POTENTIALLY NILPOTENT PATTERNS
AND THE NILPOTENT-JACOBIAN METHOD

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Abstract. A nonzero pattern is a matrix with entries in \{0, *\}. A pattern is potentially nilpotent if there is some nilpotent real matrix with nonzero entries in precisely the entries indicated by the pattern. We develop ways to construct some potentially nilpotent patterns, including some balanced tree patterns. We explore the index of some of the nilpotent matrices constructed, and observe that some of the balanced trees are spectrally arbitrary using the Nilpotent-Jacobian method. Inspired by an argument in [R. Pereira, Nilpotent matrices and spectrally arbitrary sign patterns. Electron. J. Linear Algebra 16 (2007), 232–236], we also uncover a feature of the Nilpotent-Jacobian method. In particular, we show that if \( N \) is the nilpotent matrix employed in this method to show that a pattern is spectrally arbitrary, then \( N \) must have full index.

1. Introduction and Terminology

A (zero-nonzero) pattern \( A \) is a square matrix whose entries come from the set \{*, 0\} where * denotes a nonzero entry. We then set 

\[ Q(A) = \{ A \in M_n(\mathbb{R}) \mid (A)_{i,j} \neq 0 \iff (A)_{i,j} = * \text{ for all } i, j \}. \]

An element \( A \in Q(A) \) is called a matrix realization of \( A \). A sign pattern \( A \) is a square matrix whose entries come from the set \{+, −, 0\}. While the results in this paper are written in terms of (zero-nonzero) patterns, each of the results also apply to sign patterns.

A pattern \( A \) is said to be potentially nilpotent if there exists a matrix \( A \in Q(A) \) such that \( A^k = 0 \) for some positive integer \( k \), that is, if there is a nilpotent matrix in \( Q(A) \). In Sections 2 and 3 we describe some new constructions of potentially nilpotent patterns. The need for constructions of potentially nilpotent patterns was described in [8]; the recent paper [2] summarizes much of the current knowledge about potentially nilpotent patterns.

If \( N \) is a nilpotent matrix, then the index of \( N \) is the smallest positive integer \( k \) such that \( N^k = 0 \). We say a pattern \( N \) allows index \( k \) if there is some nilpotent matrix \( N \in Q(N) \) with index \( k \). We say an \( n \times n \) pattern \( N \) allows full index if \( N \) allows index \( n \). Recent work [7] has focused on indices allowed by (sign) patterns.

A pattern \( A \) is spectrally arbitrary if any self-conjugate multiset of complex numbers is the spectrum of some \( A \in Q(A) \). Colloquially, \( A \) is spectrally arbitrary if \( A \) allows any set of eigenvalues.

A pattern \( A \) is symmetric if \( A^T = A \). Note that it is possible to have a nonsymmetric matrix \( A \in Q(A) \) when \( A \) is symmetric; \( A \in Q(A) \) is said to be combinatorially symmetric if \( A \) is symmetric. If a pattern \( A \) is symmetric, the graph of \( A \), denoted \( G(A) \), is the simple graph of the adjacency matrix obtained from \( A \) by replacing each * by 1. A pattern \( A \) is said to have a loop at vertex \( i \) if
Some of the constructions in this paper result in potentially nilpotent symmetric patterns whose graphs are trees. Tridiagonal patterns of order \( T_n \)

\[
T_n = \begin{bmatrix}
* & * & 0 & \cdots & 0 \\
* & 0 & \cdots & \cdots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \cdots & \cdots & \cdots & \vdots \\
0 & \cdots & 0 & * & *
\end{bmatrix},
\]

correspond to a path graph on \( n \) vertices with loops on each leaf. These patterns are known \([5, 6]\) to be spectrally arbitrary for \( 2 \leq n \leq 16 \), and are conjectured \([5]\) to be spectrally arbitrary (and hence potentially nilpotent) for all \( n \geq 2 \). Patterns of order \( n \) whose graph is a star with loops at each leaf,

\[
Z_n = \begin{bmatrix}
0 & * & \cdots & \cdots & * \\
* & 0 & \cdots & \cdots & 0 \\
\vdots & \cdots & \cdots & \cdots & \vdots \\
\vdots & \cdots & \cdots & \cdots & \vdots \\
* & 0 & \cdots & 0 & *
\end{bmatrix},
\]

are known \([11]\) to be spectrally arbitrary and hence potentially nilpotent for all \( n \geq 3 \). In Section 4 we describe some potentially nilpotent tree patterns which generalize the star patterns.

Determining if a pattern is potentially nilpotent is often a key step in determining if a pattern is spectrally arbitrary. The paper \([5]\) initiated the study of spectrally arbitrary patterns and described a method for determining that a pattern is spectrally arbitrary called the Nilpotent-Jacobian method (see Section 5 below). Finding an appropriate nilpotent matrix is the first step in the method and has been described as the Achilles’ heel of the Nilpotent-Jacobian method in \([3]\). Pereira \([12]\), using the Nilpotent-Jacobián method, demonstrated that certain patterns (with lots of nonzero entries) that allow full index were spectrally arbitrary. In Section 5 we demonstrate the necessity of the full index hypothesis for successfully employing the Nilpotent-Jacobián method on a spectrally arbitrary pattern (regardless of the quantity of nonzero entries):

**Theorem 5.8** Let \( A \) be a \( n \times n \) pattern. Suppose that one can show that \( A \) is spectrally arbitrary using the Nilpotent-Jacobián method. If \( N \) is the nilpotent matrix in \( Q(A) \) used in the Nilpotent-Jacobián method, then \( N \) has index \( n \).

### 2. A Construction for Potentially Nilpotent Patterns

In this section we introduce a way to construct new potentially nilpotent patterns based upon known potentially nilpotent patterns. Let \( A_m(N) \) be the \( n \times n \) pattern

\[
A_m(N) = \begin{bmatrix}
0 & \ell^T & \cdots & \ell^T \\
\ell & N & 0 & \cdots & 0 \\
\ell & 0 & \cdots & \cdots & \vdots \\
\ell & \vdots & \ddots & \cdots & 0 \\
\ell & 0 & \cdots & 0 & N
\end{bmatrix}
\]

where \( \ell = [0 \cdots 0] \), \( N \) is a \( s \times s \) pattern and \( n = ms + 1 \).
Let \( A_m(N_1, N_2, \ldots, N_m) \in \mathcal{Q}(A_m(N)) \) be the \( n \times n \) matrix

\[
A_m(N_1, N_2, \ldots, N_m) = \begin{bmatrix}
0 & e^T & \cdots & e^T \\
e & N_1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
e & 0 & \cdots & 0 & N_m
\end{bmatrix}
\]

where \( N_1, N_2, \ldots, N_m \) are \( s \times s \) nilpotent matrices, \( m \geq 2 \), \( e^T = [1 \ 0 \ \cdots \ 0] \), and \( a = (1 - m)e \). In the results developed in this paper, we explore nilpotent realizations \( A_m(N_1, \ldots, N_m) \in \mathcal{Q}(A_m(N)) \) with \( N_1 = N_2 = \cdots = N_m \). We will adopt the notation \( A_m(N) \) to denote the matrix \( A_m(N, N, \ldots, N) \). The patterns \( A_m(T_2) \) with \( m \geq 3 \) are called 2-generalized star patterns in [9].

While the next theorem follows from the more detailed result Theorem 2.3, we include a direct argument here. The patterns \( A_2(Z_3), A_2(Z_4) \) and \( A_2(Z_5) \) with graphs in Fig. 1 satisfy this theorem; in fact we will see in Section 5 that they are spectrally arbitrary patterns.

**Theorem 2.1.** If \( m \geq 2 \) and \( N \) is potentially nilpotent, then \( A_m(N) \) is potentially nilpotent.

**Proof.** Let \( N \in \mathcal{Q}(N) \) be a nilpotent matrix. Consider the matrix \( A = A_m(N) \in \mathcal{Q}(A_m(N)) \). Let \( N' \) be the matrix obtained from \( N \) by replacing the first row with \( e^T = [1 \ 0 \ \cdots \ 0] \). Then we can determine \( p_A = p_A(x) \), the characteristic polynomial of \( A \), by cofactor expansion along the first column. Note that for \( 0 \leq k \leq (m - 1) \), the \((ks + 2, 1)\) cofactor of \( xI - A \) is

\[
(-1)^{(ks+2)+1} \det(xI - A)_{ks+2,1}
\]

where the \((ks + 2, 1)\)-minor is

\[
\det(xI - A)_{ks+2,1} = (-1)^{ks} (p_{N'}) (p_N)^{m-1}.
\]

Thus

\[
p_A(x) = \det(xI - A) = x(p_N)^m + (-1)^{2+1}(-a)(p_{N'}) (p_N)^{m-1} + (m - 1)(p_{N'}) (p_N)^{m-1}
\]

\[
= x(p_N)^m \quad \text{since} \ a = (1 - m) \\
= x^n \quad \text{since} \ N \text{ is nilpotent}
\]

Thus \( A \) is nilpotent and so \( A_m \) is potentially nilpotent. \( \square \)

Note that if \( N \) is a potentially nilpotent pattern, then so is \( PN'PT \) for any permutation matrix \( P \), but \( A_m(PN'PT) \) is not permutationally equivalent to \( A_m(PN'PT) \) in general. Thus Theorem 2.1 provides a wide class of potentially nilpotent patterns.

We will next develop an observation about the index of the matrix \( A_m(N) \) when \( N \) is nilpotent.
Lemma 2.2. Suppose \( m \geq 2 \), and \( A = A_m(N) \). Then for all \( t \geq 1 \),

\[
A^t = \begin{bmatrix} 0 & f_t \\ g_t & B_t \end{bmatrix}
\]

where \( f_t = [e^T N^{t-1} \cdots e^T N^{t-1}] \),

\[
g_t = \begin{bmatrix} aN^{t-1}e \\ N^{t-1}e \\ \vdots \\ N^{t-1}e \end{bmatrix}, \quad B_t = (-m) \begin{bmatrix} W_t & \cdots & W_t \\ \vdots & \ddots & \vdots \\ W_t & \cdots & W_t \end{bmatrix} + \begin{bmatrix} N^t & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ddots & N^t \end{bmatrix},
\]

\( a = (1 - m) \), \( W_t = \sum_{i=0}^{t-2} N^i ee^T N^{(t-2)-i} \) for \( t \geq 2 \) and \( W_1 = 0 \).

Proof. We proceed by induction on \( t \). Observe that \( A = A^1 \) satisfies (3) since \( W_1 = 0 \). To minimize notation, we let \( L = ee^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \).

We observe that \( A^{t+1} = AA^t \) satisfies (3) because

\[
f_1 g_t = ae^T N^{t-1} e + (m-1)e^T N^{t-1} e = 0,
\]

\( f_1 B_t = f_{t+1}, B_1 g_t = g_{t+1} \) and

\[
g_1 f_t + B_1 B_t = \begin{bmatrix} aLN^{t-1} & \cdots & aLN^{t-1} \\ LN^{t-1} & \cdots & LN^{t-1} \\ \vdots & \ddots & \vdots \\ LN^{t-1} & \cdots & LN^{t-1} \end{bmatrix} + \begin{bmatrix} N & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ddots & N \end{bmatrix} = B_t = B_{t+1}
\]

since \( LN^{t-1} + NW_t = W_{t+1} \).

Theorem 2.3. Suppose \( m \geq 2 \), \( A = A_m(N) \), and \( N \) is nilpotent of index \( k \). Then \( A \) is nilpotent of index \( r \) for some \( r \leq 2k + 1 \). Further, if both the first row and column of \( N^{k-1} \) contain at least one nonzero entry, then \( A \) has index \( r = 2k + 1 \).

Proof. Let \( L = ee^T \). Since \( N^k = 0 \), it follows from Lemma 2.2 that \( A^{2k} = \begin{bmatrix} 0 & 0^T \\ 0 & B_{2k} \end{bmatrix} \) with

\[
B_{2k} = (-m) \begin{bmatrix} W_{2k} & \cdots & W_{2k} \\ \vdots & \ddots & \vdots \\ W_{2k} & \cdots & W_{2k} \end{bmatrix}.
\]

But \( W_{2k} = N^{k-1} LN^{k-1} \neq 0 \) if the first row and column of \( N^{k-1} \) both contain a nonzero entry. Hence \( A^{2k} \neq 0 \), but \( A^{2k+1} = 0 \) since \( W_{2k+1} = 0 \).

3. Another Construction for Potentially Nilpotent Patterns

We next describe a slight variation of the general construction given in Section 2 to give another construction which builds potentially nilpotent patterns from known potentially nilpotent patterns.
Let $N$ by an $s \times s$ pattern, and $C_m(N)$ be the $ms \times ms$ pattern

$$C_m(N) = \begin{bmatrix} N & L & \cdots & L \\ L & N & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ L & 0 & \cdots & 0 \\ & & & N \end{bmatrix}$$

where $L = \ell \ell^T$ and $\ell = \begin{bmatrix} * & 0 & \cdots & 0 \end{bmatrix}$. The pattern $C_2(\mathbb{Z}_s)$ is an example of a double star pattern. A recent paper [10] discusses the potential nilpotence of some double star (sign) patterns.

Let $N$ by an $s \times s$ matrix, $m \geq 3$ and $C_m(N) \in Q(C_m(N))$ be the $ms \times ms$ matrix

$$C_m(N) = \begin{bmatrix} N & L & \cdots & L \\ aL & N & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ L & 0 & \cdots & 0 \\ & & & N \end{bmatrix}$$

where $a = 2 - m$, and $L = ee^T$.

**Lemma 3.1.** Suppose $m \geq 3$, $R_2 = R_1 = 0$, $W_1 = W_0 = 0$, and $W_t = \sum_{i=0}^{t-2} N^i LN^{t-2-i}$ for $t \geq 2$.

If $C = C_m(N)$, then for all $t \geq 1$,

$$C^t = \begin{bmatrix} N^t & 0 \\ \cdots & \cdots \\ 0 & N^t \end{bmatrix} + \begin{bmatrix} 0 & W_{t+1} & \cdots & W_{t+1} \\ aW_{t+1} & aK_t & \cdots & aK_t \\ W_{t+1} & K_t & \cdots & K_t \\ \vdots & \vdots & \ddots & \vdots \\ W_{t+1} & K_t & \cdots & K_t \end{bmatrix}$$

where $K_t = LW_t + R_t$, and for $t \geq 3$,

$$R_t = \sum_{i=0}^{t-3} \sum_{j=0}^{t-3-i} N^{j+1} LN^{t-3-j-i} LN^i.$$

**Proof.** Let $C = C_m(N)$. We sketch a proof by induction on $t$. Note that $C = C^1$ has the form (4) since $K_1 = LW_1 + R_1 = 0$ and $W_2 = L$. To check that $C^{t+1}$ is of form (4) one can use the induction hypothesis with $C^{t+1} = CC^t$, noting that $W_{t+2} = NW_{t+1} + LN^t$ and $R_{t+1} = N(LW_t + R_t)$. □

**Theorem 3.2.** If $m \geq 3$ and $N$ is potentially nilpotent, then $C_m(N)$ is potentially nilpotent. In fact, if $N$ allows index $k$, then $C_m(N)$ allows index $r$ for some $r \leq 3k$.

**Proof.** Let $C = C_m(N) \in Q(C_m(N))$ for some matrix $N$ which is nilpotent of index $k$.

Note that each summand in $R_t$ (from line (5)) is of the form $N^a LN^b LN^c$ such that $a + b + c = t - 2$. If $t - 2 \geq 3k - 2$, then $a \geq k$ or $b \geq k$ or $c \geq k$. Hence $R_t = 0$ when $t \geq 3k$.

Since $W_t = 0$ for $t > 2k$, Lemma 3.1 implies that $C^{3k} = 0$. □
4. Potentially Nilpotent Balanced Tree Patterns

In this section we explore some examples of tree patterns which can be shown to be potentially nilpotent using the construction in Section 2. We call a tree \(T\) balanced if there exists a designated root \(r\) with \(\deg(r) \geq 2\) and \(\deg(x) = \deg(y)\) for all \(x, y \in V(T)\) with \(\text{dist}(x, r) = \text{dist}(y, r)\). In this paper, we call a pattern \(A\) a balanced tree pattern if \(G(A)\) is a balanced tree and a vertex of \(G(A)\) has a loop if and only if the vertex is a leaf of \(G(A)\). Note that \(T_n\) is a balanced tree pattern for \(n\) odd (by designating the middle vertex of the path as the root). Likewise, for \(n \geq 3\), the star patterns \(Z_n\) with a loop at each leaf and no loop at the root are examples of balanced tree patterns. The star pattern \(Z_n\) has been shown to be potentially nilpotent for \(n \geq 3\) (see [11, Theorem 3.1]). We demonstrate that other balanced tree patterns are also potentially nilpotent.

![Graphs of balanced trees: \(A_4(A_3(Z_3))\) and \(A_4(T_5^p)\)](image)

Note that the pattern of \(A_m(N)\) can be used to construct some balanced tree patterns recursively. We say that \(A\) is a recursive star pattern if \(A = A_m(N)\) and \(N\) is a recursive star pattern (with root at vertex 1), or \(A = Z_s\) is a star pattern with \(s \geq 3\). The graph of \(A_4(A_3(Z_3))\) in Figure 2 is a recursive star pattern, but \(A_4(T_5^p)\) is not a recursive star pattern where

\[
T_5^p = P T_5 P^T = \begin{bmatrix}
0 & * & 0 & * & 0 \\
* & 0 & * & 0 & 0 \\
0 & * & 0 & * & 0 \\
* & 0 & 0 & 0 & * \\
0 & 0 & 0 & * & * \\
\end{bmatrix}, \quad \text{with} \quad P = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}.
\]

The next result is a corollary of Theorem 2.11.

**Theorem 4.1.** If \(A\) is a recursive star pattern, then \(A\) is potentially nilpotent.

Theorem 4.1 gives a whole class of potentially nilpotent balanced trees but there are also other potentially nilpotent balanced trees. For example the balanced tree pattern \(A_4(T_5^p)\) (represented by the second graph in Figure 2) is a potentially nilpotent balanced tree by Theorem 2.1 since \(T_5^p\) is potentially nilpotent (\(T_5^p\) is permutationally equivalent to \(T_5\)).

**Theorem 4.2.** If the star pattern \(Z = Z_s\) allows index \(k\), then \(A_m(Z)\) allows index \(2k + 1\), and each row and column of \(Z^{k-1}\) has a nonzero entry.

**Proof.** Let \(N\) be a star pattern of order \(s\) and let \(N \in Q(N)\) have index \(k\). By Theorem 2.3 it is enough to show that each row and column of \(N^{k-1}\) contains a nonzero entry. Suppose that row 1 of \(N^{k-1}\) is \(0^T\). Since \(N\) has index \(k\), \(N^{k-1}_{ij} \neq 0\) for some row \(i \neq 1\) and some column \(j\). Then \(N_{ij}^k \neq 0\) since \(N_{ij}^k = (NN^{k-1})_{ij} = N_{ii} N_{ij}^{k-1} \neq 0\). This contradicts the fact that \(N_{ii} = 0\). Thus row 1 of \(N^{k-1}\) is not \(0^T\) and contains a nonzero entry.

Suppose that for some \(i, 2 \leq i \leq s\), row \(i\) of \(N^{k-1}\) is \(0^T\). Then since \(N^k = 0\) it follows that for each column \(j\), \(0 = N^k_{ij} = (NN^{k-1})_{ij} = N_{ii} N_{ij}^{k-1} + N_{ij} N_{ij}^{k-1} = N_{ij}^{k-1}\). That is, row 1 of \(N^{k-1}\) is \(0^T\). But this contradicts what was observed above. Thus every row of \(N^{k-1}\) contains a nonzero entry.

Since \(N\) is symmetric and \(N^k = N^{k-1} N\), a similar argument show that every column of \(N^{k-1}\) contains a nonzero entry. \(\square\)
The following corollary follows from Theorem 2.3 and Theorem 4.2.

**Corollary 4.3.** If the star pattern $Z_s$ allows full index $s$, then $A_2(Z_s)$ allows full index.

We will explore the significance of allowing full index in Section 5.

5. **Spectrally arbitrary patterns, full index, and the Nilpotent-Jacobian method.**

In this section we observe that it is necessary that a nilpotent matrix have full index if it is to be successfully employed in the Nilpotent-Jacobian method to demonstrate that a pattern is spectrally arbitrary. We will first describe the Nilpotent-Jacobian method and give some examples.

Let $A$ be an $n \times n$ sign pattern, and suppose that there exists a nilpotent matrix $N \in Q(A)$ with at least $n$ non-zero entries, say $a_{i_1,j_1}, \ldots, a_{i_n,j_n}$. Let $X = X(x_1, \ldots, x_n)$ denote the matrix obtained from $N$ with the $(i_k,j_k)$-th position replaced with the variable $x_k$ for $k = 1, \ldots, n$, and let

$$p_X(x) = x^n + f_1 x^{n-1} + f_2 x^{n-2} + \cdots + f_{n-1} x + f_n$$

be the characteristic polynomial of $X(x_1, \ldots, x_n)$, where each $f_i = f_i(x_1, \ldots, x_n)$ is a polynomial in $x_1, \ldots, x_n$. Let $J$ be the order $n \times n$ Jacobian matrix with

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}.$$  

Setting $x = (x_1, \ldots, x_n)$ and $a = (a_{i_1,j_1}, \ldots, a_{i_n,j_n})$, we let

$$J' = J|_{x=a}$$

denote the Jacobian matrix evaluated at $x = a$. The **Nilpotent-Jacobian method** is to seek a nilpotent realization of $A$ for which $J'$ is nonsingular: the following theorem, first developed in [5], shows that $A$ is spectrally arbitrary if such a realization is found.

Recall that $B$ is a superpattern of a pattern $A$ if $A_{i,j} \neq 0$ implies $B_{i,j} \neq 0$. Note that $A$ is a superpattern of itself.

**Theorem 5.1** ([5]). If $J'$ is nonsingular, then every superpattern of $A$ is spectrally arbitrary.

We apply the Nilpotent-Jacobian method to the balanced tree patterns $A_2(Z_3)$, $A_2(Z_4)$ and $A_2(Z_5)$ to demonstrate that these are spectrally arbitrary patterns. The examples will refer to the matrices $N \in Q(A_2(Z_s))$ of the form

$$N = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ x_1 & 0 & 1 & \cdots & 1 \\ 0 & x_2 & x_{s+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & x_s & x_{2s-1} & \cdots & 0 \\ y_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & y_{s-1} & y_{2s-3} \\ 0 & \cdots & \cdots & y_{s-1} & y_{2s-3} \end{bmatrix}.$$  

**Example 5.2.** Every superpattern of $A_2(Z_3)$ is spectrally arbitrary.
Proof. Take \( N = A_2(Z) \in Q(A_2(Z_3)) \) with

\[
Z = \begin{bmatrix}
0 & 1 & 1 \\
-1/2 & 1 & 0 \\
-1/2 & 0 & -1
\end{bmatrix}
\]

and \( x_1 = -y_1 = -1 \). Since \( Z \) is nilpotent, it follows from Theorem 2.3 and Corollary 4.2 that \( N \) is nilpotent. Let \( X \) be the matrix obtained from \( N \) by setting \( x_1, \ldots, x_7 \) as variables. One can check that the Jacobian matrix \( J' = J|_{(x_1, \ldots, x_7)} = (-1, -1/2, -1/2, -1/2, -1/2, -1/2, -1) \) has a nonzero determinant. Thus by Theorem 5.1 every superpattern of \( A_2(Z_3) \) is spectrally arbitrary.

Example 5.3. Every superpattern of \( A_2(Z_4) \) is spectrally arbitrary.

Proof. Take \( N = A_2(Z) \in Q(A_2(Z_4)) \) with

\[
Z = \begin{bmatrix}
0 & 1 & 1 & 1 \\
1/4 & 1 & 0 & 0 \\
-16/5 & 0 & 2 & 0 \\
-81/20 & 0 & 0 & -3
\end{bmatrix}
\]

and \( x_1 = -y_1 = -1 \). Since \( Z \) is nilpotent, it follows from Theorem 2.3 and Corollary 4.2 that \( N \) is nilpotent. Let \( X \) be obtained from \( N \) by taking \( x_1, \ldots, x_9 \) as variables. One can check that the Jacobian matrix \( J' = J|_{(x_1, \ldots, x_9)} = (-1, -1/4, -16/5, -81/20, 1, 2, -3, 1/4, 1) \) has nonzero determinant. Thus \( A_2(Z_4) \) is a spectrally arbitrary pattern.

Example 5.4. Every superpattern of \( A_2(Z_5) \) is spectrally arbitrary.

Proof. Take \( N = A_2(Z) \in Q(A_2(Z_5)) \) with

\[
Z = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 \\
-1/14 & 1 & 0 & 0 & 0 \\
4 & 0 & 2 & 0 & 0 \\
-27/2 & 0 & 0 & 3 & 0 \\
-108/7 & 0 & 0 & 0 & -6
\end{bmatrix}
\]

and \( x_1 = -y_1 = -1 \). Since \( Z \) is nilpotent, it follows from Theorem 2.3 and Corollary 4.2 that \( N \) is nilpotent. Let \( X \) be obtained from \( N \) by taking \( x_1, \ldots, x_{11} \) as variables. One can check that the Jacobian matrix \( J' = J|_{(x_1, \ldots, x_{11})} = (-1, -1/14, -27/2, -108/7, 1, 2, 3, -6, -1/14, 1) \) has nonzero determinant. Thus \( A_2(Z_5) \) is a spectrally arbitrary pattern.

In Examples 5.2, 5.3, 5.4 the matrix \( N \) that we constructed has full index. We then used this matrix in the Nilpotent-Jacobian method. If one examines other cases where the Nilpotent-Jacobian method is used (see for example the pattern \( W_n(k) \) in [1] and \( D_{n,r} \) in [4]), one will also find the initial nilpotent matrix has full index. As we will see in Theorem 5.8 this is not a coincidence, but a necessary condition about nilpotent matrices used in the Nilpotent-Jacobian method.

We first formalize a couple of observations of Pereira [12] proof of Theorem 2.2] about the entries of the \( \operatorname{adj}(xI - N) \). For \( 1 \leq i, j, \leq n \), we let

\[
p_{i,j}(x) = \left[ \operatorname{adj}(xI - N) \right]_{j,i}.
\]

Lemma 5.5. The \( k^{\text{th}} \) column of the Jacobian matrix \( J' \) described in line (6) consists of the coefficients of the polynomial \(-p_{i_k,j_k}(x)\) for all \( 1 \leq k \leq n \).

Proof. For \( 1 \leq k \leq n \), let

\[
p_{i_k,j_k}(x) = c_{k,0} + c_{k,1}x + \cdots + c_{k,n-1}x^{n-1}.
\]
Example 5.6. Consider the nilpotent matrix

\[ \begin{pmatrix} 0 & 1 & 1 \\ -1/2 & 1 & 0 \\ -1/2 & 0 & -1 \end{pmatrix} \]

with

\[ X = \begin{pmatrix} x_1 & x_2 & 0 \\ x_3 & 0 & -1 \end{pmatrix}. \]

Then \( p_X(x) = x^3 + (1 - x_2)x^2 + (-x_1 - x_2 - x_3)x + x_2x_3 - x_1 \) and

\[ J = \begin{pmatrix} 0 & -1 & 0 \\ -1 & -1 & -1 \\ -1 & x_3 & x_2 \end{pmatrix} \]

with

\[ J' = \begin{pmatrix} 0 & -1 & 0 \\ -1 & -1 & -1 \\ -1 & -1 \frac{1}{2} & 1 \end{pmatrix}. \]

On the other hand

\[ \text{adj}(xI - N) = \begin{pmatrix} x^2 - 1 & x + 1 & x - 1 \\ -x - \frac{1}{2} & x^2 + x + \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2}x + \frac{1}{2} & -\frac{1}{2} & x^2 - x + \frac{1}{2} \end{pmatrix}. \]

Hence \( p_{i_1,j_1}(x) = 0x^2 + x + 1, p_{i_2,j_2}(x) = x^2 + x + \frac{1}{2} \) and \( p_{i_3,j_3}(x) = 0x^2 + x - 1 \). The coefficients of these polynomials appear as the columns of \((-J')\).

The following lemma, formalizing another observation made in [12], is a corollary of Lemma 5.6.

Lemma 5.7. The Jacobian matrix \( J' \) described in line (i) is nonsingular if and only if the set of polynomials \( \{p_{i_1,j_1}(x), p_{i_2,j_2}(x), \ldots, p_{i_n,j_n}(x)\} \) is linearly independent.

The next result was inspired by a careful reading of a proof of Pereira, [12] Theorem 2.2, and in particular, trying to determine a converse of Pereira’s result.

Theorem 5.8. Let \( \mathcal{A} \) be a \( n \times n \) pattern. Suppose that one can show that \( \mathcal{A} \) is SAP using the Nilpotent-Jacobian method. If \( N \) is the nilpotent matrix in \( Q(\mathcal{A}) \) used in the Nilpotent-Jacobian method, then \( N \) has index \( n \).

Proof. By our hypotheses, we can prove that the pattern \( \mathcal{A} \) is SAP by using the Nilpotent-Jacobian method. Thus, by Theorem 5.3, we can find a nilpotent matrix \( N \in Q(\mathcal{A}) \) that has the required properties. We wish to show that the index of \( N \) is \( n \).

As in [12] Theorem 2.2, we consider the vector space

\[ V = \text{span}\{\text{adj}(xI - N) \mid x \neq 0\}. \]
If \( k \) is the index of \( N \), then arguing as in [12], we have \( \dim V = k \) since

\[
\text{span}\{\text{adj}(xI - N) \mid x \neq 0\} = \text{span}\{(xI - N)^{-1} \mid x \neq 0\} = \text{span}\{N^i\}_{i=0}^{k-1}.
\]

Our strategy is to show that \( k = n \) by first finding an \( n \)-dimensional vector space \( S \), and then showing that \( V = S \).

By Lemma 5.7 the polynomials \( \{p_{i_1,j_1}(x), \ldots, p_{i_n,j_n}(x)\} \) form a basis for \( P_{n-1} \), the set of polynomials of degree at most \( n - 1 \).

Let \( W_x = \text{adj}(xI - N) \). Since each entry of \( W_x \) is a polynomial of degree at most \( n - 1 \), each entry of \( W_x \) can be rewritten in terms of the basis \( \{p_{i_1,j_1}(x), \ldots, p_{i_n,j_n}(x)\} \). In other words, we can find matrices \( D_1, \ldots, D_n \) with entries in \( \mathbb{R} \) such that

\[
W_x = p_{i_1,j_1}(x)D_1 + \cdots + p_{i_n,j_n}(x)D_n.
\]

We claim that the matrices \( \{D_1, \ldots, D_n\} \) are linearly independent. In particular, the \((i_k,j_k)\)-th entry of \( W_x \) is \( p_{i_k,j_k}(x) \), hence the \((i_k,j_k)\)-th entry of \( D_k \) is 1, but for all \((i_l,j_l)\) with \( l \neq j \), we have \( (D_k)_{i_l,j_l} = 0 \). As a consequence, the matrices must be linearly independent.

Each polynomial \( p_{i_k,j_k}(x) \) can be written as

\[
p_{i_k,j_k}(x) = c_{k,0} + c_{k,1}x + \cdots + c_{k,n-1}x^{n-1}.
\]

Thus, if \( W_x = p_{i_1,j_1}(x)D_1 + \cdots + p_{i_n,j_n}(x)D_n \), we can rewrite \( W_x \) as

\[
W_x = x^{n-1}E_{n-1} + x^{n-2}E_{n-2} + \cdots + xE_1 + E_0
\]

where

\[
E_l = c_{1,l}D_1 + c_{2,l}D_2 + \cdots + c_{n,l}D_n \quad \text{for} \quad l = 0, \ldots, n - 1.
\]

This is simply a matter of expanding \( W_x \) and regrouping.

Let \( S = \text{span}\{E_0, \ldots, E_{n-1}\} \).

We now claim that the matrices \( \{E_0, \ldots, E_{n-1}\} \) are linearly independent. Suppose that \( a_0E_0 + \cdots + a_{n-1}E_{n-1} = 0 \) (where 0 denotes the zero matrix of size \( n \)). Note that this would imply that

\[
[b_1 \quad b_2 \quad \cdots \quad b_n] = -[a_{n-1} \quad a_{n-2} \quad \cdots \quad a_0] J'
\]

where, by Lemma \( 5.5 \), \( J' \) is the nonsingular Jacobian matrix. So \( a_0E_0 + \cdots + a_{n-1}E_{n-1} = 0 \) if and only if \( a_0 = \cdots = a_{n-1} = 0 \). Hence the matrices are linearly independent and \( \dim S = n \).

For each nonzero \( x \), we have shown that \( W_x = \text{adj}(xI - N) \) can be written has \( W_x = x^{n-1}E_{n-1} + \cdots + xE_1 + E_0 \in S \). Hence \( V = \text{span}\{\text{adj}(xI - N) \mid x \neq 0\} \subseteq S \).

We now show that \( S \subseteq V \). Take \( M \in S \). So, there exists constants \( r_1, \ldots, r_n \) such that

\[
M = r_0E_0 + \cdots + r_{n-1}E_{n-1}.
\]

We want to show that \( M \in V \), i.e., it is enough to show that we can find constants \( d_1, d_2, \ldots, d_n \) and nonzero constants \( z_1, z_2, \ldots, z_n \) such that

\[
M = d_1W_{z_1} + d_2W_{z_2} + \cdots + d_nW_{z_n} \quad \text{with} \quad W_{z_i} = \text{adj}(z_iI - N)
\]

As noted above, each matrix \( W_{z_i} \) can be written as

\[
W_{z_i} = z_i^{n-1}E_{n-1} + z_i^{n-2}E_{n-2} + \cdots + z_iE_1 + E_0.
\]
So, if we expand out (9) and then compare to (8), we need to be able to solve for constants
\( d_1, d_2, \ldots, d_n \) and nonzero constants \( z_1, z_2, \ldots, z_n \) in the system
\[
\begin{align*}
    d_1 z_1^{n-1} + d_2 z_2^{n-1} + \cdots + d_n z_n^{n-1} &= r_{n-1} \\
    d_1 z_1^{n-2} + d_2 z_2^{n-2} + \cdots + d_n z_n^{n-2} &= r_{n-2} \\
    \vdots \\
    d_1 z_1 + d_2 z_2 + \cdots + d_n z_n &= r_1 \\
    d_1 + d_2 + \cdots + d_n &= r_0.
\end{align*}
\]

But this system of equations is a Vandermonde system of equations, i.e.,
\[
\begin{bmatrix}
    z_1^{n-1} & z_2^{n-1} & \cdots & z_n^{n-1} \\
    z_1^{n-2} & z_2^{n-2} & \cdots & z_n^{n-2} \\
    \vdots & \vdots & \ddots & \vdots \\
    z_1 & z_2 & \cdots & z_n \\
    1 & 1 & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
    d_1 \\
    d_2 \\
    \vdots \\
    d_{n-1} \\
    d_n
\end{bmatrix}
= \begin{bmatrix}
    r_{n-1} \\
    r_{n-2} \\
    \vdots \\
    r_1 \\
    r_0
\end{bmatrix}.
\]

Because the determinant of the Vandermonde matrix is given by
\[
\prod_{1 \leq i < j \leq n} (z_j - z_i),
\]
the system can be solved for \( d_1, d_2, \ldots, d_n \) given any distinct choices of \( z_1, \ldots, z_n \). Therefore \( M \subseteq V \) and \( S \subseteq V \).

We have thus shown that the vector spaces \( S \) and \( V \) are the same, and \( k = \dim V = \dim S = n \).
Therefore the index of the nilpotent matrix \( N \) is \( n \).

Part of our interest in this result is the following corollary:

**Corollary 5.9.** Let \( A \) be an \( n \times n \) sign pattern. If \( Q(A) \) has no nilpotent matrix of index \( n \), then the Nilpotent-Jacobian method cannot be used to prove that \( A \) is SAP.

We have shown that it is a necessary condition that an \( n \times n \) pattern \( A \) allow a nilpotent matrix of index \( n \) in order for the Nilpotent-Jacobian method to successfully demonstrate that \( A \) is spectrally arbitrary. Pereira [12, Theorem 2.2] showed that if a pattern \( A \) has at most \( n - 2 \) zero entries, all of which are on the diagonal, then it is sufficient to check if \( N \) allows full index to demonstrate a pattern is spectrally arbitrary. As noted already by Pereira [12, Example 1.1], allowing full index is not sufficient for a pattern to be spectrally arbitrary.

**Corollary 5.10.** The pattern \( A_2(Z_s) \) allows full index for all \( s \geq 3 \).

**Proof.** Let \( s \geq 3 \). We note that the Nilpotent-Jacobian method was employed in [11, Theorem 4.4] to show that superpatterns of certain star patterns (including \( Z_s \)) were spectrally arbitrary. Hence by Theorem 4.8, \( Z_s \) allows full index. The result now follows from Corollary 4.3.

6. **Concluding Comments and Open Problems**

We have shown that the pattern \( A_2(Z_s) \) is spectrally arbitrary for all \( s = 3, 4, \) and \( 5 \). We expect that the pattern \( A_2(Z_s) \) is spectrally arbitrary for all \( s > 5 \), but we leave this as an open question. Given \( s \geq 3 \), we observed that \( A_m(Z_s) \) is potentially nilpotent for \( m \geq 2 \) and that \( A_m(Z_s) \) allows full index for \( m = 2 \). It would be interesting to determine if such patterns can allow full index when \( m > 2 \) and if any such pattern is spectrally arbitrary.

In this paper we have demonstrated that some classes of tree patterns (as well as some other constructions) are potentially nilpotent. It would be interesting to eventually classify potentially nilpotent
tree patterns, or even potentially nilpotent balanced tree patterns. This may be a difficult problem, since even determining potential nilpotence for the tridiagonal pattern $T_n$ is still open for $n > 16$.

We have noted that a couple of the balanced tree patterns are spectrally arbitrary. Much work could be done to develop more tools to determine what makes a balanced tree pattern spectrally arbitrary. Such tools might shed some light on the tridiagonal pattern $T_n$, to determine if $T_n$ is spectrally arbitrary for $n > 16$.

It is an open question (see, for example [13]) whether every irreducible pattern that is spectrally arbitrary can be shown to be spectrally arbitrary using the Nilpotent-Jacobian method. Theorem 5.8 suggests that one way to answer the question in the negative is to find a spectrally arbitrary pattern which does not allow a nilpotent matrix of full index. On the other hand, one might ask if every spectrally arbitrary pattern allows a nilpotent matrix of full index.

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