HODGE NUMBERS OF GENERALISED BORCEA-VOISIN THREEFOLDS

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Abstract. We shall reproof formulas for the Hodge numbers of Calabi-Yau threefolds of Borcea-Voisin type constructed by A. Cattaneo and A. Garbagnati, using the orbifold cohomology formula and the orbifold Euler characteristic.

1. Introduction

One of the many reasons behind the interest in non-symplectic automorphisms of K3 surfaces is the mirror symmetry construction of C. Borcea ([4]) and C. Voisin ([14]). They independently constructed a family of Calabi-Yau threefolds using a non-symplectic involutions of K3 surfaces and elliptic curves. Moreover C. Voisin gave a construction of explicit mirror maps.

1.1. Theorem ([4] and [14]). Let $E$ be an elliptic curve with an involution $\alpha_E$ which does not preserve $\omega_E$. Let $S$ be a K3 surface with a non-symplectic involution $\alpha_S$. Then any crepant resolution of the variety $(E \times S)/(\alpha_E \times \alpha_S)$ is a Calabi-Yau manifold with

$$h^{1,1} = 11 + 5N' - N \quad \text{and} \quad h^{2,1} = 11 + 5N - N',$$

where $N$ is a number of curves in $S^{\alpha_S}$ and $N'$ is a sum of their genera.

In [6] A. Cattaneo and A. Garbagnati generalised Borcea-Voisin construction using purely non-symplectic automorphisms of order 3, 4 and 6. They obtained the following theorem

1.2. Theorem ([6]). Let $S$ be a K3 surface admitting a purely non-symplectic automorphism $\alpha_S$ of order $n$. Let $E$ be an elliptic curve admitting an automorphism $\alpha_E$ such that $\alpha_E(\omega_E) = \zeta_n \omega_E$. Then $n \in \{2, 3, 4, 6\}$ and $(S \times E)/(\alpha_S \times \alpha_E^{n-1})$ is a singular variety which admits a crepant resolution which is a Calabi-Yau manifold.

Any crepant resolution of $(S \times E)/(\alpha_S \times \alpha_E^{n-1})$ will be called a Calabi-Yau 3-fold of Borcea-Voisin type. For all possible orders they computed the Hodge numbers of this varieties and constructed an elliptic fibrations on them. Their computations are more technical and rely on a detailed study of a crepant resolutions of threefolds.

In this paper we give shorter computations of the Hodge numbers using orbifold cohomology introducted by W. Chen and Y. Ruan in [7] and orbifold Euler characteristic (cf. [12]). The main advantage of our approach is that the computations are carried out on $S \times E$.

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2. Preliminaries

A Calabi-Yau manifold $X$ is a complex, smooth, projective $d$-fold $X$ satisfying

1. $K_X = \mathcal{O}_X$,
2. $H^i\mathcal{O}_X = 0$ for $0 < i < d$.

A non-trivial generator $\omega_X$ of $H^{4,0}(X) \cong \mathbb{C}$ is called a period of $X$. For any automorphism $\alpha_X \in \text{Aut}(X)$, the induced mapping $\alpha_X^*$ acts on $H^{4,0}(X)$ and $\alpha_X^*(\omega_X) = \lambda \omega_X$, for some $\lambda \in \mathbb{C}^*$.

If additionally $\alpha_X$ does not preserve a period then it is called symplectic. If $\alpha_S$ does not preserve a period then it is called non-symplectic. If additionally $\alpha_S$ is of finite order $n$ and $\alpha_S^*(\omega_S) = \zeta_n \omega_S$, where $\zeta_n$ is a primitive $n$-th root of unity, then it is called purely non-symplectic.

We have the following characterisation of orders of purely non-symplectic automorphisms of elliptic curves and $K3$ surfaces.

2.1. Theorem ([13]). Let $E$ be an elliptic curve. If $\alpha_E$ is an automorphism of $E$ which does not preserve the period, then $\alpha_E(\omega_E) = \zeta_n \omega_E$ where $n = 2, 3, 4, 6$.

2.2. Theorem ([11]). Let $S$ be a $K3$ surface and $\alpha_S$ be a purely non-symplectic automorphism of order $n$. Then $n \leq 66$ and if $n = p$ is a prime number, then $p \leq 19$.

Let $\sigma$ be a purely non-symplectic automorphism of order $p$ of a $K3$ surface $S$. We will denote by $S^\sigma$ the set of fixed points of $\sigma$. The action of $\sigma$ may be locally linearized and diagonalized at $p \in S^\sigma$ (cf. Cartan [5]), so the possible local actions are
\[
\left(\begin{array}{cc}
p^t & 0 \\
0 & \zeta_{p-t}
\end{array}\right), \quad \text{for } t = 0, 1, \ldots, p-2.
\]

Clearly, if $t = 0$, then $p$ belongs to a smooth curve fixed by $\sigma$, otherwise $p$ is an isolated point. We have the following description of the fixed locus of $\sigma$.

2.3. Theorem ([3], Lemma 2.2, p. 5). Let $S$ be a $K3$ surface and let $\alpha_S$ be a non-symplectic automorphism of $S$ of order $n$. Then there are three possibilities

- $S^{\alpha_S} = \emptyset$; in this case $n = 2$,
- $S^{\alpha_S} = E_1 \cup E_2$, where $E_1$, $E_2$ are disjoint smooth elliptic curves; in this case $n = 2$,
- $S^{\alpha_S} = C \cup R_1 \cup R_2 \cup \ldots \cup R_{k-1} \cup \{p_1, p_2, \ldots, p_k\}$, where $p_i$ are isolated fixed points, $R_i$ are smooth rational curves and $C$ is the curve with highest genus $g(C)$.

We refer to [3] for a proof and more precise description of the fixed locus for particular values of $n$.

3. Orbifold’s cohomology

In [7] W. Chen and Y. Ruan introduced a new cohomology theory for orbifolds. We consider varieties $X/G$, where $X$ is a projective variety and $G$ is a finite group acting on $X$ viewed as orbifold.

3.1. Definition. For $G \in GL_n(\mathbb{C})$ of order $m$, let $e^{2\pi i a_1}, e^{2\pi i a_2}, \ldots, e^{2\pi i a_n}$ be eigenvalues of $G$ for some $a_1, a_2, \ldots, a_n \in [0, 1) \cap \mathbb{Q}$. The value of the sum $a_1 + a_2 + \ldots + a_n$ is called the age of $G$ and is denoted by $\text{age}(G)$.

The age of $G$ is an integer if and only if $\det G = 1$ i.e. $G \in SL_n(\mathbb{C})$. 

3.2. Definition. For a variety \( X/G \) define the Chen-Ruan cohomology by

\[
H_{\text{orb}}^{i,j}(X/G) := \bigoplus_{[g] \in \text{Conj}(G)} \bigoplus_{U \in \Lambda(g)} H^{i-\text{age}(g), j-\text{age}(g)}(U),
\]

where \( \text{Conj}(G) \) is the set of conjugacy classes of \( G \) (we choose a representative \( g \) of each conjugacy class), \( C(g) \) is the centralizer of \( g \), \( \Lambda(g) \) denotes the set of irreducible connected components of the set fixed by \( g \in G \) and \( \text{age}(g) \) is the age of the matrix of linearized action of \( g \) near a point of \( U \).

The dimension of \( H_{\text{orb}}^{i,j}(X/G) \) will be denoted by \( h_{\text{orb}}^{i,j}(X/G) \).

3.3. Remark. If the group \( G \) is cyclic of a prime order \( p \), then we can pick a generator \( \alpha \) and the above formula simplifies to

\[
H_{\text{orb}}^{i,j}(X/G) = H^{i,j}(X) \oplus \bigoplus_{U \in \Lambda(\alpha)} H^{i-\text{age}(\alpha^k), j-\text{age}(\alpha^k)}(U).
\]

We have the following theorem.

3.4. Theorem ([15], Theorem 1.1, p. 2). Let \( G \) be a finite group acting on an algebraic smooth variety \( X \). If there exists a crepant resolution \( \tilde{X}/G \) of variety \( X/G \), then the following equality holds

\[
h^{i,j}(\tilde{X}/G) = h_{\text{orb}}^{i,j}(X/G).
\]

4. Orbifold Euler characteristic

Let \( G \) be a finite group acting on an algebraic variety \( X \). In a similar manner as in the case of Hodge numbers, we can use an orbifold formula to compute the Euler characteristic of a crepant resolution of \( X/G \) (for details see [12]).

4.1. Definition. The orbifold Euler characteristic of \( X/G \) is defined as

\[
e_{\text{orb}}(X/G) := \frac{1}{\#G} \sum_{(g,h) \in G \times G, \, gh = hg} e(X^g \cap X^h).
\]

4.2. Theorem ([12], Theorem 2, p. 534). For any finite abelian group \( G \) acting on smooth algebraic variety \( X \). If there exists a crepant resolution \( \tilde{X}/G \) of variety \( X/G \), then the following equality holds

\[
e(\tilde{X}/G) = e_{\text{orb}}(X/G).
\]

5. Computations of Hodge numbers

5.1. Order 2. Let \((S, \alpha_S)\) be a K3 surface admitting a non-symplectic involution \( \alpha_S \). Consider an elliptic curve \( E \) with non-symplectic involution \( \alpha_E \) (any elliptic curve \( E \) admits such an automorphism). Let us denote by \( H^2(S, \mathbb{C})^{\alpha_S} \) the invariant part of cohomology \( H^2(S, \mathbb{C}) \) under \( \alpha_S \) and by \( r \) the dimension \( r = \dim H^2(S, \mathbb{C})^{\alpha_S} \).

We also denote the eigenspace for \(-1\) of the induced action \( \alpha_E^* \) on \( H^1(S, \mathbb{C}) \) by \( H^2(S, \mathbb{C})_{-1} \) and by \( m \) the dimension \( m = \dim H^2(S, \mathbb{C})_{-1} \).

We see that

\[
H^2(S, \mathbb{C}) = H^2(S, \mathbb{C})^{\alpha_S} \oplus H^2(S, \mathbb{C})_{-1},
\]

hence the Hodge diamonds of the respective eigenspaces have the following forms

The Hodge diamonds of eigenspaces of the induced action of \( \alpha_E^* \) on \( H^1(E, \mathbb{C}) \) have forms
By Künneth’s formula the Hodge diamond of $H^3(S \times E, \mathbb{C})^{C_2}$ is given by

| $H^{i,j}(S, \mathbb{C})^\alpha_S$ | $H^{i,j}(S, \mathbb{C})_{-1}$ |
|----------------------------------|-----------------------------|
| 1                               | 0                           |
| 0                               | 0                           |
| $r$                             | $m-2$                       |
| 0                               | 0                           |
| 1                               | 0                           |

| $H^{i,j}(E, \mathbb{C})^\alpha_E$ | $H^{i,j}(E, \mathbb{C})_{-1}$ |
|----------------------------------|-----------------------------|
| 1                               | 0                           |
| 0                               | 0                           |
| $m-1$                           | $m-1$                       |
| 0                               | 0                           |
| 1                               | 0                           |

The local action of $-1$ on curve may be linearized to matrix

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

with age equal to 1. Thus by $4.2$

\[
h^{1,1}(S \times \widetilde{E}/C_2) = h^{1,1}_\text{orb}(S \times E/C_2) = h^{1,1}(S \times E)^{C_2} \oplus \bigoplus_{U \in \Lambda(-1)} h^{i-1,j-1}(U),
\]

which gives formulas

\[
h^{1,1}(S \times \widetilde{E}/C_2) = r + 1 + 4N \quad \text{and} \quad h^{2,1}(S \times \widetilde{E}/C_2) = m - 1 + 4N'.
\]

Since the quotient $S/\alpha_S$ is a smooth surface with Euler characteristic

\[
e(S/\alpha_S) = 12 + N - N',
\]

we recover formulas from Thm. 1.1 of [6].

5.2. **Order 3.** Let $(S, \alpha_S)$ be a $K3$ surface admitting a purely non-symplectic automorphism $\alpha_S$ of order 3. Eigenvalues of induced mapping $\alpha_S^* \in H^2(S, \mathbb{C})$ belong to $\{1, \zeta_3^2 \}$. Let us denote by $H^2(S, \mathbb{C})_{\zeta_3^2}$ the eigenspace of the eigenvalue $\zeta_3^2$. For $i = 1,2$ the dimension of $H^2(S, \mathbb{C})_{\zeta_3^2}$ does not depend on $i$ and will be denoted by $m$. Moreover let $r$ be a dimension of $H^2(S, \mathbb{C})^\alpha_S$ — invariant part of $H^2(S, \mathbb{C})$ under $\alpha_S$.

Consider an elliptic curve $E$ with the Weierstrass equation $y^2 = x^3 + 1$ together with a non-symplectic automorphism $\alpha_E$ of order 3 such that $\alpha_E(x, y) = (\zeta_3 x, y)$.

We see that

\[
H^2(S, \mathbb{C}) = H^2(S, \mathbb{C})^\alpha_S \oplus H^2(S, \mathbb{C})_{\zeta_3^2} \oplus H^2(S, \mathbb{C})_{\zeta_3^2}.
\]
Because \( \alpha^*_{S|H^2}([\omega]) = \zeta_3[\omega] \) for any \([\omega] \in H^{2,0}(S), \) we get \( H^{2,0}(S) \subset H(S,\mathbb{C})_{\zeta_3} \). The complex conjugation yields \( H^{0,2}(S) \subset H^2(S,\mathbb{C})_{\zeta_3^2} \). Finally

\[ \alpha^*_{E}([\omega_S \wedge \omega_E]) = \bar{\zeta}_3 \zeta_3 [\omega_S \wedge \omega_E] = [\omega_S \wedge \omega_E], \]

hence the Hodge diamonds of the respective eigenspaces have the following forms

\[
\begin{array}{ccc}
H^{i,j}(S,\mathbb{C})_{\alpha S} & H^{i,j}(S,\mathbb{C})_{\zeta_3} & H^{i,j}(S,\mathbb{C})_{\zeta_3^2} \\
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & r & 0 \\
0 & 0 & 1 \\
1 & 0 & 0 \\
\end{array}
\]

Similar analysis gives the Hodge diamonds of eigenspaces of action on the Hodge groups.

\[
\begin{array}{ccc}
H^{i,j}(E,\mathbb{C})_{\alpha^2 E} & H^{i,j}(E)_{\zeta_3} & H^{i,j}(E)_{\zeta_3^2} \\
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0 \\
\end{array}
\]

By Künneth’s formula the Hodge diamond of the invariant part of \( H^3(S \times E,\mathbb{C}) \) has the same form as in the case of order 2.

We denote the automorphism \( \alpha_S \times \alpha^2_E \) by \( \alpha \). Let us now consider possible actions of elements of \( \langle \alpha \rangle \simeq \mathbb{C}_3 \) on \( S \) and \( E \).

The action of \( \alpha \). The action of the automorphism \( \alpha \) on \( E \) is given by

\[ E \ni (x,y) \mapsto (\zeta^2 x, y) \in E, \]

hence it has three fixed points. Locally the action of \( \alpha \) on components of the fixed locus can be diagonalised to

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & \zeta_3 & 0 \\
0 & 0 & \zeta_3^2
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
\zeta_3^2 & 0 & 0 \\
0 & \zeta_3^2 & 0 \\
0 & 0 & \zeta_3^2
\end{pmatrix}.
\]

It follows that ages are equal to 1 and 2 respectively.

The action of \( \alpha^2 \). Analogously we get possible diagonalised matrices

\[
\begin{pmatrix}
\zeta_3 & 0 & 0 \\
0 & \zeta_3 & 0 \\
0 & 0 & \zeta_3
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & 0 & 0 \\
0 & \zeta_3^2 & 0 \\
0 & 0 & \zeta_3
\end{pmatrix},
\]

with ages 1 and 1.

Thus decomposing \( (S \times E)^\alpha = \mathcal{C} \cup \mathcal{R} \cup \mathcal{P} \), where

\[
\mathcal{C} := \{3 \text{ curves with highest genus } g(C)\},
\mathcal{R} := \{3k - 3 \text{ rational curves}\},
\mathcal{P} := \{3n \text{ isolated points}\},
\]
the orbifold formula implies that

\[ H_{\text{orb}}^{i,j}(S \times E/C_3) = H^{i,j}(S \times E)^{C_3} \oplus \bigoplus_{U \in \Lambda(C_3)} \bigoplus_{i=1}^{2} H^{i-\text{age}(\alpha'), j-\text{age}(\alpha')}(U) = \]

\[ = H^{i,j}(S \times E)^{C_3} \oplus \left( \bigoplus_{U \in \mathcal{E}} H^{i-1, j-1}(U) \oplus H^{i-1, j-1}(U) \right) \oplus \]

\[ \oplus \left( \bigoplus_{U \in \mathcal{R}} H^{i-1, j-1}(U) \oplus H^{i-1, j-1}(U) \right) \oplus \left( \bigoplus_{U \in \mathcal{P}} H^{i-1, j-1}(U) \right). \]

Therefore by (3.4)

\[ h^{1,1}(X/C_3) = r + 1 + 6 \cdot 1 + 2 \cdot (3k - 3) \cdot 1 + 3n \cdot 1 = r + 1 + 3n + 6k. \]

\[ h^{1,2}(X/C_3) = m - 1 + 2 \cdot 3 \cdot g(C) + (3k - 3) \cdot 2 \cdot 0 + 3n \cdot 0 = m - 1 + 6g(C). \]

Hence we proved the following theorem:

5.1. Theorem. If \( S^{\alpha_S} \) consists of \( k \) curves together with a curve with highest genus \( g(C) \) and \( n \) isolated points, then for any crepant resolution of the variety \( (S \times E)/(\alpha_S \times \alpha_S^2) \) the following holds

\[ h^{1,1} = r + 1 + 3n + 6k \quad \text{and} \quad h^{2,1} = m - 1 + 6g(C). \]

5.3. Order 4. Let \((S, \alpha_S)\) be a K3 surface with purely non-symplectic automorphism \( \alpha_S \) of order 4. Consider an elliptic curve \( E \) with the Weierstrass equation \( y^2 = x^3 + x \) together with a non-symplectic automorphism \( \alpha_E \) of order 4 such that

\[ \alpha_E(x, y) = (-x, iy). \]

Additionally, suppose that \( S^{\alpha_S^2} \) is not a union of two elliptic curves.

We shall keep the notation of [9],

\[ X = S \times E, \]

\[ P - \text{the identity point of } E, \]

\[ r = \dim H^2(S, \mathbb{C})^{\alpha_S}, \]

\[ m = \dim H^2(S, \mathbb{C})_{\alpha_S^i} \text{ for } i \in \{1, 2, \ldots, 5\}, \]

\[ N - \text{number of curves which are fixed by } \alpha_S^2, \]

\[ k - \text{number of curves which are fixed by } \alpha_S(g) \text{ (curves of the first type)}, \]

\[ b - \text{number of curves which are fixed by } \alpha_S^2 \text{ and are invariant by } \alpha_S \text{ (curves of the second type)}, \]

\[ a - \text{number of pairs } (A, A') \text{ of curves which are fixed by } \alpha_S^2 \text{ and } \alpha_S(A) = A' \text{ (curves of the third type)}, \]

\[ D - \text{the curve of the highest genus in } S^{\alpha_S^3}, \]

\[ n_1 - \text{number of curves which are fixed by } \alpha_S \text{ not laying on the curve } D, \]

\[ n_2 - \text{number of curves which are fixed by } \alpha_S \text{ laying on the curve } D. \]

For the same reasons as in the previous cases

\[ h_{\text{orb}}^{1,1}(X)^{C_4} = r + 1 \quad \text{and} \quad h_{\text{orb}}^{2,1}(X)^{C_4} = m - 1. \]

We denote an automorphism \( \alpha_S \times \alpha_S^2 \) by \( \alpha \). Furthermore for any \( g \in C_4 \) let

\[ M_g := \bigoplus_{U \in \Lambda(g)} H^{1-\text{age}(g), 1-\text{age}(g)}(U). \]

We will consider all possible cases.

The action of \( \alpha \) and \( \alpha^3 \). The action of automorphism \( \alpha \) on \( E \) is given by

\[ E \ni (x, y) \mapsto (-x, -iy) \in E, \]
hence it has two fixed points — $P$ and $(0,0)$. The fixed locus of $\alpha$ on $S$ consists of $k$ curves and $n_1 + n_2$ isolated points. Since locally the action of $\alpha$ on $X$ along the curve can be diagonalized to

$$\begin{pmatrix}
1 & 0 & 0 \\
0 & i & 0 \\
0 & 0 & -i
\end{pmatrix},$$

we infer that its age equals 1. Near a fixed point we have a matrix

$$\begin{pmatrix}
-1 & 0 & 0 \\
0 & -i & 0 \\
0 & 0 & -i
\end{pmatrix},$$

hence the age equals 2.

In case of the action of $\alpha^3$ we observe that the fixed locus consists of $k$ curves and $n_1 + n_2$ points with ages 1. We see that the summand of $h_{\text{orb}}^{1,1}$ from both actions is equal to $2k + 2k + 2(n_1 + n_2) = 4k + 2(n_1 + n_2)$.

| Element of $C_4$ | $\alpha$            | $\alpha^3$            |
|------------------|---------------------|-----------------------|
| Irreducible components | $2k$ curves, $2n_1 + 2n_2$ points | $2k$ curves, $2n_1 + 2n_2$ |
| The age          | curve: 1, point: 2  | curve: 1, point: 1    |
| Summand of $h_{\text{orb}}^{1,1}$ | $2k$                          | $2k + 2(n_1 + n_2)$    |

The action of $\alpha^2$. The automorphism $\alpha^2$ acts on $E$ as

$$E \ni (x, y) \mapsto (x, -y) \in E,$$

hence it has four fixed points — $P$, $(0,0)$, $(i,0)$, $(-i,0)$ from which only two are invariant under the action of $\alpha$ and the other two are permuted. After identifying $M_{\zeta^2}$ with the vector space spanned by irreducible components we will find the action of induced map $\alpha^*$ on it.

Because the matrix of the action of $\alpha^*_E$ on $M_{\zeta^2}$ is

$$\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix} \sim \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix},$$

it follows that it has 3-dimensional eigenspace for $+1$ and 1-dimensional eigenspace for $-1$.

The fixed locus of $\alpha_s^2$ consists of $N$ curves with $a$ pairs permuted by $\alpha_s$. Hence $\alpha_s^2$ on $M_{\zeta^2}$ has $(N-a)$-dimensional eigenspace for $+1$ and $a$-dimensional eigenspace for $-1$. One can see that $N = k + b + 2a$, so the total effect on $h_{\text{orb}}^{1,1}$ equals $3(N-a) + a = 3k + 3b + 4a$.

| Element of $C_4$ | $\alpha^2$            |
|------------------|-----------------------|
| Irreducible components | $4N$ curves          |
| The age          | curves: 1             |
| Summand of $h_{\text{orb}}^{1,1}$ | $3k + 3b + 4a$       |

From the orbifold formula follows that

$$h_{\text{orb}}^{1,1} = r + 1 + 4k + 2(n_1 + n_2) + 3k + 3b + 4a = 1 + r + 7k + 3b + 2(n_1 + n_2) + 4a.$$
In order to compute $h_{\text{orb}}^{1,2}$ we will use the orbifold Euler characteristic. In the table below we collect all possible intersections $X^g \cap X^h$, where $(g, h) \in C_4^2$.

| $\{g\}$ | $1$ | $\zeta_4$ | $\zeta_4^2$ | $\zeta_4^3$ |
|-----------|-----|-----------|-----------|-----------|
| $1$       | $X$  | $X^{\zeta_4}$ | $X^{\zeta_4^2}$ | $X^{\zeta_4^3}$ |
| $\zeta_4$ | $X^{\zeta_4}$ | $X^{\zeta_4^2}$ | $X^{\zeta_4^3}$ | $X^{\zeta_4}$ |
| $\zeta_4^2$ | $X^{\zeta_4^2}$ | $X^{\zeta_4^3}$ | $X^{\zeta_4^2}$ | $X^{\zeta_4}$ |
| $\zeta_4^3$ | $X^{\zeta_4^3}$ | $X^{\zeta_4}$ | $X^{\zeta_4^3}$ | $X^{\zeta_4}$ |

By [4,2] we obtain the formula

$$e(X/C_4) = c_{\text{orb}}(X/C_4) = \frac{1}{4} \left( 12e(X^{\zeta_4}) + 3e(X^{\zeta_4^2}) \right) = 6e(S^{\zeta_4}) + 3e(S^{\zeta_4^2}).$$

If $D$ is of the first type, then by the Riemann-Hurwitz formula we conclude that

$$e(S^{\zeta_4}) = 2 - 2g(D) + 2(k - 1) + n_1 \quad \text{and} \quad e(S^{\zeta_4^2}) = 2 - 2g(D) + 2(N - 1).$$

Since $X/C_4$ is Calabi-Yau we get

$$h^{1,2}(X/C_4) = h^{1,1}(X/C_4) - \frac{1}{2} e(X/C_4) = 1 + r + 7k + 3b + 2(n_1 + n_2) + 4a - (-9g(D) + 6k + 3N + 3n_1) = 1 + r + k + 3b + 2n_2 - n_1 + 4a + 9g(D) - 3N.$$

By (2), Thm. 1.1 and (2), Prop. 1 we have the following relations

$$r = \frac{1}{2}(12 + k + 2a + b - g(D) + 4h) \quad \text{and} \quad m = \frac{1}{2}(12 - k - 2a - b + g(D)),$$

where $h = \sum_{C \subseteq S^{g,s}} (1 - g(C))$. Moreover since $D$ is of the first type $h = k - g(D)$, $n_2 = 0$, $n_1 = 2h + 4$ and $b = \frac{n_1}{2}$, thus

$$h^{1,2}(X/C_4) = 1 + r + k + 3b + 2n_2 - n_1 + 4a + 9g(D) - 3N = m - 1 + 7g(D).$$

If $D$ is of the second type, then analogously the Riemann-Hurwitz formula yields

$$e(S^{\zeta_4}) = 2h + n_1 + n_2 \quad \text{and} \quad e(S^{\zeta_4^2}) = 2 - 2g(D) + 2(N - 1).$$

Thus

$$h^{1,2}(X/C_4) = h^{1,1}(X/C_4) - (6h + 3n_1 + 3n_2 + 3N - 3g(D)) = 1 + r + 7k + 3b - n_1 - n_2 + 4a - 6h - 3N + 3g(D).$$

Using the additional relations $h = k$, $n_1 + n_2 = 2h + 4$ and $b = \frac{n_1}{2} + 1$, we get

$$h^{1,2}(X/C_4) = m + 2g(D) - \frac{n_2}{2}.$$

Hence we proved the following theorem:

**5.2. Theorem** [5, Proposition 6.3]. If $S^{g,s}$ is not a sum of two elliptic curves, then for any crepant resolution of variety $X/C_4$ the following formulas hold

- If $D$ is of the first type, then
  $$h^{1,1} = 1 + r + 7k + 3b + 2(n_1 + n_2) + 4a,$$
  $$h^{1,2} = m - 1 + 7g(D).$$

- If $D$ is of the second type, then
  $$h^{1,1} = 1 + r + 7k + 3b + 2(n_1 + n_2) + 4a,$$
  $$h^{1,2} = m + 2g(D) - \frac{n_2}{2}.$$
5.4. **Order 6.** Let \((S, \gamma_S)\) be a \(K3\) surface with purely non-symplectic automorphism \(\gamma_S\) of order 6. Consider an elliptic curve \(E\) with the Weierstrass equation \(y^2 = x^3 + 1\) together with a non-symplectic automorphism \(\gamma_E\) of order 6 such that
\[
\gamma_E(x, y) = (\zeta_6^2 x, -y).
\]

We shall keep the notation of [6].

\[
X = S \times E,
\]

\(P\) – the infinity point of \(E\),

\(r = \dim H^2(S, \mathbb{C})^{\gamma_S}\),

\(m = \dim H^2(S, \mathbb{C})_{\gamma_S}^{\langle \zeta_6 \rangle}\) for \(i \in \{1, 2, \ldots, 5\}\),

\(l\) – number of curves fixed by \(\gamma_S\),

\(k\) – number of curves fixed by \(\gamma_S^5\),

\(N\) – number of curves fixed by \(\gamma_S^3\),

\(p_{(2,5)} + p_{(3,4)}\) – number of isolated points fixed by \(\gamma_S\) of type \((2, 5)\) and \((3, 4)\) i.e. the action of \(\gamma_S\) near the point linearizes to respectively

\[
\begin{pmatrix}
\zeta_6^2 & 0 \\
0 & \zeta_6^5
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
\zeta_6^3 & 0 \\
0 & \zeta_6^4
\end{pmatrix}
\]

\(n\) – number of isolated points fixed by \(\gamma_S^2\),

\(2n'\) – number of isolated points fixed by \(\gamma_S^5\) and switched by \(\gamma_S\),

\(a\) – number of triples \((A, A', A'')\) of curves fixed by \(\gamma_S^2\) such that \(\gamma_S(A) = A'\) and \(\gamma_S(A') = A''\),

\(b\) – number of pairs \((B, B')\) of curves fixed by \(\alpha_6^3\) such that \(\gamma_S(B) = B'\),

\(D\) – the curve with the highest genus in the fixed locus of \(\gamma_S\),

\(G\) – the curve with the highest genus in the fixed locus of \(\gamma_S^5\),

\(F_1, F_2\) – the curves with the highest genus in the fixed locus of \(\gamma_S^3\).

5.3. **Remark.** From \((9)\), Thm. 4.1) follows that \(g(D) \in \{0, 1\}\). Moreover by [3] we see that if \(g(F_1) \neq 0\), \(g(F_2) \neq 0\), then \(g(F_1) = g(F_2) = 1\). Clearly if \(g(D) = 1\), then \(D = G \in \{F_1, F_2\}\).

Denote by \(\gamma\) an automorphism \(\gamma_S \times \gamma_E^5\). Clearly \(\langle \gamma \rangle \simeq C_6\). Similar computations as in the previous cases imply that

\[
h_{1,1}^{\text{orb}}(X)^{C_6} = r + 1 \quad \text{and} \quad h_{2,2}^{\text{orb}}(X)^{C_6} = m - 1.
\]

For any \(g \in C_6\) let

\[
M_g := \bigoplus_{U \in \Lambda(g)} H^{1 - \text{age}(g), 1 - \text{age}(g)}(U).
\]

The action of \(\gamma\) and \(\gamma^5\). An automorphism \(\gamma\) acts on \(E\) by \(\gamma(x, y) = (\zeta_6^4 x, -y)\), hence it has only one fixed point — \(P\). The fixed locus of \(\gamma\) on \(S\) consists of \(l\) curves and \(p_{(2,5)} + p_{(3,4)}\) isolated points. Locally the action of \(\gamma\) on \(S \times E\) along the curve can be diagonalised to a matrix

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & \zeta_6 & 0 \\
0 & 0 & \zeta_6^5
\end{pmatrix}
\]

with age equal to 1. In the fixed point of type \((2, 5)\) and \((3, 4)\) we get respectively a matrices

\[
\begin{pmatrix}
\zeta_6^2 & 0 & 0 \\
0 & \zeta_6^5 & 0 \\
0 & 0 & \zeta_6^4
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
\zeta_6^3 & 0 & 0 \\
0 & \zeta_6^4 & 0 \\
0 & 0 & \zeta_6^3
\end{pmatrix}
\]

hence their ages equal 2.
In case of the action of \( \gamma^5 \) we observe that locus consists of \( l \) curves and \( p_{(2,5)} + p_{(3,4)} \) isolated points. Along the curve we have a matrix

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & \zeta_6^3 & 0 \\
0 & 0 & \zeta_6
\end{pmatrix},
\]

while in fixed points we have a matrices

\[
\begin{pmatrix}
\zeta_6^4 & 0 & 0 \\
0 & \zeta_6 & 0 \\
0 & 0 & \zeta_6
\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}
\zeta_6^2 & 0 & 0 \\
0 & \zeta_6^3 & 0 \\
0 & 0 & \zeta_6
\end{pmatrix},
\]

hence their ages are equal to 1. The above analysis shows that the effect on \( h_{\text{orb}}^{1,1} \) from both actions equals \( l + l + p_{(2,5)} + p_{(3,4)} = 2l + p_{(2,5)} + p_{(3,4)} \).

| Element of \( C_6 \) | \( \gamma \) | \( \gamma^5 \) |
|-----------------------|------------|------------|
| Irr. comp.            | 3\( l \) curves, 3\( p_{(2,5)} \) + 3\( p_{(3,4)} \) pts. | 3\( l \) curves, 3\( p_{(2,5)} \) + 3\( p_{(3,4)} \) pts. |
| The age               | curve: 1, point: 2 | curve: 1, point: 2 |
| Summand of \( h_{\text{orb}}^{1,1} \) | \( l \) | \( l + p_{(2,5)} + p_{(3,4)} \) |

The action of \( \gamma^2 \) and \( \gamma^4 \). Automorphisms \( \gamma^2 \) and \( \gamma^4 \) act on \( E \) by \( \gamma^2(x, y) = (\zeta_6^2 x, y) \) and \( \gamma^4(x, y) = (\zeta_6^4 x, y) \), they have three fixed points — \( \{ P, (0, i), (0, -i) \} \) from which only \( P \) is invariant under \( \gamma \) and the remaining two are switched. Identifying \( M_{\zeta_6} \) and \( M_{\zeta_6^2} \) with the vector space spanning by irreducible components we will find the action of induced map \( \gamma^* \) on it.

The matrix of the action of \( \gamma^* \) on \( M_{\zeta_6^2} \) is

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix} \sim \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix}
\]

hence it produces a 2-dimensional eigenspace for +1 and 1-dimensional eigenspace for −1. We have similar decomposition in the case of the action \( \gamma^* \) on \( M_{\zeta_6^2} \).

The fixed locus of \( \gamma^5_3 \) consists of \( p_{(2,5)} \) and \( n' \) pairs of points switched by \( \gamma_S \). Locally, the action of \( \gamma^2 \) at point has a matrix

\[
\begin{pmatrix}
\zeta_6^4 & 0 & 0 \\
0 & \zeta_6^4 & 0 \\
0 & 0 & \zeta_6^4
\end{pmatrix},
\]

with age equals 2. In the case of automorphism \( \gamma^4 \) we have a matrix

\[
\begin{pmatrix}
\zeta_6^2 & 0 & 0 \\
0 & \zeta_6^2 & 0 \\
0 & 0 & \zeta_6^2
\end{pmatrix},
\]

with age equal 1.

Notice that both \( \gamma^3_3 \) and \( \gamma^3_2 \) has the same fixed points but with different ages. Thus \( \gamma^3_5 \) on \( M_{\zeta_6} \) has \( (p_{2,5}^2 + n_1) \)-dimensional eigenspace for +1 and \( n_1' \)-dimensional eigenspace for −1, while \( \alpha^3_5 \) on \( M_{\zeta_6^2} \) has \( (p_{2,5}^2 + n_2) \)-dimensional eigenspace for +1 and \( n_2' \)-dimensional eigenspace for −1, where \( p_{2,5}^1, p_{2,5}^2, n_1 \) and \( n_2' \) are naturals defined as \( n_1 + n_1' = n \) and \( p_{2,5}^1 + p_{2,5}^2 = p_{2,5} \).

Moreover, a pairs of curves in the locus of \( \gamma^2_5 \) are switched by \( \gamma \). Hence \( \gamma^5_5 \) has \( (k - b) \)-dimensional eigenspace for +1 and \( b \)-dimensional eigenspace for −1. The same decomposition we will obtain in case of the action of \( \gamma^4 \).
By the Künneth formula both actions add to \( h_{\text{orb}}^{1,1} \)
\[
2(2(k - b) + b) + 2(p_{2,5}^1 + n') + n_1 + 2(p_{2,5}^2 + n') + n_2 = \\
= 4k - 2b + 2p_{(2,5)} + 3n'.
\]

| Element of \( C_6 \) | \( \gamma^2 \) | \( \gamma^4 \) |
|------------------------|-------------|-------------|
| Irreducible components | 3k curves, 3n points | 3k curves, 3n points |
| The age                | curve: 1, point: 2 | curve: 1, point: 1 |
| Summand of \( h_{\text{orb}}^{1,1} \) | 4k - 2b + 2p_{(2,5)} + 3n' |

The action of \( \gamma^3 \). An automorphism \( \gamma^3 \) acts on \( E \) by \( \gamma^3(x, y) = (x, -y) \), hence it has four fixed points — \( \{ P, (1, 0), (\zeta_3, 0), (\zeta_3^2, 0) \} \) from which only \( P \) is invariant under \( \gamma \) and the remaining three form a 3-cycle. Thus the action of \( \gamma^3_\Sigma \) on \( M_{C_6} \) has the matrix
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \zeta_3 & 0 \\
0 & 0 & 0 & \zeta_3^2
\end{pmatrix}
\]
so it produces 2-dimensional eigenspace for 1, 1-dimensional eigenspace for \( \zeta_3 \) and 1-dimensional eigenspace for \( \zeta_3^2 \).

Clearly the local action of \( \gamma^3 \) on curve has age equal to 1. The locus of \( \gamma^3_\Sigma \) consists of \( N \) curves with 4 triples of curves permuted by \( \gamma_3 \). Thus \( \gamma^3_\Sigma \) on \( M_{C_6} \) has \( (N - 2a) \)-dimensional eigenspace for +1 and two \( a \)-dimensional eigenspaces for \( \zeta_3 \) and \( \zeta_3^2 \).

The effect on \( h_{\text{orb}}^{1,1} \) equals \( 2(N - 2a) + a + a = 2N - 2a \).

| Element of \( C_6 \) | \( \gamma^3 \) |
|------------------------|-------------|
| Irreducible components | 4N curves |
| The age                | curve: 1 |
| Summand of \( h_{\text{orb}}^{1,1} \) | \( 2N - 2a \) |

Consequently, by the orbifold formula we see that
\[
h_{\text{orb}}^{1,1} = r + 1 + l + l + p_{(2,5)} + p_{(3,4)} + 4k - 2b + 2p_{(2,5)} + 3n' + 2N - 2a = \\
= r + 1 + 2l + 2N - 2b + 4k - 2a + 3n' + 3p_{(2,5)} + p_{(3,4)}.
\]

By the orbifold cohomology formula we see that non-zero contribution to \( h_{\text{orb}}^{1,2} \) have only curves in \( \Lambda(g) \) for any \( g \in C_6 \).

If \( g(D) \geq 1 \), by \[ \text{we can assume that } D = G = F_1. \] We see that contributions of \( \gamma \) and \( \gamma^3 \) equal to \( 2g(D) \). The automorphisms \( \gamma^3 \) and \( \gamma^4 \) have three fixed points on \( E \) with one 2-point orbit, hence by Künneth’s formula the effect on \( h_{\text{orb}}^{1,2} \) in this case equals \( 4g(D) \). Since \( \gamma^3 \) has four fixed points with 3-points orbit, we find that it’s contribution is equal to \( 2g(D) + g(F_2) + g(F_2/\gamma_3) \). Thus
\[
h_{\text{orb}}^{1,2}(\widetilde{X}/C_6) = m - 1 + 8g(D) + g(F_2) + g(F_2/\gamma_3).
\]

Now consider the case \( g(D) = 0 \). By the same argument as above, we see that contribution of \( \gamma^2 \) and \( \gamma^4 \) equals \( 2g(G) + 2g(G/\gamma_S) \), while the summand from \( \gamma^3 \) equals \( g(F_2) + g(F_1/\gamma_S) + g(F_2) + g(F_2/\gamma_S) \), hence
\[
h_{\text{orb}}^{1,2}(\widetilde{X}/C_6) = m - 1 + 2g(G) + 2g(G/\gamma_S) + g(F_1) + g(F_1/\gamma_S) + g(F_2) + g(F_2/\gamma_S).
\]

We proved the following theorem:
Thus the following formulas hold

\[ h^{1,1} = r + 1 + 2l + 2N - 2h + 4k - 2a + 3n' + 3p(2,5) + p(3,4), \]
\[ h^{1,2} = \begin{cases} 
  m - 1 + 8g(D) + g(F_2) + g(F_2/\gamma_S) & \text{if } g(D) \geq 1, \\
  m - 1 + 2g(G) + 2g(G/\gamma_S) + g(F_1) + g(F_1/\gamma_S) \\
  + g(F_2) + g(F_2/\gamma_S) & \text{if } g(D) = 0.
\end{cases} \]

Now, we will compute \( h^{2,1}_{\text{orb}} \) using orbifold Euler characteristic. In the table below we collect all possible intersections \( X^g \cap X^h \), where \((g, h) \in \mathbb{C}^2\).

| \( g \) | \( h \) | \( X^{(g)} \) | \( X^{(g)} \) | \( X^{(g)} \) | \( X^{(g)} \) | \( X^{(g)} \) | \( X^{(g)} \) |
|---|---|---|---|---|---|---|---|
| 1 | \( \beta_0 \) | \( \beta_0 \) | \( \beta_0 \) | \( \beta_0 \) | \( \beta_0 \) | \( \beta_0 \) | \( \beta_0 \) |
| \( \beta_0 \) | \( \beta_0 \) | \( \beta_0 \) | \( \beta_0 \) | \( \beta_0 \) | \( \beta_0 \) | \( \beta_0 \) | \( \beta_0 \) |
| \( \beta_0 \) | \( \beta_0 \) | \( \beta_0 \) | \( \beta_0 \) | \( \beta_0 \) | \( \beta_0 \) | \( \beta_0 \) | \( \beta_0 \) |
| \( \beta_0 \) | \( \beta_0 \) | \( \beta_0 \) | \( \beta_0 \) | \( \beta_0 \) | \( \beta_0 \) | \( \beta_0 \) | \( \beta_0 \) |
| \( \beta_0 \) | \( \beta_0 \) | \( \beta_0 \) | \( \beta_0 \) | \( \beta_0 \) | \( \beta_0 \) | \( \beta_0 \) | \( \beta_0 \) |

From [12] we see that
\[ e(X/\mathbb{C}) = \frac{1}{6} \left( 24e(X^{(g)}) + 8e(X^{(g)} \cap X^{(g)}) \right) = 4e(S^{(g)}) + 4e(S^{(g)}) + 2e(S^{(g)}).
\]

Thus
\[ h^{1,2}(X/\mathbb{C}) = h^{1,1}(X/\mathbb{C}) - 2e(S^{(g)}) - 2e(S^{(g)}) - e(S^{(g)}). \]

By the Riemann-Hurwitz formula we obtain
\[ e(S^{(g)}) = 2(l - 1) + 2 - 2g(D) + p(2,5) + p(3,4), \]
\[ e(S^{(g)}) = 2(k - 1) + 1 - g(G) + 1 - g(G/\gamma_S) + n, \]
\[ e(S^{(g)}) = 2(N - 2) + 1 - g(F_1) + 1 - g(F_1/\gamma_S) + 1 - g(F_2) + 1 - g(F_2/\gamma_S) \]

hence after simplifying
\[ h^{1,2}(X/\mathbb{C}) = r + 1 + 2l - 2b - 2a + 3n' + p(2,5) - p(3,4) - 2n + 4g(D) + 2g(G) + \\
+ 2g(G/\gamma_S) + g(F_1) + g(F_1/\gamma_S) + g(F_2) + g(F_2/\gamma_S). \]

Comparing both formulas we get:

5.5. **Corollary.** With the notation above, the following relation holds
\[ -m + r + 2 - 2l - 2b - 2a + 3n' + p(2,5) - p(3,4) - 2n + 4g(D) = 0. \]

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