Transition Form Factor in Extended AdS/QCD models

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Abstract

The $\gamma^*\rho^0 \rightarrow \pi^0$ transition form factor is extracted from recent result for the $\gamma^*\gamma^*\pi^0$ form factor obtained in the extended hard-wall AdS/QCD model with a Chern-Simons term. In the large momentum region, the form factor exhibits a $1/Q^4$ behavior, in accordance with the perturbative QCD analysis, and also with the Light-Cone Sum Rule (LCSR) result if the pion wave function exhibits the same endpoint behavior as the asymptotic one. The appearance of this power behavior from the AdS side and the LCSR approach seem to be rather similar: both of them come from the “soft” contributions. Comparing the expressions for the form factor in both sides, one can obtain the duality relation $z \propto \sqrt{u(1-u)}$, which is compatible with one of the most important relations of the Light-Front holography advocated by Brodsky and de Teramond. In the moderate $Q^2$ region, the comparison of the numerical results from both approaches also supports an asymptotic-like pion wave function, in accordance with previous studies for the $\gamma^*\gamma^*\pi^0$ form factor. The form factor at zero momentum transfer gives the $\gamma^*\rho^0\pi^0$ coupling constant, from which one can determine the partial width for the $\rho^0(\omega) \rightarrow \pi^0\gamma$ decay. We also calculate the form factor in the time-like region, and study the corresponding Dalitz decays $\rho^0(\omega) \rightarrow \pi^0e^+e^-$, $\pi^0\mu^+\mu^-$. Although all these results are obtained in the chiral limit, numerical calculations with finite quark masses show that the corrections are extremely small. Some of these calculations are repeated in the Hirn-Sanz model and similar results are obtained.

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I. INTRODUCTION

In recent years, the phenomenological bottom-up approach to describing strong interaction based on the AdS/CFT correspondence \[1–3\], now known as AdS/QCD, has offered much insight into various low-energy aspects of QCD. In the simplest setup, various hadron states are considered to be dual to different string modes propagating in a slice of 5D AdS space \[4–6\]. High-energy scattering of glueballs naturally exhibits QCD-like power behavior due to the warped geometry of the dual theory \[4\]. Spectra of low-lying hadron states are well reproduced \[6, 7\]. Chiral symmetry and its spontaneous breaking are also well implemented \[8–10\]. Up to now there have been extensive studies on various dynamical quantities such as decay constants, coupling constants and form factors, e.g., \[11–17\]. Furthermore, a novel relation between the string modes and the Light-Cone wave functions of the mesons was found in Ref. \[7\], from which the so-called Light-Front holography was established.

To reproduce the Wess-Zumino-Witten term in the chiral Lagrangian, a Chern-Simons (CS) term must be added \[18, 19\]. The CS term naturally introduces baryon density \[20\], since baryons are related to the instantons in the 5D model. The effect of this term to the baryon properties were later studied in Ref. \[21\]. Furthermore, with the CS term turned on, the anomalous form factor of the pion coupling to two virtual photon can be well reproduced \[22\]. Interestingly, the predictions for the form factor in the limit of large photon virtualities coincide with those of perturbative QCD (pQCD) calculated using the asymptotic form of the pion distribution amplitude. In this paper we attempt to extend this calculation to the form factor of $\gamma^* \rho^0 \rightarrow \pi^0$ transition, and then compare the results with those of the traditional approaches.

In pQCD, the asymptotic behavior of the $\gamma^* \rho \pi$ form factor has been predicted to be $1/Q^4$ \[23\]. A simple expression for this form factor in large and moderate momentum region can be obtained in the Light-Cone Sum Rules (LCSR) approach \[24, 25\], which gives the same asymptotic behavior if the pion distribution amplitude is asymptotic-like at the endpoint. However, the dominant contribution is quite different from the pQCD analysis. At zero momentum transfer the form factor defines the $\gamma\rho\pi$ coupling which determines the width of the radiative decay $\rho \rightarrow \pi\gamma$. This coupling was extracted from the traditional three-point QCD sum rule \[26\], and also from QCD sum rules in the presence of external field \[27\]. In this paper we will try to give a unified description of the form factor in the
whole region. We will mainly focus on the results in the standard hard-wall model \cite{8, 9}, and repeat part of the calculations in the Hirn-Sanz model \cite{10} as a check.

The organization of the paper is as follows. In the next section we will briefly introduce the hard-wall AdS/QCD model, and review the calculation of the $\gamma^*\gamma^*\pi^0$ form factor. The extraction of $\gamma^*\rho^0\pi^0$ form factor and comparison with other approaches will be presented in Sec. III. In Sec. IV we give the $\gamma^*\rho^0\pi^0$ form factor in the Hirn-Sanz model and compare it to that in the hard-wall model. The last section is reserved for the summary.

II. EXTENDED ADS/QCD MODEL WITH CHERN-SIMONS TERM

A. hard-wall AdS/QCD model

In the hard-wall model \cite{8}, the background is given by a slice of AdS space with the metric:

$$ds^2 = g_{MN}dx^Mdx^N = \frac{1}{z^2} \left( \eta_{\mu\nu}dx^\mu dx^\nu - dz^2 \right),$$

where $\eta_{\mu\nu} = \text{Diag} (1, -1, -1, -1)$ and $\mu, \nu = (0, 1, 2, 3)$, $M, N = (0, 1, 2, 3, z)$. The SU ($N_f$)×SU ($N_f$) chiral symmetry is realized through the gauge symmetry of two sets of gauge fields $A_L$ and $A_R$. To breaking the chiral symmetry to the vector part, an additional scalar field $X$ is introduced. The whole action is then given by:

$$S_{AdS} = \text{Tr} \int d^5x \left[ \frac{1}{z^3} (D^M X) (D_M X) + \frac{3}{z^5} X^\dagger X - \frac{1}{4g_5^2 z} (F_{MN}^{(L)} F_{MN}^{(L)} + F_{MN}^{(R)} F_{MN}^{(R)}) \right],$$

where $A = A^a t^a$, $F_{MN} = \partial_M A_N - \partial_N A_M - i[A_M, A_N]$, $D_X = \partial X - iA_L X + iXA_R$, and the generators are normalized as $\text{Tr} \ t^a t^b = \delta^{ab}/2$. The vacuum solution $\langle X(x, z) \rangle = v(z)/2 = (m_q z + \sigma z^3)/2$ then breaks the chiral symmetry to the vector part, and the phase of the fluctuations of $X(x, z)$ gives the pion field: $X(x, z) = \langle X \rangle e^{2i\pi^a(x, z)}$. The vector combination $V = (A_L + A_R)/2$ corresponds to the vector mesons. Taking the axial gauge $V_z = 0$ and Fourier transforming to the 4D momentum space, the transverse components $V^T_\mu$ then satisfies the following equation:

$$\partial_z \left( \frac{1}{z} \partial_z V^T_\mu (q, z) \right) + \frac{q^2}{z} V^T_\mu (q, z) = 0.$$
With Neumann boundary condition chosen at the cutoff, the normalized solution is simply given by

$$\psi_n^V(z) = \sqrt{\frac{2}{z_0 J_1(\gamma_{0,n})}} z J_1(M_n z)$$  \hspace{1cm} (4)$$

with $\gamma_{0,n}$ being the $n^{th}$ zero of the Bessel function $J_0(x)$ and $M_n = \gamma_{0,n}/z_0$. Matching to the experimental $\rho$ mass fixes $z_0 = (323 \text{ MeV})^{-1}$. The coupling constants, which are defined by $\langle 0 | J^a_\mu | \rho^n \rangle = f_n \delta^{ab} \epsilon_\mu$, can be obtained by analyzing the two-point correlation function derived from the action. The results are expressed through $\psi_n^V(z)$ as

$$f_n = \frac{1}{g_5} \left[ \frac{1}{z} \partial_z \psi_n^V(z) \right]_{z=0} = \frac{\sqrt{2} M_n}{g_5 z_0 J_1(\gamma_{0,n})}. \hspace{1cm} (5)$$

The non-normalized solution, or the bulk-to-boundary propagator, can also be derived analytically:

$$\mathcal{J}(Q, z) = Q z \left[ K_1(Q z) + I_1(Q z) \frac{K_0(Q z_0)}{I_0(Q z_0)} \right], \hspace{1cm} (6)$$

where $\mathcal{J}(Q, z)$ is taken at a spacelike momentum $q$ with $q^2 = -Q^2$ and satisfies the boundary condition $\mathcal{J}(Q, 0) = 1$. $I_n$ and $K_n$ are the order-$n$ modified Bessel functions of the first and second kind, respectively. It can be shown that $\mathcal{J}(Q, z)$ has the following decomposition formula \cite{28, 29}:

$$\mathcal{J}(Q, z) = g_5 \sum_{n=1}^{\infty} \frac{f_n \psi_n^V(z)}{Q^2 + M_n^2}. \hspace{1cm} (7)$$

From $\mathcal{J}(Q, z)$ the vector current correlator can be derived, whose asymptotic behavior determines the 5D coupling $g_5 = 2\pi$.

The axial combination $A = (A_L + A_R)/2$ is a little complicated because the longitudinal part will be entangled with the chiral field. The equations are as follows:

$$\partial_z \left( \frac{1}{z} \partial_z A^T_\mu \right) + \frac{q^2}{z} A^T_\mu - g_5^2 v^2 \frac{A^T_\mu}{z^3} = 0; \hspace{1cm} (8)$$

$$\partial_z \left( \frac{1}{z} \partial_z \varphi \right) + \frac{g_5^2 v^2}{z^3} (\pi - \varphi) = 0; \hspace{1cm} (9)$$

$$- q^2 \partial_z \varphi + \frac{g_5^2 v^2}{z^2} \partial_z \pi = 0. \hspace{1cm} (10)$$

where $\partial_\mu \varphi = A_\mu - A^T_\mu$. The normalization of $\varphi$ and $\pi$ is fixed by the pion kinetic term:

$$\int_0^{z_0} \mathrm{d}z \left( \frac{\varphi'(z)^2}{g_5^2 z} + \frac{\pi(z)^2 (\pi - \varphi)^2}{z^3} \right) = f_\pi^2. \hspace{1cm} (11)$$
This normalization naturally leads to the charge conservation constraint for the electromagnetic form factor of the pion, since the pion form factor is given by \[ F_\pi(Q^2) = \frac{1}{f_\pi^2} \int_0^{z_0} \text{d}z \mathcal{J}(Q, z) \left( \frac{\varphi'(z)^2}{g_5^2 z} + \frac{v(z)^2 (\pi - \varphi)^2}{z^3} \right). \] (12)

Notice that both the equations and the normalization condition is invariant if the \( \varphi \) and \( \pi \) fields are shifted by a constant simultaneously.

In the chiral limit \( m_q = 0 \), the pion decay constant can be derived from the residue of the axial current correlator at \( q^2 = 0 \) \[ f_\pi^2 = -\frac{1}{g_5^2} \frac{\partial_z A_c(0, z)}{z} \bigg|_{z=\epsilon}, \] (13)

where \( A_c(0, z) \) is the nonnormalizable solution to Eq. (8) at \( q^2 = 0 \), satisfying \( A_c'(0, z_0) = 0 \) and \( A_c(0, 0) = 1 \). We use the subscript “c” to indicate that the solution is obtained in the chiral limit. The explicit form of \( A_c(0, z) \) is given by

\[ A_c(0, z) = z \Gamma \left( \frac{2}{3} \right) \left( \frac{\alpha}{2} \right)^{1/3} \left[ I_{-1/3} \left( \alpha z^3 \right) - I_{1/3} \left( \alpha z^3 \right) \frac{I_{2/3} \left( \alpha z_0^3 \right)}{I_{-2/3} \left( \alpha z_0^3 \right)} \right], \] (14)

where \( \alpha \equiv g_5 \sigma/3 \). Matching to the experimental value of \( f_\pi \), one obtains \( \alpha = (424 \text{ MeV})^3 \) \[22\], or \( \sigma = (332 \text{ MeV})^3 \). In this case \( \pi_c(z) \) is just a constant and can be shifted to zero. Then \( \varphi_c(z) \) satisfies the same equation as \( A_c(0, z) \) with the same boundary condition at \( z = z_0 \), so we have \( \varphi_c(z) = \varphi_c(0)A_c(0, z) \). The normalization condition (11) finally fixes \( \varphi_c(0) \) to be one.

When \( m_q \neq 0 \), \( A(q^2, z) \) will generally develop a \( z \log z \) term, unless \( q^2 = g_5^2 m_q^2 \). This should not be identified with the pion pole. Away from this point, \( \partial_z A(q^2, z)/z \) is divergent as \( z \to 0 \), which makes the generalization of Eq. (13) unfeasible. This can be overcome if we choose \( \varphi(z) \), rather than \( A(0, z) \), to define the pion decay constant. These two are identical in the chiral limit, but different now. Thus we have

\[ f_\pi^2 = -\frac{1}{g_5^2} \frac{\partial_z \varphi(z)}{z} \bigg|_{z=\epsilon}. \] (15)

One may worry that if this definition is consistent with the normalization (11). Notice that in this case \( (\varphi - \pi) \) is forced to vanish at the ultraviolet boundary. Integration by parts and imposing the equation of motion with the boundary condition, Eq. (11) becomes

\[ \frac{m_\pi^2}{g_5^2} \int_0^{z_0} \frac{z}{v(z)^2} \varphi'(z)^2 \text{d}z = f_\pi^2, \] (16)
Using the same trick as in ref. [8] and ref. [17], one can show that the above condition together with Eq. (15) leads to the Gell-Mann-Oakes-Renner relation.

To solve the equations for $\varphi(z)$ and $\pi(z)$, one has to employ numerical methods. We choose the boundary value $\varphi(0) = \pi(0) = 1$, for the convenience of comparing with the chiral solution. Adjusting $m_\pi$ and $f_\pi$ to the experimental value $m_{\pi_0} = 135.0$ MeV and $f_\pi = 92.4$ MeV, one obtains $m_q = 2.14$ MeV and $\sigma = (329$ MeV$)^3$. The explicit form of $\varphi(z)$ and $\pi(z)$ are plotted in Fig. (1), together with the solution $\varphi_c(z)$. Surprisingly, the curves of $\varphi(z)$ and $\varphi_c(z)$ almost coincide, and one can hardly distinguish them. $\pi(z)$ is also very close to $\pi_c(z) = 0$, except for a peak near $z = 0$. Thus one may expect that the quark mass correction to any physical observable would be small. To check this, we recalculate the pion form factor according to Eq. (12), which has already been done in ref. [14] for finite quark mass, and in ref. [15] in the chiral limit. The result confirmed our expectation, see Fig. (2). We also calculated some other observable and obtained similar results in both cases.

![Graph of the solution $\varphi(z)$ and $\pi(z)$](image)

**FIG. 1:** *Explicit form of the solution $\varphi(z)$ and $\pi(z)$ (solid curves), together with $\varphi_c(z)$ (dashed line), the solution in the chiral limit.*
FIG. 2: The pion electromagnetic form factor calculated in the hard-wall model, the solid line denotes the result with finite quark mass and the dashed one in the chiral limit.

B. Extended hard-wall model with Chern-Simons term

Now let us turn to the derivation for the $\gamma^* \gamma^* \pi^0$ form factor by adding a CS term to the hard-wall model, pioneered by Grigoryan and Radyushkin [22]. First we should enlarge the previous considered $SU(2)_L \otimes SU(2)_R$ gauge group into $U(2)_L \otimes U(2)_R$. To do this, we replace the gauge fields $t^a A^a_\mu$ in the action by $A_\mu = t^a A^a_\mu + \hat{I}\frac{A_\mu}{2}$. The gauge field $\hat{A}_\mu$ will couple to the isosinglet current in the boundary theory. The cubic part of the CS term can be expressed in the axial gauge as

$$S^{CS}_0[A] = k \frac{N_c}{48\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{Tr} \int d^4x dz (\partial_z A_\mu) \left[ F_{\nu\rho} A_\sigma + A_\nu F_{\rho\sigma} \right],$$

with $k$ an integer. For the $U(2)_L \otimes U(2)_R$ gauge group, the corresponding cubic action reads:

$$S^{AdS}_{CS}[A(L), A(R)] = S^{CS}_0[A(L)] - S^{CS}_0[A(R)].$$

The relevant term for the anomalous $\pi^0 \gamma^* \gamma^*$ form factor can be found to be

$$S^{anom} = k \frac{N_c}{8\pi^2} \epsilon^{\mu\nu\rho\sigma} \int d^4x \int_0^{z_0} dz \left( \partial_z \varphi^a \right) \left( \partial_\rho V^a_\mu \right) \left( \partial_\sigma \tilde{V}_\nu \right).$$
Based on the holographic dictionary, one then obtains the bare form factor as

$$ F_{\gamma^*\gamma^*\pi^0}(Q_1^2, Q_2^2) = \frac{N_c}{12\pi^2 f_\pi} \cdot \frac{k}{2} \int_{z_0}^{z_0} \mathcal{J}(Q_1, z) \mathcal{J}(Q_2, z) \partial_z \varphi(z) \, dz. \quad (20) $$

In QCD, the axial anomaly determines the value of the form factor with real photons to be $F_{\gamma^*\gamma^*\pi^0}(0, 0) = \frac{N_c}{12\pi^2 f_\pi}$. To reproduce this result, a surface term must be added, and the integer $k$ must be taken to be 2. The final result for the normalized function $K(Q_1^2, Q_2^2)$ is then

$$ K(Q_1^2, Q_2^2) = F_{\gamma^*\gamma^*\pi^0}(Q_1^2, Q_2^2)/F_{\gamma^*\gamma^*\pi^0}(0, 0) $n
$$
$$ = \varphi(z_0) \mathcal{J}(Q_1, z_0) \mathcal{J}(Q_2, z_0) - \int_{z_0}^{z_0} \mathcal{J}(Q_1, z) \mathcal{J}(Q_2, z) \partial_z \varphi(z) \, dz. \quad (21) $$

This result has very interesting properties \[22\]. When one photon is real, the form factor has the following expansion at low momentum:

$$ K(0, Q^2) = 1 - a_\pi \frac{Q^2}{m_\pi^2}, \quad (22) $$

with $a_\pi \approx 0.031$ in perfect agreement with the experimental value: $a_{\exp} \simeq 0.032 \pm 0.004$. This indicates a strong Vector Meson Dominance (VMD) in this channel, which will lead to the result $a = \frac{m_\pi^2}{m_\rho^2} \simeq 0.03$. At large virtuality for one or both photons, the asymptotic behavior of the form factor can be found to be:

$$ K(0, Q^2) \to \frac{\tilde{s}}{Q^2}, $n
$$
$$ K(Q_1^2, Q_2^2) \to \frac{\tilde{s}}{3Q^2} \int_{1/2}^{1} \frac{6x(1-x)}{1+\omega(2x-1)} \, dx, \quad (23) $$

where $\tilde{s} = 8\pi^2 f_\pi^2$. Both coincide with the leading-order pQCD results calculated for the asymptotic form of the pion distribution amplitude. However, the origins of the power behavior are quite different. The power behavior appears only after we have integrated out the meson wave function in the holographic direction. This is very similar to the “soft” contributions described in the LCSR approach, which will be discussed in the following section.

### III. FORM FACTOR OF $\gamma^*\rho^0 \to \pi^0$ TRANSITION

It will be illuminating to further study the $\rho^0 \to \pi^0$ transition form factor based on the previous result. One starts with the dispersion relation for the amplitude $F_{\gamma^*\gamma^*\pi^0}(Q_1^2, Q_2^2)$ in
the variable $Q_2^2$ and at fixed $Q_1^2$. In the standard QCD sum rule approach, one assumes that the spectral density in the dispersion relation can be approximated by the ground states $\rho^0$, $\omega$ and the the higher states with an effective threshold $s_0$:

$$F_{\gamma^*\rho^0}(Q_1^2, Q_2^2) = \frac{\sqrt{2} f_{\rho} F_{\rho^0\pi^0}(Q_1^2)}{m_{\rho}^2 + Q_2^2} + \int_{s_0}^{\infty} ds \frac{\rho^h(Q_1^2, s)}{s + Q_2^2}. \quad (24)$$

Here, the $\gamma^*\rho^0(\omega) \rightarrow \pi^0$ form factor is defined as

$$\frac{1}{3} \langle \pi^0 | j_{\mu}(q_1) | \omega(q_2) \rangle = \langle \pi^0 | j_{\mu}(q_1) | \rho^0(q_2) \rangle = F_{\rho^0\pi^0}(Q_1^2) m_{\rho}^{-1} \epsilon_{\mu\nu\alpha\beta} e^{\nu} q_1^\alpha q_2^\beta, \quad (25)$$

and the decay constants have the relation:

$$3 \langle \omega | j_{\mu} | 0 \rangle = \langle \rho^0 | j_{\mu} | 0 \rangle = \frac{f_{\rho}}{\sqrt{2}} m_{\rho} e_{\nu}, \quad (26)$$

e_{\mu} being the polarization vector of the $\rho(\omega)$ meson. Since we are working in the $U(2)$V symmetric limit, the above relations are exact. On the other hand, the dispersion relation can be carried out explicitly using the decomposition formula (7). Extracting the lowest $\rho^0$ and $\omega$ pole contributions, one immediately obtains the expression for the $\gamma^*\rho_0\pi_0$ form factor:

$$F_{\rho^0\pi^0}(Q^2) = \frac{N_c}{12\pi^2 f_{\pi}} \frac{g_5 m_{\rho}}{2} \left[ \mathcal{J}(Q, z_0) \psi_{1\gamma}^V(z_0) \varphi(z_0) - \int_{z_0}^{z_0} \mathcal{J}(Q, z) \psi_{1\gamma}^V(z) \partial_z \varphi(z) dz \right]. \quad (27)$$

As discussed before, the quark mass correction is very small, so we mainly work in the chiral limit. The results for finite quark mass are listed only in Table I for comparison.

A. Large $Q^2$ region

First let us focus on the large-$Q^2$ asymptotic behavior of the form factor. The large-$Q^2$ behavior of $\mathcal{J}(Q, z)$ is dominated by the term $zQK_1(zQ)$ in Eq. (6), which behaves like $e^{-Qz}$. Thus the first term in Eq. (27) will vanish exponentially $\sim e^{-Qz_0}$ in the asymptotic region, hence can be neglected. Due to the exponential factor of $\mathcal{J}(Q, z)$, only small values of $z$ are important in the remaining integral, and the outcome is determined by the small-$z$ behavior of the wave function $\partial_z \varphi(z)$ and $\psi_{1\gamma}^V(z)$. From the previous discussion, we know that when $z \rightarrow 0$,

$$\partial_z \varphi(z) \sim -f_{\pi} g_5^2 z, \quad (28)$$

and

$$\psi_{1\gamma}^V(z) \sim \frac{m_{\rho} z^2}{\sqrt{2} z_0 J_1(\gamma_{0,n})}. \quad (29)$$
Utilizing all these facts one finds:

\[
F^{\rho\pi^0}(Q^2) \rightarrow \frac{N_em_\rho m_\rho}{12f_\pi \sqrt{2}z_0 J_1(\gamma_{0,1})} (-f_\pi^2 g_5^2) \int_0^{z_0} z^3 \ast z Q K_1(zQ) dz = \frac{\pi f_\pi m_\rho^3}{\sqrt{2} \gamma_{0,1} J_1(\gamma_{0,1})} K_1(\chi) d\chi
\
= \frac{8\sqrt{2} \pi f_\pi m_\rho^3}{\gamma_{0,1} J_1(\gamma_{0,1}) Q^4} = \frac{1.23 \text{GeV}^4}{Q^4}
\]

(30)

Using the holographic expression for \(m_\rho\) and \(f_\rho\), one may further express the result as

\[
F^{\rho\pi^0}(Q^2) \rightarrow \frac{8\sqrt{2}\pi f_\pi f_\rho m_\rho^2}{Q^4}.
\]

(31)

Although this power behavior is the same as the pQCD prediction [23], the underlying mechanism is rather different. The appearance of the power behavior in this way is similar to the LCSR analysis [24, 25], where the form factor is given by the following expression:

\[
F^{\rho\pi}_{\text{LC}}(Q^2) = \frac{f_\pi}{3f_\rho} \int_0^{1 - u} \frac{du}{u^2} \left( \varphi_\pi(u) + \frac{u}{Q^2} \frac{d\varphi^{(4)}(u)}{du} \right) \exp \left( -\frac{Q^2(1 - u)}{uM^2} + \frac{m_\rho^2}{M^2} \right)
\]

(32)

\[
= \frac{f_\pi}{3f_\rho} \frac{\varphi_\pi'(1)}{Q^4} \exp \left( \frac{m_\rho^2}{M^2} \right) \int_0^{s_0} s e^{-s/M^2} ds + O(1/Q^6).
\]

(33)

Here \(\varphi_\pi(u)\) and \(\varphi^{(4)}(u)\) are the leading twist and the twist-4 distribution amplitudes, and \(M^2\) the Borel parameter. In deriving the asymptotic behavior, we have assumed that \(\phi_\pi(u) \sim 1\)

\[
\varphi_\pi'(1)(1 - u).\quad \text{That is to say, in order to obtain the same power behavior as in the holographic model, } \phi_\pi(u) \text{ must has the same end-point behavior as the asymptotic one, namely } \phi_\pi as (u) = 6u(1 - u). \text{ This is also in accordance with the general analysis for the end-point behavior of the pion wave function [23].}
\]

Moreover, in the holographic approach \(\mathcal{J}(Q, z) \rightarrow e^{-Qz}\) tells us in the large \(Q^2\) limit we are actually probing the \(0 < z < 1/Q\) interval of the AdS slice, while in the LCSR [32] the endpoint region, \(0 < 1 - u < 1/Q^2\), or \(0 < \sqrt{1 - u} < 1/Q\) dominates. Taking into account the symmetry \(u \leftrightarrow 1 - u\) of the light quark system, one may expect that \(z\) should be dual to \(\sqrt{u(1 - u)b}\) with \(b\) a light-cone distance parameter, at least in the high energy region. This is just one of the key relations of the Light-Front holography [7, 13].

Since the absolute normalization of the asymptotic behavior is not known in both the pQCD and LCSR approaches, one can only compare their predictions to the form factor at moderate \(Q^2\) with ours. At \(Q^2 \simeq 10 \text{ GeV}^2\), direct calculation from Eq. (27) gives
$F_{\rho_0\pi^0}(Q^2) = 8.8 \times 10^{-3}$, as shown in Fig. 3. This is much larger than the pQCD result $F_{\rho_0\pi^0}^{pQCD}(Q^2) \approx 3 \times 10^{-3}$ [23]. In the LCSR approach, the result strongly depends on the shape of the leading twist distribution amplitude of the pion meson, which can be seen from Fig. 4. For the asymptotic distribution amplitude, one obtains $F_{\rho_0\pi^0}^{as}(Q^2) \approx 6.6 \times 10^{-3}$ [1]. For some non-asymptotic distribution amplitudes, the results are much larger, e.g., input of the Chernyak-Zhitnitsky (CZ) [23] and Braun-Filyanov (BF) [31] distribution amplitudes give: $F_{\rho_0\pi^0}^{CZ}(Q^2) \approx 0.014$, $F_{\rho_0\pi^0}^{BF}(Q^2) \approx 0.017$, respectively. Thus our result indicates that the true pion distribution amplitude should be asymptotic-like, in accordance with the conclusion made from the studies of the $\gamma^*\gamma^*\pi^0$ form factor [22].

![Graph](image)

FIG. 3: $\gamma^*\rho_0 \rightarrow \pi^0$ form factor calculated in the extended hard-wall AdS/QCD model, the result for finite quark mass (in solid curve) and that in the chiral limit (dashed line) almost coincide.

\(^1\)In Ref. [24] an alternative light-cone sum rule was derived for this form factor, from which a much smaller value $F_{\rho_0\pi^0}^{as}(Q^2 \approx 10\text{GeV}^2) \approx 3 \times 10^{-3}$ (see Fig. 4) was obtained. However, with the aid of the technique in Ref. [30] one can show that a boundary term was missing in their calculations. After including this term, a similar result $F_{\rho_0\pi^0}^{as}(Q^2) \approx 7.0 \times 10^{-3}$ will be obtained.
FIG. 4: LCSR results for $\gamma^* \rho^0 \rightarrow \pi^0$ form factor excerpted from [25]. The solid line corresponds to the result calculated with the asymptotic pion wave function, while the long-dashed and the short-dashed lines with the CZ and BF wave functions respectively. In comparison, the predictions of the three-point QCD sum rule (dotted) [32] and an alternative light-cone sum rule for the $\gamma^* \rho^0 \rightarrow \pi^0$ form factor [24] (dash-dotted) were also plotted.

B. Low $Q^2$ region

The $\gamma \rho^0 \pi^0$ form factor at zero momentum transfer defines the coupling constant $g_{\rho^0 \pi^0 \gamma}$ through the effective Lagrangian [33]

$$\mathcal{L}_{\rho \pi \gamma}^{\text{eff}} = g_{\rho \pi \gamma} m_{\rho}^{-1} \varepsilon_{\mu \alpha \beta} \partial^\mu \rho^0 \partial^\alpha \partial^\beta \pi^0. \quad (34)$$

Since $J(Q, 0) = J(0, z) = 1$, one immediately obtains the coupling constant:

$$g_{\rho \pi^0 \gamma} = \frac{N_c m_{\rho}}{12\pi f_{\pi}} \left[ \psi_V^1(z_0) \varphi(z_0) - \int_0^{z_0} \psi_V^1(z) \partial_z \varphi(z) dz \right] = 0.56. \quad (35)$$

This result is very close to the value extracted from the analysis of $\rho^0$ and $\omega$ photoproduction reactions through pseudoscalar exchange, which gives rise to $g_{\rho \pi^0 \gamma} = 0.54$ [34]. Also it is consistent with the QCD sum rule prediction $g_{\rho \pi^0 \gamma} = 0.63 \pm 0.07$ [26]. Based on the effective
Experiment [35] \[ m_q = 0 \] \[ m_q \neq 0 \]

| Decay Width | \[ m_q = 0 \] | \[ m_q \neq 0 \] |
|-------------|-----------------|-----------------|
| \( \Gamma(\rho^0 \to \pi^0 \gamma) \) | 0.090 ± 0.013 | 0.06735 | 0.06740 |
| \( \Gamma(\omega \to \pi^0 \gamma) \) | 0.76 ± 0.03 | 0.6125 | 0.6129 |
| \( \Gamma(\rho^0 \to \pi^0 e^+e^-) \) | 6.167 \times 10^{-4} | 6.172 \times 10^{-4} |
| \( \Gamma(\rho^0 \to \pi^0 \mu^+\mu^-) \) | 6.422 \times 10^{-5} | 6.427 \times 10^{-5} |
| \( \Gamma(\omega \to \pi^0 e^+e^-) \) | (6.5 ± 0.8) \times 10^{-3} | 5.629 \times 10^{-3} | 5.634 \times 10^{-3} |
| \( \Gamma(\omega \to \pi^0 \mu^+\mu^-) \) | (8.2 ± 2.0) \times 10^{-4} | 6.015 \times 10^{-4} | 6.019 \times 10^{-4} |

**TABLE I: Predictions of the partial decay widths (in MeV) of \( \rho^0 \) and \( \omega \) in the present approach, both in the chiral limit and with finite quark mass.**

Substituting the physical masses of the mesons and taking \( \alpha = 1/137 \), the partial widths \( \Gamma(\rho^0 \to \pi^0 \gamma) \) and \( \Gamma(\omega \to \pi^0 \gamma) \) can be obtained. We can further extrapolate the form factor to the time-like region by analytically continuing \( J(Q, z) \) to the region \( q^2 = -Q^2 > 0 \),

\[
J(q, z) = -\frac{\pi}{2} q z \left[ Y_1(q z) - J_1(q z) \frac{Y_0(q z)}{J_0(q z)} \right]
\]  

with \( Y_n \) the second kind Bessel function. From the resulting \( \gamma^* \rho^0(\omega)\pi^0 \) form factor in the time-like region, one obtains the decay widths for the \( \rho^0(\omega) \to \pi^0 e^+e^- \) and \( \rho^0(\omega) \to \pi^0 \mu^+\mu^- \) decays. In Table I we list these results together with those for the radiative decays. The only reason we keep four digits for our predictions is to show the corrections due to finite quark mass.

**IV. \( \gamma^* \rho^0 \pi^0 \) FORM FACTOR IN THE EXTENDED HIRN-SANZ MODEL**

Spontaneous chiral symmetry breaking can also be implemented through the boundary conditions at the IR cutoff, without employing the scalar field, as proposed by Hirn and Sanz [10]. Specifically, the axial combination of the left-handed and right-handed vector fields was chosen to satisfy Dirichlet boundary condition, rather than the Neumann one for
the vector part. That is to say, we require:

\[ F_{(R)\mu} (x, z = z_0) + F_{(L)\mu} (x, z = z_0) = 0. \]  
\[ A_{(R)\mu} (x, z = z_0) - A_{(L)\mu} (x, z = z_0) = 0. \]  

Then the 5D gauge transformations for \( R_\mu \) and \( L_\mu \) at the point \( z = z_0 \) must be equal. The chiral field can be defined as

\[ U (x) \equiv \xi_R (x, z_0) \xi_L^\dagger (x, z_0), \]

where the Wilson line is defined as

\[ \xi_{R(L)} (x, z) \equiv P \left\{ e^{i \int_0^z dx A_{(R(L))\mu} (x, z)} \right\}, \]

with \( P \) denoting path-ordered integral. The equality of the 5D gauge symmetry at \( z = z_0 \) enforces the following transformation law for the chiral field

\[ U (x) \rightarrow g_R (x) U (x) g_L^\dagger (x). \]

where \((g_R, g_L)\) represent the 5D gauge symmetries located on the UV brane, which are then interpreted as the 4D \( \text{SU}(N_f) \times \text{SU}(N_f) \) chiral symmetry. A vacuum state with \( U = 1 \) naturally leads to the spontaneous breaking of the symmetry group to the vector part.

To separate the dynamical fields and external sources from \( A_{(L)} \) and \( A_{(R)} \), one should first make a gauge transformation using the above Wilson lines:

\[ \hat{V}_M, \hat{A}_M \equiv i \left\{ \xi_L^\dagger (\partial_M - iA_{(L)\mu}) \xi_L \pm (L \rightarrow R) \right\}. \]

After making this transformation we are then working in the axial gauge, \( \hat{V}_z = \hat{A}_z = 0 \). For the vector part, one can simply make the following substraction

\[ V_\mu (x, z) \equiv \hat{V}_\mu (x, z) - \hat{V}_\mu (x, z_0), \]

and the dynamics of \( V_\mu (x, z) \) is completely the same as in the original hard-wall model. However, to remove the effect of the UV source of the axial field on the IR, a function \( \alpha(z) \) has to be introduced with the boundary values

\[ \alpha(0) = 1, \alpha(z_0) = 0. \]
Subsequently, the axial field can be decomposed as
\[ A_\mu (x, z) \equiv \hat{A}_\mu (x, z) - \alpha (z) \hat{A}_\mu (x, z_0). \] (46)

Since \( \hat{A}_\mu (x, z_0) \) contains the derivative of the chiral field \( U \), \( \alpha (z) \) will play the role of the 5D wave function of the pion. Moreover, to eliminate the mixing of the dynamical axial field and the pion, \( \alpha (z) \) must satisfy
\[ \partial_z \left( \frac{1}{z} \partial_z \alpha \right) = 0. \] (47)
Together with the aforementioned boundary conditions, this fixes \( \alpha (z) \) to be of the form
\[ \alpha (z) = 1 - z^2/z_0^2. \] (48)

Substituting these decompositions into the original action, one can naturally deduce the chiral lagrangian, with all the low energy constants given by simple integrals of \( \alpha (z) \). Most importantly, one has
\[ f_\pi^2 = \frac{1}{g_5^2} \int_0^{z_0} \frac{dz}{z} (\partial_z \alpha)^2 = \frac{2}{g_5^2 z_0^2}. \] (49)

If \( g_5 \) and \( z_0 \) were fixed as before, we would have \( f_\pi \simeq 72.7 \) MeV, which is somewhat smaller than the experimental value. Due to this drawback, we will only repeat some of the previous calculations in this model.

Again we start from the \( \gamma^*\gamma^*\pi^0 \) form factor in this model, which has already been derived in Ref. [36] along the same line as in the hard-wall model:
\[ F_{\gamma^*\gamma^*\pi^0} (Q_1^2, Q_2^2) = -\frac{N_c}{12\pi^2 f_\pi} \int_0^{z_0} \mathcal{J}(Q_1, z) \mathcal{J}(Q_2, z) \partial_z \alpha (z) \, dz, \] (50)
where the normalization constant \( k \) of the CS term has also been chosen to be 2. This is enough to ensure the anomaly relation since \( \alpha(z_0) = 0 \). No surface term at the IR boundary needs to be introduced. From the above expression we see that \( \alpha (z) \) indeed plays the role of pion wave function, as \( \Psi (z) \) does in the original hard-wall model. Moreover, the behavior of these two functions near the UV boundary are also the same, since
\[ \partial_z \alpha (z) = -2z/z_0^2 = -f_\pi^2 g_5^2 z. \] (51)
From this one can conclude that the asymptotic behavior of the \( \gamma^*\gamma^*\pi^0 \) form factor must be the same as in the hard-wall model, which was found in Ref. [36].
The \( \gamma^* \rho^0 \pi^0 \) form factor can be derived as in previous sections, which is given by

\[
F_{\rho^0 \pi^0}(Q^2) = -\frac{N_c}{12\pi^2 f_\pi} \frac{g_5 m_\rho}{2} \left[ \int_0^{z_0} \mathcal{J}(Q, z) \psi_1^V(z) \partial_z \alpha(z) dz \right].
\]

(52)

For the same reason as preceding discussion, its asymptotic behavior is the same as Eq. (31). The \( \rho^0 \pi^0 \gamma \) coupling can also be obtained

\[
g_{\rho^0 \pi^0 \gamma} = -\frac{N_c}{12\pi^2 f_\pi} \frac{g_5 m_\rho}{2} \int_0^{z_0} \psi_1^V(z) \partial_z \alpha(z) dz.
\]

(53)

Substituting the experimental value of \( f_\pi \) in the normalization factor, we get \( g_{\rho^0 \pi^0 \gamma} = 0.65 \), in reasonable agreement with the hard-wall result and those derived from other approaches. In ref. [37], an exhaustive list of the three-point and four-point couplings was given for the Hin-Sanz model. The corresponding value for the \( \rho \pi \gamma \) vertex is \( g_{\rho \pi \gamma} = 0.06 \, f_\pi^{-1} \text{GeV} \approx 0.649 \), confirming our result. Similar result was also obtained in refs. [21, 38].

V. SUMMARY

In this work, the \( \gamma^* \rho^0 \rightarrow \pi^0 \) transition form factor has been extracted from the \( \gamma^* \gamma^* \pi^0 \) form factor, which has been obtained in the extended hard-wall AdS/QCD model including a Chern-Simons term. As expected from pQCD, the form factor exhibits the \( 1/Q^4 \) asymptotic behavior, but with a rather different mechanism. It comes out only after we integrate the meson solution with the bulk-to-boundary propagator along the holographic direction. The power is then determined by the \( z \rightarrow 0 \) behavior of the meson solution. The appearance of this power behavior is very similar to that in the LCSR approach, where the power is dictated by the end-point behavior of the Light-Cone distribution amplitude. Comparing the corresponding expressions, one can deduce the dual relation \( z = \sqrt{u(1-u)b} \) with \( b \) a light-cone distance parameter, which is just one of the important relations in the Light-Front holography. Since the numerical results of the form factor in the LCSR approach strongly depend on the profile of the pion distribution amplitude \( \phi_\pi(u) \), the present analysis can help to discriminate between various models for \( \phi_\pi(u) \). As in the discussion for the \( \gamma^* \gamma^* \pi^0 \) form factor, our result favors an asymptotic-like pion distribution amplitude. From the form factor at \( Q^2 = 0 \) we obtains the partial width of the radiative decays \( \rho^0(\omega) \rightarrow \pi^0 \gamma \). We also extend our analysis by analytically continuing the bulk-to-boundary propagator to the time-like region. The Dalitz decays \( \rho^0(\omega) \rightarrow \pi^0 e^+e^-, \pi^0 \mu^+\mu^- \) are then studied. All these
decay rates are roughly consistent with the available measured values. The quark mass corrections are found to be very small, as expected.

Some of the calculations have been performed in the Hirn-Sanz model, which successfully describes the spontaneous chiral symmetry breaking in a simple way. Just as in the case of the $\gamma^*\gamma^*\pi^0$ form factor, the asymptotic behavior of the $\gamma^*\rho^0\pi^0$ form factor in this model is exactly the same as in the standard hard-wall model. The $\gamma\rho^0\pi^0$ coupling is also in reasonable agreement with the hard-wall result.

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