varphi_t(x, y)\). We also remark that Lemma 4 and Proposition 5 imply the existence of \(\hat{\varphi}_t(x) := \lim_{n \to \infty} \frac{\varphi_t(nx)}{n}\) for any \(x \in \mathbb{R}^d\) such that \(\langle t, x \rangle > 0\).
Lemma 17 For any $x \in \mathbb{R}^d$ and $t \in S_{x}^{d-1}$, there exists $\lim_{n \to \infty} \frac{\phi_t([n\hat{x}])}{n} = \varphi(x)$.

Proof. For any $t \in S_{x}^{d-1}$, $\phi_t$ is superadditive and consequently $\tilde{\phi}_t(x) := \lim_{n \to \infty} \frac{\phi_t([n\hat{x}])}{n}$ exists for any $x \in \mathbb{R}^d$ such that $\langle t, x \rangle > 0$. Furthermore, $\tilde{\phi}_t$ is a homogeneous-of-order-one function.

Given $t \in S_{x}^{d-1}$ and $x \in \mathbb{Z}^d$ such that $\langle t, x \rangle > 0$, by the definition of $\phi_t$ and $\varphi_t$, it follows that $\phi_t(x) \leq \varphi_t(x)$. Moreover, by the convexity of $\varphi$, if $t$ and $x$ are chosen to be in polar relation with respect to $\varphi$, we have $\tilde{\phi}_t(x) = \varphi_t(x)$. Hence, $\forall x \in \mathbb{R}^d$ and $t \in S_{x}^{d-1}$, $\tilde{\phi}_t(x) \leq \varphi_t(x)$.

On the other hand, for any $\hat{x} \in S_{x}^{d-1}$ and $t \in S_{x}^{d-1}$, let us consider the slab $S^t_{[0,[n\hat{x}]]}$, with $n$ a large integer. Given $\delta \in \left(0, \frac{1}{2(d-1)}\right)$, $N \in \mathbb{N}$, and $K_n^N(\delta) := \left[\frac{n^{1-\delta}}{N}\right]$, we can decompose $S^t_{[0,[n\hat{x}]]}$ as follows

$$S^t_{[0,[n\hat{x}]]} = S^t_{[0,[K^N_n(\delta)N\hat{x}]]} \cup S^t_{\{[K^N_n(\delta)N\hat{x}] - [K^N_n(\delta)N\hat{x}]\} \cup S^t_{\{[n\hat{x}] - [K^N_n(\delta)N\hat{x}]\}}$$

(74)

where

$$S^t_{[0,[K^N_n(\delta)N\hat{x}]]} = \bigcup_{i=0,\ldots,K^N_n(\delta)-1} S^t_{\{iN\hat{x}] - (i+1)N\hat{x}]\}$$

(75)

$$S^t_{\{[n\hat{x}] - [K^N_n(\delta)N\hat{x}]\} = \bigcup_{i=0,\ldots,K^N_n(\delta)-1} S^t_{\{[n\hat{x}] - (i+1)N\hat{x}] - [n\hat{x}] - [iN\hat{x}]\}}$$

(76)

For any configuration in $\{\omega \in \Omega : \overline{\partial}C_{[0,[n\hat{x}]]}(\omega) = \varphi_t([n\hat{x}])\}$, since there exists a constant $c_+$ such that $\varphi_t([n\hat{x}]) \leq c_+ n$, there is at least one slab in the collection $\{S^t_{\{[iN\hat{x}] - (i+1)N\hat{x}]\}}_{i \in \{0,\ldots,K^N_n(\delta)-1\}}$, which we denote by $S^t_{\{[n\hat{x}] - (i+1)N\hat{x}]\}$, such that

$$|\{e^* \in S_{[0,[n\hat{x}]]} : e^* \subset S^t_{\{[iN\hat{x}] - (i+1)N\hat{x}]\}\} | \leq c'_4 n^{2\delta},$$

(77)

with $c'_4 = c'_4(N)$. The same argument also apply to the slabs in the collection $\{S^t_{\{[n\hat{x}] - (i+1)N\hat{x}] - [n\hat{x}] - [iN\hat{x}]\}}_{i \in \{0,\ldots,K^N_n(\delta)-1\}}$, hence there exists $l \in \{0,\ldots,K^N_n(\delta)-1\}$ such that

$$|\{e^* \in S_{[0,[n\hat{x}]]} : e^* \subset S^t_{\{[n\hat{x}] - (l+1)N\hat{x}] - [n\hat{x}] - [lN\hat{x}]\}\} | \leq c'_4 n^{2\delta}.$$  

(78)

Let $C^t_{j,l}(\omega)$ be a connected component of $C_{[0,[n\hat{x}]]}(\omega) \cap S^t_{\{[n\hat{x}] - [lN\hat{x}]\}$ connecting $H^t_{[jN\hat{x}] - [lN\hat{x}]$, with $H^t_{[n\hat{x}] - [lN\hat{x}]$. Since the sum of the diameters of the components of the subgraph of $\mathfrak{G}$ induced by $\{e^* \in S_{[0,[n\hat{x}]]} : e^* \cap H^t_{[lN\hat{x}] > 0}\}$, with $y = [jN\hat{x}]$, $[n\hat{x}] - [lN\hat{x}]$, is smaller than $c'n^{2\delta}$, there exists $c'_4 = c'_4(N)$ such that

$$\left|\left\{e^* \in (\overline{\partial}C^t_{j,l}(\omega))^* : e^* \cap H^t_{[jN\hat{x}] > 0}\right\}\right| \leq c'_4 n^{2\delta(d-1)},$$

(79)

$$\left|\left\{e^* \in (\overline{\partial}C^t_{j,l}(\omega))^* : e^* \cap H^t_{[n\hat{x}] - [lN\hat{x}] > 0}\right\}\right| \leq c'_4 n^{2\delta(d-1)}.$$  

(80)

Therefore,

$$\varphi_t([n\hat{x}] - 2 [K^N_n(\delta)N\hat{x}] \leq \phi_t([n\hat{x}]) + 2c''n^{2\delta(d-1)}.$$  

(81)

Dividing by $n - 2n^{1-\delta}$ and taking the limit for $n \to \infty$, we obtain $\bar{\varphi}(x) \leq \bar{\phi}_t(x)$. ■
3.1 Proof of Theorem 1

For any \( x \in \mathbb{Z}^d \) and \( p \in \left( 1 - \frac{1}{c_3}, 1 \right) \), by Proposition 7 we are left with the estimate of the probability that there exists a finite clusters \( C_{(0,x)} \) such that \( \|S C_{(0,x)}\| = \|S_{(0,x)}\| \leq (1 + \delta) \varphi (x) \), for \( \delta \) larger than the value \( \delta^* \) given in (39).

Let \( t \in \mathbb{S}^{d-1} \). Moreover, given \( N \in \mathbb{N} \) larger than 1, let us set \( K^N (x) := \left\lceil \frac{\|x\|}{N} \right\rceil - 1 \) and

\[
\mathcal{H}^t_i := \mathcal{H}^t_{i|N\mathbb{Z}} ; \mathcal{H}^t_i^\pm := \mathcal{H}^t_{i|N\mathbb{Z}}^\pm , i = 0, \ldots, K^N (x) ;
\]

\[
\mathcal{H}^t_{(K^N(x)+1)|N\mathbb{Z}} := \mathcal{H}^t_x ; \quad \mathcal{H}^t_{i}^\pm := \mathcal{H}^t_{(K^N(x)+1)|N\mathbb{Z}}^\pm ,
\]

\[
\mathcal{S}^t_i := \mathcal{H}^t_i^+ \cap \mathcal{H}^t_{i+1}^- .
\]

With a slight notational abuse we still denote by \( S_{(0,x)} \) its representation as a hypersurface in \( \mathbb{R}^d \) and define

\[
C^t_i := C_{(0,x)} \cap \mathcal{S}^t_i ; \quad S^t_i := S_{(0,x)} \cap \mathcal{S}^t_i .
\]

Hence, \( C_{(0,x)} = \bigcup_{i=0}^{\left\lfloor \frac{\|x\|}{N} \right\rfloor - 1} C^t_i \) and \( S_{(0,x)} \cap \mathcal{S}_{(0,x)} \subseteq \bigcup_{i=0}^{\left\lfloor \frac{\|x\|}{N} \right\rfloor - 1} S^t_i \).

We say that a slab \( S^t_i \) is bad if \( S^t_i \) is not connected, otherwise we call it good, and call crossing any connected component \( s \) of \( S^t_i \) such that, denoting by \( T(s) \) the compact subset of \( S^t_i \) whose boundary is \( s \), there is a path in \( \mathbb{L}^d \cap T(s) \) connecting \( \mathcal{H}^t_i \) with \( \mathcal{H}^t_{i+1} \).

We remark that since \( C_{(0,x)} \) is connected, the existence of two crossings in \( S^t_i \) implies the existence of two disjoint paths connecting \( \mathcal{H}^t_i \) and \( \mathcal{H}^t_{i+1} \) while the converse does not hold true in general.

Let \( \eta \) be the fraction of slabs where there are at least two crossings. Since any crossing is composed by at least \( \phi (\lfloor N \mathbb{x} \rfloor) \) plaquettes, we have

\[
\frac{\|x\|}{N} (\eta 2 \phi (\lfloor N \mathbb{x} \rfloor) + (1 - \eta) \phi (\lfloor N \mathbb{x} \rfloor)) = \frac{\|x\|}{N} (1 + \eta) \phi (\lfloor N \mathbb{x} \rfloor) < (1 + \delta) \varphi (x) .
\]

Moreover, given \( \varepsilon > 0 \), there exists \( R_\varepsilon > 0 \) such that, for any \( x \in \mathbb{Z}^d \cap (R_\varepsilon B)^c \), \( \varphi (x) \leq \varphi (x) (1 + \varepsilon) \). Hence, choosing \( N \) sufficiently large such that \( \phi (\lfloor N \mathbb{x} \rfloor) \leq \phi (\lfloor N \mathbb{x} \rfloor) (1 + \varepsilon) \), since \( t \in \mathbb{S}^{d-1} \), \( \phi (N \mathbb{x}) = \phi (N \mathbb{x}) \) and, by the previous inequality, we get \( \eta < \delta \).

Then, there are at most \( \frac{\|x\|}{N} - \eta \frac{\|x\|}{N} \) 3-tuple of consecutive slabs containing a single crossing and therefore at most the same number of bad slabs containing a single crossing. Choosing, \( \eta \leq \frac{1}{6} \), it is possible to modify the configuration of at most \( c_5 N^d \) bonds, with \( c_5 = c_5 (d, \delta) \), inside any 3-tuple of consecutive slabs containing a single crossing in such a way that the resulting cluster will have at least one \( t \)-bond inside each of these slabs. Since these modifications can be performed independently, this fact and the previous proposition imply that there exists a positive constant \( c_6 = c_6 (p) \) such that

\[
\mathbb{P}_p \left\{ 0 < |C_{(0,x)}| < \infty \right\} \leq e^{-\|x\|^c_6}
\]
and consequently the mass-gap condition $f_t^{(p)}(x) \leq e^{-c_7\|x\|} h_t^{(p)}(x)$, $c_7 = c_7(p) > 0$, uniformly in $t \in S^{d-1}_{x}$.

Thus, from (54) we have

$$\mathbb{P}_p \left\{ 0 \leftrightarrow x , \left| C_{\{0,x\}} \right| < \infty \right\} \leq c_8 \, ,$$

(88)

with $c_8 = c_8(d) > 0$, for any $t \in S^{d-1}_{x}$.

We now proceed as in Section 4.3 of [CI]. Given $t \in S^{d-1}_{x}$, we extend $f_t^{(p)}$ to a function defined on the whole lattice by setting it equal to zero where it is non defined and set

$$\mathbb{R}^d \ni s \mapsto -\overrightarrow{F}_t^{(p)}(s) := \sum_{x \in \mathbb{Z}^d} f_t^{(p)}(x) e^{\langle s,x \rangle} \in \mathbb{R} \, .$$

(89)

The renewal equation (55) imply

$$H_t^{(p)}(s) = \frac{1}{1 - F_t^{(p)}(s)} \, .$$

(90)

For $s \in K^p_t$, since

$$(s,x) \leq \max_{s \in K^p_t} (s,x) = \tau_t^{(p)}(x) \leq 1$$

and $h_t^{(p)}(x) \leq e^{-\tau_t^{(p)}(x)}$, $F_t^{(p)}(s)$ is finite, moreover it is continuous, then, $\forall s \in \partial K^p_t$,

$$\mathbb{Z}^d \ni x \mapsto q_t^{(p)}(x) := f_t^{(p)}(x) e^{\langle s,x \rangle} \in \mathbb{R} \, .$$

(92)

is the density of the probability measure $Q^{(p)}_{t,s}$ on $(\mathbb{Z}^d, \mathcal{B}(\mathbb{Z}^d))$ which, has exponentially decaying tails:

$$f_t^{(p)}(x) e^{\langle s,x \rangle} \leq e^{-c_7\|x\|} h_t^{(p)}(x) e^{\langle s,x \rangle} \leq e^{-c_7\|x\|} \, .$$

(93)

If $X$ is a random vector with probability distribution $Q^{(p)}_{t,s}$, denoting by $\mathbb{E}^{t_{s}}_p$ the expectation of a random variable under $Q^{(p)}_{t,s}$, we set

$$\mu_t^{(p)}(s) := \mathbb{E}^{t_{s}}_p [X] = \text{grad log } F_t^{(p)}(s) \, ,$$

(94)

while

$$C_t^{(p)}(s) := \text{Hess log } F_t^{(p)}(s)$$

(95)

denotes the covariance matrix of $X$. Since $f_t^{(p)}(x) > 0$ on a whole half-space, $C_t^{(p)}(s)$ is non degenerate. Hence,

$$\partial K^p_t = \left\{ s \in \mathbb{R}^d : F_t^{(p)}(s) = 1 \right\} \subseteq \mathbb{R}^d \setminus D_t^{p}$$

(96)
Hence, there exist two positive constant $c_9 = c_9(p)$ and $c_{10} = c_{10}(p)$ such that
\[
\left\| n\mu - \sum_{i=1}^k \mathbb{E}^{s \nu}_{p} [X_i] \right\| \geq n c_9 \| \mu - \mu^p_t (s) \| + |n - k| c_{10} \| \mu^p_t (s) \| .
\]
(98)
Therefore, the standard large deviation upper bound for
\[
\prod_{i=1}^k Q^{(p)}_{t,s} \left\{ \sum_{i=1}^k X_i = \lfloor n\mu \rfloor \right\} \leq e^{-c_1 \frac{|n-k|}{\kappa} \cdot \| n\mu - k\mu^p_t (s) \|^2 - c_2 \cdot \| n\mu - k\mu^p_t (s) \|^2}
\]
(99)
leads to
\[
h^{(p)}_t (\lfloor n\mu \rfloor) \leq c_{13} \sqrt{n} e^{-\left[ \frac{(\| n\mu \| + c_{14} n)}{\kappa} \| \mu - \mu^p_t (s) \|^2 \right]}
\]
(100)
with $c_{13} = c_{13}(p), c_{14} = c_{14}(p)$ positive constants. Then, (96) and (100) imply
\[
\tau^t_p (\mu) \geq \langle \mu, s \rangle + c_{14} \| \mu - \mu^p_t (s) \|^2 ,
\]
(101)
that is the strict convexity of $\tau^t_p$. Moreover, because $\tau^t_p$, being an equivalent norm in $\mathbb{R}^d$, is lower semicontinuous, from (101) it follows that $\mu^p_t (s)$ and $s$ are in polar relation with respect to $\tau^t_p$, namely $\langle s, \mu^p_t (s) \rangle = \tau^t_p (\mu^p_t (s))$. Furthermore, for any $x \in \mathbb{Z}^d$, let $t \in S_{x}^{d-1}$. There exist $r' > 0, s \in B_{r'} (t) \cap \partial \mathcal{K}^p_t$, and $n_x \in \mathbb{N}$ such that
\[
\| x - n_x \mu^p_t (s) \| \leq c_{15} ,
\]
(102)
where $c_{15} = c_{15}(p) > 0$. Hence, choosing $\alpha \in \left( 0, \frac{1}{2} \right)$, since $Q^{(p)}_{t,s}$ is a centered lattice distrib-
tion, we can apply the local central limit theorem for \( \bigotimes_{i=1}^{k} Q_{t,s}^{(p)} \left\{ \sum_{i=1}^{k} X_i = x \right\} \) to get,

\[
\sum_{k \in \mathbb{N} : |k-n_x| < n^{\frac{1}{2}+\alpha}} \bigotimes_{i=1}^{k} Q_{t,s}^{(p)} \left\{ \sum_{i=1}^{k} X_i = x \right\} = \sum_{k \in \mathbb{N} : |k-n_x| < n^{\frac{1}{2}+\alpha}} \exp \left\{ -\frac{(n_x-k)^2}{2k} \left( \left( C_t^p \right)^{-1}(s) \mu_t^p(s), \mu_t^p(s) \right) \right\} \sqrt{2\pi k \det C_t^p(s)} (1 + o(1))
\]

\[
= \sum_{k \in \mathbb{N} : |k-n_x| < n^{\frac{1}{2}+\alpha}} \exp \left\{ -\frac{(n_x-k)^2}{2n_x \left( 1 + O \left( \frac{n_x^{-\frac{1}{2}}}{n_x} \right) \right)} \left( \left( C_t^p \right)^{-1}(s) \mu_t^p(s), \mu_t^p(s) \right) \right\} \sqrt{2\pi n_x \left( 1 + O \left( \frac{n_x^{-\frac{1}{2}}}{n_x} \right) \right)} \det C_t^p(s) (1 + o(1))
\]

\[
= \frac{1}{\sqrt{(2\pi n_x)^{d-1} \det C_t^p(s) \left( \left( C_t^p \right)^{-1}(s) \mu_t^p(s), \mu_t^p(s) \right)}} (1 + o(1)) .
\]

On the other hand, in the complementary range of \( k \)'s, proceeding as in (100) we obtain

\[
\sum_{k \in \mathbb{N} : |k-n_x| \geq n^{\frac{1}{2}+\alpha}} \bigotimes_{i=1}^{k} Q_{t,s}^{(p)} \left\{ \sum_{i=1}^{k} X_i = x \right\} \leq e^{-c_1 n_x^{2\alpha}}
\]

which, by (102), (97) and (88), since \( s = s(\hat{x}) \), gives back

\[
\mathbb{P}_p \left\{ 0 \leftrightarrow x , \ |C_{\{0,x\}}| < \infty \right\} = \frac{\Phi_p(\hat{x})}{\sqrt{(2\pi \|x\|)^{d-1}}} e^{-\tau^s_p(x)} (1 + o(1)) ,
\]

where

\[
S^{d-1} \ni \hat{x} \mapsto \Phi_p(\hat{x}) := \frac{\|\mu_t^p(s(\hat{x}))\|^{d-1}}{\det C_t^p(s(\hat{x})) \left( \left( C_t^p \right)^{-1}(s(\hat{x})) \mu_t^p(s(\hat{x})), \mu_t^p(s(\hat{x})) \right)} \in \mathbb{R}^+ .
\]

is a real analytic function on \( S^{d-1} \).

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