The multiple sum formulas for 9j and 12j coefficients of SU(2) and \( u_q(2) \)

Sigita Ališauskas

Institute of Theoretical Physics and Astronomy, A. Goštauto 12, Vilnius 2600, Lithuania

(March 30, 2022)

Seven different triple sum formulas for 9j coefficients of the quantum algebra \( u_q(2) \) are derived, using for these purposes the usual expansion of \( q \)-9j coefficients in terms of \( q \)-6j coefficients and recent summation formulas of twisted \( q \)-factorial series (resembling the very well-poised basic hypergeometric \( 5\phi_4 \) series) as \( q \)-generalizations of Dougall’s summation formula of the very well-poised hypergeometric \( 4F3(-1) \) series. This way for \( q = 1 \) the new proof of the known triple sum formula is proposed, as well as six new triple sum formulas for 9j coefficients of the SU(2) group, in the angular momentum theory. The mutual rearrangement possibilities of the derived triple sum formulas by means of the Chu–Vandermonde summation formulas are considered and applied to derive several versions of double sum formulas for the stretched \( q \)-9j coefficients, which give new rearrangement and summation formulas of special Kampé de Fériet functions and their \( q \)-generalizations. Several fourfold sum formulas [with each sum of the \( 5F1 \) or \( 5\phi_4 \) type] for the 12j coefficients of the second kind (without braiding) of the SU(2) and \( u_q(2) \) are proposed, as well as expressions with five sums [of the \( 4F3 \) or \( 3F2 \) or \( 3\phi_3 \) or \( 3\phi_2 \) type] for the 12j coefficients of the first kind (with braiding) instead of the usual expansion in terms of \( q \)-6j coefficients. Stretched and doubly stretched \( q \)-12j coefficients [as triple, double or single sums, related to composed or separate hypergeometric \( 4F3 \) or \( 5F1 \) or \( 5\phi_3 \) or \( 5\phi_4 \) series, respectively] are considered.

I. INTRODUCTION

The Wigner 9j coefficients arise as the recoupling coefficients of the four irreducible representations (irreps) of the SU(2) group and play the important roles in the quantum mechanical angular momentum theory. Many applications have also the 12j coefficients of both kinds as the recoupling coefficients of the five irreps of the SU(2) group. There are many known expressions for 9j coefficients as multiple series. Nevertheless, the most compact formula (however, which does not represent any symmetry of 9j-symbol) was derived originally by Ališauskas and Jucys as a triple sum series, in frames of resolution of the multiplicity-free (semistretched) coupling problems for the states of irreducible representations of the Sp(4) [SO(5)] group restricted to SU(2) \( \times \) SU(2). In Ref. it was also proved after tedious rearrangement of the fourfold sum in frames of the usual angular momentum [SU(2) representation theory] technique, by means of the Chu–Vandermonde summation formulas. Different computational, polynomial, rearrangement, specification
and other aspects of these multiple sum series were considered. Their analytical continuation was also adapted for the isoscalar factors of the Clebsch–Gordan (CG) coefficients of the Lorentz or $SL(2,C)$ group.

Recently Rosengren proposed two new proofs of the triple sum formula for $9j$ coefficients of SU$(1,1)$. His first proof was based on the use of the explicit coupling kernels in the $\mathfrak{su}(1,1)$ algebra, rather then in the $\mathfrak{su}(2)$ algebra, when in the second case the usual expansion of $9j$ coefficients of $\mathfrak{su}(1,1)$ in terms of $6j$ coefficients was rearranged using the appropriate expressions for the Racah coefficients in terms of the balanced hypergeometric $4F_3(1)$ series and Dougall’s summation formula of the very well-poised $4F_3(-1)$ series [which, in other words, corresponds to the factorial sum weighted with factor $(2j + 1)$].

For the quantum algebra $u_q(2)$, the expansion of the $q$-$9j$ coefficients in terms of $q$-$6j$ coefficients was generalized by Nomura and Smirnov et al and extended to $q$-$3nj$ coefficients (particularly, of the first and the second kind) by Nomura, who discussed their role in frames of the Yang–Baxter Relations. The corresponding summation formula of the twisted $q$-factorial series [generalizing Dougall’s summation formula and resembling (but not equivalent with) the very well-poised basic hypergeometric $5_3\phi_4$ series, depending on 3 parameters] needed for our purpose was derived by Ališauskas and the twisted very well-poised $q$-factorial series, resembling the basic hypergeometric $7\phi_6$ series (depending on 5 parameters) appear in a new approach to the Clebsch–Gordan coefficients of $u_q(2)$. In the $u_q(3)$ context, Ališauskas also used the summation formula of the $q$-factorial series depending on 4 parameters which correspond to Dougall’s summation formula of the very well-poised hypergeometric $5F_4(1)$ or basic hypergeometric $6\phi_5$ series.

The main purpose of the present paper is to derive the all independent expressions with the triple sums for the $q$-$9j$ and the usual $9j$ coefficients, as well as to rearrange expressions for the $q$-$12j$ and the usual $12j$ coefficients of the both kinds into more convenient forms, with minimal number of sums, or at least eliminating the cumbersome factorial sums weighted with factors $[2j + 1]$ or $(2j + 1)$ from the compositions of the $q$-$6j$ or usual $6j$ coefficients expanded in different forms.

In Section II, the appropriate expressions for the $6j$ coefficients of SU$(2)$ and $u_q(2)$ are presented, as well as the rearranged expansions of $9j$ coefficients of $u_q(2)$ in terms of $6j$ coefficients, allowing us to generalize Rosengren’s approach with help of the new summation formulas, weighted with factor $[2j + 1]$ (see Appendix A). In Section III, seven different triple sum formulas for $9j$ coefficients of $u_q(2)$ are derived, their summation intervals and other properties are compared. For $q = 1$ they turn either to known expression of Ališauskas and Jucys or to six new triple sum formulas for $9j$ coefficients of SU$(2)$. In Section IV, the mutual rearrangement possibilities of new expressions by means of the Chu–Vandermonde summation formulas (see Appendix B) are considered. Particularly, several versions (of different classes) of the double sum formulas for the stretched $9j$ coefficients of $u_q(2)$ and SU$(2)$ are derived, which enable to get new relations and summation formulas (presented in Appendix C) for special Kampé de Féret functions and their $q$-generalizations (cf. Refs. 16, 17).

Section V is devoted to rearrangement of the usual expansion formula (in terms of $6j$ and $q$-$6j$ coefficients) of $12j$ and $q$-$12j$ coefficients of the second kind (i.e., without braiding) into the fourfold sums, using Dougall’s summation formula of the very well-poised $5F_4(1)$ series, depending on 4 parameters. Also, specific stretched and doubly stretched $12j$ and
q-12j coefficients of the second kind are studied. In Sec. VI, expressions of 12j and q-12j coefficients of the first kind (with braiding) in terms of q-6j coefficients are rearranged using the transformation formula of the very well-poised $\phi_6(-1)$ series or q-factorial sums (depending on 5 parameters and weighted with factors ($2j + 1$) or $[2j + 1]$), resembling the very well-poised basic hypergeometric $\tau\phi_q$ series. Variety of the stretched and doubly stretched 12j and q-12j coefficients of the first kind are also considered.

## II. PRELIMINARIES

**A. Expressions for the 6j coefficients of SU(2) and $u_q(2)$**

The appropriate for our purpose expressions for the 6j (Racah) coefficients of SU(2) (with the different or coinciding signs of summation parameters in 3 numerator q-factorial arguments) were derived originally by Bandzaitis et al. (see also Refs. 3,8), when Smirnov et al. rederived them for the Racah coefficients of $u_q(2)$. These two expressions, each in two different versions, may be written as follows:

\[
\begin{aligned}
\{ a \ b \ e \ \\
\{ d \ c \ f \}_q &= \frac{\nabla [acf] \nabla [dbf]}{\nabla [abc] \nabla [dce] \sum_z} (-1)^{a+b+c+d+z} [c+f-a+z]!
\sum_z \frac{[b+f-d+z][a+d+e-f-z][b+d-f-z]}{(e+f-a-d+z)[2f+z+1]!}
\]
\end{aligned}
\]  
(2.1a)

and

\[
\begin{aligned}
\{ a \ b \ e \ \\
\{ d \ c \ f \}_q^{(\theta)} &= \frac{\nabla [eab] \nabla [fbd]}{\nabla [ecd] \nabla [fac] \sum_z} (-1)^{b+c+e+f+z} [2b-z]!
\sum_z \frac{[b+c+e+f-z][b+c+e+f-z+1][b+d+f-z+1]!}{[a+b+e-z+1]![b+f-d-z]!}
\]
\end{aligned}
\]  
(2.2a)

where $\nabla [abc]$ is asymmetric triangle coefficient,

\[
\begin{aligned}
\nabla [abc] &= \left( \frac{[a+b-c]![a-b+c]![a+b+c+1]!}{[b+c-a]!} \right)^{1/2}
\end{aligned}
\]  
(2.3)

and Eqs. (2.2a) – (2.2b) are less effective than (2.1a) and (2.1b). Here and in what follows $[x]$, $[x]_q$, $(\alpha|q)_n$, and $[\alpha]_q$ are, respectively, the $q$-numbers, $q$-factorials, $q$-Pochhammer symbols, and $q$-binomial coefficients.
\[ [x] = (q^x - q^{-x})/(q - q^{-1}), \quad [x]! = [x][x-1]...[2][1], \quad (2.4) \]

\[ (\alpha|q)_n = \prod_{k=0}^{n-1} [\alpha + k], \quad [1]! = [0]! = (\alpha|q)_0 = 1, \quad (2.5) \]

\[ \binom{n}{r}_q = \frac{[n]!q^{n(n-r)}}{[r]![n-r]!} \quad (2.6) \]

where (2.4) and (2.5) are invariant under substitution \( q \leftrightarrow q^{-1} \) and turn into usual integers \( x \), factorials \( x! \) and binomial coefficients \( \binom{n}{r} \) for \( q = 1 \).

We see that each parameter \( b, c, \) or \( e \) appears only twice in the factorial arguments under the summation sign in (2.1a), as well as parameters \( b, c, \) or \( f \) in (2.1b) [which is obtained after some shift of summation parameter in (2.1a)]. Similarly each parameter \( a, c, \) or \( d \) appears only twice in the factorial arguments under the summation sign in (2.2a), as well as parameters \( b, c, \) or \( d \) in (2.2b). Otherwise, each parameter \( a \) or \( d \) appears four times in the factorial arguments under the summation sign in (2.1a) and (2.2a), as well as parameters \( e \) or \( f \) in (2.2a) and (2.2b) and all the parameters in the most symmetric (Racah,\(^3\)) and the remaining expressions for 6\( j \) and 9\( j \) coefficients\(^2\) which include only usual symmetric triangle coefficients \( \Delta_{abc} \) in the numerator and denominator before the summation sign. Note that some triangular conditions restrict the summation intervals in (2.1a) and (2.2a), or they are represented by definite differences of factorial arguments in numerator and denominator, for example, \( (c + f - a + z) - (e + f - a - d + z) \geq 0 \) in (2.1a), or \( (b + d + f - z + 1) - (b + e + f - c - z) - 1 \geq 0 \) in (2.2a).

It should be also noted, that only the expressions presented above (2.1a) and (2.2a) are correlated with the Racah polynomials as introduced by Askey and Wilson, see Ref. 34. In contrast, the most symmetric and the remaining expressions for the 6\( j \) and 9\( j \) coefficients turn into the Racah polynomials only after some Whipple (Bailey) or Sears transform\(^8\) of the balanced hypergeometric \( 4F_3(1) \) or \( 4\phi_3 \) series are used.

**B. Rearrangement of expansions for q-9j coefficients**

We use here the definition of the q-9j coefficients of \( u_q(2) \) introduced by Nomura\(^{24,25}\) in contrast with definition by Smirnov et al\(^{34}\), when the substitution \( q \to q^{-1} \) is necessary. These coefficients (q-9j symbols) may be extracted from the recoupling-braiding coefficients of the states of four irreps and are invariant under even permutations of their rows or columns and under transposition of \( 3 \times 3 \) array (interchange of their rows and columns),

\[ \begin{bmatrix} a & b & e \\ c & d & f \\ h & k & g \end{bmatrix}_q = \begin{bmatrix} e & a & b \\ f & c & d \\ g & h & k \end{bmatrix}_q = \begin{bmatrix} c & d & f \\ h & k & g \\ a & b & e \end{bmatrix}_q = \begin{bmatrix} a & c & h \\ b & d & k \\ e & f & g \end{bmatrix}_q = \text{etc.} \quad (2.7) \]

Taking into account the braiding, in the case of odd permutations of their rows or columns, the q-9j coefficients obey\(^{25,26}\)

\[ \begin{bmatrix} a & b & e \\ c & d & f \\ h & k & g \end{bmatrix}_q = A \begin{bmatrix} a & e & b \\ c & f & d \\ h & g & k \end{bmatrix}_{q^{-1}} = A \begin{bmatrix} a & b & e \\ c & d & f \\ h & k & g \end{bmatrix}_{q^{-1}} = \text{etc.,} \quad (2.8) \]
where

$$A = (-1)^{a+b+c+d+e+f+g+h+k} q^{Z_{deh} + Z_{bo} + Z_{af}}$$

and

$$Z_{deh} = -d(d+1) - e(e+1) - h(h+1).$$

Let us consider some different versions of expansions\(^{23,24,25}\) of the \(q\)-9j coefficients of \(u_q(2)\), written after applying some symmetries of \(q\)-6j coefficients,

\[
\begin{aligned}
\left\{ \begin{array}{ccc}
    a & b & e \\
    c & d & f \\
    h & k & g
\end{array} \right\}_q &= \sum_j (-1)^{2j} q^{Z_{deh} - j(j+1)} [2j + 1] \left\{ \begin{array}{ccc}
    a & c & h \\
    k & g & j \\
    f & d & b
\end{array} \right\}_q \left\{ \begin{array}{ccc}
    k & j & c \\
    f & d & b \\
    a & g & j
\end{array} \right\}_q \left\{ \begin{array}{ccc}
    a & g & j \\
    f & b & e
\end{array} \right\}_q \\
&= \sum_j (-1)^{2j} q^{Z_{deh} - j(j+1)} [2j + 1] \left\{ \begin{array}{ccc}
    k & g & h \\
    a & c & j \\
    d & c & k
\end{array} \right\}_q \left\{ \begin{array}{ccc}
    j & b & f \\
    d & c & k \\
    a & b & j
\end{array} \right\}_q \left\{ \begin{array}{ccc}
    a & b & j \\
    f & b & e
\end{array} \right\}_q \\
&= \sum_j (-1)^{2j} q^{Z_{deh} - j(j+1)} [2j + 1] \left\{ \begin{array}{ccc}
    h & c & a \\
    j & g & k \\
    e & g & a
\end{array} \right\}_q \left\{ \begin{array}{ccc}
    k & j & c \\
    f & d & b \\
    a & g & e
\end{array} \right\}_q \left\{ \begin{array}{ccc}
    a & g & j \\
    f & b & e
\end{array} \right\}_q \\
&= \sum_j (-1)^{2j} q^{Z_{deh} - j(j+1)} [2j + 1] \left\{ \begin{array}{ccc}
    k & j & c \\
    f & d & b \\
    g & e & a
\end{array} \right\}_q \left\{ \begin{array}{ccc}
    b & e & a \\
    f & b & e
\end{array} \right\}_q \left\{ \begin{array}{ccc}
    a & g & j \\
    g & j & f
\end{array} \right\}_q \\
\end{aligned}
\]

where the summation parameters \(j\) are restricted by the triangular conditions,

\[
\max(|a - g|, |f - b|, |k - c|) \leq j \leq \min(a + g, b + f, c + k).
\] (2.10)

When we use expressions (2.1a) or (2.1b) for nonprimed \(q\)-6j coefficients and expressions (2.2a) or (2.2b) for primed \(q\)-6j coefficients, the asymmetric triangle coefficients depending on the summation parameter \(j\) are distributed in their numerators and denominators in expansions (2.9a), (2.9b), and (2.9c)–(2.9g) as follows:

\[
\frac{\nabla[a g j] \nabla[k c j]}{\nabla[k c j] \nabla[a g j]} \times \frac{\nabla[f b j]}{\nabla[k c j] \nabla[a g j] \nabla[f b j]},
\] (2.11a)

when in expansions (2.9c) and (2.9d) as follows:

\[
\frac{\nabla[k c j]}{\nabla[a g j]} \times \frac{\nabla[f b j]}{\nabla[k c j]} \times \frac{\nabla[a g j]}{\nabla[f b j]},
\] (2.11b)
Particularly, they cancel if we express all but the first \( q - 6 \) coefficients in (2.9a) by means of (2.1a), as well as the first and the last \( q - 6 \) coefficients in (2.9b) and (2.9d), the second \( q - 6 \) coefficient in (2.9c), and the first two \( q - 6 \) coefficients in (2.9e), when the remaining (primed) \( q - 6 \) coefficients \( \{ \cdot : \cdot \}^\prime_q \) in (2.9b)–(2.9e) are expressed by means of (2.2a) and the first \( q - 6 \) coefficient of (2.9a) by means of (2.1b). It is expedient to use the inverse order of summation [with substituted by \( z \to a + c - f - z \) parameters in (2.1a)] for the second \( q - 6 \) coefficients in (2.9a), (2.9c), (2.9e), and the first and last \( q - 6 \) coefficients in (2.9d), with \( j \) appearing in the upper middle position of the corresponding \( 6 \)–\( j \)-symbol.

Now the summation formulas (A2) or (A1) of the twisted very well-poised \( q \)-factorial series (see Appendix A) may be used in (2.9a) or in (2.9b) and (2.9c), respectively, if the summation parameters \( j \) are restricted naturally by the non-negative integer values of the denominator factorial arguments,

\[
\begin{align*}
\max(a - g, |f - b|, k - c) & \leq j \leq \min(a + g, c + k), \quad (2.12a) \\
\max(f - b, a - g) & \leq j \leq \min(a + g, b + f, c + k), \quad (2.12b) \\
\max(b - f, k - c) & \leq j \leq \min(a + g, c + k), \quad (2.12c)
\end{align*}
\]

respectively. In (2.9d) and (2.9e), parameters \( j \) are, respectively, restricted by the natural limits,

\[
\begin{align*}
\max(|a - g|, f - b, c - k) & \leq j \leq b + f, \quad (2.12d) \\
\max(k - c, b - f) & \leq j \leq \min(a + g, c + k). \quad (2.12e)
\end{align*}
\]

In these two last cases, the summation formulas (A2) or (A4), respectively, may be used.

However, the formal summation intervals (2.12a–2.12e) may exceed the interval (2.10), determined by triangular conditions. Of course, separate \( q - 6 \) coefficients with spoiled triangular conditions in (2.9b)–(2.9e) vanish, but such vanishing is not evident for the corresponding pure \( q \)-factorial sums of the type (2.1a) or (2.2a). We need to consider each case separately, for example, when \( j = b + f + 1, b + f + 2, \ldots \) [i.e., for \( b + f < j \leq \min(a + g, c + k) \)], or \( \max(g - a, c - k) > j \geq \max(a - g, |f - b|, k - c) \), the second or the first sum of the type (2.1a) or (2.1b) in expansion of (2.9a) turns into 0, in accordance with Karlsson’s summation formula \( \sum \) or its \( q \)-version \( \sum_q \).

\[
\sum s \frac{(-1)^s q^{(n-m-1)s}}{[s]! [n-s]!} \prod_{j=1}^m [A_j - s] = \delta_{m,n} q^{(n+1)/2 + \sum_{j=1}^m A_j} q^{-n(n+1)/2 + \sum_{j=1}^m A_j} \quad (2.13)
\]

where \( m \leq n \) are integers [cf. applications of (2.13) for the multiplicity-free isoscalar factors \( \sum_{\pi} \) of SU(\( n \)) and \( u_q (n) \)]. [Note that the third factorial sum in expansion of (2.9a) may be nonvanishing in spite of spoiling of the triangular conditions.]

### III. NEW EXPRESSIONS FOR 9j COEFFICIENTS OF SU(2) AND \( u_q (2) \)

**A. Expressions with the full triangle restrictions of summation intervals**

Hence, using the expansions (2.9a)–(2.9e), alternative expressions for the \( q - 6 \) coefficients, and summation formulas (A1)–(A5), at first we obtained five different expressions for the \( q - 9 \) coefficients,
\[
\begin{align*}
\begin{cases}
  a & b & e \\
  c & d & f \\
  h & k & g
\end{cases}
&= (-1)^{c+g-h} \nabla [abe] \nabla [fg] \nabla [kbd] \\
& \quad \times q^{(f+h-e-k)-(a+d-e-k+1)-(a-e-f)+(a-e+f+1)+Z_{deh}} \\
& \quad \times \sum_{z_1, z_2, z_3} \frac{(-1)^{z_1+z_2+z_3} [g-h+k+z_1][a+c-h+z_1]}{[z_1][g+h-k-z_1][c-a+h-z_1]} \\
& \quad \times [2h-z_1][2d-z_2][c-d+f+z_2] \\
& \quad \times \frac{[b+e-a+z_3][e-f+g+z_3]}{[b+e-a+z_3][e-f+g-z_3]}q^{z_1(a+d-e-k-z_2-z_3+1)} \\
& \quad \times \frac{[a+d+f-h+z_1+z_2]![e-f-h+k+z_1+z_3]}{[a+d+f-h+z_1+z_2]![e-f-h+k+z_1+z_3]} \quad (3.1a)
\end{align*}
\[
\begin{align*}
\begin{cases}
  a & b & e \\
  c & d & f \\
  h & k & g
\end{cases}
&= (-1)^{e-f-h+k} \nabla [abe] \nabla [fg] \nabla [kbd] \\
& \quad \times q^{(b+e-a)(e-f-h+k)-(a-e+f+1)(a-e+f+1)+Z_{deh}} \\
& \quad \times \sum_{z_1, z_2, z_3} \frac{(-1)^{z_1+z_2+z_3} [a+c-h+z_1][g-h+k+z_1]}{[z_1][c+g-h-z_1][g+h-k-z_1][z_2]} \\
& \quad \times [2h-z_1][2b-z_2][b-c+f+k-z_2] \\
& \quad \times \frac{[b-d+k-z_2][b+d+k-z_2+1][z_3]![a+b-e-z_3]}{[b+d+k-z_2+1][z_3]![a+b-e-z_3]} \\
& \quad \times \frac{[f+g-e-z_3][e+k-f-h+z_1+z_3]}{[f+g-e-z_3][2e+z_3+1][e+k-f-h+z_1+z_3]}q^{z_1(b+e-a-z_2+z_3)+z_3(a+b+f-h+k-z_2+1)-z_2(e+k-f-h)} \\
& \quad \times \frac{[b+e-a+z_3][a+b+f-h+k+z_1+z_2+1]}{[b+e-a+z_3][a+b+f-h+k+z_1+z_2+1]} \quad (3.1b)
\end{align*}
\[
\begin{align*}
\begin{cases}
  a & b & e \\
  c & d & f \\
  h & k & g
\end{cases}
&= (-1)^{c-d+f} \nabla [abc] \nabla [fg] \nabla [kbd] \\
& \quad \times q^{(c-d+f)(a+g)-(d-f+k)(c+k+1)+Z_{deh}} \\
& \quad \times \sum_{z_1, z_2, z_3} \frac{(-1)^{z_1+z_2} [a+c-g+k-z_1][a+c+g+k-z_1+1]}{[z_1][a+c-h-z_1][a+c+h-z_1+1][z_2]} \\
& \quad \times [2c-z_1][2d-z_2][c-d+f+z_2] \\
& \quad \times \frac{[d-b+k-z_2][c+d-f-z_2][b+d+k-z_2+1][z_3]}{[d-b+k-z_2][c+d-f-z_2][b+d+k-z_2+1][z_3]} \\
& \quad \times \frac{[a-b+f+g-z_3][a+b+f+g-z_3+1]}{[a-b+f+g-z_3][a+b+f+g-z_3+1]}q^{z_1(a-d+f+g-k-z_2-z_3+z_2(a+b+c+g+k-z_3+1)-z_3(c-d+f)} \\
& \quad \times \frac{[a-d+f+g-k+z_2-z_3][a+c+g+k-z_1-z_3+1]}{[a-d+f+g-k+z_2-z_3][a+c+g+k-z_1-z_3+1]} \quad (3.1c)
\end{align*}
\]
Expression (3.1a) for $q = 1$ is equivalent to the known triple sum formula\[^{[3]}\] for the $9j$ coefficients of SU(2). The numerator-denominator distributions of factorials, depending on the summation parameters $z_1$, $z_2$, $z_3$, are different in all expressions \([3.1a]–[3.1e]\). All the terms in the last sum of \([3.1a]\), in the second sum of \([3.1c]\), and in the first sum of \([3.1d]\) are of the same sign. The separate sums correspond to the finite basic hypergeometric series,

$$
p_{+1}P_{p}^m[\alpha_1, \alpha_2, \ldots, \alpha_{p+1}; \beta_1, \ldots, \beta_p; q, x] = \sum_k \frac{(\alpha_1|q)_k(\alpha_2|q)_k \cdots (\alpha_{p+1}|q)_k}{(\beta_1|q)_k \cdots (\beta_p|q)_k(1|q)_k^k} x^k,
$$

with $p = 3$, $x = q^{\pm(c+1)}$, $c = \sum_{i=1}^{p+1} \alpha_i - \sum_{j=1}^{p} \beta_j$, as defined (with a minor correction) by Álvarez-Nodarse and Smirnov\[^{[2]}\], instead of the standard basic hypergeometric functions $p_{+1}\phi_p$ (see Gasper and Rahman\[^{[3]}\]). Parameters $c = -1$ and $x = 1$ for the balanced basic hypergeometric series, which appear in expressions for $q$-$6j$ coefficients.\[^{[3]}\]

The intervals for summation parameters $z_i$ ($i = 1, 2, 3$) are mainly restricted by six \([3.1a]\) and \([3.1c]\), five \([3.1b]\) and \([3.1d]\), or four \([3.1e]\) triangle linear combinations of the type $a + b - c$, respectively. As result of their vanishing we may write 23 different (independent) expressions as double sums for the stretched $9j$ coefficients as compositions of $_4F_3[\cdot; q, x]$ and $_3F_2[\cdot; q, x]$ series [with latter in the 10 cases corresponding to the CG coefficients of $u_q(2)$]. Although Minton’s summation formulas \([33]\) or \([34]\) (see Ref. \[^{[3]}\]) may be used (12 times) for separate alternating sums in \([3.1a]–[3.1d]\), satisfying special conditions, in the case of some stretched triangles \[e.g., for $d = b + k$ or $e = b - a$ in Eq. \([3.1a]\), but the expressions obtained are equivalent to some derived previously (although using the different triple sum expressions). In contrast with Eq. (32.13) of Ref. \[^{[4]}\] and its $q$-generalization\[^{[4]}\] [appearing, e.g., in context of the stretched isofactors of $u_q(3)$], which are expressed as compositions of two generic $3F_2[\cdot; q, x]$ series, they are less symmetric.
and more complicate. Although these $3F_2[\cdots; q, x]$ series in all 23 new expressions may be rearranged into other forms separately [cf. Refs. 39, 36] in such ways that the double sums turn into compositions of two generic $3F_2[\cdots; q, x]$ series, we use more universal approach in Sec. IV.

The doubly stretched $q$-9$j$ coefficients with the adjacent consecutive stretched triangles may be expressed without sum,

\[
\begin{aligned}
\left\{ \begin{array}{ccc}
c+h & b & b+c+h \\
c & d & f \\
h & k & g \\
\end{array} \right\}_q = \frac{(-1)^{c+d-f}q^{2bc+Z_{fhnk}\nabla[e/fg]}\nabla[hgk]/\nabla[cdk]/\nabla[bdk]}{[2b]![2c]![2h]! /[2a+1][2e+1]!}^{1/2},
\end{aligned}
\]

where $a = c+h$ and $e = a+b$ [cf. Eq. (32.21) of Ref. 3], taking into account that in related 11 cases two summation parameters are fixed and the last summation may be performed using either the Chu–Vandermonde formula [9] in 9 cases, e.g., in (3.1a) for $k = g+h = b-d$, or in (3.1b) for $k = g+h = d-b$, see Appendix B] or Karlsson's [14] summation formula (2.13) [in (3.1d) for $g = e+f = k-h$, or in (3.1c) for $g = e+f = k-h$].

The different versions of the doubly stretched $q$-9$j$ coefficients with single sums in expressions [mainly as generalizations of Eqs. (32.15), (32.17), (32.17a), (32.18), and (32.20) of Ref. 3] may be obtained straightforwardly from (3.1a)–(3.1e) with fixed couples of summation parameters.

Additional restrictions for $z_i \pm z_j$ in generic expressions (3.1a)–(3.1e) may be represented as some couples of triangle linear combinations. No formula does represent any usual symmetry of 9$j$-symbol, but expressions (3.1a)–(3.1e) are mutually related by some “mirror reflection” ($j \to -j-1$) symmetries.

**B. Expressions with the partial triangle restrictions of summation intervals**

Summation formulas (A2) and (A3) also may be used, when the first (primed) $q$-6$j$ coefficients in (2.9a) and (2.9b) are expressed by means of (2.21f) and remaining (primed or nonprimed) ones by means of (2.2a) or (2.1a), respectively. It is impossible to get the definite summation interval for $j$ when expressing all three $q$-6$j$ coefficients by means of primed $q$-6$j$ coefficients (2.21f) or (2.2b) with the numerator – denominator distribution of the type (2.11b). Hence we derive in addition two more triple sum expressions for $q$-9$j$ coefficients,

\[
\begin{aligned}
\left\{ \begin{array}{ccc}
a & b & e \\
c & d & f \\
h & k & g \\
\end{array} \right\}_q = (-1)^{a+c-h}\nabla[abe]/\nabla[efg]/\nabla[kbd]/\nabla[ach]/\nabla[fcg]/\nabla[kgh] \\
\times q^{Z_{doh}-(a+b+f-g)(f+g-e+1)-(g-a)(g-a+1)} \\
\times \sum_{z_1, z_2, z_3} \frac{(-1)^{z_1+z_2+z_3}[g-h+k+z_1][g+h+k+z_1+1]}{[z_1][g+k-a-c+z_1][c+g+k-a+z_1+1]} \\
\times [2b-z_2][b+f+k-c-z_2][b+f+k+c-z_2+1] \\
\times [2g+z_1+1][z_2][b+k-d-z_2][b+d+k-z_2+1] \\
\times [b+e-a+z_3][e-f+g+z_3]q^{-z_1(b+e-a-z_2+z_3)} \\
\times [z_3][a+b-e-z_3][f+g-e-z_3][2e+z_3+1]
\end{aligned}
\]
Here the summation intervals for $z_2$ and $z_3$ are restricted by one or two triangle conditions, but $z_1 + z_2$ restricted only both together by some couples of triangular linear combinations. In these cases we may write 6 more expressions for the stretched $q$-$9j$ coefficients as double sums, but only five of them correspond to compositions of generic $4F_3[\cdots; q, x]$ and $3F_2[\cdots; q, x]$ series, and only four times all the summation intervals in these expressions are restricted by some triangle conditions. In the remaining cases some couples of triangular linear combinations appear as the summation intervals.

### IV. REARRANGEMENT OF THE TRIPLE SUM EXPRESSIONS FOR $q$-$9j$ COEFFICIENTS AND STRETCHED $q$-$9j$ COEFFICIENTS

#### A. Search for other rearrangement of the triple sum expressions

We may identify such three blocks (quintuplets) of factorials under the summation sign in numerators and denominators of each expression (3.1a)–(3.1e), which may be expanded using the Chu–Vandermonde summation formulas given in Appendix B. For example, expressions (3.1a), (3.1b), and (3.1d) may be expanded as follows:

\[
\begin{align*}
\left\{ \begin{array}{c}
a \ b \ c \\
\ d \ f \\
h \ k \ g \\
\end{array} \right\}_q = (-1)^{c+h-a} \frac{\nabla[abe] \nabla[fg] \nabla[kbd]}{\nabla[ach] \nabla[fcd] \nabla[kgh]} q^{(f+g-e-k)(a+d-e+k+1)-(a-e+f)(a-e+f+1)} \\
\times q^{-(g+h-k)(f+g-e)-(c-a+h)(c+d-f)+(b+a-e+1)(d-b+k)+Z_{dch}} \\
\times \sum_{z_1, z_2, z_3, s_1, s_2, s_3} q^{-z_1(a-c-g+k+1)-z_2(b-c+f-k+1)+z_3(a-b+f+g)} [2h - z_1]! [2d - z_2]! \\
\times [d - b + k - z_2 - s_1]! [2b + s_1 + 1]! [g + h - k - z_1 - s_2]! \\
\times [f + g - e - z_3 - s_2]! [c - a + h - z_1 - s_3]! [c + d - f - z_2 - s_3]!
\end{align*}
\]
\[
= (-1)^{e-f-h+k} \frac{\nabla[abe] \nabla[feq] \nabla[kbd]}{\nabla[ach] \nabla[fcd] \nabla[kgh]} q^{(c+h-a)(b-c+f+k+1)} \\
\times q^{(b+c-a)(e-f-h+k)-(f+g-e)(g+h-k)-(a-e+f+1)(a-e+f)+Z_{deh}} \\
\times \sum_{s_1, s_2, s_3} \frac{(s)}{[s_1]} [b - d + k - z_2][b + d + k - z_2 + 1][z_1]^2 [2h - z_1]!
\times \sum_{s_1, s_2, s_3} \frac{(s)}{[s_1]} [a - b - e - z_3 - s_1][s_2][g + h - k - z_1 - s_2]!
\times \frac{q^{-s_3}(a+b+f-h+k+z_1+z_2+2)[2c - s_3][b - c + f + k - z_2 + s_3]}{[f + g - e - z_3 - s_2][s_3][c + h - a - z_1 - s_3]!}
\times q^{(a-c+h)(c-a+g+k+2)+Z_{deh}} \\
\times \sum_{s_1, s_2, s_3} \frac{q^{-s_1(s)}(+a+g+k+2)+z_2(c+b+f+k-1)-z_1(a-b+f+g+1)}{[s_1]} [b - d + k - z_2][b + d + k - z_2 + 1][z_3]!
\times \sum_{s_1, s_2, s_3} \frac{(s)}{[s_1]} [b - a - e - z_3 - s_1][s_2][e + f - g - z_3 - s_2]!
\times q^{-(a+b-e-z_2+z_3+1)-s_2(k-e-f-h+z_1+z_3)-s_3(b+f-h+k-a+z_1+z_2+1)} \\
\times \frac{[g - h + k + z_1 + s_2][b + c + f + k - z_2 + s_3 + 1]}{[2g + s_2 + 1][s_3]!}[a - c + h - z_1 - s_3][2c + s_3 + 1]!.
\]

The summations over \(s_1, s_2, s_3\) give original expressions (3.4a), (3.4b), and (3.4c), when the summations of (4.1a), over \(z_1, z_2, z_3\) give another expression for \(q\)-9\(j\) coefficient, equivalent to (3.1a) after transpositions of two last rows and two last columns,

\[
\begin{pmatrix}
a & b & e \\
c & d & f \\
h & k & g \\
\end{pmatrix}_q = \begin{pmatrix}
a & e & b \\
h & g & k \\
c & f & d \\
\end{pmatrix}_q.
\]

(4.2a)

Otherwise, the summations of (4.1b) over \(z_1, z_2, z_3\) give expression, equivalent to (3.1c), after changing the summation parameters and taking into account the same relation (4.2a), as well as the summations of (4.1c) over \(z_1, z_2, z_3\) give expression, equivalent to (3.1c), again after change of summation parameters and applying the relation

\[
\begin{pmatrix}
a & b & e \\
c & d & f \\
h & k & g \\
\end{pmatrix}_q = \begin{pmatrix}
f & d & c \\
e & b & a \\
g & k & h \\
\end{pmatrix}_q.
\]

(4.2b)

Hence only three from these expressions for the \(q\)-9\(j\) coefficients are independent with respect to elementary rearrangements.

The quintuplet expansion by means of the Chu–Vandermonde summation formulas of expressions (3.4a) or (3.4b) leads to vanishing of the summation limit for \(z_1\) and, therefore, it is not helpful for the rearrangement of \(q\)-9\(j\) coefficients.
B. Different expressions for the stretched $q$-9$j$ coefficients

In the stretched cases, e.g., for $k = g + h$ in (4.1a) and (4.1b), or for $c = a + h$ in (4.1c) some couples of parameters $z_i$ and $s_j$ are fixed and summation over $z_l$ and $s_l$ (where $i, j, l$ is some permutation of $1, 2, 3$) is possible, using the Chu–Vandermonde formulas (see Appendix B). Hence we may derive 14 versions of expressions (from which at least 13 are independent) for the stretched $q$-9$j$ coefficients as double sums over parameters $z_j$ and $s_i$ (where further the subscripts of the summation parameters will be omitted) as compositions of the both generic $3F_2[\cdots;q,x]$ series. For example, from (4.1a) and (4.1b) with $a = c + h$ and $z_1 = s_3 = 0$, from (4.1a) with $k = h + g$ and $z_1 = s_2 = 0$ or with $e = f + g$ and $z_3 = s_2 = 0$, and from (4.1c) with $c = a + h$ and $z_1 = s_3 = 0$ [using some symmetries (2.7) of the $q$-9$j$ coefficients and, in the last case, some change of summation parameter] we obtain, respectively, the following expressions:

\[
\left\{ \begin{array}{ccc}
a & b & e \\
c & d & f \\
h & k & e+f \\
\end{array} \right\}_q = \left( \frac{[2e]![2f]!}{[2g+1]!} \right)^{1/2} \frac{\nabla [g] \nabla [b] \nabla [c] \nabla [d] \nabla [f] \nabla [e] \nabla [a] \nabla [h] \nabla [k]}{q^{(h-b-c-e)(b+c-f-k)+2bc+Z_{efhk}}}
\]

\[
\times \sum_{s,z} (-1)^{s+z}[a-c+h+s][k-b+d+z]!
\]

\[
\times q^{s(b+c+f-k-z)+z(b+c+e-h)}[h-g+k+s+z]!
\]

\[
\times [z][b+d-k-z][c-b-f+k+z][2k+z+1]!
\]

\[
= (-1)^{d+f-c} \left( \frac{[2e]![2f]!}{[2g+1]!} \right)^{1/2} \frac{\nabla [c] \nabla [a] \nabla [h] \nabla [k]}{q^{(b+c+e+1)+(e+c+1)+Z_{bcg}}}
\]

\[
\times \sum_{s,z} (-1)^{s+z}[a-c+h+s][b+c+e-h-s]!
\]

\[
\times q^{s(e-b+f+k-z)+z(b-c-e+h+1)}[b+d-k+z][2k-z]!
\]

\[
\times [z][d+k-b-z][b+c-f+k+z]!
\]

\[
= q^{2af-(a+b-c)(a+d+f-h)+Z_{bdgh}} \left( \frac{[2e]![2f]!}{[2g+1]!} \right)^{1/2} \frac{\nabla [a] \nabla [b] \nabla [f] \nabla [e] \nabla [d] \nabla [c] \nabla [h] \nabla [k]}{q^{2bc+Z_{efhk}}}
\]

\[
\times \sum_{z,s} (-1)^{a+c-h+s+z}[2b-z][b+d-g+h-z]!
\]

\[
\times [z][b+d-k-z][b+d+k-z+1][s]!
\]
\[ q^{\frac{z(a + d + f - h - s - (g + h - b - d + 1))}{[a + b - e - z - s]!} \cdot \frac{[2a - s]!}{[h - a + c + s]!}} \]
\[ \times q^{k(2f - 2b + k - 1) + (g + h + k)(a + b - f + h - k) + Z_{cdh}} \]
\[ \times \left( \frac{[2e]![2f]!}{[2g + 1]!} \right)^{1/2} \frac{\nabla [ghk] \nabla [kbd] \nabla [ach]}{\nabla [fcd] \nabla [eab]} \]
\[ \times \sum_{s,z} \frac{(-1)^{c - d + e + g + z} q^{-s(a + b + e + z + 1) - z(h - g + k)}}{[s]! [g - h + k - s]! [a + c - h - s]! [2h + s + 1]!} \]
\[ \times \frac{[c + h - a + s]! [b + d - k + z]! [b + e - a + z]!}{[z]! [d + k - b - z]! [b - a - f + h - k + s + z]! [2b + z + 1]!} \]  \quad (4.3d)

Expression (4.3a) is invariant with respect to simultaneous permutations of the two first columns and rows of the stretched \(q\)-9j coefficients in accordance with (2.8) and is related to the particular case of Eqs. (26)–(27) of Ref. 3, that appeared in context of the stretched isoscalar factors of the \(Sp(4)\) or \(SO(5)\) group restricted to \(SU(2)\times SU(2)\), but (4.3b) and (4.3c) are more convenient, since separate sums are not alternating.

The linear combinations of parameters \(a + b - e\), \(d - b + k\), and \(c + d - f\) restrict the both summation parameters in expressions (4.3b), (4.3c), and (4.3d)-(4.3e), respectively, for the \(q\)-9j coefficients with the couples of adjacent consecutive stretched triangles [cf. Eq. (32.21) of Ref. 3]. Otherwise, the summation of expression (4.3a) for \(g + h - k = 0\), or \(g - h + k = 0\) is nontrivial, as well as Eq. (4.3b) for \(g - h = k = 0\), Eq. (4.3c) for \(a + b - e = 0\), or \(g + h = k = 0\), and Eqs. (4.3d)-(4.3e) for \(g + h = k = 0\).

The both separate sums only in Eqs. (4.3a) and (4.3d) correspond to the CG coefficients of \(u_q(2)\). Hence it may be included [using Eq. (5.17) of Ref. 4] into the Clebsch–Gordan coefficients of \(u_q(2)\), reexpressed by means of Eq. (41a) of Ref. 4 or of (3.5) with changed summation parameters) and the following expressions for the stretched \(q\)-9j coefficients may be derived:

\[
\begin{pmatrix}
a & b & e \\
c & d & f \\
h & k & e + f
\end{pmatrix}_q = (-1)^{a + b + c + d + h - k} \frac{[2e]![2f]![g + h - k]! [g + h + k + 1]!}{[(2g + 1)!(2k + 1)]^{1/2} \nabla [hac] \nabla [fcd] \nabla [eab]} \times q^{2af - (a + b - e)(a + d + f - h) + (b + d - k)(b + d + k + 1)/2 + Z_{bdgh}} \\
\times \sum_m \frac{(-1)^{a - e + m} q^{-m(g + h + 1)} [a + e - m]! [c - e + h + m]!}{[a - e + m]! [c + e - h - m]!} \times \left( \frac{[d + g - h - m]! [b + m]!}{[d - g + h + m]! [b - m]!} \right)^{1/2} \begin{pmatrix} d & b & k \\ d - g & h & m \end{pmatrix}_q
\]
\[= \frac{[2e]![2f]![h + k - g]! [g + h + k]! [g + h - k]! [g + h + k + 1]!}{[(2g + 1)!(2k + 1)]^{1/2} \nabla [hac] \nabla [fcd] \nabla [eab] \nabla [kbd]} \times q^{(b + d - k)(a + b - f - h + k + 1) - (a + b - e)(a + d + f - h) + 2af + Z_{bdgh}} \times \sum_{s,z} \frac{(-1)^{a + b + c + d + h - k + s + z} q^{-s(g + h - k + 1) - z(g - h + k + 1)}}{[s]! [a + c - h - s]! [z]! [b + d - k - z]!} \times \frac{[2a - s]! [c + h - a + s]! [2d - z]! [b - d + k + z]!}{[d - a - f + h + s - z]! [a - d - e + k - s + z]!} \times \left( \frac{d}{g - h - m} \right)^{1/2} \begin{pmatrix} d & b & k \\ d - g & h & m \end{pmatrix}_q
\]
\[= \frac{[2e]![2f]![h + k - g]! [g + h + k]! [g + h - k]! [g + h + k + 1]!}{[(2g + 1)!(2k + 1)]^{1/2} \nabla [hac] \nabla [fcd] \nabla [eab] \nabla [kbd]} \times q^{(b + d - k)(a + b - f - h + k + 1) - (a + b - e)(a + d + f - h) + 2af + Z_{bdgh}} \times \sum_{s,z} \frac{(-1)^{a + b + c + d + h - k + s + z} q^{-s(g + h - k + 1) - z(g - h + k + 1)}}{[s]! [a + c - h - s]! [z]! [b + d - k - z]!} \times \frac{[2a - s]! [c + h - a + s]! [2d - z]! [b - d + k + z]!}{[d - a - f + h + s - z]! [a - d - e + k - s + z]!} \times \left( \frac{d}{g - h - m} \right)^{1/2} \begin{pmatrix} d & b & k \\ d - g & h & m \end{pmatrix}_q \]  \quad (4.4a)
\[(4.4c)\] also from expansion

\[
\frac{[2c]![2f]![a-b+c]![a+b-e]!}{[2g+1]![e-a+b]![a+b+e+1]!} \frac{\nabla[ghk]\nabla[bdk]}{\nabla[fc]d\nabla[ach]}
\times q^{(c+d-f)(a+b+e+1)-(a+b+e+1)(b+d-k)+2ed+Z_{a,c}h}
\times \sum_{s,z} \frac{[2c-s]![a-c+h+s]![k-b+d+z]!}{[s]![c+h-a-s]![z]![b+d-k-z]![2k+z+1]!}
\times \frac{(-1)^{c+k-b-f+sz}(a+b+e+1)+(a+b+e+1)}{[c-b-f+k+z-s]![a-c+g-k-z+s]!}
\]

Expression \((4.4b)\) satisfies symmetry relation \((2.8)\) for permutations of the two first columns or rows of the stretched \(q\)-9\(j\) coefficient and is a \(q\)-generalization of standard formula \((32.13)\) of Ref. 3 for the stretched 9\(j\) coefficients. The linear combinations of parameters \(a+b-e\) or \(c+d-f\) restrict the both summation parameters in expression \((4.4b)\), when \(g+h-k\) or \(c+d-f\) restrict the summation limits in expression \((4.4c)\). Hence expressions \((4.4b)\) and \((4.4c)\) for these cases of adjacent consecutive stretched triangles also turn into single terms [cf. Eq. (3.8)]. Note that the Chu–Vandermonde or Karlsson summation formulas are needed for the last summation of 11 doubly stretched cases of triple sum expressions \((3.1a)\)–\((3.1e)\), \((3.3)\). Note that the Chu–Vandermonde or Karlsson summation formulas are needed also from expansion \((4.4c)\) for these cases of adjacent consecutive stretched triangles also turn into single terms [cf. Eq. (3.8)]. Note that the Chu–Vandermonde or Karlsson summation formulas are needed for the last summation of 11 doubly stretched cases of triple sum expressions \((3.1a)\)–\((3.1e)\), e.g., in \((3.1a)\) for \(k = g + h = b - d\), or in \((3.1a)\) for \(g = e + f = k - h\).

For the diverging adjacent stretched triangles, e.g., with \(e = b - a = g - k\), summation parameters in \((4.4c)\) are dependent and the doubly stretched \(q\)-9\(j\) coefficients may be expressed as single sums with the alternating terms [cf. Eq. (32.18) of Ref. 3], when using Eq. \((3.1a)\) for \(c = f - d = a - h\) an equivalent formula may be written directly. Again, for the merging adjacent stretched triangles, e.g., with \(g = h + k = e + f\), the doubly stretched \(q\)-9\(j\) coefficients may be expressed as single sums with the fixed sign of all terms [cf. Eq. (32.20) of Ref. 3] by means of formula \((4.4b)\) [as well as for \(b = d + k = a + e\) by means of Eq. (3.1c) in contrast with the remaining formulas of this paper]. These single sums in the both cases are related to the generic \(4F_3[\cdot \cdot \cdot q, x]\) series.

The \(q\)-9\(j\) coefficients with the both stretched triangles appearing in the different layers, or rows of the \(q\)-9\(j\)-symbol are related to generic \(3F_2[\cdot \cdot \cdot q, x]\) series. In the case of two parallel stretched triangles [e.g., for \(h = a + c\) in \((4.3a)\) or \((4.4b)\)] they are proportional [cf. Eq. (32.15a) of Ref. 3] to the CG coefficients of \(u_q(2)\) (see Refs. 29,45,46), with appearing two different types of expressions. Otherwise, special \(q\)-9\(j\) coefficients with two antiparallel stretched triangles may be expressed in four different forms [see \((4.3b)\) for \(a = c+h\), or \((4.3c)\) for \(b = d + k\) as generalizations of Eqs. (32.17a) and (32.17) of Ref. 3, as well as Eq. (3.1e) for \(h = a + c\) and \(b = d + k\), or Eq. (3.1c) for \(b = d + k\) and \(g = e + f\) and correspond to the CG coefficients of \(u_q(1,1)\), with the expressions including either the alternating terms [with diverse distribution of summation parameter signs in two numerator \(q\)-factorial arguments, in analogy with CG coefficients of \(u_q(2)\)], or the fixed sign terms (with one or three numerator \(q\)-factorial arguments). Note, that expressions for the triply stretched \(q\)-9\(j\) coefficients with the three mutually antiparallel stretched triangles [e.g., \((4.3a)\) or \((4.4c)\) for \(a = c + h\) and \(d = b + k\)] are not summable.

Expression related to \((4.4a)\) may be derived [in contrast with the intermediate version of \((4.4c)\)] also from expansion \[44\] [cf. Ref. 47 in the SU(2) case] of the \(q\)-9\(j\) coefficients in terms of the Clebsch–Gordan coefficients of \(u_q(2)\) [cf. Eq. (3.12) of Ref. 3] which in the stretched case with \(h = a + c\) (and with extreme CG coefficient for coupling \(a \times c \to h\) in the r.h.s. equal to 1) may be written as follows:

\[\]
\[
\begin{align*}
\left\{ \frac{a}{c} \frac{b}{d} \frac{e}{f} \frac{g}{h} \right\}_{q^{-1}} &= q^{-Z_{e,f;h,k}} (2e+1)(2f+1)(2h+1)(2k+1)^{1/2} \left\{ \frac{h}{k} \frac{g}{g} \right\}_{q^{-1}} \\
&\times \sum_m \left[ \frac{a}{a} \frac{b}{m-a} \frac{e}{e} \frac{f}{m} \right]_q \left[ \frac{c}{c} \frac{d}{g} \frac{g}{g} \right]_q \\
&\times \left[ \frac{b}{m-a} \frac{d}{g-m-c} \frac{k}{g-h} \right]_q (R_{\alpha}^{eb})_{c,m-a}, \tag{4.5}
\end{align*}
\]

where

\[ (R_{\alpha}^{eb})_{c,m-a} = q^{2c(m-a)} \]

is a diagonal extreme element of triangular braiding \( R \)-matrix and all the CG coefficients with exception of the last one may be expressed without sum. It should be noted that only in special stretched case (4.5) the summation over non-diagonal elements of \( R \)-matrix may be escaped.

Both expressions (4.4b) and (4.4c) correspond to \( q \)-generalizations of the Kampé de Fériet function \( F_{1,1}^{1,2} \), which is defined as follows:

\[
\pm F_{C:D}^{A:B} \left[ \begin{array}{c} (a) \\ (c) \\ (d) \\ (d') \\ x, y; q \end{array} \right] = \sum_{s,t} \prod_{j=1}^A (a_j | q)_{s+t} \prod_{j=1}^B (b_j | q)_{s+b_j | q} x^{\pm \delta t} y^{\pm (1-2\delta) t} \\
\prod_{j=1}^C (c_j | q)_{s+t} \prod_{j=1}^D (d_j | q)_{s+d_j | q} [s]! [t]! q^{\pm (A-C) st}, \tag{4.6}
\]

with special parameters

\[
x = q^{p+1}, \quad p = \sum_{j=1}^A a_j + \sum_{j=1}^B b_j - \sum_{j=1}^C c_j - \sum_{j=1}^D d_j,
\]

\[
y = q^{p'+1}, \quad p' = \sum_{j=1}^A a_j + \sum_{j=1}^B b' - \sum_{j=1}^C c_j - \sum_{j=1}^D d'\]

\[
\delta = \delta_{AC}
\]

for \( A + B = C + D + 1 \) and \( |A - C| \leq 1 \). Of course, series (4.6) turn into usual Kampé de Fériet function \( F_{C:D}^{A:B}[\ldots; 1, 1] \) for \( q = 1 \). Unfortunately, the standard definition\( [4, 17] \)

\[
\Phi_{C:D}^{A:B} \left[ \begin{array}{c} (\alpha) \\ (\beta) \\ (\gamma) \\ (\delta') \\ x, y; q \end{array} \right] \tag{4.7}
\]

(cf. Refs. [16, 17]) in terms of the asymmetric \( q \)-factorials\( [36] \)

\[
(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}), \quad n = 1, 2, \ldots,
\]

derived after substitution

\[
q \rightarrow q^{1/2}, \quad [n]! \rightarrow (q; q)_n(q^{-1/2} - q^{1/2})^{-n} q^{-n(n+1)/4},
\]

\[
(a|q)_n \rightarrow (q^a; q)_n(q^{-1/2} - q^{1/2})^{-n} q^{-n(2a+n-1)/4}, \tag{4.8}
\]

\[15\]
may be not convenient in the double sums that appear in (4.4b) and (4.4c), since arguments \( x \) and \( y \) do not turn into \( q \) both together, but turn either into \( q \) and \( q^{-p'} \) or into \( q^{-p} \) and \( q \), respectively.

Furthermore, the \( q = 1 \) versions of Eqs. (4.3a)–(4.3e) correspond to the Kampé de Fériet functions \( F_{1;2}^{0:2} \) or [after reversing the order of summations, associated with spoiling some natural restrictions for both summation parameters in (4.3b)–(4.3d)] to \( F_{1;1}^{0:3} \) (cf. Refs. 16,17). Otherwise, in the generic \( q \neq 1 \) case they can be expressed in terms of our (4.6) as

\[
-F_{1;2}^{0:2} \left[ \cdots ; x, y, q \right] \quad \text{or} \quad +F_{0;3}^{1:1} \left[ \cdots ; x, y, q \right],
\]

and for \(-F_{C;D}^{A:B} \left[ \cdots ; x, y, q \right]\) with \( A \neq C \) the factors \( q^{mn} \), spoiling standard definition of \( \Phi_{A:B}^{C:D} \) functions, cannot be eliminated in the new expansion [cf. Eq. (9) of Ref. 16], unless transition \( q \to q^{-1} \) is performed preliminary.

Special rearrangement and summation formulas of the double \( q \)-factorial series and related Kampé de Fériet functions are given in Appendix C.

V. EXPRESSIONS FOR 12j COEFFICIENTS OF THE SECOND KIND OF SU(2) AND \( u_q(2) \)

A. Generic properties

The 3\( nj \) coefficients of the second kind (\( n \geq 4 \)) whose graphs are planar [hence without braiding, in contrast with the 3\( nj \) coefficients of the first kind (\( n \geq 3 \)) whose graphs are possible only on the Möbius strip] usually are expanded in terms of the factorized \( n \) different \( 6j \) coefficients,

\[
\begin{bmatrix}
  j_1 & j_2 & \cdots & j_n \\
  l_1 & l_2 & \cdots & l_n \\
  k_1 & k_2 & \cdots & k_n 
\end{bmatrix} = \sum_x (2x + 1)(-1)^{R_1 + nx} \\
\times \left\{ \begin{array}{c}
  j_1 \\
  k_1
\end{array} x \right\} \left\{ \begin{array}{c}
  j_2 \\
  k_2
\end{array} x \right\} \cdots \left\{ \begin{array}{c}
  j_{n-1} \\
  k_{n-1}
\end{array} x \right\} \left\{ \begin{array}{c}
  j_n \\
  k_n
\end{array} x \right\} \left\{ \begin{array}{c}
  k_1 \\
  l_1
\end{array} \right\} \left\{ \begin{array}{c}
  k_2 \\
  l_2
\end{array} \right\} \cdots \left\{ \begin{array}{c}
  k_{n-1} \\
  l_{n-1}
\end{array} \right\} \left\{ \begin{array}{c}
  k_n \\
  l_n
\end{array} \right\},
\]

(5.1)

where

\[
R_n = \sum_{i=1}^{n} (j_i + k_i + l_i),
\]

and the triangular conditions are satisfied by the triplets of the nearest neighbors as \( l_i, j_i, j_{i+1}, \) or \( l_i, k_i, k_{i+1} \) (\( i = 1, 2, ..., n - 1 \)), or \( l_n, j_n, j_1, \) or \( l_i, k_n, k_1, \) respectively.

The 12\( j \) coefficients of the second kind, which may be extracted from the recoupling coefficients of the five irreps without braiding, hence, with the cubic graph.
were introduced by Elliott and Flowers⁴ and redefined by Vanagas and Čiplys⁵. These $12j$ coefficients and their $q$-generalizations satisfy 24 symmetries⁶ generated by the following substitutions:

\[
\begin{bmatrix}
  j_1 & j_2 & j_3 & j_4 \\
  l_1 & l_2 & l_3 & l_4 \\
  k_1 & k_2 & k_3 & k_4 \\
\end{bmatrix}_q = (-1)^{j_1-j_2-j_3+j_4+k_1-k_2+k_3+k_4} \begin{bmatrix}
  j_1 & k_1 & j_2 & k_3 \\
  l_1 & l_3 & l_4 & l_2 \\
  j_3 & k_2 & j_4 & k_4 \\
\end{bmatrix}_q
\]

\[
\begin{bmatrix}
  j_4 & j_3 & j_2 & j_1 \\
  l_4 & l_3 & l_2 & l_1 \\
  k_4 & k_3 & k_2 & k_1 \\
\end{bmatrix}_q = \begin{bmatrix}
  l_1 & l_2 & l_3 & l_4 \\
  k_1 & k_2 & k_3 & k_4 \\
  j_1 & j_2 & j_3 & j_4 \\
\end{bmatrix}_q
\]

\[
\begin{bmatrix}
  k_4 & k_2 & k_3 & k_1 \\
  l_4 & l_2 & l_3 & l_1 \\
  j_4 & j_2 & j_3 & j_1 \\
\end{bmatrix}_q = \begin{bmatrix}
  j_1 & j_2 & j_3 & j_4 \\
  l_2 & l_1 & l_4 & l_3 \\
  k_3 & k_4 & k_1 & k_2 \\
\end{bmatrix}_q
\]

Eight triangular conditions may be visualized⁶ by means of the extended array

\[
\begin{bmatrix}
  j_1 & j_2 & j_3 & j_4 \\
  l_1 & l_2 & l_3 & l_4 \\
  k_1 & k_2 & k_3 & k_4 \\
\end{bmatrix}
\]

and are satisfied by the triplets of parameters in the first and fourth columns, as well as by the skew triplets descending from some parameter of the first or fourth column, e.g., by $l_1, k_2, j_3$, or by $j_4, l_3, k_2$.

Let’s restrict ourselves to the following rearrangements of the $q$-$6j$ coefficients in expressions⁶⁷ for the $q$-$12j$ coefficients of the second kind:

\[
\begin{bmatrix}
  j_1 & j_2 & j_3 & j_4 \\
  l_1 & l_2 & l_3 & l_4 \\
  k_1 & k_2 & k_3 & k_4 \\
\end{bmatrix}_q = (-1)^{l_1-l_2-l_3+l_4} \sum_x (2x+1) \begin{bmatrix}
  k_1 & j_1 & l_1 \\
  j_3 & k_2 & x \\
\end{bmatrix}_q
\]

\[
\times \begin{bmatrix}
  k_3 & k_4 & x \\
  j_3 & j_1 & l_2 \\
\end{bmatrix}_q \begin{bmatrix}
  k_3 & j_2 & l_4 \\
  j_4 & k_4 & x \\
\end{bmatrix}_q \begin{bmatrix}
  k_1 & k_2 & x \\
  j_4 & j_2 & l_3 \\
\end{bmatrix}_q
\]

\[
(5.4a)
\]
\[ = (-1)^{l_1-l_2-l_3+l_4} \sum_x [2x+1] \left\{ \begin{array}{ccc} k_1 & j_1 & l_1 \\
_j & k_2 & x \\
l_1 & l_2 & l_3 \\
k_3 & k_4 & l_4 \end{array} \right\}_q \]
\[ \times \left\{ \begin{array}{ccc} j_3 & x & j_1 \\
_j & k_3 & k_4 \\
l_1 & l_2 & l_4 \\
k_3 & k_4 \end{array} \right\}_q \times \left\{ \begin{array}{ccc} k_1 & x & k_2 \\
_j & j_4 & j_2 \\
l_3 & l_3 & j_2 \\
k_1 & l_3 & j_2 \end{array} \right\}_q \] (5.4b)
\[ = (-1)^{l_1-l_2-l_3+l_4} \sum_x [2x+1] \left\{ \begin{array}{ccc} k_1 & j_1 & l_1 \\
_j & k_2 & x \\
l_1 & l_2 & l_3 \\
k_3 & k_4 \end{array} \right\}_q \]
\[ \times \left\{ \begin{array}{ccc} j_3 & k_4 & x \\
_j & j_4 & j_2 \\
l_1 & l_2 & k_4 \\
k_3 & l_4 & k_4 \end{array} \right\}_q \times \left\{ \begin{array}{ccc} k_1 & x & k_2 \\
_j & j_4 & j_2 \\
l_3 & l_4 & j_2 \\
k_1 & l_4 & j_2 \end{array} \right\}_q \] (5.4c)
\[ = (-1)^{l_1-l_2-l_3+l_4} \sum_x [2x+1] \left\{ \begin{array}{ccc} j_3 & x & j_1 \\
_j & k_1 & l_2 \\
l_1 & l_2 & j_1 \\
k_3 & k_4 \end{array} \right\}_q \]
\[ \times \left\{ \begin{array}{ccc} k_3 & x & k_4 \\
_j & j_4 & j_2 \\
l_3 & l_4 & k_4 \\
k_3 & l_4 & k_4 \end{array} \right\}_q \times \left\{ \begin{array}{ccc} l_3 & j_2 & k_1 \\
m_{l_2} & x & j_4 \\
l_3 & j_4 & j_4 \end{array} \right\}_q \] (5.4d)

with the asymmetric triangle coefficients depending on the summation parameter \( x \) distributed separately in the numerators or denominators of each \( q \)-6 coefficient in expansion (5.4a), the mixed distribution of asymmetric triangle coefficients in the numerators and denominators of \( q \)-6 coefficients in expansions (5.4b) and (5.4c), and resembling (2.11b) distribution in (5.4d). Using expression (2.1b) for \( q \)-6 coefficients with summation parameter \( x \) in the right lower position, Eq. (2.1a) with inverted summation parameter for \( q \)-6 coefficients with \( x \) in the middle column, and Eq. (2.1a) in the remaining cases, with exception of Eq. (2.2a), used for the last \( q \)-6 coefficient in (5.4d), the asymmetric triangle coefficients depending on the summation parameter \( x \) cancel. Then we may use the summation formula (4.1a) for expansion (5.4a) and formula (4.1) for expansions (5.4b)–(5.4d).

B. General expressions with fourfold sums

This way we derived four different expressions for the \( q \)-12 coefficients of the second kind,

\[
\left[ \begin{array}{cccc} j_1 & j_2 & j_3 & j_4 \\
l_1 & l_2 & l_3 & l_4 \\
k_1 & k_2 & k_3 & k_4 \end{array} \right] = (-1)^{l_2-l_3+k_1-k_3-j_3+j_4} \frac{\nabla [k_3j_1l_2] \nabla [j_3k_4l_3] \nabla [k_1j_2l_3] \nabla [j_1k_2l_3]}{\nabla [k_1j_1l_1] \nabla [j_3k_2l_1] \nabla [k_3j_2l_4] \nabla [j_4k_4l_4]} \times \sum_{z_1, z_2, z_3, z_4} (-1)^{z_1+z_2+z_3+z_4} [k_2 + j_3 - l_1 + z_1] [k_1 + j_1 - l_1 + z_1] [k_1 + j_1 - l_1 + z_1] [k_1 + j_1 - l_1 + z_1] [k_1 + j_1 - l_1 + z_1] \times \frac{[2l_1 - z_1]}{[j_1 + l_2 - k_3 + z_2]} [l_2 - j_3 + k_4 + z_2] \times \frac{[j_1 - l_2 + k_3 - z_2]}{[j_3 + k_4 - l_2 - z_2]} [2l_2 + z_2 + 1] \times \frac{[j_4 + k_4 - l_4 + z_3]}{[j_2 + k_3 - l_4 + z_3]} [2l_4 - z_3] \times \frac{[k_4 + l_4 - j_4 - z_3]}{[j_2 - k_3 + l_4 - z_3]} [2l_3 + z_4 + 1] \times \frac{[j_2 + l_3 - k_1 + z_4]}{[k_2 + l_3 - j_4 + z_4]} [k_2 - l_3 + j_4 - z_3] [k_1 + k_3 + j_3 + j_4 - l_1 - l_4 + z_1 + z_3 + 1]!
\]
\[\begin{align*}
\times & \quad \frac{[k_1 + k_3 + j_3 + j_4 - l_2 - l_3 - z_2 - z_4]}{[k_1 + l_2 - l_1 - k_3 + z_1 + z_2][l_3 - l_1 + j_3 - j_4 + z_1 + z_4]} \\
\times & \quad \frac{[l_2 - l_4 - j_3 + j_4 + z_2 + z_3]}{[k_3 - k_1 + l_3 - l_4 + z_3 + z_4]} \\
= & \quad (-1)^{j_1 - j_3 - k_1 + k_2 - l_1 - l_2 + l_3 + l_4 + l_5} \sum_{z_1, z_2, z_3, z_4} \frac{[-1]^{z_2 + z_3 + z_4}[k_2 + j_3 - l_1 + z_1][l_1 + k_3 - j_3 - z_1]}{[2l_1 - z_1][2l_2 - z_2][j_1 - l_2 + k_3 + z_3]} \\
\times & \quad \frac{[j_2 - k_3 + l_4 + z_3]}{[k_4 + l_4 - j_4 - z_3]} \\
\times & \quad \frac{[j_4 + k_3 - l_2 - l_3 - z_2]}{[l_2 + j_2 - k_3 + k_4 + z_2 - z_4]}
\end{align*}\] (5.5a)

\[\begin{align*}
\times & \quad \frac{[k_2 + l_3 - j_4 - z_4]}{[k_1 + k_3 - l_1 - l_2 + z_3 + z_4]} \\
\times & \quad \frac{[j_3 + j_4 + l_2 - l_4 + z_2 - z_3]}{[k_1 + k_3 + l_3 - l_4 - z_3 - z_4]} \\
\times & \quad \frac{[j_3 - j_4 + k_1 - k_3 + l_2 + l_3 - z_2 - z_4]}{[l_1 + l_2 + l_3 - l_4 - z_2 - z_3 - z_4 - z_4]} \\
= & \quad (-1)^{k_1 - k_2 + l_1 - l_2 + l_3 + l_4 - j_2 + j_4} \sum_{z_1, z_2, z_3, z_4} \frac{[-1]^{z_2 + z_3 + z_4}[j_3 + k_2 - l_1 + z_1]}{[2l_1 - z_1][2l_2 - z_2][j_1 - k_1 - l_1 + z_1]} \\
\times & \quad \frac{[j_1 - l_2 + k_3 - z_2]}{[j_3 + k_4 - l_2 - z_2][2l_2 + z_2 + 1][j_4 - k_3 + l_4 - z_3]} \\
\times & \quad \frac{[j_2 - k_3 + l_4 - z_3]}{[j_1 + j_4 + l_4 - z_3 + 1][k_1 - j_2 + l_3 - z_4]} \\
\times & \quad \frac{[k_2 + l_3 - j_4 - z_4]}{[k_1 + j_3 + l_3 - z_4 + 1][k_1 - k_3 - l_1 + l_2 + z_1 + z_2]} \\
\times & \quad \frac{[j_3 + j_4 - l_2 + l_4 - z_2 - z_3]}{[j_3 - j_4 + k_1 + k_3 - l_2 + l_3 + z_2 - z_4]} \\
\times & \quad \frac{[j_3 - j_4 + k_1 + k_3 - l_1 - l_4 + z_1 + z_3]}{[j_3 + j_4 - l_1 - l_3 + z_1 + z_4]} \\
\times & \quad \frac{[k_1 - k_3 + l_3 + l_4 - z_2 - z_3]}{[l_1 - l_2 + l_3 + l_4 - z_1 - z_2 - z_3 - z_4]} \\
= & \quad (-1)^{k_3 + k_4 - l_1 - l_2 + l_3 + l_4 - j_3} \sum_{z_1, z_2, z_3, z_4} \frac{[z_1][l_1 - k_2 + j_3 - z_1]}{[2l_2 - z_2][k_4 + j_3 - l_2 + z_2]} \\
\times & \quad \frac{[z_2][k_3 + l_2 - j_1 - z_2]}{[2l_4 - z_4][j_2 + k_3 - l_4 + z_3]} \\
\times & \quad \frac{[z_3][z_4][j_4 + l_4 - k_4 - z_3]}{[j_2 - k_3 + l_4 - z_3]} \\
\end{align*}\] (5.5b)
Hence, we obtain
\[
\frac{[2j_2 - z_4]! [j_2 + j_4 + k_1 - k_2 - z_4]! [j_2 + j_4 + k_1 + k_2 - z_4 + 1]!}{[k_1 + j_2 - l_3 - z_4]! [k_1 + j_2 + l_3 - z_4 + 1]! [j_2 + k_3 - l_4 + z_3 - z_4]!}
\times \frac{[l_1 + l_2 - k_1 + k_3 - z_1 - z_2]! [j_3 + j_4 - k_1 + l_1 + l_4 - z_1 - z_3]!}{[j_2 - j_3 + j_4 + k_1 - l_1 + z_1 - z_4]! [j_2 + j_3 + j_4 - k_3 - l_2 + z_2 - z_4]!}
\times \frac{[l_2 + l_4 - j_3 + j_4 - z_2 - z_3]!}{[l_1 + l_2 + l_4 - k_1 - j_2 - z_1 - z_2 - z_3 + z_4]!}.
\] (5.5d)

The numerator–denominator distribution of factorials, depending on the summation parameters \(z_1, z_2, z_3, z_4\), is different in each expression (5.5a)–(5.5d). No single formula exhibits the full symmetry (5.3b)–(5.3c) of the \(q\)-12\(j\)-symbol, but (5.5a) is invariant with respect to the transition from the main notation to the left array of (5.3c), as well as under transposition
\[
\begin{bmatrix} j_1 & j_2 & j_3 & j_4 \\ l_1 & l_2 & l_3 & l_4 \\ k_1 & k_2 & k_3 & k_4 \end{bmatrix}_q = \begin{bmatrix} k_2 & k_4 & k_1 & k_3 \\ l_1 & l_3 & l_2 & l_4 \\ j_3 & j_1 & j_4 & j_2 \end{bmatrix}_q,
\] (5.6)
which, in turn, is a composition of symmetry relations (5.3b)–(5.3c). Expression (5.5d) is invariant with respect to the same symmetry (5.6), but (5.5a) and (5.5c) do not satisfy any symmetry relations. Since all the sums in these expressions correspond to the balanced hypergeometric functions, the \(q\)-phases are also trivial.

All the terms in the first sum of (5.5b) and (5.5c) are of the same sign, as well as in the last sum of (5.5d). Each separate sum corresponds in these expressions to the finite balanced basic hypergeometric series \(5F_4(q, 1)\) (5.2), which also appeared in the elementary overlap coefficients of the definite biorthogonal coupled states of \(u_q(3)\) and \(SU(3)\). The summation intervals are mainly restricted by 8 [in (5.5a)–(5.5c)], or 7 [in (5.5d)] triangle linear combinations of parameters, respectively. In addition to correspondence of numerator and denominator factorials, determined by Eq. (A4a) or (A4b), definite correlation between the factorials under summation signs reveals itself in four quintuplets of factorials of each expression (5.5a)–(5.5d), depending on the couples of adjacent summation parameters \((z_i, z_{i+1})\), where \(i = 1, 2, 3,\) or \((z_1, z_4)\), although their expansion using the Chu–Vandermonde formulas is not helpful for further rearrangement of the generic expressions.

**C. Stretched cases of the \(q\)-12\(j\) coefficients of the second kind**

Let us consider the stretched cases of \(q\)-12\(j\) coefficients. For definite stretched triangles some summation parameters in (5.5a)–(5.5d) are either fixed (31 times), or expressions are partially summable (in the 11 cases) by means of Minton’s summation formulas (B3a) or (B3b) (see Ref. 36). One of three remaining sums turns into balanced basic hypergeometric series \(4F_3[q, 1]\), the rearrangement of which enables us to transform a \(5F_4[q, 1]\) type series into \(4F_3[q, 1]\) type series, with only the last one remaining of the \(5F_4[q, 1]\) type. Particularly, for \(j_1 + l_1 = k_1\) with \(z_1 = 0\), the sum over \(z_2\) in expression (5.5a) corresponds to a \(q\)-6\(j\) coefficient, which may be reexpressed in such a form (using Regge symmetry and change of the summation parameter) that the sum over \(z_3\) also corresponds to a \(q\)-6\(j\) coefficient. Hence, we obtain
The defining relations (5.4a)–(5.4d) of the $l_q(B2a)$ and (5.4b) in this doubly stretched case. 

The all three sums in (5.8) correspond to the balanced basic hypergeometric de Fèriet each depending on 10 parameters and corresponding to the $3F_2$ series.

After the summation over $z_3$ of the balanced $3F_2(q,1]$ series is carried out [see Eqs. (3.23) and (3.24)] in this doubly stretched case of $q$-12j coefficient with $k_1 = j_1 + l_1$ and $l_3 = k_1 + j_2$ [i.e., for adjacent consecutive stretched triangles in graph (5.2)], we recognize some $q$-6j coefficients, which may also be obtained using the symmetries (5.3b)–(5.3c) and the defining relations (5.4a)–(5.4d) of the $q$-12j coefficients. In this way, we derive following the relation:

$$\begin{pmatrix}
  j_1 & j_2 & j_3 & j_4 \\
  l_1 & l_2 & k_2 & k_4 \\
  j_1 + l_1 & k_2 & k_3 & k_4
\end{pmatrix}_q = \frac{(-1)^{j_1+2} \nabla[k_3j_2l_3] \nabla[j_4k_2l_4] \nabla[l_1k_2j_3]}{\nabla[j_1l_2k_3] \nabla[j_2k_2l_4] \nabla[j_1k_4d_4] \nabla[l_1k_2j_3]} \left( \frac{[2l_1][2j_1][2j_2]}{[2k_1 + 1]} \right)^{1/2} \times \sum_{z_1, z_3, z_4} \frac{(-1)^{z_1+2} [l_2 - j_3 + k_4 + z_1][j_1 + j_3 + k_3 - k_1 - z_1]}{[z_1][z_3][z_4][l_2 + j_3 - k_4 - z_1][j_3 - j_4 - l_1 + l_3 + z_4 - z_1]} \times \frac{[2j_3 - z_1][j_4 + k_3 - l_4 + z_3][j_2 + k_3 - l_4 + z_3][2l_4 - z_3]}{[j_2 - k_3 + l_4 - z_3][k_3 + j_1 + j_3 + j_4 - l_4 - z_1 + z_3 + 1]} \times \frac{[k_4 + l_1 - l_3 + l_4 - j_3 - z_3 - z_4][k_3 - k_1 + l_3 - l_4 + z_3 + 4]}{[k_1 + k_3 + k_4 + j_4 - l_3 - z_4 + 1]} \times \frac{[2l_4 + z_4 + 1]}{[2l_2 - z_3 + j_4 - z_4]}, \quad (5.7)$$

which is the composition of two balanced $4F_3[q,1]$ series and the third balanced $5F_4[q,1]$ series.

After the summation over $z_3$ of the balanced $3F_2[q,1]$ series is carried out [see Eqs. (3.23) and (3.24)] in this doubly stretched case of $q$-12j coefficient with $k_1 = j_1 + l_1$ and $l_3 = k_1 + j_2$ [i.e., for adjacent consecutive stretched triangles in graph (5.2)], we recognize some $q$-6j coefficients, which may also be obtained using the symmetries (5.3b)–(5.3c) and the defining relations (5.4a)–(5.4d) of the $q$-12j coefficients. In this way, we derive following the relation:

$$\begin{pmatrix}
  j_1 & j_2 & j_3 & j_4 \\
  l_1 & l_2 & k_2 & k_4 \\
  j_1 + l_1 & k_2 & k_3 & k_4
\end{pmatrix}_q = \frac{(-1)^{j_1+2} \nabla[l_3j_2k_2] \nabla[j_1+j_2, l_2, l_4] \nabla[l_1k_2j_3] \nabla[j_1+j_2, j_3, j_4]}{\nabla[j_1l_2k_3] \nabla[j_2k_2l_4] \nabla[l_1k_2j_3] \nabla[j_1+j_2, j_3, j_4]} \left( \frac{[2l_1][2j_1][2j_2]}{[2k_1 + 1][2l_3 + 1]} \right)^{1/2} \times \begin{cases}
  j_1 + j_2 & l_4 & l_2 \\
  k_4 & j_3 & j_4
\end{cases}. \quad (5.8)$$

In the doubly stretched case of the $q$-12j coefficient for $j_1 = k_1 - l_1 = l_2 - k_3$ [i.e., when the adjacent stretched triangles in graph (5.2) are diverging], we obtain from Eq. (5.5a) or (5.5c), and from Eq. (5.3c) with fixed $z_1 = z_2 = 0$, two different double sum expressions, each depending on 10 parameters and corresponding to the $q$-generalizations of the Kampé de Fériet function $F_{1,3}^{1,4}$, defined as (4.6). Each separate sum corresponds to the balanced basic hypergeometric $5F_4[q,1]$ series. Again, we may identify the couples of quintuplets of factorials under summation signs in the numerator and denominator, each depending on the summation parameters $z_3$ and $z_4$.

Otherwise, in the case of the merging adjacent stretched triangles (e.g., for $k_1 = j_1 + l_1 = j_2 + l_3$), the straightforwardly derived expressions include the triple sums; in particular all three sums in (5.8) correspond to the balanced basic hypergeometric $4F_3[q,1]$ series. The $4F_3[q,1]$ type sum over $z_4$ may be rearranged in analogy with expressions for the $q$-6j coefficients into another form (cf. Ref. 38) in a such way that the sum over $z_3$ turns
into summable balanced basic hypergeometric $\mathbf{3F}_2[q, 1]$ series. Hence we obtain the doubly stretched $q$-$12j$ coefficient in terms of the double sum:

$$
\begin{align*}
\left[ \begin{array}{cccc}
\frac{j_1}{l_1} & \frac{j_2}{l_2} & \frac{j_3}{l_3} & \frac{j_4}{l_4} \\
1 & k_2 & k_3 & k_4 \\
\frac{1}{j_1+l_1} & k_2 & k_3 & k_4 \\
\end{array} \right]_q \\
\left( \frac{\eta}{(l_3k_2j_1)\eta[j_1l_3k_2]\eta[k_4j_3l_2]\eta[j_2k_3l_4]l_1k_2j_3} \right)
\frac{1}{\eta^4}
\frac{[2j_3-z]!(-1)^{z+u}[j_1-k_4+l_4+u]![j_4+l_3-k_2+u]![2j_4+u+1]!}{[j_1+j_2+j_3-j_4-z-u]![l_1+l_3-j_3+j_4+z+u+1]!} \times
\sum_{z,u} \frac{[j_1-j_3+k_4+z]![l_1+k_2-j_3+z]![j_1+j_3-k_4-j_1-j_3+z]!}{[l_2-j_3+k_4-z]![l_2+j_3-k_4-z]![l_2-l_1+j_3-z]![k_3+k_4-j_1-j_3+z]!}
\times
\frac{[2j_3-z]!(-1)^{z+u}[j_4+j_2-j_1+u]![2j_4+u+1]!}{[j_1+j_2+j_3-j_4-z-u]![l_1+l_3-j_3+j_4+z+u+1]!}
\end{align*}
$$

which again depends on 10 parameters and corresponds to the $q$-generalization of the Kampé de Fériet\footnote{Kampé de Fériet function $F^{1}_{1,3}$} function $F^{1}_{1,3}[q, 1]$, with each separate sum corresponding to the balanced basic hypergeometric $\mathbf{5F}_5[q, 1]$ series. Perhaps this expression is related to the above mentioned special case of (5.5c) with fixed $z_1 = z_2 = 0$ and the adjacent diverging stretched triangles of $q$-$12j$ coefficient with respect to some composition of the usual and “mirror reflection” ($j \rightarrow -j - 1$) symmetries\footnote{Symmetry properties}.\footnote{Symmetry properties}

In the doubly stretched case of the $q$-$12j$ coefficient with $k_1 = j_1 + l_1$ and $j_4 = k_4 + l_4$ [i.e., for antipode stretched triangles of graph (5.2)], we derive from Eq. (5.7), with fixed $z_3 = 0$ and $z_4 = l_1 - l_3 - j_3 + j_4 + z_1$, an expression with a single sum, which corresponds to the balanced basic hypergeometric $\mathbf{6F}_5[q, 1]$ series and depends on 10 parameters:

$$
\begin{align*}
\left[ \begin{array}{cccc}
\frac{j_1}{l_1} & \frac{j_2}{l_2} & \frac{j_3}{l_3} & \frac{l_4+k_4}{l_4} \\
1 & l_2 & l_3 & l_4 \\
\frac{1}{j_1+l_1} & k_2 & k_3 & k_4 \\
\end{array} \right]_q \\
\left( \frac{\eta}{(j_1l_2k_2)\eta[k_4j_3l_2]\eta[l_4k_3j_2]\eta[j_2k_3l_4]l_1k_2j_3} \right)
\frac{1}{\eta^4}
\frac{[2j_4]![2j_1]![2j_3]![2k_4]!}{[2k_1+1]![2j_4+1]!}
\times
\sum_{z_1} \frac{(-1)^{z_1}l_2-j_3+k_4+z_1]![j_4-j_3+j_2-j_1+z_1]}{[l_2+j_3-k_4-z_1]![j_1+j_2+j_3-j_4-z_1]}
\times
\frac{[l_1+k_2-j_3+z_1]![2j_3-z_1]!}{[l_1-l_3-j_3+j_4+z_1]![l_1+l_3-j_3+j_4+z_1+1]!}
\times
\frac{[k_2+j_3-l_1-z_1]![k_3+k_4-j_1-j_3+z_1]}{[j_1+j_3+k_3-k_4-z_1]!}
\end{align*}
$$

In the doubly stretched case with $k_1 = j_1 + l_1$ and $k_4 = j_4 + l_4$ (again for antipode stretched triangles) from (5.5c), after the summation over $z_2$ of the balanced $\mathbf{3F}_2[q, 1]$ series (see Appendix B), we obtain

$$
\left[ \begin{array}{cccc}
\frac{j_1}{l_1} & \frac{j_2}{l_2} & \frac{j_3}{l_3} & \frac{j_4}{l_4} \\
1 & k_2 & k_3 & k_4 \\
\frac{1}{j_1+l_1} & k_2 & k_3 & j_4 + l_4 \\
\end{array} \right]_q
$$

\footnote{Note: Additional details and derivations related to the above expressions are provided in the appendix.}
the relation

\[ \text{the adjacent (a) type, e.g., with 4-cycle, or (d) both these momenta may be outside of the 4-cycle. For adjacent, (b) antiparallel, (c) the first one may be inside and the second one outside of the 4-cycle). There are four possible different distributions of the summarized angular momenta. Both expressions (5.10) and (5.11) satisfy some Regge type symmetry relations.}

Finally, expression (5.5a) with fixed \( j_1 = j_1 + l_1 \) and \( l_3 = k_2 + j_4 \), we derive from Eq. (5.7) the relation

\[
\begin{bmatrix}
  j_1 & j_2 & j_3 & j_4 \\
  l_1 & l_2 & k_2 + j_4 & l_4 \\
 j_1 + l_1 & k_2 & k_3 & k_4
\end{bmatrix}_q
\]

\[ = \frac{(-1)^{j_1 + l_2 - k_2 + k_4} \nabla[k_1 j_2 j_3] \nabla[k_4 j_3 j_4]}{\nabla[j_1 l_2 k_3] \nabla[k_4 j_3 j_2] \nabla[j_4 l_4] \nabla[k_1 k_2 j_3]} \left( \frac{[2l_1]!! [2j_1]!! [2k_2]!! [2j_4]!!}{[2k_1 + 1]!! [2l_4 + 1]!!} \right)^{1/2}
\]

\[ \times \sum_{z_1, z_3} (-1)^{z_1 + z_3} \left[ z_1 [l_2 + j_3 - k_4 - z_1] [j_1 + j_3 + k_3 - k_4 - z_1]!
\right]
\[ \times \frac{[2j_3 - z_1] [2l_4 - z_3] [j_4 + k_4 - l_4 + z_3]}{[k_3 - k_1 + l_3 - l_4 + z_3] [k_3 + j_1 + j_3 + j_4 - l_4 - z_1 + z_3 + 1]!}
\]

The double sum depends on 9 (from 10 free) parameters and corresponds to the \( q \)-generalization of the Kampé de Fériet\textsuperscript{13} function \( F_{1:2}^{1:3} \), defined as \( F_{1:2}^{1:3} \) with \( b_1 + b_1' = c_1 \), and each separate sum corresponding to the balanced \( 4F_3[q, 1] \) or \( 4F_3 \) series. Different (i.e. not equivalent) expressions of the (a) type appear also for \( k_3 = j_1 + l_2 \) and \( j_2 = k_1 + l_3 \) from Eq. (5.5c) and for \( j_3 = l_1 + k_2 \) and \( k_4 = j_4 + l_4 \) from Eq. (5.5a). Furthermore, the doubly stretched \( q-12j \) coefficients of the “antiparallel” (b) type, with \( k_1 = j_1 + l_1 \) and \( k_3 = j_2 + l_4 \), expressions (5.5a), (5.5c), and (5.5d) (with fixed parameters \( z_1 \) and \( z_3 \)) also turn into (mutually different) double sums, again depending on 9 (from 10 free) parameters and related to the \( F_{1:2}^{1:3} \) type functions. This is also the case for expression (5.5b) (with fixed \( z_1 = z_3 = 0 \) for the doubly stretched \( q-12j \) coefficients of the “inside–outside” (c) type, with \( k_1 = j_1 + l_1 \) and \( l_4 = j_2 + k_3 \) (or, expression (5.5d) with \( l_3 = j_1 + l_3 \) and \( j_2 = k_1 + l_1 \)). Finally, expression (5.5a) with fixed \( z_2 = z_4 = 0 \) and \( l_2 = j_1 + k_3 \) and \( l_3 = k_1 + j_2 \) again
turns into the double sums depending on 9 (from 10 free) parameters and related to the $F^{13}_{12}$ type function for the doubly stretched $q$-12$j$ coefficients with the both summarized angular momenta of the “outside” (d) type. These 8 independent expressions should be related to (5.12) by means of some compositions of the usual and “mirror reflection” ($j \to -j - 1$) symmetries. Otherwise, many special versions of (5.5a)–(5.5d) with fixed $z_i = z_{i+1} = 0$ ($i = 1, 2, 3$) or $z_1 = z_4 = 0$ give expressions for the doubly stretched $q$-12$j$ coefficients with remote stretched triangles in terms of the double sums, related to compositions of the balanced $F_3[q, 1]$ and $F_4[q, 1]$ series.

Equation (5.5b) also turns into a single term for $l_1 + l_2 + l_3 - l_4 = 0$ (when the all summation parameters $z_i$ are fixed),

$$
\begin{array}{c}
\left[ \begin{array}{cccc}
 j_1 & j_2 & j_3 & j_4 \\
 l_1 & l_2 & l_3 & l_1 + l_2 + l_3 \\
 k_1 & k_2 & k_3 & k_4 
\end{array} \right]_q \\
= \frac{(-1)^{j_1 + j_3 - k_1 + k_2}[2l_1]![2l_2]![2l_3]![4k_3,j_2]![4j_4,k_4]}{[2l_4 + 1]![l_1,j_1,k_1]![l_1,k_2,k_3]![l_2,j_2,k_3]![l_2,k_3,k_4]![l_3,j_2,k_1]![l_3,k_2,k_1].}
\end{array}
$$

(5.13)

For this special $q$-12$j$ coefficient [as well as in (5.5c) for $l_1 - l_2 + l_3 + l_4 = 0$ and in (5.5d) for $l_1 + l_2 - l_3 + l_4 = 0$], four linearly dependent angular momenta appear as disconnected in certain positions on a Hamilton line of graph (5.2). Actually, the single term expression of this virtually stretched case appears in accordance with symmetries (5.3a)–(5.3c) from expansion (5.4a) with $j_1 + j_2 - j_3 + j_4 = 0$ and fixed $x = j_3 - j_1 = j_2 + j_4$.

VI. EXPRESSIONS FOR 12$j$ COEFFICIENTS OF THE FIRST KIND

A. Generic properties

Next we consider the rearrangement of expressions for the $q$-12$j$ coefficients of the first kind whose graphs are not planar:

(include some braiding). These coefficients satisfy 16 symmetries, generated by the following substitutions:
\[
\begin{align*}
\begin{array}{c}
\left\{ \begin{array}{cccc}
  j_1 & j_2 & j_3 & j_4 \\
  l_1 & l_2 & l_3 & l_4 \\
  k_1 & k_2 & k_3 & k_4
\end{array} \right\}_q & = \left\{ \begin{array}{cccc}
  j_2 & j_3 & j_4 & k_1 \\
  l_2 & l_3 & l_4 & l_1 \\
  k_2 & k_3 & k_4 & j_1
\end{array} \right\}_q \\
& = \left\{ \begin{array}{cccc}
  k_1 & j_4 & j_3 & j_2 \\
  l_4 & l_3 & l_2 & l_1 \\
  j_1 & k_4 & k_3 & k_2
\end{array} \right\}_q .
\end{array}
\end{align*}
\] (6.2a)

There expression in terms of the factorized four differently rearranged \(q\)-6\(j\) coefficients is

\[
\begin{align*}
\begin{array}{c}
\left\{ \begin{array}{cccc}
  j_1 & j_2 & j_3 & j_4 \\
  l_1 & l_2 & l_3 & l_4 \\
  k_1 & k_2 & k_3 & k_4
\end{array} \right\}_q & = \sum_x (2x + 1)(-1)^{R_4-x} q^{(x+1) + Z_{j_1j_2j_3j_4}} + Z_{k_1k_2k_3k_4} \\
	imes \left\{ \begin{array}{cccc}
  j_1 & j_2 & l_1 \\
  k_2 & k_1 & x
\end{array} \right\}_q \left\{ \begin{array}{cccc}
  j_3 & k_3 & x \\
  k_2 & j_2 & l_2 \phantom{x}
\end{array} \right\}_q \left\{ \begin{array}{cccc}
  j_4 & j_3 & l_3 \\
  k_4 & k_3 & x \phantom{x}
\end{array} \right\}_q \left\{ \begin{array}{cccc}
  k_1 & j_1 & x \\
  k_4 & j_4 & l_4 \phantom{x}
\end{array} \right\}_q 
\end{array} .
\end{align*}
\] (6.3a)

where the triangular conditions are to be satisfied by all the triplets of the nearest neighbors such as \(l_i, j_i, j_{i+1}\), or \(l_i, k_i, k_{i+1}\) (\(i = 1, 2, 3\)), or \(l_4, j_1, k_4\), or \(l_4, k_1, j_4\), respectively.

After using (2.1b) for the \(q\)-6\(j\) coefficients with the summation parameter \(x\) in the right lower position, Eq. (2.1a) with inverted summation parameter for the \(q\)-6\(j\) coefficients with \(x\) in the middle column, and (2.1a) directly in the remaining cases, the depending on the summation parameter \(x\) asymmetric triangle coefficients [distributed separately in the numerators or denominators of each \(q\)-6\(j\) coefficient in expansions (6.3a) and (6.3b)] cancel, with the exception of the factors

\[
\nabla[j_1 k_1 x] = \frac{[j_1 - k_1 + x]!}{[k_1 - j_1 + x]!}.
\]

Then, the sums over \(x\) correspond to the \(q\)-generalization of the very well-poised classical hypergeometric \(\phi_5(-1)\) series (resembling the basic hypergeometric \(\gamma\phi_6\) series) and may be rearranged into the \(3\phi_2\) or \(3\phi_2[q, x]\) type series using the following two formulas:

\[
\begin{align*}
\sum_j (-1)^{p_2+j+1} q^{(j+1)} [2j + 1][j - p_1 - 1][j - p_2 - 1][j - p_3 - 1]! \\
\frac{1}{[p_1 + j + 1]![p_2 + j + 1]![p_3 + j + 1]![p_4 - j]![p_4 + j + 1]![p_5 - j]![p_5 + j + 1]!} \\
= q^{-p_4(p_4+1)-p_2(p_4+p_5+1)} [p_1 - p_3 - 2]! \\
[p_1 + p_4 + 1]![p_2 + p_5 + 1]![p_3 + p_4 + 1]! \\
\times \sum_u (-1)^u q^{u(p_2+p_5+1)} [p_4 - p_3 - 1 - u][p_4 - p_1 - 1 - u]! \\
[u]![p_1 - p_3 - 2 - u][p_2 + p_4 + 1 - u][p_4 + p_5 + 1 - u]! 
\end{align*}
\] (6.4a)
with parameters

\[ p_1 = k_1 - j_1 - 1, \quad p_4 = j_1 + k_2 - l_1 + z_1, \quad p_5 = j_3 + k_4 - l_3 + z_3, \]
\[ p_2 = l_2 - k_2 - j_3 + z_2 - 1, \quad p_3 = l_4 - k_1 - k_4 + z_4 - 1; \]

\[
\sum_j \frac{q^{(j+1)}[2j+1][j-p_1-1][j-p_2-1][j-p_3-1][j-p_4-1][j-p_5-1]}{[p_1+j+1][p_2+j+1][p_3+j+1][p_4+j+1][p_5+j+1]}
\]
\[
= q^{- (p_4+1)(p_5+1)-p_2(p_4+p_5+1)} \frac{(-1)^nq^{u(p_2+p_5+1)}[p_4-p_3-1-u][p_4-p_1-1-u]}{[p_1+p_4+1][p_3+p_4+1]}
\times \sum_u \frac{(-1)^n q^{u(p_2+p_5+1)}[p_4-p_3-1-u][p_4-p_1-1-u][p_2+p_4+1-u]}{[p_2+p_4+1-u][p_2+p_4+1-u][p_1+p_4+1-u]}
\]

(6.4b)

with parameters

\[ p_1 = k_1 - j_1 - 1, \quad p_2 = j_3 - k_3 - z_2 - 1, \quad p_3 = j_1 - k_1 - z_4 - 1, \]
\[ p_4 = j_1 + k_2 - l_1 + z_1, \quad p_5 = l_3 - j_3 - k_4 + z_3 - 1. \]

Equation (6.4a) corresponds to (5.5) [or (5.6), when \( q = 1 \)] of Ref. 34, with the r.h.s. replaced using less symmetric expressions \( \sum \) instead of the most symmetric (Van der Waerden) expression \( \sum \) for the Clebsch–Gordan coefficients of SU(2) and \( u_q(2) \) (cf. also Refs. 55, 41), when (6.4b) is derived from (6.4a) using the analytical continuation technique.

**B. General expressions with five sums**

Substituting the summation parameter \( u \), which appeared after using (6.4a) and (6.4b) in (3.3a) and (3.3b), by \( u + z_1 \), we obtain the following expressions for the \( q \)-12j coefficients of the first kind:

\[
\begin{align*}
\left\{ \begin{array}{cccc}
\frac{j_1}{l_1} & \frac{j_2}{l_2} & \frac{j_3}{l_3} & \frac{j_4}{l_4} \\
k_1 & k_2 & k_3 & k_4
\end{array} \right\}_q &= (-1)^{j_1+k_2-k_3-j_2+j_3+j_4-l_4} \nabla [j_3j_2l_2] \nabla [k_2k_3l_3] \nabla [k_1j_4l_4] \nabla [k_4j_1l_1] \\
\sum_{z_1,z_2,z_3,z_4,u} &\frac{(-1)^{z_1+z_2+z_3+z_4}u[j_1+j_2-l_1+z_1][2l_1-z_1]}{[z_1][z_2][z_3][z_4][k_1-k_2+l_1-z_1][l_1-j_1+j_2-z_1]} \\
\times q^{-z_2(j_1+k_2+k_4+l_1-l_3+z_3+1)}[l_2+j_2-j_3+z_2][l_2+k_3-k_2+z_2] &\frac{(-1)^{z_3+z_4}u[j_3+j_4-j_3+z_3][l_3+j_4-l_3+z_3][2l_3-z_3]}{[j_2+j_3-l_2-z_2][k_2+k_3-l_2-z_2][2l_2+z_2+1][2l_4+z_4+1]} \\
\times \frac{(-1)^{z_2}q^{z_2[l_2+j_2-j_3+z_2][l_2+k_3-k_2+z_2]}[k_3+k_4-l_3+z_3][l_2+j_2-j_3+z_2][l_2+k_3-k_2+z_2]}{[l_2-l_3-k_4-z_3][l_2-j_2-j_3+z_3][k_3+k_4-l_3+z_3][l_2-j_2+j_3-z_3][k_3+k_4-l_3+z_3][l_2-l_3-k_4-z_3][l_2-j_2-j_3+z_3][k_3+k_4-l_3+z_3][l_2-j_2+j_3-z_3][k_3+k_4-l_3+z_3][l_2-l_3-k_4-z_3][l_2-j_2-j_3+z_3][k_3+k_4-l_3+z_3][l_2-j_2+j_3-z_3][k_3+k_4-l_3+z_3][l_2-l_3-k_4-z_3][l_2-j_2-j_3+z_3][k_3+k_4-l_3+z_3][l_2-j_2+j_3-z_3][k_3+k_4-l_3+z_3][l_2-l_3-k_4-z_3][l_2-j_2-j_3+z_3][k_3+k_4-l_3+z_3][l_2-j_2+j_3-z_3][k_3+k_4-l_3+z_3][l_2-l_3-k_4-z_3]<6.4b>
\end{align*}
\]

26
\[ \times \frac{q^{u(k_4-k_2-l_3+l_2+z_2+z_3)}[j_1+k_1+k_2+k_4-l_1-l_2-z_4-u]}{[u+z_1][j_1+k_4-l_4-z_1-z_4-u][j_1-j_3-l_1+l_2+z_2-u]}! \]

\[ \times \frac{2j_1-k_1+k_2-l_1-1!}{[j_1+j_3+k_2+l_1-1+2+z_4-u]}! \]

\[ = (-1)^{z_1+j_3+j_4+k_2-k_4-l_4-l_3+z_3-u+1} \]

\[ \times q^{(j_1+k_2-l_4+1)(l_2+k_3-l_3+1)-(l_2+k_2-j_4+1)(j_4+k_4-l_4+1)+z_{1234}} \]

\[ \times \sum_{z_{1234}} \frac{(-1)^{z_2+z_3+z_4+z_1}j_1+j_2-l_1+z_1}[2l_1-z_1]! \]

\[ \times \frac{l_2+k_2-k_3-z_2}[j_2-j_3+l_3-z_2][l_2+k_2+z_2+1]! \]

\[ \times \frac{2l_4-z_4}[j_1+k_4-l_4+z_4][l_2+k_4-z_2-1]! \]

\[ \times q^{-u(k_2+k_4+l_4-z_2-z_3+1)}[j_1-k_1+k_2-l_1-z_4-u]! \]

\[ \times \frac{j_1+j_3-k_2-l_1+1}[j_1-j_3+k_2-k_4-l_1+l_3+z_4-u]! \]

\[ \times \frac{[j_1+j_3-l_1-l_2+z_2-u][j_1-j_3+k_2-k_4-l_1+l_3+z_3-u]}{6.5a} \]

\[ \times \frac{[j_1+j_3+k_2-k_4-l_1+l_4-z_4-u]}{6.5b} \]

Each of expressions (6.5a) and (6.5b) includes 5 summations, with 4 separate sums (over \( z_1, z_2, z_3, \) and \( z_4 \)) corresponding to the finite (balanced in the first and last cases) basic hypergeometric series \( {}_3\phi_3 \) or \( {}_4\psi_3 \) \( {}_4\rangle \) \( q \)-hypergeometric series \( {}_2\phi_2 \) or \( {}_3\psi_2(1) \), related in the case of (6.5a) to the Clebsch–Gordan coefficients of \( u_q(2) \) or \( SU(2) \). However, it is impossible to rearrange all 5 sums together into standard basic hypergeometric series \( {}_{p+1}\phi_p \). Some correlation between the factorial terms under the summation signs reveals itself in two quintuplets of factorials of each expression (6.5a) and (6.5b), depending on the couples of summation parameters \( z_2, z_3 \) and \( z_1, z_4 \). Definite correspondences may be observed in Eqs. (6.5a) and (6.5b) between the \( q \)-phase structure and three particular factorial arguments, depending on the couples of summation parameters \( z_2, z_3, \) and \( u \), respectively [as well as in \( q=9j \) coefficients (3.1a)–(3.1e) between the \( q \)-phases and three factorial arguments, depending on the couples of summation parameters \( z_1, z_2, \) and \( z_3 \)]. The summation intervals in (6.5a) and (6.5b) are mainly restricted by 8 triangle linear combinations of parameters, respectively, but in the stretched cases only (6.5a) for \( k_4 = l_4 - j_1 \) and (6.5b) for \( k_4 = l_4 + j_1 \) (with \( z_4 = u + z_1 = 0 \) in the both cases) turn into the triple sums.

C. Stretched cases of the \( q-12j \) coefficients of the first kind

When the total (maximal) angular momentum in a stretched triangle of the \( q-12j \) coefficient of the first kind corresponds to a crossbar of the Möbius strip (6.1) \( [\) in the middle row of standard array (6.2a), e.g., for \( l_4 = k_4 + j_1 \), we obtain the following expression:
\[
\left\{ \begin{array}{cccc}
  j_1 & j_2 & j_3 & j_4 \\
  l_1 & l_2 & l_3 & l_4 + j_1 \\
  k_1 & k_2 & k_3 & k_4 \\
\end{array} \right\}_q \\
= (-1)^{k_1+k_2-k_3-j_2+j_3+j_4-l_4} \frac{\nabla[j_3,j_2,l_2]\nabla[k_2,k_3,l_2]\nabla[l_4,j_4,k_1]}{\nabla[j_1,j_2,l_2]\nabla[k_2,k_1,l_1]\nabla[j_3,j_4,l_3]\nabla[k_4,k_3,l_3]} \left( \frac{[2k_4]![2j_1]!}{[2l_4]!} \right)^{1/2} \\
\times q^{-(j_1+k_2-l_1+1)(k_1-k_2+l_2-l_3)-(l_2-k_2-j_2-j_3-1)(j_1+k_4-l_3)+Z_{j_1,j_2,j_3,j_4}+Z_{k_1,k_2,k_3,k_4}} \\
\times \sum_{z_1,z_2,z_3} \frac{(-1)^{z_1+z_2}[j_1+j_2-l_1+z_1][k_1+k_2-l_1-z_1][l_2-z_1]}{[z_1]![z_2]![z_3]![k_1-k_2+l_1-z_1][(l_1-j_1+j_2-z_1)]!} \\
\times \frac{[l_2+j_2-j_3+z_2]![l_2+k_3-k_2+z_2]}{[j_2+j_3-l_2-z_2]![k_2+k_3-l_2-z_2][2l_2+2l_2+1]!} \\
\times \frac{[k_3+k_4-l_3+z_3][j_3+j_4-l_3+z_3][2l_3-z_3]}{[k_3+l_3-k_4+z_3]![l_3-j_3+j_4-z_3][(j_1-j_3-l_1+1)!]} \\
\times q^{z_1(l_1-j_1+j_3)+z_2(z_3-1)(k_2-k_1-l_2-l_3+l_4+z_3+1)-z_3(l_3-k_2+k_4+z_2+z_3)} \frac{[l_2-l_3-k_2+k_4+z_2+z_3][j_3+k_2-l_1+l_3+z_4+z_3]}{[(j_1-j_3-l_1+z_1)]!}. \\
\] 

(6.6)

This special case of (5.5a) with 3 separate sums corresponds to the finite basic hypergeometric series \(3F_2[\cdots; q, x]\). Expression (6.6) does not simplify noticeably for two adjacent merging stretched triangles (with \(l_4 = k_4 + j_1 = k_1 + j_4\) in the same q-6j coefficient of expansion (5.3a), but one of the turns into a \(3F_2[\cdots; q, x]\) series for two adjacent diverging stretched triangles (e.g., with \(j_1 = l_4 = k_4 = l_1 - j_2\), or with \(j_1 = l_4 = k_4 = j_2 - j_3\)). The Chu–Vandermonde summation formula (31er) may be used in Eq. (6.6) for \(j_1 = 0\), \(l_1 = j_2\), \(k_4 = l_4\) and for \(k_4 = 0\), \(l_4 = j_1\), \(k_3 = l_3\). In these cases, the couples of the q-6j coefficients appear in accordance with Eq. (33.21) of Ref. 3 as well as the consequences of expansions (5.3a)–(5.3b) for fixed \(x\).

When the total angular momentum in a stretched triangle of the q-12j coefficient of the first kind is located along the Möbius strip (6.4) (e.g., for \(k_4 = l_4 + j_1\)), we obtain from (5.5b), after change of summation parameter \(z_2 \rightarrow l_2 + k_2 - k_3 - z_2\), the following expression:

\[
\left\{ \begin{array}{cccc}
  j_1 & j_2 & j_3 & j_4 \\
  l_1 & l_2 & l_3 & l_4 \\
  k_1 & k_2 & k_3 & l_4 + j_1 \\
\end{array} \right\}_q \\
= (-1)^{j_3+j_4-l_3+l_2+k_2-k_3} \frac{\nabla[j_3,j_4,l_3]\nabla[k_4,k_3,l_3]\nabla[k_2,k_3,l_2]}{\nabla[l_4,j_4,k_1]\nabla[k_2,k_1,l_1]\nabla[j_3,j_2,l_2]} \left( \frac{[2j_1]![2l_4]!}{[2k_4]!} \right)^{1/2} \\
\times q^{(j_1+k_2-l_1+1)(k_1-k_2+l_2-l_3)-(l_2-k_2-j_2-j_3-1)(j_1+k_4-l_3)+Z_{j_1,j_2,j_3,j_4}+Z_{k_1,k_2,k_3,k_4}} \\
\times \sum_{z_1,z_2,z_3} \frac{[j_1+j_2-l_1+z_1][k_1+k_2-l_1-z_1][l_2-z_1]}{[z_1]![z_2]![z_3]![k_1-k_2+l_1-z_1][(l_1-j_1+j_2-z_1)]!} \\
\times \frac{[l_2+j_2-j_3+z_2]![l_2+k_3-k_2+z_2]}{[j_2+j_3-l_2-z_2]![k_2+k_3-l_2-z_2][2l_2+2l_2+1]!} \\
\times \frac{[k_3+k_4-l_3+z_3][j_3+j_4-l_3+z_3][2l_3-z_3]}{[k_3+l_3-k_4+z_3]![l_3-j_3+j_4-z_3][(j_1-j_3-l_1+1)!]} \\
\times q^{z_1(k_2+k_4+l_2-l_3-z_2+1)} \frac{[l_2+k_2-k_3+z_2][j_2+j_3+l_2-z_2]![l_2+k_2+k_3-z_2+1]!}{[k_3+k_4+l_3+z_3][j_4-j_3+l_3+z_3]!} \\
\times q^{z_2(k_2-j_3-j_1-l_3-l_4+z_3)-z_3(j_1+j_3-l_1-j_2)} \frac{[k_2+k_4+l_2-l_3-z_2-z_3]!}{[j_1+j_3-l_1-l_2+z_2][(l_2-k_2+k_4+z_2+z_3)!]} \right). \\
\] 

(6.7)
Expression (6.7) does not simplify for two couples of adjacent diverging stretched triangles [with \( l_4 = k_4 - j_1 = k_1 - j_4 \), or with \( l_4 = k_4 - j_1 = j_4 - k_1 \), respectively, again in the same \( q \)-6\( j \) coefficient of expansion (5.33a)], but for \( l_4 = 0 \), \( k_4 = j_1 \), \( j_4 = k_1 \) (after substitution \( q \rightarrow q^{-1} \)) it corresponds to the general expression (3.1a) for the \( q \)-9\( j \) coefficients, in accordance with Eq. (33.20) of Ref. 3. The triple sums in (6.6) and (6.7) resemble expressions for the \( q \)-9\( j \) coefficients and definite correspondences may be observed between the \( q \)-phases and three factorial arguments, depending on the couples of summation parameters \( z_1, z_2 \), and \( z_3 \), respectively. Expansion of the present couples of the factorial quintuplets (depending on parameters \( z_1, z_2 \) and \( z_2, z_3 \), respectively) using the Chu–Vandermonde summation formulas enables one to perform the summation over \( z_2 \) and thus obtain expressions for the stretched \( q \)-12\( j \) coefficients of the first kind as fourfold sums, related to compositions of \( _3F_2[\cdots; q, x] \) series.

Again, one of the sums in (6.7) turns into a \( _3F_2[\cdots; q, x] \) series for two adjacent diverging stretched triangles in two adjacent \( q \)-6\( j \) coefficients of expansion (5.33a) (e.g., with \( j_1 = k_4 - l_4 = l_1 - j_2 \), or with \( j_1 = k_1 - l_4 = j_2 - l_1 \)), as well as for two adjacent merging stretched triangles (e.g., with \( k_3 = l_4 + j_1 = k_3 + l_3 \)). The possible rearrangement of the \( _3F_2[\cdots; q, x] \) series is not helpful for reducing the remaining \( _4F_3[\cdots; q, x] \) series.

The doubly stretched \( q \)-12\( j \) coefficients of the first kind with the adjacent consecutive stretched triangles may be expressed for \( k_3 = l_4 + j_1 \) and \( j_1 = l_1 + j_2 \) or \( l_3 = k_3 + k_4 \) by means of (5.7), as well as for \( l_4 = k_4 + j_1 \) and \( j_1 = l_1 + j_2 \) or \( k_4 = k_3 + l_3 \) by means of (5.9), as the double sums, related to the stretched \( q \)-9\( j \) coefficients, respectively, of the type (3.1a) or (3.1b) (with some “reflected” parameters in the last cases). In this way we derive, for \( k_4 = l_4 + j_1 \) and \( j_1 = l_1 + j_2 \) [comparing (5.7) and (3.1a)], the following relation:

\[
\begin{align*}
&\left\{ \begin{array}{cccc}
    l_1 + j_2 & j_2 & j_3 & j_4 \\
    l_1 & l_2 & l_3 & l_4 \\
    k_1 & k_2 & k_3 & l_4 + j_1 \\
\end{array} \right\}_q \\
&\quad = (-1)^{j_3 + j_4 - l_3 + l_2 + k_3 - k_4} q^{z_{j_1, j_2, j_3, j_4} + z_{k_1, k_2, k_3, k_4} - z_{l_1, l_4, l_4 + j_1}} \nabla[l_1 + l_4, j_4, k_2] / \nabla[l_1 k_1 k_2] / \nabla[l_4 k_1 j_1] \\
&\quad \times \left( \frac{[2 l_1] [2 l_4]}{[2 l_1 + 2 l_4] [2 j_1 + 1]} \right)^{1/2} \left\{ \begin{array}{ccc}
    k_4 & k_3 & l_3 \\
    j_2 & l_2 & j_3 \\
    l_1 + l_4 & k_2 & j_4 \\
\end{array} \right\}_{q-1}. \tag{6.8}
\end{align*}
\]

Similarly, for \( k_4 = l_4 + j_1 \) and \( l_3 = k_3 + k_4 \) we obtain

\[
\begin{align*}
&\left\{ \begin{array}{cccc}
    j_1 & j_2 & j_3 & j_4 \\
    l_1 & l_2 & k_3 + k_4 & l_4 \\
    k_1 & k_2 & k_3 & l_4 + j_1 \\
\end{array} \right\}_q \\
&\quad = q^{z_{j_1, j_2, j_3, j_4} + z_{k_1, k_2, k_3, k_4} - z_{l_2, j_1 + k_3, j_1}} (-1)^{j_3 + j_4 - l_4 - l_2 - k_2 - j_2 + l_1} \nabla[l_3, j_3, j_4] / \nabla[l_4 k_1 j_4] / \nabla[j_1 + k_3, j_3, k_1] \\
&\quad \times \left( \frac{[2 l_4] [2 j_1 + 2 k_3 + 1]}{[2 l_3 + 1] [2 l_4 + 1]} \right)^{1/2} \left\{ \begin{array}{ccc}
    j_1 & k_3 & j_1 + k_3 \\
    j_2 & l_2 & j_3 \\
    l_1 & k_2 & k_1 \\
\end{array} \right\}_{q-1}. \tag{6.9}
\end{align*}
\]

We remark that the doubly stretched \( q \)-12\( j \) coefficients of the first kind for \( l_4 = k_4 + j_1 \) and \( l_2 = j_2 + j_3 \) or \( l_2 = k_2 + k_3 \) [expressed by means of (6.8)], as obtained from (6.6), as
As for \( k_1 = l_4 + j_1 \) and \( j_3 = l_2 + j_2 \) or \( k_3 = l_2 + k_2 \) as obtained from (6.7), turn into double sums equivalent to compositions of \( 4F_3[\ldots; q, x] \) and \( 3F_2[\ldots; q, x] \) series. However, although the \( 3F_2[\ldots; q, x] \) series may be rearranged into other forms, these double sums are not equivalent to any composition of two generic \( 3F_2[\ldots; q, x] \) series and, moreover, they are not related to any \( q \)-9j coefficient. Minton’s summation formula \( (33a) \) (see Ref. [33]) may be helpful in the analogical position of the stretched triangles in (6.6) for \( l_4 = k_4 + j_1 \) and \( j_2 = l_2 + j_3 \) or \( k_2 = k_2 + l_2 \), as well as in (6.7) for \( k_4 = l_4 + j_1 \) and \( l_2 = k_2 + k_3 \). Otherwise, the \( 3F_2[\ldots; q, x] \) series appearing in the triple sums, which remain in Eq. (6.6) for \( l_3 = j_2 + l_3 \) or \( k_2 = l_2 + k_3 \), and in Eq. (6.7) for \( j_2 = l_2 + j_3 \) or \( k_2 = l_2 + k_3 \), may be rearranged, but it is more reasonable in such a case to use the different expansions of the type \( (3.3a)-(3.3b) \) with inserted stretched \( q \)-6j coefficients and adapted [e.g., expansion (5.3a) for \( l_4 = k_4 + j_1 \) and \( j_3 = j_3 + l_2 \) or \( j_2 + j_3 \) or \( k_2 = l_2 + k_3 \), with transposed the middle and the right columns of the third \( q \)-6j coefficients in the both cases] for summation by means of \( (A2) \) or \( (A3) \).

Furthermore, expression (5.6) turns into double sums equivalent to compositions of two \( 4F_3[\ldots; q, x] \) series for the remote stretched triangles of graph (6.1) for \( l_4 = k_4 + j_1 \) and \( j_3 = l_3 + j_4 \) or \( k_2 = k_1 + l_1 \) (when two couples of the touching angular momenta form the 4-cycles, or quadrangles), as well as expression (6.7) for \( k_4 = l_4 + j_1 \) and \( l_3 = j_3 + j_4 \) or \( k_2 = k_1 + l_1 \).

Again, Minton’s summation formula \( (33a) \) may be used in (6.7) for \( j_4 = j_3 + l_3 \) and the \( 3F_2[\ldots; q, x] \) series may be rearranged in the triple sums, which remain in Eq. (6.6) for \( l_1 = k_1 + k_2 \) or \( l_3 = j_3 + j_4 \) and in Eq. (6.7) for \( j_4 = j_3 + j_4 \) or \( k_1 = k_2 + l_1 \). Expansion (6.3a) with inserted stretched \( q \)-6j coefficients may be used for summation by means of (A1) for \( l_4 = k_4 + j_1 \) and \( l_1 = k_1 + k_2 \), as well as expansion (6.3b) for summation by means of (A2) for \( k_4 = l_4 + j_1 \) and \( k_1 = k_2 + l_1 \), after in the both cases the middle and the right columns of the second \( q \)-6j coefficients are transposed.

In general, the triply stretched cases of the \( q \)-12j coefficients of the first kind may always be expressed in terms of the double sums, applying some “mirror reflection” \((j \rightarrow −j −1)\) operations to the above mentioned expressions, or using the above mentioned rearrangements. In particular, the double sums (compositions of two \( 3F_2[\ldots; q, x] \) series) related to the most generic Kampé de Fériet functions of the type \( F_{13}^{01} \) (with 9 independent parameters) appear instead of (6.7) in the triply stretched cases with \( k_4 = l_4 + j_1 = k_3 + l_3 \) and \( j_3 = j_2 + l_2 \), or with \( j_1 = k_4 + k_3 = l_4 = j_2 - j_1 \) and \( k_3 = k_2 + l_2 \).

Let us return to the generic expressions (6.5a) and (6.5b). In both formulas, for \( j_2 = 0 \), \( l_1 = j_1 \), \( l_2 = j_3 \), the summation parameters \( z_1, z_2 \) and \( u \) are fixed, the two remaining separate sums (over \( z_3, z_4 \)) correspond to the balanced \( 4F_3[q, 1] \) series, and the couples of the \( q \)-6j coefficients appear straightforwardly, in accordance with the expansion (3.3a)-(3.3b) for fixed \( x \). Furthermore, in both (6.5a) and (6.5b), the parameters \( z_1, z_2 \) and \( u \) are also fixed for \( j_1 + j_2 + k_1 - k_2 = 0 \), i.e., for the adjacent consecutive stretched triangles \( k_2 = k_1 + l_1 = k_1 + j_1 + j_2 \) with the intermediate angular momentum corresponding to a crossbar of the Möbius strip (6.1). Two remaining separate sums (over \( z_3, z_4 \)) correspond to the balanced (Saalschützian) \( 4F_3[q, 1] \) [related to \( q \)-6j coefficient of the type (2.1a)] and summable [see Eq. (B2a)] \( \delta F_2[q, 1] \) series. In this way, we derived the following relation:

\[
\begin{pmatrix}
  j_1 \\
  j_1 + j_2 \\
  l_2 \\
  l_3 \\
  l_4 \\
  k_1 \\
  k_1 + l_1 \\
  k_3 \\
  k_4
\end{pmatrix}
\]
\[
\begin{align*}
&= (-1)^{k_1 + j_3 - l_3 - l_4 + 2k_2} \frac{q^{2j_1 k_1 + Z_{j_2 j_3 j_4} + Z_{k_2 k_3 k_4}} \nabla[k_2 k_3 l_2] \nabla[k_3 + j_1, j_4, k_4]}{
\nabla[j_1 k_3 l_1] \nabla[j_2 k_2 j_3] \nabla[j_1 j_4 l_4] \nabla[k_1 + j_1, j_3, k_3]}
\times \left( \frac{[2j_2]!![2j_1]!![2k_1]!!}{[2k_2 + 1]!![2l_1 + 1]!!} \right)^{1/2} \left\{ \begin{array}{ccc}
j_3 & j_4 & l_3 \\
k_4 & k_3 & k_1 + j_1
\end{array} \right\}_q .
\end{align*}
\]

(6.10)

For 4 mutual positions of the couples of the stretched triangles in the graph of the \(q\)-12\(j\) coefficient of the first kind, there are 22 different orientations of the total (maximal) angular momenta. In seven cases, the expressions include triple sums, twice they are proportional to the stretched \(q\)-9\(j\) coefficients, and once to the \(q\)-6\(j\) coefficient. In the remaining 12 cases, double sums may be obtained. Otherwise, for 3 mutual positions of the stretched triangles in the \(q\)-12\(j\) coefficient of the second kind, only 9 different orientations of the total angular momenta are possible, and in 6 cases double sums appear, and in 3 cases the expressions are single sums (only once proportional to the \(q\)-6\(j\) coefficient).

Finally, the summation parameters \(z_1, z_3\) and \(u\) in (6.51) are fixed for \(j_1 - j_3 + k_1 + k_3 = 0\), as well as for \(x = j_3 - k_3 = j_1 + k_1\) in expansion (6.3a), in which cases 4 linearly dependent angular momenta appear (disconnected) on the Hamilton line of graph (6.1). In this case, two remaining separate sums (over \(z_2, z_4\)) are summable Saalschützian series \(3F_2[q, 1]\) [see Eqs. (B2a) and (B2b)], and the following expression may be derived:

\[
\begin{align*}
&= (-1)^{j_2 + j_4 - l_1 - k_1 - j_1 - l_4} \\
&\times \frac{q^{2j_1 k_1 + Z_{j_2 j_3 j_4} + Z_{k_2 k_3 k_4}} [2j_1]!![2k_1]!![2k_3]!! \nabla[j_3 k_2 l_2] \nabla[j_1 k_4 l_4] \nabla[k_1 j_4 l_4] \nabla[k_3 k_2 l_2] \nabla[k_3 k_4 l_3]}{[2j_3 + 1]! \nabla[j_1 l_1] \nabla[k_1 k_2 l_1] \nabla[j_1 k_4 l_4] \nabla[j_1 j_4 l_4] \nabla[k_3 k_2 l_2] \nabla[k_3 k_4 l_3]} .
\end{align*}
\]

(6.11)

As for (5.13), the virtually stretched triangles are seen only in the \(q\)-6\(j\) coefficients, which appear in expansion of the \(q\)-12\(j\) coefficient of the first kind with extreme parameters.

VII. CONCLUDING REMARKS

Using Dougall’s summation formula\(^{25}\) of the very well-poised \(4F_3(-1)\) series and its generalization for the \(q\)-factorial series we derived six new (independent) triple sum expressions for \(9j\) coefficients of SU(2) and seven independent triple sum expressions for \(q\)-9\(j\) coefficients of the quantum algebra \(u_q(2)\). Rearrangement technique of the multiple sum expressions give several classes of double sum expressions for the stretched \(9j\) coefficients of SU(2) and \(u_q(2)\), related to the Kampé de Fériet functions. Hence the new multiple basic hypergeometric series are introduced (cf. Ref. \(^{55}\)). Otherwise, Dougall’s summation formula\(^{25}\) of the very well-poised \(5F_4(1)\) series and the transformation formula\(^{13}\) of the very well-poised \(6F_5(-1)\) series, allowed us to eliminate the factorial sums weighted with factor \((2j + 1)\) and the very well-poised series in the traditional expressions of the \(12j\) coefficients of SU(2) in terms of \(6j\) coefficients, as well as in the expressions of the \(q\)-12\(j\) coefficients of \(u_q(2)\) weighted with factor \([2j + 1]\). Although the obtained generic expressions for the \(q\)-12\(j\) coefficients of the first kind include fivefold sums and the generic expressions for the \(q\)-12\(j\) coefficients of the second kind contain fourfold sums, the stretched and doubly stretched \(q\)-12\(j\) coefficients of both become considerably simpler and present some new versions of the triple, double, or...
single basic hypergeometric series, but more unusual and complicated than appear in
the manuals of the angular momentum theory. The single term expressions for the $q$-$12j$
coefficients of the both kinds also embrace the virtually stretched cases with four extreme linearly
dependent disconnected angular momenta parameters appearing on the Hamilton lines of
graphs (5.2) or (7.1).

The symmetry properties and variety of expressions for the $12j$ and $q$-$12j$ coefficients
of the both kinds may inspire new possibilities of rearrangement of the multiple and usual
classical and basic hypergeometric series. Expressions for $3nj$ coefficients of SU(2) (with
$n > 4$) in terms of $6j$ coefficients also may be rearranged using, e.g., Watson’s transformation
formula of the very well-poised $\gamma F_6(1)$ series [see Eq. (2.5.1) of Ref. 36 or Eq. (6.10) of Ref.
56]. One may also use Eq. (5.3) of Ref. 34 for rearrangement of the very well-poised
$8F_7(1)$ series, or Eqs. (A5) of Ref. 57 for rearrangement of the very well-poised
$9F_8(1)$ series. The number of sums in two last cases increases with elimination of the sums dependent on
$(2j + 1)$.

APPENDIX A: GENERALIZATION OF DOUGALL’S $4F_3(−1)$ AND $5F_4(1)$
SUMMATION FORMULAS FOR $q$-FACTORIAL SERIES

We present here 3 summation formulas of the twisted very well-poised $q$-factorial series,
as generalizations of Dougall’s summation formula (2.3.4.8) of Ref. 28 of the very well-poised
$4F_3(−1)$ series. In particular, equation

$$
\sum_j (-1)^{p_1+j+1}q^{j(j+1)-p_1(p_1+1)}[2j+1][j-p_1-1]!
\frac{[p_1+j+1][p_2-j][p_2+j+1][p_3-j][p_3+j+1]!}{[p_1+p_2+1][p_1+p_3+1][p_2+p_3+1]!}
= \frac{q^{-(p_1+p_2+1)(p_1+p_3+1)}}{[p_1+p_2+1][p_1+p_3+1][p_2+p_3+1]!}.
$$

(A1)

(valid when the summation parameter $j$ is restricted naturally by the non-negative integer
values of the denominator factorial arguments) was derived by Ališauskas.

Two other summation formulas,

$$
\sum_j q^{j(j+1)-p_1(p_1+1)}[2j+1][j-p_1-1]![j-p_3-1]!
\frac{[p_1+j+1][p_2-j][p_2+j+1][p_3+j+1]!}{[p_1+p_2+1][p_1+p_3+1][p_2+p_3+1]!}
= \frac{q^{-(p_1+p_2+1)(p_1+p_3+1)}[-p_1-p_3-2]!}{[p_1+p_2+1][p_2+p_3+1]!}
$$

(A2)

and

$$
\sum_j (-1)^{p_2-j}q^{j(j+1)}[2j+1][j-p_1-1][j-p_3-1][-p_3-j-2]!
\frac{[p_1+j+1][p_2-j][p_2+j+1]!}{[p_1+p_2+1][p_2+j+1]!}
= \frac{q^{p_1(p_1+1)-(p_1+p_2+1)(p_1+p_3+1)}}{[p_1+p_2+1][p_2+p_3+1]!}
$$

(A3)

and

$$
\sum_j (-1)^{p_2-j}q^{j(j+1)}[2j+1][j-p_1-1][j-p_3-1][p_3-j-2]![p_2-p_3-2]!
\frac{[p_1+j+1][p_2-j][p_2+j+1]!}{[p_1+p_2+1][p_2+j+1]!}
= \frac{q^{p_1(p_1+1)-(p_1+p_2+1)(p_1+p_3+1)}}{[p_1+p_2+1][p_2+j+1]!}
$$

(A4)

correspond to the analytical continuation of (A1).
Besides we present here two summation formulas of the very well-poised $q$-factorial series,

$$\sum_j \frac{[2j+1][j-p_1-1][j-p_2-1]}{[p_1+j+1][p_2+j+1][p_3-j][p_3+j+1][p_4-j][p_4+j+1]} = \frac{[-p_1-p_2-2][p_1+p_2+p_3+p_4+2]}{[p_1+p_4+1][p_2+p_4+1][p_3+p_4+1][p_2+p_3+1][p_1+p_3+1]},$$

(A4a)

and

$$\sum_j \frac{(-1)^{p_1-j}[2j+1][j-p_1-1][j-p_2-1][j-p_3-1]}{[p_1+j+1][p_2+j+1][p_3+j+1][p_4-j][p_4+j+1]} = \frac{[-p_1-p_2-2][-p_2-p_3-2][-p_1-p_3-2]}{[p_1+p_4+1][p_2+p_4+1][p_3+p_4+1][-p_1-p_2-p_3-p_4-3]}.$$  

(A4b)

[Cf. Dougall’s summation theorem of special very well-poised hypergeometric series $_5F_4$(1) as
(2.3.4.5) of Ref. 28 and special very well-poised basic hypergeometric series $\phi_5$ as (2.4.2) of Ref. 36. Note that very well-poised basic hypergeometric series

$$2\text{Ref. 36. Note that very well-poised basic hypergeometric series}$$

in the CG (cf. Ref. 35) and $6j$ in terms of $j$ and $6j$ denominators of (6.4a) and (6.4b), respectively, for (A1) and (A2) may be obtained after cancelling some factorials in the numerators and denominators of (6.4a) and (6.4b), respectively, for $p_3 = -p_1 - 2$.

**APPENDIX B: CHU–VANDERMONDE, SAALSCHÜTZIAN, AND MINTON’S SUMMATION FORMULAS**

We present here the Chu–Vandermonde–Gauss–Heine summation formulas,\[3\[36\]

$$\sum_s \frac{q^{s(a+b+c)}}{[s]![b-s]![c-s]![a+s]} = \frac{q^{bc}[a+b+c]!}{[b]![c]![a+b]![a+c]!}.$$  

(B1a)

$$\sum_s \frac{(-1)^s q^{s(b+c-a-1) [a-s]}}{[s]![b-s]![c-s]!} = \frac{q^{bc}[a-b]![a-c]!}{[b]![c]![a-b-c]!}.$$  

(B1b)

for $a \geq b, c$,

$$\sum_s \frac{(-1)^s q^{s(b-a+c-1) [a-s]}}{[s]![b-s]![c-s]!} = (-1)^c q^{bc} \frac{[a-c]![b-a+c-1]!}{[c]![b]![b-a-1]!}.$$  

(B1c)

for $b > a \geq c$, and

$$\sum_s q^{s(a+b-c+2) [a-s]![b+s]!/[s]![c-s]!} = q^{(b+1)c} \frac{[a-c]![b]![a+b+1]!}{[c]![a+b-c+1]!},$$  

(B1d)
valid for finite $q$-factorial series and needed for rearrangements of Section IV.

Under condition $c + d = a + b + e$, we may use also the summation formulas of the balanced (Saalschützian) $3F_2[q, 1]$ series [cf. Refs. 28, 36],

$$
\sum_s \frac{(-1)^s[c + s][d - s]}{[s][a - s][b - s][c + s + 1]} = \frac{[c][d - a][d - b][e + d + 1]}{[a][b][a + e + 1][b + e + 1][e - c]}
$$

for $e - c \geq 0$ and

$$
\sum_s \frac{(-1)^s[c - s][d - s]}{[s][a - s][b - s][e - s + 1]} = \frac{[c - a][c - b][d - b][e - d]}{[a][b][e - c][e - d][e + 1]}
$$

for $e - c \geq 0$ and $e - d \geq 0$.

It is sometimes useful to implement the $q$-version of Minton’s summation formula or its inverse [cf. Eq. (1.9.6) of Ref. 38]:

$$
\sum_s \frac{(-1)^s q^{s(n - \sum_{i=1}^r m_i)}}{[s][n - s][S + s]} \prod_{j=1}^m (b_j + s)[q]_{m_j} = \frac{q^{S(\sum_{i=1}^r m_i - n)}}{(S)_n} \prod_{j=1}^m (b_j - S)[q]_{m_j},
$$

if $S + s \neq 0$ for $s = 0, 1, ..., n$, and $n \geq \sum_{i=1}^r m_i$;

$$
\sum_s \frac{(-1)^s q^{a+b+c-m}[c - s]}{[s][a - s][b - s][S - s + 1]} \prod_{j=1}^m [A_j - s] = (-1)^{a+b+c-m} q^{(S+1)(a+b+c-m)} \frac{(S - a)[S - b][S - c][S - c + 1]}{[S - c][S + 1]} \prod_{j=1}^m [S - A_j + 1],
$$

which is valid if $S - s + 1 \neq 0$ for $s = 0, 1, ..., \min(a, b)$ and $a + b - c - m \geq 0$. Note that the analytical continuation of the summation formulas (B3a) and (B3b) of the alternating series to related series with the fixed sign of all terms is impossible.

**APPENDIX C: REARRANGEMENT FORMULAS OF SOME DOUBLE SUMS AND KAMPÉ DE FÉRRIET FUNCTIONS**

Comparing the double finite $q$-factorial series that appear in the most symmetric expression (4.4b) for the stretched $q$-$9j$ coefficients with its counterparts in Eqs. (4.4c) and (4.3a)–(4.3d), respectively, and using single nonnegative integers $a, b, c, d, m, n, e - a, f - b$ (as the parameters restricting summation intervals in different situations) instead of the triangular linear combinations of angular momenta (after changing some summation parameters), the following rearrangement formulas may be written:

$$
\sum_{s, z} (-1)^{s+z} q^{s(b+c+e-m-n-1)-z(a+d+f-m-n+1)} [c + s][e - s][d + z][f - z] [a - s][b - s][n - s - z][m - a - b + s + z]
$$

$$
\frac{[e - a][a + c - m][d + f + 1][m + n - a - b]}{[a][m + n - a - b]} \sum_{s, z} [c + s][e - s][f - z] [e - a - s][z][b - z]
$$

34
\[\times (\frac{(-1)^{n+s} q^{n(b+c+1) - s(m+1) - z(a+b+c+d-m+1)}}{[d + f + 1 - z!] [n - s - z!] [a + c - m - n + s + z]!}
\]
\[= q^{a(c+e-n+1)+bn} \frac{[d!] [d + f + 1]! [e - a]! [c + e + 1]!}{[m + n - a - b]!}
\]
\[\times \sum_{s,z} \frac{(-1)^{a+s+z} q^{s(e-n-z) - z(b+d-m)} [a + c - s!] [f - z!] [m + n - s - z]!}{[s!] [a - s!] [m - s!] [c + e + 1 - s!] [z!] [b - z!] [n - z!] [d + f + 1 - z]!}
\]
\[= q^{n(c+1) - b(b+d-m) - (b+e-a)(c+e-n+1)} \frac{[d!] [d + f + 1]! [e - a]! [c + e + 1]!}{[m + n - a - b]!}
\]
\[\times \sum_{s,z} \frac{(-1)^{a+e+n+s} [c + e - a - s!] [c + e - m - s!] [f - b + z]!}{[s!] [a - s!] [c + e + 1 - s!] [b + c + e - m - n - s - z]!}
\]
\[\times q^{s(b+e-n-z)+z(b+c+d+e-m+1)} \frac{[z!] [b - z!] [n - b + z!] [d + f - b + 1 + z]!}{[b!] [m + n - a - b]!} \sum_{s,z} \frac{(-1)^{s+z} q^{s(e-z) - z(m-a+1)}}{[a - s!] [c + e - a + 1 + s]!}
\]
\[= q^{a(c+1) - n(a+d-m)} \frac{[e - a]! [c + e + 1]! [f - b]!}{[b!] [m + n - a - b]!} \sum_{s,z} \frac{(-1)^{a+b+s+z} [e - a + s]!}{[s!] [a - s]! [c + e - a + 1 + s]!}
\]
\[\times \frac{[c + s!] [a + b + d - m - s!] [f - n + z!] [d + n - z]!}{[a + b - d - m - s - z!] [z!] [n - z!] [f - b - n + z]!}
\]
\[= q^{a(c+e-m-n) - b(d+f-n+1)} \frac{[f - b]! [d + f + 1]!}{[m + n - a - b]!} \sum_{z,s} \frac{(-1)^{a+b+s+z} [e - a + s]!}{[a - s]! [c + e - a + s]!}
\]
\[\times \frac{[a + c - s!] [b + d - z!] [m + n - a - z!] q^{z(a+f-n-s) - s(c+e-m-n+1)}}{[n - a + s!] [z!] [b - z!] [d + f + 1 + z]! [m - s - z]!}
\]
\[= q^{a(c+e+1) - n(a+d+1) + (a+b-f)(b-m)} \frac{[c!] [d!] [f - b]! [d + f + 1]! [c + e + 1]!}{[b!] [m + n - a - b]!}
\]
\[\times \sum_{s,z} \frac{(-1)^{b-f+n+z} q^{s(a+b+c+d-m+z+1) - z(m-n-a-b)}}{[a - s!] [a + d + f - m - n - s!] [c + e + a + s + 1]!}
\]
\[\times \frac{[e - a + s!] [b + z!] [b + d - m + z]!}{[z!] [f - b - z!] [b - a - f + n + s + z!] [b + d + z + 1]!}
\]

We have only single terms in (C1a), (C1b), and (C1e) for \(n = 0\), as well as in (C1a) and (C1f) for \(m = 0\). The bizarre restrictions \(b + c + e - m - n \geq 0\) for (C1b) and (C1d) and \(a + d + f - m - n \geq 0\) for (C1d) and (C1g) also correspond to some triangular conditions with remaining double series summable for their limit values. Otherwise, restrictions \(c + e - m \geq 0\) in (C1d), \(a + b + d - m \geq 0\) in (C1e), or \(d + f - m \geq 0\) in (C1g) correspond to some sums of triangular conditions.

Using above derived expressions for the stretched \(q\)-9j coefficients, we may write in the notations (H1) the following rearrangement formulas for the \(q\)-generalizations of special Kampé de Fériet functions \(F_{1:1}^{0:3}, F_{1:1}^{0:3},\) and \(F_{1:1}^{1:2} Fraktur{C2a}.

\[
\begin{align*}
+F_{1:1}^{0:3} &\begin{bmatrix} b_1, b_2, -m, b'_1, b'_2, -n \end{bmatrix} \quad ; x_a, y_a; q \\
\end{align*}
\]
\[
= -F_{0:2}^{1:2} \begin{bmatrix} 1 - b_1 - b'_1 - m - n, -m - d + 1, -m, -d' - n + 1, -n \end{bmatrix} \quad ; x_b, y_b; q
\]

\]

35
\begin{align}
\times (-1)^{m+n}q^{m(b_2-b_1'-d'-m-1)+n(b_2' -b_1-d'-m-n+1)}
\times (b_1[q]_m(b_2[q]_m(b_1'[q]_n(b_2'[q]_n)
\times (d[q]_m(d'[q]_n(b_1 + b_1'[q]_m(n)) (m+n))
\times (-1)^{b_1' + b_2 + d + m - 1}q^{n(b_2' - d') + b_2(d + m - 2 - 1) + b_1'(2b_1 + b_1' + 2n - 1)}
\times \left(\frac{b_2'[q]_m(d-b_2[q]_m(-b_2'[q]_m(d'[q]_n)^{-1}}{b_2' - b_2 - 1} \right)_{q} \left[\begin{array}{c}
b_2' - b_1' - 1 \\
- b_1
\end{array}\right]_{q}
\times (-1)^{b_2' + b_2 + d + m + 1}q^{n(b_2' - d') + b_2(d + m - 2 - 1) + b_1'(2b_1 + b_1' + 2n - 1)}
\times \left(\frac{b_2'[q]_m(d-b_2[q]_m(-b_2'[q]_m(d'[q]_n)^{-1}}{b_2' - b_2 - 1} \right)_{q} \left[\begin{array}{c}
b_2' - b_1' - 1 \\
- b_1
\end{array}\right]_{q}
\end{align}

(C2b)

(C2c)

(C2d)

(C2e)

(C2f)

(C2g)

(C2h)

(C2i)
\[ \times q^{mb_2+nb'_2}(d-b_2|q)_m(b'_2-d'-n+1|q)_n \]
\[ \frac{(d|q)_m(-n-d'+1|q)_n}{(d'|q)_m(m+1|q)_n(-d-m-1|q)_n} \]
\[ = \pm F_{1:2}^{1:2}
\]
\[ \left[ \begin{array}{c}
 b_2-b_1-b'_1-n+1 \\
 b_2-d+d+1, m+1 \\
 -d-b_1-n, 1-b'_2-n
\end{array} \right]\]
\[ \times (-1)^{b_1+b_2+d+m+1} q^2(b_1-b'_1-n+1)(b_2-b'_2-d+1) q^2(b_2-d-m+1)+n(b'_2-d')+(b_2-b'_1+1) \]
\[ \times \left( \begin{array}{c}
 b_1+n-d \\
 b_2-b'_1-d+1
\end{array} \right) \left[ \begin{array}{c}
 -b_1-b'_1 \end{array} \right]^{-1}_q, \]
\[ \text{(C2j)} \]

where

\[ x_a = q^{b_2-b'_1-d-m+1} \quad \text{and} \quad y_a = q^{b'_2-b_1-d'-n+1}, \]
\[ x_b = q^{b_2-b'_1-d-m-n+1} \quad \text{and} \quad y_b = q^{b'_2-b_1-d'-m-n+1}, \]
\[ x_c = q^{b_2-b'_1-d-m-n+1} \quad \text{and} \quad y_c = q^{b_2-b_1+b'_2-d'-n+1}, \]
\[ x_d = q^{b_2-b'_1-d-m+1} \quad \text{and} \quad y_d = q^{-b_1+1}, \]
\[ x_e = q^{b_1+b'_1-b_2+n} \quad \text{and} \quad y_e = q^{d'-b'_1}, \]
\[ x_f = q^{b_1+b'_1-b_2} \quad \text{and} \quad y_f = q^{d'-b'_1-m}, \]
\[ x_g = q^{b_1+b'_1-b_2+n} \quad \text{and} \quad y_g = q^{b_1+b'_1-b'_2+m}, \]
\[ x_h = q^{-b_1+1} \quad \text{and} \quad y_h = q^{d-d'-b_1+b'_2-n}, \]
\[ x_j = q^{d'-b'_1} \quad \text{and} \quad y_j = q^{-b_1}, \]
\[ \text{and} \quad x_k = q^{d'-d+b_2-b'_1+1} \quad \text{and} \quad y_k = q^{-n-b_1+1}, \]

respectively, where only parameters \( m, n, -b_1, -b'_1 \) are apparently correlated with some triangular conditions. Special Kampé de Fériet functions \((C2a)\) and \((C2b)\) correspond, respectively, to the inverse and direct sums in \((4.3a)\), when function \((C2c)\) corresponds to the direct sum in \((4.3b)\), function \((C2d)\) corresponds to the inverse sum in \((4.3c)\), and functions \((C2e)\) and \((C2f)\) correspond, respectively, to the inverse and direct sums in \((4.3d)\). Further, functions \((C2g)\), \((C2h)\), and \((C2i)\) correspond, respectively, to the sums that appeared in \((4.4a)\) and \((4.4b)\), as well as in \((C1a)\) and \((C1b)\). The two last functions \((C2j)\) and \((C2k)\) are derived from \((C2a)\) and the direct sum in \((4.3c)\), respectively, after using the symmetry of \( F_{1:1}^{1:2} \) function in \((C2g)\) with fixed \( b_1 \) and \( b'_1 \) under interchange of two sets,

\[ b_2, m, d, d' \quad \text{and} \quad b'_2, n, b'_2-d'-n+1, b_2-d-m+1, \]

together with transition to \( F_{1:1}^{1:2} \) and \( q \to q^{-1} \).

Finiteness of the Kampé de Fériet series \((C2a)\) is ensured either by the non-negative integer values of \( m \) and \( n \), or by the non-positive integer values of \( b_1 \) and \( b'_1 \), or by some their couples \((m \text{ and } -b'_1, \text{ or } -b_1 \text{ and } n)\). The both summation parameters are also restricted by the non-negative integer values of \( m \) and \( n \) in series \((C2a), (C2c)-(C2g), \text{ and } (C2i)\), as well as by the non-negative values of \( m \) and \( -b'_1 \) in series \((C2a)\) and \((C2d)\), or by the non-negative values of \( n \) and \( d-b_2-1 \) in series \((C2c), (C2h), (C2i), \text{ and } (C2k)\). Furthermore, the parameter \( b'_1 \) with the non-positive integer values restricts the double series in \((C2h)\) and \((C2i)\), as well as separate series in \((C2a), (C2c), \text{ and } (C2i)\). Series \((C2i)\) are finite for the non-positive integer values of single parameter \( b_1 \), as well as \((C2a), (C2h), \text{ and } (C2k)\) for the non-positive integer values of \( b_2-b_1-b'_1-n+1 \). Hence, special Kampé de Fériet
functions \((C2a)\)–\((C2k)\) are summable for \(b_1 = 0\), or \(b'_1 = 0\), or \(b_2 - b_1 - b'_1 - n + 1 = 0\) (i.e., for \(c = b_1 - b'_1 = b_2 - n + 1\)) and, taking into account the symmetry of \((C2a)\) with respect to the interchange of two sets, \(b_1, b_2, -m; d'\) and \(b'_1, b'_2, -n; d\), for \(b'_2 - b'_1 - b_1 - m + 1 = 0\) (i.e., for \(c = b_1 - b'_1 = b'_2 - m + 1\)). The subscripts of the \(q\)-Pochhammer symbols in the proportionality coefficients are accepted as non-negative integers, when they perform the restricting role or correspond to definite non-negative linear combinations of \(9j\) parameters. Otherwise, for the negative integer subscripts \((-n)\) the following substitution may be used:

\[
\frac{(\alpha + n|q)(-n)}{(\beta + n|q)(-n)} \rightarrow \frac{(\beta|q)_n}{(\alpha|q)_n}.
\]

Hence in the \(q = 1\) case up to 5 or 6 parameters may be complex in the rearrangement formulas \((C2a)\)–\((C2j)\) of special Kampé de Fériet series, with exception of \((C2c), (C2i), (C2k)\), and \((C2k)\). In these three cases, which ensure the summability of the remaining series for \(b_1 - b'_1 = b'_2 - m + 1\), only \(b'_2\) and \(d\) definitely may be taken the complex numbers. Extension problem to infinite series is open, since it is impossible to ensure the non-negative values of the all denominator arguments of \(1:3\) series \([\text{cf. Eqs. (5.9c) and (5.9d) of Ref. 34, or Eqs. (3.7) and (3.14) of Ref. 59}]\), formulas \((C2a)\)–\((C2j)\) of special Kampé de Fériet series, with exception of \((C2c), (C2i), (C2k)\). In these three cases, which ensure the summability of the remaining series for \(b_1 - b'_1 = b'_2 - m + 1\), only \(b'_2\) and \(d\) definitely may be replaced by the complex numbers. Analogically, the double finite series \((C2b), (C2e), (C2g)\)–\((C2j), (C2k)\) may be transformed into standard functions \(\Phi_{A:B}^{ab}\) after substituting \(\mp \rightarrow \pm\) in the superscripts of \(\pm F_{C:D}^{ab}\) series and \(q^{-1} \rightarrow q\) in the corresponding \(q\)-phases.

Using the substitution \((4.8)\) and different strategy for each mutual relation, we may transform the double finite series \((C2a), (C2d), (C2e), (C2g)\)–\((C2i), (C2k)\) into standard functions \(\Phi_{A:B}^{ab}\). Preliminary in these situations only \(q^{b'_2}\) and \(q^{d'}\) can always be replaced by the complex numbers. Analogically, the double finite series \((C2b), (C2e), (C2g)\)–\((C2j)\) may be transformed into standard functions \(\Phi_{A:B}^{ab}\) after substituting \(\mp \rightarrow \pm\) in the superscripts of \(\pm F_{C:D}^{ab}\) series and \(q^{-1} \rightarrow q\) in the corresponding \(q\)-phases.

Note, that the summable Kampé de Fériet series \(F_{1:1}^{0,3}\) (that appeared in Refs. 16, 17) cannot be embedded into above presented versions of \(F_{1:1}^{0,3}\) series, or be derived from the expressions of the stretched \(9j\) coefficients given in Sec. IV. Actually, expansion \((4.1a)\) does not simplify under condition \(a + b - c = 0\), but the \(\pm F_{1:1}^{0,3}\) series \([34, 54]\) appear from expression \((3.1a)\) in the doubly stretched case with \(a + b - c = 0\) and \(g = k + h\), when \(9j\) coefficients are proportional to the Clebsch–Gordan coefficients. In this particular case of expression \((3.1a)\), we may also identify quintuplet of factorials under the summation sign in the numerator and denominator and reexpend it using the Chu–Vandermonde summation formulas given in Appendix B. As result of two alternative summations we obtain a \(3F_2[\cdot \cdot \cdot ; q, x]\) series, which is completely summable for \(k = b + d\).

Special cases of \(\pm F_{0:2}^{3,2}\) functions should be mentioned in context of the double sums (with 7 independent parameters) that appear in the extreme \(u_q(3)\) canonical seed isofactors \([25, 50]\) and as definite matrix elements of the \(u_q(3)\) algebra (see Section 5 of Ref. 34) and are related to some \(q\)-factorial series resembling the very well-poised \(9\phi_8\) basic hypergeometric series [which for \(q = 1\) are equivalent to the very well-poised \(8\phi_7(-1)\) classical hypergeometric series]. Further, the extreme denominator (normalization) functions of the \(u_q(3)\) and \(SU(3)\) canonical tensor operators (with 5 independent parameters) may be expressed in terms of \(\pm F_{1:2}^{1,3}\) functions [cf. Eqs. (5.9c) and (5.9d) of Ref. 34] or Eqs. (3.7) and (3.14) of Ref. 23, or in terms of \(\pm F_{2:1}^{2,2}\) functions [see Eq. (2.8) and Section II of Ref. 60], taking into account...
the definite controversies of the \( q \)-extension from the classical SU(3) case]. Besides, the summation possibilities for these special Kampé de Fériet functions are elementary.

**APPENDIX D: CLEBSCH–GORDAN COEFFICIENTS OF SU(2) AND \( U_q(2) \) AND TWISTED VERY WELL-POISED SERIES**

The very well-poised \( \phi F_5(-1) \) and \( \phi F_6(1) \) series appear in context of the Clebsch–Gordan and \( 6j \) coefficients of SU(2) as presented in Ref. 11 (see also Ref. 3), as well as their \( q \)-analog in the CG (cf. Ref. 35, where the dual Hahn \( q \)-polynomials are considered) and \( 6j \) coefficients (cf. Ref. 54) of \( u_q(2) \).

We deduce here a new expression for the Clebsch–Gordan coefficients of SU(2) and \( u_q(2) \) directly from the recoupling relation:

\[
\begin{align*}
&\left[ \frac{(j_2 + m_2)/2}{j_1/m_1} \frac{(j_2 - m_2)/2}{j_2/m_2} \right]_q \left[ \begin{array}{ccc}
j_1 & j_2 & j \\
m_1 & m_2 & m \\
\end{array} \right]_q \\
= & \sum_x (-1)^{j_1 + j_2 + j}[2x + 1][2j_2 + 1]^{1/2} \left\{ \frac{(j_2 + m_2)/2}{j_2/m_2} \frac{(j_2 - m_2)/2}{j_2/m_2} \right\}_q \\
&\times \left[ \begin{array}{ccc}
j_1 & (j_2 + m_2)/2 & x \\
m_1 & (j_2 + m_2)/2 & m' \\
\end{array} \right]_q \left[ \begin{array}{ccc}
x & (j_2 - m_2)/2 & j \\
m' & (j_2 - m_2)/2 & m \\
\end{array} \right]_q,
\end{align*}
\]

where \( m' = m_1 + \frac{1}{2}(j_2 + m_2) = m + \frac{1}{2}(j_2 - m_2) \). Inserting the stretched \( 6j \) and extreme CG coefficients expressed without sums, we obtain the following expression,

\[
\begin{align*}
&\left[ \begin{array}{ccc}
j_1 & j_2 & j \\
m_1 & m_2 & m \\
\end{array} \right]_q = \nabla[j_2,j_1,j] \left( \frac{[2j + 1][j_2 + m_2][j_2 - m_2][j_1 - m_1][j - m]}{[j_1 + m_1][j + m]} \right)^{1/2} \\
&\times q^{j_2(j_2 + 1) - j_1(j_1 + 1) - j(j + 1)/2 - m_1j_2 - (j_2 + m_2)(j_2 + m_2 + 2)/4} \\
&\times \sum_x (-1)^{j_1 + j_2 + m_2}/2 - x q^{x(x + 1)/2} x^{[2j + 1][x + m']}! \\
&\times \nabla^2[\frac{1}{2}(j_2 + m_2), j_1, x] \nabla^2[\frac{1}{2}(j_2 - m_2), j, x][x - m']!.
\end{align*}
\]

where the right-hand side is related to the left-hand side of Eq. (6.4a) with parameters

\[
\begin{align*}
p_1 &= \frac{1}{2}(j_2 - m_2) - j - 1, \quad p_2 = -m' - 1, \quad p_3 = \frac{1}{2}(j_2 + m_2) - j_1 - 1, \\
p_4 &= \frac{1}{2}(j_2 - m_2) + j, \quad p_5 = \frac{1}{2}(j_2 + m_2) + j_1.
\end{align*}
\]

Expression (D2) is invariant under 12 relations of the Regge symmetry, corresponding to the permutations in the sets \( p_1, p_2, p_3 \) or \( p_4, p_5 \). After expressing the CG coefficient of \( u_q(2) \) by means of Eq. (5.17) of Ref. 41 [which after some cyclic permutation, is for \( q = 1 \), related to Eq. (13.1c) of Ref. 3], and using the symmetry relation (4.13) of Ref. 38 (which allows one to interchange the parameters \( j_2, m_2 \) and \( j, -m \) in the CG coefficients), we derive our Eq. (6.4a) straightforwardly. The remaining very well-poised series with different numerator and denominator distributions of \( q \)-factorial arguments [e.g., the non-alternating left-hand side of (6.4b) or the non-alternating right-hand side of Eq. (5.3) of Ref. 41 with \( p_3 = -p_3 - 2 \), and their other analytical continuations] are not related to the Clebsch–Gordan coefficients of \( u_q(2) \), although sometimes they may be related to the CG coefficients of \( u_q(1,1) \).
REFERENCES

1. A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, N. J., 1957).
2. A. P. Jucys, I. B. Levinson, and V. V. Vanagas, *Mathematical Apparatus of the Theory of Angular Momentum* (Israel Program for Scientific Translations, Jerusalem, 1962) [Russ. Original, Gospolitnauchizdat, Vilnius, 1960].
3. A. P. Jucys and A. A. Bandzaitis, *Theory of Angular Momentum in Quantum Mechanics*, 2nd ed. (Mokslas, Vilnius, 1977) (in Russian).
4. L. C. Biedenharn and J. D. Louck, *Angular Momentum in Quantum Physics, Theory and Applications*, Encyclopedia of Mathematics and its Applications (Addison–Wesley, Reading, 1981), Vol. 8.
5. L. C. Biedenharn and J. D. Louck, *The Racah–Wigner Algebra in Quantum Theory*, Encyclopedia of Mathematics and its Applications (Addison–Wesley, Reading, 1981), Vol. 9.
6. S. Ališauskas and A. P. Jucys, J. Math. Phys. 12, 594 (1971); Err., ibid, 13, 575 (1972).
7. S. Ališauskas and A. P. Jucys, J. Math. Phys. 10, 2227 (1969).
8. A. P. Jucys and A. A. Bandzaitis, *Theory of Angular Momentum in Quantum Mechanics* (Mintis, Vilnius, 1965, in Russian). Note the misleading reference (concerning the triple sum formula for the 9j coefficients) to this edition in Ref. [4].
9. D. Q. Zhao and R. N. Zare, Molec. Phys. 65, 1263 (1988).
10. K. Srinivasa Rao, V. Rajeswary, and C. B. Chiu, Comput. Phys. Commun. 56, 231 (1989).
11. S. T. Lai and Y. N. Chiu, Comput. Phys. Commun. 70, 544 (1992).
12. C. C. J. Roothaan, Intern. J. Quant. Chem., Symp. S27, 13 (1993).
13. K. Srinivasa Rao and V. Rajeswary, J. Phys. A: Math. Gen. 21, 4255 (1988).
14. K. Srinivasa Rao, S. N. Pitre, and J. Van der Jeugt, Rev. Mex. Fiz. 42, 179 (1996).
15. K. Srinivasa Rao and J. Van der Jeugt, J. Phys. A: Math. Gen. 27, 3083 (1994).
16. J. Van der Jeugt, S. N. Pitre, and K. Srinivasa Rao, J. Phys. A: Math. Gen. 27, 5251 (1994).
17. S. N. Pitre and J. Van der Jeugt, J. Math. Anal. Appl. 202, 121 (1996).
18. K. Srinivasa Rao, in *Special Functions and Differential Equations*, Proceed. of the Workshop (WSSF97), Madras, 1997, Eds. K. Srinivasa Rao, R. Jagannathan, G. Vanden Berghe, and J. Van der Jeugt (Allied Publ., New Delhi, 1998), p. 165.
19. J. Van der Jeugt, S. N. Pitre, and K. Srinivasa Rao, in *Special Functions and Differential Equations*, Proceed. of the Workshop (WSSF97), Madras, 1997, Eds. K. Srinivasa Rao et al (Allied Publ., New Delhi, 1998), p. 171.
20. K. Srinivasa Rao and J. Van der Jeugt, Intern. J. Theor. Phys. 37, 891 (1998).
21. A. C. T. Wu, J. Math. Phys. 14, 1222 (1973).
22. K. Srinivasa Rao and V. Rajeswary, J. Math. Phys. 30, 1016 (1989).
23. M. Nomura, J. Phys. Soc. Jpn. 58, 2677 (1989).
24. S. Ališauskas, Liet. Fiz. Rink. [Litov. Fiz. Sb.] 13, 829 (1973) (in Russian).
25. H. Rosengren, J. Math. Phys. 39, 6730 (1998).
26. H. Rosengren, *Multivariable Orthogonal Polynomials as Coupling Coefficients for Lie and Quantum Algebra Representations*, Lund University, Doctoral Theses in Math. Sci. 1999:2.
27. H. Rosengren, J. Math. Phys. 40, 6689 (1999).
28. L. J. Slater, *Generalized Hypergeometric Series*, (Cambridge U. P., Cambridge, 1966).
29. M. Nomura, J. Math. Phys. 30, 2397 (1989).
30. M. Nomura, J. Phys. Soc. Jpn. 58, 2694 (1989).
31. M. Nomura, J. Phys. Soc. Jpn. 59, 3851 (1990).
32. M. Nomura, J. Phys. Soc. Jpn. 60, 1906 (1991).
33. Yu. F. Smirnov, V. N. Tolstoy, and Yu. I. Kharitonov, Preprint LIYaF No 1665, Leningrad, 1990; Yad. Fiz. 55, 2863 (1992) [Sov. J. Nucl. Phys. 55, 1599 (1992)].
34. S. Ališauskas, J. Phys. A: Math. Gen. 30, 4615 (1997). Note that the degree of $q$ in the r.h.s. of Eq. (4.11) should be changed to opposite.
35. R. Álvarez-Nodarse and Yu. F. Smirnov, J. Phys. A: Math. Gen. 29, 1435 (1996).
36. G. Gasper and M. Rahman, Basic Hypergeometric Series, Vol. 35, Encyclopedia of Mathematics and Its Applications, edited by G.-C. Rota (Cambridge U. P., Cambridge, 1990).
37. A. A. Bandzaitis, K. P. Žukauskas, A. J. Matulis, and A. P. Jucys, Liet. Fiz. Rink. [Litov. Fiz. Sb.] 4, 35 (1964) (in Russian).
38. Yu. F. Smirnov, V. N. Tolstoy, and Yu. I. Kharitonov, Yad. Fiz. 53, 959 (1991) [Sov. J. Nucl. Phys. 53, 593 (1991)].
39. R. M. Asherova, Yu. F. Smirnov, and V. N. Tolstoy, Yad. Fiz. 59, 1859 (1996) [Phys. Atom. Nucl. 59, 1795 (1996)]; Czech. J. Phys. 46, 127 (1996).
40. G. Racah, Phys. Rev. 62, 438 (1942).
41. S. Ališauskas, Liet. Fiz. Rink. 14, 545 (1974) [Sov. Phys. Collection (Litov. Fiz. Sb.) 14, (4), 1 (1974)].
42. P. W. Karlsson, J. Math. Phys. 12, 270 (1971).
43. S. Ališauskas, A.-A. A. Jucys, and A. P. Jucys, J. Math. Phys. 13, 1329 (1972).
44. S. Ališauskas and Yu. F. Smirnov, J. Phys. A: Math. Gen. 27, 5925 (1994).
45. V. A. Groza, I. I. Kachurik, and A. U. Klimyk, J. Math. Phys. 31, 2769 (1990).
46. Yu. F. Smirnov, V. N. Tolstoy, and Yu. I. Kharitonov, Yad. Fiz. 53, 959 (1991) [Sov. J. Nucl. Phys. 53, 593 (1991)].
47. R. T. Sharp, Nucl. Phys. A 95, 222 (1967).
48. J. Kampé de Fériet, C.R. Acad. Sc. Paris 173, 489 (1921).
49. I. B. Levinson and V. V. Vanagas, Opt. Spektrosk. 2, 10 (1957).
50. J. P. Elliott and B. H. Flowers, Proc. R. Soc. London, Ser. A 229, 536 (1955).
51. V. V. Vanagas and J. V. Čiplys, Trudy AN Lit. SSR, Ser. B 3 (15), 17 (1958).
52. H. A. Jahn and J. Hope, Phys. Rev. 93, 318 (1954).
53. R. J. Ord-Smith, Phys. Rev. 94, 1227 (1954).
54. H. Rueg, J. Math. Phys. 31, 1085 (1990).
55. V. K. Dobrev, A. D. Mitov, and P. Truini, J. Math. Phys. 41, 7752 (2000). Note that the new summation formula (5.5) of the double $q$-hypergeometric functions [derived in frames of $u_q(3)]$ also corresponds to a double series appearing in the (triply) stretched $q$-9$j$ coefficient as presented by Eq. (1.34); and finally related to the stretched $q$-6$j$ coefficient.
56. M. A. Lohe and L. C. Biedenharn, SIAM J. Math. Anal. 25, 218 (1994).
57. S. Ališauskas, J. Math. Phys. 33, 1983 (1992).
58. S. Ališauskas and J. P. Draayer, J. Phys. A: Math. Gen. 31, 7461 (1998). Note, that $(q|q)_k$ in Eq. (2.11) should be corrected to $(1|q)_k$, as well as in definitions of Refs. 34 and 35.
59. S. Ališauskas, J. Math. Phys. 37, 5719 (1996).
60. S. Ališauskas, J. Math. Phys. 40, 5939 (1999). Note, that the term “$+ks$” in the denominator of the last row of Eq. (1.8) should be written as “$+k+s$” and the all terms in the first row of the r.h.s. of Eq. (1.12) should be in exponent of $q$. 

41