On the open Dicke-type model generated by an infinite-component vector spin

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Abstract
We consider an open Dicke model made of a single infinite-component vector spin and a single-mode harmonic oscillator assuming a Jaynes-Cummings type interaction between them. We study its algebraic structure and dynamics based on superoperator formalism. It is shown that by an explicit invertible superoperator its Liouvillian is transformed into a sum of two independent Liouvillians that are generated by a dressed spin and a dressed harmonic oscillator, respectively.

Keywords: open Dicke model, infinite-component vector spin, superoperator formalism, Liouvillian

1 Introduction
Investigation of open quantum dynamical systems described by the Gorini-Kossakowski-Sudarshan-Lindblad-Davies type master equation [K, GKS, L, D2] is of fundamental importance in the research of quantum optics [BP]. Although it is in general difficult to understand dynamical properties for concrete open quantum models even qualitatively, there are some exceptions. A notably simple example is given by open harmonic oscillator models whose dynamics has been studied by various methods, see e.g. [AJP, CS, NVZ]. We also refer to the Dicke model [D] that caricatures the interaction between light (namely harmonic oscillators) and matter, see e.g. its review [G] and references therein. We are interested in dissipative Dicke-type models where the time evolution is given by Lindblad generator rather than Hamiltonian. We propose a concrete model as below and study its algebraic structure and dynamics.

We now briefly sketch our model. Assume that the radiation field is given by a single-mode harmonic oscillator as in the original Dicke model. On the other hand, we take a single infinite-component vector spin for its matter; this special setup enables rigorous algebraic approach as we will present in this work. To introduce dissipation into the model we assume that the harmonic oscillator is implicitly connected to a thermal bath, whereas the matter may or may not be connected to a thermal bath. Then by adding a Jaynes-Cummings type interaction between the harmonic oscillator and the spin, we

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obtain an open infinite-spin Dicke model. Hereafter this will be called OISD model for abbreviation.

Let us provide some background of OISD model. We first note that infinite-component vector spins can be extracted from \( n \)-compound quantum spin systems by certain limiting procedure \( n \to \infty \) \[D1, HL\]. ISD (infinite-component spin Dicke) model is a conserved (non-dissipative) quantum model that is generated by an infinite-component vector spin and a single-mode harmonic oscillator as described above. It has been investigated in \[HL, D1, BZT\]. As far as we have noticed, however, OISD model has not yet been seriously considered in the literature. Notably OISD model has both dissipation (induced by a hidden thermal bath) and non-trivial interaction (between the matter and the radiation). We will investigate their interplay and effects upon the open quantum dynamics.

This paper is divided into four parts. In \( \S 2 \) we consider the open harmonic oscillator model. Based on commutation relations satisfied by superoperators, we derive some basic properties on this model in a self-contained manner, though some of them are well known or can be easily derived from the results in the literature \[BE, CS, EFS, HNY, TP\]. Our formulation and results on the open harmonic oscillator model will be useful to investigate OISD model as well, since OISD model and the open harmonic oscillator have certain similarities in their algebraic structure. In other words, due to the special algebraic structure of the infinite-component spin system, the analysis on OISD model can be mostly reduced to the open harmonic oscillator model; such convenience cannot be expected for the (usual) finite spin system. In \( \S 3 \) we introduce the infinite-component spin system and show that dissipation makes the system unstable. In \( \S 4 \) we define OISD model and investigate its Liouvillian in detail making use of preliminaries in \( \S 2 \); \( \S 3 \). We obtain the following as our main result: The Liouvillian of OISD model is decomposed into a sum of two independent Liouvillians by the similarity transformation of a completely-positive trace-preserving [CPTP] superoperator, which is also referred to as a dynamical map in \[AL\]. In more detail, the first one is written by a dressed spin only, and the other is written by a dressed harmonic oscillator only.

We now introduce mathematical notations that will be used throughout this paper. Let \( \mathcal{H} \) be an infinite dimensional Hilbert space. Let \( \mathcal{B}(\mathcal{H}) \) be the set of all bounded linear operators on \( \mathcal{H} \) and \( \mathcal{C}_{1}(\mathcal{H}) \) be the Banach algebra of the trace class operators on \( \mathcal{H} \) with trace norm. For (unbounded) linear operators \( A, B \) and \( C \) acting in \( \mathcal{H} \), we define superoperators \( \mathcal{K}_{A} \) and \( \mathcal{D}_{B\circ C} \) acting in \( \mathcal{C}_{1}(\mathcal{H}) \) by

\[
\mathcal{K}_{A}(\rho) = [A, \rho], \quad \mathcal{D}_{B\circ C}(\rho) = 2B\rho C - \{CB, \rho\} \quad \text{for} \quad \rho \in \mathcal{C}_{1}(\mathcal{H}).
\]

We can verify the following relations by straightforward calculations:

\[
[K_{A}, K_{B}] = K_{[A, B]}, \quad [K_{A}, D_{B\circ C}] = D_{[A, B]\circ C} + D_{B\circ [A, C]}, \quad (2)
\]

\[
[D_{A\circ B}, D_{C\circ D}] = D_{[A, C]\circ [B, D]} - D_{[A, C]\circ [B, D]} + D_{[DC, A]\circ B} + D_{A\circ [B, DC]} - D_{[BA, C]\circ D} - D_{C\circ [D, BA]} + K_{[BA, DC]}, \quad (3)
\]

\[
D_{A\circ 1} = K_{A}, \quad D_{1\circ B} = -K_{B}. \quad (4)
\]
We will investigate the open quantum dynamics on $\mathcal{H}$ determined by the following type of master equation for $\rho(t) \in \mathcal{E}_1(\mathcal{H})$:

$$\frac{d}{dt}\rho(t) = -i\mathcal{K}_H(\rho(t)) + \sum_j \mathcal{D}_{A_j^\dagger A_j}(\rho(t)),$$

where $\mathcal{K}_H$ represents the infinitesimal change induced by a Hamiltonian $H$ of the corresponding closed quantum system, and the term $\sum_j \mathcal{D}_{A_j^\dagger A_j}$ with a set of operators $\{A_j, A_j^\dagger\}$ gives an effect of dissipation.

## 2 Algebraic structure of the open harmonic oscillator

We consider a one-mode photon system. Let $a$ and $a^\dagger$ denote the annihilation and the creation operators acting on the one-mode Fock space $\mathcal{F}$. The number operator is denoted as $N = a^\dagger a$. It is known that the Fock space $\mathcal{F}$ has the following complete orthonormal system $\{|n\rangle\}_{n \in \mathbb{N} \cup \{0\}}$, such that

$$N|n\rangle = n|n\rangle \quad (n \in \mathbb{N} \cup \{0\}),$$

and

$$a|0\rangle = 0, \quad a^\dagger|0\rangle = |1\rangle, \quad a|n\rangle = \sqrt{n}|n - 1\rangle, \quad a^\dagger|n\rangle = \sqrt{n + 1}|n + 1\rangle \quad (n \in \mathbb{N}).$$

Let us consider the master equation

$$\frac{d}{dt}\rho(t) = \mathcal{L}_o(\rho(t)) \quad \text{for} \quad \rho(t) \in \mathcal{E}_1(\mathcal{F})$$

with its Liouvillian

$$\mathcal{L}_o = -i\omega\mathcal{K}_N + \gamma \left((J + 1)\mathcal{D}_{a^oa^\dagger} + J\mathcal{D}_{a^\dagger o a}\right),$$

where $\omega > 0$ denotes the angular frequency of the oscillator, $\gamma \geq 0$ denotes the strength of dissipation, and $J \geq 0$ is a parameter related to the temperature (of the hidden thermal bath) [BP].

From (2-4) and $[a, a^\dagger] = 1$, we have

$$[\mathcal{K}_a, \mathcal{K}_a^\dagger] = 0, \quad [\mathcal{K}_a, \mathcal{K}_N] = \mathcal{K}_a, \quad [\mathcal{K}_a^\dagger, \mathcal{K}_N] = -\mathcal{K}_a^\dagger,$$

$$[\mathcal{K}_a, \mathcal{D}_{a^oa^\dagger}] = \mathcal{K}_a, \quad [\mathcal{K}_a^\dagger, \mathcal{D}_{a^oa^\dagger}] = \mathcal{K}_a^\dagger,$$

$$[\mathcal{K}_a, \mathcal{D}_{a^\dagger o a}] = -\mathcal{K}_a, \quad [\mathcal{K}_a^\dagger, \mathcal{D}_{a^\dagger o a}] = -\mathcal{K}_a^\dagger,$$

and

$$[\mathcal{K}_N, \mathcal{D}_{a^oa^\dagger}] = [\mathcal{K}_N, \mathcal{D}_{a^\dagger o a}] = 0,$$

$$[\mathcal{D}_{a^oa^\dagger}, \mathcal{D}_{a^\dagger o a}] = -2(\mathcal{D}_{a^oa^\dagger} + \mathcal{D}_{a^\dagger o a}).$$
We will investigate the eigenvalue problem of the Liouvillian \((9)\) making use of the algebraic relations listed above. For convenience, we define \(\Phi_{n,m} = K^m_n K^m_n \langle 0 \rangle \langle 0 |\) for \(n, m = 0, 1, 2, \ldots\). We see that the set \(\{\Phi_{n,m}\}_{n,m=0}^\infty\) is total in \(\mathcal{C}_1(\mathcal{F})\), namely its linear span is a dense subspace of \(\mathcal{C}_1(\mathcal{F})\). In fact, for each non-negative integer \(k\), the identity
\[
\text{Lin. Span} \{ \Phi_{n,m} | n \geq 0, k \geq m \geq 0 \} = \text{Lin. Span} \{ |n\rangle \langle m| |n \geq 0, k \geq m \geq 0 \}
\]
holds. We can show this set of identities by the induction on \(k\) noting \(\Phi_{n,0} = \sqrt{n} |n\rangle \langle 0|\) and 
\[
\sqrt{m+1} |n\rangle \langle m+1| = \sqrt{n} |n-1\rangle \langle m| - K^a |n\rangle \langle m|.
\]
Using \((10)\) and \((11)\) inductively we have
\[
\mathcal{K}_N(\Phi_{n,m}) = (n-m)\Phi_{n,m}, \quad \mathcal{D}_{a^\dagger a}^t(\Phi_{n,m}) = -(n+m)\Phi_{n,m} \quad (n, m \in \mathbb{N} \cup \{0\}). \quad (15)
\]
Thus all \(\Phi_{n,m}\)s are common eigenvectors for both \(\mathcal{K}_N\) and \(\mathcal{D}_{a^\dagger a}\), however, not for \(\mathcal{D}_{a^\dagger}\).

In the next paragraph, we focus on \(\mathcal{D}_{a^\dagger a}\). Set
\[
S_t(\rho) = \sum_{n=0}^\infty \frac{(1-e^{-2t})^n n!}{n!} a^\dagger n e^{-t a^\dagger} \rho e^{-ta^\dagger} a^n \quad (16)
\]
for \(t \geq 0\) and \(\rho \in \mathcal{C}_1(\mathcal{F})\). It is straightforward to derive the following properties
\[
\frac{d}{dt} S_t(\rho) = \mathcal{D}_{a^\dagger a}(S_t(\rho)), \quad S_{t_1+t_2}(\rho) = S_{t_1}(S_{t_2}(\rho)), \quad S_0(\rho) = \rho. \quad (17)
\]
The semigroup \(S_t\) defined above is completely positive and trace preserving [CPTP], since by definition \((16)\) it has the following form of CPTP maps, see e.g. [NC] §8.2.4.:
\[
S_t(\rho) = \sum_{n=0}^\infty E_n(t) \rho E_n^\dagger(t) \quad (18)
\]
with bounded operators \(E_n(t)\) satisfying the normalization condition
\[
\sum_{n=0}^\infty E_n^\dagger(t) E_n(t) = 1 \quad (19)
\]
for every \(t \geq 0\). Precisely the conditions \((18)\) and \((19)\) are satisfied by
\[
E_n(t) = \frac{(1-e^{-2t})^{n/2}}{\sqrt{n!}} a^n e^{-ta^\dagger} \in \mathfrak{B}(\mathcal{F}). \quad (20)
\]
Similarly \(\mathcal{D}_{a^\dagger a}\) generates a one-parameter semigroup of CPTP maps on \(\mathcal{C}_1(\mathcal{F})\), as the following formula holds:
\[
e^{t\mathcal{D}_{a^\dagger a}}(\rho) = \sum_{n=0}^\infty \frac{(e^{2t}-1)^n n!}{n!} a^n e^{-ta^\dagger} \rho e^{-ta^\dagger} a^\dagger n. \quad (21)
\]
We will show the following identity of superoperators:

\[ e^{\tau D_{a^\dagger o}} D_{o^\dagger a} = (e^{2\tau} D_{a^\dagger o}) + (e^{2\tau} - 1)D_{a^\dagger o} \) \quad (t \geq 0). \] (22)

Denote the right-hand side of (22) by \( X_\tau \). Then we immediately see that \( X_0 = D_{a^\dagger o} \) and \( \frac{d}{dt} X_\tau = D_{a^\dagger o} X_\tau \) due to (14).

Setting \( \tau \geq 0 \) by \( e^{2\tau} = J + 1 \) (23)
together with (13), we have

\[ e^{\tau D_{a^\dagger o}} \left(-i\omega K_N + \gamma D_{a^\dagger o}\right) = L_0 e^{\tau D_{a^\dagger o}}. \] (24)

Due to (15) and (24) we obtain

\[ L_0 e^{\tau D_{a^\dagger o}} (\Phi_{n,m}) = (-i\omega (n - m) - \gamma (n + m)) e^{\tau D_{a^\dagger o}} (\Phi_{n,m}) \quad (n,m \in \mathbb{N} \cup \{0\}). \] (25)

To see that this gives the complete solution of the eigenvalue problem for \( L_0 \), it is enough to show that the range of \( e^{\tau D_{a^\dagger o}} \) is dense in \( \mathfrak{C}(\mathcal{F}) \), since \( \{\Phi_{n,m}\}_{n,m=0}^\infty \) is total in \( \mathfrak{C}(\mathcal{F}) \). We note that the right-hand side of (16) is still well defined for \( t < 0 \) and gives a densely defined operator as the (left) inverse of \( e^{t|D_{a^\dagger o}} \) when \( |t| \) is small (i.e., \( |1 - e^{-2t}| < 1 \)). Hence the range of \( e^{\tau D_{a^\dagger o}} \) is dense in \( \mathfrak{C}(\mathcal{F}) \) for small \( \tau > 0 \). The bounded semigroup property ensures the same for arbitrary \( \tau > 0 \). We refer to the previous works [BE, CS, EFS, HNY] for the information of the eigenvalue problem of \( L_0 \).

Let us consider the semigroup \( e^{tL_0} \). Due to the commutativity (13), we have

\[ e^{tL_0} = e^{-it\omega K_N} e^{t\gamma (J+1) D_{a^\dagger o} + JD_{a^\dagger o}}. \] (26)

We have also

\[ e^{t\gamma (J+1) D_{a^\dagger o} + JD_{a^\dagger o}} = e^{t\gamma (J+1) D_{a^\dagger o} + JD_{a^\dagger o}}, \] (27)

where

\[ \gamma(s) := \frac{1}{2} \log(J + 1 - Je^{-2s}) \quad \text{for} \quad s \geq 0. \] (28)

To see (27), we put its right-hand side as \( Y(t) \). Then \( Y(0) = 1 \) holds and its derivative satisfies

\[ \frac{dY(t)}{dt} = \gamma' \left( t\gamma \right) D_{a^\dagger o} e^{t\gamma (J+1) D_{a^\dagger o} + JD_{a^\dagger o}} \]
\[ + e^{t\gamma (J+1) D_{a^\dagger o} + JD_{a^\dagger o}} \gamma (1 + t\gamma) D_{a^\dagger o} e^{t\gamma (J+1) D_{a^\dagger o} + JD_{a^\dagger o}} \]
\[ = \gamma \left( t\gamma \right) D_{a^\dagger o} + (1 + t\gamma) e^{2t\gamma (J+1) D_{a^\dagger o} + JD_{a^\dagger o}} Y(t) \]
\[ = \gamma (J + 1) D_{a^\dagger o} + JD_{a^\dagger o} Y(t), \]

where we have used (22). With (26) (27) (28) we obtain the following decomposition formula of \( e^{tL_0} \):

\[ e^{tL_0} = e^{-it\omega K_N} e^{t\gamma (J+1) D_{a^\dagger o} + JD_{a^\dagger o}}. \] (29)
Since $e^{-it\omega K N}$ (that generates a unitary evolution) is obviously a CPTP semigroup on $\mathcal{C}_1(\mathcal{F})$, and both $e^{tD_{a^{\dagger}a}}$ and $e^{tD_{a a^{\dagger}}}$ are CPTP maps as we have seen, $e^{tL_0}$ given as the composition of these CPTP maps in (29) is also a CPTP map.

We will see the asymptotic behavior of the dynamical semigroup $\{e^{tL_0}\}_{t \geq 0}$. From (16) and (23) we have

$$e^{tD_{a^{\dagger}a}}(\Phi_{0,0}) = S_t(\Phi_{00}) = \sum_{n=0}^{\infty} e^{-\beta \omega n} \frac{1}{Z} |n\rangle \langle n|, \quad \beta = \omega^{-1} \log(1 + J^{-1}). \quad (30)$$

The right-hand side of the above equality is the Gibbs state at the inverse temperature $\beta$. It follows from (25) that $\rho(t)$ approaches to the Gibbs state (30) in trace norm as $t \to \infty$ from arbitrary initial state $\rho(0)$. To see this asymptotic property, we note that the trace preserving property of $e^{tD_{a^{\dagger}a}}$ and the trace property yield

$$\text{Tr}[e^{tD_{a^{\dagger}a}}(\Phi_{n,m})] = \text{Tr}[\Phi_{n,m}] = \delta_{n,0}\delta_{m,0} \quad (n, m \in \mathbb{N} \cup \{0\}). \quad (31)$$

As the trace preserving map $e^{tD_{a^{\dagger}a}} : \mathcal{C}_1(\mathcal{F}) \to \mathcal{C}_1(\mathcal{F})$ has dense range with respect to the trace norm, any density matrix $\rho_0$ is approximated by the following finite linear combination

$$\sum_{n,m \geq 0} c_{n,m} e^{tD_{a^{\dagger}a}}(\Phi_{n,m}), \quad c_{n,m} \in \mathbb{C},$$

in $\mathcal{C}_1(\mathcal{F})$. Note that $c_{0,0} = 1$ due to $\text{Tr}(\rho_0) = 1$ and (31). From (25), (30) we obtain

$$\lim_{t \to \infty} e^{tL_0}(\rho_0) = \sum_{n,m \geq 0} c_{n,m} \lim_{t \to \infty} e^{tL_0}(e^{tD_{a^{\dagger}a}}(\Phi_{n,m}))$$

$$= e^{tD_{a^{\dagger}a}}(\Phi_{0,0}) = \sum_{n=0}^{\infty} e^{-\beta \omega n} \frac{1}{Z} |n\rangle \langle n|. \quad (32)$$

The above heuristic limiting procedure can be made rigorous by using the uniform boundedness of $e^{tL_0}$.

### 3 Infinite-component vector spin system

For the matter we consider an infinite-component spin system [D1]. We shall introduce this system by employing the algebraic formulation as in §2. This infinite-component spin system will be coupled to the open harmonic oscillator stated in §2.

Let $\mathcal{G}$ be a Hilbert space and $\{|n\rangle\}_{n \in \mathbb{Z}}$ be a complete orthonormal system of $\mathcal{G}$. As operators on $\mathcal{G}$, we consider $l_{\pm}$ and $M$ defined by

$$M |n\rangle = n |n\rangle, \quad l_{\pm} |n\rangle = |n \pm 1\rangle \quad (n \in \mathbb{Z}).$$

Note that the relations

$$[M, l_{\pm}] = \pm l_{\pm}, \quad l_{\perp}l_{\perp} = l_{\perp}l_{\perp} = 1$$

(33)
hold. Here, we consider the master equation

\[
\frac{d}{dt}\rho(t) = L_s(\rho(t)) \quad \text{for} \quad \rho(t) \in \mathcal{C}_1(\mathcal{G}),
\]

(34)

where its Liouvillian is given as

\[
L_s = -i\mu K_M + \alpha_- D_{l_- ol_+} + \alpha_+ D_{l_+ ol_-}
\]

(35)

with constants \(\mu > 0, \alpha_\pm \geq 0\). We shall mainly consider the case of \(\alpha_\pm = 0\), i.e., no dissipation for the spin as a component of OISD model in §4. However, it will become clear that treating both cases with or without dissipation on an equal footing is helpful for our discussion.

From (2), (3) and (33), we have

\[
[K_M, D_{l_- ol_+}] = [K_M, D_{l_+ ol_-}] = [D_{l_- ol_+}, D_{l_+ ol_-}] = 0.
\]

(36)

We have also

\[
e^{-i\mu t K_M}(\rho) = e^{-i\mu t M}\rho e^{i\mu t M},
\]

(37)

\[
e^{t\alpha_- D_{l_- ol_+}}(\rho) = \sum_{n=0}^{\infty} \frac{(2t\alpha_-)^n}{n!} e^{-2t\alpha_- l_-^n} \rho l_+^n
\]

(38)

\[
e^{t\alpha_+ D_{l_+ ol_-}}(\rho) = \sum_{m=0}^{\infty} \frac{(2t\alpha_+)^m}{m!} e^{-2t\alpha_+ l_+^m} \rho l_-^m
\]

(39)

for \(\rho \in \mathcal{C}_1(\mathcal{G})\). From these expressions combined with the commutativity relations (36), the solution of the master equation is given as

\[
e^{tL_s}(\rho) = e^{t\alpha_- D_{l_- ol_+}} e^{t\alpha_+ D_{l_+ ol_-}} e^{-i\mu t K_M}(\rho)
\]

\[
= \sum_{n,m=0}^{\infty} \frac{(2t\alpha_-)^n(2t\alpha_+)^m}{n!m!} e^{-2t(\alpha_- + \alpha_+)} l_-^{n-m} \rho l_+^m e^{-i\mu t M} \rho e^{i\mu t M} l_-^{n-m}
\]

\[= \sum_{k=-\infty}^{\infty} c_k(t) l_+^k \rho e^{i\mu t M} l_+^k,
\]

(40)

where

\[
c_k(t) \equiv \sum_{n,m=0}^{\infty} \delta_{n-m,k} \frac{(2t\alpha_-)^n(2t\alpha_+)^m}{n!m!} e^{-2t(\alpha_- + \alpha_+)}.
\]

(41)

The set of coefficients \(\{c_k(t)\}_{k \in \mathbb{Z}}\) may be considered as a time dependent probability distribution in the sense

\[
c_k(t) \geq 0, \quad \sum_{k=-\infty}^{\infty} c_k(t) = 1,
\]
from which it follows that the evolution (40) has CPTP property. On the other hand, the behavior of its mean and variance
\[
\sum_{k=-\infty}^{\infty} kc_k(t) = 2(\alpha_+ - \alpha_-)t,
\]
\[
\sum_{k=-\infty}^{\infty} k^2 c_k(t) - \left(\sum_{k=-\infty}^{\infty} kc_k(t)\right)^2 = 2(\alpha_+ + \alpha_-)t
\]
exhibits the floating and diffusive nature of the evolution. Moreover, it can be shown that \(\rho(t) = e^{dt}\) for arbitrary initial state \(\rho\) unless \((\alpha_+, \alpha_-) = (0, 0)\). In particular, with such non-trivial dissipation, there exists no eigenstate for \(L_s\) and there exists no steady (i.e. temporal invariant) state for the dynamical semigroup \(\{e^{dt}\}_{t\geq 0}\).

4 Open infinite-spin Dicke model

In this section, we investigate the open infinite-spin Dicke model (OISD model), an open quantum model generated by the infinite-component spin in [3] and the open harmonic oscillator in [2] with the Jaynes-Cummings interaction between them.

We will provide the precise formulation of OISD model in the following. The Hilbert space of the system is \(\mathcal{H} = \mathcal{G} \otimes \mathcal{F}\). Operators acting in \(\mathcal{H}\) such as \(M \otimes 1, 1 \otimes a^\dagger a, l_+ \otimes a\) will be simply denoted \(M, a^\dagger a, l_+ a\) by obvious embedding. We will introduce the following shorthand notations
\[
K_s = K_M, \quad K_{\text{int}} = K_{l_+ a^\dagger}, \quad K_{\text{int}} - K_{l_+ a^\dagger}, \quad K_{\text{int}} + K = K_{l_+ a^\dagger} + K_{l_+ a^\dagger}, \quad K_0 = K_N,
\]
and for nonnegative constant \(J\),
\[
D_s = (J + 1)D_{l_+ a^\dagger} + J D_{a^\dagger a}, \quad D_0 = (J + 1)D_{a^\dagger a} + J D_{a^\dagger a},
\]
\[
D_{\text{int}} - = (J + 1)D_{l_+ a^\dagger} + J D_{a^\dagger a}, \quad D_{\text{int}}^+ = (J + 1)D_{a^\dagger a} + J D_{a^\dagger a}.
\]
With the above notation, the Liouvillian of OISD model is defined by
\[
L := -i\mu K_s - i\lambda K_{\text{int}} - i\omega K_0 + \gamma D_0
\]
with positive constants \(\omega, \mu\) and nonnegative constant \(\gamma\). As we have anticipated, \(K_{\text{int}}\) is the Jaynes-Cummings interaction between the spin and the oscillator, and the constant \(\lambda \in \mathbb{R}\) denotes the strength of this interaction. The time evolution on the composed system is governed by the above Liouvillian as:
\[
\frac{d}{dt} \rho(t) = L(\rho(t)) \quad \text{for} \quad \rho(t) \in \mathcal{E}_1(\mathcal{H}).
\]

Remark that in the Liouvillian (44) of OISD model, dissipation is induced only through the harmonic oscillator (not through the spin). Later we shall discuss a more general Liouvillian that has dissipation terms both for the harmonic oscillator and the infinite-component spin. It turns out that the analysis for such a general model is essentially
reduced to the simple case (41). Hence we shall focus on this special setup for our OISD model.

Let us analyze algebraic structure of the Liouvillian \( L \) in (41). We can straightforwardly check the following commutation relations:

\[
[\mathcal{K}_o, \mathcal{K}^{\int}_{++}] = -[\mathcal{K}_o, \mathcal{K}^{\int}_{+-}] = [\mathcal{D}_o, \mathcal{D}^{\int}_{++}] = \pm \mathcal{K}^{\int}_{++},
\]

\[
[\mathcal{K}_o, \mathcal{D}^{\int}_{++}] = \pm \mathcal{D}^{\int}_{++},
\]

(46)

\[
[\mathcal{K}_o, \mathcal{D}_o] = [\mathcal{K}_o, \mathcal{D}_o] = [\mathcal{K}^{\int}_{++}, \mathcal{K}^{\int}_{++}] = [\mathcal{D}_o, \mathcal{K}^{\int}_{++}] = [\mathcal{D}_o, \mathcal{K}^{\int}_{++}] = [\mathcal{D}_o, \mathcal{D}_o] = 0.
\]

(47)

For \( \eta \in \mathbb{C} \), let us put \( W(\eta) = e^{\eta \mathcal{K}^{\int}_{++} - \eta \mathcal{K}^{\int}_{--}} \). Then from (46), (47), we get

\[
W(\eta) = \begin{pmatrix}
\mathcal{K}_o \\
\mathcal{K}^{\int}_{++} \\
\mathcal{K}^{\int}_{+-} \\
\mathcal{D}_o \\
\mathcal{D}^{\int}_{++} \\
\mathcal{D}^{\int}_{+-}
\end{pmatrix},
\]

\[
W(-\eta) = \begin{pmatrix}
\mathcal{K}_o + \eta \mathcal{K}^{\int}_{++} + \bar{\eta} \mathcal{K}^{\int}_{--} \\
\mathcal{K}^{\int}_{++} + \eta \mathcal{K}^{\int}_{+-} \\
\mathcal{K}^{\int}_{+-} + \eta \mathcal{K}^{\int}_{--} \\
\mathcal{D}_o \\
\mathcal{D}^{\int}_{++} - \eta \mathcal{D}_o \\
\mathcal{D}^{\int}_{+-} - \eta \mathcal{D}_o
\end{pmatrix},
\]

(48)

where the adjoint action \( \text{Ad}(W(\eta)) \) acts on each component. We obtain for any \( \sigma, t \geq 0 \)

\[
e^{\sigma \mathcal{D}_o} \left( \begin{pmatrix}
\mathcal{K}^{\int}_{++} \\
\mathcal{K}^{\int}_{+-} \\
\mathcal{K}^{\int}_{--} \\
\mathcal{D}^{\int}_{++} \\
\mathcal{D}^{\int}_{+-} \\
\mathcal{D}^{\int}_{--}
\end{pmatrix} \right) = \left( \begin{pmatrix}
\mathcal{K}^{\int}_{++} \cosh \sigma + \mathcal{D}^{\int}_{+-} \sinh \sigma \\
\mathcal{K}^{\int}_{+-} \cosh \sigma - \mathcal{D}^{\int}_{+-} \sinh \sigma \\
\mathcal{K}^{\int}_{--} \cosh \sigma + \mathcal{D}^{\int}_{--} \sinh \sigma \\
\mathcal{D}^{\int}_{++} \cosh \sigma - \mathcal{D}^{\int}_{+-} \sinh \sigma \\
\mathcal{D}^{\int}_{+-} \cosh \sigma - \mathcal{D}^{\int}_{+-} \sinh \sigma
\end{pmatrix} \right) e^{\sigma \mathcal{D}_o}
\]

(49)

and

\[
e^{-it(\mu \mathcal{K}_o + \omega \mathcal{K}_o)} \left( \begin{pmatrix}
\mathcal{K}^{\int}_{++} \\
\mathcal{K}^{\int}_{+-} \\
\mathcal{K}^{\int}_{--}
\end{pmatrix} \right) = \left( \begin{pmatrix}
e^{-it(\omega - \mu)} \mathcal{K}^{\int}_{++} \\
e^{it(\omega - \mu)} \mathcal{K}^{\int}_{+-}
\end{pmatrix} \right) e^{-it(\mu \mathcal{K}_o + \omega \mathcal{K}_o)}.
\]

(50)

Those can be verified by repeating a similar argument used in [22].

We can see that \( e^{L} \) is a CPTP map on \( \mathcal{C}_1(\mathcal{H}) \) for each \( t \geq 0 \) as follows. Note that \( e^{\gamma \mathcal{D}_o} \) is identical to \( 1 \otimes \text{exp}(t \gamma (J + 1) \mathcal{D}_{aoa \dagger} + t \gamma J \mathcal{D}_{a \dagger a}) \). Since the second factor is a completely positive on \( \mathcal{C}_1(\mathcal{F}) \) as we have seen in [22], \( e^{\gamma \mathcal{D}_o} \) is inevitably a completely positive map on \( \mathcal{C}_1(\mathcal{H}) = \mathcal{C}_1(\mathcal{F}) \otimes \mathcal{C}_1(\mathcal{F}) \). The trace preserving property follows from the tensor-product structure as well. Similarly \( e^{t \gamma' \mathcal{D}_s} \) and thereby \( e^{t \gamma' \mathcal{D}_s} e^{t \gamma' \mathcal{D}_s} \) are CPTP maps for \( \gamma' > 0 \). We have the following decomposition formula of the semigroup generated by \( L \):

\[
e^{L} = e^{-it(\mu \mathcal{K}_o + \omega \mathcal{K}_o)} W(\eta_1(t)) e^{t \gamma \mathcal{D}_o + \tau_2(t) \mathcal{D}_s} W(\eta_2(t)),
\]

(51)

where \( \eta_1(t), \eta_2(t) \) and \( \tau_2(t) \) are the solutions of the differential equations:

\[
\eta'_2(t) \sinh t \gamma - \gamma \eta_1(t) = 0
\]

(52)

\[
(\eta'_1(t) + \gamma \eta_1(t) \coth t \gamma) e^{-it(\omega - \mu)} + i \lambda = 0
\]

(53)

\[
\tau'_2(t) - \gamma |\eta_1(t)|^2 = 0
\]

(54)
with initial condition \( \eta_1(0) = \eta_2(0) = \tau_2(0) = 0 \). The solution for \( \eta_1(t) \) is explicitly given by

\[
\eta_1(t) = \frac{i\lambda}{(\omega - \mu)^2 + \gamma^2} \left( i(\omega - \mu)e^{i(\omega - \mu)t} + \frac{\gamma(1 - e^{i(\omega - \mu)t} \cosh \gamma t)}{\sinh \gamma t} \right),
\]

and \( \eta_2(t) \) and \( \tau_2(t) \) are obtained readily from (52) and (54). The identity can be shown as in (27). Namely by putting the right hand-side by \( Z(t) \) we have

\[
Z(0) = 1, \quad \frac{dZ(t)}{dt} = L Z(t)
\]

with the help of (48), (49) and (50). Since each factor in the right-hand side of (51) is CPTP, \( e^{t L} \) is also CPTP.

Now we will consider the decomposition of the Liouvillian \( L \). From (48) and (49), by setting \( V(\sigma) = W(\zeta_1)e^{\sigma D_o}W(\zeta_2) \) \( \sigma > 0 \) (55) we obtain

\[
\left( -i\mu K_s - i\lambda \kappa_{\text{int}} - i\omega K_o + \gamma D_o \right)V(\sigma) = V(\sigma)\left( -i\mu K_s + \lambda \gamma \delta D_s - i\omega K_o + \gamma D_o \right), \tag{56}
\]

where

\[
\delta = \frac{\lambda}{\gamma^2 + (\omega - \mu)^2}, \quad \zeta_1 = -(\omega - \mu + i\gamma \coth \sigma)\delta, \quad \zeta_2 = \frac{i\gamma \delta}{\sinh \sigma}. \tag{57}
\]

We notice that \( W(\eta) \) and \( W(\zeta) \) are CPTP maps having bounded inverses \( W(-\eta) \) and \( W(-\zeta) \), respectively, and that \( e^{\sigma D_o} \) is also CPTP having dense range and (unbounded) inverse because of the eigenvectors of \( D_o \), which can be seen in (25) with \( \omega = 0 \) and \( \gamma = 1 \). Hence \( V(\sigma) \) is an invertible CPTP map on \( \mathcal{C}_1(\mathcal{H}) \) with dense range. From (56), (57) the transformation by \( V(\sigma) \) erases the Jaynes-Cummings interaction, and at the same time, produces dissipation in the spin.

Some remarks are in order. Note that \( V(\sigma) \) for any \( \sigma > 0 \) works for the same transformation identity (56). This freedom of choice of \( \sigma \) reflects the fact that \( -i\mu K_s + \lambda \gamma \delta D_s - i\omega K_o + \gamma D_o \) in the right-hand side of (56) commutes with \( D_s \) and \( D_o \). From (57) if \( \gamma \), i.e. the strength of the dissipation of harmonic oscillator is large, then the strength of the dissipation of the dressed spin after \( V(\sigma) \), i.e. \( \lambda \gamma \delta \) is suppressed.

Let us rewrite (56) in terms of the tensor-product as

\[
LV(\sigma) = V(\sigma) \tilde{L}, \quad \tilde{L} := \tilde{L}_o \otimes 1 + 1 \otimes \tilde{L}_o, \tag{58}
\]

where \( \tilde{L}_o \) is the Liouvillian (2) in \( \mathcal{C}_1(\mathcal{F}) \), \( \tilde{L}_s \) is the Liouvillian (55) in \( \mathcal{C}_1(\mathcal{G}) \) with \( \alpha_- = \gamma(J + 1) \), \( \alpha_+ = \gamma J \). The solution for the master equation (45) of OISD model is formally written as

\[
\rho(t) = e^{t \tilde{L}_o}(\rho(0)) = V(\sigma)\left( e^{t \tilde{L}_s} \otimes e^{t \tilde{L}_o} \right)V(\sigma)^{-1}(\rho(0)). \tag{59}
\]

So far we have discussed the open quantum dynamics where dissipation is assigned to the harmonic oscillator only as given in (44). As we have anticipated, we now discuss
a more general OISD model, where each of the harmonic oscillator and the infinite-component spin has its dissipation term. Namely it has the following Liouvillian

$$ L(\gamma, \bar{\gamma}) = -i\mu K_s - i\lambda K_{\text{int}} - i\omega K_o + \gamma D_o + \bar{\gamma} D_s $$  

(60)

with positive constants $\omega, \mu, \gamma, \bar{\gamma}$ and real $\lambda$. By removing the interaction from the above Liouvillian we define the decoupled Liouvillian as

$$ \hat{L}(\gamma, \bar{\gamma}) = -i\mu K_s - i\omega K_o + \gamma D_o + \bar{\gamma} D_s $$  

(61)

Since $D_s$ commutes with any of $K_s, K_{\text{int}}, K_o, D_s, D_{\pm}, D_o$ by (47), by applying the same $V(\sigma)$ as in (55) we obtain

$$ L(\gamma, \bar{\gamma})V(\sigma) = V(\sigma)\hat{L}(\gamma, \lambda\gamma\delta + \bar{\gamma}) . $$  

(62)

By acting the transformation $V(\sigma)$ the Jaynes-Cummings interaction in the original Liouvillian is erased, whereas extra dissipation is added to the infinite-component spin, but not to the harmonic oscillator. To see this somewhat unbalanced effect of dissipation, let us compare $L(\gamma, 0)$ and $L(0, \bar{\gamma})$. For $L(\gamma, 0)$ we have already shown (56), which is obviously the special case of (62). For $L(0, \bar{\gamma})$ that corresponds to the case where dissipation is assigned to the spin only, it follows from (62) that

$$ L(0, \bar{\gamma})V(\sigma) = V(\sigma)\hat{L}(0, \bar{\gamma}) . $$  

(63)

Therefore by the same transformation law, the interaction term is erased as before, however, no dissipation appear in the harmonic oscillator.

Finally let us discuss steady states for OISD models. From the argument given in the end of §3 together with our main result (62) no eigenstate exists for $L(\gamma, \bar{\gamma})$ “unless $(\gamma, \bar{\gamma}) = (0, 0)$”. Thus no steady state exists for the quantum semigroup $\{e^{tL(\gamma, \bar{\gamma})}\}_{t \geq 0}$ unless $(\gamma, \bar{\gamma}) = (0, 0)$.

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**A Appendix**

In the text of the note, there are some identities whose derivations are tedious. We show here the details of such derivations to facilitate the reader’s understanding.

**Derivation of (2)**:

$$ [K_A, K_B](\rho) = K_A K_B(\rho) - K_B K_A(\rho) = [A, [B, \rho]] - [B, [A, \rho]] $$

$$ = A(B\rho - \rho B) - (B\rho - \rho B)A - B(A\rho - \rho A) + (A\rho - \rho A)B $$

$$ = A\rho B + \rho BA - BA\rho - \rho AB = [A, B]\rho - \rho[A, B] $$

$$ = K_{[A,B]}(\rho) , $$
\[ [K_A, D_{B0C}] (\rho) = K_A D_{B0C} (\rho) - D_{B0C} K_A (\rho) \]
\[ = A(2B \rho C - CB \rho - \rho CB) - (2B \rho C - CB \rho - \rho CB)A \]
\[ - 2B(A \rho - \rho A)C + CB(A \rho - \rho A) + (A \rho - \rho A)CB \]
\[ = 2[A, B] \rho C + 2B \rho [A, C] - ACB \rho - A \rho CB + CB \rho A \]
\[ + \rho CBA + CBA \rho - CB \rho A + A \rho CB - \rho ACB \]
\[ = 2[A, B] \rho C + 2B \rho [A, C] - (ACB - CBA) \rho - \rho (ACB - CBA) \]
\[ = 2[A, B] \rho C + 2B \rho [A, C] - ([A, C]B + C[A, B]) \rho - \rho ([A, C]B + C[A, B]) \]
\[ = D_{[A,B]0C}(\rho) + D_{B0[A,C]}(\rho). \]

**Derivation of (3)**

Applying the each side of the equality to \( \rho \in C_1(\mathcal{H}) \), we have

\[ \[D_{A0B}, D_{C0D}] (\rho) \]
\[ = D_{A0B}(D_{C0D}(\rho)) \]
\[ - D_{C0D}(D_{A0B}(\rho)) \]
\[ = 2A(2C \rho D - \{DC, \rho\})B - \{BA, (2C \rho D - \{DC, \rho\})\} \]
\[ - 2C(2A \rho B - \{BA, \rho\})D + \{DC, (2A \rho B - \{BA, \rho\})\} \]
\[ = 4AC \rho DB - 2A\{DC, \rho\}B - 2\{BA, C \rho D\} + \{BA, \{DC, \rho\}\} \]
\[ - 4CA \rho BD + 2C\{BA, \rho\}D + 2\{DC, A \rho B\} - \{DC, \{BA, \rho\}\} \]
\[ = ([A, C] + [A, C]) \rho(\{B, D\} - \{B, D\}) \]
\[ - ([A, C] - [A, C]) \rho(\{B, D\} + \{B, D\}) \]
\[ + 2[DC, A] \rho B + 2A \rho [B, DC] - 2[BA, C] \rho D - 2C \rho [D, BA] \]
\[ + BADC \rho + \rho DCBA - DCBA \rho - \rho BADC \]
\[ = 2[A, C] \rho [B, D] - 2\{A, C\} \rho [B, D] \]
\[ + 2[DC, A] \rho B + 2A \rho [B, DC] - 2[BA, C] \rho D - 2C \rho [D, BA] \]
\[ + [[BA, DC], \rho], \quad (64) \]

where the terms on the underlines in the 4-th member correspond to those in the 5-th member, and

\[ (D_{[A,C]}0\{B,D\} - D_{[A,C]}0\{B,D\} \]
\[ + D_{[DC,A]}0B + D_{[A],[B,DC]} \]
\[ - D_{[BA,C]}0D - D_{[C,0D,BA]} \]
\[ + K_{[BA,DC]}(\rho) \]
\[ = 2[A, C] \rho [B, D] - \{\{B, D\} [A, C], \rho\} - 2\{A, C\} \rho [B, D] + \{\{B, D\} [A, C], \rho\} \]
\[ + 2[DC, A] \rho B - \{B[DC, A], \rho\} + 2A \rho [B, DC] - \{\{B, DC\} A, \rho\} \]
\[ - 2[BA, C] \rho D + \{D[BA, C], \rho\} - 2C \rho [D, BA] + \{D[BA, C], \rho\} \]
\[ + [[BA, DC], \rho]. \quad (65) \]
Then the difference of both sides is

\[\begin{align*}
(65) - (64) &= -\{\{B, D\}[A, C], \rho\} + \{[B, D]\{A, C\}, \rho\} \\
&- \{B[DC, A], \rho\} - \{[B, DC]A, \rho\} \\
&+ \{D[BA, C], \rho\} + \{[D, BA]C, \rho\} \\
&= \{-\{B, D\}[A, C] + [B, D]\{A, C\}\} \\
&- B[DC, A] - [B, DC]A + D[BA, C] + [D, BA]C, \rho\} \\
&= \{2BDC\{A, C\} - 2DBAC - BDCA + BADC - BDCA + DCBA \\
&+ DBAC - DCBA + DBAC - BADC, \rho\} = 0. \\
\end{align*}\]

Thus we get (3).

**Derivation of the first equation in (17)**

By differentiate (16) with respect to \(t\), we have

\[\begin{align*}
\frac{d}{dt} S_t(\rho) &= \sum_{n=1}^{\infty} \frac{(1 - e^{-2t})^{n-1}}{(n-1)!} 2e^{-2t}a^\dagger na - e^{-ta\dagger} \rho e^{-ta\dagger} a^n \\
&+ \sum_{n=0}^{\infty} \frac{(1 - e^{-2t})^n}{n!} a^\dagger n \{ - aa^\dagger, e^{-ta\dagger} \rho e^{-ta\dagger} \} a^n \\
&= \sum_{n=1}^{\infty} \frac{(1 - e^{-2t})^{n-1}}{(n-1)!} 2e^{-2t}a^\dagger na - e^{-ta\dagger} \rho e^{-ta\dagger} a^n \\
&+ \sum_{n=0}^{\infty} \frac{(1 - e^{-2t})^n}{n!} 2na^\dagger ne^{-ta\dagger} \rho e^{-ta\dagger} a^n \\
&+ \sum_{n=0}^{\infty} \frac{(1 - e^{-2t})^n}{n!} \{ - aa^\dagger, a^\dagger ne^{-ta\dagger} \rho e^{-ta\dagger} a^n \} \\
&= \sum_{n=1}^{\infty} \frac{(1 - e^{-2t})^{n-1}}{(n-1)!} (2e^{-2t} + 2(1 - e^{-2t})) a^\dagger na - e^{-ta\dagger} \rho e^{-ta\dagger} a^n \\
&+ \sum_{n=0}^{\infty} \frac{(1 - e^{-2t})^n}{n!} \{ - aa^\dagger, a^\dagger ne^{-ta\dagger} \rho e^{-ta\dagger} a^n \} \\
&= 2a^\dagger \left( \sum_{n=0}^{\infty} \frac{(1 - e^{-2t})^n}{n!} a^\dagger n a - e^{-ta\dagger} \rho e^{-ta\dagger} a^n \right) \\
&- \{ aa^\dagger, \sum_{n=0}^{\infty} \frac{(1 - e^{-2t})^n}{n!} a^\dagger n a - e^{-ta\dagger} \rho e^{-ta\dagger} a^n \} \\
&= D_{a^\dagger o a}(S_t(\rho)). \\
\end{align*}\]
Derivation of the second equation in (17):

By the use of (16) in the right-hand side, we have

\[ S_{t_1}(S_{t_2}(\rho)) = \sum_{n=0}^{\infty} \frac{(1 - e^{-2t_1})^n}{n!} a^n \ e^{-t_1 a^a} \left( \sum_{m=0}^{\infty} \frac{(1 - e^{-2t_2})^m}{m!} a^m \ e^{-t_2 a^a} \rho \ e^{-t_2 a^a} a^m \right) e^{-t_1 a^a} a^n \]

\[ = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(1 - e^{-2t_1})^n (1 - e^{-2t_2})^m}{n! m!} a^{(n+m)} e^{-t_1 (a^a + m)} e^{-t_2 a^a} \rho \ e^{-t_2 a^a} e^{-t_1 (a^a + m)} a^{n+m} \]

\[ = \sum_{n,m=0}^{\infty} \frac{(1 - e^{-2t_1})^n (1 - e^{-2t_2})^m e^{-2t_1 m}}{n! m!} a^{(n+m)} e^{-(t_1+t_2)a^a} \rho e^{-(t_1+t_2)a^a} a^{n+m} \]

\[ = \sum_{k=0}^{\infty} \sum_{n=0}^{k} \frac{k!(1 - e^{-2t_1})^n (e^{-2t_1} - e^{-2(t_1+t_2)})^{k-n}}{n! (k-n)!} a^k e^{-(t_1+t_2)a^a} \rho e^{-(t_1+t_2)a^a} a^k \]

\[ = \sum_{k=0}^{\infty} \frac{(1 - e^{-2(t_1+t_2)})^k}{k!} a^k e^{-(t_1+t_2)a^a} \rho e^{-(t_1+t_2)a^a} a^k = S_{t_1+t_2}(\rho). \]

Derivation of (19):

Making use of (7) and (20), we see the action of the left handside of (19) on \(|m\).

\[ \sum_{n=0}^{\infty} E_n(t)\ d E_n(t) |m\rangle = \sum_{n=0}^{\infty} \frac{(1 - e^{-2t})^n}{n!} a^n a^\dagger e^{-t a^a} |k\rangle \]

\[ = \sum_{n=0}^{\infty} \frac{(n + m)!}{n! m!} (1 - e^{-2t})^n e^{-2t(m+1)} |m\rangle \]

\[ = \frac{e^{-2t(m+1)}}{1 - e^{-2t} m+1} |m\rangle = |m\rangle. \]

Here we have used the Maclaurin expansion formula

\[ \frac{1}{(1 - x)^{m+1}} = \sum_{n=0}^{\infty} \frac{(n + m)!}{n! m!} x^n \quad (|x| < 1) \]

in the fourth equality. Since \( \{ |m\rangle \}_{m \in \mathbb{N} : \{0\}} \) forms C.O.N.S. in \( \mathcal{F} \), we get (19).

For the semigroup property of \( e^{tD_{a a^a}} \), we put

\[ \tilde{S}_t(\rho) = \sum_{n=0}^{\infty} \tilde{E}_n(t) \rho \tilde{E}_n(t)^\dagger, \]

where

\[ \tilde{E}_n(t) = \frac{(e^{2t} - 1)^{n/2}}{\sqrt{n!}} a^n e^{-t a^a}. \]
Derivation of \( \frac{d}{dt} \tilde{S}_t = D_{aoa} \tilde{S}_t \):

\[
\frac{d}{dt} \tilde{S}_t(\rho) = \sum_{n=1}^{\infty} \frac{(e^{2t} - 1)^{n-1}}{(n-1)!} 2e^{2t} a^n e^{-ta^a} \rho e^{-ta^a} a^{\dagger n} \]
\[
+ \sum_{n=0}^{\infty} \frac{(e^{2t} - 1)^n}{n!} a^n \{ -a^\dagger a, e^{-ta^a} \rho e^{-ta^a} \} a^{\dagger n} \]
\[
= \sum_{n=1}^{\infty} \frac{(e^{2t} - 1)^{n-1}}{(n-1)!} 2e^{2t} a^n e^{-ta^a} \rho e^{-ta^a} a^{\dagger n} \]
\[
- \sum_{n=0}^{\infty} \frac{(e^{2t} - 1)^n}{n!} 2na^n e^{-ta^a} \rho e^{-ta^a} a^{\dagger n} \]
\[
+ \sum_{n=0}^{\infty} \frac{(e^{2t} - 1)^n}{n!} \{ -a^\dagger a, a^n e^{-ta^a} \rho e^{-ta^a} a^{\dagger n} \} \]
\[
= \sum_{n=1}^{\infty} \frac{(e^{2t} - 1)^{n-1}}{(n-1)!} (2e^{2t} - 2(e^{2t} - 1)) a^n e^{-ta^a} \rho e^{-ta^a} a^{\dagger n} \]
\[
+ \sum_{n=0}^{\infty} \frac{(e^{2t} - 1)^n}{n!} \{ -a^\dagger a, a^n e^{-ta^a} \rho e^{-ta^a} a^{\dagger n} \} \]
\[
= 2a \left( \sum_{n=0}^{\infty} \frac{(e^{2t} - 1)^n}{n!} a^n e^{-ta^a} \rho e^{-ta^a} a^{\dagger n} \right) a^{\dagger} \]
\[
- \{ a^\dagger a, \sum_{n=0}^{\infty} \frac{(e^{2t} - 1)^n}{n!} a^n e^{-ta^a} \rho e^{-ta^a} a^{\dagger n} \} \]
\[
= D_{aoa} \tilde{S}_t(\rho). \]

Derivation of \( \tilde{S}_{t_1} \tilde{S}_{t_2} = \tilde{S}_{t_1+t_2} \):

\[
\tilde{S}_{t_1} (\tilde{S}_{t_2}(\rho)) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(e^{2t_1} - 1)^n (e^{2t_2} - 1)^m}{n! m!} a^n e^{-t_1 a^a} a^m e^{-t_2 a^a} \rho e^{-t_2 a^a} a^{m} e^{-t_1 a^a} a^{\dagger n} \]
\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(e^{2t_1} - 1)^n (e^{2t_2} - 1)^m}{n! m!} a^n e^{-t_1 (a^a)^m} e^{-t_2 a^a} \rho e^{-t_2 a^a} e^{-t_1 (a^a)^m} a^{\dagger (n+m)} \]
\[
= \sum_{n=m=0}^{\infty} \frac{(e^{2t_1} - 1)^n (e^{2t_2} - 1)^m e^{2t_1 m}}{n! m!} a^{n+m} e^{-(t_1 + t_2) a^a} \rho e^{-(t_1 + t_2) a^a} a^{\dagger (n+m)} \]
\[
= \sum_{k=0}^{\infty} \sum_{n=0}^{k} \frac{k! (e^{2t_1} - 1)^n (e^{2(t_1+t_2)} - e^{2t_1})^k}{k! n!(k-n)!} a^k e^{-(t_1 + t_2) a^a} \rho e^{-(t_1 + t_2) a^a} a^{\dagger k} \]
\[
= \sum_{k=0}^{\infty} \frac{(e^{2(t_1+t_2)} - 1)^k}{k!} a^k e^{-(t_1 + t_2) a^a} \rho e^{-(t_1 + t_2) a^a} a^{\dagger k} = \tilde{S}_{t_1+t_2}(\rho). \]
Derivation of $\sum_n \tilde{E}_n \tilde{E}_n^\dagger = 1$ by applying it on each $|m\rangle$ ($m \in \mathbb{N} \cup \{0\}$).

$$\sum_{n=0}^{\infty} \tilde{E}_n(t)^\dagger \tilde{E}_n(t) |m\rangle = \sum_{n=0}^{\infty} \frac{(e^{2t} - 1)^n}{n!} e^{-ta^\dagger a} a^n a^\dagger e^{-ta^\dagger a} |m\rangle$$

$$= \sum_{n=0}^{m} \frac{(e^{2t} - 1)^n}{n!} \frac{m!}{(m-n)!} e^{-2tm} |m\rangle$$

$$= (1 + e^{2t} - 1)^m e^{-2tm} |m\rangle = |m\rangle .$$

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