Existence of solutions for the surface electromigration equation

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Abstract
We consider a model that describes electromigration in nanoconductors known as surface electromigration (SEM) equation. Our purpose here is to establish local well-posedness for the associated initial value problem in Sobolev spaces from two different points of view. In the first one, we study the pure Cauchy problem and establish local well-posedness in $H^s(\mathbb{R}^2), s > 1/2$. In the second one, we study the Cauchy problem on the background of a Korteweg–de Vries solitary traveling wave in a less regular space. To obtain our results we make use of the smoothing properties of solutions for the linear problem corresponding to the Zakharov–Kuznetsov equation for the latter problem. For the former problem we use bilinear estimates in Fourier restriction spaces introduced in [24].

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1. Introduction

In this article we consider the initial value problem (IVP) for the surface electromigration (SEM) equation,

\begin{align}
\begin{cases}
  u_t + \partial_x \Delta u + \frac{1}{2} \left( uu_x + u_x \phi_x + u_y \phi_y \right) = 0, & (x, y) \in \mathbb{R}^2, \ t \in \mathbb{R}, \\
  \Delta \phi = u_x.
\end{cases}
\end{align}

(1.1)
Here \( u = u(x, y, t) \) and \( \phi = \phi(x, y, t) \) are real-valued functions, \( \Delta \) represents the two-dimensional Laplacian operator and subscripts stand for partial derivatives. The above system was derived by Bradley [2, 3] to describe electromigration in nanoconductors and it couples the Zakharov–Kuznetsov (ZK) equation

\[
v_t + \partial_x \Delta v + vv_x = 0, \quad (x, y) \in \mathbb{R}^2, \quad t \in \mathbb{R},
\]

(1.2)

with a potential equation.

The function \( u \) represents the surface displacement and \( \phi \) is the electrostatic potential on the surface conductor. The physical situation is as follows: when an unidirectional electrical current passes through a piece of solid metal, collisions between the conduction electrons and the metal atoms at the surface lead to drift of these atoms. This is known as SEM and it may cause a solid metal surface to move and deform producing undesirable surface instabilities. The equation for SEM differs from ZK equation through the coupling to the electrical potential \( \phi \).

The ZK equation (1.2) describes the propagation of nonlinear ion-acoustic waves in a magnetized plasma. It was formally derived by Zakharov and Kuznetsov in [30]. Recently, a rigorous derivation of the ZK was given in [18] as a long wave limit of the Euler–Poisson system. It may also be viewed as a two-dimensional version of the Korteweg–de Vries (KdV) equation,

\[
u_t + u_{xxx} + uu_x = 0,
\]

(1.3)

which accounts weak lateral dispersion given by the term \( v_{xyy} \). The phenomena of the moving free surface of a metal film in response to the electrical current flowing through the bulk of the film is reminiscent of the way the flow in the bulk of a fluid affects the motion of its surface. This analogy, however, does not match since the boundary conditions are very different in the two problems. Bradley [3], Schimschak et al [29] and Gungor et al [10] considered the propagation of solitons over a free surface of a current-carrying metal film. Jorge et al in [13] investigated the evolution of lump solutions for the ZK and SEM equations. They derived approximate equations including the important effect of the radiation shed by the lumps as they evolve and studied the evolution of the lump disturbances asymptotically and numerically. Since SEM may cause electrical failure of a current carrying metal line it is interesting to have a better understanding of the properties of the equation because it will be an important factor limiting the reliability of integrated circuits. At this point we shall recall that the ZK equation is not integrable by the inverse scattering transform method. It was found that the solitary-wave solutions of the ZK equation are inelastic (see [12]).

**Remark 1.1.** From the mathematical point of view, (1.1) and (1.2) may be viewed as a two-dimensional generalization of (1.3). Indeed, it is clear if \( v \) does not depend on the transverse variable \( y \), then (1.2) reduces immediately to (1.3). Also, if \( u \) and \( \phi \) do not depend on \( y \) in (1.1) and \( v \) and \( \phi \) have a suitable decay to zero as \( x \to \pm \infty \) then it follows from the second equation in (1.1) that \( \phi_x = u \). Substituting this in the first equation, one sees that it also reduces to (1.3).

The notion of local well-posedness through this paper includes the properties of existence, uniqueness, persistence and continuous dependence upon the initial data.

The IVP associated with the ZK equation (1.2) has been widely studied in recent years. Indeed, the first result in this direction is due to Faminskii [4]; he proved that (1.2) is locally-in-time well-posed in the usual \( L^2 \)-based Sobolev spaces \( H^s(\mathbb{R}^2) \), \( s \geq 1 \). In [19], Linares and Pastor established that the IVP is locally well-posed for \( s > 3/4 \). The Sobolev index was pushed down independently by Grunrock and Herr ([7]) and Molinet and Pilod ([24]), where the authors showed the local well-posedness for \( s > 1/2 \). Recently, Kinoshita in [16] established a sharp local well-posedness for data in \( H^s(\mathbb{R}^2) \), \( s > -1/4 \). We shall also mention that the study of
IVP associated to generalizations of the ZK equation has gained a lot of attention recently. We refer the reader to [1, 5, 8, 9, 11, 17, 20, 23, 27, 28] and references therein.

In this work we focus on the study of the IVP associated with (1.1) from two different points of view. In the first one, we study the pure IVP; thus we couple (1.1) with the initial condition

\[ u(x, y, 0) = u_0(x, y) \]

and study the IVP in the Sobolev spaces \( H^s(\mathbb{R}^2) \). Our goal will be to establish a local well-posedness theory in \( H^s(\mathbb{R}^2) \), \( s > 1/2 \). Since (1.1) couples a ZK equation with a potential \( \phi \) we will exploit the properties of solutions of the ZK equation to reach our purpose.

The second point of view is closely related with the study of transverse instability of one-dimensional solitons. To put the set up forward, note that (1.1) has solitary-wave solutions given by

\[ u(x, y, t) = \varphi(x - \omega t) = 3 \omega \sech^2 \left( \frac{\sqrt{\omega}}{2} (x - \omega t) \right), \quad \omega > 0, \quad (1.4) \]

\[ \phi(x, y, t) = \psi(x - \omega t) = 6\sqrt{\omega} \tanh \left( \frac{\sqrt{\omega}}{2} (x - \omega t) \right), \quad \omega > 0. \quad (1.5) \]

Remark 1.2. Note that \( \varphi \) is exactly the solitary-wave solution of the KdV equation (1.3) and \( \psi_t = \varphi \).

To investigate the transverse instability of the traveling waves (1.4) and (1.5) under localized perturbations, one needs to study the evolution of \( v := u - \varphi \) and \( w := \phi - \psi \). Substituting this transformations into (1.1), we obtain the system

\[
\begin{aligned}
    v_t + \partial_x \Delta v + \frac{1}{2} (v v_x + v_x w_x + v_y w_y) + \frac{1}{2} (\varphi_x v + \varphi v_x) + \frac{1}{2} (\psi_x v + \varphi_x w_x) &= 0, \\
    \Delta w &= v_x.
\end{aligned}
\]

Thus, we are led to study (1.6) coupled with an initial condition \( v(x, y, 0) = v_0(x, y) \) in \( H^s(\mathbb{R}^2) \). This step is then necessary to understand the dynamics of the SEM equation on the background of a nonlocalized solitary traveling wave. It should be noted that in [2] the author established that the traveling waves (1.4) and (1.5) are unstable under sinusoidal perturbation with wave-vector perpendicular to the direction of propagation.

Next we describe the strategy to solve our problems. First we define the operator \( \mathcal{L} := (\Delta)^{-1} \partial_x \) and rewrite systems (1.1) and (1.6) as a single equation. Thus we will consider the following equivalent problems:

\[
\begin{aligned}
    u_t + \partial_x \Delta u + \frac{1}{2} \left( uu_x + u_x \partial_x \mathcal{L}(u) + u_y \partial_y \mathcal{L}(u) \right) &= 0, \\
    u(x, y, 0) &= u_0(x, y)
\end{aligned}
\]

and

\[
\begin{aligned}
    u_t + \partial_x \Delta u + \frac{1}{2} \left( uu_x + u_x \partial_x \mathcal{L}(u) + u_y \partial_y \mathcal{L}(u) \right) + \frac{1}{2} (\varphi_x u + 2 \varphi u_x) + \frac{1}{2} \varphi_x \partial_x \mathcal{L}(u) &= 0, \\
    u(x, y, 0) &= u_0(x, y).
\end{aligned}
\]
In order to solve the IVP (1.7) (a similar approach will take place for (1.8)) we will use its integral equivalent equation, i.e.,

$$u(t) = U(t)u_0 - \frac{1}{2} \int_{0}^{t} U(t - t') \left( u\partial_x u + u\partial_y \mathcal{L}(u) + u\partial_y \mathcal{L}(u) \right)(t') \, dt', \quad (1.9)$$

where $U(t) = \exp(\tau(\partial_x^2 + \partial_y \partial_x \partial_y))$ is the unitary group associated to the linear problem

$$\begin{aligned}
\left\{ u_t + \partial_x \Delta u &= 0, \quad (x, y) \in \mathbb{R}^2, \quad t \in \mathbb{R}, \\
 u(x, y, 0) &= u_0(x, y). \right. \quad (1.10)
\end{aligned}$$

In particular, it is not difficult to see that $\widehat{U(t)u_0(\xi, \mu)} = e^{i\mu(\xi^2 + \xi \mu \partial_x \partial_y)}\hat{u}_0(\xi, \mu)$, where the hat represents the Fourier transform.

Our local well-posedness result for (1.7) reads as follows (see notation below).

**Theorem 1.3.** Let $u_0 \in H^s(\mathbb{R}^2)$, $s > 1/2$. There exist $T = T(\|u_0\|_{H^s})$ and a unique solution $u$ of the IVP (1.7) satisfying

$$u \in C([0, T] : H^s(\mathbb{R}^2)) \cap X^s_T.$$

Moreover, for any $T' \in (0, T)$, the data-to-solution map, $u_0 \mapsto u(t)$, defined from a neighborhood of $u_0 \in H^s(\mathbb{R}^2)$ into the class $C([0, T'] : H^s(\mathbb{R}^2)) \cap X^s_{T'}$ is smooth.

The idea to prove theorem 1.3 is to use the dispersive properties of the equation and the contraction mapping principle to obtain a fixed point associated to the integral equation (1.9). The main tool to succeed in our purpose is a bilinear estimate in Bourgain spaces introduced in [24].

**Remark 1.4.** As far as we know the only conserved quantities satisfied by solutions of (1.1) are

$$I_1(u) = \int_{\mathbb{R}^2} u(x, y, t) \, dx \, dy \quad \text{and} \quad I_2(u) = \int_{\mathbb{R}^2} u^2(x, y, t) \, dx \, dy.$$

Thus the lack of a conservation law in high-order Sobolev spaces does not allow us to extend the solutions obtained in theorem 1.3 globally in time. Since the flow associated to the SEM equation is conserved in $L^2(\mathbb{R}^2)$ one expects to establish a local theory in this space. Moreover, a scaling argument suggests to obtain local well-posedness for $s > -1$. We do not know whether the method developed by Kinoshita in [16] may apply to lower the regularity in our case. As we mentioned before, his result shows sharp local well-posedness for the IVP associated to the ZK equation for initial data in $H^s(\mathbb{R}^2)$ for $s > -1/4$. In Kinoshita’s analysis there are some key estimates that depend on the structure of his bilinear estimate. It does not seem obvious to establish similar estimates for the nonlinearities in our case other than for the $\partial_t u \partial_x u$ term. To obtain a $L^2$ local theory for the IVP (1.7) is indeed a challenging open problem.

Regarding the IVP (1.8) our main theorem reads as follows.

**Theorem 1.5.** Let $\varphi(x - ct)$ be the soliton of the KdV equation in (1.4). Let $u_0 \in H^1(\mathbb{R}^2)$. There exist $T = T(\|u_0\|_{H^1})$ and a unique solution $u$ of the IVP (1.8) satisfying

$$u \in C([0, T] : H^1(\mathbb{R}^2)), \quad (1.11)$$

$$\|\partial_t^2 u\|_{L^2_T L^2_x} < \infty, \quad (1.12)$$
\[ \|\nabla u\|_{L^3_tL^\infty_x} < \infty, \quad (1.13) \]

and

\[ \|u\|_{L^2_tL^\infty_x} < \infty. \quad (1.14) \]

Moreover, for any \( T' \in (0, T) \) the map \( u_0 \mapsto u(t) \) defined in a neighborhood of \( u_0 \in H^1(\mathbb{R}^2) \) into the class defined by (1.11)–(1.14) is smooth.

Remark 1.6. It will be clear from the proof of theorem 1.5 below that we do not use the particular explicit form of the traveling waves in (1.4) and (1.5). Instead, we only use the properties that \( \varphi(x - \omega t) \) belongs to \( L^2_tL^\infty_x \), and \( \varphi(x - \omega t) \) and \( \varphi_x(x - \omega t) \) are uniformly bounded in \( x, t \). In particular theorem 1.5 holds if \( \varphi \) is any traveling-wave solution of the KdV equation with these two properties. For instance, \( \varphi \) may be any \( N \)-soliton of the KdV equation.

Remark 1.7. We point out that any \( N \)-soliton of the KdV equation is also a solution of both the ZK equation (1.2) and the Kadomtsev–Petviashvili equation,

\[ (u_t + u_{xxx} + uu_x)_x \pm u_{xy} = 0. \]

Similar results as the one in theorem 1.5 for these two equations have appeared in [21, 25, 26]. The local results were also extended to global ones due to the conservation of the energy at the \( H^1(\mathbb{R}^2) \) level.

Remark 1.8. Using the symmetrization introduced by Grünrock and Herr in [7] it may be possible to establish local well-posedness in \( H^s(\mathbb{R}^2), s > 3/4 \), (see for instance [22]). We choose to follow the approach presented here for two reasons. First, the aforementioned result is not sharp. Secondly, after applying the change of variables in [7] the nonlinear terms in the new equation increases and the estimating process turns out to be tedious.

This paper is organized as follows. Below we introduce the basic notation and give the linear estimates we need. Section 3 is devoted to prove theorem 1.3. The proof relies on a fixed point argument in the Bourgain spaces. We focus on proving the main bilinear estimate necessary to establish theorem 1.3. In the last section, section 4, we give the proof of theorem 1.5. The tools are based on the Strichartz estimate, Kato’s smoothing effect, and the maximal function presented in lemma 2.2.

2. Notation and linear estimates

Here we will introduce the main notation used throughout the paper and give some preliminary linear estimates. Besides the standard notation in partial differential equations, we use \( c \) to denote various constants that may vary line by line. If \( A \) and \( B \) are two positive constants, the notation \( A \lesssim B \) means that there is \( c > 0 \) such that \( A \leq cB \). Also, \( A \wedge B \colon= \min\{A, B\} \) and \( A \vee B \colon= \max\{A, B\} \). Given any \( r \in \mathbb{R} \) we write \( r^+ \) for \( r + \varepsilon \), where \( \varepsilon > 0 \) is a sufficiently small number. The mixed space-time norm is defined as (if \( 1 \leq p, q, r < \infty \))

\[ \|f\|_{L^p_tL^q_xL^r_T} = \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \left( \int_0^T |f(x, y, t)|^r \, dt \right)^{q/r} \, dy \right)^{p/q} \, dx \right)^{1/p}. \]
with standard modifications if either $p = \infty$, $q = \infty$, or $r = \infty$. Similar spaces appear if one interchanges the order of integration. If two indices are equal we put them into the same space; for instance, if $q = r$ we denote the norm \( \| \cdot \|_{L^q_t L^r_x} \) by \( \| \cdot \|_{L^q_t L^r_x} \). For short, we set \[
abla u \|_{L^q_t L^r_x} := \| \partial_x u \|_{L^q_t L^r_x} + \| \partial_t u \|_{L^q_t L^r_x} + \| \partial^2_t u \|_{L^q_t L^r_x}
\]
and \[
abla u \|_{L^q_t L^r_x} := \| \partial_x u \|_{L^q_t L^r_x} + \| \partial_t u \|_{L^q_t L^r_x}.
\]

The space-time Fourier transform of \( u = u(x, y, t) \) will be denoted by \( \mathcal{F}u = \mathcal{F}u(\xi, \mu, \tau) \) or \( \hat{u}(\xi, \mu, \tau) \), whereas the Fourier transform in space will be denoted by \( \mathcal{F}_{xy}u \). As usual, \( \mathcal{F}^{-1} \) and \( \mathcal{F}_{xy}^{-1} \) represent the respective inverse Fourier transforms.

Let \( \eta \in \mathcal{C}_0^\infty(\mathbb{R}) \) be such that \( 0 \leq \eta \leq 1 \), \( \eta \equiv 1 \) on the interval \([-5/4, 5/4]\) and \( \text{supp}(\eta) \subset [-8/5, 8/5] \). Now we set \( \zeta(\xi) = \eta(\xi) - \eta(2\xi) \) and for \( k \in \mathbb{N}^+ \{1, 2, \ldots \} \) we define
\[
\zeta_k(\xi, \mu) := \zeta(2^{-k}|(\xi, \mu)|) \quad \text{and} \quad \rho_k(\xi, \mu, \tau) := \zeta(2^{-k}(\tau - (\xi^3 + \xi\mu^2))).
\]
For convenience we set \( \zeta_0(\xi, \mu) = \eta(|(\xi, \mu)|) \) and \( \rho_0(\xi, \mu, \tau) = \eta(\tau - (\xi^3 + \xi\mu^2)) \). It is easy to see that
\[
\sum_N \zeta_N(\xi, \mu) = 1.
\]

Here and throughout the paper, any summation over the variables \( N, L, K, M \) are suppose to be dyadic with \( N, K, L, M \geq 1 \). The Littlewood–Paley multipliers on frequencies and modulations are defined as
\[
P_Nu := \mathcal{F}_{xy}^{-1} \left( \zeta_N(\mathcal{F}_{xy}u) \right) \quad \text{and} \quad Q_Lu := \mathcal{F}^{-1} \left( \rho_L(\mathcal{F}u) \right).
\]

Given \( s, b \in \mathbb{R} \), the Bourgain spaces \( X^{s,b} \) are defined as the completion of the Schwartz space \( \mathcal{S}(\mathbb{R}^3) \) under the norm
\[
\| u \|_{X^{s,b}} := \left( \int_{\mathbb{R}} \left( \tau - (\xi^3 + \xi\mu^2) \right)^{2b} \left( \int \| \hat{u}(\xi, \mu, \tau) \|^2 \, d\xi \, d\mu \, d\tau \right)^2 \, d\tau \right)^{\frac{1}{2}},
\]
where \( \langle x \rangle = 1 + |x| \). For \( T > 0 \), if \( u \) is defined on \( \mathbb{R}^2 \times [0, T] \) we set
\[
\| u \|_{X^{s,b}} := \inf \left\{ \| \tilde{u} \|_{X^{s,b}} \mid \tilde{u} : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{C}, \quad \tilde{u}|_{\mathbb{R}^2 \times [0,T]} = u \right\}.
\]

Next we state useful linear estimates to prove theorems 1.3 and 1.5. The first result concerns linear estimates in Bourgain spaces.

**Lemma 2.1.** Let \( s \in \mathbb{R} \) and \( b > 1/2 \). Then,
\[
\| \eta(t)U(t)f \|_{X^{s,b}} \lesssim \| f \|_{H^s} \quad (\text{homogeneous estimate})
\]
and
\[
\left\| \eta(t) \int_0^t U(t-t')g(t') \, dt' \right\|_{X^{s+b-\frac{1}{2}+s}} \lesssim \| g \|_{X^{s+b-\frac{1}{2}}}, \quad (\text{non-homogeneous estimate})
\]
where \(0 < \delta < 1/2\). In addition, given \(T > 0\) and \(-1/2 < b' \leq b < 1/2\) we have
\[
\|u\|_{X^{s,b}_T} \lesssim T^{b-b'} \|u\|_{X^{s,b'}_T}.
\]

**Proof.** These estimates are classical by now. See, for instance, [6].

The second result provides the linear estimates which are sufficient to prove theorem 1.5.

**Lemma 2.2.** Let \(u_0 \in L^2(\mathbb{R}^2)\). Then
\[
\|U(t)u_0\|_{L^2_{x,y}T} \lesssim \|u_0\|_{L^2_{x,y}} \quad \text{(Strichartz’s estimate)}
\]
and
\[
\|\nabla U(t)u_0\|_{L^2_{x,y}T} \lesssim \|u_0\|_{L^2_{x,y}} \quad \text{(Kato’s smoothing)}
\]
In addition, if \(u_0 \in H^s(\mathbb{R}^2), s > 3/4,\) and \(T > 0\) is fixed then there exists a constant \(c(s, T)\) depending only on \(s\) and \(T\) such that
\[
\|U(t)u_0\|_{L^2_{x,y}T} \leq c(s, T) \|u_0\|_{H^s}. \quad \text{(Maximal function)}
\]

**Proof.** These estimates were proved by Faminskii [4] inspired in those ones obtained for the KdV equation by Kenig et al [14].

**3. Proof of theorem 1.3**

This section is devoted to prove theorem 1.3. As we observed in the introduction we will use the equivalent integral equation (1.9) and the contraction mapping principal to obtain our result. The next bilinear estimates are the main tools in our analysis regarding the proof of theorem 1.3.

**Proposition 3.1.** Let \(s > 1/2\). There exists a small \(\delta > 0\) such that
\[
\|u_1 v\|_{X^{s,-1/2+\delta}} \lesssim \|u\|_{X^{s,1/2+\delta}} \|v\|_{X^{s,1/2+\delta}}
\]
and
\[
\|u_1 v\|_{X^{s,-1/2+\delta}} \lesssim \|u\|_{X^{s,1/2+\delta}} \|v\|_{X^{s,1/2+\delta}}.
\]

Before proving proposition 3.1 we remind some relevant estimates in our arguments.

**Lemma 3.2.** The following estimate holds
\[
\|u\|_{L^4_{x,y}} \lesssim \|u\|_{X^{s,1/2}_r}.
\]

**Proof.** See [24, corollary 3.2].

**Lemma 3.3.** Let \(N_1, N_2, L_1, L_2\) be dyadic numbers. Then,
\[
\|(P_{N_1}Q_{L_1}u)(P_{N_2}Q_{L_2}v)\|_{L^2} \lesssim (N_1 \wedge N_2)(L_1 \wedge L_2)^{1/2} \|P_{N_1}Q_{L_1}u\|_{L^2} \|P_{N_2}Q_{L_2}v\|_{L^2}.
\]
If, in addition, \( N_2 \geq 4N_1 \) or \( N_1 \geq 4N_2 \) then
\[
\|(P_{N_1} Q_{L_1} u)(P_{N_2} Q_{L_2} v)\|_{L^2} \lesssim \begin{cases} (N_1 \wedge N_2)^{\frac{1}{2}} (L_1 \vee L_2)^{\frac{1}{2}} (L_1 \wedge L_2)^{\frac{1}{2}} \|P_{N_1} Q_{L_1} u\|_{L^2} \|P_{N_2} Q_{L_2} v\|_{L^2} & \text{if} \ N_1 \leq N_2, \\
[N_1 \vee N_2]^{\frac{1}{2}} (L_1 \vee L_2)^{\frac{1}{2}} (L_1 \wedge L_2)^{\frac{1}{2}} \|P_{N_1} Q_{L_1} u\|_{L^2} \|P_{N_2} Q_{L_2} v\|_{L^2} & \text{if} \ N_2 \leq N_1. 
\end{cases}
\] (3.4)

**Proof.** See [24, proposition 3.6].

**Proof of proposition 3.1.** We will prove only (3.1). It will be clear from the proof itself that (3.2) may be established in a similar fashion. The proof follows the same arguments as the ones in [24, proposition 4.1]. So, we only give the main steps and follow closely the notation in [24]. It suffices, by duality, to show the estimate
\[
I \lesssim \|u\|_{L^{2}_{xy}} \|v\|_{L^{2}_{xy}} \|w\|_{L^{2}_{xy}},
\]
where
\[
I = \int_{\mathbb{R}^6} \Gamma^{\xi,\mu,\tau}_{\xi',\mu',\tau'} \hat{w}(\xi, \mu, \tau) \hat{u}(\xi_1, \mu_1, \tau_1) \hat{v}(\xi_2, \mu_2, \tau_2) d\nu,
\]
the functions \( \hat{w}, \hat{u}, \hat{v} \) are nonnegative, and
\[
\Gamma^{\xi,\mu,\tau}_{\xi',\mu',\tau'} = |\xi| \left( |(\xi, \mu)| \langle \sigma \rangle^{\frac{1}{2}+2\delta} \langle |(\xi_1, \mu_1)| \rangle^{\frac{1}{2}} \langle |(\xi_2, \mu_2)| \rangle^{\frac{1}{2}} \langle (\sigma_1)^{\frac{1}{2}} \delta^{\frac{1}{2}} \langle (\sigma_2)^{\frac{1}{2}} \rangle^{\frac{1}{2}} \right).
\]

The main difference when compared our situation and that of proposition 4.1 in [24] is that we have \( |\xi| \) in the definition of \( \Gamma^{\xi,\mu,\tau}_{\xi',\mu',\tau'} \) instead of \( |\xi| \). By using dyadic decomposition, one may rewrite \( I \) as
\[
I = \sum_{N \leq N_1 \leq N_2} I_{N,N_1,N_2},
\]
with
\[
I_{N,N_1,N_2} = \int_{\mathbb{R}^6} \Gamma^{\xi,\mu,\tau}_{\xi',\mu',\tau'} \hat{P}_{N} \hat{w}(\xi, \mu, \tau) \hat{P}_{N_1} \hat{u}(\xi_1, \mu_1, \tau_1) \hat{P}_{N_2} \hat{v}(\xi_2, \mu_2, \tau_2) d\nu.
\]

Using that \( (\xi, \mu) = (\xi_1, \mu_1) + (\xi_2, \mu_2) \) one decomposes the sum into five cases.

(a) Low \times Low \rightarrow Low interactions:
\[
I_{LL-LL} = \sum_{N \geq 2N_1 \leq 4N_2 \leq 4} I_{N,N_1,N_2}.
\]

(b) Low \times High \rightarrow High interactions:
\[
I_{LH-LH} = \sum_{4 \leq N_2 \leq N_1 \leq 2 \text{ or } N_1 \geq 4N_2} I_{N,N_1,N_2}.
\]
(c) High × Low → High interactions:

\[ I_{\text{HL-HL}} = \sum_{4 \leq N_1, N_2 \leq N_1/4} I_{N_1, N_2}. \]

(d) High × High → Low interactions:

\[ I_{\text{HH-LH}} = \sum_{4 \leq N_1, N_2 \leq N_1/4} I_{N_1, N_2}. \]

(e) High × High → High interactions:

\[ I_{\text{HH-HL}} = \sum_{N \sim N_1 - N_2} I_{N, N_1, N_2}. \]

With the above decomposition we see that

\[ I = I_{\text{LL-LH}} + I_{\text{HH-HL}} + I_{\text{HH-LH}} + I_{\text{HH-HL}} + I_{\text{HH-HH}}, \]

and our task is then to show that each term on the right-hand side is bounded by \( \|u\|_{L^2} \langle v \rangle_{L^2} \|w\|_{L^2} \).

**Estimate for** \( I_{\text{LL-LH}} \). Since all frequencies are bounded this is the simplest case. Indeed, by Parseval’s identity, Hölder’s inequality and lemma 3.2,

\[
I_{N_1, N_2} \lesssim \left\| \left( \frac{P_{N_1} u}{(\sigma_1)^{\frac{1}{2} + \delta}} \right)^{\vee} \right\|_{L^4} \left\| \left( \frac{P_{N_2} v}{(\sigma_2)^{\frac{1}{2} + \delta}} \right)^{\vee} \right\|_{L^4} \|P_{N_2} w\|_{L^2},
\]

giving

\[ I_{\text{LL-LH}} \lesssim \|u\|_{L^2} \langle v \rangle_{L^2} \|w\|_{L^2}. \]

**Estimate for** \( I_{\text{HH-HL}} \). By using dyadic decomposition on the modulation variables one writes

\[ I_{N_1, N_2} = \sum_{L_1, L_2} I_{N_1, N_2}^{L_1, L_2}, \]

where

\[
I_{N_1, N_2}^{L_1, L_2} = \int_{B^6} \Gamma_{\xi, \mu, \tau} \langle P_{N_1} Q_{L_1} w(\xi, \mu, \tau) P_{N_2} Q_{L_2} u(\nu, \mu, \tau) \rangle_{L^2} \|P_{N_2} Q_{L_2} v\|_{L^2} \, d\nu.
\]

By using Cauchy–Schwarz in \( (\xi, \mu, \tau) \) and (3.4) we obtain

\[
I_{N_1, N_2}^{L_1, L_2} \lesssim N_1 L_{1}^{\frac{1}{4} + 2\delta} N_2 L_{2}^{\frac{1}{4} + \delta} N_1^{-\delta} N_2^{-\delta} \|P_{N_1} Q_{L_1} u\|_{L^2} \|P_{N_2} Q_{L_2} v\|_{L^2} \|P_{N_2} Q_{L_2} w\|_{L^2},
\]

\[
\lesssim N_1 L_{1}^{\frac{1}{4} + 2\delta} N_2 L_{2}^{\frac{1}{4} + \delta} N_1^{-\delta} N_2^{-\delta} \frac{N_2^\frac{1}{4}}{N_2^\frac{1}{2}} L_{1}^\frac{1}{4} L_{2}^\frac{1}{4}
\times \|P_{N_1} Q_{L_1} u\|_{L^2} \|P_{N_2} Q_{L_2} v\|_{L^2} \|P_{N_2} Q_{L_2} w\|_{L^2},
\]

\[
\lesssim L_{1}^{\frac{1}{4} + 2\delta} L_{2}^{-\delta} N_1^{-\delta} \|P_{N_1} Q_{L_1} u\|_{L^2} \|P_{N_2} Q_{L_2} v\|_{L^2} \|P_{N_2} Q_{L_2} w\|_{L^2}.
\]

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where in the last inequality we used that \( N \sim N_2 \) and \( \frac{1}{N_2^2} \lesssim \frac{1}{N_1} \). Consequently,

\[
I_{LH-41} \lesssim \sum_{L_1, L_2} \left( L^{\frac{3}{2}+2\delta} L_1^{-\delta} L_2^{-\delta} \right)
\]

\[
\left( \sum_{N_1 \leq N_2<4N_1} N_1^{-2} \right) \| P_{N_1} Q_{L_1} u \|_{L_2} \| P_{N_2} Q_{L_2} v \|_{L_2} \| P_N Q_L w \|_{L_2}
\]

\[
\lesssim \| u \|_{L_2} \sum_{N_2} \| P_{N_2} v \|_{L_2} \| P_N w \|_{L_2}
\]

\[
\lesssim \| u \|_{L_2} \| v \|_{L_2} \| w \|_{L_2} .
\]

**Estimate for \( I_{LH-41} \).** This case is similar to the last one, because now we have \( N \sim N_1 \) and \( N_2 \leq N_1 \), so that

\[
I_{N_1, N_2}^{L_1, L_2} \lesssim L^{-\frac{3}{2}+2\delta} L_1^{-\delta} L_2^{-\delta} N_2 \left( L_1 \right) \| P_{N_1} Q_{L_1} u \|_{L_2} \| P_{N_2} Q_{L_2} v \|_{L_2} \| P_N Q_L w \|_{L_2} .
\]

**Estimate for \( I_{HH-4} \).** We set \( f(\xi, \mu, \tau) = f(-\xi, -\mu, -\tau) \). In addition, interpolation between (3.3) and (3.4) (see [24, equation (4.20)]) gives, for \( \theta \in [0, 1] \),

\[
\| (P_{N_1} Q_{L_1} u)(P_N Q_L w) \|_{L_2} \lesssim \frac{(N_1 \land N_2)^{1+\theta}}{(N_1 \lor N_2)^{1-\theta}} (L_1 \lor L) \left( L_1 \land L \right)^{\frac{1}{2}+\theta} \| P_{N_1} Q_{L_1} u \|_{L_2} \| P_{N_2} Q_{L_2} v \|_{L_2} .
\]

As in the last two cases we obtain

\[
I_{N_1, N_2}^{L_1, L_2} \lesssim L^{-\frac{3}{2}+2\delta} L_1^{-\delta} L_2^{-\delta} N_2 \left( L_1 \right) \| P_{N_1} Q_{L_1} u \|_{L_2} \| P_{N_2} Q_{L_2} v \|_{L_2} \| P_N Q_L w \|_{L_2} .
\]

Assuming without loss of generality that \( L = L_1 \lor L \) and using that \( N \leq N_1 \) and \( N_2 \sim N_1 \) we deduce

\[
I_{N_1, N_2}^{L_1, L_2} \lesssim L^{\frac{3}{2}+2\delta} L_1^{-\delta} L_2^{-\delta} \left( L_1 \right) \| P_{N_1} Q_{L_1} u \|_{L_2} \| P_N Q_L w \|_{L_2} .
\]

Now we choose \( \theta \in (0, 1) \) and \( \delta > 0 \) small enough such that \( 0 < 2\theta < s - \frac{1}{2} \) and \( \delta < \frac{2}{3} \). Consequently, using Cauchy–Schwarz in \( N \) and \( N_1 \),

\[
I_{HH-4} \lesssim \sum_{N \leq N_1 < 4N_\sim N_1} N_1^{-\left( s - \frac{1}{2} - 2\theta \right)} \| P_{N_1} Q_{L_1} u \|_{L_2} \| P_N Q_L w \|_{L_2} \| P_{N_2} Q_{L_2} v \|_{L_2}
\]

\[
\lesssim \| u \|_{L_2} \| v \|_{L_2} \| w \|_{L_2} .
\]
Estimate for $I_{HH-H1}$. Note in this case we have $N \sim N_1 \sim N_2$. So it does not matter if in the definition of $\Gamma^{N_1,N_2}_{\xi_1,\xi_2}$ appears $|\xi_1|$ or $|\xi_2|$ the estimate is the same, because we can always replace $N_1$ by $N$. Thus this estimate is exactly the same one given in [24, p 358], that is,

$$I_{HH-H1} \lesssim \|u\|_{L^2} \|v\|_{L^2} \|w\|_{L^2}.$$ 

Collecting all estimates above, the proof of the proposition is completed.

With proposition 3.1 in hand we are able to apply the fixed point theorem in a closed ball of $X^{s-\frac{1}{2}+\delta}_T$ in order to prove theorem 1.3. The proof is quite standard so we omit the details. We only point out that after localization (in time) of the right-hand side of (1.9) and using lemma 2.1 we only need the estimate

$$\|u \partial_x u + u, \partial_x L(u) + u_2 \partial_x L(u)\|_{X^{s-\frac{1}{2}+\delta}} \lesssim \|u\|^2_{X^{s-\frac{1}{2}+\delta}}.$$ (3.5)

To see that (3.5) holds we note that

$$m_1(\xi, \eta) = \frac{\xi^2}{\xi^2 + \eta^2} \quad \text{and} \quad m_2(\xi, \eta) = \frac{\xi \eta}{\xi^2 + \eta^2}$$ (3.6)

are Fourier multipliers in $X^{s-\frac{1}{2}+\delta}, s \geq 0$. In particular, the operators $\partial_x L$ and $\partial_y L$ are bounded in $X^{s-\frac{1}{2}+\delta}$, that is,

$$\|\partial_x L(u)\|_{X^{s-\frac{1}{2}+\delta}} \lesssim \|u\|_{X^{s-\frac{1}{2}+\delta}}$$

and

$$\|\partial_y L(u)\|_{X^{s-\frac{1}{2}+\delta}} \lesssim \|u\|_{X^{s-\frac{1}{2}+\delta}}.$$ 

Consequently, after applying proposition 3.1, we get

$$\|u \partial_x u + u_2 \partial_x L(u) + u_1 \partial_x L(u)\|_{X^{s-\frac{1}{2}+\delta}} \lesssim \|u\|^2_{X^{s-\frac{1}{2}+\delta}} + \|u\|_{X^{s-\frac{1}{2}+\delta}} \times (\|\partial_x L(u)\|_{X^{s-\frac{1}{2}+\delta}} + \|\partial_x L(u)\|_{X^{s-\frac{1}{2}+\delta}}) \lesssim \|u\|^2_{X^{s-\frac{1}{2}+\delta}}.$$ 

The proof of theorem 1.3 is thus completed.

4. Proof of theorem 1.5

This section is devoted to prove theorem 1.5. First we will establish all the nonlinear estimates to use in the proof. For simplicity, let us define

$$E_1(u)(t) := \int_0^t \int_0^\infty U(t - \tau')(u, u_x)(\tau') \, d\tau'$$ (4.1)

$$E_2(u)(t) := \int_0^t \int_0^\infty U(t - \tau')(u, \partial_x L(u))(\tau') \, d\tau'$$ (4.2)

$$E_3(u)(t) := \int_0^t \int_0^\infty U(t - \tau')(u, \partial_x L(u))(\tau') \, d\tau'$$ (4.3)
\[ E_4(u)(t) := \int_0^t U(t - t') (\phi_t u + 2\phi u_x) \, dt', \quad (4.4) \]

and

\[ E_5(u)(t) := \int_0^t U(t - t') (\phi_t \partial_L u) \, dt'. \quad (4.5) \]

Next, we estimate \( E_i, i = 1, \ldots, 5 \) in the norms needed to close the argument in the fixed point theorem. Before proceeding we remark that functions \( m_1 \) and \( m_2 \) in (3.6) are also Fourier multipliers in \( H^s(\mathbb{R}^2), s \geq 0 \). Thus the operators \( \partial_x \mathcal{L} \) and \( \partial_y \mathcal{L} \) are bounded in \( H^s(\mathbb{R}^2), s \geq 0 \).

**Lemma 4.1.** Let \( E_1 \) be defined as in (4.1). Then,

\[
\| E_1(u) \|_{L^2 H^1} \lesssim T^{2/3} \| \partial_t u \|_{L^2_x L^2_T} \| u \|_{L^2_x L^2_T} + T^{1/2} \| \partial^2 u \|_{L^2_x L^2_T} \| u \|_{L^2_x L^2_T}, \quad (4.6)
\]

\[
\| \partial^2 E_1(u) \|_{L^2_x L^2_T} \lesssim T^{2/3} \| \partial_t u \|_{L^2_x L^2_T} \| u \|_{L^2_x L^2_T} + T^{1/2} \| \partial^2 u \|_{L^2_x L^2_T} \| u \|_{L^2_x L^2_T}, \quad (4.7)
\]

\[
\| \nabla E_1(u) \|_{L^2_x L^2_T} \lesssim T^{2/3} \| \partial_t u \|_{L^2_x L^2_T} \| u \|_{L^2_x L^2_T} + T^{1/2} \| u \|_{L^2_x L^2_T} \| \partial^2 u \|_{L^2_x L^2_T}, \quad (4.8)
\]

and

\[
\| E_1(u) \|_{L^2_x L^2_T} \lesssim c(1, T) \left\{ T^{2/3} \| \partial_t u \|_{L^2_x L^2_T} \| u \|_{L^2_x L^2_T} + T^{1/2} \| u \|_{L^2_x L^2_T} \| \partial^2 u \|_{L^2_x L^2_T} \right\}. \quad (4.9)
\]

**Proof.** First we show estimate (4.6). From (4.1), Minkowski’s inequality, group properties and Leibniz’ rule we have

\[
\| E_1(u)(t) \|_{H^1} \leq \int_0^T \| uu_x \|_{L^2_x} \, dt + \int_0^T \left( \| uu_{xx} \|_{L^2_x} + \| u_x u_t \|_{L^2_x} \right) \, dt + \int_0^T \left( \| uu_{xy} \|_{L^2_x} + \| u_x u_s \|_{L^2_x} \right) \, dt. \quad (4.10)
\]

The first term on the right-hand side of (4.10) can be estimate as follows:

\[
\int_0^T \| uu_x \|_{L^2_x} \, dt \lesssim \int_0^T \| u \|_{L^2_x} \| u_x \|_{L^2_T} \, dt \lesssim T^{2/3} \| \partial_t u \|_{L^2_x L^2_T} \| u \|_{L^2_x L^2_T}, \quad (4.11)
\]

where we have used Hölder’s inequality in space and then in time.

Next we estimate the second term on the right-hand side of (4.10). An argument similar to the one applied in (4.11) yields

\[
\int_0^T \| uu_{xx} \|_{L^2_x} \, dt \lesssim \int_0^T \| u_x \|_{L^2_x} \| u_{xx} \|_{L^2_T} \, dt \lesssim T^{2/3} \| \partial_t u \|_{L^2_x L^2_T} \| u_x \|_{L^2_x L^2_T}. \quad (4.12)
\]

Hölder’s inequality in time allows us to obtain

\[
\int_0^T \| uu_{xx} \|_{L^2_x} \, dt \lesssim T^{1/2} \| uu_{xx} \|_{L^2_x L^2_T} \lesssim T^{1/2} \| u \|_{L^2_x L^2_T} \| u_{xx} \|_{L^2_x L^2_T}. \quad (4.13)
\]
The estimate of the third term on the right-hand side of (4.10) follows the same argument as in (4.12) and (4.13). Thus
\[
\int_0^T \left( \|u, u_x\|_{L^2_t L^2_y} + \|uu_y\|_{L^2_t L^2_y} \right) \, dt' \lesssim T^{2/3} \|\partial_x u\|_{L^6_t L^2_y} \|u|_{L^2_t L^2_y} + T^{1/2} \|u\|_{L^2_t L^2_y} \|uu_y\|_{L^2_t L^2_y}.
\] (4.14)

Combining (4.10) with inequalities (4.11)–(4.14) we obtain (4.6).

To establish estimate (4.7) we first apply Minkowski’s inequality, the smoothing effect (2.2) and the argument above to get
\[
\|\partial^2 E_1(u)\|_{L^2_t L^2_y} \leq \int_0^T \left( 2\|\partial_x (uu_u)\| + \|\partial_x (uu_x)\| \right) \, dt' \\
\lesssim T^{2/3} \|\partial_x u\|_{L^6_t L^2_y} \left( \|u_x\|_{L^2_t L^2_y} + \|u|_{L^2_t L^2_y} \right) + T^{1/2} \|u\|_{L^2_t L^2_y} \left( \|u_{xx}\|_{L^2_t L^2_y} + \|u_y\|_{L^2_t L^2_y} \right).
\]

The inequality (4.8) follows using Minkowski’s inequality, the smoothing effect (2.1) and the argument used to obtain (4.6):
\[
\|\nabla E_1(u)\|_{L^2_t L^2_y} \leq \int_0^T \left( \|\partial_x (uu_u)\| + \|\partial_x (uu_x)\| \right) \, dt' \\
\lesssim T^{2/3} \|\partial_x u\|_{L^6_t L^2_y} \left( \|u_x\|_{L^2_t L^2_y} + \|u|_{L^2_t L^2_y} \right) + T^{1/2} \|u\|_{L^2_t L^2_y} \left( \|u_{xx}\|_{L^2_t L^2_y} + \|u_y\|_{L^2_t L^2_y} \right).
\]

Finally, to get inequality (4.9) we use group properties, the maximal function estimate (2.3) and the argument above,
\[
\|E_1(u)\|_{L^2_t L^2_y} = \left\| U(t) \left( \int_0^T U(-t') (uu_u(t')) \, dt' \right) \right\|_{L^2_t L^2_y} \\
\leq c(1, T) \left\| \int_0^T U(-t') (uu_u(t')) \, dt' \right\|_{H^1} \\
\leq c(1, T) \int_0^T \|uu_u\|_{H^1} \, dt' \\
\leq c(1, T) T^{2/3} \|\partial_x u\|_{L^6_t L^2_y} \left( \|u_x\|_{L^2_t L^2_y} + \|u|_{L^2_t L^2_y} \right) + c(1, T) T^{1/2} \|u\|_{L^2_t L^2_y} \left( \|u_{xx}\|_{L^2_t L^2_y} + \|u_y\|_{L^2_t L^2_y} \right).
\]

The proof of the lemma is thus completed. \(\Box\)

**Lemma 4.2.** Let \(E_2\) be defined as in \((4.2)\). Then,
\[
\|E_2(u)\|_{L^p_t H^1} \lesssim T^{2/3} \|\partial_x u\|_{L^6_t L^2_y} \|u|_{L^2_t H^1} + T^{1/2} \|\partial u\|_{L^2_t L^2_y} \|\partial_x \mathcal{L}(u)\|_{L^2_t L^2_y},
\] (4.15)
\[ \| \partial^2 E_2(u) \|_{L^\infty_t L^2_x} \leq T^{2/3} \| \partial_\tau u \|_{L^2_{t\tau} L^2_x} \| u \|_{L^2_{t\tau}} + T^{1/2} \| \partial^2 u \|_{L^\infty_t L^2_x} \| \partial_\tau \mathcal{L}(u) \|_{L^2_{t\tau}}, \] (4.16)

\[ \| \nabla E_2(u) \|_{L^2_{t\tau} L^2_x} \leq T^{1/2} \| \partial^2 u \|_{L^2_{t\tau} L^2_x} \| \partial_\tau \mathcal{L}(u) \|_{L^2_{t\tau}} \leq T^{2/3} \| \partial_\tau u \|_{L^2_{t\tau} L^2_x} \| \partial_\tau \mathcal{L}(u) \|_{L^2_{t\tau}} \] (4.17)

and

\[ \| E_3(u) \|_{L^2_{t\tau} L^2_x} \leq c(1, T) \left\{ T^{2/3} \| \partial_\tau u \|_{L^2_{t\tau} L^2_x} \| u \|_{L^2_{t\tau}} + T^{1/2} \| \partial^2 u \|_{L^2_{t\tau} L^2_x} \| \partial_\tau \mathcal{L}(u) \|_{L^2_{t\tau}} \right\} . \] (4.18)

**Proof.** The inequalities (4.15)–(4.18) are obtained using the arguments applied to show the corresponding estimates in lemma 4.1, so we will omit the details. \( \square \)

**Lemma 4.3.** Let \( E_3 \) be defined as in (4.3). Then,

\[ \| E_3(u) \|_{L^2_{t\tau} L^2_x} \leq T^{2/3} \| \partial_\tau u \|_{L^2_{t\tau} L^2_x} \| u \|_{L^2_{t\tau}} + c T^{1/2} \| \partial^2 u \|_{L^2_{t\tau} L^2_x} \| \partial_\tau \mathcal{L}(u) \|_{L^2_{t\tau}}, \] (4.19)

\[ \| \partial^2 E_3(u) \|_{L^2_{t\tau} L^2_x} \leq T^{2/3} \| \partial_\tau u \|_{L^2_{t\tau} L^2_x} \| u \|_{L^2_{t\tau}} + T^{1/2} \| \partial^2 u \|_{L^2_{t\tau} L^2_x} \| \partial_\tau \mathcal{L}(u) \|_{L^2_{t\tau}}, \] (4.20)

\[ \| \nabla E_3(u) \|_{L^2_{t\tau} L^2_x} \leq T^{1/2} \| \partial^2 u \|_{L^2_{t\tau} L^2_x} \| \partial_\tau \mathcal{L}(u) \|_{L^2_{t\tau}} + T^{2/3} \| \partial_\tau u \|_{L^2_{t\tau} L^2_x} \| \partial_\tau \mathcal{L}(u) \|_{L^2_{t\tau}}, \] (4.21)

and

\[ \| E_3(u) \|_{L^2_{t\tau} L^2_x} \leq c(1, T) \left\{ T^{2/3} \| \partial_\tau u \|_{L^2_{t\tau} L^2_x} \| u \|_{L^2_{t\tau}} + T^{1/2} \| \partial^2 u \|_{L^2_{t\tau} L^2_x} \| \partial_\tau \mathcal{L}(u) \|_{L^2_{t\tau}} \right\} . \] (4.22)

**Proof.** To obtain estimates (4.19)–(4.22) we apply similar arguments to the ones used to prove the corresponding estimates in lemma 4.1. Thus we will also omit the details. \( \square \)

**Lemma 4.4.** Let \( E_1, E_2 \) and \( E_3 \) be defined as above. Then,

\[ \| \partial_\tau \mathcal{L}(E_1) \|_{L^2_{t\tau} L^2_x} + \| \partial_\tau \mathcal{L}(E_2) \|_{L^2_{t\tau} L^2_x} \leq c(1, T) \left\{ T^{2/3} \| \partial_\tau u \|_{L^2_{t\tau} L^2_x} \| u \|_{L^2_{t\tau}} + T^{1/2} \| \partial^2 u \|_{L^2_{t\tau} L^2_x} \| \partial_\tau \mathcal{L}(u) \|_{L^2_{t\tau}} \right\} . \] (4.23)

\[ \| \partial_\tau \mathcal{L}(E_2) \|_{L^2_{t\tau} L^2_x} + \| \partial_\tau \mathcal{L}(E_3) \|_{L^2_{t\tau} L^2_x} \leq c(1, T) \left\{ T^{2/3} \| \partial_\tau u \|_{L^2_{t\tau} L^2_x} \| u \|_{L^2_{t\tau}} + T^{1/2} \| \partial^2 u \|_{L^2_{t\tau} L^2_x} \| \partial_\tau \mathcal{L}(u) \|_{L^2_{t\tau}} \right\} . \] (4.24)

and

\[ \| \partial_\tau \mathcal{L}(E_3) \|_{L^2_{t\tau} L^2_x} + \| \partial_\tau \mathcal{L}(E_3) \|_{L^2_{t\tau} L^2_x} \leq c(1, T) \left\{ T^{2/3} \| \partial_\tau u \|_{L^2_{t\tau} L^2_x} \| u \|_{L^2_{t\tau}} + T^{1/2} \| \partial^2 u \|_{L^2_{t\tau} L^2_x} \| \partial_\tau \mathcal{L}(u) \|_{L^2_{t\tau}} \right\} . \] (4.25)

**Proof.** We will only prove (4.23) since the other inequalities follow similarly. We first observe that \( \partial_\tau \mathcal{L} \) and \( \partial_\tau \mathcal{L} \) commute with the operator \( U(t) \). Then using group properties, (2.3),
the remark before lemma 4.1, and Minkowski’s inequality we get
\[ \|\partial_t \mathcal{L}(E_1)\|_{L^2_t L^\infty_x} \lesssim \left\| U(t) \left( \partial_t \mathcal{L} \int_0^T U(-r')(uu_x)(r') \, dr' \right) \right\|_{L^2_t L^\infty_x} \]
\[ \lesssim c(1, T) \left\| \partial_t \mathcal{L} \int_0^T U(-r')(uu_x)(r') \, dr' \right\|_{H^1} \]
\[ \lesssim c(1, T) \int_0^T \|uu_x\|_{H^1} \, dr'. \]
The estimate (4.23) now follows by using the argument employed to obtain (4.6). This completes the proof. □

We have completed all the estimates for the terms \( E_1, E_2 \) and \( E_3 \) essential for the proof of theorem 1.5. Note that terms \( E_4 \) and \( E_5 \) contain the functions \( \varphi \) and \( \varphi_x \). Here a careful estimate need to be performed because the function \( \varphi \) does not belong to \( L^2(\mathbb{R}^2) \). The crucial property is that \( \varphi(x - \omega t) \) belongs to \( L^2_x L^\infty_T \), and \( \varphi(x - \omega t) \) and \( \varphi_x(x - \omega t) \) belong to \( L^\infty_x \).

**Lemma 4.5.** Let \( E_4 \) be defined as in (4.4). Then,
\[ \|E_4(u)\|_{L^2_x L^\infty_T} \lesssim T\|u\|_{L^3_x H^1} + T^{1/2}\|\partial^2 u\|_{L^2_x L^2_T}, \quad (4.26) \]
\[ \|\nabla E_4(u)\|_{L^2_t L^\infty_x} \lesssim T\|u\|_{L^3_x H^1} + T^{1/2}\|\partial^2 u\|_{L^2_x L^2_T}, \quad (4.27) \]
\[ \|\partial^2 E_4(u)\|_{L^2_x L^2_T} \lesssim T\|u\|_{L^3_x H^1} + T^{1/2}\|\partial^2 u\|_{L^2_x L^2_T}, \quad (4.28) \]
\[ \|E_4(u)\|_{L^2_t L^\infty_x} \lesssim c(1, T) \left\{ T\|u\|_{L^3_x H^1} + T^{1/2}\|\partial^2 u\|_{L^2_x L^2_T} \right\}, \quad (4.29) \]
and
\[ \|\partial_t \mathcal{L}(E_4(u))\|_{L^2_t L^\infty_x} + \|\partial_t \mathcal{L}(E_4(u))\|_{L^2_t L^\infty_x} \lesssim c(1, T) \left\{ T\|u\|_{L^3_x H^1} + T^{1/2}\|\partial^2 u\|_{L^2_x L^2_T} \right\}. \quad (4.30) \]

**Proof.** For (4.26), we have
\[ \|E_4(u)(t)\|_{L^2_x} \lesssim \int_0^T \|\varphi_x u\|_{L^2_x} \, dt' + \int_0^T \|\varphi u_x\|_{L^2_x} \, dt' =: I_1 + I_2. \quad (4.31) \]
Now, from Hölder’s inequality,
\[ I_1 = \int_0^T \|\varphi_x u\|_{L^2_x} \, dt' \leq \int_0^T \|\varphi_x\|_{L^2_T} \|u\|_{L^2_T} \, dt' \]
\[ \leq \int_0^T \|\varphi_x\|_{L^2_T} \|u\|_{L^2_T} \, dt' \leq \|\varphi_x\|_{L^2_T} \|u\|_{L^3_x H^1} T \lesssim T\|u\|_{L^3_x H^1}, \quad (4.32) \]
and similarly,
\[ I_2 \leq \|\varphi\|_{L^\infty_T} \|u\|_{L^3_x H^1}, \quad (4.33) \]
Note that the implicit constant in the estimates for $I_1$ and $I_2$ is independent of $T$ (because $\varphi(x - ct)$ and $\varphi(x - ct)$ are uniformly bounded in $x, t$). Thus, from (4.31)–(4.33),

$$\|E_4(u)(t)\|_{L^2_y} \lesssim T\|u\|_{L^2_y H^1}.$$  (4.34)

Also,

$$\|\partial_x E_4(u)(t)\|_{L^2_y} \lesssim \int_0^T \|\partial_x(\varphi_x u + \varphi u_x)\|_{L^2_y} \, dt' \lesssim \int_0^T \|\varphi_{xx} u + 2\varphi_x u_x\|_{L^2_y} + \int_0^T \|\varphi u_{xx}\|_{L^2_y} =: J.$$  

The first integral above can be estimated as in (4.32). Thus,

$$J \lesssim T\|u\|_{L^\infty_T H^1} + T^{1/2}\|\varphi u_{xx}\|_{L^2_T}$$

$$\lesssim T\|u\|_{L^\infty_T H^1} + T^{1/2}\|\varphi\|_{L^\infty_T} \|u_{xx}\|_{L^2_T}$$

$$\lesssim T\|u\|_{L^\infty_T H^1} + T^{1/2}\|\varphi\|_{L^\infty_T} \|u_{xx}\|_{L^2_T}$$

$$\lesssim T\|u\|_{L^\infty_T H^1} + T^{1/2} \|\partial^2 u\|_{L^\infty_T H^0}.$$  (4.35)

In the last inequality we have used that $\varphi(x - ct) \in L^1_T L^\infty$. Hence, from (4.35), we obtain

$$\|\partial_t E_4(u)(t)\|_{L^2_y} \lesssim T\|u\|_{L^\infty_T H^1} + T^{1/2}\|\partial^2 u\|_{L^\infty_T L^2_T}.$$  (4.36)

Similarly, we get

$$\|\partial_y E_4(u)(t)\|_{L^2_y} \lesssim T\|u\|_{L^\infty_T H^1} + T^{1/2}\|\partial^2 u\|_{L^\infty_T L^2_T}.$$  (4.37)

Therefore, from (4.34), (4.36) and (4.37), we deduce the bound (4.26). For (4.27), we have

$$\|\nabla E_4(u)\|_{L^2_y L^\infty_T} \lesssim \int_0^T \|\partial_x(\varphi_x u + \varphi u_x)\|_{L^2_y} + \int_0^T \|\partial_y(\varphi_x u + \varphi u_x)\|_{L^2_y}.$$  (4.38)

The first integral is estimated as that for $J$. Hence,

$$\|\nabla E_4(u)\|_{L^2_y L^\infty_T} \lesssim T\|u\|_{L^\infty_T H^1} + T^{1/2}\|\partial^2 u\|_{L^\infty_T L^2_T} + \int_0^T \|\varphi_{xx} u + \varphi u_{xx}\|_{L^2_y}$$

$$\lesssim T\|u\|_{L^\infty_T H^1} + T^{1/2}\|\partial^2 u\|_{L^\infty_T L^2_T}$$

$$+ T\|\varphi_{xx}\|_{L^\infty_T L^2_y} \|u_{xx}\|_{L^2_T} + T^{1/2}\|\varphi_{xx}\|_{L^\infty_T L^2_y} \|u_{xx}\|_{L^2_T}$$

$$\lesssim T\|u\|_{L^\infty_T H^1} + T^{1/2}\|\partial^2 u\|_{L^\infty_T L^2_T}.$$
As in (4.38), we estimate
\[
\|\partial^2 E_4(u)\|_{L^2_x L^\infty_T} \lesssim \int_0^T \|\partial_t (\varphi_x u + \varphi u_x)\|_{L^2_T} + \int_0^T \|\partial_x (\varphi_x u + \varphi u_x)\|_{L^\infty_x L^2_T} \\
\lesssim T\|u\|_{L^\infty_x H^1} + T^{1/2}\|\partial^2 u\|_{L^2_x L^2_T}.
\]
This proves (4.28). Now, for (4.29), observe that
\[
\|\partial^2 L(E_4(u))\|_{L^2_x L^\infty_T} \lesssim \int_0^T \|\partial_x (\varphi_x u + \varphi u_x)\|_{L^\infty_x L^2_T} \\
\|\partial_y (\varphi_x u + \varphi u_x)\|_{L^\infty_x L^2_T}.
\]
So, (4.29) follows by a similar analysis as before.

Finally, since \(\partial_x L\) and \(\partial_y L\) are bounded operators in \(H^1(\mathbb{R}^2)\),
\[
\|\partial_x L(E_4(u))\|_{L^2_x L^\infty_T} \lesssim \int_0^T \|\partial_x L(\varphi_x u + 2\varphi u_x)\|_{L^\infty_x L^2_T} \\
\|\partial_y L(E_4(u))\|_{L^2_x L^\infty_T} \lesssim \int_0^T \|\partial_y L(\varphi_x u + 2\varphi u_x)\|_{L^\infty_x L^2_T}.
\]
This proves (4.30). Thus, (4.30) also follows from arguments already used. This completes the proof of the lemma.

**Lemma 4.6.** Let \(E_5\) be defined as in (4.5). Then,
\[
\|E_5(u)\|_{L^\infty_x H^1} \lesssim T\|u\|_{L^\infty_x H^1}, \quad (4.39)
\]
\[
\|\nabla E_5(u)\|_{L^2_x L^\infty_T} \lesssim T\|u\|_{L^\infty_x H^1}, \quad (4.40)
\]
\[
\|\partial^2 E_5(u)\|_{L^2_x L^\infty_T} \lesssim T\|u\|_{L^\infty_x H^1}, \quad (4.41)
\]
\[
\|E_5(u)\|_{L^2_x L^\infty_T} \lesssim c(1, T)\|u\|_{L^\infty_x H^1}, \quad (4.42)
\]
and
\[
\|\partial_x L(E_5(u))\|_{L^2_x L^\infty_T} \lesssim \|\partial_y L(E_5(u))\|_{L^2_x L^\infty_T} \lesssim c(1, T)\|u\|_{L^\infty_x H^1}. \quad (4.43)
\]

**Proof.** We only estimate (4.39). The other estimates are similar to those in lemma 4.5. Since \(\partial_x L\) is a bounded operator on \(L^2(\mathbb{R}^2)\), we have
\[
\|E_5(u)(t)\|_{L^2_T} \lesssim \int_0^T \|\varphi_x \partial_x L(u)\|_{L^2_T} \|u\|_{L^\infty_x H^1} \\
\lesssim \int_0^T \|\varphi_x \|_{L^\infty_T} \|\partial_x L(u)\|_{L^2_T} \|u\|_{L^\infty_x H^1} \\
\lesssim \int_0^T \|\varphi_x \|_{L^\infty_T} \|u\|_{L^\infty_x H^1} \|u\|_{L^\infty_x H^1} \lesssim T\|u\|_{L^\infty_x H^1}.
\]
Now, since $\partial_x$ and $\partial_y$ commute with $\partial_t L$ and using that $\partial_t L$ is a bounded operator on $L^2(\mathbb{R}^2)$ once again, we deduce

$$\|\partial_t E_5(u)(t)\|_{L^2_T} \leq \int_0^T \|\partial_t (\varphi_x \partial_t L(u))\|_{L^2_T} \, dt'$$

$$\leq \int_0^T \|\varphi_x \partial_t L(u) + \varphi_x \partial_t (\partial_t u)\|_{L^2_T} \, dt'$$

$$\leq T(\|\varphi_x\|_{L^\infty_T}\|u\|_{L^\infty_T} + \|\varphi_x\|_{L^\infty_T}\|\partial_t u\|_{L^2_T})$$

$$\lesssim T\|u\|_{L^p_T H^1}$$

and

$$\|\partial_y E_5(u)(t)\|_{L^2_T} \leq \int_0^T \|\varphi_x \partial_y (\partial_y u)\|_{L^2_T} \, dt'$$

$$\leq T\|\varphi_x\|_{L^\infty_T}\|\partial_y u\|_{L^p_T L^2_T}$$

$$\lesssim T\|u\|_{L^p_T H^1}.$$

Combining the above estimates we obtain (4.39). The proof of the lemma is thus completed. □

With the above estimates in hand, we are able to prove theorem 1.5.

**Proof of theorem 1.5.** As we already mentioned, we use the fixed point theorem. Let us define the operator

$$\Phi(u)(t) = U(t)u_0 - \frac{1}{2} \sum_{j=1}^5 E_j(u)(t),$$

and the closed ball

$$X^u_T := \{ v \in C([0, T] : H^1(\mathbb{R}^2)) : \|v\| \leq a \},$$

where

$$\|v\| := \|v\|_{L^p_T H^1} + \|\partial^2 v\|_{L^2_T L^2_T} + \|\nabla v\|_{L^2_T L^\infty_T}$$

$$+ \|v\|_{L^2_T L^2_T} + \|\partial_x L(v)\|_{L^2_T L^\infty_T} + \|\partial_y L(v)\|_{L^2_T L^\infty_T}.$$

By using lemmas 4.1–4.6, we are able to show that there exist positive constants $a$ and $T$ such that $\Phi : X^u_T \mapsto X^u_T$ is well defined and is a contraction. From this point on, the arguments to complete the proof of the theorem are standard. So we will not give the details. □

**Remark 4.7.** By using the same strategy as in the proof of theorem 1.5 we may show that (1.7) is locally well-posed in the anisotropic Sobolev space $H^{s,1}(\mathbb{R}^2)$, $s > 3/4$, which is defined through the norm

$$\|v\|^2_{H^{s,1}} = \|v\|_{L^2_T H^s}^2 + \|D^s v\|_{L^2_T L^2_T}^2 + \|\partial_x v\|_{L^2_T L^2_T}^2 + \|\partial_y v\|_{L^2_T L^2_T}^2.$$

Here $D^s_x$ is defined through its Fourier transform as $\mathcal{F}_{xy}(D^s_x u_0)(\xi, \mu) = |\xi|^s \mathcal{F}_{xy}(u_0)(\xi, \mu)$. 

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To see this, it suffices to show that the integral equation (1.9) has a unique fixed point in the ball

\[ X^a_T := \{ v \in C([0,T]: H^{s,1}(\mathbb{R}^2)) : \| v \| \leq a \} \]

where now

\[ \| v \| := \| v \|_{L^2_t H^{s,1}} + \| \nabla D_s^t v \|_{L^2_t H^{s-\frac{1}{2}}} + \| \nabla \partial_s v \|_{L^2_t H^{s-\frac{1}{2}}} + \| \nabla v \|_{L^2_t H^{s+\frac{1}{2}}} + \| v \|_{L^2_t L^{\infty}} + \| \partial_s \mathcal{L}(v) \|_{L^2_t H^{s-\frac{1}{2}}} + \| \partial_s \mathcal{L}(v) \|_{L^2_t H^{s-\frac{1}{2}}} \]

The nonlinear estimates are similar to those ones for the terms \( E_1, E_2 \) and \( E_3 \) above, except the ones including the operator \( D_s^t \), where we need to use the one-dimensional fractional Leibniz rule (see [15]):

\[ \| D_s^t (fg) - f D_s^t g - g D_s^t f \|_{L^2_t} \leq c \| g \|_{L^2_t} \| D_s^t f \|_{L^2_t}, \quad s \in (0, 1), \quad (4.44) \]

and the estimate

\[ \| U(t)u_0 \|_{L^2_t H^{s,1}} \leq c \| D_s^{-\varepsilon} u_0 \|_{L^2_t}, \quad (4.45) \]

which holds for any \( 0 \leq \varepsilon < 1/2 \). Estimate (4.45) was established in [19].

For the sake of clearness we estimate \( E_1 \) in the norms \( \| D_s^t (E_1(u)) \|_{L^2_t} \) and \( \| \partial_s E_1(u) \|_{L^2_t L^{\infty}} \), assuming \( s \in (3/4, 1) \). Indeed, from group properties, Minkowski and Hölder’s inequalities, we obtain

\[ \| D_s^t (E_1(u)) \|_{L^2_t} \leq \int_0^T \| D_s^t (u u_s) \|_{L^2_t} \, dt \leq T^{1/2} \| D_s^t (u u_s) \|_{L^2_t} \, dt. \quad (4.46) \]

Now, we use (4.44) to obtain

\[ \| D_s^t (u u_s) \|_{L^2_t} \leq \| u D_s^t (u u_s) - u D_s^t u - u D_s^t u \|_{L^2_t} + \| u D_s^t u \|_{L^2_t} + \| D_s^t u \|_{L^2_t} \]

\[ \leq \| u \|_{L^2_t H^{s,1}} \| D_s^t u \|_{L^2_t} + c \| u \|_{L^2_t L^{\infty}} \| D_s^t u \|_{L^2_t} \]

The last two inequalities combine to give

\[ \| D_s^t (E_1(u)) \|_{L^2_t} \leq T^{1/2} \| u \|^2. \]

Also, in view of (4.45), we deduce

\[ \| \partial_s (E_1(u)) \|_{L^2_t L^{\infty}} \leq \int_0^T \| D_s^{-\varepsilon/2} \partial_s u \|_{L^2_t} \, dt. \quad (4.47) \]

Since \( s > 3/4 \), we can choose \( \varepsilon \) sufficiently close to 1/2 in such a way that \( 1 - \varepsilon/2 \leq s \). As a consequence,

\[ \| \partial_s (E_1(u)) \|_{L^2_t L^{\infty}} \leq \int_0^T \| uu_s \|_{H^{s-1}} \, dt. \]

From this point on, one uses the same arguments as in (4.46) and in the proof of theorem 1.5.
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