Proto-exact categories of matroids, Hall algebras, and K-theory

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Abstract
This paper examines the category $\text{Mat}_\bullet$ of pointed matroids and strong maps from the point of view of Hall algebras. We show that $\text{Mat}_\bullet$ has the structure of a finitary proto-exact category - a non-additive generalization of exact category due to Dyckerhoff-Kapranov. We define the algebraic K-theory $K_n(\text{Mat}_\bullet)$ of $\text{Mat}_\bullet$ via the Waldhausen construction, and show that it is non-trivial, by exhibiting injections

$$\pi^n_*(\mathbb{S}) \hookrightarrow K_n(\text{Mat}_\bullet)$$

from the stable homotopy groups of spheres for all $n$. Finally, we show that the Hall algebra of $\text{Mat}_\bullet$ is a Hopf algebra dual to Schmitt’s matroid-minor Hopf algebra.

Keywords Matroid · Matroid strong maps · Matroid-minor Hopf algebra · Hall algebra · Proto-exact category · K-theory

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1 Introduction

In this paper we examine the category of pointed matroids and strong maps from the perspective of Hall algebras. This perspective sheds new light on certain combinatorial Hopf algebras built from matroids and opens the door to defining algebraic K-theory of matroids. This introduction is devoted to introducing the main actors.

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1.1 Hall algebras of Abelian and exact categories

The study of Hall algebras is now well-established, with several applications in representation theory and algebraic geometry; see [22] for an overview. We now recall the most basic version of this construction. Given an abelian category $C$ such that both $\text{Hom}(M, M')$ and $\text{Ext}^1(M, M')$ are finite sets for any pair of objects $M, M' \in C$ (i.e. so that $C$ is finitary), one may define the Hall algebra of $C$ as follows. As a vector space

$$H_C = \mathbb{Q}_c[\text{Iso}(C)],$$

where the right-hand side denotes the space of $\mathbb{Q}$-valued functions on isomorphism classes in $C$ with finite support. The (associative) multiplication on $H_C$ is given by

$$(f \cdot g)([R]) = \sum_{Q \subseteq R} f([R/Q])g([Q]),$$

where $[R]$ denotes the isomorphism class of $R \in C$ and the sum is over all sub-objects $Q \subseteq R$.

The Hall algebra $H_C$ is spanned by $\delta$-functions $\delta_{[M]}$, $[M] \in \text{Iso}(C)$ supported on individual isomorphism classes. The product (1) can then be explicitly written

$$\delta_{[M]} \cdot \delta_{[N]} = \sum_{R \in \text{Iso}(C)} P^R_{M,N} \delta_R,$$

where

$$P^R_{M,N} := \# \{ L \subseteq R \mid L \simeq N, R/L \simeq M \}.$$ 

The number

$$P^R_{M,N} | \text{Aut}(M) || \text{Aut}(N)|$$

counts the isomorphism classes of short exact sequences of the form

$$0 \rightarrow N \rightarrow R \rightarrow M \rightarrow 0,$$

where $\text{Aut}(M)$ is the automorphism group of $M$. Thus, the product in $H_C$ encodes the structure of extensions in $C$.

The recipe above extends more generally to the case where $C$ is a finitary Quillen exact category (see [14]). Exact categories can be viewed as strictly full extension-closed subcategories of Abelian categories, and can be equivalently described in terms of classes $(\mathcal{M}, \mathcal{E})$ of admissible mono/epi-morphisms. For example, the category of vector bundles (i.e. locally free sheaves) on a smooth projective curve $X/\mathbb{F}_q$ is an exact category with $(\mathcal{M}, \mathcal{E})$ consisting of those monos/epis which are locally split. It is not Abelian, as the cokernel of a morphism of locally free sheaves may be a coherent sheaf that is not locally free. In this example, $\text{Iso}(C)$ is the domain of definition of automorphic forms for general linear groups over the function field $\mathbb{F}_q(X)$; moreover the Hall multiplication encodes the action of Hecke operators. Thus the theory makes contact with the Langlands program over function fields (see the beautiful papers [15,19]). We also refer the reader, who is interested in the Hall algebra of a function field, to [1].
1.2 Hall algebras in a non-additive setting

A closer examination of the construction of $H_C$ outlined above shows the assumption that $C$ is additive is unnecessary. The product (1) makes sense in any category with a well-behaved notion of exact sequences. In [7] proto-exact categories are introduced as non-additive generalizations of Quillen exact categories, and shown to suffice for the construction of an associative Hall algebra. As in the additive case, such a category is defined in terms of a pair $(\mathcal{M}, \mathcal{E})$ of admissible mono/epis which are required to satisfy certain properties. The simplest example of a non-additive proto-exact category is the category $\text{Set}^{\text{fin}}_*$ of finite pointed sets with $\mathcal{M}$ all pointed injections and $\mathcal{E}$ the pointed surjections which are isomorphisms away from the base-point.

Many non-additive proto-exact categories $C$ arise in combinatorics. In such cases $\text{Ob}(C)$ typically consist of combinatorial structures equipped with operations of “inserting” and “collapsing” sub-structures corresponding to $(\mathcal{M}, \mathcal{E})$. Examples include trees, graphs, posets, semigroup representations in $\text{Set}^{\text{fin}}_*$, and quiver representations in $\text{Set}^{\text{fin}}_*$ among others (see [17,24–27]). The product in $H_C$, which counts all extensions between two objects, thus enumerates all combinatorial structures which can be assembled from two others. Here $H_C$ is typically (dual to) a combinatorial Hopf algebra in the sense of [18]. Many combinatorial Hopf algebras, including the Hopf algebra of symmetric functions and the Connes-Kreimer Hopf algebras of rooted trees and Feynman graphs, arise in this way.

1.3 The Waldhausen $S$-construction and K-theory of proto-exact categories

Naturally one wonders what advantages there are to thinking of combinatorial objects in terms of proto-exact categories and Hall algebras. One answer is that certain constructions are only visible at the categorical level. In [7], the authors use the Waldhausen $S$-construction to associate a simplicial groupoid $S_* C$ to a proto-exact category $C$ where $S_n C$ parametrizes $n$-step flags of objects in $C$; this structure $S_* C$ has a number of interesting properties, including the structure of a 2-Segal space—a form of higher associativity with $H_C$ as a shadow. Other papers including [9–11] have studied this structure from a somewhat different perspective.

As in [7], $S_* C$ may be used to define the algebraic K-theory of $C$ by

$$K_n(C) = \pi_{n+1}|S_* C|$$

where $|S_* C|$ denotes the geometric realization of $S_* C$. These groups contain interesting homotopy-theoretic information even for simple categories like $\text{Set}^{\text{fin}}_*$, as evidenced by the following result.

**Theorem 1.1** [3,5] One has $K_*(\text{Set}^{\text{fin}}_*) \simeq \pi_*(S)$, where the right hand side denotes the stable homotopy groups of the sphere spectrum.

1.4 Matroids as a proto-exact category

Matroids are combinatorial structures which abstract various notions of independence across mathematics. A matroid $M$ consists of a finite set $E_M$ (the ground set) together with a collection of flats $\mathcal{F}(M) \subseteq 2^{E_M}$ satisfying certain natural properties. The prototypical example is obtained by taking $E_M$ to be a set of vectors in some vector space $V$, and taking $\mathcal{F}(M) \subseteq 2^{E_M}$ to be the subsets $S \subseteq E_M$ with $S = \text{Span}(S) \cap E_M$. Matroids and their generalizations have a
deep connection to several areas of mathematics; for example, in tropical geometry \textit{valuated matroids} play the role of linear spaces.

Matroids form a category \textbf{Mat} with \textit{strong maps}, a generalization of linear map in this setting. The category \textbf{Mat} is the object of study in this paper. Other aspects of \textbf{Mat} have also been studied in [13]. For technical reasons, we prefer to work with \textit{pointed matroids}, where the ground set \( E_M \) is pointed by a distinguished \textit{loop}. Starting with a minor-closed collection \( \mathcal{M} \) of pointed matroids, and denoting by \( \textbf{Mat}(\mathcal{M}) \) the full subcategory of \( \textbf{Mat} \) it generates, we show the following:

\textbf{Theorem 1} (Theorem 5.11) The category \( \textbf{Mat}(\mathcal{M}) \) is a finitary proto-exact category, with \textit{Matroid restrictions} and \textit{Matroid contractions}.

We give a matroid-theoretic proof of this theorem, which reduces to verifying the existence of certain special pushouts/pull-backs in \( \textbf{Mat} \). We then proceed to define and study the \textit{algebraic K-theory} of \( \textbf{Mat} \) via Definition 6.1. The category \( \textbf{Mat} \) has an exact forgetful functor to \( \textbf{Set}^{\text{fin}} \) with an exact left adjoint sending \( E \in \textbf{Set}^{\text{fin}} \) to the free pointed matroid on \( E \). These functors can be used to relate \( K_*(\textbf{Set}^{\text{fin}}) \) and \( K_*(\textbf{Mat}) \). We show the following two results.

\textbf{Theorem 2} (Theorem 6.3) One has \( K_0(\textbf{Mat}) \simeq \mathbb{Z} \oplus \mathbb{Z} \).

\textbf{Theorem 3} (Theorem 6.4) For all \( n \geq 0 \) there is an injective group homomorphism

\[ \pi_n(\mathbb{S}) \simeq K_n(\textbf{Set}^{\text{fin}}) \hookrightarrow K_n(\textbf{Mat}). \]

In particular, \( K_n(\textbf{Mat}) \) is non-trivial for \( n \geq 0 \).

As a corollary of Theorem 1, we may define the Hall algebra \( H_{\textbf{Mat}(\mathcal{M})} \). The product \([M]\ast [N]\) in this algebra enumerates all classes of matroids \([L] \in \mathcal{M}_{\text{iso}}\) with \( L|S \simeq M \) and \( L/S \simeq N \) for some subset \( S \subseteq E_L \). This Hall algebra \( H_{\textbf{Mat}(\mathcal{M})} \) turns out to be dual to the \textit{matroid-minor Hopf algebra} introduced by Schmitt in [23]. We obtain the following.

\textbf{Theorem 4} (Theorem 7.3) Let \( \mathcal{M} \) be a collection of pointed matroids closed under taking pointed minors and direct sums. Then the Hall algebra \( H_{\textbf{Mat}(\mathcal{M})} \) has the structure of a graded, connected, co-commutative Hopf algebra, dual to Schmitt’s matroid-minor Hopf algebra.

\textbf{1.5 Outline of this paper}

In Sects. 2 and 3 we recall basic notions regarding pointed matroids, Schmitt’s matroid-minor Hopf algebra, and its dual. Proto-exact categories are reviewed in Sect. 4. Section 5 contains a proof of Theorem 1. In Sect. 6 we define the K-theory of \( \textbf{Mat} \) and prove Theorems 2 and 3. Finally in Sect. 7 we recall the notion of a Hall algebra of a proto-exact category and prove Theorem 4.

\textbf{2 Matroids and strong maps}

This section provides a short introduction to the basic terminology and results of matroid theory we use in this paper. The reader is encouraged to see [21] for more details.
Matroids are combinatorial abstractions of various properties of linear independence among finitely many vectors in a vector space. These objects admit a number of equivalent definitions; i.e., there are a variety of ways of formulating the axioms for matroids, any of which is equivalent to any other by what amounts to an isomorphism of categories (called a “cryptomorphism” among matroid theorists). In this paper, we will use only one of these perspectives, namely that of flats.

We shall now define matroids via the flats axioms. Let $E$ be a finite set. Given a collection $C \subseteq 2^E$ of subsets of $E$ and $A, B \in C$, we say $B$ covers $A$ when $A \subset B$ and for all $C \in C$ we have $A \subseteq C \subseteq B$ implies either $C = A$ or $C = B$. A collection $\mathcal{F} \subseteq 2^E$ is the set of flats of a matroid on $E$ when it satisfies the following axioms.

(F1) The set $E \in \mathcal{F}$.
(F2) For all $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$.
(F3) For all $A \in \mathcal{F}$ the set $\{B \setminus A \mid B \text{ covers } A\}$ is a partition of $E \setminus A$.

We say $E$ is the ground set of the matroid $M$ given by flats $\mathcal{F} = \mathcal{F}(M)$.

**Example 2.1** The following are the prototypical examples of flats.\(^1\)

1. Let $E$ be a finite family of vectors in a vector space $V$. We let $S \subseteq E$ be a flat (i.e. $S \in \mathcal{F}$) when $E \cap \langle S \rangle = S$ where $\langle S \rangle$ denotes the the subspace of $V$ spanned by $S$. Then $\mathcal{F}$ is the set of flats of a vectorial matroid on $E$.
2. Let $\Gamma = (V, E)$ be a combinatorial graph with finite edge set. A set $S \subseteq E$ is a flat when the following condition holds: for all $e \in E$, if there is a path in the subgraph $(V, S)$ connecting the ends of $e$, then $e \in S$. Then $\mathcal{F}$ is the set of flats of a graphic matroid on $E$.

**Remark 2.2** We note the following.

1. Flats combinatorially abstract the vector subspace lattice in light of Example 2.1.1.
2. The graphs in Example 2.1.2 may have loops and multiple edges.

We now define the basic operations restriction and contraction for matroids.

**Definition 2.3** Let $M$ be a matroid on $E$ and $S \subseteq E$.

1. The restriction of $M$ to $S$ is the matroid $M|S$ on $S$ with flats

$$\mathcal{F}(M|S) = \{A \cap S \mid A \in \mathcal{F}(M)\}.$$  

2. The contraction of $M$ by $S$ is the matroid $M/S$ on $E \setminus S$ with flats

$$\mathcal{F}(M/S) = \{A \setminus S \mid S \subseteq A \in \mathcal{F}(M)\}.$$  

3. Any matroid which can be obtained from $M$ by a sequence of restrictions and contractions is a minor of $M$.

The following examples are meant to give the reader an idea of matroid restriction and contraction in our two running examples.

**Example 2.4** Let $M$ be a vectorial matroid obtained from a set $E \subseteq V$ as in Example 2.1.1, and let $S \subseteq E$.

\(^1\) Not all matroids arise in this way. For more details, see [21].
1. The restriction $M|S$ is obtained again as in Example 2.1.1 from $S$ in the vector space $\langle S \rangle$. In particular, matroid restriction corresponds to restriction to a subspace in the vectorial case.

2. The contraction $M/S$ is obtained again as in Example 2.1.1 from $E\setminus S$ in the vector space $V/\langle S \rangle$. In particular, matroid contraction corresponds to subspace quotient in the vectorial case.

**Example 2.5** Let $M$ be a graphic matroid obtained from graph $\Gamma = (V, E)$ as in Example 2.1.2, and let $S \subseteq E$.

1. The restriction $M|S$ is obtained again as in Example 2.1.2 from the restriction graph $\Gamma|S = (V, S)$. Thus matroid restriction corresponds to graph restriction in the graphic case.

2. The contraction $M/S$ is obtained again as in Example 2.1.2 from the contracted graph $\Gamma/S$ in which the edges of $S$ are removed and their endpoints identified. Thus matroid contraction corresponds to graph contraction in the graphic case.

The following is a basic result concerning minors which we will use later. We refer the reader to [21] for a proof.

**Lemma 2.6** For every matroid $M$ and every $T \subseteq S \subseteq E_M$ we have $(M|S)/T = (M/T)|S$.

Next we define the notion of direct sums of matroids, which will be necessary when defining the matroid-minor Hopf algebra.

**Definition 2.7** The direct sum $M_1 \oplus M_2$ of matroids $M_1$ and $M_2$ is the matroid on ground set $E_1 \sqcup E_2$ with flats of the form $F_1 \sqcup F_2$ where $F_1 \in \mathcal{F}(M_1)$ and $F_2 \in \mathcal{F}(M_2)$.

**Example 2.8** Let $M_1$ and $M_2$ be matroids on ground sets $E_1$ and $E_2$ respectively.

1. If $M_1$ and $M_2$ are obtained from subsets $E_i \subseteq V_i$ for $i = 1, 2$ of vector spaces $V_1$, $V_2$ over the same field as in Example 2.1.1, then $M_1 \oplus M_2$ is obtained from $E_1 \sqcup E_2 \subseteq V_1 \oplus V_2$. In particular, the direct sum of matroids corresponds to the direct sum of vector spaces in this special case.

2. If $M_1$ and $M_2$ are obtained from graphs $\Gamma_1$ and $\Gamma_2$ as in Example 2.1.2, then $M_1 \oplus M_2$ is the corresponding matroid of the disjoint union $\Gamma_1 \sqcup \Gamma_2$. In particular, the direct sum of matroids corresponds to the disjoint union in the graphic case.

A loop of a matroid is an element of the ground set which belongs to the (unique) minimal flat. A pointed matroid is a pair $(M, *_M)$ where $M$ is a matroid and $*_M$ is a distinguished loop (a.k.a. the base-point). We think of the base-point as a zero element. We often suppress the base-point in our notation and say $M$ is a pointed matroid with base-point $*_M$.

**Remark 2.9** For pointed matroids $(M, *_M)$, we adopt the following conventions to simplify notation and ensure that the result of various operations is also a pointed matroid. Let $\tilde{E}_M = E_M \setminus \{*_M\}$ so that $E_M = \tilde{E}_M \sqcup \{*_M\}$, and let $S \subseteq \tilde{E}_M$.

1. The pointed restriction $M|S$ is the usual restriction $M|(S \cup \{*_M\})$ with the base-point preserved.

2. The pointed contraction $M/S$ is the usual contraction $M/S$ with the base-point preserved; we do not allow contraction of the base-point.

This is exactly a loop of the graph in the graphic case, Example 2.1.2.
3. For pointed matroids \((M_1, *,_1), (M_2, *,_2)\), the pointed direct sum \(M_1 \oplus M_2\) is defined as the pointed matroid on the ground set \(E_1 \cup E_2\) with the same flats as the usual direct sum, up to identifying base-points \(*_1\) and \(*_2\) to form the new base-point.

4. For a pointed matroid \((M, *,_M)\), a pointed minor of \(M\) is a pointed matroid obtained from \((M, *,_M)\) by a sequence of restrictions and contractions as above.

With the idea that pointed matroids are generalizations of vector spaces (the base-point being the zero-vector), one naturally seeks an appropriate analogue of linear maps.

**Definition 2.10** Let \(M\) and \(N\) be pointed matroids. A (pointed) strong map \(f : M \to N\) is a function \(f : E_M \to E_N\) with \(f(*_M) = *_N\) and \(f^{-1}A \in \mathcal{F}(M)\) for all \(A \in \mathcal{F}(N)\).

**Example 2.11** The following are the prototypical examples of strong maps.\(^3\)

1. Let \(M\) and \(N\) be pointed vectorial matroids arising from \(k\)-vector spaces \(V_M\) and \(V_N\) respectively for some common field \(k\) as in Example 2.1.1. Every linear map \(L : V_M \to V_N\) such that \(L(E_M) \subseteq E_N\) yields a strong map \(M \to N\).

2. Let \(\Gamma\) and \(\Lambda\) be graphs with distinguished loops \(*_{\Gamma}\) and \(*_{\Lambda}\), and let \(M_{\Gamma}\) and \(M_{\Lambda}\) be the pointed graphic matroids arising from \(\Gamma\) and \(\Lambda\) as in Example 2.1.2. Every graph morphism \(f : \Gamma \to \Lambda\) with \(f(*_{\Gamma}) = *_{\Lambda}\) yields a strong map \(M_{\Gamma} \to M_{\Lambda}\).

3. Let \(M\) be a pointed matroid, and \(S \subset \tilde{E}_M\). The inclusion of pointed sets \(S \cup \{*_M\} \hookrightarrow E_M\) and contraction \(E_M \twoheadrightarrow E_M/S\) induce strong maps \(M|S \hookrightarrow M\) and \(M \twoheadrightarrow M/S\).

We denote by \(\textbf{Set}_\bullet\) the category of pointed sets (with pointed maps as morphisms), and by \(\textbf{Set}_\text{fin}_\bullet\) the full subcategory of \(\textbf{Set}_\bullet\) with objects the pointed finite sets. The following lemma is easy to verify; it can also be found in \([13]\).

**Lemma 2.12** We have the following:

1. Pointed matroids and strong maps form a category \(\textbf{Mat}_\bullet\).

2. The category \(\textbf{Mat}_\bullet\) has finite coproducts given by direct sums of pointed matroids.

3. There is a forgetful functor \(\mathbb{F} : \textbf{Mat}_\bullet \hookrightarrow \textbf{Set}_\text{fin}_\bullet\) which takes a pointed matroid \((M, *_M)\) to its underlying pointed ground set \((E_M, *_M)\).

### 3 The Matroid-Minor Hopf algebra and its dual

In this section we recall the matroid-minor Hopf algebra introduced by Schmitt \([23]\) and its Hopf dual, adapted to the case of pointed matroids.

Let \(\mathcal{M}\) be a collection of pointed matroids, closed under taking pointed minors and direct sums, and let \(\mathcal{M}_{\text{iso}}\) be the set of isomorphism classes of pointed matroids in \(\mathcal{M}\). Denote by \([M]\) the isomorphism class of a pointed matroid \(M\) in \(\mathcal{M}\). The set \(\mathcal{M}_{\text{iso}}\) is equipped with a natural commutative monoid structure, via

\[
[M_1] \cdot [M_2] := [M_1 \oplus M_2],
\]

with the identity \([(\{\}, \{\})]\), the equivalence class of the zero pointed matroid. Let \(k[\mathcal{M}_{\text{iso}}]\) be the monoid algebra of \(\mathcal{M}_{\text{iso}}\) over a field \(k\).

In \([23]\) Schmitt constructs a comultiplication and counit, given below.

---

\(^3\) Not all strong maps between matroids arise in this way (even if the matroids themselves do).
(Coproduct)
\[ \Delta : k[\mathcal{M}_{\text{iso}}] \to k[\mathcal{M}_{\text{iso}}] \otimes_k k[\mathcal{M}_{\text{iso}}], \quad [M] \mapsto \sum_{S \subseteq \tilde{E}_M} [M|S] \otimes [M/S] \]

(Counit)
\[ \varepsilon : k[\mathcal{M}_{\text{iso}}] \to k, \quad [M] \mapsto \begin{cases} 1 & \text{if } \tilde{E}_M = \emptyset \\ 0 & \text{otherwise}, \end{cases} \]

The algebra \( k[\mathcal{M}_{\text{iso}}] \) carries a natural grading, where \( \deg(M, \ast_M) = \# \tilde{E}_M \). With the above maps and grading, \( k[\mathcal{M}_{\text{iso}}] \) becomes a graded connected bialgebra. We let \( k[\mathcal{M}_{\text{iso}}]_n \) be the \( n \)-th graded piece of \( k[\mathcal{M}_{\text{iso}}] \). From the result of Takeuchi [28], \( k[\mathcal{M}_{\text{iso}}] \) has a unique Hopf algebra structure with a unique antipode \( S \) given by
\[
S = \sum_{i \in \mathbb{N}} (-1)^i m^{-i-1} \circ \pi \otimes \Delta^{i-1}, \tag{5}
\]
where \( m^{-1} \) is a canonical injection from \( k \) to \( k[\mathcal{M}_{\text{iso}}] \), \( \Delta^{-1} := \varepsilon \), and \( \pi : k[\mathcal{M}_{\text{iso}}] \to k[\mathcal{M}_{\text{iso}}] \) is the projection map defined by
\[
\pi|_{k[\mathcal{M}_{\text{iso}}]} = \begin{cases} \text{id} & \text{if } n \geq 1 \\ 0 & \text{if } n = 0 \end{cases}
\]
and extended linearly to \( k[\mathcal{M}_{\text{iso}}] \). We further note that the maps \( m^r \) and \( \Delta^r \) for \( r \geq 1 \) are defined inductively as follows:
\[
m^r := m \circ (\text{id} \otimes m^{r-1}), \quad \Delta^r := (\text{id} \otimes \Delta^{r-1}) \circ \Delta.
\]

**Remark 3.1** The requirement that \( \mathcal{M} \) is closed under direct sums is needed to define the algebra structure, and closure under taking minors is required for the coalgebra structure. If \( \mathcal{M}^1 \subseteq \mathcal{M}^2 \), then \( k[\mathcal{M}_{\text{iso}}]^1 \) is a Hopf subalgebra of \( k[\mathcal{M}_{\text{iso}}]^2 \).

The Hopf algebra \( (k[\mathcal{M}_{\text{iso}}], m, \Delta, \varepsilon, S) \) in the case of ordinary (unpointed) matroids is called the matroid-minor Hopf Algebra over the field \( k \).

The graded dual Hopf algebra \( k[\mathcal{M}_{\text{iso}}]^* \) of \( k[\mathcal{M}_{\text{iso}}] \) is described explicitly in [2,16]. As a vector space, \( k[\mathcal{M}_{\text{iso}}]^* \) may be identified with the space of \( k \)-valued functions on \( \mathcal{M}_{\text{iso}} \) with finite support, i.e.
\[
k[\mathcal{M}_{\text{iso}}]^* \simeq \{ f : \mathcal{M}_{\text{iso}} \to k \mid \# \text{supp}(f) < \infty \}
\]
The product in \( k[\mathcal{M}_{\text{iso}}]^* \) is given by the convolution
\[
f \diamond g([M]) = \sum_{S \subseteq \tilde{E}_M} f([M|S])g([M/S]) \tag{6}
\]
and coproduct
\[
\Delta(f)([M], [N]) := f([M \oplus N]).
\]

**Remark 3.2** We will re-interpret \( k[\mathcal{M}_{\text{iso}}]^* \) in the framework of Hall algebras in Sect. 5.
4 Proto-exact categories

In this section, we recall the definition of proto-exact category $E$. This is a generalization of a Quillen exact category allowing $E$ to be non-additive while still providing enough structure to define an associative Hall algebra by counting distinguished exact sequences in $E$. We direct the interested reader to [7] for details and proofs. We denote monomorphisms in $E$ by $\hookrightarrow$ and epimorphisms by $\twoheadrightarrow$.

A commutative square

$$
\begin{array}{ccc}
A & \xrightarrow{i} & B \\
\downarrow{i'} & & \downarrow{j'} \\
A' & \xrightarrow{i'} & B'
\end{array}
$$

(7)

is biCartesian if it is both Cartesian and co-Cartesian.

**Definition 4.1**  A proto-exact category is a category $E$ equipped with two classes of morphisms $M$ and $E$, called admissible monomorphisms and admissible epimorphisms respectively, satisfying the following properties.

1. The category $E$ has a zero object $0$. Any morphism $0 \to A$ is in $M$, and any morphism $A \to 0$ is in $E$.
2. The classes $M$ and $E$ are closed under composition and contain all isomorphisms.
3. A commutative square (7) in $E$ with $i, i' \in M$ and $j, j' \in E$ is Cartesian iff it is co-Cartesian.
4. Every diagram in $E$

$$
\begin{array}{ccc}
A' & \xleftarrow{i'} & B' \\
\xrightarrow{j'} & & \xrightarrow{j} \\
A & \xleftarrow{i} & B
\end{array}
$$

with $i' \in M$ and $j' \in E$ can be completed to a biCartesian square (7) with $i \in M$ and $j \in E$.
5. Every diagram in $E$

$$
\begin{array}{ccc}
A' & \xleftarrow{j} & A \\
\xleftarrow{i} & & \xrightarrow{i'} B
\end{array}
$$

with $i \in M$ and $j \in E$ can be completed to a biCartesian square (7) with $i' \in M$ and $j' \in E$.

Setting $A' = 0$ in Property (5) implies that every $A \to B$ in $M$ has a cokernel $B \twoheadrightarrow B/A$. A biCartesian square with horizontal maps in $M$, vertical maps in $E$, and having the form

$$
\begin{array}{ccc}
A & \xleftarrow{i} & B \\
\downarrow & & \downarrow \\
0 & \xleftarrow{} & C
\end{array}
$$

is an admissible short exact sequence or an admissible extension of $C$ by $A$, and will also be denoted

$$
A \hookrightarrow B \twoheadrightarrow C.
$$

(8)

Note that $C \simeq B/A$, as both are colimits of the same diagram. A functor $F : E \leftrightarrow D$ between proto-exact categories is exact when it preserves admissible short exact sequences.
Two extensions \( A \hookrightarrow B \rightarrow C \) and \( A \hookrightarrow B' \rightarrow C \) of \( C \) by \( A \) are \textit{equivalent} if there is a commutative diagram

\[
\begin{array}{ccc}
A & \hookrightarrow & B \\
\downarrow{\text{id}} & & \downarrow{\cong} \\
A & \hookrightarrow & B' \\
\end{array}
\]

The set of equivalence classes of such sequences is denoted \( \text{Ext}_E(C,A) \). Two admissible monomorphisms \( i_1: A \hookrightarrow B \) and \( i_2: A' \hookrightarrow B \) are \textit{isomorphic} if there is an isomorphism \( f: A \rightarrow A' \) with \( i_1 = i_2 \circ f \). The isomorphism classes in \( \mathfrak{M} \) are \textit{admissible subobjects}.

**Definition 4.2** A proto-exact category \( E \) is \textit{finitary} if, for every pair of objects \( A \) and \( B \), the sets \( \text{Hom}_E(A,B) \) and \( \text{Ext}_E(A,B) \) are finite.

**Example 4.3** The following are typical examples of proto-exact categories.

1. Any Quillen exact category is proto-exact, with the same exact structure. In particular, any Abelian category \( E \) is proto-exact with \( \mathfrak{M} \) all monomorphisms and \( \mathfrak{E} \) all epimorphisms respectively. The category \( \text{Rep}(Q, \mathbb{F}_q) \) of representations of a quiver \( Q \) over a finite field \( \mathbb{F}_q \), and the category \( \text{Coh}(X/\mathbb{F}_q) \) of coherent sheaves on a smooth projective variety over \( \mathbb{F}_q \) are both finitary Abelian.

2. The simplest example of a non-additive proto-exact category is the category \( \text{Set}_\bullet \) whose objects are pointed sets with pointed maps as morphisms. Here \( \mathfrak{M} \) consists of all pointed injections, and \( \mathfrak{E} \) all pointed surjections \( p: (S,*) \rightarrow (T,*) \) such that \( p|_{S\setminus p^{-1}(*)} \) is injective. The full subcategory \( \text{Set}_\bullet^{\text{fin}} \) of finite pointed sets is finitary.

**Remark 4.4** Another non-additive generalization of an exact category is that of a \textit{quasi-exact} category, introduced in [5]. In a quasi-exact category \( C \), the coproduct \( A \oplus B \) of any two objects \( A, B \in C \) is assumed to exist, and sequences of the form

\[ A \leftrightarrow A \oplus B \rightarrow B \]

are required to be exact. There are examples of proto-exact categories (see [12]) which do not satisfy these properties, and which are therefore not quasi-exact. One consequence of this is that the Grothendieck group of a proto-exact category (defined in Sect. 6) need not be Abelian.

## 5 Mat\(_\bullet\) as a proto-exact category

Our goal in this section is to show that \( \text{Mat}_\bullet \) (and more generally, any minor-closed collection of pointed matroids) has the structure of a finitary proto-exact category in the sense of [7]. We begin by exhibiting the classes of admissible monos/epis.

**Definition 5.1** Let \( \mathfrak{M} \) consist of all strong maps in \( \text{Mat}_\bullet \) that can be factored as

\[ N \sim M|S \hookrightarrow M, \]

and let \( \mathfrak{E} \) consist of all strong maps in \( \text{Mat}_\bullet \) that can be factored as

\[ M \twoheadrightarrow M/S \sim N \]

for some \( S \subseteq \widetilde{E}_M \).
We show \((\text{Mat}_*, \mathcal{M}, \mathcal{E})\) as above is a finitary proto-exact category. We first state some basic lemmas which will be used in what follows. For ease of notation, for a function \(f : A \to B\) and \(S \subseteq A\) we write \(fS\) for the image \(f(S)\) whenever there is no confusion. All matroids are pointed.

**Lemma 5.2** ([13]*Lemma 3.4) Let \(M\) and \(N\) be pointed matroids. A function \(f : E_M \to E_N\) is an isomorphism in \(\mathcal{E}\) precisely when \(f\) is a pointed bijection satisfying

\[
A \in \mathcal{F}(M) \iff fA \in \mathcal{F}(N).
\]

Given \(T \subseteq S \subseteq E_M\) we have as a corollary of [21]*Proposition 3.3.7 that

\[
\mathcal{F}((M|S)/T) = \{ (F \cap S)/T \mid T \subseteq F \in \mathcal{F}(M) \}. 
\]

Moreover, the following is a straightforward consequence of Lemma 5.2.

**Lemma 5.3** Let \(f : M \to N\) be an isomorphism in \(\text{Mat}_*\).

1. For all \(S \subseteq \tilde{\mathcal{E}}_M\) the map \(f|_S : M|S \to N|f S\) is an isomorphism.
2. For all \(S \subseteq \tilde{\mathcal{E}}_M\) the map \(f|_{E_M\setminus S} : M/S \to N/f S\) is an isomorphism.

**Lemma 5.4** ([13]*Lemmas 3.3 and 3.7) The following hold:

1. A morphism \(f\) is monic in \(\text{Mat}_*\) precisely when \(f\) is injective.
2. A morphism \(f\) is an epic in \(\text{Mat}_*\) precisely when \(f\) is surjective.

We further have the following. See Remark 2.9 for the notation for restriction and contraction for pointed matroids.

**Lemma 5.5** Let \(M\) be a matroid on \(E_M\) and \(S \subseteq \tilde{\mathcal{E}}_M\).

1. There is a canonical map \(i_S : M|S \hookrightarrow M\) in \(\text{Mat}_*\).
2. There is a canonical map \(c_S : M \twoheadrightarrow M/S\) in \(\text{Mat}_*\).

Now we are in a position to show \((\text{Mat}_*, \mathcal{M}, \mathcal{E})\) is a proto-exact category. The following are straightforward to verify.

1. Category \(\text{Mat}_*\) is equipped with a zero object, namely the pointed matroid \((\{\ast\}, \ast)\); furthermore, all morphisms both to and from the zero object are admissible.
2. The classes \(\mathcal{M}\) and \(\mathcal{E}\) are closed under composition and contain all isomorphisms.

We appeal frequently to the following Lemma, which we prove for its importance.

**Lemma 5.6** For all \(T \subseteq S \subseteq E_M\) with \(\ast_M \in S \setminus T\), the following is a biCartesian square in \(\text{Mat}_*\):

\[
\begin{array}{ccc}
M|S & \xrightarrow{i_S} & M \\
\downarrow c_T & & \downarrow c_T' \\
(M|S)/T & \xrightarrow{i'_S} & M/T 
\end{array}
\]

**Proof** Notice trivially that the above square commutes.
\textbf{Cartesian:} Suppose we have the following commutative diagram in $\text{Mat}_\bullet$.

\[
\begin{array}{ccc}
M \setminus S & \xrightarrow{i_S} & M \\
c_T \downarrow & & \downarrow c_T' \\
(M \setminus S) / T & \xrightarrow{i'_S} & M / T \\
\alpha & & \beta \\
\end{array}
\]

We claim $\gamma = \beta|_{EM \setminus T} : M / T \to N$ is the desired pushout morphism. Note $\beta = \gamma c_T'$ and $\alpha = \gamma i_S$ as pointed set maps. As $\mathbb{F}_\gamma$ is the pushout morphism of the corresponding square in $\text{Set}_\bullet^{\text{fin}}$, $\gamma$ is uniquely determined. Thus we need only show $\gamma$ is a strong map.

Let $F \in \mathcal{F}(N)$ be arbitrary. By assumption $\beta^{-1}F \in \mathcal{F}(M)$ and $\alpha^{-1}F \in \mathcal{F}((M \setminus S) / T)$; there is an $A \in \mathcal{F}(M)$ with $T \subseteq A$ and $\alpha^{-1}F = (A \setminus T) \cap S$. Moreover $\beta i_S = \alpha c_T$ yields $S \cap \beta^{-1}F = i_S^{-1} \beta^{-1}F = c_T^{-1} \alpha^{-1}F = T \cup \alpha^{-1}F = T \cup ((A \setminus T) \cap S)$.

Thus $T \subseteq S \cap \beta^{-1}F$ yields $T \subseteq \beta^{-1}F$, and so $\gamma^{-1}F = (\beta^{-1}F) \setminus T \in \mathcal{F}(M / T)$.

\textbf{Co-Cartesian:} Suppose we have the following commutative diagram in $\text{Mat}_\bullet$.

\[
\begin{array}{ccc}
N & \xrightarrow{\beta} & M \\
\alpha & & \downarrow c_T \\
(M \setminus S) / T & \xrightarrow{i'_S} & M / T \\
\end{array}
\]

Next we show $\beta(E_N) \subseteq S$. If $x \in E_N \setminus \beta^{-1}S$, then $\beta(x) \in E_M \setminus S$ and $\beta(x) = c_T^{-1} \beta(x)$ as $T \subseteq S$. But $\beta(x) = i'_S \alpha(x) = \alpha(x) \in S$, contradicting our initial assumption.

We claim $\gamma : E_N \to S : x \mapsto \beta(x)$ is the desired pullback morphism. Note $i_S \gamma = \beta$ and $\alpha = c_T \gamma$ as pointed set maps. As $\mathbb{F}_\gamma$ is the pullback morphism of the corresponding square in $\text{Set}_\bullet^{\text{fin}}$, $\gamma$ is uniquely determined (by uniqueness of $\mathbb{F}_\gamma$ in $\text{Set}_\bullet^{\text{fin}}$). Thus we need only show $\gamma$ is a strong map; letting $F \in \mathcal{F}(M)$, we see $\gamma^{-1}(F \cap S) = \beta^{-1}(F \cap S) = \beta^{-1}F \in \mathcal{F}(N)$ as $\beta$ is a strong map.

We can now complete the proof that $\text{Mat}_\bullet$ is proto-exact.

\textbf{Proposition 5.7} (Verifying Property 4) Every diagram $\begin{array}{c} P \xrightarrow{i'} Q \xleftarrow{j'} N \end{array}$ in $\text{Mat}_\bullet$ with $i'$ an admissible monic and $j'$ an admissible epic can be completed to a biCartesian square

\[
\begin{array}{ccc}
M & \xrightarrow{i} & N \\
\downarrow i & & \downarrow j' \\
P & \xleftarrow{i'} & Q \\
\end{array}
\]

for some $M \in \mathcal{E}$, $i \in \mathcal{M}$, $j \in \mathcal{E}$.

\textbf{Proof} Since $i' \in \mathcal{M}$ and $j' \in \mathcal{E}$, there exist $S \subseteq E_Q$ and $T \subseteq E_N$ such that $i' = i_S g$ and $j' = f c_T'$, where $g : P \to Q | S$ and $f : N / T \to Q$ are isomorphisms and $i_S : Q | S \to Q$ and $c_T : N \to N / T$ (as in Lemma 5.5). We will show that the matroid $M = N | (j')^{-1}(S)$ and canonical maps $i$ and $j$ yield the desired biCartesian square.

\[\square\] Springer
Let $A := (j')^{-1}(S)$. One observes $T \subseteq A$ and $*_{N} \in A \setminus T$; hence we obtain the following commutative diagram.

Now the following square commutes in $\text{Mat}_{\bullet}$, where $i = i_{A}$ and $j = g^{-1}f|_{A}cT$:

Using Lemma 2.6, we factor $j$ as $N|A \xrightarrow{c_{T}} (N|A)/T \xrightarrow{g^{-1}f|_{A}} P$. Thus $j$ is an admissible epic, noting $g^{-1}f|_{A}$ is an isomorphism by Lemma 5.3.

To see this square is Cartesian, note that every commuting diagram determines a corresponding commuting diagram

which admits a unique map $\delta: N/T \to M$ such that the diagram commutes by Lemma 5.6. Thus $\gamma = \delta f^{-1}$ is the pushout of the original square.

To see this square is coCartesian, note that every commuting diagram
determines a corresponding commuting diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\alpha} & N \\
\downarrow f|_{A}^{-1} \circ \beta & & \downarrow \xi_A \\
(N|A)/T & \xrightarrow{i_A} & N/T \\
\end{array}
\]

which admits a unique map \( \gamma : M \to N|A \) such that the diagram commutes by Lemma 5.6. Thus \( \gamma \) is the pullback of the original square. \( \square \)

**Proposition 5.8** [Verifying Property 5] Every diagram \( P \xleftarrow{i} M \xrightarrow{j} N \) in \( \text{Mat}_* \) with \( i \) an admissible monic and \( j \) and admissible epic can be completed to a biCartesian square

\[
\begin{array}{ccc}
M & \xrightarrow{i} & N \\
\downarrow j & & \downarrow j' \\
P & \xrightarrow{i'} & Q \\
\end{array}
\]

for some \( Q \in \text{Mat}_*, i' \in \mathcal{M}, j' \in \mathcal{E} \).

**Proof** We factor \( i \) and \( j \) to obtain the following commuting diagram:

\[
\begin{array}{ccc}
M & \xrightarrow{f} & N|S \\
\downarrow c_T & & \downarrow c_{jT} \\
M/T & \xrightarrow{f|_{E_M\setminus T}} & (N|S)/fT \\
\downarrow g & & \downarrow g|_{E_M\setminus fT} \\
P & \xrightarrow{id} & P \\
\end{array}
\]

Now the following square commutes in \( \text{Mat}_* \), where \( i' = i'_S f|_{E_M\setminus fT} g^{-1} \) and \( j' = c'_{jT} \):

\[
\begin{array}{ccc}
M & \xrightarrow{i} & N \\
\downarrow j & & \downarrow j' \\
P & \xrightarrow{i'} & N/fT \\
\end{array}
\]

We have factorizations \( N \xrightarrow{c_{jT}} N/fT \xrightarrow{id} N/fT \) and \( P \xrightarrow{f|_{E_M\setminus fT} g^{-1}} (N|S)/fT \xrightarrow{i'_S} N/fT \). Moreover \( f|_{E_M\setminus fT} g^{-1} \) is an isomorphism by Lemma 5.3. Hence \( i' \) is an admissible monic and \( j' \) is an admissible epic. What remains is similar to the proof of Proposition 5.7. \( \square \)
Proposition 5.9 (Verifying Property 3) A commuting square in $\text{Mat}_*$

$$
\begin{array}{ccc}
M & \xrightarrow{i} & N \\
\downarrow & & \downarrow'
\\
P & \xrightarrow{i'} & Q
\end{array}
$$

with $i, i'$ admissible monics and $j, j'$ admissible epics is Cartesian if and only if it is co-Cartesian.

**Proof** Suppose the above square is either Cartesian or co-Cartesian. By the previous propositions, both $P \xleftarrow{j} M \xrightarrow{i} N$ and $P \xrightarrow{i'} Q \xleftarrow{j'} N$ can be completed to biCartesian squares in $\text{Mat}_*$ having all arrows from $\mathfrak{M}$ and $\mathfrak{E}$. As pullback and pushout are natural constructions, the original square is necessarily biCartesian. $\square$

Note that the argument above is quite general; indeed, together Properties 4 and 5 imply Property 3. We have thus verified that $(\text{Mat}_*, \mathfrak{M}, \mathfrak{E})$ is a proto-exact category.

Remark 5.10 We note the following, which are easily verified.

1. The admissible sub-objects and quotient objects of $M \in \text{Mat}_*$ correspond respectively to matroids $M|S$ and $M/S$ for subsets $S \subseteq \mathfrak{E}M$.
2. The indecomposable objects of $\text{Mat}_*$ are precisely the connected pointed matroids.
3. The admissible sub-quotients of $M \in \text{Mat}_*$ are precisely the pointed minors of $M$.
4. The forgetful functor $F: \text{Mat}_* \hookrightarrow \text{Set}_*$ (or to $\text{Set}_{fin}^*$) is an exact functor of proto-exact categories.

We note also that $\text{Mat}_*$ is finitary. This follows immediately from the fact that $F$ is faithful, and that there are finitely pointed matroid structures on a finite pointed set.

The proofs in this section also show that the biCartesian completions of the diagrams from Definition (4.1) in $\text{Mat}_*$ are minors of the matroids in the diagrams. We may therefore strengthen our result as follows. Let $\mathcal{M}$ be a collection of pointed matroids which is closed under taking pointed minors, and let $\text{Mat}_*(\mathcal{M})$ denote the full sub-category of $\text{Mat}_*$ generated by objects in $\mathcal{M}$. We then obtain the following result.

**Theorem 5.11** The category $\text{Mat}_*(\mathcal{M})$ is a finitary proto-exact category. It is a full sub-category of $\text{Mat}_*$.

6 Algebraic K-theory of matroids

6.1 K-theory of proto-exact categories

We begin by recalling the construction of the algebraic K-theory of a proto-exact category following [7,12]. Let $\mathcal{E}$ be a proto-exact category and let $S_n = S_n(\mathcal{E})$ denote the maximal groupoid in the category of diagrams of the form...
where all horizontal maps are in $\mathfrak{M}$, all vertical maps are in $\mathcal{E}$, and all squares are biCartesian. For every $0 \leq k \leq n$, there is a functor

$$\partial_k: S_n \to S_{n-1}$$

obtained by omitting in the diagram (9) the objects in the $k$th row and $k$th column and forming the composite of the remaining morphisms. Similarly, for every $0 \leq k \leq n$, there is a functor

$$\sigma_k: S_n \to S_{n+1}$$

given by replacing the $k$th row by two rows connected via identity maps and replacing the $k$th column by two columns connected via identity maps. $S_\bullet(\mathcal{E})$ together with the $\partial_\bullet$, $\sigma_\bullet$ forms a simplicial object in the category of groupoids.

**Definition 6.1** The K-theory of $\mathcal{E}$ is defined by

$$K_n(\mathcal{E}) = \pi_{n+1}|S_\bullet\mathcal{E}|,$$

where $|S_\bullet\mathcal{E}|$ denotes the geometric realization of $S_\bullet\mathcal{E}$.

**Remark 6.2** One may also develop K-theory of proto-exact categories via a version of Quillen’s Q-construction. This approach leads to isomorphic K-groups as shown in [12] (or see, [3]). In particular, Theorem 1.1 remains valid if the K-theory of $\mathcal{E}_{fin}^\bullet$ is defined as in Definition 6.1, despite the fact that its original proof used the Q-construction.

The Grothendieck group $K_0(\mathcal{E})$ can be described explicitly as the free group on symbols $[A], A \in \text{Iso}(\mathcal{E})$, modulo the relations $[B] = [A][C]$ for each admissible short exact sequence

$$A \hookrightarrow B \rightarrow C.$$

If $\mathcal{E}$ has finite coproducts (where we denote by $A \oplus B$ the coproduct of $A$ and $B$), and admits split admissible short exact sequences of the form

$$A \hookrightarrow A \oplus B \rightarrow B,$$

then $K_0(\mathcal{E})$ is Abelian and has the familiar description

$$K_0(\mathcal{E}) = \mathbb{Z}[\text{Iso}(\mathcal{E})]/\sim,$$
where $\sim$ is generated by the relations $[B] = [A] + [C]$ for all admissible short exact sequences (8).

An exact functor between proto-exact categories $F : \mathcal{E} \mapsto \mathcal{D}$ induces group homomorphisms

$$F_* : K_n(\mathcal{E}) \mapsto K_n(\mathcal{D})$$

This assignment is functorial in the sense that if $G : \mathcal{D} \mapsto \mathcal{C}$ is exact, then $(G \circ F)_* = G_* \circ F_*$.  

**6.2 K-theory of Mat\textbullet**  

We begin by calculating the Grothendieck group of $\mathcal{E} = \text{Mat}\textbullet$. Here

$$A \hookrightarrow A \oplus B \twoheadrightarrow B \quad \text{and} \quad B \hookrightarrow A \oplus B \twoheadrightarrow A$$

are both admissible, so $K_0(\text{Mat}\textbullet)$ is the free Abelian group on $[A] \in \text{Iso}(\text{Mat}\textbullet)$ modulo the relations $[B] = [A] + [C]$ as above.

Given a non-zero pointed matroid $M$ and $e \in \tilde{E}_M$, we have an admissible short exact sequence

$$M|e \hookrightarrow M \twoheadrightarrow M/e.$$ 

Iterating this procedure, the class of any $M \in \text{Mat}\textbullet$ can be expressed as a sum of pointed matroids with two elements: the base-point and one additional element. There are two non-isomorphic such matroids $a$ and $b$ on the set $\{\ast, e\}$; in particular, $\mathcal{F}(a) = \{\ast, \{\ast, e\}\}$ and $\mathcal{F}(b) = \{\{\ast, e\}\}$.

Let $\text{rk}(M)$ denote the maximum length $r$ of a chain $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_r = E_M$ in $\mathcal{F}(M)$; this is the rank of the matroid $M$. Note that $\text{rk}(a) = 1$ and $\text{rk}(b) = 0$.

The matroids $a$ and $b$ span $K_0(\text{Mat}\textbullet)$ and are independent because rank and ground set cardinality is additive in admissible short exact sequences. We have thus proved the following.

**Theorem 6.3** There is an isomorphism

$$K_0(\text{Mat}\textbullet) \rightarrow \mathbb{Z}[a] \oplus \mathbb{Z}[b] \simeq \mathbb{Z}^\oplus 2$$

given by

$$M \rightarrow (\text{rk}(M), |E_M| - \text{rk}(M))$$

for each pointed matroid $M$.

The forgetful functor $\mathbb{F} : \text{Mat}\textbullet \rightarrow \text{Set}^{\text{fin}}$ sending a pointed matroid to its ground set has a left adjoint (see [13]) $\mathbb{G} : \text{Set}^{\text{fin}} \rightarrow \text{Mat}\textbullet$ sending a pointed set $E$ to the free pointed matroid on $E$. More precisely, $\mathbb{G}(E)$ is the pointed matroid with flats consisting of all subsets of $E$ containing the point. We have $\mathbb{F} \circ \mathbb{G} = I$, which implies the following.

**Theorem 6.4** There are injective group homomorphisms

$$\mathbb{G}_* : \pi_n^\text{gr}(\mathbb{G}) \simeq K_n(\text{Set}^{\text{fin}}) \hookrightarrow K_n(\text{Mat}\textbullet)$$

for all $n \geq 0$.

This shows $K_n(\text{Mat}\textbullet)$ is in general non-trivial for $n > 0$, and contains interesting information of a homotopy-theoretic nature.
Remark 6.5 As shown in [13], \( \mathbb{F} \) also has a right adjoint \( \mathbb{H} : \text{Set}^{fin}_* \to \text{Mat}_* \) with \( \mathbb{F} \circ \mathbb{H} = I \). Thus \( \mathbb{F} \) preserves both limits and colimits, and we obtain a second injection \( \mathbb{H}_* : K_n(\text{Set}^{fin}_*) \hookrightarrow K_n(\text{Mat}_*) \). \( \mathbb{G}_* \) sends the generator of \( K_0(\text{Set}^{fin}_*) \simeq \mathbb{Z} \) to \([a] \), while \( \mathbb{H}_* \) maps it to \([b] \), showing that these embeddings are distinct. We thus obtain a homomorphism 
\[
\mathbb{G}_* \times \mathbb{H}_* : K_n(\text{Set}^{fin}_*) \times K_n(\text{Set}^{fin}_*) \to K_n(\text{Mat}_*)
\]
which is an isomorphism for \( n = 0 \). It is not clear whether it is injective or surjective for \( n > 0 \).

7 The Hall algebra of matroids

7.1 Hall algebras of finitary proto-exact categories

Let \( \mathcal{E} \) be a finitary proto-exact category, and \( k \) a field of characteristic zero. Define the Hall algebra \( H_\mathcal{E} \) over \( k \) as 
\[
H_\mathcal{E} := \{ f : \text{Iso}(\mathcal{E}) \to k \mid f \text{ has finite support} \},
\]
where \( \text{Iso}(\mathcal{E}) \) denotes the set of isomorphism classes in \( \mathcal{E} \). The Hall algebra \( H_\mathcal{E} \) is an associative \( k \)-algebra under the convolution product 
\[
f \star g([B]) := \sum_{A \subseteq B} f([B/A])g([A]),
\]
where the summation \( \sum_{A \subseteq B} \) is taken over isomorphism classes of admissible sub-objects \( i : A \hookrightarrow B, i \in \mathcal{M} \), and \( [-] \) denotes isomorphism classes in \( \mathcal{E} \). Note this sum is finite by finitariness of \( \mathcal{E} \). The algebra \( H_\mathcal{E} \) has a basis of \( \delta \)-functions \( \{ \delta_{[B]} \mid [B] \in \text{Iso}(\mathcal{E}) \} \), where 
\[
\delta_{[B]}([A]) = \begin{cases} 1 & A \simeq B \\ 0 & \text{otherwise} \end{cases}.
\]
The multiplicative unit of \( H_\mathcal{E} \) is \( \delta_{[0]} \). The structure constants of this basis are given by 
\[
\delta_{[A]} \star \delta_{[C]} = \sum_{[B] \in \text{Iso}(\mathcal{E})} g^B_{A,C} \delta_{[B]},
\]
where 
\[
g^B_{A,C} = \# \{ D \subseteq B \mid D \simeq C, B/D \simeq A \}.
\]
Thus \( g^B_{A,C} \) counts the number of admissible subobjects \( D \) of \( B \) isomorphic to \( C \) such that \( B/D \) is isomorphic to \( A \).

Whether \( H_\mathcal{E} \) carries a co-product making it into a bialgebra depends on further properties of \( \mathcal{E} \). If \( \mathcal{E} \) is finitary, Abelian, linear over \( \mathbb{F}_q \), and hereditary, then \( H_\mathcal{E} \) carries the so-called Green’s co-product (see [22]). We will be concerned with situations where \( \mathcal{E} \) is not additive, but where an alternative construction applies. To this end we assume that \( \mathcal{E} \) has the following additional properties:

1. \( \mathcal{E} \) has finite co-products. This implies in particular that for \( A, B \in \mathcal{E} \), there exist morphisms \( \pi_A : A \sqcup B \hookrightarrow A \) and \( \pi_B : A \sqcup B \hookrightarrow B \) such that the composition 
\[
A \hookrightarrow A \bigsqcup B \xrightarrow{\pi_A} A
\]
is id$_A$, and the composition

$$A \mapsto A \bigsqcup B \overset{\pi_B}{\twoheadrightarrow} B$$

is 0.

2. The map $A \mapsto A \bigsqcup B$ is in $\mathcal{M}$, and $\pi_A \in \mathcal{E}$

3. The only admissible sub-objects of $A \bigsqcup B$ are of the form $A' \bigsqcup B'$, where $A' \subseteq A$, $B' \subseteq B$

We will denote $A \bigsqcup B$ by $A \oplus B$. We note that $\text{Set}_\ast$ and $\text{Mat}_\ast$ satisfy (1)—(3), but that (3) generally fails for Abelian categories. Define

$$\Delta: H_{\mathcal{E}} \to H_{\mathcal{E}} \otimes H_{\mathcal{E}}, \quad \Delta(f([A], [B])) \mapsto f([A \oplus B]). \quad (12)$$

Under the assumptions (1) — (3) on $\mathcal{E}$, $\Delta$ equips $H_{\mathcal{E}}$ with a bialgebra structure. $\Delta$ is co-commutative by (12), since $[A \oplus B] = [B \oplus A]$, and the subspace of primitive elements (i.e. those satisfying $\Delta(x) = x \otimes 1 + 1 \otimes x$) of $H_{\mathcal{E}}$ is spanned by the set of all $\delta_{[B]}$ for $B$ indecomposable (i.e. those $B$’s which cannot be written as a non-trivial co-product). Furthermore, $H_{\mathcal{E}}$ is naturally graded by $K_0(\mathcal{E})^+ \subseteq K_0(\mathcal{E})$, where $K_0(\mathcal{E})^+$ denotes the sub-semigroup generated by the effective classes, with $\deg(\delta_{[B]}) = [B] \in K_0(\mathcal{E})^+$. As any graded, connected, and co-commutative bialgebra is a Hopf algebra isomorphic to the enveloping algebra of the Lie algebra of its primitive elements by the Milnor-Moore theorem, we obtain the following.

**Theorem 7.1** Let $\mathcal{E}$ be a finitary proto-exact category $\mathcal{E}$ satisfying conditions (1) — (3) above. Then $H_{\mathcal{E}}$ has the structure of a $K_0(\mathcal{E})$-graded, connected, co-commutative Hopf algebra over $k$ with co-product (12). Moreover $H_{\mathcal{E}}$ is isomorphic as a Hopf algebra to the enveloping algebra $U(\delta_{[B]})$, where $B$ is indecomposable.

**Example 7.2** Consider the category $\mathcal{E} = \text{Set}_{\ast}^{fin}$, i.e. the category of finite dimensional vector spaces over “the field with one element”. Co-products in $\text{Set}_{\ast}^{fin}$ correspond to wedge sums, and $\text{Set}_{\ast}^{fin}$ satisfies the conditions of the theorem. We have $H_{\mathcal{E}} \simeq k[x]$, with

$$\Delta(x) = x \otimes 1 + 1 \otimes x.$$

where $x = \delta_{\{e, s\}}$ is the delta-function supported on the pointed set with one non-zero element.

### 7.2 The Hall algebra of $\text{Mat}_\ast$ and the matroid-minor Hopf algebra

Let $\mathcal{M}$ be a collection of pointed matroids closed under taking minors and direct sums. By Theorem 5.11, we may associate to $\mathcal{M}$ the finitary proto-exact category $\text{Mat}_\ast(\mathcal{M})$ and corresponding Hall algebra $H_{\text{Mat}_\ast(\mathcal{M})}$. As $\text{Mat}_\ast(\mathcal{M})$ is easily seen to satisfy properties (1) — (3) above, $H_{\text{Mat}_\ast(\mathcal{M})}$ is a Hopf algebra by Theorem 7.1. We have the following result:

**Theorem 7.3** Let $\mathcal{M}$ be a collection of pointed matroids closed under taking pointed minors and direct sums. Then $H_{\text{Mat}_\ast(\mathcal{M})} \simeq k[\mathcal{M}_{iso}]^\ast$, where $k[\mathcal{M}_{iso}]^\ast$ denotes the Hopf dual of the matroid-minor Hopf algebra of the collection $\mathcal{M}$. Moreover

$$H_{\text{Mat}_\ast(\mathcal{M})} \simeq U(\delta_{[M]}), \quad [M] \in \mathcal{M}_{iso}^{ind},$$

where $\mathcal{M}_{iso}^{ind}$ denotes the isomorphism classes of connected pointed matroids in $\mathcal{M}_{iso}$.
Proof We have identified both $H_{\mathbf{Mat}}(\mathcal{M})$ and $k[\mathcal{M}_{\text{iso}}]^*$ with 

$$\{f : \mathcal{M}_{\text{iso}} \to k \mid \# \text{supp}(f) < \infty\}.$$ 

The unit, co-unit, and co-multiplication agree, while comparing the products (11) and (6) reveals that $(H_{\mathbf{Mat}}(\mathcal{M}))^{op} \simeq k[\mathcal{M}_{\text{iso}}]^*$ as algebras. As every enveloping algebra possesses an algebra anti-automorphism fixing the unit, co-unit, and coproduct, the result follows.

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