RING THEORETIC PROPERTIES OF PARTIAL SKEW GROUPOID RINGS WITH APPLICATIONS TO LEAVITT PATH ALGEBRAS

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Abstract. Let \( \alpha = (A_g, \alpha_g)_{g \in G} \) be a group-type partial action of a connected groupoid \( G \) on a ring \( A = \bigoplus_{z \in G_0} A_z \) and \( B := A \ast_\alpha G \) the corresponding partial skew groupoid ring. In the first part of this paper we investigate the relation of several ring theoretic properties between \( A \) and \( B \). For the second part, using that every Leavitt path algebra is isomorphic to a partial skew groupoid ring obtained from a partial groupoid action \( \lambda \), we characterize when \( \lambda \) is group-type. In such a case, we obtain ring theoretic properties of Leavitt path algebras from the results on general partial skew groupoid rings. Several examples that illustrate the results on Leavitt path algebras are presented.

1. Introduction

Let \( G \) be a groupoid and \( A \) an associative and unital ring. For a partial action \( \alpha \) of \( G \) on \( A \) corresponds the partial skew groupoid ring \( B := A \ast_\alpha G \) (cf. Definition 2.9). The partial skew groupoid ring is a generalization of both partial skew group ring and groupoid algebra. Precisely, if \( G \) is a group then \( B \) is the partial skew group ring while if \( \alpha \) is the trivial partial action \( \alpha_g = id_A \), for all \( g \in G \), then \( B \) is the usual groupoid algebra.

Partial skew groupoid rings were firstly considered in [6] for ordered groupoids and, after that, in [7] for general groupoids. Basic properties of \( B \) were given in [6, 7]. For instance, if \( \alpha = (A_g, \alpha_g)_{g \in G} \) then \( B \) is unital if and only if \( A_z \) is unital, for all \( z \in G_0 \) and \( G_0 \) is finite, where \( G_0 \) denotes the set of objects of \( G \). Also, if \( A_g \) is a unital ring for each \( g \in G \), then \( B \) is an associative ring. Partial skew groupoid rings appear naturally in the partial Galois theory of groupoids [7] and in the context of groupoid graded rings [23]. Also, it was proved in [16] that every Leavitt path algebra is a partial skew groupoid ring.

A groupoid \( G \) is a disjoint union of its connected components which are full connected subgroupoids of \( G \). We use this decomposition of \( G \) to reduce partial groupoid actions to the connected case; see Remark 3.3 of [8]. The structure of a connected groupoid is well-known. Any connected groupoid \( G \) is isomorphic to \( G_0^x \times G(x) \), being \( G(x) \) the isotropy group of an object \( x \in G_0 \). If \( A = \bigoplus_{y \in G_0} A_y \) and \( \alpha \) is a group-type partial action (see Definition 2.6) then the factorization of \( G \) induces a factorization of \( B \). In
fact, it was proved in [8] that there are a global action $\beta$ of $G^2$ on $A$ and a partial action $\gamma$ of $G(x)$ on $A \ast \beta G_0^2$ such that $B \simeq (A \ast \beta G_0^2) \ast_\gamma G(x)$. In other words, if $\alpha$ is group-type then the partial skew groupoid ring $B$ is a partial skew group ring.

In the first part of this paper we investigate ring theoretic properties of $B$ for the case that it admits the factorization $B \simeq (A \ast \beta G_0^2) \ast_\gamma G(x)$. The properties of $B$ that are studied are the following: noetherianity, von Neumann regularity, the Jacobson radical, perfect, semiprimary and Krull dimension. One of the strategies used to prove the results is to apply the ring theoretic properties proved in [11] to $B$ since it is a skew partial group ring.

In the second part, we work with Leavitt path algebras. Precisely, let $L_k(E)$ be the Leavitt path algebra associated to a directed graph $E$ over a field $k$. By [16], we have that $L_k(E) \simeq D(X) \ast_\lambda G(E)$, where $D(X)$ is $k$ algebra and $\lambda$ is a partial action of the free path groupoid $G(E)$ on $D(X)$. A general characterization for $\lambda$ to be group-type is given in Theorem 4.7. A refinement of this result is given in Proposition 4.9 for the case that $E$ has a sink, $E^0$ is finite and $|E^0| \geq 2$. Assuming that $\lambda$ is group-type, we present ring theoretic properties of $L_k(E)$. Examples to illustrate the results on Leavitt path algebras are also presented.

The paper is organized as follows. The background on groupoids and partial groupoid actions will be given in Section 2. The results on ring theoretic properties of the partial skew groupoid ring are presented in Section 3. Particularly, for noetherianity and regularity properties it is convenient to establish first some general results of groupoid graded rings. For this, we introduce the notion of von Neumann groupoid graded ring and, in Proposition 3.5, we relate this concept with von Neumann regularity. The results related to Leavitt path algebras are given in Section 4.

**Conventions.** Throughout this work, by ring we mean an associative ring with identity. A ring with identity will be called unital. The Jacobson radical of a ring $A$ will be denoted by $J(A)$. The group of invertible elements of a unital ring $A$ is denoted by $U(A)$ and the characteristic of a field $k$ by $\text{ch}(k)$.

2. Preliminaries

2.1. **On connected groupoids.** A groupoid $G$ is a small category in which every morphism is an isomorphism. The set of objects of $G$ is denoted by $G_0$. The source and the target maps $s, t: G \to G_0$ are defined by $s(g) = x$ and $t(g) = y$, for any morphism $g: x \to y$ of $G$. The set $G(x,y)$ contains the morphisms $g$ of $G$ for which $s(g) = x$ and $t(g) = y$. Every object $x$ of $G$ is identified with the identity morphism, that is, $x = \text{id}_x$.

Thus $G_0 \subseteq G$, $s(g) = g^{-1}y$ and $t(g) = gg^{-1}$, for all $g \in G$. For each $x \in G_0$, the isotropy group associated to $x$ is $G(x) := G(x,x)$ and we will fix $G(x, \cdot) := \{g \in G : s(g) = x \}$ and $G(\cdot, x) := \{g \in G : t(g) = x \}$.

We denote the composition of morphisms of a groupoid by concatenation. Notice that, for $g, h \in G$, there exists $gh$ if and only if $t(h) = s(g)$. In this case, we say that $g$ and $h$ are composable. We denote by $G_s \times_t G$ the subset of $G \times G$ of composable pairs, that is, $G_s \times_t G$ is the pullback of $s$ and $t$ in the category of groupoids. Observe that $s(gh) = s(h)$ and $t(gh) = t(g)$, for all $(g,h) \in G_s \times_t G$. 

A groupoid $G$ is called connected if $G(x, y) \neq \emptyset$, for any $x, y \in G_0$. For an arbitrary groupoid $G$ the equivalence relation on $G_0$, given by $x \sim y$ if and only if $G(x, y) \neq \emptyset$, induces a decomposition of $G$ as a disjoint union of connected subgroupoids. Explicitly, for each equivalence class $X \subset G_0$ corresponds the full subgroupoid $G_X$ of $G$ whose set of objects is $X$. Hence $G$ is the disjoint union of the subgroupoids $G_X$, with $X \in G_0/\sim$.

In order to present the structure of a connected groupoid, we recall that the coarse groupoid $Y^2$ (or Brandt groupoid) associated to any non-empty set $Y$ is the groupoid whose objects are the set $Y$ and satisfies $s(y, y') = y$, $t(y, y') = y'$ and $(y', y'')(y, y') = (y, y'')$. The next result is well known and its proof will be omitted; see, for instance, Proposition 2.1 of [8].

**Proposition 2.1.** Let $G$ be a connected groupoid. Then $G \simeq G_G^0 \times G(x)$ as groupoids.

2.2. Partial groupoid action. Throughout this subsection, $G$ denotes a groupoid and $A$ denotes a ring. We start by recalling the definition of partial action given in [7].

**Definition 2.2.** A partial action of $G$ on $A$ is a set of pairs $\alpha = (A_g, \alpha_g)_{g \in G}$ that satisfies:

- (i) $A_g$ is an ideal of $A_{t(g)}$, $A_{t(g)}$ is an ideal of $A$ and $\alpha_g : A_{g^{-1}} \to A_g$ is a ring isomorphism, for all $g \in G$,
- (ii) $\alpha_x = \text{id}_{A_x}$, for all $x \in G_0$,
- (iii) $\alpha_{gh}^{-1}(A_g \cap A_h) \subset A_{(gh)^{-1}}$, for all $(g, h) \in G_s \times G_t$,
- (iv) $\alpha_{gh}(a) = \alpha_g\alpha_h(a)$, for all $a \in \alpha_{h}^{-1}(A_g \cap A_h)$, $(g, h) \in G_s \times G_t$.

We also recall other definitions that will be used later.

**Definition 2.3.** A partial action $\alpha = (A_g, \alpha_g)_{g \in G}$ of a groupoid $G$ on a ring $A$ is called

- (i) global, if $\alpha_g\alpha_h = \alpha_{gh}$, for all $(g, h) \in G_s \times G_t$,
- (ii) unital if each $A_g$ is a unital ring, that is, there exists a central idempotent element $1_g$ in $A$ such that $A_g = A1_g$, for all $g \in G$,
- (iii) finite-type, if for any $z \in G_0$ there are $g_1, \ldots, g_n \in G(., z)$ such that $A_{t(g)} = \sum_{i=1}^n A_{g_{g_i}}$, for any $g \in G(z, )$.

For the convenience of the reader, we recall Lemma 1.1 of [7] which gives some useful properties of partial groupoid actions.

**Lemma 2.4.** Let $\alpha = (A_g, \alpha_g)_{g \in G}$ be a partial action of a groupoid $G$ on a ring $A$. The following statements are true:

- (i) $\alpha_{g^{-1}} = \alpha^{-1}_g$, for all $g \in G$,
- (ii) $\alpha_g(A_g \cap A_h) = A_g \cap A_{gh}$, for all $(g, h) \in G_s \times G_t$,
- (iii) $\alpha$ is global if and only if $A_g = A_{t(g)}$, for all $g \in G$.

**Remark 2.5.** Let $\alpha = (A_g, \alpha_g)_{g \in G}$ be a partial action of a groupoid $G$ on a ring $A$. Note that $\alpha$ induces a partial action $\alpha_{G(x)} = (A_h, \alpha_h)_{h \in G(x)}$ of the isotropy group $G(x)$ on the ring $A_x$, for each $x \in G_0$. Moreover, using the decomposition of $G = \bigcup_{X \in G_0/\sim} G_X$ in connected components, it follows by Proposition 2.3 of [10] that partial actions of $G$ on $A$ are uniquely determined by partial actions of the connected groupoids $G_X$, $X \in G_0/\sim$, on $A$. 
Let $G$ be a connected groupoid and $x \in G_0$. Consider the following equivalence relation on $G(x, \cdot)$:  
$$g \sim_x l \text{ if and only if } t(g) = t(l).$$
A transversal $\tau(x) = \{\tau_y \mid y \in G_0\}$ for $\sim_x$ such that $\tau_x = x$ will be called a transversal for $x$. Observe that $\tau_y \in G(x, y)$, for all $x \neq y \in G_0$, and $\tau_x = x \in G(x)$. The next definition was given in subsection 3.2 of [8].

**Definition 2.6.** A partial action $\alpha = (A_y, \alpha_g)_{g \in G}$ of a connected groupoid $G$ on a ring $A$ will be called group-type if there exist $x \in G_0$ and a transversal $\tau(x) = \{\tau_y \mid y \in G_0\}$ for $x$ such that

1. $A_{\tau_y^{-1}} = A_x$ and $A_{\tau_y} = A_y$, for all $y \in G_0$.

**Remark 2.7.** (i) By Lemma 2.4 (iii), any global groupoid action is group-type. Also, the notion of group-type partial action does not depend on the choice of object $x$, cf. Remark 3.4 of [8]. In fact, if $\tau(x)$ is a transversal for $x$ and $y \in G_0$ then $\gamma(y) := \{\tau_z\tau_y^{-1} \mid z \in G_0\}$ is a transversal for $y$.

(ii) The name group-type is suggested by Corollary 3.6 of [9] in which it is observed that this kind of partial actions can be obtained from partial group actions.

Assume that $\alpha = (A_y, \alpha_g)_{g \in G}$ is a group-type partial action of a connected groupoid $G$ on a ring $A$. From (1) follows $A_{g^{-1}} = A_{g^{-1}} \cap A_{\alpha(g)} = A_{g^{-1}} \cap A_{\alpha(g)}$ which implies that $A_g = \alpha_g(A_{g^{-1}}) = \alpha_g(A_{g^{-1}} \cap A_{\alpha(g)}) = A_g \cap A_{\alpha(g)}$, for any $g \in G$. Hence

2. $A_g \subseteq A_{\alpha(g)}$, $g \in G$.

**Remark 2.8.** Let $\alpha$ be a partial action of a groupoid $G$ on a ring $A$. Consider an $\alpha$-invariant ideal $I$ of $A$, that is, $\alpha_g(I \cap A_{g^{-1}}) \subseteq I$, for all $g \in G$. In this case the family $\alpha|_I = (I_g, \alpha_g)_{g \in G}$, where $I_g := I \cap A_g$, is a partial action of $G$ on $I$. It is clear that if $\alpha$ is group-type then $\alpha|_I$ is group-type.

2.3. **The partial skew groupoid ring.** Throughout this subsection, $\alpha = (A_g, \alpha_g)_{g \in G}$ denotes a unital partial action of a groupoid $G$ on a ring $A$. We will assume that $A_g = A_{1_g}$, where $1_g$ is a central idempotent of $A$, for all $g \in G$.

**Definition 2.9.** The **partial skew groupoid ring** $A \ast \alpha G$ associated to $\alpha$ is the set of formal sums $\sum_{g \in G} a_g \delta_g$, where $a_g \in A_g$, with the usual addition and multiplication induced linearly by the following rule

3. $$(a_g \delta_g)(a_h \delta_h) = \begin{cases} a_g \alpha_g(a_h 1_{g^{-1}}) \delta_{gh}, & \text{if } (g, h) \in G_x \times_f G, \\ 0, & \text{otherwise,} \end{cases}$$

for all $g, h \in G$, $a_g \in A_g$ and $a_h \in A_h$.

The partial skew groupoid ring $A \ast \alpha G$ is an associative ring. It was proved in Section 3 of [6] that if $G_0$ is finite then $A \ast \alpha G$ is unital with identity $1_{A \ast \alpha G} = \sum_{g \in G_0} 1_g \delta_g$.

On the other hand it was shown in [8] that, if $G$ is connected then the factorization of $G$ given by Proposition 2.1 induces, under suitable conditions, a factorization of the partial skew groupoid ring. In order to review this construction, assume that $G$ is
connected, \(G_0\) is finite, \(\alpha\) is group-type and \(A = \bigoplus_{x \in G_0} A_x\). We also fix \(x \in G_0\) and \(\tau(x) = \{\tau_y \mid y \in G_0\}\) a transversal for \(x\) such that (1) is satisfied.

For each \((y, z) \in G_0^2\), we consider
\[
B_u = A_{t(u)} = A_z, \quad \beta_u = \alpha_{t(u)} \alpha_{s(u)} : A_y \to A_z.
\]
By Lemma 4.1 of [8], the family \(\beta = (B_u, \beta_u)_{u \in G_0^2}\) is a global action of \(G_0^2\) on \(A\). Hence we may consider the skew groupoid ring \(C := A \ast \beta G_0^2\). There is a partial action \(\gamma\) of \(G(x)\) on \(C\) defined in the following way. Given \(z \in G_0\) and \(h \in G(x)\), set \(C_{z,h} := \alpha_{t_z}(A_h)\) and
\[
C_h := \bigoplus_{u \in G_0^2} C_{t(u),h} \delta_u = \bigoplus_{u \in G_0^2} \alpha_{t(u)}(A_h) \delta_u.
\]
Also, \(\gamma_{z,h} : C_{z,h} \to C_{z,h}\) is defined by \(\alpha_{t_z}(a) \mapsto \alpha_{t_z}(\alpha_h(a))\), for all \(a \in A_{h^{-1}}\). These maps induce the following ring isomorphism
\[
\gamma_h : C_{h^{-1}} \to C_h, \quad \gamma_h(\alpha_{t(u)}(a) \delta_u) = \alpha_{t(u)}(\gamma_{h,u}(a)) \delta_u, \quad a \in A_{h^{-1}}, \quad u \in G_0^2.
\]
From Lemmas 4.2 and 4.3 of [8] follow that \(\gamma = (C_h, \gamma_{h})_{h \in G(x)}\) is a unital partial action of \(G(x)\) on \(C\). Moreover \(C_h = C_{1_h}\), where \(1_h = \sum_{z \in G_0} \alpha_{t_z}(1_2) \delta_{(z, z)}\), for all \(h \in G(x)\). In order to enunciate Theorem 4.4 of [8], which give us the factorization of \(A \ast \alpha G\), we consider the notation: \(g_x = \tau_{(t(g), s(g))} g \in G(x)\), for all \(g \in G\).

**Theorem 2.10.** Suppose that \(G\) is connected, \(G_0\) is finite, \(\alpha\) is group-type and \(A = \bigoplus_{x \in G_0} A_x\). Then the map \(\psi : A \ast \alpha G \to (A \ast \beta G_0^2) \ast \gamma G(x)\) given by
\[
\psi(a \delta_g) = a \delta_{(s(g), t(g))} \delta_e, \quad a \in A_g, \quad g \in G,
\]
is a ring isomorphism.

The factorization of \(A \ast \alpha G\) given above will be useful in the rest of the paper. We end the background with the following.

**Lemma 2.11.** Suppose that \(\alpha\) is group-type. If \(\alpha\) is finite-type, then \(\gamma\) defined in (6) is finite-type.

**Proof.** Since \(\alpha\) is finite-type, there are \(g_1, \ldots, g_n \in G(1, x)\) such that \(A_x = \sum_{i=1}^n A_{g_i}\). For each \(h \in G(x)\) we take \(\{hg_1 \tau_{(g_1)}, \ldots, hg_n \tau_{(g_n)}\} \subseteq G(x)\). Then
\[
A \ast \beta G_0^2 = \bigoplus_{u \in G_0^2} A_{t(u)} \delta_u = \bigoplus_{u \in G_0^2} \alpha_{t(u)}(A_x) \delta_u = \bigoplus_{u \in G_0^2} \sum_{i=1}^n \alpha_{t(u)}(A_{h_{g_i}}) \delta_u = \sum_{i=1}^n \bigoplus_{u \in G_0^2} \alpha_{t(u)}(A_{h_{g_i}}) \delta_u \subseteq \sum_{i=1}^n \bigoplus_{u \in G_0^2} \alpha_{t(u)}(A_{h_{g_i}} \tau_{(h_{g_i})}) \delta_u \subseteq \bigoplus_{h \in G_0^2} C_{h_{g_i} \tau_{(h_{g_i})}} \subseteq A \ast \beta G_0^2.
\]
Hence \(A \ast \beta G_0^2 = \sum_{i=1}^n C_{g_i \tau_{(g_i)}}\) and we conclude that \(\gamma\) is finite-type. \(\square\)
3. Ring theoretic properties of \( A \star \alpha G \)

In all what follows in this section, \( G \) is a connected groupoid such that \( G_0 \) is finite and \( \alpha = (A_g, \alpha_g)_{g \in G} \) is a group-type unital partial action of \( G \) on \( A = \bigoplus_{z \in G_0} A_z \). We will assume that there is \( x \in G_0 \) and \( \tau(x) = \{ \tau_y : y \in G_0 \} \) a transversal for \( x \) that satisfies (1) and \( A_g = A_1 g \), where \( 1_g \) is a central idempotent of \( A \), for all \( g \in G \). We also assume that \( \beta \) and \( \gamma \) are as in the previous section.

The aim in this section is to establish relations between ring theoretic properties of \( A \) and \( A \star \alpha G \). First, observe that

\[
\varphi : A \rightarrow A \star \alpha G \text{ given by } \varphi(a) = \sum_{z \in G_0} (a_1 z) \delta_z, \quad a \in A,
\]

is an injective ring homomorphism and whence we identify \( A \) as a unital subring of \( A \star \alpha G \). Moreover \( \varphi \) induces the following ring isomorphisms \( A \simeq \bigoplus_{z \in G_0} A_z \delta_z \simeq \bigoplus_{w \in (G_0)^2} A_w \delta_w \).

Consequently

\[
A \simeq (A \star \alpha G)_0 \simeq (A \star \beta G^2)_0.
\]

3.1. Noetherianity. The relation of the artinian property between \( A \) and \( A \star \alpha G \) was given in Theorem 1.3 of [21], for an arbitrary groupoid \( G \). In our context, this result was refined in Theorem 5.11 of [8]. Here we explore the relation of the noetherian property between \( A \) and \( A \star \alpha G \).

We recall some notions and facts on group and groupoid graded rings. We say that a ring \( R \) is called \( G \)-graded if there is a set \( \{ R_g \}_{g \in G} \) of additive subgroups of \( R \) such that \( R = \bigoplus_{g \in G} R_g \), \( R_g R_h \subseteq R_{gh} \) if \( (g, h) \in G_s \times_G G \) and \( R_g R_h = \{ 0 \} \) otherwise. A \( G \)-graded ring \( R = \bigoplus_{g \in G} R_g \) that satisfies \( R_g R_h = R_{gh} \) for all \( (g, h) \in G_s \times_G G \) is said strongly graded. Then by setting \( (A \star \alpha G)_g = A_g \delta_g \) for any \( g \in G \), it follows from (3) that \( A \star \alpha G \) is a \( G \)-graded ring. Moreover, by Proposition 3.4 of [6], \( A \star \alpha G \) is strongly graded if and only if \( \alpha \) is global. If \( G = G \) is a group we say that \( G \)-graded ring \( R = \bigoplus_{g \in G} R_g \) is epsilon-strongly graded if \( R_g R_h = R_g R_{g^{-1}} R_{gh} = R_{gh} R_{h^{-1}} R_h \), for all \( g, h \in G \). It is not difficult to see that unital partial crossed products are epsilon-strongly graded, see Theorem 35 of [22] for details.

**Lemma 3.1.** Let \( G \) be a finite groupoid, \( R = \bigoplus_{g \in G} R_g \) be a strongly \( G \)-graded unital ring and \( R_0 = \bigoplus_{z \in G_0} R_z \). If \( R_0 \) is left (right) noetherian, then \( R \) is left (right) noetherian.

**Proof.** Since \( R_g R_{g^{-1}} = R_{t(g)} \), for all \( g \in G \), we obtain a \((R_g, R_{g^{-1}})\)-bimodule epimorphism \( \mu : R_g \otimes_{R_{s(g)}} R_{g^{-1}} \rightarrow R_{t(g)} \). Similarly, there exists a \((R_{g^{-1}}, R_{g})\)-bimodule epimorphism \( \tau : R_{g^{-1}} \otimes_{R_{s(g)}} R_{g} \rightarrow R_{s(g)} \). Then, the Morita context \((R_{t(g)}, R_{s(g)}, R_g, R_{g^{-1}}, \mu, \tau)\) is strict. Thus \( R_g \) is a left projective and finitely generated \( R_{t(g)} \)-module and \( R_g \) is a right projective and finitely generated \( R_{s(g)} \)-module. Consequently, \( R_g \) is a left projective and finitely generated right \( R_0 \)-module. Using that \( G \) is finite we conclude that \( R \) is a left and right finitely generated \( R_0 \)-module. The fact that \( R_0 \) is left (right) noetherian implies that \( R \) is a left and right noetherian \( R_0 \)-module. Finally, from \( R_0 \subseteq R \) we conclude that \( R \) is a left and right noetherian \( R \)-module and the result follows. \( \square \)
In order to prove the main result of this subsection, we recall the following. A group $G$ is called a \textit{polycyclic-by-finite} if there exists a subnormal series
\[
\{1\} = G_0 \leq G_1 \leq \ldots \leq G_n \leq G_{n+1} = G
\]
such that $G/G_n$ is finite and $G_{i+1}/G_i$ is cyclic, for all $0 \leq i \leq n - 1$.

\textbf{Theorem 3.2.} The following assertions hold.

(i) If $A \ast_\alpha G$ is left noetherian then $A$ and $A \ast_\beta G_0^2$ are left noetherian.

(ii) If $A$ is left noetherian and $G(x)$ is a polycyclic-by-finite group then $A \ast_\alpha G$ is left noetherian.

\textbf{Proof.} (i) From Theorem 2.10 and Proposition 3.4 of [21] follow that $A \ast_\beta G_0^2$ is left noetherian. Applying again Proposition 3.4 of [21] for $A \ast_\beta G_0^2$ we conclude that $A$ is left noetherian.

(ii) By Lemma 3.1 the ring $C := A \ast_\beta G_0^2$ is left noetherian. Since $C \ast_\gamma G(x)$ is epsilon strongly graded Theorem 3.7 of [17] implies that it is left noetherian and whence the result follows from Theorem 2.10. \hfill \Box

\textbf{3.2. Von Neumann regularity.} Recall that a unital associative ring $R$ is von Neumann regular if $a \in aRa$, for all $a \in R$. There are several equivalent conditions to the notion of von Neumann regularity. Here, we will use the following equivalences: $R$ is von Neumann regular $\iff$ every principal left ideal is a direct summand of the left $R$-module $R$ $\iff$ if every finitely generated left ideal is a direct summand of the left $R$-module $R$.

The next definition is inspired in the group case which was considered in Section 2.2 of [18]. A G-graded ring $R = \bigoplus_{g \in G} R_g$ will be called a \textit{graded von Neumann regular} if and only if $a \in aRa$, for each $a \in R_g$ and $g \in G$. Clearly, a G-graded ring that is von Neumann regular is graded von Neumann regular. The converse is not true even in the group case; see Example 2.4 of [18].

\textbf{Remark 3.3.} Let $B := A \ast_\alpha G$ and $C := A \ast_\beta G_0^2$. By Theorem 2.10, we have $B \simeq C \ast_\gamma G(x)$. Notice that if $B$ is $G(x)$-graded von Neumann regular then $B$ is $G$-graded von Neumann regular. In fact, let $g \in G$ and $a_g\delta_g \in A \ast_\alpha G$. By (7), $\psi(a_g\delta_g) \in (C \ast_\gamma G(x))_{g\gamma}$. Then $\psi(a_g\delta_g) \in \psi(a_g\delta_g)(C \ast_\gamma G(x))\psi(a_g\delta_g)$ which implies that $a_g\delta_g \in (a_g\delta_g)(A \ast_\alpha G)(a_g\delta_g)$. Therefore $A \ast_\alpha G$ is G-graded von Neumann regular.

The next auxiliary result has immediate proof which will be omitted.

\textbf{Lemma 3.4.} Let \{$I_j\}_{1 \leq j \leq n}$ be a family of ideals of a ring $R$ that satisfies $I_iI_j = I_jI_i = 0$ for all $i \neq j$. Then $\sum_{j=1}^{n} I_j$ is von Neumann regular if and only if $I_j$ is von Neumann regular, for all $1 \leq j \leq n$.

Now we present an extension of Theorem 3 of [26] for the context of groupoid graded rings.

\textbf{Proposition 3.5.} Let $R = \bigoplus_{g \in G} R_g$ be a G-graded ring and $R_0 = \bigoplus_{z \in G_0} R_z$. Then the following assertions hold.

(i) If $R$ is graded von Neumann regular then $R_0$ is von Neumann regular.
(ii) If \( R_0 \) is von Neumann regular, \( R \) is strongly \( G \)-graded and unital, then \( R \) is graded von Neumann regular.

Proof. (i) By Lemma 3.4, it is enough to show that \( R_z \) is von Neumann regular, for all \( z \in G_0 \). Let \( a \in R_z \). Then there is \( r \in R \) such that \( a = ara \). Assume that \( r = \sum_{g \in G} r_g \), with \( r_g \in R_g \) for all \( g \in G \). Notice that \( R_z \mathcal{R} R_z \subseteq R_z \). Hence \( ara - ars \in R_z \) and consequently

\[
ara - ars = \sum_{g \in G \setminus \{z\}} ar_g a \in R_z \cap \left( \sum_{g \in G \setminus \{z\}} R_g \right) = 0.
\]

Thus \( a = ara = ars \in aR_z a \) and whence \( R_z \) is von Neumann regular.

(ii) Let \( a \in R_g \) and \( g \in G \). Since \( R \) is strongly \( G \)-graded, \( R_g \) is a left \( R_{t(g)} \)-module projective and finitely generated (see the proof of Lemma 3.1). Now, \( R_g - 1 a = R_{s(g)} \), that is, \( R_g - 1 a \) is a left ideal of \( R_{s(g)} \). Since \( R_{s(g)} \) is von Neumann regular, there is an idempotent \( u \in R_{s(g)} \) such that \( R_g - 1 a = R_{s(g)} u \). Note also that, by Proposition 2.1.1 of [19], \( 1_R = \sum_{g \in G_0} 1_z \) and \( R_{s(g)} \) is a unital ring with identity element \( 1_{s(g)} \). Consequently \( u = 1_{s(g)} u = ra \), for some \( r \in R_{g - 1} \). Moreover \( R_{t(g)} a = R_g R_{g - 1} a = R_g R_{s(g)} u = R_g u \) which implies that \( a = 1_{t(g)} a = bu \) for some \( b \in R_g \). Then \( ara = au = bu \cdot bu \). Hence \( R \) is graded von Neumann regular. \( \square \)

For the rest of this subsection we will assume that \( G \) is finite. Recall that the trace map \( \text{tr}_\alpha : A \to A \) associated to the partial action \( \alpha \) is defined by

\[
\text{tr}_\alpha(a) = \sum_{g \in G} \alpha_g(a1_{g^{-1}}), \quad a \in A.
\]

Since any group is a groupoid, the trace map is defined to partial group actions. Notice that \( \text{tr}_\alpha(1_A) = \sum_{g \in G} 1_g \). The trace map has a main role in the Galois theory for partial groupoid actions. More details can be seen in [7].

In the sequel we prove two auxiliary results.

Lemma 3.6. Let \( (C, \gamma) \) be the partial action of \( G(x) \) on \( C = A \ast \beta G_0^2 \) given by (5) and (6) and \( a \in A_x \). Then:

(i) \( \text{tr}_\gamma(1_C) = \sum_{z \in G_0} \alpha_{\tau_z}(\text{tr}_{\alpha_{G(z)}}(1_x))\delta_{z(z)} \).

(ii) An element \( b = \sum_{z \in G_0} \alpha_{\tau_z}(a)\delta_{z(z)} \in C \) is invertible in \( C \) if and only if \( a \) is invertible in \( A_x \). In this case, \( b^{-1} = \sum_{z \in G_0} \alpha_{\tau_z}(a^{-1})\delta_{z(z)} \).

Proof. Since \( C_h = C_1 \) with \( 1'_h = \sum_{z \in G_0} \alpha_{\tau_z}(1_h)\delta_{z(z)} \) and \( 1_C = \sum_{y \in G_0} 1_y\delta_{y(y)} \) we have

\[
\text{tr}_\gamma(1_C) = \sum_{h \in G(x)} \sum_{z \in G_0} \alpha_{\tau_z}(1_h)\delta_{z(z)} = \sum_{z \in G(x)} \alpha_{\tau_z} \left( \sum_{h \in G(x)} 1_h \right)\delta_{z(z)} = \sum_{z \in G_0} \alpha_{\tau_z} \left( \sum_{h \in G(x)} \alpha_{G(x)}(1_x 1_{h^{-1}}) \right)\delta_{z(z)} = \sum_{z \in G_0} \alpha_{\tau_z}(\text{tr}_{\alpha_{G(x)}}(1_x))\delta_{z(z)},
\]

and whence (i) follows. For (ii), assume that \( a \) is invertible in \( A_x \) and consider the element \( c = \sum_{z \in G_0} \alpha_{\tau_z}(a^{-1})\delta_{z(z)} \in C \). Thus, \( bc = \sum_{z \in G_0} \alpha_{\tau_z}(1_x)\delta_{z(z)} = \sum_{z \in G_0} 1_x\delta_{z(z)} = 1_C \).
and whence \( c = b^{-1} \). Conversely, consider \( c \in C \) the inverse of \( b \). Then, there are elements \( a_u \in B_u = A_{t(u)} \) such that \( c = \sum_{u \in G_0^2} a_u \delta_u \). As \( \alpha_{t(u)} : A_x \to A_{t(u)} \) is a ring isomorphism, for each \( u \in G_0^2 \), there is \( a_{x,u} \in A_x \) such that \( a_u = \alpha_{t(u)}(a_{x,u}) \). Hence,

\[
\sum_{z \in G_0} 1_z \delta_{(z,z)} = 1_C = bc = \sum_{z \in G_0} \sum_{u \in G(z,z)} \alpha_{t_z}(a_{x,u})\delta_u
\]

\[
= \sum_{z \in G_0} \sum_{y \in G_0} \alpha_{t_z}(a_{x,(y,z)})\delta_{(y,z)}.
\]

Thus, \( a_{x,(y,z)} = 0 \) if \( y \neq z \) and \( a_{x,(z,z)} = 1_x \). Then \( a_{x,(z,z)} = a^{-1} \) and we have that \( c = \sum_{z \in G_0} \alpha_{t_z}(a^{-1})\delta_{(z,z)} \).

**Lemma 3.7.** Let \((A, \beta)\) be the global action of \(G_0^2\) on \(A\) given in the previous section. Then \(tr_\beta(1_A) = |G_0|1_A\).

**Proof.** It is clear that

\[
tr_\beta(1_A) = \sum_{u \in G_0^2} \beta_u(1_{s(u)}) = \sum_{u \in G_0^2} 1_{t(u)} = \sum_{(y,z) \in G_0^2} 1_z = |G_0| \sum_{z \in G_0} 1_z = |G_0|1_A,
\]

because \(A = \bigoplus_{z \in G_0} A_z\). \qed

Now we can prove the main result of this subsection.

**Theorem 3.8.** Assume that \(G\) is finite and let \(\beta\) be the global action of \(G\) on \(A\) given in (4). Then the following assertions hold.

(i) If \(A\) is von Neumann regular and \(I\) is a left principal ideal of \(A \ast \beta G_0^2\), then \(I\) is an \(A\)-direct summand of \(A \ast \beta G_0^2\).

(ii) If \(|G_0|1_A\) is invertible in \(A\) then the following statements are equivalent:

(a) \(A\) is von Neumann regular,

(b) \(A \ast \beta G_0^2\) is graded von Neumann regular,

(c) \(A \ast \beta G_0^2\) is von Neumann regular,

(d) \(A \ast \alpha G\) is graded von Neumann regular.

Moreover, under the assumption that at least one of the above statements holds and \(tr_{\alpha G(\nu)}(1_x)\) is invertible in \(A_x\) the following additional statement holds:

(e) \(A \ast \alpha G\) is von Neumann regular.

**Proof.** For (i), consider \(u, v \in G_0^2\) such that \(s(v) = t(u)\) and \(a \in B_u = A_{t(u)}\). Then \(A_{t(v)}\delta_v \cdot a\delta_u = A_{t(v)}\delta_v(a1_{s(v)})\delta_{vu}\). Suppose that \(I = (A \ast \beta G_0^2)l\), with \(l = \sum_{u \in G_0^2} a_u \delta_u\). Then

\[
I = \sum_{u,v \in G_0^2} A_{t(v)}\delta_v a_u \delta_u = \sum_{t(u)=s(v)} A_{t(v)}\delta_v(a_u1_{s(v)})\delta_{vu}.
\]

For each \(u \in G_0^2\), let

\[
I_u := \sum_{s(v)=t(u)} A_{t(v)}\delta_v(a_u1_{s(v)}) = \sum_{s(v)=t(u)} A1_{t(v)}\delta_v(a_u1_{s(v)}).
\]
It is clear that $I_u$ is a finitely generated left ideal of $A$. Since $A$ is von Neumann, $I_u$ is a direct summand of the left $A$-module $A$. Hence $I$ is an $A$-direct summand of $C := A \ast \beta G^2_0$.

For (ii), using the identification map (8) we have by (9) that $C$ is strongly graded and $(A \ast \beta G^2_0)_0 = C_0 = A$. Thus, the equivalence $(a) \iff (b)$ follows from Proposition 3.5. In order to prove $(a) \implies (c)$, consider a principal left ideal $I$ of $C$. By (i), $I$ is an $A$-direct summand of $C$. Also, by Lemma 3.7, $\text{tr}_\beta(1_A) = |G_0|1_A$. Then, Theorem 5.5 of [14] implies that $I$ is a $C$-direct summand of $C$. Thus $C$ is von Neumann regular. For $(c) \implies (d)$, observe that $C \ast_\gamma G(x)$ is an epsilon strongly $G(x)$-graded ring such that $(C \ast_\gamma G(x))_x = C$. Then, from Corollary 3.11 of [18] we obtain that $C \ast_\gamma G(x)$ is $G(x)$-graded von Neumann regular. By Remark 3.3, $A \ast_\alpha G$ is $G$-graded von Neumann regular. Finally, note that $(d) \implies (a)$ follows from Proposition 3.5 because $(A \ast_\alpha G)_0 = \bigoplus_{\alpha \in G_0} A_\alpha \delta_\alpha \simeq A$. To finish the proof we suppose also that $\text{tr}_{G(x)}(1_x)$ is invertible in $A_x$ and prove (e). Indeed, it follows from items (i) and (ii) of Lemma 3.6 that $\text{tr}_\gamma(1_C)$ is an invertible element in the von Neumann regular ring $A \ast \beta G^2_0$. Hence, Theorem 4.3 of [13] implies that $(A \ast \beta G^2_0) \ast_\gamma G(x)$ is von Neumann regular. Thus, we obtain from Theorem 2.10 that $A \ast_\alpha G$ is von Neumann regular. 

**Remark 3.9.** Notice that the assumption $|G_0|1_A$ is invertible in $A$ in Theorem 3.8 was only used to prove $(a) \implies (c)$.

### 3.3. Jacobson radical

Here we investigate the relation between $J(A)$ and $J(A \ast_\alpha G)$. The following properties of $J(A)$ will be useful later.

**Lemma 3.10.** The following statements hold.

(i) $J(A)$ is an $\alpha$-invariant ideal of $A$, that is, $\alpha_g(J(A) \cap A_{g^{-1}}) \subset J(A)$ for all $g \in G$.
(ii) $J(A_{t(g)})1_g = J(A_g)$, for all $g \in G$; in particular $J(A_g)$ is unital.
(iii) $J(A) = \bigoplus_{\alpha \in G_0} J(A_\alpha)$.
(iv) $U(A) = U(A \ast_\alpha G) \cap A$.
(v) $J(A) = A \cap J(A \ast_\alpha G)$.

**Proof.** Let $g \in G$. As $J(A)$ is an hereditary radical and $\alpha_g$ is a ring isomorphism one has

$$\alpha_g(J(A) \cap A_{g^{-1}}) = \alpha_g(J(A_{g^{-1}})) = J(\alpha_g(A_{g^{-1}})) = J(A_g) = J(A) \cap A_g,$$

and whence (i) follows. Notice that (iii) is immediate because $A = \bigoplus_{\alpha \in G_0} A_\alpha$. To prove (ii), observe that $A_{t(g)} = A_g \oplus A_{t(g)}(1_{t(g)} - 1_g)$. Consequently, we have that $J(A_{t(g)}) = J(A_g) \oplus J(A_{t(g)}(1_{t(g)} - 1_g))$ which implies that $J(A_{t(g)})1_g = J(A_g)$.

The inclusion $\subseteq$ in (iv) is clear. For the reverse, consider the $(A, A)$-bimodule epimorphism

$$(10)\quad \pi_A: A \ast_\alpha G \to A, \quad \pi_A\left(\sum_{g \in G} a_g \delta_g\right) = \sum_{e \in G_0} a_e \delta_e.$$

Let $u \in U(A \ast_\alpha G) \cap A$. Then there is $v \in A \ast_\alpha G$ such that $uv = vu = 1_{A \ast_\alpha G}$. Since $\pi_A(u) = u$ it follows that $u\pi_A(v) = \pi_A(v)u = 1_A$ and consequently $u \in U(A)$. 

□
(v) Let $a = \sum_{g \in G} a_g \delta_g \in A \ast_{\alpha} G$, where $a_g \in A_g$, for all $g \in G$. Observe that $a_g \delta_g = (1_g \delta_g)(\alpha_g^{-1}(a) \delta_{s(g)}) = (1_g \delta_g)\varphi(\alpha_g^{-1}(a)) \in (1_g \delta_g)A$, being $\varphi$ the inclusion map defined in (8). Hence $a \in \sum_{g \in G}(1_g \delta_g)A$ and $A \ast_{\alpha} G = \sum_{g \in G}(1_g \delta_g)A$. Let $M$ be a simple right $A \ast_{\alpha} G$-module. Then, given a non-zero element $m \in M$, we have $M = m(A \ast_{\alpha} G) = \sum_{g \in G} (m(1_g \delta_g))A$, thus $M$ is a finitely generated $A$-module. Fix $N := MJ(A)$. We shall check that $N = \{0\}$. If $g \in G$ and $t \in A$, then

$$N(1_t \delta_g) = M(J(A)t)(1_g \delta_g) \subseteq MJ(A)(1_g \delta_g) \overset{(iii)}{=} MJ(A_{t(g)})(1_g \delta_g)$$

$$\overset{(ii)}{=} MJ(A_g)(1_g \delta_g) \overset{(*)}{=} M(1_g \delta_g)J(A_{g^{-1}}) \subseteq M(1_g \delta_g)J(A_{s(g)})$$

$$\subseteq MJ(A) = N.$$ 

Note that $(*)$ is true because $b(1_g \delta_g) = (1_g \delta_g)(\alpha_{g^{-1}}(b) \delta_{s(g)})$ and $\alpha_{g^{-1}}(b) \in J(A_{g^{-1}})$. Therefore $N(1_t \delta_g) = N$, for all $t \in A$ and $g \in G$, which implies that $N$ is a right $A \ast_{\alpha} G$-submodule of $M$. Since $M$ is simple, $N = M$ or $N = \{0\}$. If $N = M$ then $MJ(A) = M$. Using that $M$ is an $A$-module finitely generated, we obtain from Lemma 2.4 of [25] that $M = \{0\}$ which is a contradiction. Therefore $N = \{0\}$ and thus $J(A)$ annihilates every simple $A \ast_{\alpha} G$-module. Consequently $J(A) \subseteq J(A \ast_{\alpha} G)$. For the reverse, we recall that $a \in J(A)$ if and only if $1 - xa \in U(A)$. Hence it follows from (iv) that $A \cap J(A \ast_{\alpha} G) \subseteq J(A)$. 

We end this subsection with the following.

**Theorem 3.11.** If $\text{tr}_{\alpha(G)}(1_x)$ is invertible in $A_x$ and $|G_0|A$ is invertible in $A$ then

$$J(A \ast_{\alpha} G) \simeq J(A) \ast_{\alpha} G.$$ 

**Proof.** It is immediate from Theorem 2.10 that $J(A \ast_{\alpha} G) \simeq J((A \ast_{\beta} G_0^2) \ast_{\gamma} G(x))$. By Lemma 3.6, $\text{tr}_{\gamma}(1_C)$ is invertible in $C = A \ast_{\beta} G_0^2$. Then, it follows from Proposition 6.7 of [13] that $J((A \ast_{\beta} G_0^2) \ast_{\gamma} G(x)) = J(A \ast_{\beta} G_0^2) \ast_{\gamma} G(x)$. Also, by Lemma 3.7 and Theorem 5.7 of [14] we have that $J(A \ast_{\beta} G_0^2) = J(A \ast_{\beta} G_0^2)$. Hence

$$J((A \ast_{\beta} G_0^2) \ast_{\gamma} G(x)) = (J(A) \ast_{\beta} G_0^2) \ast_{\gamma} G(x).$$ 

By Lemma 3.10 (i) and Remark 2.8 the restriction of $\alpha$ to $J(A)$, which will be denoted by $\alpha|_{J(A)}$, is also a group-type partial action. Moreover, Lemma 3.10 (ii) implies that $\alpha|_{J(A)}$ is unital. Applying Theorem 2.10 for the partial groupoid action $\alpha|_{J(A)}$ we obtain that $J(A) \ast_{\alpha} G \simeq (J(A) \ast_{\beta} G_0^2) \ast_{\gamma} G(x)$. Consequently $J(A \ast_{\alpha} G) \simeq J(A) \ast_{\alpha} G$, as desired. 

### 3.4. Right perfect property.

A ring $R$ is said to be **right perfect** if every right $R$-module has a projective cover. It is known that $R$ is right perfect if and only if $R/J(R)$ is right artinian and $J(R)$ is right $T$-nilpotent (that is, for every infinite sequence of elements of $J(R)$, there is an $n$ such that the product of first $n$ terms is zero). Also, $R$ is right perfect if and only if $R$ satisfies the descending chain condition (DCC) on principal left ideals. These equivalences for the notion of right perfect ring can be seen, for instance, in [5].

The next result has the same proof of Lemma 3.4 of [25].
Lemma 3.12. Let $R$ be a ring and $\theta$ a unital global action of $G$ on $R$. Assume that $I$ is a $\theta$-invariant ideal of $R$. Then the following assertions hold.

(i) If $I$ is nilpotent, then $I\ast_{\theta} G$ is nilpotent.
(ii) If $I$ is right $T$-nilpotent, then $I\ast_{\theta} G$ is right $T$-nilpotent.

We need one more auxiliary result.

Lemma 3.13. Let $\theta = (S_{g}, \theta_{g})$ be a global action of a groupoid $G$ on a ring $S$ and $I$ a $\theta$-invariant ideal of $S$. Then $\theta$ induces a global action $\overline{\theta}$ of $G$ on $S/I$. Moreover, the map
\[ \Psi : (S \ast_{\theta} G)/(I \ast_{\theta} G) \rightarrow (S/I) \ast_{\overline{\theta}} G, \quad a\delta_{g} + (I \ast_{\theta} G) \mapsto (a + I)\delta_{g}, \quad a \in S_{s(g)}, \]
is a well-defined ring isomorphism.

Proof. Let $g \in G$. Put $B_{g} = B_{t(g)} := (S_{t(g)} + I)/I$. Clearly, $B_{g} = B_{t(g)}$ is an ideal of $S/I$. As in the proof of Lemma 2.2 of [13], the ring isomorphism $\theta_{g} : S_{g}^{-1} \rightarrow S_{g}$ induces the following ring isomorphism
\[ \overline{\theta}_{g} : B_{g}^{-1} \rightarrow B_{g}, \quad \overline{\theta}_{g}(a + I) = \theta_{g}(a) + I, \quad a \in S_{s(g)}. \]

Since $I$ is $\theta$-invariant it follows that $\overline{\theta}_{g}$ is well-defined ring isomorphism, for each $g \in G$. It is straightforward to check that $\overline{\theta} = (B_{g}, \overline{\theta}_{g})_{g \in G}$ is a global action of $G$ on $S/I$. Finally, consider the surjective ring homomorphism $\psi : S \ast_{\theta} G \rightarrow (S/I) \ast_{\overline{\theta}} G$ given by $\psi(a\delta_{g}) = (a + I)\delta_{g}$, for all $a \in S_{s(g)}$. It is clear that $\ker \psi = I \ast_{\theta} G$ and whence we obtain the ring isomorphism $\Psi$. \hfill \Box

The main result of this subsection is the following.

Theorem 3.14. Suppose that $\alpha$ is finite-type. Then $A \ast_{\alpha} G$ is right perfect if and only if $A$ is right perfect and $G$ is finite.

Proof. Suppose that $A \ast_{\alpha} G$ is right perfect. Observe that by (8), $A$ is a subring of $A \ast_{\alpha} G$. Moreover $A$ is a right $A$-direct summand of $A \ast_{\alpha} G$. From Proposition 2.1 of [24] it follows that $(A \ast_{\alpha} G)I \cap A = I$ for every left ideal $I$ of $A$. Since $A \ast_{\alpha} G$ satisfies the (DCC) for left principal ideals it follows that $A$ satisfies the (DCC) for left principal ideals. Thus, $A$ is right perfect. Also, by Theorem 2.10 we have that $(A \ast_{\beta} G_{0}^{2}) \ast_{\gamma} G(x)$ is right perfect, where $\gamma$ is finite-type (by Lemma 2.11). Then, Theorem 34 of [11] implies that $G(x)$ is finite. Therefore, by Proposition 2.1 we conclude that $G$ is finite.

Conversely, suppose that $A$ is right perfect and $G$ is finite. We claim that $C := A \ast_{\beta} G_{0}^{2}$ is right perfect. In fact, fix $B := A/J(A)$. Notice that $G_{0}^{2}$ acts globally on $A$ by $\beta$ and $J(A)$ is $\beta$-invariant. Hence, by Lemma 3.13, $\beta$ induces a global action $\overline{\beta}$ on $B$. Also, $B$ is right artinian because it is semisimple. As $G_{0}^{2}$ is finite we have by Theorem 1.2 of [21] that $B \ast_{\overline{\beta}} G_{0}^{2}$ is right artinian. Using Lemma 3.13 we obtain that
\[ (11) \quad B \ast_{\overline{\beta}} G_{0}^{2} \simeq C/(J(A) \ast_{\beta} G_{0}^{2}) \]
and $C/(J(A) \ast_{\beta} G_{0}^{2})$ is right artinian. We claim that $C/J(C)$ is a ring epimorphic image of $C/(J(A) \ast_{\beta} G_{0}^{2})$. In fact, for $u \in G_{0}^{2}$ and $a_{u}\delta_{u} \in J(A) \ast_{\beta} G_{0}^{2}$ we have $a_{u} \in J(A) \cap A_{t(u)}$. From Lemma 3.10 (v) it follows that $a_{u}1_{C} \in J(C)$. Since $J(C)$ is an ideal of $C$ we obtain that $a_{u}\delta_{u} = (a_{u}1_{C})(1_{u}\delta_{u}) \in J(C)$ and whence $J(A) \ast_{\beta} G_{0}^{2} \subset J(C)$. Consequently,
Lemma 6.1.14 of [20] implies that $C$ module. Moreover, it is easy to see that $K\dim$ has right Krull dimension and $A$ dimension as a right $R$ module we conclude that $A$ has right Krull dimension. Theorem 3.16. Assume that $A$ is right artinian and $J$ is nilpotent. Since $J$ nilpotent. Let $\pi$ be the canonical homomorphism from $C$ onto $C/(J(A) \ast_\beta G_0^2)$. Notice that

$$\pi(J(C)) = J((J(A) \ast_\beta G_0^2)) \subseteq J\left(C/(J(A) \ast_\beta G_0^2)\right).$$

On the other hand, (11) implies that $J(B \ast_\gamma G_0^2) \simeq J\left(C/(J(A) \ast_\beta G_0^2)\right)$, consequently $J(C)/(J(A) \ast_\beta G_0^2)$ is nilpotent because it is embedded in the nilpotent ring $J(B \ast_\gamma G_0^2)$. Also, using that $A$ is right $T$-nilpotent, it follows from Lemma 3.12 (ii) that $J(A) \ast_\beta G_0^2$ is right $T$-nilpotent. Therefore $J(A \ast_\beta G_0^2)$ is right $T$-nilpotent. Hence, $C$ is right perfect. Then by Theorem 3.10 and Proposition 2.1 of [24] we obtain that $A \ast_\alpha G$ is right perfect, as desired. □

3.5. The semiprimary property. Recall that a ring $A$ is called semiprimary if $A/J(A)$ is right artinian and $J(A)$ is nilpotent. Clearly every semiprimary ring is right perfect.

Theorem 3.15. Suppose that $\alpha$ is finite-type. Then $A \ast_\alpha G$ is semiprimary if and only if $A$ is semiprimary and $G$ is finite.

Proof. Assume that $A \ast_\alpha G$ is semiprimary. Using Theorem 3.10 and Lemma 2.11, it follows from Theorem 35 of [11] that $A \ast_\beta G_0^2$ is semiprimary and $G(x)$ is finite. Then $G$ is finite thanks to Proposition 2.1. Since $A \ast_\beta G_0^2$ is semiprimary it is right perfect. Hence, Theorem 3.14 implies that $A$ is right perfect and whence $A/J(A)$ is right artinian. Also, using (v) of Lemma 3.10 we conclude that $J(A) \subseteq J(A \ast_\beta G_0^2)$. Consequently $J(A)$ is nilpotent.

Conversely, suppose that $A$ is semiprimary and $G$ is finite. Then $A$ is right perfect. As we saw in the proof of Theorem 3.14, $C := A \ast_\beta G_0^2$ is right perfect. Thus $C/J(C)$ is right artinian. Since $J(C)$ is nilpotent it follows from Lemma 3.12 (i) that $J(A) \ast_\beta G_0^2$ is nilpotent. Moreover, as in the proof of Theorem 3.14, $J(C)/(J(A) \ast_\beta G_0^2)$ is nilpotent. Then $J(C)$ is nilpotent and whence $C$ is semiprimary. Finally, since $G(x)$ is finite, the same argument shows that $C \ast_\gamma G(x)$ is semiprimary. Thus, the result follows from Theorem 2.10. □

3.6. Krull dimension. Let $R$ be a unital ring and $M$ a right $R$-module. The Krull dimension of $M$, usually denoted by $\text{Kdim}_R M$ is the deviation of the lattice $\mathcal{L}_R(M)$ of $R$-submodules of $M$. The right Krull dimension of a ring $R$ is defined as $\text{Kdim}_R R$. It is well-known that the right modules of Krull dimension 0 are the nonzero right artinian modules and that every right noetherian module has Krull dimension.

Theorem 3.16. Assume that $G$ is finite. If $A$ has right Krull dimension then $A \ast_\alpha G$ has right Krull dimension and $\text{Kdim}_A A \ast_\alpha G \leq \text{Kdim}_A$.

Proof. Let $u \in G_0^2$. Since $A$ has Krull dimension, it follows that $B_u = A_{t(u)}$ has Krull dimension as a right $A$-module. Using that $A_u$ and $A_{t(u)}\delta_u$ are isomorphic as right $A$-modules we conclude that $A_{t(u)}\delta_u$ has Krull dimension. Since $A \ast_\beta G_0^2 = \bigoplus_{u \in G_0^2} A_{t(u)}\delta_u$, Lemma 6.1.14 of [20] implies that $C := A \ast_\beta G_0^2$ has Krull dimension as a right $A$-module. Moreover, it is easy to see that $\text{Kdim}_A C_A = \text{Kdim}_A A_A = \text{Kdim}_A$. As in the
proof of Proposition 41 of [11], we have that
\[ \mathcal{L}_C(C) \to \mathcal{L}_A(C), \quad N \mapsto N, \]
is strictly increasing. Thus, \( C \) has Krull dimension and \( \text{Kdim} C \leq \text{Kdim} C_A = \text{Kdim} A \).
From Theorem 2.10 and Proposition 41 of [11] we obtain that \( A \ast \alpha G \) has right Krull dimension and \( \text{Kdim} A \ast \alpha G \leq \text{Kdim} C \leq \text{Kdim} A \). \hfill \Box

4. Applications to Leavitt path algebras

The Leavitt path algebra associated to a directed graph was introduced in [1] and [2] as follows. Denote a directed graph by \( E = (E^0, E^1, r, d) \), where \( E^0 \) and \( E^1 \) are countable sets and \( r, d : E^1 \to E^0 \) are maps. The elements of \( E^0 \) are called vertices and the elements of \( E^1 \) are called (real) edges. A sink is a vertex such that no edge emerges out of it. A path \( \mu \) of length \( n \) in \( E \) is a sequence of edges \( \mu = \mu_1 \cdots \mu_n \) such that \( r(\mu_i) = d(\mu_{i+1}) \) for \( i \in \{1, \ldots, n-1\} \). In such a case, we will write \( |\mu| = n \), \( d(\mu) := d(\mu_1) \) is the source of \( \mu \) and \( r(\mu) := r(\mu_n) \) is the target of \( \mu \), respectively, a vertex is considered as a path of length 0. We are using \( r, d, \) instead \( t, s \) to denote the target and source maps of a path because we want to avoid confusion with the notation used for the target and source maps in groupoids. We shall denote \( E^1(v, \cdot) := \{ f \in E^1 : d(f) = v \} \).

Definition 4.1. Let \( E = (E^0, E^1, r, d) \) be a directed graph and \( k \) a field. The Leavitt path \( k \)-algebra \( L_k(E) \) of \( E \) with coefficients in \( k \) is the free associative \( k \)-algebra generated by the set \( \{ v, f, f^* : v \in E^0, f \in E^1 \} \) with the following relations:

(i) for all \( v, w \in E^0, v^2 = v \) and \( vw = 0 \) if \( v \neq w \),
(ii) \( d(f)f = fr(f) = f \), for all \( f \in E^1 \),
(iii) \( r(f)f^* = f^*d(f) = f^* \), for all \( f \in E^1 \),
(iv) for all \( f, f' \in E^1, f^*f = r(f) \) and \( f^*f' = 0 \) if \( f \neq f' \),
(v) \( v = \sum_{f \in E^1(v, \cdot)} ff^* \), for every \( v \in E^0 \) such that \( d^{-1}(v) \) is non-empty and finite.

The symbols \( f^* \) for \( f \in E^1 \) are called ghost edges, also condition (v) above is called the Cuntz-Krieger relation. Notice that \( L_k(E) = \bigoplus_{v \in E^0} L_k(E)v \) and thus \( L_k(E) \) is unital if and only if \( E^0 \) is finite. In this case, \( 1_{L_k(E)} = \sum_{v \in E^0} v \).

In what follows in this section, \( E = (E^0, E^1, r, d) \) denotes a directed graph and \( L_k(E) \) is the corresponding Leavitt path algebra, where \( k \) is a field. Our purpose in this section is to apply the results of the previous section to \( L_k(E) \).

4.1. \( L_k(E) \) as a partial skew groupoid ring. Leavitt path algebras were realized as partial skew groupoid rings in Theorem 3.11 of [16]. For readers convenience, here we recall this realization.

Firstly, the free groupoid \( G(E) \) associated to \( E \) is constructed in the following way. The set of objects \( G(E)_0 \) of \( G(E) \) is \( E^0 \). We extend the maps \( r, d \) from \( E^1 \) to \( E^1 \cup (E^1)^* \) by putting: \( d(f^*) = r(f) \) and \( r(f^*) = d(f) \), for all \( f \in E^1 \). Then, for a finite path \( \mu = \mu_1 \cdots \mu_n \) in \( E \) we put \( \mu^* := \mu_n^* \cdots \mu_1^* \). Denote by
\[ P := \{ \xi_1 \xi_2 \cdots \xi_n : n \in \mathbb{N}, \xi_i \in E^1 \cup (E^1)^* \cup E^0 \text{ and } r(\xi_i) = d(\xi_{i+1}) \}. \]
An element \( \xi_1 \ldots \xi_k f_1 \ldots f_n \xi_{k+1} \ldots \xi_n \) in \( P \) can be reduced to \( \xi = \xi_1 \ldots \xi_k \ldots \xi_n \). Similarly, \( \xi \) is the reduction of \( \xi_1 \ldots \xi_k d(\xi_{k+1}) \xi_{r+1} \ldots \xi_n = \xi_1 \ldots \xi_r(\xi_k) \xi_{r+1} \ldots \xi_n \).

The elements of \( P \) that cannot be reduced are called irreducible. The set of irreducible elements of \( P \) will be denoted by \( \text{Irr}(P) \) and the reduction of an element \( \mu \in P \) will be denoted by \( \text{irr}(\mu) \). The set of morphisms of the \( G(E) \) is \( \text{Irr}(P) \). Given \( \mu, \xi \in \text{Irr}(P) \) such that \( \rho(\mu) = d(\xi) \), the composition is \( \mu \cdot \xi = \text{irr}(\mu \xi) \). It is clear that \( \xi^{-1} = \xi^* \), for all \( \xi \in \text{Irr}(P) \).

**Remark 4.2.** Notice that in the directed graph we compose from left to right while in the free path groupoid \( G(E) \) we compose from right to left. Thus, we have that \( t(\xi) = d(\xi) \) and \( s(\xi) = r(\xi) \), for all \( \xi \in G(E) \). Precisely, if \( \bullet^v \xrightarrow{\xi} \bullet^w \) is an edge in \( E \) then \( d(\xi) = v \) and \( r(\xi) = w \). However, \( \xi \) as an element of \( G(E) \) has \( s(\xi) = w \) and \( t(\xi) = v \). Clearly, given \( \xi, \eta \in E^1 \) we have that \( \xi \eta \in E \) if and only if \( \eta \xi \in G(E) \).

Let \( W \) be the set of all finite paths in \( E \). Denote by \( W^\infty \) the set of all infinite paths in \( E \), that is, \( \xi \in W^\infty \) if and only if \( \xi = \xi_1 \xi_2 \xi_3 \ldots \) with \( r(\xi_i) = d(\xi_{i+1}) \) and \( i \in \mathbb{N} \). Also, consider

\[
W^{(1)} = \{ a \in W \setminus E^0 : \text{irr}(a) = a \} \quad \text{and} \quad W^{(2)} = \{ (a, b) \in (W \setminus E^0)^2 : r(a) = r(b) \text{ and } \text{irr}(ab^{-1}) = ab^{-1} \}.
\]

We define a partial action \( \theta \) of \( G(E) \) on the set

\[
X = \{ \xi \in W : r(\xi) \text{ is a sink} \} \cup W^\infty,
\]

in the following way. For each \( g \in G(E) \), let \( X_g \) be defined by:

- \( X_v := \{ \xi \in X : d(\xi) = v \} \), for all \( v \in E^0 \),
- \( X_a := \{ \xi \in X : \xi_1 \xi_2 \ldots \xi_{|a|} = a \} \), for all \( a \in W^{(1)} \),
- \( X_{a^{-1}} := \{ \xi \in X : d(\xi) = r(a) \} \), for all \( a \in W^{(1)} \),
- \( X_{ab^{-1}} := X_a \), for all \( (a, b) \in W^{(2)} \),
- \( X_g := \emptyset \), for all other \( g \in G(E) \).

The bijections \( \theta_g \), for \( g \in G(E) \), are defined by: \( \theta_v := \text{id}_{X_v} \), for all \( v \in E^0 \), and

\[
\theta_a : X_{a^{-1}} \rightarrow X_a, \quad \theta_a(\xi) = a\xi, \quad a \in G(E) \setminus E^0
\]

\[
\theta_{ab^{-1}} : X_{ba^{-1}} \rightarrow X_{ab^{-1}}, \quad \theta_{ab^{-1}}(\xi) = a\xi_{(|b|+1)}\xi_{(|b|+2)} \ldots,
\]

for all \( (a, b) \in W^{(2)} \). It is clear that \( \theta_a \) and \( \theta_{ab^{-1}} \) have inverses maps given respectively by:

\[
\theta_a^{-1}(\xi) = \begin{cases} r(a), & \text{if } r(a) \text{ is a sink}, \\ \xi_{|a|+1}\xi_{|a|+2} \ldots, & \text{otherwise}, \end{cases}
\]

\[
\theta_{ba^{-1}}(\xi) = b\xi_{(|a|+1)}\xi_{(|a|+2)} \ldots.
\]
By Proposition 3.7 of [16], the family of pairs \( \theta = (X_g, \theta_g)_{g \in G(E)} \) is a partial action of \( G(E) \) on \( E \). Following [16], denote the algebra of functions from \( E \) to \( k \) by \( \mathcal{F}(E) \) and put \( \mathcal{F}(X_g) := \{ f \in \mathcal{F}(E) : f \text{ vanishes outside of } X_g \} \) for each \( X_g \neq \emptyset \) and \( \mathcal{F}(X_g) := \{ \text{null function} \} \) for each \( X_g = \emptyset \). Also, define \( \bar{\theta}_g(f) = f \circ \theta_{g^{-1}} \), for all \( f \in \mathcal{F}(X_{g^{-1}}) \). It follows from Proposition 3.8 of [16] that \( \bar{\theta} = (\mathcal{F}(X_g), \bar{\theta}_g)_{g \in G(E)} \) is a partial action of \( G(E) \) on \( \mathcal{F}(E) \).

For each \( g \in G(E) \), \( 1_g \) denotes the characteristic of \( X_g \), that is, \( 1_g(x) = 1 \) if \( x \in X_g \) and \( 1_g(x) = 0 \) if \( x \notin X_g \). Notice that \( 1_g \) is the identity element of \( \mathcal{F}(X_g) \). Let \( D(E) \) be the \( k \)-subalgebra of \( \mathcal{F}(E) \) generated by \( \{ 1_g : g \in G(E) \} \). Fix the ideal \( D(E)_g := 1_g D(E) \) of \( \mathcal{F}(X_g) \), for each \( g \in G \). Observe that \( D(E)_g \) is generated (as a subalgebra of \( D(X) \)) by \( \{ 1_g 1_h : h \in G(E) \} \). It is clear that the restriction of \( \bar{\theta}_g \) to \( D(E)_g \) is a bijection onto \( D(E)_g \). We will denote by \( \lambda_g \) the restriction of \( \bar{\theta}_g \) to \( D(E)_g \), that is,

\[
\lambda_g : D(E)_{g^{-1}} \to D(E)_g, \quad \lambda_g = \bar{\theta}_g|_{D(E)_{g^{-1}}}. 
\]

Then \( \lambda = (D(E)_g, \lambda_g)_{g \in G(E)} \) is a partial action of \( G(E) \) on \( D(E) \).

Now we can enunciate Theorem 3.11 of [16].

**Theorem 4.3.** \( L_k(E) \) is isomorphic (as a \( k \)-algebra) to \( D(X) \star_\lambda G(E) \).

**Remark 4.4.** Let \( E \) be a directed graph. From Proposition 1.2.14 of [4] follows that \( L_k(E) \simeq \bigoplus_{i \in I} L_k(E_i) \), where \( E = \bigsqcup_{i \in I} E_i \) is the decomposition of \( E \) in its connected components. For each \( E_i \), we denote the set given in \( (12) \) by \( X_i \), the free path groupoid by \( G(E_i) \) and the action of \( G(E_i) \) on \( D(X_i) \) by \( \lambda_i \). Thus, Theorem 4.3 implies that \( L_k(E) \simeq \bigoplus_{i \in I} D(X_i) \star_{\lambda_i} G(E_i) \).

**Remark 4.5.** The Leavitt path algebra \( L_k(E) \) can be realized as a partial skew group ring cf. Proposition 3.2 of [15]. In this case, the group that acts partially is the free group \( \mathbb{F} \) generated by \( E^1 \). On the other hand, suppose that the partial action \( \lambda \) given in \( (13) \) is group-type, that is, there exist \( v \in E^0 \) and a transversal \( \tau(v) \) for \( v \) such that \( \lambda \) satisfies \( (1) \). Assume that \( E^0 \) is finite. From Theorem 2.10 it follows that \( L_k(E) \) can be realized again as a partial skew group ring. In this case, the group acting partially is the isotropy group \( G(E)(v) \) which in general is easier to deal than \( \mathbb{F} \). Hence, sometimes it is more convenient to realize \( L_k(E) \) as a partial skew group ring using the partial action \( \lambda \); see Remark 4.21.

4.2. **Determining when \( \lambda \) is group-type.** Let \( E, D(X), \lambda \) and \( G(E) \) be as in the previous Subsection. In order to apply the results of Section 3 for a Leavitt path algebra \( L_k(E) \), we need to find under what conditions \( \lambda \) is a group-type partial action of \( G(E) \) on \( D(X) \). Thanks to Remark 4.4, we will assume from now on in this Subsection that \( E \) is connected and \( E^0 \) is finite.

**Lemma 4.6.** The following assertions hold.

(i) \( D(X) = \bigoplus_{v \in E^0} D(X)_v \).

(ii) \( X_v \neq \emptyset \), for all \( v \in E^0 \).
(iii) For every \( v \in E^0 \) such that \( E^1(v) \) is non-empty and finite set, we have that

\[
1_v = \sum_{f \in E^1(v)} 1_f \quad \text{and} \quad D(X)_v = \bigoplus_{f \in E^1(v)} D(X)_f.
\]

**Proof.** (i) We claim that \( D(X) = \sum_{v \in E^0} D(X)_v \). In fact, consider an element \( a = 1_{g_1}1_{g_2}\cdots 1_{g_n} \in D(X) \) with \( g_1, g_2, \ldots , g_n \in G(E) \). Since \( 1_g1_{t(g)} = 1_g \) for all \( g \in G(E) \), we have that \( a = 1_{t(g_1)}a \). Thus \( a \in D(X) \) is a direct combination of elements \( a \) as above, we conclude that \( D(X) = \sum_{v \in E^0} D(X)_v \). Also, the sum is direct because \( D(X)_v = D(X)1_v \) and \( 1_v1_{v_j} = 1_{v_j} \), for all \( v, v_1, v_2 \in E^0 \).

(ii) Let \( v \in E^0 \). If \( v \) is a sink then \( v \in X_v \) and \( X_v \) is non-empty. Suppose that \( v \) is not a sink. In this case, there exists \( \xi_1 \in E^1_\text{sink} \) such that \( d(\xi_1) = v \). If \( r(\xi_1) = v_1 \) is a sink \( \xi_2 \in X_v \). Otherwise, there exists \( \xi_2 \in E^1 \) such that \( d(\xi_2) = v_1 \). If \( r(\xi_2) = v_2 \) is a sink then \( \xi_1\xi_2 \in X_v \). Otherwise, we repeat the process. If this process is finite then we obtain an element \( \xi_1\xi_2\cdots\xi_n \in X_v \). If the process is infinite then we have an infinite path \( \xi = \xi_1\xi_2\cdots \) for which \( d(\xi) = v \), that is, \( \xi \in X_v \).

(iii) Let \( v \in E^0 \) such that \( E^1(v) \) is non-empty and finite. Given \( f, f' \in E^1 \), with \( f \neq f' \) and \( f, f' \in d^{-1}(v) \), we have that \( X_f \cap X_{f'} = \emptyset \). Hence, in order to show the equality on the left side in (iii) it is enough to verify that \( X_v = \bigcup_{f \in E^1(v)} X_f \). The inclusion \( \supseteq \) is immediate. For the reverse, consider \( \xi \in X_v \). Then there are \( f \in E^1 \) and \( \eta \in W \cup W^\infty \) with \( r(f) = d(\eta) \) such that \( \xi = f\eta \). Then \( \xi \in X_f \) and \( d(f) = d(\xi) = v \) which implies the inclusion \( \subseteq \). The equality \( D(X)_v = \bigoplus_{f \in E^1(v)} D(X)_f \) follows from the fact that \( 1_f1_p = \delta_{f,p}1_f \), for all \( f, p \in E^1 \).

For the next result, we recall that if \( \xi \) is an edge in \( E \) such that \( d(\xi) = v \) and \( r(\xi) = w \) then \( \xi \) as an element of \( G(E) \) has \( s(\xi) = w \) and \( t(\xi) = v \); see Remark 4.2. We also denote \( W^{(1)} := \{ a^{-1} : a \in W^{(1)} \} \).

**Theorem 4.7.** Suppose that \( E \) is connected. Then \( \lambda \) is a group-type partial action of \( G(E) \) on \( D(X) \) if and only if there are a vertex \( v \in E^0 \) of \( E \) and a transversal \( \tau(v) = \{ \tau_w : v \to w : w \in E^0 \} \) for \( v \) in \( G(E) \) such that \( \tau_w \in W^{(1)} \cup W^{(1)} \cup W^{(2)} \) for all \( w \in E^0, w \neq v \) and

- if \( \tau_w \in W^{(1)} \) then

\[
\{ \xi \in X : \xi_1\xi_2\cdots\xi_{|r^{-1}_w|} = \tau^{-1}_w \} = \{ \xi \in X : d(\xi) = v \},
\]

- if \( \tau_w \in W^{(1)} \) then

\[
\{ \xi \in X : \xi_1\xi_2\cdots\xi_{|\tau|} = \tau_w \} = \{ \xi \in X : d(\xi) = w \},
\]

- if \( \tau_w = a_wb_w^{-1} \) with \( (a_w, b_w) \in W^{(2)} \) then

\[
\{ \xi \in X : \xi_1\xi_2\cdots\xi_{|b_w|} = b_w \} = \{ \xi \in X : d(\xi) = v \},
\]

- if \( \tau_w = a_wb_w^{-1} \) with \( (a_w, b_w) \in W^{(2)} \) then

\[
\{ \xi \in X : \xi_1\xi_2\cdots\xi_{|a_w|} = a_w \} = \{ \xi \in X : d(\xi) = w \},
\]

and

\[
\tau_w \in W^{(1)} \cup W^{(1)} \cup W^{(2)} \quad \text{for all} \quad w \neq v.
\]
In both cases, we have that $\eta$ is acyclic. and only if $1_{\tau} = 1_w$, for all $w \in E^0$. Consequently, $\lambda$ is a group-type if and only if the transversal $\tau(v)$ satisfies (14), (15), (16) and (17). □

**Remark 4.8.** Notice that the inclusions $\subseteq$ in equations (14), (15), (16) and (17) always hold. Also, suppose that $v \in E^0$ is a sink and that there exists a transversal $\tau(v) = \{ \tau_w : v \rightarrow w : w \in E^0 \}$ in $G(E)$ for $v$ contained in $W$. In this case $\tau(v) \setminus \{v\} \subseteq W^{(1)}$ and $\lambda$ is a group-type partial action of $G(E)$ on $D(X)$ with respect to the transversal $\tau(v)$ if and only if (15) holds.

For the reader’s convenience we recall that a path $\xi = \xi_1 \xi_2 \ldots \xi_n$ in $E$ is *closed* if $d(\xi) = r(\xi)$. If $\xi$ is closed and $d(\xi_i) \neq d(\xi_j)$, for every $i \neq j$, then $\xi$ is called a *cycle*. For a cycle $\xi$ in $E$, $\xi^\infty$ denotes the infinity path $\xi \xi \ldots$ in $E$. A directed graph is said *acyclic* if it has no cycle.

The next result gives necessary and sufficient conditions on $E$ for $\lambda$ to be group-type in the case that $E$ has a sink and $|E^0| \geq 2$.

**Proposition 4.9.** Suppose that $|E^0| \geq 2$ and that $E$ has a sink $v$. Then $\lambda$ is a group-type partial action of $G(E)$ on $D(X)$ if and only if $|d^{-1}(w)| = 1$, for all $w \in E^0 \setminus \{v\}$ and $E$ is acyclic.

*Proof.* Let $v \in E^0$ a sink. Since $\lambda$ is group-type, by (i) of Remark 2.7 and Theorem (4.7), we can take a transversal $\tau(v) = \{ \tau_w : v \rightarrow w : w \in E^0 \}$ in $G(E)$ for $v$ such that $\tau_w \in W^{(-1)} \cup W^{(1)} \cup W^{(2)}$ for all $w \in E^0$, $w \neq v$, and satisfies (14), (15), (16) and (17). Let $w \in E^0 \setminus \{v\}$. If $\tau_w \in W^{(-1)}$ then $\tau_w^{-1} \in W$ and $t(\tau_w^{-1}) = v$. Hence, there exists $\xi \in E^1$ such that $d(\xi) = v$ which is a contradiction because $v$ is a sink. If $\tau_w = a_w b_w^{-1}$ with $(a_w, b_w) \in W^{(2)}$ then $t(b_w) = v$. Again, there is $\xi \in E^1$ such that $d(\xi) = v$ which is an absurd. Hence, $\tau_w \in W^{(1)}$. In particular $d(\tau_w) = w$ and $|d^{-1}(w)| \geq 1$. Suppose that there exists $w_0 \in E^\infty$ with $|d^{-1}(w_0)| > 1$. If $\tau_{w_0} = \xi_1 \ldots \xi_r \in W^{(1)}$ then there is $\eta \in E^1$ with $\eta \neq \xi_1$ and $d(\eta) = w_0$. If $r(\eta_1)$ is a sink then $\eta_1 \in X$. Otherwise, there is $\eta_2 \in E^1$ with $d(\eta_2) = r(\eta_1)$. If $r(\eta_2)$ is a sink then $\eta_1 \eta_2 \in X$. Otherwise, we repeat the process. If the process is finite then we obtain an element $\eta = \eta_1 \ldots \eta_r \in X$ such that $d(\eta) = w_0$. If the process is infinite, then there is an infinite path $\eta = \eta_1 \eta_2 \ldots \in X$ such that $d(\eta) = w_0$. In both cases, we have that $\eta \in \{ \xi \in X : d(\xi) = w_0 \} \setminus \{ \xi \in X : \xi_1 \xi_2 \ldots \xi_{|\tau_{w_0}|} = \tau_{w_0} \}$ which implies that (15) does not hold and we have a contradiction. Now suppose that $\nu = \nu_1 \nu_2 \ldots \nu_n$ is a cycle in $E$. Thus $\nu^\infty \in X$. If $\nu^\infty = \tau(\nu_1) \eta$ for some path $\eta$ in $E$ then $v = r(\tau(\nu_1)) = d(\eta)$, which is a contradiction. Hence $\nu^\infty \in \{ \xi \in X : d(\xi) = d(\nu_1) \} \setminus \{ \xi \in X : \xi_1 \xi_2 \ldots \tau(\nu_1) = \tau(\nu_1) \}$ and again we have a contradiction. So $E$ is acyclic.

Conversely, since $E^0$ is finite, $E$ is acyclic and $|d^{-1}(w)| = 1$, for every $w \in E^0 \setminus \{v\}$, it follows that there are no infinite paths in $E$. Also, for any $w \in E^0 \setminus \{v\}$, there is a
unique (finite) path $\tau_w$ in $E$ with $d(\tau_w) = w$ and $r(\tau_w) = v$. Indeed, write $w = w_1$ and let $\eta_1 \in E^1$ with $d(\eta_1) = w_1$. As $E$ is acyclic we have $w_2 = r(\eta_1) \neq w_1$. If $w_2 = v$ we take $\tau_w = \eta_1$ or else we repeat the process. Since $E^0$ is finite, the process is also finite. Thus, there are vertices $w_1 = w, \ldots, w_n = v$ with $w_i \neq w_j$ for all $1 \leq i \neq j \leq n$ and edges $\eta_1, \ldots, \eta_n$ with $d(\eta_i) = w_i$ and $r(\eta_i) = w_{i+1}$, for all $1 \leq i \leq n-1$. Hence $\tau_w = \eta_1 \cdots \eta_n$ satisfies $d(\tau_w) = w$ and $r(\tau_w) = v$. Suppose that $\xi$ is a path in $E$ such that $d(\xi) = w$ and $r(\xi) = v$. Using that $E$ has no infinite paths, we conclude that $\xi = \xi_1 \cdots \xi_r$. Since $|d^{-1}(w)| = 1$, we have that $\eta_1 = \xi_1$. Hence, $d(\eta_2) = r(\eta_1) = r(\xi_1) = d(\xi_2)$ which implies $\xi_2 = \eta_2$. In this way one obtains $r = n$ and $\eta_i = \xi_i$, for all $1 \leq i \leq n$. Therefore $\xi = \tau_w$. Finally, notice that $\{\mu \in X : \mu_1 \mu_2 \cdots \mu_1 = \mu_w\} = \{\mu \in X : d(\mu) = w\} = \{\tau_w\}$ and whence (15) holds. Thus, $\lambda$ is a group-type partial action of $G(E)$ on $D(X)$ with transversal $\tau(v) = \{\tau_w\}_{w \in E^0}$ for $v$.

**Corollary 4.10.** Suppose that $|E^0| \geq 2$ and that $E$ has a sink $v$. If $\lambda$ is a group-type partial action of $G(E)$ on $D(X)$ then

$$L_k(E) \simeq D(X) \star \beta (E^0)^2 \simeq M_n(k).$$

**Proof.** Let $v$ be the unique sink of $E$. By (i) of Remark 2.7 and Theorem (4.7), there exists a transversal $\tau(v) = \{\tau_{w_j} : 1 \leq j \leq n\}$ in $G(E)$ for $v$ that satisfies (14), (15), (16) and (17). Since $G(E)(v)$ is the trivial group $\{v\}$, it follows from Theorem 2.10 and Theorem 4.3 that $L_k(E)$ is isomorphic to $D(X) \star \beta (E^0)^2$, where $\beta = (B_u, \beta_u)_{u \in (E^0)^2}$ is the global action of $G(E)^0 = (E^0)^2$ on $D(X)$ given by (4). Explicitly, if $u = (w_i, w_j) \in (E^0)^2$ then $B_u = D(X)w_j = D(X)w_i$. As we saw in the proof of Proposition 4.9, $X_{\tau_w} = \{\tau_w\}$, for all $w \in E^0$. Thus $1_{w_j}1_g = 0$ if $\tau_{w_j} \not\in X_g$ or $1_{w_j}1_g = 1$ if $\tau_{w_j} \in X_g$, for all $g \in G(E)$. Hence $D(X)1_{w_j} = k\langle 1_{w_j} \rangle$ which is isomorphic (as algebra) to $k$. Also, notice that $\beta_u : k\langle 1_{w_i} \rangle \to k\langle 1_{w_j} \rangle$ is given by $\beta_u(a1_{w_i}) = a1_{w_j}$, for all $a \in k$. Hence it is clear that

$$\psi : D(X) \star \beta (E^0)^2 \to M_n(k), \quad \psi(a1_{w_j} \delta_{(w_i, w_j)}) = ae_{ji}, \quad a \in k,$$

is an algebra isomorphism, where $e_{ji}$ is the elementary matrix that has 1 in the $(j, i)$-entry and 0 in the other entries. \hfill $\square$

**Remark 4.11.** The result of the previous corollary is known; see, for instance, Theorem 2.6.17 of [4].

**Example 4.12.** Consider the Toeplitz graph $E_T$ given by

```
  u     v
  |     |
  \_     \
  \_     \_  
  \_     \
  \_     \
  \_     \
  \_     \
```

Then follows by Proposition 4.9 that $\lambda$ is not a group-type partial action of $G(E_T)$ on $D(X)$.

Notice that Proposition 4.9 is not true if $E$ has no sink. Indeed, in the next example, $\lambda$ is a group-type partial action of $G(E)$ on $D(X)$ and $|d^{-1}(u)| = 2$, for some vertex $u$. 
Example 4.13. Consider the directed graph $E$ given by

- $w_1 \xrightarrow{\xi_1} \bullet \xleftarrow{\eta} \bullet \xleftarrow{\xi_2} u \xrightarrow{\xi_3} \bullet \xleftarrow{\xi_4} \bullet \xrightarrow{\xi_5} w_2$

Consider the set $\tau(v) = \{ \tau_v = v, \tau_{w_1} = \xi_1, \tau_u = \eta^{-1}, \tau_{w_2} = \xi_2 \}$. It is clear that $\tau_{w_1}$ and $\tau_{w_2}$ satisfy (15) while $\tau_u$ verifies (14). Hence, $\lambda$ is a group-type partial action of $G(E)$ on $D(X)$ and $|d^{-1}(u)| = 2$.

Recall that $E$ is finite if both $E^0$ and $E^1$ are finite sets.

Remark 4.14. If $E$ is finite and $E$ has no sinks, then $E$ contains a cycle. Thus $L_k(E)$ is neither semisimple (Corollary 4.2.13 of [4]) nor von Neumann regular (Theorem 3.4.1 of [4]).

Example 4.15. Let $n \in \mathbb{N}$, $n > 1$ and consider the directed graph $E = A_n$:

- $v_1 \xrightarrow{\xi_1} v_2 \xrightarrow{\xi_2} \cdots v_{n-1} \xrightarrow{\xi_{n-1}} v_n$

By Corollary 4.10, we have that $L_k(E) \simeq D(X) \ast_{\beta} (E^0)^2 \simeq M_n(k)$.

Example 4.16. In the previous example, we take a transversal of a vertex $v$ which is a sink. Now we present an example where the vertex is not a sink. Consider the directed graph $E$:

- $v_2 \xrightarrow{\xi_2} v_1 \xrightarrow{\xi_3} v_3 \xrightarrow{\xi_4} v_4 \xrightarrow{\xi_5} \cdots$

Clearly, $G(E)$ is connected. Let $\tau(v_1) = \{ \tau_{v_i} : v_1 \to v_i \}_{1 \leq i \leq 5}$ be a transversal for $v_1$ given by

- $\tau_{v_1} = v_1, \quad \tau_{v_2} = \xi_2, \quad \tau_{v_i} = \xi_i \xi_{i-1}^{-1}, \quad \text{for all } 3 \leq i \leq 5.$

It is directly to verify that $\tau(v_1) \setminus \{ v_1 \} \subset W^{(1)} \cup W^{(2)}$ and (15), (16) and (17) are satisfied. Then, by Proposition 4.7, $\lambda$ is a group-type partial action of $G(E)$ on $D(X)$. Notice that $G(E)(v_1)$ is the infinite cyclic group generated by $\xi_1 \xi_5 \xi_1^{-1}$ which is isomorphic (as a group) to $\mathbb{Z}$. By Theorem 2.10, $L_k(E) \simeq C \ast_{\gamma} \mathbb{Z}$, where $C = D(X) \ast_{\beta} (E^0)^2$. We will calculate $C$ explicitly. Denote by $\xi_5^\infty$ the infinite path $\xi_5 \xi_5 \xi_5 \cdots$. Then $X_{v_i} = \{ \xi_i \xi_5^\infty \}$ for all $i \neq 2$ and $X_{v_2} = \{ \xi_2 \xi_1 \xi_5^\infty \}$. Hence, as in Example 4.15, we get $C \simeq M_5(k)$ and $L_k(E) \simeq M_5(k) \ast_{\gamma} \mathbb{Z}$. 

**Example 4.17.** Let \( n \) be a positive integer and \( R_n \) the rose with \( n \) petals, that is, \( R_n \) is the directed graph that has one vertex \( v \) and \( n \) loops \( \xi_i \), with \( 1 \leq i \leq n \). Here \( G(E) = G(E)(v) \) is a group. Thus the only transversal possible to \( v \) is \( \tau(v) = \{v\} \).

In this case, the action \( \beta \) of \( G(E) \) on \( D(X) \) is trivial and \( L_k(R_n) \cong D(X) \ast \lambda G(E) \).

Moreover, \( X = X_v = X_{\xi_i} = W^\infty \) and, by Lemma 4.6, \( D(X) = \bigoplus_{i=1}^n D(X)_{\xi_i} \). It is straightforward to check that

\[
D(X)_{\xi_i}^{-1} = D(X), \quad \lambda_{\xi_i}(1_{D(X)}) = 1_{\xi_i} \quad \text{and} \quad \lambda_{\xi_i}^{-1}(1_{\xi_i}) = 1_{D(X)}.
\]

Since \( G(E)_0 = E^0 = \{v\} \), it is easy to verify that the partial action \( \gamma \) of \( G(E)_v \) on \( D(X) \) given in Lemma 4.3 of [8] is equal to \( \lambda \). Thus \( L_k(R_n) \cong D(X) \ast \lambda G(E) \).

On the other hand, let \( L_k(1, n) \) be the free associative \( k \)-algebra with \( 2n \) generators \( \{x_1, \ldots, x_n, y_1, \ldots, y_n\} \) subject to the relations \( y_j x_i = \delta_{i,j}^1 \text{id} \) and \( \sum_{i=1}^n x_i y_i = \text{id} \). Consider the linear map \( \varphi \) from the free associative algebra \( \mathbb{k}\langle x_1, \ldots, x_n, y_1, \ldots, y_n \rangle \) onto \( D(X) \ast \lambda G(E) \) given by

\[
\varphi(y_j) = 1_{D(X)} \delta_{\xi_i}^{-1}, \quad \varphi(x_i) = 1_{\xi_i} \delta_{\xi_i}.
\]

By (19), we have that

\[
\sum_{i=1}^n \varphi(x_i) \varphi(y_i) = \sum_{i=1}^n (1_{\xi_i} \delta_{\xi_i})(1_{D(X)} \delta_{\xi_i}^{-1}) = \sum_{i=1}^n 1_{\xi_i} \delta_1 = 1_{D(X) \ast \lambda G(E)}.
\]

Hence, \( \varphi \) induces an algebra homomorphism \( \psi : L_k(1, n) \to D(X) \ast \lambda G(E) \) which is bijective. Then \( L_k(R_n) \cong L_k(1, n) \) and we recover Proposition 1.3.2 of [4].

### 4.3. Ring theoretic properties of \( L_k(E) \).

Given a directed graph \( E \), we apply the results of Section 3 to relate algebraic properties of the ring \( D(X) \) to algebraic properties of the Leavitt path algebra \( L_k(E) \).

#### 4.3.1. Noetherianity of \( L_k(E) \).

We shall relate noetherian properties of \( L_k(E) \) and \( D(X) \).

We start with the following.

**Lemma 4.18.** The ring \( D(X) \) is noetherian if and only if \( E^0 \) is finite and \( D(X)_v \) is a noetherian ring, for all \( v \in E^0 \).

**Proof.** Assume that \( D(X) \) is noetherian and suppose that \( E^0 = \{v_0, v_1, \ldots\} \) is infinite. Consider, for each \( j \geq 0 \), the left ideal \( I_j := D(X)_{v_0} \oplus \cdots \oplus D(X)_{v_j} \) of \( D(X) \). Since \( 1_{v_i} = \delta_{ij} 1_{v_i} \), we have that \( 1_{v_j+1} \notin I_{j+1} \setminus I_j \). Hence, \( I_j \subseteq I_{j+1} \) for all \( j \geq 0 \) and we have a contradiction because \( D(X) \) is noetherian. Now consider \( v \in E^0 \) and \( I \) a left ideal of \( D(X)_v \). By Lemma 4.6, \( I \) is a left ideal of \( D(X) \) and hence \( I \) is finitely generated as left \( D(X) \)-module. Then \( I \) is finitely generated as left \( D(X)_v \)-module because \( D(X)_v \) is a direct summand of \( D(X) \). The converse is immediate from well-known results of noetherian rings.

From now on we shall assume that \( G(E) \) is connected, \( E^0 \) is finite and that there are a vertex \( v \in E^0 \) and a transversal \( \tau(v) = \{\tau_w : v \rightarrow w : w \in E^0\} \) for \( v \) such that \( \tau(v) \setminus \{v\} \subset W^{(-1)} \cup W^{(1)} \cup W^{(2)} \) and (14), (15), (16) and (17) are satisfied.
Observe that as $\lambda$ is a group-type partial action of $G(E)$ on $D(X)$ we obtain from Theorem 2.10 that $L_k(E) \simeq D(X) \star_\lambda G(E)$. Also, if $G(E)(v)$ is finite then $G(E)$ is finite by Proposition 2.1.

For the reader’s convenience, we recall some extra notions on a directed graph $E$. Let $\xi = \xi_1 \xi_2 \ldots \xi_n$ be a path in $E$. An element $\eta \in E^1$ is called an exit for $\xi$ if there exists $1 \leq i \leq n$ such that $d(\eta) = d(\xi_i)$ and $\eta \neq \xi_i$ and we say that $E$ satisfies Condition (NE) if there is no cycle in $E$ with an exit.

**Proposition 4.19.** The following assertions are valid.

(i) If $L_k(E)$ is left noetherian then $D(X)$ and $D(X) \star_\beta (E^0)^2$ are left noetherian.

(ii) Suppose that $G(E)(v)$ is a polycyclic-by-finite group. The following statements are equivalent:

(a) $L_k(E)$ is left noetherian,

(b) $D(X)$ is left noetherian,

(c) $D(X)_v$ is left noetherian, for all $v \in E^0$,

(d) $E$ is finite and it satisfies Condition (NE).

**Proof.** Item (i) follows directly from Theorem 3.2 (i). The equivalence (a) ⇔ (d) in (ii) follows from Theorem 3.10 of [3] and (b) ⇔ (c) is Lemma 4.18. From Theorem 3.2 (ii) we obtain that (b) ⇒ (a). Notice that (a) ⇒ (b) by item (i). □

**Remark 4.20.** The equivalence between (a) and (d) in (ii) of Proposition 4.19 was proved for general directed graphs in Theorem 3.10 of [3].

**Remark 4.21.** Since $\mathbb{Z}$ is a polycyclic-by-finite and $M_5(\mathbb{k})$ is left noetherian it follows from Proposition 4.19 (ii) that the algebra $L_k(E)$ from Example 4.16 is left noetherian.

If we realize $L_k(E)$ as a partial skew group ring as in Proposition 3.2 of [15], we obtain that $L_k(E)$ is isomorphic to the partial skew group ring $D_0 \star_\theta \mathbb{F}_5$, where $\mathbb{F}_5$ is the free group generated by $E^1$, $D_0$ is a subalgebra of the algebra of functions on a set $X$ and $\theta$ is a partial action of $\mathbb{F}_5$ on $D_0$. It was proved in [12] that a partial skew group ring of a noetherian ring by a partial action of a polycyclic-by-finite group is noetherian. It is well-known that $\mathbb{F}_5$ is not polycyclic-by-finite. Hence we can not use [12] for to conclude that $L_k(E)$ is noetherian.

4.3.2. **Regularity of $L_k(E)$**. There are several results in the literature on group graded von Neumann regularity for Leavitt path algebras; see, for instance, Theorems 1.1, 1.3 and 1.4 in [18]. In particular, $L_k(E)$ with the standard $\mathbb{Z}$-grading is graded von Neumann regular. Our next result relates groupoid graded von Neumann regularity of $L_k(E)$ with von Neumann regularity of the rings $D(X)$ and $L_k(E)$ itself.

**Proposition 4.22.** Suppose that $\text{ch}(\mathbb{k})$ does not divide $|E^0|1_k$.

(i) The following statements are equivalent:

(a) $D(X)$ is von Neumann regular,

(b) $D(X) \star_\beta G_0^2$ is graded von Neumann regular,

(c) $D(X) \star_\beta G_0^2$ is von Neumann regular,

(d) $L_k(E)$ is $G(E)$-graded von Neumann regular.
(ii) If \( G(E)(v) \) is finite and \( \text{tr}_{\lambda G(E)(v)}(1_v) \) is invertible in \( D(X)_v \) then:

(a) The ring \( D(X) \) is semiprimitive, that is \( J(D(X)) = \{0\} \).

(b) If \( L_k(E) \) is \( G(E) \)-graded von Neumann regular then it is von Neumann regular.

Proof. Firstly, notice that \( |E^0|_k \) is invertible in \( D(X) \). Also, by Theorem 2.10, we have that \( L_k(E) \simeq D(X) \star_{\lambda} G(E) \) because \( \lambda \) is a group-type partial action of \( G(E) \) on \( D(X) \).

Thus (i) follows from Theorem 3.8.

(ii) From Theorem 3.11, \( J(L_k(E)) \simeq J(D(X)) \star_{\lambda} G(E) \). However, by Proposition 6.3 of [2], \( J(L_k(E)) = \{0\} \) and whence \( J(D(X)) = \{0\} \). Hence \( D(X) \) is semiprimitive. For (b), observe that from the equivalence (a) \( \iff \) (d) in part (i) we obtain that \( D(X) \) is von Neumann regular. Hence (b) follows by (e) in Theorem 3.8.

\[ \square \]

Example 4.23. Assume that \( \text{ch}(k) \neq 5 \) and consider the Leavitt path algebra given in Example 4.16, that is, \( L_k(E) \simeq M_5(k) \star_{\gamma} \mathbb{Z} \). Since \( M_5(k) \) is von Neumann regular, it follows from Proposition 3.10 of [18] and Remark 3.3 that \( L_k(E) \) is \( G(E) \)-graded von Neumann regular. Hence, the equivalences in (i) of the previous proposition are satisfied. Also, Remark 3.3 implies that \( L_k(E) \) is \( G(x) \)-graded von Neumann. However, by Theorem 3.4.1 of [4], \( L_k(E) \) is not von Neumann regular because \( E \) has one cycle.

Remark 4.24. We have by (i) of Lemma 4.6 that \( D(X) \) is a direct sum of orthogonal idempotents rings. Thus, by Lemma 3.4, \( D(X) \) is von Neumann regular if and only if \( D(X)_v \) is von Neumann regular, for all \( v \in E^0 \).

4.3.3. Right perfect, semiprimary and Krull dimension. We end the paper with the following remark about other ring theoretic properties of \( L_k(E) \).

Remark 4.25. (i) \( L_k(E) \) is right perfect if and only if \( D(X) \) is right perfect and \( G(E)(v) \) is finite; this follows from Theorem 3.14.

(ii) \( L_k(E) \) is semiprimary if and only if \( D(X) \) is semiprimary and \( G(E)(v) \) is finite; this follows from Theorem 3.15.

(iii) If \( G(E)(v) \) is finite and \( D(X) \) has right Krull dimension then \( L_k(E) \) has right Krull dimension and \( \text{Kdim} L_k(E) \leq \text{Kdim} D(X) \). In fact, this is an immediate consequence of Theorem 3.16 since \( G(E) \) is finite.

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