Most numbers are not normal

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Abstract

We show, from a topological viewpoint, that most numbers are not normal in a strong sense. More precisely, the set of numbers \( x \in (0, 1] \) with the following property is comeager: for all integers \( b \geq 2 \) and \( k \geq 1 \), the sequence of vectors made by the frequencies of all possible strings of length \( k \) in the \( b \)-adic representation of \( x \) has a maximal subset of accumulation points, and each of them is the limit of a subsequence with an index set of nonzero asymptotic density. This extends and provides a streamlined proof of the main result given by Olsen (2004) in this Journal. We provide analogues in the context of analytic P-ideals and regular matrices.

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1. Introduction

A real number \( x \in (0, 1] \) is normal if, informally, for each base \( b \geq 2 \), its \( b \)-adic expansion contains every finite string with the expected uniform limit frequency (the precise definition is given in the next few lines). It is well known that most numbers \( x \) are normal from a
measure theoretic viewpoint, see e.g. [5] for history and generalisations. However, it has been recently shown that certain subsets of nonnormal numbers may have full Hausdorff dimension, see e.g. [1, 4]. The aim of this work is to show that, from a topological viewpoint, most numbers are not normal in a strong sense. This provides another nonanalogue between measure and category, cf. [25].

For each \( x \in (0, 1] \), denote its unique nonterminating \( b \)-adic expansion by

\[
x = \sum_{n \geq 1} \frac{d_{b,n}(x)}{b^n},
\]

with each digit \( d_{b,n}(x) \in \{0, 1, \ldots, b - 1\} \), where \( b \geq 2 \) is a given integer. Then, for each string \( s = s_1 \cdots s_k \) with digits \( s_j \in \{0, 1, \ldots, b - 1\} \) and each \( n \geq 1 \), write \( \pi_{b,s,n}(x) \) for the proportion of strings \( s \) in the \( b \)-adic expansion of \( x \) which start at some position \( \leq n \), i.e.,

\[
\pi_{b,s,n}(x) := \frac{\# \{ i \in \{1, \ldots, n\} : d_{b,i+j-1}(x) = s_j \text{ for all } j = 1, \ldots, k \} }{n}.
\]

In addition, let \( S_b^k \) be the set of all possible strings \( s = s_1 \cdots s_k \) in base \( b \) of length \( k \), hence \( \# S_b^k = b^k \), and denote by \( \pi_{b,n}(x) \) the vector \( (\pi_{b,s,n}(x) : s \in S_b^k) \). Of course, \( \pi_{b,n}(x) \) belongs to the \((b^k - 1)\)-dimensional simplex for each \( n \). However, the components of \( \pi_{b,n}(x) \) satisfy an additional requirement: if \( k \geq 2 \) and \( s = s_1 \cdots s_{k-1} \) is a string in \( S_b^{k-1} \), then

\[
\pi_{b,s,n}(x) = \sum_{s_{k}} \pi_{b,ss_{k},n}(x) = \sum_{s_{0}} \pi_{b,s_{0}ss_{k},n}(x) + O(1/n) \quad \text{as } n \to \infty,
\]

where \( ss_{k} \) and \( ss_{k} \) stand for the concatenated strings (indeed, the above identity is obtained by a double counting of the occurrences of the string \( s \) as the occurrences of all possible strings \( ss_{k} \)); or, equivalently, as the occurrences of all possible strings \( s_{0}ss_{k} \), with the caveat of counting them correctly at the two extreme positions, hence with an error of at most 1). It follows that the set \( L_b^k(x) \) of accumulation points of the sequence of vectors \( (\pi_{b,n}(x) : n \geq 1) \) is contained in \( \Delta_b^k \), where

\[
\Delta_b^k := \left\{ (p_s)_{s \in S_b^k} \in \mathbb{R}^{b^k} : \sum_s p_s = 1, p_s \geq 0 \text{ for all } s \in S_b^k, \right. \]

\[
\quad \text{and } \sum_{s_{0}} p_{s_{0}ss_{k}} = \sum_{s_{k}} p_{ss_{k}} \text{ for all } s \in S_b^{k-1} \left. \right\}.
\]

Then \( x \) is said to be normal if

\[
\forall b \geq 2, \forall k \geq 1, \forall s \in S_b^k, \lim_{n \to \infty} \pi_{b,s,n}(x) = 1/b^k.
\]

Hence, if \( x \) is normal, then \( L_b^k(x) = \{(1/b^k, \ldots, 1/b^k)\} \). Olsen proved in [23] that the subset of nonnormal numbers with maximal set of accumulation points is topologically large:

**Theorem 1.1.** The set \( \{ x \in (0, 1] : L_b^k(x) = \Delta_b^k \text{ for all } b \geq 2, k \geq 1 \} \) is comeager.

First, we strenghten Theorem 1.1 by showing that the set of accumulation points \( L_b^k(x) \) can be replaced by the much smaller subset of accumulation points \( \eta \) such that every neighbourhood of \( \eta \) contains “sufficiently many” elements of the sequence, where “sufficiently many” is meant with respect to a suitable ideal \( \mathcal{I} \) of subsets of the positive integers \( \mathbb{N} \); see Theorem 2.1. Hence, Theorem 1.1 corresponds to the case where \( \mathcal{I} \) is the family of finite sets.
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Then, for certain ideals \( I \) (including the case of the family of asymptotic density zero sets), we even strengthen the latter result by showing that each accumulation point \( \eta \) can be chosen to be the limit of a subsequence with “sufficiently many” indexes (as we will see in the next Section, these additional requirements are not equivalent); see Theorem 2.3. The precise definitions, together with the main results, follow in Section 2.

2. Main results

An ideal \( I \subseteq \mathcal{P}(\mathbb{N}) \) is a family closed under finite union and subsets. It is also assumed that \( I \) contains the family of finite sets \( \text{Fin} \) and it is different from \( \mathcal{P}(\mathbb{N}) \). Every subset of \( \mathcal{P}(\mathbb{N}) \) is endowed with the relative Cantor-space topology. In particular, we may speak about \( G_\delta \)-subsets of \( \mathcal{P}(\mathbb{N}) \), \( F_\sigma \)-ideals, meager ideals, analytic ideals, etc. In addition, we say that \( I \) is a P-ideal if it is \( \sigma \)-directed modulo finite sets, i.e., for each sequence \( (S_n) \) of sets in \( I \) there exists \( S \in I \) such that \( S_n \setminus S \) is finite for all \( n \in \mathbb{N} \). Lastly, we denote by \( Z \) the ideal of asymptotic density zero sets, i.e.,

\[
Z = \{ S \subseteq \mathbb{N} : d^*(S) = 0 \}
\]

where \( d^*(S) := \limsup_n \frac{1}{n} \#(S \cap [1, n]) \) stands for the upper asymptotic density of \( S \), see e.g. [20]. We refer to [14] for a recent survey on ideals and associated filters.

Let \( x = (x_n) \) be a sequence taking values in a topological vector space \( X \). Then we say that \( \eta \in X \) is an \( I \)-cluster point of \( x \) if \( \{ n \in \mathbb{N} : x_n \notin U \} \notin I \) for all open neighbourhoods \( U \) of \( \eta \). Note that \( \text{Fin} \)-cluster points are the ordinary accumulation points. Usually \( Z \)-cluster points are referred to as statistical cluster points, see e.g. [13]. It is worth noting that \( I \)-cluster points have been studied much before under a different name. Indeed, as it follows by [19, theorem 4.2] and [16, lemma 2.2], they correspond to classical “cluster points” of a filter (depending on \( x \)) on the underlying space, cf. [7, definition 2, p.69].

With these premises, for each \( x \in (0, 1] \) and for all integers \( b \geq 2 \) and \( k \geq 1 \), let \( \Gamma_b^k(x, I) \) be the set of \( I \)-cluster points of the sequence \( (\pi_{b,n}^k(x) : n \geq 1) \).

**Theorem 2.1.** The set \( \{ x \in (0, 1] : \exists b \geq 2, k \geq 1, \Gamma_b^k(x, I) = \Delta_b^k \} \) is comeager, provided that \( I \) is a meager ideal.

The class of meager ideals is really broad. Indeed, it contains \( \text{Fin}, Z \), the summable ideal \( \{ S \subseteq \mathbb{N} : \sum_{n \in S} 1/n < \infty \} \), the ideal generated by the upper Banach density, the analytic P-ideals, the Fubini sum \( \text{Fin} \times \text{Fin} \), the random graph ideal, etc.; cf. e.g. [3, 14]. Note that \( \Gamma_b^k(x, I) = L_b^k(x) \) if \( I = \text{Fin} \). Therefore Theorem 2.1 significantly strengthens Theorem 1.1.

**Remark 2.2.** It is not difficult to see that Theorem 2.1 does not hold without any restriction on \( I \). Indeed, if \( I \) is a maximal ideal (i.e., the complement of a free ultrafilter on \( \mathbb{N} \)), then for each \( x \in (0, 1] \) and all integers \( b \geq 2, k \geq 1 \), we have that the sequence \( (\pi_{b,n}^k(x) : n \geq 1) \) is bounded, hence it is \( I \)-convergent so that \( \Gamma_b^k(x, I) \) is a singleton.

On a similar direction, if \( x = (x_n) \) is a sequence taking values in a topological vector space \( X \), then \( \eta \in X \) is an \( I \)-limit point of \( x \) if there exists a subsequence \( (x_{n_k}) \) such that \( \lim_k x_{n_k} = \eta \) and \( \mathbb{N} \setminus \{ n_1, n_2, \ldots \} \in I \). Usually \( Z \)-limit points are referred to as statistical limit points, see e.g. [13]. Similarly, for each \( x \in (0, 1] \) and for all integers \( b \geq 2 \) and \( k \geq 1 \), let \( \Lambda_b^k(x, I) \) be the set of \( I \)-limit points of the sequence \( (\pi_{b,n}^k(x) : n \geq 1) \). The analogue of Theorem 2.1 for \( I \)-limit points follows.
We claim there exists a nonempty open set nowhere dense. To this aim, fix \( t \subset (0, 1] \) that is nonempty and open in \((0, 1]\). There exists a string \( G \) of \( \sigma \)-ideals including, in particular, the ideal \( \mathcal{I} \) generated by the upper Banach density (which is known to not be a \( P \)-ideal, see e.g. [12, p.299]).

It is remarkable that there exist \( F_{\sigma} \)-ideals which are not \( P \)-ideals, see e.g. [11, section 1.11]. Also, the family of analytic \( P \)-ideals is well understood and has been characterised with the aid of lower semicontinuous submeasures, cf. Section 3. The results in [6] suggest that the study of the interplay between the theory of analytic \( P \)-ideals and their representability may have some relevant yet unexploited potential for the study of the geometry of Banach spaces.

Finally, recalling that the ideal \( \mathcal{I} \) defined in (2) is an analytic \( P \)-ideal, an immediate consequence of Theorem 2.3 (as pointed out in the abstract) follows:

**Corollary 2.4.** The set of \( x \in (0, 1] \) such that, for all \( b \geq 2 \) and \( k \geq 1 \), every vector in \( \Delta^k_b \) is a statistical limit point of the sequence \( (\pi_{b,n}(x) : n \geq 1) \) is comeager.

It would also be interesting to investigate to what extend the same results for nonnormal points belonging to self-similar fractals (as studied, e.g., by Olsen and West in [24] in the context of iterated function systems) are valid.

We leave as open question for the interested reader to check whether Theorem 2.3 can be extended for all \( F_{\sigma} \)-ideals including, in particular, the ideal \( \mathcal{I} \) generated by the upper Banach density (which is known to not be a \( P \)-ideal, see e.g. [12, p.299]).

3. Proofs of the main results

**Proof of Theorem 2.1.** Let \( \mathcal{I} \) be a meager ideal on \( \mathbb{N} \). It follows by Talagrand’s characterisation of meager ideals [28, theorem 21] that it is possible to define a partition \( \{I_1, I_2, \ldots\} \) of \( \mathbb{N} \) into nonempty finite subsets such that \( S \notin \mathcal{I} \) whenever \( I_n \subset S \) for infinitely many \( n \). Moreover, we can assume without loss of generality that \( \max I_n < \min I_{n+1} \) for all \( n \in \mathbb{N} \).

The claimed set can be rewritten as \( \bigcap_{b \geq 2} \bigcap_{k \geq 1} X^k_b \), where \( X^k_b := \{ x \in (0, 1] : \Gamma^k_b(x, \mathcal{I}) = \Delta^k_b \} \). Since the family of meager subsets of \((0,1]\) is a \( \sigma \)-ideal, it is enough to show that the complement of each \( X^k_b \) is meager. To this aim, fix \( b \geq 2 \) and \( k \geq 1 \) and denote by \( \| \cdot \| \) the Euclidean norm on \( \mathbb{R}^{2^b} \). Considering that \( \{\eta_1, \eta_2, \ldots\} := \Delta^k_b \cap \mathbb{Q}^{2^b} \) is a countable dense subset of \( \Delta^k_b \) and that \( \Gamma^k_b(x, \mathcal{I}) \) is a closed subset of \( \Delta^k_b \) by [19, lemma 3.1(iv)], it follows that

\[
(0, 1) \setminus X^k_b = \bigcup_{t \geq 1} \{ x \in (0, 1] : \eta_t \notin \Gamma^k_b(x, \mathcal{I}) \} = \bigcup_{t \geq 1} \{ x \in (0, 1] : \exists \varepsilon > 0, \{ n \in \mathbb{N} : \| \pi^k_{b,n}(x) - \eta_t \| < \varepsilon \} \in \mathcal{I} \} \leq \bigcup_{t,p,m \geq 1} \{ x \in (0, 1] : \forall q \geq p, \exists n \in I_q, \| \pi^k_{b,n}(x) - \eta_t \| \geq 1/m \}.
\]

Denote by \( S_{t,p,m} \) the set in the latter union. Thus it is sufficient to show that each \( S_{t,p,m} \) is nowhere dense. To this aim, fix \( t, p, m \in \mathbb{N} \) and a nonempty relatively open set \( G \subset (0, 1] \). We claim there exists a nonempty open set \( U \) contained in \( G \) and disjoint from \( S_{t,p,m} \). Since \( G \) is nonempty and open in \((0, 1]\), there exists a string \( \tilde{s} = s_1 \cdots s_j \in S^j_b \) such that \( x \in G \) whenever \( d_{b,i}(x) = s_i \) for all \( i = 1, \ldots, j \). Now, pick \( x^* \in (0, 1] \) such that \( \lim_n \pi^k_{b,n}(x^*) = \eta_t \),
which exists by [22, theorem 1]. In addition, we can assume without loss of generality that \( d_{b,i}(x^*) = s_i \) for all \( i = 1, \ldots, j \). Since \( \pi_{b,n}^k(x^*) \) is convergent to \( \eta_i \), there exists \( g \geq p + j \) such that \( \| \pi_{b,n}^k(x^*) - \eta_i \| < 1/m \) for all \( n \geq \min I_q \). Define \( V := \{ x \in (0, 1) : d_{b,i}(x) = d_{b,i}(x^*) \text{ for all } i = 1, \ldots, \max I_q + k \} \) and note that \( V \subseteq G \) because \( d_{b,i}(x) = s_i \) for all \( i \leq j \) and \( x \in V \), and \( V \cap S_{t,m,p} = \emptyset \) because, for each \( x \in V \), the required property is not satisfied for this choice of \( g \) since \( \pi_{b,n}^k(x) = \pi_{b,n}^k(x^*) \) for all \( n \leq \max I_q \). Clearly, \( V \) has nonempty interior, hence it is possible to choose such \( U \subseteq V \).

This proves that each \( S_{t,m,p} \) is nowhere dense, concluding the proof.

Before we proceed to the proof of Theorem 2.3, we need to recall the classical Solecki’s characterisation of analytic P-ideals. A lower semicontinuous submeasure (in short, lscsm) is a monotone subadditive function \( \varphi : \mathcal{P}(N) \to [0, \infty] \) such that \( \varphi(\emptyset) = 0 \), \( \varphi([n]) < \infty \), and \( \varphi(A) = \lim_m \varphi(A \cap [1, m]) \) for all \( A \subseteq N \) and \( n \in N \). It follows by [26, theorem 3.1] that an ideal \( \mathcal{I} \) is an analytic P-ideal if and only if there exists a lscsm \( \varphi \) such that

\[
\mathcal{I} = \{ A \subseteq N : \| A \|_\varphi = 0 \}, \quad \| N \|_\varphi = 1, \quad \varphi(N) < \infty.
\]  

(3)

Here, \( \| A \|_\varphi := \lim_n \varphi(A \setminus [1, n]) \) for all \( A \subseteq N \). Note that \( \| A \|_\varphi = \| B \|_\varphi \) whenever the symmetric difference \( A \Delta B \) is finite, cf. [11, lemma 1.3.3(b)]. Easy examples of lscsms are \( \varphi(A) := \#A \) or \( \varphi(A) := \sup_n (1/n) \#(A \cap [1, n]) \) for all \( A \subseteq N \) which lead, respectively, to the ideals \( \text{Fin} \) and \( \mathcal{Z} \) through the representation (3).

**Proof of Theorem 2.3.** First, let us suppose that \( \mathcal{I} \) is an \( F_\sigma \)-ideal. We obtain by [2, theorem 2.3] that \( \Lambda^k_b(x, \mathcal{I}) = \Gamma^k_b(x, \mathcal{I}) \) for each \( b \geq 2 \), \( k \geq 1 \), and \( x \in (0, 1) \). Therefore the claim follows by Theorem 2.1.

Then, we assume hereafter that \( \mathcal{I} \) is an analytic P-ideal generated by a lscsm \( \varphi \) as in (3). Fix integers \( b \geq 2 \) and \( k \geq 1 \), and define the function

\[
u : (0, 1] \times \Delta^k_b \longrightarrow \mathbb{R} : (x, \eta) \longmapsto \lim_{t \to \infty} \| \{ n \in N : \| \pi_{b,n}^k(x) - \eta \| \leq 1/t \} \|_\varphi,
\]

where \( \| \cdot \| \) stands for the Euclidean norm on \( \mathbb{R}^{b^k} \). It follows by [2, lemma 2.1] that every section \( \nu(x, \cdot) \) is upper semicontinuous, so that the set

\[
\Lambda^k_b(x, \mathcal{I}, q) := \{ \eta \in \Delta^k_b : \nu(x, \eta) \geq q \}
\]

is closed for each \( x \in (0, 1] \) and \( q \in \mathbb{R} \).

At this point, we prove that, for each \( \eta \in \Delta^k_b \), the set \( X(\eta) := \{ x \in (0, 1] : \nu(x, \eta) \geq 1/2 \} \) is comeager. To this aim, fix \( \eta \in \Delta^k_b \) and notice that

\[
(0, 1] \setminus X(\eta) = \bigcup_{t \geq 1} \{ x \in (0, 1) : \| \{ n \in N : \| \pi_{b,n}^k(x) - \eta \| \leq 1/t \} \|_\varphi < 1/2 \}
\]

\[
= \bigcup_{t \geq 1} \{ x \in (0, 1) : \lim_{h \to \infty} \varphi(\{ n \geq h : \| \pi_{b,n}^k(x) - \eta \| \leq 1/t \}) < 1/2 \}
\]

\[
= \bigcup_{t,h \geq 1} \{ x \in (0, 1) : \varphi(\{ n \geq h : \| \pi_{b,n}^k(x) - \eta \| \leq 1/t \}) < 1/2 \}.
\]

Denoting by \( Y_{t,h} \) the inner set above, it is sufficient to show that each \( Y_{t,h} \) is nowhere dense. Hence, fix \( G \subseteq (0, 1], s \in S^b_p \), and \( x^* \in (0, 1] \) as in the proof of Theorem 2.1. Considering that \( \| \cdot \|_\varphi \) is invariant under finite sets, it follows that

\[
\varphi(\{ n \geq j' : \| \pi_{b,n}^k(x^*) - \eta \| \leq 1/t \}) \geq \| \{ n \geq j' : \| \pi_{b,n}^k(x^*) - \eta \| \leq 1/t \} \|_\varphi = \nu(x^*, \eta) = 1,
\]
where \( j' := j + h \). Since \( \varphi \) is lower semicontinuous, there exists an integer \( j'' > j' \) such that

\[
\varphi(\{ n \in [j', j''] : \| \pi_{b,n}^k(x^*) - \eta \| \leq 1/t \}) \geq 1/2.
\]

Define \( V := \{ x \in (0, 1] : d_{b,i}(x) = d_{b,i}(x^*) \text{ for all } i = 1, \ldots, j'' \} \). Similarly, note that \( V \subseteq G \) because \( d_{b,i}(x) = s_i \) for all \( i \leq j \) and \( x \in V \), and \( V \cap Y_{t,b} = \emptyset \) because \( \varphi(\{ n \geq h : \| \pi_{b,n}^k(x) - \eta \| \leq 1/t \}) \) is at least \( \varphi(\{ n \in [j', j''] : \| \pi_{b,n}^k(x) - \eta \| \leq 1/t \}) \geq 1/2 \) for all \( x \in V \). Since \( V \) has nonempty interior, it is possible to choose \( U \subseteq V \) with the required property.

Finally, let \( E \) be a countable dense subset of \( \Delta_b^k \). Considering that \( X := \{ x \in (0, 1] : E \subseteq \Lambda_b^k(x, I, 1/2) \} \) is equal to \( \bigcap_{\eta \in E} X(\eta) \), it follows that the set \( X \) is comeager. However, considering that

\[
\Lambda_b^k(x, I) = \bigcup_{q > 0} \Lambda_b^k(x, I, q)
\]

by [2, theorem 2.2] and that \( \Lambda_b^k(x, I, 1/2) \) is a closed subset such that \( E \subseteq \Lambda_b^k(x, I, 1/2) \subseteq \Lambda_b^k(x, I) \subseteq \Delta_b^k \) for all \( x \in X \), we obtain that \( \Lambda_b^k(x, I, 1/2) = \Delta_b^k(x, I) = \Delta_b^k \) for all \( x \in X \). In particular, the claimed set contains \( X \), which is comeager. This concludes the proof.

4. Applications

4.1. Hausdorff and packing dimensions

We refer to [10, chapter 3] for the definitions of the Hausdorff dimension and the packing dimension.

**Proposition 4.1.** The sets defined in Theorem 2.1 and Theorem 2.3 have Hausdorff dimension 0 and packing dimension 1.

**Proof.** Reasoning as in [23], the claimed sets are contained in the corresponding ones with ideal \( \text{Fin} \), which have Hausdorff dimension 0 by [22, theorem 2.1]. In addition, since all sets are comeager, we conclude that they have packing dimension 1 by [10, corollary 3.10(b)].

4.2. Regular matrices

We extend the main results contained in [15, 27]. To this aim, let \( A = (a_{n,i} : n, i \in \mathbb{N}) \) be a regular matrix, that is, an infinite real-valued matrix such that, if \( z = (z_n) \) is a \( \mathbb{R}^d \)-valued sequence convergent to \( \eta \), then \( A_n z := \sum_i a_{n,i} z_i \) exists for all \( n \in \mathbb{N} \) and \( \lim_n A_n z = \eta \), see e.g. [9, chapter 4]. Then, for each \( x \in (0, 1] \) and integers \( b \geq 2 \) and \( k \geq 1 \), let \( \Gamma_b^k(x, I, A) \) be the set of \( I \)-cluster points of the sequence of vectors \( (A_n \pi_{b,n}^k(x) : n \geq 1) \), where \( \pi_{b,n}^k(x) \) is the sequence \( (\pi_{b,n}^k(x) : n \geq 1) \).

In particular, \( \Gamma_b^k(x, I, A) = \Gamma_b^k(x, I) \) if \( A \) is the infinite identity matrix.

**Theorem 4.2.** The set \( \{ x \in (0, 1] : \Gamma_b^k(x, I, A) \supseteq \Delta_b^k \text{ for all } b \geq 2, k \geq 1 \} \) is comeager, provided that \( I \) is a meager ideal and \( A \) is a regular matrix.

**Proof.** Fix a regular matrix \( A = (a_{n,i}) \) and a meager ideal \( I \). The proof goes along the same lines as the proof of Theorem 2.1, replacing the definition of \( S_{t,m,p} \) with \( S'_{t,m,p} := \{ x \in (0, 1] : \forall q \geq p, \exists n \in I_q, \| A_n \pi_{b,n}^k(x) - \eta \| \geq 1/m \} \).
Recall that, thanks to the classical Silverman–Toeplitz characterisation of regular matrices, see e.g. [9, theorem 4-1, II] or [8], we have that $\sup_n \sum_i |a_{n,i}| < \infty$. Since $\lim_n \pi^k_{b,n}(x^*) = \eta_i$, it follows that there exist sufficiently large integers $q \geq p + j$ and $j_A \geq j$ such that, if $d_{b,i}(x) = d_{b,i}(x^*)$ for all $i = 1, \ldots, j_A + k$, then

$$
\|A_n \pi^k_{b}(x) - \eta_i\| \leq \|A_n \pi^k_{b}(x^*) - \eta_i\| + \left\| \sum_i a_{n,i} (\pi^k_{b,i}(x) - \pi^k_{b,i}(x^*)) \right\|
\leq \|A_n \pi^k_{b}(x^*) - \eta_i\| + \sum_i |a_{n,i}| \|\pi^k_{b,i}(x) - \pi^k_{b,i}(x^*)\|
\leq \|A_n \pi^k_{b}(x^*) - \eta_i\| + \sum_{i > j_A} |a_{n,i}| < \frac{1}{m}
$$

(4)

for all $n \in I_q$. We conclude analogously that $S'_{i,m,p}$ is nowhere dense.

The main result in [27] corresponds to the case $\mathcal{I} = \text{Fin}$ and $k = 1$, although with a different proof; cf. also Example 4-10 below.

At this point, we need an intermediate result which is of independent interest. For each bounded sequence $x = (x_n)$ with values in $\mathbb{R}^k$, let $\text{K-core}(x)$ be the Knopp core of $x$, that is, the convex hull of the set of accumulation points of $x$. In other words, $\text{K-core}(x) = \text{co} \ L_x$, where $\text{co} \ S$ is the convex hull of $S \subseteq \mathbb{R}^k$ and $L_x$ is the set of accumulation points of $x$. The ideal version of the Knopp core has been studied in [16, 18]. The classical Knopp theorem states that, if $k = 2$ and $A$ is a nonnegative regular matrix, then

$$
\text{K-core}(Ax) \subseteq \text{K-core}(x)
$$

(5)

for all bounded sequences $x$, where $Ax = (A_n x : n \geq 1)$, see [17, p. 115]; cf. [9, chapter 6] for a textbook exposition. A generalisation in the case $k = 1$ can be found in [21]. We show, in particular, that a stronger version of Knopp’s theorem holds for every $k \in \mathbb{N}$.

**Proposition 4-3.** Let $x = (x_n)$ be a bounded sequence taking values in $\mathbb{R}^k$, and fix a regular matrix $A$ such that $\lim_n \sum_i |a_{n,i}| = 1$. Then inclusion (5) holds.

**Proof.** Define $\kappa := \sup_n \|x_n\|$ and let $\eta$ be an accumulation point of $Ax$. It is sufficient to show that $\eta \in K := \text{K-core}(x)$. Possibly deleting some rows of $A$, we can assume without loss of generality that $\lim Ax = \eta$. For each $m \in \mathbb{N}$, let $K_m$ be the closure of $\text{co}\{x_m, x_{m+1}, \ldots\}$, hence $K \subseteq K_m$. Define $d(a, C) := \min_{b \in C} \|a - b\|$ for all $a \in \mathbb{R}^k$ and nonempty compact sets $C \subseteq \mathbb{R}^k$. In addition, for each $m \in \mathbb{N}$, let $Q_m(a) \in K_m$ be the unique vector such that $d(a, K_m) = \|a - Q_m(a)\|$. Similarly, let $Q(a)$ be the vector in $K$ which minimizes its distance with $a$. Then, notice that, for all $n, m \in \mathbb{N}$, we have

$$
d(A_n x, K) \leq \inf_{b \in K} \inf_{c \in \mathbb{R}^k} (\|A_n x - c\| + \|c - b\|)
\leq \inf_{c \in K_m} \inf_{b \in K} (\|A_n x - c\| + \|c - b\|)
\leq \inf_{c \in K_m} \|A_n x - c\| + \sup_{y \in K_m} \inf_{b \in K} \|y - b\|
= d(A_n x, K_m) + \sup_{y \in K_m} d(y, K).
$$

Since $d(\eta, K) = \lim_n d(A_n x, K)$ by the continuity of $d(\cdot, K)$, it is sufficient to show that both $d(A_n x, K_m)$ and $\sup_{y \in K_m} d(y, K)$ are sufficiently small if $n$ is sufficiently large and $m$ is chosen properly.

To this aim, fix $\varepsilon > 0$ and choose $m \in \mathbb{N}$ such that $\sup_{y \in K_m} d(y, K) \leq \varepsilon/2$. Indeed, it is sufficient to choose $m \in \mathbb{N}$ such that $d(x_n, L_x) < \varepsilon/2$ for all $n \geq m$: indeed, in the opposite,
the subsequence \((x_j)_{j \in J}\), where \(J := \{ n \in \mathbb{N} : d(x_n, L_x) \geq \varepsilon / 2 \}\), would be bounded and without any accumulation point, which is impossible. Now pick \(y \in K_m\) so that \(y = \sum_j \lambda_j x_j\) for some strictly increasing sequence \((ij)\) of positive integers such that \(i_1 \geq m\) and some real nonnegative sequence \((\lambda_{ij})\) with \(\sum_j \lambda_{ij} = 1\). It follows that

\[
d(y, K) \leq \left\| y - \sum_j \lambda_{ij} Q(x_j) \right\| \leq \sum_j \lambda_{ij} \left\| x_j - Q(x_j) \right\| \leq \sum_j \lambda_{ij} d(x_j, L_x) \leq \frac{\varepsilon}{2}.
\]

Suppose for the moment that \(A\) has nonnegative entries. Since \(A\) is regular, we get\(\lim_n \sum_i a_{n,i} = 1\) and \(\lim_n \sum_{i<m} a_{n,i} = 0\) by the Silverman–Toeplitz characterisation, hence \(\lim_n \sum_{i \geq m} a_{n,i} = 1\) and there exists \(n_0 \in \mathbb{N}\) such that \(\sum_{i \geq m} a_{n,i} \geq 1/2\) for all \(n \geq n_0\). Thus, for each \(n \geq n_0\), we obtain that \(d(A_n x, K_m) = \|A_n x - Q_m(A_n x)\| \leq \alpha_n + \beta_n + \gamma_n\), where

\[
\alpha_n := \left\| A_n x - \frac{A_n x}{\sum_i a_{n,i}} \right\|, \quad \beta_n := \left\| \frac{A_n x}{\sum_i a_{n,i}} - Q_m \left( \frac{A_n x}{\sum_i a_{n,i}} \right) \right\|
\]

and

\[
\gamma_n := \left\| Q_m \left( \frac{A_n x}{\sum_i a_{n,i}} \right) - Q_m(A_n x) \right\|.
\]

Recalling that \(\kappa = \sup_n \|x_n\|\), it is easy to see that

\[
\gamma_n \leq \alpha_n \leq \kappa \sum_i |a_{n,i}| \cdot \left( 1 - \frac{1}{\sum_i a_{n,i}} \right).
\]

In addition, setting \(t_n := \sum_{i \geq m} a_{n,i} / \sum_i a_{n,i} \in [0, 1]\) for all \(n \geq n_0\), we get

\[
\beta_n \leq \frac{\sum_i * \sum_i a_{n,i} - \sum_i a_{n,i} *}{\sum_i a_{n,i}} \leq \frac{1}{\sum_i a_{n,i}} \left( \sum_{i \geq m} a_{n,i} \left( \sum_i * + \sum_i * \right) - \sum_i a_{n,i} \sum_i * \right) \leq \frac{1}{\sum_i a_{n,i}} \left( t_n \sum_{i < m} |a_{n,i}| + (1 - t_n) \sum_i |a_{n,i}| \right).
\]

where \(\sum_{i \in I}^\star\) stands for \(\sum_{i \in I} a_{n,i} x_i\). Note that the hypothesis that the entries of \(A\) are nonnegative has been used only in the first line of (6), so that \(\sum_{i \geq m} / \sum_{i \geq m} a_{n,i} \in K_m\). Since \(\lim_n \sum_{i < m} |a_{n,i}| = 0\), \(\lim_n t_n = 1\), and \(\sup_n \sum_i |a_{n,i}| < \infty\) by the regularity of \(A\), it follows that all \(\alpha_n, \beta_n, \gamma_n\) are smaller than \(\varepsilon / 6\) if \(n\) is sufficiently large. Therefore \(d(A_n x, K) \leq \varepsilon\) and, since \(\varepsilon\) is arbitrary, we conclude that \(\eta = \lim_n A_n x \in K\).

Lastly, suppose that \(A\) is a regular matrix such that \(\lim_n \sum_i |a_{n,i}| = 1\) and let \(B = (b_{n,i})\) be the nonnegative regular matrix defined by \(b_{n,i} = |a_{n,i}|\) for all \(n, i \in \mathbb{N}\). Considering that

\[
d(A_n x, K_m) \leq \|A_n x - B_n x\| + d(B_n x, K_m) \leq \kappa \sum_i |a_{n,i} - |a_{n,i}|| + \varepsilon,
\]

and that \(\lim_n \sum_i |a_{n,i} - |a_{n,i}|| = 0\) because \(\lim_n \sum_i a_{n,i} = \lim_n \sum_i |a_{n,i}| = 1\), we conclude that \(d(A_n x, K_m) \leq 2\varepsilon\) whenever \(n\) is sufficiently large. The claim follows as before.
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The following corollary is immediate:

**Corollary 4.4.** Let \( x = (x_n) \) be a bounded sequence taking values in \( \mathbb{R}^k \), and fix a nonnegative regular matrix \( A \). Then inclusion (5) holds.

**Remark 4.5.** Inclusion (5) fails for an arbitrary regular matrix: indeed, let \( A = (a_{n,i}) \) be the matrix defined by \( a_{n,2n} = 2, a_{n,2n-1} = -1 \) for all \( n \in \mathbb{N} \), and \( a_{n,i} = 0 \) otherwise. Set also \( k = 1 \) and let \( x \) be the sequence such that \( x_n = (-1)^n \) for all \( n \in \mathbb{N} \). Then \( A \) is regular and \( \lim Ax = 3 \notin [-1, 1] = \text{K-core}(x) \).

**Remark 4.6.** Proposition 4.3 keeps holding on a (possibly infinite dimensional) Hilbert space \( X \) with the following provisions: replace the definition of \( \text{K-core}(x) \) with the closure of \( \text{co} \, L_x \) (this coincides in the case that \( X = \mathbb{R}^k \)) and assume that the sequence \( x \) is contained in a compact set (so that \( \text{K-core}(x) \) is also nonempty).

With these premises, we can strengthen Theorem 4.2 as follows.

**Theorem 4.7.** The set \( \{ x \in (0, 1] : \Gamma_b^k(x, \mathcal{I}, A) = \Delta_b^k \text{ for all } b \geq 2, k \geq 1 \} \) is comeager, provided that \( \mathcal{I} \) is a meager ideal and \( A \) is a regular matrix such that \( \lim_n \sum_i |a_{n,i}| = 1 \).

**Proof.** Let us suppose that \( A = (a_{n,i}) \) is nonnegative regular matrix, i.e., \( a_{n,i} \geq 0 \) for all \( n, i \in \mathbb{N} \), and fix a meager ideal \( \mathcal{I} \), a real \( x \in (0, 1] \), and integers \( b \geq 2, k \geq 1 \). Thanks to Theorem 4.2, it is sufficient to show that every accumulation point of the sequence \( (A_n \pi_{b,n}^k(x) : n \geq 1) \) is contained in the convex hull of the set of accumulation points of \( (\pi_{b,n}^k(x) : n \geq 1) \), which is in turn contained into \( \Delta_b^k \). This follows by Proposition 4.3.

Since the family of meager sets is a \( \sigma \)-ideal, the following is immediate by Theorem 4.7.

**Corollary 4.8.** Let \( \mathcal{A} \) be a countable family of regular matrices such that \( \lim_n \sum_i |a_{n,i}| = 1 \). Then the set \( \{ x \in (0, 1] : \Gamma_b^k(x, \mathcal{I}, A) = \Delta_b^k \text{ for all } b \geq 2, k \geq 1, \text{ and all } A \in \mathcal{A} \} \) is comeager, provided that \( \mathcal{I} \) is a meager ideal.

It is worth to remark that the main result [15] is obtained as an instance of Corollary 4.8, letting \( \mathcal{A} \) be the set of iterates of the Cesàro matrix (note that they are nonnegative regular matrices), and setting \( k = 1 \) and \( \mathcal{I} = \text{Fin} \). The same holds for the iterates of the Hölder matrix and the logarithmic Riesz matrix as in [24, sections 3 and 4].

Next, we show that the hypothesis \( \lim_n \sum_i |a_{n,i}| = 1 \) for the entries of the regular matrix in Theorem 4.7 cannot be removed.

**Example 4.9.** Let \( A = (a_{n,i}) \) be the matrix such that \( a_{n,(2n-1)!} = -1 \) and \( a_{n,(2n)!} = 2 \) for all \( n \in \mathbb{N} \), and \( a_{n,i} = 0 \) otherwise. It is easily seen that \( A \) is regular. Then, set \( b = 2, k = 1 \), and \( \mathcal{I} = \text{Fin} \). We claim that the set of all \( x \in (0, 1] \) such that 2 is an accumulation point of the sequence \( \pi_{2,1}(x) = (\pi_{2,1,n}(x) : n \geq 1) \) is comeager. Indeed, its complement can be rewritten as \( \bigcup_{m,p} S_{m,p} \), where

\[
S_{m,p} := \{ x \in (0, 1] : |A_n \pi_{2,1}(x) - 2| \geq 1/m \text{ for all } n \geq p \}.
\]

Let \( x^* \in (0, 1] \) such that \( d_{2,n}(x^*) = 1 \) if and only if \((2i-1)! \leq n < (2i)! \) for some \( i \in \mathbb{N} \). Then it is easily seen that \( \lim_n \pi_{2,1,n}(x^*) = 2 \). Along the same lines of the proof of Theorem 4.2, it follows that each \( S_{m,p} \) is meager. We conclude that \( \{ x \in (0, 1] : \Gamma_2^1(x, \text{Fin}, A) = \Delta_2^1 \} \) is meager.
which proves that the condition $\lim_n \sum_i |a_{n,i}| = 1$ in the statement of Theorem 4.7 cannot be removed.

In addition, the main result in [27] states that Theorem 4.2, specialised to the case $\mathcal{I} = \text{Fin}$ and $k = 1$, can be further strengthened so that the set $\{x \in (0, 1] : \Gamma^1_b(x, \text{Fin}, A) \supseteq \Delta^1_b \}$ for all $b \geq 2$ and all regular $A$ is comeager. Taking into account the argument in the proof of Theorem 4.7, this would imply that the set

$$\{x \in (0, 1] : \Gamma^1_b(x, \text{Fin}, A) = \Delta^1_b \text{ for all } b \geq 2 \text{ and all nonnegative regular } A\}$$

should be comeager. However, this is false as it is shown in the next example.

**Example 4.10.** For each $y \in (0, 1]$, let $(e_{y,k} : k \geq 1)$ be the increasing enumeration of the infinite set $\{n \in \mathbb{N} : d_{2,n}(y) = 1\}$. Then, let $\mathcal{A} = \{A_y : y \in (0, 1]\}$ be family of matrices $A_y = \left(a_{n,i}^{(y)}\right)$ with entries in $\{0, 1\}$ so that $a_{n,i}^{(y)} = 1$ if and only if $e_{y,n} = i$ for all $y \in (0, 1]$ and all $n, i \in \mathbb{N}$. Then each $A_y$ is a nonnegative regular matrix. It follows, for each ideal $\mathcal{I}$,

$$\{x \in (0, 1] : \Gamma^1_2(x, \mathcal{I}, A) = \Delta^1_2 \text{ for all } A \in \mathcal{A}\} = \emptyset.$$

Indeed, for each $x \in (0, 1]$, the sequence $\mathbf{\pi}^1_2(x) = (\mathbf{\pi}^1_{2,n}(x) : n \geq 1)$ has an accumulation point $\eta \in \Delta^1_2$. Hence there exists a subsequence $(\mathbf{\pi}^1_{2,n_k}(x) : k \geq 1)$ which is convergent to $\eta$. Equivalently, $\lim A_n \mathbf{\pi}^1_2(x) = \eta$, where $y \in (0, 1]$ is defined such that $e_{y,k} = n_k$ for all $k \in \mathbb{N}$. Therefore $\{\eta\} = \Gamma^1_2(x, \mathcal{I}, A_y) \neq \Delta^1_2$, in particular, the set defined in (7) is empty.

Lastly, the analogues of Theorem 4.2 and Theorem 4.7 hold for $\mathcal{I}$-limit points, if $\mathcal{I}$ is an $F_\sigma$-ideal or an analytic $P$-ideal. Indeed, denoting with $\Lambda^k_\mathcal{I}(x, \mathcal{I}, A)$ the set of $\mathcal{I}$-limit points of the sequence $(A_n \mathbf{\pi}^k_\mathcal{I}(x) : n \geq 1)$, we obtain:

**Theorem 4.11.** Let $A$ be a regular matrix and let $\mathcal{I}$ be an $F_\sigma$-ideal or an analytic $P$-ideal. Then the set $\{x \in (0, 1] : \Lambda_\mathcal{I}^k(x, \mathcal{I}, A) \supseteq \Delta^k_b \text{ for all } b \geq 2 \text{ and } k \geq 1\}$ is comeager.

Moreover, the set $\{x \in (0, 1] : \Lambda_\mathcal{I}^k(x, \mathcal{I}, A) = \Delta^k_b \text{ for all } b \geq 2 \text{ and } k \geq 1\}$ is comeager if, in addition, $A$ satisfies $\lim_n \sum_i |a_{n,i}| = 1$.

**Proof.** The first part goes along the same lines of the proof of Theorem 2.3. Here, we replace $\mathbf{\pi}^k_\mathcal{I}(x)$ with $(A_n \mathbf{\pi}^k_\mathcal{I}(x) : n \geq 1)$ and using the chain of inequalities (4): more precisely, we consider $j'' \in \mathbb{N}$ such that $\varphi((n \in [j', j'') : \|A_n \mathbf{\pi}^k_\mathcal{I}(x') - \eta\| \leq 1/2t)) \geq 1/2$, and, taking into considering (4), we define $V := \{x \in (0, 1] : d_{b,j'}(x) = d_{b,j'}(x^*) \text{ for all } i = 1, \ldots, k + j''\}$, where $j''$ is a sufficiently large integer such that $\sum_{i>j''} |a_{n,i}| \leq 1/2t$ for all $n \in [j', j'')$.

The second part follows, as in Theorem 4.7, by the fact that every accumulation point of $(A_n \mathbf{\pi}^k_\mathcal{I}(x) : n \geq 1)$ belongs to $\Delta^k_b$.

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