LOG CANONICAL DEGENERATIONS OF DEL PEZZO SURFACES IN $\mathbb{Q}$-GORENSTEIN FAMILIES

YURI PROKHOROV

ABSTRACT. We classify del Pezzo surfaces of Picard number one with log canonical singularities admitting $\mathbb{Q}$-Gorenstein smoothings.

1. INTRODUCTION

Throughout this paper we work over the complex number field $\mathbb{C}$. A smoothing of a surface $X$ is a flat family $X \rightarrow \mathcal{D}$ over a unit disk $0 \in \mathcal{D} \subset \mathbb{C}$ such that the fiber $X_0$ is isomorphic to $X$ and the general fiber is smooth. In this situation $X$ can be considered as a degeneration of a fiber $X_t$, $0 \neq t \in \mathcal{D}$. A smoothing is said to be $\mathbb{Q}$-Gorenstein if so the total family $X$ is. Throughout this paper a del Pezzo surface means a normal projective surface whose anticanonical divisor is $\mathbb{Q}$-Cartier and ample. We study $\mathbb{Q}$-Gorenstein smoothings of del Pezzo surfaces with log canonical singularities. This is interesting for applications to birational geometry and the minimal model program (see e.g. [MP09], [Pro16]) as well as to moduli problems [KSB88], [Hac04]. Smoothings of del Pezzo surfaces with log terminal singularities were considered in [Man91], [HP10], [Pro15].

1.1. Theorem. Let $X$ be a del Pezzo surface with only log canonical singularities and $p(X) = 1$. Assume that $X$ admits a $\mathbb{Q}$-Gorenstein smoothing and there exists at least one non-log terminal point ($o \in X$). Let $\eta : Y \rightarrow X$ be the minimal resolution. Then there is a rational curve fibration $\varphi : Y \rightarrow \mathbb{P}^1$ such that a component $C_1$ of the $\eta$-exceptional divisor dominating $\mathbb{P}^1$ is unique, it is a section of $\varphi$, and its discrepancy equals $-1$. Moreover, $o$ is the only non-log terminal singularity and singularities of $X$ outside $o$ are at worst Du Val of type A. The surface $X$ and singular fibers of $\varphi$ are described in the table below.

All the cases except possibly for $3o$ with $n = 5, 6$ and $4o$ with $5 \leq n \leq 8$ occur.
singularities | $\rho(Y)$ | $K_X^2$ | singular fibers of $\varphi$ | condition
--- | --- | --- | --- | ---
$1^0$ | Ell$_n$ | $\emptyset$ | 2 | $n$ | $n \leq 9$
$2^0$ | $[n; [2]^4]$ | 4 A$_1$ | 10 | $n - 2$ | $4(I_2)$ | $3 \leq n \leq 6$
$3^0$ | $[n, 2, 2; [2]^4]$ | 2 A$_1$ | 10 | $n - 2$ | $2(I_2)(II)$ | $3 \leq n \leq 8$
$4^0$ | $[2, 2, n, 2, 2; [2]^4]$ | $\emptyset$ | 10 | $n - 2$ | $2(II)$ | $3 \leq n \leq 10$
$5^0$ | $[n; [3]^3]$ | 3 A$_2$ | 11 | $n - 1$ | $3(I_3)$ | 2, 3, 4
$6^0$ | $[n; [2], [4]^2]$ | A$_1$, 2 A$_3$ | 12 | $n - 1$ | $(I_2)2(I_4)$ | 2, 3
$7^0$ | $[2; [2], [3], [6]]$ | A$_1$, A$_2$, A$_5$ | 13 | 1 | $(I_2)(I_3)(I_6)$ |

For a precise description the surfaces that occur in our classification we refer to Sect. 8.

To show the existence of Q-Gorenstein smoothings we use unobstructedness of deformations (see Proposition 7.5) and local investigation of Q-Gorenstein smoothability of log canonical singularities:

1.2. Theorem. Let $(X \ni P)$ be a strictly log canonical surface singularity of index $I > 1$ admitting a Q-Gorenstein smoothing. Then it belongs to one of the following types:

| $I$ | $(X \ni P)$ | condition | $\mu_P$ | $- K^2$ |
|---|---|---|---|---|
$1^*$ | 2 | $[n_1, \ldots, n_s; [2]^4]$ | $\sum (n_i - 3) \leq 3$ | $4 - \sum (n_i - 3)$ | $\sum (n_i - 2)$ |
$2^*$ | 3 | $[n; [3], [3], [3]]$ | $n = 2, 3, 4$ | $4 - n$ | $n$ |
$3^*$ | 4 | $[n; [2], [4], [4]]$ | $n = 2, 3$ | $3 - n$ | $n + 1$ |
$4^*$ | 6 | $[2; [2], [3], [6]]$ | 0 | 4 |

where $\mu_P$ is the Milnor fiber of the smoothing.

Q-Gorenstein smoothings exist in cases $2^*$, $3^*$, $4^*$ as well as in the case $1^*$ for singularities of types $[n; [2]^4]$ with $n \leq 6$, $[n_1, \ldots, n_s; [2]^4]$ with $\sum (n_i - 2) \leq 2$, $[4, 3; [2]^4]$, and $[3, 3, 3; [2]^4]$. In all other cases the existence of Q-Gorenstein smoothings is unknown.

Smoothability of log canonical singularities of index 1 were studied earlier (see e.g. [LW86, Ex. 6.4], [Wah80, Corollary 5.12].

As a bi-product we construct essentially canonical threefold singularities of index 5 and 6.

We say that a canonical singularity $(\mathfrak{X} \ni o)$ is essentially canonical if there exist a crepant divisor with center $o$. V. Shokurov conjectured that essentially canonical singularities of given dimension have bounded indices. This is well-known in dimension two: canonical surface singularities are Du Val and their index equals 1. Shokurov’s conjecture was proved in dimension three by M. Kawakita [Kaw15]. More precisely, he proved that the index of an essentially canonical threefold singularity is at most 6. The following theorem supplements Kawakita’s result.
1.3. **Theorem.** For any $1 \leq I \leq 6$ there exist a three-dimensional essentially canonical singularity of index $I$.

In fact our result is new only for $I = 5$ and $6$: [HT87] classified threefold canonical hyperquotient singularities and among them there are examples satisfying conditions of our theorem with $I \leq 4$. Theorem 1.3 together with [Kaw15] gives the following

1.4. **Theorem.** Let $\mathcal{I}$ be the set of indices of three-dimensional essentially canonical singularities. Then

$$\mathcal{I} = \{1, 2, 3, 4, 5, 6\}.$$

**Acknowledgments.** I thank Brendan Hassett whose questions encouraged me to write up my computations. The questions were asked during Simons Symposia “Geometry Over Nonclosed Fields, 2016”. I am grateful to the organizers of this activity for the invitation and creative atmosphere.

## 2. Log canonical singularities

For basic definitions and terminology of the minimal model program, we refer to [KM98] or [Kol92].

2.1. Let $(X \ni o)$ be a log canonical surface singularity. The **index** of $(X \ni o)$ is the smallest positive integer $I$ such that $IK_X$ is Cartier. We say that $(X \ni o)$ is **strictly log canonical** if it is log canonical but not log terminal.

2.2. **Definition.** A normal Gorenstein surface singularity is said to be **simple elliptic** if the exceptional divisor of the minimal resolution is a smooth elliptic curve. We say that a simple elliptic singularity is of type $\text{Ell}_n$ if the self-intersection of the exceptional divisor equals $-n$.

A normal Gorenstein surface singularity is called a **cusp** if the exceptional divisor of the minimal resolution is a cycle of smooth rational curves or a rational nodal curve.

2.3. We recall a notation on weighted graphs. Let $(X \ni o)$ be a rational surface singularity, let $\eta : Y \to X$ be its minimal resolution, and let $E = \sum E_i$ be the exceptional divisor. Let $\Gamma = \Gamma(X, o)$ be the dual graph of $(X \ni o)$, that is, $\Gamma$ is a weighted graph whose vertices correspond to exceptional prime divisors $E_i$ and edges join vertices meeting each other. In the usual way we attach to each vertex $E_i$ the number $-E_i^2$. Typically, we omit 2 if $-E_i^2 = 2$.

If $(X \ni o)$ is a cyclic quotient singularity of type $\frac{1}{r}(1, q)$, $\gcd(r, q) = 1$, then the graph $\Gamma$ is a chain:

$$(2.3.1) \quad \circ \quad _{n_1} \quad \circ \quad _{n_2} \quad \cdots \quad _{n_k} \quad \circ$$
We denote it by \([n_1, \ldots, n_k] = \langle r, q \rangle\). The numbers \(n_i\) are determined by the expression of \(r/q\) as a continued fraction \([\text{Bri68}]\). For positive integers \(n, r, q\), \(\gcd(r, q) = 1\), \(i = 1, \ldots, s\), the symbol
\[
\langle n; r_1, \ldots, r_s; q_1, \ldots, q_s \rangle
\]
denotes the following graph
\[
\langle r_2, q_2 \rangle \quad \cdots \quad \langle r_{s-1}, q_{s-1} \rangle
\]
\[
\langle r_1, q_1 \rangle \quad \langle r_s, q_s \rangle
\]
For short, we will omit \(q_i\)'s: \(\langle n; r_1, \ldots, r_s \rangle\). If \(\langle r_i, q_i \rangle = [n_{i,1}, n_{i,2}, \ldots]\), then we also denote
\[
\langle n; r_1, \ldots, r_s; q_1, \ldots, q_s \rangle = [n; [n_{1,1}, n_{1,2}, \ldots], \ldots, [n_{s,1}, n_{s,2}, \ldots]].
\]
For example, \(\langle n; 3, 3, 3; 1, 1, 2 \rangle = [n; [3], [3], [2, 2]]\) is the graph:
\[
\begin{array}{c}
\circ \\
\circ \quad \circ \\
\circ \quad \circ
\end{array}
\]
The graph
\[
\begin{array}{c}
\circ \\
\circ \quad \circ \\
\circ
\end{array}
\]
will be denoted by \([n_1, \ldots, n_s; [2]^4]\).

2.4. Theorem (\([\text{Kaw88} \S 9]\)). Let \((X \ni o)\) be a strictly log canonical surface singularity of index \(I\). Then one of the following holds:

(i) \(I = 1\) if and only if \((X \ni o)\) is either a simple elliptic singularity or a cusp,

(ii) \(I = 2\) if and only if \(\Gamma(X, o)\) is of type \([n_1, \ldots, n_s; [2]^4], s \geq 1\),

(iii) \(I = 3\) if and only if \(\Gamma(X, o)\) is of type \([n; 3, 3, 3]\),

(iv) \(I = 4\) if and only if \(\Gamma(X, o)\) is of type \([n; 2, 4, 4]\),

(v) \(I = 6\) if and only if \(\Gamma(X, o)\) is of type \([n; 2, 3, 6]\).

2.4.1. Corollary. A strictly log canonical surface singularity is not rational if and only if it is of index 1.

2.5. Let \((X \ni o)\) be a strictly log canonical surface singularity of index \(I\), let \(\eta: Y \rightarrow X\) be its minimal resolution, and let \(E = \sum E_i\) be the exceptional divisor. Let us contract all the components of \(E\) with discrepancies > \(-1\):

\[
\eta: Y \xrightarrow{\tilde{\eta}} \tilde{X} \xrightarrow{\sigma} X.
\]
Let $\tilde{C} = \sum \tilde{C}_i := \tilde{\eta}_* E$ be the $\sigma$-exceptional divisor. Then the pair $(\tilde{X}, \tilde{C})$ has only divisorial log terminal singularities (dlt) and the following relation holds

$$K_{\tilde{X}} = \sigma^* K_X - \tilde{C}.$$  

The extraction $\sigma : \tilde{X} \to X$ is called the dlt modification of $(X \ni o)$.

2.5.1. **Corollary** (see [Kaw88, §9], [Kol92, §3], [KM98, §4.1], [Pro01, §6.1]). In the above notation one of the following holds:

(i) $I = 1$, $\tilde{X} = Y$ is smooth, and $(X \ni o)$ is either a simple elliptic or a cusp singularity;

(ii) $I = 2$, $\tilde{C} = \sum_{i=1}^s \tilde{C}_i$ is a chain of smooth rational curves meeting transversely at smooth points of $\tilde{X}$ so that $\tilde{C}_i \cdot \tilde{C}_{i+1} = 1$, and the singular locus of $\tilde{X}$ consists of two Du Val points of type $A_1$ lying on $\tilde{C}_1$ and two Du Val points of type $A_1$ lying on $\tilde{C}_s$ (the case $s = 1$ is also possible and then $\tilde{C} = \tilde{C}_1$ is a smooth rational curve containing four Du Val points of type $A_1$);

(iii) $I = 3$, 4, or 6, $\tilde{C}$ is a smooth rational curve, the pair $(\tilde{X}, \tilde{C})$ has only purely log terminal singularities (plt), and the singular locus of $\tilde{X}$ consists of three cyclic quotient singularities of types $\frac{1}{r_i}(1, q_i)$, $\gcd(r_1, q_1) = 1$ with $\sum 1/r_i = 1$. In this case $I = \text{lcm}(r_1, r_2, r_3)$.

2.6. Let $(X \ni o)$ be a log canonical singularity of index $I$ (of arbitrary dimension). Recall (see e.g. [KM98, Definition 5.19]) that the index one cover of $(X \ni o)$ is a finite morphism $\pi : X^2 \to X$, where

$$X^2 := \text{Spec} \left( \bigoplus_{i=0}^{I-1} \mathcal{O}_X(-iK_X) \right).$$  

Then $X^2$ is irreducible, $\mathcal{O}^2 = \pi^{-1}(o)$ is one point, $\pi$ is étale over $X \setminus \{o\}$ and $K_{X^2} = \pi^* K_X$ is Cartier. In this situation, $(X^2 \ni \mathcal{O}^2)$ is a log canonical singularity of index 1. Moreover if $(X \ni o)$ is log terminal (resp. canonical, terminal), then so the singularity $(X^2 \ni \mathcal{O}^2)$ is.

2.6.1. **Corollary.** A strictly log canonical surface singularity of index $I > 1$ is a quotient of a simple elliptic or cusp singularity $(X^2 \ni \mathcal{O}^2)$ by a cyclic group $\mu_I$ of order $I = 2, 3, 4$ or 6 whose action on $X^2 \setminus \{\mathcal{O}^2\}$ is free.

2.7. **Construction** (see [Kaw88, Proof of Theorem 9.6]). Let $(X \ni o)$ be a strictly log canonical surface singularity of index $I > 1$, let $\pi : (X^2 \ni \mathcal{O}^2) \to (X \ni o)$ be the index one cover, and let $\tilde{\pi} : (\tilde{X}^2 \ni \tilde{C}^2) \to (X^2 \ni \mathcal{O}^2)$ be the minimal resolution. The action of $\mu_I$ lifts to $\tilde{X}^2$ so that the induced action on $\mathcal{O}_{\tilde{X}^2}(K_{\tilde{X}^2} + \tilde{C}^2) = \tilde{\pi}^* \mathcal{O}_{X^2}(K_{X^2})$ and $H^0(\tilde{C}^2, \mathcal{O}_{\tilde{C}^2}(K_{\tilde{C}^2}))$ is faithful. Let
\((\tilde{X} \supset \tilde{C}) := (\tilde{X} \supset \tilde{C}^2)/\mu_f\). Thus we obtain the following diagram

\[
\begin{array}{ccc}
\tilde{X}^2 & \xrightarrow{\tilde{\pi}} & \tilde{X} \\
\downarrow{\tilde{\sigma}} & & \downarrow{\tilde{\sigma}} \\
X^2 & \xrightarrow{\pi} & X
\end{array}
\]

Here \(\sigma : (\tilde{X} \supset \tilde{C}) \to (X \ni o)\) is the dlt modification.

The following definition can be given in arbitrary dimension. For simplicity we state it only for dimension two which is sufficient for our needs.

2.8. **Adjunction.** Let \(X\) be a normal surface and \(D\) be an effective \(\mathbb{Q}\)-divisor on \(X\). Write \(D = C + B\), where \(C\) is a reduced divisor on \(X\), \(B\) is effective, and \(C\) and \(B\) have no common component. Let \(\nu : C' \to C\) be the normalization of \(C\). One can construct an effective \(\mathbb{Q}\)-divisor \(\text{Diff}_C(B)\) on \(C'\), called the different, as follows; see [Kol92, Chap. 16] or [Sho93, §3] for details. Take a resolution of singularities \(f : X' \to X\) such that the proper transform \(C'\) of \(C\) on \(X'\) is also smooth. Clearly, \(C'\) is nothing but the normalization of the curve \(C\). Let \(B'\) be the proper transform of \(B\) on \(X'\). One can find an exceptional \(\mathbb{Q}\)-divisor \(A\) on \(X'\) such that \(K_{X'}+C'+B' \equiv fA\). The different \(\text{Diff}_C(B)\) is defined as the \(\mathbb{Q}\)-divisor \((B' - A)|_{C'}\).

2.8.2. **Theorem** (Inversion of Adjunction [Sho93, Kaw07]). The pair \((X, C + B)\) is lc (resp. plt) near \(C\) if and only if the pair \((C', \text{Diff}_C(B))\) is lc (resp. klt).

2.8.3. **Proposition.** Let \((X \ni P)\) be a surface singularity and let \(o \in C \subset X\) be an effective reduced divisor such that the pair \((X, C)\) is plt. Then \((P \in C \subset X)\) is analytically isomorphic to

\[
(0 \in \{x_1 - \text{axis}\} \subset \mathbb{C}^2)/\mu_r(1, q), \quad \gcd(r, q) = 1.
\]

In particular, \(C\) is smooth at \(P\) and \(\text{Diff}_C(0) = (1 - 1/r)P\). The dual graph of the minimal resolution of \((X \ni P)\) is a chain \((3.3.1)\) and the proper transform of \(C\) is attached to one of its ends.

3. **\(\mathbb{Q}\)-Gorenstein smoothings of log canonical singularities**

In this section we prove the classificational part of Theorem 1.2.

3.1. **Notation.** Let \((X \ni P)\) be a normal surface singularity, let \(\eta : Y \to X\) be the minimal resolution and let \(E = \sum E_i\) be the exceptional divisor. Write

\[
(3.1.1) \quad K_Y = \eta^*K_X - \Delta,
\]
where $\Delta$ is an effective $\mathbb{Q}$-divisor with $\text{Supp}(\Delta) = \text{Supp}(E)$. Thus one can define the self-intersection $K^2_{(X,P)} := \Delta^2$ which is a well-defined natural invariant. We usually write $K^2$ instead of $K^2_{(X,P)}$ if no confusion is likely. The value $K^2$ is non-positive and it equals zero if and only if $(X \ni P)$ is a Du Val point.

- We denote by $\varsigma_P$ the number of exceptional divisors over $P$.

3.2. Lemma. Let $(X \ni P)$ be a normal surface singularity and let $\mathfrak{X} \to \mathcal{O}$ be its $\mathbb{Q}$-Gorenstein smoothing. If $(X \ni P)$ is log terminal, then the pair $(\mathfrak{X},X)$ is plt and the singularity $(\mathfrak{X} \ni P)$ is terminal.

If $(X \ni P)$ is log canonical, then the pair $(\mathfrak{X},X)$ is lc and the singularity $(\mathfrak{X} \ni P)$ is isolated canonical.

Proof. By the higher-dimensional version of the inversion of adjunction (see [KM98, Th. 5.50], [Kaw07] and Theorem 2.8.2) the singularity $(\mathfrak{X} \ni P)$ is log terminal (resp. log canonical) if and only if the pair $(\mathfrak{X},X)$ is plt (resp. lc) at $P$. Since $X$ is a Cartier divisor on $\mathfrak{X}$, the assertion follows. $\square$

3.3. Lemma ([Kol91, Proposition 6.2.8]). Let $(X \ni P)$ be a rational surface singularity. If $(X \ni P)$ admits a $\mathbb{Q}$-Gorenstein smoothing, then $K^2$ is an integer.

3.4. Theorem ([KSB88 Proposition 3.10], [LW86 Proposition 5.9]). Let $(X \ni P)$ be a log terminal surface singularity. The following are equivalent:

(i) $(X \ni P)$ admits a $\mathbb{Q}$-Gorenstein smoothing,

(ii) $K^2 \in \mathbb{Z}$,

(iii) $(X \ni P)$ is either Du Val or a cyclic quotient singularity of the form

$$\frac{1}{m}(q_1,q_2)$$

with

$$(q_1 + q_2)^2 \equiv 0 \ mod \ m, \ \ \ \gcd(m,q_i) = 1.$$ 

A log terminal singularity satisfying equivalent conditions above is called a T-singularity.

3.4.1. Remark (see [KSB88]). It easily follows from (iii) that any non-Du Val T-singularity can be written in the form

$$\frac{1}{dm^2}(1,dma - 1)$$

Below we describe log canonical singularities with integral $K^2$. Note however, that in general, the condition $K^2 \in \mathbb{Z}$ is necessary but not sufficient for the existence of $\mathbb{Q}$-Gorenstein smoothing (cf. Theorem 1.2 and Proposition 3.5 (DV)).

3.5. Proposition. Let $(X \ni P)$ be a rational strictly log canonical surface singularity. Then in the notation of Theorem 2.3 the invariant $K^2$ is integral if and only if $X$ is either of type $[n_1,\ldots,n_s;[2]^4]$ or of type $\langle n; r_1,r_2,r_3; \varepsilon,\varepsilon,\varepsilon \rangle$, where $(r_1,r_2,r_3) = (3,3,3),(2,4,4),(2,3,6)$ and $\varepsilon = 1$ or $-1$. Moreover, we have:
if $X$ is of type $[n_1, \ldots, n_s; [2]^4]$ or $\langle n; r_1, r_2, r_3; -1, -1, -1 \rangle$, then

$$-K^2 = n - 2,$$

where in the case $[n_1, \ldots, n_s; [2]^4]$, we put $n := \sum (n_i - 2) + 2$;

(nDV) if $X$ is of type $\langle n; r_1, r_2, r_3; 1, 1, 1 \rangle$, then

$$-K^2 = n - 9 + \sum r_i.$$

For the proof we need the following lemma.

3.5.1. **Lemma.** Let $V$ be a smooth surface and let $C, E_1, \ldots, E_m \subset V$ be proper smooth rational curves on $V$ whose configuration is a chain:

$$\circ C \to \cdots \to \circ E_1.$$

Let $D = C + \sum \alpha_i E_i$ be a $\mathbb{Q}$-divisor such that $(K_V + D) \cdot E_j = 0$ for all $j$.

(i) If all the $E_i$’s are $(-2)$-curves, then $D^2 = C^2 + \sum \delta_i = C^2 + \frac{2m}{m+1}$.

(ii) If $m = 1$ and $E_1^2 = -r$, then $D^2 = C^2 = \frac{(r-1)(3-r)}{r}$.

**Proof.** Assume that $E_i^2 = -2$ for all $i$. It is easy to check that $D = C + \sum_{i=1}^{m} \frac{i}{m+1} E_i$. Then

$$D^2 - C^2 = \frac{2m}{m+1} + \left( \sum_{i=1}^{m} \frac{i}{m+1} E_i \right)^2 = \frac{2m}{m+1} + \frac{2}{(m+1)^2} \left( -\sum_{i=1}^{m} i^2 + \sum_{i=1}^{m-1} i(i+1) \right) = \frac{m}{m+1}.$$  

Now let $m = 1$ and $E_1^2 = -r$. Then $D = C + \frac{r-1}{r} E_1$. Hence

$$D^2 - C^2 = \frac{2(r-1)}{r} - \frac{(r-1)^2}{r} = \frac{(r-1)(3-r)}{r}.$$  

**Proof of Proposition 3.5.** Let $\Delta$ be as in (3.1.1) and let $C := |\Delta|$. Write $\Delta = C + \sum \Delta_i$, where $\Delta_i$ are effective connected $\mathbb{Q}$-divisors. By Lemma 3.5.1 we have

$$\delta_i := \left( (C + \Delta_i)^2 - C^2 \right) = \begin{cases} 1 - \frac{1}{r_i} & \text{if } \Delta_i \text{ is of type } \frac{1}{r_i}(1, -1), \\ 4 - r_i - \frac{3}{r_i} & \text{if } \Delta_i \text{ is of type } \frac{1}{r_i}(1, 1). \end{cases}$$

Then

$$K^2 = \left( C + \sum \Delta_i \right)^2 = C^2 + \sum \delta_i.$$  

If $(X \ni P)$ is of type $[n_1, \ldots, n_s, [2], [2], [2], [2]]$, then

$$K^2 = C^2 + 2 = -\sum (n_i - 2).$$

Assume that $C$ is irreducible and $(X \ni P)$ is of type $\langle n; r_1, r_2, r_3 \rangle$, where $\sum 1/r_i = 1$. 

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If all the $\text{Supp}(\Delta_i)$'s are Du Val chains, then
\[ K^2 = C^2 + \sum \left(1 - \frac{1}{r_i}\right) = -n + 2. \]

If $(X \ni P)$ is of type $\langle n; r_1, r_2, r_3; 1, 1, 1 \rangle$, then
\[ K^2 = C^2 + \sum \left(4 - r_i - \frac{2}{n}\right) = -n + 9 - \sum r_i. \]

It remains to consider the “mixed” case. Assume for example that $(X \ni P)$ is of type $\langle n; 3, 3, 3 \rangle$. Then $\delta_1 \in \{0, 2/3\}$. Since $\sum \delta_i$ is an integer, the only possibility is $\delta_1 = \delta_2 = \delta_3$, i.e. all the chains $\Delta_i$ are of the same type. The cases $\langle n; 2, 4, 4 \rangle$ and $\langle n; 2, 3, 6 \rangle$ are considered similarly. □

3.5.2. Corollary. Let $(X \ni P)$ be a strictly log canonical surface singularity of index $I \geq 2$ admitting a $\mathbb{Q}$-Gorenstein smoothing. Let $(X^\sharp \ni P^\sharp) \to (X \ni P)$ be the index one cover. Then
\[ -K^2_{(X^\sharp \ni P^\sharp)} = \begin{cases} I(n - 2) & \text{in the case (DV),} \\ I(n - 1) & \text{in the case (nDV).} \end{cases} \]

3.5.3. Remark. In the above notation we have (see e.g. [KM98, Theorem 4.57])
\[ \text{mult}(X^\sharp \ni P^\sharp) = \max \left(2, -K^2_{(X^\sharp \ni P^\sharp)}\right), \]
\[ \text{emb dim}(X^\sharp \ni P^\sharp) = \max \left(3, -K^2_{(X^\sharp \ni P^\sharp)}\right). \]

The following proposition is the key point in the proof of of Theorem 1.2.

3.6. Proposition. Let $(X \ni P)$ be a strictly log canonical rational surface singularity of index $I \geq 3$ admitting a $\mathbb{Q}$-Gorenstein smoothing. Then $(X \ni P)$ is of type $\langle n; [r_1], [r_2], [r_3] \rangle$.

Proof. By Lemma 3.3 the number $K^2$ is integral and by Proposition 3.5 $(X \ni P)$ is either of type nDV or of type DV.

Let $f : X \to \mathcal{D}$ be a $\mathbb{Q}$-Gorenstein smoothing. By Lemma 3.2 the pair $(\mathcal{X}, X)$ is log canonical and $(P \in \mathcal{X})$ is an isolated canonical singularity. Let $\pi : (\mathcal{X}^\sharp \ni P^\sharp) \to (\mathcal{X} \ni P)$ be the index one cover (see 2.6) and let $X^\sharp := \pi^*X$. Then $X^\sharp$ is a Cartier divisor on $\mathcal{X}^\sharp$, the singularity $(\mathcal{X}^\sharp \ni P^\sharp)$ is canonical (of index 1), and the pair $(\mathcal{X}^\sharp, X^\sharp)$ is lc. Moreover, $\mathcal{X}^\sharp$ is CM, $X^\sharp$ hence normal, and the canonical divisor $K_{X^\sharp}$ is Cartier. Therefore, $\pi$ induces the index one cover $\pi_X : (X^\sharp \ni P^\sharp) \to (X \ni P)$. In particular, the index of $(P \in \mathcal{X})$ equals $I$. Since $I \geq 3$, the singularity $(X^\sharp \ni P^\sharp)$ is simple elliptic and the dlt modification coincides with the minimal resolution.

3.7. First we consider the case where $(P \in \mathcal{X})$ is terminal. Below we essentially use the classification of terminal singularities (see e.g. [Rei87]).
In our case, \((\mathfrak{X}^x \ni P^x)\) is either smooth or an isolated cDV singularity. In particular,

\[
\text{emb dim}(X^x \ni P^x) \leq \text{emb dim}(\mathfrak{X}^x \ni P^x) \leq 4.
\]

On the other hand, by Corollary \(3.5.2\) and Remark \(3.5.3\)

\[
\text{emb dim}(X^x \ni P^x) = I(n - 2).
\]

If \(\text{emb dim}(\mathfrak{X}^x \ni P^x) = 3\), i.e. \((\mathfrak{X}^x \ni P^x)\) is smooth, then \(\text{emb dim}(X^x \ni P^x) = 3\), mult \((X^x \ni P^x) = 3\), and \(I = n = 3\). In this case \((\mathfrak{X} \ni P)\) is a cyclic quotient singularity of type \(\frac{1}{3}(1,1,-1)\) [Rei87]. We may assume that \((\mathfrak{X}^x, P^x) = (\mathbb{C}^3,0)\) and \(X^x\) is given by an invariant equation \(\psi(x_1,x_2,x_3) = 0\) with \(\text{mult}_0 \psi = 3\). Since \((X^x \ni P^x)\) is a simple elliptic singularity, the cubic part \(\psi_3\) of \(\psi\) defines a smooth elliptic curve on \(\mathbb{P}^2\). Hence we can write \(\psi_3 = x_3^3 + \tau(x_1,x_2)\), where \(\tau(x_1,x_2)\) is a cubic homogeneous polynomial without multiple factors. The minimal resolution \(\tilde{X}^x \to X^x\) is the blowup of the origin. In the affine chart \(\{x_2 \neq 0\}\) the surface \(\tilde{X}^x\) is given by the equation \(\tau(x_1,1) + x_3^3 + x_2(x_1, \ldots) = 0\) and the action of \(\mu_3\) is given by the weights \((0,1,1)\). Then it is easy to see that \(\tilde{X}\) has three singular points of type \(\frac{1}{3}(1,1,1)\). This proves our assertion.

Thus we may assume that \(\text{emb dim}(\mathfrak{X}^x \ni P^x) = 4\), i.e. \((\mathfrak{X}^x \ni P^x)\) is a hypersurface singularity. Then \(I \leq 4\). We may assume that \((\mathfrak{X}^x \ni P^x) \subset (\mathbb{C}^4 \ni 0)\) is a hypersurface given by an equation \(\phi(x_1, \ldots, x_4) = 0\) with \(\text{mult}_0 \phi = 2\) and \(X^x\) is cut out by an invariant equation \(\psi(x_1, \ldots, x_4) = 0\). Furthermore, we may assume that \(x_1, \ldots, x_4\) are semi-invariants with \(\mu_1\)-weights \((1,1,-1,b)\), where either \(b = 0\) or \(I = 4 = 2b\) (see [Rei87]).

Consider the case \(\text{mult}_0 \psi = 1\). Since \(\psi\) is invariant, we have \(\psi = x_4 + (\text{higher degree terms})\) and \(b = 0\). In this case the only quadratic invariants are \(x_1x_3, x_2x_3\), and \(x_1^2\). Thus \(\phi_2\) is a linear combination of \(x_1x_3, x_2x_3, x_1^2\). If \(\phi\) contains either \(x_1x_3\) or \(x_2x_3\), then the eliminating \(x_4\) we see that \((X^x \ni P^x)\) is a hypersurface singularity whose equation has quadratic part of rank \(\geq 2\). In this case \((X^x \ni P^x)\) is a Du Val singularity of type \(A_n\), a contradiction. Thus, \(\phi_2 = x_3^2\) and \(I = 3\) by the classification of terminal singularities [Rei87]. By eliminating \(x_4\) we see that \((X^x \ni P^x)\) is isomorphic to

\[
\{\phi_3(x_1,x_2,x_3) + (\text{higher degree terms})\}/\mu_3(1,1,-1)
\]

and we can argue as in the case \(\text{emb dim}(\mathfrak{X}^x \ni P^x) = 3\) (see above).

Now let \(\text{mult}_0 \psi > 1\). Then

\[
\text{emb dim}(X^x \ni P^x) = -K_{(X^x \ni P^x)} = \text{mult}(X^x \ni P^x) = 4 = I
\]

(see Remark \(3.5.3\)). According to [KM98 Theorem 4.57] the curve given by quadratic parts of \(\phi\) and \(\psi\) in the projectivization \(\mathbb{P}(T_{P^x,H})\) of the tangent space is a smooth elliptic curve. According to the classification [Rei87] there are two cases.
Case: $b = 0$ and $\phi$ is an invariant. In this case, as above, $\phi_2$ and $\psi_2$ are linear combination of $x_1x_3$, $x_2x_3$, $x_4^2$ and so $\{\phi_2 = \psi_2 = 0\}$ cannot be smooth, a contradiction.

Case: $b = 2$ and $\phi$ is a semi-invariant of weight 2. Then, up to linear coordinate change of $x_1$ and $x_2$, we can write

$$\phi_2 = a_1x_1x_2 + a_2x_1^2 + a_3x_2^2 + a_4x_3^2, \quad \psi_2 = b_1x_1x_3 + b_2x_4^2.$$  

Since $\phi_2 = \psi_2 = 0$ defines a smooth curve, $a_1x_1x_2 + a_2x_1^2 + a_3x_2^2$ has no multiple factors, so we may assume that $\phi_2 = x_1x_2 + x_3^2$. Similarly, $b_1 \neq 0$ and $b_2 \neq 0$ and we may assume that $\psi_2 = x_1x_3 + x_4^2$. Then easy computations (see e.g. [KM92].7.7.1) show that $(X^2 \ni P^2)$ is a singularity of type $[2; 2, [4]^2]$. This proves our assertion.

3.8. Now we assume that $(P \in X)$ is strictly canonical. Let $\gamma : \tilde{X} \to X$ be the crepant blowup of $(P \in X)$. By definition $\tilde{X}$ has only $\mathbb{Q}$-factorial terminal singularities and $K_{\tilde{X}} = \gamma^*K_X$. Let $E = \sum E_i$ be the exceptional divisor and let $\tilde{X}$ be the proper transform of $X$. Since the pair $(\tilde{X}, X)$ is log canonical, we can write

$$K_{\tilde{X}} + \tilde{X} + E = \gamma^*(K_X + X), \quad \gamma^*X = \tilde{X} + E.$$  

The pair $(\tilde{X}, \tilde{X} + E)$ is log canonical and $\tilde{X}$ has isolated singularities, so $\tilde{X} + E$ has generically normal crossings along $\tilde{X} \cap E$. Hence $C := \tilde{X} \cap E$ is a reduced curve. By the adjunction we have

$$K_{\tilde{X}} + C = (K_{\tilde{X}} + \tilde{X} + E)|_{\tilde{X}} = \gamma^*(K_X + X)|_{\tilde{X}} = \gamma_{\tilde{X}}^*K_X.$$  

Thus $\gamma_{\tilde{X}} : \tilde{X} \to X$ is a dlt modification of $(X \ni P)$. Since $I \geq 3$, there is only one divisor over $P \in X$ with discrepancy $-1$. Hence this divisor coincides with $C$ and so $C$ is irreducible and smooth. In particular, $\tilde{X}$ meets only one component of $E$.

Claim. Let $Q \in \tilde{X}$ be a point at which $E$ is not Cartier. Then in a neighborhood of $Q$ we have $\tilde{X} \sim K_{\tilde{X}}$. In particular, $Q \in C$.

Proof. We are going to apply the results of [Kaw15]. The extraction $\gamma : \tilde{X} \to X$ can be decomposed in a sequence of elementary crepant blowups

$$\gamma_i : \tilde{X}_{i+1} \to \tilde{X}_i, \quad i = 0, \ldots, N,$$

where $\tilde{X}_0 = \tilde{X}$, $\tilde{X}_N = \tilde{X}$, for $i = 1, \ldots, N$ each $\tilde{X}_i$ has only $\mathbb{Q}$-factorial canonical singularities, and the $\gamma_i$-exceptional divisor $E_{i+1,i}$ is irreducible. [Kaw15] defined a divisor $F$ with $\text{Supp}(F) = E$ on $\tilde{X}_N = \tilde{X}$ inductively: $F_1 = E_{1,0}$ on $\tilde{X}_1$ and $F_{i+1} = [\gamma_i^*F_i]$. In our case, by (3.8.1) the divisor $F$ is reduced, i.e. $F = E$. Then by [Kaw15] Theorem 4.2 we have $E \sim -K_{\tilde{X}}$ near $Q$. Since $\tilde{X} + E$ is Cartier, $\tilde{X} \sim K_{\tilde{X}}$ near $Q$. $\square$
Claim. The singular locus of $\tilde{X}$ near $C$ consists of three cyclic quotient singularities $P_1, P_2, P_3$ of types $\frac{1}{r_i}(1,-1,b_i)$, where $\gcd(b_i,r_i) = 1$ and $(r_1, r_2, r_3) = (3,3,3), (2,4,4), \text{and} (2,3,6)$ in cases $I = 3, 4, 6$, respectively.

Proof. Let $P_1, P_2, P_3 \in C$ be singular points of $\tilde{X}$. Since $C = \tilde{X} \cap E$ is smooth, $E$ is not Cartier at $P_i$'s. Hence $P_1, P_2, P_3 \in \tilde{X}$ are (terminal) non-Gorenstein points. Now the assertion follows by [Kaw15, Theorem 4.2].

Therefore, $P_i \in \tilde{X}$ is a point of index $\frac{r_i}{\gcd(2, r_i)}$. Hence the singularities of $\tilde{X}$ are of types $\frac{1}{r_i}(1,1)$. This proves Proposition 3.6.

3.9. Let $(X \ni P)$ be a normal surface singularity admitting a $\mathbb{Q}$-Gorenstein smoothing $f : X \to D$. Let $M_P$ be the Milnor fiber of $f$. Thus, $(M_P, \partial M_P)$ is a smooth 4-manifold with boundary. Denote by $\mu_P = b_2(M_P)$ the Milnor number of the smoothing. In our case we have (see [GS83])

$$(3.9.1) b_1(M_P) = 0, \quad \text{Eu}(M_P) = 1 + \mu_P.$$

3.9.2. Proposition (cf. [HP10, §2.3]). Let $(X \ni P)$ be a rational surface singularity. Assume that $(X \ni P)$ admits a $\mathbb{Q}$-Gorenstein smoothing. Then for the Milnor number $\mu_P$ we have

$$(3.9.3) \mu_P = K^2_{(X,P)} + \varsigma_P.$$

Proof. Obviously, $K^2_{(X,P)} + \varsigma_P$ depends only on the analytic type of the singularity $(X \ni P)$. According to [Loo86 Appendix], for $(X \ni P)$ there exists a projective surface $Z$ with a unique singularity isomorphic to $(X \ni P)$ and a $\mathbb{Q}$-Gorenstein smoothing $3/(\mathfrak{T} \ni 0)$. Let $\eta : Y \to Z$ be the minimal resolution. Write

$$K_Y = \eta^*K_Z - \Delta, \quad K^2_Y = K^2_Z + \Delta^2.$$

Let $Z'$ be the general fiber. Since

$$\text{Eu}(Y) = \text{Eu}(Z) + \varsigma_P, \quad \chi(\mathcal{O}_Y) = \chi(\mathcal{O}_Z),$$

by Noether's formula we have

$$0 = K^2_Y + \text{Eu}(Y) - 12\chi(\mathcal{O}_Z) = K^2_Z + \Delta^2 + \text{Eu}(Z) + \varsigma_P - 12\chi(\mathcal{O}_{Z'}) =$$

$$= \Delta^2 + \varsigma_P + \text{Eu}(Z) + K^2_{Z'} - 12\chi(\mathcal{O}_{Z'}) = \Delta^2 + \varsigma_P + \text{Eu}(Z) - \text{Eu}(Z').$$

By (3.9.1) we have $\mu_P = \Delta^2 + \varsigma_P$. 

3.9.4. Corollary (see [Man91, Proposition 13]). If $(X \ni P)$ is a $T$-singularity of type $\frac{1}{dm^2}(1, dma - 1)$, then

$$(3.9.5) \mu_P = d - 1, \quad -K^2 = \varsigma_P - d + 1.$$

Proposition 3.9.2 implies the following.
3.9.6. **Corollary.** Let \((X \ni P)\) be a strictly log canonical surface singularity of index \(I > 1\) admitting a \(\mathbb{Q}\)-Gorenstein smoothing. Then

\[
\mu_P = \begin{cases} 
4 - \sum (n_i - 3) & \text{in the case (DV) with } I = 2, \\
13 - n - \sum r_i & \text{in the case (nDV)}. 
\end{cases}
\]

**Proof of the classificational part of Theorem 1.2** Let \(\pi : (X^\sharp \ni P^\sharp) \to (X \ni P)\) be the index one cover. A \(\mathbb{Q}\)-Gorenstein smoothing \((X \ni P)\) is induced by an equivariant smoothing of \((X^\sharp \ni P^\sharp)\). In particular, \((X^\sharp \ni P^\sharp)\) is smoothable. Assume that \((X \ni P)\) is of type \([n_1, \ldots, n_s; [2]^4]\) with \(s > 1\).

Then \((X^\sharp \ni P^\sharp)\) is a cusp singularity. By [Wah81, Th. 5.6] its smoothability implies

\[
\mu_P \geq 0.
\]

□

The existence of \(\mathbb{Q}\)-Gorenstein smoothings follows from examples and discussions in the next two sections.

4. **Examples of \(\mathbb{Q}\)-Gorenstein smoothings**

4.1. **Proposition ([Ste91, Cor. 19]).** A rational surface singularity of index 2 and multiplicity 4 admits a \(\mathbb{Q}\)-Gorenstein smoothing.

Note that for a log canonical surface singularity \((X \ni P)\) of index 2 by [Art66, Cor. 6] we have

\[
\mu_P = \max(4, 2 - K^2) = \max(4, 2 + \sum (n_i - 2)).
\]

4.1.1. **Corollary.** A log canonical singularity of type \([n_1, \ldots, n_s; [2]^4]\) with \(\sum (n_i - 2) \leq 2\) admits a \(\mathbb{Q}\)-Gorenstein smoothing.

Let us consider explicit examples.

4.1.2. **Example.** Let \(\mathfrak{X} = \mathbb{C}^3/\mathbb{Z}_2(1, 1, 1)\) and

\[
f : \mathfrak{X} \to \mathbb{C}, \quad (x, x, 2, x_3) \mapsto x_1^2 + (x_2^2 + c_1 x_3^{2k}) \left( x_3^2 + c_2 x_2^{2m} \right),
\]

where \(k, m \geq 1\) and \(c_1, c_2\) are constants. The central fiber \(X = \mathfrak{X}_0\) is a log canonical singularity of type \([2, \ldots, 2; [2]^4]\).
Indeed, the $\frac{1}{2}(1,1,1)$-blowup of $X' \to X \ni 0$ has irreducible exceptional divisor. If $k,m \geq 3$, then the singular locus of $X'$ consists of two Du Val singularities of types $D_{k+1}$ and $D_{m+1}$. Other cases are similar.

4.1.3. Example. Let $\mu_2$ act on $\mathbb{C}^4_{x_1, \ldots, x_4}$ diagonally with weights $(1,1,1,0)$ and let $\phi(x_1, \ldots, x_4)$ and $\psi(x_1, \ldots, x_4)$ be invariants such that $\text{mult}_0 \phi = \text{mult}_0 \psi = 2$ and the quadratic parts $\phi(2), \psi(2)$ define a smooth elliptic curve in $\mathbb{P}^3$. Let $X := \{ \phi = 0 \}/\mu_2(1,1,1,0)$. Consider the family

$$f : \mathcal{X} \to \mathbb{C}, \quad (x_1, \ldots, x_4) \mapsto \psi.$$ 

The central fiber $X = \mathcal{X}_0$ is a log canonical singularity of type $[4; [2]^4]$.  

4.1.4. Proposition (cf. [Kaw92, Ex. 4.2]). *Singularities of types* $[5; [2]^4], \ldots, [3,3,3; [2]^4]$ *admit* $\mathbb{Q}$-Gorenstein smoothings.

Now consider singularities of index $> 2$.

4.2. Example (cf. [Kaw92, 6.7.1]). Let $\mathcal{X} = \mathbb{C}^3/\mu_3(1,1,2)$ and

$$f : (x_1, x_2, x_3) \mapsto x_1^3 + x_2^3 + x_3^3.$$ 

The central fiber $X = \mathcal{X}_0$ is a log canonical singularity of type $[2; [3]^3]$.  

4.3. Example. Let $\mathcal{X} = \mathbb{C}^3/\mu_9(1,4,7)$ and

$$f : (x_1, x_2, x_3) \mapsto x_1 x_2^2 + x_2 x_3^2 + x_3 x_1^2.$$ 

The central fiber $X = \mathcal{X}_0$ is a log canonical singularity of type $[4; [3]^3]$. The total space has a canonical singularity at the origin.

4.4. Example (cf. [Kaw92, 7.7.1]). Let

$$\mathcal{X} = \{ x_1 x_2 + x_3^2 + x_4^{2k+1} = 0 \}/\mu_4(1,1,-1,2), \quad k \geq 1.$$ 

Consider the family

$$f : \mathcal{X} \to \mathbb{C}, \quad (x_1, \ldots, x_4) \mapsto x_1^2 + x_2(x_1 + x_2) + \psi_3(x_1, \ldots, x_4),$$

where $\psi_3$ is an invariant with $\text{mult}(\psi_3) \geq 3$. The central fiber $X = \mathcal{X}_0$ is a log canonical singularity of type $[2; [2],[4]^2]$. The singularity of the total space is terminal of type $\text{cAx}/4$.

4.5. Example. Let $\mathcal{X} := \{ x_1 x_2 + x_3^2 + x_4^2 = 0 \}/\mu_8(1,5,3,7)$. Consider the family

$$f : \mathcal{X} \to \mathbb{C}, \quad (x_1, \ldots, x_4) \mapsto x_1 x_4 + x_2 x_3.$$ 

The central fiber $X = \mathcal{X}_0$ is a log canonical singularity of type $[3; [2],[4]^2]$. The singularity of the total space $\mathcal{X}$ is canonical $[HT87]$.

More examples of $\mathbb{Q}$-Gorenstein smoothings will be given in the next section.
5. INDICES OF CANONICAL SINGULARITIES

5.1. Notation. Let \( S = S_d \subset \mathbb{P}^d \) be a smooth del Pezzo surface of degree \( d \geq 3 \). Let \( Z \) be the affine cone over \( S \) and let \( z \in Z \) be its vertex. Let \( \delta : \tilde{Z} \to Z \) be the blowup the maximal ideal of \( z \) and let \( \tilde{S} \subset \tilde{Z} \) be the exceptional divisor. The affine variety \( Z \) can be viewed as the spectrum of the anti-canonical graded algebra:

\[
Z = \text{Spec } R(-K_S), \quad R(-K_S) := \bigoplus_{n \geq 0} H^0(S, \mathcal{O}_S(-nK_S))
\]

and the variety \( \tilde{Z} \) can be viewed as the total space \( \text{Tot}(\mathcal{L}) \) of the line bundle \( \mathcal{L} := \mathcal{O}_S(K_S) \). Here \( \tilde{S} \) is the negative section. Denote by \( \gamma : \tilde{Z} \to S \) the natural projection.

5.2. Lemma. The map \( \delta \) is a crepant morphism and \((Z \ni z)\) is a canonical singularity.

Proof. Write \( K_{\tilde{Z}} = \delta^*K_Z + a\tilde{S} \). Then

\[
K_{\tilde{S}} = (K_{\tilde{Z}} + \tilde{S})|_{\tilde{S}} = (a+1)\tilde{S}|_{\tilde{S}}.
\]

Under the natural identification \( S = \tilde{S} \) one has \( \mathcal{O}_\tilde{S}(K_{\tilde{S}}) \simeq \mathcal{O}_S(-1) \simeq \mathcal{O}_\tilde{S}(\tilde{S}) \).

Hence, \( a = 0 \). \( \square \)

5.3. Construction. Assume that \( S \) admits an action \( \varsigma : G \to \text{Aut}(S) \) of a finite group \( G \). The action naturally extends to an action on the algebra \( R(-K_S) \), the cone \( Z \), and its blowup \( \tilde{Z} \). We assume that

(A) \( G \simeq \mu_1 \) is a cyclic group of order 1,

(B) the action \( G \) on \( S \) is free in codimension one, and

(C) the quotient \( S/G \) has only Du Val singularities.

Let \( G_P \) be the stabilizer of a point \( P \in S \). Since \( \mathcal{L} = \mathcal{O}_S(K_S) \), the fiber \( \mathcal{L}_P \) of \( \gamma : \tilde{Z} = \text{Tot}(\mathcal{L}) \to S \) is naturally identified with \( \wedge^2 T_{P,S}^\vee \), where \( T_{P,S} \) is the tangent space to \( S \) at \( P \). By our assumption\(^{[B]}\) in suitable analytic coordinates \( x_1, x_2 \) near \( P \), the action of \( G_P \) is given by

\[
(x_1, x_2) \mapsto (\varsigma_{I_P}^{b_P} \cdot x_1, \varsigma_{I_P}^{-b_P} \cdot x_2),
\]

where \( \varsigma_{I_P} \) is a primitive \( I_P \)-th root of unity, \( \gcd(I_P, b_P) = 1 \), and \( I_P \) is the order of \( G_P \). Therefore, the action of \( G_P \) on \( \mathcal{L}_P \simeq T_{P,S}^\vee \) is trivial. Let \( \mathcal{P} := \mathcal{L}_P \cap \tilde{S} \).

The algebra \( R(-K_S) \) admits also a natural \( \mathbb{C}^* \)-action compatible with the grading. Thus \( \gamma : \tilde{Z} \to S \) is a \( \mathbb{C}^* \)-equivariant \( \mathbb{A}^1 \)-bundle, where \( \mathbb{C}^* \)-action on \( S \) is trivial and the induced action \( \lambda : \mathbb{C}^* \to \text{Aut}(\tilde{Z}) \) is just multiplication in fibers. Fix an embedding \( G = \mu_1 \subset \mathbb{C}^* \). Then two actions \( \varsigma \) and \( \lambda \) commute and so we can define a new action of \( G \) on \( \tilde{Z} \) by

\[
\varsigma'(\alpha) = \lambda(\alpha)\varsigma(\alpha), \quad \alpha \in G.
\]

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Take local coordinates $x_1, x_2, x_3$ in a neighborhood of $\tilde{P} \in \tilde{Z}$ compatible with the decomposition $T_{\tilde{P}, \tilde{Z}} = T_{\tilde{P}, \tilde{S}} \oplus T_{\tilde{P}, \tilde{S}}$, of the tangent space and \((5.3.1)\). Then the action of $G_P$ is given by

\[(5.3.3) \quad (x_1, x_2, x_3) \mapsto (\zeta_{b_P}^{a_P} x_1, \zeta_{b_P}^{-a_P} x_2, \zeta_{a_P}^{b_P} x_3), \quad \gcd(a_P, I_P) = 1.\]

5.4. Claim. The quotient $\tilde{X} := \tilde{Z}/\varsigma'(G)$ has only terminal singularities.

**Proof.** All the points of $\tilde{Z}$ with non-trivial stabilizers lie on the negative section $\tilde{S}$. The image of such a point $\tilde{P}$ on $\tilde{X}$ is a cyclic quotient singularity of type $\Gamma_{I_P}(b_P, -b_P, a_P)$ by \((5.3.3)\). □

By the universal property of quotients, there is a contraction $\varphi : \tilde{X} \to \bar{X}$ contracting $E$ to a point, say $o$, where $\bar{X} := Z/G$ and $E := \tilde{S}/G$. Thus we have the following diagram:

\[(5.4.1)\]

5.5. **Proposition.** $(\bar{X} \ni o)$ is an isolated canonical non-terminal singularity of index $|G|$. 

**Proof.** Since the action $\varsigma'$ is free in codimension one, the contraction $\varphi$ is crepant by Lemma 5.2. The index of $(\bar{X} \ni o)$ is equal to the l.c.m. of $|G_P|$ for $P \in S$. On the other hand, by the holomorphic Lefschetz fixed point formula $G$ has a fixed point on $S$. Hence, $G = G_P$ for some $P$. □

5.6. Now we construct explicit examples of del Pezzo surfaces with cyclic group actions satisfying the conditions \([A][C]\).

5.6.1. **Example.** Recall that a del Pezzo surface $S$ of degree 6 is unique up to isomorphism and can be given in $\mathbb{P}_{u_0; u_1}^1 \times \mathbb{P}_{v_0; v_1}^1 \times \mathbb{P}_{w_0; w_1}^1$ by the equation

$$u_1 v_1 w_1 = u_0 v_0 w_0.$$ 

Let $\alpha \in \text{Aut}(S)$ be the following element of order 6:

$$\alpha : (u_0 : u_1; v_0 : v_1; w_0 : w_1) \mapsto (v_1 : v_0; w_1 : w_0; u_1 : u_0).$$

Points with non-trivial stabilizers belong to one of three orbits and representatives are the following:

- $P = (1 : 1; 1 : 1; 1 : 1), \quad |G_P| = 6$,
- $Q = (1 : \zeta_3; 1 : \zeta_3; 1 : \zeta_3), \quad |G_Q| = 3$,
- $R = (1 : 1; 1 : -1; 1 : -1), \quad |G_R| = 2$.

It is easy to check that they give us Du Val points of type $A_5$, $A_2$, $A_1$, respectively.
5.6.2. Example. A del Pezzo surface $S$ of degree 5 is obtained by blowing up four points $P_1, P_2, P_3, P_4$ on $\mathbb{P}^2$ in general position. We may assume that $P_1 = (1 : 0 : 0)$, $P_2 = (0 : 1 : 0)$, $P_3 = (0 : 0 : 1)$, $P_4 = (1 : 1 : 1)$. Consider the following Cremona transformation:

$$\alpha : (u_0 : u_1 : u_2) \mapsto (u_0(u_2 - u_1) : u_2(u_0 - u_1) : u_0u_2).$$

It is easy to check that $\alpha^5 = \text{id}$ and the indeterminacy points are exactly $P_1, P_2, P_3$. Thus $\alpha$ lifts to an element $\alpha \in \text{Aut}(S)$ of order 5.

**Claim.** Let $\alpha \in \text{Aut}(S)$ be any element of order 5. Then $\alpha$ has only isolated fixed points and the singular locus of the quotient $S/\langle \alpha \rangle$ consists of two Du Val points of type $A_4$.

**Proof.** For the characteristic polynomial of $\alpha$ on $\text{Pic}(S)$ there is only one possibility: $t^5 - 1$. Therefore, the eigenvalues of $\alpha$ are $1, \zeta_5, \ldots, \zeta_5^4$. This implies that every invariant curve is linearly proportional (in $\text{Pic}(S)$) to $-K_S$. In particular, this curve must be an ample divisor.

Assume that there is a curve of fixed points. By the above it meets any line. Since on $S$ there are at most two lines passing through a fixed point, all the lines must be invariant. In this case $\alpha$ acts on $S$ identically, a contradiction.

Thus the action of $\alpha$ on $S$ is free in codimension one. By the topological Lefschetz fixed point formula $\alpha$ has exactly two fixed points, say $Q_1$ and $Q_2$. We may assume that actions of $\alpha$ in local coordinates near $Q_1$ and $Q_2$ are diagonal:

$$(x_1, x_2) \mapsto (\zeta_5^r x_1, \zeta_5^k x_2), \quad (y_1, y_2) \mapsto (\zeta_5^l y_1, \zeta_5^m y_2),$$

where $r, k, l, m$ are not divisible by 5. Then by the holomorphic Lefschetz fixed point formula

$$2 = (1 - \zeta_5^{-1})^{-1}(1 - \zeta_5^{-1})^{-1} + (1 - \zeta_5^{-1})^{-1}(1 - \zeta_5^{-1})^{-1}.$$ 

Easy computations with cyclotomics show that up to permutations and modulo 5 there is only one possibility: $r = 1, k = 4, l = 2, m = 3$. This means that the quotient has only Du Val singularities of type $A_4$. \hfill \Box

5.6.3. Example. Let $\mu_3$ act on $S = \mathbb{P}^2$ diagonally with weights $(0, 1, 2)$. The quotient has three Du Val singularities of type $A_2$.

5.6.4. Example. Let $\mu_4$ act on $S = \mathbb{P}^1_{u_0 : u_1} \times \mathbb{P}^1_{v_0 : v_1}$ by

$$(u_0 : u_1; v_0 : v_1) \mapsto (v_0 : v_1; u_1 : u_0).$$

The quotient has three Du Val singularities of types $A_1, A_3, A_3$.

Note that in all examples above the group generated by $\alpha^n$ also satisfies the conditions $[\text{A}][\text{C}]$. We summarize the above information in the following table. Together with Proposition 5.5 this proves Theorem 1.3.
Therefore, $\tilde{X}$ on replacing $\lambda$ local equation of $\tilde{X}$ 5.7.  

Let $\tilde{X} := \gamma^{-1}(C)$, $X^t := \delta(X^t)$, and $X := \pi(X^t)$. Then the singularity $(X \ni o)$ is log canonical of index $|G|$. Moreover, replacing $\lambda$ with $\lambda^{-1}$ is necessary we may assume that $X$ is a Cartier divisor on $\mathfrak{X}$.

Proof. Put $\tilde{X} := X^t/G$. Since the divisor $\tilde{X} + \tilde{S}$ is trivial on $\tilde{S}$, the contraction $\delta$ is log crepant with respect to $K_{\tilde{X}} + \tilde{X} + \tilde{S}$ and so $\varphi$ is with respect to $K_{\tilde{X}} + \tilde{X} + E$. By construction $X^t$ is a cone over the elliptic curve $C$ and $X = X^t/G$. Therefore, $(X \ni o)$ is a log canonical singularity. Comparing with (2.7) we see that the index of $(X \ni o)$ equals $|G|$. We claim that $\tilde{X} + E$ is a Cartier divisor on $\mathfrak{X}$. Let $\tilde{C} := \gamma^{-1}(C) \cap \tilde{S} = \tilde{S} \cap X^t$. Pick a point $\tilde{P} \in \tilde{Z}$ with non-trivial stabilizer $G_P$. By our assumptions $\tilde{P} \in \tilde{C}$. Take local coordinates $x_1, x_2, x_3$ as in (5.3.3). Moreover, we can take them so that $x_1$ is a local coordinate along $C$. By (5.3.3) the action near $\tilde{P}$ has the form $\frac{1}{I_p}(b_P,-b_P,a_P)$. Since $\gcd(a_P,I_P) = \gcd(b_P, I_P) = 1$, in our case we have $a_P, b_P \in \{\pm 1\}$. By the holomorphic Lefschetz fixed point formula applied to the action of $G_P$ on $C$ the value of $b_P$ does not depend on $P$. Thus by (5.3.2) replacing $\lambda$ with $\lambda^{-1}$ we may assume that $a_P = b_P$. In our coordinates the local equation of $\tilde{S}$ is $x_3 = 0$ and the local equation of $X^t$ is $x_2 = 0$. Now it is easy to see that the local equation $x_2x_3 = 0$ of $\tilde{S} + X^t$ is $G_P$-invariant. Therefore, $\tilde{X} + E$ is Cartier. Since it is $\varphi$-trivial, the divisor $X = \varphi_*(\tilde{X} + E)$ on $\mathfrak{X}$ is Cartier as well. □

| No. | $K_S^2$ | Ref. | $G$ | $I$ | $\text{Sing}(\tilde{X})$ |
|-----|---------|------|-----|-----|------------------------|
| 1#  | 6       | 5.6.1| $\langle \alpha \rangle$ | 6 | $\frac{1}{6}(1, -1, 1), \frac{1}{3}(1, -1, 1), \frac{1}{2}(1, 1, 1)$ |
| 2#  | 5       | 5.6.2| $\langle \alpha \rangle$ | 5 | $\frac{1}{5}(1, -1, 1), \frac{1}{5}(2, -3, 1)$ |
| 3#  | 8       | 5.6.1| $\langle \alpha \rangle$ | 4 | $2 \times \frac{1}{4}(1, -1, 1), \frac{1}{2}(1, 1, 1)$ |
| 4#  | 6       | 5.6.1| $\langle \alpha^2 \rangle$ | 3 | $3 \times \frac{1}{3}(1, -1, 1)$ |
| 5#  | 9       | 5.6.3| $\langle \alpha \rangle$ | 3 | $3 \times \frac{1}{3}(1, -1, 1)$ |
| 6#  | 6       | 5.6.1| $\langle \alpha^3 \rangle$ | 2 | $4 \times \frac{1}{2}(1, 1, 1)$ |
| 7#  | 8       | 5.6.1| $\langle \alpha^2 \rangle$ | 2 | $4 \times \frac{1}{2}(1, 1, 1)$ |

Note that our table agrees with the corresponding one in [Kaw15].

Now we apply the above technique to construct examples of $\mathbb{Q}$-Gorenstein smoothings.

5.7. Theorem. Let $(X \ni o)$ be a surface log canonical rational singularity of one of the following types

$[2; [2, 3, 6]], [3; [2, 4, 4]], [n; [3, 3, 3]], n = 3, 4, [n; [2, 2, 2]], n = 5, 6.$

Then $(X \ni o)$ admits a $\mathbb{Q}$-Gorenstein smoothing.

5.7.1. Lemma. In the notation of (5.3.1), let $C \subset S$ be a smooth elliptic $G$-invariant curve passing through all the points with non-trivial stabilizers. Let $X^t := \gamma^{-1}(C)$, $X^t := \delta(X^t)$, and $X := \pi(X^t)$. Then the singularity $(X \ni o)$ is log canonical of index $|G|$. Moreover, replacing $\lambda$ with $\lambda^{-1}$ is necessary we may assume that $X$ is a Cartier divisor on $\mathfrak{X}$. 

Proof. Put $\tilde{X} := X^t/G$. Since the divisor $\tilde{X} + \tilde{S}$ is trivial on $\tilde{S}$, the contraction $\delta$ is log crepant with respect to $K_{\tilde{X}} + \tilde{X} + \tilde{S}$ and so $\varphi$ is with respect to $K_{\tilde{X}} + \tilde{X} + E$. By construction $X^t$ is a cone over the elliptic curve $C$ and $X = X^t/G$. Therefore, $(X \ni o)$ is a log canonical singularity. Comparing with (2.7) we see that the index of $(X \ni o)$ equals $|G|$. We claim that $\tilde{X} + E$ is a Cartier divisor on $\mathfrak{X}$. Let $\tilde{C} := \gamma^{-1}(C) \cap \tilde{S} = \tilde{S} \cap X^t$. Pick a point $\tilde{P} \in \tilde{Z}$ with non-trivial stabilizer $G_P$. By our assumptions $\tilde{P} \in \tilde{C}$. Take local coordinates $x_1, x_2, x_3$ as in (5.3.3). Moreover, we can take them so that $x_1$ is a local coordinate along $C$. By (5.3.3) the action near $\tilde{P}$ has the form $\frac{1}{I_p}(b_P,-b_P,a_P)$. Since $\gcd(a_P,I_P) = \gcd(b_P, I_P) = 1$, in our case we have $a_P, b_P \in \{\pm 1\}$. By the holomorphic Lefschetz fixed point formula applied to the action of $G_P$ on $C$ the value of $b_P$ does not depend on $P$. Thus by (5.3.2) replacing $\lambda$ with $\lambda^{-1}$ we may assume that $a_P = b_P$. In our coordinates the local equation of $\tilde{S}$ is $x_3 = 0$ and the local equation of $X^t$ is $x_2 = 0$. Now it is easy to see that the local equation $x_2x_3 = 0$ of $\tilde{S} + X^t$ is $G_P$-invariant. Therefore, $\tilde{X} + E$ is Cartier. Since it is $\varphi$-trivial, the divisor $X = \varphi_*(\tilde{X} + E)$ on $\mathfrak{X}$ is Cartier as well. □
Proof of Theorem 5.7. It is sufficient to embed $X$ to a canonical threefold singularity $(X \ni o)$ as a Cartier divisor. Let $(X^3 \ni o^3) \to (X \ni o)$ be the index one cover. Then $(X^3 \ni o^3)$ is a simple elliptic singularity (see 2.6). In the notation of Examples 5.6 consider the following $\mu_1$-invariant elliptic curve $C \subset S$:

\[
\begin{align*}
1# & \quad \zeta_3 (u_0w_1 - u_1w_0)(v_0 + v_1) + (u_0v_1 - u_1v_0)(w_0 + w_1) \\
3# & \quad (u_1^2 - u_0^2)v_0v_1 + \zeta_4 u_0u_1(v_1^2 - v_0^2) \\
5# & \quad u_0^2u_1 + u_1^2u_2 + u_2^2u_0 \\
6# & \quad c_1(u_0w_1 - u_1w_0)(v_0 + v_1) + c_2(u_0v_1 - u_1v_0)(w_0 + w_1) \\
7# & \quad c_1(u_0^2v_0^2 - u_1^2v_1^2) + c_2v_0v_1(u_0^2 - u_1^2) + c_3(u_0^2v_0^2 - u_1^2v_1^2) + c_5u_0u_1(v_0^2 - v_1^2)
\end{align*}
\]

where $c_i$'s are constants and $\zeta_n$ is a primitive $n$-th root of unity. Then we apply Lemma 5.7.1. □

6. Noether’s Formula

6.1. Proposition ([HP10]). Let $X$ be a projective rational surface with only rational singularities. Assume that every singularity of $X$ admits a $\mathbb{Q}$-Gorenstein smoothing. Then

\[K_X^2 + \rho (X) + \sum_{P \in X} \mu_P = 10.\]

Proof. Let $\eta : Y \to X$ be the minimal resolution. Since $X$ has only rational singularities, we have

\[\text{Eu}(Y) = \text{Eu}(X) + \sum P \zeta_P, \quad \chi(\mathcal{O}_Y) = \chi(\mathcal{O}_X).\]

Further, we can write

\[K_Y = \eta^* K_X - \sum P \Delta_P, \quad K_Y^2 = K_X^2 + \sum P \Delta_P^2.\]

By the usual Noether formula for smooth surfaces

\[12\chi(\mathcal{O}_X) = K_Y^2 + \text{Eu}(Y) = K_X^2 + \text{Eu}(X) + \sum P (\Delta_P^2 + \zeta_P).\]

Now the assertion follows from (3.9.3). □

6.2. Let $X$ be an arbitrary normal projective surface, let $\eta : Y \to X$ be the minimal resolution, and let $D$ be a Weil divisor on $X$. Write $\eta^* D = D_Y + D^\bullet$, where $D_Y$ is the proper transform of $D$ and $D^\bullet$ is the exceptional part of $\eta^* D$. Define the following number

\[c_X(D) = -\frac{1}{2} \langle D^\bullet \rangle \cdot (|\eta^* D| - K_Y).\]
6.2.2. Proposition ([Blu95, §1]). In the above notation we have
\[
\chi(\mathcal{O}_X(D)) = \frac{1}{2}D \cdot (D - K_X) + \chi(\mathcal{O}_X) + c_X(D) + c'_X(D),
\]
where
\[
c'_X(D) := h^0(R^1\eta_*\mathcal{O}_Y([\eta^*D])) - h^0(R^1\eta_*\mathcal{O}_Y).
\]

6.2.4. Remark. Note that \(c_X(D)\) can be computed locally:
\[
c_X(D) = \sum_{P \in X} c_{P,X}(D),
\]
where \(c_{P,X}(D)\) is defined by the formula (6.2.1) for each germ \((X \ni P)\).

6.2.5. Lemma. Let \((X \ni P)\) be a rational log canonical surface singularity.
Then
\[
c_{P,X}(-K_X) = \Delta^2 - [\Delta]^2 - 3.
\]
where, as usual, \(\Delta\) is defined by \(K_Y = \eta^*K_X - \Delta\).

Proof. Put \(D := -K_X\) and write
\[
\eta^*D = -K_Y - \Delta, \quad \langle D^* \rangle = \langle -\Delta \rangle = [\Delta] - \Delta,
\]
\[
[\eta^*D] - K_Y = -2K_Y - [\Delta] = -2\eta^*K_X + 2\Delta - [\Delta].
\]
Therefore,
\[
c_{P,X}(D) = \frac{1}{2}(\Delta - [\Delta]) \cdot (-2\eta^*K_X + 2\Delta - [\Delta]) = \frac{1}{2}([\Delta] - \Delta) \cdot ([\Delta] - 2\Delta).
\]
Since \((X \ni P)\) is a rational singularity, we have
\[
-2 = 2p_a([\Delta]) - 2 = ([\Delta] - \Delta) \cdot [\Delta], \quad [\Delta]^2 + 2 = \Delta \cdot [\Delta]
\]
and the equality follows. \(\square\)

6.2.6. Corollary. Let \((X \ni P)\) be a rational log canonical surface singularity such that \(K^2\) is integral. Then
\[
c_{P,X}(-K_X) = \begin{cases} -1 & \text{in the case (DV)}, \\ 0 & \text{if } (X \ni P) \text{ is log terminal or in the case (nDV)}. \end{cases}
\]

6.2.8. Corollary. Let \(X\) be a del Pezzo surface with log canonical rational singularities and \(\rho(X) = 1\). Assume that for any singularity of \(X\) the invariant \(K^2\) is integral. Then \(H^i(X, \mathcal{O}_X) = 0\) for \(i > 0\) and \(\dim |-K_X| \geq K^2_X - 1\).

Proof. By the Serre duality \(H^2(X, \mathcal{O}_X) = H^0(X, K_X) = 0\). If the singularities of \(X\) are rational, then the Albanese map is a well defined morphism \(\text{alb} : X \to \text{Alb}(X)\). Since \(\rho(X) = 1\), we have \(\dim \text{Alb}(X) = 0\) and so \(H^1(X, \mathcal{O}_X) = 0\). The last inequality follows from (6.2.3) because \(c'_X(-K_X), c_X(-K_X) \geq 0\) (see (6.2.7)). \(\square\)
7. DEL PEZZO SURFACES

7.1. Assumption. From now on let $X$ be a del Pezzo surface satisfying the following conditions:

(i) the singularities of $X$ are log canonical and $X$ has at least one non-log terminal point $o \in X$,
(ii) $X$ admits a $\QQ$-Gorenstein smoothing,
(iii) $\rho(X) = 1$.

7.2. Lemma. In the above assumptions the following holds:

(i) $\dim | - K_X | > 0$,
(ii) $X$ has exactly one non-log terminal point.

Proof. (i) is implied by semicontinuity. (ii) follows from Shokurov’s connectedness theorem [Sho93, Lemma 5.7], [Kol92, Th. 17.4]. □

7.3. Lemma. Let $D \in | - K_X |$ be any member.

(i) All components of $D$ pass through $o$ and does not meet each other outside.
(ii) Any component $D_i \subset \text{Supp}(D)$ is a smooth rational curve.

Proof. Since $D \sim -K_X$ is not Cartier at $o$, we have $o \in \text{Supp}(D)$. If $D_i \not= o$, then $D \not= D_i$ and the locus of log canonical singularities of the pair $(X, D_i)$ consists of two connected components: $o$ and $D_i$. Since $-(K_X + D_i)$ is ample, this contradicts Shokurov’s connectedness theorem [Sho93, Lemma 5.7], [Kol92, Th. 17.4]. The contradiction proves (ii).

Let $D_i \subset \text{Supp}(D)$ be an irreducible component. Assume that $D \not= D_i$. Then $-(K_X + D_i)$ is ample. By the adjunction formula

$$(K_X + D_i)|_{D_i} = K_{D_i} + \text{Diff}_{D_i}(0).$$

Hence, $\deg K_{D_i} < \deg \text{Diff}_{D_i}(0) \leq 0$ and so $D_i$ is a smooth rational curve.

If $D = D_i$, then $D$ is reduced and irreducible and again by the adjunction formula $K_D + \text{Diff}_D(0) = 0$. Since $o \in X$ is a singular point and $o \in D$, we have $\text{Diff}_D(0) \not= 0$ and so $p_a(D) = 0$. □

7.4. Construction. Let $\sigma : \tilde{X} \to X$ be a dlt modification and let

$$\tilde{C} = \sum_{i=1}^{s} \tilde{C}_i = \sigma^{-1}(o)$$

be the exceptional divisor. Thus $\rho(\tilde{X}) = s + 1$.

For some large $k$ the divisor $-kK_X$ is very ample. Let $H \in | - kK_X |$ be a general member and let $\Theta := \frac{1}{k}H$. Then $K_X + \Theta \equiv 0$ and the pair $(X, \Theta)$ is lc at $o$ and klt outside $o$. We can write

$$(7.4.1) \quad K_{\tilde{X}} + \tilde{C} = \sigma^* K_X, \quad K_{\tilde{X}} + \tilde{\Theta} + \tilde{C} = \sigma^* (K_X + \Theta),$$

21
where \( \tilde{\Theta} \) is the proper transform of \( \Theta \) on \( \tilde{X} \). Clearly \( \tilde{C} \cap \text{Supp}(\tilde{\Theta}) = \emptyset \) and \( \tilde{\Theta} \) is nef and big. Note also that \( K_{\tilde{X}} \) is \( \sigma \)-nef.

Let \( D \in |-K_{\tilde{X}}| \) be any member. We can take \( D \) so that \( o \in \text{Supp}(D) \). This holds automatically if \( I > 1 \). We have

\[
(7.4.2) \quad K_{\tilde{X}} + \sum m_i \tilde{C}_i + \tilde{D} \sim 0, \quad m_i \geq 2 \quad \forall i.
\]

We distinguish two cases that will be treated in Sect. 8 and 9 respectively:

(A) there exists a fibration \( \tilde{X} \to T \) over a smooth curve,

(B) \( \tilde{X} \) has no dominant morphism to a curve.

To show the existence of \( \mathbb{Q} \)-Gorenstein smoothings we use unobstructedness of deformations:

7.5. Proposition ([HP10, Proposition 3.1]). Let \( Y \) be a projective surface with log canonical singularities such that \( -K_Y \) is big. Then there are no local-to-global obstructions to deformations of \( Y \). In particular, if the singularities of \( Y \) admit \( \mathbb{Q} \)-Gorenstein smoothings, then the surface \( Y \) admits a \( \mathbb{Q} \)-Gorenstein smoothing.

However, in some cases the corresponding smoothings can be constructed explicitly:

7.5.1. Example. Consider the hypersurface \( X \subset \mathbb{P}(1,1,2,3) \) given by \( z^2 = y\phi_4(x_1, x_2) \). Then \( X \) is a del Pezzo surface with \( K_X^2 = 1 \). The singular locus of \( X \) consists of the point \( (0 : 0 : 1 : 0) \) of type \( [3; [2]^4] \) and four points \( \{ z = y = \phi_4(x_1, x_2) = 0 \} \) of types \( A_1 \). Therefore, \( X \) is of type \( [2]^n \) with \( n = 3 \).

7.5.2. Example. Consider the hypersurface \( X \subset \mathbb{P}(1,1,2,3) \) given by \( (x_1^3 - x_2^3)z + y^3 = 0 \). Then \( X \) is a del Pezzo surface with \( K_X^2 = 1 \). The singular locus of \( X \) consists of the point \( (0 : 0 : 1 : 0) \) of type \( [2; [3]^2] \) and three points \( (1 : \zeta_k^k : 0 : 0), \ k = 0, 1, 2 \) of types \( A_2 \). Therefore, \( X \) is of type \( [2]^n \) with \( n = 2 \).

8. Proof of Theorem 1.1: Fibrations

In this section we consider the case (A) of 7.4. First we describe quickly the singular fibers that occur in our classification.

8.1. Let \( Y \) be a smooth surface and let \( Y \to T \) be a rational curve fibration. Let \( \Sigma \subset Y \) be a section and let \( F \) be a singular fiber. We say that \( F \) is of type \( (I_k) \) or (II) if its dual graph has the following form, where \( \Box \) corresponds to \( \Sigma \) and \( \bullet \) corresponds to a \((-1)\)-curve:

\[
(I_k) \quad \Box \quad \bullet \quad \circ \quad \cdots \quad \circ \quad \circ
\]
Assume that \( T \simeq \mathbb{P}^1 \) and \( Y \) has only fibers of these types (I\(_k\)) or (II). Let \( Y \to \bar{X} \) be the contraction of all curves in fibers having self-intersections less than \(-1\), i.e. corresponding to white vertices. Then \( \rho(\bar{X}) = 2 \) and \( \bar{X} \) has a contraction \( \theta : \bar{X} \to T \).

8.1.1. **Remark.** Let \( \bar{C} \subset \bar{X} \) be the image of \( \Sigma \). Assume that \( \bar{X} \) is projective, \( \bar{C}^2 < 0 \), i.e. \( \bar{C} \) is contractible, and \( (K_{\bar{X}} + \bar{C}) \cdot \bar{C} = 0 \). For a general fiber \( F \) of \( \theta \) we have \( (K_{\bar{X}} + \bar{C}) \cdot F = -1 \). Therefore, \( -(K_{\bar{X}} + \bar{C}) \) is nef. Now let \( \tilde{X} \to X \) be the contraction of \( \bar{C} \). Then \( X \) is a del Pezzo surface with \( \rho(X) = 1 \).

8.2. Recall that we use the notation of 7.1 and 7.4. In this section we assume that \( \tilde{X} \) has a rational curve fibration \( \tilde{X} \to T \), where \( T \) is a smooth curve (the case \([A]\)). Since \( \rho(X) = 1 \), the curve \( \bar{C} \) is not contained in the fibers. A general fiber \( \tilde{F} \subset \tilde{X} \) is a smooth rational curve. By the adjunction formula \( K_{\tilde{X}} \cdot \tilde{F} = -2 \). By \([7.4.2]\) we have \( \tilde{F} \cdot \sum m_i \tilde{C}_i = 2 \) and so \( \tilde{F} \cdot \tilde{D} = 0 \). Hence there exists exactly one component of \( \bar{C} \), say \( \tilde{C}_1 \), such that \( \tilde{F} \cdot \tilde{C}_1 = 1 \), \( m_1 = 2 \), and for \( i \neq 1 \) we have \( \tilde{F} \cdot \tilde{C}_i = 0 \). This means that the divisor \( \tilde{D} \) and the components \( \tilde{C}_i \) with \( i \neq 1 \) are contained in the fibers and \( \tilde{C}_1 \) is a section of the fibration \( \tilde{X} \to T \).

Let us contract all the vertical components of \( \bar{C} \), i.e. the components \( \tilde{C}_i \) with \( i \neq 1 \). We get the following diagram:

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\nu} & X \\
\sigma \downarrow & & \theta \downarrow \\
X & & T
\end{array}
\]

Let \( C := \nu_* \tilde{C} = \nu_* \tilde{C}_1, \Theta = \nu_* \tilde{\Theta} \), and \( D = \nu_* \tilde{D} \). By \([7.4.1]\) and \([7.4.2]\) we have

\[
(8.2.1) \quad K_{\bar{X}} + \tilde{C} + \tilde{\Theta} \equiv 0, \quad K_{\bar{X}} + 2\tilde{C} + \tilde{D} \sim 0.
\]

Moreover, the pair \((\bar{X}, \bar{C} + \bar{\Theta})\) is lc and if \( I > 1 \), then \( \dim |\tilde{D}| > 0 \).

8.3. **Lemma.** If the singularity \((X \ni o)\) is not rational, then \( T \) is an elliptic curve, \( \bar{X} \simeq \tilde{X} \) is smooth, and \( X \) is a generalized cone over \( T \).

**Proof.** By Theorem \([2.3][1]\) the surface \( \tilde{X} \) is smooth along \( \tilde{C} \). Since \( \tilde{C}_1 \) is a section, we have \( \tilde{C}_1 \simeq T \) and \( \bar{C} \) cannot be a combinatorial cycle of smooth rational curves. Hence both \( \tilde{C}_1 \) and \( T \) are smooth elliptic curves. Then
\( \tilde{C} = \tilde{C}_1 \) and \( \rho(\tilde{X}) = \rho(X) + 1 = 2 \). Hence any fiber \( \tilde{F} \) of the fibration \( \tilde{X} \to T \) is irreducible. Since \( \tilde{F} \cdot \tilde{C}_1 = 1 \), any fiber is not multiple. This means that \( \tilde{X} \to T \) is a smooth morphism. Therefore, \( \tilde{X} \) is a geometrically ruled surface over an elliptic curve. \( \square \)

From now on we assume that the singularities of \( X \) are rational. In this case, \( T \simeq \mathbb{P}^1 \) and \( \dim |D| > 0 \).

**8.4. Lemma.** Let \( \tilde{F} \) be a degenerate fiber (with reduced structure). Then the dual graph of \( \tilde{F} \) has one of the forms described in 8.1:

(I) \( k \) with \( k = 2, 3, 4 \) or \( 6 \), or (II).

**Proof.** Let \( Y \to \tilde{X} \) be the minimal resolution. Since \( \rho(\tilde{X}) = 2 \), every degenerate fiber of the composition \( Y \to T \) contains exactly one \((-1)\)-curve.

Let \( \tilde{P} := \tilde{C} \cap \tilde{F} \). Since \( -(K_{\tilde{X}} + \tilde{C} + \tilde{F}) \) is \( \theta \)-ample, the pair \( (\tilde{X}, \tilde{C} + \tilde{F}) \) is plt outside \( \tilde{C} \) by Shokurov’s connectedness theorem. Let \( m \) be the multiplicity of \( \tilde{F} \). Since \( \tilde{C} \) is a section of \( \theta \), we have \( \tilde{C} \cdot \tilde{F} = 1/m < 1 \) and so the point \( \tilde{P} \in \tilde{X} \) is singular.

If the pair \( (\tilde{X}, \tilde{F}) \) is plt at \( \tilde{P} \), then \( \tilde{X} \) has on \( \tilde{F} \) two singular points and these points are of types \( \frac{1}{n}(1, q) \) and \( \frac{1}{n}(1, -q) \) (see e.g. [Pro01, Th. 7.1.12]). We may assume that \( \tilde{P} \in \tilde{X} \) is of type \( \frac{1}{n}(1, q) \). In this case, \( m = n \) and the pair \( (\tilde{X}, \tilde{C} + \tilde{F}) \) is lc at \( \tilde{P} \) because \( \tilde{C} \cdot \tilde{F} = 1/n \). By Theorem 1.2, we have \( n = 2, 3, 4, \) or \( 6 \) and \( q = 1 \). We get the case (I) \( k \). From now on we assume that \( (\tilde{X}, \tilde{F}) \) is not plt at \( \tilde{P} \). In particular, \( (\tilde{X} \not\ni \tilde{P}) \) is not of type \( \frac{1}{n}(1, 1) \). Then again by Theorem 1.2 the singularity \( \tilde{P} \) is of type \([n_1, \ldots, n_k; 2]^4\). Hence the part of the dual graph of \( \tilde{F} \) attached to \( \tilde{C}_1 \) has the form

\[
\begin{array}{cccccc}
\circ & n_1 & \cdots & n_k & \circ \\
\bigcirc & & & & \\
\end{array}
\]

where \( k \geq 1 \). Then \( K_{\tilde{X}} + \tilde{C} \) is of index 2 at \( \tilde{P} \) (see [Kol92, Prop. 16.6]). Since \( (K_{\tilde{X}} + \tilde{C}) \cdot m\tilde{F} = -1 \), the number \( 2(K_{\tilde{X}} + \tilde{C}) \cdot \tilde{F} = -2/m \) must be an integer. Therefore, \( m = 2 \). Assume that \( \tilde{X} \) has a singular point \( \tilde{Q} \) on \( \tilde{F} \setminus \{\tilde{P}\} \). We can write \( \text{Diff}_F(0) = \alpha_1 \tilde{P} + \alpha_2 \tilde{Q} \), where \( \alpha_1 \geq 1 \) (by the inversion of adjunction) and \( \alpha_2 \geq 1/2 \). Then \( \text{Diff}_F(\tilde{C}) = \alpha'_1 \tilde{P} + \alpha_2 \tilde{Q} \), where \( \alpha'_1 = \alpha_1 + \tilde{F} \cdot \tilde{C} \geq 3/2 \). On the other hand, the divisor

\[-(K_{\tilde{F}} + \text{Diff}_F(\tilde{C})) = -(K_{\tilde{X}} + \tilde{F} + \tilde{C})|_F \]

is ample. Hence, \( \deg \text{Diff}_F(\tilde{C}) < 2 \), a contradiction. Thus \( \tilde{P} \) is the only singular point of \( \tilde{X} \) on \( \tilde{F} \). Then contracting \((-1)\)-curves successively, we see that \( \bullet \) is attached to one of the \((-2)\)-curves at the end of the graph and \( n_k = n_{k-1} = 2 \) and \( k = 2 \), i.e. we get the case (II). \( \square \)
Proof of Theorem 1.1 in the case 7.4[A]. If all the fibers are smooth, then by Lemma 8.3 we have the case 1°. If there exist a fiber of type (I_k) with k > 2, then I > 2 and by Theorem 1.2 we have cases 5°, 6°, 7°. If all the fibers are of types (I_2) or (II), then I = 2 and we have cases 2°, 3°, 4°. □

9. Proof of Theorem 1.1: Birational contractions

9.1. In this section we assume that \( \tilde{X} \) has no dominant morphism to a curve (case 7.4[B]). It will be shown that this case does not occur.

Run the \( K_{\tilde{X}} \)-MMP on \( \tilde{X} \). Since \( -K_{\tilde{X}} \) is big, on the last step we get a Mori fiber space \( \bar{X} \rightarrow T \) and by our assumption \( T \) cannot be a curve. Hence \( T \) is a point and \( \bar{X} \) is a del Pezzo surface with \( \rho(\bar{X}) = 1 \). Moreover, the singularities of \( \bar{X} \) are log terminal and so \( \bar{X} \not\simeq X \). Thus we get the following diagram

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\sigma} & X \\
\downarrow{\nu} & & \downarrow{

}\end{array}
\]

Put \( \bar{C} := \nu_{*}\tilde{C} \) and \( \bar{C}_i := \nu_{*}\tilde{C}_i \). By (7.4.2) we have

\[(9.1.1) \quad K_{\bar{X}} + \sum m_i\bar{C}_i + \bar{D} \sim 0, \quad m_i \geq 2.\]

Since \( \rho(X) = \rho(\bar{X}) \) and \( \bar{C} \) is the \( \sigma \)-exceptional divisor, the whole \( \bar{C} \) cannot be contracted by \( \nu \).

9.2. Lemma. Any fiber \( \nu^{-1}(\bar{P}) \) of positive dimension meets \( \bar{C} \).

Proof. Since \( \bar{X} \) is normal, \( \nu^{-1}(\bar{P}) \) is a connected contractible divisor. Since all the components of \( \bar{C} \) are \( K_{\bar{X}} \)-non-negative, \( \nu^{-1}(\bar{P}) \not\subset \tilde{C} \). Since \( \rho(X) = 1 \), we have \( \nu^{-1}(\bar{P}) \cap \tilde{C} \neq \emptyset \). □

9.3. Lemma. If \( \nu \) is not an isomorphism over \( \bar{P} \), then \( (\bar{X}, \bar{C}) \) is plt at \( \bar{P} \). In particular, \( \bar{C} \) is smooth at \( \bar{P} \).

Proof. Since \( K_{\bar{X}} + \bar{C} + \bar{\Theta} \equiv 0 \), the pair \( (\bar{X}, \bar{C} + \bar{\Theta}) \) is lc. By the above lemma there exists a component \( \bar{E} \) of \( \nu^{-1}(\bar{P}) \) meeting \( \bar{C} \). By Kodaira’s lemma the divisor \( \bar{\Theta} - \sum \alpha_i\bar{C}_i \) is ample for some \( \alpha_i > 0 \). Hence \( \bar{E} \) meets \( \bar{\Theta} \) and so \( \text{Supp}(\bar{\Theta}) \) contains \( \bar{P} \). Therefore, \( (\bar{X}, \bar{C}) \) is plt at \( \bar{P} \). □

9.3.1. Corollary. \( (X, C) \) is dlt.

9.4. Lemma. (i) \( \bar{C} \) is an irreducible smooth rational curve;
(ii) \( \bar{X} \) has at most two singular points on \( \bar{C} \);
(iii) the singularities of \( X \) are rational.

Proof. [i] Let \( \bar{C}_1 \subset \bar{C} \) be any component meeting \( \bar{D} \) and let \( \bar{C}' := \bar{C} - \bar{C}_1 \). Assume that \( \bar{C}' \neq 0 \). By 9.3.1 any point \( \bar{P} \in \bar{C}_1 \cap \bar{C}' \) is a smooth point of \( \bar{X} \). Hence \( \text{Diff}_{\bar{C}_1}(\bar{C}') \) contains \( \bar{P} \) with positive integral coefficient and \( \deg \text{Diff}_{\bar{C}_1}(\bar{D} + \bar{C}') \geq 2 \) because \( \text{Supp}(\bar{D}) \cap \bar{C} \neq \emptyset \). On the other hand,
We have \(-K_{X} + \mathcal{C} + \mathcal{D}\) is ample by (9.1.1). Thus contradicts the adjunction formula. Thus \(\mathcal{C}\) is irreducible. Again by the adjunction
\[
\deg K_{\mathcal{C}} + \deg \text{Diff}_{\mathcal{C}}(0) < 0.
\]
Hence, \(p_{\alpha}(\mathcal{C}) = 0\).

(ii) Assume that \(\bar{X}\) is singular at \(\bar{P}_{1}, \ldots, \bar{P}_{N} \in \bar{\mathcal{C}}\). Write
\[
\text{Diff}_{\mathcal{C}}(0) = \sum_{i=1}^{N} \left(1 - \frac{1}{b_{i}}\right) \bar{P}_{i}
\]
for some \(b_{i} \geq 2\). The coefficient of \(\text{Diff}_{\mathcal{C}}(\bar{D})\) at points of the intersection \(\text{Supp}(\bar{D}) \cap \bar{C}\) is at least 1. Since \(\text{Supp}(\bar{D}) \cap \bar{C} \neq \emptyset\), we have \(N \leq 2\).

(iii) If \((X \ni o)\) is a non-rational singularity, then \(p_{\alpha}(\mathcal{C}) = 1\) and \(\bar{X}\) is smooth along \(\bar{\mathcal{C}}\). Hence \(p_{\alpha}(\mathcal{C}) \geq 1\). This contradicts (i). \(\square\)

9.5. Lemma. Let \(\varphi : S \rightarrow S'\) be a birational Mori contraction of surfaces with log terminal singularities and let \(E \subset S\) be the exceptional divisor. Then \(-K_{S} \cdot E \leq 1\) and the equality holds if and only if the singularities of \(S\) along \(E\) are at worst Du Val.

Proof. Let \(\psi : S_{\text{min}} \rightarrow S\) be the minimal resolution and let \(\hat{E} \subset S_{\text{min}}\) be the proper transform of \(E\). Write \(K_{S_{\text{min}}} = \psi^{*}K_{S} - \Delta\). Since \(K_{S_{\text{min}}} : \psi^{*}E < 0\), the divisor \(K_{S_{\text{min}}}\) is not nef over \(Z\). Hence, \(K_{S_{\text{min}}} : \hat{E} = -1\) and so \(-K_{S} : E + \hat{E} : \Delta = 1\). \(\square\)

9.6. Lemma. Let \(\nu' : \bar{X} \rightarrow X'\) be the first extremal contraction in \(\nu\) and let \(\hat{E}\) be its exceptional divisor. Then \(\hat{E} \not\subset \mathcal{C}\). Moreover, \(\hat{E} \cap \mathcal{C}\) is a singular point of \(\bar{X}\) and smooth point of \(\bar{C}\).

Proof. Since \(\rho(X) = 1\), \(\hat{E} \cap \bar{C} \neq \emptyset\). Since \(K_{\bar{X}}\) is \(\sigma\)-nef, \(\hat{E} \not\subset \mathcal{C}\). Since \(\mathcal{C}\) is a smooth rational curve, \(\hat{E}\) meets \(\mathcal{C}\) at a single point, say \(\hat{P}\). Further, \(\sigma(\hat{E})\) meets \(\text{Supp}(\Theta)\) outside \(o\). Hence, \(\Theta \cdot \hat{E} > 0\). By Lemma 9.5 \(K_{\bar{X}} : \hat{E} \geq -1\). Since \(K_{\bar{X}} + \mathcal{C} + \Theta \equiv 0\), we have \(\hat{C} \cdot \hat{E} < 1\). Hence \(\hat{C} \cap \hat{E}\) is a singular point of \(\bar{X}\). Since \((\hat{X}, \hat{C})\) is \(\text{dlt}\), \(\hat{C} \cap \hat{E}\) is a smooth point of \(\hat{C}\) (see e.g. [Ko92, 16.6]). \(\square\)

9.7. Proposition. \(\rho(\bar{X}) = 2\) and \(\hat{C}\) is irreducible. Moreover, \(\bar{X}\) has exactly two singular points on \(\bar{C}\) and \(I > 2\).

Proof. Assume the converse, i.e. \(\hat{C}\) is reducible. By Lemma 9.4 the curve \(\hat{C}\) is irreducible. Let \(s\) be the number of components of \(\hat{C}\). So, \(\rho(\bar{X}) = s + 1\). Hence \(\nu\) contracts \(s - 1\) components of \(\hat{C}\) and exactly one divisor, say \(\hat{E}\) such that \(\hat{E} \not\subset \mathcal{C}\). By Lemma 9.6 the curve \(\hat{E}\) is contracted on the first step. Note that \(\hat{C}\) is a chain \(\hat{C}_{1} + \cdots + \hat{C}_{s}\), where both \(\hat{C}_{1}\) and \(\hat{C}_{s}\) contain two points of type \(A_{1}\) and the middle curves \(\hat{C}_{2}, \ldots, \hat{C}_{s-1}\) are contained in the smooth locus. By Lemma 9.6 we may assume that \(\hat{E}\) meets \(\hat{C}_{1}\). Then \(\nu\)
contracts $\tilde{C}_1, \ldots, \tilde{C}_{s-1}$. However $\tilde{C}_s$ contains two points of type $A_1$ and it is not contracted. Thus $\tilde{X}$ has two singular points of type $A_1$ on $\tilde{C}$. Again by Lemma 9.4 the surface $\tilde{X}$ has no other singular points on $\tilde{C}$. In particular, $2\tilde{C}$ is Cartier, $\tilde{X}$ has only singularities of type $T$, and $K^2_{\tilde{X}}$ is an integer.

On the other hand, we have $-K_{\tilde{X}} = m\tilde{C} + \tilde{D}$, $m \geq 2$. By the adjunction formula

\[-1 = \deg(K_{\tilde{X}} + \text{Diff}_{\tilde{C}}(0)) = (K_{\tilde{X}} + \tilde{C}) \cdot \tilde{C} = -\tilde{D} \cdot \tilde{C} - (m-1)\tilde{C}^2.\]

This gives us $\tilde{D} \cdot \tilde{C} = \tilde{C}^2 = 1/2$, $m = 2$, and $K^2_{\tilde{X}} = 9/2$, a contradiction.

Finally, by Lemmas 9.4 and 9.6 the surface $\tilde{X}$ (resp. $\bar{X}$) has exactly three (resp. two) singular points on $\tilde{C}$.

By Theorem 1.2 the surface $X$ has at least one non-Du Val singularity lying on $\tilde{C}$. Thus Theorem 1.1 is implied by the following.

9.8. **Proposition.** $\tilde{X}$ has only Du Val singularities on $\tilde{C}$.

**Proof.** Assume that the singularities of $\tilde{X}$ at points lying on $\tilde{C}$ are of types $1/n_1(1,1)$ and $1/n_2(1,1)$ with $n_1 \geq n_2$ and $n_1 > 2$. In this case near $\tilde{C}$ the divisor $H := -(K_{\tilde{X}} + 2\tilde{C})$ is Cartier. By the adjunction formula

$$K_{\tilde{C}} + \text{Diff}_{\tilde{C}}(0) = (K_{\tilde{X}} + \tilde{C})|_{\tilde{C}} = -(H + \tilde{C})|_{\tilde{C}}$$

Hence,

$$\deg \text{Diff}_{\tilde{C}}(0) < 2 - H \cdot \tilde{C} \leq 1.$$ In particular, $\tilde{X}$ has at most one singular point on $\tilde{C}$, a contradiction. \qed

**References**

[Art66] Michael Artin. On isolated rational singularities of surfaces. *Amer. J. Math.*, 88:129–136, 1966.

[Bla95] R. Blache. Riemann-Roch theorem for normal surfaces and applications. *Abh. Math. Semin. Univ. Hamburg.*, 65:307–340, 1995.

[Bri68] Egbert Brieskorn. Rationale Singularitäten komplexer Flächen. *Invent. Math.*, 4:336–358, 1967/1968.

[dv92] Theo de Jong and Duco van Straten. A construction of $\mathbb{Q}$-Gorenstein smoothings of index two. *Int. J. Math.*, 3(3):341–347, 1992.

[GS83] Gert-Martin Greuel and Joseph and Steenbrink. On the topology of smoothable singularities. In *Singularities, Part 1 (Arcata, Calif., 1981)*, volume 40 of *Proc. Sympos. Pure Math.*, pages 535–545. Amer. Math. Soc., Providence, R.I., 1983.

[Hac04] Paul Hacking. Compact moduli of plane curves. *Duke Math. J.*, 124(2):213–257, 2004.

[HP10] Paul Hacking and Yuri Prokhorov. Smoothable del Pezzo surfaces with quotient singularities. *Compositio Math.*, 146(1):169–192, 2010.

[HT87] Takayuki Hayakawa and Kiyohiko Takeuchi. On canonical singularities of dimension three. *Japan. J. Math. (N.S.)*, 13(1):1–46, 1987.

[Kaw88] Yujiro Kawamata. Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces. *Ann. of Math. (2)*, 127(1):93–163, 1988.
[Kaw07] Masayuki Kawakita. Inversion of adjunction on log canonicity. *Invent. Math.*, 167:129–133, 2007.

[Kaw15] Masayuki Kawakita. The index of a threefold canonical singularity. *Amer. J. Math.*, 137(1):271–280, 2015.

[KM92] János Kollár and Shigefumi Mori. Classification of three-dimensional flips. *J. Amer. Math. Soc.*, 5(3):533–703, 1992.

[KM98] János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

[Kol91] János Kollár. Flips, flops, minimal models, etc. In *Surveys in differential geometry (Cambridge, MA, 1990)*, pages 113–199. Lehigh Univ., Bethlehem, PA, 1991.

[Kol92] János Kollár, editor. *Flips and abundance for algebraic threefolds*. Société Mathématique de France, Paris, 1992. Papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991, Astérisque No. 211 (1992).

[KSB88] J. Kollár and N. I. Shepherd-Barron. Threefolds and deformations of surface singularities. *Invent. Math.*, 91(2):299–338, 1988.

[Loo86] Eduard Looijenga. Riemann-Roch and smoothings of singularities. *Topology*, 25:293–302, 1986.

[LW86] Eduard Looijenga and Jonathan Wahl. Quadratic functions and smoothing surface singularities. *Topology*, 25(3):261–291, 1986.

[Man91] Marco Manetti. Normal degenerations of the complex projective plane. *J. Reine Angew. Math.*, 419:89–118, 1991.

[MP09] S. Mori and Yu. Prokhorov. Multiple fibers of del Pezzo fibrations. *Proc. Steklov Inst. Math.*, 264(1):131–145, 2009.

[Pro01] Yu. Prokhorov. *Lectures on complements on log surfaces*, volume 10 of *MSJ Memoirs*. Mathematical Society of Japan, Tokyo, 2001.

[Pro15] Yu. Prokhorov. A note on degenerations of del Pezzo surfaces. *Annales de l’institut Fourier*, 65(1):369–388, 2015.

[Pro16] Yu. Prokhorov. Q-Fano threefolds of index 7. *Proc. Steklov Inst. Math.*, 294:139–153, 2016.

[Rei87] Miles Reid. Young person’s guide to canonical singularities. In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, volume 46 of *Proc. Sympos. Pure Math.*, pages 345–414. Amer. Math. Soc., Providence, RI, 1987.

[Sho93] V. V. Shokurov. 3-fold log flips. *Russ. Acad. Sci., Izv., Math.*, 40(1):95–202, 1993.

[Ste91] Jan Stevens. Partial resolutions of rational quadruple points. *Int. J. Math.*, 2(2):205–221, 1991.

[Wah80] Jonathan M. Wahl. Elliptic deformations of minimally elliptic singularities. *Math. Ann.*, 253(3):241–262, 1980.

[Wah81] Jonathan Wahl. Smoothings of normal surface singularities. *Topology*, 20(3):219–246, 1981.

Steklov Mathematical Institute, Russia
Moscow State Lomonosov University, Russia
National Research University Higher School of Economics, Russia
E-mail address: prokhor@mi.ras.ru