Twisted homology of quantum $SL(2)$

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Abstract

We calculate the twisted Hochschild and cyclic homology (in the sense of Kustermans, Murphy and Tuset) of the coordinate algebra of the quantum $SL(2)$ group relative to twisting automorphisms acting by rescaling the standard generators $a, b, c, d$. We discover a family of automorphisms for which the “twisted” Hochschild dimension coincides with the classical dimension of $SL(2, \mathbb{C})$, thus avoiding the “dimension drop” in Hochschild homology seen for many quantum deformations. Strikingly, the simplest such automorphism is the canonical modular automorphism arising from the Haar functional. In addition, we identify the twisted cyclic cohomology classes corresponding to the three covariant differential calculi over quantum $SU(2)$ discovered by Woronowicz.

1 Introduction

Cyclic homology and cohomology were independently discovered by Alain Connes [1] and Boris Tsygan [21] in the early 1980’s, and should be thought of as extensions of de Rham (co)homology to various categories of noncommutative algebras. Quantum groups also appeared in the same period, with the first example of a “compact quantum group” in the C*-algebraic setting being Woronowicz’s “quantum $SU(2)$” [24].

The noncommutative differential geometry (in the sense of Connes) of quantum $SU(2)$ was thoroughly investigated by Masuda, Nakagami

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and Watanabe [15]. They calculated the Hochschild and cyclic homology of the coordinate algebra $\mathcal{A}(SL_q(2))$ of quantum $SL(2)$ as well as the K-theory and K-homology of the C*-algebra of the compact quantum $SU(2)$ group. This work was extended by Feng and Tsygan [5], who computed the Hochschild and cyclic homology of the standard quantized coordinate algebra $\mathcal{A}(G_q)$ associated to an arbitrary complex semisimple Lie group $G$. The homologies are roughly speaking those of a classical space labelling the symplectic leaves of the Poisson-Lie group $G$ (the semiclassical limit of $\mathcal{A}(G_q)$). In particular, the Hochschild dimension of $\mathcal{A}(G_q)$ equals the rank of $G$. This “dimension drop” had already been observed for other quantizations of Poisson algebras. Many authors regarded it as an unpleasant feature and asked for generalizations of cyclic homology which detect the quantized parts of quantum groups as well.

One candidate is twisted Hochschild and cyclic (co)homology defined by Kustermans, Murphy and Tuset [12], relative to a pair of an algebra $\mathcal{A}$ and automorphism $\sigma$. This reduces to ordinary Hochschild and cyclic (co)homology of $\mathcal{A}$ on taking $\sigma$ to be the identity. The standard theory is intimately related with the idea of considering tracial functionals on noncommutative algebras as analogues of integrals, whereas the twisted theory arises naturally from functionals whose tracial properties are of the form $h(ab) = h(\sigma(b)a)$. Noncommutative spaces equipped with such functionals include duals of nonunimodular groups, type III von Neumann algebras and compact quantum groups. The aim of [12] was to adapt Connes’ constructions relating cyclic cohomology and differential calculi to covariant differential calculi in the sense of Woronowicz, since the volume forms of such calculi define in general twisted cocycles rather than usual ones [17]. The possibility of pairing twisted cyclic cocycles (e.g. over quantum homogeneous spaces) with equivariant K-theory was demonstrated in [16], and it seems an interesting problem to investigate how far this original motivation of cyclic cohomology extends to the twisted setting.

In this paper we compute the twisted Hochschild and cyclic homologies $HH^n_\sigma(\mathcal{A})$, $HC^n_\sigma(\mathcal{A})$ for the coordinate algebra $\mathcal{A} = \mathcal{A}(SL_q(2))$ of the quantum $SL(2)$ group, with generic deformation parameter $q$. We consider all automorphisms $\sigma$ of the form $a,b,c,d \mapsto \lambda a, \mu b, \mu^{-1} c, \lambda^{-1} d$, where $a, b, c, d$ are the standard generators, and $\lambda, \mu$ are nonzero elements of $k$. As an overview we collect the dimensions of $HH^n_\sigma(\mathcal{A})$ as a $k$-vector space, see the main text for explicit formulas for generators:

**Theorem 1.1** We have

$$\dim HH^n_\sigma(\mathcal{A}) = \begin{cases} 0 & n > 3, \\
N + 1 & \lambda = q^{-(N+2)}, \mu = 1,
0 & \text{otherwise},
\end{cases}$$

$$\dim HH^1_\sigma(\mathcal{A}) = \begin{cases} N + 1 & \lambda = q^{-(N+2)}, \mu = 1, \\
2 & \lambda = q^{-(N+1)}, \mu = q^{\pm(M+1)},
0 & \text{otherwise},
\end{cases}$$

$$\dim HH^0_\sigma(\mathcal{A}) = \begin{cases} \infty & \mu = 1, \\
\infty & \mu \neq 1, \mu \notin q^\mathbb{Z},
\infty & \lambda = q^{-(N+1)}, \mu = q^{\pm(M+1)},
0 & \text{otherwise},
\end{cases}$$

$$\dim HH^2_\sigma(\mathcal{A}) = \begin{cases} \infty & \mu = 1, \\
4 & \lambda = q^{-(N+1)}, \mu = q^{\pm(M+1)},
0 & \text{otherwise},
\end{cases}$$
for $M, N \in \mathbb{N}$.

Strikingly (Theorem 4.12), there exists a family of automorphisms for which the twisted Hochschild dimension takes the classical value three (note also that the homological dimension of $A(SL_q(2))$ is three [13] - the twisted theory avoids the “dimension drop”. Remarkably, the simplest such automorphism ($\lambda = q^{-2}, \mu = 1$) is the canonical modular automorphism associated to the Haar functional on $A$. Similar results were obtained for Podleś quantum spheres [7] and quantum hyperplanes [20].

In [5], Feng and Tsygan considered formal quantizations, with $A(G_q)$ a Hopf algebra over $\mathbb{C}[[\hbar]]$ with $q = e^\hbar$. They showed that for a Hopf algebra $A$ over a field $k$, with coproduct $\Delta$, counit $\varepsilon$ and antipode $S$, and an $A$-bimodule $M$, there is an isomorphism

$$H_n(A, M) \cong \text{Tor}_n^A(M', k) \quad (1)$$

Here, $M'$ is $M$ as a linear space with right action given by

$$m \triangleright a := \sum S(a(2))ma(1) \quad (2)$$

using Sweedler’s notation for the coproduct, and $k = A/\ker \varepsilon$ is the trivial left $A$-module. Then they computed these Tor-groups using the spectral sequence associated to the filtration induced by $\hbar$.

In this paper we compute $\text{Tor}_n^A(M', k)$ from a Koszul-type free resolution

$$0 \to A \to A^3 \to A^3 \to A \to k \to 0 \quad (3)$$

of $k$. Noncommutative Koszul resolutions were studied by several authors, in particular Wambst [22], but as far as we know were not applied to quantum groups. In our opinion this resolution shows very clearly the geometric mechanisms behind the computations. We will see that the maps of the resulting complex computing the twisted Hochschild homology become zero for $q = 1$, so one obtains the Hochschild-Kostant-Rosenberg theorem for $SL(2)$ (the algebraic cotangent bundle of $SL(2)$ is trivial). However, for $q \neq 1$ this does not happen for any twisting automorphism.

A summary of this paper is as follows. In section 2 we recall how the twisted theory was discovered [12], then give the definitions of $HH^\sigma_*(A)$ and $HC^\sigma_*(A)$, and the underlying cyclic object. We specialize to Hopf algebras and explain the methods adapted from [5]. We then present the general scheme of the noncommutative Koszul complexes used here. In section 3 we introduce the quantum $SL(2)$ group. In section 4 we present our calculations of $HH^\sigma_*(A)$ for $A = A(SL_q(2))$.

Twisted cyclic homology is defined as the total homology of Connes’ mixed $(b, B)$-bicomplex coming from the underlying cyclic object, as in [14]. In section 5 we compute this homology via a spectral sequence.

Finally, in section 6 we discuss the relation of our results to previously known twisted cyclic cocycles coming from the three covariant differential calculi over $A(SL_q(2))$ discovered by Woronowicz. The twisted cyclic 3-cocycle arising from the three dimensional left covariant calculus was given explicitly in [12] and [17]. We show (Theorem 6.1) that this 3-cocycle is a trivial element of twisted cyclic cohomology. Further, the twisted 4-cocycles arising from the two bicovariant four dimensional calculi both correspond to the twisted 0-cocycle coming from the Haar functional (as elements of even periodic twisted cyclic cohomology).
2 Twisted cyclic homology

2.1 MOTIVATION

Twisted cyclic (co)homology arose from the study of covariant differential calculi over quantum groups \[12\].

Let \( A \) be an algebra over \( \mathbb{C} \). Given a differential calculus \((\Omega, d)\) over \( A \), with \( \Omega = \bigoplus_{n=0}^{\infty} \Omega_n \), Connes \[13\] considered linear functionals \( \int : \Omega_N \to \mathbb{C} \), which are closed and graded traces on \( \Omega \), meaning

\[
\int d\omega = 0 \quad \forall \ \omega \in \Omega_{N-1}
\]

\[
\int \omega_m \omega_n = (-1)^{mn} \int \omega_n \omega_m \quad \forall \ \omega_m \in \Omega_m, \ \omega_n \in \Omega_n
\]  

(4)

Connes found that such linear functionals are in one to one correspondence with cyclic \( N \)-cocycles \( \tau \) on the algebra, via

\[
\tau(a_0, a_1, \ldots, a_N) = \int a_0 \ da_1 \ da_2 \ldots da_N
\]  

(5)

which led directly to his simplest formulation of cyclic cohomology \[13\].

If \( A \) is the coordinate algebra of a quantum group, then Woronowicz proposed to study covariant differential calculi, for which the left coaction of \( A \) on \( A \) given by the coproduct \( \Delta : A \to A \otimes A \) extends to a coaction \( \Delta_L : \Omega \to A \otimes \Omega \) compatible with the differential \[24\], \[25\]. For such calculi the natural linear functionals \( \int : \Omega_N \to \mathbb{C} \) are no longer graded traces, but twisted graded traces, meaning that

\[
\int \omega_m \omega_n = (-1)^{mn} \int \Delta_L(\omega_n) \omega_m \quad \forall \ \omega_m \in \Omega_m, \ \omega_n \in \Omega_n
\]  

(6)

for some degree zero automorphism \( \sigma \) of \( \Omega \). In particular, \( \sigma \) restricts to an automorphism of \( A \), and, for any \( a \in A, \omega_N \in \Omega_N \) we have

\[
\int \omega_N a = \int \sigma(a) \omega_N
\]  

(7)

Hence for each covariant calculus there is a natural automorphism of \( A \). Motivated by this observation, Kustermans, Murphy and Tuset defined “twisted” Hochschild and cyclic cohomology for any pair of an algebra \( A \) and automorphism \( \sigma \), and showed that the one-to-one correspondence between graded traces and cyclic cocycles generalizes to this setting. The next section recalls their definitions, transposed to homology.

2.2 TWISTED HOCHSCHILD AND CYCLIC HOMOLOGY

Let \( A \) be a unital, associative algebra over a field \( k \) (assumed to be of characteristic zero) and \( \sigma \) an automorphism. We define the cyclic object \[2\], \[14\] underlying twisted cyclic homology \( HC^\sigma_n(A) \) of \( A \) relative to \( \sigma \). Set \( C_n := A^{\otimes(n+1)} \). For clarity, we will denote \( a_0 \otimes a_1 \otimes \cdots \otimes a_n \in C_n \) by \( (a_0, a_1, \ldots, a_n) \). Define

\[
d_{n,i}(a_0, a_1, \ldots, a_n) = (a_0, \ldots, a_i a_{i+1}, \ldots, a_n) \quad 0 \leq i \leq n - 1
\]

\[
d_{n,n}(a_0, a_1, \ldots, a_n) = (\sigma(a_n) a_0, a_1, \ldots, a_{n-1})
\]

\[
s_{n,i}(a_0, a_1, \ldots, a_n) = (a_0, \ldots, a_i, 1, a_{i+1}, \ldots, a_n) \quad 0 \leq i \leq n
\]
\[ \tau_n(a_0, a_1, \ldots, a_n) = (\sigma(a_n), a_0, \ldots, a_{n-1}) \]  

(8)

For \( \sigma = \text{id} \) these are the face, degeneracy and cyclic operators of the standard cyclic object associated to \( \mathcal{A} \) \[14\]. For general \( \sigma \) the operator \( T_n := \tau_n^{n+1} \) is not equal to the identity, but all other relations of the cyclic category are fulfilled. Hence \( C_n \) becomes what is called a paracyclic object \[5\]. To obtain a cyclic object, we pass to the cokernels \( C_n := C_n/\sigma_n^{1} \), \( C_n := \text{im}(\text{id} - T_n) \). Dualizing \[12\], we call the cyclic homology of this cyclic object the \( \sigma \)-twisted cyclic homology \( HC_n^\sigma(\mathcal{A}) \) of \( \mathcal{A} \). Hence \( HC_n^\sigma(\mathcal{A}) \) is the total homology of Connes’ mixed \((b, B)\)-bicomplex

\[
\begin{array}{c}
\cdots \\
b_4 \downarrow & b_3 \downarrow & b_2 \downarrow & b_1 \\
C_3^\sigma & C_2^\sigma & C_1^\sigma & C_0^\sigma \\
b_3 \downarrow & b_2 \downarrow & b_1 \\
C_2^\sigma & C_1^\sigma & C_0^\sigma \\
b_2 \\
C_1^\sigma \\
b_1 \\
C_0^\sigma
\end{array}
\]  

(9)

The maps \( b_n \) and \( B_n \) are given by

\[
b_n = \sum_{i=0}^{n} (-1)^i d_{n,i}, \quad B_n = (1 + (-1)^n \tau_n^{n+1}) s_n N_n,
\]

(10)

with \( N_n = \sum_{j=0}^{n} (-1)^j \tau_n^j \), and \( s_n : C_n^\sigma \to C_{n+1}^\sigma \) the “extra degeneracy”

\[
s_n(a_0, a_1, \ldots, a_n) = (1, a_0, a_1, \ldots, a_n)
\]

(11)

We calculate \( HC_n^\sigma(\mathcal{A}) \) via the spectral sequence associated to the mixed complex. Let \( HH_n^\sigma(\mathcal{A}) \) denote the entries of its first page, that is, \( HH_n^\sigma(\mathcal{A}) := H_n(C_n^\sigma, b_n) \) (the homologies of the columns). For \( \sigma = \text{id} \) these are the Hochschild homologies \( HH_n(\mathcal{A}) = H_n(\mathcal{A}, \mathcal{A}) \). Hence we call \( HH_n^\sigma(\mathcal{A}) \) as in \[12\] the \( \sigma \)-twisted Hochschild homology of \( \mathcal{A} \).

To compute \( HH_n^\sigma(\mathcal{A}) \) consider the mixed complex \[9\] with \( C_n^\sigma \) replaced by the original \( C_n \). This is not a bicomplex: the commutation relations in a paracyclic object imply that the (lifts of the) operators \( b_n \) and \( B_n \) anticommute according to (see \[9\], Theorem 2.3)

\[
b_n+1 B_n + B_{n-1} b_n = \text{id} - T_n.
\]

(12)

But the columns form the complex \((C_n, b_n)\) which computes the Hochschild homology \( H_n(\mathcal{A}, \alpha \mathcal{A}) \) of \( \mathcal{A} \) with coefficients in the bimodule \( \alpha \mathcal{A} \) which is \( \mathcal{A} \) as a vector space with bimodule structure

\[
a \triangleright b \triangleleft c := \sigma(a) bc
\]

(13)

In many cases \( C_n = C_n^0 \oplus C_n^1 \), \( C_n^0 := \ker(\text{id} - T_n) \), for example when \( \sigma \) is diagonalizable. In this case, \( T_n = \sigma^{\otimes (n+1)} \) is also diagonalizable, and \( C_n^0 \) and \( C_n^1 \) are the eigenspace of \( T_n \) corresponding to the eigenvalue 1 and the direct sum of all other eigenspaces, respectively. Then:
PROPOSITION 2.1 If \( C_n = C^0_n \oplus C^1_n \), then \( H_\ast(\mathcal{A},\sigma\mathcal{A}) \cong HH^\ast(\mathcal{A}) \).

Proof. Note that (12) implies that \( b_\ast \) commutes with \( \text{id} - T_\ast \), so the decomposition \( C_n = C^0_n \oplus C^1_n \) defines a decomposition of complexes, and we can identify \( HH^\ast(\mathcal{A}) \) with the homologies of the subcomplex \((C^0_n, b_\ast) \subset (C_n, b_\ast) \). Hence \( H_\ast(\mathcal{A},\sigma\mathcal{A}) \) is the direct sum of \( HH^\ast(\mathcal{A}) \) and the homologies of \((C^1_n, b_\ast) \). But \((\text{id} - T_\ast)|_{C^1_n} \) is a bijection under these assumptions, and we have on \( C^1_n \) again by (12) the relation

\[
b_{n+1}(1 - T_n)^{-1}B_n + (1 - T_n)^{-1}B_{n-1}b_n = \text{id}.
\]

So \((\text{id} - T_\ast)^{-1}B_\ast \) is a contracting homotopy for \((C^1_n, b_\ast) \) and the claim follows. \( \square \)

This will allow us to calculate \( HH^\ast(\mathcal{A}) \) using standard techniques of homological algebra.

The spectral sequence calculation is most efficiently done by passing to the normalized mixed complex (see for example [24], Application 9.8.4). This leaves the first page unchanged. The second step is to calculate the horizontal homology of the rows relative to the maps \( B_n \) which in the normalized complex are given explicitly by

\[
B_n(a_0, \ldots, a_n) = \sum_{i=0}^{n} (-1)^i(1, \sigma(a_i), \ldots, \sigma(a_n), a_0, \ldots, a_{i-1}). \tag{14}
\]

For quantum \( SL(2) \), we find that everything stabilises at the second page, and we can then read off the twisted cyclic homology.

For later use we note that by using the Hochschild-Kostant-Rosenberg theorem applied to an appropriate subalgebra, we obtain:

LEMMA 2.2 If \( x, y \) are commuting elements of \( \mathcal{A} \), with \( \sigma(x) = x, \sigma(y) = y \), then for any \( s, t \geq 0 \) we have

\[
B_0[x^s y^t] = t[(x^s y^{t-1}, y)] + s[(x^{s-1} y^t, x)] \in HH_1^\ast(\mathcal{A})
\]

From now on, we will drop the suffices and write \( b_n \) as \( b \).

2.3 HOCHSCHILD HOMOLOGY OF HOPF ALGEBRAS

For arbitrary algebras, the Hochschild homologies are derived functors in the category of \( \mathcal{A} \otimes \mathcal{A}^{op} \)-modules, and working with explicit resolutions usually involves lengthy calculations. But if \( \mathcal{A} \) is a Hopf algebra then we can describe \( H_\ast(\mathcal{A},\mathcal{M}) \) for an arbitrary \( \mathcal{A} \)-bimodule \( \mathcal{M} \) as a derived functor in the category of \( \mathcal{A} \)-modules. Define a right \( \mathcal{A} \)-module \( \mathcal{M}' \) which is \( \mathcal{M} \) as a vector space with right action given by

\[
m \triangleleft a := \sum S(a(2)) \triangleright m \triangleleft a(1), \quad a \in \mathcal{A}, m \in \mathcal{M}. \tag{15}
\]

Consider \( k \) as the trivial \( \mathcal{A} \)-module \( \mathcal{A}/\ker \varepsilon \). Feng and Tsygan proved:

PROPOSITION 2.3 \( \square \) There is an isomorphism of vector spaces

\[
H_\ast(\mathcal{A},\mathcal{M}) \cong \text{Tor}_{\mathcal{A}}^\ast(\mathcal{M}', k).
\]

Proof. The \( \text{Tor}_{\mathcal{A}}^\ast(\mathcal{M}', k) \) are computed from the complex \((C_\ast, d) \) (with zeroth tensor component now being \( \mathcal{M}' \)) with boundary map \( d \) given by

\[
d = d_0 + \sum_{i=1}^{n-1} (-1)^i d_i + (-1)^n \tilde{d}_n, \tag{16}
\]
where the $d_i$ are as above and
\[
\begin{align*}
\tilde{d}_0(a_0, a_1, \ldots, a_n) & := (a_0 \triangleright a_1, a_2, \ldots, a_n), \\
\tilde{d}_n(a_0, a_1, \ldots, a_n) & := (\varepsilon(a_n) a_0, a_1, \ldots, a_{n-1}).
\end{align*}
\] (17)

We define two linear maps $\xi, \xi' : C_n \to C_n$ by
\[
\begin{align*}
\xi(a_0, a_1, \ldots, a_n) & := (S((a_1 \ldots a_n)(2)) \triangleright a_0, (a_1)(1), \ldots, (a_n)(1)) \\
\xi'(a_0, \ldots, a_n) & := ((a_1 \ldots a_n)(2)) \triangleright a_0, (a_1)(1), \ldots, (a_n)(1)).
\end{align*}
\] (18)

Then $\xi \circ \xi' = \xi' \circ \xi = \text{id}_{C_n}$. It is easily checked that $\xi$ commutes with $d_i$ for $1 \leq i \leq n-1$ and that $\xi \circ d = d_i \circ \xi$, $i = 0, n$. Hence $\xi \circ d = b \circ \xi$ and $\xi$ is an isomorphism of complexes of $k$-vector spaces. \hfill \Box

Let $\pi : M' \to H_0(A, M)$ be the canonical projection. Then we have $\pi(m \trianglelefteq a) = \varepsilon(a)\pi(m)$, and if we consider $H_0(A, M)$ as trivial right $A$-module, then $\pi \otimes \text{id}_{A^\otimes n}$ induces a morphism $H_n(A, M) \to H_0(A, M) \otimes_k \text{Tor}_n^A(k, k)$.

If $A$ is commutative and $M = A$ with the standard bimodule structure, then $H_0(A, M) = A$. $\pi$ is the identity, and the above map is the isomorphism of the Hochschild-Kostant-Rosenberg theorem. For $M = A$ the map defines a “classical shadow” of twisted Hochschild homology.

### 2.4 Noncommutative Koszul Resolutions

Propositions 2.1 and 2.3 allow us to compute $HH^\bullet(A)$ for Hopf algebras $A$ and diagonalizable $\sigma$ from a resolution of the trivial $A$-module $k$. In the commutative case, such a resolution can be constructed in form of a Koszul complex associated to a minimal set of generators of $\ker \varepsilon$. In many examples the construction of the resolution is as follows.

Let $A$ be an algebra and $d$ be a positive integer. Let $x_{i,j}$, $1 \leq i, j \leq d$ be elements of $A$ satisfying
\[
x_{i,j}x_{i-1,k} = x_{i,k}x_{i-1,j}.
\] (19)

In the commutative case one can take $x_{i,j} = x_{1,j}$, and in many examples the $x_{i,j}$ will be uniquely determined by the $x_{1,j}$.

For $0 \leq n \leq d$, let $K_n(x_{i,j})$ be the $A$-bimodule $A^{d\choose n}$, which we identify for $n > 0$ with the submodule of $A^d \otimes_A \cdots \otimes_A A^d$ ($n$ factors) spanned over $A$ by $e_{i_1} \otimes_A \cdots \otimes_A e_{i_n}$, $1 \leq i_1 < \ldots < i_n \leq d$, where $e_i$ is a basis of $A^d$. For $n > d$ we set $K_n(x_{i,j}) := 0$. For an $A$-bimodule $N$, set $K_n(x_{i,j}, N) := K_n(x_{i,j}) \otimes_A N$ and define $A$-module maps
\[
k_m : K_n(x_{i,j}, N) \to K_{n-1}(x_{i,j}, N), \quad m = 1, \ldots, n
\]
(we suppress the index $n$ at the $k_m$) by
\[
k_m : e_{i_1} \otimes_A \cdots \otimes_A e_{i_n} \otimes_A y \mapsto e_{i_1} \otimes_A \cdots \otimes_A e_{i_{m-1}} \otimes_A e_{i_m} \otimes_A \cdots \otimes_A e_{i_n} \otimes_A y \otimes x_{n,m}.
\]

Then for $r < s$:
\[
(k_s k_r - k_{s-1} k_r)(e_{i_1} \otimes_A \cdots \otimes_A e_{i_s} \otimes_A y)
= e_{i_1} \otimes_A \cdots \otimes_A e_{i_r} \otimes_A e_{i_{r+1}} \otimes_A \cdots \otimes_A e_{i_s} \otimes_A y \otimes (x_{n,i_r} x_{n-1,i_r} - x_{n,i_r} x_{n-1,i_r}).
\]

The last bracket vanishes by (19). Thus we get
The zeroth homology of this complex is obviously the quotient of \( N \) by the submodule generated by all elements of the form \( y \triangleleft x_{1,i} \), \( y \in N \), \( i = 1, \ldots, d \). The classical application of Koszul complexes is to produce resolutions of this quotient, but the Koszul complex is not always acyclic (see [19] for the commutative case). In our application we will take \( N = A \) to be a Hopf algebra with the standard bimodule structure, and the \( x_{1,j} \) \((1 \leq j \leq d)\) will generate \( \ker \varepsilon \) as an (left or right) \( A \)-module. The associated Koszul complex will be checked by hand to be acyclic (see Proposition 4.1 below), so it provides a resolution of \( A \). The map \( \varepsilon \) is then given by

\[
\varepsilon_1 \otimes_A \cdots \otimes_A \varepsilon_n \mapsto x_{1,i_1} \land \cdots \land x_{1,i_n} := \sum_{s \in S_n} (-1)^{|s|} x_{n,i_{(s)}} \otimes \cdots \otimes x_{1,i_{(1)}}.
\]

(20)

3 Quantum \( SL(2) \)

In this section, we introduce the main facts on the standard quantized coordinate ring \( A = A(SL_q(2)) \) that will be used below.

3.1 THE HOPF ALGEBRA \( A(SL_q(2)) \)

Let \( k \) be a field of characteristic zero, and \( q \in k \) some nonzero parameter, which we assume is not a root of unity. The coordinate algebra \( A = A(SL_q(2)) \) of the quantum group \( SL_q(2) \) over \( k \) is the \( k \)-algebra generated by symbols \( a, b, c, d \) with relations

\[
ab = qba, \quad ac = qca, \quad bd = qdb, \quad cd = qdc, \quad bc = cb
\]

\[
ad - qbc = 1, \quad da - q^{-1}bc = 1
\]

(21)

There is a unique Hopf algebra structure on \( A \) such that

\[
\Delta(a) = a \otimes a + b \otimes c, \quad \Delta(b) = a \otimes b + b \otimes d,
\]

\[
\Delta(c) = c \otimes a + d \otimes c, \quad \Delta(d) = c \otimes b + d \otimes d,
\]

\[
\varepsilon(a) = \varepsilon(d) = 1, \quad \varepsilon(b) = \varepsilon(c) = 0,
\]

\[
S(a) = d, \quad S(b) = -q^{-1}b, \quad S(c) = -qc, \quad S(d) = a.
\]

(22)

A vector space basis of \( A \) is given by the monomials

\[
e_{i,j,k} := a^ib^jc^k, \quad i \in Z, \quad j, k \in N, \quad a^i := d^{-i} \quad \text{for} \quad i < 0,
\]

(23)

(use the convention that \( x^0 = 1 \), for \( x \in A, x \neq 0 \)). We have

\[
e_{i,j,k} e_{l,m,n} = q^{-(j+k)} e_{i+l,j+m,k+n} + \sum_{r > 0} \lambda_{i,j,k,l,m,n}(r) e_{i+l,j+m+r,k+n+r}
\]

for some constants \( \lambda_{i,j,k,l,m,n}(r) \). It follows that \( A \) admits a \( \mathbb{Z} \)-grading and three separating decreasing \( \mathbb{N} \)-filtrations

\[
A = \bigoplus_{i \in \mathbb{Z}} A_i, \quad A = A_0 \supset A_1 \supset \ldots, \quad x = b, c, bc,
\]

(24)
where $\mathcal{A}^0 = \text{span}\{e_{i,j,k}\}_{i,j,k}$ and $\mathcal{A}^i$ is the span of $e_{i,j,k}$ with $j, k, j + k \geq n$ for $x = b, c, bc$, respectively. For $x \in \mathcal{A}$, let $x_i$ be its component in $\mathcal{A}_i^0$. Set $\mathcal{A}_{i,k} := \mathcal{A}_i^0 \cap \mathcal{A}_k^0$. Then $\mathcal{A}_{i,n} \mathcal{A}_{j,m} = \mathcal{A}_{i+j,n+m}$. Define a Hermitian inner product on $\mathcal{A}$ by requiring that $e_{i,j,k}$ are orthonormal and let $\pi_x, \pi_{i,j,k}$, denote the orthogonal projections onto $(\mathcal{A}_i^0)^\perp, e_{i,j,k}$. We freely consider $\pi_{i,j,k}$ as a map $\mathcal{A} \rightarrow k$. Note that $\pi_x(y) = \pi_x(y)$ for all $y \in \mathcal{A}$.

Finally, $\mathcal{A}$ has a $\mathbb{Z}^2$-grading given by

$$\text{deg}(e_{i,j,k}) = (i, j - k)$$

This grading extends to $\mathcal{A}^\otimes (n+1)$ and is preserved by the Hochschild boundary and the maps $B_n$. Hence $HH^*_n(\mathcal{A})$ and $HC^*_n(\mathcal{A})$ are naturally $\mathbb{Z}^2$-graded.

### 3.2 The Haar Functional

The Hopf algebra $\mathcal{A}$ is cosemisimple, that is, there is a unique linear functional $h : \mathcal{A} \rightarrow k$ satisfying $h(1) = 1$ and

$$(h \otimes \text{id})\Delta(x) = h(x)1 = (\text{id} \otimes h)\Delta(x) \quad \forall x \in \mathcal{A}$$

If $k = \mathbb{C}$ and $q \in \mathbb{R}$, then $\mathcal{A}$ can be made into a Hopf $*$-algebra whose $C^*$-algebraic completion is Woronowicz’s quantum $SU(2)$ group. The functional $h$ extends to the Haar state of this compact quantum group. Hence (with slight abuse of terminology) we also in the general case call $h$ the Haar functional of $\mathcal{A}$. For any $x, y \in \mathcal{A}$, we have

$$h(xy) = h(y\sigma_{\text{mod}}(x))$$

where $\sigma_{\text{mod}}$ is the so-called modular automorphism of $\mathcal{A}$ given by

$$\sigma_{\text{mod}}(a) = q^{-2}a, \quad \sigma_{\text{mod}}(d) = q^2d, \quad \sigma_{\text{mod}}(b) = b, \quad \sigma_{\text{mod}}(c) = c$$

So $h$ is a $\sigma_{\text{mod}}^{-1}$-twisted cyclic 0-cocycle.

### 3.3 The Automorphism Group of $\mathcal{A}(SL_q(2))$

For $\lambda, \mu \in k^\times$ there are unique automorphisms $\sigma_{\lambda, \mu}, \tau_{\lambda, \mu}$ of $\mathcal{A}$ with

$$\sigma_{\lambda, \mu}(a) = \lambda a, \quad \sigma_{\lambda, \mu}(b) = \mu b, \quad \sigma_{\lambda, \mu}(c) = \mu^{-1}c, \quad \sigma_{\lambda, \mu}(d) = \lambda^{-1}d,$$

$$\tau_{\lambda, \mu}(a) = \lambda a, \quad \tau_{\lambda, \mu}(b) = \mu c, \quad \tau_{\lambda, \mu}(c) = \mu b, \quad \tau_{\lambda, \mu}(d) = \lambda^{-1}d.$$ 

It is easy to check that this list is complete, although we do not know a reference where this was pointed out explicitly:

**Proposition 3.1** If $\sigma$ is an automorphism of $\mathcal{A}(SL_q(2))$, then either $\sigma = \sigma_{\lambda, \mu}$ or $\sigma = \tau_{\lambda, \mu}$ for some $\lambda, \mu$.

**Proof.** Using the $\mathbb{Z}$-grading and the $\mathbb{N}$-filtrations mentioned above it is a straightforward calculation to check that up to rescaling and exchanging $b$ and $c$ the original generators are the only elements of the algebra that fulfill the defining relations. \hfill \Box

The $\sigma_{\lambda, \mu}$ act diagonally with respect to the generators $a, b, c, d$. The $\tau_{\lambda, \mu}$ are also diagonalizable. For fixed $\lambda, \mu$ define $x_{\pm} = c \pm \mu b$. Then $\tau_{\lambda, \mu}(x_{\pm}) = \pm x_{\pm}$, and $a, x_+, x_-, d$ generate $\mathcal{A}$. So by Proposition 3.1

**Corollary 3.2** For $\mathcal{A} = \mathcal{A}(SL_q(2))$, and for each $n$ and every automorphism $\sigma$, we have $HH^*_n(\mathcal{A}) \cong H_n(\mathcal{A}, \sigma \mathcal{A})$. 


4 Twisted Hochschild homology of \(\mathcal{A}(SL_q(2))\)

4.1 A Koszul Resolution of \(\mathcal{A}/\ker \varepsilon\)

Using the above facts it is easy to see that \(\ker \varepsilon\) is generated as both a left and right \(\mathcal{A}\)-module by \(x_{1,1} := a - 1, x_{1,2} := b, x_{1,3} := c\). For these elements there exists a Koszul complex \((K_*, k), K_n := K_n(x_{i,j}, \mathcal{A})\), in the sense of section 4.1, with the \(x_{i,j}\) given by

\[
\begin{pmatrix}
  a - 1 & b & c \\
  q^{-1}a - 1 & b & c \\
  q^{-2}a - 1 & b & c
\end{pmatrix}.
\]

We check by explicit calculation that this Koszul complex is acyclic:

**Proposition 4.1** The left \(\mathcal{A}\)-module \(\mathcal{A}/\ker \varepsilon\) possesses a resolution \((K_*, k)\) of the form

\[
0 \to \mathcal{A} \to \mathcal{A} \to \mathcal{A} \to \mathcal{A} \to \mathcal{A}/\ker \varepsilon \to 0.
\]

The augmentation map \(K_0 = \mathcal{A} \to k = \mathcal{A}/\ker \varepsilon\) is given by the counit \(\varepsilon\). The left \(\mathcal{A}\)-module morphisms \(k_n : K_n \to K_{n-1}, n = 1, 2, 3\), are given by

\[
k_1 : \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mapsto a - 1, b, c
\]

\[
k_2 : \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} b \\ 1 - q^{-1}a \\ 0 \end{pmatrix}, \begin{pmatrix} c \\ 0 \\ 1 - q^{-1}a \end{pmatrix}, \begin{pmatrix} 0 \\ c \\ -b \end{pmatrix}
\]

\[
k_3 : 1 \mapsto \begin{pmatrix} c \\ -b \\ q^{-2}a - 1 \end{pmatrix}.
\]

**Proof.** It follows from Proposition 2.1 (or directly) that this is a complex. Let \((x, y, z)^t \in \ker(k_1), \text{i.e.} \ x(a - 1) + yb + zc = 0. \) Then \(\pi_{bc}(x(a - 1)) = 0\).

Using the \(\mathbb{Z}\)-grading we have \(\pi_{bc}(x) = 0, \text{so} \ x = x'b + x'c\). Subtracting \(k_2(x', x'', 0)^t\) from \((x, y, z)^t\) we get a new element of \(\ker(k_1)\) with \(x = 0\).

Hence \(\pi_{c}(y) = \pi_{c}(z) = 0, \text{so} \ y = y', z = z', b = b' = -y'\) and this element is a multiple of \(k_3(0, 0, 1)^t\). In a similar manner

\[
x \left( \begin{pmatrix} b \\ 1 - q^{-1}a \\ 0 \end{pmatrix} \right) + y \left( \begin{pmatrix} c \\ 0 \\ 1 - q^{-1}a \end{pmatrix} \right) + z \left( \begin{pmatrix} 0 \\ c \\ -b \end{pmatrix} \right) = 0
\]

implies \(x = x'c, y = -x'b, x'(q^{-2}a - 1) = z\) for some \(x' \in \mathcal{A}\).

**Corollary 4.2** If \(n > 3\), then \(HH^*_\sigma(\mathcal{A}) = 0\) for all automorphisms \(\sigma\).

The morphism between the resulting short complex \(\varepsilon \mathcal{A} \otimes \mathcal{A} K_*\) and the standard complex for \(\text{Tor}^\mathcal{A}_n(\varepsilon \mathcal{A}, k)\) yielding an isomorphism in homology is given explicitly by:

1. The map \(\varphi_0 : \mathcal{A} \to C_0 = \mathcal{A}\) is the identity.
2. The map $\varphi_1 : A^3 \to C_1 = A^\otimes 2$ is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto (x, a-1) + (y, b) + (z, c).$$  \hfill (31)

Since $d(x, 1, 1) = (x, 1)$ for any $x$, we have $[(x, a-1)] = [(x, a)]$ in $\text{Tor}_4^A (\sigma A', k)$.

3. The map $\varphi_2 : A^3 \to C_2 = A^\otimes 3$ is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto (x, b, a-1) - (x, q^{-1}a - 1, b) +$$

$$+ (y, c, a-1) - (y, q^{-1}a - 1, c) + (z, c, b) - (z, b, c)$$ \hfill (32)

4. Finally, the map $\varphi_3 : A \to C_3 = A^\otimes 4$ in the complex for the $\text{Tor}$-groups is given by $x \mapsto x \otimes v$, where

$$v = -(q^{-2}a - 1, b, c) + (q^{-2}a - 1, c, b) - (c, q^{-1}a - 1, b) +$$

$$+ (c, b, a-1) - (b, c, a-1) + (b, q^{-1}a - 1, c).$$  \hfill (33)

One sees by direct computation that this is a morphism of complexes, and by the comparison theorem (see [23], Theorem 2.2.6) this is a quasi-isomorphism.

4.2 Computation of $HH_n^\sigma (A)$, $n \leq 3$

All automorphisms arising from finite-dimensional calculi are of the form $\sigma = \sigma_{\lambda, \mu}$, and from now on we will only consider automorphisms of this type. In fact, they are of the form $\sigma(x) = \sigma_{\mu, f}(f + x)$, where $f$ is a functional in the dual Hopf algebra $A^\circ$ acting on $x$ by $f \ast x = \sum f(x^i) x^i$ acting on $x$ (see Theorems 4.1, 4.3 and 4.8 in [24]). It is clear that such automorphisms do not exchange $b$ and $c$. By Corollary 4.2 we have $HH_n^\sigma (A) \cong H_n (A, \sigma A)$, and the homologies $H_n (A, \sigma A)$ can be calculated via our noncommutative Koszul resolution.

So let $\lambda, \mu \in k^\times$ and $\sigma = \sigma_{\lambda, \mu}$. We apply $\sigma A \otimes A \cdot$ to our resolution and obtain the complex $(F, f)$ of vector spaces

$$0 \to A \to f_0 A \to f_1 A \to f_2 A \to f_3 A \to f_4 A \to 0,$$  \hfill (34)

with morphisms $f_n$ given by

$$f_1 \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = x_1 \triangleright a - x_1 + y_1 \triangleright b + z_1 \triangleright c,$$

$$f_2 \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_2 \triangleright b + y_2 \triangleright c \\ x_2 - q \cdot x_2 \triangleright a + z_2 \triangleright c \\ y_2 - q \cdot y_2 \triangleright a - z_2 \triangleright b \end{pmatrix},$$

$$f_3 (x_3) = \begin{pmatrix} x_3 \triangleright c \\ -x_3 \triangleright b \\ q \cdot x_3 \triangleright a - x_3 \end{pmatrix}.$$  \hfill (35)
Writing $\varepsilon_{i,j,k} := q^{i+j+k+2}\lambda\mu^{-1}$ we have

$$\lambda q^{j+k}e_{i,j,k} \triangleleft a = e_{i,j,k} + q^{-1-i-j}(1 - \varepsilon_{i,j,k})e_{i,j+1,k+1},$$

$$\lambda^{-1}e_{i,j,k} \triangleleft b = (1 - \varepsilon^{-1}_{i,j,k})e_{i+1,j+1,k}$$

$$+ \left\{ \begin{array}{ll}
0 & \text{if } i > 0 \\
q^{-2i-1}(1 - \varepsilon^{-1}_{i,j,k})e_{i+1,j+2,k+1} & \text{if } i < 0
\end{array} \right.,$$

$$\lambda e_{i,j,k} \triangleleft c = (1 - \varepsilon_{i,j,k})e_{i-1,j,k+1}$$

$$+ \left\{ \begin{array}{ll}
q^{-2i+1}(1 - \varepsilon_{i,j,k})e_{i-1,j+1,k+2} & \text{if } i > 0 \\
0 & \text{if } i \leq 0
\end{array} \right. \quad (36)$$

For $q = \lambda = \mu = 1$ we have $f_\alpha = 0$ and we recover the Hochschild-Kostant-Rosenberg theorem for $SL(2, k)$. The cotangent bundle of an algebraic group is trivial, so in the classical case $HH_n(A) = A \otimes \Lambda^n k^3$.

It is clear, however, that for $q \neq 1$ there is no twisting automorphism for which this happens.

The calculations lead to five distinct cases:

1. $\mu = 1$, $\lambda \notin \{q^{-N+2}\}_{N \geq 0}$, and $\mu \neq 1$, $\lambda = 1$.
2. $\mu = 1$, $\lambda = q^{-N+2}$, $N \geq 0$.
3. $\mu = q^M+1$, $\lambda = q^{-N+1}$, $M$, $N \geq 0$.
4. $\mu = q^{-(M+1)}$, $\lambda = q^{-N+1}$, $M$, $N \geq 0$.
5. $\mu = q^{2(M+1)}$, $M \geq 0$, $\lambda \notin q^{-N}$, and $\mu \neq q^{2}$, $\lambda \neq 1$.

The computation of $HH^\sigma_0(A)$ and $HH^\sigma_k(A)$ is done most easily “by hand” using the original Hochschild complex, but for $HH^\sigma_2(A)$ and $HH^\sigma_3(A)$ the calculations are done via the Koszul resolution.

### 4.3 $HH^\sigma_0(A)$

We calculate from first principles the twisted Hochschild homology $HH^\sigma_0(A)$ for all automorphism $\sigma = \sigma_{\lambda,\mu}$. We start with the observation that:

$$b(a_1, a_2a_3) = b(a_1a_2, a_3) + b(\sigma(a_2), a_1, a_2) \quad \forall \ a_1, a_2, a_3 \in A$$

So for any $a_1, a_2 \in A$, there exist $x_a, x_b, x_c, x_d \in A$ such that

$$b(a_1, a_2) = b[(x_a, a) + (x_b, b) + (x_c, c) + (x_d, d)]$$

Hence the image of the twisted Hochschild boundary is spanned by

$$A_{i,j,k} := e_{i,j,k}a - \lambda ae_{i,j,k}$$

$$= (q^{-(j+k)} - \lambda)e_{i+1,j+1,k}$$

$$+ \left\{ \begin{array}{ll}
0 & \text{if } i \geq 0 \\
(q^{-(j+k+1)} - \lambda q^{-2i-1})e_{i+1,j+1,k+1} & \text{if } i < 0
\end{array} \right.,$$

$$B_{i,j,k} := e_{i,j,k}b - \mu be_{i,j,k} = (1 - \mu q^{-1})e_{i,j+1,k},$$

$$C_{i,j,k} := e_{i,j,k}c - \mu^{-1}ce_{i,j,k} = (1 - \mu^{-1} q^{-1})e_{i,j,k+1},$$

$$D_{i,j,k} := e_{i,j,k}d - \lambda^{-1}de_{i,j,k}$$

$$= (q^{j+k} - \lambda^{-1})e_{i-1,j,k}$$

$$+ \left\{ \begin{array}{ll}
0 & \text{if } i \leq 0 \\
(q^{j+k+1} - \lambda^{-1} q^{-2i+1})e_{i-1,j+1,k+1} & \text{if } i > 0
\end{array} \right..$$
For given \((i, j, k)\), the elements \(B_{i,j,k}\) and \(C_{i,j,k}\) both vanish if and only if \(i = 0\) and \(\mu = 1\). Therefore, for all \(\lambda\), \(\mu\), \(\text{im} \ b\) contains the basis elements
\[
e_{i,j,k}, \quad i \neq 0, \ j, k > 0.
\] (37)

Omitting the span of these terms from the above list of generators we see that \(\text{im} \ b\) is spanned by \([37]\) together with
\[
\begin{align*}
A_{-1,j,k} &= (q^{-(j+k)} - \lambda) e_{0,j,k} + (q^{-(j+k+1)} - \lambda q) e_{0,j+1,k+1} \\
\tilde{A}_{i,j,k} &= (q^{-(j+k)} - \lambda) e_{i+1,j,k}, \quad i \neq -1 \\
B_{i,j,0} &= (1 - \mu q^{-i}) e_{i,j+1,0}, \quad B_{0,j,k} = (1 - \mu) e_{0,j+1,k}, \\
C_{i,0,k} &= (1 - \mu^{-1} q^{-i}) e_{i,0,k+1}, \quad C_{0,j,k} = (1 - \mu^{-1}) e_{0,j,k+1}, \\
D_{1,j,k} &= (q^{j+k} - \lambda^{-1}) e_{0,j,k} + (q^{j+k+1} - \lambda^{-1} q^{-1}) e_{0,j+1,k+1} \\
\tilde{D}_{i,j,k} &= (q^{j+k} - \lambda^{-1}) e_{i-1,j,k}, \quad i \neq -1.
\end{align*}
\]

Since \(\tilde{D}_{i+2,j,k}\) is proportional to \(A_{i,j,k}\) and both vanish if and only if \(\lambda = q^{-(j+k)}\), we can omit \(\tilde{D}_{i,j,k}\) from this list. We also have
\[
A_{-1,j,k} = -\lambda q^{-(j+k)} D_{1,j,k},
\]
so the \(D_{1,j,k}\) can be omitted as well. Finally, \(C_{0,j,k}\) is for \(j > 0\) a nonzero multiple of \(B_{-1,j-1,k+1}\) and can be omitted. Thus the degree 0 part (with respect to the \(\mathbb{Z}\)-grading) of \(\text{im} \ b\) is spanned by
\[
(1 - \lambda q^{j+k}) e_{0,j,k} + (q^{-1} - \lambda q^{j+k+1}) e_{0,j+1,k+1},
\]
\[(1 - \mu) e_{0,r,s}, \quad r + s > 0
\]
and the nonzero degrees by \([37]\) together with
\[
(1 - \lambda q^{j+k}) e_{i,j,k}, \quad (1 - \mu q^{-i}) e_{i,j+1,0}, \quad (1 - \mu^{-1} q^{-i}) e_{i,0,k+1}.
\]
where \(i \neq 0\), and \(j, k \geq 0\). Dually,
\[
HH^0_b(A) = \{ \text{linear } h : A \to k : h(a_1 a_2) = h(\sigma(a_2) a_1) \}
\]

For \(\lambda \notin q^{-N}\) we have \(h(e_{i,j,k}) = 0\) for \(i \neq 0\), and
\[
h(b^i c^j) = \begin{cases} (-q)^{-k} \frac{f(k-j)}{f(k)} h(b^{-i-k}) : & j \geq k \\ (-q)^{-j} \frac{f(k-i)}{f(k)} h(c^{k-j}) : & j \leq k \end{cases} \] (38)

where \(f(n) = \lambda - q^{-n}\).

We now present the generating twisted 0-cycles, together with dual twisted 0-cocycles. Our calculations now break down into five cases:

**Case 1:** \(\mu = 1, \lambda \notin \{q^{-(N+2)}\}_{N \geq 0}\) and \(\mu \neq 1, \lambda = 1\). Then
\[
HH^0_b(A) = k[1] \oplus \bigoplus_{x \in \{a, b, c, d\}, \sigma(x) = x} (\sum_{r \geq 0} k[x^{r+1}]) \] (39)

For \(\mu = 1 = \lambda\) (i.e. \(\sigma = \text{id}\)) this agrees with \([15]\). The dual \(\sigma\)-twisted 0-cocycles are defined on basis elements \(x = e_{i,j,k}\) with \(\sigma(x) = x\) as follows:
\[
h_{[1]}(x) = \begin{cases} 1 & : x = 1 \\ (-q)^{j+1} \frac{f(0)}{f(j+2k-2)} : & x = (bc)^{j+1} \\ 0 & : \text{otherwise} \end{cases} \] (40)
(if \( \lambda = 1 \), obviously \( f(0) = 0 \)). For \( y = [a^{r+1}] \), \([d^{r+1}] \) define

\[
h_{[y]}(x) = \begin{cases} 1 & : x = y \\ 0 & : \text{otherwise} \end{cases}
\]

(41)

For \( y = [b^{s+1}] \), \([c^{t+1}] \) define

\[
h_{[b^{s+1}]}(x) = \begin{cases} (-q)^k \frac{f(s+1)}{f(s+2k)} & : x = b^{s+1} (bc)^k \\ 0 & : \text{otherwise} \end{cases}
\]

(42)

\[
h_{[c^{t+1}]}(x) = \begin{cases} (-q)^j \frac{f(t+1)}{f(t+2j)} & : x = (bc)^j c^{t+1} \\ 0 & : \text{otherwise} \end{cases}
\]

(43)

These all satisfy (38). For any \( x \), \( y \) in (39), we have \( h_{[y]}(x) = \delta_{[x],[y]} \), so the 0-cycles given in (39) are linearly independent, hence a basis.

**Case 2:** \( \mu = 1 \), \( \lambda = q^{-(N+2)} \), \( N \geq 0 \). We have

\[
HH^{0}_{[\mu]}(A) = \left( \sum_{k \in S} k[b^{k}] \right) \oplus \left( \sum_{t \in S} k[c^{t}] \right) \oplus \left( \sum_{0 \leq i \leq N+2} k[b^{i} c^{N+2-i}] \right)
\]

(44)

where \( S = \{ \text{integers} \geq N + 3 \} \cup \{ N + 1, N - 1, N - 3, \ldots \geq 0 \} \), with the convention that if \( 0 \in S \), we include only one copy of \( k[1] \). Dual 0-cocycles are \( h_{[y]} \), defined for \( [y] = [b^{i} c^{N+2-i}] \) on the basis \( e_{i,j,k} \) by

\[
h_{[y]}(x) = \begin{cases} 1 & : x = y \\ 0 & : \text{otherwise} \end{cases}
\]

(45)

and for \( [y] = [b^{s}] \), \([c^{t}] \), \( s \), \( t \in S \) by

\[
h_{[b^{s}]}(x) = \begin{cases} (-q)^k \frac{f(s)}{f(s+2k)} & : x = b^{s} (bc)^k \\ 0 & : \text{otherwise} \end{cases}
\]

\[
h_{[c^{t}]}(x) = \begin{cases} (-q)^j \frac{f(t)}{f(t+2j)} & : x = (bc)^j c^{t} \\ 0 & : \text{otherwise} \end{cases}
\]

(46)

So for each pair \([x],[y] \) appearing in (44), we have \( h_{[y]}(x) = \delta_{[x],[y]} \).

**Case 3:** \( \mu = q^{M+1} \), \( \lambda = q^{-(N+1)} \), \( M \), \( N \geq 0 \).

\[
HH^{0}_{[\mu]}(A) \cong k^2 = k[a^{M+1} c^{N+1}] \oplus k[a^{M+1} b^{N+1}]
\]

(47)

Also \( HH^{0}_{[\mu]}(A) \cong k^2 \), with basis the twisted 0-cocycles \( h_{[y]} \), \([y] = [a^{M+1} c^{N+1}] \), \([a^{M+1} b^{N+1}] \), defined on elements \( x = e_{i,j,k} \), with \( \sigma(x) = x \) by

\[
h_{[y]}(x) = \begin{cases} 1 & : x = y \\ 0 & : \text{otherwise} \end{cases}
\]

(48)

**Case 4:** \( \mu = q^{-(M+1)} \), \( \lambda = q^{-(N+1)} \), \( M \), \( N \geq 0 \). We have

\[
HH^{0}_{[\mu]}(A) \cong k^2 = k[a^{M+1} b^{N+1}] \oplus k[a^{M+1} c^{N+1}]
\]

(49)

with \( HH^{0}_{[\mu]}(A) \cong k^2 \) with basis \( h_{[y]} \), \([y] = [a^{M+1} b^{N+1}],[a^{M+1} c^{N+1}] \), defined as in (48).

**Case 5:** \( \mu = q^{\pm(M+1)} \), \( M \geq 0 \), \( \lambda \notin q^{n} \), and \( \mu \notin q^{n} \), \( \lambda \neq 1 \). Then

\[
HH^{0}_{[\mu]}(A) = 0 = HH^{0}_{[\lambda]}(A)
\]
4.4 Twisted cocycles defined by derivations

Before proceeding with $HH^*_\sigma(A)$, we present a general construction of $\sigma$-twisted Hochschild $n$-cocycles. It is essentially a variant of the characteristic map of $[4,10]$.

Let $A$ be a $k$-algebra and $\sigma$ be an automorphism. A $\sigma$-derivation of $A$ is a $k$-linear map $\partial : A \to A$, such that $\partial(a_0a_1) = a_0\partial(a_1) + \partial(a_0)\sigma(a_1)$. The following is straightforward:

**Proposition 4.3** If $h$ is a $\sigma_1$-twisted 0-cocycle, $\partial_1$, ..., $\partial_{n-1}$ derivations of $A$, and $\partial_n$ is a $\sigma_2$-derivation, then

$$\phi_n(a_0,a_1,\ldots,a_n) = h(a_0\partial_1(a_1)\ldots\partial_n(a_n)) \quad (50)$$

is a $\sigma_1 \circ \sigma_2$-twisted Hochschild $n$-cocycle

In general there is no reason for such a cocycle to represent a nontrivial element of Hochschild cohomology, nor for it also to be cyclic. However:

**Lemma 4.4** Suppose $A$ is a unital algebra, $h$ a $\sigma_1$-twisted 0-cocycle and $\partial$ a $\sigma_2$-derivation of $A$. Defining $\phi_1$ by $\phi_1(x,y) = h(x\partial(y))$, then $\phi_1$ is a $\sigma_1 \circ \sigma_2$-twisted cyclic cocycle if and only if $h(\partial(a)) = 0$ for all $a \in A$.

For $A = A(SL_q(2))$ there are obvious derivations $\partial_h$, $\partial_b$ defined by

$$\partial_h(a) = a, \quad \partial_h(b) = 0, \quad \partial_h(c) = 0, \quad \partial_h(d) = -d$$
$$\partial_b(a) = 0, \quad \partial_b(b) = b, \quad \partial_b(c) = -c, \quad \partial_b(d) = 0 \quad (51)$$

and extended via the Leibniz rule. For any $x \in A$ define an inner derivation $\partial'_x$ by $\partial'_x(y) = [x,y] = xy - yx$. The following is straightforward:

**Proposition 4.5** The vector space of all derivations of $A(SL_q(2))$ is spanned by $\partial_h$, $\partial_b$ together with the inner derivations.

In the sequel we will use the derivation $\partial_b = \partial_h + \partial_b$, which satisfies

$$\partial_b(a) = a, \quad \partial_b(b) = b, \quad \partial_b(c) = -c, \quad \partial_b(d) = -d \quad (52)$$

and also the $\sigma_{\lambda,1}$-derivation defined by

$$\partial(a) = a, \quad \partial(b) = 0 = \partial(c), \quad \partial(d) = -\lambda^{-1}d. \quad (53)$$

4.5 $HH^*_\sigma(A)$

The second twisted Hochschild boundary is given by

$$b(a_1,a_2,a_3) = (a_1a_2,a_3) - (a_1,a_2a_3) + (\sigma(a_3)a_1,a_2).$$

In particular (take $a_2 = a_3 = 1$) the image of $b$ contains all elementary tensors of the form $(a_1,1)$, and the residue classes of

$$(e_{i,j,k},a), \quad (e_{i,j,k},b), \quad (e_{i,j,k},c), \quad (e_{i,j,k},d)$$

generate $A \otimes A / \text{im } b$. Now, $HH^*_\sigma(A)$ is the kernel of the map $A \otimes A / \text{im } b \to A$ induced by the first twisted Hochschild boundary. This sends the classes of the above elements to $A_{i,j,k}, B_{i,j,k}, C_{i,j,k}, D_{i,j,k}$ from the previous section. It is straightforward to check for triviality and linear dependence.

We now present generators of $HH^*_\sigma(A)$ and dual twisted 1-cocycles. From Proposition [13] for any $\sigma$-twisted 0-cocycle $h$ and derivation $\partial$, defining $\phi_1(x,y) = h(x\partial(y))$ gives a $\sigma$-twisted Hochschild 1-cocycle. For
$A(SL_q(2))$ all automorphisms $\sigma_{\lambda, \mu}$ commute with the derivations $\partial_a, \partial_b, \partial_c, \partial_d$, and the $c_{i,j,k}$ are eigenvectors for these derivations. By Lemma 4.3, $\phi_1$ is cyclic if and only if $h \circ \partial = 0$. We take $\partial_0 = \partial_a + \partial_b$ defined by (52).

Case 1: $\mu = 1$, $\lambda \notin \{q^{-(N+2)}\}_{N \geq 0}$ and $\mu \neq 1$, $\lambda = 1$. Then

$$HH^1_q(A) = k[\omega_1] \oplus \bigoplus_{x \in \{a,b,c,d\}, \sigma(x) = x} \left( \sum_{r \geq 0} k([x^r, x]) \right)$$

(54)

where $\omega_1 = (\mu - 1)(d, a) + (q - q^{-1})(b, c)$. We note that, for all $\mu$ and for $\lambda \neq q^{-2}$, we have $[(c, b)] = -\mu[(b, c)], [(d, a)] = -\lambda[(a, d)]$. For $\mu = 1 = \lambda$ this is in agreement with (53), apart from the sign change in $\omega_1$. Now recall the 0-cocycles $h_{[x]}$ defined in (40)-(43). Given such an $x$, define a Hochschild 1-cocycle $\phi_{[x]}$ by

$$\phi_{[x]}(y, z) = h_{[x]}(y \partial_0(z))$$

(55)

Then the Hochschild 1-cocycles dual to the generators of $HH^1_q(A)$ are:

$$\phi_{[x^{r+1}]} \leftrightarrow (x^r, x), \quad x \in \{a, b, c, d\}, \quad \sigma(x) = x.$$ Dual to $\omega_1$ we have

$$\phi_{\omega_1}(x, y) = h_{[y]}(x \partial_0(y))$$

(56)

with $h_{[1]}$ defined in (40) and $\partial_0$ defined in (52). Then for $\lambda = 1$ we have $\phi_{\omega_1}(d, a) = 1$, and for $\lambda \neq q^{-2}$ we have $\phi_{\omega_1}(b, c) = \frac{q(1-\lambda)}{1-q^2}$ Since $h_{[1]} \circ \partial_0 = 0$, by Lemma 4.3 $\phi_{\omega_1}$ is in fact a $\sigma$-twisted cyclic 1-cocycle.

Case 2. $\mu = 1$, $\lambda = q^{-(N+2)}, N \geq 0$. Then

$$HH^1_q(A) = \bigoplus_{x \in S'} k[[b^x, b]] \oplus \bigoplus_{t \in S'} k[[c^t, c]] \oplus \bigoplus_{0 \leq i \leq N} k[[b^i c^{N+1-i}, b]] \oplus \bigoplus_{0 \leq i \leq N} k[[b^{i+1} c^{N-i}, c]] \oplus k[\omega_1]$$

(57)

Here $S' = \{\text{integers} \geq N\} \cup \{N-2, N-4, \ldots \geq 0\}$, and for $N \geq 1$ we have $[\omega_1] = [(c, b)] = -[(b, c)]$ for $N$ odd, $[\omega_1] = 0$ for $N$ even. For $\lambda = q^{-2}$, $[(c, b)]$ and $[(b, c)]$ are linearly independent.

Recall the 0-cocycles $h_{[x]}$, with $[x] = [b^x], [c^t]$, defined in (42), (43). The dual Hochschild 1-cocycles (56) are

$$\phi_{[b^{r+1}]} \leftrightarrow (b^r, b), \quad \phi_{[c^{t+1}]} \leftrightarrow (c^t, c)$$

together with the twisted cyclic 1-cocycle $\phi_{\omega_1} (55)$ dual to $\omega_1$. To define twisted 1-cocycles dual to $[(b^i c^{N+1-i}, b)], [(b^{i+1} c^{N-i}, c)]$ we need to work a little harder. It is straightforward to show that any $\sigma$-twisted Hochschild 1-cocycle is uniquely defined by its values $\phi(y, t)$, for $t = a, b, c, d$ and basis elements $y = e_{i,j,k}$.

**Lemma 4.6** For $\lambda = q^{-(N+2)}$, $\mu = 1$ defining $\phi_1, \phi_2$ on basis elements $y = e_{i,j,k}$ by

$$\phi_1(b^i c^{N+1-i}, b) = \beta_{N,i}, \quad \phi_1(d b^i c^{N-i}, a) = q^{-(N+1)} \beta_{N,i},$$

$$\phi_1(y, b) = 0 = \phi_1(y, a) \quad \text{otherwise}, \quad \phi_1(y, c) = 0 = \phi_1(y, d) \quad \forall y$$

$$\phi_2(ab^i c^{N-i}, d) = q^{N+1} \gamma_{N,i}, \quad \phi_2(b^{i+1} c^{N-i}, c) = \gamma_{N,i}$$

$$\phi_2(y, d) = 0 = \phi_2(y, c) \quad \text{otherwise}, \quad \phi_2(y, a) = 0 = \phi_1(y, b) \quad \forall y$$

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for arbitrary $\beta_{N,i}, \gamma_{N,i}, 0 \leq i \leq N$, gives well-defined $\sigma$-twisted Hochschild 1-cocycles.

Setting each $\beta_{N,i}, \gamma_{N,i}$ to 1 in turn, and all others to zero, we see that the twisted Hochschild 1-cycles $[[b^i c^{N+1-i}, b]], [[b^{i+1} c^{N-i}, c]]$ are nontrivial and linearly independent. This extends to $N + 1$ linearly independent twisted cyclic 1-cocycles. Define $\phi = \phi_1 + \phi_2$. Cyclicity requires

$$\phi(xc, b) = -\phi(b, xc) = -(N + 1 - i)\phi(bx, c)$$

where $x = b^i c^{N-i}$. Hence for $\phi$ to be cyclic, we need

$$(i + 1)\beta_{N,i} = -(N + 1 - i)\gamma_{N,i} \quad 0 \leq i \leq N$$

i.e. $\gamma_{N,i} = \frac{-(i + 1)}{(i + 1)} \beta_{N,i}$ for $0 \leq i \leq N$. Then:

**Lemma 4.7** For each $0 \leq i \leq N$, defining $\phi_{N,i} = \phi_1 + \phi_2$ with

$$\beta_{N,i} = N + 1 - i, \quad \gamma_{N,i} = -(i + 1)$$

gives a well-defined twisted cyclic 1-cocycle satisfying

$$\phi_{N,i}(xc, b) = N + 1 - i, \quad \phi_{N,i}(bx, c) = -(i + 1)$$

$$\phi_{N,i}(ax, d) = -q^{N+1}(i + 1), \quad \phi_{N,i}(dx, a) = q^{-(N+1)}(N + 1 - i)$$

for $x = b^i c^{N-i}$, and $\phi_{N,i}(y, t) = 0$ for all basis elements $y$ and $t = a, b, c, d$ otherwise.

**Case 3.** $\mu = q^{M+1}, \lambda = q^{-(N+1)}, M, N \geq 0$. Then $HH^*_A(A) \cong k^4$, with basis given by the Hochschild cycles

$$(a^M b^{N+1}, a), \quad (a^{M+1} b^N, b), \quad (d^{M+1} c^N, c), \quad (d^M c^{N+1}, d)$$

(58)

The dual basis for $HH^{M}_A(A) \cong k^4$ is given by the 1-cocycles $\phi_{[x], t}$, for $[x] = [a^{M+1} b^{N+1}], [d^{M+1} c^{N+1}]$, and $t = a, b$ defined by

$$\phi_{[x], t}(y, z) = h_{[x]}(y \delta_t(z))$$

(59)

where the $h_{[x]}$ were defined in (53) and $\delta_a, \delta_b$ in (51). We have

$$\phi_{[a^{M+1} b^{N+1}], a}(a^M b^{N+1}, a) = q^{-(N+1)}$$

$$\phi_{[a^{M+1} b^{N+1}], b}(a^M b^{N+1}, b) = 1$$

$$\phi_{[d^{M+1} c^{N+1}], c}(d^{M+1} c^{N+1}, c) = -1$$

$$\phi_{[d^{M+1} c^{N+1}], d}(d^M c^{N+1}, d) = -q^{N+1}$$

with all other pairings being zero.

**Case 4.** $\mu = q^{-(M+1)}, \lambda = q^{-(N+1)}, M, N \geq 0$. $HH^*_A(A) \cong k^4$, with basis given by the Hochschild cycles

$$(a^M b^{N+1}, a), \quad (d^{M+1} b^N, b), \quad (a^{M+1} c^N, c), \quad (d^M b^{N+1}, d)$$

(60)

Analogously to Case 3, the dual basis for $HH^{M}_A(A) \cong k^4$ is given by the 1-cocycles $\phi_{[x], t}$ for $[x] = [a^{M+1} c^{N+1}], [d^{M+1} b^{N+1}]$, and $t = a, b$.

**Case 5.** For $\mu = q^{\pm (M+1)}, M \geq 0, \lambda \notin q^{-N}$, and $\mu \notin q^Z, \lambda \neq 1$, $HH^*_A(A) = 0$ and $HH^*_A(A) = 0$. 17
4.6 $HH^2_2(\mathcal{A})$

We now compute $HH^2_2(\mathcal{A})$. The first step is to describe the kernel of

$$\psi : \mathcal{A}^2 \to \mathcal{A}, \quad (x, y) \mapsto \psi_+ (x) + \psi_-(y)$$

If $\ker \psi_+ \subset \mathcal{A}$ are fixed complements to $\ker \psi_+$ and

$$\phi_\pm : \im \psi_\pm \to \ker \psi_\pm$$

are the inverses of $\psi_+ |_{\ker \psi_+}$, then the linear map

$$\ker \psi_+ \oplus (\im \psi_+ \cap \im \psi_-) \oplus \ker \psi_- \to \ker \psi,$$

$$(x, y, z) \mapsto (x + \phi_+(y), z - \phi_-(y))$$

is an isomorphism of vector spaces. Using (59) one determines a basis of $\im \psi_+ \cap \im \psi_-$ and obtains:

**Proposition 4.8** The set

$$\mathcal{B}_{\ker \psi} := \{(x_+, 0, (0, x_-) | x_\pm \in \mathcal{B}_{\ker \psi_\pm}\}

\cup \{(-\lambda^{-2} e_{-1,j,k} + \varepsilon_{-1,j,k+1} e_{-i,j-2,j,k+1} + \varepsilon_{-i,j-2,j,k}, i \geq 2, j \geq 1, k \geq 0, \varepsilon_{-i,j-2,j,k} \neq 1) \}

\cup \{(-\lambda^{-2} q e_{-1,j,0} + \varepsilon_{-1,j,0} e_{-i,j+2,j,0} + \varepsilon_{-i,j+2,j,0}, i \leq -2, j \geq 0, \varepsilon_{-i,j,0} = 1) \}

\cup \{(\varepsilon_{-1,j,k} e_{-1,j,k+1} + e_{-i,j+1,k+1}, i \geq 0, k \geq 1, \varepsilon_{-i,j,k} = 1) \}

\cup \{(0, 0, e_{0,j,k}, i \geq 0, j \geq 0, \varepsilon_{0,j} = 1) \}

\cup \{(\lambda^{-2} q e_{-1,j,k} + \varepsilon_{-1,j,k} e_{-i,j+2,j,k}, i \geq 2, k \geq 0, \varepsilon_{-i,j+2,j,k} = 1) \}

\cup \{(\varepsilon_{-1,j,k} e_{-1,j,k+1} + e_{-i,j+1,k+1}, i \geq 0, k \geq 1, \varepsilon_{-i,j,k} = 1) \}

form a vector space basis of $\ker \psi + \im \varphi$.

Now we can compute which of these remain nontrivial and linearly independent modulo the image of the map $\varphi : x \mapsto (\psi_-(x), -\psi_+(x))$.

**Proposition 4.9** The classes of

$$\{e_{i,j,0} \mid i \geq 0, j \geq 0, \varepsilon_{i,j} = 1\}$$

$$\cup \{\lambda^{-2} q e_{-1,j,0} e_{-i,j+2,j,0} + e_{-i,j+2,j,0}, i \geq -2, j \geq 0, \varepsilon_{-i,j,0} = 1\}$$

$$\cup \{(\varepsilon_{-1,j,k} e_{-1,j,k+1}, i \geq 0, k \geq 1, \varepsilon_{-i,j,k} = 1)\}$$

form a vector space basis of $\ker \psi + \im \varphi$.

Next we check for which linear combinations $(x, y) \neq (0, 0)$ of these there exists $z$ with $(x, y, z) \in \ker f_2$, and determine those $z$ with $(0, 0, z) \in \ker f_2$, giving a generating set for $HH^2_2(\mathcal{A})$:

**Proposition 4.10** The classes of

$$\{0, 0, e_{0,j,k}, j, k \geq 0, \varepsilon_{0,j} = 1\}$$

$$\cup \{e_{i,j,0}, 0, 0, i \geq 0, j \geq 0, \lambda = q^{-j-1}, \mu = q^{i+1}\}$$

$$\cup \{\lambda^{-2} q e_{-1,j,k} e_{-i,j+2,j,k}, i \geq 2, k \geq 0, \varepsilon_{-i,j+2,j,k} = 1, \mu = q^{i+1}\}$$

$$\cup \{e_{i,j,0}, \lambda^{-2} q e_{-1,j,k} e_{-i,j+2,j,k}, i \leq -2, j \geq 0, \varepsilon_{-i,j+2,j,k} = 1, \mu = q^{i+1}\}$$

$$\cup \{e_{-1,j,k}, \lambda^{-2} e_{-1,j,k+1}, j \geq 0, k \geq 1, \lambda = q^{-j-k}, \mu = 1\}$$

generate $HH^2_2(\mathcal{A})$.  

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The classes of the elements in the first line are trivial for $\mu \neq 1$, and for $\mu = 1$ they contain those from the last line. It follows directly from the definition of $f_3$ that the remaining classes are independent. Hence:

$$HH_{2}^{\sigma}(A) \cong \begin{cases} k^{N+1} & : \lambda = q^{-(N+2)}, N \geq 0, \mu = 1, \\ k^2 & : \lambda = q^{-(N+1)}, \mu = q^{\pm(M+1)}, M, N \geq 0, \\ 0 & : \text{otherwise}. \end{cases}$$ (61)

To calculate $HC_{2}^{\sigma}(A)$ we need generators in the original Hochschild complex. In Case 2 we compute generators from the above using $\phi_2$ and $\xi$. In Cases 3 and 4 we use simpler generators that are directly verified to be homologous to those obtained from the above ones:

**Case 2.** $\mu = 1$, $\lambda = q^{-(N+2)}$, $N \geq 0$. Then $HH_{2}^{\sigma}(A) \cong k^{N+1}$. Taking $x = b^i c^{N-i}$ ($0 \leq i \leq N$), a basis is given by

$$\omega_2(N,i) = (bcx, a, d) - (bxc, d, a) - q(dbx, a, c) + q(bdx, c, a)$$

$$+ (dax, b, c) - (adx, c, b) - q^{-1}(cax, b, d) + q^{-1}(acx, d, b)$$ (62)

**Case 3.** $\mu = q^{M+1}$, $\lambda = q^{-(N+1)}$, $M, N \geq 0$. Then $HH_{2}^{\sigma}(A) \cong k^2$, with basis given by the Hochschild cycles

$$\omega_2 = (a^M b^N, b, a) - q^{-1}(a^M b^N, a, b)$$

$$\omega'_2 = (d^M c^N, c, d) - q(d^M c^N, d, c)$$ (63)

**Case 4.** $\mu = q^{-(M+1)}$, $\lambda = q^{-(N+1)}$, $M, N \geq 0$. $HH_{2}^{\sigma}(A) \cong k^2$, with basis given by the Hochschild cycles

$$\omega_2 = (a^M c^N, c, a) - q^{-1}(a^M c^N, a, c)$$

$$\omega'_2 = (d^M b^N, b, d) - q(d^M b^N, d, b)$$ (64)

Finally, $HH_{2}^{\sigma}(A) = 0$ for all other $\sigma = \sigma_{\lambda, \mu}$.

### 4.7 $HH_{3}^{\sigma}(A)$

The third homology $HH_{3}^{\sigma}(A)$ can be determined easily using the Koszul resolution. We abbreviate:

$$\psi_+ : A \to A, \quad x \mapsto x \triangleright b, \quad \psi_- : A \to A, \quad x \mapsto x \triangleright c.$$ 

From (36) we obtain in a straightforward way:

**Proposition 4.11** The sets

$$B_{\ker \psi_\pm} := \{ e_{i,j,k} \mid \pm i \geq 0, \ e_{i,j,k} = 1 \}$$

are vector space bases of $\ker \psi_\pm$. Hence the sets

$$B_{\text{im } \psi_\pm} := \{ E_{i,j,k}^\pm \mid e_{i,j,k} \notin \ker \psi_\pm \}, \quad E_{i,j,k}^\pm := \psi_\pm(e_{i,j,k})$$

are vector space bases of $\text{im } \psi_\pm$. 

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If \( x \in HH_3^3(A) = \ker f_3 \), then \( x \in \ker \psi_3 \cap \ker \psi_- \), so by Proposition 4.11 \( \pi_{i,j,k}(x) \neq 0 \) implies \( i = 0 \). Insertion in \( q^{-2}x \uparrow a = x \) gives

\[
x \in \begin{cases} \text{span } \{ b^i c^{N-i} \}_{i=0,...,N} : \lambda = q^{-N-2}, \mu = 1, \\ 0 : \text{otherwise}. \end{cases}
\]

Conversely, all these monomials are elements of \( \ker f_3 \). Hence:

**Case 2:** For \( \mu = 1, \lambda = q^{-(N+2)} \), \( N \geq 0 \) we have \( HH_3^3(A) \cong k^{N+1} \).

**Cases 1, 3, 4, 5:** \( HH_3^3(A) = 0 \).

It is also straightforward that \( HH_3^3(A) = 0 \) for all \( \sigma = \tau_{\lambda,\mu} \). Therefore:

**Theorem 4.12** For any automorphism \( \sigma \), we have

\[
HH_3^3(A) = \begin{cases} k^{N+1} & \sigma = \sigma_{\lambda,\mu}, \lambda = q^{-(N+2)}, N \geq 0, \mu = 1, \\ 0 & \text{otherwise}. \end{cases}
\]

Note that the \( N = 0 \) case \( (\lambda = q^{-2}, \mu = 1) \) is precisely the modular automorphism \( (\kappa) \). For \( \lambda = q^{-(N+2)} \) we translate the generators back to the original Hochschild complex using the maps \( \varphi_3 \) \((19)\) and \( \xi \) \((15)\), giving

\[
\omega_3(N,i) = A(N,i) - B(N,i), \quad 0 \leq i \leq N,
\]

with \( x = b^i c^{N-i} \), and the terms “\( a_0 \wedge a_1 \wedge a_2 \)” are given by:

\[
\begin{align*}
a \wedge b \wedge c &= (a, b, c) - (a, c, b) + q(c, a, b) - q^2(c, b, a) + q^2(b, c, a) - q(b, a, c) \\
b \wedge a \wedge d &= (b, a, d) - (b, d, a) + q(d, b, a) - (d, a, b) + (a, d, b) - q^{-1}(a, b, d) \\
1 \wedge a \wedge c &= (1, a, c) - q(1, c, a) + q(c, 1, a) - q(c, a, 1) + (a, c, 1) - (a, c, 1) \\
1 \wedge b \wedge d &= (1, b, d) - q(1, d, b) - (b, 1, d) + (b, d, 1) - q(d, b, 1) + q(d, 1, b) \\
1 \wedge b \wedge c &= (1, b, c) - (1, c, b) - (b, 1, c) + (b, c, 1) + (c, 1, b) - (c, b, 1) \\
1 \wedge a \wedge d &= (1, a, d) - (1, d, a) + (d, 1, a) - (d, a, 1) + (a, d, 1) - (a, 1, d)
\end{align*}
\]

and throughout we denote \( a_0 \otimes a_1 \otimes a_2 \) by \( (a_0, a_1, a_2) \).

In the normalized complex this becomes \( \omega_3(N,i) = A(N,i) \) since \( B(N,i) \) is degenerate.

**5 Twisted cyclic homology of \( A(SL_q(2)) \)**

We calculate the twisted cyclic homology of \( A(SL_q(2)) \) as the total homology of Connes’ mixed \( (b, B) \)-bicomplex \((20)\) coming from the underlying cyclic object, as in section \((22)\). Having found the twisted Hochschild homology, we can now complete the spectral sequence calculation. We remind the reader that throughout we are working with the normalized mixed complex.
5.1 Case 1

Proposition 5.1 In case 1, \( \mu = 1, \lambda \neq 1 \), \( HC_{n}^{\sigma}(A) \) is infinite-dimensional, \( HC_{2n+1}^{\sigma}(A) = k[\omega_1] \), and \( HC_{2n+2}^{\sigma}(A) = k[1] \), where \( [\omega_1] \) is the distinguished generator of \( HH_2^{\sigma}(A) \).

Proof. By definition, \( HC_{n}^{\sigma}(A) = HH_0^{\sigma}(A) \), generated by \([1]\), together with \( [x^{r+1}] \) \((r \geq 0)\), for those \( x \in \{a, b, c\} \) with \( \sigma(x) = x \), while \( HH_2^{\sigma}(A) \) is infinite-dimensional, \( HH_3^{\sigma}(A) = \im(B_0) \oplus k[\omega_1] \).

Further \( HH_n^{\sigma}(A) = 0 \) for \( n \geq 2 \) in each case. Hence the spectral sequence stabilizes at the second page:

\[
\begin{array}{c}
0 \\
\downarrow \\
0 \\
\downarrow \\
k[\omega_1] \\
\downarrow \\
0 \\
\downarrow \\
HH_0^{\sigma}(A)
\end{array}
\]

with all further maps being zero. The result follows. \( \square \)

5.2 Case 2

Proposition 5.2 In case 2, \( \mu = 1, \lambda = q^{-(N+2)} \), \( N \geq 0 \), we have \( HC_{n}^{\sigma}(A) \) infinite dimensional, while

\[
HC_{1}^{\sigma}(A) \cong \begin{cases} 
k^{N+1} & : N \text{ even} \\
k^{N+2} & : N \text{ odd} \end{cases}, \quad HC_{2}^{\sigma}(A) \cong k^{N+2}, \quad N \text{ odd}
\]

Proof. Recall from (14) that

\[
HH_0^{\sigma}(A) = \left( \sum_{s \in S} k[b^s] \right) \oplus \left( \sum_{t \in S'} k[e^t] \right) \oplus \left( \sum_{0 \leq i \leq N+2} k[b^c e^{N+2-i}] \right)
\]

where \( S = \{ \text{integers} \geq N+3 \} \cup \{ N+1, N-1, N-3, \ldots \geq 0 \} \), with the convention that if \( 0 \in S \), we include only one copy of \( k[1] \). From (15)

\[
HH_1^{\sigma}(A) = \left( \sum_{s \in S} k[(b^s, b)] \right) \oplus \left( \sum_{t \in S'} k[(e^t, c)] \right) \oplus \left( \sum_{0 \leq i \leq N} k[(b^c e^{N+1-i}, b)] \right) \oplus \left( \sum_{0 \leq i \leq N} k[(b^{i+1} e^{N-i}, c)] \right) \oplus k[\omega_1]
\]
Here \( S' = \{ \text{integers} \geq N \} \cup \{ N - 2, N - 4, \ldots \geq 0 \} \), and \( [\omega] = [(c, b)] = -[(c, b)] \) for \( N \) odd, \( [\omega] = 0 \) for \( N \) even. Now, for \( s, t \geq 0 \),

\[
B_0[b^{s+1}] = (s + 1)[b^s, b], \quad B_0[c^{t+1}] = (t + 1)[c^t, c]
\]

Note that, for \( s \geq 0, s + 1 \in S \) if and only if \( s \in S' \). We also have

\[
B_0[1] = [(1, 1)] = [b(1, 1, 1)] = 0
\]

By Lemma 5.2 for \( 0 \leq i \leq N \)

\[
B_0[b^{i+1} c^{N+1-i}] = (N + 1 - i)[(b^{i+1} c^{N-i}), c] + (i + 1)[(b^i c^{N+1-i}), b]
\]

Hence \( \ker(B_0) = k[1] \) if \( N \) is odd, 0 if \( N \) is even. Further,

\[
HH^1_\mathbb{Z} \circ \text{im}(B_0) \cong \left\{ \begin{array}{ll}
{k^{N+1} = \sum_{0 \leq i \leq N} k[\omega_1(N, i)]} & : N \text{ even} \\
{k^{N+2} = \sum_{0 \leq i \leq N} k[\omega_1(N, i)]} & : N \text{ odd}
\end{array} \right.
\]

with generators

\[
[\omega_1(N, i)] = [(xc, b)] = [(bx, c)], \quad x = b^i c^{N-i}, \quad 0 \leq i \leq N
\]

(67) together with (if \( N \) is odd) \( [\omega] = [(c, b)] = -[(b, c)] \).

**Proposition 5.3** For \( N \) odd, \( B_1 = 0 \). For \( N \) even, \( \text{im}(B_1) \) is at most one-dimensional, spanned by \([B_1(\omega_1(N, \frac{1}{2} N))]\).

**Proof.** Recall that \( HH^2_\mathbb{Z} \cong k^{N+1} \), with generators \( \omega_2(N, i), 0 \leq i \leq N \) given in (62). We use the construction of \( \sigma \)-twisted Hochschild \( n \)-cocycles of Proposition 5.1. Using (60–64), for each \( n \in \mathbb{Z} \) define a trace \( h_n \) by

\[
h_n(b^{i+1}) = 0 = h_n(d^{i+1}), \quad h_n(b^i) = \delta_{n,j}, \quad h_n(c^j) = \delta_{-n,j}
\]

(68) for \( i, j \geq 0 \). For the derivation \( \partial_b \) and \( \sigma \)-derivation \( \partial \) define a \( \sigma \)-twisted Hochschild 2-cocycle \( \phi_{2,n} \) by

\[
\phi_{2,n}(x, y, z) = h_n(x \partial_b(y) \partial(z))
\]

(69)

**Lemma 5.4** \( \phi_{2,n}, \omega_2(N, i) \geq 0 \), unless \( n = 2i - N \). For \( i \neq \frac{1}{2} N \), we have \( \phi_{2,2i-N}, \omega_2(N, i) > 0 \).

**Proof.** Directly, \( \phi_{2,n}, \omega_2(N, i) \geq \phi_{2,n}, q(bdx, c, a) - q^{-1}(cax, b, d) > 0 \) (considering only potentially nonzero terms)

\[
= h_n(bdx(-c)a - q^{-1}cax(-c^{-1}d)) = h_n(q^{-1}caxb - qbdxc) = h_n(q^N c^N b c^x d - q^{-1}a) = q^{N+2}(q^N - q^{-1})h_n(bcx(1 + qbc)) = \left( \frac{q^2 - 1}{q^{N+4} - 1} \right) h_n(bcx)
\]

Since \( bcx = b^{i+1} c^{N+1-i} \), it’s clear from (62), (63) that \( h_n(bcx) = 0 \) unless \( n = (i+1) - (N-i+1) = 2i - N \) and \( h_{2i-N}(bcx) \neq 0 \) unless \( 2i = N \).

To find \( \text{im}(B_1) \), since \( B_1 \circ B_0 = 0 \), we need only consider \( B_1 \) applied to \( \omega_1 \) and the \( \omega_1(N, i) \). For \( [\omega_1] = [(c, b)] = -[(b, c)] \), which is nonzero if and only if \( N \) is odd, we have \( \deg(\omega_1) = (0, 0) \) for the \( \mathbb{Z}^2 \)-grading (20).
The maps $B_n$ preserve the grading, so $\deg(B_1(\omega_1)) = (0, 0)$ also. Now, $\deg(\omega_2(N, i)) = (0, 2i - N) \neq (0, 0)$ for any $N, i$ since $N$ is odd. The Hochschild boundary maps $b$ also preserve the grading, so $B_1(\omega_1)$ cannot be cohomologous to any nontrivial element of $HH^2_0(A)$.

Now consider the generators $\omega_1(N, i)$ (14). These contain only $b$'s and $c$'s, so combining this with (13), it is immediate that each $\phi_{2,n}$ vanishes on $\text{im}(B_1)$. So for $i \neq 0, N$, we have $[B_1(\omega_1(N, i))]) = 0$. □

The second page of the spectral sequence reads:

\[
\begin{array}{ccccccccc}
0 & \leftarrow & HH_0^0(A)/\text{im}(B_2) & \leftarrow & \text{ker}(B_2)/\text{im}(B_1) & \leftarrow & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & HH_2^0(A)/\text{im}(B_2) & \leftarrow & \text{ker}(B_2)/\text{im}(B_1) & \leftarrow & \text{ker}(B_1)/\text{im}(B_0) & \leftarrow & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & HH_2^0(A)/\text{im}(B_1) & \leftarrow & \text{ker}(B_1)/\text{im}(B_0) & \leftarrow & \text{ker}(B_0) & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & HH_0^0(A)/\text{im}(B_0) & \leftarrow & \text{ker}(B_0) & & & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & HH_0^0(A) & & & & & & \\
\end{array}
\]

The only potentially nonzero differential is $f : \ker(B_0) \rightarrow HH_0^0(A)/\text{im}(B_2)$, and after this step the spectral sequence stabilises, giving

- $HC_0^0(A) = HH_0^0(A)$,
- $HC_1^0(A) = HH_1^0(A)/\text{im}(B_0)$,
- $HC_2^0(A) = (HH_2^0(A)/\text{im}(B_1)) \oplus \ker(B_0)$,
- $HC_2^{2n+1}(A) = ((HH_2^0(A)/\text{im}(B_2))/\text{im}(f)) \oplus (\ker(B_1)/\text{im}(B_0))$,
- $HC_2^{2n+2}(A) = (\ker(B_2)/\text{im}(B_1)) \oplus \ker(f)$

Hence $HC_0^0(A)$ is infinite-dimensional, given by (14), whilst $HC_1^0(A)$ is given by (22). Now, $\ker(B_0) = k[1]$, which is nonzero if and only if $N$ is odd. So for $N$ even, $\ker(f) = 0 = \text{im}(f)$. For $N$ odd, $\text{im}(B_1) = 0$, hence $HC_2^1(A) = HH_2^0(A) \oplus k[1] \cong k^{N+2}$. This completes the proof of Proposition 5.3. □

Recall that $HH_2^0(A)$ and $HH_2^1(A)$ are both isomorphic to $k^{N+1}$ with generators $\omega_2(N, i) \quad (0 \leq i \leq N)$ respectively. By symmetry and the $\mathbb{Z}^2$-grading ($\deg(\omega_2(N, i)) = 0, 2i - N = \deg(\omega_3(N, i))$) we expect, but do not have a proof for:

**Conjecture 5.5**

1. For $N$ even, $[B_1(\omega_1(N, \frac{1}{2} N))] \neq 0 \in HH_2^0(A)$, and is proportional to $[\omega_2(N, \frac{1}{2} N)]$.

2. For all $N, B_2 : HH_2^0(A)/\text{im}(B_1) \rightarrow HH_2^0(A)$ is injective. It follows that for $N$ odd, $f : \ker(B_0) \rightarrow HH_2^0(A)$ is the zero map.

From this it would follow that, for $N$ even, $HC_2^0(A) \cong k^N$, generated by $[\omega_2(N, i)]$ for $i \neq 0, N$, $HC_2^{2n+1}(A) \cong k^{N+2}$, generated by $[\omega_3(N, \frac{1}{2} N)]$ together with (16), and $HC_2^{2n+2}(A) \cong k[\omega_2(N, \frac{1}{2} N)]$. For $N$ odd we would have $HC_2^{2n+1}(A) \cong k^{N+2}$, given by (16), and $HC_2^{2n+4}(A) \cong k[1]$. 23
5.3 Cases 3, 4 and 5

Proposition 5.6 For case 3, \( \mu = q^{M+1}, \lambda = q^{-(N+1)} \), \( M, N \geq 0 \),

1. \( HC^n_0(A) \cong k^2 \), with generators \([a^{M+1}b^{N+1}], [a^{M+1}b^{N+1}]\).

2. \( HC^n_1(A) \cong k^2 \), with generators \([(a^{M+1}b^N), [(d^{M+1}c^N), c]], \) equivalently \([(d^M c^{N+1}), d]], [(a^{M+1}b^{N+1}), a]]\).

3. \( HC^n_2(A) = 0 \), \( n \geq 2 \)

Proof. Recall that \( HH^n_0(A) \cong k^2 \), generated by \([a^{M+1}b^{N+1}], [a^{M+1}b^{N+1}]\),
\( HH^n_1(A) \cong k^2 \), generated by \([a^{M+1}b^N], [(d^{M+1}c^N), c]], [(d^M c^{N+1}), d]]\) and \([(a^{-b^{N+1}}, a)], \) and \( HH^n_2(A) \cong k^2 \) with generators \( \omega_2 \) and \( \omega_2 \).

Lemma 5.7 For \( B_0 : HH^n_0(A) \to HH^n_1(A) \) we have:

\[
B_0[a^{M+1}b^{N+1}] = (N + 1)[(d^{M+1}c^N), c] + (M + 1)q^{-(N+1)}[(d^{M+1}c^N), c]]
\]

\[
B_0[a^{M+1}b^{N+1}] = (N + 1)[(a^{M+1}b^N), [(d^{M+1}c^N), c]], [(d^M c^{N+1}), d]]
\]

Proof. We treat only \([d^{M+1}c^{N+1}], \) the calculations for \([a^{M+1}b^{N+1}]\) are completely analogous. By considering \( b_1, d^{M+1}c^N, c) \), we find that

\[
B_0[a^{M+1}b^{N+1}] = [(1, d^{M+1}c^N), c)] + [q^{-(M+1)}(c, d^{M+1}c^N)]
\]

Now, for any \( x, y \in A \), and for all \( r \geq 0 \), a simple induction shows that:

\[
[x, y^{r+1}] = \sum_{j=0}^{r} [(\sigma(y))^j x y^{r-j}, y]
\] (70)

Also, \([(c, d^{M+1}c^N)] = q^{M+1}[(d^{M+1}c^N), c] + q^{-(M+1)}[(d^{M+1}c^N), c]]\). It follows from (70) that \([(d^{M+1}c^N), c]) = N[(d^{M+1}c^N), c] \) and \([(c^N, d^{M+1})] = (M + 1)q^{M(N+1)}[(d^{M}c^{N+1}), d]]. \) Hence the result.

Corollary 5.8 It follows that:

1. \( \ker(B_0) = 0 \).

2. \( HH^n_1(A)/\im(B_0) \cong k^2 \), generated by \([(a^{M+1}b^N), [(d^{M+1}c^N), c]], \) equivalently \([(d^M c^{N+1}), d]], [(a^{M+1}b^{N+1}), a]]\).

Lemma 5.9 \( B_1 : HH^n_1(A) \to HH^n_2(A) \) is surjective.

Proof. Using (63) we have

\[
B_1[a^{M}b^{N+1}, a] = (1, a^{-b^{N+1}}, a) - q^{-(N+1)}(1, a^{-b^{N+1}})
\]

\[
B_1[d^{M}c^{N+1}, d] = (1, d^{M}c^{N+1}, d) - q^{N+1}(1, d, d^{M}c^{N+1})
\]

Consider the twisted Hochschild 2-cocycles given by

\[
\phi_2 (x, y, z) = h_{[a^{M+1}b^{N+1}]}(x \partial_a(y) \partial_b(z))
\]

\[
\phi_2' (x, y, z) = h_{[d^{M}c^{N+1}]}(x \partial_a(y) \partial_b(z))
\]

with \( h_{[a^{M+1}b^{N+1}]} \), \( h_{[d^{M}c^{N+1}]} \) defined in (68). Then

\[
\phi_2(B_1(a^{M}b^{N+1}, a)) = - (N + 1)q^{-(N+1)}
\]

\[
\phi_2'(B_1(a^{M}b^{N+1}, a)) = 0 = \phi_2(B_1(d^{M}c^{N+1}, d))
\]

\[
\phi_2(B_1(d^{M}c^{N+1}, d)) = - q^{N+1}(N + 1)
\]
It follows that \( B_1(a^M b^{N+1}, a) \) and \( B_1(d^M c^{N+1}, d) \) are nontrivial and linearly independent, and hence span \( HH_2^2(A) \cong k^2 \).

It follows that \( \ker(B_1)/\im(B_0) = 0 \), and \( HH_2^2(A)/\im(B_1) = 0 \). So:

1. \( HC_0^0(A) = HH_0^2(A) \)
2. \( HC_1^0(A) = HH_1^2(A)/\im(B_0) \cong k^2 \)
3. \( HC_2^2(A) = (HH_2^2(A)/\im(B_1)) \oplus \ker(B_0) = 0 \)
4. \( HC_2^2(A) = \ker(B_1)/\im(B_0) = 0 \)

This completes the proof of Proposition 5.6.

Dually, we have \( HC_0^0(A) \cong k^2 \), generated by the two 0-cocycles \( h_{[a M+1, b N+1]} \), \( h_{[d M+1, c N+1]} \) defined in (18). To give the generators of \( HC_1^0(A) \cong k^2 \), define a new derivation \( \partial' = (N + 1) \partial_a - (M + 1) \partial_b \). We have

\[
\partial'(a^M b^{N+1}) = 0 = \partial'(d^M c^{N+1})
\]

so by Lemma 4.4, the twisted Hochschild 1-cocycles \( \phi_1, \phi_1' \) defined by

\[
\phi_1(x, y) = h_{[a M+1, b N+1]}(x \partial'(y)), \quad \phi_1'(x, y) = h_{[d M+1, c N+1]}(x \partial'(y))
\]

are also twisted cyclic 1-cocycles, and satisfy

\[
\phi_1(a^M b^{N+1}, a) = N + 1, \quad \phi_1(d^M c^{N+1}, d) = 0
\]

\[
\phi_1'(a^M b^{N+1}, a) = 0, \quad \phi_1'(d^M c^{N+1}, d) = -(N + 1)
\]

In fact \( \phi_1, \phi_1' \) are a basis for \( HC_1^2(A) \cong k^2 \).

**Proposition 5.10**  
In case 4, \( \mu = q^{-(M+1)}, \lambda = q^{-(N+1)}, M, N \geq 0 \),

1. \( HC_0^0(A) \cong k^2 \), with generators \( [d^M b^{N+1}], [a^M c^{N+1}] \).
2. \( HC_1^0(A) \cong k^2 \), generated by \( [(d^M b^{N+1}, b)], [(a^M c^{N+1}, c)] \), equivalently \( [(d^M b^{N+1}, d)], [(a^M c^{N+1}, a)] \).
3. \( HC_2^2(A) = 0 \); \( n \geq 2 \).

The proof is completely analogous to that of Proposition 5.6. We also have \( HC_0^0(A) \cong k^2 \), generated by the two 0-cocycles \( h_{[a M+1, c N+1]} \), \( h_{[d M+1, b N+1]} \), and \( HC_1^0(A) \cong k^2 \), generated by \( \phi_1, \phi_1' \) defined by

\[
\phi_1(x, y) = h_{[a M+1, c N+1]}(x \partial'(y)), \quad \phi_1'(x, y) = h_{[d M+1, b N+1]}(x \partial'(y))
\]

where \( \partial' = (N + 1) \partial_a + (M + 1) \partial_b \).

The remaining case is the trivial one:

**Proposition 5.11**  
In case 5 (\( \mu = q^{2(M+1)}, M \geq 0, \lambda \notin q^{2-N}, \) and \( \mu \notin q^2 \), \( \lambda \neq 1 \)), we have \( HC_n^0(A) = 0 \) for all \( n \geq 0 \).

**Proof.** In each case \( HH_n^2(A) = 0 \) for all \( n \geq 0 \), so the spectral sequence stabilizes at the first page, with all entries being zero. \( \square \)
6 Covariant differential calculi

In this section we identify the classes in twisted cyclic cohomology of \( \mathcal{A}(SL_q(2)) \) of the twisted cyclic cocycles arising from the three and four dimensional covariant differential calculi originally discovered for quantum \( SU(2) \) by Woronowicz.

6.1 THREE DIMENSIONAL LEFT-COVARIANT CALCULUS

The automorphism of \( \mathcal{A}(SL_q(2)) \) corresponding to Woronowicz’s three-dimensional left-covariant calculus over quantum \( SU(2) \) is

\[
\sigma(a) = q^{-2}a, \quad \sigma(b) = q^4b, \quad \sigma(c) = q^{-4}c, \quad \sigma(d) = q^2d
\]  

(71)

The twisted cyclic 3-cocycle over \( \mathcal{A}(SL_q(2)) \) arising from this calculus was written down in [17] section 3 (denoted by \( \tau_{\omega,h} \)) and [12] section 5 (corresponding to the linear functional \( \int \)).

Theorem 6.1 For \( A = \mathcal{A}(SL_q(2)) \), we have \( HC^3_\sigma(A) = 0 \) for the automorphism (71), hence the twisted cyclic 3-cocycle corresponding to \( \tau_{\omega,h} \) and \( \int \) is a trivial element of twisted cyclic cohomology.

Specializing Proposition 5.6 to the automorphism (71), we obtain:

Proposition 6.2 For \( \sigma = \sigma_{\lambda,\mu} \), with \( \lambda = q^{-2}, \mu = q^4 \), we have

1. \( HC^0_{\sigma}(A) = HH^0_{\sigma}(A) \cong k^2 \), with generators \( [d^i c^2], [a^i b^2] \).
2. \( HC^1_{\sigma}(A) \cong k^2 \) generated by \( [(a^i b, b)], [(d^i c, c)] \),
   equivalently \( [(d^i c^2, d)], [(a^i b^2, a)] \).
3. \( HC^n_{\sigma}(A) = 0 \), \( n \geq 2 \)

By duality between twisted cyclic homology and cohomology we have

Corollary 6.3 \( HC^0_{\sigma}(A) \cong k^2 \cong HC^1_{\sigma}(A) \), \( HC^n_{\sigma}(A) = 0 \) for \( n \geq 2 \).

So the twisted cyclic 3-cocycles coming from Woronowicz’s three-dimensional calculus that appear in [17] section 3 (denoted by \( \tau_{\omega,h} \)) and [12] section 5 (corresponding to the linear functional \( \int \)) are necessarily trivial elements of twisted cyclic cohomology, thus proving Theorem 6.1.

6.2 FOUR DIMENSIONAL BICOVARIANT CALCULI

It is well-known (see [17] for example) that the twisted cyclic 4-cocycles on \( \mathcal{A}(SL_q(2)) \) coming from both Woronowicz’s four-dimensional bicovariant calculi over quantum \( SU(2) \) are both simply \( S^2h \), the promotion of the twisted 0-cocycle given by the Haar functional \( h \) to a 4-cocycle via the periodicity operator \( S \). Explicitly (up to a normalising constant),

\[
(S^2h)(a_0, a_1, a_2, a_3, a_4) = h(a_0 a_1 a_2 a_3 a_4)
\]  

(72)

On basis elements, \( h \) is given by

\[
h(a^{i+1}b^j c^k) = 0 = h(d^{i+1}b^j c^k)
\]

\[
h(b^j c^k) = \begin{cases} 
(-q)^{-k}(1-q^{-2})(1-q^{-2(k+1)})^{-1} & : j = k \\
0 & : j \neq k
\end{cases}
\]  

(73)

From [29] we see that \( h \) is a well-defined \( \sigma_{\min}^{-1} \)-twisted cyclic 0-cocycle, given by \( \lambda = q^2, \mu = 1 \), and hence corresponds to Case 1. By inspection, we see that \( h \) is exactly the twisted 0-cocycle \( h_{[1]} \) defined in [10].
7 Conclusions

The original motivation for this work was the belief that calculating twisted cyclic cohomology would give new insight into existing classification results \cite{8,9} for covariant differential calculi over quantum $SL(2)$ and quantum $SU(2)$. However, we see from the Woronowicz four-dimensional calculi that nonisomorphic calculi can give rise to cohomologous cocycles, and as Theorem 6.1 shows, interesting differential calculi can correspond to trivial elements of twisted cyclic cohomology.

The striking result that twisting via the modular automorphism overcomes the dimension drop in Hochschild homology seems to offer the most promising direction for future work. Similar results have been obtained by the first author for Podleś quantum spheres \cite{7}, and by Sitarz for quantum hyperplanes \cite{20}. It seems natural to ask whether the modular automorphism can overcome the dimension drop in Hochschild homology for larger classes of quantum groups, and look for applications of these results.

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References

[1] A. Connes, Spectral sequence and homology of currents for operator algebras, Tagungsbericht 42/81, Mathematisches Forschungszentrum Oberwolfach (1981).
[2] A. Connes, Cohomologie cyclique et foncteurs Ext$^n$, C. R. Acad. Sci. Paris Sér. I Math. 296, no. 23, 953-958 (1983).
[3] A. Connes, Noncommutative differential geometry, Inst. Hautes Études Sci. Publ. Math. 62, 257-360 (1985).
[4] A. Connes, H. Moscovici, Cyclic cohomology and Hopf symmetry, Conférence Moshé Flato 1999, Vol. I (Dijon), Math. Phys. Stud., 21, 121-147, Kluwer Acad. Publ., Dordrecht (2000).
[5] P. Feng, B. Tsygan, Hochschild and cyclic homology of quantum groups, Comm. Math. Phys. 140, no. 3, 481-521 (1991).
[6] E. Getzler, J.D.S. Jones, The cyclic homology of crossed product algebras, J. Reine Angew. Math. 445, 161-174 (1993).
[7] T. Hadfield, Twisted cyclic homology of all Podleś quantum spheres, arXiv:math.QA/0405243 (2004).
[8] I. Heckenberger, Classification of left-covariant differential calculi over the quantum group $SL_q(2)$, J. Algebra 237, 203-237, (2001).
[9] I. Heckenberger, K. Schmudgen, Classification of bicovariant differential calculi on the quantum groups $SL_q(n+1)$ and $Sp_q(2n)$, J. Reine Angew. Math. 502, 141-162 (1998).
[10] M. Khalkhali, B. Rangipour, *Cup products in Hopf-cyclic cohomology*, C. R. Math. Acad. Sci. Paris **340**, no. 1, 9-14 (2005).

[11] A. Klimyk, K. Schmüdgen, *Quantum groups and their representations*, Springer (1997).

[12] J. Kustermans, G. Murphy, L. Tuset, *Differential calculi over quantum groups and twisted cyclic cocycles*, J. Geom. Phys. **44**, no. 4, 570-594 (2003).

[13] T. Levasseur, J.T. Stafford, *The quantum coordinate ring of the special linear group*, J. Pure and Applied Algebra **86**, 181-186 (1993).

[14] J.-L. Loday, *Cyclic homology*, Grundlehren der mathematischen Wissenschaften **301**, Springer-Verlag, Berlin (1998).

[15] T. Masuda, Y. Nakagami, J. Watanabe, *Noncommutative differential geometry on the quantum SU(2)*, I, K-theory **4**, no. 2, 157-180 (1990).

[16] S. Neshveyev, L. Tuset, *Hopf algebra equivariant cyclic cohomology, K-theory and index formulas*. K-Theory **31**, no. 4, 357-378 (2004).

[17] K. Schmüdgen, E. Wagner, *Examples of twisted cyclic cocycles from covariant differential calculi*, Lett. Math. Phys. **64**, no. 3, 245-254 (2003).

[18] K. Schmüdgen, E. Wagner, *Dirac operator and a twisted cyclic cocycle on the standard Podleś quantum sphere*, J. Reine Angew. Math. **574**, 219-235 (2004).

[19] J.-P. Serre, *Local algebra*, Springer (2000).

[20] A. Sitarz, *Twisted Hochschild homology of quantum hyperplanes*, to appear in K-theory, [arXiv:math.OA/0405240] (2004).

[21] B.L. Tsygan, *The homology of matrix Lie algebras over rings and the Hochschild homology*, (Russian) Uspekhi Mat. Nauk **38**, 217-218 (1983), Russ. Math. Survey **38**, no. 2, 198-199 (1983).

[22] M. Wambst, *Complexes de Koszul quantiques*, Annales de l’Institut Fourier, **43**, no. 4, 1089-1156 (1993).

[23] C. Weibel, *An introduction to homological algebra*, Cambridge University Press (1995)

[24] S.L. Woronowicz, *Twisted SU(2) group : an example of a noncommutative differential calculus*, Publ. R.I.M.S. (Kyoto University) **23**, 117-181 (1987).

[25] S.L. Woronowicz, *Compact matrix pseudogroups*, Comm. Math. Phys. **111**, 613-665 (1987).