PARAQUATERNIONIC CR-SUBMANIFOLDS OF PARAQUATERNIONIC KÄHLER MANIFOLDS AND SEMI-RIEMANNIAN SUBMERSIONS

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ABSTRACT. In this paper we introduce paraquaternionic CR-submanifolds of almost paraquaternionic hermitian manifolds and state some basic results on their differential geometry. We also study a class of semi-Riemannian submersions from paraquaternionic CR-submanifolds of paraquaternionic Kähler manifolds.

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1. INTRODUCTION

The notion of CR-submanifold of a Kähler manifold was introduced by Bejancu [6] as a generalization both of totally real and of holomorphic submanifolds of Kähler manifolds. Then many papers appeared studying geometry of CR-submanifolds of Kähler manifolds and this notion was further extended to other ambient spaces; in the monographs [7, 12, 35] we can find the most significant results concerning CR-submanifolds. A class of examples of CR-submanifolds of manifolds endowed with different geometric structures is given in [30].

On the other hand, the paraquaternionic structures, firstly named quaternionic structures of second kind, have been introduced by P. Libermann in [26]. The differential geometry of manifolds endowed with this kind of structures is a very interesting subject and these manifolds have been intensively studied by many authors (see, e.g., [2, 3, 10, 14, 17, 23, 24, 36]). The study of submanifolds of a paraquaternionic Kähler manifold is also of interest and several types of such submanifolds can be found in the recent literature: paraquaternionic submanifolds [31], Kähler and para-Kähler submanifolds [2, 27], normal semi-invariant submanifolds [18], lightlike submanifolds [19, 20], \( F \)-invariant submanifolds [32]. In this note we define a new class of submanifolds of paraquaternionic Kähler manifolds, which we call paraquaternionic CR-submanifolds, as a natural extension of CR-submanifolds in paraquaternionic setting.

The paper is organized as follows: in Section 2 we collect basic definitions, some formulas and results for later use. In Section 3 we introduce the concept of paraquaternionic CR-submanifold and show that on the normal bundle of a paraquaternionic CR-submanifold of a paraquaternionic Kähler manifold exist two complementary orthogonal distributions. In Section 4 we investigate the integrability of the distributions involved in the definition of a paraquaternionic CR-submanifold. In Section 5, following the same techniques as in [18], we study the canonical foliation induced on a paraquaternionic CR-submanifold; conditions
on total geodesicity are derived. We also obtain necessary and sufficient conditions for a paraquaternionic CR-submanifold of a paraquaternionic Kähler manifold to be a ruled submanifold with respect to the canonical foliation. In Section 6 we define the paraquaternionic CR-submersions (in the sense of Kobayashi [25]) as semi-Riemannian submersions from paraquaternionic CR-submanifolds onto an almost paraquaternionic hermitian manifold and obtain some properties concerning their geometry. In Section 7 we discuss curvature properties of fibers and base manifold for paraquaternionic CR-submersions.

2. Preliminaries

Let $M$ be a smooth manifold endowed with an almost paraquaternionic structure $\sigma$, that is, a rank-3 subbundle $\sigma$ of $\text{End}(TM)$ which admits a local basis $\{J_1, J_2, J_3\}$ on a coordinate neighborhood $U \subset M$ such that we have:

$$J_a^2 = -\epsilon_a \text{Id}, \quad J_a J_{a+1} = -J_{a+1} J_a = \epsilon_{a+2} J_{a+2}$$

where $\epsilon_1 = 1$, $\epsilon_2 = \epsilon_3 = -1$ and the indices are taken from $\{1, 2, 3\}$ modulo 3. Moreover, the pair $(M, \sigma)$ is said to be an almost paraquaternionic manifold and $\{J_1, J_2, J_3\}$ is called a canonical local basis of $M$.

In an almost paraquaternionic manifold $(M, \sigma)$ we take intersecting coordinate neighborhoods $U$ and $U'$. Let $\{J_1, J_2, J_3\}$ and $\{J'_1, J'_2, J'_3\}$ be canonical local bases of $\sigma$ in $U$ and $U'$ respectively. Then $\{J'_1, J'_2, J'_3\}$ are linear combinations of $\{J_1, J_2, J_3\}$ in $U \cap U'$:

$$J'_a = \sum_{\beta=1}^{3} a_{\alpha\beta} J_{\beta}, \quad \alpha = 1, 2, 3,$$

where $a_{\alpha\beta}$ are functions in $U \cap U'$, $\alpha, \beta = 1, 2, 3$ and $A = (a_{\alpha\beta})_{\alpha, \beta = 1, 2, 3} \in \text{SO}(2, 1)$.

Let $(M, g)$ be a semi-Riemannian manifold and let $\sigma$ be an almost paraquaternionic structure on $M$. The metric $g$ is said to be adapted to the almost paraquaternionic structure $\sigma$ if it satisfies:

$$g(J_{\alpha} X, J_{\alpha} Y) = \epsilon_{\alpha} g(X, Y), \quad \alpha \in \{1, 2, 3\},$$

for all vector fields $X, Y$ on $M$ and any local basis $\{J_1, J_2, J_3\}$ of $\sigma$; or, equivalently, if all endomorphisms of $\sigma$ are skew-symmetric with respect to $g$. In this case, $(M, \sigma, g)$ is said to be an almost paraquaternionic hermitian manifold. The existence of paraquaternionic hermitian structures on manifolds and tangent bundles has been recently investigated in [21].

It is easy to see that any almost paraquaternionic hermitian manifold is of dimension $4m$, $m \geq 1$, and any adapted metric is necessarily of neutral signature $(2m, 2m)$.

Let $\{J_1, J_2, J_3\}$ be a canonical local base of $\sigma$ in a coordinate neighborhood $U$ of an almost paraquaternionic hermitian manifold $(M, \sigma, g)$. If we denote by:

$$\Omega_{J_{\alpha}}(X, Y) = g(X, J_{\alpha} Y), \quad \alpha = 1, 2, 3$$

for any vector fields $X$ and $Y$, then, by means of (2), we see that:

$$\Omega = \Omega_{J_1} \wedge \Omega_{J_1} - \Omega_{J_2} \wedge \Omega_{J_2} - \Omega_{J_3} \wedge \Omega_{J_3}$$

is a globally well-defined 4-form on $M$, called the fundamental 4-form of the manifold.
If \((M, \sigma, g)\) is an almost paraquaternionic hermitian manifold such that the bundle \(\sigma\) is parallel with respect to the Levi-Civita connection \(\nabla\) of \(g\), then \((M, \sigma, g)\) is said to be a paraquaternionic Kähler manifold. Equivalently, locally defined 1-forms \(\omega_1, \omega_2, \omega_3\) exist such that we have for all \(\alpha \in \{1, 2, 3\}\):

\[
\nabla_X J_\alpha = -\epsilon_\alpha [\omega_\alpha + 2(X)J_{\alpha+1} - \omega_{\alpha+1}(X)J_\alpha + 1]
\]

for any vector field \(X\) on \(M\), where the indices are taken from \(\{1, 2, 3\}\) modulo 3 (see [17]). Moreover, it can be proved that an almost paraquaternionic hermitian manifold of dimension strictly greater than 4 is paraquaternionic Kähler if and only if \(\nabla \Omega = 0\) (see [22]).

We remark that any paraquaternionic Kähler manifold is an Einstein manifold, provided that \(\operatorname{dim} M > 4\) (see [10, 17, 23]).

Let \((M, g)\) be a semi-Riemannian manifold and let \(N\) be an immersed submanifold of \(M\). Then \(N\) is said to be a non-degenerate submanifold of \(M\) if the restriction of the semi-Riemannian metric \(g\) to \(T N\) is non-degenerate at each point of \(N\). We denote by the same symbol \(g\) the semi-Riemannian metric induced by \(g\) on \(N\) and by \(T N^\perp\) the normal bundle to \(N\).

For the rest of this section we will assume that the induced metric on \(N\) is non-degenerate.

Then we have the following orthogonal decomposition:

\[ TM = TN \oplus TN^\perp. \]

Also, we denote by \(\nabla\) and \(\nabla\) the Levi-Civita connection on \(M\) and \(N\), respectively. Then the Gauss formula is given by:

\[
\nabla_X Y = \nabla_X Y + B(X, Y)
\]

for any \(X, Y \in \Gamma(TN)\), where \(B : \Gamma(TN) \times \Gamma(TN) \to \Gamma(TN^\perp)\) is the second fundamental form of \(N\) in \(M\).

On the other hand, the Weingarten formula is given by:

\[
\nabla_X \xi = -A_\xi X + \nabla_X^\perp \xi
\]

for any \(X \in \Gamma(TN)\) and \(\xi \in \Gamma(TN^\perp)\), where \(-A_\xi X\) is the tangent part of \(\nabla_X \xi\) and \(\nabla_X^\perp \xi\) is the normal part of \(\nabla_X \xi\); \(A_\xi\) and \(\nabla^\perp\) are called the shape operator of \(N\) with respect to \(\xi\) and the normal connection, respectively. Moreover, \(B\) and \(A_\xi\) are related by:

\[
g(B(X, Y), \xi) = g(A_\xi X, Y)
\]

for any \(X, Y \in \Gamma(TN)\) and \(\xi \in \Gamma(TN^\perp)\) (see [29]).

If we denote by \(\overline{\mathcal{R}}\) and \(\mathcal{R}\) the curvature tensor fields of \(\nabla\) and \(\nabla\) we have the Gauss equation:

\[
\overline{\mathcal{R}}(X, Y, Z, U) = R(X, Y, Z, U) - g(B(X, Z), B(Y, U)) + g(B(Y, Z), B(X, U)),
\]

for all \(X, Y, Z, U \in \Gamma(TN)\).

### 3. Paraquaternionic CR-submanifolds

**Definition 3.1.** Let \(N\) be an \(n\)-dimensional non-degenerate submanifold of an almost paraquaternionic hermitian manifold \((M, \sigma, g)\). We say that \((N, g)\) is a **paraquaternionic CR-submanifold** of \(M\) if there exists a non-degenerate distribution...
\[ D : x \to D_x \subseteq T_x N \text{ such that on any } U \cap N \text{ we have:} \]
i. \( D \) is a paraquaternionic distribution, i.e.
\[ J_\alpha D_x = D_x, \ \alpha \in \{1, 2, 3\} \]
\[ \text{and} \]
ii. \( D^\perp \) is a totally real distribution, i.e.
\[ J_\alpha D^\perp_x \subseteq T^\perp_x N, \ \alpha \in \{1, 2, 3\} \]
for any local basis \( \{J_1, J_2, J_3\} \) of \( \sigma \) on \( U \) and \( x \in U \cap N \), where \( D^\perp \) is the orthogonal complementary distribution to \( D \) in \( T N \).

**Definition 3.2.** A non-degenerate submanifold \( N \) of an almost paraquaternionic hermitian manifold \( (M, \sigma, g) \) is called a paraquaternionic (respectively, totally real) submanifold if \( D^\perp = 0 \) (respectively, \( D = 0 \)). A paraquaternionic CR-submanifold is said to be proper if it is neither paraquaternionic nor totally real.

**Example 3.3.** i. The canonical immersion of \( P^n \mathbb{B}(c) \) into \( P^m \mathbb{B}(c) \), where \( n \leq m \), provides us a very natural example of paraquaternionic submanifold (see [27]).

ii. The real projective space \( P^n \mathbb{B}(c) \) is a totally-real submanifold of the paraquaternionic projective space \( P^n \mathbb{B}(c) \), where \( s \in \{0, ..., n\} \) denotes the index of the manifold, defined as the dimension of the largest negative definite vector subspace of the tangent space.

iii. Let \((M_1, g_1, \sigma_1)\) and \((M_2, g_2, \sigma_2)\) be two paraquaternionic Kähler manifolds. If \( U_1 \) and \( U_2 \) are open subsets of \( M_1 \) and \( M_2 \) respectively, on which local basis \( \{J_1^{(1)}, J_2^{(1)}, J_3^{(1)}\} \) and \( \{J_1^{(2)}, J_2^{(2)}, J_3^{(2)}\} \) for \( \sigma_1 \) and \( \sigma_2 \) respectively, are defined, then the product manifold \( U = U_1 \times U_2 \) can be endowed with an almost paraquaternionic hermitian non-Kähler structure \((g, \sigma)\) (see [32]). Now, if \( N_1 \) is a paraquaternionic submanifold of \( U_1 \) and \( N_2 \) is a totally-real submanifold of \( U_2 \), then \( N = N_1 \times N_2 \) is a proper paraquaternionic CR-submanifold of the almost paraquaternionic hermitian manifold \((U, g, \sigma)\).

iv. A large class of examples of proper paraquaternionic CR-submanifolds can be constructed using the paraquaternionic momentum map [33] and the technique from [30]. Suppose that a Lie group \( G \) acts freely and isometrically on the paraquaternionic Kähler manifold \((M, \sigma, g)\), preserving the fundamental 4-form \( \Omega \) of the manifold. We denote by \( \mathfrak{g} \) the Lie algebra of \( G \), by \( \mathfrak{g}^* \) its dual and by \( V \) the unique Killing vector field corresponding to a vector \( V^* \in \mathfrak{g} \). Then there exists a unique section \( f \) of bundle \( \mathfrak{g}^* \otimes \sigma \) such that (see [33])
\[ \nabla f_{V^*} = \theta_{V^*}, \]
for all \( V^* \in \mathfrak{g} \), where the section \( \theta_{V^*} \) of the bundle \( \Omega^1(\sigma) \) with values in \( \sigma \) is well defined globally by
\[ \theta_{V^*}(X) = \sum_{\alpha=1}^{3} \omega_\alpha(V^*, X) J_\alpha, \ \forall X \in TM. \]
Moreover, the group \( G \) acts by isometries on the pre-image \( f^{-1}(0) \) of the zero-section \( 0 \in \mathfrak{g}^* \otimes \sigma \). Similarly as in [30], we have the decomposition
\[ T_x(f^{-1}(0)) = T_x(G \cdot x) \oplus H_x, \forall x \in M, \]
where \( G \cdot x \) represents the orbit of \( G \) through \( x \), supposed to be non-degenerate, and \( H_x \) is the orthogonal complementary subspace of \( T_x(G \cdot x) \) in \( T_x(f^{-1}(0)). \) Because
Proposition 3.4. If $f^{-1}(0)$ is a smooth submanifold of a paraquaternionic Kähler manifold $(M, \sigma, g)$, then $f^{-1}(0)$ is a proper paraquaternionic CR-submanifold of $M$.

We remark that, in general, $f^{-1}(0)$ is not a differentiable submanifold of $M$, but always we can take a subset $N \subset f^{-1}(0)$ which is invariant under the action of $G$ and which is a submanifold of $M$. A particular example is given in [11], as a paraquaternionic version of the example constructed by Galicki and Lawson in [16]: if $p$ and $q$ are distinct and relatively prime natural numbers, then we have the action of the Lie group $G = \{e^{jt}|t \in \mathbb{R}\}$ on $P^2\mathbb{B}$ defined by

$$\phi_{p,q}(t) \cdot [u_0, u_1, u_2] := [e^{jpt}u_0, e^{jpt}u_1, e^{jpt}u_2],$$

where $e^{jt} = \cosh t + js\sinh t$ and $[u_0, u_1, u_2]$ are homogenous coordinates on $P^2\mathbb{B}$. We can see that this action is free, isometric and preserves the para-quaternionic structure on $P^2\mathbb{B}$ and, moreover, we have that the pre-image by the momentum map $f_{p,q} : P^2\mathbb{B} \to \Im\mathbb{B}$ of the zero-section $0 \in \Im\mathbb{B}$ is (see also [33]):

$$f_{p,q}^{-1}(0) = \{[u_0, u_1, u_2] \in P^2\mathbb{B}|q\bar{u}_0u_0 + p\bar{u}_1u_1 + p\bar{u}_2u_2 = 0\}.$$ 

Finally, we conclude that the subset $N$ of the regular points of $f_{p,q}^{-1}(0)$, given by

$$N = \{[u_0, u_1, u_2] \in f_{p,q}^{-1}(0)|q^2|u_0|^2 + p^2|u_1|^2 + p^2|u_2|^2 \neq 0\}$$

is a a proper paraquaternionic CR-submanifold of $P^2\mathbb{B}$.

Definition 3.5. Let $N$ be a paraquaternionic CR-submanifold of an almost paraquaternionic hermitian manifold $(M, \sigma, g)$. Then we say that:

i. $N$ is $\mathcal{D}$-geodesic if $B(X, Y) = 0$, $\forall X, Y \in \Gamma(\mathcal{D})$;

ii. $N$ is $\mathcal{D}^\perp$-geodesic if $B(X, Y) = 0$, $\forall X, Y \in \Gamma(\mathcal{D}^\perp)$;

iii. $N$ is mixed geodesic if $B(X, Y) = 0$, $\forall X \in \Gamma(\mathcal{D}), Y \in \Gamma(\mathcal{D}^\perp)$;

iv. $N$ is mixed foliated if $N$ is mixed geodesic and $\mathcal{D}$ is integrable.

We may easily prove the next result (see [4, 31]):

Proposition 3.6. Any paraquaternionic submanifold of a paraquaternionic Kähler manifold is a totally geodesic paraquaternionic Kähler submanifold.

By using this proposition, we deduce the next consequences.

Corollary 3.7. Let $(N, g)$ be a paraquaternionic submanifold of a paraquaternionic Kähler manifold $(M, \sigma, g)$. Then:

i. $\dim N = 4n, n \geq 1$ and the signature of $g|_{TN}$ is $(2n, 2n)$;

ii. $N$ is an Einstein manifold, provided that $\dim N > 4$.

Corollary 3.8. The paraquaternionic submanifolds of $\mathbb{R}_{2m}^4$ and of paraquaternionic projective space $P^n\mathbb{B}$ are locally isometric with $\mathbb{R}_{2n}^4$ and $P^n\mathbb{B}$, respectively, where $n \leq m$.

Next, let $(N, g)$ be a paraquaternionic CR-submanifold of a paraquaternionic Kähler manifold $(M, \sigma, g)$. We put $\nu_{\alpha x} = J_\alpha(D_x^\perp)$, $\alpha \in \{1, 2, 3\}$, and $\nu_x^\perp = \nu_{1x} \oplus \nu_{2x} \oplus \nu_{3x}$, and remark that $\nu_{1x}$, $\nu_{2x}$, $\nu_{3x}$ are mutually orthogonal non-degenerate vector subspaces of $T_x N^\perp$, for any $x \in U \cap N$. We also note that the subspaces $\nu_{\alpha x}$ depends on the choice of the local base $(J_\alpha)_x$, while $\nu_x^\perp$ does’nt depend from it.
Proposition 3.9. Let \((N, g)\) be a paraquaternionic CR-submanifold of a paraquaternionic \(\mathbb{K}\)ähler manifold \((M, \sigma, g)\). Then we have:

i. \(J_{\alpha}(\nu_{\alpha}) = D_{\alpha}^x\), \(\forall x \in U \cap N, \alpha \in \{1, 2, 3\}\);

ii. \(J_{\alpha}(\nu_{\beta\gamma}) = \nu_{\gamma\alpha}\), for any even permutation \((\alpha, \beta, \gamma)\) of \((1, 2, 3)\) and \(x \in U \cap N\);

iii. The mapping \(\nu^\perp : x \in N \to \nu_x^\perp\) defines a non-degenerate distribution of dimension 3s, where \(s = \dim D_x^\perp\);

iv. \(J_{\alpha}(\nu_x) = \nu_x\), \(\forall x \in U \cap N, \alpha \in \{1, 2, 3\}\), where \(\nu\) is the complementary orthogonal distribution to \(\nu^\perp\) in \(TN^\perp\).

Proof. The assertions i. and ii. are obvious from (1). The assertion iii. follows from the Definition 3.1, since \(J_{\alpha}, \alpha \in \{1, 2, 3\}\), are automorphisms of \(T_xN\), for any \(x \in U \cap N\). Concerning the proof of (iv.), firstly one can remark that \(\nu_x\) doesn’t depend from the choice of the local base \((J_{\alpha})_x\) and secondly, one could observe that the subspace \(TN^\perp_x = D_x \oplus D_x^+ \oplus \nu_x^\perp\) is the paraquaternionic subspace generated by \(TN_x\), hence its orthogonal \(\nu\) is also a paraquaternionic subspace, i.e. closed under \(J_{\alpha}, \alpha \in \{1, 2, 3\}\). This completes the proof of the proposition. \(\Box\)

4. Integrability of Distributions

Theorem 4.1. The distribution \(D^\perp\) is integrable.

Proof. For any \(U, V \in \Gamma(D^\perp)\) and \(X \in \Gamma(D)\) we have:

\[
g(\nabla_U V, X) = g(\nabla_V U, X) = \epsilon_{\alpha} g(J_{\alpha} \nabla_U V, J_{\alpha} X) = \epsilon_{\alpha} g(\epsilon_{\alpha} [\omega_{\alpha+2}(U)J_{\alpha+1} V - \omega_{\alpha+1}(U)J_{\alpha+2} V] + \nabla_U J_{\alpha} V, J_{\alpha} X) = \epsilon_{\alpha} g(\nabla_U J_{\alpha} V, J_{\alpha} X)
\]

and by using (4) we obtain:

\[	ag{12} g(\nabla_U V, X) = -\epsilon_{\alpha} g(A_{J_{\alpha} V} U, J_{\alpha} X).
\]

On the other hand, if we take \(U, V \in \Gamma(D^\perp)\) and \(C \in \Gamma(TN)\) we have:

\[
g(A_{J_{\alpha} V} U, C) = -g(\nabla_C J_{\alpha} V, U) = g(\epsilon_{\alpha} [\omega_{\alpha+2}(C) J_{\alpha+1} V - \omega_{\alpha+1}(C) J_{\alpha+2} V], U) - g(J_{\alpha} \nabla_C V, U)
\]

hence

\[
g(A_{J_{\alpha} V} U, C) = -g(J_{\alpha} \nabla_C V, U).
\]

Moreover, from \(C \cdot g(J_{\alpha} U, V) = 0\), one has

\[
g(J_{\alpha} \nabla_C U, V) = g(J_{\alpha} \nabla_C V, U).
\]

We conclude:

\[	ag{13} g(A_{J_{\alpha} V} U, C) = g(A_{J_{\alpha} V} U, C).
\]

From (12) and (13) we deduce that for any \(U, V \in \Gamma(D^\perp)\) and \(X \in \Gamma(D)\) we have:

\[
g(\nabla_U V - \nabla_V U, X) = 0,
\]

which implies \([U, V] \in \Gamma(D^\perp), \forall U, V \in \Gamma(D^\perp)\). Thus \(D^\perp\) is integrable. \(\Box\)

Theorem 4.2. The paraquaternionic distribution \(D\) is integrable if and only if \(N\) is \(D\)-geodesic.
Proof. Similarly as in above theorem, we obtain:

$$g(∇_X Y, U) = \epsilon_\alpha g(∇_X J_\alpha Y, J_\alpha U)$$

for any $X, Y \in \Gamma(D)$ and $U \in \Gamma(D^\perp)$, and taking into account (5) we obtain:

(14)

$$g(∇_X Y, U) = \epsilon_\alpha g(B(X, J_\alpha Y), J_\alpha U).$$

If we suppose that $N$ is $D$-geodesic, from (14) we derive $∇_X Y \in \Gamma(D)$, which implies $[X, Y] \in \Gamma(D)$. Thus $D$ is integrable.

Conversely, if we suppose that $D$ is integrable, then the leaves are invariant of the paraquaternionic structure and so they are totally geodesic in $N$. In particular, $N$ is $D$-geodesic. □

Remark 4.3. If $Q$ is a non-degenerate distribution on a semi-Riemannian manifold $(M, g)$, then we can consider a well-defined $Q^\perp$-valued vector field on $N$, called the mean curvature vector of $Q$ (see [9]), given by:

$$H_Q = \frac{1}{q} \sum_{i=1}^q \theta_i h_Q(E_i, E_i),$$

where $h_Q$ is the second fundamental forms of $Q$, $q = \text{dim} Q$, $\{E_1, ..., E_q\}$ is a pseudo-orthonormal basis of $Q$ and $\theta_i = g(E_i, E_i) \in \{-1, 1\}$, $\forall i \in \{1, ..., q\}$.

The distribution $Q$ is said to be minimal if the mean curvature vector $H_Q$ of $Q$ vanishes identically.

Theorem 4.4. The paraquaternionic distribution $D$ is minimal.

Proof. For any $X \in \Gamma(D)$ and $U \in \Gamma(D^\perp)$ we obtain similarly as in above theorems:

(15)

$$g(∇_X X, U) = \epsilon_\alpha g(A_{J_\alpha U} J_\alpha X, X)$$

and

(16)

$$g(∇_{J_\alpha X} J_\alpha X, U) = -g(A_{J_\alpha U} X, J_\alpha X) = -g(A_{J_\alpha U} J_\alpha X, X),$$

for all $\alpha \in \{1, 2, 3\}$.

From (15) and (16) we deduce:

$$g(∇_X X + \epsilon_\alpha ∇_{J_\alpha X} J_\alpha X, U) = 0$$

and so

(17)

$$h^D(X, X) + \epsilon_\alpha h^D(J_\alpha X, J_\alpha X) = 0, \ \forall \alpha \in \{1, 2, 3\}.$$ 

From (17) it follows that

$$h^D(X, X) = -h^D(J_1 X, J_1 X) = -h^D(J_2 J_3 X, J_2 J_3 X) = -h^D(J_3 X, J_3 X) = -h^D(X, X)$$

and so

(18)

$$h^D(X, X) = 0, \ \forall X \in \Gamma(D).$$

We obtain $H^D = 0$ and the assertion follows. We can remark that the relation (18) not imply $h^D = 0$, since $h^D$ is not symmetric in general, unless $D$ is integrable. □
Remark 4.5. For any paraquaternionic CR-submanifold \((N, g)\) of an almost paraquaternionic hermitian manifold \((M^{4m}, \sigma, g)\), having \(\text{dim} \mathcal{D} = 4r\) and \(\text{dim} \mathcal{D}^\perp = p\), we can choose a local pseudo-orthonormal frame in \(M\):
\[
\{e_1, \ldots, e_r, e_{r+1}, \ldots, e_{r+p}, e_{r+p+1}, \ldots, e_m, J_1e_1, \ldots, J_1e_m, J_2e_1, \ldots, J_2e_m, J_3e_1, \ldots, J_3e_m\}
\]
such that restricted to \(N\), \(\{e_i, J_1e_i, J_2e_i, J_3e_i\}_{i \in \{1, \ldots, r\}}\) are in \(\mathcal{D}\) and \(\{e_{r+1}, \ldots, e_{r+p}\}\) are in \(\mathcal{D}^\perp\).

Let \(\{\omega^1, \ldots, \omega^{4r}\}\) be the \(4r\) 1-forms on \(N\) satisfying:
\[
\omega^j(Z) = 0, \quad \omega_i(E_j) = \delta_{ij}, \quad i, j \in \{1, \ldots, 4r\},
\]
for any \(Z \in \Gamma(\mathcal{D}^\perp)\), where \(E_j = e_j, E_{r+j} = J_1e_j, E_{2r+j} = J_2e_j\) and \(E_{3r+j} = J_3e_j, j \in \{1, \ldots, r\}\). Then \(\omega = \omega^1 \wedge \ldots \wedge \omega^{4r}\) does not depend on the particular base \(\{J_1, J_2, J_3\}\) and defines a \(4r\)-form on \(N\). Therefore, we have
\[
d\omega = \sum_{i=1}^{4r} (-1)^i \omega^1 \wedge \ldots \wedge d\omega^i \ldots \wedge \omega^{4r}
\]
and taking account of (19), we deduce that \(d\omega = 0\) if and only if
\[
d\omega(Z_1, Z_2, X_1, \ldots, X_{4r-1}) = 0
\]
and
\[
d\omega(Z_1, X_1, \ldots, X_{4r}) = 0
\]
for any \(Z_1, Z_2 \in \Gamma(\mathcal{D}^\perp)\) and \(X_1, \ldots, X_{4r} \in \Gamma(\mathcal{D})\). We remark now easily that (21) holds if and only if \(\mathcal{D}^\perp\) is integrable and (22) holds if and only if \(\mathcal{D}\) is minimal (see also [13]). On another hand, if \(\{\omega^{4r+1}, \ldots, \omega^{4r+p}\}\) is the dual frame to the pseudo-orthonormal frame \(\{e_{r+1}, \ldots, e_{r+p}\}\) of \(\mathcal{D}^\perp\), we can define a \(p\)-form \(\omega^\perp\) on \(M\) by \(\omega^\perp = \omega^{4r+1} \wedge \ldots \wedge \omega^{4r+p}\). Similarly we find that \(\omega^\perp\) is closed if \(\mathcal{D}\) is integrable and \(\mathcal{D}^\perp\) is minimal.

Consequently, from Theorems 4.1 and 4.3 we obtain the following result.

Theorem 4.6. Let \(N\) be a closed paraquaternionic CR-submanifold of a paraquaternionic \(\mathbb{K}\)ähler manifold \((M, \sigma, g)\). Then the \(4r\)-form \(\omega\) is closed and defines a canonical de Rham cohomology class \(\lbrack\omega\rbrack\) in \(H^{4r}(M, \mathbb{R})\). Moreover, this cohomology class is non-trivial if \(\mathcal{D}\) is integrable and \(\mathcal{D}^\perp\) is minimal.

5. Canonical foliations on paraquaternionic CR-submanifolds

Since the totally real distribution \(\mathcal{D}^\perp\) of a paraquaternionic CR-submanifold \(N\) of a paraquaternion \(\mathbb{K}\)ähler manifold \((M, \sigma, g)\) is always integrable we conclude that we have a foliation \(\mathcal{F}^\perp\) on \(N\) with structural distribution \(\mathcal{D}^\perp\) and transversal distribution \(\mathcal{D}\) (see [9]). We say that \(\mathcal{F}^\perp\) is the canonical totally real foliation on \(N\).

Theorem 5.1. Let \(N\) be a paraquaternionic CR-submanifold of a paraquaternionic \(\mathbb{K}\)ähler manifold \((M, \sigma, g)\). The next assertions are equivalent:

i. \(\mathcal{F}^\perp\) is totally geodesic;

ii. \(B(X, Y) \in \Gamma(\nu), \forall X \in \Gamma(\mathcal{D}), Y \in \Gamma(\mathcal{D}^\perp)\);

iii. \(A_N X \in \Gamma(\mathcal{D}^\perp), \forall X \in \Gamma(\mathcal{D}^\perp), N \in \Gamma(\nu^\perp)\);

iv. \(A_N Y \in \Gamma(\mathcal{D}), \forall Y \in \Gamma(\mathcal{D}), N \in \Gamma(\nu^\perp)\).
Proof. By using (3), (4), (5) and (6) we obtain for any \( X, Z \in \Gamma(D^\perp) \) and \( Y \in \Gamma(D) \):

\[
\begin{align*}
g(J_\alpha(\nabla_X Z), Y) &= -g(\nabla_X Z, J_\alpha Y) \\
&= g(\epsilon_\alpha[\omega_{\alpha+2}(X)J_{\alpha+1}Z - \omega_{\alpha+1}(X)J_{\alpha+2}Z] + \nabla_X J_\alpha Z, Y) \\
&= g(-A_{J_\alpha Z}X + \nabla_X^\perp J_\alpha Z, Y) \\
&= -g(A_{J_\alpha Z}X, Y)
\end{align*}
\]

and taking into account (7) we derive:

\[
(23) \quad g(J_\alpha(\nabla_X Z), Y) = -g(B(X, Y), J_\alpha Z).
\]

i. \( \Rightarrow \) ii. If \( \mathfrak{F}^\perp \) is totally geodesic, then \( \nabla_X Z \in \Gamma(D^\perp) \), for \( X, Z \in \Gamma(D^\perp) \) and from (23) we derive:

\[
g(B(X, Y), J_\alpha Z) = 0
\]

and the implication is clear.

ii. \( \Rightarrow \) i. If we suppose \( B(X, Y) \in \Gamma(\mu) \), \( \forall X \in \Gamma(D) \), \( Y \in \Gamma(D^\perp) \), then from (23) we derive:

\[
g(J_\alpha(\nabla_X Z), Y) = 0
\]

and we conclude \( \nabla_X Z \in \Gamma(D^\perp) \). Thus \( \mathfrak{F}^\perp \) is totally geodesic.

ii. \( \Leftrightarrow \) iii. This equivalence is clear from (7).

iii. \( \Leftrightarrow \) iv. This equivalence is true because \( A_N \) is a self-adjoint operator. 

\[\square\]

Corollary 5.2. If \( N \) is a mixed-geodesic paraquaternionic CR-submanifold of a paraquaternionic Kähler manifold \( (M, \sigma, g) \), then the canonical totally real foliation \( \mathfrak{F}^\perp \) on \( N \) is totally geodesic.

Corollary 5.3. Let \( N \) be a paraquaternionic CR-submanifold of a paraquaternionic Kähler manifold \( (M, \sigma, g) \) with \( \nu = 0 \). Then the canonical totally real foliation \( \mathfrak{F}^\perp \) on \( N \) is totally geodesic if and only if \( N \) is mixed geodesic.

We note that the condition \( \nu = 0 \) in Corollary 5.3 characterizes the following interesting class of paraquaternionic CR-submanifolds (compare with the definition in the complex case given in [35], pag. 78).

Definition 5.4. A paraquaternionic CR-submanifold \( N \) of a paraquaternionic Kähler manifold \( (M, \sigma, g) \) is called generic if the restriction \( TM|_N \) to \( N \) of the tangent bundle of ambient manifold \( M \) is generated over the paraquaternions by the tangent bundle of \( N \), i.e. if \( TN + \sigma TN = TM|_N \) or, equivalently, if \( \nu = 0 \).

Remark 5.5. The proper paraquaternionic CR-submanifolds given by Proposition 3.4 are generic because we have

\[
T_x^\perp(f^{-1}(0)) = \sigma T_x(G \cdot x).
\]

Proposition 5.6. Let \( N \) be a generic paraquaternionic CR-submanifold of a paraquaternionic Kähler manifold \( (M, \sigma, g) \). Then the paraquaternionic distribution \( D \) is geodesic.

Proof. \( N \) generic means that

\[
TM|_N = D \oplus D^\perp \oplus \sigma D^\perp
\]

where

\[
\nu^\perp = \sigma D^\perp, \quad TN = D \oplus D^\perp.
\]
From (3) and (14) we have
\[
g(J^\alpha \nabla_X Y, J^\alpha U) = g(X, J^\alpha J^\alpha U) = g(X, U),
\]
for all \(X, Y \in \Gamma(D), U \in \Gamma(D^\perp)\) and since \(TN^\perp = \nu^\perp\),
\[
(J^\alpha \nabla_X Y)^\perp = B(X, J^\alpha Y)
\]
where \(W^\perp\) means the component of the vector \(W \in TN\) in the orthogonal to \(TN\), that is in \(\nu^\perp\).

More explicitly, we have
\[
(J^\alpha \nabla^D_X Y + J^\alpha \nabla^D^\perp_X Y)^\perp = B(X, J^\alpha Y)
\]
where \(\nabla^D_X Y, \nabla^D^\perp_X Y\) are the \(D, D^\perp\) components of \(\nabla_X Y\) respectively.

Equivalently, since \(J^\alpha \nabla^D_X Y \in D\), we have
\[
J^\alpha \nabla^D^\perp_X Y = B(X, J^\alpha Y)
\]
that is
\[
(24) \quad \nabla^D^\perp_X Y = -\epsilon_\alpha J^\alpha B(X, J^\alpha Y)
\]
As a consequence, we have
\[
\epsilon_\alpha J^\alpha B(X, J^\alpha Y) = \epsilon_\beta J^\beta B(X, J^\beta Y), \ \forall \alpha, \beta
\]
and, for \(J^\alpha J^\beta = \epsilon_\gamma J^\gamma\),
\[
B(X, J^\alpha Y) = -\epsilon_\beta \epsilon_\gamma B(X, J^\beta Y).
\]
Hence
\[
(25) \quad B(X, Y) = -\epsilon_\alpha \epsilon_\beta J^\alpha B(X, J^\beta Y)
\]
and also
\[
(26) \quad B(X, J^\gamma Y) = J^\gamma B(X, Y)
\]
since \(\epsilon_\gamma \epsilon_\alpha \epsilon_\beta = 1\).

Note that from (26) and symmetry of \(B\) it follows
\[
B(X, J^\gamma Y) = B(Y, J^\gamma X)
\]
from which we deduce:
\[
(27) \quad B(X, Y) = -\epsilon_\gamma B(J^\alpha X, J^\beta Y).
\]
By applying repeatedly the (27) we find
\[
B(X, Y) = -B(J_1 X, J^1 Y) = -B(J_2 J_3 X, J_2 J_3 Y) = -B(J_3 X, J_3 Y) = -B(X, Y)
\]
hence,
\[
B = 0.
\]

\[\square\]

**Definition 5.7.** [9] A submanifold \(N\) of a semi-Riemannian manifold \((M, g)\) is said to be a ruled submanifold if it admits a foliation whose leaves are totally geodesic submanifolds immersed in \((M, g)\).

**Definition 5.8.** A paraquaternionic CR-submanifold of a paraquaternionic Kähler manifold which is a ruled submanifold with respect to the foliation \(\mathfrak{F}^\perp\) is called totally real ruled paraquaternionic CR-submanifold.
Theorem 5.9. Let \( N \) be a paraquaternionic CR-submanifold of a paraquaternionic Kähler manifold \((M, \sigma, g)\). Then the following assertions are mutually equivalent:

i. \( N \) is a totally real ruled paraquaternionic CR-submanifold.

ii. \( N \) is \( D^\perp \)-geodesic and:

\[
B(X, Y) \in \Gamma(\nu), \forall X \in \Gamma(D), \ Y \in \Gamma(D^\perp).
\]

iii. The subbundle \( \nu^\perp \) is \( D^\perp \)-parallel, i.e:

\[
\nabla_X J_\alpha Z \in \Gamma(\nu^\perp), \forall X, Z \in \Gamma(D^\perp), \alpha \in \{1, 2, 3\}
\]

and the second fundamental form satisfies:

\[
B(X, Y) \in \Gamma(\nu), \forall X \in \Gamma(D^\perp), \ Y \in \Gamma(T_N).
\]

iv. The shape operator satisfies:

\[
A_{J_\alpha Z} X = 0, \forall X, Z \in \Gamma(D^\perp), \alpha \in \{1, 2, 3\}
\]

and

\[
A_N X \in \Gamma(D), \forall X \in \Gamma(D^\perp), N \in \Gamma(\nu).
\]

Proof. i. \( \Leftrightarrow \) ii. This equivalence follows from Theorem 5.1 since for any \( X, Z \in \Gamma(D^\perp) \) we have:

\[
\nabla_X Z = \nabla_X Z + B(X, Z)
\]

and thus the leaves of \( D^\perp \) are totally geodesic immersed in \( M \) if and only if \( h^\perp = 0 \) and \( N \) is \( D^\perp \)-geodesic.

i. \( \Leftrightarrow \) iii. If \( X, Z \in \Gamma(D^\perp) \) and \( N \in \Gamma(\nu) \), then we have:

\[
g(\nabla_X Z, N) = \epsilon_\alpha g(J_\alpha \nabla_X Z, J_\alpha N)
\]

\[
= \epsilon_\alpha g(\epsilon_\alpha [\omega_{\alpha+2}(X)J_{\alpha+1}Z - \omega_{\alpha+1}(X)J_{\alpha+2}Z] + \nabla_X J_\alpha Z, J_\alpha N)
\]

\[
= \epsilon_\alpha g(-A_{J_\alpha Z} X + \nabla_X J_\alpha Z, J_\alpha N)
\]

and thus we obtain:

\[
g(\nabla_X Z, N) = \epsilon_\alpha g(\nabla_X J_\alpha Z, J_\alpha N).
\]

Similarly we find:

\[
g(\nabla_X Z, U) = -\epsilon_\alpha g(B(X, J_\alpha U), J_\alpha Z), \forall X, Z \in \Gamma(D^\perp), U \in \Gamma(D).
\]

On the other hand, from (5) we deduce for any \( X, Z, W \in \Gamma(D^\perp) \):

\[
g(\nabla_X Z, J_\alpha W) = g(B(X, Z), J_\alpha W).
\]

But \( M \) is a totally real ruled paraquaternionic CR-submanifold iff \( \nabla_X Z \in \Gamma(D^\perp), \forall X, Z \in \Gamma(D^\perp) \) and by using (28), (29) and (30) we deduce the equivalence.

ii. \( \Leftrightarrow \) iv. This equivalence follows from (7). \( \Box \)

Corollary 5.10. Let \( N \) be a paraquaternionic CR-submanifold of a paraquaternionic Kähler manifold \((M, \sigma, g)\). If \( N \) is totally geodesic, then \( N \) is a totally real ruled paraquaternionic CR-submanifold.
6. SEMI-RIEMANNIAN SUBMERSIONS FROM PARAQUATERNIONIC CR-SUBMANIFOLDS

Semi-Riemannian submersions were introduced by O’Neill [29]. Let \((M, g)\) and \((M', g')\) be two connected semi-Riemannian manifolds of index \(s\) \((0 \leq s \leq \dim M)\) and \(s'\) \((0 \leq s' \leq \dim M')\) respectively, with \(s' \leq s\). Roughly speaking, a semi-Riemannian submersion is a smooth map \(\pi : M \to M'\) which is onto and satisfies the following conditions:

(i) \(\pi_* : T_p M \to T_{\pi(p)} M'\) is onto for all \(p \in M\);

(ii) The fibres \(\pi^{-1}(p')\), \(p' \in M'\), are semi-Riemannian submanifolds of \(M\);

(iii) \(\pi_*\) preserves scalar products of vectors normal to fibres.

The vectors tangent to fibres are called vertical and those normal to fibres are called horizontal. We denote by \(V\) the vertical distribution, by \(H\) the horizontal distribution and by \(v\) and \(h\) the vertical and horizontal projection. An horizontal vector field \(X\) on \(M\) is said to be basic if \(X\) is \(\pi\)-related to a vector field \(X'\) on \(M'\).

It is clear that every vector field \(X'\) on \(M'\) has a unique horizontal lift \(X\) to \(M\) and \(X\) is basic.

Remark 6.1. If \(\pi : (M, g) \to (M', g')\) is a semi-Riemannian submersion and \(X, Y\) are basic vector fields on \(M\), \(\pi\)-related to \(X'\) and \(Y'\) on \(M'\), then we have the next properties (see [28]):

(i) \(g(X, Y) = g'(X', Y') \circ \pi\);

(ii) \(h[X, Y]\) is a basic vector field and \(\pi_* h[X, Y] = [X', Y'] \circ \pi\);

(iii) \(h(\nabla_X Y)\) is a basic vector field \(\pi\)-related to \(\nabla'_X Y'\), where \(\nabla\) and \(\nabla'\) are the Levi-Civita connections on \(M\) and \(M'\);

(iv) \([E, U] \in \Gamma(V), \forall U \in \Gamma(V)\) and \(\forall E \in \Gamma(TM)\).

Remark 6.2. A semi-Riemannian submersion \(\pi : M \to M'\) determines, as well as in the Riemannian case (see [15]), two \((1,2)\) tensor field \(T\) and \(A\) on \(M\), by the formulas:

\[
\tag{31}
T(E, F) = h\nabla_{vE}vF + v\nabla_{vE}hF
\]

and respectively:

\[
\tag{32}
A(E, F) = v\nabla_{hE}hF + h\nabla_{hE}vF
\]

for any \(E, F \in \Gamma(TM)\). We remark that for \(U, V \in \Gamma(V)\), \(T(U, V)\) coincides with the second fundamental form of the immersion of the fibre submanifolds and for \(X, Y \in \Gamma(H), A(X, Y) = \frac{1}{2}v[X, Y]\) characterizes the complete integrability of the horizontal distribution \(H\).

It is easy to see that \(T\) and \(A\) satisfy:

\[
\tag{33}
T(U, V) = T(V, U),
\]

\[
\tag{34}
A(X, Y) = -A(Y, X)
\]

\[
\tag{35}
\nabla_X Y = h\nabla_X Y + A(X, Y),
\]

\[
\tag{36}
\nabla_U V = v\nabla_U V + T(U, V),
\]

for any \(X, Y \in \Gamma(H)\) and \(U, V \in \Gamma(V)\).
Remark 6.3. In [24], S. Kobayashi observed the next similarity between a Riemannian submersion and a CR-submanifold of a Kähler manifold: both involve two distributions (the vertical and horizontal distribution), one of them being integrable. Then he introduced the concept of CR-submersion, as a Riemannian submersion from a CR-submanifold to an almost hermitian manifold. Next, we’ll consider CR-submersions from paraquaternionic CR-submanifolds of a paraquaternionic Kähler manifold.

Definition 6.4. Let $N$ be a paraquaternionic CR-submanifold of an almost paraquaternionic hermitian manifold $(M, \sigma, g)$ and $(M', \sigma', g')$ be an almost hermitian manifold. A semi-Riemannian submersion $(\pi, \nu, \sigma)$ from a CR-submanifold to an almost hermitian manifold. Next, we’ll consider CR-submersions from paraquaternionic CR-submanifolds of a paraquaternionic Kähler manifold.

Proposition 6.5. Let $N$ be a paraquaternionic CR-submanifold of a paraquaternionic Kähler manifold $(M, \sigma, g)$ and $(M', \sigma', g')$ be an almost paraquaternionic hermitian manifold. If $\pi : N \to M'$ is a paraquaternionic CR-submersion if the following conditions are satisfied:

i. $\nu = D^\perp$;

ii. For each $p \in N$, $\pi_* : D_p \to T_{\pi(p)}M'$ is an isometry with respect to each complex and product structure of $D_p$ and $T_{\pi(p)}M'$, where $T_{\pi(p)}M'$ denotes the tangent space to $M'$ at $\pi(p)$.

Proposition 6.6. Let $N$ be a paraquaternionic CR-submanifold of a paraquaternionic Kähler manifold $(M, \sigma, g)$ and $(M', \sigma', g')$ be an almost paraquaternionic hermitian manifold. If $\pi : N \to M'$ is a paraquaternionic CR-submersion, then

$$h \nabla_X J_\alpha Y = J_\alpha h \nabla_X Y - \epsilon_\alpha [\omega_{\alpha+2}(X)J_{\alpha+1}Y - \omega_{\alpha+1}(X)J_{\alpha+2}Y]$$

(37)

$$A(X, J_\alpha Y) = J_\alpha \tau B(X, Y)$$

(38)

$$\tau B(X, J_\alpha Y) = J_\alpha A(X, Y)$$

(39)

$$\overline{h} B(X, J_\alpha Y) = J_\alpha \overline{\tau} B(X, Y)$$

(40)

for any $X, Y \in \Gamma(D)$, where $\nabla$ is the Levi-Civita connection on $N$ and $\overline{\tau}, \tau$ denote the canonical projections on $\nu$ and $\nu^\perp$, respectively.

Proof. From Gauss equations and (35), we have

$$\nabla_X Y = h \nabla_X Y + A(X, Y) + \overline{h} B(X, Y) + \tau B(X, Y),$$

(41)

for any $X, Y \in \Gamma(D)$, where $\nabla$ is the Levi-Civita connection on $M$.

From (41) we obtain:

$$\nabla_X J_\alpha Y = h \nabla_X J_\alpha Y + A(X, J_\alpha Y) + \overline{h} B(X, J_\alpha Y) + \tau B(X, J_\alpha Y) - J_\alpha h \nabla_X Y - J_\alpha A(X, Y) - J_\alpha \overline{h} B(X, Y) - J_\alpha \tau B(X, Y).$$

(42)

On another hand, from (41) we have:

$$\nabla_X J_\alpha Y = -\epsilon_\alpha [\omega_{\alpha+2}(X)J_{\alpha+1}Y - \omega_{\alpha+1}(X)J_{\alpha+2}Y] \in \Gamma(D),$$

(43)

for any $X, Y \in \Gamma(D)$.

Comparing now (42) and (43) and identifying the components from $\Gamma(D), \Gamma(D^\perp)$, $\Gamma(\nu)$ and $\Gamma(\nu^\perp)$, we obtain the wanted identities. \qed

Proposition 6.6. Let $N$ be a paraquaternionic CR-submanifold of a paraquaternionic Kähler manifold $(M, \sigma, g)$ and $(M', \sigma', g')$ be an almost paraquaternionic hermitian manifold. If $\pi : N \to M'$ is a paraquaternionic CR-submersion, then

$$\tau \nabla_{U^\perp} J_\alpha V = J_\alpha v \nabla_{U^\perp} V - \epsilon_\alpha [\omega_{\alpha+2}(U)J_{\alpha+1}V - \omega_{\alpha+1}(U)J_{\alpha+2}V]$$

(44)

$$J_\alpha T(U, V) = -h A_{J_\alpha V} U$$

(45)
\[(46)\]
\[J_\alpha \pi B(U,V) = -vA_{J_\alpha}v U\]

\[(47)\]
\[\bar{\nabla}_U J_\alpha V = J_\alpha \pi B(U,V)\]

for any \(U, V \in \Gamma(D^\perp)\), where \(\nabla\) is the Levi-Civita connection on \(N\) and \(\nabla^\perp\) is the normal connection.

**Proof.** From Gauss equations and (46), we have:

\[(48)\]
\[\bar{\nabla}_U V = T(U,V) + v \nabla_U V + \pi B(U,V) + \tilde{\pi} B(U,V),\]

for any \(U, V \in \Gamma(D^\perp)\), where \(\bar{\nabla}\) is the Levi-Civita connection on \(M\).

On the other hand, from Weingarten formula, we have:

\[(49)\]
\[\bar{\nabla}_U J_\alpha V = -h A_{J_\alpha} v U - v A_{J_\alpha} v U + \bar{\nabla}_U J_\alpha V + \tilde{\pi} \nabla_U J_\alpha V,\]

for any \(U, V \in \Gamma(D^\perp)\).

From (48) and (49) we deduce:

\[(50)\]
\[(\bar{\nabla}_U J_\alpha) V = -h A_{J_\alpha} v U - v A_{J_\alpha} v U + \bar{\nabla}_U J_\alpha V + \tilde{\pi} \nabla_U J_\alpha V - J_\alpha T(U,V) - J_\alpha v \nabla_U V - J_\alpha \pi B(U,V) - J_\alpha \tilde{\pi} B(U,V),\]

On another hand, from (47) we have:

\[(51)\]
\[(\bar{\nabla}_U J_\alpha) V = -\epsilon_\alpha [\omega_{\alpha+2}(U) J_{\alpha+1} V - \omega_{\alpha+1}(U) J_{\alpha+2} V] \in \Gamma(\mu^\perp),\]

for any \(U, V \in \Gamma(D^\perp)\).

Comparing now (50) and (51) and identifying the components from \(\Gamma(D), \Gamma(D^\perp), \Gamma(\nu)\) and \(\Gamma(\nu^\perp)\), we obtain the wanted identities. \(\square\)

**Theorem 6.7.** Let \(N\) be a paraquaternionic CR-submanifold of a paraquaternionic Kähler manifold \((M, \sigma, g)\) and \((M', \sigma', g')\) be an almost paraquaternionic hermitian manifold. If \(\pi : N \to M'\) is a paraquaternionic CR-submersion, then \((M', \sigma', g')\) is a paraquaternionic Kähler manifold.

**Proof.** We have from (47):

\[(52)\]
\[h \nabla_X J_\alpha Y - J_\alpha h \nabla_X Y = -\epsilon_\alpha [\omega_{\alpha+2}(X) J_{\alpha+1} Y - \omega_{\alpha+1}(X) J_{\alpha+2} Y]\]

for any local basic vector fields \(X, Y\) on \(N\).

We define the local 1-forms \(\omega'_\alpha\) on \(M'\) by:

\[(53)\]
\[\omega'_\alpha(X') \circ \pi = \omega_\alpha(X)\]

for any local vector field \(X'\) on \(M'\), \(\pi\)-related with an horizontal vector field \(X\) on \(N\).

On the other hand, from the definition of a paraquaternionic CR-submersion, we deduce that for any local bases \(\{J'_1, J'_2, J'_3\}\) of \(\sigma'\) we have a corresponding local basis \(\{J_1, J_2, J_3\}\) of \(\sigma\) such that:

\[(54)\]
\[\pi \circ J_\alpha = J'_\alpha \circ \pi, \quad \alpha \in \{1, 2, 3\}.\]

Using (52), (53), (54) and Remark 6.1 we obtain:

\[(55)\]
\[(\nabla_X J'_\alpha) Y = -\epsilon_\alpha [\omega'_{\alpha+2}(X') J'_{\alpha+1} Y' - \omega'_{\alpha+1}(X') J'_{\alpha+2} Y']\]

for any local vector fields \(X', Y'\) on \(M'\), \(\pi\)-related with two local basic vector fields \(X, Y\) on \(N\), where \(\nabla'\) is the Levi-Civita connection on \(M'\). Hence \((M', \sigma', g')\) is a paraquaternionic Kähler manifold. \(\square\)
Theorem 6.8. Let $N$ be a mixed foliated paraquaternionic CR-submanifold of a paraquaternionic Kähler manifold $(M, \sigma, g)$ and $(M', \sigma', g')$ be an almost paraquaternionic hermitian manifold. If $\pi : N \to M'$ is a paraquaternionic CR-submersion, then $N$ is locally a semi-Riemannian product of a paraquaternionic submanifold and a totally real submanifold of $M$. In particular, if $N$ is complete and simply connected then it is a global semi-Riemannian product.

Proof. Because $N$ is a mixed foliated paraquaternionic CR-submanifold, in particular $N$ is a mixed geodesic submanifold, and from (28) we obtain that the distribution $\mathcal{D}^\perp$ is parallel. On another hand, $\mathcal{D}$ being integrable, we have $\mathcal{A}(X, Y) = v\nabla_X Y = 0$, for any $X, Y \in \Gamma(\mathcal{D})$ and therefore the distribution $\mathcal{D}^\perp$ is also parallel. Hence $N$ is locally a semi-Riemannian product $(N_1, g_1) \times (N_2, g_2)$, where $N_1$ and $N_2$ are leaves of $\mathcal{D}$ and $\mathcal{D}^\perp$.

Finally, if $N$ is complete and simply connected, applying the decomposition theorem for semi-Riemannian manifolds (see [34]) we obtain the last part of the theorem.

Example 6.9. We remarked in Section 3 that

$$N = \{ [u_0, u_1, u_2] \in f_{p, q}^{-1}(0) | q^2 |u_0|^2 + p^2 |u_1|^2 + p^2 |u_2|^2 \neq 0 \}$$

is a a proper paraquaternionic CR-submanifold of $\mathbb{P}^2 \mathbb{B}$. Moreover, the Lie group $G = \{ e^{it} \mid t \in \mathbb{R} \}$ acts freely and isometrically on $N$. Using now the paraquaternionic Kähler reduction (see Theorem 5.2 from [33]) we obtain that the manifold $M' = N/G$ equipped with the submersed metric (i.e. the one $g'$ which makes the projection $\pi : (N, g) \to (M', g')$ a semi-Riemannian submersion) is again a paraquaternionic Kähler manifold with respect to the structure $\sigma'$ induce on $M'$ from the structure $\sigma$ by the projection $\pi$. Moreover, $\pi : N \to N/G$ is a paraquaternionic CR-submersion.

7. Some curvature properties of paraquaternionic CR-submersions

Let $(M, g)$ be a semi-Riemannian manifold. The sectional curvature $K$ of a 2-plane in $T_p M$, $p \in M$, is defined by:

$$K(X \wedge Y) = \frac{\mathcal{R}(X, Y, X, Y)}{g(X, X)g(Y, Y) - g(X, Y)^2}.$$  \hspace{1cm} (56)

It is clear that the above definition makes sense only for non-degenerate planes, i.e. those satisfying $Q(X \wedge Y) = g(X, X)g(Y, Y) - g(X, Y)^2 \neq 0$.

If $\pi : (N, g) \to (M', g')$ is a semi-Riemannian submersion, then the tensor fields $\mathcal{A}$ and $T$ defined in the above Section play a fundamental role in expressing the curvatures of the $(N, g)$, $(M', g')$ and of any fibre $(\pi^{-1}(p'), \tilde{g}_{p'})$, $p' \in M'$, because we have the following formulas stated in [28]:

$$\mathcal{R}(U, V, W, W') = \tilde{\mathcal{R}}(U, V, W, W') - g(T(U, W), T(V, W')) + g(T(V, W), T(U, W'))$$ \hspace{1cm} (57)

$$\mathcal{R}(X, Y, Z, Z') = \mathcal{R}^*(X, Y, Z, Z') - 2g(A(X, Y), A(Z, Z')) + g(A(Y, Z), A(Z, Z'))$$ \hspace{1cm} (58)

for any $U, V, W, W' \in \Gamma(\mathcal{V})$ and $X, Y, Z, Z' \in \Gamma(\mathcal{H})$, where $\mathcal{R}$ is the Riemannian curvature of $(N, g)$, $\tilde{\mathcal{R}}$ is the Riemannian curvature of fibre and $\mathcal{R}^*(X, Y, Z, Z') = g(\mathcal{R}^*(Z, Z')Y, X)$, $\mathcal{R}^*(\cdot, \cdot)$ being the (1,3)-tensor field on $\Gamma(\mathcal{H})$ with values in $\Gamma(\mathcal{H})$.
which associates to any \(X, Y, Z \in \Gamma(H)\) and \(p \in N\), the horizontal lift \((R^*(X, Y)Z)_p\) of \(R'_{\pi(p)}(\pi_\alpha(x_p), \pi_\alpha(y_p))\pi_\alpha(z_p)\), \(R'\) denoting the Riemannian curvature of \((M', g')\).

**Theorem 7.1.** Let \(N\) be a paraquaternionic CR-submanifold of a paraquaternionic Kähler manifold \((M, \sigma, g)\) and \((M', \sigma', g')\) be an almost paraquaternionic hermitian manifold. If \(\pi: N \to M'\) is a paraquaternionic CR-submersion, then the sectional curvatures of \(M\) and the fibres are related by:

\[
\kappa(U \land V) = \tilde{\kappa}(U \land V) - \epsilon_\alpha \theta_U \theta_V [g(A_{J_\alpha}U, A_{J_\alpha}V) - \theta_T(U, W)T(V, W') - g(T(U, W), T(U, W'))]
\]

(59)

for any unit space-like or time-like orthogonally vector fields \(U, V\) and, respectively,

\[
\kappa(U \land V) = \theta_U \theta_V \tilde{\kappa}(U, V, U, V)
\]

and, respectively,

(60)

\[\kappa(U \land V) = \tilde{\kappa}(U \land V) - \epsilon_\alpha \theta_U \theta_V [g(T(U, W), T(V, V')) - g(B(U, W), B(U, W'))],\]

(62)

where \(\theta_U = g(U, U) \in \{-1, 1\}\) and \(\theta_V = g(V, V) \in \{-1, 1\}\).

From (61), (62) and (60), using (53), we obtain:

\[
\kappa(U \land V) = \tilde{\kappa}(U \land V) - \epsilon_\alpha \theta_U \theta_V [g(T(U, W), T(V, V')) - g(B(U, W), B(U, V))]
\]

(63)

for any unit space-like or time-like orthogonally vector fields \(U, V \in \Gamma(D^\perp)\).

Finally, using (55), (60) and (63) in (63), we obtain (59).

**Theorem 7.2.** Let \(N\) be a paraquaternionic CR-submanifold of a paraquaternionic Kähler manifold \((M, \sigma, g)\) and \((M', \sigma', g')\) be an almost paraquaternionic hermitian manifold. If \(\pi: N \to M'\) is a paraquaternionic CR-submersion, then for any unit space-like or time-like horizontal vector field \(X\) one has:

\[
\overline{\kappa}_\alpha(X) = H'_\alpha(\pi_*X) - 4g(\pi B(X, X), \pi B(X, X)) + 2g(\overline{\kappa}B(X, X), \overline{\kappa}B(X, X))
\]

(64)

for \(\alpha \in \{1, 2, 3\}\), where \(\overline{\kappa}_\alpha\) and \(H'_\alpha\) are the holomorphic sectional curvatures of \(M\) and \(M'\), defined by \(\overline{\kappa}_\alpha(X) = K(X \land J_\alpha X)\) and \(H'_\alpha(X) = K'(X \land J_\alpha X)\), respectively.

**Proof.** From (58) and Gauss equation we have:

\[
\overline{\kappa}(X, Y, Z, Z') = R^*(X, Y, Z, Z') - 2g(A(X, Y), A(Z, Z'))
\]

(65)

\[+ g(A(Y, Z), A(Z', Z')) - g(A(X, Z), A(Y, Z')) + g(B(Y, Z), B(X, Z')) - g(B(Y, Z'), B(X, Z))\]

for any \(X, Y, Z, Z' \in \Gamma(D)\).

From (63) and (66) we see that the holomorphic sectional curvature of \(M\) is:

\[
\overline{\kappa}_\alpha(X) = \epsilon_\alpha \overline{\kappa}(X, J_\alpha X, X, J_\alpha X)
\]

(66)
for any unit space-like or time-like vector field $X \in \Gamma(D)$.

On the other hand we can easily see that:
\begin{equation}
H'_\alpha(\pi_*X) = \epsilon_\alpha R^*(X, J_\alpha X, X, J_\alpha X)
\end{equation}
for any unit space-like or time-like vector field $X \in \Gamma(D)$.

From (65), (66) and (67), using (34), we obtain:
\begin{equation}
H_\alpha(X) = H'_\alpha(\pi_*X) - 3 \epsilon_\alpha g(A(X, J_\alpha X), A(X, J_\alpha X))
+ \epsilon_\alpha [g(B(X, J_\alpha X), B(X, J_\alpha X)) - g(B(X, X), B(J_\alpha X, J_\alpha X))].
\end{equation}

From (34) and (39) one has:
\begin{equation}
\bar{v} B(X, J_\alpha X) = 0.
\end{equation}

Similarly, from (34), (38) and (39) we derive:
\begin{equation}
\bar{v} B(J_\alpha X, J_\alpha X) = \epsilon_\alpha \bar{v} B(X, X).
\end{equation}

and from (40), using the symmetry of $B$, we obtain:
\begin{equation}
\bar{h} B(J_\alpha X, J_\alpha X) = - \epsilon_\alpha \bar{h} B(X, X).
\end{equation}

Finally, using (69), (70) and (71) in (68) we obtain (64). □

**Corollary 7.3.** Let $N$ be a totally geodesic paraquaternionic CR-submanifold of a paraquaternionic Kähler manifold $(M, \sigma, g)$ and $(M', \sigma', g')$ be an almost paraquaternionic hermitian manifold. If $\pi : N \rightarrow M'$ is a paraquaternionic CR-submersion one has:
\begin{equation}
\overline{\nabla} \alpha(X) = H'_\alpha(\pi_*X),
\end{equation}
for any unit space-like or time-like horizontal vector field $X$.

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