HUNTING FOR CURVES WITH MANY POINTS

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Abstract. We construct curves with many points over finite fields by using
the class group.

1. Introduction

The question how many rational points a curve of given genus over a finite field
of given cardinality can have is an attractive challenge and the table of curves with
many points ([10]) has been a reference point for progress for genus \( \leq 50 \) and small
fields of characteristic \( p = 2 \) and \( p = 3 \). The tables record there for a pair \((q, g)\) an
interval \([a, b]\) where \( b \) is the best upper bound for the maximum number of points
of a curve of genus \( g \) over \( \mathbb{F}_q \) and \( a \) gives a lower bound obtained from an explicit
example of a curve \( C \) defined over \( \mathbb{F}_q \) with \( a \) (or at least \( a \)) rational points. So
\( N_q(g) \in [a, b] \) with \( N_q(g) \) the maximum number of rational points on a smooth
connected projective curve defined over \( \mathbb{F}_q \). For progress on the upper bounds we
refer to the papers of Howe and Lauter, [1] and to the references given in the tables.
It is the purpose of this little paper to record recent improvements of the table and
at the same time give constructions for many of the present records in the table.
The methods employ the class group and are variations on well-known themes and
do not involve new ideas. We hope it inspires people to take up the challenge to
improve the tables.

2. Using the Class Group

Let \( C \) be a curve of genus \( g \) defined over \( \mathbb{F}_2 \) with ‘many’ rational points. If \( f \)
is a rational function not of the form \( h^2 + h \) for \( h \) in the function field \( \mathbb{F}_2(C) \) of \( C \)
then in the Artin-Schreier cover of \( C \) defined by \( w^2 + w = f \) all the rational points
of \( C \) where \( f \) vanishes will be split. If the contribution of the poles of \( f \) is limited
and \( f \) vanishes in many points this may yield curves over \( \mathbb{F}_2 \) with many points.
Serre applied this method (cf. [7]) to construct curves over \( \mathbb{F}_2 \) of genus \( g = 11 \)
with 14 rational points (resp. \( g = 13 \) with \( N = 15 \) and \( g = 10 \) with \( N = 13 \) thus
determining \( N_2(10) = 13 \), \( N_2(11) = 14 \) and \( N_2(13) = 15 \). To find suitable \( f \) one
uses the class group. Serre used an elliptic curve, but we may of course use higher
genus curves as well. We illustrate the method by constructing many curves that
reach or improve the present ‘records’.

2.1. Base Curve of Genus 1 over \( \mathbb{F}_2 \). Let \( C \) be the (projective smooth) curve
of genus 1 defined by the affine equation \( y^2 + y = x^3 + x \). It has five rational points
\( P_0 = \infty \), \( P_1 = (0, 0) \), \( P_2 = (1, 0) \), \( P_3 = (1, 1) \) and \( P_4 = (0, 1) \). Since we have an
isomorphism \( C(\mathbb{F}_2) \cong \mathbb{Z}/5\mathbb{Z} \) of abelian groups such that under the isomorphism \( P_i \)
corresponds to \( i \pmod{5} \), the divisor \( \sum a_i P_i \) is of degree 0 and linearly equivalent to 0 if and only if
\[
\sum_{i=0}^{4} a_i = 0 \quad \text{and} \quad \sum_{i=0}^{4} a_i i \equiv 0 \pmod{5}.
\] (1)
So there exists a function \( f \in \mathbb{F}_2(C) \) with divisor \((f)\) equal to a given divisor \( \sum a_i P_i \) if and only if the 5-tuple \((a_0, a_1, \ldots, a_4) \in \mathbb{Z}^5\) satisfies (1).

Often we shall denote a relation for linear equivalence \( \sum_i a_i P_i \sim 0 \) simply by \([a_0, a_1, \ldots, a_4]\) for shortness. Two examples of such relations for the present curve are \([-3, -1, 2, 1, 1]\) and \([-1, -3, 1, 1, 2]\) with functions \( f_1 \) and \( f_2 \). This gives curves \( C_{f_1} \) and \( C_{f_2} \) of genera \( g = 4 \) with \( N = 8 \) rational points. Note that \( N_2(4) = 8 \), so these curves realize the maximum for genus 4. The fibre product \( C_{f_1} \times_C C_{f_2} \) has genus \( g = 11 \) with \( N = 14 \) rational points. Since the upper bound for \( N_2(11) \) is 14, Serre (cf. [7]) thus showed that \( N_2(11) = 14 \).

Another example is the relation \(-7P_0 + 2P_1 + 3P_2 + P_4 + P_5 \sim 0\). If \( f_3 \) is a corresponding function then the Artin-Schreier cover is a curve \( C_{f_3} \) with genus \( g = 5 \) and with \( N = 9 \) rational points as one readily checks. Note that \( N_2(5) = 9 \). The fibre product \( C_{f_1} \times_C C_{f_3} \) is a curve of genus \( g = 13 \) with \( N = 15 \) rational points, hence \( N_2(13) = 15 \), again due to Serre.

Combining \( f_1 \) with the function \( f_4 \) corresponding to the relation \([-3, 2, 1, -1, 1]\) gives \( C_{f_1} \times_C C_{f_4} \) of genus 10 with \( N = 13 \) rational points. Note that the divisor of \( f_1 + f_4 \) is \(-2P_0 - P_1 + 2P_2 + 2P_4 + P_5 \). Artin-Schreier reduction shows that the conductor of this cover is \( 2P_0 + 2P_1 + 2P_3 \). This determines \( N_2(10) = 13 \), a result due to Serre.

We list here eight relations that can be used to obtain more good curves;

| \( n \) | relation | \( n \) | relation |
|---|---|---|---|
| 1 | \([-3, -1, 2, 1, 1]\) | 5 | \([-5, -1, 3, 2, 1]\) |
| 2 | \([-1, -3, 1, 1, 2]\) | 6 | \([-9, 1, 3, 2, 3]\) |
| 3 | \([-7, 2, 3, 1, 1]\) | 7 | \([-11, 1, 4, 3, 3]\) |
| 4 | \([-3, 2, 1, -1, 1]\) | 8 | \([-13, 2, 4, 3, 4]\) |

and reap the fruits by associating to a tuple \( f_1, \ldots, f_i \) the corresponding fibre product \( C_{f_1} \times_C C_{f_2} \times \cdots \times_C C_{f_i} \).

| space | \( g \) | \# \( C(\mathbb{F}_2) \) | interval | space | \( g \) | \# | interval |
|---|---|---|---|---|---|---|---|
| \( \langle f_1 \rangle \) | 4 | 8 | [8] | \( \langle f_1, f_3, f_4 \rangle \) | 29 | 25 | [25 \(-27] |
| \( \langle f_3 \rangle \) | 9 | [9] | [13] | \( \langle f_1, f_2, f_3 \rangle \) | 32 | 27 | [26 \(-29] |
| \( \langle f_1, f_4 \rangle \) | 10 | 13 | [13] | \( \langle f_1, f_3, f_4 \rangle \) | 34 | 27 | [27 \(-30] |
| \( \langle f_1, f_2 \rangle \) | 11 | 14 | [14] | \( \langle f_1, f_3, f_5 \rangle \) | 35 | 29 | [29 \(-31] |
| \( \langle f_1, f_3 \rangle \) | 13 | 15 | [15] | \( \langle f_1, f_3, f_6 \rangle \) | 39 | 33 | [33] |
| \( \langle f_3, f_5 \rangle \) | 14 | 15 | [15 \(-16] | \( \langle f_3, f_6, f_7 \rangle \) | 43 | 33 | [33 \(-36] |
| \( \langle f_3, f_6 \rangle \) | 15 | 17 | [17] | \( \langle f_3, f_6, f_8 \rangle \) | 45 | 33 | [33 \(-37] |
| \( \langle f_1, f_2, f_3 \rangle \) | 28 | 25 | [25 \(-26] | \( \langle f_3, f_7, f_8 \rangle \) | 45 | 33 | [33 \(-37] |

This reproduces many records of the tables and gives one improvement of the tables for curves over \( \mathbb{F}_2 \), namely for \( g = 32 \).

2.2. Base of Genus 2 over \( \mathbb{F}_2 \). Let \( C \) be the curve of genus 2 defined by the affine equation \( y^2 + y = (x^2 + x)/(x^3 + x^2 + 1) \) over \( \mathbb{F}_2 \). It has six rational points
$P_i$ ($i = 1, \ldots, 6$) and class group $\text{Jac}(C)(\mathbb{F}_2) \cong \mathbb{Z}/19\mathbb{Z}$. Let $\phi : C \to \text{Jac}(C)$ be the Abel-Jacobi map given by $Q \mapsto -\deg(Q)P_1$. For a suitable numbering of the $P_i$ the images of the $P_i$ in $\text{Jac}(C)(\mathbb{F}_2) \cong \mathbb{Z}/19\mathbb{Z}$ are as follows

| $i$ | 1  | 2  | 3  | 4  | 5  | 6  |
|-----|----|----|----|----|----|----|
| $\phi(P_i)$ | 0  | 1  | 14 | 6  | 16 | 4  |

From this table we see that we have the following linear equivalence

$$-3P_1 - 3P_2 + 2P_3 + 2P_4 + P_5 + P_6 \sim 0.$$  \hspace{1cm} (2)

If $f$ is the function with the left hand side of (2) as divisor the curve $C_f$ obtained as the double cover of $C$ defined by $w^2 + w = f$ has genus $g = 7$ and 10 rational points. This is optimal. Similarly the relation

$$-9P_1 + 2P_2 + 2P_3 + P_4 + 2P_5 + 2P_6 \sim 0$$

gives rise to a curve with $g = 8$ and 11 rational points, again an optimal curve over $\mathbb{F}_2$. Let $Q$ be the divisor of zeros of the polynomial $x^3 + x^2 + 1$ on $C$. Then $\phi(Q) = 3$ and $-Q + \sum_{i=1}^{6} P_i \sim 0$ which gives us a curve of genus 9 with 12 points, again an optimal curve.

We list a number of divisors of functions $f_n$ ($n = 1, \ldots, 10$):

| $n$ | relation | $n$ | relation |
|-----|-----------|-----|-----------|
| 1   | $[-3, -3, 2, 2, 1, 1]$ | 6   | $[11, 3, 2, 2, 3, 1]$ |
| 2   | $[-9, 2, 2, 1, 2, 2]$  | 7   | $[1, 0, -5, 2, 1, 1]$ |
| 3   | $[1, 1, 1, 1, 1, 1] \sim Q$ | 8   | $[3, -3, -3, 1, 1, 1]$ |
| 4   | $[-1, -5, 1, 2, 2, 1]$ | 9   | $[-7, 0, 2, 2, 2, 1]$ |
| 5   | $[-5, -1, 2, 1, 1, 2]$ | 10  | $[-13, 3, 1, 1, 3, 3]$ |

and we find as results:

| space   | $g$ | $\#C(\mathbb{F}_2)$ | interval | space   | $g$ | $\#C(\mathbb{F}_2)$ | interval |
|---------|-----|----------------------|----------|---------|-----|----------------------|----------|
| $\langle f_1 \rangle$ | 7   | 10                   | [10]     | $\langle f_2, f_5 \rangle$ | 20  | 19                   | [19 - 21]|
| $\langle f_2 \rangle$ | 8   | 11                   | [11]     | $\langle f_2, f_6 \rangle$ | 22  | 21                   | [21 - 22]|
| $\langle f_3 \rangle$ | 9   | 12                   | [12]     | $\langle f_2, f_{10} \rangle$ | 24  | 21                   | [21 - 23]|
| $\langle f_4, f_8 \rangle$ | 17  | 17                   | [17 - 18]| $\langle f_1, f_4, f_5 \rangle$ | 43  | 34                   | [33 - 36]|
| $\langle f_1, f_4 \rangle$ | 18  | 18                   | [18 - 19]| $\langle f_2, f_5, f_9 \rangle$ | 44  | 33                   | [33 - 37]|

Again, we find one improvement, namely for $g = 43$.

We can also employ the class group for making an unramified cover of $C$ of degree 19 in which $P_1$ splits completely. The genus of this cover is $g = 20$ and it contains 19 rational points. The interval is $[19 - 21]$.

2.3. **Base of Genus 3 over $\mathbb{F}_2$.** Consider now the curve $C$ of genus 3 defined over $\mathbb{F}_2$ by the homogeneous equation

$$x^3 y + x^3 z + x^2 y^2 + xz^3 + y^3 z + y^2 z^2 = 0.$$ 

It has 7 rational points and its class group $\text{Jac}(C)(\mathbb{F}_2)$ is isomorphic to $\mathbb{Z}/71\mathbb{Z}$. We can number the rational points $P_i$ ($i = 1, \ldots, 7$) such that their images $\phi(P_i) = \text{class of } (P_i - P_1)$ under the Abel-Jacobi map are as in the following table.

| $i$ | 1  | 2  | 3  | 4  | 5  | 6  | 7  |
|-----|----|----|----|----|----|----|----|
| $\phi(P_i)$ | 0  | 1  | 34 | 55 | 10 | 14 | 49 |
There is a divisor \(Q_3\) of degree 3 with \(\phi(Q_3 - 3P_0) = 9\) and one of degree 5, say \(Q_5\), with \(\phi(Q_5 - 5P_0) = 35 \pmod{71}\); there is also a divisor \(Q_7\) of degree 7 with \(\phi(Q_7 - 7P_0) = 21\). We list a number of relations:

| \(n\) | divisor | \(\phi\) | \(n\) | divisor |
|---|---|---|---|---|
| 1 | \([-3, 2, 1, 1, 2, 1, -4]\) | 5 | \([-1, 3, 1, 1, 1, 2, -7]\) |
| 2 | \([-11, 1, 2, 1, 2, 3, 2]\) | 6 | \([2, -1, 1, 2, 2, 2, 1]\) | \(\sim 3Q_3\) |
| 3 | \([-13, 2, 1, 2, 2, 3, 3]\) | 7 | \([1, 1, 1, 1, 1, 1]\) | \(\sim Q_7\) |
| 4 | \([1, -5, 1, 2, 2, 1, 1]\) | 8 | \([-15, 4, 1, 3, 4, 1, 2]\) |

that yield the following results:

| curve | \(g\) | \#C(\(\mathbb{F}_2\)) | interval | curve | \(g\) | \#C(\(\mathbb{F}_2\)) | interval |
|---|---|---|---|---|---|---|---|
| \((f_1)\) | 9 | 12 | [12] | \((f_2, f_3)\) | 29 | 25 | [25 - 27] |
| \((f_7)\) | 12 | 14 | [14 - 15] | \((f_4, f_6)\) | 31 | 25 | [27 - 28] |
| \((f_1, f_3)\) | 24 | 22 | [21 - 23] | \((f_2, f_3, f_8)\) | 69 | 49 | [49 - 52] |

Note that for \(f_1\) we have to do Artin-Schreier reduction. If the conductor would be \(3P_0\) (resp. \(3P_0 + P_6\)) then \(C_{f_1}\) would give \(g = 7, N = 11\) (resp. \(g = 8, 12\)), and these are impossible. So the conductor is \(3P_0 + 3P_6\) giving \((g = 9, N = 12)\), an optimal curve. Again we find one improvement (for \(g = 24\)) for our tables. The interval for genus 69 comes from page 121 of [6].

### 2.4. Base of Genus 5 over \(\mathbb{F}_2\)

Consider the fibre product \(C\) of \(C_1\) given by \(y^2 + y = x^3 + x\) and \(C_2\) given by \(y^2 + y = x^5 + x^3\) over the \(x\)-line. This is an optimal curve of genus 5 with 9 points. Its Jacobian is isogenous with the product of the three Jacobians of the curves \(C_1\) and \(C_2\) and the curve \(C'_2\) defined by \(y^2 + y = x^5 + x\). The corresponding \(L\)-polynomials are \(2t^2 + 2t + 1, 4t^4 + 4t^3 + 2t^2 + 2t + 1\) and \(4t^4 + 4t^3 + 2t^2 + 2t + 1\). The class groups are \(\mathbb{Z}/5\mathbb{Z}, \mathbb{Z}/13\mathbb{Z}\) and \(\mathbb{Z}/15\mathbb{Z}\). The nine rational points are \(P_\infty\) and eight points that can be identified by their \((x, y_1, y_2, y_3)\)-coordinates. Writing the class group of \(C\) as \(\mathbb{Z}/65\mathbb{Z} \times \mathbb{Z}/15\mathbb{Z}\) we have the following table:

| \(i(P_i)\) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| \(\phi\) | \((0, 0)\) | \((1, 1)\) | \((51, 14)\) | \((64, 1)\) | \((14, 14)\) | \((57, 11)\) | \((47, 4)\) | \((8, 11)\) | \((18, 4)\) |

The relation \([1, 2, -13, 1, 2, 2, 2, 2]\) gives a curve with \(g = 16\) and 17 points (interval \([17 - 18]\)). The relation \([1, 1, -11, 1, -1, 3, 2, 1, 3]\) gives a curve with \(g = 16\) and 16 points. The fibre product has genus \(g = 39\) with 31 points. The combination of the relations \([1, 2, -13, 1, 2, 2, 2, 2, 1]\) and \([3, 3, -17, 1, 2, 1, 3, 1, 3]\) gives a curve with \(g = 42\) and 33 points (interval \([33 - 35]\)).

### 2.5. Genus 1 over \(\mathbb{F}_3\)

Consider the elliptic curve defined by the equation \(y^2 = x^3 + 2x + 1\) over \(\mathbb{F}_3\). It has 7 points \(P_i\) with \(i = 0, \ldots, 7\) such that \(P_i \rightarrow i \pmod{7}\) defines an isomorphism \(C(\mathbb{F}_3) \cong \mathbb{Z}/7\mathbb{Z}\). The relations \([-4, -1, 0, 1, 2, 1, 1]\) and \([-4, -2, 1, 2, 1, 1, 1]\) give a fibre product of genus \(g = 30\) with 38 points that improves the interval \([37 - 46]\) slightly.

### 2.6. Curves of Genus 2 over \(\mathbb{F}_3\)

The curve \(C\) of genus 2 over \(\mathbb{F}_3\) given by \(y^2 = x^5 + x^3 + x + 1\) has zeta function \((9t^4 + 9t^3 + 5t^2 + 3t + 1)/((1-t)(1-3t))\). The curve has 7 rational points and class group \(\mathbb{Z}/27\mathbb{Z}\). The 7 rational points map to \(0, 1, 26, 17, 10, 23, 4\) in the group. We have the relations \([1, 1, 1, 1, -2, 2, -4]\) and
This gives a new entry for the table, albeit not a very strong one (resulting interval [47 – 61]).

2.7. Base Curve of Genus 3 over \( \mathbb{F}_3 \). The curve \( C \) of genus 3 given by

\[
2x^4 + x^3z + 2x^2y^2 + x^2yz + x^2z^2 + 2xz^3 + 2y^4 + 2y^3z + 2y^2z^2 = 0
\]

has 10 rational points and class group \( \mathbb{Z}/204 \). The 10 points map to

\[
0, 72, 129, 59, 182, 121, 172, 45, 47, 26
\]
in \( \mathbb{Z}/204 \). We have the relation \([1, 2, 1, 1, 1, 1, 1, −10, 1]\) leading to an Artin-Schreier cover with genus 18 with 28 points improving the interval \([26 – 31]\) to \([28 – 31]\).

3. Subgroups of the Class Group

Let \( C \) be a smooth complete irreducible curve over \( \mathbb{F}_q \) with Jacobian \( \text{Jac}(C) \) and class group \( G = \text{Jac}(C)/\langle \mathbb{F}_q \rangle \). We choose a rational point \( P \), provided there is one, and consider the morphism \( \phi : C \to \text{Jac}(C) \) given by \( Q \mapsto Q − \deg(Q)P \). If \( H \) is a subgroup of \( G \) of index \( d \) containing the images \( \{\phi(P_i) : i \in I\} \) then there exists an unramified degree \( d \) cover \( \tilde{C} \) of \( C \) in which the points \( P_i \) with \( i \in I \) split completely. Choosing \( C \) and \( I \) appropriately can produce curves with many points.

3.1. Base of Genus 4 over \( \mathbb{F}_2 \). The reader might think that it is necessary to start with a curve with many points. To dispel this idea start with the genus 4 curve given by the equation

\[
y^2 + y = (x^7 + x^5 + 1)/(x^2 + x)
\]

over \( \mathbb{F}_2 \) which has three rational points \( P_1, P_2, P_3 \). The \( L \)-polynomial is \( 16t^8 + 4t^6 + 4t^5 + 4t^4 + 2t^3 + t^2 + 1 \), hence the class number is 32. In fact, the class group is \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/16\mathbb{Z} \) and the differences \( P_1 − P_3 \) map to \((0, 0), (1, 0) \) and \((1, 8) \). Hence there exists an étale cover of degree 8 in which the three points split completely. This gives a curve of genus 25 with 24 points. This is optimal.

Or start with the hyperelliptic curve of genus 5 given by

\[
y^2 + y = (x^9 + x^7 + x^3 + x + 1)/(x^2 + x)
\]

again with three rational points and class group \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/30\mathbb{Z} \) with the points now mapping to \((0, 0), (1, 0) \) and \((0, 15) \). This gives an étale cover of degree 15 with genus \( g = 61 \) with 45 points, very close to the best upper bound 47 and improving the interval \([41 – 47]\) of [6], p. 121.

3.2. Genus 2 over \( \mathbb{F}_3 \). We consider the curve \( C \) of genus 2 over \( \mathbb{F}_3 \) given by the affine equation \( y^3 − y = x − 1/x \). The zeta function is \( (9t^4 + 12t^3 + 10t^2 + 4t + 1)/(1 − t)(1 − 3t) \) and its class group is isomorphic to \( \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \). The images of the 8 rational points under the Abel-Jacobi map are

\[
[0, 0], [2, 4], [1, 0], [1, 4], [0, 1], [2, 3], [5, 5], [3, 5]
\]
in \( \mathbb{Z}/6\mathbb{Z} \). The subgroup of the class group of index 12 given by \( x_1 + x_2 \equiv 0 \pmod{3} \) and \( x_1 \equiv x_2 \equiv 0 \pmod{2} \) contains \([0, 0] \) and \([2, 4] \) and thus leads to a cover of degree 12 of genus \( g = 13 \) and with 24 rational points (interval \([24 – 25]\)). There is a place \( Q_2 \) of degree 2 which maps to \([2, 1]\) in the class group. The relation
\[ \sum_{j=1}^{8} P_j \sim 4Q_2 \] leads to an Artin-Schreier cover with \( g = 14 \) with 24 points (interval [24 – 25]).

An alternative case: consider \( y^2 = x(x^2 + 1)(x^2 - x - 1) \) over \( \mathbb{F}_3 \) of genus 2 with 6 points and class group \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z} \). The points map under an Abel-Jacobi map to
\[ [0, 0], [1, 0], [0, 9], [0, 1], [1, 7], [1, 3]. \]
So the index 10 subgroup defined by \( x \) has 6 rational points and zeta function \( (9t - 3t^2) \) genus 11 with 20 points. (Interval [20 – 22].) The curve has 10 points over \( \mathbb{F}_3 \) and class group \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/15\mathbb{Z} \). The curve possesses 20 rational points over \( \mathbb{F}_9 \). The images of these points under a suitable Abel-Jacobi map are given by
\[ [0, 0], [1, 0], [0, 9], [0, 1], [1, 7], [1, 3], [2, 9], [13, 12], [4, 7], [11, 14], [9, 6], [6, 0], [4, 8], [11, 13], [0, 6], [0, 0]. \]

The equation \( 5a + 5b = 0 \pmod{15} \) defines a subgroup of index 3 in \( \mathbb{Z}/15\mathbb{Z} \times \mathbb{Z}/15\mathbb{Z} \) containing the images of 10 points, hence we get a curve \( C \), an unramified cover of \( C \) of degree 3 with genus 4 with 30 points, the maximum possible.

Similarly, the six points \( P_i \) corresponding to pairs \( [a, b] \) with \( a \equiv b \equiv 0 \pmod{3} \) lie in a subgroup of index 9. So there is an unramified cover \( \tilde{C} \) of degree 9 and genus 10 with \( 9 \times 6 = 54 \) rational points. The interval in the tables is [54 – 55].

### 3.3. Base Curve of Genus 3 over \( \mathbb{F}_3 \)
Consider the plane curve of degree 4 over \( \mathbb{F}_3 \) given by
\[ 2x^3y + 2x^3z + x^2y^2 + xz^3 + 2y^3z + yz^3 = 0. \]
This curve has 10 points over \( \mathbb{F}_3 \) and has class group isomorphic to \( \mathbb{Z}/14\mathbb{Z} \) and the 10 points go to
\[ [0, 0], [1, 0], [6, 13], [7, 3], [13, 7], [4, 12], [11, 11], [6, 2], [4, 7] \]
The subgroup defined by \( x_2 \equiv 0 \pmod{7} \) has index 7 and contains the images of 4 points. The corresponding curve over \( \mathbb{F}_3 \) has genus 15 with 28 rational points; this is optimal.

Or consider the hyperelliptic curve given by the equation
\[ y^2 + (x^3 - x)y = x^7 - x^2 + x \]
over \( \mathbb{F}_3 \). It is of genus 3, has five rational points mapping in the class group \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z} \) to \( (0, 0, 0), (0, 0, 6), (0, 1, 0), (0, 0, 11), (0, 0, 1) \). The index 12 subgroup containing the first three points gives rise to a cover of genus \( g = 25 \) with 36 points (the interval being [36 – 40]).

The hyperelliptic curve \( y^2 = x^7 - x^2 + x \) over \( \mathbb{F}_3 \) has similarly a class group \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/14\mathbb{Z} \) and an index 14 subgroup containing three points giving rise to a curve of genus 29 with 42 points (interval [42 – 44]).
The hyperelliptic curve of genus 4 over \( \mathbb{F}_3 \) given by \( y^2 + xy = x^9 - x \) has class group \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/126\mathbb{Z} \) and the six rational points map to

\[
(0,0), (1,0), (0,125), (0,1), (0,14), (0,112)
\]

and so the index 14 subgroup containing four of these points gives rise to a cover of genus 43 with 56 points improving the interval \([55 - 60]\).

3.4. **Examples over \( \mathbb{F}_4 \)**. Let \( C \) be the hyperelliptic curve of genus 2 over \( \mathbb{F}_4 \) given by \( y^2 + y = x^5 + x^3 + x \) with class group \( \mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z} \). It has a cover of degree 7 in which three points split completely giving a curve of genus 8 with 21 rational points (interval \([21 - 24]\)).

3.5. **Genus 3 over \( \mathbb{F}_q \)**. Let \( \alpha \) be a generator of the multiplicative group \( \mathbb{F}_q^* \) and consider the curve \( C \) of genus 3 given by \( y^3 - y = \alpha^2 x^4 \) over the field \( \mathbb{F}_q \). This curve has 28 rational points, the bitangent points of a plane model. The class group is of the form \( (\mathbb{Z}/4\mathbb{Z})^6 \). The 28 points map under an Abel-Jacobi map to the following elements in \( (\mathbb{Z}/4\mathbb{Z})^6 \).

\[
[0,0,0,0,0,0], [3,1,3,3,0,1], [3,2,1,0,3,0], [2,0,3,1,1,3], [2,1,1,1,1,3],
[2,0,1,1,3], [3,1,0,1,1,3], [1,1,0,0,1,3], [2,1,0,2,1,3], [1,3,3,2,0,3],
[1,2,2,3,2,0], [2,2,3,2,1,2], [2,0,0,0,1,12], [2,1,1,3,4,2], [1,3,1,3,2,0],
[0,0,1,3,2,1], [1,3,1,2,2,2], [2,0,2,3,1,2], [3,1,3,0,3,2], [3,1,0,2,1,6],
[0,3,2,0,0,2], [1,1,1,2,1,1], [1,2,0,3,3,3], [0,0,3,2,3,1], [2,1,0,1,1,6],
[2,1,0,1,2,2], [2,1,0,1,2,3], [2,1,0,1,1,3].
\]

The index 2 subgroup of the class group defined by the equation \( 2x_1 \equiv 0 \pmod{4} \) contains the images of 16 points. The corresponding étale covering of \( C \) of degree 2 is of genus 5 and has 32 points (interval \([32 - 35]\)).

The index 4 subgroup of the class group defined by the equation \( x_2 + x_3 + x_4 + x_5 \equiv 0 \pmod{4} \) contains the images of 12 points. We thus find a degree 4 étale cover of \( C \) of genus 9 with 48 points (interval \([48 - 50]\)).

4. **Using the class group over extension fields**

Let \( C \) be a smooth projective curve defined over \( \mathbb{F}_q \) of genus \( g \) and with \( m \) rational points. Let \( J \) be its Jacobian variety. It is easy to see that if \( Z(C,t) \) is the zeta function of \( C/\mathbb{F}_q \) and \( Z_n(C,t) \) the zeta function of \( C \) considered as a curve over \( \mathbb{F}_q^n \) then we have \( Z_n(C,t^n) = \prod_{\zeta} Z(C,\zeta t) \), where the product is taken over the \( n \)th roots \( \zeta \) of 1. If we write \( Z(C,t) = L(C,t)/(1-t)(1-qt) \) and \( Z_n(C,t) = L_n(C,t)/(1-t)(1-q^n t) \) we get \( L_n(C,1) = \prod_{\zeta} L(C,\zeta) \). Moreover, we know that \( \#J(\mathbb{F}_q) = L(C,1) \) and \( \#J(\mathbb{F}_q^n) = L_n(C,1) \) and we thus get

\[
[J(\mathbb{F}_q^n) : J(\mathbb{F}_q)] = \prod_{\zeta^n = 1, \zeta \neq 1} L(C,\zeta).
\]

Since under the Abel-Jacobi map the \( \mathbb{F}_q \)-rational points of \( C \) map to \( J(\mathbb{F}_q) \) we conclude that there exists an unramified cover of \( C \) defined over \( \mathbb{F}_q \) of degree \( d = \prod_{\zeta^n = 1, \zeta \neq 1} L(C,\zeta) \) in which all the \( \mathbb{F}_q \)-rational points split completely. Thus we find a curve \( \tilde{C} \) of genus \( d(g-1) + 1 \) with at least \( dm \) rational points. This idea was exploited very successfully by Niederreiter and Xing in their papers [2] up to [9].

Here we employ it more systematically by going through all isomorphism classes of curves of low genera. This improves the tables at many places.

We now list all possible \( L \)-functions of genus 2 curves over \( \mathbb{F}_2 \). By \( N_i \) we mean \( \#C(\mathbb{F}_2^i) \).
If we now take $n = 2$ we get curves defined over $\mathbb{F}_4$; for example we find the following cases that do not give improvements of the table for $\mathbb{F}_4$ but good realizations of the present records.

| $f$ | $[N_1, N_2]$ | $\mathbb{C}$ | $L$ |
|-----|---------------|---------------|-----|
| $(x^2 + x)/(x^4 + x^2 + 1)$ | [6, 6] | $\mathbb{Z}/19\mathbb{Z}$ | $4t^4 + 6t^3 + 5t^2 + 3t + 1$ |
| $(x^3 + x + 1)/(x^4 + x^2 + 1)$ | [0, 6] | $\mathbb{Z}/6\mathbb{Z}$ | $4t^4 - 6t^3 + 5t^2 - 3t + 1$ |
| $1/(x^4 + x^2 + 1)$ | [2, 6] | $\mathbb{Z}/3\mathbb{Z}$ | $4t^4 - 2t^3 + t^2 - t + 1$ |
| $x/(x^4 + x^2 + 1)$ | [4, 6] | $\mathbb{Z}/9\mathbb{Z}$ | $4t^4 + 2t^3 + t^2 + t + 1$ |
| $x^2/(x^4 + x^2 + 1)$ | [4, 10] | $\mathbb{Z}/11\mathbb{Z}$ | $4t^4 + 2t^3 + 3t^2 + t + 1$ |
| $(x^3 + 1)/(x^4 + x^2 + 1)$ | [2, 10] | $\mathbb{Z}/5\mathbb{Z}$ | $4t^4 - 2t^3 + 3t^2 + t + 1$ |
| $1/(x^2 + x + 1)$ | [3, 7] | $\mathbb{Z}/6\mathbb{Z}$ | $4t^4 + t^2 + 1$ |
| $(x + 1)/(x^2 + x + 1)$ | [5, 7] | $\mathbb{Z}/13\mathbb{Z}$ | $4t^4 + 4t^3 + 3t^2 + 2t + 1$ |
| $(x^3 + x^2 + 1)/(x^2 + x + 1)$ | [3, 3] | $\mathbb{Z}/25 \times \mathbb{Z}/27$ | $4t^4 - t^2 + 1$ |
| $1/x + x^3$ | [4, 4] | $\mathbb{Z}/8\mathbb{Z}$ | $4t^4 + 2t^3 + t + 1$ |
| $1/x + x^2 + x^3$ | [2, 8] | $\mathbb{Z}/4\mathbb{Z}$ | $4t^4 - 2t^3 + 2t^2 - t + 1$ |
| $1/x + 1 + x^3$ | [2, 4] | $\mathbb{Z}/2\mathbb{Z}$ | $t^4 - 2t^3 - t + 1$ |
| $1/x + 1 + x^2 + x^3$ | [4, 8] | $\mathbb{Z}/10\mathbb{Z}$ | $4t^4 + 2t^3 + 2t^2 + t + 1$ |
| $x^5$ | [3, 5] | $\mathbb{Z}/5\mathbb{Z}$ | $4t^4 + 1$ |
| $x^5 + x^3 + x$ | [3, 9] | $\mathbb{Z}/7\mathbb{Z}$ | $4t^4 + 2t^2 + 1$ |
| $x^5 + x$ | [5, 9] | $\mathbb{Z}/15\mathbb{Z}$ | $4t^4 + 4t^3 + 4t^2 + 2t + 1$ |
| $x^5 + x + 1$ | [1, 5] | $\mathbb{Z}/3\mathbb{Z}$ | $4t^4 - 4t^3 + 4t^2 - 2t + 1$ |
| $x^5 + x^3 + 1$ | [5, 5] | $\mathbb{Z}/13\mathbb{Z}$ | $4t^4 + 4t^3 + 2t^2 + 2t + 1$ |
| $x^5 + x + 1$ | [1, 9] | $\mathbb{Z}/17\mathbb{Z}$ | $4t^4 - 4t^3 + 2t^2 - 2t + 1$ |

With $n = 3$ we get the following cases:

| $[N_1, N_2]$ | $g$ | $\#C(\mathbb{F}_4)$ | Interval |
|---------------|------|----------------------|----------|
| [5, 9]        | 4    | 15                   | [15]     |
| [4, 10]       | 6    | 20                   | [20]     |
| [3, 9]        | 8    | 21                   | [21 - 24]|

Here the case $g = 14$ with $\#C(\mathbb{F}_4) = 65$ is optimal; the case $g = 8$ with 35 rational points equals the lower bound of the interval [35 – 42] of the tables while $g = 20$ with 76 points improves the interval [68 – 83] of the tables.

Starting with the curves of genus 2 over $\mathbb{F}_4$ gives the following results:

| $[N_1, N_2]$ | $g$ | $\#C(\mathbb{F}_4)$ | Interval |
|---------------|------|----------------------|----------|
| [9, 24]       | 9    | 72                   | [72 – 81]|
| [9, 25]       | 10   | 81                   | [81 – 87]|
| [8, 24]       | 11   | 80                   | [80 – 91]|
| [8, 26]       | 12   | 88                   | [83 – 97]|
| [7, 27]       | 15   | 98                   | [98 – 113]|
| [7, 31]       | 17   | 112                  | [112 – 123]|

And the same can be done starting with the curves of genus 2 over $\mathbb{F}_8$. Here there are still many empty places in the tables due to the lack of good examples.
We can use curves of higher genus too. Going through all non-hyperelliptic curves of genus 3 over $\mathbb{F}_2$ gives two improvements for the table over $\mathbb{F}_4$:

| $N_1, N_2$ | $g$ | $\#C(\mathbb{F}_4)$ | interval |
|------------|-----|---------------------|----------|
| [18, 54]   | 20  | 342                 |          |
| [17, 63]   | 25  | 408                 |          |
| [17, 65]   | 26  | 425                 |          |
| [16, 64]   | 27  | 416                 |          |
| [16, 70]   | 30  | 464                 |          |
| [15, 71]   | 33  | 480                 |          |
| [14, 66]   | 34  | 462                 |          |
| [15, 75]   | 35  | 510                 |          |
| [14, 70]   | 36  | 490                 |          |
| [15, 79]   | 37  | 540                 |          |
| [14, 74]   | 38  | 518                 |          |
| [13, 67]   | 39  | 494                 |          |
| [14, 78]   | 40  | 546                 |          |
| [14, 80]   | 41  | 560                 |          |
| [14, 82]   | 42  | 574                 |          |
| [12, 66]   | 44  | 516                 |          |
| [13, 79]   | 45  | 572                 |          |
| [13, 81]   | 46  | 585                 |          |
| [13, 83]   | 47  | 598                 |          |
| [12, 74]   | 48  | 564                 |          |
| [13, 87]   | 49  | 624                 |          |
| [12, 78]   | 50  | 588                 |          |

and considering these curves over $\mathbb{F}_4$ gives three improvements for the table over $\mathbb{F}_{16}$:

| $N_1, N_2, N_3$ | $g$ | $\#C(\mathbb{F}_4)$ | interval |
|-----------------|-----|---------------------|----------|
| [5, 9, 5]       | 15  | 35                  | [33 - 37]|
| [5, 11, 5]      | 17  | 40                  | [40]     |
| [4, 12, 7]      | 27  | 52                  | [50 - 56]|

and considering these curves over $\mathbb{F}_4$ gives three improvements for the table over $\mathbb{F}_{16}$:

| $N_1, N_2, N_3$ | $g$ | $\#C(\mathbb{F}_{16})$ | interval |
|-----------------|-----|------------------------|----------|
| [4, 12, 7]      | 27  | 156                    | [145 - 176]|
| [3, 11, 6]      | 35  | 187                    |          |
| [5, 11, 5]      | 41  | 220                    | [216 - 249]|

Going through the hyperelliptic curves of genus 4 over $\mathbb{F}_2$ yields one improvement of the table over $\mathbb{F}_4$: the curve with $[N_1, N_2, N_3, N_4] = [6, 10, 6, 26]$ yields a curve of genus 28 with 54 points over $\mathbb{F}_4$; the interval was $[53 - 58]$.

Taking the curve of genus 4 over $\mathbb{F}_2$ with $[N_1, N_2, N_3, N_4] = [8, 8, 8, 16]$ over $\mathbb{F}_2$ and $L$-polynomial $16t^8 + 40t^7 + 56t^6 + 56t^5 + 44t^4 + 28t^3 + 14t^2 + 5t + 1$ gives a degree 13 cover with genus 40 and 104 points, improving the interval $[103 - 141]$.

Similarly, we can list all pairs $[N_1, N_2]$ with $N_i = \#C(\mathbb{F}_{3^i})$ occurring for curves of genus 2 over $\mathbb{F}_3$. This yields the following harvest:

| $N_1, N_2$ | $g$ | $\#C(\mathbb{F}_3)$ | interval |
|------------|-----|---------------------|----------|
| [8, 14]    | 5   | 32                  | [32 - 35]|
| [7, 15]    | 6   | 35                  | [35 - 40]|
| [6, 16]    | 8   | 42                  | [40 - 47]|
| [6, 18]    | 9   | 48                  | [48 - 50]|
| [6, 20]    | 10  | 54                  | [54]     |

Doing the same for curves of genus 2 over $\mathbb{F}_9$ gives the following.
Similarly, the genus 2 curves over $\mathbb{F}_3$ give with $n = 3$:

| $[N_1, N_2]$ | $g$ | $\#C(\mathbb{F}_{27})$ | interval |
|--------------|-----|-------------------|----------|
| $[8, 10]$    | 26  | 200               |          |
| $[7, 13]$    | 29  | 196               |          |
| $[7, 11]$    | 40  | 273               | $[244 − 346]$ |

For the case of non-hyperelliptic curves of genus 3 over $\mathbb{F}_3$ we find the following cases:

| $m$ | $\#C(\mathbb{F}_3)$ | $L(-1)$ | $g$ | $\#\tilde{C}(\mathbb{F}_9)$ | interval |
|-----|---------------------|---------|-----|-----------------------------|----------|
| 687439 | 8 | 12 | 25 | 96 | $[82 − 108]$ |
| 787452 | 8 | 13 | 27 | 104 | $[91 − 114]$ |
| 687411 | 7 | 17 | 35 | 119 | $[116 − 139]$ |
| 787567 | 7 | 18 | 37 | 126 | $[120 − 145]$ |
| 884286 | 7 | 20 | 41 | 140 | $[128 − 158]$ |
| 687541 | 7 | 21 | 43 | 147 | $[120 − 164]$ |

Here we use the following notation for plane curves of degree 3 over $\mathbb{F}_3$. We write the polynomials in $x, y, z$ in pure lexicographic order and use then the coefficients $c_i$ ($i = 1, \ldots, 15$) to associate the natural number $m = \sum_{i=1}^{15} c_i 3^{i-1}$ to the curve. So $x^4$ corresponds to 1 and the Klein curve $x^3y + y^3z + z^3x = 0$ to 196833.
## 5. Tables

Table p=2.

| g \ q | 2  | 4  | 8  | 16 | 32 | 64 | 128 |
|-------|----|----|----|----|----|----|-----|
| 1     | 5  | 9  | 14 | 25 | 44 | 81 | 150 |
| 2     | 6  | 10 | 18 | 33 | 53 | 97 | 172 |
| 3     | 7  | 14 | 24 | 39 | 64 | 113| 192 |
| 4     | 8  | 15 | 25 | 45 | 71 | 129| 215 |
| 5     | 9  | 17 | 29-30 | 49-53 | 83-85 | 132-145 | 227-234 |
| 6     | 10 | 20 | 33-35 | 65 | 86-96 | 161 | 243-258 |
| 7     | 10 | 21-22 | 34-38 | 63-69 | 98-107 | 177 | 262-283 |
| 8     | 11 | 21-24 | 35-42 | 62-75 | 97-118 | 169-193 | 276-302 |
| 9     | 12 | 26 | 45 | 72-81 | 108-128 | 209 | 288-322 |
| 10    | 13 | 27 | 42-49 | 81-87 | 113-139 | 225 | 296-345 |
| 11    | 14 | 26-29 | 48-53 | 80-91 | 120-150 | 201-236 | 294-366 |
| 12    | 14-15 | 29-31 | 49-57 | 88-97 | 129-161 | 257 | 321-388 |
| 13    | 15 | 33 | 56-61 | 97-102 | 129-172 | 225-268 |
| 14    | 15-16 | 32-35 | 65 | 97-107 | 146-183 | 241-284 | 353-437 |
| 15    | 17 | 35-37 | 57-67 | 98-113 | 158-194 | 258-300 | 386-455 |
| 16    | 17-18 | 36-38 | 56-71 | 95-118 | 147-204 | 267-316 |
| 17    | 17-18 | 40 | 63-74 | 112-123 | 154-212 |
| 18    | 18-19 | 41-42 | 65-77 | 113-129 | 161-220 | 281-348 |
| 19    | 20 | 37-43 | 60-80 | 129-134 | 172-228 | 315-364 |
| 20    | 20 | 21 | 40-45 | 76-83 | 127-139 | 177-236 | 342-380 |
| 21    | 21 | 44-47 | 72-86 | 129-145 | 185-243 | 281-396 | 321-412 |
| 22    | 22-23 | 42-48 | 74-89 | 129-150 | 321-412 |
| 23    | 22-23 | 45-50 | 68-92 | 126-155 |
| 24    | 24-25 | 49-52 | 81-95 | 129-161 | 225-267 | 337-444 | 513-653 |
| 25    | 24 | 51-53 | 86-97 | 144-165 | 408-460 |
| 26    | 24-25 | 55 | 82-100 | 150-171 | 425-476 |
| 27    | 24-25 | 52-56 | 96-103 | 156-176 | 213-290 | 416-492 |
| 28    | 25-26 | 54-58 | 97-106 | 145-181 | 257-298 | 513 | 577-745 |
| 29    | 25-27 | 52-60 | 97-109 | 161-186 | 227-305 |
| 30    | 25-27 | 53-61 | 96-112 | 162-191 | 273-313 | 464-535 | 609-784 |
| 31    | 27-28 | 60-63 | 89-115 | 165-196 | 450-547 | 578-807 |
| 32    | 27-29 | 57-65 | 90-118 |
| 33    | 28-29 | 65-66 | 97-121 | 193-207 | 480-570 |
| 34    | 27-30 | 65-68 | 98-124 | 183-212 | 402-581 |
| 35    | 29-31 | 64-69 | 112-127 | 187-217 | 253-351 | 510-593 |
| 36    | 30-31 | 64-71 | 112-130 | 185-222 | 490-604 | 705-917 |
| 37    | 30-32 | 66-72 | 121-132 | 208-227 | 540-616 |
| 38    | 30-33 | 64-74 | 129-135 | 193-233 | 291-375 | 518-627 |
| 39    | 33 | 65-75 | 120-138 | 194-238 | 494-638 |
| 40    | 32-34 | 75-77 | 103-141 | 225-243 | 293-390 | 546-649 |
| 41    | 33-35 | 65-78 | 118-144 | 220-249 | 308-398 | 560-661 |
| 42    | 33-35 | 75-80 | 129-147 | 209-254 | 307-405 | 574-672 |
| 43    | 34-36 | 72-81 | 116-150 | 226-259 | 306-413 | 546-684 |
| 44    | 33-37 | 68-83 | 130-153 | 226-264 | 325-420 | 516-695 |
| 45    | 33-37 | 80-84 | 144-156 | 242-268 | 313-428 | 572-706 |
| 46    | 34-38 | 81-86 | 129-158 | 243-273 | 585-717 |
| 47    | 36-38 | 73-87 | 126-161 | 598-729 |
| 48    | 34-39 | 80-89 | 128-164 | 243-282 | 564-740 |
| 49    | 36-40 | 81-90 | 130-167 | 213-286 | 621-751 |
| 50    | 40 | 91-92 | 130-170 | 255-291 | 588-762 | 913-1207 |
Table \( p=3 \).

| \( g \) | \( q \) | 3     | 9     | 27    | 81    |
|-------|-------|-------|-------|-------|-------|
| 1     | 7     | 16    | 38    | 100   |       |
| 2     | 8     | 20    | 48    | 118   |       |
| 3     | 10    | 28    | 56    | 136   |       |
| 4     | 12    | 30    | 64    | 154   |       |
| 5     | 13    | 32–35 | 72–75 | 160–172 |     |
| 6     | 14    | 35–40 | 76–85 | 190   |       |
| 7     | 16    | 40–43 | 82–95 | 180–208 |     |
| 8     | 17–18 | 42–47 | 92–105 | 226   |       |
| 9     | 19    | 48–50 | 99–113 | 244   |       |
| 10    | 20–21 | 54    | 94–123 | 226–262 |     |
| 11    | 20–22 | 55–58 | 100–133 | 220–280 |     |
| 12    | 22–23 | 56–62 | 109–143 | 298   |       |
| 13    | 24–25 | 64–65 | 136–153 | 256–312 |     |
| 14    | 24–26 | 56–69 | 278–330 |     |       |
| 15    | 28    | 64–73 | 136–170 | 292–348 |     |
| 16    | 27–29 | 74–77 | 144–178 | 370   |       |
| 17    | 25–30 | 74–81 | 128–185 | 288–384 |     |
| 18    | 28–31 | 67–84 | 148–192 | 306–401 |     |
| 19    | 32    | 84–88 | 145–199 |       |       |
| 20    | 30–34 | 70–91 |       |       |       |
| 21    | 32–35 | 88–95 | 163–213 | 352–455 |     |
| 22    | 30–36 | 78–98 |       |       |       |
| 23    | 32–37 | 92–101 |       |       |       |
| 24    | 31–38 | 91–104 | 208–234 |     |       |
| 25    | 36–40 | 96–108 | 196–241 | 392–527 |     |
| 26    | 36–41 | 110–111 | 200–248 | 500–545 |     |
| 27    | 39–42 | 104–114 |       |       |       |
| 28    | 37–43 | 105–117 |       |       |       |
| 29    | 42–44 | 104–120 | 196–269 |     |       |
| 30    | 38–46 | 91–123 | 196–276 | 551–617 |     |
| 31    | 40–47 | 120–127 |       | 460–635 |     |
| 32    | 40–48 | 92–130 |       |       |       |
| 33    | 46–49 | 128–133 | 220–297 | 570–671 |     |
| 34    | 46–50 | 111–136 | 594–689 |     |       |
| 35    | 47–51 | 119–139 | 612–707 |     |       |
| 36    | 48–52 | 118–142 | 244–318 | 730   |       |
| 37    | 52–54 | 126–145 | 236–325 | 648–742 |     |
| 38    | 105–149 | 629–755 |     |       |       |
| 39    | 48–56 | 140–152 | 271–340 | 646–708 |     |
| 40    | 56–57 | 118–155 | 273–346 | 663–781 |     |
| 41    | 50–58 | 140–158 |       | 680–795 |     |
| 42    | 52–59 | 122–161 | 280–360 | 697–808 |     |
| 43    | 56–60 | 147–164 | 672–821 |     |       |
| 44    | 47–61 | 119–167 | 278–374 |     |       |
| 45    | 54–62 | 136–170 | 704–847 |     |       |
| 46    | 55–63 | 162–173 | 720–859 |     |       |
| 47    | 54–65 | 154–177 | 299–395 | 690–872 |     |
| 48    | 55–66 | 163–180 | 325–402 | 752–885 |     |
| 49    | 64–67 | 168–183 | 316–409 | 768–898 |     |
| 50    | 63–68 | 182–186 | 312–416 | 784–911 |     |
For exhaustive references we refer to the bibliography of the tables, [10]. For general background on curves over finite fields we refer to the book by Stichtenoth [9] and for general background on curves over finite fields with many points to Serre’s notes [8] and for an overview of the methods of Niederreiter and Xing to [6].

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