ALMOST ALL STANDARD LAGRANGIAN TORI IN $\mathbb{C}^n$ ARE NOT HAMILTONIAN VOLUME MINIMIZING

HIROSHI IRIYEH AND HAJIME ONO

Abstract. In 1993, Y.-G. Oh proposed a problem whether standard Lagrangian tori in $\mathbb{C}^n$ are volume minimizing under Hamiltonian isotopies of $\mathbb{C}^n$. In this article, we prove that most of them do not have such property if the dimension $n$ is greater than two. We also discuss the existence of Hamiltonian non-volume minimizing Lagrangian torus orbits of compact toric Kähler manifolds.

1. Introduction

In his two famous papers [6] and [7], Y.-G. Oh introduced the notions of Hamiltonian minimality, Hamiltonian stability and Hamiltonian volume minimizing of Lagrangian submanifolds. First of all, let us review these definitions. Let $(M, \omega, J)$ be a Kähler manifold. A submanifold $i : L \to M$ is called Lagrangian if $i^* \omega = 0$ and $\dim_{\mathbb{R}} L = (1/2) \dim_{\mathbb{R}} M$. This condition is equivalent to the existence of an orthogonal decomposition

$$T_{i(p)} M = i_* T_p L \oplus J(i_* T_p L)$$

for any $p \in L$. Throughout this article all Lagrangian submanifolds are assumed to be connected, embedded, closed (i.e., compact and without boundary) and equipped with the induced Riemannian metric from the ambient manifold $M$. We denote by $\text{Vol}(L)$ the volume of $L$ with respect to the metric. Then we have the linear isomorphism defined by

$$\Gamma(T^\perp L) \ni V \mapsto \alpha_V := i^* (\omega(V, \cdot)) \in \Omega^1(L),$$

where $T^\perp L$ denotes the normal bundle of the embedding $i$ and $\Omega^1(L)$ the set of all one-forms on $L$.

A variational vector field $V \in \Gamma(T^\perp L)$ of $i$ is called a Hamiltonian variation if $\alpha_V$ is exact. It implies that the infinitesimal deformation of $i$ with the vector field $V$ preserves the Lagrangian constraint.

2000 Mathematics Subject Classification. Primary 53D12; Secondary 53D10.

Key words and phrases. Lagrangian torus; toric manifold; Hamiltonian stability; Hamiltonian volume minimization.
Definition 1 (Oh [7]). A Lagrangian embedding \( i : L \to (M, \omega, J) \) is said to be Hamiltonian minimal (H-minimal) if it satisfies
\[
\frac{d}{dt} \operatorname{Vol}(i_t(L)) \bigg|_{t=0} = 0
\]
for any smooth deformation \( \{i_t\}_{-\varepsilon < t < \varepsilon} \) of \( i = i_0 \) with a Hamiltonian variation.

We can easily check, using the first variation formula (see [7, p. 178]), that \( i : L \to M \) is H-minimal if and only if the equation \( \delta \alpha_H = 0 \) holds on \( L \), where \( \delta \) and \( H \) are the codifferential operator on \( L \) and the mean curvature vector of \( L \), respectively. Next, we explain the notion of Hamiltonian stability of H-minimal Lagrangian submanifolds.

Definition 2 (Oh [7, 6]). Suppose that a Lagrangian embedding \( i : L \to (M, \omega, J) \) is H-minimal. Then \( i \) (or Lagrangian submanifold \( L \)) is said to be Hamiltonian stable (H-stable) if it satisfies
\[
\frac{d^2}{dt^2} \operatorname{Vol}(i_t(L)) \bigg|_{t=0} \geq 0
\]
for any smooth deformation \( \{i_t\}_{-\varepsilon < t < \varepsilon} \) of \( i = i_0 \) with a Hamiltonian variation.

Let us now consider the linear complex space \( \mathbb{C}^n \) endowed with the standard symplectic structure \( \omega_0 := dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n \) and the standard complex structure. The Lagrangian torus
\[
T(b_1, \ldots, b_n) := S^1(b_1) \times \cdots \times S^1(b_n) \subset \mathbb{C}^n
\]
is a typical example of Lagrangian submanifolds of \( \mathbb{C}^n \) and called a standard or elementary torus. Here \( S^1(b) \subset \mathbb{R}^2 \cong \mathbb{C} \) denotes the boundary of a round disc with area \( b \) centered at the origin, i.e., the radius of \( S^1(b) \) is \( \sqrt{b/\pi} \). Using his second variation formula [7, Theorem 3.4], Oh proved the following

Theorem 3 ([7], Theorem 4.1). The torus \( T(b_1, \ldots, b_n) \subset \mathbb{C}^n \) is an H-minimal and H-stable Lagrangian submanifold.

Based on this result, Oh proposed the following conjecture.

Conjecture 4 (Oh [7], p. 192). The Lagrangian torus \( T(b_1, \ldots, b_n) \) in \( \mathbb{C}^n \) satisfies
\[
\operatorname{Vol}(\phi(T(b_1, \ldots, b_n))) \geq \operatorname{Vol}(T(b_1, \ldots, b_n))
\]
for any \( \phi \in \operatorname{Ham}_c(\mathbb{C}^n, \omega_0) \).
A Lagrangian submanifold of $M$ with this property is said to be Hamiltonian volume minimizing. A symplectic diffeomorphism $\phi$ of $(M, \omega)$ is called Hamiltonian if $\phi$ is the time-one map of the flow $\{\phi^t_H\}$, $\phi^0_H = \text{id}_M$, of the Hamiltonian vector field $X_H$ defined by a compactly supported Hamiltonian function $H \in C^\infty([0,1] \times M)$. The isotopy $\{\phi^t_H\}_{0 \leq t \leq 1}$ is called a Hamiltonian isotopy of $M$. We denote the set of all compactly supported Hamiltonian diffeomorphisms of $M$ by $\text{Ham}_c(M, \omega)$. Note that a (time-independent) Hamiltonian vector field on $M$ gives rise to a Hamiltonian variation of a Lagrangian embedding $i : L \to M$.

Conjecture 4 is false for $n \geq 3$. Indeed, C. Viterbo has already pointed out that $T(1,2,2)$ and $T(1,2,3)$ are Hamiltonian isotopic, but the first has volume $16\pi^{3/2}$ and the second $8\sqrt{6}\pi^{3/2}$ (see [10, p. 419]). Therefore, the second one is not Hamiltonian volume minimizing. This fact is a consequence of a result by Chekanov [3, Theorem A] concerning symplectic topology of the Lagrangian tori (see Section 2).

The first result of this article is that almost all standard tori are not Hamiltonian volume minimizing. Before we state it we prepare some notations. Consider the standard action of the complex torus $(\mathbb{C}^\times)^n$ on $\mathbb{C}^n$. Then the action by the real torus $T^n \subset (\mathbb{C}^\times)^n$ on it is Hamiltonian and its moment map $\mu_0 : \mathbb{C}^n \cong \mathbb{R}^{2n} \to (\mathbb{R}_{\geq 0})^n$ is given by

$$\mu_0(x_1, \ldots, x_n, y_1, \ldots, y_n) = \left(\frac{1}{2}(x_1^2 + y_1^2), \ldots, \frac{1}{2}(x_n^2 + y_n^2)\right).$$

Notice that a standard torus $T(b_1, \ldots, b_n)$ in $\mathbb{C}^n$ is represented by $\mu_0^{-1}(b_1/2\pi, \ldots, b_n/2\pi)$.

**Theorem 5.** Suppose $n \geq 3$ and we denote by $D_n$ the open dense subset of $(\mathbb{R}_{> 0})^n$ defined as follows:

$$\left\{(a_1, \ldots, a_n) \in (\mathbb{R}_{> 0})^n \mid \begin{array}{ll} \text{There exists } (i, j, k) \in \mathbb{Z}^3 \text{ such that} \\
1 \leq i < j < k \leq n \text{ and} \\
(a_i - a_j)(a_j - a_k)(a_k - a_i) \neq 0 \end{array} \right\}.$$

Then for any $a = (a_1, \ldots, a_n) \in D_n$ there exists $a' \in (\mathbb{R}_{> 0})^n$ satisfying the following two properties:

1. $\phi(\mu_0^{-1}(a)) = \mu_0^{-1}(a')$ for some $\phi \in \text{Ham}_c(\mathbb{C}^n, \omega_0)$.
2. $\text{Vol}(\mu_0^{-1}(a)) > \text{Vol}(\mu_0^{-1}(a'))$.

In particular, a Lagrangian torus $\mu_0^{-1}(a)$ is not Hamiltonian volume minimizing in $(\mathbb{C}^n, \omega_0)$.

We use the Chekanov’s theorem to prove this result (see Section 2).

In general, Darboux’s theorem says that any point in a symplectic manifold $(M, \omega)$ possesses a neighbourhood which is isomorphic to a
neighbourhood of the origin of \((\mathbb{C}^n, \omega_0)\). Then Chekanov’s theorem ensures any symplectic manifold the existence of a pair of Lagrangian tori which are mutually Hamiltonian isotopic and not intersect. Furthermore, in the class of compact toric symplectic manifolds, we can regard Theorem 5 as a local model of a \(T^n\)-fixed point of such a manifold. This observation yields an interesting application. From now on, let \((M, \omega, J)\) be a complex \(n\)-dimensional compact toric Kähler manifold, i.e., \(M\) admits an effective holomorphic action of the real torus \(T^n\) which is Hamiltonian with respect to the Kähler form \(\omega\). Its moment map is denoted by \(\mu: M \to \Delta = \mu(M) \subset \mathbb{R}^n\). We may assume, without loss of generalities, that the moment polytope \(\Delta\) satisfies
\[
\Delta = \{ a \in (\mathbb{R}_{\geq 0})^n \mid l_r(a) := \langle a, \mu_r \rangle - \lambda_r \geq 0, \lambda_r < 0, r = n+1, \ldots, d \},
\]
where each \(\mu_r\) is a primitive element of the lattice \(\mathbb{Z}^n \subset \mathbb{R}^n\) and inward-pointing normal to the \(r\)-th \((n-1)\)-dimensional face of \(\Delta\). The interior of \(\Delta\) is denoted by \(\Delta^0\). It is known that each fibre \(\mu^{-1}(b), b \in \Delta^0\), is a Lagrangian torus and H-minimal (see [8, Proposition 3.1]). Then we obtain

**Theorem 6.** Let \((M, \omega, J)\) be a compact toric Kähler manifold equipped with the moment map \(\mu: M \to \Delta \subset \mathbb{R}^n\) which is specified as above. Assume that \(\dim M = n \geq 3\). Then for any \(a \in D_n\) there exists a constant \(c_a > 0\) satisfying the following properties: For any \(c \in (0, c_a)\)
\[
(1) \ ca, ca' \in \Delta^0, \\
(2) \ \phi_c(\mu^{-1}(ca)) = \mu^{-1}(ca') \text{ for some } \phi_c \in \text{Ham}(M, \omega), \\
(3) \ \text{Vol}(\mu^{-1}(ca)) > \text{Vol}(\mu^{-1}(ca'))
\]
hold, where the vector \(a' \in (\mathbb{R}_{>0})^n\) is the one given by Theorem 5.

In particular, for any \(a \in D_n\) the Lagrangian tori \(\mu^{-1}(ca), c \in (0, c_a)\), are not Hamiltonian volume minimizing in \(M\).

A proof of Theorem 6 is based on Theorem 5 and given in Section 3. It gives a remarkable consequence in the case of complex projective space \(\mathbb{C}P^n\) endowed with the standard Fubini-Study Kähler form \(\omega_{FS}\). Let us review the following

**Proposition 7.** A Lagrangian torus \(T^{n_1} \times \cdots \times T^{n_k}\) in \((\mathbb{C}P^{n_1}, \omega_{FS}, J) \times \cdots \times (\mathbb{C}P^{n_k}, \omega_{FS}, J)\) is H-minimal and H-stable, where each factor \(T^{n_i} \subset (\mathbb{C}P^{n_i}, \omega_{FS}, J)\) is a Lagrangian torus orbit with the induced flat metric from \(\mathbb{C}P^{n_i}\).

Note that Proposition 7 is proved in [8, Section 4] only in the case where \(k = 1, 2\), however, its proof is easily extended to the general case. Combining it with Theorem 6 we obtain
Corollary 8. There exist infinitely many Lagrangian tori $T^{n_1} \times \cdots \times T^{n_k}$ in $(\mathbb{C}P^{n_1}, \omega_{FS}, J) \times \cdots \times (\mathbb{C}P^{n_k}, \omega_{FS}, J)$ which are $H$-minimal and $H$-stable but not Hamiltonian volume minimizing if $n_1 + \cdots + n_k \geq 3$.

In particular, Conjecture 1.4 in [8] is false for $n \geq 3$.

2. Chekanov’s Theorem and the proof of Theorem 5

Let $L$ and $L'$ be Lagrangian submanifolds of $(\mathbb{C}^n, \omega_0)$. Then $L$ is said to be \textit{Hamiltonian isotopic} to $L'$ if there exists $\phi \in \text{Ham}_c(\mathbb{C}^n, \omega_0)$ such that $\phi(L) = L'$. For a standard torus $L = T(b_1, \ldots, b_n)$, we define the following three invariants:

$$s(L) := \min_i b_i, \quad m_s(L) := \#\{ i \mid b_i = s(L) \},$$

$$\Gamma(L) := \text{span}_\mathbb{Z}(b_1 - s(L), \ldots, b_n - s(L)) \subset \mathbb{R}.$$

These invariants are sufficient to determine whether given two standard tori are Hamiltonian isotopic or not.

\textbf{Theorem 9 (Chekanov [3]).} For standard tori $L$ and $L'$ of $(\mathbb{C}^n, \omega_0)$, $L$ is Hamiltonian isotopic to $L'$ if and only if

$$s(L) = s(L'), \quad m_s(L) = m_s(L'), \quad \Gamma(L) = \Gamma(L')$$

hold.

Using it we shall prove our main theorem.

\textit{Proof of Theorem 5} It suffices to consider the case where $a_1 \leq a_2 \leq \cdots \leq a_n$, because $L := \mu_0^{-1}(a_1, \ldots, a_n)$ and $\mu_0^{-1}(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$ are Hamiltonian isotopic and have the same volume for any permutation $\sigma$. By assumption, there exists indices $k, l \in \{1, 2, \ldots, n-1\}$ ($k < l$) such that

$$a := a_1 = \cdots = a_k < a_{k+1}, \quad b := a_l < a_{l+1} := c.$$

Let us first suppose that $(c-a)/(b-a) \in \mathbb{Q}$. We represent it as $q/p$, where $p$ and $q$ are coprime positive integers. Then there exist integers $M$ and $N$ satisfying $Mp + Nq = 1$, and hence

$$\begin{pmatrix} M & N \\ M - q & N + p \end{pmatrix} \in SL(2; \mathbb{Z}).$$

Let

$$\begin{pmatrix} b' \\ c' \end{pmatrix} := \begin{pmatrix} M & N \\ M - q & N + p \end{pmatrix} \begin{pmatrix} b - a \\ c - a \end{pmatrix} + \begin{pmatrix} a \end{pmatrix}.$$

Since $p \geq 1$, we obtain

$$b' = c' = \frac{b - a}{p} + a \in (a, b].$$
Here we put $a' := (a_1, \ldots, a_{l-1}, b', c', a_{l+2}, \ldots, a_n)$ and consider another standard torus $L' := \mu_0^{-1}(a')$. Then we have $s(L) = s(L') = 2\pi a$, $m_s(L) = m_s(L') = k$ and $\Gamma(L) = \Gamma(L')$. By Theorem 9, $L$ is Hamiltonian isotopic to $L'$. Moreover, $b'c' < bc$ yields $\text{Vol}(L) = (2\pi)^n \prod_{i=1}^n a_i > (2\pi)^n \frac{b'c'}{bc} \prod_{i=1}^n a_i = \text{Vol}(L')$.

Therefore $L$ is not Hamiltonian volume minimizing.

Now we are going to the case where $(c - a)/(b - a) \notin \mathbb{Q}$. It is represented by the binary notation:

\[
\frac{c - a}{b - a} = \sum_{i=0}^{\infty} 2^{-i} m_i, \quad m_0 \in \mathbb{N}, \quad m_i \in \{0, 1\} \text{ for all } i \geq 1.
\]

Since $(c - a)/(b - a)$ is an irrational number, there are infinitely many indices $k$ such that $m_k = 0$ and $m_{k+1} = 1$. We fix such an index $k$ and set

\[
\sum_{i=0}^{k} 2^{-i} m_i = \sum_{i=0}^{k-1} 2^{-i} m_i =: \frac{q}{p}, \quad \sum_{i=k+1}^{\infty} 2^{-i} m_i =: r,
\]

where $p$ and $q$ are coprime positive integers. Then we obtain

\[
\frac{c - a}{b - a} = \frac{q}{p} + r, \quad 1 \leq p \leq 2^{k-1} \quad \text{and} \quad 0 < r < 2^{-k}.
\]

Next, let $(M_0, N_0)$ be a pair of integers satisfying $M_0 p + N_0 q = 1$. Then all solutions $(M, N) \in \mathbb{Z}^2$ of the equation $M p + N q = 1$ are given by $(M_l, N_l) := (M_0 + l q, N_0 - l p)$, $l \in \mathbb{Z}$.

**Lemma 10.** There exists $l \in \mathbb{Z}$ satisfying

\[
(2.1) \quad 0 < M_l (b - a) + N_l (c - a) < \frac{b - a}{2}
\]

and

\[
(2.2) \quad 0 < M_l (b - a) + N_l (c - a) + pr (b - a) < b - a.
\]

**Proof.** Notice that

\[
M_l (b - a) + N_l (c - a) = \frac{b - a}{p} (M_l p + N_l \frac{c - a}{b - a} p) = \frac{b - a}{p} (1 + N_l pr).
\]

Since $N_l$ has period $p$ and $1 + N_l pr$ is irrational for each $l$, there exists $l \in \mathbb{Z}$ such that

\[
0 < 1 + N_l pr < p^2 r
\]

holds. For such $l$, we have

\[
0 < M_l (b - a) + N_l (c - a) < pr (b - a).
\]
Since $0 < pr < 1/2$, (2.1) holds, and it immediately yields (2.2). □

Finally, for the integer $l$ in Lemma [10] if we put
\[
\begin{pmatrix} b' \\ c' \end{pmatrix} := \begin{pmatrix} M_l & N_l \\ M_l - q & N_l + p \end{pmatrix} \begin{pmatrix} b - a \\ c - a \end{pmatrix} + \begin{pmatrix} a \\ a \end{pmatrix},
\]
then we can easily verify that
\[
a < b' < \frac{b - a}{2} + a < b, \quad a < c' < b < c.
\]
The rest of the argument is completely same as the case where $(c - a)/(b - a)$ is rational. □

3. The case of toric Kähler manifolds

In this section, we keep all notations as in Section 1. For a complex $n$-dimensional compact toric Kähler manifold $(M, \omega, J)$, the point $\mu^{-1}(0) \in M$ is a fixed point of the $(\mathbb{C}^\times)^n$-action. By the construction, there exists a toric affine neighbourhood $U$ of $\mu^{-1}(0)$ such that $(U, \mu^{-1}(0))$ is isomorphic to $(\mathbb{C}^n, 0)$ as $(\mathbb{C}^\times)^n$-spaces. Using this identification we can define the standard complex coordinates $(w_1, \ldots, w_n)$ on $U$. Their polar coordinates are given by $w_i = r_i e^{\sqrt{-1}\theta_i}, i = 1, \ldots, n$.

As a set $U$ is described as
\[
U = M \setminus \mu^{-1}(F), \quad F := \bigcup_{F: \text{facet of } \Delta, 0 \notin F} F.
\]
The restriction of the Kähler form $\omega$ on $U$ can be expressed as
\[
\omega|_U = 2\sqrt{-1}\partial \bar{\partial} \varphi,
\]
where $\varphi$ is a real-valued function defined on $(\mathbb{R}_{\geq 0})^n$ (see [1], [2]). Then the moment map $\mu : M \to \Delta$ is represented as
\[
\mu(p) = \left( r_1 \frac{\partial \varphi}{\partial r_1}, \ldots, r_n \frac{\partial \varphi}{\partial r_n} \right)(p) =: (x_1, \ldots, x_n) = x.
\]
Putting $u_i := \sqrt{2x_i} \cos \theta_i$ and $v_i := \sqrt{2x_i} \sin \theta_i$, a straightforward calculation yields
\[
\omega = \sum_{i=1}^n du_i \wedge dv_i = \sum_{i=1}^n dx_i \wedge d\theta_i, \quad \mu = \left( \frac{1}{2}(u_1^2 + v_1^2), \ldots, \frac{1}{2}(u_n^2 + v_n^2) \right)
\]
on $U$. Thus $(U, \omega|_U, \mu|_U)$ is isomorphic as Hamiltonian $T^n$-spaces to $(V, \omega_0|_V, \mu_0|_V)$, where $V := \mu_0^{-1}(\Delta \setminus F)$.

Now we are in a position to prove our second result.
Proof of Theorem 6. Given a vector \( a \in D_n \), according to Theorem 5 we can take \( a' \in (\mathbb{R}_{>0})^n \) and \( \phi \in \text{Ham}_c(\mathbb{C}^n, \omega_0) \) which satisfy
\[
\phi(\mu_0^{-1}(a)) = \mu_0^{-1}(a')
\]
and
\[
\text{Vol}(\mu_0^{-1}(a)) > \text{Vol}(\mu_0^{-1}(a')).
\]
Then we choose a Hamiltonian isotopy \( \{ \phi^t_H \}_{0 \leq t \leq 1} \) with \( \phi^1_H = \phi \) and consider another one defined by
\[
\mathbb{C}^n \to \mathbb{C}^n, \quad p \mapsto \sqrt{c} \phi^t_H(p/\sqrt{c})
\]
for a constant \( c > 0 \). Since the map \( \vartheta : p \mapsto \sqrt{c} p \) is conformally symplectic, we have
\[
\vartheta \circ \phi^t_H \circ \vartheta^{-1} = \phi^t_{cH_\vartheta},
\]
where \( H_\vartheta(t, p) := H(t, \vartheta^{-1}(p)) = H(t, p/\sqrt{c}) \). For the proof, see [5, p. 152]. Moreover, we easily see from (3.3) and (3.5) that
\[
\phi^1_{cH_\vartheta}(\mu_0^{-1}(c a)) = \mu_0^{-1}(c a')
\]
and \( \text{supp}(c(H_\vartheta)_t) = c \text{supp}(H_t) \). Hence, if we choose a constant \( c_\alpha > 0 \) sufficiently small, then \( \text{supp}(c(H_\vartheta)_t) \subset V \) holds for any \( c \in (0, c_\alpha) \).

Using the action angle coordinates \((x_1, \ldots, x_n, \theta_1, \ldots, \theta_n)\) on \( U \) explained before, let us identify \((U, \omega|_U, \mu|_U)\) with \((V, \omega_0|_V, \mu_0|_V)\), and extend the Hamiltonian function \( cH_\vartheta \) on \( U \) to \( M \) as
\[
\hat{H}(t, p) := \begin{cases} 
 cH_\vartheta(t, p), & p \in U \\
 0, & p \in M \setminus U (= \mu^{-1}(\mathcal{F})).
\end{cases}
\]
Then \( \hat{H} \in C^\infty_c([0,1] \times M) \) and hence we obtain
\[
\phi^1_{\hat{H}}(\mu^{-1}(c a)) = \mu^{-1}(c a').
\]

In order to complete the proof of Theorem 6 we have to compare the volumes of two flat tori \( \mu^{-1}(c a) \) and \( \mu^{-1}(c a') \) with respect to the induced metric from the toric manifold \((M, \omega, J)\).

In general, all \( \omega \)-compatible toric complex structures on \((M, \omega)\) can be parametrized by smooth functions on \( \Delta^o \), which is shown by Abreu in [4, Section 2]. More precisely, we can choose a strictly convex function \( g \in C^\infty(\Delta^o) \) whose Hessian \( \text{Hess}_x(g) \) describes the complex structure \( J \) on \( M \). Moreover, the determinant of \( \text{Hess}_x(g) \) is given by
\[
\left\{ \delta(x) \prod_{r=1}^d l_r(x) \right\}^{-1},
\]
where $\delta \in C^\infty(\Delta)$ is a strictly positive function (see [1, Theorem 2.8]). Then the Riemannian metric of the fibre $\mu^{-1}(x) \subset M$ of $x \in \Delta^c$ is given by the $(n \times n)$-matrix $(\text{Hess}_x(g))^{-1}$, and hence

$$\text{Vol}(\mu^{-1}(ca))^2 - \text{Vol}(\mu^{-1}(ca'))^2 = (2\pi \sqrt{c})^{2n} \left\{ \delta(ca) \prod_{i=1}^{n} a_i \prod_{r=n+1}^{d} l_r(ca) - \delta(ca') \prod_{i=1}^{n} a'_i \prod_{r=n+1}^{d} l_r(ca') \right\}$$

holds. The value at $c = 0$ of the quantity of the inside of the brackets above is

$$\delta(0) \prod_{r=n+1}^{d} (-\lambda_r) \left( \prod_{i=1}^{n} a_i - \prod_{i=1}^{n} a'_i \right),$$

which is positive due to equation (3.4). Therefore, there exists a constant $d_a > 0$ such that for any $d \in (0, d_a)$

$$\text{Vol}(\mu^{-1}(da)) - \text{Vol}(\mu^{-1}(da')) > 0$$

holds. Then the positive constant $\min\{c_a, d_a\}$ fulfills all the required properties. \qed

4. Remained open problems

Finally, let us discuss the remained part of Oh’s conjecture and add some remarks. According to Theorem 5 the unsolved part of Conjecture 4 is as follows.

**Problem 11.** Let $a \leq b$ and $k = 1, 2, \ldots, n$. Is a standard torus $T(a, \ldots, a; b, \ldots, b)$ in $\mathbb{C}^n$ Hamiltonian volume minimizing?

This problem had already been considered by Anciaux in the case where $n = 2$, and he gave a partial answer to it. More precisely, he showed in [2, Main Theorem] that $T(a, a) \subset \mathbb{C}^2$ has the least volume among all $H$-minimal Lagrangian tori of its Hamiltonian isotopy class. However, this result does not imply that $T(a, a)$ is Hamiltonian volume minimizing in $\mathbb{C}^2$.

Next we turn to the case of complex projective space. In this case, each Lagrangian torus constructed in Theorem 6 is the preimage of a point $ca \in \Delta^c$ located far away from the barycentre $q$ of $\Delta$. Its fibre $\mu^{-1}(q) \subset \mathbb{C}P^n$ is a minimal Lagrangian torus and called the Clifford torus. Thus the following question raised by Oh is still open.

**Problem 12** ([6], p. 516). Is the Clifford torus in $\mathbb{C}P^n$ Hamiltonian volume minimizing?
We point out that Urbano proved that the only H-stable minimal Lagrangian torus in $\mathbb{C}P^2$ is the Clifford one (see [9, Corollary 2]).

**ACKNOWLEDGEMENTS**

The authors would like to thank Professor Yoshihiro Ohnita for suggesting us to work on this subject at GeoQuant 2013 Summer School, ESI in Vienna. The first author was partly supported by the Grant-in-Aid for Young Scientists (B) (No. 24740049), JSPS. The second author was partly supported by the Grant-in-Aid for Scientific Research (C) (No. 24540098), JSPS.

**REFERENCES**

[1] M. Abreu, *Kähler geometry of toric manifolds in symplectic coordinates*, Fields Institute Communications 35 (2003), 1–24.

[2] H. Anciaux, *An isoperimetric inequality for Hamiltonian stationary Lagrangian tori in $\mathbb{C}^2$ related to Oh’s conjecture*, Math. Z. 241 (2002), 639–664.

[3] Yu. V. Chekanov, *Lagrangian tori in a symplectic vector space and global symplectomorphisms*, Math. Z. 223 (1996), 547–559.

[4] V. Guillemin, *Kaehler structures on toric varieties*, J. Differ. Geom. 40 (1994), 285–309.

[5] H. Hofer and E. Zeihnder, *Symplectic Invariants and Hamiltonian Dynamics*, Birkhäuser (1994)

[6] Y.-G. Oh, *Second variation and stabilities of minimal lagrangian submanifolds in Kähler manifolds*, Invent. Math. 101 (1990), 501–519.

[7] Y.-G. Oh, *Volume minimization of Lagrangian submanifolds under Hamiltonian deformations*, Math. Z. 212 (1993), 175–192.

[8] H. Ono, *Hamiltonian stability of Lagrangian tori in toric Kähler manifolds*, Ann. Glob. Anal. Geom. 31 (2007), 329–343.

[9] F. Urbano, *Index of Lagrangian submanifolds of $\mathbb{C}P^n$ and the Laplacian of 1-forms*, Geom. Dedicata 48 (1993), 309–318.

[10] C. Viterbo, *Metric and isoperimetric problems in symplectic geometry*, J. Amer. Math. Soc. 13 (2000), 411–431.

H. Iriyeh
School of Science and Technology for Future Life
Tokyo Denki University
5 Senju-Asahi-Cho, Adachi-Ku
Tokyo, 120-8551 Japan
*E-mail:* hirie@im.dendai.ac.jp

H. Ono
Department of Mathematics
Saitama University
255 Shimo-Okubo, Sakura-Ku
Saitama, 380-8570 Japan
*E-mail:* hono@rimath.saitama-u.ac.jp